Coupler curves of moving graphs and counting realizations of rigid graphs

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Abstract

A calligraph is a graph that for almost all edge length assignments moves with one degree of freedom in the plane, if we fix an edge and consider the vertices as revolute joints. The trajectory of a distinguished vertex of the calligraph is called its coupler curve. To each calligraph we uniquely assign a vector consisting of three integers. This vector bounds the degrees and geometric genera of irreducible components of the coupler curve. A graph, that up to rotations and translations admits finitely many, but at least two, realizations into the plane for almost all edge length assignments, is a union of two calligraphs. We show that this number of realizations is equal to a certain inner product of the vectors associated to these two calligraphs. As an application we obtain an improved algorithm for counting numbers of realizations, and by counting realizations we characterize invariants of coupler curves.

Keywords: minimally rigid graphs, Laman graphs, number of realizations, coupler curves, infinitely near base points, algebraic series of planar curves, divisor classes

MSC Class: 52C25, 70B15, 14C20

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1 Introduction

In this article we count realizations of rigid graphs into the plane and investigate invariants of the trajectories of vertices of graphs that move in the plane. We start with a warm up example inspired by [4, Figure 5] in order to explain our main result and the state of art. The quoted definitions in this introduction are meant for building some intuition, and are made precise in §2. We conclude this introduction with an overview of the remaining article.

Warm up example. Let us consider the graph $C_3$ in Figure 1 and assign to the edges $\{0, 3\}$, $\{1, 3\}$ and $\{2, 3\}$ some lengths $\lambda_{03}$, $\lambda_{13}$ and $\lambda_{23}$, respectively. We now consider all possible ways to assign coordinates to each of the vertices such that vertex 1 is sent to $(0, 0)$, vertex 2 is sent to $(1, 0)$ and the distance between vertex $i$ and vertex 3 is equal to $\lambda_{i3}$ for all $i \in \{0, 1, 2\}$:

$$(x_i - x_3)^2 + (y_i - y_3)^2 = \lambda_{i3}^2.$$  

The “coupler curve” of $C_3$ is defined as the set of all possible coordinates for vertex 0 and thus consists of two circles. The center of the second circle is the reflection of vertex 3 along the line spanned by the edge $\{1, 2\}$. The marked graphs $C_3$ and $\mathcal{L}$ in Figure 1 are examples of “calligraphs”. Informally, when fixing vertices 1 and 2 of a calligraph, we require that the vertex 0 draws a curve.

![Figure 1: The coupler curves of the two calligraphs intersect in four different points.](image)

The union $C_3 \cup \mathcal{L}$ of the two calligraphs in Figure 1 can be realized in the plane in four different ways up to translations and rotations. We see in Figure 2 how the four realizations of $C_3 \cup \mathcal{L}$ are related to the number of intersections between the coupler curves of $C_3$ and $\mathcal{L}$.

![Figure 2: The intersections of coupler curves of two calligraphs are related to the number of realizations of a minimally rigid graph into the plane.](image)
The graph $C_3 \cup \mathcal{L}$ is called “minimally rigid graph” and its “number of realizations” $c(C_3 \cup \mathcal{L})$ is equal to four for almost all choices of edge length assignments. This number can be expressed as the number of complex solutions of quadratic equations. In Figure 2 all solutions are real, but in general there may be non-real solutions.

**Main result.** In this article we assign to a calligraph $\mathcal{G}$ its “class” $[\mathcal{G}]$, namely a triple $(a, b, c)$ of integers that satisfies two axioms. Axiom A1 states that $[\mathcal{L}] = (1, 1, 0)$, $[\mathcal{R}] = (1, 0, 1)$ and $[C_3] = (2, 0, 0)$, where $\mathcal{R}$ is defined in Figure 3. Axiom A2 essentially states that if $\mathcal{G} \cup \mathcal{G}'$ is a minimally rigid graph for some calligraphs $\mathcal{G}$ and $\mathcal{G}'$, then its number of realizations equals $c(\mathcal{G} \cup \mathcal{G}') = [\mathcal{G}] \cdot [\mathcal{G}'] = 2(a \cdot a' - b \cdot b' - c \cdot c')$ (see Definition 1). For example, we verify that $c(C_3 \cup \mathcal{L}) = [C_3] \cdot [\mathcal{L}] = 2(2 \cdot 1 - 0 \cdot 1 - 0 \cdot 0) = 4$. Our main result is Theorem I, which asserts that the class of a calligraph exists and is unique. By its Corollary I the class satisfies six properties P1—P6 that characterize invariants of coupler curves. For example, the coupler curve of $C_3$ is of degree 4 by P2 and P5. Property P6 is more technical and bounds the geometric genera and degrees of each of the irreducible components of a coupler curve. As an application of our methods we obtain an improved algorithm for computing the number of realizations of minimally rigid graphs that uses the algorithm in [5] as a fallback algorithm (see Remark 5).

**State of the art: coupler curves.** A calligraph is a special case of a linkage with revolute joints and thus its origins can be traced back to at least 1785, when James Watt used a calligraph to convert a rotational motion to an approximate straight-line motion for his steam engine. Kempe described in 1876 a method that constructs for any given planar algebraic curve a linkage that traces out a portion of this curve [17]. Coupler curves of linkages have been studied extensively by engineers and we refer to [14, Sections 7 and 15.3.7] for more details and further references. Recently, a method was provided for constructing a linkage, with a small number of links and joints, that has a given rational curve as trajectory [10, 19]. There exist minimally rigid graphs such that for a carefully chosen edge length assignment, a vertex in this graph traces out a trajectory, while fixing some edge into place. We refer to [24] for an overview of the classification of such paradoxically moving graphs.

**State of the art: number of realizations.** The investigation of rigid structures can be traced back to James Clerk Maxwell [20] and there is currently again a considerable interest in rigidity theory due to various applications in natural science and engineering [27]. Minimally rigid graphs (also known as Laman graphs) have been classified in [22] by Pollaczek-Geiringer and independently in [18]. Bounds on the number of realizations of such graphs have been investigated in [1–4, 9, 12, 26]. Algorithms and theory for computing the precise number of realizations for
minimally rigid graphs have been investigated in [5, 15, 23].

**Theoretical context.** The number of realizations of the union of two calligraphs is equal to the number of complex intersections of their coupler curves for almost all choices of edge length assignments. In Figure 2, for example, all these intersections are real. It follows from Bézout’s theorem that this number is equal to the product of the degrees of the coupler curves minus the complex intersections at infinity counted with multiplicities. If we fix a calligraph and vary its edge length assignments, then we obtain an “algebraic series” of coupler curves in the projective plane. If all curves in this series meet the same complex points, then these points are called “base points”. We show that the complex intersections at infinity of coupler curves are exactly at such base points. In this light, our main result is that a base point associated to a calligraph coincides with a base point associated to either the calligraph \( C_3 \), \( L \) or \( R \). The degree of a coupler curve of a calligraph and its multiplicities at the base points remains constant for almost all choices of edge length assignments and is encoded by the class of a calligraph. The number of complex intersections of the coupler curves, minus the number of intersections at infinity, is equal to a certain inner product between the classes (see Axiom A2).

To prepare the reader, let us consider the base points of calligraphs in a bit more detail. Almost each curve in the algebraic series of \( C_3 \) is a union of two circles that meet complex conjugate “cyclic” points at infinity with multiplicity 2. The algebraic series of \( L \) contains all circles that are centered around vertex 1. These circles meet aside the cyclic base points, also the “1-centric” base points. Similarly, the algebraic series of \( R \) contains all circles that are centered at vertex 2 and meet aside the cyclic base points the “2-centric” base points. The 1-centric and 2-centric points are “infinitely near” to the cyclic base points. This terminology is made precise in §4.

The multiplicities of general curves in the algebraic series of a calligraph at the cyclic, 1-centric and 2-centric points correspond to the three numbers in the class of this calligraph. In particular, \([C_3] = (2, 0, 0)\), \([L] = (1, 1, 0)\) and \([R] = (1, 0, 1)\) (see Axiom A1). We encode the algebraic series of calligraphs as so called divisor classes, and obtain the two axioms and six properties for classes using standard algebro geometric methods. The multiplicity at the cyclic points (classically known as cyclicity) was used in [28] to recover the degree of coupler curves for a certain type of linkages. To our best knowledge the 1-centric and 2-centric base points have not been considered before and allow us to characterize the coupler curves of any calligraph.

We want to emphasize that in this article we count the number of realizations over the complex numbers, and thus we obtain in general only an upper bound for the number of real realizations (see [15, Section 8, page 5]).
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Overview. In §2 we start with introducing terminology and conclude with a precise statement of the main result, namely Theorem I, together with some conjectures and open problems.

We clarify in §3 how the main result can be applied to determine the number of realizations of minimally rigid graphs and invariants of coupler curves such as degree and geometric genus. In particular, we discuss the correctness and heuristics of an algorithm for computing the number of realizations. In Appendix B we give a detailed example illustrating the recursive structure of this algorithm.

In §4 we start by recalling the notion of infinitely near base points and with a proof for Proposition 12, which states that Theorem I and Corollary I hold under the assumption that calligraphs are “centric”, namely that the base points of the algebraic series of a calligraph are either cyclic, 1-centric or 2-centric.

In §5 we show that the procedure for detecting infinitely near base points can be done directly on a system of quadric equations defined by a calligraph. This translates into sufficient conditions for the centricity of calligraphs in terms of symbolic modifications of these equations.

In §6 we show that all calligraphs are centric by using the sufficient conditions from §5 and thereby conclude the proof of Theorem I and Corollary I. We also need Proposition 33, but we moved its proof to Appendix C as this part is independent and may distract from the logical path.

We have made an effort to make this article accessible to a wide audience. The main results and algorithmic applications do not require many preliminaries, however, for the proofs we assume familiarity with algebraic geometry. In order to prepare the reader we list in Appendix A the notation that is used across different sections.

2 Statements of main result and open problems

In §2.1 we state the definition of the class of a calligraph and Theorem I. In §2.2 we introduce definitions related to coupler curves in order to state Corollary I. We conclude in §2.3 with some conjectures and open problems.
2.1 Classes of calligraphs

Let $\mathcal{G}$ denote a graph with vertices $v(\mathcal{G}) \subset \mathbb{Z}_{\geq 0}$ and edges $e(\mathcal{G})$. In what follows, all graphs are undirected and simple. We call a graph $\mathcal{G}$ marked if $\{1, 2\} \in e(\mathcal{G})$. The graphs $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{C}_v$ for $v \in \mathbb{Z} \setminus \{0, 1, 2\}$ are defined as in Figure 3.

![Figure 3: Three calligraphs that play a central role.](image)

We call $\mathcal{G}$ a minimally rigid graph if $|e(\mathcal{G})| = 2|v(\mathcal{G})| - 3$ and $|e(\mathcal{H})| \leq 2|v(\mathcal{H})| - 3$ for all subgraphs $\mathcal{H} \subset \mathcal{G}$ such that $|v(\mathcal{H})| > 1$. For example, $C_3 \cup \mathcal{L}$ in Figure 1 and the triangle $\mathcal{L} \cup \mathcal{R}$ are both minimally rigid graphs.

We call $\mathcal{G}$ a calligraph if $v(\mathcal{G}) \cap v(\mathcal{C}_v) = \{0, 1, 2\}$, $e(\mathcal{G}) \cap e(\mathcal{C}_v) = \{\{1, 2\}\}$ and $\mathcal{G} \cup \mathcal{C}_v$ is a minimally rigid graph for all $v \not\in v(\mathcal{G})$. For example, $\mathcal{L}$, $\mathcal{R}$ and $C_3$ are all calligraphs.

By counting the vertices and edges we find that a calligraph is a minimally rigid graph minus one edge.

Suppose that $\text{Prop}: X \rightarrow \{\text{True}, \text{False}\}$ is a proposition about some algebraic set $X$ such that $\{x \in X : \text{Prop}(x) = \text{False}\}$ is contained in an algebraic set $Y$. If each irreducible component of $Y$ forms a lower dimensional subset of some irreducible component of $X$, then we call any element in $X \setminus Y$ general and we say that $\text{Prop}(x) = \text{True}$ for almost all $x \in X$. Informally, we may think of a general element in $X$ as a random element. Notice that if $X \subseteq \mathbb{C}^n$, then there exists a non-zero $n$-variate complex polynomial $f$ such that $f(y) = 0$ for all $y \in Y$. See Remark 35 for the relation to the notion of “generic”.

If $\mathcal{G}$ is a marked graph, then the set of edge length assignments $\Omega_\mathcal{G}$ is defined as the set of maps $\omega: e(\mathcal{G}) \rightarrow \mathbb{C}$ such that $\omega(\{1, 2\}) = 1$. The set of realizations $\Xi_\mathcal{G}$, that are compatible with the edge length assignment $\omega \in \Omega_\mathcal{G}$, is defined as the set of maps $\xi: v(\mathcal{G}) \rightarrow \mathbb{C}^2$ such that $\xi(1) = (0, 0)$, $\xi(2) = (1, 0)$ and

$$(x_i - x_j)^2 + (y_i - y_j)^2 = \omega(\{i,j\})^2$$

for all $\{i,j\} \in e(\mathcal{G})$, where $\xi(i) = (x_i, y_i)$ for all $i \in v(\mathcal{G})$.

It follows from [15, Theorem 3.6] or [5, Corollary 1.11] that $|\Xi_\mathcal{G}| = |\Xi_\mathcal{G}'|$ for almost all edge length assignments $\omega, \omega' \in \Omega_\mathcal{G}$. If $\omega \in \Omega_\mathcal{G}$ is general, then we define the number of realizations of $\mathcal{G}$ as

$$c(\mathcal{G}) := |\Xi_\mathcal{G}| \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$
Notice that the definition “general” means in this case that the set \( \{ \omega \in \Omega_G : |\Xi^G_\omega| \neq c(G) \} \) is contained in a lower dimensional set of the algebraic set \( \Omega_G \). The number of realizations does not depend on the choice of the marked edge \{1, 2\} and thus is well-defined for graphs that are not marked. It follows from [22] or [18] that \( e(G) \) is a minimally rigid graph if and only if \( e(G) \) is well-defined for graphs that are not marked. We call \( G \) minimally rigid graphs admit a calligraphic split of the form \( e(G) \). We define the bilinear map \( \cdot : Z^3 \times Z^3 \rightarrow Z \) as

\[
(a, b, c) \cdot (a', b', c') = 2 (a a' - b b' - c c').
\]

**Definition 1.** A class for calligraphs is a function that assigns to each calligraph \( G \) an element \( [G] \) of \( Z^3 \) such that the following two axioms are fulfilled.

A1. \( [L] = (1, 1, 0) \), \( [R] = (1, 0, 1) \) and \( [C_v] = (2, 0, 0) \) for all \( v \in Z^3 \).

A2. If \( (G, G') \) is a calligraphic split, then \( c(G \cup G') = [G] \cdot [G'] \).

**Theorem 1.** The class for calligraphs exists and is unique.

The class \( [G] \) of a calligraph \( G \) is uniquely determined by the numbers \( c(G \cup L) \), \( c(G \cup R) \) and \( c(G \cup C_v) \) for some \( v \notin v(G) \) (see forward Algorithm 1). We see in Corollary 1 that the class reveals key properties about coupler curves. However, for this we need to introduce in §2.2 some additional concepts.

### 2.2 Invariants of coupler curves

In this article we assume that curves are real algebraic, namely the complex solution sets of polynomial equations with real coefficients. If \( C \subset C^2 \) is a curve, then we denote by \( \deg C \) its degree. If \( C \) is also irreducible, then \( g(C) \) denotes its geometric genus (see for example [21]). The singular locus of \( C \) is denoted by \( \text{sing} C \). We call \( C \) a circle if \( \deg C = 2 \) and \( C \cap R^2 \) is a circle.

The coupler curve of a calligraph \( G \) with respect to edge length assignment \( \omega \in \Omega_G \) is defined as follows (“t” stands for “trajectory”):

\[
t_\omega(G) := \{ \xi(0) : \xi \in \Xi^G_\omega \}.
\]

We call \( (T_1, \ldots, T_n) \) its coupler decomposition if \( t_\omega(G) = T_1 \cup \cdots \cup T_n \) and \( T_i \) is an irreducible curve for all \( 1 \leq i \leq n \). For example, in Figure 1 we see that the
coupler curve of $C_3$ for some choice of edge length assignment consist of two circles $T_1$ and $T_2$. The \textit{coupler multiplicity} of $G$ is defined as

$$m(G) := |\{\xi \in \Xi_\omega^G : \xi(0) = p\}|,$$

where both $\omega \in \Omega_G$ and $p \in t_\omega(G)$ are general. In other words, $m(G)$ equals the number of realizations sending vertices 1, 2 and 0 to $(0, 0)$, $(1, 0)$ and $p$, respectively. General coupler curves are indeed curves by Lemma 14(d) and the coupler multiplicity is well-defined by Lemma 16.

In Figure 4 we consider the real points of the coupler curves and the coupler multiplicities of three calligraphs $L$, $M$ and $Q$, for some general choice of edge length assignments $\tilde{\omega} \in \Omega_L$, $\omega \in \Omega_M$ and $\hat{\omega} \in \Omega_Q$. Each coupler curve is a circle centered at $(0, 0)$ and the points $p \in t_\omega(M)$ and $q \in t_{\hat{\omega}}(Q)$ are assumed to be general. We see that $m(M) = 2$, since there exist two realizations $\xi, \xi' \in \Xi_{\omega}^M$ such that $\xi(0) = \xi'(0) = p$, where $\xi'(3)$ is obtained by flipping $\xi(3)$ along the edge $\{1, 2\}$. Similarly, $m(Q) = 2$, as there exist two real realizations $\xi, \xi' \in \Xi_{\hat{\omega}}^Q$ such that $\xi(0) = \xi'(0) = q$, where $\xi'(3)$ is obtained by reflecting $\xi(3)$ along the line spanned by $\{0, 2\}$. The thick arcs on the coupler curve of $Q$ correspond to the subset $\{\xi(0) : \xi \in \Xi_{\hat{\omega}}^Q \text{ and } \xi(0), \xi(3) \in \mathbb{R}^2\}$, namely the real trace of vertex 0 when we consider the graph as a moving mechanism.

![Figure 4: Three calligraphs $L$, $M$ and $Q$ together with their coupler curves and coupler multiplicities. For each $r \in t_{\hat{\omega}}(Q)$ that lies on one of the two thick arcs there exists $\xi \in \Xi_{\hat{\omega}}^Q$ such that $\xi(0) = r$ and $\xi(3)$ is real.
](image)

A connected graph is called \textit{k-vertex connected} if it has more than $k$ vertices and remains connected whenever fewer than $k$ vertices are removed. We call a calligraph $G$ \textit{thin}, if $G' := G \cup L \cup R$ is 3-vertex connected and $G'$ minus the edge $\{v_1, v_2\}$ is minimally rigid for all edges $\{v_1, v_2\} \in e(G')$. For example, the calligraph $H$ in Figure 6 is thin. We remark that if $G$ is thin, then $G \cup L \cup R$ is “generic globally rigid” (see [6, Corollary 1.7] or [15, Theorem 5.1]) and we shall see in Corollary I that this property characterizes coupler multiplicity of $G$.

We call $(\alpha_1, \ldots, \alpha_n)$ a \textit{class partition} for a calligraph $G$ if $\alpha_i = (\alpha_{i0}, \alpha_{i1}, \alpha_{i2}) \in \mathbb{Z}^3$ such that $\alpha_{i0} \geq \alpha_{i1} \geq 0$, $\alpha_{i0} \geq \alpha_{i2} \geq 0$ and $\alpha_1 + \cdots + \alpha_n = \frac{1}{m} \cdot [G]$ for all $1 \leq i \leq n$, where $m$ denotes the coupler multiplicity of $G$. 
Corollary I. The following six properties are satisfied for all calligraphs $G$ and $G'$ and almost all edge length assignments $\omega \in \Omega_G$ and $\omega' \in \Omega_{G'}$, where we use the following notation:

$$T := t_\omega(G), \quad m := m(G), \quad T' := t_{\omega'}(G'), \quad m' := m(G').$$

P1. If $[G] = (a_0, a_1, a_2)$, then $a_0 \geq a_1 \geq 0$ and $a_0 \geq a_2 \geq 0$.

P2. If $G$ is thin, then $m(G) = 1$.

P3. If $G \cup L$ is not minimally rigid, then $[G] = (m, m, 0)$.

P4. If $(G, G')$ is a calligraphic split, then $m \cdot m' \cdot |T \cap T'| = [G] \cdot [G']$.

P5. If $[G] = (a_0, a_1, a_2)$, then $\deg T = 2a_0/m$.

P6. If $(T_1, \ldots, T_n)$ is a coupler decomposition for $T$, then there exists a class partition $(\alpha_1, \ldots, \alpha_n)$ for $G$ such that for all $1 \leq i \leq n$ we have

$$\deg T_i = \alpha_i \cdot (1, 0, 0) \quad \text{and} \quad g(T_i) \leq \frac{1}{2} \cdot \alpha_i \cdot \left( \alpha_i - (2, 1, 1) \right) + 1 - |\text{sing} T_i|.$$

Before we continue let us mention some conjectures and open problems for the reader to keep in mind as we clarify Theorem I, Corollary I and its applications in §3. The proof of our main results are prepared in §4, §5 and Appendix C, and concluded in §6.

