Survival of a static target in a gas of diffusing particles with exclusion

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I. STATEMENT OF THE PROBLEM

Suppose that at \( t = 0 \) a gas of diffusing particles of constant density \( n_0 \) be brought in contact with a spherical absorber of radius \( R \) in \( d \) dimensions. The particles are absorbed upon hitting the absorber. Remarkably, this simple setting captures the essence of many diffusion-controlled chemical kinetic processes [1–7]. The evolution of the average coarse-grained particle density of the gas is described by the diffusion equation

\[
\partial_t n = \nabla \cdot [D(n) \nabla n],
\]

where \( D(n) \) is the gas diffusivity. Here we will be interested in large fluctuations rather than in the average behavior. One important fluctuating quantity is the number of particles \( N \) that is absorbed during a long time \( T \). We will focus on two questions: (i) What is the probability that \( N = 0 \), that is no particle hit the absorber until time \( T \)? (ii) What is the most likely history of the particle density of the gas conditional on the non-hitting until time \( T \)?

These questions also appear in the context of search for an immobile target by a swarm of diffusing searchers, see e.g. Ref. [8] and references therein. This process has been extensively studied in the simplest case when the searchers are non-interacting random walkers (RWs). In this case \( D(n) = D_0 = \text{const} \), and the probability that the target survives until time \( T \), \( \mathcal{P}_{\text{RW}}(T) \) was found to exhibit the following long-time behavior [3–11]:

\[
- \ln \frac{\mathcal{P}_{\text{RW}}(T)}{n_0} \approx \begin{cases}
\frac{2(D_0 T)^{1/2}}{\sqrt{\pi}}, & d = 1, \\
\frac{4 \pi D_0 T}{\ln(D_0 T/R^2)}, & d = 2, \\
(d - 2) \Omega_d R^{d-2} D_0 T, & d > 2,
\end{cases}
\]

where \( \Omega_d = 2 \pi^{d/2}/\Gamma(d/2) \) is the surface area of the \( d \)-dimensional unit sphere, and \( \Gamma(z) \) is the gamma function.

Equations (3) and (4) give the leading terms of the corresponding asymptotics at long times, when \( \ell = R/\sqrt{D_0 T} \ll 1 \), i.e., the characteristic diffusion length \( \sqrt{D_0 T} \) is very large compared to the target radius \( R \). Equation (2) is independent of \( R \), and the parameter \( \ell \) is irrelevant. As a result, Eq. (2) becomes valid as soon as \( T \) is much larger than the inverse microscopic hopping rate.

The target survival problem is a particular case of a more general problem of finding the complete statistics of particle absorption by the absorber. For the RWs, this problem has been recently studied in Ref. [18].

Here we extend the target problem in several directions. First, we consider a lattice gas of interacting searchers. Throughout most of the paper we assume that the searchers interact via exclusion. This can be a good simplistic model for studying diffusion-controlled chemical reactions in crowded environments such as living cell [19]. Specifically, we will consider a lattice gas described by the symmetric simple exclusion process (SSEP). In this process each particle can hop to a neighboring lattice site if that site is unoccupied by another particle. If it is occupied, the move is disallowed. The average behavior of this gas is still described by the diffusion equation with \( D = D_0 = \text{const} \) [20], so the SSEP and the RWs are indistinguishable at the level of averages. However, as we show here, the long-time asymptotic of the target survival probability \( \mathcal{P}(T) \) for the SSEP behaves differently than that for the RWs:

\[
- \ln \mathcal{P} \simeq (d - 2) \Omega_d R^{d-2} D_0 T \arcsin^2 \sqrt{n_0}, \quad d > 2.
\]

This expression has the same structure as Eq. (4), but increases much faster with the gas density \( n_0 \) [21], see Fig. 1. We note that previous results for the SSEP only included bounds on \( \mathcal{P} \) [22].
Second, we show that, for $d = 1$, the survival probability $\mathcal{P}(T)$ strongly depends on the initial condition. This effect does not require inter-particle interaction, it also occurs for the RWs as we show below. In particular, the asymptotic (2) is only valid after averaging over random initial distributions of particles, that is, for the annealed setting [23, 24]. We find a different result for a deterministic initial condition, also called quenched setting [23, 24]. For the RWs the two results for $\ln \mathcal{P}(T)$ differ by a numerical factor. For the SSEP even their $n_0$-dependence is different for $d = 1$.

Third, we demonstrate that the two basic one-dimensional solutions, annealed and quenched, play a central role in higher dimensions when one is interested in intermediate asymptotics of $\mathcal{P}(T)$ for $\ell \gg 1$, that is when the diffusion length $\sqrt{D_0 T}$ is much longer than the lattice constant, but much shorter than the absorber radius $R$.

Fourth, in addition to evaluating $\mathcal{P}(T)$ in different regimes, we also find the most likely history of the gas density conditional on the target survival until time $T$. We achieve this result, and most of the others, by employing the macroscopic fluctuation theory (MFT) [22]. This coarse-grained large-deviation formalism was unavailable when most of the studies of the target survival probability were performed. The MFT is well suited for the analysis of large deviations in lattice gases, including the class of additional interacting diffusive gases.

In the next section we present the MFT formulation of the target survival problem. Sec. IV deals with $d \geq 2$ for $\ell \ll 1$. Here $\ln \mathcal{P}(T)$ is mostly contributed to by a stationary solution of the MFT equations, independently of whether the setting is annealed or quenched. We derive these solutions, evaluate $\ln \mathcal{P}$ and verify the results for $d = 3$ by solving the MFT problem numerically. In Sec. V we study analytically and numerically the survival probability in non-stationary settings, deterministic and random, in all dimensions and at different densities. In Sec. VI we extend our results for $d \geq 2$ to a broad class of interacting lattice gases. Our main results are summarized in Sec. VII. In Appendix we present, for non-interacting RWs, exact microscopic derivations of $\mathcal{P}(T)$ for the annealed and quenched settings and for $d = 1, 2$ and 3. Both the microscopic derivation and the MFT calculations show that, for $\ell \ll 1$, the leading contribution to $\ln \mathcal{P}(T)$ is sensitive to the initial condition only in one dimension.

FIG. 1: (Color online) The function $\arcsin^2 \sqrt{n_0}$ (the solid line) which describes the density dependence of $-\ln \mathcal{P}(T)$ for the SSEP at $d > 2$. The straight line shows the corresponding density dependence for the gas of random walkers (RW).

II. MACROSCOPIC FLUCTUATION THEORY OF TARGET SURVIVAL

The macroscopic fluctuation theory (MFT) was developed for the analysis of non-equilibrium steady states of diffusive lattice gases [26–30]. Subsequently it was extended to a host of non-stationary settings [18, 23, 31–33]. The MFT, and its extensions to reacting particle systems [36, 37], have proven to be highly efficient and versatile. Here we outline the MFT formulation, refereing the reader to the above references for further details.

