Inner Nonlinear Waves and Inelastic Light Scattering of Fractional Quantum Hall States as Evidence of the Gravitational Anomaly

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(Dated: February 23, 2018)

We develop the quantum hydrodynamics of inner waves in the bulk of fractional quantum Hall states. We show that the inelastic light scattering by inner waves is a sole effect of the gravitational anomaly. We obtain the formula for the oscillator-strength, or mean energy of optical absorption expressed solely in terms of an independently measurable static structure factor. The formula does not explicitly depend on a model interaction potential.

PACS numbers: 73.43.Cd, 73.43.Lp, 73.43.-f, 02.40.-k, 11.25.Hf

Introduction Excitations in the bulk of fractional quantum Hall states (FQH) are neutral collective modes of density modulations. These modes are generally gapped. Evidence of collective modes were seen in inelastic light scattering [1] and in optical absorption by surface acoustic waves [2]. The numerically obtained spectrum of a small system [3] and [4], also shows a dispersive branch of a collective excitation.

The experimental accessibility of the dispersion of neutral modes of FQHE states calls for a better understanding of inner waves. There is a renewed interest in the subject. Some recent papers are collected in Ref. [5].

In this paper we show that inelastic light scattering by inner FQH waves is a sole effect of the gravitational anomaly. This observation gives a geometric interpretation to inner waves, and also, a new analytic formula for the "oscillation strength" of optical absorption $\Delta_k$. The hydrodynamic description of inner FQHE waves faces a long-standing problem of the quantizing of incompressible hydrodynamics, specifically the chiral flows, that are flows with an extensive vorticity. Accounting for the gravitational anomaly described below represents perhaps the first consistent quantization of incompressible flows, whose applications go beyond the QHE.

Before we proceed, an important comment about the spectrum of incompressible waves is in order. The GMP theory [5] adopted a variational approach initially developed by Feynman for the superfluid helium [6]. The GMP approach assumes that a certain two-body Hamiltonian $H = \sum_q V_q \rho_q \rho - q$, where $\rho_q$ is the electronic density mode, indeed, delivers a FQH state. Then it assumes that excitations include a single-mode density modulation $|k\rangle = \rho_k |0\rangle$, and interprets the diagonal matrix element of the Hamiltonian

$$\Delta_k = \frac{\langle k|H|k\rangle}{\langle k|k\rangle}.$$  \hfill (1)

as a variational approximation to the excitation spectrum. The net result is expressed in terms of a model potential $V_q$. 

The gravitational anomaly only recently entered the subject are in Ref. [10]. The gravitational anomaly comes to the stage to prevent a quantization scheme from violation diffeomorphism invariance, the relabeling symmetry of the fluid. It is quite remarkable, that optical probes directly test this fundamental symmetry. The hydrodynamic description of inner FQHE waves corrects the Helmholtz law through the gravitational anomaly. The inelastic light scattering is the effect of this correction.

Our main observation is that the quantization subtly corrects the Helmholtz law through the gravitational anomaly. We argue that the gravitational anomaly governs one of the major observables in FQH, the inelastic light scattering.

The gravitational anomaly only recently entered the QHE literature. Some papers on the subject are in Ref. [10]. The gravitational anomaly is an elusive phenomenon which appeared as a higher-order gradient correction to bulk transport coefficients [22]. What would be the clean, experimentally accessible effects of the gravitational anomaly? We argue that the gravitational anomaly corrects the Helmholtz law (see, e.g., [12]): Vortices are frozen (or passively dragged by) the flow. Since vortices represent electrons they could be probed by light. Then, the Helmholtz law forbids inelastic light scattering. Being perturbed by light, vortices instantaneously change the flow and remain frozen into a new flow. They cannot accelerate against the flow.

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Such an approach is justified for compressible fluids, like helium, where atomic density modulation, is a linear wave. In this case a single-mode $|k\rangle = \rho_k|0\rangle$ is a long-lived state. Contrary to GMP’s major assumption, a single-mode state does not approximate a long-lived excitation in incompressible fluids, such as the FQHE. A reason for it is that incompressible waves are essentially nonlinear. A single-mode state decays into multiple modes, and does not have a spectrum, and $\Delta_k$ has no direct relation to true excitations, as it seems commonly accepted in the literature.

Still, we argue that $\Delta_k$ could be measured in optical absorption and give a new formula for $\Delta_k$ in terms of the structure factor. It refines the GMP formula which expresses $\Delta_k$ in terms of model potential $V_q$.

