Stability of a Charged Particle Beam in a Resistive Plasma Channel

S. J. Han

P.O. Box 4684, Los Alamos, NM 87544-4684
e-mail: sjhan@cybermesa.com

Abstract

A self-focusing of a coasting relativistic beam in a plasma channel that is confined by an external magnetic field is studied as a means of reconditioning the beam emerging from a beam injector [a radio frequency quadrupole (RFQ)] for a linac. A detailed study of the beam stability in the self-focused beam has been carried out. In order to explain beam filaments and the resistive hose instability in a unified way, we treat all the azimuthal modes in the derivation of the dispersion relation in a finite plasma channel that exhibit many unstable modes, which are classified by Weinberg’s scheme [Steven Weinberg, J. Math. 8, 614 (1967)]. To overcome the energy requirement of a beam injector for a high-current, high energy linac, we suggest to add an energy booster of a compact synchrotron to the RFQ. The analysis is then applicable to the charged particle beam transport in a proton accelerator, such as Large Hadron Collider (LHC) at CERN or APT at LANL.

PACS numbers:

* This study was supported by LDRD fund and was circulated as a technical report (LA-UR-93-2146) at Los Alamos.
I. INTRODUCTION

The application of electron beam transport in a cold plasma has been slow, but has made great strides in recent years for a beam focusing device \[6\], laser-guided beam transport problems \[7\], and heavy ion fusion \[8\]. In these applications, the \( m = 1 \) hose instability, if it occurs, displaces the beam position from the center of beam propagation. In both a beam focusing device and heavy ion fusion, this can lead to the displacement of the focal point, which may cause the loss of the beam to the wall of the device. Furthermore, instabilities with higher modes (e.g., \( m \gg 1 \)) can lead to beam filament.

Weinberg \[24, 25\] studied a relativistic beam instability with perturbations on a particle orbit which treats all \( m \) (azimuthal quantum number) values to account for beam deformation, and examines various types of unstable modes in the entire frequency range.

Resistive instabilities of a charged particle beam penetrating into a plasma channel were studied first by Longmire \[1\]. They are complex phenomena and are still not well understood for realistic situations. The hose (\( m = 1 \) mode) instability was studied by a number of authors \[1, 2, 3\]. They have shown that a relativistic electron beam (REB) penetrating into a cold plasma becomes unstable to the \( m = 1 \) mode. However, their analysis remains valid for a background plasma whose skin depth (\( c^2/4\pi\sigma\omega \))\(^{1/2} \) much larger than the beam radius.

Since then a significant progress has been made, the REB propagating in neutral plasmas has been studied extensively by many authors \[4\]. Because of the mathematical complexity of the problem, it has been necessary for many authors to adopt simple models which are valid for specific problems, so that the applicability of models remains limited. In general there are serious difficulties in comparing theoretical analysis of an instability with experiments (e.g., either a z-pinch or a plasma produced by a gas discharge, which is then confined by a magnetic field). We incorporate a feasible experimental condition of interest in the theory and compare with experiment. In particular, we are interested in the feasibility of applying the theory of a beam-focusing (i.e., reconditioning) in a plasma channel, which is expected to be far more effective than the conventional quadrupole magnet focusing in accelerator technology. The basic idea of the technique involves the self-pinching of a high-current of charged particles by self-magnetic fields \[5\]. However, the self-pinched beam is subject to numerous instabilities associated with particle motion as described below.

The previous analysis of beam instability was carried out with a model of a beam pene-
trating into a cold plasma which is infinite in spatial extent for mathematical convenience. However, this approach can not be applied to the self-focusing problems for which the background plasma must be finite. This permits one to impose the boundary condition and thereby to obtain the dispersion relation that gives qualitative information on the stability of the REB. Moreover, it does not apply to the beam focusing by a plasmas in a drift tube confined by an external axial magnetic field in accelerator technology, nor does it tell how the boundaries of a finite plasma channel are formed.

A similar situation occurs in an ion-beam transport in the accelerator technology. The excessive space-charge is the origin of the beam divergence in a high-current, high-energy proton linac. One important new problem arises in this connection, however, is how do we overcome the space-charge effect in the absence of a energy booster before injecting the beam into a linac, which has not been addressed in the previous work. The principal difficulty encountered in the development of the high-current proton accelerator is that a high-current beam must reach a critical energy to overcome the space-charge effect. This difficulty occurs in a situation for any high-current, high-energy proton accelerator with the direct application of a radio frequency quadrupole (RFQ) as a beam injector [10, 13, 15].

The RFQ by Kapchinski and Teplyakov [13] makes use of a strong focusing with rf-electrical field, based on the same principle as is used in a quadrupole mass spectrometer, and is basically a homogeneous transport channel with additional acceleration. By the geometrical modulation of quadrupole electrodes, one generates the axial field. Thus the RFQ has a linac structure which accelerates and focuses the beam with the same rf fields. However, the very fact that such axial acceleration is feasible by the geometrical modulation brings with it new difficulties which seem to be formidable.

Yet the acceleration by the RFQ is independent of beam velocity with a large radial acceptance, which is a great advantage in the design of a proton linac. Thus the devise offers the possibility of utilizing it as a beam injector for a high-current linac. In order to make use of it as an alternative to dc acceleration where a beam injector is required, it is necessary to demonstrate the possibility of reaching energies to about 7Mev in the RFQ before injecting the proton beam for a high-current (80mA) linac such as Large Hadron Collider (LHC) at CERN as shown in Figure 1. This prerequisite is determined by solving the equations of motion in the presence of self-field in a quadrupole structure in a linac [9]. The energy requirements impose additional restrictions on the feasibility of the LHC at
The requirements have been appreciated for some time now because of our inability to invent a new device that overcome the beam energy requirement for a beam injector for a high-current, high-energy accelerator; however it was not difficult to overcome the limitation for a low-current accelerator such as the proton accelerator at Los Alamos (LAMPF).

From a theoretical point of view, the reason for this limitation can be understood, to a great extent, with the observations that the beam focusing force by a quadrupole magnet in a linac is globally a second-order effect, and the geometrical requirements of a high focusing and a large magnetic aperture cannot be achieved simultaneously for a diverging high-current proton beam [9, 21].

There are several review papers on the progress of RFQ [15] and its wide application in many laboratories. The maximum attainable beam energy still remains about the same in spite of intense effort to overcome the limitation posed by a high-current, high-energy accelerator.

For example, we give here the numbers for a figure of merits for the LHC planned at CERN [19]; with 100 mA input to the RFQ, 80 mA proton beam was accelerated to 520 KeV by RFQ, of which 65 mA proton beam was accelerated through the first linac tank. Subsequent linac tank cannot accelerate the high current beam and loses a significant amount of the beam as shown in Fig. 1 which was obtained by solving Eqs. 13 a-b of Courant, Livingston and Snyder [9]. It is this fundamental limitation of RFQ that makes it exceedingly difficult to build a high-current, high-energy accelerator such as LHC at CERN. In general, the higher the current density of a proton beam, the higher the injection energy will be in order for quadrupole magnets in a proton linac to transport the beam with a sufficient focusing force.

The calculation of the above energy requirement is most simply carried out by introducing the condition that the particles remain in oscillating orbits in one direction in the presence of self-fields and quadrupole magnetic fields with a given field gradient:

\[
\frac{d^2 y}{dx^2} + (K^2 - K_1^2)y = 0,
\]

where \((K^2 - K_1^2) \geq 0\), \(K^2 = (dB_z/dy)/(BR)\), \(K_1^2 = \frac{2\pi^2 m_0}{e^2 p^2 M}\) with \(\gamma \approx 1.0\), \(dB_z/dy = 3.94 \times 10^3 Gauss/cm\) [9], and \(P_\perp = 3.0 \times 10^{-4} BR(Gauss - cm)\) [22].

The Figure 1 shows the critical beam injection energy for a given current density, and shows the domain of beam stability. The occurrence of beam divergence due to the space
charge effect is the origin of the beam divergence which results from our attempt to accelerate a high-current beam in a linac, since the focusing force by a quadrupole magnet is not sufficiently strong to overcome the space-charge, which poses the fundamental limitation of the current accelerator technology [14, 15].

Although the demonstration of successful operation of RFQ in low-current proton accelerator at Los Alamos (LAMPF) has raised a hope of developing a high-energy, high-current proton accelerator, the fundamental difficulty in the development of a high-energy, high-current accelerator is remains unsolved; that the high-current beam diverges (bursts) as it emerges from RFQ in a linac is a real challenge to overcome. This difficulty occurs even if the best possible design parameters of RFQ to overcome the Coulomb repulsion in the beam are employed [15]. Yet the problem of space-charge of a high-current beam was the original motivation of Kapchinski and Teplyakov’s concept of RFQ as a possible beam injector for a linac.

