Nuclear Reactions and Superfluid Time Dependent Density Functional Theory

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Abstract: The extension of Time Dependent Density Functional Theory (TDDFT) to superfluid systems is discussed in the context of nuclear reactions and large amplitude collective motion.

keywords: density functional theory, pairing, nuclear reactions, Coulomb excitation.

Time Dependent Density Functional Theory - general remarks

Density Functional Theory (DFT) has become nowadays a standard theoretical tool for studies of interacting many-body Fermi systems\cite{1, 2, 3, 4}. It offers a universal and formally exact approach, which had enormous practical successes. In the field of condensed matter it is widely used whenever properties of electronic systems need to be determined \cite{5, 6, 7, 8}. There is however a significant difference between DFT and other theoretical tools of quantum many-body physics. The latter are usually designed in a way which allows to estimate their applicability and also provide a method to systematically improve the predictions and obtain greater accuracy. On the contrary, the central object in DFT is the energy density functional which is merely proved to exist by means of the Hohenberg-Kohn theorem \cite{9}. The theorem states that certain unique energy density functional provides the energy of the ground state and spatial density distribution as a result of some minimum principle. But it neither offers any method of construction of such functional eg. starting from interparticle interactions, nor it guaranties that the functional can be written in an analytic form, which is crucial for any practical calculations. In electronic systems the situation is nevertheless much simpler as the interelectronic interaction is well known. Consequently, guided by Hartree-Fock approximation, one may construct the main component of the functional and only the exchange and correlation energy contributions need to be specified. Their form is usually extracted from ab-initio Quantum Monte Carlo (QMC) calculations for uniform systems \cite{10, 11}.

The case of atomic nuclei is far more complicated. Two types of particles: neutrons and protons, need to be taken into account in the description of the system. The nuclear interaction is quite complex, involving many terms, including also the three-body part. Moreover, an atomic nucleus is a selfbound system, which prevents the straightforward application of the Hohenberg-Kohn theorem. Therefore it is more difficult to quantitatively justify a given choice of the functional. The nuclear energy density functionals have various forms, the most popular being the Skyrme functional (including variety of parametrizations), which despite of known shortcomings is still widely used (see eg. \cite{12, 13, 14, 15, 16} and references therein). Although DFT is based on rigorous theorems and a hierarchy of increasingly accurate approximations can be constructed, such as the local density approximation (LDA), generalized gradient approximations (GGA) and hybrids of exact exchange with GGA\cite{17, 18, 19, 20}, there is no general method of calculating corrections involving the information about inter-
particle interaction. More systematic approach to construction of the energy functional can be found in Refs. [21, 22, 23, 24].

Nevertheless the simple scheme offered by the energy density functional theory is very attractive, as in the DFT instead of searching for the wave function of an $N$–particle system, which depends on $3N$ variables, one solves a system of $N$ nonlinear, coupled partial differential equations. This simplification is achieved by introducing the Kohn-Sham (K-S) scheme, where the density is expressible through the set of orbitals which are determined from variational principle [25]. Consequently the minimization of the functional leads to set of equations for the orbitals defining the density distribution. The strict formulation of DFT limits its applicability to the ground-state properties of the system. However in the context of nuclear reactions the proper treatment of excited states is crucial. It can be achieved with an extension of the DFT to include time evolution. Time-dependent density functional theory (TDDFT) is an universal approach to the quantum many-body dynamics (see [26, 27, 28] and references therein). It means that TDDFT can be used to describe nonstationary situations in systems consisting of nuclei, atoms, molecules, solids, or nanostructures. TDDFT applies the same philosophy as DFT to time-dependent problems. The Runge-Gross theorem, which is the time-dependent counterpart of the Hohenberg-Kohn theorem proves that if two $N$-fermion systems evolve from the same initial state, but are subject to two different time-dependent potentials, their respective time-dependent densities will be different [29]. There is however an important problem which troubled TDDFT over many years and is related to the definition of the exchange potential, which may exhibit nonlocality in time and in principle violate causality principle. The so-called causality paradox has been resolved in a series of papers [30, 31, 32, 33, 34], but it also implies that the exact expression of the exchange potential as a functional of the density is unknown and requires certain approximations. It is however the only fundamental approximation in TDDFT.

Similarly to static K-S equations one can define time dependent Kohn-Sham equations that describe non-interacting particles that evolve in a time-dependent Kohn-Sham potential, and produce the same density as that of the interacting system of interest. Thus, just as in the ground-state case, the time dependent Schroedinger equation is replaced by much simpler set of equations to solve. TDDFT can be, and usually is, used to obtain excited states be means of linear response theory ([35] and references therein). However, there are important differences as well, and there are features of TDDFT that are unique to the time-dependent case. First of all the ground-state DFT is based on the variational minimum principle. In the time-dependent case, there is no analogous minimum principle. It is possible to derive the formal framework of TDDFT from a stationary-action principle, but in contrast to DFT, where the ground-state energy is the quantity of central importance, the action is practically of no interest in itself[1]. Another new feature of TDDFT are currents, which need to be added to the description of evolving system [36, 37]. Moreover the feature that have no counterpart in static DFT is that the time-dependent exchange-correlation potential contains memory term [38, 39]. Namely, the potential at time $t$ depends on densities $\rho(r', t')$ at earlier times, where $t' \leq t$. This memory is, in principle, infinitely long-ranged. Unfortunately very little is known about the memory term and the most common approximation is the adiabatic approximation, which ignores all memory effects. This is obviously very convenient, as it allows to switch easily from DFT to TDDFT, but imposes serious limitations on the theory. The effects which are incorrectly described as a result of this approximation are collective energy dissipation processes [20].

In the context of nuclear reactions the typical situation in which the TDDFT can be used is the following: a system is initially in a ground state (obtained within the standard DFT) and is then acted upon by a perturbation that drives it out of equilibrium. The external perturbations in the nuclear system can be of various origins: they can be caused by photon absorption, by neutron capture, or the perturbation can arise as an interaction between the projectile and the target nucleus, which are initially in their ground states. It has to be emphasized that the resulting deviations from equilibrium, can be arbitrarily strong.

[1] The uniqueness of the stationary-action point remains unproven.
TDDFT can be applied both in the linear-response regime (where it provides information about excitation energies and spectral properties) as well as in the nonlinear regime, where the external perturbations can be strong enough to compete with, or even override the internal interactions that provide the structure and stability of matter. This is of particular interest for the induced nuclear fission processes, which one would like to describe within TDDFT.

