Anisotropy universe in doubly warped product scheme

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Abstract

We study the GMGHS spacetimes to analyze the evolution of the anisotropy universe, which can be treated as a doubly warped products manifold possessing warping functions (or scale factor) having the Kantowski-Sachs solution which represents homogeneous but anisotropically expanding(contracting) cosmology. We investigate the curvature associated with three phases in the evolution of the universe.

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1 Introduction

Since the cosmic microwave background was discovered, there have been many ideas and proposals to figure out how the universe has evolved. The standard big bang cosmological model based on the Friedmann-Robertson-Walker (FRW) spacetimes has led to the inflationary cosmology [1] and nowadays to the M-theory cosmology with bouncing universes [2]. These spacetimes are foliated by a special set of spacelike hypersurfaces such that each hypersurface corresponds to an instant of time.

The warped product manifold was developed to point out that several of the well-known exact solutions to Einstein field equations are pseudo-Riemannian warped products [3]. Furthermore, a general theory Lorentzian multiply warped were applied to discuss the Schwarzschild spacetime in the interior of the event horizon [4]. From a physical point of view, these warped product spacetimes are interesting since they include classical examples of space-time such as the FRW manifold and the intermediate zone of Reissner-Nordström (RN) manifold [5, 6]. The FRW cosmological model was studied to investigate non-smooth curvatures associated with multiple discontinuities involved in the evolution of the universe [7]. Einstein postulated that the universe is homogeneous and isotropic at each moment of its evolution. The FRW metric is an exact solution of Einstein’s field equations of general relativity. This describes a homogeneous, isotropic expanding or contracting universe that may be simply connected or multiply connected. The universe is approximately homogeneous and isotropy, but not exactly so. In homogeneous and anisotropy model at any point the universe is expanding at different rates in different directions.

The GMGHS solution of the Einstein field equations represents the geometry exterior to a spherically symmetric static charged black hole and GMGHS metric in the doubly warped product spacetime has the same form with the Kantowski-Sachs solution [8]. By turning antisymmetric tensor gauge fields off, the static charged black hole solution was found by Gibbons, Maeda [9], and by Garfinkle, Horowitz, Strominger [10], independently.

In this paper, as a cosmological model we will exploit the GMGHS spacetimes to analyze of the anisotropy universe, which can be treated as a doubly warped products manifold possessing warping functions (or scale factor) having the Kantowski-Sachs solution which represents homogeneous but
anisotropically expanding(contracting) cosmology. We will also analyze the nonsmooth features of the spatially flat GMGHS universe by introducing double discontinuities occurred at the radiation-matter and matter-lambda phase transitions in the astrophysical phenomenology. We also investigate the curvature on the doubly warped products. We shall use geometrized units, i.e., \( G = c = 1 \), for notational convenience.

2 The GMGHS solution of the anistropy spacetime

The GMGHS solution of the Einstein field equations represents the geometry exterior to a spherically symmetric static charged black hole. In the Schwarzschild coordinates, the line element for the GMGHS metric in the exterior region \( r > 2m \) has the form as follows

\[
ds^2 = -(1 - \frac{2m}{r})dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2 \left(1 - \frac{\alpha}{r}\right)\left(d\theta^2 + \sin^2 \theta d\phi^2\right),
\]

where

\[
e^{2\Phi} = 1 - \frac{\alpha}{r}, \quad F_{rt} = \frac{Q}{r^2}
\]

with \( \alpha = Q^2/m \). Here, \( \Phi \) is the dilaton field and \( F_{rt} \) is the electric field strength. The parameters \( m \) and \( Q \) are mass and charge respectively. Note that the metric in the \( t-r \) plane is identical to the Schwarzschild case. As like in the Schwarzschild spacetime, the GMGHS has an event horizon at \( r = 2m \). We also note that the area of the sphere of the GMGHS black hole, defined by \( \int d\theta d\phi \sqrt{g_{\theta\theta}g_{\phi\phi}} \), is smaller than the Schwarzschild spacetime by an amount depending on the charge.

