Non-equilibrium Effects in the Thermal Switching of Underdamped Josephson Junctions

Juan José Mazo,1 Fernando Naranjo,1,2 and David Zueco1

1Departamento de Física de la Materia Condensada and Instituto de Ciencia de Materiales de Aragón, CSIC-Universidad de Zaragoza, 50009 Zaragoza, Spain
2Universidad Pedagógica y Tecnológica de Colombia, Tunja, Colombia

(Dated: April 16, 2010)

We study the thermal escape problem in the low damping limit. We find that finiteness of the barrier is crucial for explaining the thermal activation results. In this regime low barrier non-equilibrium corrections to the usual theories become necessary. We propose a simple theoretical extension accounting for these non-equilibrium processes which agrees numerical results. We apply our theory to the understanding of switching current curves in underdamped Josephson junctions.

PACS numbers: 74.50.+r, 05.40.-a

In 1940 Kramers derived his famous formulas describing rates in chemical reactions [1]. The theoretical framework for his calculation was the escape of a Brownian particle over a potential barrier. Far from being a particular case, noise-activated escape is applicable in a wide number of problems in science, going from biology to quantum information processing [2]. Due to the many fields involved, intense activity emerged in the subject proposing better theories for this, nowadays, old problem [2–4].

In particular, studies on thermal switching in Josephson junctions (JJ) benefit from this effort [5,12]. Experimental results are affected by thermal fluctuations and measurements in the lab allow to predict junctions parameters by fitting the switching with available expressions. Also, some fundamental issues as the quantum-classical transition have been addressed by means of rates measurements [9,11,13]. It is clear, that such a measurements need to be compared with appropriate theoretical results. Needless to say the exact formula does not exist and many theories are available in the literature, which starting from the Kramers seminal work, cover different set of parameters [3,4].

In a recent numerical work [14], for very low values of the damping parameter, it has been found a significant deviation from the expected result. Here we present a theory that is able to give account for the observed deviation. Moreover, we predict that this effect will appear in any biased system where the damping over force ramp ratio is not large. In such a case, the usual theories are not suitable, and, as we show, it is needed to include non-equilibrium effects and finite barrier correction in a full description of the problem.

To be definite, the dynamics for the phase difference in the junction is usually described by the, so called, resistively and capacitively shunted junction (RCSJ) model, which is equivalent to the more general problem of a Brownian particle in a metastable potential:

\[ m \ddot{x} + m \gamma \dot{x} = -\frac{dV}{dx} + \xi(t), \]

where the potential \( V(x) = V_0(1 - \cos x) - Ix \) and \( \xi(t) \) is the stochastic force describing the thermal fluctuations. Here we consider white thermal noise, \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t)\xi(t') \rangle = 2m\gamma k_B T \delta(t - t') \).

For moderate to low values of the damping parameter there exists a temperature dependent critical current (force) for the system to switch from a superconducting or locked state (\( \langle \dot{x} \rangle = 0 \)) to a resistive or running one \( (m\gamma \dot{x} = I) \). Such a situation corresponds to the problem of escape from a metastable well. In switching current experiments many current-voltage (force-velocity) curves are performed to obtain the switching current probability distribution function, \( P(I) \). The measured \( P(I) \) can be directly related to the thermal activation rate \( \gamma \) and experimental results can be understood in terms of such parameter.

For very weak damping, the Kramers result for the activation rate is

\[ r_{KLD} = \frac{\gamma |I_b|/k_B T}{\omega_a/2\pi} e^{-\Delta U/k_B T}. \]

There we recognize the transition-state-theory result multiplied by a prefactor valid in the very low damping regime. For our system the action at the barrier \( J_b \) is usually approached by the cubic potential result \( J_b = 7.2A U/\omega_a \). Then:

\[ r_{KLD} \approx \frac{7.2 \gamma}{2\pi} \frac{\Delta U}{k_B T} e^{-\Delta U/k_B T}. \]

This equation shows that the rate scales linearly with the damping and depends only on the barrier height over thermal energy factor, \( \Delta U/k_B T \). This expression is only valid in the low damping and infinite barrier limit (\( |I_b|/k_B T \ll 1 \) and \( \Delta U/k_B T \gg 1 \)).

Many theories have extended the Kramers result to the moderate-to-small damping regime [3,15–18] following the infinite barrier approximation. Given its simplicity, the result of Büttiker et al. (BHL) [15] has been usually applied in the JJ literature. Finite barrier corrections have been studied in [19,20]. More recently, Drozdov and Hayashi (DH) proposed a new theory which is not perturbative in the barrier height [21].

We are interested in the dynamics of the system in the low damping limit. In this limit the coupling to the bath is very weak and the time to reach thermal equilibrium very long (\( \sim 1/\gamma \)). This fact has important consequences: For biased systems, escape occurs at very low values of the \( \Delta U/k_B T \) ratio and junctions may escape before thermal equilibrium is estab-
lished and thus non-equilibrium effects dominate the process. In order to study such effects we need first to know the importance of finite barrier effects in particle activation problem at low damping and take into account the average energy of junctions before each switching event.

