Epi-completely regular topological spaces

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Abstract. The purpose of this work is to introduce and study a new topological property called epi-complete-regularity. A space \((X, T)\) is called an epi-completely-regular space if there exists a topology \(T'\) on \(X\) which is coarser than \(T\) such that \((X, T')\) is Tychonoff. This new property is investigated and some examples are presented in this work to illustrate its relationships with other kinds of normality and complete-regularity.

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1. Introduction

The notion of epi-normality was introduced by Arhangel’skii during his visiting to Department of Mathematics in King Abdulaziz University, Saudi Arabia on 2012. The notion of epi-normality has been studied by Kalantan and Alzahrani in 2016 [15]. Then, Alzahrani studied the notion of epi-regularity in 2018 [5]. Kalantan and Alshammari studied the notion of epi-mild normality in 2018 [18]. At the beginning of 2020, Alshammari studied the notion of epi-almost normality [3]. Thabit studied the notion of epi-quasi normality [31]. At the end of 2021, Thabit and others studied the notion of epi-partial normality [32]. The space \(X\) means a topological space in whole paper. We need to recall that: a subset \(A\) of a space \(X\) is said to be a closed domain subset if it is the closure of its own interior [20]. The complement of a closed domain subset is called open domain. A subset \(A\) of a space \(X\) is called \(\pi\)-closed if it is a finite intersection of closed domain subsets [33]. The complement of a \(\pi\)-closed subset is called \(\pi\)-open. Two subsets \(A\) and \(B\) of a space \(X\) are said to be separated if there exist two disjoint open subsets \(U\) and \(V\) of \(X\) such that \(A \subseteq U\) and \(B \subseteq V\) [11, 12, 23]. If \(T\) and \(T'\) are two topologies on \(X\) such that \(T' \subseteq T\), then \(T'\) is called a topology coarser than \(T\), and \(T\) is called finer [12]. A \(T_1\)-space is a \(T_1\) normal space, a \(T_3\)-space is a \(T_1\) regular space and a Tychonoff space is a

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$T_1$ completely regular space. A space $X$ is said to be $\pi$-normal [14], if any pair of disjoint closed subsets $A$ and $B$ of $X$, one of which is $\pi$-closed, can be separated. A space $X$ is said to be almost-normal [14, 28], if any pair of disjoint closed subsets $A$ and $B$ of $X$, one of which is closed domain, can be separated. A space $X$ is said to be mildly normal [29], if any pair of disjoint closed domain subsets $A$ and $B$ of $X$ can be separated. A space $X$ is said to be partially normal [4], if any pair of disjoint closed subsets $A$ and $B$ of $X$, one of which is closed domain and the other is $\pi$-closed, can be separated. A space $(X, T)$ is said to be epi-normal [15] (resp. epi-mildly normal [18], epi-almost normal [3], epi-regular [5], epi-quasi normal [31], epi-partially normal [32]), if there exists a topology $T'$ on $X$ coarser than $T$ such that $(X, T')$ is a $T_3$ (resp. Hausdorff mildly-normal, Hausdorff almost-normal, $T_3$, Hausdorff-quasi-normal, Hausdorff partially-normal) space. A space $X$ is said to be Hausdorff or a $T_2$-space, if for each distinct two points $x, y \in X$ there exist two open subsets $U$ and $V$ of $X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$ [12]. A space $X$ is said to be completely Hausdorff or Urysohn [12, 30], if for each distinct two points $x, y \in X$ there exist two open subsets $U$ and $V$ of $X$ such that $x \in U$, $y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. A space $X$ is said to be almost completely regular if for each $x \in X$ and each closed domain subset $F$ of $X$ such that $x \notin F$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$ [28]. A space $X$ is said to be almost-regular if for each $x \in X$ and each closed domain subset $F$ of $X$ such that $x \notin F$, there exist two disjoint open subsets $U$ and $V$ such that $x \in U$ and $F \subseteq V$ [27]. A space $X$ is said to be sub-metrizable [13], if there exists a metric $d$ on $X$ such that the topology $T_d$ on $X$ generated by $d$ is coarser than $T$. The topology on $X$ generated by the family of all open domain subsets of $X$, denoted by $T_\alpha$, is coarser than $T$, and $(X, T_\alpha)$ is called the semi-regularization of $X$. A space $(X, T)$ is called semi-regular if $T = T_\alpha$ [22]. A space $X$ is called H-closed [12], if $X$ Hausdorff almost-compact [19, 24]. A space $X$ is called $C$-normal [8] (resp. $C$-regular [6], $C$-Tychonoff [7]) if there exist a normal (resp. regular, Tychonoff) space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A space $X$ is called $L$-normal [16] (resp. $CC$-normal [17]) if there exist a normal space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf (resp. countably compact) subspace $A \subseteq X$. A space $X$ is called $L$-regular [6] (resp. $L$-Tychonoff [7]) if there exist a regular (resp. Tychonoff) space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each Lindelöf subspace $A \subseteq X$. The basic definitions and any undefined terms in this article can be found in [31] and [32].

