Quantum storage in quantum ferromagnets

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Quantum data is inherently fragile and must be protected to unlock the potential of quantum technologies. A pertinent concern in schemes for quantum storage is their potential to be implemented in the near future. While Heisenberg ferromagnets are readily available and can be potentially implemented, quantum storage in them has never before been addressed. We address this issue by considering the storage of quantum data within a special quantum ferromagnet, where every pair of spins interacts with equal strength. We analyze the storage error for a unital and local noise model, and optimize the memory lifetime with respect to system size. Our analysis relies on Taylor decompositions of unitary evolutions in terms of Fréchet matrix derivatives, and uses Davis’ divided difference representation for these Fréchet derivatives, and the recursive structure of these divided differences. We thereby obtain upper bounds on the error for passive quantum storage. With our bounds, we numerically study the potential to enhance memory lifetimes. Our approach lays the foundation for optimization of the memory lifetime based on the spectral structure of any physical system.

I. INTRODUCTION

Reliable storage of quantum data is an essential enabling agent for quantum technologies, ranging from quantum repeater networks to general purpose quantum computers. However, reliably storing quantum data is difficult, because decoherence on any quantum system is unavoidable. Given the promise of quantum technologies, the robust storage of quantum data in a multitude of physical systems has unsurprisingly been extensively studied. It is well-known that errors occurring in quantum systems can be mitigated using a multitude of different schemes. One can for instance use active control methods such as dynamical decoupling \cite{1,2} or by continuously performing quantum error correction \cite{3}. Alternatively, one might consider storing quantum data in a physical system that self-corrects to some degree, so that quantum error correction need only be performed once, before the data is accessed. This draws our attention to self-correcting quantum memories.

A self-correcting quantum memory is a scheme where the lifetime of stored quantum data can be enhanced simply by increasing size of the physical system \cite{4,5}. Self-correcting quantum memories \cite{6,13}, while attractive, have their design fraught with difficulties. In particular, self-correcting quantum memories have been shown to be challenging to implement in a multitude of desirable settings \cite{8,14,17}. Moreover, previous proposals invariably utilize physical systems involving unphysical multi-body interactions, which can only be approximated \cite{6,13,20}. Here, we study the extent in which self-correction in a Heisenberg ferromagnet is possible.

There are several reasons why the Heisenberg ferromagnet is an attractive medium for quantum storage for several reasons. First, its spectral gap, the energy difference between its ground space and the first excited state, can grow with the size of the system \cite{21}. This spectral gap, when large, can suppress excitation rates from the ground state \cite{22}. From this perspective, storing quantum data within the ground space of a physical system with a growing spectral gap is advantageous. Second in \cite{14}, Marvian and Lidar prove that commuting 2-local Hamiltonians are unsuitable for robust quantum storage. Moreover, the Heisenberg Hamiltonian can potentially have a large spectral gap which ought to be good for quantum storage. Third, the Heisenberg ferromagnet is naturally abundant, not least because of the ubiquitous exchange interaction that arises in many-electron systems. Even in many physical systems that cannot be naturally interpreted as ferromagnets, the Heisenberg ferromagnet can nonetheless be engineered. For example, systems dominated by dipole interactions can be symmetrized using dynamic pulse sequences to become effective Heisenberg ferromagnets. Similar techniques allow effective Heisenberg ferromagnet to be engineered in systems including quantum dots and superconducting circuits. For these reasons, we consider Heisenberg ferromagnets as prime candidates for quantum storage.

Given our choice of the Heisenberg ferromagnet as a physical system, where should we store the quantum data? If we store quantum data into the ground state, we can benefit from (1) suppression of excitations induced by the spectral gap, and (2) being in a decoherence-free subspace and thereby avoiding phase errors induced by the natural dynamics of the physical system. In the case of the Heisenberg ferromagnet, its ground space necessarily contains the symmetric subspace, which in turn comprises of all qubit-states that are left invariant under arbitrary permutations of the qubits. An immediate concern in using symmetric states as quantum memories is whether they can support quantum error correction. The first symmetric states that come to mind are

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GHZ states. However, GHZ states are known to be completely vulnerable to phase errors, and hence terrible for quantum storage. Fortunately, there are quantum error-correcting codes residing within the symmetric subspace and in principle, they can be robustly initialized in cavity QED-based systems by inducing Rabi-like oscillations between Dicke states. One could for instance use permutation-invariant quantum codes that can correct $t$ errors.

To model the degradation of quantum data, we use quantum channels, which map quantum states to quantum states. We restrict our attention to unital quantum channels, which are probabilistic ensembles of unitary evolutions and describe how a physical system interacts with a classical environment. The unitary evolutions could arise from unwanted physical interactions, such as spurious local fields afflicting each particle independently. A unital noise model can (1) model dephasing errors, which are a dominant source of errors in many physical systems, and (2) render the subsequent analysis to be analytically tractable, because perturbative techniques for matrix functions can be used.

