1. Introduction

Let $G$ be a finite group and $k$ a field of characteristic $p$. Let $C^*(BG;k)$ be the cochains on the classifying space $BG$. Using the machinery of Elmendorf, Kříž, Mandell and May [8], one can regard $C^*(BG;k)$ as a strictly commutative $S$-algebra over the field $k$. The derived category $D(C^*(BG;k))$ has thus a structure of a tensor triangulated category via the left derived tensor product $-\otimes L C^*(BG;k) -$. The unit for the tensor product is $C^*(BG;k)$.

In this paper we apply techniques and results from [3–6] to classify the localising subcategories of $D(C^*(BG;k))$. More precisely, there is a notion of stratification for triangulated categories via the action of a graded commutative ring which implies that the localising subcategories are parameterised by sets of homogeneous prime ideals [4]. For $D(C^*(BG;k))$ we use the natural action of the endomorphism ring of the tensor identity which is isomorphic to the cohomology algebra $H^*(G,k)$ of the group $G$.

**Theorem 1.1.** The derived category $D(C^*(BG;k))$ is stratified by the ring $H^*(G,k)$. This yields a one to one correspondence between the localising subcategories of $D(C^*(BG;k))$ and subsets of the set of homogeneous prime ideals of $H^*(G,k)$. 

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It is proved in [6] that there is an equivalence of tensor triangulated categories between $D(C^*(BG; k))$ and the localising subcategory of $K(K\text{ln}kG)$ generated by the tensor identity. Here, $K(K\text{ln}kG)$ is the homotopy category of complexes of injective (= projective) $kG$-modules, studied in [6,9].

The main theorem of [5] states that $K(K\text{ln}kG)$ is stratified as a tensor triangulated category by $H^*(G, k)$. Theorem 1.1 is a consequence of a more general result concerning tensor triangulated categories, which is described below.

Let $(T, \otimes, \mathbb{1})$ be a compactly generated tensor triangulated category, as described in [3, §8], and $R$ a graded commutative Noetherian ring acting on $T$ via a homomorphism $R \to \text{End}_T^*(\mathbb{1})$. In this case, for each homogeneous prime ideal $p$ of $R$ there exists a local cohomology functor $\Gamma_p: T \to T$; see [3]. The support of an object $X$ in $T$ is then defined to be

$$\text{supp}_R X = \{ p \in \text{Spec} R \mid \Gamma_p X \neq 0 \}.$$ 

The condition that $T$ is stratified by the action of $R$ means that assigning a subcategory $S$ of $T$ to its support

$$\text{supp}_R S = \bigcup_{X \in S} \text{supp}_R X$$

yields a bijection between tensor ideal localising subcategories of $T$ and subsets of the homogeneous prime ideal spectrum $\text{Spec} R$ contained in $\text{supp}_T T$; see [4, Theorem 4.2]. Theorem 1.1 is thus a special case of the result below that relates tensor ideal localising subcategories of $T$ and the localising subcategories of $\text{Loc}_T(\mathbb{1})$, the localising subcategory of $T$ generated by the tensor unit. We note that $\text{Loc}_T(\mathbb{1})$ is a compactly generated tensor triangulated category in its own right and that $R$ acts on it as well.

**Theorem 1.2.** Suppose that the Krull dimension of $R$ is finite. If $T$ is stratified by $R$ as a tensor triangulated category, then so is $\text{Loc}_T(\mathbb{1})$, and there is a bijection

$$\begin{align*}
\{ \text{Tensor ideal localising subcategories of } T \} & \sim \{ \text{Localising subcategories of } \text{Loc}_T(\mathbb{1}) \}.
\end{align*}$$

It assigns each tensor ideal localising subcategory $S$ of $T$ to $S \cap \text{Loc}_T(\mathbb{1})$.

**Remark 1.3.** The theorem is not true without the assumption that $T$ is stratified by $R$. For example, let $T$ be the derived category of quasi-coherent sheaves on the projective line $\mathbb{P}^1$. The tensor unit is $\mathcal{O}$. In this example there are no proper localising subcategories of $\text{Loc}_T(\mathcal{O})$ since $\text{End}_T^*(\mathcal{O}) = k$, while there are many tensor ideal localising subcategories of $T$.

