Special Orthogonal Group SO(3), Euler Angles, Angle-axis, Rodriguez Vector and Unit-Quaternion: Overview, Mapping and Challenges

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Abstract

The attitude of a rigid-body in the three dimensional space has a unique and global definition on the Special Orthogonal Group SO(3). This paper gives an overview of the rotation matrix, attitude kinematics and parameterization. The four most frequently used methods of attitude representations are discussed with detailed derivations, namely Euler angles, angle-axis parameterization, Rodriguez vector, and unit-quaternion. The mapping from one representation to others including SO(3) is given. Also, important results which could be useful for the process of filter and/or control design are given. The main weaknesses of attitude parameterization using Euler angles, angle-axis parameterization, Rodriguez vector, and unit-quaternion are illustrated.

Keywords: Special Orthogonal Group SO(3), Euler angles, Angle-axis, Rodriguez Vector, Unit-quaternion, Mapping, Parameterization, Attitude, Control, Filter, Observer, Estimator, Rotation, Rotational matrix, Transformation matrix, Orientation, Transformation, Roll, Pitch, Yaw.

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1. Introduction

Automated or semi-automated systems have been proven to be indispensable in the majority of contemporary engineering applications. Rigid-bodies rotating and/or moving in space constitute a fundamental part of these applications, which include unmanned aerial vehicles (UAVs), satellites, spacecrafts, rotating radars, underwater vehicles, ground vehicles, robotic systems, etc. **Rotational matrix**, also termed "attitude", describes the orientation of a rigid body rotating in the three dimensional (3D) space. Thus, in this article the words attitude, orientation or rotational matrix will be used interchangeably. The estimation and/or control process of a rigid-body moving and rotating in space is defined by pose (attitude and position) problem. Obviously, attitude is a major part of the pose problem. The successful estimation and/or control process of a rigid-body rotating and/or moving in space depends primarily on the precise representation of the attitude of the rigid-body. There are several approaches to attitude parameterization of a rigid-body in the 3D space. Some of those approaches use $3 \times 3$ parameterization, while others use three or four components. The attitude can be parameterized through a $3 \times 3$ orthogonal matrix, which is known to follow the Special Orthogonal Group $\text{SO}(3)$. Attitude parameterization could also be achieved by three components through making use of Rodriguez vector or Euler angles (roll, pitch, and yaw). Additionally, the attitude can be parameterized using four components, for instance, unit-quaternion and angle-axis parameterization. The mapping of the three components (Euler angles or Rodriguez vector) or four components (unit-quaternion or angle-axis) to $\text{SO}(3)$ must have orthogonal configuration with the determinant equal to one. The main problem of the $3 \times 3$ configuration is that it is neither Euclidean nor has a vector form. Accordingly, the majority of researchers prefer to consider the vector form in the process of estimation and/or control of a rigid-body rotating and/or moving in space. The Euclidean parameterization includes Rodriguez vector and Euler angles which lie in $\mathbb{R}^3$. Unit-quaternion vector is used extensively in the estimation and/or control process of a rigid-body rotating and/or moving in space, in spite of the fact that it is non-Euclidean and lies in the three-sphere $S^3$.

A more detailed discussion of the above-mentioned methods of attitude parameterization and mapping from one method to the other have been discussed in several journal articles, for instance [1], [2], [3], and [4]. The main advantage of representing attitude directly on $\text{SO}(3)$ is that a $3 \times 3$ set creates a global and unique representation of the attitude problem. This implies that any orientation of a rigid-body in 3D space has a unique rotational matrix. Rodriguez vector, Euler angles and angle-axis parameterization fail to capture the attitude at certain configurations and could be limited to singularity, called “unstable equilibria”. On the other hand, unit-quaternion does not have the problem of singularity, but it nonetheless suffers from non-uniqueness in representation of the attitude. The above-named deficiencies of the attitude representation methods have been considered in [1] and [3]. These shortcomings will be elaborated on in the subsequent sections. The control and/or the estimation process of a rigid-body rotating and/or moving in space could possibly fail if the vector associated with the parameterization of the rigid body started at certain configurations.

Tracing control of any rigid-body is mainly based on attitude. Over the last few decades, tracking control of a rigid-body has gained popularity and aroused interest of researchers in the control community. These applications included spacecrafts ([5], [6], and [7]), satellites [8], mobile robots [6, 9], etc. Attitude and pose filtering problem is an essential task in robotics applications. This filtering problem requires a set of measurements made with respect to the inertial frame and obtained from the sensors attached to the rigid-body. In the past, this problem used to be tackled through vehicles equipped with high quality sensors [10, 11, 12]. High quality sensors are quite expensive and may not be an optimal fit for small scale systems. With the introduction of micro electro-mechanical systems (MEMSs) a range of inertial measurement units (IMUs) was proposed. These units are small, inexpensive, and light-weight, allowing them to be easily attached to small drones, mini UAVs, spacecrafts, radar, satellites, mobile robots and other applications. Despite the advantages that IMUs offer, their quality is fairly low and the measurements they provide are prone to bias and noise.

The environment is unpredictable and the uncertainties present in measurements are characterized by irregular behavior [12, 11] and additionally complicated by the fact that the initial conditions may not be accurately known [13, 14]. Therefore, the attitude and pose filter should be particularly robust against
uncertainties in measurements and any initialized error, and should be able to converge to the desired solution. Over the past few decades, several successful filters have been proposed providing high quality estimation of the true attitude or pose. The attitude estimation problem used to be addressed either by a Gaussian filter or a nonlinear deterministic filter. The majority of Gaussian filters consider the unit-quaternion in the problem formulation and filter structure, for instance [15], [16], [17], [11], [18], and [19]. Nonlinear deterministic filters are directly developed on $SO(3)$ such as, [12], and [20]. Other filters were evolved directly on $SO(3)$ and $SE(3)$, considering Rodriguez vector and angle-axis parameterization in stability analysis such as [21, 22, 20, 23, 24, 25, 26]. The filters in [21, 22] are nonlinear stochastic filters.

According to literature, the crucial factor of success of any control and/or estimation process of a rigid-body rotating and/or moving in space is primarily attributed to attitude parameterization. Therefore, the first and main focus of this article is to give an overview of different methods of attitude parameterization and ways to transition between those methods. In fact, transitioning between methods could help to avoid the complexities of the design process. The second purpose is to outline important results associated with $SO(3)$, angle-axis parameterization, Rodriguez vector, and unit-quaternion which could be very beneficial in the control and/or the estimation process of a rigid-body rotating and/or moving in space. These results could significantly simplify the process of filter and/or control design.

The remainder of the paper is organized in the following manner: Section 2 presents an overview of all mathematical and attitude notation, preliminaries to $SO(3)$, angle-axis, Rodriguez vector, and unit-quaternion parameterization, and some useful identities. Section 3 gives an overview of $SO(3)$, attitude kinematics and some important results. Section 4 shows the mapping from/to Euler angles to/from other methods of attitude representation, Euler rate, angular velocity transformation and the problem of attitude parameterization using Euler angles. Section 5 illustrates the mapping from/to angle-axis parameterization to/from other methods of attitude representation, some important results and the problem of attitude parameterization using angle-axis parameterization. Section 6 parameterizes the attitude using Rodriguez vector and shows the mapping from/to Rodriguez vector to/from other methods of attitude representation, and Rodriguez vector dynamics. Also, Section 6 presents some important results and the problem of attitude parameterization using Rodriguez vector. Section 7 gives an overview of unit-quaternion, shows the mapping from/to unit-quaternion to/from other attitude representations, attitude kinematics, and the problem of attitude parameterization using unit-quaternion. Section 9 concludes the work by summarizing attitude parameterization and mapping from one attitude representation to other representations in Table 7, 8, 9, and 10.
2. Preliminaries and Math Notation

Table 1 presents the important math notation used throughout the paper.

| Symbol | Description |
|--------|-------------|
| \(\mathbb{N}\) | The set of integer numbers |
| \(\mathbb{R}_+\) | The set of nonnegative real numbers |
| \(\mathbb{R}^n\) | Real \(n\)-dimensional vector |
| \(\mathbb{R}^{n \times m}\) | Real \(n\)-by-\(m\) dimensional matrix |
| \(\|\cdot\|\) | Euclidean norm, for \(x \in \mathbb{R}^n, \|x\| = \sqrt{x^\top x}\) |
| \(S^1\) | Unit-circle, \(S^1 = \{x = [x_1, x_2]^\top \in \mathbb{R}^2 \mid \sqrt{x_1^2 + x_2^2} = 1\}\) |
| \(S^2\) | Two-sphere, \(S^2 = \{x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3 \mid \|x\| = 1\}\) |
| \(S^n\) | \(n\)-sphere, \(S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}\) |
| \(\top\) | Transpose of a component |
| \(\times\) | Cross multiplication |
| \(\mathbf{I}_n\) | Identity matrix with dimension \(n\)-by-\(n\), \(\mathbf{I}_n \in \mathbb{R}^{n \times n}\) |
| \(\mathbf{0}_n\) | Zero matrix with dimension \(n\)-by-\(n\), \(\mathbf{0}_n \in \mathbb{R}^{n \times n}\) |
| \(\det (\cdot)\) | Determinant of a component |
| \(\text{Tr} \{\cdot\}\) | Trace of a component |

Table 2 provides some important definitions and notation related to attitude.
Table 2: Attitude Notation

| Symbol | Explanation |
|--------|-------------|
| \{I\} | Inertial-frame of reference |
| \{B\} | Body-frame of reference |
| GL (3) | The 3 dimensional General Linear Group |
| O (3) | Orthogonal Group |
| SO (3) | Special Orthogonal Group |
| so (3) | The space of 3 \times 3 skew-symmetric matrices, and Lie-algebra of SO (3) |
| \| \times \| | Skew-symmetric of a matrix |
| \mathcal{P}_a | Anti-symmetric projection operator |
| R | Attitude/Rotational matrix/Orientation of a rigid-body, \( R \in \text{SO}(3) \) |
| \Omega | Angular velocity vector with \( \Omega = [\Omega_x, \Omega_y, \Omega_z]^\top \in \mathbb{R}^3 \) |
| \nu^I | Vector in the inertial-frame, \( \nu^I \in \mathbb{R}^3 \) |
| \nu^B | Vector in the body-frame, \( \nu^B \in \mathbb{R}^3 \) |
| \mathcal{R}_\xi | Attitude representation obtained using Euler angles \( (\phi, \theta, \psi) \), \( \mathcal{R}_\xi \in \text{SO}(3) \) |
| \xi | Euler angle vector with, \( \xi = [\phi, \theta, \psi]^\top \in \mathbb{R}^3 \) |
| J | Transformation matrix between Euler rate and angular velocity vector, \( J \in \mathbb{R}^{3 \times 3} \) |
| \mathcal{R}_a | Attitude representation obtained using angle-axis parameterization, \( \mathcal{R}_a \in \text{SO}(3) \) |
| a | Angle of rotation, \( a \in \mathbb{R} \) |
| u | Unit vector (axis of parameterization), \( u = [u_1, u_2, u_3]^\top \in \mathbb{S}^2 \subset \mathbb{R}^3 \) |
| \mathcal{R}_\rho | Attitude representation obtained using Rodriguez vector, \( \mathcal{R}_\rho \in \text{SO}(3) \) |
| \rho | Rodriguez vector, \( \rho = [\rho_1, \rho_2, \rho_3]^\top \in \mathbb{R}^3 \) |
| \mathcal{R}_Q | Attitude representation obtained using unit-quaternion vector, \( \mathcal{R}_Q \in \text{SO}(3) \) |
| Q | Unit-quaternion vector, \( Q = [q_0, q_1, q_2, q_3]^\top \in \mathbb{S}^3 \subset \mathbb{R}^4 \) |
| Q' | Complex conjugate of unit-quaternion, \( Q' \in \mathbb{S}^3 \subset \mathbb{R}^4 \) |
| \odot | Multiplication operator of two unit-quaternion vectors |

Let \( x \in \mathbb{R}^3 \) and \( M \in \mathbb{R}^{3 \times 3} \), and consider \( f(x, M) \) function such as

\[
f(x, M) = x^\top M x \in \mathbb{R}
\]

where \( ^\top \) is the transpose of a component. Hence, the arbitrary transformation through mapping is

\[
f : \mathbb{R}^3 \times \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}
\]

2.1. SO(3) Preliminaries

Let GL(3) stand for the 3 dimensional general linear group that is a Lie group characterized with smooth multiplication and inversion. O(3) refers to the Orthogonal Group and is a subgroup of the general linear group

\[
O(3) := \ \{ A \in \mathbb{R}^{3 \times 3} \mid A^\top A = A A^\top = I_3 \} \tag{1}
\]
The associated Lie-algebra of $\text{SO}(3)$ is termed $\mathfrak{so}(3)$ and is defined by

$$\mathfrak{so}(3) := \left\{ A : \mathbb{R}^{3 \times 3} \mid A^\top = -A \right\}$$

with $A \in \mathbb{R}^{3 \times 3}$ being the space of skew-symmetric matrices. Define the map $[\cdot]_\times : \mathbb{R}^3 \to \mathfrak{so}(3)$ such that

$$[a]_\times = a \times \beta$$

where $\times$ is the cross product between the two vectors. Let the vext operator be the inverse of a skew-symmetric matrix $[\cdot]_\times$ to vector form, denoted by $\text{vex} : \mathfrak{so}(3) \to \mathbb{R}^3$ such that

$$\text{vex}(A) = a \in \mathbb{R}^3$$

for all $a \in \mathbb{R}^3$ and $A \in \mathfrak{so}(3)$ as given in (4). Let $\mathcal{P}_a$ denote the anti-symmetric projection operator on the Lie-algebra $\mathfrak{so}(3)$, defined by $\mathcal{P}_a : \mathbb{R}^{3 \times 3} \to \mathfrak{so}(3)$ such that

$$\mathcal{P}_a(B) = \frac{1}{2} (B - B^\top) \in \mathfrak{so}(3)$$

for all $B \in \mathbb{R}^{3 \times 3}$. Hence, one obtains

$$\text{vex}(\mathcal{P}_a(B)) = \beta \in \mathbb{R}^3$$

$$|\beta|_\times = \mathcal{P}_a(B) = \frac{1}{2} (B - B^\top)$$

2.2. Angle-axis parameterization preliminaries

The attitude of a rigid body can be established given angle of rotation $\alpha \in \mathbb{R}$ and a unit-axis $u = [u_1, u_2, u_3]^\top \in \mathbb{S}^2$. This method of attitude reconstruction is termed angle-axis parameterization [3]. The mapping of angle-axis-parameterization to $\text{SO}(3)$ is governed by $\mathcal{R}_\alpha : \mathbb{R} \times \mathbb{S}^2 \to \text{SO}(3)$ such that

$$\mathcal{R}_\alpha(\alpha, u) = \exp(-\alpha [u]_\times) = \mathbb{I}_3 + \sin(\alpha) [u]_\times + (1 - \cos(\alpha)) [u]_\times^2$$

2.3. Rodriguez vector preliminaries

The attitude can be obtained through Rodriguez parameters vector $\rho = [\rho_1, \rho_2, \rho_3]^\top \in \mathbb{R}^3$. The related map to $\text{SO}(3)$ is given by $\mathcal{R}_\rho : \mathbb{R}^3 \to \text{SO}(3)$ with

$$\mathcal{R}_\rho = \frac{1}{1 + \|\rho\|^2} \left( \left(1 - \|\rho\|^2\right) \mathbb{I}_3 + 2\rho \rho^\top + 2 [\rho]_\times \right)$$
2.4. Unit-quaternion preliminaries

The unit-quaternion is defined by

\[ Q = [q_0, q^\top] \in S^3 \]

where \( q_0 \in \mathbb{R} \) and \( q = [q_1, q_2, q_3]^\top \in \mathbb{R}^3 \) such that

\[ S^3 = \{ Q \in \mathbb{R}^4 \mid \|Q\| = 1 \} \] (11)

Let \( Q = [q_0, q^\top] \in S^3 \), hence, \( Q^* = Q^{-1} \in S^3 \) can be defined as follows

\[ Q^* = Q^{-1} = \left[ q_0 - q \right] \in S^3 \] (12)

where \( Q^* \) and \( Q^{-1} \) are complex conjugate and inverse of the unit-quaternion, respectively.

2.5. Useful math identities

This subsection presents a list of identities which are going to prove subsequently useful in the article.