The typical procedure used in the context of TDDFT is the following:

- Prepare the initial state, which is usually the ground state (in principle, one can start from any initial state, but non-ground states or even non-stationary initial states are rarely considered and more difficult to obtain in practice). This can be achieved by solving static Kohn-Sham equations for a nucleus (or nuclei if more than one is involved in the reaction process), to get a set of ground-state Kohn-Sham orbitals and orbital energies.

- The time evolution can be obtained by applying certain external fields simulating, e.g., the photon absorption, or through generating nonzero velocities of nuclei towards each other by performing global phase change of orbitals corresponding to transformation to a moving inertial frame. Then one solves the time-dependent Kohn-Sham equation from the initial time to the desired final time. The time propagation of the orbitals gives the time-dependent density.

- During time evolution one may calculate the desired observable(s) as functionals of $\rho(r,t)$.

### Superfluidity and Density Functional Theory

The existence of superfluidity has been experimentally confirmed in a large number of systems, including various condensed matter systems, $^3$He and $^4$He liquids, nuclei and neutron stars, and both fermionic and bosonic cold atoms in traps. It is also predicted to show up in dense quark matter. In the case of low energy nuclear reactions, in particular when non-magic medium or heavy nuclei are involved, the proper treatment of superfluidity is crucial and the conventional DFT description has to be extended. The first attempt to develop the formal framework of DFT for superconductors has been triggered by the discovery of high-temperature superconductivity [40, 41]. The extension requires an introduction of an anomalous density $\chi(r,\sigma, r', \sigma') = \langle \hat{\psi}^{\sigma'}(r') \hat{\bar{\psi}}^{\sigma}(r) \rangle$ ($\sigma$ denotes the spin degrees of freedom), which play the role of the superconducting order parameter. The pairing potential is then formally defined as a functional derivative of the energy functional with respect to $\chi$:

$$\Delta(r, \sigma, r', \sigma') = -\frac{\delta E(\rho, \chi)}{\delta \chi^*(r, \sigma, r', \sigma')}.$$  

Introducing Bogoliubov transformation, which allows to express both normal and anomalous densities in a form similar to the orbital expansion in conventional DFT, one arrives at Kohn-Sham scheme for superfluid fermion systems, which formally resembles the Bogoliubov-de Gennes equations. Unfortunately they form set of integro-differential equations in coordinate space and their solution is extremely difficult in practice. This is a consequence of the nonlocality of the pairing potential $\Delta(r, \sigma, r', \sigma')$. It is possible however to formulate the problem using local pairing field [42]. The justification for the so-called SLDA (Superfluid Local Density Approximation) has been developed in a series of papers (see Refs. [43, 44, 45, 46, 47, 48]) and was shown to be very accurate for nuclei and cold atomic gases. The prescription involves the renormalization of the pairing coupling constant, which is a function of the momentum cutoff. In the case of the spherical cutoff the analytic formula can be derived (spin indices are omitted for clarity):

$$\Delta(r) = g_{eff}(r) \chi_c(r),$$  

$$\frac{1}{g_{eff}(r)} = \frac{1}{g(r)} - \frac{nk_c(r)}{2\pi^2\hbar^2} \left( 1 - \frac{k_F(r)}{2k_c(r)} \ln \frac{k_c(r) + k_F(r)}{k_c(r) - k_F(r)} \right).$$
where anomalous density $\chi_c$ is defined within the truncated space and

$$E_c + \mu = \frac{h^2 k_c^2(r)}{2m(r)} + \Gamma(r) \tag{4}$$

$$\mu = \frac{h^2 k_F^2(r)}{2m(r)} + \Gamma(r) \tag{5}$$

In the above formula $E_c$ defines the corresponding energy cutoff.

In a similar manner like in the conventional DFT, SLDA can be extended to describe time dependent phenomena. In this case the adiabatic approximation is applied, which neglects possible memory effects in the time evolution. As a result the time-dependent Kohn-Sham equations have formal structure of the time dependent Bogliubov-de Gennes (TDBdG) or Hartree-Fock-Bogoliubov (TDHFB) equations (spin indices are omitted):

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} U_\mu(r, t) \\ V_\mu(r, t) \end{pmatrix} = \begin{pmatrix} h(r, t) & \Delta(r, t) \\ \Delta^*(r, t) & -h^*(r, t) \end{pmatrix} \begin{pmatrix} U_\mu(r, t) \\ V_\mu(r, t) \end{pmatrix}, \tag{6}$$

where $h(r, t) = -\frac{\hbar^2}{2m} \nabla^2 + \Gamma(r, t)$.

Using the above framework which is an extension of the DFT to the real-time dynamics of Fermi superfluids it has been possible to describe a completely different physical system, where pairing plays an important role - cold atomic gas in the so-called unitary regime. Apart from being able to describe correctly known experimental facts, this approach leads also to new qualitative predictions including supercritical flow, quantum shock waves and domain walls, Higgs modes, vortex crossings, etc. (see Ref. [49, 50, 51, 52]).

In the case of nuclear system the set of equations have to be solved for both protons and neutrons, which are coupled through the potential $\Gamma(r, t)$ depending on both neutron and proton densities. For the case of Skyrme parametrization of the density functional the following local densities and currents are used as building blocks (time variable is omitted):

- density: $\rho(r) = \rho(r, r')|_{r=r'}$,
- spin density: $\tilde{s}(r) = \tilde{s}(r, r')|_{r=r'}$,
- current: $\tilde{j}(r) = \frac{1}{2i}(\tilde{\nabla} - \tilde{\nabla}^\dagger)\rho(r, r')|_{r=r'}$,
- spin current (2nd rank tensor): $\tilde{J}(r) = \frac{1}{2i}(\tilde{\nabla} - \tilde{\nabla}^\dagger) \otimes \tilde{s}(r, r')|_{r=r'}$,
- kinetic energy density: $\tau(r) = \tilde{\nabla} \cdot \tilde{\nabla}^\dagger \rho(r, r')|_{r=r'}$,
- spin kinetic energy density: $\tilde{T}(r) = \tilde{\nabla} \cdot \tilde{\nabla}^\dagger \tilde{s}(r, r')|_{r=r'}$,

where both normal and anomalous densities are defined as:

$$\rho(r, r') = \sum_\mu V_{\mu}^\ast(r, r') \rho_{\mu}(r', r'), \tag{7}$$

$$\chi(r, r') = \sum_\mu V_{\mu}^\ast(r, r') \chi_{\mu}(r', r'), \tag{8}$$

and the $U$ and $V$ components can be thought of as coefficients of the Bogoliubov transformation between single-particle ($\tilde{\psi}$) and quasiparticle ($\tilde{\alpha}$) bases:

$$\left( \begin{array}{c} \tilde{\psi} \\ \tilde{\psi}^\dagger \end{array} \right) = \mathcal{B} \left( \begin{array}{c} \tilde{\alpha} \\ \tilde{\alpha}^\dagger \end{array} \right) ; \mathcal{B} = \left( \begin{array}{cc} U & V^\ast \\ V & U^\ast \end{array} \right). \tag{9}$$

