DYNAMICS OF TIMOSHENKO SYSTEM WITH TIME-VARYING WEIGHT AND TIME-VARYING DELAY

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Abstract. This paper is concerned with the well-posedness of global solution and exponential stability to the Timoshenko system subject with time-varying weights and time-varying delay. We consider two problems: full and partially damped systems. We prove existence of global solution for both problems combining semigroup theory with the Kato’s variable norm technique. To prove exponential stability, we apply the Energy Method. For partially damped system the exponential stability is proved under assumption of equal-speed wave propagation in the transversal and angular directions. For full damped system the exponential stability is obtained without the hypothesis of equal-speed wave propagation.

1. Introduction. Time delay is the property of a physical system by which the response to an applied force is delayed in its effect, and the central question is that delays source can destabilize a system that is asymptotically stable in the absence of delays, see [5]. In fact, an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used, see references [4],[7] and [23] for details.

In this paper we investigate global existence and decay properties of solutions for Timoshenko system with time-varying weight and time-varying delay. The first case we consider is the full damped system

\[
\begin{align*}
\begin{cases}
\rho_1 \ddot{\varphi} - k(\dot{\varphi} + \psi)_x + \mu_1(t) \dot{\varphi} + \mu_2(t) \varphi(x, t - \tau_1(t)) &= 0, \\
\rho_2 \ddot{\psi} - b \psi_{xx} + k(\dot{\varphi} + \psi) + \tilde{\mu}_1(t) \dot{\psi} + \tilde{\mu}_2(t) \psi(x, t - \tau_2(t)) &= 0,
\end{cases}
\end{align*}
\]

where \(\mu_1(t), \tilde{\mu}_1(t), \mu_2(t), \tilde{\mu}_2(t)\) are time-varying weights and \(\tau_1(t), \tau_2(t)\) are time-varying delays.

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For the second case we consider the following partially damping system
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2 \psi_t(x, t - \tau_2(t)) &= 0,
\end{align*}
\]
where the weights \(\tilde{\mu}_1(t), \tilde{\mu}_2(t)\) and the delay \(\tau_2(t)\) acting only the angular direction.

Here \(\varphi = \varphi(x, t), \psi = \psi(x, t)\) model the transverse displacement of the beam and the angular direction of the filament of the beam respectively and \(\rho_1, \rho_2, k, b\) are positive real numbers. The systems are subject to the Dirichlet boundary conditions
\[
\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0, \quad t > 0,
\]
and the initial conditions
\[
\begin{align*}
\varphi(x, 0) &= \varphi_0, \quad \varphi_t(x, 0) = \varphi_1 & \text{on } [0, L], \\
\psi(x, 0) &= \psi_0, \quad \psi_t(x, 0) = \psi_1 & \text{on } [0, L], \\
\varphi_t(x, t - \tau_1(0)) &= f_{01}(x, t - \tau_1(0)) & \text{in } [0, L] \times [0, \tau_1(0)], \\
\psi_t(x, t - \tau_2(0)) &= f_{02}(x, t - \tau_2(0)) & \text{in } [0, L] \times [0, \tau_2(0)],
\end{align*}
\]
where \((x, t) \in [0, L] \times [0, \infty[,\) the initial datum \((\varphi_0, \varphi_1, \psi_0, \psi_1, f_{01}, f_{02})\) belong to a suitable Sobolev space.

We are interested in proving the exponential stability for such of each problem. From mathematical point of view, the problem (1) is very different of problem (2) because the partially damped Timoshenko system is exponentially stable if and only if the coefficients satisfy
\[
\frac{\rho_1}{\rho_2} = \frac{k}{b}.
\]

This assumption (4), so called equal-speed wave propagation, is due to existence of a second spectrum of frequencies in the context of the stabilization of dissipative models of the Timoshenko type, see for instance [18] and [20]. Regarding consequences of the second spectrum, for some authors this constitutes a non-physical scenario, see references thes [21] and [22] for more details.

To the best of our knowledge, the first contribution in literature for dissipative Timoshenko systems with a time delay was given by Said-Hoauri and Larskri [19]. More precisely, in [19] the problem
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2 \psi_t(x, t - \tau) &= 0,
\end{align*}
\]
with constant weights \(\mu_1, \mu_2\) and time delay \(\tau > 0\) was studied. Under an appropriate assumption on the weights of the two feedbacks \((\mu_1 < \mu_2)\), it was proved the well-posedness of the system and, under condition of equal-speed wave propagation (4), it was established an exponential decay result.

The result in [19] were improved by Kirane et al. [11]. There, the authors considered the problem with a time-varying delay \(\tau(t)\) on angular direction of the form
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2 \psi_t(x, t - \tau(t)) &= 0,
\end{align*}
\]
and proved the global existence and exponential stability under suitable assumptions on the delay term and assuming equal velocity of wave propagations.

Feng and Pelicer [6] considered the following nonlinear Timoshenko system with delay of the form
\[
\begin{align*}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \mu_1 \psi_t + \mu_2 \psi_t(x, t - \tau) + f(\psi) &= 0
\end{align*}
\]
and proved the global existence and exponential stability under the usual equal wave speeds assumption.

The nonlinear Timoshenko system with \( h, g \) nonlinear terms and source terms \( f \) subject to variable delay was considered by Yang et al. [24]

\[
\begin{cases}
\rho_1 \phi_{tt} - k(\phi_x + \psi)_x = h, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\phi_x + \psi) + \mu_1 \phi_t + \mu_2 \psi_t(x, t - \tau(t)) + f(\psi) = g.
\end{cases}
\]

The authors proved the well-posedness of global solution and existence of global attractor subject the waves are assumed to propagate under the same speed in the transversal and angular direction.

For waves with time-varying delay and time-varying weights we cite the recent work of Barros et al. [3] where was studied the equation given by

\[
u_t(x, t) - u_{xx}(x, t) + \mu_1(t)u_t(x, t) + \mu_2(t)u_t(x, t - \tau(t)) = 0
\]
in a bounded domain. Under proper conditions on time-varying weights \( \mu_1(t), \mu_2(t) \) and time-varying delay \( \tau(t) \), the authors proved global existence and estimative the decay rate for the energy.

This manuscript is written as follows. In section 2, we introduce some notations and preliminary results. In section 3, using the Kato variable norm technique, and under some restriction on the time-varying weights and on the time-varying delay, the system showed to be well-posed for both problems. In section 4, we present the result of exponential stability. For the first case (full damped system) we consider suitable premises about the delay and the hypothesis of equal-speed wave propagation (4). The result of exponential stability are proved by Energy Method, using suitably sophisticated estimates for multipliers to construct an appropriated Lyapunov functional.

2. Preliminaries. As in [14], for the time-varying delay, we assume

\( \tau_i(t) \in W^{2,\infty}([0, T]), \quad T > 0, \quad i = 1, 2 \)

and there exist positive constants \( \tau_{01}, \tau_{02}, \tau_{12}, d_1 \) and \( d_2 \) satisfying

\[
\begin{cases}
0 < \tau_{01} \leq \tau_1(t) \leq \tau_{11}, \quad \forall t > 0, \\
0 < \tau_{02} \leq \tau_2(t) \leq \tau_{12}, \quad \forall t > 0
\end{cases}
\]

and

\[
\begin{cases}
\tau'_1(t) \leq d_1 < 1, \quad \forall t > 0, \\
\tau'_2(t) \leq d_2 < 1, \quad \forall t > 0.
\end{cases}
\]

We start by setting the following hypotheses:

- **(H1)** \( \mu_1, \tilde{\mu}_1 : \mathbb{R}_+ \to [0, +\infty] \) are a non-increasing functions of class \( C^1(\mathbb{R}_+) \) satisfying

\[
\begin{cases}
\frac{\mu_1'(t)}{\mu_1(t)} \leq M_1, \quad 0 < a_0 \leq \mu_1(t), \quad \forall t \geq 0, \\
\frac{\tilde{\mu}_1'(t)}{\tilde{\mu}_1(t)} \leq \tilde{M}_1, \quad 0 < \tilde{a}_0 \leq \mu_1(t), \quad \forall t \geq 0,
\end{cases}
\]

where \( a_0, \tilde{a}_0, M_1 > 0 \) and \( \tilde{M}_1 > 0 \) are constants.

- **(H2)** \( \mu_2, \tilde{\mu}_2 : \mathbb{R}_+ \to \mathbb{R} \) are functions of class \( C^1(\mathbb{R}_+) \), which is not necessarily positives or monotones, such that
Then there is a constant $c$ that there are two constants $\sigma > 0$, $\tau > 0$, $\omega > 0$, where $\phi$, $\tau$, and $\omega$ are non-increasing functions and assume that there are two constants $\sigma > -1$ and $\omega > 0$ such that

$$E(t) = 0 \forall t \geq \frac{E^\sigma(0)}{\omega^{\sigma}}$$

$$E(t) \leq E(0) \left( \frac{1 + \sigma}{1 + \omega \sigma t} \right)^{\frac{1}{\sigma}} \forall t \geq 0$$

$$E(t) \leq E(0)e^{(1-\omega)t} \forall t \geq 0$$  

We introduce now two lemmas needed later.

**Lemma 2.1** (Sobolev-Poincare’s inequality). Let $q$ be a number with $2 \leq q \leq +\infty$. Then there is a constant $c_\sigma = c_\sigma([0,L],q)$ such that

$$\|\Psi\|_q \leq c_\sigma \|\Psi_x\|_2, \text{ for } \Psi \in H^1_0([0,L]).$$

**Lemma 2.2** ([8][12]). Let $E: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a non increasing function and assume that there are two constants $\sigma > -1$ and $\omega > 0$ such that

$$\int_0^{+\infty} E^{1+\sigma}(t) dt \leq \frac{1}{\omega} E^\sigma(0) E(S), 0 \leq S < +\infty.$$  

Then

$$E(t) = 0 \forall t \geq \frac{E^\sigma(0)}{\omega^{\sigma}}$$

$$E(t) \leq E(0) \left( \frac{1 + \sigma}{1 + \omega \sigma t} \right)^{\frac{1}{\sigma}} \forall t \geq 0$$

$$E(t) \leq E(0)e^{(1-\omega)t} \forall t \geq 0$$

As in Nicaise and Pignotti [15] we introduce the new variables

$$\begin{cases}
  z_1(x,\rho,t) = \varphi_t(x,t-\tau_1(t)\rho) & \text{in } [0,L][0,1][0,\infty], \\
  z_2(x,\rho,t) = \psi_t(x,t-\tau_2(t)\rho) & \text{in } [0,L][0,1][0,\infty]
\end{cases}$$

(12)

to prove the existence of a unique solution of problems (1) and (2) which satisfies

$$\tau_j(t)z_{jt}(x,\rho,t) + (1 - \tau_j(t)\rho)z_j(x,\rho,t) = 0, \text{ for } j = 1,2.$$ 

(13)

3. **Global solution.** In this section, our goal is to prove existence and uniqueness of solutions for the problems (1) and (2). We use semigroup theory and Kato’s variable norm technique.

3.1. **First case: Full damped system.** We consider the problem with frictional damping acting on transversal and angular direction.

$$\begin{cases}
  \rho_1 \varphi_{tt} - k(\varphi_x + \psi_x)x + \mu_1(t)\varphi_t + \mu_2(t)\varphi_t(x,t-\tau_1(t)) = 0, \\
  \rho_2 \psi_{tt} - b\psi_x + k(\varphi_x + \psi) + \tilde{\mu}_1(t)\psi_t + \tilde{\mu}_2(t)\psi_t(x,t-\tau_2(t)) = 0,
\end{cases}$$

(14)

where $\varphi = \varphi(x,t)$, $\psi = \psi(x,t)$, $(x,t) \in [0,L][0,\infty]$, $\mu_1, \mu_2, \tilde{\mu}_1, \tilde{\mu}_2$ are non-constant weights, $0 < \tau_i(t) (i = 1,2)$ is a non-constant time delay and $\rho_1, \rho_2, k, b$ are positive real numbers. This system is subject to the Dirichlet boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \quad t > 0$$

(15)

and to the initial conditions

$$\begin{cases}
  \varphi(x,0) = \varphi_0, \varphi_t(x,0) = \varphi_1 & \text{on } [0,L], \\
  \psi(x,0) = \psi_0, \psi_t(x,0) = \psi_1 & \text{on } [0,L], \\
  \varphi_t(x,t-\tau_1(0)) = f_{01}(x,t-\tau_1(0)) & \text{in } [0,L][0,\tau_1(0)], \\
  \psi_t(x,t-\tau_2(0)) = f_{02}(x,t-\tau_2(0)) & \text{in } [0,L][0,\tau_2(0)].
\end{cases}$$


The initial data \((\varphi_0, \varphi_1, \psi_0, \psi_1, f_{01}, f_{02})\) belong to a suitable Sobolev space.

