Fixed-Parameter Tractability of Graph Deletion Problems over Data Streams

Arijit Bishnu * Arijit Ghosh * Sudeshna Kolay † Gopinath Mishra *
Saket Saurabh ‡

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Abstract

In this work, we initiate a systematic study of parameterized streaming complexity of graph deletion problems – $\mathcal{F}$-Subgraph Deletion, $\mathcal{F}$-Minor Deletion and Cluster Vertex Deletion in the four most well-studied streaming models: the Ea (edge arrival), Dea (dynamic edge arrival), Va (vertex arrival) and Al (adjacency list) models. We also consider the streaming complexities of a collection of widely-studied problems that are special variants of $\mathcal{F}$-Subgraph Deletion, namely Feedback Vertex Set, Even Cycle Transversal, Odd Cycle Transversal, Triangle Deletion and Cluster Vertex Deletion. Except for the Triangle Deletion and Cluster Vertex Deletion problems, we show that none of the other problems have space-efficient streaming algorithms when the problems are parameterized by $k$, the solution size. In fact, we show that these problems admit $\Omega(n \log n)$ lower bounds in all the four models stated above. This improves the lower bounds given by Chitnis et al. [CCE+16] for the Ea model in SODA’16. For the Triangle Deletion and Cluster Vertex Deletion problems, the question of lower bounds for the problems parameterized by $k$ is open for the Al model. For all other models, we show an improved lower bound of $\Omega(n \log n)$ for Triangle Deletion. With regards to Cluster Vertex Deletion, we extend the results of Chitnis et al. (SODA’16) in the Ea model to the Dea and Va models.

Faced with these lower bound results, our goal is to obtain parameterized space-efficient streaming algorithms. We exploit the power of parameterization – a usual approach taken in parameterized algorithms – to study a problem with respect to parameters greater than the solution size or consider some structural parameters. We apply this approach to parameterized streaming algorithms and consider the structural parameter of vertex cover size $K$ that is always larger than the solution size $k$ for all the above problems.

Our study shows an interesting set of results. Parameterized by vertex cover size $K$, some of the problems on some of the graph streaming models do not admit space-efficient streaming algorithms, while it does so for others. The main highlights are as follows.

- In SODA’16, Chitnis et al. showed that $\mathcal{F}$-Subgraph Deletion admits a lower bound of $\Omega(n)$ in the Ea model. As a first step towards positive results for $\mathcal{F}$-Subgraph Deletion, we show that $\mathcal{F}$-Subgraph Deletion parameterized by vertex cover $K$ can be solved using $O(\Delta(\mathcal{F}) K^{\Delta(\mathcal{F})+1})$ space in the Al model, where $\Delta(\mathcal{F})$ denotes the upper bound on the degree of any vertex of the graphs in $\mathcal{F}$. To rule out the possibility of efficient algorithmic results for $\mathcal{F}$-Subgraph Deletion (parameterized by vertex cover) in Va, Dea, and Ea model, we give a set of lower bounds by using reduction from communication complexity.

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*Indian Statistical Institute, Kolkata, India
†Ben-gurion University, Israel
‡The Institute of Mathematical Sciences, HBNI, India
• If we consider special variants of $F$-Subgraph Deletion, namely Feedback Vertex Set, Even Cycle Transversal, Odd Cycle Transversal and Triangle Deletion, then it follows that when parameterized by vertex cover $K$ all these problems have efficient streaming algorithms in the Al model. However, we are able to show lower bounds for the Feedback Vertex Set, Even Cycle Transversal, Odd Cycle Transversal and Triangle Deletion problems (parameterized by $K$) in the other three models.

• Surprisingly, although Cluster Vertex Deletion is very similar to the other special variants of $F$-Subgraph Deletion considered in this paper, the problem can be solved in Dea, Ea, Va, Al models using $O(K^2 \log^4 n)$ space.

• Lastly, we show that $F$-Minor Deletion parameterized by vertex cover $K$ can be solved using $O(\Delta(F)K^{\Delta(F)+1})$ space in the Al model. However, we show lower bounds for the other three models.

1 Introduction

Nowadays, very often large graphs are represented as a sequence, or a stream, of edges. Therefore, a promising prospect to deal with problems on large graphs is the study of streaming algorithms, where a compact sketch of the subgraph whose edges have been streamed/revealed so far, is stored and computations are done on this sketch. Algorithms that can access the sequence of edges of the input graph, $p$ times in the same order, are defined as $p$-pass streaming algorithms. For simplicity, we refer to 1-pass streaming algorithms as streaming algorithms. The space used by a ($p$-pass) streaming algorithm, is defined as the streaming complexity of the algorithm. The algorithmic model to deal with streaming graphs is determined by the way the graph is revealed. Streaming algorithms for graph problems are usually studied in the following models [CDK18, McG14, MVV16]. For the upcoming discussion, $V(G)$ and $E(G)$ will denote the vertex and edge set, respectively of the graph $G$ having $n$ vertices.

(i) Edge Arrival (Ea) model: The stream consists of edges of $G$ in an arbitrary order.

(ii) Dynamic Edge Arrival (Dea) model: Each element of the input stream is a pair $(e, \text{state})$, where $e \in E(G)$ and state $\in \{\text{insert, delete}\}$ describes whether $e$ is being inserted into or deleted from the current graph.

(iii) Vertex Arrival (Va) model: The vertices of $V(G)$ are exposed in an arbitrary order. After a vertex $v$ is exposed, all the edges between $v$ and a neighbor of $v$ that has already been exposed, are revealed. This set of edges are revealed one by one in an arbitrary order.

(iv) Adjacency List (Al) model: The vertices of $V(G)$ are exposed in an arbitrary order. When a vertex $v$ is exposed, all the edges that are incident to $v$, are revealed one by one in an arbitrary order. Note that in this model each edge is exposed twice, once for each exposure of an endpoint.

The downside of streaming complexity for graph problems is that very few optimization problems are known to have efficient streaming algorithms, i.e., where the streaming complexity of the problem is $O(n \log^{O(1)} n)$ [AKL16, GVV17, KKS17]. One way of avoiding this bottleneck is to study input graph classes that have some common structural property (e.g., the solution size for all the input graphs has a common upper bound, or all the input graphs have bounded vertex cover, etc.) and not for general graphs. Recently, there has been an endeavour to study streaming complexity of graph problems under the lens of parameterized complexity [CCE+16, CCE+15, CCHM15, FK14].

The goal of parameterized complexity is to restrict the combinatorial explosion to a parameter that is hopefully much smaller than the input size. Formally, a parameterization of a problem is assigning a non-negative integer $k$ to each input instance and we say that a parameterized problem is fixed-parameter tractable (FPT) if there is an algorithm that solves the problem in time $f(k) \cdot n^{O(1)}$
time, where \( n \) is the size of the input and \( f \) is an arbitrary computable function depending on the parameter \( k \) only. Readers are requested to refer to \( \text{CFK}^{+15} \) for details on parameterized complexity. In the context of streaming complexity, we wish to study parameterized graph problems in order to design streaming algorithms with space complexity \( O(f(k) \log O(1) n) \), where \( f \) is as earlier – this is the notion of efficiency of streaming algorithms for parameterized graph problems. In other words, parameterized streaming algorithms restrict the non-logarithmic space explosion to be dependent only on the input parameter, which is often much smaller than the input instance size.

**General Notation.** The set \( \{1, \ldots, n\} \) is denoted as \([n]\). Without loss of generality, we assume that the number of vertices in the graph is \( n \), which is a power of 2. Given an integer \( i \in [n] \) and \( r \in [\log_2 n] \), \( \text{bit}(i, r) \) denotes the \( r \)-th bit in the bit expansion of \( i \). The union of two graphs \( G_1 \) and \( G_2 \) with \( V(G_1) = V(G_2) \), is \( G_1 \cup G_2 \), where \( V(G_1 \cup G_2) = V(G_1) = V(G_2) \) and \( E(G_1 \cup G_2) = E(G_1) \cup E(G_2) \). For \( X \subseteq V(G) \), \( G \setminus X \) is the subgraph of \( G \) induced by \( V(G) \setminus X \). The degree of a vertex \( u \in V(G) \), is denoted by \( \text{deg}_G(u) \). The maximum and average degrees of the vertices in \( G \) are denoted as \( \Delta(G) \) and \( \Delta_{\text{av}}(G) \), respectively. For a family of graphs \( F \), \( \Delta(F) = \max_{F \in F} \Delta(F) \). A graph \( F \) is a subgraph of a graph \( G \) if \( V(F) \subseteq V(G) \) and \( E(F) \subseteq E(G) \). A graph \( F \) is said to be a minor of a graph \( G \) if \( F \) can be obtained from \( G \) by deleting edges and vertices and by contracting edges. The neighborhood of a vertex \( v \in V(G) \) is denoted by \( N_G(v) \). For \( S \subseteq V(G) \), \( N_G(S) \) denotes the set of vertices in \( V(G) \setminus S \) that are neighbors of every vertex in \( S \). A vertex \( v \in N_G(S) \) is said to be a common neighbor of \( S \) in \( G \). The size of any minimum vertex cover in \( G \) is denoted as \( \text{VC}(G) \). A cycle on the sequence of vertices \( v_1, \ldots, v_n \), is denoted as \( C(v_1, \ldots, v_n) \). For a matching \( M \) in \( G \), the vertices in the matching are denoted by \( V(M) \). \( C_t \) denotes a cycle of length \( t \). \( P_t \) denotes a path having \( t \) vertices. A graph \( G \) is said to be a cluster graph if \( G \) is a disjoint union of cliques, that is, no three vertices of \( G \) can form an induced \( P_3 \).

**Problem Definition.** We study the streaming complexity of parameterized versions of \( F \)-**Subgraph Deletion**, \( F \)-**Minor Deletion** and \( CVD \). The parameters we consider in this paper are (i) the solution size \( k \) and (ii) the size \( K \) of the vertex cover of the input graph \( G \). In \( F \)-**Subgraph Deletion**, \( F \)-**Minor Deletion** and \( CVD \) the objective is to decide whether there exists \( X \subseteq V(G) \) of size at most \( k \) such that \( G \setminus X \) has no graphs in \( F \) as a subgraph, has no graphs in \( F \) as a minor and has no induced \( P_3 \), respectively. **Feedback Vertex set (FVS)**, **Even Cycle Transversal (ECT)**, **Odd Cycle Transversal (OCT)** and **Triangle Deletion (TD)** are special cases of \( F \)-**Subgraph Deletion** when \( F = \{2, 3, 4, 5, \ldots\} \), \( F = \{3, 4, 5, \ldots\} \), \( F = \{4, 5, \ldots\} \) and \( F = \{3\} \), respectively. FVS is also a special case of \( F \)-**Minor Deletion** when \( F = \{3\} \). **Cluster vertex deletion (CVD)** is different as we are looking for induced structures.

