Abstract. A fractafold, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold. For compact fractafolds based on the Sierpiński gasket, it was shown by the first author how to compute the discrete spectrum of the Laplacian in terms of the spectrum of a finite graph Laplacian. A similar problem was solved by the second author for the case of infinite blowups of a Sierpiński gasket, where spectrum is pure point of infinite multiplicity. Both works used the method of spectral decimations to obtain explicit description of the eigenvalues and eigenfunctions. In this paper we combine the ideas from these earlier works to obtain a description of the spectral resolution of the Laplacian for noncompact fractafolds. Our main abstract results enable us to obtain a completely explicit description of the spectral resolution of the fractafold Laplacian. For some specific examples we turn the spectral resolution into a “Plancherel formula”. We also present such a formula for the graph Laplacian on the 3-regular tree, which appears to be a new result of independent interest. In the end we discuss periodic fractafolds and fractal fields.

Acknowledgments. The second author is very grateful to Stanislav Molchanov, Peter Kuchment and Daniel Lenz for very helpful discussions, and to Eugene B. Dynkin for asking questions about the periodic fractal structures.
# Contents

| Section                                                                 | Page |
|------------------------------------------------------------------------|------|
| Abstract                                                               | 2    |
| Acknowledgments                                                        | 2    |
| 1. Introduction                                                        | 4    |
| 2. Sierpiński fractafolds                                              | 5    |
| 2.1. Infinite Sierpiński gaskets                                      | 5    |
| 2.2. Laplacian on the Sierpiński gasket                               | 7    |
| 2.3. Sierpiński gasket: spectral decimation and the eigenfunction extension map | 10   |
| 3. Periodic Fractafolds                                                | 12   |
| 4. The Tree Fractafold                                                 | 17   |
| 5. General infinite fractafolds and graphs                             | 22   |
| 6. Technical details                                                  | 25   |
| 6.1. Underlying graph assumptions and Sierpiński fractafolds           | 25   |
| 6.2. Eigenfunction extension map on fractafolds                        | 25   |
| 6.3. Spectral decomposition (resolution of the identity)               | 26   |
| References                                                             | 28   |
1. Introduction

Our aim is a “Plancherel formula”:

\[ P_\lambda f(x) = \int P(\lambda, x, y)f(y)d\mu(y) \]

\[ f = \int_{\sigma(-\Delta)} P_\lambda f dm(\lambda) \]

\[ -\Delta P_\lambda f = \lambda P_\lambda f \]

\[ ||f||^2 = \int_{\sigma(-\Delta)} ||P_\lambda f||^2_\lambda dm(\lambda). \]

Our plan:
find a continuation from graphs to fractafolds.
find the explicit spectral resolution of the graph Laplacian on \( \Gamma \);
describe explicitly a Hilbert space of \( \lambda \)-eigenfunctions with norm \( || \cdot ||_\lambda \);
2. Sierpiński fractafolds

2.1. Infinite Sierpiński gaskets.

Figure 2.1. A part of an infinite Sierpiński gasket.
Figure 2.2. An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathcal{R}(\cdot)$, the vertical axis contains the spectrum of $\sigma(-\Delta_{\Gamma_0})$ and the horizontal axis contains the spectrum $\sigma(-\Delta)$.

**Theorem 2.1.** On the Barlow-Perkins infinite Sierpiński fractafold the spectrum of the Laplacian consists of a dense set of eigenvalues $\mathcal{R}^{-1}(\Sigma_0)$ of infinite multiplicity and a singularly continuous component of spectral multiplicity one supported on $\mathcal{R}^{-1}(J_R)$. [T98, Quint09]
2.2. Laplacian on the Sierpiński gasket. Let $\mu_{SG}$ be the normalized Hausdorff probability measure on $SG$.

The Laplacian $\Delta_{SG}$ on $SG$ is self-adjoint on $L^2(SG, \mu_{SG})$ with appropriate boundary conditions and, using Kigami’s resistance (or energy) form,

$$\mathcal{E}(f, f) = \lim_{n \to \infty} \left( \frac{5}{3} \right)^n \sum_{x, y \in V_n, x \sim y} (f(x) - f(y))^2 = -\frac{3}{2} \int_{SG} f \Delta_{SG} f d\mu_{SG}$$

for functions in the corresponding domain of the Laplacian (Dirichlet or Neumann).
Example 2.2. Spectral decimation for the unit interval $[0,1]$. $\Delta = \Delta_{[0,1]} = \frac{d^2}{dx^2}$ is the standard Laplacian on $[0,1]$, $\mu = \mu_{[0,1]}$ is the Lebesgue measure on $[0,1]$, and

$$E(f,f) = \lim_{n \to \infty} 2^n \sum (f(\frac{k}{2^n}) - f(\frac{k+1}{2^n}))^2 = \int_0^1 (f'(x))^2 \, dx = -\int_0^1 f \Delta f \, d\mu$$

for functions in the domain of the Dirichlet or Neumann self-adjoint Laplacian. The “eigenfunction extension map” is

$$\psi_{v,\lambda}(x) = \cos(\sqrt{\lambda} |x-v|) - \frac{\cos(\sqrt{\lambda})}{\sin(\sqrt{\lambda})} \sin(\sqrt{\lambda} |x-v|)$$

where $v$ is 0 or 1. See [Post2008].

To compute the spectrum of $-\Delta_{[0,1]}$ one can use the spectral decimation method with inverse iterations of the polynomial

$$R(z) = z(4 - z).$$

Each positive eigenvalue can be written as

$$\lambda = \lim_{m \to \infty} 4^m \lambda_m$$

for a sequence $\{\lambda_m\}_{m=m_0}^{\infty}$ such that

$$\lambda_m = R(\lambda_{m+1})$$
and $\lambda_{m_0} \in \{0, 4\}$. Then

$$\Re(z) = \lim_{k \to \infty} R^{ok}(4^{-k}z) = 2 - 2 \cos(\sqrt{z})$$

satisfies the functional equation $R(\Re(z)) = \Re(4z)$ and $\sigma(-\Delta_{[0,1]}) \subset \Re^{-1}\{0, 4\}$.
2.3. Sierpiński gasket: spectral decimation and the eigenfunction extension map. Fukushima-Shima-Stichartz-T [FS] [St03] [St06book] [T98]. Each positive eigenvalue for $-\Delta_{SG} u = \lambda u$ can be written as

$$\lambda = \lim_{m \to \infty} 5^m \lambda_m = 5^{m_0} \lim_{k \to \infty} 5^k \lambda_{k+m_0}$$

for a sequence $\{\lambda_m\}_{m=m_0}^\infty$ such that $\lambda_m = R(\lambda_{m+1})$ and $\lambda_m \in \{2, 5, 6\}$ where

$$R(z) = z(5 - z).$$

With solutions of Poincare functional equations

$$R(z) = \lim_{k \to \infty} R^{\circ k}(5^{-k} z) \quad R(R(z)) = R(5z).$$

we obtain

$$\Sigma_D = 5 \left( R^{-1}\{2, 5\} \cup 5R^{-1}\{5\} \bigcup_{m_0=2}^\infty 5^{m_0}R^{-1}\{3, 5\} \right)$$

and

$$\Sigma_N = \{0\} \cup 5 \left( R^{-1}\{3\} \cup \bigcup_{m_0=1}^\infty 5^{m_0}R^{-1}\{3, 5\} \right).$$