Using the new variables (12), the problem (14) is equivalent to

\[
\begin{align*}
\rho_1 \varphi_t - k(\varphi_x + \psi)_x + \mu_1(t) \varphi_t + \mu_2(t) z_1(x, 1, t) &= 0, \\
\tau_1(t) z_1(x, \rho, t) + (1 - \tau_1(t) \rho) z_1(x, \rho, t) &= 0, \\
\rho_2 \psi_t - b \psi_{xx} + k(\varphi_x + \psi) + \mu_1(t) \psi_t + \mu_2(t) z_2(x, 1, t) &= 0, \\
\tau_2(t) z_2(x, \rho, t) + (1 - \tau_2(t) \rho) z_2(x, \rho, t) &= 0,
\end{align*}
\]

where \(x \in [0, L], \, \rho \in [0, 1] \) and \( t > 0 \). The above system subjected to the following initial and boundary conditions

\[
\begin{align*}
\varphi(0, t) &= \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0 \quad \text{on } [0, \infty], \\
\varphi(x, 0) &= \varphi_0, \, \varphi_1(x, 0) = \varphi_1 \quad \text{on } [0, L], \\
\psi(x, 0) &= \psi_0, \, \psi_1(x, 0) = \psi_1 \quad \text{on } [0, L], \\
z_1(x, \rho, 0) &= \varphi_1(x, -\tau_1(0) \rho) = f_{01}(x, -\tau_1(0) \rho) \quad \text{in } [0, L] \times [0, 1], \\
z_2(x, \rho, 0) &= \psi_1(x, -\tau_2(0) \rho) = f_{02}(x, -\tau_2(0) \rho) \quad \text{in } [0, L] \times [0, 1].
\end{align*}
\]

For the semigroup setup, we define the vector function \( U = (\varphi, \psi, \varphi_1, \psi_1, z_1, z_2)^T \).

The system (16)-(17) can be written as

\[
\begin{align*}
U_t & = \mathcal{A}(t) U, \\
U(0) & = U_0 = (\varphi_0, \psi_0, \varphi_1, \psi_1, f_{01}(\cdot, \tau(0)), f_{02}(\cdot, \tau(0)))^T,
\end{align*}
\]

where the operator \( \mathcal{A}(t) \) is defined by

\[
\mathcal{A}(t) = \begin{pmatrix}
\frac{1}{\rho_1} \left[ k(\varphi_x + \psi)_x - \mu_1(t) u - \mu_2(t) z_1(x, 1, t) \right] \\
\frac{1}{\rho_2} \left[ b \psi_{xx} - k(\varphi_x + \psi) - \mu_1(t) v - \mu_2(t) z_2(x, 1, t) \right] \\
v \\
u \\
\tau_1(t) \rho - 1 \frac{z_1(\rho, x, \rho, t)}{\tau_1(t)} \\
- \frac{z_2(\rho, x, \rho, t)}{\tau_2(t)}
\end{pmatrix}.
\]

We introduce the phase space

\[
\mathcal{H} = H_0^1([0, L])^2 \times L^2([0, L])^2 \times L^2([0, L] \times [0, 1])^2.
\]

The domain of \( \mathcal{A} \) is defined by

\[
D(\mathcal{A}(t)) = \{ (\varphi, \psi, u, v, z_1, z_2)^T \in H/u = \varphi_1 = z_1(\cdot, 0), \\
v = \psi_1 = z_2(\cdot, 0) \quad \text{in } [0, L],
\]

where \( H = (H^2([0, L]) \cap H_0^1([0, L]))^2 \times H_0^1([0, L])^2 \times L^2([0, L] ; H_0^1([0, 1]))^2. \)

Notice that the domain of the operator \( \mathcal{A}(t) \) does not depend on time \( t \), i.e.,

\[
D(\mathcal{A}(t)) = D(\mathcal{A}(0)), \quad \forall t > 0.
\]

\( \mathcal{H} \) is a Hilbert space provided with the inner product

\[
\langle U, \tilde{U} \rangle_{\mathcal{H}} = \int_0^L \rho_1 u \tilde{u} + \rho_2 v \tilde{v} + k(\varphi_x + \psi)(\varphi_x + \tilde{\psi}) + b \psi_x \tilde{\psi}_x \\
+ \sum_{j=1}^2 \xi_j(t) \tau_j(t) \int_0^1 z_j \tilde{z}_j \, d\rho \, dx,
\]

for \( U = (\varphi, \psi, u, v, z_1, z_2)^T \) and \( \tilde{U} = (\varphi, \tilde{\psi}, \tilde{u}, \tilde{v}, \tilde{z}_1, \tilde{z}_2)^T \), where

\[
\xi_1(t) = \xi_1 \mu_1(t),
\]

\[
\xi_2(t) = \xi_2 \mu_2(t).
\]
\[ \xi_2(t) = \xi_2 \hat{\mu}_1(t) \]  

(24)

are non-increasing functions of class \( C^1(\mathbb{R}_+) \) and \( \hat{\xi}_1 \) and \( \hat{\xi}_2 \) be a positive constant such that

\[
\frac{\beta_1}{\sqrt{1 - d_1}} < \hat{\xi}_1 < 2 - \frac{\beta_1}{\sqrt{1 - d_1}},\]

(25)

\[
\frac{\beta_2}{\sqrt{1 - d_2}} < \hat{\xi}_2 < 2 - \frac{\beta_2}{\sqrt{1 - d_2}},\]

(26)

A general theory for equations of type (18) has been developed using semigroup theory, see the references [9], [10] and [17] and references therein. The simplest way to prove existence and uniqueness results in to show that the triplet \( \{A, \mathcal{H}, Y\} \), with \( A = \{A(t)/t \in [0, T]\} \), for some fixed \( T > 0 \) and \( Y = A(0) \), forms a CD-systems (or constant domain system, see [9] and [10]). More precisely, the following theorem, which is due to Tosio Kato (Theorem 1.9 of [9]) gives the existence and uniqueness results and is proved in Theorem 1.9 of [9] (see also Theorem 2.13 of [10] or [1]). For convenience let states Kato’s result here.

**Theorem 3.1.** Assume that

(i) \( Y = D(A(0)) \) is dense subset of \( \mathcal{H} \),

(ii) (21) holds,

(iii) for all \( t \in [0, T] \), \( A(t) \) generates a strongly continuous semigroup on \( \mathcal{H} \) and the family \( A(t) = \{A(t)/t \in [0, T]\} \) is stable with stability constants \( C \) and \( \beta \) independent of \( t \) (i.e. the semigroup \( (S_t(s))_{s \geq 0} \) generated by \( A(t) \) satisfies \( \|S_t(s)u\|_{\mathcal{H}} \leq C e^{\beta s}\|u\|_{\mathcal{H}} \) for all \( u \in \mathcal{H} \) and \( s \geq 0 \)),

(iv) \( \partial_t A(t) \) belongs to \( L^\infty([0, T], B(Y, \mathcal{H})) \), which is the space of equivalent classes of essentially bounded, strongly measurable functions from \( [0, T] \) into the set \( B(Y, \mathcal{H}) \) of bounded operators from \( Y \) into \( \mathcal{H} \).

Then, problem (18) has a unique solution \( U \in C([0, T], Y) \cap C^1([0, T], \mathcal{H}) \) for any initial datum in \( Y \).

Using the time-dependent inner product (22) and the Theorem 3.1 we get the following result of existence and uniqueness of global solutions to the problem (18).

**Theorem 3.2** (Global solution). For any initial datum \( U_0 \in \mathcal{H} \) there exists a unique solution \( U \) satisfying

\[ U \in C([0, +\infty[, \mathcal{H}) \]

for problem (18).

Moreover, if \( U_0 \in D(A(0)) \), then

\[ U \in C([0, +\infty[, D(A(0))) \cap C^1([0, +\infty[, \mathcal{H}). \]

**Proof.** Our goal is then to check the above assumptions for problem (18). Firstly, we prove \( D(A(0)) \) is dense in \( \mathcal{H} \). The proof is the same as the one Lemma 2.2 of [16], we give it for the sake of completeness.

Let \( (\varphi, \psi, \hat{u}, \hat{v}, \hat{z}_1, \hat{z}_2)^T \) be orthogonal to all elements of \( D(A(0)) \), namely

\[
0 = \langle (\varphi, \psi, u, v, z_1, z_2)^T, (\varphi, \psi, \hat{u}, \hat{v}, \hat{z}_1, \hat{z}_2)^T \rangle_\mathcal{H}
\]

\[ = \int_0^L \left[ \rho_1 u \hat{u} + \rho_2 v \hat{v} + k(\varphi_x + \psi)(\hat{\varphi}_x + \hat{\psi}) + b \psi_x \hat{\psi}_x \right] dx \]
for all \((\hat{\varphi}, \hat{\psi}, \hat{u}, \hat{v}, \hat{z}_1, \hat{z}_2)^T \in D(A(0)).\

We first take \(\varphi = \psi = u = v = z_1 = 0\) and \(z_2 \in D([0, L] \times [0, 1])\). As \((0, 0, 0, 0, z_2)^T \in D(A(0))\), we get
\[
\int_0^L \int_0^1 z_2 \hat{z}_2 \, d\rho \, dx = 0.
\]

Since \(D([0, L] \times [0, 1])\) is dense in \(L^2([0, L] \times [0, 1])\), we deduce that \(\hat{z}_2 = 0\). In the same manner, by taking \(\varphi = \psi = u = v = z_2 = 0\) and \(z_1 \in D([0, L] \times [0, 1])\) we see that \(\hat{z}_1 = 0\).

Next we take \(\varphi = \psi = u = z_1 = z_2 = 0\) and \(v \in D([0, L])\). As \((0, 0, 0, v, 0, 0)^T \in D(A(0))\), then
\[
\int_0^L v \hat{v} \, dx = 0.
\]

Since \(D([0, L])\) is dense in \(L^2([0, L])\), we deduce that \(\hat{v} = 0\). Similarly, we find \(\hat{u} = 0\).

Now, we take \(\varphi = u = v = z_1 = z_2 = 0\) and \(\psi \in D([0, L])\). As \((0, \psi, 0, 0, 0)^T \in D(A(0))\), then we obtain
\[
k \int_0^L \psi \hat{\psi} \, dx + b \int_0^L \psi x \hat{\psi}_x \, dx = 0.
\]

Since \(D([0, L])\) is dense in \(H_0^2([0, L])\) (equipped with the inner product \((\cdot, \cdot)_{H_0^2([0, L])}\)), we deduce that \(\hat{\psi} = 0\). Similarly, we find \(\hat{\varphi} = 0\).

We consequently
\[
D(A(0)) \text{ is dense in } H. \quad (27)
\]

Secondly, we notice that
\[
\frac{\|\Phi\|_t}{\|\Phi\|_s} \leq e^{\frac{c}{2}[t-s]}, \quad \forall t, s \in [0, T],
\]
where \(\Phi = (\varphi, \psi, u, v, z_1, z_2)^T\), \(\tau_0 = \min\{\tau_{01}, \tau_{02}\}\), \(c\) is a positive constant and \(\| \cdot \|\) is the norm associated the inner product (22). For all \(t, s \in [0, T]\), we have
\[
\|\Phi\|^2_t - \|\Phi\|^2_s e^{\frac{c}{2}[t-s]} = \left(1 - e^{-\frac{c}{2}[t-s]}\right) \int_0^L \left[\rho_1 u^2 + \rho_2 v^2 + k(\rho_x + \psi)^2 + bu_x^2\right] \, dx \\
+ \sum_{j=1}^2 \left(\xi_j(t)\tau_j(t) - \xi_j(s)\tau_j(s) e^{\frac{c}{2}[t-s]}\right) \int_0^1 \int_\Omega \int_0^1 z_j^2(x, \rho) \, d\rho \, dx.
\]

It is clear that \(1 - e^{-\frac{c}{2}[t-s]} \leq 0\). Now we will prove
\[
\xi_j(t)\tau_j(t) - \xi_j(s)\tau_j(s) e^{\frac{c}{2}[t-s]} \leq 0
\]
for \(j \in \{1, 2\}\) and for some \(c > 0\). To do this, we have
\[
\tau_j(t) = \tau_j(s) + \tau'_j(r)(t-s),
\]
where \(r \in [s, t]\) and for \(j \in \{1, 2\}\).

Hence \(\xi_j\) is a non-increasing function and \(\xi_j > 0\), we get
\[
\xi_j(t)\tau_j(t) \leq \xi_j(s)\tau_j(s) + \xi_j(s)\tau'_j(r)(t-s),
\]
for \( j \in \{1, 2\} \), which implies
\[
\frac{\xi_j(t)\tau_j(t)}{\xi_j(s)\tau_j(s)} \leq 1 + \frac{|\tau_j'(r)|}{\tau_j(s)}|t - s|,
\]
for \( j \in \{1, 2\} \). Using (9) and \( \tau_j' \) is bounded, we deduce that
\[
\frac{\xi_j(t)\tau_j(t)}{\xi_j(s)\tau_j(s)} \leq 1 + \frac{C}{\tau_0}|t - s| \leq e^{\frac{C}{\tau_0}|t - s|},
\]
for \( j \in \{1, 2\} \), which proves (28) and therefore (iii) follows.

