The second homology group of the commutative case of
Kontsevich’s symplectic derivation Lie algebra

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Abstract

The symplectic derivation Lie algebras defined by Kontsevich are related to various geo-
metric objects including moduli spaces of graphs and of Riemann surfaces, graph homologies,
Hamiltonian vector fields, etc. Each of them and its Chevalley-Eilenberg chain complex have
a $\mathbb{Z}_{\geq 0}$-grading called weight. We consider one of them $\mathfrak{c}_g$, called the “commutative case”,
and its positive weight part $\mathfrak{c}_g^+ \subset \mathfrak{c}_g$. The symplectic invariant homology of $\mathfrak{c}_g^+$ is closely re-
lated to the commutative graph homology, hence there are some computational results from
the viewpoint of graph homology theory. However, the entire homology group $H_\bullet(\mathfrak{c}_g^+)$ is not
known well. We determined $H_2(\mathfrak{c}_g^+)$ by using classical representation theory of $\text{Sp}(2g; \mathbb{Q})$ and
the decomposition by weight.

1 Introduction

In [8, 10], Kontsevich defined the three symplectic Lie algebras $\mathfrak{l}_g, \mathfrak{a}_g, \mathfrak{c}_g$. They have deep
relations to geometric objects below, therefore they are studied from various viewpoints.
The Lie algebras $\mathfrak{l}_g$ and $\mathfrak{a}_g$ correspond to the moduli spaces of graphs and of Riemann
surfaces respectively. The Lie algebra $\mathfrak{c}_g$ is studied to describe invariants of 3-dimen-
sional manifolds and some kinds of characteristic classes of foliations by an identification of the
scalar extension of $\mathfrak{c}_g$ with Hamiltonian vector fields satisfying certain conditions (e.g. [7,
12] [11]). $\mathfrak{c}_g$ is also related to supergeometry and mathematical physics through the map to
the set of characteristic classes of $Q$-manifolds, which is an important kind of supermanifolds
[11] [12].

They are interpreted in terms of graph homology theory. We can take the direct limit of
each Lie algebra e.g. $\mathfrak{c}_\infty = \lim_{g \to \infty} \mathfrak{c}_g$, and see its homology is endowed with the Hopf algebra
structure. Kontsevich’s theorem described the primitive part of the stable homology group of
each Lie algebra as a certain kind of graph homology [8, Theorem 1.1.]. The homology of $\mathfrak{c}_g$
is tied to the commutative graph homology in the theorem. This version of graph homology is
used in perturbative Chern-Simons theory and provides an extension of Vassiliev invariants
[11] [12].

The Lie algebra $\mathfrak{c}_g$ has a $\mathbb{Z}_{\geq 0}$-grading called weight, and its positive weight part is denoted
by $\mathfrak{c}_g^+$. It is known that an argument using a spectral sequence shows that

$$H_\bullet(\mathfrak{c}_g) \cong H_\bullet(\text{sp}(2g; \mathbb{Q})) \otimes H_\bullet(\mathfrak{c}_g^+)_{\text{Sp}}$$

holds in the stable range. Here $H_\bullet(\mathfrak{c}_g^+)_{\text{Sp}}$ is the Sp-invariant part of $H_\bullet(\mathfrak{c}_g^+)$. This iso-
morphism makes the computation of $H_\bullet(\mathfrak{c}_g)$ relatively easier. Moreover, it is one way to
systematically construct cohomology classes of higher degree in $\mathfrak{c}_g$ from ones of lower degree
by taking duals and cup products. This method of taking positive weight part is applied to
$\mathfrak{l}_g$ and $\mathfrak{a}_g$ in [12].
Kontsevich’s theorem shows a relationship between \( H_\bullet(c_g) \) and the commutative graph homology. However, the problem of the computation of these groups still remains. There are some computational results from the viewpoint of graph (co)homology theory [1, 3, 2]. Willwacher and Živković gave the generating function of the Euler characteristic of the commutative graph homology and displayed it up to weight 60 in [16, p. 575, Table 1] as \( \chi_b^{\text{odd}} \). The commutative graph homology itself is determined up to weight 12 by Conant, Gerlits, and Vogtmann [4].

The homology group \( H_\bullet(c_g^+) \) is a direct sum of the subspaces \( H_\bullet(c_g^+)_w \) generated by homogeneous elements of weight \( w \). In this paper, we prove the following.

**Theorem 1.1.** \( H_2(c_g^+)_w = 0 \) if \( g, w \geq 4 \).

It is easy to see that \( H_1(c_g^+) = S^3Q^{2g} \) (see Proposition 2.1), however, it has yet to be known about the higher degree of \( H_\bullet(c_g^+) \). Theorem 1.1 is proved in Section 5 by an application of classical representation theory and weight. The parts of weight 1–3 are also determined in terms of \( \text{Sp} \)-modules, in conclusion, so is \( H_2(c_g^+) \) as a corollary.

In Section 2, we first recall the Lie algebra \( c_g \) and \( c_g^+ \).

In Section 3, we review the classical representation theory of \( \text{Sp}(2g; \mathbb{Q}) \).

In Section 5, we prove Theorem 1.1 by using the facts proved or introduced in Section 3-4.

In Section 6, we give some corollaries of Theorem 1.1 in the case that weight is 1–3. Here we obtain the description of entire \( H_2(c_g^+) \).

We use the word “\( \text{Sp} \)” as a shorthand for \( \text{Sp}(2g; \mathbb{Q}) \) when no confusion can arise.

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## 2 The Lie algebra \( c_g^+ \)

We assume \( g \geq 4 \). Let \( H := Q^{2g} \) be the fundamental representation of the symplectic group \( \text{Sp}(2g; \mathbb{Q}) \), and \( \mu : H \otimes H \to \mathbb{Q} \) be the symplectic form on \( H \). We fix a symplectic basis \( a_1, \ldots, a_g, b_1, \ldots, b_g \) with respect to \( \mu \), i.e. for any \( 1 \leq i, j \leq g \), \( \mu(a_i, a_j) = \mu(b_i, b_j) = 0 \) and \( \mu(a_i, b_j) = -\mu(b_j, a_i) = \delta_{ij} \) where \( \delta_{ij} \) is the Kronecker delta.

For \( w \geq 0 \), define

\[
\mathfrak{c}_g(w) := S^{w+2}H \quad \text{and} \quad \mathfrak{c}_g^+ := \bigoplus_{w \geq 1} \mathfrak{c}_g(w) \subset \bigoplus_{w \geq 0} \mathfrak{c}_g(w) \equiv \mathfrak{c}_g.
\]

(2.1)

An element \( \xi \in \mathfrak{c}_g(w) \subset \mathfrak{c}_g \) is said to be of weight \( w \).