**Escape at small barrier**—We will show here that small barrier effects are very important in the low damping case, the convergence to the infinite barrier result is very slow and the DH theory reproduces the numerical results at any barrier.

We have numerically integrated the Langevin equation (1) of the system for different values of damping and barrier height. In our simulations we have computed the mean time for the system to first reach the potential barrier. For low values of the damping such mean time (the first-passage-time problem) corresponds to the inverse of the escape rate. According to theory, simulations are started with particles placed in the metastable potential well and zero velocity. Some issues about the initial conditions problem will be addressed below. At any point the numerical result is obtained from $10^4$ escaping events. We show results for $V_0 = 0.31$, $m = 0.35$ and different values of $F$, damping and temperature. The results are summarized in Figs. 1 and 2 where we plot the activation rate as a function of barrier and damping respectively and compare to some existing theories.

Figure 1 shows the rate dependence on the barrier for different values of damping and barrier height. In order to see deviations from the Kramers low damping result we divide the obtained rates by Eq. 2. We recall that (2) is obtained assuming weak damping and high barrier. For comparison, we also plot the exact result for arbitrary barrier in the limit of vanishing damping in (3), $r(\gamma \to 0) = r_{HTB}$ with

$$r_{HTB} = \gamma k_B T \left[ \int_0^{J_b} dJ e^{-\beta E(J)} \int_0^{J_b} dJ' \frac{\omega(J')}{2\pi} e^{\beta E(J')} \right]^{-1},$$

where $J_b$ is the action at the barrier and $\beta = 1/k_B T$.

Remarkably, the approach between both results is slow, meaning that the high barrier approximation is accurate only at very high barrier values indeed [Fig. 1(a)]. For instance $r_{HTB}/r_{KLD} = 0.72$ for $\Delta U/k_B T = 5$ and 0.85 for $\Delta U/k_B T = 10$. As a consequence, all theories which try to extend the Kramers result to the moderate-to-small damping region [15,16] also fail at low damping values unless very large barriers are considered.

We know about two main attempts to include finite barrier effects in this limit: the first one is due to Melnikov (MFB) [19] and fails at small barriers, see Fig. 1(b). The second one was proposed by Drozdov and Hayashi (DH) for
vide a particle energy enough to overcome the barrier. This problem. Escape occurs as soon as thermal fluctuations provide energy enough to surmount the barrier but also kinetic energy at the bottom. In order to study the importance of this issue, in Fig. 3 we plot the rates with two initial conditions, $v = \pm \sqrt{k_B T/m}$ and compare to the one with zero velocity. As expected, we see that for small barriers initial kinetic energy speed up the escape times.

When particles are placed with zero velocity at the bottom of the well, the activation time is $r^{-1}$. However, if particles have extra initial energy $E_{in}$ the escape time is given by $r^{-1} - \tau$ where $\tau$ is the activation time up to this extra energy, which can be computed at low damping from Eq. (3) replacing $\Delta U$ by $E_{in}$. Putting all together we generalize the rate formulas as,

$$r_{in} = \frac{1}{r^{-1} - \tau}.$$  

This equation shows that as soon as $\tau(E_{in}) \sim r^{-1}$ the initial conditions problem affect the escape rates. In Fig. 3 where $E_{in} = k_B T/2$, this correction becomes important for $\Delta U/k_B T \lesssim 2$. If $\Delta U \leq E_{in}$ the passage time is almost a deterministic process which depends on the initial position and velocity of the system. Figure 3 illustrates this effect and confirms our theoretical prediction.

Switching current—In a typical JJ experiment the probability distribution function of the junction switching current $P(I)$ is measured performing many current-voltage curves where current is continuously increased at a given rate. From these results the mean switching current $I_{sw}$ and its standard deviation can be trivially computed. Such $P(I)$ can be easily related to the escape rates $r(I)$ as

$$P(I) = r(I) \left( \frac{dl}{dr} \right)^{-1} \left( 1 - \int_0^l P(u) du \right).$$  

Alike, escape rates can be computed from measured $P(I)$.

Figure 4(a) shows our numerical results for the average switching current and compares them to theoretical predictions. We integrate Eq. 1 for an ensemble of thermalized junctions. Current is increased at a given ramp and switching events are recorded. As expected BHL based predictions fail in the very low damping regime. However, surprisingly, also DH is unable to explain our numerical results, which lie in between both theories. This is due to the competition between the equilibrium time of the system, given by $\gamma^{-1}$, and the time order for the change of the current, given by the inverse of the current ramp. Thus, switching in the very low damping regime is a non-equilibrium process. The coupling to the external bath is so weak that other junctions are not able to reach the thermal energy before switching. Thus, junctions escape in an evaporative cooling way where more energetic junctions switch first and the ensemble is effectively cooled. This picture is confirmed in Fig. 4(c) where for a given damping we show the mean energy for the trapped junctions as a
FIG. 4: (color online) (a) Average switching current at different values of the damping for $d_{\text{avg}}$ over ramp ratio [23]. (c) Mean energy divided by $k_B T$ and $\dot{\gamma}$ function of the current and the fraction of particles which have switched.

