Based on the Liouville-von Neumann equation, we obtain closed system of equations for the description of a qutrit or coupled qutrits in arbitrary time-dependent external magnetic field. The dependence of the dynamics on the initial states and magnetic field modulation is studied analytically and numerically. We compare the relative entanglement measure’s dynamics in the bi-qutrit system with permutation particle symmetry. We find the magnetic field modulation which retains the entanglement in the system of two coupled qutrits. Analytical formulas for entanglement measures in the chain from 2 to 6 qutrits are presented.

PACS numbers: 03.67.Bg Entanglement production and manipulation
03.67.Mn Entanglement measures, witnesses, and other characterizations

I. INTRODUCTION

Multi-level quantum systems are studied intensively, since they have wide applications. Some of the existent analytical results for spin 1 [1] are derived in terms of the coherent vector [2]. The class of exact solutions for a three-level system is given in Ref. [3]. The application of coupled multi-level systems in quantum devices is actively studied [4]. The study of these systems is topical in view of possible applications for useful work in microscopic systems [5]. Exact solutions for two uncoupled qutrits interacting with vacuum are obtained in Ref. [6]. For the case of the qutrits interacting with stochastic magnetic field exact solutions are obtained in Ref. [7]. Exact solutions for coupled qutrits in magnetic field as far as we know were not found.

The entanglement in multi-particle coupled systems is an important resource for many problems in quantum information science, but its quantitative value is difficult because of different types of entanglement. Multi-dimensional entangled states are interesting both for the study of the foundations of quantum mechanics and for the topicality of developing new protocols for quantum communication. For example, it was shown that for maximally entangled states of two quantum systems, the qudits break the local realism stronger than the qubits [8], and that the entangled qudits are less influenced by the noise than the entangled qubits. Using entangled qudits or qudits instead of qubits is more protective from interception. From the practical point of view, it is clear that generating and saving the entanglement in the controlled manner is the primary problem for the realization of the quantum computers. The maximally entangled states are best suited for the protocols of quantum teleportation and quantum cryptography. The entanglement and the symmetry are the basic notions of the quantum mechanics. We study the dynamics of multipartite systems, which are invariant at any subsystem permutation. The aim of our work is finding exact solutions for the dynamics of coupled qutrits interacting with alternating magnetic field as well as the comparative analysis of the entanglement measures in the chain of qutrits.

The rest of the paper is organized as following. The Hamiltonian of the anisotropic qutrit in arbitrary alternating magnetic field is described in Sec. II. Then the system of equations for the description of the qutrit dynamics is derived in the Bloch vector representation. We introduce the consistent magnetic field, which describe entire class of field forms. In section III we derive the system of equations for the description of the dynamics of two coupled qutrits in the consistent field and find the analytical solution for the density matrix in the case of anisotropic interaction. Analytical formulas, which describe the entanglement in spin chains from 2 to 6 qutrits, are presented in Sec. IV. The results are demonstrated graphically in Sec. V for concrete parameters. The brief conclusions are given in Sec. VI. The auxiliary analytical results are presented in the Appendices.
II. QUTRIT

A. Qutrit Hamiltonian

We take the qutrit Hamiltonian (for the spin-1 particle) in the space of one qutrit $C^3$ in the basis $|1\rangle = (1,0,0)$, $|0\rangle = (0,1,0)$, and $|-1\rangle = (0,0,1)$, in external magnetic field $\mathbf{h} = (h_1,h_2,h_3)$ with anisotropy, in the form

$$\hat{H} = h_1S_1 + h_2S_2 + h_3S_3 + Q(S_3^2 - 2/3E) + d(S_1^2 - S_2^2),$$

(1)

where $h_1$, $h_2$, $h_3$ are the Cartesian components of the external magnetic field in the frequency units (we assume $\hbar = 1$, Bohr magneton $\mu_B = 1$); $S_1$, $S_2$, $S_3$ are the spin-1 matrices (see Appendix A); $E$ stands for the $3 \times 3$ unity matrix; $Q$, $d$ are the anisotropy constants. When the constants $Q$, $d$ are zeros, then the two Hamiltonian eigenvalues are symmetrically placed in respect to the zero level.

B. Liouville-von Neumann equation

The qutrit dynamics in the magnetic field we describe in the density matrix formalism with the Liouville-von Neumann equation

$$i\partial_t \rho = [\hat{H}, \rho], \quad \rho(t = 0) = \rho_0.$$  

(2)

It is convenient to rewrite Eq. (2) presenting the density matrix $\rho$ in the decomposition with the full set of orthogonal Hermitian matrices $C_\alpha$ (further the summation over Greek indices will be from 0 to 8 and over the Latin ones from 1 to 8, see Appendix A)

$$\rho = \frac{1}{\sqrt{6}} C_\alpha R_\alpha = \begin{pmatrix}
\frac{1}{\sqrt{6}} + \frac{R_1}{\sqrt{18}} + \frac{R_6}{\sqrt{18}} & \frac{R_1 + i(R_2 + R_3)}{\sqrt{12}} & \frac{R_1 - i(R_2 - R_3)}{\sqrt{12}} \\
\frac{R_4 + R_5}{\sqrt{6}} & \frac{1}{3} - \frac{R_6}{\sqrt{18}} & \frac{i(R_4 + R_5)}{\sqrt{12}} \\
\frac{R_4 - R_5}{\sqrt{6}} & \frac{R_1 - i(R_2 - R_3)}{\sqrt{12}} & \frac{1}{3} + \frac{R_6}{\sqrt{18}}
\end{pmatrix}.$$  

(3)

Since $\text{Tr} C_i = 0$ for $1 \leq i \leq 8$, then from the condition $\text{Tr} \rho = R_0$ it follows that $R_0 = 1$. And although the results are independent of the basis choice, in this basis the functions $R_i = \text{Tr} \rho C_i$ have the concrete physical meaning [10]. The values $R_1, R_2, R_3$ are the polarization vector Cartesian components; $R_4$ is the two-quantum coherence contribution in $R_2$; $R_5$ is the one-quantum anti-phase coherence contribution in $R_2$; $R_6$ is the contribution of the rotation between the phase and anti-phase one-quantum coherence; $R_7$ is the one-quantum anti-phase coherence contribution in $R_1$; $R_8$ is the two-quantum coherence contribution in $R_1$.