Chitnis et al. \( \text{CCE}^{+16} \) studied parameterized streaming algorithms for optimization problems on graphs and showed that a number of problems admit \( \Omega(n) \) lower bound in the \( \text{EA} \) model. In particular, they showed the above mentioned lower bound holds for \( F \)-**Subgraph Deletion**. To go beyond the negative results of Chitnis et al. \( \text{CCE}^{+16} \), we exploit the power of parameterization and study a problem with respect to parameters larger than the solution size or consider some structural parameters. For parameterized streaming algorithms, we consider the structural parameter of vertex cover size \( K \) that is always larger than the solution size \( k \) for all the above problems considered here. Our main conceptual contribution is to introduce the concept of structural parameterizations to the study of parameterized streaming algorithms. Apart from \( F \)-**Subgraph Deletion**, note that, we also consider \( F \)-**Minor Deletion** and **Cluster Vertex Deletion (CVD)** in this paper.

Let graph \( G \) and a non-negative integer \( k \) be the inputs to the graph problems we consider. Notice that for \( F \)-**Subgraph Deletion**, \( F \)-**Minor Deletion** and \( CVD \), \( K \geq k \). Therefore, our
motivation was to look at hard problems in the streaming setting under the natural parameter $k$ and see if they become streamable under the larger parameter $K$. Interestingly, the parameter $K$ has different effects on the above mentioned problems in the different streaming models. We show that structural parameters help to obtain efficient parameterized streaming algorithms for some of the problems, while no such effect is observed for other problems. This throws up the more general and deeper question in parameterized streaming complexity of classification of problems based on the different graph streaming models and different parameterization. We believe that our results and concepts will be instrumental in opening up the avenue for such studies in future. To contextualize our results, we first introduce the notions of hardness and streamability.

**Hardness and Streamability**  Let $\Pi$ be a parameterized graph problem that takes as input a graph on $n$ vertices and a parameter $k$. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a computable function. For a model $\mathcal{M} \in \{\text{Dea}, \text{EA}, \text{VA}, \text{AL}\}$, whenever we say that an algorithm $\mathcal{A}$ solves $\Pi$ with complexity $f(n, k)$ in model $\mathcal{M}$, we mean $\mathcal{A}$ is a randomized algorithm that for any input instance of $\Pi$ in model $\mathcal{M}$ gives the correct output with probability 2/3 and has streaming complexity $f(n, k)$.

**Definition 1.1.** A parameterized graph problem $\Pi$, that takes an $n$-vertex graph and a parameter $k$ as input, is $\Omega(f)$ $p$-pass hard in the Edge Arrival model, or in short $\Pi$ is $(\text{EA}, f, p)$-hard, if there does not exist any $p$-pass streaming algorithm of streaming complexity $O(f(n, k))$ bits that can solve $\Pi$ in model $\mathcal{M}$.

Analogously, $(\text{Dea}, f, p)$-hard, $(\text{VA}, f, p)$-hard and $(\text{AL}, f, p)$-hard are defined.

**Definition 1.2.** A graph problem $\Pi$, that takes an $n$-vertex graph and a parameter $k$ as input, is $O(f)$ $p$-pass streamable in Edge Arrival model, or in short $\Pi$ is $(\text{EA}, f, p)$-streamable if there exists a $p$-pass streaming algorithm of streaming complexity $O(f(n, k))$ words\(^1\) that can solve $\Pi$ in Edge Arrival model.

$(\text{Dea}, f, p)$-streamable, $(\text{VA}, f, p)$-streamable and $(\text{AL}, f, p)$-streamable are defined analogously. For simplicity, we refer to $(\mathcal{M}, f, 1)$-hard and $(\mathcal{M}, f, 1)$-streamable as $(\mathcal{M}, f)$-hard and $(\mathcal{M}, f)$-streamable, respectively, where $\mathcal{M} \in \{\text{Dea}, \text{EA}, \text{VA}, \text{AL}\}$.

**Definition 1.3.** Let $\mathcal{M}_1, \mathcal{M}_2 \in \{\text{Dea}, \text{EA}, \text{VA}, \text{AL}\}$ be two streaming models, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a computable function, and $p \in \mathbb{N}$.

(i) If for any parameterized graph problem $\Pi$, $(\mathcal{M}_1, f, p)$-hardness of $\Pi$ implies $(\mathcal{M}_2, f, p)$-hardness of $\Pi$, then we say $\mathcal{M}_1 \leq_h \mathcal{M}_2$.
(ii) If for any parameterized graph problem $\Pi$, $(\mathcal{M}_1, f, p)$-streamability of $\Pi$ implies $(\mathcal{M}_2, f, p)$-streamability of $\Pi$, then we say $\mathcal{M}_1 \leq_s \mathcal{M}_2$.

Recall the descriptions of Dea, Ea, Va, and Al. Now, from Definitions 1.1, 1.2, and 1.3 we have the following Observation.

**Observation 1.4.** $\text{AL} \leq_h \text{EA} \leq_h \text{DEA}$; $\text{VA} \leq_h \text{EA} \leq_h \text{DEA}$; $\text{DEA} \leq_s \text{EA} \leq_s \text{VA}$; $\text{DEA} \leq_s \text{EA} \leq_s \text{AL}$.

The above observation has the following implication. If we prove a lower (upper) bound result for some problem $\Pi$ in model $\mathcal{M}$, then it also holds in any model $\mathcal{M}'$ such that $\mathcal{M} \leq_h \mathcal{M}'$ ($\mathcal{M} \leq_s \mathcal{M}'$). For example, if we prove a lower bound result in AL or VA model, it also holds in EA and DEA model; if we prove an upper bound result in DEA model, it also holds in EA, VA and AL model. In

\(^1\)It is usual in streaming that the lower bound results are in bits, and the upper bound results are in words.
general, there is no direct connection between $\text{Al}$ and $\text{Va}$. In $\text{Al}$ and $\text{Va}$, the vertices are exposed in an arbitrary order. However, we can say the following when the vertices arrive in a fixed (known) order.

**Observation 1.5.** Let $\text{Al}'$ ($\text{Va}'$) be the restricted version of $\text{Al}$ ($\text{Va}$), where the vertices are exposed in a fixed (known) order. Then $\text{Al}' \leq_h \text{Va}'$ and $\text{Va}' \leq_s \text{Al}'$.

**Our results** All the results have been summed up in Table 1.

| Problem | Parameter | $\text{Al}$ model | $\text{Va}$ model | $\text{EA}$/DEA model |
|---------|-----------|-------------------|-------------------|----------------------|
| $\mathcal{F}$-Subgraph deletion | $k$ | $(\text{Al}, n \log n)$-hard | $(\text{Va}, n \log n)$-hard | $(\text{EA}, n \log n)$-hard |
| | | $(\text{Al}, n/p, p)$-hard | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| | $K$ | $(\text{Al}, \Delta(\mathcal{F}) \cdot K^{\Delta(\mathcal{F})+1})$-str.* | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| $\mathcal{F}$-Minor deletion | $k$ | $(\text{Al}, n \log n)$-hard | $(\text{Va}, n \log n)$-hard | $(\text{EA}, n \log n)$-hard |
| | | $(\text{Al}, n/p, p)$-hard | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| | $K$ | $(\text{Al}, \Delta(\mathcal{F}) \cdot K^{\Delta(\mathcal{F})+1})$-str.* | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| FVS, ECT, OCT | $k$ | $(\text{Al}, n \log n)$-hard | $(\text{Va}, n \log n)$-hard | $(\text{EA}, n \log n)$-hard |
| | | $(\text{Al}, n/p, p)$-hard | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| TD | $k$ | OPEN | $(\text{Va}, n \log n)$-hard | $(\text{EA}, n \log n)$-hard |
| | | | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| CVD | $k$ | OPEN | $(\text{Va}, n \log n)$-hard | $(\text{EA}, n \log n)$-hard |
| | | | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| | $K$ | $(\text{Al}, K^2)$-str.* | $(\text{Va}, n \log n)$-hard | $(\text{EA}, n/p, p)$-hard |
| | | | $(\text{Va}, n/p, p)$-hard | $(\text{EA}, n/p, p)$-hard |
| | | | $(\text{Va}, K^2 \log^2 n)$-str. | $(\text{DEA}, K^2 \log^2 n)$-str. |

Table 1: A summary of our results. “str.” means streamable. See Remark 1.

**Remark 1.** The algorithmic results marked * are deterministic. The lower bound results in $\text{Va}$ and $\text{Al}$ hold even if we know the sequence in which vertices are exposed, and the upper bound results hold even if the vertices arrive in an arbitrary order. In general, the lower bound in the $\text{Al}$ model for some problem $\Pi$ does not imply the lower bound in the $\text{Va}$ model for $\Pi$. However, our lower bound proofs in the $\text{Al}$ model hold even if we know the order in which vertices are exposed. So, the lower bound for FVS, ECT, OCT in the $\text{Al}$ model implies the lower bound in the $\text{Va}$ model. By Observation 1.4 and 1.5, we will be done by showing a subset of the algorithmic and lower bound results mentioned in the above table. The algorithmic results for CVD, $\mathcal{F}$-SUBGRAPH DELETION and $\mathcal{F}$-MINOR DELETION are proved in Theorems 2.1, 3.4 and 3.7 respectively. The other algorithmic results are mentioned in Corollary 3.5. The lower bound results for FVS, ECT, OCT are proved in Theorem 4.1. The lower bound results for TD and CVD are proved in Theorem 4.2 and Theorem 4.3 respectively.

Here are the highlights of our dichotomy results when we move from one streaming model to another and one problem to another, as presented in Table 1.

1. Chitnis et al. [CCE+16] showed that $\mathcal{F}$-SUBGRAPH DELETION is $(\text{EA}, n/p, p)$-hard. As a first algorithmic result for $\mathcal{F}$-SUBGRAPH DELETION, we show that $\mathcal{F}$-SUBGRAPH DELETION parameterized by $K$, that is vertex cover, is $(\text{Al}, \Delta(\mathcal{F}) \cdot K^{\Delta(\mathcal{F})+1})$-streamable (Theorem 3.4). We also extend our result to $\mathcal{F}$-MINOR DELETION and show that $\mathcal{F}$-MINOR DELETION parameterized by $K$ is $(\text{Al}, \Delta(\mathcal{F}) \cdot K^{\Delta(\mathcal{F})+1})$-streamable (Theorem 3.7). So, FVS, ECT, OCT and TD parameterized by $K$ are $(\text{Al}, K^2)$-streamable. (Corollary 3.5). For lower bounds, the FVS, ECT and
OCT problems parameterized by solution size \(k\) have (AL, \(n \log n\))-hardness and (AL, \(n/p, p\))-hardness (Theorem \[4.1\] (I)). Note that by Observation \[1.4\] this implies \((\mathcal{M}, n \log n)\)-hardness and \((\mathcal{M}, n/p, p)\)-hardness for all models \(\mathcal{M} \in \{\text{VA, EA, DEA}\}\). The streaming complexity for FVS, ECT and OCT were shown to be (EA, \(n/p, p\))-hard in \[CCE+16\]. Note that we give our lower bounds in the AL model. To the best of our knowledge, this is the first set of results on hardness in the AL model. Also, for 1-pass streaming complexity, we improve the lower bound to \(\Omega(n \log n)\) space complexity from \(\Omega(n)\) given in \[CCE+16\]. We further studied the problems parameterized by vertex cover size \(K\), hoping to obtain more efficient parameterized streaming complexity. Our results show that parameterized by vertex cover size \(K\), the above problems are still (VA, \(n/p, p\))-hard (Theorem \[4.1\] (III)). By Observation \[1.4\] this also implies \((\mathcal{M}, n/p, p)\)-hardness for all models \(\mathcal{M} \in \{\text{EA, DEA}\}\). However, in the AL model, the FVS, ECT, OCT parameterized by \(K\) are (AL, \(K^3\))-streamable (Corollary \[3.5\]).

