Pseudo-differential operators and related additive geometric stable processes

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Abstract

Additive processes are obtained from Lévy ones by relaxing the condition of stationary increments, hence they are spatially (but not temporally) homogeneous. By analogy with the case of time-homogeneous Markov processes, one can define an infinitesimal generator, which is, of course, a time-dependent operator. Additive versions of stable and Gamma processes have been considered in the literature. We introduce here time-inhomogeneous generalizations of the well-known geometric stable process, defined by means of time-dependent versions of fractional pseudo-differential operators of logarithmic type. The local Lévy measures are expressed in terms of Mittag-Leffler functions or $H$-functions with time-dependent parameters.

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1 Introduction

Geometric stable random variables (GS r.v.’s) have been studied since the Eighties and widely applied, in particular, in modelling data with heavy-tails behavior, in mathematical finance and other fields of research (see [20] and [24], for the univariate and multivariate cases, respectively, and also [22]). Indeed, the GS laws are characterized by heavy tails, unboundedness at zero and by stability properties (with respect to geometric summation). The GS process can be defined by means of the $\alpha$-stable process, as follows. Let us consider an $\alpha$-stable process $S_{\alpha,\theta} := \{S_{\alpha,\theta}(t), t \geq 0\}$ which has (according to Feller’s parametrization) the following characteristic function

$$Ee^{i\xi S_{\alpha,\theta}(t)} = \exp\{-t|\xi|^\alpha e^{i\text{sign}(\xi)\pi\theta/2}\} = \exp\{-t\psi_{\alpha,\theta}(\xi)\} \quad \alpha \in (0, 2), \alpha \neq 1.
$$

where $|\theta| \leq \min\{\alpha, 2 - \alpha\}$. For $\alpha = 1$, the characteristic function can be written in the form [1] in the symmetric case only, i.e. for $\theta = 0$. So, in order to have unified formulas, we

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exclude the case \( \alpha = 1 \) from our exposition. We will denote by
\[ G_{\alpha, \theta} := \{ G_{\alpha, \theta}(t), t \geq 0 \} \]
the univariate GS process; which can be represented (see [14]) as
\[ G_{\alpha, \theta}(t) := S_{\alpha, \theta}(\Gamma(t)), \quad t \geq 0, \quad (2) \]
where \( \Gamma := \{ \Gamma(t), t \geq 0 \} \) is an independent Gamma subordinator, with density
\[ f_{\Gamma(t)}(x) = \frac{e^{-x/b}x^{at-1}}{\Gamma(at)\theta^a}, \quad x, t \geq 0, \quad a, b > 0 \quad (3) \]
and characteristic function
\[ \mathbb{E}e^{i\xi \Gamma(t)} = (1 - i\xi b)^{-at}. \]
We recall that the Lévy measure of \( \Gamma \) is given by \( \nu_{\Gamma}(dx) = ax^{-1}e^{-x/b}dx. \) We will put for simplicity \( a = 1. \) As a consequence of (2), \( G_{\alpha, \theta} \) is a Lévy process (see, for example, [39]), with characteristic exponent
\[ \eta_{G_{\alpha, \theta}}(\xi) := \frac{1}{t} \ln \mathbb{E}e^{i\xi G_{\alpha, \theta}(t)} = -\ln (1 + b\psi_{\alpha, \theta}(\xi)), \quad \xi \in \mathbb{R}. \quad (4) \]
We note that, for \( \theta = 0 \), the process \( G_{\alpha, \theta} \) is symmetric; in particular, for \( \alpha = 2 \), it reduces to the well-known variance gamma (VG) process, which, by (2), can be represented as \( G_{2,0}(t) := B(\Gamma(t)), t \geq 0, \) where \( B := \{ B(t), t \geq 0 \} \) is a standard Brownian motion, independent of \( \Gamma. \) The VG is applied in option pricing, since it allows for a wider modelling of skewness and kurtosis than the Brownian motion does. Moreover the variance gamma process has been successfully applied in the modelling of credit risk in structural models. The pure jump nature of the process and the possibility to control skewness and kurtosis of the distribution allow the model to price correctly the risk of default of securities having a short maturity, something that is generally not possible with structural models in which the underlying assets follow a Brownian motion (for more details, see, for example, [33] and [9]). For \( \alpha \neq 2 \) the symmetric GS law is also called Linnik distribution (see [21]).

On the other hand, in the completely positively asymmetric case, i.e. for \( \theta = -\alpha \) and for \( \alpha \in (0, 1) \), the GS law reduces to the so-called Mittag-Leffler distribution (21), while the corresponding process is called GS subordinator (since it is increasing). Its Lévy density, being of order \( \alpha/x \) for \( x \) near the origin, is almost integrable near zero, so that the subordinator is very slow. Thus it can be used for time-changing another process, in order to slow it down (see [39], [25]).

In this paper, we present time-inhomogeneous versions of the GS process. There is a wide literature inspiring this subject. In [26] and [11] the authors studied the so called multistable processes, namely inhomogeneous extensions of stable Levy ones, which are obtained by letting the stability parameter vary in time. Such processes turned out to be very useful in financial and physical applications, where the data display jumps with varying intensity (for financial applications of additive processes see, for example [19]). Moreover, time inhomogeneous versions of Gamma subordinators and VG processes can be respectively found in [8] and [28]. On the same line of research, in [32] the authors defined the so called inhomogeneous subordinators, i.e. non decreasing additive processes, which are used as models of random time change to extend the theory of Bochner subordination. A remarkable
case is that of the multistable subordinator considered in [31] (for an application of Markov processes time-changed by multistable subordinators see [6]).

We will define two, alternative, time-inhomogeneous versions of the GS process. The first one is obtained by letting the parameters $\alpha$ and $\theta$ vary in time; then the generator of the process is defined by means of a Riesz-Feller space-fractional derivative with time-varying parameters (i.e. $\alpha(t)$ and $\theta(t)$). We will obtain some results on the tails’ behavior of the density and the Lévy measure (at least in the subordinator case, i.e. for $\alpha(t) \in (0, 1)$ and $\theta(t) = -\alpha(t)$, $\forall t$) for the new process. However, as we will see, we cannot recover, as special case, an inhomogeneous version of the VG process, since only the standard VG process can be obtained by putting $\alpha(t) = 2$ and $\theta(t) = 0$, for any $t$. Moreover, no subordinating relationship, similar to (2) can be established. Therefore we consider a second inhomogeneous version of the GS process, by letting the variance parameter $b$ depend on $t$ and keeping $\alpha$ and $\theta$ constant in time. The two previous drawbacks are then overcome, since this new process can be constructed as a stable process time-changed by an inhomogeneous Gamma subordinator. Moreover the Lévy measure can be evaluated, in this case, for any value of $\theta$ and it will expressed in terms of H-functions.

The plan of the paper is the following. In Section 2 we aim at a twofold scope: firstly, we recall some basic facts on additive processes and propagators, then we present some new results related to them, which complete the analysis of this topic given in [32] and [3]. In Section 3 we write the generators of the GS processes as fractional operators of logarithmic type, which have been treated, in [3], [4] and [5], in the homogeneous case. In Section 4 and 5 we construct the two additive geometric stable processes and present all the related results, together with some relevant particular cases.

2 Notation and preliminary results

Additive processes are obtained from Lévy ones by relaxing the condition of stationarity of the increments (see, for example, [36], p.47). Indeed a process $X := \{X(t), t \geq 0\}$ is said to be additive if

- $X(0) = 0$ almost surely (a.s.)
- $X$ has independent increments
- $X$ is stochastically continuous

Thus $X$ is a spatially (but not temporally) homogeneous Markov process. For any $0 \leq s \leq t$, the distribution $\mu_{s,t}$ of the increment $X(t) - X(s)$ is such that

$$
\mathbb{E} e^{ip(X(t)-X(s))} = \int_{\mathbb{R}} e^{ipx} \mu_{s,t}(dx) = e^{\int_s^t \eta(p,\tau)d\tau} \quad 0 \leq s \leq t, \quad p \in \mathbb{R},
$$

where

$$
\eta(p, t) = ibtp - \frac{1}{2}ztp^2 + \int_{\mathbb{R}} (e^{ipy} - 1 - ipy1_{[-1,1]}(y)) \nu_t(dy).
$$
Here \( (b_t, z_t, \nu_t) \) is called the characteristic triplet of \( X \). In particular, \( b_t \in \mathbb{R}, z_t > 0 \) and \( \nu_t \) is the time-dependent Lévy measure, such that

\[
\int_{\mathbb{R}} (y^2 \wedge 1) \nu_t(dy) < \infty \quad \forall t \geq 0.
\]

An additive process is completely determined by the set of measures \( \mu_s, t, 0 \leq s \leq t \), since, from (6), all the finite-dimensional distributions are completely specified. Indeed, let \( 0 = t_0 < t_1 < t_2 < \ldots < t_n \) and \( \xi_j \in \mathbb{R} \), for \( j = 1, \ldots, n \), then the \( n \)-times characteristic function can be written as

\[
\mathbb{E} e^{i \sum_{j=1}^{n} \xi_j X(t_j)} = \mathbb{E} \exp \left\{ i \sum_{k=1}^{n} \sum_{j=k}^{n} \xi_j (X(t_k) - X(t_{k-1})) \right\},
\]

where the right-hand side is obtained by simple algebraic manipulations. Then, by independence of the increments, formula (7) reduces to

\[
\prod_{k=1}^{n} e^{i \eta_k \sum_{j=k}^{n} \xi_j \tau^j} = \exp \left\{ \int_{\mathbb{R}} \eta \left( \sum_{j=1}^{n} \xi_j 1_{[0,t_j]}(\tau) \right) d\tau \right\}.
\]

An interesting example of additive process is the so-called multistable process (here denoted \( S_{\alpha,\theta}^t \)), recently studied in [11] and [26]. It can be defined by letting the parameters \( \alpha \) and \( \theta \) in (1) be time-dependent. Indeed, by assigning the two functions \( \alpha(t) \) and \( \theta(t) \) such that \( \alpha(t) \in (0,1) \) and \( |\theta(t)| \leq \min(\alpha(t), 2 - \alpha(t)) \), for any \( t \geq 0 \), the characteristic function reads

\[
\mathbb{E} e^{i \xi S_{\alpha,\theta}^t} = e^{- \int_0^t \psi_{\alpha(\tau)\theta(\tau)}(\xi) d\tau}
\]

Hence \( S_{\alpha,\theta}^t \) is a time-inhomogeneous extension of stable processes, having independent and non-stationary increments, which turned out to be very useful in applications. In the symmetric case, where \( \theta(t) = 0 \forall t \) and \( \psi_{\alpha(t)\theta(t)} = |\xi|^\alpha(t) \), the joint distribution has the following characterization (see [26], formula (6))

\[
\mathbb{E} e^{i \xi S_{\alpha,\theta}^t} = e^{- \int_0^t \psi_{\alpha(\tau)}(\xi) d\tau}
\]

for \( 0 < t_1 < \ldots < t_n \).