The nuclear components of the potentials $\Gamma$ and $\Delta$ in the Skyrme parametrization read:

$$h(r, t) = -\nabla \cdot \left( B(r, t) + \tilde{\sigma} \cdot \tilde{C}(r, t) \right) \nabla + U(r, t) + U_{\Delta}(r, t) \cdot \tilde{\sigma}$$

$$+ \frac{1}{2i} \left[ \tilde{W}(r, t) \cdot (\nabla \times \tilde{\sigma}) + \nabla \cdot (\tilde{\sigma} \times \tilde{W}(r, t)) \right] + \frac{1}{4} \left( \nabla \cdot \tilde{U}_{\Delta}(r, t) + \tilde{U}_{\Delta}(r, t) \cdot \nabla \right) \tag{10}$$

\[\text{In the nuclear physics community the term 'HFB' is more frequently used, although BdG eqs. have formally the structure of HFB eqs. in coordinate representation.}\]
\[ B(r,t) = \frac{\hbar^2}{2m} + C^\tau \rho \]  
(11)
\[ C(r,t) = C^{\tau T} \hat{s} \]  
(12)
\[ U(r,t) = 2C^\rho \rho + 2C^\Delta \rho \nabla^2 \rho + C^\tau \tau + C^{\tau J} \cdot \vec{J} + C^\gamma (\gamma + 2) \rho^{\gamma + 1} \]  
(13)
\[ \vec{W}(r,t) = -C^{\nabla J} \vec{\nabla} \rho \]  
(14)
\[ \vec{W}_s(r,t) = 2C^s \rho + 2C^{\Delta s} \nabla^2 \rho + C^{s T} \vec{T} + C^{\nabla J} \vec{\nabla} \times \vec{j} \]  
(15)
\[ \vec{U}_\Delta(r,t) = C^\gamma \vec{j} + \frac{1}{2} C^{\nabla J} \vec{\nabla} \times \vec{s} \]  
(16)

and pairing potential (spins are omitted):
\[ \Delta(r,t) = g_{eff}(r,t) \chi(r,t). \]  
(17)

Consequently the energy density functional reads
\[ E = \int d^3r \mathcal{H}(r) \]  
(18)

where
\[ \mathcal{H}(r) = C^\rho \rho^2 + C^s \hat{s} \cdot \hat{s} + C^\Delta \rho \nabla^2 \rho + C^{\Delta s} \hat{s} \cdot \nabla^2 \hat{s} + C^\tau (\rho \vec{J} - \vec{J} \cdot \vec{J}) + C \nabla^s (\vec{\nabla} \cdot \hat{s})^2 + C^{s T} (\vec{s} \cdot \vec{T} - \vec{J}^2) + C^{\nabla J} (\rho \vec{\nabla} \cdot \vec{J} + \hat{s} \cdot (\vec{\nabla} \times \vec{j})) + C^\gamma \rho^\gamma - g_{eff} |\chi|^2 \]  
(19)

\[ J_i = \sum_{k,l} \epsilon_{ikl} J_{kl} \]  
(20)
\[ J^2 = \sum_{k,l} J_{kl}^2. \]  
(21)

Apart from nuclear components the Coulomb term has to be added for protons.

One may express the equations using the generalized density matrix which (in an arbitrary basis) is defined as:
\[ \mathcal{R}(t) = \begin{pmatrix} \rho & \chi \\ \chi^* & 1 - \rho^* \end{pmatrix} \]  
(22)

and fulfills the equation of motion:
\[ i\hbar \frac{\partial}{\partial t} \mathcal{R}(t) = [\mathcal{H}, \mathcal{R}], \]  
(23)

where
\[ \mathcal{H}(t) = \begin{pmatrix} h(t) & \Delta(t) \\ \Delta^+(t) & -h^*(t) \end{pmatrix}. \]  
(24)

The time evolution of the superfluid system governed by eqs. exhibits fundamental differences as compared to the conventional TDDFT. In the latter case one needs to evolve the number of orbitals equal to the number of particles forming the system and consequently the complexity of the problem scales with the number of particles. In the TDSLDA however, all orbitals, that span Hilbert space defined by the Bogoliubov transformation, have to be evolved. This ensures that the total energy of the system is conserved during time evolution i.e. \( \frac{d}{dt} E = 0 \). If the space is truncated eg. by introducing a momentum or energy cutoffs at some initial time, then certain properties of the Bogoliubov transformation do not hold.
Namely, the Bogoliubov transformation becomes noninvertible and the closure relation is violated: $BB^* \neq 1$. As a consequence:

\[ \chi \neq -\chi^T \]
\[ 1 - \rho \neq UU^*, \]

and the total energy is no longer conserved. However this problem becomes important only for a sufficiently long time evolution, and depends on both the strength of the pairing potential and the size of the subspace. Weaker pairing admits longer evolution without violation of the energy conservation. The total energy is still conserved to high accuracy if the evolution of the system is short enough. The requirement for the truncated subspace to be large enough indicates that the complexity of TDSLDA increases rapidly and does not scale with the number of evolved particles as in the case of standard TDDFT. If the equations are solved in a box with certain discretization of spatial coordinates then it is the size of the box which determines the complexity of the problem. Consequently the TDSLDA equations are usually 2 – 3 orders of magnitude more computationally demanding than standard TDDFT.

**Numerical implementation**

The last decade has witnessed a proliferation of methods and techniques for numerically solving the Kohn-Sham equations (see Ref. [53] and references therein). Among these methods real-space and real-time methods applied to TDDFT turned out to be extremely efficient [54]. The mathematical structure of TDSLDA requires solving a system of coupled, complex, time-dependent nonlinear partial differential equations. In various applications of TDSLDA, a spatial three-dimensional Cartesian grid in coordinate space with periodic boundary conditions has been used, with derivatives evaluated in momentum (Fourier-transformed) space. This method represents a flexible tool to describe large amplitude nuclear motion as it contains the coupling to the continuum and between single-particle and collective degrees of freedom. The solutions are $U$ and $V$ components of the Bogoliubov transformation represented on a discrete three-dimensional spatial lattice. From the wave functions one can extract various observables using the usual quantum mechanical rules (velocity and pairing fields, density distributions, etc.). The time evolution is performed using the fifth-order predictor-corrector-modifier Adams-Bashforth-Milne (ABM) method, which provides a combination of high accuracy and numerical stability. The time step is usually chosen so the relative truncation error in the ABM method is between $10^{-7}$ and $10^{-15}$. The present computational abilities allow to consider boxes of sizes up to $50^3 - 60^3$, which with the lattice constant of 1 fm allow to consider dynamics of arbitrarily heavy nuclei. The time step of the order of 0.07 – 0.08 fm/c ensures that the evolution for time intervals of the order of 10000 fm/c will keep the numerical accuracy within the accepted range.

