Observation of two $\mathcal{PT}$ transitions in an electric circuit with balanced gain and loss*

Tishuo Wang$^1$, Jianxiong Fang$^1$, Zhongyi Xie$^1$, Nenghao Dong$^1$, Yogesh N. Joglekar$^{2,a}$, Zixin Wang$^{4,b}$, Jiaming Li$^{1,c}$, and Le Luo$^{1,d}$

$^1$ School of Physics and Astronomy, Sun Yat-sen University, Zhuhai, Guangdong 519082, P.R. China
$^2$ Department of Physics, Indiana University Purdue University Indianapolis (IUPUI), Indianapolis, IN 46202, USA
$^3$ School of Electronics and Information Technology, Sun Yat-sen University, Guangzhou, Guangdong 510006, P.R. China

Received 2 March 2020 / Received in final form 3 June 2020
Published online 4 August 2020
© EDP Sciences / Società Italiana di Fisica / Springer-Verlag GmbH Germany, part of Springer Nature, 2020

Abstract. We investigate $\mathcal{PT}$-symmetry breaking transitions in a dimer comprising two LC oscillators, one with loss and the second with gain. The electric energy of this four-mode model oscillates between the two LC circuits, and between capacitive and inductive energy within each LC circuit. Its dynamics are described by a non-Hermitian, $\mathcal{PT}$-symmetric Hamiltonian with three different phases separated by two exceptional points. We systematically measure the eigenfrequencies of energy dynamics across the three regions as a function of gain-loss strength. In addition to observe the well-studied $\mathcal{PT}$ transition for oscillations across the two LC circuits, at higher gain-loss strength, transition within each LC circuit is also observed. With their extraordinary tuning ability, $\mathcal{PT}$-symmetric electronics are ideally suited for classical simulations of non-Hermitian systems.

1 Introduction

Open classical systems with gain and loss, described by non-Hermitian Hamiltonians that are invariant under combined operations of parity and time ($\mathcal{PT}$) reversal have attracted considerable attention over the past two decades. This interest started with the seminal work by Carl Bender and co-workers [1,2], and blossomed after their first realization in the optical domain [3,4]. A $\mathcal{PT}$-symmetric Hamiltonian has a purely real spectrum when the non-Hermiticity is small and its eigenvectors are simultaneous eigenvectors of the antilinear $\mathcal{PT}$ operator with eigenvalue one. When the non-Hermiticity exceeds a finite threshold, the spectrum changes into complex conjugate pairs, and the $\mathcal{PT}$ operator maps an eigenstate into the eigenstate of its complex-conjugate eigenvalue [5]. This transition, called $\mathcal{PT}$-symmetry breaking transition occurs at an exceptional point (EP), where both eigenvalues and corresponding eigenvectors of the non-Hermitian Hamiltonian coalesce. Over the past decade, systems with gain and loss, represented by positive and negative purely imaginary potentials, have been realized in a multitude of “wave systems” including electric circuits [6–11], synthetic photonic lattices [12], and micro-ring resonators [13,14]. On the other hand, due to the quantum limits on noise in linear amplifiers [15], effective $\mathcal{PT}$-symmetric systems in the quantum regime have only been recently realized across multiple platforms such as ultracold atoms [16], NV centers [17], superconducting qubit [18], single photons [19,20], and atomic assemblies [21].

Compared with the more challenging quantum realizations, classical setups with balanced gain and loss, or with mode-selective losses have many advantages. They are conceptually simpler, experimentally more accessible, and can have extraordinary tuning abilities that include interaction-induced nonlinearities, memory effects, and time delays. Thus, classical simulations of interesting quantum systems, especially systems with losses, gain, and noise [22], are of great interest. Among classical platforms, electronic circuits have experimentally advanced the studies of non-Hermitian physics with their low-cost configurations and enhanced sensitivity [11,23], novel devices and applications [10], and the ability to simulate topological condensed matter phenomena [24]. In particular, the simplest $\mathcal{PT}$-symmetric electric dimer, i.e. one dissipative RLC oscillator coupled to a –RLC oscillator with gain, has been used to demonstrate many novel concepts in non-Hermitian, $\mathcal{PT}$-symmetric systems such as the $\mathcal{PT}$-symmetry breaking transition and scattering in static or time-periodic (Floquet) $\mathcal{PT}$-symmetric circuits [6–8,25–27].

Figure 1 illustrates a $\mathcal{PT}$-symmetric dimer [6]. It consists of two LC oscillators, one with resistance $R$ (loss) and

---

* Contribution to the Topical Issue “Topological Ultracold Atoms and Photonic Systems”, edited by G. Juzeliūnas, R. Ma, Y.-J. Lin and T. Calarco.

