On the Isomorphism Problem of $p$-Endomorphisms

By

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Abstract

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Let \( X = (X, \mathcal{B}, \mu, T) \) be a measure-preserving system on a Lebesgue probability space. Given a fixed probability vector \( p = (p_1, \ldots, p_s) \), we say that \( X = (X, \mathcal{B}, \mu, T) \) is a \( p \)-endomorphism if \( T \) is \( s \)-to-1 a.e. and the conditional probabilities of the preimages are precisely the components of \( p \). Two measure-preserving systems \( X = (X, \mathcal{B}, \mu, T) \) and \( Y = (Y, \mathcal{C}, \nu, S) \) are isomorphic if there exist a measure-preserving bijective map \( \varphi : X \rightarrow Y \) such that \( \varphi T = S \varphi \) a.e.

This thesis considers the isomorphism problem of \( p \)-endomorphisms, generalizing the work of Hoffman and Rudolph [H,R] which treats the case when \( p \) is a uniform probability vector, i.e. \( p = (1/p, \ldots, 1/p) \). In particular, we generalize the tvwB criterion introduced in Hoffman and Rudolph to prove two results.

The first result is Theorem 2.4.1, which generalizes the main theorem in [H,R] to \( p \)-endomorphisms. We paraphrase this as follows:

**Theorem 2.4.1.** Let \( X = (X, \mathcal{B}, \mu, T) \) be a \( p \)-endomorphism. Then \( X = (X, \mathcal{B}, \mu, T) \) is one-sided Bernoulli if and only if \( X = (X, \mathcal{B}, \mu, T) \) is tvwB.

We give two proofs of this result. The first follows Ornstein’s classical proof of his famous theorem that two shifts of equal entropy are isomorphic, and a second proof which follows the joinings proof as given in [H,R]. As a corollary of the joinings proof, we show that there are uncountably many automorphisms of the one-sided Bernoulli shift \( B^+(p) \) unless the components of \( p \) are pairwise distinct. We also give examples of tvwB \( p \)-endomorphisms such as mixing one-sided Markov shifts and a generalization of the \([T, Id]\) transformation.

The second main result is Theorem 5.1.1, which in view of Theorem 2.4.1, reduces to the statement that for any two tvwB finite group extensions of one-sided Bernoulli shifts, there is an isomorphism between them in a stronger sense than that asserted in Theorem 2.4.1. Specifically, we have the following theorem in Chapter 5 which we paraphrase as follows:

**Theorem 5.1.1’.** Let \( G \) be a finite group. For any two tvwB \( G \)-extensions of the one-sided shift \( B^+(p) \), there is an isomorphism which preserves the Bernoulli factor algebra and maps fibres over points in the factor to other such fibres by group rotations.
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Chapter 1:
Introduction

1.1. Background

Let $X$ be a compact metric space and $\mathcal{B}$ be its Borel $\sigma$-algebra. Suppose $(X, \mathcal{B}, \mu)$ is a nonatomic Lebesgue probability space (a probability measure space isomorphic to the unit interval with Lebesgue measure). An endomorphism of $X$ is a measure-preserving map $T : X \to X$, i.e. for all $B \in \mathcal{B}$, $\mu(T^{-1}B) = \mu(B)$. For us, a measure-preserving system (m.p.s.) is a quadruple $(X, \mathcal{B}, \mu, T)$ where $(X, \mathcal{B}, \mu)$ is a nonatomic Lebesgue probability space defined on a compact metric space $X$, and $T$ is an endomorphism.

This thesis is mainly concerned with the isomorphism problem of measure-preserving systems. Two measure-preserving systems $X = (X, \mathcal{B}, \mu, T)$ and $Y = (Y, \mathcal{C}, \nu, S)$ are (measure-theoretically) isomorphic if after deleting null sets $X_0$ and $Y_0$ from $X$ and $Y$ respectively, there is a bijection $\phi : X \setminus X_0 \to Y \setminus Y_0$ such that $\phi$ is measure-preserving and $\phi T = S\phi$ on $X \setminus X_0$. We say that $\phi$ is an isomorphism from $X$ to $Y$. In case $X = Y$, we say that $\phi$ is an automorphism of $X$. More generally, a factor map $\phi : X \to Y$ is a measure-preserving map $\phi : X \to Y$ such that $\phi T = S\phi$ a.e. In this case, we say that $Y$ is a factor of $X$, and $X$ is an extension of $Y$.

The classical isomorphism problem in ergodic theory is the classification of two-sided Bernoulli shifts. To define a two-sided Bernoulli shift, fix a probability vector $p = \ldots$
Consider the finite set $I = \{1, \ldots, s\}$ and a measure $m$ on $I$ defined by $m(j) = p_j$. Construct the product space $I^\mathbb{Z}$. Let $\mathcal{B}$ denote the Borel sigma-algebra and $\sigma$ denote the left-shift transformation defined by $\sigma(x)_j = x_{j+1}$, where $x_t$ is the $t$-th coordinate of $x$. The two-sided Bernoulli shift $B(p)$ is the m.p.s. $(I^\mathbb{Z}, \mathcal{B}, m^\mathbb{Z}, \sigma)$.

The breakthrough in the classification problem of two-sided Bernoulli shifts came when Kolmogorov introduced the concept of entropy into ergodic theory. It is not difficult to show that the entropy of a m.p.s. is invariant under isomorphism. Moreover, the entropy of a Bernoulli shift is easy to compute. Indeed, for a probability vector $p = (p_1, \ldots, p_s)$, define the entropy of $p$, denoted $h(p)$, to be $\sum_{j=1}^s -p_j \log_2 p_j$. It can be shown that the entropy of the two-sided Bernoulli shift $B(p)$ is just $h(p)$. It thus follows that the two-sided Bernoulli shift on two symbols with equal weights, which has entropy $\log_2 2$, is not isomorphic to the two-sided Bernoulli shift on three symbols with equal weights, which has entropy $\log_2 3$.

The solution of the classification problem was finally achieved around 1970 when Ornstein showed that entropy is in fact a complete invariant for 2-sided Bernoulli shifts; that is, two 2-sided Bernoulli shifts are isomorphic if and only if they have the same entropy. More generally, it can be shown that entropy is also a complete invariant for two-sided shift spaces with finitely determined measures. Since Ornstein’s original proof, other criteria such as the weak Bernoulli and the very weak Bernoulli conditions have been developed which also turn out to be sufficient for a m.p.s. to be isomorphic to a two-sided Bernoulli shift. The reader is referred to Shields for an excellent account of the proof of Ornstein’s theorem as well as a discussion of finitely determined measures. Examples of weak Bernoulli and very weak Bernoulli systems such as ergodic toral automorphisms and two-sided mixing Markov shifts are discussed in Petersen.