### 2.1 Propagators and time-dependent generators

Let \( X \) be a univariate additive process and let

\[
p_{s,t}(dy) = P(X(t) \in d(x+y)|X(s) = x)
\]

be its transition probability, which is independent of \( x \) since \( X \) is space-homogeneous. Clearly \( X \) defines a propagator (namely, a two parameters semigroup)

\[
T_{s,t}f(x) = \int_{\mathbb{R}} f(x+y)p_{s,t}(dy) \quad 0 \leq s \leq t
\]
for \( f \) in the Banach space \( C(\mathbb{R}) \), equipped with the sup-norm. Of course, \( T_{t,t} \) is the identity operator and, for any \( 0 \leq r \leq s \leq t \), the chain rule \( T_{s,t}T_{r,s} = T_{r,t} \) holds. The time-dependent generator of \( T_{s,t} \) is defined (see [35], page 48, and [36]) as the operator

\[
\mathcal{A}_t f = \lim_{h \to 0^+} \frac{T_{t,t+h}f - f}{h}
\]
on a subset of \( C(\mathbb{R}) \) where the limit (meant in the sup-norm of \( C(\mathbb{R}) \)) exists.

Let \( \mathcal{S}(\mathbb{R}) \) be the Schwartz space of infinitely differentiable functions on \( \mathbb{R} \), decreasing at infinity (together with all their derivatives) faster than any power. In analogy to the standard theory of Lévy processes, we show that, by restricting the domain of \( T_{s,t} \) to \( \mathcal{S}(\mathbb{R}) \subset C(\mathbb{R}) \), we can easily find the form of the time-dependent generator of \( T_{s,t} \), by means of the representation as pseudo-differential operator. Indeed, the crucial point is that the Fourier transform

\[
\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int e^{-ipx} f(x) dx
\]
is a bijective operation on \( \mathcal{S}(\mathbb{R}) \) and the inverse transform is defined as

\[

f(x) = \frac{1}{\sqrt{2\pi}} \int e^{ipx} \tilde{f}(p) dp.
\]

In the following lemma we extend well-known results on Lévy processes to the additive case.

**Lemma 1** Let \( X \) be an additive process with characteristic exponent \( \eta(p,t) \) (given in (6)) continuous in \( t \) and such that \( |\eta(p,t)| \) is bounded in \( t \). Let \( T_{s,t} \) be the associated propagator. Then, for any \( f \in \mathcal{S}(\mathbb{R}) \),

i) the propagator has the following representation

\[
T_{s,t}f(x) = \frac{1}{\sqrt{2\pi}} \int e^{ipx} \tilde{f}(p) dp
\]

ii) the time-dependent generator has the form

\[
\mathcal{A}_t f(x) = \frac{1}{\sqrt{2\pi}} \int e^{ipx} \eta(p,t) \tilde{f}(p) dp
\]

iii) the generator can be equivalently written as

\[
\mathcal{A}_t f(x) = \frac{1}{2} z_t \frac{\partial^2}{\partial x^2} f(x) + b_t \frac{\partial f(x)}{\partial x} + \int \left[ f(x + y) - f(x) - y \frac{\partial}{\partial x} f(x) 1_{|y| \leq 1} \right] \nu_t(dy),
\]

**Proof.** i) The Fourier transform of \( T_{s,t}f(x) \) is

\[
\frac{1}{\sqrt{2\pi}} \int dx e^{-ipx} \int \int f(x + y)p_{s,t}(dy) = \frac{1}{\sqrt{2\pi}} \int p_{s,t}(dy) \int dx f(x + y)e^{-ip(x+y)}
\]

\[
= \exp \left( \int_s^t \eta(p, \tau) d\tau \right) \tilde{f}(p)
\]
which proves (i), by Fourier inversion.

ii) Let \( f \) be in the domain of the generator, i.e. suppose that \( \frac{T_{t+h}f-f}{h} \) converges uniformly to \( \mathbf{A}_t f \) as \( h \to 0^+ \). Then it is sufficient to use the representation (10) and to apply the pointwise limit

\[
\mathbf{A}_t f(x) = \lim_{h \to 0^+} T_{t+h} f(x) - f(x) = \lim_{h \to 0^+} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ipx} e^{f_{t+h}^\eta(p,\tau)d\tau} - 1 h f(p)dp
\]

The result immediately follows by exchanging the limit and the integral, which is permitted by the dominated convergence theorem. Indeed, by the mean value theorem,

\[
\left| e^{ipx} e^{f_{t+h}^\eta(p,\tau)d\tau} - 1 h f(p) \right| \leq \eta(p, \tau^*) f(p) \leq \eta(p, \tau_{\text{max}}) f(p) \leq C(1 + |p|^2) f(p)
\]

where \( \tau_{\text{max}} \) is the point where \( |\eta(p, \tau)| \) has its maximum in \( \tau \) and, in the last inequality, we used [1, p.31]. Moreover

\[
\int_{\mathbb{R}} C(1 + |p|^2) f(p)dp < \infty
\]

since \( |C(1 + |p|^2) f(p)| \) is clearly a Schwartz function.

iii) It is sufficient to insert (6) into (11) and the result is obtained by Fourier inversion.  

Some of the processes considered here are symmetric. Recall that an additive process is symmetric if the transition probability is such that \( p_{s,t}(x, dy)dx = p_{s,t}(y, dx)dy \), for any \( s, t \in \mathbb{R}^+ \) and \( x, y \in \mathbb{R} \). In this case the characteristic function of its increments (5) reads

\[
\mathbb{E} e^{i\xi(X(t) - X(s))} = e^{f_t^\eta(\xi, \tau)d\tau} \quad 0 \leq s \leq t.
\] (12)

The following result extends to symmetric additive processes a well-known property enjoyed by symmetric Lévy ones (see [1] p.178)

**Lemma 2** Let \( X \) be a symmetric additive process. Then the associated propagator \( T_{s,t} \) is self-adjoint in \( \mathbb{L}_2 \).

**Proof.** By (9) we can write

\[
T_{s,t} f(x) = \int_{\mathbb{R}} f(y) p_{s,t}(x, dy).
\]

It is enough to show that \( < T_{s,t} f; g > = < f; T_{s,t} g > \), for any \( f, g \in \mathbb{L}_2 \) (where \( < \cdot ; \cdot > \) denotes the scalar product in \( \mathbb{L}_2 \)). Then

\[
<T_{s,t} f; g > = \int_{\mathbb{R}} T_{s,t} f(x) g(x)dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x) p_{s,t}(x, dy)dx
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) g(x) p_{s,t}(y, dx)dy = < f; T_{s,t} g >
\]

where the order of integration has been inverted by the Fubini theorem.  

■
2.2 Time-change by inhomogeneous subordinators

The so-called non-homogeneous subordinators (i.e. non-decreasing additive processes, which can be used as random-time) have been studied in [32]. Their transition measure $\mu_{s,t}$ has Laplace transform

$$\int_0^\infty e^{-\eta z} \mu_{s,t}(dz) = e^{-\int_s^t f(\eta, \tau) d\tau}, \quad \eta > 0,$$

where $f(\eta, \tau) = \int_0^\infty (1 - e^{-s\eta}) \nu_{\tau}(ds)$ is a Bernstein function. A remarkable example is the multistable subordinator studied in [31], which corresponds to $f(\eta, \tau) = \eta^{\alpha(\tau)}$, where $\alpha(\tau) \in (0, 1)$.

In [32] the authors considered propagators defined by the Bochner integral

$$T_{s,t}f = \int_0^\infty T_{\tau}f \mu_{s,t}(dz), \quad (13)$$

where $T_t$ is a contraction semigroup (not necessarily associated to a stochastic process) acting on a generic Banach space, and $\mu_{s,t}$ is the increment law of a non-homogeneous subordinator. The operator (13) is a subordinated propagator (not necessarily associated to a stochastic process) acting on a generic Banach space. In [32], Theorem 4.1, the form of the time-dependent generator of (13) is found, by considering an Hilbert space as domain of $T_{s,t}$ and assuming $T_{s,t}$ to be self-adjoint. This result generalizes the Phillips theorem (see [36]) holding for one-parameter subordinated semigroups.

In view of what follows, it is useful to prove a more general result which is valid for not necessarily self-adjoint propagators. The price we pay to make this generalization is to narrow the attention to propagators acting on $S(\mathbb{R})$ only. Moreover we only consider propagators associated to time-changed to Lévy processes.