Now we calculate \( \langle A(t)U, U \rangle_t \) for a fixed \( t \). Take \( U = (\varphi, \psi, u, v, z_1, z_2)^T \in D(A(t)) \). Then
\[
\langle A(t)U, U \rangle_t = \int_0^L \left[ k(\varphi_x + \psi)_x - \mu_1(t)u - \mu_2(t)z_1(x, 1) \right] u \, dx
+ \int_0^L \left[ b\psi_{xx} - k(\varphi_x + \psi) - \tilde{\mu}_1(t)v - \tilde{\mu}_2(t)z_2(x, 1) \right] v \, dx
+ k \int_0^L (u_x + v)(\varphi_x + \psi) \, dx + b \int_0^L v_x \psi_x \, dx
- \sum_{j=1}^2 \xi_j(t) \int_0^L \int_0^1 (1 - \tau_j'(t)\rho) z_{j\rho}(x, \rho)z_j(x, \rho) \, d\rho \, dx.
\]
Integrating by parts, we obtain
\[
\langle A(t)U, U \rangle_t = -\mu_1(t) \int_0^L u^2 \, dx - \tilde{\mu}_1(t) \int_0^L v^2 \, dx
- \mu_2(t) \int_0^L z_1(x, 1)u \, dx - \tilde{\mu}_2(t) \int_0^L z_2(x, 1)v \, dx
- \sum_{j=1}^2 \frac{\xi_j(t)}{2} \int_0^L \int_0^1 (1 - \tau_j'(t)\rho) \frac{\partial}{\partial \rho} z_j^2(x, \rho) \, d\rho \, dx.
\]
Since
\[
(1 - \tau_j'(t)\rho) \frac{\partial}{\partial \rho} z_j^2(x, \rho) = \frac{\partial}{\partial \rho} \left( (1 - \tau_j'(t)\rho) z_j^2(x, \rho) \right) + \tau_j'(t)z_j^2(x, \rho),
\]
we have
\[
\int_0^1 (1 - \tau_j'(t)\rho) \frac{\partial}{\partial \rho} z_j^2(x, \rho) \, d\rho = (1 - \tau_j'(t)) z_j^2(x, 1) - z_j^2(x, 0) + \tau_j'(t) \int_0^1 z_j^2(x, \rho) \, d\rho.
\]
So we get
\[
\langle A(t)U, U \rangle_t = -\mu_1(t) \int_0^L u^2 \, dx - \tilde{\mu}_1(t) \int_0^L v^2 \, dx
- \mu_2(t) \int_0^L z_1(x, 1)u \, dx - \tilde{\mu}_2(t) \int_0^L z_2(x, 1)v \, dx
+ \sum_{j=1}^2 \frac{\xi_j(t)}{2} \int_0^L z_j^2(x, 0) \, dx - \sum_{j=1}^2 \frac{\xi_j(t)(1 - \tau_j'(t))}{2} \int_0^L z_j^2(x, 1) \, dx
- \sum_{j=1}^2 \frac{\xi_j(t)\tau_j'(t)}{2} \int_0^L \int_0^1 z_j^2(x, \rho) \, d\rho \, dx.
\]
(29)
Due to Young’s inequality, we have

\[
\begin{align*}
\mu_2(t)\int_0^L z_1(x, 1)u\,dx &\leq \frac{|\mu_2(t)|}{2\sqrt{1-d_1}} \int_0^L u^2\,dx \\
\tilde{\mu}_2(t)\int_0^L z_2(x, 1)v\,dx &\leq \frac{|\tilde{\mu}_2(t)|}{2\sqrt{1-d_2}} \int_0^L v^2\,dx + \frac{|\tilde{\mu}_2(t)|}{2}\int_0^L z_2^2(x, 1)\,dx,
\end{align*}
\]

Inserting (30) into (29), we obtain

\[
\langle A(t)U, U \rangle_t \leq -\left(\mu_1(t) - \frac{\xi_1(t)}{2} - \frac{|\mu_2(t)|}{2\sqrt{1-d_1}}\right)\int_0^L u^2\,dx \\
- \left(\tilde{\mu}_1(t) - \frac{\xi_2(t)}{2} - \frac{|\tilde{\mu}_2(t)|}{2\sqrt{1-d_2}}\right)\int_0^L v^2\,dx \\
- \left(\xi_1(t) - \frac{\xi_1(t)\tau_1'(t)}{2} - \frac{|\mu_2(t)|}{2}\sqrt{1-d_1}\right)\int_0^L z_1^2(x, 1)\,dx \\
- \left(\xi_2(t) - \frac{\xi_2(t)\tau_2'(t)}{2} - \frac{|\tilde{\mu}_2(t)|}{2}\sqrt{1-d_2}\right)\int_0^L z_2^2(x, 1)\,dx \\
+ \sum_{j=1}^2 \frac{\xi_j(t)|\tau_j'(t)|}{2\tau_j(t)} \tau_j(t) \int_0^1 \int_0^1 \sqrt{1+\tau_j'(t)^2}\,d\rho\,dx.
\]

Then, we have

\[
\langle A(t)U, U \rangle_t \leq -\mu_1(t) \left(1 - \frac{\xi_1(t)}{2} - \frac{\beta_1}{2\sqrt{1-d_1}}\right)\int_0^L u^2\,dx \\
- \tilde{\mu}_1(t) \left(1 - \frac{\xi_2(t)}{2} - \frac{\beta_2}{2\sqrt{1-d_2}}\right)\int_0^L v^2\,dx \\
- \mu_1(t) \left(\xi_1(t) - \frac{\xi_1(t)\tau_1'(t)}{2} - \frac{\beta_1}{2}\sqrt{1-d_1}\right)\int_0^L z_1^2(x, 1)\,dx \\
- \tilde{\mu}_1(t) \left(\xi_2(t) - \frac{\xi_2(t)\tau_2'(t)}{2} - \frac{\beta_2}{2}\sqrt{1-d_2}\right)\int_0^L z_2^2(x, 1)\,dx \\
+ \kappa(t)\langle U, U \rangle_t,
\]

where

\[
\kappa(t) = \sum_{j=1}^2 \frac{\sqrt{1+\tau_j'(t)^2}}{2\tau_j(t)}
\]

which implies that

\[
\langle A(t)U, U \rangle_t - \kappa(t)\langle U, U \rangle_t \leq 0
\]

which means that the operator \( \tilde{A} = A(t) - \kappa(t)I \) is dissipative.

Moreover \( \kappa' = \frac{\tau'(t)\tau''(t)}{2\tau(t)\sqrt{1+\tau'(t)^2}} - \frac{\tau'(t)\sqrt{1+\tau'(t)^2}}{2\tau(t)^2} \) is bounded on \([0, T]\) for all \( T > 0 \) (by (9), (25) and (26)) and we have
\[
\frac{d}{dt} A(t) U = \begin{pmatrix}
0 \\
0 \\
-\rho_1^{-1}[\mu_1'(t)u + \mu_2'(t)z_1(t,1)] \\
-\rho_2^{-1}[\mu_1'(t)u + \mu_2'(t)z_2(t,1)] \\
\tau_1'(t)\tau_1(t)\rho^{-1}(\tau_1'(t)\rho^{-1})z_1\rho(x,\rho) \\
\tau_2'(t)\tau_2(t)\rho^{-1}(\tau_2'(t)\rho^{-1})z_2\rho(x,\rho)
\end{pmatrix},
\]

with \( \frac{\tau_j'(t)\tau_s(t)\rho^{-1}(\tau_j'(t)\rho^{-1})}{\tau_j(t)^2} \) bounded on \([0, T]\) by (9), (25) and (26). Thus

\[
\frac{d}{dt} \dot{A}(t) \in L_\infty([0, T], B(D(A(0)), H)),
\]

where \( L_\infty([0, T], B(D(A(0)), H)) \) is the space of equivalence classes of essentially bounded, strongly measurable functions from \([0, T]\) into \( B(D(A(0)), H) \). Here \( B(D(A(0)), H) \) is the set of bounded linear operators from \( D(A(0)) \) into \( H \).

Now, we will show that \( \lambda I - A(t) \) is surjective for fixed \( t > 0 \) and \( \lambda > 0 \). For this purpose, let \( F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in H \), we seek \( U = (\varphi, \psi, u, v, z_1, z_2)^T \in D(A(t)) \) solution of

\[
(\lambda I - A(t))U = F,
\]

that is verifying following system of equations

\[
\begin{align*}
\lambda \varphi - u &= f_1, \\
\lambda \psi - v &= f_2, \\
\lambda u - \frac{1}{\rho_1^2} [k(\varphi_x + \psi)_x - \mu_1(t)u - \mu_2(t)z_1(x, 1)] &= f_3, \\
\lambda v - \frac{1}{\rho_2^2} [b\psi_{xx} - k(\varphi_x + \psi) - \mu_1(t)v - \mu_2(t)z_2(x, 1)] &= f_4, \\
\lambda z_1 + \frac{1}{\tau_1(t)} \tau_1\rho(x, \rho) &= f_5, \\
\lambda z_2 + \frac{1}{\tau_2(t)} \tau_2\rho(x, \rho) &= f_6.
\end{align*}
\]

Suppose that we have found \( \varphi \) and \( \psi \) with the appropriated regularity. Therefore, the first and second equation in (33) give

\[
\begin{align*}
u &= \lambda \varphi - f_1, \\
v &= \lambda \psi - f_2.
\end{align*}
\]

It is clear that \( u \in H^1_0([0, L]) \) and \( v \in H^1_0([0, L]) \). Furthermore, by (33) we can find \( z_j \) \((j = 1, 2)\).

Following the same approach as in [16], we obtain, by using equation for \( z_j \) in (33), we have

\[
\begin{align*}
z_1(x, \rho) &= u(x)e^{-\varphi(t, x)} + \tau_1(t)e^{-\mu_1(t)\rho} \int_0^\rho f_3(x, s) e^{\varphi(t, s)} ds, \\
z_2(x, \rho) &= v(x)e^{-\varphi(t, x)} + \tau_2(t)e^{-\mu_2(t)\rho} \int_0^\rho f_6(x, s) e^{\varphi(t, s)} ds,
\end{align*}
\]

if \( \tau_j'(t) = 0 \), where \( \varphi_j(\ell,t) = \lambda \ell \tau_j(t) \), for \( j \in \{1, 2\} \) and

\[
\begin{align*}
z_1(x, \rho) &= u(x)e^{\varphi(t, x)} + \varphi(t, x) \int_0^\rho \tau_1(t)f_4(x, s) e^{-\varphi(t, s)} ds, \\
z_2(x, \rho) &= v(x)e^{\varphi(t, x)} + \varphi(t, x) \int_0^\rho \tau_2(t)f_6(x, s) e^{-\varphi(t, s)} ds,
\end{align*}
\]

otherwise, where \( \zeta_j(\ell, t) = \lambda \frac{\tau_j(t)}{\tau_j' \rho} \ln(1 - \ell \tau_j'(t)) \), for \( j \in \{1, 2\} \), satisfying

\[
z_1(x, 0) = u(x), \quad z_2(x, 0) = v(x) \quad \text{for} \ x \in [0, L].
\]
For all \( j \in \{1, 2\} \), from (34), we obtain

\[
\begin{aligned}
z_1(x, \rho) &= \lambda \varphi(x) e^{-\vartheta_1(\rho, t)} - f_1(x, \rho) e^{-\vartheta_1(\rho, t)} + \tau_1(t) e^{-\vartheta_1(\rho, s)} \int_0^\rho S_1(x, s) e^{\vartheta_1(s, t)} ds, \\
z_2(x, \rho) &= \lambda \psi(x) e^{-\vartheta_2(\rho, t)} - f_2(x, \rho) e^{-\vartheta_2(\rho, t)} + \tau_2(t) e^{-\vartheta_2(\rho, s)} \int_0^\rho S_2(x, s) e^{\vartheta_2(s, t)} ds,
\end{aligned}
\]  
(36)

if \( \tau_j(t) = 0 \), and

\[
\begin{aligned}
z_1(x, \rho) &= \lambda \varphi(x) e^{\zeta_1(\rho, t)} - f_1(x, \rho) e^{\zeta_1(\rho, t)} + \zeta_1(t) e^{\zeta_1(\rho, s)} \int_0^\rho S_1(x, s) e^{-\zeta_1(s, t)} ds, \\
z_2(x, \rho) &= \lambda \psi(x) e^{\zeta_2(\rho, t)} - f_2(x, \rho) e^{\zeta_2(\rho, t)} + \zeta_2(t) e^{\zeta_2(\rho, s)} \int_0^\rho S_2(x, s) e^{-\zeta_2(s, t)} ds,
\end{aligned}
\]  
(37)

otherwise.

In particular, for all \( j \in \{1, 2\} \), if \( \tau'(t) = 0 \) and from (36), we have

\[
\begin{aligned}
z_1(x, 1) &= \lambda \varphi(x) e^{-\vartheta_1(1, t)} - f_1(x, 1) e^{-\vartheta_1(1, t)} + \tau_1(t) e^{-\vartheta_1(1, s)} \int_0^1 S_1(x, s) e^{\vartheta_1(s, t)} ds, \\
z_2(x, 1) &= \lambda \psi(x) e^{-\vartheta_2(1, t)} - f_2(x, 1) e^{-\vartheta_2(1, t)} + \tau_2(t) e^{-\vartheta_2(1, s)} \int_0^1 S_2(x, s) e^{\vartheta_2(s, t)} ds,
\end{aligned}
\]  
(38)

and if \( \tau'(t) \neq 0 \) and from (37), we have

\[
\begin{aligned}
z_1(x, 1) &= \lambda \varphi(x) e^{\zeta_1(1, t)} - f_1(x, 1) e^{\zeta_1(1, t)} + \zeta_1(t) e^{\zeta_1(1, s)} \int_0^1 S_1(x, s) e^{-\zeta_1(s, t)} ds, \\
z_2(x, 1) &= \lambda \psi(x) e^{\zeta_2(1, t)} - f_2(x, 1) e^{\zeta_2(1, t)} + \zeta_2(t) e^{\zeta_2(1, s)} \int_0^1 S_2(x, s) e^{-\zeta_2(s, t)} ds,
\end{aligned}
\]  
(39)

By using (33) and (34), the functions \( \varphi \) and \( \psi \) satisfying the following system

\[
\begin{aligned}
\lambda^2 \varphi - \frac{1}{\rho_2} [k(\varphi_x + \psi)_{xx} - \mu_1(t) u - \mu_2(t) z_1(x, 1)] &= f_3 + \lambda f_1, \\
\lambda^2 \psi - \frac{1}{\rho_2} [b \psi_x - k(\varphi_x + \psi) - \mu_1(t) v - \mu_2(t) z_2(x, 1)] &= f_4 + \lambda f_2.
\end{aligned}
\]  
(40)

Solving the system (40) is equivalent to finding \( (\varphi, \psi) \in (H^2([0, L]) \cap H^1_0([0, L]))^2 \) such that

\[
\begin{aligned}
\int_0^L \left[ \rho_1 \lambda^2 \varphi \eta + k(\varphi_x + \psi) \eta_x + \mu_1(t) \eta u + \mu_2(t) z_1(x, 1) \right] dx \\
= \int_0^L \rho_1 (f_3 + \lambda f_1) \eta dx, \\
\int_0^L \left[ \rho_2 \lambda^2 \psi \theta + b \psi_x \theta_x + k(\varphi_x + \psi) \theta + \mu_1(t) \eta v + \mu_2(t) z_2(x, 1) \right] dx \\
= \int_0^L \rho_2 (f_4 + \lambda f_2) \theta dx,
\end{aligned}
\]  
(41)

for all \( (\eta, \theta) \in H^1_0([0, L])^2 \).