In this paper, I introduce and study a new topological property called epi-complete regularity. I show that this new property is different from epi-normality, epi-regularity, epi-mild normality, epi-quasi normality, epi-partial normality and epi-almost normality. Some properties, counterexample and relationships of this property are investigated. This paper contains three main sections starting from section 2. In section 2, the definition of epi-complete regularity is introduced and some examples are presented. Some properties of epi-complete regularity are studied and given in section 3.
2. Preliminaries

First, I present the main definition of this study:

**Definition 1.** A space \((X, \mathcal{T})\) is called an *epi-completely-regular* space if there exists a topology \(\mathcal{T}'\) on \(X\) which is coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is Tychonoff.

From Definition 1, note that: every epi-completely regular space is Hausdorff and any Tychonoff space is epi-completely-regular, but the converses are not true in general, for example: the irregular lattice topology, Example 6 is a Hausdorff space which is not epi-completely regular. The Smirnov’s deleted sequence topology, Example 10, and the half disc topology, Example 5, are epi-completely regular spaces which are not Tychonoff. Now, I present the next results:

**Theorem 1.** Every epi-completely-regular space is Urysohn.

*Proof.* Let \((X, \mathcal{T})\) be an epi-completely-regular space. Then, there exists a topology \(\mathcal{T}'\) on \(X\) that is coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is \(T_1\)-completely-regular. Thus, \((X, \mathcal{T}')\) is Tychonoff. Hence, \((X, \mathcal{T}')\) is Uryshon (completely Hausdorff). Since \(\mathcal{T}' \subseteq \mathcal{T}\), we conclude: \((X, \mathcal{T})\) is Urysohn.

Observe that: any Urysohn space is not necessary to be epi-completely regular. For example, the Tychonoff corkscrew topology, Example 9, and the irregular lattice topology, Example 6, are Urysohn spaces which are not epi-completely-regular. Thus, the converse of Theorem 1 is not true in general.

**Theorem 2.** Every epi-completely-regular space is epi-regular.

*Proof.* Let \((X, \mathcal{T})\) be an epi-completely-regular space. Then, there exists a topology \(\mathcal{T}'\) on \(X\) coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is \(T_1\)-completely-regular. Since every completely-regular space is regular [12], we get: \((X, \mathcal{T}')\) is a \(T_1\)-regular space. Hence, \((X, \mathcal{T}')\) is \(T_3\)-space. Therefore, \((X, \mathcal{T})\) is epi-regular.

Note that: the converse of Theorem 2 is not necessarily true in general. For example, the Tychonoff corkscrew topology, Example 9, is an epi-regular space which is not epi-completely-regular. Also, complete regularity and epi-complete regularity are different from each other, for example, the half disc topology, Example 5, is an epi-completely-regular space, which is not completely-regular and any uncountable indiscrete space is a completely-regular space which is not epi-completely-regular.

**Theorem 3.** Every epi-almost-normal space is epi-completely-regular.

*Proof.* Let \((X, \mathcal{T})\) be an epi-almost-normal space. Then, there exists a topology \(\mathcal{T}'\) on \(X\) which is coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is a Hausdorff almost-normal space. Since every almost-normal \(T_1\)-space is almost-regular [27], we have: \((X, \mathcal{T}')\) is Hausdorff almost-normal almost-regular. Since every almost-normal almost-regular space is almost-completely regular [28], we get: \((X, \mathcal{T}')\) is Hausdorff almost-completely regular. Let the
semi regularization of \((X, T')\) be \((X, T'_s)\). Then, \((X, T'_s)\) is a Hausdorff completely-regular space because the semi regularization of a Hausdorff almost completely regular space is Hausdorff completely regular [22]. Since \(T'_s \subseteq T' \subseteq T\), we conclude: \(T'_s\) is a topology on \(X\) that is coarser than \(T\) such that \((X, T'_s)\) is Hausdorff completely-regular and hence Tychonoff. Therefore, \((X, T)\) is epi-completely-regular.

Since every epi-completely-regular space is epi-regular (Theorem 2), every sub-metrizable space is epi-normal and every epi-normal space is epi-almost-normal [3, 15], we obtain:

**Corollary 1.**

1. Every sub-metrizable space is epi-completely-regular.
2. Every epi-normal space is epi-completely-regular.

Thus, we conclude the following implications:

\[
\text{epi-normal} \implies \text{epi-almost-normal} \implies \text{epi-completely-regular} \implies \text{epi-regular}
\]

The next example is an epi-completely regular space which is not epi-normal.