**Lemma 3** Let $M$ be a Lévy process associated to the semigroup $T_t$ on $S(\mathbb{R})$, having characteristic function $\mathbb{E}e^{i\xi M(t)} = e^{i\eta(\xi)}$, $\xi \in \mathbb{R}$ and let $H$ be an inhomogeneous subordinator with Laplace transform $\mathbb{E}e^{-p(H(t) - H(s))} = e^{-\int_s^t f(p, \tau) d\tau}$, $p \in \mathbb{R}^+$. Then the additive process $M(H(t))$ has time-dependent generator

$$L_t h(x) = \int_0^\infty (T_{\tau}h(x) - h(x)) \nu_{\tau}(dz), \quad h \in S(\mathbb{R}).$$

**Proof.** By a standard conditioning argument, we have

$$\mathbb{E}e^{ipM(H(t))} = e^{-\int_0^t f(-\eta(p), \tau) d\tau}.$$ 

It is now sufficient to apply Lemma 1 to the characteristic exponent $-f(-\eta(p), \tau)$. Indeed, by using expression (13), we have

$$L_t h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \left(-f(-\eta(p), t)\right) \tilde{h}(p) dp = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \left(\int_0^\infty (e^{\eta(p)s} - 1) \nu_{\tau}(ds)\right) \tilde{h}(p) dp$$

and recalling that

$$T_{\tau}h(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} e^{\eta(p)s} \tilde{h}(p) dp$$

we obtain the result. □
3 Fractional derivatives with time-varying parameters

3.1 Multistable processes and their generators

The Riesz-Feller (RF) fractional derivative is a pseudo-differential operator defined in [29] by means of its Fourier transform. For any $f \in S(\mathbb{R})$, the RF fractional derivative $D^\alpha,\theta$ (with $\alpha \in (0, 2]$, and $|\theta| \leq \min\{\alpha, 2 - \alpha\}$), is defined by

$$
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} D^\alpha,\theta f(x) dx = -|\xi|^{\alpha} e^{-i \text{sign}(\xi) \theta \pi/2} \hat{f}(\xi),
$$

namely its "symbol" reads

$$
\hat{D}^\alpha,\theta(\xi) = -|\xi|^{\alpha} e^{-i \text{sign}(\xi) \theta \pi/2}.
$$

The adjoint of $D^\alpha,\theta$ is defined as the operator $\overline{D}^\alpha,\theta$ such that

$$
\int_{\mathbb{R}} D^\alpha,\theta f(x) g(x) dx = \int_{\mathbb{R}} f(x) \overline{D}^\alpha,\theta g(x) dx \quad \forall f, g \in S(\mathbb{R})
$$

and it is easy to check that $\overline{D}^\alpha,\theta$ is such that

$$
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \overline{D}^\alpha,\theta f(x) dx = -\psi_{\alpha,\theta}(\xi) \tilde{f}(\xi) = -|\xi|^{\alpha} e^{i \text{sign}(\xi) \theta \pi/2} \tilde{f}(\xi).
$$

Indeed, by writing $D^\alpha,\theta f(x)$ and $\overline{D}^\alpha,\theta g(x)$ as the inverse Fourier transforms of (14) and (17), we can verify that (16) is satisfied.

By inserting (17) into (11) (restricted to the time-homogeneous case), we see that the operator $\overline{D}^\alpha,\theta$ is the generator of the stable process $S_{\alpha,\theta}$. Therefore, if $T_t$ is the stable semigroup, we have that $q(x, t) = T_t f(x) = \mathbb{E}(f(S_{\alpha,\theta}(t)) | S_{\alpha,\theta}(0) = x)$ solves

$$
\frac{\partial}{\partial t} q(x, t) = \overline{D}^\alpha,\theta q(x, t) \quad q(x, 0) = f(x),
$$

while the stable density $p_{\alpha,\theta}(x, y, t)$ (which obviously depends on $y - x$ only, since $S_{\alpha,\theta}$ is a Lévy process) solves the backward equation

$$
\frac{\partial}{\partial t} p_{\alpha,\theta}(x, y, t) = \overline{D}^\alpha,\theta p_{\alpha,\theta}(x, y, t) \quad p_{\alpha,\theta}(x, y, 0) = \delta(x - y).
$$

The picture is completed by the forward equation (where the operator on the right-hand side acts on the forward variable $y$)

$$
\frac{\partial}{\partial t} p_{\alpha,\theta}(x, y, t) = D^\alpha,\theta p_{\alpha,\theta}(x, y, t) \quad p_{\alpha,\theta}(x, y, 0) = \delta(x - y)
$$

\footnote{To avoid confusion with formulas reported in [29], we point out that our definition of Fourier transform of a function $h$ is $\hat{h}(\xi) = 1/\sqrt{2\pi} \int_{\mathbb{R}} e^{-i\xi x} h(x) dx$, while the authors in [29] define the Fourier transform as $\hat{h}_M(\xi) = \int_{\mathbb{R}} e^{i\xi x} h(x) dx$, hence we have $\hat{h}(\xi) = \hat{h}_M(-\xi)/\sqrt{2\pi}$. Then, for $h(x) = D^\alpha,\theta f(x)$, the authors in [29] write $\hat{h}_M(\xi) = -|\xi|^{\alpha} e^{i \text{sign}(\xi) \theta \pi/2} \hat{f}_M(\xi)$, which, according to our definition of Fourier transform, becomes $\hat{h}(\xi) = -\frac{1}{\sqrt{2\pi}} |\xi|^{\alpha} e^{-i \text{sign}(\xi) \theta \pi/2} \hat{f}(\xi) = -|\xi|^{\alpha} e^{-i \text{sign}(\xi) \theta \pi/2} \hat{f}(\xi)$.
which has been studied in [29] (although with a different notation and in a different setting).

The previous facts can be extended to the case of a time-varying fractional index \( \alpha(t) \) and skewness parameter \( \theta(t) \) (throughout the paper, we will assume that \( \alpha(t) \) e \( \theta(t) \) are continuous functions); to this aim we give the following definition.

**Definition 4 (RF fractional derivative with varying parameters)** Let \( \alpha(t) \in (0, 2] \) and \( |\theta(t)| \leq \min\{\alpha(t), 2 - \alpha(t)\}, \forall t \geq 0 \), we define \( D_{x}^{\alpha(t), \theta(t)} \) by its Fourier transform

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} D_{x}^{\alpha(t), \theta(t)} f(x) dx = -|\xi|^{\alpha(t)} e^{-i\text{sign}(\xi) \theta(t) \pi/2} \tilde{f}(\xi), \quad f \in S(\mathbb{R}),
\]

for any \( t \geq 0 \), so that its symbol can be written as

\[
D_{x}^{\alpha(t), \theta(t)}(\xi) = -|\xi|^{\alpha(t)} e^{-i\text{sign}(\xi) \theta(t) \pi/2}.
\]  

Thanks to a suitable regularization of hyper-singular integrals, the fractional derivative \( D_{x}^{\alpha(t), \theta(t)} \) defined above can be also represented as follows (see [29]):

\[
D_{x}^{\alpha(t), \theta(t)} = \frac{\Gamma(1 + \alpha(t))}{\pi} \sin \left( \frac{(\alpha(t) + \theta(t)) \pi}{2} \right) \int_{0}^{\pi/2} f(x + z) - f(x) dz + \frac{\Gamma(1 + \alpha(t))}{\pi} \sin \left( \frac{(\alpha(t) - \theta(t)) \pi}{2} \right) \int_{0}^{\pi/2} f(x - z) - f(x) dz,
\]

The adjoint of the time-varying RF fractional derivative, say \( \overline{D_{x}^{\alpha(t), \theta(t)}} \), is defined by

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \overline{D_{x}^{\alpha(t), \theta(t)}} f(x) dx = -\psi_{\alpha(t), \theta(t)}(\xi) \tilde{f}(\xi) = -|\xi|^{\alpha(t)} e^{i\text{sign}(\xi) \theta(t) \pi/2} \tilde{f}(\xi).
\]  

By (11), \( \overline{D_{x}^{\alpha(t), \theta(t)}} \) is the time-dependent generator of the so-called multistable process \( S_{\alpha, \theta}^{t} \) studied in [26] and [11], since its characteristic function reads

\[
\mathbb{E}e^{i\xi S_{\alpha, \theta}^{t}} = \exp \left\{ -\int_{0}^{t} \psi_{\alpha(s), \theta(s)}(\xi) ds \right\}.
\]

In the special case where \( \alpha(t) \leq 1 \) and \( \theta(t) = -\alpha(t) \) for each \( t \geq 0 \), the process \( S_{\alpha, \theta}^{t} \) coincides with the multistable subordinator given in [31] and the generator reduces to the variable order (left-sided) Riemann-Liouville derivative,

\[
\overline{D_{x}^{\alpha(t), -\alpha(t)}} f(x) := \begin{cases} \frac{(-1)}{\Gamma(1 - \alpha(t))} \frac{d}{dx} \int_{x}^{\pi} \frac{f(z)}{(x - z)^{\alpha(t)}} dz, & \alpha(t) \in (0, 1) \\ \frac{d}{dx} f(x), & \alpha(t) = 1, \ \forall t, \end{cases}
\]

with symbol \( -(i\xi)^{\alpha(t)} \).