We focus on Laughlin’s series of states, where a filling fraction $1/\nu$ is an integer.

Correspondence between FQH states and fast rotating superfluid The analogy between FQHE and a superfluid was suggested in Refs. 3, 4 and developed to a correspondence in Ref. 8. We briefly review it. In short, a drift of vortices in a fast rotating superfluid and a motion of electrons in FQH regime are governed by same equations.

Fast rotating superfluid is a dense media of same sense vortices with a quantized circulation, which we denote by $2\pi \Gamma$. The total vorticity of the fluid is compensated by a solid rotation with a frequency $\Omega$, such that the mean density of vortices is $\rho_0 = \Omega/(\pi \Gamma)$. We assume that the vortices are in a liquid phase (do not crystallize). The frequency of rotation $\Omega$ corresponds to the Larmor frequency $\Omega = eB/2m_e$ with an effective mass $m_e$. The "mass" is the only phenomenological parameter of the theory determined by the spectral gap. Its energy scale is the Coulomb interaction $2\hbar\Omega = \hbar^2/2m_e\ell^2 \sim e^2/\ell$, where $\ell = \sqrt{\hbar/eB}$ is the magnetic length. Then vortices correspond to electrons if the vortex circulation in units of $\hbar/m_e$ is the inverse of the filling fraction and the gap in the spectrum is of the order of $\hbar \Omega$

$$\Gamma = \left( \frac{\hbar}{m_e} \right) \nu^{-1}, \quad \Omega = \frac{eB}{2m_e}. \quad (2)$$

This correspondence differs from that of GMP 3. The authors of Ref. 3 referred to the work of Feynman 8, who considered atomic density modes of a compressible superfluid at rest. Rather, we discuss the modes of vorticity of a rotating incompressible superfluid 22. We will measure the distance in units of magnetic length and the energy (the bulk gap) in units of the $2\hbar\Omega$, setting $\ell = \hbar = m_e = 1$. In these units, the mean density $\rho_0 = 1/(2\pi \Gamma) = \nu/2\pi$. We restore the parameters in final formulas. The filling fraction $\nu$ plays a role of a semiclassical parameter.

Helmholtz law The hydrodynamics of a 2D incompressible flow can be cast in a compact Helmholtz form:

The material derivative of vorticity vanishes. If $\mathbf{u} = (u_x, u_y)$ is the velocity of a flow, $\omega = \nabla \times \mathbf{u}$ is the vorticity, $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the material derivative, and the fluid is incompressible $\nabla \cdot \mathbf{u} = 0$, then the Euler equation in the Helmholtz form reads

$$D_t \omega = 0. \quad (3)$$

In the context of FQH vorticity is identified with the electronic density. In a rotating frame with no net vorticity the correspondence between electronic density and vorticity is

$$\rho(r) = \rho_0 + \frac{1}{2\pi \Gamma} \omega(r). \quad (4)$$

The velocity of the flow $\mathbf{u}$ does not have a measurable analog in the FQHE. It could be thought as a transversal part of the fictitious gauge field attaching a flux of magnetic field to electrons.

It is quite remarkable that essential features of Laughlin’s states are encapsulated in the quantum version of the Helmholtz equation. We will see some of it now.

The Helmholtz law reflects a geometric meaning of hydrodynamics: Incompressible flows are generated by a successive action of volume-preserving diffeomorphisms. In the QHE this concept has been suggested in Ref. 13. Therefore, the dynamics of FQH inner waves, and the equivalent problem of a quantum hydrodynamics, both are seen as a more abstract problem of the quantization of the group of volume-preserving diffeomorphisms. This group is generated by density mode operators $\bar{\rho}_k = \int e^{ikr} \rho(r) \ d^2r$, with the algebra

$$[\bar{\rho}_k, \bar{\rho}_{k'}] = i e^{ik \cdot k'} \bar{\rho}_{k+k'} \quad (5)$$

with the structure constants (in the long-wave approximation)

$$e_{kk'} = k \times k'$$

On the torus the structure constants are $e_{kk'} = 2e^2/2||k \times k'|| \sin(k \times k')$. We used here bar to emphasize the quantization as in Ref. 3. The classical limit of $5$ is the Poisson brackets of hydrodynamics (see e.g., 14).

Nonlinear waves Few important properties already follow from 3. One is that inner density waves are essentially nonlinear. A well-known fact is that the 2D incompressible hydrodynamics does not assume linear waves. In the language of the quantum theory, this means that single density modes are not long-lived states.

