ON RIESZ TRANSFORMS CHARACTERIZATION OF $H^1$ SPACES ASSOCIATED WITH SOME SCHRÖDINGER OPERATORS

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Abstract. Let $L f(x) = -\Delta f(x) + V(x) f(x)$, $V \geq 0$, $V \in L^1_{loc}(\mathbb{R}^d)$, be a non-negative self-adjoint Schrödinger operator on $\mathbb{R}^d$. We say that an $L^1$-function $f$ belongs to the Hardy space $H^1_L$ if the maximal function
\[ M_L f(x) = \sup_{t>0} |e^{-tL} f(x)| \]
belongs to $L^1(\mathbb{R}^d)$. We prove that under certain assumptions on $V$ the space $H^1_L$ is also characterized by the Riesz transforms $R_j = \frac{\partial}{\partial x_j} L^{-1/2}$, $j = 1, ..., d$, associated with $L$. As an example of such a potential $V$ one can take any $V \geq 0$, $V \in L^1_{loc}$, in one dimension.

In memory of Andrzej Hulanicki.

1. Introduction.

On $\mathbb{R}^d$ we consider a Schrödinger operator $L = -\Delta + V(x)$, where $V(x)$ is a locally integrable nonnegative potential, $V \neq 0$. It is well known that $-L$ generates the semigroup $\{T_t\}_{t>0}$ of linear contractions on $L^p(\mathbb{R}^d)$, $1 \leq p < \infty$. The Feynman-Kac formula implies that the integral kernels $T_t(x,y)$ of this semigroup satisfy
\[ 0 \leq T_t(x,y) \leq P_t(x-y) = (4\pi t)^{-d/2} \exp(-|x-y|^2/4t). \]

We say that an $L^1$-function $f$ belongs to the Hardy space $H^1_L$ if the maximal function
\[ M_L f(x) = \sup_{t>0} |T_t f(x)| \]
belongs to $L^1(\mathbb{R}^d)$. We set\[ \|f\|_{H^1_L} = \|M_L f\|_{L^1(\mathbb{R}^d)}. \]

Let $Q = \{Q_j\}_{j=1}^\infty$ be a family that consists of closed cubes with disjoint interiors such that $\mathbb{R}^d$ is the closure of $\bigcup_{j=1}^\infty Q_j$. We shall always assume that there exist constants $C, \beta > 0$ such that if $Q_i^{***} \cap Q_j^{***} \neq \emptyset$ then $d(Q_i) \leq Cd(Q_j)$, where $d(Q)$ denotes the diameter of $Q$, and $Q^*$ is the cube with the same center as $Q$ such that

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\(d(Q^*) = (1 + \beta)d(Q)\). Clearly, there is a constant \(C > 0\) such that

\[\sum_{j=1}^{\infty} 1_{Q_j^*}(x) \leq C.\]  

In order to state results from [4] we recall the notion of the local Hardy space associated with the collection \(Q\). We say that a function \(a\) is an \(H^1_{Q}\)-atom if there exists \(Q \in Q\) such that either \(a = |Q|^{-1}1_Q\) or \(a\) is the classical atom with support contained in \(Q^*\) (that is, there is a cube \(Q' \subset Q^*\) such that \(\text{supp } a \subset Q'\), \(\int a = 0\), \(|a| \leq |Q'|^{-1}\)).

The atomic space \(H^1_Q\) is defined by

\[H^1_Q = \{f : f = \sum_j \lambda_j a_j, \sum_j |\lambda_j| < \infty\},\]

where \(\lambda_j \in \mathbb{C}\), \(a_j\) are \(H^1_Q\)-atoms. We set

\[\|f\|_{H^1_Q} = \inf \{\sum_j |\lambda_j|\},\]

where the infimum is taken over all representations of \(f\) as in (1.3).

Following [4] we will also impose two additional assumptions on the potential \(V\) and the collection \(Q\) of cubes, mainly:

\[\text{(D)} \quad (\exists C, \varepsilon > 0) \sup_{y \in Q^*} \int T_{2^n d(Q)^2}(x, y) dx \leq C n^{-1-\varepsilon} \quad \text{for } Q \in Q, n \in \mathbb{N};\]

\[\text{(K)} \quad (\exists C, \delta > 0) \int_0^{2t} (1_{Q^*}-V) * P_s(x) ds \leq C \left(\frac{t}{d(Q)^2}\right) \delta \quad \text{for } x \in \mathbb{R}^d, Q \in Q, t \leq d(Q)^2.\]

