Quantum queer superalgebra and crystal bases

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Abstract: In this paper, we develop the crystal basis theory for the quantum queer superalgebra \(U_q(q(n))\). We define the notion of crystal bases, describe the tensor product rule, and present the existence and uniqueness of crystal bases for \(U_q(q(n))\)-modules in the category \(\mathcal{O}_{\text{int}}^{20}\).

Key words: Quantum queer superalgebra; crystal bases; odd Kashiwara operators.

1. Introduction. The crystal bases are one of the most prominent discoveries of the modern combinatorial representation theory. Immediately after its first appearance in 1990 in [3], the crystal basis theory developed rapidly and attracted considerable mathematical attention. Many important and deep results for symmetrizable Kac-Moody algebras have been established in the last 20 years following the original works [3–5]. In particular, an explicit combinatorial realization of crystal bases for classical Lie algebras was given in [6].

In contrast to the case of Lie algebras, the crystal base theory for Lie superalgebras is still in its beginning stage. A major difficulty in the superalgebra case arises from the fact that the category of finite-dimensional representations is general not semisimple. Nevertheless, there is an interesting category of finite-dimensional \(U_q(g)\)-modules which is semisimple for the two superanalogues of the general linear Lie algebra \(g\) of \(q\)-modules: \(g = gl(m|n)\) and \(g = q(n)\). This is the category \(\mathcal{O}_{\text{int}}^{20}\) of representations that appear as subrepresentations of tensor powers \(V^{\otimes N}\) of the natural representation \(V\) of \(U_q(g)\). The semisimplicity of \(\mathcal{O}_{\text{int}}^{20}\) is verified in [1] for \(g = gl(m|n)\) and in [2] for \(g = q(n)\).

The crystal basis theory of \(\mathcal{O}_{\text{int}}^{20}\) for the general linear Lie superalgebra \(g = gl(m|n)\) was developed in [1]. In this case the irreducible objects in \(\mathcal{O}_{\text{int}}^{20}\) are indexed by partitions having so-called \((m,n)\)-hook shapes. This combinatorial description enables us to index the crystal basis of any irreducible object \(V(\lambda)\) in \(\mathcal{O}_{\text{int}}^{20}\) with highest weight \(\lambda\) by the set \(B(Y)\) of semistandard tableaux \(Y\) of shape \(\lambda\). In addition to the existence of the crystal basis, the decompositions of \(V(\lambda) \otimes V\) and \(B(Y) \otimes B\), where \(B\) is the crystal basis for \(V\), have been found in [1].

In this paper we focus on the second superanalogue of the general linear Lie algebra: the queer Lie superalgebra \(q(n)\). It has been known since its inception that the representation theory of \(q(n)\) is more complicated compared to the other classical Lie superalgebra theories. A distinguished feature of \(q(n)\) is that any Cartan subsuperalgebra has a nontrivial odd part. As a result, the highest weight space of any highest weight \(q(n)\)-module has a structure of a Clifford module. In particular, every \(gl(n)\)-component of a finite-dimensional \(q(n)\)-module appears with multiplicity larger than one (in fact, a power of two). Important results related to the representation theory of \(q(n)\) include the \(q(n)\)-analogue of the celebrated Schur-Weyl duality discovered by Sergeev in 1984 [8], and character formulae for all simple finite-dimensional representations found by Penkov and Serganova in 1997 [7].

The foundations of the highest weight representation theory of the quantum queer superalgebra \(U_q(q(n))\) have been established in [2]. An interesting observation in [2] is that the classical limit of a simple highest weight \(U_q(q(n))\)-module is a simple highest weight \(U(q(n))\)-module or a direct sum of two highest weight \(U(q(n))\)-modules.

In view of the above remarks, it is clear that developing a crystal basis theory for the category \(\mathcal{O}_{\text{int}}^{20}\) of \(U_q(q(n))\) is a challenging problem. The purpose of this paper is to announce the results that lead to a solution of this problem. Take the base
field to be $\mathbf{C}(q)$. Our main theorem is the existence and uniqueness of the crystal bases of $U_q(q(n))$-modules in $\mathcal{O}_{\text{int}}^{\mathcal{O}_{\text{int}}}$.