However, the Euler equation can be linearized about an inhomogeneous background. Example are Tkachenko linear modes of a vortex crystal 12, 13. Also, if we impose a periodic density modulation $|k_0\rangle$, then on top of it there are linear waves $\bar{\rho}_{q-k_0} |k_0\rangle = \bar{\rho}_{q-k_0} \bar{\rho}_{k_0} |0\rangle$. 


This suggests that, in contrast to a single mode, the two-modes states do have a spectrum. This assertion agrees with the interpretation of the inelastic light scattering experiments of Pinczuk et al [1] as a Raman type two-modes process by Platzman and He [1]. In the case of two-modes excitations the mean energy \( \Delta_{q,k_0} = \langle q,k_0 \rangle H(q,k_0) \) is true variational approximation of the spectrum. In the limit \( k_0 \to 0 \) the waves become nonlinear, and the variational approach fails. We address the spectrum of inner waves elsewhere.

Another consequence mentioned already is that the Helmholtz law prohibits the absorption of light. The central point of this paper is to show how this problem is resolved by the quantization.

**Quantization of Euler equation** Quantization of the Euler equation meets essential difficulties. The advection term \( u \cdot \nabla \omega = \nabla (u \cdot \omega) \) where two operators sit at the same point requires a regularization. The problem in a general setting has a long history of failures and commonly considered nearly impossible. A scheme of regularization where points are split \( u(r + \frac{k}{2}) \omega(r - \frac{k}{2}) \) leads to inconsistencies. The difficulty is that the points-splitting distance itself depends on the flow \( \epsilon \), that is, a regularization scheme is specific to the flow and cannot be practical to all varieties of flows at once. However, in a special case, when the flow consists of a dense media of vortices, quantization can be achieved. In this case, a variable short-distance cutoff is the distance between vortices \( \epsilon \sim 1/\sqrt{\rho} \). Below we present a heuristic, but an economic approach to quantization.

We will use the complex notations. We denote the complex velocity by \( u_z = u_x - iu_y \) and use the stream function \( \psi \) and the traceless part of the fluid momentum flux tensor \( \Pi_{ij} = u_i u_j - \langle 1/2 \rangle \delta_{ij} u^2 \). In complex coordinates \( u_z = 2i\partial_\psi \) and \( \Pi_{zz} = u_z^2 \). We will write the advection term as

\[
u \cdot \nabla \omega = i [\partial_z^2 \Pi_{zz} - \partial_z^2 \Pi_{zz}], \quad (6)
\]

Hence, we have to understand the quantum meaning of

\[
\Pi_{zz} \equiv u_z^2 = -4(\partial_z \psi)^2, \quad (7)
\]

For that we recall a notion of the projected density operator.

**Normal ordering and quantization** States on the lowest Landau level (LLL), and also flows of rotating superfluid, are realized as Bargmann space [2, 17]. It is a space of holomorphic functions with the inner product \( \langle g|f \rangle = \int e^{-1/2|z|^2} g^*(z)f(z)dzd\bar{z} \). The density operators acting in the Bargmann space obeying the algebra [3] are realized by the normally ordered operator

\[
\hat{\rho}_k = \sum_i e^{-\frac{i}{2}kz_i^1} e^{-\frac{i}{2}k\bar{z}_i}, \quad (8)
\]

where \( k = k_x + ik_y \) is a complex wave vector and

\[
z_i^1 = 2\partial_{z_i}.
\]

In [3] it was called a projected (onto LLL) density operator. It is organized such that states \( |k\rangle = \hat{\rho}_k|0\rangle \) is holomorphic and, hence belongs to LLL. It is also chiral \( \rho_{-k} = \rho_k \). The projected density mode operator is obtained from the density mode \( \rho_k = \sum_i e^{-ikr_i} \) by positioning the anti-holomorphic coordinates to the left to holomorphic coordinates, and replacing them by the holomorphic differentiating operators \( \bar{z}_i \to 2\partial_{z_i} \). As a result matrix elements between states on LLL (with respect to Bargmann inner product) are the same as that of density modes. Similarly, the two-modes operator that entered the momentum flux tensor on the Bargmann space is represented by a normal ordered string

\[
\rho_k \rho_{k'} = \sum_{i,j} e^{-\frac{i}{2}kz_i^1} e^{-\frac{i}{2}k'\bar{z}_j} e^{-\frac{i}{2}kz_i^1} e^{-\frac{i}{2}k'z_j} \cdot (9)
\]

The projected density modes generate coherent states of LLL and also the states of rotating superfluid if \( \bar{z}_i \) is the coordinate of a vortex.