The starting point for the derivation of the MFT can be a Langevin equation that provides a faithful large-scale description to a broad family of diffusive gases:

$$\partial_t n = \nabla \cdot [D(n)\nabla n] + \nabla \cdot \left[ \sqrt{\sigma(n)} \eta(x,t) \right], \quad (7)$$

where $\eta(x,t)$ is a zero-average Gaussian noise, delta-correlated both in space and in time [20]. As one can see, a fluctuating diffusive gas is fully characterized by $D(n)$ and another coefficient, $\sigma(n)$, that comes from the shot noise and is equal to twice the mobility of the gas [20]. Essentially, the MFT formalism is a WKB theory (after Wentzel, Kramers and Brillouin) of the functional Fokker-Planck equation following from the Langevin equation (7). The WKB theory employs, in a smart way, the typical number of particles in the relevant region of space as a large parameter $[23, 24, 27, 32]$. In the MFT formalism, the particle number density field $n(x,t)$ and the canonically conjugate “momentum” den-
sity field \( p(x, t) \) obey Hamilton equations

\[
\partial_t q = \nabla \cdot [D(q)\nabla q - \sigma(q)\nabla p], \quad (8)
\]
\[
\partial_t p = -D(q)\nabla^2 p - \frac{1}{2} \sigma'(q)(\nabla p)^2, \quad (9)
\]

where the prime denotes the derivative with respect to the argument. Equations (8) and (9) can be written in terms of variational derivatives:

\[
\partial_t q = \frac{\delta H}{\delta p}, \quad \partial_t p = -\frac{\delta H}{\delta q}. \quad (10)
\]

Here

\[
H[q(x, t), p(x, t)] = \int dx \mathcal{H}
\]

is the Hamiltonian, and

\[
\mathcal{H}(q, p) = -D(q)\nabla q \cdot \nabla p + \frac{1}{2} \sigma(q)(\nabla p)^2
\]

is the Hamiltonian density. The spatial integration in Eq. (11), and everywhere in the following, is performed over the whole space outside the target. Because of the rotational symmetry of the problem, we assume that the solution only depends on the radial coordinate and time. We will consider the target survival problem in an arbitrary dimension \( d \). The boundary conditions on the target are \( q(r = R, t) = p(r = R, t) = 0 \) \cite{18}, where the condition on \( p(r = R, t) \) just fixes an arbitrary constant. Far away from the target the gas is unperturbed, so \( q(r = \infty, t) = n_0 \). The boundary conditions in time are the following. At \( t = 0 \) we prescribe

\[
q(r > R, t = 0) = n_0, \quad (13)
\]

where, for the SSEP, \( 0 < n_0 < 1 \). This is a deterministic, or quenched initial condition, see Refs. \cite{18 23 24 32 33}. A random initial condition (that is, annealed setting) is considered in Sec. IV C. Before focusing on the target survival problem, let us consider for a moment a slightly different setting where \( N \), the specified number of absorbed particles by time \( t = T \), is arbitrary. This condition,

\[
\Omega_d \int_R^\infty dt r^{d-1} [n_0 - q(r, T)] = N, \quad (14)
\]

imposes an integral constraint on the solution. This constraint is identical to the one arising in the problem of statistics of integrated current during a specified time \cite{18 23 32 34 35}. A similar derivation yields the following boundary condition for \( p \) at \( t = T \):

\[
p(r, t = T) = \lambda \theta(r - R), \quad (15)
\]

where \( \theta(\ldots) \) is the Heaviside step function, and \( \lambda \) is an a priori known Lagrange multiplier that is ultimately set by Eq. (14) \cite{18 23}. Accordingly, we demand \( p(r = \infty, t) = \lambda \). The particular case of \( N = 0 \) that we are interested in here corresponds to \( \lambda \to +\infty \) \cite{18}. In this case the total particle flux to the target vanishes at all times \( 0 < t < T \).

The solution of the MFT equations for \( q(r, t) \) yields the optimal trajectory: the most likely density history of the system conditional on the number of absorbed particles \( N \). Once \( q(r, t) \) and \( p(r, t) \) are found, we can calculate the mechanical action \( S \) which yields \( \ln \mathcal{P}(N) \) up to a pre-exponential factor:

\[
-\ln \mathcal{P} \simeq S = \Omega_d \int_0^T dt \int_R^\infty dr r^{d-1} (p\partial_t q - \mathcal{H})
\]

\[
= \frac{1}{2} \Omega_d \int_0^T dt \int_R^\infty dr r^{d-1} \sigma(q)(\partial_r p)^2. \quad (16)
\]

For the SSEP \( D(q) = D_0 = \text{const} \) and \( \sigma(q) = 2D_0q(1-q) \) \cite{20}, and Eq. (10) becomes

\[
-\ln \mathcal{P} \simeq S = \Omega_d D_0 \int_0^T dt \int_R^\infty dr r^{d-1} q(1-q)(\partial_r p)^2. \quad (17)
\]

Upon rescaling \( t \) by \( T \) and \( r \) by \( \sqrt{D_0 T} \) \cite{38}, we can effectively put \( T = 1 \) in Eqs. (13) and (15) and replace \( R \) by \( \ell / \sqrt{D_0 T} \) and \( N \) by \( \nu = N/(D_0 T)^{d/2} \) everywhere. Equation (17) for the SSEP becomes

\[
-\ln \mathcal{P} \simeq (D_0 T)^{d/2} s(\ell, \nu, n_0), \quad (18)
\]

where

\[
s = \Omega_d \int_0^{1/\ell} dt \int_\ell^\infty dr r^{d-1} q(1-q)(\partial_r p)^2. \quad (19)
\]

We are interested in the limit of \( s(\ell, \nu, n_0) \) as \( \nu \to 0 \). In one spatial dimension, \( d = 1 \), the parameter \( R \) (and hence \( \ell = R/\sqrt{D_0 T} \)) is irrelevant because of the translational symmetry of the ensuing MFT problem. We will consider this case in Sec. IV A. For \( d \geq 2 \) there are two natural limiting cases: of small and large \( \ell \).

**III. \( \ell \ll 1 \): QUASI-STATIONARY FLUCTUATIONS**

**A. \( d > 2 \)**

A small \( \ell \) in the deterministic theory, described by Eq. (11), means that \( T \) is much longer than the characteristic diffusion time \( R^2/D_0 \) needed for the gas density to approach a steady state around the target. As a result, the average particle flux to the target can be determined by using the stationary solution of the diffusion equation. For \( D(n) = D_0 = \text{const} \) this reduces to solving the Laplace equation \( \nabla^2 n = 0 \) with the boundary conditions \( n(r = R) = 0 \) and \( n(r = \infty) = n_0 \), leading to

\[
n(r) = n_0 \left( 1 - \frac{r^{d-2}}{r_0^{d-2}} \right), \quad d > 2. \quad (20)
\]

We argue that same logic holds for fluctuations, including those responsible for the survival probability. Hence,
when $\ell \ll 1$, the leading order contribution to the action $S$ from Eq. (10) comes from the stationary solution of the MFT equations that obeys the boundary conditions in space, but not the boundary conditions in time. For such solutions Eqs. (2) and (3) become

$$-D(q) \frac{dq}{dr} + \sigma(q) v = j = \text{const},$$

(21)

$$\frac{D(q)}{r^{d-1}} \frac{d}{dr} (r^{d-1} v) + \frac{1}{2} \sigma'(q) v^2 = 0,$$

(22)

where $v(r) \equiv dp/dr$. The target survival implies that the particle flux at $r = \ell$ vanishes at all times $0 < t < T$. Therefore $j = 0$ and from Eq. (21) $v = (D/\sigma)(dq/dr)$. Plugging this into Eq. (22) we obtain

$$\nabla_r^2 q + \left( \frac{D'}{D} - \frac{\sigma'}{2\sigma} \right) \left( \frac{dq}{dr} \right)^2 = 0,$$

(23)

where

$$\nabla_r^2 = \frac{1}{r^{d-1}} \frac{d}{dr} \left( r^{d-1} \frac{d}{dr} \right)$$

is the spherically symmetric Laplace operator in $d$ dimensions. For the SSEP Eq. (24) reads

$$\nabla_r^2 q + \frac{2q - 1}{2q(1 - q)} \left( \frac{dq}{dr} \right)^2 = 0.$$  

(24)

Remarkably, the substitution $q(r) = \sin^2 u(r)$ reduces the nonlinear ordinary differential equation (24) to the spherically symmetric Laplace equation in $d$ dimensions:

$$\nabla_r^2 u = 0.$$  

(25)