(ii) The CVD problem behaves very differently. We show that the problem is (VA, \(n/p, p\))-hard (Theorem \[4.3\]). By Observation \[1.4\] this also implies \((\mathcal{M}, n/p, p)\)-hardness for all models \(\mathcal{M} \in \{\text{EA, DEA}\}\). In \[CCE+16\], the (EA, \(n/p, p\))-hardness for the problem was shown, and we are able to extend this result to the VA model. However, we were not able to resolve the parameterized streaming complexity of CVD parameterized by \(k\) in the AL model. Surprisingly, when we parameterize by the vertex cover size \(K\), CVD is (DEA, \(K^2 \log^4 n\))-streamable (Theorem \[2.1\]). By Observation \[1.4\] this also implies \((\mathcal{M}, K^2 \log^4 n)\)-streamability for \(\mathcal{M} \in \{\text{AL, VA, EA}\}\).

**Our methods.** Our hardness results are obtained from reductions from well-known problems in communication complexity. The problems we reduced from are INDEX\(_n\), DISJ\(_n\) and PERM\(_n\) (Please refer to Section \[4.1\] for details).

On the algorithmic front, our results on CVD uses a sampling technique similar to that for VERTEX COVER \[CCE+16\], but our analysis is different as it exploits the structure of a cluster graph.

Our algorithms for \(\mathcal{F}\)-SUBGRAPH DELETION and \(\mathcal{F}\)-MINOR DELETION parameterized by vertex cover size \(K\), need an algorithm for an auxiliary problem, COMMON NEIGHBOR. In this problem, the objective is to obtain a subgraph \(H\) of the input graph \(G\) such that the subgraph contains a maximal matching \(M\) of \(G\). Also, for each pair of vertices \(a, b \in V(M)\), the edge \((a, b)\) is present in \(H\) if and only if \((a, b) \in E(G)\), and enough \(2^\Delta(F)\) common neighbors of all subsets of at most \(\Delta(F)\) vertices of \(V(M)\) are retained in \(H\). Using structural properties of such a subgraph, called the common neighbor subgraph, we show that it is enough to solve \(\mathcal{F}\)-SUBGRAPH DELETION and \(\mathcal{F}\)-MINOR DELETION on the common neighbor subgraph.

**Related Work** Problems in class P have been extensively studied in streaming complexity in the last decade \[MCG14\]. Recently, there has been a lot of interest in studying streaming complexity of NP-hard problems like HITTING SET, SET COVER, MAX CUT and MAX CSP \[GVV17, KKS17\]. Some notable results include Kapralov et al.’s \[KKSV17\] resolution of 1-pass streaming complexity in the EA model of approximating MAX CUT in graphs and Assadi et al.’s \[AKL16\] resolution of the 1-pass streaming complexity in the EA model of approximating SET COVER. Assadi et al. \[AKL16\] also showed an interesting dichotomy of approximating the size of the optimal set cover and outputting an approximately optimal set cover. Fafianie and Kratsch \[FK14\] were the first to study parameterized streaming complexity of NP-hard problems like \(d\)-HITTING SET and EDGE DOMINATING SET in graphs. Chitnis et al. \[CCE+16, CCE+15, CCHM15\] over a series of papers

\[2\] By enough, we mean \(O(K')\) in this case.
developed a sampling technique to design efficient parameterized streaming algorithms for promised variants of Vertex Cover, $d$-Hitting Set problem, $b$-Matching etc. They also proved lower bounds for problems like $G$-Free Deletion, $G$-Editing, Cluster Vertex Deletion, Co-graph Vertex Deletion etc. \cite{CCE+16}.

**Organisation of the paper.** Our algorithm for CVD is described in Section 2. The algorithms for $F$-Subgraph deletion and $F$-Minor deletion are given in Section 3. The lower bound results are in Section 4. Appendix A has all formal problem definitions.

## 2 CVD in the DEA model

In this Section, we show that CVD parameterized by vertex cover size $K$, is $(\text{Dea}, K^2 \log^4 n)$-streamable. By Observation 1.4, this implies $(\mathcal{M}, K^2 \log^4 n)$-streamability for all $\mathcal{M} \in \{\text{Ea, Va, Al}\}$. The sketch of the algorithm for CVD parameterized by vertex cover size $K$ in the DEA model is in Algorithm 1. The algorithm is inspired by the streaming algorithm for Vertex Cover \cite{CCE+16}.

Before discussing the algorithm, let us discuss some terms.

A family of hash functions of the form $h : [n] \to [m]$ is said to be *pairwise independent hash family* if for a pair $i, j \in [n]$ and a randomly chosen $h$ from the family, $P(h(i) = h(j)) \leq \frac{1}{m}$. Such a hash function $h$ can be stored efficiently by using $O(\log n)$ bits \cite{MR95}.

**$\ell_o$-sampler** \cite{CF14}: Given a dynamic graph stream, an $\ell_o$-sampler does the following: with probability at least $1 - \frac{1}{n^c}$, where $c$ is a positive constant, it produces an edge uniformly at random from the set of edges that have been inserted so far but not deleted. If no such edge exists, $\ell_o$-sampler reports Null. The total space used by the sampler is $O(\log^3 n)$.

**Algorithm 1: CVD**

**Input:** A graph $G$ having $n$ vertices in the DEA model, with vertex cover size at most $K \in \mathbb{N}$, solution parameter $k \in \mathbb{N}$, such that $k \leq K$.

**Output:** A set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ is a cluster graph if such a set exists. Otherwise, the output is Null.

```plaintext
begin
    From a pairwise independent family of hash functions that map $V(G)$ to $[\beta K]$, choose $h_1, \ldots, h_{\alpha \log n}$ such that each $h_i$ is chosen uniformly and independently at random, where $\alpha$ and $\beta$ are suitable large constants.
    For each $i \in [\alpha \log n]$ and $r, s \in [\beta K]$, initiate an $\ell_o$ sampler $L_i^{r,s}$.
    for (each $(u, v)$ in the stream) do
        Irrespective of $(u, v)$ being inserted or deleted, give the respective input to the $\ell_o$-samplers $L_i^{h_i(u), h_i(v)}$ for each $i \in [\alpha \log n]$.
    For each $i \in [\alpha \log n]$, construct a subgraph $H_i$ by taking the outputs of all the $\ell_o$-samplers corresponding to the hash function $h_i$.
    Construct $H = H_1 \cup \cdots \cup H_{\alpha \log n}$.
    Run the classical FPT algorithm for CVD on the subgraph $H$ and solution size bound $k$ \cite{CFK+15}.
    if $(H$ has a solution $S$ of size at most $k)$ then
        Report $S$ as the solution to $G$.
    else
        Report Null
end
```
Theorem 2.1. CVD, parameterized by vertex cover size $K$, is (DEA, $K^2 \log^4 n$)-streamable.

Proof. Let $G$ be the input graph of the streaming algorithm and by assumption $VC(G) \leq K$. Let $h_1, \ldots, h_{\alpha \log n}$ be a set of $\alpha \log n$ pairwise independent hash functions such that each $h_i$ chosen uniformly and independently at random from a pairwise independent family of hash functions, where $h : V(G) \to [\beta K]$, $\alpha$ and $\beta$ are suitable constants. For each hash function $h_i$ and pair $r, s \in [\beta K]$, let $G^i_{r,s}$ be the subgraph of $G$ induced by the vertex set $\{v \in V(G) : h_i(v) \in \{r, s\}\}$. For the hash function $h_i$ and for each pair $r, s \in [\beta K]$, we initiate an $\ell_o$ sampler for the dynamic stream restricted to the subgraph $G^i_{r,s}$. Therefore, there is a set of $O(K^2)$ $\ell_o$-samplers $\{L^i_{r,s} : r, s \in [\beta K]\}$ corresponding to the hash function $h_i$. Now, we describe what our algorithm does when an edge is either inserted or deleted. A pseudocode of our algorithm for CVD is given in Algorithm 1. When an edge $(u, v)$ arrives in the stream, that is $(u, v)$ is inserted or deleted, we give the respective input to $L^i_{h_i(u),h_i(v)}$, where $i \in [\alpha \log n]$. At the end of the stream, for each $i \in [\alpha \log n]$, we construct a subgraph $H_i$ by taking the outputs of all the $\ell_o$-samplers corresponding to the hash function $h_i$. Let $H = H_1 \cup \cdots \cup H_{\alpha \log n}$. We run the classical FPT algorithm for CVD on the subgraph $H$ and solution size bound $k \,(\text{CFK+15})$, and report YES to CVD if and only if we get YES as answer from the above FPT algorithm on $H$. If we output YES, then we also give the solution on $H$ as our solution to $G$.

The correctness of the algorithm needs an existential structural result on $G$ (Claim 2.2) and the fact that if there exists a set $X \subset V(G)$ whose deletion turns $H$ into a cluster graph, then the same $X$ deleted from $G$ will turn it into a cluster graph with high probability (Claim 2.3).

Claim 2.2. There exists a partition $\mathcal{P}$ of $V(G)$ into $Z_1, \ldots, Z_t$, $I$ such that the subgraph induced in $G$ by each $Z_i$, is a clique with at least 2 vertices, and the subgraph induced by $I$ is the empty graph.

Proof. We start with a partition which may not have the properties of the claim and modify it iteratively such that the final partition does have all the properties of the Claim. Let us start with a partition $\mathcal{P}$ that does not satisfy the given condition. First, if there exists a part $Z_i$ having one vertex $v$, we create a new partition by adding $v$ to $I$. Next, if there exists a part $Z_i$ having at least two vertices and the subgraph induced by $Z_i$ is not a clique, then we partition $Z_i$ into smaller parts such that each smaller part is either a clique having at least two vertices or a singleton vertex. We create a new partition by replacing $Z_i$ with the smaller cliques of size at least 2 and adding all the singleton vertices to $I$. Now, let $\mathcal{P}'$ be the new partition of $V(G)$ obtained after all the above modifications. In $\mathcal{P}'$, each part except $I$ is a clique of at least two vertices. If the subgraph induced by $I$ has no edges, $\mathcal{P}'$ satisfies the properties in the Claim and we are done. Otherwise, there exists $u, v \in I$ such that $(u, v) \in E(G)$. In this case, we create a new part with $\{u, v\}$, and remove both $u$ and $v$ from $I$. Note that in the above iterative description, each vertex goes to a new part at most 2 times - (i) it can move at most once from a part $Z_i$ to a smaller part $Z_j$ that is a clique on at least 2 vertices and such a vertex will remain in the same part in all steps afterwards, or it can move at most once from a $Z_i$ to $I$, and (ii) a vertex can move at most once from $I$ to become a part of a clique $Z_i$ with at least 2 vertices and such a vertex will remain in the same part in all steps after that. Therefore, this process is finite and there is a final partition that we obtain in the end. This final partition has all the properties of the claim.

The correctness of the algorithm, as claimed in Theorem 2.1, follows from the following Claim along with the fact that $H$ is a subgraph of $G$.