We denote the Wick contraction \( \bar{A}\bar{B} = \bar{A}\bar{B} - \bar{A}\bar{B} \) and compute \( \bar{u}z\bar{u}_z \). The Wick contraction of two density modes follows from [9]

\[
\rho_k \rho_{k'} = \hat{\rho}_{k+k'} - (1 - e^{\frac{i}{2}k-k'}) \cdot (10)
\]

The next step is to express the momentum flux tensor [7] (acting on the Bargmann space) in terms of the generators \( \hat{\rho}_k \). We get insight by computing it for the ground state where the density is uniform \( \rho_k = N\delta_{k0} \) and there is no flow.

Equation [10] gives the contraction of two stream functions. At the uniform state it reads

\[
\begin{align*}
\psi(r)\psi(r') &= \frac{2\pi}{\nu} \int e^{ik(r-r')} \left( 1 - e^{-\frac{k^2}{k^4}} \right) \frac{d^2k}{(2\pi)^2} \cdot (11)
\end{align*}
\]

Now we can compute \( u_z(r)u_z(r') \) without any difficulty. In the hydrodynamic limit

\[
|r - r'| \gg \ell : \quad u_z(r)u_z(r') \sim (z - z')^{-2} \cdot (12)
\]

As \( r \to r' \), the divergency cutoff by the magnetic length, but the result is zero anyway in the state with the rotational symmetry. The effect of short-distance divergence does not show up. It is nulled by averaging over the angle.

**Gravitational anomaly in hydrodynamics** Now we evaluate \( u_z^2 \) on a flow state where the density and the cutoff \( \epsilon|u| \) at \( r \to r' \) are not uniform. The result follows from the geometric interpretation of the fluid flow. In this picture the distance between particles (vortices) is interpreted as a metric \( ds^2 = d|z|^2 \) of an auxiliary Riemann surface, and a flow as an evolving surface. The (scalar) curvature of the auxiliary surface is

\[
R = -4\rho^{-1} \partial_z \partial_\bar{z} \log \rho \cdot (13)
\]
The distance between particles is invariant under a change of coordinates, which can be seen as a relabeling of fluid particles. In hydrodynamics, this fictitious symmetry is typically applied to fluid atoms. In our approach, it is a relabeling symmetry of vortices. We want to keep this major symmetry intact in quantization.

To proceed, we first notice that in the hydrodynamic limit the contraction of stream functions is the Green function of the Laplace operator

$$\psi(r)\psi(r') = \frac{2}{\nu}G(r,r').$$  \hspace{1cm} (14)

It is natural to assume that in a flow state the contraction is the Green function of the Laplace-Beltrami operator in the metric $\rho d\Omega^2$. The Green function diverges at $r \to r'$ and requires a regularization. There is only one covariant regularization. It identifies the short-distance cutoff with the geodesic distance $d(r,r')$. With this prescription, we define the Wick contraction of the momentum flux tensor \[ \rho \vec{u} \cdot \vec{\nabla} \vec{\omega} \] as a limit:

$$\vec{u}_z \vec{u}_z = \frac{4\pi}{\nu} \lim_{r \to r'} \partial_z \partial_z \left[ G(r, r') + \frac{1}{2\pi} \log d(r, r') \right].$$  \hspace{1cm} (15)

The result of this limit is well known (Supplemented Material): The short distance expansion of \[ (16) \] is the Schwarzian of the metric

$$\vec{u}_z \vec{u}_z = \frac{1}{\nu} \left( \partial_z^2 \log \rho - \frac{1}{2} (\partial_z \log \rho)^2 \right).$$

Then the Wick contraction of the advection term \[ (19) \] is expressed through the curvature \[ (21) \] of the auxiliary Riemann surface

$$\vec{u} \cdot \vec{\nabla} \vec{\omega} = \frac{1}{96\pi} \vec{\nabla} \mathcal{R} \times \vec{\nabla} \omega.$$  \hspace{1cm} (16)

This is the main result of the quantization \[ (24) \]. We can now treat the hydrodynamics as a field theory, with a constant cutoff, independent of the flow. The cutoff is the magnetic length. The price for this is that the material derivative is no longer zero. With the help of \[ (21) \] we obtain

$$D_\mu \rho = \frac{1}{96\pi} \mathcal{R} \times \vec{\nabla} \rho.$$  \hspace{1cm} (17)

If waves are small, $\mathcal{R} \approx -\rho_0^2 \Delta \rho$, the correction to the Helmholtz law could be treated in the harmonic approximation

$$D_\mu \rho_k = \frac{\pi}{24\nu^2} \sum q q^2(k \cdot q) \rho_k \rho_{k-q}.$$  \hspace{1cm} (18)

We emphasize that Eq. \[ (24) \] incurs higher order gradients and powers of curvature, both suppressed by a small magnetic length. Later we show how to generate higher gradient corrections in the harmonic approximation.