Under the unitary evolution the length of the generalized Bloch vector

$$b = \sqrt{R_2^2}$$

(4)

is conserved. The length of the generalized vector [4] for pure states equals to $\sqrt{2}$. Since $i\partial_t \rho^n = [\hat{H}, \rho^n]$ $(n = 1, 2, 3, \ldots)$, then under unitary evolution there is countable number of the conservation laws $\text{Tr} \rho = c_1 = 1$, $\text{Tr} \rho^2 = c_2, \ldots$, from which only $c_2, c_3$ are algebraically independent [11]. Additional quadric invariants of motion can be easily obtained after equating the matrix elements in defining the pure state. For example, two of these invariants, which follow from the expression $(\rho^2 - \rho)_{13} = 0$, have the form

$$R_1^2 - R_2^2 + R_5^2 - R_6^2 - 2\sqrt{\frac{2}{3}(1 - \sqrt{2}R_6)}R_8 = 0, \quad R_5R_7 - R_1R_2 + \frac{2}{\sqrt{3}}(\frac{1}{\sqrt{2}} - R_6)R_4 = 0.$$  

(5)

For numerical calculations, these invariants control also the signs of the values $R_i$ and thus the using of the invariants is useful when the analytical solutions are difficult to find. According to the Kelly-Hamilton theorem, the density matrix $\rho$ satisfies to its characteristic equation

$$\rho^3 - \rho^2 + \frac{2 - b^2}{6} \rho - \text{det} \rho E = 0.$$  

(6)
From equation (9) it follows that the density matrix determinant $\det \rho = (\text{Tr} \rho^3 - \text{Tr} \rho^2) / 3 + (2 - b^2) / 18$ is also the motion invariant. The Liouville-von Neumann equation in terms of the functions $R_i$ takes the form of the closed system of 8 real differential first-order equations. This system of equations in the compact form can be written as following [11 12]:

$$\partial_t R_i = e_{ij} h_i R_j,$$

(7)

where $e_{ij}$ are the antisymmetrical structure constants, $h_i = 2(h_1, h_2, h_3, 0, 0, 0, \sqrt{3}, 0, d)$ are the Hamiltonian components in the basis $C_n$ (see Appendix A).

C. The consistent field

Consider the qutrit dynamics in the alternating field of the form

$$\tilde{h}(t) = (\omega_1 \text{cn}(\omega t|k), \omega_1 \text{sn}(\omega t|k), \omega_0 \text{dn}(\omega t|k)),$$

(8)

where $\text{cn}, \text{sn}, \text{dn}$ are the Jacobi elliptic functions [13]. Such field modulation under the changing of the elliptic modulus $k$ from 0 to 1 describes the whole class of field forms from trigonometric ($\text{cn}(\omega t|0) = \cos \omega t$, $\text{sn}(\omega t|0) = \sin \omega t$, $\text{dn}(\omega t|0) = 1$) [14] to the exponentially impulse ones ($\text{cn}(\omega t|1) = \cosh \omega t$, $\text{sn}(\omega t|1) = \sinh \omega t$, $\text{dn}(\omega t|1) = \cosh \omega t$) [15]. The elliptic functions $\text{cn}(\omega t|k)$ and $\text{sn}(\omega t|k)$ have the real period $\frac{4K}{\omega}$, while the function $\text{dn}(\omega t|k)$ has the two times smaller period. Here $K$ is the full elliptic integral of the first kind [13]. In other words, even though the field is periodic with common real period $\frac{4K}{\omega}$, but as we can see, the frequency of the longitudinal field amplitude modulation is two times higher than the one of the transverse field. Such field we call consistent.

Let us make use of the substitution $\rho = \alpha_1^{-1} \alpha_1$. Then we obtain the equation for the matrix $r$ in the form

$$i \partial_t r = [\alpha_1 \tilde{h}, \alpha_1^{-1} - i \alpha_1 \partial_t (\alpha_1^{-1})],$$

(9)

with the matrix

$$\alpha_1 = \begin{pmatrix} f & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f^{-1} \end{pmatrix},$$

(10)

where $f(\omega t|k) = \text{cn}(\omega t|k) + i \text{sn}(\omega t|k)$. Since

$$\alpha_1 S_1 \alpha_1^{-1} = S_1 \text{cn}(\omega t|k) - S_2 \text{sn}(\omega t|k), \quad \alpha_1 S_2 \alpha_1^{-1} = S_1 \text{sn}(\omega t|k) + S_2 \text{cn}(\omega t|k), \quad \alpha_1 S_3 \alpha_1^{-1} = S_3,$$

(11)

then the equation for the matrix $r$ without taking into account the anisotropy can be written as following

$$i \partial_t r = [\omega_1 S_1 + \delta \text{dn}(\omega t|k) S_3, r], \quad r(t = 0) = \rho_0, \quad \delta = \omega_0 - \omega.$$  

(12)

At $k = 0$ equation (12) describes the dynamics of the qutrit in the circularly polarized field [14 16 17]. The exact solutions of this equation are known and at some initial conditions the explicit formulas are given in Ref. [18]. At exact resonance, $\omega = \omega_0$ it is straightforward to present Eq. (2) in the deformed field ($k \neq 0$) (8) for the given initial condition $\rho = \rho_0$:

$$\rho(t) = \alpha_1^{-1} e^{-\omega_1 t S_1} \rho_0 e^{\omega_1 t S_1} \alpha_1.$$  

(13)

Explicit solutions for some specific initial conditions are given in the Appendix B, Eqs. (39) - (44). From the explicit exact solutions in the deformed field at resonance $\delta = 0$ one can see that the populations and the transition probabilities do not depend on the field deformation (it is independent of the $k$ modulus). Consider the solution of Eq. (12) far from the resonance in the form of $\delta$ power expansion

$$r(t) = r^{(0)}(t) + r^{(1)}(t) + \cdots.$$  

(14)

Then we put the expansion (14) in Eq. (12) and equate the same degree terms. As the result we obtain the system of equations for finding $r^{(1)}(t)$:

$$i \partial_t r^{(0)} = \omega_1 [S_1, r^{(0)}],$$  

(15a)
We multiply Eq. (15) to the left by the matrix $e^{-\rightarrow \hbar_{t}S_{1}}$ and to the right by the matrix $e^{-\rightarrow \hbar_{t}S_{1}}$ for formation of the integrating multiplier [19]. Now finding the terms $\nu^{(l)}$ in the series (14) is defined by the previous ones $\nu^{(l-1)}$ as following

$$\nu^{(l)}(t) = -i\bar{\delta} \int_{0}^{t} d\tau e^{\hbar_{t-\tau}S_{1}} \delta \nu^{(l-1)}(\nu_{t'}|k)\nu^{(l-1)}(\nu_{t'|}k)e^{-\hbar_{t-\tau}S_{1}}.$$  \hspace{1cm} (16)