Claim 2.3. Let $X \subset V(H)$ be such that $H \setminus X$ is a cluster graph. Then $G \setminus X$ is a cluster graph with high probability.
Proof. Consider a partition $\mathcal{P}$ of $V(G)$ into $Z_1, \ldots, Z_t$, as mentioned in Claim 2.2. Note that our algorithm does not need to find such a partition. The existence of $\mathcal{P}$ will be used only for the analysis purpose. Let $Z = \bigcup_{i=1}^{t} Z_i$. Note that since $VC(G) \leq K$, each $Z_i$ can have at most $K + 1$ vertices, and it must be true that $t \leq VC(G) \leq K$. In fact, we can obtain the following stronger bound that $|Z| \leq 2K$. The total number of vertices in $Z$ is at most $VC(G) + t$. Since $t \leq VC(G) \leq K$, the total number of vertices in $Z$ is at most $2K$.

A vertex $u \in V(G)$, is said to be of high degree if $deg_G(u) \geq 40K$, and low degree, otherwise. Let $V_h \subseteq V(G)$ be the set of all high degree vertices and $V_l$ be the set of low degree vertices in $G$. Let $E_l$ be the set of edges in $G$ having both the endpoints in $V_l$. It can be shown [CCE+16] that

(i) Fact-1: $|V_h| \leq K$, $E_l = O(K^2)$;

(ii) Fact-2: $E_l \subseteq E(H)$, and $deg_H(u) \geq 4K$ for each $u \in V_h$, with probability at least $1 - \frac{1}{n^{O(1)}}$.

Note that Fact-2 makes our algorithmic result for CVD probabilistic.

Let $cvd(G) \subseteq V(G)$ denote a minimum set of vertices such that $G \setminus cdv(G)$ is a cluster graph. Our parametric assumption says that $|cdv(G)| \leq VC(G) \leq K$. Now consider the fact that a graph is a cluster graph if and only if it does not have any induced $P_3$. First, we show that the high degree vertices in $G$ surely need to be deleted to make it a cluster graph, i.e., $V_h \subseteq cdv(G)$. Let us consider a vertex $u \in V_h$. As the subgraph induced by $I$ has no edges and $|Z| \leq 2K$, each vertex in $I$ is of degree at most $|Z| \leq 2K$. So, $u$ must be in some $Z_i$ in the partition $\mathcal{P}$. As $deg_G(u) \geq 40K$, using $|Z| \leq 2K$, $u$ must have at least $38K$ many vertices from $I$ as its neighbors in $G$. Thus, there are at least $19K$ edge disjoint induced $P_3$’s that are formed with $u$ and its neighbors in $I$. If $u \notin cdv(G)$, then more than $K$ neighbors of $u$ that are in $I$ must be present in $cdv(G)$. It will contradict the fact that $|cdv(G)| \leq VC(G) \leq K$. Similarly, we can also argue that $V_h \subseteq cdv(H) = X$ as $deg_H(u) \geq 4K$ by Fact-2.

Next, we show that an induced $P_3$ is present in $G \setminus V_h$ if and only if it is present in $H \setminus V_h$. Removal of $V_h$ from $G$ (or $H$) removes all the induced $P_3$’s in $G$ (or $H$) having at least one vertex in $V_h$. Any induced $P_3$ in $G \setminus V_h$ (or $H \setminus V_h$) must have all of its vertices as low degree vertices. Now, using Fact-2, note that all the edges, in $G$, between low degree vertices are in $H$. In other words, an induced $P_3$ is present in $G \setminus V_h$ if and only if it is present in $H \setminus V_h$. Thus for a set $X \subseteq V(G)$, if $(H \setminus V_h) \setminus X$ is a cluster graph then $(G \setminus V_h) \setminus X$ is also a cluster graph.

Putting everything together, if $X \subseteq V(G)$ is such that $H \setminus X$ is a cluster graph, then $G \setminus X$ is also a cluster graph.

Coming back to the proof of Theorem 2.1, we are using $O(\log n)$ many hash functions, and each hash function requires a storage of $O(\log n)$ bits. There are $O(K^2)$ many $\ell_p$-samplers for each hash function and each $\ell_p$-sampler needs $O(\log^3 n)$ bits of storage. Putting everything together, the total space used by our algorithm is $O(K^2 \log^4 n)$.

3 Deterministic algorithms in the AL model

In this Section, we show that $\mathcal{F}$-SUBGRAPH DELETION is $(\mathcal{AL}, \Delta(\mathcal{F}) \cdot K^{\Delta(\mathcal{F})+1})$-streamable when the vertex cover of the input graph is parameterized by $K$. This will imply that FVS, ECT, OCT and TD parameterized by vertex cover size $K$, are $(\mathcal{AL}, K^3)$-streamable. This complements the results in Theorems 4.1 and 4.2 (in Section 1) that show that the problems parameterized by vertex cover size $K$ are $(\mathcal{VA}, n/p, p)$-hard (see also Table 1). Note that by Observation 1.4 this also implies that the problems parameterized by vertex cover size $K$ are $(\mathcal{M}, n/p, p)$-hard when
$\mathcal{M} \in \{\text{Ea,Dea}\}$. Finally, we design an algorithm for $\mathcal{F}$-MINOR DELETION that is inspired by the algorithm for $\mathcal{F}$-SUBGRAPH DELETION.

For the algorithm for $\mathcal{F}$-SUBGRAPH DELETION, we define an auxiliary problem COMMON NEIGHBOR and a streaming algorithm for it. This works as a subroutine for our algorithm for $\mathcal{F}$-SUBGRAPH DELETION.

For a graph $G$ and a parameter $\ell \in \mathbb{N}$, $H$ will be called a common neighbor subgraph for $G$ if

(i) $V(H) \subseteq V(G)$ such that $H$ has no isolated vertex;
(ii) $E(H)$ contains the edges

- of a maximal matching $M$ of $G$ along with the edges where both the endpoints are from $V(M)$,
- such that for each subset $S \subseteq V(M)$, $|S| \leq d$, $|N_H(S) \setminus V(M)| = \min\{|N_G(S) \setminus V(M)|, \ell\}$,

that is, $E(H)$ contains edges to at most $\ell$ common neighbors of $S$ in $N_G(S) \setminus V(M)$.

In simple words, a common neighbor subgraph $H$ of $G$ contains the subgraph of $G$ induced by $V(M)$ as a subgraph of $H$ for some maximal matching $M$ in $G$. Also, for each subset $S$ of at most $d$ vertices in $V(M)$, $H$ contains edges to sufficient common neighbors of $S$ in $G$. The parameters $d \leq K$ and $\ell$ are referred to as the degree parameter and common neighbor parameter, respectively.

The COMMON NEIGHBOR problem is formally defined as follows. It takes as input a graph $G$ with $\text{VC}(G) \leq K$, degree parameter $d \leq K$ and common neighbor parameter $\ell$ and produces a common neighbor subgraph of $G$ as the output. COMMON NEIGHBOR parameterized by vertex cover size $K$, has the following result.

**Algorithm 2: COMMON NEIGHBOR**

**Input:** A graph $G$, with $\text{VC}(G) \leq K$, in the AL model, a degree parameter $d \leq K$, and a common neighbor parameter $\ell$.

**Output:** A common neighbor subgraph $H$ of $G$.

1. **begin**
2. Initialize $M = \emptyset$ and $V(M) = \emptyset$, where $M$ denotes the current maximal matching.
3. Initialize a temporary storage $T = \emptyset$.
4. for (each vertex $u \in V(G)$ exposed in the stream) do
5. for (each $(u,x) \in E(G)$ in the stream) do
6. if $(u \notin V(M)$ and $x \notin V(M))$ then
7. Add $(u,x)$ to $M$ and both $u,x$ to $V(M)$.
8. if $(x \in V(M))$ then
9. Add $(u,x)$ to $T$.
10. if (If $u$ is added to $V(M)$ during the exposure of $u$) then
11. Add all the edges present in $T$ to $E(H)$.
12. else
13. for (each $S \subseteq V(M)$ such that $|S| \leq d$ and $(u,z) \in T \forall z \in S)$ do
14. if ($N_H(S)$ is less than $\ell$) then
15. Add the edges $(u,z) \forall z \in S$ to $E(H)$.
16. Reset $T$ to $\emptyset$.
17. **end**

**Lemma 3.1.** COMMON NEIGHBOR, with a common neighbor parameter $\ell$ and parameterized by vertex cover size $K$, is ($\text{AL}, K^2\ell$)-streamable.
Proof. We start our algorithm by initializing \( M = \emptyset \) and construct a matching in \( G \) that is maximal under inclusion; see Algorithm 2. As \( |VC(G)| \leq K, |M| \leq K \). Recall that we are considering the At. model here. Let \( M_u \) and \( M_u' \) be the maximal matchings just before and after the exposure of the vertex \( u \) (including the processing of the edges adjacent to \( u \)), respectively. Note that, by construction these partial matchings \( M_u \) and \( M_u' \) are also maximal matchings in the subgraph exposed so far. The following Lemma will be useful for the proof.

Claim 3.2. Let \( u \in N_G(S) \setminus V(M) \) for some \( S \subseteq V(M) \). Then \( S \subseteq V(M_u) \), that is, \( u \) is exposed, after all the vertices in \( S \) are declared as vertices of \( V(M) \).

Proof. Observe that if there exists \( x \in S \) such that \( x \notin V(M_u) \), then after \( u \) is exposed, there exists \( y \in N_G(u) \) such that \((u,y)\) is present in \( M_u' \). This implies \( u \in V(M_u') \subseteq V(M) \), which is a contradiction to \( u \in N_G(S) \setminus V(M) \).

Now, we describe what our algorithm does when a vertex \( u \) is exposed. A complete pseudocode of our algorithm for COMMON NEIGHBOR is given in Algorithm 2. When a vertex \( u \) is exposed in the stream, we try to extend the maximal matching \( M_u \). Also, we store all the edges of the form \((u,x)\) such that \( x \in V(M_u) \), in a temporary memory \( T \). As \( |M_u| \leq K \), we are storing at most \( 2K \) many edges in \( T \). Now, there are the following possibilities.

- If \( u \in V(M_u') \), that is, either \( u \in V(M_u) \) or the matching \( M_u \) is extended by one of the edges stored in \( T \), then we add all the edges stored in \( T \) to \( E(H) \).
- Otherwise, for each \( S \subseteq V(M_u) \) such that \( |S| \leq d \) and \( S \subseteq N_G(u) \), we check whether the number of common neighbors of the vertices present in \( S \), that are already stored, is less than \( \ell \). If yes, we add all the edges of the form \((u,z)\) such that \( z \in S \) to \( E(H) \); else, we do nothing. Now, we reset \( T \) to \( \emptyset \).

As \( |M| \leq K, |V(M)| \leq 2K \). We are storing at most \( \ell \) common neighbors for each \( S \subseteq V(M) \) with \( |S| \leq d \) and the number of edges having both the endpoints in \( M \) is at most \( O(K^2) \), the total amount of space used is at most \( O(K^d \ell) \).

We call our algorithm described in the proof of Lemma 3.1 and given in Algorithm 2 as \( A_{cn} \). The following structural Lemma of the common neighbor subgraph of \( G \), obtained by algorithm \( A_{cn} \) is important for the design and analysis of streaming algorithms for \( \mathcal{F} \)-SUBGRAPH DELETION. The proof of this structural result is similar to that in [FJPT].

