Complete loop quantization of a dimension 1+2

Lorentzian gravity theory

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Abstract. De Sitter Chern-Simons gravity in $D = 1 + 2$ spacetime is known to possess an extension with a Barbero-Immirzi like parameter. We find a partial gauge fixing which leaves a compact residual gauge group, namely SU(2). The compacticity of the residual gauge group opens the way to the usual LQG quantization techniques. We recall the exemple of the LQG quantization of SU(2) CS theory with cylindrical space topology, which thus provides a complete LQG of a Lorentzian gravity model in 3-dimensional space-time.

1. Introduction
The gauge group of $D$-dimensional Lorentzian spacetime, the Lorentz group $SO(1,D-1)$, is noncompact. In the loop quantization framework\(^1\), this is known to make difficult a proper definition of the internal product in the vector space of quantum states.

For space-time dimension 1 + 3, the usual way out is, starting from the Palatini-Holst action [2], to partially fix the gauge (the “time gauge”) in such a way that the residual group be the compact group $SO(3)$ or $SU(2)$. The presence of the Barbero-Immirzi parameter [3] $\gamma$ appears to be crucial. However, for other dimensions, with the Lorentz group being $SO(D,1)$, the time gauge does not lead to the compact gauge group $SO(D)$. But a recent proposal has been given by the authors of Ref. [4], who have shown that it exists a Hamiltonian framework where the gauge group may be chosen to be compact, e.g., $SO(D)$, even in the case of a Lorentzian theory with $(1, D − 1)$ signature. In the latter case, however, as these authors have pointed out, their construction is not possible in a Lagrangian framework. The aim of the present talk is to show that a compact gauge group quantization does exist in a Lorentzian space-time of dimension 1+2 – where also the time gauge does not help – provided there is a positive cosmological constant.

Our starting point is a formulation, due to Bonzom and Livine [5], of $(1 + 2)$-gravity with cosmological constant, in the presence of a Barbero-Immirzi like parameter. The gauge group of this theory is the de Sitter group $SO(1,3)$, and we shall show the existence of a partial gauge-fixing which reduces the theory to a usual Chern-Simons theory with the compact $SU(2)$ gauge group. One can then apply known results in order to quantize the theory. One may recall, e.g., the LQG results of [6], obtained by explicitly solving all the constraints, in a particular topology of 2-dimensional space.

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\(^1\) See [1] for general references on the loop quantization of General Relativity.
Our case is somewhat different from that of the authors of [4] (second paper). On the one hand, their simplicity constraints are not needed in 1 + 2 dimensions; on the other hand, we rely strongly on the existence of the Barbero-Immirzi like parameter of Bonzom and Livine, a feature very peculiar to that dimension. Moreover, here we have – and must have – a non-zero cosmological constant.

2. (2+1)-Gravity with a cosmological constant as a Chern-Simons theory with a Barbero-Immirzi parameter
The variables of 1+2 gravity in the first order formalism are the triad forms $e^I = e^I_{\mu} dx^\mu$ and the Lorentz conection forms $\omega_I = \frac{1}{2} \varepsilon_{IJK} \omega^{JK} = \omega_{I\mu} dx^\mu$.

We suppose that we have a positive cosmological constant $\Lambda > 0$. The gauge invariance group of the theory is then the de Sitter gauge group SO(1,3). Note that if $\Lambda$ were negative, the gauge group would be SO(2,2) and our procedure for obtaining a compact residual gauge group would fail. The basis for the Lie algebra so(1,3) is given by the three Lorentz generators $J^I \equiv \frac{1}{2} \varepsilon^{IJK} J_{JK}$ and the three “translation” generators $P_I$, obeying the commutation rules

$$[J^I, J^J] = \varepsilon^{IJK} J_K, \quad [J^I, P_J] = \varepsilon^{IJK} P_K, \quad [P_I, P_J] = \Lambda \varepsilon_{IJK} J^K.$$

We shall denote by $A = \omega_I J^I + e^I P_I$ the SO(1,3) connection, obeying the gauge transformation rules $A'(x) = g^{-1}(x)dg(x) + g^{-1}(x)A(x)g(x), g \in \text{SO}(1,3)$ which, when written in terms of the component fields, reproduce the well-known de Sitter transformation rules.