The proofs will appear in full detail in a forthcoming paper. To overcome the challenges described above, we modify the notion of a crystal basis and introduce the so-called abstract $q(n)$-crystal. To do so we first define odd Kashiwara operators $\tilde{e}_i, \tilde{f}_i, \tilde{k}_i$, where $\tilde{k}_i$ corresponds to an odd element in the Cartan subalgebra of $\mathfrak{g}(n)$.

Then, a crystal basis for a $U_q(q(n))$-module $M$ in the category $\mathcal{O}_{\text{int}}^{\mathcal{O}_{\text{int}}}$ is a triple $(L, B, (b_i)_{i\in\mathbb{Z}})$, where the crystal lattice $L$ is a free $\mathbf{C}[|q|]$-submodule of $M$, $B$ is a finite $\mathfrak{gl}(n)$-crystal, $(b_i)_{i\in\mathbb{Z}}$ is a family of vector spaces such that $L/qL = \bigoplus_{b_i\in B} b_i$, with a set of commutation properties for the action of the Kashiwara operators imposed in addition. The definition of a crystal basis leads naturally to the notion of an abstract $q(n)$-crystal an example of which is the $\mathfrak{gl}(n)$-crystal $B$ in any crystal basis $(L, B, (b_i)_{i\in\mathbb{Z}})$. The modified notion of a crystal allows us to consider the multiple occurrence of $\mathfrak{g}(n)$-crystals corresponding to a highest weight $U_q(q(n))$-module $M$ in $\mathcal{O}_{\text{int}}^{\mathcal{O}_{\text{int}}}$ as a single $q(n)$-crystal.

It is worth noting that $M$ is not necessarily a simple module and that the $q(n)$-crystal $B$ of $M$ depends only on the highest weight $\lambda$ of $M$, hence we may write $B = B(\lambda)$. In order to find the highest weight vector of $B(\lambda)$, we use the action of the Weyl group on $B(\lambda)$ and define odd Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ for $i = 2, \ldots, n - 1$. Then the highest weight vector of $B(\lambda)$ is simply the unique vector annihilated by the $2n - 2$ Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$.

In addition to the existence and uniqueness of the crystal basis of $M$, we establish an isomorphism $B \otimes B(\lambda) \cong \bigsqcup_{\lambda+\epsilon_i, \text{strict}} B(\lambda + \epsilon_i)$ and explicitly describe the highest weight vectors of $B \otimes B(\lambda)$ in terms of the even Kashiwara operators $f_i$ and the highest weight vector of $B(\lambda)$. We conjecture that the highest weight vectors of $B(\lambda) \otimes B$ can be found in an analogous way with the aid of the odd Kashiwara operators $\tilde{f}_i$.

2. The quantum queer superalgebra. For an indeterminate $q$, let $\mathbf{F} = \mathbf{C}(q)$ be the field of formal Laurent series in $q$ and let $\mathbf{A} = \mathbf{C}[|q|]$ be the subring of $\mathbf{F}$ consisting of formal power series in $q$.

Let $P' = \mathbb{Z}k_1 \oplus \cdots \oplus \mathbb{Z}k_n$ be a free abelian group of rank $n$ and let $\mathfrak{h} = \mathbf{C} \otimes_{\mathbb{Z}} P'$. Define the linear functionals $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(k_j) = \delta_{ij}$, $i, j = 1, \ldots, n$, and set $P = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$. We denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ the simple roots.

Definition 2.1. The quantum queer superalgebra $U_q(q(n))$ is the superalgebra over $\mathbf{F}$ with 1 generated by $e_i, f_i, e_{-i}, f_{-i}$ for $i = 1, \ldots, n - 1$, $q^h$ for $h \in P'$, $k_j$ for $j = 1, \ldots, n$ with the following defining relations:

\[
\begin{align*}
q_i^h &= 1, & q_i^h q_j^h &= q_j^{h+h} & (h_1, h_2 \in P'), \\
q_i^h e_j^h &= q_j^{h+h} e_j^h & (h \in P'), \\
q_i^h f_j^h &= q_j^{-h} f_j^h & (h \in P'), \\
q_i^h k_j^h &= k_j^{h} & (h \in \mathbb{Z}), \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{q_j^{k_i} - q_j^{-k_i}}{q_j - q_j^{-1}}, \\
e_i e_j - e_j e_i &= f_i f_j - f_j f_i = 0 & (|i - j| > 1), \\
e_i^2 e_j &= (q + q^{-1})e_i e_j e_i + e_i e_j e_i^2 = f_i f_j - (q + q^{-1})f_j f_i f_j + f_j f_i^2 = 0 & (|i - j| = 1), \\
k_i^2 &= q^{2k_i} - q^{-2k_i}, \\
k_i k_j + k_j k_i &= 0 & (i \neq j), \\
k_i e_i - q e_i k_i &= e_i q^{-k_i}, \\
k_i f_i - q f_i k_i &= -f_i q^{k_i}, \\
e_i f_j - f_j e_i &= \delta_{ij}(k_j q_j^{k_i} - q_j^{k_i}), \\
e_i e_j - e_j e_i &= \delta_{ij}(k_j q_j^{k_i} - q_j^{k_i}), \\
e_i e_{i+1} - q e_{i+1} e_i &= \epsilon_j e_i e_{i+1} + q e_{i+1} e_i f_i, \\
e_i f_{i+1} f_i - f_i f_{i+1} &= f_i f_{i+1} + q f_{i+1} f_i, \\
e_i^2 &= (q + q^{-1})e_i e_j e_i + e_i e_j e_i^2 = f_i f_j - (q + q^{-1})f_j f_i f_j + f_j f_i^2 = 0, & (|i - j| = 1).
\end{align*}
\]
(i = 1, . . . , n − 1), and let U0 be the subalgebra generated by qh (h ∈ P+) and k1 (j = 1, . . . , n). In [2], it was shown that the algebra Uq(g(n)) has the triangular decomposition:

\[ U^- \otimes \mathbb{C} \otimes U^+ \cong U_q(g(n)). \]

Hereafter, a Uq(g(n))-module is understood as a Uq(g(n))-supermodule. A Uq(g(n))-module M is called a weight module if M has a weight space decomposition \( M = \bigoplus_{\mu} \mathbb{C} M_\mu \), where

\[ M_\mu := \{ m \in M; q^h m = q^{\mu(h)} m \} \text{ for all } h \in P^+. \]

The set of weights of M is defined to be

\[ \text{wt}(M) = \{ \mu \in P; M_\mu \neq 0 \}. \]

**Definition 2.2.** A weight module V is called a highest weight module with highest weight \( \lambda \in P \) if V is generated by a finite-dimensional U0-module \( \mathfrak{v}_\lambda \) satisfying the following conditions:

(a) \( \epsilon_i v = \epsilon_i u = 0 \) for all \( v \in \mathfrak{v}_\lambda, i = 1, \ldots, n - 1, \)

(b) \( q^h v = q^{\lambda(h)} v \) for all \( v \in \mathfrak{v}_\lambda, h \in P^+. \)

There is a unique irreducible highest weight module with highest weight \( \lambda \in P \) up to parity change. We denote it by \( V(\lambda) \).

Set

\[ P^{20} = \{ \lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n \in P; \lambda_j \in \mathbb{Z}_{\geq 0} \} \]

\[ \Lambda^+ = \{ \lambda = \lambda_1 e_1 + \cdots + \lambda_n e_n \in P^{20}; \lambda_i \geq \lambda_{i+1} \text{ and } \lambda_i = \lambda_{i+1} \text{ implies } \lambda_i = \lambda_{i+1} = 0 \text{ for all } i = 1, \ldots, n - 1 \}. \]

Note that each element \( \lambda \in \Lambda^+ \) corresponds to a strict partition \( \lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0) \). Thus we will call \( \lambda \in \Lambda^+ \) a strict partition.

We define \( \mathcal{O}_{\mathfrak{g}1}^{\geq 0} \) to be the category of finite-dimensional weight modules such that \( \text{wt}(M) \subseteq P^{20} \) and \( k_i|\mathcal{O}_{\mathfrak{g}1}^{\geq 0} = 0 \) for any \( i \in \{1, \ldots, n\} \) and \( \mu \in P^{20} \) satisfying \( \langle k_i, \mu \rangle = 0 \). The fundamental properties of the category \( \mathcal{O}_{\mathfrak{g}1}^{\geq 0} \) are summarized in the following proposition.

