OPTIMAL PARTITION PROBLEMS FOR THE FRACTIONAL LAPLACIAN

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ABSTRACT. In this work, we prove an existence result for an optimal partition problem of the form

\[ \min \{ F_s(A_1, \ldots, A_m) : A_i \in \mathcal{A}_s, A_i \cap A_j = \emptyset \text{ for } i \neq j \} , \]

where \( F_s \) is a cost functional with suitable assumptions of monotonicity and lower semicontinuity, \( \mathcal{A}_s \) is the class of admissible domains and the condition \( A_i \cap A_j = \emptyset \) is understood in the sense of Gagliardo \( s \)-capacity, where \( 0 < s < 1 \).

Examples of this type of problem are related to fractional eigenvalues. As the main outcome of this article, we prove some type of convergence of the \( s \)-minimizers to the minimizer of the problem with \( s = 1 \), studied in [5].

1. Introduction

Throughout this article, we consider \( \Omega \subset \mathbb{R}^n \) to be a fixed Lipschitz domain, that is an open bounded subset of \( \mathbb{R}^n \) with Lipschitz boundary. Fix \( 0 < s < 1 \) and \( m \in \mathbb{N} \). We consider optimal partition problems of the form

\[ (1.1) \quad \min \{ F_s(A_1, \ldots, A_m) : A_i \in \mathcal{A}_s(\Omega), A_i \cap A_j = \emptyset \text{ for } i \neq j \} , \]

where \( F_s \) is a cost functional which satisfies some lower semicontinuity and monotonicity assumptions and \( \mathcal{A}_s(\Omega) \) denotes the class of admissible domains.

Optimal partition problems were studied by several authors: Bucur, Buttazzo and Henrot [5], Bucur and Velichkov [6], Caffarelli and Lin [8], Conti, Terracini and Verzini [9, 10], Helffer, Hoffmann-Ostenhof and Terracini [19], among others.

In [8], Caffarelli and Lin established the existence of classical solutions to an optimal partition problem for the Dirichlet eigenvalue, as well as the regularity of free interfaces. One more recent work about regularity of solutions to optimal partition problems involving eigenvalues of the Laplacian is [23], where Ramos, Tavares and Terracini used the existence result of [5] and proved that the free boundary of the optimal partition is locally a \( C^{1,\alpha} \)- hypersurface up to a residual set.

Conti, Terracini and Verzini proved in [9] the existence of the minimal partition for a problem in \( N \)-dimensional domains related to the method of nonlinear eigenvalues introduced by Nehari in [21]. Moreover, they showed some connections between the variational problem and the behavior of competing species systems with large interaction.

2010 Mathematics Subject Classification. 35R11, 49Q10.

Key words and phrases. Fractional partial differential equations, optimal partition.
Tavares and Terracini proved in [26] the existence of infinitely many sign-changing solutions for the system of \( m \)-Schrödinger equations with competition interactions and the relation between the energies associated and an optimal partition problem which involves \( m \)-eigenvalues of the Laplacian operator.

In a recent work [16], we studied a general shape optimization problem where \( m = 1 \).

To mention some references which have to do with optimal partition problems involving fractional operators, we suggest to look through [27], [29], and references therein too.

A class of optimal partition problems involving the half-Laplacian operator and a subcritical cost functional was considered by Zilio in [29]. That work encompasses findings about optimal regularity of the density-functions which characterize the partitions, for the entire set of minimizers. Besides, a numerical related scheme and its consequences are shown.

In [27], Terracini-Verzini-Zilio consider a class of competition-diffusion nonlinear systems involving the half-Laplacian, including the fractional Gross-Pitaevskii system.

For more references related to optimal partition problems see, for instance, [1, 2, 4, 7, 10, 18, 22, 25]

The goal of this article is to prove the existence of an optimal partition for the problem (1.1), where \( F_s \) is decreasing in each coordinate and lower semicontinuous for a suitable notion of convergence in \( \mathcal{A}_s(\Omega) \), which is the set of admissible domains. This existence result is carried out in Section 3. The dependence on \( s \) is related to the Gagliardo \( s \)-capacity measure and the fractional Laplacian operator \(( -\Delta )^s \), we will detail that and other preliminaries in Section 2.

We follow the ideas given by Bucur, Buttazzo and Henrot in [5], where the existence of solution to (1.1) in the case \( s = 1 \) was proved.

Furthermore, we prove convergence of minimums and optimal partition shapes to those of the case \( s = 1 \), studied in [5]. This last aim is accomplished in Section 4 and we consider it the most interesting contribution of this work.

At the end, we include an Appendix with useful properties of \( s \)-capacity. Most of those results we suppose are well-known. Despite of that, we decided to incorporate them for completeness.

2. Preliminaries and statements

2.1. Notations and preliminaries. Given \( s \in (0, 1) \) we consider the fractional Laplacian, that for smooth functions \( u \) is defined as

\[
(-\Delta)^s u(x) := c(n, s) \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy = -\frac{c(n, s)}{2} \int_{\mathbb{R}^n} \frac{u(x + z) - 2u(x) + u(x - z)}{|z|^{n+2s}} \, dz.
\]

where \( c(n, s) := (\int_{\mathbb{R}^n} \frac{1 - \cos \zeta_1}{|\zeta|^{n+2s}} \, d\zeta)^{-1} \) is a normalization constant.
The constant \( c(n, s) \) is chosen in such a way that the following identity holds,

\[
(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}(u)),
\]

for \( u \) in the Schwarz class of rapidly decreasing and infinitely differentiable functions, where \( \mathcal{F} \) denotes the Fourier transform. See \cite[Proposition 3.3]{17}.

The natural functional setting for this operator is the fractional Sobolev space \( H^s(\mathbb{R}^n) \) defined as

\[
H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) : \frac{u(x) - u(y)}{|x - y|^{\frac{n+2s}{2}}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}
\]

\[
= \left\{ u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^{2s})|\mathcal{F}(u)(\xi)|^2 d\xi < \infty \right\}
\]

which is a Banach space endowed with the norm \( \|u\|_s^2 := \|u\|_2^2 + [u]_s^2 \), where the term

\[
[u]_s^2 := \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy
\]

is the so-called Gagliardo semi-norm of \( u \).

To contemplate the boundary condition, we work in \( H^s_0(\Omega) \), which is the closure of \( C_c^\infty(\Omega) \) in the norm \( \| \cdot \|_s \). As we are dealing with a Lipschitz domain \( \Omega \), \( H^s_0(\Omega) \) coincides with the space of functions vanishing outside \( \Omega \), i.e.,

\[
H^s_0(\Omega) = \left\{ u \in H^s(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega \right\},
\]

See \cite[Corollary 1.4.4.5]{17} for a proof of the identity above.