For any Lie bracket \([A, B]\), we have

\[ [A, B] = AB - BA, \quad A, B \in \mathbb{R}^{3 \times 3} \] (13)

\[ [Av, Bv] = (AB - BA) v, \quad A, B \in \mathbb{R}^{3 \times 3}, v \in \mathbb{R}^3 \] (14)

\[ \text{Tr} \{ [A, B] \} = 0, \quad A, B \in \mathbb{R}^{3 \times 3} \] (15)

where \( \text{Tr} \{ \cdot \} \) refers to the trace of a matrix. Let \( R \in SO(3) \) and \( \omega, \upsilon \in \mathbb{R}^3 \), then the following identities hold

\[ [Rv]_x = R [v]_x R^\top \] (16)

\[ -[\omega]_x \omega = [0, 0, 0]^\top \] (17)

\[ [\omega]_x \upsilon = -[v]_x \omega \] (18)

\[ -[\omega]_x [v]_x = (\omega^\top v) I_3 - v \omega^\top \] (19)

\[ \text{Tr} \{ A [v]_x \} = 0, \quad A = A^\top \in \mathbb{R}^{3 \times 3} \] (20)

\[ A^\top [v]_x + [v]_x A = [(\text{Tr} \{ A \}) I_3 - A] [v]_x, \quad A \in \mathbb{R}^{3 \times 3} \] (21)

\[ \text{Tr} \{ A [v]_x \} = \text{Tr} \{ \mathcal{P}_a (A) [v]_x \} = -2 \text{vex} (\mathcal{P}_a (A))^\top v, \quad A \in \mathbb{R}^{3 \times 3} \] (22)

Before we proceed further, any \( R \) is a rotational matrix and it is important to note that

\[ R = R_{\xi} = R_\alpha = R_\rho = R_Q \in SO(3) \]

with respect to the method of attitude parameterization.

3. Special Orthogonal Group \( SO(3) \)

The matrix \( R \in \mathbb{R}^{3 \times 3} \) is said to represent the attitude of a rigid-body if and only if \( R \in SO(3) \) which, in turn, is true when the following two conditions are satisfied:

1. \( \det (R) = +1 \).

2. \( R^\top R = RR^\top = I_3 \), which means that \( R^{-1} = R^\top \).
$R$ is called rotation by inversion if $\det(R) = -1$, which does not belong to $\text{SO}(3)$ and therefore, will not be considered in our analysis. The focus of this analysis is $R \in \text{SO}(3)$, which satisfies both of the above-mentioned conditions. The main advantage of the $R \in \text{SO}(3)$ representation is that the attitude $R$ is global and unique, implying that each physical orientation of a rigid-body corresponds to a unique rotational matrix.

3.1. Attitude kinematics

Let $R \in \text{SO}(3)$ denote the attitude (rotational matrix). The relative orientation of the body-frame $\{B\}$ with respect to the inertial-frame $\{I\}$ is given by the rotational matrix $R$ as illustrated in Figure 1.

![Figure 1: The orientation of a rigid-body in body-frame relative to inertial-frame in the 3D-space [22].](image)

The inertial-frame is fixed and commonly known as the world or the global coordinate, while the origin of the inertial-frame is denoted by $o^I$. The body-frame is fixed to the moving rigid-body and the origin of the body-frame is denoted by $o^B$. For a body-frame vector $v^B \in \mathbb{R}^3$ and inertial-frame vector $v^I \in \mathbb{R}^3$, the translation from body-frame to inertial-frame is defined by the attitude matrix $R \in \text{SO}(3)$

$$v^B = R^\top v^I$$  \hspace{1cm} (23)

The orientation of the inertial-frame is always fixed which means that $\dot{v}^I = 0$. The velocity at the body-frame in (23) is expressed by

$$\dot{v}^B = R^\top \dot{v}^I = R^\top Rv^B$$  \hspace{1cm} (24)

At every time instant $t$, there exists a unique angular velocity vector $\Omega = [\Omega_x, \Omega_y, \Omega_z]^\top \in \mathbb{R}^3$ such that, for every particle in the body, one has

$$\dot{v}^B = v^B \times \Omega = [v^B]_\times \Omega$$  \hspace{1cm} (25)
where $\Omega$ is the body-referenced angular velocity. Hence, combining (24) and (25) yields

$$
\dot{R}^T R v^B = -[\Omega]_x v^B
$$
$$
\dot{R}^T = [\Omega]_x^T R^T
$$

(26)

where $[v^B]_x \Omega = -[\Omega]_x v^B$. Thus, the attitude kinematics $\dot{R} \in \mathbb{R}^{3 \times 3}$ of a rigid-body are given by

$$
\dot{R} = R [\Omega]_x
$$

(27)

From (27), one can find that the anti-symmetric projection operator of the attitude kinematics $P_a : \mathbb{R}^{3 \times 3} \rightarrow so(3)$ is

$$
P_a (\dot{R}) = P_a (R [\Omega]_x)
$$
$$
= \frac{1}{2} \left( R [\Omega]_x + [\Omega]_x R^T \right)
$$
$$
= \frac{1}{2} \left( \text{Tr} \{ R \} [\Omega]_x - [R^T \Omega]_x \right)
$$
$$
= \frac{1}{2} \left( [\text{Tr} \{ R \} I_3 - R]^T \Omega \right)_x \in so(3)
$$

(28)

where $R [\Omega]_x + [\Omega]_x R^T = \left( [\text{Tr} \{ R \} I_3 - R]^T \Omega \right)_x$ as defined in identity (21). Thus, the dynamics in (27) can be defined in vector form $\text{vex} (P_a (R)) \in \mathbb{R}^3$ as

$$
\frac{d}{dt} \text{vex} (P_a (R)) = \text{vex} (P_a (R [\Omega]_x))
$$
$$
= \frac{1}{2} \left( [\text{Tr} \{ R \} I_3 - R]^T \Omega \right)
$$

(29)

Similarly, for $A = A^T \in \mathbb{R}^{3 \times 3}$, one obtains

$$
\frac{d}{dt} \text{vex} (P_a (AR)) = \text{vex} (P_a (AR [\Omega]_x))
$$
$$
= \frac{1}{2} \left( [\text{Tr} \{ AR \} I_3 - AR]^T \Omega \right)
$$

(30)

### 3.2. Normalized Euclidean distance

Since, $-1 \leq \text{Tr} \{ R \} \leq 3$ for $R \in \text{SO} (3)$, the normalized Euclidean distance of a rotational matrix on $\text{SO} (3)$ is defined as

$$
\| R \|_I = \frac{1}{4} \text{Tr} \{ I_3 - R \} \in [0, 1]
$$

(31)
3.3. Normalized Euclidean distance dynamics

From the normalized Euclidean distance in (31), the attitude dynamics in (27), and the identity in (22), one can find that the dynamics of the normalized Euclidean distance of (31) are equal to

\[
\frac{d}{dt} \| R \|_I = \frac{d}{dt} \frac{1}{4} \operatorname{Tr} \{ I_3 - R \} \\
= -\frac{1}{4} \operatorname{Tr} \{ R \} \\
= -\frac{1}{4} \operatorname{Tr} \{ R [\Omega]_\times \} \\
= -\frac{1}{4} \operatorname{Tr} \{ R \mathcal{P}_a ([\Omega]_\times) \} \\
= \frac{1}{2} \operatorname{vex} (\mathcal{P}_a (R))^T \Omega 
\]

(32)

3.3.1. Discrete Attitude Kinematics

The attitude kinematics defined in continuous form in (27) could be expressed in discrete form using exact integration by

\[
R[k+1] = R[k] \exp ([[\Omega[k]]_\times] \Delta t) 
\]

(33)

where \( k \in \mathbb{N} \) refers to the \( k \)th sample, \( \Delta t \) is a time step which is normally small, and \( R[k] \) and \( \Omega[k] \) refer to the true attitude and angular velocity at the \( k \)th sample, respectively.

Now we introduce important properties which will be helpful in the process of designing attitude filter and/or attitude control.

**Lemma 1.** Let \( R \in \text{SO}(3) \), \( u \in S^2 \) denote a unit vector such that \( \| u \| = 1 \), and \( \alpha \in \mathbb{R} \) denote the angle of rotation about \( u \). Then, the following holds:

\[
\operatorname{vex} (\mathcal{P}_a (R)) = 2 \cos \left( \frac{\alpha}{2} \right) \sin \left( \frac{\alpha}{2} \right) u 
\]

(34)

\[
\| R \|_I = \frac{1}{2} (1 - \cos (\alpha)) = \sin^2 \left( \frac{\alpha}{2} \right) 
\]

(35)

\[
\| \operatorname{vex} (\mathcal{P}_a (R)) \|^2 = 4 \cos^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\alpha}{2} \right) 
\]

(36)

**Proof.** See (92), (90), and (93).

**Lemma 2.** Let \( R \in \text{SO}(3) \) and \( \rho \in \mathbb{R}^3 \) be the Rodriguez parameters vector. Then, the following holds:

\[
\operatorname{vex} (\mathcal{P}_a (R)) = 2 \frac{\rho}{1 + \| \rho \|^2} 
\]

(37)

\[
\| R \|_I = \frac{\| \rho \|^2}{1 + \| \rho \|^2} 
\]

(38)

\[
\| \operatorname{vex} (\mathcal{P}_a (R)) \|^2 = 4 \frac{\| \rho \|^2}{\left( 1 + \| \rho \|^2 \right)^2} 
\]

(39)

**Proof.** See (111), (109), and (112).

The following Lemma (Lemma 3) is true if either Lemma 1 or Lemma 2 holds:

**Lemma 3.** Let \( R \in \text{SO}(3) \). Then, the following holds:

\[
\| \operatorname{vex} (\mathcal{P}_a (R)) \|^2 = 4 (1 - \| R \|_I) \| R \|_I 
\]

(40)
Proof. See (94), or (112).

The following Lemma (Lemma 4) is true if both Lemma 1 and Lemma 2 hold:

**Lemma 4.** Let \( \rho \in \mathbb{R}^3 \) be the Rodriguez parameters vector, \( u \in S^2 \) denote a unit vector such that \( \|u\| = 1 \), and \( \alpha \in \mathbb{R} \) denote the angle of rotation about \( u \). Then, the following holds:

\[
\begin{align*}
\rho &= \tan \left( \frac{\alpha}{2} \right) u \\
\alpha &= 2 \tan^{-1} (\|\rho\|) \\
u &= \cot \left( \frac{\alpha}{2} \right) \rho
\end{align*}
\]

(41) \hspace{1cm} (42) \hspace{1cm} (43)

**Proof.** See (118), (119), and (120).

**Lemma 5.** Let \( R \in SO(3) \) and \( Q = [q_0, q^\top]^\top \in S^3 \) be a unit-quaternion vector with \( q_0 \in \mathbb{R} \) and \( q \in \mathbb{R}^3 \). Then, the following holds:

\[
\text{vex} (\mathcal{P}_a (R)) = 2q_0q
\]

(44)

\[
\|R\| = 1 - q_0^2
\]

(45)

\[
\|\text{vex} (\mathcal{P}_a (R))\|^2 = 4q_0^2 \|q\|^2
\]

(46)

**Proof.** See (153), (155), and (156).

The following Lemma (Lemma 6) is true if both Lemma 1 and Lemma 5 hold:

**Lemma 6.** Let \( Q = [q_0, q^\top]^\top \in S^3 \) be a unit-quaternion vector with \( q_0 \in \mathbb{R} \) and \( q \in \mathbb{R}^3 \), and let \( u \in S^2 \) denote a unit vector such that \( \|u\| = 1 \), and \( \alpha \in \mathbb{R} \) denote the angle of rotation about \( u \). Then, the following holds:

\[
\begin{align*}
\alpha &= 2 \cos^{-1} (q_0) \\
u &= \frac{1}{\sin (\alpha/2)} q \\
q_0 &= \cos (\alpha/2) \\
q &= u \sin (\alpha/2)
\end{align*}
\]

(47) \hspace{1cm} (48) \hspace{1cm} (49) \hspace{1cm} (50)

**Proof.** See (162), (163), and (161).

The following Lemma (Lemma 7) is true if Lemma 2 and Lemma 5 hold:

**Lemma 7.** Let \( \rho \in \mathbb{R}^3 \) be the Rodriguez parameters vector and let \( Q = [q_0, q^\top]^\top \in S^3 \) be a unit-quaternion vector with \( q_0 \in \mathbb{R} \) and \( q \in \mathbb{R}^3 \). Then, the following holds:

\[
\begin{align*}
q_0 &= \pm \frac{1}{\sqrt{1 + \|\rho\|^2}} \\
q &= \pm \frac{\rho}{\sqrt{1 + \|\rho\|^2}} \\
\rho &= \frac{q}{q_0}
\end{align*}
\]

(51) \hspace{1cm} (52) \hspace{1cm} (53)

**Proof.** See Lemma 2 and Lemma 5. It should be remarked that Equation (53) is only valid for \( q_0 \neq 0 \).
Lemma 8. Let $R \in \text{SO}(3)$ and $\rho \in \mathbb{R}^3$ be the Rodriguez parameters vector. Define $M = M^\top \in \mathbb{R}^{3 \times 3}$, where $M$ is with rank 3, and $\text{Tr} \{M\} = 3$. Define $\bar{M} = \text{Tr} \{M\} I_3 - M$ and let the minimum singular value of $\bar{M}$ be $\lambda := \lambda(\bar{M})$. Then, the following holds:

\[
\|MR\|_I = \frac{1}{2} \frac{\rho^\top M \rho}{1 + \|\rho\|^2} \quad (54)
\]

\[
\text{vex}(\mathcal{P}_a(MR)) = \frac{(I_3 + [\rho]_x)^\top \bar{M}}{1 + \|\rho\|^2} \rho \quad (55)
\]

\[
\|\text{vex}(\mathcal{P}_a(MR))\|^2 = \frac{\rho^\top M (I_3 - [\rho]_x^2) M \rho}{(1 + \|\rho\|^2)^2} \quad (56)
\]

\[
\|MR\|_I \leq \frac{2}{\lambda} \frac{\|\text{vex}(\mathcal{P}_a(MR))\|^2}{1 + \text{Tr} \{M^{-1} MR\}} \quad (57)
\]

Proof. See (126), (127), (128), and (131).

Some of the above mentioned results and mapping to/from $\text{SO}(3)$ from/to other attitude representations could be found in [1], [2], [3], and [22, 23, 25].

3.4. Attitude error and attitude error dynamics

Consider the attitude dynamics in (27)

\[
\dot{R} = R [\Omega]_x
\]

Let us introduce desired/estimator attitude dynamics

\[
\dot{R}_\star = R_\star [\Omega_\star]_x \quad (58)
\]

Consider the error in attitude to be given by

\[
\tilde{R} = R^\top R_\star \quad (59)
\]

The attitude filter/control aims to drive $\tilde{R} \rightarrow I_3$. The dynamics of the attitude error can be found to be

\[
\dot{\tilde{R}} = \dot{R}^\top R_\star + R^\top \dot{R}_\star
\]

\[
= [\Omega]_x^\top \dot{R}_\star + R^\top \dot{R}_\star [\Omega_\star]_x
\]

\[
= -[\Omega]_x \tilde{R} + \tilde{R} [\Omega_\star]_x \quad (60)
\]

where \([\Omega]_x^\top = -[\Omega]_x\) as given in (3).

4. Euler Parameterization

The set of Euler angles is extensively used and widely known for attitude representation of the rigid-body. These angles are easy to visualize and understand allowing many researchers to use Euler angles for attitude parameterization such as [27, 28]. The naming convention is to label the angle based on the axis of rotation associated with. Roll angle represents rotation about the $x$-axis and is denoted by $\phi$, pitch angle refers to rotation about the $y$-axis and is denoted by $\theta$, and yaw angle is the rotation about the $z$-axis and is denoted by $\psi$. Figure 2 illustrates the orientation of the rigid-body relative to the three axes and Euler angles associated with each axis of rotation.
The names of these angles: roll, pitch and yaw are widely used in all aircraft applications. The rotational matrix could be described by a direction cosine matrix [29]. Let the world axis of the inertial-frame be \( p^I = [x^I, y^I, z^I] \) and the body axis of the rigid-body fixed frame be \( p^B = [x^B, y^B, z^B] \). Then, the direction cosine matrix of an angle \( \gamma \) is given by

\[
R = \begin{bmatrix}
\cos(\gamma_{x^B,x^I}) & \cos(\gamma_{x^B,y^I}) & \cos(\gamma_{x^B,z^I}) \\
\cos(\gamma_{y^B,x^I}) & \cos(\gamma_{y^B,y^I}) & \cos(\gamma_{y^B,z^I}) \\
\cos(\gamma_{z^B,x^I}) & \cos(\gamma_{z^B,y^I}) & \cos(\gamma_{z^B,z^I})
\end{bmatrix}
\]  (61)

For instance, consider the cosine matrix in (61), the rotation by an angle \( \phi \) about the \( x^I \)-axis is given by the following cosine matrix

\[
R = \begin{bmatrix}
\cos(0) & \cos(\pi/2) & \cos(\pi/2) \\
\cos(\pi/2) & \cos(\phi) & \cos(\pi/2 + \phi) \\
\cos(\pi/2) & \cos(\pi/2 - \phi) & \cos(\phi)
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos(\phi) & -\sin(\phi) \\
0 & \sin(\phi) & \cos(\phi)
\end{bmatrix}
\]

More details on the direction cosine matrix visit [29].