Now we observe that solving the system (41) is equivalent to solve the problem

\[
\Upsilon((\varphi, \psi), (\eta, \theta)) = L(\eta, \theta),
\]  
(42)

where the bilinear form

\[
\Upsilon : (H^1_0([0, L]) \times H^1_0([0, L]))^2 \to \mathbb{R}
\]

and the linear form

\[
L : H^1_0([0, L]) \times H^1_0([0, L]) \to \mathbb{R}
\]

are defined by

\[
\Upsilon((\varphi, \psi)(\eta, \theta)) = \int_0^L \left[ \rho_1 \lambda^2 \varphi \eta + k(\varphi_x + \psi) (\eta_x + \theta) \right] dx + \int_0^L \left( \rho_2 \lambda^2 \psi \theta + b \psi_x \theta_x \right) dx
\]
\[ + \int_0^L \lambda \varphi (\mu_1(t) + \mu_2(t)N_0) \eta \, dx + \int_0^L \lambda \psi (\mu_1(t) + \mu_2(t)N_0) \theta \, dx \]

and

\[ L(\eta, \theta) = \int_0^L (\mu_1(t)f_1 + \mu_2(t)N_1) \eta \, dx + \int_0^L (\mu_1(t)f_2 + \mu_2(t)N_2) \theta \, dx + \int_0^L \rho_1(f_3 + \lambda f_1) \eta \, dx + \int_0^L \rho_2(f_4 + \lambda f_2) \theta \, dx, \]

where, for all \( j \in \{1, 2\}, \)

\[
\begin{cases} 
\eta, \theta & \text{if } \tau'(t) = 0, \\
\eta(1), \theta(1) & \text{if } \tau'(t) \neq 0
\end{cases}
\]

and

\[
\begin{cases} 
N_1 = -f_1(x, 1)e^{-\varphi_1(1,t)} + \tau_1(t)e^{-\varphi_1(1,t)} \int_0^1 f_5(x, s)e^{\psi_1(s,t)} \, ds, \\
N_2 = -f_2(x, 1)e^{-\varphi_2(1,t)} + \tau_2(t)e^{-\varphi_2(1,t)} \int_0^1 f_6(x, s)e^{\psi_2(s,t)} \, ds,
\end{cases}
\]

if \( \tau'(t) = 0, \) and

\[
\begin{cases} 
N_1 = -f_1(x, 1)e^{\eta_1(t)+1} + e^{\eta_1(t)+1} \int_0^1 \frac{\tau_1(t)f_3(x,s)}{1-s\tau_1(t)} e^{-\eta_1(s,t)} \, ds, \\
N_2 = -f_2(x, 1)e^{\eta_2(t)+1} + e^{\eta_2(t)+1} \int_0^1 \frac{\tau_2(t)f_4(x,s)}{1-s\tau_2(t)} e^{-\eta_2(s,t)} \, ds,
\end{cases}
\]

otherwise. It is easy to verify that \( \mathbf{Y} \) is continuous and coercive, and \( L \) is continuous. So applying the Lax-Milgram Theorem, we deduce that for all \((\eta, \theta) \in H_0^1([0, L])^2\) the problem (42) admits a unique solution

\[(\varphi, \psi) \in H_0^1([0, L])^2.\]

Applying the classical elliptic regularity, it follows from (41) that

\[(\varphi, \psi) \in H^2([0, L])^2.\]

Therefore, the operator \(\lambda I - \tilde{A}(t)\) is surjective for any \(\lambda > 0\) and \(t > 0\). Again as \(\kappa(t) > 0\), this prove that

\[\lambda I - \tilde{A}(t) = (\lambda + \kappa(t)) I - \tilde{A}(t) \text{ is surjective,} \quad (43)\]

for any \(\lambda > 0\) and \(t > 0\).

Then, (28), (31) and (43) imply that the family \(\mathbf{A} = \{\tilde{A}(t)/t \in [0, T]\}\) is a stable family of generators in \(\mathcal{H}\) with stability constants independent of \(t\), by Proposition 1.1 from [9]. Therefore, the assumptions (i) – (iv) of Theorem 3.1 are verified by (21), (27), (28), (31), (32) and (43). Thus, the problem

\[
\{ 
\tilde{U}_0 = \tilde{A}(t)\tilde{U}, \\
\tilde{U}(0) = U_0 = (\varphi_0, \psi_0, \varphi_1, \psi_1, f_{01}(\cdot, -), f_{02}(\cdot, -), \tau(0)))^T
\]

has a unique solution \(\tilde{U} \in C([0, +\infty[; D(A(0))) \cap C^1([0, +\infty[, \mathcal{H})\) for \(U_0 \in D(A(0))\).

The requested solution of (18) is then given by

\[U(t) = \int_0^t \kappa(s) \, ds \tilde{U}(t)\]

because

\[U(t) = \kappa(t) \int_0^t \kappa(s) \, ds \tilde{U}(t) + \int_0^t \kappa(s) \, ds \tilde{U}_i(t) = \int_0^t \kappa(s) \, ds \left(\kappa(t) + \tilde{A}(t)\right) \tilde{U}(t) = \tilde{A}(t) \int_0^t \kappa(s) \, ds \tilde{U}(t)\]
We introduce the phase space

\[ H = H^1_0([0, L]) \times L^2([0, L]) \times L^2([0, L]) \times L^2([0, L]) \]

which concludes the proof. \[ \square \]

### 3.2. Second case: Partially damped system

Now we consider the problem with just one frictional damping acting on angular direction.

\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \tilde{\mu}_1(t)\psi_t + \tilde{\mu}_2(t)\psi_t(x, t - \tau_2(t)) &= 0,
\end{aligned}
\]

where \( \varphi = \varphi(x, t) \), \( \psi = \psi(x, t) \), \( (x, t) \in [0, L] \times [0, \infty] \), \( \tilde{\mu}_1, \tilde{\mu}_2 \) are non-constant weights, \( 0 < \tau_2(t) \) is a non-constant time delay and \( \rho_1, \rho_2, k, b \) are positive real numbers.

This system is subject to the Dirichlet boundary conditions

\[ \varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0, \quad t > 0 \]

and to the initial conditions

\[
\begin{aligned}
\varphi(x, 0) &= \varphi_0, \quad \varphi_t(x, 0) = \varphi_1 \quad \text{on } [0, L], \\
\psi(x, 0) &= \psi_0, \quad \psi_t(x, 0) = \psi_1 \quad \text{on } [0, L], \\
\psi_t(x, t - \tau_2(0)) &= f_{02}(x, t - \tau_2(0)) \quad \text{in } [0, L] \times (0, \tau_2(0)].
\end{aligned}
\]

The initial data \( (\varphi_0, \varphi_1, \psi_0, \psi_1, f_{02}) \) belong to a suitable Sobolev space.

Using only the second equation from (12), the problem (45) is equivalent to

\[
\begin{aligned}
\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x &= 0, \\
\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \tilde{\mu}_1(t)\psi_t + \tilde{\mu}_2(t)\psi_t(x, 1, t) &= 0, \\
\tau_2(t)\psi_t(x, \rho, t) + (1 - \tau_2^2(t)\rho)\psi_t(x, \rho, t) &= 0,
\end{aligned}
\]

where \( x \in [0, L], \rho \in [0, 1] \) and \( t > 0 \). The above system subjected to the following initial and boundary conditions

\[
\begin{aligned}
\varphi(0, t) &= \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0 \quad \text{on } [0, +\infty], \\
\varphi(x, 0) &= \varphi_0, \quad \varphi_t(x, 0) = \varphi_1 \quad \text{on } [0, L], \\
\psi(x, 0) &= \psi_0, \quad \psi_t(x, 0) = \psi_1 \quad \text{on } [0, L], \\
\psi_t(x, \rho, 0) &= \psi_t(x, -\tau_2(0)\rho) = f_{02}(x, -\tau_2(0)\rho) \quad \text{in } [0, L] \times [0, 1].
\end{aligned}
\]

For the semigroup setup \( U = (\varphi, \psi, \varphi_t, \psi_t, z_2)^T \) and rewrite (47)-(48) as

\[
\begin{aligned}
U(t) &= A(t)U, \\
U(0) &= U_0 = (\varphi_0, \psi_0, \varphi_1, \psi_1, f_{02}(\cdot, -\tau(0)))^T,
\end{aligned}
\]

where the operator \( A(t) \) is defined by

\[
A(t) = \begin{pmatrix}
\varphi \\
\psi \\
u \\
v \\
z_2
\end{pmatrix} = \begin{pmatrix}
u \\
u
\end{pmatrix} \begin{pmatrix}
1 \rho \frac{(\varphi_x + \psi)_x}{1} \\
\rho^2 b\psi_{xx} - k(\varphi_x + \psi) - \tilde{\mu}_1(t)\psi_t - \tilde{\mu}_2(t)\psi_t(x, 1, t)
\end{pmatrix}.
\]

We introduce the phase space

\[ H = H^1_0([0, L]) \times L^2([0, L] \times [0, 1]) \]

and the domain of \( A \) is defined by

\[
D(A(t)) = \{ (\varphi, \psi, u, v, z_2)^T \in H/\psi = \psi_t = z_2(\cdot, 0) \text{ in } [0, L] \},
\]

where

\[ H = (H^2([0, L]) \cap H^1_0([0, L]))^2 \times (H^1_0([0, L]))^2 \times L^2([0, L]; H^1_0([0, 1])) \].
Notice that the domain of the operator $A(t)$ is independent of the time $t$, i.e.
\[ D(A(t)) = D(A(0)), \quad \forall t > 0. \]  
(52)

$\mathcal{H}$ is a Hilbert space provided with the inner product
\[ \langle U, \hat{U} \rangle_{\mathcal{H}} = \int_0^L \left[ \rho_1 u \hat{u} + \rho_2 v \hat{v} + k(\phi_x + \psi)(\hat{\phi}_x + \hat{\psi}) + b\psi_x \hat{\psi}_x + \xi_2(t) \int_0^1 z_2 \hat{z}_2 \, d\rho \right] \, dx, \]
(53)
for $U = (\phi, \psi, u, v, z_2)^T$ and $\hat{U} = (\hat{\phi}, \hat{\psi}, \hat{u}, \hat{v}, \hat{z}_2)^T$, satisfying (24) and (26).

Proceeding in a similar way to what was done for Problem (18) we prove Theorem 3.1 and then guarantee the existence and uniqueness of the solution of Problem (49).

4. Exponential stability.

4.1. First case: Full damped system. First we define the energy associated to the solution of problem (16) by the following formula
\[ E(t) = \frac{1}{2} \int_0^L \left[ \rho_1 \phi_x^2 + \rho_2 \psi_t^2 + k(\phi_x + \psi)^2 + b\psi_x^2 \right] \, dx 
+ \sum_{j=1}^2 \frac{\xi_j(t) \tau_j(t)}{2} \int_0^1 z_j(x, \rho, t) \, d\rho \, dx, \]
(54)
satisfying (23)-(26).

In the next result we will show that the energy of the system is a non-increasing function.