The explicitly computed multiplicities grow exponentially fast.
If we define
\[ \Sigma_{ext} = 5 \left( \mathbb{R}^{-1}\{2\} \cup \bigcup_{m=0}^{\infty} 5^m \mathbb{R}^{-1}\{5\} \right) \subset \mathbb{R}^{-1}\{0, 6\}. \]
then

**Proposition 2.3.** For any \( v \in \partial SG \) and any complex number \( \lambda \notin \Sigma_{ext} \) there is a unique continuous function \( \psi_{v,\lambda}(\cdot) : SG \to \mathbb{R} \), called the **eigenfunction extension map**, such that \( \psi_{v,\lambda}(v) = 1 \), \( \psi_{v,\lambda} \) vanishes at the other two boundary points, and the pointwise eigenfunction equation \(-\Delta \psi_{v,\lambda}(x) = \lambda \psi_{v,\lambda}(x)\) holds at every point \( x \in SG \setminus \partial SG \).
3. Periodic FractaFolds

Figure 3.1. A part of the periodic triangular lattice finitely ramified Sierpiński fractal field and the graph $\Gamma_0$. 
Proposition 3.1. The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum.

The absolutely continuous spectrum is $\mathcal{A}^{-1}[0, \frac{16}{3}]$.

The pure point spectrum consists of two infinite series of eigenvalues of infinite multiplicity. The series $5\mathcal{A}^{-1}\{3\} \subsetneq \mathcal{A}^{-1}\{6\}$ consists of isolated eigenvalues, and the series $5\mathcal{A}^{-1}\{5\} = \mathcal{A}^{-1}\{0\}\setminus\{0\}$ is at the gap edges of the a.c. spectrum. The eigenfunction with compact support are complete in the p.p. spectrum. The spectral resolution is given in the main theorem.
Remark 3.2. Note that on a periodic graph, linear combinations of compactly supported eigenfunctions are dense in an eigenspace.

(see [Kuchment05, Theorem 8], [Kuchment93], [KuchmentPost, Lemma 3.5])

The computation of compactly supported 5- and 6-series eigenfunctions is discussed in detail in [St03, T08], and the eigenfunctions with compact support are complete in the corresponding eigenspaces. In particular, [St03, T08] show that any 6-series finitely supported eigenfunction on $\Gamma_{n+1}$ is the continuation of any finitely supported function on $\Gamma_n$, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathcal{F}$ can be computed using the eigenfunction extension map on fractafolds (see Subsection 6.2).

Similarly, any 5-series finitely supported eigenfunction on $\Gamma_{n+1}$ can be described by a cycle of triangles (homology) in $\Gamma_n$, and the corresponding continuous eigenfunction on the Sierpiński fractafold $\mathcal{F}$ is computed using the eigenfunction extension map on fractafolds.
Example 3.3. The Ladder Fractafold.

It is easy to see that the spectrum of $-\Delta_\Gamma$ is $[0, 6]$, $-\Delta_{\Gamma_0}$ has absolutely continuous spectrum $[0, 6]$ with multiplicity 2 in $[0, 2]$ and $[4, 6]$ and multiplicity 4 in $[2, 4]$. 

Figure 3.3. The graphs $\Gamma$ and $\Gamma_0$ for the Ladder Fractafold
Example 3.4. The Honeycomb Fractafold.

Figure 3.4. A part of the infinite periodic Sierpiński fractafold based on the hexagonal (honeycomb) lattice.
In this section we study in detail the spectrum of the Laplacian on the tree fractafold whose cell graph $\Gamma$ is the 3-regular tree. In a sense this example is the “universal covering space” of all the other examples.
It is easy to see from the 6-eigenvalue equation that $F_z$ is the unique (up to a constant multiple) function in $E_6$ that is radial about $z$ (a function of $d(x, z)$). Let $\tilde{P}_6(x, y) = \frac{1}{\sqrt{3}}F_x(y) = \frac{1}{3}(-\frac{1}{2})^{d(x, y)}$ and define $\tilde{P}_6 F(x) = \sum_y \tilde{P}_6(x, y) F(y)$.

**Theorem 4.1.** $\tilde{P}_6$ is the orthogonal projection $\ell^2(\Gamma_0) \to E_6$; $\{F_z\}$ is not an orthonormal basis of $E_6$, since $\langle F_z, F_y \rangle = \sqrt{3}F_z(y)$, but it is a tight frame

$$\sum_z |\langle F, F_z \rangle|^2 = 3\|F\|_{\ell^2(\Gamma_0)}^2.$$
The solution of problem \((a)\) is due to Cartier [Cartier]. We outline the solution following [F-TN].

**Definition 4.2.** Let \(z \in \mathbb{C}\) with \(2^{2z-1} \neq 1\). Let \(c(z) = \frac{1}{3} \frac{2^{2z-2} - 2}{2^{2z-2} - 2z-1}\), \(c(1 - z) = \frac{1}{3} \frac{2^{2z-2} - 2}{2^{2z-2} - 2z-1}\) and \(\varphi_z(n) = c(z)2^{-nz} + c(1 - z)2^{-n(1-z)}\).

**Remark 4.3.** Note that \(c(z)\) and \(c(1 - z)\) are characterized by the identities \(c(z) + c(1 - z) = 1\) and \(c(z)2^{z} + c(1 - z)2^{z-1} = c(z)2^{z} + c(1 - z)2^{1-z}\) which imply \(\varphi_z(0) = 1\) and \(\varphi_z(1) = \varphi_z(-1)\).

**Theorem 4.4.** For any fixed \(y \in \Gamma\), let \(f_y(x) = \varphi_z(d(x, y))\). Then

\[-\Delta f_y = (3 - 2z - 2^{1-z})f_y\]

and \(f_y\) may be characterized as the unique \((3 - 2z - 2^{1-z})\)-eigenfunction that is radial about \(y\) and satisfying \(f_y(y) = 1\).