One important new possibility that arises in this connection, however, is a utilization of self-focusing by a neutral plasma channel which has not been treated in the previous papers. The existence of the current limitation in the RFQ has made it necessary to study the possibility of space-charge compensated beam transport [16] in the development of a high-current, high-energy proton accelerator such as a large hadron collider (LHC) at CERN.

An important question one should ask: is there any method that can be applied to reconditioning the perturbed beam emerging from RFQ before injecting it to a linac? A well-known application of space-charge compensated beam [16] is an effective, useful concept to overcome the problem of space-charge effect which is the major stumbling block in a high-current beam transport. The decisive advantage of this approach was demonstrated in a high-current electron accelerator by shielding the space-charge with a quiescent plasma [6]. The focusing force may reach to the value, that is greater than that of a super-conducting magnet by several order. Here we assume the presence of an over-dense plasma in which the plasma density is much higher than that of a beam. Thus the Coulomb repulsive force due to space-charge in the beam is balanced by the self-field of a bunched beam maintaining a constant radius. However, the effectiveness of the space-charge neutralized beam transport depends on $\beta = v/c$. Consequently, since the beam emerging from RFQ is in non-relativistic domain ($\beta = v/c \ll 1$), the feasibility of a beam reconditioning is out of the question. This technique can be applied only to a charged particle beam with the velocity $\beta = v/c \geq 1$ by
the combined use of the synchrotron as the energy booster.

For some time it has been realized that it might not be possible to make use of the RFQ as a beam injector for a high-current high-energy linac \[15\] \[19\]. The simplest and probably the best, way of accelerating the high-current proton beam is to apply the phase-locking method in the synchro-cyclotron \[17\] \[18\], but it will be difficult to efficiently extract the high-density beam. Yet it seemed feasible to attempt the experiment with a combination of a synchrotron with the RFQ, instead of the betatron injection \[20\], to accelerate the beam to the relativistic domain, in spite of the fact that the beam is being lost in the extraction process. Thus the role of the RFQ is promising as a pre-injector for a high-current, high-energy accelerator provided that the beam emerging from the RFQ can be reconditioned by a plasma channel described below. In PHERMEX facility at Los Alamos, the high-energy γ-rays are produced by short pulsed high-energy electron beam. To make a uniformly diverging x-ray source, it is essential to have a highly focused beam, a pencil beam, that can be achieved only by a plasma focusing.

Finally, the dependence of $\beta = v/c$ on the effectiveness of space-charge compensated beam transport arises from the form of self magnetic filed which is the primary focusing force. Since the stability due to beam modulation has been studied in a plasma channel \[24\] and a somewhat more detailed study of the model has been already made by Weinberg himself \[25\], here we study the beam reconditioning with the assumption that the beam is accelerated somehow to the relativistic domain in which there is no beam modulation and is coasting in a plasma channel confined by an external magnetic field. In this process the beam is being reconditioned and is to be transferred to a linac in the next stage of acceleration \[19\]. As we shall see, this simple model calculation presents a rather formidable mathematical challenge which defines the ultimate fate of LHC at CERN.

We shall be interested here primarily in the stability of the relativistic beam, in particular, the hose instability of a coasting relativistic beam in a plasma channel confined by the external magnetic field since this is the one which may lead to the loss of charged particles before the beam is injected to the main linac. An elegant formalism of beam instability is provided by the use of the Lagrangian displacement vectors which have been introduced by Bernstein, Frieman, Kruskal and Kulsrud in their study of a Rayleigh type energy principle for hydro-magnetic stability problems \[29\] and is applied to the Boltzman-Vlasov equation for plasmas by Low \[32\]. The treatment given by these authors has shown that the method
is indeed very powerful in treating stability problems. Here we make use of the concept of Lagrangian displacement vectors \[29\] to give an alternative, more mathematically elegant method leading to Weinberg’s results \[25\] in the limit of vanishing external magnetic field \[36\].

The present model is essentially equivalent to that of Weinberg \[25\], but is different in one major respect, that is, we incorporate the external magnetic field to confine the necessary plasma inside the drift tube, so that we may impose proper boundary conditions. The space-charge compensated beam transport is essentially the same for both the high-current electron rf linac (ILC) and the high-current proton linac (LHC), although it is far more useful to overcome the difficulty of focusing the beam in a proton linac.

With these qualitative remarks as an introduction, we proceed to the development of the space-charge compensated beam transport in linac. We therefore focus on the stability of the REB penetrating into a plasma channel confined by a weak external axial magnetic field (\(i.e., B_{ext}^0 \approx 1.0 KG\)), consistent with the experiment \[6\]. Since we introduce perturbations in particle orbits which are determined by the external magnetic field and the self-field, it is important to include all the relevant fields in the equations of motion of a particle.

It is the purpose of this paper to present the essence of our results and some of the details of how the dispersion relation is obtained. The present work is a generalization of Weinberg’s work on the general resistive instability \[25\]. To make the discussion of our model clearer, we derive the dispersion relation by Weinberg \[25\] by taking the limit \(\omega_c = 0\) which implies that no external magnetic field applied to the system. This will illustrate the versatility of the formal application of the Lagrangian displacement vectors in the study of beam instability. It is shown here that, even in the presence of an axial guide magnetic field, the REB remains unstable to various perturbations. These include the \(m = 1\) hose instability and the set of the higher modes that are unstable to various perturbations (type A, B, \(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\), and \(\mathcal{D}\)).

II. BASIC EQUATIONS

There are two avenues of approach to this problem. One is to pursue the approach of Weinberg \[25\] and consider the the first-order effects by the external magnetic field. Though some details of his theory are open to criticism, there is no doubt that the paper points to
the right direction for this difficult problem. However I must confess that I have been unable to follow the early part, "Introduction", of his paper. A brief study of Weinberg’s paper has convinced me to take a more formal approach that is transparent. Indeed it is much more preferable mathematically to apply a straightforward technique to a complex perturbation problem of a many-body system. Moreover, without a confining external magnetic field, Weinberg’s model does not allow one to impose the necessary boundary conditions that are essential to derive his dispersion relations, Eq. 1-9 [25] and is therefore inconsistent with his object of the problem that he set out to solve. Hence the object of this paper is also to establish what is correct in Weinberg’s analysis, and to remove some of ambiguities.

To obtain a proper dispersion relation, it is first necessary to specify the unperturbed motion of an electron in the beam. The REB is assumed to move in the $z$-direction with the average velocity $v$. The self-field is then given by

$$B_s = \frac{4\pi e\beta}{r} \int_0^r r'n(r')dr',$$

where $n(r)$ is the particle density of the REB.

Then the pinch effect can be described by the force (MHD) equation

$$\frac{dp}{dr} = \frac{d}{dr} \left( \frac{B_s^2}{8\pi} \right) + \frac{B_s^2}{4\pi r},$$

where $p$ is the hydrodynamic pressure on the beam. The solution of Eq. (3) is then

$$p(r) = p_0 - \frac{1}{8\pi} \int_0^r \frac{1}{r^2} \frac{d}{dr}(r^2 B_s^2)dr,$$

with the axial pressure $p_0$

$$p_0 = \frac{1}{8\pi} \int_0^R \frac{1}{r^2} \frac{d}{dr}(r^2 B_s^2)dr.$$

Hence Eq. (4) can be written

$$p(r) = \frac{1}{8\pi} \int_r^R \frac{1}{r^2} \frac{d}{dr}(r^2 B_s^2)dr.$$  

In addition we define

$$\alpha^2 \equiv \frac{4\pi e^2\beta^2}{\gamma M} \int_0^r r'n(r')dr',$$

which is equal to $\alpha^2 \equiv \omega^2_\beta = e\beta/(\gamma M)(B_s/r)$, where $(r^2\alpha^2)' = (\beta^2r)\omega^2_p$ and $\omega^2_p = 4\pi e^2n/(\gamma M)$. 

8
The rotational motion of a coasting beam can be described by the equation of motion in the presence of an external and a self magnetic fields:

\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) (v_{\theta} \hat{e}_\theta + v_z \hat{e}_z) = \frac{e}{\gamma M} (\beta \times \mathbf{B}^{ext} + \beta \times \mathbf{B}_s),
\]

where \( \omega_c = eB^{ext}/(\gamma Mc) = ecB^{ext}/E \) and \( v_z = \text{constant} \).

One can easily show from Eq. (8) that the angular frequencies are given by

\[
\omega_\pm = \frac{1}{2} [ -\omega_c \pm \sqrt{\omega_c^2 + 4\alpha^2} ].
\]

Hence the transverse motion of a particle in the polar coordinates can be described by the equation:

\[
\ddot{r} = \dot{r} \times \omega_c - \alpha^2 r,
\]

where \( \omega_c = eB^{ext}/(\gamma Mc) \). Here \( \gamma \) and \( M \) are the Lorentz factor and the mass of a charged particle respectively.