2.3 Conjectures and open problems

The following conjecture states that for almost all edge length assignments each component of a coupler curve has the same degree and geometric genus.

Conjecture 2. In P6 we additionally have for all $1 \leq i, j \leq n$:

$$\deg T_i = \deg T_j \quad \text{and} \quad g(T_i) = g(T_j) = \frac{1}{2} \cdot \alpha_i \cdot \left( \alpha_i - (2, 1, 1) \right) + 1 - |\text{sing} T_i|.$$

Note that the equality for the geometric genus follows if the delta invariant of any isolated singularity of the coupler curve equals one (see the proof of Proposition 12). Even if this conjecture is confirmed, we still do not know how to determine efficiently the number $|\text{sing} T_i|$ of singularities of an irreducible component $T_i$ of the coupler curve, and the number $n$ of irreducible components.

Open problem 3. For any given calligraph, determine the number of irreducible components of the coupler curve and for each such component its degree and geometric genus.

On the other hand, we refer to [15, Section 8] for open problems concerning the number of realizations of minimally rigid graphs.
3 Applications of the main result

In this section we shall consider two applications of our main result Theorem I and
its Corollary I.

- If \((G, G')\) is a calligraphic split, then we reduce the computation of \(c(G \cup G')\)
to the computation of the numbers of realizations of six minimally rigid graphs
with at most \(\max(|v(G)|, |v(G')|) + 1\) vertices. If \((G, G')\) is non-trivial, then we
confirm experimentally that this reduction leads to a significant speed up.

- If \(G\) is a calligraph with coupler curve \(T\) and coupler decomposition \((T_1, \ldots, T_n)\),
then we determine \(\deg T\) and give non-trivial bounds for \(n, \deg T_i\) and \(g(T_i)\) for
all \(1 \leq i \leq n\). Furthermore, we determine the number of complex intersections
of coupler curves of two calligraphs.

The first application is clarified in §3.1 and Appendix B. The second application is
clarified in §3.2 and §3.3.

3.1 Counting the number of realizations using classes

Suppose we would like to determine the number of realizations \(c(U)\) of the minimally
rigid graph \(U\) in Figure 5. We remove the vertex of degree 2 from \(U\) and obtain \(V\).
It is straightforward to see that \(c(U) = 2 \cdot c(V)\).

\[ \text{Figure 5: } c(U) = 2 \cdot c(V) \text{ and thus it is sufficient to compute } c(V). \]

Since \(V\) does not contain any degree 2 vertices, we are now going to apply Axioms A1
and A2. For this we need to find a non-trivial calligraphic split for \(V\). First we
choose an edge and a vertex in \(V\). After relabeling the vertices we obtain the
rightmost marked graph in Figure 5 such that \(\{1, 2\}\) and 0 are the chosen edge and
vertex, respectively. We consider the subgraph of \(V\) induced by \(v(V) \setminus \{0, 1, 2\}\) as
is illustrated in Figure 6. This subgraph consists of two components defined by the
edges \(\{3, 4\}\) and \(\{5, 6\}\). These components define the calligraphs \(H\) and \(I\) which
are induced by the vertices \(\{3, 4\} \cup \{0, 1, 2\}\) and \(\{5, 6\} \cup \{0, 1, 2\}\), respectively. We
verify that \((H, I)\) is a non-trivial calligraphic split for \(V\).

Our next goal is to determine the class of \([H]\). We first compute the number of
realizations for the minimally rigid graphs in Figure 7 using the algorithm described in [5, Section 5]. It follows from Axioms A1 and A2 that $[\mathcal{H}] \cdot [\mathcal{L}] = [\mathcal{H}] \cdot [\mathcal{R}] = 8$ and $[\mathcal{H}] \cdot [C_8] = 24$, where $[\mathcal{L}] = (1, 1, 0)$, $[\mathcal{R}] = (1, 0, 1)$ and $[C_8] = (2, 0, 0)$. We solve the resulting system of linear equations, where the three entries of $[\mathcal{H}]$ are the indeterminates, and obtain

$$[\mathcal{H}] = (6, 2, 2).$$

In this particular example $\mathcal{H}$ and $\mathcal{I}$ are isomorphic as calligraphs, so $[\mathcal{H}] = [\mathcal{I}]$. However, in general we need to compute $c(\mathcal{I} \cup \mathcal{L})$, $c(\mathcal{I} \cup \mathcal{R})$ and $c(\mathcal{I} \cup C_8)$ as well and thus in total the numbers of realizations of six graphs with at most $|v(\mathcal{H})| + 1$ vertices. We conclude from Axiom A2 that

$$c(\mathcal{U}) = 2 \cdot c(\mathcal{V}) = 2 \cdot c(\mathcal{H} \cup \mathcal{I}) = 2 \cdot [\mathcal{H}] \cdot [\mathcal{I}] = 112.$$

Remark 4 (algorithms). Our example works in general and can be translated into Algorithm 1 and Algorithm 2. We shall refer to the algorithm described in [5, Section 5] for computing numbers of realizations by “Algorithm [5]”. A graph can be determined to be minimally rigid in polynomial time in the number of vertices [16].

We find calligraphic splits for a minimally rigid graph $\mathcal{U}$ by trying out each pair consisting of an edge and a vertex (there are less than $4 \cdot v(\mathcal{U})^2$ possibilities). In order to determine whether a calligraphic split $(\mathcal{A}, \mathcal{B})$ is non-trivial in Algorithm 2, it is sufficient to verify that $|v(\mathcal{A})| \geq |v(\mathcal{B})| \geq 5$. In this case $\mathcal{A}, \mathcal{B} \notin \{\mathcal{L}, \mathcal{R}, C_v\}$ for all $v \in \mathbb{Z}_{\geq 0}$ and thus Algorithm 1 and Algorithm 2 do not end up in an infinite recursion. Hence, the correctness of Algorithm 1 and Algorithm 2 is a straightforward consequence of A1, A2 and P3. In particular, each linear system of equations
in Algorithm 1 always has a unique solution. See Appendix B for an example of the recursive execution tree of Algorithm 2.

**Algorithm 1 getClass**

- **Input.** A calligraph $G$.
- **Output.** Its class $\left[G\right]$.
- **Method.**

```
if $G \cup L$ is not minimally rigid then
  return $[G] := (m, m, 0)$ where $m := \frac{1}{2} \cdot \text{getNoR}(G \cup R)$.

if $G \cup R$ is not minimally rigid then
  return $[G] := (m, 0, m)$ where $m := \frac{1}{2} \cdot \text{getNoR}(G \cup L)$.
```

Determine $[G]$ by solving the linear system: $\left\{ \text{getNoR}(G \cup F) = [G] \cdot [F] : F \in \{L, R, C\} \right\}$.

**Algorithm 2 getNoR**

- **Input.** A minimally rigid graph $U$.
- **Output.** The number of realizations $c(U)$.
- **Method.**

```
if $v \in v(U)$ has degree 2 then
  return $2 \cdot \text{getNoR}(V)$ where $V$ is obtained from $U$ by removing $v$.

Find a non-trivial calligraphic split $(A, B)$ for $U$.

if no such split was found then
  return $c(U)$ where $c(U)$ is computed using Algorithm [5].
```

**Remark 5** (experimental results). We call a graph *splittable* if it is a minimally rigid graph that admits a non-trivial calligraphic split. Notice that if an input graph $U$ is splittable, then Algorithm 2 reduces the computation of $c(U)$ to the computation of the numbers of realizations of six minimally rigid graphs with at most $|v(U)| - 1$ vertices. To our best knowledge Algorithm [5] is currently the fastest symbolic algorithm for counting complex realizations. We still expect that Algorithm [5] is super exponential in $|v(U)|$, and indeed Figure 8 confirms heuristically that Algorithm 2 leads to a significant speed up for splittable graphs $U$ such that $|v(U)| \geq 15$. In Figure 9 we charted the relative number of splittable minimally rigid graphs $U$ such that $7 \leq |v(U)| \leq 12$. We used the C++ version of Algorithm [5] as a fallback for graphs that are either unsplittable or have less than 13 vertices. We remark that if instead we use the slower MATHEMATICA version of Algorithm [5], then our MATHEMATICA implementation of Algorithm 2 is also faster for splittable graphs with less than 13 vertices. The implementations of the algorithms can be found at [11].

## 3.2 Invariants of coupler curves and their intersections

Suppose that we would like to determine the degrees and geometric genera of the components of the coupler curve of the calligraph $H$ defined in Figure 6.
Figure 8: The timing for Algorithm 2 relative to Algorithm [5] using a set of 100 pseudo random graphs that are splittable. The 100% line represents the timing for Algorithm [5]. We see that Algorithm 2 is faster for splittable graphs with at least 15 vertices.

Figure 9: Amount of splittable graphs with respect to all minimally rigid graphs whose vertices are of degree at least 3 (thus each vertex belongs to at least 3 edges). If the number of vertices is at most 12, then at least 50% of such graphs is splittable.

The equation of the coupler curve of \( \mathcal{H} \) can be determined explicitly and was done in [14, Section 7.3]. Hence, its components, degrees and intersections with other coupler curves can be computed in a straightforward manner. However, for larger calligraphs this quickly becomes infeasible and therefore we present an alternative method that works in purely combinatorial manner without the need to perform computations in polynomial rings. We assume that we can obtain an arbitrary good approximation of the real part of the coupler curve of a calligraph for some given edge length assignment. In practice this means that we have a drawing of the coupler curve as for example in Figure 10.

We apply Algorithm 1 and find that \([\mathcal{H}] = (6, 2, 2)\). In Figure 10 we illustrate the coupler curve \( T := t_\omega(\mathcal{H}) \) for edge length assignment \( \omega \in \Omega_{\mathcal{H}} \). We observe from this approximation that \( T \) admits the coupler decomposition \((T_1, T_2)\) such that \(|\text{sing} T_1|, |\text{sing} T_2| \geq 3 \) and \( \deg T_1, \deg T_2 \geq 6 \). As \( \mathcal{H} \) is thin and \( \deg T = 12 \) by P2 and P5, we find that \( \deg T_1 = \deg T_2 = 6 \). By P6 there exists a class partition \((\alpha_1, \alpha_2)\) that is equal to either \((3, 0, 0), (3, 2, 2), (3, 2, 0), (3, 0, 2)\), \((3, 1, 2), (3, 1, 0)\) or \((3, 1, 1), (3, 1, 1)\). If \( \alpha_1 \in \{(3, 0, 0), (3, 2, 0), (3, 1, 2)\}, \) then
Figure 10: The coupler curve $T$ of $\mathcal{H}$ consists of two components $T_1$ and $T_2$ such that $\deg T_1 = \deg T_2 = 6$, $0 \leq g(T_1) \leq 1$ and $0 \leq g(T_2) \leq 1$.

$g(T_1) < 0$, which is impossible. Therefore, $\alpha_1 = \alpha_2 = (3, 1, 1)$ so that for $i \in \{1, 2\}$ we have

$$0 \leq g(T_i) \leq 4 - |\text{sing } T_i| \leq 1.$$ 

We illustrate in Figure 11, for some choice of edge length assignment, the intersections of the coupler curves $T$ and $T'$ of the calligraphs $\mathcal{H}$ and $\mathcal{C}_8$, respectively. The coupler multiplicities are equal to one in both cases by $P2$ and thus it follows from $A1$, $A2$ and $P4$ that

$$|T \cap T'| = [\mathcal{H}] \cdot [\mathcal{C}_8] = (6, 2, 2) \cdot (2, 0, 0) = 24 = c(\mathcal{H} \cup \mathcal{C}_8).$$

Figure 11: The 20 real intersections between the coupler curves of $\mathcal{H}$ and $\mathcal{C}_8$ in $\mathbb{C}^2$. There are $[\mathcal{H}] \cdot [\mathcal{C}_8] = 24$ complex intersections in total.

We observe that 20 of the complex intersection points in $T \cap T'$ are real. See [4, Figure 5] for an example where all 24 intersections are real.
3.3 Invariants of a calligraph with 10 vertices

Let us consider the calligraph \( \mathcal{F} \) as defined in Figure 12. Suppose that \( \omega \in \Omega_{\mathcal{F}} \) is a general edge length assignment and let \( T = t_{\omega}(\mathcal{F}) \) be a coupler curve of the calligraph \( \mathcal{F} \) with coupler decomposition \( (T_1, \ldots, T_n) \). Our goal is to determine \( \deg T \) and a lower bound for \( n \). Moreover, we determine upper bounds for the degrees and the geometric genera of the irreducible components \( T_1, \ldots, T_n \).

We apply Algorithm 1 and find that \( [\mathcal{F}] = (272, 0, 0) \), where \( 272 = 2^4 \cdot 17 \). We verify that \( \mathcal{F} \) is thin and thus \( \deg T = 2 \cdot 272 = 544 \) by P2 and P5.

The largest minimally rigid subgraph \( \mathcal{F} \) that contains the vertex 0 is induced by the vertices \( \{0, 6, 8, 9\} \). Let \( \mathcal{F}' \) be defined by this graph, but with vertices 6 and 8 relabeled to 1 and 2, respectively, so that \( \mathcal{F}' \) is marked. The largest minimally rigid subgraph of \( \mathcal{F}'' \subset \mathcal{F} \) that contains the edge \( \{1, 2\} \) is induced by \( \{1, 2, 3, 4\} \).

It is straightforward to see that it is not possible to continuously move vertex 0 of \( \mathcal{F} \) while fixing vertices 1, 2, 3 and 4 and arrive at a position where vertex 0 is flipped along the edge \( \{8, 6\} \) or \( \{8, 9\} \). With flipping a vertex along an edge we mean that we reflect the vertex along the line that is spanned by this edge, and in this manner obtain another realization of \( \mathcal{F} \). Such an almost everywhere continuous movement, without flipping any vertices along the way, defines a component of the coupler curve \( T \) (we believe that such a component must be irreducible). Since \( c(\mathcal{F}') = 4 \) and \( c(\mathcal{F}'') = 4 \) we obtain in this manner 16 possibly reducible components and their union coincides with the coupler curve. Thus, we established the following lower bound for the number of irreducible components: \( n \geq 16 \).

By P5 the degree of a coupler curve is uniquely determined by the class of a calligraph and therefore does not depend on the starting position of the vertices before we move the graph. From this we deduce that the components that are related by flipping a vertex in our example are of equal degree. It follows that each of the obtained possibly reducible components is of degree \( d \). Since \( \deg T = 544 = 16 \cdot d \) we find that
We established an upper bound for the degrees of irreducible components: \( \deg T_i \leq 34 \) for all \( 1 \leq i \leq n \).

In order to determine an upper bound for the geometric genera of the irreducible components, we apply P6: there exists a class partition \((\alpha_1, \ldots, \alpha_n)\) such that \( \alpha_i = (t, 0, 0) \) for some \( t \leq 17 \) so that for all \( 1 \leq i \leq n \) the following holds:

\[
g(T_i) \leq \frac{1}{2} \cdot (t, 0, 0) \cdot (t - 2, -1, -1) + 1 \leq 17 \cdot 15 + 1 = 256.
\]

In comparison, the geometric genus of a curve of degree 34 is at most \((34 - 1)(34 - 2)/2 = 528\) by the genus formula at [13, Exercise I.7.2b].

### 4 Base points and pseudo classes of calligraphs

If we vary the edge length assignments for a given calligraph \( \mathcal{G} \), then we obtain a set of coupler curves. The set of defining polynomials of these coupler curves can be represented in terms of a single polynomial \( F \) with coefficients in the edge lengths. After embedding the coupler curves of \( \mathcal{G} \) into the projective plane, we observe that all their Zariski closures pass through certain complex points at infinity. In this section we show how to recover from \( F \) such base points and their multiplicities in terms of substitutions and polynomial quotients. We conclude this section by proving Proposition 12, which states that the 3-tuple consisting of multiplicities at three distinguished base points is under a certain hypothesis a class, and that we can recover from such classes invariants of coupler curves.

Suppose that \( \mathcal{F} \subset \mathbb{C}[x, y] \) is a subset of polynomials. In this article, we assume that \( \mathcal{F} = \{ F(c_1, \ldots, c_n, x, y) : c_1, \ldots, c_n \in \mathbb{C} \} \subset \mathbb{C}[x, y] \) for some polynomial \( F \in \mathbb{C}[u_1, \ldots, u_n, x, y] \). We say that \( \mathcal{F} \) has a base point at \( p := (x_p, y_p) \) in \( \mathbb{C}^2 \) if \( f(p) = 0 \) for all \( f \in \mathcal{F} \). This base point is called \( m \)-fold if for all \( f \in \mathcal{F} \) the lowest-degree monomial term of \( f(x + x_p, y + y_p) \) has degree \( \geq m \) and there exists \( g \in \mathcal{F} \) such that the lowest-degree monomial term of \( g(x + x_p, y + y_p) \) has degree \( m \).

**Definition 6.** Let \( \alpha_p^m, \beta_p^m : \mathbb{C}[x, y] \to \mathbb{C}[x, y] \) for \( p := (x_p, y_p) \in \mathbb{C}^2 \) and \( m \in \mathbb{Z}_{\geq 0} \) be the maps

\[
\alpha_p^m(f) := f(x + x_p, y + y_p) \div x^m \quad \text{and} \quad \beta_p^m(f) := f(xy + x_p, y + y_p) \div y^m,
\]

where \( f \div g \) for polynomials \( f, g \in \mathbb{C}[x, y] \) denotes the polynomial quotient.

A base point \( q := (x_q, y_q) \in \mathbb{C}^2 \) is infinitely near to an \( m \)-fold base point \( p \) of \( \mathcal{F} \) if either

- \( q \) is a base point of \( \alpha_p^m(\mathcal{F}) \) such that \( x_q = 0 \), or
- \( q \) is a base point of \( \beta_p^m(\mathcal{F}) \) such that \( x_q = y_q = 0 \).
Similarly, we define base points that are infinitely near to \( q \) and so on. See forward Remark 13 for an algebro geometric interpretation of infinitely near base points.

We denote by \( i \) the \textit{imaginary unit}.

**Example 7.** Let \( \mathfrak{F} := \{(1 - x)^2 + y^2 - \ell^2x^2 : \ell \in \mathbb{C}\} \) be a subset of polynomials in \( \mathbb{C}[x,y] \). The base points \( p := (0,i) \) and \( \overline{p} := (0,-i) \) of \( \mathfrak{F} \) are both 1-fold. For determining whether \( p \) admits infinitely near base points we consider

\[
\alpha_p^{1}(\mathfrak{F}) = \{-2 + x + 2iy + xy^2 - \ell^2x : \ell \in \mathbb{C}\} \quad \text{and} \quad \beta_p^{1}(\mathfrak{F}) = \{2i - 2x + y + x^2y - \ell^2x^2y : \ell \in \mathbb{C}\}.
\]

We find that \( q := (0,-i) \) is the 1-fold base point of \( \alpha_p^{1}(\mathfrak{F}) \). Notice that the complex conjugate \( \overline{q} = (0,i) \) is the 1-fold base point of \( \alpha_p^{1}(\mathfrak{F}) \). We verify that both \( \alpha_q^{1} \circ \alpha_p^{1}(\mathfrak{F}) \) and \( \beta_q^{1} \circ \alpha_p^{1}(\mathfrak{F}) \) are base point free and thus we identified all base points of \( \mathfrak{F} \). <

The \textit{projective plane} \( \mathbb{P}^{2} \) is defined as \( \left( \mathbb{C}^{3} \setminus \{(0,0,0)\} \right) / \sim \), where

\[
(z_0 : z_1 : z_2) \sim (\lambda z_0 : \lambda z_1 : \lambda z_2) \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \{0\}.
\]

If \( h \in \mathbb{C}[z_0,z_1,z_2] \) is homogeneous, then we denote by \( V(h) \subset \mathbb{P}^{2} \) its \textit{zero set}. The \textit{line at infinity} is defined as \( V(z_0) \). We define \( \gamma_{0}, \gamma_{1}, \gamma_{2} : \mathbb{C}[z_0,z_1,z_2] \to \mathbb{C}[x,y] \) as

\[
\gamma_{0}(h) := h(1,x,y), \quad \gamma_{1}(h) := h(x,1,y) \quad \text{and} \quad \gamma_{2}(h) := h(y,x,1).
\]

The embeddings \( \gamma_{0}^{*}, \gamma_{1}^{*}, \gamma_{2}^{*} : \mathbb{C}^{2} \hookrightarrow \mathbb{P}^{2} \) are defined as

\[
\gamma_{0}^{*}(x,y) := (1 : x : y), \quad \gamma_{1}^{*}(x,y) := (x : 1 : y) \quad \text{and} \quad \gamma_{2}^{*}(x,y) := (y : x : 1).
\]

Suppose that \( \mathfrak{H} \subset \mathbb{C}[z_0,z_1,z_2] \) is a subset of homogeneous polynomials of the same degree. The \textit{series associated to} \( \mathfrak{H} \) is defined as the following set of projective curves:

\[
\Gamma_{\mathfrak{H}} := \left\{ V(h) \subset \mathbb{P}^{2} : h \in \mathfrak{H} \right\}.
\]

We call \( r := (r_0 : r_1 : r_2) \in \mathbb{P}^{2} \) a \textit{base point} of the series \( \Gamma_{\mathfrak{H}} \) if \( r = \gamma_{j}^{*}(p) \) for some \( p := (p_x,p_y) \) such that either

- \( j = 0 \) and \( p \) is a base point of \( \gamma_{0}(\mathfrak{H}) \),
- \( j = 1 \) and \( p \) is a base point of \( \gamma_{1}(\mathfrak{H}) \) such that \( p_x = 0 \), or
- \( j = 2 \) and \( p \) is a base point of \( \gamma_{2}(\mathfrak{H}) \) such that \( p_x = p_y = 0 \).