**Proposition 2.3** [2].

(a) Every Uq(g(n))-module in \( \mathcal{O}_{\mathfrak{g}1}^{\geq 0} \) is completely reducible.

(b) Every irreducible object in \( \mathcal{O}_{\mathfrak{g}1}^{\geq 0} \) has the form \( V(\lambda) \) for some \( \lambda \in \Lambda^+ \).

3. Crystal bases. Let M be a Uq(g(n))-module in \( \mathcal{O}_{\mathfrak{g}1}^{\geq 0} \). For \( i = 1, 2, \ldots, n - 1 \), we define the even Kashiwara operators on M in the usual way. That is, for a weight vector \( u \in M_\lambda \), consider the i-string decomposition of u:

\[ u = \sum_{k \geq 0} f_i^{(k)} u_k, \]

where \( \epsilon_i u_k = 0 \) for all \( k \geq 0 \), \( f_i^{(k)} = f_i^k / [k]! \), \( [k] = \frac{q^k - q^{-k}}{q - q^{-1}} \), \( [k] = [k][k-1] \cdots [2][1] \), and we define the even Kashiwara operators \( \tilde{e}_i, \tilde{f}_i \) (\( i = 1, \ldots, n - 1 \)) by

\[ \tilde{e}_i u = \sum_{k \geq 0} f_i^{(k-1)} u_k, \]

\[ \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k. \]

On the other hand, we define the odd Kashiwara operators \( \kappa, \tilde{e}_i, \tilde{f}_i \) by

\[ \kappa = q^{\Lambda - 1} k, \]

\[ \tilde{e}_i = -q^{\lambda_i - 1} qk \tilde{e}_1 q^{-\lambda_i - 1}, \]

\[ \tilde{f}_i = -qf_1 q^{-1} - qf_i q^{-1}. \]

Recall that an abstract g(n)-crystal is a set B together with the maps \( \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\}, \varphi, \varepsilon : B \to \mathbb{Z} \cup \{-\infty\} \) (\( i = 1, \ldots, n - 1 \)), and \( \text{wt} : B \to P \) satisfying the conditions given in [5]. In this paper, we say that an abstract g(n)-crystal is a g(n)-crystal if it is realized as a crystal basis of a finite-dimensional integrable Uq(g(n))-module. In particular, we have \( \varepsilon_i(b) = \max \{ n \in \mathbb{Z}^{\geq 0}; \varepsilon_i b \neq 0 \} \) and \( \varphi_i(b) = \max \{ n \in \mathbb{Z}^{\geq 0}; f_i^n b \neq 0 \} \) for any b in a g(n)-crystal B.

**Definition 3.1.** Let \( M = \bigoplus_{\mu \in P^{\geq 0}} M_\mu \) be a Uq(g(n))-module in the category \( \mathcal{O}_{\mathfrak{g}1}^{\geq 0} \). A crystal basis of M is a triple \( (L, B, \ell_B = (b)_{b \in B}) \), where

(a) \( L \) is a free \( A \)-submodule of M such that

(i) \( F \otimes_A L \cong M \),

(ii) \( L = \bigoplus_{\mu \in P^{\geq 0}} L_\mu \), where \( L_\mu = L \cap M_\mu \),

(iii) \( L \) is stable under the Kashiwara operators \( \tilde{e}_i, \tilde{f}_i \) (\( i = 1, \ldots, n - 1 \)), \( \kappa, \tilde{e}_i, \tilde{f}_i \).

(b) \( B \) is a g(n)-crystal together with the maps \( \tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\} \) such that

(i) \( \text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i, \text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i \),

(ii) for all \( b, b' \in B, \tilde{f}_i b = b' \) if and only if \( b = \tilde{e}_i b' \),

(iii) \( \ell_B(b) \) is a family of non-zero C-vector spaces such that

(iv) for \( i = 1, \ldots, n - 1, \) we have

1. if \( \tilde{e}_i b = 0 \) then \( \tilde{e}_i l_B(b) = 0 \), and otherwise \( \tilde{e}_i \) induces an isomorphism \( l_B(b) \cong l_B(b) \).

