Algebraic reverse Khovanskii–Teissier inequality via Okounkov bodies

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Abstract
Let \(X\) be a projective variety of dimension \(n\) over an algebraically closed field of arbitrary characteristic and let \(A, B, C\) be nef divisors on \(X\). We show that for any integer \(1 \leq k \leq n-1\),

\[
(B^k \cdot A^{n-k}) \cdot (A^k \cdot C^{n-k}) \geq \frac{k!(n-k)!}{n!} (A^n) \cdot (B^k \cdot C^{n-k}).
\]

The same inequality in the analytic setting was obtained by Lehmann and Xiao for compact Kähler manifolds using the Calabi–Yau theorem, while our approach is purely algebraic using (multipoint) Okounkov bodies. We also discuss applications of this inequality to Bézout-type inequalities and inequalities on degrees of dominant rational self-maps.

Keywords Reverse Khovanskii–Teissier inequality · Okounkov bodies · Intersection numbers · Volumes

Mathematics Subject Classification 14C20 · 14M25 · 14C17

1 Introduction

In [23], Lehmann and Xiao have proved the so called reverse Khovanskii–Teissier inequality on compact Kähler manifolds. Namely, for nef \((1, 1)\)-classes \(\alpha, \beta\) and \(\gamma\) on a compact Kähler manifold of dimension \(n\), we have

\[
(\beta^k \cdot \alpha^{n-k}) \cdot (\alpha^k \cdot \gamma^{n-k}) \geq \frac{k!(n-k)!}{n!} (\alpha^n) \cdot (\beta^k \cdot \gamma^{n-k}).
\]  

(1.1)

In fact such an inequality was first observed by Xiao [30] with a weaker constant \(\frac{k!(n-k)!}{4n!}\) which can be improved to \(\frac{k!(n-k)!}{n!}\) by the technique of Popovici [26] (cf. [30, Remark 3.1]). Such
an inequality between intersection numbers plays an important role in the proof of Morse type inequalities (see also [26, 30]) and it also has interesting applications to dynamical degrees of rational maps [1, 7]. Besides, it has an interesting analogue in convex geometry [23, Theorem 5.9] and applications in the study of convolution of convex valuations [8, Theorem 2.9].

The proof of Popovici and Lehmann–Xiao depends on solving Monge–Ampère equations which deeply relies on the Calabi–Yau theorem, and it has been asked in [21, Remark 9.3] (the arXiv version of [22]) whether there is an algebraic approach working for projective varieties defined over arbitrary fields. In [7, Theorem 3.4.3], by studying intersections of numerical cycles, Dang has proved a weaker form of (1.1) in the algebraic geometry setting with the constant \( \frac{k!(n-k)!}{n!} \) replacing by \( \frac{1}{(n-k+1)^k} \).

In this paper, we use the (multipoint) Okounkov bodies to prove this optimal inequality in the algebraic geometry setting. The main result is

**Theorem 1.1** Let \( X \) be a projective variety of dimension \( n \) over an algebraically closed field of arbitrary characteristic. Let \( A, B, C \) be nef divisors on \( X \). Then for any integer \( 1 \leq k \leq n-1 \),

\[
(B^k \cdot A^{n-k}) \cdot (A^k \cdot C^{n-k}) \geq \frac{k!(n-k)!}{n!} (A^n) \cdot (B^k \cdot C^{n-k}).
\]

Moreover, if \( A, B, C \) are ample, then this inequality is strict.

We shall mention that the constant \( \frac{k!(n-k)!}{n!} \) in (1.2) is optimal. For instance, if \( X = \mathbb{P}^k \times \mathbb{P}^{n-k} \), one can take \( A, B, C \) to be the divisors on \( X \) of type \((1,1),(1,0),(0,1)\), respectively, then the equality holds in (1.2). It is interesting and natural to ask for the characterization of the equality case of (1.2), but the answer might be very complicated. One naive guess is that if the equality holds while both sides are non-zero, then \( B \) and \( C \) have numerical dimensions \( k \) and \( n-k \) respectively and \( A \) lies on the plane spanned by \( B \) and \( C \) in the Néron–Severi group \( \text{NS}(X) \otimes \mathbb{Z} \mathbb{R} \). We will give such a characterization when \( X \) is a surface in Proposition 5.1.

Our proof makes use of the connection between volumes of big divisors and volumes of (multipoint) Okounkov bodies discovered in [15, 20, 29]. The desired inequality follows from a dedicated comparison between the Okounkov bodies associated to an admissible flag and the convex bodies associated to sub-flags obtained by cutting the admissible flag with very general hyperplanes. In this way, we actually obtain a more general result for restricted volumes, see Theorems 3.4, 3.5, and 3.8 for details. Another advantage of this method is that we only need to deal with the intersection theory of divisors, which is much easier to handle than that of algebraic cycles (cf. [7]). Very recently, Hu and Xiao [12] give a purely combinatorial proof of Theorem 1.1.

As a consequence, we can get a Bézout-type inequality which was proved by Xiao [31, Theorem 1.6] when the base field is \( \mathbb{C} \), generalizing the classical Bézout theorem for hypersurfaces. Here we get a better constant than [31, Theorem 1.6].