In this paper, we study the potential of Heisenberg ferromagnets to be used for quantum storage. Namely, we obtain an upper bound on the storage error of quantum data in terms of the storage time, the noise strength, the sizes of the exchange constants in the Heisenberg model, and the number of qubits used. In our scheme, quantum data can be stored within the ground space of Heisenberg ferromagnets using quantum codes that lie within the symmetric subspace.

Because the quantum codes that we use lie within the symmetric subspace and correct a non-trivial number of errors, they are non-stabilizer codes and cannot be readily analyzed using techniques suitable for studying stabilizer codes. Hence in this paper, we introduce new techniques for studying quantum memories. Namely, by describing the error model as a unital quantum channel, we reduce the problem to that of studying perturbations of a unitary evolution. Such perturbations can be studied using matrix derivatives, which in turn admit a representation in terms of divided differences. Because these divided differences are functions of the spectral structure of the underlying Hamiltonian, we are able to obtain an upper bound on the storage error that depends not just on the noise strength, but also the internal interaction strength of the unperturbed Hamiltonian. We find that in our scheme is akin to a partial self-correcting quantum memory, where storage lifetime cannot be arbitrarily increased with system system. Instead, we obtain a lower bound on the storage lifetime and find the optimal system size.

II. THE PHYSICAL SETTING

Before we can make precise the specifications of the physical system and the noise model, we need to define some standard terminology. Let $|0\rangle$ and $|1\rangle$ denote a spin-up and spin-down state respectively, and let $I = |0\rangle\langle 0| + |1\rangle\langle 1|$, $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ denote the identity, the bit-flip and the phase-flip operator on a single qubit respectively. Let $Y = iXZ = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$ denote the Hermitian bit-flip and phase-flip operator. Together, $I, X, Y, Z$ are single-qubit Pauli matrices, and the set of $n$ qubit Pauli matrices is $\{I, X, Y, Z\}^\otimes n$. For multi-qubit systems, let $X_j, Y_j$ and $Z_j$ represent multi-qubit operators that apply $X, Y$ and $Z$ respectively on the $j$th qubit and leaves the remaining qubits unchanged. Let $1 = I^\otimes n$ denote an $n$-qubit identity operator, and let $Z$ denote a single-qubit identity channel.

The physical system that we consider here is the Heisenberg ferromagnet, which is a quantum model for ferromagnetism. When there is no external magnetic field and every pair of spins interacts identically, the corresponding Heisenberg Hamiltonian on $n$ spin-half particles is

$$H = -J \sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k + Y_j Y_k + Z_j Z_k - 1.$$ (II.1)

The Hamiltonian $H$ is essentially a sum of swap operators because $\pi_{j,k} = X_j X_k + Y_j Y_k + Z_j Z_k + 1$ is a swap operator that swaps qubits $j$ and $k$. The key advantage of using this Hamiltonian is that it is exactly solvable [1]. Namely, the Hamiltonian admits the spectral decomposition $H = \sum_{j=0}^{\lfloor n/2 \rfloor} \lambda_j \Pi_j$ where its ground state energy is $\lambda_0 = 0$ with multiplicity $n + 1$, and its positive energy eigenvalues are $\lambda_j = J_j (n + 1 - j)$ with multiplicities $(n + 1 - 2j) \left( \binom{n}{j} - \binom{n-1}{j-1} \right)$ where $j = 1, \ldots, \lfloor n/2 \rfloor$ [2]. The ground space projector $\Pi_0$ is simply the projector onto the symmetric subspace. In general, the eigenprojectors $\Pi_j$ are fairly complicated; they are linear combinations of generalized adjacency matrices of Johnson graphs [3]. In this paper we only use the fact that (1) $\lambda_k$ increases monotonically with $k$, and (2) $\lambda_k - \lambda_j \geq J (k - j) (n + 2 - 2k) \geq 2J (\frac{n}{2} - k + 1)$ for $j < k \leq n/2$, and the symmetry of the ground space of this Hamiltonian.

A unital quantum channel $\mathcal{N}$ maps density matrices $\rho$ to density matrices $U_\rho U_\rho^\dagger$ with probability $p_\rho$. Here, $U_\rho = e^{-i(H + A_\rho)\tau}$ are unitary operators corresponding to perturbations $A_\rho$, and $\tau$ denotes the time duration of the storage. We remark that $\mathcal{N}$ can describe phase errors; $\mathcal{N}$ can be a dephasing noise model when $A_\rho$ are diagonal operators in the eigenbasis of $H$. Here, the perturbations $A_\rho$ are local in the sense that they model the coupling of classical fields to each qubit independently, and can thus be written as a linear combination of multi-qubit Pauli operators that act non-trivially on exactly one qubit. Such coherent noise models have been recently been considered with respect to quantum tolerance [4].