**Definition 2.1.** Given \( A \subset \Omega \), for any \( 0 < s < 1 \), we define the Gagliardo \( s \)-capacity of \( A \) relative to \( \Omega \) as

\[
\text{cap}_s(A, \Omega) = \inf \left\{ [u]_s^2 : u \in C_c^\infty(\Omega), \ u \geq 1 \text{ in a neighborhood of } A \right\}.
\]

We say that a subset \( A \) of \( \Omega \) is an \( s \)-quasi open subset of \( \Omega \) if there exists a decreasing sequence \( \{G_k\}_{k \in \mathbb{N}} \) of open sets such that \( \lim_{k \to \infty} \text{cap}_s(G_k, \Omega) = 0 \) and \( A \cup G_k \) is an open set.

We denote by \( \mathcal{A}_s(\Omega) \) the class of all \( s \)-quasi open subsets of \( \Omega \).

In the case \( s = 1 \) the definitions are completely analogous with \( \|\nabla u\|_2^2 \) instead of \([u]_s^2\).

We say that a property \( P(x) \) holds \( s \)-quasi everywhere on \( E \subset \Omega \) (\( s \)-q.e. on \( E \)), if \( \text{cap}_s(\{x \in E : P(x) \text{ does not hold}\}, \Omega) = 0 \).

A function \( u : \mathbb{R}^n \to \mathbb{R} \) is said \( s \)-quasi-continuous if there exists a decreasing sequence \( \{G_k\}_{k \in \mathbb{N}} \) of open sets such that \( \lim_{k \to \infty} \text{cap}_s(G_k, \Omega) = 0 \) and \( u|_{\mathbb{R}^n \setminus G_k} \) is continuous.

The following theorem allows us to work with \( s \)-quasi continuous functions instead of the classical fractional Sobolev ones.

**Theorem 2.2** (Theorem 3.7, \cite{28}). For every function \( u \in H^s_0(\Omega) \) there exist a unique \( \tilde{u} : \mathbb{R}^n \to \mathbb{R} \) \( s \)-quasi-continuous function such that \( u = \tilde{u} \) a.e. in \( \mathbb{R}^n \).

From this point, we identify a function \( u \in H^s_0(\Omega) \) with its \( s \)-quasi continuous representative.
For $A \in \mathcal{A}_s(\Omega)$, we consider the fractional Sobolev space

$$H^s_0(A) := \{ u \in H^s_0(\Omega) : u = 0 \text{ s.a.e. in } \mathbb{R}^n \setminus A \}.$$ 

To go into detail about $s$-capacity we refer the reader, for instance, to [24, 28].

2.2. Statements. Given $A \in \mathcal{A}_s(\Omega)$, we denote by $u_A^s \in H^s_0(A)$ the unique weak solution to

$$(2.1) \quad (-\Delta)^s u_A^s = 1 \quad \text{in } A, \quad u_A^s = 0 \quad \text{in } \mathbb{R}^n \setminus A.$$ 

We say that $u_A^s \to u_A^s$ strongly in $L^2(\Omega)$. In the weak $L^2(\Omega)$-convergence, the $L^2(\Omega)$-limit function $u$ of the sequence $(u_A^s)_{k \in \mathbb{N}}$ is not required to be a solution of (2.1) in $A$ (the weak $\gamma_s$-limit), i.e., it is not required that $u \neq u_A^s$. That is the main hassle we should get through to arrive at the compactness result on $\mathcal{A}_s(\Omega)$, in Section 3.1.

Let $m \in \mathbb{N}$ be fixed and $0 < s \leq 1$. Let $F_s : \mathcal{A}_s(\Omega)^m \to [0, \infty)$ be such that

- $F_s$ is weak $\gamma_s$-lower semicontinuous, that is,
  $$F_s(A_1, \ldots, A_m) \leq \liminf_{k \to \infty} F_s(A_1^k, \ldots, A_m^k),$$
  for every $\{(A_1^k, \ldots, A_m^k)_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \ldots, A_m) \in \mathcal{A}_s(\Omega)^m$ such that $(A_1^k, \ldots, A_m^k) \xrightarrow{\gamma_s} (A_1, \ldots, A_m)$.

- $F_s$ is decreasing, that is, for every $(A_1, \ldots, A_m), (B_1, \ldots, B_m) \in \mathcal{A}_s(\Omega)^m$ such that $A_i \subset B_i$ for $i = 1, \ldots, m$, we have
  $$F_s(A_1, \ldots, A_m) \geq F_s(B_1, \ldots, B_m).$$

Under these assumptions, we are able to recover the existence result of [5], for the fractional case. Rigorously speaking, we have the following theorem.

**Theorem 2.6.** Let $F_s : \mathcal{A}_s(\Omega)^m \to [0, \infty)$ be a decreasing and weak $\gamma_s$-lower semicontinuous functional. Then, there exists a solution to

$$(2.2) \quad \min \{ F_s(A_1, \ldots, A_m) : A_i \in \mathcal{A}_s(\Omega), \text{ cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j \}.$$ 

The proof of Theorem 2.6 is carried out in Section 3 and we use ideas from [5] and [16].

Now, we present the main point of this article, that is the convergence of minimums and optimal partition shapes to those of the case $s = 1$. 

\[ \text{ } \]
Theorem 2.8. Let  

where  

are verified. Then,  

\( m = \lim_{s \uparrow 1} m_s \),

where  

is defined in (2.3).

Moreover, if  

is a minimizer of (2.3), then, there exist a subsequence  

such that  

and  

where  

is a minimizer of (2.3) with  

The proof of Theorem 2.8 is carried out in Section 4 and we use again ideas from [16].
Moreover, the following variational characterization holds for the eigenvalues whose corresponding eigenfunctions form an orthogonal basis in $L^2(\Omega)$. Moreover, the following variational characterization holds for the eigenvalues

\begin{equation}
\lambda_k^s(A) = \min_{u \perp W_{k-1}} \frac{c(n,s)}{2} \frac{\|u\|_2^2}{\|u\|_1^2},
\end{equation}

where $W_k$ is the space spanned by the first $k$ eigenfunctions $u_1^s, \ldots, u_k^s$.

Due to (2.6) and the stability result proved in [3, Theorem 1.2], we know that $\lambda_k^s(A) \to \lambda_k^1(A)$, when $s \uparrow 1$, for every $k \in \mathbb{N}$.