4.1. Attitude parameterization through Euler angles

Take, for example, the inertial-frame and the body-frame as defined in Figure 2. For simplicity, consider the following notation

\[
s = \sin, \quad c = \cos, \quad t = \tan
\]

With respect to the cosine matrix in (61), the rotation about \( x^I, y^I, \) and \( z^I \), respectively, is given by
In order to obtain Euler angles from a given rotational matrix, from (65) one has the inertial-frame and body-frame is equivalent to

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & c\phi & -s\phi \\
  0 & s\phi & c\phi
\end{bmatrix}
\begin{bmatrix}
  x^B \\
  y^B \\
  z^B
\end{bmatrix}
= R_{x,\phi}
\begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}
\Rightarrow R_{x,\phi} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & c\phi & -s\phi \\
  0 & s\phi & c\phi
\end{bmatrix}
\tag{62}
\]

- **rolling** about \( x^I \) by an angle \( \phi \)

\[
\begin{bmatrix}
  x^I \\
  y^I \\
  z^I
\end{bmatrix}
= \begin{bmatrix}
  c\theta & 0 & s\theta \\
  0 & 1 & 0 \\
  -s\theta & 0 & c\theta
\end{bmatrix}
\begin{bmatrix}
  x^B \\
  y^B \\
  z^B
\end{bmatrix}
\Rightarrow R_{y,\theta} =
\begin{bmatrix}
  c\theta & 0 & s\theta \\
  0 & 1 & 0 \\
  -s\theta & 0 & c\theta
\end{bmatrix}
\tag{63}
\]

- **pitching** about \( y^I \) by an angle \( \theta \)

\[
\begin{bmatrix}
  x^I \\
  y^I \\
  z^I
\end{bmatrix}
= \begin{bmatrix}
  c\psi & s\psi & 0 \\
  -s\psi & c\psi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x^B \\
  y^B \\
  z^B
\end{bmatrix}
\Rightarrow R_{z,\psi} =
\begin{bmatrix}
  c\psi & -s\psi & 0 \\
  s\psi & c\psi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\tag{64}
\]

- **yawing** about \( z^I \) by an angle \( \psi \)

The transformation matrix is obtained by

1. Transformation about the fixed \((x, \phi)\).
2. Next, transformation about the fixed \((y, \theta)\).
3. Next, transformation about the fixed \((z, \psi)\).

to yield

\[
R_S = R_{z,\psi}R_{y,\theta}R_{x,\phi} = R_{(\phi,\theta,\psi)}
\]

\[
= \begin{bmatrix}
  c\phi & -s\psi & 0 \\
  s\psi & c\phi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  c\theta & 0 & s\theta \\
  0 & 1 & 0 \\
  -s\theta & 0 & c\theta
\end{bmatrix}
\begin{bmatrix}
  1 \\
  0 \\
  0
\end{bmatrix}
\Rightarrow R_S =
\begin{bmatrix}
  c\phi & -s\psi & 0 \\
  s\psi & c\phi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

which is equivalent to

\[
R_S = \begin{bmatrix}
  c\phi c\psi & -c\phi s\psi + s\psi c\phi c\psi & s\psi + c\phi s\psi c\phi c\psi \\
  c\phi s\psi & c\phi c\psi + s\phi s\psi c\psi & -s\psi c\phi + c\phi s\psi c\phi c\psi \\
  -s\theta & s\phi & c\phi
\end{bmatrix} \in SO(3)
\tag{65}
\]

with \( R_S \in SO(3) \) being the attitude representation using Euler angles \((\phi, \theta, \psi)\). Thus, the relation between the inertial-frame and body-frame is equivalent to

\[
\begin{bmatrix}
  x^I \\
  y^I \\
  z^I
\end{bmatrix}
= \begin{bmatrix}
  c\phi c\psi & -c\phi s\psi + s\psi c\phi c\psi & s\psi + c\phi s\psi c\phi c\psi \\
  c\phi s\psi & c\phi c\psi + s\phi s\psi c\psi & -s\psi c\phi + c\phi s\psi c\phi c\psi \\
  -s\theta & s\phi & c\phi
\end{bmatrix}
\begin{bmatrix}
  x^B \\
  y^B \\
  z^B
\end{bmatrix}
\]

\[
= R_Sv^B
\]

\[v^I = R_Sv^B\]

4.2. **Euler angles from SO(3)**

In order to obtain Euler angles from a given rotational matrix, from (65) one has

\[
\frac{R_{(3,2)}}{R_{(3,3)}} = \frac{\sin(\phi) \cos(\theta)}{\cos(\phi) \cos(\theta)}
\]

\[\tag{66}\]
Hence, the roll angle can be found to be
\[
\phi = \arctan \left( \frac{R_{(3,2)}}{R_{(3,3)}} \right) \tag{67}
\]

Next, from (65) one has
\[
\sin(\theta) = -R_{(3,1)} \tag{68}
\]
\[
\cos(\theta) = \sqrt{R_{(3,2)}^2 + R_{(3,3)}^2} \tag{69}
\]

Thus, the pitch angle is equivalent to
\[
\theta = \arctan \left( \frac{-R_{(3,1)}}{\sqrt{R_{(3,2)}^2 + R_{(3,3)}^2}} \right) \tag{70}
\]

Finally, from (65) one has
\[
\frac{R_{(2,1)}}{R_{(1,1)}} = \cos(\theta) \sin(\psi) \tag{71}
\]
\[
\cos(\theta) \cos(\psi)
\]

Thereby, the yaw angle could be found to be
\[
\psi = \arctan \left( \frac{R_{(2,1)}}{R_{(1,1)}} \right) \tag{72}
\]

Summarizing the above-mentioned results, Euler angles can be obtained from a given rotational matrix \( R \) (the mapping from \( \text{SO}(3) \) to \( \xi \)) in the following manner:
\[
\begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix} =
\begin{bmatrix}
\arctan \left( \frac{R_{(3,2)}}{R_{(3,3)}} \right) \\
\arctan \left( \frac{-R_{(3,1)}}{\sqrt{R_{(3,2)}^2 + R_{(3,3)}^2}} \right) \\
\arctan \left( \frac{R_{(2,1)}}{R_{(1,1)}} \right)
\end{bmatrix} \tag{73}
\]

4.3. Angular velocities transformation

Let us define the Euler angle vector by \( \mathbf{\xi} = [\phi, \theta, \psi]^T \) and the body-fixed angular velocity vector by \( \mathbf{\Omega} = [\Omega_x, \Omega_y, \Omega_z]^T \). The Euler rate vector \( \dot{\mathbf{\xi}} = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T \) is related to the body-fixed angular velocity vector (\( \mathbf{\Omega} \)) through a transformation matrix \( \mathcal{J} \) such that:
\[
\dot{\mathbf{\xi}} = \mathcal{J} \mathbf{\Omega} \tag{74}
\]
The body-fixed angular velocity vector \( \Omega \) is related to the Euler rate vector \( \dot{\xi} \) by

\[
\Omega = J^{-1} \dot{\xi}
\]

\[
= \begin{bmatrix}
\dot{\phi} \\
0 \\
0
\end{bmatrix} + R_{x,\phi} \begin{bmatrix}
0 \\
\dot{\theta} \\
0
\end{bmatrix} + R_{x,\phi} R_{y,\theta} \begin{bmatrix}
0 \\
0 \\
\psi
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\dot{\phi} \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
c\phi & s\phi & 0 \\
0 & c\phi & s\phi \\
0 & -s\phi & c\phi
\end{bmatrix} \begin{bmatrix}
0 \\
\dot{\theta} \\
0
\end{bmatrix} + \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix} \begin{bmatrix}
c\phi & s\phi & 0 \\
0 & -s\phi & c\phi \\
0 & s\theta & 0 & c\theta
\end{bmatrix} \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\]

which is equivalent to

\[
\Omega = J^{-1} \dot{\xi} = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
s\theta \\
c\phi & s\phi & 0 \\
0 & -s\phi & c\phi
\end{bmatrix} \begin{bmatrix}
\dot{\phi} \\
\theta \\
\psi
\end{bmatrix}
\]

(75)

From (75), and for a given initial Euler angles vector \( \xi(0) \), the Euler rate can be found to be [29]

\[
\dot{\xi} = J \Omega = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \begin{bmatrix}
s\phi \theta & c\phi \theta & 0 \\
0 & c\phi & s\phi \theta & 0 \\
0 & -s\phi & c\phi \theta & 0
\end{bmatrix} \begin{bmatrix}
\Omega_x \\
\Omega_y \\
\Omega_z
\end{bmatrix}
\]

(76)

which means that the transformation from the body-fixed angular velocity vector \( \Omega \) to Euler rate vector \( \dot{\xi} \) is given by

\[
J = \begin{bmatrix}
1 & s\phi \theta & c\phi \theta \\
0 & c\phi & s\phi \theta & 0 \\
0 & -s\phi & c\phi \theta & 0
\end{bmatrix}
\]

(77)

4.4. Detailed derivation of Euler rate

Let us detail the transformation matrix \( J \). Recall the attitude dynamics in (27)

\[
\dot{R} = R [\Omega]_x
\]

Also, recall the transformation matrix in (65)

\[
R = \begin{bmatrix}
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} \\
r_{31} & r_{32} & r_{33}
\end{bmatrix} = \begin{bmatrix}
c\theta c\psi & -c\phi s\psi + s\phi c\theta c\psi & s\phi c\psi + c\phi s\psi c\theta \\
c\theta s\psi & c\phi c\psi + s\phi s\psi s\theta & -s\phi s\psi + c\phi s\psi s\theta c\psi \\
-s\theta & s\phi c\psi & c\phi c\theta
\end{bmatrix}
\]

The derivative of the above equations is as follows:

\[
\dot{R} = \begin{bmatrix}
\dot{r}_{11} & \dot{r}_{12} & \dot{r}_{13} \\
\dot{r}_{21} & \dot{r}_{22} & \dot{r}_{23} \\
\dot{r}_{31} & \dot{r}_{32} & \dot{r}_{33}
\end{bmatrix}
\]

(78)
such that

\[
\begin{align*}
  r_{11} &= -\dot{\theta}s\phi c\psi - \dot{\psi}c\theta s\psi \\
  r_{12} &= \dot{\phi} (s\phi s\psi + c\phi c\theta c\psi) + \dot{\theta}s\phi c\theta c\psi - \psi (c\phi c\psi + s\phi s\theta s\psi) \\
  r_{13} &= \dot{\phi} (c\phi s\psi - s\phi s\theta c\psi) + \dot{\theta}c\phi c\theta c\psi + \psi (s\phi c\psi - c\phi s\theta s\psi) \\
  r_{21} &= -\dot{\theta}s\theta s\psi + \psi c\theta c\psi \\
  r_{22} &= \dot{\phi} (c\phi s\theta s\psi - s\phi c\psi) + \dot{\theta}s\phi c\theta s\psi + \psi (s\phi s\theta c\psi - c\phi c\psi) \\
  r_{23} &= -\dot{\phi} (c\phi s\psi + s\phi s\theta s\psi) + \dot{\theta}c\phi c\theta s\psi + \psi (s\phi s\psi + c\phi c\theta c\psi) \\
  r_{31} &= -\dot{\psi}c\theta \\
  r_{32} &= \dot{\phi}c\phi c\theta - \dot{\theta}s\phi s\theta \\
  r_{33} &= -\dot{\phi}s\phi c\theta - \dot{\theta}c\phi s\theta \\
\end{align*}
\]

From (65) and (78) one can find

\[
R^\top \dot{R} = \begin{bmatrix}
  0 & -\psi c\phi c\theta + \theta s\phi & \dot{\theta}c\phi + \psi s\phi c\theta \\
  -\psi c\phi c\theta - \theta s\phi & 0 & -\dot{\phi} + \psi s\theta \\
  -\dot{\theta}c\phi - \psi s\phi c\theta & \phi - \psi s\theta & 0
\end{bmatrix} (79)
\]

From (27) and (79) one has

\[
R^\top \dot{R} = [\Omega]_x
\]

Thus, the body-fixed angular velocity vector can be expressed as

\[
\mathbf{vex} ([\Omega]_x) = \mathbf{vex} (R^\top \dot{R})
\]

\[
\begin{bmatrix}
  \Omega_x \\
  \Omega_y \\
  \Omega_z
\end{bmatrix} = \begin{bmatrix}
  \phi - \psi s\theta \\
  \dot{\theta}c\phi + \psi s\phi c\theta \\
  -\psi c\phi c\theta + \theta s\phi
\end{bmatrix}
\]

\[
\Omega = J^{-1} \hat{\xi} = \begin{bmatrix}
  1 & 0 & -s\phi \\
  0 & c\phi & s\phi c\theta \\
  0 & -s\phi & c\phi c\theta
\end{bmatrix} \begin{bmatrix}
  \phi \\
  \theta \\
  \psi
\end{bmatrix} (80)
\]

Therefore, the Euler rate becomes

\[
\hat{\xi} = J \Omega = \begin{bmatrix}
  1 & s\phi t\theta & c\phi t\theta \\
  0 & c\phi & -s\phi \\
  0 & s\phi s\sec\theta & c\phi s\sec\theta
\end{bmatrix} \begin{bmatrix}
  \Omega_x \\
  \Omega_y \\
  \Omega_z
\end{bmatrix} (81)
\]

and the transformation matrix is equivalent to

\[
J = \begin{bmatrix}
  1 & s\phi t\theta & c\phi t\theta \\
  0 & c\phi & -s\phi \\
  0 & s\phi s\sec\theta & c\phi s\sec\theta
\end{bmatrix} (82)
\]
4.5. Euler parameterization problem

Euler angles are kinematically singular. In other words, the transformation of the Euler angles rates \(\dot{\xi}\) to the angular velocity \(\Omega\) is locally defined, while there is no global definition for the transformation matrix \(J\). At certain configurations, the mapping from \(SO(3)\) to Euler angles \(\xi\) could result in infinite solutions. For example, consider the following orthogonal matrix

\[
R = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix} \in SO(3)
\]

\(\det (R) = +1\)  \[R^\top R = RR^\top = I_3\] (83)

It is impossible to obtain the true Euler angles of the orthogonal matrix in (83), and there exist infinite solutions, since

\[
R_{(\phi, \theta, \psi)} = R_{(90, 90, 90)} = R_{(0, 90, 0)} = R_{(45, 90, 45)} = R_{(\star, 90, \star)} = \cdots
\]

where \(\star\) denotes any angle. Hence, for \(R = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{bmatrix}\), there exists infinite number of solutions for the Euler angles, and as a result the true Euler angles cannot be obtained. Thus, despite Euler angles being capable of describing every attitude, the representation produced by the set of Euler angles is not unique. Consequently, these representation is limited to local attitude maneuvers, due to the fact that continuous control maneuvers are not guaranteed for certain configurations. In addition, there are angular velocities \(\Omega\) that cannot be obtained by the means of the time derivatives of Euler angles \(\dot{\xi}\).

5. Angle-axis Parameterization

5.1. Mapping: From/To unit vector and angle of rotation to/from other representations

The relative orientation of any two frames can always be expressed in terms of a single rotation about a given normalized vector with a given rotation angle. Let \(u = [u_1, u_2, u_3]^\top \in S^2\) denote a unit vector such that \(\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2} = 1\), and \(\alpha \in \mathbb{R}\) denote the angle of rotation about \(u\). Then for a given rotation angle \(\alpha\) and unit axis \(u\), the corresponding rotational matrix \(R_\alpha: \mathbb{R} \times S^2 \rightarrow SO(3)\) (the mapping from \((\alpha, u)\) to \(SO(3)\)) can be expressed by the following formula:

\[
R_\alpha (\alpha, u) = \exp (\alpha [u]_\times) \in SO(3)
\]

(84)

Figure 3 visualizes the angle-axis parameterization such that any point on a unit sphere can be described by a rotation angle \(\alpha\) and the unit axis \(u\).

This isomorphism lies at the core of the Euler’s theorem, which states that the axis of rotation \(u \in S^2\) with rotational angle \(\alpha \in \mathbb{R}\) yields the rotational matrix \(R_\alpha (\alpha, u) = \exp (\alpha [u]_\times) \in SO(3)\). The isomorphism in (84) can be expressed with respect to Rodriguez formula by [30]

\[
R_\alpha (\alpha, u) = I_3 + \sin (\alpha) [u]_\times + (1 - \cos (\alpha)) [u]_\times^2 \in SO(3)
\]

(85)
Figure 3: The orientation of a rigid-body in body-frame relative to inertial-frame.

