EXPONENTIAL UPPER BOUNDS ON THE SPECTRAL GAPS AND HOMOGENEOUS SPECTRUM FOR THE NON-CRITICAL EXTENDED HARPER’S MODEL

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Abstract. For non-critical extended Harper’s model with Diophantine frequency, we establish the exponential decay of the upper bounds on the spectral gaps and prove the spectrum is homogeneous. Especially we give a relationship between the decaying rate and Lyapunov exponent in non-self-dual region.

1. Introduction. The extended Harper’s model (EHM) proposed by D.J. Thouless in 1983 is a model from solid state physics defined by the following quasi-periodic Jacobi operator acting on $\ell^2(\mathbb{Z})$:

$$(H_{\lambda,\alpha,\theta}u)_n = \tilde{c}(\theta + n\alpha)u_{n+1} + c(\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_n$$

where $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the phase, $\alpha \in \mathbb{T}$ is the frequency and

$$\tilde{c}(\theta) := \lambda_1 e^{-2\pi i(\theta + \frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{2\pi i(\theta + \frac{\alpha}{2})},$$

$$c(\theta) := \lambda_1 e^{2\pi i(\theta + \frac{\alpha}{2})} + \lambda_2 + \lambda_3 e^{-2\pi i(\theta + \frac{\alpha}{2})}, \quad \lambda := (\lambda_1, \lambda_2, \lambda_3).$$

We call the constant $\lambda$ coupling constant.

Physically, the extended Harper’s model describes the influence of a transversal magnetic field of flux $\alpha$ on a single tight-binding electron in a 2-dimensional crystal layer. The 2-dimensional electron is permitted to hop to both nearest (expressed through $\lambda_2$) and next-nearest neighboring (NNN) interaction between lattice sites (expressed through $\lambda_1$ and $\lambda_3$). Without loss of generality, one may assume $0 \leq \lambda_2$, $0 \leq \lambda_1 + \lambda_3$ and at least one of $\lambda_1, \lambda_2, \lambda_3$ to be positive. When $\lambda_1 = \lambda_3 = 0$ the model reduces to the well known almost Mathieu operator (AMO) (in the physics literature also known as Harper’s model).

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Just like the Schrödinger case we can study EHM by global theory of quasiperiodic cocycles [7] in the regime of “large” and “small” constants which can be described through purely dynamical properties. By the complexified Lyapunov exponent we have a partition of the spectrum into the following [2]:

1. **Supercritical**, characterized by positive Lyapunov exponent.
2. **Subcritical**, characterized by the Lyapunov exponent vanishing in a strip of complexified phases.
3. **Critical**, characterized as being neither of the two above.

For EHM, depending on the duality transformation \( \sigma : \lambda = (\lambda_1, \lambda_2, \lambda_3) \rightarrow \hat{\lambda} = (\lambda_3, \lambda_2, \lambda_1) \), we could divide the parameter space into three regions as shown in figure 1.

- **Region I**: \( 0 < \lambda_1 + \lambda_3 < 1, 0 < \lambda_2 < 1 \),
- **Region II**: \( 0 < \lambda_1 + \lambda_3 < \lambda_2, 1 < \lambda_2 \),
- **Region III**: \( \max\{1, \lambda_2\} < \lambda_1 + \lambda_3, \lambda_2 > 0 \).

![Figure 1. Parameter partition](image)

We can see from Figure 1 that region I and region II are dual to each other and region III is self-dual region. Jitomirskaya-Marx gave an exact formula for Lyapunov exponent for all coupling constants [27]. The dual behavior in the non-self dual region is similar to the case of the AMO.

In [27], for EHM Jitomirskaya and Marx explicitly computed the complexified Lyapunov exponent (c.f. Section 2.3) for all \( \lambda \) and all irrational \( \alpha \). We can conclude from their results that the type of behavior (subcritical, supercritical, critical) only depends on \( \lambda \) (all the energies in the spectrum share the same behavior) and are independent of \( \alpha \). And one can give the classification precisely by choosing the corresponding cocycle in \( \text{SL}(2, \mathbb{R}) \) [7]:

1. **Supercritical region**: \( I \cup \{\lambda_1 + \lambda_3 = 0, 0 < \lambda_2 < 1\} \),
2. **Subcritical region**: \( II \cup \{\lambda_1 + \lambda_3 = 0, \lambda_2 > 1\} \) and \( III \cap \{\lambda_1 \neq \lambda_3\} \),
Critical region: $L_I \cup L_{II} \cup L_{III}$ and $III \cap \{\lambda_1 = \lambda_3\}$.

For convenience, let $NC$ to be the union of the supercritical region and the subcritical region defined above.

For general Jacobi operators, scholars are mainly interested in their spectral measures and spectrums. While the spectral measure of EHM has been well studied in [7], in this paper, we will focus on the property of its spectrum.

1.1. Upper bounds of spectral gaps. Our first result is to establish exponential decaying upper bounds for the spectral gaps of the EHM with Diophantine frequency. Firstly we introduce the concept of the gaps. It is well known that the spectrum of $H_{\lambda, \alpha, \theta}$, denoted by $\Sigma_{\lambda, \alpha}$, is a compact subset of $\mathbb{R}$, independent of $\theta$ if $\alpha$ is rationally independent. Any bounded connected component of $\mathbb{R} \setminus \Sigma_{\lambda, \alpha}$ is called a spectral gap. By the Gap Labeling theorem [28], each gap corresponds to one integer $k$ in the way that $N(E) = k\alpha \mod \mathbb{Z}$, where $N(E)$ is the integrated density of states (IDS) (one can refer to (4) for its definition). Therefore it is natural to define $G_k := \{E \in \mathbb{R} \setminus \Sigma_{\lambda, \alpha} | N(E) = k\alpha \mod \mathbb{Z}\}$. We also recall that $\alpha \in \mathbb{R}^d$ is called Diophantine (denoted by $\alpha \in DC_d(\gamma, \tau)$) if there exist $\gamma, \tau > 0$ such that

$$\inf_{j \in \mathbb{Z}} |\langle n, \alpha \rangle - j| > \frac{\gamma}{|n|^\tau}. \tag{1}$$

Denote $DC_d := \bigcup_{k \geq 0, \tau > 0} DC_d(\gamma, \tau)$ as the union set. In particular, when $d = 1$, we simplify the above notations as $DC(\gamma, \tau)$ and $DC$ respectively. Once we have these, now we can introduce our precise results:

**Theorem 1.1.** Let $\alpha \in DC$, if $\lambda \in NC$, then there exist constants $r = r(\lambda) > 0$ and $C = C(\lambda, \alpha, r) > 0$, such that

$$|G_k| \leq Ce^{-2\pi r|k|}, \forall k \in \mathbb{Z} \setminus \{0\}. \tag{1}$$

For the non-self dual region, we can have a better control of the exponential decay rate. Jitomirskaya and Marx [27] have calculated that the Lyapunov exponent on the spectrum is zero in both the region $II$ and $III$. In the region $I$ it is given by (c.f. Theorem 2.1):

$$\epsilon_1(\hat{\lambda}) = \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{\max(\lambda_1 + \lambda_3, 1) + \sqrt{\max(\lambda_1 + \lambda_3, 1)^2 - 4\lambda_1\lambda_3}} > 0, \tag{2}$$

where $\hat{\lambda} \in I$. Let $\hat{\lambda} := \begin{cases} \lambda, & \text{if } \lambda \in I \\ \hat{\lambda}, & \text{if } \lambda \in II \end{cases}$ then we have the following:

**Theorem 1.2.** Let $\alpha \in DC$ and $\lambda \in I \cup II$, for any $0 < r < \frac{\epsilon_1(\hat{\lambda})}{2\pi}$, there exists a constant $C = C(\lambda, \alpha, r) > 0$, such that

$$|G_k| \leq Ce^{-2\pi r|k|}, \forall k \in \mathbb{Z} \setminus \{0\}. \tag{3}$$

Considering the estimates of gaps, the investigation may arise from a question that whether the spectrum is a Cantor set for the AMO case. This is so called Ten Martini problem dubbed by Barry Simon after an offer of Mark Kac in 1981 [35]. Many mathematicians have been working on this problem. Finally the problem has been solved by Avila-Jitomirskaya [5]. But there is a more difficult problem known as the Dry Ten Martini problem says that whether $G_k$ is non-empty for any $k \in \mathbb{Z}$. In fact, under the assumption that all gaps are open for AMO, Thouless and his coauthors [38] gave a theoretic explanation of the quantization of the Hall
conductance by Laughlin’s argument i.e., the Hall conductance is quantized whenever the Fermi energy lies in an energy gap. This problem was partly resolved in [12, 32, 5, 6]. Recently, Avila-You-Zhou [8] solved the Dry Ten Martini problem for the non-critical case. For EHM, Han proved all the gaps are open for a weak Diophantine frequency in the non-self dual region [23].

Considering the upper bound estimates of the spectral gaps, it has its own interest. The first result about upper bounds of gaps was given by Moser-Pöschel in [31]. It was proved by them that given any analytic potential $V : T^d \to \mathbb{R}$, $d \geq 2$ and $\omega \in DC_d$, considering the continuous quasi-periodic Schrödinger operator on $L^2(\mathbb{R})$:

$$(L_{V, \omega})(t) = -y''(t) + V(\omega t)y(t),$$

then $|G_k(V)|$ is exponentially small with respect to $|k|$ provided that $|\langle k, \omega \rangle|$ is large enough and $\langle k, \omega \rangle \in \mathcal{R}(k)$ where

$$\mathcal{R}(k) := \{ \langle k, \omega \rangle \in \mathbb{R} : \inf_{j \in \mathbb{Z}} |\langle m - k, \omega \rangle - j| \geq \gamma \frac{1}{|m|^2} \}, \quad \forall m \in \mathbb{Z}^d \setminus \{k\}.$$  

Afterwards Amor [1] proved if the potential is small, the spectral gaps have sub-exponential decaying for any $k \in \mathbb{Z}^d \setminus \{0\}$ with Diophantine frequency. Damanik-Goldstein [13] obtained a stronger result about the upper bounds of spectral gaps: if $V \in C^\omega_r(T^d, \mathbb{R})$, $\epsilon := \sup_{|x| < r} |V(x)|$ is small enough, $|G_k| \leq \epsilon e^{-\frac{C}{r} |k|}$, which means they get an exponential decay of all $G_k$. Furthermore they showed a relationship between the rate of decaying and the analytic radius of $V$. Recently, Leguil-You-Zhao-Zhou [29] got the exponential asymptotic estimation of the gaps for quasiperiodic Schrödinger operator with subcritical potential and Diophantine frequency. Liu-Shi [30] proved that the size of the spectral gaps of quasi-periodic Schrödinger operators with small potential decays exponentially with $0 \leq \beta(\alpha) < \infty$. The definition of $\beta(\alpha)$ can be found in Section 2.1.

Furthermore, Shi and Yuan [34] extended Liu-Shi’s results to EHM, they got the exponential upper bounds of the spectral gaps with $0 \leq \beta(\alpha) < \infty$ in region $I$ and $II$. In this paper, we give a different method based on KAM scheme. We get sharp upper bounds of the spectral gaps in region $I$ and region $II$. We describe that the decaying rate of length of spectral gaps is close to the Lyapunov exponent. This kind of phenomenon was proved for AMO in [26, 29]. We also obtain the upper bounds of the spectral gaps in regime $NC$ relies on a general dynamical result known as almost reducibility theorem (ART) which shows equivalence between subcritical behavior of analytic $SL(2, \mathbb{R})$-cocycles and a certain dynamical property known as almost reducibility. A proof of ART is announced in [2], to appear in [4]. There are two key technical points in our paper. The first one is the global almost reducibility based on almost localization. The second one is quantitative local reducibility. The key point is that in these two kinds of reducibility, we just need to shrink the analytical radius a little. That is why we can get the sharp upper bounds for spectral gaps in region $I$ and $II$.