We define a Lie bracket \([ , ]\) on \( \mathfrak{c}_g \) by

\[
\begin{array}{c}
\mathfrak{c}_g(k) \otimes \mathfrak{c}_g(l) \\
\xrightarrow{\mu(x_i, y_j)x_1 \cdots \hat{x}_i \cdots x_{k+2} \ y_1 \cdots \hat{y}_j \cdots y_{l+2}} \\
\sum_{1 \leq i \leq k+2 \atop 1 \leq j \leq l+2} \mu(x_i, y_j)x_1 \cdots \hat{x}_i \cdots x_{k+2} \ y_1 \cdots \hat{y}_j \cdots y_{l+2}
\end{array}
\]

for \( k, l \geq 0 \) and monomials \( x_1 \cdots x_{k+2} \in \mathfrak{c}_g(k) \), \( y_1 \cdots y_{l+2} \in \mathfrak{c}_g(l) \). Here \( \hat{\cdot} \) means the absence of that component. This bracket is interpreted as the classical Poisson bracket on \( C^\infty(\mathbb{R}^{2g}) \).
restricted to polynomial functions whose coefficients in \( \mathbb{Q} \). Note that this bracket preserves weights on \( c_g \) and is \( \text{Sp} \)-equivariant. Then the subspace \( c_g^+ \subset c_g \) becomes a Lie subalgebra.

The differential \( \partial_* \) on the Chevalley-Eilenberg chain complex \( \wedge \mathbf{c}_g \) with respect to this Lie bracket is described as

\[
\partial_n(f_1 \wedge \cdots \wedge f_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} [f_i, f_j] \wedge f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge \hat{f}_j \wedge \cdots \wedge f_n
\]

for any \( f_1, \ldots, f_n \in c_g \). We also introduce weights on \( \wedge \mathbf{c}_g \) by declaring that \( f_1 \wedge \cdots \wedge f_n \) is of weight \( k_1 + \cdots + k_n \) if \( f_1 \in c_g(k_1), \ldots, f_n \in c_g(k_n) \). We denote by \( (\wedge^w c_g)_w \) the subspace of \( \wedge^w c_g \) generated by elements of weight \( w \). Similarly, we introduce a notation \( (\wedge^w c_g^+)_w \subset (\wedge^w c_g)_w \).

Since the bracket map \([,]\) is \( \text{Sp} \)-equivariant, so is the differential \( \partial_* \). The weight on \( \wedge \mathbf{c}_g \) defined above is also preserved by \( \partial_* \). Therefore for \( n \geq 1 \), we can decompose the chain space \( \wedge^n c_g^+ \) by weight and moreover into \( \text{Sp} \)-irreducible components because each \( (\wedge^n c_g^+)_w \) is a finite dimensional \( \text{Sp} \)-module.

Hereafter we concentrate on \( c_g^+ \) rather than \( c_g \).

**Remark 2.1.** We regard the vector space \( \wedge \mathbf{c}_g^+ \) as the quotient space of \( \bigoplus_{n \geq 1} (c_g^+) \otimes^n \). Let \( \pi: \bigoplus_{n \geq 1} (c_g^+) \otimes^n \to \wedge \mathbf{c}_g^+ \) be the quotient map. We write

\[ c_g(k_1) \wedge \cdots \wedge c_g(k_n) = \pi(c_g(k_1) \otimes \cdots \otimes c_g(k_n)) \]

for \( n \geq 2 \). We easily see that \( c_g(k) \wedge c_g(l) = c_g(l) \wedge c_g(k) \cong c_g(k) \otimes c_g(l) \) if \( k, l \geq 1 \) and \( k \neq l \). This notation might be unusual, however we use it for simplicity. For example,

\[
\begin{align*}
c_g(2) \wedge c_g(1) \wedge c_g(1) &= \begin{pmatrix} \text{The vector subspace of } \wedge^3 c_g^+ \text{ generated by} \\ \text{the elements of the form } f_1 \wedge f_2 \wedge f_3 \\ \text{for } f_1 \in c_g(2), f_2, f_3 \in c_g(1) \end{pmatrix} \\ &\cong c_g(2) \otimes (\wedge^2 c_g(1)) \end{align*}
\]

We use the following natural way to regard \( c_g(k) \) and \( c_g(k) \wedge c_g(l) \) as \( \text{Sp} \)-submodules of \( H^{\otimes n} \). Consider an injective map

\[
\begin{array}{ccc}
c_g(k) & \longrightarrow & H^\otimes(k+2) \\
x_1 \cdots x_{k+2} & \longmapsto & \sum_{\sigma \in \Sigma_{k+2}} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k+2)}
\end{array}
\]

for \( k \geq 1 \). We always see degree \( n \) polynomials of \( H \) as elements in \( H^\otimes n \) by this map. Then for \( k > l \geq 1 \), we see \( c_g(k) \wedge c_g(l) \subset \wedge^2 c_g^+ \) as a submodule of \( H^\otimes(k+l+4) \) by an injective map

\[
\begin{array}{ccc}
c_g(k) \wedge c_g(l) & \longrightarrow & H^\otimes(k+2) \otimes H^\otimes(l+2) = H^\otimes(k+l+4) \\
f \wedge h & \longmapsto & f \otimes h
\end{array}
\]

and for \( k \geq 1 \),

\[
\begin{array}{ccc}
\wedge^2 c_g(k) & \longrightarrow & H^\otimes(2k+4) \\
f \wedge h & \longrightarrow & f \otimes h - h \otimes f
\end{array}
\]

using the same notation \( \iota \). For \( m,n \geq 1 \), \( \alpha \in H^\otimes m \) and \( \beta \in H^\otimes n \), define

\[
\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha \in H^\otimes(m+n), \quad \alpha \odot \beta := \alpha \otimes \beta + \beta \otimes \alpha \in H^\otimes(m+n).
\]

Let us refer to \( H_1(c_g^+) \) before we discuss \( H_2(c_g^+) \).
Proposition 2.1. (1) The map $\partial_2 = [\cdot]: (\wedge^2 c_g^+) \otimes (\wedge^1 c_g^+) = c_g(w)$ is surjective if $g, w \geq 2$.

$H_1(c_g^+) = S^3 H$ if $g \geq 2$.

**Proof.** For (1), we see that this map is zero or surjective because it is Sp-equivariant and $c_g(w) = S^{w+2} H$ is Sp-irreducible. Now, we have $\partial_2(a^w_g \wedge a^2_b g) = [a^w_g a_g, a^2_b b_g] = a^{w+2}_g$. This leads to surjectivity of this map.

(2) follows from (1) and $\wedge^2 c_g^+ = \bigoplus_{w \geq 2}(\wedge^2 c_g^+)$. Contrary to this case, the chain space $(\wedge^2 c_g^+)$ is not Sp-irreducible for general $w$. In order to apply the similar method of confirming all the cycles are boundaries, we must know how the chain space $(\wedge^2 c_g^+)_w$ decomposes into Sp-irreducible components.

3 Representation theory of $\text{Sp}(2g; \mathbb{Q})$

The following is a classical theorem in representation theory (see e.g. [5]).

