CONFORMALLY OSSERMAN MANIFOLDS AND SELF-DUALITY
IN RIEMANNIAN GEOMETRY

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Abstract. We study the spectral geometry of the conformal Jacobi operator on a 4-dimensional Riemannian manifold \((M, g)\). We show that \((M, g)\) is conformally Osserman if and only if \((M, g)\) is self-dual or anti self-dual. Equivalently, this means that the curvature tensor of \((M, g)\) is given by a quaternionic structure, at least pointwise.

Let \(R_g\) and \(W_g\) be the curvature operator and the Weyl curvature operator associated to a Riemannian manifold \((M, g)\). The Jacobi operator \(J_{R_g}\) and the conformal Jacobi operator \(J_{W_g}\) are defined by:

\[
J_{R_g}(x) : y \mapsto R_g(y, x) x
\]

and

\[
J_{W_g}(x) : y \mapsto W_g(y, x) x.
\]

In contrast to the Jacobi operator, the conformal Jacobi operator is conformally invariant \([1]\); if \(h = e^{\alpha} g\) is a conformally equivalent Riemannian metric, then

\[
J_{W_h} = J_{W_g}.
\]

One says that \((M, g)\) is Osserman (resp. conformally Osserman) if the eigenvalues of \(J_{R_g}\) (resp. \(J_{W_g}\)) are constant on the bundle of unit tangent directions.

In a series of papers started by Chi \([3]\) and continued by Nikolayevsky \([5, 6, 7]\) it was shown that Osserman manifolds of dimension \(n \neq 16\) are two-point homogeneous spaces; for \(m \neq 16\), this gives an affirmative answer to a question raised by Osserman (see, for example, \([8]\)).

Previous work \([1]\) has shown that conformally Osserman manifolds are conformally flat if \(m \equiv 1 \mod 2\) and are either conformally flat or conformally equivalent to a complex space form (i.e. to complex projective space with the Fubini-Study metric or to the non-compact dual) if \(m \equiv 2 \mod 4\). In this present paper, we study the 4-dimensional setting motivated by work of Sekigawa and Vanhecke \([9]\). Our main result is the following.

**Theorem 1.** Let \((M, g)\) be a 4-dimensional oriented Riemannian manifold. The following conditions are equivalent:

1. \((M, g)\) is conformally Osserman.
2. \((M, g)\) is self-dual or anti-self dual.

It is useful to work in a purely algebraic setting. Let \(V\) be a finite dimensional vector space equipped with a positive definite inner product \((\cdot, \cdot)\). We say that \(R \in \otimes^4 V^*\) is an algebraic curvature tensor if \(R\) has the usual symmetries:

\[
R(x, y, z, w) = -R(y, x, z, w) = R(z, w, x, y), \quad \text{and}
\]

\[
R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0.
\]

The associated curvature operator \(\mathcal{R}(x, y)\) is then characterized by the identity

\[
(\mathcal{R}(x, y) z, w) = R(x, y, z, w).
\]

Let \(\tau\) be the scalar curvature and let \(\rho\) be the Ricci

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operator. Set
\[
\mathcal{R}^0_g(x, y)z := g(y, z)x - g(x, z)y, \quad \text{and} \\
\mathcal{L}(x, y)z := g(py, z)x - g(px, z)y + g(y, z)px - g(x, z)py.
\]

The associated Weyl operator given by
\[
\mathcal{W}(x, y) := \mathcal{R}(x, y) + \frac{1}{(m - 1)(m - 2)} \tau \mathcal{R}^0_g(x, y) + \frac{1}{m - 2} \mathcal{L}(x, y)
\]

We say that \( R \) is Osserman (resp. conformally Osserman) if the eigenvalues of \( J_R \) (resp. \( J_W \)) are constant on the unit sphere in \( V \).

Let \( \Phi \) be a skew-symmetric endomorphism of \( V \) with \( \Phi^2 = -1 \). Define
\[
R_\Phi(x, y)z := (\Phi y, z)\Phi x - (\Phi x, z)\Phi y - 2g(\Phi x, y)\Phi z.
\]

Then \( R_\Phi \) is an algebraic curvature tensor. We say that \( \{ \Phi_1, \Phi_2, \Phi_3 \} \) is a unitary quaternion structure on \( V \) if the \( \Phi_i \) are self-adjoint and if the usual structure equations are satisfied:
\[
\Phi_i\Phi_j + \Phi_j\Phi_i = -2\delta_{ij} \text{ id}.
\]

Theorem 1 is a consequence of the following purely algebraic fact:

**Theorem 2.** Let \( R \) be an algebraic curvature tensor on a 4-dimensional vector space \( V \). The following assertions are equivalent:

1. \( R \) is conformally Osserman.
2. \( R \) is self-dual or anti-self dual.
3. There exists a unitary quaternion structure on \( V \) so that
   \[
   W = \lambda_1 R_{\Phi_1} + \lambda_2 R_{\Phi_2} + \lambda_3 R_{\Phi_3} \quad \text{where} \quad \lambda_1 + \lambda_2 + \lambda_3 = 0.
   \]

We begin by showing that the first assertion implies the second assertion in Theorem 2. Let \( R \) be a conformally Osserman algebraic curvature tensor on \( \mathbb{R}^4 \). Zero is always an eigenvalue of \( J_W \) since \( J_W(x)x = 0 \). Let \( e_1 \) be a unit vector. Since \( J_W(\cdot) \) is symmetric, it has an orthonormal basis of real eigenvectors. Thus we may extend \( e_1 \) to an orthonormal basis \( \{ e_1, e_2, e_3, e_4 \} \) so that
\[
J_W(e_1)e_2 = ae_2, \quad J_W(e_1)e_3 = be_3, \quad J_W(e_1)e_4 = ce_4.
\]

Since \( \text{Tr}(J_W) = 0 \), we have \( a + b + c = 0 \).