**Lemma 4.1.** Let $(\phi, \psi, z_1, z_2)$ be a solution to the system (16)-(17). Then the energy functional defined by (54) satisfies
\[ E'(t) \leq -\mu_1(t) \left( 1 - \frac{\xi_1}{2} - \frac{\beta_1}{2(1 - d_1)} \right) \int_0^L \phi_x^2 \, dx 
- \dot{\mu}_1(t) \left( 1 - \frac{\xi_2}{2} - \frac{\beta_2}{2(1 - d_2)} \right) \int_0^L \psi_t^2 \, dx 
- \mu_1(t) \left( \frac{\xi_1(1 - \tau_1(t))}{2} - \frac{\beta_1(1 - d_1)}{2} \right) \int_0^L z_1^2(x, 1, t) \, dx 
- \dot{\mu}_1(t) \left( \frac{\xi_2(1 - \tau_2(t))}{2} - \frac{\beta_2(1 - d_2)}{2} \right) \int_0^L z_2^2(x, 1, t) \, dx 
\leq 0. \]
(55)

**Proof.** Multiplying the first equation (16) by $\phi_x$, the third equation by $\psi_t$, integrating on $[0, L]$ and using integration by parts, we get
\[ \frac{\rho_1}{2} \frac{d}{dt} \int_0^L \phi_x^2 \, dx + k \int_0^L (\phi_x + \psi) \phi_{xt} \, dx = -\mu_1(t) \int_0^L \phi_x^2 \, dx 
- \mu_2(t) \int_0^L z_1(x, 1, t) \phi_t \, dx, \]
\[ \frac{\rho_2}{2} \frac{d}{dt} \int_0^L \psi_t^2 \, dx + b \int_0^L \psi_x \psi_{xt} \, dx + k \int_0^L (\phi_x + \psi) \psi_t \, dx = -\dot{\mu}_1(t) \int_0^L \psi_t^2 \, dx. \]
Then

\[
\frac{1}{2} \frac{d}{dt} \int_0^L \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k(\varphi_x + \psi)^2 + b\psi_x^2 \right] \, dx = -\mu_1(t) \int_0^L \varphi_t^2 \, dx \\
- \mu_2(t) \int_0^L z_1(x, t) \varphi_t \, dx \\
- \tilde{\mu}_1(t) \int_0^L \psi_t^2 \, dx \\
- \tilde{\mu}_2(t) \int_0^L z_2(x, t) \psi_t \, dx. \quad (56)
\]

Multiplying (13) by \( \xi_j(t)z_j(x, \rho, t) \) and integrate on \([0, L] \times [0, 1]\), to obtain

\[
\tau_j(t)\xi_j(t) \int_0^L \int_0^1 z_{tt}(x, \rho, t)z_j(x, \rho, t) \, d\rho \, dx \\
= -\frac{\xi_j(t)}{2} \int_0^L \int_0^1 (1 - \tau_j(t)\rho) \frac{\partial}{\partial \rho}(z_j(x, \rho, t))^2 \, d\rho \, dx.
\]

Consequently,

\[
\frac{d}{dt} \left( \frac{\xi_j(t)\tau_j(t)}{2} \int_0^L \int_0^1 z_j^2(x, \rho, t) \, d\rho \, dx \right) \\
= -\frac{\xi_j(t)}{2} \int_0^L \int_0^1 (1 - \tau_j(t)\rho) \frac{\partial}{\partial \rho}(z_j(x, \rho, t))^2 \, d\rho \, dx \\
+ \frac{\xi_j(t)}{2} \tau_j(t) \int_0^L \int_0^1 z_j^2(x, \rho, t) \, d\rho \, dx \\
= \frac{\xi_j(t)}{2} \int_0^L (z_j^2(x, 0, t) - z_j^2(x, 1, t)) \, dx \\
+ \frac{\xi_j(t)}{2} \tau_j(t) \int_0^L \int_0^1 z_j^2(x, 1, t) \, d\rho \, dx \\
+ \frac{\xi_j(t)}{2} \tau_j(t) \int_0^L \int_0^1 z_j^2(x, \rho, t) \, d\rho \, dx. \quad (57)
\]

From (54), (56) and (57) we obtain

\[
E'(t) = \frac{\xi_1(t)}{2} \int_0^L \varphi_t^2 \, dx + \frac{\xi_2(t)}{2} \int_0^L \psi_t^2 \, dx \\
- \sum_{j=1}^2 \frac{\xi_j(t)}{2} \int_0^L z_j^2(x, 1, t) \, dx + \frac{\xi_j(t)}{2} \int_0^L \tau_j(t) \int_0^L z_j^2(x, 1, t) \, dx \\
+ \sum_{j=1}^2 \frac{\xi_j(t)}{2} \tau_j(t) \int_0^L \int_0^1 z_j^2(x, \rho, t) \, d\rho \, dx - \mu_1(t) \int_0^L \varphi_t^2 \, dx - \tilde{\mu}_1(t) \int_0^L \psi_t^2 \, dx \\
- \mu_2(t) \int_0^L z_1(x, t) \varphi_t \, dx - \tilde{\mu}_2(t) \int_0^L z_2(x, t) \psi_t \, dx. \quad (58)
\]
By Young’s inequality, we have
\[
E'(t) \leq \left( \mu_1(t) - \frac{\xi_1(t)}{2} - \frac{|\mu_2(t)|}{2\sqrt{1 - d_1}} \right) \int_0^L \varphi_t^2 \, dx
- \left( \tilde{\mu}_1(t) - \frac{\xi_2(t)}{2} - \frac{|\tilde{\mu}_2(t)|}{2\sqrt{1 - d_2}} \right) \int_0^L \psi_t^2 \, dx
- \left( \frac{\xi_1(t)}{2} - \frac{\xi_1(t) \tau_1(t)}{2} - \frac{|\mu_2(t)| \sqrt{1 - d_1}}{2} \right) \int_0^L z_1^2(x, 1, t) \, dx
- \left( \frac{\xi_2(t)}{2} - \frac{\xi_2(t) \tau_2(t)}{2} - \frac{|\tilde{\mu}_2(t)| \sqrt{1 - d_2}}{2} \right) \int_0^L z_2^2(x, 1, t) \, dx
+ \sum_{j=1}^2 \frac{\xi_j(t) \tau_j(t)}{2} \int_0^L \int_0^1 z_j(x, \rho, t) \, d\rho \, dx
\leq -\mu_1(t) \left( 1 - \frac{\xi_1}{2} - \frac{\beta_1}{2\sqrt{1 - d_1}} \right) \int_0^L \varphi_t^2 \, dx
- \tilde{\mu}_1(t) \left( 1 - \frac{\xi_2}{2} - \frac{\beta_2}{2\sqrt{1 - d_2}} \right) \int_0^L \psi_t^2 \, dx
- \mu_1(t) \left( \frac{\xi_1(1 - \tau_1)}{2} - \frac{\beta_1 \sqrt{1 - d_1}}{2} \right) \int_0^L z_1^2(x, 1, t) \, dx
- \tilde{\mu}_1(t) \left( \frac{\xi_2(1 - \tau_2)}{2} - \frac{\beta_2 \sqrt{1 - d_2}}{2} \right) \int_0^L z_2^2(x, 1, t) \, dx
\leq 0.
\]

We have the following theorem.

**Theorem 4.2.** Let \( U_0 \in D(\mathcal{A}(0)) \). Assume that the hypotheses (9)-(11), (H1) and (H2) holds. Then problem (14)-(17) admits a unique solution
\[
\varphi \in C \left( [0, +\infty[; H^1_0([0, L]) \cap C^1 \left( [0, +\infty[; L^2([0, L]) \right) \right),
\psi \in C \left( [0, +\infty[; H^1_0([0, L]) \cap C^1 \left( [0, +\infty[; L^2([0, L]) \right) \right),
z_1, z_2 \in C \left( [0, +\infty[; L^2([0, L]) \cap C^1 \left( [0, +\infty[; L^2([0, L]) \right) \right).
\]

Moreover, for some positive constants \( c, \alpha \), we obtain the following decay property
\[
E(t) \leq ce^{\alpha t}, \quad \forall t \geq 0.
\]

We state and prove two lemmas that will be needed to establish the asymptotic behavior. The next result was adapted from Lemma 4.1 in [13].

**Lemma 4.3.** There exists a positive constant \( C \) such that the following inequality holds for every \((\varphi, \psi) \in H([0, L])^2\),
\[
\int_0^L (|\varphi_x|^2 + |\psi_x|^2) \, dx \leq C \int_0^L (k|\varphi| + \psi| + b|\psi_x|^2) \, dx \leq E(t).
\]

**Proof.** We will argue by contradiction. Indeed, let us suppose that (60) is not true. So, we can find a sequence \( \{(\varphi_\nu, \psi_\nu)\}_{\nu \in \mathbb{N}} \) in \( H^1_0([0, L])^2 \) satisfying
\[
\int_0^L (k|\varphi_x + \psi_x|^2 + b|\psi_x|^2) \, dx \leq \frac{1}{\nu}
\]
and
\[ \int_0^L (|\varphi_{xx}|^2 + |\psi_{xx}|^2) \, dx = 1. \] (62)

From (62), the sequence \( \{(\varphi_\nu, \psi_\nu)\}_{\nu \in \mathbb{N}} \) is bounded in \( H_1^0([0, L]) \).

Since the embedding \( H_1^0([0, L]) \to L^2([0, L]) \) is compact, then the sequence \( \{(\varphi_\nu, \psi_\nu\}_\nu \in \mathbb{N} \) converge strongly in \( L^2([0, L]) \).

From (61)
\[ \psi_{xx} \to 0 \quad \text{strongly in} \quad L^2([0, L]). \] (63)

Using Poincare's inequality we can conclude that
\[ \psi_\nu \to 0 \quad \text{strongly in} \quad L^2([0, L]). \] (64)

Now, from (61), we have
\[ \varphi_{xx} + \psi_\nu \to 0 \quad \text{strongly in} \quad L^2([0, L]). \] (65)

Then, from (63), we get
\[ \varphi_{xx} \to 0 \quad \text{strongly in} \quad L^2([0, L]). \] (66)

Using Poincare's inequality, we conclude that
\[ \varphi_\nu \to 0 \quad \text{strongly in} \quad L^2([0, L]), \]
contradicting (62).

\[ \square \]

**Lemma 4.4.** Let \((\varphi, \psi, z_1, z_2)\) be a solution to the problem (16)-(17). Then, the energy functional defined by (54) satisfies
\[ \int_0^L \varphi_i^2 \, dx < -\frac{1}{\sigma_1} E'(t), \] (64)
\[ \int_0^L \psi_i^2 \, dx < -\frac{1}{\sigma_2} E'(t), \] (65)
where \( \sigma_1 = a_0 \left( 1 - \frac{\xi_1}{2} - \frac{b_1}{2\sqrt{1-d_1}} \right) \) and \( \sigma_2 = a_0 \left( 1 - \frac{\xi_2}{2} - \frac{b_2}{2\sqrt{1-d_2}} \right). \)

**Proof.** From Lemma 4.1, we have that
\[ -E'(t) \geq \mu_1(t) \left( 1 - \frac{\xi_1}{2} + \frac{b_1}{2\sqrt{1-d_1}} \right) \int_0^L \varphi_i^2 \, dx \]
\[ + \tilde{\mu}_1(t) \left( 1 - \frac{\xi_2}{2} - \frac{b_2}{2\sqrt{1-d_2}} \right) \int_0^L \psi_i^2 \, dx \]
\[ + \mu_1(t) \left( \frac{\xi_1}{2} (1 - \tau_i^1(t)) + \frac{\beta_1 \sqrt{1-d_1}}{2} \right) \int_0^L z_1(x, 1, t) \, dx \]
\[ + \tilde{\mu}_1(t) \left( \frac{\xi_2}{2} (1 - \tau_i^2(t)) - \frac{\beta_2 \sqrt{1-d_2}}{2} \right) \int_0^L z_2^2(x, 1, t) \, dx \]
\[ \geq 0. \]

From (H1), we obtain
\[ 0 \leq a_0 \left( 1 - \frac{\xi_1}{2} - \frac{b_1}{2\sqrt{1-d_1}} \right) \int_0^L \varphi_i^2 \, dx \]
\[ \leq \mu_1(t) \left( 1 - \frac{\xi_1}{2} - \frac{b_1}{2\sqrt{1-d_1}} \right) \int_0^L \varphi_i^2 \, dx \]
\[ \leq -E'(t) \]
and
\[ 0 \leq \bar{a}_0 \left( 1 - \frac{\xi_2}{2} - \frac{\beta_2}{2\sqrt{1 - d_2^2}} \right) \int_0^L \psi_1^2 \, dx \]
\[ \leq \bar{\mu}_1(t) \left( 1 - \frac{\xi_2}{2} - \frac{\beta_2}{2\sqrt{1 - d_2^2}} \right) \int_0^L \psi_1^2 \, dx \]
\[ \leq -E'(t) \]
and the lemma is proved. \( \square \)

**Proof of Theorem 4.2.** From now on, we denote by \( c \) various positive constants which may be different at different occurrences.

Given \( 0 \leq S < T < \infty \) we start by multiplying the first equation of (16) by \( \varphi E^q(t) \) and then integrating over \( [S, T] \times [0, L] \), we obtain
\[ \int_S^T \int_0^L E^q(t) \varphi_1 \varphi_t \, dx \, dt = 0. \]

Notice that
\[ \varphi_1 \varphi_t = (\varphi_t \varphi)_t - \varphi_t^2. \]

Using integration by parts and the boundary conditions we know that
\[ 0 = \rho_1 \left[ E^q(t) \int_0^L \varphi_1 \varphi_t \, dx \right]_S^T - \rho_1 \int_S^T qE^{q-1}(t)E'(t) \int_0^L \varphi_1 \varphi_t \, dx \, dt \]
\[ - \int_S^T E^q(t) \int_0^L \rho_1 \varphi_t^2 \, dx \, dt + \int_S^T E^q(t) \int_0^L k(\varphi_x + \psi) \varphi_x \, dx \, dt \]
\[ + \int_S^T E^q(t) \int_0^L \varphi_1 \varphi_t \, dx \, dt + \int_S^T E^q(t) \int_0^L \mu_2(t) \varphi z_1(x, 1, t) \, dx \, dt. \]  \( (66) \)

Now multiplying the third equation of (16) by \( \psi E^q(t) \) and then integrating over \( [S, T] \times [0, L] \), we obtain
\[ \int_S^T \int_0^L E^q(t) \psi [\rho_2 \psi_t - b \psi_{xx} + k(\varphi_x + \psi) + \bar{\mu}_1(t) \psi_1 + \bar{\mu}_2(t) \varphi z_2(x, 1, t)] \, dx \, dt = 0. \]

Similarly to what was done previously, we obtain that
\[ 0 = \rho_2 \left[ E^q(t) \int_0^L \psi \psi_t \, dx \right]_S^T - \rho_2 \int_S^T qE^{q-1}(t)E'(t) \int_0^L \psi \psi_t \, dx \, dt \]
\[ - \int_S^T E^q(t) \int_0^L \rho_2 \psi_t^2 \, dx \, dt + b \int_S^T E^q(t) \int_0^L \psi_t^2 \, dx \, dt \]
\[ + \int_S^T E^q(t) \int_0^L k(\varphi_x + \psi) \psi_x \, dx \, dt + \int_S^T E^q(t) \int_0^L \bar{\mu}_1(t) \psi \psi_t \, dx \, dt \]
\[ + \int_S^T E^q(t) \int_0^L \bar{\mu}_2(t) \psi z_2(x, 1, t) \, dx \, dt. \]  \( (67) \)