**Theorem 3.1.** There is a bijection

| Isomorphism classes of finite dimensional polynomial irreducible $\text{Sp}(2g; \mathbb{Q})$ representations | Young diagrams with at most $g$ rows |
|-------------------------------------------------|-----------------------------------|
| $[V_\lambda]$ | $\lambda$ |

where $V_\lambda$ is a certain submodule of $(\wedge^{2g} H) \otimes \cdots \otimes (\wedge^{2g} H)$. $V_\lambda$ is generated by the element $a_\lambda := (a_1 \wedge \cdots \wedge a_{2g}) \otimes \cdots \otimes (a_1 \wedge \cdots \wedge a_{2g}) \in (\wedge^{2g} H) \otimes \cdots \otimes (\wedge^{2g} H)$ as an $\text{Sp}(2g; \mathbb{Q})$-module. Here $\lambda = [\lambda_1' \cdots \lambda_g']$ ($g \geq \lambda_1' \geq \cdots \geq \lambda_g' \geq 1$) is the transpose of $\lambda$, and $[V_\lambda]$ is the isomorphism class of $V_\lambda$. This $a_\lambda$ is called the highest weight vector with respect to the ordered symplectic basis $a_1, \ldots, a_g, b_1, \ldots, b_g$. For example,

$$a_{[2]} = (a_1 \wedge a_2) \otimes (a_1 \wedge a_2) \otimes a_1, \quad a_{[22]} = (a_1 \wedge a_2 \wedge a_1) \otimes (a_1 \wedge a_2).$$

(3.1)

Now we describe each Sp-irreducible component of $\wedge^* c_g^+$ by a Young diagram. We simply write an Sp-module isomorphic to $V_\lambda$ as $\lambda$ when we focus mainly on its isomorphism class.

**Lemma 3.1.** (i) If $k > l \geq 1$,

$$c_g(k) \otimes c_g(l) \cong \bigoplus_{0 \leq \lambda_2 \leq l+2} \bigoplus_{0 \leq \rho \leq l+2-\lambda_2} [(k+l+4-\lambda_2 - 2\rho) \cdot \lambda_2]$$

In particular, if $V \subset c_g(k) \otimes c_g(l)$ is an Sp-irreducible submodule corresponding to a Young diagram $\lambda = [\lambda_1 \lambda_2]$, then the following conditions hold:

$$l + 2 \geq \rho + \lambda_2, \quad k + 2 \leq \rho + \lambda_1 \quad \text{where} \quad \rho := \frac{1}{2}(k + l + 4 - \lambda_1 - \lambda_2) \in \mathbb{Z}_{\geq 0}$$

(3.2)

(ii) If $k \geq 1$,

$$c_g(k) \wedge c_g(k) \cong \bigoplus_{0 \leq \lambda_2 \leq k+2} \bigoplus_{0 \leq \rho \leq k+2-\lambda_2} \bigoplus_{\rho + \lambda_2 \text{ is odd}} [(2k+4-\lambda_2 - 2\rho) \cdot \lambda_2]$$

In particular, if $V \subset c_g(k) \otimes c_g(l)$ is an Sp-irreducible submodule corresponding to a Young diagram $\lambda = [\lambda_1 \lambda_2]$, then the following conditions hold:

$$\rho + \lambda_2 \text{ is odd}, \quad \rho + \lambda_2 \leq k+2 \leq \rho + \lambda_1 \quad \text{where} \quad \rho := \frac{1}{2}(2k + 4 - \lambda_1 - \lambda_2) \in \mathbb{Z}_{\geq 0}$$

(3.3)
Proof. (i) follows from the Littlewood-Richardson rule and branching rules. 
(ii) follows from the same rules and the plethysm of symmetric polynomials
\[ e_2 \circ h_{k+2} = \sum_{1 \leq j \leq 2k+4-j, \ j \text{ is odd}} s_{\{(2k+4-j) \ j\}} \]
where \( e_2 \) is the second elementary symmetric function, \( h_{k+2} \) is the \( (k+2) \)-nd complete symmetric function and \( s_{\lambda} \) is the Schur function corresponding to a Young diagram \( \lambda \) (see [13]).

This lemma shows, for fixed \( k \) and \( l \), each Young diagram \( \lambda \) satisfying (3.2) or (3.3) corresponds to exactly one \( Sp \)-irreducible component in \( \zeta_g(k) \land \zeta_g(l) \), so that we identify such Young diagrams with them. From now, we denote simply by \( \lambda \) an \( Sp \)-irreducible component \( V \subset \zeta_g(k) \land \zeta_g(l) \) isomorphic to \( V_\lambda \), like \( \lambda \subset \zeta_g(k) \land \zeta_g(l) \).

4 Detecting the highest weight vector

We introduce some notations. Define, for \( n, k \geq 2 \) and \( 1 \leq i, j, i_1, \ldots, i_k \leq n, \)
\[
\mu_{ij} : H^\otimes n \longrightarrow H^\otimes(n-2),
\]
\[ x_1 \otimes \cdots \otimes x_n \longmapsto \mu(x_i, x_j)x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes \hat{x}_j \otimes \cdots \otimes x_n \] (4.1)
\[ A_{i_1 \cdots i_k} : H^\otimes n \longrightarrow (\wedge^k H)^\otimes(n-k),
\]
\[ x_1 \otimes \cdots \otimes x_n \longmapsto (x_{i_1} \land \cdots \land x_{i_k}) \otimes x_1 \otimes \cdots \otimes \hat{x}_{i_1} \otimes \cdots \otimes \hat{x}_{i_k} \otimes \cdots \otimes x_n \] (4.2)

By regarding \( \zeta_g(k) \) as an \( Sp \)-submodule of \( H^\otimes(k+2) \), we have the following.

Lemma 4.1. Let \( k \geq l \geq 1, f \in \zeta_g(k) \) and \( h \in \zeta_g(l) \).

(1) Let \( 1 \leq i \leq k + 2 \) and \( 1 \leq j \leq l + 2 \). If \( k > l \), then \( \mu_{ij} \circ (f \otimes h) = \mu_{1,k+l+4}(f \otimes h) \) and \( A_{ij} \circ (f \otimes h) = A_{1,k+l+4}(f \otimes h) \). If \( k = l \), then \( \mu_{ij} \circ (f \land h) = \mu_{1,2k+4}(f \land h) \) and \( A_{ij} \circ (f \land h) = A_{1,2k+4}(f \land h) \).

(2) For \( f = x_1 \cdots x_{k+2}, h = y_1 \cdots y_{l+2}, \) we have
\[
\mu_{1,k+l+4}(f \land h) = \sum_{1 \leq i \leq l+2} x_i \otimes (x_1 \cdots \hat{x}_i \cdots x_{k+2}) \otimes (y_1 \cdots \hat{y}_j \cdots y_{l+2}) \] (4.3)
Moreover these formulae hold if we switch \( \land \) and \( \odot \) each other.