In particular, the area of the sphere approaches zero as \( r \to \alpha \), leading to a surface singularity as

\[
R = \frac{\alpha^2(r - 2m)}{2r^3(r - \alpha)^2}.
\]

As far as the case of \( \alpha \leq 2m \), the singular surface remains inside the event horizon so that the Penrose diagram is identical to the Schwarzschild spacetime. Also the case of \( \alpha = 2m \), which implies \( Q^2 = 2m^2 \) is the extremal limit which the event horizon and the surface singularity meet.
On the other hand, the line element for the GMGHS metric for the interior region $r < 2m$ can be described by

$$ds^2 = -\left(\frac{2m}{r} - 1\right)^{-1}dr^2 + \left(\frac{2m}{r} - 1\right)dt^2 + r^2\left(1 - \frac{\alpha}{r}\right)(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $r$ and $t$ are now new temporal and spatial variables, respectively. A multiply warped product manifold, denoted by $M = (B \times F_1 \times \cdots \times F_n, g)$, consists of the Riemannian base manifold $(B, g_B)$ and fibers $(F_i, g_i)$ $(i = 1, \ldots, n)$ associated with the Lorentzian metric [4]. In particular, for the specific case of $(B = R, g_B = -d\mu^2)$, the GMGHS metric (2.4) can be rewritten as two warped products $(a, b) \times_f R \times_f S^2$ by making use of a lapse function

$$N^2 = \frac{r_H - r}{r},$$

as well as warping functions given by $f_1$ and $f_2$ as follows

$$f_1(\mu) = \left(\frac{2m}{F^{-1}(\mu)} - 1\right)^{1/2},$$

$$f_2(\mu) = \left(F^{-1}(\mu)^2 - \alpha F^{-1}(\mu)^2\right)^{1/2}.$$

The lapse function (2.5) is well defined in the region $r < r_H (= 2m)$ to rewrite it as two warped products spacetime by defining a new coordinate $\mu$ as follows

$$\mu = \int_0^r \frac{dx}{(r_H - x)^{1/2}} = F(r).$$

Setting the integration constant zero as $r \to 0$, we have

$$\mu = 2m \cos^{-1}\left(\frac{r_H - r}{r_H}\right) - [(r_H - r)r]^{1/2},$$

which has boundary conditions as follows

$$\lim_{r \to r_H} F(r) = (2n - 1)m\pi, \quad \lim_{r \to 0} F(r) = 0,$$

for positive integer $n$, and $dr/d\mu > 0$ implies that $F^{-1}(\mu)$ is well-defined function. We can thus rewrite the GMGHS metric (2.4) with the lapse function (2.5)

$$ds^2 = -d\mu^2 + \left(\frac{2m}{F^{-1}(\mu)} - 1\right)dr^2 + \left(F^{-1}(\mu)^2 - \alpha F^{-1}(\mu)^2\right)d\Omega^2$$

$$= -d\mu^2 + f_1(\mu)^2 dr^2 + f_2(\mu)^2 d\Omega^2.$$
by using the warping functions (2.7). This GMGHS metric in the doubly warped product spacetime has the same form with the Kantowski-Sachs solution [8] which represents homogeneous but anisotropically expanding (contracting) cosmology. In the case of the interior region \( r < 2m \), the GMGHS metric has been rewritten as doubly warped product spacetime having two warping functions in terms of \( f_1 \) and \( f_2 \), which have the same form with the Ricci curvature of the multiply warped interior Schwarzschild metric [4]. The only difference from the Schwarzschild is the \( \alpha \) term in the warping function \( B \) in Eq. (2.7).

