Parametrization of the QCD coupling in Hard and Regge processes

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We examine the parametrization of the QCD coupling in the Bethe-Salpeter equations for the hard and Regge processes and determine the argument of $\alpha_s$ of the factorized gluon. Our analysis shows that for the hard processes $\alpha_s = \alpha_s(k_1^2/(1-\beta))$ where $k_1^2$ and $\beta$ are the longitudinal and transverse moment of the soft parton. On the other hand, in the Regge processes $\alpha_s = \alpha_s(k_1^2/\beta)$. We have also shown that the well-known parametrization $\alpha_s = \alpha_s(k_2^2)$ in the DGLAP equations stands only if the lowest integration limit $\mu^2$ over $k_2^2$ (the starting point of the $Q^2$-evolution) obeys the relation $\mu \gg \Lambda_{QCD} \exp(\pi/2)$, otherwise the coupling should be replaced by the more complicated expression.

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I. INTRODUCTION

Total resummations of radiative corrections in QCD are often performed with composing and solving evolution equations. Quite often such equations are of the Bethe-Salpeter type, with one gluon (or one ladder rung) being factorized. In particular, both the DGLAP and BFKL equations are of that type. As is well-known, the argument of equations. Quite often such equations are of the Bethe-Salpeter type, with one gluon (or one ladder rung) being factorized. In particular, both the DGLAP and BFKL equations are of that type. As is well-known, the argument of

\begin{equation}
\alpha_s = \alpha_s(k_1^2) \tag{1}
\end{equation}

in each rung of the ladder Feynman graphs. Such a dependence originally was used in analogy to the parametrization of $\alpha_s$ in the hard kinematics according to the results of Refs. [1, 2]. Later, the proof of the parametrization (1) in the DGLAP context was suggested in Ref. [3]. However, that proof was not done accurate enough. In our recent paper Ref. [4] we have revised Ref. [3] and showed that $\alpha_s$ in the Bethe-Salpeter equations, including DGLAP, is basically replaced by the effective coupling $\alpha_s^{\text{eff}}$ given by much more complicated expression than Eq. (1):

\begin{equation}
\alpha_s^{\text{eff}} = \alpha_s(\mu^2) + \frac{1}{\pi b} \arctan \left( \frac{\pi [\ln(k_1^2/\beta)^2] - \ln(\mu^2/\Lambda^2)]}{\pi^2 + \ln(k_1^2/\beta^2) \ln(\mu^2/\Lambda^2)} \right) \tag{2}
\end{equation}

where the Sudakov variable $\beta$ is the fraction of the longitudinal of the ladder parton, $\Lambda = \Lambda_{QCD}$ and $b = [11N - 2n_f]/(12\pi)$. However when $\mu$, the starting point of the $Q^2$-evolution is chosen large enough, namely when

\begin{equation}
\mu \gg \Lambda_{QCD} \exp(\pi/2), \tag{3}
\end{equation}

Eq. (2) can be simplified down to

\begin{equation}
\alpha_s^{\text{eff}} \approx \alpha_s(k_1^2/\beta). \tag{4}
\end{equation}

When Eq. (2) is applied to the DIS structure functions at $x \sim 1$ and Eq. (3) is also fulfilled, Eqs. (24) can be approximated by the DGLAP expression of Eq. (1).

II. PARAMETRIZATION OF $\alpha_s$ FOR QCD PROCESSES IN THE HARD KINEMATICS

Let us consider the contribution $M_t$ of the Feynman graph depicted in Fig. 1. The cases with $u$ and $s$-channel gluons factorized can be considered quite similarly. The solid lines in Fig. 1 denote quarks, though the generalization to the case of gluons is obvious. Through the paper we will assume that the lower particles, with momenta $p_1, p'_1$, have small virtualities $\sim \mu^2$ whereas virtualities of the upper partons, with momenta $q, q'$ are large: $-q^2 \sim -q'^2 \sim Q^2 \gg \mu^2$. Applying the Feynman rules to Fig. 1 we obtain:

\begin{equation}
M_t = \frac{s^2}{4\pi^2} \int d\alpha d\beta dk_1^2 \frac{M(s, Q^2, s\alpha, s\beta, k_1^2)}{s\beta - Q^2 - s\alpha\beta - k_1^2 + i\epsilon}[s\alpha - s\alpha\beta - k_1^2 + i\epsilon](s\alpha\beta + k_1^2 - i\epsilon) \alpha_s(-s\alpha\beta - k_1^2). \tag{5}
\end{equation}
We have used the standard Sudakov variables $k = -\alpha(q + xp) + \beta p + k_\perp$ and we have dropped the color factors as unessential for our analysis. $M$ corresponds to the blob in Fig. 1. In Eq. (3) we have also neglected the virtuality $p^2 = \mu^2$ of the initial parton and denoted $s = 2pq, \; x = Q^2/2pq, \; Q^2 = -q^2$. Amplitude $M$ is unknown, so it is impossible to perform the integration over any of the variables in Eq. (3). However, if we assume the leading logarithmic (LL) accuracy, we can use the QCD generalization of the bremsstrahlung Gribov theorem. According to it, $M$ does not depend on $\alpha$ and $\beta$. Integrating over $\alpha$ in Eq. (3) is conventionally performed with closing the integration contour down and taking the residue at $\alpha = (+k^2 / \mu^2)/(1 - \beta)$. It converts Eq. (3) into

$$M_t = -\frac{1}{2\pi} \int_\mu^2 \frac{dk^2}{k^2_\perp} \int_{\beta_0}^1 d\beta \frac{(1 - \beta)}{\beta} M(s, Q^2, k^2_\perp) \alpha_s \left(-\frac{k^2}{1 - \beta}\right)$$

(6)

where $\beta_0 = x + k^2_\perp / s$. Obviously, $\beta_0 \approx x = Q^2/2pq$ when $x \sim 1$ and the upper limit $s$ of the integration over $k^2_\perp$ can be changed for $Q^2$. The minus sign of the $\alpha_s$-argument in Eq. (6) indicates explicitly that the argument is space-like and for the space-like argument $\alpha_s$ is given by the well-known expression:

$$\alpha_s \left(-\frac{k^2}{1 - \beta}\right) = \frac{1}{b \ln \left(k^2 / ((1 - \beta)\Lambda^2)\right)}$$

(7)

With the LL accuracy, $k^2/(1 - \beta) \approx k^2_\perp$ and therefore in Eq. (6) $\alpha_s \approx \alpha_s(k^2_\perp)$. The minus sign of the argument of $\alpha_s$ is traditionally dropped, which drives us back to the standard expression of Eq. (1).

### III. Parametrization of $\alpha_s$ in the Bethe-Salpeter Equations

In this section we study the parametrization of $\alpha_s$ in the Bethe-Salpeter equation for the forward scattering amplitude $A$. Let us assume that $A$ obeys the following Bethe-Salpeter equation:

$$A = A_0 + \frac{1}{4\pi^2} \int dk^2_\perp d\beta dm^2 M(s\beta, Q^2, (m^2\beta + k^2_\perp)) \frac{(1 - \beta)k^2_\perp}{(m^2\beta + k^2_\perp - \mu^2)^2} \alpha_s(m^2)$$

(8)

The second term in the rhs of Eq. (6) is depicted in Fig. 2. We have used the standard Sudakov variables: $k = -\alpha(q + xp) + \beta p + k_\perp$. Following Ref. [3], we have replaced the Sudakov variable $\alpha = 2pk/2pq$ by the new variable $m^2 = (p - k)^2$. $M$ in Eq. (6) denotes the upper blob in Fig. 2. It includes both the off-shell amplitude $A$ and a kernel. Now we just notice that Eq. (6) can be solved only after $M$ has been known. $A_0$ stands for an inhomogeneous term. We focus on integrating over $\alpha$ in Eq. (6) and introduce

$$I = \int_{-\infty}^{\infty} dm^2 M(s\beta, Q^2, (m^2\beta + k^2_\perp)) \frac{(1 - \beta)k^2_\perp}{(m^2\beta + k^2_\perp - \mu^2)^2} \alpha_s(m^2)$$

(9)
The integrand of Eq. (9) has the singularities in $m^2$. First, there are two poles from the propagators:

$$m^2 = -k^2_\perp / \beta + i\varepsilon$$

and

$$m^2 = 0 - i\varepsilon.$$  \hspace{1cm} (10)  \hspace{1cm} (11)

