"INFINITE" PROPERTIES OF CERTAIN LOCAL COHOMOLOGY MODULES OF DETERMINANTAL RINGS

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Dedicated to Nguyen Tu Cuong on the occasion of his 70th birthday
and for a long friendship

Abstract. For given integers \( m, n \geq 2 \) there are examples of ideals \( I \) of complete determinantal local rings \( (R, \mathfrak{m}) \), \( \dim R = m + n - 1 \), \( \text{grade} \ I = n - 1 \), with the canonical module \( \omega_R \) and the property that the socle dimensions of \( H^m_{\mathfrak{m}^{n-2}}(\omega_R) \) and \( H^n_{\mathfrak{m}^{n-1}}(\omega_R) \) are not finite. In the case of \( m = n \), i.e. a Gorenstein ring, the socle dimensions provide further information about the \( \tau \)-numbers as studied in [10]. Moreover, the endomorphism ring of \( H^n_{\mathfrak{m}^{n-1}}(\omega_R) \) is studied and shown to be an \( R \)-algebra of finite type but not finitely generated as \( R \)-module generalizing an example of [15].

1. Introduction

Let \( I \) denote an ideal of a local ring \( (R, \mathfrak{m}) \) with \( k = R/\mathfrak{m} \) its residue field. Let \( M \) be a finitely generated \( R \)-module, and let \( H^i_I(M) \), \( i \in \mathbb{Z} \), denote the local cohomology modules of \( M \) with respect to \( I \) (see [3] or [1] for definitions). By \( \text{grade}(I, M) \) we denote the length of the largest regular sequence contained in \( I \). In the case of \( M = R \), an equicharacteristic complete regular local ring, the following is known:

(a) The Bass numbers \( \dim_R \text{Ext}_R^i(k, H^i_I(R)) \) are finite (see [8] and [9]).

(b) The natural homomorphism \( R \to \text{Hom}_R(H^d_I(R), H^c_I(R)) \), \( c = \text{grade} \ I \), is an isomorphism if and only if \( H^i_I(R) = 0 \) for \( i = d - 1, d \), and \( c < d - 1, d = \dim R \) (see [15]).

The socle \( \text{Hom}_R(k, H^d_I(M)) \) is in general not finite dimensional as has been shown at first by R. Hartshorne (see [5]) by disproving a question by A. Grothendieck about cofiniteness of local cohomology, i.e. the finiteness of \( \text{Hom}_R(R/I, H^d_I(R)) \). In their paper Huneke and Koh (see [7]) studied cofiniteness of various ideals. Moreover they showed that for a field \( k \) of characteristic zero and a polynomial ring \( R = k[x_{ij}, 1 \leq i \leq 2, 1 \leq j \leq 3] \) the local cohomology \( H^3_R(R) \) is not \( I \)-cofinite for \( I \) the ideal generated by the maximal minors of the \( 2 \times 3 \)-matrix \( (x_{ij}) \). A large class of local cohomology modules with infinite socle has been constructed by T. Marley and C. Vassilev (see [11]).

The main topic of the present note is to discuss and relate the properties (a) and (b) above in the non-smooth situation of certain determinantal rings. More precisely:

Theorem 1.1. Let \( R \) denote the completion of the coordinate ring of the vanishing ideal of the \( 2 \times 2 \) minor of an \( m \times n \)-matrix of variables and therefore \( \dim R = m + n - 1 \). For the ideal \( I \) generated by the first \( n - 1 \) columns we have \( \dim R/I = m \) and \( \text{grade} \ I = n - 1 \). Let \( \omega_R \) denote the canonical module of \( R \).

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(a) $\text{Hom}_R(k, H^{m+n-2}_{\omega}(\omega))$ and $\text{Hom}_R(k, H^{m+n-2}(\omega))$ are not finite dimensional, that is $I$ is not cofinite and $H^i_\omega(\omega) = 0$ for $i \neq n-1, m+n-2$.

(b) $\text{Hom}_R(H^{-1}_i(\omega), H^{-1}(\omega))$ is an $R$-algebra of finite type, not finitely generated over $R$.

