GLOBAL REGULARITY AND DECAY BEHAVIOR FOR LERAY EQUATIONS WITH CRITICAL-DISSIPATION AND ITS APPLICATION TO SELF-SIMILAR SOLUTIONS

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Abstract. In this paper, we show the global regularity and the optimal decay of weak solutions to the generalized Leray problem with critical dissipation. Our method is based on the maximal smoothing effect, $L^p$-type elliptic regularity of linearization, and the action of the heat semigroup generated by the fractional powers of Laplace operator on distributions with Fourier transforms supported in an annulus. As a by-product, we shall construct a self-similar solution to the three-dimensional incompressible Navier-Stokes equations, and more importantly, prove the global regularity and the optimal decay without additional requirement of existing literatures.

1. Introduction

In this paper, we study the regularity and decay behavior of weak solutions to the generalized Leray problem in $\mathbb{R}^3$ governed by

\[
\begin{cases}
(-\Delta)^\alpha U - \frac{2\alpha - 1}{2\alpha} U - \frac{1}{2\alpha} x \cdot \nabla U + \nabla P + U \cdot \nabla U = \text{div} F, \\
\text{div} U = 0.
\end{cases}
\]

(1.1)

It arises from studying the forward self-similar solutions of Cauchy problem to the three-dimensional incompressible fractional Navier-Stokes equations with $\alpha \in (0, 1]$, which reads as follows

\[
\begin{cases}
\dot{u} + u \cdot \nabla u + (-\Delta)^\alpha u + \nabla \pi = 0, \\
\text{div} u = 0,
\end{cases}
\]

(1.2)

complemented with the initial condition

\[
u(x, 0) = u_0(x) \quad \text{in} \quad \mathbb{R}^3.
\]

(1.3)

When $0 < \alpha < 1$, the fractional Laplacian $(-\Delta)^\alpha$ is also called the Lévy operator

\[
(-\Delta)^\alpha u(x) = c_\alpha \text{p.v.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2\alpha}} \, dy, \quad c_\alpha = \frac{\alpha(1 - \alpha)4^\alpha \Gamma(\frac{3}{2} + \alpha)}{\Gamma(2 - \alpha)\pi^{\frac{3}{2}}}.
\]

To illustrate the self-similar solutions to the system (1.2) firstly raised by Leray in [12], let us review the well-known scaling property, that is, if $(u, \pi)$ is a solution to the system (1.2), then so is $(u_\lambda, \pi_\lambda)$ for each $\lambda > 0$, where

\[
u_\lambda(x, t) = \lambda^{2\alpha - 1} u(\lambda x, \lambda^2 t) \quad \text{and} \quad \pi_\lambda(x, t) = \lambda^{2(2\alpha - 1)} \pi(\lambda x, \lambda^2 t).
\]

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It has been shown that the scaling invariance plays an essential role in the regularity theory of the incompressible Navier-Stokes equations. This scaling indicates that the ratio $\frac{|x|^2}{t}$ is important and suggests that we search for a special solution which is scale invariant with respect to the natural scaling, 

$$u_\lambda = u \quad \text{and} \quad \pi_\lambda = \pi \quad \text{for each} \quad \lambda > 0,$$

which are often called self-similar solutions. Taking $\lambda(t) = t^{-\frac{\alpha}{2\alpha}}$ with $t > 0$ (or $\lambda(t) = (-t)^{-\frac{\alpha}{2\alpha}}$ with $t < 0$), we obtain the forward self-similar solutions (or backward self-similar solutions, resp.) to the system (1.2) having the form

$$u(x, t) = \lambda^{2\alpha-1}(t)u(\lambda(t)x, 1) \quad \text{and} \quad \pi(x, t) = \lambda^{2(2\alpha-1)}(t)\pi(\lambda(t)x, 1).$$

Denoting $U(x) = u(x, 1)$ and $P(x) = \pi(x, 1)$, one easily verifies that the profile pair $(U, P)$ solves either the generalized Leray equations in $\mathbb{R}^3$

$$\begin{cases}
(-\Delta)^\alpha U - \frac{2\alpha - 1}{2\alpha} U - \frac{1}{2\alpha} x \cdot \nabla U + \nabla P + U \cdot \nabla U = 0, \\
\text{div } U = 0,
\end{cases}
$$

(1.4)

or

$$\begin{cases}
(-\Delta)^\alpha U + \frac{2\alpha - 1}{2\alpha} U + \frac{1}{2\alpha} x \cdot \nabla U + \nabla P + U \cdot \nabla U = 0, \\
\text{div } U = 0,
\end{cases}
$$

(1.5)

Existence results for the backward self-similar solutions to the system (1.5) were obtained in, see, e.g., [13, 19]. In this paper, we are interested in the forward self-similar solutions to the problem (1.4). Due to the singularity arising from self-similarity, we cannot directly construct a solution in the Sobolev space. Thus, one decomposes

$$U = U_0 + V$$

with $U_0(x) = e^{(-\Delta)^\alpha}u_0(x)$ and the difference $V$ satisfies in $\mathbb{R}^3$

$$\begin{cases}
(-\Delta)^\alpha V - \frac{2\alpha - 1}{2\alpha} V - \frac{1}{2\alpha} x \cdot \nabla V + \nabla P = -V \cdot \nabla V + L_{U_0}(V) + \text{div } F_0, \\
\text{div } V = 0,
\end{cases}
$$

(1.6)

where $F_0 = -U_0 \otimes U_0$ and

$$L_{U_0} V = -U_0 \cdot \nabla V - V \cdot \nabla U_0.$$

For such system with $\alpha = 1$, existence of weak solutions $V \in H^1(\mathbb{R}^3)$ was firstly shown by Korobkov and Tsai in [9]. But they said they did not prove regularity and decay estimate for the weak solutions. This problem was later solved in the following theorem by Lai, Miao and Zheng in [10].

**Theorem 1.1** (Theorem 3.6, [10]). Let $\alpha \in (\frac{5}{3}, 1]$ and $u_0 = \frac{\sigma(x)}{|x|^{\alpha}}$ with $\sigma(x) = \sigma(x/|x|) \in C^{1,0}(S^2)$. Then there exists $C = C\left(\|\sigma\|_{W^{1,\infty}(S^2)}\right) > 0$ such that the system (1.1) admits at least one weak solution $V \in H^\alpha(\mathbb{R}^3)$ with

$$\|V\|_{H^\alpha(\mathbb{R}^3)} \leq C\left(\|\sigma\|_{W^{1,\infty}(S^2)}\right).$$
In particular, for $\alpha = 1$,
$$|V|(x) \leq C\left(\|\sigma\|_{W^{1,\infty}(S^2)}\right) (1 + |x|)^{-\frac{3}{2}} \log(e + |x|).$$

In Theorem 1.1 it is also proved that the existence of weak solutions $V \in H^\alpha(\mathbb{R}^3)$ to the system (3.1) with $\alpha \in (\frac{5}{6}, 1)$ by using the blowup argument and Leray’s degree theory. Recently, we further develop some estimates in the weighted framework, and then we obtain the regularity and decay estimate of weak solutions for the case where $\alpha \in (\frac{5}{6}, 1)$, but it is not optimal.

**Theorem 1.2** ([11]). Let $\alpha \in (\frac{5}{6}, 1]$ and $u_0 = \frac{\sigma(x)}{|x|^3}$ with $\sigma(x) = \sigma(|x|)$ be a solution to the generalized Navier-Stokes equations in $\Omega$. To do this, it makes us think of a very similar problem, the stationary solutions to the generalized Leray problem with critical dissipation. Theorem 1.2

$$\|\sqrt{1+|\cdot|}V\|_{H^{1+\alpha}(\mathbb{R}^3)} \leq C(\|\sigma\|_{W^{1,\infty}(S^2)}).$$

Moreover, one has
$$|V(x)| \leq C(\|\sigma\|_{W^{1,\infty}(S^2)}) (1 + |x|)^{-\frac{3}{2}}.$$

**Remark 1.3.** Now we try to explain why $\alpha > 5/6$ by the scaling analysis, namely, if $V$ is the solution to the system (1.6), then so does $V_\lambda$ for each $\lambda > 0$, where
$$V_\lambda(x) = \lambda^{2\alpha - 1}V(\lambda x).$$

A direct calculation yields
$$\|V_\lambda\|_{H^{\alpha}(\mathbb{R}^3)} = \lambda^{2\alpha - 1 + \frac{3}{2}}\|V\|_{H^{\alpha}(\mathbb{R}^3)} = \lambda^{3\alpha - \frac{5}{2}}\|V\|_{H^{\alpha}(\mathbb{R}^3)},$$

which implies $\alpha = \frac{5}{6}$ is “the critical index”. Thus the problem (1.6) is classified as “critical” in the fractional powers of Laplace operator $\alpha = \frac{5}{6}$.

Therefore, the goal of the present paper is to study the global regularity and decay behaviour of weak solution to the generalized Leray problem with critical dissipation. To do this, it makes us think of a very similar problem, the stationary solutions to the four-dimensional Navier-Stokes equations in $\Omega \subset \mathbb{R}^4$

$$\begin{cases}
-\nabla V + V \cdot \nabla V + \nabla P = F, \\
\text{div } V = 0.
\end{cases}
$$

In 1979, Gerhardt [6] proved the regularity of weak solutions to the system (1.7) and his proof strongly relies on two ingredients. One is the so-called “compactness lemma” that if $U \in L^2(\Omega)$ and $V \in H^1(\Omega)$, we have

$$\int_\Omega |U| \cdot |V|^2 \text{d}x \leq \varepsilon \|V\|_{H^1(\Omega)}^2 + C\varepsilon \|V\|_{L^2(\Omega)}^2 \quad \text{for each } \varepsilon > 0.$$ 

Another is $L^p$-theory of Stokes equations, which is the linear part of the model (1.7), that if $V$ satisfies the Dirichlet problem of Stokes equations

$$\begin{cases}
-\nabla V + \nabla P = F, & x \in \Omega, \\
V = 0, & x \in \partial\Omega, \\
\text{div } V = 0, & x \in \Omega,
\end{cases}
$$

(1.8)
we have by using bounds for Calderón-Zygmund integral singular operators [16] that
\[ \|V\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C(p, \Omega) \|F\|_{L^p(\Omega)} \quad \text{for each } p \in (1, +\infty). \]

Based on the mathematical similarity between (1.1) and (1.8), we would like to borrow this idea to apply to the weak solutions to the Leray problem (1.1) to get global regularity, but it does not work. The first difficulty we meet is that \(L^p\) elliptic regularity is no longer available for solution of linearization
\[ (1.10) \]
\[
\begin{cases}
(-\Delta)^{\alpha} V - \frac{2\alpha - 1}{2\alpha} V - \frac{1}{2\alpha} x \cdot \nabla V + \nabla P = F, \\
\text{div } V = 0,
\end{cases}
\]
due to the coefficient \(x\) is unbounded over whole space \(\mathbb{R}^3\). Indeed, if one estimates the solution \(V\) to the system (1.10) directly by making use of (1.9) for each \(p \in (1, +\infty)\),
\[ (1.11) \quad \|V\|_{W^{2\alpha,p}(\mathbb{R}^3)} + \|P\|_{W^{1,p}(\mathbb{R}^3)} \leq C(p, \alpha) \left( \|F\|_{L^p(\mathbb{R}^3)} + \|V\|_{L^p(\mathbb{R}^3)} + \|x \cdot \nabla V\|_{L^p(\mathbb{R}^3)} \right), \]

it is not clear how to control the last term. Hence, to proceed further, we need to look for new methods to obtain \(W^{2\alpha,p}(\mathbb{R}^3)\)-regularity for weak solutions to the problem (1.10).

Viewed from the perspective of microlocal analysis, it can be thought of as the maximal smoothing effect in the \(L^p(\mathbb{R}^3)\) framework. This observation suggests us to consider localization of solution in frequency space. Performing \(\hat{\Delta}_q\) to both sides of the first equation of (1.10), we easily find that the couple \((V_q, P_q) := (\hat{\Delta}_q V, \hat{\Delta}_q P)\) is smooth and solves the following system in \(\mathbb{R}^3\)
\[ (1.12) \]
\[
\begin{cases}
(-\Delta)^{\alpha} V_q - \frac{2 - \alpha}{2\alpha} V_q - \frac{1}{2\alpha} x \cdot \nabla V_q + \nabla P_q = F_q + R_q, \\
\text{div } V_q = 0,
\end{cases}
\]
where the commutator \([\hat{\Delta}_q, x \otimes] V\) is defined by
\[ [\hat{\Delta}_q, x \otimes] V = \hat{\Delta}_q (x \otimes V) - x \otimes \hat{\Delta}_q V. \]

Next, taking the standard \(L^p\)-estimate of \(V_q\) with \(p \geq 2\), one has
\[ (1.13) \]
\[
\int_{\mathbb{R}^3} (-\Delta)^{\alpha} V_q |V_q|^{p-2} V_q \, dx = \frac{2\alpha - 1}{2\alpha} \|V_q\|_{L^p(\mathbb{R}^3)}^p - \frac{1}{2\alpha} \int_{\mathbb{R}^3} x \cdot \nabla V_q |V_q|^{p-2} V_q \, dx
= \int_{\mathbb{R}^3} (-\nabla P_q + F_q) |V_q|^{p-2} V_q \, dx + \int_{\mathbb{R}^3} R_q |V_q|^{p-2} V_q \, dx.
\]

Formally, we will meet a new difficulty caused by the commutator, excluding the difficulty arising from \(x \cdot \nabla V\). In order to overcome this new difficulty, we see that commutator can be rewritten as
\[ [\hat{\Delta}_q, x \otimes] V = \int_{\mathbb{R}^3} \varphi_q(x - y) y \otimes V(x - y) \, dy - x \otimes \int_{\mathbb{R}^3} \varphi_q(x - y) V(y) \, dy
= \int_{\mathbb{R}^3} (x - y) \varphi_q(x - y) \otimes V(y) \, dy. \]
Taking Fourier transform of commutator and using the following property of Fourier transform which is the key point to overcome the first difficulty

\[(x \cdot f)\hat{\chi}(\xi) = i\div \hat{f}(\xi),\]

we immediately get

\[
F \left( [\hat{\Delta}_q, x]V \right)(\xi) = i2^{-q}\nabla_{\xi}\hat{\varphi}(2^{-q}\xi) \otimes \hat{V}(\xi) = i2^{-q}\nabla_{\xi}\hat{\varphi}(2^{-q}\xi) \otimes \hat{\Delta}_q V(\xi),
\]

which means that

\[ [\hat{\Delta}_q, x]V = 2^{-q}\hat{\varphi} \ast (\hat{\Delta}_q V)(x) \quad \text{with} \quad \varphi(x) = x\varphi(x) \in \mathcal{S}(\mathbb{R}^3). \]

This equality enables us to bound the commutator as follows

\[
\int_{\mathbb{R}^3} R_q |V_q|^{p-2}V_q \, dx \leq \frac{C}{2\alpha} \left\| \tilde{\Delta}_q V \right\|_{L^p(\mathbb{R}^3)} \left\| V_q \right\|_{L^p(\mathbb{R}^3)}^{p-1}.
\]

Next, we deal with integral term involving \( x \cdot \nabla V \). Integrating by parts, it can bounded as follows

\[
-\frac{1}{2\alpha} \int_{\mathbb{R}^3} x \cdot \nabla V_q |V_q|^{p-2}V_q \, dx = -\frac{1}{2p\alpha} \int_{\mathbb{R}^3} x \cdot \nabla |V_q|^p \, dx
\]

\[
= \frac{3}{2p\alpha} \int_{\mathbb{R}^3} |V_q|^p \, dx.
\]

It is surely a basic point that we employ the localization method so as to make this happen. This can help us to make best use of the available structure of \( x \cdot \nabla V \) in the equations (1.10), not as perturbation in (1.11).

Inserting the estimate (1.14) and the following lower bound concerning dissipation from the Bernstein inequality in Lemma 2.9

\[
\int_{\mathbb{R}^3} (-\Delta)^\alpha |V_q|^{p-2}V_q \, dx \geq C_p^{2q\alpha} \left\| V_q \right\|_{L^p(\mathbb{R}^3)}, \quad \forall p \in [2, +\infty)
\]

into the equality (1.13) leads to

\[
c_p^{2q\alpha} \left\| V_q \right\|_{L^p(\mathbb{R}^3)} \leq \left\| \nabla P_q \right\|_{L^p(\mathbb{R}^3)} + \left\| F_q \right\|_{L^p(\mathbb{R}^3)} + C \left\| \tilde{\Delta}_q V \right\|_{L^p(\mathbb{R}^3)}.
\]

Since

\[-\Delta P_q = \div F_q \quad \text{in} \ \mathbb{R}^3,
\]

we have the classical elliptic estimate that for each \( p \in [2, \infty) \),

\[
\left\| P_q \right\|_{W^{2,p}(\mathbb{R}^3)} \leq C \left\| \div F_q \right\|_{L^p(\mathbb{R}^3)}.
\]

This estimate together with the estimate (1.15) yields the \( L^p \)-type elliptic regularity that for each \( p \in [2, +\infty) \),

\[
\left\| V \right\|_{B^p_{2,p}(\mathbb{R}^3)} + \left\| P \right\|_{B^p_{1,p}(\mathbb{R}^3)} \leq C_{p,\alpha} \left( \left\| F \right\|_{B^p_{2,p}(\mathbb{R}^3)} + \left\| V \right\|_{B^p_{0,p}(\mathbb{R}^3)} \right).
\]
In addition, with the help of Bony’s paraproduct decomposition, we develop a Besov type compactness lemma: for the divergence free vector field \( \mathbf{U} \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \) and \( \mathbf{V} \in B^{3}_{p,p}(\mathbb{R}^3) \) with \( 2 \leq p < \frac{9}{2} \), we have

\[
(1.17) \quad \| \text{div} (\mathbf{U} \otimes \mathbf{V}) \|_{B^{3}_{p,p}(\mathbb{R}^3)} \leq \varepsilon \| \mathbf{V} \|_{\dot{B}^{\frac{3}{2}}_{p,p}(\mathbb{R}^3)} + C_{\varepsilon,p,\mathbf{U}} \| \mathbf{V} \|_{B^{3}_{p,p}(\mathbb{R}^3)} \quad \text{for each } \varepsilon > 0.
\]

With both estimates (1.16) and (1.17) in hand, we establish our first results stated as follows:

**Theorem 1.4.** Let \( \alpha \in (0, 1] \) and \( \mathbf{F} \in \dot{H}^{-1}(\mathbb{R}^3) \). Then the problem (1.1) admits at least one weak solution \( (\mathbf{U}, P) \in H^{\alpha}(\mathbb{R}^3) \times L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3) \) such that

(i) for each the divergence-free vector field \( \phi \in \mathcal{D}(\mathbb{R}^3) \),

\[
\int_{\mathbb{R}^3} \Lambda^\alpha \mathbf{U} \cdot \Lambda^\alpha \phi \, dy + \frac{2-\alpha}{\alpha} \int_{\mathbb{R}^3} \mathbf{U} \cdot \phi \, dy + \frac{1}{2\alpha} \int_{\mathbb{R}^3} \mathbf{y} \cdot \nabla \phi \cdot \mathbf{U} \, dy
= \int_{\mathbb{R}^3} P \text{div} \phi \, dy + \int_{\mathbb{R}^3} \mathbf{F} : \nabla \phi \, dy;
\]

(ii) when \( p \in [5/6, 1] \), we have \( (\mathbf{U}, P) \in H^{2\alpha}(\mathbb{R}^3) \times H^{1}(\mathbb{R}^3) \). Moreover,

1. if \( \mathbf{F} \in \dot{B}^{s-1}_{p,p}(\mathbb{R}^3) \) with \( s \geq 0 \) and \( p \in (2, +\infty) \), we have

\[
(\mathbf{U}, P) \in B^{s+2\alpha}_{p,p}(\mathbb{R}^3) \times B^{s+1}_{p,p}(\mathbb{R}^3).
\]

2. if \( \mathbf{F} \in H^{s+1}_{(x)^{2\alpha}}(\mathbb{R}^3) \) with \( s \geq 0 \) and \( \beta \in (0, \alpha) \), we have

\[
(\mathbf{U}, P) \in H^{s+\beta}_{(x)^{2\alpha}}(\mathbb{R}^3) \times H^{s+\alpha}_{(x)^{2\alpha}}(\mathbb{R}^3).
\]

3. if \( \sup_{x \in \mathbb{R}^3} ||x|^{4\alpha-2} \mathbf{F}(x)| + \sup_{x \in \mathbb{R}^3} ||x|^{4\alpha-1} \nabla \mathbf{F}(x) < +\infty \), we have

\[
(1.18) \quad ||\mathbf{U}(x)||_{C(\mathbb{R}^3)} \leq C(1)^{-4\alpha-1} \log \langle x \rangle \quad \text{for each } x \in \mathbb{R}^3.
\]

Moreover, if there exists some \( \gamma \in (0, 1) \) such that

\[
||| \cdot ||^{4\alpha-1+\gamma} \text{div} \mathbf{F} ||_{\dot{C}^\gamma(\mathbb{R}^3)} < \infty,
\]

we have

\[
(1.19) \quad ||\mathbf{U}(x)||_{C(\mathbb{R}^3)} \leq C(1)^{-4\alpha-1} \quad \text{for each } x \in \mathbb{R}^3.
\]

**Remark 1.5.** The existence of weak solutions can be obtained by performing the same argument such as used in [11, 19], so we omit it and mainly focus on the global regularity and the optimal decay estimate through our paper.

**Remark 1.6.** In the proof of the decay estimate (1.18), we firstly find that the solution \( \mathbf{U} \) can be expressed by

\[
(1.20) \quad \mathbf{U}(x) = \int_0^1 e^{-(\Delta)^{s}(1-s)} \mathbb{P}(\text{div}_x \left( s^{\frac{1}{2}-2} G(\cdot / s^{\frac{1}{2}-2}) \right) \) ds,
\]
where \( G = U \otimes U + F \) and \( P \) is the Leray projector. Secondly, we develop the action of the heat semigroup generated by the fractional powers of Laplace operator on distributions with Fourier transforms supported in an annulus

\[
(1.21) \quad \left\| \left( | \cdot |^m D^m_1 \Delta P G \right) * f \right\|_{L^p(\mathbb{R}^3)} \leq C_2^q(\gamma - m) e^{-ct \omega^p} \left\| f \right\|_{L^p(\mathbb{R}^3)}.
\]

**Remark 1.7.** Without the additional assumptions, the logarithmic loss in decay estimate (1.18) is caused by Leray projector \( P \), which seems inevitable.

Similar to the problem (1.1), we can show the global regularity and decay behaviour for weak solutions to the problem (1.6) following from Theorem 1.3 that

**Theorem 1.8.** Let \( \alpha \in \left[ \frac{5}{6}, 1 \right] \) and \( u_0(x) = \frac{\sigma(x)}{|x|^{2\alpha - 1}} \) with \( \sigma(x) = \sigma(x/|x|) \in C^{1,0}(\mathbb{S}^2) \). Assume that \((V, P) \in H^\alpha(\mathbb{R}^3) \times L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)\) is a weak solution to the problem (1.6) constructed in Theorem 1.4. Then we have that for each \( s \geq 0 \) and \( p \in (2, +\infty) \),

\[
(V, P) \in \left( B_{p,p}^s(\mathbb{R}^3) \times B_{p,p}^s(\mathbb{R}^3) \right) \cap \left( H_{(\infty)^{\lambda}}(\mathbb{R}^3) \times H_{(\infty)^{\lambda}}(\mathbb{R}^3) \right).
\]

More importantly, we have the optimal decay estimate

\[
(1.22) \quad \left| V(x) \right| \leq C(x)^{-4(\alpha - 1)} \text{ for each } x \in \mathbb{R}^3.
\]

**Remark 1.9.** Compared with the force in (1.1), the problem (1.6) has the very special. Without the additional assumptions, the logarithmic loss in decay estimate (1.22) can help us to obtain the optimal decay estimate (1.22) without logarithmic loss.

Since the forward self-similar solution \( u(x, t) \) takes the form

\[
u(x, t) = t^{-\frac{2\alpha - 1}{2\alpha}} U(x/t^\frac{1}{\alpha}) = t^{-\frac{2\alpha - 1}{2\alpha}} \left( V + e^{\Delta} u_0 \right)(x/t^\frac{1}{\alpha}),
\]

we immediately get by using Theorem 1.8 that

**Corollary 1.10.** Let \( \alpha \in \left[ \frac{5}{6}, 1 \right] \) and \( u_0(x) = \frac{\sigma(x)}{|x|^{2\alpha - 1}} \) with \( \sigma(x) = \sigma(x/|x|) \in C^{1,0}(\mathbb{S}^2) \). Assume that the \( u \in BC_w([0, +\infty), L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)) \) is the forward self-similar solution to Cauchy problem (1.20)-(1.3) established in [11]. Then

1. For \( \forall p \in [2, +\infty) \) and \( \forall s \in [0, +\infty) \), we have that

\[
u(x, t) - e^{-t(-\Delta)^\alpha} u_0(x) \in B_{p,p}^s(\mathbb{R}^3)
\]

and \( \forall k \in \mathbb{N} \),

\[
sup_{|\alpha| = k} \left\| D^\alpha \nu(x, t) - e^{-t(-\Delta)^\alpha} u_0 \right\|_{L^\infty(\mathbb{R}^3)} \leq C t^{\frac{2\alpha + 1 - k}{2\alpha}},
\]

which implies that the \( u(x, t) \) is smooth in \( \mathbb{R}^3 \times (0, +\infty) \);

2. we have the following pointwise estimates

\[
u(x, t) \leq C \frac{1}{|x|^{4\alpha - 1} + t^{\frac{1}{2\alpha}}}
\]

and

\[
u(x, t) - e^{t(-\Delta) u_0} \leq C \frac{1}{|x|^{4\alpha - 1} + t^{\frac{4\alpha - 1}{2\alpha}}}
\]

for all \( (x, t) \in \mathbb{R}^3 \times (0, +\infty) \).
Remark 1.11. (i) For the subcritical case $\alpha \in (5/6, 1]$, we here establish the global regularity
\[
\mathbf{u}(x, t) - e^{-t(-\Delta)^\alpha} \mathbf{u}_0(x) \in B_{p,p}^s(\mathbb{R}^3)
\]
for each $s \geq 0$ and $p \geq 2$. This improve the known global regularity
\[
\mathbf{u}(x, t) - e^{-t(-\Delta)^\alpha} \mathbf{u}_0(x) \in H^{1+\alpha}(\mathbb{R}^3)
\]
shown in [10, 11].
(ii) By developing the $L^p$-type elliptic regularity and “compactness lemma”, we firstly prove global regularity and decay behaviour of the forward self-similar solution for the critical problem where $\alpha = \frac{5}{4}$.
(iii) Let us point out that we establish the optimal decay estimate (1.22) under the natural assumption $\sigma(x) \in C^{1,0}(\mathbb{S}^2)$, and we drop the additional condition in the existing literature, for example, that $\sigma(x) \in C^{1,\gamma}(\mathbb{S}^2)$ with $\gamma > 0$ in [4].

Notations. We first agree that $(x) = (e + |x|^2)^{\frac{1}{2}}$, $B_r(x_0) = \{x \in \mathbb{R}^3| |x - x_0| < r \}$ and
\[
\mathcal{C}(x_0, r_1, r_2) = \{x \in \mathbb{R}^3| r_1 < |x - x_0| < r_2 \}
\]
For $s \in \mathbb{R}$, we denote $s = [s] + \{s\}$, where $[s]$ is the integer part and $\{s\}$ is the decimal part. Let $\mathcal{D}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the space of smooth functions with compact support, the Schwartz space of rapidly decreasing smooth functions and the space of tempered distributions on $\mathbb{R}^n$ respectively.

The rest of our paper is organized as follows: In the next section, we give some preliminary lemmas and $B_{p,p}^{2\alpha}(\mathbb{R}^3)$-regularity of linearization to the Leray problem which is the key point in our proof. In Section 3 we are devoted to proving the global regularity of weak solutions to the Leray problem. Finally, the optimal decay estimates of weak solutions to Leray problem and the forward self-similar solutions to the Navier-Stokes equations are established in Section 4.

2. PRELIMINARY AND $L^p$-THEORY OF LINEARIZATION

2.1. Littlewood-Paley theory. In this subsection, we will review some analysis statement including the Littlewood-Paley theory, see for example [1, 2, 14]. First of all, let us recall Fourier transform that for each $f \in \mathcal{S}'(\mathbb{R}^n)$,
\[
\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx, \quad \forall \xi \in \mathbb{R}^n,
\]
and its inverse Fourier transform
\[
\hat{f}(x) = \mathcal{F}^{-1}(f)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) \, d\xi, \quad \forall x \in \mathbb{R}^n.
\]
There exist two smooth radial functions $\hat{h}(\xi) \in \mathcal{D}(B_{\frac{3}{2}}(0))$ and $\hat{\phi}(\xi) \in \mathcal{D}(\mathcal{C})$ with $\mathcal{C} = \mathcal{C}(0; \frac{3}{4}, \frac{8}{3})$ such that $0 \leq \hat{h}(\xi), \hat{\phi}(\xi) \leq 1$ and
\[
\hat{h}(\xi) + \sum_{j \geq 0} \hat{\phi}(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n,
\]
Therefore, we can write the following Littlewood-Paley decompositions:

\[ \sum_{j \in \mathbb{Z}} \hat{\phi}(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \]

Under the framework of dyadic partition of unity in frequency-space domain, we introduce some notations concerning the Littlewood-Paley theory.

- **Homogeneous decomposition:** the dyadic blocks

\[ \hat{\Delta}_j f(x) = 2^{jn} \int_{\mathbb{R}^n} \hat{\phi}(2^jy) f(x-y) \, dy, \quad \hat{\Delta}_j = \sum_{i=-1}^{1} \hat{\Delta}_{j+i}, \quad \forall j \in \mathbb{Z} \]

and the low-frequency cut-off operators

\[ \hat{S}_j f(x) = 2^{jn} \int_{\mathbb{R}^n} h(2^jy) f(x-y) \, dy, \quad \forall j \in \mathbb{Z}. \]

- **Inhomogeneous decomposition:**

\[ \Delta_{-1} f(x) = \hat{S}_0 f(x) = \int_{\mathbb{R}^n} h(y) f(x-y) \, dy, \]

\[ \Delta_j f(x) = \hat{\Delta}_j f(x) \quad \text{and} \quad S_j f(x) = \hat{S}_j f(x), \quad \forall j \geq 0. \]

Let us remark that for each \( f \in \mathcal{S}'(\mathbb{R}^n), \)

\[ S_j f(x) = \sum_{j' \leq j-1} \Delta_{j'} f(x), \quad \forall j \in \mathbb{Z}, \]

but for each polynomial \( f \in \mathcal{S}'(\mathbb{R}^n) \) whose compact support of \( \hat{f}(\xi) \) in \( \{0\}, \)

\[ \hat{S}_j f(x) \neq \sum_{j' \leq j-1} \hat{\Delta}_{j'} f(x), \quad \forall j \in \mathbb{Z}. \]

In order to remedy this defect, we introduce a new space of tempered distributions

\[ \mathcal{S}'_h(\mathbb{R}^n) := \mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{P}, \]

which entails that for each \( f \in \mathcal{S}'_h(\mathbb{R}^n), \) we have

\[ \hat{S}_j f(x) = \sum_{j' \leq j-1} \hat{\Delta}_{j'} f(x), \quad \forall j \in \mathbb{Z}. \]

Therefore, we can write the following Littlewood-Paley decompositions:

\[ I_d = \sum_{j \geq -1} \Delta_j \quad \text{and} \quad \hat{I}_d = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j, \]

which make sense in \( \mathcal{S}'(\mathbb{R}^n) \) and \( \mathcal{S}'_h(\mathbb{R}^n), \) respectively. Also, we note that the above dyadic block and low-frequency cut-off operator map \( L^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \) with norms independent of \( j \) and \( p, \) which is contained in the following Bernstein lemma.

**Lemma 2.1** ([14], Bernstein inequality). Let \( 1 \leq p \leq q \leq \infty \) and \( f \in L^p(\mathbb{R}^n). \) Then there exists a constant \( C \) such that for all \( j \in \mathbb{Z} \) and \( k \in \mathbb{N}, \)

\[ \sup_{|\alpha|=k} \| D^\alpha \hat{S}_j f \|_{L^q(\mathbb{R}^n)} \leq C^{k2^j(k+n\left(\frac{1}{p}-\frac{1}{q}\right))} \| \hat{S}_j f \|_{L^p(\mathbb{R}^n)}, \]

\[ C^{-k2^j} \| \hat{\Delta}_j f \|_{L^q(\mathbb{R}^n)} \leq \sup_{|\alpha|=k} \| D^\alpha \hat{\Delta}_j f \|_{L^q(\mathbb{R}^n)} \leq C^{k2^j(k+n\left(\frac{1}{p}-\frac{1}{q}\right))} \| \hat{\Delta}_j f \|_{L^p(\mathbb{R}^n)}. \]
Remark 2.2. The proof of the inequality (2.1) and the right inequality of (2.1) heavily relies on the Young inequality, that is, \( \forall p, q, r \in [1, \infty], \)
\[
\| f \ast g \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^q(\mathbb{R}^n)}\| g \|_{L^r(\mathbb{R}^n)},
\]
where \( 1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \). This inequality could also be applied in the Lorentz spaces first developed by R. O’Neil in [17], and a more general Young inequality in the Lorentz spaces such as in [7], whose effect is obviously better than the classical Young inequality (2.1).

Lemma 2.3 ([4], Theorem 1.2.13). Let \( 1 \leq \ell \leq s \leq \infty, 1 \leq r < \infty \) and \( 1 < p, q < \infty \) with
\[
1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r},
\]
If \( f \in L^{q,\ell}(\mathbb{R}^n) \) and \( g \in L^r(\mathbb{R}^n) \), then \( f \ast g \in L^{p,s}(\mathbb{R}^n) \), and there exists a constant \( C \) depending only on \( r, q, s \) and \( \ell \) such that
\[
\| f \ast g \|_{L^{p,s}(\mathbb{R}^n)} \leq C \| f \|_{L^{q,\ell}(\mathbb{R}^n)} \| g \|_{L^r(\mathbb{R}^n)}.
\]

Applying the above generalized Young inequality instead of (2.1) in the proof of the Bernstein inequality, we would have the generalized Bernstein inequality: Assume \( 1 < p, q < \infty \) and \( p < q \), then
\[
\sup_{|\alpha|=k} \| D^\alpha \hat{\Delta}_j f \|_{L^\infty(\mathbb{R}^n)} \leq C 2^{j(k\frac{n+1}{p})} \| \hat{\Delta}_j f \|_{L^{p,\infty}(\mathbb{R}^n)},
\]
\[
\sup_{|\alpha|=k} \| D^\alpha \hat{\Delta}_j f \|_{L^{q,1}(\mathbb{R}^n)} \leq C 2^{j(\frac{n+1}{p} - \frac{1}{q})} \| \hat{\Delta}_j f \|_{L^{p,\infty}(\mathbb{R}^n)},
\]
so does the low-frequency cut-off operator \( \hat{S}_j \).