The performance of a quantum code as a quantum memory is traditionally quantified in terms how well it preserves entanglement. Here the maximally entangled state between an ancillary qudit and the codespace is $|\Psi_{E}\rangle = \sum_{j=0}^{M-1} |j\rangle \otimes |j\rangle / \sqrt{M}$ Decoherence occurs on half of this entangled state, where the ancillary qudit is left alone, and $\mathcal{N}$ applies to the codespace. The net effect is the
application of the quantum channel \( \mathcal{N} = \mathcal{I} \otimes \mathcal{N} \), where \( \mathcal{I} \) models the identity channel on the ancillary qutit. The storage error with respect to an \( n \)-qubit code \( \mathcal{C} \) can then be represented as the minimum trace distance between the initial entangled state and the entangled state after being exposed to decoherence and has a recovery map subsequently applied, where the minimization is over all possible recovery maps. Explicitly, this storage error is

\[
\epsilon_c = \min_{\mathcal{R}} \frac{1}{2} \| \Psi_c \langle \Psi_c | - \mathcal{R}(\| \Psi_c \langle \Psi_c | ) \|_1 ,
\]

where \( \mathcal{R} = \mathcal{I} \otimes \mathcal{R} \), the minimization is over all quantum channels \( \mathcal{R} \) mapping \( n \) qubits to \( n \) qubits, and \( \| \cdot \|_1 \) denotes the trace norm. There are two advantages of using the trace distance over the entanglement fidelity to quantify the storage error. First, fidelity and distance metrics are asymptotically equivalent in terms of noiseless transmission rates over noisy channels [31]. Second, fidelity, by not being a distance metric, suffers from some problems as pointed out in [32].

\section{Bounding the Storage Error}

We begin by analyzing the exactly computable errors incurred by the quantum repetition code. This not only allows us to evaluate the performance of using GHZ-type states, but also allows us to compute a baseline error when no quantum error correction is used. This baseline error crucially allows us to determine the parameter regimes where quantum error correction may enhance memory lifetimes. Repetition codes are known to be completely vulnerable to phase noise, because any single phase error takes induces a logical bit-flip between the logical codewords. To see how this happens explicitly, let us have a repetition code \( \text{rep}(n) \) on \( n \) qubits, with orthonormal codewords \( |0\rangle_R = \frac{|0\rangle \otimes \cdots \otimes |0\rangle}{\sqrt{2^n}} \) and \( |1\rangle_R = \frac{|1\rangle \otimes \cdots \otimes |1\rangle}{\sqrt{2^n}} \). Then \( Z_j |0\rangle_R = |1\rangle_R \) for any \( j = 1, \ldots, n \). Hence, with respect to quantum repetition, phase errors are the worst possible errors and cannot be error-corrected. Using the Trotter–Szegedy expansion on \( e^{-i(H+A)\tau} \), the fact that \( A \) preserves the codespace, and that \( H \) leaves all states in the codespace invariant, we can show that \( \epsilon_{\text{rep}(n)} = |\sin(\alpha \tau)| \), where \( \alpha = \max_{\lambda} \|A\|_2 \) to describe the noise strength of our noise model, and \( \| \cdot \| \) denotes the operator norm of a matrix. This will aid our comparison of the storage error of our scheme that uses quantum error correction with the baseline storage error for an unprotected qubit, which is simply \( |\sin \alpha \tau| \).

To correct \( t \) errors in the symmetric subspace, we can use a code on \( (d-1)(2t+1)^2 \) qubits with logical codewords \( |j_{\text{pl}}\rangle = \sum_{x} f_x |x\rangle \), for \( j = 0, \ldots, d-1 \) and where \( f_x \) are coefficients of the polynomial \( \sum_x f_x x^2 = (1 + x + \cdots + x^{d-1})^{2t+1} \). Let us denote \( c \) as the storage error of using such a code. Each unitary process can be written as \( U_n = G_u + B_u \), where \( G_u \) and \( B_u \) denotes the correctible and uncorrectible parts respectively. From this, the storage error can be bounded in terms of the Euclidean norm \( B_u |j_{\text{pl}}\rangle \).

Whenever the largest of these norms \( b \) is no larger than 0.01, then \( \epsilon \leq 1.01b \). We proceed to bound \( b \) by utilizing the Taylor series \( g(H + A) = g(H) + \sum_{j=1}^{\infty} \frac{d^j g(H, A)}{j!} \), where \( g(x) = e^{-i x^2} \) is an exponential function and \( \frac{d^j g(H, A)}{j!} = \frac{d^j}{dx^j} |_{x=0} g(H + t A) \) are Frechet derivatives [33, 34]. At this point, Deadman and Relton result’s can be used to bound the uncorrectable part of the unitary processes so that \( \epsilon_c \leq \frac{\|H\| + \tau^{-1}}{\|t\|} |(an)^{j+1} \). However this simple bound, cannot elucidate the relationship between the Hamiltonian’s spectrum and the storage error. Hence, we instead use Davis’ representation to make precise the connection between these Frechet derivatives and the Hamiltonian’s spectral decomposition \( H = \sum \lambda_i \Pi_i \). Namely, given a vector \( y = (y_1, \ldots, y_k) \) and denoting \( g(y) = g(y_1, \ldots, y_k) \) as a divided difference of \( g \) of order \( k - 1 \), Davis’ representation of Frechet derivatives is

\[
\frac{D^j g(H, A)}{j!} = \sum_{i_0, \ldots, i_j} g(\lambda_{i_0}, \ldots, \lambda_{i_j}) (\Pi_{i_j} A) \cdots (\Pi_{i_1} A) \Pi_{i_0}.
\]