Proof. (1) holds because elements of \( S^{m+2}H(\subset H^\otimes(m+2)) \) are invariant under a permutation of factors for any \( m \geq 1 \).

(2) is shown by a direct computation. Note that for all \( x_1, \ldots, x_n \in H, \)
\[ x_1 \cdots x_n = \sum_{1 \leq i \leq n} x_i \otimes (x_1 \cdots \hat{x}_i \cdots x_n) \] (4.4)
hold as an element of \( H^\otimes n \) by the definition of \( \iota. \)
We write $\mu_{i,n}$, $\Lambda_{i,n}$ on $H^{\otimes n}$ as $\mu_{\text{end}}$, $\Lambda_{\text{end}}$ respectively. Based on this lemma, when we want to apply $\mu_{i,j}$s and $\Lambda_{i,j}$s to an element of $\wedge^k q_{ij}$, it is enough to consider only $\mu_{\text{end}}$ and $\Lambda_{\text{end}}$. In [4.2], for example, we can further apply $\mu_{i,j}$ or $\Lambda_{i_1\ldots i_l}$ ($1 \leq j_1, \ldots, j_l \leq n - k$) on the latter tensor component $H^{\otimes (n - k)}$, again on $H^{\otimes (n - k - 2)}$ or $H^{\otimes (n - k - 1)}$, and so on. For $n, p \geq 1$, $k_1, \ldots, k_p \geq 2$ and $1 \leq i, j, i_1, \ldots, i_l \leq n$,

$$\text{id}_{(\wedge^k H) \otimes \cdots \otimes (\wedge^p H)} \otimes \mu_{i,j} : (\wedge^k H) \otimes \cdots \otimes (\wedge^p H) \otimes H^{\otimes n} \rightarrow (\wedge^k H) \otimes \cdots \otimes (\wedge^p H) \otimes H^{\otimes (n - 2)} \quad (4.5)$$

is also denoted by the same symbol $\mu_{i,j}$, and

$$\text{id}_{(\wedge^k H) \otimes \cdots \otimes (\wedge^p H)} \otimes \Lambda_{i_1\ldots i_l} : (\wedge^k H) \otimes \cdots \otimes (\wedge^p H) \otimes H^{\otimes n} \rightarrow (\wedge^k H) \otimes \cdots \otimes (\wedge^p H) \otimes (\wedge^l H) \otimes H^{\otimes (n - l)} \quad (4.6)$$

by $\Lambda_{i_1\ldots i_l}$ for short. Under this notation we can think of a composition of $\mu_{i,j}$s and $\Lambda_{i_1\ldots i_l}$s.

Next lemma gives a method to detect the highest weight vector of an Sp-irreducible submodule of $H^{\otimes n}$. For a completely reducible Sp-module $V$, its Sp-irreducible submodule $V$ and $w \in W$, the image of $w$ by the canonical projection $W \rightarrow V$ is denoted by $v|_{V}$.

**Lemma 4.2.** Let $\xi \in \zeta_{\rho}(k_1) \wedge \cdots \wedge \zeta_{\rho}(k_s)$ and let $V \subset \zeta_{2}(k_1) \wedge \cdots \wedge \zeta_{2}(k_s)$ be an Sp-irreducible component, which corresponds to a Young diagram $\lambda$. Assume $\xi$ is mapped to $a_\lambda \in V_\lambda$ by some compositions of $\mu_{i,j}$s and $\Lambda_{i_1\ldots i_l}$s. Then $\xi|_{V}$ is nonzero and spans $V$ as an Sp-module.

**Proof.** $\mu_{i,j}$s and $\Lambda_{i_1\ldots i_l}$s are Sp-equivalent homomorphisms because Sp($2g; \mathbb{Q}$) diagonally acts on $H^{\otimes n}$ and on its quotient modules. Since both $V$ and $V_\lambda$ are Sp-irreducible, if there is a nonzero Sp-equivalent homomorphism $V \rightarrow V_\lambda$, then it is an isomorphism.

In the proof of [Theorem 1.1] we frequently use the following.

**Lemma 4.3.** Let $k \geq l \geq 1$ and $\lambda = [\lambda_1, \lambda_2] \subset \zeta_2(k) \wedge \zeta_2(l)$. Set $\rho := \frac{1}{2}(k + l + 4 - \lambda_1 - \lambda_2)$. Then $\left(a_{1}^{k+2-\rho}a_{3}^{2} \wedge a_{1}^{l+2-\lambda_2-\rho}a_{2}^{\lambda_2}a_{3}^{\rho}\right)_{\lambda}$ generates $\lambda \subset \zeta_2(k) \wedge \zeta_2(l)$ as an Sp-module.

**Proof.** We use **Lemma 4.2**. If $k > l$, then

$$a_{1}^{k+2-\rho}a_{3}^{2} \wedge a_{1}^{l+2-\lambda_2-\rho}a_{2}^{\lambda_2}a_{3}^{\rho} \stackrel{\mu_{\text{end}}}{\longrightarrow} a_{1}^{k+2-\rho}a_{3}^{2} \otimes a_{1}^{l+2-\lambda_2-\rho}a_{2}^{\lambda_2}a_{3}^{\rho-1} \otimes a_{1}^{l+2-\lambda_2-\rho}a_{2}^{\lambda_2}a_{3}^{\rho-1}$$

$$\vdots$$

$$\vdots$$

$$\stackrel{\Lambda_{\text{end}}}{\longrightarrow} (\rho!)^{2} a_{1}^{l+2-\lambda_2-\rho}a_{2}^{\lambda_2}$$

$$\vdots$$

$$\vdots$$

$$\stackrel{\Lambda_{\text{end}}}{\longrightarrow} (\rho!)^{2} (k + 2 - \rho)\lambda_2 \cdot (a_{1} \wedge a_{2}) \otimes a_{1}^{k+1-\rho} \otimes a_{1}^{l+2-\lambda_2-\rho}a_{2}^{\lambda_2}$$

where $\mu_{\text{end}}$ is applied $\rho$ times and $\Lambda_{\text{end}}$ is applied $\lambda_2$ times. We used (1.4) to find coefficients in each $\mu_{\text{end}}$ or $\Lambda_{\text{end}}$. In the last equality, we used the fact that $x^{m} = m!x^{\otimes m} \in H^{\otimes m}$ for any $m \geq 1$ and any $x \in H$. This is shown by the definition of $\iota$. Since $(a_{1} \wedge a_{2}) \otimes a_{1}^{\otimes \lambda_1-\lambda_2} = a_{\lambda}$, the statement follows.
If \( k = l \), then by a similar procedure,

\[
\begin{align*}
\rho_{\text{end}}^\rho &\colon a_1^{k+2-\rho}a_3^\rho \wedge a_1^{l+2-\lambda_2-\rho}a_2^{\lambda_2} \\
\Lambda_{\text{end}}^{\lambda_2} &\colon (\rho)^2 \cdot a_1^{k+2-\rho} \wedge a_1^{l+2-\lambda_2-\rho}a_2^{\lambda_2} \quad \text{(if } \rho \text{ is even)} \\
&\quad \cdot a_1^{k+2-\rho} \circ a_1^{l+2-\lambda_2-\rho}a_2^{\lambda_2} \quad \text{(if } \rho \text{ is odd)}
\end{align*}
\]

Here \( \rho_{\text{end}}^\rho \) is the \( \rho \)-time compositions of \( \mu_{\text{end}} \), and \( \Lambda_{\text{end}}^{\lambda_2} \) is the \( \lambda_2 \)-time compositions of \( \Lambda_{\text{end}} \). Note that \( \rho + \lambda_2 \) is always odd by \( \text{(3)} \) and that \( x^{\otimes 1} \circ x^{\otimes m} = 2x^{\otimes (l+m)} \in H^{\otimes (l+m)} \) holds for \( l, m \geq 1 \) and \( x \in H \) from the definition of \( \circ \).