The boundary conditions $q(\ell) = 0$ and $q(\infty) = n_0$ become $u(\ell) = 0$ and $u(\infty) = \arcsin \sqrt{n_0}$. Solving this problem and returning to $q$, we obtain Eq. (6). This is the most likely density profile conditional on survival of the target until time $T$. Now we can calculate $v(r)$:

$$v(r) = \frac{1}{2q(1 - q)} \frac{dq}{dr} = \frac{2(d - 2) \ell^{d-2} \arcsin \sqrt{n_0}}{r^{d-1} \sin \left[ \frac{2}{(1 - \frac{\ell^{d-2}}{r^{d-2}}) \arcsin \sqrt{n_0}} \right].}$$

(26)

In particular, for $d = 3$

$$q(r) = \sin^2 \left[ \frac{1 - \ell}{r} \arcsin \sqrt{n_0} \right],$$

(27)

$$v(r) = \frac{2 \ell \arcsin \sqrt{n_0}}{r^2 \sin \left[ \frac{2}{(1 - \frac{\ell}{r}) \arcsin \sqrt{n_0}} \right].}$$

(28)

The asymptotic of $q(r)$ near the target,

$$q(r - \ell \ll \ell) \simeq (d - 2) \ell^2 \arcsin^2 \sqrt{n_0} \left( \frac{r}{\ell} - 1 \right)^2,$$

(29)

is quadratic in $r - \ell$. Also notable is a diverging asymptotic of $v(r) = dp/dr$ near the target:

$$v(r - \ell \ll \ell) \simeq \frac{1}{r - \ell}$$

(30)

which is independent of $n_0$. The asymptotic behaviors near the target assure that the particle flux to the target vanishes. Furthermore, each of the two terms in the flux, see Eq. (21), vanish separately. As it turns out, these features, including the 'one over the distance' asymptotic (30), are quite universal: they are observed, for $0 < t < 1$, in the quenched and annealed settings and in all dimensions (including $d = 1$ where the MFT solution is non-stationary) for all lattice gases that behave as non-interacting RWs at low densities. An example of the stationary gas density profile for $d = 3$ is shown in Fig. 2.

In spite of the singularity of $v(r)$ at $r = \ell$, the action (19) is bounded, and we obtain

$$s = (d - 2) \Omega_d \ell^{d-2} \arcsin^2 \sqrt{n_0}, \quad d > 2,$$

(31)

and arrive at Eq. (15). In particular, for $d = 3$

$$s = 4\pi \ell \arcsin^2 \sqrt{n_0}$$

(32)

and

$$- \ln \mathcal{P} \simeq S = 4\pi R\ell T \arcsin^2 \sqrt{n_0}. \quad (33)$$

Notice that as $n_0$ approaches 1, the asymptotic survival probability goes down rapidly but remains non-zero.

As the solution (6) and (26) is stationary, the survival probability is independent, in the leading order, of whether the particle are distributed randomly or deterministically at $t = 0$. Here for very long times, $D_{\ell} T \gg R^2$, the optimal fluctuation becomes unconstrained by the process duration, and details of the initial condition become irrelevant. As we will see in Sec. IV, the situation changes for $d = 1$, and for any $d$ when $\ell \ll 1$.

For $n_0 \ll 1$, Eq. (33) reduces to Eq. (4) for the RWs. Further, Eqs. (6) and (26) become

$$q(r) = n_0 \left( 1 - \frac{\ell^{d-2}}{r^{d-2}} \right)^2,$$

(34)

$$v(r) = \frac{d - 2}{r} \left[ \frac{r}{\ell} \right]^{d-2} - 1 \right].$$

(35)

These low-density asymptotics for the SSEP represent exact solutions for the RWs, where $D(q) = D_0 = \text{const}$ and $\sigma(q) = 2D_0 q$ (24).

The stationary solution (6) and (26), or (34) and (35), does not satisfy the boundary conditions in time. To accommodate these boundary conditions, the full time-dependent solutions of the MFT problem must develop narrow boundary layers in time at $t = 0$ and $t = 1$, cf. Ref. [42]. The boundary layers only give a subleading
A. $d = 1$, deterministic initial condition

For the SSEP in one dimension Eqs. (8) and (9) can be written as

$$\partial_t q = \partial_x^2 q - 2\partial_x [q(1-q)\partial_x p],$$

$$\partial_t p = -\partial_x^2 p + (2q - 1)(\partial_x p)^2,$$

whereas the Hamiltonian density (12) becomes

$$\mathcal{H}(q, p) = -\partial_x p \partial_x q + q(1-q)(\partial_x p)^2.$$
Here, and in most of the following exposition on the SSEP and RW, we put \( D_0 = 1 \). We will consider a one-sided problem and put the absorbing wall at \( x = 0 \), so that \( q(x = 0, t) = p(x = 0, t) = 0 \). We assume a deterministic initial condition,

\[
q(x > 0, t = 0) = n_0, \quad 0 < n_0 < 1,
\]

and demand \( q(x = \infty, t) = n_0 \). Upon rescaling \( t \) by \( T \), Eq. (15) becomes

\[
p(x, t = 1) = \lambda \theta(x),
\]

and we also have \( p(x = \infty, t) = \lambda \). We remind the reader that \( \lambda \) is ultimately set by the number of absorbed particles: when this number goes to zero, \( \lambda \to \infty \).

Once \( q(x, t) \) and \( p(x, t) \) are found, we obtain

\[
- \ln P \simeq \sqrt{T} s_1(n_0),
\]

\[
s_1(n_0) = \int_0^1 dt \int_0^\infty dx \, q(1 - q) (\partial_x p)^2,
\]

where the subscript in \( s_1 \) refers to \( d = 1 \).

We have been unable to solve this problem exactly for arbitrary \( n_0 \). In the following we solve it in the limit of \( n_0 \ll 1 \), when the SSEP reduces to RWs. Based on these results, we then compute the next-order correction in \( n_0 \) perturbatively. At the end of this subsection we solve the problem numerically for a range of values of \( n_0 \).

1. Low-density limit: non-interacting random walkers

In the limit of \( n_0 \ll 1 \) we can drop \( h_1 = -q^2(\partial_x p)^2 \) in the Hamiltonian density \( \mathcal{H} \), and the corresponding terms in the MFT equations, arriving at the RW model. As in other examples \[18, 22, 32\], the MFT problem for the RW is solvable by the Hopf-Cole transformation \( Q = q e^{-p} \) and \( P = e^p \). This is because, in the new variables, the Hamilton equations are decoupled:

\[
\partial_t Q = \partial_x^2 Q, \quad \partial_t P = -\partial_x^2 P.
\]

We can solve the anti-diffusion equation \[17\] backward in time, with the initial condition \( P(x, T) = 1 + (e^\lambda - 1) \theta(x) \) and the boundary conditions \( P(0, t) = 1 \) and \( P(\infty, t) = e^\lambda \). The solution is

\[
P(x, t) = 1 + (e^\lambda - 1) \text{erf} \left( \frac{x}{\sqrt{1-t}} \right),
\]

where \( X = x/2 \), and \( \text{erf} z = (2/\sqrt{\pi}) \int_0^z e^{-u^2} du \) is the error function. At \( t = 0 \) we obtain

\[
Q(x, 0) = \frac{q(x, 0)}{P(x, 0)} = \frac{n_0}{1 + (e^\lambda - 1) \text{erf} X}.
\]

This expression is the initial condition for the diffusion equation \[40\] forward in time. The boundary conditions are \( Q(0, t) = q(0, t)/P(0, t) = 0 \) and \( Q(\infty, t) = 0 \). The solution is

\[
Q(x, t) = \frac{1}{\sqrt{\pi t}} \int_0^\infty d\mu \, e^{-\frac{(X-\mu)^2}{1 + (e^\lambda - 1) \text{erf} \mu}}.
\]