1.2. Homogeneous spectrum. The upper bound estimates of the spectral gaps provide an efficient way for proving the homogeneity of the spectrum. The concept of homogeneous set was introduced by Carleson [11], and is defined as follows:

Definition 1.1. Given $\mu > 0$, a closed set $S \subset \mathbb{R}$ is called $\mu$-homogeneous if for any $0 < \epsilon \leq \text{diam} S$ and any $E \in S$, we have

$$|S \cap (E - \epsilon, E + \epsilon)| > \mu \epsilon.$$
Here $\cdot$ means the Lebesgue measure for a measurable set. Our main result is:

**Theorem 1.3.** Let $\alpha \in \text{DC}$ and $\lambda \in \text{NC}$, the spectrum $\Sigma_{\lambda,\alpha}$ is $\mu$-homogeneous for some $0 < \mu = \mu(\lambda, \alpha) < 1$.

The concept of homogeneous spectrum was originated in the inverse spectral theory, where people want to find a characterization of potentials under some assumptions of spectral measures or spectrum. It plays an important role in the theory as the fundamental work of Sodin-Yuditskii [36, 37]. Under the assumption of finite total gap length, homogenous spectrum and reflectionless imply the almost periodicity of the corresponding potential [36] and purely absolutely continuous spectrum [22].

The concept of homogeneous spectrum plays an important role in recent study of Deift’s conjecture [18, 19], which asks whether for almost periodic initial datum, the solution of the KDV equation is almost periodic in the time variable. One can consult [20, 39] for the criterion of Deift’s conjecture and consult [9, 14, 16, 29] for recent advances on Deift’s conjecture with small quasi-periodic initial datum.

Let us recall recent results on the homogeneity of the spectrum. For the discrete case, Damanik-Goldstein-Schlag-Voda [17] proved that the spectrum is homogeneous for strong Diophantine $\alpha$ in the positive Lyapunov exponent region. Leguil-You-Zhao-Zhou [29] gave a global description of the homogeneity of the spectrum for quasi-periodic Schrödinger operators. They proved the spectrum is homogeneous for a (measure-theoretically) typical analytic potential. For the continuous Schrödinger operator $L_{V,\omega}$ with multi-frequency $\omega \in \mathbb{R}^d$, it was proved by Damanik-Goldstein-Lukic [15] the spectrum is homogeneous with $\omega \in \text{DC}_d$ and $V$ is sufficient small. Liu-Shi obtained the homogeneity of the spectrum for quasi-periodic Schrödinger operators with small potential and $0 \leq \beta(\alpha) < \infty$ [30]. For EHM, Shi and Jian [33] got the homogeneous with $0 \leq \beta(\alpha) < \infty$ in region II.

Note that the condition in our result $\lambda \in \text{NC}$ is necessary for the homogeneity of the spectrum. From the point of view in [2], the critical energy lies in a Lebesgue zero measure set. So there is no homogeneous spectrum in the critical region.

2. **Preliminaries.** For a given function $f$ defined on a strip $\{|\Im z| < h\}$, we define $|f|_h := \sup_{|\Im z| < h} |f(z)|$. Analogously, for $f$ defined on $\mathbb{T}$, we set $|f|_\mathbb{T} := \sup_{x \in \mathbb{T}} |f(x)|$. For any $f : \mathbb{T} \to \mathbb{C}$, let $[f] := \int_{\mathbb{T}} f(\theta) d\theta$. When $\theta \in \mathbb{R}$, we also set $\|\theta\|_\mathbb{T} := \inf_{j \in \mathbb{Z}} |\theta - j|$.

2.1. **Continued fraction expansion.** Let $\alpha \in \mathbb{T}\setminus\mathbb{Q}$. Suppose $(\frac{p_n}{q_n})_n$ to be the best rational approximations of $\alpha$, we have $\|k\alpha\|_\mathbb{T} \geq \|q_{n-1}\alpha\|_\mathbb{T}$, $\forall 1 \leq k < q_n$, and

$$\frac{1}{2q_{n+1}} \leq \|q_n\alpha\|_\mathbb{T} \leq \frac{1}{q_{n+1}}.$$ 

Denote $\beta(\alpha) := \limsup_{n \to \infty} \frac{\ln q_{n+1}}{q_n}$. Equivalently, we have

$$\beta(\alpha) = \limsup_{n \to \infty} \frac{1}{|n|} \ln \frac{1}{\|n\alpha\|_\mathbb{T}}.$$ 

We can deduce if $\alpha \in \text{DC}$ then $\beta(\alpha) = 0$. 


2. Extended Harper’s model. For any $\psi \in \ell^2(\mathbb{Z})$, let $\mu_{\lambda,\alpha,\theta}^\psi$ be the spectral measure of $H_{\lambda,\alpha,\theta}$ corresponding to $\psi$:

$$
\langle (H_{\lambda,\alpha,\theta} - E)^{-1}\psi, \psi \rangle = \int_{\mathbb{R}} \frac{1}{E - E'} d\mu_{\lambda,\alpha,\theta}^\psi(E'), \quad \forall E \in \mathbb{C} \setminus \Sigma_{\lambda,\alpha}.
$$

We denote $\mu_{\lambda,\alpha,\theta} := \mu_{\lambda,\alpha,\theta}^\psi$, where $\{e_n\}_{n \in \mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. This measure can serve as a universal spectral measure for $H_{\lambda,\alpha,\theta}$ because of the ergodic property.

We introduce another operator $\tilde{H}_{\lambda,\alpha,\theta}$ which is unitary equivalent to $H_{\lambda,\alpha,\theta}$ but its entries are real as follows

$$(\tilde{H}_{\lambda,\alpha,\theta})_{n} = |c|((\theta + n\alpha)u_{n+1} + |c|((\theta + (n-1)\alpha)u_{n-1} + 2\cos 2\pi(\theta + n\alpha)u_{n})$$

with $|c| := \sqrt{c(\theta)c(\theta)}$ and $\tilde{H}_{\lambda,\alpha,\theta} = UH_{\lambda,\alpha,\theta}U^{-1}$, where $U$ is a unitary operator defined as follows:

$$
\langle e_i, Ue_j \rangle = \delta_{ij} \prod_{n=0}^{j-1} e^{2\pi i \arg c(\theta + n\alpha)}.
$$

Thus they share the same spectrum. So if we want to study the property of spectrum (from topology or measure theory) we could study $\tilde{H}_{\lambda,\alpha,\theta}$ instead of $H_{\lambda,\alpha,\theta}$.

The *integrated density of states* (IDS) is the function $N_{\lambda,\alpha} : \mathbb{R} \to [0,1]$ defined by

$$
N_{\lambda,\alpha}(E) = \int_{\mathbb{T}} \mu_{\lambda,\alpha,\theta}(-\infty,E)d\theta.
$$

It is a continuous non-decreasing surjective function. Let $\tilde{N}_{\lambda,\alpha}$ be the IDS of $\{\tilde{H}_{\lambda,\alpha,\theta}\}_{\theta \in \mathbb{T}}$. We have $N_{\lambda,\alpha}(E) = \tilde{N}_{\lambda,\alpha}(E)$ (see [24]).

2.3. Quasi-periodic cocycles. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A \in C^0(\mathbb{T}, M_2(\mathbb{C}))$ be measurable with $\log \|A(x)\| \in L^1(\mathbb{T})$. The quasi-periodic *cocycle* $(\alpha, A)$ is the dynamical system on $\mathbb{T} \times \mathbb{C}^2$ defined by $(\alpha, A)(x, v) = (x + \alpha, A(x)v)$. The iterates of $(\alpha, A)$ are of the form $(\alpha, A)^n = (\alpha, A_n)$, where

$$
A_n(x) := \begin{cases} A(x + (n-1)\alpha) \cdots A(x + \alpha)A(x), & n \geq 0 \\ A^{-1}(x + n\alpha)A^{-1}(x + (n+1)\alpha) \cdots A^{-1}(x - \alpha), & n < 0 \end{cases}.
$$

The *Lyapunov exponent* is defined by $L(\alpha, A) := \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{T}} \ln \|A_n(x)\| dx$.

**Remark 2.1.** By subadditivity, for any compact set $K \subset (\mathbb{R} \setminus \mathbb{Q}) \times C^0(\mathbb{R}/\mathbb{Z}, \text{SL}(2,\mathbb{C}))$, for every $\delta > 0$ there exists $C_{K,\delta} > 0$ such that for every $k \geq 0$,

$$
\sup_{(\alpha, A) \in K} \sup_{x \in \mathbb{R}/\mathbb{Z}} \ln \|A_k(x)\| \leq C_{K,\delta} + k( \sup_{(\alpha, A) \in K} L(\alpha, A) + \delta ) \tag{5}
$$

We say that $(\alpha, A)$ is *uniformly hyperbolic* if for every $x \in \mathbb{T}$, there exists a continuous splitting $\mathbb{C}^2 = E^s(x) \oplus E^u(x)$ such that for every $n \geq 0$,

$$
|A_n(x)v| \leq Ce^{-cn}|v|, \quad v \in E^s(x),
$$

$$
|A_n(x)^{-1}v| \leq Ce^{-cn}|v|, \quad v \in E^u(x + n\alpha),
$$

for some constants $C, c > 0$. This splitting is invariant by the dynamics, i.e.,

$$
A(x)E^s(x) = E^s(x + \alpha), \quad * = \"s\" \text{ or } \"u\", \quad \forall x \in \mathbb{T}.$$
Given two cocycles \((\alpha, A_1), (\alpha, A_2) \in \mathbb{T} \times C^k(\mathbb{T}, \text{SL}(2, \mathbb{R}))\), one says they are \(C^k\) conjugated if there exists \(Z \in C^k(\mathbb{T}, \text{PSL}(2, \mathbb{R}))\) such that
\[
Z(\theta + \alpha)^{-1} A_1(\theta) Z(\theta) = A_2(\theta).
\]

We say \((\alpha, A)\) is \(C^{k', k}\) almost reducible, if \(A \in C^{k'}(\mathbb{T}, \text{SL}(2, \mathbb{R}))\), and the closure of its \(C^k\) conjugacies contains a constant. The notion of analytic almost reducible is the same with \(C^\omega\) instead of \(C^k\) and \(C^{k'}\). We say \((\alpha, A)\) is \(*\) reducible if it is conjugate to a constant cocycle by a \(*\) conjugacy where \(*\) can be \(C^k\) or \(C^\omega\).

Given \(\theta \in \mathbb{T}\), let
\[
R_\theta = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix}.
\]

Any \(A \in C^0(\mathbb{T}, \text{PSL}(2, \mathbb{R}))\) is homotopic to \(x \to R_{\frac{k}{2}x}\) for some \(k \in \mathbb{Z}\) called the degree of \(A\), denoted by \(\text{deg} A = k\).

