2017

Comparison of ambiguity function of eigenwaveform to wideband and pulsed radar waveforms: a comprehensive tutorial

Nieh, Jo-Yen; Romero, Ric A.

IET

Nieh, Jo-Yen, and Ric A. Romero. "Comparison of ambiguity function of eigenwaveform to wideband and pulsed radar waveforms: a comprehensive tutorial." The Journal of Engineering 2018.4 (2018): 203-221.
http://hdl.handle.net/10945/61053

Downloaded from NPS Archive: Calhoun
Comparison of ambiguity function of eigenwaveform to wideband and pulsed radar waveforms: a comprehensive tutorial

Jo-Yen Nieh¹, Ric A. Romero²

¹Department of Electrical and Electronic Engineering, Chung Cheng Institute of Technology, National Defense University, Taoyuan, Taiwan
²Department of Electrical and Computer Engineering, Naval Postgraduate School, Monterey, CA, USA
E-mail: rromero@nps.edu

Published in The Journal of Engineering; Received on 22nd September 2017; Revised on 29th November 2017; Accepted on 17th December 2017

Abstract: The authors present a comprehensive research tutorial on the ambiguity function (AF) of eigenwaveform for extended targets compared to AFs of common radar waveforms (e.g. wideband and pulsed waveforms). They present new findings of AF properties that contradict classical AF results for the point target assumption. It is shown that the AF properties (peak and volume) for an extended target are not constant thereby contradicting AF properties of waveforms for a point target. They investigate corresponding AFs and note many advantages (or few disadvantages) of using eigenwaveform compared to classical wideband and rectangular-pulsed waveforms. They investigate unambiguous range, range resolution, Doppler resolution, and detection probability of various waveforms for insightful engineering trade-offs. For illustration, they utilise two extended target models to show that these parameters are not only functions of the transmit waveform but also of the nature of the target. They use both single-pulse and pulse-train waveforms to produce AFs to illustrate the effect on Doppler and range ambiguities and resolutions. Finally, they investigate range-Doppler map (RDM) applications of the traditional waveforms and compared them with eigenwaveform RDM. They conclude that the eigenwaveform is superior in probability of detection and Doppler considerations compared to wideband and rectangular waveforms.

1 Introduction

Optimum waveform design for extended target using the signal-to-noise ratio (SNR) criterion for a deterministic response and mutual information (MI) for a stochastic response in additive Gaussian noise was first tackled in [1]. For the problem of matched waveform design for a deterministic extended target in signal-dependent interference, Pillai et al. [2] proposed an algorithm to form a finite duration transmit waveform for a known target. In [3], the SNR-based optimum matched waveform for a known target response in a signal-dependent interference was derived in the frequency domain using the SNR criterion. For both deterministic and stochastic target responses in signal-dependent interference, optimum waveform design in the frequency domain using SNR and MI criteria were also addressed. The optimum waveform for known extended target in Gaussian noise derived in [1] was termed eigenwaveform in [3]. In [4], Kay derived in the frequency domain the optimum waveform for Gaussian-distributed point target in Gaussian clutter. In [5], eigenwaveform design was considered in improving or mitigating the effects of moving extended targets in terms of detection probability. Both works in [5, 6] discussed the detection probability with the use of eigenwaveform for known extended target in Gaussian noise. The works in [7, 8] applied the notion of adaptive matched waveforms (e.g. eigenwaveform) in a cognitive radar platform.

Our preliminary work on investigating ambiguity function (AF) of extended targets using the eigenwaveform was shown in [6]. Traditional AF analysis for point target has a rich literature. As such we do not attempt to cite all relevant works but we point out excellent texts in [9–12]. Our interest, however, is extended targets. Unlike traditional waveforms for a point target whose transmit responses totally dictate the shape of the AF, both transmit waveform and extended target response contribute to the shape of the AF. To illustrate this important point in this tutorial, we create two target types in this paper (oscillatory and non-oscillatory). We also look at the properties (peak and volume) of the AF for an extended target which turn out to be different compared to the AF peak and volume with the point target assumption. AF analysis for point target states that the AF peak and the volume under the AF from traditional matched filter analysis are constant. We show new findings that the AF for an extended target can have various peaks and that the volumes are different for various waveforms. The converse is also true, i.e. given a transmit waveform, various target responses yield AFs that have different peaks and volumes. We feature the eigenwaveform as an excellent waveform for providing large AF peaks. It is also an excellent waveform in terms of detection performance. It is also shown to have a nice AF sidelobe response especially in the Doppler spread domain. Coherent pulse train (multiple-pulse waveform) is a common pulse-Doppler waveform. As such, we study both one-pulse and multiple-pulse AFs with the use of eigenwaveform. We calculate various AFs as a function of varying parameters such as time-on-target, pulse-repetition frequency, number of pulses, increasing signal-to-noise ratios and so on. Also, range-Doppler map (RDM) application is considered. Various performance parameters of the eigenwaveform are compared to other waveforms such as wideband and rectangular waveforms. While other wideband waveforms are of course possible, we specifically consider the wideband impulse waveform since our interest here is pulsed waveforms (e.g. pulsed Doppler waveforms). Among the newer AF results for eigenwaveform and insightful experiments, the paper’s main contribution is the comprehensive investigation and exposition of the theory and application of eigenwaveform-based AF for extended targets as compared to the two classical radar waveforms.

This paper is organised as follows. In Section 2, we derive the AF for an extended target given a transmit waveform. We show that various waveforms can have differing peak values and that the eigenwaveform has the largest AF peak compared to all possible waveforms. In Section 3, we discuss a minimum target range (or delay) separation for various waveforms (specially eigenwaveform) and its relation to range resolution. We also consider AF of various one-pulse waveforms (including the eigenwaveform) such that we can qualitatively gauge and compare parameters such as range
resolution, Doppler resolution, and AF peaks. In radar, coherent pulse transmission is a common technique so in Section 4 we consider AFs produced by multiple-pulse eigenwaveform, wideband, and rectangular waveforms. Comparison and trade-offs are discussed. In Section 5, we consider the detection performance of the eigenwaveform as compared to other waveforms. We also present RDM application of the multiple-pulse eigenwaveform. In Section 6, we present our conclusions.

2 Properties of the AF for extended targets and the eigenwaveform

In classical AF studies, the received signal is assumed to be a replica of the transmit waveform. In other words, targets are assumed to be point targets. Again, AF analysis of various popular waveforms such as pulsed, phase-coded, and chirp waveforms assuming a point target is mature. Aside from the excellent texts cited above, we point out a few select ones such as [13–17]. Of course AFs due to MIMO (multiple-input, multiple-output) radar waveforms are possible [18, 19]. Our interest, however, is extended targets and how various transmit waveforms affect AFs. More specifically, our interest is how both an extended target and its corresponding eigenwaveform design affect the AF.

In radar application, the AF features the output of the matched filter as a function of Doppler shift and time delay is given by

\[ |\mathcal{X}(\tau, f_d)|^2 = \left| \int_{-\infty}^{\infty} x(t) e^{-j2\pi f_d t} dt \right|^2, \tag{1} \]

where \(x(t)\) is the transmit waveform, \(f_d\) is the Doppler shift, and \(\tau\) is the time delay. Notice that this definition where squared-magnitude of the matched filter response is used in [11]. Sometimes the magnitude of the matched filter response is used as in [9]. To avoid confusion, we will choose one and use the squared-magnitude definition of the AF. Again, it is clear from (1) that the received signal is the exact replica of the transmit waveform (assuming no noise or interference) where the target is assumed to be a point target. Unfortunately, (1) does not work when the target has a finite extent. In the case of an extended target, the AF is not only a function of transmit waveform but is also a function of the target response. If the target response is given by \(h(t)\) then the AF is given by

\[ |\mathcal{X}(\tau, f_d)|^2 = \left| \int_{-\infty}^{\infty} x(t) h(t) e^{-j2\pi f_d t} dt \right|^2, \tag{2} \]

where \(x(t)\) is the convolution return of transmit signal \(x(t)\) and target response \(h(t)\). It is clear in (2) that for extended targets, both transmit waveform and target response dictate the AF. Our interest is to form the AF of the eigenwaveform given an extended target response and compare that function with AFs produced by traditional waveform such as wideband pulse and rectangular pulse waveforms.