III. BI-QUTRIT

In the space $C^{3} \otimes C^{3}$ the two-qutrit density matrix can be written in the Bloch representation

$$\rho = \frac{1}{6} R_{\alpha\beta} C_{\alpha} \otimes C_{\beta}, \; R_{00} = 1, \; \rho(t = 0) = \rho_{0}, \hspace{1cm} (17)$$

where $\otimes$ denotes the direct product. The functions $R_{m0}, R_{0m}$ characterise the individual qutrits and functions $R_{mn}$ characterise their correlations. The length of the generalized Bloch vector $\sqrt{R_{\alpha\beta} - 1}$ for pure states equals $2\sqrt{2}$. Consider the Hamiltonian of the system of two qutrits with anisotropic and exchange interaction in magnetic field in the following form

$$H_{2} = (\hbar \vec{S} + Q(S_{3}^{2} - 2/3E) + d(S_{1}^{2} - S_{2}^{2})) \otimes E + E \otimes (\hbar \vec{S} + \tilde{Q}(S_{3}^{2} - 2/3E) + \tilde{d}(S_{1}^{2} - S_{2}^{2}) + JS_{1} \otimes S_{1} = \frac{1}{2} h_{\alpha\beta} C_{\alpha} \otimes C_{\beta}, \hspace{1cm} (18)$$

where $\hbar$ and $\vec{h}$ are the magnetic field vectors in frequency units, which operate on the first and the second qudits respectively, and $J$ is the constant of isotropic exchange interaction.

The system of equations for two qutrits takes the real form in terms of the functions $R_{m0}, R_{0m}, R_{mn}$ as the closed system of 80 differential equations [12], supplemented by the initial conditions

$$\partial_{t}R_{m0} = \sqrt{\frac{2}{3}} e_{pim}(h_{p0}R_{0p} + h_{p1}R_{1p}), \; \partial_{t}R_{0m} = \sqrt{\frac{2}{3}} e_{pim}(h_{0p}R_{0i} + h_{tp}R_{it}), \hspace{1cm} (19a)$$

$$\partial_{t}R_{mn} = e_{pim} \left[ \sqrt{\frac{2}{3}}(h_{pm}R_{0p} + h_{p0}R_{pm}) + g_{rm}h_{pr}R_{ri} \right] + e_{pin} \left[ \sqrt{\frac{2}{3}}(h_{mp}R_{0i} + h_{0p}R_{mi}) + g_{rln}h_{rp}R_{li} \right], \hspace{1cm} (19b)$$

where by definition

$$\text{Tr} \rho C_{\alpha} \otimes C_{\beta} = \frac{2}{3} R_{\alpha\beta} \hspace{1cm} (20)$$

and $h_{p0} = \sqrt{6}(\hbar_{x}, 0, 0, \tilde{Q}_{x}, 0, 0, 0, d)$, $h_{0p} = \sqrt{6}(\hbar_{x}, 0, 0, \frac{\tilde{Q}_{y}}{\sqrt{3}}, 0, 0, 0, d)$, $h_{11} = h_{22} = h_{33} = 2J$ are the Hamiltonian expansion coefficients in the basis $C_{\alpha} \otimes C_{\beta}$ (other coefficients equal to zero). In equations (19) Latin indices $m, n$ take the values from 1 to 8. Numerical values for the structure constants $e_{pim}, g_{rln}$ are given in Appendix A.

The energy of the coupled qutrits in terms of the correlation functions has the following form

$$E(t) = \frac{1}{3}(h_{p0}R_{p0} + h_{0p}R_{0p} + \sum_{i=1}^{3} h_{ii}R_{ii}). \hspace{1cm} (21)$$

We study the dynamics of two qutrits in the magnetic field $\vec{h} = (\omega_{1}\text{cn}(\omega_{t}|k)), \; \omega_{1}\text{sn}(\omega_{t}|k), \; \omega_{0}\text{dn}(\omega_{t}|k), \; \vec{h} = (\omega_{2}\text{cn}(\omega_{t}|k), \; \omega_{2}\text{sn}(\omega_{t}|k), \; \omega_{0}\text{dn}(\omega_{t}|k))$ at the anisotropy constants equal to 0. We transform the matrix density $\rho = \alpha_{2}^{-1}r_{2} \alpha_{2}$ with the matrix $\alpha_{2} = \alpha_{1} \otimes \alpha_{1}$. Then equation for the matrix $r_{2}$ takes the form $i\partial_{t}r_{2} = [\vec{H}, r_{2}]$ with the transformed Hamiltonian
However, the Hamiltonian eigenvalues cannot be found in the simple analytic form because of the lowering of the system.

Given the exact solution, one can find the negative eigenvalues of the partly transposed matrix.

Conserved during the evolution, since the initial state and Hamiltonian are symmetric in respect to the particle permutation.

Functions is given in Appendix C. The correlation functions have the property.

Coefficients can be done by choosing, for example, the transformation matrix for spin-3/2 and spin-2 in the form.

For larger number of the qutrits with pairwise isotropic interaction, the generalization is evident. In the case of interaction of qudits with different dimensionality, the reduction of the original system to the system with constant coefficients can be done by choosing, for example, the transformation matrix for spin-3/2 and spin-2 in the form.

The absolute value of the sum of these eigenvalues is given by.

In the consistent field at resonance \( \omega = \omega_0 = \omega_1 = \hbar \) at equal \( \omega_1 = \omega_1 \), the Hamiltonian eigenvalues equal to 

\(-2J, -J, J, -2\omega_1, -J - \omega_1, J - \omega_1, -J + \omega_1, J + \omega_1, J + 2\omega_1\).

This allows to find the exact solution in the closed form for any initial condition, since the matrix exponent \( e^{iHt} \) in this case can be calculated analytically.

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However, the Hamiltonian eigenvalues cannot be found in the simple analytic form because of the lowering of the system symmetry.