Transforming back to \( q \) and \( p \), and taking the limit of \( \lambda \to \infty \), we obtain

\[
q(x, t) = \frac{n_0}{\sqrt{\pi t}} \text{erf} \left( \frac{X}{\sqrt{1-t}} \right) \times \int_0^\infty d\mu \, e^{-\frac{(X-\mu)^2}{1 + (e^\lambda - 1) \text{erf} \mu}} e^{-\frac{(X+\mu)^2}{1 + (e^\lambda - 1) \text{erf} \mu}},
\]

\[
v(x, t) = \partial_x p(x, t) = \frac{e^{-\frac{X^2}{1-t}}}{\sqrt{\pi(1-t)} \text{erf} \left( \frac{X}{\sqrt{1-t}} \right)}.
\]

Figure 3 depicts the density history of the system as described by Eq. \[50\]. The lower panel shows a density “void” that forms immediately. Also noteworthy is a density peak that accompanies the void formation. To the right of the density peak \( v \) is very small, and the dynamics is essentially governed by the deterministic equation \[1\] and corresponds to a diffusive outflow of the gas. At \( t = 1 \) we obtain

\[
q(x, 1) = \frac{n_0}{\sqrt{\pi}} \int_0^\infty d\mu \, e^{-\frac{(X-\mu)^2}{1 + (e^\lambda - 1) \text{erf} \mu}} e^{-\frac{(X+\mu)^2}{1 + (e^\lambda - 1) \text{erf} \mu}},
\]

As one can see, \( q(x, 1) \) behaves linearly in \( x \) at small \( x \). At \( 0 < t < 1 \), however, \( q(x, 1) \) is quadratic at small \( x \), as in the stationary solution derived above. Now, \( v(x, t) \), as described by Eq. \[51\], again exhibits the universal ‘one over the distance’ asymptotic. Indeed, at \( x \ll \sqrt{1-t} \),

\[
v \simeq \frac{1}{x},
\]

independent of time. This asymptotic already holds at \( t = 0 \). The character of singularity at \( x = 0 \) only changes at \( t = 1 \), as \( v(x, t = 1) \) is equal to \( \delta(x) \) with an infinite prefactor \( \lambda \to \infty \).

To compute the rescaled action \( s_1^\text{RW} \) we write

\[
s_1^\text{RW} = \int_0^1 \int_0^\infty dx \, q(x, t) v(x, t)^2,
\]

which follows from Eq. \[45\] at \( q \ll 1 \). It is more convenient, however, to use the formula (derived in \[18\]) which only includes spatial integration. For \( N = 0 \) this formula simplifies to

\[
s_1^\text{RW} = \int_0^\infty dx \left[ q(x, 1) \ln P(x, 1) - q(x, 0) \ln P(x, 0) \right].
\]

After cancelations we obtain

\[
s_1^\text{RW} = -2n_0 \int_0^\infty d\mu \ln \text{erf} \mu = -\Lambda_1 n_0,
\]
2. Finite-density correction

Now let us go back to the SSEP and consider a small but finite $n_0$. We can calculate a small correction $\delta s$ to the action [50] by treating the term $h_1 = -q_1^2(\partial_x p)^2$ of the SSEP Hamiltonian [11] perturbatively. In the first order of perturbation theory we have

$$\delta s = - \int_0^1 dt \int_0^\infty dx h_1[q_0(x, t), p_0(x, t)]$$
$$= \int_0^1 dt \int_0^\infty dx q_0^2(x, t)v_0^2(x, t),$$  \hspace{1cm} \text{(58)}

where $q_0(x, t)$ and $v_0(x, t)$ are the unperturbed solutions, given by the RW formulas [50] and [51]. Plugging Eqs. [50] and [51] into Eq. [58] we obtain

$$\delta s = \frac{2n_0^2}{\pi^2} \int_0^1 dt \int_0^\infty dX \frac{e^{-\frac{3x^2}{t(1-t)}}}{\sqrt{t(1-t)}}$$
$$\times \left[ \int_0^\infty d\mu \frac{e^{-(x-\mu)^2}(\sqrt{2\mu})}{\sqrt{\pi}} \right]^2$$  \hspace{1cm} \text{(59)}

where $X = x/2$. To evaluate the above integral, we first replace the square of the integral over $\mu$ by a product of two identical integrals over $\mu_1$ and $\mu_2$. The integration over $X$ reduces to calculating Gaussian integrals:

$$\int_0^\infty dX e^{-\frac{3x^2}{t(1-t)}} \left[ e^{-(x-\mu_1)^2} - e^{-(x+\mu_1)^2} \right]$$
$$\times \left[ e^{-(x-\mu_2)^2} - e^{-(x+\mu_2)^2} \right]$$
$$= \frac{\pi}{\sqrt{2t(1-t)}} e^{\frac{1}{2}(\mu_1 + \mu_2)^2} \left( e^{2\mu_1\mu_2} - e^{2\mu_1\mu_2} \right)$$  \hspace{1cm} \text{(60)}

Now we perform integration over $t$ in Eq. [59]:

$$\int_0^1 dt \frac{\pi}{\sqrt{2t(1-t)}} e^{\frac{1}{2}(\mu_1 + \mu_2)^2} \left( e^{2\mu_1\mu_2} - e^{2\mu_1\mu_2} \right)$$
$$= \frac{\pi^{3/2}}{\sqrt{2}} \left[ e^{-\frac{1}{2}(\mu_1 - \mu_2)^2} \text{erf} \left( \frac{\mu_1 - \mu_2}{\sqrt{2}} \right) \right]$$
$$- e^{-\frac{1}{2}(\mu_1 - \mu_2)^2} \text{erf} \left( \frac{\mu_1 + \mu_2}{\sqrt{2}} \right) \equiv I(\mu_1, \mu_2),$$  \hspace{1cm} \text{(61)}

where $\text{erf} z = 1 - \text{erf} z$. The remaining double integral over $\mu_1$ and $\mu_2$ is evaluated numerically to yield

$$\delta s = \frac{2n_0^2}{\pi^2} \int_0^\infty \int_0^\infty d\mu_1 d\mu_2 \frac{I(\mu_1, \mu_2)}{\text{erf} \mu_1 \text{erf} \mu_2} = \Lambda_2 n_0^2,$$  \hspace{1cm} \text{(62)}

where $\Lambda_2 = 1.08337\ldots$. Therefore,

$$-\ln \mathcal{P} = \sqrt{T} s_1(n_0), \quad s_1(n_0) = \Lambda_1 n_0 + \Lambda_2 n_0^2 + \ldots.$$  \hspace{1cm} \text{(63)}
3. Numerical solution

We solved the MFT equations using a modification of the iteration algorithm, originally developed by Chernykh and Stepanov [11] for evaluating the probability density of large negative velocity gradients in the Burgers turbulence. Variants of this algorithm have been used in the context of MFT of lattice gases, with and without on-site reactions [32, 33, 34, 37, 42]. The algorithm iterates the diffusion-type equation (8) forward in time from \( t = 0 \) to \( t = 1 \), and the anti-diffusion-type equation (9) backward in time from \( t = 1 \) to \( t = 0 \). As in Ref. [42], our implementation of this algorithm involved an implicit finite difference scheme, which is beneficial for iteration convergence. At fixed \( n_0 \) and \( \lambda \) we continued iterations until local convergence of the solutions were achieved with a high accuracy. Then we increased \( \lambda \) and repeated the solution until the action (19) converged to 1 per cent. We also verified that, for large \( \lambda \) that we achieved, the mass loss to the absorber was negligible.

Figure 4 shows an example of our numerical solution for the deterministic initial condition and \( d = 1 \). At small and moderately large \( n_0 \), the density history of the system is similar to that for RWs, with a rapidly forming density void accompanied by a density peak. The density peak is lower than for the RWs, and it becomes progressively lower and broader as \( n_0 \) approaches 1. The numerically found \( v(x, t) = \partial_x p(x, t) \) exhibits, at small \( x \), the universal asymptotic [6,5].