Lemma 3.3. Let \( G \) be a graph with \( VC(G) \leq K \) and let \( F \) be a connected graph with \( \Delta(F) \leq d \leq K \). Let \( H \) be the common neighbor subgraph of \( G \) with degree parameter \( d \) and common neighbor parameter \( (d+2)K \), obtained by running the algorithm \( A_{cn} \). Then the following holds in \( H \): For any subset \( X \subseteq V(H) \), where \( |X| \leq K \), \( F \) is a subgraph of \( G \setminus X \) if and only if \( F' \) is a subgraph of \( H \setminus X \), such that \( F \) and \( F' \) are isomorphic.

Proof. Let the common neighbor subgraph \( H \), obtained by algorithm \( A_{cn} \), contain a maximal matching \( M \) of \( G \). First, observe that since \( VC(G) \leq K \), the size of a subgraph \( F \) in \( G \) is at most \( dK \). Now let us consider a subset \( X \subseteq V(H) \) such that \( |X| \leq K \). First, suppose that \( F' \) is a subgraph of \( H \setminus X \) and \( F' \) is isomorphic to \( F \). Then since \( H \) is a subgraph of \( G \), \( F' \) is also a subgraph of \( G \setminus X \). Therefore, \( F = F' \) and we are done.

Conversely, suppose \( F \) is a subgraph of \( G \setminus X \) that is not a subgraph in \( H \setminus X \). We show that there is a subgraph \( F' \) of \( H \setminus X \) such that \( F' \) is isomorphic to \( F \). Consider an arbitrary ordering \( \{e_1, e_2, \ldots, e_s\} \subseteq (E(G) \setminus E(H)) \cap E(F) \); note that \( s \leq |E(F)| \). We describe an iterative subroutine
that converts the subgraph $F$ to $F'$ through $s$ steps, or equivalently, through a sequence of isomorphic subgraphs $F_0,F_1,F_2,...,F_s$ in $G$ such that $F_0 = F$ and $F_s = F'$.

Let us discuss the consequence of such an iterative routine. Just before the starting of step $i \in [s]$, we have the subgraph $F_{i-1}$ such that $F_{i-1}$ is isomorphic to $F$ and the set of edges in $(E(G) \setminus E(H)) \cap E(F_{i-1})$ is a subset of $\{e_i,e_{i+1},...,e_s\}$. In step $i$, we convert the subgraph $F_{i-1}$ into $F_i$ such that $F_{i-1}$ is isomorphic to $F_i$. Just after the step $i \in [s]$, we have the subgraph $F_i$ such that $F_i$ is isomorphic to $F$ and the set of edges in $(E(G) \setminus E(H)) \cap E(F_i)$ is a subset of $\{e_{i+1},e_{i+2},...,e_s\}$. In particular, in the end $F_s = F'$ is a subgraph both in $G$ and $H$.

Now consider the instance just before step $i$. We show how we select the subgraph $F_i$ from $F_{i-1}$. Let $e_i = (u,v)$. Note that $e_i \notin E(H)$. By the definition of the maximal matching $M$ in $G$, it must be the case that $|\{u,v\} \cap V(M)| \geq 1$. From the construction of the common neighbor subgraph $H$, if both $u$ and $v$ are in $V(M)$, then $e_i = (u,v) \in E(H)$. So, exactly one of $u$ and $v$ is present in $V(M)$. Without loss of generality, let $u \in V(M)$. Observe that $v$ is a common neighbor of $N_G(v)$ in $G$.

Because of the maximality of $M$, each vertex in $N_G(v)$ is present in $V(M)$. Now, as $(u,v) \notin E(H)$, $v$ is not a common neighbor of $N_G(v)$ in $H$. From the construction of the common neighbor subgraph, $H$ contains $(d+2)K$ common neighbors of all the vertices present in $N_G(v)$. Of these common neighbors, at most $(d+1)K$ common neighbors can be vertices in $X \cup F_i$. Thus, there is a vertex $v'$ that is a common neighbor of all the vertices present in $N_G(v)$ in $H$ such that $F_{i+1}$ is a subgraph that is isomorphic to $F_i$. Moreover, $(E(G) \setminus E(H)) \cap E(F_{i+1}) \subseteq \{e_{i+2},e_{i+3},...,e_s\}$. Thus, this leads to the fact that there is a subgraph $F'$ in $H \setminus X$ that is isomorphic to the subgraph $F$ in $G \setminus X$. 

Now we are ready for our streamability results.

**Theorem 3.4.** \(\mathcal{F}\)-Subgraph Deletion parameterized by vertex cover size $K$ is \((\text{AL},d \cdot K^{d+1})\)-streamable, where $d = \Delta(\mathcal{F}) \leq K$.

**Proof.** Let $(G,k,K)$ be an input for \(\mathcal{F}\)-Subgraph Deletion, where $G$ is the input graph, $k \leq K$ is the size of the solution of \(\mathcal{F}\)-Subgraph Deletion, and the parameter $K$ is at least VC($G$).

Now, we describe the streaming algorithm for \(\mathcal{F}\)-Subgraph Deletion. First, we run the Common Neighbor streaming algorithm described in Lemma 3.1 (and given in Algorithm 2) with degree parameter $d$ and common neighbor parameter $(d+2)K$, and let the common neighbor subgraph obtained be $H$. We run a traditional FPT algorithm for \(\mathcal{F}\)-Subgraph Deletion on $H$ and output YES if and only if the output on $H$ is YES.

Let us argue the correctness of this algorithm. By Lemma 3.3, for any subset $X \subseteq V(H)$, where $|X| \leq K$, $F \in F$ is a subgraph of $G \setminus X$ if and only if $F'$, such that $F'$ is isomorphic to $F'$, is a subgraph of $H \setminus X$. In particular, let $X$ be a $k$-sized vertex set of $G$. As mentioned before, $k \leq K$. Thus, by Lemma 3.3, $X$ is a solution of \(\mathcal{F}\)-Subgraph Deletion in $H$ if and only if $X$ is a solution of \(\mathcal{F}\)-Subgraph Deletion in $G$. Therefore, we are done with the correctness of the streaming algorithm for \(\mathcal{F}\)-Subgraph Deletion.

The streaming complexity of \(\mathcal{F}\)-Subgraph Deletion is same as the streaming complexity for the algorithm $\mathcal{A}_\text{cn}$ from Lemma 3.1 with degree parameter $d = \Delta(\mathcal{F})$ and common neighbor parameter $(d+2)K$. Therefore, the streaming complexity of \(\mathcal{F}\)-Subgraph Deletion is $O(d \cdot K^{d+1})$. \(\square\)

**Corollary 3.5.** FVS, ECT, OCT and TD parameterized by vertex cover size $K$ are \((\text{AL},K^3)\)-streamable due to deterministic algorithms.

Finally, we describe a streaming algorithm for \(\mathcal{F}\)-Minor Deletion.

**Proposition 3.6.** [FJP14] Let $G$ be a graph with $F$ as a minor and VC($G$) $\leq K$. Then there exists a subgraph $G^*$ of $G$ that has $F$ as a minor such that $\Delta(G^*) \leq \Delta(F)$ and $V(G^*) \leq V(F)+K(\Delta(F)+1)$. 

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Due to the above Proposition, the algorithm for $\mathcal{F}$-MINOR DELETION works similar to that of $\mathcal{F}$-SUBGRAPH DELETION.

**Theorem 3.7.** $\mathcal{F}$-MINOR DELETION parameterized by vertex cover size $K$ are $(A_l, d \cdot K^{d+1})$-streamable, where $d = \Delta(\mathcal{F}) \leq K$.

**Proof.** Let $(G, k, K)$ be an input for $\mathcal{F}$-MINOR DELETION, where $G$ is the input graph, $k$ is the size of the solution of $\mathcal{F}$-MINOR DELETION we are looking for, and the parameter $K$ is such that $VC(G) \leq K$. Note that, $k \leq K$.

Now, we describe the streaming algorithm for $\mathcal{F}$-MINOR DELETION. First, we run the COMMON NEIGHBOR streaming algorithm described in Lemma 3.1 with degree parameter $d$ and common neighbor parameter $(d+2)K$, and let the common neighbor subgraph obtained be $H$. We run a traditional FPT algorithm for $\mathcal{F}$-MINOR DELETION and output YES if and only if the output on $H$ is YES.

Let us argue the correctness of this algorithm, that is, we prove the following for any $F \in \mathcal{F}$. $G \setminus X$ contains $F$ as a minor if and only if $H \setminus X$ contains $F'$ as a minor such that $F$ and $F'$ are isomorphic, where $X \subseteq V(G)$ is of size at most $K$. For the only if part, suppose $H \setminus X$ contains $F'$ as a minor. Then since $H$ is a subgraph of $G$, $G \setminus X$ contains $F'$ as a minor. For the if part, let $G \setminus X$ contains $F$ as a minor. By Proposition 3.6, $G \setminus X$ contains a graph $G'$ such that $G'$ contains $F$ as a minor and $\Delta(G') \leq \Delta(F)$. Now, Lemma 3.3 implies that $H \setminus X$ also contains a graph $G'$ that is isomorphic to $G'$. Hence, $H \setminus X$ contains $F'$ as a minor such that $F'$ is isomorphic to $F$.

The streaming complexity of the streaming algorithm for $\mathcal{F}$-MINOR DELETION is same as the streaming complexity for the algorithm $A_{cn}$ from Lemma 3.1 with degree parameter $d = \Delta(\mathcal{F})$ and common neighbor parameter $(d+2)K$. Therefore, the streaming complexity for $\mathcal{F}$-MINOR DELETION is $O(d \cdot K^{d+1})$. \qed

### 4 The Lower Bounds

Before we prove the lower bound results presented in Table 1, note that a lower bound on FEEDBACK VERTEX SET is also a lower bound for $\mathcal{F}$-SUBGRAPH DELETION (deletion of cycles as subgraphs) and $\mathcal{F}$-MINOR DELETION (deletion of 3-cycles as minors). Thus, we will be done by proving the following theorems; Observations 1.4 and 1.5 imply the other hardness results.

**Theorem 4.1.** FEEDBACK VERTEX SET, EVEN CYCLE TRANSVERSAL and ODD CYCLE TRANSVERSAL are

(I) $(A_l, n \log n)$-hard parameterized by solution size $k$ and even if $\Delta_{av}(G) = O(1)$,

(II) $(A_l, n/p, p)$-hard parameterized by solution size $k$ and even if $\Delta(G) = O(1)$, and

(III) $(V_a, n/p, p)$-hard parameterized by vertex cover size $K$ and even if $\Delta_{av}(G) = O(1)$.

**Theorem 4.2.** TD is

(I) $(V_a, n \log n)$-hard parameterized by solution size $k$ and even if $\Delta_{av}(G) = O(1)$,

(II) $(V_a, n/p, p)$-hard parameterized by solution size $k$ and even if $\Delta(G) = O(1)$, and

(III) $(V_a, n/p, p)$-hard parameterized by vertex cover size $K$ and even if $\Delta_{av}(G) = O(1)$.

