MAPPING CLASS GROUPS OF SURFACES OF GENUS $\geq 3$
DO NOT VIRTUALLY SURJECT TO $\mathbb{Z}$

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ABSTRACT. We prove a well known conjecture of Nikolai Ivanov which states that if $X$ is a surface of genus $\geq 3$ (with any number of punctures and boundary components), $\text{Mod}(X)$ is the mapping class group of $X$, and $K < \text{Mod}(X)$ is a finite-index subgroup, then $K$ does not virtually surject to $\mathbb{Z}$. As a corollary of this we get that $H_1(Z; \mathbb{Q}) = 0$ whenever $Z$ is a finite cover of $\mathcal{M}_{g,n}$, the moduli space of complex algebraic curves of genus $g \geq 3$ with $n$ marked points.

1. Introduction

Let $X = X_{g,b}^p$ be an oriented surface with of genus $g$ with $b$ boundary components and $p$ punctures. Let $\text{Mod}(X) = \text{Mod}_{g,b}^p$ be its mapping class group, that is - the group of orientation preserving diffeomorphisms $X \rightarrow X$ that fix the punctures and boundary components pointwise, up to isotopies that fix the punctures and boundary components pointwise.

If $g \geq 3$ then $\text{Mod}(X)$ is a perfect group - it has no nontrivial abelian quotients. Finite-index subgroups $\text{Mod}(X)$ need not have this property. It is simple to construct such subgroups that have finite but nontrivial abelianization. In [14], problem 7, Ivanov made the following conjecture.

Conjecture 1.1. (Ivanov) Let $g \geq 3$, $b, p \geq 0$. Let $K < \text{Mod}(X)$ be a finite-index subgroup. Then $H^1(K; \mathbb{Q}) = 0$.

Ivanov’s conjecture has several equivalent formulations. It is equivalent to the conjecture that $H_1(K; \mathbb{Q}) = 0$ for all finite-index $K < \text{Mod}(X)$, to the conjecture that $H_1(K; \mathbb{Z})$ is finite for all such $K$, and to the conjecture that $K$ does not surject to $\mathbb{Z}$ for all such $K$. The last of these forms is the one that most commonly appears in the literature, and it is the form of the conjecture we will use.

Ivanov’s conjecture is one of the most well known and long-standing open questions about mapping class groups. It has appeared in numerous compilations of open problems about mapping class groups over the years (cf. [15])

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problem 2.11A, [13] problem 7, [21] question 6.1), and has generated a great deal of interest and work.

Ivanov’s conjecture is known to be false for surfaces of genus $< 3$. This is simple to see in genus $0, 1$. It is due to Taherkhani ([28]) and McCarthy ([22]) in genus 2.

In genus $\geq 3$ there are several classes of finite-index subgroups $K \leq \text{Mod}(X)$ for which it is known that $H_1(K; \mathbb{Z})$ is finite. This is known for $K = \text{Mod}(X)$ (Mumford, [24], for $K$ containing the Torelli group $I$ of Mod($X$) (Hain, [11]), for $K$ containing the Johnson Kernel $J$ of Mod($X$) (Putman, [26]), and for $K$ containing any term of the Johnson filtration of Mod($X$) (Ershov and He, [4]).

In [27], Putman and Wieland showed that Ivanov’s conjecture is equivalent to a question about a certain collection of representations of mapping class groups which they called higher Prym representations (and which we will describe below). In this paper we answer Putman and Wieland’s representation theoretic question. This allows us to prove Ivanov’s conjecture.

**Theorem 1.2.** Let $X = X_{g,b}^p$. Suppose that $g \geq 3$, $b, p \geq 0$. Let $K \leq \text{Mod}(X)$ be a finite-index subgroup. Then $H_1(K; \mathbb{Q})$ is finite.

1.0.1. An application to algebraic geometry. Theorem [1.2] has an important interpretation regarding the topology of $\mathcal{M}_{g,n}$, the moduli space of Riemann surfaces of genus $g$ with $n$ punctures (or equivalently - the moduli space of complex algebraic curves of genus $g$ with $n$ marked points).

Recall that $\mathcal{M}_{g,n} \cong \mathcal{T}_{g,n}/\text{Mod}(X)$, where $\mathcal{T}_{g,n}$ is the Teichmüller space of genus $g$ Riemann surfaces with $n$ punctures. Every finite cover $Z \to \mathcal{M}_{g,n}$ corresponds to a finite-index subgroup $K \leq \text{Mod}(X)$. Since $\mathcal{T}_{g,n}$ is contractible, we have that $H_1(Z; \mathbb{Q}) \cong H_1(K; \mathbb{Q})$. This gives the following corollary of Theorem [1.2]

**Corollary 1.3.** Let $g \geq 3$ and $n \geq 0$. If $Z \to \mathcal{M}_{g,n}$ is a finite cover then $H_1(Z; \mathbb{Q}) = 0$.

1.0.2. An application to actions on $S^1$. It is a classical result of Nielsen that $\text{Mod}(X_{g,0}^1)$ admits a faithful continuous action on $S^1$ without global fixed points (c.f. [7] section 8.2.6). Mann and Wolff showed that if $g \geq 3$ then every faithful continuous action of $\text{Mod}(X_{g,0}^1)$ on $S^1$ is either trivial or semi-conjugate to Nielsen’s action ([20]).

The question of whether or not (and to what degree) this action can be smoothed has attracted significant attention. Parwani proved that if $X$ is a surface of genus $\geq 6$, then the mapping class group $\text{Mod}(X)$ admits no non-trivial $C^1$ actions on $S^1$ ([25]). A different proof of this result appears in Mann and Wolff’s paper.
One can now ask similar questions about actions of finite-index subgroups of $\text{Mod}(X)$ on $S^1$. Recall that the complexity of the surface $X = X_{g,b}^p$ is defined to be $3g - 3 + b + p$. Baik, Kim, and Koberda showed in [1] that a finite-index subgroup of $\text{Mod}(X)$ admits a faithful action on $S^1$ by $C^2$-diffeomorphisms if and only if $c(X) \leq 1$.

Remark 3.1 in Parwani’s paper shows that an analog of Baik, Kim, and Koberda’s result for $C^1$ diffeomorphisms holds, assuming Ivanov’s conjecture is true for surfaces of genus $\geq 3$. Together with our Theorem 1.2 this yields the following corollary.

**Corollary 1.4.** If $g \geq 6$, and $K < \text{Mod}(X)$ is a finite index subgroup, then there is no faithful homomorphism $K \to \text{Diff}^+_1(S^1)$.

### 1.1. Prym representation and the No Finite Orbits criterion.

Let $f \in \text{Mod}(X)$. The map $f$ induces a linear transformation $f_* : H_1(X; \mathbb{Q}) \to H_1(X; \mathbb{Q})$. The assignment $f \to f_*$ gives a representation $\text{Mod}(X) \to \text{GL}(H_1(X; \mathbb{Q}))$ called the *standard homological representation*. When $g \geq 1$ and $b + p \leq 1$ then the image of this representation is a symplectic group, and it is often called the *symplectic representation*.

This representation is just one of a large family of homological representations, which are assigned to finite covers $\pi : Y \to X$. Given such a cover, there is a finite-index subgroup $L_{Y \to X} \leq \text{Mod}(X)$ of maps that can be lifted to maps $Y \to Y$.

A given $f$ may be lifted to $Y$ in several ways. To avoid ambiguity, we add a puncture $\star \in X$, and consider the mapping class group $\Gamma = \text{Mod}(X \setminus \{\star\}) \cong \text{Mod}^{p+1}_{g,b}$. We consider only $f \in \Gamma$ that can be lifted to $Y$ and that fix the preimage of $\star$. Any such lift is unique. This lift of $f$ induces an map $\rho_\pi(f) : H_1(Y; \mathbb{Q}) \to H_1(Y; \mathbb{Q})$. The assignment $f \to \rho_\pi(f)$ is called the *homological representation associated to $\pi$*. Note that this representation is only defined on $\Gamma_\pi$, the finite-index subgroup of $\Gamma$ consisting of elements that lift to the cover $Y$. To circumvent this problem, we can either induce the representation to all of $\Gamma$, or restrict ourselves to only consider characteristic covers $\pi$ (in which case $\Gamma = \Gamma_\pi$).

Let $B \subset H_1(Y; \mathbb{Q})$ be the subspace spanned by all boundary components and all loops parallel to punctures in $Y$. Let $V = V_\pi = H_1(Y; \mathbb{Q})/B$. Since $B$ is a $\rho_\pi(\Gamma)$-invariant subspace, $V$ is a $\Gamma$-representation. We call these *higher Prym representations* of $\Gamma$.

Putman and Wieland denoted by $\text{NFO}(g, b, p)$ (or *No Finite Orbits for $X = X_{g,b}^p$*) the condition that for every finite cover $\pi : Y \to X$, $V$ does not have any finite $\Gamma$-orbits aside from the orbit of the zero vector. They proved the following theorem:
Theorem 1.5. (Putman-Wieland, [27]) Fix $b, p \geq 0$ and $g \geq 3$. If NFO($g - 1, b + 1, p$) holds then the group Mod($X^p_{g,b}$) does not virtually surject to $\mathbb{Z}$. If Mod($X^p_{g,b+1}$) does not virtually surject to $\mathbb{Z}$ then NFO($g, b, p$) holds.

Putman and Wieland’s theorem provides a direction for resolving Ivanov’s conjecture. Unfortunately, homological (and higher Prym) representations are difficult to work with and are not very well understood. It was not clear whether or not this equivalent version of Ivanov’s conjecture described by the Putman-Wieland theorem was any easier to prove than the original version. Nevertheless, Theorem 1.5 lead to several approaches to proving Ivanov’s conjecture.