If we now define the rotational velocities as \( \mathbf{u}_\pm = \dot{r} - \omega_\pm e_z \times r \), then we can show that \( (\omega_+ - \omega_-) r = -e_z \times (\mathbf{u}_+ - \mathbf{u}_-) \), which describes a uniform circular motion of an electron with frequencies \( \omega_\pm \) in the \( r-\theta \) plane. Furthermore, by taking the Landau gauge \( \mathbf{A} = \frac{1}{2} B_0^{ext} \times \mathbf{r} \), the canonical momentum becomes \( \mathbf{p} = \gamma M (\mathbf{u}_+ + \mathbf{u}_-)/2 \). The rotational velocities are not a constant of motion; but the canonical momentum \( \mathbf{p} \) can be related to the Hamiltonian of the system \[23\] and hence we employ \( \mathbf{p} \) in the derivation of a dispersion relation.

If a beam is in a relativistic motion in the presence of electric and magnetic fields, then its translational motion of a particle in a fluid element must be in a manifestly covariant form \[22\]:

\[
\frac{d\beta}{dt} = \frac{e}{\gamma Mc} [ \mathbf{E} + \beta \times \mathbf{B} - \beta (\beta \cdot \mathbf{E}) ],
\]

where \( \beta = \mathbf{v}/c \).

Next for a uniform beam with density profile,

\[
n(r) = \begin{cases} 
n & \text{if } r < a \\
0 & \text{if } r > a,
\end{cases}
\]

the rotation frequency is given by

\[
\alpha^2(r) = \begin{cases} 
\omega_\beta^2 & \text{if } r < a, \\
\omega_\beta^2 \alpha^2/r^2 & \text{if } r > a.
\end{cases}
\]

We often simply write for a uniform beam \( \alpha^2 \equiv \omega_\beta^2 = 2\pi e^2 n\beta^2/\gamma M \) if there is no ambiguity.
III. FIRST-ORDER EQUATIONS

The treatment of the first-order equations of motion in Weinberg’s paper did not appear to be in the most convenient form which is difficult to follow through [25]. In fact it is almost impossible to introduce an external magnetic field in a formal analysis in Weinberg’s approach, which is necessary for a systematic study of the beam instabilities in a finite plasma channel.

A. Calculation of Perturbed Fields

In this section, first we write down the first-order field equations from the Maxwell equations assuming that all variables vary as \( f_i(r) \exp(i m \theta + i k z - i \omega t) \):

\[
q^2 E_{1r} = -\frac{m \omega}{r c} B_{1z} + i k \frac{\partial E_{1z}}{\partial r} - 4\pi i \frac{\omega}{c^2} J_{1r}, \quad (14a)
\]

\[
q^2 E_{1\theta} = -\frac{m}{r} k E_{1z} - i \frac{\omega}{c} \frac{\partial B_{1z}}{\partial r} - 4\pi i \frac{\omega}{c^2} J_{1\theta}, \quad (14b)
\]

\[
q^2 B_{1\theta} = i \frac{\omega}{c} \frac{\partial E_{1z}}{\partial r} - k \frac{m}{r} B_{1z} - 4\pi i \frac{k}{c} J_{1r}, \quad (14c)
\]

\[
q^2 B_{1r} = \frac{m \omega}{r c} E_{1z} + i k \frac{\partial B_{1z}}{\partial r} + 4\pi i \frac{k}{c} J_{1\theta}, \quad (14d)
\]

where \( q^2 = \omega^2/c^2 - k^2 \).

Second from Eqs. (14) we obtain the decoupled field equations:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_{1z}}{\partial r} \right) - \frac{m^2}{r^2} E_{1z} + q^2 E_{1z} = \frac{4\pi \omega}{c^2 (k^2 + q^2)} \left[ -i q^2 J_{1z} + i \frac{m}{r} k J_{1\theta} + \frac{k}{r} \frac{\partial}{\partial r} (r J_{1r}) \right], \quad (15a)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial B_{1z}}{\partial r} \right) - \frac{m^2}{r^2} B_{1z} + q^2 B_{1z} = -\frac{4\pi}{r c} \left[ -i m J_{1r} + \frac{\partial}{\partial r} (r J_{1\theta}) \right]. \quad (15b)
\]

Here we shall examine in detail the first-order equations of motion by introducing the displacement vector in a particle orbit \( \xi(r_0, t) \), where \( r_0 \) describes the unperturbed trajectory of a charged particle (an electron or a proton) and \( t \) is the time. Here the displacement \( \xi \) is defined by the equation \( r = r_0 + \xi(r_0, t) \). Upon introducing the Lagrangian displacement vector \( \xi \), it is possible to expand the velocity in terms of \( \xi \). This makes it simple to derive the first-order equations of motion in the presence of an external magnetic field and the perturbed fields. In particular, we must modify the equation of motion for a fluid element that moves with relativistic speed in the presence of electric and magnetic fields as described in Eq. (11).
Next we expand the velocity to the first-order in $\xi$ defined by $r = r_0 + \xi$, limiting to the fast mode for the time being,

$$v(r_0 + \xi) = v_0 + v_1(\xi),$$

(16)

where $v_1$ is given by the following equation.

$$v_1 = \frac{\partial \xi}{\partial t} + v_0 \cdot \nabla \xi,$$

$$= -i(\Omega - m\omega_+)\xi + \omega_+ (\xi r - \xi \theta),$$

(17)

where we have limited to the fast mode $\omega_+$ in Eq. (9) and will repeat the same calculations later for the slow mode $\omega_-$ to check algebras.

We then derive the first-order equation of motion from Eqs. (11) and (14) after a short algebra:

$$\frac{\partial^2 \xi}{\partial t^2} + 2v_0 \cdot \nabla \frac{\partial \xi}{\partial t} + (v_0 \cdot \nabla)(v_0 \cdot \nabla)\xi = \frac{e}{\gamma M} [E_1 + \beta \times B_1 + \beta_1 \times B_0^{ext}$$

$$+ \beta_1 \times B_0^s + \beta \times (\xi \cdot \nabla)B_0^s - \beta(\beta \cdot E_1)],$$

(18)

where $\beta = v/c$ and the subscript 1 denotes the first-order.

### B. Calculation of Perturbed Density

To calculate the perturbed current density, we must still find a way of expressing the perturbed beam density in terms of displacement vectors. A simple alternative method to that of Weinberg [25] is to linearize the equation of continuity:

$$\frac{\partial n}{\partial t} + \nabla (nv) = 0,$$

(19)

with care on $x \to x + \xi$ and $n(x + \xi, t) = n_0 + (\xi \cdot \nabla)n_0 + n_1$ and then to pick up the first-order terms in the expansion of Eq. (19):

$$\frac{\partial n_1}{\partial t} + \nabla \cdot (n_0 v_1) + \nabla \cdot (n_1 v_0) + (v_0 \cdot \nabla)(\xi \cdot \nabla)n_0 - \nabla(\xi \cdot \nabla) \cdot (n_0 v_0) = 0,$$

(20)

where we have used $\nabla \to \nabla_0 - \nabla_0 \xi \cdot \nabla_0$ in the expansion and then dropped the subscript from $\nabla_0$.

Returning to Eq. (20) with $\xi_i = \xi_i(r)e^{i(kz+m\theta-\omega t)}$, and expressing $\xi_r$ and $\xi_\theta$, we obtain

$$- i(\Omega - m\omega_+)n_1 - i(\Omega - m\omega_+) \nabla \cdot (n_0 \xi) = 0,$$

(21)
which yields the perturbed density $n_1 = -\nabla \cdot (n_0 \xi)$. This is exactly the same as Weinberg’s derivation of the perturbed density which is much more elegant a method of deriving the perturbed density (Eq. (5.10) of Weinberg’s [25]).

Hence we write the density perturbation as

$$n(x + \xi) = n_0(1 - \nabla \cdot \xi),$$

for a uniform beam.

This is perhaps the most crucial equation that describes the collective effects. We can impose an appropriate jump condition at the beam-plasma boundary with the equation and derive the first-order equations of motion in terms of the displacement vector $\xi$ in the many-particle system. As mentioned earlier, there is a domain of parameters in which we can find a tractable solution for the hose instability in a plasma channel, for which we refer the reader to Weinberg’ paper [25].

IV. DISPERSION RELATIONS FOR A UNIFORM BEAM

Since the hose instability in a modulated beam that includes the beam bunching in the non-relativistic domain is exceedingly complicated, we limit the hose instability in the relativistic domain in which a model illustrates a charge-compensated beam transport consistent with the experiment [6]. Moreover, we set aside the question of whether a reconditioning of a relativistic beam emerging from a beam injector can be studied quantitatively with the aid of the coasting beam model in a plasma channel. Hence we leave the question of an appropriate beam injector open for future discussion.