Assume now that \(A \in C^0(\mathbb{T}, \text{SL}(2, \mathbb{R}))\) is homotopic to the identity. Then there exist \(\phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}\) and \(v : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}^+\) such that
\[
A(x) \begin{pmatrix} \cos 2\pi y \\ \sin 2\pi y \end{pmatrix} = v(x, y) \begin{pmatrix} \cos 2\pi(y + \phi(x, y)) \\ \sin 2\pi(y + \phi(x, y)) \end{pmatrix}.
\]
The function \(\phi\) is called a lift of \(A\). Let \(\mu\) be any probability on \(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\) which is invariant under the continuous map \(T : (x, y) \to (x + \alpha, y + \phi(x, y))\), projecting over Lebesgue measure on the first coordinate. Then the number
\[
\rho(\alpha, A) = \int \phi d\mu \mod \mathbb{Z}
\]
is independent of the choices of \(\phi\) and \(\mu\), and is called the fibered rotation number of \((\alpha, A)\). From definition we can prove directly that

**Lemma 2.1.** If \(A : \mathbb{T} \to \text{SL}(2, \mathbb{R})\) is homotopic to the identity, then
\[
|\rho(\alpha, A) - \theta| < |A - R_\theta|_{\mathbb{T}}.
\]

The fibered rotation number is invariant under real conjugacies which are homotopic to the identity. More generally, if \((\alpha, A_1)\) is conjugated to \((\alpha, A_2)\), i.e., \(B(\cdot + \alpha)^{-1} A_1(\cdot) B(\cdot) = A_2(\cdot)\), for some \(B : \mathbb{T} \to \text{PSL}(2, \mathbb{R})\) with \(\text{deg} B = n\), then
\[
\rho(\alpha, A_1) = \rho(\alpha, A_2) + \frac{n\alpha}{2}.
\]

We consider a typical example associated with extended Harper’s model \(\{H_{\lambda, \alpha, \theta} : \theta \in \mathbb{T}\}\). Any formal solution of \(H_{\lambda, \alpha, \theta} u = Eu\) satisfies
\[
\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = A_{\lambda,E}(\theta + n\alpha) \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}, \quad \forall n \in \mathbb{Z},
\]
where
\[
A_{\lambda,E}(\theta) = \frac{1}{\tilde{c}(\theta)} \begin{pmatrix} E - 2\cos 2\pi \theta & -c(\theta - \alpha) \\ c(\theta) & 0 \end{pmatrix}.
\]

We introduce a normalized cocycle \((\alpha, \tilde{A}_{\lambda,E})\), where
\[
\tilde{A}_{\lambda,E}(\theta) = \frac{1}{\sqrt{|c(\theta)|} |c(\theta - \alpha)|} \begin{pmatrix} E - 2\cos 2\pi \theta & -|c(\theta - \alpha)| \\ |c(\theta)| & 0 \end{pmatrix}
= Q_\lambda(\theta + \alpha)^{-1} A_{\lambda,E}(\theta) Q_\lambda(\theta),
\]
where \(Q_\lambda\) is analytic on \(|\Im \theta| \leq h(\lambda)\) for some \(h\) that is independent of \(E\) when \(\lambda\) belongs to the subcritical region. Especially if \(\lambda \in \mathcal{H}; h(\lambda)\) can be chosen to be \(\frac{h}{2\pi}\) which is defined in (2) (see [24]).
Remark 2.2. When \( \lambda \) belongs to region \( II \), \( c(x) \) is analytic and non-zero on \( |S(x)| < \frac{2\pi}{2\pi} \). Furthermore, the winding number of \( c(\cdot + i\epsilon) \) is equal to zero when \( |\epsilon| < \frac{2\pi}{2\pi} \) (see [24]). So \( c(\theta) \) is analytic and non-zero on \( |S(x)| < \frac{2\pi}{2\pi} \). But when \( \lambda \) belongs to region \( III \) and \( \lambda_1 \neq \lambda_3 \), \( c(x) \) is analytic and non-zero on a small strip where \( c(x)e^{\epsilon x} \in \mathbb{C}\{(-\infty, 0) \} \).

The spectrum of \( \tilde{H}_{\lambda,\alpha,\theta} \) does not depend on \( \theta \) because of the unique ergodicity. The spectral properties of \( \tilde{H}_{\lambda,\alpha,\theta} \) and the dynamics of \((\alpha, \tilde{A}_{\lambda,E})\) are closely related by the well-known fact: \( E \in \Sigma_{\lambda,\alpha} \) if and only if \((\alpha, \tilde{A}_{\lambda,E})\) is not uniformly hyperbolic.

For any fixed \( E \in \mathbb{R} \), the map \( x \rightarrow \tilde{A}_{\lambda,E}(x) \) is homotopic to the identity, hence the rotation number \( \rho(\alpha, \tilde{A}_{\lambda,E}) \) is well defined. The relation between the IDS of \( \{H_{\lambda,\alpha,\theta}\}_{\theta \in \mathbb{T}} \) and rotation number of \((\alpha, \tilde{A}_{\lambda,E})\) yields
\[
\tilde{N}_{\lambda,\alpha}(E) = 1 - 2\rho(\alpha, \tilde{A}_{\lambda,E}).
\]

2.4. Aubry duality. We want to study the supercritical region \( \hat{\lambda} \) where \( \lambda \in region II \) by its dual region. It turns out the spectrum \( \Sigma_{\lambda,\alpha} \) is related to the spectrum \( \Sigma_{\lambda,\alpha} \) of \( H_{\lambda,\alpha,\theta} \) in the following way
\[
\Sigma_{\lambda,\alpha} = \lambda_2 \Sigma_{\lambda,\alpha}. \tag{6}
\]

The IDS \( N_{\lambda,\alpha}(E) \) of \( \{H_{\lambda,\alpha,\theta}\}_{\theta \in \mathbb{T}} \) is equivalent to the IDS \( N_{\lambda,\alpha}(E/\lambda) \) of \( \{H_{\lambda,\alpha,\theta}\}_{\theta \in \mathbb{T}} \).

Assume \( \lambda \) in region \( II \). The decaying rate of spectral gaps has a close relationship with Lyapunov exponent. So we introduce the following Theorem in [27]. For a matrix-valued function \( M(\theta) \), let \( M(\theta) = M(\theta + i\epsilon) \) be the phase-complexified matrix.

**Theorem 2.1.** Extended Harper’s model is supercritical in region \( I \) and subcritical in region \( II \). Indeed
- when \( \lambda \) belongs to region \( II \), \( L(\alpha, A_{\lambda,E}, \epsilon) = L(\alpha, \tilde{A}_{\lambda,E}, \epsilon) = 0 \) on \( |\epsilon| \leq \frac{1}{2\pi} \epsilon_1(\lambda) \),
- when \( \lambda \) belongs to region \( II \), we have \( \hat{\lambda} = \left( \frac{\lambda_1}{\lambda_2}, \frac{1}{\lambda_2}, \frac{\lambda_2}{\lambda_1} \right) \) belongs to region \( I \) and \( L(\alpha, \tilde{A}_{\lambda,E}) = \epsilon_1(\hat{\lambda}), \) where
\[
\epsilon_1(\hat{\lambda}) = \ln \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4\lambda_1\lambda_3}}{\max(\lambda_1 + \lambda_3, 1) + \sqrt{\max(\lambda_1 + \lambda_3, 1)^2 - 4\lambda_1\lambda_3}} > 0. \tag{7}
\]

3. Upper bounds of spectral gaps of EHM.

3.1. Some reducibility results. In this subsection, we introduce some reducibility results. Firstly we introduce Avila’s Almost Reducibility Theorem (ART):

**Theorem 3.1** (Avila.A [4]). Given \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) with \( \beta(\alpha) = 0 \) and \( A \in C^r(\mathbb{T}, \text{SL}(2, \mathbb{R})) \), if \((\alpha, A)\) is subcritical, then it is almost reducible.

By ART we have

**Corollary 3.1.** Let \( \alpha \in DC(\gamma, \tau) \) and \( \lambda \) belong to subcritical region. There exists \( h_1 = h_1(\lambda, \alpha) > 0 \) such that for any \( E \in \Sigma_{\lambda,\alpha} \) satisfying \( 2\rho(\alpha, \tilde{A}_{\lambda,E}) - k\alpha \in \mathbb{Z} \) with \( k \in \mathbb{Z}\{0\} \), for any \( 0 < r < h_1 \), there exists \( U \in C^r(\mathbb{T}, \text{PSL}(2, \mathbb{R})) \) such that
\[
U(\cdot + \alpha)^{-1}\tilde{A}_{\lambda,E}(\cdot)U(\cdot) = \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}. \tag{8}
\]

Moreover, there exist \( C_2, C_3 > 0 \), depending on \( \lambda, \alpha, r \), such that \( |\varphi| \leq C_2 e^{-2\pi r|k|} \) and for any \( r'' \in (0, r] \), \( |U|_{r''} \leq C_3 e^{2\pi r'' |k|} \).
Proof. Note that subcriticality implies almost reducibility by Theorem 3.1. We just need to replace \( S_E^r \) with \( \tilde{A}_{\lambda,E} \) in the proof of Corollary 5.1 in [29]. 

In fact if \( \lambda \in II \), we have even stronger results about the \( h_1 \) in Corollary 3.1. The constant \( h_1 \) is very important because it describes the decay of the size of gaps. The key is to use the following theorem.

**Theorem 3.2.** Let \( \alpha \in \mathbb{R} \) satisfy \( \beta(\alpha) = 0 \). If \( \lambda \in II \), then for any \( r \in (0, \frac{c_1(\lambda)}{2\pi}) \), \( E \in \Sigma_{\lambda,\alpha} \), the following holds on \( \{|z| < r\} \):

1. either \( (\alpha, \tilde{A}_{\lambda,E}) \) is almost reducible to \( (\alpha, R_\theta) \) for some \( \theta = \theta(E) \in \mathbb{R} \). i.e. for any \( \varepsilon > 0 \) there exists \( W \in C_r(\mathbb{T}, \text{PSL}(2,\mathbb{R})) \) such that
   \[
   |W(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E}(\cdot) W(\cdot) - R_\theta|_r < \varepsilon,
   \]

2. or \( (\alpha, \tilde{A}_{\lambda,E}) \) is reducible. i.e. there exists \( W \in C_r(\mathbb{T}, \text{PSL}(2,\mathbb{R})) \) such that
   \[
   W(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E}(\cdot) W(\cdot) = A',
   \]

where \( A' \in \text{PSL}(2,\mathbb{R}) \).

This Theorem is proved by the technique developed in [6]. We get almost reducibility through almost localization. The first quantitative (almost)-reducible estimate for EHM is proved by Han in [23]. But the difference between the technique in this article and [23] is that we focus on the analytic radius in the almost reducibility or the decay rate in almost localization. In the process of constructing conjugate transform we must keep the analytic radius close to Lyapunov exponent.

Then we can use the Theorem 3.2 to replace Lemma 5.1 in [29]. With the same proof as Corollary 3.1, we can get the following

**Corollary 3.2.** Let \( \alpha \in \text{DC}(\gamma, \tau), \lambda \in II \). For any \( E \in \Sigma_{\lambda,\alpha} \) satisfying \( 2\rho(\alpha, \tilde{A}_{\lambda,E}) - k\alpha \in \mathbb{Z} \) with \( k \in \mathbb{Z} \setminus \{0\} \), for any \( 0 < r < \frac{c_1(\lambda)}{2\pi} \), there exist \( U \in C_r(\mathbb{T}, \text{PSL}(2,\mathbb{R})) \) and \( \varphi \in \mathbb{R} \setminus \{0\} \) such that

\[
U(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E}(\cdot) U(\cdot) = \begin{pmatrix} 1 & \varphi \\ 0 & 1 \end{pmatrix}.
\]

Moreover, there exist \( C_4, C_5 > 0 \), depending on \( \lambda, \alpha, r \), such that \( |\varphi| \leq C_4 e^{-2\pi r |k|} \) and for any \( r'' \in (0, r], |U|_{r''} \leq C_5 e^{\frac{2\pi r''}{r} |k|} \).

### 3.2. Criterion for upper bounds of spectral gaps.

Let \( B_h(\mathbb{T}) := \{ \theta + i\varepsilon | \theta \in \mathbb{T}, |\varepsilon| < h \} \). We consider the case that \( \lambda \) belongs to subcritical region. In this region, we can choose \( h = h(\lambda) \) such that \( |\varepsilon| \) is analytic in \( B_h(\mathbb{T}) \). Because \( \lambda_1 \neq \lambda_3 \), we know \( c(\theta) \tilde{c}(\theta) > 0 \) for \( \theta \in \mathbb{T} \). By the continuity of \( c \tilde{c} \) we can always choose such \( h \). Especially, \( h \) could be chosen to be \( \frac{c_1(\lambda)}{2\pi} \) when \( \lambda \in II \). Let

\[
\kappa := \min \left\{ \inf_{z \in B_h(\mathbb{T})} |c(z)|, \quad \inf_{z \in B_h(\mathbb{T})} |\tilde{c}(z)|, \quad \left( \sup_{z \in B_h(\mathbb{T})} |c(z)| \right)^{-1}, \quad \left( \sup_{z \in B_h(\mathbb{T})} |\tilde{c}(z)| \right)^{-1} \right\}.
\]

Then we have \( \kappa = \kappa(\lambda, h) > 0 \).

