ON MONODROMY IN FAMILIES OF ELLIPTIC CURVES OVER $\mathbb{C}$

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ABSTRACT. We show that, in a smooth family of elliptic curves over $\mathbb{C}$ with a smooth base $B$, the general fiber of the mapping $J: B \to \mathbb{A}^1$ (assigning $j$-invariant of the fiber to a point) has $m$ connected components, then the monodromy group of the family (acting on $H^1(\cdot, \mathbb{Z})$ of the fibers) has index at most $2m$ in $\text{SL}(2, \mathbb{Z})$. Some applications are given, including that to monodromy of hyperplane sections of del Pezzo surfaces.

INTRODUCTION

It is believed that if, in a family of algebraic varieties, the fibers “vary enough” then the monodromy group acting on the cohomology of the fiber should be in some sense big. Quite a few results have been obtained in this direction. See for example [5] for families of elliptic curves, [9] for families of hyperelliptic curves, [2] for families of abelian varieties (see also [10] for abelian varieties in the arithmetic situation). In the cited papers cohomology means “étale cohomology with finite coefficients”. In this paper we address the question of “big monodromy” for families of elliptic curves over $\mathbb{C}$ and singular cohomology.

The main result of the paper (Proposition 2.3) asserts that if $\pi: \mathcal{X} \to B$ is a smooth non-isotrivial family of elliptic curves over $\mathbb{C}$ and if the general fiber of its “$J$-map” $J_{\mathcal{X}}: B \to \mathbb{A}^1$ (assigning to each point of the base the $j$-invariant of the fiber) has $m$ connected components, then the monodromy group of the family $\mathcal{X}$ is a subgroup of index at most $2m$ in $\text{SL}(2, \mathbb{Z})$. Here, by monodromy group we mean that acting on $H^1(\cdot, \mathbb{Z})$ of the fiber.

An immediate consequence of this proposition is that, in any non-isotrivial family of elliptic curves, the monodromy group has finite index in $\text{SL}(2, \mathbb{Z})$ (Corollary 2.7). This requires some comments.

The above assertion is similar to a well-known result about elliptic curves over number fields, viz. to Serre’s Theorem 3.2 from Chapter IV of [15]. It is quite probable that one can prove our Corollary 2.7 by imitating, mutatis mutandis, Serre’s proof of this theorem or even derive it from Serre’s theorem or similar arithmetical results. One merit of the approach presented in this

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paper is that the proofs are very simple and elementary. One should add that
the similarity between arithmetic and geometric situations is not absolute.
For example, Theorem 5.1 from [9] suggests that, over $\mathbb{C}$, the monodromy
group for some families of hyperelliptic curves of genus $g$ should be the
entire $\text{Sp}(2g, \mathbb{Z})$. However, as we show in Proposition 4.2 for any family of
hyperelliptic curves of genus at least 3 over $\mathbb{C}$, the monodromy group acting
on $H^1(\cdot, \mathbb{Z})$ of the fiber is a proper subgroup of $\text{Sp}(2g, \mathbb{Z})$.

Our main result has three simple consequences, which are presented in
Section 3. First, any smooth (i.e., without degenerate fibers) family of ellip-
tic curves over a smooth base with commutative fundamental group, must
be isotrivial (Proposition 3.1). Second, for non-isotrivial families we obtain
an upper bound on the index of the monodromy group in $\text{SL}(2, \mathbb{Z})$. Third,
in the case of smooth elliptic surfaces we use Miranda’s results from [13] to
to obtain an upper bound on the index of monodromy group in terms of
of singular fibers (it turns out that only fibers of the types $I_n$ and $I_{n'}$ count); see Proposition 3.6.

Finally, in Section 5, we derive from our main result that the hyperplane
monodromy group of a smooth Del Pezzo surface is always the entire $\text{SL}(2, \mathbb{Z})$
(Proposition 5.1). (I realize that it is not the only way to obtain this result.)
Observe that, in view of Proposition 4.2 Proposition 5.1 cannot be extended
to surfaces with hyperelliptic hyperplane sections.

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Notation and conventions. All our algebraic varieties are defined over $\mathbb{C}$
and reduced, so they are essentially identified with their sets of closed points;
the only exception is the proof of Lemma 2.5. If $X$ is an algebraic variety,
then $X_{\text{sm}}$ is its smooth locus and $X_{\text{sing}}$ is its singular locus.

When we say “a general $X$ has property $Y$”, this always means “property
$Y$ holds for a Zariski open and dense set of $X$’s”. The word “generic” is
used in the scheme-theoretic sense (only once).

If $B$ is an algebraic variety and $\pi: \mathcal{X} \to B$ is a proper and flat morphism
such that a general fiber of $\pi$ is, say, a smooth curve of genus 1, we will say
that $\pi$ is a family of curves of genus 1. If, in addition, the morphism $f$ is
smooth, we will say that $\pi$ (or just $\mathcal{X}$ if there is no danger of confusion) is
a smooth family, or a family of smooth varieties. If $\pi: \mathcal{X} \to B$ is a family
over $B$ and $f: B' \to B$ is a morphism, then by $p': f^*\mathcal{X} \to B'$ we mean the
the pull-back of $\mathcal{X}$ along $f$.

By $\pi_1$ of an algebraic variety over $\mathbb{C}$ we always mean fundamental group
in the classical (complex) topology.

As usual, we put $\Gamma(2) = \{ A \in \text{SL}(2, \mathbb{Z}) : A \equiv \text{Id} \ (\text{mod} \ 2) \}$.

Finally, we fix some terminology and notation concerning elliptic curves.
Following Miranda [13], we distinguish between curves of genus 1 and
elliptic curves: by elliptic curve over a field $K$ we mean a smooth projective
curve over $K$ of genus 1 with a distinguished $K$-rational point.
Similarly, by a smooth family of curves of genus 1 we will mean a smooth family \( \pi : X \to B \) such that its fibers are curves of genus 1, and by a smooth family of elliptic curves we mean a pair \((X, s)\), where \(X \to B\) is a smooth family of curves of genus 1 and \(s : B \to X\) is a section.

To each curve \(C\) of genus 1 over a field \(K\) one can assign its \(j\)-invariant \(j(C) \in K\); recall that if \(C\) is (the smooth projective model of) the curve defined by the Weierstrass equation \(y^2 = x^3 + px + q\), then

\[
j(C) = 1728 \cdot \frac{4p^3}{4p^3 + 27q^2}.
\]

Two curves of genus 1 over \(\mathbb{C}\) are isomorphic if and only if their \(j\)-invariants are equal.

We say that a family over \(B\) is isotrivial if it becomes trivial after a pullback along a generically finite morphism \(B_1 \to B\). For families of curves of genus 1 this is equivalent to the condition that \(j\)-invariants of all fibers are the same.

1. Generalities on monodromy groups

Suppose that \(B\) is an irreducible variety and \(\pi : X \to B\) is a family of smooth varieties.