What we will deal with in this thesis is the isomorphism problem in the case when the endomorphism $T : X \rightarrow X$ is not invertible. Throughout this thesis, $\mathbb{N}^*$ will denote the set of nonnegative integers. One example of a non-invertible endomorphism is a one-sided Bernoulli shift, which is derived from a two-sided Bernoulli shift by restricting the shift space to $\{1, \ldots, s\}^{\mathbb{N}^*}$. Let $B^+(p)$ denote the m.p.s. obtained from $B(p)$ in this way. In this connection, we remark that Ornstein’s theorem does not hold for one-sided Bernoulli shifts. Indeed, it is easy to construct two probability vectors with different numbers of components which have the same entropy. This leads to two 1-sided shifts having the same entropy; however, they are not isomorphic because they have distinct numbers of inverse images.
In a recent paper, Hoffman and Rudolph made a fundamental contribution to the isomorphism problem of measure-preserving systems with non-invertible maps. In [H,R], Hoffman and Rudolph considered a class of endomorphisms called the uniformly p-to-1 endomorphisms, and introduced a condition called tree very weak Bernoulli (tvwB) on this class. They then proved that the tvwB condition is necessary and sufficient for a uniformly p-to-1 endomorphism to be isomorphic to the one-sided Bernoulli shift $B^+(p)$, where $p$ is the uniform vector $(1/p, \ldots, 1/p)$. As the authors remarked, entropy turns out to have no role in this theory. Perhaps as a result of this, many of the arguments are much simpler than those in Ornstein’s proof. This thesis will consider generalizations of the tvwB criterion which will allow us to prove isomorphism theorems for more general measure-preserving systems with non-invertible transformations.

1.2. Organization and Contributions of Thesis

Without delving into definitions and details, which will be presented after this section and in subsequent chapters, we outline below the contents and main results in each chapter of this thesis.

Chapter 1 is this introduction. In §1.3, we will define the class of objects of interest in this thesis, the $p$-endomorphisms, and extend the tvwB criterion to a general probability vector $p$ (i.e. $p$ not necessarily uniform).

Chapters 2 and 3 give two proofs of one of the main results in this thesis, Theorem 2.4.1, which generalizes the main result in [H,R] to $p$-endomorphisms for a general probability vector $p$:

**Theorem 2.4.1.** Let $p = (p_1, \ldots, p_s)$ be a probability vector. If $X = (X, \mathcal{B}, \mu, T)$ is a p-endomorphism, then $X$ is tvwB if and only if $X \cong B^+(p)$.

In Chapter 2, we give a proof which follows the classical argument used by Ornstein in his isomorphism theorem. The proof is based on a joint paper by del Junco and me [J,J] which treats the case when $p$ is a uniform probability vector. In particular, it does not require the machinery of joinings introduced in [H,R]. The main additional ingredient which enables us to extend the proof in [J,J] is proposition 2.1.3, which essentially says that under certain factor maps (tree-adapted factor maps) between $p$-endomorphisms, conditional probabilities
of inverse images are preserved. In §2.5, we give an entirely different and quite elementary proof of Theorem 2.4.1 in the special case when the components of \( p \) are pairwise distinct.

In Chapter 3, we will mirror the proof of the main result in [H,R] and extend their notion of one-sided joinings to give a proof of Theorem 2.4.1. The new ingredient needed is an additional condition in the definition of one-sided joinings which is trivial in the case of a uniform probability vector. We will also show that the joinings proof implies that there are uncountably many automorphisms of \( B^+(p) \), unless the components of \( p \) are pairwise distinct (in which case the identity is the only one).

Chapter 4 illustrates some examples of tvwB \( p \)-endomorphisms such as certain classes of one-sided Markov shifts and certain extensions of one-sided Bernoulli shifts. It follows that these are all one-sided Bernoulli by Theorem 2.4.1.

Chapter 5 proves an isomorphism theorem for tvwB finite group extensions of one-sided Bernoulli shifts. The main result is Theorem 5.1.1, which in view of Theorem 2.4.1 reduces to the following result which we paraphrase as follows:

**Theorem 5.1.1’.** For any two tvwB finite group extensions of the one-sided shift \( B^+(p) \), there exists an isomorphism which preserves the Bernoulli factor algebra and maps fibres over points in the factor to other such fibres by group rotations.

### 1.3. The Tree Very Weak Bernoulli Condition

Suppose that \((X, \mathcal{B}, \mu)\) is a nonatomic Lebesgue space and \( T : X \to X \) is measure-preserving. Fix a probability vector \( p \) with finitely many components. Let \( |p| \) denote the number of components of \( p \). A m.p.s. \( X = (X, \mathcal{B}, \mu, T) \) is a \( p \)-endomorphism if \( T \) is \( |p| \) to 1 a.e., the conditional probabilities of the \( |p| \) inverse images of \( x \) are the components of \( p \) for a.a. \( x \), and the entropy of \( X \) is \( h(p) \). Throughout this thesis, we will let \( \text{End}(p) \) denote the collection of \( p \)-endomorphisms.

The standard element in \( \text{End}(p) \) is the one-sided Bernoulli shift \( B^+(p) \). Let us recall its definition here. For the finite set \( I = \{1, \ldots, |p|\} \), define a measure \( m \) on \( I \) by \( m(j) = p_j \). Consider the product space \( I^{\mathbb{N}^*} \) with the Borel sigma-algebra \( \mathcal{B} \), product measure \( m^{\mathbb{N}^*} \) and shift transformation \( \sigma \). We then define \( B^+(p) \) to be the m.p.s. \((I^{\mathbb{N}^*}, \mathcal{B}, m^{\mathbb{N}^*}, \sigma)\). (The entropy condition in the definition of \( p \)-endomorphism is thus a natural one to make since \( B^+(p) \) has
In [H,R], Hoffman and Rudolph introduced a condition called tree very weak Bernoulli (tvwB) which they proved to be necessary and sufficient for a uniformly p-to-one endomorphism, i.e. a p-endomorphism with \( p = (1/p, \ldots, 1/p) \), to be isomorphic to the one-sided Bernoulli shift on \( p \) symbols with equal weights. We will now extend the various definitions in [H,R] in order to handle the case that the components of the probability vector are not all equal.