5 Proof of the main theorem

Fix a weight \( w \geq 4, k \geq l \geq 1 \) such that \( k + l = w \), and an \( \text{Sp} \)-irreducible submodule \( V \) corresponding to a Young diagram \( \lambda = [k, \lambda_2] \subset \epsilon_g(k) \wedge \epsilon_g(l) \). Set \( \rho := \frac{1}{2}(k + l + 4 - \lambda_1 - \lambda_2) \).

Here is our strategy of the proof. First, since \( \epsilon_g(w) = S^{w+2}H \) itself is \( \text{Sp} \)-irreducible, we have \( \partial_2(V) = 0 \) if \( \lambda \neq [w + 2] \). Therefore we focus respectively on the case \( \lambda \neq [w + 2] \) and the case \( \lambda = [w + 2] \).

Next we take an element \( \omega_3 \in \wedge^3 c_g^+ \). We denote by \( \omega_3|_\lambda \) the image of \( \omega_3 \) by the projection to the isotypical component of \( \wedge^3 c_g^+ \) corresponding to \( \lambda \). For example, if \( \partial_3(\omega_3) \in \epsilon_g(k) \wedge \epsilon_g(l) \) and it is mapped by \( \mu_{\text{end}}^{\otimes 8} \) and \( \Lambda_{\text{end}}^{\otimes 8} \) to a nonzero constant multiple of \( a_3 \in (\wedge^2 H)^{\otimes \lambda_2} \otimes H^{\otimes (\lambda_1 - \lambda_2)} \), then \( \partial_3(\omega_3)|_V \) generates \( V \) as an \( \text{Sp} \)-module. Since \( \partial_3 \) is \( \text{Sp} \)-equivariant, we have \( \partial_3(\omega_3)|_\lambda = \partial_3(\omega_3)|_V \). In this case, \( \partial_3(\omega_3)|_V \), which is a generator of \( V \), is in \( \text{Im}(\partial_3) \).

Consequently whole \( V \) is in \( \text{Im}(\partial_3) \). This can happen if \( \lambda \neq [w + 2] \). For \( \lambda = [w + 2] \), we have to determine \( \text{Ker} \partial_2 \) restricted to the isotypical component of \( \wedge^3 c_g^+ \) corresponding to \( \lambda = [w + 2] \), and show that its generators are in \( \text{Im}(\partial_3) \) by a similar argument.

We divide our argument for the following cases:

(I) the case \( \rho = 0 \),

(II) the case \( \rho = 1 \) and (i) \( \lambda_2 \geq 1 \), (ii) \( \lambda_2 = 0 \),

(III) the case \( \rho \geq 2 \) and (i) \( k - \rho \geq 1 \), (ii) \( k - \rho = 0 \), (iii) \( k - \rho \leq -1 \).

These cases clearly cover all possible patterns.

5.1 The case (I)

Suppose \( \rho = 0 \). We set

\[
\omega_3 := a_1^k a_4 \wedge a_1^2 b_4 \wedge a_1^{\lambda_1-k-2} a_2^{\lambda_2} \in \epsilon_g(k-1) \wedge \epsilon_g(1) \wedge \epsilon_g(l). \tag{5.1}
\]

Since \( \partial_3(\omega_3) = a_1^{k+2} a_4^{\lambda_1-k-2} a_2^{\lambda_2} \), by using Lemma 4.3 the element \( \partial_3(\omega_3)|_\lambda = \partial_3(\omega_3)|_V \) generates \( \lambda \in \epsilon_g(k) \wedge \epsilon_g(l) \).

5.2 The case (II)(i)

Lemma 5.1. Let \( p, q, r \geq 0 \) with \( p \geq r \).

(1) \( \Lambda_{\text{end}}^{\otimes (r+1)}(a_1^p a_2 \otimes a_1^q a_2^r) = -p!q!(r+1)! (a_1 \wedge a_2)^{(r+1)} \otimes a_1^{p+q-r-1} \).
(2) \( \Lambda_{\text{end}}^{(r+1)}(a^3_1 a_2 \otimes a^3_2 a^3_2) = -2 \cdot p!q!(r+1)![(a_1 \wedge a_2)^{(r+1)} \otimes a_1^{p+q-r-1}] \) if \( r \) is even.

**Proof.** Easily shown by induction on \( r \). □

Suppose \( p = 1 \) and \( \lambda_2 \geq 1 \). We set

\[
\omega_3 := a^2_1 a_4 \wedge a^{k-1}_1 a_2 b_4 \wedge a^2_1 a^{l+1-\lambda_2} a^{\lambda_2} a_2 \wedge b_3 - a^2_1 a_4 \wedge a^{k-2}_1 a_2 a_3 b_4 \wedge a^{l+2-\lambda_2} a^{\lambda_2} a_2 \wedge b_3 \\
\in \mathfrak{c}_g(1) \wedge \mathfrak{c}_g(k-1) \wedge \mathfrak{c}_g(l).
\]  

(5.2)

Note that all exponents here are certainly nonnegative because \( l + 1 - \lambda_2 = l + 2 - (\lambda_2 + \rho) \geq 0 \).

If \( k > l \), then

\[
\begin{align*}
\omega_3 & \xrightarrow{\partial_3} a^{k+1}_1 a_3 \wedge a^{l+1-\lambda_2} a^{\lambda_2} a_2 b_3 - a^k_1 a_2 a_3 \wedge a^{l+2-\lambda_2} a^{\lambda_2} a_2 b_3 \wedge b_3 \\
& \in \mathfrak{c}_g(k) \wedge \mathfrak{c}_g(l)
\end{align*}
\]

(5.3)

and if \( k = l \), then

\[
\begin{align*}
\omega_3 & \xrightarrow{\partial_3} a^{k+1}_1 a_3 \wedge a^{l+1-\lambda_2} a^{\lambda_2} a_2 b_3 - a^k_1 a_2 a_3 \wedge a^{l+2-\lambda_2} a^{\lambda_2} a_2 b_3 \wedge b_3 \\
& \in \mathfrak{c}_g(k) \wedge \mathfrak{c}_g(l)
\end{align*}
\]

(5.4)

We used Lemma 5.1 and the proof of Lemma 4.3 at the last rows in (5.3) and (5.4).

