A GENERALIZATION OF ALMOST SCHUR LEMMA ON CR MANIFOLDS

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ABSTRACT. In this paper, we study a general almost Schur Lemma on pseudo-Hermitian (2n+1)-manifolds \((M, J, \theta)\) for \(n \geq 2\). When the equality of almost Schur inequality holds, we derive the contact form \(\theta\) is pseudo-Einstein and the pseudo-Hermitian scalar curvature is constant.

1. INTRODUCTION

In Riemannian manifolds, the classical Schur Lemma states that the scalar curvature of an Einstein manifold of dimension \(n \geq 3\) must be constant. So it is interesting to see the relation between scalar curvature and Einstein condition. Recently, De Lellis and C. Topping \cite{LT}\ proved an almost Schur Lemma assuming the nonnegative of Ricci curvature. Their result can be seen as a quantitative version or a stability property of the Schur Lemma. Later, in \cite{B, C1, C2} and \cite{GW}, the authors considered general closed Riemannian manifolds, and obtained a generalization of the De Lellis-Topping’s theorem.

However, in the pseudo-Hermitian manifold, the pseudo-Einstein condition does not imply the constant pseudo-Hermitian scalar curvature. This is because of the appearance of torsion terms in the contracted Bianchi identity \((2.3)\). Hence there is a natural question to ask under which condition a pseudo-Einstein manifold has constant pseudo-Hermitian scalar curvature. More general, how does the pseudo-Hermitian scalar curvature change when the manifold is close to the pseudo-Einstein manifold. In \cite{CSW}, the authors addressed to this question.

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and shown that if $\text{Im} \left( A_{\alpha \beta}^{, \alpha \beta} \right) = 0$, where $A_{\alpha \beta}$ is the pseudo-Hermitian torsion, then the answer is affirmative. In fact, the answer came from the following CR almost Schur theorem in [CSW] on a closed pseudo-Hermitian $(2n + 1)$-manifold $M$ for $n \geq 2$.

**Theorem 1.1.** (CSW) For $n \geq 2$, if $(M, J, \theta)$ is a closed pseudo-Hermitian $(2n + 1)$-manifold with $\text{Im} \left( A_{\alpha \beta}^{, \alpha \beta} \right) = 0$ and

\begin{equation}
(Ric - \frac{n+1}{2} \text{Tor})(Z, Z) \geq 0 \quad \text{for all } Z \in T_{1,0}(M),
\end{equation}

then

\begin{equation}
\int_M (R - \overline{R})^2 \leq \frac{2n(n+1)}{(n-1)(n+2)} \int_M \sum_{\alpha, \beta} |Ric_{\alpha \beta} - \frac{R}{n} h_{\alpha \beta}|^2,
\end{equation}

where $\overline{R}$ is the average value of the pseudo-Hermitian scalar curvature $R$ over $M$. Moreover, equality holds then the contact form $e^{\frac{1}{n+1} \tau} \theta$ will be pseudo-Einstein.

In this paper, motivated by [C1], we are interested in a more general curvature condition with respect to (1.1). We prove a similar inequality to (1.2) with the inequality constant depending on the lower bound of Webster Ricci tensor minus $\frac{n+1}{2}$ times torsion tensor and also on the value of the first positive eigenvalue of the sub-Laplacian.

**Theorem 1.2.** For $n \geq 2$, if $(M, J, \theta)$ is a closed pseudo-Hermitian $(2n + 1)$-manifold with

\begin{equation}
(Ric - \frac{n+1}{2} \text{Tor})(Z, Z) \geq -2K |Z|^2,
\end{equation}

for all $Z \in T_{1,0}(M)$ and for some nonnegative constant $K$, then

\begin{equation}
||R - \overline{R}||_{L^2} \leq \sqrt{\kappa} \left( \int_M \sum_{\alpha, \beta} |Ric_{\alpha \beta} - \frac{R}{n} h_{\alpha \beta}|^2 \right)^{1/2} + \frac{2n}{\lambda_1} ||\text{Im}(A_{\alpha \beta}^{, \alpha \beta})||_{L^2},
\end{equation}

where $\kappa = \frac{2n(n+1)}{(n-1)(n+2)} (1 + \frac{2nK}{(n+1)\lambda_1})$ and $\lambda_1$ is the first positive eigenvalue of the sub-Laplacian. Moreover, if the equality holds then the contact form $\theta$ is pseudo-Einstein, $\text{Im}(A_{\alpha \beta}^{, \alpha \beta}) = 0$, and the pseudo-Hermitian scalar curvature $R = \overline{R}$ is a constant.
We observe that when $K = 0$ in Theorem 1.2, we obtain Theorem 1.1. Moreover, equality holds in (1.2), we know that the contact form $\theta$ is pseudo-Einstein and $R = \overline{R}$ is a constant. This result is stronger than we gave in Theorem 1.1.