If \(b \in B_{\text{sm}}, k \in \mathbb{N}, \) and \(G\) is an abelian group, then the fundamental group \(\pi_1(B_{\text{sm}}, b)\) acts on \(H^k(p^{-1}(b), G)\).

**Definition 1.1.** The image (corresponding to this action) of \(\pi_1(B_{\text{sm}}, b)\) in \(\text{Aut}(H^k(p^{-1}(b), G))\) will be called monodromy group of the family \(X\) at \(b\) and denoted \(\text{Mon}(X, b)\) (we suppress the mention of \(k\) and \(G\); there will be no danger of confusion).

Since \(B\) is irreducible, monodromy groups at \(b\) are isomorphic for all \(b \in B_{\text{sm}}\). If \(b, b_1 \in B_{\text{sm}}, \) the isomorphisms \(\text{Aut}(H^k(p^{-1}(b), \mathbb{Z})) \to \text{Aut}(H^k(p^{-1}(b_1), \mathbb{Z}))\) induced by two paths from \(b\) to \(b_1\) lying in \(B_{\text{sm}}\), differ by an inner automorphism of \(\text{Aut}(H^k(p^{-1}(b), G))\). Thus, if we fix once and for all the group \(A = \text{Aut}(H^k(p^{-1}(b_0), G))\) for some \(b_0 \in B_{\text{sm}}, \) then all the groups \(\text{Mon}(X, b)\) define the same conjugacy class of subgroups of \(A;\) this class (or, abusing the language, any subgroup belonging to this class) will be denoted by \(\text{Mon}(X)\).

In the sequel we will be working with families of smooth curves of genus \(g\) (in most cases, \(g = 1\)) as fibers and monodromy action on \(H^1\) of the fiber. Since monodromy preserves the intersection form, the subgroups \(\text{Mon}(X)\), where \(X\) is such a family, will be defined up to an inner automorphism of the group \(\text{Sp}(2g, \mathbb{Z})\) (\(\text{SL}(2, \mathbb{Z})\) if \(g = 1\)).

**Convention 1.2.** If \(\pi : X \to B\) is a non-smooth family, then by \(\text{Mon}(X)\) we mean \(\text{Mon}(X|_U)\), where \(U \subset B\) is the Zariski open subset over which \(\pi\) is smooth.

Below we list some simple properties of monodromy groups.
Proposition 1.3. Suppose that $B$ is an irreducible variety, $U \subset B$ is a non-empty Zariski open subset, and $\mathcal{X}$ is a smooth family over $B$. Then $\text{Mon}(\mathcal{X}|_U) = \text{Mon}(\mathcal{X})$.

Proof. The result follows from the fact that, for any $b \in U \cap B_{\text{sm}}$, the natural homomorphism $\pi_1(U \cap B_{\text{sm}}, b) \to \pi_1(B_{\text{sm}}, b)$ is epimorphic (see for example [8, 0.7(B) ff.]).

Proposition 1.4. Suppose that $B'$ and $B$ are smooth irreducible varieties and $\mathcal{X}$ is a smooth family over $B$. If $f: B' \to B$ is a dominant morphism such that a general fiber of $f$ has $m$ connected components, then $\text{Mon}(f^*\mathcal{X})$ is conjugate to a subgroup of $\text{Mon}(\mathcal{X})$, of index at most $m$.

Corollary 1.5. Suppose that $B'$ and $B$ are smooth irreducible varieties and $\mathcal{X}$ is a smooth family over $B$. If $f: B' \to B$ is a dominant morphism such that a general fiber of $f$ is connected, then $\text{Mon}(f^*\mathcal{X})$ is conjugate to $\text{Mon}(\mathcal{X})$.

Proof of Proposition 1.4. It follows from [17, Corollary 5.1] and the algebraic version of Sard’s theorem that there exists a Zariski open non-empty $U \subset B$ such that all the fibers of $f$ over points of $U$ are smooth and the induced mapping $f': f^{-1}U \to U$ is a locally trivial bundle in the complex topology. Proposition 1.3 implies that $\text{Mon}(\mathcal{X}|_U) = \text{Mon}(\mathcal{X})$. Since $f^{-1}(U)$ is (path) connected, each fiber of this bundle has $m$ connected components, and the base is locally path connected, $f'_*(\pi_1(f^{-1}U, x))$ is a subgroup of index $m$ in $\pi_1(U, f(x))$ for any $x \in f^{-1}(U)$. This implies the proposition.

2. The main result

We begin with a folklore result for which I do not know an adequate reference.

Proposition 2.1. Suppose that $\pi: \mathcal{X} \to B$ is a smooth family of curves of genus 1, where $B$ is a variety (i.e., a reduced scheme of finite type over $\mathbb{C}$). Then the mapping from $B$ to $\mathbb{C}$ that assigns $j$-invariant $j(f^{-1}(b))$ to a point $b \in B$, is induced by a morphism from $B$ to $\mathbb{A}^1$.

Proof. If $\mathcal{X}' = \text{Pic}^0(\mathcal{X}/B)$ (relative Picard variety, see [11, Section 5]), then the family $\rho': \mathcal{X}' \to \mathbb{C}$ has a section (to wit, the zero section) and induces the same mapping from $B$ to $\mathbb{C}$ since $\text{Pic}^0(\mathbb{C}) \cong \mathbb{C}$ if $\mathbb{C}$ is a smooth curve of genus 1 over $\mathbb{C}$. Thus, without loss of generality one may assume that the family in question has a section; in this case see [7, §5].

Notation 2.2. If $\mathcal{X} \to B$ is a smooth family of curves of genus 1, then the morphism $B \to \mathbb{A}^1$ assigning the $j$-invariant $j(p^{-1}(b))$ to a point $b \in B$, will be denoted by $J_{\mathcal{X}}$. Following Miranda [13, Lecture V], we will say that $J_{\mathcal{X}}$ is the $J$-map of the family $\mathcal{X}$.

In a family of smooth projective curves of genus 1, the monodromy group acting on $H^1$ of the fiber is contained in $\text{SL}(2, \mathbb{Z})$. 

Proposition 2.3. Suppose that $\pi: X \to B$ is a smooth family of curves of genus 1 over a smooth and connected base $B$ (the ground field is $\mathbb{C}$); let $J_X: B \to \mathbb{A}^1$ be the morphism attaching to any point $a \in B$ the $j$-invariant of the fiber of $X$ over $a$. If the morphism $J_X$ is dominant and its general fiber has $m$ connected components, then the monodromy group acting on $H^1(\cdot, \mathbb{Z})$ of fibers of $p$ is a subgroup of index at most $2m$ in $\text{SL}(2, \mathbb{Z})$.

We begin with two lemmas.

Lemma 2.4. Suppose that $\pi: X \to B$ is a smooth family of curves of genus 1. Then there exists a smooth family of elliptic curves $p': X' \to B$ such that the induced mappings $J_X, J_{X'}: B \to \mathbb{A}^1$ are the same and $\text{Mon}(\text{Aut}(X')) \subset \text{SL}(2, \mathbb{Z})$ is conjugate to $\tau(\text{Mon}(X))$, where $\tau: \text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z})$ is the automorphism defined by the formula $M \mapsto (M^t)^{-1}$.