Previous approaches to Ivanov’s conjecture One approach is due to Farb and Hensel. The groups Aut($F_n$), the automorphism groups of free groups, also have homological representations (which are constructed in much the same way as above). Farb and Hensel proved in [5] that there are no nontrivial finite Aut($F_n$) orbits in any homological representation of Aut($F_n$) when $n \geq 3$.

Given any $0 \neq v$ in such a representation, Farb and Hensel construct an automorphism $f$ of a very simple type (called a slide map) such that the $\langle f \rangle$-orbit of $v$ is infinite. It should be noted that in the Aut($F_n$) case the NFO condition is not known to be equivalent to any type of result about abelianization of finite-index subgroups. Putman and Wieland’s result relies on Hodge theoretic techniques that are not available in that setting.

Farb and Hensel’s elegant proof is relatively short and elementary. Unfortunately, slide maps do not exist in Mod($X$). To apply Farb and Hensel’s ideas, one needs to find an analog these maps. The most obvious candidates for such an analog are Dehn twists. However, in order to use these one needs to show that given $0 \neq v \in V$, there is a $u \in V$ represented by a connected component of the pre-image of a simple closed curve in $X$ such that $\hat{i}(u, v) \neq 0$, where $\hat{i}(\cdot, \cdot)$ is the algebraic intersection form.

The latter condition would be easy to satisfy if the collection of all $u$ represented by a connected component of the pre-image of a simple closed curve in $X$ spanned all of $V$. Farb and Hensel called this space the simple closed curve homology subspace in [6]. For some time it was unclear whether this subspace is equal to all of $V$. Koberda and Santharoubane proved in 2015 ([16]) that these two spaces need not be equal in the case where we take homology with $\mathbb{Z}$ coefficients. Malestein and Putman proved in 2017 ([19]) that the two spaces need not be equal even in the rational coefficients case. Malestein and Putman’s result makes it unfeasible to use Dehn twists as a slide map analog when attempting to prove Ivanov’s conjecture.
A second approach to proving the conjecture is more ambitious. It involves calculating the image of all of $\Gamma$ in $H_1(Y;\mathbb{Q})$ and checking to see if any nontrivial vectors in $V$ have finite orbits. This image was calculated for for abelian covers of closed surfaces by Looijenga in [18] and for a class of covers of closed surfaces called redundant covers by Grünwald, Larsen, Lubotzky, and Malestein in [8]. In both cases the image of $\Gamma$ can be described explicitly. It is commensurable with the subgroup of $\text{Aut}(H_1(Y;\mathbb{Z}),\hat{i}) \cong \text{Sp}(H_1(Y;\mathbb{Z}))$ composed of matrices that intertwine the deck group. This is an arithmetic group, and it can be shown to have no finite orbits.

Unfortunately, the methods used in [8] and [18] are not applicable for all covers or surfaces. Calculating the image of general homological representations appears to lie out of reach of our current understanding. What’s more, there’s also reason to believe that when we consider non-closed surfaces (which we must when trying to use the Putman-Wieland theorem) the image of homological representations can become quite complicated. For example, in [23] McMullen produced infinitely many covers of punctured spheres where the image of the corresponding Prym representation is not an arithmetic group.

In this paper we use the Putman-Wieland theorem to prove Ivanov’s conjecture by proving the following.

**Theorem 1.6.** The condition $\text{NFO}(g,b,p)$ is satisfied for every $p,b$ when $g \geq 2$. That is, whenever the genus of $X$ is $\geq 2$, $\pi : Y \to X$ is a finite cover, and $0 \neq c \in V$, the $\Gamma$-orbit of $c$ is infinite.

Theorem 1.2 follows directly from Theorem 1.6 by applying Theorem 1.5. The rest of this paper is devoted to proving Theorem 1.6. Note that if $Z \to Y \to X$ is a tower of covers, there is a $\Gamma$-equivariant injection $H_1(Y;\mathbb{Q}) \to H_1(Z;\mathbb{Q})$ given by the transfer map. Every cover of $X$ is itself covered by a characteristic cover. Thus, it is enough to prove Theorem 1.6 for characteristic covers $Y \to X$, and this will always be our assumption.

Our approach is more akin to Farb and Hensel’s. Given $0 \neq c \in V$ as in the statement of Theorem 1.6 we find a sequence $f_1, f_2, \ldots, \in \rho_\pi(\Gamma)$ such that $f_i(c) \neq f_j(c)$ for $i \neq j$.

Instead of using Dehn twists, we use a type of map called curve-pushing maps. These are a similar to the well known class of point-pushing maps for punctured surface.

Intuitively, a point-pushing map is the map that results from pushing a puncture along a closed curve and dragging the surface along with it. In the same intuitive vein, a curve-pushing map is the map that results from pushing a simple closed curve in the surface along a thickened closed curve that contains it and dragging the surface along with it. These classes of maps are much more flexible than Dehn twists, and much less tied to how lifts of
simple closed curves in $X$ behave in $Y$. This flexibility allows us to circumvent the pitfalls that are inherent when trying to use Dehn twists to prove theorem 1.2.

1.2. Outline of the proof of Theorem 1.6. In Sections 2, 3 and 4 we define curve-pushing maps, and describe an algorithm (which we call the iterated pushing algorithm) that describes how to find the image of a homology class under a curve-pushing map or the lift of a curve-pushing map to a finite cover. One important feature of this algorithm is that given a homology class, the image of this homology class under the curve-pushing map only depends on its intersection numbers with a specific family of curves called the top and bottom curves of the curve-pushing map. These sections also contain several important definitions that we use to describe curve-pushing maps.

In section 5 we discuss the central trick in our proof, which we call the folding trick. This trick is based on a somewhat surprising phenomenon which we now describe. A curve-pushing map $P$ is defined by two objects - the simple closed curve that is being pushed, and the thickened curve that it is being pushed along. This is similar to the point-pushing case where the map is defined by a puncture and a closed curve along which the puncture is pushed. However, unlike the point-pushing case, the isotopy class of a curve-pushing map depends on the specific embedding of the thickened curve in the surface. Homotopic thickened curves can give rise to different isotopy classes of push maps.

We exploit this phenomenon by associating to every curve-pushing map $P$ and every $k \in \mathbb{N}$ a deformation of $P$ that we denote $P_{+k_0}$. Essentially, this deformation adds $k$ homotopically trivial loops to the thickened curve which defines $P$.

Denote by $\tilde{\phi}$, and $\tilde{\phi}_{+k_0}$ the maps induced by $P, P_{+k_0}$ on $V$. The folding trick provides a relationship between these maps. It shows that if $c \in V$ and $c_{+k_0} \in V$ is a vector satisfying certain intersection number conditions with the top and bottom curves then

$$\tilde{\phi}_{+k_0}(c) = \tilde{\phi}(c_{+k_0}).$$

This easily implies that the orbit of $v$ is infinite, assuming one can find an infinite collection of vectors $c_{+k_0}$ satisfying the conditions of the folding trick. For some curve-pushing maps, it may not be possible to find $c_{+k_0}$ satisfying these conditions. The remainder of the proof is dedicated to finding curve-pushing maps for which this is possible.

Section 6 provides a description of $V$ as a deck group module. Section 7 uses the results of Section 6 to associate to each $0 \neq c \in V$ a collection of curve-pushing maps (which we call $c$-suitable curve-pushing maps) for which
there is an infinite collection of vectors $c_{k_0}$ satisfying the conditions of the folding trick. The proof is then completed in Section 8.

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2. **Push Maps**

In this section we define curve-pushing maps, which are central to our proof of Theorem 1.6. Curve-pushing maps are closely related to the more commonly used point-pushing and boundary-pushing maps.

There are two ways of viewing these maps. One standard way is to apply the Birman exact sequence ([2]). Let $X$ be a surface with punctures, $p$ be one of the punctures of $X$, and $\overline{X}$ be the surface obtained from $X$ by filling in the puncture $p$. The Birman exact sequence is the sequence:

1. $\pi_1(X, p) \rightarrow \text{Mod}(X) \rightarrow \text{Mod}(\overline{X}) \rightarrow 1.$

The image of the map $\pi_1(X, p) \rightarrow \text{Mod}(X)$ is called the point-pushing subgroup, and its elements are called point-pushing maps.

Suppose $X$ is a surface with boundary components. Let $\overline{X}$ be the surface obtained from $X$ by filling in one of the boundary components with a disk. There is a version of the Birman exact in this case that is due to Johnson ([12]):

1. $\pi_1(U\overline{X}) \rightarrow \text{Mod}(X) \rightarrow \text{Mod}(\overline{X}) \rightarrow 1.$

Where $U\overline{X}$ is the unit tangent bundle of $\overline{X}$. The image of the map $\pi_1(U\overline{X}) \rightarrow \text{Mod}(X)$ is called the boundary-pushing subgroup and its elements are called boundary-pushing maps.

Boundary-pushing maps can be used to define a more general class of map called curve-pushing maps. Given a surface $X$ and a non-separating simple closed curve $\delta$ in $X$, we can form a new surface $X_{\delta}$ by cutting along $\delta$. The surface $X_{\delta}$ has genus one less than the genus of $X$, and has two additional boundary components. A curve-pushing map is the map of $X$ we get by taking a boundary-pushing map on $X_{\delta}$ that pushes one of the added boundary components, and then regluing the two added boundary components.

In our proof of Theorem 1.6 we will require a more explicit construction of boundary pushing and curve pushing maps that uses a well known theorem in
differential topology called the *isotopy extension theorem*. We will also require some further definitions and structures to describe curve-pushing maps which will be essential for our calculations. This construction and these definitions and structures are described in this section.