Hence

$$ R_\alpha (\alpha, u) = I_3 + \sin (\alpha) \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} + (1 - \cos (\alpha)) \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} $$

Consider the following properties

$$ \sin^2 (\alpha) = \frac{1}{2} - \frac{1}{2} \cos (2\alpha) $$

$$ 2\sin^2 (\alpha/2) = 1 - \cos (\alpha) $$

from (86) one has

$$ R_\alpha (\alpha, u) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin (\alpha) \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} + 2\sin^2 (\alpha/2) \begin{bmatrix} -u_1^2 + u_3^2 \\ u_1u_2 \\ u_1u_3 \end{bmatrix} $$

$$ + (1 - \cos (\alpha)) \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} -u_1^2 + u_3^2 \\ u_1u_2 \\ u_1u_3 \end{bmatrix} $$
which is equivalent to
\[
R_\alpha (\alpha, u) = \begin{bmatrix}
1 - 2 (u_2^2 + u_3^2) \sin^2 (\alpha/2) & - \sin (\alpha) u_3 + 2u_2 u_3 \sin^2 (\alpha/2) & \sin (\alpha) u_2 + 2u_1 u_3 \sin^2 (\alpha/2) \\
\sin (\alpha) u_3 + 2u_1 u_2 \sin^2 (\alpha/2) & 1 - 2 (u_1^2 + u_3^2) \sin^2 (\alpha/2) & - \sin (\alpha) u_1 + 2u_2 u_3 \sin^2 (\alpha/2) \\
- \sin (\alpha) u_2 + 2u_1 u_3 \sin^2 (\alpha/2) & \sin (\alpha) u_1 + 2u_2 u_3 \sin^2 (\alpha/2) & 1 - 2 (u_1^2 + u_2^2) \sin^2 (\alpha/2)
\end{bmatrix}
\] (87)

The angle of rotation \( \alpha \) and the unit axis \( u \) can be easily obtained knowing the orientation matrix \( R_\alpha (\alpha, u) \in SO (3) \) in (87). From (86), the angle \( \alpha \) \( \text{(the mapping from } SO (3) \text{ to } (\alpha, u) \text{)} \) is given by
\[
\alpha = \cos^{-1} \left( \frac{1}{2} (\text{Tr } R) - 1 \right)
\] (88)

Knowing \( \alpha \), the unit vector \( u \) can be obtained by
\[
u = \frac{1}{\sin (\alpha)} \text{vex} \left( \frac{1}{2} (R - R^\top) \right)
\]
\[
= \frac{1}{\sin (\alpha)} \text{vex} (\mathcal{P}_\alpha (R))
\] (89)

From (86), the normalized Euclidean distance can be defined in terms of angle-axis components by
\[
\|R\|_I = \frac{1}{2} (1 - \cos (\alpha))
\]
\[
= \sin^2 \left( \frac{\alpha}{2} \right)
\] (90)

This shows (35) in Lemma 1. Hence, the following holds
\[
\cos^2 \left( \frac{\alpha}{2} \right) = 1 - \sin^2 \left( \frac{\alpha}{2} \right)
\]
\[
= 1 - \|R\|_I
\] (91)

From (91) the anti-symmetric operator of the rotational matrix can be expressed with regards to angle-axis components by
\[
\mathcal{P}_\alpha (R) = \sin (\alpha) [u]_\times
\]
\[
= 2 \cos \left( \frac{\alpha}{2} \right) \sin \left( \frac{\alpha}{2} \right) [u]_\times
\]

As such, the vex operator can be found to be
\[
\text{vex} (\mathcal{P}_\alpha (R)) = 2 \cos \left( \frac{\alpha}{2} \right) \sin \left( \frac{\alpha}{2} \right) u
\] (92)

This proves (34) in Lemma 1. The norm square of the result in (92) is
\[
\|\text{vex} (\mathcal{P}_\alpha (R))\|^2 = \text{vex} (\mathcal{P}_\alpha (R))^\top \text{vex} (\mathcal{P}_\alpha (R))
\]
\[
= u^\top u \sin^2 (\alpha), \quad \|u\|^2 = 1
\]
\[
= 4 \cos^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\alpha}{2} \right)
\] (93)
Considering the results of (90) and (91), one can find
\[
\|\text{vex} (\mathcal{P}_d (R))\|^2 = 4\cos^2 \left( \frac{\alpha}{2} \right) \sin^2 \left( \frac{\alpha}{2} \right)
\]
\[
= 4 \left( 1 - \sin^2 \left( \frac{\alpha}{2} \right) \right) \sin^2 \left( \frac{\alpha}{2} \right)
\]
\[
= 4 \left( 1 - \|R\|_{I} \right) \|R\|_{I}
\]
(94)

This confirms (40) in Lemma 3.

5.2. Problems of angle-axis parameterization:

The problem of expressing the attitude using angle-axis components is that some rotations are not defined in terms of angle-axis parameterization. For instance, consider the following results
\[
\alpha = \cos^{-1} \left( \frac{1}{2} \left( \text{Tr} \{R\} - 1 \right) \right)
\]
\[
u = \frac{1}{2\sin(\alpha)} \left[ \begin{array}{ccc}
R_{32} - R_{23} \\
R_{13} - R_{31} \\
R_{21} - R_{12}
\end{array} \right]
\]

Thus, the following four orientations cannot use angle-axis components in their definition
\[
\text{at } R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \in \text{SO}(3) \Rightarrow \alpha = 0 \in \mathbb{R}, \quad \nu = [\infty, \infty, \infty]^T \notin S^2
\]
(95)

\[
\text{at } R = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} \in \text{SO}(3) \Rightarrow \alpha = \pi \in \mathbb{R}, \quad \nu = [\infty, \infty, \infty]^T \notin S^2
\]
(96)

\[
\text{at } R = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix} \in \text{SO}(3) \Rightarrow \alpha = \pi \in \mathbb{R}, \quad \nu = [\infty, \infty, \infty]^T \notin S^2
\]
(97)

\[
\text{at } R = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix} \in \text{SO}(3) \Rightarrow \alpha = \pi \in \mathbb{R}, \quad \nu = [\infty, \infty, \infty]^T \notin S^2
\]
(98)

Therefore, \(\alpha\) should not be a multiple of \(\pi\), while the representation of angle-axis parameterization is valid for all \(\alpha \neq k\pi \forall k = 0, 1, 2, 3, \ldots\).

6. Rodriguez Vector Parameterization

6.1. Mapping: From/To Rodriguez vector to/from other representations

The Rodriguez vector \(\rho = [\rho_1, \rho_2, \rho_3]^T \in \mathbb{R}^3\), proposed by Olinde Rodriguez [31], can be employed for attitude representation. It should be noted, that naming conventions vary and in some articles Rodriguez vector is termed Gibbs vector [32, 33]. The orthogonal matrix \(R_{\rho}\) is obtained through mapping the vector on \(\mathbb{R}^3\) to \(\text{SO}(3)\) such that \(R_{\rho} : \mathbb{R}^3 \to \text{SO}(3)\) [34]. The Cayley transform is given by [34]
\[
R_{\rho} = \left( I_3 + [\rho]_x \right) \left( I_3 - [\rho]_x \right)^{-1}, \quad R_{\rho} \in \text{SO}(3)
\]
(99)

It is important to present the following properties [35]:

\[
\]
1. The matrix multiplication in (99) is commutative, so $R_\rho$ can be alternatively defined as

$$R_\rho = (I_3 + [\rho]_\times) (I_3 - [\rho]_\times)^{-1} \in SO(3)$$

$$= (I_3 - [\rho]_\times)^{-1} (I_3 + [\rho]_\times) \in SO(3)$$

and

$$R_\rho^\top = (I_3 - [\rho]_\times) (I_3 + [\rho]_\times)^{-1} \in SO(3)$$

$$= (I_3 + [\rho]_\times)^{-1} (I_3 - [\rho]_\times) \in SO(3)$$

2. The determinant of $R_\rho$ must always be +1.

3. For $\det (R_\rho) = -1$, Rodriguez vector cannot be defined.

4. $R_\rho^\top R_\rho = R_\rho R_\rho^\top = R_\rho^{-1} R_\rho = R_\rho R_\rho^{-1} = I_3$.

The related map from Rodriguez parameters vector form to $SO(3)$ ($R_\rho : \mathbb{R}^3 \rightarrow SO(3)$) is [33]

$$R_\rho = \frac{1}{1 + \|\rho\|^2} \left( \left(1 - \|\rho\|^2\right) I_3 + 2 \rho \rho^\top + 2 [\rho]_\times \right)$$  \hspace{1cm} (100)

To proof the result in (100), we solve for (99). The following derivation shows mapping $R_\rho : \mathbb{R}^3 \rightarrow SO(3)$

$$(I_3 - [\rho]_\times)^{-1} = \frac{1}{1 - \rho_3^2} \begin{bmatrix} 1 & \rho_3 & -\rho_2 \\ -\rho_3 & 1 & \rho_1 \\ -\rho_2 & -\rho_1 & 1 \end{bmatrix}$$

$$= \frac{1}{1 + \rho_1^2 + \rho_2^2 + \rho_3^2} \begin{bmatrix} 1 + \rho_1^2 & \rho_1 \rho_2 & \rho_1 \rho_3 - \rho_2^2 \\ \rho_2 \rho_1 & 1 + \rho_2^2 & \rho_2 \rho_3 + \rho_1^2 \\ \rho_3 \rho_1 & \rho_3 \rho_2 & 1 + \rho_3^2 \end{bmatrix}$$

$$= \frac{1}{1 + \|\rho\|^2} \left( I_3 + \rho \rho^\top + [\rho]_\times \right)$$

Considering the identities in (5) and (8), we have $[\rho]_\times \rho = 0$ and $[\rho]_\times^2 = \rho \rho^\top - \|\rho\|^2 I_3$ such that

$$R_\rho = (I_3 + [\rho]_\times) (I_3 - [\rho]_\times)^{-1}$$

$$= \frac{1}{1 + \|\rho\|^2} \left( I_3 + [\rho]_\times \right) \left( I_3 + \rho \rho^\top + [\rho]_\times \right)$$

$$= \frac{1}{1 + \|\rho\|^2} \left( I_3 + 2 [\rho]_\times + [\rho]_\times \rho \rho^\top + \rho \rho^\top + [\rho]_\times \right)$$

$$= \frac{1}{1 + \|\rho\|^2} \left( I_3 + 2 [\rho]_\times + 2 \rho \rho^\top \right)$$

The end result is $R_\rho : \mathbb{R}^3 \rightarrow SO(3)$ (the mapping from $\rho$ to $SO(3)$)

$$R_\rho = \frac{1}{1 + \|\rho\|^2} \left( \left(1 - \|\rho\|^2\right) I_3 + 2 [\rho]_\times + 2 \rho \rho^\top \right)$$  \hspace{1cm} (101)
which can be written in detailed form as follows

\[
R_\rho = \frac{1}{1 + \|\rho\|^2} \begin{bmatrix}
1 + \rho_1^2 - \rho_2^2 - \rho_3^2 & 2(\rho_1\rho_2 - \rho_3) & 2(\rho_1\rho_3 + \rho_2) \\
2(\rho_1\rho_2 + \rho_3) & 1 + \rho_3^2 - \rho_1^2 & 2(\rho_2\rho_3 - \rho_1) \\
2(\rho_1\rho_3 - \rho_2) & 2(\rho_2\rho_3 + \rho_1) & 1 + \rho_2^2 - \rho_1^2 - \rho_3^2
\end{bmatrix}
\]  

(102)

Now, let us present the proof of orthogonality

\[
R_\rho^T R_\rho = \left( (I_3 - [\rho]_\times) (I_3 + [\rho]_\times) \right)^T \left( (I_3 + [\rho]_\times) (I_3 - [\rho]_\times) \right)^{-1}
\]

\[
= (I_3 - [\rho]_\times) (I_3 + [\rho]_\times)^{-1} (I_3 + [\rho]_\times) (I_3 - [\rho]_\times)^{-1}
\]

\[
= (I_3 - [\rho]_\times) (I_3 - [\rho]_\times)^{-1}
\]

\[
= I_3
\]

(103)

Hence

\[
R_\rho^T R_\rho = I_3
\]

(104)

Let us prove the inverse mapping: \( \rho : SO(3) \rightarrow \mathbb{R}^3 \)

\[
R_\rho = \left( I_3 + [\rho]_\times \right) (I_3 - [\rho]_\times)^{-1}
\]

\[
R_\rho (I_3 - [\rho]_\times) = I_3 + [\rho]_\times
\]

\[
R_\rho - I_3 = R_\rho [\rho]_\times + [\rho]_\times
\]

\[
R_\rho - I_3 = (R_\rho + I_3) [\rho]_\times
\]

\[
[\rho]_\times = (R_\rho + I_3)^{-1} (R_\rho - I_3)
\]

Hence, one can find \( \rho : SO(3) \rightarrow \mathbb{R}^3 \) (the mapping from \( SO(3) \) to \( \rho \)) as [31]

\[
\rho = \text{vex} \left( (R_\rho + I_3)^{-1} (R_\rho - I_3) \right)
\]

(105)

which is also equivalent to

\[
\rho = \frac{1}{1 + \text{Tr} \{R\}} \begin{bmatrix}
R_{32} - R_{23} \\
R_{13} - R_{31} \\
R_{21} - R_{12}
\end{bmatrix}
\]

(106)

From, (65), (73), and (100), the related map from Rodriguez vector to Euler angles \( \xi : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) (the mapping from \( \rho \) to \( \xi \)) can be defined by

\[
\begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix} = \begin{bmatrix}
\arctan \left( \frac{2\rho_3\rho_2 + 2\rho_1}{1 + \rho_2^2 - \rho_1^2 - \rho_3^2} \right) \\
\arctan \left( \frac{2\rho_2 - 2\rho_1\rho_3}{\sqrt{4(\rho_2^2\rho_3 + \rho_1)^2 + (1 + \rho_2^2 - \rho_1^2 - \rho_3^2)^2}} \right) \\
\arctan \left( \frac{2\rho_1\rho_2 + 2\rho_3}{1 + \rho_2^2 - \rho_1^2 - \rho_3^2} \right)
\end{bmatrix}
\]

(107)

Let us define the relationship between \( \mathcal{P}_a (R) \) and the normalized Euclidean distance \( \|R\|_1 \). With direct
substitution of (100) in (31) one obtains
\[
\|R\|_I = \frac{1}{4} \text{Tr} \left\{ I_3 - \frac{1}{1 + \|\rho\|^2} \left( \left( 1 - \|\rho\|^2 \right) I_3 + 2 [\rho]_\times + 2 \rho \rho^\top \right) \right\}
\]
\[
= \frac{1}{4} \frac{1}{1 + \|\rho\|^2} \text{Tr} \left\{ 2 \|\rho\|^2 I_3 - 2 \rho \rho^\top - 2 [\rho]_\times \right\}
\]
\[
= \frac{1}{4} \frac{1}{1 + \|\rho\|^2} \left( 6 \|\rho\|^2 - 2 \|\rho\|^2 \right)
\] (108)

Where \(\text{Tr} \{ [\rho]_\times \} = 0\) and \(\text{Tr} \{ \rho \rho^\top \} = \|\rho\|^2\). Thus, the normalized Euclidean distance \(\|R\|_I\) in terms of Rodriguez vector is equivalent to
\[
\|R\|_I = \frac{\|\rho\|^2}{1 + \|\rho\|^2}
\] (109)

This proves (38) in Lemma 2. Likewise, the anti-symmetric projection operator of attitude \(R\) in (100) can be defined as
\[
\mathcal{P}_a (R) = \frac{1}{2} \frac{1}{1 + \|\rho\|^2} \left( \mathcal{R}_\rho - \mathcal{R}_\rho \right)
\]
\[
= \frac{1}{2} \frac{1}{1 + \|\rho\|^2} \left( 2 [\rho]_\times + 2 [\rho]_\times \right)
\]
which is equivalent to
\[
\mathcal{P}_a (R) = 2 \frac{1}{1 + \|\rho\|^2} [\rho]_\times
\] (110)

Thereby, the vex operator of (110) becomes
\[
\text{vex} (\mathcal{P}_a (R)) = 2 \frac{\rho}{1 + \|\rho\|^2}
\] (111)

This proves (37) in Lemma 2. The square norm of (111) is
\[
\|\text{vex} (\mathcal{P}_a (R))\|^2 = 4 \frac{\|\rho\|^2}{\left( 1 + \|\rho\|^2 \right)^2}
\] (112)

This shows (39) in Lemma 2. Also, from (109), and (111), one can find that
\[
\rho = \frac{\text{vex} (\mathcal{P}_a (R))}{2 \left( 1 + \|\rho\|^2 \right)}
\] (113)

and from (109), \(\|\rho\|^2\) can be defined in terms of normalized Euclidean distance \(\|R\|_I\) by
\[
\|\rho\|^2 = \frac{\|R\|_I}{1 - \|R\|_I}
\] (114)
Thereby, from (113) and (114), one has \( \rho : \text{SO}(3) \to \mathbb{R}^3 \) (the mapping from \( \text{SO}(3) \) to \( \rho \))

\[
\rho = \frac{\text{vex}(\mathcal{P}_a(R))}{2(1 - \| R \|)}
\]  