2.1 Discrete or vector signal modelling

Due to the advent of arbitrary waveform generators, modern waveforms are designed in discrete-time. Moreover, the modern waveform design is usually simulated by modern computing, which utilises discrete-frequency techniques. Proper digital-to-analogue conversion (DAC) easily converts the waveform into a practical continuous waveform. Assuming a proper sampling rate through Nyquist theory, we can utilise discrete-time and discrete-frequency models to form a discrete version of the AF for extended target which is given by

\[ |\mathcal{X}(\tau_n, f_{d_n})|^2 = \left| \sum_{i=0}^{N-1} s[i] x[i - \tau_n] W_{N}^{f_{d_n}} \right|^2 \]

where \(\tau_n\) is the discrete delay and \(f_{d_n}\) is the Doppler spread index, \(s[i]\), \(h[i]\) and \(s[i]\) are the discrete transmit, target response and return signals, respectively. \(N\) is the length of return vector \(s[i]\), \(f_{d_n} = 0, 1, N - 1\), and

\[ W_{N}^{f_{d_n}} = e^{j2\pi f_{d_n}/N}. \tag{3} \]

In general, the length of \(f_{d_n}\) may be desired to be different from \(N\). That extension is easily incorporated into (3).

The peak of the AF is located at zero-delay and zero-Doppler (\(f_{d_n} = 0; \tau_n = 0\)). That is

\[ |\mathcal{X}(\tau_n, f_{d_n})|_{peak}^2 = |x[0; 0]|^2 = \sum_{i=0}^{N} s[i] x[i] \tag{4} \]

It seems that this peak may not be constant given an extended target considering various transmit waveforms but this requires proof. Before going into the extended target case, let’s examine the classical point target case. If the target is a (non-fluctuating) point target then \(h[i]\) has an impulse response (i.e. no extent) and thus the peak is given by

\[ |\mathcal{X}(\tau_n, f_{d_n})|_{peak}^2 = E_{p} = E_{p} \sum_{i=0}^{N} x[i] x[i] = E_{p}^2. \tag{5} \]

where \(E_{p}\) is the energy associated with the point target and it is clear from (5) that the AF peak is always a constant no matter what kind of waveform is used [9, 11]. In fact for point targets (non-fluctuating), the volume under the AF for all possible transmit waveforms is also constant and is given by \(E_{p}^2 E_{p}^2\). If the received point target energy and transmit energy are normalised (\(E_{p}^2 = 1, E_{p}^2 = 1\)) then the peak is 1 [9, 11]. The properties that AF peak and volume are constant regardless of transmit waveform are the basis of designing waveforms on how to manipulate ambiguity side lobes in AF. We will show in this work that for an extended target, various AFs arise depending on the transmit waveform and that these properties may not hold for extended targets.

2.2 Properties of the AF

2.2.1 Property 1: Given an extended target, the peak of the AF is different for different transmit waveforms: In other words, the AF peak is not constant for all waveforms, i.e. \(E_{p}^2\) in (4) is not constant unlike in the point target case whose peak is constant as given by (5). We will now prove Property 1.

For compact representation, let \(h\) be the complex-valued discrete-time target vector of length \(n\), then we can easily form \(h = \sqrt{E_{h}} h\) where \(E_{h}\) is the energy of the target response and the normalised target response \(h\) clearly has an energy of 1. Let \(x\) be an arbitrary transmit signal vector of length \(n\). From [5], we can form a

This is an open access article published by the IET under the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0/).
normalised target convolution matrix given by

\[ H = \begin{bmatrix}
  h & 0 & 0 & 0 \\
  . & h & 0 & 0 \\
  . & . & \ddots & h \\
  . & . & . & h
\end{bmatrix} \]

where \( H \) is a \((2n - 1) \times n\) matrix. Thus, any arbitrary transmit waveform \( x \) yields a corresponding AF whose peak is given by

\[ |\lambda_n(f; x)|_{\text{peak}} = \max_{\forall i; j} |x^H h[i]|^2 \]

\[ = \left| x^H \bar{x} \right|^2 = \frac{1}{E_h} \left| \sum_{i} h_i^* h_i \right|^2 \]

\[ = \frac{1}{E_h} \left| \sum_{i} h_i^* h_i \right|^2 \]

(6)

It is clear in (6) that various peaks are possible depending on what \( x \) is. In fact, it should be clear that the value of the peak depends on the how \( x \) correlates with the target response (represented by \( \sqrt{E} \bar{x} \)). If various peaks are possible, then it is our goal to find the AF (from all possible \( x \)) which yields the maximum peak among all possible peaks.

The largest peak can be found by maximising the AF peak for all possible transmit waveforms \( x \) given \( H \) via

\[ \max_{\forall x; H} |\lambda_n(f; x)|_{\text{peak}} = \max_{\forall x; H} |x^H \bar{x}|^2 \]

where we let

\[ \bar{R}_h = H^H H \]

Based on the structure of \( H, \bar{R}_h \) will have the form

\[ \bar{R}_h = \frac{1}{E_h} \begin{bmatrix}
  \sum_i |h_i|^2 & \sum_i h_i^* h_{i+1} & \cdots & \sum_i h_i^* h_{N-1} \\
  \sum_i h_{i+1} h_i^* & \sum_i |h_i|^2 & \cdots & \sum_i h_i^* h_{N-1} \\
  \vdots & \ddots & \ddots & \vdots \\
  \sum_i h_i^* h_{N-1} & \cdots & \sum_i |h_i|^2 & \sum_i h_i^* h_{N-1}
\end{bmatrix} \]

(7)

or equivalently

\[ \bar{R}_h = \frac{1}{E_h} \begin{bmatrix}
  \sum_i |h_i|^2 & \sum_i h_i^* h_{i+1} & \cdots & \sum_i h_i^* h_{N-1} \\
  \sum_i h_{i+1} h_i^* & \sum_i |h_i|^2 & \cdots & \sum_i h_i^* h_{N-1} \\
  \vdots & \ddots & \ddots & \vdots \\
  \sum_i h_i^* h_{N-1} & \cdots & \sum_i |h_i|^2 & \sum_i h_i^* h_{N-1}
\end{bmatrix} \]

where all the diagonal components have the same energy as \( H \) (which is 1). It can be shown with some effort that the matrix \( \bar{R}_h \) is conjugate symmetric, Toeplitz and positive definite. That is

\[ \bar{R}_h \bar{R}_h^H = (H^H H)^H = H^H \bar{R}_h \]

The \( \bar{R}_h \) matches exactly the definition of sample correlation function from [20] but without the normalisation. We may refer to \( \bar{R}_h \) as the normalised target response autocorrelation matrix.

Notice that the only design parameter in (6) is the transmit waveform \( x \). In other words, maximising the peak in (6) is equivalent to maximising the argument \( x^H \bar{R}_h x \) which is given by

\[ \max_{\forall x; H} x^H \bar{R}_h x \]

Of course, the energy constraint is given by \( E_x \leq x^H x \). Notice the argument inside the squared magnitude in (7) is an eigenvalue problem [21, 22] where the eigenvalue and eigenvector relation are given by

\[ \bar{R}_h q_i = \lambda_i q_i \]

(8)

and

\[ \lambda_i = \frac{q_i^H \bar{R}_h q_i}{q_i^H q_i} = q_i^H \bar{R}_h q_i \]

(9)

where \( \lambda_i \) is any eigenvalue of the autocorrelation matrix \( \bar{R}_h \) and \( q_i \) is the corresponding unit-energy eigenvector of length \( n \).