Proof. Put $X' = \text{Pic}^0(X/B)$. As we have seen in the proof of Proposition 2.1, $J_X = J_{X'}$ and the family $p': X' \to B$ has a section. Finally, $R^1p'_*\mathbb{Z} \cong \text{Hom}(R^3p_*\mathbb{Z}, \mathbb{Z})$, where by $\mathbb{Z}$ we mean the constant sheaf with the stalk $\mathbb{Z}$ (see for example [14, §9]), and this implies the assertion about monodromy. □

Lemma 2.5. Suppose that $p_1: X_1 \to B$ and $p_2: X_2 \to B$ are families of elliptic curves over the same irreducible base $B$ such that $J_{X_1} = J_{X_2}$ and the morphism $J_{X_1} = J_{X_2}$ is not constant; put $G_i = \text{Mon}(X_i), i = 1, 2$. Then there exist subgroups $H_1 \subset G_1$ and $H_2 \subset G_2$ such that $(G_i: H_i) \leq 2$ and $H_1$ is conjugate to $H_2$.

Proof. Put $K = \mathbb{C}(B)$ (the field of rational functions). For $m = 1, 2$ denote by $\tilde{X}_m$ the generic fiber of $p_m$ (over $\text{Spec} \mathbb{K}$). The curves $\tilde{X}_1$ and $\tilde{X}_2$ are elliptic curves over $K$ with the same $j$-invariant, and this $j$-invariant is not equal to 0 or 1728 since the $J$-map $J_{X_1} = J_{X_2}$ is not constant. Hence, either $\tilde{X}_1 \cong \tilde{X}_2$ or there exists a quadratic extension $L \supset K$ such that $\tilde{X}_1 \times_{\text{Spec} \mathbb{K}} \text{Spec} L \cong \tilde{X}_2 \times_{\text{Spec} \mathbb{K}} \text{Spec} L$ (see [16, Chapter X, Proposition 5.4]). This implies that there exists a Zariski open $U \subset B$ and a finite étale morphism $\alpha: V \to U$ of degree 2 that $(i \circ \alpha)^*X_1 \cong (i \circ \alpha)^*X_2$, where $i: U \to B$ is the natural inclusion. If $b \in V$, then, for $m = 1, 2$, the group $\text{Mon}((i \circ \alpha)^*X_m, b)$ is a subgroup of index at most 2 in $\text{Mon}(X_m, \alpha(b))$. In view of Proposition 1.3 this implies the lemma. □

Proof of Proposition 2.3. Since a subgroup of index 2 in $\text{SL}(2, \mathbb{Z})$ is unique (see Remark 2.9), the automorphism $M \mapsto (M^t)^{-1}$ maps this subgroup onto itself, so Lemma 2.4 implies that we may assume that the family $\pi: X \to B$ in question has a section. Assuming that, put

$$V = \{(p, q) \in \mathbb{A}^2 : 4p^3 + 27q^2 \neq 0 \}$$

and consider the smooth family of elliptic curves $B \to V$ in which the fiber over $(p, q)$ is the smooth projective model $C_{p, q}$ of the curve with equation $y^2 = x^3 + px + q$ and the section assigns to $(p, q)$ the “point at infinity” of...
This model. It is well known (see for example [1, Corollary to Theorem 1]) that $\text{Mon}(B) = \text{SL}(2, \mathbb{Z})$.

Now put $A_0^1 = A^1 \setminus \{0\}$, $A_{0,1728}^1 = A^1 \setminus \{0, 1728\}$ and

$$V' = J^{-1}_B(A_{0,1728}^1) = \{(p, q) \in \mathbb{A}^2 : 4p^3 + 27q^2 \neq 0, p \neq 0, q \neq 0\}.$$

Let $B'$ be the restriction of the family $B$ to $V'$; put $B' = J^{-1}_X(A_{0,1728}^1)$, and let $X'$ be the restriction of $X$ to $B'$. Proposition [1.3] implies that $\text{Mon}(X') = \text{Mon}(X)$ and $\text{Mon}(B') = \text{Mon}(B) = \text{SL}(2, \mathbb{Z})$.

Observe that there exists an isomorphism $g: V' \to A_0^1 \times A_{0,1728}^1$ such that the diagram

$$
\begin{array}{ccc}
V' & \xrightarrow{g} & A_0^1 \times A_{0,1728}^1 \\
\downarrow J_X & & \downarrow \text{pr}_2 \\
A_{0,1728}^1 & &
\end{array}
$$

is commutative. Indeed, one can define $g$ by the formula $(p, q) \mapsto \left(\frac{\lambda^2}{4} \left(\frac{1728}{j} - 1\right), \frac{\lambda^3}{4} \left(\frac{1728}{j} - 1\right)\right)$.

Hence, in the fibered product

$$
\begin{array}{ccc}
W & \xrightarrow{f} & V' \\
\downarrow u & & \downarrow J_{B,0} \\
B' & \xrightarrow{J_{X,0}} & A_{0,1728}^1
\end{array}
$$

(we mean fibered product in the category of reduced algebraic varieties, so $W$ is the scheme theoretic fibered product modulo nilpotents) the variety $W$ is isomorphic to $(A_0^1) \times B'$ (in particular, $W$ is smooth and irreducible) and and fibers of $f$ are isomorphic to fibers of $J_{X,0}$. Thus, the hypothesis implies that a general fiber of the morphism $J_{X,0}$ has $m$ connected components. On the other hand, any fiber of the morphism $u$ is irreducible since it is isomorphic to $A_0^1$. Now Proposition [1.4] and Corollary [1.5] imply that for the pull-back families $f^*B'$ and $u^*X'$ on $W$, the group $\text{Mon}(f^*B')$ is a subgroup of index at most $m$ in $\text{Mon}(B) = \text{SL}(2, \mathbb{Z})$ and $\text{Mon}(u^*X) = \text{Mon}(X)$ (as usual, this equation holds up to a conjugation).

It is clear that $J_{f^*B'} = J_{B'} \circ f = J_{X'} \circ u = J_{u^*X'}$. Applying Lemma [2.5] to the families $f^*B'$ and $u^*X'$, one obtains the result.

Remark 2.6. I do not know whether there exists a family satisfying the hypotheses of Proposition [2.8] such that the monodromy group is not the entire $\text{SL}(2, \mathbb{Z})$.

Proposition [2.8] implies the following fact.
Corollary 2.7. If $\mathcal{X} \to B$ is a non-isotrivial smooth family of curves of genus 1, then its monodromy group is a subgroup of finite index in $\text{SL}(2, \mathbb{Z})$.

Another corollary of Proposition 2.3 is as follows.

Corollary 2.8. If, under the hypotheses of Proposition 2.3, a general fiber of $J_{\mathcal{X}}$ is connected, then the monodromy group acting on $H^1(\cdot, \mathbb{Z})$ of fibers of $p$ is either the entire $\text{SL}(2, \mathbb{Z})$ or a subgroup of index 2 in $\text{SL}(2, \mathbb{Z})$.