Consider functionals $F_s(A_1, \ldots, A_m) = \Phi_s(\lambda^s_k(A_1), \ldots, \lambda^s_k(A_m))$. Theorem 2.6 claims that for every $(k_1, \ldots, k_m) \in \mathbb{N}^m$, the minimum

\[ \min \{ \Phi_s(\lambda^s_k(A_1), \ldots, \lambda^s_k(A_m)) : A_i \in A_s(\Omega), \text{cap}_s(A_i \cap A_j, \Omega) \text{ for } i \neq j \} \]

is achieved, where $\Phi_s : \mathbb{R}^m \to \mathbb{R}$, is increasing in each coordinate and lower semi-continuous.

Moreover, if $\Phi_s(t_1, \ldots, t_m) \to \Phi_1(t_1, \ldots, t_m)$ for every $(t_1, \ldots, t_m) \in \mathbb{R}^m$ and

\[ \Phi_1(t_1, \ldots, t_m) \leq \liminf_{k \to \infty} \Phi_{k}^{s}(t_1^k, \ldots, t_m^k), \]

for every $(t_1^k, \ldots, t_m^k) \to (t_1, \ldots, t_m)$, then Theorem 2.8 together with the existence result of [5] imply that

\[ \min \{ \Phi_1(\lambda^s_k(A_1), \ldots, \lambda^s_k(A_m)) : A_i \in A_1(\Omega), \text{cap}_1(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j \} \]

\[ = \liminf_{s \uparrow 1} \min \{ \Phi_s(\lambda^s_k(A_1), \ldots, \lambda^s_k(A_m)) : A_i \in A_s(\Omega), \text{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j \}. \]

3. Proof of Theorem 2.6

In this section, we adapted the ideas from [5], where the authors consider the Laplacian operator, to recover their results for the fractional case. Despite the similarity of the proofs, we include them for the reader’s convenience and recalling that in the context of this article we need the nonlocal tools proved in [16].

3.1. Certain compactness on $A_s(\Omega)$. Consider $\mathcal{K}_s$ given by

\begin{equation}
\mathcal{K}_s := \{ w \in H_0^s(\Omega) : w \geq 0, (-\Delta)^s w \leq 1 \text{ in } \Omega \}. \end{equation}

**Proposition 3.1** (Proposition 3.3 and Lemma 3.5, [16]). $\mathcal{K}_s$ is convex, closed and bounded in $H_0^s(\Omega)$. Moreover, if $u, v \in \mathcal{K}_s$, then, $\max\{u, v\} \in \mathcal{K}_s$.

**Proposition 3.2** (Lemma 3.2, [16]). Given $A \in \mathcal{A}_s(\Omega)$, $u_A^s$ is the solution to

\[ \max \{ w \in H_0^s(\Omega) : w \leq 0 \text{ in } \mathbb{R}^n \setminus A, (-\Delta)^s w \leq 1 \text{ in } \Omega \}. \]

Moreover, $u_A^s \in \mathcal{K}_s$, for every $A \in \mathcal{A}_s(\Omega)$.

From now on, we understand the identity $A = \{ u_A^s > 0 \}$ in the sense of the Gagliardo $s$-capacity, thanks to Proposition A.5.
Remark 3.3. The class $\mathcal{A}_s(\Omega)$ is sequentially pre-compact with respect to the weak $\gamma_s$-convergence. Indeed, given a sequence $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$, we know that $\{u_{A_k}\}_{k \in \mathbb{N}} \subset \mathcal{K}_s$. By Proposition 3.1, there exist a subsequence $\{u_{A_{k_j}}\}_{j \in \mathbb{N}} \subset \{u_{A_k}\}_{k \in \mathbb{N}}$ and a function $u \in \mathcal{K}_s$ such that $u_{A_{k_j}} \rightharpoonup u$ strongly in $L^2(\Omega)$. Denote by $A := \{u > 0\}$. Then, $A_{k_j} \rightharpoonup A$.

Next proposition allows us to pass from the weak $\gamma_s$-convergence to the strong one, if we are willing to enlarge the sequence involved.

**Proposition 3.4.** Let $\{A_k\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ and $A, B \in \mathcal{A}_s(\Omega)$ be such that $A_{k \rightharpoonup} A \subset B$.

Then, there exists a subsequence $\{A_{k_j}\}_{j \in \mathbb{N}} \subset \{A_k\}_{k \in \mathbb{N}}$ and a sequence $\{B_{k_j}\}_{j \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ such that $A_{k_j} \subset B_{k_j}$ and $B_{k_j} \rightharpoonup B$.

**Proof.** Since $A_{k_j} \rightharpoonup A \subset B$, we know that $u_{A_{k_j}} \rightarrow u$ strongly in $L^2(\Omega)$, where $\{u > 0\} = A$. As a consequence of Proposition 3.1, $u \in \mathcal{K}_s$. Moreover, by Proposition 3.2, $u \leq u_A$. Then, $u \leq u_{A_k}$. Since $A \subset B$, $u_A \leq u_B$. Thus, $u \leq u_B$.

Denote by $B^\varepsilon = \{u_B > \varepsilon\}$ and consider $\{u_{A_{k_j} \cup B^\varepsilon}\}_{k \in \mathbb{N}} \subset \mathcal{K}_s$. Again by Proposition 3.1, there exists a subsequence $\{A_{k_j}\}_{j \in \mathbb{N}} \subset \{A_k\}_{k \in \mathbb{N}}$ such that $u_{A_{k_j} \cup B^\varepsilon} \rightarrow u^\varepsilon$ strongly in $L^2(\Omega)$.

Due to the convergence $u_{A_{k_j}} \rightarrow u$ strongly in $L^2(\Omega)$ and $u \leq u_B$, we conclude from [16, Lemma 3.6], $u^\varepsilon \leq u_B$.

Inside the proof of [16, Lemma 3.7], it was shown that $(u_B^\varepsilon - \varepsilon)^+ \leq u_{B^\varepsilon}$. Since $B^\varepsilon \subset A_{k_j} \cup B^\varepsilon$, it follows that $u_{B^\varepsilon} \leq u_{A_{k_j} \cup B^\varepsilon}$. So, taking the limit $j \rightarrow \infty$, we obtain

$$(u_B^\varepsilon - \varepsilon)^+ \leq u_{B^\varepsilon} \leq u^\varepsilon \leq u_B^\varepsilon.$$

The sequence $\{u^\varepsilon\}_{\varepsilon > 0}$ is contained in $\mathcal{K}_s$. So, by Proposition 3.1, up to a subsequence, we know it has a weak limit in $H^s_0(\Omega)$. But, the previous inequality tells that this weak limit should be $u_B^\varepsilon$. In addition, $u^\varepsilon \rightarrow u_B^\varepsilon$ strongly in $L^2(\Omega)$.