Now we multiply the equation (13) by \( E^q(t)\xi_j(t)e^{-2\rho_j(t)}z_j(x, \rho, t) \) and then integrate over \( [0, L] \times [0, 1] \times [S, T] \) to see that
\[ 0 = \int_S^T \int_0^L E^q(t)\xi_j(t)e^{-2\rho_j(t)}z_j(t) (\tau_j(t)z_{jt} + (1 - \rho_j'(t))z_{j\rho}) \, d\rho \, dx \, dt \]
\[
= \frac{1}{2} \int_0^L \int_0^1 \int_S^T E^q(t)\xi_j(t)\tau_j(t)e^{-2\rho\tau_j(t)} \frac{\partial}{\partial t} z_j^2 dt \, d\rho \, dx \\
+ \frac{1}{2} \int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} (1 - \rho\tau_j(t)) \frac{\partial}{\partial \rho} z_j^2 \, d\rho \, dx \, dt.
\]
Using integration by parts and the boundary conditions we know that
\[
0 = \left[ \frac{\xi_j(t)\tau_j(t)}{2} E^q(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \right]_S^T \\
- \frac{1}{2} \int_0^T qE^{q-1}(t)E'(t)\xi_j(t)\tau_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt \\
- \frac{1}{2} \int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt \\
+ \frac{1}{2} \int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt.
\]
Using integration by parts and the boundary conditions we know that
\[
0 = \left[ \frac{\xi_j(t)\tau_j(t)}{2} E^q(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \right]_S^T \\
- \frac{1}{2} \int_0^T qE^{q-1}(t)E'(t)\xi_j(t)\tau_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt.
\]
Using the boundary conditions we know that
\[
0 = \left[ \frac{\xi_j(t)\tau_j(t)}{2} E^q(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \right]_S^T
\]
Moreover, as
\[
- \frac{1}{2} \int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} (1 - \tau_j'(t)) z_j^2(x, 1, t) \, dx \, dt \leq 0.
\]
Using the boundary conditions we know that
\[
0 = \left[ \frac{\xi_j(t)\tau_j(t)}{2} E^q(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \right]_S^T \\
- \frac{1}{2} \int_0^T qE^{q-1}(t)E'(t)\xi_j(t)\tau_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt
\]
then, from (68), (69) and (70), we have that
\[
\int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt \leq 0,
\]
then, from (68), (69) and (70), we have that
\[
\int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt
\]
then, from (68), (69) and (70), we have that
\[
\int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt
\]
then, from (68), (69) and (70), we have that
\[
\int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt
\]
then, from (68), (69) and (70), we have that
\[
\int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt
\]
then, from (68), (69) and (70), we have that
\[
\int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt
\]
then, from (68), (69) and (70), we have that
\[
\int_0^T E^q(t)\xi_j(t) \int_0^L \int_0^1 e^{-2\rho\tau_j(t)} z_j^2 \, d\rho \, dx \, dt
\]
where $\gamma_0 = 2 \min \{e^{-2\rho_{r_1}}, e^{-2\rho_{r_2}}, 1\}$.

Using the Young and Sobolev-Poincaré inequalities and Lemma 4.3, we find that

$$\left[ E^q(t) \int_0^L e^{-2\rho_{j_1}(t)} z_j^2(t) \, d\rho \, dx \right]_{T} \leq E^q(S) \int_0^L \varphi(x, S) \varphi_{i_1}(x, S) \, dx - E^q(T) \int_0^L \varphi(x, T) \varphi_{i_1}(x, T) \, dx \leq c E^{q+1}(S).$$

Similarly, we have that

$$\left[ E^q(t) \int_0^L \psi_{i_1} \, dx \right]_{T} \leq c E^{q+1}(S).$$

Now, we know that

$$- \sum_{j=1}^{2} \left[ \frac{\xi_j(t) \tau_j(t)}{2} E^q(t) \int_0^L \int_0^1 e^{-2\rho_{j_1}(t)} z_j^2 \, d\rho \, dx \right]_{T} \leq \sum_{j=1}^{2} \frac{\xi_j(S) \tau_j(S)}{2} E^q(S) \int_0^L \int_0^1 e^{-2\rho_{j_1}(t)} z_j^2(x, \rho, S) \, d\rho \, dx \leq c E^{q}(S) \sum_{j=1}^{2} \xi_j(S) \tau_j(S) \int_0^L \int_0^1 z_j^2(x, \rho, S) \, d\rho \, dx \leq c E^{q+1}(S).$$

By (55), we have

$$q \int_0^T E^{q-1}(t) E'(t) \left( \rho_1 \varphi_{i_1} \, dx + \rho_2 \psi_{i_1} \right) \, dx \, dt \leq c \int_0^T (-E'(t)) E^q(t) \, dt \leq c E^{q+1}(S).$$

Similarly,

$$q \int_0^T E^{q-1}(t) E'(t) \sum_{j=1}^{2} \frac{\xi_j(t) \tau_j(t)}{2} \int_0^L \int_0^1 e^{-2\rho_{j_1}(t)} z_j^2 \, d\rho \, dx \, dt \leq c E^{q+1}(S).$$
From Lemma 4.4, we deduce that
\[ \int_S^T E^q(t) \int_0^L \varphi_i^2 \, dx \, dt \leq -c \int_S^T E^q(t) E'(t) \, dt \leq cE^{q+1}(S). \]
Similarly,
\[ \int_S^T E^q(t) \int_0^L \varphi_i^2 \, dx \, dt \leq cE^{q+1}(S). \]
Now from (H1) and by Young’s and Poincaré’s inequalities, we obtain
\[ \left| \int_S^T E^q(t) \int_0^L \mu_1(t) \varphi_i \, dx \, dt \right| \leq \mu_1(0) \left| \int_S^T E^q(t) \int_0^L \varphi_i \, dx \, dt \right| \]
\[ \leq c(\varepsilon_1) \left| \int_S^T E^q(t) \int_0^L \varphi_i^2 \, dx \, dt \right| + \varepsilon_1 \int_S^T E^q(t) E(t) \, dt \]
\[ \leq c(\varepsilon_1)E^{q+1}(S) + \varepsilon_1 \int_S^T E^{q+1}(t) \, dt. \] (73)

Similarly,
\[ \left| \int_S^T E^q(t) \int_0^L \tilde{\mu}_1(t) \psi_i \, dx \, dt \right| \leq c(\varepsilon_2)E^{q+1}(S) + \varepsilon_2 \int_S^T E^{q+1}(t) \, dt. \] (74)
From (H2), we obtain that
\[ \left| \int_S^T E^q(t) \int_0^L \mu_2(t) \varphi z_1(x, 1, t) \, dx \, dt \right| \leq \beta_1 \mu_1(0) \left| \int_S^T E^q(t) \int_0^L \varphi z_1(x, 1, t) \, dx \, dt \right| \]
\[ \leq c(\varepsilon_3)E^{q+1}(S) + \varepsilon_3 \int_S^T E^{q+1}(t) \, dt. \] (75)

Similarly,
\[ \left| \int_S^T E^q(t) \int_0^L \tilde{\mu}_2(t) \psi z_2(x, 1, t) \, dx \, dt \right| \leq c(\varepsilon_4)E^{q+1}(S) + \varepsilon_4 \int_S^T E^{q+1}(t) \, dt. \] (76)
Finally,
\[ \frac{1}{2} \int_S^T E^q(t) \xi_1(t) \int_0^L z_1^2(x, 0, t) \, dx \, dt \leq \frac{\tilde{\xi}_1 \mu_1(0)}{2} \left| \int_S^T E^q(t) \int_0^L \varphi_i^2 \, dx \, dt \right| \]
\[ \leq c \int_S^T E^q(t)(-E'(t)) \, dt \leq cE^{q+1}(S). \]
Similarly,
\[ \frac{1}{2} \int_S^T E^q(t) \xi_2(t) \int_0^L z_2^2(x, 0, t) \, dx \, dt \leq cE^{q+1}(S). \]
Choosing \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and \( \varepsilon_4 \) small enough, we deduce from (73)-(76) that
\[ \int_S^T E^{q+1} \, dt \leq \frac{1}{\gamma} E^{q+1}(S). \]
Since $E(S) \leq E(0)$ for $S \geq 0$, we have that

$$\int_S^T E^{q+1} dt \leq \frac{1}{\gamma} E(0) E^q(S).$$

We choose $q = 0$, we conclude from Lemma 2.2 that

$$E(t) \leq E(0)e^{1-\gamma t}.$$  

This ends the proof of Theorem 4.2.

4.2. **Second case: Partially damped system.** We define the energy associated to the solution of problem (47) by the following formula

$$E(t) = \frac{1}{2} \int_0^L \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k (\varphi_x + \psi)^2 + b\psi_x^2 \right] dx + \frac{\xi_2(t)\tau_2(t)}{2} \int_0^L \int_0^1 z_2^2(x, \rho, t) d\rho dx,$$  \hspace{1cm} (77)

satisfying (24) and (26).

As was done in the first case, we will now prove that energy of the system is a non-increasing function.

Lemma 4.5. Let $(\varphi, \psi, z_2)$ be a solution to the system (47)-(48). Then the energy functional defined by (77) satisfies

$$E'(t) \leq -\bar{\mu}_1(t) \left( 1 - \frac{\xi_2}{2} \right) \frac{\beta_2}{2\sqrt{1-d_2}} \int_0^L \psi_t^2 dx$$

$$- \bar{\mu}_1(t) \left( \frac{\xi_2(1-\tau_2'(t))}{2} - \frac{\beta_2(1-d_2)}{2} \right) \int_0^L z_2^2(x, 1, t) dx \leq 0.$$

Proof. Multiplying the first equation of (47) by $\varphi_t$, the third equation by $\psi_t$, integrating on $[0, L]$ and using integration by parts, we get

$$\frac{\rho_1}{2} \frac{d}{dt} \int_0^L \varphi_t^2 dx + k \int_0^L (\varphi_x + \psi)\varphi_{xt} dx = 0,$$

$$\frac{\rho_2}{2} \frac{d}{dt} \int_0^L \psi_t^2 dx + b \int_0^L \psi_x\psi_{xt} dx + k \int_0^L (\varphi_x + \psi)\psi_t dx = -\bar{\mu}_1(t) \int_0^L \psi_t^2 dx$$

$$- \bar{\mu}_2(t) \int_0^L z_2(x, 1, t)\psi_t dx. \hspace{1cm} (78)$$

Then

$$\int_0^L \left[ \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k (\varphi_x + \psi)^2 + b\psi_x^2 \right] dx = -\bar{\mu}_1(t) \int_0^L \psi_t^2 dx$$

$$- \bar{\mu}_2(t) \int_0^L z_2(x, 1, t)\psi_t dx. \hspace{1cm} (78)$$

Multiplying the third equation of (47) by $\xi_2(t)\tau_2(t)$ and integrate on $[0, L] \times [0, 1]$, from (57), to $j = 2$ we get

$$\frac{d}{dt} \left( \frac{\xi_2(t)\tau_2(t)}{2} \int_0^L \int_0^1 z_2^2(x, \rho, t) d\rho dx \right).$$
\begin{align*}
= \frac{\xi_2(t)}{2} & \int_0^L (z_2^2(x,0,t) - z_2^2(x,1,t)) \, dx \\
+ \frac{\xi_2(t)\tau_2'(t)}{2} & \int_0^L \int_0^1 z_2^2(x,1,t) \, d\rho \, dx \\
+ \frac{\xi_2(t)\tau_2(t)}{2} & \int_0^L \int_0^1 z_2^2(x,\rho,t) \, d\rho \, dx.
\end{align*}

From (77), (78) and (79) we obtain

\[
E'(t) = \frac{\xi_2(t)}{2} \int_0^L \psi_t^2 \, dx - \frac{\xi_2(t)}{2} \int_0^L z_2^2(x,1,t) \, dx
\]

\[
+ \frac{\xi_2(t)\tau_2'(t)}{2} \int_0^L \int_0^1 z_2^2(x,1,t) \, d\rho \, dx
\]

\[
+ \frac{\xi_2(t)\tau_2(t)}{2} \int_0^L \int_0^1 z_2^2(x,\rho,t) \, d\rho \, dx
\]

\[
- \bar{\mu}_1(t) \int_0^L \psi_t^2 \, dx - \bar{\mu}_2(t) \int_0^L z_2(x,1,t) \, \psi_t \, dx.
\]

By Young’s inequality, we have that

\[
E'(t) \leq - \left( \bar{\mu}_1(t) - \frac{\xi_2(t)}{2} - \frac{|\bar{\mu}_2(t)|}{2\sqrt{1 - d_2}} \right) \int_0^L \psi_t^2 \, dx
\]

\[
- \left( \frac{\xi_2(t)}{2} - \frac{\xi_2(t)\tau_2(t)}{2} - \frac{|\bar{\mu}_2(t)|}{2}\sqrt{1 - d_2} \right) \int_0^L z_2^2(x,1,t) \, dx
\]

\[
+ \frac{\xi_2(t)\tau_2(t)}{2} \int_0^L \int_0^1 z_2^2(x,\rho,t) \, d\rho \, dx
\]

\[
\leq - \bar{\mu}_1(t) \left( 1 - \frac{\xi_2(t)}{2} - \frac{\beta_2}{2\sqrt{1 - d_2}} \right) \int_0^L \psi_t^2 \, dx
\]

\[
- \bar{\mu}_1(t) \left( \frac{\xi_2(t)}{2} \tau_2(t) - \frac{\beta_2\sqrt{1 - d_2}}{2} \right) \int_0^L z_2^2(x,1,t) \, dx
\]

\[
\leq 0.
\]

Now we show that under the assumption

\[
\rho_1 = \frac{\rho_2}{k},
\]

the solution of problem (47)-(48) decays exponentially to the steady state with an exponential decay rate. Our method builds on a suitable Lyapunov functional that can be obtained by the energy method.