(c) $\text{Hom}_R(k, H^m_i(\omega))$ and $\text{Hom}_R(k, H^0_m(H^{m+n-2}(\omega)))$ are not finite dimensional.

In the case of $m = n$ the ring $R$ is a Gorenstein ring and $\omega_R \cong R$. For $m = n = 2$ we recover R. Harthorne’s example (see [5]) by completely different arguments. For an ideal $I$ the cohomological dimension is defined by $\text{cd} \ I = \sup \{ i \in \mathbb{N} | H^i(I) \neq 0 \}$ (introduced by R. Hartshorne (see [4])). Whence, $\text{cd} \ I = m + n - 2$ for the examples above. That is, possible non-cofinite ideals could occur in the highest non-vanishing cohomological level. In the case of a $d$-dimensional Gorenstein ring $(R, m)$ the socle dimension $\text{Hom}_R(k, H^d_m(\omega_R))$ are called $\tau$-numbers of type $(i, j)$ of $I$ (see [10]). In the case of a regular local ring containing a field they coincide with the Lyubeznik numbers introduced by Lyubeznik in [9] (see [10, 3.5] for the details). Therefore, the above results yield further information about these $\tau$-numbers.

In Section 2 we prove the essentials about local cohomology for our purposes. Theorem 2.2 is the needed technical result for our constructions. Furthermore, we introduce the basics for the construction of our examples, based about some results of Segre varieties. In Section 3 we prove the statements of the examples introduced in Section 2. For the needed results about Commutative Algebra we refer to Matsumura’s book [12]. For a few homological arguments resp. some facts about determinantal ideals we refer to [16] resp. to [2].

2. Preliminaries and constructions

At the beginning let us recall a few basics from local duality.

Notations and Recalls 2.1. (A) Let $(R, m)$ denote a $d$-dimensional Cohen-Macaulay ring with $E = E_R(R/m)$ the injective hull of its residue field. Suppose that $R$ possesses a normalized dualizing complex $D$ in the sense of [16, 11.4.6]. Then $R$ admits a canonical module $\omega_R$ (see [13]). Moreover, for an arbitrary $R$-module $X$ there is the following Local Duality Theorem

$$H^i_m(X) \cong \text{Hom}_R(\text{Ext}^d_{\omega}(X, \omega), E)$$

for all $i \geq 0$ (see e.g. [16, 10.4.3]). For further properties of $\omega_R$ we refer to [16] and [13].

(B) With the notation of (A) let $M$ denote a finitely generated $R$-module. Then $\omega_R(M)$, the canonical module of $M$ (in the sense of [13]), exists and there is an isomorphism $\text{Ext}^c_R(M, \omega_R) \cong \omega_R(M)$, where $c = \dim R - \dim M$ (see also [13] for more details).

(C) With the notation of (A) it follows that $\omega_R \rightarrow D$, a minimal injective resolution of $\omega_R$, is a normalized dualizing complex in the sense of [16, 11.4.6]. Let $M$ denote a finitely generated $R$-module. Then there is a natural morphism

$$M \rightarrow \text{Hom}_R(\text{Hom}_R(M, D), D)$$

that is an isomorphism in cohomology. Let $c = d - \dim M$, then it induces a natural homomorphism

$$M \rightarrow \text{Ext}^c_R(M, \omega_R), \omega_R) = \omega_R(\omega_R(M)).$$

(D) It is known that for an $R$-module $M$ the module $\omega_R(\omega_R(M))$ is the $S_2$-ification of $M$ (see [13]). Moreover, the natural homomorphism $M \rightarrow \omega_R(\omega_R(M))$ induces a short exact sequence

$$0 \rightarrow M/u(M) \rightarrow \omega_R(\omega_R(M)) \rightarrow C \rightarrow 0,$$

where $u(M)$ denotes the intersection of those primary components of $M$ that are of highest dimension $\dim M$ and $\dim C \leq \dim M - 2$ for $C$ the cokernel. Let $I \subset R$ an ideal of
grade $c$ in the Cohen-Macaulay ring $R$. Then we get an inverse system of homomorphisms $R/I^a \to \Ext_R^a(Ext_R^a(R/I^a, \omega_R), \omega_R)$. By passing to the inverse limit there is a homomorphism
\[
\hat{R}^I \to B := \varprojlim \Ext_R^a(Ext_R^a(R/I^a, \omega_R), \omega_R) \cong \Ext_R^a(H^a_I(\omega_R), \omega_R),
\]
where $\hat{R}^I$ denotes the $I$-adic completion of $I$. Note that the inverse maps are induced by $R/I^a+1 \to R/I^a$. By a slight modification of the arguments of [15, Section 3] it follows that $B$ is a commutative ring.