Thus, the largest peak of the AF is achieved by taking the eigenvector \( q_{\text{max}} \) corresponding to the largest eigenvalue \( \lambda_{\text{max}} \) as the transmit waveform. In other words, the AF that produces the maximum peak is the return that convolves the eigenwaveform with the target response. Since \( q_{\text{max}} \) is the unit-energy, we need to incorporate any transmit energy constraint \( E_x \), i.e. the transmit waveform is \( x = \sqrt{E_x} q_{\text{max}} \). Thus, the maximum AF given a target response correlation matrix is

\[ \max_{\forall i; j} |\lambda_n(f; x)|_{\text{peak}} = \max_{\forall i; j} \left| x^H \bar{R}_h x \right|^2 \]

\[ = E_x^2 E_{\text{max}}^2 \left| q_{\text{max}}^H \bar{R}_h q_{\text{max}} \right|^2 \]

(10)

which says that the maximum peak is given by the squared-product of maximum eigenvalue, target energy and transmit energy. Moreover, this maximum peak comes from the AF produced by using the eigenwaveform.

In conclusion, given any specific target response, the maximum peak value of AF will be attained by taking eigenwaveform corresponding to maximum eigenvalue as transmit signal instead of any arbitrary waveform (and this includes traditional pulsed or wideband waveform). The maximum peak value is \( \lambda_{\text{max}} E_x^2 E_{\text{max}}^2 \) as opposed to \( E_x^2 \) in (4) whose value depends on both how well the transmit waveform matches with the target. Notice that a peak of \( E_x^2 E_{\text{max}}^2 \) can be attained by a special waveform. This AF peak is attained by an idealised impulse waveform [5]. In our paper, we implement the wideband waveform as practical implementation of the impulse waveform.

2.2.2 Property 2: Given an extended target, the volume of the AF is different for different transmit waveforms: In other words, the AF volume is not constant for all transmit waveforms. We will now prove Property 2. The volume of the extended target AF is given by

\[ V_{\text{AF}} = \frac{|\Delta_{\text{AF}}|}{N} \frac{\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |x^H \bar{R}_h x|^2}{N} \]

It can be shown by Parseval’s theorem for discrete Fourier transform that

\[ \sum_{i=0}^{N-1} |x^H \bar{R}_h x|^2 = \frac{1}{N} \sum_{i=0}^{N-1} |\lambda_n(f; x)|^2 \]

Thus, the AF volume becomes

\[ V_{\text{AF}} = \frac{1}{N} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |x^H \bar{R}_h x|^2 \]

\[ = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |x^H \bar{R}_h x|^2 \]

By substituting \( i_1 \) for \( i \) and \( i_2 \) for \( j \) (for \( i \neq j \)), where \( i_1 \) and \( i_2 \) cover the range from 0 to \( (N-1) \), the total volume of the AF is...
given by
\[
V_{\text{AF}} = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} |s[i_1] \times s^*[i_2]|^2
= \left( \sum_{i=0}^{N-1} |s[i]|^2 \right) \times \left( \sum_{j=0}^{N-1} |s^*[j]|^2 \right) = E_i^2
= E_i^2 |x|^2 R x^2
\]
where we use (4) and (6) complete the last line of the proof. It is clear that the AF volume for extended target is not constant and various waveforms can produce various volumes (and indeed interesting shapes) depending on the target response itself (not just the transmit waveform like in the point target case).

One of the two common radar waveforms, we will use in our paper is the wideband (impulse) waveform. An extended target illuminated by the wideband waveform has a volume that is given by
\[
V_{\text{AF}} = |\Delta t_s| \left| \frac{\Delta f_{\text{to}}}{N} \right| \sum_{\tau_s} \sum_{f_{\text{do}}} |X[\tau_s; f_{\text{do}}]|^2 = E_i^2 E_s^2.
\]
This is very interesting since this is the same volume produced if we illuminate a point target with any transmit waveform. In other words, for extended targets, only the wideband waveform can produce this volume!

With the use of the eigenwaveform, the largest AF volume possible is given by
\[
V_{\text{AF}} = |\Delta t_s| \left| \frac{\Delta f_{\text{to}}}{N} \right| \sum_{\tau_s} \sum_{f_{\text{do}}} |X[\tau_s; f_{\text{do}}]|^2 = \lambda_{\text{max}}^2 E_i^2 E_s^2.
\]
In other words, the volume is the same value as the largest peak. This volume is amplified by \(\lambda_{\text{max}}^2\) compared to that of the volume produced by the wideband waveform.

3 Ambiguity functions of one-pulse waveforms

3.1 Minimum target separation

For point targets with the use of basic one-pulse waveforms, the minimum (range or time delay) separation of two targets is dictated by the length of transmit waveform. It is defined as the minimum time separation needed so that the return pulses do not overlap each other, i.e. it is the minimum corresponding range separation required such that two point targets can be resolved. This is easily illustrated in Fig. 1a. Let \(T_x\) be the time duration of the transmit waveform. For the return waveforms not to overlap each other, two targets need to be separated by \(T_x = T_x/2\). For basic pulsed waveform (not chirp or waveforms employing compression), the corresponding minimum range in which two targets can be resolved is given by
\[
R_p = c T_x = c T_x/2
\]
where \(c\) is the speed of light (in free space). For basic one-pulsed waveforms, minimum range separation is sometimes referred to as range resolution. In practice, target resolution can be improved by waveform design (using compression techniques) that can potentially increase the bandwidth (\(B\)) of the waveform. As such, range resolution is usually given by
\[
R_p = c T_x = \frac{c}{2B}. \quad \text{(11)}
\]

Using minimum separation requirement, if the target has a response with duration \(T\) and that the eigenwaveform has same time duration \(T_x = T\), the falling edge of one target return has to be separated by \(T/2\) from the leading edge of another target return as illustrated in Fig. 1b. Thus, the minimum time needed in order for two extended targets to be resolved is given by \(T_p = 1.5 T\). Thus, the minimum range separation such that two targets can be resolved with the use of the eigenwaveform is given by
\[
R_p = c T_p = 1.5 c T. \quad \text{(12)}
\]
For an extended target, both the eigenwaveform and the target response dictate the effective bandwidth of the return signal. As such, a simple equation corresponding to (11) in terms of bandwidth does not easily apply. This is also the big difference between point target and extended target where waveform bandwidth is easily manipulated if a target is truly a point target. It is clear that for traditional waveforms (for point targets with wide bandwidths) the resolution promised by (11) will not be realised if used for extended targets (especially if a target has a narrowband response). This is due to the well-known fact that the Fourier transform of the return signal is the result of multiplication of the frequency responses of the transmit waveform and target response. Since a practical extended target does not have an idealised target frequency response like a point target, the frequency response of the return signal will surely be different from the frequency response of the transmit waveform. As such, large bandwidths may be decreased.

3.2 Eigenwaveform, wideband, rectangular waveforms

Traditionally for point targets, a pulse (usually shaped) is a basic choice for transmit signal. For extended targets in this paper, we investigate two pulsed waveform types and compare that to the eigenwaveform. The waveforms are wideband waveform (i.e. a very narrow pulse compared to the length of the extended target) and rectangular waveform (i.e. a rectangular pulse with length less or equal to that of the extended target). Given a target response, we derive and illustrate AFs for these two waveforms and compare them to the AF of the eigenwaveform. In Fig. 2, we illustrate the finite-length of the three waveforms (wideband, rectangular, and eigenwaveform) as they interact with an extended target whose response has finite support \(T\). Let \(T_x\) be the time duration of any transmit waveform. As mentioned earlier, we assume the extent of the wideband pulse is much less than the target response duration. As such, in Fig. 2a we may idealise the wideband pulse to be an impulse \((T_x \sim 0)\). For rectangular waveform, the pulse width is \(T_r = r\) as shown in Fig. 2b. The length \(r\) may be thought of as time-on-target (ToT) and the ratio of \(r\) to \(T\) may be defined as the
ToT ratio (ToTR) and is given by

\[ \text{ToTR} = \frac{r}{T}. \]

In Fig. 2c, the eigenwaveform is shown to have the same length as the target duration and thus the ToTR is equal to 1. For a wideband waveform, ToTR = \( T_x/T \), where \( T_x \ll T \). In the example simulations (in this paper) using the wideband waveform, \( T_x/T \) is 1/31. For this section, we concentrate on one-pulse waveforms to gain insight into the AFs of these three waveforms as they convolve with target responses.