(115)

And from (109) and (114), we have

\[
\| \text{vex}(\mathcal{P}_a(R)) \|^2 = 4 \times \frac{\| \rho \|^2}{1 + \| \rho \|^2} \times \frac{1}{1 + \| \rho \|^2} = 4 \| R \| I_1 (1 - \| R \|)
\]  

(116)

This confirms (40) in Lemma 3 and (39) in Lemma 2. According to the results in (113), (91), and (92), one can find

\[
\rho = \frac{\text{vex}(\mathcal{P}_a(R))}{2(1 - \| R \|)} = 2 \cos \left( \frac{\alpha}{2} \right) \sin \left( \frac{\alpha}{2} \right) u = \tan \left( \frac{\alpha}{2} \right) u
\]  

(117)

As such, the related map from angle-axis parameterization to Rodriguez vector \( \rho : \mathbb{R} \times S^2 \to \mathbb{R}^3 \) (the mapping from \( (\alpha, u) \) to \( \rho \)) can be defined by [31, 33, 32]

\[
\rho = \tan \left( \frac{\alpha}{2} \right) u
\]  

(118)

which proves (41). In order to find the inverse mapping of (118), recall \( \| R \|_I = \sin^2 \left( \frac{\alpha}{2} \right) \) from (90) and \( \| \rho \|^2 = \frac{\| R \|^2}{1 - \| R \|^2} \) from (114). Hence, the rotation angle \( \alpha \) can be obtained by

\[
\| \rho \|^2 = \frac{\sin^2 \left( \frac{\alpha}{2} \right)}{\cos^2 \left( \frac{\alpha}{2} \right)} = \frac{\sin^2 \left( \frac{\alpha}{2} \right)}{\cos^2 \left( \frac{\alpha}{2} \right)} = \tan^2 \left( \frac{\alpha}{2} \right)
\]

Therefore, \( \alpha \) becomes

\[
\alpha = 2 \tan^{-1}(\| \rho \|) = 2 \sin^{-1}\left(\frac{\| \rho \|}{\sqrt{1 + \| \rho \|^2}}\right)
\]

and from (118) the unit axis \( u \in S^2 \) is

\[
u = \cot \left( \frac{\alpha}{2} \right) \rho
\]

Thus, the related map from Rodriguez vector to angle-axis parameterization \( \rho : \mathbb{R}^3 \to \mathbb{R} \times S^2 \) (the mapping from \( \rho \) to \( (\alpha, u) \)) can be expressed as [31, 33, 32]

\[
\alpha = 2 \tan^{-1}(\| \rho \|) = 2 \sin^{-1}\left(\frac{\| \rho \|}{\sqrt{1 + \| \rho \|^2}}\right)
\]  

(119)

\[
u = \cot \left( \frac{\alpha}{2} \right) \rho
\]  

(120)

the above mentioned results show (42) and (43).
6.2. Attitude measurements

Considering the body-frame vector \( v_i^B \in \mathbb{R}^3 \) and the inertial-frame vector \( v_i^I \in \mathbb{R}^3 \) of the \( i \)th measurement, the relationship between these two frames of the \( i \)th measurement is given by

\[
v_i^B = R^\top v_i^I (121)
\]

where \( R \in \text{SO}(3) \) is the attitude matrix. Let \( n \) denote the number of body-frame and inertial-frame vectors available for measurement for \( i = 1, \ldots, n \). In general, three body-frame and inertial-frame non-collinear vector measurements are required for attitude estimation or reconstruction \( (n \geq 3) \). However, if two non-collinear vectors are available for measurements \( (n = 2) \), the third vector can be defined by \( v_3^B = v_1^B \times v_2^B \) and \( v_3^I = v_1^I \times v_2^I \), such that the three vectors are non-collinear. Define

\[
M^B = \sum_{i=1}^{n} \left( \frac{v_i^B (v_i^B)^\top}{\|v_i^B\|^2} \right) \in \mathbb{R}^{3 \times 3} (122)
\]

\[
M^I = \sum_{i=1}^{n} \left( \frac{v_i^I (v_i^I)^\top}{\|v_i^I\|^2} \right) \in \mathbb{R}^{3 \times 3} (123)
\]

with \( n \geq 2 \), one of the above-mentioned equations in (122) or (123) is normally employed for attitude estimation. Let \( M := M^* \), such that \( * \) refers to \( I \) or \( B \) and \( \bar{M} = \text{Tr} \{ M \} I_3 - M \). Consider three measurements \( (n = 3) \) such that \( \text{Tr} \{ M \} = 3 \) and the normalized Euclidean distance of \( MR \|MR\|_I = \frac{1}{4} \text{Tr} \{ M (I_3 - R) \} \). According to angle-axis parameterization in (9), one obtains

\[
\|MR\|_I = \frac{1}{4} \text{Tr} \left\{ -M \left( \sin(\theta) [u]_x + (1 - \cos(\theta)) [u]^2_x \right) \right\}
\]

\[
= -\frac{1}{4} \text{Tr} \left\{ (1 - \cos(\theta)) M [u]^2_x \right\} (124)
\]

where \( \text{Tr} \{ M [u]^2_x \} = 0 \) as given in identity (20). One has [30]

\[
\|R\|_I = \frac{1}{4} \text{Tr} \{ I_3 - R \} = \frac{1}{2} (1 - \cos(\theta)) = \sin^2 \left( \frac{\alpha}{2} \right) (125)
\]

and the Rodriguez parameters vector with respect to angle-axis parameterization is [3]

\[
u = \cot \left( \frac{\alpha}{2} \right) \rho
\]

From identity (19) \( [u]^2_x = -u^\top u I_3 + uu^\top \), the expression in (124) becomes

\[
\|MR\|_I = \frac{1}{2} \|R\|_I u^\top M u = \frac{1}{2} \|R\|_I \cos^2 \left( \frac{\alpha}{2} \right) \rho^\top M \rho
\]

From (125), one can find \( \cos^2 \left( \frac{\alpha}{2} \right) = 1 - \|R\|_I \) which means

\[
\tan^2 \left( \frac{\alpha}{2} \right) = 1 - \|R\|_I
\]

Therefore, the normalized Euclidean distance can be expressed with respect to Rodriguez parameters vector as

\[
\|MR\|_I = \frac{1}{2} (1 - \|R\|_I) \rho^\top M \rho = \frac{1}{2} \frac{\rho^\top M \rho}{1 + \|\rho\|^2} (126)
\]
This proves (54) in Lemma 8. The anti-symmetric projection operator may be expressed in terms of Rodriguez parameters vector using the identity in (6) and (10) by

\[
P_a(MR) = \frac{M\rho \rho^T - \rho \rho^T M + M [\rho]_\times + [\rho]_\times M}{1 + \|\rho\|^2}
\]

It follows that the vex operator of the above expression is

\[
\text{vex}(P_a(MR)) = \frac{(I_3 + [\rho]_\times)^T M \rho \rho^T}{1 + \|\rho\|^2}
\]

This shows (55) in Lemma 8. Also, one can find

\[
\text{vex}(P_a(MR)) \text{vex}(P_a(MR))^T = \frac{(I_3 + [\rho]_\times)^T M \rho \rho^T M (I_3 + [\rho]_\times)}{(1 + \|\rho\|^2)^2}
\]

The 2-norm of (127) can be obtained by

\[
\|\text{vex}(P_a(MR))\|^2 = \frac{\rho^T \bar{M} \left(I_3 - [\rho]_\times^2\right) \bar{M} \rho}{(1 + \|\rho\|^2)^2}
\]

This proves (56) in Lemma 8. Using identity (19) \([\rho]_\times^2 = -\rho^T \rho I_3 + \rho \rho^T\), one obtains

\[
\|\text{vex}(P_a(MR))\|^2 = \frac{\rho^T \bar{M} \left(I_3 - [\rho]_\times^2\right) \bar{M} \rho}{(1 + \|\rho\|^2)^2}
\]

\[
\geq \frac{\lambda (\bar{M})}{\lambda (\bar{M})} \left(1 - \frac{\|\rho\|^2}{1 + \|\rho\|^2}\right) \rho^T \bar{M} \rho
\]

\[
\geq 2\lambda (1 - \|R\|_I) \|MR\|_I
\]

where \(\lambda = \lambda (M)\) is the minimum singular value of \(M\) and \(\|R\|_I = \|\rho\|^2 / \left(1 + \|\rho\|^2\right)\) as defined in (109). It can be found that

\[
1 - \|R\|_I = \frac{1}{4} \left(1 + \text{Tr} \left\{M^{-1}MR\right\}\right)
\]

Therefore, from (129) and (130) the following inequality holds

\[
\|\text{vex}(P_a(MR))\|^2 \geq \frac{\lambda}{2} \left(1 + \text{Tr} \left\{M^{-1}MR\right\}\right) \|MR\|_I
\]

which confirms (57) in Lemma 2. It should be remarked that both \(M^{-1}\) and \(MR\) can be obtained by a set
of vectorial measurements helping the designer to avoid the process of attitude reconstruction. For more details visit [21, 20, 23, 25].

6.3. Rodriguez vector kinematics

The kinematic relationship between Rodriguez vector and angular velocity can be expressed as follows [3]

\[
\frac{d}{dt} \rho = \frac{1}{2} (\Omega - \Omega \times \rho + (\Omega \cdot \rho) \rho) \\
= \frac{1}{2} (\Omega + [\rho]_\times \Omega + \rho \rho^T \Omega) \\
= \frac{1}{2} \left( I_3 + [\rho]_\times + \rho \rho^T \right) \Omega
\]

Hence, the kinematics are governed by

\[
\frac{d}{dt} \rho = \frac{1}{2} \left( I_3 + [\rho]_\times + \rho \rho^T \right) \Omega \tag{132}
\]

6.4. Attitude error and attitude error dynamics in the sense of Rodriguez vector

Consider the attitude dynamics in (27)

\[
\dot{R} = R \left[ \Omega \right]_\times
\]

consider

\[
\Omega_* = \Omega + \beta
\]

where \( \Omega_* \) could represent the uncertain measurements of \( \Omega \) and \( \beta \) could be considered as an unknown variable. Let us introduce desired attitude dynamics (estimator attitude dynamics ) by

\[
\dot{R}_* = R_* \left[ \Omega_* - \hat{\beta} \right]_\times
\]

where \( \hat{\beta} \) is the estimate of \( \beta \). Let the error in attitude be given by

\[
\tilde{R} = RR_*^T
\]

and define the error in \( \beta \) by \( \tilde{\beta} = \beta - \hat{\beta} \). Hence, the dynamics of the attitude error can be found to be

\[
\dot{\tilde{R}} = \tilde{R} R_*^T + R \tilde{R}_*^T \\
= R \left[ \Omega \right]_\times R_*^T - R \left[ \Omega + \beta - \hat{\beta} \right]_\times R_*^T \\
= -R \left[ \tilde{\beta} \right]_\times R_*^T \tag{135}
\]

where \( \left[ \Omega + \beta - \hat{\beta} \right]_\times = - \left[ \Omega + \beta - \hat{\beta} \right]_\times \) as defined in (3). One can find

\[
\dot{\tilde{R}} = -R \left[ \tilde{\beta} \right]_\times R_*^T \\
= -R I_3 [\tilde{\beta}]_\times I_3 R_*^T \\
= -R R_*^T R_* [\tilde{\beta}]_\times R_* R_*^T \\
= -R R_*^T [R_* \tilde{\beta}]_\times \\
= -R \left[ R_* \tilde{\beta} \right]_\times \tag{136}
\]
with \([R, \tilde{\beta}]\times = R \times [\tilde{\beta}] \times R^\top\) being given in identity (16). Thus, in view of (27) and (132), one can write (136) in terms of Rodriguez vector dynamics as

\[
\frac{d}{dt} \tilde{\rho} = -\frac{1}{2} \left( I_3 + [\tilde{\rho}] \times + \tilde{\rho} \tilde{\rho}^\top \right) R \times \tilde{\beta}
\]

where \(\tilde{\rho}\) is the error in Rodriguez vector associated with \(\tilde{R}\). For more detailed derivations visit [22, 23]. The objective of attitude filter/control is to drive the error in Rodriguez vector to zero (\(\tilde{\rho} \to 0\)). Driving \(\tilde{\rho} \to 0\) implies that \(\tilde{R} \to I_3\), since we have

\[
R_{\tilde{\rho}} (\tilde{\rho}) = \frac{1}{1 + \|\tilde{\rho}\|^2} \left( \left( 1 - \|\tilde{\rho}\|^2 \right) I_3 + 2 \tilde{\rho} \tilde{\rho}^\top + 2 [\tilde{\rho}] \times \right)
\]

\[
\tilde{\rho} = 0 \iff R_{\tilde{\rho}} (\tilde{\rho}) = I_3
\]

6.5. Problems of Rodriguez vector parameterization:

Although Rodriguez vector provides a unique representation of the attitude, the Rodriguez vector and the modified Rodriguez vector are not defined for 180° of rotation. To be more specific, the parameters of the Rodriguez vector are not defined for any of the following three rotational matrices

\[
\text{at } R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \text{SO} (3) \Rightarrow \rho = [\infty, \infty, \infty]^\top \notin \mathbb{R}^3
\]

(139)

\[
\text{at } R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in \text{SO} (3) \Rightarrow \rho = [\infty, \infty, \infty]^\top \notin \mathbb{R}^3
\]

(140)

\[
\text{at } R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \text{SO} (3) \Rightarrow \rho = [\infty, \infty, \infty]^\top \notin \mathbb{R}^3
\]

(141)

which means that as \(\alpha \to \pi\), the Rodriguez vector \(\rho\) \(\to \infty\). Therefore, the mapping from SO (3) to Rodriguez vector cannot be achieved for any rotational matrix in (139), (140), or (141). In this regard, for \(R\) equivalent to (139), (140), or (141), continuous control laws cannot be globally defined using the three-parameter vector.

7. Unit-quaternion

Unit-quaternion has proven to be an effective tool for the tracking control of UAVs, such as [36, 37, 38]. Unit-quaternion also showed impressive results in attitude estimators. The unit-quaternion vector has been employed in deterministic attitude filters, for instance [12, 39]. However, unit-quaternion has been used more extensively in Gaussian attitude filters, namely Kalman filter [17], extended Kalman filter [15, 40], multiplicative extended Kalman filter [10], unscented Kalman filter [18], and invariant extended Kalman filter [19]. The main advantage of the unit-quaternion vector is that it gives nonsingular representation of the attitude. The vector \(Q\) is said to be a unit-quaternion such that \(Q \in S^3\) if the following two conditions are met [35, 3]

1. \(Q \in \mathbb{R}^4\).
2. \(\|Q\| = 1\).
The unit-quaternion is a four-element representation of the attitude, and these four elements do not have intuitive physical meanings. The unit-quaternion is denoted by

\[ Q = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T = \begin{bmatrix} q_0 \\ q \end{bmatrix} \in S^3 \] (142)

where \( q = [q_1, q_2, q_3]^T \in \mathbb{R}^3 \) and \( q_0 \in \mathbb{R} \). The unit-quaternion vector \( Q \) is defined by

\[ S^3 = \left\{ Q \in \mathbb{R}^4 \mid \|Q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1 \right\} \]

where \( Q \) is non-Euclidean, lies in the three-sphere \((S^3)\) and is given by

\[ Q = q_0 + iq_1 + jq_2 + kq_3 \]

with \( i, j, \) and \( k \) being hyper-imaginary numbers which satisfy the following rules

\[ i^2 = j^2 = k^2 = ijk = -1 \]
\[ ij = i \times j = -j \times i = -ji = k, \quad ji = -k \]
\[ jk = j \times k = -k \times j = -kj = i, \quad kj = -i \]
\[ ki = k \times i = -i \times k = -ik = j, \quad ik = -j \]

where multiplication follows the “natural order” convention \([3, 41]\). Quaternion multiplication is in general associative, that is,

\[ Q_1 \circ Q_2 \circ Q_3 = (Q_1 \circ Q_2) \circ Q_3 = Q_1 \circ (Q_2 \circ Q_3) \] (143)

where \( Q_1, Q_2, Q_3 \in S^3 \) and \( \circ \) denotes multiplication operator between two quaternion vectors. Quaternion multiplication is not commutative such that

\[ Q_1 \circ Q_2 \neq Q_2 \circ Q_1, \quad \forall Q_1, Q_2 \in S^3 \] (144)

Hence

\[
\begin{align*}
P &= p_0 + p \\
  &= p_0 + ip_1 + jp_2 + kp_3 \\
M &= m_0 + m \\
  &= m_0 + im_1 + jm_2 + km_3 \\
PM &= (p_0 + ip_1 + jp_2 + kp_3)(m_0 + im_1 + jm_2 + km_3) \\
  &= p_0m_0 - p \cdot m + p_0m + m_0p + p \times m
\end{align*}
\]