In Section 3, we consider a closed pseudo-Hermitian $(2n+1)$-manifold $M$ with zero pseudo-Hermitian torsion and we derive a lower bound estimate for the first positive eigenvalue $\lambda_1$ of the sub-Laplacian $\Delta_b$ by using the diameter of $M$ and lower bound of Webster Ricci tensor (see Proposition 3.1). This estimate is also an independent interesting result. As a consequence, we have the following Corollary.

**Corollary 1.3.** Under the same conditions as in the Theorem 1.2, We also assume that $M$ is torsion free, then

\[
\int_M (R - \overline{R})^2 \leq C(Kd^2) \int_M \sum_{\alpha, \beta} |Ric_{\alpha \beta} - \frac{R}{n} h_{\alpha \beta}|^2,
\]

where $d$ is the diameter of $M$ with respect to the Carnot-Carathéodory distance and $C(Kd^2)$ is a constant only depending on $Kd^2$. Moreover, the equality holds if and only if the contact form $\theta$ is pseudo-Einstein.

**2. The Proof of Theorem 1.2**

In this section, we take the method used in [C1] to prove the inequality (1.4). But in the equality case, our proof is more different from [C1].

We need the following two integral formulas. The first integral is equation (3.4) in [CSW], for $n \geq 2$ and a smooth real-valued function $\varphi$,

\[
\int_M (\Delta_b \varphi)^2 - \frac{1}{n} \varphi \nabla^\gamma \gamma h_{\alpha \beta} - \int_M (Ric - \frac{n+1}{2} Tor)(((\nabla_b \varphi)_C, (\nabla_b \varphi)_C).
\]

The second integral comes from Lemma 2.2 in [CC] with its last equation in P. 268,

\[
\frac{n^2}{2} \int_M \varphi_0^2 = \int_M \sum_{\alpha, \beta} (\varphi_{\alpha \beta} \varphi_{\alpha \beta} - \varphi_{\alpha \beta} \varphi_{\alpha \beta}) + \int_M (Ric + \frac{n}{2} Tor)(((\nabla_b \varphi)_C, (\nabla_b \varphi)_C),
\]

where $\varphi_0 = T \varphi$ and $T$ is the characteristic vector field of the contact form $\theta$. 
Proof of Theorem 1.2

Proof. We denote the traceless Webster Ricci tensor by $\hat{Ric}_{\alpha\beta} = Ric_{\alpha\beta} - \frac{R}{n} h_{\alpha\beta}$, then the contracted Bianchi identity yields

$$\hat{Ric}_{\alpha\beta} \gamma = (Ric_{\alpha\beta} - \frac{R}{n} h_{\alpha\beta}) \gamma = \frac{n-1}{n} (R_\alpha - i n A_{\alpha\beta}, \gamma).$$

Let $f$ be the unique solution of $\Delta_b f = R - \overline{R}$ with $\int_M f = 0$. According to (2.3), we compute

$$\int_M (R - \overline{R})^2$$

$$= \int_M (R - \overline{R}) \Delta_b f = - \int_M \langle \nabla_b R, \nabla_b f \rangle = - \int_M (R_\alpha f^\alpha + R_{\overline{\alpha}} f_{\overline{\alpha}})$$

$$= \left( -\frac{n}{n-1} \int_M \hat{Ric}_{\alpha\beta} \gamma f^\alpha - i n \int_M A_{\alpha\beta}, \gamma f^\alpha \right) + \text{complex conjugate}$$

$$= \left( -\frac{n}{n-1} \int_M \hat{Ric}_{\alpha\beta} f_{\alpha\beta}^\gamma + i n \int_M A_{\alpha\beta} f_{\alpha\beta} \right) + \text{complex conjugate}$$

$$= \left( \frac{n}{n-1} \int_M \hat{Ric}_{\alpha\beta} f_{\alpha\beta}^\gamma - \frac{1}{2} f_{\gamma\gamma} h_{\alpha\beta} \right) + i n \int_M (A_{\alpha\beta} f_{\alpha\beta} - A_{\overline{\alpha}\overline{\beta}} f_{\overline{\alpha}\overline{\beta}})$$