**Theorem 2.1.** *(The isotopy extension theorem)* Let $X$ be a compact manifold (possibly with boundary) and $N$ a submanifold (without boundary). Let $H : N \times [0, 1] \to X$ be a smooth homotopy such that $H_t(x) = H(x, t) : N \to X$ is an embedding for each $t$ and $H_0$ is the inclusion of $N$ into $X$. Then $H$ can be extended to a smooth isotopy $\tilde{H} : X \times [0, 1] \to X$ where $\tilde{H}_t(x) = \tilde{H}(x, t) : X \to X$ is a diffeomorphism for each $t$, and $\tilde{H}_0$ is the identity map.

Extensions of homotopies $H$ that have the added property that $H_0 = H_1$ allow us to construct several important families of mapping classes.

For our purposes, the manifold $X$ will always be a surface. Our proof of Theorem 1.6 require us to consider surfaces that may have punctures. As it is stated, Theorem 2.1 requires the underlying manifold to be compact. We circumvent this by treating the punctures of $M$ as marked points whenever applying Theorem 2.1 and requiring all homotopies to fix them pointwise.

2.1. **Point-pushing maps.** Suppose first that $N = \{p\}$ is comprised of a single point. A homotopy $H$ as in Theorem 2.1 such that $H_0 = H_1$ is simply a parametrized based closed curve $\gamma : S^1 \to M$, based at the point $p$.

Let $\tilde{H}_1 : X \to X$ be the map defined in the isotopy extension theorem. Let $P_\gamma = \tilde{H}_1|_{X \setminus N}$. The map $P_\gamma$ is well defined, since $\tilde{H}_1(p) = p$. We call $P_\gamma$ the *point pushing map* that pushed the point $p$ about the curve $\gamma$.

By its definition, the map $\tilde{H}_1$ isotopic to the identity. However, the map $P_\gamma$ needn’t be (and generally isn’t) isotopic to the identity.

Intuitively, we think of point pushing maps as the result of placing our finger on the puncture $p$, and moving it along the curve $\gamma$, pushing the surface as we go.

2.2. **Boundary-pushing maps.** Suppose that $X$ is a surface with boundary. Let $N = \delta$ be one of the boundary curves of $X$. Let $\overline{X}$ be the surface obtained from $X$ by gluing a disk $D$ to the curve $\delta$. Let $H : S^1 \times [0, 1] \to \overline{X}$ be a map such that $H_t(x) = H(x, t)$ is an embedding for each $t$, and such that $H_0 = H_1$ is the inclusion of $\delta$ into $X$.

Let $\tilde{H}_1 : \overline{X} \to \overline{X}$ be the map provided by Theorem 2.1. Write $X = X \cup_\delta D$. Since $\tilde{H}_1|_{\delta}$ is the identity map, and $\tilde{H}_1$ is a diffeomorphism, it must send $X$ to $X$, and $D$ to $D$. Thus, the map $P_{\delta,H} = \tilde{H}_1|_{X}$ is a well defined diffeomorphism $X \to X$. We call $P_{\delta,H}$ the *boundary pushing map* of $\delta$ through the homotopy $H$. 
Similarly to the point pushing case, the map $\tilde{H}_1 : \overline{X} \to \overline{X}$ is homotopic to the identity, but the map $P_{\delta,H}$ is generally not.

We will confine ourselves to a specific type of homotopy. Let $n$ be an integer and let $A_n$ be a $n+2$ holed sphere. We think of $A_n$ as the result of removing $n$ disks $D_1', \ldots, D_n'$ from the interior of the annulus $A = \mathbb{R}/\mathbb{Z} \times [0,1]$. Let $f : A_n \to X$ be an immersion that sends the boundaries of each disk $D_i'$ homeomorphically to the boundary component $\delta$. The map $f$ can be extended to a map $\overline{f} : A \to \overline{X}$ by sending the each disk $D_i'$ homeomorphically to the disk $D$.

We can think of the image of $\overline{f}$ as a describing a thickened curve in $\overline{X}$ that contains the disk $D$. We call the curve $t \to f(t,1)$ the top curve of the thickened curve and denote it $\tau = \tau_f$. We call the curve $t \to f(t,0)$ the bottom curve of the thickened curve and denote it $\beta = \beta_f$.

Let $g = (g^1, g^2) : S^1 \to A_n$ be an orientation preserving homeomorphism from $S^1$ to the boundary of $D_1'$ in $A_n$. We can use the map $\overline{f}$ to define a homotopy $H : S^1 \times [0,1] \to \overline{X}$ by setting:

$$H(x,t) = \overline{f}(g^1(x) + 2\pi t, g^2(x)).$$

This homotopy satisfies $H_0 = H_1$. We can think of it intuitively as moving the curve $\delta$ along the thickened curve bounded between $\beta$ and $\tau$. Whenever we use boundary pushing maps, they will always be boundary pushing maps of this form. We will also always assume that the maps $f, g$ are differentiable.

2.2.1. Roads and Junctures. We now give some definitions regarding curve pushing maps that we will need later one.

**Definition 2.2.** A self intersection of the thickened curve $\gamma$ is called a juncture.

**Definition 2.3.** Let $\chi$ be a juncture that does not contain $\delta$. There is an open set $U$ containing it such that $U$ is orientation preserving diffeomorphic to $\mathbb{R}^2$, and under this diffeomorphism $U \cap \gamma$ is the union of finitely many thickened straight line segments whose intersection is the juncture. We call these segments roads.

We require a slightly different definition for roads passing through the juncture that contains $\delta$.

**Definition 2.4.** Suppose $\chi$ is a juncture that contains $\delta$. We have a similar description to the one in the previous definition except that $U \cap \gamma$ is the union of finitely many thickened line segments with a disk removed from their interior. In this case, the word roads will refer to the connected components of $(U \cap \gamma) \setminus \chi$. 
In both definitions, if the curve $\gamma$ passes through the same road multiple times, we will count them as separate roads.

Note that these definitions are local. The roads are part of the open neighborhoods containing the junctures, and do not continue past these neighborhoods.

Let $\mathcal{R}$ be the set of all roads. Let $\mathcal{J}$ be the set of all junctures. Given $r \in \mathcal{R}$ and $\chi \in \mathcal{J}$, we say that $r$ is passes through $\chi$ and write $r \in \chi$ if $r$ is one of the roads in the juncture $\chi$.

**Definition 2.5.** Given a juncture $\chi$ and two roads $r_1, r_2 \in \chi$, we denote by $\hat{i}(r_1, r_2)$ the oriented intersection number of the portions of the top (or bottom) curves that run along these roads.

**Definition 2.6.** We have an ordering $<$ on the set $\mathcal{R}$ where $r_1 < r_2$ if $\gamma$ passes through $r_1$ before it passes through $r_2$.

**Definition 2.7.** Let $r, r' \in \mathcal{R}$. We say that $r'$ precedes $r$, and denote $r' \prec r$ if:

(a) The roads $r'$ and $r$ both pass through the same juncture $\chi \in \mathcal{J}$.
(b) The roads $r'$ and $r$ are transverse.
(c) The two roads satisfy $r' < r$.
(d) There is no road $r''$ that passes through $\chi$ and is transverse to $r'$ which satisfies $r' < r'' < r$.

In order to unify the notation in our discussion, we make the following definition.

**Definition 2.8.** Add a juncture $\chi_0$ that contains the simple closed curve $\delta$ and a road $r_0$ such that $r_0$ is a minimal element for the ordering $<$. We think of $r_0, \chi_0$ as the beginning of the thickened curve $\gamma$. We call $r_0$ the initial road of $\gamma$ and $\chi_0$ the initial juncture. The parametrized bottom and top curves $\beta, \tau$ are always parametrized such that $\beta(0) = \beta(1)$ and $\tau(0) = \tau(1)$ occur in the road $r_0$.

To simplify our discussions, we will always make the following assumptions about thickened curves $\gamma$.

(a) The immersion assumption: The thickened curve $\gamma$ is immersed.
(b) The one exit assumption: There is an open neighborhood $U$ of $\chi_0$ such that $U \cap \gamma$ is homeomorphic to a single thickened line segment with a disk removed.
(c) The no triple intersections assumption: No juncture $\chi \in \mathcal{J}$ contains 3 roads, each one of which is transverse to the other two.

Note that we can always modify any thickened curve by a homotopy so that satisfies the above conditions. The one condition to notice is the no triple
intersection condition. A triple intersection at any juncture other than $\chi_0$ be pushed (as illustrated in Figure 1) and replaced by three junctures without such intersections. The juncture $\chi_0$ does not contain a triple intersection by the one exit assumption.

![Figure 1. Pushing a triple intersection.](image)

2.3. Curve-pushing maps. Let $X$ be a surface and $N = \delta \subset X$ be an essential simple closed curve.

Suppose first that the curve $\delta$ is separating, and write $X = X_1 \cup_\delta X_2$. The curve $\delta$ is a boundary curve of $X_1$. Any boundary pushing map $P_{\delta,H} : X_1 \to X_1$ can be extended to a map, which we also call $P_{\delta,H}$, from $X$ to $X$ by gluing $X_2$ to $X_1$ along $\delta$ and setting $P_{\delta,H}|_{X_2} = \text{Id}$. We call this extended map a curve pushing map and say that it pushes the curve $\delta$ through the homotopy $H$.

Suppose now that the curve $\delta$ is non-separating. Let $X_0$ be the surface obtained by cutting $X$ along $\delta$. The genus of the surface $X_0$ is one less than the genus of $X$ and it has two more boundary components. Call the two additional boundary components $\delta_1, \delta_2$. The surface $X$ is obtained from $X_0$ by gluing $\delta_1$ to $\delta_2$.