Theorem 2.2 of [4] states that if we assume (D) and (K) then we have the following atomic characterization of the Hardy space \(H^1_{L}\):

\[f \in H^1_{L} \iff f \in H^1_{Q}. \quad \text{Moreover, } C^{-1} \|f\|_{H^1_Q} \leq \|f\|_{H^1_L} \leq C \|f\|_{H^1_Q}.\]

For \(j = 1, \ldots, d\), let

\[R_j f(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/\varepsilon} \frac{\partial}{\partial x_j} T_t f(x) dt \sqrt{t}\]

be the Riesz transform \(\frac{\partial}{\partial x_j} \mathcal{L}^{-1/2}\) associated with \(\mathcal{L}\), where the limit is understood in the sense of distributions (see Section 2). The main result of this paper is to prove that, under these conditions, the operators \(R_j\) characterize the space \(H^1_L\), that is, the following theorem holds.

**Theorem 1.5.** Assume that a potential \(V \geq 0\) and a collection of cubes \(Q\) are such that (D) and (K) hold. Then there exists a constant \(C > 0\) such that

\[C^{-1} \|f\|_{H^1_L} \leq \|f\|_{L^1(\mathbb{R}^d)} + \sum_{j=1}^{d} \|R_j f\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{H^1_L}.\]
Remark 1.7. For $\ell > 0$ denote by $Q_\ell(\mathbb{R}^n)$ a partition of $\mathbb{R}^n$ into cubes whose diameters have length $\ell$. Assume that for a locally integrable nonnegative potential $V_1$ on $\mathbb{R}^d$ and a collection $Q$ of cubes the conditions (D) and (K) hold. Consider the potential $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$, $x_1 \in \mathbb{R}^d$, $x_2 \in \mathbb{R}^n$, and the family $\mathcal{Q} = \{Q_1 \times Q_2 : Q_1 \in Q, Q_2 \in \mathcal{Q}_{d(Q_1)}(\mathbb{R}^n)\}$ of cubes in $\mathbb{R}^{d+n}$. It is easily seen that the pair $(V, \mathcal{Q})$ fulfills (D) and (K).

Remark 1.8. One can check that Theorem 2.2 of [4] (see (1.4)) and Theorem 1.5 together with their proofs remain true if we replace cubes by rectangles in the definition of atoms and in the conditions (D) and (K), provided the rectangles have side-lengths comparable to min$(\ell)$, and the conditions (D) and (K) hold for relevant collections $Q$ of rectangles. As a corollary of this observation we obtain that if $V$ satisfies (D) and (K) for relevant collections $Q$ of cubes, then $V$ satisfies (D) and (K) for relevant collections $Q$ of rectangles if we replace cubes by rectangles in the definition of atoms and in the conditions (D) and (K), provided the rectangles have side-lengths comparable to min$(\ell)$.

Examples. We finish the section by recalling some examples of nonnegative potentials $V$ considered in [1] and [4], such that the semigroups generated by $\Delta - V$ satisfy (D) and (K) for relevant collections $Q$ of cubes.

• The Hardy space $H^1_\mathcal{L}$ associated with one-dimensional Schrödinger operator $-\mathcal{L}$ was studied in Czaja-Zienkiewicz [1]. It was proved there that for any nonnegative $V \in L^1_{\text{loc}}(\mathbb{R})$ the collection $Q$ of maximal dyadic intervals $Q$ of $\mathbb{R}$ that are defined by the stopping time condition

$$|Q| \int_{16Q} V(y) \, dy \leq 1,$$

fulfills (D) for certain small $\beta > 0$ (see [1, Lemma 2.2]). The authors also remarked that (K) is satisfied. Indeed,

$$\int_{0}^{2t} (1_Q \cdots V) \, * P_s(x) \, ds \leq \int_{0}^{2t} \|1_Q \cdots V\|_{L^1} \|P_s\|_{L^\infty} \, ds \leq \int_{0}^{2t} |Q|^{-1} \frac{ds}{\sqrt{4\pi s}} \leq C \frac{t^{1/2}}{|Q|},$$

where in the second inequality we have used (1.9).

• $V(x) = |x|^{-d}$, $d \geq 3$, $\gamma > 0$. Then for $Q$ being the Whitney decomposition of $\mathbb{R}^d \setminus \{0\}$ that consists of dyadic cubes the conditions (D) and (K) hold (see Theorem 2.8 of [4]).

• $d \geq 3$, $V$ satisfies the reverse Hölder inequality with exponent $q > d/2$, that is,

$$\left( \frac{1}{|B|} \int_B V(y)^q \, dy \right)^{1/q} \leq C \frac{1}{|B|} \int_B V(y) \, dy$$

for every ball $B$. Define the family $Q$ by: $Q \in Q$ if and only if $Q$ is the maximal dyadic cube for which $d(Q)^2|Q|^{-1} \int_Q V(y) \, dy \leq 1$. Then the conditions (D) and (K) are true (see [4, Section 8]).
Let us finally mention that the Riesz transforms characterization of the Hardy spaces associated with Schrödinger operators with potentials satisfying the reverse Hölder inequality was proved in [2].