**Example 1.** Consider the Example 10 in [26], let \(G = D^{\omega_1}\), where \(D = \{0, 1\}\) with the discrete topology. Let \(H\) be a subspace of \(G\) consisting of all points of \(G\) with at most countably many non zero coordinates. Put \(X = G \times H\). Raushan Buzyakova proved that \(X\) cannot be mapped onto a normal space \(Y\) by a bijective continuous function [9]. It can be observed that: \(H\) is a \(T_2\)-Fréchet space and hence it is a \(k\)-space. \(G\) is also a \(T_2\)-compact space. Hence, \(X = H \times G\) is a \(k\)-space [26]. Since \(X\) is Tychonoff, we get \(X\) is epi-completely regular. The space \(X\) is not \(C\)-normal [26]. Since every \(C\)-Tychonoff Fréchet Lindelöf space is \(C\)-normal, we conclude: \(X\) is not Lindelöf. Since \(X\) is not \(C\)-normal, we obtain \(X\) is neither \(CC\)-normal, sub-metrizable nor epi-normal. The space \(X\) is not a locally compact space as well. Thus, the space \(X\) is an epi-completely regular space which is neither \(C\)-normal, \(CC\)-normal, epi-normal, sub-metrizable nor locally compact.

Observe that: any \(C\)-Tychonoff (resp. \(C\)-normal) space is not necessary to be epi-completely regular. Here is a counterexample:

**Example 2.** The countable complement topology \((\mathbb{R}, CC)\) is both \(C\)-Tychonoff and \(C\)- regular space [6, 7], which is neither epi-completely-regular, epi-regular nor epi-mildly normal because it is not Hausdorff.

The following example is a normal space, which is not epi-completely regular.

**Example 3.** The left ray topology \((\mathbb{R}, \mathcal{L})\), the right ray topology \((\mathbb{R}, \mathcal{R})\) [30] are normal spaces, which are not epi-completely regular because they are not Hausdorff.

Note that: complete regularity (resp. \(L\)-regularity) does not imply to epi-complete-regularity in general as shown by the next example.
**Example 4.** The double pointed reals topology [30, Example 62], is both a regular and completely regular space [30], which is not epi-completely regular because it is not Hausdorff.

The next example is an epi-completely-regular space which is neither Tychonoff nor completely-regular.

**Example 5.** The half disc topology [30, Example 78] is not Tychonoff. The semi regularization of $X$ is the closed upper half plane with the Euclidean topology $U$ on $\mathbb{R}$ that is a topology coarser than $T$ and $(X, U)$ is a $T_4$-space. Thus, $X$ is epi-normal. Hence, $X$ is epi-completely-regular. Since $(X, T)$ is an almost-completely regular space if and only if $(X, T_s)$ is completely-regular [22], we get: the half disc topology is almost-completely regular. Therefore, the half disc topology is an epi-completely-regular space, which is neither completely-regular, Tychonoff nor epi-normal.

A Urysohn epi-mildly normal Lindelöf space is not necessary to be epi-completely-regular, for example:

**Example 6.** The irregular lattice topology [30, Example 79], is a Urysohn Lindelöf space, which is neither normal, completely regular nor semi-normal [30]. It is also a mildly-normal space, which is not partially-normal [4]. Hence, it is neither quasi-normal, almost-normal nor semi-normal. Since every almost-regular Lindelöf space is quasi-normal [21], and $X$ is a Lindelöf non quasi-normal space, it is not almost-regular. Since $(X, T)$ is a Hausdorff mildly-normal space, it is epi-mildly normal. Hence, the irregular lattice topology is a Urysohn epi-mildly-normal space, which is neither epi-almost-normal, epi-regular nor epi-completely-regular.

An almost-completely regular space is not necessarily epi-completely-regular. For example:

**Example 7.** The telophase topology [30, Example 73], is a $T_1$-compact, paracompact space, which is neither Hausdorff, normal nor semi-regular [30]. Clearly that: $X$ is an almost-regular space. Since it is an almost-regular paracompact space, it is almost-normal. Since every almost-regular $T_1$ space is almost-completely regular, we have: the telophase topology is $T_1$-almost-completely regular. Since the telophase topology is not Hausdorff, it is neither epi-completely-regular, epi-mildly normal nor epi-regular. Therefore, the telophase topology is an almost-completely regular space, which is neither epi-completely-regular, epi-mildly-normal nor epi-regular.