Any boundary pushing map $P_{\delta_1,H} : X_0 \to X_0$ satisfies $P_{\delta_1,H}|_{\delta_1} = \text{Id}$, $P_{\delta_1,H}|_{\delta_2} = \text{Id}$. Thus, by gluing $\delta_1$ to $\delta_2$, we get a well defined map $P_{\delta,H} : X \to X$. We say that this is a curve pushing map that pushes the curve $\delta$ along the homotopy $H$.

3. The action of curve-pushing maps on homology

In this section we describe the map induced by a curve-pushing map $P \in \text{Mod}(X)$ on $H_1(X;\mathbb{Z})$. This description appears as Proposition 3.7. While we will not require the full strength of this description in our proof of Theorem 1.6 we will require the concepts and methods that are involved in it.

A portion of the discussion below appears in our paper [9]. We include it here in the interest of providing a self contained explanation.

Let $P \in \text{Mod}(X)$ be a curve pushing map that pushes the simple closed curve $\delta$ through the thickened curve $\gamma$. As in section 2.2 let $\beta$ and $\tau$ be the
bottom and top curves of $\gamma$ respectively. Let $b, t$ be the homology classes of $\beta, \tau$ respectively. Let $\phi = P_\gamma : H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ be the map induced by $P$ on $H_1(X; \mathbb{Z})$.

Let $a \in H_1(X; \mathbb{Z})$. Let $\alpha$ be a representative of $a$. The general principle is that every time the thickened curve $\gamma$ intersects $\alpha$, the map $P$ pushes this intersection along the curve $\gamma$. We separate the intersections of $\alpha$ and $\gamma$ into two types.

Case 1: The portion of $\alpha$ intersecting $\gamma$ does not also intersect $\delta$. Figure 2 below shows this intersection being pushed along $\gamma$.

Case 2: The portion of $\alpha$ intersecting $\gamma$ does intersect $\delta$. Figure 3 below shows this intersection being pushed along $\gamma$.

Figure 2. Pushing an intersection with $\beta$ and $\tau$ along $\gamma$.

Figure 3. Pushing an intersection with $\beta$ along $\gamma$.

Denote by $N_b = N_b(a) = \hat{i}(b, a)$, where $\hat{i}(\cdot, \cdot)$ is the oriented intersection form on $H_1(X; \mathbb{Z})$. Similarly let $N_t(a) = \hat{i}(t, a)$.

An intersection of the first type illustrated above contributes $\pm 1$ to both $N_b$ and $N_t$. An intersection of the second type contributes either $\pm 1$ to $N_t$.

Denote $\phi(a) = a + w$. One principle that we can see in Figures 2 and 3 can be stated in the following way: a positive intersection with $b$ contributes $b$ to $w$, a positive intersection with $t$ contributes $-t$ to $w$. 

We will use this principle, together with a second principle which we elucidate below to construct \( w \) in a iterative manner. Let

\[
    w_0 = N_b \cdot b - N_t \cdot t.
\]

The class \( a + w_0 \) may have further intersections with the thickened curve \( \gamma \) that \( a \) did not. This happens when \( \beta, \tau \) have self intersections (see Figure 4 below). To each such intersection, one must apply the surgery depicted in Figure 5. These surgeries may lead to further self intersections, which lead to further surgeries, etc. Repeating this process iteratively will allow us to find a formula for \( w \).

**Figure 4.** Adding \( w_0 \) may add further intersections.

**Definition 3.1.** A \( \gamma \)-curve is a curve in \( X \) composed of segments that lie inside the thickened curve \( \gamma \) and which are parallel to portions of the curves \( \beta, \tau \).

The class \( w_0 \), by definition can be represented by a \( \gamma \)-curve. The iterated process we describe below will produce a sequence of \( \gamma \)-curves.

3.1. **The iterated pushing algorithm.** Write \( \mathcal{R} = \{r_1, \ldots, r_k\} \), where the roads in \( \mathcal{R} \) are indexed according to the order < described in \[2.2.1\].

Start with \( i = 1 \), and perform the following:

**The iterated pushing algorithm:** Suppose that there exists \( j < i \) such that \( r_j < r_i \). Suppose that there are \( p_1 \) positively oriented strands and \( n_1 \) negatively oriented strands of \( w_{i-1} \) passing through \( r_j \). Add \( p_1 + n_1 \) positively oriented strands and \( p_1 + n_1 \) negatively oriented strands, parallel to the portions of \( \beta, \tau \) that exit \( r_i \), and perform the surgery described in the right portion of Figure 5 to get the \( \gamma \)-curve \( w_i \). After performing this surgery for all \( r \in \mathcal{R} \) that satisfies \( r < r_i \), increment \( i \) by 1 and repeat.
The algorithm described above shows us how $P$ pushes the all the strands parallel to previous roads, as $\gamma$ passes through the $i^{th}$ road.

Note that in the iterated pushing algorithm, it’s important to use the relation $\prec$. If $r_j$ and $r_i$ don’t pass through the same juncture, then $P$ does not push the strands in $r_j$ as $\gamma$ passes through $r_i$. The same problem occurs if they pass through the same juncture, but $r_j > r_i$. If $r_j < r_i$ pass through the same juncture, but they are not transverse, then no pushing occurs. Finally, if $r_j < r_i$ pass through $\chi$ and are transverse, but there exists $r_j < r_k < r_i$ with $r_k \in \chi$ and $r_k$ transverse to $r_j$ then the strands in $r_j$ will be already be pushed out of the juncture when $\gamma$ passes through $r_k$ and will not need to be pushed out again when it passes through $r_i$.

![Figure 5. Pushing past an intersection in the iterated pushing algorithm.](image)

### 3.2. Describing $w_j$

Denote $w = w_k$. By construction we have that: $\phi(a) = a + w$. We wish to understand the homology classes $w_1, \ldots, w_k$ described above. We first need to define several terms.

**Definition 3.2.** Let $r \in \mathcal{R}$ be a road that is incident at the juncture $\chi$. Let $u$ be a $\gamma$-curve. Outside of $\chi$, The intersection of $r$ with $u$ consists of $r^+(u)$ strands oriented in the positive direction (that is, the direction of $r$) and $r^-(u)$ strands in the opposite direction. The *discrepancy of $u$ at $r$* or $\Delta_r(u)$ is defined to be $r^+(u) - r^-(u)$. When $u$ is the curve $w_0$ described in the iterated pushing algorithm, then the discrepancy of the initial road is $N_b - N_t$.

**Definition 3.3.** Let $r \in \mathcal{R}$, and $\chi$ be a juncture such that $r \in \chi$. Let $p_r$, or the *the posterior curve of $r$* be the curve made of the following four segments:

- a positively oriented strand parallel to the portion of $\beta$ that starts as $r$ exits $\chi$ and ends at the end of $\beta$,
- a negatively oriented strand parallel to $\tau$ that ends as $r$ exits $\chi$ and starts at the end of $\tau$,
- two short segments connecting these two (See Figure 6). The homology class of the posterior curve of the initial road is just $b - t$. 
Note that the definition of the top and bottom curve in Section 2.2 came with a parametrization \([0, 1] \to X\). If \(r\) is a road that leaves the intersection \(\chi\) at time \(t_r\), then the segments forming the posterior curve are the restrictions of \(\beta, \tau\) to the interval \([t_r, 1]\), together with two short segments connecting them.

**Definition 3.4.** The homology class \(p_r\) of \(p_r\) is always of the form \(md\), where \(d\) is the homology class of \(\delta\) and \(m \in \mathbb{Z}\). We call the number \(m\) the posterior winding number of \(u\) at \(r\) and denote it \(\omega_r(u)\).

Now consider the class \(w_i\) described above. This class is described by a \(\gamma\) curve. The curve \(w_{i+1}\) is associated to the road \(r_{i+1}\). Say that this road passes through the juncture \(\chi\). Let \(s_1, \ldots, s_l\) be the collection of all roads that precede \(r_{i+1}\), that is \(s_j \prec r_{i+1}\) for every \(j\).

**Lemma 3.5.** In the notation above:

\[
w_{i+1} = w_i + \sum_{j=1}^{l} \hat{i}(s_j, r_{i+1}) \Delta_{s_j}(w_i)p_{r_{i+1}}.
\]

*Proof.* For each \(1 \leq j \leq l\), the intersection of \(\chi\) with \(w_i\) contains \(s_j^+(w_i)\) positively oriented strands parallel to \(s_j\) and \(s_j^-(w_i)\) negatively oriented strands. The iterated pushing algorithm will push these strands to create \(\hat{i}(s_j, r_{i+1}) s_j^+(w_i)\) positively oriented copies of \(p_{r_{i+1}}\) and \(\hat{i}(s_j, r_{i+1}) s_j^-(w_i)\) negatively oriented copies. All in all, this push contributes \(\hat{i}(s_j, r_{i+1}) \Delta_{s_j}(w_i)p_{r_{i+1}}\).

No other strands will be pushed as \(\gamma\) passes through \(r_{i+1}\). Summing over all \(j\) gives the result.

The next lemma describes a phenomenon of curve pushing maps that we call *no propagation of discrepancy*. This phenomenon does not always hold for lifts of curve pushing maps to finite covers.
Lemma 3.6. In the notation above, for \( r \in \mathcal{R} \) and every \( 1 \leq i \leq k = \#\mathcal{R} \) we have that:
\[
\Delta_r(w_i) = \Delta_r(w_0).
\]

Proof. In each stage of the iterative pushing algorithm, an equal amount of positively and negatively oriented strands are added. These strands are always parallel to each other from beginning to end. Thus, performing each stage of the iterative pushing algorithm does not change the discrepancy at any road. \( \square \)

Note that for every \( r \): \( \Delta_r(w_0) = N_b - N_t = \hat{i}(b, a) - \hat{i}(t, a) \).