Note that \( \tilde{A}_{\lambda,E}(\theta) \) belongs to \( \text{SL}(2,\mathbb{R}) \). Based on Moser-Pöschel argument, we will estimate the size of spectral gap \( G_k = (E_k^-, E_k^+) \) via quantitative reducibility of the corresponding Schrödinger cocycle at its edge points.

We know the gaps can only exist where the rotation number \( 2\rho(\alpha, \tilde{A}_{\lambda,E}) = \langle k, \alpha \rangle \) (Gap Labeling). Then we know the cocycle is reducible on the closure of the gaps
i.e. there exist $X \in C^\omega_r(T, \text{PSL}(2, \mathbb{R}))$ for some $0 < r < h$ and a constant $B \in \text{SL}(2, \mathbb{R})$ such that

$$X(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E_k^+}(\cdot)X(\cdot) = B.$$ 

Since $E_k^+ \in \Sigma_{\lambda,\alpha}$ is a right edge point of $G_k$, $B$ has the form that $B = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$ with $0 \leq \zeta < \frac{1}{2}$. We will show that the upper bound is decided by $X$ and $\zeta$. For any $0 < \delta < 1$, a direct calculation yields

$$X(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E_k^+ - \delta}(\cdot)X(\cdot) = B - \delta P,$$

with

$$P(\cdot) = \frac{1}{|c|} \left( X_{11}(\cdot)X_{12}(\cdot) - \zeta X_{11}(\cdot)^2 - \zeta X_{11}(\cdot)X_{12}(\cdot) + X_{12}(\cdot)^2 \right).$$

Obviously,

$$|P|_{r} \leq \kappa^{-1}(1 + \zeta)|X|_{r}^2 < 2\kappa^{-1}|X|_{r}^2. \quad (11)$$

We know the rotation number of $(\alpha, A_{\lambda,E})$ is constant on the gap. Thus $E_k^+ - \delta \in G_k$ is equivalent to

$$\rho(\alpha, X(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E_k^+ - \delta}(\cdot)X(\cdot)) = \rho(\alpha, X(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E_k^+}(\cdot)X(\cdot)) = \rho(\alpha, B) = 0.$$ 

If we find some $\delta > 0$ such that $\rho(\alpha, B - \delta P) \neq 0$, we get $|G_k| < \delta$. For $\alpha \in DC(\gamma, \tau)$, let

$$D_\tau := 2^{4\tau+9} \Gamma(4\tau + 2) \quad (12)$$

where $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ is Gamma function.

In the following, we use one standard KAM step to emphasize the leading term of the cocycle which accounts for the rotation number. The proof could be found in [29], [21] and [1]. Here we omit the proof.

**Lemma 3.1.** Let $\alpha \in DC(\gamma, \tau)$. If $0 < \delta < \kappa D^{-1}_\tau \gamma^3 r^{4\tau+1} |X|_{r}^{-2}$, then there exist $\tilde{X} \in C^\omega_\frac{r}{2}(T, \text{SL}(2, \mathbb{R}))$ and $P_1 \in C^\omega_\frac{r}{2}(T, \text{gl}(2, \mathbb{R}))$ such that

$$\tilde{X}(\cdot + \alpha)^{-1}(B - \delta P(\cdot))\tilde{X}(\cdot) = e^{b_0 - \delta b_1 + \delta^2 P_1(\cdot)} \quad (13)$$

where $b_0 := \begin{pmatrix} 0 & \zeta \\ 0 & 0 \end{pmatrix}$ and

$$b_1 := \begin{pmatrix} \frac{X_{11}X_{12}}{|c|} - \frac{\zeta}{2} \left[ \frac{X_{11}^2}{|c|} \right] \\ -\frac{X_{12}^2}{|c|} \end{pmatrix} - \zeta \left[ \frac{X_{11}X_{12}}{|c|} \right] + \left[ \frac{X_{11}^2}{|c|} \right]$$

with $|c| = |c|(-\alpha)$ and estimates

$$|\tilde{X} - Id|_{\frac{r}{2}} \leq 2\kappa^{-1}D_\tau \gamma^{-3} r^{-4\tau+1} |X|_{r}^{-2}, |P_1|_{\frac{r}{2}} \leq 2\kappa^{-2}D_\tau^2 \gamma^{-6} r^{-2(4\tau+1)} |X|_{r}^4. \quad (14)$$

After the preparation, we can state a theorem which builds a connection between the bounds of gaps and the relative size of $X$ and $\zeta$.

**Theorem 3.3.** Let $\alpha \in DC(\gamma, \tau)$, $\nu \in (0, \frac{1}{2})$ and $\lambda$ belong to subcritical region. Let $E$ be an edge point of the spectral gap $G(E)$. Assume that there are $\zeta \in (0, \frac{1}{2})$ and $X \in C^\omega_r(T, \text{PSL}(2, \mathbb{R}))$ such that

$$X(\cdot + \alpha)^{-1} \tilde{A}_{\lambda,E_k^+}(\cdot)X(\cdot) = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}. \quad (15)$$
Then if we have
\[ |X|^4 \zeta^\nu \leq 10^{-5} \kappa^8 D^{-4} \gamma^{12} r^{4(4\tau + 1)} \] (16)
then \(|G(E)| \leq \zeta^{1-\nu},\) where \(D)\) is the constant defined in (12).

We prove Theorem 3.3 by classic Moser-Pöschel argument. Before proving the theorem we do some technical calculation.

**Lemma 3.2.** For any \(X \in C^\omega(T, \text{PSL}(2, \mathbb{R})), \left| \frac{X^2}{|c|} \right| \geq \frac{1}{4}\kappa^2 |X|^{-2}. \)

**Proof.** Let
\[ u_1(\theta) := \begin{pmatrix} X_{11}(\theta) \\ X_{21}(\theta) \end{pmatrix}, u_2(\theta) := \begin{pmatrix} X_{12}(\theta) \\ X_{22}(\theta) \end{pmatrix} \]

Since \(|\det(X(\theta))| = 1,\) we have \(\|u_1\|_{L^2(T)} \|u_2\|_{L^2(T)} > 1,\) which implies that
\(\|X_{11}\|_{L^2(T)} + \|X_{21}\|_{L^2(T)} \geq \|u_1\|_{L^2(T)} > \|u_2\|_{L^2(T)} \geq |X|^{-1}.\)

By (15), we know \(X_{21}(\cdot + \alpha) = \sqrt{|c|} \left| \frac{X}{c} \right| (\cdot - \alpha) X_{11}(\cdot).\) So \(X_{11} = \|X_{11}\|^2_{L^2(T)} \geq \frac{1}{4}\kappa^2 |X|^{-2}.\) We can finish the proof. \(\square\)

We still need the following key observation for the transformation \(X(\cdot).\)

**Lemma 3.3.** For any \(\nu \in (0, \frac{1}{4}),\) if
\[ |X| \zeta^{\frac{\nu}{2}} \leq \frac{1}{4}\kappa^2 \] (17)
then the following hold
\[ 0 < \frac{|X_{11}|^2}{|c|} - \frac{|X_{11}X_{12}|^2}{|c|} \leq \frac{1}{2} \zeta^{-\nu}, \] (18)
\[ \frac{|X_{11}|^2}{|c|} - \frac{|X_{11}X_{12}|^2}{|c|} \geq 8 \zeta^{2\nu}. \] (19)

**Proof.** Assume (18) does not hold. So \(X_{12}\) and \(X_{11}\) are almost linearly dependent. Let
\[ \sigma := X_{12} - \frac{X_{11}X_{12}}{|X_{12}|} X_{11}, \]
and we have
\[ \left| \frac{\sigma^2}{|c|} \right| < 2 \zeta^\nu \] (20)

By (15), we can get
\[ \sqrt{\frac{|c(\cdot)|}{|c(\cdot - \alpha)|}} (X_{11}(\cdot + \alpha)\sigma(\cdot) - X_{11}(\cdot)\sigma(\cdot + \alpha)) = 1 + \zeta \sqrt{\frac{|c(\cdot)|}{|c(\cdot - \alpha)|}} X_{11}(\cdot + \alpha) X_{11}(\cdot). \]
By the bound of $|c|$ which is $\kappa$ defined in (10), we have
\[
\left\| \sqrt{\frac{|c|}{|c|}} (X_{11}(\cdot + \alpha) \sigma(\cdot) - X_{11}(\cdot) \sigma(\cdot + \alpha)) \right\| 
\leq \kappa^{-1} \left\| \frac{X_{11}(\cdot + \alpha)}{|c|} \sigma(\cdot) \right\| + \kappa^{-\frac{1}{2}} \left\| \frac{X_{11}(\cdot) \sigma(\cdot + \alpha)}{|c|} \right\| 
\leq \kappa^{-1} \left| \frac{X_{11}(\cdot + \alpha)}{|c|} \right| + \kappa^{-\frac{1}{2}} \left| \frac{X_{11}(\cdot) \sigma(\cdot + \alpha)}{|c|} \right| 
\leq \kappa^{-\frac{1}{2}} |X| \sqrt{\alpha^2 + \kappa^{-2} |X|^2} 
\leq \frac{\sqrt{2}}{2}.
\]

The second inequality due to the Hölder inequality with measure $\frac{d\theta}{|\theta - \alpha|}$ and $\frac{d\theta}{|\sigma|}$. The third inequality due to (17) and (20).

On the other hand, $\zeta |\sqrt{\frac{|c|}{|c|}} X_{11}(\cdot + \alpha) X_{11}(\cdot)|_T \leq \frac{1}{10} \zeta^{1-\nu}$, which implies
\[
|1 + \zeta \sqrt{\frac{|c|}{|c|}} X_{11}(\cdot + \alpha) X_{11}(\cdot)| > 1 - \frac{1}{16} \zeta^{1-\nu}.
\]

We reach a contradiction by (21) and (22).

Combing with Lemma 3.2, we get $|X|_{\nu} \geq 4 \zeta^\nu$, which implies (19).

Proof of Theorem 3.3. Let $d(\delta) := \det (b_0 - \delta b_1)$. Because the $\bar{X}$ in Lemma 3.1 is close to the identity, we have
\[
\rho(\alpha, B - \delta P(\cdot)) = \rho(\alpha, e^{b_0 - \delta b_1} + \delta^2 P_1(\cdot)).
\]

So we can see the rotation number is mainly decided by $d(\delta)$. By a direct calculation, we get
\[
d(\delta) = -\delta \left[ \frac{X_{11}^2}{|c|} \right] \zeta + \delta^2 \left[ \frac{X_{11}^2}{|c|} \right] \left[ \frac{X_{12}^2}{|c|} \right] - \left[ \frac{X_{11} X_{12}}{|c|} \right]^2 - \frac{\zeta^2}{4} \left[ \frac{X_{11}^2}{|c|} \right]^2 
= \delta \left[ \frac{X_{11}^2}{|c|} \right] \left[ \frac{X_{12}^2}{|c|} \right] - \left[ \frac{X_{11} X_{12}}{|c|} \right]^2 \left( \delta - \left[ \frac{X_{11}^2}{|c|} \right] \zeta + \frac{\delta}{4} \left[ \frac{X_{11}^2}{|c|} \right]^2 \zeta^2 \right). 
\]

Fix $\nu \in (0, \frac{1}{2})$ and let $\delta_1 = \zeta^{1-\nu}$. Let $\zeta > 0$ satisfies (16). By Lemma 3.1, we can conjugate the system to the cocycle $(\alpha, e^{b_0 - \delta_1 b_1} + \delta_1^2 P_1)$.