An epi-completely-regular space need not be almost-normal nor quasi-normal. Here is an example:

**Example 8.** The Thomas’ plank topology [30, Example 93], Let $X = \bigcup_{i=0}^{\infty} L_i$, where $L_0 = (0, 1) \times \{0\}$ and $L_i = [0, 1) \times \{\frac{i}{2}\}$ for each $i \geq 1$. For each $i \geq 1$, each point $(x, \frac{i}{2}) \in L_i$, $x \neq 0$, we have $\{(x, \frac{i}{2})\}$ is an open subset of $X$. For each $i \geq 1$, the basic open subset of the
points \((0, \frac{1}{n}) \in L_i\) is a subset \(W_i\) of \(L_i\) such that \(L_i - W_i\) is finite. The basic open subset of any point \((x, 0) \in L_0\) is of the form \(U_i(x, 0) = \{(x, 0)\} \cup \{(x, \frac{1}{n}) : n > i\}\). It can be observed that: each basic open subsets of \(X\) is clopen (closed-and-open). Hence, \((X, T)\) is a zero-dimensional, Hausdorff, regular, completely-regular, semi-regular, Urysohn, locally-compact and Tychonoff space, and it is neither normal nor paracompact \([30]\). Hence, the Thomas’ plank topology is an almost-regular and almost-completely regular space. Since it is Hausdorff, we have: the Thomas’ plank topology is epi-completely-regular and epi-regular space. Since \(X\) is Hausdorff locally-compact, we obtain: \(X\) is a \(k\)-space. Thus, \(X\) is \(C\)-normal. It can be observed that: each \(L_i, i \geq 1\) is open because \(L_0\) is closed \([30]\). Also, \(A = \{(0, \frac{1}{n}) : n \geq 1\}\) is a closed subset of \(X\) \([30]\). Since \(A \cap L_0 = \emptyset\), we get: \(A\) and \(L_0\) are disjoint closed subsets of \(X\), which cannot be separated \([30]\). Let \(U = \bigcup_{n \in \mathbb{N}} L_{2n}\) and \(V = \bigcup_{n \in \mathbb{N}} L_{2n+1}\). Then, \(U\) and \(V\) are disjoint open subsets of \(X\). Thus, \(\overline{U} = U \cup L_0\) and \(\overline{V} = V \cup L_0\). Hence, \(\overline{U}\) and \(\overline{V}\) are closed-domains in \(X\) such that \(\overline{U} \cap \overline{V} = L_0\). Therefore, \(L_0\) is a \(\pi\)-closed subset of \(X\). Since \(A\) and \(L_0\) cannot be separated, we obtain: \(X\) is not \(\pi\)-normal.

**Claim 1:** Any singleton \(\{(x, 0)\}\) is \(\pi\)-closed and any singleton \(\{(0, \frac{1}{n})\}\), \(i \geq 1\) is also \(\pi\)-closed in \(X\).

**Proof of the Claim 1:** Let \(U_x = \{(x, \frac{1}{n}) : n \in \mathbb{N}\}\) and \(V_x = \{(x, \frac{1}{2n+1}) : n \in \mathbb{N}\}\). Then, \(U_x\) and \(V_x\) are disjoint open subsets of \(X\) such that \(\overline{U_x} = U_x \cup \{(x, 0)\}\) and \(\overline{V_x} = V_x \cup \{(x, 0)\}\). Therefore, \(\overline{U_x}\) and \(\overline{V_x}\) are closed domain subsets of \(X\) and \(\overline{U_x} \cap \overline{V_x} = \{(x, 0)\}\). Thus, \(\{(x, 0)\}\) is \(\pi\)-closed in \(X\) for each \(x \in (0, 1)\). Now, fix a sequence \(\langle (x_k, \frac{1}{n}) : k \in \mathbb{N}\rangle\) of distinct points of \(L_i\). Consider the two subsequences \(U_i = \{(x_{2k}, \frac{1}{n}) : k \in \mathbb{N}\}\) and \(V_i = \{(x_{2k+1}, \frac{1}{n}) : k \in \mathbb{N}\}\). Then, \(U_i\) and \(V_i\) are disjoint open subsets of \(X\), \(U_i, V_i \subset L_i\) for each \(i \geq 1\), \(\overline{U_i} = U_i \cup \{(0, \frac{1}{n})\}\) and \(\overline{V_i} = V_i \cup \{0, \frac{1}{n}\}\). Since \(\overline{U_i}\) and \(\overline{V_i}\) are closed-domains of \(X\), we get: \(\{(0, \frac{1}{n})\}\) is \(\pi\)-closed for each \(i \geq 1\). Now, let \(G = \bigcup_{i \geq 1} U_i\) and \(H = \bigcup_{i \geq 1} V_i\). Then, \(G\) and \(H\) are disjoint open subsets of \(X\) such that \(\overline{G} = G \cup A \cup \{(x_{2k}, 0) : k \in \mathbb{N}\}\) and \(\overline{H} = H \cup A \cup \{(x_{2k+1}, 0) : k \in \mathbb{N}\}\), where \(A = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}\). Then, \(\overline{G}\) and \(\overline{H}\) are closed-domains in \(X\) such that \(\overline{G} \cap \overline{H} = A\). Hence, \(A\) is \(\pi\)-closed. Since \(A \cap L_0 = \emptyset\) and they cannot be separated \([30]\), we obtain that: \(X\) is not quasi-normal. It is easy to show that \(X\) cannot be semi-normal.