If \( q \in \mathbb{C}^{2} \) is a base point that is infinitely near to \( p \), then we say that \( q \) is also \textit{infinitely near to} \( r \). We call \( r \in \mathbb{P}^{2} \) \textit{cyclic} if it is equal to either \((0:1:i)\) or \((0:1:-i)\).

Suppose that \( r \in \mathbb{P}^{2} \) is a base point of the series \( \Gamma_{\mathfrak{H}} \) such that \( r = \gamma_{1}^{*}(p) \), where \( p = (0,\pm i) \) is an \( m \)-fold base point of \( \gamma_{1}(\mathfrak{H}) \), and \( q \in \mathbb{C}^{2} \) is a base point of \( \alpha_{p}^{m} \circ \gamma_{1}(\mathfrak{H}) \). In this case we call

- \( r \) a \textit{cyclic base point} of \( \Gamma_{\mathfrak{H}} \),
• a 1-centric base point of $\Gamma_\mathcal{H}$ if $q = (0, 0)$, and
• a 2-centric base point of $\Gamma_\mathcal{H}$ if $q = (0, \mp i)$ such that $y_p = -y_q$.

Notice that 1-centric and 2-centric base points are infinitely near to a cyclic base point. We call the series $\Gamma_\mathcal{H}$ centric if all its base points are either cyclic, 1-centric or 2-centric, and if almost all curves in the series meet the line at infinity only at the cyclic points.

**Example 8.** Let $\mathcal{H} := \{(z_1 - z_0)^2 + z_2^2 - \ell^2 z_0^2 : \ell \in \mathbb{C}\}$. Then

\[
\gamma_0(\mathcal{H}) = \{(x - 1)^2 + y^2 - \ell^2 : \ell \in \mathbb{C}\}, \\
\gamma_1(\mathcal{H}) = \{(1 - x)^2 + y^2 - \ell^2 x^2 : \ell \in \mathbb{C}\}, \\
\gamma_2(\mathcal{H}) = \{(x - y)^2 + 1 - \ell^2 y^2 : \ell \in \mathbb{C}\}.
\]

We observe that $\gamma_0(\mathcal{H})$ is base point free and defines a pencil of circles in $\mathbb{C}^2$ that are centered around $(1, 0)$. We deduce from Example 7 that $\gamma_1(\mathcal{H})$ has two complex conjugate base points whose $\gamma^*_1$-images in $\mathbb{P}^2$ are cyclic, and that the complex conjugate infinitely near base points are 2-centric. Finally, we check that $(0, 0)$ is not a base point of $\gamma_2(\mathcal{H})$. Thus there are in total four complex base points and each of them is 1-fold. A conic in the series $\Gamma_\mathcal{H}$ meets the line at infinity at exactly the two cyclic points, and thus we established that $\Gamma_\mathcal{H}$ is centric.

Suppose that $\mathcal{G}$ is a calligraph. We consider the embedding of its coupler curves into the projective plane, namely we let $T_\omega \subset \mathbb{P}^2$ be the Zariski closure of the curve $\gamma_0^*(t_\omega(\mathcal{G}))$. Define $\mathcal{H} \subset \mathbb{C}[z_0, z_1, z_2]$ as the set of square free homogeneous polynomials such that

\[
\{V(h) \subset \mathbb{P}^2 : h \in \mathcal{H}\} = \{T_\omega \subset \mathbb{P}^2 : \omega \in \Omega_\mathcal{G}\}.
\]

We shall see in Lemma 14(c) that $\mathcal{H}$ can be represented similarly as in Example 8, namely in terms of a square free homogeneous polynomial in $z_0, z_1$ and $z_2$, whose coefficients are polynomials in the variables for the edge lengths. The series $\Gamma_\mathcal{H}$ associated to $\mathcal{H}$ is called the series associated to the calligraph $\mathcal{G}$. We call $\mathcal{G}$ centric if its associated series $\Gamma_\mathcal{H}$ is centric.

**Definition 9.** The pseudo class for calligraphs assigns to each calligraph $\mathcal{G}$ the tuple

\[
[\mathcal{G}] := (m_a, m_b, m_c) \in \mathbb{Z}^3,
\]

where $m$ equals the coupler multiplicity $m(\mathcal{G})$ and where the series $\Gamma_\mathcal{H}$ associated to $\mathcal{G}$ has an $a$-fold cyclic base point, a $b$-fold 1-centric base point, and a $c$-fold 2-centric base point. If $\Gamma_\mathcal{H}$ has no cyclic, 1-centric or 2-centric base points, then $a = 0$, $b = 0$ and $c = 0$, respectively.

**Remark 10.** If a cyclic base point of the series associated to a calligraph has multiplicity $n$, then both cyclic base points must have multiplicity $n$. This statement
also holds if we replace “cyclic” with “1-centric” or “2-centric”. The reason is that for real edge length assignments, the coupler curves of a calligraph are real and thus the multiplicities at complex conjugate base points of the associated series must be equal (see Example 7).

Example 11. Suppose that \( \Gamma \) is the series associated to the calligraph \( \mathcal{R} \) as defined in Figure 3. The coupler multiplicity of \( \mathcal{R} \) is equal to 1. The coupler curves of \( \mathcal{R} \) define circles centered around the point \((1, 0)\) and thus \( \mathcal{H} \) is defined as in Example 8. Hence, \( \Gamma \) is centric with two complex conjugate 1-fold cyclic base points, and two complex conjugate 1-fold 2-centric base points. We conclude that \( \mathcal{R} \) has pseudo class \((1, 0, 1)\).

The following proposition shows that the pseudo class is a class under the assumption that calligraphs are centric. We show in §6 that calligraphs are indeed centric.

**Proposition 12.** If calligraphs are centric, then the following holds:

- The coupler multiplicity and the pseudo class for calligraphs are well-defined.
- The pseudo class satisfies the Axioms A1, A2, and the six Properties P1—P6.
- A class for calligraphs must be the unique pseudo class.

For its proof we assume that the reader is familiar with intersection theory for surfaces (see for instance [13, Sections V.1 and V.3]). Before we prove Proposition 12 let us in preparation first give the algebro geometric interpretation of infinitely near base points in Remark 13 and prove Lemmas 14 to 17. Lemma 17 is based on a result about “global rigidity” in [6, Corollary 1.7].

**Remark 13.** Our definition for infinitely near base points is motivated as follows. Let 

\[ W_i := \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 : z_i \neq 0\} \]

so that \( \mathbb{P}^2 = W_0 \cup W_1 \cup W_2 \) and \( \gamma_i^*(\mathbb{C}^2) = W_i \). Suppose that \( V(h) \subset \mathbb{P}^2 \) is a curve that has multiplicity \( m > 0 \) at some point \( r \) (see [13, Exercise I.5.3] for multiplicities of curves at points). Let us assume that \( r \not\in W_0 \) and \( r \in W_1 \) so that \( r = \gamma_1^*(p) \) for some \( p \in \mathbb{C}^2 \). In this case the preimage of \( V(h) \) with respect to \( \gamma_1^* \) is defined as the zero set \( V(f) \subset \mathbb{C}^2 \), where \( f := h \circ \gamma_1^* = h(x, 1, y) \) is a polynomial in \( \mathbb{C}[x, y] \). Notice that \( V(f) \) has multiplicity \( m \) at \( p \), since \( \gamma_1^* \) defines an isomorphism \( \mathbb{C}^2 \to W_1 \). We translate \( p \) to the origin and consider the corresponding curve 

\[ C := V(f(x + x_p, y + y_p)). \]

The blowup of \( \mathbb{C}^2 \) at the origin is defined as follows (see [13, Section I.4]):

\[ U := \{(x, y; t_0 : t_1) \in \mathbb{C}^2 \times \mathbb{P}^1 : x t_1 = y t_0\}. \]
We find that $U = U_0 \cup U_1$ where

$$U_i := \{(x, y; t_0 : t_1) \in U : t_i \neq 0\}.$$ 

In the chart $U_0$ we have the relation $y = x\tilde{y}$ with $\tilde{y} := t_1/t_0$. Thus the map $\phi : \mathbb{C}^2 \to U_0$ that sends $(x, \tilde{y})$ to $(x, \tilde{y}x; 1 : \tilde{y})$ is an isomorphism. Let $\rho : U_0 \to \mathbb{C}^2$ be the projection that sends $(x, y; t_0 : t_1)$ to $(x, y)$. We find that the composition $\rho \circ \phi$ sends $(x, y)$ to $(x, yx)$ and is an isomorphism when $x \neq 0$. The Zariski closure of the fiber $\rho^{-1}(0, 0)$ is called an exceptional curve and the Zariski closure of $\rho^{-1}(C)$ contains this exceptional curve with multiplicity $m$ (see [13, Example I.4.9.1 and Proposition V.3.6]). This exceptional curve corresponds via $\phi$ to the line $V(x) \subset \mathbb{C}^2$.

The $(\rho \circ \phi)$-preimage of $C$ is defined as the zero set of $f(x + x_p, yx + y_p) = g \circ \rho \circ \phi$ with $g(x, y) := f(x + x_p, y + y_p)$. After removing the $(x^m)$-component, this preimage is defined as the zero set of $\alpha^m_p(f)$. As $\rho \circ \phi$ is an isomorphism outside $V(x)$, we are only interested in $(x_q, y_q) \in V(\alpha^m_p(f))$ such that $x_q = 0$. In this case $y_q$ corresponds to a tangent direction of $C$ at the origin and thus gives a geometric interpretation of an infinitely near point (see [13, Example I.4.9.1 and Figure I.3]).

For the remaining chart $U_1$ we consider $\beta^m_p(f)$ instead of $\alpha^m_p(f)$. Because $U_1 \setminus U_0 = \{(x, y; t_0 : t_1) \in U : t_0 = 0, t_1 = 1\}$, it follows that to complete the analysis we only need to check the multiplicity of $(0, 0)$ in $V(\beta^m_p(f))$. Indeed, the definitions in this section are chosen such that each point in the overlapping charts $W_0 \cap W_1$, $W_0 \cap W_2$, $W_1 \cap W_2$ and $U_0 \cap U_1$ is only considered once.

**Lemma 14.** Suppose that $\mathcal{G}$ is a calligraph. Let $\mathcal{V} := \nu(\mathcal{G}) \setminus \{1, 2\}$, $\mathcal{E} := \varepsilon(\mathcal{G}) \setminus \{1, 2\}$, $(x_1, y_1) := (0, 0)$ and $(x_2, y_2) := (1, 0)$.

(a) If $V_\mathcal{G} \subset \mathbb{C}^{2|\mathcal{V}|} \times \mathbb{C}^{|\mathcal{E}|}$ denotes the zero set of

$$\{(x_i - x_j)^2 + (y_i - y_j)^2 - \ell^2_{i,j} : \{i, j\} \in \mathcal{E}\} \subset \mathbb{C}[\ell_e, x_i, y_i : e \in \mathcal{E}, i \in \mathcal{V}],$$

then $\dim V_\mathcal{G} = 1 + |\mathcal{E}|$. In particular, if $\lambda = (\lambda_e)_{e \in \mathcal{E}} \in \mathbb{C}^{\mathcal{E}}$ is general and $H_\lambda \subset \mathbb{C}^{2|\mathcal{V}|} \times \mathbb{C}^{\mathcal{E}}$ is the zero set of $\{\ell_e - \lambda_e : e \in \mathcal{E}\}$, then $\dim(V_\mathcal{G} \cap H_\lambda) = 1$.

(b) There exists a square free polynomial $F \in \mathbb{C}[\ell_e, x_0, y_0 : e \in \mathcal{E}]$ such that for all edge length assignments $\omega \in \Omega_\mathcal{G}$, the coupler curve $t_\omega(\mathcal{G})$ is equal to the zero set of $F(\lambda, x_0, y_0)$, where $\lambda = (\omega(e))_{e \in \mathcal{E}}$.

(c) There exists a square free polynomial $H \in \mathbb{C}[\ell_e, z_0, z_1, z_2 : e \in \mathcal{E}]$ that is homogeneous in the variables $z_0$, $z_1$ and $z_2$ such that the series $\Gamma_\omega$ associated to the calligraph $\mathcal{G}$ is defined as the series associated to the following subset of homogeneous polynomials:

$$\mathcal{H} := \{H(\lambda, z_0, z_1, z_2) : \lambda \in \mathbb{C}^{\mathcal{E}}\} \subset \mathbb{C}[z_0, z_1, z_2].$$

(d) The coupler curve $t_\omega(\mathcal{G})$ is 1-dimensional for almost all $\omega \in \Omega_\mathcal{G}$.
Proof. (a) Recall that the calligraph \( G \) can be obtained by removing an edge \( \{u, v\} \) from some minimally rigid graph \( \mathcal{M} \). We define \( V_\mathcal{M} := V_\mathcal{G} \cap M \), where \( M \subset \mathbb{C}^{2|V|} \) denotes the zero set of the polynomial \((x_u - x_v)^2 + (y_u - y_v)^2 - \lambda_{\{u, v\}}^2 \) for some general edge length \( \lambda_{\{u, v\}} \in \mathbb{C} \). By construction, a point in \( V_\mathcal{M} \cap H_\lambda \) corresponds to a realization of \( \mathcal{M} \) for some general edge lengths determined by \( \lambda = (\lambda_e)_{e \in \mathcal{E}} \) and \( \lambda_{\{u, v\}} \). Recall that for a general choice of edge lengths, a minimally rigid graph admits at least one, but only finitely many realizations. Therefore, \( V_\mathcal{M} \cap H_\lambda \) is non-empty and \( \dim(V_\mathcal{M} \cap H_\lambda) = 0 \). This implies that \( \dim V_\mathcal{M} = |\mathcal{E}| \). Moreover, \( \dim V_\mathcal{G} > |\mathcal{E}| \) as \( G \) is not minimally rigid. It follows from [8, Theorem 0.2] that
\[
\dim V_\mathcal{M} \geq \dim V_\mathcal{G} - \text{codim} M = \dim V_\mathcal{G} - 1.
\]
Hence, \( \dim V_\mathcal{G} = 1 + |\mathcal{E}| \) and \( \dim(V_\mathcal{G} \cap H_\lambda) = 1 \) as was to be shown.

(b) Let \( \rho: \mathbb{C}^{2|V|} \times \mathbb{C}^{|\mathcal{E}|} \rightarrow \mathbb{C}^2 \times \mathbb{C}^{|\mathcal{E}|} \) denote the linear projection that sends
\[
(x_i, y_i, \ell_e)_{i \in \mathcal{V}, e \in \mathcal{E}}
\]
to \( (x_0, y_0, \ell_e)_{e \in \mathcal{E}} \). We know from [7, Theorem 3 in Section 3.2] that the ideal of \( \rho(V_\mathcal{G}) \) is obtained by eliminating the variables \( \{x_i, y_i : i \in \mathcal{V} \setminus \{0\}\} \) from the ideal \( \langle V_\mathcal{G} \rangle \) of \( V_\mathcal{G} \). By assertion (a), \( V_\mathcal{G} \cap H_\lambda \) is a non-linear curve and thus its projection to the \( (x_0, y_0) \)-plane is again a curve. As a direct consequence of the definitions, this planar curve corresponds to the coupler curve \( t_\omega(G) \), where \( \lambda = (\omega(e))_{e \in \mathcal{E}} \). This implies that the elimination ideal \( \langle V_\mathcal{G} \rangle \cap \mathbb{C}[\ell_e, x_0, y_0] \) is generated by a single polynomial. We define \( F(\ell, x_0, y_0) \) to be the square free part of this polynomial. Notice that the zero set of \( F(\lambda, x_0, y_0) \) is equal to \( t_\omega(G) \), which was to be shown.

(c) This assertion is a direct consequence of assertion (b), since the square free polynomial \( H \) is the homogenization of \( F \) with respect to the variables \( x_0 \) and \( y_0 \).

(d) This assertion is a direct consequence of assertion (b). \( \square \)

**Lemma 15.** If the calligraphs \( G \) and \( G' \) are centric, then the coupler curves \( t_\omega(G) \) and \( t_{\omega'}(G') \) are curves that intersect transversally for almost all edge length assignments \( \omega \in \Omega_G \) and \( \omega' \in \Omega_{G'} \).

**Proof.** Let \( F(\ell, z) \) denote the square free polynomial in Lemma 14(b), where we renamed the variables \((x_0, y_0)\) to \( z := (z_1, z_2) \). Thus, the coupler curve \( t_\omega(G) \) is equal to the zero set of \( F(a, z) \), where \( a_e \) equals the edge length \( \omega(e) \) for all \( e \in \mathcal{E} \). Let \( F_i \) for \( i \in \{1, 2\} \) denote the partial derivative \( \partial_{z_i} F \) with respect to \( z_i \). Analogously, we obtain a polynomial \( G(b, z) \) for \( G' \) and its partial derivatives \( G_1 \) and \( G_2 \). Since for general edge length assignments the coupler curves of \( G \) and \( G' \) do not have a common component, it follows that \( F \) and \( G \) have no common factor.
For all $p \in \mathbb{C}^2$ we consider the set

$$U_p := \left\{ (\alpha, \beta) \in \mathbb{C}^m \times \mathbb{C}^n : \right. \left. F(\alpha,p) = G(\beta,p) = F_1(\alpha,p) \cdot G_2(\beta,p) - F_2(\alpha,p) \cdot G_1(\beta,p) = 0 \right\},$$

where $m := |e(G)| - 1$ and $n := |e(G')| - 1$.

First suppose that both $G$ and $G'$ admit only a 1-dimensional family of coupler curves. Thus, the coupler curves of $G$ and $G'$ are without loss of generality the circles centered at $(0,0)$ and $(1,0)$, respectively. In this case, the main assertion holds and thus we may assume in the remainder of the proof that $G$ admits a $d$-dimensional family of coupler curves such that $d \geq 2$. This implies that $m \geq 2$ and thus $m + n \geq 3$.

We suppose by contradiction that for general $\omega \in \Omega_G$ and $\omega' \in \Omega_{G'}$, the coupler curves $t_\omega(G)$ and $t_{\omega'}(G)$ intersect non-transversally at some point $p \in \mathbb{C}^2$. Algebraically, this means that for all $(a, b) \in \mathbb{C}^m \times \mathbb{C}^n$ there exists $p \in \mathbb{C}^2$ such that $(a, b) \in U_p$. Indeed, the equations $F(a,p) = 0$ and $G(b,p) = 0$ indicate that both coupler curves pass through the point $p$ and the Jacobian determinant equation $F_1(a,p) \cdot G_2(b,p) - F_2(a,p) \cdot G_1(b,p) = 0$ means that the tangent vectors of the coupler curves are either linear dependent at $p$, or $p$ is a singular point of at least one of the coupler curves. Hence, we established that

$$\dim \bigcup_{p \in \mathbb{C}^2} U_p = m + n.$$

Let $A_p := \{ \alpha \in \mathbb{C}^m : F(\alpha,p) = 0 \}$ and $B_p := \{ \beta \in \mathbb{C}^n : G(\beta,p) = 0 \}$. Since the calligraphs are centric, the series associated to these calligraphs only admit base points at infinity. It follows that there does not exist a point $p \in \mathbb{C}^2$ such that $A_p = \mathbb{C}^m$ and $B_p = \mathbb{C}^n$. This implies that for all $p \in \mathbb{C}^2$ we have

$$\dim U_p \leq \dim A_p \times B_p = m + n - 2.$$

**Claim 1.** For general $p \in \mathbb{C}^2$, $a \in A_p$ and $b \in B_p$ we have

$$F_1(a,p) \cdot G_2(b,p) - F_2(a,p) \cdot G_1(b,p) \neq 0.$$

Let the coupler curves $C_\alpha$ and $D_\beta$ denote the zero sets in $\mathbb{C}^2$ of $F(\alpha,z)$ and $G(\beta,z)$, respectively. We observe that $\dim A_q = \dim A_p$ for almost all $q \in \mathbb{C}^2$ and thus instead of first fixing a general $p \in \mathbb{C}^2$ and afterwards a general $a \in A_p$, we may equivalently first fix a general $a \in \mathbb{C}^m$ and afterwards a general $p \in C_a$ so that $a$ is general in $A_p$ as well. It follows that $p \notin \text{sing} C_a$ as a general point in $C_a$ is smooth. The analogous argument shows that $p \notin \text{sing} D_\beta$. This implies that $(F_1(a,p), F_2(a,p)) \neq (0,0)$ and $(G_1(b,p), G_2(b,p)) \neq (0,0)$. We set $s_\alpha(z) := F_1(\alpha,z)/F_2(\alpha,z)$ and $t_\beta(z) := G_1(\beta,z)/G_2(\beta,z)$. Now suppose by contradiction that $F_1(a,p) \cdot G_2(b,p) - F_2(a,p) \cdot G_1(b,p) = 0$. In this case, the curves $C_a$ and
Lemma 14(b) that we know from Lemma 14(a) that well, we established that line at Lemma 16. 