For a probability vector \( p \), we define the p-tree to be the set, denoted \( \mathcal{T} \), consisting of all finite sequences (including the empty sequence) of integers in \( \{1, \ldots, |p|\} \). We define a node to be an element of the p-tree. The length of a node \( v \), denoted \( |v| \), is the number of integers in the sequence \( v \) (thus, the empty sequence has length zero). We will refer to the empty sequence as the root node and denote it by \( \emptyset \). Given any two nodes \( u \) and \( v \), we define the node \( uv \) by concatenating the sequence \( u \) to the left of the sequence \( v \). We will use \( \mathcal{T}' \) to denote the set \( \mathcal{T} \setminus \emptyset \). Define the map \( \sigma : \mathcal{T}' \to \mathcal{T} \) by setting \( \sigma(v) \) to be the sequence obtained by deleting the leftmost symbol in the sequence \( v \). Note that if \( |v| = 1 \), then \( \sigma(v) = \emptyset \). Moreover, \( \sigma \) is a \( |p| \) to 1 surjection. It is helpful to picture a p-tree as a tree in the graph-theoretic sense with vertices corresponding to the nodes and with an edge between \( u \) and \( \sigma(u) \) for each \( u \neq \emptyset \). The picture we get is the usual \( |p| \)-ary tree. In view of this pictorial representation, we refer to the set \( \{uv \mid 0 \leq |u| \leq N\} \) as the subtree of height \( N \) rooted at \( v \) and the set \( \{uv \mid |u| \geq 0\} \) as the subtree rooted at \( v \).

To each node \( v \) in \( \mathcal{T} \), we assign a weight, denoted \( w_v \), as follows. For the root node \( \emptyset \), we set \( w_\emptyset = 1 \). For any other node \( v = (a_1, \ldots, a_j) \), set \( w_v = \prod_{i=1}^j p_{a_i} \). We define a tree automorphism to be a bijection \( A : \mathcal{T} \to \mathcal{T} \) such that \( A \circ \sigma(v) = \sigma \circ A(v) \) and \( w_v = w_{Av} \) for \( v \in \mathcal{T}' \) (i.e. \( A \) preserves the tree structure and weights). Note that this implies \( A(\emptyset) = \emptyset \).

It is obvious that the set \( A \) of tree automorphisms forms a group under composition. For \( N \in \mathbb{N} \), let \( \mathcal{T}_N \subseteq \mathcal{T} \) denote the set of nodes of length \( \leq N \) and let \( \mathcal{T}_N' \) denote the set \( \mathcal{T}_N \setminus \emptyset \). Let \( A_N \) be the subgroup of bijections of \( \mathcal{T}_N \) which is \( A|_{\mathcal{T}_N} \).

Given a fixed compact metric space \((R,d)\), let us say that a R-tree name is a function \( g : \mathcal{T}' \to R \), and for each \( N \in \mathbb{N} \), a R,N-tree name is a function \( g' : \mathcal{T}'_N \to R \). As in [H,R], we define a distance function, \( \bar{t}_N \), on the space of R,N-tree names as follows: if \( h : \mathcal{T}_N \to R \) and \( h' : \mathcal{T}_N' \to R \), let
\[ \tilde{t}_n(h, h') = \frac{1}{N} \inf_{A \in A_N} \sum_{0 < |v| \leq N} d(h(v), h'(Av))w_v. \]

Note that \( \tilde{t}_n \) is not a metric but it does satisfy the triangle inequality. We say that two \( R, N \)-tree names \( h \) and \( h' \) are the same up to tree automorphism if there exists some tree automorphism \( A \in A_N \) such that \( h(v) = h'(Av) \) for all \( v \). Note that \( \tilde{t}_n(h, h') = 0 \) if and only if \( h \) and \( h' \) are the same up to tree automorphism. We shall let \( R^{N^\tau} \) denote the set of \( R, N \)-tree names and \( R^{N^\vee} \) denote the equivalence classes of \( R, N \)-tree names modulo tree automorphism.

Suppose that \( X = (X, \mathcal{B}, \mu, T) \in \text{End}(p) \). We wish to construct \( R \)-tree names for each point in \( X \). To do this, fix a measurable partition \( K_X : X \to \{1, \ldots, |p|\} \) such that for a.a. \( x \), the \( |p| \) inverse images of \( x \) have distinct \( K_X \) values and the conditional probability of the inverse image \( x' \) of \( x \) with \( K_X(x') = j \) is \( p_j \). We will refer to \( K_X \) as a tree partition of \( X \). Note that unless the components of \( p \) are pairwise distinct, there will in general be many different tree partitions.

We now use \( K_X \) to define a collection of partial inverses \( T_v \) for each node \( v \in T' \). For each node \( v \) of length one (i.e., \( v \in \{1, \ldots, |p|\} \)), define the map \( T_v : X \to X \) by setting \( T_v x \) to be the inverse image \( x' \) of \( x \) with \( K_X(x') = v \). We may then extend the definition of \( T_v \) to an arbitrary node \( v \in T' \) as follows: if \( v = (a_1, \ldots, a_n) \), set \( T_v(x) = T_{a_1}(T_{a_2}(\ldots(T_{a_n}x)\ldots)) \).

Note that \( T_v \) is injective and \( v \mapsto T_v x \) maps \( T' \) to \( \{T^{-j}x \mid j > 0\} \) for a.a. \( x \). If \( y \in T^{-j}x \) for some \( j \in \mathbb{N} \), then there exists a unique \( v \) of length \( j \) such that \( y = T_v x \), and \( w_v \) is the conditional probability of the preimage \( y \) of \( x \) under the map \( T^j \).

Suppose we have some function \( g : X \to R \). Using the tree partition \( K_X \), we may now associate to a.a. \( x \) in \( X \) the \( R \)-tree name \( \tau_x^g : T' \to R \) by setting \( \tau_x^g(v) = g(T_v x) \). We shall refer to the \( R \)-tree name \( \tau_x^g \) as the \( g \)-tree name of \( x \) and the restriction \( \tau_x^g|_{T_x^g} \) as the \( g \)-\( N \)-tree name of \( x \). For any function \( f : X \to R \) and \( n \in \mathbb{N} \), define \( f^{n^\tau} : X \to R^{n^\tau} \) by sending \( x \) to its \( f \)-\( n \)-tree name and \( f^{n^\vee} : X \to R^{n^\vee} \) by sending \( x \) to the equivalence class in \( R^{n^\vee} \) containing its \( f \)-\( n \)-tree name.