**Corollary 1.2** (cf. [31, Theorem 1.6]) Let \( X \) be a projective variety with an ample divisor \( H \) as polarization and let \( A_1, \ldots, A_r \) be nef divisor classes on \( X \). Assume that \( a_1, \ldots, a_r \in \mathbb{Z}_{>0} \) and \( |a| := \sum_{i=1}^{r} a_i \leq n \). Let \( Y_1, \ldots, Y_r \) be subvarieties of cycle classes \( A_1^{a_1}, \ldots, A_r^{a_r} \), and assume that they have proper intersection, then

\[
\deg_H(Y_1 \cap \cdots \cap Y_r) \leq \min_k \left\{ \frac{\prod_{i=1}^{r} \binom{|a_i|}{a_i}}{(H^n)^{r-1}} \frac{k!(n-k)!}{n!} \right\} \prod_{i=1}^{r} \deg_H(Y_i).
\]
There is another application on degrees of the iterations of rational self-maps on projective varieties. Let \( f: X \rightarrow X \) be a dominant rational self-map of a projective variety \( X \) of dimension \( n \) defined over an algebraically closed field of arbitrary characteristic and let \( H \) be a nef and big divisor on \( X \). For any integer \( 0 \leq i \leq n \), the \( i \)-th degree of \( f \) with respect to \( H \) is defined by

\[
\deg_{i,H}(f) = (\pi_1^*H^{n-i} \cdot \pi_2^*H^i),
\]

where \( \pi_1 \) and \( \pi_2 \) are the projections from the normalization of the graph of \( f \) in \( X \times X \) onto the first and the second factor, respectively. In [7], Dang proved a weaker form of (1.2) in order to study the sequence of intermediate degrees of the iterations of a dominant rational self-map and to recover the results in [1, 9, 28]. As an application of Theorem 1.1, we give a new and simple proof of [7, Theorem 1] with better constants.

**Corollary 1.3** (cf. [7, Theorem 1]) Let \( X \) be a projective variety of dimension \( n \) and let \( H \) be a nef and big divisor on \( X \). Fix an integer \( 0 \leq i \leq n \).

1. For any dominant rational self-maps \( f, g \) on \( X \),

\[
\deg_{i,H}(f \circ g) \leq \left( \frac{n}{H^n}\right) \deg_{i,H}(f) \cdot \deg_{i,H}(g).
\]

2. For any nef and big divisor \( L \) on \( X \) and any dominant rational self-maps \( f \) on \( X \),

\[
\deg_{i,H}(f) \leq \left( \frac{n}{L^n}\right)^2 (H^{n-i} \cdot L^i) \cdot (L^{n-i} \cdot H^i) \cdot \deg_{i,L}(f).
\]

This result is essential in the definition of the dynamical degree of a self-rational map (cf. [1, 7, 9, 29]).

This paper is organized as the following. In Sect. 2, we introduce definitions and basic knowledge on (multipoint) Okounkov bodies. In Sect. 3, we develop properties of Okounkov bodies and prove Theorem 1.1 along with several general statements. In Sect. 4, we give the applications of the main theorem. In Sect. 5, we give a characterization of the equality case of Theorem 1.1 on surfaces.

### 2 Preliminaries

#### 2.1 Notation and conventions

We work over an algebraically closed field of arbitrary characteristic. We adopt the standard notation and definitions in [18–20]. A variety is reduced and irreducible. A divisor on a projective variety always means a Cartier divisor. When the base field is uncountable, a property holds for a very general choice of data if it is satisfied away from a countable union of proper closed subvarieties of the relevant parameter space.

#### 2.2 Volumes

Let \( X \) be a projective variety of dimension \( n \) and let \( D \) be a divisor on \( X \). The volume of \( D \) is the real number

\[
\vol_X(D) = \lim_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.
\]
We say that $D$ is big if $\text{vol}_X(D) > 0$. For more details and properties of volumes, we refer to [18, 2.2.C] and [19, 11.4.A]. By the homogeneity of volumes, this definition can be extended to $\mathbb{Q}$-divisors. Note that if $D$ is a nef divisor, then $\text{vol}_X(D) = (D^n)$.

### 2.3 Restricted volumes

We recall the notation in [20, §2.4]. Let $V$ be a projective variety and let $D$ be a big divisor on $V$. The augmented base locus $B_+(D) \subset V$ is defined to be $B_+(D) = B(D - A)$ for any sufficiently small ample $\mathbb{Q}$-divisor $A$, where $B(D - A)$ is the stable base locus of $D - A$. Let $X$ be an irreducible closed subvariety of $V$ of dimension $n$, then the restricted volume of $D$ from $V$ to $X$ is defined by

$$
\text{vol}_{V|X}(D) = \lim_{m \to \infty} \frac{\dim \left( \text{Im} \left( H^0(V, mD) \xrightarrow{\text{restr}} H^0(X, mD|_X) \right) \right)}{m^n / n!}.
$$

For a sufficiently divisible integer $m > 0$, consider $\pi_m : V_m \to V$ to be the blowing-up of $V$ along the base ideal of $|mD|$, then we have

$$
\pi_m^* |mD| = |M_m| + E_m
$$

where $M_m$ is free, and $E_m$ is the fixed part. If $X \not\subset B_+(D)$, the asymptotic intersection number of $D$ and $X$ is defined to be

$$
\|D^n \cdot X\| = \limsup_{m \to \infty} \frac{(M_m^n \cdot X_m)}{m^n},
$$

where $X_m$ is the strict transform of $X$ on $V_m$ (cf. [10, Definition 2.6]). Recall that we have the following Fujita’s approximation theorem of restricted volumes.

**Theorem 2.1** ([10, Theorem 2.13], [20, Remark 3.6]) Let $V$ be a projective variety and let $D$ be a big divisor on $V$. Let $X$ be an irreducible closed subvariety of $V$ of dimension $n$ such that $X \not\subset B_+(D)$. Then

$$
\text{vol}_{V|X}(D) = \|D^n \cdot X\|.
$$

### 2.4 Okounkov bodies

Okounkov bodies of big divisors were introduced in [15, 20] motivated by earlier works of Okounkov [24, 25]. There have been many interesting applications of Okounkov bodies in the study of geometric properties of divisors, for example, [2–6, 13, 14, 16, 17, 27].