IV. ENTANGLEMENT IN THE QUTRITS

A. Entanglement in the bi-qutrit

For the initial maximally entangled state, which is symmetrical at the particle permutation,

\[ |\psi> = \frac{1}{\sqrt{3}} \sum_{i=-1}^{1} |i> \otimes |i> , \]

in the consistent field at the resonance \( \omega = \omega_0 = \omega_1 = \hbar \), the exact solution for the correlation functions is given in Appendix C. The correlation functions have the property \( R_{\alpha \beta} = R_{\beta \alpha} \), i.e. the symmetry is conserved during the evolution, since the initial state and Hamiltonian are symmetric in respect to the particle permutation.

Given the exact solution, one can find the negative eigenvalues of the partly transposed matrix \( \varrho^{PT} = (T \otimes E)\varrho \) (here \( T \) denotes the transposition):

\[ \epsilon_1 = \epsilon_2 = -\frac{1}{27} \sqrt{69 + 28 \cos 3Jt - 16 \cos 6Jt}, \quad \epsilon_3 = -\frac{1}{27} (5 + 4 \cos 3Jt) . \]

The absolute value of the sum of these eigenvalues \( m_{W} \) defines the entanglement measure (negativity) between the qutrits \[20\].

The entanglement between the qutrits can be described quantitatively with the measure \[21\]

\[ m_{SM} = \sqrt{\frac{1}{8} (R_{ij} - R_{i0}R_{0j})^2}. \]

This measure equals to 0 for the separable state and to 1 for the maximally entangled state, and it is applicable for both pure and mixed states.

That is why for the maximally entangled initial state of two qutrits, the entanglement in the consistent field is defined by the formula with the found solution for the density matrix.

\[ m_{SM} = \frac{1}{\sqrt{6561}} \sqrt{4457 + 2776 \cos 3Jt - 632 \cos 6Jt - 56 \cos 9Jt + 16 \cos 12Jt} . \]
This measure is numerically equivalent to the measure $m_{VW}$ \cite{20, 22}, which is defined by the absolute value of the sum of the negative eigenvalues \cite{20} of the partly transposed matrix.

According to the definition \cite{23} for 2-qutrit pure state, the entanglement measure equals to

$$
\eta_2 = \frac{1}{2} \sum_{i=1}^{2} S_i,
$$

where $S_i = -\Tr \rho_i \log_3 \rho_i$ is the reduced von Neumann entropy, the index $i$ numerates the particles, i.e. the other particle are traced out.

Since the qutrit reduced matrix eigenvalues equal to $\lambda_1 = \lambda_2 = \frac{1}{27}(5 + 4 \cos 3Jt)$, $\lambda_3 = \frac{1}{27}(17 - 8 \cos 3Jt)$, then the entanglement measure in the bi-qutrit takes the form

$$
\eta_2 = -\sum_{i=1}^{3} \lambda_i \log_3 \lambda_i.
$$

Normalized by the unity, the measure I-concurrence, which is easy to calculate, is defined by the formulae \cite{24}

$$
m_I = \frac{\sqrt{3}}{2} \sqrt{2(1 - \Tr \rho_i^2)} = \frac{1}{9} \sqrt{57 + 32 \cos 3Jt - 8 \cos 6Jt},
$$

where $\rho_1 = \frac{1}{\sqrt{6}} C_{a} R_{a0}$ is the reduced qutrit matrix.

The time-dependence of the measure $m_{SM}$ for the symmetrical initial state

$$
|s> = \frac{1}{\sqrt{12}} \sum_{i\neq j} (|i > \otimes |j > + |j > \otimes |i >)
$$

takes the form

$$
m_{SM}^{|s>} = \frac{1}{\sqrt{209952}} \sqrt{102679 + 19136 \cos 3Jt + 29312 \cos 6Jt - 1024 \cos 9Jt + 800 \cos 12Jt};
$$

at $t = 0$ this measure equals to $\sqrt{23}/32$.

The measures $m_{VW}$, $m_{SM}$, $\eta_2$, $m_I$, $m_{SM}^{|s>}$ do not depend on the parameters of the consistent field, sign of the exchange constant at zero anisotropy parameters. It should be noted that the Wooters entanglement measure (concurrence) in the system of two qubits with the isotropic interaction in the circularly polarized field at resonance is also independent of the alternating field amplitude \cite{23}, but depends on the exchange constant magnitude and the initial conditions only.

At zero external field the entanglement measure \cite{24} takes the analytic form at equal non-zero anisotropy parameters $Q = d = \tilde{d} = \tilde{Q}$

$$
m_{SM}(Q) = \frac{1}{(9J^2 + 8QJ + 16Q^2)^2} \sum_{k=0}^{4} q_k \cos \left( k\sqrt{9J^2 + 8QJ + 16Q^2} t \right),
$$

where $q_0 = 4457J^8 + 11616QJ^7 + 47392Q^2J^6 + 85888Q^3J^5 + 163072Q^4J^4 + 194560Q^5J^3 + 221184Q^6J^2 + 131072Q^7J + 65536Q^8$; $q_1 = 8J^2(J + 2Q)^2(347J^4 + 518QJ^3 + 1440Q^2J^2 + 1504Q^3J + 1024Q^4)$; $q_2 = -8J^2(J + 2Q)^2(79J^4 + 76QJ^3 + 320Q^2J^2 + 448Q^3J + 256Q^4)$; $q_3 = -8J^3(7J - 4Q)(J + 2Q)^2(J + 4Q)$, $q_4 = 16J^4(J + 2Q)^4$.

### B. Entanglement in the chain of qutrits

We consider the Hamiltonian of the chain of $N$ qutrits with the pairwise isotropic interaction in the magnetic field $\overrightarrow{J}$ in the following form

$$
H_N = \sum \left( \overrightarrow{J}_1 \overrightarrow{S} \otimes \overrightarrow{E} \otimes \cdots \otimes \overrightarrow{E} + \overrightarrow{J}_2 \overrightarrow{S} \otimes \overrightarrow{E} \otimes \cdots \otimes \overrightarrow{E} \right),
$$

(34)
where the summation is over different possible positions of $\vec{S}$ in the direct products. Because the maximally entangled state of $N$ qutrits

$$|\phi> = \frac{1}{\sqrt{3}} \sum_{i=1}^{1} |i>^\otimes N,$$

and the Hamiltonian (34) have the permutation symmetry, it follows that the density matrix of $N$ qutrits has the symmetric correlation functions. The length of the generalized Bloch vector for pure states equals $\sqrt{3^{N-1}}$.