Finally, in the symmetric case \( \theta(t) = 0 \ \forall t > 0 \), the propagator is self-adjoint (in agreement to Lemma [2]), and its generator is given by the self-adjoint RF derivative (which is also known as Riesz derivative) \( D_{x}^{\alpha(t), 0} = \overline{D_{x}^{\alpha(t), 0}} \), with symbol \( -|\xi|^\alpha(t) \), having the following representation

\[
D_{x}^{\alpha(t), 0} f(x) = \begin{cases} \frac{\Gamma(1 + \alpha(t))}{\pi} \sin \left( \frac{\alpha(t) \pi}{2} \right) \int_{0}^{\pi} f(x + z) + f(x - z) - 2f(x) dz, & \alpha(t) \neq 2 \\
\frac{d^{2}}{dx^{2}} f(x), & \alpha(t) = 2, \end{cases}
\]
3.2 Fractional logarithmic operator with time-varying parameters

By the Phillips’ theorem, the classical Geometric Stable process $S_{\alpha,\theta}(\Gamma)$ has generator

$$G_{b}^{\alpha,\theta} u = \int_{0}^{\infty} (T_s u - u) \frac{e^{-s/b}}{s} ds$$

where $T_t$ is the semigroup associated to $S_{\alpha,\theta}$. Since the characteristic function reads

$$\mathbb{E} e^{i\xi S_{\alpha,\theta}(\Gamma(t))} = \exp\{-t \ln(1 + b\psi_{\alpha,\theta}(\xi))\},$$

where $-\psi_{\alpha,\theta}(\xi) = -|\xi|^\alpha e^{i\frac{\pi}{2}\theta \text{sgn}(\xi)}$ is the symbol of $D_{\alpha,\theta}$, then, in the spirit of operational functional calculus, we wonder if the generator can be also written in the form of the fractional operator

$$G_{b}^{\alpha,\theta} = -\ln(1 - b\overline{D}^{\alpha,\theta}). \quad (25)$$

But, in order to give sense to (25), we need a broader discussion, which regards a large class of subordinated semigroups. Let $T_t$ be a contraction semigroup generated by $\mathcal{A}$ and let $T^f_t$ be the time changed semigroup, where $f(x) = \int_{0}^{\infty} (1 - e^{-sx}) \nu(ds)$ is the Bernstein function of the underlying subordinator with Lévy measure $\nu$. The Phillips’ theorem states that the generator of $T^f_t$ is

$$\mathcal{A}^f u = \int_{0}^{\infty} (T_s u - u) \nu(ds).$$

where $\text{Dom}(\mathcal{A}) \subset \text{Dom}(\mathcal{A}^f)$. We now wonder whether it is possible to write $\mathcal{A}^f = -f(-\mathcal{A})$ in the sense of operational functional calculus. According to [37], the answer is affirmative if $f$ is a complete Bernstein function.

We recall that $f$ is said to be a complete Bernstein function if and only if has the following representation

$$f(z) = \int_{0}^{\infty} \frac{z}{z + \tau} \sigma(d\tau)$$

for a suitable measure $\sigma$ such that $\int_{0}^{\infty} \frac{\sigma(d\tau)}{\tau + 1} < \infty$ (for further details see [38], ch.6).

By taking $\sigma(d\tau) = \frac{d\tau}{\tau} 1_{[b^{-1},\infty]}$, we have that $z \rightarrow \ln(1 + bz)$ is a complete Bernstein function with the following representation

$$\ln(1 + bz) = \int_{b^{-1}}^{\infty} \frac{z}{\tau + z} d\tau \quad \forall z \in \mathbb{C}.$$ 

Hence, by using [37], p.455, we obtain a nice integral representation of $G_{b}^{\alpha,\theta}$ involving the adjoint of the RF derivative $\overline{D}^{\alpha,\theta}$ and its resolvent $R_\tau := (\tau - \overline{D}^{\alpha,\theta})^{-1}$ only:

$$G_{b}^{\alpha,\theta} = -\ln(1 - b\overline{D}^{\alpha,\theta}) = \int_{b^{-1}}^{\infty} \frac{1}{\tau} \overline{D}^{\alpha,\theta}(\tau - \overline{D}^{\alpha,\theta})^{-1} d\tau \quad (26)$$

which is valid on $\text{Dom}(\overline{D}^{\alpha,\theta})$. In the spirit of [3], [4] and [5], we call (26) fractional logarithmic operator. Analogously, by considering a generic semigroup $T_t$ generated by $\mathcal{A}$, subordination to Gamma process produces the new generator

$$-\ln(1 - b\mathcal{A}) = \int_{b^{-1}}^{\infty} \frac{1}{\tau} \mathcal{A}(\tau - \mathcal{A})^{-1} d\tau. \quad (27)$$
Remark 5 In order to strengthen the connection to fractional calculus, we observe that for functions \( f \in S(\mathbb{R}) \) such that \( \hat{f}(\xi) \) is compactly supported in \( |\xi| < 1/b^{1/\alpha} \), the symbol of the generator can be expanded as

\[
\hat{G}_b^{\alpha,\theta}(\xi)\hat{f}(\xi) = -\ln(1 + b\psi_{\alpha,\theta}(\xi))\hat{f}(\xi) = \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n} \psi_{\alpha,\theta}^n(\xi)\hat{f}(\xi).
\]

Therefore \( G_b^{\alpha,\theta} \) has the form of a powers’ series of fractional operators, i.e.

\[
G_b^{\alpha,\theta} f(x) = \sum_{n=1}^{\infty} \frac{b^n}{n} \overline{D}_x^{\alpha,\theta} \overline{D}_x^{\alpha,\theta} f(x) \quad \text{for any } j \in \mathbb{N}, \text{Formula (28) can be checked by observing that}
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\xi x} \sum_{n=1}^{\infty} \frac{b^n}{n} \overline{D}_x^{\alpha,\theta} \overline{D}_x^{\alpha,\theta} f(x) dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{b^n}{n} \int_{-\infty}^{+\infty} e^{-i\xi x} \overline{D}_x^{\alpha,\theta} \overline{D}_x^{\alpha,\theta} f(x) dx
\]

\[
= \left[ \text{by (27)} \right]
\]

\[
= -\frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{b^n}{n} \psi_{\alpha,\theta}(\xi)\hat{f}(\xi) \int_{-\infty}^{+\infty} e^{-i\xi x} \overline{D}_x^{\alpha,\theta} \overline{D}_x^{\alpha,\theta} f(x) dx
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n b^n}{n} \psi_{\alpha,\theta}(\xi)^n \hat{f}(\xi)^n.
\]

In the following sections, we study propagators generated by two possible time-inhomogeneous extensions of (26). The first one is obtained by letting \( \alpha \) and \( \theta \) depend on \( t \), as follows.

**Definition 6** Let \( f \in S(\mathbb{R}) \), \( \alpha(t) \in (0, 2] \), \( |\theta(t)| \leq \min\{\alpha(t), 2 - \alpha(t)\} \), then, for any \( b > 0 \), \( t \geq 0 \),

\[
P_b^{\alpha(t),\theta(t)} := -\ln(1 - bD^{\alpha(t),\theta(t)}) = \int_{b-1}^{\infty} \frac{1}{z} D^{\alpha(t),\theta(t)}(z - D^{\alpha(t),\theta(t)})^{-1} dz. \quad (29)
\]

In the second case, instead, we allow the parameter \( b \) to be time-dependent and thus we use the following operator.

**Definition 7** Let \( f \in S(\mathbb{R}) \), \( \alpha \in (0, 2] \), \( |\theta| \leq \min\{\alpha, 2 - \alpha\} \), then, for any \( b(t) > 0 \), \( t \geq 0 \),

\[
P_{b(t)}^{\alpha,\theta} := -\ln(1 - b(t)D^{\alpha,\theta}) = \int_{b(t)-1}^{\infty} \frac{1}{z} D^{\alpha,\theta}(z - D^{\alpha,\theta})^{-1} dz. \quad (30)
\]
we have, from (34), that
\[ \mathcal{P}_{\alpha(t)}^{\theta(t)}(\xi) = -\ln(1 + b\psi_{\alpha(t)}^{\theta(t)}(\xi)) \] (31)
and
\[ \mathcal{P}_{\theta(t)}^{\alpha(t)}(\xi) = -\ln(1 + b(t)\psi_{\alpha(t)}^\theta(\xi)) \] (32)
respectively.

4 First-type inhomogeneous GS process

A natural way of defining a non-homogeneous generalization of the GS process is by considering a time-varying fractional index \( \alpha(t) \), with \( \alpha(t) \in (0, 2) \), \( \alpha(t) \neq 1 \), for any \( t \geq 0 \), similarly to what is done for defining the multistable process in [26]. This will influence the thickness of the tails, which are no more power-law as in the homogeneous case (see [25]). We also let the skewness parameter \( \theta \) vary with \( t \), under the assumption that \( |\theta(t)| \leq \min\{\alpha(t), 2 - \alpha(t)\} \), \( \forall t \geq 0 \). By assigning the two functions \( t \to \alpha(t) \) and \( t \to \theta(t) \) with the above properties, and assuming their continuity, we have the following:

**Definition 8 (Inhomogeneous GS process - I)** The process \( G^{I}_{\alpha,\theta}(t) := \{G^{I}_{\alpha,\theta}(t), t \geq 0\} \) is defined by the following joint characteristic function:

\[ \mathbb{E}e^{i \sum_{j=1}^{d} \xi_j G^{I}_{\alpha,\theta}(t_j)} := \exp \left\{- \int_{\mathbb{R}} \ln \left(1 + b(\pm i)^{\alpha(z)} \left(\sum_{j=1}^{d} \xi_j 1_{[0,t_j]}(z)\right)^{\alpha(z)}\right) dz \right\}, \tag{33} \]

for \( \xi_j \in \mathbb{R}, t_j \geq 0, d \in \mathbb{N} \).