3.3. The push map formula. Putting the above discussion together, we get the following description of \( \phi \). As we noted in the beginning of the section, we do not need this description in our proof of Theorem 1.6. We do, however, require all of the tools and concepts that are involved in it.

Proposition 3.7. (The push map formula) For every \( a \in H_1(X; \mathbb{Z}) \):
\[
\phi(a) = a + (N_b - N_t) \left( \sum_{\chi \in \mathcal{J}} \sum_{r \in \chi} \sum_{r' \in \chi, r' < r} \hat{i}(r', r) \omega_r \right) d.
\]

3.4. A note concerning Figures. In all of the figures we provided above, intersections of \( \alpha \) with \( \gamma \) always appear along the initial road \( r_0 \). This is justifiable by the one exit and immersion assumptions that appear in Section 2.2.1.

Indeed, by these assumptions, we can divide the portion of \( \gamma \) outside \( \chi_0 \) into segments bounded on one side by a road which runs parallel to \( r_0 \) as it exits \( \chi_0 \) for the first time, and on the other side by a road which enters \( \chi_0 \) from the other direction.

To each such segment we can associate a closed curve by taking the portion of \( \beta \) that runs along this segment in the positive orientation, the portion of \( \tau \) that runs along the segment in the negative orientation, and two short segments connecting them.

None of these curves pass through \( \chi_0 \). If \( A_n, f \) are the \( n \)-holed annulus and the map used to define \( \gamma \) as in Section 2.2, the pre-image of each of the curves above bounds a rectangle in \( A_n \). Thus, all of these curves are homotopically trivial in \( X \).

Thus, by adding these curves to \( \alpha \) we get that \( \alpha \) is homotopic to a curve \( \alpha' \) with the following property. Every intersection of \( \alpha \) with \( \beta \) or \( \tau \) corresponds to a single intersection of the same sign of \( \alpha' \) with the curves \( \beta, \tau \) that occurs in \( r_0 \). Any further intersections of \( \alpha' \) with \( \beta, \tau \) come in pairs of strands of opposite orientation that pass through roads other than \( r_0 \). These pairs do
not affect the pushing algorithm as they do not contribute to the discrepancy at any road.

This justifies the depiction of the intersections of $\alpha$ and $\gamma$ that appear in our figures.

4. Lifts of curve pushing maps

Let $\pi : Y \to X$ be a characterisite finite cover with deck group $G$. Let $P \in \Gamma$ be a curve pushing map that pushes the curve $\delta$ through the thickened curve $\gamma$ with top and bottom curves $\tau, \beta$ respectively. Let $\tilde{P} \in \text{Mod}(Y)$ be a lift of $P$ to $\text{Mod}(Y)$. Let $\tilde{\phi} = \tilde{P}^* : H_1(Y; \mathbb{Z}) \to H_1(Y; \mathbb{Z})$.

Let $\alpha$ in $Y$ be a closed curve. As in the previous section, the curve $\tilde{\phi}(\alpha)$ can be obtained from $\alpha$ by repeatedly performing the local surgeries detailed in the iterated pushing algorithm. However, in the non trivial cover case there are several complications.

The first complication is that the curves $\beta, \tau$ may not lift to closed curves in $Y$. If they do, they will have multiple lifts. Every road will have its own top and bottom curves, and different roads may have different top and bottom curves.

The second complication arises from the fact at any given road, the top and bottom curves associated to that road run alongside each other as they pass through the road, but may not run alongside to each other from start to finish. This causes two issues. The first is that for any given juncture, the posterior curves defined in Section 3.2 may not be closed in $Y$. The second complication is that Lemma 3.6 may no longer hold. The discrepancies at each road might change as we proceed with the iterated pushing algorithm.

The final complication that arises is that $\tilde{P}$ may no longer be the identity outside of the pre-image of $\gamma$. There may be some twisting that arises from the deck group action.

These complication can give rise to very different descriptions than the one given in Proposition 3.7. This is not surprising. The linear transformation $\phi$ given in Proposition 3.7 is unipotent. For sufficiently complicated thickened curves $\gamma$, the map $P$ has positive topological entropy. By [10] and [17] we know that for such maps, there must be regular covers $\pi : Y \to X$ where the map $\tilde{\phi}$ has eigenvalues off the unit circle, and thus cannot be virtually unipotent.

In our proof of Theorem 1.6 we will not need a detailed description of $\tilde{\phi}$ and we can completely circumvent the complications detailed above. In a future paper we will provide such an explicit and detailed description for lifts of curve-pushing and point-pushing maps to finite covers. For our purposes, we only require that the iterated pushing algorithm can be applied in this case as well.
4.1. **Notation.** Given a closed curve $\beta$ in $X$, an **elevation of** $\beta$ is a connected component of $\pi^{-1}(\beta)$.

The pre-image $\tilde{\gamma} = \pi^{-1}(\gamma)$ may not be a thickened curve, but it admits the same local description. It’s a collection of roads that meet at junctures, and whose ends are extended and glued to each other.

Let $\tilde{R}$ be the collection of roads of $\tilde{\gamma}$ and $\tilde{J}$ be the collection of junctures. Each $r \in \tilde{R}$ has an associated elevation of $\beta$ and an associated elevation of $\tau$ - these are the elevations of $\beta$ and $\tau$ that run along the road $r$. We will denote these $\tilde{\beta}_r$ and $\tilde{\tau}_r$ respectively. We denote their homology classes by $b_r, t_r$ respectively.

Note that in the definition of $\prec$ for roads in the cover case, all the roads through any pre-image of $\chi_0$ are considered to be non-transverse to each other. No pushing occurs in the juncture $\chi_0$ due to the one exit assumption, and this holds in the cover $Y$ as well.

Any road $r \in \tilde{R}$ projects to a road $\bar{r} \in R$. Let $\tilde{p}_r$ be the lift of the posterior curve of the road $\bar{r}$ at $r$. Note that $\tilde{p}_r$ is not necessarily a closed curve.

As in Section 3, we define a $\tilde{\gamma}$-curve as a union of strands that run parallel to the sides of the roads in $\tilde{R}$. For each $\tilde{\gamma}$-curve $u$ and for each $r \in \tilde{R}$ we can define the integers $r^+(u), r^-(u)$, and the discrepancy $\Delta_r(u) = r^+(u) - r^-(u)$ as in Section 3.2.

5. **The folding trick**

Let $P \in \text{Mod}(X)$ be a curve-pushing map that pushes the curve $\delta$ through the thickened curve $\gamma$. Let $\phi = P_* : H_1(X; \mathbb{Z}) \to H_1(X; \mathbb{Z})$, and $a \in H_1(X; \mathbb{Z})$.

One feature of Proposition 3.7 which might be initially surprising is that $\phi(a)$ is determined by $\gamma$, but not by the homotopy classes of $\beta, \tau$. That is, it’s possible to find two thickened curves $\gamma, \gamma'$ with homotopic top and bottom curves such that the maps induced on $H_1(X; \mathbb{Z})$ by their corresponding push maps are not equal. This feature will lead us to the central trick in our proof of Theorem 1.6.

We illustrate this in Figure 7. The thickened curve in the bottom half of the figure is obtained from the thickened curve in the top by adding a homotopically trivial loop. We show the image under both of the corresponding push maps of a portion of a given curve. If we set $a$ to be the homology class of this curve, $\gamma$ to be the thickened curve in the top portion of the figure, $\gamma'$ to be the thickened curve in the bottom portion, and $d = [\delta]$, we have that:

$$\phi_{\gamma}(a) = \phi_{\gamma'}(a) + d.$$ 

Our proof of Theorem 1.6 uses an observation which we call the folding trick. We begin by setting up some notation and stating the folding trick for the trivial cover $X \to X$. 
Figure 7. The image of a homology under a curve-pushing map isn’t invariant under homotopies of $\gamma$.

**Definition 5.1.** For any $k \in \mathbb{N}$, let $\gamma_{+k\circ}$ be the thickened curve obtained from $\gamma$ by adding $k$ homotopically trivial loops at the beginning of the curve with oriented intersection number 1, as illustrated in Figure 8.

Figure 8. The thickened curve $\gamma_{+k\circ}$.

**Definition 5.2.** Let $P_{+k\circ}$ be the map that pushes $\delta$ through the thickened curve $\gamma_{+k\circ}$ and let $\phi_{+k\circ}$ be the map induced by $P_{+k\circ}$ on $H_1(X;\mathbb{Z})$. 
Lemma 5.3. (The folding trick) Let $P, b, t$ be as above. Fix $k \in \mathbb{N}$. Let $c \in H_1(X; \mathbb{Z})$. Let $\Delta = \hat{i}(b, c) - \hat{i}(t, c)$. Suppose that $c_{+k_0} \in H_1(X; \mathbb{Z})$ is an element such that $\hat{i}(b, c_{+k_0}) = \hat{i}(b, c) + k\Delta$, $\hat{i}(t, c_{+k_0}) = \hat{i}(t, c) + k\Delta$. Then:

$$\phi_{+k_0}(c) = \phi(c_{+k_0}).$$

Proof. We begin with the case $k = 1$. The thickened curve $\gamma_{+o}$ has one more juncture than $\gamma$ that comes right after the initial juncture. The intersection number at this juncture is 1.

The posterior curves of the initial roads of $P, P_{+o}$ are homotopic. Indeed, the posterior curve of $P_{+k_0}$ is obtained from the posterior curve of $P$ by adding the portions of the top and bottom curves as they pass through the homotopically trivial loop and connecting them.

Let $p_0$ be the homology class of the posterior curve of the initial road of $\gamma$, which is the same as the homology class of the posterior curve of the initial road of $\gamma_{+o}$, both of which are equal to $b - t$.