Figure 5 shows the numerically found \( s_1(n_0) \) for the deterministic initial condition and \( d = 1 \). For small \( n_0 \), there is an excellent agreement with the RW asymptotic [66]. For moderate \( n_0 \), the results agree with the weakly-nonlinear asymptotic [63]. As \( n_0 \) continues to grow, \( s_1 \) grows more rapidly. It must diverge at \( n_0 = 1 \), because in this case \( -\ln p(T) \) scales with time as \( T \) rather than \( \sqrt{T} \), as follows from simple microscopic arguments. Our numerical solution becomes prohibitive at \( n_0 \) very close to 1. The available data indicate the \((1 - n_0)^{-1/2}\) divergence of \( s_1 \) as \( n_0 \to 1 \).

A spherically symmetric three-dimensional version of the iteration algorithm was used for the verification of the stationary solution for \( d = 3 \), presented in Sec. [31 A]

B. \( d > 1 \), deterministic initial condition

When \( \ell \gg 1 \), Eq. (19) simplifies to

\[
s \simeq \Omega_d \ell^{d-1} \int_0^1 dt \int_t^\infty dr \, q(1 - q) \, (\partial_r p)^2.
\]

The remaining double integral is equal to the rescaled action \( s_1(n_0) \) in the (rescaled) one-dimensional problem, \( r \equiv x \), with an absorber at \( x = \ell \). Because of the translational invariance, \( s_1 = s_1(n_0) \) is independent of \( \ell \).

![Figure 4](image_url) (Color online) Numerically computed most likely density history of the SSEP for \( d = 1 \), conditional on the zero flux to the target (located at \( x = 0 \)) for a deterministic initial condition with density \( n_0 = 0.8 \). Upper panel: \( q \) versus \( x \) at \( t = 0 \) (dashed line), 0.25 (dash-dotted line), 0.5 (dotted line) and 1 (solid line). Lower panel: numerically computed \( v = \partial_x p \) versus \( x \) at \( t = 0.25 \) (dash-dotted line), 0.5 (dotted line) and 0.75 (thin solid line). The thick solid line shows the universal asymptotic \( v = 1/x \). The coordinate \( x \) is rescaled by \( \sqrt{T} \) (here \( D_0 = 1 \)), time is rescaled by \( T \).

![Figure 5](image_url) (Color online) The function \( s_1(n_0) \) found numerically for the deterministic initial condition. Shown are numerical data (points), the RW asymptotic [69] (the dotted line) and the finite-density asymptotic [63] (the dashed line).
As a result,
\[ s \simeq \Omega_d \ell^{d-1} s_1(n_0) \]  
(64)
\[ -\ln \mathcal{P} \simeq S \simeq \Omega_d s_1(n_0) R^{d-1} \sqrt{T}. \]  
(65)

Here \( \ln \mathcal{P} \) is proportional to \( \sqrt{T} \), rather than \( T \). In particular, for \( d = 3 \)
\[ s \simeq 4\pi \ell^2 s_1(n_0) \]  
(66)
\[ -\ln \mathcal{P} \simeq S \simeq 4\pi s_1(n_0) R^2 \sqrt{T}. \]  
(67)

The case of \( d = 2 \) is not special here, and Eq. (65) holds:
\[ -\ln \mathcal{P} \simeq \frac{8\sqrt{\pi}}{3} s_1(n_0) R \sqrt{T} \text{ when } R \gg \sqrt{T}. \]  
(67)

\section*{C. \( d = 1 \), Random initial condition}

In the annealed setting, that we consider here, one allows equilibrium fluctuations in the initial condition and averages over them. In a stochastic realization of the process, the initial density profile is chosen from the equilibrium probability distribution corresponding to density \( n_0 \). As a consequence, the most likely initial density profile, conditional on the target survival until time \( T \), is different from the flat profile \( q = n_0 \). The “cost” of optimal fluctuation now includes the cost of creating the optimal initial density profile. Still, the total cost is less than the cost for the quenched (deterministic) initial condition, so the survival probability for the annealed setting is higher than for the quenched setting.

In the MFT formalism, the annealed setting is described, in one dimension, by the initial condition that involves a combination of \( q(x, t = 0) \) and \( p(x, t = 0) \) [23]:
\[ p(x, 0) - 2 \int_{n_0}^{q(x,0)} dq_1 \frac{D(q_1)}{\sigma(q_1)} = \lambda \theta(x). \]  
(68)

For the SSEP, \( D = D_0 = 1 \) and \( \sigma(q) = 2q(1 - q) \), this becomes
\[ p(x, 0) - \ln \frac{1 - n_0}{n_0[1 - q(x, 0)]} = \lambda \theta(x). \]  
(69)

For the RWs, \( D = D_0 = 1 \) and \( \sigma(q) = 2q \), we have
\[ p(x, 0) - \ln \frac{q(x, 0)}{n_0} = \lambda \theta(x). \]  
(70)

Equation (70) replaces Eq. (12) in Sec. IV A. When \( q(x, t) \) and \( p(x, t) \) are found, one can evaluate
\[ -\ln \mathcal{P} \simeq \sqrt{T}(s_0 + s_1). \]

Here \( s_1 \) is the action given by Eq. (45) [but with a different \( q(x, t) \), see below], whereas \( s_0 \) is the cost of creating the optimal initial condition \( q_0(x) \). This cost is given by the Boltzmann-Gibbs formula [23, 32, 33]. For the SSEP
\[ s_0 = \int_0^\infty dx \left\{ \ln \frac{1 - q_0(x)}{1 - n_0} + q_0(x) \ln \frac{q_0(x)(1 - n_0)}{n_0[1 - q_0(x)]} \right\}, \]  
(71)

whereas for the RWs
\[ s_0 = \int_0^\infty dx \left[ n_0 - q_0(x) + q_0(x) \ln \frac{q_0(x)}{n_0} \right], \]  
(72)

1. Low-density limit: non-interacting random walkers

For the RWs, the annealed problem can be solved via the Hopf-Cole transformation. In the new variables \( Q \) and \( P \), the initial condition (70) yields:
\[ Q(x > 0, t = 0) = n_0 e^{-\lambda}. \]  
(73)

Solving the diffusion equation (16) with this initial condition and the boundary conditions \( Q(0, t) = 0 \) and \( Q(\infty, t) = n_0 e^{-\lambda} \), we obtain
\[ Q(x, t) = n_0 e^{-\lambda} \text{erf} \left( \frac{X}{\sqrt{t}} \right), \]  
(74)

where \( X = x/2 \) as before. Now, \( P(x, t) \) is still described by Eq. (15). Therefore, we can calculate \( q(x, t) = Q(x, t)P(x, t) \). Sending \( \lambda \) to infinity, we arrive at
\[ q(x, t) = n_0 \text{erf} \left( \frac{X}{\sqrt{t}} \right) \text{erf} \left( \frac{X}{\sqrt{1 - t}} \right), \]  
(75)
a symmetric function of \( t - 1/2 \). There is no density peak in the annealed setting: the density is monotonically increasing with \( x \) at all times. Interestingly, at \( t = 1 \) and \( t = 0 \) the optimal density
\[ q(x, 1) = q(x, 0) = n_0 \text{erf} X, \]  
(76)
is the same as predicted by the deterministic theory, Eq. (1), at \( t = 1 \). At times \( 0 < t < 1 \), the optimal density profile \( q(x, t) \) is a quadratic function of \( x \) at small \( x \) as before.