$^a$ e-mail: jojoglek@iupui.edu
$^b$ e-mail: wangzix@mail.sysu.edu.cn
$^c$ e-mail: lijiam29@mail.sysu.edu.cn
$^d$ e-mail: luole5@mail.sysu.edu.cn
In Section 3. We conclude the paper with a brief discussion in Section 4.

2 Theoretical analysis

The dynamics of currents and voltages in Figure 1 are determined by Kirchoff laws [6],

\[
0 = I_n^R + I_n^C + I_n^L, \quad V_n = (-1)^n I_n^R R - \frac{1}{C} \int_0^t I_n^C(\tau) \, d\tau, \quad -V_1 = L \left( \frac{dI_1^L}{dt} + M \frac{dI_2^L}{dt} \right), \quad -V_2 = L \left( \frac{dI_2^L}{dt} + M \frac{dI_1^L}{dt} \right),
\]

where \( n = 1(2) \) represents the loss (gain) oscillator, and the currents across the resistor, capacitor, and inductor are denoted by \( I_n^R, I_n^C \) and \( I_n^L \) respectively. Equations (1)–(4) can be rewritten into a set of four, linear, coupled differential equations,

\[
\frac{dV_1}{dt} = -\gamma \omega_0 V_1 + I_1^L C, \quad \frac{dV_2}{dt} = \gamma \omega_0 V_2 + I_2^L C, \quad \frac{dI_1^L}{dt} = -V_1 + \mu V_2 - \frac{1}{L(1-\mu^2)}, \quad \frac{dI_2^L}{dt} = \mu V_1 - V_2 - \frac{1}{L(1-\mu^2)}.
\]

Equations (5)–(8) can be rewritten into a Schrödinger-like equation for the column vector \( |\psi(t)\rangle = (V_1, V_2, I_1^L, I_2^L)^T \), i.e. \( i \frac{d|\psi\rangle}{dt} = h|\psi\rangle \) where \( h \) is a \( 4 \times 4 \) non-Hermitian matrix with purely imaginary entries [6],

\[
h = i \begin{pmatrix} -\gamma \omega_0 & 0 & \frac{1}{\gamma} & 0 \\ 0 & \gamma \omega_0 & 0 & \frac{1}{\gamma} \\ -\frac{1}{L(1-\mu^2)} & L(1-\mu^2) & 0 & 0 \\ \frac{\mu}{L(1-\mu^2)} & 0 & 0 & 0 \end{pmatrix}.
\]

We note that casting the Kirchoff-law equations into a Schrödinger-like form is neither physically transparent not very useful, because the matrix \( h \) does not become Hermitian even in the no-gain, no-loss limit (\( \gamma = 0 \)). Moreover, since the vector \( |\psi\rangle \) has entries with two different engineering dimensions, so does the matrix \( h \). To make a connection with the \( P\bar{T} \)-symmetric quantum theory, we need to identify a quantity that remains conserved in the \( \gamma = 0 \) limit and study its dynamics. For two inductively coupled LC oscillators, that quantity is given by the circuit energy, i.e. \( Q(t) = \langle \psi(t)|A|\psi(t)\rangle \) where the positive-definite,
bilinear form $A$ is given by

$$
A = \begin{pmatrix}
\frac{C}{\gamma} & 0 & 0 & 0 \\
0 & \frac{C}{\gamma} & 0 & 0 \\
0 & 0 & \frac{L}{\gamma} & \frac{1}{2} \mu L \\
0 & 0 & \frac{L}{2} & \frac{1}{2} \mu L
\end{pmatrix}.
$$

Thus, an analogy with the standard quantum theory (where the norm a state is conserved in the Hermitian limit), we consider a state whose norm is given by the energy, i.e. $|\phi\rangle = A^{1/2}|\psi\rangle$ where $A^{1/2}$ denotes the positive, Hermitian square root of the matrix $A$.