\( D_b \) must intersect tangentially at \( p \). The slope \( s_a(p) = t_b(p) \) of their mutual tangent line at \( p \) does not depend on \( a \) or \( b \), as \( (a, b) \in A_p \times B_p \) is general. As \( p \) is general as well, we established that \( s_a(z) = t_{\beta}(z) \) for almost all \( (z, \alpha, \beta) \in C^2 \times A_p \times B_p \). Therefore, the rational function \( u(z) := s_a(z) = t_{\beta}(z) \) only depends on \( z \in C^2 \) and the \( \alpha \)'s and \( \beta \)'s cancel out. We established that \( u(z) \) defines a slope field that is continuous for almost all \( z \in C^2 \), and in particular, \( u(z) \) is continuous at \( p \). By construction, the curve \( C_a \) is an integral curve for the slope field that passes through \( p \). By the implicit function theorem we may assume without loss of generality that \( C_a \) is in a complex analytic neighborhood \( W_p \) around \( p \) defined by the graph of the function \( g(z_1) \). It follows from the Picard-Lindelöf theorem that \( g(z_1) \) is locally the unique solution to the differential equation \( f'(z_1) = u(z_1, f(z_1)) \) with initial condition \( f(p_1) = p_2 \), where \( p = (p_1, p_2) \). Hence, \( C_{\tilde{a}} \cap W_p = C_a \cap W_p \) for all \( \tilde{a} \in A_p \). Now recall that, by assumption, the family of coupler curves is \( d \)-dimensional with \( d \geq 2 \). Thus, the family of coupler curves that pass through the point \( p \) is \((d-1)\)-dimensional. We arrived at a contradiction, as the coupler curves containing \( p \) cannot all be equal in the neighborhood \( W_p \). This concludes the proof of Claim 1.

It follows from Claim 1 that \( U_p \neq A_p \times B_p \) for almost all \( p \in C^2 \). Notice that \( U_p \subset A_p \times B_p \) is also not a component of maximal dimension, since \( (a, b) \in A_p \times B_p \) was assumed general. Thus, for almost all \( p \in C^2 \) the algebraic set \( U_p \subset A_p \times B_p \) is of codimension 1 so that

\[
\dim U_p \leq m + n - 3.
\]

We arrived at a contradiction since \( \dim \cup_{p \in C^2} U_p \leq m + n - 1 \). This concludes the proof as the coupler curves \( t_\omega(G) \) and \( t_{\omega'}(G) \) must intersect transversally. \( \square \)

**Lemma 16.** Suppose that \( G \) and \( G' \) are centric calligraphs.

- The coupler multiplicity and pseudo class of \( G \) are well-defined.
- If \((G, G')\) is a calligraphic split, then for almost all the edge length assignments \( \omega \in \Omega_G \) and \( \omega' \in \Omega_{G'} \), we have

\[
c(G \cup G') = m \cdot m' \cdot |T \cap T'|,
\]

where \( T := t_\omega(G) \), \( m := m(G) \), \( T' := t_{\omega'}(G') \) and \( m' := m(G') \).

**Proof.** Let \( E, V \) and \( V_G, H_\lambda \subset C^{2|V|} \times C^{|E|} \) be defined as in Lemma 14(a). Let the linear projection \( \phi: V_G \rightarrow \phi(V_G) \subset C^2 \times C^{|E|} \) send \((x_i, y_i, \ell_e)_{e \in V, e \in E} \) to \((x_0, y_0, \ell_e)_{e \in E} \). We suppose that \((\omega(e))_{e \in E} = (\lambda_e)_{e \in E} \), where \( \omega \in \Omega_G \) is general by assumption. We know from Lemma 14(a) that \( C_\lambda := V_G \cap H_\lambda \) is a curve and it follows from Lemma 14(b) that

\[
\phi(C_\lambda) = \{(p, \lambda) \in C^2 \times C^{|E|} : p \in T\}.
\]

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Since the projection $\phi(C_\lambda)$ is a curve, the restricted map $\phi|_{C_\lambda}$ is generically finite. From this it follows that $\phi$ itself is generically finite. As a straightforward consequence of the definitions, we find that for almost all $p \in T$ the following holds:

$$m = |\phi^{-1}(p, \lambda)| = |\{\xi \in \Xi_\omega^G : \xi(0) = p\}| < \infty.$$  

We established that the coupler multiplicity $m$ of $G$ is well-defined as it does not depend on the general choices of $\omega \in \Omega_G$ and $p \in T$.

Suppose that $|G| = m \cdot (a, b, c)$ is the pseudo class of $G$. As a consequence of Lemma 14(c), the multiplicities $(a, b, c)$ do not depend on the general choice of edge length assignments. We established that the pseudo class is well-defined.

The number of realizations $c(G \cup G') \in \mathbb{Z}_{>0}$ of the minimally rigid graph $G \cup G'$ does not depend on the general choices of edge length assignments by [15, Theorem 3.6] (see alternatively [5, Corollary 1.11]). We know from Lemma 14(d) that the coupler curves $T$ and $T'$ are indeed curves. Each point in $T \cap T'$ corresponds to the realization of the vertex 0 in a realization of $G \cup G'$ (see Figures 2 and 4). Thus, if $T \cap T' = \{p_1, \ldots, p_k\}$, then

$$c(G \cup G') = \sum_{1 \leq i \leq k} |\{\xi \in \Xi_\omega^G : \xi(0) = p_i\}| \cdot |\{\xi' \in \Xi_\omega^{G'} : \xi'(0) = p_i\}|.$$  

This implies that $c(G \cup G') = m \cdot m' \cdot |T \cap T'|$.  

**Lemma 17.** If $G$ is a thin calligraph, then for general $p \in t_\omega(G)$ and general edge length assignment $\omega \in \Omega_G$ there exists a unique realization $\xi \in \Xi_\omega^G$ such that $\xi(0) = p$. In other words, $|\{\xi \in \Xi_\omega^G : \xi(0) = p\}| = 1$.

**Proof.** Let us assign general coordinates to the vertices of $G' := G \cup L \cup R$. We may assume up to rotations, translations and scalings that vertices 1 and 2 have coordinates $(0, 0)$ and $(1, 0)$, respectively. Suppose that $\omega' \in \Omega_{G'}$ is the corresponding edge length assignment. Since $G'$ is thin, it follows that $G'$ is 3-vertex connected and “generically redundantly rigid”, so $G'$ is “generically globally rigid” by [6, Corollary 1.7]. In other words, $G'$ admits up to rotations, translations and reflections only one realization in the plane that is compatible with the edge length assignment $\omega'$. This implies that $|\Xi_{G'}| = 2$, since we do not identify reflections. We now fix $\xi' \in \Xi_{G'}$ to be one of the two realizations. Let the edge length assignment $\omega \in \Omega_G$ and realization $\xi \in \Xi_\omega^G$ be induced by $\omega'$ and $\xi'$, respectively. By construction, we have that $p := \xi(0)$ and $\omega$ are general and thus satisfy our main hypothesis. The distances $\delta(\xi(0) - \xi(1))$ and $\delta(\xi(0) - \xi(2))$ with $\delta(x, y) := (x^2 + y^2)^{1/2}$ are equal to the edge lengths $\omega'\{(0, 1)\}$ and $\omega'\{(0, 2)\}$, respectively. Hence, $|\Xi_{G'}| = 2$ implies that $|\{\xi \in \Xi_\omega^G : \xi(0) = p\}| = 1$.  

\[\square\]
Proof of Proposition 12. Suppose that $G$ and $G'$ are centric calligraphs and let the edge length assignments $\omega \in \Omega_G$ and $\omega' \in \Omega_{G'}$ be general. We follow the shorthand notation of Corollary I, where $T := t_\omega(G)$, $m := m(G)$, $T' := t_{\omega'}(G')$ and $m' := m(G')$. We know from Lemma 14(d) that the coupler curves $T$ and $T'$ are indeed curves. Let the pseudo classes of $G$ and $G'$ be $[G] = m \cdot (a, b, c)$ and $[G'] = m' \cdot (a', b', c')$. We know from Lemma 16 that these coupler multiplicities and pseudo classes are well-defined. We proceed to show that axioms and properties for pseudo classes of centric calligraphs.

Recall from Example 11 that $R$ has pseudo class $(1, 0, 1)$. Similarly, we find that $[L] = (1, 1, 0)$ and $[C_v] = (2, 0, 0)$ for all $v \in \mathbb{Z}_{\geq 3}$. Hence, the pseudo class satisfies Axiom A1.

If $G \cup L$ is not minimally rigid, then we observe that the coupler curve $T$ of $G$ is a circle that is centered around vertex 1, which implies that $[G] = m \cdot [L] = (m, m, 0)$. The analogous statement holds in case $G \cup R$ is not minimally rigid, and thus P3 holds true.

Property P2 follows from Lemma 17 and the definition of coupler multiplicity.

Let $\Gamma_H$ be the associated series of $G$. Since $\Gamma_H$ is centric by assumption, almost all curves in $\Gamma_H$ meet the line at infinity at only the complex conjugate cyclic points with multiplicity $a$ and thus it follows from Bézout’s theorem that a curve in $\Gamma_H$ has degree $2a$ so that P5 is fulfilled.

Suppose that $Y$ is the blowup of $\mathbb{P}^2$ at the cyclic points and the infinitely near 1-centric and 2-centric points. We remark that after blowing up $\mathbb{P}^2$ at the cyclic points $u$ and $\overline{u}$, the infinitely near points lie on the two complex conjugate exceptional curves that contract to $u$ and $\overline{u}$, respectively (see Remark 13). Let $v, \overline{v}$ and $w, \overline{w}$ denote the pairs of 1-centric and 2-centric points, respectively. Thus the centers of blowup consist of three pairs of complex conjugate points. Therefore, $Y$ admits a real structure, namely an antiholomorphic involution $\sigma_Y : Y \to Y$ (see Remark 10).

In the following we use the algebro geometric concepts of divisor classes and canonical classes, which should not be confused with classes or pseudo classes of calligraphs.

Let $\Theta := \{u, \overline{u}, v, \overline{v}, w, \overline{w}\}$ and for $p \in \Theta$ let $\varepsilon_p$ denote the divisor class of the exceptional curve that is centered at $p$. We denote by $\varepsilon_0$ the divisor class of the pullback of a general line in $\mathbb{P}^2$. The Néron-Severi lattice of $Y$ is generated by the group lattice $N(Y) = \langle \varepsilon_0, \varepsilon_p : p \in \Theta \rangle \mathbb{Z}$, where the non-zero intersection products between the generators are $\varepsilon_0^2 = -\varepsilon_p^2 = 1$ for $p \in \Theta$ (see [13, Proposition V.3.2]). The real structure $\sigma_Y$ induces a unimodular involution $\sigma_* : N(Y) \to N(Y)$ such that
\(\sigma_*(\varepsilon_0) = \varepsilon_0\) and \(\sigma_*(\varepsilon_p) = \varepsilon_p\) for \(p \in \Theta\) (see [25, chapter I]). Therefore, the real group lattice is defined as

\[N_\mathbb{R}(Y) = \langle \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle_\mathbb{Z}\]

where \(\varepsilon_1 := \varepsilon_c + \varepsilon_\pi, \varepsilon_2 := \varepsilon_u + \varepsilon_\pi\) and \(\varepsilon_3 := \varepsilon_v + \varepsilon_\pi\).

The only nonzero intersection products between the generators are

\[\varepsilon_0^2 = 1 \quad \text{and} \quad \varepsilon_1^2 = \varepsilon_2^2 = \varepsilon_3^2 = -2.\]

The canonical class of \(Y\) is as follows (see [13, Example II.8.20.3 and Proposition V.3.3]):

\[\kappa := -3\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3.\]

If \(C \subset \mathbb{C}^2\) is a real curve, then its class \([C] \in N_\mathbb{R}(Y)\) is defined as the divisor class of the strict transform of \(\gamma^*_0(C) \subset \mathbb{P}^2\) along the blowup map \(Y \to \mathbb{P}^2\). Notice that the coupler curve \(T\) is real, if the edge length assignment \(\omega\) is real. We have \(\deg T = 2a = [T] \cdot \varepsilon_0\) by \(P5\) so that

\[ [T] := 2a \varepsilon_0 - a \varepsilon_1 - b \varepsilon_2 - c \varepsilon_3 \quad \text{and} \quad [T'] := 2a' \varepsilon_0 - a' \varepsilon_1 - b' \varepsilon_2 - c' \varepsilon_3.\]

Let \(\tilde{T}\) be the strict transform of \(\gamma^*_0(T) \subset \mathbb{P}^2\) to \(Y\) along the blowup map \(Y \to \mathbb{P}^2\). By construction, \(\varepsilon_1 - \varepsilon_2\) and \(\varepsilon_1 - \varepsilon_3\) are divisor classes of the exceptional curves that are contracted to points by the blowup map \(Y \to \mathbb{P}^2\). Since these curves are not components of \(\tilde{T}\), we require that \([T] \cdot (\varepsilon_1 - \varepsilon_2) \geq 0\) and \([T] \cdot (\varepsilon_1 - \varepsilon_3) \geq 0\), which implies that \(a \geq b \geq 0\) and \(a \geq c \geq 0\) so that \(P1\) holds.

In order to verify \(P6\), let us consider the coupler decomposition \((T_1, \ldots, T_n)\) of \(T\) and let us denote the corresponding irreducible components of the strict transform \(\tilde{T}\) to \(Y\) by \((\tilde{T}_1, \ldots, \tilde{T}_n)\). We set \(\alpha_i := [T_i]\) for all \(1 \leq i \leq n\) so that \(\alpha_1 + \ldots + \alpha_n = [T] = \frac{1}{m} \cdot [G]\).

If \(\alpha_i = (\alpha_{i0}, \alpha_{i1}, \alpha_{i2})\), then \(\alpha_{i0} \geq \alpha_{i1} \geq 0\) and \(\alpha_{i0} \geq \alpha_{i2} \geq 0\) by the same argument used for proving \(P1\). Using the same argument for proving \(P5\) we find that

\[\deg T_i = 2\alpha_{i0} = \alpha_i \cdot (1, 0, 0).\]

It follows from the genus formula (see [13, Proposition V.1.5 and Examples V.3.9.2 and V.3.9.3]) that:

\[g(T_i) = g(\tilde{T}_i) = \frac{1}{2}([T_i]^2 + [T_i] \cdot \kappa) + 1 - \sum_{p \in \text{sing} \tilde{T}_i} \delta_p(\tilde{T}_i),\]

where the delta invariant \(\delta_p(\tilde{T}_i)\) of a singular point \(p \in \tilde{T}_i\) is at least one so that

\[|\text{sing} T_i| \leq \sum_{p \in \text{sing} \tilde{T}_i} \delta_p(\tilde{T}_i).\]

It is now a straightforward to verify that \(P6\) is satisfied.

If \((G, G')\) is a calligraphic split, then \(c(G \cup G') = m \cdot m' \cdot |T \cap T'|\) by Lemma 16. It follows from Lemma 15 that the curves \(T\) and \(T'\) intersect transversally and thus
we know from [13, Theorem V.1.1] that
\[ |T \cap T'| = |T| \cdot |T'| = 4aa' - 2aa' - 2bb' - 2cc' = \frac{1}{m \cdot m'} \cdot |G| \cdot |G'|. \]
This concludes the proof for P4 and as a consequence Axiom A2 holds as well.

We established that all axioms and properties for pseudo classes are true under the assumption that calligraphs are centric. It follows from Algorithm 1 and the discussion in Remark 4, that if the class \([G]\) of the calligraph \(G\) exist, then by A1, A2 and P3 it can be uniquely recovered from the number of realizations \(c(G \cup L)\), \(c(G \cup R)\) and \(c(G \cup C_v)\). Therefore, the class for calligraphs is unique and we conclude the proof of Proposition 12.

The following lemma is only needed in §5 and its 8 conditions correspond to the 8 conditions of Proposition 30. However, now is the right time to understand its statement.

**Lemma 18.** If \(\mathcal{H} \subset \mathbb{C}[z_0, z_1, z_2]\) is a subset of homogeneous polynomials of the same degree with real coefficients, then its associated series \(\Gamma_{\mathcal{H}}\) is centric if the following 8 conditions hold:

1. We have \(V(f) \cap V(z_0) = \{(0 : 1 : i), (0 : 1 : -i)\}\) for a general \(f \in \mathcal{H}\).
2. The set \(\gamma_0(\mathcal{H})\) has no base points.
3. The set \(\gamma_1(\mathcal{H})\) has an \(m\)-fold base point at \(p := (0, i)\) for some \(m > 0\).
4. The set \(\gamma_2(\mathcal{H})\) does not have \((0, 0)\) as a base point.
5. If \(\alpha_m^p \circ \gamma_1(\mathcal{H})\) has an \(n\)-fold base point \(q\) for some \(n > 0\), then \(q \in \{(0, 0), (0, -i)\}\).
6. The set \(\beta_m^p \circ \gamma_1(\mathcal{H})\) does not have \((0, 0)\) as a base point.
7. If \(q\) is an \(n\)-fold base point, then \(\alpha_n^q \circ \alpha_m^p \circ \gamma_1(\mathcal{H})\) has no base points.
8. If \(q\) is an \(n\)-fold base point, then \((0, 0)\) is not a base point of \(\beta_n^q \circ \alpha_m^p \circ \gamma_1(\mathcal{H})\).

**Proof.** Since \(\mathcal{H}\) consist of polynomials with real coefficients, the multiplicities at conjugate base points are equal, so that we only need to check \(p := (0, i)\). The remaining statements are a straightforward consequence of the definitions and left to the reader. It may be instructive to verify that if \(\mathcal{H}\) is as defined as in Example 8, then each of the eight conditions hold. 

We now proceed with showing that the hypothesis of Proposition 12 holds, namely that indeed all calligraphs are centric.
5 Sufficient conditions for centricity of calligraphs

In Lemma 18 of the previous section, we gave eight sufficient conditions for a calligraph to be centric. We conclude this section with equivalent conditions by instead performing operators on the quadratic polynomials that are associated to edges. Since we know the quadratic polynomials, the resulting conditions can be shown to be valid for every calligraph.

The methods of this and the remaining sections should be accessible to the non-expert, but are rather technical due to coordinate dependence. For this reason let us start with giving an informal explanation. If $G$ is a calligraph with associated series $\Gamma_H$, then in Lemma 18 we gave sufficient conditions for $G$ to be centric in terms of $H$. In particular, it is required that the base points of $\alpha_m \circ \gamma_1(H)$ are either $(0, 0)$ or $(0, -i)$ for some $m > 0$ with $p = (0, i)$. However, $H$ is not directly accessible from the combinatorial data of $G$ and thus will not lead to a proof which states that all calligraphs are centric. Instead, we assign to each edge $e = \{i, j\}$ of the calligraph $G$ a quadratic polynomial $(x_i - x_j)^2 + (y_i - y_j)^2 - \ell_e^2$, where $(x_i, y_i)$ is the coordinate of a realization of vertex $i$ and $\ell_e$ is the edge length of $e$. The polynomial representing $H$ is obtained by first homogenizing and then eliminating from the set of quadratic polynomials the variables $x_i$ and $y_i$ for all vertices $i > 0$, except vertex 0. Geometrically, this means that the coupler curves of $G$ are linear projections of curves defined by the zero set of quadratic polynomials. We introduce operators such as $H$, $M$ and $T$, that take as input a finite set of polynomials and output the set of polynomials that are the result of substitutions and removing certain factors. We apply compositions of these maps to the set of quadratic polynomials and show in particular that the above condition for $\alpha_m \circ \gamma_1(H)$ is equivalent to the condition $B \circ G \circ T \circ M \circ H \circ \mu(E) \subseteq \{(0, 0), (0, -i)\}$, where $\mu(E)$ is the set of quadratic polynomials assigned to edges $E$ and the expression $B \circ G$ informally means “the set of base points after elimination”. This will be the fifth of the eight conditions at Proposition 30. Let us now proceed with formally defining the notation.

Suppose that $G$ is a calligraph with $V := v(G) \setminus \{1, 2\}$ and $E := e(G) \setminus \{1, 2\}$. We define $R := \mathbb{C}[x_i, y_i, \ell_e : i \in V, e \in E]$ and $S := \mathbb{C}[x_0, y_0, \ell_e : e \in E]$ to be polynomial rings. We denote the set of variables of these rings by

$$\Upsilon_R := \{x_i, y_i, \ell_e : i \in V, e \in E\} \quad \text{and} \quad \Upsilon_S := \{x_0, y_0, \ell_e : e \in E\}.$$ 

The map $\mu : E \to R$ is defined as

$$\mu(e) := (x_i - x_j)^2 + (y_i - y_j)^2 - \ell_e^2,$$

where $e = \{i, j\}$, $(x_1, y_1) := (0, 0)$ and $(x_2, y_2) := (1, 0)$.

Let $f \in R$ be a polynomial. We denote by $f \downarrow \{a_0 \to b_0, \ldots\}$ the result of sub-
stuting the variable $a_i \in \mathcal{T}_R$ in $f$ with the rational function $b_i \in \mathbb{C}(\mathcal{T}_R \cup \{z_0\})$ for all $i$. The variable $z_0$ will serve a homogenization variable for polynomials that belong to $R$.