Remark 2.9. It is well known that a subgroup of index 2 in $\text{SL}(2, \mathbb{Z})$ is unique. To wit, this is the subgroup of automorphisms of $\mathbb{Z}^2$ with determinant 1 that, after reduction modulo 2, induce even permutations of non-zero elements of $(\mathbb{Z}/2\mathbb{Z})^2$. The uniqueness of such a subgroup follows from the fact that an epimorphism from $\text{SL}(2, \mathbb{Z})$ onto $\mathbb{Z}/2\mathbb{Z}$ is unique since the abelianization of $\text{SL}(2, \mathbb{Z})$ is the cyclic group of order 12.

3. Applications

Proposition 3.1. If $B$ is a smooth algebraic variety with abelian fundamental group, then any smooth family of curves of genus 1 over $X$ must be isotrivial.

Proof. Suppose that $\pi: \mathcal{X} \to B$ is a smooth family of curves of genus 1, where $B$ is smooth and irreducible and $\pi_1(B) = G$ is abelian.

We are to show that the $J$-map $J_{\mathcal{X}}: B \to \mathbb{A}^1$ is constant. If this is not the case, then Corollary 2.7 asserts that the monodromy group $\text{Mon}(\mathcal{X})$ of the family $\mathcal{X}$ has finite index in $\text{SL}(2, \mathbb{Z})$. Since $\Gamma(2)$ has finite index in $\text{SL}(2, \mathbb{Z})$, one has $(\Gamma(2) : \Gamma(2) \cap \text{Mon}(\mathcal{X})) < \infty$. If $G$ is the image of $\Gamma(2) \cap \text{Mon}(\mathcal{X})$ in $\Gamma(2)/\{\pm \text{Id}\}$, then $G$ is an abelian subgroup of finite index in $\Gamma(2)/\{\pm \text{Id}\}$. The latter group is isomorphic to the free group $F_2$ with two generators, and Schreier’s theorem on subgroups of free groups implies that $F_2$ contains no abelian subgroup of finite index. We arrived at the desired contradiction. □

For the case of non-commutative $\pi_1$ of the base, one can obtain a lower bound on the order of monodromy groups in non-isotrivial families.

Proposition 3.2. Suppose that $\mathcal{X} \to B$ is a smooth non-isotrivial family of curves of genus 1 over a smooth base $B$ and that $\pi_1(B)$ can be generated by $r \geq 2$ elements. Then $(\text{SL}(2, \mathbb{Z}) : \text{Mon}(\mathcal{X})) \leq 12(r - 1)$.

Corollary 3.3. Suppose that $\mathcal{X} \to C$ is a non-isotrivial family of elliptic curves over a smooth curve of genus $g$, with $s$ degenerate fibers. Then $(\text{SL}(2, \mathbb{Z}) : \text{Mon}(\mathcal{X})) \leq 12(2g + s - 1)$.

Proposition 3.2 is a consequence of the following elementary lemma.

Lemma 3.4. Suppose that $G \subset \text{SL}(2, \mathbb{Z})$ is a subgroup of finite index and that $G$ can be generated by $r$ elements. Then $(\text{SL}(2, \mathbb{Z}) : G) \leq 12(r - 1)$. 

Proof of the lemma. Throughout the proof, free group with \( m \) generators will be denoted by \( F_m \).

Since \( G \) can be generated by \( r \) elements, there exists an epimorphism \( \pi: F_r \to G \). Putting \( H = p^{-1}(\Gamma(2)) \), one obtains the following commutative diagram of embeddings and surjections:

\[
\begin{align*}
H & \to G \cap \Gamma(2) \to \Gamma(2) \\
\text{index}=d & \quad \text{index}=d & \text{index}=6 \\
F_r & \to G' \to \text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{Z})/\Gamma(2) \cong S_3.
\end{align*}
\]

If \( d \leq 6 \) is the order of \( \text{Im}(j \circ i) \), then \( (G : G \cap \Gamma(2)) = (F_r : H) = d \), so by Schreier’s theorem \( H \cong F_{d(r-1)+1} \). Since the morphism \( p' \) is surjective, the group \( G \cap \Gamma(2) \) can be generated by \( d(r-1)+1 \) elements. Put \( F = \Gamma(2)/\{\pm \text{Id}\} \), and let \( \pi: \Gamma(2) \to F \) be the natural projection. The subgroup \( \pi(G \cap \Gamma(2)) \subset F \) can also be generated by \( d(r-1)+1 \) elements; since \( F \cong F_2 \), Schreier’s theorem implies that \( \pi(G \cap \Gamma(2)) \cong F_m \), where \( m \leq d(r-1)+1 \) (indeed, the free group \( F_n \) cannot be generated by \( k < n \) elements).

Applying Schreier’s for the third time, we obtain that \( (F : \pi(H \cap \Gamma(2))) = m - 1 \leq d(r-1) \), whence \( (\Gamma(2) : G \cap \Gamma(2)) \leq 2(m-1) \leq 2d(r-1) \). It follows from the right-hand square of the diagram (2) that

\[
(\text{SL}(2, \mathbb{Z}) : G) = \frac{(\text{SL}(2, \mathbb{Z}) : \Gamma(2)) \cdot (\Gamma(2) : G \cap \Gamma(2))}{(G : G \cap \Gamma(2))} \leq \frac{12d(r-1)}{d},
\]

whence the result. \( \square \)

Proof of Proposition 3.2. Put \( \text{Mon}(\mathcal{X}) = G \subset \text{SL}(2, \mathbb{Z}) \). Since \( \pi_1(B) \) can be generated by \( r \) elements, the same is true for \( G \); now Corollary 2.7 implies that \( (\text{SL}(2, \mathbb{Z}) : G) < +\infty \), and Lemma 3.4 applies. \( \square \)

Using Proposition 2.3 one can obtain other lower bounds for monodromy groups. Observe first that the named proposition immediately implies the following corollary, in the statement of which we use Convention 1.2.

Corollary 3.5. If \( \pi: \mathcal{X} \to C \) is a family of elliptic curves over a smooth projective curve \( C \) and if \( J_{\mathcal{X}}: C \to \mathbb{A}^1 \) is its \( J \)-map, and if \( J_{\mathcal{X}} \) is not constant, then \( (\text{SL}(2, \mathbb{Z}) : \text{Mon}(\mathcal{X})) \leq 2 \text{deg} J_{\mathcal{X}} \).

If \( \mathcal{X} \) is smooth and \( \pi \) has a section, one can be more specific.

Proposition 3.6. Suppose that \( \pi: \mathcal{X} \to C \) is a minimal smooth elliptic surface with section (it means that \( \mathcal{X} \) is a smooth projective surface, \( C \) is a smooth projective curve, the general fiber of \( \pi \) is a smooth curve of genus 1, no fiber of \( \pi \) contains a rational \((−1)\)-curve, and \( p \) has a section) and that \( J_{\mathcal{X}} \) is not constant.