We also note that all elements of the column vector $|\phi\rangle$ now have the same engineering dimensions. With this change of basis, the Kirchoff-law equations can be written as

$$
d|\phi\rangle dt = H|\phi\rangle, \tag{11}
$$

$$
H(\gamma) = \frac{\omega_0}{2} \begin{pmatrix}
-2i\gamma & 0 & i\gamma_0 & -i\gamma_{PT} \\
2i\gamma & -i\gamma_0 & 0 & 0 \\
-i\gamma_0 & i\gamma_{PT} & 0 & 0 \\
i\gamma_{PT} & -i\gamma_0 & 0 & 0
\end{pmatrix}, \tag{12}
$$

where $\gamma_{PT} = 1/\sqrt{1-\mu} - 1/\sqrt{1+\mu}$ and $\gamma_0 = 1/\sqrt{1-\mu} + 1/\sqrt{1+\mu}$ denote the locations of the two EPs, as we will discuss in the following paragraphs. Thus, the time-evolution of the “square-root of energy” state vector $|\phi\rangle$ is given by the Hamiltonian $H(\gamma) = A^{1/2}hA^{-1/2}$ that becomes Hermitian in the limit $\gamma = 0$, and satisfies $[PT, H] = 0$. Here the parity operator exchanges the labels $1 \leftrightarrow 2$, and the time-reversal operator, in addition to complex conjugation ($\ast$), reverses the currents $I_n^L \rightarrow -I_n^L$.

In matrix representation, they are given by

$$
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}. \tag{13}
$$

The eigenvalues of the $PT$-symmetric Hamiltonian $H(\gamma)$, equation (12), determine whether the system is in the $PT$-symmetric phase (all real eigenvalues) or broken phase (emergence of complex conjugate eigenvalues), and in the latter case, the degree of $PT$-symmetry breaking i.e. the fraction of eigenvalues that have become complex [28]. They are given by

$$
\omega_{1,2} = \frac{\omega_0}{2} \left( \sqrt{\gamma_0^2 - \gamma^2 \pm \sqrt{\gamma_0^4 - \gamma^2 \gamma_{PT}^2 - \gamma^2}} \right), \tag{14}
$$

$$
\omega_{3,4} = -\omega_{1,2}. \tag{15}
$$

In the Hermitian limit $\gamma \rightarrow 0$, the eigenvalues reduce to $\omega_1 = \omega_0/\sqrt{1-\mu} \geq \omega_2 = \omega_0/\sqrt{1+\mu}$ as is expected for two coupled LC oscillators. They approach each other, and become degenerate at the first EP, $\gamma_{PT}$. Then they turn into complex-conjugate pairs, as do $\omega_{3,4} = -\omega_{1,2}$. As $\gamma$ is increased further, the amplifying mode frequencies $\omega_{1,2}$ become degenerate at the second EP, $\gamma = \gamma_0$, as do the decaying-mode frequencies $\omega_{3,4}$.

We note that for weakly coupled LC oscillators, $\mu = M/L \ll 1$, a Taylor expansion gives $\omega_{1,2} \approx \omega_0(1 \pm \mu/2)$, a Hermitian gap $\omega_1 - \omega_2 = \omega_0\mu$ and a dimensionless $\gamma_{PT} \sim \mu$ thus recovering the $PT$-threshold result for a prototypical dimer Hamiltonian $H_D = \omega_0((\sigma_3 + i\gamma_0\sigma_2)/2$. For $\mu \ll 1$, the second threshold gives $\gamma_0 \approx 2$ or equivalently $R_0 = \sqrt{L/C}/2$. This is precisely the resistance at which a parallel RLC circuit is critically damped. In the opposite limit, $\mu \rightarrow 1$, the two LC circuits are strongly coupled, leading to a divergent $\gamma_{PT} \approx \sqrt{L/\Delta L}$ and $\gamma_0 = \gamma_{PT} \mp \sqrt{2}$, where $\Delta L = L - M = L(1 - \mu)$ denotes the small difference between the self-inductance and the mutual inductance.

In this limit, the frequencies $\omega_{1,2}$ are asymmetrically located around $\omega_0$, with their difference diverging as $\omega_1 - \omega_2 = \omega_0\sqrt{L/\Delta L}$.

In the parameter range $\gamma_{PT} < \gamma < \gamma_0$, the energy dynamics across the two LC circuits is in the $PT$-broken phase while the dynamics within each LC circuit is in the $PT$-symmetric phase. Past the second EP at $\gamma = \gamma_0$, all four eigenvalues are purely imaginary, which indicates overdamped (dissipative) modes for the RLC circuit and their gain analog for the $\pm$RLC circuit. In this regime, dynamics across the two LC circuits, as well as within each LC circuit are in the $PT$-broken phase. In our experiments, discussed in the following section, we only focus on the positive real and imaginary parts of the eigenvalues, as the remaining two eigenvalues are determined by the particle-hole symmetric nature of the spectrum [29].