**Definition 19.** We define the following operators $R \to R$, where $f \in R$, $r \in \mathcal{T}_R$, $c \in \mathbb{C}$, $\deg f$ is the degree of $f$ when considered as a polynomial in $x_0$, $y_0$, and $\delta(r, f)$ is the maximal integer such that $r^{\delta(r, f)}$ is a monomial factor of $f$:

$$\hat{H}_r(f) := \left(z_{\deg f}^{\deg f} \cdot f \downarrow \{x_0 \rightarrow \frac{x_0}{z_0^m}, y_0 \rightarrow \frac{y_0}{z_0^n}\}\right) \downarrow \{r \rightarrow 1, z_0 \rightarrow r\},$$

$$\hat{F}_r(f) := r^{-\delta(r, f)} \cdot f,$$

$$\hat{M}_c(f) := f \downarrow \{y_0 \rightarrow x_0y_0 + c\},$$

$$\hat{N}_c(f) := f \downarrow \{y_0 \rightarrow y_0 + c, x_0 \rightarrow x_0y_0\}.$$

We shall denote $\hat{H}_{x_0}$, $\hat{M}$, $\hat{N}$ by $\hat{H}$, $\hat{M}$, $\hat{N}$, respectively.

The polynomial $\hat{H}_r(f)$ is obtained by first homogenizing a polynomial $f$ and then dehomogenizing with respect to a variable determined by $r$. This was used in §4 to determine base points at infinity for algebraic series associated to calligraphs. In order to determine the infinitely near base points, we apply the compositions $\hat{F}_r \circ \hat{M}_C$ and $\hat{F}_r \circ \hat{N}_C$, where $\hat{F}_r$ removes a monomial factor (see Remark 13 for the algebro geometric interpretation of these factors). We make this more explicit in the following example.

**Example 20.** Suppose that $\mathcal{G}$ is defined by the calligraph $\mathcal{R}$ in Figure 3 with associated series $\Gamma_\mathcal{G}$. Let $p := (0, i)$ and $e := \{0, 2\}$. We identify $x$ and $y$ in §4 with $x_0$ and $y_0$, respectively. Recall from Example 11, together with Examples 7 and 8, that

$$\gamma_i(\delta) = \{f_i \downarrow \{\ell_e \rightarrow \lambda\} : \lambda \in \mathbb{C}\},$$

$$\alpha_p^1 \circ \gamma_i(\delta) = \{g \downarrow \{\ell_e \rightarrow \lambda\} : \lambda \in \mathbb{C}\}, \quad \beta_p^1 \circ \gamma_i(\delta) = \{h \downarrow \{\ell_e \rightarrow \lambda\} : \lambda \in \mathbb{C}\},$$

where

$$f_0 := (x_0 - 1)^2 + y_0^2 - \ell_e^2, \quad f_1 := (1 - x_0)^2 + y_0^2 - \ell_e^2 x_0^2,$$

$$f_2 := (x_0 - y_0)^2 + 1 - \ell_e^2 y_0^2, \quad g := -2 + x_0 + 2i y_0 + x_0 y_0^2 - \ell_e^2 x_0 \quad \text{and} \quad h := 2i - 2x_0 + y_0 + x_0 y_0^2 - \ell_e^2 x_0 y_0.$$

We now verify that

$$\mu(e) = f_0, \quad \hat{H}(f_0) = f_1, \quad \hat{H}_{y_0}(f_0) = f_2, \quad \hat{F}_{x_0} \circ \hat{M} \circ \hat{H}(f_0) = g, \quad \hat{F}_{y_0} \circ \hat{N} \circ \hat{H}(f_0) = h.$$

In particular, $\hat{H}(f_0)$ is equal to $f_1 = f_{\text{hom}}(x_0, 1, y_0)$, where

$$f_{\text{hom}} \in \mathbb{C}[z_0, z_1, z_2, \ell_e : e \in \mathbb{E}]$$

is the homogenization of $f_0$ with respect to the variables $x_0$ and $y_0$. Indeed, this corresponds to $\gamma_i(\delta)$. Similarly, $\hat{M}(f_1) = \hat{M} \circ \hat{H}(f_0) = f_1(x_0, x_0 y_0 + i)$. Since
$p$ is a base point of $\gamma_1(\mathcal{S})$ with multiplicity $m = 1$, we find that $f_1(x_0, x_0y_0 + i)$ has $x_0^m$ as a factor. We remove this monomial factor using the operator $\hat{F}_{x_0}$ so that $\hat{F}_{x_0} \circ \hat{M} \circ \hat{H}(f_0)$ corresponds to $\alpha_1^p \circ \gamma_1(\mathcal{S})$. Notice that we can choose for $m$ in Definition 6 the maximal integer such that the remainders of the polynomial quotients vanish identically. The reason is that the polynomial $f_0$ has no $(x_0)$-factor or $(y_0)$-factor, as this would mean that two general coupler curves of $\mathcal{G}$ have a common component.

We recover the base points of coupler curves by eliminating certain variables from an ideal generated by quadratic polynomials. For that purpose, we consider the lexicographic monomial ordering on $R$ that is induced by the following ordering on the variables for all $i, j \in \mathbb{N}$ and $\{a, b\}, \{a', b'\} \in \mathbb{E}$ such that $i < j$, $a > b$, $a' > b'$ and either $(a < a')$ or $(a = a' \text{ and } b < b')$:

$$\ell_{(a,b)} < \ell_{(a',b')}, \quad \ell_{(a,b)} < x_i, \quad x_i < x_j, \quad x_i < y_i, \quad y_i < x_j \text{ and } y_i < y_j.$$ If $A = \{z_0, \ldots, z_n\}$ and $B = \{z_0, \ldots, z_m\}$ are ordered sets of variables such that $B \subset A$ with an ordering $z_0 < \ldots < z_n$, then we denote by $A \upharpoonright B$ the linear projection $\mathbb{C}^{|A|} \to \mathbb{C}^{|B|}$ that sends $(z_0, \ldots, z_n)$ to $(z_0, \ldots, z_m)$. We set $\pi := \Upsilon_R \upharpoonright \Upsilon_S$.

**Example 21.** If $\mathcal{G}$ is equal to the calligraph $\mathcal{C}_3$, then

$$\mathbb{V} = \{0, 3\}, \quad \mathbb{E} = \{\{3, 0\}, \{3, 1\}, \{3, 2\}\}, \quad R = \mathbb{C}[x_0, y_0, x_3, y_3, \ell_{30}, \ell_{31}, \ell_{32}]$$

and $\ell_{30} < \ell_{31} < \ell_{32} < x_0 < y_0 < x_3 < y_3$. In this case $\pi = \Upsilon_R \upharpoonright \Upsilon_S$ is the map $\mathbb{C}^7 \to \mathbb{C}^5$ that sends $(\ell_{30}, \ell_{31}, \ell_{32}, x_0, y_0, x_3, y_3)$ to $(\ell_{30}, \ell_{31}, \ell_{32}, x_0, y_0)$. \hfill \triangledown

Let $\varphi(R)$ denote the set of all finite subsets of $R$ and let $\langle P \rangle$ denote the ideal in $R$ generated by $P \in \varphi(R)$. The map $\hat{G}: \varphi(R) \to \varphi(R)$ assigns to $P$ the reduced Gröbner basis $\hat{G}(P)$ for the elimination ideal $\langle P \rangle \cap S$ with respect to the above lexicographic ordering on $R$. The map $V: \varphi(R) \to \text{powerset} (\mathbb{C}^{\Upsilon_R})$ assigns to $P$ its zero set $V(P)$. We shall denote $V(\{f_1, \ldots, f_r\})$ by $V(f_1, \ldots, f_r)$.

**Notation 22.** By abuse of notation we consider the operator $\hat{H}: R \to R$ also as the map $\hat{H}: \varphi(R) \to \varphi(R)$ so that $\hat{H}(P) = \{\hat{H}(p) : p \in P\}$; we use the same notation for the other operators in Definition 19 and thus the composition of $\hat{G}$ with an operator is defined. Similarly, we consider $\mu: \mathbb{E} \to R$, as the map $\mu: \varphi(\mathbb{E}) \to \varphi(R)$ so that for example the composition $\hat{H} \circ \mu$ is defined. We shall denote by $\pi \circ V(P)$ the Zariski closure of $\{\pi(p) : p \in V(P)\}$, where we recall that $\pi$ denotes the linear projection $\Upsilon_R \upharpoonright \Upsilon_S$. \hfill \triangledown

We now introduce a map, which recovers base points from $\hat{G}(P)$. For all polynomials $f \in S$ there exist polynomials $c_\alpha \in \mathbb{C}[x_0, y_0]$ such that

$$f = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^\mathbb{E}} c_\alpha(x_0, y_0) \prod_{e \in \mathbb{E}} \ell_{e}^{a_e}.$$
Notice that $\alpha: \mathbb{E} \to \mathbb{Z}_{\geq 0}$ is a map and that $\alpha_e = \alpha(e)$ is its evaluation at $e$. We call $p$ a base point of $f$ if $c_\alpha(p) = 0$ for all $\alpha \in \mathbb{Z}_{\geq 0}$. The map $B_t: \mathcal{P}(R) \to \text{powerset}(\mathbb{C}^2)$ with $t \in \Upsilon_S \cup \{0\}$ is defined as

$$B_t(P) := \{ p \in \mathbb{C}^2 \cap V(t) : p \text{ is a base point of some } f \in P \cap S \}.$$  

We denote $B_{x_0}$ by $B$. Notice that in the definition of $B_0$ we have that $\mathbb{C}^2 \cap V(0) = \mathbb{C}^2$.

**Remark 23.** If $p \in \mathbb{C}^2$ is a base point of a polynomial $f \in S$, then it is a base point as defined in §4 for the following subset:

$$\{ f \downarrow \{ \ell_e \to \lambda_e : e \in \mathbb{E} \} : \lambda \in \mathbb{C}^E \} \subset \mathbb{C}[x_0, y_0],$$

where $\mathbb{C}^E$ defines the set of maps with domain $\mathbb{E}$ and codomain $\mathbb{C}$. Geometrically this means that the base point $p$ is for all edge length assignments $\lambda \in \mathbb{C}^E$ contained in the coupler curve defined by $V(f \downarrow \{ \ell_e \to \lambda_e : e \in \mathbb{E} \})$.  

**Example 24.** Suppose that $\mathcal{G}$ is defined by the calligraph $\mathcal{C}_3$. We show that $\mathcal{G}$ is centric. First notice that

$$\mu(\mathbb{E}) = \{(x_0 - x_3)^2 + (y_0 - y_3)^2 - \ell_{30}^2, \quad (x_3 - 1)^2 + y_3^2 - \ell_{32}^2, \quad \ell_{30}^2 - \ell_{31}^2 \}.$$  

We have $\hat{G} \circ \mu(\mathbb{E}) := \{ f \}$, where we consider $f \in S$ as a polynomial in $x_0$ and $y_0$ with coefficients depending on the $\{ \ell_e \}_{e \in \mathbb{E}}$. Thus, for each choice of edge length assignment $\lambda \in \mathbb{C}^E$, the zero set of $f \downarrow \{ \ell_e \to \lambda_e : e \in \mathbb{E} \}$ defines a coupler curve of $\mathcal{G}$. We can factor the polynomial $f$ as follows:

$$f = \left((x_0 - c_x)^2 + (y_0 - c_y)^2 - \ell_{30}^2\right) \cdot \left((x_0 - c_x)^2 + (y_0 + c_y)^2 - \ell_{30}^2\right),$$

where $c_y := (\ell_{31}^2 - \ell_{32}^2)^{1/2}$ and $c_x := \frac{1}{2}(\ell_{31}^2 - \ell_{32}^2 + 1)$. We verify using the Pythagorean theorem and the law of cosines that indeed $(c_x, \pm c_y)$ are the centers of the two circles as depicted in Figure 1. Similarly to Example 20, the set $\gamma_1(\mathfrak{F})$ corresponds to $\hat{H} \circ \hat{G} \circ \mu(\mathbb{E}) = \{ f_1 \}$, where

$$f_1 := \left((1 - c_x x_0)^2 + (y_0 - c_y y_0)^2 - \ell_{30}^2 x_0^2\right) \cdot \left((1 - c_x x_0)^2 + (y_0 + c_y y_0)^2 - \ell_{30}^2 y_0^2\right).$$

The 1-fold base points of $\gamma_1(\mathfrak{F})$ are $p := (0, i)$ and $\overline{p} = (0, -i)$ and indeed we find that $B \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E}) = B(\{ f_1 \}) = \{ p, \overline{p} \}$. It is straightforward to verify also the relations in Table 1 for this example. Therefore, it follows from Lemma 18 that $\mathcal{G}$ is

**Table 1:** See Example 24. We denote by $\text{bp}(\mathfrak{F})$ the base points of the subset $\mathfrak{F} \subset \mathbb{C}[x, y]$.

| $\text{bp}(\gamma_0(\mathfrak{F}))$ | $B_0 \circ \hat{G} \circ \mu(\mathbb{E})$ | $\emptyset$ |
| $\text{bp}(\gamma_1(\mathfrak{F}))$ | $B \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})$ | $\{(0, i), (0, -i)\}$ |
| $\text{bp}(\gamma_2(\mathfrak{F}))$ | $B \circ \hat{H}_{y_0} \circ \hat{G} \circ \mu(\mathbb{E})$ | $\emptyset$ |
| $\text{bp}(\alpha^1_p \circ \gamma_1(\mathfrak{F}))$ | $B \circ \hat{F}_{x_0} \circ \hat{M} \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})$ | $\emptyset$ |
| $\text{bp}(\beta^2_p \circ \gamma_1(\mathfrak{F}))$ | $B \circ \hat{F}_{y_0} \circ \hat{N} \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})$ | $\emptyset$ |
Lemma 27. In Lemma 18 and that it follows from the fact that \((\hat{H} \circ \mu(B)) \downarrow \{x_0 \to 0\} \supseteq \{1+y_0^2\}\). Hence, the first condition can be shown without computing a Gröbner basis. \(\Box\)

The approach in Example 24 does not directly lead to a general proof that all calligraphs are centric, as for most calligraphs it is not feasible to compute the Gröbner basis \(\hat{G} \circ \mu(B)\). To overcome this obstacle we shall modify the operators so that instead of for example \(B \circ \hat{F}_{x_0} \circ \hat{M} \circ \hat{H} \circ \hat{G} \circ \mu(B)\) we consider \(B \circ \hat{G}(P)\), where \(P\) is obtained from \(\hat{F}_{x_0} \circ \hat{M} \circ \hat{H} \circ \mu(B)\) using substitutions and removing factors. In other words, we want the operators \(\hat{H}, \hat{M}, \hat{N}\) to commute with the map \(\hat{G}\) up to removing some factors. This is the content of Lemma 27 below, but first we account for these factors in the following definition.

**Definition 25.** We follow Notation 22 and consider the following compositions of operators in Definition 19 as either \(R \to R\) or \(\varphi(R) \to \varphi(R)\):

\[
\begin{align*}
G &:= \hat{F}_{y_0} \circ \hat{F}_{x_0} \circ \hat{G}, & H_c &:= \hat{F}_{y_0} \circ \hat{F}_{x_0} \circ \hat{H}_r, \\
M_c &:= \hat{F}_{y_0} \circ \hat{F}_{x_0} \circ \hat{M}_c, & N_c(\cdot) &:= (\hat{F}_{y_0} \circ \hat{F}_{x_0} \circ \hat{N}_c(\cdot)) \downarrow \{x_0 \to y_0, y_0 \to x_0\},
\end{align*}
\]

where \(r \in \{x_0, y_0\}\) and \(c \in \mathbb{C}\). We shall denote \(H_{x_0}, M_i, N_i\) by \(H, M, N\), respectively. In addition, we define the operator \(T\): \(\varphi(R) \to \varphi(R)\) as

\[
T(P) := \{f \downarrow \{x_0 \to x_0 \cdot s, \; \ell_e \to \ell_e \cdot s : e \in \mathbb{E}\} : f \in P\},
\]

where \(s := x_0 \prod_{x \in U}(y_0 - u)\) and \(U := \{u : (0, u) \in B \circ G(P)\}\). \(\Box\)

Notice that in the operator \(N_c\) we interchange \(x_0\) and \(y_0\) so that always the \(x_0\) coordinate of infinitely near base points vanishes identically. The motivations for the new operator \(T\) are clarified in Lemma 28 and Example 29. Before stating Lemma 27 we first assert in Lemma 26 that the elimination of variables in an ideal corresponds geometrically to a linear projection.

**Lemma 26.** Suppose that \(A\) is a set of variables and \(B \subset A\). If \(\kappa = A \setminus B\) and \(Z \subset \mathbb{C}^{|A|}\) is a variety with ideal \(\langle P \rangle \subset \mathbb{C}[A]\), then the ideal of \(\kappa(Z)\) is \(\langle P \rangle \cap \mathbb{C}[B]\). In particular, for all \(P \in \varphi(R)\) we have

\[V \circ \hat{G}(P) = \pi \circ V(P)\]

*Proof.* See [7, Theorem 3 in Section 3.2]. \(\Box\)

**Lemma 27.** If \(P, Q \in \varphi(R)\) such that \(|G(P)| = |G(Q)| = 1\) and \(V(P) = V(Q)\), then for all \(c \in \mathbb{C}, r \in \{x_0, y_0\}\) and \(s \in S\) we have

\[
\begin{align*}
V \circ M_c \circ G(P) &= V \circ G \circ M_c(P), & V \circ H_r \circ G(P) &= V \circ G \circ H_r(P), \\
V \circ N_c \circ G(P) &= V \circ G \circ N_c(P), & V \circ T \circ G(P) &= V \circ G \circ T(P),
\end{align*}
\]

and \(B \circ G(P) = B \circ G(Q)\).
Proof. Recall that \( \pi = \Upsilon_R \circ \Upsilon_S \) and let \( m = |\Upsilon_R| - |\Upsilon_S| \) and \( n = |\Upsilon_S| \) such that \( \pi: \mathbb{C}^{[\Upsilon_R]} \to \mathbb{C}^{[\Upsilon_S]} \) can be restated as the projection \( \pi: \mathbb{C}^m \times \mathbb{C}^n \to \mathbb{C}^n \) to the second component. Let \( \eta: \mathbb{C}^m \times \mathbb{C}^n \longrightarrow \mathbb{C}^m \times \mathbb{C}^n \) be a map of the form \( \text{id} \times \nu \) for some rational map \( \nu: \mathbb{C}^n \longrightarrow \mathbb{C}^n \), where \( \text{id}: \mathbb{C}^m \to \mathbb{C}^m \) is the identity map so that the following diagram is commutative.

\[
\begin{array}{c}
\mathbb{C}^m \times \mathbb{C}^n \xrightarrow{\eta} \mathbb{C}^m \times \mathbb{C}^n \\
\downarrow \pi \quad \downarrow \pi \\
\mathbb{C}^n \xrightarrow{\nu} \mathbb{C}^n
\end{array}
\]

Same as \( \pi \circ V(\cdot) \) in Notation 22, we let \( \eta \circ V(\cdot) \) and \( \nu \circ V(\cdot) \) denote the Zariski closures of \( \{\eta(p) : p \in V(\cdot)\} \) and \( \{\nu(p) : p \in V(\cdot)\} \), respectively. Since \( \eta \) sends a fiber of \( \pi \) to a fiber we find that

\[ \nu \circ \pi \circ V(P) = \pi \circ \nu \circ V(Q). \] \hspace{1cm} (1)

We know from Lemma 26 that

\[ V \circ \hat{G} = \pi \circ V. \] \hspace{1cm} (2)

Suppose that \( \hat{O} \in \{\hat{M}_c, \hat{N}_c, T\} \) and notice that if \( \eta \) is birational with a polynomial inverse, then \( \eta \circ V(A) = V(\{f \circ \eta^{-1} : f \in A\}) \) for all \( A \in \wp(R) \). Thus, there exist birational maps \( \nu_\circ \) and \( \eta_\circ \) such that \( \eta_\circ = \text{id} \times \nu_\circ \) and

\[ V \circ \hat{O} = \eta_\circ \circ V \quad \text{and} \quad V \circ \hat{O} \circ \hat{G} = \nu_\circ \circ V \circ \hat{G}. \] \hspace{1cm} (3)

For example, if \( \hat{O} = \hat{M}_c \), then \( \nu_\circ^{-1} \) and \( \nu_\circ \) maps \((x_0, y_0, \ldots)\) to \((x_0, y_0x_0 + c, \ldots)\) and \((x_0, (y_0 - c)x_0^{-1}, \ldots)\), respectively. We apply (1), (2) and (3) for all \( \hat{O} \in \{\hat{M}_c, \hat{N}_c, T\} \) and obtain the following sequence of equalities:

\[ V \circ \hat{O} \circ \hat{G}(P) = \nu_\circ \circ V \circ \hat{G}(P) = \nu_\circ \circ \pi \circ V(P) \]
\[ = \pi \circ \nu_\circ \circ V(Q) = \pi \circ V \circ \hat{O}(Q) = V \circ \hat{G} \circ \hat{O}(Q). \]

We set \( O := F_{y_0} \circ \hat{x}_0 \circ \hat{O} \) and \( D := V(x_0y_0) \), and we observe that

\[ V \circ G(\cdot) \setminus D = \pi \circ V(\cdot) \setminus D, \]
\[ V \circ O(\cdot) \setminus D = \eta_\circ \circ V(\cdot) \setminus D. \] \hspace{1cm} (4)

We apply (1) and the identities at (4) and deduce that

\[ V \circ O \circ G(P) \setminus D = \nu_\circ \circ V \circ G(P) \setminus D = \nu_\circ \circ \pi \circ V(P) \setminus D \]
\[ = \pi \circ \eta_\circ \circ V(Q) \setminus D = \pi \circ V \circ O(Q) \setminus D = V \circ G \circ O(Q) \setminus D. \]

But this implies that for all \( O \in \{M_c, N_c\} \) we have

\[ V \circ O \circ G(P) = V \circ G \circ O(Q). \]

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Let us now consider the case where \( \hat{O} = \hat{H}_r \) and \( O = H_r \). Recall the embeddings \( \gamma_i^*: \mathbb{C}^2 \to \mathbb{P}^2 \) for \( i \in \{0,1,2\} \) as defined in §4 and let \( \eta_r := \text{id} \times \nu_r \) with \( \nu_r := \left( (\gamma_i^*)^{-1} \circ \gamma_0^{-1}\right) \) and \( (r,i) \in \{(x_0,1),(y_0,2)\} \). For example, if \( r = x_0 \), then both \( \nu_r \) and \( \nu_r^{-1} \) send \((x_0,y_0,\ldots)\) to \((x_0^{-1},y_0x_0^{-1},\ldots)\). If \( r \neq 0 \), then \( \eta_r \circ V(A) = V(\{r^\deg f \cdot (f \circ \eta_r^{-1}) : f \in A\}) \) for all \( A \in \wp(R) \). This implies that \( V \circ H_r(\cdot) \setminus D = \eta_r \circ V(\cdot) \setminus D \) and thus
\[
V \circ H_r(\cdot) \setminus D = \eta_r \circ V(\cdot) \setminus D.
\]
Applying the same arguments as before we confirm the following assertion
\[
V \circ H_r \circ G(P) = V \circ G \circ H_r(Q).
\]
Since \( G(P) = \{f\} \) and a base point of \( f \in S \) is expressed in Remark 23 in terms of zero sets, we find that \( V(P) = V(Q) \) implies that \( B \circ G(P) = B \circ G(Q) \). We verified all assertions and concluded the proof. \( \square \)

The following lemma shows that the map \( T \) modifies a set of polynomials such that the base point candidates associated to the input remain candidates for the output. The motivation for this map is that only after the transformation we can characterize the base point candidates associated to the input remain candidates for the output. We clarify this in more detail in Example 29.