$$\leq \frac{2n}{n-1} \int_M \| \hat{Ric} \|_{L^2} \left( \int_M \sum_{\alpha, \beta} |f_{\overline{\alpha}\overline{\beta}} - \frac{1}{2} f_{\gamma\gamma} h_{\alpha\beta}|^2 \right)^{\frac{1}{2}} + i n \int_M (A_{\alpha\beta} f_{\alpha\beta} - A_{\overline{\alpha}\overline{\beta}} f_{\overline{\alpha}\overline{\beta}}),$$

here we used $\int_M \hat{Ric}_{\alpha\beta} f_{\gamma\gamma} h_{\alpha\beta} = 0$ and $f_{\alpha\beta} - \frac{1}{2} f_{\gamma\gamma} h_{\alpha\beta} = f_{\beta\alpha} - \frac{1}{2} f_{\gamma\gamma} h_{\beta\alpha}$ is symmetric in $\alpha, \beta$.

Now from (2.1) and the assumption on the curvature condition (1.3), we obtain

$$\frac{n+2}{n-1} \int_M \sum_{\alpha, \beta} |f_{\overline{\alpha}\overline{\beta}} - \frac{1}{2} f_{\gamma\gamma} h_{\alpha\beta}|^2 \leq \frac{n+1}{2n} \int_M (\Delta_b f)^2 + K \int_M |\nabla_b f|^2.$$

Besides, by using integration by parts and Hölder inequality,

$$i \int_M (A_{\alpha\beta} f_{\alpha\beta} - A_{\overline{\alpha}\overline{\beta}} f_{\overline{\alpha}\overline{\beta}}) = i \int_M (A_{\alpha\beta} f_{\alpha\beta} - A_{\overline{\alpha}\overline{\beta}} f_{\overline{\alpha}\overline{\beta}}) \leq 2 \| \text{Im}(A_{\alpha\beta}, \gamma) \|_{L^2} \| f \|_{L^2}.$$
Then

\[(2.7) \quad \lambda_1 \int_M |\nabla_b f|^2 \leq \|R - \overline{R}\|^2_{L^2} \quad \text{and} \quad \lambda_1^2 \int_M |f|^2 \leq \|R - \overline{R}\|^2_{L^2}.\]

Due to (2.7), we can rewrite (2.5) and (2.6) as

\[\frac{n+2}{n-1} \int_M \sum_{\alpha, \beta} |f_{\alpha\beta} - \frac{1}{n} f_{\gamma\gamma} h_{\alpha\beta}|^2 \leq \left( \frac{n+1}{2n} + \frac{K}{\lambda_1} \right) \|R - \overline{R}\|^2_{L^2}\]

and

\[i \int_M (A_{\alpha\beta} f^{\alpha\beta} - A_{\alpha\beta\overline{\alpha}\overline{\beta}} f^{\overline{\alpha}\overline{\beta}}) \leq \frac{2}{\lambda_1} \|\text{Im}(A_{\alpha\beta}, \overline{\alpha\beta})\|_{L^2} \|R - \overline{R}\|_{L^2},\]

which combine with (2.4), we then give the equation (1.4).

Moreover, if the equality of (1.4) holds, then \(f\) will satisfy

(i) \((\text{Ric} - \frac{n+1}{n} Tor + 2K) ((\nabla_b f)_\mathbb{C}, (\nabla_b f)_\mathbb{C}) = 0,\)

(ii) \(f_{\alpha\beta} = 0\) for all \(\alpha, \beta,\)

(iii) \(R - \overline{R} = c_1 f\) and \(\text{Im}(A_{\alpha\beta}, \overline{\alpha\beta}) = c_2 f\) for some real constants \(c_1\) and \(c_2,\)

(iv) \(f_{\alpha\overline{\beta}} - \frac{1}{n} f_{\gamma\gamma} h_{\alpha\overline{\beta}} = \mu \text{Ric}\overline{\alpha\overline{\beta}}\) for some constant \(\mu,\) and

(v) \(\lambda_1 \int_M f^2 = \int_M |\nabla_b f|^2 \quad \text{and} \quad \lambda_1 \int_M |\nabla_b f|^2 = \int_M (R - \overline{R})^2.\)

Simple computation shows that

\[(2.8) \quad \Delta_b f = R - \overline{R} = -\lambda_1 f\]

and

\[(2.9) \quad f_{\alpha\overline{\beta}} - \frac{1}{n} f_{\gamma\gamma} h_{\alpha\overline{\beta}} = \mu \text{Ric}\overline{\alpha\overline{\beta}}\]

with \(\mu = \frac{n+1}{n+2} (1 + \frac{2nK}{(n+1)\lambda_1}).\)