**Nuclear reactions within TDSLDA framework**

There are both bad and good news in regard to the application of TDDFT, and TDSLDA in particular, to nuclear reactions and in general to any quantum scattering problem. The good news is that TDSLDA is naturally suited to describe large amplitude collective motion of nuclear system and offers a valuable insight into the dynamics described through the evolution of spatial density distribution. This clear picture is missing in approaches operating within the energy representation. TDSLDA offers a computational framework which mimics closely the way how the low energy nuclear scattering experiments are performed. Namely, it simulates the evolution of the system in real time, where spatio-temporal coordinates of the collision/reaction can be easily extracted. It provides information about the energy distribution among various degrees of freedom, e.g. various types of nuclear deformations.

The bad news is related to the general difficulty of TDDFT to address questions concerning many-body wave functions. Observables that require knowledge of the many-body
wave function are not easy to extract. For example the state-to-state transition probability:

\[ S_{if} = \lim_{t \to \infty} \langle \Phi_f | \Phi(t) \rangle, \quad \text{and} \quad \lim_{t \to -\infty} |\Phi(t)\rangle = |\Phi_i\rangle \]  

(27)

is an important quantity, which however in the case of TDDFT requires a special procedure to compute\[55\]. Another difficult quantities are e.g. the momentum distribution or the transitional densities which require more information than just the local densities which enter the expression for the energy functional \[56, 69\]. In general, all more than one-body observables, including various types of conditional probabilities are difficult to determine within the presented formalism. Adiabatic approximation is another limitation preventing us from the proper description of dissipation effects, except for the one-body dissipation processes.

**Coulomb excitation and gamma absorption**

The simplest type of reaction which can be studied within the TDSLDA formalism is the Coulomb scattering of two nuclei. It is the simplest case, as it does not involve nuclear interaction between colliding nuclei and can serve as a textbook example of a nucleus being a subject of an external time dependent perturbations originating from electromagnetic field.

The Coulomb excitation can naturally lead to the excitation of various collective modes including giant dipole and giant quadrupole resonances \[57, 58, 59, 60, 61, 62\]. It is can also provide a tool for studies of multiphonon nuclear states \[63, 64, 65, 66, 67, 68\].

In order for a collision to lead to nonadiabatic nuclear processes, the external potential have to vary in time fast enough, which implies that the collision has to occur at relativistic energies. The interaction time needs to be relatively short, and for an efficient excitation of nuclear modes of frequency \(\omega\), the collision time \(\tau_{\text{coll}}\) has to fulfill the condition that \(\omega \tau_{\text{coll}} \simeq 1\). Here \(b\) is the impact parameter, \(v\) is the projectile velocity, and \(\gamma = (1 - v^2/c^2)^{-1/2}\) is the Lorentz factor. For example, in the collision process: \(^{238}\text{U} + ^{238}\text{U}\) at 700 MeV/n, studied in Ref. \[70\] the collision time was of the order of 10 fm/c for impact parameters of the order of the nucleus diameter.

In the Coulomb scattering the excitation process is governed by the electromagnetic field which has to be built into the framework of the TDDFT. This can be achieved using the requirement of the gauge invariance of the energy density functional. Namely, the coupling of the nuclear system to the electromagnetic field:

\[
\vec{E} = -\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t},
\]

\[
\vec{B} = \vec{\nabla} \times \vec{A}
\]

(28)

(29)

is realized through the following transformation:

\[
\vec{\nabla} \psi \rightarrow \vec{\nabla}_{A} \psi = \left( \vec{\nabla} - i \frac{e}{c \hbar} \vec{A} \right) \psi,
\]

\[
\vec{\nabla} \psi^* \rightarrow \vec{\nabla}_{-A} \psi^* = \left( \vec{\nabla} + i \frac{e}{c \hbar} \vec{A} \right) \psi^*,
\]

\[
i \hbar \frac{\partial}{\partial t} \psi \rightarrow \left( i \hbar \frac{\partial}{\partial t} - e \phi \right) \psi,
\]

(31)

(32)

(33)

which implies that \(\vec{\nabla} \psi \psi^* \rightarrow \vec{\nabla}_{A} \psi \psi^*\). Consequently it requires the following transformation of proton densities and currents (subscript \(A\) denotes the quantities in the presence of electromagnetic field):

- density: \(\rho_A(r) = \rho_A(r)\),
- spin density: \(s_A(r) = s(r)\)
- current: \(j_A(r) = j(r) - \frac{1}{c} \vec{A} \rho(r)\),
\* spin current (2nd rank tensor): \( J_A(r) = \mathbf{J}(r) - \frac{i}{\hbar c} \vec{A} \otimes \vec{s}(r) \),

\* spin current (vector): \( \vec{J}_A(r) = \vec{J}(r) - \frac{i}{\hbar c} \vec{A} \times \vec{s}(r) \),

\* kinetic energy density: \( \tau_A(r) = \left( \vec{\nabla} - i \frac{\hbar}{\mbox{e} c} \vec{A} \right) \cdot \left( \vec{\nabla}' + i \frac{\hbar}{\mbox{e} c} \vec{A} \right) \rho(r, r') |_{r = r'} 
= \tau(r) - 2 \frac{\hbar}{\mbox{e} c} \cdot \vec{A} \cdot \vec{j}(r) + \frac{e^2}{\hbar^2 c^2} |\vec{A}|^2 \rho(r), \)

\* spin kinetic energy density: \( \bar{T}_A(r) = \left( \vec{\nabla} - i \frac{\hbar}{\mbox{e} c} \vec{A} \right) \cdot \left( \vec{\nabla}' + i \frac{\hbar}{\mbox{e} c} \vec{A} \right) \bar{s}(r, r') |_{r = r'} 
= \bar{T}(r) - 2 \frac{\hbar}{\mbox{e} c} \vec{A}^T \cdot \vec{J}(r) + \frac{e^2}{\hbar^2 c^2} |\vec{A}|^2 \bar{s}(r). \)