Then

\[
(\text{SL}(2, \mathbb{Z}) : \text{Mon}(\mathcal{X})) \leq 2 \sum_{s \in C} e(s),
\]

where \( e(s) \) is the number of fibers of \( \pi \) containing a rational \((−1)\)-curve at \( s \).
where \( e(s) = n \) if the fiber over \( s \) is a cycle of \( n \) smooth rational curves or the nodal rational curve if \( n = 1 \) (type \( \text{I}_n \) in Kodaira’s classification [12, 13]), \( e(s) = n \) if the fiber over \( s \) consists of \( n + 5 \) smooth rational curves with intersection graph isomorphic to the extended Dynkin graph \( \tilde{D}_{n+4} \), \( n \geq 1 \) (type \( \text{I}^*_n \) in Kodaira’s classification), and \( e(s) = 0 \) otherwise.

Proof. In view of Corollary 3.5 the index in the left-hand side of (3) is less or equal to \( 2 \deg J_X \), and \( \deg J_X \) equals \( \sum_{s \in C} e(s) \) by virtue of Corollary IV.4.2 from [13]. \( \square \)

Similarly, one can express \( \deg J_X \) (and obtain a lower bound for \( \text{Mon}(X) \)) using the information about the points where \( j \)-invariant of the fiber (smooth or not) equals 0 or 1728, see for example [13, Lemma IV.4.5, Table IV.3.1] and Table. In the notation if [13], \( j \)-invariant is 1728 times less than that defined by (1); of course, this does not affect multiplicities of poles.

4. A remark on families of hyperelliptic curves

**Proposition 4.1.** If \( \pi : X \to B \) is a smooth family of hyperelliptic curves of genus \( g > 2 \), then

\[
(\text{Sp}(2g, \mathbb{Z}) : \text{Mon}(X)) \geq \frac{2^g (2^{2g} - 1)(2^{2(g-1)} - 1) \cdots (2^2 - 1)}{(2g + 2)!}.
\]

**Corollary 4.2.** If \( \pi : X \to B \) is a smooth family of hyperelliptic curves of genus \( g > 2 \), then \( \text{Mon}(X) \) is a proper subgroup of \( \text{Sp}(2g, \mathbb{Z}) \).

**Proof of Proposition 4.1.** In this proof, \( \text{Mon}(X, \mathbb{Z}) \) will denote the monodromy group acting on the integer \( H^1 \) of a fiber of \( X \), and \( \text{Mon}(X, \mathbb{Z}/2\mathbb{Z}) \) will stand for the monodromy group acting on cohomology with coefficients in \( \mathbb{Z}/2\mathbb{Z} \).

Since the reduction modulo 2 mapping \( \text{Sp}(2g, \mathbb{Z}) \to \text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) \) is surjective, one has

\[
(\text{Sp}(2g, \mathbb{Z}) : \text{Mon}(X, \mathbb{Z})) \geq (\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) : \text{Mon}(X, \mathbb{Z}/2\mathbb{Z})),
\]

so it suffices to show that

\[
(\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) : \text{Mon}(X, \mathbb{Z}/2\mathbb{Z})) \geq \frac{2^g (2^{2g} - 1)(2^{2(g-1)} - 1) \cdots (2^2 - 1)}{(2g + 2)!}.
\]

To that end, let \( X \) be a hyperelliptic curve of genus \( g \geq 2 \) that is a fiber of \( \mathcal{X} \); denote its Weierstrass points by \( P_1, \ldots, P_{2g+2} \). It is well known (see for example [6, Lemma 2.1]) that the 2-torsion subgroup \( (\text{Pic}(X))_2 \subset \text{Pic}(X) \) is generated by classes of divisors \( P_i - P_j \). Since \( \text{Pic}(X)_2 \cong H^1(X, \mathbb{Z}/2\mathbb{Z}) \), the action of \( \pi_1(B_{\text{sm}}) \) on \( H^1(X, \mathbb{Z}/2\mathbb{Z}) \) is completely determined by the permutations of the Weierstrass points \( P_1, \ldots, P_{2g+2} \) it induces. Thus, order of \( \text{Mon}(X, \mathbb{Z}/2\mathbb{Z}) \) is at most \( (2g + 2)! \). Since

\[
(\text{Sp}(2g, \mathbb{Z}/2\mathbb{Z}) : 1) = 2^g (2^{2g} - 1)(2^{2(g-1)} - 1) \cdots (2^2 - 1),
\]
Remark 4.3. The bound in Proposition 4.1 is sharp, which follows from A'Campo's paper [1]. To wit, for any \( g \geq 2 \) let us regard \( \mathbb{A}^{2g+1} \) as the space of polynomials

\[
P(x) = x^{2g+2} + a_{2g}x^{2g} + \cdots + a_1x + a_0,
\]

and denote the space of polynomials \( P \in \mathbb{A}^{2g+1} \) with a multiple root by \( \Delta \subset \mathbb{A}^{2g+1} \). If \( \mathcal{X} \) is a family over \( \mathbb{A}^{2g+1} \setminus \Delta \) in which the fiber over \( P \) is the smooth projective model of the curve with equation \( y^2 = P(x) \) (which is a hyperelliptic curve of genus \( g \)), then part of the corollary on page 319 of [1] can be restated to the effect that index \( (\text{Sp}(2g, \mathbb{Z}) : \text{Mon}(\mathcal{X})) \) is equal to the right-hand side of (4). Actually, not only the order of \( \text{Mon}(\mathcal{X}) \) is known: a description of this group can be found in the appendix to [4], which (the appendix) is devoted to the exposition of results of A.Varchenko.

5. AN APPLICATION TO DEL PEzzo SURFACES

In this section we prove the following fact.

**Proposition 5.1.** If \( X \subset \mathbb{P}^n \) is a del Pezzo surface embedded by (a subsystem of) the anticanonical linear system \(|-K_X|\), then the monodromy group acting on \( H^1(\cdot, \mathbb{Z}) \) of its smooth hyperplane sections is the entire \( \text{SL}(2, \mathbb{Z}) \).

First recall some notation and definitions.

If \( X \) is an algebraic variety and \( \mathcal{R} \) is a coherent sheaf of reduced \( \mathcal{O}_X \)-algebras, we denote its relative spectrum (which is a scheme over \( X \)) by \( \text{Spec} \mathcal{R} \) (under our assumptions \( \text{Spec} \mathcal{R} \) is an algebraic variety and the canonical morphism \( \text{Spec} \mathcal{R} \to X \) is finite).

If \( p \in \mathbb{P}^n \) is a point and \( L \subset \mathbb{P}^n \) is a linear subspace, then \( p, L \) denotes the linear span of \( \{p\} \cup L \).