2. if \( \tilde{f}_i b = 0 \) then \( \tilde{f}_i l_B(b) = 0 \), and otherwise \( \tilde{f}_i \) induces an isomorphism \( l_B(b) \cong l_B(b) \).
Note that one can prove that $\bar{e}_i^2 = \bar{f}_i^2 = 0$ as endomorphisms of $L/qL$ for any crystal basis $(L, B, l_B)$.

**Example 3.2.** Let

$$V = \bigoplus_{j=1}^{n} F_{v_j} \oplus \bigoplus_{j=1}^{n} F_{v_j^*}$$

be the vector representation of $U_q(\mathfrak{g}(n))$. The action of $U_q(\mathfrak{g}(n))$ on $V$ is given as follows:

$$e_i v_j = \delta_{i,j+1} v_j, \quad e_i v_j^* = \delta_{i,j+1} v_j^*, \quad f_i v_j = \delta_{i,j+1} v_j, \quad f_i v_j^* = \delta_{i,j+1} v_j^*$$

Set

$$L = \bigoplus_{j=1}^{n} A v_j \oplus \bigoplus_{j=1}^{n} A v_j^*,$$

and let $B$ be the crystal graph given below.

Here, the actions of $\bar{f}_i$ $(i = 1, \ldots, n-1, \mathbf{T})$ are expressed by $i$-arrows. Then $(L, B, l_B = (l_j^n)_{j=1}^{n})$ is a crystal basis of $V$.

**Theorem 3.3.** Let $M_j$ be a $U_q(\mathfrak{g})$-module in $\mathcal{O}^{\leq 0}_{\text{int}}$ with crystal basis $(L_j, B_j, l_B_j)$ $(j = 1, 2)$. Set $B_1 \otimes B_2 = B_1 \times B_2$ and

$$l_{B_1 \otimes B_2} = (l_{B_1} \otimes l_{B_2})_{b_1 \in B_1, b_2 \in B_2}.$$

Then

$$(L_1 \otimes A L_2, B_1 \otimes B_2, l_{B_1 \otimes B_2})$$

is a crystal basis of $M_1 \otimes M_2$, where the action of the Kashiwara operators on $B_1 \otimes B_2$ are given as follows:

$$\bar{e}_i (b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varphi_i (b_1) \geq e_i (b_2), \\ b_1 \otimes \bar{e}_i b_2 & \text{if } \varphi_i (b_1) < e_i (b_2), \end{cases}$$

$$\bar{f}_i (b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2 & \text{if } \varphi_i (b_1) > e_i (b_2), \\ b_1 \otimes \bar{f}_i b_2 & \text{if } \varphi_i (b_1) \leq e_i (b_2), \end{cases}$$

$$\bar{e}_T (b_1 \otimes b_2) = \begin{cases} \bar{e}_T b_1 \otimes b_2 & \text{if } \langle k_1, \text{wt } b_2 \rangle = 0, \\ b_1 \otimes \bar{e}_T b_2 & \text{otherwise}, \end{cases}$$

$$\bar{f}_T (b_1 \otimes b_2) = \begin{cases} \bar{f}_T b_1 \otimes b_2 & \text{if } \langle k_1, \text{wt } b_2 \rangle = 0, \\ b_1 \otimes \bar{f}_T b_2 & \text{otherwise}. \end{cases}$$

**Sketch of Proof.** Our assertion follows from the following comultiplication formulas.

$$\Delta(\bar{e}_i) = \bar{e}_i \otimes q^{2k_i} + 1 \otimes \bar{e}_i,$$

$$\Delta(\bar{f}_i) = \bar{f}_i \otimes q^{h_i+k_i} + 1 \otimes \bar{f}_i$$

Motivated by the properties of crystal bases, we introduce the notion of abstract crystals.

**Definition 3.4.** An abstract $q(n)$-crystal is a $\mathfrak{gl}(n)$-crystal together with the maps $\bar{e}_T, \bar{f}_T : B \rightarrow B \sqcup \{0\}$ satisfying the following conditions:

(a) $\text{wt}(B) \subset \mathbb{Z}^\geq 0$,

(b) $\text{wt}(\bar{e}_T b) = \text{wt}(b) + \alpha_1$, $\text{wt}(\bar{f}_T b) = \text{wt}(b) - \alpha_1$,

(c) for all $b, b' \in B$, $\bar{f}_T b = b'$ if and only if $b = \bar{e}_T b'$.