Note that for any \( g : X \to R \), the \( R \)-tree name \( \tau_x^g \) depends on the choice of the tree partition \( K_X \). However, the tvwB condition, which we now define for \( p \)-endomorphisms, is not affected by the choice (since for any \( g : X \to R \), different choices yield the same \( g \)-\( N \)-tree name of \( x \) up to tree automorphism). Note that this definition is essentially the same as the definition in [H,R]. The only difference is that our definition of \( \tilde{t}_n \) is slightly different as the
group of tree automorphisms is more restrictive in the case of a general probability vector $p$.

**Definition 1.3.1.** Let $X = (X, \mathcal{B}, \mu, T) \in \text{End}(p)$ and $g : X \to R$ for a compact metric space $(R, d)$. We say that $(X, g)$ is **tree very weak Bernoulli** (tvwB) if for each $\varepsilon > 0$, there exists some $N$ such that, whenever $n \geq N$, we have some set $G \subseteq X$ of measure at least $1 - \varepsilon$ with $\bar{t}_n(\tau_g^x, \tau_g^y) < \varepsilon$ for all $x$ and $y$ in $G$.

We shall say that the a.m.p.s. $X \in \text{End}(p)$ is **tvwB** if for all measurable functions $g : X \to R$ for a compact metric space $R$, $(X, g)$ is tvwB. It is immediate that the tvwB property is preserved under isomorphism.
Chapter 2:
Tvwb p-Endomorphisms on Lebesgue Spaces

The goal in this chapter is to prove the following theorem stated in §2.4, which generalizes the main result in [H,R] to an arbitrary probability vector.

**Theorem 2.4.1.** Let \( p = (p_1, \ldots, p_s) \) be a probability vector. If \( X = (X, \mathcal{B}, \mu, T) \in \text{End}(p) \), then \( X \) is tvwb if and only if \( X \cong B^+(p) \).

This chapter is organized into several sections. The first three sections establish some basic tools that will be necessary in the proof of Theorem 2.4.1. In §2.1, we define the concept of a tree-adapted factor map and establish a crucial property shared by these factor maps. §2.2 proves the tree ergodic theorem, which may be considered as a “backward” version of Birkhoff’s ergodic theorem. §2.3 establishes the tree Rokhlin lemma and its strong form, which are analogues of the standard Rokhlin lemma for invertible transformations. §2.4 contains the proof of the main theorem. The proof in §2.4 does not require the machinery of one-sided joinings introduced in [H,R] and is patterned on the proof of Theorem 2.4.1 given in a joint paper by del Junco and me [J,J] in the case that \( p \) is a uniform probability vector. §2.5 gives a completely different, and quite simple, proof of Theorem 2.4.1 in the special case that \( p \) has pairwise distinct components.
Throughout this chapter, \( p \) will denote the probability vector \((p_1, \ldots, p_s)\).

### 2.1. Tree-Adapted Factors

**Definition 2.1.1.** Let \( X = (X, \mathcal{B}, \mu, T) \in \text{End}(p) \), then a m.p.s. \( Y = (Y, \mathcal{C}, \nu, S) \) is a tree-adapted factor of \( X \) if there is a factor map \( \phi : X \to Y \) such that for a.a. \( x \) in \( X \), \( \phi \) gives a bijection of the inverse images of \( x \) and those of \( \phi(x) \). We will refer to the factor map \( \phi : X \to Y \) as a tree-adapted factor map.

Note that this definition is more restrictive than in [H,R], which requires only that \( \phi \) maps \( T^{-1}x \) one-to-one into \( S^{-1}\phi(x) \). Nonetheless, this definition will suffice since the factor maps which we construct in the proof of the isomorphism theorem (in particular, in Proposition 2.4.9) is tree-adapted in our sense.

The following proposition shows that a tree-adapted factor of a \( p \)-endomorphism is itself a \( p \)-endomorphism. It is the extension of Lemma 2.3 in [H,R] to \( p \)-endomorphisms. For a \( p \)-endomorphism \( X = (X, \mathcal{B}, \mu, T) \), define the function \( p_X : X \to (0,1) \) by setting \( p_X(x) \) to be the conditional probability of the preimage \( x \) of \( Tx \). We shall refer to \( p_X \) as the \( p \)-function on \( X \).

**Proposition 2.1.2.** A tree-adapted factor of a \( p \)-endomorphism is a \( p \)-endomorphism.

**Proof:** Let \( X = (X, \mathcal{B}, \mu, T) \in \text{End}(p) \). Suppose \( Y = (Y, \mathcal{C}, \nu, S) \) is a tree-adapted factor of \( X \) and \( \phi : X \to Y \) is a tree-adapted factor map. By definition, \( S \) is also \([p] \)-to-one a.e. Fix a point \( y \in Y \), and suppose \( y_1, \ldots, y_s \) are the inverse images of \( y \). For each \( x \in \phi^{-1}y \), the conditional measure on \( T^{-1}x \) given \( x \) pushes forward via \( \phi \) to a measure on \( S^{-1}y \). The conditional measure on \( S^{-1}y \) given \( y \) is just an average of these image measures on \( S^{-1}y \) over all \( x \in \phi^{-1}y \). As \( \phi \) is tree-adapted and \( X \in \text{End}(p) \), these image measures assign the \( s \) inverse images of \( y \) with measures equal to the components of \( p \). Thus, \((p_Y(y_1)), \ldots, p_Y(y_s))\) is an average of probability vectors \((p_{\sigma(1)}, \ldots, p_{\sigma(s)})\) for a permutation \( \sigma : \{1, \ldots, s\} \to \{1, \ldots, s\} \).

The fact that the function \( h(t) = -t \log_2 t \) is strictly concave implies that the entropy of the probability vector \((p_Y(y_1)), \ldots, p_Y(y_s))\) is at least \( h(p) \) with equality holding if and only if this probability vector is some permutation of \( p \). The fact that the entropy of \( X \) is \( h(p) \) implies that equality must hold for a.a. \( y \) and so the entropy of \( Y \) is \( h(p) \). But this is precisely saying that \( Y = (Y, \mathcal{C}, \nu, S) \) is a \( p \)-endomorphism. ■
The following proposition gives an important property shared by tree-adapted factor maps; namely, they preserve the $p$-function.

**Proposition 2.1.3.** Suppose $X = (X, \mathcal{B}, \mu, T) \in \text{End}(p)$. If $\phi : X \to Y$ is a tree-adapted factor map, then for a.a. $x$ in $X$, $p_X(x) = p_Y(\phi(x))$.