5.3 The case (II) (ii)

Suppose \( p = 1 \) and \( \lambda_2 = 0 \). In this case, we have \( \lambda = [w+2] \cong \wedge^1 \mathfrak{c}_g(w) \), so that we have to see \( \text{Ker}(\partial_2) \) in detail. Only here we do not fix \( k \) and \( l \). \( k \) runs over \( \left\lfloor \frac{w}{2} \right\rfloor \leq k \leq w - 1 \) because \( k \geq 1 \) and \( k + l = w \).

For each such \( k \), the space \( \mathfrak{c}_g(k) \wedge \mathfrak{c}_g(l) \) contains an \( \Lambda \)-irreducible component \( \lambda = [w+2] \), which is generated by \( v^{(k)} := a^{k+1}_1 a_2 \wedge a^{l+1}_1 b_2 \in \mathfrak{c}_g(k) \wedge \mathfrak{c}_g(l) \) by the similar argument to Lemma 4.2. Let \( \pi^{(k)} : \mathfrak{c}_g(k) \wedge \mathfrak{c}_g(l) \to [w+2] \) be the projection to the component \([w+2]\) and set \( u^{(k)} := \pi^{(k)}(v^{(k)}) \in [w+2] \subset \mathfrak{c}_g(k) \wedge \mathfrak{c}_g(l) \). We easily see that \( \partial_2(u^{(k)}) = a^{k+1}_1 a_2 \wedge a^{l+1}_1 b_2 \) for any \( k \). Therefore \( K := \text{Ker}(\partial_2)_{[w+2]} \) is generated by \( \{u^{(k)} - u^{(m)} \mid \left\lfloor \frac{w}{2} \right\rfloor \leq k < m \leq w - 1 \} \) as an \( \Lambda \)-module. Then it is enough to show the following:

**Lemma 5.2.** \( \{v^{(k)} - v^{(m)} \mid \left\lfloor \frac{w}{2} \right\rfloor \leq k < m \leq w - 1 \} \subset \text{Im}(\partial_3) \)

**Proof.** Let \( \left\lfloor \frac{w}{2} \right\rfloor \leq k < m \leq w - 1 \). We set

\[
\omega_3 := -a^{k+1}_1 a_2 \wedge a^{m-k}_1 a_2 b_2 \wedge a^{w-m+1}_1 b_2 + a^{k+1}_1 a_3 \wedge a^{m-k}_1 a_2 b_2 \wedge a^{w-m+1}_1 b_3
\]

then obtain

\[
\partial_3(\omega_3) = - \left( a^{m-1}_1 a_2 \wedge a^{w-m+1}_1 b_2 - a^{w+2-m+k}_1 a^{m-k}_1 a_2 b_2 - a^{k+1}_1 a_2 \wedge a^{w-m+1}_1 b_3 \right) \\
- a^{w+2-m+k}_1 a^{m-k}_1 a_2 b_2 \\
= v^{(k)} - v^{(m)}.
\]

(5.6)

Hence \( v^{(k)} - v^{(m)} \in \text{Im}(\partial_3) \). □

We also see that \( K \subset \text{Im}(\partial_3) \) by restricting each \( \omega_3 \) in the proof above to the isotypical component of \( \wedge^3 \mathfrak{c}_g \) corresponding to \([w+2]\).
5.4 The case (III)(i)

Suppose \( \rho \geq 2 \) and \( k - \rho \geq 1 \). We set

\[
\omega_3 := (\rho - 1) \cdot a_1^2 a_4 \wedge a_1^{k-\rho} a_2^b b_4 \wedge a_1^{i+2-\lambda_2-\rho} a_2^{\lambda_2} b_3^2
\]

\[
- \rho \cdot a_1^2 a_4 \wedge a_1^{k-\rho-1} a_2^b b_4 \wedge a_1^{i+3-\lambda_2-\rho} a_2^{\lambda_2-1} b_3^2 \in \epsilon_g(1) \wedge \epsilon_g(k-1) \wedge \epsilon_g(l).
\]

All exponents here are certainly nonnegative by the assumption here. Then

\[
\partial_3(\omega_3) = (\rho - 1) \cdot a_1^{k-\rho+2} a_2^{\rho} \cdot a_1^{i+2-\lambda_2-\rho} a_2^{\lambda_2} b_3^2
\]

\[
- \rho \cdot a_1^{k-\rho+1} a_2^b b_3 \wedge a_1^{i+3-\lambda_2-\rho} a_2^{\lambda_2-1} b_3^2 \in \epsilon_g(k) \wedge \epsilon_g(l).
\]

The second term of \( \partial_3(\omega_3) \) vanishes in the \( \rho \)-th contraction, while the first term survives by Lemma 4.3.

5.5 The case (III)(ii)

Suppose \( k = \rho \geq 2 \). This case is further divided into the following:

(a) the case \( k \geq 4 \),

(b) the case \( k = 3 \) and \( (1) \lambda_2 \geq 1 \), \( (2) \lambda_2 = 0 \),

(c) the case \( k = 2 \).

In the case (a), we set

\[
\omega_3 := a_1^2 a_4 \wedge a_1^{k-2-\lambda_2} a_2^{\lambda_2-1} b_3^2
\]

\[
\partial_3 \rightarrow a_1^2 a_4 \wedge a_1^{k-2-\lambda_2} a_2^{\lambda_2-1} b_3^2 - k^2 \cdot a_1^2 a_4 \wedge a_1^{k-2-\lambda_2} a_2^{\lambda_2-1} b_3^2 \]

\[= 0.\] (5.7)

The second term of (5.7) is an element of \( \epsilon_g(1) \wedge \epsilon_g(k + l - 1) \), which does not contain an Sp-irreducible component corresponding to \( \lambda \) because \( 1 + 2 \neq \rho + \lambda_2 \geq \rho = k \geq 4 \) violates the condition (3.3). On the other hand, the first term of (5.7) is mapped to

\[(\rho^2-2)!((k-\rho+2)!(l+2-\lambda_2-\rho)!(a_1 \wedge a_2)^{\lambda_2} \otimes a_1^{\lambda_1-\lambda_2} (5.8)\]

by \( \Lambda^{\lambda_2}_a \wedge \mu^{\rho}_a \circ \iota \). Therefore \( \partial_3(\omega_3|_{\lambda_2}) \) generates \( \lambda \in \epsilon_g(k) \wedge \epsilon_g(l) \).