**Deviation from Helmholtz law** The implication of quantum corrections is that the Helmholtz law held for quantum operators does not hold for their matrix elements: The material derivative for the projected density mode \[ (24) \] does not vanish. Acceleration of particles against the flow appears in higher derivatives, and it is a quantum correction, but, as we will see, it is the only source for the light scattering.

One can ask whether model specific effects beyond the hydrodynamics could change the coefficient 1/24 in \[ (24) \]. We do not see such possibility. The coefficient has a topological origin and is quantized in units of 1/24 or 1/48 depending on a FQHE state.

The universal departure from the Helmholtz law is the central result of the paper.

**Hamiltonian and current** Now we are in a position to determine the Hamiltonian which yields Eq. \[ (24) \]. We write the Helmholtz equation as a continuity equation

$$\dot{\rho} + \nabla \cdot \vec{J} = 0.$$  \hspace{1cm} (19)

and cast it in the Hamiltonian $\dot{\rho} = \frac{i}{\hbar}\left[ H, \rho \right]$. Using the commutation relation for two functionals of density

$$[F, G] = \frac{\nu}{2\pi^2} \int \left[ \nabla \frac{\delta F}{\delta \rho} \times \nabla \frac{\delta G}{\delta \rho} \right] \rho dV,$$

followed form \[ (25) \], we obtain the formula for the current

$$\vec{J} = \frac{1}{2\pi^2} \rho \nabla \times \frac{\delta H}{\delta \rho}.$$  \hspace{1cm} (20)

Then we determine the current from \[ (24) \] and compute the Hamiltonian. We write the formulas in a semiclassical manner treating $\rho$ as the mean density of the flow (not an operator). The result for the current is

$$\vec{J} = \rho \left( \vec{u} + \frac{1}{96\pi} \nabla \times \mathcal{R} \right) - \frac{\Gamma}{4} (1-2\nu) \nabla \times \vec{\omega}. $$  \hspace{1cm} (21)

The last term here is the divergent-free part of the current which does not enter the equation \[ (19) \] and is determined separately. We comment on its origin later.

From there we determine the Hamiltonian (in the Eulerian specification). We write it by separating classical and quantum contributions and restoring the units

$$H = \int \left( \mathcal{H} - \hbar S \right) \rho_0 d^2 \vec{r},$$  \hspace{1cm} (22)

$$\mathcal{H} = \frac{m_s}{2} \left( \vec{u}^2 + 2 \vec{u} \cdot \vec{w} - \pi^2 \rho \log \rho \right),$$  \hspace{1cm} (23)

$$S = -\pi \Gamma \left( \rho \log \rho + \frac{1}{96\pi} (\nabla \log \rho)^2 \right).$$  \hspace{1cm} (24)

For references we write the Hamiltonian in the form which separates overall scale and emphasize $\nu$ as a semiclassical parameter

$$H = \frac{\hbar^2}{2\nu m_s} \int \left[ 2\pi \rho(r)G(r,r')\rho(r') d^2 \vec{r} + (\nu - \frac{1}{2}) \rho \log \rho + \frac{\nu}{96\pi} (\nabla \log \rho)^2 \right] d^2 \vec{r}.$$
Here $G$ is the Green function of the Laplace operator.

The first two terms in the classical part (23) are the kinetic and centrifugal energies, $\nabla \times \mathbf{u}_0 = 2\bar{\mathbf{r}}$. The last term in (23) regularizes the divergency of the kinetic energy at vortex cores. It was known in the theory of superfluid since the 1961 paper of Kemoklidze and Khalatnikov [18], see also more recent Ref. [19]. This term is the Casimir invariant, whose Poisson bracket with all other local fields vanishes. For this reason, does not show in Eqs. (17, 19). It enters the current as a divergence-free term.

The quantum part (24) at a fixed vortex circulation $\Gamma$ does not depend on $m_\star$. It also consists of two terms. The first term is a quantum correction to the Kemoklidze-Khalatnikov term. The second term in (24) (also the second term in (21) and the RHS of (17)) is the effect of the gravitational anomaly.