With [III.1], we leverage on the remarkable properties of divided differences in our error analysis. First, divided differences are invariant under any permutation of their arguments. Second, divided differences of a vector with \( k \) identical arguments is related to the \( (k-1) \)th derivative of the underlying function. For the exponential function, we have \( g(y_1, \ldots, y_k) = \frac{(k-1)!}{(k-1)!} \), when \( y_1, \ldots, y_k \) are identical. Third, a divided difference when not evaluated on identical arguments can be recursively defined; whenever \( y_i \neq y_j \), \( g(y) = \frac{(k-1)!}{(k-1)!} g(y_{\text{not }i}, y_{\text{not }j}) \); where \( y_{\text{not }i} \) denotes a vector obtained from \( y \) by deleting its \( i \)th component.

Using Davis’ representation of Frechet derivatives, we can see that \( b \leq \sum_{j+1}^{\infty} (an)^j (h_j + c_j) \), where the codespace error is \( c_j = \frac{\sigma}{\sqrt{T}} \), and the high energy error \( h_j \) is the sum of all \( g(\lambda_{n_1}, \ldots, \lambda_{n_j}, 0) \) for which \( n_1 + \cdots + n_j > 0 \). The coefficient of \( h_j \) arises from the operator norms of the perturbation, and the coefficient of \( c_j \) arises when all arguments of the divided difference are equal to zero. Using bounds on the difference of distinct eigenvalues of \( H \) and the recursive relation on divided difference, we can show that

\[
|g(\lambda_{n_1}, \ldots, \lambda_{n_j}, 0)| \leq \max_{\Omega} \frac{\tau^{j-|\Omega|}/(j - |\Omega|)!}{\prod_{\ell \in \Omega} (|J(|\frac{\tau}{2}|)| + 1 - \nu) - \nu},
\]

where the maximization is taken over all subsets \( \Omega \) of \( \{0, \ldots, j\} \) with at most \( j \) elements, and \( y_u = y_v \) for all \( u, v \in \Omega \) and \( y_u = \lambda_{u+1} \) for \( u = 0, \ldots, j - 1 \) and \( y_j = 0 \).

To evaluate a bound on \( h_j \), we overcount and use the permutation symmetry of the input arguments of divided differences. Generally, the summands in \( h_j \) have repeated arguments that are either non-zero or zero. When the repeated arguments are non-zero, (1) there are \( |n/2| \) ways of assigning non-zero values to the repeated indices, (2)
there is a factor of $O(n^{-1})$ from popping the last argument which must be zero, (3), there are $r$ repeated arguments which contribute a factor of $\tau^{r-1}/(r-1)!$, and (4) there are $j-r$ non-zero non-repeating arguments for the divided difference. The summands of $h_j$ with non-repeating arguments can be accounted for because for any $\alpha = 1, \ldots, \lceil n/2 \rceil + 1$, we have

$$\sum_{n_1, \ldots, n_k = 1}^{[n/2]+1} \prod_{j=1}^{k} n_j^{-d[n_j \neq \alpha]} = \left( S - \frac{1}{\alpha} + 1 \right)^k \leq S^k, \quad (III.3)$$

where $S = S_{\lceil n/2 \rceil} + 1 = 1 + \frac{1}{2} + \cdots + \frac{1}{\lceil n/2 \rceil} + 1$ are harmonic numbers. Using known bounds for harmonic numbers [36], we get $S_n \leq \ln n + \frac{1}{2} \ln n + 0.57722$, so for $n \geq 9$, we have $S \leq \ln 2n$. On the other hand, when the $r$ repeated arguments are equal to zero, there are $j + 1 - r$ non-repeating arguments for the divided difference. Together with the fact that the binomial coefficients are at most $2^j$, we can show that when $n \geq 9$, we have $h_j \leq \frac{2j}{S^2} (2S/J)^j$. It follows from using properties of the geometric sum and optimizing over the Hölder’s inequality for the codespace error that

$$\epsilon \leq \frac{3\epsilon^2}{2} \left( c_1 x_1^{t+1} + \inf_{0 < \theta < 1/(an)} c_2 x_2^{t+1} \right), \quad (III.4)$$

where $x_1 = 2San/J$, $x_2 = an\theta$, $c_1 = \frac{2j}{S^2(1-x_1)}$, and $c_2 = \frac{\epsilon^{(\theta-1)}}{1-x_2}$. By setting $\theta = 1$, we obtain a lower bound for $\tau$ in terms of the noise $a$ given by

$$\tau \geq \log \left( \frac{2\epsilon}{3} \right) - (t+1) \log(1/(x_1 + 1/(x_2) x_2)). \quad (III.5)$$

Using [III.5], we calculate the enhancement of the memory lifetime $\tau$ over the baseline lifetime $\tau_0 = \arcsin(\epsilon)/a$, which we define as $\tau / \tau_0$. When the storage error for a qubit into a permutation-invariant code that corrects $t$ errors is at most $10^{-4}$ with the exchange constants $J = 10$. We summarize these results in Figure 1. As can be seen, this enhancement factor, which suggests that for every sufficiently small $a$, there would be an optimal $J$ for our scheme.