For illustration in this paper, we form two targets with very different frequency responses and set both the transmit energy and the target energy to be unit-energy (\( E_x = 1 \) and \( E_h = 1 \)). We will keep using these two targets for comparison to illustrate the fact that unlike in the case of point targets, the target responses play important roles in the formation of AFs as well as detection performances (which will be covered later). In Fig. 3, we present the magnitudes of frequency responses of two different targets (top panel), the corresponding frequency responses of the eigenwaveforms (middle panel), and the frequency responses of their return echoes (bottom panel). From Fig. 3, we can conclude that choosing the eigenvector with the maximum eigenvalue of target autocorrelation matrix as the transmit signal is tantamount to choosing a band of frequencies (Fig. 3 middle panel) where the echoes (Fig. 3 bottom panel) guarantee the largest returns in terms of magnitudes [3, 7]. Interestingly, the magnitudes of the return frequency responses exhibit suppression of the less dominant frequency bands of the target response. This turns out to be important since this suppression effect will translate to sidelobe suppression in the AF of target returns illuminated by eigenwaveforms. While the two targets illustrated here are contrived, each represents something practically meaningful. Target 1 (a name which we will now use for consistency throughout the paper) illustrated in Fig. 3 (left top panel) with two dominant bands represents targets with oscillatory tendencies in time domain while Target 2 (also to be used consistently herein) in Fig. 3 (right top panel) with a low-pass shape represents targets exhibiting less oscillatory tendencies in the time domain. Target 2 is actually a low-pass shaped pulse in the time domain which represents a set of target responses which have a strong initial return but decays off in time. In other words, we have two target responses which represent potential practical target responses and thus we can illustrate how each extended target type will respond to the three waveforms we are to investigate.

### 3.3 Eigenwaveform versus wideband AFs

A quick and very insightful trade-off comparison is to use an eigenwaveform and the wideband waveform as transmit waveforms. In
this scenario, we utilise Target 1 \((n = 31\) samples). In Fig. 4, the AFs (3D and contour plots) of wideband transmit waveform and eigenwaveform are calculated and illustrated. It is clear that the AF of eigenwaveform is more compact in terms of Doppler (frequency) response compared with the wideband waveform. The ‘cleaner’ Doppler response is the result of two factors. The major factor is the fact that eigenwaveform exhibits narrowband characteristics which suppresses the less dominant target frequencies. The minor factor is the fact the length of the return echo for the eigenwaveform is about twice that of the wideband waveform. Increasing length of transmission results in the increase of Doppler resolution. Doppler resolution is usually referred to as the effective mainlobe width in the Doppler domain given a fixed delay (usually when delay is zero). In fact, delay (or range) resolution is usually taken to be the effective mainlobe width in the time-delay domain given a fixed Doppler frequency (usually \(f_0 = 0\)). Sidelobe suppression in the AF is usually a desired design characteristic for traditional waveforms for point targets. Here, the eigenwaveform has an inherent feature of being able to suppress sidelobes in the AF. Since the length of the return for the eigenwaveform is larger than that of the wideband waveform (recall Figs. 2 and 4 bottom panel) the mainlobe length when looking at the delay spread of the eigenwaveform is larger compared to the mainlobe length of the wideband AF. In other words, the range resolution of the wideband waveform is slightly better than the eigenwaveform. However, the eigenwaveform AF contour is clearly much cleaner (due to suppression effect and the Doppler resolution effect of longer return) than the wideband waveform. Concentrated energy in the origin of the AF is desired in radar waveform design for avoiding ambiguities. Also bigger sidelobes can cause a threshold to be crossed which can cause an increase in probability of false alarms (PFA). Compared to the wideband waveform, the reduction in range resolution is a small price to pay considering better Doppler resolution and the qualitative sidelobe reduction. Indeed, it is a small price to pay when we consider the peak value of eigenwaveform AF compared to wideband AF (since peak value translates to improved detection performance). In Fig. 4 where Target 1 is used \((E_1 = 1, E_s = 1)\), the peak of the eigenwaveform AF is 33.52 \((\lambda_{\text{max}} = 5.79)\) while the peak of the wideband AF is 1.

3.4 Rectangular waveform AF for oscillatory target (Target 1)

The AF from the use of rectangular transmit waveform depends on \(r\) relative to \(T\) (i.e. ToTR) in Fig. 2. In this section, we consider two rectangular waveforms: a low-ToTR and a high-ToTR. Using a high-ToTR of 0.64, we illustrate in Fig. 5a AF, Fig. 5b AF contour, Fig. 5c Target 1 frequency response (magnitude), Fig. 5d transmit signal frequency response (magnitude), and Fig. 5e return signal frequency response (magnitude). Using a low-ToTR of 0.09, we illustrate in Fig. 6a, AF, Fig. 6b AF contour, Fig. 6c Target 1 frequency response (magnitude), Fig. 6d transmit signal frequency response (magnitude), and Fig. 6e return signal frequency response (magnitude). The AF peak of high-ToTR in Fig. 5 is lower \((0.0296)\) compared to the AF peak \((0.2482)\) of the low-ToTR rectangular waveform. This is because transmit spectrum’s main lobe (sinc-function in frequency domain shown in Fig. 5d) barely overlaps the two dominant frequency bands of Target 1 (in Fig. 5c) thereby reducing the energy return as shown in Fig. 5e. Notice that there are also some subpeaks and sidelobes in AF for the high-ToTR rectangular pulse. On the other hand, low ToTR has a better peak value in the AF but a qualitatively worse Doppler response. This is because the low-to-ToTR rectangular waveform has a wider response (in frequency domain) than the high-ToTR rectangular waveform and therefore allows for more frequencies to appear. To conclude, given unit-transmit energy \((E_1 = 1)\) and unit-target energy \((E_s = 1)\), we note that the AF peak values for both rectangular waveforms are both lower than 1. Recall that the wideband waveform AF has a peak value of 1 while the eigenwaveform has a peak value of \(\lambda_{\text{max}}\), which is 33.52 for Target 1. In other words, rectangular waveforms (low-pass shape in frequency domain) are not good waveforms for Target 1 when it comes to AF peaks.

3.5 Rectangular waveform AF for non-oscillatory target (Target 2)

In this section, we consider a less-oscillatory target or non-oscillatory target in which Target 2 is a good example. Again, we let \(E_1 = 1\) and \(E_s = 1\). From the previous example, we have already gained the insight that the wideband AF peak is 1 and that it has a slightly better range resolution than the eigenwaveform and rectangular waveform. Thus we are able to make comparison with rectangular waveform AF without generating the wideband AF. While the rectangular waveforms (low and high ToTRs) do not work well for oscillatory targets, the rectangular waveforms actually perform well in terms of AF peaks for the non-oscillatory Target 2 compared the wideband waveform. For this example scenario, it is instructive to just use one of the rectangular waveforms.
Fig. 5  AF analysis of rectangular waveform with Target 1 (ToTR = 0.64)
a AF
b AF contour plot
c Target 1 frequency response (magnitude)
d Transmit signal frequency response (magnitude)
e Return signal frequency response (magnitude)

Fig. 6  AF analysis of rectangular waveform with Target 1 (ToTR = 0.09)
a AF
b AF contour plot
c Target 1 frequency response (magnitude)
d Transmit signal frequency response (magnitude)
e Return signal frequency response (magnitude)
waveforms to compare with the eigenwaveform. We choose to look at the high-ToTR rectangular waveform AF and compared that to the eigenwaveform AF for Target 2. Using a high-ToTR of 0.64, we illustrate in Fig. 7a. AF, Fig. 7b. AF contour, Fig. 7c Target 2 frequency response (magnitude), Fig. 7d transmit signal frequency response (magnitude), and Fig. 7e return signal frequency response (magnitude). Using the eigenwaveform, we illustrate in Fig. 8a. AF, Fig. 8b. AF contour, Fig. 8c Target 2 frequency response (magnitude) of target response, Fig. 8d transmit signal frequency response (magnitude) of rectangular waveform, and Fig. 8e return signal frequency response (magnitude) of rectangular waveform. Notice that the high-ToTR rectangular waveform AF has a decent peak (232.38). In hindsight, such a result may not be surprising since Target 2 is a low-pass shaped response and that a rectangular pulse should return a large echo specially if the durations of target and rectangular waveform are comparable. Looking at rectangular waveform AF contour in Fig. 7b, the overall sidelobe level is also low (of course this is very much influenced by the fact that both target and transmit waveform response are concentrated near the zero Hertz frequency). Notice however that AF peak resulting from the eigenwaveform is still higher (377.45 which is the square of $\lambda_{max} = 19.4281$). Looking at the eigenwaveform AF contour in Fig. 8b, the Doppler spread is narrower than that of Fig. 7b. In this case of non-oscillatory target, the rectangular AF peak is clearly much higher than the wideband waveform AF peak (which is 1).