Thus, there exists a sequence $\varepsilon_j \downarrow 0$ such that $u_{A_{k_j} \cup B^\varepsilon_j} \rightarrow u_B^\varepsilon$ strongly in $L^2(\Omega)$. That is, $A_{k_j} \cup B_j := B_{k_j} \rightharpoonup B$, where $\{B_{k_j}\}_{j \in \mathbb{N}}$ is the enlarged sequence. \hfill $\square$

### 3.2. An auxiliary functional.

Fix $m \in \mathbb{N}$ and $0 < s < 1$. Let $F_s: \mathcal{A}_s(\Omega)^m \rightarrow [0, \infty]$ be a decreasing and strong $\gamma_s$-lower semicontinuous functional.

We define a functional $G_s: \mathcal{K}_s^m \rightarrow [0, \infty]$

$$G_s(w_1, \ldots, w_m) := \inf \left\{ \liminf_{k \rightarrow \infty} J_s(w_1^k, \ldots, w_m^k) : w_i^k \rightarrow w_i \text{ strongly in } L^2(\Omega) \right\},$$

where $J_s: \mathcal{K}_s^m \rightarrow [0, \infty]$ is defined as

$$J_s(w_1, \ldots, w_m) := \inf \{ F_s(A_1, \ldots, A_m) : A_i \in \mathcal{A}_s(\Omega), u_{A_i}^s \leq w_i \text{ for } i = 1, \ldots, m \}$$

and $\mathcal{K}_s$ was given in (3.1).

We will show that $G_s$ satisfies the following properties:

$(G_1)$ $G_s$ is decreasing on $\mathcal{K}_s^m$, that is $G_s(u_1, \ldots, u_m) \geq G_s(v_1, \ldots, v_m)$, if $u_i \leq v_i$ for every $i = 1, \ldots, m$. 

With the notation above, Proposition 3.5.

Proposition 3.5. With the notation above, $G_s$ satisfies (G1) and (G2).

Proof. By construction, it is clear that $G_s$ verifies (G2).

To prove (G1), let $(u_1, \ldots, u_m), (v_1, \ldots, v_m) \in K_s^m$ such that $u_i \leq v_i$ for every $i = 1, \ldots, m$.

Take $\{u_i^k\}_{k \in \mathbb{N}} \subset K_s$ such that $u_i^k \to u_i$ strongly in $L^2(\Omega)$ for every $i = 1, \ldots, m$ and

$$G_s(u_1, \ldots, u_m) = \lim_{k \to \infty} J_s(u_1^k, \ldots, u_m^k).$$

Consider $v_i^k := \max\{v_i, u_i^k\}$ for every $i = 1, \ldots, m$ and $k \in \mathbb{N}$. By Proposition 3.1, we obtain that $v_i^k \in K_s$. In addition, $v_i^k \to \max\{v_i, u_i\} = v_i$ strongly in $L^2(\Omega)$, for every $i = 1, \ldots, m$. Thus, noticing that $J_s$ is decreasing, we have

$$G_s(v_1, \ldots, v_m) \leq \liminf_{k \to \infty} J_s(u_1^k, \ldots, v_m^k) \leq \liminf_{k \to \infty} J_s(u_1^k, \ldots, u_m^k) = G_s(u_1, \ldots, u_m).$$

Now, we prove the most important property of $G_s$, which is the connection with the cost functional $F_s$.

Proposition 3.6. The functional $G_s$ satisfies (G3).

Proof. By definition of $G_s$ (3.2), it is clear that $G_s(u_{A_1}^1, \ldots, u_{A_m}^m) \leq F_s(A_1, \ldots, A_m)$, for every $(A_1, \ldots, A_m) \in \mathcal{A}_s(\Omega)^m$.

To obtain the other inequality, it is enough to prove that for every sequence $\{u_i^k\}_{k \in \mathbb{N}} \subset K_s(\Omega)$ such that $u_i^k \to u_{A_i}^*$ strongly in $L^2(\Omega)$ for $i = 1, \ldots, m$, we have

$$F_s(A_1, \ldots, A_m) \leq \liminf_{k \to \infty} J_s(u_1^k, \ldots, u_m^k).$$

By definition of $J_s$, there exists $\{(A_1^k, \ldots, A_m^k)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ such that

$$(3.3) \quad u_{A_i^k}^* \leq u_i^k \quad \text{for } i = 1, \ldots, m, \quad \text{and} \quad F_s(A_1^k, \ldots, A_m^k) \leq J_s(u_1^k, \ldots, u_m^k) + \frac{1}{k}.$$

By Remark 3.3, there exists $v_i \in K_s$ such that $u_{A_i^k}^* \to v_i$ strongly in $L^2(\Omega)$, up to a subsequence. That is, $A_i^k \xrightarrow{\gamma_s} B_i := \{v_i > 0\} \in \mathcal{A}_s(\Omega)$, for every $i = 1, \ldots, m$.

Moreover, taking the limit in $u_{A_i^k}^* \leq u_i^k$, we obtain that $v_i \leq u_{A_i}$ for every $i = 1, \ldots, m$. In addition, we have $B_i \subset A_i = \{u_{A_i}^* > 0\}$. We are able to apply Proposition 3.4, to obtain the existence of subsequences $\{A_i^{k_j}\}_{j \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)$ such that $A_i^{k_j} \subset B_i^{k_j}$ and $B_i^{k_j} \xrightarrow{\gamma_s} A_i$. 
Now, by using the strong $\gamma_s$-lower semicontinuity and decreasing property of $F_s$ and (3.3), we conclude

$$F_s(A_1, \ldots, A_m) \leq \liminf_{j \to \infty} F_s(B^k_{1}, \ldots, B^k_{m})$$

$$\leq \liminf_{j \to \infty} F_s(A^k_{1}, \ldots, A^k_{m})$$

$$\leq \liminf_{j \to \infty} J_s(u^k_{1}, \ldots, u^k_{m}),$$

which implies the remaining inequality $F_s(A_1, \ldots, A_m) \leq G_s(u^*_{A_1}, \ldots, u^*_{A_m}).$  \hfill \Box

The decreasing property of a functional $F_s$ makes equivalent its weak and strong $\gamma_s$-lower semicontinuity, which is the content of next theorem.

**Theorem 3.7.** Let $m \in \mathbb{N}$ and $0 < s < 1$. Let $F_s: \mathcal{A}_s(\Omega)^m \to [0, \infty]$ be a decreasing functional. Then, the following assertions are equivalent

1. $F_s$ is weakly $\gamma_s$-lower semicontinuous.
2. $F_s$ is strong $\gamma_s$-lower semicontinuous.

**Proof.** Since every strongly $\gamma_s$-convergent sequence $\{A_k\}_{k \in \mathbb{N}}$ is, in addition, weakly $\gamma_s$-convergent, (1) $\Rightarrow$ (2) is clear. (See definitions and Proposition A.5). Let us see the converse.