**Theorem 4.3.** CVD is $(V_a, n/p, p)$-hard parameterized by solution size $k$ and even if $\Delta(G) = O(1)$.

We prove the above theorems by reduction from communication complexity problems discussed below.
4.1 Communication complexity results

Lower bounds of communication complexity have been used to provide lower bounds for the streaming complexity of problems. In Yao’s two party communication model, Alice and Bob get inputs and the objective is to compute a function of their inputs with minimum bits of communication. In one way communication, only Alice is allowed to send messages and Bob produces the final output; whereas in two way communication both Alice and Bob can send messages.

**Definition 4.4.** The one (two) way communication complexity of a problem $\Pi$ is the minimum number of bits that must be sent by Alice to Bob (exchanged between Alice and Bob) to solve $\Pi$ on any arbitrary input with success probability $2/3$.

The following problems are very fundamental problems in communication complexity and we use these problems in showing lower bounds on the streaming complexity of problems considered in this paper.

(i) $\text{INDEX}_n$: Alice gets as input $x \in \{0,1\}^n$ and Bob has an index $j \in [n]$. Bob wants to determine whether $x_j = 1$. Formally, $\text{INDEX}_n(x, j) = 1$ if $x_j = 1$ and 0, otherwise.

(ii) $\text{DISJ}_n$: Alice and Bob get inputs $x, y \in \{0,1\}^n$, respectively. The objective is to decide whether there exists an $i \in [n]$ such that $x_i = y_i = 1$. Formally, $\text{DISJ}_n(x, y) = 0$ if there exists an $i \in [n]$ such that $x_i = y_i = 1$ and 1, otherwise.

(iii) $\text{PERM}_n$ [SW15]: Alice gets a permutation $\pi : [n] \to [n]$ and Bob gets an index $j \in [n \log n]$. The objective of Bob is to decide the value of $\text{PERM}_n(\pi, j)$, defined as the $j$-th bit in the string of 0’s and 1’s obtained by concatenating the bit expansions of $\pi(1) \ldots \pi(n)$. In other words, let $\Phi : [n \log n] \to [n] \times [\log n]$ be a bijective function defined as

$$
\Phi(j) = \left(\left\lceil \frac{j}{\log n} \right\rceil, j + \log n - \left\lceil \frac{j}{\log n} \right\rceil \times \log n\right)
$$

For a permutation $\pi : [n] \to [n]$, Bob needs to determine the value of the $\gamma$-th bit of $\pi\left(\left\lceil \frac{j}{\log n} \right\rceil\right)$, where $\gamma = \left(j + \log n - \left\lceil \frac{j}{\log n} \right\rceil \times \log n\right)$.

**Proposition 4.5 (KN97, SW15).** (i) The one way communication complexity of $\text{INDEX}_n$ is $\Omega(n)$.

(ii) The two way communication complexity of $\text{DISJ}_n$ is $\Omega(n)$.

(iii) The one way communication complexity of $\text{PERM}_n$ is $\Omega(n \log n)$.

A note on reduction from $\text{INDEX}_n$, $\text{DISJ}_n$, $\text{PERM}_n$: A reduction from a problem $\Pi_1$ in one/two way communication complexity to a problem $\Pi_2$ in streaming algorithms is typically as follows: The two players Alice and Bob device a communication protocol for $\Pi_1$ that uses a streaming algorithm for $\Pi_2$ as a subroutine. Typically in a round of communication, a player gives inputs to the input stream of the streaming algorithm, obtains the compact sketch produced by the streaming algorithm and communicates this sketch to the other player. This implies that a lower bound on the communication complexity of $\Pi_1$ also gives a lower bound on the streaming complexity of $\Pi_2$.

The following Proposition summarizes a few important consequences of reductions from problems in communication complexity to problems for streaming algorithms:

**Proposition 4.6.** (i) If we can show a reduction from $\text{INDEX}_n$ to a problem $\Pi$ in model $\mathcal{M}$ such that the reduction uses a 1-pass streaming algorithm of $\Pi$ as a subroutine, then $\Pi$ is $(\mathcal{M}, n)$-hard.

(ii) If we can show a reduction from $\text{DISJ}_n$ to a problem $\Pi$ in model $\mathcal{M}$ such that the reduction uses a 1-pass streaming algorithm of $\Pi$ as a subroutine, then $\Pi$ is $(\mathcal{M}, n/p, p)$-hard, for any $p \in \mathbb{N}$ [CCE15, BGMS18, AMP+06].

(iii) If we can show a reduction from $\text{PERM}_n$ to a problem $\Pi$ in model $\mathcal{M}$ such that the reduction uses a 1-pass streaming algorithm of $\Pi$ as a subroutine, then $\Pi$ is $(\mathcal{M}, n \log n)$-hard.
4.2 Proofs of Theorems 4.1, 4.2, 4.3

Proof of Theorem 4.1. The proofs for all three problems are similar. We first consider Feedback Vertex Set. To begin with, we show the hardness results of FVS for solution size $k = 0$.

![Illustration of Proof of Theorem 4.1 (I).](image)

Figure 1: Illustration of Proof of Theorem 4.1 (I). Consider $n = 4$. Let $\pi : [4] \to [4]$ such that $\pi(1) = 3, \pi(2) = 4, \pi(3) = 2$ and $\pi(4) = 1$. So the concatenated bit string is $11001001$. In (a), $j = 5$, $\Phi(j) = (\psi, \gamma) = (3, 1)$, $\text{PERM}_n(\pi, j) = 1$, and $G$ contains a cycle. In (b), $j = 4$, $\Phi(j) = (\psi, \gamma) = (2, 2)$, $\text{PERM}_n(\pi, j) = 0$, and $G$ does not contain a cycle.

Proof of Theorem 4.1 (I). We give a reduction from $\text{PERM}_n$ to FVS in the AL model when the solution size parameter $k = 0$. The idea is to build a graph $G$ with $\Delta_{av}(G) = O(1)$ and construct edges according to the input of $\text{PERM}_n$, such that the output of $\text{PERM}_n$ is 0 if and only if $G$ is cycle-free.

Let $\mathcal{A}$ be a one pass streaming algorithm that solves FVS in AL model using $o(n \log n)$ space. Let $G$ be a graph with $4n + 2$ vertices $u_1, \ldots, u_n, v_1, \ldots, v_n, u'_1, \ldots, u'_n, v'_1, \ldots, v'_n, w, w'$. Let $\pi$ be the input of Alice for $\text{PERM}_n$. See Figure 1 for an illustration.

**Alice’s input to $\mathcal{A}$:** Alice inputs the graph $G$ first by exposing the vertices $u_1, \ldots, u_n, v_1, \ldots, v_n$, sequentially. (i) While exposing the vertex $u_i$, Alice gives as input to $\mathcal{A}$ the edges $(u_i, u'_i), (u_i, v_{\pi(i)})$; (ii) while exposing the vertex $v_i$, Alice gives the edges $(v_i, v'_i), (v_i, u_{\pi^{-1}(i)})$ to the input stream of $\mathcal{A}$.

After the exposure of $u_1, \ldots, u_n, v_1, \ldots, v_n$ as per the AL model, Alice sends the current memory state of $\mathcal{A}$, i.e. the sketch generated by $\mathcal{A}$, to Bob. Let $j \in [n \log n]$ be the input of Bob and let $(\psi, \gamma) = \Phi(j)$.

**Bob’s input to $\mathcal{A}$:** Bob exposes the vertices $u'_1, \ldots, u'_n, v'_1, \ldots, v'_n, w, w'$, sequentially. (i) While exposing a vertex $u'_i$ where $i \neq \psi$, Bob gives the edge $(u'_i, u_i)$ to the input stream of $\mathcal{A}$; (ii) while exposing $u'_\psi$, Bob gives the edges $(u'_\psi, u_\psi)$ and $(u'_\psi, w')$; (iii) while exposing a vertex $v'_i$, Bob gives the edge $(v'_i, v_i)$, and the edge $(v'_i, w)$ if and only if $\text{bit}(i, \gamma) = 1$; (iv) while exposing $w$, Bob gives the edge $(w, w')$, and the edge $(w, v'_i)$ if and only if $\text{bit}(i, \gamma) = 1$; (v) while exposing $w'$, Bob gives the edges $(w', w)$ and $(w', u'_\psi)$.

Observe that $\Delta_{av}(G) = O(1)$. Now we show that the output of FVS is NO if and only if $\text{PERM}_n(\pi, j) = 1$. Recall that $k = 0$.

---

\footnote{Recall that we take $n$ as a power of 2. For $1 \leq i \leq n - 1$, the bit expansion of $i$ is the usual bit notation of $i$ using $\log_2 n$ bits; the bit expansion of $n$ is $\log_2 n$ many consecutive zeros. For example: Take $n = 32$. The bit expansion of 32 is 100000. We ignore the bit 1 and say that the bit expansion of 32 is 00000.}
From the construction, observe that \((w, w'), (w', u'_\psi), (u'_\psi, u_\psi), (u_\psi, v_{\pi(\psi)}), (v_{\pi(\psi)}, v'_{\pi(\psi)}) \in E(G)\). When \(\text{PERM}_n(\pi, j) = 1\), the edge \((v'_{\pi(\psi)}, w)\) is present in \(G\). So, \(G\) contains the cycle \(C(w, w', u'_\psi, u_\psi, v_{\pi(\psi)}, v'_{\pi(\psi)})\), that is, the output of FVS is NO.

On the other hand, if the output of FVS is NO, then there is a cycle in \(G\). From the construction, the cycle is \(C(w, w', u'_\psi, u_\psi, v_{\pi(\psi)}, v'_{\pi(\psi)})\). As \((v'_{\pi(\psi)}, w)\) is an edge, the \(\gamma\)-th bit of \(\pi(\psi)\) is 1, that is \(\text{PERM}_n(\pi, j) = 1\). Now by Propositions 4.5 and 4.6(iii), we obtain that Feedback Vertex Set is \((\text{AL}, n \log n)\)-hard even if \(\Delta_{\text{rel}}(G) = \mathcal{O}(1)\) and when \(k = 0\).

\[\begin{array}{cccccc}
\bullet & u_11 & u_12 & u_21 & u_22 & \bullet \\
\bullet & u_13 & u_14 & u_23 & u_24 & \bullet \\
\bullet & u_31 & u_32 & u_33 & u_34 & \bullet \\
\bullet & u_41 & u_42 & u_43 & u_44 & \bullet \\
\end{array}\]

\[\begin{array}{cccccc}
\bullet & u_11 & u_12 & u_21 & u_22 & \bullet \\
\bullet & u_13 & u_14 & u_23 & u_24 & \bullet \\
\bullet & u_31 & u_32 & u_33 & u_34 & \bullet \\
\bullet & u_41 & u_42 & u_43 & u_44 & \bullet \\
\end{array}\]

Figure 2: Illustration of Proof of Theorem 4.1 (II). Consider \(n = 4\). In (a), \(x = 1001\) and \(y = 0100\), that is, \(\text{Disj}_n(x, y) = 1\), and \(G\) does not contain a cycle. In (b), \(x = 1100\) and \(y = 0110\), that is, \(\text{Disj}_n(x, y) = 0\), and \(G\) contains a cycle.