Next, we define the homogeneous Besov spaces in terms of the Bernstein inequality.

Definition 2.1 (Homogeneous Besov space). Assume \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \), one defines
\[
\dot{B}_{p,r}^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{\dot{B}_{p,r}^s(\mathbb{R}^n)} < \infty \right\},
\]
where
\[
\| f \|_{\dot{B}_{p,r}^s(\mathbb{R}^n)} := \left\| \left( 2^{js} \| \hat{\Delta}_j f \|_{P_{r}(\mathbb{Z})} \right)_{j \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})}.
\]

Similar to the standard function spaces (e.g., Sobolev spaces \( H^s(\mathbb{R}^n) \) or \( L^p(\mathbb{R}^n) \) spaces), the homogeneous Besov spaces have some properties illustrated by the following proposition.

Proposition 2.4. \( (i) \) For each \( p \geq 1 \), \( \dot{B}_{p,r}^s(\mathbb{R}^n) \) with \( s < \frac{n}{p} \) or \( \dot{B}_{p,1}^{n/p}(\mathbb{R}^n) \) is Banach space.

\( (ii) \) Inclusion: For each \( 0 \leq r_1 \leq r_2 \leq \infty \),
\[
\| f \|_{\dot{B}_{p,r_2}^s(\mathbb{R}^n)} \leq C \| f \|_{\dot{B}_{p,r_1}^s(\mathbb{R}^n)}.
\]

\( (iii) \) Embedding: For each \( 1 \leq p_1 \leq p_2 \leq \infty \),
\[
\| f \|_{\dot{B}_{p_2,r}^s(\mathbb{R}^n)} \leq C \| f \|_{\dot{B}_{p_1,r}^{s+n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}(\mathbb{R}^n)}.
\]
(iv) The interpolation inequality: If \( s_2 \geq s_1 \), there exists \( \theta \in [0,1] \) such that
\[
\| f \|_{\dot{B}_{p,r}^{s_2,1+(1-\theta)s_2}(\mathbb{R}^n)} \leq C \| f \|_{\dot{B}_{p,r}^{s_1,1}(\mathbb{R}^n)} \| f \|_{\dot{B}_{p,r}^{1-\theta,1}(\mathbb{R}^n)}^{1-\theta}.
\]
(v) The sharp interpolation inequality: If \( s_2 > s_1 \), there exists \( \theta \in (0,1) \) such that
\[
\| f \|_{\dot{B}_{p,r}^{s_2,1+(1-\theta)s_2}(\mathbb{R}^n)} \leq C \frac{1}{s_2 - s_1} \left( \frac{1}{\theta} + \frac{1}{1 - \theta} \right) \| f \|_{\dot{B}_{p,r}^{s_1,1}(\mathbb{R}^n)} \| f \|_{\dot{B}_{p,r}^{1-\theta,1}(\mathbb{R}^n)}^{1-\theta}.
\]
(vi) For \( s \in \mathbb{R} \), \( \dot{H}^s(\mathbb{R}^n) = \dot{B}_{2,2}^s(\mathbb{R}^n) \), and their norms are equivalent
\[
\frac{1}{C|s|+1} \| f \|_{\dot{B}_{2,2}^s(\mathbb{R}^n)} \leq \| f \|_{\dot{H}^s(\mathbb{R}^n)} \leq C|s|+1 \| f \|_{\dot{B}_{2,2}^s(\mathbb{R}^n)}.
\]

Next, we will give the definition of the nonhomogeneous Besov spaces, which enjoys most properties which have been proven for the homogeneous Besov spaces.

**Definition 2.2 (Inhomogeneous Besov space).** Let \( s \in \mathbb{R} \), \( p, r \geq 1 \), the inhomogeneous Besov space \( \dot{B}_{p,r}^s(\mathbb{R}^n) \) defined as follows
\[
\dot{B}_{p,r}^s(\mathbb{R}^n) = \left\{ f(x) \mid f \in \mathcal{S}'(\mathbb{R}^n) \text{ satisfies } \| f \|_{\dot{B}_{p,r}(\mathbb{R}^n)} < \infty \right\},
\]
where
\[
\| f \|_{\dot{B}_{p,r}(\mathbb{R}^n)} := \left\| \left( 2^j s \| \Delta_j f \|_p \right)_{j \geq -1} \right\|_{L^p([0,1])}.
\]

Now we can state some useful properties of the nonhomogeneous Besov spaces.

**Proposition 2.5.** Assume \( s \in \mathbb{R} \), \( 1 \leq p, r \leq \infty \), then there hold
(i) \((\dot{B}_{p,r}^s(\mathbb{R}^n), \| \cdot \|_{\dot{B}_{p,r}^s(\mathbb{R}^n)})\) is the Banach space.
(ii) The interpolation inequality: If \( s_1 < s_2 \), there exists a constant \( C > 0 \) such that for each \( \theta \in (0,1) \),
\[
\| f \|_{\dot{B}_{p,r}^{s_2,1+(\theta s_2-1)(s_2-s_1)}(\mathbb{R}^n)} \leq C \frac{1}{(s_2 - s_1) \theta(1 - \theta)} \| f \|_{\dot{B}_{p,r}^{s_1,1}(\mathbb{R}^n)} \| f \|_{\dot{B}_{p,r}^{1-\theta,1}(\mathbb{R}^n)}^{1-\theta}.
\]
(iii) Density: For \( p, r < \infty \), \( \mathcal{D}(\mathbb{R}^n) \) is dense in \( \dot{B}_{p,r}^s(\mathbb{R}^n) \).
(iv) Duality: For each \( 1 \leq p, r < \infty \), we have \( \dot{B}_{p,r}^{-s}(\mathbb{R}^n) = \left( \dot{B}_{p,r}^s \right)'(\mathbb{R}^n) \).
(v) For \( \forall s > 0 \), we have from [2] Chapter 6 that
\[
\dot{B}_{p,r}^s(\mathbb{R}^n) = L^p(\mathbb{R}^n) \cap \dot{B}_{p,r}^s(\mathbb{R}^n).
\]
(vi) The general inclusions:
\[
\dot{B}_{p,r}^s(\mathbb{R}^n) \hookrightarrow \dot{B}_{p,r}^{s_1,1}(\mathbb{R}^n), \quad s_1 < s \quad \text{or} \quad s_1 = s, r_1 \geq r,
\]
\[
\dot{B}_{p,r}^s(\mathbb{R}^n) \hookrightarrow \dot{B}_{p_1,r_1}^{s-s_1,1}(\mathbb{R}^n), \quad p_1 \geq p,
\]
\[
\dot{B}_{p,r}^{\infty,1}(\mathbb{R}^n) \hookrightarrow C_b(\mathbb{R}^n) = C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).
\]

After stating the Besov type spaces, we will discuss the simpler Hölder spaces which are defined as follows.
Definition 2.3 (Hölder space $C^{k,\gamma}(\Omega)$). Assume $k \in \mathbb{N}$, $\gamma \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$. Let us define the Hölder space $C^{k,\gamma}(\Omega)$ as follows:

$$C^{k,\gamma}(\Omega) = \{ f(x) \in C^k_b(\Omega) \mid \| f \|_{C^{k,\gamma}(\Omega)} < \infty \},$$

where

$$\| f \|_{C^{k,\gamma}(\Omega)} = \| f \|_{C^k(\Omega)} + \sup_{x \neq y} \sup_{|\alpha| = k} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x - y|^\gamma} =: \| f \|_{C^k(\Omega)} + \| f \|_{\dot{C}^{k,\gamma}(\Omega)}.$$

Remark 2.6. Let us point out that if $\gamma = r - [r] > 0$, we have

$$\frac{1}{C} \| f \|_{B^r_{s,\infty}(\mathbb{R}^n)} \leq \| f \|_{\dot{C}([r];\gamma)(\mathbb{R}^n)} \leq C_r \| f \|_{B^r_{s,\infty}(\mathbb{R}^n)},$$

where $C_r = C_r \left( \frac{1}{\gamma} - \frac{1}{1 - \gamma} \right)$.

Next, we introduce the definition of a weighted Hilbert space.

Definition 2.4. Let $s \geq 0$ and $w(x)$ be the nonzero weighted function. Then we define the weighted space $H^s_w(\mathbb{R}^n)$ as

$$H^s_w(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{H^s_w(\mathbb{R}^n)} < \infty \},$$

where

$$\| f \|_{H^s_w(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^2 w(x) \, dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} f(x)|^2 w(x) \, dx \right)^{\frac{1}{2}}.$$

Lastly we recall Bony’s paraproduct algorithm [3], which is one of the most powerful tools of paradifferential calculus. In terms of the Littlewood-Paley decomposition,

$$f = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f \quad \text{and} \quad g = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j g,$$

so that, we can write formally,

$$fg = \sum_{j \in \mathbb{Z}} \left( \hat{\Delta}_{j+1} f \hat{\Delta}_j g - \hat{\Delta}_j f \hat{\Delta}_{j+1} g \right).$$

After some simplifications, we readily get

$$fg = \sum_{j \in \mathbb{Z}} \left( \hat{\Delta}_j f \hat{\Delta}_j g + \hat{\Delta}_j g \hat{\Delta}_j f + \hat{\Delta}_j f \hat{\Delta}_j g \right)$$

$$= \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f \hat{\Delta}_j g + \sum_{j \in \mathbb{Z}} \hat{\Delta}_j g \hat{\Delta}_{j-1} f + \sum_{|j - j'| \leq 1} \hat{\Delta}_j f \hat{\Delta}_j g.$$

As a result, the product of two tempered distributions is decomposed into two paraproducts, respectively, the paraproduct term

$$\tilde{T}_f g = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j f \hat{\Delta}_{j-1} g$$

and the reminder term

$$\tilde{R}(f, g) = \sum_{|j - j'| \leq 1} \hat{\Delta}_j f \hat{\Delta}_j g = \sum_{j \in \mathbb{Z}} \tilde{\Delta}_j f \tilde{\Delta}_j g.$$
In summary, the product of two tempered distributions can be split into three parts as follows

\[ fg = \hat{T}_f g + \hat{T}_g f + \hat{R}(f, g). \]

2.2. Some useful lemmas. In this subsection, we are devoted to showing some useful lemmas which play an important role in our proof. Let us begin with a compactness lemma which is a vital cornerstone for the critical case where \( \alpha = \frac{5}{6} \).

**Lemma 2.7 (Compactness Lemma).** Assume the divergence free vector field \( U \in \dot{H}^{\frac{5}{6}}(\mathbb{R}^3) \) and \( V \in B^{\frac{5}{9}}_{p, r}(\mathbb{R}^3) \) with \( 2 \leq p < \frac{9}{2} \) and \( 1 \leq r \leq \infty \). Then, for any \( \varepsilon > 0 \), there exists a constant \( C > 0 \) depending only on \( \varepsilon, p \) and \( U \) such that

\[ \| \text{div} (U \otimes V) \|_{B^{\frac{5}{9}}_{p, r}(\mathbb{R}^3)} \leq C \varepsilon \| V \|_{B^{\frac{5}{9}}_{p, r}(\mathbb{R}^3)} + C_{\varepsilon, p} \| V \|_{B^{\frac{5}{9}}_{p, r}(\mathbb{R}^3)}. \]

**Proof.** According to the Bony paraproduct decomposition, we split \( U \otimes V \) into three parts as follows

\[ U \otimes V = \hat{T}_U V + \hat{T}_V U + \hat{R}(U, V). \]

By the Hölder inequality, we can infer that for \( 2 \leq p < \frac{9}{2} \),

\[ \| \text{div} \hat{T}_U V \|_{B^{\frac{5}{9}}_{p, r}(\mathbb{R}^3)} \leq C \| \hat{S}_{k-1} U \|_{L^\infty(\mathbb{R}^3)} \| \hat{\Delta}_k V \|_{L^p(\mathbb{R}^3)} \|_{\ell^r(\mathbb{Z})}. \]

Inserting

\[ U = \sum_{k < N} \hat{\Delta}_k U + \sum_{k \geq N} \hat{\Delta}_k U =: U^1 + U^2, \]

into (2.5) gives

\[ \| \text{div} \hat{T}_U V \|_{B^{\frac{5}{9}}_{p, r}(\mathbb{R}^3)} \leq C \| 2^k \| \hat{S}_{k-1} U^2 \|_{L^\infty(\mathbb{R}^3)} \| \hat{\Delta}_k V \|_{L^p(\mathbb{R}^3)} \|_{\ell^r(\mathbb{Z})} \]

\[ \quad + C \| 2^k \| \hat{S}_{k-1} U^1 \|_{L^\infty(\mathbb{R}^3)} \| \hat{\Delta}_k V \|_{L^p(\mathbb{R}^3)} \|_{\ell^r(\mathbb{Z})}. \]

On one hand,

\[ \| 2^k \| \hat{S}_{k-1} U^1 \|_{L^\infty(\mathbb{R}^3)} \| \hat{\Delta}_k V \|_{L^p(\mathbb{R}^3)} \|_{\ell^r(\mathbb{Z})} \leq C \| \hat{S}_N U \|_{L^\infty(\mathbb{R}^3)} \| 2^k \| \hat{\Delta}_k V \|_{L^p(\mathbb{R}^3)} \|_{\ell^r(\mathbb{Z})}. \]

By the sharp interpolation inequality, one has

\[ \| U \|_{B^{\frac{5}{6}}_{p, 1}(\mathbb{R}^3)} \leq \| U \|_{B^{\frac{5}{9}}_{p, \infty}(\mathbb{R}^3)} \| U \|_{B^{\frac{5}{6}}_{p, \infty}(\mathbb{R}^3)} \]

We furthermore have by using Young’s inequality that

\[ \| 2^k \| \hat{S}_{k-1} U^2 \|_{L^\infty(\mathbb{R}^3)} \| \hat{\Delta}_k V \|_{L^p(\mathbb{R}^3)} \|_{\ell^r(\mathbb{Z})} \leq C \| U \|_{B^{\frac{5}{6}}_{p, \infty}(\mathbb{R}^3)} \| U \|_{B^{\frac{5}{9}}_{p, 2}(\mathbb{R}^3)} \| U \|_{B^{\frac{5}{6}}_{p, 2}(\mathbb{R}^3)} \]

\[ \leq C_2 \| V \|_{B^{\frac{5}{9}}_{p, 2}(\mathbb{R}^3)} \| U \|_{B^{\frac{5}{9}}_{p, 2}(\mathbb{R}^3)} \| U \|_{B^{\frac{5}{9}}_{p, 2}(\mathbb{R}^3)} \leq C_2 \| V \|_{B^{\frac{5}{9}}_{p, 2}(\mathbb{R}^3)} \| U \|_{B^{\frac{5}{9}}_{p, 2}(\mathbb{R}^3)} + \frac{\varepsilon}{2} \| U \|_{B^{\frac{5}{9}}_{p, 2}(\mathbb{R}^3)}. \]
On the other hand,
\[
\left\| 2^k \| \dot{S}_{k-1} U \|_{L^\infty(\mathbb{R}^3)} \| \dot{\Lambda}_k V \|_{L^p(\mathbb{R}^3)} \right\|_{\ell^r(\mathbb{Z})} \leq C \left\| \| U \|_{\dot{B}_{\infty,\infty}^{\frac{3}{2}}(\mathbb{R}^3)} \right\| \left\| \| U \|_{\dot{B}_{\infty,\infty}^{\frac{3}{2}}(\mathbb{R}^3)} \right\|
\leq C \sup_{k \geq N} 2^{\frac{3k}{2}} \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|.
\] (2.8)

Plugging (2.7) and (2.8) into (2.6), we obtain
\[
\left\| \text{div} \hat{T} U V \right\|_{\dot{B}^0_{p,r}(\mathbb{R}^3)} \leq C \sup_{k \geq N} 2^{\frac{3k}{2}} \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|
\]
\[+ C \cdot 2^{\frac{3k}{2}} \| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} + \frac{\varepsilon}{2} \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|.
\] (2.9)

The divergence-free condition allows us to write
\[
\text{div} \hat{T} U V = \hat{T} \nabla U.
\]

Moreover, we have by the Hölder inequality that
\[
\left\| \text{div} (\hat{T} U V) \right\|_{\dot{B}^0_{p,r}(\mathbb{R}^3)} \leq C \left\| \| \dot{\Lambda}_k U \|_{L^p(\mathbb{R}^3)} \| \| \dot{S}_{k-1} \nabla V \|_{L^\infty(\mathbb{R}^3)} \right\|_{\ell^r(\mathbb{Z})}
\]
\[+ C \left\| \| \dot{\Lambda}_k U \|_{L^p(\mathbb{R}^3)} \| \| \dot{S}_{k-1} \nabla V \|_{L^\infty(\mathbb{R}^3)} \right\|_{\ell^r(\mathbb{Z})}.
\] (2.10)

The Bernstein inequality in Lemma 2.1 can be applied to tackle the low-frequency part
\[
\left\| \| \dot{\Lambda}_k U \|_{L^p(\mathbb{R}^3)} \| \| \dot{S}_{k-1} \nabla V \|_{L^\infty(\mathbb{R}^3)} \right\|_{\ell^r(\mathbb{Z})} \leq C \left\| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \| \| \nabla V \|_{\dot{B}_{\infty,r}^{\frac{3}{2}}(\mathbb{R}^3)} \| \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| V \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|.
\] (2.11)

By the Hölder inequality, we obtain
\[
\left\| \| \dot{\Lambda}_k U \|_{L^p(\mathbb{R}^3)} \| \| \dot{S}_{k-1} \nabla V \|_{L^\infty(\mathbb{R}^3)} \right\|_{\ell^r(\mathbb{Z})} \leq C \left\| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \| \| \nabla V \|_{\dot{B}_{\infty,r}^{\frac{3}{2}}(\mathbb{R}^3)} \| \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| V \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|.
\] (2.12)

Putting (2.11) and (2.12) into (2.10) to get
\[
\left\| \text{div} (\hat{T} U V) \right\|_{\dot{B}^0_{p,r}(\mathbb{R}^3)} \leq C \sup_{k \geq N} 2^{\frac{3k}{2}} \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| V \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|
\]
\[+ C \cdot 2^{\frac{3k}{2}} \| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \| \| V \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} + \frac{\varepsilon}{2} \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|.
\] (2.13)

For the remainder term, the Bernstein inequality in Lemma 2.1 and the Hölder inequality help us to conclude that
\[
\left\| \text{div} \left( \hat{R}(U, V) \right) \right\|_{\dot{B}^0_{p,r}(\mathbb{R}^3)} \leq C \left\| \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| \dot{\Lambda}_k V \|_{L^p(\mathbb{R}^3)} \right\|_{\ell^r(\mathbb{Z})} + C \left\| \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| \dot{\Lambda}_k V \|_{L^p(\mathbb{R}^3)} \right\|_{\ell^r(\mathbb{Z})}
\]
\[\leq C \cdot 2^{\frac{3k}{2}} \| \| U \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \| \| V \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} + C \sup_{k \geq N} 2^{\frac{3k}{2}} \| \dot{\Lambda}_k U \|_{L^2(\mathbb{R}^3)} \| \| V \|_{\dot{B}_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)} \|.
\]
This estimate together with (2.9) and (2.13), we immediately obtain that

\[ \| \text{div} \left( U \otimes V \right) \|_{L^p(B^r_{\infty}(R^3))} \leq \frac{\varepsilon}{2} \| U \|_{L^2(B^r_{\infty}(R^3))} + C \sup_{k \geq N} 2^{\frac{k}{r}} \| \hat{\Delta}_k U \|_{L^2(B^r_{\infty}(R^3))} \]

(2.14)

\[ + C \varepsilon \frac{2^{\frac{k}{r}}}{2} \| U \|_{L^2(B^r_{\infty}(R^3))} \| U \|_{L^p(B^r_{\infty}(R^3))}. \]

Since \( \ell'(Z) \subset \ell^\infty(Z) \) for each \( r \in [1, \infty) \) and \( U \in \dot{H}^{\frac{3}{2}}(R^3) \), we have that, for any \( \varepsilon > 0 \), there exists a suitable integer \( N \) such that

\[ \sup_{k \geq N} 2^{\frac{k}{r}} \| \hat{\Delta}_k U \|_{L^2(B^r_{\infty}(R^3))} \leq \left( \sum_{k \geq N} 2^{\frac{k}{r}} \| \hat{\Delta}_k U \|_{L^2(B^r_{\infty}(R^3))} \right)^\frac{1}{2} \leq \frac{\varepsilon}{2C}, \]

which together with (2.14) implies the desired result (2.4) in Lemma 2.7.

□

The action of the heat semigroup \( e^{-t(-\Delta)^\alpha} \) on distributions with Fourier transforms supported in an annulus is described in following lemma.

**Lemma 2.8.** Let \( \alpha \in (0, 1] \) and \( p, r \in [1, \infty] \). Then there exists two positive constants \( c \) and \( C \) such that for \( p \geq r \),

\[ \| (\cdot \cdot \cdot |m D^\gamma O_{\alpha,q} \ast f \|_{L^p(R^3)} \leq C2^{q[|\gamma|-m]+(1/r-1/p)}e^{-ct2^\gamma_\alpha} \| f \|_{L^r(R^3)}, \]

where \( G_{\alpha,q} \) is the heat kernel with fractional diffusion \( (-\Delta)^\alpha \) and \( O_{\alpha,q} = \hat{\Delta}_q \mathbb{P}G_{\alpha}. \)

**Proof.** First of all, we take the Fourier transform of \( D^\gamma O_{\alpha,q}f \) yielding

\[ \mathcal{F}(D^\gamma O_{\alpha,q}f) = 2^{q[|\gamma|-} \mathcal{F} \left( \mathbb{P}D^\gamma \varphi \right) \left( 2^a \xi \right) e^{-t|\xi|^{2a}} \hat{f}. \]

Denoting \( \phi(x) := \mathbb{P}D^\gamma \varphi \), we easily verify that the \( \phi(x) \) is a smooth function with Fourier transform compactly supported in the ring \( \mathbb{R}(0; 3/4, 8/3) \). Hence we have

\[ D^\gamma O_{\alpha,q}f = 2^{q[|\gamma|} \mathbb{P}D^\gamma \varphi \left( 2^a \xi \right) e^{-t|\xi|^{2a}} \hat{f}, \]

where

\[ g(t, \xi) = \mathcal{F}^{-1} \left( \phi(\xi)e^{-t|\xi|^{2a}} \right). \]

When \( m = 2k_0 \) is nonnegative even number, we take the Fourier transform to obtain

\[ \| x \|^m g(t, x) = \frac{1}{(1 + |x|^2)^{\frac{3}{2}}} \int_{R^3} e^{ix \cdot \xi} (I_d - \Delta)^3 (\Delta)^{k_0} \left( \phi(\xi)e^{-t|\xi|^{2a}} \right) d\xi. \]

Thanks to the Leibniz rule, one has

\[ (\Delta)^{k_0} \left( \phi(\xi)e^{-t|\xi|^{2a}} \right) = \sum_{j=1}^{3} \sum_{i=0}^{m} C^i_j \left( \partial^m_{x_j} \phi(\xi) \right) \left( \partial^i_{x_j} e^{-t|\xi|^{2a}} \right). \]

Moreover, we have

\[ (I_d - \Delta)^3(\Delta)^{k_0} (\phi(\xi)e^{-t|\xi|^{2a}}) \]

\[ = \sum_{\beta_1 \leq \beta, |\beta| \leq 6} C^\beta_{\beta_1} \left( \sum_{j=1}^{3} \sum_{i=0}^{m} C^i_j \partial^\beta - \beta_1 \left( \partial^m_{x_j} \phi(\xi) \right) \partial^\beta_1 \left( \partial^i_{x_j} e^{-t|\xi|^{2a}} \right) \right). \]
On the other hand, we have by the Faà-di-Bruno formula that
\[ e^{|x|^{2m}} \partial_{x_j}^{\beta} \left( \partial_{x_j}^i e^{-t|x|^{2m}} \right) = \sum_{\beta_1 + \cdots + \beta_{|\beta|} = |\beta|} (-t)^{|\beta|+1} \prod_{k=1}^{\ell} \partial_{x_k}^{\beta_k} \left( |x|^{2m} \right). \]

Since \( \text{supp} \hat{\phi}(\xi) \subset C(0; 3/4, 8/3) \), we have that for each \( \xi \in \text{supp} \hat{\phi}(\xi) \),
\[
\left| \partial^{\beta} \phi \left( \partial_{x_j}^{m-i} \hat{\phi}(\xi) \right) \partial_{x_j}^{\beta} \left( \partial_{x_j}^i e^{-t|x|^{2m}} \right) \right| \leq C \left( 1 + t \right)^{|\beta|+i} e^{-t|x|^{2m}} \leq C \left( 1 + t \right)^{|\beta|+i} e^{-\frac{a}{m} t}.
\]

Since
\[
\lim_{t \to +\infty} (1 + t)^{6+m} e^{-\frac{a}{m} t} = 0,
\]
we immediately have the following pointwise estimate that for each even nonnegative integer \( m = 2k_0 \),
\[ (2.17) \]
\[ |x|^m g(t, x) \leq C e^{-ct} \frac{1}{(1 + |x|^2)^{\frac{3}{4}}}. \]

If the real number \( m \) is not an even integer, there exists \( \theta \in (0, 1) \) such that
\[ m = \theta m + (1 - \theta)(m + 2), \]
where \( m = 2[m/2] \).

Furthermore, we get by the interpolation inequality and (2.17) that for each \( m \geq 0 \),
\[ (2.18) \]
\[ \| \cdot |^m g(t, \cdot) \|_{L^p(\mathbb{R}^d)} \leq C e^{-\frac{a}{m} t} \quad \text{for each } p \in [1, \infty]. \]

Recall from (2.16), one writes
\[ (\cdot |^m D^\gamma O_{\alpha, q}) * f = (\cdot |^m 2^{|\gamma|} |^2 q^m \left( 2^{2\alpha q} t, q \right) ) * f \]
\[ = 2^{|\gamma| - m} |^2 q^m (\cdot |^m g) \left( 2^{2\alpha q} t, 2q \right) * f. \]

This equality together with (2.18) enables us to conclude that for each \( 1 \leq r \leq p \leq \infty \),
\[ \| (\cdot |^m D^\gamma O_{\alpha, q}) * f \|_{L^p(\mathbb{R}^d)} \leq C 2^{|\gamma| - m + 3(1/r - 1/p)} e^{-ct2^{2\alpha q}} \| f \|_{L^r(\mathbb{R}^d)}. \]

So we finish the proof of Lemma 2.8.

\[ \square \]

**Lemma 2.9** (\[5\], New Bernstein’s inequality). Let \( p \in [2, +\infty) \) and \( \alpha \in (0, 1] \). Then there exist two positive constants \( c_p \) and \( C_p \) such that for any \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( j \in \mathbb{Z} \),
\[ c_p 2^{\frac{2\alpha q}{p}} \| \partial^j f \|_{L^p(\mathbb{R}^n)} \leq \left\| \Lambda^\alpha \left( |\partial^j f|^{\frac{p}{q}} \right)^{\frac{q}{p}} \right\|_{L^q(\mathbb{R}^n)} \leq C_p 2^{\frac{2\alpha q}{p}} \| \partial^j f \|_{L^p(\mathbb{R}^n)}. \]

**Lemma 2.10** (\[11\]). Let \( \alpha \in (0, 1) \), \( f \in L^2(\mathbb{R}^d) \) and \( \phi \in \dot{C}^{0, \beta}(\mathbb{R}^3) \cap \dot{W}^{1, \infty}(\mathbb{R}^3) \). Then there exists a constant \( C > 0 \) such that for each \( \beta \in (0, \alpha) \),
\[ \| \Lambda^\alpha, \phi \|_{L^2(\mathbb{R}^d)} \leq C \max \left\{ \| \phi \|_{\dot{C}^{0, \beta}(\mathbb{R}^3)}, \| \phi \|_{\dot{W}^{1, \infty}(\mathbb{R}^3)} \right\} \| f \|_{L^2(\mathbb{R}^d)}. \]
**Lemma 2.11.** Let \( s \in (0, 1), \beta \in (0, 1) \) and \( f \in L^2_{\langle \cdot \rangle 2^s}(\mathbb{R}^3) \). Then we have

(i) for each \( \beta < s \),
\[
\| [\langle \cdot \rangle^\beta, \Lambda^s] f \|_{L^2(\mathbb{R}^3)} \leq C \| f \|_{L^2(\mathbb{R}^3)};
\]
(ii) for each \( \beta \geq s \),
\[
\| [\langle \cdot \rangle^\beta, \Lambda^s] f \|_{L^2(\mathbb{R}^3)} \leq C \| \langle \cdot \rangle^\beta f \|_{L^2(\mathbb{R}^3)}.
\]

**Proof.** If \( \beta < s \), we immediately have by Lemma 2.10 that
\[
\| [\langle \cdot \rangle^\beta, \Lambda^s] f \|_{L^2(\mathbb{R}^3)} \leq C \| f \|_{L^2(\mathbb{R}^3)}.
\]
Now, it remains for us to consider the case where \( \beta \geq s \). Since \( \Lambda = \sqrt{-\Delta} \), we have in terms of definition of fractional Laplacian that
\[
[\langle x \rangle^\beta, \Lambda^s] f(x) = -cs \int_{\mathbb{R}^3} \frac{\langle x \rangle^\beta - \langle y \rangle^\beta}{|x - y|^{3+s}} f(y) \, dy.
\]
We decompose it into the following two parts
\[
\int_{\mathbb{R}^3} \frac{\langle x \rangle^\beta - \langle y \rangle^\beta}{|x - y|^{3+s}} f(y) \, dy = \left( \int_{B_{100}(x)} + \int_{\mathbb{R}^3 \setminus B_{100}(x)} \right) \frac{\langle x \rangle^\beta - \langle y \rangle^\beta}{|x - y|^{3+s}} f(y) \, dy
\]
\[=: I + II.
\]
The first term \( I \) can be bounded by
\[
|I| \leq \beta \int_{B_{100}(x)} \frac{1}{|x - y|^{2+s}} |f(y)| \, dy,
\]
which together with the Young inequality yields
\[
(2.19) \quad \| I \|_{L^2(\mathbb{R}^3)} \leq C \| f \|_{L^2(\mathbb{R}^3)}.
\]
As for term II, we split it into two parts as follows
\[
II = \left( \int_{B_{\frac{c}{2}}(0) \setminus B_{100}(x)} + \int_{B_{\frac{c}{2}}(0) \setminus B_{100}(x)} \right) \frac{\langle x \rangle^\beta - \langle y \rangle^\beta}{|x - y|^{3+s}} f(y) \, dy
\]
\[=: II_1 + II_2,
\]
where \( B_{\frac{c}{2}}(0) = \mathbb{R}^3 \setminus B_{\frac{c}{2}}(0) \).
Since
\[
|\langle x \rangle^\beta - \langle y \rangle^\beta| \leq |x| |x|^{\beta} \leq |x|^{\beta} + |y|^{\beta},
\]
we see that
\[
|II_2| \leq (1 + 2^\beta) \int_{\mathbb{R}^3 \setminus B_{100}(x)} \frac{1}{|x - y|^{3+s}} \langle y \rangle^\beta |f(y)| \, dy.
\]
Moreover, by the Young inequality again, we get
\[
(2.20) \quad \| II_2 \|_{L^2(\mathbb{R}^3)} \leq C \| \langle \cdot \rangle^\beta f \|_{L^2(\mathbb{R}^3)}.
\]
Now, we turn to bound the term $\Pi_1$. For $y \in B_{\frac{1}{10}}(0)$, we observe by the triangle inequality that

$$\frac{1}{2} |x| \leq |x| - |y| \leq |x - y| \leq |x| + |y| \leq \frac{3}{2} |x|.$$  

(2.21)

From this inequality, we have

$$|\Pi_1| \leq 4 \int_{\mathbb{R}^3 \setminus B_{100}(x)} \frac{1}{|x - y|^{3 + s - \beta}} |f(y)| \, dy.$$ 

By the Hölder inequality, the Young inequality and the fact $3 + s - \beta > 2$, one obtains

$$\|\Pi_1\|_{L^2(B_1(0))} \leq C \|\Pi_1\|_{L^\infty(B_1(0))} \leq C \|f\|_{L^2(\mathbb{R}^3)}.$$  

(2.22)

In terms of (2.21), we find that

$$|\Pi_1| \leq 4 \int_{B_{\frac{1}{10}}(0)} \frac{1}{|x - y|^{3 + s - \beta}} |f(y)| \, dy \leq C \frac{1}{|x|^{3 + s - \beta}} \int_{B_{\frac{1}{10}}(0)} |f(y)| \, dy.$$ 

By the Cauchy-Schwarz inequality, one gets

$$|\Pi_1| \leq \frac{C}{|x|^{3 + s - \beta}} \int_{B_{\frac{1}{10}}(0)} \frac{1}{|y|^2} \left( |y|^2 |f| \right) (y) \, dy \leq \frac{C}{|x|^{3 + s - \beta}} \left( \int_{B_{\frac{1}{10}}(0)} \frac{1}{|y|^2} \, dy \right)^{\frac{1}{2}} \|\cdot \|^2_{L^2(\mathbb{R}^3)} \leq \frac{C}{|x|^{3 + s}} \|\cdot \|^2_{L^2(\mathbb{R}^3)}.$$ 

Hence, we have

$$\|\Pi_1\|_{L^2(\mathbb{R}^3 \setminus B_1(0))} \leq C \left( \int_{\mathbb{R}^3 \setminus B_1(0)} \frac{1}{|x|^{3 + 2s}} \, dx \right)^{\frac{1}{2}} \leq C \|\cdot \|^2_{L^2(\mathbb{R}^3)}.$$ 

(2.23)

Collecting the above estimates (2.11), (2.20), (2.22) and (2.23), we eventually obtain

$$\| \left[ \cdot \beta, \Lambda^s \right] f \|_{L^2(\mathbb{R}^3)} \leq C \|\cdot \|^2_{L^2(\mathbb{R}^3)},$$

which implies the second desired estimate of Lemma 2.11. \qed

**Lemma 2.12.** Let $\beta \in (0, 1]$ and $f \in \mathcal{D}(\mathbb{R}^3)$. There exists a constant $C > 0$ such that

$$\frac{1}{C} \|\cdot \Lambda f\|_{L^2(\mathbb{R}^3)} \leq \|\cdot \|^2_{L^2(\mathbb{R}^3)} \leq C \|\cdot \|_{L^2(\mathbb{R}^3)}.$$ 

(2.24)

**Proof.** According to the fact that

$$\sum_{i=1}^{3} \frac{(-i\xi)(i\xi)}{|\xi|^2} = 1,$$

we write

$$\Lambda f = -\sum_{i=1}^{3} \mathcal{R}_i \partial_{x_i} f,$$

where $\mathcal{R}_i$ is the Riesz operator.