The discrepancy at the added juncture is $\Delta$. Applying the iterated pushing algorithm gives:

$$\phi_{+o}(a) = \phi(a) + \Delta(b - t) = \phi(a_{+o})$$

Repeating this same argument for $\gamma_{k+o}$ and $\gamma_{k+2o}, \gamma_{k+2o}, \ldots$ gives the general result.

The intuitive reasoning behind the folding trick is that adding a homotopically trivial loop at the beginning of $\gamma$ adds an additional juncture in the beginning of the loop with the same posterior curve as the initial juncture. The discrepancy at this juncture is the same as the discrepancy at the original juncture because discrepancy does not propagate. As far as the iterated pushing algorithm is concerned, this is entirely equivalent to simply adding $\Delta$ strands transverse to the initial road of $\gamma$. We think think of this intuitively as folding the discrepancy in $r_0$ back onto itself.

Now let $\tilde{P} \in \text{Mod}(Y)$ be a lift of $P$ to the characteristic cover $Y \to X$. Let $d$ be the degree of this cover. Let $\tilde{\phi}, \tilde{\phi}_{+k_0}$ be induced maps on $H_1(Y; \mathbb{Z})$.

A homotopically trivial loop in $X$ lifts to a homotopically trivial loop in $Y$. The $k$ homotopically trivial loops added to the beginning of $\gamma$ to form $\gamma_{+k_0}$ will lift to $d$ copies of $k$ homotopically trivial loops. In each such copy, the top and bottom curve run alongside to each other, and thus discrepancy does not propagate in the junctures added by these loops.

Let $s_1, \ldots, s_d$ be the lifts to $Y$ of $r_0$. Let $b_i, t_i$ be the homology classes of the elevations of the bottom and top curves of $s_i$. The exact same argument as in Lemma 5.3 gives following.
Lemma 5.4. (The folding trick for covers) Fix $k \in \mathbb{N}$. Let $c \in H_1(Y;\mathbb{Z})$. Let $\Delta_i = \tilde{i}(b_i, c) - \tilde{i}(t_i, c)$. Suppose that $c_{+k\circ} \in H_1(Y,\mathbb{Z})$ is a homology class that satisfies $\tilde{i}(b_j, c_{+k\circ}) = \tilde{i}(b_j, c) + k\Delta_j$ and $\tilde{i}(t_j, c_{+k\circ}) = \tilde{i}(t_j, c) + k\Delta_j$ for every $j$. Then:

$$\tilde{\phi}_{+k\circ}(c) = \tilde{\phi}(c_{+k\circ}).$$

Comment: The quantifiers that appear in the folding trick and the folding trick for covers may seem initially awkward. The reason for these quantifiers is that it’s possible that there are not vectors $c_{+k\circ}$ which satisfy the conditions in these lemmas. Finding the circumstances under which such vectors exist will take up the majority of the remainder of our proof.

6. Describing homology as a deck group module.

Let $Y \to X$ be a characteristic cover with deck group $G$. Let $B \subset H_1(Y;\mathbb{Q})$ be the subspace spanned by the homology classes of all the boundary components of $Y$ and all the small loops about the punctures of $Y$. Let $V = H_1(Y;\mathbb{Q})/B$. The spaces $V$ and $H_1(Y;\mathbb{Q})$ are both $G$-representations. In this section we wish to describe their $G$-representation structure.

When the surface $X$ has boundary components or punctures, the description of $H_1(Y;\mathbb{Q})$ is due to Gaschütz.

Theorem 6.1. (Gaschütz) Suppose $\pi_1(X) \cong F_n$ for $n \geq 2$. Then as a $G$-representation:

$$H_1(Y;\mathbb{Q}) \cong 1 \oplus \mathbb{Q}[G]^{n-1}$$

where 1 is the trivial representation of $G$.

We provide a topological proof of Gaschütz’s theorem that is due to Grünewald, Larsen, Lubotzky, and Malestein ([8]). The proof is short and we include it here since it provides some insight for the result we need.

Proof. Let $R$ be a wedge of $n$ circles meeting at a single point $\star$. We have that $\pi_1(R) \cong F_n$. Let $K < F_n$ be the subgroup corresponding to the cover $Y$, and let $\tilde{R} \to R$ be the corresponding cover of $R$.

The space $\tilde{R}$ is a graph, and thus a simplicial complex. There are $\#G$ pre-images of the point $\star$, and $G$ acts freely on the set of preimages. Thus $C_0(\tilde{R},\mathbb{Q})$, the space of simplicial 0 chains, is isomorphic to $\mathbb{Q}[G]$ as a $G$-representation. The same argument holds for the preimage of the $n$ edges in $\tilde{R}$. Thus, $C_1(\tilde{R},\mathbb{Q}) \cong \mathbb{Q}[G]^n$.

Fix a preimage $\tilde{\pi}$ of $\star$. This gives a correspondence between the preimages of $\star$ and the elements of $G$. Let $e$ be an oriented edge in $\tilde{R}$ connecting the vertex $h$ to the vertex $g$ (with $h, g \in G$). Let $\delta : C_1(\tilde{R},\mathbb{Q}) \to C_0(\tilde{R})$ be the boundary map. Then $\delta(e) = g - h$. 
Let $a \subset \mathbb{Q}[G]$ be the augmentation ideal, that is:
\[ a = \{ \sum_{g \in G} a_g g \mid \sum a_g = 0 \}. \]

The description of $\delta$ above gives that $\text{Im}(\delta) \subset a$.

As a $G$-representation, $\mathbb{Q}[G] \cong a \oplus 1$. Since $\dim H_0(\tilde{R}, \mathbb{Q}) = 1$, we must have that $\text{Im}(\delta)$ is a codimension 1 subspace of $\mathbb{Q}[G]$, and thus $\text{Im}(\delta) = a$.

Since $\tilde{R}$ is a graph, it has no 2-chains, and thus the only 1-chain that is a boundary is the trivial chain. Thus, we get that $H_1(Y; \mathbb{Q}) \cong H_1(\tilde{R}; \mathbb{Q})$ is the kernel of a surjection $\mathbb{Q}[G]^n \to a$.

Since $\mathbb{Q}[G]^n$ is a semi-simple $\mathbb{Q}[G]$ module, and $\mathbb{Q}[G] \cong a \oplus 1$, we get the desired result. $\square$

When the surface $X$ is a closed surface, the corresponding result is due to Chevalley and Weil (3).

**Theorem 6.2.** (Chevalley-Weil) Suppose that $X$ is a closed surface of genus $g$ for $g \geq 2$. Then as a $G$-representation,
\[ H_1(Y; \mathbb{Q}) \cong 1^2 \oplus \mathbb{Q}[G]^{2g-2}. \]

The proof of the Chevalley-Weil theorem is very similar to the proof Gaschütz’s theorem. The only difference is that one needs to add a 2-chain to $X$ and to keep track of the resulting boundaries.

We will need a weaker but more generally applicable version of these theorems which gives some information about $V$ when $X$ is not closed.

**Lemma 6.3.** Suppose that $X$ is a surface of genus $g \geq 2$ (possibly with punctures or boundary components). Let $Y, G, V$ be as above. Then $V$ contains $\mathbb{Q}[G]^{2g-2}$ as a sub-representation.

Proof. The homology groups $H_1(Y; \mathbb{Q}), V$ do not distinguish between punctures and boundary components, so we can assume that if $X$ is not closed then it has boundary components but no punctures. Let $b$ be the number of boundary components.

Equip $X$ with a CW-structure. Form a new space $\overline{X}$ by adding $d$ disks to fill in the $b$ boundary components. The cover $Y \to X$ can be extended to a branched cover $\overline{Y} \to \overline{X}$ where $\overline{Y}$ is a closed surface, and $H_1(\overline{Y}; \mathbb{Q})$ is isomorphic to $V$ as a $G$-module.

Let $C_*(Y, \mathbb{Q})$ be the CW-chain complex of $\overline{Y}$. We can write $C_2(\overline{Y}, \mathbb{Q}) \cong C_2(Y, \mathbb{Q}) \oplus U$ where $U$ is the subspace spanned by the lifts of the $b$ disks added to $X$ to form $\overline{X}$. Since $G$ permutes the preimages of each such disk, the space $U$ is isomorphic to a subspace of $\mathbb{Q}[G]^b$.

Note that $C_1(\overline{Y}, \mathbb{Q}) \cong C_1(Y, \mathbb{Q})$ and $C_0(\overline{Y}, \mathbb{Q}) \cong C_0(Y, \mathbb{Q})$. Furthermore, the boundary map $\delta_1 : C_1(\overline{Y}, \mathbb{Q}) \to C_0(\overline{Y}, \mathbb{Q})$ is just the boundary map.
$C_1(Y, \mathbb{Q}) \to C_0(Y, \mathbb{Q})$ and the restriction of the boundary map $\delta_2$ to $C_2(Y, \mathbb{Q})$ is equal to the boundary map $\delta_2$ on the 2-chains in $Y$.

Thus, we have that $V \cong H_1(Y; \mathbb{Q})/W$, where $W$ is the image of $\delta_2(U)$ in $H_1(Y; \mathbb{Q})$. Since $U$ is a sub-representation of $\mathbb{Q}[G]^b$, we have that $W$ is also a sub-representation of $\mathbb{Q}[G]^b$.