**Proof of Theorem 4.1 (II).** We give a reduction from \(\text{Disj}_n\) to FVS in the AL model when the solution size parameter \(k = 0\). The idea is to build a graph \(G\) with \(\Delta(G) = \mathcal{O}(1)\) and construct edges according to the input of \(\text{Disj}_n\), such that the output of \(\text{Disj}_n\) is 1 if and only if \(G\) is cycle-free.

Let \(\mathcal{A}\) be a one pass streaming algorithm that solves FVS in AL model, such that \(\Delta(G) = \mathcal{O}(1)\), and the space used is \(o(n)\). Let \(G\) be a graph with \(4n\) vertices \(u_{11}, u_{12}, u_{13}, u_{14}, \ldots, u_{n1}, u_{n2}, u_{n3}, u_{n4}\). Let \(x, y\) be the input of Alice and Bob for \(\text{Disj}_n\), respectively. See Figure 2 for an illustration.

**Alice’s input to \(\mathcal{A}\)**: Alice inputs the graph \(G\) by exposing the vertices \(u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{n1}, u_{n2}\), sequentially. (i) While exposing \(u_{i1}\), Alice gives as input to \(\mathcal{A}\) the edge \((u_{i1}, u_{i3})\). Also, Alice gives the edge \((u_{i1}, u_{i2})\) as input to \(\mathcal{A}\) if and only if \(x_i = 1\); (ii) while exposing \(u_{i2}\), Alice gives the edge \((u_{i2}, u_{i4})\) as input to \(\mathcal{A}\). Also, Alice gives the edge \((u_{i2}, u_{i1})\) as input to \(\mathcal{A}\) if and only if \(x_i = 1\).

After the exposure of \(u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{n1}, u_{n2}\) as per the AL model, Alice sends current memory state of \(\mathcal{A}\), i.e., the sketch generated by \(\mathcal{A}\), to Bob.

**Bob’s input to \(\mathcal{A}\)**: Bob exposes the vertices \(u_{13}, u_{14}, u_{23}, u_{24}, \ldots, u_{n3}, u_{n4}\) sequentially. (i) While exposing \(u_{i3}\), Bob gives the edge \((u_{i3}, u_{i1})\) as input to \(\mathcal{A}\), and gives the edge \((u_{i3}, u_{i4})\) if and only if \(y_i = 1\); (ii) while exposing \(u_{i4}\), Bob gives the edge \((u_{i4}, u_{i2})\) as input to \(\mathcal{A}\), and gives the edge \((u_{i4}, u_{i3})\) if and only if \(y_i = 1\).

Observe that \(\Delta(G) \leq 4\). Recall that \(k = 0\). Now we show that the output of FVS is NO if and only if \(\text{Disj}_n(x, y) = 0\).
From the construction, \((u_1, u_3), (u_2, u_4) \in E(G)\), for each \(i \in [n]\). If \(\text{Dis}J_n(x, y) = 0\), there exists \(i \in [n]\) such that \(x_i = y_i = 1\). This implies the edges \((u_1, u_2)\) and \((u_3, u_4)\) are present in \(G\). So, the cycle \(C(u_1, u_2, u_3, u_4)\) is present in \(G\), that is, the output of FVS is NO.

Conversely, if the output of FVS is NO, there exists a cycle in \(G\). From the construction, the cycle must be \(C(u_1, u_2, u_3, u_4)\) for some \(i \in [n]\). As the edges \((u_1, u_2)\) and \((u_3, u_4)\) are present in \(G\), \(x_i = y_i = 1\), that is, \(\text{Dis}J_n(x, y) = 0\).

Now by Propositions 4.5 and 4.6(ii), we obtain that Feedback Vertex Set is \((AL, n/p, p)\)-hard even if \(\Delta(G) = O(1)\) and when \(k = 0\).

**Proof of Theorem 4.1 (III).** We give a reduction from \(\text{Dis}J_n\) to FVS in the \(VA\) model when the solution size parameter \(k = 0\). The idea is to build a graph \(G\) with vertex cover size bounded by \(K\) and \(\Delta(G) = O(1)\), and construct edges according to the input of \(\text{Dis}J_n\), such that the output of \(\text{Dis}J_n\) is 1 if and only if \(G\) is cycle-free.

Let \(\mathcal{A}\) be a one pass streaming algorithm that solves FVS in \(VA\) model, such that \(VC(G) \leq K\) and \(\Delta_{av}(G) = O(1)\), and the space used is \(o(n)\). Let \(G\) be a graph with \(n + 3\) vertices \(u_a, v_1, \ldots, v_n, u_b, w\). Let \(x, y\) be the input of Alice and Bob for \(\text{Dis}J_n\), respectively. See Figure 3 for an illustration.

**Alice’s input to \(\mathcal{A}\):** Alice inputs the graph \(G\) first by exposing the vertices \(u_a, v_1, \ldots, v_n\), sequentially. (i) While exposing \(u_a\), Alice does not give any edge; (ii) while exposing \(v_i\), Alice gives the edge \((v_i, u_a)\), as input to \(\mathcal{A}\), if and only if \(x_i = 1\).

After the exposure of \(u_a, v_1, \ldots v_n\) as per \(VA\) model, Alice sends the current memory state of \(\mathcal{A}\), i.e., the sketch generated by \(\mathcal{A}\), to Bob.

**Bob’s input to \(\mathcal{A}\):** Bob first exposes \(u_b\) and then exposes \(w\). (i) While exposing \(u_b\), Bob gives the edge \((u_b, v_i)\) if and only if \(y_i = 1\); (ii) while exposing \(w\), Bob gives the edges \((w, u_a)\) and \((w, u_b)\), as inputs to \(\mathcal{A}\).

From the construction, observe that \(VC(G) \leq 2 \leq K\) and \(\Delta_{av}(G) = O(1)\). Recall that \(k = 0\). Now we show that the output of FVS is NO if and only if \(\text{Dis}J_n(x, y) = 0\).
From the construction, \((u_a, w), (u_b, w) \in E(G)\). If \(\text{Dis}_a(x, y) = 0\), there exists \(i \in [n]\) such that \(x_i = y_i = 1\). This implies the edges \((u_a, v_i)\) and \((u_b, v_i)\) are present in \(G\). So, the cycle \(C(u_a, v_i, u_b, w)\) is present in \(G\), that is, the output of FVS is NO.

Conversely, if the output of FVS is NO, there exists a cycle in \(G\). From the construction, the cycle must be \(C(u_a, v_i, u_b, w)\) for some \(i \in [n]\). As the edges \((u_a, v_i)\) and \((u_b, v_i)\) are present in \(G\), \(x_i = y_i = 1\), that is, \(\text{Dis}_a(x, y) = 0\).

Now by Propositions 4.5 and 4.6(ii), we obtain that Feedback Vertex Set parameterized by vertex cover size \(K\) is \((VA, n/p, p)\)-hard even if \(\Delta_{av}(G) = O(1)\), and when \(k = 0\).

In each of the above three cases, we can make the reduction work for any \(k\), by adding \(k\) many vertex disjoint cycles of length 4, i.e. \(C_4\)'s, to \(G\). In Theorem 4.1 (III), the vertex cover must be bounded. In the given reduction for Theorem 4.1 (III), the vertex cover of the constructed graph is at most 2. Note that by the addition of \(k\) many edge disjoint \(C_4\)'s, the vertex cover of the constructed graph in the modified reduction is at most \(2k + 2\), and is therefore still a parameter independent of the input instance size.

This completes the proof of the Theorem 4.1 with respect to FVS.

If the graph constructed in the reduction, in any of the above three cases for Feedback Vertex Set, contains a cycle, then it is of even length. Otherwise, the graph is cycle free. Hence, the proof of this Theorem with respect to ECT is same as the proof for FVS.

Similarly, a slight modification can be made to the constructed graph, in all three of the above cases, such that a cycle in the graph is of odd length if a cycle exists. Thereby, the proof of this Theorem with respect to OCT also is very similar to the proof for FVS.

**Proof of Theorem 4.2.** We first show the hardness results of TD for \(k = 0\) in all three cases.

![Figure 4: Illustration of Proof of Theorem 4.2 (I). Consider \(n = 4\). Let \(\pi : [4] \to [4]\) such that \(\pi(1) = 3, \pi(2) = 4, \pi(3) = 2,\) and \(\pi(4) = 1\). So the concatenated bit string is 11001001. In (a), \(j = 5, \Phi(j) = (\psi, \gamma) = (3, 1), \text{PERM}_n(\pi, j) = 1\) and \(G\) contains a triangle. In (b), \(j = 4, \Phi(j) = (\psi, \gamma) = (2, 2), \text{PERM}_n(\pi, j) = 0,\) and \(G\) does not contain any triangle.](image)

**Proof of Theorem 4.2 (I).** We give a reduction from \(\text{PERM}_n\) to TD when the solution size parameter \(k = 0\). Let \(A\) be a one pass streaming algorithm that solves TD in VA model, such that \(\Delta_{av}(G) = O(1)\), and the space used is \(o(n \log n)\). Let \(G\) be a graph with \(2n + 1\) vertices \(u_1, \ldots, u_n, v_1, \ldots, v_n, w\). Let \(\pi\) be the input of Alice for \(\text{PERM}_n\). See Figure 4 for an illustration.

**Alice's input to \(A\):** Alice inputs the graph \(G\) by exposing the vertices \(u_1, \ldots, u_n, v_1, \ldots, v_n, w\), sequentially. (i) While exposing the vertex \(u_i\), Alice does not give any edge; (ii) while exposing the vertex \(v_i\), Alice gives the edges \((v_{\pi(i)}, u_i)\) as an input to the stream of \(A\).
After the exposure of \( u_1, \ldots, u_n, v_1, \ldots, v_n \) as per the VA model, Alice sends the current memory state of \( \mathcal{A} \), i.e. the sketch generated by \( \mathcal{A} \), to Bob. Let \( j \in [\lceil n \log n \rceil] \) be the input of Bob and let \( (\psi, \gamma) = \Phi(j) \).

**Bob’s input to \( \mathcal{A} \):** Bob exposes only the vertex \( w \). Bob gives the edge \((w, u_\psi)\), and the edge \((w, v_\gamma)\) if and only if \( \text{bit}(i, \gamma) = 1 \), as input to the stream of \( \mathcal{A} \).

From the construction, note that if \( \Delta_{av}(G) = \Theta(1) \). Recall that \( k = 0 \). Now we show that, the output of TD is NO if and only if \( \text{PERM}_n(\pi, j) = 1 \).

From the construction, the edges \((u_\psi, \pi(\psi))\) and \((w, u_\psi)\) are present in \( G \). If \( \text{PERM}_n(\pi, j) = 1 \), then \((\pi(\psi), w) \in E(G) \). So, there exists a triangle in \( G \), that is, the output of TD is NO.

On the other hand, if the output of TD is NO, then there exists a triangle in \( G \). From the construction, the triangle is formed with the vertices \( u_\psi, \pi(\psi) \) and \( w \). As \((\pi(\psi), w) \in E(G)\), the \( \gamma \)-th bit of \( \pi(\psi) \) is 1, that is, \( \text{PERM}_n(\pi, j) = 1 \).

Now by Propositions 4.5 and 4.6 (iii), we obtain that TD is \((\text{VA}, n \log n)\)-hard even if \( \Delta_{av}(G) = \Theta(1) \), and when \( k = 0 \).