Now, suppose $F_s$ is strongly $\gamma_s$-lower semicontinuous. To arrive at the weakly $\gamma_s$-lower semicontinuity of $F_s$ from the strong one, the strategy is to take into account the auxiliary functional $G_s$ defined in (3.2) and its properties.

Fix $\{(A^k_1, \ldots, A^k_m)\}_{k \in \mathbb{N}} \subset \mathcal{A}_s(\Omega)^m$ and $(A_1, \ldots, A_m) \in \mathcal{A}_s(\Omega)^m$ such that

$$(A^k_1, \ldots, A^k_m) \xrightarrow{\gamma_s} (A_1, \ldots, A_m).$$

That means, $u^*_{A^k_i} \to u_i$ strongly in $L^2(\Omega)$ and $A_i = \{u_i > 0\}$, for $i = 1, \ldots, m$.

Since for every $i = 1, \ldots, m$, $\{u^*_{A^k_i}\}_{k \in \mathbb{N}} \subset \mathcal{K}_s$, by Proposition 3.1, $u_i \in \mathcal{K}_s$. Moreover, by Proposition 3.2, $u_i \leq u^*_{A_i}$. Then, we can use $(G_3)$, the decreasing property of $G_s$, so that we obtain

$$(3.4) \quad G_s(u^*_{A_1}, \ldots, u^*_{A_m}) \leq G_s(u_1, \ldots, u_m).$$

On the other hand, by recalling $(G_1)$, the relationship between $F_s$ and $G_s$, we get the following identities

$$(3.5) \quad F_s(A_1, \ldots, A_m) = G_s(u^*_{A_1}, \ldots, u^*_{A_m}) \quad \text{and} \quad F_s(A^k_1, \ldots, A^k_m) = G_s(u^*_{A^k_1}, \ldots, u^*_{A^k_m}),$$

for every $k \in \mathbb{N}$.

Now, due to $(G_2)$ (the $L^2(\Omega)$-lower semicontinuity of $G_s$) in addition to $u^*_{A^k_i} \to u_i$ strongly in $L^2(\Omega)$ for every $i = 1, \ldots, m$, we connect (3.4) and (3.5) to conclude that

$$F_s(A_1, \ldots, A_m) \leq \liminf_{k \to \infty} G_s(u^*_{A^k_1}, \ldots, u^*_{A^k_m})$$

\begin{align*}
&\leq \liminf_{k \to \infty} G_s(u^*_{A^k_1}, \ldots, u^*_{A^k_m}) \\
&= \liminf_{k \to \infty} F_s(A^k_1, \ldots, A^k_m).
\end{align*}
Moreover, since cap\(F\) with the same index, which converge \(F\), and recalling (3.6) the result is proved.

3.3. Existence of an optimal partition. With the help of the previous outcomes of this section, we are able to prove existence of a minimal partition shape for (2.2).

Proof of Theorem 2.6. Denote by

\[\alpha := \inf \{F_s(A_1, \ldots, A_m) : A_i \in A_s(\Omega), \text{cap}_s(A_i \cap A_j, \Omega) = 0 \text{ for } i \neq j\}.\]

Let \(\{A_i^k, \ldots, A_m^k\}_{k \in \mathbb{N}} \subset A_s(\Omega)^m\) be such that

\[\text{cap}_s(A_i^k \cap A_j^k, \Omega) = 0 \text{ for } i \neq j, \quad \text{and } \lim_{k \to \infty} F_s(A_1^k, \ldots, A_m^k) = \alpha.\]

By Remark 3.3, there exist \(A_1 \in A_s(\Omega)\) and a subsequence \(\{A_i^{k_j}\}_{j \in \mathbb{N}} \subset \{A_i^k\}_{k \in \mathbb{N}}\) such that \(A_i^{k_j} \rightharpoonup A_i\). Now, consider \(\{A_2^{k_j}\}_{j \in \mathbb{N}} \subset \{A_2^k\}_{k \in \mathbb{N}}\) and \(\{A_3^{k_j}\}_{j \in \mathbb{N}} \subset \{A_3^k\}_{k \in \mathbb{N}}\) such that \(A_i^{k_j} \rightharpoonup A_i\) for \(i = 1, 2, 3\). Repeating this argument, we find a sequence \(\{(A_1^k, \ldots, A_m^k)\}_{k \in \mathbb{N}}\) and \((A_1, \ldots, A_m) \in A_s(\Omega)\) such that \(A_i^k \rightharpoonup A_i\) for every \(i = 1, \ldots, m\).

Since \(F_s\) is weak \(\gamma_s\)-lower semicontinuous, we obtain

\[F_s(A_1, \ldots, A_m) \leq \liminf_{k \to \infty} F_s(A_1^k, \ldots, A_m^k) = \alpha.\]

To finish the proof, let us see \(\text{cap}_s(A_i \cap A_j, \Omega) = 0\) for \(i \neq j\) be satisfied.

Let \(i, j \in \{1, \ldots, m\}\) be such that \(i \neq j\). Notice that this product \(u_{A_i^k}^s \cdot u_{A_j^k}^s\) is an \(s\)-continuous function too, by Lemma A.1, and \(u_{A_i^k}^s \cdot u_{A_j^k}^s = 0\) s-q.e. in \(\mathbb{R}^n \setminus (A_i^k \cap A_j^k)\).

Moreover, since \(\text{cap}_s(A_i^k \cap A_j^k, \Omega) = 0\), we have \(u_{A_i^k}^s \cdot u_{A_j^k}^s = 0\) s-q.e. in \(\mathbb{R}^n\).

By [28, Lemma 3.8], there exist subsequences \(\{u_{A_i^k}^s\}_{k \in \mathbb{N}}\) and \(\{u_{A_j^k}^s\}_{k \in \mathbb{N}}\), denoted with the same index, which converge s-q.e. to \(u_i\) and \(u_j\) respectively. Then, passing to the limit, we obtain \(u_i \cdot u_j = 0\) s-q.e. in \(\mathbb{R}^n\). That is \(\text{cap}_s(\{u_i \cdot u_j \neq 0\}, \Omega) = 0\). But, \(\{u_i \cdot u_j \neq 0\} = A_i \cap A_j\).

We have shown that \((A_1, \ldots, A_m)\) is admissible for the minimization problem (2.2) and recalling (3.6) the result is proved.

Due to Theorems 3.7 and 2.6, we can establish the next immediate corollary.

Corollary 3.8. Let \(F_s : A_s(\Omega)^m \to [0, \infty]\) be a decreasing and strong \(\gamma_s\)-lower semicontinuous functional. Then, there exists a solution to (2.2).