The action \( s_1 \) is given by Eq. (54) with the same \( v(x, t) \) as in the quenched case, Eq. (57), and with \( q(x, t) \) given by Eq. (56). As in the quenched setting, it is more convenient to calculate \( s_1 \) using Eq. (55) that is equally valid in the annealed case. The cost of the initial condition \( s_0 \) can be evaluated from Eq. (72). Adding up \( s_0 \) and \( s_1 \), we obtain after cancelations
\[ s_0 + s_1 = 2n_0 \int_0^\infty d\mu \text{erfc} \mu = \frac{2n_0}{\sqrt{\pi}}, \]  
(77)

so
\[ -\ln \mathcal{P}_{\text{RW}} \simeq \Lambda_0 n_0 \sqrt{T}, \]  
(78)

with \( \Lambda_0 = 2/\sqrt{\pi} \), in agreement with previous results [16], see also Appendix. To our knowledge, the optimal density history [74] that contributes most to this survival probability, has been previously unknown.
2. Finite-density correction

Now we return to the SSEP. Assuming $n_0 \ll 1$, we can calculate a small correction $\mathcal{O}(n_0^2)$ to the expression $s_0 + s_1$ from Eq. (77). The correction to $s_1$ is again calculated from Eq. (58), where $v_0(x,t)$ is still given by Eq. (51), but $v_0(x,t)$ is now given by the annealed history, Eq. (75). We obtain

$$\delta s_1 = \frac{n_0^2}{\pi} \int_0^1 dt \int_0^\infty dx \frac{e^{-\frac{x^2}{4t}}}{1-t} \text{erf}^2 \left( \frac{x}{\sqrt{4t}} \right).$$

(79)

The integral over $x$ can be evaluated using the formula

$$\int_0^\infty d\mu e^{-b\mu^2} \text{erf}^2 \mu = \left. \frac{1}{\pi b} \arctan \left( \frac{1}{b(b+2)} \right) \right|_{b=0}, \quad b > 0.$$

The remaining integral over $t$ is elementary,

$$\int_0^1 \frac{dt}{\sqrt{1-t}} \arctan \frac{1-t}{\sqrt{4t}} = (3 - 2\sqrt{2})\pi,$$

and we obtain

$$\delta s_1 = \frac{(3\sqrt{2} - 4)n_0^2}{\sqrt{\pi}}.$$

(80)

There is also a small correction to $s_0$ that comes from the difference of free energies of the SSEP and the RWs. We calculate this correction by expanding the integrand of Eq. (71) in small $n_0$ and $q_0(x)$ up to, and including, the quadratic terms. The resulting correction is

$$\delta s_0 = \frac{1}{2} \int_0^\infty dx (q_0(x) - n_0)^2 = \frac{(2 - \sqrt{2})n_0^2}{\sqrt{\pi}}.$$

(81)

where we used the zero-order result (70) for $q_0(x)$. Adding up $\delta s_0$ and $\delta s_1$, we finally obtain, for the annealed setting,

$$\ln \mathcal{P}_{\text{ann}} \simeq \sqrt{T} s_1^{\text{ann}}(n_0),$$

$$s_1^{\text{ann}}(n_0) = \frac{2}{\sqrt{\pi}} \left[ n_0 + (\sqrt{2} - 1) n_0^2 + \ldots \right].$$

(82)

The $n_0^2$ correction agrees with the results of Santos and Schütz [43]. They solved a different problem for the SSEP, which involved particle injection from the boundary into a semi-infinite line. Remarkably, that problem can be mapped, already at the exact microscopic level, into the target survival problem we are dealing with here. As a result, the $n_0^2$ correction in the annealed setting, described by Eq. (82), corresponds to the second cumulant of the statistics of the total number of injected particles at time $t = T$, when the system is empty at $t = 0$ [44].

Overall, Eqs. (63) and (82) show that, in one dimension, the survival probability exhibits different $n_0$-dependences in the quenched and annealed settings.

V. EXTENSION TO GENERAL INTERACTING LATTICE GASES

Importantly, the steady-state equation (23) can be solved analytically for general $D(q)$ and $\sigma(q)$, thus extending our long-time results for $d \geq 2$ to a whole family of diffusive gases of interacting particles. Indeed, by denoting

$$u(r) = r^{d-1} \frac{d\sigma(r)}{dr} \quad \text{and} \quad f(r) = \ln \frac{D[q(r)]}{\sqrt{\sigma[q(r)]}}$$

we can recast Eq. (23) into a linear first order ODE,

$$\frac{du}{dr} + \frac{df}{dr} u = 0,$$

whose general solution is

$$u(r) = C \exp[-f(r)],$$

(86)

where $C = \text{const}$. Using Eq. (85), we obtain one more first-order ODE that can be easily integrated. Using the boundary conditions $q(\ell) = 0$ and $q(\infty) = n_0$ to determine the two integration constants, we obtain the solution for $q(r)$ in implicit form:

$$\frac{\int_0^q \frac{D(z)}{\sqrt{\sigma(z)}} dz}{\int_0^{n_0} \frac{D(z)}{\sqrt{\sigma(z)}} dz} = 1 - \left( \frac{\ell}{r} \right)^{d-2}, \quad d > 2.$$

(87)

This solution exists for all lattice gases for which the integrals in Eq. (87) are bounded. This puts a limitation on the behaviors of $D(q)$ and $\sigma(q)$ at $q \to 0$. For example, let $D(q \to 0) \sim q^\alpha$ and $\sigma(q \to 0) \sim q^\beta$. Then the integrals converge at $q = 0$ if and only if

$$2\alpha - \beta + 2 > 0.$$

(88)

For the SSEP and RWs one has $\alpha = 0$ and $\beta = 1$. Therefore, the condition (88) is satisfied, and the solution (87) is valid.
exists. The condition \( [88] \) is also satisfied for a family of repulsion processes \([45]\).

When the solution \([87] \) exists, the action is bounded leading to a nonzero target survival probability. The rescaled action is the following:

\[
s = \frac{1}{2} \Omega_d \int_0^{\ell_0} dr r^{d-1} \sigma v^2 \\
= \frac{1}{2} \Omega_d \int_0^{n_0} dq dq \int_0^{\ell_0} dr r^{d-1} \sigma v^2 \\
= \frac{1}{2} \left( d - 2 \right) \Omega_d \ell^{d-2} \left[ \int_0^{n_0} \frac{D(q)}{\sqrt{\sigma(q)}} dq \right]^2, \tag{89}
\]

where Eq. \([89] \) and the steady-state relation \( v = (D/\sigma)(dq/dr) \) have been used in the last step. As a result,

\[
- \ln \mathcal{P} \simeq \frac{1}{2} \left( d - 2 \right) \Omega_d R^{d-2} T \left[ \int_0^{n_0} \frac{D(q)}{\sqrt{\sigma(q)}} dq \right]^2. \tag{90}
\]

This closed-form result solves the target survival problem for a broad class of diffusive lattice gases. It has the same structure as Eq. \([4] \) except the \( n_0 \)-dependence which is model-specific. When specialized to the RW and SSEP, Eq. \([89] \) yields Eqs. \([11] \) and \([11] \), respectively.

