Estimation of Ideal Configuration and Dimension of Pico Hydropower using Euler-Lagrange Equation and Runge-Kutta Method

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Abstract. Pico hydropower is a type of hydropower that can produce electric power under or equal to 5000 Watt. Consisting of one turbine to rotate one generator, it has been used in remote areas to provide for electricity supply. Although the system serves quite well, sometimes the necessity for electricity supply is larger than the system can supply. The solution would be to build two pico hydropower systems. However, the cost of such a solution will also double. In addition to that, the water source and landscape may not sustain more than one pico hydropower. An alternative solution that is being proposed to resolve the problem is by using one turbine to rotate two electric generators in a unity pico hydropower system. The objective of the study concerning the proposed solution is to obtain a configuration and dimension of the energy converter. They are sought by determining the equation of motion for energy converter which is configured in several wheels. The equation of motion, which is in differential form, is obtained using the Euler-Lagrange procedure in classical mechanics. Both configuration and dimension of energy converter are solved using numerical methods. The numerical method used in this study is the fourth-order of the Runge-Kutta method for analyzing some physical parameters i.e.: angular velocity of the energy converter, friction coefficient, driving force, radius and inertia of the wheels. By employing varying numbers in the computational process using the fourth-order of the Runge-Kutta method in Matlab, we seek that the pico hydropower system with ideal configuration and dimension of energy converter based on the radius of the wheels are configured as \( r_1 > r_2 > r_3 > r_4 = r_5 = r_6 \). The configuration yield an optimum rotation of the coupled wheels \( (r_5 \text{ and } r_6) \) as a representation of the electric generator’s axis although the thrust is assumed lower. The design expected to be a solution for optimizing a pico hydropower, especially in remote areas with lower water flow.
1. Introduction

A hydropower producing electric power under or equal to 5000 Watt is classified as a pico hydropower. This type of hydropower is usually developed in remote areas using one turbine to rotate one generator. As a natural electric power, a pico hydropower has some typical advantages, i.e: easy to maintain, low water consumption, durability, and low construction cost. Therefore, it is suitable for remote areas [1,2,3]. There have been no significant issues in the implementation of the pico hydropower using one turbine to rotate one generator, except that the system cannot accommodate the increasing demand, particularly for two times as large electric power consumption, when only one unit is being used.

A solution for the problem is using two adjacent units of pico hydropower to generate electricity twice the power of one unit can generate [1]. However, it is significantly expensive for operation and maintenance. In addition to that, building two units may not be possible in some areas due to the water source availability and landscape feature which do not computable for more than one unit of pico hydropower to be installed [2]. A novel innovation that may be potential to overcome the problem, particularly in terms of cost and natural resource is to develop a pico hydro power using one turbine to rotate two electric generators. Such a system is assuredly not only effective and efficient to operate but also cost-effective to build because the system did not need additional costs for both converter and generator. Also, the proposed system requires water source only as much as the previous system does, so that it does not cost more natural resources.

Obtaining an optimal pico hydropower using one turbine to rotate two electric generators requires some analyses concerning its components. A pico hydropower consists of some main components, i.e: forebay, penstock pipe, energy converter, electric generator, and electricity stabilizer [1,2,4]. The energy converter, as one of the most important components of pico hydropower, is analyzed in this work to seek the condition for optimum rotation speed. It converts the potential energy of water at the end of the penstock pipe to the rotational kinetic energy of the turbine. The energy converter consists of a turbine, several wheels, and an electric generator [5,6,7].

The rotating system can be analyzed through its equation of motion, derived using Euler-Lagrange formalism from the corresponding Lagrangian. Such an approach was taken in the analysis of the rotation of bicycle wheels [8]. This phenomenon is similar to the rotation of the pico hydropower in this study. The Euler-Lagrange procedure yields the equation of motions in differential equations [8,9]. Several differential equations require numerical methods to obtain their solutions, along with the corresponding influencing parameters relating to the phenomenon being analyzed [9,10].

The physical parameters for optimizing the energy converter in the system under investigation are sought in the attempt to numerically solve the equation of motion relating to the rotation of the energy converter. In this study, the Runge-Kutta method is applied to determine the solution of the equation of motion describing the hydraulic converter [11,12]. The method is iterative, making it easy for computer programming, and does not require a differential form to determine its numerical solution [13]. Therefore, the Runge-Kutta method is used in this work to seek the numerical solution of the equation of motion explaining the energy converter obtained from Euler-Lagrange formalism. The result is used in estimating the ideal configuration and dimension of the pico hydropower of interest. The design of the pico hydropower in this study is intended to be a solution in the development of a small scale of electricity supply, especially in remote areas.