2. Auxiliary estimates

Lemma 2.1. For every \( \alpha > 0 \) there exists a constant \( C > 0 \) (independent of \( V \)) such that for \( j = 1, \ldots, d \) and \( y \in \mathbb{R}^d \) we have

\[
\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right|^2 \exp(\alpha |x - y|/\sqrt{t}) \, dx \leq Ct^{-d/2-1},
\]

(2.2)

\[
\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| \exp(\alpha |x - y|/\sqrt{t}) \, dx \leq Ct^{-1/2}.
\]

(2.3)

The lemma is known. For reader’s convenience we give a sketch of a proof in Section 4.

For \( \varepsilon > 0, j = 1, \ldots, d \), we define the operator

\[ R_j^\varepsilon f(x) = \int R_j^\varepsilon(x, y) f(y) \, dy, \]

where \( R_j^\varepsilon(x, y) = \int_{\varepsilon}^1 \frac{\partial}{\partial x_j} T_t(x, y) \frac{dt}{\sqrt{t}} \). It is not difficult to see that for \( f \in L^1(\mathbb{R}^d) \) the limits \( \lim_{\varepsilon \to 0} R_j^\varepsilon f(x) \) exist in the sense of distributions and define tempered distributions which will be denoted by \( R_j f \). Moreover, for \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) we have

\[
\|\langle R_j f, \varphi \rangle\| \leq C \|f\|_{L^1(\mathbb{R}^d)} \left( \|\varphi\|_{L^2(\mathbb{R}^d)} + \left\| \frac{\partial}{\partial x_j} \varphi \right\|_{L^\infty(\mathbb{R}^d)} \right).
\]

(2.4)

To see this we write

\[ R_j^\varepsilon \varphi(y) = \int_{\varepsilon}^1 \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} T_t(x, y) \varphi(x) \, dx \frac{dt}{\sqrt{t}} - \int_{\varepsilon}^1 \int_{\mathbb{R}^d} T_t(x, y) \frac{\partial}{\partial x_j} \varphi(x) \, dx \frac{dt}{\sqrt{t}}. \]

Since

\[ \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right|^2 \, dx \right]^{1/2} \frac{dt}{\sqrt{t}} \leq C \int_{\mathbb{R}^d} \varphi(x) \, dx \frac{dt}{\sqrt{t}} \]

and

\[ \int_{\mathbb{R}^d} \int_{\varepsilon}^1 T_t(x, y) \frac{dt}{\sqrt{t}} \, dx \leq 2 \]

(see Lemma 2.1), we conclude that \( R_j^\varepsilon \varphi(y) \) converges uniformly, as \( \varepsilon \to 0 \), to a bounded function which will be denoted by \( R_j^\varepsilon \varphi(y) \), and

\[ |R_j^\varepsilon \varphi(y)| \leq C \left( \|\varphi\|_{L^2(\mathbb{R}^d)} + \left\| \frac{\partial}{\partial x_j} \varphi \right\|_{L^\infty(\mathbb{R}^d)} \right). \]

For fixed \( Q \in \mathcal{Q} \) and \( 0 < \varepsilon < 1 \), let

\[ R_{j,Q,0}^\varepsilon(x, y) = \begin{cases} \int_{\varepsilon}^{d(Q)} \frac{\partial}{\partial x_j} T_t(x, y) \frac{dt}{\sqrt{t}} & \text{if } \varepsilon < d(Q)^2 < 1/\varepsilon; \\ \int_{\varepsilon}^{1/\varepsilon} \frac{\partial}{\partial x_j} T_t(x, y) \frac{dt}{\sqrt{t}} & \text{if } d(Q)^2 \geq 1/\varepsilon; \\ 0 & \text{if } d(Q)^2 \leq \varepsilon; \end{cases} \]

for
Lemma 2.5. Assume (D) holds. Then there exists a constant 

\[ R_{j,Q,\infty}(x,y) = \begin{cases} \int_{d(Q)^2}^{1/\varepsilon} \frac{\partial}{\partial x_j} T_t(x,y) \frac{dt}{\sqrt{t}} & \text{if } \varepsilon < d(Q)^2 < 1/\varepsilon; \\ 0 & \text{if } d(Q)^2 \geq 1/\varepsilon; \\ \int_{\varepsilon}^{d(Q)^2} \frac{\partial}{\partial x_j} T_t(x,y) \frac{dt}{\sqrt{t}} & \text{if } d(Q)^2 \leq \varepsilon. \end{cases} \]