Recall from [18, Chapter V-6.4] that the function $|x|^a$ belong to the $A_p$ class with $p > 1$, if and only if $-n < a < n(p - 1)$. Thus, the function $|x|^{2\beta}$ belongs to the $A_2$ class as
long as $\beta \in (0,1]$, and then we immediately have by the weighted inequality for singular integral established in [18, Chapter V-4.2] that

$$
(\star) \quad \|\langle \cdot \rangle^\beta \Lambda f\|_{L^2(\mathbb{R}^3)} \leq C \sum_{i=1}^{3} \|\langle \cdot \rangle^\beta \mathcal{R}_i \partial_x f\|_{L^2(\mathbb{R}^3)},
$$

which gives the first inequality in (2.24).

To see the inverse inequality, we write $\partial_x f$ in terms of Fourier transform as

$$
\mathcal{F}(\partial_x f)(\xi) = -i\xi \hat{f}(\xi) = -\frac{i\xi}{|\xi|}|\xi|\hat{f}(\xi) = -\mathcal{F}(\mathcal{R}_i \Lambda f)(\xi)
$$

for each $i \in \{1, 2, 3\}$.

In parallel with (2.25), we conclude that for each $i \in \{1, 2, 3\}$,

$$
\|\langle \cdot \rangle^\beta \partial_x f\|_{L^2(\mathbb{R}^3)} \leq \|\langle \cdot \rangle^\beta \Lambda f\|_{L^2(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle^\beta \Lambda f\|_{L^2(\mathbb{R}^3)}.
$$

The sum of all $i$ is the second inequality in (2.24). $\Box$

### 2.3. $L^p$-type theory of the linearized Leray problem

In this subsection, we are going to investigate the high regularity of the Leray operator in $\mathbb{R}^3$.

Let the couple $(V, P) \in H^\alpha(\mathbb{R}^3) \times L^{\frac{3}{\alpha}}(\mathbb{R}^3)$ be a weak solution to the system (2.26) which satisfies

$$
\begin{cases}
(\Delta)^\alpha V - \frac{2\alpha - 1}{2\alpha} V - \frac{1}{2\alpha} x \cdot \nabla V + U \cdot \nabla V + \nabla P = F, \\
\text{div} V = 0.
\end{cases}
$$

Let the couple $(V, P) \in H^\alpha(\mathbb{R}^3) \times L^{\frac{3}{\alpha}}(\mathbb{R}^3)$ be a weak solution to the system (2.26) with $\alpha \in (0,1]$. Then there exists a constant $C = C_{p,\alpha} > 0$ such that

$$
\|V\|_{\dot{B}^{\alpha+2\alpha}_{p,\infty}(\mathbb{R}^3)} + \|P\|_{\dot{B}^{\alpha+1}_{p,\infty}(\mathbb{R}^3)} \leq C_{p,\alpha} \left(\|F\|_{\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^3)} + \|V\|_{\dot{B}^{\alpha}_{p,\infty}(\mathbb{R}^3)}\right).
$$

**Proof.** Taking $\phi(y) = e_i \varphi_q(y)$ with $\varphi_q(y) = 2^{2q} \varphi(2^q y)$ in the above equality and denoting $f_q = \Delta_q f$ for each $f$, it yields by a simple calculation that for each $i = 1, 2, 3$,

$$
(-\Delta)^\alpha V_q^{(i)} - \frac{2\alpha - 1}{2\alpha} \dot{\Delta}_q V_q^{(i)} - \frac{1}{2\alpha} x \cdot \nabla V_q^{(i)} + \partial_x P_q = F_q^{(i)} + R_q^{(i)}.
$$

This implies that the couple $(V_q, P_q) := (\Delta_q V, \dot{\Delta}_q P)$ is smooth and solves the following system in the whole space

$$
\begin{cases}
(\Delta)^\alpha V_q - \frac{2\alpha - 1}{2\alpha} V_q - \frac{1}{2\alpha} x \cdot \nabla V_q + \nabla P_q = F_q + R_q, \\
\text{div} V_q = 0,
\end{cases}
$$

where $F_q = \sum_{i=1}^3 F_q^{(i)}$ and $R_q = \sum_{i=1}^3 R_q^{(i)}$. [18, Chapter V-4.2]
where the commutator \( R_q \) defined by

\[
R_q = [\hat{\Delta}_q, \mathbf{x} \otimes \mathbf{V}] = \hat{\Delta}_q (\mathbf{x} \otimes \mathbf{V}) - \mathbf{x} \otimes \hat{\Delta}_q \mathbf{V}.
\]

Since the Fourier transform \( \mathcal{F} : \mathcal{S}(\mathbb{R}^3) \to \mathcal{S}(\mathbb{R}^3) \) is a bounded linear operator, we know that each term in (2.29) is a Schwarz function. So, multiplying the first equation of (2.34) by \( |V_q|^{p-2}V_q \) with \( p \geq 2 \) and then integrating the resulting equality in space variable over the whole space, we get

\[
\int_{\mathbb{R}^3} (-\Delta)^\alpha V_q |V_q|^{p-2} V_q \, dx - \frac{2\alpha - 1}{2\alpha} \|V_q\|_{L^p(\mathbb{R}^3)}^p
\]

(2.30)

\[
- \frac{1}{2\alpha} \int_{\mathbb{R}^3} \mathbf{x} \cdot \nabla V_q |V_q|^{p-2} V_q \, dx
\]

\[
= \int_{\mathbb{R}^3} (-\nabla P_q + F_q) |V_q|^{p-2} V_q \, dx + \int_{\mathbb{R}^3} R_q |V_q|^{p-2} V_q \, dx.
\]

The new Bernstein inequality in Lemma (2.9) enables us to infer the lower bound of the first quantity in the left side of equality (2.30) that for each \( p \in [2, \infty) \),

\[
\int_{\mathbb{R}^3} (-\Delta)^\alpha |V_q|^{p-2} V_q \, dx \geq c_p 2^{2\alpha} \|V_q\|_{L^p(\mathbb{R}^3)}^p.
\]

(2.31)

Integrating by parts yields

\[
- \frac{2\alpha - 1}{2\alpha} \|V_q\|_{L^p(\mathbb{R}^3)}^p - \frac{1}{2\alpha} \int_{\mathbb{R}^3} \mathbf{x} \cdot \nabla V_q |V_q|^{p-2} V_q \, dx
\]

(2.32)

\[
= - \frac{2\alpha - 1}{2\alpha} \|V_q\|_{L^p(\mathbb{R}^3)}^p + \frac{3}{2p\alpha} \|V_q\|_{L^p(\mathbb{R}^3)}^p = \frac{3 + p - 2p\alpha}{2p\alpha} \|V_q\|_{L^p(\mathbb{R}^3)}^p.
\]

By the Hölder inequality, one has

\[
\int_{\mathbb{R}^3} (\nabla P_q + F_q) |V_q|^{p-2} V_q \, dx \leq \left( \|\nabla P_q\|_{L^p(\mathbb{R}^3)} + \|F_q\|_{L^p(\mathbb{R}^3)} \right) \|V_q\|_{L^{p-1}(\mathbb{R}^3)}^{p-1}.
\]

(2.33)

Now we turn to deal with the integral term involving commutator which takes

\[
[\hat{\Delta}_q, \mathbf{x} \otimes \mathbf{V}] = \int_{\mathbb{R}^3} \varphi_q(\mathbf{x} - \mathbf{y}) \mathbf{y} \otimes \mathbf{V}(\mathbf{x} - \mathbf{y}) \, d\mathbf{y} - \mathbf{x} \otimes \int_{\mathbb{R}^3} \varphi_q(\mathbf{x} - \mathbf{y}) \mathbf{V}(\mathbf{y}) \, d\mathbf{y}
\]

\[
= - \int_{\mathbb{R}^3} (\mathbf{x} - \mathbf{y}) \varphi_q(\mathbf{x} - \mathbf{y}) \otimes \mathbf{V}(\mathbf{y}) \, d\mathbf{y}.
\]

Letting \( \tilde{\varphi}(\mathbf{x}) = \mathbf{x} \varphi(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^3) \), taking the Fourier transform and using the support property of \( \varphi \), we find

\[
\mathcal{F} \left( [\hat{\Delta}_q, \mathbf{x} \otimes \mathbf{V}] \right)(\xi) = 2^{-q} \tilde{\varphi}_q(\xi) \tilde{\mathbf{V}}(\xi) = i 2^{-q} \text{div}_\xi \varphi \left( 2^{-q} \xi \right) \otimes \tilde{\mathbf{V}}(\xi)
\]

\[
= i 2^{-q} \text{div}_\xi \varphi \left( 2^{-q} \xi \right) \otimes \hat{\Delta}_q \mathbf{V}(\xi),
\]

which means that

\[
[\hat{\Delta}_q, \mathbf{x} \otimes \mathbf{V}] = 2^{-q} \tilde{\varphi}_q * \left( \hat{\Delta}_q \mathbf{V} \right)(\mathbf{x}).
\]

(2.34)
With the equality (2.34) in hand, we infer by the Hölder inequality and the Bernstein inequality in Lemma 2.1 that

\[
\int_{\mathbb{R}^3} \nabla q \cdot 2^{-1} \nabla \left( \frac{1}{2} \nabla q \right) \| \nabla q \|_{L^p(\mathbb{R}^3)}^2 \leq \frac{C}{2 \alpha} \| \Delta q \|_{L^p(\mathbb{R}^3)}^2.
\]  

(2.35)

Inserting estimates (2.31), (2.32), (2.33) and (2.35) into the equality (2.30) leads to

\[
\| \nabla q \|_{L^p(\mathbb{R}^3)} + \| \mathbf{F}_q \|_{L^p(\mathbb{R}^3)} + \| \Delta q \|_{L^p(\mathbb{R}^3)}.
\]  

(2.36)

Our task is now to bound the quantity concerning the pressure \( P \). Applying the divergence operator to the first equation of (2.29), we readily obtain by a simple calculation that

\[
- \Delta P_q = \text{div} \mathbf{F}_q \quad \text{in} \quad \mathbb{R}^3.
\]  

(2.37)

The standard \( L^p \)-estimate for elliptic equations for each \( p \in [2, \infty) \),

\[
\| P_q \|_{W^{2,p}(\mathbb{R}^3)} \leq C \| \text{div} \mathbf{F}_q \|_{L^p(\mathbb{R}^3)}
\]

means that

\[
2^q \| P_q \|_{L^p(\mathbb{R}^3)} \leq C \| \mathbf{F}_q \|_{L^p(\mathbb{R}^3)} \quad \text{for each} \quad p \in [2, \infty).
\]  

(2.38)

Plugging estimate (2.38) into (2.36), we immediately have

\[
c_p 2^q \| \mathbf{F}_q \|_{L^p(\mathbb{R}^3)} + \frac{3 + p - 2p\alpha}{2p\alpha} \| \mathbf{F}_q \|_{L^p(\mathbb{R}^3)} + 2^q \| P_q \|_{L^p(\mathbb{R}^3)}
\]

\[
\leq C \| \mathbf{F}_q \|_{L^p(\mathbb{R}^3)} + \frac{C}{2 \alpha} \| \Delta q \|_{L^p(\mathbb{R}^3)}.
\]  

(2.39)

Multiplying (2.39) by \( 2^q \) and then taking the \( \ell^p \)-norm of the resulting inequality with respect to \( q \), we eventually obtain that for each \( p \in [2, \infty) \),

\[
\| \mathbf{V} \|_{B^{2q,3}_{\alpha}(\mathbb{R}^3)} + \| P \|_{B^{3+3q}_{\alpha}(\mathbb{R}^3)} \leq C_{p,\alpha} \left( \| \mathbf{F} \|_{B^{3+3q}_{\alpha}(\mathbb{R}^3)} + \| \mathbf{V} \|_{B^{\alpha}_{\alpha}(\mathbb{R}^3)} \right),
\]

which implies the desired estimate in Theorem 2.13.

Next, we give the estimate of weak solution \( \mathbf{V} \) in the weighted Hilbert space. In forthcoming part of this subsection, we always assume that \( \mathbf{F} = \mathbf{F}_1 + \text{div} \mathbf{F}_2 \).

**Theorem 2.14.** Assume the divergence free vector field \( \mathbf{U} \in L^\infty(\mathbb{R}^3) \cap W^{1,3}(\mathbb{R}^3) \). Let the couple \( (\mathbf{V}, P) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \) be a weak solution to the system (2.23) with \( \alpha \in (0, 1] \). Then there exists a constant \( C > 0 \) such that
(i) if \( \alpha \in (0, 1) \), \( F_1 \in L^2(\mathbb{R}^3) \) and \( F_2 \in H^1(\mathbb{R}^3) \) satisfying
\[
\| (\cdot)^\beta \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} < \infty,
\]
we have that for each \( \beta < \alpha \),
\[
\| (\cdot)^\beta V \|_{H^\alpha(\mathbb{R}^3)} \leq C \left( \| F_1 \|_{L^2(\mathbb{R}^3)} + \| (\cdot)^\beta \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \| F_2 \|_{H^1(\mathbb{R}^3)} \right);
\]
(ii) if \( \alpha = 1 \), \( F_1 \in L^2(\mathbb{R}^3) \) and \( F_2 \in H^1(\mathbb{R}^3) \) satisfying
\[
\| (\cdot)^\beta \div F_2 \|_{H^{-1}(\mathbb{R}^3)} < \infty,
\]
we have
\[
\| (\cdot)V \|_{H^1(\mathbb{R}^3)} \leq C \left( \| F_1 \|_{L^2(\mathbb{R}^3)} + \| (\cdot) \div F_2 \|_{H^{-1}(\mathbb{R}^3)} + \| F_2 \|_{H^1(\mathbb{R}^3)} \right).
\]

Proof. Since the proof of (2.41) was shown in [11], so we focus on the proof of the case where \( \alpha \in (0, 1) \). Denoting \( g_R(x) = g(x/R) \) with \( g(x) = \frac{1}{(1+|x|^2)^{\frac{\alpha}{4} + \frac{\beta}{2}}} \) and \( R \gg 1 \),
\[
V_R := g_RV, \quad P_R := g_RP \quad \text{and} \quad F_R := g_RF,
\]
we easily find by taking the test function as \( (x)^{2\beta}g_R(x)\) in (2.27) that
\[
\int_{\mathbb{R}^3} \Lambda^\alpha \cdot \Lambda^\alpha \left( g_R(x)^\beta V_{\beta,R} \right) \, dx = \frac{2 - \alpha}{2\alpha} \| V_{\beta,R} \|_{L^2(\mathbb{R}^3)}^2
\]
\[
- \frac{1}{2\alpha} \int_{\mathbb{R}^3} x \cdot \nabla \left( g_R^2(x)^{2\beta}V \right) \, dx + \int_{\mathbb{R}^3} U \cdot \nabla \left( g_R^2(x)^{2\beta}V \right) \, dx
\]
\[
= \int_{\mathbb{R}^3} F_R(x)^{2\beta}V_R \, dx - \int_{\mathbb{R}^3} g_R(x)^\beta \nabla PV_{\beta,R} \, dx,
\]
where and what in follows, we denote \( f_{\beta,R}(x) = g_R(x)^\beta f(x) \).

Firstly, we rewrite the first term in the left side of the above equality as
\[
\int_{\mathbb{R}^3} \Lambda^\alpha \cdot \Lambda^\alpha \left( g_R(x)^\beta V_{\beta,R} \right) \, dx
= \int_{\mathbb{R}^3} \left( g_R(x)^\beta \right) \Lambda^\alpha \cdot \Lambda^\alpha V_{\beta,R} \, dx + \int_{\mathbb{R}^3} \left[ \Lambda^\alpha, g_R(x)^\beta \right] \Lambda^\alpha V \cdot V_{\beta,R} \, dx
= \| \Lambda^\alpha V_{\beta,R} \|_{L^2(\mathbb{R}^3)}^2 + \int_{\mathbb{R}^3} \left[ \Lambda^\alpha, g_R(x)^\beta \right] \Lambda^\alpha V \cdot V_{\beta,R} \, dx
+ \int_{\mathbb{R}^3} \left[ \Lambda^\alpha, g_R(x)^\beta \right] \Lambda^\alpha V \cdot V_{\beta,R} \, dx.
\]

By the Cauchy-Schwarz inequality and Lemma (2.11) we find that for each \( \beta < \alpha \),
\[
\int_{\mathbb{R}^3} \left[ \Lambda^\alpha, g_R(x)^\beta \right] V \cdot \Lambda^\beta V_{\beta,R} \, dx \leq \| \left[ \Lambda^\alpha, g_R(x)^\beta \right] V \|_{L^2(\mathbb{R}^3)} \| \Lambda^\alpha (\cdot)^\beta V_R \|_{L^2(\mathbb{R}^3)} \leq \| V \|_{L^2(\mathbb{R}^3)} \| V_{\beta,R} \|_{H^\alpha(\mathbb{R}^3)}.
\]

Also, we have that for each \( \beta < \alpha \),
\[
\int_{\mathbb{R}^3} \left[ \Lambda^\alpha, g_R(x)^\beta \right] \Lambda^\alpha V \cdot V_{\beta,R} \, dx \leq \| \left[ \Lambda^\alpha, g_R(x)^\beta \right] \Lambda^\alpha V \|_{L^2(\mathbb{R}^3)} \| V_{\beta,R} \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \| V \|_{L^2(\mathbb{R}^3)} \| V_{\beta,R} \|_{H^\alpha(\mathbb{R}^3)}.
\]
Thus, we have
\[(2.43) \int_{\mathbb{R}^3} \Lambda^\alpha \mathbf{V} \cdot \Lambda^\alpha (g_R(x)^{2\beta} \mathbf{V}_R) \, dx \geq \|\Lambda^\alpha \mathbf{V}_{\beta,R}\|_{L^2(\mathbb{R}^3)}^2 - \frac{1}{4\alpha} \int_{\mathbb{R}^3} x \cdot \nabla g_R^2(x)^{2\beta} \mathbf{V} \, dx,\]

Noting that
\[-\frac{2\alpha - 1}{2\alpha} \|\mathbf{V}_{\beta,R}\|_{L^2(\mathbb{R}^3)}^2 - \frac{1}{2\alpha} \int_{\mathbb{R}^3} x \cdot \nabla g_R^2(x)^{2\beta} \mathbf{V} \, dx\]
\[= \frac{5 - 4\alpha}{4\alpha} \|\langle \cdot \rangle^\beta \mathbf{V}_R\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{4\alpha} \int_{\mathbb{R}^3} x \cdot \nabla \langle \cdot \rangle^\beta \mathbf{V}_R^2 \, dx + \frac{1}{4\alpha} \int_{\mathbb{R}^3} x \cdot \nabla g_R^2(x)^{2\beta} \mathbf{V}^2 \, dx,\]

A simple calculation yields
\[\frac{1}{4\alpha} \int_{\mathbb{R}^3} x \cdot \nabla \langle \cdot \rangle^\beta \mathbf{V}_R^2 \, dx = \frac{\beta}{2\alpha} \int_{\mathbb{R}^3} |x|^2 (\langle \cdot \rangle^\beta)^2 \mathbf{V}_R^2 \, dx = \frac{\beta}{2\alpha} \|\mathbf{V}_{\beta,R}\|_{L^2(\mathbb{R}^3)}^2 - \frac{\epsilon}{2\alpha} \int_{\mathbb{R}^3} \langle \cdot \rangle^2 \mathbf{V}_R^2 \, dx,\]

and
\[\frac{1}{4\alpha} \int_{\mathbb{R}^3} x \cdot \nabla g_R^2(x)^{2\beta} \mathbf{V}^2 \, dx = -\frac{1 + 2\beta}{4\alpha} \int_{\mathbb{R}^3} \frac{|x|^2}{1 + |x|^2} \mathbf{V}_R^2 \, dx \geq -\frac{1 + 2\beta}{4\alpha} \|\mathbf{V}_{\beta,R}\|_{L^2(\mathbb{R}^3)}^2.\]

Therefore, we have
\[(2.44) \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla \left(g_R^2(x)^{2\beta} \mathbf{V}\right) \, dx = \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla g_R^2 \left(\langle x \rangle^2 \mathbf{V}\right) \, dx + \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla \langle x \rangle^2 \left(g_R^2 \mathbf{V}\right) \, dx.\]

On one hand, we see that
\[\int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla g_R^2 \left(\langle x \rangle^2 \mathbf{V}\right) \, dx = -\left(1 + 2\beta\right) \int_{\mathbb{R}^3} \frac{1}{1 + |x|^2} \frac{x \cdot \mathbf{U}}{R^2} \mathbf{V}_{\beta,R}^2 \, dx \leq \frac{1 + 2\beta}{2R} \|\mathbf{U}\|_{L^\infty(\mathbb{R}^3)}^2 \|\mathbf{V}_{\beta,R}\|_{L^2(\mathbb{R}^3)}^2.\]
On the other hand, we get by using $\beta \in (0,1)$ that
\[
\int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla \langle \mathbf{x} \rangle^{2\beta} \left( g_R^2 \mathbf{V}^2 \right) \, dx = 2\beta \int_{\mathbb{R}^3} \langle \mathbf{U} \cdot \mathbf{x} \rangle \langle \mathbf{x} \rangle^{2(\beta-1)} \left( g_R^2 \mathbf{V}^2 \right) \, dx \\
\leq 2\beta \| \mathbf{U} \|_{L^\infty(\mathbb{R}^3)} \| \mathbf{V} \|_{L^2(\mathbb{R}^3)} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)}.
\]
Thus we have
\[
(2.45) \quad -\int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla \left( g_R^2 \langle \mathbf{x} \rangle^{2\beta} \mathbf{V} \right) \, dx \leq \frac{C}{R} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)}^2 + \| \mathbf{V} \|_{L^2(\mathbb{R}^3)} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)}.
\]
By the Cauchy-Schwarz inequality, one has
\[
(2.46) \quad \int_{\mathbb{R}^3} F_{1R}(\mathbf{x})^{2\beta} \mathbf{V} \, dx \leq \| \langle \cdot \rangle^{\beta} F_1 \|_{L^2(\mathbb{R}^3)} \| \langle \cdot \rangle^{\beta} \mathbf{V} \|_{L^2(\mathbb{R}^3)}.
\]
While
\[
\int_{\mathbb{R}^3} \text{div} F_{2R}(\mathbf{x})^{2\beta} \mathbf{V} \, dx \leq \| \langle \cdot \rangle^{\beta} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} \| g_R \langle \cdot \rangle^{\beta} \mathbf{V} \|_{H^\alpha(\mathbb{R}^3)} \\
\leq C \| g_R \|_{W^{0,\infty}(\mathbb{R}^3)} \| \langle \cdot \rangle^{\beta} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} \| \langle \cdot \rangle^{\beta} \mathbf{V} \|_{H^\alpha(\mathbb{R}^3)},
\]
and this estimate implies
\[
\int_{\mathbb{R}^3} \text{div} F_{2R}(\mathbf{x})^{2\beta} \mathbf{V} \, dx \leq C \| \langle \cdot \rangle^{\beta} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} \| \langle \cdot \rangle^{\beta} \mathbf{V} \|_{H^\alpha(\mathbb{R}^3)}.
\]
Recall that
\[
(2.47) \quad -\Delta P = -\text{div} (\mathbf{U} \cdot \nabla \mathbf{V} - F_1 - \text{div} F_2) =: -\Delta P_1 - \Delta P_2 - \Delta P_3.
\]
Since $\beta \in (0,1)$, the weight $|\mathbf{x}|^{2\beta}$ belongs to the $A_2$-class, and then we obtain
\[
(2.48) \quad -\int_{\mathbb{R}^3} g_R(\mathbf{x})^{\beta} \nabla P_2 \mathbf{V}_{\beta,R} \, dx \leq \| \langle \cdot \rangle^{\beta} \nabla P_2 \|_{L^2(\mathbb{R}^3)} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)} \\
\leq C \| \langle \cdot \rangle^{\beta} F_1 \|_{L^2(\mathbb{R}^3)} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)}.
\]
We have by the integration by parts that
\[
-\int_{\mathbb{R}^3} g_R(\mathbf{x})^{\beta} \nabla P_1 \mathbf{V}_{\beta,R} \, dx = \int_{\mathbb{R}^3} (\mathbf{V} \cdot \nabla g_R^2) \langle \mathbf{x} \rangle^{2\beta} P_1 \, dx + \int_{\mathbb{R}^3} (\mathbf{V} \cdot \nabla \langle \mathbf{x} \rangle^{2\beta}) g_R^2 P_1 \, dx.
\]
By the Hölder inequality and the fact the weight $|\mathbf{x}|^{2\beta}$ belongs to $A_2$-class, we have
\[
\int_{\mathbb{R}^3} (\mathbf{V} \cdot \nabla g_R^2) \langle \mathbf{x} \rangle^{2\beta} P_1 \, dx = - (1 + 2\beta) \int_{\mathbb{R}^3} \frac{1}{1 + |\mathbf{x}|^2} \frac{\mathbf{x} \cdot \mathbf{V}}{R^2} \langle \mathbf{x} \rangle^{2\beta} P_1 \, dx \\
\leq \frac{C}{R^{1-\beta}} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)} \| P_1 \|_{L^2(\mathbb{R}^3)} \\
\leq \frac{C}{R^{1-\beta}} \| \mathbf{U} \|_{L^\infty(\mathbb{R}^3)} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)} \| \mathbf{V} \|_{L^2(\mathbb{R}^3)}
\]
and
\[
\int_{\mathbb{R}^3} (\mathbf{V} \cdot \nabla \langle \mathbf{x} \rangle^{2\beta}) g_R^2 P_1 \, dx = 2\beta \int_{\mathbb{R}^3} (\mathbf{V} \cdot \mathbf{x}) \langle \mathbf{x} \rangle^{2(\beta-1)} (g_R^2 P_1) \, dx \\
\leq C \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)} \| P_1 \|_{L^2(\mathbb{R}^3)} \\
\leq C \| \mathbf{U} \|_{L^\infty(\mathbb{R}^3)} \| \mathbf{V}_{\beta,R} \|_{L^2(\mathbb{R}^3)} \| \mathbf{V} \|_{L^2(\mathbb{R}^3)}.
\]
Therefore
\begin{equation}
- \int_{\mathbb{R}^3} g_R(x)^{\beta} \nabla P_1 V_{\beta,R} \, dx \leq C \|U\|_{L^\infty(\mathbb{R}^3)} \|V_{\beta,R}\|_{L^2(\mathbb{R}^3)} \|V\|_{L^2(\mathbb{R}^3)}.
\end{equation}

In the same way as in the proof of (2.49), the term involving \(P_3\) can be bounded as follows:
\begin{equation}
- \int_{\mathbb{R}^3} g_R(x)^{\beta} \nabla P_3 V_{\beta,R} \, dx \leq C \|V_{\beta,R}\|_{L^2(\mathbb{R}^3)} \|P_3\|_{L^2(\mathbb{R}^3)}
\end{equation}
\begin{equation}
\leq C \|V_{\beta,R}\|_{L^2(\mathbb{R}^3)} \|F_2\|_{L^2(\mathbb{R}^3)}.
\end{equation}

Inserting estimates (2.43) - (2.50) into the equality (2.42), we readily get that for large \(R\) and \(\forall \varepsilon > 0\),
\begin{equation}
\|\Lambda^\alpha V_{\beta,R}\|_{L^2(\mathbb{R}^3)}^2 + \frac{1 - \alpha}{\alpha} \|\langle \cdot \rangle^{\beta} V_R\|_{L^2(\mathbb{R}^3)}^2
\end{equation}
\begin{equation}
\leq \|\langle \cdot \rangle^{\beta} F_R\|_{L^2(\mathbb{R}^3)} \|\langle \cdot \rangle^{\beta} V_R\|_{L^2(\mathbb{R}^3)} + C \|V\|_{H^\alpha(\mathbb{R}^3)} \|V_{\beta,R}\|_{H^{\alpha}(\mathbb{R}^3)} + \frac{\varepsilon^{\beta}}{2\alpha} \|V_R\|_{L^2(\mathbb{R}^3)}^2
\end{equation}
\begin{equation}
\leq C \left( \|V\|_{H^\alpha(\mathbb{R}^3)}^2 + \|\langle \cdot \rangle^{\beta} F_1\|_{L^2(\mathbb{R}^3)}^2 + \|\langle \cdot \rangle^{\beta} \text{div} F_2\|_{H^{-\alpha}(\mathbb{R}^3)}^2 + \|F_2\|_{L^2(\mathbb{R}^3)}^2 \right).
\end{equation}

Since \(\beta \in (0, 1)\), we know that the coefficient \(\frac{1 - \alpha}{\alpha} > 0\) and \(g_R(x)^{\beta} V\) meets the requirement of test function (2.42). Moreover, (2.51) becomes
\begin{equation}
\|\Lambda^\alpha V_{\beta,R}\|_{L^2(\mathbb{R}^3)}^2 + \|\langle \cdot \rangle^{\beta} V_R\|_{L^2(\mathbb{R}^3)}^2
\end{equation}
\begin{equation}
\leq C \left( \|V\|_{H^\alpha(\mathbb{R}^3)}^2 + \|\langle \cdot \rangle^{\beta} F_1\|_{L^2(\mathbb{R}^3)}^2 + \|\langle \cdot \rangle^{\beta} \text{div} F_2\|_{H^{-\alpha}(\mathbb{R}^3)}^2 + \|F_2\|_{L^2(\mathbb{R}^3)}^2 \right).
\end{equation}

According to this estimate and the following estimate from Theorem 2.13
\[ \|V\|_{H^{2\alpha}(\mathbb{R}^3)} \leq C(\|F\|_{L^2(\mathbb{R}^3)}), \]
we can show the desired estimate (2.43) by performing the above process again and sending \(R \to \infty\).

\begin{thm}
Let \(s > 0\), \(\alpha \in (0, 1]\), \(F_1 \in H^s(\mathbb{R}^3)\) and \(F_2 \in H^{s+1}(\mathbb{R}^3)\) and \(U \equiv 0\). Assume that \((V, P) \in H^{s+1}(\mathbb{R}^3) \times H^{s+1}(\mathbb{R}^3)\) is a weak solution to the system (2.26).