**Proof:** By proposition 2.1.2, $Y = (Y, \mathcal{C}, \nu, S) \in \text{End}(p)$. Let $y \in Y$ and let $\mu_y$ be the conditional measure on $\phi^{-1}(y)$ given $y$. With no loss of generality, suppose $p_1 \geq \ldots \geq p_s$. Let $y_1, \ldots, y_s$ be the inverse images of $y$ such that $p_Y(y_j) = p_j$. Since $\phi$ is tree-adapted, for each $x \in \phi^{-1}(y)$, let $x_{y_j}$ be the unique inverse image of $x$ such that $\phi(x_{y_j}) = y_j$. Now, $p_1 = p_Y(y_1)$ is an average of $p_X(x_{y_1})$ over $x \in \phi^{-1}(y)$. As $p_X(x_{y_1}) \leq p_1$, it follows that $p_Y(y_1) = p_1 = p_X(x_{y_1})$ for $\mu_y$-a.a. $x$. Next, note that since $x_{y_1} \neq x_{y_2}$ by tree-adaptedness, $p_X(x_{y_2}) \leq p_2$ for $\mu_y$-a.a. $x$. Hence, $p_Y(y_2) = p_2 = p_X(x_{y_2})$ for $\mu_y$-a.a. $x$. Inductively, we see that $p_Y(y_j) = p_X(x_{y_j})$ for $\mu_y$-a.a. $x$ for each $1 \leq j \leq s$. As $y$ is arbitrary, there is a set $G$ of full measure in $X$ such that for each $x \in G$, if $x' \in T^{-1}x$, then $p_X(x') = p_Y(\phi(x'))$. Then $T^{-1}G$ is also a set of full measure satisfying the statement of the proposition. □

If $X = (X, \mathcal{B}, \mu, T) \in \text{End}(p)$ and $x \in X$, define the $p_X$-name of $x$ to be the sequence $(p_X(x), p_X(Tx), \ldots)$. Proposition 2.1.3 implies that if $\phi : X \to Y$ is a tree-adapted factor map, then the $p_X$-name of $x$ equals the $p_Y$-name of $\phi(x)$ for a.a. $x$.

The fact that the $p$-function is preserved under tree-adapted factor maps leads to the following proposition which will be used in the proof of Theorem 2.4.1.

**Proposition 2.1.4.** Suppose $X = (X, \mathcal{B}, \mu, T) \in \text{End}(p)$ and $Y = (Y, \mathcal{C}, \nu, S)$ is a tree-adapted factor of $X$. Consider a tree-adapted factor map $\phi : X \to Y$. For any measurable function $f : Y \to R$ and any $m \in \mathbb{N}$, we have for a.a.$x$,

$$(f \circ \phi)^m \nu(x) = f^m \nu(\phi(x)).$$

**Proof:** For $x \in X$, define the map $\pi_x : \mathcal{T}_m' \to \mathcal{T}_m'$ by setting $\pi_x(v) = u$ if $\phi(T_v x) = S_u \phi(x)$ and $|v| = |u|$. Note that this map is well-defined and is a bijection as $\phi$ is tree-adapted. Moreover, since $\phi$ is a factor map, it follows that $\pi_x \circ \sigma = \sigma \circ \pi_x$. By the remarks following proposition 2.1.3, $p_X$-name of $z = p_Y$-name of $\phi(z)$ for a.a. $z$ and thus $\pi_x$ preserves the weights of nodes so that $\pi_x$ defines a tree automorphism. It follows that for every $v \in \mathcal{T}_m'$,

$$\tau_x^{f \circ \phi}(v) = f \circ \phi(T_v x) = f(S_{\pi_x(v)} \phi(x)) = \tau_x^{f \phi}(\pi_x(v)).$$

□
2.2. The Tree Ergodic Theorem

The goal in this section is to demonstrate a tree version of the ergodic theorem (proposition 2.2.3). To begin, notice that the following is a trivial consequence of the definition of tvwB of a m.p.s. $X \in \text{End}(p)$.

**Proposition 2.2.1.** Suppose $X \in \text{End}(p)$ is tvwB, then $X$ is ergodic.

**Proof:** Consider the characteristic function $g = \chi_G$ for an invariant set $G$. Then $(X, \chi_G)$ can only be tvwB if $G$ has measure zero or one. ■

For a m.p.s. $X = (X, \mathcal{B}, \mu, T)$ and a real-valued function $g$ on $X$ and $x \in X$, set

$$A_N(g)(x) = \frac{1}{N} \sum_{i=0}^{N-1} g(T^i x).$$

The following proposition is proved using Birkhoff’s ergodic theorem.

**Proposition 2.2.2.** Suppose $X = (X, \mathcal{B}, \mu, T) \in \text{End}(p)$ is ergodic. Let $B \subseteq X$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have a set $G$ of measure $\geq 1 - \varepsilon$ such that for each $x \in G$,

$$\left| \frac{1}{n} \sum_{0 < |v| \leq n} w_v \chi_B(T_v x) - \mu(B) \right| < \varepsilon.$$

**Proof:** Note that

$$\frac{1}{n} \sum_{0 < |v| \leq n} w_v \chi_B(T_v x) = \sum_{|v|=n} w_v A_n(\chi_B)(T_v x). \quad (1)$$

Since $X$ is ergodic, we have for $a.a. x$, $A_n(\chi_B)(x) \to \mu(B)$ as $n \to \infty$. In particular, for $\delta > 0$, there exists some $N$ such that for all $n \geq N$ and for all $y$ in a set $G$ of measure $\geq 1 - \delta$,

$$|A_n(\chi_B)(y) - \mu(B)| < \delta. \quad (2)$$

Hence, if $\mu_x$ is the conditional measure on $T^{-n}x$ given $x$, then

$$\mu_x(G) \geq 1 - \sqrt{\delta}. \quad (3)$$

for all $x$ in a set $G'$ of measure at least $1 - \sqrt{\delta}$. For each $x \in G'$, by (2) and (3), we have

$$\left| \sum_{|v|=n} w_v A_n(\chi_B)(T_v x) - \mu(B) \right| \leq \left| \sum_{T_v x \in G \atop |v|=n} w_v A_n(\chi_B)(T_v x) - \sum_{T_v x \in G \atop |v|=n} w_v \mu(B) \right|$$

$$+ \left| \sum_{T_v x \in G \atop |v|=n} w_v A_n(\chi_B)(T_v x) - \sum_{T_v x \in G \atop |v|=n} w_v \mu(B) \right|$$

$$< \delta + 2 \sqrt{\delta}.$$
Since \( \delta \) can be chosen to be arbitrarily small, the result now follows from (1).