The argument given by Chi in his analysis of the 4-dimensional setting was in part purely algebraic. This algebraic argument extends without change to this setting to show that, after possibly replacing \( \{ e_2, e_3, e_4 \} \) by \( \{-e_2, -e_3, -e_4\} \) that the non-vanishing components of the Weyl curvature on this basis are given by:
\[
\begin{align*}
W_{1221} &= W_{3443} = -W_{1234} = a, \\
W_{1331} &= W_{2442} = -W_{1342} = b, \\
W_{1441} &= W_{2332} = -W_{1423} = c.
\end{align*}
\]

Let \( e^i := e^i \wedge e^i \) where \( \{ e^i \} \) is the dual basis for \( V^* \). We consider the following bases for \( \Lambda^2_\pm(V) \):
\[
f_1^\pm = e^{12} \pm e^{34}, \quad f_2^\pm = e^{13} \pm e^{24}, \quad f_3^\pm = e^{14} \pm e^{23}.
\]

Since \( \mathcal{W}(e^{pq}) = \frac{1}{4} W_{pqij} e^{ij} \),
\[
\begin{align*}
\mathcal{W}(f_1^+) &= 0, & \mathcal{W}(f_1^-) &= 0, & \mathcal{W}(f_2^-) &= 0, \quad \mathcal{W}(f_3^-) &= 0, \\
\mathcal{W}(f_1^+) &= -2a f_1^+, & \mathcal{W}(f_2^-) &= -2b f_2^+, & \mathcal{W}(f_3^-) &= -2c f_3^+.
\end{align*}
\]

Thus conformally Osserman algebraic curvature tensors are self-dual.

Next we show that Assertion (2) implies Assertion (3) in Theorem 2. Suppose that \( R \) is a self-dual algebraic curvature tensor on \( \mathbb{R}^4 \). Let \( e_1 \) be a unit vector. Choose an orthonormal basis \( \{ e_1, e_2, e_3, e_4 \} \) for \( \mathbb{R}^4 \) so that
\[
J_W(e_1)e_2 = ae_2, \quad J_W(e_1)e_3 = be_3, \quad J_W(e_1)e_4 = ce_4.
\]
We then have
\[
W_{1221} = a, \quad W_{1231} = 0, \quad W_{1241} = 0, \\
W_{1321} = 0, \quad W_{1331} = b, \quad W_{1341} = 0, \\
W_{1421} = 0, \quad W_{1431} = 0, \quad W_{1441} = c.
\]
By replacing \(\{e_2, e_3, e_4\}\) by \(\{-e_2, -e_3, -e_4\}\) if necessary, we can assume that \(\{e_1, e_2, e_3, e_4\}\) is an oriented orthonormal basis. We have
\[
\mathcal{W}(e^{12} - e^{34}) = (W_{1212} - W_{3412})e^{12} + (W_{1234} - W_{3434})e^{34}, \\
+ (W_{1213} - W_{3413})e^{13} + (W_{1214} - W_{3414})e^{14}, \\
+ (W_{1223} - W_{3423})e^{23} + (W_{1224} - W_{3424})e^{24}.
\]
Since \(W\) is self-dual, \(\mathcal{W}(e^{12} - e^{34}) = 0\). Equation (2) then implies
\[
W_{3412} = W_{1212} = -a, \quad W_{3434} = W_{1234} = -a, \quad W_{3413} = W_{1213} = 0, \\
W_{3414} = W_{1214} = 0, \quad W_{3423} = W_{1223} = 0, \quad W_{3424} = W_{1224} = 0.
\]
We argue similarly using \(e^{13} + e^{24}\) and \(e^{14} - e^{23}\) to see that the formulas of Equation (1) hold. We define a unitary quaternion structure by defining \(\Phi_3 := \Phi_1 \Phi_2\) where
\[
\Phi_1 : e_1 \mapsto e_2, \quad \Phi_1 : e_2 \mapsto -e_1, \quad \Phi_1 : e_3 \mapsto e_4, \quad \Phi_1 : e_4 \mapsto -e_3, \\
\Phi_2 : e_1 \mapsto e_3, \quad \Phi_2 : e_3 \mapsto -e_1, \quad \Phi_2 : e_4 \mapsto -e_2, \quad \Phi_2 : e_2 \mapsto e_4.
\]
It is then immediate that the formulas of Equation (1) hold for \(\tilde{W} := aW_{\Phi_1} + bW_{\Phi_2} + cW_{\Phi_3}\) and thus \(W = \tilde{W}\). This shows the second assertion implies the third assertion.

Finally, if \(W\) is given by a unitary quaternion structure, then the discussion of Equation (1) shows that \(W\) is Osserman. This completes the proof of Theorem 2 and thereby of Theorem 1 as well. \(\Box\)

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