The inverse of the rotation is defined by the complex conjugate or inverse of a unit-quaternion, which is given by

\[ Q^* = Q^{-1} = \begin{bmatrix} q_0 \\ -q \end{bmatrix} \in S^3 \] (145)

7.1. Quaternion multiplication

Analogously to linear matrix multiplication of rotation matrices, the composition of successive rotations represented by unit-quaternion is obtained by the distributive and associative, but not commutative, quaternion multiplication. To define this operation, consider two unit-quaternion vectors

\[ Q_1 = \begin{bmatrix} q_{01} \\ q_1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} q_{02} \\ q_2 \end{bmatrix}, \quad \forall Q_1, Q_2 \in S^3 \]
The quaternion product between $Q_1$ and $Q_2$, denoted by $Q_3 \in S^3$, is given by

$$Q_3 = Q_1 \odot Q_2 = \begin{bmatrix} q_{01} \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \odot \begin{bmatrix} q_{02} \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_{01}q_{02} - q_1^2q_2 \\ q_0q_{12} + q_0q_3 + [q_1]_\times q_2 \\ q_0q_{23} - [q_1]_\times q_3 \\ q_0q_{31} - [q_2]_\times q_1 \end{bmatrix}$$

(146)

The neutral element of the unit-quaternion is denoted by $Q_t \in S^3$, which is defined by

$$Q_t = Q \odot Q^* = \begin{bmatrix} q_0 \\ q \\ 0 \end{bmatrix} \odot \begin{bmatrix} q_0 \\ -q \\ 0 \end{bmatrix} = \begin{bmatrix} q_0q_0 + q^\top q \\ -q_0q + qoq - [q]_\times q \\ q_0^2 + ||q||^2 \\ -[q]_\times q \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(147)

where $[q]_\times q = [0, 0, 0]^\top$ as given in (17). The inverse of quaternion multiplication is equivalent to

$$(Q_1 \odot Q_2)^{-1} = Q_2^{-1} \odot Q_1^{-1}, \quad \forall Q_1, Q_2 \in S^3$$

7.2. Mapping: From/To unit-quaternion to/from other representations

Using the quaternion product, the unit-quaternion may also be utilized to give the coordinates of a vector in multiple frames of reference. Actually, having the property $v^B = R_Q^{T} (Q) v^I$, the vector $v^B$ may be obtained through the quaternion product by the following operation [1, 3]:

$$\begin{bmatrix} 0 \\ v^B \end{bmatrix} = \begin{bmatrix} q_0 \\ -q \end{bmatrix} \odot \begin{bmatrix} 0 \\ v^I \end{bmatrix} \odot \begin{bmatrix} q_0 \\ q \end{bmatrix} = \begin{bmatrix} qq^\top v^I - q_0q^\top x_q - (v^I)^\top (q \times q) \\ qq^\top v^I + q_0q^\top x_q + qoq + qoq - [q]_\times v^I + [q]_\times v^I \times x_q \\ qq^\top v^I - q_0 (v^I)^\top x_q - (v^I)^\top v^I + [q]_\times v^I \times x_q \\ qq^\top v^I + q_0q^\top x_q - q_0q \times v^I + q_0q \times v^I + qoq \times v^I \times x_q \end{bmatrix} = \begin{bmatrix} 0 \\ qq^\top v^I + q_0q^\top x_q - 2q_0q \times v^I - q \times v^I \times q \\ qq^\top v^I + q_0q^\top x_q - 2q_0q \times v^I - ((q \cdot q) v^I - (q \cdot v^I) q) \\ qq^\top v^I + q_0q^\top x_q - 2q_0q \times v^I - q \cdot q \cdot v^I - q \cdot v^I \cdot q \\ (q_0^2 - ||q||^2) v^I - 2q_0 [q]_\times v^I + 2qq^\top v^I \end{bmatrix}$$

(148)
The translation from body-frame \((v^B)\) to inertial-frame \((v^I)\) is given in (23) by \(v^I = R^T v^B\). Thus the result in (148) indicates that

\[
\begin{bmatrix}
0 \\
v^B
\end{bmatrix} = \begin{bmatrix}
0_{3\times3} \\
R_Q^T (Q)
\end{bmatrix} v^I
\]  
\[(149)\]

with

\[
R_Q^T (Q) = \left(q_0^2 - \|q\|^2\right) I_3 + 2q q^T - 2q_0 [q]_x
\]

Therefore, the coordinate of a moving frame with respect to a reference-frame (the mapping from \(Q\) to \(SO(3)\)) \(R_Q : S^3 \rightarrow SO(3)\) is governed by

\[
R_Q (Q) = \left(q_0^2 - \|q\|^2\right) I_3 + 2q q^T + 2q_0 [q]_x
\]
\[
= I_3 + 2q_0 [q]_x + 2 [q]^2
\]  
\[(150)\]

In order to find the normalized Euclidean distance in terms of unit-quaternion components, it is necessary to find the trace of the rotational matrix in (151). One can find that the trace of \(R_Q (Q)\) is

\[
\text{Tr} \{ R_Q (Q) \} = 1 - 2 \left(q_1^2 + q_3^2\right) + 1 - 2 \left(q_2^2 + q_3^2\right) + 1 - 2 \left(q_1^2 + q_2^2\right)
\]
\[
= 3 - 4 \left(1 - q_0^2\right)
\]
\[
= 4q_0^2 - 1
\]  
\[(152)\]

Hence, from (31) and (152), it can be found that the normalized Euclidean distance in terms of unit-quaternion is

\[
\|R\|_f = \frac{1}{4} \text{Tr} \{ I_3 - R_Q (Q) \}
\]

which is equivalent to

\[
\|R\|_f = 1 - q_0^2
\]  
\[(153)\]

The result in (153) proves (45). Considering the identity in (6) and the mapping from quaternion \(Q\) to \(SO(3)\) in (151), the anti-symmetric projection operator \(P_a (R)\) can be defined in terms of unit-quaternion as follows

\[
P_a (R_Q (Q)) = 2q_0 \begin{bmatrix}
0 & -q_3 & q_2 \\
q_3 & 0 & -q_1 \\
-q_2 & q_1 & 0
\end{bmatrix}
\]  
\[(154)\]

Consequently, one can find the vex operator with respect to unit-quaternion:

\[
\text{vex} (P_a (R_Q (Q))) = 2q_0 q \in \mathbb{R}^3
\]  
\[(155)\]

The result in (155) justifies (44). Thus, the norm of the vex operator with regards to unit-quaternion is equivalent to

\[
\|\text{vex} (P_a (R_Q (Q)))\|^2 = 4q_0^2 \|q\|^2 \in \mathbb{R}
\]  
\[(156)\]
This proves (46). Recalling $\mathcal{R}_Q (Q)$ from (150) and (151), one can find $\mathcal{R}_Q : \text{SO} (3) \rightarrow S^3$ (the mapping from $\text{SO} (3)$ to $Q$) [3]

\[
q_0 = \frac{1}{2} \sqrt{1 + R_{(1,1)} + R_{(2,2)} + R_{(3,3)}}
\]
\[
q_1 = \frac{1}{4q_0} \left( R_{(3,2)} - R_{(2,3)} \right)
\]
\[
q_2 = \frac{1}{4q_0} \left( R_{(1,3)} - R_{(3,1)} \right)
\]
\[
q_3 = \frac{1}{4q_0} \left( R_{(2,1)} - R_{(1,2)} \right)
\]  
(157)

Or

\[
q_1 = \pm \frac{1}{2} \sqrt{1 + R_{(1,1)} - R_{(2,2)} - R_{(3,3)}}
\]
\[
q_2 = \frac{1}{4q_1} \left( R_{(1,2)} + R_{(2,1)} \right)
\]
\[
q_3 = \frac{1}{4q_1} \left( R_{(1,3)} + R_{(3,1)} \right)
\]
\[
q_0 = \frac{1}{4q_1} \left( R_{(3,2)} - R_{(2,3)} \right)
\]  
(158)

Or

\[
q_2 = \pm \frac{1}{2} \sqrt{1 - R_{(1,1)} + R_{(2,2)} - R_{(3,3)}}
\]
\[
q_1 = \frac{1}{4q_2} \left( R_{(1,2)} + R_{(2,1)} \right)
\]
\[
q_3 = \frac{1}{4q_2} \left( R_{(2,3)} + R_{(3,2)} \right)
\]
\[
q_0 = \frac{1}{4q_2} \left( R_{(1,3)} - R_{(3,1)} \right)
\]  
(159)

Or

\[
q_3 = \pm \frac{1}{2} \sqrt{1 - R_{(1,1)} - R_{(2,2)} + R_{(3,3)}}
\]
\[
q_1 = \frac{1}{4q_3} \left( R_{(1,3)} + R_{(3,1)} \right)
\]
\[
q_2 = \frac{1}{4q_3} \left( R_{(2,3)} + R_{(3,2)} \right)
\]
\[
q_0 = \frac{1}{4q_3} \left( R_{(2,1)} - R_{(1,2)} \right)
\]  
(160)

For a given $R$, at least one of $q_0$, $q_1$, $q_2$, and $q_3$ is non-zero at any time instant, while the singularity can always be avoided through the proper choice of one of the above-mentioned formulas (157), (158), (159), or (160). The unit-quaternion is often considered to be an angle-axis representation. Indeed, from (86), (87), and (89), the rotation by an angle $\alpha \in \mathbb{R}$ about an arbitrary unit-length vector $u \in S^2$ can be described by
the unit-quaternion \( Q : \mathbb{R} \times S^2 \rightarrow S^3 \) (the mapping from \((\alpha, u)\) to \(Q\))

\[
Q = \begin{bmatrix}
\cos(\alpha/2) \\
u \sin(\alpha/2)
\end{bmatrix}
= \begin{bmatrix}
\cos(\alpha/2) & u_1 \sin(\alpha/2) & u_2 \sin(\alpha/2) & u_3 \sin(\alpha/2)
\end{bmatrix}^T
\] (161)

And from (161) the mapping from angle-axis representation to unit-quaternion \( Q : S^3 \rightarrow \mathbb{R} \times S^2 \) (the mapping from \(Q\) to \((\alpha, u)\)) can be accomplished by

\[
\begin{align*}
\alpha &= 2\cos^{-1}(q_0) \\
u &= \frac{1}{\sin(\alpha/2)}q
\end{align*}
\] (162, 163)

Also, unit-quaternion can be mapped to Euclidean vector in the sense of Euler angles. From, (65), (73), and (151), Euler angles can be defined in terms of unit-quaternion components \( \xi : S^3 \rightarrow \mathbb{R}^3 \) (the mapping from \(Q\) to \(\xi\)) by

\[
\begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix} = \begin{bmatrix}
\arctan\left(\frac{2(q_0q_2 + q_0q_1)}{1 - 2(q_1^2 + q_2^2)}\right) \\
\arcsin\left(2(q_0q_2 - q_3q_1)\right) \\
\arctan\left(\frac{2(q_2q_1 + q_0q_3)}{1 - 2(q_1^2 + q_3^2)}\right)
\end{bmatrix}
\] (164)

The mapping from unit-quaternion to Rodriguez vector and vice-versa is proven in Lemma 7. Let us get back to quaternion multiplication, let \(Q_1, Q_2 \in S^3\)

\[
Q_1 \odot Q_2 = \begin{bmatrix}
q_{01} \\
q_{1}
\end{bmatrix} \odot \begin{bmatrix}
q_{02} \\
q_{2}
\end{bmatrix} = \begin{bmatrix}
q_{01}q_{02} - q_1^\top q_2 \\
q_1q_{02} + q_2q_1 + |q_1| \times q_2
\end{bmatrix}
\]

hence, one can easily find

\[
\mathcal{R}_Q(Q_1) \mathcal{R}_Q(Q_2) = \mathcal{R}_Q(Q_1 \odot Q_2)
\] (165)

One can find that for \(X \in \mathbb{R}^4\) that

\[
Q \odot X^* \odot X \odot Q^* = X^* \odot X, \quad Q \in S^3, X \in \mathbb{R}^4
\] (166)

Let \(Q_1 = [q_{01}, q_1^\top]^T = 2Q \odot X^*\) and \(Q_2 = [q_{02}, q_2^\top]^T = 2X \odot Q^*\), for all \(Q \in S^3\) and \(X \in \mathbb{R}^4\), then

\[
q_1 = -q_2, \quad q_1, q_2 \in \mathbb{R}^3
\] (167)

7.3. Attitude measurements

Consider the body-frame vector \(v_i^B \in \mathbb{R}^3\) and the inertial-frame vector \(v_i^I \in \mathbb{R}^3\). The relationship between a body-frame and an inertial-frame of the \(i\)th measurement is given in (23)

\[
v_i^B = R^\top v_i^I
\] (168)

where \(R \in SO(3)\) is the attitude matrix for \(i = 1, \ldots, n\). The body-frame vector measurement in (168) in terms of unit-quaternion is equivalent to

\[
\begin{bmatrix}
0 \\
v_i^B
\end{bmatrix} = Q^{-1} \odot \begin{bmatrix}
0 \\
v_i^I
\end{bmatrix} \odot Q
\] (169)

\[
\begin{bmatrix}
\tau_i^B \\
\tau_i^I
\end{bmatrix} = Q^{-1} \odot \tau_i^I \odot Q
\] (170)
where

\[ \mathbf{v}_I^i = \begin{bmatrix} 0 \\ v_i^i \end{bmatrix} \in \mathbb{R}^4, \quad \mathbf{v}_B^i = \begin{bmatrix} 0 \\ v_i^B \end{bmatrix} \in \mathbb{R}^4 \]

7.4. Unit-quaternion attitude kinematics and measurements

Let \( \Omega = [\Omega_x, \Omega_y, \Omega_z]^T \in \mathbb{R}^3 \) be the angular velocity defined relative to the body-frame \( \Omega \in \{B\} \), and let \( Q \in S^3 \) be a unit-quaternion vector. Consider the following representations

\[ \bar{\Omega} = \begin{bmatrix} 0 \\ \Omega \end{bmatrix} \]

\[ \Gamma (\Omega) = \begin{bmatrix} 0 & -\Omega^T \\ \Omega & -[\Omega]_x \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_x & -\Omega_y & -\Omega_z \\ \Omega_x & 0 & \Omega_z & -\Omega_y \\ \Omega_y & -\Omega_z & 0 & \Omega_x \\ \Omega_z & \Omega_y & -\Omega_x & 0 \end{bmatrix} \] (171)

\[ \Xi (Q) = \begin{bmatrix} -q^T \\ q_0 I_3 + [q]_x \end{bmatrix} = \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \] (172)

\[ \Psi (Q) = \begin{bmatrix} 0 & -q^T \\ q & q_0 I_3 + [q]_x \end{bmatrix} = \begin{bmatrix} 0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_3 & -q_2 & q_1 & q_0 \\ q_1 & q_0 & -q_3 & q_2 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \] (173)

\[ \bar{\Psi} (Q) = \begin{bmatrix} 0 & q^T \\ q & q_0 I_3 + [q]_x \end{bmatrix} = \begin{bmatrix} 0 & q_1 & q_2 & q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_3 & q_0 & q_1 & q_0 \\ q_1 & q_0 & -q_3 & q_2 \\ q_3 & q_0 & q_1 & q_0 \end{bmatrix} \] (174)

where the relationship between (172) and (173) is given in [15].