In conclusion, we notice that for oscillatory targets (target responses with resonances in particular bands) the wideband waveform tends to result in a larger peak than the rectangular waveform. For non-oscillatory targets, the rectangular waveform tends to result in a larger peak than the wideband waveform. However, regardless of the target response, the eigenwaveform provides the largest AF peak compared to both waveforms. Qualitatively, the overall AF sidelobe suppression of the eigenwaveform is clearly superior to both waveforms.

3.6 AF zero-delay and zero Doppler cuts for extended targets

In AF analysis, two cross sections are usually of interest: the AF zero-delay and the zero-Doppler cuts. The zero-delay cut is simply the AF cross section when $r_s = 0$ and zero-Doppler cut is the AF cross section when $f_i = 0$. The cross sections are important since they usually convey delay (or range) resolution and Doppler (or velocity) resolution. For classical waveforms, the zero-delay cut is given by

$$|x(0; f_{dn})|^2 = \sum_{n=1}^{N} |x[n] \times x^*[n] \times W^f_{n}\|^2$$

where

$$W^f_{n} = e^{2j\pi f_{dn}/N}$$

and $x[n]$ is the transmit signal. Unfortunately, this does not work for extended targets. However, recall that we incorporated the fact that the target has finite extent in (3) which simplifies to

$$|x(0; f_{dn})|^2 = \sum_{n=1}^{N} |x[n]/h[n]|^2 \times W^f_{n}\|^2$$

where $x[n] = x[n]/h[n]$ is the return echo. Notice that we can simplify (14) to

$$|x(0; f_{dn})|^2 = \sum_{n=1}^{N} |x[n]|^2 \times W^f_{n}\|^2$$

which states that the AF zero-delay is simply the squared-magnitude of inverse (fast) Fourier transform of the squared-magnitude of the transmit waveform–target response convolution.

![Fig. 7 AF analysis of rectangular transmit waveform with Target 2 (AF peak value = 232.38)](image.png)

- a AF
- b AF contour plot
- c Target 2 frequency response (magnitude)
- d Transmit signal frequency response (magnitude)
- e Return signal frequency response (magnitude)
The corresponding zero-Doppler cut is then the squared autocorrelation function of the return echo (rather than just the transmit signal) and is therefore given by

$$|x[t_n;0]|^2 = \sum_{n=1}^{N} s[n] \times s^*[n - t_n]$$

The significance of the AF zero-delay and zero-Doppler cuts is illustrated using transmissions with multiple pulses which is the topic of the next section.

4 Coherent multiple-pulse transmission

When multiple-pulse transmission or coherent pulse train is used, the AF is the superposition of the AF of a single transmission with some amplitude re-scaling. From [11], the AF for a pulse train return is given by

$$|x(t, f_d)|^2 = \sum_{q=-L-1}^{L-1} \left| x_1(t - qT_R; f_d) \frac{\sin[\pi f_d(L - |q|)T_R]}{\sin(\pi f_d T_R)} \right|^2$$

(15)

where $|x_1|^2$ is the AF of a single transmit waveform-target response pair. $L$ is the number of pulses in a transmission and $T_R$ is the separation between pulses or pulse repetition interval (PRI). The pulse repetition frequency (PRF) is defined as

$$f_R = \frac{1}{T_R}.$$ 

In pulsed-Doppler radar system, one of the key design elements is PRF. In practise, the definition of low, medium, and high PRFs is truly application driven. This is because the PRI (or PRF) dictates the desired or specified unambiguous range. The unambiguous range is given by

$$R_{ua} = \frac{cT_R}{2} = \frac{c}{2PRF}$$

or the unambiguous delay is given by

$$T_{ua} = \frac{T_R}{2} = \frac{1}{2PRF}.$$ 

When using coherent pulses, the Doppler resolution is basically the length of the mainlobe of the zero-Doppler cut of (15). It is given by

$$f_{d, \rho} \simeq \frac{2}{LT_R}.$$ 

In practise, a simple rule of thumb is that medium PRF is a decade larger than low PRF and the high PRF is a decade larger than medium PRF but such rule is easily broken depending on application. For the convenient illustration of the PRF concepts in our work when illuminating an extended target, we will simply define the following: (a) high PRF is when $T_R$ is four to ten times $T$ (target duration), (b) medium PRF is when $T_R$ is forty times $T$, and (c) low PRF is when $T_R$ is a hundred times $T$. Here our definition of high PRF is pretty high. For example, even with $T_R = 10T$, the unambiguous range is $T_{ua} = (T_R/2) = (10T/2) = 5T$. In other words, the unambiguous delay for this PRF can accommodate about five target responses. This choice is not motivated by maximising or minimising the unambiguous range from some specific application but rather to show what happens when a (very)

---

**Fig. 8** AF analysis of eigenwaveform with Target 2 (AF peak value = 377.45)

a AF
b AF contour plot
c Target 2 frequency response (magnitude)
d Transmit signal frequency response (magnitude)
e Return signal frequency response (magnitude)
high PRF is lowered into medium PRF and then lowered again to a low PRF. In other words, we vary the length of $T_R$ to investigate how the different PRFs affect the AF in terms of how the volume inside the AF changes and how range and Doppler resolution may be affected as the PRI or PRF is changed. As pointed out earlier, the volume of the AF may be manipulated depending on the waveform choice.

Another parameter of interest in a pulse train is the duty cycle where

$$d_t = \frac{T_x}{T_R} = T_x \times f_R.$$

Multiple-pulse transmissions convolving with finite-duration targets resulting in multiple echoes are illustrated in Fig. 9a using wideband waveform, Fig. 9b using rectangular waveform, and Fig. 9c using eigen waveform.

In Figs. 10 and 11 using $L = 3$ pulses, $T_R = 124$ delay samples, AF contours are shown using the wideband, high-ToTR rectangular, and eigen waveform pulse trains for Targets 1 and 2, respectively. It is clear all the waveforms (rectangular, wideband waveforms, and eigen waveform) are very high PRF (AF contours are very close in delay spread since $T_R = 4T$) regardless whether it is Target 1 or 2. In Fig. 10 (with Target 1), we can deduce from the AF contour that the very high-PRF rectangular waveform yields a slightly cleaner sidelobe suppression response compared to the wideband waveform. However, its AF peak (0.0298) is lower than the AF wideband waveform peak (1). We can also deduce that the wideband waveform has the tightest range (delay) response. The eigen waveform produces the largest peak (33.5), tightest Doppler response, and we can also see that it qualitatively produces the cleanest overall sidelobe suppression level. In Fig. 11 with Target 2, all the waveforms have low overall sidelobe levels with the eigen waveform having the tightest AF response in the range (delay) domain but it also has the lowest peak (1). The rectangular waveform has a decent peak (232.38) but the eigen waveform has the largest peak (377.45).