**Theorem 4.5.** For any \(F \in \ell^2(\Gamma_0)\) we have the explicit spectral resolution

\[F = \tilde{P}_0F + \int_{\Sigma} \tilde{P}_\lambda F dm(\lambda)\]

for

\[\tilde{P}_\lambda F(x) = \frac{1}{3(6 - \lambda)} \sum_y \psi_{\frac{1}{2}+\lambda}(d(x, y))F(y).\]
An explicit Plancherel formula on $\Gamma$ is given in terms of the modified mean inner product

$$< f, g >_M = \lim_{N \to \infty} \frac{1}{N} \sum_{d(x, x_0) \leq N} f(x) \overline{g(x)}.$$ 

We deal with eigenspaces for which the limit exists and is independent of the point $x_0$. This is not the usual mean on $\Gamma$, since the cardinality of the ball $\{x : d(x, x_0) \leq N\}$ is $O(2^n)$, but it is tailor made for functions of growth rate $O(2^{-d(x, x_0)/2})$, which is exactly the growth rate of our generalized eigenfunctions.

**Theorem 4.6.** Suppose $f$ has finite support. Then

$$< P_\lambda f, f > = 12b(\lambda)^{-1} < P_\lambda f, P_\lambda f >_M$$

and

$$\|f\|_{L^2(\Gamma)}^2 = \int_\Sigma < P_\lambda f, P_\lambda f >_M 12b(\lambda)^{-1} d\mu(\lambda).$$
Figure 4.2. A part of $\Gamma_1$ with a 5-eigenfunction (values not shown are equal to zero).
5. General infinite fractafolds and graphs

Let \( \Gamma \) be the cell graph, an arbitrary infinite 3-regular graph. Then \( \Sigma = \sigma(-\Delta_{\Gamma}) \subset [0,6] \), and for \( \mu - a.e. \lambda \)

\[ -\Delta_{\Gamma} P_{\lambda}(\cdot, b) = \lambda P_{\lambda}(\cdot, b) \]

\[ P_{\lambda} f(a) = \sum_{b \in \Gamma} P_{\lambda}(a, b) f(b) \]

\[ f = \int_{\Sigma} P_{\lambda} f d\mu(\lambda) \]

\[ -\Delta_{\Gamma} P_{\lambda} f = \lambda P_{\lambda} f, \]
Let $\Gamma_0$ denote the 4-regular edge graph of $\Gamma$ and

$$\tilde{P}_\lambda(x, y) = \frac{1}{6 - \lambda} \sum_{a \in x} \sum_{b \in y} P\lambda(a, b)$$

(there are 4 terms in the sum).

**Theorem 5.1.** The spectral resolution of $-\Delta_{\Gamma_0}$ is given by

$$F = \tilde{P}_0 F + \int_\Sigma \tilde{P}_\lambda F d\mu(\lambda)$$

where

$$-\Delta_{\Gamma_0} \tilde{P}_\lambda F = \lambda \tilde{P}_\lambda F$$

for $\mu - a.e. \lambda$, and

$$\tilde{P}_\lambda F(x) = \sum_{y \in \Gamma_0} \tilde{P}_\lambda(x, y) F(y).$$

In particular, $\sigma(-\Delta_{\Gamma_0}) = \Sigma$ or $\Sigma \cup \{6\}$. 
Problems:
(a) Find an explicit formula for $P_\lambda(a, b)$;
(b) Give an explicit description of the projection operator $\tilde{P}_6$;
(c) Find an explicit description of the generalized eigenspace $\xi_\lambda$ and its inner product, and transfer this to $\tilde{\xi}_\lambda$ of $\Gamma_0$.

Conjecture 5.2. For $\mu - a.e. \lambda$ there exists a Hilbert space of $\lambda$-eigenfunctions $\xi_\lambda$ with inner product $<,>_\lambda$ such that $P_\lambda f \in \xi_\lambda$ for $\mu - a.e. \lambda$ for every $f \in \ell^2(\Gamma)$, and

$$ < P_\lambda f, f > = < P_\lambda f, P_\lambda f >_\lambda . $$

Moreover a similar statement holds for $< \tilde{P}_\lambda F, F >$. 

6. Technical details

6.1. Underlying graph assumptions and Sierpiński fractafolds. Let $\Gamma_0 = (V_0, E_0)$ be a finite or infinite graph. To define a Sierpiński fractafold, we assume that $\Gamma_0$ is a 4-regular graph which is a union of complete graphs of 3 vertices. It can be said that $\Gamma_0$ is a regular 3-hyper-graph in which every vertex belongs to two hyper-edges.

We define a Sierpiński fractafold $\mathfrak{F}$ by replacing each cell of $\Gamma_0$ by a copy of $SG$.

6.2. Eigenfunction extension map on fractafolds. For any function $f_0$ on $\Gamma_0$ (and any $\lambda$ as above), we define the eigenfunction extension map by

$$\Psi_\lambda f_0(x) = \sum_{v \in V_0} f_0(v) \psi_{v,\lambda}(x).$$

By definition, $f = \Psi_\lambda f_0$ is a continuous extension of $f_0$ to the Sierpiński fractafold $\mathfrak{F}$ which is a pointwise solution to the eigenvalue equation

$$-\Delta \psi_{v,\lambda}(x) = \lambda \psi_{v,\lambda}(x)$$

for all $x \in \mathfrak{F} \setminus V_0$. $\Psi_\lambda : \ell^2(V_0) \to L^2(\mathfrak{F}, \mu)$ is a bounded linear operator for any $\lambda \notin \mathcal{R}^{-1}\{2, 5, 6\}$, and its adjoint $\Psi_\lambda^* : L^2(\mathfrak{F}, \mu) \to \ell^2(V_0)$ is

$$\left(\Psi_\lambda^* g\right)(v) = \int_{\mathfrak{F}} g(x) \psi_{v,\lambda}(x) d\mu(x).$$
6.3. Spectral decomposition (resolution of the identity). Let the self-adjoint discrete Laplacian $\Delta_{\Gamma_0}$ on $\Gamma_0$ have a spectral decomposition

$$-\Delta_{\Gamma_0} = \int_{\sigma(-\Delta_{\Gamma_0})} \lambda dE_{\Gamma_0}(\lambda)$$

$$-\Delta_{\Gamma_0} f_0(v) = \int_{\sigma(-\Delta_{\Gamma_0})} \lambda \sum_{u \in V_0} P_{\Gamma_0}(\lambda, u, v) f_0(u) dm_{\Gamma_0}(\lambda).$$

We define

$$M(\lambda) = \prod_{m=1}^{\infty} \frac{(1 - \frac{1}{5} \lambda_m)(1 - \frac{1}{2} \lambda_m)}{(1 - \frac{1}{6} \lambda_m)(1 - \frac{1}{3} \lambda_m)},$$

where $\lambda = \lim_{m \to \infty} 5^m \lambda_m$ and $\lambda_m = R(\lambda_{m+1})$. This function does not depend on the fractafold, but only on the Sierpiński gasket.