As a result the proton single-particle hamiltonian acquires the form:

\[
\hat{h}_A(r, t) = -\vec{\nabla}_A \cdot \left( \hat{B}(r, t) + \vec{\sigma} \cdot \hat{C}(r, t) \right) \vec{\nabla}_A + \frac{1}{2\hbar} \left( \hat{\nabla}(r, t) \cdot (\vec{\nabla}_A \times \vec{\sigma}) + \vec{\nabla}_A \cdot (\vec{\sigma} \times \hat{\nabla}(r, t)) \right) + U_A(r, t) + \bar{U}_A(r, t) \cdot \vec{\sigma} + \frac{1}{\hbar} \left( \vec{\nabla}_A \cdot \bar{U}_A(r, t) \right) + \bar{U}_A(r, t) \cdot \vec{\nabla}_A + e\phi, \tag{34}
\]

and

\[
\begin{align*}
U_A(r, t) &= U(r, t) - C^{\mu J} \frac{\hbar}{\mbox{e} c} \vec{\nabla} \cdot [\vec{A} \times \vec{s}] - \left( C^{\tau} \frac{2}{\hbar} \frac{\hbar^2}{\mbox{e} c^2} \vec{A}^2 \rho \right), \tag{35} \\
\bar{U}_A(r, t) &= \bar{U}_A(r, t) - C^{\sigma J} \frac{\hbar}{\mbox{e} c} \vec{\nabla} \cdot [\vec{\rho}] - C^{\sigma T} \left( \frac{2}{\hbar} \frac{\hbar^2}{\mbox{e} c^2} \vec{J} + \frac{e^2}{\hbar^2 c^2} \vec{A}^2 \bar{s} \right), \tag{36} \\
\bar{U}_A(r, t) &= \bar{U}_A(r, t) - C^{\sigma J} \frac{\hbar}{\mbox{e} c} \vec{A} \rho, \tag{37} \\
\vec{\nabla} \cdot \left( B(r, t) + \vec{\sigma} \cdot \hat{C}(r, t) \right) \vec{\nabla} A &= \left[ \vec{\nabla} \cdot \left( B(r, t) + \vec{\sigma} \cdot \hat{C}(r, t) \right) \right] \cdot \vec{\nabla} A + \\
\left( B(r, t) + \vec{\sigma} \cdot \hat{C}(r, t) \right) \left[ \Delta - i \frac{\hbar}{\mbox{e} c} (\vec{A} \cdot \vec{\nabla} A + \vec{\nabla} A \cdot \vec{A}) + \frac{e^2}{\hbar^2 c^2} |\vec{A}|^2 \right], \tag{38}
\end{align*}
\]

where spatial and time variables of quantities: \( \rho, \vec{s}, \vec{J}, \vec{A}, \phi \) have been omitted.

Apart from the coupling of proton charges and currents to the electromagnetic field, there is also a component which describes the interaction of magnetic field with the nucleon spin: \( \mu_l \vec{\sigma} \cdot \vec{B}, \) where \( \mu_p = 5.8588 \hbar / 2 m_p c \) and \( \mu_n = -3.8263 \hbar / 2 m_p c \) for protons and neutrons, respectively. This correction however is small and will be neglected.

One of the most straightforward applications of the TDDFT is the calculation of the nuclear photoabsorption cross section. The process is an example of a perturbation of a nuclear system induced by a photon absorption, which can be described within the linear regime and although it does not require the whole machinery of TDDFT, it is not a trivial effect, as it combines in a nutshell several challenging aspects of physics of an atomic nucleus [74]. A perturbed nucleus exhibits large amplitude nonadiabatic motion and damping effects lead to a collective energy dissipation [72]. In the classical picture the isovector Giant Dipole Resonance (IGDR) is formed by two types of fluids representing neutrons and protons vibrating around a common center of mass. In the Steinwedel-Jensen and Goldhaber-Teller models the mass dependence of the excitation energy reads: \( A^{-1/3} \) and \( A^{-1/6} \), respectively [25]. A reasonably good estimation of the IGDR vibrational frequency is \( \omega \approx 80 \text{MeV} A^{-1/3} \) for spherical nuclei. For deformed nuclei, the IGDR reveals the splitting of characteristic frequencies which, roughly speaking, measures the aspect ratio of the nuclear shape. Due to the fact that IGDR is not an eigenstate of nuclear Hamiltonian, it is characterized by a spreading width which cannot be extracted from the hydrodynamical approach and has to be reproduced by including a reliable microscopic model of the atomic nucleus. The total width of IGDR possesses two components: one related to the coupling of the IGDR to more complex nuclear configurations \( \Gamma^i \), and the second associated with coupling to the continuum, e.g. related to emission lifetimes of particles (neutrons) \( \Gamma^i \). These two widths contribute to the total width of the IGDR, \( \Gamma = \Gamma^i + \Gamma^i \), and their relative contributions vary depending on the mass number and the \( N/Z \) ratio. The escape width is usually more important for light nuclei. The physical mechanism related to \( \Gamma^i \) may be quite complicated.
and depends on the energy. It involves coupling to low energy surface vibrations, Landau damping and collisional damping \cite{72}.

In contemporary approaches, the description of the atomic nucleus is provided by DFT and the IGDR has to be described within the same framework. The linear response formalism has been intensively used in the past due to the relative computational simplicity. It originates from the small amplitude perturbation of the static DFT solution. This approach gives rise to the well known \((Q)\)RPA equations that have been solved for a variety of functionals and forces, though only very recently for deformed systems \cite{73,75,76,77,78,79}. The application of full TDDFT formally incorporates the description of nonequilibrium phenomena and therefore covers both nonadiabatic and anharmonic effects. In order to calculate the cross section for the photoabsorption cross section for deformed nuclei one needs to average results over various orientations. It is usually a computationally demanding task especially in the case of triaxial nuclei like \(^{188}\text{Os}\) which need to be averaged at least over three different orientations. The results obtained for various deformed systems: \(^{172}\text{Yb}\), \(^{188}\text{Os}\) and \(^{238}\text{U}\) reveal a good agreement with experimentally extracted behaviour of the cross section as a function of energy \cite{80}. The calculation of the width of the resonance requires however the proper treatment of the dissipation effects and therefore cannot be reproduced.