(E) For an arbitrary $R$-module $X$ there is a natural homomorphism $R \to \Hom_R(X, X), r \mapsto \phi(r)$, the multiplication by $r \in R$ on $X$. Its kernel is $\Ann_R X$. In general the endomorphism ring $\Hom_R(X, X)$ is not commutative.

For the property of the ring $B$ in 2.1 (D) we need a few more intrinsic properties.

**Theorem 2.2.** Let $(R, m, k)$ be a $d$-dimensional complete local Cohen-Macaulay ring. Let $\omega_R$ denote its canonical module. For an ideal $I \subset R$ with $c = \text{grade } I$ we have:

(a) There are isomorphisms
\[
\Hom_R(H^d_I(\omega_R), H^d_I(\omega_R)) \cong \Ext^d_R(H^d_I(\omega_R), \omega_R) \cong \Hom_R(H^d_{m^c}(H^d_I(\omega_R)), E).
\]

(b) The endomorphism ring $\Hom_R(H^d_I(\omega_R), H^d_I(\omega_R))$ is a finitely generated $R$-module if and only if $\dim_k \Hom_R(k, H^d_{m^c}(H^d_I(\omega_R)))$ is finite.

(c) The natural ring homomorphism $\rho : R \to \Hom_R(H^d_I(\omega_R), H^d_I(\omega_R))$ is onto if and only if $\dim_k \Hom_R(k, H^d_{m^c}(H^d_I(\omega_R))) = 1$

**Proof.** (a): In order to prove the first isomorphism we refer to [14, 2.2 (d)]. In order to proof the second one we have that
\[
\Ext^d_R(H^d_I(\omega_R), \omega_R) \cong \Hom_R(H^d_{m^c}(H^d_I(\omega_R)), E)
\]
as follows by the Local Duality for the complete local Cohen-Macaulay ring $R$ (see 2.1 (A)).

(b): By virtue of (a) we have isomorphisms $B/m^n \cong \Hom_R(\Hom_R(R/m^n, H^d_{m^c}(H^d_I(\omega_R))), E)$ that form an inverse system. By passing to the inverse limit there are isomorphisms
\[
\hat{B}^n = \varprojlim \Hom_R(\Hom_R(R/m^n, H^d_{m^c}(H^d_I(\omega_R))), E) \cong \Hom_R(H^d_{m^c}(H^d_I(\omega_R)), E) \cong B
\]
because of $\text{Supp}_R H^d_{m^c}(H^d_I(\omega_R)) \subseteq V(m)$. That is, $B$ is $m$-adic complete. Then $B$ is a finitely generated $R$-module if and only if $\dim_k B/mB < \infty$ (see e.g. [12, Theorem 8.4]).

(c): Since $H^d_I(\omega_R) \neq 0$ it follows that the natural homomorphism
\[
\rho \otimes 1_k : R \otimes_R k \to \Hom_R(H^d_I(\omega_R), H^d_I(\omega_R)) \otimes_R k
\]
is not zero. Then the statement follows by (b). \qed

**Construction 2.3.** (A) Let $k$ denote a field. Let $m, n \geq 2$ denote integers and let
\[
X = \begin{pmatrix}
x_{11} & \cdots & x_{1n} \\
\vdots & \ddots & \vdots \\
x_{m1} & \cdots & x_{mn}
\end{pmatrix}
\]
denote a $m \times n$ matrix of mn variables over $k$. Let $P = k[[X]]$ denote the formal powers series ring in the mn variables of $X$. We define $R = P/I_2$, where $I_2 = I_2(X)$ denotes the ideal generated by the $2 \times 2$ minors of $X$. Then $R$ is a local Cohen-Macaulay domain of dimension $m + n - 1$ (see [6]).