To obtain the desired dispersion relation for a coasting beam for which wake fields are negligible, we assume the first-order quantities vary as $f_i e^{-i(\omega t - m\theta - kz)}$ as before. Here we limit our calculation to the fast mode $\omega_+$. Then Eq. (18) yields after a brief algebra,

$$-[(\Omega - m\omega_+)^2 - r(a^2)\xi_r + i(\Omega - m\omega_+)(2\omega_+ + \omega_c)\xi_\theta] = \frac{e}{\gamma M}(E_{1r} - \beta B_{1\theta}),$$

(23a)

$$-i(\Omega - m\omega_+)(2\omega_+ + \omega_c)\xi_r - (\Omega - m\omega_+)^2\xi_\theta = \frac{e}{\gamma M}(E_{1\theta} + \beta B_{1r}),$$

(23b)

$$-(\Omega - m\omega_+)^2\xi_z = \frac{e}{\gamma^3 M} E_{1z},$$

(23c)

where $\Omega = \omega - kv$. 


The significance of the right hand-side of Eq. (23c) is easily seen if we compare it with those of Eq. (23a) and Eq. (23b). In the relativistic domain for which $\gamma \gg 1$, Eq. (18) is essentially reduced to a two-dimensional ($r$ and $\theta$) problem, since $\xi_z$ is too small to retain and may be dropped in the analysis. Hence there is no beam bunching in the relativistic domain. This observation is consistent with the order-analysis of Weinberg [24, 25] and gives a hint that there is a domain in which we may find a tractable solution for the beam instability. The straightforward solution of these equations is still very difficult although perhaps not impossible. Hence we follow Weinberg’s analysis very closely to obtain approximate solutions in the limiting cases.

With the same set of basic assumptions as in Weinberg’s analysis $|\omega|a \ll 1$, $|k|a \ll 1$, and $|q|a \approx 1$, we look for the parameters of the beam and plasma $c^2/(4\pi\sigma v_0) \ll 1$ and $\omega a/v_0 \ll 1$ such that $\omega a \ll v_0$ and $|k|a \ll 1$. Repeating the same order analysis of $J_1r$, $J_1\theta$ and $J_1z$ in Eq. (15a), we arrive at the equation:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial E_{1z}}{\partial r} \right) - \frac{m^2}{r^2} E_{1z} + q^2 E_{1z} = \frac{-4\pi ie\beta \omega}{c} n_1$$  (24)

In the following analysis we follow closely the procedure of Weinberg [25] and make the same basic assumptions, $|\omega|a \ll v$, $|k|a \ll 1$, and $|q|a \sim 1$, where $q^2 = -k^2 + (i\omega/c^2)(4\pi\sigma - i\omega) \rightarrow 4\pi i\sigma \omega/c^2$. Here $a$ is the beam radius.

By trivial extension of the arguments leading to Eq. (3.30) of Weinberg [25], we obtain

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dE_{1z}}{dr} \right) - \frac{m^2}{r^2} E_{1z} + q^2 E_{1z} = \frac{-4\pi iev\omega}{c^2} n_1,$$  (25)

where $E_{1z} = E(r)e^{-i\omega t + ikz + im\theta}$. Here we have taken the plasma current $J_p = \sigma E_1$, where $\sigma$ is a scalar. In a dense plasma, Ohm’s law in its simple form remains valid in a wide frequency range. Hence the Hall effect in the REB is negligible. This assumption of scalar conductivity in our model is reasonable, because $\omega_c/\nu_{ei} \ll 1$, where $\omega_c \sim 5.0 \times 10^8\sec^{-1}$ and $\nu_{ei} \sim 2.2 \times 10^9\sec^{-1}$. Here $\sigma \sim 1.4 \times 10^{12}\sec^{-1}$ for $n_p \sim 10^{12}/\text{cm}^3$ at $kT = 12\text{eV}$ [22].

We rewrite Eqs. (23) to calculate $n_1$

$$- [(\Omega - m\omega_+)^2 - r(\alpha^2)] \xi_r + i(\Omega - m\omega_+)(2\omega_+ + \omega_c) \xi_\theta = \frac{F_r(r)}{\gamma M}$$  (26a)

$$+ i(\Omega - m\omega_+)(2\omega_+ + \omega_c) \xi_r + (\Omega - m\omega_+)^2 \xi_\theta = \frac{F_\theta(r)}{\gamma M},$$  (26b)

where $\Omega = \omega - kv$ and $(\alpha^2)' = 0$ in Eq. (26a) for a uniform beam.
Here we have extended Weinberg’s analysis on the right-hand sides of Eq. (23a) and Eq. (23b) to write
\[ F_{1r} = \frac{e}{\gamma M} (E_{1r} - \beta B_{1\theta}) = -\frac{iev}{\gamma M \omega} E_{1z} = \exp \left[-i(\omega t - m\theta - ikz)\right] \frac{\mathcal{F}_r(r)}{\gamma M}, \]  
(27a)
\[ F_{1\theta} = \frac{e}{\gamma M} (E_{1\theta} + \beta B_{1r}) = \frac{ev}{\gamma M} E_{1z} = \exp \left[-i(\omega t - m\theta - ikz)\right] \frac{\mathcal{F}_\theta(r)}{\gamma M}, \]  
(27b)
where we take approximate values of \( F_i \) in terms of the electric field \( E_{1z} \). That is,
\[ \mathcal{F}_r(r) = -\frac{iev}{\omega} \mathcal{E}(r)', \]  
(28a)
\[ \mathcal{F}_\theta(r) = \frac{evm \mathcal{E}(r)}{\omega r}. \]  
(28b)
Solving for \( \xi \) in Eq. (26) in terms of the perturbed field \( E_{1z} = e^{-i\omega t - im\theta + ikz} \mathcal{E} \), we have
\[ \xi_r = \frac{2iev}{\gamma M \omega \Xi(\omega)} \left[ \mathcal{E}'(r) - \frac{(2\omega_+ + \omega_c)}{(\Omega - m\omega_+)} \left( \frac{m}{r} \right) \mathcal{E}(r) \right], \]  
(29a)
\[ \xi_\theta = -\frac{ev}{\gamma M \omega \Xi(\omega)} \left[ -\frac{(2\omega_+ + \omega_c)}{(\Omega - m\omega_+)} \mathcal{E}'(r) + [1 - \frac{r(\alpha')^2}{(\Omega - m\omega_+)^2}] \left( \frac{m}{r} \right) \mathcal{E}(r) \right], \]  
(29b)
where \( \Xi(\omega) = (\Omega - m\omega_+)^2 - 4\alpha^2 - r(\alpha')^2 - \omega_c^2 \).

Making use of the identities \( n(r) = (r^2\alpha^2')/(\gamma M)/(4\pi e^2\beta^2 r) \) and \( 2\omega_+ + \omega_c = \sqrt{\omega_c^2 + \alpha^2} \)
and defining \( n_1 = e^{i(m\theta + kz - \omega)} N(r) = -\nabla \cdot (n \xi) \), we obtain the perturbed density in terms of \( \mathcal{E} \) as
\[ N(r) = \frac{ic^2}{4\pi ev \omega} \left[ \frac{1}{r} \frac{d}{dr} \left( r f + \frac{\mathcal{E}}{dr} \right) - \frac{m^2}{r^2} f_+ \mathcal{E} - g_+ \mathcal{E} \right], \]  
(30)
where
\[ f_+ = \frac{2\alpha^2 + r(\alpha')^2}{\omega_c^2 + 4\alpha^2 - (\Omega - m\omega_+)^2 + r(\alpha')^2} \]  
(31a)
\[ g_+ = \left( \frac{m}{r} \right) \frac{1}{\Omega - m\omega_+} \frac{d}{dr} \left[ \sqrt{\omega_c^2 + 4\alpha^2} f_+ \right]. \]  
(31b)

For the angular frequency \( \omega_- = \frac{1}{2}[-\omega_c - \sqrt{\omega_c^2 + 4\alpha^2}] \), an exactly parallel calculation yields \( f_- \) and \( g_- \). This result shows an interesting symmetry that \( f_+ \rightarrow f_- \) by means of the substitution \( \omega_+ \rightarrow \omega_- \) or vice versa. It should stressed that this calculation for \( f_- \) and \( g_- \)
should be carried out to make sure our algebras are indeed correct, although it is somewhat tedious.

Since the electron motion in the \( r-\theta \) plane has slow and fast rotations, and the generalized momentum (canonical momentum) is the average of the mechanical momenta in the presence of a static magnetic field, we take \( f = [f_+ + f_-]/2 \) which is given by
\[ f = \left[ 2\alpha^2 + r(\alpha^2)' \right] \frac{\Theta^2}{\Theta^4 - m^2(\omega_e^2 + 4\alpha^2)(\Omega + m\omega_e/2)^2}, \quad (32) \]

where \( \Theta^2 = (1 - m^2/4)(\omega_e^2 + 4\alpha^2) - (\Omega + m\omega_e/2)^2 + r(\alpha^2)' \).