The entanglement measures for the many-particle multi-level quantum systems are not studied enough and difficult to calculate in the analytic form, that is why we will present only analytic formulas for the entropy measure $\eta_N$ [23], which is defined by the eigenvalues of the reduced one-particle matrices for each qutrit. As the result of the mentioned symmetry, the reduced matrices are equal to each other. Therefore the entanglement measure for $N$ qutrits is defined by the formulae

$$\eta_N = -\sum_{i=1}^{3} r_i \log_3 r_i.$$

The eigenvalues of the reduced matrices for 3, 4, 5, and 6 qutrits are presented in the table below

| $N$ | $r_1 = r_2$ | $r_3$ |
|-----|-------------|--------|
| 3   | $29 - 4 \cos 5Jt$ | $17 + 8 \cos 5Jt$ |
| 4   | $905 - 98 \cos 3Jt - 72 \cos 7Jt$ | $295 + 196 \cos 3Jt + 144 \cos 7Jt$ |
| 5   | $16919 - 1944 \cos 3Jt - 800 \cos 9Jt$ | $2285 - 8688 \cos 5Jt + 1600 \cos 9Jt$ |
| 6   | $21977 - 1694 \cos 3Jt - 1936 \cos 7Jt - 560 \cos 11Jt$ | $42525 - 3388 \cos 3Jt + 3872 \cos 7Jt + 1120 \cos 11Jt$ |

The measures $\eta_3, \eta_4, \eta_5, \eta_6$ do not depend on sign of the exchange constant like the measure $\eta_2$.

V. NUMERICAL RESULTS

In Fig. 1 we present the populations of the upper and middle levels in the qutrit averaged over the time interval $\tau \to \infty$: $P^+ = \frac{1}{\tau} \int_0^\tau dt \left( \frac{1}{\tau} + \frac{1}{\sqrt{6}} R_3(t) + \frac{1}{\sqrt{2}} R_6(t) \right)$, $P^0 = \frac{1}{\tau} \int_0^\tau dt (\frac{1}{2} - \frac{1}{\sqrt{2}} \sqrt{2} R_6(t))$ in dependence on the normalized Larmor frequency $\omega_0/\omega$. The population of the upper level in qutrit coincides in form with the upper level occupation in a two-level system [19], i.e. this demonstrates the magnetic resonance position stabilization and the presence of the parametric resonances.

In Fig. 2 we note the considerable suppression of the qutrit spin oscillations $S_y = c_n(\omega t|k) \sin \omega_1 t$ and $S_z = -\cos \omega_1 t$ by the environment (fluctuator) in the case of the resonance $\omega = \omega_0$, $\overline{\omega} = 0$. 

FIG. 1: The time-averaged populations for the initial pure state $| - 1 >$ versus the normalized Larmor frequency $\omega_0/\omega$ at the parameters $k = 0.85$ (solid line), $k = 0.2$ (dashed line), $d = Q = 0$, $\omega_1 = 1/3$, $\omega = 1$ (I shows the upper level $| 1 >$ population; II shows the middle level $| 0 >$ population).
FIG. 2: Dynamics of the spin vector components $S_y$, $S_z$ for the initial pure state $|−1⟩$ (dashed lines) in the circularly polarized field with the parameters: $k = 0$, $ω_1 = 0.02$, $ω = ω_0 = 1$, $d = Q = 0$. Solid lines demonstrate the deformation of the spin components due to the influence of the second spin (the fluctuator) with $J = 0.1$ for the initial pure state $|−1⟩ ⊗ |−1⟩$.

FIG. 3: Disentanglement dynamics of the initially maximally entangled state in the bi-qutrit: in the zero external field with equal anisotropy constants $Q = d = d = Q = 0.02507$, $J = −0.1$ (curve 1) and for $J = 0.1$ (curve 2); in the consistent field the curve 3 (thick line) demonstrates complete coincidence of the measures $m_{VW}$ and $m_{SM}$ at $J=0.1$ and zero anisotropy constants; the curve 4 demonstrates the entropy measure $η_2$; $I$-concurrence is presented by the curve 5 at $J=0.1$.

The bi-qutrit energy $E$ in the consistent field at isotropic interaction in the case of the solution (46) is constant and equal to $\frac{2}{3}J$.

Although the analytic expressions for the measures in a bi-qutrit $m_{VW}$, $m_{SM}$ are different, but the numerical values are practically identical. Maximal deviation in the rectangle $(1 ≥ J ≥ 0.01) \times (100 ≥ t ≥ 0)$ equals 0.014, where $\times$ denotes the Cartesian product.

Measures $η_2$ and $m_I$ qualitatively coincide with the measures $m_{VW}$, $m_{SM}$.

We have found that the anisotropy of the qutrits disentangles them, namely the entanglement is decreased down to 0.0010 (see graphs 1 and 2 in Fig. 3).

In the constant longitudinal field $ω = \vec{ω} = (0, 0, ω_0)$ (the bi-qutrit Hamiltonian eigenvalues are equal to $J, J, x_1, x_2, x_3, −p, −p, p, p$, where $x_1, x_2, x_3$ are the roots of the equation $x^3 + 2x^2J − p^2x − 2J^3 = 0$, $p = \sqrt{J^2 + ω_0^2}$) the Hamiltonian contains the antisymmetric part, thus it follows that the density matrix for the initial symmetric state will not be symmetric because of the breaking the symmetry of the particle permutations. The analytic solution is cumbersome. In the constant longitudinal impulse field $\vec{ω} = \vec{ω} = (0, 0, 2(θ((t−17)(t−60))+θ((40−t)(57−t)(t−60))))$ the entanglement dynamics is blocked at $ω_0 ≫ J$ (Fig. 4). This points to the possibility to control the entanglement.

In Fig. 5 we present the comparative dynamics of the entropy entanglement measure for 2 to 6 qutrits. The disentanglement dynamics of the measures $η_3, η_4, η_5, η_6$ is similar to the one in the case of two qutrits, but with smaller oscillation amplitude, i.e. larger number of the qutrits disentangles less than two qutrits ($0.889 ≤ η_3 ≤ 1$).

VI. CONCLUSION

We have shown that the time-averaged upper-level occupation probability for the qutrit in the consistent field in dependence on the normalized Larmor frequency $ω_0/ω$ coincides in form with the upper-level occupation probability in the two-level system and the parametric resonances appear (Fig. 1). In the qutrit coupled to another qutrit
FIG. 4: Disentanglement of the maximally entangled state \[ |2\rangle \otimes |4\rangle \] (solid line) in the impulse field \( \omega_1 = \omega_2 = 0 \), \( \omega_0 = -\omega_0 = 2 \), \( J = 0.178 \). The dashed curve presents the evolution in zero external field.