(B) We put $\bar{x}_i = x_{i1}, \ldots, x_{in}, i = 1, \ldots, n$, the elements of the $i$-th column of $X$ and $\bar{x} = \bar{x}_i$ the elements of the last column. We define $I = (\bar{x}_1, \ldots, \bar{x}_{n-1})R$ the ideal generated by the elements
of the first $n - 1$ columns of $X$. Then $\dim R/I = m$ and height $l = \text{grade } I = n - 1$. In the following we recall the form ring $G_I(R)$ of $I \subset R$. First we define a $m \times n$-matrix

$$
\mathcal{Y} = \begin{pmatrix}
T_{11} & \cdots & T_{1,m-1} & x_{1n} \\
\vdots & \ddots & \vdots & \vdots \\
T_{m1} & \cdots & T_{m,n-1} & x_{mn}
\end{pmatrix}
$$

where $T_{ij}, 1 \leq i \leq m, 1 \leq j \leq n - 1$, are variables over $R$. Let $J = I_2(\mathcal{Y})$ the ideal generated by the $2 \times 2$-minors of $\mathcal{Y}$. Let $T$ denote the set of all variables $T_{ij}, 1 \leq i \leq m, 1 \leq j \leq n - 1$ of degree 1 and let $\mathcal{G} = (R/I)[T]$. Then consider the induced homogeneous map

$$
\Theta : \mathcal{G}/J \rightarrow \text{Gr}_1(R), \quad T_{ij} + J \mapsto x_{ij} + I/\hat{I}^2, \quad 1 \leq i \leq m, 1 \leq j \leq n - 1.
$$

Note that it is well defined and surjective. Clearly, $\mathcal{G}/J$ is a Cohen-Macaulay determinantal domain with $\dim \mathcal{G}/J = m + n - 1$, i.e. $J$ is a prime ideal. Moreover $\text{Gr}_1(R)$ is a ring with $\dim \text{Gr}_1(R) = m + n - 1$. Let $J'$ be the preimage of $\ker \Theta$ in $\mathcal{G}$. Let $\mathcal{P} \subset \text{Spec} \mathcal{G}$ denote the preimage of a highest dimensional prime ideal in $\text{Gr}_1(R)$. Then $J \subseteq J' \subseteq \mathcal{P}$ and $\dim \mathcal{G}/J = \dim \mathcal{G}/\mathcal{P}$ and therefore $J = \ker \Theta$. That is, $\Theta$ is an isomorphism and $\text{Gr}_1(R)$ is a domain.

(C) Next we introduce the ring $S = k[[x, \underline{a}]]$, where $x = x_1, \ldots, x_m$ with $x_i = x_m, i = 1, \ldots, m$ and $\underline{a} = a_1, \ldots, a_n$ are sequences of variables over $k$. Then $S$ is a regular local ring with $\underline{a} = a_1, \ldots, a_n$ a regular sequence. Therefore $\text{Gr}_{(\underline{a})}(S) \cong (S/\underline{a})[T] \cong k[[x]][T]$ with $T = T_1, \ldots, T_n$ variables of degree 1. We define a ring homomorphism

$$
\phi : R \rightarrow S, \quad x_i \mapsto a_i x_j, \quad i = 1, \ldots, n - 1, j = 1, \ldots, m.
$$

Note that $S$ is a domain and $\dim R = \dim S = m + n + 1$. Therefore $\phi$ is injective. The homomorphism $\phi$ extends to a homomorphism

$$
\psi : \text{Gr}_1(R) \rightarrow \text{Gr}_{(\underline{a})}(S), \quad T_{ij} + J \mapsto x_j T_i, \quad i = 1, \ldots, m, j = 1, \ldots, n - 1.
$$

Since both $\text{Gr}_1(R)$ and $\text{Gr}_{(\underline{a})}(S)$ are domains of the same dimension $\psi$ is injective.