In order to show \(\theta\) is pseudo-Einstein, \(R = \overline{R}\) is a constant and \(\text{Im}(A_{\alpha\beta}, \overline{\alpha\beta}) = 0,\) it suffices to claim that \(f\) is identically zero. So we need to derive some equations from (i)~(v). First, we claim that

\[(2.10) \quad \text{Ric}\overline{\alpha\overline{\beta}} f^{\alpha} - i(n + 2)\mu A_{\alpha\overline{\beta}} f^{\overline{\alpha}} - i(n + 1)A_{\alpha\overline{\beta}} f^{\overline{\alpha}} f^{\overline{\beta}} + 2K f^{\overline{\beta}} = 0.\]
We differentiate (ii) and use (2.9), we have

\[ 0 = f_{\alpha \beta \gamma} = f_{\alpha \beta \gamma} + i h_{\beta \gamma} f_{\alpha 0} + R_{\alpha \rho \beta \gamma} f_{\rho} \]

\[ = \frac{1}{n} f_{\sigma \beta} h_{\alpha \gamma} + \mu \hat{Ric}_{\alpha \beta} + i h_{\beta \gamma} f_{\alpha 0} + R_{\alpha \rho \beta \gamma} f_{\rho} . \]

Contracting with \( h^{\beta \gamma} \), we obtain

\[ (2.11) \quad 0 = \frac{1}{n} f_{\sigma \alpha} + \mu \hat{Ric}_{\alpha \beta} f_{\beta} + i f_{\alpha 0} + \hat{Ric}_{\alpha \beta} f_{\beta} . \]

By differentiating the equation (2.8) yields

\[ -\lambda_1 f_{\alpha} = f_{\sigma \alpha} + f_{\sigma} \overline{\sigma}_{\alpha} = f_{\sigma \alpha} + P_{\alpha} f - i n A_{\alpha \beta} f_{\beta} \]

\[ = f_{\sigma \alpha} + \frac{n}{n-1} (f_{\alpha} \overline{\beta} - \frac{1}{n} f_{\gamma} h_{\alpha \beta}) \overline{\beta} - i n A_{\alpha \beta} f_{\beta} \]

\[ = f_{\sigma \alpha} + \frac{n}{n-1} \hat{Ric}_{\alpha \beta} \overline{\beta} - i n A_{\alpha \beta} f_{\beta} , \]

here the operator \( P_{\alpha} f \) is defined by \( P_{\alpha} f = f_{\overline{\sigma} \alpha} + i n A_{\alpha \beta} f_{\beta} \) and the second equation follows from equation (3.3) in [GL]. Thus, the contracted Bianchi identity (2.3) and \( R_{\alpha} = -\lambda_1 f_{\alpha} \) which follows from (2.8) imply

\[ f_{\sigma \alpha} = -\lambda_1 f_{\alpha} - \frac{n}{n-1} \hat{Ric}_{\alpha \beta} \overline{\beta} + i n A_{\alpha \beta} f_{\beta} \]

\[ = -\lambda_1 f_{\alpha} - \mu (R_{\alpha} - i n A_{\alpha \beta} \beta) + i n A_{\alpha \beta} f_{\beta} \]

\[ = (\mu - 1) \lambda_1 f_{\alpha} + i n A_{\alpha \beta} \beta + i n A_{\alpha \beta} f_{\beta} . \]

Also, by the commutation relations ([Le, Lemma 2.3]), we have

\[ i n f_{\alpha 0} = i n f_{\alpha 0} - i n A_{\alpha \beta} f_{\beta} = (f_{\sigma \alpha} - f_{\sigma} \overline{\sigma}_{\alpha}) \overline{\alpha} - i n A_{\alpha \beta} f_{\beta} \]

\[ = f_{\sigma \alpha} - P_{\alpha} f = f_{\sigma \alpha} - \frac{n}{n-1} \hat{Ric}_{\alpha \beta} \overline{\beta} \]

\[ = (2\mu - 1) \lambda_1 f_{\alpha} + 2 i n A_{\alpha \beta} \beta + i n A_{\alpha \beta} f_{\beta} . \]