Let

$$\Sigma'_\infty = 5 \left( \bigcup_{m=1}^{\infty} 5^m R^{-1}\{3,5\} \right) \subseteq \Sigma_\infty = 5 \left( R^{-1}\{2\} \cup \bigcup_{m=0}^{\infty} 5^m R^{-1}\{3,5\} \right).$$
Theorem 6.1. The Laplacian $\Delta$ is self-adjoint and

$$\mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma'_{\infty} \subset \sigma(-\Delta) \subset \mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \cup \Sigma_{\infty}.$$ 

Moreover, the spectral decomposition $-\Delta = \int_{\sigma(-\Delta)} \lambda dE(\lambda)$ can be written as

$$-\Delta = \int_{\mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \setminus \Sigma_{\infty}} \lambda M(\lambda) \Psi_{\lambda}^* d\left( E_{\Gamma_0}(\mathcal{R}(\lambda)) \right) \Psi_{\lambda} + \sum_{\lambda \in \Sigma_{\infty}} \lambda E\{\lambda\}.$$ 

Here $E\{\lambda\}$ denotes the eigenprojection if $\lambda$ is an eigenvalue. All eigenvalues and eigenfunctions of $\Delta$ can be computed by the spectral decimation method. Furthermore, the Laplacian $\Delta$ on the Sierpiński fractal $\mathcal{F}$ has the spectral decomposition of the form

$$-\Delta f(x) = \int_{\mathcal{R}^{-1}(\sigma(-\Delta_{\Gamma_0})) \setminus \Sigma_{\infty}} \lambda \left( \int_{\mathcal{F}} P(\lambda, x, y) f(y) d\mu(y) \right) dm(\lambda) + \sum_{\lambda \in \Sigma_{\infty}} \lambda E\{\lambda\} f(x)$$

where $m = m_{\Gamma_0} \circ \mathcal{R}$ and

$$P(\lambda, x, y) = M(\lambda) \sum_{u, v \in V_0} \psi_{v, \lambda}(x) \psi_{u, \lambda}(y) P_{\Gamma_0}(\mathcal{R}(\lambda), u, v).$$
References

[1] R.S. Strichartz and A. Teplyaev, *Spectral analysis on infinite Sierpinski fractafolds*, Journal d’Analyse Mathematique, Vol 116 (2012)

[2] B. Steinhurst and A. Teplyaev, *Existence of a meromorphic extension of spectral zeta functions on fractals*, preprint, (2011) arXiv:1011.5485.

[3] K. Hare, B. Steinhurst, A. Teplyaev, D. Zhou *Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified regular fractals*, preprint (2010).

[4] N Bajorin, T Chen, A Dagan, C Emmons, M Hussein, M Khalil, P Mody, B Steinhurst, A Teplyaev, *Vibration modes of 3n-gaskets and other fractals*, J. Phys. A: Math Theor. 41 (2008) 015101 (21pp); *Vibration Spectra of Finitely Ramified, Symmetric Fractals*, Fractals 16 (2008), 243–258.

[DGV] G. Derfel, P. Grabner and F. Vogl, *The zeta function of the Laplacian on certain fractals*. Trans. Amer. Math. Soc. 360 (2008), 881–897.

[DGV2] G. Derfel, P. Grabner and F. Vogl, *Complex asymptotics of Poincaré functions and properties of Julia sets*. Math. Proc. Cambridge Philos. Soc. 145 (2008), no. 3, 699–718.

[DGV3] G. Derfel, P. J. Grabner, and F. Vogl, *Laplace Operators on Fractals and Related Functional Equations*, submitted to the J. Phys. A: Math. Gen.

[F-TN] A. Figa-Talamanca and C. Nebbia, *Harmonic analysis and representation theory for groups acting on homogeneous trees*. London Mathematical Society Lecture Note Series, 162. Cambridge University Press, Cambridge, 1991.

[Post2008] O. Post, *Equilateral quantum graphs and boundary triples*. Analysis on Graphs and its Applications, Proceedings of Symposia in Pure Mathematics, Amer. Math. Soc., 77 (2008), 469–490.
[Quint09] J.-F. Quint, *Harmonic analysis on the Pascal graph*. J. Funct. Anal. **256** (2009), 3409–3460.

[St89] R. S. Strichartz, *Harmonic analysis as spectral theory of Laplacians*. J. Funct. Anal. **87**, 51–148 (1989). Corrigendum to “Harmonic analysis as spectral theory of Laplacians”. J. Funct. Anal. **109**, 457–460 (1992).

[St98] R. S. Strichartz, *Fractals in the large*. Canad. J. Math. **50** (1998), 638–657.

[St03] R. S. Strichartz, *Fractafolds based on the Sierpinski and their spectra*. Trans. Amer. Math. Soc. **355** (2003), 4019–4043.

[St2005gaps] R. S. Strichartz, *Laplacians on fractals with spectral gaps have nicer Fourier series*. Math. Res. Lett. **12** (2005), 269–274.

[St06book] R. S. Strichartz, *Differential equations on fractals: a tutorial*. Princeton University Press, 2006.

[St10] R. S. Strichartz, *Transformation of spectra of graph Laplacians*, Rocky Mountain J. Math., to appear.

[T98] A. Teplyaev, *Spectral Analysis on Infinite Sierpiński Gaskets*, J. Funct. Anal., **159** (1998), 537-567.

[T07] A. Teplyaev, *Spectral zeta functions of fractals and the complex dynamics of polynomials*, Trans. Amer. Math. Soc. **359** (2007), 4339–4358. MR 2309188 (2008j:11119)

[MT03] L. Malozemov and A. Teplyaev, *Self-similarity, operators and dynamics*. Math. Phys. Anal. Geom. **6** (2003), 201–218.

[Kuchment91] P. Kuchment, *On the Floquet theory of periodic difference equations*. Geometrical and algebraical aspects in several complex variables (Cetraro, 1989), 201–209, Sem. Conf., **8**, EditEl, Rende, 1991.
[Kuchment93] P. Kuchment, *Floquet theory for partial differential equations*. Operator Theory: Advances and Applications 60, Birkhäuser Verlag, Basel, 1993.