For \( d = 2 \) we obtain, with logarithmic accuracy, the long-time asymptotic

\[
- \ln \mathcal{P} \simeq \frac{\pi T}{\ln \frac{\sqrt{\sigma(q)}}{\sigma(q)}} \left[ \int_0^{n_0} \frac{D(q)}{\sqrt{\sigma(q)}} dq \right]^2. \tag{91}
\]

As an additional illustration of the general results \([87] \) and \([89] \), we consider a family of zero range processes (ZRP). A ZRP describes interacting (but not excluding) random walkers on a lattice: A particle at site \( i \) can hop to a neighboring site with a rate \( R(n_i) \) that only depends on the number of particles \( n_i \) on the departure site \( i \). Naturally, \( R(0) = 0 \). If \( R'(n_i) > 0 \), the ZRP is described at the macroscopic level by \( D(q) = R'(q) \) and \( \sigma(q) = 2R(q) \), see e.g. Ref. \([23] \). Therefore, \( D(q)/\sqrt{\sigma(q)} = R'(q)/\sqrt{2R(q)} = (dq/dr)/\sqrt{2R(q)} \). Evaluating the integrals in Eq. \([87] \), we obtain for the stationary density profile:

\[
\frac{R(q)}{R(n_0)} = \left[ 1 - \left( \frac{\ell}{r} \right)^{d-2} \right]^2. \tag{92}
\]

In its turn, Eq. \([11] \) yields the long-time asymptotic of the target survival probability for the ZRP:

\[
- \ln \mathcal{P} \simeq \left( d - 2 \right) \Omega_d R(n_0) R^{d-2} T, \quad d > 2. \tag{93}
\]

VI. DISCUSSION

In this work we evaluated the survival probability \( \mathcal{P}(T) \) of a spherical target of radius \( R \) in a gas of unbiased diffusive particles (“searchers”), with density \( n_0 \), that interact with each other via exclusion as described by the SSEP. We also determined the most likely particle density history conditional on the target survival until time \( T \). The results depend on the dimension of space \( d \) and on the basic rescaled parameter \( \ell = R/\sqrt{D_0T} \). When \( \ell \) is small and \( d > 2 \), \( \mathcal{P}(T) \) is mostly contributed to by an exact stationary solution of the macroscopic fluctuation theory (MFT) that we obtained. For large \( \ell \), and for any \( \ell \) in one dimension, the relevant MFT solutions are non-stationary. In this case \( \ln \mathcal{P}(T) \) scales differently with \( T, R, d \) and \( n_0 \), and also depends on whether the initial condition is deterministic or random. These effects (for large \( \ell \), and for any \( \ell \) in one dimension) are also observed in the absence of exclusion: for non-interacting random walkers. In the special case of \( \ell \ll 1 \) and \( d = 2 \) logarithmic corrections to \( \mathcal{P}(T) \) appear. Table \( \text{I} \) can serve as a quick guide to our main results for the survival probability for the SSEP in different limits. The long-time asymptotics of the survival probability for a whole class of interacting lattice gases for \( d > 2 \) and \( d = 2 \) are given by Eqs. \([90] \) and \([90] \), respectively.

| Dimension | \( \ell \ll 1 \) | \( \ell \gg 1 \) |
|-----------|----------------|----------------|
| \( d = 1 \) deterministic | Eq. \([86] \) with \( d = 1 \) | Eq. \([53] \) with \( d = 1 \) |
| \( d = 1 \) random | Eq. \([60] \) | Eq. \([53] \) |
| \( d = 2 \) deterministic | Eq. \([86] \) | Eq. \([53] \) with \( d = 2 \) |
| \( d = 2 \) random | Eq. \([60] \) | Eq. \([60] \) |
| \( d > 2 \) deterministic | Eq. \([86] \) | Eq. \([53] \) |
| \( d > 2 \) random | Eq. \([86] \) | Eq. \([53] \) |

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Here we present microscopic derivations of the target survival probability $P(T)$ for the non-interacting RWs in one, two and three dimensions. We verify that, for $d = 1$, $P(T)$ depends on the initial conditions (random or deterministic). We also show that, for $d = 2$ and $3$, the leading term of the asymptotic of $P(T)$ is independent of the initial conditions. For $d = 3$ we reproduce, both in microscopic calculations and in the MFT framework, an exact result [13] for $P(T)$ in the random setting. Finally, we derive, for $d = 3$, a more accurate asymptotic of $P(T)$ for the deterministic setting. In all these calculations we put $D_0 = 1$.

1. $d = 1$

We start with random initial conditions and first consider RWs on a large but finite interval $(0, L)$. A RW starts in the interval $(x, x + dx)$, where $0 < x < L$, with probability $dx/L$. The probability that this RW does not hit the target (which is at the origin) until time $T$ is $\text{erf}(x/\sqrt{4T})$. Averaging over random particle locations at $t = 0$ we obtain the average single-particle non-hitting probability

$$\int_0^L \frac{dx}{L} \text{erf} \left( \frac{x}{\sqrt{4T}} \right) = 1 - \frac{1}{L} \int_0^L dx \text{erf} \left( \frac{x}{\sqrt{4T}} \right). \quad (A.1)$$

For sufficiently large $L \gg \sqrt{T}$ we can assume that the number of RWs on the interval is equal to $n_0L$. Since all the $n_0L$ particles are independent, the probability that none of them hits the target is

$$\left[ 1 - \frac{1}{L} \int_0^L dx \text{erf} \left( \frac{x}{\sqrt{4T}} \right) \right]^{n_0L}$$

which in the $L \to \infty$ limit becomes

$$P_{\text{RW}}(T) = \exp \left[ -n_0 \int_0^\infty dx \text{erf} \left( \frac{x}{\sqrt{4T}} \right) \right] = \exp \left( -\frac{2n_0\sqrt{T}}{\sqrt{\pi}} \right), \quad (A.2)$$

in agreement with Eq. (A.7) and Ref. [10].

Now let the initial positions of our RWs be deterministic. One example of deterministic setting is a periodic one, with exactly one particle on each site $k = 1, 2, \ldots$ of a lattice with lattice constant $1/n_0$. The probability that the particles which is initially located at site $k$s does not hit the target until time $T$ is $\text{erf}(k/(n_0\sqrt{4T}))$. The probability that neither of the particles hit the target is

$$P_{\text{RW}}(T) = \prod_{k=1}^\infty \text{erf} \left( \frac{k}{n_0\sqrt{4T}} \right) \quad (A.3)$$

Taking the logarithm we obtain

$$\ln P_{\text{RW}}(T) = \sum_{k=1}^\infty \ln \text{erf} \left( \frac{k}{n_0\sqrt{4T}} \right). \quad (A.4)$$

We are interested in the regime of $n_0\sqrt{T} \gg 1$, when the characteristic diffusion length is much larger than the lattice constant. The leading-order result can be obtained by replacing the summation in Eq. (A.4) by integration. Here we present a more accurate result that also includes a pre-exponential factor. We use the asymptotic [10]:

$$\sum_{k=1}^\infty \ln \text{erf}(ku) \approx -\frac{\Lambda_1}{2u} - \ln \sqrt{u} + \ln \pi^{3/4}, \quad 0 < u \ll 1.$$ 

Here $\Lambda_1 = 2 \int_0^\infty d\mu \ln \text{erf}(\mu) = 2.06883\ldots$, see Eq. (56). As a result,

$$P_{\text{RW}}(T) \approx \sqrt{2} \pi^{3/4} n_0^{1/2} T^{1/4} \exp \left( -\Lambda_1 n_0 \sqrt{T} \right). \quad (A.5)$$

The pre-exponential factor is independent of details of the deterministic initial condition. It coincides with our MFT result (77) and differs from the annealed result, Eqs. (A.2) and (78) and Ref. [10].

The pre-exponential factor is non-universal: it depends on details of the initial condition. This dependence is quite sensitive, as can be seen if we change the periodic arrangement of RWs at $t = 0$ by putting exactly 2 particles on each even lattice cite $2k$, $k = 1, 2, \ldots$ of the same lattice as before, leaving all odd sites empty. Repeating the calculations, we arrive at

$$P_{\text{RW}}(T) \approx \sqrt{2} \pi^{3/4} n_0^{1/2} T^{1/4} \exp \left( -\Lambda_1 n_0 \sqrt{T} \right), \quad (A.6)$$

with the same exponent as in Eq. (A.5) but a much larger pre-exponent.