Another feature that one observes in a large amplitude nuclear motion studied within superfluid TDDFT (TDSLDA) is that the occupation probabilities of proton and neutron quasiparticle states vary in time considerably. Namely, from the TDSLDA solutions one can extract the occupation probabilities for both proton and neutron quasiparticle states as follows:

\[
n_k(t) = \sum_{\sigma=\uparrow,\downarrow} \int d^3r |V_k(r,\sigma,t)|^2, \tag{39}
\]

where \(k\) labels the proton and neutron quasiparticle wave functions respectively, which are solutions of the TDSLDA equations. These occupation probabilities vary in time rather strongly and therefore the assumption which is usually made, to simplify the calculations, that they are frozen to their ground state values is violated \cite{80}.

The advantage of the TDDFT approach is that it can be used also beyond the linear regime when the perturbation of a nucleus is arbitrarily strong. Such a case occurs in ultrarelativistic Coulomb collisions at impact parameters of the order of nuclear diameters. The question which can be easily provided by TDDFT is the amount of energy transferred into internal degrees of freedom as a result of the collision. In the coulex reaction involving \(^{238}\text{U}\) studied in Ref. \cite{70} it turned out that approximately half of the total energy has been transferred to internal degrees of freedom at impact parameters: \(14 - 20\) fm. The other half was responsible for translational motion of a target nucleus. It is instructive to compare this fully microscopic result with the simple model based on the Goldhaber-Teller model (GT). Within the model it is assumed that both protons and neutrons are represented by rigid density distributions which can oscillate harmonically against each other. Thus the target nucleus possesses only two types of degrees of freedom: those related to the center-of-mass (CM) motion and those describing the internal harmonic excitation of GDR. The comparison between the average energy transferred to the internal motion of the target nucleus obtained within TDSLDA and also within the simplified Goldhaber-Teller model shows that significantly more energy is deposited by the projectile within the TDSLDA. Namely, for impact parameters \(14 - 20\) fm GT model predicts only \(40 - 60\%\) of TDSLDA energy transferred to internal degrees of freedom. The Goldhaber-Teller model is equivalent to the linear regime, assuming that all isovector transition strength is concentrated in two sharp lines, corresponding to an axially deformed target. An exact linear response approach would therefore severely underestimate the amount of the internal energy deposited, one reason being the non-linearity of the response, naturally incorporated in TDSLDA. The other reason being the fact that the present microscopic framework describing the target allows for many degrees of freedom, apart from pure dipole oscillations, to be excited. At the same time, the CM target energy alone is approximately the same as obtained in a simplified point particles Coulomb recoil model of both the target and projectile.

After collision the excited nucleus will subsequently emit radiation and neutrons. Part of the radiation emitted right after collision can be described within the TDDFT. Namely,
having the proton densities and currents extracted from TDSLDA

$$\rho(r, t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \rho(r, \omega) \exp(-i\omega t),$$  \hspace{1cm} (40)$$

$$\vec{j}(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \vec{j}(r, \omega) \exp(-i\omega t),$$  \hspace{1cm} (41)$$

one may evaluate the frequency distribution of emitted radiation:

$$\frac{dE}{d\omega} = \frac{4e^2}{c} \sum_{l,m} |\vec{b}_{lm}(k, \omega)|^2$$  \hspace{1cm} (42)$$

and the radiated power

$$P(t + r/c) = \frac{e^2}{\pi c} \sum_{l,m} \left| \int_{-\infty}^{\infty} \vec{b}_{lm}(k, \omega) \exp(-i\omega t) d\omega \right|^2$$  \hspace{1cm} (43)$$

where

$$\vec{b}_{lm}(k, t) = \int d^3 r \vec{b}(r, t) j_l(kr) Y_{lm}^*(\hat{\vec{r}})$$  \hspace{1cm} (44)$$

$$\vec{b}_{lm}(k, \omega) = \int_{-\infty}^{\infty} \vec{b}_{lm}(k, t) \exp(i\omega t) dt$$  \hspace{1cm} (45)$$

describe the corresponding multipole components of radiation. In the calculations presented in Ref. [70] the nuclear evolution has been followed during approximately 2500 fm/c after collision and two components of the electromagnetic radiation can be distinguished. The one which arises from the CM acceleration as a result of collision (Bremsstrahlung), and takes part only during the relatively short time interval $\tau_{coll} = b/v\gamma$. This contribution is of the order of 0.1% of the total radiation emitted within the first 2500 fm/c. The radiation emitted from the internal motion has much longer time scale. It was concluded that the main part of radiation coming from the target nucleus is due to the excitation of IGDR. The smaller fraction is related to the Giant Quadrupole Resonance (GQR) and also can be attributed to the low lying mode (pygmy resonance). One needs to remember, however, that the amount of energy emitted during this time interval is of the order of 1% of the total absorbed energy during the collision. It turned out that although the intensity of radiation decreases with increasing impact parameter, the ratio between the intensities, due to the internal modes with that of the CM motion, remains fairly constant.

The evolution of the total dipole moment of the target nucleus can be easily extracted from TDSLDA. It exhibits damping which is a consequence of one-body dissipation processes, leading to the transfer of the collective energy to single particle degrees of freedom. This dissipation process can be studied in TDDFT, even within the adiabatic approximation. It provides however only a small fraction of the total width of the collective mode. Assuming that the energy of the dipole mode is proportional to the square of its amplitude one finds out that the damping rate is almost perfectly described by the exponential decay law: $E_{coll}(t) \propto \exp(-t/\tau)$ with $\tau \approx 500$ fm/c and it does not depend on the nucleus orientation during the collision [70].

### Nuclear reactions, fusion and fission

The most interesting processes, although still to large extent unexplored, are related to nuclear collisions and in particular to nuclear fission and fusion. Indeed the role of pairing correlations is regarded as the key ingredient, which allows to properly describe an induced nuclear fission and therefore the superfluid extension of TDDFT seems to be the natural candidate for a microscopic theory of induced fission. Fully microscopic description of nuclear fission, which is a long standing goal, is a research topic more than seven decades old, a problem of great practical and fundamental interest. Its complexity due to the large
The number of strongly coupled degrees of freedom made this problem computationally challenging. It has been pointed out in the papers by Meitner and Frisch and Bohr and Wheeler in 1939 [81, 82] that nuclear fission can be regarded as the evolution of the nuclear shape leading eventually to splitting into two or more fragments, although its dynamics is still not well established. The process of transforming the compound nucleus into fragments is not well understood and usually the simplest adiabatic approximation is applied. The adiabatic approximation is however a questionable assumption around the scission point where the collective motion of the nucleus speeds up. Moreover, the currently applied phenomenological methods introduce several parameters whose values are determined from adjustments to various observables. In most of these approaches, the nuclear shape evolves on the nuclear potential energy landscape, being a subject to both conservative and dissipative forces until the scission point is reached. The location of this point is to certain extent arbitrary and have rather obscure meaning in quantum mechanics. At this point various observables including the kinetic energy of fragments and their masses are extracted. As a consequence, phenomenological approaches of this type have rather limited predictive power. The superfluid TDDFT, and TDSLDA in particular, offers the possibility to describe the fission process around the scission point without such assumptions, in particular the one concerning adiabaticity, and thus can provide the quantitative predictions for the kinetic energy distributions of fragments.