Clearly, \( R_{j,Q,0}(x,y) = R_{j,Q,0}(x,y) + R_{j,Q,\infty}(x,y) \) for every \( Q \in \mathbb{Q} \) and \( 0 < \varepsilon < 1 \). For \( f \in L^1(\mathbb{R}^d) \) denote

\[ R_{j,Q,0} f(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} R_{j,Q,0,\varepsilon}(x,y) f(y) dy, \quad R_{j,Q,\infty} f(x) = \lim_{\varepsilon \to \infty} \int_{\mathbb{R}^d} R_{j,Q,\infty,\varepsilon}(x,y) f(y) dy, \]

which of course exist in the sense of distributions.

For \( Q \in \mathbb{Q} \) we define

\[ Q'(Q) = \{ Q' \in \mathbb{Q} : Q'' \cap (Q')^c \neq \emptyset \}, \quad Q''(Q) = \{ Q'' \in \mathbb{Q} : Q''' \cap (Q'')^c = \emptyset \}. \]

\[ \int_{\mathbb{R}^d} \sup_{0 < \varepsilon < 1} |R_{j,Q,\infty}(x,y)| dx \leq C \quad \text{for } y \in \bigcup_{Q' \in Q'(Q)} Q'^{c}. \]

Proof. Fix \( y \in \bigcup_{Q' \in Q'(Q)} Q'^{c} \). Let \( Q' \in Q'(Q) \) be such that \( y \in Q'^{c} \). Denote by \( S \) the left-hand side of (2.6). Then

\[
S \leq \int_{\mathbb{R}^d} \int_{\min(d(Q), d(Q'))^2} |\partial_{x_j} T_{t}(x,y)| \frac{dt}{\sqrt{t}} dx + \int_{\mathbb{R}^d} \int_{d(Q')^2} |\partial_{x_j} T_{t}(x,y)| \frac{dt}{\sqrt{t}} dx \\
= S_1 + S_2.
\]

Recall that \( d(Q) \sim d(Q') \). Using (2.3), we get

\[ S_1 \leq C \int_{\min(d(Q), d(Q'))^2} t^{-1} dt \leq C. \]

Applying (2.3) and (D), we obtain

\[
S_2 = \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \int_{2^n d(Q')^2} \int_{2^{n+1} d(Q')^2} \frac{dt}{\sqrt{t}} dx \\
\leq C \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \int_{2^n d(Q')^2} \int_{2^{n+1} d(Q')^2} \frac{dt}{\sqrt{t}} dx \\
\leq C \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} \frac{dt}{\sqrt{t}} dx \\
\leq C \sum_{n=0}^{\infty} \int_{\mathbb{R}^d} T_{2^n d(Q')^2}(z,y) dz \leq C + C \sum_{n=1}^{\infty} n^{-1-\varepsilon} \leq C.
\]

For \( 0 \leq \varepsilon < d(Q)^2 \) let

\[ W_{j,Q}(x,y) = \int_{\varepsilon}^{d(Q)^2} \frac{\partial}{\partial x_j} (T_t(x,y) - P_t(x-y)) \frac{dt}{\sqrt{t}}. \]
Set \( W^e_{j,Q} f(x) = \int W^e_{j,Q}(x,y) f(y) \, dy, \) \( W_{j,Q} f = W^0_{j,Q} f. \)

**Lemma 2.8.** Assuming (K) there exists a constant \( C > 0 \) such that for every \( Q \in \mathcal{Q} \) one has

\[
\sup_{y \in Q^*} \int_{\mathbb{R}^d} \int_0^1 \frac{d(Q)^2}{\sqrt{t}} \left| \frac{\partial}{\partial x_j} \left( T_t(x,y) - P_t(x,y) \right) \right| \, dt \, dx \leq C.
\]

**Proof.** The proof borrows some ideas from [1, Lemma 2.3]. Fix \( j \in \{1, \ldots, d\} \) and denote

\[
J_Q(x,y) = \int_0^1 \frac{d(Q)^2}{\sqrt{t}} \left| \frac{\partial}{\partial x_j} \left( T_t(x,y) - P_t(x,y) \right) \right| \, dt.
\]

The perturbation formula asserts that

\[
T_t - P_t = - \int_0^t P_{t-s} VT_s \, ds.
\]