*Static structure factor.* Now we show that the hydrodynamic equations encapsulate independently known long wave expansion of the structure factor. This fact could serve as a check and justification of the hydrodynamic equations (17, 24).

According to the general theory of linear response, the structure factor

$$s_k = \frac{1}{N} \langle 0 | \rho_{-k} \rho_k | 0 \rangle = \frac{1}{N} \langle 0 | \bar{\rho}_{-k} \bar{\rho}_k | 0 \rangle,$$

appears in the harmonic approximation of the current, or as a the rigidity of density modes in the Hamiltonian (see Supplemental Material)

$$J_k \approx \frac{1}{N} \sum_{q \neq 0} i q^q s_q^{-1} \bar{\rho}_{-q} \rho_q,$$

$$H \approx \frac{1}{2N} \sum_{q \neq 0} s_q^{-1} \bar{\rho}_{-q} \rho_q.$$  \hspace{1cm} (25)

(26)

Here $q^q$ is the vector normal to $q$, and the ground state energy is set to zero.

We compute the inverse structure factor by expanding (22). The result is

$$s_q^{-1} = \frac{2}{q^2} - \left( \frac{1}{2\nu} - 1 \right) + [s_q^{-1}]_+,\hspace{1cm} (27)$$

where $[s_q^{-1}]_+$ is the part of the expansion which consists of positive powers of the momentum. The leading term in $[s_q^{-1}]_+$ is the effect of the gravitational anomaly

$$[s_q^{-1}]_+ = \frac{q^2}{24\nu} + \mathcal{O}(q^4).$$  \hspace{1cm} (28)

Inverting (27), we obtain the first three terms of the small $q$ expansion of the structure factor

$$s_q = \frac{q^2}{2} + \frac{q^4}{8\nu} (1 - 2\nu) + \frac{q^6}{8\nu^2} (\frac{3}{4} - \nu) (\frac{1}{3} - \nu) \hspace{1cm} (29)$$

Each of the three terms in these formulas is independently known, has a universal meaning, and reflects symmetries of the hydrodynamics and Laughlin states. The term $q^2$ in (29) corresponds to the kinetic energy of the fluid, $\frac{1}{2}u^2$, and referred as the ‘perfect screening’ sum rule; $q^4$ corresponds to the $\rho \log \rho$ term in (23, 24) and is referred as the ”compressibility” sum rule. Finally, the $q^6$ term represents the gravitational anomaly. It was first obtained in Ref. [20] directly from the Laughlin wave function. In equivalent forms, it appeared independently in Ref. [10].

Using (29) and (10) we obtain the long-wave expansion of the projected structure factor

$$\bar{s}_k = \frac{1}{N} \langle 0 | \bar{\rho}_{-k} \bar{\rho}_k | 0 \rangle.$$  \hspace{1cm} (30)

From (10) we have $\bar{s}_q = s_q - (1 - e^{-\frac{1}{2}\lambda \bar{\rho}})$. Hence,

$$\bar{s}_q = (1 - \nu) \frac{q^4}{8\nu} \left( 1 + \frac{1}{6\nu} (3 - 10\nu) q^2 \right) + \ldots \hspace{1cm} (30)$$

There is no apparent reasons to think that higher terms in the expansion, but the first three, are universal.

*Harmonic approximation.* We can now express the correction to the Helmholtz law in terms of the structure factor, refining Eq. (15). Let us compute $[H, \bar{\rho}_k]$ with the Hamiltonian (20). The first term in the expansion of $s_{q}$ (27) forms the material derivative, and the second does not contribute. The correction to the Helmholtz law is due to the positive part of the expansion (27), whose leading term is the gravitational anomaly (28). We obtain a refine form of the Eq. (18) valid at all $k$:

$$D_t \bar{\rho}_k = \frac{\pi}{\nu} \sum_q e_{kq} [s_q^{-1}]_+ \bar{\rho}_q \bar{\rho}_{k-q}.$$  \hspace{1cm} (31)

At $k \to 0$ (31) reduces to (18).

We comment, that the harmonic approximation does not mean that waves are linear, or that the state $\bar{\rho}_k | 0 \rangle$ is an excitation, as it seems suggested in [3].

*Optical absorption by nonlinear waves.* Now we are ready to compute the optical absorption. Absorption occurs when light accelerates particles against the flow, i.e., due to a departure from the Helmholtz law.