FIG. 5: Disentanglement of the maximally entangled state \[ |2\rangle \otimes |8\rangle \] in the chain of 2, 3, 4, 5 and 6 qutrits with \( J = 0.1 \).

(fluctuator), the spin oscillations are essentially suppressed.

The comparative analysis of the bi-qutrit entanglement measures on the base of the analytic solution for the density matrix demonstrates that, in spite of the different approaches to the derivation of the formulas for the entanglement, all the formulas give quite close results (Fig. 3), and the measures \( m_{\text{VW}} \) and \( m_{\text{SM}} \) are practically equal. This is in accordance with the general results for the entanglement in the systems with the permutational symmetry [22].

The analytical formulas for the entanglement measures \( \eta_3, \eta_4, \eta_5, \eta_6 \) are similar to the disentanglement measure for two qutrits \( \eta_2 \), but with numerically smaller oscillation amplitude, i.e. the larger number of the qutrits disentangles less than two qutrits.

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VII. APPENDIX A

The matrix representation of the full set of Hermitian orthogonal operators for spin-1 has the form

\[
C_1 = S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_2 = S_2 = i \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_3 = S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[ (38a) \]
\[
C_4 = i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
C_5 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
C_6 = \sqrt{3}(S_3^2 - 2/3E) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
(38b)

\[
C_7 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},
C_8 = S_2^2 - S_1^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
C_9 = \sqrt{\frac{2}{3}} E,
\]
(38c)

where \( E \) is the unity \( 3 \times 3 \) matrix. These matrices have the property of the trace equal to zero \( \text{Tr} C_a = 0 \) and orthogonality \( \text{Tr} C_a C_b = 2\delta_{ab}, \) \( 1 \leq a, b \leq 8. \) The connection between the basis \( C_a \) and the Gell-Mann basis \( \lambda_a \) is the following:

\[
C_1 = \frac{1}{\sqrt{2}}(\lambda_1 + \lambda_6),
C_2 = \frac{1}{\sqrt{2}}(\lambda_2 + \lambda_7),
C_3 = \frac{1}{\sqrt{2}}\lambda_3 + \frac{\sqrt{3}}{2}\lambda_8, C_4 = \lambda_5,
C_5 = \frac{1}{\sqrt{2}}(\lambda_2 - \lambda_7),
C_6 = \frac{\sqrt{3}}{2}\lambda_3 - \frac{1}{\sqrt{2}}\lambda_8,
C_7 = \frac{1}{\sqrt{2}}(\lambda_1 - \lambda_6),
C_8 = \lambda_4.
\]

Non-zero structure constants \( \varepsilon_{abc} (g_{abc}) \) antisymmetric (symmetric) in respect to the permutation of any pair of indices for the commutators \( [C_a, C_b] = 2i\varepsilon_{abc} C_c \) (anticommutators \( \{C_a, C_b\} = \frac{1}{2}E\delta_{ab} + 2i\varepsilon_{abc} C_c \)) are respectively equal according to the definitions \( \varepsilon_{abc} = \frac{1}{4i} \text{Tr} [C_a, C_b] C_c :\)

\[
\varepsilon_{123} = \epsilon_{145} = \epsilon_{156} = -\epsilon_{245} = \epsilon_{278} = -\epsilon_{345} = \frac{1}{2};
\]

\[
\varepsilon_{156} = \epsilon_{267} = \frac{\sqrt{3}}{2},
\varepsilon_{348} = -1; g_{abc} = \frac{1}{4} \text{Tr} \{C_a, C_b\} C_c; g_{336} = g_{446} = -g_{666} = g_{888} = \frac{1}{\sqrt{3}},
\]

\[
g_{116} = g_{226} = g_{556} = g_{777} = -\frac{1}{2\sqrt{3}},
\]

\[
g_{118} = g_{124} = g_{137} = -g_{228} = g_{235} = -g_{457} = g_{558} = -g_{778} = \frac{1}{2}.
\]

Hence, the product of the generators is equal to \( C_a C_b = \frac{1}{2}E\delta_{ab} + (g_{abc} + ie_{abc})C_c. \)

**VIII. APPENDIX B**

For the initial state \(| -1 >\) at non-zero detuning \( \delta = \omega_0 - \omega \) the density matrix elements in the circularly polarized field have the form

\[
\rho_{11} = \omega_1^4 \sin^4 \frac{\Omega t}{2},
\rho_{12} = -\frac{\sqrt{2} \omega_1^3}{\Omega^4} \sin^3 \frac{\Omega t}{2} e^{-i\omega t} \left( i\Omega \cos \frac{\Omega t}{2} + \delta \sin \frac{\Omega t}{2} \right),
\]
(39)

\[
\rho_{13} = -\frac{\omega_1^2}{\Omega^4} \sin^2 \frac{\Omega t}{2} e^{-2i\omega t} \left( \omega_1^2 - 2i\delta \Omega \sin \Omega t + (2\delta^2 + \omega_1^2) \cos \Omega t \right),
\]
(40)

\[
\rho_{22} = \frac{\omega_1^2 \sin^2 \frac{\Omega t}{2}}{\Omega^4} (2\delta^2 + \omega_1^2 (1 + \cos \Omega t)),
\rho_{23} = -\omega_1^4 \frac{1}{2\Omega^4} e^{-i\omega t} \left( 2\delta^2 + \omega_1^2 (1 + \cos \Omega t) \right) \left( \delta \sin \frac{\Omega t}{2} + i\frac{\Omega}{2} \sin \Omega t \right),
\]
(41)

\[
\rho_{33} = \frac{1}{4\Omega^4} \left( (2\delta^2 + \omega_1^2 (1 + \cos \Omega t))^2 \right),
\rho_{ik} = \rho_{ki}^*.
\]
(42)

where \( \Omega = \sqrt{\omega_1^2 + \delta^2} \) is the Rabi frequency.