3. Examples and Remarks

In the following we denote by $\omega_R$ the canonical module of a Cohen-Macaulay ring $R$. For basic definitions and properties of $\omega_R$ we refer to [2, 3, 3], the generalization in [13] and the summary in 2.1. For a local ring $(R, m)$ we denote by $E = E_R(R/m)$ the injective hull of the residue field.

**Example 3.1.** We fix the notation of 2.3, that is, $R$ is the determinantal Cohen-Macaulay ring of dimension $m + n - 1$ with $I = (x_i, \ldots, x_{i-1})R$ the ideal generated by the elements of the first $n - 1$ columns of $X$. Then there are the following results:

(a) $H^I_i(\omega_R) \neq 0, i = n - 1, m + n - 2$, that is, $\text{cd}(I) = m + n - 2$.

(b) $\dim_k \text{Hom}_R(k, H^{m+n-2}_I(\omega_R))$ and $\dim_k \text{Hom}_R(k, H^{m+n-2}_I(R))$ are not finite, that is $I$ is not cofinite.

(c) $\text{Hom}_R(H^{n-1}_I(\omega_R), H^{n-1}_I(\omega_R))$ is a commutative Noetherian ring, not finitely generated over $R$.

(d) $\dim_k \text{Hom}_R(k, H^m_m(H^{n-1}_I(\omega_R)))$ and $\dim_k \text{Hom}_R(k, H^0_m(H^{m+n-2}_I(\omega_R)))$ are not finite.

**Proof.** The ring homomorphism $\phi$ of 2.3 (C) induces ring homomorphisms

$$
\phi^a : R/I^a \rightarrow S/(\underline{a})^a S \quad \text{for all } a \geq 1.
$$
which are well-defined. We claim that $\phi^a$ is injective for all $\alpha \geq 1$. The restriction $\psi^a$ of the injection $\psi$ of 2.3 (C) to degree $\alpha$ yields injective homomorphisms

$$\psi^a : I^\alpha / I^{\alpha+1} \to (\overline{a})^a S / (\overline{a})^{a+1} S \text{ for all } \alpha \geq 0.$$ 

Let $D_\alpha := \Coker \psi^a$ and let $\{x_i a_j : i = 1, \ldots, n-1, j = 1, \ldots, m\}$. We define $C_\alpha = \Coker \phi^a = S / ((\overline{a})^\alpha S + S[[x \cdot a]])$ because of $\Im \phi = S[[x \cdot a]]$. Now it follows that

$$D_\alpha \cong ((\overline{a})^\alpha S + S[[x \cdot a]]) / ((\overline{a})^{\alpha+1} S + S[[x \cdot a]]).$$

as a consequence of the following commutative diagram with exact rows and the snake lemma

$$
\begin{array}{ccccccc}
0 & \to & I^\alpha / I^{\alpha+1} & \to & R / I^{\alpha+1} & \to & R / I^\alpha & \to & 0 \\
\downarrow \psi^a & & \downarrow \phi^a+1 & & \downarrow \phi^a & & \\
0 & \to & (\overline{a})^\alpha S / (\overline{a})^{\alpha+1} S & \to & S / (\overline{a})^{\alpha+1} S & \to & S / (\overline{a})^\alpha S & \to & 0.
\end{array}
$$

Since $\psi^a$ is injective for all $\alpha \geq 0$ as the restriction of $\phi^a$ it follows by induction that $\phi^a$ is injective too. Therefore $D_\alpha$ is spanned by $\{x_i a_j \}^a$ with $0 \leq i < \alpha$ over $k$. As an $R$-module it is finitely generated. For the radical of the annihilator it follows that $\text{Rad}_S \text{Ann}_S D_\alpha = (x, a)$. So that $C_\alpha, \alpha \geq 1,$ is an $R$-module of finite length. There is the short exact sequence

$$0 \to R / I^\alpha \to S / (\overline{a})^\alpha S \to C_\alpha \to 0 \text{ for all } \alpha \geq 1.$$ 