It should be stressed that an argument similar to that of Weinberg [25] for taking an average based on the probability of rotation in either (positive or negative) direction may not hold in the presence of an external field. Since the equation of motion can be written as \( dp/dt = e[E + \beta \times B] \), it is indeed correct to take the canonical momentum in our analysis. This clearly shows the inadequacy of Weinberg’s model. And yet in the limit of \( B^\text{ext} \rightarrow 0 \), our final results reduce to those of Weinberg [25].

Similarly, we take \( g = [g_+ + g_-]/2 \) and then is given by

\[
g = -\left( \frac{m^2}{r} \right) \frac{(\Omega + m\omega_e/2)^2}{(\Omega^2 - m^2\alpha^2 + m\omega_e\Omega)} \left[ \frac{(\omega_e^2 + 4\alpha^2)f(r)}{\Theta^2} \right]' + \frac{1}{2} \left( \frac{m^2}{r} \right) \frac{(\omega_e^2 + 4\alpha^2)^{1/2}}{\Omega^2 - m^2\alpha^2 + m\omega_e\Omega} \left[ (\omega_e^2 + 4\alpha^2)^{1/2}f(r) \right]' \quad (33)\]

It should be noticed that, for a uniform beam, \((\alpha^2)' = 0\) which simplifies algebras immensely. Henceforth we will assume that a uniform beam is coasting in the over-dense plasmas and that the beam is completely neutralized. For the special case of a uniform beam density profile, it is possible to obtain a wave equation similar to that of Weinberg [25].

By taking \( n(r) = n\theta(r - a) \), it follows from Eqs. (32) and (33)

\[
f(r) = \begin{cases} 
1 - \eta^2 & \text{if } r < a, \\
0 & \text{if } r > a. 
\end{cases} \quad (34)\]

Here \( \eta^2 \) is defined as

\[
\eta^2 = 1 - \frac{2[(4 - m^2) + (1 - m^2/4)\lambda_c^2 - \bar{\lambda}^2]}{[(4 - m^2) + (1 - m^2/4)\lambda_c^2 - \lambda^2]^2 - m^2(4 + \lambda_c^2)\bar{\lambda}^2} \quad (35)\]

Similarly,

\[
g(r) = -\zeta^2 \frac{\delta(r - a)}{a}, \quad (36)\]

where

\[
\zeta^2 = 2m^2(1 - \eta^2) \frac{1 + \lambda_c^2/4}{\lambda^2 - m^2(1 + \lambda_c^2/4)} \left[ 1 + \frac{2\bar{\lambda}^2}{\lambda^2 - (1 - m^2/4)\lambda_c^2 - (4 - m^2)} \right], \quad (37)\]
with \( \bar{\lambda} = \lambda + m\lambda_c/2 \), \( \lambda = \Omega/\alpha \), and \( \lambda_c = \omega_c/\alpha \). Here \( \alpha^2 = \omega_{\beta}^2 = 2\pi e^2 n\beta^2/(\gamma M) \) for a uniform beam density.

Substitution of these expressions for the perturbed density in Eq. (25) yields

\[
\frac{1}{r} \frac{d}{dr} \left( r[1 - f(r)] \frac{d\mathcal{E}}{dr} \right) - \frac{m^2}{r^2} (1 - f(r)) \mathcal{E} + \left[ q^2 + g(r) \right] \mathcal{E} = 0.
\]

The detailed algebra leading to this wave equation is straightforward, but it is somewhat tedious. Similar to that of a two-dimensional vibrating membrane problem, the solution of Eq. (38), which is finite and satisfies the boundary conditions for a finite plasma channel \[25\], is given by Hankel functions \[35\].

\[
\mathcal{E}(r) = \begin{cases} 
J_m(qr/\eta) & \text{if } r < a, \\
H_m^1(qr) - \alpha_m J_m(qr) & \text{if } r > a.
\end{cases}
\]

Here \( H_m^1(qr) = J_m(qr) + iN_m(qr) \)

\( \mathcal{E}(r) \) by Eq. (39), together with the boundary conditions at the surface of the plasma channel of radius \( b \) specifies the eigenvalue \( \alpha_m \),

\[
\alpha_m = \begin{cases} 
H_{m-1}^1(qb)/J_{m-1}(qb) & \text{if } m > 1, \\
H_1^1(qb)/J_1(qb) & \text{if } m = 0.
\end{cases}
\]

Integrating Eq. (38) over \( a - \varepsilon \) and \( a + \varepsilon \), we obtain

\[
a \mathcal{E}'(a + \varepsilon) - a\eta^2 \mathcal{E}'(a - \varepsilon) = \zeta^2 \mathcal{E}(a),
\]

Substitution of appropriate solutions of the wave equation from Eq. (39) into the left hand side of Eq. (41) and rearrangement of terms in the limit \( \varepsilon \to 0 \) yields the dispersion relation:

\[
\eta \left( \frac{J_m'(qa/\eta)}{J_m(qa/\eta)} \right) + \frac{\zeta^2}{qa} = \frac{[H_m^1(qa)]'}{H_m^1(qa)} - \alpha_m(qb)J_m'(qa)/(\alpha_m(qb)J_m(qa)),
\]

where \( m \geq 0 \).

This dispersion relation is identical in form to that of Weinberg \[25\], but differs in \( \eta \) and \( \zeta \). In the limit of vanishing \( B_{\text{ext}} \), the dispersion relation goes over into that of Weinberg \[25\]. But it should be noticed that the boundary condition at the edge of the plasma column in Weinberg’s analysis \[25\] is not valid, since there is no external magnetic field that confines the plasma channel in experiments. It is therefore apparent that his dispersion relation Eq. (1.9) [and Eq. (12)] is inconsistent with the problem he has posed in his paper and is in self-contradiction.
V. ANALYSIS OF THE DISPERSION RELATION FOR A UNIFORM BEAM

We shall be interested here mainly in the resistive hose \((m = 1)\) instability in the low and high frequency limits since the resistive hose mode is the one that leads to the loss of a beam. We note from Eq. (42) that, since \(q^2 = 4\pi i\sigma \omega/c^2\), it may be possible to obtain a rather simple solution which holds to a higher order of approximation in the low and high frequency limits. The method of making such an approximation lies in the realization that the conductivity of the plasma remains fixed for a given plasma density and the approximation of Bessel functions in the asymptotic limits as a function of \(\omega\) readily available.

Thus the analytic solutions of Eq. (42) can be studied in low and high frequency regimes with the asymptotic limits of Bessel functions; the two asymptotic solutions should then be connected smoothly by analytic continuation. For each \(m\), we may classify the modes as in Weinberg [25]. This classification is not entirely trivial, since the external magnetic field introduces new modes by removing the degeneracy found in Weinberg’s analysis [25]. The central question in determining the efficiency of beam transport by means of plasma focusing is to find which instability would affect the beam transport most significantly. We present here only an outline of the classification similar to that of Weinberg [25] with emphasis on the resistive hose mode \((|\omega| \ll \sigma)\) that affects the beam transport most dangerously, since if the \(m = 1\) hose mode occurs, the entire beam can be lost. Moreover the two-stream mode for which \(\omega \sim \sigma\) has been already treated in detail [27].

Returning to Eq. (42) we expand the left-hand side (LHS) and the right-hand side (RHS), using the following identities in Bessel functions:

\[
H_m^{(1)}(qa) = J_m(qa) + iY_m(qa),
\]
\[
J_m'(qa) = (1/q)[qJ_{m-1}(qa) - (m/a)J_m(qa)],
\]
\[
H_m^{(1)'}(qa) = (1/q)[qH_{m-1}(qa) - (m/a)H_m(qa)],
\]

as

\[
LHS = \eta \left[ J_{m-1}(qa/\eta) - m(qa/\eta) \right], \quad \text{(44a)}
\]
\[
RHS = -\frac{m}{qa} \frac{Y_{m-1}(qa)J_{m-1}(qb) - Y_{m-1}(qb)J_{m-1}(qa)}{Y_m(qa)J_{m-1}(qb) - Y_{m-1}(qb)J_m(qa)}. \quad \text{(44b)}
\]
With Eqs. (44a) and (44b), Eq. (42) becomes
\[
\eta \left[ \frac{J_{m-1}(qa/\eta)}{J_m(qa/\eta)} \right] = \frac{-m}{qa} (1 - \eta^2) - \frac{\zeta^2}{qa} \\
- \frac{Y_{m-1}(qa) J_{m-1}(qb) - Y_{m-1}(qb) J_{m-1}(qa)}{Y_m(qa) J_{m-1}(qb) - Y_{m-1}(qb) J_m(qa)}.
\]  
(45)

Here \( \eta^2 \) and \( \zeta^2 \) are defined by Eq. (35) and Eq. (37).