We also see in Fig. 4(c) that particles escape with an initial energy which goes from $E_{\text{in}} = K_B T$ to zero when current is increased. The simplest way to introduce this fact in the theory is to assume an average value of $E_{\text{in}} = K_B T/2$ and use our Eq. (5). Figure 4(a) shows that in this way we are able to reproduce quite accurately the numerical results. This correction turns out to be important when the average barrier at the switching current is of the order of the thermal energy. See that in the figure it is also plotted the value of the barrier at the mean switching current (open symbols). Finally, using Eqs. (4), (5) and (6) it can be seen that in this region of the parameter space, the results depend on the $\gamma/\dot{\gamma}$ ratio, as confirmed in Fig. 4(b). Therefore our theory allows to estimate the values for $\gamma/\dot{\gamma}$ where non equilibrium corrections are necessary.

Although presented in the framework of the JJ switching current measurements, our results are further more general and apply to any experiment where an activation rate is measured as a function of an external parameter which can be controlled at will. An important issue to study, it is the influence of the observed competition between two different time scales on results for biased systems at higher values of the damping and thus transfer our theoretical scheme from the energy-diffusion regime to the phase-diffusion one. This is the typical case for many of the current biological-physics experiments [24, 25].

We thanks F. Falco and L.M. Flora for discussions and critical reading of the manuscript. This work was supported by Spain MICINN project FIS2008-01240, co-financed by FEDER funds.

[1] H. Kramers, Physica 7, 284 (1940).
[2] E. Pollak and P. Talkner, Chaos 15, 026116 (2005).
[3] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. 62, 251 (1990).
[4] V. I. Melnikov, Phys. Rep. 209, 1 (1991).
[5] V. Ambegaokar and B. I. Halperin, Phys. Rev. Lett. 22, 1364 (1969).
[6] M. J. Stephen, Rev. Mod. Phys. 81, 393 (1969).
[7] J. Kurkijärvi, Phys. Rev. B 6, 832 (1972).
[8] T. A. Fulton and L. N. Dunkleberger, Phys. Rev. B 9, 4760 (1974).
[9] S. Washburn, R. A. Webb, R. F. Voss, and S. M. Faris, Phys. Rev. Lett. 54, 2712 (1985).
[10] M. H. Devoret, J. M. Martinis, and J. Clarke, Phys. Rev. Lett. 55, 1908 (1985).
[11] J. M. Martinis, M. H. Devoret, and J. Clarke, Phys. Rev. B 35, 4682 (1987).
[12] P. Silvestrini, O. Liengme, and K. E. Gray, Phys. Rev. B 37, 1525 (1988).
[13] A. Wallraff, A. Lukashenko, J. Lisenfeld, A. Kemp, M. V. Fis-tul, Y. Koval, and A. V. Ustino, Nature 425, 155 (2003).
[14] J. J. Mazo, F. Naranjo, and K. Segall, Phys. Rev. B 78, 174510 (2008).
[15] M. Böttiker, E. P. Harris, and R. Landauer, Phys. Rev. B 28, 1268 (1983).
[16] H. Grabert, Phys. Rev. Lett. 61, 1683 (1988).
[17] E. Pollak, H. Grabert, and P. Hänggi, J. Chem. Phys. 91, 4073 (1989).
[18] V. I. Melnikov and S. Meshkov, J. Chem. Phys. 85, 1018 (1986).
[19] V. I. Melnikov, Phys. Rev. E 48, 3271 (1993).
[20] R. Ferrando, R. Spadacini, G. E. Tommei, and V. I. Melnikov, Phys. Rev. E 51, 1645 (1995).
[21] A. N. Drozdov and S. Hayashi, Phys. Rev. E 60, 3804 (1999).
[22] r_{\text{DH}} = A(r_{\text{HTB}}/\Gamma_{\text{ST}}) \times \Gamma_{\text{ST}}$ where $r_{\text{HTB}}$ is given by $61$. $\Gamma_{\text{ST}}$ is the Transition-State-Theorey rate, $\Gamma_{\text{ST}} = \frac{1}{\sqrt{2\pi m}} \int_{-\infty}^{\infty} dx e^{-B(x)}$ and $A(\gamma)$ is the interpolation function given by Melnikov and Meshkov [18]. $A(\gamma) = \exp\left[\frac{1}{2} \int_0^1 dx \ln \{1 - \exp[-\gamma(x^2 + 1)]\} / (x^2 + 1)\right]$. $A(\gamma)$
[23] To normalize Eq. (1) we divide it by $V_0$ and time by $\omega^{-1} = \sqrt{m/V_0}$. Then , the adimensionalized parameters are $\bar{\gamma} = \gamma/\omega$ and $\bar{t} = k_B T/V_0 = 0.645 k_B T / \dot{I}$ and $\bar{I} = I/(\omega V_0) = 9.70 I$. $\bar{I}$
[24] C. Hyeon and D. Thirumalai, Proc. Nat. Acad. Sci. USA 100, 10249 (2003).
[25] O. K. Dudko, Proc. Nat. Acad. Sci. USA 106, 8795 (2009).