2. Method

The equation of motion of the system is formulated using the Euler-Lagrange formalism. The formulation is started by designing an energy converter system that is configured in several wheels. The wheels configurations are a representation of the turbine, several wheels, and electric generators. In representation, the general coordinate system is used for convenience in determining the parameters involved. The energy converter designed in this study is configured of six wheels connected linearly or coaxially (See figure.1). This configuration resembles that of bicycle wheels [8]. All the wheels are
connected from one to the next by a belt, except for the fifth and sixth, which are coaxial. Two sources of electric power are simultaneously rotated by this pair, which is each wheel is connected to one electric generator.

**Figure 1.** The coordinate system of energy converter for pico hydropower with six wheels. The important wheels are the first as turbine representation to convert the potential energy of water flow and both fifth and sixth as the electric generator for converting the kinetic energy of wheels cycle to electric power.

The general form of Lagrange function is a representation of linear summation between kinetics and potential energies as:

\[ L = T - V, \]

where \( L \) is Lagrange function, \( T \) is kinetic energy, and \( V \) is potential energy. The equation of motion for the energy converter is obtained by using the non-conservative Euler-Lagrange equation:

\[
\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = Q_n. \tag{2}
\]

In equation (2) three parameters are involved, i.e: general coordinate \( q_n \), time \( t \), and external force \( Q_n \). The external force in this case, is a force that rotates the system and a force in the opposite direction, resisting the motion [8]. This procedure of formulation is a modification of the mechanical principle which is used in the analysis of the motion of wheels in a bicycle [9].

The first wheel is acted on by a torque, resulting from a driving force that applies in perpendicular to the line connecting the rotation axis and the point of contact [8,9]. The torque

\[ \tau = Fr_\perp \tag{3} \]

pushes the turbine so the system can rotate. There is also friction force as an external force acting in the opposite direction to the driving force. The value of the friction force depends on the kinetic friction coefficient \( b \) and the rate of angular displacement. These forces yield a resultant external force

\[ Q_n = Fr_\perp - \dot{\theta}rb \tag{4} \]

The pivot axis of the wheels is set as the reference height, hence the potential energy of the converter system is zero \( V=0 \). That is not the case for kinetic energy \( T \). The rotational kinetic energy of the pico hydropower consists of six components, each of which is contributed by the corresponding wheels, which are expressed in the following expressions:

\[ T_i = \frac{1}{2}I_i\dot{\theta}_i^2 \tag{5} \]

where the \( i \)-index means the number starts from 1 to 6. Using equation (5) as kinetic energy and \( V=0 \) as potential energy, the real Lagrangian for the converter system of the Pico hydropower is obtained as the following.

\[ L = \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 + I_3\dot{\theta}_3^2 + I_4\dot{\theta}_4^2 + I_5\dot{\theta}_5^2 + I_6\dot{\theta}_6^2). \tag{6} \]
Where the rotation speeds of every wheel have certain relations:
\[
\dot{\theta}_2 = \dot{\theta}_3 = \dot{\theta}_4 \left( \frac{r_1}{r_2} \right)
\]
\[
\dot{\theta}_4 = \dot{\theta}_5 = \dot{\theta}_6 = \dot{\theta}_1 \left( \frac{r_3 r_2}{r_1 r_4} \right)
\]  
(7)
(8)

Inserting both equations (7) and (8) into equation (6), the Lagrangian involving inertial and radius of the system of wheels has the following form:
\[
L = \frac{\theta^2_1}{2} \left[ I_1 + I_2 \left( \frac{r_1}{r_2} \right)^2 + I_3 \left( \frac{r_2}{r_3} \right)^2 + I_4 \left( \frac{r_3 r_2}{r_1 r_4} \right)^2 + I_5 \left( \frac{r_3}{r_2 r_4} \right)^2 + I_6 \left( \frac{r_4}{r_2 r_4} \right)^2 \right]
\]  
(9)

Furthermore, the equation of motion describing the energy converter of the pico hydropower is obtained by using the equation (2) as the Euler-Lagrange formalism involving a Lagrangian, external force of equation (6), with general coordinate \( q_i=0 \). These conditions result in a Euler-Lagrange equation in the following form:
\[
\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = F_{r_i} - \dot{\theta}_i b
\]  
(10)