Substituting these into (2.11) and using the fact \( \mu = \frac{n+1}{n+2} (1 + \frac{2nK}{(n+1) \lambda_1}) \), we final get

\[ 0 = \frac{1}{n} f_{\sigma \alpha} + \mu \hat{Ric}_{\alpha \beta} \overline{\beta} + i n f_{\alpha 0} + \hat{Ric}_{\alpha \beta} f_{\beta} \]

\[ = 2 K f_{\alpha} + i (n + 2) \mu A_{\alpha \beta} \beta + i (n + 1) A_{\alpha \beta} f_{\beta} + \hat{Ric}_{\alpha \beta} f_{\beta} \]

which is (2.10) as claimed.
Next, we want to show

\[(2.12) \quad \int_M A_{\alpha\beta} f^\alpha f^\beta = \int_M A_{\overline{\alpha\beta}} f^{\overline{\alpha}} f^{\overline{\beta}} = 0 \quad \text{and} \quad \int_M \text{Ric}_{\alpha\overline{\beta}} f^\alpha f^{\overline{\beta}} + K \int_M |\nabla_b f|^2 = 0.\]

From (2.10) we know that

\[\text{Ric}_{\alpha\overline{\beta}} f^\alpha f^{\overline{\beta}} + i (n+2) \mu A_{\alpha\beta}, f^{\alpha} + i(n+1)A_{\alpha\overline{\beta}} f^\alpha f^\beta + 2K f_\alpha f^\alpha = 0.\]

But compare this with (i)

\[(2.13) \quad \text{Ric}_{\alpha\overline{\beta}} f^\alpha f^{\overline{\beta}} + \frac{n+1}{2} i (A_{\alpha\beta} f^\alpha f^\beta - A_{\overline{\alpha\overline{\beta}}} f^{\overline{\alpha}} f^{\overline{\beta}}) + 2K f_\alpha f^\alpha = 0,\]

one gets

\[\frac{2(n+2)}{n+1} \mu A_{\alpha\beta}, f^{\alpha} = -(A_{\alpha\beta} f^\alpha f^\beta + A_{\overline{\alpha\overline{\beta}}} f^{\overline{\alpha}} f^{\overline{\beta}}).\]

Then integral it yields

\[\int_M A_{\alpha\beta} f^\alpha f^\beta + \int_M A_{\overline{\alpha\overline{\beta}}} f^{\overline{\alpha}} f^{\overline{\beta}} = 0\]

due to \(\int_M A_{\alpha\overline{\beta}} f^\alpha f^\beta = \int_M (A_{\alpha\beta} f^\alpha), f^\beta = 0\), by (ii). Also by the reality of \(A_{\alpha\beta}, f^\alpha\), we know

\[\int_M A_{\alpha\beta} f^\alpha f^\beta = -\int_M A_{\overline{\alpha\overline{\beta}}} f^{\overline{\alpha}} f^{\overline{\beta}} = \int_M A_{\overline{\alpha\overline{\beta}}}, f^{\overline{\alpha}} f^{\overline{\beta}}\]

is real. Hence, the integral of (2.13),

\[\int_M \text{Ric}_{\alpha\overline{\beta}} f^\alpha f^{\overline{\beta}} + (n+1) i \int_M A_{\alpha\overline{\beta}} f^\alpha f^\beta + K \int_M |\nabla_b f|^2 = 0\]

will imply (2.12) as we wanted.

Now, by applying (ii) and (2.12) to the equation (2.2), we final obtain

\[\frac{n^2}{2} \int_M f_0^2 + \int_M \sum_{\alpha, \beta} f_{\alpha\beta} f_{\overline{\alpha\overline{\beta}}} + K \int_M |\nabla_b f|^2 = 0.\]

It implies that \(f = 0\) as desired. This completes the proof of Theorem 1.2. \(\square\)
3. First eigenvalue estimate of the sub-Laplacian

Let \((M, J, \theta)\) be a closed pseudo-Hermitian \((2n + 1)\)-manifold with vanishing pseudo-Hermitian torsion. In this section, by applying the argument of the CR analogous Li-Yau’s gradient estimate in [CKL], we derive a lower bound estimate for the first positive eigenvalue \(\lambda_1\) of the sub-Laplacian \(\Delta_b\) using the diameter of \(M\) and lower bound of Webster Ricci tensor. In the last, we prove Corollary 1.3.

**Proposition 3.1.** Let \((M, J, \theta)\) be a closed pseudo-Hermitian \((2n + 1)\)-manifold with vanishing pseudo-Hermitian torsion and the Webster Ricci tensor is bounded from below by a nonpositive constant \(-K\). Let \(f\) be an eigenfunction of \(\Delta_b\) with respect to the first eigenvalue \(\lambda_1\). Then there exists constants \(C_1(n), C_2(n) > 0\) depending on \(n\) alone, such that

\[
\lambda_1 \geq \frac{C_1}{d^2} \exp(-C_2d\sqrt{K}).
\]

Here \(d\) is the diameter of \(M\) with respect to the Carnot-Carathéodory distance (see definition 2.3 in [CKL]).