(i) When \(\alpha \in (0, 1]\), if
\[ \sup_{|\gamma|=|s|} \left( \|\langle \cdot \rangle^{\beta} (I_d + \Lambda^{(s)} D^\gamma) F_1\|_{L^2(\mathbb{R}^3)} + \|\langle \cdot \rangle^{\beta} \Lambda^{(s)} D^\gamma \text{div} F_2\|_{H^{-\alpha}(\mathbb{R}^3)} \right) < \infty, \quad \forall \beta \in (0, \alpha), \]
we have
\begin{equation}
\sup_{|\gamma|=|s|} \|\langle \cdot \rangle^{\beta} \Lambda^{(s)} D^\gamma V\|_{H^{\alpha}(\mathbb{R}^3)} \leq C \|F_2\|_{H^{1+\alpha}(\mathbb{R}^3)} + C \sup_{|\gamma|=|s|} \left( (I_d + \Lambda^{(s)} D^\gamma) F_1\right)_{L^2(\mathbb{R}^3)}^2
\end{equation}
\begin{equation}
+ C \sup_{|\gamma|=|s|} \|\langle \cdot \rangle^{\beta} \Lambda^{(s)} D^\gamma \text{div} F_2\|_{H^{-\alpha}(\mathbb{R}^3)}. \]
\end{thm}
(ii) When $\alpha = 1$, if
$$\sup_{|\gamma|=[s]} \left( \left\| \langle \cdot \rangle (I_d + \Lambda^{(s)} D^\gamma) F_1 \right\|_{L^2(\mathbb{R}^3)} + \left\| \langle \cdot \rangle \Lambda^{(s)} D^\gamma \text{div} F_2 \right\|_{H^{-1}(\mathbb{R}^3)} \right) < \infty,$$

we have
$$\sup_{|\gamma|=[s]} \left\| \langle \cdot \rangle \Lambda^{(s)} D^\gamma V \right\|_{H^1(\mathbb{R}^3)} \leq C \left\| F_2 \right\|_{H^{1+s}(\mathbb{R}^3)} + C \sup_{|\gamma|=[s]} \left( \left\| (I_d + \Lambda^{(s)} D^\gamma) F_1 \right\|_{L^2(\mathbb{R}^3)} + \sup_{|\gamma|=[s]} \left\| \langle \cdot \rangle \Lambda^{(s)} D^\gamma \text{div} F_2 \right\|_{H^{-1}(\mathbb{R}^3)} \right).$$

(2.53)

Proof. First of all, we consider the following approximate system in the whole space
$$(-\Delta)^{\alpha} V_N - \frac{2\alpha-1}{2\alpha} V_N - \frac{1}{2\alpha} x \cdot \nabla V_N + \nabla P_N = S_N F,$$
$$\text{div} V_N = 0.$$  

(2.54)

Since $F \in H^s(\mathbb{R}^3)$, we know that $S_N F \in H^\infty(\mathbb{R}^3) := \cap_{s \geq 0} H^s(\mathbb{R}^3)$ belongs to the Schwarz class. Moreover, following the proof of Theorem 2.13, we can get $V_N \in H^\infty(\mathbb{R}^3)$ is a smooth solution to the system (2.54). Performing differential operator $D^\gamma$ with $|\gamma| = [s]$ to the approximate system yields
$$(-\Delta)^{\alpha} D^\gamma V_N - \frac{2\alpha-1}{2\alpha} D^\gamma V_N - \frac{1}{2\alpha} x \cdot \nabla D^\gamma V_N + \nabla D^\gamma P_N$$
$$= D^\gamma S_N F + \sum_{|\gamma|=|s|-1} C_\gamma^{(s)} (D x) \cdot \nabla D^\gamma V_N,$$
$$\text{div} D^\gamma V_N = 0.$$  

(2.55)

Denoting $\{s\} = s - [s]$, one easily verifies that
$$(-\Delta)^{\alpha} \Lambda^{(s)} D^\gamma V_N - \frac{2\alpha-1}{2\alpha} \Lambda^{(s)} D^\gamma V_N - \frac{1}{2\alpha} \Lambda^{(s)} (x \cdot \nabla D^\gamma V_N)$$
$$= - \nabla \Lambda^{(s)} D^\gamma P_N + \Lambda^{(s)} D^\gamma S_N F + \sum_{|\gamma|=|s|-1} C_\gamma^{(s)} \Lambda^{(s)} ((D x) \cdot \nabla D^\gamma V_N),$$
$$\text{div} \Lambda^{(s)} D^\gamma V_N = 0.$$  

(2.56)

In terms of Fourier transform, we easily find that
$$\mathcal{F} \left( \Lambda^{(s)} (x \cdot \nabla D^\gamma V_N) \right) = |\xi|^{\{s\}} \text{div}_x (\xi \mathcal{F} (D^\gamma V_N)(\xi))$$
$$= \text{div}_x \left( \xi |\xi|^{\{s\}} \mathcal{F} (D^\gamma V_N)(\xi) - \{s\} |\xi|^{\{s\}} \mathcal{F} (D^\gamma V_N)(\xi) \right).$$

Plugging this equality into (2.56) leads to
$$(-\Delta)^{\alpha} \Lambda^{(s)} D^\gamma V_N - \frac{2\alpha-1}{2\alpha} \Lambda^{(s)} D^\gamma V_N - \frac{1}{2\alpha} x \cdot \nabla \Lambda^{(s)} D^\gamma V_N + \nabla \Lambda^{(s)} D^\gamma P_N$$
$$= \Lambda^{(s)} D^\gamma S_N F + \sum_{|\gamma|=|s|-1} C_\gamma^{(s)} \Lambda^{(s)} ((D x) \cdot \nabla D^\gamma V_N) + \frac{\{s\}}{2\alpha} \Lambda^{(s)} D^\gamma V_N,$$
$$\text{div} \Lambda^{(s)} D^\gamma V_N = 0.$$  

(2.57)
In particular, when \( s \in (0, 1) \), problem (2.57) can be reduced to

\[
\begin{cases}
(-\Delta)^s \Lambda^s \mathbf{V}_N - \frac{2\alpha - 1}{2\alpha} \Lambda^s \mathbf{V}_N - \frac{1}{2\alpha} \mathbf{x} \cdot \nabla \Lambda^s \mathbf{V}_N + \nabla \Lambda^s P_N \\
= \Lambda^s \mathbf{S}_N \mathbf{F} + s \Lambda^s \mathbf{V}_N,
\end{cases}
\]

\[
\text{div} \Lambda^s \mathbf{V}_N = 0.
\]

(2.58)

Furthermore, we apply (2.40) to \( \Lambda^s \mathbf{V}_N \) in (2.58) to get

\[
\| \langle \cdot \rangle^\beta \Lambda^s \mathbf{V}_N \|_{H^\alpha(\mathbb{R}^3)} \leq C \| \Lambda^s \mathbf{V}_N \|_{H^\alpha(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta S_N \Lambda^s \mathbf{F}_1 \|_{L^2(\mathbb{R}^3)} + C \| \Lambda^s \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}
\]

\[
+ C \| \langle \cdot \rangle^\beta S_N \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta \Lambda^s \mathbf{V}_N \|_{L^2(\mathbb{R}^3)}
\]

\[
+ C \| \langle \cdot \rangle^\beta [\Lambda^s, \mathbf{U} \cdot \nabla] \mathbf{V}_N \|_{L^2(\mathbb{R}^3)}.
\]

(2.59)

Since

\[
\| |x|^\beta S_N \Lambda^s \mathbf{F}_1 | = |x|^\beta \int_{\mathbb{R}^3} h_N(x - y) (\Lambda^s \mathbf{F}_1)(y) \, dy \\
\leq \int_{\mathbb{R}^3} |x - y|^\beta |h_N| (x - y) |\Lambda^s \mathbf{F}_1| (y) \, dy \\
+ \int_{\mathbb{R}^3} |h_N| (x - y) |y|^\beta |\Lambda^s \mathbf{F}_1| (y) \, dy,
\]

we have

\[
\| \langle \cdot \rangle^\beta S_N \Lambda^s \mathbf{F}_1 \|_{L^2(\mathbb{R}^3)} \leq C 2^{-\beta N} \| \Lambda^s \mathbf{F}_1 \|_{L^2(\mathbb{R}^3)} + C \| \cdot |^\beta \Lambda^s \mathbf{F}_1 \|_{L^2(\mathbb{R}^3)}.
\]

Since

\[
S_N \text{div} \Lambda^s \mathbf{F}_2 = \text{div} \Lambda^s \mathbf{F}_2 + \sum_{k \geq N} \hat{\Delta}_k \text{div} \Lambda^s \mathbf{F}_2,
\]

we have by the triangle inequality that

\[
\| \langle \cdot \rangle^\beta S_N \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} \leq \| \langle \cdot \rangle^\beta \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \sum_{k \geq N} \| \langle \cdot \rangle^\beta \hat{\Delta}_k \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)}.
\]

Furthermore, we calculate

\[
\sum_{k \geq N} \| \langle \cdot \rangle^\beta \hat{\Delta}_k \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)}
\]

\[
\leq \sum_{k \geq N} \| \hat{\Delta}_k \langle \cdot \rangle^\beta \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \sum_{k \geq N} \| \hat{\Delta}_k, \langle \cdot \rangle^\beta \| \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)}
\]

\[
\leq C \| \langle \cdot \rangle^\beta \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \sup_{k \geq N} \| \hat{\Delta}_k, \langle \cdot \rangle^\beta \| \text{div} \Lambda^s \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}
\]

\[
\leq C \| \langle \cdot \rangle^\beta \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + C \sup_{k \geq N} 2^{-k} \| \text{div} \Lambda^s \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}.
\]

Thus, it follows that

\[
\| \langle \cdot \rangle^\beta S_N \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \| \langle \cdot \rangle^\beta \text{div} \Lambda^s \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + C \| \text{div} \Lambda^s \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}.
\]

(2.61)
Inserting (2.60) and (2.61) into (2.59) yields
\begin{equation}
\|\langle \cdot \rangle^\beta \Lambda^s V_N\|_{H^\alpha(R^3)} \leq C\|\Lambda^s V_N\|_{H^\alpha(R^3)} + C\|\langle \cdot \rangle^\beta \Lambda^s F_1\|_{L^2(R^3)} + C\|\Lambda^s F_2\|_{H^1(R^3)} + C\|\langle \cdot \rangle^\beta \Lambda^s V_N\|_{L^2(R^3)}.
\end{equation}
(2.62)

Since $F_1 \in H^s(R^3)$ and $F_2 \in H^{s+1}(R^3)$, we get from Theorem 2.13 and Theorem 2.14 that
\begin{equation}
\|V_N\|_{H^{s+2\alpha}(R^3)} + \|\nabla P_N\|_{H^s(R^3)} \leq C\|F_1\|_{H^s(R^3)} + C\|F_2\|_{H^{s+1}(R^3)}.
\end{equation}
(2.63)
and
\begin{equation}
\|\langle \cdot \rangle^\beta V_N\|_{H^\alpha(R^3)} \leq C\left(\|V_N\|_{H^{s}(R^3)} + \|\langle \cdot \rangle^\beta F_1\|_{L^2(R^3)} + \|\langle \cdot \rangle^\beta \nabla F_2\|_{H^{-\alpha}(R^3)} + \|F_2\|_{L^2(R^3)}\right).
\end{equation}
(2.64)

For each $s \in (0,1)$, we have by Lemma 2.11 that
\begin{equation}
\|\langle \cdot \rangle^\beta \Lambda^s V_N\|_{L^2(R^3)} \leq C\|\Lambda^s \langle \cdot \rangle^\beta V_N\|_{L^2(R^3)} + C\|\langle \cdot \rangle^\beta V_N\|_{L^2(R^3)}.
\end{equation}
(2.65)

Inserting (2.63), (2.64) and (2.65) into (2.62), we obtain that for each $s \in [0,\alpha]$, \begin{equation}
\|\langle \cdot \rangle^\beta \Lambda^s V_N\|_{H^\alpha(R^3)} \leq C\|\langle \cdot \rangle^\beta (I_d + \Lambda^s)F_1\|_{L^2(R^3)} + C\|\Lambda^s F_2\|_{H^1(R^3)} + C\|\langle \cdot \rangle^\beta (I_d + \Lambda^s) \nabla F_2\|_{H^{-\alpha}(R^3)}.
\end{equation}
(2.66)

For each $s \in (\alpha,\min\{1,2\alpha\})$, we have by Lemma 2.11 again that
\begin{equation}
\|\langle \cdot \rangle^\beta \Lambda^s V_N\|_{L^2(R^3)} \leq C\|\Lambda^{s-\alpha} \langle \cdot \rangle^\beta \Lambda^s V_N\|_{L^2(R^3)} + C\|\langle \cdot \rangle^\beta V_N\|_{L^2(R^3)}.
\end{equation}
(2.67)

Plugging (2.63), (2.66) and (2.67) into (2.62) leads to that for each $s \in (\alpha,\min\{1,2\alpha\})$, \begin{equation}
\|\langle \cdot \rangle^\beta \Lambda^s V_N\|_{H^\alpha(R^3)} \leq C\|\langle \cdot \rangle^\beta (I_d + \Lambda^s)F_1\|_{L^2(R^3)} + C\|\Lambda^s F_2\|_{H^1(R^3)} + C\|\langle \cdot \rangle^\beta (I_d + \Lambda^s) \nabla F_2\|_{H^{-\alpha}(R^3)}.
\end{equation}

Proceeding step by step, we can show that for each $s \in (0,1)$, \begin{equation}
\|\langle \cdot \rangle^\beta \Lambda^s V_N\|_{H^\alpha(R^3)} \leq C\|\langle \cdot \rangle^\beta (I_d + \Lambda^s)F_1\|_{L^2(R^3)} + C\|\Lambda^s F_2\|_{H^1(R^3)} + C\|\langle \cdot \rangle^\beta (I_d + \Lambda^s) \nabla F_2\|_{H^{-\alpha}(R^3)}.
\end{equation}
(2.68)

Next, we consider the case where $s \geq 1$. Suppose there exists an integer $k \in \mathbb{N}^+$ such that for $0 \leq s < k$, \begin{equation}
sup_{|\gamma|=|s|} \|\langle \cdot \rangle^\beta \Lambda^{\{s\} D^\gamma} V_N\|_{H^\alpha(R^3)} \leq C\|\langle \cdot \rangle^\beta (I_d + \Lambda^{\{s\} D^\gamma})F_1\|_{L^2(R^3)} + C\|\Lambda^s F_2\|_{H^1(R^3)} + C\|\langle \cdot \rangle^\beta (I_d + \Lambda^{\{s\} D^\gamma}) \nabla F_2\|_{H^{-\alpha}(R^3)}.
\end{equation}
(2.69)
Employing (2.40) in the system (2.55), we readily have
\[
\sup_{|\gamma|=k} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{H^a(\mathbb{R}^3)} 
\leq C \sup_{|\gamma|=k} \left( \| D^{\gamma} V_N \|_{H^\alpha(\mathbb{R}^3)} + \| \langle \cdot \rangle^{\beta} S_N D^{\gamma} F_1 \|_{L^2(\mathbb{R}^3)} + \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{L^2(\mathbb{R}^3)} \right)
+ C \sup_{|\gamma|=k} \left( \| \langle \cdot \rangle^{\beta} S_N D^{\gamma} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \| D^{\gamma} F_2 \|_{H^1(\mathbb{R}^3)} \right).
\]

From (2.60), it is easy to infer that
\[
\| \langle \cdot \rangle^{\beta} S_N D^{\gamma} F_1 \|_{L^2(\mathbb{R}^3)} \leq C 2^{-\beta \alpha} \| D^{\gamma} F_1 \|_{L^2(\mathbb{R}^3)} + \| \cdot \|^{\beta} D^{\gamma} F_1 \|_{L^2(\mathbb{R}^3)}.
\]

The same way as deriving (2.61) enables us to conclude that
\[
\| \langle \cdot \rangle^{\beta} S_N D^{\gamma} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \| \langle \cdot \rangle^{\beta} D^{\gamma} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + C \| D^{\gamma} \text{div} F_2 \|_{L^2(\mathbb{R}^3)}.
\]

Since \( \alpha \in (0,1) \) and \( k \in \mathbb{N}^+ \), we have by the Leibniz rule that for each \( M \in \mathbb{N}^+ \),
\[
\| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{L^2(\mathbb{R}^3)} \leq \sum_{k \geq M} \| \hat{A}_k \left( \langle \cdot \rangle^{\beta} D^{\gamma} V_N \right) \|_{L^2(\mathbb{R}^3)} + \| S_M \left( \langle \cdot \rangle^{\beta} D^{\gamma} V_N \right) \|_{L^2(\mathbb{R}^3)}
\leq C 2^{-M \alpha} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{H^\alpha(\mathbb{R}^3)} + \sup_{|\gamma|=k-1} \| S_M D \left( \langle \cdot \rangle^{\beta} D^{\gamma} V_N \right) \|_{L^2(\mathbb{R}^3)}
+ \sup_{|\gamma|=k-1} \| S_M \left( D \langle \cdot \rangle^{\beta} D^{\gamma} V_N \right) \|_{L^2(\mathbb{R}^3)}.
\]

which implies
\[
\sup_{|\gamma|=k} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{L^2(\mathbb{R}^3)}
\leq C \sup_{|\gamma|=k-1} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{L^2(\mathbb{R}^3)} + C 2^{-M \alpha} \sup_{|\gamma|=k} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{H^\alpha(\mathbb{R}^3)}.
\]

Inserting (2.71)–(2.73) into (2.70) and then taking the suitable \( M \), we get
\[
\sup_{|\gamma|=k} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{H^\alpha(\mathbb{R}^3)} \leq C \sup_{|\gamma|=k} \left( \| D^{\gamma} V_N \|_{H^\alpha(\mathbb{R}^3)} + \| \langle \cdot \rangle^{\beta} D^{\gamma} F_1 \|_{L^2(\mathbb{R}^3)} \right)
+ C \sup_{|\gamma|=k} \left( \| \langle \cdot \rangle^{\beta} D^{\gamma} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \| D^{\gamma} F_2 \|_{H^1(\mathbb{R}^3)} \right)
+ C \sup_{|\gamma|=k-1} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{L^2(\mathbb{R}^3)}.
\]

With estimates (2.63) and (2.73) in hand, the estimate (2.74) becomes
\[
\sup_{|\gamma|=k} \| \langle \cdot \rangle^{\beta} D^{\gamma} V_N \|_{H^\alpha(\mathbb{R}^3)} \leq C \sup_{|\gamma|=k} \left( \| \langle \cdot \rangle^{\beta} (I_d + D^{\gamma}) F_1 \|_{L^2(\mathbb{R}^3)} + \| D^{\gamma} F_2 \|_{H^1(\mathbb{R}^3)} \right)
+ C \sup_{|\gamma|=k} \| \langle \cdot \rangle^{\beta} D^{\gamma} \text{div} F_2 \|_{H^{-\alpha}(\mathbb{R}^3)}.
\]
Employing (2.40) in the system (2.57), we readily have that for each \( s \in (k, k + 1) \),
\[
\sup_{|\gamma|=k} \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{H^0(\mathbb{R}^3)}
\leq C \sup_{|\gamma|=k} \left( \|\Lambda^{(s)} D^\gamma V_N \|_{H^0(\mathbb{R}^3)} + \|\langle \cdot \rangle \beta S_N \Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right)
\leq C \sup_{|\gamma|=k} \left( \sum_{|\gamma|=k} \left( \|\langle \cdot \rangle \beta S_N \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \|\Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right) \right)
\leq C \sup_{|\gamma|=k} \left( \sum_{|\gamma|=k} \left( \|\langle \cdot \rangle \beta S_N \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \|\Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right) \right).
\]
By mimicking (2.60), we can show
\[
\|\langle \cdot \rangle \beta S_N \Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \leq C 2^{-\beta N} \|\Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} + C \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{L^2(\mathbb{R}^3)}.
\]
From (2.61), we get
\[
\|\langle \cdot \rangle \beta S_N \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + C \|\Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{L^2(\mathbb{R}^3)}.
\]
In the similar method as in proof of (2.73), we infer that for each \( k \in \mathbb{N}^+ \),
\[
\sup_{|\gamma|=k} \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{L^2(\mathbb{R}^3)}
\leq C \sup_{|\gamma|=k} \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{L^2(\mathbb{R}^3)} + C 2^{-M_s} \sup_{|\gamma|=k} \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{H^0(\mathbb{R}^3)}.
\]
Inserting (2.77)-(2.79) into (2.70) and then taking the suitable integer \( M \), we immediately get that for each \( s \in (k, k + 1) \),
\[
\sup_{|\gamma|=k} \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{H^0(\mathbb{R}^3)}
\leq C \sup_{|\gamma|=k} \left( \|\Lambda^{(s)} D^\gamma V_N \|_{H^0(\mathbb{R}^3)} + \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right)
\leq C \sup_{|\gamma|=k} \left( \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \|\Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right)
\leq C \sup_{|\gamma|=k} \left( \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \|\Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right).
\]
Plugging (2.63) and (2.73) into the above inequality leads to that for each \( s \in (k, k + 1) \),
\[
\sup_{|\gamma|=k} \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{H^0(\mathbb{R}^3)}
\leq C \sup_{|\gamma|=k} \left( \|\langle \cdot \rangle \beta (I_d + \Lambda^{(s)} D^\gamma) F_1 \|_{L^2(\mathbb{R}^3)} + \|\Lambda^{(s)} D^\gamma F_1 \|_{H^1(\mathbb{R}^3)} \right)
\leq C \sup_{|\gamma|=k} \left( \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \|\Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right).
\]
Using the estimate (2.75), we get from (2.80) that for each \( s \in [k, k + 1] \),
\[
\sup_{|\gamma|=k} \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma V_N \|_{H^0(\mathbb{R}^3)}
\leq C \sup_{|\gamma|=k} \left( \|\langle \cdot \rangle \beta \Lambda^{(s)} D^\gamma \nabla \div F_2 \|_{H^{-\alpha}(\mathbb{R}^3)} + \|\Lambda^{(s)} D^\gamma F_1 \|_{L^2(\mathbb{R}^3)} \right).
\]
By induction, we eventually get that for each $s$, there exists a constant $C$ such that

$$\sup_{|\gamma|=k} \left\| \langle \cdot \rangle^\beta (I_d + \Lambda^s D^\gamma) F_1 \right\|_{L^2(\mathbb{R}^3)} + C \sup_{|\gamma|=k} \left\| \Lambda^s D^\gamma F_2 \right\|_{H^1(\mathbb{R}^3)}$$

$$+ C \sup_{|\gamma|=k} \left\| \langle \cdot \rangle^\beta \Lambda^s D^\gamma \text{div} \ F_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)}.$$ 

By induction, we eventually get that for each $s \geq 0$,

$$\sup_{|\gamma|=|s|} \left\| \langle \cdot \rangle^\beta \Lambda^s D^\gamma V_N \right\|_{H^\alpha(\mathbb{R}^3)}$$

$$\leq C \sup_{|\gamma|=|s|} \left\| \langle \cdot \rangle^\beta (I_d + \Lambda^s D^\gamma) F_1 \right\|_{L^2(\mathbb{R}^3)} + C \sup_{|\gamma|=|s|} \left\| \Lambda^s D^\gamma F_2 \right\|_{H^1(\mathbb{R}^3)}$$

$$+ C \sup_{|\gamma|=|s|} \left\| \langle \cdot \rangle^\beta \Lambda^s D^\gamma \text{div} \ F_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)}.$$ 

Taking $N \to \infty$ implies the desired estimate (2.52). 

In the same way as in the proof of (2.52), we can show the desired estimate (2.53) of Theorem 2.15. $\square$

3. High Regularity for Weak Solutions to the Leray Problem

In this section, we are going to show the high regularity for weak solutions to the perturbed Leray problem (1.1) and the perturbed Leray problem (1.3). For simplicity, we just need to consider the following hybrid problem in the whole space

$$\begin{cases} 
(-\Delta)^\alpha V - \frac{2\alpha - 1}{2\alpha} V - \frac{1}{2\alpha} x \cdot \nabla V + \nabla P = -V \cdot \nabla V + LU_0(V) + \text{div} \, F, \\
\text{div} \, V = 0, 
\end{cases}$$

where

$$LU_0 \, V = -U_0 \cdot \nabla V - V \cdot \nabla U_0.$$ 

Let $F_1 = -V \cdot \nabla U_0 + \text{div} \, F$. Then the system (3.1) can be rewritten as

$$\begin{cases} 
(-\Delta)^\alpha V - \frac{2\alpha - 1}{2\alpha} V - \frac{1}{2\alpha} x \cdot \nabla V + (U_0 + V) \cdot \nabla V + \nabla P = F_1, \\
\text{div} \, V = 0. 
\end{cases}$$

3.1. High Regularity for Weak Solutions to the Perturbed Leray Problem

This subsection is devoted to proving $L^p$-type elliptic regularity for the weak solutions to the problem (3.1).

**Theorem 3.1.** Assume $\alpha \in [5/6, 1]$, $p \in [2, +\infty)$, $\text{div} \, F \in L^2(\mathbb{R}^3) \cap \dot{B}^{0}_{p,p}(\mathbb{R}^3)$ and $\sigma \in C^{1,0}(\mathbb{S}^2)$. Let $(V, P) \in H^{\alpha}(\mathbb{R}^3) \times L^{3\alpha/2}(\mathbb{R}^3)$ be a weak solution to the system (3.1). Then there exists a constant $C = C_{p,\sigma} > 0$ such that

$$\|U\|_{\dot{B}^{\frac{3}{2\alpha}}_{p,p}(\mathbb{R}^3)} + \|P\|_{\dot{B}^{1}_{p,p}(\mathbb{R}^3)} \leq C_{p,\sigma} \left(1 + \|\text{div} \, F\|_{\dot{B}^{0}_{p,p}(\mathbb{R}^3)} + \|\text{div} \, F\|_{L^2(\mathbb{R}^3)}\right).$$
Proof. Let \( V_k = J_k V \) be a sequence of smooth functions approximating \( V \), and \( W_k \in H^{\alpha} (R^3) \) be a weak solution to the corresponding linearized equations in the whole space
\[
(-\Delta)^\alpha W_k - \frac{2\alpha - 1}{2\alpha} W_k - \frac{1}{2\alpha} x \cdot \nabla W_k + J_k (U_0 + V_k) \cdot \nabla J_k W_k + \nabla P_k = F_1,
\]
\( \text{div } W_k = 0. \)

Since \( U_0 \in L^2 (R^3) \), \( V \in H^{\alpha} (R^3) \) and \( \nabla U_0 \in L^\infty (R^3) \), we readily have that \( F_1 \in L^2 (R^3) \), moreover, we get the following estimate by the standard \( L^2 \)-estimate
\[
\| W_k \|_{H^{\alpha} (R^3)} \leq C_\alpha \| F_1 \|_{L^2 (R^3)}.
\]
Taking \( s = 0 \) in Theorem 2.13, we find that weak solution \( (W_k, P_k) \) fulfils
\[
\| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} + \| P_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} \leq C_p \left( \| \text{div} (U_0 + V_k) \|_{L^\infty (R^3)} + \| F_1 \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} + \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} \right).
\]
Let us first consider the case where \( p \in [2, 9/2) \). Since \( V \in H^{\alpha} (R^3) \) with \( \alpha \in [5/6, 1] \), we obtain by Lemma 2.7 that for each \( p \in [2, 9/2) \),
\[
\| \text{div} (V_k \otimes J_k W_k) \|_{\dot{B}^{\alpha}_{p,p} (R^3)} \leq \varepsilon \| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} + C \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)}.
\]
Since \( \alpha \geq \frac{5}{6} \), one has by using the interpolation inequality and the Young inequality that
\[
\| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} \leq \| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} \leq \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} + C \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)}.
\]
This estimate together with (3.7) leads to
\[
\| \text{div} (V_k \otimes J_k W_k) \|_{\dot{B}^{\alpha}_{p,p} (R^3)} \leq \varepsilon \| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} + C \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)}.
\]
Using the Leibniz estimate, we see that
\[
\| \text{div} (U_0 \otimes J_k W_k) \|_{\dot{B}^{0}_{p,p} (R^3)} \leq C \| U_0 \|_{L^\infty (R^3)} \| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} + C \| \nabla U_0 \|_{L^\infty (R^3)} \| W_k \|_{\dot{B}^{0}_{p,p} (R^3)}.
\]
Since \( \alpha \in [5/6, 1] \), we get by the interpolation inequality and the Young inequality that
\[
\| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} \leq C \| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} + C \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)}.
\]
Plugging this estimate into (3.9) leads to
\[
\| \text{div} (U_0 \otimes J_k W_k) \|_{\dot{B}^{\alpha}_{p,p} (R^3)} \leq \varepsilon \| W_k \|_{\dot{B}^{\alpha}_{p,p} (R^3)} + C \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)}.
\]
Inserting estimates (3.8) and (3.10) into (3.6), we have
\[
\| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} + \| P_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} \leq C_p \left( \| F_1 \|_{\dot{B}^{0}_{p,p} (R^3)} + C \| W_k \|_{\dot{B}^{2\alpha}_{p,p} (R^3)} \right) + \varepsilon C \| U_0 \|_{\dot{B}^{0}_{p,p} (R^3)}.
\]
Since $p \geq 2$, we obtain the following sharp inequality
\[
\|W_k\|_{\dot{B}^{0}_{p,1}(\mathbb{R}^3)} \leq \sum_{q \leq N} \|\hat{\Delta}_q W_k\|_{L^p(\mathbb{R}^3)} + \sum_{q > N} \|\hat{\Delta}_q W_k\|_{L^p(\mathbb{R}^3)}
\]
\[\leq C 2^{\frac{N-p}{p}} \|W_k\|_{L^2(\mathbb{R}^3)} + 2^{-2\alpha N} \|W_k\|_{\dot{B}^{2\alpha}_{p,\infty}(\mathbb{R}^3)},
\]
where $N$ is an integer to be chosen later.

Plugging this sharp inequality into estimate (3.11) allows us to conclude
\[
\|W_k\|_{\dot{B}^{2\alpha}_{p,\infty}(\mathbb{R}^3)} + \|P_k\|_{\dot{B}^{1}_{p,\infty}(\mathbb{R}^3)} \leq C_p \left( \|F_1\|_{\dot{B}^{0}_{p,1}(\mathbb{R}^3)} + 2^{-2\alpha N} \|W_k\|_{L^2(\mathbb{R}^3)} \right) + C_p \left( \varepsilon + C_\varepsilon 2^{-\frac{4N}{p}} \right) \|W_k\|_{L^2(\mathbb{R}^3)}.
\]

Choosing a suitable number $\varepsilon$ and a sufficiently large integer $N$ in (3.13), one has
\[
\|W_k\|_{\dot{B}^{2\alpha}_{p,\infty}(\mathbb{R}^3)} + \|P_k\|_{\dot{B}^{1}_{p,\infty}(\mathbb{R}^3)} \leq C_p \left( \|F_1\|_{\dot{B}^{0}_{p,1}(\mathbb{R}^3)} + \|W_k\|_{L^2(\mathbb{R}^3)} \right).
\]

Plugging (3.15) into (3.11), we immediately obtain
\[
\|W_k\|_{\dot{B}^{2\alpha}_{p,\infty}(\mathbb{R}^3)} + \|P_k\|_{\dot{B}^{1}_{p,\infty}(\mathbb{R}^3)} \leq C_p \left( \|F_1\|_{\dot{B}^{0}_{p,1}(\mathbb{R}^3)} + \|W_k\|_{L^2(\mathbb{R}^3)} \right).
\]

According to $F_1 = -V \cdot \nabla U_0 + \text{div} F$, we have by the H"older inequality that
\[
\|F_1\|_{L^2(\mathbb{R}^3)} \leq \|\nabla U_0\|_{L^\infty(\mathbb{R}^3)} \|V\|_{L^2(\mathbb{R}^3)} + \|\text{div} F\|_{L^2(\mathbb{R}^3)},
\]
and by the Leibniz estimate that for each $p \in [2, 9/2),$
\[
\|F_1\|_{\dot{B}^{p/2}_{p,1}(\mathbb{R}^3)} \leq \|\nabla U_0\|_{L^\infty(\mathbb{R}^3)} \|V\|_{L^p(\mathbb{R}^3)} + \|\text{div} F\|_{L^p(\mathbb{R}^3)}
\leq C \|\nabla U_0\|_{L^\infty(\mathbb{R}^3)} \|V\|_{H^{\sigma}(\mathbb{R}^3)} + \|\text{div} F\|_{L^p(\mathbb{R}^3)}.
\]

Plugging (3.16) and (3.17) into (3.15), we immediately have that for each $p \in [2, 9/2),$
\[
\|W_k\|_{\dot{B}^{2\alpha}_{p,\infty}(\mathbb{R}^3)} + \|P_k\|_{\dot{B}^{1}_{p,\infty}(\mathbb{R}^3)} \leq C_{p, \sigma} \left( 1 + \|F_1\|_{\dot{B}^{0}_{p,1}(\mathbb{R}^3)} + \|\text{div} F\|_{L^2(\mathbb{R}^3)} \right).
\]

The above uniform estimate (3.18) uniformly in $k$ enables us to infer that subsequences of $W_k$ and $P_k$ converge locally weakly in $H^{2\alpha}(\mathbb{R}^3)$ and in $H^1(\mathbb{R}^3)$ to a solution $(W, P)$ of Cauchy problem
\[
(\Delta)^\sigma W - \frac{2\alpha - 1}{2\alpha} \div W - \frac{1}{2\alpha} \mathbf{x} \cdot \nabla W + (U_0 + V) \cdot \nabla W + \nabla P = F_1
\]
\[\div W = 0.
\]
Since $V$ and $W$ are both weak solutions to the equations (3.19), we get by uniqueness that $V = W$, and we have
\[
\|V\|_{\dot{B}^{2\alpha}_{p,\infty}(\mathbb{R}^3)} + \|P\|_{\dot{B}^{1}_{p,\infty}(\mathbb{R}^3)} \leq C_{p, \sigma} \left( 1 + \|\text{div} F\|_{\dot{B}^{0}_{p,1}(\mathbb{R}^3)} + \|\text{div} F\|_{L^2(\mathbb{R}^3)} \right).
\]

Next we consider the case where $\text{div} F \in L^2(\mathbb{R}^3) \cap \dot{B}^{0}_{p,p}(\mathbb{R}^3)$ with $\frac{3}{2} \leq p < \infty$. By the interpolation theorem, we have that for each $r \in (2, p),$
\[
\|\text{div} F\|_{\dot{B}^{r}_{p,r}(\mathbb{R}^3)} \leq \|\text{div} F\|_{L^2(\mathbb{R}^3)} \|\text{div} F\|_{\dot{B}^{0}_{p,p}(\mathbb{R}^3)}^{\frac{r-2}{r}} \|\text{div} F\|_{L^2(\mathbb{R}^3)}^{\frac{2}{r}},
\]
which implies that $\text{div } \mathbf{F} \in L^2(\mathbb{R}^3) \cap \dot{B}^{\theta}_{p,p}(\mathbb{R}^3)$ for each $p \in [2, 9/2)$. We know from (3.20) that for each $p \in [2, 9/2)$,

$$
\mathbf{V} \in H^{2\alpha}(\mathbb{R}^3) \cap \dot{B}^{2\alpha}_{p,p}(\mathbb{R}^3) \quad \text{with } \alpha \in [5/6, 1].
$$

Hence, by using the Sobolev embedding theorem, we have $\mathbf{V} \in L^\infty(\mathbb{R}^3)$ and

$$
\nabla \mathbf{V} \in L^p(\mathbb{R}^3) \quad \text{for each } p \in [2, \infty).
$$

These facts imply the Leibniz estimate

$$
\|(\mathbf{V} + \mathbf{U}_0) \cdot \nabla \mathbf{V}\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} + \|\mathbf{V} \cdot \nabla \mathbf{U}_0\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} + \|\text{div } \mathbf{F}\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)}
\leq C \left( \|\mathbf{V}\|_{L^\infty(\mathbb{R}^3)} + \|\mathbf{U}_0\|_{L^\infty(\mathbb{R}^3)} \right) \|\nabla \mathbf{V}\|_{L^p(\mathbb{R}^3)} + C \|\mathbf{V}\|_{L^\infty(\mathbb{R}^3)} \|\nabla \mathbf{U}_0\|_{L^p(\mathbb{R}^3)}
+ \|\text{div } \mathbf{F}\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} < \infty.
\tag{3.21}
$$

Since $\mathbf{V}$ is a weak solution to the system (3.1) with $\alpha \in [5/6, 1]$, we have by Theorem 2.13 that

$$
\|\mathbf{V}\|_{\dot{B}^{2\alpha}_{p,p}(\mathbb{R}^3)} + \|\mathbf{P}\|_{\dot{B}^{1\alpha}_{p,p}(\mathbb{R}^3)}
\leq C_p \left( \|\text{div } (\mathbf{V} + \mathbf{U}_0) \otimes \mathbf{V}\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} + \|\mathbf{F}\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} + \|\mathbf{W}_k\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} \right). \tag{3.22}
$$

Inserting (3.12) and (3.21) into (3.22) and then choosing the suitable $N$, we finally obtain Theorem 3.1 for the case where $p > \frac{9}{2}$.