**Remark 1.4.** The assumption that the Krull dimension of $R$ is finite is artificial, and is used only to ensure that for each $X \in T$ and $p \in \text{Spec} R$ the object $\Gamma_p X$ belongs to $\text{Loc}_T(X)$. One can replace this condition by, for instance, the assumption that $T$ arises as the homotopy category of a Quillen model category [10, §6].

2. Localising subcategories of $\text{Loc}_T(\mathbb{1})$

In this section $T$ is a triangulated category with set-indexed coproducts and the tensor product $\otimes$ provides a symmetric monoidal structure with unit $\mathbb{1}$ on $T$, which is exact in each variable and preserves set-indexed coproducts.

The proof of Theorem 1.2 is based on a sequence of elementary lemmas. The first one describes the tensor ideal localising subcategory of $T$ which is generated by a class $C$ of objects; we denote this by $\text{Loc}_{C^\otimes}(T)$.

**Lemma 2.1.** Let $C$ be a class of objects of $T$. Then

$$\text{Loc}_{C^\otimes}(T) = \text{Loc}_T(\{ X \otimes Y \mid X \in C, Y \in T \}).$$

**Proof.** Set $S = \text{Loc}_T(\{ X \otimes Y \mid X \in C, Y \in T \})$. It suffices to show that $S$ is tensor ideal. This means that $FS \subseteq S$ for each tensor functor $F = - \otimes Y$, which is an immediate consequence of Lemma 2.2 below. □

**Lemma 2.2.** Let $F: U \to V$ be an exact functor between triangulated categories that preserves set-indexed coproducts. If $C$ is a class of objects of $U$, then

$$F \text{ Loc}_U(C) \subseteq \text{Loc}_V(FC).$$

**Proof.** The preimage $F^{-1} \text{ Loc}_V(FC)$ is a localising subcategory of $U$ containing $C$. Thus it contains $\text{Loc}_U(C)$, and one gets

$$F \text{ Loc}_U(C) \subseteq FF^{-1} \text{ Loc}_V(FC) \subseteq \text{Loc}_V(FC).$$
Lemma 2.3. Let \( \Gamma : T \to T \) be a colocalisation functor that preserves set-indexed coproducts. Then for any \( X \in T \) and \( Y \in \text{Loc}_T(\mathbb{1}) \), there is a natural isomorphism
\[
\Gamma X \otimes Y \sim \Gamma (X \otimes Y).
\]

Remark 2.4. There is an analogous result for a localisation functor \( L : T \to T \) that preserves set-indexed coproducts: For any \( X \in T \) and \( Y \in \text{Loc}_T(\mathbb{1}) \), there is a natural isomorphism \( L(X \otimes Y) \sim L(X \otimes Y) \).

Proof of Theorem 1.2. A colocalisation functor \( \Gamma \) comes with a natural morphism \( \Gamma X \to X \). Tensoring this with an object \( Y \in \text{Loc}_T(\mathbb{1}) \) gives a morphism \( \Gamma X \otimes Y \to X \otimes Y \) that factors through the natural morphism \( \Gamma (X \otimes Y) \to X \otimes Y \). Here, one uses that \( \Gamma X \otimes Y \) belongs to \( \Gamma T \), since the objects \( Y' \in T \) with \( \Gamma X \otimes Y' \in \Gamma T \) form a localising subcategory containing \( \mathbb{1} \). The induced morphism \( \phi_Y : \Gamma X \otimes Y \to \Gamma (X \otimes Y) \) is an isomorphism. To see this, observe that the objects \( Y' \in T \) such that \( \phi_Y \) is an isomorphism form a localising subcategory containing \( \mathbb{1} \).