Because of $C_\alpha \cong \oplus_{\beta=1}^{\alpha-1} D_\beta$ it follows that it is an $R$-module of finite length. Since $S / (\overline{a})^\alpha S$ is a finitely generated Cohen-Macaulay $R$-module it implies that $H^1_m(R / I^\alpha) \cong C_\alpha$ and $H^i_m(R / I^\alpha) = 0$ for all $i \neq 1, m$. The above short exact sequences form an inverse system of short exact sequences. By passing to the inverse limit we get a short exact sequence

$$0 \to R / I^\alpha \to S \to \lim_{\leftarrow i} H^1_m(R / I^\alpha) \to 0.$$ 

Note that the first family is given by surjective maps. By virtue of the Local Duality Theorem for a Cohen-Macaulay ring (see 2.1 (A)) there are isomorphisms

$$H^i_m(R / I^\alpha) \cong \text{Hom}_R(\text{Ext}^{m+n-1-i}(R / I^\alpha, \omega_R), E)$$

for all $\alpha \geq 1$ and by passing to the inverse limit $\lim_{\leftarrow i} H^i_m(R / I^\alpha) \cong \text{Hom}_R(H^{m+n-1-i}_I(\omega_R), E)$ for all $i$. Therefore $H^i_I(\omega_R) = 0$ for $i \neq n-1, m+n-2$ and

$$\lim_{\leftarrow i} H^i_m(R / I^\alpha) \cong \text{Hom}_R(H^{m+n-2}_I(\omega_R), E) \neq 0.$$ 

Because $S$ is not a finitely generated $R$-module $\text{Hom}_R(H^{m+n-2}_I(\omega_R), E)$ is not finitely generated too. By the isomorphism

$$k \otimes_R \text{Hom}_R(H^0_I(\omega_R), E) \cong \text{Hom}_R(\text{Hom}_R(k, H^{m+n-2}_I(\omega_R)), E)$$

and by Matlis Duality $\dim_k \text{Hom}_R(k, H^{m+n-2}_I(\omega_R))$ is not finite. Moreover, $H^{m+n-2}_I(\omega_R) \cong H^{m+n-2}_I(R) \otimes_R \omega_R$ so that $\text{Hom}_R(H^{m+n-2}_I(\omega_R), E) \cong \text{Hom}_R(\omega_R, \text{Hom}_R(H^{m+n-2}_I(R), E))$. Because $\omega_R$ is a finitely generated $R$-module $\text{Hom}_R(H^{m+n-2}_I(R), E)$ is not finitely generated and $0$ as above $\dim_k \text{Hom}_R(k, H^{m+n-2}_I(R))$ is not finite. That is, (a) and (b) of 3.1 are proved.

Now recall that $S / (\overline{a})^\alpha S$ is an $m$-dimensional Cohen-Macaulay ring, finitely generated over $R$ and $\dim_R C_\alpha = 0$. Therefore

$$S / (\overline{a})^\alpha S \cong \omega_R(R / I^\alpha) \cong \text{Ext}^{n-1}_R(\text{Ext}^{n-1}_R(R / I^\alpha, \omega_R), \omega_R)$$

(by view of 2.1 (D)). By passing to the inverse limit of the corresponding inverse and by Theorem 2.2 (a) it yields that $S \cong \text{Hom}_R(H^{n-1}_I(\omega_R), H^{n-1}_I(\omega_R))$, whence (c) is shown. The first
claim in (d) follows by 2.2 (b) since $S$ is not finitely generated over $R$. The second one is clear by (b) since $\text{Supp}_R H^n_{\mathfrak{m}}(R) = V(\mathfrak{m})$. \hfill \Box$

For the case of $m = n$ in 3.1 we get that $R$ is a Gorenstein ring and therefore $\omega_R \cong R$. For $m = n = 2$ we recover Hartshorne’s example in this different context with additional properties.

**Question 3.2.** Let $I \subset R$ denote an ideal of a local ring $(R, \mathfrak{m})$ with $c = \text{grade } I$. We do not know whether the endomorphism ring $\text{Hom}_R(H^c_I(R), H^c_I(R))$ is in general commutative and Noetherian.

For further results about the endomorphism ring $\text{Hom}_R(H^c_I(R), H^c_I(R))$ we refer also to [14].

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