The process defined above (which, for brevity, we will call \( GS^{I} \)), is an additive process, as can be checked by comparing (33) with (8) and taking into account the symbol of the usual GS given in (4).

**Remark 9** For the reader’s convenience, we present the two main special cases of (33), i.e. i) for \( \alpha(t) \in (0, 1) \), \( \theta(t) = \pm \alpha(t) \) (completely asymmetric case)

\[ \mathbb{E}e^{i \sum_{j=1}^{d} \xi_j G^{I}_{\alpha,\pm \alpha}(t_j)} := \exp \left\{- \int_{\mathbb{R}} \ln \left(1 + b(\pm i)^{\alpha(z)} \left(\sum_{j=1}^{d} \xi_j 1_{[0,t_j]}(z)\right)^{\alpha(z)}\right) dz \right\}, \tag{34} \]

\[ \mathbb{E}e^{i \sum_{j=1}^{d} \xi_j G^{I}_{\alpha,\theta}(t_j)} := \exp \left\{- \int_{\mathbb{R}} \ln \left(1 + b(\pm i)^{\alpha(z)} \left(\sum_{j=1}^{d} \xi_j 1_{[0,t_j]}(z)\right)^{\alpha(z)}\right) dz \right\}. \tag{35} \]

It is easy to check the independence of increments: indeed, in the first case, for \( \theta(t) = \pm \alpha(t) \), we have, from (34), that

\[ \mathbb{E}e^{i [G^{I}_{\alpha,\pm \alpha}(t_2) - G^{I}_{\alpha,\pm \alpha}(t_1)]} = \exp \left\{- \int_{\mathbb{R}} \ln \left(1 + b(\pm i)^{\alpha(z)} \xi^{\alpha(z)} 1_{[0,t_2]}(z) - 1_{[0,t_1]}(z)\right) dz \right\} \]

\[ = \exp \left\{- \int_{\mathbb{R}} \ln \left(1 + b(\pm i)^{\alpha(z)} \xi^{\alpha(z)} 1_{[t_1,t_2]}(z)\right) dz \right\} \]

\[ = \exp \left\{- \int_{t_1}^{t_2} \ln \left(1 + b(\pm i)^{\alpha(z)} \xi^{\alpha(z)}\right) dz \right\} \]
and thus
\[
E e^{iξ(G^I_{α,±α}(t_2) - G^I_{α,±α}(t_1))} = E e^{iξ(G^I_{α,±α}(t_2))} E e^{iξ(G^I_{α,±α}(t_1))}.
\]

A similar check can be done for \(θ(t) = 0\).

**Theorem 10** Let \(f \in \mathcal{S}(\mathbb{R})\) and let \(\mathcal{P}^b_{θ(t)}\) be the operator defined in Def. 6, then, for any \(b > 0, t ≥ s\), the following initial value problem
\[
\begin{cases}
\frac{∂}{∂t} u(x,t) = \mathcal{P}^b_{θ(t)} u(x,t) \\
u(x,s) = f(x)
\end{cases}
\]  
(36)
is satisfied by \(T^{G^I_{α,θ}}_{s,t} f(x) := E [f(G^I_{α,θ}(t)) | G^I_{α,θ}(s) = x]\).

**Proof.** The process \(G^I_{α,θ} := \{G^I_{α,θ}(t), t ≥ 0\}\) has characteristic function
\[
E e^{iξG^I_{α,θ}(t)} = \exp \left\{ - \int_0^t \ln \left( 1 + bψ_{α(t)}(ξ) \right) dξ \right\}, \quad ξ ∈ \mathbb{R}, \ t ≥ 0, \ b > 0. \tag{37}
\]
By taking the Fourier transform with respect to \(x\), we can rewrite the first equation in (36) as follows:
\[
\frac{∂}{∂t} \widehat{u}(ξ,t) = \mathcal{P}^b_{θ(t)}(ξ) \widehat{u}(ξ,t) = \left[ \text{by (37)} \right] = -\ln(1 + bψ_{α(t)}(ξ)) \widehat{u}(ξ,t),
\]
and Lemma 1 (ii) gives the result. ■

**Remark 11** One could observe that (36) does not apparently coincide with the so-called "forward" equation for propagators (consult, for example, [34]). Indeed, by a simple calculation, we have
\[
\frac{d}{dt} T^{G^I_{α,θ}}_{s,t} f = \lim_{h→0^+} \frac{T^{G^I_{α,θ}}_{s,t+h} - T^{G^I_{α,θ}}_{s,t}}{h} = \lim_{h→0^+} \frac{T^{G^I_{α,θ}}_{s,t+h} - T^{G^I_{α,θ}}_{s,t}}{h} = \lim_{h→0^+} \frac{T^{G^I_{α,θ}}_{s,t+h} - I}{h} f = T^{G^I_{α,θ}}_{s,t} \mathcal{A}_t f
\]
Since, in general, \(T^{G^I_{α,θ}}_{s,t}\) and \(\mathcal{A}_t\) do not commute, it is not true that \(u(x,t) = T^{G^I_{α,θ}}_{s,t} f(x)\) solves the non-autonomous equation \(\frac{d}{dt} u(x,t) = \mathcal{A}_t u(x,t)\) under \(u(x,s) = f(x)\). However, in our case, \(f\) lives in the Schwartz space, and it is easy to check that \(T^{G^I_{α,θ}}_{s,t}\) and \(\mathcal{A}_t\) (which are given by (10) and (11)), commute.

**4.0.1 On the subordinator**
In the special case where \(θ(t) = -α(t)\) and \(α(t) ∈ (0, 1)\) for any \(t > 0\), we can easily evaluate the Lévy measure of the process \(G^I_{α,-α}\), which is an inhomogeneous subordinator in the sense
of \[32\]. Recall that \(\psi_{\alpha(t),-\alpha(t)}(\xi) = (-i\xi)^{\alpha(t)}\) and thus the Laplace transform of \(G_{\alpha,-\alpha}^I\) can be written as follows

\[
\mathbb{E}e^{-\lambda G_{\alpha,-\alpha}^I(t)} = \exp\left(-\int_0^t \ln(1 + b\lambda^{\alpha(s)})ds\right), \quad \lambda > 0. \tag{38}
\]

The integral in (38) is finite, since

\[
\int_0^t \ln(1 + b\lambda^{\alpha(s)})ds \leq b \int_0^t \lambda^{\alpha(s)}ds < \infty,
\]

for \(\alpha(s) \in (0, 1)\), for any \(s\).

**Lemma 12** The time dependent Lévy measure of \(G_{\alpha,-\alpha}^I\) is given by

\[
\nu_{t}^{G_{\alpha,-\alpha}}(dx) = x^{-1} \alpha(t) E_{\alpha(t)}(-x^{\alpha(t)}/b)dx, \tag{39}
\]

where, for any \(s \geq 0\), \(E_{\alpha(s)}(x)\) denotes the Mittag-Leffler function \(E_{\alpha(s)}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha(s)j+1)}\).

**Proof.** From (39), by applying formula (1.9.13), p.47 in [17], for any fixed \(s\), we can write

\[
\int_0^\infty (e^{-\lambda x} - 1) \nu_{t}^{G_{\alpha,-\alpha}}(x)dx = -\alpha(t) \int_0^\infty \left(\int_0^\lambda e^{-zx}dz\right) E_{\alpha(t)}(-x^{\alpha(t)}/b)dx
\]

\[
= -\alpha(t) \int_0^\lambda \left(\int_0^\infty e^{-zx} E_{\alpha(t)}(-x^{\alpha(t)}/b)dx\right)dz
\]

\[
= -b\alpha(t) \int_0^\lambda \frac{z^{\alpha(t)-1}}{b z^{\alpha(t)} + 1}dz
\]

\[
= -\ln(1 + b\lambda^{\alpha(t)}),
\]

which agrees with (38). We now check that the condition \(\int_0^\infty \frac{x}{1 + x} \nu_{t}^{G_{\alpha,-\alpha}}(x)dx < \infty\) is satisfied, as follows

\[
\int_0^\infty \frac{x}{1 + x} \nu_{t}^{G_{\alpha,-\alpha}}(x)dx = \alpha(t) \int_0^\infty \frac{1}{1 + x} E_{\alpha(t)}(-x^{\alpha(t)}/b)dx
\]

\[
= \frac{1}{b} \alpha(t) M^+ \int_0^\infty \frac{1}{1 + x b + x^{\alpha(t)}}dx < \infty,
\]

where \(M^+\) is a positive constant. \(\blacksquare\)

It is easy to check that, letting \(\alpha(t) \to 1\), for any \(t\), we obtain from (39) that

\[
\lim_{\alpha(t) \to 1} \nu_{t}^{G_{\alpha,-\alpha}}(x)dx = x^{-1} e^{-x/b} = \nu_I(dx)
\]
while, for $\alpha(t) = \alpha$, for any $t \geq 0$, we get the Lévy measure of the standard GS process, which reads $\nu^G_{\alpha,-\alpha}(x)dx = x^{-1}\alpha E_\alpha(-x^\alpha/b)$ (see [7]).

Finally, we show how the tails' behavior of the density of the process $G_{\alpha, -\alpha}^I$, for any fixed $t$, differs from those holding for both the stable and geometric stable random variables (see [35], p. 17, and [23] respectively). The latter is, obviously, obtained in the special case where $\alpha(s) = \alpha$, for any $s$.