We can further unravel how the storage error varies with $\tau$ and $n$ in Figure 2. In Figure 2 we plot $\log_{10} (\frac{1}{2})$, and find that for when the targeted storage error is low enough, our scheme can substantially enhance the memory lifetime. This is consistent with the conclusion one may draw from Figure 1.

![FIG. 2: Here, contour plots depict the negative of the logarithm of the storage error, namely $-\log_{10}(\epsilon)$. The vertical and horizontal axes depict the memory lifetime and number of correctible errors respectively. The larger the value on the heat map, the smaller the storage error. Here, $J = 10$ and $a = 4(10^{-5})$.](image)

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We can draw some connections between the parameters of our scheme and the parameters of what is experimentally achievable. Recently, a superconducting qubit was stored between 12ns to 20ns with a fidelity of 0.9995 [37]. If our noise model can approximate such experimental noise, these experimental parameters can be recast into a baseline storage error of $5(10^{-4})$ with a memory lifetime of 12ns, which corresponds to an effective noise with $a = 4(10^{-5})$GHz. Using (1), we can see that if a 625 qubit permutation-invariant code that corrects 12 errors is used, a modest enhancement of the qubit’s storage lifetime to 30ns can be achieved when $J = 1000$GHz with perfect error correction. Similarly, if $J$ can be increased to an ultrastrong coupling regime with $J = 1000$GHz, the qubit’s storage lifetime can be enhanced to 50ns using 2209 physical qubits. From this, we can see how increasing $J$ in a Heisenberg ferromagnet enhances the storage lifetime.

IV. DISCUSSIONS

Schemes for quantum memories have been extensively studied, but often require the design of physical systems that are challenging to implement in practice, because of the oft required many-body physical interactions which do not naturally exist in nature. In contrast, our scheme is based on a physically abundant physical system, the Heisenberg model, and should be easier to implement.
To analyze our scheme, we introduce a novel approach, which departs from previous methods used to study the performance of quantum memories. Namely, we directly use matrix perturbation theory and its connection with divided differences on the Hamiltonians spectrum to obtain a new type of analytical bound on the storage error. From this, we can understand the storage error that can be achieved under the assumption of perfect quantum error correction at the end of the storage time.

Because our analysis does not rely on any specific property of permutation-invariant codes apart from their containment in one of the Hamiltonian’s eigenspace, our analysis also applies to any permutation-invariant code that corrects at least a single error. For example, we can use matrix perturbation theory and its connection with divided differences on the Hamiltonians spectrum to obtain a new type of analytical bound on the storage error. Moreover, our analysis technique could extend to any physical system with a completely understood spectral structure, such as those with Hamiltonians constructed from the stabilizers of stabilizer codes. This lays the foundations for analyzing using stabilizer codes as quantum memories using our techniques.

Several questions remain. It would be interesting to study the extent to which partial self-correction can hold for Heisenberg models with different geometries for the interactions, such as those with only nearest neighbor interactions on a square grid. Also, there are still no explicit protocols for the decoding of any permutation-invariant invariant code that corrects at least a single error. For our scheme to be brought closer to experimental implementation, this needs to be addressed.

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Appendix A: Storage error for quantum repetition codes

The impact of a unitary evolution $e^{-i(H+A)t}$ on an arbitrary state $|\psi\rangle$ in $\text{rep}(n)$ when $A = \sum_{j=1}^{n} z_j Z_j$ comprises of only phase errors can be analyzed using the Trotter-Suzuki decomposition $e^{-i(H+A)t}/|\psi\rangle = \lim_{N \to \infty} (e^{-iHt/N}e^{-iAt/N})|\psi\rangle$. Since $e^{-iHt/N}$ leaves symmetric states invariant, and $\text{rep}(n)$ comprises only of symmetric states, it follows that $e^{-i(H+A)t} |\psi\rangle = e^{-iAt/N} |\psi\rangle$. The implication of this is that the system Hamiltonian is irrelevant in the dynamics induced by phase noise evolutions. Hence $e^{-i(H+A)t} |\psi\rangle = e^{-iAt/N} |\psi\rangle$. By counting the parity of phase errors, we get $e^{-iAtt} |jR\rangle = \cos \theta |jR\rangle + \sin \theta |(j\oplus 1)R\rangle$, where $\theta = (z_1 + \cdots + z_n)\tau$ and $\oplus$ denotes binary addition. Hence $e^{-iAt/N}$ always maps states from $\text{rep}(n)$ to $\text{rep}(n)$. It is now straightforward to evaluate $\epsilon_{\text{rep}(n)}$, especially since we need only evaluate the trace distance between the pure states $\sqrt{\frac{1}{2}}(|0\rangle\langle 0|) + \sqrt{\frac{1}{2}}(|1\rangle\langle 1|)$ and $\sqrt{\frac{1}{2}}(|0\rangle\langle 0|e^{-iAt}|R\rangle + |1\rangle\langle 1|e^{-iAt}|R\rangle)^2/4$. This simplifies to yield $\epsilon_{\text{rep}(n)} = \sqrt{1 - \cos^2 \theta} = |\sin \theta|$ where $\theta = (z_1 + \cdots + z_n)\tau$. Hence the storage error for using the quantum repetition code exposed to phase errors is