Let \( X = (X, B, \mu, T) \in \text{End}(p) \) and consider a partition \( P : X \to C \) for some finite set \( C \). Given \( M \in \mathbb{N} \) and \( x \in X \), define a measure \( \theta_{x, M, P} \) on \( C \) by

\[
\theta_{x, M, P}(c) = \frac{1}{M} \sum_{v \in T^i_x | P(T^i x) = c} w_v
\]

for each \( c \in C \). We say that \( x \in X \) is \( \varepsilon, M \)-generic for \( P \) if

\[
\sum_{c \in C} |\theta_{x, M, P}(c) - \mu(P^{-1}(c))| < \varepsilon.
\]

The following is the Tree Ergodic Theorem and follows easily from proposition 2.2.2.

**Proposition 2.2.3 (Tree Ergodic Theorem).** Let \( X = (X, B, \mu, T) \) be an ergodic \( p \)-endomorphism and \( P : X \to C \) be a finite partition. For every \( \varepsilon > 0 \), we have for all large \( M \), a set \( G \) of measure \( 1 - \varepsilon \) such that for each \( x \in G \), \( x \) is \( \varepsilon, M \)-generic for \( P \).

### 2.3. The Tree Rokhlin Lemma

Let \( X = (X, B, \mu, T) \) be an ergodic m.p.s. A \( \varepsilon \)-tree Rokhlin tower of height \( N + 1 \) in \( X \) is a collection of pairwise disjoint measurable sets \( \{B_0, B_1, \ldots, B_N\} \) in \( X \) such that \( B_j = T^{-j}B_0 \) and whose union \( \bigcup_{i=0}^N B_i \) has measure \( > 1 - \varepsilon \). If \( M \) is such a tree Rokhlin tower, we will let \( \bigcup M \) denote the union of the sets in \( M \). We will refer to the set \( B_0 \) as the base of the tower and the set \( B_j = T^{-j}B_0 \) as the \( j \)-th level of the tower. If \( B' \subseteq B \), we will refer to the union \( \bigcup_{i=1}^N T^{-i}B' \) as the column of the tower over \( B' \). We now establish the analogues for \( p \)-endomorphisms of the standard Rokhlin lemma and the Strong Rokhlin lemma. Their proofs follow along the same lines as propositions 5.2 and 5.3 in [H,R], although their results were only stated for uniformly \( p \)-to-1 endomorphisms. For completeness, we include the proofs below.

**Proposition 2.3.1 (Tree Rokhlin Lemma).** Let \( X = (X, B, \mu, T) \) be ergodic, then for each \( \varepsilon > 0 \) and \( N \in \mathbb{N} \), there exists a \( \varepsilon \)-tree Rokhlin tower of height \( N+1 \) in \( X \).

**Proof:** For a set \( D \subseteq X \) of positive measure, define the set

\[
B = \{x \in X \mid \min(i \geq 0 \mid T^i x \in D) \equiv 0 \text{ mod}(N + 1)\} \setminus D
\]
Note that $B$ is well-defined since by ergodicity, for a.a. $x$ in $X$, $T^ix \in D$ for some $i \in \mathbb{N}$. It suffices to show that $B \cap T^{-j}B = \emptyset$, for each $1 \leq j \leq N$. Suppose on the contrary, $x \in B \cap T^{-j}B$ for some $1 \leq j \leq N$. Then there exists $y \in B$ such that $y = T^jx$. By the definition of $B$, the set $\{x, Tx, ..., T^{j-1}x, T^jx = y\}$ is disjoint from $D$, hence we have
\[
\min(t \geq 0 \mid T^tx \in D) = \min(t \geq 0 \mid T^ty \in D) + j.
\]
Since $x \in B$ and $y \in B$, taking mod$(N+1)$ of both sides forces $j = 0$, which is a contradiction. Hence $B \cap T^{-j}B = \emptyset$. Let $\mathbf{M} = \{T^{-i}B \mid 0 \leq i \leq N\}$.

Now, as $X$ is ergodic, $\mu(B) \geq 1/(N+1) - \mu(D)$ and so
\[
\mu(\cup_{i=0}^{N} T^{-i}B) = (N+1)\mu(B) \geq 1 - (N+1)\mu(D).
\]
Choosing $D$ such that $\mu(D) < \varepsilon/(N+1)$ shows that the Rokhlin tower $\mathbf{M}$ has the desired property. $lacksquare$

Before we prove the Strong Tree Rokhlin lemma, we introduce some notations which we will use in the rest of this thesis concerning distributions of partitions on a probability measure space. For a finite set $C$, define the total variation norm on the probability measures on $C$ by
\[
|\rho - \theta| = \sum_{c \in C} |\rho(c) - \theta(c)|.
\]
We write $\rho \sim \theta$ if $|\rho - \theta| < \varepsilon$. Suppose $(\Omega, \mathcal{B}, \lambda)$ is a probability measure space. For a subset $G \subseteq \Omega$ of positive measure and a measurable finite partition $P : \Omega \to C$, let $P|G$ denote the restriction of $P$ to $G$. Let $\lambda_G$ be a probability measure on $\mathcal{B}$ defined by $\lambda_G(B) = \lambda(B \cap G)/\lambda(G)$ for $B \subseteq \Omega$. Obviously, $\lambda = \lambda_G$ if $G = \Omega$. For a partition $Q : G \to C$, let $\dist_{\lambda_G}(Q)$ be the probability measure on $C$ defined by $\dist_{\lambda_G}(Q)(c) = \lambda_G(Q^{-1}(c))$ for each $c \in C$. We will refer to the sets $Q^{-1}(c) \subseteq G$, $c \in C$, as the atoms of $Q$. Where there is no ambiguity as to the measure on the domain of the partition, we will generally omit the subscript and just write $\dist(Q)$. For a measure preserving transformation $T$ on $(\Omega, \mathcal{B}, \lambda)$ and a partition $P : \Omega \to C$, we define the $P$-name of $x$ to be the infinite sequence $(P(x), P(Tx), \ldots)$ and the $P$-name of $x$ to be the finite sequence $(P(x), \ldots, P(T^{n-1}x))$.

If $P : \Omega \to C$ and $Q : \Omega \to C'$ are partitions into finite sets $C$ and $C'$, we say that $Q \leq P$ if there exists a map $\pi : C \to C'$ such that $Q = \pi \circ P$, i.e. knowing $P(x)$ determines $Q(x)$.

We will need the following technical lemma that guarantees the measurability of the atoms of the various partitions that will be constructed in the proof of the Strong Tree Rokhlin lemma and in the proof of Theorem 2.4.1.