Following [5], we shall start from the fact that the most general background independent and gauge invariant action depending only on $A$ is of the Chern-Simons form and has two independent terms:

$$S = -\frac{\kappa}{2} \int_{\mathcal{M}} \langle A, dA + \frac{2}{3} A A \rangle - \frac{\kappa}{\gamma} \int_{\mathcal{M}} \langle A, dA + \frac{2}{3} A A \rangle^2,$$

(∧ symbol omitted) where $\langle \cdot, \cdot \rangle_{\alpha}, \alpha = 1, 2$ are the two invariant quadratic forms corresponding to the two quadratic Casimir operators of SO(1,3) [7]: $C(1) = P_I J^I, C(2) = \eta_{IJ} (\frac{1}{2} P^I P^J - J^I J^J)$. The parameter $\kappa$ is proportional to the inverse of the gravitation constant, whereas $\gamma$ is a parameter which shares with the usual Barbero-Immirzi parameter the property of not appearing in the classical field equations, which read, indeed:

$$F(A) = dA + A^2 = 0,$$

or, in components:

$$R^I \equiv d\omega^I + \varepsilon^I_{\phantom{I}JK} \omega^J \omega^K = \frac{A}{2} \varepsilon^I_{\phantom{I}JK} e^J e^K, \quad T^I \equiv de^I + \varepsilon^I_{\phantom{I}JK} \omega^J e^K = 0.$$

One recognizes the Einstein equation with cosmological constant and the null torsion condition.

3. Decomposition of SO(1,3) in “rotations” and “boosts”
We shall now introduce new variables corresponding to a decomposition of SO(1,3) – which is isomorphic to the 4-dimensional Lorentz group – in “rotations” and “boosts”. We call $L_i$ the “rotation” generators and $K_i$ the “boost” generators ($i = 1, 2, 3$).

$$[L_i, L_j] = \varepsilon_{ijk} L_k, \quad [L_i, K_j] = \varepsilon_{ijk} K_k, \quad [K_i, K_j] = -\varepsilon_{ijk} L_k.$$

$^2$ The tangent space indices $I, J, \cdots = 0, 1, 2$ are lowered and raised with the Minkowski metric $\eta_{IJ} = \text{diag}(-1, 1, 1)$ and with its inverse $\eta^{IJ}$. $\mu, \nu, \cdots = t, x, y$ are world coordinates indices, and the space-time metric reads $g_{\mu\nu} = \eta_{IJ} e^I_{\mu} e^J_{\nu}$.

$^3$ The Levi Civita tensor $\varepsilon_{IJK}$ is normalized as $\varepsilon_{012} = 1$. Moreover, $e^I_{\phantom{I}K} = \eta^{IM} \eta^{JN} \varepsilon_{MNK}$, etc. Beware of the signs!
The new variables are the 1-forms $A^i$ and $B^i$ appearing in the expression of the SO(1,3) connection: $A = A^i L_i + B^i K_i$. One recognizes in $A^i$ an SO(3) (or SU(2)) connection.

The relations between the old and new generators and variables read:

\[(L_1, L_2, L_3) = (P_2/\sqrt{\Lambda}, -P_1/\sqrt{\Lambda}, -J^0), \quad (K_1, K_2, K_3) = (J^2, -J^1, P_0/\sqrt{\Lambda})\]
\[(A^1, A^2, A^3) = (\sqrt{\Lambda} e^2, -\sqrt{\Lambda} e^1, -\omega_0), \quad (B^1, B^2, B^3) = (\omega_2, -\omega_1, \sqrt{\Lambda} e^0)\]

Our partial gauge fixing will consist in freezing the “boost” gauge degrees of freedom, keeping SU(2) as a residual gauge invariance.

Let us first write the action in the new variables:\n
\[S = -\frac{\kappa}{2} \int_{\mathbb{R}} dt \int_{\Sigma} (\dot{B}^i(A^i + B^i/\gamma) + \dot{A}^i(B^i - A^i/\gamma)) - \int_{\mathbb{R}} dt H,\]

where $H$ is the Hamiltonian $H = \int d^2x (A_i G_A(x) + B_i G_B(x))$. The kinetic terms of the action determine the symplectic structure of the theory, or equivalently its Poisson bracket algebra, the non-vanishing brackets being:

\[\{A^i_a(x), A^j_b(y)\} = \frac{1}{\kappa} \epsilon_{ab} \delta^{ij} \gamma^2 \delta^2(x - y), \quad \{B^i_a(x), B^j_b(y)\} = \frac{1}{\kappa} \epsilon_{ab} \delta^{ij} \gamma^2 \delta^2(x - y),\]
\[\{B^i_a(x), A^j_b(y)\} = -\frac{1}{\kappa} \epsilon_{ab} \delta^{ij} \gamma^2 \delta^2(x - y) .\]