In order to show $|G_{k}| \leq \delta_1$, it is sufficient to show that $\rho(\alpha, e^{b_0 - \delta_1 b_1} + \delta_1^2 P_1) > 0$. By (16), one has $|X|_{\nu} \zeta^{\frac{1}{2}} \leq \frac{1}{4} \kappa^2$. Then we apply Lemma 3.3, and get
\[
\left[ \frac{X_{11}^2}{|c|} \right] \zeta + \frac{\delta}{4} \left[ \frac{X_{11}^2}{|c|} \right]^2 \zeta^2 \leq \frac{2}{3} \delta_1.
\]

Hence, for $d(\delta_1) = \det (b_0 - \delta_1 b_1)$, we have
\[
d(\delta_1) \geq \zeta^{1-\nu} \cdot S \zeta^{2\nu} \cdot \frac{1}{3} \zeta^{1-\nu} = \frac{8}{3} \zeta^2.
\]

Following the expressions of $b_0$ and $b_1$, we have
\[
|b_0 - \delta_1 b_1| \leq \zeta + \kappa^{-1} \delta_1 (1 + \zeta) |X|^2 \leq 2 \kappa^{-1} \zeta^{1-\nu} |X|^2.
\]
In view of Lemma 8.1 in [25], there exists $\mathcal{P} \in \text{SL}(2, \mathbb{R})$ with $|\mathcal{P}| \leq 2(e^{0\delta_1} - 1) \frac{1}{\sqrt{d(\delta_1)}}$ such that

$\mathcal{P}^{-1}e^{0\delta_1}\mathcal{P} = R\sqrt{d(\delta_1)}$.

Combining with the estimation of $d(\delta_1)$, we have

$|b_{0\delta_1}| \leq \frac{2\kappa^{-1}\zeta^{1-\nu} |X|^2}{\sqrt{D_\gamma^2}} \leq 2\kappa^{-1} |X|^2 \zeta^{1-\nu}$.

Then, due to Lemma 2.1 and Lemma 3.1,

$|\rho(\alpha, e^{b_{0\delta_1}} + \delta^2 P_1(\cdot)) - \sqrt{d(\delta_1)}| \leq \delta_1^2 |\mathcal{P}|^2 |P_1|_T$

$\leq 16\kappa^{-4} D_\gamma^2 \gamma^{-6} r^{-2(4\tau + 1)} |X|^2 \zeta^{2-2\nu} \leq \zeta^{2-4\nu},$

the last inequality is obtained by the assumption (10). Combining with (23), we have

$\rho(\alpha, e^{b_{0\delta_1}} + \delta^2 P_1) \geq \sqrt{d(\delta_1)} - |\rho(\alpha, e^{b_{0\delta_1}} + \delta^2 P_1)) - \sqrt{d(\delta_1)}| > 0.$

This concludes the proof.

3.3. Upper bounds of the sizes of gaps. Proof of Theorem 1.1. By Aubry duality, it is enough to consider the subcritical case because the fact that $\Sigma_{\lambda, \alpha} = \lambda_2\Sigma_{\lambda, \alpha}$. So we consider the case of $\lambda$ belonging to subcritical region. If $2\rho(\alpha, \hat{A}_{\lambda, \epsilon} - k\alpha, \mathbb{Z}$, then by Corollary 3.1, there exists $h_1 = h_1(\lambda, \alpha) > 0$, for any $0 < \hat{r} < h_1$, there exists $X \in C_{r}^\omega(\mathbb{T}, \text{PSL}(2, \mathbb{R}))$ and we have

$X(\cdot + \alpha)^{-1}\hat{A}_{\lambda, \epsilon}(\cdot)X(\cdot) = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$

with $\zeta \leq C_2(\lambda, \alpha, \hat{r}) e^{-2\pi r |k|}$ and $|X(r)^| | \leq C_3(\lambda, \alpha, \hat{r}) e^{-2\pi \epsilon |k|}$ for any $r'' \in (0, \hat{r})$.

For any $0 < r < h_1$, let $\hat{r} := \frac{1}{2}(r + h_1)$, $\nu := \frac{\hat{r} - r}{\hat{r}}$ and $\nu'' := \frac{h_1 - r}{60}$. By a direct calculation, there exists some $N = N(\lambda, \alpha, r)$ such that if $k > N$, we have

$|X(\nu)^| | \leq C_3^4 C_2^4 e^{-\frac{1}{2} \pi (h_1 - r)|k|} \leq 10^{-5} \kappa \gamma^{-4} r^{4(4\tau + 1)}.$

Hence, by Theorem 3.3, we have

$|G_k| \leq \zeta^{1-\nu} \leq C_2^4 \nu e^{-2\pi r |k|}, \forall |k| > N.$

Let $C_6 := \max\{\max_{|k| \leq N} \{|G_k|, C_2^{1-\nu}\}$, then we get the exponential upper bound for all $k \in \mathbb{Z}\{0\}$.

If $\lambda \in I$, the proof is same as above. One only needs to replace Corollary 3.1 with Corollary 3.2 and $h_1$ with $\frac{\epsilon_1(\lambda)}{2\pi}$ respectively.

4. Homogeneous spectrum. Let us recall the definition of homogeneous spectrum. It describes a phenomenon that the spectrum is not so thin. In other words the gaps are relatively thin. If we want to point out the gaps are thin we just need to say that “two relatively large gaps could not be too close”. So we just need to estimate the relative size between the distance of gaps and the upper bounds of gaps. The idea first appeared in [1]. More precisely, we express this by the following Theorem
Theorem 4.1. Let $H$ be an bounded adjoint operator and $\Sigma$ be the spectrum of $H$. Let $G_0$ be the union of two unbounded gaps. If there exist labels $G_k, k \neq 0$ which labeling the gaps such that the follows hold:

(H1) there exist constants $C_8$ and $r$ such that
$$\text{dist}(G_{k_1}, G_{k_2}) \geq C_8 e^{-r|k_1 - k_2|}, \text{if } k_1 \neq k_2, \quad (25)$$

(H2) there exist constants $C_9$ and $r'$ such that
$$|G_k| \leq C_9 e^{-r'|k|}, \text{ for } k,$$

(H3) $r < \frac{1}{2} r'$.

Then there exists $\mu = \mu(C_8, C_9, r, r', \text{diam}\Sigma) \in (0, 1)$, such that
$$|(E - \epsilon, E + \epsilon) \cap \Sigma| > \mu \epsilon, \text{ for } \forall \ E \in \Sigma, \forall 0 < \epsilon \leq \text{diam}\Sigma.$$

Proof. Given $E \in \Sigma$ and $\epsilon > 0$, let
$$\mathcal{N} = \mathcal{N}(E, \epsilon) := \{k \in \mathbb{Z} : G_k \cap (E - \epsilon, E + \epsilon) \neq \emptyset\},$$
and let $k_0 \in \mathcal{N}$ be such that $|k_0| = \min_{k \in \mathcal{N}}$. Let us first assume that $\epsilon < \min\left\{\frac{1}{4}\text{diam}\Sigma, \epsilon_0\right\}$ where $\epsilon_0 = \epsilon_0(C_8, C_9, r, r')$ such that
$$\frac{32 \cdot 2^{\frac{5}{2}} C_9^2}{C_8^2 r^2} \cdot \frac{2^{-2}}{\epsilon_0} < \frac{1}{4}. \quad (26)$$

Since $E \in \Sigma$, it is easy to see that
$$|G_{k_0} \cap (E - \epsilon, E + \epsilon)| \leq \epsilon.$$

We consider the following two cases.

Case 1.
$$C_9 r'^{-1} e^{-r'|k_0|} \leq \frac{\epsilon}{8} \quad (27)$$

We have
$$\Sigma_{k \in \mathcal{N} \setminus \{k_0\}} |G_k \cap (E - \epsilon, E + \epsilon)| \leq C_9 \Sigma_{|k| \geq k_0} e^{-r'|k|} \leq \frac{\epsilon}{4},$$
by (27). So we have
$$|(E - \epsilon, E + \epsilon) \cap \Sigma| \geq \frac{3}{4} \epsilon, \quad \forall 0 < \epsilon \leq \epsilon_0.$$

Case 2.
$$C_9 r'^{-1} e^{-r'|k_0|} \geq \frac{\epsilon}{8}, \quad (28)$$

Then we let $k_1 \in \mathcal{N}$ be such that $|k_1| = \min_{k \in \mathcal{N} \setminus \{k_0\}}$. By the definition of $\mathcal{N}$, we have
$$\text{dist}(G_{k_1}, G_{k_0}) \leq 2\epsilon.$$
Combining with (H1), we get
$$e^{-r|k_1|} \leq \frac{2}{C_8} \epsilon e^{r|k_0|}.$$

Thus
$$\Sigma_{k \in \mathcal{N} \setminus \{k_0\}} |G_k \cap (E - \epsilon, E + \epsilon)| \leq C_9 \Sigma_{|k| \geq k_1} e^{-r'|k|} \leq \frac{\epsilon}{4},$$
by (26) and (28).

Then we have
$$|(E - \epsilon, E + \epsilon) \cap \Sigma| \geq \frac{3}{4} \epsilon, \forall 0 < \epsilon \leq \epsilon_0.$$
that

From Theorem 4.2 we have there exists

Proof. Lemma 4.1. Let \( \alpha \in \text{DC} \), \( \lambda \in \mathcal{NC} \). Then IDS of \( H_{\lambda,\alpha,\theta} \) is \( \frac{1}{2} \)-Hölder continuous.

Proof. By Aubry duality, it is enough for us to consider the case where \( \lambda \) belongs to subcritical region. We can use the following theorem

**Theorem 4.2** ([10]). Let \( \alpha \in \text{DC}(\gamma, \tau) \), there exists a numerical constant \( D_0 > 0 \), such that if \( (\alpha, A) \) is \( C^{k, k'} \) almost reducible with \( k' > k \geq D_0 \tau \), then for any continuous map \( B : \mathbb{T}^d \to \text{SL}(2, \mathbb{C}) \), we have

\[
|L(\alpha, A) - L(\alpha, B)| \leq C_0 \|B - A\|^\frac{1}{2},
\]

where \( C_0 \) is a constant depending on \( d, \gamma, \tau \).

The cocycle \( (\alpha, \tilde{A}_{\lambda,E}) \) is almost reducible by Theorem 3.1 because it is subcritical. From Theorem 4.2 we have there exists \( C_0 = C_0(\alpha) \) such that for any \( \epsilon > 0 \),

\[
|L(\alpha, \tilde{A}_{\lambda,E+\epsilon}) - L(\alpha, \tilde{A}_{\lambda,E})| < \kappa^{-1} C_0 \epsilon^\frac{1}{2}. \tag{29}
\]

On the other hand, by Thouless formula, there exists an absolutely constant \( c' > 0 \) such that for any \( \epsilon > 0 \),

\[
|L(\alpha, \tilde{A}_{\lambda,E+\epsilon}) - L(\alpha, \tilde{A}_{\lambda,E})| = \frac{1}{2} \int \ln(1 + \frac{\epsilon^2}{(E - E')^2}) dN_{\lambda,\alpha}(E') \geq c'(N_{\lambda,\alpha}(E + \epsilon) - N_{\lambda,\alpha}(E - \epsilon)).
\]

Combing the last estimate with (29), we deduce that

\[
N_{\lambda,\alpha}(E + \epsilon) - N_{\lambda,\alpha}(E - \epsilon) \leq \kappa^{-1} c'^{-1} C_0 \epsilon^\frac{1}{2},
\]

for \( E \in \Sigma_{\lambda,\alpha} \). Since \( N_{\lambda,\alpha} \) is locally constant on the complement of \( \Sigma_{\lambda,\alpha} \), we have that \( N_{\lambda,\alpha} \) is \( \frac{1}{2} \)-Hölder continuous.