Second, there are two cuts. The first cut is originated by the $k^2$ -dependence of $M$. In particular, it can be the logarithmic dependence. The cut begins at

$$m^2 = -k^2_\perp / \beta + i\varepsilon$$

and goes to the left. The second cut is related to $\alpha_s$. It begins at

$$m^2 = 0 - i\varepsilon$$

and goes to the right. The singularities (10-13) are depicted in Fig. 3. The integration over $m^2$ in Eq. (9) runs along

\begin{align*}
   \Re m^2 &< m^2 < \Re m^2 \\
   \Im m^2 &< m^2 < \Im m^2
\end{align*}

FIG. 3: Singularities of $I$ given by Eqs. (10-13)

the $\Re m^2$ -axis from $-\infty$ to $\infty$, so the integral can be calculated with choosing an appropriate closed integration contour $C$ and taking residues. The contour $C$ should include the line $-\infty < m^2 < \infty$ and a semi-circle $C_R$ with radius $R$. The contour $C_R$ may be situated either in the upper or in the right semi-plane of the $m^2$ -plane. However, if we choose $C_R$ to be in the upper semi-plane, we should deal with the cut (12) of an unknown amplitude $M$, which is impossible without making assumptions about $M$. Alternatively, choosing the contour $C_R$ in the lower semi-plane
involves analysis of the cut (13) of $\alpha_s$ and $\alpha_s$ is known. By this reason, we choose the latter option for $C_R$. the contour $C_{cut}$ which runs along both sides of the cut (13). According to the Cauchy theorem,

$$I_C \equiv \int_C dm^2 M(s\beta, Q^2, k^2) \frac{(1-\beta)k^2}{(m^2\beta + k^2_\perp - i\epsilon)^2} = -2\pi i \frac{(1-\beta)}{k^2_\perp} M(s\beta, Q^2, -k^2_\perp/(1-\beta))\alpha_s(\mu^2).$$  \(14\)

The rhs of Eq. (14) is the residue at the pole (11) and $\mu$ is introduced to regulate the IR singularity for $\alpha_s$. It should be chosen as large as $\mu >> \Lambda$ to guarantee applicability of the perturbative expression for $\alpha_s$. When the initial partons are quarks, $\mu$ should also obey $\mu \gg$ the quark mass. Obviously,

$$I_C = I + I_{cut} + I_R$$  \(15\)

where $I$ is defined in Eq. (9), $I_R$ stands for the integration over the lower semi-circle and $I_{cut}$ refers to the integration along the cut (13). $I_R$ can be dropped because $I_R \to 0$ when $R \to \infty$. Now we specify $I_{cut}$:

$$I_{cut} = -2\pi i \int_{\mu^2}^{\infty} dm^2 \frac{(1-\beta)k^2}{(m^2\beta + k^2_\perp - i\epsilon)^2} \frac{\alpha_s(m^2)}{m^2}. \tag{16}$$

The integration in Eq. (16) cannot be done precisely because it involves the unknown amplitude $M$ depending on $m^2$. Nevertheless, it is possible to estimate $I_{cut}$. Indeed, the $m^2$- dependence of $M$ in Eq. (16) can be neglected in the region $m^2 \ll k^2_\perp/\beta$. Doing so, we obtain the following estimate of $I$:

$$I \approx -2\pi i \frac{(1-\beta)}{k^2_\perp} M(s\beta, Q^2, k^2_\perp) \int_{\mu^2}^{k^2_\perp/\beta} dm^2 \frac{\alpha_s(m^2)}{m^2} = -\frac{2\pi i (1-\beta)}{k^2_\perp} M(s\beta, Q^2, k^2_\perp) \alpha^{eff}_s, \tag{17}$$

with $\alpha^{eff}_s$ given by Eq. (4). When $\mu$ is chosen as large that Eq. (3) is fulfilled, we can drop $\pi^2$ in Eq. (17) and arrive at the estimate

$$I \approx - \frac{2\pi i (1-\beta)}{k^2_\perp} M(s\beta, Q^2, k^2_\perp) \alpha_s(k^2_\perp/\beta). \tag{18}$$

Finally, we give several estimates of $R = |\alpha^{eff}_s - \alpha_s|/\alpha^{eff}_s$, assuming that $\Lambda \approx 0.1\text{GeV}$: $R = 5\%$ at $\mu^2 \approx 30\text{GeV}^2$; then $R = 10\%$ at $\mu^2 \approx 2.4\text{GeV}^2$ and $R = 50\%$ at $\mu^2 \approx 0.8\text{GeV}^2$.

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