### 3 Experimental setup and results

The experimental setup is similar to that in references [6,30], although in those works, only the transition across the first EP at $\gamma_{PT}$ was studied. However, the circuit in Figure 1 inherently allows exploration of all three regions, separated by the two EPs. Our LC circuits have $L = 7.91 \, \text{mH}$ and $C = 10.14 \, \text{nF}$, and a fundamental frequency of $\omega_0 = 2\pi \times 17.77 \, \text{kHz}$. The two oscillators are moderately coupled, i.e. $\mu = 0.6$, leading to $\gamma_{PT} = 0.79$ and $\gamma_0 = 2.37$. To measure the eigenfrequencies across these regions, we adopt two methods.

First, the inherent, resistive losses of the two inductors on the gain and the loss sides are canceled by two negative resistors. Without this cancellation, the experimental results do not match the theoretical analysis presented in the preceding section. In our setup, this resistance in each inductor, shown in Figure 1, is $R_L = 16.80 \, \Omega$. It is compensated, in each isolated $\pm$RLC circuit, by a negative impedance converter (NIC), which is based on a non-inverting amplifier [6,30]. When each isolated LC circuit reaches this balance, the output voltage in each, i.e. $V_{1,2}(t)$ start self-oscillations [30]. In our experiments, the negative resistance compensates the inherent resistance of the inductors to within 2% accuracy. We emphasize that although $R_L$ is tens of Ohms, its effects become important at large $\gamma$, which is achieved by reducing the loss-gain resistance $\pm R$. For example, to traverse across the second EP at $\gamma_0$, the loss-gain resistances $\pm R$ are reduced to 350 $\Omega$. With an uncompensated inductor resistance,
In this paper, we have presented theoretical analysis and experimental observation of PT transitions in a minimal electric circuit with balanced gain and loss. In addition to presenting a transparent way to map the Kirchhoff law equations into a Schrödinger equation for the circuit energy dynamics, we have emphasized the four-mode nature of the PT-symmetric electric circuit which gives rise to three regions separated by two exceptional points. We have then presented experimental data for the complex eigenfrequencies of this model, which are in good agreement with the theoretical model.

Recently, a passive (without gain) PT-symmetric electric circuit is also studied for the slowly decaying mode without exceptional point [8]. The time evolution of the voltage (and current) in the passive case is limited by the noise floor of the data acquisition due to lack of gain (continuous energy injection). Therefore, it is difficult to experimentally determine the phases without a sufficiently long-time evolution result, particularly in the large γ regime. In comparison with this passive scheme, our balanced gain and loss PT-symmetric electric circuit overcomes the unbalanced resistances of the RLC pairs and provides a better platform to explore the phase diagram. The tricky thing of the balanced setup is the resistances in the gain and loss sides should be well matched during each measurement.

PT-symmetric electric circuit is a hot topic not only in the PT physical researches, but also in the engineering fields. Recently, there are a plenty of fantastical applications on the PT-symmetric platforms, such as robust wireless power transfer with nonlinear or high-order schemes [10,31], sensitive detection near the exceptional points [32,33]. In our setup, the two exceptional points are not higher order ones, they are corresponding to the dynamics of dimer and LC oscillation respectively. We
first experimentally mapped out the both $PT$-symmetry breaking. We hope our studies inspire some potential applications in the wireless transfer energy as well as sensitive detections.

We thank Zhenhua Yu for discussions. JLi received supports from National Natural Science Foundation of China (NSFC) under grant No. 11804406, Fundamental Research Funds for Sun Yat-sen University. Science and Technology Program of Guangzhou 2019-030105-3001-0035. LL received supports from NSFC under grant No. 11774436, Guangdong Province Youth Talent Program under grant No. 2017GC010656, Sun Yat-sen University Core Technology Development Fund, and the Key-Area Research and Development Program of Guangdong Province under grant No. 2019B030330001.

**Author contribution statement**

T. Wang and J. Fang contributed equally to this work. Correspondence and requests for materials should be addressed to Y.N.J., J.L., or L.L.

**Publisher’s Note** The EPJ Publishers remain neutral with regard to jurisdictional claims in published maps and institutional affiliations.