**Theorem 3.2.** Assume $\alpha \in [5/6, 1]$, $s \geq 0$, $p \in [2, +\infty)$, $\text{div } \mathbf{F} \in L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,p}(\mathbb{R}^3)$ and $\sigma \in C^{1,0}(\mathbb{S}^2)$. Let $\mathbf{V}$ be a weak solution to the system (3.1). Then there exists a constant $C$ such that

$$
\|\mathbf{V}\|_{\dot{B}^{2s+2\alpha}_{p,p}(\mathbb{R}^3)} + \|\mathbf{P}\|_{\dot{B}^{s+1\alpha}_{p,p}(\mathbb{R}^3)} \leq C \left( p, \sigma, \|\text{div } \mathbf{F}\|_{L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,p}(\mathbb{R}^3)} \right). \tag{3.23}
$$

**Proof.** Applying Theorem 2.13 to the equations (3.1), we get that for all $s > 0$,

$$
\|\mathbf{V}\|_{\dot{B}^{s+2\alpha}_{p,p}(\mathbb{R}^3)} + \|\mathbf{P}\|_{\dot{B}^{s+1\alpha}_{p,p}(\mathbb{R}^3)} \leq C_p \left( \|\mathbf{F}\|_{\dot{B}^{s}_{p,p}(\mathbb{R}^3)} + \|\mathbf{V}\|_{\dot{B}^{s}_{p,p}(\mathbb{R}^3)} \right),
\tag{3.24}
$$

where

$$
\mathbf{F} = -\text{div } (\mathbf{V} + \mathbf{U}_0) \otimes \mathbf{V} - \text{div } (\mathbf{V} \otimes \mathbf{U}_0) + \text{div } \mathbf{F}.
$$

Since $\text{div } \mathbf{F} \in L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,p}(\mathbb{R}^3)$ with $s \geq 0$, we have by the interpolation theorem that there exists $\theta = \frac{2ps}{2ps + 3(p-2)}$ such that

$$
\|\text{div } \mathbf{F}\|_{\dot{B}^\theta_{p,p}(\mathbb{R}^3)} \leq \|\text{div } \mathbf{F}\|_{L^2(\mathbb{R}^3)}^{\theta} \|\text{div } \mathbf{F}\|_{\dot{B}^{1-\theta}_{p,p}(\mathbb{R}^3)} < \infty.
$$

Moreover, it follows from Theorem 3.1 that

$$
\mathbf{V} \in H^{2\alpha}(\mathbb{R}^3) \cap \dot{B}^{2\alpha}_{p,p}(\mathbb{R}^3).
\tag{3.25}
$$

Thanks to the Bony paraproduct decomposition, we have by the Leibniz estimates that

$$
\|\text{div } (\mathbf{V} + \mathbf{U}_0) \otimes \mathbf{V}\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} \leq C \|\mathbf{V}\|_{L^\infty(\mathbb{R}^3)} \|\mathbf{V}\|_{\dot{B}^{s+1\alpha}_{p,p}(\mathbb{R}^3)} + C \|\mathbf{V}\|_{L^p(\mathbb{R}^3)} \|\mathbf{U}_0\|_{W^{s+1,\infty}(\mathbb{R}^3)} + C \|\mathbf{U}_0\|_{L^\infty(\mathbb{R}^3)} \|\mathbf{V}\|_{\dot{B}^{s+1\alpha}_{p,p}(\mathbb{R}^3)},
$$

and

$$
\|\text{div } (\mathbf{V} \otimes \mathbf{U}_0)\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} \leq C \|\nabla \mathbf{U}_0\|_{L^\infty(\mathbb{R}^3)} \|\mathbf{V}\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)} + C \|\mathbf{V}\|_{L^\infty(\mathbb{R}^3)} \|\nabla \mathbf{U}_0\|_{\dot{B}^\alpha_{p,p}(\mathbb{R}^3)}.
$$
Both estimates above imply
\[
\|F\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} \leq \|\text{div} F\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} + C\|V\|_{L^\infty(\mathbb{R}^3)}\|\dot{V}\|_{\dot{B}^{s+1}_{p,p}(\mathbb{R}^3)} + C\|\nabla U_0\|_{W^{s+1,\infty}(\mathbb{R}^3)} \\
+ C\|\nabla U_0\|_{L^\infty(\mathbb{R}^3)}\|V\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} + C\|\nabla U_0\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} \cdot
\]

Inserting this estimate into (3.24) yields
\[
\|V\|_{\dot{B}^{s+2\alpha}_{p,p}(\mathbb{R}^3)} + \|P\|_{\dot{B}^{s+1}_{p,p}(\mathbb{R}^3)} \\
\leq C_p\|\text{div} F\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} + C\|V\|_{L^\infty(\mathbb{R}^3)}\|\dot{V}\|_{\dot{B}^{s+1}_{p,p}(\mathbb{R}^3)} + C\|\nabla U_0\|_{L^p(\mathbb{R}^3)}\|U_0\|_{W^{s+1,\infty}(\mathbb{R}^3)} \\
+ C\|\nabla U_0\|_{L^\infty(\mathbb{R}^3)}\|V\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} + C\|\nabla U_0\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} + C_p\|\nabla U_0\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} .
\]

Since \(\sigma \in C^{1,0}(S^2)\), we have by Lemma 2.8 that for each \(s \geq 0\),
\[
\|U_0\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} \leq C\|\Delta U_0\|_{L^p(\mathbb{R}^3)} + \sum_{q \geq 0} e^{-2^{q\alpha}}\|\Delta^q U_0\|_{L^p(\mathbb{R}^3)} \\
\leq C\|\sigma\|_{L^\infty(S^2)}\|\Delta^{-1} U_0\|_{L^p(\mathbb{R}^3)} \\
\leq C\|\sigma\|_{L^\infty(S^2)}\|\Delta^{-1} U_0\|_{L^p(\mathbb{R}^3)} \quad \text{for each } q > \frac{3}{2\alpha}
\]
for each \(p > \frac{3}{2\alpha - 1}\) and
\[
\|\nabla U_0\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} \leq C\|\sigma\|_{W^{1,\infty}(S^2)}\|\Delta^{-1} U_0\|_{L^p(\mathbb{R}^3)} \quad \text{for each } q > \frac{3}{2\alpha - 1}
\]

Thanks to (3.25), (3.27) and (3.28), we easily find by condition \(\dot{F} \in L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,p}(\mathbb{R}^3)\) that all terms in the right side of (3.26) are bound as long as \(s \leq 1 - \alpha\), that is, for each \(0 \leq s \leq 1 - \alpha\),
\[
\|U\|_{\dot{B}^{s+2\alpha}_{p,p}(\mathbb{R}^3)} + \|P\|_{\dot{B}^{s+1}_{p,p}(\mathbb{R}^3)} \leq C\left(p, \sigma, \|\text{div} F\|_{L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,p}(\mathbb{R}^3)} \right) .
\]

With estimate (3.22) in hand, we can show by repeating the above step that for each \(0 \leq s \leq 2(1 - \alpha)\),
\[
\|U\|_{\dot{B}^{s+2\alpha}_{p,p}(\mathbb{R}^3)} + \|P\|_{\dot{B}^{s+1}_{p,p}(\mathbb{R}^3)} \leq C\left(p, \sigma, \|\text{div} F\|_{L^2(\mathbb{R}^3) \cap \dot{B}^s_{p,p}(\mathbb{R}^3)} \right) .
\]

Repeat the above process until the termination condition of Theorem 3.2 is reached.

Taking \(\sigma = 0\) in Theorem 3.1 and Theorem 3.2, they become Theorem 1.4 (ii)-(i). Now we come back to show the high regularity of weak solutions in Theorem 1.8. Since \(\sigma \in C^{1,0}(S^2)\), we have by the Leibniz estimate, (3.27) and (3.28) that for each \(s \geq 0\) and \(p \geq 2\),
\[
\|\text{div}(U_0 \otimes U_0)\|_{\dot{B}^s_{p,p}(\mathbb{R}^3)} \\
\leq C\|U_0\|_{L^p(\mathbb{R}^3)}\|\nabla U_0\|_{\dot{B}^{s+1}_{2,1}(\mathbb{R}^3)} + C\|\nabla U_0\|_{L^{3p/2}(\mathbb{R}^3)}\|U_0\|_{\dot{B}^{3p/2}_{3,1}(\mathbb{R}^3)} < \infty.
\]

Taking \(\text{div} F = \text{div}(U_0 \otimes U_0)\) in Theorem 3.1 and Theorem 3.2 we immediately obtain the high regularity of weak solutions in Theorem 1.8.
3.2. Regularity in the framework of the weighted Hilbert space. In this subsection, we will study the regularity in the framework of the weighted Hilbert space, which helps us to analysis the behaviour of weak solution for large $|x|$.

**Theorem 3.3.** Let $s \geq 0$, $\sigma \in C^{1,0}(\mathbb{S}^2)$ and $F \in H^{s+1}(\mathbb{R}^3)$. Assume that $V$ is a weak solution to the system (3.3). Then there exists a constant $C$ such that

(i) if $\alpha \in [\frac{3}{2}, 1]$, $\text{div } F \in H^s(\mathbb{R}^3)$, we have that for each $\beta \in (0, \alpha)$,

$$\|V\|_{H^{s+\alpha}(\mathbb{R}^3)} + \|P\|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C \left( \|\sigma\|_{H^s(\mathbb{R}^3)}, \|\text{div } F\|_{H^{s}(\mathbb{R}^3)} \right).$$

(ii) if $\alpha = 1$, $\text{div } F \in H^s(\mathbb{R}^3)$, we have

$$\|V\|_{H^{s+1}(\mathbb{R}^3)} + \|P\|_{H^{s+1}(\mathbb{R}^3)} \leq C \left( \|\sigma\|_{H^s(\mathbb{R}^3)}, \|\text{div } F\|_{H^{s}(\mathbb{R}^3)} \right).$$

**Proof.** First of all, we have from Theorem 3.1 and Theorem 3.2 that for each $s \geq 0$,

$$\|V\|_{H^{s+2\alpha}(\mathbb{R}^3)} + \|\nabla P\|_{H^{s}(\mathbb{R}^3)} \leq C \|\text{div } F\|_{H^{s}(\mathbb{R}^3)}.$$  

Since $\alpha \in [5/6, 1]$, we have that

$$V + U_0 \in L^\infty(\mathbb{R}^3) \cap W^{1, \frac{3}{2}}(\mathbb{R}^3).$$

By Theorem 2.14, we readily have

$$\left\| \langle \cdot \rangle^{\beta} V \right\|_{H^{s}(\mathbb{R}^3)} \leq C \left( \left\| \langle \cdot \rangle^{\beta} V \cdot \nabla U_0 \right\|_{L^2(\mathbb{R}^3)} + \left\| \langle \cdot \rangle^{\beta} \text{div } F \right\|_{L^2(\mathbb{R}^3)} \right),$$

$$\leq C \left( \left\| \langle \cdot \rangle^{\beta} \nabla U_0 \right\|_{L^\infty(\mathbb{R}^3)}, \|V\|_{L^2(\mathbb{R}^3)} + \left\| \langle \cdot \rangle^{\beta} \text{div } F \right\|_{L^2(\mathbb{R}^3)} \right).$$

This estimate together with the fact $U_0 = \frac{\sigma}{|x|^{3/2-1}}$ and $\sigma \in C^{1,0}(\mathbb{S}^2)$ implies

$$\left\| \langle \cdot \rangle^{\beta} \nabla U_0 \right\|_{L^\infty(\mathbb{R}^3)} \leq C,$$

which together with (3.32) gives

$$\left\| \langle \cdot \rangle^{\beta} V \right\|_{H^{s}(\mathbb{R}^3)} \leq C_{\sigma} \left\| \langle \cdot \rangle^{\beta} \text{div } F \right\|_{L^2(\mathbb{R}^3)}.$$
Since \( \alpha \in [5/6, 1] \), we have by the Leibniz estimate and \((3.32)\) that
\[
\| \mathbf{F}_2 \|_{H^{1+s}(\mathbb{R}^3)} \leq C \| (\mathbf{V} + \mathbf{U}_0) \otimes \mathbf{V} \|_{H^{1+s}(\mathbb{R}^3)}
\]
\[
\leq C \left( \| \mathbf{V} \|_{L^\infty(\mathbb{R}^3)} + \| \mathbf{U}_0 \|_{W^{1,\infty}(\mathbb{R}^3)} \right) \| \mathbf{V} \|_{H^{1+s}(\mathbb{R}^3)}
\]
\[
\leq C \| \text{div} \mathbf{F} \|_{H^s(\mathbb{R}^3)} + C_\sigma \| \text{div} \mathbf{F} \|_{H^s(\mathbb{R}^3)}.
\]

(3.36)

Now we deal with the term involving \( \tilde{\mathbf{F}}_1 \). For the case \( \{ s \} = 0 \), it is obvious that
\[
\sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta (I_d + \Lambda^{(s)} D^\gamma) \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)} = \sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta (I_d + D^\gamma) \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \left\| \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)} + \sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}.
\]

The Leibniz formula enables us to conclude that
\[
\sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \sup_{|\gamma| = s} \left\| D^\gamma \left( \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \right) \right\|_{L^2(\mathbb{R}^3)} + C \sum_{|\gamma_1| = 1}^s \sup_{|\gamma_2| = s} \left\| \partial^{\gamma_1} \langle \cdot \rangle^\beta \partial^{\gamma_2} \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \left\| \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \right\|_{H^s(\mathbb{R}^3)} + C \| \tilde{\mathbf{F}}_1 \|_{H^{s-1}(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \right\|_{H^s(\mathbb{R}^3)}.
\]

As for the case \( \{ s \} \in (0, 1) \), we know by the triangle inequality that
\[
\sup_{|\gamma| = [s]} \left\| \langle \cdot \rangle^\beta (I_d + \Lambda^{(s)} D^\gamma) \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)} + \sup_{|\gamma| = [s]} \left\| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}.
\]

By Lemma \((2.11)\) we get
\[
\sup_{|\gamma| = [s]} \left\| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \sup_{|\gamma| = [s]} \left\| \Lambda^{(s)} \left( \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \right) \right\|_{L^2(\mathbb{R}^3)} + \sup_{|\gamma| = [s]} \left\| \left[ \Lambda^{(s)}, \langle \cdot \rangle^\beta \right] D^\gamma \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \sup_{|\gamma| = [s]} \left\| \Lambda^{(s)} \left( \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \right) \right\|_{L^2(\mathbb{R}^3)} + C \sup_{|\gamma| = [s]} \left\| \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \right\|_{L^2(\mathbb{R}^3)}.
\]

According to the Leibniz formula, we get by Leibniz estimates that
\[
\| \Lambda^{(s)} \left( \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \right) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| \Lambda^{(s)} D^\gamma \left( \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \right) \|_{L^2(\mathbb{R}^3)} + C \sum_{|\gamma_1| = 1}^s \sup_{|\gamma_2| = [s]} \| \Lambda^{(s)} \left( \partial^{\gamma_1} \langle \cdot \rangle^\beta \partial^{\gamma_2} \tilde{\mathbf{F}}_1 \right) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| \Lambda^{(s)} D^\gamma \left( \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \right) \|_{L^2(\mathbb{R}^3)} + C \| \tilde{\mathbf{F}}_1 \|_{H^s(\mathbb{R}^3)},
\]

and
\[
\sup_{|\gamma| = [s]} \| \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \|_{L^2(\mathbb{R}^3)} \leq C \| \langle \cdot \rangle^\beta \tilde{\mathbf{F}}_1 \|_{H^s(\mathbb{R}^3)} + \sup_{|\gamma| = [s]} \| \langle \cdot \rangle^\beta D^\gamma \tilde{\mathbf{F}}_1 \|_{L^2(\mathbb{R}^3)}.
\]
Hence, we have that for each \( s > 0 \),
\[
\sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta (I_d + \Lambda^{(s)} D^\gamma) \tilde{F}_1 \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta \tilde{F}_1 \right\|_{H^s(\mathbb{R}^3)}.
\]

Inserting (3.36) and (3.37) into (3.35) and using \( \text{div} \mathbf{F} \in H^s(\mathbb{R}^3) \), we get
\[
\sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \mathbf{V} \right\|_{H^s(\mathbb{R}^3)} \leq C \sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta (I_d + \Lambda^{(s)} D^\gamma) \text{div} \mathbf{F} \right\|_{L^2(\mathbb{R}^3)} + C \left\| \langle \cdot \rangle^\beta \tilde{F}_1 \right\|_{H^s(\mathbb{R}^3)} + C
\]
\[
+ C \sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \text{div} \mathbf{F}_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)}.
\]

Since
\[
\langle \mathbf{x} \rangle^\beta \tilde{F}_1 = -\langle \mathbf{x} \rangle^\beta \mathbf{V} \cdot \nabla U_0,
\]
we get by the Leibniz estimate that for each \( s \geq 0 \),
\[
\left\| \langle \cdot \rangle^\beta \mathbf{V} \cdot \nabla U_0 \right\|_{H^s(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta \mathbf{V} \right\|_{H^s(\mathbb{R}^3)} \left\| \nabla \mathbf{U}_0 \right\|_{L^\infty(\mathbb{R}^3)} + C \left\| \langle \cdot \rangle^\beta \mathbf{V} \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \mathbf{U}_0 \right\|_{W^{s, \infty}(\mathbb{R}^3)}
\]
\[
\leq C \left\| \sigma \right\|_{W^{s+1, \infty}(\mathbb{S}^2)} \left\| \langle \cdot \rangle^\beta \mathbf{V} \right\|_{H^s(\mathbb{R}^3)}.
\]

Hence, we have
\[
\sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta (I_d + \Lambda^{(s)} D^\gamma) \tilde{F}_1 \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta \mathbf{V} \right\|_{H^s(\mathbb{R}^3)}.
\]

We turn to deal with the term involving \( \mathbf{F}_2 \). As long as \( \{s\} = 0 \), we see that
\[
\sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \text{div} \mathbf{F}_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)} = \sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta D^\gamma \text{div} \mathbf{F}_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)}
\]
\[
\leq \sum_{|\gamma| = s} \left\| D^\gamma \langle \cdot \rangle^\beta \text{div} \mathbf{F}_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)} + C \sum_{|\gamma| = s} \sup_{|\gamma_1| = 1, |\gamma_1| + |\gamma_2| = s} \left\| \partial^{\gamma_1} \langle \cdot \rangle^\beta \partial^{\gamma_2} \text{div} \mathbf{F}_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)}.
\]

By the Hölder inequality, one has
\[
\sum_{|\gamma| = s} \sup_{|\gamma_1| = 1, |\gamma_1| + |\gamma_2| = s} \left\| \partial^{\gamma_1} \langle \cdot \rangle^\beta \partial^{\gamma_2} \text{div} \mathbf{F}_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \sum_{|\gamma| = s} \sup_{|\gamma_1| = 1, |\gamma_1| + |\gamma_2| = s} \left\| \partial^{\gamma_1} \langle \cdot \rangle^\beta \partial^{\gamma_2} \text{div} \mathbf{F}_2 \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \mathbf{F}_2 \right\|_{H^s(\mathbb{R}^3)}.
\]

Therefore, we have that for the case \( \{s\} = 0 \),
\[
\sup_{|\gamma| = s} \left\| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \text{div} \mathbf{F}_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta \text{div} \mathbf{F}_2 \right\|_{H^s(\mathbb{R}^3)} + C \left\| \mathbf{F}_2 \right\|_{H^s(\mathbb{R}^3)}.
\]
For the case \( \{s\} \in (0, 1) \), we observe that
\[
\sup_{|\gamma|=|s|} \| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)}
\]
\[
\leq \sup_{|\gamma|=|s|} \| \text{div} \left( \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \mathbf{F}_2 \right) \|_{H^{-\alpha}(\mathbb{R}^3)} + \sup_{|\gamma|=|s|} \| \nabla \langle \cdot \rangle^\beta \cdot \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)}
\]
\[
\leq C \sup_{|\gamma|=|s|} \| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{H^{1-\alpha}(\mathbb{R}^3)} + C \sup_{|\gamma|=|s|} \| \nabla \langle \cdot \rangle^\beta \cdot \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}.
\]

By the Hölder inequality, we get
\[
\| \nabla \langle \cdot \rangle^\beta \cdot \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)} \leq \| \nabla \langle \cdot \rangle^\beta \|_{L^\infty(\mathbb{R}^3)} \| \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)} \leq C \| \mathbf{F}_2 \|_{H^\epsilon(\mathbb{R}^3)}.
\]

By Lemma 2.11 we obtain
\[
\| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{H^{1-\alpha}(\mathbb{R}^3)}
\]
\[
\leq \| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)} + \| \Lambda^{1-\alpha} \left( \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| \Lambda^{(s)} \left( \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + \| \left[ [\langle \cdot \rangle^\beta, \Lambda^{(s)}] D^\gamma \mathbf{F}_2 \right] \|_{L^2(\mathbb{R}^3)}
\]
\[
+ \| \langle \cdot \rangle^\beta \Lambda^{1+(s)-\alpha} D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)} + \| [\Lambda^{1-\alpha}, \langle \cdot \rangle^\beta] \left( \Lambda^{(s)} D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| \Lambda^{(s)} \left( \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}
\]
\[
+ \| \langle \cdot \rangle^\beta \Lambda^{1+(s)-\alpha} D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta \left( \Lambda^{(s)} D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)}.
\]

Our task is now to bound the first term in the last line of (3.33). By Lemma 2.11 again, we have that for each \( \{s\} \in (0, \alpha) \),
\[
\| \langle \cdot \rangle^\beta \Lambda^{1+(s)-\alpha} D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| \Lambda^{1+(s)-\alpha} \left( \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + \| \left[ [\langle \cdot \rangle^\beta, \Lambda^{1+(s)-\alpha}] D^\gamma \mathbf{F}_2 \right] \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| \Lambda^{1+(s)-\alpha} \left( \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}
\]

Similarly, we can show that
\[
\| \langle \cdot \rangle^\beta \left( \Lambda^{(s)} D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} \leq \| \Lambda^{(s)} \left( \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + \| [\langle \cdot \rangle^\beta, \Lambda^{(s)}] D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| \Lambda^{(s)} \left( \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}.
\]

Inserting (3.44) and (3.45) into (3.33) leads to that for each \( \{s\} \in (0, \alpha) \),
\[
\sup_{|\gamma|=|s|} \| \langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \text{div} \mathbf{F}_2 \|_{H^{-\alpha}(\mathbb{R}^3)}
\]
\[
\leq \sup_{|\gamma|=|s|} \| \Lambda^{1+(s)-\alpha} \left( \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + C \sup_{|\gamma|=|s|} \| \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}.
\]

In terms of the Leibniz rule, one has
\[
\| \langle \cdot \rangle^\beta D^\gamma \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)} \leq \| D^\gamma \left( \langle \cdot \rangle^\beta \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + C \sum_{|\gamma_1|=1,|\gamma_2|=|s|} \sup_{|\gamma_1|+|\gamma_2|+|\gamma|=|s|} \| \partial^{\gamma_1} \langle \cdot \rangle^\beta \partial^{\gamma_2} \mathbf{F}_2 \|_{L^2(\mathbb{R}^3)}
\]
\[
\leq \| D^\gamma \left( \langle \cdot \rangle^\beta \mathbf{F}_2 \right) \|_{L^2(\mathbb{R}^3)} + C \| \mathbf{F}_2 \|_{H^\epsilon(\mathbb{R}^3)}.
\]
Also, we have
\[
\left\| \langle \cdot \rangle^\beta D_x^\gamma F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} \leq D_x^\gamma \left( \langle \cdot \rangle^\beta F_2 \right) \bigg\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + C \sum_{|\gamma_1|=1}^{[s]} \sup_{|\gamma_1|+|\gamma_2|=|s|} \left\| \partial^{\gamma_1} \langle \cdot \rangle^\beta \partial^{\gamma_2} F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + C \| F_2 \|_{H^s(\mathbb{R}^3)}.
\]

Inserting (3.47) and (3.48) into (3.46) yields that for each \( \{s\} \in (0, \alpha) \),
\[
\sum_{|\gamma|=|s|} \left\| \langle \cdot \rangle^\beta A^{s} D^\gamma \text{div} F_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + C \| F_2 \|_{H^s(\mathbb{R}^3)}.
\]

Now we consider the case where \( \{s\} \in [\alpha, 1) \), which means \( 1 + \{s\} - \alpha \geq 1 \). In terms of Lemma 2.12 we immediately have
\[
\left\| \langle \cdot \rangle^\beta A^{1+\{s\}-\alpha} D^\gamma F_2 \right\|_{L^2(\mathbb{R}^3)} \leq C \sum_{i=1}^{3} \left\| \langle \cdot \rangle^\beta \partial_{x_i} A^{\{s\}-\alpha} D^\gamma F_2 \right\|_{L^2(\mathbb{R}^3)}.
\]

Moreover, we have by Lemma 2.11 that
\[
\left\| \langle \cdot \rangle^\beta \partial_{x_i} A^{\{s\}-\alpha} D^\gamma F_2 \right\|_{L^2(\mathbb{R}^3)} \leq \left\| A^{\{s\}-\alpha} \left( \langle \cdot \rangle^\beta \partial_{x_i} D^\gamma F_2 \right) \right\|_{L^2(\mathbb{R}^3)} + \left\| \left( A^{\{s\}-\alpha}, \langle \cdot \rangle^\beta \partial_{x_i} D^\gamma F_2 \right) \right\|_{L^2(\mathbb{R}^3)} \\
\leq \left\| A^{\{s\}-\alpha} \left( \langle \cdot \rangle^\beta \partial_{x_i} D^\gamma F_2 \right) \right\|_{L^2(\mathbb{R}^3)} + C \left\| \langle \cdot \rangle^\beta \partial_{x_i} D^\gamma F_2 \right\|_{L^2(\mathbb{R}^3)}.
\]

By the Leibniz rule, one has
\[
\left\| A^{\{s\}-\alpha} \left( \langle \cdot \rangle^\beta \partial_{x_i} D^\gamma F_2 \right) \right\|_{L^2(\mathbb{R}^3)} \leq \left\| A^{\{s\}-\alpha} \right\|_{L^2(\mathbb{R}^3)} \left\| \langle \cdot \rangle^\beta \partial_{x_i} D^\gamma F_2 \right\|_{L^2(\mathbb{R}^3)} + C \left\| D^\gamma \right\|_{L^2(\mathbb{R}^3)} \left\| F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + C \| F_2 \|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)}.
\]

and
\[
\left\| \langle \cdot \rangle^\beta \partial_{x_i} D^\gamma F_2 \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta F_2 \right\|_{H^{1+\{s\}-1}(\mathbb{R}^3)} + C \| F_2 \|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)}.
\]

Inserting (3.51) and (3.52) into (3.50), we have that for each \( \{s\} \in [\alpha, 1) \),
\[
\left\| \langle \cdot \rangle^\beta A^{1+\{s\}-\alpha} D^\gamma F_2 \right\|_{L^2(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + C \| F_2 \|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)}.
\]

Plugging (3.52), (3.49) and (3.53) into (3.44) yields that for each \( \{s\} \in (0, 1) \),
\[
\sup_{|\gamma|=|s|} \left\| \langle \cdot \rangle^\beta D^\gamma \text{div} F_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + C \| F_2 \|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)}.
\]

Collecting both estimates (3.44) and (3.54), we obtain that for each \( s > 0 \),
\[
\sup_{|\gamma|=|s|} \left\| \langle \cdot \rangle^\beta A^{\{s\}} D^\gamma \text{div} F_2 \right\|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \left\| \langle \cdot \rangle^\beta F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + C \| F_2 \|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)}.
\]

Since \( F_2 = -(V + U_0) \otimes V \), we have
\[
\left\| \langle \cdot \rangle^\beta F_2 \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} \leq \left\| \langle \cdot \rangle^\beta V \otimes V \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)} + \left\| \langle \cdot \rangle^\beta U_0 \otimes V \right\|_{H^{1+\{s\}-\alpha}(\mathbb{R}^3)}.
\]
and
\begin{equation}
\|F_2\|_{H^s(\mathbb{R}^3)} \leq C \|V\|_{L^\infty(\mathbb{R}^3)} \|V\|_{H^s(\mathbb{R}^3)} + C \|U_0\|_{W^{s,\infty}(\mathbb{R}^3)} \|V\|_{H^s(\mathbb{R}^3)}.
\end{equation}

Thanks to the Leibniz estimate, we get by using the fact $\alpha \in [5/6, 1]$ that
\begin{equation}
\|\langle \cdot \rangle^\beta V \otimes V\|_{H^{1+\alpha}(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle^\beta V\|_{H^{1-\alpha}(\mathbb{R}^3)} \|V\|_{H^{2+\alpha}(\mathbb{R}^3)} + C \|\langle \cdot \rangle^\beta V\|_{H^{1-\alpha}(\mathbb{R}^3)} \|V\|_{L^\infty(\mathbb{R}^3)} \leq C \|V\|_{H^{3+\alpha}(\mathbb{R}^3)} \|\langle \cdot \rangle^\beta V\|_{H^{1+\alpha}(\mathbb{R}^3)}.
\end{equation}

Also, we have
\begin{equation}
\|\langle \cdot \rangle^\beta U_0 \otimes V\|_{H^s(\mathbb{R}^3)} \leq C \|U_0\|_{H^{3+\alpha}(\mathbb{R}^3)} \|\langle \cdot \rangle^\beta V\|_{H^{1+\alpha}(\mathbb{R}^3)}.
\end{equation}

Plugging (3.58) and (3.59) into (3.56) leads to
\begin{equation}
\|\langle \cdot \rangle^\beta F_2\|_{H^{1+\alpha}(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle^\beta V\|_{H^{1+\alpha}(\mathbb{R}^3)}.
\end{equation}

Inserting this estimate and (3.57) into (3.55) and then using (3.32), we get
\begin{equation}
\sup_{|\gamma| = |s|} \|\langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \text{div} F_2\|_{H^{-\alpha}(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle^\beta V\|_{H^{1+\alpha}(\mathbb{R}^3)} + C.
\end{equation}

Then we put (3.39) and (3.60) into (3.38) to yield
\begin{equation}
\sup_{|\gamma| = |s|} \|\langle \cdot \rangle^\beta \Lambda^{(s)} D^\gamma \text{div} F\|_{H^{\alpha}(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle^\beta V\|_{H^{1+\alpha}(\mathbb{R}^3)} + C \sup_{|\gamma| = |s+\alpha|} \|\langle \cdot \rangle^\beta (I_d + \Lambda^{(s)} D^\gamma) \text{div} F\|_{L^2(\mathbb{R}^3)} + C.
\end{equation}

By the same argument as used in the proof of (3.49), we can show that
\begin{equation}
\sup_{|\gamma| = |s|} \|\langle \cdot \rangle^\beta (I_d + \Lambda^{(s)} D^\gamma) F\|_{L^2(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle^\beta F\|_{H^{s}(\mathbb{R}^3)} + C \|F\|_{H^s(\mathbb{R}^3)}.
\end{equation}

Thanks to the definition of the inhomogeneous Hilbert space, one has
\begin{equation}
\|\langle \cdot \rangle^\beta V\|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C \|\langle \cdot \rangle^\beta V\|_{L^2(\mathbb{R}^3)} + C \sup_{|\gamma| = |s+\alpha|} \|\Lambda^{(s+\alpha)} D^\gamma (\langle \cdot \rangle^\beta V)\|_{L^2(\mathbb{R}^3)}.
\end{equation}

In terms of the Leibniz rule, we find that
\begin{equation}
\sup_{|\gamma| = |s+\alpha|} \|\Lambda^{(s+\alpha)} D^\gamma (\langle \cdot \rangle^\beta V)\|_{L^2(\mathbb{R}^3)} \leq \sup_{|\gamma| = |s+\alpha|} \|\Lambda^{(s+\alpha)} (\langle \cdot \rangle^\beta D^\gamma V)\|_{L^2(\mathbb{R}^3)} + C \sum_{|\gamma_1| = |s+\alpha|} \sup_{|\gamma_1| = |s+\alpha|} \|\partial^\gamma_1 (\langle \cdot \rangle^\beta D^\gamma V)\|_{H^{(s+\alpha)}(\mathbb{R}^3)} \leq \sup_{|\gamma| = |s+\alpha|} \|\Lambda^{(s+\alpha)} (\langle \cdot \rangle^\beta D^\gamma V)\|_{L^2(\mathbb{R}^3)} + C \|V\|_{H^{s+\alpha}(\mathbb{R}^3)}.
\end{equation}

For \(\{s\} \in [0, 1 - \alpha)\), we have \(|s + \alpha| = |s|\) and
\[\alpha \leq \{\alpha + s\} = \alpha + \{s\} < 1.\]

By Lemma 2.11 we further get
\[\|\Lambda^{(s+\alpha)} (\langle \cdot \rangle^\beta D^\gamma V)\|_{L^2(\mathbb{R}^3)} \leq \sup_{|\gamma| = |s+\alpha|} \|\Lambda^{(s+\alpha)} (\langle \cdot \rangle^\beta D^\gamma V)\|_{L^2(\mathbb{R}^3)} + C \|V\|_{H^{s+\alpha}(\mathbb{R}^3)}.
\]
plugging this estimate into (3.63) yields that for each \(s \in [0, 1 - \alpha)\),
\[
\|\langle\cdot\rangle^\beta D\gamma V\|_{L^2(\mathbb{R}^3)} \leq C \|\langle\cdot\rangle^\beta \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)} + C \|\Lambda^\alpha \langle\cdot\rangle^\beta L^s D\gamma V\|_{L^2(\mathbb{R}^3)} + C \|\|V\|_{H^\alpha(\mathbb{R}^3)}
\]
(3.64)
\[
\leq C \|\langle\cdot\rangle^\beta D\gamma V\|_{L^2(\mathbb{R}^3)} + C \sup_{[\gamma]\in[s]} \|\langle\cdot\rangle^\beta \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)} + C \|\|V\|_{H^\alpha(\mathbb{R}^3)}
\]
For the case where \(s \in [\alpha, 1)\), we have
\[
\{s + \alpha\} = \{s\} + 1 \text{ and } \{s + \alpha\} = \{s\} + \alpha - 1.
\]
In view of Lemma 2.11 and Lemma 2.12, we have
\[
\sup_{[\gamma]\in[s]} \|\langle\cdot\rangle^\beta \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)} \leq \sup_{[\gamma]\in[s] + 1} \|\langle\cdot\rangle^\beta \Lambda_0 L^{s+1} D\gamma V\|_{L^2(\mathbb{R}^3)} + \sup_{[\gamma]\in[s] + 1} \|\Lambda^{\alpha+1} \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \sup_{[\gamma]\in[s]} \|\langle\cdot\rangle^\beta \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)} + \sup_{[\gamma]\in[s]} \|\Lambda^{\alpha+1} \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)}
\]
By Lemma 2.11 again, one has
\[
\sup_{[\gamma]\in[s]} \|\langle\cdot\rangle^\beta \Lambda_0 L^{s+1} D\gamma V\|_{L^2(\mathbb{R}^3)} \leq C \sup_{[\gamma]\in[s]} \|\langle\cdot\rangle^\beta \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)}
\]
\[
\leq C \sup_{[\gamma]\in[s]} \|\Lambda^{\alpha+1} \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)} + C \sup_{[\gamma]\in[s]} \|\Lambda^{\alpha+1} \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)}
\]
From this estimate, it follows that
\[
\sup_{[\gamma]\in[s] + 1} \|\langle\cdot\rangle^\beta D\gamma V\|_{L^2(\mathbb{R}^3)} \leq C \sup_{[\gamma]\in[s]} \|\langle\cdot\rangle^\beta \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)} + C \|\|V\|_{H^\alpha(\mathbb{R}^3)}
\]
Furthermore, we get by the fact \(\alpha \in [5/6, 1]\) that for each \(s \in [\alpha, 1)\),
\[
\sup_{[\gamma]\in[s] + 1} \|\Lambda^{\alpha+1} L^s D\gamma V\|_{L^2(\mathbb{R}^3)} \leq C \sup_{[\gamma]\in[s]} \|\langle\cdot\rangle^\beta \Lambda_0 L^s D\gamma V\|_{L^2(\mathbb{R}^3)} + C \|\|V\|_{H^\alpha(\mathbb{R}^3)}
\]
Plugging this estimate into (3.68) gives that for each \( s \in [\alpha, 1) \),
\[
\| \langle \cdot \rangle^\beta V \|_{H^{s+\alpha}(\mathbb{R}^3)} 
\leq C \| \langle \cdot \rangle^\beta |V| \|_{L^2(\mathbb{R}^3)} + C \sup_{|\gamma|=|s|} \| \langle \cdot \rangle^\beta A^{(s)}D^\gamma V \|_{H^{s}(\mathbb{R}^3)} + C \| V \|_{H^s(\mathbb{R}^3)}.
\] (3.65)

Inserting (3.62), (3.64) and (3.65) into (3.61), we readily have for each \( s > 0 \),
\[
\| \langle \cdot \rangle^\beta V \|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C \| \langle \cdot \rangle^\beta V \|_{H^{s+1-\alpha}(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta \text{div} F \|_{H^{s}(\mathbb{R}^3)} + C.
\] (3.66)

Using estimate (3.33), we get from (3.66) that for each \( s \in [0, 2\alpha - 1] \),
\[
\| \langle \cdot \rangle^\beta V \|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C_\sigma \| \langle \cdot \rangle^\beta \text{div} F \|_{L^2(\mathbb{R}^3)} + C \| \langle \cdot \rangle^\beta \text{div} F \|_{H^{s}(\mathbb{R}^3)} + C,
\] (3.67)

which means that for each \( s \in [0, 2\alpha - 1] \),
\[
\| \langle \cdot \rangle^\beta V \|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C \left( \sigma, \| \langle \cdot \rangle^\beta \text{div} F \|_{H^{s}(\mathbb{R}^3)} \right).
\]

Plugging (3.67) into (3.66) yields that for each \( s \in [2\alpha - 1, 2(2\alpha - 1)] \),
\[
\| \langle \cdot \rangle^\beta V \|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C \left( \sigma, \| \langle \cdot \rangle^\beta \text{div} F \|_{H^{s}(\mathbb{R}^3)} \right).
\]

By repeating this process, we can conclude that for each \( s \geq 0 \),
\[
\| \langle \cdot \rangle^\beta V \|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C \left( \sigma, \| \langle \cdot \rangle^\beta \text{div} F \|_{H^{s}(\mathbb{R}^3)} \right).
\] (3.68)

Lastly, we turn to estimate pressure. Since
\[
-\Delta P = \text{div div}(V \otimes V + U_0 \otimes V + V \otimes U_0 - F)
\]

and the fact \( \langle x \rangle^{2\beta} \) belongs to \text{A}_2 class, we get by (3.27) and (3.32) that
\[
\| \langle \cdot \rangle^{\beta} P \|_{H^{s+\alpha}(\mathbb{R}^3)} \leq C \| \langle \cdot \rangle^{\beta} (V \otimes V + U_0 \otimes V + V \otimes U_0 - F) \|_{H^{s}(\mathbb{R}^3)}
\leq C \| \langle \cdot \rangle^{\beta} V \|_{H^{s+\alpha}(\mathbb{R}^3)} + C \| \langle \cdot \rangle^{\beta} F \|_{H^{s+\alpha}(\mathbb{R}^3)} < \infty.
\]

This estimate together with (3.68) implies the estimate (3.30).