$$\epsilon_{\text{rep}(n)} = |\sin(a\tau)|,$$

(A.1)

where $a = \|A\|/n$. By setting $n = 1$, we can conclude that the baseline error for an unprotected qubit is $|\sin a\tau|$. For short storage times, increasing the number of qubits using the quantum repetition code only makes the error worse.

Appendix B: Storage error for quantum codes correcting $t$ errors

Now we bound the storage error in terms of the correctible and uncorrectable parts of unitary matrices. Each unitary matrix in the unital channel $\mathcal{N}$ can be written as $U = G + B$, where $G$ is promised to be correctible. From the linearity of the quantum channel $\mathcal{R}$, and the triangle inequality for the trace norm, and the monotonicity of the trace distance under the partial trace, we get $\epsilon_c \leq \sum_{u} p_u \sum_{d=0}^{d-1} \|M_{J,U}\|_1$, where $M_{J,U} = |jL\rangle\langle jL| - \mathcal{R}(U)|jL\rangle\langle jL|U\rangle$. By writing $M_{J,U}$ into a size 2 block matrix induced by the projectors onto the correctible and uncorrectable subspaces respectively, we can use properties of the trace-norm and a block matrix generalization of the Gersgorin circle theorem [] to show that

$$\epsilon_c \leq \sum_{u} p_u \frac{d}{2} \sum_{j=0}^{d-1} \left(\|b_{u,j}\| + b_{u,j}\right),$$

(B.1)
where \( b_{u,j} = \sqrt{\langle j_L | B_u^j B_u | j_L \rangle} \). When \( b_{u,j} \leq 0.01 \), (B.1) simplifies to yield

\[
\epsilon_c \leq 1.01 \sum_a \frac{b_u}{d} \sum_{j=0}^{d-1} b_{u,j}.
\]

(B.2)

1. **Trace norm of \( M_{j,U} \)**

By writing \( M_{j,U} \) into a size 2 block matrix induced by the projectors \( \Pi_g \) and \( \Pi_b \) onto the correctible and uncorrectible subspaces respectively, we can write

\[
M_{j,U} = \left( \begin{array}{ccc}
\Pi_G |j_L\rangle \langle j_L| \Pi_G & -\Pi_G \mathcal{R}(U|j_L\rangle \langle j_L|U^\dagger) \Pi_G & \Pi_G \mathcal{R}(U|j_L\rangle \langle j_L|U^\dagger) \Pi_B \\
\Pi_B \mathcal{R}(U|j_L\rangle \langle j_L|U^\dagger) \Pi_G & \Pi_B |j_L\rangle \langle j_L|B^\dagger \Pi_B & \Pi_B \mathcal{R}(B|j_L\rangle \langle j_L|B^\dagger) \Pi_B \\
\Pi_B \mathcal{R}(B|j_L\rangle \langle j_L|U^\dagger) \Pi_G & \Pi_B \mathcal{R}(B|j_L\rangle \langle j_L|U^\dagger) \Pi_B & \Pi_B \mathcal{R}(B|j_L\rangle \langle j_L|U^\dagger) \Pi_G
\end{array} \right).
\]

(B.3)

Using the fact that \( \mathcal{R} \) corrects errors on the correctible subspace and does not induce interactions between the correctible and uncorrectible subspaces, and the fact that \( \Pi_t G|j_L\rangle = 0 \), we get

\[
M_{j,U} = \left( \begin{array}{ccc}
|j_L\rangle \langle j_L|B^\dagger & \Pi_B |j_L\rangle \langle j_L|B^\dagger \Pi_B & \Pi_B |j_L\rangle \langle j_L|B^\dagger \Pi_B \\
\Pi_B |j_L\rangle \langle j_L|B^\dagger \Pi_B & |j_L\rangle \langle j_L|B^\dagger \Pi_B & \Pi_B |j_L\rangle \langle j_L|B^\dagger \Pi_B \\
\Pi_B |j_L\rangle \langle j_L|B^\dagger \Pi_B & \Pi_B |j_L\rangle \langle j_L|B^\dagger \Pi_B & |j_L\rangle \langle j_L|B^\dagger \Pi_B
\end{array} \right).
\]

(B.4)

By the isometric invariance of the trace-norm, we get

\[
\|M_{j,U}\|_1 = \left\| \begin{array}{ccc}
\Pi_G & -\Pi_G \mathcal{R}(U) & \Pi_G \\
\Pi_B \mathcal{R}(U) & \Pi_B & \Pi_B \\
\Pi_B \mathcal{R}(U) & \Pi_B & \Pi_B
\end{array} \right\|_1.
\]