We recall the definition of Okounkov bodies from [20]. Let $X$ be a projective variety of dimension $n$. Consider an admissible flag $X_\bullet$ on $X$

$$
X_\bullet : X = X_0 \supsetneq X_1 \supsetneq \cdots \supsetneq X_{n-1} \supsetneq X_n = \{x\}
$$

where each $X_i$ is an irreducible closed subvariety of $X$ which is non-singular at the point $x$ and codim $X_i = i$. For a big divisor $D$ on $X$, we consider the $\mathbb{Q}$-linear system

$$
|D|_\mathbb{Q} = \{D' \mid D \sim_\mathbb{Q} D' \geq 0\}
$$

and a valuation-like function

$$
\nu_{X_\bullet} : |D|_\mathbb{Q} \to \mathbb{R}_{\geq 0},
$$

$$
D' \mapsto \nu_{X_\bullet}(D') = (\nu_1, \nu_2, \ldots, \nu_n),
$$

where $\nu_i$ is the valuation at $X_i$.
where $v_i$ are defined inductively as follows:

1. define $v_1 := \mult_{X_1} D' \in \mathbb{Q}_{\geq 0}$ and $D'_1 := D' - v_1 X_1$ on $X_0$, and inductively,
2. assuming that $v_i \in \mathbb{Q}_{\geq 0}$ and $D'_i$ on $X_{i-1}$ are defined, then define $v_{i+1} := \mult_{X_{i+1}} (D'_i |_{X_i}) \in \mathbb{Q}_{\geq 0}$ and $D'_{i+1} = D'_i |_{X_i} - v_{i+1} X_{i+1}$ on $X_i$.

Here we remark that as $X_{i-1}$ is non-singular at $x$, $D'_i |_{X_i}$ is a well-defined $\mathbb{Q}$-divisor in a neighborhood of $x$ and $\mult_{X_{i+1}} (D'_i |_{X_i})$ can be well-defined. The Okounkov body of $D$ with respect to $X_\bullet$ is defined as

$$\Delta_{X_\bullet}(D) := \text{the convex closure of } v_{X_\bullet}(|D|_{\mathbb{Q}}) \text{ in } \mathbb{R}_\geq 0.$$ 

Here the definition is equivalent to [20, Definition 1.8] but we use $\mathbb{Q}$-linear systems instead of global sections as in [20] to make the notation simpler. Such a formulation appears for example in [2]. By [20, Theorem A], we have

$$\text{vol}_{\mathbb{R}^n} (\Delta_{X_\bullet}(D)) = \frac{1}{n!} \text{vol}_X(D) \tag{2.1}$$

for every admissible flag $X_\bullet$ on $X$.

### 2.5 Multipoint Okounkov bodies

In this paper, we will study the property of the Okounkov body of a given admissible flag cutting by very general hyperplanes (see Proposition 3.2). For example, given an admissible flag $X_\bullet$ and a very general hyperplane $H$ on $X$, in order to naturally define an admissible flag $H_\bullet$, we need to pick a closed point in $X_{n-1} \cap H$. But in this way we lost the information of other points in $X_{n-1} \cap H$. So the natural idea is to consider the whole set of points $X_{n-1} \cap H$ instead of just picking one. To this end, we need to extend the definition of admissible flags and Okounkov bodies to multipoint admissible flags and multipoint Okounkov bodies.

Now we recall the definition of multipoint Okounkov bodies in [29]. For our purpose, we only introduce a special case. Let $Z$ be a projective variety of dimension $k$. Consider a multipoint admissible flag $Z_\bullet$ on $Z$

$$Z_\bullet : Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_{k-1} \supseteq Z_k = \{p_1, \ldots, p_N\}$$

where $Z_k$ consists of $N$ distinct points and for $0 \leq i < k$, each $Z_i$ is an irreducible closed subvariety of $Z$ which is non-singular at the points $p_1, \ldots, p_N$ and codim $Z_i = i$. For $1 \leq j \leq N$, denote by $Z_\bullet(p_j)$ the admissible flag

$$Z = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_{k-1} \supseteq \{p_j\}.$$

Then for a big divisor $D$ on $Z$, we can consider the functions $v_{Z_\bullet(p_j)}(D') (1 \leq j \leq N)$ for $D' \in |D|_{\mathbb{Q}}$. Note that $v_{Z_\bullet(p_j)}(D')$ only differs on the last coordinate in $\mathbb{R}^k$. So the lexicographical order of $\{v_{Z_\bullet(p_j)}(D') \mid 1 \leq j \leq N\}$ is just the order of the last coordinates. We define the subset $V_j(D) \subseteq |D|_{\mathbb{Q}}$ by

$$V_j(D) = \{D' \in |D|_{\mathbb{Q}} \mid v_{Z_\bullet(p_j)}(D') < v_{Z_\bullet(p_i)}(D') \text{ for all } i \neq j\}. \tag{2.2}$$

The multipoint Okounkov body of $D$ with respect to $Z_\bullet$ and $p_j$ is defined in [29, Definition 3.4] as

$$\Delta_{Z_\bullet,j}(D) := \text{the convex closure of } v_{Z_\bullet(p_j)}(V_j(D)) \text{ in } \mathbb{R}_\geq 0.$$
By [29, Theorem 1.2], we have
\[ \sum_{j=1}^{N} \operatorname{vol}_{R^k}(\Delta_{Z_{\cdot},j}(D)) = \frac{1}{k!} \operatorname{vol}_{Z}(D) \] (2.3)
for every multipoint admissible flag $Z_{\cdot}$ on $Z$.