**Lemma 4.6.** If \((\varphi, \psi, z_2)\) is a solution of (47)-(48), then the functional \(I_1\), defined by

\[
I_1(t) = - \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) \, dx
\]

satisfies the estimate

\[
\frac{d}{dt} I_1(t) \leq - \int_0^L \left( \rho_1 \varphi_t^2 + \left( \rho_2 - \frac{C_p \bar{\mu}_1(0)^2}{2\varepsilon} \right) \psi_t^2 \right) \, dx + k \int_0^L (\varphi_x + \psi)^2 \, dx
\]

\[
+ (b + \varepsilon) \int_0^L \psi_x^2 \, dx + \frac{C_p \beta_2^{(0)} \mu_1(0)^2}{2\varepsilon} \int_0^L z_2^2(x,1,t) \, dx.
\]
for any $\varepsilon > 0$ and $C_p$ is the Poincaré’s constant.

Proof. Differentiating $I_1(t)$ and using (47), we obtain

$$\frac{d}{dt} I_1(t) = -\int_0^L \left( \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 \right) dx + k \int_0^L (\varphi_x + \psi_x)^2 dx + b \int_0^L \psi_x^2 dx$$

$$+ \tilde{\mu}_1(t) \int_0^L \psi \varphi_t dx + \tilde{\mu}_2(t) \int_0^L z_2(x, 1, t) \psi dx.$$ 

From hypothesis (H1) and (H2), we have

$$\frac{d}{dt} I_1(t) \leq -\int_0^L \left( \rho_1 \varphi_t^2 + \rho_2 \psi_t^2 \right) dx + k \int_0^L (\varphi_x + \psi_x)^2 dx + b \int_0^L \psi_x^2 dx$$

$$+ \mu_1(0) \int_0^L \psi \varphi_t dx + \beta_2 \tilde{\mu}_1(0) \int_0^L z_2(x, 1, t) \psi dx.$$ 

By the Poincaré’s and Young’s inequalities we conclude the lemma. \qed

As in [2] we introduce the multiplier $\omega$ given by the solution of the Dirichlet problem

$$- \omega_{xx} = \psi_x, \quad \omega(0) = \omega(L) = 0. \quad (84)$$

We have the following result.

Lemma 4.7. If $(\varphi, \psi, z_2)$ is a solution of (47)-(48), then the functional $I_2$, defined by

$$I_2(t) = \int_0^L (\rho_2 \psi \varphi_t + \rho_1 \varphi_t \omega) \, dx \quad (85)$$

satisfies the estimate

$$\frac{d}{dt} I_2(t) \leq (\tilde{\mu}_1(0) + kC_p - b) \int_0^L \psi_x^2 dx + \left( \rho_2 + \tilde{\mu}_1(0)^2 \right) \int_0^L \psi_t^2 dx$$

$$+ \rho_2 \int_0^L \varphi_t^2 dx + \frac{C_p \rho_1}{4 \varepsilon_1} \int_0^L z_2^2(x, 1, t) dx, \quad (86)$$

for any constants $\varepsilon_1, \varepsilon_2 > 0$ and $C_p$ is the Poincaré’s constant.

Proof. Differentiating $I_2(t)$ and using (45), we obtain

$$\frac{d}{dt} I_2(t) = \rho_2 \int_0^L \psi_t^2 dx - b \int_0^L \psi_x^2 dx - k \int_0^L \psi^2 dx - \tilde{\mu}_1(t) \int_0^L \psi \varphi_t dx$$

$$- \tilde{\mu}_2(t) \int_0^L \psi z_2(x, 1, t) dx + k \int_0^L \omega_x^2 dx + \rho_1 \int_0^L \varphi_t \omega_t dx$$

$$\leq \rho_2 \int_0^L \psi_t^2 dx - b \int_0^L \psi_x^2 dx - k \int_0^L \psi^2 dx + \tilde{\mu}_1(t) \int_0^L \psi \varphi_t dx$$

$$+ \tilde{\mu}_2(t) \int_0^L \psi z_2(x, 1, t) dx + k \int_0^L \omega_x^2 dx + \rho_1 \int_0^L \varphi_t \omega_t dx.$$ 

From hypothesis (H1) and (H2), we have

$$\frac{d}{dt} I_2(t) \leq \rho_2 \int_0^L \psi_t^2 dx - b \int_0^L \psi_x^2 dx - k \int_0^L \psi^2 dx + \tilde{\mu}_1(0) \int_0^L \psi \varphi_t dx.$$
Applying Young’s inequality, we obtain (90).

From (84), we have the inequalities

\[
\int_0^L \omega_2^2 \, dx \leq \int_0^L \psi^2 \, dx \leq C_p \int_0^L \psi_\omega^2 \, dx,
\]

(87)

\[
\int_0^L \omega_1^2 \, dx \leq \int_0^L \omega_{\omega x}^2 \, dx \leq C_p \int_0^L \psi_t^2 \, dx,
\]

(88)

where \(C_p\) is the Poincaré’s constant. By using the above estimates and Poincaré’s and Young’s inequalities, we conclude the lemma.

**Lemma 4.8.** If \((\varphi, \psi, z_2)\) is a solution of (47)-(48), then the functional \(J(t)\), defined by

\[
J(t) = \rho_2 \int_0^L \psi_t \left( \varphi_x + \psi \right) \, dx + \rho_2 \int_0^L \psi_x \varphi_t \, dx
\]

(89)

satisfies the estimate

\[
\frac{d}{dt} J(t) \leq [b \varphi_x \psi_x]_{x=0}^{x=L} - \frac{k}{2} \int_0^L (\varphi_x + \psi)^2 \, dx
\]

\[
+ \left( \rho_2 + \frac{\tilde{\mu}_1(0)^2}{k} \right) \int_0^L \psi_t^2 \, dx + \frac{\beta_2^2 \tilde{\mu}_1(0)^2}{k} \int_0^L z_2^2(x,1,t) \, dx.
\]

(90)

**Proof.** Differentiating \(J(t)\), using (45) and (81), we obtain

\[
\frac{d}{dt} J(t) = [b \varphi_x \psi_x]_{x=0}^{x=L} - k \int_0^L (\varphi_x + \psi)^2 \, dx + \rho_2 \int_0^L \psi_t^2 \, dx
\]

\[
- \tilde{\mu}_1(t) \int_0^L \psi_t (\varphi_x + \psi) \, dx - \tilde{\mu}_2(t) \int_0^L z_2(x,1,t) (\varphi_x + \psi) \, dx
\]

\[
\leq [b \varphi_x \psi_x]_{x=0}^{x=L} - k \int_0^L (\varphi_x + \psi)^2 \, dx + \rho_2 \int_0^L \psi_t^2 \, dx
\]

\[
+ \left| \tilde{\mu}_1(t) \right| \int_0^L \psi_t (\varphi_x + \psi) \, dx + \left| \tilde{\mu}_2(t) \right| \int_0^L z_2(x,1,t) (\varphi_x + \psi) \, dx.
\]

From hypothesis (H1) and (H2), we have

\[
\frac{d}{dt} J(t) \leq [b \varphi_x \psi_x]_{x=0}^{x=L} - k \int_0^L (\varphi_x + \psi)^2 \, dx + \rho_2 \int_0^L \psi_t^2 \, dx
\]

\[
+ \tilde{\mu}_1(0) \left| \int_0^L \psi_t (\varphi_x + \psi) \, dx \right| + \beta_2 \tilde{\mu}_1(0) \left| \int_0^L z_2(x,1,t) (\varphi_x + \psi) \, dx \right|.
\]

Applying Young’s inequality, we obtain (90).

We now need to deal with the boundary term \([b \varphi_x \psi_x]_{x=0}^{x=L}\). Setting

\[
q(x) = 2 - 4 \frac{x}{L}, \quad x \in [0, L],
\]

then \(|q(x)| \leq 2\). Hence, the following lemma is obtained. We have the following lemma.
Lemma 4.9. \([b \varphi_x \psi_x]_{x=0}^{x=L}\) satisfies the following estimates

\[
[b \varphi_x \psi_x]_{x=0}^{x=L} \leq -\frac{\varepsilon \rho_1}{k} \frac{d}{dt} \int_0^L q \varphi_t \varphi_x \, dx - \frac{\rho_2 b}{4\varepsilon} \frac{d}{dt} \int_0^L q \psi_1 \psi_x \, dx
\]
\[
+ \frac{3\varepsilon}{L} \int_0^L \varphi_x^2 \, dx + \left( \varepsilon L + \frac{b^2}{2} + \frac{b^2}{2\varepsilon L} \right) \int_0^L \psi_t^2 \, dx
\]
\[
+ \frac{2\rho_1 \varepsilon}{kL} \int_0^L \varphi_t^2 \, dx + \left( \frac{\rho_2 b}{2\varepsilon L} + \frac{\hat{\mu}_1(0)^2}{4\varepsilon^2} \right) \int_0^L \psi_t^2 \, dx
\]
\[
+ \frac{k^2 \varepsilon}{4} \int_0^L (\varphi_x + \psi)^2 \, dx + \frac{\beta_2^2 \hat{\mu}_1(0)^2}{4\varepsilon^2} \int_0^L \varepsilon \varphi_x^2 (x,1,t) \, dx,
\]

for any \(\varepsilon > 0\).

Proof. By using \(\varepsilon\)-Young inequality, we see that

\[
[b \varphi_x \psi_x]_{x=0}^{x=L} \leq \varepsilon \left[ \varphi_x^2 (L,t) + \varphi_x^2 (0,t) \right] + \frac{b^2}{4\varepsilon} \left[ \psi_x^2 (L,t) + \psi_x^2 (0,t) \right].
\]

Also, one verifies that

\[
\frac{d}{dt} \rho_2 b \int_0^L q \psi_t \psi_x \, dx = \frac{b^2}{2} \left[ q \psi_x^2 \right]_{x=0}^{x=L} - \frac{b^2}{2} \int_0^L q \psi_x^2 \, dx
\]
\[
- \frac{\rho_2 b}{2} \int_0^L q_x \psi_t^2 \, dx - kb \int_0^L q \psi_x (\varphi_x + \psi) \, dx
\]
\[
- b \hat{\mu}_1(t) \int_0^L q \varphi_x \psi_1 \, dx - b \hat{\mu}_2(t) \int_0^L q \psi_x \varphi_2 (x,1,t) \, dx
\]
\[
\leq -b^2 \left[ \varphi_x^2 (L,t) + \varphi_x^2 (0,t) \right] + \frac{2b^2}{L} \int_0^L \psi_x^2 \, dx + \frac{2\rho_2 b}{L} \int_0^L \psi_t^2 \, dx
\]
\[
+ \varepsilon^2 b^2 \int_0^L \varphi_x^2 \, dx + \frac{b^2}{\varepsilon} \int_0^L \psi_x^2 \, dx + 2\varepsilon b^2 \int_0^L \psi_x^2 \, dx
\]
\[
+ \frac{\hat{\mu}_1(0)^2}{\varepsilon} \int_0^L \psi_t^2 \, dx + \frac{\beta_2^2 \hat{\mu}_1(0)^2}{\varepsilon} \int_0^L \varepsilon \varphi_x^2 (x,1,t) \, dx.
\]

Similarly, we have

\[
\frac{d}{dt} \rho_1 \int_0^L q \varphi_t \varphi_x \, dx \leq -k \left[ \varphi_x^2 (L,t) + \varphi_x^2 (0,t) \right] + \frac{3k}{L} \int_0^L \psi_x^2 \, dx
\]
\[
+ kL \int_0^L \psi_x^2 \, dx + \frac{2\rho_1}{L} \int_0^L \varphi_t^2 \, dx.
\]

Combining (93) and (94), we obtain

\[
\varepsilon \left[ \varphi_x^2 (L) + \varphi_x^2 (0) \right] + \frac{b^2}{4\varepsilon} \left[ \psi_x^2 (L) + \psi_x^2 (0) \right]
\]
\[
\leq -\frac{\varepsilon \rho_1}{k} \frac{d}{dt} \int_0^L q \varphi_t \varphi_x \, dx + \frac{3\varepsilon}{L} \int_0^L \varphi_x^2 \, dx + \varepsilon L \int_0^L \psi_x^2 \, dx + \frac{2\rho_1 \varepsilon}{kL} \int_0^L \varphi_t^2 \, dx
\]
\[
- \frac{\rho_2 b}{4\varepsilon} \frac{d}{dt} \int_0^L q \psi_t \psi_x \, dx + \frac{b^2}{2\varepsilon L} \int_0^L \psi_x^2 \, dx + \frac{\rho_2 b}{2\varepsilon L} \int_0^L \psi_t^2 \, dx
\]
\[
+ \frac{k^2 \varepsilon}{4} \int_0^L (\varphi_x + \psi)^2 \, dx + \frac{b^2}{4\varepsilon^3} \int_0^L \psi_x^2 \, dx + \frac{b^2}{2} \int_0^L \psi_t^2 \, dx.
\]
satisfies the estimate

\[
\text{Theorem 4.11.}
\]