Moreover, we can write down the Ricci curvature on the two warped product as

\[
\begin{align*}
R_{tt} &= -\frac{f''_1}{f_1} - \frac{2f''_2}{f_2}, \\
R_{rr} &= f_1 f''_1 + \frac{2f_1 f''_1}{f_2}, \\
R_{\theta\theta} &= \frac{f'_1 f_2 f'_2}{f_1} + f'^2_2 + f_2 f''_2 + 1, \\
R_{\phi\phi} &= \left( \frac{f'_1 f_2 f'_2}{f_1} + f'^2_2 + f_2 f''_2 + 1 \right) \sin^2 \theta, \\
R_{mn} &= 0, \text{ for } m \neq n,
\end{align*}
\]

\( (2.12) \)

### 3 Anisotropy universe in the frame of warped products space

The FRW metric describes a homogeneous, isotropic expanding. In the spatially flat FRW cosmology with \( k = 0 \), the early universe was radiation dominated, the adolescent universe was matter dominated, and the present universe is now entering into dark energy dominated phase in the absence of vacuum energy. If the universe underwent inflation, there was a very early period when the stress-energy was dominated by vacuum energy. The Friedmann equation may be integrated to give the age of the universe in terms of present cosmological parameters.
With the above astrophysical phenomenology in mind, consider the homogeneous but anisotropically expanding (contracting) cosmology in the two warped product spacetime. In physical cosmology, the radiation dominated era was the first of the three phases of the known universe, the other two being the matter dominated era and the dark energy dominated era. For a dark energy dominated universe the evolution of the scale factor in the FRW metric is obtained solving the Friedmann equations. We have the scale factor \( f_1(t) \) as a function of time \( t \) which scales as
\[
f_1(t) \propto t^{1/2}
\]
for a radiation dominated (RD) universe, and scales as
\[
f_1(t) \propto t^{2/3}
\]
for a matter dominated (MD) universe, and scales as
\[
f_1(t) \propto e^{Kt}
\]
for a dark energy dominated (DED) universe. Where the coefficient \( K \) in the exponential, the Hubble constant, \( K = \sqrt{8\pi G \rho_{\text{full}}/3} = \sqrt{\Lambda/3} \). This exponential dependence on time makes the spacetime geometry identical to the de Sitter Universe, and only holds for a positive sign of the cosmological constant, the sign that was observed to be realized in Nature anyway. The current density of the observable universe is of the order of \( 9.44 \times 10^{-27} \text{ kg m}^{-3} \) and the age of the universe is of the order of 13.8 billion years, or \( 4.358 \times 10^{17} \text{ s} \). The Hubble parameter, \( K \), is \( \sim 70.88 \text{ km s}^{-1} \text{ Mpc}^{-1} \). (The Hubble time is 13.79 billion years.) The value of the cosmological constant, \( \Lambda \), is \( \sim 2 \times 10^{-35} \text{ s}^{-2} \).

**Definition 3.1** A \( C^0 \)-Lorentzian metric on \( M \) is a nondegenerate \((0,2)\) tensor of Lorentzian signature such that
\[
(i) \quad g \in C^0 \text{ on } S
\]
\[
(ii) \quad g \in C^\infty \text{ on } M \cap S^c
\]
\[
(iii) \quad \text{For all } p \in S, \text{ and } U(p) \text{ partitioned by } S, g|_{U^+_p} \text{ and } g|_{U^-_p} \text{ have smooth extensions to } U. \text{ We call } S \text{ a } C^0\text{-singular hypersurface of } (M, g).
\]

Consider \( M_0 \) as a \( C^0\)-singular hypersurface of \((M, g)\). In the GMGHS spacetime, \( f_1 > 0 \) is smooth functions on \( M_0 = (t_0, t_\infty) \) except at \( t \neq t_i \) \((i = 1, 2)\), that is \( f \in C^\infty(S) \) (where \( S = \{t_i\} \times H^\perp \)) for \( t \neq t_i \) and \( f_1 \in C^0(S) \) at \( t = t_i \) in \( M_0 \) to yield
\[
f_1 = \begin{cases}
  c_0 t_1^{1/2}, & t < t_1 \\
  c_1 t_1^{2/3}, & t_1 \leq t \leq t_2 \\
  c_2 e^{Kt_2}, & t > t_2
\end{cases}
\]  
(3.1)

with the boundary conditions
\[
c_0 t_1^{1/2} = c_1 t_1^{2/3}, \quad c_1 t_2^{2/3} = c_2 e^{Kt_2}.
\]  
(3.2)
Experimental values for $t_1$ and $t_2$ are given by $t_1 = 4.7 \times 10^4$ yr and $t_2 = 9.8$ Gyr. Moreover $c_1$ and $c_2$ are given in terms of $c_0$, $t_1$ and $t_2$ as follows