7.4.1. Continuous Unit-quaternion Attitude Kinematics

Thus, the attitude kinematics can be defined by

\[ \dot{Q} = \frac{1}{2} Q \odot \bar{\Omega} = \frac{1}{2} \Gamma (\Omega) Q = \frac{1}{2} \Xi (Q) \Omega = \frac{1}{2} \Psi (Q) \Omega \] (175)

According to the expression in (175), the attitude dynamics are linear and time-variant by the following representation

\[ \dot{Q} = \frac{1}{2} \Gamma (\Omega) Q \] (176)

7.4.2. Discrete Unit-quaternion Attitude Kinematics

The continuous form of the attitude kinematics given in (176) could be defined in discrete form through exact integration by

\[ Q [k + 1] = \frac{1}{2} \exp (\Gamma (\Omega [k]) \Delta t) Q [k] \] (177)

where \( k \) denotes the \( k \)th sample, \( \Delta t \) denotes a time step which is normally small, and \( Q [k] \) and \( \Omega [k] \) refer to the true unit-quaternion and angular velocity at the \( k \)th sample, respectively.
7.4.3. Sensor Measurements

Equation (169) is considered to be the measured output obtained from sensors attached to the moving body

\[
\begin{bmatrix}
0 \\
v^B_i
\end{bmatrix} = Q^{-1} \odot \begin{bmatrix}
0 \\
v^I_i
\end{bmatrix} \odot Q
\]  

(178)

The expression in (178) is nonlinear. Extended Kalman filter [15] is the earliest described filter that took into account both linear dynamics in (176) and the nonlinear dynamics in (178). However, the nonlinear representation in (178) can be reformulated as follows [17]

\[
\begin{bmatrix}
0 \\
v^B_i
\end{bmatrix} = Q^{-1} \odot \begin{bmatrix}
0 \\
v^I_i
\end{bmatrix} \odot Q
\]

\[
Q \odot \begin{bmatrix}
0 \\
v^B_i
\end{bmatrix} = \begin{bmatrix}
0 \\
v^I_i
\end{bmatrix} \odot Q
\]

\[
\begin{bmatrix}
0 & - (v^B_i - v^I_i)^\top \\
v^B_i - [v^I_i \times v^B_i]
\end{bmatrix} \begin{bmatrix}
q_0 \\
q
\end{bmatrix} = \begin{bmatrix}
0 & - (v^I_i)^\top \\
v^I_i - [v^I_i \times v^I_i]
\end{bmatrix} \begin{bmatrix}
q_0 \\
q
\end{bmatrix}
\]  

(179)

Thus, the expression in (179) is equivalent to

\[
0 = \begin{bmatrix}
0 & 0 \\
v^B_i - v^I_i & - (v^B_i - v^I_i)^\top - [v^I_i \times v^B_i]
\end{bmatrix} \begin{bmatrix}
q_0 \\
q
\end{bmatrix}
\]  

(180)

Therefore, the linear and time-variant state-space representation of the attitude problem in terms of quaternion is equivalent to

\[
\dot{Q} = \frac{1}{2} \begin{bmatrix}
0 & -\Omega^\top \\
\Omega & -[\Omega]_x
\end{bmatrix} \begin{bmatrix}
q_0 \\
q
\end{bmatrix}
\]  

(181)

\[
Y = 0 = \begin{bmatrix}
0 & 0 \\
v^B_i - v^I_i & - (v^B_i - v^I_i)^\top - [v^I_i \times v^I_i]
\end{bmatrix} \begin{bmatrix}
q_0 \\
q
\end{bmatrix}
\]  

(182)

where \(Q \in S^3\) is the state vector, and \(Y \in \mathbb{R}^4\) is the output vector. Both \(\Gamma (\Omega)\) and \(H (v^I_i, v^B_i)\) are time-variant known matrices obtained from the measurements of the sensors attached to the moving body. The representation in (182) is valid only for \(v^B_i\) free of noise and bias components. This is the case because equation (182) disregards noise and bias attached to \(v^B_i\). Nonetheless, this approach produced impressive results for an uncertain \(v^B_i\) as described in [17].
7.5. Rotational acceleration

The relationship between rotational acceleration $\dot{\Omega}$ and the quaternion derivative can be defined as follows:

$$Q^* \circ Q = Q_1$$
$$Q^* \circ Q + Q^* \circ \dot{Q} = 0$$
$$2Q^* \circ Q + 2Q^* \circ \dot{Q} = 0$$
$$2Q^* \circ Q + \dot{\Omega} = 0$$
$$\dot{\Omega} = -2Q^* \circ Q$$

Now one can find

$$\ddot{Q} = \frac{1}{2} \left( Q \circ \dot{\Omega} + Q \circ \ddot{\Omega} \right)$$
$$= \frac{1}{2} \left( -2Q \circ \dot{Q}^* \circ Q + Q \circ \ddot{\Omega} \right)$$
$$Q^* \circ \ddot{Q} = \frac{1}{2} \left( -2Q^* \circ Q \circ \dot{Q}^* \circ Q + \dot{\Omega} \circ Q_1 \right)$$
$$= \frac{1}{2} \left( -2Q^* \circ Q \circ \dot{Q}^* \circ Q + \dot{\Omega} \right)$$

$$\dot{\Omega} = 2 \left( Q^* \circ \ddot{Q} + Q^* \circ \dot{Q} \circ \dot{Q}^* \circ Q \right)$$

as $Q^* \circ \ddot{Q} \circ \dot{Q}^* \circ Q = \dot{Q} \circ Q^*$

$$\dot{\Omega} = 2 \left( Q^* \circ \ddot{Q} + Q \circ \ddot{\Omega} \right)$$

(183)

Note that $\|\dot{Q}\|$ is not necessarily equal to 1.

7.6. Unit-quaternion update

The rotational matrix can be constructed knowing unit-quaternion if and only if $\|Q\| = 1$. During the control/estimation process, the unit-quaternion may lose precision and thereby, $\|Q\| \neq 1$. In order to achieve valid mapping from $Q$ to $SO(3)$, it is necessary to maintain $\|Q\| = 1$ at each time instant which can be accomplished through the substitution of $Q$ by

$$Q = \frac{Q}{\|Q\|}$$

(184)

7.7. Unit-quaternion error and error derivative

Let $Q \in S^3$ be the true unit-quaternion vector and let the desired/estimator unit-quaternion vector be given by $Q_* = [q_*, q_*^T]^T \in S^3$ for all $q_{*,0} \in \mathbb{R}$ and $q_* \in \mathbb{R}^3$. The true and desired/estimator unit-quaternion dynamics are given by

$$\dot{Q} = \frac{1}{2} \Psi (Q) \dot{\Omega} = \frac{1}{2} \begin{bmatrix} 0 & -q^T \\ q & q_0 I_3 + [q]_x \end{bmatrix} \dot{\Omega} = \frac{1}{2} \begin{bmatrix} -q^T \Omega \\ (q_0 I_3 + [q]_x) \dot{\Omega} \end{bmatrix}$$

(185)

$$\dot{Q}_* = \frac{1}{2} \Psi (Q_*) \dot{\Omega}_* = \frac{1}{2} \begin{bmatrix} 0 & -q_*^T \\ q_* & q_{*,0} I_3 + [q_*]_x \end{bmatrix} \dot{\Omega}_* = \frac{1}{2} \begin{bmatrix} -q_*^T \Omega_* \\ (q_{*,0} I_3 + [q_*]_x) \dot{\Omega}_* \end{bmatrix}$$

(186)
Let the error between the desired and the true unit-quaternion be defined by

$$\bar{\mathbf{Q}} = \mathbf{Q}^\dagger = \mathbf{Q}^{-1} \circ \mathbf{Q}$$

where $\bar{\mathbf{Q}} = [\bar{q}_0, \bar{q}^\top]^\top \in S^3$ for all $\bar{q}_0 \in \mathbb{R}$ and $\bar{q} \in \mathbb{R}^3$. The objective of attitude filter/control in terms of unit-quaternion is to drive the error in unit-quaternion $\bar{\mathbf{Q}} \rightarrow \mathbf{Q}_1 = [1, 0, 0, 0]^\top$. Driving $\bar{\mathbf{Q}} \rightarrow \mathbf{Q}_1$ implies $\bar{R} \rightarrow \mathbf{I}_3$, since we have

$$\mathcal{R}_Q (\bar{\mathbf{Q}}) = \left( \bar{q}_0^2 - \bar{q}^\top \bar{q} \right) \mathbf{I}_3 + 2\bar{q}\bar{q}^\top + 2\bar{q}_0 [\bar{q}] \times$$

$$\bar{\mathbf{Q}} = \mathbf{Q}_1 \Leftrightarrow \mathcal{R}_Q (\bar{\mathbf{Q}}) = \mathbf{I}_3$$

(189)

Accordingly, the unit-quaternion error dynamics are given by

$$\bar{\mathbf{Q}} = \mathbf{Q}_*^{-1} \circ \mathbf{Q} + \mathbf{Q}_*^{-1} \circ \mathbf{Q}$$

(190)

Consequently, from (185) and (187) one has

$$\dot{\mathbf{Q}}_*^{-1} \circ \mathbf{Q} = \frac{1}{2} \left[ \begin{matrix} -q_0 \vec{q}_0 \Omega_* \times [\bar{q}_0] \times \Omega_* \times \mathbf{Q} & q_0^\top \Omega_* \times \mathbf{Q} \\ -q_0 \vec{q}_0 \Omega_* \times [\bar{q}_0] \times \Omega_* \times \mathbf{Q} & q_0 \vec{q}_0 \Omega_* \times \mathbf{Q} \end{matrix} \right]$$

(191)

where $\Omega_* = [0, \Omega_*]^\top \in \mathbb{R}^4$ and $\Omega_* \in \mathbb{R}^3$. From (186), one could find

$$\dot{Q}_*^{-1} = \frac{1}{2} \Psi (Q_*) \Omega_* = \frac{1}{2} \left[ \begin{matrix} 0 & q_0 \vec{q}_0 \mathbf{I}_3 + [\bar{q}_0] \times \\ q_0 \vec{q}_0 \mathbf{I}_3 + [\bar{q}_0] \times \mathbf{Q} & q_0^\top \Omega_* \times \mathbf{Q} \end{matrix} \right] \Omega_* = \frac{1}{2} \left[ \begin{matrix} (q_0 \vec{q}_0 \mathbf{I}_3 + [\bar{q}_0] \times) \Omega_* \times \mathbf{Q} & q_0^\top \Omega_* \times \mathbf{Q} \\ q_0 \vec{q}_0 \mathbf{I}_3 + [\bar{q}_0] \times \mathbf{Q} & q_0 \vec{q}_0 \Omega_* \times \mathbf{Q} \end{matrix} \right]$$

(187)
From (191) and (192) the unit-quaternion error dynamics in (190) are equivalent to

\[
\dot{Q} = \dot{Q}_a^{-1} \circ Q + Q_a^{-1} \circ \dot{Q}
\]

\[
= \frac{1}{2} \left[ \begin{array}{c}
-\tilde{q}^T \\
-\tilde{q}_0 \mathbf{I}_3 + [\tilde{q}]_x
\end{array} \right] \Omega^* + \frac{1}{2} \left[ \begin{array}{c}
-\tilde{q}^T \\
\tilde{q}_0 \mathbf{I}_3 + [\tilde{q}]_x
\end{array} \right] \Omega
\]

\[
= \frac{1}{2} \left[ \tilde{q}_0 (\Omega - \Omega^*) + [\tilde{q}]_x (\Omega^* + \Omega) \right]
\]  

(193)

Therefore, the problem can be summarized as follows

\[
\begin{align*}
\dot{Q} & = \left[ \begin{array}{c}
\tilde{q}_0 \\
\tilde{q}
\end{array} \right] \\
& = \left[ \begin{array}{c}
q_0 \omega q_0 + \tilde{q}^T q \\
q_0 \omega q_0 - q_0 \omega q_* - [\tilde{q}]_x \omega
\end{array} \right] \\
\dot{Q} & = \frac{1}{2} \left[ \tilde{q}_0 (\Omega - \Omega^*) + [\tilde{q}]_x (\Omega^* + \Omega) \right]
\end{align*}
\]

The above-mentioned expression can be simplified if the error in angular velocity is selected using \( \Omega = \Omega - \mathcal{R}_Q (\dot{Q}) \Omega^* \). Therefore, one can find

\[
\dot{q}^T (\Omega^* - \Omega) = \dot{q}^T \Omega^* - \dot{q}^T \mathcal{R}_Q (\dot{Q}) \Omega + \dot{q}^T \mathcal{R}_Q (\dot{Q}) \Omega + \dot{q}^T \Omega
\]

\[
= \dot{q}^T (\mathcal{R}_Q (\dot{Q}) \Omega^* - \Omega) + \dot{q}^T (\mathbf{I}_3 - \mathcal{R}_Q (\dot{Q})) \Omega^*
\]

\[
= \dot{q}^T (\mathcal{R}_Q (\dot{Q}) \Omega^* - \Omega) + \dot{q}^T (\mathcal{I}_3 - \mathcal{R}_Q (\dot{Q})) \Omega^*
\]

\[
= -\dot{q}^T (\Omega - \mathcal{R}_Q (\dot{Q}) \Omega^*)
\]

\[
[\tilde{q}]_x (\Omega^* + \Omega) + \tilde{q}_0 (\Omega - \Omega^*) = [\tilde{q}]_x \Omega^* + [\tilde{q}]_x \Omega + [\tilde{q}]_x \mathcal{R}_Q (\dot{Q}) \Omega^* - [\tilde{q}]_x \mathcal{R}_Q (\dot{Q}) \Omega^* + \tilde{q}_0 \Omega - \tilde{q}_0 \Omega^* + \tilde{q}_0 \mathcal{R}_Q (\dot{Q}) \Omega^* - \tilde{q}_0 \mathcal{R}_Q (\dot{Q}) \Omega^*
\]

\[
= [\tilde{q}]_x (\Omega - \mathcal{R}_Q (\dot{Q}) \Omega^*) + \tilde{q}_0 (\Omega - \mathcal{R}_Q (\dot{Q}) \Omega^*)
\]

hence

\[
\dot{Q} = \left[ \begin{array}{c}
\tilde{q}_0 \\
\tilde{q}
\end{array} \right] = \frac{1}{2} \left[ \begin{array}{c}
-\dot{q}^T (\Omega - \mathcal{R}_Q (\dot{Q}) \Omega^*) \\
([\tilde{q}]_x + \tilde{q}_0 \mathbf{I}_3) (\Omega - \mathcal{R}_Q (\dot{Q}) \Omega^*)
\end{array} \right]
\]

(194)

which is equivalent to

\[
\dot{Q} = \frac{1}{2} \Psi (\dot{Q}) \ddot{\Omega}
\]

(195)

where \( \ddot{\Omega} = [0, \dddot{\Omega}]^T \).

7.8. Problem of unit-quaternion

Despite providing a global representation of the attitude and being free of non-singularity in the attitude parameterization, unit-quaternion vector is non-unique. Two different unit-quaternion vectors can result
in the same rotational matrix such that
\[ R_Q (Q) = R_Q (-Q) \quad \forall Q \in S^3 \] (196)
such as
\[ R_Q (S) = R_Q (P), \quad \forall S = [q_0, q_1, q_2, q_3]^T, P = [-q_0, -q_1, -q_2, -q_3]^T \] (197)

For example, consider the following two unit-quaternion vectors

\[ S = \begin{bmatrix} 0.7794 \\ -0.1440 \\ 0.4623 \\ -0.3976 \end{bmatrix} \in S^3, \quad P = -S = \begin{bmatrix} -0.7794 \\ 0.1440 \\ -0.4623 \\ 0.3976 \end{bmatrix} \in S^3 \]

One can easily find \( R_Q : S^3 \rightarrow SO (3) \)
\[ R_Q (S) = R_Q (P) = \begin{bmatrix} 0.2563 & 0.4867 & 0.8351 \\ -0.7529 & 0.6423 & -0.1433 \\ -0.6061 & -0.5921 & 0.5311 \end{bmatrix} \in SO (3) \]

Hence, the controller should be carefully designed when using unit-quaternion for attitude parameterization [42]. In brief, physical attitude \( R \in SO (3) \), which has a unique orientation, is represented by a pair of antipodal quaternions \( \pm Q \in S^3 \), such that \( R = R_Q (\pm Q) \in SO (3) \).

8. Simulation

In this section, two examples are presented to illustrate that not any angular velocity (\( \Omega \)) can be obtained by the means of the time derivatives of Euler angles (\( \dot{\xi} \)). It also shows that Rodriguez vector has a unique representations, in spite of the fact, that it cannot achieve some configurations. It should be noted that the true attitude dynamics follow [3]
\[ \dot{R} = R [\Omega] \times \]
The attitude \( R \) obtained from the above mentioned dynamics is the true attitude. Thus, any method of attitude parameterization, such as Euler angles, Rodriguez vector, or unit-quaternion obtained from the dynamics in (132), (132) and (175), respectively, should be suitable for acquiring the true representation of the true attitude \( R \).