By the homogeneity (cf. [29, Proposition 3.10]), the definition of (multipoint) Okounkov bodies can be extended to big $\mathbb{Q}$-divisors. In particular, (2.1) and (2.3) hold for big $\mathbb{Q}$-divisors.

**Remark 2.2** In [29], it has been assumed that the base field is $\mathbb{C}$ and $Z$ is smooth. But as the proof of [29, Theorem 1.2] follows the line of [20, Theorem A], it is not hard to check that [29, Theorem 1.2] holds for any projective variety over any algebraically closed field. Here as in [20, Remark 3.7], we do not need to assume that the base field is uncountable.

### 2.6 Cutting a divisor by general hyperplanes

We will use the following lemma which is a direct consequence of Bertini’s theorem.

**Lemma 2.3** Let $X$ be a projective variety. Let $D$ be a $\mathbb{Q}$-divisor on $X$ and let $P$ be a prime divisor whose generic point lies in the smooth locus of $X$. Then for a general very ample divisor $H$ on $X$, the following statements hold:

1. if $\dim X > 2$, then $P|_H$ is a prime divisor whose generic point lies in the smooth locus of $H$;
2. if $\dim X = 2$, then $P|_H$ is a reduced divisor (consisting of points) lies in the smooth locus of $H$;
3. $\operatorname{mult}_P D = \operatorname{mult}_{P'} D|_H$ for each irreducible component $P'$ of $P|_H$.

**Proof** (1) and (2) follows from Bertini’s theorem [11, Theorem II.8.18]. To get (3), we just need to choose $H$ general so that

- for each irreducible component $P_1$ of $D$, $P_1|_H$ is reduced, and
- for irreducible components $P_1$ and $P_2$ of $D$, $P_1|_H$ and $P_2|_H$ has no common irreducible component.

This is again by Bertini’s theorem. $\square$.

### 3 Inequalities between volumes via Okounkov bodies

#### 3.1 A comparision result on Okounkov bodies

We define $\operatorname{pr}_{\geq k} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ to be the projection of the last $n-k$ coordinates and $\operatorname{pr}_{\leq k} : \mathbb{R}^n \to \mathbb{R}^k$ to be the projection of the first $k$ coordinates.

The following lemma gives a comparison of the Okounkov bodies of a certain admissible flag and its sub-flag via the natural projections.

**Lemma 3.1** Let $X$ be a projective variety of dimension $n$, let $D$ be a big divisor on $X$, and let $X_{\cdot}$ be an admissible flag. Fix an integer $1 \leq k \leq n$. Denote $Y = X_k$ and $Y_{\cdot} = X_{k+\cdot}$. Suppose that $D|_Y$ is big and $\mathcal{O}_{X_{i-1}}(X_i)|_Y$ is a semiample line bundle on $Y$ for $1 \leq i \leq k$. Then

$$\operatorname{pr}_{\geq k}(\Delta_{X_{\cdot}}(D)) \subset \Delta_{Y_{\cdot}}(D|_Y).$$
**Proof** Fix any $D' \in |D|_{Q}$. Suppose that $\nu_{X_{\bullet}}(D') = (v_{1}, \ldots, v_{n})$. Recall that for $0 \leq i \leq n - 1$ we define by induction that $v_{i+1} = \text{mult}_{X_{i+1}} D'_{|X_{i}}$ and $D'_{i+1} = D'_{|X_{i}} - v_{i+1}X_{i+1}$. Then by definition $\nu_{Y_{\bullet}}(D'_{k}|y) = (v_{k+1}, \ldots, v_{n})$. By construction,

$$D'_{k}|y \sim_{Q} D|y \sim \sum_{i=1}^{k} v_{i}X_{i}|y,$$

where we view $X_{i}|y$ as a divisor defined by the line bundle $\mathcal{O}_{X_{i-1}}(X_{i})|y$. So by our assumption, $D|y - D'_{k}|y$ is semiample. Therefore, we can find $\bar{D} \in |D|_{Q}$ such that $\nu_{Y_{\bullet}}(\bar{D}) = \nu_{Y_{\bullet}}(D'_{k}|y) = (v_{k+1}, \ldots, v_{n})$. This concludes the desired inclusion. \hfill \Box

The following proposition describes the behavior of the Okounkov body of an admissible flag when cutting by very general hyperplanes.

**Proposition 3.2** Let $X$ be a projective variety of dimension $n$ over an uncountable algebraically closed field, let $D$ be a big divisor on $X$, and let $X_{\bullet}$ be an admissible flag. Fix an integer $1 \leq k \leq n - 1$. For very general very ample divisors $H_{1}, \ldots, H_{n-k}$ on $X$, denote $Z = H_{1} \cap \cdots \cap H_{n-k}$, then there is a natural multipoint admissible flag $Z_{\bullet}$ on $Z$ given by $Z_{i} = X_{i} \cap Z$ for $0 \leq i \leq k - 1$, and $Z_{k} = X_{k} \cap Z = \{p_{1}, \ldots, p_{N}\}$. Then

$$\text{pr}_{\leq k}(\Delta_{X_{\bullet}}(D)) \subset \bigcap_{m \in \mathbb{Z}_{>0}} \Delta_{Z_{\bullet},j}(D|z + \frac{1}{m}A_{Z})$$

for all $1 \leq j \leq N$ and any ample divisor $A_{Z}$ on $Z$.