(B.5)

Subsequently evaluating upper bounds on the trace norms of the block matrices and using the block matrix generalization of the Gersgorin circle theorem [], we get

\[
\|M_{j,U}\|_1 \leq \left( b_{u,j}^2 + b_{u,j} \right).
\]

(B.6)

2. **Deadman and Relton bounds**

The bound for storage errors in [B.2] allows us to directly express the storage error in terms of the Frechet derivatives. Given a unitary \( g(H + A) \), its uncorrectible component when we use a quantum code that corrects \( t \) errors is just the remainder term \( R_0^t(H, A) = \sum_{j=t+1}^{\infty} \frac{1}{j!} \varepsilon_0^j \langle H, A \rangle \). Using the results of Deadman and Relton [6][Lemma 3.1], this remainder term can be expressed as a contour integral, and has its operator norm \( \| \cdot \| \) bounded by

\[
\|R_0^t(H, A)\| \leq \frac{2e}{2\pi} \frac{2(\|H\| + \tau^{-1})(2\tau^{-1})}{\tau^{-k-2}} \|A\|^{t+1} = \frac{2e}{\pi} (\|H\| + \tau^{-1})(\|A\|\tau)^{t+1}.
\]

(B.7)

Here \( 2(\|H\| + \tau^{-1})(2\tau^{-1}) \) is the perimeter of a rectangular contour used, where the length and width of the rectangle are \( \|H\| + 1/\tau \) and 2/\( \tau \) respectively. Consequently, we have

\[
\epsilon_c \leq \frac{3e}{\pi} (\|H\| + \tau^{-1})(a\tau)^{t+1}.
\]

(B.8)

**Appendix C: Divided difference bound**

Here we will proceed to show that

\[
|g(\lambda_{n_1}, \ldots, \lambda_{n_j}, 0)| \leq \max_{\Omega} \frac{\tau^j - |\Omega| / (j - |\Omega|)!}{\prod_{i \in \Omega} (J_{\left\lfloor \frac{n}{2} \right\rfloor + 1 - y_i})}.
\]

(C.1)

We first use the eigenvalue properties of the mean-field Heisenberg Hamiltonian \( H \) to show that

\[
|g(\bar{y})| \leq \frac{1}{2J} \left( \frac{|g(y[not \ i])|}{\frac{1}{2} + 1 - n_i} + \frac{|g(y[not \ j])|}{\frac{1}{2} + 1 - n_j} \right),
\]

(C.2)
where \( \mathbf{y} = (\lambda_{n_1}, \ldots, \lambda_{n_j}, \lambda_{n_{j+1}}) \) and \( \left\lceil \frac{n}{2} \right\rceil + 1 = \lfloor n/2 \rfloor + 1 \) denotes the number of unique eigenvalues of the mean-field Heisenberg Hamiltonian. What is most noteworthy about this recursive relationship is that each term on the right hand side depends only on the component of \( \mathbf{y} \) that is removed. While this bound is certainly not tight, it nonetheless allows divided differences to be computed solely based on which components of \( \mathbf{y} \) were removed. This thereby avoids the need to evaluate an exponential number of terms in order to determine a bound for a single divided difference.

After a binary tree is recursively constructed using this algorithm, we can use the values of the leaves of this binary tree to obtain an upper bound on \( |g(\mathbf{y})| \). We assign this vertex with the value of 1, and mark it as non-terminating. With the assignment of the root vertex done, we now describe the recursive step. Any vertex in a binary tree is a leaf if its vertex degree is 0, 1. At this point, the root vertex is also a leaf.

1. Examine a given non-terminating vertex that is also a leaf.

2. If this vertex is labeled by a divided difference \( g(\mathbf{y}) \) where all components of \( \mathbf{y} \) are identical and \( \mathbf{y} \) has at least 2 components, then mark this vertex as a terminating vertex. Also assign this vertex value to be its previous vertex value multiplied by \( \tau^{r-1}/(r-1)! \), where \( r \) is the number of components of \( \mathbf{y} \).

3. If this vertex \( v \) is labeled by a divided difference \( g(\mathbf{y}) \) where not all components of \( \mathbf{y} \) are identical and \( \mathbf{y} \) has at least two components, then select two non-identical components of \( \mathbf{y} \) to pop.

   (a) Let \( y_i \) and \( y_j \) denote the components of \( \mathbf{y} \) to pop.

   (b) Create a child vertices \( \eta_1 \) and \( \eta_2 \), with parent vertex \( v \).

   (c) Label \( \eta_1 \) and \( \eta_2 \) with the divided differences \( g(\mathbf{y}[\text{not } i]) \) and \( g(\mathbf{y}[\text{not } j]) \) respectively.

   (d) Assign \( \eta_1 \) with a vertex value of \( \frac{1}{J(\lfloor \frac{n}{2} \rfloor + 1) - y_i} \) times the vertex value of \( v \)

   (e) Assign \( \eta_2 \) with a vertex value of \( \frac{1}{J(\lfloor \frac{n}{2} \rfloor + 1) - y_j} \) times the vertex value of \( v \).