Therefore

\[
J_Q(x,y) \leq \int_0^1 \frac{d(Q)^2}{\sqrt{t}} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-z) \right| V_1(z) T_s(z,y) \, dz \, ds \, \frac{dt}{\sqrt{t}}
\]

\[
+ \int_0^1 \frac{d(Q)^2}{\sqrt{t}} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-z) \right| V_1(z) T_{s'}(z,y) \, dz \, ds \, \frac{dt}{\sqrt{t}}
\]

\[
+ \int_0^1 \frac{d(Q)^2}{\sqrt{t}} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-z) \right| V_2(z) T_s(z,y) \, dz \, ds \, \frac{dt}{\sqrt{t}}
\]

\[
= J'_1(x,y) + J''_1(x,y) + J_2(x,y),
\]

where \( V_1(x) = V(x) 1_{Q^+}, V_2(x) = V(x) - V_1(x). \)

To evaluate \( J'_1 \) observe that

\[
\int_{\mathbb{R}^d} \left| \frac{\partial}{\partial x_j} P_{t-s}(x-y) \right| \, dx \leq C t^{-1/2} \quad \text{for } 0 < s < t/2.
\]

Thus, using (K), we get

\[
\int_{Q^+} J'_1(x,y) \, dx \leq C \int_0^{d(Q)^2} \int_0^{t/2} \int_{\mathbb{R}^d} t^{-1/2} V_1(z) P_{s}(z-y) \, dz \, ds \, \frac{dt}{\sqrt{t}}
\]

\[
\leq C \int_0^{d(Q)^2} t^{-1/2} \left( \frac{t}{d(Q)^2} \right)^\delta \, dt \leq C.
\]

Similarly,

\[
\int_{Q^+} J''_1(x,y) \, dx \leq C \int_0^{d(Q)^2} \int_0^{t/2} \int_{\mathbb{R}^d} (t-s)^{-1/2} V_1(z) P_{t-s}(z-y) \, dz \, ds \, \frac{dt}{\sqrt{t}}
\]

\[
= C' \int_0^{d(Q)^2} \int_{\mathbb{R}^d} V_1(z) P_t(z-y) \, dz \, dt \leq C.
\]

In order to estimate \( J_2 \) we notice that

\[
(2.9) \quad \left| \frac{\partial}{\partial x_j} P_{t-s}(x-z) \right| \leq C d(Q)^{d-1} e^{-c(t|x-z|/d(Q))^2}
\]
for $0 < s < t < d(Q)^2$, $z \notin Q^{**}$, $x \in Q^{**}$. Lemma 3.10 of [4] asserts that
\[
\sup_{y \in \mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} V(z) T_s(z, y) \, dz \, ds \leq C.
\]
Hence, by (2.9), we obtain
\[
\int_{Q^{**}} J_2(x, y) \, dx \leq C d(Q)^{-1} \int_0^{d(Q)^2} \int_0^t \int_{\mathbb{R}^d} V_2(z) T_s(z, y) \, dz \, ds \, \frac{dt}{\sqrt{t}} \leq C d(Q)^{-1} \int_0^{d(Q)^2} \int_0^t \int_{\mathbb{R}^d} V(z) T_s(z, y) \, dz \, ds \, \frac{dt}{\sqrt{t}} \leq C.
\]
We now turn to estimate $J_Q(x, y)$ for $x \notin Q^{**}$ and $y \in Q^*$. Clearly,
\[
\int_{Q^{**}} J_Q(x, y) \, dx \leq \int_{Q^{**}} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| \left| \frac{\partial}{\partial x_j} P_t(x - y) \right| \, \frac{dt}{\sqrt{t}} \, dx + \int_{Q^{**}} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} P_t(x - y) \right| \, \frac{dt}{\sqrt{t}} \, dx = J'_Q + J''_Q.
\]
Using (2.2) combined with the Cauchy-Schwarz inequality we get
\[
J'_Q \leq \int_0^{d(Q)^2} \left( \int_{Q^{**}} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right|^2 e^{2|x-y|/\sqrt{t}} \, dx \right)^{1/2} \left( \int_{Q^{**}} e^{-2|x-y|/\sqrt{t}} \, dx \right)^{1/2} \, \frac{dt}{\sqrt{t}} \leq C \int_0^{d(Q)^2} t^{-d/4-1/2} \left( \int_{Q^{**}} \left( \frac{\sqrt{t}}{|x-y|} \right)^N \, dx \right)^{1/2} \, \frac{dt}{\sqrt{t}} \leq C.
\]
The estimates for $J''_Q$ go in the same way. Hence
\[
\sup_{y \in Q^*} \int_{Q^{**}} J_Q(x, y) \, dx \leq C.
\]

Let $\{\phi_Q\}_{Q \in \mathcal{Q}}$ be a family of smooth functions that form a resolution of identity associated with $\{Q^*\}_{Q \in \mathcal{Q}}$, that is, $\phi_Q \in C_c^\infty(Q^*)$, $0 \leq \phi_Q \leq 1$, $|\nabla \phi_Q(x)| \leq C d(Q)^{-1}$, $\sum_{Q \in \mathcal{Q}} \phi_Q(x) = 1$ a.e.