**Claim 2:** The Thomas’ plank topology is not almost-normal.

**Proof of the Claim 2:** It can be observed that, \(A_1 = \{(0, \frac{1}{2n}) : n \in \mathbb{N}\}\) is a closed subset of \(X\) and \(U = \bigcup_{n \in \mathbb{N}} L_{2n}\) is an open-domain subset of \(X\) such that \(A_1 \subset U\). Then, for each open subset \(W\) of \(X\) such that \(A_1 \subset W\), we have: \(A_1 \subset W \subset \overline{W} \not\subset U\) because there are some points \((x, 0) \in \overline{W}\), and \((x, 0) \not\in U\) for each \((x, 0) \in L_0\). Hence, \(X\) is not almost-normal. Note that: \(A = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}\) and \(L_0\) are disjoint \(\pi\)-closed subsets that cannot be separated. If \(U = \bigcup_{n \in \mathbb{N}} L_n\) is \(\pi\)-open subset of \(X\) such that \(A \subset U\), then, for each open set \(W\) of \(X\), we have: \(A \subset W \subset \overline{W} \not\subset U\) and \(A \subset W \subset \text{int}(\overline{W}) \not\subset U\). Thus, \(X\) is neither quasi-normal nor semi-normal. Therefore, the Thomas’ plank topology is an epi-completely-regular space, which is neither almost-normal, semi-normal nor quasi-normal.
Note that: an epi-regularity does not imply to epi-complete-regularity as shown by the next example:

**Example 9. The Tychonoff corkscrew topology:** [30, Example 90], Let $X = S \cup \{a^+, a^-\}$, where $(S, T)$ is homeomorphic to the deleted Tychonoff plank topology [30]. The basic open subset $U$ of $a^+$ contains all points of $X$ which lies above a certain level $k$. That means: $U = \{x \in X : L(x) > k+1\}$. The basic open subset $V$ of $a^-$ contains all points of $X$ which lies below a certain level $k$. That means: $V = \{x \in X : L(x) < k+1\}$. The space $(X, T)$ is a Hausdorff, regular and semi-regular space, which is neither Tychonoff, Urysohn, locally-compact, Lindelöf, first-countable, normal nor completely-regular [30]. Since $(X, T)$ is a Hausdorff regular space, it is epi-regular. Since every regular almost-normal space is completely-regular [28], and $X$ is regular non completely-regular, we obtain: $(X, T)$ is not almost-normal.

**Claim 1:** Any Hausdorff topology $T'$ on $X$, which is coarser than $T$, cannot be completely-regular.

**Proof of the Claim 1:** Let $T'$ be any Hausdorff topology on $X$ which is coarser than $T$. I show $(X, T')$ is not a completely-regular space. Let $A$ be any closed subset of $(X, T')$ and $a^+ \notin A$. Then, $A$ is a closed subset of $(X, T)$ and $a^+ \notin A$. Thus, $X \setminus A$ is an open subset of $(X, T)$ containing $a^+$. But $a^+$ cannot be separated by a continuous function from a closed subset $A$ of $X$ consisting the complement of the basis neighborhood of $a^+$ [30]. Thus, $(X, T')$ is not a completely-regular space. Therefore, any Hausdorff topology $T'$ on $X$, which is coarser than $T$ cannot be completely-regular. Hence, $(X, T)$ is not epi-complete-regular. Hence, $X$ is not epi-almost-normal. Since every $T_1$-semi-regular almost-regular regular space is epi-complete-regular (Corollary 7), and $X$ is $T_1$-semi-regular non epi-complete-regular, we obtain that: $X$ is not almost-regularly regular. Therefore, the Tychonoff corkscrew topology is an epi-regular space, which is neither epi-complete-regular, almost-regularly regular nor epi-almost-normal.

Note that: the Mrówka space $\Psi(A)$ [15, Example 2.10], is a Tychonoff, first-countable and locally compact space, which is neither normal, countably-compact nor epi-normal. Hence, it is an epi-complete-regular space, which is not epi-normal. The space presented in [15, Example 3.1], is a sub-metrizable, epi-normal, Tychonoff and $C$-normal space, which is not mildly-normal. The space presented in [6, Example 2.8], is an epi-complete-regular space, which is neither $C$-normal nor epi-normal. Now, since every Hausdorff locally compact space is Tychonoff [12], we get:

**Corollary 2.** Every Hausdorff locally-compact space is epi-complete-regular.

The converse of Corollary 2 cannot be true in general. Here is a counterexample:

**Example 10. The Smirnov’s deleted sequence topology** [30, Example 64], is a Urysohn space, which is neither semi-regular, completely-regular, locally-compact nor almost-normal. Since any closed domain subset of $X$ is just the closed domain in the Euclidean topology and $U \subseteq T$ [30], we obtain: $X$ is both almost-regular and almost-completely regular. The Smirnov’s deleted sequence topology is not almost-normal because the closed domain subset $B = [-1, 0]$ is disjoint from the closed subset $A = \{\frac{1}{n} : n \in \mathbb{N}\}$, and they cannot be
separated. Since \( U \subseteq T \), \( U \) is the Euclidian topology on \( \mathbb{R} \), which is coarser than \( T \), and \( (\mathbb{R}, U) \) is a \( T_4 \)-space, we obtain: \( X \) is epi-normal (in fact it is sub-metrizable [5]). Since the Smirnov’s deleted sequence topology is a Lindelöf non regular space, it is not \( L \)-regular [6]. Therefore, the Smirnov’s deleted sequence is an epi-completely-regular space, which is neither completely-regular, almost-normal, \( L \)-regular nor locally-compact. The Niemytzki plane topology, the sorgenfrey line square and the Michael line are Tychonoff and hence epi-completely-regular spaces [30], which are not locally-compact.

**Example 11.** The deleted Tychonoff plank [30, Example 87], is a Tychonoff locally-compact space. Hence, it is an epi-completely-regular space. The deleted Tychonoff plank is neither almost-normal nor sub-metrizable [6, 8]. Therefore, the deleted Tychonoff plank topology is an epi-completely-regular space, which is not sub-metrizable.

**Example 12.** The odd-even topology [30, Example 6], is a completely regular and normal space, which is not epi-completely regular being not Hausdorff.

Every Hausdorff semi-regular almost-compact (resp. \( H \)-closed) space is not necessary to be epi-completely regular. Here is a counterexample:

**Example 13.** The minimal Hausdorff topology [30, Example 100], is a Hausdorff, semi-regular, second-countable and almost-compact space, which is neither Urysohn, regular, normal nor compact [30]. Since \( X \) is a semi-regular non regular space, we have: \( X \) is not almost-regular. Since \( X \) is a \( T_1 \) non almost-regular space, it is not almost-normal. Hence, \( X \) is a quasi-normal space, which is not semi-normal [31]. Since \( X \) is not Urysohn, it is neither epi-almost-normal, epi-regular, epi-completely-regular nor epi-normal. Therefore, the minimal Hausdorff topology is a semi-regular, Hausdorff and epi-quasi-normal almost-compact \( H \)-closed space [31], which is neither almost-regular, epi-regular, epi-completely-regular nor Urysohn.

Observe that: a normal compact space need not be epi-completely regular. For example: the excluded point topology [30, Example 15], and the either-or-topology [30, Example 17], are normal compact spaces, which are neither epi-completely-regular, epi-regular nor epi-normal.

### 3. Some properties of epi-complete regularity

In this section, I present the following results:

**Theorem 4.** Epi-complete regularity is a topological property.

_Proof._ Let \( (X, T) \cong (Y, S) \) and \( (X, T) \) be an epi-completely-regular space. There is a homeomorphism \( f : X \rightarrow Y \) and a topology \( T' \) on \( X \) that is coarser than \( T \) such that \((X, T') \) is Tychonoff. Define \( S' \) on \( Y \) by \( S' = \{ f(U) : U \in T' \} \). Then, \( S' \) is a topology on \( Y \), which is coarser than \( S \), and \((Y, S') \) is Tychonoff. Thus, \((Y, S) \) is epi-completely-regular.

**Theorem 5.** Epi-complete regularity is an additive property.
Proof. Let $X_s$ be an epi-completely-regular space for each $s \in S$. Then, there exists a topology $T'_s$ on $X_s$, which is coarser than $T_s$, such that $(X_s, T'_s)$ is a $T_1$-completely-regular space. Since both $T_1$ and complete-regularity are additive properties, we obtain: $(X, \oplus T'_s) \neq (X, T_s)$ is $T_1$-completely regular (Tychonoff). Since $\oplus T'_s$ is a topology coarser than $\oplus T_s$, we get: $(X, \oplus T_s)$ is epi-completely-regular.

**Theorem 6.** Epi-complete regularity is a hereditary property.

Proof. Let $(X, T)$ be an epi-completely-regular space, and $(M, T_M)$ be a subspace of $X$. Then, there exists a topology $T'$ on $X$ that is coarser than $T$ such that $(X, T')$ is $T_1$-completely-regular. To show $(M, T_M)$ is epi-completely-regular, define $T_M'$ on $M$ by: $T_M' = \{ U \cap M : U \in T' \}$. Then, $T_M' \subseteq T_M$. Hence, $T_M'$ is a topology on $M$ which is coarser than $T_M$. Since $(X, T')$ is a $T_1$-completely-regular space and $(M, T_M')$ is a subspace of $X$, we obtain: $(M, T_M')$ is a $T_1$-completely-regular subspace. Therefore, $(M, T_M)$ is epi-completely-regular.

**Theorem 7.** A product space $X = \prod_{\alpha \in \Lambda} X_\alpha$, $X_\alpha \neq \emptyset$ for each $\alpha \in \Lambda$, is an epi-completely-regular space if and only if each factor $X_\alpha$ is epi-completely-regular for each $\alpha \in \Lambda$.