[Kuchment05] P. Kuchment, *Quantum graphs II. Some spectral properties of quantum and combinatorial graphs*. J. Phys. A. 38 (2005), 4887–4900.

[KuchmentPost] P. Kuchment and O. Post, *On the spectra of carbon nano-structures*. Comm. Math. Phys. 275 (2007), no. 3, 805-826.

[KuchmentVainberg] P. Kuchment and B. Vainberg, *On the structure of eigenfunctions corresponding to embedded eigenvalues of locally perturbed periodic graph operators*. Comm. Math. Phys. 268 (2006), 673–686.

[FS] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket*. Potential Analysis 1 (1992), 1-35.

[Cartier] P. Cartier, *Harmonic analysis on trees*. (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), pp. 419–424. Amer. Math. Soc., Providence, R.I., 1973.

Department of Mathematics, Cornell University, Ithaca, NY 14853
E-mail address: str@math.cornell.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009
E-mail address: teplyaev@math.uconn.edu
VECTOR ANALYSIS ON FRACTALS AND APPLICATIONS

MICHAEL HINZ\textsuperscript{1} AND ALEXANDER TEPLYAEV\textsuperscript{2}

Abstract. The paper surveys some recent results concerning vector analysis on fractals. We start with a local regular Dirichlet form and use the framework of 1-forms and derivations introduced by Cipriani and Sauvageot to set up some elements of a related vector analysis in weak and non-local formulation. This allows to study various scalar and vector valued linear and non-linear partial differential equations on fractals that had not been accessible before. Subsequently a stronger (localized, pointwise or fiberwise) version of this vector analysis can be developed, which is related to previous work of Kusuoka, Kigami, Eberle, Strichartz, Hino, \textit{Ionescu}, Rogers, Röckner, and the authors.

\textit{Date:} July 16, 2012.

\textsuperscript{1}Research supported in part by NSF grant DMS-0505622 and by the Alexander von Humboldt Foundation Feodor (Lynen Research Fellowship Program).

\textsuperscript{2}Research supported in part by NSF grant DMS-0505622.
1. Introduction

(1.1) \[ \text{div}(a(\nabla u)) = f \]

(1.2) \[ \Delta u + b(\nabla u) = f \]

(1.3) \[ i \frac{\partial u}{\partial t} = (-i\nabla - A)^2 u + Vu. \]
2. **Navier-Stokes equations**

Assume that the space $X$ is compact, connected and topologically one-dimensional of arbitrarily large Hausdorff and spectral dimensions.

**Theorem 2.1** (The Hodge theorem). A 1-form $\omega \in \mathcal{H}$ is harmonic if and only if it is in $(\text{Im} \partial)^\perp$, that is $\text{div} \ \omega = 0$.

Using the classical identity $\frac{1}{2} \nabla|u|^2 = (u \cdot \nabla)u + u \times \text{curl} \ u$ we obtain

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p = 0, \\
\text{div} \ u = 0,
\end{array} \right.
\end{align*}$$

(2.1)

**Theorem 2.2.** Any weak solution $u$ of (2.1) is harmonic and stationary, i.e. $u$ is independent of $t \in [0, \infty)$. Given an initial condition $u_0$ the corresponding weak solution is uniquely determined.
**Theorem 2.3.** Assume that points have positive capacity (i.e. we have a resistance form in the sense of Kigami) and the topological dimension is one. Then a nontrivial solution to (2.1) exists if and only if the first Čech cohomology $\check{H}^1(X)$ of $X$ is nontrivial.

**Remark 2.4.** We conjecture that any set that carries a regular resistance form is a topologically one-dimensional space when equipped with the associated resistance metric.
3. Magnetic Schrödinger equations

\[ \mathcal{E}^{a,V}(f,g) = \langle (-i\partial - a)f, (-i\partial - a)g \rangle_{H} + \langle fV,g \rangle_{L^2(X,m)}, \quad f, g \in \mathcal{C}, \]

**Theorem 3.1.** Let \( a \in \mathcal{H}_\infty \) and \( V \in L_\infty(X,m) \).

(i) The quadratic form \((\mathcal{E}^{a,V},\mathcal{F}_C)\) is closed.

(ii) The self-adjoint non-negative definite operator on \( L^2_C(X,m) \) uniquely associated with \((\mathcal{E}^{a,V},\mathcal{F}_C)\) is given by

\[ H^{a,V} = (-i\partial - a)^*(-i\partial - a) + V, \]

and the domain of the operator \( A \) is a domain of essential self-adjointness for \( H^{a,V} \).

Note: related Dirac operator is well defined and self-adjoint

\[ D = \begin{pmatrix} 0 & -i\partial^* \\ -i\partial & 0 \end{pmatrix} \]
Let $X$ be a locally compact separable metric space and $m$ a Radon measure on $X$ such that each nonempty open set is charged positively. We assume that $(\mathcal{E}, \mathcal{F})$ is a symmetric local regular Dirichlet form on $L_2(X, m)$ with core $\mathcal{C} := \mathcal{F} \cap C_0(X)$. Endowed with the norm $\|f\|_{\mathcal{E}} := \mathcal{E}(f)^{1/2} + \sup_X |f|$ the space $\mathcal{C}$ becomes an algebra and in particular,

$$(4.1) \quad \mathcal{E}(fg)^{1/2} \leq \|f\|_{\mathcal{E}} \|g\|_{\mathcal{E}}, \quad f, g \in \mathcal{C},$$

see [16]. For any $g, h \in \mathcal{C}$ we can define a finite signed Radon measure $\Gamma(g, h)$ on $X$ such that

$$2 \int_X f \, d\Gamma(g, h) = \mathcal{E}(fg, h) + \mathcal{E}(fh, g) - \mathcal{E}(gh, f), \quad f \in \mathcal{C},$$

the mutual energy measure of $g$ and $h$. By approximation we can also define the mutual energy measure $\Gamma(g, h)$ for general $g, h \in \mathcal{F}$. Note that $\Gamma$ is symmetric and bilinear, and $\Gamma(g) \geq 0, g \in \mathcal{F}$. For details we refer the reader to [28]. We provide some examples.
Examples

(i) **Dirichlet forms on Euclidean domains.** Let $X = \Omega$ be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ and

$$E(f, g) = \int_{\Omega} \nabla f \nabla g \, dx, \quad f, g \in C^\infty(\Omega).$$

If $H_0^1(\Omega)$ denotes the closure of $C^\infty(\Omega)$ with respect to the scalar product $E_1(f, g) := E(f, g) + \langle f, g \rangle_{L^2(\Omega)}$, then $(E, H_0^1(\Omega))$ is a local regular Dirichlet form on $L^2(\Omega)$. The mutual energy measure of $f, g \in H_0^1(\Omega)$ is given by $\nabla f \nabla g \, dx$.