Proposition 2.5. Suppose that the unit \( \mathbb{1} \) is compact in \( T \) and let \( \Gamma : T \to \text{Loc}_T(\mathbb{1}) \) denote the right adjoint of the inclusion \( \text{Loc}_T(\mathbb{1}) \to T \). If \( S \) is a localising subcategory of \( \text{Loc}_T(\mathbb{1}) \), then
\[
\text{Loc}_T^\otimes(S) \cap \text{Loc}_T(\mathbb{1}) = \Gamma(\text{Loc}_T^\otimes(S)) = S.
\]

Proof. We verify each of the following inclusions
\[
S \subseteq \text{Loc}_T^\otimes(S) \cap \text{Loc}_T(\mathbb{1}) \subseteq \Gamma(\text{Loc}_T^\otimes(S)) \subseteq S.
\]
The first one is clear. Composing the functor \( \Gamma \) with the inclusion \( \text{Loc}_T(\mathbb{1}) \to T \) yields a colocalisation functor that preserves set-indexed coproducts, since \( \mathbb{1} \) is compact. For an object \( X \) in \( \text{Loc}_T^\otimes(S) \cap \text{Loc}_T(\mathbb{1}) \), we have \( \Gamma X \cong X \). This gives the second inclusion. Applying Lemma 2.3 together with the description of \( \text{Loc}_T^\otimes(S) \) from Lemma 2.1 yields the third inclusion.

Corollary 2.6. Suppose that the unit \( \mathbb{1} \) is a compact object in \( T \). Assigning each localising subcategory \( S \) of \( \text{Loc}_T(\mathbb{1}) \) to \( \text{Loc}_T^\otimes(S) \) gives a bijection
\[
\left\{ \text{Localising subcategories } \text{of } \text{Loc}_T(\mathbb{1}) \right\} \sim \left\{ \text{Tensor ideal localising subcategories of } T \text{ generated by objects from } \text{Loc}_T(\mathbb{1}) \right\}.
\]

Proof. The inverse map sends \( U \subseteq T \) to \( U \cap \text{Loc}_T(\mathbb{1}) \).

We are now ready to prove Theorem 1.2. Note that in this \( T \) is a compactly generated tensor triangulated category, which entails a host of additional requirements; see [3, §8] for a list.

Proof of Theorem 1.2. It follows from Proposition 2.5 that the assignment
\[
S \mapsto \text{Loc}_T^\otimes(S)
\]
is an injective map from the localising subcategories of \( \text{Loc}_T(\mathbb{1}) \) to the tensor ideal localising subcategories of \( T \). In general, it is not bijective, as the example of Remark 1.3 shows. However, since \( T \) is stratified by \( R \) as a tensor triangulated category, it follows from [4, §7] that each tensor ideal localising subcategory is generated by a set of objects of the form \( \Gamma_p \mathbb{1} \). Since \( R \) has finite Krull dimension, [4, Theorem 3.4] yields that \( \Gamma_p \mathbb{1} \) is in \( \text{Loc}_T(\mathbb{1}) \). Therefore, given a tensor ideal localising subcategory \( U \) of \( T \), the localising subcategory
\[
U' = \text{Loc}_T((\Gamma_p \mathbb{1} \mid p \in \text{Supp}_R U)) \subseteq \text{Loc}_T(\mathbb{1})
\]
satisfies \( \text{Loc}_T^\otimes(U') = U \). This proves the surjectivity of the assignment. Moreover, we have shown that each localising subcategory of \( \text{Loc}_T(\mathbb{1}) \) is generated by objects of the form \( \Gamma_p \mathbb{1} \), so \( \text{Loc}_T(\mathbb{1}) \) is stratified by the action of \( R \); see [4, Theorem 4.2].

3. The cohomological nucleus

Let \( T(\otimes, \mathbb{1}) \) be a compactly generated tensor triangulated category and let \( R \) be a graded commutative Noetherian ring acting on \( T \) via a homomorphism \( R \to \text{End}_T(\mathbb{1}) \). Suppose in addition that \( R \) has finite Krull dimension.

We define the cohomological nucleus of \( T \) as the set of homogeneous prime ideals \( p \) of \( R \) such that there exists an object \( X \in T \) satisfying \( \text{Hom}_T(\mathbb{1}, X) = 0 \) and \( \Gamma_p X \neq 0 \). This definition is motivated by work of Benson, Carlson, and Robinson in the context of modular group representations [2].