It is important to emphasize several aspects of superfluid TDDFT which need to be considered during the studies of nuclear reactions:

- The preparation of the initial condition needs to take into account the property that the \( U \) components of the Bogoliubov transformation are not localized within a nucleus and extend over the whole space. Therefore, the preparation of the initial configuration is more complicated, as in the static solution the properties of \( U \) components depend on the mutual arrangement of nuclei. Moreover, one cannot simply apply the Galilean boost to one nucleus without affecting the other through the change of the distribution of \( U_\mu(r) \). The strategy which can be applied in this case involves separation of two fragments by a nonpenetrable potential wall, which is subsequently removed during the evolution. This ensures that two nuclei are completely disentangled at the initial time.

- The quantities like the cross section, which can be extracted from the calculations require an additional degree of freedom to be averaged over. Namely, various relative phases of the pairing fields of two colliding nuclei need to be considered. It is a priori not known whether this effect will have a significant impact on a collision process, but clearly certain observables, e.g. the particle transfer rate, may turn out to be seriously affected.

- The description of the particle transfer and the emission of more than one nucleon would be an interesting test for the quality of the nuclear energy density functional. It is well known for example that an analogue of such process in atomic physics, namely, the multiple ionization is hard to describe within TDDFT [83].

Conclusions

Nuclear reactions and fission in particular play an important role in applications to energy production, astrophysics, etc. and in recent years the nuclear fission research has undergone a renaissance worldwide. Particularly important for applications are the properties of prompt fission neutrons and gamma rays, which are emitted before the weak decays of the fission fragments toward stability. Due to the complexity of the nuclear many body problem, the computationally realistic description of either nuclear reactions or nuclear structure properties required methods based on different assumptions which allowed to simplify the problem and made it tractable. Therefore, the unified description of nuclei, both their static and dynamic properties is dramatically called for, as otherwise one would not be able to gain a deep physical insight into nuclear processes. The superfluid TDDFT, e.g. in the
framework of TDSLDA is a perfect candidate to provide a fully microscopic description of nuclear fission and low energy nuclear reactions. With increasing computational abilities it is likely to become soon a standard tool in the field, having the advantage of being a fully microscopic theory and treating all nucleonic degrees of freedom on the same footing. In the next years we will certainly witness the growing importance of methods based on time-dependent DFT, probing its advantages and limitations in the application to large amplitude nuclear dynamics - one of the greatest unsolved nuclear many-body problems.

Acknowledgments
I would like to thank my collaborators: Aurel Bulgac, Carlos Bertulani, Michael M. Forbes, Kenneth J. Roche, Ionel Stetcu, and Gabriel Wlazłowski. I acknowledge support of Polish National Science Centre (NCN) Grant, decision no. DEC-2013/08/A/ST3/00708. This work has been also partly supported by the ERANET NuPNET grant SARFEN of the Polish National Centre for Research and Development (NCBiR), by Polish National Science Centre (NCN) Grant under Contract no. UMO- 2012/07/B/ST2/03907, and U.S. Department of Energy (DOE) Grant No. DE-FG02-97ER41014.

References
[1] W. Kohn, *Nobel Lecture: Electronic structure of matter – wave functions and density functionals*, Rev. Mod. Phys. 71, 1253 (1999).
[2] R. M. Dreizler, E. K. U. Gross, *Density Functional Theory: An Approach to the Quantum Many–Body Problem*, Springer-Verlag, Berlin, (1990).
[3] H. Eschrig, *The Fundamentals of Density Functional Theory*, B.G. Teubner Verlagsgesellschaft Stuttgart - Leipzig (1996).
[4] R. G. Parr, W. Yang, *Density-Functional Theory of Atoms and Molecules*, Oxford University Press, New York (1989).
[5] W. E. Picket, Rev. Mod. Phys. 61, 433 (1989).
[6] M. Brack, Rev. Mod. Phys. 65, 677 (1993).
[7] G. B. Brivio, M. I. Trioni, Rev. Mod. Phys. 71, 231 (1999).
[8] C. Freysoldt, B. Grabowski, T. Hickel, J. Neugebauer, G. Kresse, A. Janotti, C. G. Van de Walle, Rev. Mod. Phys. 86, 253 (2014).
[9] P. Hohenberg, W. Kohn, Phys. Rev. 136 B864 (1964).
[10] D. M. Ceperley, Phys. Rev. B18, 3126 (1978).
[11] D. M. Ceperley, B. J. Alder, Phys. Rev. Lett. 45, 566 (1978).
[12] D. Tarpanov, J. Dobaczewski, J. Toivanen, B. G. Carlsson, Phys. Rev. Lett. 113, 252501 (2014).
[13] R. Rodriguez-Guzman, L. M. Robledo, Phys. Rev. C89, 054310 (2014).
[14] M. Baldo, L. M. Robledo, P. Schuck, and X. Vinas, Phys. Rev. C87, 064305 (2013).
[15] S. A. Fayans, D. Zawischa, Int. J. Mod. Phys. B15, 1684 (2001).
[16] S. Bogner et al., [arXiv:1304.3713](https://arxiv.org/abs/1304.3713)
[17] A. D. Becke, Phys. Rev. A38 3098 (1988).
[18] C. Lee, W. Yang, R. G. Parr, Phys. Rev. B37 785 (1988).
[19] J. P. Perdew, K. Burke, M. Ernzerhof, Phys. Rev. Lett. 77 3865 (1996); 78, 1396 (1997) (E).

[20] A. D. Becke, J. Chem. Phys. 98, 5648 (1993).

[21] F. Raimondi, K. Bennaceur, J. Dobaczewski, J. Phys. G: Nucl. Part. Phys. 41 055112 (2014).

[22] J. Dobaczewski, K. Bennaceur, F. Raimondi, J. Phys. G: Nucl. Part. Phys. 39, 125103 (2012).

[23] B.G. Carlsson, J. Dobaczewski, M. Kortelainen, Phys. Rev. C78 044326 (2008).

[24] J.E. Drut, R.J. Furnstahl, L. Platter, Prog. Part. Nucl. Phys. 64 120-168 (2010).