(ii) **Dirichlet forms on Riemannian manifolds.** Let $X = M$ be a smooth compact Riemannian manifold and

$$E(f, g) = \int_M \langle df, dg \rangle_{T^*M} \, dvol, \quad f, g \in C^\infty(M).$$

Here $dvol$ denotes the Riemannian volume measure. Similarly as in (i) the closure of $E$ in $L^2(M, dvol)$ yields a local regular Dirichlet form. The mutual energy measure of two energy finite functions $f, g$ is given by $\langle df, dg \rangle_{T^*M} \, dvol$.

(iii) **Dirichlet forms induced by resistance forms on fractals.**
Consider $\mathcal{C} \otimes \mathcal{B}_b(X)$, where $\mathcal{B}_b(X)$ is the space of bounded Borel functions on $X$ with the symmetric bilinear form

\begin{equation}
\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} := \int_X b d \Gamma(a, c),
\end{equation}

$a \otimes b, c \otimes d \in \mathcal{C} \otimes \mathcal{B}_b(X)$, let $\|\cdot\|_{\mathcal{H}}$ denote the associated seminorm on $\mathcal{C} \otimes \mathcal{B}_b(X)$ and write

Define space of differential 1-forms on $X$

\[ \mathcal{H} = \mathcal{C} \otimes \mathcal{B}_b(X)/\ker \|\cdot\|_{\mathcal{H}} \]

we

The space $\mathcal{H}$ becomes a bimodule if we declare the algebras $\mathcal{C}$ and $\mathcal{B}_b(X)$ to act on it as follows: For $a \otimes b \in \mathcal{C} \otimes \mathcal{B}_b(X)$, $c \in \mathcal{C}$ and $d \in \mathcal{B}_b(X)$ set

\begin{equation}
(c(a \otimes b)) := (ca) \otimes b - c \otimes (ab)
\end{equation}

and

\begin{equation}
(a \otimes b)d := a \otimes (bd).
\end{equation}
A derivation operator \( \partial : \mathcal{C} \to \mathcal{H} \) can be defined by setting
\[
\partial f := f \otimes 1.
\]

It obeys the Leibniz rule,
\[
(5.4) \quad \partial(fg) = f \partial g + g \partial f, \quad f, g \in \mathcal{C},
\]
and is a bounded linear operator satisfying
\[
(5.5) \quad \|\partial f\|_\mathcal{H}^2 = \mathcal{E}(f), \quad f \in \mathcal{C}.
\]

On Euclidean domains and on smooth manifolds the operator \( \partial \) coincides with the classical exterior derivative (in the sense of \( L^2 \)-differential forms). Details can be found in [21, 22, 39, 40, 46].
Being Hilbert, \( \mathcal{H} \) is self-dual. We therefore regard 1-forms also as vector fields and \( \partial \) as the gradient operator. Let \( \mathcal{C}^* \) denote the dual space of \( \mathcal{C} \), normed by
\[
\|w\|_{\mathcal{C}^*} = \sup \{|w(f)| : f \in \mathcal{C}, \|f\|_{\mathcal{C}} \leq 1\}.
\]
Given \( f, g \in \mathcal{C} \), consider the functional
\[
u \mapsto \partial^*(g\partial f)(\nu) := -\langle \partial \nu, g\partial f \rangle_{\mathcal{H}} = -\int_{\mathcal{X}} g \, d\Gamma(\nu, f)
\]
on \( \mathcal{C} \). It defines an element \( \partial^*(g\partial f) \) of \( \mathcal{C}^* \), to which we refer as the divergence of the vector field \( g\partial f \).

**Lemma 5.1.** The divergence operator \( \partial^* \) extends continuously to a bounded linear operator from \( \mathcal{H} \) into \( \mathcal{C}^* \) with \( \|\partial^* v\|_{\mathcal{C}^*} \leq \|v\|_{\mathcal{H}}, \; v \in \mathcal{H} \). We have
\[
\partial^* v(\nu) = -\langle \partial \nu, v \rangle_{\mathcal{H}}
\]
for any \( \nu \in \mathcal{C} \) and any \( v \in \mathcal{H} \).
The Euclidean identity
\[
\text{div } (g \, \text{grad } f) = g \Delta f + \nabla f \nabla g
\]
has a counterpart in terms of \(\partial\) and \(\partial^*\). Let \((A, \text{dom } A)\) denote the infinitesimal \(L_2(X, \mu)\)-generator of \((E, \mathcal{F})\).

**Lemma 5.2.** We have
\[
\partial^*(g \partial f) = gAf + \Gamma(f, g),
\]
for any simple vector field \(g \partial f, f, g \in \mathcal{E}\), and in particular, \(Af = \partial^* \partial f\) for \(f \in \mathcal{E}\).

**Corollary 5.3.** The domain \(\text{dom } \partial^*\) agrees with the subspace
\[
\{v \in \mathcal{H} : v = \partial f + w : f \in \text{dom } A, w \in \ker \partial^*\}.
\]
For any \(v = \partial f + w\) with \(f \in \text{dom } A\) and \(w \in \ker \partial^*\) we have \(\partial^* v = Af\).
[1] E. Akkermans, G. Dunne, A. Teplyaev Physical Consequences of Complex Dimensions of Fractals. Europhys. Lett. 88, 40007 (2009).

[2] E. Akkermans, G. Dunne, A. Teplyaev Thermodynamics of photons on fractals. Phys. Rev. Lett. 105(23):230407, 2010.

[3] N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst, and A. Teplyaev, Vibration modes of $3^n$-gaskets and other fractals. J. Phys. A, 41(1):015101.

[4] N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst, and A. Teplyaev, Vibration Spectra of Finitely Ramified, Symmetric Fractals, Fractals 16 (2008), 243–258.

[5] M. T. Barlow, Diffusions on fractals. Lectures on Probability Theory and Statistics (Saint-Flour, 1995), 1–121, Lecture Notes in Math., 1690, Springer, Berlin, 1998.

[6] M.T. Barlow, R.F. Bass, The construction of Brownian motion on the Sierpinski carpet, Ann. Inst. H. Poincaré 25 (1989), 225-257.

[7] M.T. Barlow, R.F.Bass, Transition densities for Brownian motion on the Sierpinski carpet, Probab. Th. Relat. Fields 91 (1992), 307-330.