As a free group, $\text{rank}(\pi_1(X)) = 2g + b - 1$. Thus, by Gaschütz’s theorem:

$$H_1(Y; \mathbb{Q}) \cong \mathbb{Q}[G]^{2g+b-2} \oplus \mathbb{1}.$$  

By semi-simplicity of $\mathbb{Q}[G]$-modules, we get that $V \cong \mathbb{Q}[G]^r \oplus H$ where $r \geq 2g - 2 = 2g + b - 2 - b$. This proves the result.

\[\square\]

7. Finding suitable curve-pushing maps.

Recall that $V = H_1(Y; \mathbb{Q})/B$, where $B \subset H_1(Y; \mathbb{Q})$ is the subspace spanned by all the homology classes of the the boundary components of $Y$ and of curves parallel to the punctures of $Y$. Fix $0 \neq c \in V$. In this section we find a curve-pushing map $P \in \text{Mod}(X)$ that we will use in our proof of Theorem 1.6 to prove that the $\Gamma$ orbit of $c$ is infinite. Throughout this section we always assume that the genus of $X$ is $\geq 2$.

**Definition 7.1.** Let $M$ be a vector space over $\mathbb{Q}$. A subset $A \subset M$ is said to be negligible if it is contained in a finite union of proper linear subspaces of $M$. The set $A$ is said to be substantial if its complement is negligible.

We will repeatedly use the fact that if $A \subset M$ is a substantial subset, and $T: M' \to M$ is a surjective linear transformation, then $T^{-1}(A)$ is substantial in $M'$.

**Definition 7.2.** A vector $v \in V$ is said to be generic if $\mathbb{Q}[G] \cdot v \cong \mathbb{Q}[G]$. A pair of vectors $(u, v) \in V^2$ is said to be generic if both $u, v$ are generic and $\mathbb{Q}[G] \cdot \text{span}\{u, v\} \cong \mathbb{Q}[G]^2$.

**Observation 7.3.** By Lemma 6.3, the space $V$ contains $\mathbb{Q}[G]^2$ as a subrepresentation. Thus, the collection of non-generic vectors $v \in V$ is negligible, as is the collection of non-generic pairs $(u, v) \in V^2$. Thus, the collection of generic pairs is substantial.

**Definition 7.4.** A pair $(u, v) \in V^2$ is called a push pair if there is a curve-pushing map $P \in \text{Mod}(X)$, a lift $\tilde{P} \in \text{Mod}(Y)$ of $P$, a lift $s$ of the initial road $r_0$ of $P$, such that $u, v$ are the homology classes of the top and bottom curves of $s$.

**Lemma 7.5.** Let $x, y \in \pi_1(X, \star)$. Let $\delta$ be a simple closed curve in $X$, and $z \in \pi_1(X, \star)$ be an element corresponding to the curve $\delta$. Let $N_\delta$ be the normal closure of $z$ in $\pi_1(X, \star)$. If $xy^{-1} \in N_\delta$ then there exists a curve-pushing map
* P that pushes the curve δ through a thickened curve, whose top and bottom curves are represented by \( x, y \in \pi_1(X, \star) \).

**Proof.** Let \( A_n \) be an annulus with \( n \) additional boundary components, as in the definition of curve pushing maps in Section 2.2. Pick a base point \( \star_0 \in A_n \).

Let \( \beta_0, \tau_0 \) be the representatives of the top and bottom curves of the annulus in \( \pi_1(A_n, \star_0) \), and \( \alpha_1, \ldots, \alpha_n \) the representatives of the \( n \) holes. We have a presentation

\[
\pi_1(A_n, \star_0) = \langle \beta_0, \tau_0, \alpha_1, \ldots, \alpha_n | \beta_0 \tau_0^{-1} \alpha_1 \ldots \alpha_n = 1 \rangle.
\]

If \( xy^{-1} \in N \), then there is a homomorphism \( \pi_1(A_n, \star_0) \to \pi_1(X, \star) \) that sends \( \beta_0 \) to \( x \), \( \tau_0 \) to \( y \) and each hole of \( A_n \) to a conjugate of \( z \). Since \( A_n \) is an Eilenberg-Maclane space, we must have a map \( f \) inducing this homomorphism. As in section 2.2, this map defines a thickened curve \( \gamma \) with bottom curve \( x \) and top curve \( y \).

Let \( K < \pi_1(X, \star) \) be the subgroup corresponding to the cover \( Y \to X \). Let \( \delta \) be a non-separating simple closed curve in \( X \), represented by the element \( z \in \pi_1(X, \star) \). Let \( d \) be the image of \( z \) in \( G \), and let \( \ell \) be the order of \( d \).

We have a map \( T : K \to V \) that factors as \( K \to H_1(Y; \mathbb{Q}) \to V \) where the first map is the abelianization map, and the second map is the quotient map.

**Lemma 7.6.** Let \( W_\delta = T(K \cap N_\delta) \). The space \( W_\delta \) contains a subspace \( L \subseteq W_\delta \) such that \( L \) is \( G \)-invariant, and as a \( G \)-representation \( L \cong \mathbb{Q}[G] \).

**Proof.** Let \( A_\delta : V \to V \) be the averaging operator:

\[
A_\delta(x) = \frac{1}{\ell} \sum_{i=1}^{\ell} d^i x.
\]

**Claim:** \( \text{Ker}(A_\delta) \subseteq W_\delta \). To see this, note that \( \text{Ker}(A_\delta) \) is spanned by all elements of the form \( v - d \cdot v \). Let \( x \in K \) such that \( T(X) = v \). Since \( K < \pi_1(X, \star) \) we have that \( x z z^{-1} z^{-1} \in K \cap N_\delta \). Since \( T(x z z^{-1} z^{-1}) = v - d \cdot v \), we get the claim.

**Claim:** Let \( E_\delta \subseteq V \) be the subspace spanned by the homology classes of all elevations of \( \delta \). Then \( E_\delta \subseteq E_\delta' \subseteq V \), where \( E_\delta' \) is a \( G \)-invariant subspace and as \( G \)-representations, \( E_\delta' \cong \mathbb{Q}[G] \). To see this, let \( u_\delta \) be the homology class of some elevation of \( \delta \). The space \( E_\delta \) is the \( G \)-representation and generated by \( u_\delta \). As such, it is isomorphic to a sub-representation of \( \mathbb{Q}[G] \). Since \( \mathbb{Q}[G] \) is a sub-representation of \( V \), and \( \mathbb{Q}[G] \)-representations are semi-simple, we get the claim.

Now consider the isomorphism \( \iota : E_\delta' \to \mathbb{Q}[G] \). The \( \iota \)-image of \( E_\delta \) in \( \mathbb{Q}[G] \) is just the subspace of all \( d \)-invariant vectors. Denote this subspace \( \mathbb{Q}[G]_d \). The
averaging operator $A_\delta$ can be restricted to $E_\delta$. Under the isomorphism $\iota$ it gives the operator $A_d : \mathbb{Q}[G] \to \mathbb{Q}[G]$ such that $A_d(h) = \sum_{i=1}^\ell d_i h$.

Let $\mathbb{Q}[G]^d = \text{Ker}A_d$. The space $\mathbb{Q}[G]^d$ consists of all $h = \sum_{g \in G} a_g \cdot g$ such that $\sum_{i=1}^\ell a_d g = 0$. We have the decomposition $\mathbb{Q}[G] \cong \mathbb{Q}[G]^d \oplus \mathbb{Q}[G]^d$.

To conclude the proof, note that $\iota^{-1}(\mathbb{Q}[G]^d) \subset W_\delta$ by the first claim, and $\iota^{-1}(\mathbb{Q}[G]_d) \subset W_\delta$ because the homology class of every elevation of $\delta$ is in $W_\delta$.

Corollary 7.7. Let $\delta$ be a non-separating simple closed curve in $X$. Then the set of generic $v \in W_\delta$ is substantial in $W_\delta$.

Proof. Let $L \subseteq W_\delta$ be the subspace provided by Lemma 7.6. The space $W_\delta$ is a $\langle d \rangle$-representation. Since $\langle d \rangle$ representations are semi-simple, we can write $W_\delta = L \oplus M$, where $M$ is a $\langle d \rangle$-representation. This decomposition gives us a surjective $\langle d \rangle$-equivariant projection $W_\delta \to \mathbb{Q}[G]$.

The space $\mathbb{Q}[G]$ can be decomposed further as

$$\mathbb{Q}[G] = \bigoplus_{\psi \in \text{Irr}(G)} \psi^\dim(\psi)$$

where $\text{Irr}(G)$ is the set of all isomorphism classes of irreducible representations of $G$. For every $\psi \in \text{Irr}(G)$ we have a projection $\mathbb{Q}[G] \to \psi^\dim(\psi)$, and $\dim(\psi)$ further projections $\psi^\dim(\psi) \to \psi$.

An element $v \in \mathbb{Q}[G]$ precisely when its image under one of the compositions $\mathbb{Q}[G] \to \psi^\dim(\psi) \to \psi$ is equal to 0. Thus, the set of all generic elements in $\mathbb{Q}[G]$ is substantial. This now implies that the set of all generic elements in $W_\delta$ is substantial.

Corollary 7.8. Let $\delta$ be a non-separating simple closed curve in $X$. Then the set of all $w \in W_\delta$ for which there exists a generic push pair pair $(u,v) \in V^2$ with $u - v = w$ is substantial in $W_\delta$.

Proof. Consider the map $D : V^2 \to V$ given by $D(u,v) = u - v$. Let $S_\delta = D^{-1}(W_\delta)$. By Lemma 7.5 any pair of integral vectors $(u,v) \in S_\delta$ is a push pair since there are $x, y \in K$ represented by the homology classes $u, v$ respectively such that $xy^{-1} \in N_\delta$.

By Corollary 7.7 the set of all generic elements in $W_\delta$ is substantial. The restriction of $D$ to $S_\delta$ is surjective to $W_\delta$. The property of being substantial is preserved under pullbacks by surjective maps. Thus, the set of all $(u,v) \in S_\delta$ such that $w = D(u,v)$ is generic is also substantial.
Let $L \subseteq W_\delta$ be the subspace provided by Lemma 7.6. Let $y \in L$ be a generic element. By Lemma 6.3 there exists $x \in V$ such that $(x,y)$ is a generic pair.