**Remark 3.3** By [29, Proof of Theorem 1.2], if $\Delta_{Z_{\bullet},j}(D|z)^{\circ} \neq \emptyset$, then

$$\bigcap_{m \in \mathbb{Z}_{>0}} \Delta_{Z_{\bullet},j}(D|z + \frac{1}{m}A_{Z}) = \Delta_{Z_{\bullet},j}(D|z).$$

But we do not need this fact in this paper.

**Proof** By definition, the set $\nu_{X_{\bullet}}(|D|_{Q})$ lies in $\mathbb{Q}^{n}_{>0}$, which is a countable set. So there exists a countable set $S \subset |D|_{Q}$ such that $\Delta_{X_{\bullet}}(D)$ is the convex closure of $\{\nu_{X_{\bullet}}(D') \mid D' \in S\}$. It suffices to show that for all $D' \in S \subset |D|_{Q}$ and for very general very ample divisors $H_{1}, \ldots, H_{n-k}$ on $X$, we have

$$\text{pr}_{\leq k}(\nu_{X_{\bullet}}(D')) \in \Delta_{Z_{\bullet},j}(D|z + \frac{1}{m}A_{Z})$$

(3.1)

for all $m \in \mathbb{Z}_{>0}$.

In fact, it suffices to show that (3.1) holds for a single $D' \in S$ and for very general very ample divisors $H_{1}, \ldots, H_{n-k}$ on $X$. More precisely, for fixed very ample linear systems $\mathcal{L}_{1}, \ldots, \mathcal{L}_{n-k}$ on $X$, if (3.1) holds for every $D' \in S$ and every $H_{i} \in \mathcal{L}_{i}|Z_{i}, D'_{i}$ where $Z_{i}, D'_{i} \subset \mathcal{L}_{i}$ is a countable union of proper closed subset (depending on $D'$) for each $i$, then (3.1) holds for every $D' \in S$ and every $H_{i} \in \mathcal{L}_{i}| \bigcup_{D' \in S} Z_{i}, D'_{i}$. Here $\bigcup_{D' \in S} Z_{i}, D'_{i}$ is again a countable union of proper closed subset of $\mathcal{L}_{i}$ as $S$ is countable.

Now we consider a fixed $D' \in S$. Suppose that $\nu_{X_{\bullet}}(D') = (v_{1}, \ldots, v_{n})$. Recall that for $0 \leq i \leq n - 1$ we define by induction that $v_{i+1} = \text{mult}_{X_{i+1}} D'_{i}|X_{i}$ and $D'_{i+1} = D'_{i}|X_{i} - v_{i+1}X_{i+1}$. By applying Lemma 2.3 inductively, by taking $H_{1}, \ldots, H_{n-k}$ general, we get that

- $v_{i+1} = \text{mult}_{Z_{i+1}} D'_{i}|Z_{i}$ and $D'_{i+1}|Z_{i} = D'_{i}|Z_{i} - v_{i+1}Z_{i+1}$ for $0 \leq i \leq k - 2$;
\( v_k = \text{mult}_{p_s} D'_{k-1}|Z_{k-1} \) for all \( 1 \leq s \leq N \).

As \( A_Z \) is ample, we can find \( A'_Z \in |AZ|_Q \) whose support is very ample which does not contain \( p_j \) and \( Z_{k-1} \) but contains \( p_s \) for all \( s \neq j \), in other words,

- \( \text{mult}_{p_j} A'_Z = \text{mult}_{Z_{k-1}} A'_Z = 0 \), and
- \( \text{mult}_{p_s} A'_Z = \mu_s > 0 \) for all \( s \neq j \).

Then from the construction,

\[
v_{Z*}(p_j) \left( D'|Z + \frac{1}{m} A'_Z \right) = (v_1, \ldots, v_k)
\]

and

\[
v_{Z*}(p_s) \left( D'|Z + \frac{1}{m} A'_Z \right) = \left( v_1, \ldots, v_{k-1}, v_k + \frac{1}{m} \mu_s \right)
\]

for \( s \neq j \). Then we have \( D'|Z + \frac{1}{m} A'_Z \in V_j(D|Z + \frac{1}{m} A_Z) \) as defined in (2.2) and hence \( (v_1, \ldots, v_k) \in \Delta_{*,j}(D|Z + \frac{1}{m} A_Z) \).

3.2 The generalized reverse Khovanskii–Teissier inequality

As the volume of Okounkov body can compute the volume of big divisors, the comparison result in Sect. 3.1 will yield a stronger version of the reverse Khovanskii–Teissier inequality as follows.

**Theorem 3.4** Let \( X \) be a projective variety of dimension \( n \) over an uncountable algebraically closed field. Fix an integer \( 1 \leq k \leq n - 1 \). Let \( D \) be a big divisor on \( X \) and let \( B_1, \ldots, B_k, C_1, \ldots, C_{n-k} \) be very general very ample divisors on \( X \). Denote \( Y = B_1 \cap \cdots \cap B_k \) and \( Z = C_1 \cap \cdots \cap C_{n-k} \). Then

\[
\text{vol}_Y(D|Y) \cdot \text{vol}_Z(D|Z) \geq \frac{k!(n-k)!}{n!} \text{vol}_X(D) \cdot (Y \cdot Z).
\]

**Proof** It is easy to construct an admissible flag \( X_\bullet \) such that \( X_i = B_1 \cap \cdots \cap B_i \) for any \( 1 \leq i \leq k \). By applying Lemma 3.1, we get

\[
\text{pr}_{\geq k}(\Delta_{X_\bullet}(D)) \subset \Delta_{Y_\bullet}(D|Y).
\]