[8] M.T. Barlow, R.F. Bass, Brownian motion and harmonic analysis on Sierpinski carpets, Canad. J. Math. 51 (1999), 673-744.

[9] M.T. Barlow, R.F. Bass, T. Kumagai, Stability of parabolic Harnack inequalities on metric measure spaces, J. Math. Soc. Japan 58(2) (2006), 485-519.

[10] M. T. Barlow, R. F. Bass, T. Kumagai, and A. Teplyaev, Uniqueness of Brownian motion on Sierpinski carpets. J. Eur. Math. Soc. 12 (2010), 655-701.
[11] M.T. Barlow, A. Grigor’yan, T. Kumagai, On the equivalence of parabolic Harnack inequalities and heat kernel estimates, to appear in J. Math. Soc. Japan.

[12] M.T. Barlow, T. Kumagai, Transition density asymptotics for some diffusion processes with multi-fractal structures, Electron. J. Probab. 6 (2001), 1-23.

[13] M.T. Barlow, E.A. Perkins, Brownian motion on the Sierpinski gasket, Probab. Th. Relat. Fields 79 (1988), 543-623.

[14] R.F. Bass, A stability theorem for elliptic Harnack inequalities, to appear in J. Europ. Math. Soc.

[15] O. Ben-Bassat, R.S. Strichartz, A. Teplyaev, What is not in the domain of Sierpinski gasket type fractals, J. Funct. Anal. 166 (1999), 197-217.

[16] N. Bouleau, F. Hirsch, Dirichlet Forms and Analysis on Wiener Space, deGruyter Studies in Math. 14, deGruyter, Berlin, 1991.

[17] J. Chen, Statistical mechanics of Bose gas in Sierpinski carpets. Submitted. arXiv:1202.1274

[18] E. Christensen and C. Ivan, Extensions and degenerations of spectral triples. Comm. Math. Phys. 285 (2009), 925–955.

[19] E. Christensen, C. Ivan and M. L. Lapidus, Dirac operators and spectral triples for some fractal sets built on curves, Adv. Math. 217 (2008), 42–78.

[20] F. Cipriani, D. Guido, T. Isola, J.-L. Sauvageot, Differential 1-forms, their integrals and potential theory on the Sierpinski gasket, preprint ArXiv (2011).

[21] F. Cipriani, J.-L. Sauvageot, Derivations as square roots of Dirichlet forms, J. Funct. Anal. 201 (2003), 78-120.

[22] F. Cipriani, J.-L. Sauvageot, Fredholm modules on p.c.f. self-similar fractals and their conformal geometry, Comm. Math. Phys. 286 (2009), 541-558.
[23] G. Derfel, P. J. Grabner, and F. Vogl, Laplace Operators on Fractals and Related Functional Equations, to appear in J. Phys. A: Math. Gen.
[24] L.C. Evans, Partial Differential Equations, Grad. Stud. Math. vol 19, AMS, Providence, RI, 1998.
[25] K.J. Falconer, Semilinear PDEs on self-similar fractals, Comm. Math. Phys. 206 (1999), 235-245.
[26] K.J. Falconer, J. Hu, Nonlinear Diffusion Equations on unbounded fractal domains, J. Math. Anal. Appl. 256 (2001), 606-624.
[27] E. Fan, Z. Khandker, R.S. Strichartz, Harmonic oscillators on infinite Sierpinski gaskets, Comm. Math. Phys. 287 (2009), 351-382.
[28] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov processes, deGruyter, Berlin, New York, 1994.
[29] M. Fukushima, T. Shima, On a spectral analysis for the Sierpinski gasket, Pot. Anal. 1 (1992), 1-35.
[30] S. Goldstein, Random walks and diffusions on fractals, in: Percolation Theory and Ergodic Theory on infinite Particle Systems (ed. H. Kesten), in: IMA Math Appl. vol. 8, Springer, New York, 1987, 121-129.
[31] A. Grigor’yan, A. Telcs, Two-sided estimates of heat kernels in metric measure spaces, Ann. Probab. 40 (2012), 1212-1284.
[32] B.M. Hambly, Brownian motion on a homogeneous random fractal, Probab. Th. Rel. Fields 94 (1992), 1-38.
[33] B.M. Hambly, Brownian motion on a random recursive Sierpinski gasket, Ann. Probab. 25 (1997), 1059–1102.
[34] B.M. Hambly, *Asymptotics for functions associate d with heat flow on the Sierpinski carpet*, Canad. J. Math. 63 (2011), 153-180.

[35] B.M. Hambly, T. Kumagai, S. Kusuoka, X.Y. Zhou, *Transition density estimates for diffusion processes on homogeneous random Sierpinski carpets*, J. Math. Soc. Japan 52 (2000), 373-408.

[36] B. M. Hambly, V. Metz and A. Teplyaev, *Self-similar energies on post-critically finite self-similar fractals*, J. London Math. Soc. 74 (2006), 93–112.

[37] K. Hare, B. Steinhurst, A. Teplyaev, D. Zhou *Disconnected Julia sets and gaps in the spectrum of Laplacians on symmetric finitely ramified fractals*, to appear in the Mathematical Research Letters (MRL), arXiv:1105.1747

[38] M. Hino, *On singularity of energy measures on self-similar sets*, Probab. Theory Relat. Fields 132 (2005), 265-290.

[39] M. Hinz, *1-forms and polar decomposition on harmonic spaces*, (2011) to appear in Potential Analysis.

[40] M. Hinz, M. Röckner, A. Teplyaev, *Vector analysis for local Dirichlet forms and quasilinear PDE and SPDE on fractals*, preprint arXiv:1202.0743 (2012).

[41] M. Hinz, A. Teplyaev, *Local Dirichlet forms, Hodge theory, and the Navier-Stokes equations on topologically one-dimensional fractals*, preprint (2012).

[42] M. Hinz, A. Teplyaev, *Dirac and magnetic Schrödinger operators on fractals*, preprint (2012).

[43] M. Hinz, M. Zähle, *Semigroups, potential spaces and applications to (S)PDE*, Pot. Anal. 36 (2012), 483-515.

[44] J. Hu, M. Zähle, *Schrödinger equations and heat kernel upper bounds on metric spaces*, Forum Math. 22 (2010), 1213-1234.