### 4.1 Zero-Doppler cut

Utilising our high-PRF definition $T_R = 10T$, three-pulse transmission, and Target 1, we illustrate the AF zero-Doppler cuts of the wideband, rectangular waveforms, and eigen waveform in Fig. 12. It is clear from Fig. 12 that the range resolution of the wideband waveform is slightly better than the other two waveforms as was already inferred from the AF contours of Fig. 10. The peak is clearly apparent (and the largest) for the eigen waveform AF. These are the advantages of looking at another perspective via zero-Doppler cuts that may not be evident from the AF contours. Moreover, when eigen waveform is used, $T_R = 4T$ (in Figs. 10c and 11c) is the smallest value so that the zero-Doppler cut matched filtered pulse returns do not overlap. Thus, $T_R = 4T$ is the minimum PRI ($T_{\text{min}}$) such that the AF matched-filtered return echoes from the same target do not overlap. If $T_R$ is increased with the use of eigen waveform, then the matched filtered pulse returns start to overlap. Thus, the maximum PRF allowable if no overlap in the eigen waveform AF zero-Doppler cut is desired is given by

$$f_R^{\text{max}} = \frac{1}{T_{\text{min}}} = \frac{1}{4T}.$$

The PRI $T_R = 4T$ translates to the actual pulse returns being separated by $2T$ from the same target. This is not to be confused from $T_R = 1.5T$ (mentioned in an earlier section) which is the minimum time separation required for two targets such that pulse returns from the two targets do not overlap (i.e. such that the two targets can be resolved).

It is interesting to note in Fig. 12 that the ratio of each first sidelobe peak to mainlobe peak for all three zero-Doppler cuts to be 0.44 for $L = 3$ scenario. The ratio 0.44 is the square of $(2/3)$. The ratio of each sidelobe peak to mainlobe peak can be formulated for any $L$ as

$$P_i = \left(\frac{L - i}{L}\right)^2, \quad (16)$$

where $P_i$ is the ratio of $i$th side peak amplitude to main peak amplitude.

---

Fig. 9 Multiple-pulse transmission for extended target setting

- **a** Wideband waveform
- **b** Rectangular waveform
- **c** Eigen waveform
Another useful comparison is to illustrate the AF zero-Doppler cuts for various PRFs given a particular waveform. Utilising the eigenwaveform and \( L = 3 \), we show the zero-Doppler cuts for high PRF, medium PRF, and low PRF in Fig. 13. Just like in traditional pulsed-Doppler waveforms, the low PRF yields the best unambiguous range. However, just like in pulsed-Doppler waveforms, the choice between low, medium, and high PRF has an impact in the Doppler domain which is considered shortly.

Also, we explore the impact of increasing the number of pulses (\( L = 1, 3, 5 \)) given a fixed PRF in Fig. 14. Again, utilising the eigen waveform, we illustrate the AF zero-Doppler cuts for Target 1 with high PRF. Of course, \( L = 1 \) is an one-pulse waveform which is shown to illustrate what is gained by coherent pulse integration when using multiple pulses. Since the PRF is fixed, the separation between lobes for \( L = 3 \) and \( L = 5 \) are the same. The main gain of coherent integration is clearly the gain which is \( L^2 \) but at the expense of higher first sidelobe-to-main peak ratio (0.44 for \( L = 3 \) and 0.64 for \( L = 5 \)) as predicted by (16).

In general (when no compression is used), the range resolution depends on resulting extent of the convolution of the transmit waveform and target response. In our paper, the wideband waveform has a slightly better range resolution than the eigen waveform or rectangular waveform due to its narrow time extent. For eigen waveform, there is a lower limit of \( T_R = 4T \) for matched filtered echoes not to overlap in AF’s range (delay) domain. Increasing \( L \) increases coherent integration gain while sacrificing sidelobe peak to main peak ratio. Increasing \( T_R \) (which translates to lowering of PRF) increases an unambiguous range and its impact on Doppler domain is covered in the next section.

### 4.2 Zero-delay cut

The zero-delay cut shows the AF Doppler spread when the delay is zero. At times, the zero-delay cut offers a perspective that is not quite apparent from an AF contour. From [9, 11], we realise that the width of the main lobe is decreased (i.e. Doppler resolution improvement) by increasing the number of pulses in a transmission by observing the AF zero-delay cut which is given by

\[
|\chi[0; f_d]|^2 = \left| \frac{1}{L} \chi_L(0; f_d) \frac{\sin[\pi f_d L T_R]}{\sin[\pi f_d T_R]} \right|^2.
\]

Employing the eigen waveform, we illustrate in Fig. 15 the AF zero-delay cuts for Targets 1 and 2 with \( L = 1, 3, 5 \) using medium PRF with \( T_R = 40T \). We include \( L = 1 \) to remind us the AF \( |\chi[0; f_d]|^2 \) corresponding to \( L = 1 \) serves as the envelope as dictated by (17). For traditional waveforms for point targets, the zero-delay cut is purely a function of the transmit waveform. From Fig. 15, it is clear that extended target AF zero-delay cuts differ from target to target. Notice that the zero-Doppler cut is the squared multiplication of \( L = 1 \) zero-Doppler cut with a sinc-train. The frequency separation of the sinc-lobes is dictated by \( \frac{1}{T_R} \) and the width of the main lobe (or any sinc lobe) is given by

\[
\text{BW}_L = \frac{2}{T_R} = f_{\text{m,p}}
\]

which is also considered as the Doppler resolution. However, in (17) notice that the separation of the sinc-lobes is also a function of \( T_R \) and \( L \). In other words, Doppler separation (to avoid ambiguity) is also a function of \( T_R \) and \( L \). Thus, by varying \( T_R \) and \( L \), we

---

**Fig. 10** Target 1 AF contour plots comparison \((L = 3, T_R = 4T)\)

*a* Wideband waveform

*b* Rectangular waveform

*c* Eigen waveform

---

*This is an open access article published by the IET under the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0/)*
Fig. 11 Target 2 AF contour plots comparison (L = 3, ToTR = 4T)

a Wideband waveform
b Rectangular waveform
c Eigenwaveform

Fig. 12 Zero-Doppler cut of Target 1 ambiguity function (L = 3, Tr=10T, High PRF)

Zero-Doppler cut of rectangular–Target 1 ambiguity function (L = 3, Tr=10T, duty cycle=0.64, High PRF)

Zero-Doppler cut of eigenwaveform–Target 1 ambiguity function (L = 3, Tr=10T, High PRF)
can improve (or degrade) Doppler resolution and change the frequency separation of the sinc-lobes. For example, we illustrate in Fig. 16 the eigenwaveform AF zero-delay cuts for Targets 1 and 2 with $L = 1, 3, 5$ but now using high PRF (smaller $T_R$). It is clear that the higher PRF (which is lower $T_R$) has wider separation but poorer resolution since the bandwidth (18) is increased by...
decreasing $T_R$. Thus, to maintain the same bandwidth or Doppler resolution in (18) while decreasing $T_R$ (increasing sinc-lobe separation), it is necessary to increase $L$. In other words, Doppler resolution is dictated by the length of total transmission $T_T = LT_R$ as observed earlier.

4.3 Range and Doppler resolution trade-off

It is now clear there is an inherent trade-off between range and Doppler resolution by varying the PRF (and $L$) of multiple pulse transmission of eigenwaveforms (or other waveforms for that matter) for extended targets. To illustrate the range and Doppler resolution trade-off, Figs. 15 and 16 compare zero-delay cuts with single, three, and five retransmissions with low and high PRF, respectively.

Fig. 15 Zero-delay cut comparison with single, three, and five retransmissions with low PRF using the eigenwaveform

Fig. 16 Zero-delay cut comparison with single, three, and five retransmissions with high PRF using the eigenwaveform
trade-off that continue to exist even for multiple-pulse waveform for extended targets, we employ a high PRF and a low PRF eigenwaveform on Target 1 and show the corresponding zero-Doppler cuts in Figs. 17a and c, respectively, and zero-delay cuts in Figs. 17b and d with various $L$ given a constant transmit energy constraint and a constant total length constraint (i.e. $E_t$ and $LT_R$ is constant). From Figs. 17a and c, it is clear that the range resolution is the same since the pulse width remains the same; the unambiguous range is worse for the high PRF than low PRF. From Figs. 17b and d, the Doppler resolution is the same since $LT_R$ is constant but the sinc-lobe separation is larger for the high PRF than low PRF.

In summary, range resolution and Doppler resolution depend on the type of transmit waveform and transmit signal length. For multiple pulse eigenwaveforms, increasing PRF (decreasing $TR$) results in increasing Doppler sinc-lobes separation but increasing PRF results in reducing the unambiguous range. When larger $L$ is used, Doppler resolution may improve but larger $L$ results in more sidelobes which results in sacrificing sidelobe peak to main-lobe peak ratio. It is a classical trade-off where a specific application dictates the right choice for a system. However, we should mention the fact that the narrowband nature of the eigenwaveform (as opposed to the wideband waveform) helps in the reduction of the sinc-lobe peaks.