In the case (b), \( \omega_3 \) to be defined and its image by \( \partial_3 \) are as below.

\[
\begin{array}{c|ccc}
(1) \lambda_2 & (2) \lambda_2 = 0 & l = 1 & l = 2, 3 \\
\omega_3 & a_1^2 a_4 \wedge a_1^{\lambda_2} b_3^2 & a_1^{i-\lambda_2-1} a_2^{\lambda_2} b_3^2 & a_1^{i+1} a_2 b_3^2 & 2a_1^2 a_4 \wedge a_1^{\lambda_2} b_3^2 \\
\partial_3(\omega_3) & a_1^2 a_4 \wedge a_1^{\lambda_2} b_3^2 & a_1^{i-\lambda_2-1} a_2^{\lambda_2} b_3^2 & a_1^{i+1} a_2 b_3^2 & -3a_1^2 a_4 \wedge a_1^{\lambda_2} b_3^2 \\
& -9a_1^2 a_4 \wedge a_1^{\lambda_2} b_3^2 & a_1^{i-\lambda_2-1} a_2^{\lambda_2} b_3^2 & -9a_1^2 a_4 \wedge a_1^{\lambda_2} b_3^2 & -3a_1^3 a_2 b_3^2 \\
& & & & 2a_1^3 a_2 b_3^2
\end{array}
\]

The second term of \( \partial_3(\omega_3) \) in the case (1) is an element of \( \epsilon_g(1) \wedge \epsilon_g(k + l - 1) \), which does not contain \( \lambda \) as its Sp-irreducible component because \( \rho \geq 4 \). Moreover the second terms of \( \partial_3(\omega_3) \) in the both cases in (2) vanish when they are mapped by \( \mu^{\rho_3}_a \circ \iota \). Since all the first terms in the cases (1) and (2) are mapped to nonzero constant multiples of \( (a_1 \wedge a_2)^{\lambda_2} \otimes a_1^{\lambda_1-\lambda_2} \)

by \( \Lambda^{\lambda_2}_a \wedge \mu^{\rho_3}_a \circ \iota \), the element \( \partial_3(\omega_3|_{\lambda_2}) \) in each case generates \( \lambda \subset \epsilon_g(3) \wedge \epsilon_g(l) \).

Finally we consider the case (c). Since \( k \geq 1 \) and \( k + l = w \geq 4 \), we get \( \rho = k = l = 2 \), so that \( \lambda_1 + \lambda_2 = 4 \). By the condition (3.3), the only possible \( \lambda \) is \( \lambda_1 = 3 \) and \( \lambda_2 = 1 \) because \( \lambda_2 \) must be an odd number. Then

\[
\omega_3 := a_1^2 a_4 \wedge a_1^{\lambda_2} b_3^2 \wedge a_1 a_2 b_3^2 - 2a_1^2 a_4 \wedge a_1 a_2 b_3^2 \wedge a_2 a_3 b_3^2
\]

\[
\partial_3 \rightarrow a_1^2 a_4 \wedge a_1 a_2 b_3^2 - 2a_1^2 a_3 \wedge a_2 a_3 b_3^2
\]

\[
\mu^{\rho_3}_a \circ \iota \rightarrow 4a_1^2 \wedge a_1 a_2 \wedge \Lambda^{\lambda_2}_a \rightarrow 8(a_1 \wedge a_2) \otimes (a_1 \circ a_1) = 16(a_1 \wedge a_2) \otimes a_1^{\rho_3}.
\]
5.6 The case (III)(iii)

Suppose $k - \rho \leq -1$. We argue in two cases: (a) the case $k \geq 3$, (b) the case $k = 2$.

In the case (a), we get

$$k \leq \rho - 1 \leq \rho + \lambda_2 - 1 \leq l + 2 - 1 = l + 1.$$  \hspace{1cm} (5.9)

Since $k \geq l \geq 1$, we see all the possible patterns are

1. $k = l + 1 = \rho - 1$ and $\lambda = [1]$,
2. $k = l = \rho - 2$ and $\lambda = [0]$,
3. $k = l = \rho - 1$ and $\lambda = [2]$,
4. $k = l = \rho - 1$ and $\lambda = [11]$.

Note that $\rho \geq k + 1 \geq 4$ in all cases.

$\omega_3$ to be defined and its image by $\partial_3$ are as below:

| | Case (1) | Case (2) |
|---|---|---|
| $\omega_3$ | $a_3 a_3^{k+1} \wedge a_3 b_3^{k-1} \wedge b_3^2 b_4$ | $a_1^2 a_3 \wedge a_1^2 a_3 \wedge b_3^{k+2}$ |
| $\partial_3(\omega_3)$ | $-a_1 a_3^{k+1} \wedge b_3^{k+1}$ | $a_1^{k+2} \wedge b_3^{k+2}$ |
| | $+(\bigtriangledown \in c_g(2k - 2) \wedge c_g(1))$ | $+(\bigtriangledown \in c_g(1) \wedge (k - 1) \wedge c_g(k + 1))$ |
| | $+(\bigtriangledown \in c_g(k + 1) \wedge c_g(k - 2))$ | $+(\bigtriangledown \in c_g(1) \wedge c_g(2k - 1))$ |

Here $\bigtriangledown$ means some element which we need not to specify.

In all cases, both $c_g(1) \wedge c_g(2k - 1)$ and $c_g(1) \wedge c_g(2k - 2)$ do not contain $\lambda$ as an Sp-irreducible component since $\rho \geq 4$. In the case (1), $\rho = k + 1$ implies $c_g(k + 1) \wedge c_g(k - 2)$ has no $\lambda$ component. In the case (2), $\rho = k + 2 \notin (k - 1) + 2$ implies $c_g(k - 1) \wedge c_g(k + 1)$ has no $\lambda$ component. In the case (4), $\rho + 1 = k + 2 \notin (k - 1) + 2$ implies $c_g(k - 1) \wedge c_g(k + 1)$ has no $\lambda$ component.

In each case of (1), (2) and (4), the remaining term which is specifically written is clearly mapped to a nonzero constant multiple of $a_3$ by $A^{\omega_3}_{\text{end}} \circ \mu^{\omega_3}_{\text{end}} \circ \iota$. This holds even for the case (3) because it is true for the first term of $\partial_3(\omega_3)$, and because the second term of $\partial_3(\omega_3)$ vanishes by the $(k + 1)$-st contraction.

From the above, $\partial_3(\omega_3)$ in each case generates $\lambda \subset c_g(k) \wedge c_g(l)$.