Proof. Let $(\prod_{\alpha \in \Lambda} X_\alpha, T)$ be an epi-completely-regular space, $X_\alpha \neq \emptyset$ for each $\alpha \in \Lambda$. There exists a topology $T'$ which is coarser than $T$ such that $(\prod_{\alpha \in \Lambda} X_\alpha, T')$ is $T_1$-completely-regular. Thus, we have each factor $(X_\alpha, T'_\alpha)$ is a $T_1$-completely-regular space [12], where $T'_\alpha$ is a topology coarser than $T_\alpha$ for each $\alpha \in \Lambda$. Thus, $(X_\alpha, T'_\alpha)$ is an epi-complete regular space for each $\alpha \in \Lambda$. Conversely, suppose that $(X_\alpha, T'_\alpha)$ is an epi-complete regular space for each $\alpha \in \Lambda$. Then, for each $\alpha \in \Lambda$, there exists a topology $T'_\alpha$ that is coarser than $T_\alpha$ such that $(X_\alpha, T'_\alpha)$ is a $T_1$-completely-regular space. Thus, the product space $(\prod_{\alpha \in \Lambda} X_\alpha, T')$ is $T_1$-completely-regular, where $T'$ is coarser than $T$. Therefore, $(\prod_{\alpha \in \Lambda} X_\alpha, T)$ is epi-complete-regular.

**Corollary 3.** Epi-complete regularity is a multiplicative property.

**Theorem 8.** Every epi-complete-regular nearly-compact (resp. nearly-paracompact) space is epi-normal.

Proof. Let $(X, T)$ be an epi-complete-regular nearly-compact (resp. nearly-paracompact) space. Then, there exists a topology $T'$ on $X$ which is coarser than $T$ such that $(X, T')$ is a Tychonoff compact (resp. paracompact) space. Thus, $(X, T')$ is a $T_1$-normal space. Hence, $(X, T')$ is a $T_4$-space. Therefore, $(X, T)$ is epi-normal.

Now, we recall the definition of the Alexandroff duplicate space. For any space $X$, let $X' = X \times \{1\}$. Clearly that $X \cap X' = \emptyset$. Let $A(X) = X \cup X'$. For an element $x \in X$, the element $(x, 1) \in X'$ and for a subset $B \subseteq X$, let $B \times \{1\} = \{(x, 1) : x \in B\} \subseteq X'$. For each $(x, 1) \in X'$, let $B((x, 1)) = \{(x, 1)\}$. For each $x \in X$, let $B(x) = \{U \cup (U \times \{1\}) \setminus \{(x, 1)\}$.
$U$ is open in $X$ with $x \in U$}. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{B(x) : x \in X\} \cup \{B((x, 1)) : (x, 1) \in X'\}$ as its neighborhood system. The space $A(X)$ with this topology is called the Alexandroff duplicate of $X$ [2].

**Theorem 9.** The Alexandroff duplicate $A(X)$ of an epi-completely-regular space $X$ is epi-completely-regular.

**Proof.** Let $(X, \mathcal{T})$ be an epi-completely-regular space. Then, there exists a topology $\mathcal{T}'$ on $X$ that is coarser than $\mathcal{T}$ such that $(X, \mathcal{T}')$ is $T_1$-completely-regular. Since $T_1$ and complete-regularity are preserved by the Alexandroff duplicate space [2], we obtain: $A(X, \mathcal{T}')$ is also a $T_1$-completely-regular space, which is coarser than $A(X, \mathcal{T})$ by the topology of the Alexandroff duplicate. Hence, $A(X)$ is epi-completely-regular.

Since every subspace of a cube is completely-regular [12], we get:

**Corollary 4.** Every $T_1$-subspace of a cube is epi-completely-regular.

Since every $C_2$-paracompact Fréchet space is epi-normal, and any Mrówka space $\Psi(A)$ is Tychonoff [18], we obtain:

**Corollary 5.**

(1) Every $C_2$-paracompact first-countable space is epi-completely-regular.

(2) Any Mrówka space $\Psi(A)$ is epi-completely-regular.

Note that: a space $(X, \mathcal{T})$ is an almost-completely regular space if and only if the semi-regularization $(X, \mathcal{T}_s)$ of $(X, \mathcal{T})$ is completely-regular [22]. Also, complete-regularity is not a semi-regularization property, but almost-complete regularity is [22]. For example, the half disc topology $(X, \mathcal{T})$ is not completely-regular [30], and its semi-regularization $(X, \mathcal{T}_s)$ is the usual topology on the closed upper half plane, which is completely-regular.

**Theorem 10.** If $(X, \mathcal{T})$ is an almost-completely regular space such that the semi-regularization $(X, \mathcal{T}_s)$ of $(X, \mathcal{T})$ is $T_1$, then $(X, \mathcal{T})$ is epi-completely-regular.