4. Proof of Theorem 2.8

This is the main part of the article, where we study the behavior of optimal partition shapes obtained in Section 3 and their minimum values. Again, we use some results from [16].

Lemma 4.1 (Lemma 4.1, [16]). Let \(0 < s_k \uparrow 1\) and let \(u_k \in K_{s_k}\). Then, there exists \(u \in H^1_0(\Omega)\) and a subsequence \(\{u_k\}_{j \in \mathbb{N}} \subset \{u_k\}_{k \in \mathbb{N}}\) such that \(u_k_j \to u\) strongly in \(L^2(\Omega)\).
Moreover, if \( u_k \in \mathcal{K}_{s_k} \) is such that \( u_k \to u \) strongly in \( L^2(\Omega) \), then \( u \in \mathcal{K}_1 \).

Next proposition gives an idea of the limit behavior of \( u^*_A \), when the domains also are varying with \( s \).

**Proposition 4.2** (Proposition 4.5, [16]). Let \( 0 < s_k \uparrow 1, A^k \in \mathcal{A}_{s_k}(\Omega) \) be such that \( u^*_{A^k} \to u \) strongly in \( L^2(\Omega) \). Then, there exist \( \tilde{A}^k \in \mathcal{A}_{s_k}(\Omega) \) such that \( A^k \subset \tilde{A}^k \) and \( \tilde{A}^k \) \( \gamma \)-converges to \( A := \{ u > 0 \} \).

Now we are ready to prove the main result of this article.

**Proof of Theorem 2.8.** First, notice that \( m_1 \) is achieved by [5, Theorem 3.2].

Let \( 0 < s_k \uparrow 1 \). By Theorem 2.6, there exists \( (A^k_1, \ldots, A^k_m) \in \mathcal{A}_{s_k}(\Omega)^m \) such that

\[
\text{cap}_{s_k}(A^k_i \cap A^k_j, \Omega) = 0 \text{ for } i \neq j \quad \text{and} \quad F_{s_k}(A^k_1, \ldots, A^k_m) = m_k,
\]

where \( m_k = m_{s_k} \) defined in (2.2).

Let \( (A_1, \ldots, A_m) \in \mathcal{A}_1(\Omega)^m \) be such that \( \text{cap}_1(A_i \cap A_j, \Omega) = 0 \) for \( i \neq j \). Since \( 0 < s_k \uparrow 1 \), we can assume \( 0 < \varepsilon_0 < s_k \uparrow 1 \), for some fixed \( \varepsilon_0 \).

Now, recalling Corollary A.7 and Remark A.8, we know that \( (A_1, \ldots, A_m) \) belongs to

\[
\{(B_1, \ldots, B_m) : B_i \in \mathcal{A}_{s_k}(\Omega), \text{cap}_{s_k}(B_i \cap B_j, \Omega) = 0 \text{ for } i \neq j\},
\]

for every \( k \in \mathbb{N} \). This fact and condition \((H_1)\) imply that

\[
\limsup_{k \to \infty} F_{s_k}(A^k_1, \ldots, A^k_m) \leq \lim_{k \to \infty} F_{s_k}(A_1, \ldots, A_m) = F_1(A_1, \ldots, A_m).
\]

It follows that

\[
\limsup_{k \to \infty} m_k \leq m_1.
\]

To see the remaining inequality, let us denote \( u^k_i := u^*_{A^k_i} \in \mathcal{K}_{s_k} \). By Lemma 4.1, there is \( u_i \in \mathcal{K}_1 \) such that, up to a subsequence, \( u^k_i \to u_i \) strongly in \( L^2(\Omega) \) and a.e. in \( \Omega \).

Denote by \( A_i := \{ u_i > 0 \} \in \mathcal{A}_1(\Omega) \) for every \( i = 1, \ldots, m \). We claim that \( \text{cap}_1(A_i \cap A_j, \Omega) = 0 \) for \( i \neq j \).

Indeed, let \( i \neq j \) be fixed. For each \( k \in \mathbb{N} \), due to Lemma A.2 and (4.1), we know that

\[
|\{u^k_i \cdot u^k_j \neq 0\}| = |A^k_i \cap A^k_j| \leq C(n, s_k) \text{cap}_{s_k}(A^k_i \cap A^k_j, \Omega) = 0.
\]

Then, \( u^k_i \cdot u^k_j = 0 \) a.e. in \( \mathbb{R}^n \). Since \( u^k_i \to u_i \) a.e. in \( \Omega \) for \( i = 1, 2 \), we conclude \( u_i \cdot u_j = 0 \) a.e. in \( \Omega \), it is still true in \( \mathbb{R}^n \) considering that they belong to \( H^s_0(\Omega) \). So, \( u_i \cdot u_j = 0 \) a.e. in \( \mathbb{R}^n \).

Reminding that we are working with 1-quasi continuous representative functions in \( H^s_0(\Omega) \), the previous identity \( u_i \cdot u_j = 0 \) a.e. in \( \mathbb{R}^n \) and [20, Lemma 3.3.30] tells that \( u_i \cdot u_j = 0 \) 1-q.e. in \( \mathbb{R}^n \). That means, \( \text{cap}_1(A_i \cap A_j, \Omega) = 0 \).

Consequently, \( (A_1, \ldots, A_m) \) is admissible to the problem 2.2 with \( s = 1 \) and we obtain \( m_1 \leq F_1(A_1, \ldots, A_m) \).

Moreover, by Proposition 4.2, there exists \( \tilde{A}^k_i \in \mathcal{A}_{s_k}(\Omega) \) such that \( A^k_i \subset \tilde{A}^k_i \) and \( (\tilde{A}^k_1, \ldots, \tilde{A}^k_m) \) \( \gamma \)-converges to \( (A_1, \ldots, A_m) \).
Finally, from condition $(H_2)$ and the decreasing property of $F_{s_k}$, we conclude that
\[ m_1 \leq F_1(A_1, \ldots, A_m) \leq \liminf_{k \to \infty} F_{s_k}(A_1^k, \ldots, A_m^k) \]
\[ \leq \liminf_{k \to \infty} F_{s_k}(A_1^k, \ldots, A_m^k) = \liminf_{k \to \infty} m_k. \]
Therefore, from previous conclusion and (4.2) we have the identity (2.4), so that the results follow. □

**Appendix A. Some useful properties of $s$-capacity**

The following lemmas address some basic properties of $s$-capacity and $s$-quasi continuous functions. We suppose those results are well-known and we include them for completeness.

**Lemma A.1.** Let $u, v : \mathbb{R}^n \to \mathbb{R}$ be $s$-quasi continuous functions. Then, the product $u \cdot v$ is also an $s$-quasi continuous function.