Let $B_1$ and $B_2$ be abstract $q(n)$-crystals. The tensor product $B_1 \otimes B_2$ of $B_1$ and $B_2$ is defined to be the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ together with the maps $\bar{e}_T, \bar{f}_T$ defined by (7). Then it is an abstract $q(n)$-crystal. Note that $\otimes$ satisfies the associative axiom.

**Example 3.5.**

(a) If $(L, B, l_B)$ is a crystal basis of a $U_q(\mathfrak{g}(n))$-module $M$ in the category $\mathcal{O}^{\leq 0}_{\text{int}}$, then $B$ is an abstract $q(n)$-crystal.

(b) The crystal graph $B$ is an abstract $q(n)$-crystal.

(c) By the tensor product rule, $B^{\otimes N}$ is an abstract $q(n)$-crystal. When $n = 3$, the $q(n)$-crystal structure of $B \otimes B$ is given below.

(d) For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$, let $Y_\lambda$ be the skew Young diagram having $\lambda_i$ many boxes in the first diagonal, $\lambda_2$ many boxes in the second diagonal, etc. For example, if $\lambda$ is given by $(7 > 6 > 4 > 2 > 0)$, then we have
Let $\mathcal{B}(Y_\lambda)$ be the set of all semistandard tableaux of shape $Y_\lambda$ with entries from $1, 2, \ldots, n$. Then by an admissible reading introduced in [1], $\mathcal{B}(Y_\lambda)$ is embedded in $\mathcal{B}^{[\lambda]}$ and it is stable under $\tilde{e}_i, \tilde{f}_i, \tilde{e}_i \tilde{f}_i$. Hence it becomes an abstract $\mathcal{O}(n)$-crystal. Moreover, the $\mathcal{O}(n)$-crystal structure thus obtained does not depend on the choice of admissible readings.

Let $B$ be an abstract $\mathcal{O}(n)$-crystal. For $i = 2, \ldots, n-1$, let $w$ be an element of the Weyl group $W$ with shortest length such that $w(a_1) = a_1$. Such an element is unique and we may choose $w = s_2 \cdots s_i s_1 \cdots s_{i-1}$. We define the odd Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ by

$$\tilde{e}_i = S_{w^{-1}} \tilde{e}_i S_w, \quad \tilde{f}_i = S_{w^{-1}} \tilde{f}_i S_w.$$

Here $S_w$ is the Weyl group action on the $\mathfrak{gl}(n)$-crystal. The operators $\tilde{e}_i, \tilde{f}_i$ do not depend on the choice of reduced expressions of $w$. We say that $b \in B$ is a highest weight vector if $\tilde{e}_i b = \tilde{f}_i b = 0$ for all $i = 1, \ldots, n-1$.

4. Existence and uniqueness. In this section, we present the main result of our paper.

**Theorem 4.1.**

(a) Let $\lambda \in \Lambda^+$ be a strict partition and let $M$ be a highest weight $U_q(\mathfrak{gl}(n))$-module in the category $\mathcal{O}_{\lambda \in \mathfrak{gl}(n)}$ with highest weight $\lambda$. If $(L, B, l_B)$ is a crystal basis of $M$, then $L_\lambda$ is invariant under $\mathfrak{k}_i := q^{k-1} \mathfrak{k}_i$ for all $i = 1, \ldots, n$. Conversely, if $M_\lambda$ is generated by a free $\mathfrak{g}$-submodule $L_\lambda^0$ invariant under $\mathfrak{k}_i$ (i.e., $1, \ldots, n$), then there exists a unique crystal basis $(L, B, l_B)$ of $M$ such that

(i) $L_\lambda = L_\lambda^0$,
(ii) $B_\lambda = \{ b_\lambda \}$,
(iii) $L_\lambda^0 / q L_\lambda^0 = b_\lambda$,
(iv) $B$ is connected.

Moreover, as an abstract $\mathcal{O}(n)$-crystal, $B$ depends only on $\lambda$. Hence we may write $B = B(\lambda)$.