The following corollary follows easily from Lemma 2.8.

**Corollary 2.11.** For $f \in L^1(\mathbb{R}^d)$ we have
\[
\lim_{\varepsilon \to 0} \|W_{\varepsilon, Q}(\phi_Q f) - W_j, Q(\phi_Q f)\|_{L^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \|W_j, Q(\phi_Q f)\|_{L^1(\mathbb{R}^d)} \leq C \|\phi_Q f\|_{L^1(\mathbb{R}^d)}
\]
with $C$ independent of $Q$ and $f$.

**Lemma 2.12.** There exists a constant $C > 0$ such that for every $Q \in \mathcal{Q}$ and every $f \in L^1(\mathbb{R}^d)$ such that $\text{supp} \, f \subset \tilde{Q} = \bigcup_{Q' \in \mathcal{Q}(Q)} Q'$ we have
\[
\|R_j(\phi_Q f) - \phi_Q R_j f\|_{L^1(\mathbb{R}^d)} \leq C \|f\|_{L^1(\tilde{Q})}.
\]

\[\square\]
Proof. Note that
\[ R_j(\phi_Q f)(x) - \phi_Q(x)R_j f(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/\varepsilon} \left( \frac{\partial}{\partial x_j} T_t(x, y) \right) (\phi_Q(y) - \phi_Q(x)) f(y) \, dy \, \frac{dt}{\sqrt{t}}. \]

From (2.2) we conclude
\[ \int_{\mathbb{R}^d} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| (\phi_Q(y) - \phi_Q(x)) \left| \frac{x - y}{\sqrt{t}} \right| dt \, dx \]
\[ \leq C \frac{d(Q)}{d(Q)^2} \int_{\mathbb{R}^d} \int_0^{d(Q)^2} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| e^{\frac{|x - y|}{\sqrt{t}}} dt \, dx \leq C. \]

Now (2.13) follows from (2.6) and (2.14). \qed

The following lemma is motivated by [4, Lemma 3.8].

**Lemma 2.15.** There exists a constant \( C > 0 \) such that
\[ \sum_{Q' \in Q} \left\| 1_{Q \ast \cdots \ast Q_j} \left( \sum_{Q'' \in Q''(Q)} \phi_{Q''} f \right) \right\|_{L^1(\mathbb{R}^d)} \leq C \| f \|_{L^1(\mathbb{R}^d)}. \]

**Proof.** Let \( S \) denote the left-hand side of (2.16). Applying (1.2), we have
\[ S \leq \sum_{Q' \in Q} \sum_{Q'' \in Q''(Q)} \left\| 1_{Q \ast \cdots \ast Q_j} (\phi_{Q''} f) \right\|_{L^1(\mathbb{R}^d)} \]
\[ = \sum_{Q'' \in Q} \sum_{Q \in Q''(Q')} \left\| 1_{Q \ast \cdots \ast Q_j} \phi_{Q''} f \right\|_{L^1(\mathbb{R}^d)} \]
\[ \leq C \sum_{Q'' \in Q} \left\| R_j(\phi_{Q''} f) \right\|_{L^1((Q'')^*)} \]
\[ \leq C \sum_{Q'' \in Q} \left\| R_{j', Q'', Q}(\phi_{Q''} f) \right\|_{L^1((Q'')^*)} + C \sum_{Q'' \in Q} \left\| R_{j', Q'', Q}(\phi_{Q''} f) \right\|_{L^1((Q'')^*)}. \]

Using (2.6) and (1.2), we get
\[ \sum_{Q'' \in Q} \left\| R_{j', Q'', Q}(\phi_{Q''} f) \right\|_{L^1((Q'')^*)} \leq C \sum_{Q'' \in Q} \| \phi_{Q''} f \|_{L^1(\mathbb{R}^d)} \leq C' \| f \|_{L^1(\mathbb{R}^d)}. \]

Identically as in (2.10) for \( y \in (Q'')^* \) we have
\[ \int_{(Q'')^*} \int_0^{d(Q'')^2} \left| \frac{\partial}{\partial x_j} T_t(x, y) \right| \, dt \, dx \leq C, \]
which implies
\[ \sum_{Q'' \in Q} \left\| R_{j', Q'', Q}(\phi_{Q''} f) \right\|_{L^1((Q'')^*)} \leq C \sum_{Q'' \in Q} \| \phi_{Q''} f \|_{L^1(\mathbb{R}^d)} \leq C \| f \|_{L^1(\mathbb{R}^d)}. \]