If \( A_1, \ldots, A_4 \) are points on the affine line with coordinates \( a_1, \ldots, a_4 \), then by their cross-ratio we mean

\[
[A_1, A_2, A_3, A_4] = \frac{a_3 - a_1}{a_3 - a_2} \cdot \frac{a_4 - a_1}{a_4 - a_2}.
\]

If \( X \subset \mathbb{P}^n \) is a smooth projective variety and \( X^* \subset (\mathbb{P}^n)^* \) is its projective dual, one can define the “universal smooth hyperplane section of \( X \),” that is, the family

\[
U_X = \{(x, \alpha) \in X \times ((\mathbb{P}^n)^* \setminus X^*): x \in H_\alpha\},
\]

where \( H_\alpha \subset \mathbb{P}^n \) is the hyperplane corresponding to the point \( \alpha \in (\mathbb{P}^n)^* \).

The morphism \( \pi: (x, \alpha) \mapsto \alpha \) makes \( \mathcal{X} \) a smooth family of \( n \)-dimensional projective varieties over \( (\mathbb{P}^n)^* \setminus X^* \); for any natural \( d \), this family induces a monodromy action of \( \pi_1((\mathbb{P}^n)^* \setminus X^*) \) on \( H^d(Y, \mathbb{Z}) \), where \( Y \) is a smooth hyperplane section of \( X \).

In the above setting, the image of \( \pi_1((\mathbb{P}^n)^* \setminus X^*) \) in the group \( \text{Aut}(H^n(Y, \mathbb{Z})) \) will be called **hyperplane monodromy group** of \( X \).
Lemma 5.2. Suppose that $X \subset \mathbb{P}^n$ is a smooth projective variety and that $p \in \mathbb{P}^n \setminus X$ is a point such that the projection with center $p$ induces an isomorphism $\pi_p: X \to X' \subset \mathbb{P}^{n-1}$. If $H \ni p$ is a hyperplane that is transversal to $X$, then, after identifying $Y = X \cap H$ with $Y' = \pi_p(Y) = X' \cap \pi_p(H)$, the hyperplane monodromy groups acting on $H^n(Y, \mathbb{Z})$ and $H^n(Y', \mathbb{Z})$, are the same.

The proof that is sketched below was suggested to me by Jason Starr.

Sketch of proof. Denote by $H_p \subset (\mathbb{P}^n)^*$ the hyperplane corresponding to the point $p \in \mathbb{P}^n$. It is clear that $H_p$ is naturally isomorphic to $(\mathbb{P}^{n-1})^*$ and that $(X')^* = X^* \cap H_p$. Moreover, the hyperplane $H_p$ is transversal to $X^*$ at any smooth point of $X^*$ (indeed, if $H_p$ is tangent to $X^*$ at a smooth point, then $p \in (X^*)^* = X$, which contradicts the hypothesis).

To prove the lemma it suffices to show that $\pi_1(H_p \setminus (X')^*)$ surjects onto $\pi_1((\mathbb{P}^n)^* \setminus X^*)$. To that end observe that there exists a line $\ell \subset H_p$ that is transversal to the smooth part of $X^* \cap H_p = (X')^*$ (in particular, $\ell$ does not pass through singular points of $X^* \cap H_p$). It follows from the transversality of $H_p$ to the smooth part of $X^*$ that $\ell$ is transversal to the smooth part of $X^*$, too. Thus, $\pi_1(\ell \setminus X^*)$ surjects both onto $\pi_1(H_p \setminus (X')^*)$ and onto $\pi_1((\mathbb{P}^n)^* \setminus X^*)$, whence the desired surjectivity. \hfill $\square$

Lemma 5.2 implies that when studying hyperplane monodromy groups one may always assume that the variety in question is embedded by a complete linear system. Recall that if a del Pezzo surface $X \subset \mathbb{P}^n$ is embedded by the complete linear system $|-K_X|$ then $\deg X = n \leq 9$; besides, if $n > 3$, $p \in X$ is a general point, and $X$ is the blow-up of $X$ at $p$, then the projection $\pi_p: X \to \mathbb{P}^{n-1}$ induces an isomorphism $\pi_p: \overline{X} \to \overline{X'} = \overline{\pi_p(X)} \subset \mathbb{P}^{n-1}$ and $X' \subset \mathbb{P}^{n-1}$ is a del Pezzo surface embedded by $|-K_{X'}|$.

Lemma 5.3. In the above setting, suppose that the hyperplane monodromy group of $X'$ is the entire $\text{SL}(2, \mathbb{Z})$. Then the hyperplane monodromy group of $X$ is the entire $\text{SL}(2, \mathbb{Z})$ as well.

Proof. Informally the proof may be summed up in one phrase: if variation of hyperplanes passing through $p$ and transversal to $X$ is enough to obtain the entire group $\text{SL}(2, \mathbb{Z})$, then a fortiori this is the case for all hyperplanes transversal to $X$. A formal argument follows.

Assume that the $\mathbb{P}^{n-1}$ into which the surface $X$ is projected is a hyperplane in $\mathbb{P}^n, \mathbb{P}^{n-1} \not\ni x$. If a hyperplane $H \subset \mathbb{P}^n$ contains the point $p \in X$, then $H = \overline{p, h}$, where $h = H \cap \mathbb{P}^{n-1}$ is a hyperplane in $\mathbb{P}^{n-1}$. If $H$ is transversal to $X$, then $\pi_p$ induces an isomorphism between the curve $H \cap X$, which is
a smooth hyperplane section of $X$, and the curve $h \cap X'$, which is a smooth hyperplane section of $X'$. Put
\[ V = \{ \alpha \in (\mathbb{P}^{n-1})^* : p, h_\alpha \text{ is transversal to } X \}, \]
where $h_\alpha \subset \mathbb{P}^{n-1}$ is the hyperplane corresponding to the point $\alpha \in (\mathbb{P}^{n-1})^*$, and set
\[ U = \{ (\alpha, x) \in V \times X' : x \in h_\alpha \}. \]
In the diagram
\[
\begin{array}{ccc}
U_{X'}, & \longrightarrow & U \\
\downarrow q' & & \downarrow q \\
(\mathbb{P}^{n-1})^* \setminus (X')^* & \underset{j}{\longrightarrow} & V \underset{r}{\longrightarrow} (\mathbb{P}^n)^* \setminus X^*
\end{array}
\]
where $U_{X'}$ and $U_X$ are universal smooth hyperplane sections of $X'$ and $X$, $j$ is an open embedding, and $r$ maps a hyperplane $h_\alpha \subset \mathbb{P}^{n-1}$ to the hyperplane $p, h_\alpha \subset \mathbb{P}^n$, both squares are Cartesian. Pick a point $\alpha \in V$; the hyperplane $p, h_\alpha \subset \mathbb{P}^n$ is $H_{r(\alpha)}$, where $r(\alpha) \in (\mathbb{P}^n)^*$. If $Y' = q^{-1}(\alpha)$ and $Y = X \cap H_{r(\alpha)} = q^{-1}(r(\alpha))$, then in the commutative diagram
\[
\begin{array}{ccc}
\pi_1((\mathbb{P}^{n-1})^* \setminus (X')^*, \alpha) & \xrightarrow{w} & \pi_1(V, \alpha) \\
\downarrow u & & \downarrow v \\
\text{Aut}(H^1(Y', \mathbb{Z})) & \longrightarrow & \text{Aut}(H^1(Y, \mathbb{Z}))
\end{array}
\]
the mapping $w$ is an epimorphism since $V$ is Zariski open in $(\mathbb{P}^{n-1})^* \setminus (X')^*$, whence $\text{Im} u \subset \text{Im} v$. This proves the lemma. □

Projecting del Pezzo surfaces in $\mathbb{P}^n$, $n > 3$, consecutively from general points on them, one arrives at a cubic in $\mathbb{P}^3$; Lemma 5.3 implies that it suffices to prove Proposition 5.1 for this surface.