2. $d = 2$

In two dimensions, the probability $P(r, T|R)$ that a RW starting at the radial coordinate $r > R$ does not hit the target by the time $T$ has a cumbersome exact expression. In the long time limit, $t \ll 1$, it suffices to use the following asymptotic that is valid with logarithmic accuracy (see e.g. [46]):

$$P(r, T|R) \approx 1 - \frac{1}{\ln \left( \frac{r}{4T} \right)} \Gamma \left( 0, \frac{r^2}{4T} \right). \quad (A.7)$$

Here $\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} dt$ is the incomplete gamma function. We now employ the same line of reasoning as in one dimension. For the random initial condition we average the probability (A.7) over the random locations of the particles in the annulus $R \leq r \leq L$ and obtain the average single-particle non-hitting probability

$$1 - \frac{1}{\ln \left( \frac{r}{4T} \right)} \int_R^L \frac{2\pi dr}{L^2 - r^2} \Gamma \left( 0, \frac{r^2}{4T} \right).$$
When $L \gg \sqrt{T} \gg R$, the number of RWs in the annulus is approximately equal to $\pi n_0 (L^2 - R^2)$. Therefore, the probability that no RW hit the target is

$$P_{\text{RW}}(T) = \left[ 1 - \frac{4T}{(L^2 - R^2) \ln \frac{4}{T}} \right]^{\pi n_0 (L^2 - R^2)} \exp \left[ -\frac{4\pi n_0 T}{\ln \frac{4}{T}} \int_0^\infty dz \Gamma(0, z) \right],$$

where we have simplified the limits of integration by recalling that $\ell \ll 1$ and taking the limit of $L \to \infty$. Computing the integral $\int_0^\infty dz \Gamma(0, z) = 1$, we recover Eq. (3).

In the deterministic setting the probability is

$$P_{\text{RW}}(T) = \prod_{r_j \geq R} \left[ 1 - \frac{1}{\ln \frac{4}{T}} \Gamma \left( 0, \frac{r_j^2}{4T} \right) \right],$$

(A.8)

The product is taken over initial positions $r_j$ which are deterministic. We assume that, at $t = 0$, there is exactly one particle on each site of a square grid with lattice spacing $n_0^{-1/2}$ outside of the circular target of radius $R$. We take the logarithm of (A.8) and, ignoring pre-exponential factors in the final result, expand the logarithm to the opposite case of $\ell \gg 1$ and taking the limit of $\ell \to \infty$. We obtain

$$\ln P_{\text{RW}}(T) \approx -\frac{n_0}{\ln \frac{4}{T}} \int_R^\infty 2\pi r \Gamma \left( 0, \frac{r^2}{4T} \right) dr$$

$$= -4\pi n_0 T \int_0^{\sqrt{T}} dz \Gamma(0, z)$$

$$\approx -4\pi n_0 T \frac{T}{\ln \frac{4}{T}},$$

again arriving at Eq. (3). That is, in contrast to $d = 1$, here the leading-order results for $P_{\text{RW}}(T)$ for random and deterministic initial conditions coincide in the limit of $\ell \ll 1$.

3. $d = 3$

In three dimensions, the probability $P(r, T|R)$ that a RW starting at the radial coordinate $r > R$ does not hit the target until time $T$ can be found by solving the backward diffusion equation (which is mathematically identical to the forward diffusion equation)

$$\frac{\partial}{\partial T} P(r, T|R) = \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) P(r, T|R)$$

subject to

$$P(r, T = 0|R) = 1, \quad P(r = R, T > 0|R) = 0.$$

In contrast to two dimensions, the solution has now a simple form:

$$P(r, T|R) = 1 - \frac{r}{R} \text{erfc} \left( \frac{r - R}{\sqrt{4T}} \right).$$

(A.9)

When the initial locations are random, we start with a spherical annulus $R \leq r \leq L$ and average (A.9) over initial locations to yield the average single-particle non-hitting probability

$$1 - \frac{3R}{L^3 - R^3} \int_R^L dr r \text{erfc} \left( \frac{r - R}{\sqrt{4T}} \right).$$

The probability $P_{\text{RW}}(T)$ that no RW hit the target is given by

$$\ln P_{\text{RW}}(T) = \frac{4\pi n_0 (L^3 - R^3)}{3} \ln \left[ 1 - \frac{3R}{L^3 - R^3} \int_R^L dr r \text{erfc} \left( \frac{r - R}{\sqrt{4T}} \right) \right]$$

$$\approx -4\pi n_0 R \sqrt{4T} \int_R^\infty dr r \text{erfc} \left( \frac{r - R}{\sqrt{4T}} \right)$$

$$= -4\pi n_0 R \sqrt{4T} \int_0^\infty dx (x \sqrt{4T} + R) \text{erfc} x$$

$$= -4\pi n_0 R T - 8\pi n_0 R^2 \sqrt{T},$$

(A.10)

in agreement with previous results [13]. Equation (A.10) is valid for any $\ell$. When $\ell = R/\sqrt{T} \ll 1$, the first term is the leading one and yields Eq. (4) with $d = 3$. In the opposite case of $\ell \gg 1$ it is the second term that is the leading one, and it yields Eq. (3) with $d = 3$.

Even in the long-time limit, $\ell = R/\sqrt{T} \ll 1$, Eq. (A.10) is more accurate than the leading-order asymptotic (3) that stems from the steady-state MFT solution. Importantly, the final result (A.10) can be also obtained from the MFT formalism if one solves the full time-dependent problem. The problem formulation is almost identical to that for $d = 1$, see Sec. IV C except that Eq. (68) is replaced by

$$p(r, 0) - 2 \int_{q_0}^{q(0)} dq_1 \frac{D(q_1)}{\sigma(q_1)} = \lambda (r - R),$$

(A.11)

and all integrations over $x$ from 0 to $\infty$ are replaced by integrations over the whole space outside the target. The calculations proceed along the lines of Sec. IV C. The most likely gas density history, in the original (not rescaled) variables, is

$$q(r, t) = n_0 \left[ 1 - \frac{R}{r} \text{erfc} \left( \frac{r - R}{\sqrt{4T}} \right) \right] \times \left\{ 1 - \frac{R}{r} \text{erfc} \left( \frac{r - R}{\sqrt{4T(t - t)}} \right) \right\}.$$

(A.12)

The calculation of the target survival probability ultimately reduces to evaluating the same integral as in
Eq. (A.10), giving the same result. In the long-time limit, $\ell \ll 1$, this integral is mostly contributed by the region $r - R \lesssim \sqrt{T}$. In this region Eq. (A.12) can be approximated, up to small corrections, $n_0(1-R/r)^2$, which coincides with the steady solution (33) for $d = 3$. The deviations from the steady-state solution are responsible for the second term on the right hand side of Eq. (A.10), which is a subleading term in this limit.

In the deterministic case the microscopic calculation, similar to that in 2d, boils down to evaluating the integral

$$
\ln \frac{P^{RW}(T)}{4\pi n_0} = \int_0^\infty dr \frac{r^2}{2} \left[ 1 - \frac{r}{R} \text{erfc}\left(\frac{r-R}{\sqrt{4T}}\right) \right]. \tag{A.13}
$$

Exactly the same expression follows from the MFT. For $\ell \ll 1$ we expand the logarithm to the second order and arrive at

$$
\ln P^{RW}(T) \simeq -4\pi n_0 RT - (4-\sqrt{2})\sqrt{\pi n_0} R^2 \sqrt{T}. \tag{A.14}
$$

The leading term coincides with that for the annealed setting, Eq. (A.10). The subleading term is different. For $\ell \gg 1$ we can replace $r$ by $R$ everywhere in the integrand of Eq. (A.13) except under the erfc, thus arriving at Eq. (65) (but for the RWs) with $s_1(n_0) = \Lambda_1 n_0$. Here the initial conditions affect the leading order result.

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