By applying Proposition 3.2 to the case \( H_i = C_i \) for \( 1 \leq i \leq n - k \), we get

\[
\text{pr}_{\geq k}(\Delta_{X_\bullet}(D)) \subset \bigcap_{m \in \mathbb{Z} \geq 0} \Delta_{Z_{\bullet,j}} \left( D|Z + \frac{1}{m} A_Z \right)
\]

for all \( 1 \leq j \leq N \), where \( Z = C_1 \cap \cdots \cap C_{n-k} \), \( N = (X_k \cdot Z) = (Y \cdot Z) \), and \( A_Z \) is an ample divisor on \( Z \). So

\[
\Delta_{X_\bullet}(D) \subset \Delta_{Z_{\bullet,j}} \left( D|Z + \frac{1}{m} A_Z \right) \times \Delta_{Y_\bullet}(D|Y) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}
\]
for all \( 1 \leq j \leq N \) and all \( m \in \mathbb{Z}_{>0} \). Combining with (2.1) and (2.3), this yields
\[
\frac{1}{n!} (Y \cdot Z) \cdot \text{vol}_X(D) = N \text{vol}_{\mathbb{R}^n}(\Delta_{X*}(D))
\]
\[
\leq \sum_{j=1}^{N} \text{vol}_{\mathbb{R}^k} \left( \Delta_{Z*} \left( D|_Z + \frac{1}{m} A_Z \right) \right) \cdot \text{vol}_{\mathbb{R}^{n-k}}(\Delta_{Y*}(D|_Y))
\]
\[
= \frac{1}{k!} \text{vol}_Z \left( D|_Z + \frac{1}{m} A_Z \right) \cdot \frac{1}{(n-k)!} \text{vol}_Y(D|_Y).
\]
We can conclude the assertion by taking \( m \to \infty \), as \( \text{vol}_Z \) is a continuous function for big \( \mathbb{Q} \)-divisors by [20, Corollary 4.12].

As a consequence, we can obtain a general form of Theorem 1.1.

**Theorem 3.5** Let \( X \) be a projective variety of dimension \( n \). Fix an integer \( 1 \leq k \leq n - 1 \). Let \( A, B_1, \ldots, B_k, C_1, \ldots, C_{n-k} \) be nef divisors on \( X \). Then
\[
(B_1 \cdots B_k \cdot A^{n-k}) \cdot (A^k \cdot C_1 \cdots C_{n-k})
\]
\[
\geq \frac{k!(n-k)!}{n!} (A^n) \cdot (B_1 \cdots B_k \cdot C_1 \cdots C_{n-k}).
\]
(3.2)
Moreover, if \( A, B_1, \ldots, B_k, C_1, \ldots, C_{n-k} \) are ample, then this inequality is strict.

**Proof** After base change we may assume that the base field is uncountable. As the inequality is homogeneous and nef divisors are the limits of ample \( \mathbb{Q} \)-divisors in the Néron–Severi group \( \text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R} \), it suffices to prove the inequality (3.2) for very ample divisors \( A, B_1, \ldots, B_k, C_1, \ldots, C_{n-k} \) on \( X \). Then the inequality (3.2) follows from Theorem 3.4 by taking \( D = A \).

For the last statement, suppose that \( A, B_1, \ldots, B_k, C_1, \ldots, C_{n-k} \) are ample. Without loss of generality, we may assume that \( A \) is very ample and is a subvariety of \( X \). Assume to the contrary that the equality holds. Note that for any sufficiently small \( t > 0 \), \( B_k - tA \) is ample. So applying (3.2) to \( A, B_1, \ldots, B_{k-1}, B_k - tA, C_1, \ldots, C_{n-k} \), we get
\[
(B_1 \cdots B_{k-1} \cdot (B_k - tA) \cdot A^{n-k}) \cdot (A^k \cdot C_1 \cdots C_{n-k})
\]
\[
\geq \frac{k!(n-k)!}{n!} (A^n) \cdot (B_1 \cdots B_{k-1} \cdot (B_k - tA) \cdot C_1 \cdots C_{n-k}).
\]
By the assumption, this implies that
\[
(B_1 \cdots B_{k-1} \cdot A^{n-k+1}) \cdot (A^k \cdot C_1 \cdots C_{n-k})
\]
\[
\leq \frac{k!(n-k)!}{n!} (A^n) \cdot (B_1 \cdots B_{k-1} \cdot A \cdot C_1 \cdots C_{n-k}).
\]
(3.3)
On the other hand, applying (3.2) to \( A|_A, B_1|_A, \ldots, B_{k-1}|_A, C_1|_A, \ldots, C_{n-k}|_A \) on the variety \( A \), we get
\[
(B_1 \cdots B_{k-1} \cdot A^{n-k+1}) \cdot (A^k \cdot C_1 \cdots C_{n-k})
\]
\[
\geq \frac{(k-1)!(n-k)!}{(n-1)!} (A^n) \cdot (B_1 \cdots B_{k-1} \cdot A \cdot C_1 \cdots C_{n-k}).
\]
(3.4)
Here we remark that we need \( k > 1 \) in order to apply (3.2), but inequality (3.4) holds trivially if \( k = 1 \). Then (3.3) and (3.4) yields
\[(A^n) \cdot (B_1 \cdot \cdots \cdot B_{k-1} \cdot A \cdot C_1 \cdot \cdots \cdot C_{n-k}) \leq 0,\]
a contradiction. \(\square\)

**Remark 3.6** From the proof of Theorem 3.5, it is easy to see that the inequality (3.2) is strict as long as \(A, B_k\) are ample and \((B_1 \cdot \cdots \cdot B_{k-1} \cdot A \cdot C_1 \cdot \cdots \cdot C_{n-k}) > 0\).