**Lemma 28.** If \( |G(P)| = 1 \), then
\[
B \circ G(P) \subseteq B \circ G \circ T(P)
\]
and \( \hat{G} \circ T(P) = \{g \cdot s\} \) for some \( g \in S \), where \( s \) is as defined in Definition 25.

**Proof.** Let \( G(P) = \{f\} \), where
\[
f = \sum_{a \in \mathbb{Z}_{\geq 0}^k} c_a(x_0,y_0) \prod_{e \in \mathbb{R}} \ell_e^{a_e}.
\]
We know from Lemma 27 that \( \langle T \circ G(P) \rangle = \langle G \circ T(P) \rangle \) and thus there exists a monomial \( \lambda := c \cdot x_0^a \cdot y_0^b \) with \( c \in \mathbb{C} \setminus \{0\} \) and \( a, b \in \mathbb{Z}_{\geq 0} \) such that
\[
\hat{G} \circ T(P) = \left\{ \lambda \sum_{a \in \mathbb{Z}_{\geq 0}^k} c_a(x_0 \cdot s,y_0) \prod_{e \in \mathbb{E}} (\ell_e \cdot s)^{a_e} \right\}.
\]
We notice that \( c_0(x_0 \cdot s,y_0) = h \cdot s \) for some polynomial \( h \in S \), since \( c_0(0,u) = 0 \) and \( s \downarrow \{y_0 \to u\} = 0 \) for all \( (0,u) \in B(\{f\}) \). From this we deduce that \( \hat{G} \circ T(P) = \{g \cdot s\} \) for some \( g \in S \). The assertion \( B \circ G(P) \subseteq B \circ G \circ T(P) \) is now a direct consequence of the definitions. \( \square \)

The purpose of the following example is to clarify the definitions and lemmas, and to prepare the reader for the proof strategy in the remaining text. We shall use Table 2 which is obtained via straightforward calculations.
Table 2: We list polynomials that are obtained after applying certain compositions of operators to $\mu(e)$, where $e := \{i,j\}$ is an edge in $\mathbb{E}$. Let $c \in \mathbb{C}$ and suppose that $s = x_0 \Pi_{i \in \mathbb{U}} (y_0 - y_i)$ as in Definition 25. For example, we read from this table that if $1 \in e$, then $T \circ M \circ H \circ \mu(e) = h_2$, where $h_2 = x_1^2 + y_i^2 - \ell_e^2 s^2$.

| $0 \in e$ | $0,1,2 \notin e$ | $1 \in e$ | $2 \in e$ |
|-----------|-----------------|----------|----------|
| $\mu(e)$  | $f_1$           | $g_1$    | $h_1$    | $k_1$    |
| $H \circ \mu(e)$ | $f_2$ | $g_1$ | $h_1$ | $k_1$ |
| $H_{y_0} \circ \mu(e)$ | $f_3$ | $g_1$ | $h_1$ | $k_1$ |
| $T \circ M \circ H \circ \mu(e)$ | $f_4$ | $g_2$ | $h_2$ | $k_2$ |
| $T \circ N \circ H \circ \mu(e)$ | $f_5$ | $g_2$ | $h_2$ | $k_2$ |
| $T \circ M_c \circ M \circ H \circ \mu(e)$ | $f_6$ | $g_2$ | $h_2$ | $k_2$ |
| $T \circ N_c \circ M \circ H \circ \mu(e)$ | $f_7$ | $g_2$ | $h_2$ | $k_2$ |

The polynomials in the above table are defined as follows:

$$
\begin{align*}
  f_1 & := (x_0 - x_i)^2 + (y_0 - y_i)^2 - \ell_e^2, \\
  f_2 & := (1 - x_i x_0)^2 + (y_0 - y_i x_0)^2 - \ell_e^2 x_0^2, \\
  f_3 & := (x_0 - x_i y_0)^2 + (1 - y_i y_0)^2 - \ell_e^2 y_0^2, \\
  f_4 & := (x_i - i y_0 + i y_i)(-2 + x_0 s(x_i + i y_0 - i y_i)) - x_0 \ell_e^2 s^3, \\
  f_5 & := (1 + i x_i y_0 - i y_i y_0)(2i + x_0 s(1 - i x_i y_0 - y_i y_0)) - y_0^2 x_0 \ell_e^2 s^3, \\
  f_6 & := 2i(c + i x_i - y_i) + x_0 s(c^2 + x_i^2 + 2i y_0 - 2 c y_i - y_i^2 + x_0 y_0 s(2c - 2 y_i + y_0 x_0 s)) - x_0 \ell_e^2 s^3, \\
  f_7 & := (c + i x_i + x_0 s - y_i)(2i + y_0 x_0 s(c - i x_i + x_0 s - y_i)) - y_0 x_0 \ell_e^2 s^3, \\
  g_1 & := (x_i - x_j)^2 + (y_i - y_j)^2 - \ell_e^2, \\
  g_2 & := (x_i - x_j)^2 + (y_i - y_j)^2 - \ell_e^2 s^2, \\
  h_1 & := x_i^2 + y_i^2 - \ell_e^2, \\
  h_2 & := x_i^2 + y_i^2 - \ell_e^2 s^2, \\
  k_1 & := (x_i - 1)^2 + y_i^2 - \ell_e^2, \\
  k_2 & := (x_i - 1)^2 + y_i^2 - \ell_e^2 s^2.
\end{align*}
$$

Example 29. Suppose that $G$ is equal to the calligraph $H$ in Figure 6. It follows from Table 2 that

$$
\begin{align*}
  \mu(\{0,3\}) &= (x_0 - x_3)^2 + (y_0 - y_3)^2 - \ell_{30}^2, \\
  \mu(\{0,4\}) &= (x_0 - x_4)^2 + (y_0 - y_4)^2 - \ell_{30}^2, \\
  \mu(\{3,4\}) &= (x_3 - x_4)^2 + (y_3 - y_4)^2 - \ell_{33}^2, \\
  \mu(\{3,1\}) &= x_3^2 + y_3^2 - \ell_{31}^2, \\
  \mu(\{4,2\}) &= (x_4 - 1)^2 + y_4^2 - \ell_{42}^2.
\end{align*}
$$

Suppose that $\Gamma_\mathcal{B}$ is the series associated to $G$ and that $(0,u)$ is a base point of $\alpha_p^m \circ \gamma_1(\mathcal{B})$, where $p = (0,i)$ is a base point of $\gamma_1(\mathcal{B})$ with multiplicity $m$. If $G$ is centric, then $u \in \{0,-i\}$ and thus

$$
Z := B \circ \hat{F} \circ \hat{M} \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E}) \subseteq \{(0,0), (0,-i)\}.
$$

Since two general curves in $\Gamma_\mathcal{B}$ do not contain a common component, we can replace $\hat{F} \circ \hat{M}, \hat{H}, \hat{G}$ by $M, H, G$ so that $B \circ M \circ H \circ G \circ \mu(\mathbb{E}) = Z$. Let $P := M \circ H \circ \mu(\mathbb{E})$,
where
\[ M \circ H \circ \mu(\{0, 3\}) = (x_3 - iy_0 + iy_3)(-2 + x_0(x_3 + iy_0 - iy_3)) - x_0\ell_{30}, \]
\[ M \circ H \circ \mu(\{0, 4\}) = (x_4 - iy_0 + iy_4)(-2 + x_0(x_4 + iy_0 - iy_4)) - x_0\ell_{40}, \]
\[ M \circ H \circ \mu(\{3, 4\}) = (x_3 - x_4)^2 + (y_3 - y_4)^2 - \ell^2_{43}. \]
\[ M \circ H \circ \mu(\{3, 1\}) = x_3^2 + y_3^2 - \ell^2_{31}, \]
\[ M \circ H \circ \mu(\{4, 2\}) = (x_4 - 1)^2 + y_4^2 - \ell^2_{42}. \]

By applying Lemma 27 two times we deduce that \( B \circ G(P) = Z. \) In this example we prepare the reader for our general strategy to show that

\[ B \circ G(P) \subseteq \{(0, 0), (0, -i)\}. \]

In order to motivate this strategy, we first investigate a more straightforward approach that leads to a problem. Let \( \kappa: \mathbb{C}^{11} \rightarrow \mathbb{C}^2 \) be defined as the linear projection \( \Upsilon_R \mid \{x_0, y_0\}. \) We set \( L := V(\{\ell_e - \lambda_e : e \in E\}) \), where \( \lambda \in \mathbb{C}^E \) is a general edge length assignment. We define \( C_\lambda \) as the Zariski closure of \( (V(P) \cap L) \setminus V(x_0) \).

Thus the section \( C_\lambda \subseteq \mathbb{C}^{11} \) of \( V(P) \) is a curve consisting of one or more irreducible components and its linear projection \( \kappa(C_\lambda) \subseteq \mathbb{C}^2 \) passes through the base points in \( B \circ G(P) \). These base points lie by definition on the line \( V(x_0) \subseteq \mathbb{C}^2 \), which is a component of the linear projection \( \kappa(V(P) \cap L) \). Thus \( V(P) \cap L \) contains aside \( C_\lambda \), additional components in the hyperplane \( V(x_0) \subseteq \mathbb{C}^{11} \) that project to the line \( V(x_0) \subseteq \mathbb{C}^2 \). We want to show that a base point has no preimage with respect to the restricted projection \( \kappa|_{C_\lambda} \) and thus it is hopeless to recover base points directly from the zero set \( V(P) \cap V(x_0) \subseteq \mathbb{C}^{11} \) with the ideal

\[ \langle P \cup \{x_0\} \rangle = \langle x_3 - iy_0 + iy_3, x_4 - iy_0 + iy_4, \]
\[ (x_3 - x_4)^2 + (y_3 - y_4)^2 - \ell^2_{43}, x_3^2 + y_3^2 - \ell^2_{31}, (x_4 - 1)^2 + y_4^2 - \ell^2_{42} \rangle. \]

Let \( U := \{u : (0, u) \in B \circ G(P)\} \) be the set of \( y_0 \)-coordinates of the base points. We define the map \( \rho: \mathbb{C}^4 \rightarrow \mathbb{C}^3 \) as

\[ (x_3, y_3, x_4, y_4) \mapsto \left( (x_3 - x_4)^2 + (y_3 - y_4)^2, x_3^2 + y_3^2, (x_4 - 1)^2 + y_4^2 \right). \]

Thus \( \rho \) is dominant and sends vertex coordinates to the squares of the corresponding edge lengths. Since \( \lambda \) was chosen general we have

\[ (\lambda^2_{33}, \lambda^2_{31}, \lambda^2_{42}) \notin \rho(V(x_3 - iu + iy_3, x_4 - iu + iy_4)), \]

for all \( u \in U \), and thus there does not exist a \( q \in C_\lambda \) such that \( \kappa(q) = (0, u) \). This means that the projection \( \kappa|_{C_\lambda} \) is not proper. For an example of an improper map, we may think of the projection of the hyperbola \( \{(x, y) \in \mathbb{C}^2 : xy = 1\} \) to the \( x \)-axis in which case the origin does not have a preimage (see [13, Example II.4.6.1]).

To avoid the above problem of base points not having preimages, we consider \( T(P) \) instead of \( P \). This modification does not exclude any candidates for base points since we know from Lemma 28 that \( B \circ G(P) \subseteq B \circ G \circ T(P) \). Let the birational
map $\eta: \mathbb{C}^{11} \longrightarrow \mathbb{C}^{11}$ be defined as

$$(x_0, y_0, x_3, y_3, x_4, y_4, \ell_{43}, \ell_{31}, \ell_{42}) \mapsto (x_0 s^{-1}, y_0, x_3, y_3, x_4, y_4, \ell_{43}s^{-1}, \ell_{31}s^{-1}, \ell_{42}s^{-1}).$$

Notice that $\eta$ sends $V(P)$ to $V \circ T(P)$, but is not defined at the zero set $V(s)$, where $s = x_0 \prod_{u \in U} (y_0 - u)$. Thus, $V \circ T(P) \cap L$ consists of the image $\eta(C)$ and additional components in $V(s)$. The preimages with respect to the linear projection $\kappa: \mathbb{C}^{11} \longrightarrow \mathbb{C}^2$ of base points in $B \circ G \circ T(P)$ are contained in the zero set $V \circ T(P) \cap V(x_0) \subset \mathbb{C}^{11}$ with the ideal:

$$\langle T(P) \cup \{x_0\} \rangle = \langle x_0, x_3 - iy_0 + iy_3, x_4 - iy_0 + iy_4, (x_3 - x_4)^2 + (y_3 - y_4)^2, x_3^2 + y_3^2, (x_4 - 1)^2 + y_4^2 \rangle.$$

The generators of this ideal factor into linear factors and thus there exist linear spaces $W_1, \ldots, W_8$ such that

$$V \circ T(P) \cap V(x_0) = W_1 \cup \cdots \cup W_8.$$

In the proof of Lemma 34 in §6 we show that for each base point $(0, u)$ in $B \circ T \circ G(P)$, there exists $1 \leq i \leq 8$ such that $\kappa(W_i) = (0, u)$. We deduce from Proposition 33 in §6 that $\kappa(W_i)$ equals either the line $V(x_0)$ or a point in $\{(0,0), (0, -i)\}$. Thus, we conclude that

$$B \circ G(P) \subseteq B \circ T \circ G(P) \subseteq \{(0,0), (0, -i)\}.$$

We remark that in Proposition 33 and Lemma 34 we consider the linear projection $\pi = \Upsilon_R \wr \Upsilon_S$ instead of $\kappa = \Upsilon_R \wr \{x_0, y_0\}$, but the translation to the setting of this example is straightforward.

**Proposition 30.** A calligraph $\mathcal{G}$ is centric if the following 8 conditions hold for all $c \in \{0, i\}$:

1. $(H \circ \mu(E)) \downarrow \{x_0 \to 0\} \supseteq \{1 + y_0^2\}$,
2. $B_0 \circ G \circ \mu(E) = \emptyset$,
3. $B \circ G \circ H \circ \mu(E) \subseteq \{(0, i), (0, -i)\}$,
4. $B \circ \mu(H_{y_0} \circ \mu(E)) \neq \{(0,0)\}$,
5. $B \circ \mu(G \circ T \circ M \circ H \circ \mu(E)) \subseteq \{(0,0), (0, -i)\}$,
6. $B \circ \mu(G \circ T \circ N \circ H \circ \mu(E)) \neq \{(0,0)\}$,
7. $B \circ \mu(G \circ T \circ M_c \circ M \circ H \circ \mu(E)) = \emptyset$,
8. $B \circ \mu(G \circ T \circ N_c \circ M \circ H \circ \mu(E)) \neq \{(0,0)\}$.

**Proof.** Let $\Gamma_\beta$ be the series associated to $\mathcal{G}$ as characterized by Lemma 14(c). It is straightforward to see that Condition 1 implies that the curves in $\Gamma_\beta$ meet the line at infinity only at the cyclic points (see Example 24). The first equality in each row of Table 3 is a direct consequence of the definitions (see Example 20). Thus, if the remaining set relations in Table 3 hold, then the assertions of this proposition follow from Lemma 18. Two general curves in $\Gamma_\beta$ do not have a common component, and
thus their defining polynomials do not contain a \((x_0)\)-factor or \((y_0)\)-factor. Hence, we can replace \(\hat{G}, \hat{H}, \hat{F}_{x_0} \circ \hat{M}, \hat{F}_{y_0} \circ \hat{N}\) by \(G, H, M, N\), respectively. Moreover, we can replace \(B_{y_0}\) with \(B\), since the operator \(N_c\) interchanges \(x_0\) and \(y_0\). The set relations in Table 3 are now a consequence of Lemmas 27 and 28, and thus we concluded the proof. 

\[\square\]

\begin{table}[h]
\centering
\caption{See the proof of Proposition 30. Let \(p := (0, i)\), \(q := (0, c)\) and \(m, n \in \mathbb{Z}_{\geq 0}\). Let \(bp(\mathcal{F})\) denote the base points of the subset \(\mathcal{F} \subset \mathbb{C}[x, y]\). The set relation \(A \propto B\) for \(A, B \subset \mathbb{C}^2\) indicates that \(A \subseteq \{(y, x) : (x, y) \in B\}\).}
\begin{tabular}{llll}
\hline
\(bp(\gamma_0(\mathcal{F}))\) &= \(B_0 \circ \hat{G} \circ \mu(\mathbb{E})\) &= \(B_0 \circ G \circ \mu(\mathbb{E})\) \\
\(bp(\gamma_1(\mathcal{F}))\) &= \(B \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})\) &= \(B \circ G \circ H \circ \mu(\mathbb{E})\) \\
\(bp(\gamma_2(\mathcal{F}))\) &= \(B \circ \hat{H} \circ \mu(\mathbb{E})\) &= \(B \circ G \circ H \circ \mu(\mathbb{E})\) \\
\(bp(\alpha_{m}^{\mu} \circ \gamma_1(\mathcal{F}))\) &= \(B \circ \hat{F}_{x_0} \circ \hat{M} \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})\) &\(\subseteq\) & \(B \circ G \circ T \circ M \circ H \circ \mu(\mathbb{E})\) \\
\(bp(\beta_{n}^{\mu} \circ \gamma_1(\mathcal{F}))\) &= \(B \circ \hat{F}_{y_0} \circ \hat{N} \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})\) &\(\propto\) & \(B \circ G \circ T \circ N \circ H \circ \mu(\mathbb{E})\) \\
\(bp(\alpha_{m}^{\mu} \circ \gamma_2(\mathcal{F}))\) &= \(B \circ \hat{F}_{x_0} \circ \hat{M} \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})\) &\(\subseteq\) & \(B \circ G \circ T \circ M \circ H \circ \mu(\mathbb{E})\) \\
\(bp(\beta_{n}^{\mu} \circ \gamma_2(\mathcal{F}))\) &= \(B \circ \hat{F}_{y_0} \circ \hat{N} \circ \hat{H} \circ \hat{G} \circ \mu(\mathbb{E})\) &\(\propto\) & \(B \circ G \circ T \circ N \circ M \circ H \circ \mu(\mathbb{E})\) \\
\hline
\end{tabular}
\end{table}

6 All calligraphs are centric

In this section we show that the eight sufficient conditions for centricity in Proposition 30 hold for all calligraphs. We then conclude the proof of the main results Theorem I and Corollary I by referring to Proposition 12 in §4. The eight conditions of Proposition 30 are proven in Lemmas 31, 32 and 34. Lemma 34 depends on Proposition 33, which is proven in Appendix C.