Consider an acoustic wave imposed through the Hall bar as an experiment [2]. It creates a state $| k \rangle = \hat{\mathbf{f}} | k \rangle$. In solids, the optical absorption measures the differential intensity $S_k(\omega) = \frac{1}{N} (k | \delta(H - \hbar\omega) | k )$ and the integrated intensity $\bar{q}_k = \hbar \int S_k(\omega) d\omega = \frac{1}{N} \langle k | \hat{\mathbf{f}} k \rangle$, the projected static structure factor. Another object of interest in spectroscopy is the oscillation strength, the first moment of the intensity

$$\bar{f}_k = \int \omega S_k(\omega) d\omega = \frac{1}{N} (k | H | k ) = \frac{i}{2N} \langle 0 | \bar{\rho}_k \bar{\rho}_{-k} | 0 \rangle.$$  \hspace{1cm} (32)
and the the mean energy \( \Delta_k \)
\[
\Delta_k = \tilde{f}_k/\tilde{s}_k = \langle k | H | k \rangle / \langle k | k \rangle.
\] (33)

In fluids, a proper definition of the intensity must be written in a coordinate system moving with the fluid. This means that the time derivative in (2) is the material derivative
\[
\tilde{f}_k = \frac{1}{2N_1} \langle 0 | \left[ D_t \tilde{\rho}_k, \tilde{\rho}_k - q \right] | 0 \rangle.
\] (34)

Hence, only the rhs of (31) enters (34).

Typically \( S_k(\omega) \) features an asymmetric peak supported by the curve \( h\omega = \Delta_k \), which is, rudimentarily interpreted as a spectrum of excitations. Such an interpretation will be valid, would \( \tilde{\rho}_k(0) \) be a long-lived state, as happens in a compressible fluid. As we commented above, in the FQHE, the state \( \tilde{\rho}_k(0) \) is one of many short-lived coherent states.

Interpretation aside, we compute \( \tilde{f}_k \). Equation (31) reduces (31) to \( 0 | \tilde{\rho}_k - q \tilde{\rho}_k - q | 0 \rangle \), which we compute with the help of the algebra (32) and the structure constants (33) for the torus. We express result for the mean energy (35) in terms of
\[
\tilde{s}_k = (1 - \nu)^{-1} e^{\frac{1}{2} k^2} \tilde{s}_k.
\]

In units \( h^2/(\pi m^* \ell^2) = 2h\Omega/\pi \) the mean energy reads
\[
\Delta_k = \tilde{s}_k^{-1} \int \sin^2 \left( \frac{1}{2} k \times q \right) e^{-\frac{q^2}{2}} [s_q^{-1}, (\tilde{s}_q - \tilde{s}_{q = q})] \, d^2 q.
\] (35)

Contrary to Eq. (4.15) of Ref. (3), our formula does not explicitly depend on a model interaction. It is expressed only through independently measured (or calculated) structure factor. We emphasize that beyond terms in (30) the structure factor depends on details of the material, an so as the mean energy (35).

**SUPPLEMENTED MATERIAL**

**Structure factor and harmonic approximation** Here we give a sketch of the derivation of harmonic approximation of the current and the Hamiltonian (25,26). In the harmonic approximation they are
\[
J_k = \frac{1}{4} \sum_{q \neq 0} iq^* s_q^{-1} \tilde{\rho}_{k - q} \tilde{\rho}_q, \quad H \approx \frac{1}{4} \sum_{q \neq 0} s_q^{-1} \tilde{\rho}_{-q} \tilde{\rho}_q.
\] (36)

Here \( q^* \) is the vector normal to \( q \), and the ground state energy is set to zero.

Let us apply a small testing external potential \( A_0 \) by adding a term \( A_0 \rho \) to the Hamiltonian (22). According to (20) the effect of the external potential shifts the current by the Lorentz drift current \( \rho E^* \), where \( E = -\nabla A_0 \) is the electric field, and \( E^* \) is the vector normal to \( E \). Hence, in a steady state \( J = \rho E^* \), the Hall effect. From here we can find the density mode triggered by a non-uniform electric field. In the harmonic approximation in density

**Magnetoroton minimum** Both \( \tilde{f}_k \) and \( \tilde{s}_k \) feature a broad asymmetric peak at \( k \ell \sim 1 \), both vanish at \( k = 0, \infty \), but their ratio, \( \Delta_k \), is finite and non-zero.