5 Detection probability and RDM

In this section, we investigate the probability of detection of a system that employs various waveforms such eigenwaveform, rectangular and wideband waveforms. We can assume the target to have zero Doppler and it’s extension to the generalised case is straightforward.

5.1 Detection probability of basic one-pulse waveforms

Let $h$ be the complex-valued target response and $w$ be the complex valued white Gaussian noise in the receiver with a sample variance of $\sigma^2$. Let $s$ be the convolution of transmit waveform $x$ and $h$. Then the detection hypotheses are given by

$H_0: y = w$

$H_1: y = s + w = Hx + w$

where $H$ is the target convolution matrix corresponding to $h$. The decision statistic using matched filter theory for a fixed threshold $\gamma$ is

$T(y) = \text{Re}\{y^Hs\} = \text{Re}\{y^HHx\}$.

When the wideband waveform is used, the detection probability [5] given a fixed false alarm probability (PFA) is given by

$P_D = Q\left(Q^{-1}(P_{FA}) - \frac{2E_s}{\sigma^2}\right)$ (19)

$= Q\left(Q^{-1}(P_{FA}) - \sqrt{2E_s\text{TNR}}\right)$ (20)

where TNR = $E_s/\sigma^2$ (received target-to-noise ratio). Alternatively, (19) can be given as

$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{2E_s\text{SNR}}\right)$ (21)

where SNR = $E_s/\sigma^2$ (transmit signal-to-noise ratio). In other words, $P_D$ is a function of both transmit energy and received target energy along with the noise variance. A more compact version of (19) which resembles detection probability of traditional matched filter analysis [21] is

$P_D = Q\left(Q^{-1}(P_{FA}) - \sqrt{2E_s\text{NR}}\right)$ (22)
where $E_{NR} = E_s/E_k/\sigma^2$, where $E_s$ specifically means return (echo) energy using the wideband waveform. It can be shown [5, 6] that the detection probability with the use of eigenwaveform is given by

$$P_D = \Pr(T \geq \gamma'; H_1) = Q\left(\frac{\gamma' - E_s}{\sqrt{\sigma^2 E_s}/2}\right) = Q\left(\frac{\gamma' - \lambda_{\max} E_s E_k}{\sqrt{2\lambda_{\max} E_s E_k/\sigma^2}}\right) = Q(Q^{-1}(P_{FA}) - \sqrt{2\lambda_{\max} E_s/\sigma^2}) = Q(Q^{-1}(P_{FA}) - \sqrt{2\lambda_{\max} E_s/\sigma^2}),$$

(23)

Notice that performance improvement with the use of eigenwaveform compared to the wideband (impulse) waveform given a fixed PFA by comparing (22) and (23) where $\lambda_{\max}$ effectively amplifies the $E_{NR}$ in (22).

For any arbitrary waveform (rectangular waveform included), it can be shown that the generalised detection probability is given by

$$P_D = \Pr(T \geq \gamma'; H_1) = Q\left(\frac{\gamma' - E_s}{\sqrt{\sigma^2 E_s}/2}\right) = Q\left(\frac{2E_s E_k/\sqrt{\sigma^2 x^H R_e x}}{\sigma^2}\right) = Q(Q^{-1}(P_{FA}) - \sqrt{2\lambda_{\max} E_s/\sigma^2}).$$

(24)

where $x$ and $R_e$ are normalised transmit waveform and target response autocorrelation matrix. It is intuitive to conclude that the value of $(x^H R_e x)$ may vary from close to zero (since $R_e$ is positive definite) to $\lambda_{\max}$. As a consequence, the detection performance pretty much depends on the how good/bad the transmit waveform matches target response’s autocorrelation matrix. In other words, if we want to compare detection probabilities of arbitrary waveforms, they have to be calculated via (24) and only then we can tell if it is greater or less than the detection probability of the wideband impulse waveform which is given in (19). In other words, stating what the transmit energy is not enough to know the detection probability of an arbitrary waveform. Its correlation with target dictates the total return energy. The eigenwaveform ensures the maximum detection probability since the term $x^H R_e x$ inside the square root in (24) is maximised which yields the detection probability in (23). The detection performance curves for various waveforms illuminating Targets 1 and 2 are shown in Figs. 18 and 19 given a fixed TNR (target-to-noise energy ratio) for various false alarm probabilities ($P_{FA}$). Considering Fig. 18, the performance of the eigenwaveform is superior to rectangular and wideband waveforms (given a fixed $P_{FA}$) as expected. For example, $P_{FA} = 0.01$ and $P_D = 0.9$, the eigenwaveform advantage over the wideband waveform is about 7.6 dB (which makes sense since this is about equal to 10log($\lambda_{\max}$)). What’s more incredible however (for oscillatory Target 1) is that the eigenwaveform with a stringent requirement of $P_{FA} = 10^{-4}$ still outperforms both the wideband and the rectangular waveforms with a looser requirement of $10^{-2}$ (for $P_D > 0.1$)! In Fig. 19, where Target 2 is considered, the performance curve for eigenwaveform still is the best compared to rectangular and wideband waveforms given a fixed $P_{FA}$. However, as can be inferred from the AFs previously studied, the detection performance of the rectangular waveform is much better than the wideband waveform (for the non-oscillatory Target 2). This reinforces the notion that was stated earlier. We know that the eigenwaveform yields the best detection performance but detection performance of other waveforms have to be calculated via (24) such that performance comparison (in terms of detection) between waveforms can be made. Finally, the detection performance comparison adds the needed final dimension to the range-Doppler tradeoff when considering various waveforms. In other words, systems which are noise-limited may opt for the eigenwaveform with a slight hit on range resolution. Other systems which may not be noise-limited may opt for other waveforms if range resolution takes precedence (but with possibly substantial cost in detection performance).

5.2 Detection probability of coherent multiple-pulse transmission and RDM application

When moving target is present, one of the AF application is the RDM which significantly demonstrates the benefit of using eigenwaveform for extended targets. Assume $L$ pulses are sent and that the return echo is received. Any target detected may be moving (or not moving) such that a Doppler component is possible. After receiving the return, the long sequence is carefully re-arranged so that each return from every pulse are aligned according to same

Fig. 18 Target 1 detection probability comparison of wideband waveform, rectangular waveform (ToTR = 0.64) and eigenwaveform (TNR = 0 dB)
delay and stored in a measurement matrix. By taking fast-Fourier transform (FFT) in Doppler direction, signal energy converges into corresponding Doppler bins (indices) due to FFT’s circular shift property. The magnitude of the measurement matrix after the FFT operation is considered the RDM since it reveals the characteristic of moving target’s delay (which corresponds to range) and Doppler shift (which corresponds to velocity). The multiple-pulse wideband, rectangular waveforms, and eigenwaveform are utilised as the transmit waveforms here. A peak value may be used to detect a target’s position (delay) and Doppler spread. Of course, the RDM can detect multiple targets with different speeds and distances and ambiguities can be avoided via discussion from earlier sections.

Moreover, when multiple pulses are used, it can be shown that the detection probability is also a function of the number pulses \( (L) \). The detection probability with \( L \) pulses with the use of eigenwaveform is be shown to be

\[
P_D = \Pr (T \geq \gamma; H_1) = \Phi \left( \frac{\gamma - \frac{E_s}{2}}{\sqrt{\sigma^2 E_s/2}} \right) = \Phi \left( \frac{\gamma - \lambda_{\max} LE_s E_s}{\sqrt{\sigma^2 E_s/2}} \right)
\]

(25)

The gain due to \( L \) (10log\( L \) in dB) is also true for other waveforms as shown in Fig. 20. In Fig. 20, we illustrate the Target 1 detection performance comparison: wideband waveform, rectangular waveform and eigenwaveform with various multiple-pulses transmission (\( L = 3, 5, 10 \)).