We recall the following CR version of Bochner formula in a pseudo-Hermitian \((2n + 1)\)-manifold ([G]). For a smooth real-valued function \(\varphi\),

\[
\frac{1}{2} \Delta_b |\nabla_b \varphi|^2 = |(\nabla^H)^2 \varphi|^2 + \langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle + 2 \langle J \nabla_b \varphi, \nabla_b \varphi_0 \rangle + [2Ric - (n - 2)Tor] \langle (\nabla_b \varphi)_C, (\nabla_b \varphi)_C \rangle.
\]

Since

\[
|\nabla^H |^2 = 2 \sum_{\alpha, \beta} (|\varphi_{\alpha \beta}|^2 + |\varphi_{\alpha \overline{\beta}}|^2) \geq 2 \sum_{\alpha} |\varphi_{\alpha \overline{\alpha}}|^2 \geq \frac{1}{2n} (\Delta_b \varphi)^2 + \frac{n}{2} \varphi_0^2
\]

and for any constant \(v > 0\),

\[
2 \langle J \nabla_b \varphi, \nabla_b \varphi_0 \rangle \leq 2 |\nabla_b \varphi| |\nabla_b \varphi_0| \leq v^{-1} |\nabla_b \varphi|^2 + v |\nabla_b \varphi_0|^2.
\]
Therefore, for a real function $\varphi$ and any $v > 0$, the Bochner formula (3.2) becomes

\begin{equation}
\Delta_b |\nabla_b \varphi|^2 \geq \frac{1}{n} (\Delta_b \varphi)^2 + n \varphi_0^2 + 2 \langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle - 2v |\nabla_b \varphi_0|^2 \\
+ 2[2Ric - (n - 2)Tor - 2v^{-1}] ((\nabla_b \varphi)_C, (\nabla_b \varphi)_C).
\end{equation}

**Proof of Proposition 3.1**

**Proof.** Let $f$ be an eigenfunction of $\Delta_b$ with respect to the eigenvalue $\lambda_1$. Since

$$\lambda_1 \int_M f = - \int_M \Delta_b f = 0,$$

$f$ must change sign. We may normalize $f$ to satisfy $\min f = -1$ and $\max f \leq 1$. Let us consider the function $\varphi = \ln(f + a)$, for some constant $a > 1$. Then the function $\varphi$ satisfies

$$\Delta_b \varphi = - |\nabla_b \varphi|^2 - \frac{\lambda_1 f}{f + a}$$

and thus

\begin{equation}
\langle \nabla_b \varphi, \nabla_b \Delta_b \varphi \rangle = - \langle \nabla_b \varphi, \nabla_b |\nabla_b \varphi|^2 \rangle - \frac{a \lambda_1}{f + a} |\nabla_b \varphi|^2.
\end{equation}

Since

$$\Delta_b \varphi_0 = (\Delta_b \varphi)_0 + 2[ (A_{\beta \beta} \varphi^\beta),_\alpha + (A_{\alpha \beta} \varphi^\beta),_\alpha ]
= \left( - |\nabla_b \varphi|^2 - \frac{\lambda_1 f}{f + a} \right)_0 = - 2 \langle \nabla_b \varphi, \nabla_b \varphi_0 \rangle - \frac{a \lambda_1}{f + a} \varphi_0.$$

Therefore, we have

\begin{equation}
\frac{1}{2} \Delta_b \varphi_0^2 = |\nabla_b \varphi_0|^2 + \varphi_0 \Delta_b \varphi_0 = |\nabla_b \varphi_0|^2 - \langle \nabla_b \varphi, \nabla_b \varphi_0^2 \rangle - \frac{a \lambda_1}{f + a} \varphi_0^2.
\end{equation}

And

\begin{equation}
\Delta_b \frac{f}{f + a} = \frac{a}{(f + a)^2} \Delta_b f - \frac{2a}{(f + a)^2} |\nabla_b f|^2 = - 2 \langle \nabla_b \varphi, \nabla_b \frac{f}{f + a} \rangle - \frac{a \lambda_1}{f + a} \frac{f}{f + a}.
\end{equation}