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Lemma 2.3.2. Suppose $X = (X, \mathcal{B}, \mu, T) \in \text{End}(p)$. Consider a measurable finite partition $P : X \to C$. Then for any $C,n$-tree name $\tau$, the set $\{x \in X \mid \tau^P_x = \tau \text{ on } T^*_n\}$ is measurable.

Proof: Note that $\tau$ induces a subset in $C^n$ whose elements are indexed by the nodes of length $n$. In fact, if $|v| = n$, we may associate the sequence $\tau(v) = (\tau(v), \ldots, \tau(\sigma^{n-1}v)) \in C^n$ to $v$. Let $\tilde{C}(v)$ denote the set $\{x \in X \mid P$-n-name of $x$ is $\tilde{\tau}(v)\}$. Let $G = \cup_{|v| = n} T_v X \cap \tilde{C}(v)$, which is clearly measurable by the definition of the partial inverses $T_v$ in §1.3. For $x \in X$, let $\mu_x$ be the conditional measure on $T^{-n}x$ given $x$. Notice that the function $g : X \to [0,1]$ defined by $g(x) = \mu_x(G)$ is measurable. In particular, the set $\{x \in X \mid g(x) = 1\}$ is measurable. However, this is precisely the set $\{x \in X \mid \tau^P_x = \tau \text{ on } T^*_n\}$. ■

Proposition 2.3.3 (Strong Tree Rokhlin lemma). Let $X \in \text{End}(p)$ be ergodic. Consider a measurable partition $P : X \to C$. Then for each $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists an $\varepsilon$-tree Rokhlin tower of height $N+1$ in $X$ whose base $B$ is independent of $P$, i.e. $\text{dist}(P|B) = \text{dist}(P)$.

Proof: Let $N'$ be an integer and $\varepsilon' > 0$, both to be specified later. Use the Tree Rokhlin lemma to build an $\varepsilon'/2$-tree Rokhlin tower $M'$ of height $N' + 1$ with base $D$. Let $\tilde{P}$ be the partition $P^{N'}|D$. By Lemma 2.3.2, $\tilde{P}$ defines a measurable partition of $D$. For each atom $\alpha$ in $\tilde{P}$, divide $\alpha$ into $N+1$ measurable disjoint sets $\beta_0^\alpha, \ldots, \beta_N^\alpha$ of equal measure. Consider the set

$$B' = \bigcup_{\alpha \in \tilde{P}} \bigcup_{0 \leq i \leq N} \bigcup_{j \equiv i \mod (N+1)} T^{-j} \beta_i^\alpha.$$ 

Notice that $B'$ is disjoint from $T^{-j}B'$ for all $0 < j \leq N$. Let $M = \{T^{-i}B' \mid 0 \leq i \leq N\}$. As $M$ contains the $(N+1)^{st}$ to $(N' - N)^{th}$ levels of $M'$, we may choose $N'$ large enough and $\varepsilon'$ small enough such that the union of the sets in $M$ has measure $1 - \varepsilon/2$.

From our definition of $B'$, we have

$$\text{dist}(P|B') = \text{dist}(P \bigcup_{i=1}^{N'-N} T^{-i}D).$$ 

For any $\eta > 0$, by decreasing $\varepsilon'$ and increasing $N'$ if necessary, we have

$$|\text{dist}(P|B') - \text{dist}(P)| \leq \eta.$$ 

If $\eta > 0$ is small enough, we may remove at most $\varepsilon/2$ fraction of $B'$ to arrive at a set $B$ such that $\text{dist}(P|B) = \text{dist}(P)$ and $B$ forms the base of an $\varepsilon$-tree Rokhlin tower of height $N + 1$. ■
2.4. Proof of the Isomorphism Theorem

Unless otherwise specified, all m.p.s. in this section are \( p \)-endomorphisms. The purpose of this section is to prove the following theorem.

**Theorem 2.4.1.** Let \( X = (X, \mathcal{B}, \mu, T) \in \text{End}(p) \), then \( X \) is tvwB if and only if \( X \cong B^+(p) \).

Our first goal is to prove that \( B^+(p) \) is tvwB, which immediately proves half of Theorem 2.4.1. To do this, we will prove an extension of Lemma 2.5 in [H,R] to \( \text{End}(p) \), which shows that any tree-adapted factor of a tvwB \( p \)-endomorphism remains tvwB. Before we prove this (Proposition 2.4.3), a technical lemma is in order.

**Lemma 2.4.2.** Given a m.p.s. \( X = (X, \mathcal{B}, \mu, T) \), for each \( n \in \mathbb{N}, \delta > 0 \) and \( B \subseteq X \) such that \( \mu(B) > 1 - \delta \), there exists a set \( G \) of measure \( > 1 - \sqrt{\delta} \) such that for each \( x \in G \),

\[
\frac{1}{n} \sum_{0 < |v| \leq n} w_v \chi_B(T_v x) > 1 - \sqrt{\delta}.
\]

**Proof:** For each \( k \geq 0 \), let

\[
w_k(x, B) = \sum_{\substack{|v| = k \\cap T_v x \in B}} w_v.
\]

Clearly, we have

\[
\mu(B) = \int w_k(x, B) \, d\mu(x).
\]

Hence,

\[
\mu(B) = \int \frac{1}{n} \sum_{i=1}^{n} w_i(x, B) \, d\mu(x) > 1 - \delta.
\]

So,

\[
\frac{1}{n} \sum_{i=1}^{n} w_i(x, B) > 1 - \sqrt{\delta}
\]

except on a set of measure at most \( \sqrt{\delta} \). We finish the result by observing that

\[
\frac{1}{n} \sum_{i=1}^{n} w_i(x, B) = \frac{1}{n} \sum_{0 < |v| \leq n} w_v \chi_B(T_v x).
\]

Suppose we have a measurable function \( g : X \to \mathbb{R} \) for a compact metric space \((R, d)\). We say that \( g \) is **tree-adapted** if for a.a. \( x \), \( g \) assigns different values to the inverse images of \( x \). If \( \mathcal{D} \) is the Borel \( \sigma \)-algebra of \((R, d)\), we say that \( g \) is **generating** if \( B = \bigvee_{i=0}^{\infty} T^{-i} g^{-1}(\mathcal{D}) \).
It is clear that a generating \( g \) is necessarily tree-adapted. We now prove the extension of Lemma 2.5 in [H,R]. The proof follows along similar lines, though we will also need the fact that tree adapted maps preserve the \( p \)-function (Proposition 2.1.3).