For the initial doubly stochastic state \( \frac{1}{\sqrt{3}}(|-1 > - |0 > + |1 >) \) and for the states \(|0 >, \frac{1}{2} |-1 > + \frac{1}{\sqrt{2}} |0 > + \frac{1}{2} |1 > |\)
at exact resonance \( \delta = 0 \) in the consistent field, the density matrices are respectively equal

\[
\begin{pmatrix}
\frac{1}{16} (\cos 2\omega t + 3) \\
\frac{1}{16} f (4 - i\sqrt{2} \sin 2\omega t) \\
\frac{1}{12} f^2 (\cos 2\omega t + 3)
\end{pmatrix},
\begin{pmatrix}
\frac{1}{8} f^{-1} (i\sqrt{2} \sin 2\omega_1 t + 4) \\
\frac{1}{8} f^{-1} (4 - i\sqrt{2} \sin 2\omega_1 t) \\
\frac{1}{12} f^{-1} (4 - i\sqrt{2} \sin 2\omega_1 t)
\end{pmatrix},
\]
(43)

\[
\begin{pmatrix}
\frac{1}{2} \sin^2 \omega_1 t \\
\frac{1}{2} \sin^2 \omega_1 t \\
\frac{1}{2} \sin^2 \omega_1 t
\end{pmatrix},
\begin{pmatrix}
\frac{1}{16} (5 - \cos 2\omega_1 t) \\
\frac{1}{8} \sin 2\omega_1 t \\
\frac{1}{8} \sin 2\omega_1 t
\end{pmatrix},
\begin{pmatrix}
\frac{1}{16} (5 - \cos 2\omega_1 t) \\
\frac{1}{8} \sin 2\omega_1 t \\
\frac{1}{8} \sin 2\omega_1 t
\end{pmatrix}.
\]
(44)

For both the initial middle level and the doubly stochastic pure initial state, the populations of the upper and bottom levels are equal. This property is fulfilled for the mixed state as well.
IX. APPENDIX C

The exact solution for the correlation functions of the initial state \( \hat{2A} \), which is symmetric under the particle permutation, in the consistent field at resonance \( \omega = \omega_0 = \omega_0 = h \) and equal \( \omega_1 = \omega_1 \) takes the form

\[
R_{0,1} = R_{0,2} = R_{0,3} = 0, \quad R_{0,4} = \frac{8}{3} \sqrt{3} \cos^2 \omega_1 t \cnu \sin^2 \frac{3Jt}{2}, \quad R_{0,5} = -\frac{4}{3} \sqrt{3} \cnu \sin^2 \frac{3Jt}{2} \sin 2\omega_1 t,
\]

\[
R_{0,6} = \frac{2}{9} \sqrt{2} (3 \cos 2\omega_1 t - 1) \sin^2 \frac{3Jt}{2}, \quad R_{0,7} = \frac{4}{3} \sqrt{3} \cnu \sin 2\omega_1 t \sin^2 \frac{3Jt}{2},
\]

\[
R_{0,8} = \frac{4}{3} \sqrt{2} \cnu \sin 2\omega_1 t (1 - 2\sin^2 \omega_1) \sin^2 \frac{3Jt}{2}, \tag{45a}
\]

\[
R_{1,1} = \frac{1}{36} \left(16 + 12(\cos 3Jt + 2) \left(\cnu^2 u - \sn^2 u\right) \cos^2 \omega_1 t + 2 \cos 3Jt - 3 \cos (3J - 2\omega_1) t \right)
- 12 \cos 2\omega_1 t - 3 \cos (3J + 2\omega_1) t, \quad R_{1,2} = \frac{2}{3} (\cos 3Jt + 2) \cnu \sin^2 \omega_1 t,
\]

\[
R_{1,3} = \frac{1}{3} (\cos 3Jt + 2) \cnu \sin 2\omega_1 t, \quad R_{1,4} = \frac{1}{3} \cnu \sin 3Jt \sin 2\omega_1 t,
\]

\[
R_{1,5} = \frac{1}{6} \left(2 \left(\cnu^2 u - \sn^2 u\right) \cos^2 \omega_1 t + 3 \cos 2\omega_1 t - 1\right) \sin 3Jt, \quad R_{1,6} = \frac{1}{\sqrt{3}} \cnu \sin 3Jt \sin 2\omega_1 t,
\]

\[
R_{1,7} = -\frac{2}{3} \cos^2 \omega_1 t \cnu \sin 3Jt, \quad R_{1,8} = \frac{1}{\sqrt{3}} R_{1,6}, \tag{45b}
\]

\[
R_{22} = \frac{1}{18} \left(6(\cos 3Jt + 2) \left(\cnu^2 u - \sn^2 u\right) \cos^2 \omega_1 t + 3 \cos 3Jt - 3(\cos 3Jt + 2) \cos 2\omega_1 t + 8\right),
\]

\[
R_{23} = -\frac{1}{3} \left(\cos 3Jt + 2\right) \cnu \sin 2\omega_1 t, \quad R_{24} = \frac{1}{\sqrt{3}} R_{16}, \quad R_{25} = -R_{17}, \quad R_{26} = \sqrt{3} R_{14},
\]

\[
R_{27} = \frac{1}{6} \left(2 \left(\cnu^2 u - \sn^2 u\right) \cos^2 \omega_1 t - 3 \cos 2\omega_1 t + 1\right) \sin 3Jt, \quad R_{28} = -\frac{1}{\sqrt{3}} R_{26}, \tag{45c}
\]

\[
R_{33} = \frac{1}{18} \left(-2 \cos 3Jt + 3 \cos (3J - 2\omega_1) t + 12 \cos 2\omega_1 t + 3 \cos (3J + 2\omega_1) t + 2\right),
\]

\[
R_{34} = -\frac{2}{3} \cos^2 \omega_1 t \left(\cnu^2 u - \sn^2 u\right) \sin 3Jt, \quad R_{35} = -R_{14}, \quad R_{36} = 0,
\]

\[
R_{37} = -\frac{1}{3} \cnu \sin 3Jt \sin 2\omega_1 t, \quad R_{38} = \frac{4}{3} \cos^2 \omega_1 t \cnu \sin 3Jt, \tag{45d}
\]

\[
R_{44} = \frac{1}{72} \left(-\left(1 - 8\cnu^2 u \sn^2 u\right) \cos^3 \omega_1 t + 8 \cos 3Jt - 12(2 \cos 3Jt + 1) \cos 2\omega_1 t + 9 \cos 4\omega_1 t + 19\right),
\]

\[
R_{45} = \frac{1}{12} \cnu \left(24 \left(\cnu^2 u - 3\sn^2 u\right) \sin \omega_1 t \cos^3 \omega_1 t + 2(2 \cos 3Jt + 1) \sin 2\omega_1 t - 3 \sin 4\omega_1 t\right),
\]