Example 1:
Consider this angular velocity
\[ \Omega = \begin{bmatrix} 0.1 \sin (0.3376t) \\ 0.07 \sin (0.6079t + \pi) \\ 0.05 \sin (0.7413t + \frac{\pi}{3}) \end{bmatrix} \text{ (rad/sec)} \] (198)

Let the initial attitude be given by
\[ R (0) = \begin{bmatrix} 0.9479 & -0.2040 & 0.2448 \\ 0.2177 & 0.9756 & -0.0297 \\ -0.2328 & 0.0814 & 0.9691 \end{bmatrix} \in SO (3) \] (199)

The initial condition of \( R (0) \) in terms of Euler angles, Rodriguez vector and unit-quaternion are listed in Table 3.
Table 3: Initial conditions - Example 1

| Representation | Mapping | Numerical values |
|----------------|---------|-----------------|
| Euler angles   | \( \xi : \text{SO}(3) \rightarrow \mathbb{R}^3 \) | \( \xi(0) = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 4.8035 \\ 13.4601 \\ 12.9329 \end{bmatrix} \) (deg) |
| Rodriguez vector | \( \rho : \text{SO}(3) \rightarrow \mathbb{R}^3 \) | \( \rho(0) = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 0.0286 \\ 0.1227 \\ 0.1083 \end{bmatrix} \) |
| Unit-quaternion | \( Q : \text{SO}(3) \rightarrow S^3 \) | \( Q(0) = \begin{bmatrix} q_0 \\ q \end{bmatrix} = \begin{bmatrix} 0.9865 \\ 0.0282 \\ 0.1210 \\ 0.1069 \end{bmatrix} \) |

The attitude \( R \) has been obtained from the dynamics in (27) \((\dot{R} = R [\Omega]_\times)\) given \( R(0) \) in (199) and the angular velocity in (198). The mapping of the attitude \( R \) from \( \text{SO}(3) \) to Euler angles, Rodriguez vector, and unit-quaternion is obtained from (73), (106), and (157), respectively, and depicted in Figure 5, 6, and 7, in blue colors. Euler angles obtained from Euler dynamics in (74) are plotted in red color in Figure 5 against blue-colored Euler angles obtained through the mapping in (73) from \( \text{SO}(3) \). Rodriguez vector obtained from Rodriguez vector dynamics in (132) is plotted in red color in Figure 6 against Rodriguez vector obtained through the mapping in (106) from \( \text{SO}(3) \) in blue colors. Unit-quaternion obtained from unit-quaternion dynamics in (175) is plotted in red color in Figure 6 against unit-quaternion obtained through the mapping in (157), from \( \text{SO}(3) \) in blue colors. Table 4 summarizes the comparison between the three representations with colors corresponding to those used in Figure 5, 6, and 7. In Example 1, low gain and rate of angular velocity is considered. Figure 5, 6, and 7 show accurate tracking between the mapping from \( \text{SO}(3) \) and the dynamics obtained from (73), (106), and (157), respectively.

Table 4: Representation, color notation and related mapping - Example 1

| Representation | Dynamics | Color notation | Mapping | Figure |
|---------------|----------|----------------|---------|--------|
| \( \text{SO}(3) \) | \( \dot{R} = R [\Omega]_\times \) | blue | \( R(t) \rightarrow \xi(t) \) | 5 |
| Euler angles | \( \dot{\xi} = J \Omega \) | Red | \( \int \dot{\xi} = \xi(t) \) | 5 |
| Rodriguez vector | \( \dot{\rho} = \frac{1}{2} (I_3 + [\rho]_\times + \rho \rho^\top) \Omega \) | Red | \( \int \dot{\rho} = \rho(t) \) | 6 |
| Unit-quaternion | \( \dot{Q} = \frac{1}{2} \Gamma(\Omega) Q \) | Red | \( \int \dot{Q} = Q(t) \) | 7 |
Figure 4: Angular velocity - Example 1

Figure 5: Euler Angles ($\dot{R} = [\Omega]_{\times}$, $R \rightarrow \xi$) vs ($\dot{\xi} = J\Omega$) - Example 1
Figure 6: Rodriguez vector \( \vec{R} = R \Omega \), \( R \rightarrow \rho \) vs \( (\rho = \frac{1}{2} (I_3 + [\rho]_{x} + \rho \rho^T) \Omega ) \) - Example 1

Figure 7: Unit-Quaternion \( \vec{R} = R \Omega \), \( R \rightarrow Q \) vs \( (Q = \frac{1}{2} \Gamma (\Omega) Q) \) - Example 1
Example 2:

In order to illustrate the problem of attitude parameterization through Euler angles (\(\xi\)) obtained from Euler rates (\(\dot{\xi}\)), consider the following angular velocity

\[
\Omega = \begin{bmatrix}
0.3\sin(0.8422t) \\
0.21\sin(0.3682t + \pi) \\
0.15\sin(1.4516t + \frac{\pi}{2})
\end{bmatrix} \text{ (rad/sec)}
\]  \(\text{(200)}\)

The angular velocity in (200) is faster in rate and higher in gains than (198). Let the initial attitude be given by

\[
R(0) = \begin{bmatrix}
0.6679 & -0.1808 & 0.7219 \\
0.6552 & 0.6030 & -0.4551 \\
-0.3530 & 0.7770 & 0.5213
\end{bmatrix} \in SO(3)
\]  \(\text{(201)}\)

The initial conditions of \(R(0)\) in terms of Euler angles, Rodriguez vector and unit-quaternion are listed in Table 5.

| Representation | Mapping | Numerical values |
|----------------|---------|------------------|
| Euler angles   | \(\xi : SO(3) \rightarrow \mathbb{R}^3\) | \(\xi(0) = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} 56.1428 \\ 20.6724 \\ 44.4471 \end{bmatrix} \text{ (deg)}\) |
| Rodriguez vector | \(\rho : SO(3) \rightarrow \mathbb{R}^3\) | \(\rho(0) = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} 0.4413 \\ 0.3850 \\ 0.2994 \end{bmatrix}\) |
| Unit-quaternion | \(Q : SO(3) \rightarrow S^3\) | \(Q(0) = \begin{bmatrix} q_0 \\ q \end{bmatrix} = \begin{bmatrix} 0.8355 \\ 0.3687 \\ 0.3216 \\ 0.2502 \end{bmatrix}\) |

The attitude \(R\) has been obtained from the dynamics in (27) \((\dot{R} = R[\Omega])\) given \(R(0)\) in (201) and the angular velocity in (200). The mapping of the attitude \(R\) from \(SO(3)\) to Euler angles, Rodriguez vector, and unit-quaternion is obtained from (73), (106), and (157), respectively, and depicted in Figure 9, 11, and 12, in blue colors. Euler angles obtained from Euler dynamics in (74) are plotted in red color in Figure 9 against Euler angles obtained through the mapping in (73) from \(SO(3)\) in blue colors. It can be clearly seen in Figure 9 that Euler angles obtained from Euler rate in (73) failed to give the true Euler angles obtained from the dynamics in (27). Let Euler angles \(\dot{\xi}\) obtained from Euler dynamics \(\dot{\xi}\) in (74) be mapped such that \(\xi : \mathbb{R}^3 \rightarrow R_3\). In spite of \(R_\xi \in SO(3)\) at every time instant, \(R_\xi\) remains far from the true \(R\). Additionally, Figure 10 illustrates the problem in Euler dynamics in terms of \(\|R\| - \|R_\xi\|\). Figure 10 shows high error in the difference between \(\|R\|\) and \(\|R_\xi\|\). On the other hand, the representation of Rodriguez vector and unit-quaternion obtained from the dynamics in (106), and (157), respectively, is accurate and produces the same results as the mapping of \(R\) to Rodriguez vector and unit-quaternion obtained from (27). Rodriguez vector obtained from Rodriguez vector dynamics in (132) is plotted in red color in Figure 9 against Rodriguez vector obtained through the mapping in (106) from \(SO(3)\) drawn in blue colors. Unit-quaternion obtained from unit-quaternion dynamics in (175) is plotted in red color in Figure 11 against unit-quaternion obtained through the mapping in (157), from \(SO(3)\) in blue colors. Table 6 summarizes the comparison between the three representations, while the color notation are the same as those used in the above presented plots.
Table 6: Representation, color notation and related mapping - Example 2

| Representation      | Dynamics                                  | Color notation | Mapping          | Figure |
|---------------------|-------------------------------------------|----------------|------------------|--------|
| SO(3)               | $\dot{R} = R [\Omega]_\times$            | blue           | $R(t) \rightarrow \xi(t)$ | 9      |
|                     |                                           |                | $R(t) \rightarrow \rho(t)$ | 11     |
|                     |                                           |                | $R(t) \rightarrow Q(t)$ | 12     |
| Euler angles        | $\dot{\xi} = J\Omega$                    | Red            | $\int \dot{\xi} = \xi(t)$ | 9      |
| Rodriguez vector    | $\dot{\rho} = \frac{1}{2} \left( I_3 + [\rho]_\times + \rho\rho^\top \right) \Omega$ | Red            | $\int \dot{\rho} = \rho(t)$ | 11     |
| Unit-quaternion     | $\dot{Q} = \frac{1}{2} \Gamma(\Omega) Q$ | Red            | $\int \dot{Q} = Q(t)$ | 12     |

Figure 8: Angular velocity - Example 2
Figure 9: Euler Angles ($R = R[\Omega]_\phi$, $R \rightarrow \xi$) vs ($\dot{\xi} = J\Omega$) - Example 2

Figure 10: Error in normalized Euclidean distance associated with Euler Angles - Example 2
9. Conclusion

This article presents the attitude configuration on the special orthogonal group \(\text{SO}(3)\) and summarizes four popular methods of attitude representation, namely Euler angles, angle-axis parameterization, Rodriguez vector, and unit-quaternion. The \(3 \times 3\) orthogonal matrix gives a global and unique representation of the attitude, however it does not have vector representation. Euler angles representation is the most commonly used type of attitude representation, since it can be easily visualized and understood. Nonetheless, it suffers from singularity at certain configurations and resulting in unsuccessful mapping from \(\text{SO}(3)\)
to Euler angles. Rodriguez vector and angle-axis parameterization are commonly used in the analysis of the attitude filter and control design, and they have unique representation of the attitude. Despite multiple advantages, both methods fail to represent the attitude at quite a few configurations. In spite of the fact that unit-quaternion does not have the singularity problem, it suffers from non-uniqueness. The article gives a clear and detailed mapping between different attitude parameterizations. A summary of the mapping is provided in Table 7, 8, 9, and 10. Also, the article provides some important results which could be used to achieve successful estimation and/or control process of a rigid-body rotating and/or moving in space.

Table 7: Mapping: SO (3) and other parameterizations

| $R: R^3 \rightarrow SO(3)$ | $\xi: SO(3) \rightarrow R^3$ |
|-----------------------------|-----------------------------|
| $R = \begin{bmatrix} s \sin, c = \cos, t = \tan \\ \cos t + s \phi \sin t \\ \cos t \sin t + c \phi \\ \phi \sin t \\ \cos t \cos t + c \phi \\ c \phi \sin t \end{bmatrix}$ | $\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \arctan \left( \frac{R_{(1,1)}}{R_{(1,3)}} \right) \\ \arctan \left( \frac{R_{(2,1)}}{\sqrt{R_{(1,1)}^2 + R_{(1,3)}^2}} \right) \\ \arctan \left( \frac{R_{(3,1)}}{R_{(1,1)}} \right) \end{bmatrix}$ |

| $R: R \times S^2 \rightarrow SO(3)$ | $a, u: SO(3) \rightarrow R \times S^2$ |
|-----------------------------|-----------------------------|
| $R_a (a, u) = I_3 + \sin (a) [u]_\times + (1 - \cos (\theta)) [u]^2_\times$ | $a = \cos^{-1} \left( \frac{1}{2} (\text{Tr} \{R\} - 1) \right)$ |
| $u = \frac{1}{\sin (\phi)} \text{vex} (P_a (R))$ | $u = \text{vex} \left( \left( R - I_3 \right) (R + I_3)^{-1} \right)$ |

| $R: S^3 \rightarrow SO(3)$ | $Q: SO(3) \rightarrow S^3$ |
|-----------------------------|-----------------------------|
| $R_Q (Q) = I_3 + 2q_0 [q]_\times + 2 [q]^2_\times$ | See (157), (158), (159), and (160) |
| $R_Q (Q) = (q_0^2 - \|q\|^2) I_3 + 2q_0 [q]_\times + 2qq^\top$ |

Table 8: Mapping: Angle-axis and other parameterizations

| $a, u: SO(3) \rightarrow R \times S^2$ | $R: R \times S^2 \rightarrow SO(3)$ |
|-----------------------------|-----------------------------|
| $a = \cos^{-1} \left( \frac{1}{2} (\text{Tr} \{R\} - 1) \right)$ | $R_a (a, u) = I_3 + \sin (a) [u]_\times + (1 - \cos (\theta)) [u]^2_\times$ |
| $u = \frac{1}{\sin (\alpha) \sqrt{1 + \|\rho\|^2}} \text{vex} (P_a (R))$ | $\rho = \text{vex} \left( \frac{\rho a (R)}{2(1 - \|R\|^2)} \right)$ |

| $a, u: R^3 \rightarrow R \times S^2$ | $R: R \times S^2 \rightarrow R^3$ |
|-----------------------------|-----------------------------|
| $a = 2\tan^{-1} (\|\rho\|) = 2\sin^{-1} \left( \frac{\|\rho\|}{\sqrt{1 + \|\rho\|^2}} \right)$ | $\rho = \tan \left( \frac{\theta}{2} \right) u$ |
| $u = \cot \left( \frac{\theta}{2} \right) \rho$ | $\rho = \text{vex} \left( \frac{\rho a (R)}{2(1 - \|R\|^2)} \right)$ |

| $a, u: S^3 \rightarrow R \times S^2$ | $Q: R \times S^2 \rightarrow S^3$ |
|-----------------------------|-----------------------------|
| $a = 2\cos^{-1} (q_0)$ | $Q = \begin{bmatrix} \cos (\alpha / 2) \\ u \sin (\alpha / 2) \end{bmatrix}$ |
| $u = \frac{1}{\sin (\alpha / 2)} q_0$ | $Q = \begin{bmatrix} \cos (\alpha / 2) \\ u \sin (\alpha / 2) \end{bmatrix}$ |
Table 9: Mapping: Rodriguez vector and other parameterizations

| $\rho : \text{SO} (3) \to \mathbb{R}^3$ | $R : \mathbb{R}^3 \to \text{SO} (3)$ |
|--------------------------------------|--------------------------------------|
| $\rho = \text{vex} \left( (R - I_3) (R + I_3)^{-1} \right)$ | $R_\rho (\rho) = \frac{(1-\|\rho\|^2) I_3 + 2\rho \rho^\top + 2[\rho]_\times}{1+\|\rho\|^2}$ |
| $\rho = \frac{\text{vex}(P_a(R))}{2(1-\|\rho\|^2)}$ | |
| $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ | |
| $\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \text{arctan} \left( \frac{2\rho_2 \rho_3 + 2\rho_1}{1 + \rho_1^2 - \rho_2^2 - \rho_3^2} \right) \\ \text{arctan} \left( \frac{2\rho_1^2 - 2\rho_1 \rho_3}{\sqrt{4(\rho_2 \rho_3 + \rho_1)^2 + (1 + \rho_1^2 - \rho_2^2 - \rho_3^2)^2}} \right) \\ \text{arctan} \left( \frac{2\rho_1 \rho_2 + 2\rho_3}{1 + \rho_1^2 - \rho_2^2 - \rho_3^2} \right) \end{bmatrix}$ | |
| $\rho : \mathbb{R} \times S^2 \to \mathbb{R}^3$ | $\alpha, u : \mathbb{R}^3 \to \mathbb{R} \times S^2$ |
| $\rho = \tan \left( \frac{\alpha}{2} \right) u$ | $\alpha = 2\tan^{-1}(\|\rho\|) = 2\sin^{-1} \left( \frac{\|\rho\|}{\sqrt{1+\|\rho\|^2}} \right)$ |
| $u = \cot \left( \frac{\alpha}{2} \right) \rho$ | |
| $\rho : S^3 \to \mathbb{R}^3$ | $Q : \mathbb{R}^3 \to S^3$ |
| $\rho = q/q_0$ | $q_0 = \pm \frac{1}{\sqrt{1+\|\rho\|^2}}$ |
| $q = q_0 \rho$ | |
Table 10: Mapping: Unit-quaternion and other parameterizations

| Mapping | Description | Equation |
|---------|-------------|----------|
| Q : SO(3) → S^3 | | See (157), (158), (159), and (160) |
| R : S^3 → SO(3) | | \[
R_Q(Q) = I_3 + 2q_0 [q]_+[2 [q]_x 
\]
| | | \[
R_Q(Q) = (q_0^2 - \|q\|^2) I_3 + 2q_0 [q]_x + 2qq^T 
\]
| \xi : S^3 → R^3 | | \[
\begin{bmatrix}
\phi \\
\theta \\
\psi
\end{bmatrix} = 
\begin{bmatrix}
\arctan \left( \frac{2(q_2q_3 + q_0q_1)}{1 - 2(q_1^2 + q_3^2)} \right) \\
\arcsin \left( \frac{2q_0q_2 - q_3q_1}{\sqrt{1 + \|q\|^2}} \right) \\
\arctan \left( \frac{2(q_2q_3 + q_0q_1)}{1 - 2(q_1^2 + q_3^2)} \right)
\end{bmatrix}
\]
| Q : R × S^2 → S^3 | a, u : S^3 → R × S^2 | \[
Q = \begin{bmatrix}
\cos (a/2) \\
\sin (a/2)
\end{bmatrix} \\
u = \frac{1}{\sin(a/2)} q
\]
| \rho : S^3 → R^3 | | \[
\rho = q/q_0
\]

| \rho : S^3 → R^3 | | \[
q_0 = \pm \frac{1}{\sqrt{1 + \|\rho\|^2}} \\
q = q_0 \rho
\]

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