The next lemma reduces the problem to the case of “del Pezzo surfaces of degree 2”, for which the anticanonical linear system defines a ramified covering of degree 2 over $\mathbb{P}^2$.

Suppose that $X \subset \mathbb{P}^3$ is a smooth cubic and $p \in X$ is a general point. Let $\tilde{X}$ be the blow-up of $X$ at $p$. Then the projection $\pi_p : \mathbb{P}^3 \longrightarrow \mathbb{P}^2$ induces a finite morphism $\tilde{\pi}_p : \tilde{X} \rightarrow \mathbb{P}^2$ of degree 2; the branch locus of this morphism is a smooth curve $C \subset \mathbb{P}^2$ of degree 4. For $\alpha \in (\mathbb{P}^2)^*$, denote the corresponding line by $\ell_\alpha \subset \mathbb{P}^2$. If $\ell_\alpha$ is transversal to $C$ (i.e., $\alpha \notin C^*$), then $\tilde{\pi}_p^{-1}(\ell_\alpha)$ is smooth, irreducible, and isomorphic to $X \cap p, \ell_\alpha$.

**Lemma 5.4.** Put
\[
X = \{ (\alpha, x) \in ((\mathbb{P}^2)^* \setminus C^*) \times \tilde{X} : \tilde{\pi}_p(x) \in \ell_\alpha \}
\]
and denote the morphism $(\alpha, x) \mapsto \alpha$ by $q : X \rightarrow (\mathbb{P}^2)^* \setminus C^*$. If $R$ is a commutative ring and $\text{Mon}(X, R) = \text{SL}(2, R)$, then the hyperplane monodromy group of $X$ with coefficients in $R$ is also equal to $\text{SL}(2, R)$. 

Proof. The proof is similar to that of Lemma 5.3. It suffices to make the following modifications in the diagram (\(\mathcal{D}\)): put \(n = 3\), replace \(U_{X'}\) by \(X'\), replace \((X')^*\) by \(C^*\), and put \(V = (\mathbb{P}^2)^* \setminus (C^* \cup \{\beta\})\), where \(\ell_\beta = T_{p,X} \cap \mathbb{P}^2\).

\[\square\]

**Lemma 5.5.** The monodromy group acting on \(H^1\) of fibers of the family (\(\mathcal{D}\)) with coefficients \(\mathbb{Z}/2\mathbb{Z}\), is \(SL(2, \mathbb{Z}/2\mathbb{Z})\).

**Proof.** Suppose that \(Y = \overline{\pi_p^{-1}(\ell)}\), where \(\ell \subset \mathbb{P}^2\) is transversal to \(C\), is a fiber of the family (\(\mathcal{D}\)). We are to show that the monodromy group in question performs all the permutations of the non-zero elements of \(H^1(Y, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^2\).

Recall that \(H^1(Y, \mathbb{Z}/2\mathbb{Z}) \cong (\text{Pic}(Y))_2\) (the subgroup of elements of order at most 2). If \(\ell \cap C = \{P_1, P_2, P_3, P_4\}\) and \(Q_j = \overline{\pi_p^{-1}(P_j)}\), then, since \(Y\) is a two-sheeted covering of \(\ell\) with ramification at \(Q_1, \ldots, Q_4\), any point of order 2 in \(\text{Pic}(Y)\) is represented by a divisor of the form \(Q_i - Q_j\). Thus, to prove the lemma it suffices to show that, as \(\ell\) varies in the family of lines transversal to \(C\), the monodromy on the set \(\ell \cap C\) performs all the permutations of the set \(\ell \cap C\). The latter assertion is a particular case of [3, Theorem on p. 906]. \(\square\)

The following lemma, which will be used in the proof of Proposition 5.7, is valid over algebraically closed fields of arbitrary characteristic.

**Lemma 5.6.** Suppose that \(W\) is a smooth irreducible variety of dimension \(n\), \(L\) is a smooth irreducible curve (we do not assume that \(W\) or \(L\) is projective), and \(\varphi: W \to L\) is a proper and surjective morphism with \((n-1)\)-dimensional fibers. Put \(Z = \text{Spec} \varphi_*\mathcal{O}_W\) and let \(v: Z \to L\) be the natural morphism.

If there exists a point \(p \in L\) such that \(\varphi^{-1}(p)\) is irreducible and the morphism \(\varphi\) has maximal rank at a general point of \(\varphi^{-1}(p)\), then the natural morphism \(v: Z \to L\) is an isomorphism.

**Proof.** It is clear that \(Z\) is an irreducible and reduced curve. Since \(\varphi\) is proper and \(\varphi^{-1}(p)\) is connected, the stalk \((\varphi_*\mathcal{O}_W)_p\) is a local ring, so \(v^{-1}(p)\) consists of one point; denote this point by \(z\). I claim that that \(z\) is a smooth point of \(Z\) and the morphism \(v\) is unramified at \(z\). Indeed, let \(\tau \in \mathcal{O}_{L,p}\) be a generator of the maximal ideal. Its image \(v^*\tau \in \mathcal{O}_{Z,z}\) can be represented by a regular function \(f \in \mathcal{O}_W(\varphi^{-1}(U))\), where \(U \subset L\) is a Zariski neighborhood of \(p\). Since the morphism \(\varphi\) has maximal rank at a general point of \(\varphi^{-1}(p)\), the function \(v^*\tau\) vanishes on the irreducible divisor \(\varphi^{-1}(p)\) with multiplicity 1. Since regular functions on \(\varphi^{-1}(U)\) must be constant on the fibers of the proper morphism \(\varphi\), any element of the maximal ideal of the local ring \(\mathcal{O}_{Z,z}\) is representable by a regular function \(g \in \mathcal{O}_W(\varphi^{-1}(V))\), where \(V\) is a Zariski neighborhood of \(p\), such that the zero locus of \(g\) in \(\varphi^{-1}(V)\) coincides with \(u^{-1}(z)\). Hence, \(v^*\tau\) generates the maximal ideal of \(\mathcal{O}_{Z,z}\), which proves our claim.
Since \( v^{-1}(p) = \{z\} \), \( Z \) is smooth at \( z \), and \( v \) is unramified at \( z \), we conclude that the finite morphism \( v \) has degree 1. Since \( L \) is smooth, Zariski main theorem implies that \( v \) is an isomorphism. \( \square \)

**Proposition 5.7.** Suppose that \( \pi : X \to \mathbb{P}^2 \) is a finite morphism of degree 2 branched over a smooth quartic \( C \subset \mathbb{P}^2 \), where \( X \) is smooth. If \( J : (\mathbb{P}^2)^* \setminus C^* \to \mathbb{A}^1 \) is the morphism \( \alpha \mapsto j(\pi^{-1}(\ell_\alpha)) \), where \( \ell_\alpha \) is the line in \( \mathbb{P}^2 \) corresponding to \( \alpha \in (\mathbb{P}^2)^* \), then a general fiber of \( J \) is irreducible.