### A. Low Frequency Regime: \( |q| \to 0 \)

For \( |qa| \ll 1 \), the dispersion relation Eq. (42) can be rewritten as
\[
\frac{\eta}{qa} \left( \frac{J'_m(qa/\eta)}{J_m(qa/\eta)} \right) = \frac{(m + \zeta^2)}{(qa)^2} + O(1),
\]  
(46)
where \( m \neq 0 \), and
\[
\frac{\eta}{qa} \left( \frac{J'_m(qa/\eta)}{J_m(qa/\eta)} \right) = \frac{1}{2} \left( b^2/a^2 - 1 \right) + O(q^2),
\]  
(47)
where \( m = 0, \ qb \ll 1 \).

Here we have used the identity \( H^{(1)}_m(qa) = J_m(qa) + iY_m(qa) \). We now classify the modes into those for which \( |qa/\eta| \to 0 \) at low frequency and designate \( C \) and \( D \) modes, otherwise we call \( A \) and \( B \) modes.

For \( m = 1, qa \ll 1, \) and \( qb \ll 1 \) in Eq. (45), we obtain the following expansion of the left hand-side of the equation:
\[
\frac{\eta}{qa} \left( \frac{J'_1(qa/\eta)}{J_1(qa/\eta)} \right) \to \frac{\eta^2}{(qa)^2} \left[ 1 - \frac{(qa)^2}{(2\eta)^2} - \frac{q^4a^4}{96\eta^4} - \cdots \right],
\]
and similarly the right hand-side can be expanded as
\[
-\frac{1}{qa} + \frac{Y_0(qa) J_0(qb) - Y_0(qb) J_0(qa)}{Y_1(qa) J_0(qb) - Y_0(qb) J_1(qa)} \\
\approx \frac{1}{qa} \left[ 1 - (qa)^2 \ln(b/a) + \frac{1}{2} (qa)^4 \ln^2(b/a) + \frac{1}{2} \ln(b/a) \right] \\
- \frac{1}{qa} \left[ \frac{1}{4} (qa)^4 (b^2 - a^2) + \cdots \right].
\]

Next we rewrite Eq. (35) as
\[
\eta^2 = 1 - \frac{[3(1 + \lambda_e^2/4) - \bar{\lambda}^2]}{[\lambda^2 - 3(1 + \lambda_e^2/4)^2 - 4(1 + \lambda_e^2/4)\bar{\lambda}^2]} \\
\approx \frac{1}{3} + \frac{1}{6} \lambda_e^2 - \frac{14}{27} (1 - \lambda^2/2 + \cdots \bar{\lambda}^2 + \cdots),
\]  
(49)
and similarly rewrite Eq. (37)

\[
\zeta^2 = \frac{2(1 + \lambda_c^2/4)}{\lambda^2 - (1 + \lambda_c^2/4)} \left[ 1 - \frac{2\bar{\lambda}^2}{3(1 + \lambda_c^2/4) - \lambda^2} \right] (1 - \eta^2) \quad (50)
\]

\[
\approx \left[ \frac{4}{3} (1 - \lambda_c^2/4) - \frac{4}{3} (1 - \lambda_c^2/4) \left( \frac{10}{9} \sigma^2 - \frac{5}{9} \sigma^4 + \cdots \right) \right], \quad (51)
\]

where \( \bar{\sigma}^2 = \bar{\lambda}^2/(1 + \lambda_c^2/4) \).

As shown by Weinberg [25], for \( |qa/\eta| \ll 1 \) we substitute the expressions Eq. (48) and Eq. (50) to obtain an implicit dispersion relations

\[
\bar{\lambda}^2 + \bar{\lambda}^4 + \cdots = \frac{1}{4} \lambda_c^2 (1 - \lambda_c^2/2) - \frac{1}{2} (1 - \lambda_c^2/2) \{ \frac{1}{4} (1 - \lambda_c^2/4) + \ln(b/a) \} (qa)^2
\]

\[
+ \frac{1}{4} (1 - \lambda_c^2) \left\{ \frac{7}{16} + \frac{1}{32} \lambda_c^2 (1 - \lambda_c^2) + \ln^2(b/a) + \ln(b/a) \right\} (qa)^4
\]

\[
- \frac{1}{8} (1 - \lambda_c^2) q^4 a^2 b^2 + \cdots. \quad (52)
\]

Here we have assumed \( |qa| \ll 1 \) and \( |qb| \ll 1 \). In the limit \( q^2 \to 0 \), \( \bar{\lambda} \) is oscillatory, which is the hose instability and shows that, in the presence of the external field, the growth rate is reduced due to the restoring force by the magnetic field. This is in agreement with the numerical result although the effects are not significant [37].

While it is possible to calculate the \( \bar{\lambda}^2 \) to the order of \( (qa)^4 \) by an iteration technique as in Weinberg [25], we just write, for the sake of simplicity, \( \bar{\lambda}^2 \) to the order of \( q^2 a^2 \) instead:

\[
\lambda^2 + \lambda_c \lambda = -\frac{1}{8} \lambda_c^4 - \frac{1}{2} (1 - \lambda_c^2/2) \{ \frac{1}{4} (1 - \lambda_c^2/4) + \ln(b/a) \} (qa)^2 + \cdots. \quad (53)
\]

Here \( \bar{\lambda} = \lambda + \lambda_c/2 \), \( \lambda = \pm(\omega - kv)/\omega_\beta \), and \( q^2 = (4\pi i \sigma \omega)/c^2 \), and \( \lambda_c^2 = \omega_c^2/\omega_\beta^2 \).

Thus,

\[
\lambda = \frac{1}{2} \left[ -\lambda_c \pm (1 - \lambda_c^2)^{1/2} \{ \lambda_c^2 - 2\left[ \frac{1}{4} (1 - \lambda_c^2/4) + \ln(b/a) \right] q^2 a^2 \}^{1/2} \right]. \quad (54)
\]

Eq. (54) is the dispersion relation for the hose instability in the limit \( q^2 a^2 \to 0 \) and it shows that, in the presence of the external magnetic field, the growth rate is reduced due to the restoring force by the magnetic field which is in agreement with numerical computations [37].
B. Unstable Modes (A and B): $|qa/\eta|$ does not converge to zero:

If $qa/\eta$ does not converge to zero, $\eta$ must then converge to zero as fast as $|qa|$. Rewriting Eq. (35) as

$$\eta^2 = \frac{\Lambda^2 - 2\Lambda - m^2(4 + \lambda^2_c)\bar{\lambda}^2}{\Lambda^2 - m^2(4 + \lambda^2_c)\lambda^2}, \quad (55)$$

where $\Lambda = (4 - m^2) + (1 - m^2/4)\lambda^2_c - \bar{\lambda}^2$.

Hence the numerator of the above equation must be zero which defines the modes of A and B modes:

$$\bar{\lambda}^2 = \begin{cases} 
(1 + m^2/4)\lambda^2_c + (3 + m^2) - \Delta_m^{1/2} & \text{(A mode)}, \\
(1 + m^2/4)\lambda^2_c + (3 + m^2) + \Delta_m^{1/2} & \text{(B mode)},
\end{cases} \quad (56)$$

where $\Delta_m = 1 + 4m^2(1 + \lambda^2_c/(3 + \lambda^2_c))$.

With the external magnetic field, the degeneracy of $m = 0$ and $m = 2$ are removed in A and B modes found in Weinberg’s analysis [25].

But as in Weinberg’s analysis, $m + \zeta^2$ in the righthand side of Eq. (46) does not vanish for $m > 0$ when $\bar{\lambda}^2$ takes either value of the above A and B modes.

Hence we may carry out similar analysis iteratively for $\bar{\lambda}^2$ from Eq. (46) for $A_{mn}$ - mode by the equation

$$\eta \rightarrow (qa)/j_{mn}, \quad (57)$$

where $j_{mn}$ is the n-th root of $J_m(x) = 0$ which yields

$$\bar{\lambda}^2_\tau \rightarrow (3 + m^2) + (1 + m^2)\lambda^2_c - \sqrt{\Delta_1} - \frac{1}{2} \frac{(qa)^2}{j^2_{mn}} \left[2 + \frac{m^2(4 + \lambda^2_c) - 2}{\sqrt{\Delta_1}}\right]. \quad (58)$$

Similarly, for $B_{mn}$ - mode we have

$$\bar{\lambda}^2 \rightarrow (3 + m^2) + (1 + m^2)\lambda^2_c + \sqrt{\Delta_1} + \frac{1}{2} \frac{(qa)^2}{j^2_{mn}} \left[2 + \frac{m^2(4 + \lambda^2_c) - 2}{\sqrt{\Delta_1}}\right], \quad (59)$$

where $\Delta_1 = m^2(4 + \lambda^2_c)(3 + \lambda^2_c) + 1$.