**Remark 3.7** Although we deduce Theorem 3.5 from Theorem 3.4, it is not hard to see that Theorem 3.5 implies Theorem 3.4 conversely, as the volume of a big divisor can be approximated by ample divisors by Fujita’s approximation theorem (cf. [20, §3.1] or Theorem 2.1). This fact was pointed out by Jian Xiao.

### 3.3 Proof of Theorem 1.1

It is just a special case of Theorem 3.5.

### 3.4 Further discussions on restricted volumes

In this subsection we will discuss the generalization of Theorem 3.4 to restricted volumes, which was suggested by Jian Xiao.

**Theorem 3.8** Let \(V\) be a projective variety over an uncountable algebraically closed field and let \(D\) be a big divisor on \(V\). Let \(X\) be an irreducible closed subvariety of \(V\) of dimension \(n\) such that \(X \not\subset B_1 + (D)\). Fix an integer \(1 \leq k \leq n-1\). Let \(B_1, \ldots, B_k, C_1, \ldots, C_{n-k}\) be very general very ample divisors on \(V\). Denote \(Y = B_1 \cap \cdots \cap B_k \cap X\) and \(Z = C_1 \cap \cdots \cap C_{n-k} \cap X\).

Then
\[
\text{vol}_V|Y| (D) \cdot \text{vol}_V|Z| (D) \geq \frac{k!(n-k)!}{n!} \text{vol}_V|X| (D) \cdot (Y \cdot Z)_X.
\]

**Proof** For any sufficiently divisible integer \(m > 0\), consider \(\pi_m : V_m \to V\) to be the blowing-up of \(V\) along the base ideal of \([mD]\), then we have
\[
\pi_m^* [mD] = |M_m| + E_m
\]
where \(M_m\) is free, and \(E_m\) is the fixed part. Denote by \(X_m\) the strict transform of \(X\) on \(V_m\). By taking \(B_1, \ldots, B_k, C_1, \ldots, C_{n-k}\) very general, we may assume that \(Y, Z \not\subset B_1 + (D)\) and for every sufficiently divisible integer \(m > 0\), \(Y_m = (\pi_m^* B_1, \ldots, \pi_m^* B_k) \cdot X_m\) and \(Z_m = (\pi_m^* C_1, \ldots, \pi_m^* C_{n-k} \cdot X_m)\) as cycles, where \(Y_m, Z_m\) are the strict transforms of \(Y, Z\) on \(V_m\) respectively. Then by applying Theorem 3.5 to
\[
M_m|X_m|, \pi_m^* B_1|X_m|, \ldots, \pi_m^* B_k|X_m|, \pi_m^* C_1|X_m|, \ldots, \pi_m^* C_{n-k}|X_m|
\]
on \(X_m\), we get
\[
\left(\pi_m^* B_1 \cdot \cdots \cdot \pi_m^* B_k \cdot M_m^{n-k} \cdot X_m\right) \cdot \left(M_m^k \cdot \pi_m^* C_1 \cdot \cdots \cdot \pi_m^* C_{n-k} \cdot X_m\right) \geq \frac{k!(n-k)!}{n!} (M_m^n \cdot X_m) \cdot (\pi_m^* B_1 \cdot \cdots \cdot \pi_m^* B_k \cdot \pi_m^* C_1 \cdot \cdots \cdot \pi_m^* C_{n-k} \cdot X_m).
\]
By taking \(m \to \infty\), we get
\[
\|D^k \cdot Y\| \cdot \|D^{n-k} \cdot Z\| \geq \frac{k!(n-k)!}{n!} \|D^n \cdot X\| \cdot (Y \cdot Z)_X.
\]
This concludes the desired inequality by Theorem 2.1. \(\square\)
4 Applications

4.1 Proof of Corollary 1.2

It is equivalent to showing that
\[(A_1^{a_1} \cdots A_r^{a_r} \cdot H^{n-|a|}) \leq \min_k \left( \prod_{i=1}^r \left( \frac{|a|}{a_i} \right) (H^n)^{r-1} \right) \prod_{i=1}^r (A_i^{a_i} \cdot H^{n-a_i}).\]

Without loss of generality, it suffices to show that
\[(H^n)^{r-1} \cdot (A_1^{a_1} \cdots A_r^{a_r} \cdot H^{n-|a|}) \leq \prod_{i=2}^r \left( \frac{|a|}{a_i} \right) \prod_{i=1}^r (A_i^{a_i} \cdot H^{n-a_i}). \tag{4.1}\]

Without loss of generality, we may assume that \(H\) is very ample. When \(r = 1\), (4.1) is trivial. Suppose that \(r \geq 2\). Take general elements \(H_1, \ldots, H_{n-|a|} \in |H|\) and take \(Z = H_1 \cap \cdots \cap H_{n-|a|}\). Applying Theorem 3.5 to \(A_r|Z\) and \(H|Z\), we get
\[(H^n) \cdot (A_1^{a_1} \cdots A_r^{a_r} \cdot H^{n-|a|}) = (H^{[a]|Z}) \cdot (A_1^{[a]}|Z) \cdots A_r^{[a]}|Z) \leq \left( \frac{|a|}{a_r} \right) (A_1^{a_1}|Z) \cdots A_{r-1}^{a_{r-1}}|Z) \cdot A_r^{a_r}|Z \cdot H^{[a]-a_r}|Z) = \left( \frac{|a|}{a_r} \right) (A_1^{a_1} \cdots A_{r-1}^{a_{r-1}} \cdot H^{n-a_r+a_r}) \cdot (A_r^{a_r} \cdot H^{n-a_r}).\]

So we can prove inequality (4.1) by induction on \(r\).