In the case (b), we get $k = l = 2$ and $\rho = 3$. Since $\rho + \lambda_2$ must be an odd number, we also see that $\lambda_1 = 2$ and $\lambda_2 = 0$. We adopt a slightly different approach from the others. Set and see

\[ \zeta_3 := a_3 a_3^{k+1} \wedge a_3 b_3^{k-1} \wedge b_3^2 b_4 - a_3^2 a_4 \wedge b_3^2 \wedge a_1^2 b_4 \in c_g(2) \wedge c_g(1) \wedge c_g(1) \]

\[ \partial_3 \rightarrow 3a_3^2 a_4^{2} a_4 \wedge b_3^2 b_4 - 6a_4 a_3^2 a_4 b_3 \wedge a_4 d_3 - 9a_3^2 a_4 a_4 b_3 \wedge a_1^2 b_4 \]

\[ - a_1 a_3^2 \wedge a_1 b_3^2 + a_1^2 a_3 \wedge b_3^2. \hspace{1cm} (5.10) \]

Set

\[ \eta := 3a_3^2 a_4^{2} a_4 \wedge b_3^2 b_4 - 6a_4 a_3^2 a_4 b_3 \wedge a_4 d_3 - 9a_3^2 a_4 a_4 b_3 \wedge a_1^2 b_4 + (a_1^2 a_3 \wedge b_3^2) \bigl|_{\lambda} \]

\[ \in c_g(3) \wedge c_g(1). \hspace{1cm} (5.11) \]
\[ \partial_2(\eta) = 0 \] is shown by a direct computation. Therefore there exists \( \zeta'_1 \in \wedge^3 \mathcal{C}_g^+ \) such that \( \partial_3(\zeta'_1) = \eta \) from the case \( k = 3 \) and \( l = 1 \), which is done in the argument so far. Hence

\[ \partial_3(\zeta_3 - \zeta'_1) = -a_1a_3^2 \wedge a_1b_3^2 + a_2a_3^2 \wedge b_3^2 - (a_2a_3^2 \wedge b_3^2)|_{\lambda}. \]  
(5.12)

Since \( (a_1^2a_2^2 \wedge b_3^2 - (a_1^2a_2^2 \wedge b_3^2)|_{\lambda}) = 0 \), the element \( \partial_4((\zeta_4 - \zeta'_4)|_{\lambda}) = -(a_1a_3^2 \wedge a_1b_2^2)|_{\lambda} \) generates \( \lambda \subset \mathcal{C}_g(k) \wedge \mathcal{C}_g(l) \).

\section{Lower weight cases}

For \( 1 \leq w \leq 3 \), the weight \( w \) part \( H_2(\mathcal{C}_g^+)_w \) is determined as follows.

**Lemma 6.1.** If \( g \geq 4 \), then \( H_2(\mathcal{C}_g^+)_{1} = 0 \), \( H_2(\mathcal{C}_g^+)_{2} = [51] + [33] + [22] + [11] + [0] \), and \( H_2(\mathcal{C}_g^+)_{3} = [1] \).

**Proof.** \( H_2(\mathcal{C}_g^+)_{1} = 0 \) is obvious because no \( k \geq l \geq 1 \) satisfy \( k + l = 1 \).

Since the weight 2 part of \( \wedge^3 \mathcal{C}_g^+ \) is zero and since \( \partial_2 = [.] : \wedge^2 \mathcal{C}_g(1) \to \mathcal{C}_g(2) = S^4H = [4] \) is surjective, we have \( H_2(\mathcal{C}_g^+)_{2} = \wedge^2 \mathcal{C}_g(1)/\mathcal{C}_g(2) \). The Sp-irreducible decomposition of \( \wedge^2 \mathcal{C}_g(1) \) is \([51] + [33] + [4] + [22] + [11] + [0] \), therefore the statement follows.

The Sp-irreducible decomposition of \( \mathcal{C}_g(2) \otimes \mathcal{C}_g(1) \) is

\[ \mathcal{C}_g(2) \otimes \mathcal{C}_g(1) = [7] + [61] + [52] + [43] + [5] + [41] + [32] + [3] + [21] + [1]. \]  
(6.1)

The space \( \wedge^3 \mathcal{C}_g(1) \) does not have \([5] \) and \([1] \) as its Sp-irreducible components. We use the same method as in the case \( w \geq 4 \) about all the other Sp-irreducible components. It is enough to define \( \omega_3 \) as the following.

\[
\begin{array}{c|cccc}
\omega_3 & [7] & [61] & [52] & [43] \\
\hline
\omega_3 & a_1^2a_4 \wedge a_1^2b_4 \wedge a_1^3 & a_1^2a_4 \wedge a_1^2b_4 \wedge a_1^2a_2 & a_1^2a_4 \wedge a_1^2b_4 \wedge a_1a_2^2 & a_1^2a_4 \wedge a_1^2b_4 \wedge a_2^3 \\
[41] & 0 & 0 & 0 & 0 \\
\omega_3 & a_1^2a_4 \wedge a_1a_3b_4 \wedge a_1a_2b_3 & a_1^2a_4 \wedge a_1a_3b_4 \wedge a_2^3b_3 & -a_1^2a_4 \wedge a_2a_3b_4 \wedge a_2b_3 & -a_1^2a_4 \wedge a_2a_3b_4 \wedge a_1a_2b_3 \\
[32] & 0 & 0 & 0 & 0 \\
\omega_3 & a_1^2a_4 \wedge a_1a_3b_4 \wedge a_1b_3^2 & 0 & 0 & 0 \\
[21] & 0 & 0 & 0 & 0 \\
\end{array}
\]

Again, since \( \partial_2 = [.] : \mathcal{C}_g(2) \wedge \mathcal{C}_g(1) \to \mathcal{C}_g(3) = S^5H = [5] \) is surjective, we have \( H_2(\mathcal{C}_g^+)_{3} = (\mathcal{C}_g(2) \otimes \mathcal{C}_g(1))/\mathcal{C}_g(3) \otimes \text{Im} (\partial_3 : \wedge^3 \mathcal{C}_g(1) \to \mathcal{C}_g(2) \wedge \mathcal{C}_g(1)) = [1]. \)

From [Theorem 1.1](#) and [Lemma 6.1](#) we obtain the following.

**Corollary 6.1.** \( H_2(\mathcal{C}_g^+) = [51] + [33] + [22] + [11] + [1] + [0] \) if \( g \geq 4 \).

Moreover, we know the Sp-irreducible decomposition of \( \wedge^3 \mathcal{C}_g(1) \):

\[
\wedge^3 \mathcal{C}_g(1) = [711] + [631] + [531] + [333] + [7] + [61] + 2[52] + [43] + [421] + [322] + 2[41] + 2[311] + 2[3].
\]  
(6.2)

We have already seen that

\[
\text{Im} (\partial_3|_{\wedge^3 \mathcal{C}_g(1)}) = [7] + [61] + [52] + [43] + [41] + [32] + [3] + [21].
\]  
(6.3)

All the components above are contained in \( \wedge^3 \mathcal{C}_g(1) \). Therefore the space \( H_3(\mathcal{C}_g^+)_{3} = \text{Ker} (\partial_3|_{\wedge^3 \mathcal{C}_g(1)}) \) consists of the remaining components. In other words, the following holds.

**Corollary 6.2.** \( H_3(\mathcal{C}_g^+)_{3} = [711] + [63] + [531] + [333] + [52] + [421] + [322] + [41] + [2][311] + 2[3] \) if \( g \geq 4 \).
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