The lemma is a consequence of (2.17)-(2.19). \qed
3. Proof of Theorem 1.5

In order to prove the second inequality of (1.6) it suffices by (2.4) and (1.4) to verify that there exists a constant $C > 0$ such that

\begin{equation}
\| R_j a \|_{L^1(\mathbb{R}^d)} \leq C
\end{equation}

for every $H^1_Q$-atom $a$ and $j = 1, \ldots, d$. Assume that $a$ is an $H^1_Q$-atom supported by a cube $Q^*$, $Q \subset Q$. Then

\begin{align*}
R_j a(x) &= \lim_{\varepsilon \to 0} \left( R_{j,Q,0}^\varepsilon a(x) + R_{j,Q,\infty}^\varepsilon a(x) \right) \\
&= \lim_{\varepsilon \to 0} \left( W_{j,Q}^\varepsilon a(x) + H_{j,Q,\infty}^\varepsilon a(x) + R_{j,Q,\infty}^\varepsilon a(x) \right),
\end{align*}

where $H_{j,Q}^\varepsilon a(x) = \int_{\varepsilon^d(Q)^2} \frac{a}{\partial x_j} (a * P_\varepsilon)(x) \frac{dx}{\sqrt{t}}$. Similarly to (2.4), the limit

\begin{equation}
H_{j,Q} a(x) = \lim_{\varepsilon \to 0} H_{j,Q}^\varepsilon a(x)
\end{equation}

exists in the sense of distributions. Moreover, by the boundedness of the local Riesz transforms on the local Hardy spaces (see [7]), we have $\| H_{j,Q} a \|_{L^1(\mathbb{R}^d)} \leq C$ with $C$ independent of $a$. Using Lemmas 2.8 and 2.5, we obtain (3.1).

We now turn to prove the first inequality of (1.6). To this end, by the local Riesz transform characterization of the local Hardy spaces (see [7, Section 2]), it suffices to show that

\begin{equation}
\sum_{Q \in Q} \| H_{j,Q} (\phi_Q f) \|_{L^1(\mathbb{R}^d)} \leq C \left( \| f \|_{L^1(\mathbb{R}^d)} + \| R_j f \|_{L^1(\mathbb{R}^d)} \right), \quad j = 1, \ldots, d.
\end{equation}

Clearly,

\begin{equation}
H_{j,Q} (\phi_Q f) = -W_{j,Q} (\phi_Q f) + R_{j,Q,0} (\phi_Q f).
\end{equation}

Lemma 2.8 together with (1.2) implies

\begin{equation}
\sum_{Q \in Q} \| W_{j,Q} (\phi_Q f) \|_{L^1(\mathbb{R}^d)} \leq C \sum_{Q \in Q} \| \phi_Q f \|_{L^1(\mathbb{R}^d)} \leq C \| f \|_{L^1(\mathbb{R}^d)}.
\end{equation}

Note that

\begin{equation}
R_{j,Q,0} (\phi_Q f) = -R_{j,Q,\infty} (\phi_Q f) + \left[ R_j \left( \phi_Q \sum_{Q' \in Q'(Q)} (\phi_{Q'} f) \right) - \phi_Q R_j \left( \sum_{Q' \in Q'(Q)} (\phi_{Q'} f) \right) \right]
\end{equation}

\begin{equation}
- \phi_Q R_j \left( \sum_{Q'' \in Q''(Q)} (\phi_{Q''} f) \right) + \phi_Q R_j f.
\end{equation}

Lemmas 2.5, 2.12, and 2.15 combined with (3.4) imply

\begin{equation}
\sum_{Q \in Q} \| R_{j,Q,0} (\phi_Q f) \|_{L^1(\mathbb{R}^d)} \leq C \left( \sum_{Q \in Q} \| \phi_Q f \|_{L^1(\mathbb{R}^d)} + \sum_{Q \in Q, Q' \in Q'(Q)} \| \phi_{Q'} f \|_{L^1(\mathbb{R}^d)} \right)
\end{equation}

\begin{equation}
+ \| f \|_{L^1(\mathbb{R}^d)} + \sum_{Q \in Q} \| \phi_Q R_j f \|_{L^1(\mathbb{R}^d)}
\end{equation}

\begin{equation}
\leq C \left( \| f \|_{L^1(\mathbb{R}^d)} + \| R_j f \|_{L^1(\mathbb{R}^d)} \right).
\end{equation}

Now (3.2) follows from (3.3) and (3.5).
4. Proof of Lemma 2.1

The proof is based on estimates of the semigroup $T_t$ acting on weighted $L^2$ spaces. This technique was utilize e.g. in [5], [8], [3].