**Theorem 13** Let $\alpha_t^* := \max_{0 \leq s \leq t} \alpha(s)$, then, for $x \to \infty$,

$$P(G_{\alpha, -\alpha}^I(t) > x) \sim \frac{b}{\Gamma(1 - \alpha_t^*)} \int_0^t \alpha(s)x^{-\alpha(s)}ds, \quad t \geq 0. \quad (40)$$

**Proof.** By (38), we can write that

$$\int_0^{+\infty} e^{-\eta x} P(G_{\alpha, -\alpha}^I(t) > x)dx = 1 - \frac{E e^{-\eta G_{\alpha, -\alpha}^I(t)}}{\eta} = 1 - \exp\left\{-\int_0^t \ln(1 + b\eta^{\alpha(s)})ds\right\} \sim b\eta^{\alpha_t^* - 1} \int_0^t \alpha(s)\eta^{\alpha(s) - \alpha_t^*}ds, \quad (41)$$

for any fixed $t$ and for $\eta \to 0$. The integral in the last line of (41) is a regularly varying function in $1/\eta$, since, for any real $k$, by the mean value theorem we have that

$$\frac{\int_0^t \alpha(s)(k\eta)^{\alpha(s) - \alpha_t^*}ds}{\int_0^t \alpha(s)\eta^{\alpha(s) - \alpha_t^*}ds} = k^{\alpha(\bar{s}_t) - \alpha_t^*},$$

where $\bar{s}_t \in (0, t]$. Then, by applying the Tauberian theorem (see Theorem XIII-5-4, p.446, in [12]), as $x \to \infty$ we obtain

$$P(G_{\alpha, -\alpha}^I(t) > x) \sim \frac{bx^{-\alpha_t^*}}{\Gamma(1 - \alpha_t^*)} \int_0^t \alpha(s)x^{\alpha_t^* - \alpha(s)}ds.$$  

**Remark 14** An analogous result is proved to hold, in [3], for the multistable symmetric process.

5 Second-type inhomogeneous GS process

We here recall the definition of time-inhomogeneous (or additive) Gamma subordinator, which we denote by $\Gamma^I := \{\Gamma^I(t), t \geq 0\}$. The latter has been studied for the first time in [8] and then considered in [32] as a remarkable example among inhomogeneous subordinators. For an assigned strictly positive and bounded function $s \to b(s)$, the process $\Gamma^I$ is completely determined by its finite dimensional distributions

$$E e^{\sum_{j=1}^d \xi_j \Gamma^I(t_j)} := \exp\left\{-\int_\mathbb{R} \ln \left(1 - ib(s)\sum_{j=1}^d \xi_j 1_{(0, t_j]}(s)\right)ds\right\}, \quad (42)$$
corresponding to the time-dependent Lévy density
\[ \nu_t^{\Gamma_I}(x) = x^{-1} e^{-x/b(t)}, \quad x > 0. \] (43)

It is evident that, in the special case \( b(t) = b \), the process \( \Gamma_I \) reduces to the standard Gamma subordinator \( \Gamma \). The function \( s \to b(s) \) must be meant as a time-dependent scale parameter, since it is proved in [8] that for a.e. path \( w \) the following construction holds
\[ \Gamma_I(t, w) = \int_0^t b(s) \Gamma(s, w) ds \] (44)

where \( \Gamma \) is the Gamma subordinator corresponding to \( b = 1 \). Formula (44) shows that \( \Gamma_I \) reduces to the \( \Gamma \) subordinator by a change of scale, and therefore \( \Gamma_I \) is statistically tractable, as opposed to the usual case of non-stationary processes.

We prove now the following result concerning the governing equation of the additive Gamma subordinator, by considering that, for \( \alpha = 1, \theta = -1 \), equations (30) and (32) reduce to
\[ P_{b(t)}(x, f(x)) = -\ln(1 - b(t) d/dx) f(x) = \int_0^{\infty} \frac{1}{z} \frac{d}{dx} (z - \frac{d}{dx})^{-1} f(x) dz. \]
and
\[ \hat{P}_{b(t)}(\xi) = -\ln(1 - ib(t)\xi) \] (45)
respectively.

**Lemma 15** Let \( f \in S(\mathbb{R}) \). Then the propagator \( T_{s,t}^{\Gamma_I} f(x) := \mathbb{E} [ f(\Gamma_I(t)) \big| \Gamma_I(s) = x ] \) associated to the process \( \Gamma_I \) satisfies the following initial value problem
\[ \begin{cases} \frac{\partial}{\partial t} u(x, t) = -\ln(1 - b(t) \frac{\partial}{\partial x}) u(x, t), & t \geq s \\ u(x, s) = f(x) \end{cases} \]
where the operator on the right side must be meant in the sense of [27].

**Proof.** We take the first time-derivative of (42), in the case \( d = 1 \) with \( t_1 = t \) and \( \xi_1 = \xi \), so that we get
\[ \frac{\partial}{\partial t} \mathbb{E}e^{i\xi \Gamma_I(t)} = -\ln(1 - ib(t)\xi) \mathbb{E}e^{i\xi \Gamma_I(t)} = \hat{P}_{b(t)}(\xi) \mathbb{E}e^{i\xi \Gamma_I(t)}, \]
and considering Lemma 1 (ii), the proof is complete. \(\blacksquare\)

In order to let this process verify the useful property of finite exponential moments, we make the further assumption that \( b(t) < K \), for any \( t \geq 0 \) and for a constant \( K < 1 \). Thus, for any fixed \( t \) and for \( |u| \leq M \), with \( M > 1 \), we have that
\[ \int_0^{+\infty} e^{ux} \mu_t^{\Gamma_I}(x) dx = \int_0^t \int_0^{+\infty} x^{-1} e^{x(u-1/b(s))} dx ds < \infty, \] (46)
since we can choose $K = 1/M$, so that $1/b(s) > M$. This is a necessary and sufficient condition for the finiteness of the moment generating function (see [10]), which, thus, is finite for any $|\gamma| \leq M$, and reads

$$
\mathbb{E} e^{\gamma \Gamma^I(t)} = \exp \left\{ - \int_0^t \ln (1 - \gamma b(s)) \, ds \right\},
$$

so that we get

$$
\mathbb{E} \Gamma^I(t) = \int_0^t b(s) \, ds \quad \text{(47)}
$$

$$
\mathbb{V} \left[ \Gamma^I(t) \right] = \int_0^t b(s)^2 \, ds
$$

As a consequence of (46), we can state that the additive Gamma process $\Gamma^I$ is a special semimartingale (see [15]) and then it is suitable for financial applications (see [16]), when $b(t) < K < 1$.

The additive Gamma process is the fundamental ingredient to construct the following.

**Definition 16 (Inhomogeneous GS process - II)** Let $\{S_{\alpha,\theta}(t), t \geq 0\}$ be an $\alpha$-stable process defined in (7) and $\Gamma^I$ be a inhomogeneous Gamma subordinator, independent from $S_{\alpha,\theta}$, we define $G_{\alpha,\theta}^{III}(t) := \{G_{\alpha,\theta}^{III}(t), t \geq 0\}$ by the following subordination (see section 2.2)

$$
G_{\alpha,\theta}^{III}(t) := S_{\alpha,\theta}(\Gamma^I(t)), \quad t \geq 0.
$$

**Theorem 17** Let $f \in S(\mathbb{R})$, then $T_{s,t}^{III} f(x) := \mathbb{E} \left[ f \left( G_{\alpha,\theta}^{III}(t) \right) \right] x$ satisfies the following initial value problem

$$
\begin{cases}
\frac{\partial}{\partial t} u(x,t) = \mathcal{P}_{b(t),x}^{\alpha,\theta} u(x,t), & t \geq s, \\
u(x,s) = f(x),
\end{cases}
$$

where the operator $\mathcal{P}_{b(t),x}^{\alpha,\theta}$ is defined in Def[7].

**Proof.** For any $t > 0$, we can evaluate the characteristic function of $S_{\alpha,\theta}(\Gamma^I(t))$, by a standard conditioning argument

$$
\mathbb{E} e^{i \xi S_{\alpha,\theta}(\Gamma^I(t))} = \mathbb{E} \left\{ \mathbb{E} \left[ e^{i \xi S_{\alpha,\theta}(\Gamma^I(t))} \Big| \Gamma^I(t) \right] \right\} = \mathbb{E} \exp\{-\psi_{\alpha,\theta}(\xi) \Gamma^I(t)\}
$$

$$
= [\text{by (1)}] = \exp \left\{ - \int_0^t \ln [1 + b(s)\psi_{\alpha,\theta}(\xi)] \, ds \right\},
$$

where $b(s) \geq 0$, for any $s \geq 0$ and $\int_0^t b(s) \, ds < \infty$. By taking the Fourier transform of the first equation in (49), we get, by (32)

$$
\frac{\partial}{\partial t} \hat{u}(\xi,t) = \mathcal{P}_{b(t)}^{\alpha,\theta}(\xi) \hat{u}(\xi,t)
$$

$$
= - \ln(1 + b(t)\psi_{\alpha,\theta}(\xi)) \hat{u}(\xi,t)
$$

and Lemma [1] (ii) gives the result. ■

Of course, the same considerations made in Remark [11] are also valid for Theorem 17.
Remark 18 As suggested by (44) for the Gamma process, we observe that the process defined in Def (16) can be obtained by a homogeneous GS by a time-dependent change of scale, and this makes $GS^{II}$ statistically tractable, just as $\Gamma^I$. Indeed, let $G_{\alpha,\theta}$ be a homogeneous geometric stable process such that
\[
\mathbb{E} e^{i\xi G_{\alpha,\theta}(t)} = e^{-t \ln (1 + \psi_{\alpha,\theta}(\xi))}
\]