Lemma 4.10. If \((\varphi, \psi, z_2)\) is a solution of \((47)-(48)\), then the functional \(I_3\), defined as

\[
I_3(t) = \xi_2 \tau_2(t) \int_0^1 \int_0^L e^{-2\tau_2(t)\rho} z_2^2(x, \rho, t) \, d\rho \, dx
\]

satisfies the estimate

\[
\frac{d}{dt} I_3(t) \leq -2I_3(t) + \xi_2 \int_0^L \psi_t^2 \, dx.
\]

Proof. Differentiating \(I_3(t)\), we obtain

\[
\frac{d}{dt} I_3(t) = \xi_2 \tau_2'(t) \int_0^L \int_0^1 e^{-2\tau_2(t)\rho} z_2^2(x, \rho, t) \, d\rho \, dx
\]

\[
-2\rho \xi_2 \tau_2(t) \tau_2'(t) \int_0^L \int_0^1 e^{-2\tau_2(t)\rho} z_2^2(x, \rho, t) \, d\rho \, dx
\]

\[
+ 2\xi_2 \tau_2(t) \int_0^L \int_0^1 e^{-2\tau_2(t)\rho} z_2(x, \rho, t) z_2(t) \, d\rho \, dx.
\]

By using the third equation in \((47)\), we have

\[
\tau_2(t) \int_0^L \int_0^1 e^{-2\tau_2(t)\rho} z_2(x, \rho, t) z_2(t) \, d\rho \, dx
\]

\[
= \int_0^1 \int_0^L e^{-2\tau_2(t)\rho} (\tau_2'(t)\rho - 1) z_2(x, \rho, t) z_2(t) \, d\rho \, dx
\]

\[
= \frac{1}{2} \int_0^1 \int_0^L \frac{\partial}{\partial \rho} \left( e^{-2\tau_2(t)\rho} (\tau_2'(t)\rho - 1) z_2^2(x, \rho, t) \right) \, d\rho \, dx
\]

\[
+ \tau_2(t) \int_0^1 \int_0^L e^{-2\tau_2(t)\rho} (\tau_2'(t)\rho - 1) z_2^2(x, \rho, t) \, d\rho \, dx
\]

\[
- \frac{\tau_2'(t)}{2} \int_0^L \int_0^1 e^{-2\tau_2(t)\rho} z_2^2(x, \rho, t) \, d\rho \, dx.
\]

From \((98)\) and \((99)\), we get

\[
\frac{d}{dt} I_3(t) = -2\xi_2 \tau_2(t) \int_0^L \int_0^1 e^{-2\tau_2(t)\rho} z_2^2(x, \rho, t) \, d\rho \, dx + \xi_2 \int_0^L \psi_t^2 \, dx
\]

\[
- \xi_2 (1 - \tau_2'(t)) e^{-2\tau_2(t)} \int_0^L z_2^2(x, 1, t) \, dx,
\]

from which immediately follows \((97)\).

The main result in this section is the following theorem.

Theorem 4.11. Let \(U_0 \in D(\mathcal{A}(0))\). Assume that the hypotheses \((9)-(11)\), \((H1)\), \((H2)\) and \((81)\) holds. Then problem \((47)-(48)\) admits a unique solution

\[
\varphi \in C \left( [0, +\infty]; H^1_0([0, L]) \right) \cap C^1 \left( [0, +\infty]; L^2([0, L]) \right),
\]

\[
\psi \in C \left( [0, +\infty]; H^1_0([0, L]) \right) \cap C^1 \left( [0, +\infty]; L^2([0, L]) \right),
\]

\[
z_2 \in C \left( [0, +\infty]; L^2([0, L]) \right),
\]

\[
z_2 \in C \left( [0, +\infty]; L^2([0, L]) \right).
Moreover, for some positive constants \( c, \alpha \), we obtain the following decay property
\[
E(t) \leq cE(0)e^{-\alpha t}, \quad \forall t \geq 0.
\] (100)

Proof. Let us define the Lyapunov functional
\[
\mathcal{L}(t) = ME(t) + \frac{1}{8}I_1(t) + NI_2(t) + J(t) + I_3(t)
\]
\[
+ \frac{\varepsilon \rho_1}{k} \int_0^L q\varphi_1\varphi_x \, dx + \frac{\rho_2}{4\varepsilon} \int_0^L q\psi_1\psi_x \, dx,
\] (101)
where \( M, N, \varepsilon, \varepsilon_1, \varepsilon_2 \) are positive real numbers which will be chosen later. By the Lemma 4.5, exists a positive constant \( C \) such that
\[
\frac{d}{dt} E(t) \leq -C \left[ \int_0^L \psi_1^2 \, dx + \int_0^L \varphi_2^2(x, 1, t) \right],
\] (102)
with
\[
C = \min \left\{ \hat{\mu}_1(t) \left( 1 - \frac{\xi_2}{2} - \frac{\beta_2}{2\sqrt{1-d_2^2}} \right), \hat{\mu}_1(t) \left( \frac{\xi_2(1-\tau_2'(t))}{2} - \frac{\beta_2\sqrt{1-d_2^2}}{2} \right) \right\}.
\]
Using the estimates (83), (86), (90), (91), (97), (102) and the following inequality
\[
\int_0^L \varphi_2^2 \, dx \leq 2 \int_0^L (\varphi_x + \psi)^2 \, dx + 2C_p \int_0^L \psi_1^2 \, dx,
\] (103)
we obtain
\[
\frac{d}{dt} \mathcal{L}(t) \leq A_1 \int_0^L \psi_1^2 \, dx + A_2 \int_0^L \varphi_2^2(x, 1, t) \, dx + A_3 \int_0^L \varphi_1^2 \, dx
\]
\[
+ A_4 \int_0^L (\varphi_x + \psi)^2 \, dx + A_5 \int_0^L \psi_2^2 \, dx - 2I_3(t),
\] (104)
where
\[
\begin{aligned}
A_1 &= -MC - \frac{1}{8} \left( \rho_2 - \frac{C_p\hat{\mu}_1(0)}{16\varepsilon} \right) + N \left( \rho_2 + \frac{\hat{\mu}_1(0)^2}{4k} + \frac{C_p\hat{\mu}_1}{4\varepsilon^2} \right) \\
&\quad + \left( \rho_2 + \frac{\hat{\mu}_1(0)^2}{k} \right) + \left( \frac{\rho_2}{2kL} + \frac{\mu(0)^2}{4\varepsilon^2} \right) + \xi_2, \\
A_2 &= -MC + \frac{C_p\beta_2\hat{\mu}_1(0)^2}{16\varepsilon} + \frac{NC_p\beta_2\hat{\mu}_1(0)^2}{4\varepsilon} + \frac{\beta_2\hat{\mu}_1(0)^2}{4\varepsilon^2} + \beta_2^2\hat{\mu}_1(0)^2 \\
A_3 &= -\frac{6\hat{\mu}_1}{8} + N\rho_1 \varepsilon_2 + 2\frac{\rho_1\varepsilon}{kE} \\
A_4 &= -\frac{3k}{8} + \frac{6\mu_0}{4\varepsilon^2} + \frac{6\nu}{L} \\
A_5 &= \frac{1}{8}(b + \varepsilon) - N(b - \hat{\mu}_1(0)\varepsilon_1 - kC_p) + \frac{6\varepsilon C_p}{L} + \varepsilon L + \frac{b^2}{2} + \frac{b^2}{2L} + \frac{\psi^2}{4\varepsilon^2}.
\end{aligned}
\]
First, let us choose \( \varepsilon_1 \) small enough such that
\[
\varepsilon_1 \leq \frac{b - kC_p}{2\hat{\mu}_1(0)}.
\]
Then, take \( \varepsilon \) small enough such that
\[
\varepsilon \leq \min \left\{ \frac{3k}{16} \left( \frac{k^2}{4} + \frac{6}{L} \right), \frac{kL}{32} \right\}.
\]
Then, we select \( N \) large enough so that
\[
\frac{N(b - \hat{\mu}_1(0)\varepsilon_1 - kC_p)}{4} \geq \frac{1}{8}(b + \varepsilon) + \frac{6\varepsilon C_p}{L} + \varepsilon L + \frac{b^2}{2} + \frac{b^2}{2L} + \frac{\psi^2}{4\varepsilon^2}.
\]
After that, we pick \( \varepsilon_2 \) so small that
\[
\varepsilon_2 \leq \frac{1}{32N}.
\]
Finally, since $\xi_2(t)\tau_2(t)$ non-negative and limited, we choose $M$ large enough so that (104) simplifies into

$$\frac{d}{dt} \mathcal{L}(t) \leq -c_1 \int_0^L \left[ \varphi_x^2 + \psi_x^2 + (\varphi + \psi)^2 + \psi_x^2 + z_2^2(x,1,t) \right. $$

$$+ \left. \int_0^1 z_2^2(x,\rho,t) \, d\rho \right] dx$$

$$\leq -c_1 \int_0^L \left[ \varphi^2 + \psi^2 + (\varphi + \psi)^2 + \psi_x^2 + \int_0^1 z_2^2(x,\rho,t) \, d\rho \right] dx$$

for a certain positive constant $c_1$.

This implies by (77) that there exists $c_2 > 0$ such that

$$\frac{d}{dt} \mathcal{L}(t) \leq -c_2 E(t), \quad \forall t \geq 0. \quad (105)$$

At this stage, we are in position to compare $\mathcal{L}(t)$ with $E(t)$. This is given in the following result.

**Lemma 4.12.** For $M$ large enough there exist two positive constants $\gamma_1$ and $\gamma_2$ depending on $M$, $N$ and $\varepsilon$ such that

$$\gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t), \quad \forall t \geq 0.$$

**Proof.** We consider the functional

$$G(t) = \frac{1}{8} I_1(t) + NI_2(t) + J(t) + I_3(t)$$

$$+ \frac{\varepsilon \rho_1}{k} \int_0^L q\varphi \varphi_x \, dx + \frac{\rho_2 b}{4\varepsilon} \int_0^L q\psi \psi_x \, dx$$

and show that

$$|G(t)| \leq \hat{C} E(t), \quad \hat{C} > 0.$$

From (82), (85), (89) and (96), we obtain

$$|G(t)| \leq \frac{1}{8} \left| - \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) \, dx \right|$$

$$+ \left| N \int_0^L (\rho_2 \psi \psi_t + \rho_1 \varphi \varphi_t) \, dx \right|$$

$$+ \left| \rho_2 \int_0^L \psi_1 (\varphi_x + \psi) \, dx + \rho_2 \int_0^L \psi_x \varphi_t \, dx \right|$$

$$+ \left| \frac{\varepsilon \rho_1}{k} \int_0^L q\varphi \varphi_x \, dx + \frac{\rho_2 b}{4\varepsilon} \int_0^L q\psi \psi_x \, dx \right|$$

$$+ \left| \bar{\xi}_2 \tau_2(t) \int_0^1 e^{-2\tau_2(t)\rho \tau_2^2(x,\rho,t)} \, d\rho \, dx \right|.$$

By using (87), (103), the relation

$$\int_0^L \varphi^2 \, dx \leq 2C_\varphi \int_0^L (\varphi_x + \psi)^2 \, dx + 2C_\psi \int_0^L \psi_x^2 \, dx$$
and Poincaré’s and Young’s inequalities, we get
\[ |G(t)| \leq \alpha_1 \int_0^L \varphi_x^2 \, dx + \alpha_2 \int_0^L \psi_x^2 \, dx + \alpha_3 \int_0^L (\varphi_x + \psi)^2 \, dx \\
+ \alpha_4 \int_0^L \psi_x^2 \, dx + \alpha_5 \int_0^L \int_0^1 \varphi_t^2(x, \rho, t) \, d\rho \, dx, \quad (106) \]
where the positive constants \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) and \( \alpha_5 \) are determined as follows:
\[
\begin{align*}
\alpha_1 &= \frac{\rho_1}{16} + \frac{N \rho_1}{2} + \frac{\rho_2}{2} + \frac{\varepsilon \rho_1}{k}, \\
\alpha_2 &= \frac{9 \rho_2}{16} + \frac{\rho_2}{2} + \frac{\varepsilon \rho_1}{k}, \\
\alpha_3 &= \frac{\rho_1 C_p}{8} + \frac{\rho_2}{2} + \frac{\varepsilon \rho_1}{k}, \\
\alpha_4 &= \frac{\rho_1 C_p^2}{8} + \frac{\rho_2 C_p}{16} + \frac{N \rho_1 C_p}{2} + \frac{\rho_2}{2} + \frac{2 \varepsilon \rho_1 C_p}{k} + \frac{\rho_2 b}{4} \\
\alpha_5 &= \left(2 - \frac{\beta}{\sqrt{1-\beta^2}}\right) \tau_1. 
\end{align*}
\]
According to (106), we have
\[ |G(t)| \leq \hat{C} E(t), \]
for
\[ \hat{C} = 2 \max \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \} \min \{ \rho_1, \rho_2, k, b \}. \]
Therefore, we obtain
\[ |\mathcal{L}(t) - M E(t)| \leq \hat{C} E(t). \]
So, we can choose \( M \) large enough so that \( \gamma_1 = M - \hat{C} > 0, \gamma_2 = M + \hat{C} > 0. \) Then
\[ \gamma_1 E(t) \leq \mathcal{L}(t) \leq \gamma_2 E(t), \quad \forall t \geq 0. \quad (107) \]
holds true.

Now, combining (105) and (107), we obtain
\[ \frac{d}{dt} \mathcal{L}(t) \leq \alpha \mathcal{L}(t), \quad \forall t \geq 0 \]
which leads to
\[ \mathcal{L}(t) \leq \mathcal{L}(0) e^{-\alpha t}, \quad t \geq 0. \quad (108) \]
The desired result (100) follows by using estimates (107) and (108). \( \square \)

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