Inserting the Lagrangian of the system from equation (9) in equation (10), the equation of motion for energy converter of pico hydropower is obtained as
\[
d^2\theta_i = \frac{F_{r_i} - \dot{\theta}_i b}{I_1 + I_2 \left( \frac{r_1}{r_2} \right)^2 + I_3 \left( \frac{r_2}{r_3} \right)^2 + I_4 \left( \frac{r_3 r_2}{r_1 r_4} \right)^2 + I_5 \left( \frac{r_3}{r_2 r_4} \right)^2 + I_6 \left( \frac{r_4}{r_2 r_4} \right)^2}
\]  
(11)

where we can define the denominator on the right-hand side of equation (2) as an inertial parameter in the form
\[
I = \left[ I_1 + I_2 \left( \frac{r_1}{r_2} \right)^2 + I_3 \left( \frac{r_2}{r_3} \right)^2 + I_4 \left( \frac{r_3 r_2}{r_1 r_4} \right)^2 + I_5 \left( \frac{r_3}{r_2 r_4} \right)^2 + I_6 \left( \frac{r_4}{r_2 r_4} \right)^2 \right]
\]  
(12)

The equation of motion represents the rotation of the turbine (the first wheel), due to the driving force from the potential energy of water at the end of the penstock pipe [2,3]. The rotation of other wheels can be analyzed using equations (9).

Next, the numerical solution of equation (11) as the equation of motion for the energy converter of pico hydropower is determined using the fourth-order of Runge-Kutta methods. In the execution of the Runge-Kutta to solve the equation of motion, the external force that drives the turbine (the first wheel) is assumed to be constant for several period of time and zero within a relatively short time range. The algorithm for the fourth-order of the Runge-Kutta method to determine the numerical solution of this case is carried out in some steps. Firstly, the equation of motion is written in the following form.
\[
\frac{d\theta}{dt} = \frac{F_{r_1} - \dot{\theta}_1 b}{I}
\]  
(13)

then, the time range \((h)\) between initial the time, \(t^{(i)}\), and the boundary final time, \(t^{(m)}\), is determined. The range is divided by the number of the time step, \(N\). Consequently, the number of iterations is written as
\[
h = \frac{t^{(m)} - t^{(i)}}{N}
\]  
(14)

In this case, the constants of the fourth-order of the Runge-Kutta method i.e. \(k_1, k_2, k_3, \) and \(k_4\) are determined with an assumption that the function of \(k\) depends on two independent variables, i.e: time \((t)\) and angular speed \((\dot{\theta})\). Correspondingly, the constants of the fourth order of Runge-Kutta methods are in these forms:
\[
k_1 = hf \left( t^{(i)}, \dot{\theta}_i^{(i)} \right),
\]  
(15)
\[
k_2 = hf \left( t^{(i)} + \frac{h}{2}, \dot{\theta}_i^{(i)} + \frac{k_1}{2} \right),
\]  
(16)
\[
k_3 = hf \left( t^{(i)} + \frac{h}{2}, \dot{\theta}_i^{(i)} + \frac{k_2}{2} \right),
\]  
(17)
\[
k_4 = hf \left( t^{(i+1)}, \dot{\theta}_i^{(i)} + k_3 \right).
\]  
(18)
In the formulation, superscripts indicate the number of iteration in numerical analysis and the subscript attaching to the variable refers to the corresponding wheel of the energy converter. Lastly, $\theta_1$, which represents the rotation of the first wheel, or the turbine, is written in a simple formulation as

$$\dot{\theta}_1^{(i+1)} = \dot{\theta}_1^{(i)} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$  \hspace{1cm} (19)$$

The algorithm for the fourth-order of the Runge-Kutta used in computing the rotational speed of the first wheel is also used for the other wheels. The computation process in determination of the rotation speed of the second, third, fourth, and paired wheels (fifth and sixth wheels) follow from the formulation in equations (7) and (8). This algorithm is implemented in Matlab by testing some values. The computation results are used to analyze the ideal configuration of the energy converter for pico hydropower [13].