We divide the interval $[0, t]$ into $n$ subintervals of length $t/n$ and write the telescopic series $G_{\alpha,\theta}(t) = \sum_{k=0}^{n-1} (G_{\alpha,\theta}(t_{k+1}) - G_{\alpha,\theta}(t_k))$. In order to change the scale, we let the increment $G_{\alpha,\theta}(t_{k+1}) - G_{\alpha,\theta}(t_k)$ be changed into $b(t_k)^{\frac{1}{\alpha}} (G_{\alpha,\theta}(t_{k+1}) - G_{\alpha,\theta}(t_k))$. Then, for any $t \geq 0$, the following limit as $n \to \infty$ holds in distribution
\[
\sum_{k=0}^{n-1} b(t_k)^{\frac{1}{\alpha}} (G_{\alpha,\theta}(t_{k+1}) - G_{\alpha,\theta}(t_k)) \xrightarrow{d} G_{\alpha,\theta}(t)
\]
(51)

Indeed, by independence of the increments,
\[
\mathbb{E} \exp \left\{ i\xi \sum_{k=0}^{n-1} b(t_k)^{\frac{1}{\alpha}} (G_{\alpha,\theta}(t_{k+1}) - G_{\alpha,\theta}(t_k)) \right\} = \prod_{k=0}^{n-1} \mathbb{E} \exp \left\{ i\xi b(t_k)^{\frac{1}{\alpha}} (G_{\alpha,\theta}(t_{k+1}) - G_{\alpha,\theta}(t_k)) \right\} = \prod_{k=0}^{n-1} \exp \left\{ -(t_{k+1} - t_k) \ln (1 + b(t_k)\psi_{\alpha,\theta}(\xi)) \right\}
\]
\[
\lim_{n \to \infty} \exp \left\{ - \int_{0}^{t} \ln (1 + b(\tau)\psi_{\alpha,\theta}(\xi)) d\tau \right\}.
\]

We derive now the tails’ behavior of the density of the process $G_{\alpha,\theta}$, for any fixed $t$. In this case, contrary to $GS^I$, we can prove the following result in the more general setting, i.e. for any $\alpha \in (0, 2]$ and $|\theta| \leq \min \{\alpha, 2 - \alpha\}$, not only in the completely positively skewed case. Indeed, we can resort here to the subordinating relation (48).

Theorem 19 Let $\alpha \in (0, 2]$ and $|\theta| \leq \min \{\alpha, 2 - \alpha\}$, then we have that
\[
\begin{align*}
\lim_{x \to \infty} x^{\alpha} P(G_{\alpha,\theta}^{II}(t) > x) &= \frac{C_{\alpha,\theta}}{\Gamma(1-\alpha)} \int_{0}^{t} b(s) ds, \quad t \geq 0, \\
\lim_{x \to -\infty} x^{\alpha} P(G_{\alpha,\theta}^{II}(t) < -x) &= \frac{\overline{C}_{\alpha,\theta}}{\Gamma(1-\alpha)} \int_{0}^{t} b(s) ds,
\end{align*}
\]
(52)

where $C_{\alpha,\theta} = \frac{1}{2} \left[ 1 - \tan(\pi \theta / 2) \tan(\pi \alpha / 2) \right]$ and $\overline{C}_{\alpha,\theta} = \frac{1}{2} \left[ 1 + \tan(\pi \theta / 2) \tan(\pi \alpha / 2) \right]$.

Proof. We prove the first relation in (52), the second one can be obtained by similar arguments. We apply Property 1.2.15 in [35], p.16, regarding the tail behavior of the stable process, which in our parametrization of stable laws reads
\[
\lim_{x \to \infty} x^{\alpha} P(S_{\alpha,\theta}(t) > x) = \frac{C_{\alpha,\theta} t}{\Gamma(1-\alpha)}.
\]

\footnote{In agreement with part of the literature, the authors in [35] express the characteristic function of stable distributions using the parameters triplet $\alpha, \sigma, \beta$ (see [35], p.7). Our parametrization with $(\alpha, \theta)$, due to Feller, can be obtained by setting $\sigma = (\cos \frac{\pi \theta}{2})^{\frac{1}{\alpha}}$ and $\theta = \frac{1}{2} \arctan(-\beta \tan \frac{\pi \alpha}{2}).$}
By Def.16, we have
\[
\lim_{x \to \infty} x^\alpha P(S_{\alpha,\theta}(\Gamma(t)) > x) = \lim_{x \to \infty} x^\alpha \int_0^\infty P(S_{\alpha,\theta}(\Gamma(t)) > x | \Gamma(t) = z) P(\Gamma(t) \in dz)
\]
\[
= \int_0^\infty \lim_{x \to \infty} x^\alpha P(S_{\alpha,\theta}(z) > x) P(\Gamma(t) \in dz)
\]
\[
= \frac{C_{\alpha,\theta}}{\Gamma(1-\alpha)} E \Gamma(t),
\]
which gives (52), by (47).

5.0.1 Computation of the Lévy measure

We now compute explicitly the time-dependent Lévy measure of the GS\textsuperscript{II} process. As far as we know, this expression is not even known for the homogeneous geometric stable process, with the exception of the case of the GS subordinator (see, for example, [7]). In the following, let \(H_{m,n}^{p,q}\) denote the H-function defined as (see [30] p.13):

\[
H_{m,n}^{p,q}[z | (a_1, A_1) \ldots (a_p, A_p), (b_1, B_1) \ldots (b_q, B_q)] = \frac{1}{2\pi i} \int_C \left\{ \prod_{j=1}^m \Gamma(b_j + B_j s) \right\} \left\{ \prod_{j=1}^n \Gamma(1 - a_j - A_j s) \right\} z^{-s} ds
\]

where \(z \neq 0\), \(m, n, p, q \in \mathbb{N}_0\), for \(0 \leq m \leq q\), \(0 \leq n \leq p\), \(a_j, b_j \in \mathbb{R}\), \(A_j, B_j \in \mathbb{R}^+\), for \(i = 1, \ldots, p\), \(j = 1, \ldots, q\) and \(C\) is a contour such that the following condition is satisfied

\[
A\lambda(b_j + \nu) \neq B_j(a\lambda - k - 1), \quad j = 1, \ldots, m, \lambda = 1, \ldots, n, \nu, k = 0, 1, \ldots (53)
\]

**Lemma 20** The Lévy measure of GS\textsuperscript{II} defined in Def.16 is given, for \(\alpha \in (0, 1)\), by

\[
\nu_t^{G_{\alpha,\theta}}(dx) = \alpha |x|^{-1} dx H_{3,2}^{1,2} \left[ \frac{b(t)}{|x|^\alpha} \right] \left( \frac{1}{1, \alpha} \right) \left( \frac{1}{1, \frac{\alpha-\theta}{2}} \right), \quad x \neq 0, |\theta| \leq \alpha (54)
\]

while, for \(\alpha \in (1, 2)\), by

\[
\nu_t^{G_{\alpha,\theta}}(dx) = \alpha b(t)^{-1/\alpha} dx H_{3,2}^{1,2} \left[ \frac{b(t)^{1/\alpha}}{|x|} \right] \left( \frac{1}{1, \frac{1}{\alpha}} \right) \left( 1 + \frac{1}{\alpha} \right) \left( \frac{\alpha-\theta}{2\alpha}, \frac{\alpha-\theta}{2\alpha} \right), \quad x \neq 0, |\theta| \leq 2-\alpha. (55)
\]

**Proof.** We apply the series representation of the stable law (see [12], Lemma 1, p.583), together with the reflection property of the Gamma function. Then, by considering the
subordinating relationship \( S_{\alpha,\theta}(\Gamma^I) \) together with (43), we can write, for \( x > 0 \) and \( \alpha \in (0, 1) \),

\[
\nu^{G_{\alpha,\theta}}_t(dx) = \int_0^{+\infty} p_\alpha(x; \theta, \tau) \tau^{-1} e^{-\tau/b(t)} d\tau
\]

\[
= \int_0^{+\infty} \frac{1}{\tau^{1/\alpha}} p_\alpha \left( \frac{x}{\tau^{1/\alpha}}; \theta, 1 \right) \tau^{-1} e^{-\tau/b(t)} d\tau
\]

\[
= \frac{dx}{\pi x} \sum_{k=1}^{\infty} \left( -x^{-\alpha} \right)^k \frac{\Gamma(k/\alpha + 1) \sin(k(\theta - \alpha)\pi/2)}{k!} \int_0^{+\infty} \tau^{k-1} e^{-\tau/b(t)} d\tau
\]

\[
= -\frac{\alpha dx}{\pi x} \sum_{k=1}^{\infty} \frac{(-b(t)x^{-\alpha})^k}{(k - 1)!} \Gamma(k(\alpha - \theta)/2) \Gamma(1 - (k\alpha - \theta)/2)
\]

\[
= \frac{adxb(t)x^{-\alpha-1}}{2\pi i} \int \frac{\Gamma(\alpha - s)\Gamma(s)\Gamma(1 - s)}{\Gamma(\alpha - \theta)(1 - s)/2} \Gamma(1 - (\alpha - \theta)(1 - s)/2) ds
\]

\[
= adxb(t)x^{-\alpha-1} H_{3,2}^{1,2} \left[ \begin{array}{c} b(t) \\ x^{-\alpha} \end{array} \right] \Gamma((\alpha - \theta)(1 - s)/2) \Gamma(1 - (\alpha - \theta)(1 - s)/2)
\]

which coincides with (54), by applying (1.60) of (30), with \( \sigma = 1 \). It is immediate to check that condition (53) is satisfied. As far as the case \( \alpha \in (1, 2) \) is concerned, we have that

\[
\nu^{G_{\alpha,\theta}}_t(dx) = \int_0^{+\infty} \frac{1}{\tau^{1/\alpha}} p_\alpha \left( \frac{x}{\tau^{1/\alpha}}; \theta, 1 \right) \tau^{-1} e^{-\tau/b(t)} d\tau
\]