(b) The $\mathcal{O}(n)$-crystal $B(\lambda)$ has a unique highest weight vector $b_\lambda$.

(c) If $b \in B \otimes B(\lambda)$ is a highest weight vector, then we have

$$b = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b_\lambda$$

for some $j$ such that $\lambda + \epsilon_j$ is a strict partition.

(d) Let $M$ be a $U_q(\mathfrak{gl}(n))$-module in the category $\mathcal{O}_{\lambda \in \mathfrak{gl}(n)}$, and let $(L, B, l_B)$ be a crystal basis of $M$. Then there exist decompositions $M = \bigoplus_{a \in \Lambda_+} M_a$ as a $U_q(\mathfrak{gl}(n))$-module, $L = \bigoplus_{a \in \Lambda_+} L_a$ as an $\mathcal{A}$-module, $B = \bigcup_{a \in \Lambda} B_a$ as a $\mathcal{A}$-crystal, parameterized by a set $A$ such that for any $a \in A$ the following conditions hold:

(i) $M_a$ is a highest weight module with highest weight $\lambda_a$ and $B_a \simeq B(\lambda_a)$ for some strict partition $\lambda_a$,
(ii) $L_a = L \cap M_a$, $L_a/q L_a = \bigoplus_{b \in B_a} l_b$,
(iii) $(L_a, B_a, l_{B_a})$ is a crystal basis of $M_a$.

(e) Let $M$ be a highest weight $U_q(\mathfrak{gl}(n))$-module in the category $\mathcal{O}_{\lambda \in \mathfrak{gl}(n)}$ with highest weight $\lambda$. Assume that $M$ has a crystal basis $(L, B(\lambda), l_{B(\lambda)})$ such that $L_\lambda / q L_\lambda = l_b$. Then we have

(i) $V \otimes M = \bigoplus \lambda \in \Lambda_+, \text{strict} M_j$, where $M_j$ is a highest weight $U_q(\mathfrak{gl}(n))$-module with highest weight $\lambda + \epsilon_j$ and $\dim(M_j)_{\lambda + \epsilon_j} = 2 \dim M_\lambda$,
(ii) $L_j = (L \otimes L) \cap M_j$,
(iii) $B \otimes B(\lambda) \simeq \bigoplus_{a \in \Lambda_+, \text{strict} B_j}$, where

$$B_j \simeq B(\lambda + \epsilon_j), \quad L_j / q L_j = \bigoplus_{b \in B_j} l_b.$$

We will prove all of our assertions at once by induction on the length of $\lambda$. The proof is involved because our theorem consists of several interlocking statements. The key ingredient is a combinatorial proof of (c).

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References

[1] G. Benkart, S.-J. Kang and M. Kashiwara, Crystal bases for the quantum superalgebra $U_q(\mathfrak{gl}(m,n))$, J. Amer. Math. Soc. 13 (2000), no. 2, 295–331.

[2] D. Grantcharov et al., Highest weight modules over quantum queer superalgebra $U_q(q(n))$, Commun. Math. Phys. 296 (2010), no. 3, 827–860.

[3] M. Kashiwara, Crystallizing the $q$-analogue of universal enveloping algebras, Commun. Math. Phys. 133 (1990), no. 2, 249–260.

[4] M. Kashiwara, On crystal bases of the $q$-analogue of universal enveloping algebras, Duke Math. J. 63 (1991), no. 2, 465–516.

[5] M. Kashiwara, The crystal base and Littelmann’s refined Demazure character formula, Duke Math. J. 71 (1993), no. 3, 839–858.

[6] M. Kashiwara and T. Nakashima, Crystal graphs for representations of the $q$-analogue of classical Lie algebras, J. Algebra 165 (1994), no. 2, 295–345.

[7] I. Penkov and V. Serganova, Characters of finite-dimensional irreducible $q(n)$-modules, Lett. Math. Phys. 40 (1997), no. 2, 147–158.

[8] A. N. Sergeev, Tensor algebra of the identity representation as a module over the Lie superalgebras $\mathfrak{gl}(n,m)$ and $Q(n)$, Mat. Sb. (N.S.) 123(165) (1984), no. 3, 422–430.