**Proof.** By definition, there exist decreasing sequences $\{A_k\}_{k \in \mathbb{N}}$ and $\{B_k\}_{k \in \mathbb{N}}$ of open sets such that $\lim_{k \to \infty} \text{cap}_s(A_k, \Omega) = \lim_{k \to \infty} \text{cap}_s(B_k, \Omega) = 0$ and $u|_{\mathbb{R}^n \setminus A_k}$, $v|_{\mathbb{R}^n \setminus B_k}$ are continuous.

Consider $C_k := A_k \cup B_k$. Then, $\{C_k\}_{k \in \mathbb{N}}$ is a decreasing sequence of open sets such that $\lim_{k \to \infty} \text{cap}_s(C_k, \Omega) = 0$, since $\text{cap}_s(C_k, \Omega) \leq \text{cap}_s(A_k, \Omega) + \text{cap}_s(B_k, \Omega)$ by [28, Proposition 3.6]. Moreover, $(u \cdot v)|_{\mathbb{R}^n \setminus C_k}$ is continuous. □

Next lemma gives a relation between the Lebesgue measure and the $s$-capacity of a subset $A \subset \Omega$. The proof is easy and follows [15, Section 4.7, Theorem 2 VI], where it was shown with the classical capacity measure ($s = 1$).

**Lemma A.2.** For every $A \subset \Omega$, $|A| \leq C(\Omega, s) \text{cap}_s(A, \Omega)$, where $C(\Omega, s)$ is the Poincaré’s constant in $H^s_0(\Omega)$.

**Proof.** For every $\varepsilon > 0$, there exists a function $u_\varepsilon \in H^s_0(\Omega)$ such that $u_\varepsilon \geq 1$ a.e. in a neighborhood of $A$ and $[u_\varepsilon]^2 = \text{cap}_s(A, \Omega) + \varepsilon$.

On the other hand, by Poincaré’s inequality,
\[ |A| = \int_A 1 \, dx \leq \int_{\mathbb{R}^n} u_\varepsilon^2 \, dx \leq C(\Omega, s) \text{cap}_s(A, \Omega) \leq C(\Omega, s) (\text{cap}_s(A, \Omega) + \varepsilon). \]
Take the limit $\varepsilon \downarrow 0$ to obtain the result. □

For every $A \in \mathcal{A}_s(\Omega)$, we will show that $A = \{u_A > 0\}$ in the sense of $\text{cap}_s(\cdot, \Omega)$. To prove this aim, we need some previous results which are modifications from [11, Lemma 2.1] and [12, Proposition 5.5].

**Lemma A.3.** Let $A \in \mathcal{A}_s(\Omega)$, Then, there exists an increasing sequence $\{u_k\}_{k \in \mathbb{N}} \subset H^s_0(\Omega)$ of non negative functions, such that $\sup_{k \in \mathbb{N}} u_k = 1_A$ s-q.e. on $\Omega$.

We omit the proof since it is completely analogous to that of [11, Lemma 2.1].

We prove a density result in $H^s_0(A)$, for $A \in \mathcal{A}_s(\Omega)$, which is similar to [12, Proposition 5.5].
Lemma A.4. Let $A \in \mathcal{A}_s(\Omega)$. Then, $\{\varphi u_A^s : \varphi \in C_c^\infty(\Omega)\}$ is dense in $H_s^0(A)$.

Proof. In order to prove the lemma, it is sufficient to see that we can approximate any non-negative function $w \in H_s^0(A)$ with $(-\Delta)^s w \in L^\infty(\Omega)$, since $L^\infty(\Omega)$ is dense in $H^{-s}(\Omega)$ and $w = w^+ - w^-$. Indeed, for an arbitrary function $w \in H_s^0(\Omega)$, we know that $(-\Delta)^s w =: f \in H^{-s}(\Omega)$.

Denote by $f := (-\Delta)^s w$. Then,

$$(-\Delta)^s w \leq \|f\|_{L^\infty(\Omega)} = \|f\|_{L^\infty(\Omega)} (-\Delta)^s u_A^s \quad \text{in} \ A.$$  

By comparison, we obtain $0 \leq w \leq cu_A^s$, where $c := \|f\|_{L^\infty(\Omega)}$.

For every $\varepsilon > 0$, consider $(w - \varepsilon)A^s \in H_s^0(\Omega)$. Thus,

$$(A.1) \quad \{(w - \varepsilon)A^s > 0\} \subset \{u_A^s > \varepsilon\}.$$  

Notice that $u_A^s \in L^\infty(\Omega)$ by [13, Theorem 4.1]. Observe that, using (A.1), $\varepsilon < u_A^s \leq \|u_A^s\|_{L^\infty(\Omega)}$ in $\{(w - \varepsilon)A^s > 0\}$. Then, the function $\frac{(w - \varepsilon)A^s}{u_A^s}$ belongs to $H_s^0(\Omega)$. So, there exists a sequence $\{\varphi_{k}^s\}_{k \in \mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_{k}^s \rightarrow \frac{(w - \varepsilon)A^s}{u_A^s}$ strongly in $H_s^0(\Omega)$, when $k \rightarrow \infty$. Therefore, $\varphi_{k}^s u_A^s \rightarrow (w - \varepsilon)A^s$ strongly in $H_s^0(\Omega)$, when $k \rightarrow \infty$.

On the other hand, $(w - \varepsilon)A^s \rightarrow w$ strongly in $H_s^0(\Omega)$, when $\varepsilon \downarrow 0$. Consequently, by a diagonal argument, there exist subsequences $\varepsilon_j \downarrow 0$ and $\{\varphi_{k_j}^s\}_{j \in \mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_{k_j}^s u_A^s \rightarrow w$ strongly in $H_s^0(\Omega)$. □

The following proposition is an essential component to relate domains and functions. It also contributes to the proofs of principal results Theorems 2.6 and 2.8.

Proposition A.5. Let $A \in \mathcal{A}_s(\Omega)$. Then, $A = \{u_A^s > 0\}$ in sense of cap$_s(\cdot, \Omega)$. That is, cap$_s(A \triangle \{u_A^s > 0\}, \Omega) = 0$.

Proof. It is clear that $u_A^s = 0$ s-q.e. on $\mathbb{R}^n \setminus A$. So, $\{u_A^s > 0\} \subset A$.

To see $A \subset \{u_A^s > 0\}$, we use the previous lemmas.

By Lemma A.3, there exists an increasing sequence $\{v_k\}_{k \in \mathbb{N}} \subset H_s^0(\Omega)$ of non-negative functions, such that $\sup_{\mathbb{N}} v_k = 1_A$ s-q.e. on $\Omega$.