We assume the notation of §5. The following lemma shows that the base points of series associated to calligraphs lie on the line at infinity.

\textbf{Lemma 31.} \(B_0 \circ G \circ \mu(\mathbb{E}) = \emptyset\).

\textit{Proof.} We suppose by contradiction that \((\alpha, \beta) \in B_0 \circ G \circ \mu(\mathbb{E})\) is a base point.

Let \(C_\lambda := V(\{\mu(e) \downarrow \{\ell_e \rightarrow \lambda_e\} : e \in \mathbb{E}\})\) for all \(\lambda \in \mathbb{C}^\mathbb{R}\) and let \(\kappa : \mathbb{C}^{2|\mathbb{V}|} \rightarrow \mathbb{C}^2\) be defined as the linear projection \(\{x_i, y_i : i \in \mathbb{V}\} \mapsto \{x_0, y_0\}\). We denote the Zariski closure of the linear projection \(\kappa(C_\lambda)\) by \(C'_\lambda\). Suppose that \(\hat{\lambda} \in \mathbb{C}^\mathbb{R}\) is a general choice of edge length assignment and let \(L_\delta := \{\lambda \in \mathbb{C}^\mathbb{R} : |\lambda_e - \hat{\lambda}_e| \leq \delta\}\) with \(\delta \in \mathbb{R}_{>0}\). Recall from Remark 23 that \(C'_\lambda \subset \mathbb{C}^2\) is a coupler curve that passes through the base point \((\alpha, \beta) \in \mathbb{C}^2\).

\textbf{Claim 1.} There exists a \(\delta \in \mathbb{R}_{>0}\) such that for all \(\lambda \in L_\delta\) and general \(q \in C'_\lambda\), we have \(|\kappa^{-1}(q) \cap C_\lambda| > 0\) and \(|\kappa^{-1}(\alpha, \beta) \cap C_\lambda| = 0\).

Since \(q\) is general in the linear projection \(C'_\lambda\), it must have a preimage in \(C_\lambda\) and thus \(|\kappa^{-1}(q) \cap C_\lambda| > 0\). For the remaining assertion, we consider the map \(\rho : \mathbb{C}^{2|\mathbb{V}|} \rightarrow \mathbb{C}^\mathbb{R}\)
that sends \((x_i,y_i)_{i \in V}\) to \(( (x_i-x_j)^2 + (y_i-y_j)^2 ))_{(i,j) \in E}\). Notice that \(C_\lambda \subset \mathbb{C}^{2|V|}\) corresponds to the fiber \(\rho^{-1}(\lambda)\). It follows from the definition of calligraphs that \(|E| = 2|V| - 1\) and we know from Lemma 14(a) that \(\dim C_\lambda = 1\). We deduce that \(\rho\) is dominant and thus the image via \(\rho\) of the codimension two set \(\{(x,y) \in \mathbb{C}^{2|V|}: x_0 = \alpha, y_0 = \beta\}\) has codimension at least one in \(\mathbb{C}^E\). In other words, the set \(W := \{\rho(x,y) : (x_0,y_0) \neq (\alpha,\beta)\}\) is Zariski dense in \(\mathbb{C}^E\). This implies that \(|\kappa^{-1}(\alpha,\beta) \cap C_\lambda| = 0\) for all \(\lambda \in W\). There exists a radius \(\delta\) such that \(L_\delta \subset W\) and thus we conclude that Claim 1 holds.

Suppose that \(p := (\tilde{x}, \tilde{y})_{i \in V}\) is a point in \(C_\lambda\) such that \(|\tilde{x}_0 - \alpha| < \varepsilon\) and \(|\tilde{y}_0 - \beta| < \varepsilon\) for some \(\varepsilon \in \mathbb{R}_{>0}\). Recall that for all \(e \in E\) such that \(0 \in e\), we have:

\[(\tilde{x}_0 - x_i)^2 + (\tilde{y}_0 - y_i)^2 = \lambda^2_e.
\]

It follows from Claim 1 with \(q = \kappa(p)\) that we can choose \(\varepsilon > 0\) arbitrary small and that \(|\kappa^{-1}(\alpha,\beta) \cap C_\lambda| = 0\) for all \(\lambda \in L_\delta\), where \(\delta > 0\) is small enough. On the other hand, there exists a \(\lambda' \in \mathbb{C}^E\) such that for all \(e \in E\) we have

\[\lambda_e' = 0 \quad \text{if} \quad 0 \notin e, \quad \text{and} \quad (\alpha - \tilde{x}_i)^2 + (\beta - \tilde{y}_i)^2 = (\tilde{\lambda}_e + \lambda_e')^2 \quad \text{if} \quad 0 \in e.
\]

By choosing \(\varepsilon\) very small we can ensure that \(\tilde{\lambda} + \lambda' \in L_\delta\). We arrived at a contradiction with Claim 1, since \(|\kappa^{-1}(\alpha,\beta) \cap C_{\tilde{\lambda} + \lambda'}| \geq 1\) by construction. Hence, the base point \((\alpha,\beta)\) cannot exist and we concluded the proof of the main assertion. \(\square\)

**Lemma 32.**

(a) \((H \circ \mu(E)) \cup \{x_0 \to 0\} \supseteq \{1 + y_0^2\}\).

(b) \(B \circ G \circ H \circ \mu(E) \subseteq \{(0,i), (0,-i)\}\).

(c) \(B \circ G \circ H_{y_0} \circ \mu(E) \nsubseteq \{(0,0)\}\).

**Proof.** (a) From Table 2 we see that \(1 + y_0^2 = f_2 \downarrow \{x_0 \to 0\} \in (H \circ \mu(E)) \downarrow \{x_0 \to 0\}\).

(b) It follows from (a) that the elimination ideal \(\langle H \circ \mu(E) \cup \{x_0\} \rangle \cap S\) contains \(1 + y_0^2\) as well. Hence, \(1 + y_0^2\) is in the ideal \(\langle G \circ H \circ \mu(E) \cup \{x_0\} \rangle \subset S\) so that if \((0,u) \in B \circ G \circ H \circ \mu(E)\), then \(u = \pm i\).

(c) Using Table 2 we find that \(1 + x_0^2 \in (H_{y_0} \circ \mu(E)) \downarrow \{y_0 \to 0\}\). Similarly as in the proof of (b) we deduce that if \((u,0) \in B_0 \circ G \circ H_{y_0} \circ \mu(E)\), then \(u = \pm i \neq 0\). \(\square\)

**Proposition 33.** If \(P := T \circ M \circ H \circ \mu(E)\), then the zero set \(V(P \cup \{x_0\})\) is a union of linear spaces \(W_i \cup \cdots \cup W_r\) and the linear projection \(\pi(W_i)\) is equal to either \(V(x_0), V(x_0, y_0)\) or \(V(x_0, y_0 + i)\), for all \(1 \leq i \leq r\).

**Proof.** See Appendix C. \(\square\)
Lemma 34.

(a) \( B \circ G \circ T \circ M \circ H \circ \mu(\mathbb{E}) \subseteq \{(0,0), (0,-i)\} \).

(b) \( B \circ G \circ T \circ N \circ H \circ \mu(\mathbb{E}) \not\subseteq \{(0,0)\} \).

(c) \( B \circ G \circ T \circ M_c \circ M \circ H \circ \mu(\mathbb{E}) = \emptyset \).

(d) \( B \circ G \circ T \circ N_c \circ M \circ H \circ \mu(\mathbb{E}) \not\subseteq \{(0,0)\} \).

Proof. (a) Suppose that \( P := T \circ M \circ H \circ \mu(\mathbb{E}) \) is characterized as in Table 2. Let \( s = x_0 \prod_{u \in U} (y_0 - u) \) and \( U = \{ u : (0, u) \in B \circ G(P) \} \) be as in Definition 25. By Lemma 28 there exists \( g \in S \) such that
\[
\hat{G}(P) = \{ g \cdot x_0 \cdot \prod_{u \in U} (y_0 - u) \}.
\]
By Lemma 26 this means geometrically that
\[
\pi \circ V(P) = V(g) \cup V(x_0) \cup \left( \bigcup_{u \in U} V(y_0 - u) \right).
\]
We know from Proposition 33 that \( V(P \cup \{ x_0 \}) = W_1 \cup \cdots \cup W_r \), where \( W_i \) is a linear space for all \( 1 \leq i \leq r \). Notice that the restriction \( \pi|_{W_i} \) is surjective for all \( 1 \leq i \leq r \). The preimage of the component \( V(x_0) \subset \pi \circ V(P) \) consists of the union of those linear spaces \( W_i \) satisfying \( \pi(W_i) = V(x_0) \). The preimage of the intersection \( V(y_0 - u) \cap V(x_0) \) in \( \pi \circ V(P) \) consists of the union of those linear spaces \( W_i \) for which \( \pi(W_i) = V(x_0, y_0 - u) \). We conclude from Proposition 33 that \( U \subseteq \{0, -i\} \) and therefore \( B \circ G(P) \subseteq \{(0,0), (0,-i)\} \) as was to be shown.

The proofs for the assertions (b), (c) and (d) are similar to the proof of (a) except that \( P \) is equal to \( T \circ N \circ H \circ \mu(\mathbb{E}), T \circ M_c \circ M \circ H \circ \mu(\mathbb{E}) \) and \( T \circ N_c \circ M \circ H \circ \mu(\mathbb{E}) \), respectively. See Table 2 for a characterization of the the polynomials in these sets. Since each polynomial in \( P \downarrow \{ x_0 \rightarrow 0 \} \) is either linear or a product of two linear factors, it follows that \( V(P \cup \{ x_0 \}) \) is a union of linear subspaces \( W_1 \cup \cdots \cup W_r \).

Thus, \( \pi|_{W_i} \) is surjective and for all \( u \in U \) there exists \( 1 \leq i \leq r \) such that \( \pi(W_i) = V(x_0, y_0 - u) \). For assertion (b) we observe that \( V(P \cup \{ x_0, y_0 \}) = \emptyset \) and thus \( 0 \not\in U \) so that \( B \circ G(P) \not\subseteq \{(0,0)\} \). For (c) and (d) we have \( \pi(W_i) = V(x_0) \) for all \( 1 \leq i \leq r \), because no polynomial in \( P \downarrow \{ x_0 \rightarrow 0 \} \) depends on \( y_0 \). Therefore, \( B \circ G(P) = \emptyset \) for assertions (c) and (d) so that we concluded the proof. \( \square \)

Proof of Theorem 1 and Corollary I. It follows from Proposition 30 in combination with Lemmas 31, 32 and 34 that all calligraphs are centric. Thus Theorem I and Corollary I are now a direct consequence of Proposition 12. \( \square \)
A  Overview notation

Below we list an overview of notation that is used across more than one section. See the assigned sections § 2, § 4 and § 5 for the precise definitions.

§ 2
\[ v(\mathcal{G}), e(\mathcal{G}) \] Vertices \( v(\mathcal{G}) \subset \mathbb{Z}_{\geq 0} \) and edges of a graph \( \mathcal{G} \).
\[ \mathcal{L}, \mathcal{R}, \mathcal{C}_v \] The calligraphs defined in Figure 3 with \( v \in \mathbb{Z}_{\geq 3} \).
\[ \Omega_\mathcal{G}, \Xi^\omega_\mathcal{G} \] Sets of edge length assignments and realizations of a marked graph \( \mathcal{G} \) with edge length assignment \( \omega \in \Omega_\mathcal{G} \).
\[ c(\mathcal{G}) \] Number of realizations of a graph \( \mathcal{G} \).
\[ \text{sing} \, C, \, \deg C, \, g(C) \] Singular locus, degree and geometric genus of a curve \( C \).
\[ [\mathcal{G}], \, t_\omega(\mathcal{G}), \, m(\mathcal{G}) \] Class, coupler curve and coupler multiplicity of a calligraph \( \mathcal{G} \).

§ 4
\[ \alpha_p^m, \beta_p^m \] Maps \( \mathbb{C}[x, y] \to \mathbb{C}[x, y] \) that perform substitutions and quotients for base point analysis.
\[ i \] Imaginary unit.
\[ \gamma_0, \gamma_1, \gamma_2 \] Maps \( \mathbb{C}[z_0, z_1, z_2] \to \mathbb{C}[x, y] \) that send a homogeneous polynomial \( h \) to \( h(1, x, y), h(x, 1, y) \) and \( h(y, x, 1) \), respectively.
\[ \Gamma_B \] The series associated to a calligraph \( \mathcal{G} \) with \( \mathcal{B} \) the set of homogeneous polynomials, whose zero sets are projectivized coupler curves of \( \mathcal{G} \).

§ 5
\[ V, \, E \] \( V = v(\mathcal{G}) \setminus \{1, 2\} \) and \( E = e(\mathcal{G}) \setminus \{\{1, 2\}\} \).
\[ S, \, T_S, \, R, \, T_R \] Polynomial rings \( S = \mathbb{C}[T_S] \) and \( R = \mathbb{C}[T_R] \) with \( T_S \subset T_R \) being sets of variables.
\[ \pi = T_R \setminus T_S \] The linear projection \( \mathbb{C}^{|T_R|} \to \mathbb{C}^{|T_S|} \) that forgets coordinates.
\[ f \downarrow \{a_0 \to b_0, \ldots\} \] Substitution operator for polynomials \( f \in R \).
\[ \varphi(R) \] The set of all finite subsets of the polynomial ring \( R \).
\[ \mu \] Map \( \mu : \varphi(E) \to \varphi(R) \). See Notation 22 as \( \mu \) is also the map \( \mathbb{E} \to R \) that assigns a quadratic polynomial to an edge!
\[ H_r, \, M_c, \, N_c, \, T, \, G \] Maps \( \varphi(R) \to \varphi(R) \) that are defined via operators in Definition 19 and Definition 25. We abbreviate \( H = H_{x_0} \), \( M = M_i \) and \( N = N_i \).
\[ B \] Map \( \varphi(R) \to \text{powerset(\mathbb{C}^2)} \) for assigning base points.
\[ V \] Map \( \varphi(R) \to \text{powerset(\mathbb{C}^{|T_R|})} \) for assigning the zero set.
\[ \pi \circ V(P) \] The Zariski closure of \( \{\pi(p) : p \in V(P)\} \), see Notation 22.

Remark 35. Our definition of “general” in § 2.1 is closely related to the notions of generic in rigidity theory (see for example [15, §2]) and generic point in scheme theory (see [13, Example II.2.3.4]). For example, a point \( p \in \mathbb{C}^2 \) is general with respect to some property, if \( f(p) \neq 0 \) for some polynomial \( f \in \mathbb{C}[u, v] \) that depends on this property. In comparison, \( p \in \mathbb{C}^2 \) is generic in the rigidity theoretic sense if \( f(p) \neq 0 \) for all \( f \in \mathbb{Z}[u, v] \).
B Case study for the getNoR algorithm

We compute the number of realizations \( c(G) \) using Algorithm 2, where the minimally rigid graph \( G \) is defined in Figure 13. We depict the recursive execution tree of Algorithm 2 in Figure 14 together with references to the figures that depict the corresponding minimally rigid graphs and calligraphs.

![Figure 13: The minimally rigid graph G has 17 vertices.](image)

We notice that \( G \) does not have degree two vertices and that \((G_1, G_2)\) is a non-trivial calligraphic split for \( G \) (see Figure 15). This step corresponds to the first vertical “split” separator in Figure 14. We have \( c(G) = [G_1] \cdot [G_2] \) by Axiom A2 and thus we would like to compute the classes \([G_1]\) and \([G_2]\) using Algorithm 1.

In order to compute \([G_1]\) we first determine the number of realizations for the minimally rigid graphs \( G_{1L} = G_1 \cup L \), \( G_{1R} = G_1 \cup R \) and \( G_{1C} = G_1 \cup C_v \) for some vertex \( v \notin v(G_1) \) as is depicted in Figure 16. This step is associated to the first vertical “glue” separator in Figure 14.

In order to compute \( c(G_{1R}) \), \( c(G_{1L}) \) and \( c(G_{1C}) \) we call Algorithm 2 three times and arrive at the second vertical “split” separator in Figure 14. We continue recursively and thus the remaining steps are analogous. The captions of the corresponding figures are self-explanatory.

A minimally rigid graph that corresponds to one of the 36 leaves of the execution tree has at most 10 vertices instead of 17. We either continue recursively or resort to the fall-back algorithm for computing the number of realizations, namely Algorithm [5]. We conclude that \( c(G) = 200192 \).
Figure 14: Execution tree for Algorithm 2 with input $G$ and output $c(G) = 200192$. 
\[ \mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \quad \text{and} \quad c(\mathcal{G}) = 200192 \]

**Figure 15:** A non-trivial calligraphic split \((\mathcal{G}_1, \mathcal{G}_2)\) for the minimally rigid graph \(\mathcal{G}\).

\[[\mathcal{G}_1] = (368, 96, 176)\]

\[ \mathcal{G}_{1L} := \mathcal{G}_1 \cup \mathcal{L} \quad \text{and} \quad c(\mathcal{G}_{1L}) = 544 \]
\[ \mathcal{G}_{1R} := \mathcal{G}_1 \cup \mathcal{R} \quad \text{and} \quad c(\mathcal{G}_{1R}) = 384 \]
\[ \mathcal{G}_{1C} := \mathcal{G}_1 \cup \mathcal{C}_v \quad \text{and} \quad c(\mathcal{G}_{1L}) = 1472 \]

**Figure 16:** The class of the calligraph \(\mathcal{G}_1\) in Figure 15.

\[ \mathcal{G}_{1L} = \mathcal{G}_{1L1} \cup \mathcal{G}_{1L2} \]
\[ \mathcal{G}_{1R} = \mathcal{R}_{1R1} \cup \mathcal{G}_{1R2} \]
\[ \mathcal{G}_{1C} = \mathcal{G}_{1C1} \cup \mathcal{G}_{1C2} \]

**Figure 17:** Non-trivial calligraphic splits for the minimally rigid graphs in Figure 16.

\[[\mathcal{G}_2] = (272, 0, 0)\]

\[ \mathcal{G}_{2L} := \mathcal{G}_2 \cup \mathcal{L} \quad \text{and} \quad c(\mathcal{G}_{2L}) = 544 \]
\[ \mathcal{G}_{2R} := \mathcal{G}_2 \cup \mathcal{R} \quad \text{and} \quad c(\mathcal{G}_{2R}) = 544 \]
\[ \mathcal{G}_{2C} := \mathcal{G}_2 \cup \mathcal{C}_v \quad \text{and} \quad c(\mathcal{G}_{2C}) = 1088 \]

**Figure 18:** The class of the calligraph graph \(\mathcal{G}_2\) in Figure 15.
Figure 19: Non-trivial calligraphic splits for the minimally rigid graphs in Figure 18.

Figure 20: The classes for three of the six calligraphs in Figure 17. For the remaining calligraphs we have \([G_{1L2}] = [G_{1R2}] = [G_{1C2}] = [H] = (6, 2, 2)\) (see §3.1).
Figure 21: The classes of four of the six calligraphs in Figure 19. For the remaining two calligraphs we have $[\mathcal{G}_{2R2}] = [\mathcal{G}_{2C2}] = 2 \cdot [\mathcal{C}_v] = (4, 0, 0)$.
C  The proof for Proposition 33

The goal of this section is to prove Proposition 33 in §6. We associate to each edge of a calligraph a product of two linear polynomials. Thus, the zero set of these polynomials form a union of linear spaces. Proposition 33 lists all possible linear projections of such a linear space. Algebraically, this projection corresponds to Gaussian elimination, which in turn can be described in terms of a procedure that depends on the combinatorics of the calligraph. We clarify this procedure with examples, but for this we need to introduce some graph theoretic terminology.

Suppose that \( \mathcal{G} \) is a calligraph and recall that \( V = v(\mathcal{G}) \setminus \{1, 2\} \) and \( E = e(\mathcal{G}) \setminus \{\{1, 2\}\} \).

A walk is defined as a sequence of vertices \( \rho = (\rho_0, \ldots, \rho_r) \) such that \( \{\rho_i, \rho_{i+1}\} \in E \) for all \( 0 \leq i < r \). We define \( \text{start}(\rho) := \rho_0 \) and \( \text{end}(\rho) := \rho_r \). If all vertices are single digits, then we write \( \rho \) as \( \rho_0\rho_1\rho_2 \cdots \rho_r \). For example, the walk \( (0, 3, 4) \) shall be written as 034. We call the walk \( \rho \) western, eastern or round if \((\text{start}(\rho), \text{end}(\rho))\) is equal to \((0, 1), (0, 2) \) or \((0, 0)\), respectively. We call \( \rho \) a route if \( \{\rho_0, \ldots, \rho_r\} = V \) and \( \text{start}(\rho) = 0 \). For example, if \( \mathcal{G} \) is the calligraph in Figure 22, then both \((0, 3, 4, 5, 6)\) and \((0, 6, 5, 4, 6, 3)\) are routes.

Suppose that \( \Lambda \) is a finite set of walks and let \( \Lambda_v := \{\rho \in \Lambda : v \in \{\text{start}(\rho), \text{end}(\rho)\}\} \) for \( v \in v(\mathcal{G}) \). We call \( \Lambda \) initial if \( |\Lambda| = |E| \) and for all \( \{i, j\} \in E \) either \( (i, j) \in \Lambda \) or \( (j, i) \in \Lambda \). For example, if \( \mathcal{G} \) is the calligraph in Figure 22, then the set of walks \( \{03, 06, 32, 34, 36, 41, 45, 46, 56\} \) is initial.