The proof of the estimate (3.31) is similar to the proof of (3.30), so we here omit it. \( \square \)

Now we come back to show the second estimate (ii)-(2) in Theorem 1.4. Taking \( U_0 \equiv 0 \) in Theorem 3.3, we immediately have
\[
(U, P) \in H^{x+s}_x(\mathbb{R}^3) \times H^{x+s}_x(\mathbb{R}^3).
\]

Lastly, we show the weighted estimate in Theorem 1.8. According to Theorem 3.3, it suffice to show that for each \( \alpha \in [5/6, 1) \),
\[
\| \langle \cdot \rangle^\beta \text{div} F_0 \|_{H^{s}(\mathbb{R}^3)} < \infty,
\] (3.69)

and for \( \alpha = 1 \),
\[
\| \langle \cdot \rangle \text{div} F_0 \|_{H^{s}(\mathbb{R}^3)} < \infty.
\] (3.70)

Since \( \text{div} F_0 = -U_0 \cdot \nabla U_0 \) and \( U_0(x) = e^{-(\cdot)^\alpha} u_0(x) \) with \( u_0(x) = \frac{\sigma(x)}{|x|^{2\alpha-1}} \), we have
\[
\| \langle \cdot \rangle^\beta \text{div} F_0 \|_{H^{s}(\mathbb{R}^3)} < \infty.
\]
On the other hand, we see by the Bernstein inequality in Lemma 2.1 that
\[
\leq \| \Delta_{-1} (\cdot)^{\beta} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} + \left( \sum_{k \geq 0} 2^{2ks} \| \Delta_k (\cdot)^{\beta} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} \right)^{\frac{1}{2}}.
\]
Since \( \beta \in (0, \alpha) \) and \( \alpha \in [5/6, 1] \), we have
\[
4\alpha - (\beta + 1) > \frac{3}{2}.
\]
Moreover, one has by the Hölder inequality and Proposition 2.8 in [10] that
\[
\| (\cdot)^{\beta} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} \leq \| (\cdot)^{2\alpha-1} \mathbf{U}_0 \|_{L^\infty(\mathbb{R}^3)} \| (\cdot)^{\beta-2\alpha+1} \nabla \mathbf{U}_0 \|_{L^2(\mathbb{R}^3)} \\
\leq \| (\cdot)^{2\alpha-1} \mathbf{U}_0 \|_{L^\infty(\mathbb{R}^3)} \| (\cdot)^{2\alpha} \nabla \mathbf{U}_0 \|_{L^\infty(\mathbb{R}^3)} \| (\cdot)^{\beta-4\alpha+1} \|_{L^2(\mathbb{R}^3)} < \infty.
\]
On the other hand, we see by the Bernstein inequality in Lemma 2.1 that
\[
\left( \sum_{k \geq 0} 2^{2ks} \| \Delta_k (\cdot)^{\beta} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} \right)^{\frac{1}{2}} \leq \left( \sum_{k \geq 0} 2^{2k(s-[s]-1)} \right)^{\frac{1}{2}} \sup_{|\gamma|=|s|+1} \| D^\gamma (\cdot)^{\beta} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} \\
\leq C \sup_{|\gamma|=|s|+1} \| D^\gamma (\cdot)^{\beta} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)}.
\]
Thanks to the Leibniz formula, we obtain
\[
\sup_{|\gamma|=|s|+1} \| D^\gamma (\cdot)^{\beta} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} \leq \sup_{|\gamma|=|s|+1} \| (\cdot)^{\beta} D^\gamma \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} + C \sum_{|\gamma_1|=1}^{[s]+1} \sup_{|\gamma_1|+|\gamma_2|=|s|+1} \| \partial^{\gamma_1} (\cdot)^{\beta} \partial^{\gamma_2} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)}.
\]
Since \( \text{div} \mathbf{F}_0 = -\mathbf{U}_0 \cdot \nabla \mathbf{U}_0 \), we obtain by Proposition 2.8 in [10] that
\[
\sup_{|\gamma|=|s|+1} \| (\cdot)^{\beta} D^\gamma \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} \leq C \sum_{|\gamma_1|=1}^{[s]+1} \sup_{|\gamma_1|+|\gamma_2|=|s|+1} \| (\cdot)^{\beta} \partial^{\gamma_1} \mathbf{U}_0 \partial^{\gamma_2} \nabla \mathbf{U}_0 \|_{L^2(\mathbb{R}^3)} \\
\leq C \sum_{|\gamma_1|=1}^{[s]+1} \sup_{|\gamma_1|+|\gamma_2|=|s|+1} \| (\cdot)^{2\alpha-1+|\gamma_1|} \partial^{\gamma_1} \mathbf{U}_0 \|_{L^\infty(\mathbb{R}^3)} \| (\cdot)^{2\alpha+|\gamma_2|} \partial^{\gamma_2} \nabla \mathbf{U}_0 \|_{L^\infty(\mathbb{R}^3)} \times \| (\cdot)^{\beta-4\alpha-|s|} \|_{L^2(\mathbb{R}^3)} < \infty.
\]
By the Hölder inequality, we get
\[
\sum_{|\gamma_1|=1}^{[s]+1} \sup_{|\gamma_1|+|\gamma_2|=|s|+1} \| \partial^{\gamma_1} (\cdot)^{\beta} \partial^{\gamma_2} \text{div} \mathbf{F}_0 \|_{L^2(\mathbb{R}^3)} < \infty.
\]
to the three-dimensional generalized Navier-Stokes system with \( \alpha \) the inhomogeneous part \( V \) with

\[
\text{So, we have (4.2)}
\]

Thus, our task is devoted to showing the optimal decay estimate for the solutions (4.3)

\[
\| \text{div } F_0 \|_{H^s(\mathbb{R}^3)} \leq C \| \text{div } F_0 \|_{H^s(\mathbb{R}^3)}.
\]

Since \( \alpha \in [5/6, 1] \), we have by the Leibniz estimate, (3.27) and (3.28) that

\[
\| \text{div } F_0 \|_{H^s(\mathbb{R}^3)} \leq C \| U_0 \|_{L^\infty(\mathbb{R}^3)} \| \nabla U_0 \|_{H^s(\mathbb{R}^3)}
+ C \| \nabla U_0 \|_{L^2(\mathbb{R}^3)} \| U_0 \|_{W^{s, \infty}(\mathbb{R}^3)} < \infty.
\]

So, we have

\[
\left( \sum_{k \geq 0} 2^{2k} \| \hat{\Delta}_k (\cdot) \text{div } F_0 \|_{L^2(\mathbb{R}^3)} \right)^{1/2} < \infty.
\]

This estimate together with (3.71) yields the claim (3.69). Next, by the same argument used in the proof of (3.69), we can show the claim (3.70). So, we complete the proof of Theorem 1.8.

4. THE OPTIMAL DECAY ESTIMATE FOR THE FORWARD SELF-SIMILAR SOLUTIONS TO THE GENERALIZED NAVIER-STOKES EQUATIONS

In this section, we are going to show the optimal decay estimate for the forward self-similar solutions to the generalized Navier-Stokes equations. Firstly, the existence of the large forward self-similar solutions

\[
(4.1) \quad u = t^{\frac{1}{2\alpha-1}} U_0(x/t^{\frac{1}{\alpha}}) + t^{\frac{1}{2\alpha-1}} V(x/t^{\frac{1}{\alpha}})
\]

to the three-dimensional generalized Navier-Stokes system with \( \alpha \in (5/8, 1] \) was shown in [10], where the homogeneous part \( U_0 = \sigma(x) \) with \( \sigma(x) = \sigma(x/|x|) \in C^{1,0}(\mathbb{S}^2) \), and the inhomogeneous part \( V \) solves

\[
(4.2) \quad \begin{cases}
(-\Delta)^\alpha V - \frac{2\alpha - 1}{2\alpha} V - \frac{1}{2\alpha} x \cdot \nabla V + \nabla P = -V \cdot \nabla V + L_{U_0}(V) - U_0 \cdot \nabla U_0, \\
\text{div } V = 0,
\end{cases}
\]

with

\[
L_{U_0} V = -U_0 \cdot \nabla V - V \cdot \nabla U_0.
\]

Since \( \sigma \in C^{1,0}(\mathbb{S}^2) \), we have by Theorem 3.1 that for each \( 2 \leq p < \infty \),

\[
(4.3) \quad \| V \|_{B^{p,\infty}_{\beta}(\mathbb{R}^3)} \leq C (\| \sigma \|_{W^{1,\infty}(\mathbb{S}^2)}).
\]

Thus, our task is devoted to showing the optimal decay estimate for the solutions \( V \) to the problem (4.2). To do this, we firstly construct the approximate solutions \((V_k, P_k)\) to the corresponding linearized equations

\[
(4.4) \quad \begin{cases}
(-\Delta)^\alpha V_k - \frac{2\alpha - 1}{2\alpha} V_k - \frac{1}{2\alpha} x \cdot \nabla V_k + \nabla P_k = f_k, \\
\text{div } V_k = 0,
\end{cases}
\]

where

\[
f_k(x) = -\phi_k(x) \left( V \otimes V_k + U_0 \otimes V_k + V_k \otimes U_0 + U_0 \otimes U_0 \right)(x)
\]
and \( \phi_k = \phi(x/k) \) with the smooth cut-off function \( \phi \) satisfying
\[
\phi(x) = \begin{cases} 
1 & x \in B_{\frac{1}{2}}(0), \\
0 & x \in \mathbb{R}^3 \setminus B_1(0).
\end{cases}
\]
By the standard degree theory as used in [11], it is easy to prove that Cauchy problem (4.4) admits one weak solutions \( V_k \in H^\alpha_\sigma(\mathbb{R}^3) \) such that
\[
\| V_k \|_{H^\sigma(\mathbb{R}^3)} \leq C(\| \sigma_0 \|_{W^{1,\infty}(\mathbb{R}^3)}).
\]
Since \( \text{supp} f_k \subset B_k(0) \), we can write the weak solutions \( V_k \) in the integral form in terms of Proposition 4.1 established in [11] that
\[
(4.5) \quad V_k(x) = \int_0^1 e^{-(\Delta)^\alpha(1-s)} \text{P div}_x \left( s^{-\frac{1}{2}} f_k(s^{-\frac{1}{2}}) \right) \, ds.
\]
Next, we are going to show the optimal decay estimate for the \( V_k \) in two steps, which implies the optimal decay estimate for \( V \) by taking \( k \to \infty \). Let us begin with the lower decay estimate for \( V \).

### 4.1. The lower decay estimate.
In this subsection, we proceed with the lower decay estimate for the \( V_k \) by using boundedness of the operator \( \nabla \text{P} e^{-(\Delta)^\alpha \cdot f} \) in the weighted space. Specifically,

**Proposition 4.1.** Let
\[
(4.6) \quad W(x) = \int_0^1 e^{-(\Delta)^\alpha(1-s)} \text{P div}_x \left( s^{-\frac{1}{2}} f(s^{-\frac{1}{2}}) \right) \, ds.
\]

(i) If \( \alpha \in [5/6, 1] \) and \( f(x) \in L^5(\mathbb{R}^3) \) satisfies
\[
\sup_{x \in \mathbb{R}^3} |x|^{4\alpha-2} |f(x)| < \infty,
\]
we have
\[
(4.7) \quad \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha-2} |W|(x) \leq C \| f \|_{L^5(\mathbb{R}^3)} + C \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha-2} |f|(x) \right).
\]

(ii) If \( \alpha \in [5/6, 1) \) and \( f(x) \in \dot{H}^1(\mathbb{R}^3) \cap \dot{W}^{1,6}(\mathbb{R}^3) \) satisfies
\[
\sup_{x \in \mathbb{R}^3} |x|^{4\alpha-1} \text{div} f(x) < \infty,
\]
we have
\[
(4.8) \quad \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha-1} |\nabla \otimes W|(x) \leq C \| f \|_{L^2(\mathbb{R}^3) \cap L^5(\mathbb{R}^3)} + C \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha-1} \text{div} f(x) \right).
\]

(iii) If \( \alpha = 1 \) and \( f(x) \in \dot{W}^{1, \frac{4}{3}}(\mathbb{R}^3) \cap \dot{W}^{1,6}(\mathbb{R}^3) \) satisfies
\[
\sup_{x \in \mathbb{R}^3} |x|^3 \text{div} f(x) < \infty,
\]
we have
\[
(4.9) \quad \sup_{x \in \mathbb{R}^3} \langle x \rangle^3 |\nabla \otimes W|(x) \leq C \| f \|_{L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)} + C \sup_{x \in \mathbb{R}^3} \left( |x|^3 \text{div} f(x) \right).
\]
Proof. From \([4.6]\), we know that

\[
W(x) = \int_0^1 \int_{\mathbb{R}^3} \nabla (O_\alpha (1 - s, x - y)) \left( s^{\frac{1}{\alpha} - 2} f(y/s^{\frac{1}{\alpha}}) \right) \, dy \, ds,
\]

where \(O_\alpha\) is the generalized Oseen kernel of operator \(\mathbb{P}G_\alpha\) with \(G_\alpha = e^{-(\Delta)^s}\).

Since \(\alpha \in [5/6, 1]\) and \(f \in L^5(\mathbb{R}^3)\), we have by the Young inequality that

\[
\sup_{x \in \mathbb{R}^3} |W(x)| \leq \int_0^1 \int_{\mathbb{R}^3} s^{\frac{1}{\alpha}} \|\nabla (\mathbb{O}_\alpha) (1 - s, \cdot)\|_{L^5(\mathbb{R}^3)} \|f(\cdot / s^{1/\alpha})\|_{L^5(\mathbb{R}^3)} s^{\frac{1}{\alpha} - 2} \, ds \leq C \left\|f\right\|_{L^5(\mathbb{R}^3)} \int_0^1 (1 - s)^{-\frac{3}{2\alpha}} s^{\frac{1}{\alpha} - 2} \, ds \leq C \left\|f\right\|_{L^5(\mathbb{R}^3)}.
\]

(4.10)

Noting that \(O_{q,\alpha} = \Delta_q O_\alpha\), we rewrite \(W\) as

\[
W(x) = \int_0^1 \int_{\mathbb{R}^3} \nabla O_\alpha (1 - s, x - y) \cdot f(\cdot / s^{1/\alpha}) \, dy \, ds = (\sum_{q < 0} + \sum_{q \geq 0}) \int_0^1 \int_{\mathbb{R}^3} \nabla O_{q,\alpha} (1 - s, x - y) \cdot f(y/s^{1/\alpha}) \, dy \, ds =: W^1 + W^2.
\]

We first proceed to bound \(W^2\), which can be decomposed into two parts

\[
W^2 = \sum_{q \geq 0} \int_0^1 \left( \int_{B_{|x|}(0)} + \int_{\mathbb{R}^3 \setminus B_{|x|}(0)} \right) \nabla O_{q,\alpha} (1 - s, x - y) \cdot f(y/s^{1/\alpha}) \, dy \, ds
\]

\[
=: W^2_1 + W^2_2.
\]

Since \(\alpha \in [5/6, 1]\), we have

\[
|W^2_2| \leq \frac{C}{|x|^{\alpha - 2}} \sum_{q \geq 0} \int_0^1 \int_{\mathbb{R}^3} |\nabla O_{q,\alpha} (1 - s, x - y)| \left\|y^{4\alpha - 2} f(\cdot / s^{1/\alpha})\right\| \, dy \, s^{\frac{1}{\alpha} - 2} \, ds
\]

(4.11)

\[
\leq C \sup_{x \in \mathbb{R}^3} \left\|y^{4\alpha - 2} f(\cdot / s^{1/\alpha})\right\| \frac{1}{|x|^{\alpha - 2}} \sum_{q \geq 0} \int_0^1 2^q e^{-c2^{2q\alpha} (1 - s)} \, ds
\]

\[
\leq C \sup_{x \in \mathbb{R}^3} \left\|y^{4\alpha - 2} f(\cdot / s^{1/\alpha})\right\| \frac{1}{|x|^{\alpha - 2}}.
\]

For the term \(W^2_1\), we see that

\[
|W^2_1| \leq \frac{C}{|x|^3} \sum_{q \geq 0} \int_0^1 \int_{B_{|x|}(0)} |x - y|^3 |\nabla O_{q,\alpha} (1 - s, x - y) \cdot f(y/s^{1/\alpha})| \, dy \, s^{\frac{1}{\alpha} - 2} \, ds.
\]

By the Young inequality and Lemma \([20]\) we have

\[
|W^2_1| \leq \frac{C}{|x|^3} \sup_{x \in \mathbb{R}^3} \left\|y^{4\alpha - 2} f(\cdot / s^{1/\alpha})\right\| \sum_{q \geq 0} \int_0^1 2^q e^{-c2^{2q\alpha} (1 - s)} \, ds \int_{B_{|x|}(0)} \frac{1}{|y|^{4\alpha - 2}} \, dy
\]

(4.12)

\[
\leq C \sup_{x \in \mathbb{R}^3} \left\|y^{4\alpha - 2} f(\cdot / s^{1/\alpha})\right\| \frac{1}{|x|^{\alpha - 2}}.
\]
Now we turn to bound $V^2$. Firstly, one writes

$$W^2 = \sum_{q<0} \int_0^1 \left( \int_{B_{1/2}(0)} + \int_{\mathbb{R}^3 \setminus B_{1/2}(0)} \right) \nabla q, (1 - s, x - y) \cdot f(y/s^{\frac{1}{2}}) dy \ s^{\frac{1}{2} - 2} ds$$

$$=: W^2_1 + W^2_2.$$

Similar to the proof of (4.11), we can show that

$$\left| W^2_2 \right| \leq \frac{C}{|x|^{4\alpha - 2}} \sum_{q<0} \int_0^1 \int_{\mathbb{R}^3 \setminus B_{1/2}(0)} \left| \nabla q, (1 - s, x - y) \right| \times$$

$$\times \left| y \right|^{4\alpha - 2} f(y/s^{\frac{1}{2}}) dy \ s^{\frac{1}{2} - 2} ds$$

(4.13)

$$\leq C \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha - 2} |f|(x) \right) \frac{1}{|x|^{4\alpha - 2}} \sum_{q<0} 2^q \int_0^1 ds$$

$$\leq C \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha - 2} |f|(x) \right) \frac{1}{|x|^{4\alpha - 2}}.$$

Lastly, we obtain

$$\left| W^2_1 \right| \leq \frac{C}{|x|^3} \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha - 2} |f|(x) \right) \sum_{q<0} 2^q \int_0^1 ds \int_{B_{1/2}(0)} \frac{1}{|y|^{4\alpha - 2}} dy$$

(4.14)

$$\leq C \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha - 2} |f|(x) \right) \frac{1}{|x|^{4\alpha - 2}}.$$

Collecting estimates (4.11), (4.12), (4.13) and (4.14), we readily have

$$\sup_{|x| \geq 1} |x|^{4\alpha - 2} |W|(x) \leq C \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha - 2} |f|(x) \right).$$

This inequality together with (4.10) leads to

$$\sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha - 2} |W|(x) \leq C \|f\|_{L^6(\mathbb{R}^3)} + C \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha - 2} |f|(x),$$

which implies the first desired estimate in Proposition 4.1.

Next, we turn to show decay estimate for $\nabla \otimes W$. Taking derivative of $W$ with respect to $x_k$, it follows from (4.6) that $\forall k \in \{1, 2, 3\}$,

$$(4.15) \quad \partial_{x_k} W(x) = \int_0^1 \int_{\mathbb{R}^3} \partial_{x_k} (\nabla q, (1 - s, x - y)) \cdot \left( s^{\frac{1}{2} - 2} f(y/s^{\frac{1}{2}}) \right) dy ds.$$
By the Young inequality, one has
\[
\sup_{x \in \mathbb{R}^3} |\partial_{x_k} W(x)| \leq \int_0^1 \left\| \partial_{x_k} \left( O_a(1-s, \cdot) \right) \right\|_{L^2(\mathbb{R}^3)} \left\| \text{div } f(\cdot/s^{\frac{1}{2n}}) \right\|_{L^2(\mathbb{R}^3)} s^{\frac{1}{2n}-2} \, ds
\]
\[
+ \int_1^\frac{1}{s} \left\| \partial_{x_k} \left( O_a(1-s, \cdot) \right) \right\|_{L^\infty(\mathbb{R}^3)} \left\| \text{div } f(\cdot/s^{\frac{1}{2n}}) \right\|_{L^6(\mathbb{R}^3)} s^{\frac{1}{2n}-2} \, ds
\]
(4.16)
\[
\leq C \int_0^1 \left\| \text{div } f \right\|_{L^2(\mathbb{R}^3)} s^{\frac{1}{2n}-2} \, ds + C \int_1^\frac{1}{s} \left\| \text{div } f \right\|_{L^6(\mathbb{R}^3)} (1-s)^{-\frac{1}{2n}} \, ds
\]
\[
\leq C \left( \left\| \text{div } f \right\|_{L^2(\mathbb{R}^3)} + \left\| \text{div } f \right\|_{L^6(\mathbb{R}^3)} \right).
\]
According to (1.15), we decompose $W$ into the following two parts
\[
\partial_{x_k} W = \int_0^1 \left( \int_{B_{\frac{1}{2}}} (1 \notin B_{\frac{1}{2}}(0)) \partial_{x_k} \left( O_a(1-s, x-y) \right) \right) \text{div } f(y/s^{\frac{1}{2n}}) s^{\frac{1}{2n}-2} \, dy ds
\]
\[
= : W_{kS} + W_{kL}.
\]
Noting that
\[
|W_{kL}| \leq \frac{C}{|x|^{4a-1}} \int_0^1 \int_{\mathbb{R}^3 \setminus B_{\frac{1}{2}}(0)} \left\| \partial_{x_k} \left( O_a(1-s, x-y) \right) \right\| |y|^{4a-1} \left\| \text{div } f(y/s^{\frac{1}{2n}}) \right\| s^{\frac{1}{2n}-2} \, dy ds,
\]
we get by the Young inequality that
\[
|W_{kL}|(x) \leq \frac{C}{|x|^{4a-1}} \left\| \partial_{x_k} O_a(1, \cdot) \right\|_{L^1(\mathbb{R}^3)} \sup_{x \in \mathbb{R}^3} |x|^{4a-1} \left\| \text{div } f(x) \right\| \int_0^1 (1-s)^{-\frac{1}{2n}} \, ds
\]
(4.17)
\[
\leq \frac{C}{|x|^{4a-1}} \sup_{x \in \mathbb{R}^3} |x|^{4a-1} \left\| \text{div } f(x) \right\|,
\]
in the last line of (4.17), we have used the fact that $\partial_{x_k} O_a(1, x) \in L^1(\mathbb{R}^3)$. Indeed, we have by the Bernstein inequality in Lemma 2.1 that
\[
\left\| \partial_{x_k} O_a(1, \cdot) \right\|_{L^1(\mathbb{R}^3)} \leq C \sum_{q \in \mathbb{Z}} 2^q \left\| \Delta_q G_a(1, \cdot) \right\|_{L^1(\mathbb{R}^3)}
\]
\[
\leq C \sum_{q \leq 0} 2^q \left\| \Delta_q G_a(1, \cdot) \right\|_{L^1(\mathbb{R}^3)} + C \sum_{q > 0} 2^{-q} \left\| D^2 \Delta_q G_a(1, \cdot) \right\|_{L^1(\mathbb{R}^3)}
\]
\[
\leq C \left\| G_a(1, \cdot) \right\|_{W^{2,1}(\mathbb{R}^3)} < \infty.
\]
As for the term $W_{kS}$, we write in terms of the high-low decomposition
\[
W_{kS} = \left( \sum_{q \leq 0} + \sum_{q > 0} \right) \int_0^1 \int_{B_{\frac{1}{2}}(0)} \partial_{x_k} \left( O_{q,a}(1-s, x-y) \right) \text{div } f(y/s^{\frac{1}{2n}}) s^{\frac{1}{2n}-2} \, dy ds
\]
\[
= : W_{kS}^h + W_{kS}^l.
\]
For the high frequency regime, we easily find that
\[
|W_{kS}^h| \leq \frac{C}{|x|^3} \sum_{q \geq 0} \int_0^1 \int_{\mathbb{R}^3} |x-y|^3 |\partial_{x_k} \left( O_{q,a}(1-s, x-y) \right)| \times
\]
Similarly, we have from (4.18) that for \( \alpha \in [5/6, 1) \) that
\[
\left| W_{kS}^\alpha \right| \leq \frac{C}{|x|^3} \sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} |\text{div} f|(x)) \sum_{q \geq 0} 2^q \int_0^1 \frac{1}{y |y|^{4\alpha - 1}} \text{div} f(y/3^q) \, dy \text{d}s.
\]
(4.18)

Thus we have that for each \( \alpha \in [5/6, 1) \),
\[
\left| W_{kS}^\alpha \right| \leq \frac{C}{|x|^3} \sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} |\text{div} f|(x)) \sum_{q \geq 0} 2^q \int_0^1 \text{div} f(y/3^q) \, dy |y|^{4\alpha - 1} \text{d}s.
\]
(4.19)

For the low frequency regime, we see that
\[
\left| W_{kS}^\alpha \right| \leq \frac{C}{|x|^3} \sum_{q \geq 0} \int_0^1 \int_{B_{3q}(0)} |x - y|^3 |\partial_{x_k} (O_{q,1}(1 - s, x - y))| \times
\]
\[
\text{div} f(y/3^q) \, dy \, s^{\frac{1}{2} - \alpha} \text{d}s.
\]

Thus we have that for each \( \alpha \in [5/6, 1) \),
\[
\left| W_{kS}^\alpha \right| \leq \frac{C}{|x|^3} \sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} |\text{div} f|(x)) \sum_{q \geq 0} 2^q \int_0^1 \text{div} f(y/3^q) \, dy |y|^{4\alpha - 1} \text{d}s.
\]
(4.18)

Combining this estimate with (4.16), (4.17) and (4.19) yields the second desired estimate.

For \( \alpha = 1 \), we apply the Young inequality and Lemma 2.8 to \( W_{kS}^\alpha \) to obtain
\[
\sum_{q > 0} \int_0^1 \int_{\mathbb{R}^3} |x - y|^3 |\partial_{x_k} (O_{q,1}(1 - s, x - y))| \, \text{div} f(y/3^q) \, s^{\frac{1}{2} - \alpha} \, dy \, ds
\]
\[
\leq C \sum_{q > 0} \int_0^1 \int_{\mathbb{R}^3} \left\| |\cdot|^{3} |\partial_{x_k} O_{q,1}(1 - s, \cdot)| \right\|_{L^2(\mathbb{R}^3)} \left\| \text{div} f(y/3^q) \right\|_{L^2(\mathbb{R}^3)} s^{\frac{1}{2} - \alpha} \, ds
\]
\[
\leq C \| \text{div} f \|_{L^2(\mathbb{R}^3)} \sum_{q > 0} 2^q - \frac{\alpha}{2} \int_0^1 s^{-\frac{3}{2}} \, ds \leq C \| \text{div} f \|_{L^2(\mathbb{R}^3)},
\]

which implies that for \( \alpha = 1 \),
\[
(4.20) \quad \left| W_{kS}^\alpha \right|(x) \leq \frac{C}{|x|^3} \| \text{div} f \|_{L^2(\mathbb{R}^3)}.
\]

Similarly, we have from (4.18) that for \( \alpha = 1 \),
\[
\left| W_{kS}^\alpha \right| \leq \frac{C}{|x|^3} \sum_{q \leq 0} \int_0^1 \left\| |\cdot|^{3} |\partial_{x_k} (O_{q,1}(1 - s, \cdot)) \right\|_{L^2(\mathbb{R}^3)} \times
\]
\[
\text{div} f(\cdot/s^q) \left\|_{L^2(\mathbb{R}^3)} s^{\frac{1}{2} - \alpha} \, ds
\]
\[
\leq \frac{C}{|x|^3} \sum_{q \leq 0} 2^q \| \text{div} f \|_{L^2(\mathbb{R}^3)} \int_0^1 s^{-\frac{3}{2}} \, ds \leq \frac{C}{|x|^3} \| \text{div} f \|_{L^2(\mathbb{R}^3)}.
\]
(4.21)
Collecting estimates (4.16), (4.17), (4.20) and (4.21) give the third desired estimate, and then we finish the proof of Proposition 4.1.

We apply Proposition 4.1 to $V_k$ to obtain

$$
(4.22) \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |V_k|(x) \leq C \|f_k\|_{L^6(\mathbb{R}^3)} + C \sup_{x \in \mathbb{R}^3} \langle |x|^{4a-2} |f_k|(x) \rangle.
$$

Since $\sigma \in C^{1,0}(\mathbb{S}^2)$, we get by using Theorem 5.1 that for all $p \in [2, \infty)$

$$
(4.23) \|V_k\|_{B^p_2(\mathbb{R}^3)} \leq C(\|\sigma\|_{W^{1,\infty}(\mathbb{S}^2)}).
$$

This estimate enables us to conclude by the Hölder inequality, (3.27) and (3.28) that

$$
\|f_k\|_{L^6(\mathbb{R}^3)} \leq C \left( \|V_k\|_{L^\infty(\mathbb{R}^3)} + \|U_0\|_{L^\infty(\mathbb{R}^3)} \|\nabla V\|_{L^2(\mathbb{R}^3)} \right.
$$
$$
+ C \left( \|V\|_{L^\infty(\mathbb{R}^3)} + \|U_0\|_{L^\infty(\mathbb{R}^3)} \|\nabla U_0\|_{L^2(\mathbb{R}^3)} \right)
$$
$$
\leq C \|V_k\|_{H^{2a}(\mathbb{R}^3)}^2 + C \|V\|_{H^{2a}(\mathbb{R}^3)}^2 + C \|U_0\|_{L^\infty(\mathbb{R}^3)}^2 + C \|\nabla U_0\|_{L^2(\mathbb{R}^3)}^2
$$
$$
\leq C(\|\sigma\|_{W^{1,\infty}(\mathbb{S}^2)}).
$$

By the Hölder inequality, one has

$$
(4.24) \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |V \otimes V_k|(x)
$$
$$
\leq R^{4n-2} \|V\|_{L^\infty(\mathbb{R}^3)} \|V_k\|_{L^\infty(\mathbb{R}^3)} + \|V\|_{L^\infty(\mathbb{R}^3 \setminus B_R(0))} \sup_{|x| \geq R} \langle x \rangle^{4a-2} |V_k|(x),
$$

where $R$ to be fixed later.