Let $u = x + y$ and $v = x - y$. The pair $(u,v)$ is a generic pair, and $u - v = 2y \in L$ so $(u,v) \in S_\delta$. This gives us that $S_\delta$ contains a generic pair.

In particular, $S_\delta$ contains a $\mathbb{Q}[G]^2$ as a sub-representation. Using semi-simplicity of $G$-representations once more, we get a surjective $G$-equivariant projection $S_\delta \rightarrow \mathbb{Q}[G]^2$.

As in the proof of Corollary 7.7, the set of all generic pairs in $\mathbb{Q}[G]^2$ is substantial. This means that the set of all generic pairs $(u,v) \in S_\delta$ is substantial.

Since the intersection of substantial sets is substantial, we get that the set of all generic pairs $(u,v) \in S_\delta$ such that $w = u - v$ is generic is a substantial set, and thus the set of all $w \in W_\delta$ satisfying the conditions of the Corollary is substantial in $W_\delta$.

\[\square\]

Definition 7.9. Let $c$ be as above. A pair of vectors $(u,v) \in V^2$ is called $c$-distinguished if $\hat{i}(c,u) \neq \hat{i}(c,v)$. The pair $(u,v)$ is called generically $c$-distinguished if it is generic and $c$-distinguished.

Definition 7.10. The vector $c \in V$ is said to be simply hidden if $\hat{i}(c,a) = 0$ whenever $a$ is an elevation of a non-separating simple closed curve in $X$.

Lemma 7.11. Let $c$ be simply hidden. Let $c^\perp = \{v \in V | \hat{i}(v,c) = 0\}$. Then there exists a non-separating simple closed curve $\delta$ in $X$ such that $c^\perp \cap W_\delta \neq W_\delta$.

Proof. We begin by showing the following claim.

\textbf{Claim:} There exists a non-separating simple closed curve $\epsilon$ in $X$ represented by the element $d \in G$ such that $c$ is not $d$-invariant.

Let $\delta$ be a non-separating simple closed curve in $X$, and let $d$ be its image in $G$. Since the images of such curve in $\pi_1(X,\star)$ generate $\pi_1(X,\star)$, the collection of all $d$'s generates $G$. If $c$ is $d$-invariant for every $d$, then it must be $G$ invariant. Since $c \neq 0$ this means that $c$ is the image of some $a \in H_1(X;\mathbb{Z})$ under the map $H_1(X;\mathbb{Z}) \rightarrow H_1(Y;\mathbb{Z}) \rightarrow V$, where the first map is the transfer map.

Let $B_X$ be the subspace of $H_1(X;\mathbb{Z})$ spanned by its loops and punctures. The image under the transfer map of $b \in B_X$ is in $B \subset H_1(Y;\mathbb{Z})$. Since $c \neq 0$, we get that $a \neq B_X$.

The intersection form $\hat{i}_X(\cdot,\cdot)$ is non-degenerate on $H_1(X;\mathbb{Z})/B_X$ and every non-zero element in this space has a non-zero intersection with the image of a simple closed curve. Since $a \notin B_X$, there must be a non-separating simple closed curve $\epsilon$ in $X$ with $\hat{i}(a,\epsilon) \neq 0$. This implies that $c$ has a nontrivial intersection with an elevation of $\epsilon$, which is a contradiction to our assumption that $c$ is simply hidden. This proves the first claim.
Let \( \delta \) be a simple closed curve in \( X \) such that \( c \) is not \( d \)-invariant, where \( d \) is the image of \( \delta \) in \( G \).

Note that since \( W_\delta \) is a \( \mathbb{Q}[\langle d \rangle] \)-module, it is enough to prove the lemma for some \( c' \in \mathbb{Q}[\langle d \rangle] \cdot d \).

Let \( \ell \) be the order of \( d \). As in the proof of Lemma 7.6, let \( A_\delta : V \to V \) be the averaging operator:

\[
A_\delta(x) = \frac{1}{\ell} \sum_{i=1}^{\ell} d^i x.
\]

Let \( c' = c - A_\delta c \). By our assumption, \( c' \neq 0 \) and \( A_\delta c' = 0 \).

**Claim:** Let \( V_\delta \subset V \) be the space of \( d \)-invariant vectors. If \( v \in V_\delta \) then \( \hat{i}(c', v) = 0 \). To see this, we calculate:

\[
\hat{i}(c', v) = \hat{i}(c', A_\delta v) = \hat{i}(A_\delta c', v) = 0.
\]

The space \( V \) can be decomposed as \( V = \text{Ker}(A_\delta) \oplus V_\delta \). As in the proof of Lemma 7.6, this is true because it holds in \( \mathbb{Q}[G] \) and \( V \) is a sub-representation of \( \mathbb{Q}[G]^n \) for some \( n \).

**Claim:** There exists \( u' \in \text{Ker}(A_\delta) \) such that \( \hat{i}(c', u') \neq 0 \). To see this, note the following. Since \( c' \neq 0 \), there exists \( u \in V \) such that \( \hat{i}(c', u) \neq 0 \). Since \( V = \text{Ker}(A_\delta) \oplus V_\delta \), we can write \( u = u' + h \) with \( u' \in \text{Ker}(A_\delta) \) and \( h \in V_\delta \). The equality

\[
\hat{i}(c', u' + h) = \hat{i}(c', u') + \hat{i}(c', h) = \hat{i}(c', u')
\]

now gives the claim.

To complete the proof, we note that as in the proof of Lemma 7.6, \( \text{Ker}(A_\delta) \subset W_\delta \).

**Corollary 7.12.** Let \( c \) be simply hidden. Then there exists a non-separating simple closed curve \( \delta \) in \( X \) such that the set of all \( u \in W_\delta \) satisfying \( \hat{i}(c, u) \neq 0 \) is substantial in \( W_\delta \).

**Definition 7.13.** Let \( c \in V \). A generic, \( c \)-distinguished, push pair \( (u, v) \) is called a \( c \)-suitable pair. The corresponding curve-pushing map is also said to be \( c \)-suitable.

Since the intersection of substantial sets is always substantial (and hence non-empty), Corollaries 7.8 and 7.12 show the following.

**Lemma 7.14.** Let \( c \) be simply hidden. Then there exists a \( c \)-suitable push map.
8. Proof of Theorem 1.6

Proof. Let \( c \in V \). We wish to prove that the \( \Gamma \)-orbit of \( c \) is infinite.

Suppose first that \( c \) is not simply hidden. This means that there is a non-separating simple closed curve \( \delta \) in \( X \) and an elevation \( \tilde{\delta} \) of \( \delta \) to \( Y \) such that \( \hat{i}(c,[\tilde{\delta}]) \neq 0 \). Take \( T_\delta \in \Gamma \) to be the Dehn twist about \( \delta \). Let \( \tilde{T}_\delta \) be its lift to \( Y \). The \( \langle \tilde{T}_\delta \rangle \) orbit of \( c \) is infinite.

Now assume that \( c \) is simply hidden. Let \( P \in \Gamma \) be a suitable curve-pushing map, as in Lemma 7.14. Let \( \tilde{P} \) be its lift to \( Y \).

Let \( m \) be the degree of the cover \( Y \to X \). Let \( r_0 \) be the initial road of \( P \), and let \( s_1, \ldots, s_m \) be the \( m \)-lifts of this initial road. For \( i = 1, \ldots, m \), let \( b_i \) be the homology class of the bottom curve of \( s_i \), and \( t_i \) be the homology class of the top curve of \( s_i \).

Since \( P \) is \( c \)-suitable, the pair \( b_1, t_1 \) is generic, which implies that the set

\[
\{b_1, \ldots, b_m, t_1, \ldots, t_m\}
\]

is comprised of \( 2m \) linearly independent vectors. Therefore, the map \( V \to \mathbb{Q}^{2m} \) given by

\[
v \mapsto (\hat{i}(v,b_1), \ldots, \hat{i}(v,t_m))
\]

is surjective.

For every \( 1 \leq j \leq m \), let \( \Delta_j = \hat{i}(c, t_j - b_j) \). The surjectivity of the map \( v \mapsto (\hat{i}(v,b_1), \ldots, \hat{i}(v,t_m)) \) means that for every \( k \) we can find vectors \( c_{+k} \in V \) such that for all \( 1 \leq j \leq m \) it’s true that \( \hat{i}(b_j, c_{+k}) = \hat{i}(b_j, c) + k\Delta_j \) and \( \hat{i}(t_j, c_{+k}) = \hat{i}(t, c) + k\Delta_j \). This collection of vectors satisfy the conditions of the folding trick detailed in Lemma 5.4.

Since \( P \) is \( c \)-suitable, some of the pairs \( (b_j, t_j) \) are \( c \)-distinguished, so for some \( j \), \( \Delta_j \neq 0 \). Thus, all the vectors \( c_{+k} \) above are different.

Let \( \phi : V \to V \) be the map induced by \( \tilde{P} \). For every \( k \), let \( \tilde{\phi}_{+k} \) be the map induced by the \( \tilde{P}_{+k} \) (as defined in Section 5).

By Lemma 5.4 we have that for every \( k \):

\[
\tilde{\phi}(c_{+k}) = \tilde{\phi}_{+k}(c).
\]

Since \( \tilde{\phi} \) is invertible, and the set of all element of the form \( c_{+k} \) is infinite, the set of all elements of the form \( \tilde{\phi}(c_{+k}) \) must be infinite as well. Therefore, the set of all elements of the form \( \tilde{\phi}_{+k}(c) \) is infinite as well. This gives that \( c \) has an infinite \( \Gamma \)-orbit, as required.


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