Next for $m = 0$, Eq. (47) can be written as

$$\eta \rightarrow (qa)/y_n, \quad (60)$$

Here $y_n$ is the n-th root of the equation

$$J'_0(y)/(yJ_0(y)) = \frac{1}{2} \left(b^2/a^2 - 1\right). \quad (61)$$
Hence for \( m = 0 \), there is no difference between the present analysis and that of Weinberg’s analysis except for the presence \( \lambda_c^2 \) term which removes the degeneracy. Solving Eq. (55) by iteration, we obtain

\[
\bar{\lambda}^2 = \lambda_c^2 + 3 \pm 1 - 2(qa)^2/y_n^2 + \cdots .
\] (62)

C. \( m = 1 \) Resistive Hose Instability: (C, D) modes

Suppose \( qa/\eta \to 0 \). For \( m \neq 0 \), the left-hand side of Eq. (46) then becomes

\[
\eta \left( \frac{J_m'(qa/\eta)}{J_m(qa/\eta)} \right) \approx \frac{m}{q^2 a^2},
\] (63)

with the aid of the identity

\[
J_m'(qa/\eta) = -J_{m+1}(qa/\eta) + (m\eta/qa)J_m(qa/\eta).
\]

And yet the condition that \( qa/\eta \to 0 \) implies that, by Eq. (35), \( \bar{\lambda} \) does not take the following value

\[
\bar{\lambda}^2 = (1 + m^2/4)(4 + \lambda_c^2) - 1 \pm [m^2(4 + \lambda_c^2)(4 + \lambda_c^2) + 1]^{1/2},
\] (64)

for which \( \eta = 0 \).

With the condition \( |qa/\eta| \to 0 \) and with Eq. (63), Eq. (46) yields the following condition:

\[
m\eta^2 + m + \zeta^2 = 0,
\] (65)

where \( m > 0 \).

The zeros of this equation in which \( \eta^2 \) and \( \zeta^2 \) are defined by Eq. (48) and Eq. (50) defines the modes of type C and D and is given by:

\[
\bar{\lambda}^2 = (m^2 - 2m + 3/2)(1 + \lambda_c^2/4) \pm (1 + \lambda_c^2/4)^{1/2} \left[ \{3m(m - 2) + 9/4\}(1 + \lambda_c^2/4) - m(m - 2) \right]^{1/2},
\] (66)

where \( \pm \) signs correspond to the modes of type C and D respectively based on Weinberg’s classification [25]. Note also that if \( \bar{\lambda} \to \lambda \) for \( \lambda_c = \omega_c/\alpha = 0 \), the results of Eq. (66) in the limits go over B14 and B15 of Weinberg [25] as they should.
D. High Frequency Regime: \(|q| \to \infty\)

1. \(A\) Mode and \(B\) Mode

For \(|qa| \gg 1\) and \(|qb| \gg 1\), with the aid of asymptotic formula of the Bessel functions of the first kind \(J_{m-1}(qa)\) and the second kind \(Y_{m-1}(qa)\) for \(qa \gg 1\),

\[
Y_{m-1}(qa) \sim \left[ \frac{2}{\pi qa} \right]^{1/2} \cos[qa - \frac{\pi}{4} - \frac{1}{2}(m - 1)\pi], \tag{67}
\]

and

\[
J_{m-1}(qa) \sim \left[ \frac{2}{\pi qa} \right]^{1/2} \sin[qa - \frac{\pi}{4} - \frac{1}{2}(m - 1)\pi], \tag{68}
\]

the dispersion relation Eq. (45) reduces to

\[
\eta \left( \frac{J_m'(qa/\eta)}{J_m(qa/\eta)} \right) + \frac{\zeta^2}{qa} \approx \mathcal{O}\left( \frac{1}{qa} \right). \tag{69}
\]

Next we divide the modes into those for which \(|qa/\eta|\) remains finite with \(|q| \to \infty\) (\(A\) and \(B\) modes) and those for which \(|qa/\eta| \to \infty\) (\(D\) mode).

With some algebraic rearrangement, we may rewrite Eq. (35) as

\[
\eta^2 = 1 - \frac{2[(1-m^2/4)(4+\lambda^2) - \bar{\lambda}^2]}{[\lambda^2 - (1+m/2)^2(4+\lambda^2)][\lambda^2 - (1-m/2)^2(4+\lambda^2)]}, \tag{70}
\]

It is easy to see from Eq. (55) that \(|qa/\eta|\) may remain finite if the following conditions are met:

\[
\bar{\lambda}^2 \to \begin{cases} 
(1+m/2)^2(4+\lambda^2) & A \text{ mode}, \\
(1-m/2)^2(4+\lambda^2) & B \text{ mode}.
\end{cases} \tag{71}
\]

For a finite value of \(|x| = |qa/\eta|\), we may rewrite Eq. (69) to obtain the \(1/q^2\) term in \(\bar{\lambda}^2\),

\[
\frac{J'_m(x)}{xJ_m(x)} + \frac{\zeta^2}{(x\eta)^2} \to 0, \tag{72}
\]

where \(x = qa/\eta\).

The \(A\) and \(B\) modes for which \(\bar{\lambda}^2\) takes \((1 \pm m/2)^2(4+\lambda^2)\), and using those in Eq. (70) and Eq. (50) we may write

\[
\frac{\zeta^2}{\eta^2} = - \left( 1 - \frac{1}{\eta^2} \right) \Gamma_{\pm}(m, \bar{\lambda}_\pm), \tag{73}
\]

22
where $\Gamma_\pm$ is defined as the following:

$$
\Gamma_\pm(m, \bar{\lambda}_\pm) = \frac{m^2}{2} \left( \frac{4(1 + \lambda_c^2/4)}{\lambda^2_\pm - m^2(1 + \lambda_c^2/4)} \right) \left[ 1 + \frac{\bar{\lambda}_\pm^2}{\lambda^2_\pm - (1 - m^2/4)\lambda_c^2 - 4(1 - m^2/4)} \right].
$$

(74)

After a brief algebra, we obtain

$$
\Gamma_\pm(m, \bar{\lambda}_\pm) = \pm \frac{3}{4} (m \pm 2).
$$

(75)

Since $1/\eta^2 \to 0$ as $\eta \to \infty$,

$$
\frac{\zeta^2}{\eta^2} = \pm \frac{3}{4} \frac{(m \pm 2)}{(m \pm 1)},
$$

(76)

which is independent of $\lambda_c$. If $\bar{\lambda}_\pm^2 = (1 + m/2)^2(4 + \lambda_c^2)$ for $A$ mode, then

$$
\frac{\zeta^2}{\eta^2} = \frac{3}{4} \frac{(m + 2)}{(m + 1)}.
$$

(77)

Hence the dispersion relation for $A$ mode is

$$
\bar{\lambda}_\pm^2 \to 4(1 + m/2)^2(1 + \lambda_c^2/4) + \frac{3}{4} \frac{(m + 2)}{(m + 1)} \frac{x^2}{(qa)^2},
$$

(78)

where $x$ is the solution of the equation

$$
x J'_m(x) = \pm \frac{3}{4} \frac{(2 + m)}{(1 + m)} J_m(x).
$$

(79)

When $\bar{\lambda}_\pm^2 = (1 - m/2)^2(4 + \lambda_c^2)$ for $B$ mode except for $m = 1$ and $m = 2$, we have

$$
\frac{\zeta^2}{\eta^2} = \frac{3}{4} \frac{(m - 2)}{(m - 1)}.
$$

(80)

Hence the dispersion relation for $B$ mode is

$$
\bar{\lambda}_\pm^2 \to 4(1 - m/2)^2(1 + \lambda_c^2/4) + \frac{3}{4} \frac{(m - 2)}{(m - 1)} \frac{x^2}{(qa)^2},
$$

(81)

where $x$ is the solution of the equation

$$
x J'_m(x) = \pm \frac{3}{4} \frac{(2 - m)}{(1 - m)} J_m(x),
$$

(82)

where both $m \neq 1$ and $m \neq 2$.

We also notice that our results do not go over to those of Weinberg’s analysis in the limit $\lambda_c \to 0$, because there was an algebraic error in his analysis [see equations (B26) and (B27)].
2. \( D \) Mode

If \( \eta^2 \) remains finite but \( \zeta^2 \to \infty \), then the mode \( D \) would take place in the beam. This happens if \( \lambda^2 \neq (1 \pm \sqrt{2})(4 \pm \lambda^2) \) and \( \lambda^2 \to m^2(1 + \lambda^2_c/4) \) except for \( m = 1 \) for which the numerator of \( \zeta^2 \) becomes zero with \( \lambda^2 = m^2(1 + \lambda^2_c/4) \). For \( m = 0 \), \( \zeta^2 \) is identically zero. Hence the type \( D \) mode begins with \( m \geq 2 \).