For every $v_k$, by Lemma A.4, there exists a sequence $\{\varphi_{k_j}^s\}_{j \in \mathbb{N}} \subset C_c^\infty(\Omega)$ such that $\varphi_{k_j}^s u_A^s \rightarrow v_k$ strongly in $H_s^0(\Omega)$ and s-q.e., when $j \rightarrow \infty$. Since $\varphi_{k_j}^s u_A^s = 0$ s-q.e. in $\{u_A^s = 0\}$, then $v_k = 0$ s-q.e. in $\{u_A^s = 0\}$. Therefore, $1_A = 0$ s-q.e. in $\{u_A^s = 0\}$, which implies $A \subset \{u_A^s > 0\}$. □

Now, we prove a key estimate used in Section 4, which is a simply remark following the proof of [14, Proposition 2.2]. Notice that we are interested in finding a positive constant connecting in some sense cap$_s(\cdot, \Omega)$ and cap$_1(\cdot, \Omega)$. But, we also want that this constant does not depend on $s$. As our goal in Section 4 is related to the limit case $s \uparrow 1$, we can assume $0 < \varepsilon_0 < s < 1$ for some $\varepsilon_0$ and that will be enough to obtain this desired and independent constant.

As we said before, the proof of next lemma follows [14, Proposition 2.2] and, despite of the similarity, it is included since we want to analyze how the constant depends on $s$. 
Lemma A.6. Let \( \varepsilon > 0 \) and \( \varepsilon_0 < s < 1 \). Then, there exits a constant \( C > 0 \) such that for every \( u \in H^1_0(\Omega) \)

\[
(1 - s)|u|^2_s \leq C\|\nabla u\|_{L^2(\Omega)}.\]

and \( C = C(\Omega, n, \varepsilon_0) \) does not depend on \( s \).

Proof. Let \( u \in C^\infty(\Omega) \), we split \([u]^2\) into two pieces.

For the first part, use the change of variable \( z = y - x \) and observe that for \( z \in B_1(0) \setminus \{0\} \) and \( \varphi(t) := u(x + tz) \) for \( t \in [0, 1] \) we estimate

\[
\left| \frac{u(x + z) - u(x)}{|z|} \right| = \left| \frac{\int_0^1 \varphi'(t)\,dt}{|z|} \right| = \left| \frac{\int_0^1 \nabla u(x + tz) \cdot z \, dt}{|z|} \right| \leq \int_0^1 |\nabla u(x + tz)| \, dt.
\]

Now, use the previous remark and Jensen’s inequality to obtain

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| < 1\}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy = \int_{\mathbb{R}^n} \int_{B_1(0)} \frac{|u(x) - u(z + x)|^2}{|z|^{n+2s}} \, dz \, dx
\]

\[
= \int_{\mathbb{R}^n} \int_{B_1(0)} \frac{|u(x) - u(z + x)|^2}{|z|^{n+2s}} \, dz \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \int_{B_1(0)} \left( \int_0^1 \frac{\nabla u(x + tz) \cdot z}{|z|^{n+2s-1}} \, dt \right)^2 \, dz \, dx
\]

\[
\leq \left\| \nabla u \right\|_{L^2(\Omega)}^2 \int_{B_1(0)} \frac{1}{|z|^{n+2s-1}} \, dz
\]

\[
= \frac{|B_1(0)|}{2(1 - s)} \left\| \nabla u \right\|_{L^2(\Omega)}.
\]

For the remaining part, use \( |a - b|^2 \leq 2(a^2 + b^2) \) and easily follows

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| \geq 1\}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \leq 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| \geq 1\}} \frac{|u(x)|^2 + |u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy
\]

\[
\leq 4 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \cap \{|y-x| \geq 1\}} \frac{|u(x)|^2}{|x - y|^{n+2s}} \, dx \, dy
\]

\[
\leq \int_{\mathbb{R}^n} |u(x)|^2 \left( \int_{|z| \geq 1} \frac{1}{|z|^{n+2s}} \, dz \right) \, dx
\]

\[
= \frac{|B_1(0)|}{2s} \left\| u \right\|_{L^2(\Omega)}
\]

\[
\leq \frac{|B_1(0)|}{2\varepsilon_0} C_1(\Omega, n) \left\| \nabla u \right\|_{L^2(\Omega)},
\]

where \( C_1(\Omega, n) \) is the constant of classical Poincaré’s inequality in \( H^1_0(\Omega) \).
Then, put together the two estimates to conclude
\[
(1 - s)[u]^2 \leq (1 - s) \left( \frac{C_1(\Omega, n)}{2\varepsilon_0} + \frac{1}{2(1 - s)} \right) |B_1(0)|\|\nabla u\|_{L^2(\Omega)}
\]
\[
\leq \left( \frac{C_1(\Omega, n)}{2\varepsilon_0} + \frac{1}{2} \right) |B_1(0)|\|\nabla u\|_{L^2(\Omega)}
\]
\[
= C(\Omega, n, \varepsilon_0)\|\nabla u\|_{L^2(\Omega)}.
\]

Automatically, we obtain an estimate relating the \(s\)-capacity and the 1-capacity.

**Corollary A.7.** Let \(\varepsilon_0 > 0\) and \(\varepsilon_0 < s < 1\). Then, there exists a constant \(C > 0\) such that for every \(A \subset \Omega\)
\[
(1 - s) \text{cap}_s(A, \Omega) \leq C \text{cap}_1(A, \Omega),
\]
and \(C = C(\Omega, n, \varepsilon_0)\) does not depend on \(s\).

We deduce other useful remark from Lemma A.6: every 1-quasi open set is also an \(s\)-quasi open, for \(0 < s < 1\).

**Remark A.8.** For every \(0 < s < 1\), \(A_1(\Omega) \subset A_s(\Omega)\). Moreover, if \(0 < s < t \leq 1\), then \(A_t(\Omega) \subset A_s(\Omega)\).

**Proof.** Let \(A \in A_1(\Omega)\). There exists a decreasing sequence of open sets \(\{G_k\}_{k \in \mathbb{N}}\) such that \(A \cup G_k\) is open and \(\text{cap}_1(G_k, \Omega) \to 0\).

Let \(0 < s < 1\). By Corollary A.7, \(\text{cap}_s(G_k, \Omega) \to 0\). Then, \(A \in A_s(\Omega)\).

To prove \(A_t(\Omega) \subset A_s(\Omega)\) for \(0 < s < t \leq 1\), use definitions of capacity and [14, Proposition 2.1].

\[\square\]

**Acknowledgements**

This paper was partially supported by grants UBACyT 20020130100283BA, CONICET PIP 11220150100032CO and ANPCyT PICT 2012-0153. The author wants to thank Prof. Julián Fernández Bonder for helpful conversations. A. Ritorto is a doctoral fellow of CONICET.

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