Now, we define $F : M \times [0, 1] \to \mathbb{R}$ by

$$F(x, t) = t(|\nabla_b \varphi|^2 - \alpha \frac{f}{f + a} + \gamma t \varphi_0^2)
= t((\alpha + 1) |\nabla_b \varphi|^2 + \alpha \Delta_b \varphi + \gamma t \varphi_0^2).$$
where $\alpha$ be a nonzero constant and $\gamma$ be a positive constant which will be chosen later. By applying the Bochner inequality (3.3) with $v = \gamma t$, and using (3.4), (3.5) and (3.6), one can derive

\begin{equation}
\Delta_b F + 2 \langle \nabla_b \varphi, \nabla_b F \rangle \geq \frac{1}{n} \left( \Delta_b \varphi \right)^2 + nt\varphi_0^2 - 2 \left( Kt + \gamma^{-1} \right) |\nabla_b \varphi|^2 - \frac{a\lambda_1}{f+a} \left[ 2 |\nabla_b \varphi|^2 - \alpha \frac{\lambda_1}{f+a} + 2\gamma t\varphi_0^2 \right] = \frac{1}{n} \left( \Delta_b \varphi \right)^2 + nt\varphi_0^2 - 2 \left( Kt + \gamma^{-1} \right) |\nabla_b \varphi|^2 - \frac{a\lambda_1}{f+a} [F + t |\nabla_b \varphi|^2 + \gamma t\varphi_0^2].
\end{equation}

(3.7)

On the other hand, from the definition of $F(x, t)$, we have

\[
\Delta_b \varphi = \alpha^{-1} [t^{-1} F - (\alpha + 1) |\nabla_b \varphi|^2 - \gamma t\varphi_0^2],
\]

thus

\[
(\Delta_b \varphi)^2 \geq (\alpha t)^{-2} F^2 - 2\alpha^{-2} t^{-1} F \left[ (\alpha + 1) |\nabla_b \varphi|^2 + \gamma t\varphi_0^2 \right].
\]

Substituting this into (3.7), we obtain

\[
\Delta_b F + 2 \langle \nabla_b \varphi, \nabla_b F \rangle \geq \frac{1}{n\alpha t} F^2 - \frac{a\lambda_1}{f+a} F + [n - (\alpha \frac{\lambda_1}{f+a} + \frac{2}{n\alpha^2} F)\gamma] t\varphi_0^2 - 2\left[ Kt + \gamma^{-1} + \frac{1}{2} \frac{a\lambda_1}{f+a} + \frac{\alpha+1}{n\alpha^2} F \right] |\nabla_b \varphi|^2.
\]

Thus, at a maximum point $p_t$ of $F(\cdot, t)$, we have

\[
0 \geq \Delta_b F(p_t, t) + 2 \langle \nabla_b \varphi, \nabla_b F \rangle (p_t, t).
\]

Hence, at $(p_t, t)$,

\begin{equation}
0 \geq \frac{1}{n\alpha t} F^2 - \frac{a\lambda_1}{f+a} F + [n - (\alpha \frac{\lambda_1}{f+a} + \frac{2}{n\alpha^2} F)\gamma] t\varphi_0^2 - 2\left[ Kt + \gamma^{-1} + \frac{1}{2} \frac{a\lambda_1}{f+a} + \frac{\alpha+1}{n\alpha^2} F \right] |\nabla_b \varphi|^2.
\end{equation}

(3.8)

We claim that there exist constants $\alpha$ depending only on $n$ with $(\alpha + 1) < 0$ and $\gamma$ depending on $\lambda_1$, $a$ and $K$ such that

\[
F(x,t) < -\frac{na^2}{\alpha+1} [K + \gamma^{-1} + \frac{1}{2} \frac{a\lambda_1}{a-1}] \]
on $M \times [0, 1]$. We prove it by contradiction. Suppose not, then

$$\max_{M \times [0, 1]} F(x, t) \geq -\frac{n\alpha^2}{\alpha + 1}[K + \gamma^{-1} + \frac{1}{2} \frac{a \lambda_1}{a - 1}].$$

Since $F$ is continuous in the variable $t$ and $F(x, 0) = 0$, thus there exists a $t_0 \in (0, 1]$ such that

$$\max_{M \times [0, t_0]} F(x, t) = -\frac{n\alpha^2}{\alpha + 1}[K + \gamma^{-1} + \frac{1}{2} \frac{a \lambda_1}{a - 1}].$$

Assume $F$ achieves its maximum at the point $(p_{t_0}, s_{t_0})$ on $M \times [0, t_0]$. Then

$$F(p_{t_0}, s_{t_0}) = -\frac{n\alpha^2}{\alpha + 1}[K + \gamma^{-1} + \frac{1}{2} \frac{a \lambda_1}{a - 1}] > 0.$$ (3.9)