In a high frequency limit, \( |q| \to \infty \), \( |qa| \gg 1 \), and \( |qb| \gg 1 \), \( D \) mode is given by

\[
\lambda^2 = (1 + \frac{\lambda^2_c}{4}) + \frac{x^2}{2q^2a^2},
\]

where \( x \) is a root of \( xJ_1'(x) = -3J_1(x) \) and \( qa/|\eta| \to \infty \).

We have carried out the above analysis to guide numerical work of solving the dispersion relation for various instabilities, Eq. (45), by a computer. The above analytical results have been borne out in our detail numerical calculations. We have found that the external magnetic field reduces the growth rate somewhat, but not significantly since \( \omega_c = ecB_{\text{ext}}/\mathcal{E} \) in the relativistic domain. The results (Figure 2) are essentially same as the Weinberg’s analysis.

VI. DISCUSSION AND CONCLUSION

In conclusion we note that it may be somewhat confusing by studying both a proton beam for a proton linac and an electron beam for PHERMEX facility together, but the stability analysis remains valid for both cases with a proper change of charge and mass of a particle. The proposed technique of combination of the synchrotron and the RFQ to accelerate a high-current beam to meet the necessary energy requirements for the beam injection may become feasible, since it takes so short a time to travel for the beam in a short plasma channel that the resistive hose instability may not develop to disrupt the integrity of the beam. Moreover, the technique has been demonstrated in an electron accelerator by Nakanishi group [6]. However, the experimental evidence we could find (including the plasma cone in a Be chamber of PHERMEX facility) is too scant to permit any safe generalization. Indeed it would be desirable to have experimental demonstrations for a high-energy, high-current proton beam. Yet the case we have discussed so far is conclusive enough to demonstrate its feasibility. Without a proper reconditioning a high-current beam by a dense plasma channel, a reliable operation of a high-current, high-energy accelerator is highly unlikely.
However, the creation of a dense plasma-channel may complicate the maintenance of a linac in a routine operation and may make it difficult to perform any reliable experiments. Still a potential difficulty in developing a reliable beam injector for a high-current, high-energy accelerator remains the major stumbling block for the the large hadron collider (LHC) at CERN or APT project at LANL. In any practical, realistic sense, the LHC is doomed to failure \cite{38, 39, 40}, but particle physicists could still turn to astrophysical observations \cite{41}.

[1] C. L. Longmire (unpublished note).
[2] M. Rosenbluth, Phys. Fluids 3, 932 (1960).
[3] R. C. Mjolsness, J. Enoch and C. L. Longmire, Phys. Fluids, 6 1741 (1963).
[4] M. Lampe, et al., Phys. Fluids, 27, 2921 (1984).
[5] W. H. Bennett, Phys. Rev. 45, 890 (1934).
[6] H. Nakanishi, et al., Phys. Rev. Lett. 66, 1870 (1991).
[7] G. J. Caporaso, et al., Phys. Rev. Lett. 57, 1591 (1986).
[8] E. Boggaschi et al., Phys. Rev. Lett. 66, 1705 (1991).
[9] E. D. Courant, M. S. Livingston, and H. S. Snyder, Phys. Rev. 88 1190 (1952); E. D. Courant and H. S. Snyder, Ann. Phys. (NY) 3, 1 (1958).
[10] M. Puglisi, The Radiofrequency Quadrupole Linear Accelerator (CERN Accelartor School, Oxford, 1985) p. 706.
[13] I. M. Kapchinskii and V. V. Vladmirskii, Proc. 1959 Int’l Conf. High-Energy Accelerators (CERN, Geneva, Switchland).
[12] I. M. Kapchinskij and Y. Telyakov, Prib. Tech. Eksp. 119, 17 (1979); Wolfgang Paul, Rev. Mod. Phys. 62, 531(1990).
[13] I. M. Kapchinskij and V. V. Vladmirskij, Proc. 1959 Int’l Conf. High-Energy Accelerators (CERN, Geneva, Switchland) p274.
[14] L. Picardi, R. Raimondi and C. Ronsivalle, Nucl. Inst. and Methods in Phys. Res. A 303 209 (1991).
[15] A. Schempp, Nucl. Inst. and Methods in Phys. Res. B 99 688 (1995).
[16] M. V. Nezlin, Plasma Physics, 10, 337 (1968).
[17] D. Bohm and L. L. Foldy, Phys. Rev. 70, 249 (1946).
[21] A magnet unit consists of two quadrupole magnets and the second magnet in a linac is rotated with respect to the first magnet by $\pi/2$, and thus the first-order focusing force $dB_z/dy$ in one direction in the first magnet becomes the defocusing force in the second magnet, and *vice versa*. Hence the focusing force globally becomes the second-order effect.

[22] J. D. Jackson, *Classical Electrodynamics* (John Wiley and Sons, New York 1975), 2-nd edition. P559 and p582.

[23] M. H. Johnson and B. A. Lippmann, Phys. Rev. **76**, 828 (1949).

[24] Steven Weinberg, J. Math. Phys. **5**, 1371 (1964).

[25] Steven Weinberg, J. Math. Phys. **8**, 614 (1967) and the references therein.

[36] This interesting study by Weinberg [J. Math. Phys. **8**, 614 (1967)], however, has not been widely known in the literature during the hey days of SDI research in the late 70's, probably because Weinberg’s paper is almost impossibly difficult and confusing to follow through in his Introduction of the paper. There is also a difficulty of reconciling with his boundary conditions and the physics problem he has posed at the outset.

[27] E. A. Frieman, M. L. Goldberger, K. M. Watson, S. Weinberg, and M. N. Rosenbluth, Phys. Fluids, **5**, 196 (1962).

[28] S. J. Han, Phys. Rev. A **44**, 5784 (1991) and the references therein.

[29] I. Bernstein, *et al.*, Proc. Royal Soc. (London) A **17**, 244(1958).

[32] C. Kittel *Quantum Theory of Solids* (John Wiley and Sons, New York, 1963) 2-nd edition, Chapt 12.

[33] F. E. Low, Proc. Roy. Soc. (London) A **248**, 282 (1958).

[34] S. J. Han and B. R. Suydam, Phys. Rev. **26**, 926 (1982).

[35] Sir Harold Jeffreys and Bertha Jeffreys *Methods of Mathematical Physics* (Cambridge University Press, Cambridge, 1978) 3-rd edition, p575.

[36] The footnote 7 on the broken symmetry in his paper may not be valid [Steven Weinberg, J. Math. Phys. **8**, 614 (1967)] since $\omega$ is the frequency of electro-magnetic wave (a photon -
an original Bose gas) in an degenerate electron gas in relativistic motion although its angular
momentum is one and obeys Bose statistics. More to the point, one has to show the appearance
of low-mass spinless particles. In other words, the system is too complicated to apply Goldstone
theorem. On the other hand, to relate to the Goldstone theorem, perhaps one has to show the
presence of a plasmon \[ \omega^2 = \omega_p^2 + 3 < u^2 > k^2 \] - a pseudo-Goldstone boson - a longitudinal
wave] in the presence of an electron fluid. Moreover, the translational symmetry in the system
is broken by the external magnetic field through the boundary conditions. See a paper by
E. S. Abers and Benjamin W. Lee. Physics Report bf 9, 1 1973 and S. J. Han, arXiv: cond-
mat/0607433, May 1, 2008, Coherent Collective Excitations in a Superfluid: Broken Symmetries
and Fluctuation-Dissipation.

[37] J. R. Sobehart and S. J. Han (unpublished).

[38] Physics Today, September, 2007, page 32.

[39] David Kestembaum, NPR, 6/24/2008.

[40] Physics Today, November, 2008, page 24.

[41] David Lindley, The End of Physics: The Myth of a Unified Theory, (Basic Books, New York,
1993) Chapt. II.
The condition $K_2^2 - K_1^2 \geq 0$ in Eq. (1) was imposed on the injection beam energy by $K_1^2 = \frac{2\pi e^2 n_0}{c^2 p M}$ for a stable particle orbit.
FIG. 2: The growth rate $\text{Im}(\Omega/\omega_\beta)$ of the hose-mode ($C_1$) as a function of frequency $4\pi\sigma^2\omega/c^2$ for various external magnetic field strengths $B_i$. The curve for $B_{ext} = 0.0 kG$ corresponds to the Weinberg’s solution for which the present model of plasma channel breaks down. As the strength of the magnetic field is increased by $\Delta B_{ext} = 0.5 kG$, the growth rate decreases, but not significantly.