[25] W. Kohn, L. J. Sham, Phys. Rev. 140 A1133 (1965).

[26] C. A. Ullrich, Time-dependent density-functional theory: concepts and applications, Oxford University Press, (2012).

[27] Fundamentals of time-dependent density functional theory, Eds. M. A. L. Marques, N. T. Maitra, F. M. S. Nogueira, E. K. U. Gross, and A. Rubio, Springer, Berlin (2012).

[28] G. Onida, L. Reining, A. Rubio, Rev. Mod. Phys. 74 601 (2002).

[29] E. Runge, E. K. U. Gross, Phys. Rev. Lett. 52 997 (1984).

[30] A. K. Rajagopal, Phys. Rev. A54, 3916 (1996).

[31] R. van Leeuwen, Phys. Rev. Lett. 80, 1280 (1998).

[32] R. van Leeuwen, Int. J. Mod. Phys. A15, 1969 (2001).

[33] S. Mukamel, Phys. Rev. A71, 024503 (2005).

[34] G. Vignale, Phys. Rev. A77 062511 (2008).

[35] P. Elliott, K. Burke, F. Furche, Excited states from time-dependent density functional theory, in Reviews in Computational Chemistry, Eds. K. B. Lipkovitz, T. R. Cundari, Vol. 26, Ch. 3, p.91 (2009).

[36] B.-X. Xu, A.K. Rajagopal, Phys. Rev. A31 2682 (1985).

[37] R. van Leeuwen, Phys. Rev. Lett. 82, 3863 (1999).

[38] J.F. Dobson, M.J. Brunner, E.K.U. Gross, Phys. Rev. Lett. 79 1905 (1997).

[39] G. Vignale, C. A. Ullrich, S. Conti, Phys. Rev. Lett. 79 4878 (1997).

[40] L. N. Oliveira, E. K. U. Gross, and W. Kohn, Phys. Rev. Lett. 60 2430 (1988).

[41] O.-J. Wacker, R. Kümmel, E.K.U. Gross, Phys. Rev. Lett. 73, 2915 (1994).

[42] S. Kurth, M. Marques, M. Lüders, E.K.U. Gross, Phys. Rev. Lett. 83 2628 (1999).

[43] A. Bulgac, Y. Yu, Phys. Rev. Lett. 88, 042504 (2002).

[44] A. Bulgac, Phys. Rev. C65, 051305 (2002).

[45] A. Bulgac, Y. Yu, Phys. Rev. Lett. 91, 190404 (2003).

[46] Y. Yu, A. Bulgac, Phys. Rev. Lett. 90, 222501 (2003); Phys. Rev. Lett. 90, 161101 (2003)

[47] A. Bulgac, Phys. Rev. A76, 040502 (2007).
[48] A. Bulgac, Annu. Rev. of Nucl. Part. Sci., 63 97 (2013).
[49] A. Bulgac, M. M. Forbes, P. Magierski, Lecture Notes in Physics, Vol. 836, Chap. 9, p.305-373 (2012).
[50] A. Bulgac, Y.-L. Luo, P. Magierski, K. J. Roche, Y. Yu, Science 332 1288 (2011).
[51] A. Bulgac, Y.-L. Luo, K. J. Roche, Phys. Rev. Lett. 108 150401 (2012).
[52] A. Bulgac, M. M. Forbes, M. M. Kelley, K. J. Roche, G. Wlazowski, Phys. Rev. Lett. 112 025301 (2014).
[53] T. L. Beck. Rev. Mod. Phys. 72, 1041 (2000).
[54] K. Yabana, G. F. Bertsch, Phys. Rev. B54, 4484 (1996).
[55] N. Rohringer, S. Peter, J. Burgdörfer, Phys. Rev. A74, 042512 (2004).
[56] F. Wilken, D. Bauer, Phys. Rev. A76, 023409 (2007).
[57] G. Baur, C. A. Bertulani, Phys. Lett. B174, 23 (1986).
[58] J. R. Beene, et al. Phys. Rev. C41, 920 (1990).
[59] J. Ritman et al. Phys. Rev. Lett. 70, 533 (1993).
[60] R. Schmidt et al. Phys. Rev. Lett. 70, 1767 (1993).
[61] P. E. Mueller et al. Nucl. Phys. A569, 123c (1994).
[62] A. Grüenschloss et al., Phys. Rev. C60, 051601 (1999)(R).
[63] H. Emling, Nucl. Phys. A553, 493c (1993).
[64] H. Emling, Prog. Part. Nucl. Phys. 33, 729 (1994).
[65] K. Boretzky, et al., Phys. Lett. B384, 30 (1996).
[66] T. Aumann, P. F. Bortignon, H. Emling, Annu. Rev. Nucl. Part. Sci. 48, 351 (1998).
[67] C. A. Bertulani, V. Yu. Ponomarev, Phys. Rep. 321, 139 (1999).
[68] K. Boretzky, et al., Phys. Rev. C68, 024317 (2003).
[69] Y. Li, C. A. Ullrich, Chem. Phys. 391, 157 (2011).
[70] I. Stetcu, C. Bertulani, A. Bulgac, P. Magierski, K.J. Roche, Phys. Rev. Lett. 114 012701 (2015).
[71] M. N. Harakeh and A. van der Woude, Giant Resonances, Oxford University Press, Oxford, (2001).
[72] G.F. Bertsch, P.F. Bortignon, and R.A. Broglia, Rev. Mod. Phys. 55 287 (1973).
[73] P. Ring and P. Schuck, The Nuclear Many-Body Problem, Springer-Verlag, New York, (1980).
[74] J. Terasaki and J. Engel, Phys. Rev. C82 034326 (2010); arXiv:1105.3817v1 (2011).
[75] C. Losa et al., Phys. Rev. C81 064307 (2010).
[76] S. Péru et al., Phys. Rev. C83 014314 (2011).
[77] M. Martini, S. Péru, and M. Dupuis, Phys. Rev. C83 034309 (2011).
[78] M. Stoitsov et al., arXiv:1107.3530v1 (2011).
[79] P. Avogadro and T. Nakatsukasa, Phys. Rev. C 84 014314 (2011).
[80] I. Stetcu, A. Bulgac, P. Magierski, K. J. Roche, Phys. Rev. C 84 051309(R) (2011).
[81] L. Meitner, O.R. Frisch, Nature (London) 143 239 (1939).
[82] N. Bohr, J.A. Wheeler, Phys. Rev. 56 426 (1939)
[83] M. Petersilka, E.K.U. Gross, Laser Physics 9 105 (1999).