### 3. Results and discussion

The computational results of parameters describing the energy converter refer to the radius, moment of inertia, and mass of all the wheels. In addition to that, friction force and driving force as external forces are also taken into the computation process. All parameters used in this modeling are only for testing the configuration and dimension of the energy converter system. The radius values representing the ideal configuration and dimensions of wheel are 0.6, 0.3, 0.4, 0.1, 0.05, and 0.05 m, respectively, from the first wheel to the coupled fifth and sixth wheels. The other parameters relating to the moment of inertia for each wheel are 0.05, 0.025, 0.015, 0.01, and 0.01 kgm$^2$, respectively. For the last two parameters, friction and driving force, the values are 0.05 and 50 N, respectively. The angular speed of rotation plotted again as a function of time for each wheel is showed in figure 2. From the graphs, it can be seen that the velocity of the coupled fifth and sixth wheels, which act as electric generators are the highest compared to other wheels in the energy converter system. It means that it requires only a low angular speed of rotation of the turbine to generate electrical power that is two times the power resulted from one unit pico hydropower [14,15].

Figure 2 shows that the energy converter system of the Pico hydropower can operate optimally. It can be seen from the angular speed of the wheels, with the first wheel acting as a turbine that rotates in low rotational speed, the system can accelerate the coupled wheels (fifth and sixth wheels) as the two electric generators. After a comparison of similar graphs resulting from other values of friction and driving forces as external forces, it agrees with the expectation that the higher the driving force to the first wheel, the faster the rotation speed of both the fifth and sixth wheels as electric generators. Moreover, the higher the friction coefficient of all axis wheels converter system, the lower the rotation speed of both the fifth and sixth wheels as electric generators [14,15].

Based on the computation using the fourth-order of the Runge-Kutta method in Matlab, both the configuration and dimension of the system of wheels that enable optimal operation may vary. Some different variations can be found after testing with some values of input parameters of the energy converter. Figure 2 is only one of the ideal characteristics that are obtained from the running program in Matlab. The ideal configuration and dimension that can be determined in this study are based on some physical parameters of wheels. The first wheel ($r_1$) is the largest, namely 0.6 m of the radius. The radius of the second wheel ($r_2$) is 0.2 m, which is less than both the first wheel and third wheel. The radius of the third wheel ($r_3$) is 0.4 m, which is lower than the first wheel but bigger than the second one. The fourth wheel has 0.3 m of the radius. The fourth is smaller than the third but bigger than the second. The coupled wheels, the fifth ($r_5$) and sixth ($r_6$) wheels have 0.05 m radius. The two wheels, which are the smallest in size, are connected directly to the axes of the electric generator. The schematic diagram of this configuration is shown in figure 3.
Figure 2. The velocity of wheels that is assumed as an ideal configuration and dimension with a radius of 0.6, 0.3, 0.4, 0.1, 0.05, and 0.05 m, respectively, from the first wheel to the coupled fifth and sixth wheels.

Figure 3. Ideal configuration of energy converter of pico hydropower as the result of Runge-Kutta calculation in matlab.

The result shown in figure 4 is compared again the parameters of the ideal converter. Similar to the ideal converter in figure 3, the coupled wheels are connected directly to the axes of the electric generator. The computation result shows that the configuration is not ideal, which is evident from the low value of the rotation speed of the coupled wheels. Although the coupled wheels rotate faster than the first wheel, their rotations are slower than other the rest of the wheels. In addition to that, a larger driving force, that is 100 N, needs to be given to this configuration. Even with this value of driving force, the rotation is slower than ideal configuration acted on by 50 N of the driving force. The configuration of some rotation speeds in figure 4 is shown out in figure 5.
Figure 4. The velocity of wheels that is assumed as an ideal configuration and dimension with a radius of 0.6, 0.3, 0.4, 0.1, 0.05, and 0.05 m, respectively, from the first wheel to the coupled fifth and sixth wheels.

Figure 5. Ideal configuration of energy converter of pico hydropower as the result of Runge-Kutta calculation in matlab.

4. Conclusion
Pico hydropower based on the radius of the wheels that are obtained through numerical analyses using the fourth-order of the Runge-Kutta method in Matlab are $r_1>r_3>r_2>r_4=r_5=r_6$. The first wheel ($r_1$), which acts as a turbine is the largest, which is 0.6 m of the radius. The second wheel ($r_2$) is 0.2 m, which is less than both first and third wheels. The radius of the third wheel ($r_3$) is 0.4 m, which is smaller than the first but larger than the second. The radius of the fourth wheel ($r_4$) is 0.3 m is smaller than the third, but larger than the second. Finally, the coupled wheels, fifth ($r_5$) and sixth ($r_6$), each have a radius of 0.05 m. The Pico hydropower that has an ideal configuration and dimension of energy converter is supposed to offer a solution in the development of a small scale of electricity supply, especially in remote areas.
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