By the Hölder inequality, the interpolation inequality and Proposition 2.8 in [10], we get

$$
\sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |U_0 \otimes V_k + U_0 \otimes V_k + U_0 \otimes U_0|(x)
$$
$$
\leq \sup_{x \in \mathbb{R}^3} \langle x \rangle^{2a-1} |U_0|(x) \sup_{x \in \mathbb{R}^3} \langle x \rangle^{2a-1} |V_k|(x) + \left( \sup_{x \in \mathbb{R}^3} \langle x \rangle^{2a-1} |U_0|(x) \right)^2
$$
$$
\leq C \|V_k\|_{L^\infty(\mathbb{R}^3)}^2 \left( \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |V_k|(x) \right)^{\frac{1}{2}} + C.
$$

Moreover, we get by the Young inequality and $\sigma \in C^{1,0}(\mathbb{S}^2)$ that

$$
(4.25) \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |U_0 \otimes V_k + U_0 \otimes V_k + U_0 \otimes U_0|(x) \leq C + \frac{1}{4} \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |V_k|(x).
$$

Inserting (4.24) and (4.25) into (4.22) gives

$$
(4.26) \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |V_k|(x) \leq C + \frac{1}{4} \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4a-2} |V_k|(x)
$$
$$
+ \|V\|_{L^\infty(\mathbb{R}^3 \setminus B_R(0))} \sup_{|x| \geq R} \langle x \rangle^{4a-2} |V_k|(x).
$$
With the help of the interpolation theorem, we have
\[
\|V\|_{L^\infty(B_r(0))} \leq \left\| \left(1 - \phi_R \right)V \right\|_{L^\infty(\mathbb{R}^3)} \\
\leq C \left\| \left(1 - \phi_R \right)V \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \left(1 - \phi_R \right)V \right\|_{L^{\frac{18}{5}}(\mathbb{R}^3)} \\
\leq C \left\| V \right\|_{L^2(\mathbb{R}^3)} \left\| \nabla \left(1 - \phi_R \right)V \right\|_{L^{\frac{18}{5}}(\mathbb{R}^3)}. \tag{4.27}
\]
Plugging this estimate into (4.27), we readily have
\[
\|\nabla \left(1 - \phi_R \right)V\|_{L^{\frac{18}{5}}(\mathbb{R}^3)} \leq \frac{C}{R} \|V\|_{L^{\frac{18}{5}}(\mathbb{R}^3)} + C \|\nabla V\|_{L^{\frac{18}{5}}(B_{\frac{3}{2}(0)})}. \tag{4.28}
\]
The Leibniz estimate allows us to conclude
\[
\|\nabla \left(1 - \phi_R \right)V\|_{L^{\frac{18}{5}}(\mathbb{R}^3)} \leq C \frac{1}{R} \|V\|_{L^{\frac{18}{5}}(\mathbb{R}^3)} + C \|\nabla V\|_{L^{\frac{18}{5}}(B_{\frac{3}{2}(0)})}. \tag{4.27}
\]
Plugging this estimate into (4.27), we readily have
\[
\|V\|_{L^\infty(B_{\frac{3}{2}(0)})} \leq \frac{C}{R^m} + C \|\nabla V\|_{L^{\frac{18}{5}}(B_{\frac{3}{2}(0)})}. \tag{4.28}
\]
Since \(V \in H^{2\alpha}(\mathbb{R}^3)\) with \(\alpha \in [\frac{5}{6}, 1]\), we get by the inclusion
\[
H^{2\alpha}(\mathbb{R}^3) \hookrightarrow W^{1, \frac{18}{5}}(\mathbb{R}^3)
\]
and (4.28) that
\[
\lim_{R \to +\infty} \|\nabla V\|_{L^\infty(B_R(0))} = 0.
\]
Thus, there exists a number \(R_0 > 0\) such that for all \(R > R_0\),
\[
\|\nabla V\|_{L^\infty(B_R(0))} < \frac{1}{4}.
\]
Then (4.11) becomes
\[
\sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha - 2} |V_k(x)| \leq C \left(\|\sigma\|_{W^{1, \infty}(\mathbb{S}^2)} \right) + \frac{1}{2} \sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha - 2} |V_k(x)|,
\]
which implies
\[
\sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha - 2} |V_k(x)| \leq C \left(\|\sigma\|_{W^{1, \infty}(\mathbb{S}^2)} \right). \tag{4.29}
\]
Since \(V\) is the limit of \(V_k\), we get from Fatou’s lemma and (4.29) that
\[
\sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha - 2} |V(x)| \leq C \left(\|\sigma\|_{W^{1, \infty}(\mathbb{S}^2)} \right). \tag{4.30}
\]
Next, we begin to show the decay estimate for \(\nabla V_k\) which fulfills
\[
\nabla V_k(x) = \int_0^1 \int_{\mathbb{R}^3} \nabla O_\alpha(1 - s, x - y) \text{div}_x \left( s^{\frac{4}{3}} f \left( \frac{y}{s^{\frac{1}{3}}} \right) \right) dy ds.
\]
Based on this equality, we readily have by Proposition 4.1 that for each \(\alpha \in [\frac{5}{6}, 1]\),
\[
\sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha - 1} |\nabla V_k(x)| \leq C \| \text{div} f_k \|_{L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)} + C \sup_{x \in \mathbb{R}^3} \left( |x|^{4\alpha - 1} \| \text{div} f_k \|_{L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)} \right)
\]
and for \(\alpha = 1\),
\[
\sup_{x \in \mathbb{R}^3} \langle x \rangle^3 |\nabla V_k(x)| \leq C \| \text{div} f_k \|_{L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)} + C \sup_{x \in \mathbb{R}^3} \left( |x|^3 \| \text{div} f_k \|_{L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)} \right). \tag{4.32}
\]
By the triangle inequality, (3.27), (3.28), (4.3) and (4.31), we easily find that
\[
\sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} | \text{div} f_k| (x)) \\
\leq \sup_{x \in \mathbb{R}^3} |x|^{4\alpha - 1} | \mathbf{V} \cdot \nabla \mathbf{k} + \mathbf{V}_k \cdot \nabla \mathbf{U}_0 + \mathbf{U}_0 \cdot \nabla \mathbf{V}_k| + \sup_{x \in \mathbb{R}^3} |x|^{4\alpha - 1} | \mathbf{U}_0 \cdot \nabla \mathbf{V}_k|
\leq C + C \sup_{x \in \mathbb{R}^3} \langle x \rangle^{2\alpha} | \nabla \mathbf{V}_k|(x).
\]
Since
\[
\sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} | \nabla \mathbf{V}_k| (x)) \leq \| \nabla \mathbf{V}_k \|_{L^\infty (\mathbb{R}^3)} \left( \sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} | \nabla \mathbf{V}_k| (x)) \right)^{\frac{2\alpha}{4\alpha - 1}},
\]
we have by the Young inequality that
\[
\sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} | \text{div} f_k| (x)) \leq C + C \left( \sup_{x \in \mathbb{R}^3} (|x|^{2\alpha - 1} | \nabla \mathbf{V}_k| (x)) \right)^{\frac{2\alpha}{4\alpha - 1}} \\
\leq C + \frac{1}{4} \sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} | \nabla \mathbf{V}_k| (x)).
\]
Inserting this estimate into (4.31) and (4.32), respectively, yields that for each \( \alpha \in [5/6, 1), \)
(4.33) \[
\sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} | \nabla \mathbf{V}_k| (x)) \leq C \| \text{div} f_k \|_{L^2 (\mathbb{R}^3) \cap L^6 (\mathbb{R}^3)} + C
\]
and for \( \alpha = 1, \)
(4.34) \[
\sup_{x \in \mathbb{R}^3} (|x|^{3} | \nabla \mathbf{V}_k| (x)) \leq C \| \text{div} f_k \|_{L^\frac{5}{3} (\mathbb{R}^3) \cap L^6 (\mathbb{R}^3)} + C.
\]
By the Hölder inequality, (3.27), (3.28) and (4.3), one has that for each \( \alpha \in [5/6, 1), \)
\[
\| \text{div} f_k \|_{L^6 (\mathbb{R}^3)} \leq \| \mathbf{V} \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{V}_k \|_{L^6 (\mathbb{R}^3)} + \| \mathbf{V}_k \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{U}_0 \|_{L^6 (\mathbb{R}^3)} \\
+ \| \mathbf{U}_0 \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{V}_k \|_{L^6 (\mathbb{R}^3)} + \| \mathbf{U}_0 \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{U}_0 \|_{L^6 (\mathbb{R}^3)}
\leq C(\| \sigma \|_{W^{1, \infty} (\mathbb{S}^2)}).
\]
and
\[
\| \text{div} f_k \|_{L^2 (\mathbb{R}^3)} \leq \| \mathbf{V} \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{V}_k \|_{L^2 (\mathbb{R}^3)} + \| \mathbf{V}_k \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{U}_0 \|_{L^2 (\mathbb{R}^3)} \\
+ \| \mathbf{U}_0 \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{V}_k \|_{L^2 (\mathbb{R}^3)} + \| \mathbf{U}_0 \|_{L^\infty (\mathbb{R}^3)} \| \nabla \mathbf{U}_0 \|_{L^2 (\mathbb{R}^3)}
\leq C(\| \sigma \|_{W^{1, \infty} (\mathbb{S}^2)}).
\]
Plugging both estimates into (4.33) gives that for each \( \alpha \in [5/6, 1), \)
\[
\sup_{x \in \mathbb{R}^3} (|x|^{4\alpha - 1} | \nabla \mathbf{V}_k| (x)) \leq C(\| \sigma \|_{W^{1, \infty} (\mathbb{S}^2)}).
\]
For \( \alpha = 1, \) we can show by the Hölder inequality (3.27), (3.28) and (4.3) again that
\[
\| \text{div} f_k \|_{L^\frac{5}{3} (\mathbb{R}^3)} \leq \| \mathbf{V} \|_{L^\frac{5}{3} (\mathbb{R}^3)} \| \nabla \mathbf{V}_k \|_{L^\frac{5}{3} (\mathbb{R}^3)} + \| \mathbf{V}_k \|_{L^\frac{5}{3} (\mathbb{R}^3)} \| \nabla \mathbf{U}_0 \|_{L^\frac{5}{3} (\mathbb{R}^3)} \\
+ \| \mathbf{U}_0 \|_{L^\frac{5}{3} (\mathbb{R}^3)} \| \nabla \mathbf{V}_k \|_{L^2 (\mathbb{R}^3)} + \| \mathbf{U}_0 \|_{L^\frac{5}{3} (\mathbb{R}^3)} \| \nabla \mathbf{U}_0 \|_{L^\frac{5}{3} (\mathbb{R}^3)}
\leq C(\| \sigma \|_{W^{1, \infty} (\mathbb{S}^2)}).
Inserting this estimate into (4.34) implies that for $\alpha = 1$,

$$
\sup_{x \in \mathbb{R}^3} (x)^3 |\nabla V_k(x)| \leq C(\|\sigma\|_{W^{1,\infty}(\mathbb{S}^2)}).
$$

Finally, we get by taking $k \to \infty$ that for each $\alpha \in [5/6, 1]$,

$$
(4.35) \quad \sup_{x \in \mathbb{R}^3} (x)^{4\alpha-1} |\nabla V| (x) \leq C(\|\sigma\|_{W^{1,\infty}(\mathbb{S}^2)}).
$$

4.2. The decay estimate with logarithmic loss. In this subsection, we are going to show the almost optimal estimate with logarithmic loss which is caused by the Leray projector $\mathbb{P}$.

**Proposition 4.2.** Let $\alpha \in [5/6, 1]$ and $f(x) \in \dot{H}^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ satisfies

$$
(4.36) \quad \sup_{x \in \mathbb{R}^3} \left| (x)^{4\alpha-2} f(x) \right| + \sup_{x \in \mathbb{R}^3} \left| (x)^{4\alpha-1} \nabla f(x) \right| < \infty,
$$

and

$$
(4.37) \quad W(x) = \int_0^1 e^{(-\Delta)^{\alpha}(1-s)} \mathbb{P} \text{div}_x \left( s^{\frac{1}{2\alpha}-2} f \left( \frac{\cdot}{s^{\frac{1}{2\alpha}}} \right) \right) \, ds.
$$

Then $W$ enjoys the decay estimate

$$
(4.38) \quad |W|(x) \leq C(\langle x \rangle)^{-(4\alpha-1)} \log \langle x \rangle \quad \text{for each} \quad x \in \mathbb{R}^3.
$$

**Proof.** Thanks to (4.10), it suffices to show the decay estimate of $W$ for large $|x|$. So we always assume that $|x| \geq 200$ in the remaining part of the proof. In terms of the low-high frequency decomposition, one splits $W$ into three parts as follows

$$
W = \left( \sum_{q \geq 0} + \sum_{N_x \leq q < 0} + \sum_{q < N_x} \right) \int_0^1 \dot{\Delta}_q \mathbb{P} e^{(-\Delta)^{\alpha}(1-s)} s^{\frac{1}{2\alpha}-2} \left( \text{div} f \left( \frac{\cdot}{s^{\frac{1}{2\alpha}}} \right) \right) \, ds
$$

$$
=: I + II + III,
$$

where $N_x = -\lfloor \log_2 |x| \rfloor + 1$.

The first term $I$ which can be rewritten as

$$
I = \sum_{q \geq 0} \int_0^1 \int_{B_{\frac{s}{2}}} O_{a,q}(1-s, x - y) s^{\frac{1}{2\alpha}-2} \left( \text{div} f \left( \frac{\cdot}{s^{\frac{1}{2\alpha}}} \right) \right) \, dy \, ds
$$

$$
+ \sum_{q \geq 0} \int_0^1 \int_{\mathbb{R}^3 \setminus B_{\frac{s}{2}}} O_{a,q}(1-s, x - y) s^{\frac{1}{2\alpha}-2} \left( \text{div} f \left( \frac{\cdot}{s^{\frac{1}{2\alpha}}} \right) \right) \, dy \, ds
$$

$$
=: I_1 + I_2.
$$

The term $I_2$ can be bounded as follows

$$
I_2 = \sum_{q \geq 0} \int_0^1 \int_{\mathbb{R}^3 \setminus B_{\frac{s}{2}}} O_{a,q}(1-s, x - y) \frac{1}{|x|^{4\alpha-1}} \left( \frac{|y|}{s^{\frac{1}{2\alpha}}} \right)^{4\alpha-1} \left( \text{div} f \left( \frac{\cdot}{s^{\frac{1}{2\alpha}}} \right) \right) \, dy \, ds
$$

$$
\leq \frac{4}{|x|^{4\alpha-1}} \sum_{q \geq 0} \int_0^1 \int_{\mathbb{R}^3} |O_{a,q}|(1-s, x - y) \left( \frac{|y|}{s^{\frac{1}{2\alpha}}} \right)^{4\alpha-1} \left| \text{div} f \left( \frac{\cdot}{s^{\frac{1}{2\alpha}}} \right) \right| \, dy \, ds.
$$
Lemma 2.8 enables us to conclude that

\[
|I_2| \leq \frac{4}{|x|^{4a-1}} \sup_{x \in \mathbb{R}^3} |x|^{4a-1} \text{div } f(x) \sum_{q \geq 0} \int_0^1 e^{-c(1-s)2^{2q\alpha}} ds
\]

\[
\leq \frac{C}{|x|^{4a-1}} \sup_{x \in \mathbb{R}^3} |x|^{4a-1} \text{div } f(x) \sum_{q \geq 0} 2^{-2q\alpha} \leq C \sup_{x \in \mathbb{R}^3} |x|^{4a-1} \text{div } f(x) \frac{1}{|x|^{4a-1}}.
\]  

(4.39)

As for $I_1$, we see that

\[
I_1 = \frac{1}{|x|^{4a-1}} \sum_{q \geq 0} \int_0^1 \int_{B_{|x|}(0)} |x|^{4a-1} O_{\alpha,q}(1-s, x-y) s^{\frac{1}{2\alpha}-2} (\text{div } f) (y/s^{\frac{1}{2\alpha}}) dy ds
\]

\[
\leq \frac{C}{|x|^{4a-1}} \sum_{q \geq 0} \int_0^1 \int_{\mathbb{R}^3} |x-y|^{4a-1} |O_{\alpha,q}|(1-s, x-y) s^{\frac{1}{2\alpha}-2} |\text{div } f| (y/s^{\frac{1}{2\alpha}}) dy ds.
\]

Moreover, we have by Lemma 2.8 that

\[
|I_1| \leq \frac{C}{|x|^{4a-1}} \sum_{q \geq 0} 2^{-\frac{q}{4}(4a-1)-\frac{1}{2}} \int_0^1 e^{-c(1-s)2^{2q\alpha}} s^{\frac{1}{2\alpha}-2} \left\| \text{div } f(\cdot/s^{\frac{1}{2\alpha}}) \right\|_{L^2(\mathbb{R}^3)} ds
\]

\[
\leq C \| \text{div } f \|_{L^2(\mathbb{R}^3)} \frac{1}{|x|^{4a-1}} \int_0^1 s^{\frac{1}{2\alpha}-2+\frac{q}{4}} ds \leq C \| f \|_{B^1(\mathbb{R}^3)} \frac{1}{|x|^{4a-1}}.
\]  

(4.40)

For the term $II$, we need to use the smooth cut-off function $\phi_{|x|}(y)$ to decompose the force $f(y/s^{\frac{1}{2\alpha}})$ into two parts as follows

\[
f(y/s^{\frac{1}{2\alpha}}) = \phi_{|x|}(y)f(y/s^{\frac{1}{2\alpha}}) + \left(1 - \phi_{|x|}(y)\right) f(y/s^{\frac{1}{2\alpha}}) =: f^1(y/s^{\frac{1}{2\alpha}}) + f^2(y/s^{\frac{1}{2\alpha}}).
\]

Thus, we can rewrite in terms of the support of $\phi_{|x|}(y)$ the second term $II$ as

\[
II = \sum_{N_k \leq q < 0} \int_0^1 \int_{B_{|x|}(0)} \nabla O_{\alpha,q}(1-s, x-y) s^{\frac{1}{2\alpha}-2} \cdot f^2(y/s^{\frac{1}{2\alpha}}) dy ds
\]

\[
+ \sum_{N_k \leq q < 0} \int_0^1 \int_{\mathbb{R}^3 \setminus B_{|x|}(0)} O_{\alpha,q}(1-s, x-y) s^{\frac{1}{2\alpha}-2} (\text{div } f^2) (y/s^{\frac{1}{2\alpha}}) dy ds
\]

\[
= II_1 + II_2.
\]  

(4.41)

Following the same method as used in the proof of (4.30), we can show that $|II_1|$ can be bounded by

\[
\frac{C}{|x|^{4a+2}} \sum_{N_k \leq q < 0} \int_0^1 \int_{B_{|x|}(0)} |x-y|^{4a+2} |\nabla O_{\alpha,q}|(1-s, x-y) s^{\frac{1}{2\alpha}-2} |f| (y/s^{\frac{1}{2\alpha}}) dy ds
\]

\[
\leq C \sup_{x \in \mathbb{R}^3} \frac{|x|^{4a-2} f(x)|}{|x|^{4a+2}} \sum_{N_k \leq q < 0} 2^{-q(4a+2-4)} \int_{B_{|x|}(0)} \frac{1}{|y|^{3a-2}} dy
\]

\[
\leq C \sup_{x \in \mathbb{R}^3} \frac{|x|^{4a-2} f(x)|}{|x|^{4a+2}} |x|^{4a+2-4} |x|^{3-4a+2} \leq C \sup_{x \in \mathbb{R}^3} \frac{|x|^{4a-2} f(x)|}{|x|^{4a+1}}.
\]
which implies

\[(4.42) \quad |\Pi_1| \leq C \sup_{x \in \mathbb{R}^3} \left| |x|^{4\alpha-2} f(x) \right| \frac{1}{|x|^{4\alpha-1}}.
\]

In the same fashion in (4.39), it is easy to show by the fact $|\nabla \phi_\xi| \leq \frac{C}{|\xi|}$ that

\[
|\Pi_2| \leq \frac{C}{|x|^{4\alpha-1}} \sup_{x \in \mathbb{R}^3} \left( \left| |x|^{4\alpha-1} \text{div} f(x) \right| + \left| |x|^{4\alpha-2} f(x) \right| \right) \sum_{\alpha,q \leq q < 0} \int_0^1 e^{-c(1-s)2^\alpha} ds
\]

\[
(4.43) \quad \leq \frac{C}{|x|^{4\alpha-1}} \sup_{x \in \mathbb{R}^3} \left( \left| |x|^{4\alpha-1} \text{div} f(x) \right| + \left| |x|^{4\alpha-2} f(x) \right| \right) \sum_{\alpha,q \leq q < 0} 1
\]

\[
\leq C \sup_{x \in \mathbb{R}^3} \left( \left| |x|^{4\alpha-1} \text{div} f(x) \right| + \left| |x|^{4\alpha-2} f(x) \right| \right) \frac{1}{|x|^{4\alpha-1}} \log(x).
\]

For the last term $\Pi_3$, we decompose it in terms of (4.41) that

\[
\Pi_3 = \sum_{q < N_x} \int_0^1 \int_{B_{\frac{|x|}{2}}(0)} \nabla O_{\alpha,q}(1-s, x-y) s^{\frac{4\alpha}{4}} \cdot f^2(y/s^{\frac{4}{4}}) dy ds
\]

\[
+ \sum_{q < N_x} \int_0^1 \int_{R^3 \setminus B_{\frac{|x|}{2}}(0)} \nabla O_{\alpha,q}(1-s, x-y) s^{\frac{4\alpha}{4}} \cdot f^2(y/s^{\frac{4}{4}}) dy ds
\]

\[
= \Pi_{12} + \Pi_{22}.
\]

We perform the process as used in deriving (4.40) to get that $|\Pi_{12}|$ can be bounded by

\[
\frac{C}{|x|^{4\alpha-1}} \sum_{q < N_x} \int_0^1 \int_{B_{\frac{|x|}{2}}(0)} |x-y|^{4\alpha-1} |\nabla O_{\alpha,q}| \left( 1-s, x-y \right) s^{\frac{4\alpha}{4}} |f^2(y/s^{\frac{4}{4}}) dy ds
\]

\[
\leq C \sup_{x \in \mathbb{R}^3} |x|^{4\alpha-2} f(x) \frac{1}{|x|^{4\alpha-1}} \sum_{q < N_x} 2^{-q(4\alpha-4)} \int_{B_{\frac{|x|}{2}}(0)} 1 |y|^{4\alpha-2} dy
\]

\[
\leq C \sup_{x \in \mathbb{R}^3} |x|^{4\alpha-2} f(x) \frac{1}{|x|^{4\alpha-1}} |x|^{4\alpha-1} |x|^{3-(4\alpha-2)} \leq C \sup_{x \in \mathbb{R}^3} \left| |x|^{4\alpha-2} f(x) \right| \frac{1}{|x|^{4\alpha-1}}.
\]

This estimate guarantees that

\[(4.44) \quad |\Pi_{12}| \leq C \sup_{x \in \mathbb{R}^3} |x|^{4\alpha-2} f(x) \frac{1}{|x|^{4\alpha-1}}.
\]

Now it remains to deal with $\Pi_{22}$. Since $|y| \geq \frac{1}{2}|x|$, we have

\[
\Pi_{22} \leq \frac{4}{|x|^{4\alpha-2}} \sum_{q < N_x} \int_0^1 \int_{R^3} |\nabla O_{\alpha,q}| \left( 1-s, x-y \right) \left( \frac{|y|}{s^{\frac{4}{4}}} \right)^{4\alpha-2} |f^2(y/s^{\frac{4}{4}}) dy ds
\]

\[
\leq C \sup_{x \in \mathbb{R}^3} |x|^{4\alpha-2} f(x) \frac{1}{|x|^{4\alpha-2}} \sum_{q < N_x} 2^q \leq C \sup_{x \in \mathbb{R}^3} \left| |x|^{4\alpha-2} f(x) \right| \frac{1}{|x|^{4\alpha-1}}.
\]

Combining this estimate with (4.39), (4.40), (4.42), (4.43) and (4.44) yields

\[
\sup_{|x| \geq 2} |W(x)| \leq C \frac{1}{|x|^{4\alpha-1}} \log(x).
\]
This inequality together with (4.10) implies the estimate (4.38).

By Proposition 4.2, we immediately get the decay estimate in Theorem 1.3. Now we turn to show the decay estimate in Theorem 1.8. With both estimates (4.33) and (4.35) in hand, we write \( V \) in the integral form in terms of Proposition 4.1 established in [11] that

\[
V(x) = \int_0^1 e^{-(-\Delta)^{\alpha}(1-s)}\mathbb{P} \text{div}_x \left( s^{\frac{1}{\alpha}-2} f \left( \cdot, \frac{x}{s^{\frac{1}{\alpha}}} \right) \right) \, ds,
\]

where

\[
f(x) = - \left( V \otimes V + U_0 \otimes V + V \otimes U_0 + U_0 \otimes U_0 \right)(x).
\]

Since \( \alpha \in [5/6, 1] \) and \( \sigma \in C^{1,0}(\mathbb{S}^2) \), it is easy to verify by using both estimates (4.33) and (4.35) again that

\[
\sup_{x \in \mathbb{R}^3} \langle x \rangle^{4\alpha-1} \text{div} \left( V \otimes V + U_0 \otimes V + V \otimes U_0 + U_0 \otimes U_0 \right)(x) < \infty,
\]

which meets the condition (4.36) in Proposition 4.2. Thus we have

\[
|V(x)| \leq C(\|\sigma\|_{W^{1,\infty}(\mathbb{S}^2)}) (1 + |x|)^{4\alpha-1} \log^{-1}(1 + |x|),
\]

which is the decay estimate in Theorem 1.8.

4.3. The optimal decay estimate. In this section, we are going to show the optimal decay estimate for \( V \) with the special force \( f \). Since the decay rate for each term of \( f \) is different, we decompose it into two parts as follows

\[
f(x) = - \left( V \otimes V + U_0 \otimes V + V \otimes U_0 \right)(x) - (U_0 \otimes U_0)(x),
\]

where \( f_1 = - \left( V \otimes V + U_0 \otimes V + V \otimes U_0 \right)(x). \)

**Proposition 4.3.** Let \( \alpha \in [\frac{5}{6}, 1] \), \( \sigma \in C^{1,0}(\mathbb{S}^2) \), and \( V \) be a weak solution to the system (1.6). Assume \( f_2(x) \in H^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3) \) satisfies

\[
\sup_{x \in \mathbb{R}^3} |x|^{4\alpha-2} f_2(x) \| + \sup_{x \in \mathbb{R}^3} |x|^{4\alpha-1} \nabla f_2(x) \| < \infty,
\]

and

\[
W(x) = \int_0^1 e^{-(-\Delta)^{\alpha}(1-s)}\mathbb{P} \text{div}_x \left( s^{\frac{1}{\alpha}-2}(f_1 + f_2)(\cdot, \frac{x}{s^{\frac{1}{\alpha}}}) \right) \, ds.
\]

If, moreover, there exists some \( \gamma \in (0, 1) \) such that

\[
\| \cdot |^{4\alpha-1+\gamma} \text{div} f_2 \|_{C^{\gamma}(\mathbb{R}^3)} < \infty,
\]

then we have

\[
|W(x)| \leq C \left( \|\sigma\|_{W^{1,\infty}(\mathbb{R}^3)} \| \cdot |^{4\alpha-1+\gamma} \text{div} f_2 \|_{C^{\gamma}(\mathbb{R}^3)} \right) (1 + |x|)^{4\alpha-1}.
\]
Proof. From (4.46), we have
\[
|W(x)| \leq C(\|\sigma\|_{L^1(\mathbb{R}^3)})(1 + |x|)^{4\alpha-1} \log^{-1}(1 + |x|).
\]
This decay estimate allows us to write
\[
W(x) = \int_0^1 e^{-(\Delta)^{(1-s)}} \mathbb{P} \text{div}_x \left( s^{\frac{1}{\alpha}-2} (f_1 + f_2) \cdot \frac{1}{s^{\frac{1}{\alpha}}} \right) \, ds
=: W^2 + W^4.
\]

For |x| ≥ 200, as in the proof of Proposition 4.2, we decompose W^2 as follows
\[
W^2 = \left( \sum_{q \geq 0} + \sum_{N_x \leq q < 0} + \sum_{q < N_x} \right) \int_0^1 \Delta_{q} \mathbb{P} e^{-(\Delta)^{(1-s)}} s^{\frac{1}{\alpha}-2} (\text{div} f) \cdot \frac{1}{s^{\frac{1}{\alpha}}} \, ds
=: I + II + III,
\]
where N_x = [log_2 |x|] + 1.

From the proof of Proposition 4.2, we know that
\[
(\|I\| + |\|I_1\| + |\|III\||)(x) \leq C(1 + |x|)^{4\alpha-1},
\]
where I_1 comes from the following decomposition
\[
I_1 = \sum_{N_x \leq q < 0} \int_0^1 B_{|x|^2(0)} \nabla O_{a,q}(1-s, x-y) s^{\frac{1}{\alpha}-2} \left( \phi_{\frac{x}{s}} f \right) \left( y/s^{\frac{1}{\alpha}} \right) \, dy \, ds
+ \sum_{i=1}^2 \sum_{N_x \leq q < 0} \int_0^1 \int_{\mathbb{R}^3 \setminus B_{|x|^2(0)}} O_{a,q}(1-s, x-y) s^{\frac{1}{\alpha}-2} \text{div} \left( \phi_{\frac{x}{s}} f_i \right) \left( y/s^{\frac{1}{\alpha}} \right) \, dy \, ds
=: I_{11} + I_{12} + I_{13}.
\]

So our task is now to bound I_{11} and I_{12}. To begin with, we estimate I_{11}. In terms of estimates (3.27), (3.28) and (4.50), it is easy to check that there exist a positive constant C such that
\[
\sup_{x \in \mathbb{R}^3} |x|^{4\alpha-1} |f_i|(x) \leq C.
\]
With the decay estimate (4.52) in hand, by using the triangle inequality, we readily have that for all |x| ≥ 200,
\[
|I_{11}| \leq \frac{C}{|x|^{4\alpha}} \sum_{N_x \leq q < 0} \int_{\mathbb{R}^3} |O_{a,q}|(1-s, x-y) s^{\frac{1}{\alpha}-2} \left| y^{4\alpha} \text{div} \left( \phi_{\frac{x}{s}} f_i \right) \left( y/s^{\frac{1}{\alpha}} \right) \right| \, dy \, ds.
\]
By the Young inequality and (4.52), we obtain
\[
|I_{11}| \leq \frac{C}{|x|^{4\alpha}} \left( \sup_{x \in \mathbb{R}^3} |x|^{4\alpha-1} |f_i|(x) + \sup_{x \in \mathbb{R}^3} |x|^{4\alpha} |\text{div} f_i|(x) \right) \sum_{N_x \leq q < 0} \int_0^1 s^{\frac{1}{\alpha}} \, ds
\leq \frac{C}{|x|^{4\alpha}} \log(x).
\]
Collecting (4.51) and (4.53) yields that for all $|x| \geq 200$,  
\begin{equation}
(4.54) \quad |\mathbf{V}^{\natural} (x)| \leq C(1 + |x|)^{4\alpha - 1}.
\end{equation}

It remains for us to bound $\Pi_{2,2}$ for large $|x|$. Firstly, we note that  
\begin{equation}
(4.55) \quad \Pi_{2,2} = \sum_{N_x \leq q < 0} \int_0^1 \int_{\mathbb{R}^3} O_{\alpha,q}(1 - s, x - y)s^{\frac{1}{\gamma} - 2} \Delta_{y} \left( \frac{\text{div} f_2}{s^{\frac{1}{\gamma}}} \right) (y/s^{\frac{1}{\gamma}}) \, dy \, ds 
\leq C \sum_{N_x \leq q < 0} \int_0^1 2^{-q\gamma} \int_{\mathbb{R}^3} \left\| O_{\alpha}(\gamma) \right\|_{L^1(\mathbb{R}^3)} \left\| \left( \text{div} f_2 \right) \left( \frac{\cdot/s^{\frac{1}{\gamma}}}{} \right) \right\|_{B_{\infty,\infty}^{\gamma}(\mathbb{R}^3)} \, ds,
\end{equation}

where 
\begin{align*}
f_2^y(y/s^{\frac{1}{\gamma}}) = \left( 1 - \phi_{\frac{|x|}{2}}(y) \right) f_2(y/s^{\frac{1}{\gamma}}).
\end{align*}

Since  
\begin{align*}
(\text{div} f_2)(y/s^{\frac{1}{\gamma}}) = -\nabla \left( \phi_{\frac{|x|}{2}}(y) \right) \cdot f_2(y/s^{\frac{1}{\gamma}}) + \left( 1 - \phi_{\frac{|x|}{2}}(y) \right) (\text{div} f_2)(y/s^{\frac{1}{\gamma}}),
\end{align*}
we further obtain by the fact the Hölder space $\dot{C}^{\gamma}(\mathbb{R}^3) = B_{\infty,\infty}^{\gamma}(\mathbb{R}^3)$ for all $\gamma \in (0,1)$ that  
\begin{align*}
\left\| \left( \text{div} f_2 \right) \left( \frac{\cdot/s^{\frac{1}{\gamma}}}{} \right) \right\|_{B_{\infty,\infty}^{\gamma}(\mathbb{R}^3)} & \leq \left\| \nabla \left( \phi_{\frac{|x|}{2}}(\cdot) \right) \cdot f_2(\cdot/s^{\frac{1}{\gamma}}) \right\|_{\dot{C}^{\gamma}(\mathbb{R}^3)} + \left\| \left( 1 - \phi_{\frac{|x|}{2}}(\cdot) \right) (\text{div} f_2)(\cdot/s^{\frac{1}{\gamma}}) \right\|_{\dot{C}^{\gamma}(\mathbb{R}^3)} =: L_1 + L_2.
\end{align*}