**Proof of Theorem 1.3.** By Aubry duality, it is enough for us to consider the subcritical case. We just need to verify (H1), (H2) and (H3) in Theorem 4.1. (H2) holds with \( r' = \frac{1}{2} h_1 \) and \( C_9 = C_9(\lambda, \alpha, r') \) where \( h_1 \) and \( C_6 \) are given in the proof of Theorem 1.1. For two different gaps \( G_{k_1}(\lambda), G_{k_2}(\lambda) \), without loss of generality, we assume that \( E_{k_1}^+ \leq E_{k_2}^- \). Hence

\[
\text{dist}(G_{k_1}(\lambda), G_{k_2}(\lambda)) = E_{k_2}^- - E_{k_1}^+.
\]

On the other hand, the \( \frac{1}{2} \)-Hölder continuity of \( N_{\lambda,\alpha} \) implies

\[
|N_{\lambda,\alpha}(E_{k_2}^-) - N_{\lambda,\alpha}(E_{k_1}^+)| \leq c_1(\lambda, \alpha)(E_{k_2}^- - E_{k_1}^+) \frac{1}{2}.
\]

Because \( \alpha \in \text{DC}(\gamma, \tau) \) there exists \( \tilde{C} = \tilde{C}(\alpha, r') > 0 \) such that

\[
|N_{\lambda,\alpha}(E_{k_2}^-) - N_{\lambda,\alpha}(E_{k_1}^+)| \geq \|(k_1 - k_2, \alpha)\|_\tau \geq \tilde{C} e^{-\frac{r'}{\tau}|k_1 - k_2|}.
\]
Combining with the above estimations, we conclude that
\[ \text{dist}(G_{k_1}(\lambda), G_{k_2}(\lambda)) \geq \left( \frac{C}{\epsilon_1} \right)^2 e^{-\omega[|k_1 - k_2|]}, \]
which satisfying (H1), (H3). The proof is finished.

5. Proof of Theorem 3.2.

5.1. Almost localization in region I.

**Definition 5.1.** Let \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), \( \theta \in \mathbb{R} \), \( \epsilon_0 > 0 \). We say that \( k \) is an \( \epsilon_0 \)-resonance of \( \theta \) if \( \|2\theta - k\alpha\| \leq e^{-\epsilon_0k} \) and \( \|2\theta - k\alpha\| = \min_{|l| \leq |k|} \|2\theta - l\alpha\| \).

**Definition 5.2.** Let \( 0 = |n_0| < |n_1| < \ldots \) be the \( \epsilon_0 \)-resonances of \( \theta \). If this sequence is infinite, we say \( \theta \) is \( \epsilon_0 \)-resonant, otherwise we say it is \( \epsilon_0 \)-non-resonant.

Firstly, we proof an almost localization lemma:

**Lemma 5.1.** Let \( \lambda \) be in region II. \( \epsilon_1 = \epsilon_1(\hat{\lambda}) \) be Lyapunov exponent of \((\alpha, A_{\lambda, E})\). Then for any \( 0 < \delta < \epsilon_1 \), there exist constants \( \epsilon_0 = \epsilon_0(\lambda, \delta) \), \( J = J(\lambda, \alpha, \delta) \) and \( \lambda_1 = \lambda_1(\lambda, \delta) \) so that for every solution \( u \) of \( \hat{H}_{\lambda, \alpha, \theta}u = Eu \) satisfying \( u(0) = 1 \) and \( |u_j| \leq 1 + |j| \) if \( A_1(|n_j| + 1) < |j| < \frac{1}{6}|n_{j+1}| + 1 \) and \( j > J \), then \( |u_j| \leq e^{-\epsilon_1(\lambda, \delta)|j|} \). Where \( \{n_j\} \) are the \( \epsilon_0 \)-resonances of \( \theta \). If \( \theta \) is \( \epsilon_0 \)-non-resonant, for \( |j| > A_1(\max\{|n_j|, J\} + 1) \) we have \( |u_j| \leq e^{-\epsilon_1(\lambda, \delta)|j|} \).

Before proving the lemma, we want to do some statements to outline the process. Suppose \( u \) is a solution of \( \hat{H}_{\lambda, \alpha, \theta}u = Eu \). For an interval \( I = [j_1, j_2] \), let \( \Gamma_I \) be the coupling operator between \( I \) and \( \bar{I} \):

\[
\Gamma_I(i, j) = \begin{cases} 
\tilde{c}(\theta + (j_1 - 1)\alpha), & (i, j) = (j_1, j_1 - 1), \\
c(\theta + (j_1 - 1)\alpha), & (i, j) = (j_1 - 1, j_1), \\
\tilde{c}(\theta + j_2\alpha), & (i, j) = (j_2 + 1, j_2), \\
c(\theta + j_2\alpha), & (i, j) = (j_2, j_2 + 1), \\
0, & \text{otherwise}.
\end{cases}
\]

Let \( H_I \) be the restriction of the operator \( \hat{H}_{\lambda, \alpha, \theta} \) to \( I \). Then we have
\[ 0 = (H - E)u = (H_I \oplus H_{\bar{I}} + \Gamma_I - E)u. \]

Let \( P_I : \ell^2(\mathbb{Z}) \to \mathbb{C}^{|I|} \) be a projection operator. For any \( j \in I \) we have
\[ u(j) = -[(H_I - E)^{-1} P_I \Gamma_I u](j) = \tilde{c}(\theta + (j_1 - 1)\alpha)G_I(j, j_1)u(j_1 - 1) + c(\theta + j_2\alpha)G_I(j, j_2)u(j_2 + 1). \] (30)

We can estimate \( u(j) \) from the above formula. If entries in \( G_I \) decay along minor diagonal exponentially, we get results we need. We call such an interval “regular” box. We give a precise definition in the below:

**Definition 5.3.** A point \( j \in \mathbb{Z} \) is called \((k, m)\)-regular if there exists an interval \([j_1, j_2]\) containing \( j \), where \( j_2 = j_1 + k - 1 \) such that
\[ |G_I(j, j_i)| \leq e^{-m|j - j_i|} \quad \text{and} \quad \text{dist}(j, j_i) \geq \frac{1}{3}k \quad \text{for} \quad i = 1, 2, \]
otherwise \( j \) is called \((k, m)\)-singular.
Although not every interval is regular, the singular boxes are sparse. In fact in this situation the singular intervals are mutually exclusive. By (30) and \( u(0) = 1 \) we can know for fixed \( m \) there exists \( k_0 = k_0(m) \) such that for \( k > k_0 \), 0 is \((m,k)\)-singular. Thus we can hope \( j \) not so close to 0 is regular. Notice \( \eta_{t+1} \gg \eta_t \). Fix \( 0 < \xi < \eta_0 \). Due to \( \beta(\alpha) = 0 \) there exists \( \tilde{N} = \tilde{N}(\xi) \) such that for \( n > \tilde{N} \) we have \( \|n\alpha\| > e^{-\xi n} \).

Thus

\[
2e^{-4\xi \lceil n_{t+1} \rceil} \leq C_\xi e^{-2\xi \lceil n_{t+1} \rceil} \leq \|(n_{t+1} - n_t)\alpha\|
\]

\[
= \|(n_{t+1} \alpha - 2\theta) - (n_t \alpha - 2\theta)\| \leq 2\|2\theta - n_t \alpha\| \leq 2e^{-\xi \lceil n_{t+1} \rceil},
\]

we have \( |n_{t+1}| \geq \frac{C_\xi}{2} |n_t| \).

Let \( \{q_i\}_{i \geq 1} \) be the sequence of denominators of best approximations of \( \alpha \). We associate with any integer \( 3(|n_t| + 1) \leq |j| \leq \frac{1}{3} |n_{t+1}| \) scales \( l \geq 0 \) and \( s \leq 1 \) such that

\[
2sq_i \leq \left\lfloor \frac{|j|}{4} \right\rfloor < \min\{2(s + 1)q_i, 2q_{i+1}\}, \tag{31}
\]

Firstly, we give a lemma which state the sparsity of singular boxes.

**Lemma 5.2.** For any \( 0 < \eta < \epsilon_1 \), there exist \( \epsilon_0 = \epsilon_0(\eta) \) and \( j_0 = j_0(\alpha, \eta) \) such that for \( 3(|n_t| + 1) \leq |j| \leq \frac{1}{3} |n_{t+1}| \) and \( |j| > j_0 \), \( j \) is \((6s(j)q_l(j) - 1, \epsilon_1 - \eta)\)-regular. If \( \theta \) is \( \epsilon_0\)-non-resonant, for \( |j| > 3(\max\{|n_l|, j_0\} + 1) \) we have \( j \) is \((6s(j)q_l(j) - 1, \epsilon_1 - \eta)\)-regular.

**Proof.** Following the proof of Lemma 3.2 in [23] with \( \epsilon_0 = \frac{n}{2007} \xi = \frac{\epsilon_0}{2007} \) where \( C_4 \) is a absolute constant, we get the result. \( \Box \)

Now, we are ready to prove Lemma 5.1. Our basic ideal is to combine many small regular boxes to be a large regular box. Then we can get the result.

**Proof of Lemma 5.1.** First of all, we set some parameters. Let \( \frac{\delta}{2} \) be the \( \eta \) in Lemma 5.2. We can get \( \epsilon_0 = \epsilon_0(\delta) \), \( j_0 = j_0(\alpha, \eta) \). We set \( A_1 = 100\epsilon_1\delta^{-1} \) and \( \xi = \frac{\epsilon_0}{2007} \).

Then there exists \( j_1 = j_1(\xi) \) such that if \( n_{t+1} > j_1 \) then \( \frac{|n_{t+1}|}{|n_t|} > 7A_1 \). Then \( A_1 |n_t| + 1 < j < \frac{1}{6} |n_{t+1}| \) is reasonable.

Next we construct a map

\[
I : \{ j \in \mathbb{Z} : 3(|n_t| + 1) \leq |j| \leq \frac{1}{3} |n_{t+1}| \} \to I(j),
\]

where \( I(j) \) is a regular box with length \( 6s(j)q_l(j) - 1 \). We can always find such a box because \( j \) is \((6s(j)q_l(j) - 1, \epsilon_1 - \frac{\delta}{2})\)-regular. Let \( \delta = I(j) = [I_-(j), I_+(j)] \). We can know from the construction of \( s(j) \) and \( q_l(j) \) that \( \frac{\delta}{8} \leq \text{dist}(j, \partial I) \leq \frac{3\delta}{8} \). We set \( \zeta := \frac{\delta}{8\epsilon_1} \), \( N := \max\{2(\log \frac{\xi}{\zeta} + 1), 9\} \).

Now we state a process to estimate \( u(j) \) where \( A_1(|n_t| + 1) < |j| < \frac{1}{6} |n_{t+1}| \). Let \( P(j) \) be the right side of (30). i.e.

\[
P(j) = \tilde{c}\theta + (I_-(j) - 1)\alpha G_{I_+(j)}(j, I_-(j))u(I_-(j) - 1) + c(\theta + I_+(j)\alpha)G_{I_+(j)}(j, I_+(j))u(I_+(j) + 1).
\]
We can see each term of \( P(j) \) consists of \( c, \tilde{c}, G_I \) and \( u \). Note that \( |G_I(j, I_{-}(j))| < e^{-(\epsilon_1 - \frac{2}{3})|j|} \) (so as \( I_+(j) \)). Let \( P_1 = P(j) \). Assume we have gotten \( P_i \) whose terms consist of \( c, \tilde{c}, G_I \) and \( u \), we going to \( P_{i+1} \):

**Step 1.** Select all \( u \)-terms appear in \( P_i \), we get \( \{u(m_k)\}_{k=1}^{\infty} \)

**Step 2.** If \( k \) satisfies \( 3(|n_k| + 1) \leq |m_k| \leq \frac{1}{3}|n_k+1| \), replace \( u(m_k) \) by \( P(m_k) \); otherwise \( u(m_k) \) remain unchanged.