Remark 4.1 The constant we obtain is slightly better than Xiao’s, which is \(\min_k \left( \prod_{i=1}^n \left( \frac{a_i}{|a|} \right) (H^n)^{n-1} \right)\), but the proof is basically the same as long as one knows Theorem 3.5. It remains interesting to find the optimal constant. We refer to [30, §3.4] for some related discussions.

4.2 Proof of Corollary 1.3

(1) For any dominant rational self-maps \(f, g\) on \(X\), take a projective normal variety \(W\) with generically finite morphisms \(p_j : W \to X\) for \(j = 1, 2, 3\) such that \(p_2 = p_1 \circ f\) and \(p_3 = p_2 \circ f\). Then by the projection formula between \(W\) and the graphs of \(f, g\),

1. \(\deg_i(f \circ g) = \frac{1}{\deg(p_1)} \cdot (p_2^*H^{n-i} \cdot p_3^*H^i);\)
2. \(\deg_i(f) = \frac{1}{\deg(p_2)} \cdot (p_2^*H^{n-i} \cdot p_3^*H^i);\)
3. \(\deg_i(g) = \frac{1}{\deg(p_3)} \cdot (p_1^*H^{n-i} \cdot p_2^*H^i).\)

If \(i = 0\) or \(n\) the statement is trivially true. If \(1 \leq i \leq n - 1\), by Theorem 1.1, we have
\[
\deg_i(f \circ g) = \frac{1}{\deg(p_1)} \cdot (p_1^*H^{n-i} \cdot p_3^*H^i) \leq \frac{1}{\deg(p_1)} \cdot \frac{\binom{n}{i}}{(p_2^*H^n) (p_1^*H^{n-i} \cdot p_3^*H^i) \cdot (p_2^*H^{n-i} \cdot p_3^*H^i)} = \frac{\binom{n}{i}}{(H^n) \deg_i(f) \cdot \deg_i(g)}.
\]
A
≡ B
2
(2) Let \(\pi_1\) and \(\pi_2\) be the projections from the normalization of the graph of \(f\) in \(X \times X\) onto the first and the second factor, respectively. If \(i = 0\) or \(n\) the statement is trivially true. If \(1 \leq i \leq n - 1\) then by applying Theorem 1.1 repeatedly, one can get
\[
\deg_{\ell, H}(f) = (\pi^n_1 H^{n-i} \cdot \pi^n_2 H^i) 
\leq \frac{n}{(\pi^n_1 L^n)} (\pi^n_1 H^{n-i} \cdot \pi^n_1 L^i) 
\leq \frac{n}{(\pi^n_1 L^n)} (\pi^n_2 L^{n-i} \cdot \pi^n_2 L^i) 
= \frac{n^2}{(L^n)^2} \deg_{\ell, L}(f).
\]

\textbf{Remark 4.2} As we apply the optimal inequality (1.2), we obtain better constants than Dang’s in [7, Theorem 1]. For example, the constant in [7, Theorem 1(i)] is \(\frac{(n-i+1)^i}{(H^n)^i}\). It remains interesting to find the optimal constants.

5 Characterization of the equality case on surfaces

In this section, we give a characterization of the equality case of Theorem 1.1 on surfaces. The proof is motivated by [1, Proposition 1.15]. For simplicity, we state the characterization for non-singular surfaces, and the general case can be easily worked out by taking a desingularization.

\textbf{Proposition 5.1} Let \(X\) be a non-singular projective surface. Let \(A, B, C\) be nef divisors on \(X\). Then
\[2(B \cdot A) \cdot (A \cdot C) = (A^2) \cdot (B \cdot C) \neq 0\]
if and only if the following conditions hold
\begin{enumerate}
\item \(A \equiv s B + t C\) in \(\text{NS}(X) \otimes \mathbb{Z} \otimes \mathbb{R}\) for some \(s, t > 0\);
\item \(B^2 = C^2 = 0\), \((B \cdot C) \neq 0\).
\end{enumerate}

\textbf{Proof} The “if” part is obvious. We only deal with the “only if” part.

Suppose that \(2(B \cdot A) \cdot (A \cdot C) = (A^2) \cdot (B \cdot C) \neq 0\). Then \((B \cdot A), (A \cdot C), (A^2), (B \cdot C)\) are all positive numbers. Set \(\Gamma = A - \frac{(A^2)}{2(A \cdot B)} B\). Then \((\Gamma \cdot C) = 0\) and \((\Gamma^2) = \frac{(A^2)^2 (B^2)}{4(A \cdot B)^2} \geq 0\).

We claim that \(\Gamma\) and \(C\) are propotional in \(\text{NS}(X) \otimes \mathbb{Z} \otimes \mathbb{R}\) by the Hodge index theorem. Fix any ample divisor \(H\) on \(X\), by our assumption, \(C\) is not numerically trivial, so \((H \cdot C) \neq 0\). Then \(((\Gamma - \frac{(\Gamma \cdot H)}{(C \cdot H)} C) \cdot H) = 0\). On the other hand, \((\Gamma - \frac{(\Gamma \cdot H)}{(C \cdot H)} C)^2 \geq 0\). So the classical Hodge index theorem (see [11, Theorem V.1.9]) implies that \(\Gamma - \frac{(\Gamma \cdot H)}{(C \cdot H)} C = 0\). As they are propotional, we can also get \(\Gamma = \frac{(\Gamma \cdot A)}{(C \cdot A)} C\), which implies that
\[
A = \frac{(A^2)}{2(B \cdot A)} B + \frac{(A^2)}{2(C \cdot A)} C.
\]
by the construction of \(\Gamma\). The fact that \((B^2) = (C^2) = 0\) can be obtained by intersecting (5.1) with \(B, C\) respectively. \(\square\)
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