$$c_1 = c_0 t_1^{-1/6}, \quad c_2 = c_0 t_1^{-1/6} t_2^{2/3} e^{-K t_2}.$$ 

Thus we have

$$f_1 = \begin{cases} c_0 t_1^{1/2}, & t < t_1 \\ c_0 t_1^{-1/6} t_2^{2/3}, & t_1 \leq t \leq t_2 \\ c_0 t_1^{-1/6} t_2^{2/3} e^{K(t-t_2)}, & t > t_2 \end{cases} \quad (3.3)$$

Note that in the GMGHS spacetime, $f_1 \in C^0(S)$ since if we assume $f_1 \in C^1(S)$ one could have the boundary conditions $\frac{1}{2} c_0 t_1^{-1/2} = \frac{2}{3} c_1 t_1^{-1/3}$ and $\frac{2}{3} c_1 t_1^{-1/3} = K c_2 e^{K t_2}$, which cannot satisfy the above boundary conditions (3.2) simultaneously.

Choose $\alpha = 0$ in (2.4). For a radiation-dominated (RD) universe era $t < t_1$, put

$$f_1(t) = c_0 t_1^{1/2}$$

we have

$$f_2(t) = F^{-1}(t) = \frac{2m}{c_0^2 t + 1} \quad (3.4)$$

thus we have the metric of radiation-dominated (RD) universe era

$$ds^2 = -dt^2 + c_0^2 t dr^2 + \left( \frac{2m}{c_0^2 t + 1} \right)^2 d\Omega^2 \quad (3.5)$$

For a matter-dominated (MD) universe era $t_1 \leq t \leq t_2$, put

$$f_1(t) = c_0 t_1^{-1/6} t_2^{2/3} = c_3 t_2^{2/3}$$

we have

$$f_2(t) = F^{-1}(t) = \frac{2m}{c_3^2 t^{4/3} + 1} \quad (3.6)$$

thus we have the metric of matter-dominated (MD) universe era

$$ds^2 = -dt^2 + c_3^2 t^{4/3} dr^2 + \left( \frac{2m}{c_3^2 t^{4/3} + 1} \right)^2 d\Omega^2 \quad (3.7)$$
For a dark-energy-dominated (DED) universe era $t > t_2$, put
\[ f_1(t) = c_0 t_1^{-1/6} t_2^{2/3} e^{K(t-t_2)} = c_4 e^{Kt} \]
we have
\[ f_2(t) = F^{-1}(t) = \frac{2m}{c_2^2 e^{2Kt} + 1} \]
thus we have
\[ ds^2 = -dt^2 + c_4^2 e^{2Kt} dr^2 + \left( \frac{2m}{c_2^2 e^{2Kt} + 1} \right)^2 d\Omega^2 \]

Now we can write the curvature of a static spherically symmetric GMGHS spacetime in radiation-dominated (RD) universe era as
\[
R_{tt} = \frac{115c_0^4 t^2 - 2c_0^2 t - 1}{4 t^2 (c_0^2 t + 1)^2}, \\
R_{rr} = -\frac{c_0^2 (5c_0^2 t + 1)}{4 t (c_0^2 t + 1)}, \\
R_{\theta\theta} = -\frac{c_0^4 t^5 + 4c_0^2 t^4 + 10c_0^4 m^2 t + 6c_0^4 t^3 - 2c_0^2 m^2 + 4c_0^2 t^2 + t}{t (c_0^2 t + 1)^4}, \\
R_{\phi\phi} = -\frac{c_0^4 t^5 + 4c_0^2 t^4 + 10c_0^4 m^2 t + 6c_0^4 t^3 - 2c_0^2 m^2 + 4c_0^2 t^2 + t}{t (c_0^2 t + 1)^4} \sin^2 \theta, \\
R_{mn} = 0, \text{ for } m \neq n, \quad (3.10)
\]

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