After the two steps, we get a new formula \( P_{i+1} \) that consists of \( c, \tilde{c}, G_I, u \). And \( P_i \) and \( P_{i+1} \) have same value. Let us consider \( P_N \). Each term in \( P_N \) has the form like

\[
\prod_{i=1}^{r+1} |\hat{c}(\theta + z'_i)G_I\hat{z}(j_i, z_i)G_I(z'_i)\cdots G_I(z'_r, z_{r+1})u(z'_{r+1})|.
\]

Where \( \hat{c} = c \) or \( \hat{c} = \tilde{c} \), \( r \leq N \), \( |z_i - z'_i| = 1 \). Either \( z'_{r+1} \notin \{j \in \mathbb{Z} : 3(|n_k| + 1) \leq |j| \leq \frac{1}{3}(|n_k+1|) \} \) or \( r = N \). There are at most \( 2^N \) terms in \( P_N \). Let \( C_\lambda : = |c|_{C^0} \). So we have the following estimation

\[
|P_N| \leq 2^N C_\lambda e^{-(\epsilon_1 - \frac{2}{3})(|i - z_i| + \sum_{i=1}^{r} |z'_i - z_i|)} u(z'_{r+1})
\]

\[
\leq C(\lambda, \delta) |j| e^{-(\epsilon_1 - \frac{2}{3})(1 - \zeta)|j|} (|z'_{r+1}| < 2^N |j|)
\]

\[
\leq C(\lambda, \delta) |j| e^{-(\epsilon_1 - \frac{2}{3})|j|} (|j| > N\zeta)
\]

\[
\leq e^{-(\epsilon_1 - \frac{2}{3})|j|} (|j| > 16\delta^{-2}(\ln(C(\lambda, \delta) + 1)).
\]

Then we choose \( J = \max\{N\zeta, 16\delta^{-2}(\ln(C(\lambda, \delta) + 1), j_0, j_1\} \). The proof is finished.

\[ \square \]

5.2. Almost reducibility in region II. Fix \( 0 < r < \frac{\epsilon_1}{2\pi}, 0 < \epsilon \), and set \( \bar{r} := \frac{1}{2}(\frac{\epsilon_1}{2\pi} + r), \eta := \frac{(\epsilon_1 - 2\pi r)}{1000} \). By Theorem 3.3 in [6] we know that for any \( E \in \Sigma_{\lambda, \alpha} \), there exist \( \theta = \theta(E) \) and \( \{\tilde{u}_j\}_{j \in \mathbb{Z}} \) satisfying \( H_{\lambda, \Theta, \alpha} \tilde{u} = \tilde{E} \tilde{u} \) with \( u_0 = 1 \) and \( |\tilde{u}_j| \leq 1 \) for every \( j \in \mathbb{Z} \). We can get \( \epsilon_0 = \frac{\delta}{\pi \bar{r}} \), \( C_4 \) is the absolutely constant that appeared in the proof of Lemma 5.2), \( A_1, J \) by Lemma 5.1 with \( \delta = \frac{1}{2}(\epsilon_1 - 2\pi r) \). Notice that \( \epsilon_0 = \epsilon_0(\lambda, \alpha, r), A_1 = A_1(\lambda, r) \) and \( J = J(\lambda, \alpha, r) \).

We consider the case that \( \theta \) is \( \epsilon_0 \)-resonant. Then we denote by \( \{n_i\} \) the resonances of \( \theta \). For \( |n_i| > J \) we have

\[
|\tilde{u}_j| \leq e^{2\pi \delta |j|}, \quad A_1 |n_i - 1| < |j| < \frac{1}{6} |n_i|.
\]

Denote \( m := |n_i - 1|, M := |n_i| \). We can select a \( l \) such that

\[
|m| \geq \max\left\{ \frac{|12| |\ln \epsilon|}{\epsilon_1 - 2\pi \bar{r}}, \frac{\epsilon_0 \eta}{4\lambda_1 \epsilon_1} \right\}
\]

where \( \tilde{N} = \tilde{N}(\xi) \) is a integer depending on \( \xi \) and \( \alpha \) such that for \( n > \tilde{N} \) we have \( |n\alpha| > e^{-\xi} \). Then we can know \( \frac{A_1 |n_i - 1|}{\eta} \leq \eta \). Let \( I := [\frac{|\eta|}{\eta} + 2, [\frac{|\eta|}{\eta}] - 2] = [x_1, x_2] \). We define \( u^l : z \to \sum_{j \in I} \tilde{u}_j e^{ijz} \) and \( U : z \to e^{2\pi i \theta u^l(z)} (u^l(z - \alpha)) \). Then

\[
A_{\lambda, E}(z) U(z) = e^{2\pi i \theta} U(z + \alpha) + \left( \begin{array}{c} g(z) \\ 0 \end{array} \right),
\]
for some analytic function $g^*$ whose Fourier coefficients are possibly nonzero only at four points $x_1, x_2, x_1 - 1$ and $x_2 + 1$. By (32), we have

$$|g^*|_r \leq 4e^{-\frac{c(\theta - \alpha)}{\sqrt{\theta(\theta - \alpha)}}}.$$  

(34)

**Lemma 5.3** ([6]). Let $l \geq 1$ and $1 \leq p \leq [q_{l+1}/q_l]$. If $P$ has essential degree at most $pq_l - 1$ and $x_0 \in \mathbb{T}$, then for some absolute constant $K_0 > 0$,

$$\|P\|_{C^0} \leq K_0 q_{l+1} \sup_{0 \leq m \leq pq_l - 1} |P(x_0 + ma)|.$$  

Let $\tilde{U}(z) = Q_\lambda(z)U(z)$, $\tilde{G}(z) = Q_\lambda(z + \alpha)G(z)$ where $G(z) = \begin{pmatrix} g^*(z) \\ 0 \end{pmatrix}$. Here we show the construction of $Q_\lambda$ which was constructed in [24]. By the Lemma 3.2 in [24], there exists function $f$ analytic in $|3\theta| < \frac{\pi}{2\theta}$ such that

$$\frac{c(\theta)}{|\theta|} = e^{f(\theta + \alpha) - f(\theta)}.$$  

Then we have

$$Q_\lambda(\theta) = \frac{e^{-f(\theta)}}{\sqrt{|\theta - \alpha|}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{1 - \frac{c(\theta - \alpha)}{c(\theta)}}}.$$  

**Lemma 5.4.** For arbitrarily $\eta > 0$, we have

$$\inf_{|3z| \leq r} |\tilde{U}(z)| \geq e^{-2\eta m},$$  

for $m > m(\alpha, \lambda, r, \eta)$.

**Proof.** We prove the result by contradiction. Suppose for some $z_0 \in \{|3z| \leq r\}$ we have $\|\tilde{U}(z_0)\| < e^{-2\eta m}$. We let $sq_k$ be minimal such that $sq_k > A_1 m$ and $sq_k - 1 < q_k + 1$ where $s, k \in \mathbb{N}$. Let $K := [-\frac{sq_k}{2}, sq_k - 1 - \frac{sq_k}{2}]$. Define $\tilde{U}^K$ as

$$\tilde{U}^K = Q_\lambda \left( e^{2\pi i \theta} u^K(z) \right).$$  

Since

$$\max\{\|Q_\lambda(x)\|_{\frac{1}{2\theta}}, \|Q_{\lambda^{-1}}\|_{\frac{1}{2\theta}}\} \leq C(\lambda),$$  

we have $\|\tilde{U} - \tilde{U}^K\|_r \leq 2C(\lambda)e^{-\frac{c(\theta - \alpha)}{\sqrt{\theta(\theta - \alpha)}}} \leq e^{-c(\lambda, \alpha, r)m}$ for $m > m(\lambda, \alpha, r)$ where $c(\lambda, \alpha, r) > 100\eta$. Just like before we have

$$\tilde{A}_{\lambda, E}(z) \tilde{U}^K(z) = e^{2\pi i \theta} \tilde{U}^K(z + \alpha) + \tilde{H}(z)$$  

where $\|\tilde{H}\|_r \leq C(\lambda)e^{-\frac{c(\theta - \alpha)}{\sqrt{\theta(\theta - \alpha)}}} \leq e^{-c(\lambda, \alpha, r)m}$ for $m > m(\lambda, \alpha, r)$. Note that for any $l \in \mathbb{N}$,

$$e^{2\pi i \theta} \tilde{U}^K(z_0 + la) = \tilde{A}_l(z_0) \tilde{U}^K(z_0) - \sum_{s=1}^l e^{2\pi i (s-1)\theta} \tilde{A}_{l-s}(z_0 + sa) \tilde{H}(z_0 + (s-1)\alpha).$$  

With consideration of (5) and $L(\alpha, \tilde{A}_{\lambda, E, r}) = 0$ for $|\epsilon| \leq \frac{\pi}{2\theta}$, this implies for $m > m(\eta)$ large enough and for any $0 \leq l \leq m$

$$\|\tilde{U}^K(z_0 + la)\| \leq C(\lambda)e^{2\eta m} (e^{-2\eta m} + e^{-c(\lambda, \alpha, r)m}) \leq e^{-\eta m},$$  

for $m > m(\lambda, \alpha, r)$. We have $\|u^K(z_0 + la)\| \leq C(\lambda)e^{-\eta m}$. By Lemma 5.3, we have

$$\|u^K(x + i\beta z_0)\|_{C^0} \leq K_0 C(\lambda)e^{2\eta m} e^{-\eta m}.$$  

For $m > m(\lambda, \alpha, r)$ such that $q_{k+1} < e^{4q_k}$, we have $\|u^K(x + i\beta z_0)\|_{C^0} < 1$. This contradicts $\tilde{u}(0) = 1$. \qed
Lemma 5.5 ([23]). For some $C > 0$ depending on $\lambda$ and $\alpha$,
\[ \| \tilde{A}_k \|_2 \leq C(1 + |k|)^C. \]

Let $B(x)$ be the matrix with columns $\tilde{U}(x)$ and $\overline{U(x)}$. We want to prove $B(x)$ is invertible and control the bound of $B(x)^{-1}$ in $|\Re x| < r$ in the following. Let $L^{-1} = \| 2\theta - n_{l-1} \alpha \|$. Notice that $e^{-\xi M} < L^{-1} \leq e^{-r_{0}m}$.

Lemma 5.6. For $B(x)$ defined above and $m > m(\lambda, \alpha, r)$, we have
\[ \inf_{x \in \mathbb{R}/\mathbb{Z}} | \det B(x) | \geq L^{-5C} \]
where $C$ is the constant appeared in Lemma 5.5.

Proof. Recall that for any $2 \times 2$ complex matrix $M$ with columns $V$ and $W$,
\[ | \det M | = \| M \| \min_{\lambda \in C} \| W - \lambda V \|, \quad (35) \]
with the minimizing $\lambda$ satisfying $\| \lambda V \| \leq \| W \|$.