---

Fig. 19 Target 2 detection probability comparison of wideband waveform, rectangular waveform (ToTR = 0.64) and eigenwaveform (TNR = 0 dB)

Fig. 20 Performance comparison: wideband waveform, rectangular waveform and eigenwaveform with various multiple-pulses transmission (\( L = 3, 5, 10 \))
performance curves for the three waveforms as a function of increasing $L$. As expected, increasing $L$ increases detection probability.

Finally, we generate the RDM (3D maps) where we utilise Target 1 and illuminate it with the wideband, rectangular (ToTR = 0.64), and eigenwaveform (duty cycle $d_t = 10^{-3}$) where $L = 31$ as a function of increasing SNR ($0$, $3$, $10$ dB) given a fixed TNR = 0 dB. We illustrate in Fig. 21 the 3D RDMs of these various waveforms as a function of increasing $E_{SR}$ (return energy-to-noise ratio). In this scenario, the target is located at range (delay) = 560 with normalised Doppler of $f_d = 0.0968$. Notice that the target is not very visible for SNR = 0 dB with the use of rectangular waveform. For the same SNR, there is a small peak corresponding to Target 1 with the use of the wideband waveform. Notice however that the Target 1 is clearly discernable with SNR = 0 dB when the eigenwaveform is used. Increasing SNR enhances the peaks for all waveforms with the eigenwaveform clearly yielding the largest peak.

RDMs are usually presented as 2D maps (magnitude or squared-magnitude). In this scenario, we place two targets (Target 1 type) in two different range-Doppler bins. Again we use the wideband, rectangular (ToTR = 0.64), and eigenwaveform ($d_t = 10^{-3}$) where $L = 31$ as a function of increasing SNR ($0$, $3$, $10$ dB) given a fixed TNR = 0 dB. The RDMs are illustrated in Fig. 22. In this

This is an open access article published by the IET under the Creative Commons Attribution License (http://creativecommons.org/licenses/by/3.0/)

J. Eng., 2018, Vol. 2018, Iss. 4, pp. 203–221
doi: 10.1049/joe.2017.0393
scenario, the targets are located at range (delay) = 450 with normalised Doppler of \( f_1 = 0.1613 \) and range (delay) = 750 with normalised Doppler of \( f_2 = -0.1290 \). Again, notice that the targets are not very discernable for SNR = 0 dB with the use of rectangular waveform. For the same SNR, there is a some faint possibility that one of the targets may be detected with the use wideband waveform. Both targets are clearly pronounced when the eigenwaveform is used even with SNR = 0 dB. Increasing SNR enhances waveform. Both targets are clearly pronounced when the eigenwaveform clearly yielding the brightest target bins.

6 Conclusion

In this comprehensive tutorial paper, we showed the two properties (peak and volume) of AF for an extended target are not constant over all transmit waveforms. This is unlike the well-known properties of AF for point targets where AF peak and volume are constant no matter what waveform is. We showed that the AF that produces the largest peak is the AF that uses the eigenwaveform as transmit waveform. To gauge the performance (e.g. detection, range, and Doppler performance parameters) of the basic one-pulse waveforms (eigenwaveform, wideband, and rectangular waveforms), we use two extended target types: oscillatory and non-oscillatory targets. We showed various AF contours and compared the peaks that correspond to various waveforms. The AF peak is larger for wideband waveform when illuminating an oscillatory target compared to rectangular waveform. The AF peak is larger for rectangular waveform compared to the wideband waveform when illuminating a non-oscillatory target. The wideband waveform resulted in slightly better range resolution (compared to other waveforms) but eigen waveform resulted in the best AF Doppler response and best low overall sidelobe level. In addition the eigenwaveform clearly produced the largest peak as well as best detection probability given a false alarm probability compared to all other waveforms. Multiple-pulse waveforms were investigated. AF zero-delay and zero-Doppler cuts were discussed. Various trades were made with PRF and the number of pulses to illustrate that classical trade-off between range and Doppler parameters remains. Finally, the multipulse waveforms (eigenwaveform, wideband, and rectangular waveforms) were used to produce RDMs. It is clear in the RDMs that the targets (with various delay and Doppler components) are easily discerned or detected with the use of the eigenwaveform compared to other waveforms even in low transmit SNRs.

7 References

[1] Bell M.R.: ‘Information theory and radar waveform design’, IEEE Trans. Inf. Theory, 1993, 39 (5), pp. 1578–1597

[2] Pillai S.U., Oh H.S., Youla D.C., et al.: ‘Optimum transimtreceiver design in the presence of signal-dependent interference and channel noise’, IEEE Trans. Inf. Theory, 2000, 46 (2), pp. 577–584

[3] Romero R.A., Ba J., Goodman N.A.: ‘Theory and application of SNR and mutual information matched illumination waveforms’, IEEE Trans. Aerosp. Electron. Syst., 2011, 47 (2), pp. 912–927

[4] Kay S.M.: ‘Optimal signal design for detection of Gaussian point targets in stationary Gaussian clutter/reverberation’, IEEE J. Sel. Top. Signal Process. Mag., 2007, 1 (1), pp. 31–41

[5] Romero R.A.: ‘Detection performance of matched transmit waveform for moving extended targets’, Asilomar Conf. Signals, Systems, and Computers, November 2013

[6] Nieh J., Romero R.A.: ‘Ambiguity function and detection probability considerations for matched waveform design’, IEEE Conf. Acoustics, Speech and Signal Processing, April 2013, pp. 4280–4284

[7] Goodman N.A., Venkata P.R., Nefeld M.A.: ‘Adaptive waveform design and sequential hypothesis testing for target recognition with active sensors’, IEEE J. Sel. Top. Signal Process., 2007, 1 (1), pp. 105–113

[8] Romero R.A., Goodman N.A.: ‘Improved waveform design for target recognition with multiple transmissions’, IEEE Conf. Waveform Diversity and Design, April 2009, pp. 26–30

[9] Levanon N.: ‘Radar principles’ (John Wiley and Sons, NJ, 1988)

[10] Levanon N., Mozeson E.: ‘Radar signals’ (John Wiley and Sons, NJ, 2004)

[11] Mahafza B.R.: ‘Radar signal analysis and processing using MATLAB’ (CRC Press, Boca Raton, FL, 2009)

[12] Peebles P.Z.: ‘Radar principles’ (John Wiley and Sons, NJ, 1998)

[13] Nutall A.H.: ‘Some windows with very good sidelobe behavior’, IEEE Trans. Acoustics, Speech, Signal Process., 1981, 29 (1), pp. 84–91

[14] Getz B., Levanon N.: ‘Weight effects on the periodic ambiguity function’, IEEE Trans. Aerosp. Electron. Syst., 1995, 31 (1), pp. 182–193

[15] Benedetto J.J., Donatelli J.J.: ‘Ambiguity function and frame-theoretic properties of periodic zero-autocorrelation waveforms’, IEEE J. Sel. Top. Signal Process., 2007, 1 (1), pp. 6–20

[16] Auslander L., Tolimieri R.: ‘Characterizing the radar ambiguity functions’, IEEE Trans. Inf. Theory, 1984, 30 (6), pp. 832–836

[17] Freedman A., Levanon N.: ‘Properties of the periodic ambiguity function’, IEEE Trans. Aerosp. Electron. Syst., 1994, 30 (3), pp. 938–941

[18] Antonio G.S., Furhmann D.R., Robey F.C.: ‘MIMO radar ambiguity functions’, IEEE J. Sel. Top. Signal Process., 2007, 1 (1), pp. 167–177

[19] Li Y., Vorobyov S.A., Koivunen V.: ‘Ambiguity function of the transmit beamspace-based MIMO radar’, IEEE Trans. Signal Process., 2015, 63, (17), pp. 4445–4457

[20] Thirrian C.W.: ‘Discrete random signals and statistical signal processing’ (Prentice-Hall Signal Processing Series, CA, 2004)

[21] Kay S.M.: ‘Fundamentals of statistical signal processing’ (Prentice-Hall, Upper Saddle River, NJ, 1993)

[22] Haykin S.: ‘Adaptive filter theory’ (Prentice-Hall, Upper Saddle River, NJ, 1996)