\[
R_{46} = -\frac{2}{3\sqrt{3}} \cos^2 \omega_1 t (2 \cos 3Jt - 9 \cos 2\omega_1 t + 7) \cnu \snu,
\]

\[
R_{47} = \frac{1}{12} \cnu \left(-24 \left(\cnu^2 u - 3\sn^2 u\right) \sin \omega_1 t \cos^3 \omega_1 t - 2(2 \cos 3Jt + 1) \sin 2\omega_1 t + 3 \sin 4\omega_1 t\right),
\]

\[
R_{48} = 4 \cos^4 \omega_1 t \cnu \snu \left(\cnu^2 u - \sn^2 u\right), \tag{45e}
\]

\[
R_{55} = \frac{1}{18} \left(6 \cos 3Jt + 6 \cos 2\omega_1 t - 4 \right) \left(\cnu^2 u - \sn^2 u\right) \cos^2 \omega_1 t - \cos 3Jt + 3(\cos 3Jt + 2) \cos 2\omega_1 t
- 9 \cos 4\omega_1 t + 1\right), \quad R_{56} = \frac{1}{6\sqrt{3}} \cnu \left(4 \sin^2 \frac{3Jt}{2} \sin 2\omega_1 t - 9 \sin 4\omega_1 t\right),
\]

\[
R_{57} = \frac{1}{6} \left(2 \cos 3Jt + \cos (3J - 2\omega_1) t + 4 \cos 2\omega_1 t + 6 \cos 4\omega_1 t + \cos (3J + 2\omega_1) t - 2\right) \cnu \snu,
\]

\[
R_{58} = \frac{1}{12} \cnu \left(-24 \left(\cnu^2 u - 3\sn^2 u\right) \sin \omega_1 t \cos^3 \omega_1 t + 2(2 \cos 3Jt + 1) \sin 2\omega_1 t - 3 \sin 4\omega_1 t\right), \tag{45f}
\]
\[ R_{66} = \frac{1}{36} (-4 \cos 3Jt + 6 \cos (3J - 2\omega_1)t - 12 \cos 2\omega_1t + 27 \cos 4\omega_1t + 6 \cos (3J + 2\omega_1)t + 13), \]
\[ R_{67} = \frac{1}{6\sqrt{3}} \text{sn} (2\cos 3Jt - 1) \sin 2\omega_1t + 9 \sin 4\omega_1t, \]
\[ R_{68} = \frac{1}{3\sqrt{3}} \cos^2 \omega_1t (-2 \cos 3Jt + 9 \cos 2\omega_1t - 7) (\text{cn}^2u - \text{sn}^2u), \] (45g)

\[ R_{77} = \frac{1}{18} ((\cos 3Jt - 1) (3\text{cn}^2u - 3\text{sn}^2u - 1) + 9 \cos 4\omega_1t (\text{cn}^2u - \text{sn}^2u - 1) + 3(\cos 3Jt + 2) \cos 2\omega_1t (\text{cn}^2u - \text{sn}^2u + 1)), \]
\[ R_{78} = \frac{1}{6} \text{sn} (6 (3\text{cn}^2u - \text{sn}^2u) \cos^2 \omega_1t + 2 \cos 3Jt - 3 \cos 2\omega_1t + 1) \sin 2\omega_1t, \] (45h)

\[ R_{88} = \frac{1}{72} (72 (1 - 8 \text{cn}^2u \text{sn}^2u) \cos^4 \omega_1t + 8 \cos 3Jt - 12(2 \cos 3Jt + 1) \cos 2\omega_1t + 9 \cos 4\omega_1t + 19), \] (45i)

where \( u = (\hbar t | k) \). It is straightforward to find the analytic solution for larger number of qutrits at the same conditions.

[1] F. Hioe, Phys. Rev. A 28, 879 (1983).
[2] F. Hioe and J. Eberly, Phys. Rev. Lett. 47, 838 (1981).
[3] A. M. Ishkhanyan, J. Phys. A: Math. Gen 33, 5041 (2000).
[4] V. E. Zobov, Shao, and A. S. Ermilov, JETP Letters 87, 334 (2008).
[5] M. O. Scully, M. S. Zubairy, G. S. Agarwal, and Waltther, Science 299, 862 (2003).
[6] L. Derkacz and L. Jakobczyk, Phys. Rev. A 74, 032313 (2006).
[7] M. Ali, arXiv:0911.0767v1 [quant-ph] (2009).
[8] D. Kaszlikowski, P. Gnacinski, M. Zukowski, W. Miklaszewski, and A. Zeilinger, Phys. Rev. Lett. 85, 4418 (2000).
[9] R. J. Morris, Phys. Rev. A 133, A740 (1964).
[10] P. Allard and T. Hard, Journal of Mag. Resonance 153, 15 (2001).
[11] J. N. Elgin, Phys. Letters A 80, 140 (1980).
[12] E. A. Ivanchenko, J. Math. Phys. 50, 042704 (2009).
[13] M. Abramovitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, New York, 1968).
[14] I. I. Rabi, Phys. Rev. 51, 652 (1937).
[15] A. Bambini and P. R. Berman, Phys. Rev. A 23, 2496 (1981).
[16] J. B. Miller, B. H. Suits, and A. N. Garroway, Journal of Mag. resonance 151, 228 (2001).
[17] M. Grifoni and P. Hanggi, Physics Reports 304, 229 (1998).
[18] M. R. Nath, S. Sen, and G. Gangopadhyay, Pramana-Journal of Physics 61, 1089 (2003).
[19] E. A. Ivanchenko, Low Temp. Phys. 31, 577 (2005).
[20] G. Vidal and R. F. Werner, Phys. Rev. A 65, 32314 (2002).
[21] J. Schienz and G. Mahler, Phys. Rev. A 52, 4396 (1995).
[22] G. Toth and O. Gührne, Phys. Rev. Letters 102, 170503 (2009).
[23] F. Pan, D. Liu, G. Y. Lu, and J. P. Draayer, Int. J. Theor. Phys. 43, 1241 (2004).
[24] F. Mintert, A. R. R. Carvalho, M. Kus, and A. Buchleitner, Physics Reports 415, 207 (2005).
[25] S.-X. Zhang, Q.-S. Zhu, and X.-Y. Kuang, Commun. Theor. Phys. (Beijing, China) 50, 883 (2008).