The Hamiltonian is purely constraints, the fields $A_i$ and $B_i$ playing the role of Lagrangian multipliers for the Gauss and curvature constraints:

\[G_A(\varepsilon) = \kappa \int_{\Sigma} \epsilon^i \left(DB - \frac{1}{\gamma} \left(F(A) - \frac{1}{2} B \times B\right)\right)^i \approx 0, \quad \text{with } D = d + A \times\]
\[G_B(\eta) = \kappa \int_{\Sigma} \eta^i \left(F(A) - \frac{1}{2} B \times B + \frac{1}{\gamma} DB\right)^i \approx 0, \quad \text{and } F(A) = dA + \frac{1}{2} A \times A.\]

These constraints are first class:

\[\{G_A(\varepsilon), G_A(\varepsilon')\} = G_A(\varepsilon \times \varepsilon'), \quad \{G_A(\varepsilon), G_B(\eta)\} = G_B(\varepsilon \times \eta), \quad \{G_B(\eta), G_B(\eta')\} = -G_A(\eta \times \eta'),\]

and generate the gauge transformations:

\[\{G_A(\varepsilon), A\} = D\varepsilon, \quad \{G_B(\eta), A\} = -\eta \times B, \quad \{G_A(\varepsilon), B\} = \varepsilon \times B, \quad \{G_B(\eta), B\} = D\eta .\]

4. Partial gauge fixing

We choose to impose the axial-like gauge condition $B^i_y = 0$, i.e., $\omega_1 = \omega_2 = \epsilon^i_y = 0$, adding to the Hamiltonian a term $H \rightarrow H + \int_{\Sigma} d^2x \mu^i B^i_y$, the field $\mu$ being a Lagrange multiplier. It turns out that the constraints $B^i_y \approx 0$ and $G_B \approx 0$ are second class, whereas $G_A \approx 0$ remains first class. Following Dirac [8], we introduce the corresponding Dirac brackets, and treat the second class constraints as strong equalities. In particular, $G_B = 0$ yields $D_y B^i_x - \gamma F^i_{xy} = 0$, which implies that $B^i_x$ is not an independent field.

The set of independent dynamical variables reduces to the pairs of conjugate variables $A^i_x$ and $A^i_y$, whose Dirac brackets read:

\[\{A^i_x(x), A^j_y(x')\}_D = \frac{1}{\kappa} \delta^{ij} \gamma^2 \delta^2(x - x').\]

4 As usual we assume that space-time $\mathcal{M}$ may be split as $\mathbb{R} \times \Sigma$, where $\Sigma$ is the 2-dimensional space manifold. Boldface letters mean forms, etc. in $\Sigma$. 

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With the field redefinition $A^i_x = A^i_x - \gamma B^i_x$, $A^i_y = A^i_y$, the brackets and the hamiltonian read

$$\{A^i_x(x), A^j_y(x')\}_D = \frac{\gamma}{\kappa} \delta^{ij} \delta^2(x-x') \quad \text{and} \quad H = -\frac{\kappa}{\gamma} \int \Sigma d^2x A^i_x F^i_{xy}(A). \quad (2)$$

These are the Hamiltonian and brackets of a Chern-Simons theory for the compact gauge group SU(2).

5. Quantization

The problem of the quantization of the present theory is thus reduced to that of the Chern-Simons theory with gauge group SU(2). We may refer for this to the literature \[7, 9, 10\].

The residual gauge symmetry group SU(2) being compact, LQG methods apply. The case of 2-dimensional space being a cylinder was treated in \[6\], with a complete solution of the constraints.

6. Conclusions

We have thus succeeded to reduce the gauge symmetry to that of a compact group, namely SU(2), through a suitable gauge fixing. We note that the presence of a Holst-like term in the action (second term in (1)), together with the Barbero-Immirzi-like parameter $\gamma$, is crucial, much in the same way as in the $D = 3 + 1$ case, where the gauge symmetry of the Palatini-Holst action \[2\] is reduced to the same SU(2) through a time gauge fixing – which is only available in that dimension!

A complete quantization of 3 dimensional gravity with a positive cosmological constant can thus be performed in the usual loop quantization scheme, thanks to the compactness of SU(2).

A more detailed account will appear in \[11\].

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