Now we prove the Lemma 5.6 by contradiction. Suppose there exist some $x_0$ and $\lambda_0$ such that $\| e^{-\pi in_{l-1}x_0} \tilde{U}(x_0) - \lambda_0 e^{-\pi in_{l-1}x_0} \tilde{U}(x_0) \| \leq L^{-4C}$. Note that
\[ \| e^{2\pi i \theta} \tilde{U}(x_0 + s\alpha) - \tilde{A}(x_0) \tilde{U}(x_0) \| \leq \sum_{k=1}^{s} | \tilde{A}_{k-1}(x_0 + k\alpha) \tilde{G}(x_0 + (k - 1)\alpha) | \leq C(1 + |s|)^{C+1} e^{-\pi(s-r)M}. \]

Then we have
\[ \| e^{-\pi in_{l-1}x_0 + s\alpha} \tilde{U}(x_0 + s\alpha) - e^{2\pi i \theta} e^{-\pi in_{l-1}x_0} \lambda_0 \tilde{U}(x_0 + s\alpha) \| \leq L^{-C} \quad 0 \leq s \leq L, \quad (36) \]
for $m > m(\lambda, \alpha)$ such that $\xi < \frac{\pi(s-r)}{2L\xi}$. Combining with $L^{-1} = \| 2\theta - n_{l-1} \alpha \|$ we have
\[ \| e^{-\pi in_{l-1}(x_0 + s\alpha)} \tilde{U}(x_0 + s\alpha) - \lambda_0 e^{-\pi in_{l-1}(x_0 + s\alpha)} \tilde{U}(x_0 + s\alpha) \| \leq L^{-C} + 2sL^{-1} \| \tilde{U} \|_0 \leq L^{-\frac{1}{2}}, \quad 0 \leq s \leq L^{\frac{1}{2}}. \]

The last inequality is due to the fact that $\| \tilde{U} \|_0 \leq C(\lambda) \frac{A_{l}}{r}$. Let $pq_k > 10A_1m$ be minimal with $pq_k - 1 < q_{k+1}$. Set $K := [-[pq_k/2], pq_k - 1 - [pq_k/2]]$ and $\tilde{U}(x) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Define $\tilde{U}^{K} := \begin{pmatrix} u^K_1 \\ u^K_2 \end{pmatrix}$. We have $\| \tilde{U} - \tilde{U}^{K} \|_0 \leq e^{-3A_1rm}$ by (32).

Using Lemma 5.3, we have
\[ \| e^{-\pi in_{l-1}x} \tilde{U}^{K}(x) - e^{-\pi in_{l-1}x} \lambda_0 \tilde{U}^{K}(x) \|_0 \leq K_{0} e^{\xi K_{0}} 20A_1m \left( L^{-1/3} + 2e^{-3A_1rm} \right), \]
where $m > m(\alpha, \xi)$ such that $q_{k+1} < e^{\xi q_k}$. Let $\kappa := \frac{1}{2} \min \{ \frac{\xi}{\xi}, 3A_1r \}$. If $m > m(\lambda, \alpha, r)$ such that $\xi < \frac{\xi}{2}$ we can get
\[ \| e^{-\pi in_{l-1}x} \tilde{U}(x) - e^{-\pi in_{l-1}x} \lambda_0 \tilde{U}^{K}(x) \|_0 \leq e^{-\kappa m}. \]

Substituting $x_1 = x_0 + so$ in (36) and taking $s = [L/2]$, we get
\[ \| ie^{-\pi in_{l-1}x_1} \tilde{U}(x_1) + ie^{\pi in_{l-1}x_1} \lambda_0 \tilde{U}(x_1) \| \leq L^{-C} + 2L^{-1} \| \tilde{U} \|_0. \]
Thus we can get $\hat{U}(x_1) \leq e^{-2km}$. We use Lemma 5.4 with $\eta = \kappa$. With $m > m(\lambda, \alpha, r, \kappa(\lambda, \alpha, r))$ we get a contradiction. Combining with (35) we get the result.

**Lemma 5.7.** Let $x_0 \in T$. Then for $m > m(\lambda, \alpha, r)$ we have

$$
\sup_{|\Delta z| < r} |\det B(z) - \det B(x_0)| \leq e^{-c(\lambda, r)M}.
$$

**Proof.** By construction of $B(z)$ and (34) we have

$$
\hat{A}_{\lambda,E}(z)B(z) = \begin{pmatrix} e^{2\pi i \theta} & 0 \\
0 & e^{-2\pi i \theta} \end{pmatrix} B(z + \alpha) + (\hat{G}(z), \overline{G(z)}),
$$

where $\|\hat{G}\|_\infty \leq C(\lambda)e^{-\frac{1}{2}z^{2\pi \alpha}}$. Thus we get

$$
|\det B(x_0 + \alpha) - \det B(x_0)| \leq 2\|\det B\|_0|\hat{G}|_0 \leq e^{-\frac{1}{2}z^{2\pi \alpha}},
$$

for $m > m(\lambda, \alpha, r)$. Let $pq_k > 4M$ be minimal with $pq_k - 1 < q_{k+1}$. Set $K := [-|pq_k/2|, pq_k - 1 - |pq_k/2|]$ and $f(x) := \det B(x) - \det B(x_0)$. By $\|f\|^{\pi(r+\varphi)}$ and $\|\hat{G}\|_\infty \leq C(\lambda)e^{-\frac{1}{2}z^{2\pi \alpha}}$ we know $\|f - f^K\|_0 \leq e^{-2\pi r(1-\eta)}$. By Lemma 5.3,

$$
\|f^K\|_0 \leq K_0e^{K_08M}e^{-\frac{1}{2}z^{2\pi \alpha}M} \leq e^{-\frac{1}{2}z^{2\pi \alpha}M}
$$

for $m$ is large enough such that $\xi$ is small. So $\|f\|_0 \leq e^{-\frac{1}{2}z^{2\pi \alpha}M}$ for $m > m(\lambda, \alpha, r)$. By the Hadamard three-circle theorem,

$$
\ln \sup_{|\Delta z| = \delta \frac{\pi}{\varphi}} |f(z)| \leq (1 - \delta) \ln \sup_{|\Delta z| = \frac{\pi}{\varphi}} |f(z)| + \delta \ln \sup_{|\Delta z| = \frac{\pi}{\varphi}} |f(z)|,
$$

We can get $\|f\| \leq e^{-c(\lambda, r)M}$ with $c(\lambda, r) = \frac{(1 - 1/\varphi)^2}{4\alpha}$.

**Proof of Theorem 3.2.** Let $S := \Re \hat{U}$ and $T := -\Im \hat{U}$. Let $W_1(x)$ be the matrix with columns $S$ and $T$. We have $B = W_1 \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. Notice $\Re \det B(x) = 0$ for $x \in T$.

If $\theta$ is $\epsilon_0$-resonance, we choose $m$ satisfying (33) and larger than all the conditions appeared in the Lemma 5.4, 5.6, 5.7 and $m > m(\lambda, \alpha, r)$ such that $\|n\alpha\|_{\mathbb{R}/\mathbb{Z}} \geq e^{-\xi n}$ for any $n \geq m$ where

$$
\xi \leq \min\{\frac{c_0\eta}{20A_1c_1C(\lambda, \alpha)}, \frac{c(\lambda, \alpha)}{2}\}.
$$

Then we have $\inf_{|\Delta z| < \tau} |\Delta \det B(z)| \geq e^{-\eta M}$ by Lemma 5.6, 5.7. So $(\det W_1)^{\frac{1}{2}} = (\frac{\det B}{2\eta})^{\frac{1}{2}}$ is analytic on $|\Delta z| < \tau$. Let $W(x) = \frac{1}{(\det W_1)^{\frac{1}{2}}}W_1$ we have

$$
\|W(\cdot + \alpha)^{-1} \hat{A}_{\lambda,E}W(\cdot) - R_\theta\| \leq e^{-\frac{1}{2\varphi}z^{2\pi \alpha}M} \leq \epsilon.
$$

If $\theta$ is $\epsilon_0$-non-resonant, by Lemma 5.1 we can get $\hat{u}_j \leq e^{-2\pi r^{[j]}}$ for $|j| > M$ where $M = M(\lambda, \alpha, r)$. Define $u : z \rightarrow \Sigma_j \hat{u}_j e^{2\pi r^{[j]}z}$ and $U : z \rightarrow (e^{2\pi i \theta} u(z))$. Then

$$
\hat{A}_{\lambda,E}(z) \hat{U}(z) = e^{2\pi i \theta} \hat{U}(z + \alpha),
$$

where $\hat{U}(z) = Q_\lambda(z)U(z)$. Let $B$ be the matrix with columns $\hat{U}(z)$ and $\overline{U(z)}$. So $B(z)$ is analytic in $|\Delta z| \leq \frac{\tau}{2\pi}$ and we have $\det B(\cdot + \alpha) = \det B(x)$. We can get $\det B(z)$ is constant.
If $\det B \neq 0$, let $W(x) = \frac{1}{\sqrt{-2\pi \cdot \det B}} B \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$, then we have

$$W(x + \alpha)^{-1} A_{\lambda,\epsilon} W(x) = R_\theta.$$ 

If $\det B = 0$, then if we denote $\tilde{U} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we can get $u_1(x) = \arg u_2(x)$ for $x \in \mathbb{T}$. Denoting $W_2(x) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ we have $\tilde{U}(x) = \phi(x)W_2(x)$ where $\phi \in C(\mathbb{T}, \mathbb{C})$ and $|\phi(x)| = 1$ for $x \in \mathbb{T}$. Let $M(x)$ be the matrix with columns $W_2(x)$ and $\frac{1}{w_1^2 + w_2^2} R_z W_2(x)$. Then $M \in C(\mathbb{T}, \text{SL}(2,\mathbb{R}))$ and we have

$$\tilde{A}_{\lambda,\epsilon}(x) M(x) = \hat{M}(x + \alpha) \begin{pmatrix} d(x) & \tilde{r}(x) \\ 0 & d(x)^{-1} \end{pmatrix},$$

where $d(x) = \frac{e^{2\pi i \phi(x+\alpha)}}{\phi(x)}$, $|d(x)| = 1$ and $d(x)$ being a real number. Without loss of generality, we assume $d(x) = 1$. So $\phi(x) = e^{2\pi ikx}$ where $\theta + k\alpha = 0$. We can see $W_2(x)$ has an analytic extension to $|\Im x| \leq \frac{1}{2\pi}$ because $W_2(x) = e^{-2\pi ikx} \tilde{U}(x)$. But $w_1^2 + w_2^2$ may have zeroes in the strip. So we need to use the following Lemma.

**Lemma 5.8 ([3]).** Let $V : \mathbb{T} \to \mathbb{C}^2$ be analytic in $|\Im z| < \eta$. Assume that $\delta_1 < \|V(z)| < \delta_2^{-1}$ holds on $|\Im z| < \eta$. Then there exists $M : \mathbb{T} \to \text{SL}(2,\mathbb{C})$ analytic on $|\Im z| < \eta$ with first column $V$ and $\|M\|_\eta \leq C\delta_1^{-2}\delta_2^{-1}(1 - \ln(\delta_1\delta_2))$.

By (37) we know $W_2(z) \neq 0$ in $|\Im z| < r$. Then we can find $(W_2, W_2') \in C_\eta^\omega(\mathbb{T}, \text{SL}(2,\mathbb{C}))$ by Lemma 5.8. Let $\hat{M}(x) := (W_2(x), \frac{1}{2}(W_2'(x) + \overline{W'_2(x)})) \in C_\eta^\omega(\mathbb{T}, \text{SL}(2,\mathbb{R}))$. We have

$$\tilde{A}_{\lambda,\epsilon}(x) \hat{M}(x) = \hat{M}(x + \alpha) \begin{pmatrix} 1 & \tau(x) \\ 0 & 1 \end{pmatrix}.$$ 

Since $\tau(x) \in C_\eta^\omega(\mathbb{T}, \mathbb{C})$ we can find $\psi(x) \in C_\eta^\omega(\mathbb{T}, \mathbb{C})$ such that

$$-\psi(x + \alpha) + \psi(x) + \tau(x) = \int_\mathbb{T} \tau(x) dx := \tau.$$ 

This implies

$$\begin{pmatrix} 1 & -\psi(x + \alpha) \\ 0 & 1 \end{pmatrix} \hat{M}^{-1}(x + \alpha) \tilde{A}_{\lambda,\epsilon}(x) \hat{M}(x) \begin{pmatrix} 1 & \psi(x) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}.$$ 

Let $W(x) = \hat{M}(x) \begin{pmatrix} 1 & \phi(x) \\ 0 & 1 \end{pmatrix}$. The proof is finished.

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