The limiting values of the structure factor are
\[
k \to 0 : \quad \tilde{s}_k \sim \frac{k^4}{8\nu}, \quad [s_q^{-1}]_+ \sim \frac{k^2}{2\nu}\]
\[
k \to \infty : \quad \tilde{s}_k = 1, \quad [s_q^{-1}]_+ = \frac{1}{q^2}.
\]

Hence, the mean-energy \( \Delta_k \) smoothly interpolates between
\[
\Delta_{k=0} = 4\nu \int q^2 [s_q^{-1}]_+ (\nabla^2 s_q) \, e^{-\frac{q^2}{2}} \, d^2 q
\]
and
\[
\Delta_{k=\infty} = \int [s_q^{-1}]_+ (s_q + 1) e^{-\frac{q^2}{2}} \, d^2 q
\]

featuring a broad minimum at \( k \ell \sim 1 \). The minimum also shown in numerically evaluated \( \Delta_k \) from model Hamiltonians (3). GMP called it magnetoroton minimum. The minimum is also featured by Eq. (35), however we failed to recognize its universal meaning and physics behind it. It relies on features of \( \tilde{s}_k \) beyond its universal part (27,30).

We are not aware of an experimental evidence of the minimum in the absorption spectrum of Laughlin's states, such as \( \nu = \frac{1}{3} \) state. A sequence of minima in optical absorption are reported in Ref. (2) for fractions other than \( \frac{1}{3} \). It is not clear whether they are related to the GMP minimum for Laughlin's states.

The author thanks A. Cappelli, A.Gromov, and G. Volovik for discussions and interest to this work. The work was supported by the NSF under Grant NSF DMR-1206648. The author thanks the Gordon and Betty Moore Foundation, EPiQS Initiative through Grant GBMF4302 for the hospitality at Stanford University, and the Brazilian Ministry of Education (MEC) and the UFRN-FUNPEC for the hospitality at IIP during the work on this paper.
mode, in the Fourier space, we write \( \delta J_k / \delta \rho_{k-q} = i q^* A_{0q} \). Further variation in \( A_0 \) gives

\[
i q^* = \frac{\delta^2 J_k}{\delta A_{0q} \delta \rho_{k-q}} = \frac{\delta \rho_q}{\delta A_{0q}} \frac{\delta^2 J_k}{\delta \rho_q \delta \rho_{k-q}} = (0| \rho_{-q} \rho_q | 0) \frac{\delta^2 J_k}{\delta \rho_q \delta \rho_{k-q}},
\]

(37)

where we used the relation \( \delta \rho_q = (0| \rho_{-q} \rho_q | 0) \delta A_{0q} \) and the notation \( s_q = \frac{1}{\delta} (0| \rho_{-q} \rho_q | 0) \). The formula \[35\] for the current follows. The Hamiltonian follows from the relation \[20\] \( J = (2 \pi \Gamma)^{-1} \rho \nabla \times \delta \rho \).

**Short distance expansion of the Green function** Here we comment on how to compute the short distance expansion of the regularized Green function on a Riemann surface with the metric \( ds^2 = \rho |dz|^2 \)

\[
G^R (r, r') = G(r, r') + \frac{1}{2 \pi} \log d(r, r').
\]

(38)

This result is standard. We included it here (and in \[16\]), because of failure to find a proper reference.

At close points the Green function behaves as \( G(r, r') \to -\frac{1}{2 \pi} \log |r-r'| \). At the same time the short distance expansion of the geodesic distance reads (see, e.g., \[21\])

\[
2 \log d(r, r') = 2 \log |r-r'| + \log \rho + \frac{1}{2} (z' - z) \partial \log \rho + \frac{1}{2} (z' - z) \bar{\partial} \log \rho
\]

\[
\quad - \frac{1}{12} |z' - z|^2 \partial \log \rho^2 + \frac{1}{48} [(z' - z) \partial \log \rho + (z' - z) \bar{\partial} \log \rho]^2
\]

\[
\quad + \frac{1}{6} [(z' - z)^2 \partial^2 + 2 (z' - z)^2 \bar{\partial} \partial \log \rho + (z' - z)^2 \bar{\partial}^2] \log \rho + ...
\]

The first term of this expansion cancels the divergent part of the Green function. Taking holomorphic derivatives in \( z \) and \( z' \) we obtain

\[
\lim_{r \to r'} \partial_z \partial_{z'} \left[ G(r, r') + \frac{1}{2 \pi} \log d(r, r') \right] = \frac{1}{24 \pi} \left( \partial_z^2 \log \rho - \frac{1}{2} \left( \partial_z \log \rho \right)^2 \right).
\]

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