Next, we define a concatenation of two walks. If \( \rho = (\rho_0, \ldots, \rho_r) \) and \( \rho' = (\rho'_0, \ldots, \rho'_s) \) are walks, then \( \rho \circ \rho' := (\rho_0, \ldots, \rho_{r-1}, \rho'_0, \ldots, \rho'_s) \) and \( -\rho := (\rho_r, \ldots, \rho_0) \). If \( \rho, \rho' \in \Lambda_v \), then \( \rho + \rho' \) is defined as the first element in the following tuple that is a walk:

\[
(\rho \circ \rho', \rho \circ -\rho', -\rho \circ \rho', -\rho \circ -\rho').
\]

For example, \( 034 + 45 = 034 + 54 = 430 + 45 = 430 + 54 = 0345 \).

We define \( \rho > \rho' \) if \( r > s \) or if \( r = s \), then \( (\rho_0, \ldots, \rho_r) >_{\text{lex}} (\rho'_0, \ldots, \rho'_s) \) according to the lexicographic ordering. Let \( \max(\Lambda) \) be the unique maximal element in the finite set of walks \( \Lambda \) with respect to the total strict order \( > \). We define

\[
\Delta_v(\Lambda) := (\Lambda \setminus \Lambda_v) \cup \{\rho + \rho' : \rho = \max(\Lambda_v \cap \Lambda_0) \text{ and } \rho' \in \Lambda_v \setminus \{\rho\}\}.
\]

For example, if \( \mathcal{G} \) is as in Figure 22 and \( \Lambda = \{06, 032, 036, 0341, 0346, 03456\} \), then \( \Delta_6(\Lambda) = \{032, 0341\} \cup \{034560, 0345630, 03456430\} \) is obtained as follows: \( \Lambda \setminus \Lambda_6 = \{032, 0341\} \), \( 034560 = \max(\Lambda_6 \cap \Lambda_0) \) and \( \Lambda_6 \setminus \{03456\} = \{06, 036, 0346\} \) so that \( 03456 + 06 = 034560, 03456 + 036 = 0345630 \) and \( 03456 + 036 = 03456430 \) are elements of \( \Delta_6(\Lambda) \).
Example 36. Suppose that $\mathcal{G}$ is defined as the calligraph in Figure 22 (in this example we may ignore the sign labeling for the edges) and let
\[ \Lambda^0 := \{03, 06, 32, 34, 36, 41, 45, 46, 56\} \]
be an initial set of walks. We consider the following sets in terms of a union that comes from the definition of $\Delta_v$ for $v$ in the route $(0, 3, 4, 5, 6)$ such that $v \neq 0$:
\[
\begin{align*}
\Lambda^3 &:= \Delta_3(\Lambda^0) = \{06, 41, 45, 46, 56\} \cup \{032, 034, 036\}. \\
\Lambda^4 &:= \Delta_4(\Lambda^3) = \{06, 56, 032, 036\} \cup \{0341, 0345, 0346\}. \\
\Lambda^5 &:= \Delta_5(\Lambda^4) = \{06, 032, 036, 0341, 0346\} \cup \{0345\}. \\
\Lambda^6 &:= \Delta_6(\Lambda^5) = \{032, 0341\} \cup \{034560, 0345630, 03456430\}.
\end{align*}
\]
We observe that each walk in $\Lambda^6$ is either western, eastern or round.

Lemma 37. If $\Lambda$ is an initial set of walks and $(v_0, \ldots, v_n)$ is a route, then each walk in $\Delta_{v_n} \circ \ldots \circ \Delta_{v_2} \circ \Delta_{v_1}(\Lambda)$ is either western, eastern or round.

Proof. We verified the assertion in Example 36 by computing $\Delta_6 \circ \Delta_5 \circ \Delta_4 \circ \Delta_3(\Lambda)$ via a deterministic procedure that halts. This procedure generalizes to any calligraph and route, and it is straightforward to see that each walk in the output must be either western, eastern or round. \hfill \Box

In what follows we assign to each walk a linear polynomial. The concatenation of walks corresponds to linear combinations of these polynomials such that common variables cancel out. We consider the ideal generated by the polynomials associated to walks in an initial set. The zero set of this ideal corresponds a linear space $W_i$ in Proposition 33 for some $1 \leq i \leq r$. In Proposition 44 we list all possible ideals that are obtained after eliminating all but one variable. The zero sets of the resulting ideals corresponds to the linear projection $\pi(W_i)$. Thus, Proposition 44 translates to a proof of Proposition 33.

Let $A := \mathbb{C}[a_i, b_i : i \in V]$ and $B := \mathbb{C}[b_i : i \in V]$. We denote the set of variables for these rings by $\Upsilon_A := \{a_i, b_i : i \in V\}$ and $\Upsilon_B := \{b_i : i \in V\}$. If $I \subset V$, then
\[ \widehat{A}(I) := \mathbb{C}[\Upsilon_A \setminus \{a_i\}_{i \in I}]. \]
We use the following short hand notation:

\[ a_{ij} := a_i - a_j \quad \text{and} \quad b_{ij} := b_i - b_j. \]

A **sign labeling** is defined as a map \( \tau : E \to \{1, -1\} \) such that \( \tau(e) = 1 \) for all edges \( e \in E \) such that \( 0 \notin e \). Let \( \hat{E} := \{(i, j) : (i, j) \in E\} \) and let \( \varphi_\tau : \hat{E} \to A \) be defined as

\[ \varphi_\tau(e) := a_{ij} + \tau(e) \cdot b_{ij}, \]

where \( a_0 := 0, (a_1, b_1) := (0, 0) \) and \( (a_2, b_2) := (-1, 0) \). The **\( \tau \)-polynomial** of a walk \( \rho = (\rho_0, \rho_1, \ldots, \rho_r) \) is defined as

\[ \tau_\rho := \varphi_\tau(\{\rho_0, \rho_1\}) + \varphi_\tau(\{\rho_1, \rho_2\}) + \cdots + \varphi_\tau(\{\rho_{r-1}, \rho_r\}), \]

Notice that \( \tau_\rho \in \hat{A}(\rho_1, \ldots, \rho_{r-1}) \) by construction. We call \( \tau_\rho \) **western**, **eastern** or **round** if \( \rho \) is as such. The **\( \tau \)-ideal** in the ring \( A \) of a finite set of walks \( \Lambda \) is defined as

\[ I_\tau(\Lambda) := \langle \tau_\rho : \rho \in \Lambda \rangle. \]

**Example 38.** Suppose that \( G, \Lambda^0, \Lambda^3, \Lambda^4, \Lambda^5 \) and \( \Lambda^6 \) are as in Example 36 and that the sign labeling \( \tau \) is defined as in Figure 22, where \(+/-\) stands for \(1/-1\). Let us consider the **\( \tau \)-ideal** \( I_\tau(\Lambda^3) = \langle \tau_{06}, \tau_{41}, \tau_{45}, \tau_{56}, \tau_{032}, \tau_{034}, \tau_{036} \rangle \), where

\[
\begin{align*}
\tau_{06} &= a_{06} + b_{06}, & \tau_{41} &= a_{14} - b_{14}, & \tau_{45} &= a_{45} + b_{45}, \\
\tau_{46} &= a_{46} - b_{46}, & \tau_{56} &= a_{56} + b_{56}, \\
\tau_{032} &= (a_{03} + b_{03}) + (a_{32} - b_{32}) = b_0 - 2b_3 + 1, \\
\tau_{034} &= (a_{03} + b_{03}) + (a_{34} - b_{34}) = -a_4 + b_0 - 2b_3 + b_4, \\
\tau_{036} &= (a_{03} + b_{03}) + (a_{36} + b_{36}) = -a_6 + b_0 - b_6. 
\end{align*}
\]

By applying Gaussian elimination we find that

\[ I_\tau(\Lambda^3) \cap \hat{A}(4) = \langle \tau_{06}, \tau_{56}, \tau_{032}, \tau_{036}, \tau_{0341}, \tau_{0345}, \tau_{0346} \rangle, \]

where

\[ \tau_{0341} = \tau_{034} + \tau_{41}, \quad \tau_{0345} = \tau_{034} + \tau_{45} \quad \text{and} \quad \tau_{0346} = \tau_{034} + \tau_{46}. \]

Since \( 034 \in \max(\Lambda^3 \cap \Lambda^3) \) and \( \tau_{034v} = \tau_{0344v} \) for \( v \in \{1, 5, 6\} \) it follows from the definition of \( \Delta_4 \) that \( I_\tau(\Lambda^4) = I_\tau(\Lambda^3) \cap \hat{A}(4) \). In fact, we find that \( I_\tau(\Lambda^i) = I_\tau(\Lambda^j) \cap \hat{A}(i) \) for all \((i, j) \in \{(0, 3), (3, 4), (4, 5), (5, 6)\}\) and thus

\[ I_\tau(\Lambda^6) = I(\Lambda) \cap \hat{A}(3, 4, 5, 6) = I(\Lambda) \cap B. \]

Let us compute the following \( \tau \)-polynomials in \( I_\tau(\Lambda^5) \) and \( I_\tau(\Lambda^6) \):

\[
\begin{align*}
\tau_{0346} &= \tau_{034} + \tau_{46} = -a_6 + b_0 - 2b_3 + b_6, \\
\tau_{0345} &= \tau_{034} + \tau_{45} + \tau_{56} = -a_6 + b_0 - 2b_3 + 2b_4 - b_6, \\
\tau_{0345630} &= \tau_{03456} - \tau_{0346} = 2b_4 - 2b_6. 
\end{align*}
\]

Notice that \( 03456, 0346 \in \Lambda^5 \) and \( 0345630 = 03456 + 0346 \in \Lambda^6 \). We choose the
walk 641 so that 03456 + 641 and 0346 + 641 are both western and
\[ \tau_{03456430} = \tau_{03456} - \tau_{0346} = \tau_{03456+641} - \tau_{0346+641}. \]
Thus, the round polynomial \( \tau_{03456430} \) can be written as a difference of western polynomials in the ideal \( I(\Lambda) \). Notice that we can replace 641 by any walk \( \rho \) such that \( \text{start}(\rho) = 6 \) and \( \text{end}(\rho) = 1 \). The remaining two round walks in \( \Lambda^6 \) can also be written as a difference of western polynomials:
\[
\tau_{03456} = \tau_{03456+641} - \tau_{0346+641} \quad \text{and} \quad \tau_{0346} = \tau_{03456+641} - \tau_{0346+641}.
\]
Since \( I(\Lambda^6) = I(\Lambda) \cap B \), we conclude that \( I(\Lambda^6) \) is generated by eastern and western polynomials:
\[
I(\Lambda^6) = \langle \tau_{032}, \tau_{0341}, \tau_{06+641}, \tau_{036+641}, \tau_{0346+641}, \tau_{03456+641} \rangle.
\]
The constructions in this example generalize to any calligraph, route and sign labeling.

**Lemma 39.** If \( \Lambda \) is a set of walks and \( \tau \) a sign labeling, then
\[
I_\tau(\Delta_v(\Lambda)) = I_\tau(\Lambda) \cap \hat{A}(v) \quad \text{for all} \quad v \in V.
\]

*Proof.* Direct application of Gaussian elimination as is explained in Example 38.

**Lemma 40.** If \( f \) is a round \( \tau \)-polynomial for some sign labeling \( \tau \), then there exist two western \( \tau \)-polynomials \( g \) and \( h \) such that \( f = g - h \).

*Proof.* Recall that if \( f = \tau_{03456430} \) as in Example 36, then \( g = \tau_{0345641} \) and \( h = \tau_{034641} \). We conclude the proof as this construction directly generalizes to any calligraph and sign labeling.

**Proposition 41.** If \( \Lambda \) is an initial set of walks and \( \tau \) is a sign labeling, then the elimination ideal \( I_\tau(\Lambda) \cap B \) is generated by the set of all \( \tau \)-polynomials that are either western or eastern.

*Proof.* Suppose that \( (v_0, \ldots, v_n) \) is a route and let \( Q = \Delta_{v_n} \circ \cdots \circ \Delta_{v_1} \circ \Delta_{v_0}(\Lambda) \). We know from Lemma 37 that each walk in \( Q \) is either western, eastern or round. We deduce from Lemma 39 that \( I_\tau(Q) = I_\tau(\Lambda) \cap B \). By Lemma 40 all polynomials in \( \{ \tau_\rho : \rho \in Q \} \) are generated by western and eastern polynomials and thus we concluded the proof.

If \( \tau \) is a sign labeling and \( \rho = (\rho_0, \ldots, \rho_r) \) an eastern or western walk, then we define the function \( \chi_{\tau,\rho} : \rho \to \{-2, 0, 2\} \) as
\[
\chi_{\tau,\rho}(\rho_i) := \begin{cases} 
\tau(\{\rho_i, \rho_{i+1}\}) - \tau(\{\rho_{i-1}, \rho_i\}) & \text{if } i \notin \{0, r\}, \\
0 & \text{otherwise}.
\end{cases}
\]
Notice that \( \chi_{\tau,\rho} \) is well defined and only attains values in \( \{-2, 0, 2\} \).
Example 42. Suppose that $G$ with sign labeling $\tau$ is defined as in Figure 22 and let $\rho := 03645641$ be a western walk. We simplify the $\tau$-polynomial of $\rho$ as follows, where we used that $a_{ij} + a_{jk} = a_{ik}$ and $b_{ij} + b_{jk} = b_{ik}$:

$$\tau_\rho = (a_{03} + b_{03}) + (a_{36} + b_{36}) + (a_{64} - b_{64}) + (a_{45} + b_{45}) + (a_{56} + b_{56}) + (a_{64} - b_{64}) + (a_{41} - b_{41}) = a_{03} + a_{36} + a_{64} + a_{45} + a_{56} + a_{64} + a_{41} + b_{03} + b_{36} - b_{64} + b_{45} + b_{56} - b_{64} - b_{41} = b_{06} - b_{64} + b_{46} - b_{61} = b_0 - 2b_6 + 2b_4 - 2b_6.$$

Notice that the places where $\chi_{\tau,\rho}$ attains $-2$ and $2$ are underlined at $03645641$ and $03645641$, respectively.

Thus the coefficients of $\tau_\rho$ are determined by $\chi_{\tau,\rho}$. The next lemma states that this holds for any eastern or western walk.

Lemma 43. If $\tau$ is a sign labeling and $\rho$ is an either eastern or western walk, then there exists $\epsilon \in \{0, 1\}$ such that

$$\tau_\rho = b_0 + \sum_{v \in \rho} \chi_{\tau,\rho}(v) b_v + \epsilon.$$

Proof. Straightforward consequence of the definitions (see Example 42). Notice that $\epsilon = 0$ or $\epsilon = 1$ if $\text{end}(\rho) = 1$ and $\text{end}(\rho) = 2$, respectively. \hfill \Box

Proposition 44. If $\Lambda$ is an initial set of walks and $\tau$ a sign labeling, then $I_{\tau}(\Lambda) \cap \mathbb{C}[b_0]$ is equal to either $\langle 0 \rangle$, $\langle 1 \rangle$, $\langle b_0 \rangle$ or $\langle b_0 + 1 \rangle$.

Proof. We know from Proposition 41 that $I_{\tau}(\Lambda) \cap \mathbb{B} = \langle Q \rangle$, where $Q$ is a set of $\tau$-polynomials that are either eastern or western. If there is an eastern or western walk $\rho$ whose edges all have sign 1, then Lemma 43 yields that $\tau_\rho = b_0 + \epsilon \in Q \subset I_{\tau}(\Lambda) \cap \mathbb{B}$ for some $\epsilon \in \{0, 1\}$ and thus $I_{\tau}(\Lambda) \cap \mathbb{C}[b_0] \in \{\langle 1 \rangle, \langle b_0 \rangle, \langle b_0 + 1 \rangle\}$. Now let us assume that no such walk exists. Let $K$ be the largest connected subgraph of $G$ which contains vertex 0 and only edges with positive sign. Notice that $1, 2 \not\in K$ by assumption. Let $f: \mathbb{V} \to \mathbb{B}$ be defined as

$$f(v) := \begin{cases} b_v + \frac{1}{2}b_0 & \text{if } v \in K \setminus \{0\}, \\ b_v & \text{otherwise.} \end{cases}$$

The linear isomorphism $\lambda: \mathbb{C}^n \to \mathbb{C}^n$ sends $(b_{v_1}, \ldots, b_{v_n})$ to $(f(v_1), \ldots, f(v_n))$, where $\mathbb{V} = \{v_1, \ldots, v_n\}$ and $v_1 < \ldots < v_n$ with $v_1 = 0$ (here we consider $\mathbb{B}$ as a set of polynomial functions). By Lemma 43 there exists for all $\tau_\rho \in Q$ an $\epsilon \in \{0, 1\}$ such
that
\[ \tau_\rho \circ \lambda = b_0 + \sum_{v \in \rho \cap K} \chi_{\tau,\rho}(v) \left( b_v + \frac{1}{2} b_0 \right) + \sum_{v \in \rho \setminus K} \chi_{\tau,\rho}(v) b_v + \epsilon \]
\[ = b_0 - 2(b_{\rho_i} + \frac{1}{2} b_0) + \sum_{v \in \rho \setminus \{\rho_i\}} \chi_{\tau,\rho}(v) b_v + \epsilon, \]

where \( i \) is the first index in \( \rho \) such that \( \rho_i \in K \) and \( \rho_{i+1} \notin K \). Indeed, if a walk goes back into \( K \) it also needs to go out again, so all further transformations of \( \lambda \) cancel out: \( |\{v \in \rho \cap K : \chi_{\tau,\rho}(v) = -2\}| = |\{v \in \rho \cap K : \chi_{\tau,\rho}(v) = 2\}| + 1 \). We deduce that \( \tau_\rho \circ \lambda \in \mathbb{C}[\Upsilon_B \setminus \{b_0\}] \) for all \( \tau_\rho \in Q \). The ideal of \( \lambda^{-1}(V_Q) \) is generated by \( \{\tau_\rho \circ \lambda : \tau_\rho \in Q\} \), where \( V_Q \) denotes the zero set of \( Q \). Hence, we observe that \( \kappa \circ \lambda^{-1}(V_Q) = \mathbb{C} \), where the linear projection \( \kappa : \mathbb{C}^n \to \mathbb{C} \) sends \( (b_{v_1}, \ldots, b_{v_n}) \) to \( b_{v_1} = b_0 \). Since \( \kappa = \kappa \circ \lambda^{-1} \) we find that \( \kappa(V_Q) = \mathbb{C} \) as well. The ideal of the projection \( \kappa(V_Q) \) is equal to the elimination ideal \( I_r(\Lambda) \cap \mathbb{C}[b_0] \) by Lemma 26 and thus \( I_r(\Lambda) \cap \mathbb{C}[b_0] = \langle 0 \rangle \). Since we considered all cases we concluded the proof. \( \square \)

Proof of Proposition 33. For all \( f \in P = T \circ M \circ H \circ \mu(\mathbb{E}) \) as listed in Table 2 we consider the substitution \( f \downarrow \{x_0 \to 0\} \). Next we make the identification \( a_0 = 0, a_i = -x_i \) and \( b_i = -iy_i \) for \( i > 0 \), where \( (x_1, y_1) := (0, 0), (x_2, y_2) := (1, 0) \) and \( x_i, y_i \in \Upsilon_R \) for \( i \in \mathbb{Z}_{\geq 0} \setminus \{1, 2\} \). We find that the ideal of \( P \cup \{x_0\} \) is generated by \( \{\varphi(e) : e \in \mathbb{E} \} \cup \{x_0\} \), where
\[
\varphi(\{i,j\}) := \begin{cases} 
  a_{0i} + b_{0i} & \text{if } 0 \in \{i,j\}, \\
  (a_{ij} + b_{ij}) \cdot (a_{ij} - b_{ij}) & \text{if } 0 \notin \{i,j\} \text{ with } i < j.
\end{cases}
\]

As each generator in \( \{\varphi(e) : e \in \mathbb{E} \} \cup \{x_0\} \) splits into linear factors, we find that \( V(P \cup \{x_0\}) \) is equal to a union of linear spaces \( W_1 \cup \cdots \cup W_r \). Let us consider the ideal \( I_r(\Lambda) \subseteq \mathbb{C}[\Upsilon_\Lambda] \) via the inclusion \( \Upsilon_\Lambda \subseteq \Upsilon_R \) as an ideal in the ring \( \mathbb{C}[\Upsilon_R] \). Under this identification there exist for all \( 1 \leq i \leq r \) a sign labeling \( \tau \) such that \( I_r(\Lambda) + \langle x_0 \rangle \) is the ideal of \( W_i \). Hence, \( (I_r(\Lambda) + \langle x_0 \rangle) \cap \mathbb{C}[b_0] \) is the ideal of \( \pi(W_i) \) by Lemma 26. We have \( (I_r(\Lambda) + \langle x_0 \rangle) \cap \mathbb{C}[b_0] = (I_r(\Lambda) \cap \mathbb{C}[b_0]) + \langle x_0 \rangle \) and thus the proof is concluded by Proposition 44. \( \square \)
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