For \( p \) in \( \text{Spec } R \) consider the tensor ideal localising subcategory
Proposition 3.1. Let \( p \) be a homogeneous prime ideal of \( R \). The following conditions are equivalent:

1. Every object \( X \) in \( T \) with \( \text{Hom}_\Gamma^1(\mathbb{1}, X) = 0 \) satisfies \( I_p X = 0 \).
2. One has \( \text{Loc}_T(I_p \mathbb{1}) = I_p T \).
3. Every localising subcategory of \( I_p T \) is a tensor ideal of \( T \).

Proof. The Krull dimension of \( R \) is finite, so \( I_p X \) is in \( \text{Loc}_T(X) \) for each \( X \) in \( T \), by [4, Theorem 3.4]. This fact is used without further comment.

(1) \( \Rightarrow \) (2): Set \( S = \text{Loc}_T(I_p \mathbb{1}) \). Note that \( S \subseteq I_p T \); we claim that equality holds. Indeed, \( S \subseteq \text{Loc}_T(\mathbb{1}) \) and also \( \text{Loc}_\Gamma^0(S) = I_p T \), since \( I_p = I_p \mathbb{1} \otimes \mathbb{1} \). Thus, for any \( X \) in \( I_p T \) from Proposition 2.5 one gets an exact triangle \( G X \rightarrow X \rightarrow X' \rightarrow \) with \( G X \in S \) and \( \text{Hom}_\Gamma^0(\mathbb{1}, X) = 0 \). Then (1) implies \( X' = 0 \) and hence \( X \in S \).

(2) \( \Rightarrow \) (3): Let \( S \) be a localising subcategory of \( I_p T \). Using (2) and the fact that \( I_p T \) is a tensor ideal of \( T \), one has \( \text{Loc}_\Gamma^0(S) \subseteq \text{Loc}_\Gamma(\mathbb{1}) \). Then it follows, again from Proposition 2.5, that \( S \) is a tensor ideal of \( T \).

(3) \( \Rightarrow \) (1): Assume \( \text{Hom}_\Gamma^1(\mathbb{1}, X) = 0 \); then \( \text{Hom}_\Gamma^1(\mathbb{1}, I_p X) = 0 \), as \( \mathbb{1} \) is compact. Condition (3) implies that \( \text{Loc}_T(I_p \mathbb{1}) = I_p T \). Thus \( I_p X \) belongs to \( \text{Loc}_T(I_p \mathbb{1}) \) and therefore also to \( \text{Loc}_T(X) \). So one obtains \( \text{Hom}_\Gamma^1(I_p X, I_p X) = 0 \), which implies \( I_p X = 0 \). □

Consider as an example for \( T \) the stable module category \( \text{StMod}kG \) of a finite group \( G \) with the canonical action of \( R = H^\ast(G, k) \). We refer to \([1,2]\) for the discussion of two variations of the nucleus, namely the group theoretic and the representation theoretic nucleus. There it is shown that \( \text{Loc}_T(\mathbb{1}) = T \) if and only if the centraliser of every element of order \( p \) in \( G \) is \( p \)-nilpotent and every block is either principal or semisimple, where \( p \) denotes the characteristic of the field \( k \).

It is convenient to define for any class \( C \) of objects of \( T \)

\[
\text{C}^\perp = \{ Y \in T \mid \text{Hom}_\Gamma^0(X, Y) = 0 \text{ for all } X \in C \}, \\
\uparrow C = \{ X \in T \mid \text{Hom}_\Gamma^0(X, Y) = 0 \text{ for all } Y \in C \}.
\]

Now let \( S = \text{Loc}_T(\mathbb{1}) \). The representation theoretic nucleus is by definition

\[
\bigcup_{X \in \uparrow S \cap T} \text{supp}_R X.
\]

Clearly, this is contained in the cohomological nucleus. It is a remarkable fact that the representation theoretic nucleus is non-empty if \( S \neq \emptyset \); this is proved in [1,2]. Moreover, Question 13 of [7] asks whether \( S = \uparrow(S^\perp \cap T) \). Note that \( S = \uparrow(S^\perp) \) follows from general principles.

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