[45] M. Ionescu, L. Rogers, R.S. Strichartz, *Pseudodifferential operators on fractals*, to appear in Rev. Math. Iberoam.
[46] M. Ionescu, L. Rogers, A. Teplyaev, *Derivations, Dirichlet forms and spectral analysis*, (2010) to appear in J. Funct. Anal.
[47] N. Kajino, *Spectral asymptotics for laplacians on self-similar sets*, J. Funct. Anal. **258** (2010), 1310-1360.
[48] N. Kajino, *Heat kernel asymptotics for the measurable Riemannian structure on the Sierpinski gasket*, Pot. Anal. **36** (2012), 67-115.
[49] N. Kajino, *On-diagonal oscillation of the heat kernels on post-critically self-similar fractals*, Probab. Th. Relat. Fields, in press.
[50] C. Kaufmann, R. Kesler, A. Parshall, E. Stamey B. Steinhurst, *Quantum Mechanics on Laakso Spaces*, arXiv:1011.3567, to appear in J. Math. Phys.
[51] J. Kigami, *Harmonic metric and Dirichlet form on the Sierpiński gasket*. Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), 201–218, Pitman Res. Notes Math. Ser., **283**, Longman Sci. Tech., Harlow, 1993.
[52] J. Kigami, *Analysis on Fractals*, Cambridge Univ. Press, Cambridge, 2001.
[53] J. Kigami, *Harmonic analysis for resistance forms*, J. Funct. Anal. **204** (2003), 525–544.
[54] J. Kigami, *Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate*, Math. Ann. **340** (4) (2008), 781–804.
[55] J. Kigami, *Volume doubling measures and heat kernel estimates on self-similar sets*, Mem. Amer. Math. Soc. vol. **199** (932), 2009.
[56] J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*, to appear in Memoirs of the AMS.
[57] J. Kigami, M.L. Lapidus, *Weyl’s problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals*, Comm. Math. Phys. **158** (1993), 93-125.
[58] J. Kigami and M. L. Lapidus, Self-similarity of volume measures for Laplacians on p.c.f. self-similar fractals. Comm. Math. Phys. 217 (2001), 165–180.
[59] T. Kumagai, Estimates of the transition densities for Brownian motion on nested fractals, Probab. Th. Relat. Fields 96 (1993), 205-224.
[60] T. Kumagai, K.-Th. Sturm, Construction of diffusion processes on fractals, d-sets, and general metric measure spaces, J. Math. Kyoto Univ. 45(2) (2005), 307-327.
[61] S. Kusuoka, A diffusion process on a fractal, in: Probabilistic Methods on Mathematical Physics (ed. K. Ito, N. Ikeda), Proc. Taniguchi Int. Symp., Katata and Kyoto, 1985, Tokyo, 1987, 251-274.
[62] S. Kusuoka, Dirichlet forms on fractals and products of random matrices, Publ. Res. Inst. Math. Sci. 25 (1989), 659-680.
[63] S. Kusuoka, Lecture on diffusion process on nested fractals. Lecture Notes in Math. 1567 39–98, Springer-Verlag, Berlin, 1993.
[64] S. Kusuoka, X.Y. Zhou, Dirichlet forms on fractals: Poincaré constant and resistance, Probab. Th. Relat. Fields 93 (1992), 169-196.
[65] N. Lal and M. L. Lapidus, Higher-Dimensional Complex Dynamics and Spectral Zeta Functions of Fractal Differential Sturm-Liouville Operators. arXiv:1202.4126
[66] M. L. Lapidus and M. van Frankenhuysen, Fractal geometry, complex dimensions and zeta functions. Geometry and spectra of fractal strings. Springer Monographs in Mathematics. Springer, New York, 2006.
[67] M.L. Lapidus, Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl-Berry conjecture, Trans. Amer. Math. Soc. 325 (1991), 465-529.
[68] T. Lindstrøm, Brownian motion on nested fractals. Mem. Amer. Math. Soc. 420, 1989.
[69] Z. Ma and M. Röckner, Introduction to the theory of (nonsymmetric) Dirichlet forms. Universitext, Springer-Verlag, Berlin, 1992.
[70] V. Metz, *Potentialtheorie auf dem Sierpinski gasket*, Math. Ann. **289** (1991), 207-237.

[71] A. Pelander and A. Teplyaev, *Infinite dimensional i.f.s. and smooth functions on the Sierpinski gasket*, Indiana Univ. Math. J. **56** (2007), 1377-1404.

[72] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol. 1*, Acad. Press, San Diego 1980.

[73] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, vol. 2*, Acad. Press, San Diego 1980.

[74] Martin Reuter and Frank Saueressig, *Fractal space-times under the microscope: a renormalization group view on Monte Carlo data*, Journal of High Energy Physics 2011, 2011:12.

[75] Rogers, L.C.G., Williams, D.: *Diffusions, Markov Processes, and Martingales. Volume one: Foundations*, 2nd ed. Wiley, 1994.

[76] L.G. Rogers, R.S. Strichartz, *Distributions on p.c.f. fractafolds*, J. Anal. Math. **112** (2010), 137-191.

[77] C. Sabot, *Existence and uniqueness of diffusions on finitely ramified self-similar fractals*, Ann. Sci. École Norm. Sup. (4) **30** (1997), 605-673.

[78] B. Steinhurst, *Uniqueness of Brownian Motion on Laakso Spaces*, arXiv:1103.0519, to appear in Potential Analysis.

[79] B. Steinhurst and A. Teplyaev, *Symmetric Dirichlet forms and spectral analysis on Barlow-Evans fractals*, preprint, 2011.

[80] R.S. Strichartz, *Taylor approximations on Sierpinski type fractals*. J. Funct. Anal. **174** (2000), 76–127.

[81] R.S. Strichartz, *Differential Equations on Fractals: A Tutorial*, Princeton Univ. Press, Princeton 2006.
[82] R.S. Strichartz, A fractal quantum mechanical model with Coulomb potential, Comm. Pure Appl. Anal. 8 (2) (2009), 743-755.

[83] A. Teplyaev, Spectral analysis on infinite Sierpinski gaskets, J. Funct. Anal. 159 (1998), 537-567.

[84] A. Teplyaev, Gradients on fractals. J. Funct. Anal. 174 (2000) 128–154.

[85] A. Teplyaev, Spectral zeta functions of fractals and the complex dynamics of polynomials, Trans. Amer. Math. Soc. 359 (2007), 4339–4358.

[86] A. Teplyaev, Harmonic coordinates on fractals with finitely ramified cell structure, Canad. J. Math. 60 (2008), 457–480.

Mathematisches Institut, Friedrich-Schiller-Universität Jena, Ernst-Abbe-Platz 2, 07737, Germany and Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009 USA

E-mail address: Michael.Hinz.1@uni-jena.de and Michael.Hinz@uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009 USA

E-mail address: Alexander.Teplyaev@uconn.edu