According to the definition of Hölder space and the support of $\phi$, we have  
\begin{align*}
L_1 & = \sup_{y, z \in \mathbb{R}^3 \setminus \{0\} \atop y \neq z} \frac{\left| \nabla \left( \phi_{\frac{|x|}{2}}(y) \right) \cdot f_2(y/s^{\frac{1}{\gamma}}) - \nabla \left( \phi_{\frac{|x|}{2}}(z) \right) \cdot f_2(z/s^{\frac{1}{\gamma}}) \right|}{|y - z|^\gamma} \\
& = \sup_{y, z \in \mathbb{R}^3 \setminus \{0\} \atop y \neq z} \frac{\left| \nabla \left( \phi_{\frac{|x|}{2}}(\cdot) \right) \cdot f_2(\cdot/s^{\frac{1}{\gamma}}) \right| - \nabla \left( \phi_{\frac{|x|}{2}}(z) \right) \cdot f_2(z/s^{\frac{1}{\gamma}})}{|y - z|^\gamma}.
\end{align*}

Moreover, one has by the triangle inequality that  
\begin{align*}
L_1 \leq \sup_{y, z \in \mathbb{R}^3 \setminus \{0\} \atop y \neq z} \frac{\left| \nabla \left( \phi_{\frac{|x|}{2}}(y) \right) \cdot f_2(y/s^{\frac{1}{\gamma}}) - \nabla \left( \phi_{\frac{|x|}{2}}(z) \right) \cdot f_2(y/s^{\frac{1}{\gamma}}) \right|}{|y - z|^\gamma} \\
+ \sup_{y, z \in \mathbb{R}^3 \setminus \{0\} \atop y \neq z} \frac{\left| \nabla \left( \phi_{\frac{|x|}{2}}(\cdot) \right) \cdot f_2(\cdot/s^{\frac{1}{\gamma}}) \right| - \nabla \left( \phi_{\frac{|x|}{2}}(z) \right) \cdot f_2(z/s^{\frac{1}{\gamma}})}{|y - z|^\gamma}.
\end{align*}

Thus, we have  
\begin{align*}
L_1 \leq \left\| \nabla \left( \phi_{\frac{|x|}{2}}(\cdot) \right) \right\|_{\dot{C}^{\gamma}(\mathbb{R}^3)} \sup_{y \in \mathbb{R}^3 \setminus \{0\}} |f_2(y/s^{\frac{1}{\gamma}})|
\end{align*}
On one hand, using the property of $\phi_{\frac{1}{2}}$, one has

$$K_1 \leq C \frac{s^{\alpha - 1}}{|x|^{1 + \gamma}} \sup_{y, z \in \mathbb{R}^3 \backslash B_{\frac{|x|}{2}}(0)} \frac{1}{|y|^{4\alpha - 2}} \sup_{y \in \mathbb{R}^3 \backslash B_{\frac{|x|}{2}}(0)} \left| \frac{y}{s^\frac{1}{2\alpha}} \right| f_2(y/s^\frac{1}{2\alpha}) - f_2(z/s^\frac{1}{2\alpha})$$

$$\leq C \frac{s^{2\alpha - 1}}{|x|^{4\alpha - 1 + \gamma}} \sup_{y \in \mathbb{R}^3} |y|^{4\alpha - 2} |f_2(y)|.$$

On the other hand, we split $K_2$ into two parts as follows:

$$K_2 = \left\| \nabla \left( \phi_{\frac{1}{2}} \right) \right\|_{L^\infty(\mathbb{R}^3)} \sup_{y, z \in \mathbb{R}^3 \backslash B_{\frac{|x|}{2}}(0)} \frac{1}{|y - z|} \left| \frac{y}{s^\frac{1}{2\alpha}} \right| f_2(y/s^\frac{1}{2\alpha}) - f_2(z/s^\frac{1}{2\alpha})$$

$$+ \left\| \nabla \left( \phi_{\frac{1}{2}} \right) \right\|_{L^\infty(\mathbb{R}^3)} \sup_{y, z \in \mathbb{R}^3 \backslash B_{\frac{|x|}{2}}(0)} \frac{1}{|y - z|} \left| \frac{y}{s^\frac{1}{2\alpha}} \right| f_2(y/s^\frac{1}{2\alpha}) - f_2(z/s^\frac{1}{2\alpha}) := K_1^2 + K_2^2.$$
Next, we deal with the term $L_2$. By the triangle inequality, we obtain

\[
L_2 \leq \sup_{y, z \in \mathbb{R}^3 \setminus B_{\frac{1}{|x|}}(0)} \left| \phi_{\frac{|x|}{2}}(y) - \phi_{\frac{|x|}{2}}(z) \right| \left| \text{div} f_2(y/s^\gamma) - \text{div} f_2(z/s^\gamma) \right| |y - z|^{-\gamma} \]

\[
+ \sup_{y, z \in \mathbb{R}^3 \setminus B_{\frac{1}{|x|}}(0), y \neq z} \frac{\phi_{\frac{|x|}{2}}(z) \left| \text{div} f_2(y/s^\gamma) - \text{div} f_2(z/s^\gamma) \right|}{|y - z|^{-\gamma}} =: L_2^1 + L_2^2.
\]

Since

\[
\| \phi_{\frac{|x|}{2}} \|_{C^\gamma(\mathbb{R}^3)} \leq C \| \phi_{\frac{|x|}{2}} \|_{L^{1-\gamma}(\mathbb{R}^3)} \| \nabla \phi_{\frac{|x|}{2}} \|_{L^\gamma(\mathbb{R}^3)} \leq \frac{C}{|x|\gamma},
\]

we have

\[
L_2^1 = \sup_{y, z \in \mathbb{R}^3 \setminus B_{\frac{1}{|x|}}(0)} \left| \phi_{\frac{|x|}{2}}(y) - \phi_{\frac{|x|}{2}}(z) \right| \frac{\left| \text{div} f_2(y/s^\gamma) - \text{div} f_2(z/s^\gamma) \right|}{|y - z|^{-\gamma}}
\]

\[
\leq C \frac{4 - \alpha}{2\alpha} \sup_{y \in \mathbb{R}^3} |y|^{4\alpha - 1} \left| \text{div} f_2(y) \right|.
\]

As for the last term $L_2^2$, we note that

\[
L_2^2 \leq \sup_{y, z \in \mathbb{R}^3 \setminus B_{\frac{1}{|x|}}(0), |y - z| \geq \frac{|x|}{2}} \frac{\left| \text{div} f_2(y/s^\gamma) - \text{div} f_2(z/s^\gamma) \right|}{|y - z|^{-\gamma}}
\]

\[
+ \sup_{y, z \in \mathbb{R}^3 \setminus B_{\frac{1}{|x|}}(0), |y - z| < \frac{|x|}{2}} \frac{\left| \text{div} f_2(y/s^\gamma) - \text{div} f_2(z/s^\gamma) \right|}{|y - z|^{-\gamma}} =: L_2^{2,1} + L_2^{2,2}.
\]

In the same way as in the proof of $K_2^1$, it is obvious

\[
L_2^{2,1} \leq C \frac{4 - \alpha - 1}{|x|^{4\alpha - 1 + \gamma}} \sup_{y \in \mathbb{R}^3} |y|^{4\alpha - 1} \left| \text{div} f_2(y) \right|.
\]

For the term $L_2^{2,2}$, we observe that

\[
L_2^{2,2} \leq \frac{C}{|x|^{4\alpha - 1 + \gamma}} \sup_{y, z \in \mathbb{R}^3 \setminus B_{\frac{1}{|x|}}(0)} |y|^{4\alpha - 1 + \gamma} \left| \text{div} f_2(y/s^\gamma) - \text{div} f_2(z/s^\gamma) \right| |y - z|^{-\gamma}
\]

By the triangle inequality, we have

\[
\sup_{y, z \in \mathbb{R}^3 \setminus B_{\frac{1}{|x|}}(0), |y - z| < \frac{|x|}{2}} \frac{\left| \text{div} f_2(y/s^\gamma) - \text{div} f_2(z/s^\gamma) \right|}{|y - z|^{-\gamma}}
\]

\[
\leq C \frac{4 - \alpha}{2\alpha} \sup_{y \in \mathbb{R}^3} |y|^{4\alpha - 1} \left| \text{div} f_2(y) \right|.
\]
Collecting both estimates for $A$ and $B$, we have

\[
\lesssim \sup_{y, z \in \mathbb{R}^3 \setminus B_{|y|}(0)} \frac{||y|^{4a-1+\gamma} \text{div} f_2(y/s^{\frac{1}{2\alpha}}) - |z|^{4a-1+\gamma} \text{div} f_2(z/s^{\frac{1}{2\alpha}})}{|y - z|^{\gamma}} + \sup_{y, z \in \mathbb{R}^3 \setminus B_{|y|}(0)} \frac{||y|^{4a-1+\gamma} - |z|^{4a-1+\gamma}| \text{div} f_2(y/s^{\frac{1}{2\alpha}}) - \text{div} f_2(z/s^{\frac{1}{2\alpha}})}{|y - z|^{\gamma}} =: A + B.
\]

According to the definition of the Hölder space, we find that

\[
A \lesssim C S^{\frac{4a-1}{2\alpha}} ||| |y|^{4a-1+\gamma} \text{div} f_2|||_{C^\gamma(R^3)}.
\]

On the other hand, the symmetry of $B$ helps us to conclude

\[
B \lesssim \sup_{y, z \in \mathbb{R}^3 \setminus B_{|y|}(0)} \frac{||y|^{4a-1+\gamma} - |z|^{4a-1+\gamma}| \text{div} f_2(y/s^{\frac{1}{2\alpha}})}{|y - z|^{\gamma}} \leq C \sup_{y, z \in \mathbb{R}^3 \setminus B_{|y|}(0)} \max\{|y|^{4a-2+\gamma}, |z|^{4a-2+\gamma}\} |y - z|^{-\gamma} \text{div} f_2\left(y/s^{\frac{1}{2\alpha}}\right) \leq C \sup_{y \in \mathbb{R}^3} |y|^{4a-1} \text{div} f_2\left(y/s^{\frac{1}{2\alpha}}\right) \leq C S^{\frac{4a-1}{2\alpha}} \sup_{y \in \mathbb{R}^3} |y|^{4a-1} \text{div} f_2\left(y\right).
\]

Collecting both estimates for $A$ and $B$, we have

\[
L_{2,2}^{2,2} \leq C S^{\frac{4a-1}{2\alpha}} \left(||| |y|^{4a-1+\gamma} \text{div} f_2|||_{C^\gamma(R^3)} + \sup_{y \in \mathbb{R}^3} |y|^{4a-1} \text{div} f_2\left(y\right)\right).
\]

This estimate together with (4.57) and (4.58) gives

\[
(4.59) \quad L_2 \leq C S^{\frac{4a-1}{2\alpha}} \left(||| |y|^{4a-1+\gamma} \text{div} f_2|||_{C^\gamma(R^3)} + \sup_{y \in \mathbb{R}^3} |y|^{4a-1} \text{div} f_2\left(y\right)\right).
\]

Inserting (4.56) and (4.59) into (4.55) and then using the conditions (4.47) and (4.49), we eventually obtain

\[
\Pi_{2,2} \leq C \frac{1}{|x|^{4a-1+\gamma}} \sum_{N_\alpha \leq q < 0} 2^{-q\gamma} \int_0^1 \left(1 + s^{-\frac{\gamma}{2\alpha}}\right) ds \leq C \frac{C}{|x|^{4a-1+\gamma}}\left(2^{-\gamma N_\alpha}\right) \leq C \frac{C}{|x|^{4a-1}}.
\]

Thus, we complete the proof of Proposition 4.3. \(\square\)

With Proposition 4.3 in hand, we turn to show the optimal estimate in Theorem 1.8. To do this, we just need to prove (4.47) and (4.49). Since $\sigma \in C^{1,0}(\mathbb{S}^2)$ and $f_2 = \text{div}(U_0 \otimes U_0)$,
we have in terms of Proposition 4.1 established in \cite{11} that
\begin{equation}
\sup_{x \in \mathbb{R}^3} |x|^{4\alpha-2} \mathcal{E}_2(x) + \sup_{x \in \mathbb{R}^3} |x|^{4\alpha-1} \nabla \mathcal{E}_2(x) \\
\leq C \left( \sup_{x \in \mathbb{R}^3} |x|^{2\alpha-1} |U_0|(x) \right)^2 + \sup_{x \in \mathbb{R}^3} |x|^{2\alpha-1} |U_0|(x) \sup_{x \in \mathbb{R}^3} |x|^{2\alpha} |\text{div} \ U_0|(x) < \infty.
\end{equation}

Thus, our task is now to show
\begin{equation}
\| | \cdot |^{4\alpha-1+\gamma} \text{div} \mathbf{f}_2 \|_{\dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^3)} < \infty.
\end{equation}

In view of the Bony paraproduct, we write
\[ |x|^{4\alpha-1+\gamma} \text{div} \mathbf{f}_2 = \hat{T} |x|^{4\alpha-3+\gamma} |x|^2 \text{div} \mathbf{f}_2 + \hat{\mathcal{T}}_2 \text{div} \mathbf{f}_2 |x|^{4\alpha-3+\gamma} + \hat{R} (|x|^{4\alpha-3+\gamma} |x|^2 \text{div} \mathbf{f}_2). \]

In terms of the property of support, we can show by using \( \alpha \in [5/6, 1] \) and \( \gamma \in (0, 1) \) that
\begin{align*}
\left\| \hat{T} \right\|_{\dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^3)} &\leq C \sup_{x \in \mathbb{R}^3} 2^{\gamma} \left\| \hat{S}_{q-1}(| \cdot |^2 \text{div} \mathbf{f}_2) \right\|_{L^\infty(\mathbb{R}^3)} \left\| \hat{\Delta}_q(| \cdot |^{4\alpha-3+\gamma}) \right\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C \left\| | \cdot |^2 \text{div} \mathbf{f}_2 \right\|_{\dot{B}^{3-4\alpha}_{\infty,\infty}(\mathbb{R}^3)} \left\| | \cdot |^{4\alpha-3+\gamma} \right\|_{\dot{B}^{4\alpha-3+\gamma}_{\infty,\infty}(\mathbb{R}^3)} \leq C \left\| | \cdot |^2 \text{div} \mathbf{f}_2 \right\|_{\dot{B}^{3-4\alpha}_{\infty,\infty}(\mathbb{R}^3)},
\end{align*}

and
\begin{align*}
\left\| \hat{\mathcal{T}}_2 \right\|_{\dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^3)} &\leq C \sup_{x \in \mathbb{R}^3} 2^{\gamma} \left\| \hat{\Delta}_q(| \cdot |^2 \text{div} \mathbf{f}_2) \right\|_{L^\infty(\mathbb{R}^3)} \left\| \hat{\Delta}_q(| \cdot |^{4\alpha-3+\gamma}) \right\|_{L^\infty(\mathbb{R}^3)} \\
&\leq C \left\| | \cdot |^2 \text{div} \mathbf{f}_2 \right\|_{\dot{B}^{3-4\alpha}_{\infty,\infty}(\mathbb{R}^3)} \left\| | \cdot |^{4\alpha-3+\gamma} \right\|_{\dot{B}^{4\alpha-3+\gamma}_{\infty,\infty}(\mathbb{R}^3)} \leq C \left\| | \cdot |^2 \text{div} \mathbf{f}_2 \right\|_{\dot{B}^{3-4\alpha}_{\infty,\infty}(\mathbb{R}^3)}.
\end{align*}

Since \( \mathbf{f}_2 = -U_0 \otimes U_0 \) and \( \sigma \in C^{1,0}(S^2) \), we have by (2.2) that
\begin{equation}
\left\| | \cdot |^2 \text{div} \mathbf{f}_2 \right\|_{\dot{B}^{3-4\alpha}_{\infty,\infty}(\mathbb{R}^3)} \leq C \left\| | \cdot |^2 \text{div} \mathbf{f}_2 \right\|_{L_{\text{loc}}^\infty(\mathbb{R}^3)} \leq C \sup_{x \in \mathbb{R}^3} |x|^{2\alpha-1} \sup_{\mathbb{R}^3} |x|^{2\alpha} |\text{div} \ U_0|(x) \left\| | \cdot |^{3-4\alpha} \right\|_{L_{\text{loc}}^\infty(\mathbb{R}^3)}.
\end{equation}

This estimate together with Proposition 4.1 established in \cite{11} means that
\begin{equation}
\left\| \hat{T} \right\|_{\dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^3)} + \left\| \hat{\mathcal{T}}_2 \right\|_{\dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^3)} + \left\| \hat{R} \right\|_{\dot{B}^\gamma_{\infty,\infty}(\mathbb{R}^3)} < \infty.
\end{equation}

Next, we turn to bound the paraproduct term \( \hat{T} |x|^{4\alpha-3+\gamma} |x|^2 \text{div} \mathbf{f}_2 \). Since
\[ \int_{\mathbb{R}^3} h_q(x - y) \, dy = \int_{\mathbb{R}^3} h(x - y) \, dy = \hat{h}(0) = 1 \quad \text{for each} \ q \in \mathbb{Z}, \]
we have
\[ \hat{S}_{q-1}(| \cdot |^{4\alpha-3+\gamma}) = \left( \hat{S}_{q-1}(| \cdot |^{4\alpha-3+\gamma} - 2^{4\alpha-3+\gamma} |x|^{4\alpha-3+\gamma}) + 2^{4\alpha-3+\gamma} |x|^{4\alpha-3+\gamma} \right). \]
\[= \int_{\mathbb{R}^3} h_q(x - y) \left( |y|^{4\alpha-3+\gamma} - 2^{4\alpha-3+\gamma} |x|^{4\alpha-3+\gamma} \right) \, dy + 2^{4\alpha-3+\gamma} |x|^{4\alpha-3+\gamma}. \]

Performing the fact \((a + b)^p \leq 2^p(a^p + b^p)\) with \(p \geq 0\), one has

\[
\left| \int_{\mathbb{R}^3} h_q(x - y) \left( |y|^{4\alpha-3+\gamma} - 2^{4\alpha-3+\gamma} |x|^{4\alpha-3+\gamma} \right) \, dy \right| \\
\leq 2^{4\alpha-3+\gamma} \int_{\mathbb{R}^3} |x - y|^{4\alpha-3+\gamma} |h_q|(x - y) \, dy \leq C 2^{-q(4\alpha-3+\gamma)}. \]

Thus, we have

\[
\left| \hat{S}_{q-1}(\xi) |\cdot|^{4\alpha-3+\gamma} \right| \leq C 2^{-q(4\alpha-3+\gamma)} + 2^{4\alpha-3+\gamma} |x|^{4\alpha-3+\gamma}. \]

With the inequality in hand, we can show by (2.2) that

\[
\left\| \hat{T}_{1,|\cdot|^{4\alpha-3+\gamma}} \right\|_{L^\infty(\mathbb{R}^3)} \leq C \sup_{q \in \mathbb{Z}} 2^{q(4\alpha-3+\gamma)} \left\| \hat{S}_{q-1}(\xi) |\cdot|^{4\alpha-3+\gamma} \hat{\Delta}_q(|\cdot|^2 \text{div} f_2) \right\|_{L^\infty(\mathbb{R}^3)} \\
\leq C \sup_{q \in \mathbb{Z}} 2^{q(4\alpha-3+\gamma)} \left\| |\cdot|^{4\alpha-3+\gamma} \hat{\Delta}_q(|\cdot|^2 \text{div} f_2) \right\|_{L^{3/4,\infty}(\mathbb{R}^3)} \\
+ C \sup_{q \in \mathbb{Z}} \left\| \hat{\Delta}_q(|\cdot|^2 \text{div} f_2) \right\|_{L^{3/4,\infty}(\mathbb{R}^3)} =: A_1 + A_2. \]

Since \(\sigma \in C^{1,0}(\mathbb{S}^2)\) and \(f_2 = \text{div}(U_0 \otimes U_0)\), we have by Proposition 4.1 established in [11] that

\[(4.64) \quad A_2 \leq C \sup_{x \in \mathbb{R}^3} |x|^{2\alpha-1} |U_0|(|x|) \sup_{x \in \mathbb{R}^3} |x|^{2\alpha} \text{div} U_0(|x|) \left\| |\cdot|^3 \text{div} f_2 \right\|_{L^{3/4,\infty}(\mathbb{R}^3)} < \infty. \]

It remains for us to bound \(A_2\). Since \(f_2 = -U_0 \otimes U_0\), we rewrite it in terms of the Bony paraproduct that

\[
f_2 = - \sum_{i,j=1}^3 \left( \hat{T}_{U_0} \hat{U}_0^i + \hat{T}_{U_0} \hat{U}_0^i + \hat{R}(\hat{U}_0^i, \hat{U}_0^j) \right). \]

Moreover, taking Fourier transform and property of support, we observe that

\[
\mathcal{F} \left( \hat{\Delta}_q(|\cdot|^2 \text{div} f_2) \right)(\xi) \\
= - \varphi(\xi/2^q) \Delta_\xi \left( i\xi \hat{f}_2(\xi) \right) \\
= \sum_{i,j=1}^3 \sum_{|k-q| \leq 5} \varphi(\xi/2^q) \Delta_\xi \left( i\xi_i \hat{S}_{k-1} \hat{U}_0^j(\xi) \hat{\Delta}_k \hat{U}_0(\xi) + i\xi_i \hat{S}_{k-1} \hat{U}_0^i(\xi) \hat{\Delta}_k \hat{U}_0(\xi) \right) \\
+ \sum_{i,j=1}^3 \sum_{k \geq q-5} \varphi(\xi/2^q) \Delta_\xi \left( i\xi_i \hat{\Delta}_k \hat{U}_0^j(\xi) \hat{\Delta}_k \hat{U}_0(\xi) \right). \]

Furthermore, we have by the triangle inequality that

\[
A_1 \leq C \sup_{q \in \mathbb{Z}} \sum_{|k-q| \leq 5} 2^{q(4\alpha-3+\gamma)} \left\| |\cdot|^{4\alpha-3+\gamma} \hat{\Delta}_q(|\cdot|^2 \hat{S}_{k-1} U_0 \cdot \nabla \hat{\Delta}_k U_0) \right\|_{L^{3/4,\infty}(\mathbb{R}^3)} \]
Then there holds that

\[ \| \cdot |^m \tilde{\Delta} f \|_{L^p, \infty(\mathbb{R}^3)} \leq C \| \tilde{\Delta} q(\cdot | |^m f) \|_{L^p, \infty(\mathbb{R}^3)} + C 2^{-q(m-3(1/r-1/p))} \| f \|_{L^r, \infty(\mathbb{R}^3)}. \]

Proof. First of all, we observe that

\[ |x|^m \tilde{\Delta}_q f = 2^m \tilde{\Delta}_q (| \cdot |^m f) + \int_{\mathbb{R}^3} h_q(x - y) \left( |x|^m - 2^m |y|^m \right) f(y) \, dy. \]

It is obvious by using the fact \((a + b)^p \leq 2^p (a^p + b^p)\) with \(p \geq 1\) that

\[ I := \left\| \int_{\mathbb{R}^3} h_q(x - y) \left( |x|^m - 2^m |y|^m \right) f(y) \, dy \right\| \leq 2^{m \gamma} 2^{\gamma q} \int_{\mathbb{R}^3} |x - y|^m |h_q(2^q(x - y))| |f(y)| \, dy. \]

Moreover, by the Young inequality, we obtain

\[ \| I \|_{L^p, \infty(\mathbb{R}^3)} \leq C 2^{-q(m-3(1/r-1/p))} \| | \cdot |^m h \|_{L^r(\mathbb{R}^3)} \| f \|_{L^r, \infty(\mathbb{R}^3)} \]

\[ \leq C 2^{-q(m-3(1/r-1/p))} \| f \|_{L^r, \infty(\mathbb{R}^3)}, \]

where \(s\) satisfies \(1 + \frac{1}{p} = \frac{1}{s} + \frac{1}{r}\).

Thus, it follows the desired estimate by the triangle inequality that

\[ \| \cdot |^m \tilde{\Delta} f \|_{L^p, \infty(\mathbb{R}^3)} \leq C \| \tilde{\Delta} q(\cdot | |^m f) \|_{L^p, \infty(\mathbb{R}^3)} + C 2^{-q(m-3(1/r-1/p))} \| f \|_{L^r, \infty(\mathbb{R}^3)}. \]

And then we complete the proof of Lemma 4.4.\( \square \)

By (1.65) in Lemma 4.4 we have

\[ A_1^1 \leq C \sum_{q \in \mathbb{Z}} \left\| \Delta_q (| |^{4q} \tilde{\Delta} \tilde{S}_{k-1} U_0 \cdot \nabla \tilde{\Delta} U_0) \right\|_{L^{\frac{3}{2q-1}, \infty}(\mathbb{R}^3)} + \sum_{q \in \mathbb{Z}} \left\| | |^{2q} \tilde{\Delta} \tilde{S}_{k-1} U_0 \cdot \nabla \tilde{\Delta} k U_0 \right\|_{L^{\frac{3}{2q-1}, \infty}(\mathbb{R}^3)} =: A_{1,1}^1 + A_{1,2}^1. \]

Since

\[ \left\| | |^{2q} \tilde{S}_{k-1} U_0 \right\|_{L^\infty(\mathbb{R}^3)} \leq \sup_{k \leq q-1} \left\| | |^{2q} \tilde{\Delta} \tilde{S}_{k-1} U_0 \right\|_{L^\infty(\mathbb{R}^3)} \]

\[ \leq C \sum_{k \leq q-1} 2^{-k(2q-1)} \| | \cdot |^{2q-1} \|_{L^\infty(\mathbb{R}^3)}. \]
By the H"older inequality, (2.2) and Lemma 2.8, we find that

\[
A_1^{1,2} \leq C \| \cdot |2^{a-1} \hat{S}_{q-1} U_0 \|_{L^\infty(\mathbb{R}^3)} \sup_{q \in Z} \| \cdot |3^{-2\alpha} \nabla \hat{\Delta}_q U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
\leq C \| \| \| \cdot |2^{a-1} \hat{S}_{q-1} U_0 \|_{L^\infty(\mathbb{R}^3)} \sup_{q \in Z} \| \cdot |3^{-2\alpha} \nabla \hat{\Delta}_q U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
\leq C \| \| \| \cdot |2^{a-1} \hat{S}_{q-1} U_0 \|_{L^\infty(\mathbb{R}^3)} \sup_{q \in Z} \| \cdot |3^{-2\alpha} \nabla \hat{\Delta}_q U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \leq C \| \sigma \|_{L^\infty(\mathbb{S}^2)}^2.
\]

By the H"older inequality, (2.2) and Lemma 2.8 we find that

\[
A_1^{1.1} \leq C \sup_{q \in Z} 2^{(4\alpha-3-\gamma)} \| \cdot |2^{a-1} \hat{S}_{q-1} U_0 \|_{L^\infty(\mathbb{R}^3)} \| \cdot |2^{a+\gamma} \hat{\Delta}_q \nabla U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
\leq C \sup_{q \in Z} 2^{(4\alpha-3-\gamma)} \| \cdot |2^{a-1} \hat{S}_{q-1} U_0 \|_{L^\infty(\mathbb{R}^3)} \sup_{q \in Z} 2^{2q(\alpha-1)} \| \hat{\Delta}_q (\sigma / |2^{a-1}|) \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
\leq C \| \sigma \|_{L^\infty(\mathbb{S}^2)} \| \cdot |2^{a-1} \hat{S}_{q-1} U_0 \|_{L^\infty(\mathbb{R}^3)} \sup_{q \in Z} 2^{2q(\alpha-1)} \| \hat{\Delta}_q (\sigma / |2^{a-1}|) \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \leq C \| \sigma \|_{L^\infty(\mathbb{S}^2)}^2.
\]

Collecting estimates for $A_1^{1,1}$ and $A_1^{1,2}$, we readily have

\[
(4.67) \quad A_1^1 \leq C \| \sigma \|_{L^\infty(\mathbb{S}^2)}^2.
\]

Similarly, we can show

\[
(4.68) \quad A_2^2 \leq C \| \sigma \|_{L^\infty(\mathbb{S}^2)}^2.
\]

Now we focus on the term $A_3^3$. By (4.65) in Lemma 4.4 we have

\[
A_3^3 \leq C \sup_{q \in Z} \sum_{k \geq q-5} 2^{(4\alpha-3-\gamma)} \| \hat{\Delta}_q \left( |2^{a-1-\gamma} \hat{\Delta}_k U_0 \cdot \nabla \hat{\Delta}_k U_0 \right) \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
+ C \sup_{q \in Z} \sum_{k \geq q-5} 2^{2(1-\alpha)} \| \cdot |2^{a-1} \hat{\Delta}_k U_0 \cdot \nabla \hat{\Delta}_k U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} =: A_3^{3,1} + A_3^{3,2}.
\]

By Lemma 2.8, we have that for each $\alpha \in [5/6, 1]$,

\[
A_3^{3,2} \leq C \sup_{q \in Z} \sum_{k \geq q-5} 2^{2(1-\alpha)} \| \cdot |2^{a-1} \hat{\Delta}_k U_0 \|_{L^{\infty}(\mathbb{R}^3)} \| \nabla \hat{\Delta}_k U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
\leq C \sup_{q \in Z} \sum_{k \geq q-5} 2^{2(1-\alpha)} \| \cdot |2^{a-1} \hat{\Delta}_k U_0 \|_{L^{\infty}(\mathbb{R}^3)} \| \nabla \hat{\Delta}_k U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
\leq C \sup_{q \in Z} 2^{2(1-2\alpha)} \| \hat{\Delta}_q U_0 \|_{L^\infty(\mathbb{R}^3)} \| \cdot |2^{a-1} \hat{\Delta}_k U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \\
\leq C \| \cdot |2^{a-1} \hat{\Delta}_k U_0 \|_{L^{\frac{3}{3-2\alpha}}(\mathbb{R}^3)} \leq C \| \| \| \sigma \|_{L^\infty(\mathbb{S}^2)}^2.
\]

As for $\alpha = 1$, we observe by (4.65) in Lemma 4.4 that

\[
A_3^3 = C \sup_{q \in Z} \sum_{k \geq q-5} 2^{(1+\gamma)} \| \cdot |1^{1+\gamma} \hat{\Delta}_q \left( |2^{a-1} \hat{\Delta}_k U_0 \cdot \nabla \hat{\Delta}_k U_0 \right) \|_{L^{3,\infty}(\mathbb{R}^3)}
\]
By the discrete Young inequality and the Hölder inequality and Lemma 2.8, we have

\[ A_1^{3,3} \leq \sup_{q \in \mathbb{Z}} 2^q \left\| \frac{1}{2} \check{\Delta}_q U_0 \cdot \nabla \check{\Delta}_q U_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \]

\[ \leq C \sup_{q \in \mathbb{Z}} 2^q \left\| \frac{1}{2} \check{\Delta}_q U_0 \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla \check{\Delta}_q U_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \]

\[ \leq C \sup_{q \in \mathbb{Z}} 2^{-q} \left\| \hat{\Delta}_q u_0 \right\|_{L^\infty(\mathbb{R}^3)} \sup_{q \in \mathbb{Z}} \left\| \nabla \check{\Delta}_q U_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \]

\[ \leq C \left\| \sigma / |\sigma| \right\|_{L^{2,\infty}(\mathbb{R}^3)} \left\| \nabla u_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \leq C \left\| \sigma \right\|_{L^\infty(\mathbb{S}^2)} \left\| \nabla u_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)}. \]

Since \( u_0(x) = \frac{\sigma}{|\sigma|} \) as \( \alpha = 1 \) and \( \sigma \in C^{1,0}(\mathbb{S}^2) \), we have by the Leibniz formula that

\[ \left\| \nabla u_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \leq C \left\| \sigma \right\|_{L^\infty(\mathbb{S}^2)} \left\| 1/|\sigma| \right\|_{L^{2,\infty}(\mathbb{R}^3)} \]

\[ + C \left\| \sigma \right\|_{W^{1,\infty}(\mathbb{S}^2)} \left\| 1/|\sigma| \right\|_{L^{2,\infty}(\mathbb{R}^3)}. \]

Thus, we have

\[ A_1^{3,3} \leq C \left\| \sigma \right\|_{W^{1,\infty}(\mathbb{S}^2)}. \]

In view of (4.65) in Lemma 4.3, we see that

\[ A_1^{3,1} \leq C \sup_{q \in \mathbb{Z}} 2^{q(4\alpha - 3\gamma)} \left\| \hat{\Delta}_q \left( |1|^{4\alpha - 1 - \frac{1}{\gamma}} (\check{\Delta}_k U_0 \cdot \nabla \check{\Delta}_k U_0) \right) \right\|_{L^{2,\infty}(\mathbb{R}^3)} \]

\[ \leq C \sup_{k \in \mathbb{Z}} 2^{k(4\alpha - 3\gamma)} \left\| |1|^{4\alpha - 1 - \frac{1}{\gamma}} (\check{\Delta}_k U_0 \cdot \nabla \check{\Delta}_k U_0) \right\|_{L^{2,\infty}(\mathbb{R}^3)} \]

\[ \leq C \sup_{k \in \mathbb{Z}} 2^{k(4\alpha - 3\gamma)} \left\| |1|^{4\alpha - 1 - \frac{1}{\gamma}} \check{\Delta}_k U_0 \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla \check{\Delta}_k U_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)}. \]

Moreover, we have by Lemma 2.8 and the Bernstein inequality in Lemma 2.1 that

\[ \left\| |1|^{4\alpha - 1 - \frac{1}{\gamma}} \check{\Delta}_k U_0 \right\|_{L^\infty(\mathbb{R}^3)} \left\| \nabla \check{\Delta}_k U_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \leq C \left( 2^{-k(4\alpha - 1 + \gamma)} 2^{k(3 - 2\alpha)} \check{\Delta}_k u_0 \right) \left\| \check{\Delta}_k U_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \]

\[ \leq C \left( 2^{-k(4\alpha - 1 + \gamma)} 2^{k(3 - 2\alpha)} \check{\Delta}_k u_0 \right) \left\| \check{\Delta}_k U_0 \right\|_{L^{2,\infty}(\mathbb{R}^3)} \leq C \left( 2^{-k(4\alpha - 1 + \gamma)} \left\| \sigma / |\sigma| \right\|_{L^{2,\infty}(\mathbb{R}^3)} \right), \]

which helps us to infer that

\[ A_1^{3,1} \leq C \left\| \sigma \right\|_{L^\infty(\mathbb{S}^2)}. \]

This estimate together with estimates for \( A_1^{3,2} \) and \( A_1^{3,3} \) leads to

\[ A_1 \leq C \left\| \sigma \right\|_{L^\infty(\mathbb{S}^2)}. \]

Collecting estimates (4.67), (4.68) and (4.69), we immediately have

\[ A_1 \leq C \left\| \sigma \right\|_{L^\infty(\mathbb{S}^2)}. \]

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This inequality together with (4.63) and (4.64) yields the claim (4.61). So, we complete the proof of Theorem 1.8.

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