**References**

1. C.M. Bender, S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998)
2. C.M. Bender, Rep. Prog. Phys. 70, 947 (2007)
3. A. Guo, G.J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G.A. Siviloglou, D.N. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009)
4. C.E. Rüter, K.G. Makris, R. El-Ganainy, D.N. Christodoulides, M. Segev, D. Kip, Nat. Phys. 6, 192 (2010)
5. Y.N. Joglekar, C. Thompson, D.D. Scott, G. Vemuri, Eur. Phys. J. Appl. Phys. 63, 30001 (2013)
6. J. Schindler, A. Li, M.C. Zheng, F.M. Ellis, T. Kottos, Phys. Rev. A 84, 040101(R) (2011)
7. M. Chitsazi, H. Li, F.M. Ellis, T. Kottos, Phys. Rev. Lett. 119, 093901 (2017)
8. R.D.J. León-Montiel, M.A. Quiroz-Juárez, J.L. Domínguez-Juárez, R. Quintero-Torres, J.L. Aragón, A.K. Harter, Y.N. Joglekar, Commun. Phys. 1, 88 (2018)
9. M. Chitsazi, S. Factor, J. Schindler, H. Ramezani, F.M. Ellis, T. Kottos, Phys. Rev. A 89, 043842 (2014)
10. S. Assawaworrarit, X. Yu, S. Fan, Nature 546, 387 (2017)
11. P.-Y. Chen, M. Sakhdari, M. Hajizadegan, Q. Cui, M.M.-C. Cheng, R. El-Ganainy, A. Aù, Nat. Electron. 1, 297 (2018)
12. A. Regensburger, C. Bersch, M.-A. Mìri, G. Onishchukov, D.N. Christodoulides, U. Peschel, Nature 488, 167 (2012)
13. B. Peng, S.K. Ozdemir, F. Lei, F. Monifi, M. Gianfreda, G.L. Long, S. Fan, F. Nori, C.M. Bender, L. Yang, Nat. Phys. 10, 394 (2014)
14. H. Xu, D. Mason, L. Jiang, J.G.E. Harris, Nature 537, 80 (2016)
15. C.M. Caves, Phys. Rev. D 26, 1817 (1982)
16. J. Li, A.K. Harter, J. Liu, L. de Melo, Y.N. Joglekar, L. Luo, Nat. Commun. 10, 855 (2019)
17. Y. Wu, W. Liu, J. Geng, X. Song, X. Ye, C.-K. Duan, X. Rong, J. Du, Science 364, 878 (2019)
18. M. Naghiloo, M. Abbasi, Y.N. Joglekar, K.W. Murch, Nat. Phys. 15, 1232 (2019)
19. L. Xiao, X. Zhan, Z.H. Bian, K.K. Wang, X. Zhang, X.P. Wang, J. Li, K. Machuziki, D. Kim, N. Kawakami, W. Yi, H. Obuse, B.C. Sanders, P. Xue, Nat. Phys. 13, 1117 (2017)
20. Z. Bian, L. Xiao, K. Wang, X. Zhan, F.A. Onanga, F. Ruzicka, W. Yi, Y.N. Joglekar, P. Xue, Phys. Rev. Research 2, 022039(R) (2020)
21. Z. Zhang, Y. Zhang, J. Sheng, L. Yang, M.-A. Mìri, D.N. Christodoulides, B. He, Y. Zhang, M. Xiao, Phys. Rev. Lett. 117, 123601 (2016)
22. J. Franson, Physics 9, 66 (2016)
23. Z. Xiao, H. Li, T. Kottos, A. Alù, Phys. Rev. Lett. 123, 213901 (2019)
24. C.H. Lee, S. Imhof, C. Berger, F. Bayer, J. Brehm, L.W. Molenkamp, T. Kiessling, R. Thomale, Commun. Phys. 1, 39 (2018)
25. Z. Lin, J. Schindler, F.M. Ellis, T. Kottos, Phys. Rev. A 85, 050101(R) (2012)
26. H. Li, B. Shapiro, T. Kottos, Phys. Rev. B 98, 121101(R) (2018)
27. H. Ramezani, J. Schindler, F.M. Ellis, U. Günther, T. Kottos, Phys. Rev. A 85, 062122 (2012)
28. Y.N. Joglekar, J.L. Barnett, Phys. Rev. A 84, 024103 (2011)
29. Y.N. Joglekar, Phys. Rev. A 82, 044101 (2010)
30. J. Schindler, Master thesis, Wesleyan University, 2013
31. M. Sakhdari, M. Hajizadegan, P.-Y. Chen, Phys. Rev. Res. 2, 013152 (2020)
32. H. Hodaei, A.U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D.N. Christodoulides, M. Khajavikhan, Nature 548, 187 (2017)
33. M. Sakhdari, M. Hajizadegan, Q. Zhong, D.N. Christodoulides, R. El-Ganainy, P.-Y. Chen, Phys. Rev. Lett. 123, 193901 (2019)