4. If this vertex \( v \) is labeled by \( g(\mathbf{y}) \) where \( \mathbf{y} = (y) \) has one component, then create one child \( \eta \) of \( v \). Assign \( \eta \) with a vertex value of \( \frac{1}{(J/2 + 1) - y} \) times the vertex value of \( v \). Assign \( \eta \) to be a terminating vertex.

After a binary tree is recursively constructed using this algorithm, we can use the values of the leaves of this binary tree to obtain an upper bound on \( g(\mathbf{y}) \). Namely, if \( L \) is the set of all leaves in the binary tree, and \( \text{val}(u) \) denotes the vertex value of a leaf \( u \in L \), then

\[
|g(\mathbf{y})| \leq \max_{u \in L} \text{val}(u), \tag{C.3}
\]

and this proves (C.1). Here \( \Omega \) corresponds to the index set of the components of \( \mathbf{y} \) that are popped for a given terminating leaf. The components of \( \mathbf{y} \) that are not popped are identical, and the divided difference evaluated on them has an absolute value exactly equal to \( \tau^{r-1}/(r-1)! \). The term \( (J(\lfloor \frac{n}{2} \rfloor + 1) - y_v) \) in the denominator arises for each \( y_v \) that is popped to eventually arrive at a terminating leaf in the binary tree.

**Appendix D: upper bound on the high energy error** \( h_j \)

The high energy error \( h_j \) is just a sum of the absolute values of divided differences. To obtain an upper bound on this sum, we use the tree-based bound on each summand, and putting each summand into one of two categories. Now note that here can only be a total of \( j \) repeats in \( j + 1 \) arguments for the divided difference. So the number of non-repeating arguments is at least 1. In the first category, we consider non-repeating arguments of divided differences that are non-zero. The second category is the complement of the first. Then, together with the fact that the binomial coefficients are at most \( 2^j \), the above implies that

\[
h_j \leq 2^j \sum_{r=2}^{j} \left( \frac{n}{2} \right) \binom{2^{r-1}}{(r-1)} (S/J)^{j-r} \tag{D.1}
\]

\[
\leq 2^j \sum_{r=2}^{j} \frac{\tau^{r-1}}{(r-1)!} \left( \frac{S}{J} \right)^{j-r} \left( \frac{1}{J} + \frac{S}{J} \right). \tag{D.1}
\]
When $S/J < 1$, we can use the Hölder inequality on the above sum to get

$$h_j \leq 2^j \left( \frac{H}{J} \right)^{j-2} e^{\tau \left( \frac{1}{J} + \frac{S}{J} \right)}.$$  \hfill (D.2)

When $n \geq 9$, we get $1 + S/H < 2$ and $h_j \leq \frac{2J}{S^2} (2S/J)^j$. In this case, we use a geometric sum when we also have $2San/J < 1$, to get the bound

$$\sum_{j=t+1}^{\infty} (an)^j h_j \leq \frac{2Je^\tau (2San/J)^{t+1}}{S^2} \frac{1}{1-2San/J}. \hfill (D.3)$$

Now for all positive $x, y, \theta$ such that $|y/\theta| < 1$, we have

$$\sum_{j=t}^{\infty} \frac{x^j y^j}{j!} = \sum_{j=t}^{\infty} \frac{(\theta x)^j}{j!} (y/\theta)^j \leq e^{\theta x} \frac{(y/\theta)^t}{1-y/\theta}$$  \hfill (D.4)

where $0 < \theta x < 1$. From this, the codespace error is bounded by

$$\sum_{j=t+1}^{\infty} (an)^j c_j = \sum_{j=t+1}^{\infty} \frac{(an\tau)^j}{j!} \leq \inf_\theta \left\{ e^{\theta \tau} \frac{(an/\theta)^{t+1}}{1-an/\theta} : 0 < \theta < 1/(an) \right\}. \hfill (D.5)$$

Together this implies that

$$\epsilon \leq \frac{3e^\tau}{2} \left( c_1 x_1^{t+1} + \inf_{0<\theta<1/(an)} c_2 x_2^{t+1} \right), \hfill (D.6)$$

where $x_1 = 2San/J$, $x_2 = an\theta$, $c_1 = \frac{2J}{S^2(1-x_1)}$, and $c_2 = \frac{e^{(\theta-1)\tau}}{1-x_2}$. By setting $\theta = 1$, and using the bound $c_1 x_1^{t+1} + c_2 x_2^{t+1} \leq (c_1^{1/(t+1)} x_1 + c_2^{1/(t+1)} x_2)^{t+1}$, we can obtain a lower bound for $\tau$ given by

$$\tau \geq \log(\epsilon) - \log(3/2) - (t+1) \log(c_1^{1/(t+1)} x_1 + c_2^{1/(t+1)} x_2). \hfill (D.7)$$

can be easily obtained. From this inequality, we can optimize $\tau$ with respect to the $t$ and thereby obtain a lower bound on $\tau$ in terms of the noise $a$. 

\[ \quad \]