By applying (3.8) at a maximum point $p_{t_0}$ of $F(\cdot, s_{t_0})$ and using (3.9), one obtain

$$0 \geq \frac{1}{n\alpha^2 s_{t_0}} F(p_{t_0}, s_{t_0})^2 - \frac{\alpha \lambda_1}{f + a} F(p_{t_0}, s_{t_0})$$
$$+ [n - \frac{\alpha \lambda_1}{f + a} s_{t_0} + \frac{2}{n\alpha^2} F(p_{t_0}, s_{t_0}) \gamma] s_{t_0} \varphi_0^2.$$ (3.10)

Now we choose

$$\alpha + 1 = -\frac{3}{n} \quad \text{and} \quad \gamma^{-1} = \frac{n + 3}{n} \frac{a \lambda_1}{a - 1} + 2K,$$

then

$$\frac{1}{n\alpha^2 s_{t_0}} F(p_{t_0}, s_{t_0})^2 - \frac{\alpha \lambda_1}{f + a} F(p_{t_0}, s_{t_0}) \geq n s_{t_0}^{-1}(K + \frac{1}{2} \frac{a \lambda_1}{a - 1}) F(p_{t_0}, s_{t_0}) > 0$$

and

$$n - \frac{\alpha \lambda_1}{f + a} s_{t_0} + \frac{2}{n\alpha^2} F(p_{t_0}, s_{t_0}) \gamma \geq \frac{n}{3} [1 - (\frac{n + 3}{n} \frac{a \lambda_1}{a - 1} + 2K) \gamma] = 0.$$ This leads to a contradiction with (3.10). Therefore, we obtain that

$$F(x, t) < (n + 3)^2 (\frac{n + 2}{2n} \frac{a \lambda_1}{a - 1} + K)$$
on $M \times [0, 1]$. In particular, at $t = 1$, we have

$$|\nabla_b \varphi|^2 + \frac{\alpha + 3}{n} \frac{a \lambda_1 f}{f + a} + (\frac{\alpha + 3}{n} \frac{a \lambda_1}{a - 1} + 2K)^{-1} \varphi_0^2 \leq (n + 3)^2 (\frac{n + 2}{2n} \frac{a \lambda_1}{a - 1} + K).$$
Thus, we obtain the subgradient estimate
\[ |\nabla \varphi|^2 + \left( \frac{n+3}{n} \frac{a\lambda_1}{a-1} + 2K \right) - \frac{1}{(n+3)^2} \leq (n+3)^2 \left( \frac{n+2}{2n} \frac{a\lambda_1}{a-1} + K \right) - \frac{n+3}{n} \frac{a\lambda_1 f}{f+a} \]
\[ \leq (n+3)^2 \left( \frac{n+3}{2n} \frac{a\lambda_1}{a-1} + K \right). \]
Therefore,
\[ |\nabla \varphi|^2 \leq (n+3)^2 \left( \frac{n+3}{2n} \frac{a\lambda_1}{a-1} + K \right). \]
By integrating \(|\nabla \varphi| = |\nabla \ln(f+a)|\) along a minimal horizontal geodesic \(\varsigma\) joining the points at which \(f = -1\) and \(f = \max f\), it follows that
\[ \ln \frac{a}{a-1} \leq \ln \left( \frac{a+\max f}{a-1} \right) = \ln(a+\max f) - \ln(a-1) \]
\[ \leq \int_{\varsigma} |\nabla \ln(f+a)| \leq (n+3)d \sqrt{\frac{n+3}{2n} \frac{a\lambda_1}{a-1} + K}, \]
for all \(a > 1\). Setting \(s = (a-1)/a\), we obtain
\[ \frac{(n+3)^3}{2n} \lambda_1 \geq (d^2 \ln s^{-1})^2 - (n+3)^2 K \]
for all \(0 < s < 1\). Maximizing the right hand side as a function of \(s\) by setting \(s = \exp(-1 - \sqrt{1 + (n+3)^2 K d^2})\), we get the estimate
\[ \lambda_1 \geq \frac{4n}{(n+3)^2 d^2} \exp(-1 - \sqrt{1 + (n+3)^2 K d^2}) \]
as claimed. This completes the proof of Proposition 3.1. \(\square\)

Now we prove Corollary 1.3.

Proof of Corollary 1.3 From the Proposition 3.1 we have
\[ \frac{K}{\lambda_1} \leq C_1 K d^2 \exp(C_2 d \sqrt{K}). \]
So we obtain the inequality in Corollary 1.3 with the constant
\[ C(K d^2) = \frac{4n^2}{(n-1)(n+2)} \left( \frac{n+1}{2n} + C_1 K d^2 \exp(C_2 d \sqrt{K}) \right). \]

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