\[
= \int_0^{+\infty} \frac{1}{\tau^{1/\alpha}} p_\alpha \left( \frac{x}{\tau^{1/\alpha}}; \theta, 1 \right) \tau^{-1} e^{-\tau/b(t)} d\tau
\]

\[
= \frac{dx}{\pi x} \sum_{k=1}^{\infty} \left( -x^{-\alpha} \right)^k \frac{\Gamma(k/\alpha + 1) \sin(k(\theta - \alpha)\pi/2)}{k!} \int_0^{+\infty} \tau^{k-1} e^{-\tau/b(t)} d\tau
\]

\[
= -\frac{\alpha dx}{\pi x} \sum_{k=1}^{\infty} \frac{(-b(t)x^{-\alpha})^k}{(k - 1)!} \Gamma(k/\alpha) \Gamma(-k/\alpha) \sin(k(\alpha - \theta)\pi/2)
\]

\[
= -\frac{adxb(t)x^{-\alpha-1}}{2\pi i} \int \frac{\Gamma((l + 1)/\alpha)\Gamma(1 - (l + 1)/\alpha)}{\Gamma((l + 1)/(\alpha - \theta)/2)\Gamma(1 - (l + 1)/(\alpha - \theta)/2)} ds
\]

\[
= \frac{adxb(t)x^{-\alpha-1}}{2\pi i} \int \frac{\Gamma((l + 1)/\alpha)\Gamma(1 - (l + 1)/\alpha)}{\Gamma((l + 1)/(\alpha - \theta)/2)\Gamma(1 - (l + 1)/(\alpha - \theta)/2)} ds
\]

\[
= adxb(t)x^{-\alpha-1} H_{2,3}^{1,2} \left[ \begin{array}{c} x/b(t)^{1/\alpha} \\ 1 - \frac{1}{\alpha} \end{array} \right] \Gamma((1 - s)/\alpha) \Gamma(1 - (1 - s)/2\alpha)
\]

which coincides with (53) by (1.58) in (30). For \( x < 0 \) similar steps lead to (54) and (55), respectively, for \( \alpha \in (0, 1) \) and \( \alpha \in (1, 2) \), by considering formula (6.4) in (12).
Remark 21 We can check that (54) coincides with the Lévy measure of the GS subordinator, for \( \theta = -\alpha \), \( b(t) = b \), for any \( t \), and \( \alpha \in (0,1) \) (see [7]). Indeed, in this case, we can write (54) as follows:

\[
\nu^{GII}_t (dx) = \frac{\alpha}{x} dx H^1_{2,1} \left[ \frac{b(t)}{x^\alpha} \left| \frac{1}{1,1} \right| \right] \]

\[
= \left[ \text{by (1.58) in [30]} \right]
\]

\[
= \frac{\alpha}{x} dx H^0_{1,2} \left[ \frac{x^\alpha}{b(t)} \left| \frac{0,0,0,0,0}{0,0,0,0,0} \right| \right] \]

\[
= \left[ \text{by (1.125) in [30]} \right]
\]

\[
= \alpha x^{-1} E_\alpha (-x^\alpha/b(t)) dx,
\]

where \( E_\alpha(z) \) denotes the Mittag-Leffler function, for \( \alpha > 0 \), \( z \in \mathbb{C} \). Moreover, for \( \theta = -\alpha \) and by letting \( \alpha \to 1^- \), we get (43): indeed

\[
\lim_{\alpha \to 1^-} \nu^{GII}_t (dx) = \frac{dx}{x} H^0_{1,0} \left[ \frac{b(t)}{x} \left| \frac{1,1}{1,1} \right| \right] = \frac{dx}{x} e^{-x/b(t)}, \quad (57)
\]

by applying (1.58) and (1.125) of [30].

Remark 22 On the other hand, we can check that (55), in the special case \( \alpha = 2 \), \( b(t) = b \), for any \( t \), and \( \theta = 0 \), coincides with the Lévy measure of the VG process (see formula (13) in [27]). Indeed, from the last line of (56), we have that

\[
\nu^{GII}_{\alpha,\theta}(dx) = 2 dx H^2_{2,3} \left[ \frac{|x|}{\sqrt{b}} \left| \frac{1}{2,1,1} \right| \left( \frac{1,1,1}{1,1,1} \right) \left( \frac{1,1,1}{1,1,1} \right) \left( \frac{1,1,1}{1,1,1} \right) \right] 
\]

\[
= 2 dx \frac{1}{\sqrt{b}} 2 \pi i \int_L \frac{\Gamma(s) \Gamma \left( \frac{s}{2} - \frac{1}{2} \right) \left( |x|/\sqrt{b} \right)^{-s}}{s-1} ds
\]

\[
= 4 dx \frac{1}{\sqrt{b}} 2 \pi i \int_L \frac{\Gamma(s) \left( |x|/\sqrt{b} \right)^{-s}}{s-1} ds
\]

\[
= \left[ \text{by (1.37)-(1.38) in [30]} \right]
\]

\[
= 4 dx \sum_{\nu=-\infty}^{\infty} \lim_{s \to \nu} (s+\nu) \frac{\Gamma(s)}{s-1} \left( \frac{|x|}{\sqrt{b}} \right)^{-s}
\]

\[
= 4 dx \sum_{\nu=-\infty}^{\infty} \lim_{s \to \nu} \frac{\Gamma(s+\nu+1)}{(s+\nu-1)...s-1} \left( \frac{|x|}{\sqrt{b}} \right)^{-s}
\]

\[
= 4 dx \sum_{\nu=-\infty}^{\infty} \frac{(-1)^{\nu+1}}{(\nu+1)!} \left( \frac{|x|}{\sqrt{b}} \right)^{\nu+1} = 4 dx \frac{e^{-|x|/\sqrt{b}}}{|x|}.
\]

Again, for \( \theta = -\alpha \) and by letting \( \alpha \to 1^+ \), we can easily obtain from (55) formula (43), by considering also (1.60) in [30].

21
5.1 Inhomogeneous Variance Gamma process

For $\theta = 0$ and $\alpha = 2$, we have the important special case represented by the time-inhomogeneous VG process (hereafter $VG^I$), which we define as $\{B(\Gamma^I(t)), t \geq 0\}$, where $B$ is a Brownian motion such that $B(t) \sim \mathcal{N}(0, 2t)$. We note that a pioneering definition of this process can be found in [28]. Its characteristic function reads

$$Ee^{i\xi B(\Gamma^I(t))} = \exp \left\{ - \int_0^t \ln \left( 1 + b(s)\xi^2 \right) ds \right\}$$

(58)

and its transition operator satisfies the following initial value problem

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) = -\ln \left( 1 - b(t) \frac{\partial^2}{\partial x^2} \right) u(x, t) \\ u(x, s) = f(x). \end{cases}$$

Thus the generator of $\{B(\Gamma^I(t)), t \geq 0\}$ can be written as $A_t = -\ln \left( 1 - b(t) \frac{\partial^2}{\partial x^2} \right)$, by recalling the representation (30) together with (24).

The mean-square displacement of $VG^I$ can be evaluated as follows:

$$E B^2(\Gamma^I(t)) = E \Gamma^I(t) = \int_0^t b(s)ds,$$

which is finite, by assumption. We note that the choice of $b(t)$ determines the asymptotic properties of the process, which can be either diffusive, sub-diffusive, or super-diffusive.

We further observe that, also in the non-homogeneous case, the $VG^I$ process can be represented as the difference of two independent inhomogeneous gamma subordinators $\Gamma^I_1$ and $\Gamma^I_2$, each having the following characteristic functions

$$E e^{i\xi \Gamma^I_1(t)} = E e^{i\xi \Gamma^I_2(t)} = \exp \left\{ - \int_0^t \ln \left( 1 - i\sqrt{b(s)}\xi \right) \right\}.$$

Indeed, we can write

$$E e^{i\xi \Gamma^I_1(t) - i\xi \Gamma^I_2(t)} = \exp \left\{ - \int_0^t \left[ \ln \left( 1 - i\sqrt{b(s)}\xi \right) + \ln \left( 1 + i\sqrt{b(s)}\xi \right) \right] ds \right\}$$

$$= \exp \left\{ - \int_0^t \ln \left[ \left( 1 - i\sqrt{b(s)}\xi \right) \left( 1 + i\sqrt{b(s)}\xi \right) \right] ds \right\}$$

$$= \exp \left\{ - \int_0^t \ln \left( 1 + b(s)\xi^2 \right) ds \right\} = E e^{i\xi B(\Gamma^I(t))}.$$

and this property is very important for financial applications, in order to model stochastic volatility.

Under the additional assumption that $b(t) < K$, for any $t \geq 0$ and for a constant $K < 1$, it is easy to check that the moment generating function of $VG^I$ is finite for any $|\gamma| \leq M$, with $M > 1$, since we can choose in this case $K = 1/M^2$, so that

$$E e^{\gamma B(\Gamma^I(t))} = \exp \left\{ \int_0^t \ln \left( 1 - \gamma^2 b(s) \right) ds \right\} < \infty.$$
As a consequence of the subordination by the gamma process, the VG process has infinitely many small jumps and a finite number of large jumps. The subordination implies the introduction of a new parameter (with respect to the Brownian case) and enables the VG model to capture the negative skewness and excess kurtosis, which are often displayed, in financial applications, by the log returns. The variance parameter of the gamma process controls the degree of randomness of subordination: indeed large values of the variance result in fatter tails of the density. This feature is confirmed, in the inhomogeneous case, by considering Theorem 13, for $\alpha = 2$ and $\theta = 0$.

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