Fix $y_0 \in \mathbb{R}^d$ and $\alpha > 0$. The semigroup $\{T_t\}_{t \geq 0}$ acting on $L^2(e^{\alpha|x-y_0|} \, dx)$ has the unique extension to a holomorphic semigroup $T_\zeta$, $\zeta \in \{\zeta \in \mathbb{C} : |\text{Arg} \zeta| < \pi/4\}$ such that

\begin{equation}
\|T_\zeta\|_{L^2(e^{\alpha|x-y_0|} \, dx) \rightarrow L^2(e^{\alpha|x-y_0|} \, dx)} \leq C'e^{c'\alpha^2 R\zeta}
\end{equation}

with $C$ and $c'$ independent of $V$ and $y_0$ (see, e.g., [3, Section 6]). Let $-\mathcal{L}_\alpha$ denote the infinitesimal generator of $\{T_t\}_{t \geq 0}$ considered on $L^2(e^{\alpha|x-y_0|} \, dx)$. The quadratic form $Q = Q_{\alpha,y_0}$ associated with $\mathcal{L}_\alpha$ is given by

\begin{equation}
Q(f,g) = \sum_{j=1}^d \int_{\mathbb{R}^d} \frac{\partial}{\partial x_j} f(x) \frac{\partial}{\partial x_j} g(x) e^{\alpha|x-y_0|} \, dx + \int_{\mathbb{R}^d} V(x) f(x) g(x) e^{\alpha|x-y_0|} \, dx
\end{equation}

\begin{equation}
+ \sum_{j=1}^d \int_{\mathbb{R}^d} f(x) g(x) \frac{\partial}{\partial x_j} e^{\alpha|x-y_0|} \, dx,
\end{equation}

$D(Q) = \{f \in L^2(e^{\alpha|x-y_0|} \, dx) : V(x)^{1/2} f(x), \frac{\partial}{\partial x_j} f(x) \in L^2(e^{\alpha|x-y_0|} \, dx), j = 1, \ldots, d\}$.

Note that

\begin{equation}
\left| \frac{\partial}{\partial x_j} e^{\alpha|x-y_0|} \right| \leq C\alpha e^{\alpha|x-y_0|} \quad \text{for } x \neq y_0.
\end{equation}

Clearly,

\begin{equation}
|Q(f,g)| \leq C_{\alpha} \|f\|_Q \|g\|_Q
\end{equation}

with $C_{\alpha}$ independent of $y_0$ and $V$, where

\begin{equation}
\|f\|^2_Q = \int_{\mathbb{R}^d} \left( \sum_{j=1}^d \left| \frac{\partial}{\partial x_j} f(x) \right|^2 + V(x)|f(x)|^2 + |f(x)|^2 \right) e^{\alpha|x-y_0|} \, dx.
\end{equation}

Moreover, there exists a constant $C > 0$ independent of $V$ and $y_0$ such that

\begin{equation}
\|f\|^2_Q \leq C Q(f,f).
\end{equation}

The holomorphy of the semigroup $T_t$ combined with (4.1) imply

\begin{equation}
\|\mathcal{L}_\alpha T_t g\|_{L^2(e^{\alpha|x-y_0|} \, dx)} \leq C't^{-1} e^{c''t\alpha^2} \|g\|_{L^2(e^{\alpha|x-y_0|} \, dx)}
\end{equation}

with constants $C'$ and $c''$ independent of $V$ and $y_0$. Setting $g(x) = T_{1/2}(x,y_0)$, $f(x) = T_{1/2} g(x) = T_1(x,y_0)$ and using (4.4), (4.5), (4.1), and (1.1), we get

\begin{equation}
\left\| \left( \frac{\partial}{\partial x_j} T_1(x,y_0) \right)^2 \right\|^2_{L^2(e^{\alpha|x-y_0|} \, dx)} \leq \|f\|^2_Q \leq C Q(f,f)
\end{equation}

\begin{equation}
\leq C\|\mathcal{L}_\alpha f\|_{L^2(e^{\alpha|x-y_0|} \, dx)} \|f\|_{L^2(e^{\alpha|x-y_0|} \, dx)} \leq C'' \|g\|^2_{L^2(e^{\alpha|x-y_0|} \, dx)} \leq C'''
\end{equation}
with $C'''$ independent of $y_0$ and $V$. Since $T_t(x, y) = t^{-d/2} \tilde{T}_1(x/\sqrt{t}, y/\sqrt{t})$, where $\{\tilde{T}_s\}_{s > 0}$ is the semigroup generated by $\Delta - tV(\sqrt{tx})$, we get (2.2) from (4.6), because $C'''$ is independent of $V$ and $y_0$. Now (2.3) follows from (2.2) and the Cauchy-Schwarz inequality.

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