**Proof.** Let us show that the morphism \( J \) extends to a morphism

\[
J_1 : (\mathbb{P}^2)^* \setminus (C^*)_{\text{sing}} \to \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}.
\]

Indeed, if \( \ell \subset \mathbb{P}^2 \) is a line and \( \ell \cap C = \{P_1, P_2, P_3, P_4\} \), then the curve \( \pi^{-1}(\ell) \) is a curve of genus 1 and

\[
j(\pi^{-1}(\ell)) = 256 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1 - \lambda)^2},
\]

where \( \lambda \) is the cross-ratio \( [P_1, P_2, P_3, P_4] \), in no matter what order (see for example [16 Chapter III, Proposition 1.7b]). If \( \alpha \) is a smooth point of \( C^* \subset (\mathbb{P}^2)^* \), then the line \( \ell_\alpha \) is tangent to \( C \) at exactly one point that is not an inflection point. Thus, as the line \( \ell \) tends to \( \ell_\alpha \), exactly two intersection points from \( \ell \cap C \) merge, so the cross-ratio of these four points tends to 0 (or 1, or \( \infty \), depending on the ordering), and formula (8) shows that \( j(\pi^{-1}(\ell)) \) tends to \( \infty \). This proves the existence of the desired extension.

Our argument shows that \( J_1^{-1}(\infty) = C^* \setminus (C^*)_{\text{sing}} \); if we regard \( J_1 \) as a rational mapping from \( (\mathbb{P}^2)^* \) to \( \mathbb{P}^1 \) and if

\[
\begin{array}{ccc}
W & \xrightarrow{\sigma} & J_2 \\
\downarrow J_1 & & \downarrow \pi \\
(\mathbb{P}^2)^* & \setminus J_1 & \to \mathbb{P}^1
\end{array}
\]

is a minimal resolution of indeterminacy for \( J_1 \), then \( J_2^{-1}(\infty) \) equals the strict transform of \( C^* \) with respect to \( \sigma \).

Now I claim that, at a general point of \( J_2^{-1}(\infty) \), derivative of \( J_2 \) has rank 1. It suffices to prove this assertion for \( J_1 \) and a general smooth point of \( C^* \). To that end it suffices to construct an analytic mapping \( \gamma : D \to (\mathbb{P}^2)^* \), where \( D \) is a disk in the complex plane with center at 0, such that \( \gamma(D \setminus \{0\}) \subset (\mathbb{P}^2)^* \setminus C^* \), \( \gamma(0) \) is a smooth point of \( C^* \), and \( |j(\pi^{-1}(\ell_\gamma(t)))| \sim \text{const} / |t| \).

Suppose that a point \( c \in C \) is not an inflection point nor a tangency point of a bitangent; if \( \ell_\alpha \subset \mathbb{P}^2 \) is the tangent line to \( C \) at \( c \), then \( \alpha \) is a smooth point of \( C^* \). Now choose affine \((x, y)\)-coordinates in \( \mathbb{P}^2 \) so that \( c = (0, 0) \), the tangent \( \ell_\alpha \) has equation \( y = 0 \), and \( \ell_\alpha \cap C = \{c, (C, 0), (D, 0)\} \), where \( C, D \neq 0 \) (so the remaining two points of \( \ell_\alpha \cap C \) are in the finite part of \( \mathbb{P}^2 \) with respect to the chosen coordinate system). If \( \ell_\gamma(t) \) is the line with affine equation \( y = t \), then, for all small enough \( t \), one has \( \ell_\gamma(t) \cap C = \{A(t), B(t), C(t), D(t)\} \), where the \( x \)-coordinates of \( A(t) \) and \( B(t) \) are...
\[ \sqrt{t} + o(\sqrt{|t|}) \] (for both values of \( \sqrt{t} \)), while the \( x \) coordinates of \( C(t) \) and \( D(t) \) tend to finite and non-zero numbers \( C \) and \( D \). Hence,

\[ |[C(t), A(t), B(t), D(t)]| \sim \frac{\text{const}}{\sqrt{|t|}} \quad \text{as } t \to 0; \]

formula (8) implies that \( |j(\pi^{-1}(\ell_t))| \sim \text{const}/|t| \), as desired.

Let

\[ W \xrightarrow{J_2} \mathbb{P}^1 \]

\[ \downarrow \quad u \quad \downarrow \]

\[ Z \]

be the Stein factorization in which \( W \) is a blow-up of \((\mathbb{P}^2)^* \) (see (9)), \( Z = \text{Spec}(J_2), \mathcal{O}_W \), and \( v \) is a finite morphism. Applying Lemma 5.6 with \( L = \mathbb{P}^1 \), \( \varphi = J_2 \), and \( p = \infty \), we conclude that \( v \) is an isomorphism. Thus, fibers of \( J_2 \) coincide with fibers of \( u \); since the latter are connected, fibers of \( J_2 \) are connected as well. Bertini theorem implies that a general fiber of \( J_2 \) is smooth; since it is connected, it must be irreducible. This implies that a general fiber of \( J \) is irreducible. \( \square \)

**Proof of Proposition 5.1.** In view of Proposition 5.2 and Lemmas 5.3 and 5.4, it suffices to prove that \( \text{Mon}(\mathcal{X}) = \text{SL}(2, \mathbb{Z}) \), where \( \mathcal{X} \) is the family defined by (7).

Applying Proposition 5.7 to the surface \( \bar{X} \) (blow-up of a cubic at a general point \( p \)) and the mapping \( \bar{\pi}_p : \bar{X} \to \mathbb{P}^2 \) (induced by the projection with center \( p \)), we see that the family \( \mathcal{X} \) defined by formula (7) satisfies the hypothesis of Proposition 2.8. Hence, \( \text{Mon}(\mathcal{X}) \) is either the entire \( \text{SL}(2, \mathbb{Z}) \) or its subgroup of index 2. In the first case we are done, and the second case is impossible: Lemma 5.5 tells us that \( \text{Mon}(\mathcal{X}) \) induces, on the cohomology with coefficients in \( \mathbb{Z}/2\mathbb{Z} \), the entire group \( \text{GL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3 \), while the only subgroup of index 2 in \( \text{SL}(2, \mathbb{Z}) \) induces only even permutations of non-zero elements of \( (\mathbb{Z}/2\mathbb{Z})^2 \) (see Remark 2.9). This completes the proof. \( \square \)

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