**Proof of Theorem 4.2 (II):** We give a reduction from \( \text{DIS}_n \) to TD when the solution size parameter \( k = 0 \). Let \( \mathcal{A} \) be a one pass streaming algorithm that solves TD in VA model, such that \( \Delta(G) = \Theta(1) \), and the space used is \( o(n) \). Let \( G \) be a graph with \( 3n \) vertices \( u_{11}, u_{12}, u_{13}, \ldots, u_{n1}, u_{n2}, u_{n3} \). Let \( x, y \) be the input of Alice and Bob for \( \text{DIS}_n \). See Figure 5 for an illustration.

**Alice’s input to \( \mathcal{A} \):** Alice inputs the graph \( G \) first by exposing the vertices \( u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, \ldots, u_{n1}, u_{n2}, u_{n3} \), sequentially. (i) While exposing \( u_{1i} \), Alice does not give any edge; (ii) while exposing \( u_{2i} \), Alice gives the edge \((u_{1i}, u_{2i})\), if and only if \( x_i = 1 \), as inputs to \( \mathcal{A} \).

After the exposure of \( u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, \ldots, u_{n1}, u_{n2} \) as per the VA model, Alice sends current memory state of \( \mathcal{A} \), i.e. the sketch generated by \( \mathcal{A} \), to Bob.

**Bob’s input to \( \mathcal{A} \):** Bob exposes the vertices \( u_{31}, u_{32}, u_{33}, \ldots, u_{n3} \), sequentially. While exposing \( u_{3i} \), Bob gives the edges \((u_{3i}, u_{1i})\) and \((u_{3i}, u_{2i})\) as two inputs to \( \mathcal{A} \) if and only if \( y_i = 1 \).
From the construction, note that $\Delta(G) \leq 2$. Recall that $k = 0$. Now we show that the output of TD is NO if and only if $\text{Disj}_n(x, y) = 0$.

If $\text{Disj}_n(x, y) = 0$, there exists $i \in [n]$ such that $x_i = y_i = 1$. From the construction, the edges $(u_{i2}, u_{i1})$, $(u_{i3}, u_{i1})$ and $(u_{i3}, u_{i2})$ are present in $G$. So, there exists a triangle in $G$, that is, the output of TD is NO.

Conversely, if the output of TD is NO, there exists a triangle in $G$. From the construction, the triangle is $(u_{i1}, u_{i2}, u_{i3})$ for some $i \in [n]$. As the edge $(u_{i2}, u_{i1}) \in E(G)$, $x_i = 1$; and as the edges $(u_{i3}, u_{i1})$ and $(u_{i3}, u_{i2})$ are in $G$, $y_i = 1$. So, $\text{Disj}_n(x, y) = 0$.

Now by Propositions 4.5 and 4.6(ii), we obtain that TD is $(V_a, n/p, p)$-hard even if $\Delta(G) = O(1)$, and when $k = 0$.

\[\square\]

![Figure 6: Illustration of Proof of Theorem 4.2 (III). Consider $n = 4$. In (a), $x = 1000$ and $y = 0101$, that is, $\text{Disj}_n(x, y) = 1$, and $G$ does not contain any triangle. In (b), $x = 0011$ and $y = 1010$, that is, $\text{Disj}_n(x, y) = 0$, and $G$ contains a triangle.](image)

**Proof of Theorem 4.2 (III).** We give a reduction from $\text{Disj}_n$ to TD parameterized by vertex cover size $K$, where $\mathcal{A}$ is a one pass streaming algorithm that solves TD parameterized by $K$ in Va model such that $\Delta_{av}(G) = O(1)$, and the space used is $o(n)$. Let $G$ be a graph with $n + 2$ vertices $u_a, v_1, \ldots, v_n, u_b$. Let $x, y$ be the input of Alice and Bob for $\text{Disj}_n$. See Figure 6 for an illustration.

**Alice’s input to $\mathcal{A}$:** Alice inputs the graph $G$ first by exposing the vertices $u_a, v_1, \ldots, v_n$ sequentially. (i) While exposing $u_a$, Alice does not give any edge; (ii) while exposing $v_i$, Alice gives the edge $(v_i, u_a)$ as input to $\mathcal{A}$ if and only if $x_i = 1$.

After the exposure of $u_a, v_1, \ldots, v_n$ as per the $\text{Va}$ model, Alice sends current memory state of $\mathcal{A}$, i.e. the sketch generated by $\mathcal{A}$, to Bob.

**Bob’s input to $\mathcal{A}$:** Bob exposes $u_b$ only. Bob gives the edge $(u_b, u_a)$ unconditionally, and an edge $(u_b, v_i)$ as input to $\mathcal{A}$ if and only if $y_i = 1$.

From the construction, observe that $\text{VC}(G) \leq 2 \leq K$ and $\Delta_{av}(G) = O(1)$. Recall that $k = 0$. Now we show that the output of TD is NO if and only if $\text{Disj}_n(x, y) = 0$.

Observe that $(u_a, u_b) \in E(G)$. If $\text{Disj}_n(x, y) = 0$, there exists an $i \in [n]$ such that $x_i = y_i = 1$. From the construction, the edges $(v_i, u_a)$ and $(u_b, v_i)$ are present in $G$. So, $G$ contains the triangle with vertices $u_a, u_b$ and $w$, i.e., the output of TD is NO.

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On the other hand, if the output of TD is NO, there exists a triangle in $G$. From the construction, the triangle is formed with the vertices $v_a, u_b$ and $v_i$. As $(v_i, u_b) \in E(G)$ implies $x_i = 1$, and $(v_i, u_a) \in E(G)$ implies $y_i = 1$. So, $\text{Disj}_n(x, y) = 0$.

Now by Propositions 4.5 and 4.6(ii), we obtain that TD parameterized by vertex cover size $K$ is $(VA, n/p, p)$-hard even if $\Delta_{av}(G) = O(1)$, and when $k = 0$.

In each of the above cases, we can make the reductions work for any $k$, by adding $k$ many vertex disjoint triangles to $G$. In Theorem 4.2 (III), the vertex cover must be bounded. In the given reduction for Theorem 4.2 (III), the vertex cover of the constructed graph is at most 2. Note that by the addition of $k$ many edge disjoint $C_4$’s, the vertex cover of the constructed graph in the modified reduction is at most $2k + 2$, and is therefore still a parameter independent of the input instance size.

Hence, we are done with the proof of the Theorem 4.2.

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**Figure 7:** Illustration of Proof of Theorem 4.3. Consider $n = 4$. In (a), $x = 0101$ and $y = 1000$, that is, $\text{Disj}_n(x, y) = 1$, and $G$ does not have any induced $P_3$. In (b), $x = 1100$ and $y = 0112$, that is, $\text{Disj}_n(x, y) = 0$, and $G$ contains an induced $P_3$.

**Proof of Theorem 4.3.** We give a reduction from $\text{Disj}_n$ to CVD for solution size parameter $k = 0$. Let $\mathcal{A}$ be a one pass streaming algorithm that solves CVD in VA model, such that $\Delta(G) = O(1)$, and the space used is $o(n)$. Consider a graph $G$ with $3n$ vertices $u_{11}, u_{12}, u_{13}, \ldots, u_{n1}, u_{n2}, u_{n3}$. Let $x, y$ be the input of Alice and Bob for $\text{Disj}_n$. See Figure 7 for an illustration.

**Alice’s input to $\mathcal{A}$:** Alice inputs the graph $G$ by exposing the vertices $u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{n1}, u_{n2}$, sequentially. (i) While exposing $u_{i1}$, Alice does not give any edge; (ii) while exposing $u_{i2}$, Alice gives the edge $(u_{i2}, u_{i1})$ as input to $\mathcal{A}$ if and only if $x_i = 1$.

After the exposure of $u_{11}, u_{12}, u_{21}, u_{22}, \ldots, u_{n1}, u_{n2}$ as per the VA model, Alice sends current memory state of $\mathcal{A}$, i.e. the sketch generated by $\mathcal{A}$, to Bob.

**Bob’s input to $\mathcal{A}$:** Bob exposes the vertices $u_{i3}, \ldots, u_{n3}$, sequentially. While exposing $u_{i3}$, Bob gives the edges $(u_{i3}, u_{i2})$ as an input to $\mathcal{A}$ if and only if $y_i = 1$.

From the construction, note that $\Delta(G) \leq 2$. Observe that, there exists a $P_3$ in $G$ if and only if there exists an $i \in [n]$ such that $x_i = y_i = 1$. Hence, the output of CVD is NO if and only if $\text{Disj}_n(x, y) = 0$.

Now by Propositions 4.5 and 4.6(ii), we obtain that CVD is $(VA, n/p, p)$-hard even if $\Delta(G) = O(1)$, and when $k = 0$.

We can make the reduction work for any $k$, by adding $k$ many vertex disjoint $P_3$’s to $G$.

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5 Conclusion

In this paper, we initiate the study of parameterized streaming complexity with structural parameters for graph deletion problem. Our study also compared the parameterized streaming complexity of several graph deletion problems in the different streaming models. In particular, our results on the $\mathcal{F}$-SUBGRAPH DELETION problem and its variants show the advantage of studying graph problems both in different streaming models as well as under different parameterizations.

In future, we wish to investigate why such a dichotomy exists for seemingly similar graph deletion problems. We also wish to conduct a systematic study of other graph deletion problems under different parameterizations and in different streaming models. Moreover, resolving the parameterized complexity of TD and CVD parameterized by solution size $k$ in the Al model remains open.
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### A Problem Definitions

In this Section we define the following problems formally.

| Problem                      | Input                                                                 | Output                                                                 |
|------------------------------|-----------------------------------------------------------------------|------------------------------------------------------------------------|
| **F-SUBGRAPH DELETION**      | A graph $G$, a family $\mathcal{F}$ of connected graphs, and a non-negative integer $k$. | Does there exist a set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ does not contain any graph from $\mathcal{F}$ as a subgraph? |
| **F-MINOR DELETION**         | A graph $G$, a family $\mathcal{F}$ of connected graphs, and a non-negative integer $k$. | Does there exist a set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ does not contain any graph from $\mathcal{F}$ as a minor? |
| **FVS**                      | A graph $G$ and a non-negative integer $k$.                           | Does there exist a set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ does not contain any cycle? |
| **ECT**                      | A graph $G$ and a non-negative integer $k$.                           | Does there exist a set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ does not contain any cycle of even length? |
| **OCT**                      | A graph $G$ and a non-negative integer $k$.                           | Does there exist a set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ does not contain any cycle of odd length, i.e., $G \setminus X$ is bipartite? |
| **TD**                       | A graph $G$ and a non-negative integer $k$.                           | Does there exist a set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ does not contain any triangle? |
| **CVD**                      | A graph $G$ and a non-negative integer $k$.                           | Does there exist a set $X \subset V(G)$ of $k$ vertices such that $G \setminus X$ is a cluster graph, i.e., $G \setminus X$ does not contain any induced $P_3$? |
| **COMMON NEIGHBOR**          | A graph $G$ with $VC(G) \leq K$, degree parameter $d \leq K$ and common neighbor parameter $\ell$. | A common neighbor subgraph of $G$. |