**Proof.** Let $(X, \mathcal{T})$ be an almost-completely regular space and the semi-regularization $(X, \mathcal{T}_s)$ of $(X, \mathcal{T})$ be $T_1$. Since the semi-regularization of an almost-completely regular space is completely-regular [22], we get: $(X, \mathcal{T}_s)$ is $T_1$-completely-regular. Thus, $(X, \mathcal{T}_s)$ is Tychonoff. Since $\mathcal{T}_s$ is a topology on $X$ which is coarser than $\mathcal{T}$, we obtain: $(X, \mathcal{T})$ is epi-completely-regular.

Since every extremally-disconnected space is $T_1$-$\pi$-normal [14], we get: every extremally-disconnected space is $T_1$-almost-completely regular. Since every extremally-disconnected semi-regular space is Tychonoff [3], we conclude:

**Corollary 6.**

(a) Every Hausdorff extremally-disconnected space is epi-completely-regular.
(b) Every extremally-disconnected semi-regular space is epi-completely-regular.

In fact, an epi-completely-regular space is not necessary to be extremally-disconnected. For example: the rational sequence topology [30, Example 65], is a semi-regular epi-completely-regular space being Tychonoff, which is not extremally disconnected. The next result is obvious:

**Theorem 11.** If the semi-regularization space \((X, T_s)\) of a space \((X, T)\) is an epi-completely-regular space, then \((X, T)\) is epi-completely-regular.

**Theorem 12.** Every Hausdorff almost-completely regular space is epi-completely-regular.

**Proof.** Let \((X, T)\) be a Hausdorff almost-completely regular space. Let \((X, T_s)\) be the semi-regularization of \((X, T)\). Then, \((X, T_s)\) is a Hausdorff completely regular space because the semi-regularization of a Hausdorff almost-completely regular space is Hausdorff completely regular [22]. Thus, \((X, T_s)\) is Tychonoff. Since \(T_s \subseteq T\), we conclude: \((X, T)\) is epi-completely regular.

The next results are obvious:

**Corollary 7.**

(1) Every \(T_1\)-semi-regular (resp. semi-normal) almost-completely regular space is epi-completely-regular.

(2) Any nearly-paracompact Hausdorff space is epi-normal.

(3) Every almost-paracompact Hausdorff Lindelöf space is epi-quasi-normal.

Since every epi-completely-regular regular space is epi-regular, we get: every epi-completely-regular space is \(C\)-regular, but the converse is not true in general. For example: the odd-even topology, Example 12, is a normal and completely-regular space [30], which is not \(T_1\). Thus, the odd-even topology is a \(C\)-regular space, which is not epi-completely regular.

**Theorem 13.** Every semi-regular almost-normal \(T_1\)-space is Tychonoff.

**Proof.** Let \(X\) be a semi-regular \(T_1\)-almost-normal space. Then, \(X\) is almost-normal. Since every semi-regular almost-normal space is regular, we obtain: \(X\) is a \(T_1\)-regular almost-normal space. Hence, \(X\) is a \(T_1\)-completely-regular space because every regular almost-normal space is completely-regular [10]. Therefore, \(X\) is Tychonoff.

From Theorem 13, we conclude the next corollary:

**Corollary 8.** Every semi-regular almost-normal \(T_1\)-space is epi-completely regular.

The next theorem has been presented in [25, Theorem 9 - 1.17, page 306]:

**Theorem 14.** [25], A space \((X, T)\) is a \(T_1\)-space if and only if \(T\) contains the finite complement topology on \(X\). i.e. \(\mathcal{C}F \subseteq T\) and \((X, \mathcal{C}F)\) is the finite complement topology on \(X\).
From Theorem 14, we conclude:

**Corollary 9.** If \((X, \mathcal{T})\) is a \(T_1\)-space, then there exists a topology \(\mathcal{T}'\) coarser than \(\mathcal{T}\) such that \((X, \mathcal{T}')\) is \(T_1\) almost completely regular (resp. almost regular). We can say that \((X, \mathcal{T})\) is epi-almost completely regular (resp. epi-almost regular).

Recall that: any closed extension space \((X^p, \mathcal{T}^*)\) of a given space \((X, \mathcal{T})\) is always connected, \(\pi\)-normal, almost normal, separable and it cannot be \(T_1\) [1]. Thus, we conclude:

**Corollary 10.** Every closed extension space \((X^p, \mathcal{T}^*)\) of a given space \((X, \mathcal{T})\) cannot be epi-completely regular.

The next problem is still open in this work:

**Problem:**
- Is there an example of an epi-completely-regular space, which is not epi-mildly-normal?.

### 4. Conclusion

A new version of complete regularity called epi-complete regularity has been studied in this work. I have shown that epi-complete regularity is different from both epi-regularity and epi-normality. I have proved that epi-complete regularity is a topological, productive, hereditary and additive property. Some properties, counterexamples and relationships with some other forms of topological properties have been presented and proved.

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