**Proposition 2.4.3.** Let \( X = (X, E, \mu, T) \in \text{End}(p) \) and let \( Y = (Y, C, \nu, S) \) be a tree-adapted factor of \( X \). Suppose the function \( h : X \to R \) is generating. If \( (X, h) \) is tvwB, then for any function \( g : Y \to R' \) into a compact metric space \( (R', d') \), \( (Y, g) \) is tvwB.

**Proof:** By normalizing \( d \) and \( d' \), we may assume that the metric spaces \( R \) and \( R' \) have unit diameter. Let \( \phi : X \to Y \) be a tree-adapted factor map. Now, \( h \) is generating and \( g \circ \phi \) is \( \mathcal{B} \)-measurable. Hence, for each \( \varepsilon > 0 \), there exist some \( s \in \mathbb{N} \), \( 0 < \delta < \varepsilon \) and a set \( G_\varepsilon \) of measure \( > 1 - \varepsilon \) such that whenever \( x \) and \( x' \) are in \( G_\varepsilon \) and \( d(h(T^i x), h(T^i x')) < \delta \) for all \( 0 \leq i \leq s \), then \( d'(g(\phi(x)), g(\phi(x'))) < \varepsilon \).

Since \( (X, h) \) is tvwB, for a sufficiently large \( N \), there exists a set \( G \subseteq X \) of measure \( > 1 - \delta^2 \) such that whenever \( z \) and \( z' \) are in \( G \),

\[
\tilde{t}_N(\tau_x, \tau_{z'}) < \delta^2. \tag{1}
\]

Moreover, by Lemma 2.4.2, we have a set \( G' \subseteq X \) of measure \( > 1 - \sqrt{\varepsilon} \) such that for each \( x \in G' \)

\[
\frac{1}{N} \sum_{0 < |v| \leq n} w_v x_G_x(T_v x) > 1 - \sqrt{\varepsilon}. \tag{2}
\]

Fix a pair of points \( x \) and \( x' \) in \( G \cap G' \). Using (1), we have some tree automorphism \( A \in \mathcal{A} \) such that

\[
\frac{1}{N} \sum_{0 < |v| \leq N} w_v d(h(T_v x), h(T_{Av} x')) < \delta^2. \tag{3}
\]

Let \( D_{\delta}^{s+} = \{ v \in T' \mid d(h(T^i(T_v x)), h(T^i(T_{Av} x'))) \geq \delta \text{ for some } 0 \leq i \leq s \} \). Then by (3),

\[
\frac{1}{N} \sum_{v \in D_{\delta}^{s+}} w_v < (s + 1)\delta + s/N. \tag{4}
\]

Let \( V = \{ v \in T' \mid v \notin D_{\delta}^{s+}, T_v x \in G_\varepsilon, T_{Av} x' \in G_\varepsilon \} \). Then notice that whenever \( v \in V \), by our choice of \( \delta \),

\[
d'(g(\phi(T_v x)), g(\phi(T_{Av} x'))) < \varepsilon. \tag{5}
\]

Note that the tree automorphism \( A \) corresponds to a bijection of the trees of inverse images of \( x \) and \( x' \). This in turn pushes down via \( \phi \) to a bijection of the trees of inverse images
of \( \phi(x) \) and \( \phi(x') \). Since \( \phi \) is tree-adapted and hence preserves the \( p \)-function (proposition 2.1.3), this last bijection in turn yields a tree automorphism \( A' \). By (2), (4) and (5), we have

\[
\frac{1}{N} \sum_{0 < |v| \leq N} w_v d'(\phi(S_{A'}(x)), g(S_{A'}(x'))) = \frac{1}{N} \sum_{0 < |v| \leq N} w_v d'(\phi(T_v(x)), g(T_{A'}(x')))
\]

\[
\leq \frac{1}{N} \sum_{0 < |v| \leq N} w_v + \frac{1}{N} \sum_{0 < |v| \leq N} w_v + \frac{1}{N} \sum_{v \in D^+_x} w_v
\]

\[
+ \frac{1}{N} \sum_{v \in V} w_v d'(g(\phi(T_v(x)), g(\phi(T_{A'}(x'))))
\]

\[
< \sqrt{\varepsilon} + \sqrt{\varepsilon} + (s + 1)\delta + s/N + \varepsilon < 5\sqrt{\varepsilon}
\]

for all sufficiently large \( N \) and small \( \delta \).

As \( \mu(G' \cap G) > 1 - 2\sqrt{\varepsilon} \), if \( \varepsilon \) is sufficiently small, then there will be a large set \( H \subseteq Y \) such that each \( y \in H \) is the image of some point in \( G' \cap G \). Thus, the result follows from the preceding calculation. ■

Proposition 2.4.3 implies that if \((X, f)\) is tvwB for some generating \( f \), then \((X, g)\) is also tvwB for any compact-valued function \( g \). We thus see that the m.p.s. \( X \) is tvwB if and only if \((X, f)\) is tvwB for some generating \( f \). Hence, \( B^+(p) \) is tvwB since its standard generator (zero coordinate partition) yields the same tree name for all points in \( B^+(p) \) by choosing the tree partition \( K_{B^+(p)} \) to be the zero coordinate partition. We have thus proved the easier half of Theorem 2.4.1, which we record as the proposition below.

**Proposition 2.4.4.** Suppose \( X = (X, B, \mu, T) \in \text{End}(p) \). If \( X \cong B^+(p) \), then \( X \) is tvwB. ■

Our goal now is to show that if a \( p \)-endomorphism is tvwB, then it is one-sided Bernoulli. It will be convenient for our presentation to work with functions defined on \( p \)-endomorphisms with range in the metric space \([0,1)\) (with the absolute value metric \(|x - y|\)). As we will see below, this gives us a fairly natural way to partition the range space and to define tree name distributions induced by a function on a \( p \)-endomorphism. For a m.p.s. \( Y = (Y, C, \nu, S) \) and two functions \( g, h : Y \to [0,1) \), let \(|g - h| = \|g - h\|_{L^1} \).

For any function \( g : Y \to [0,1) \) and \( N \in \mathbb{N} \), we will construct a “discretize” version of \( g \) which closely approximates it. Specifically, for each \( N \), let \( P_N \) be the set of dyadic intervals \( \{[0, 1/2^N), \ldots; [2^N - 1)/2^N, 1) \} \) of length \( 2^{-N} \). Let \( D_N \) be the set of midpoints of the intervals in \( P_N \), i.e. \( D_N = \{(2t + 1)/2^{N+1}, 0 \leq t < 2^{N} \} \). Define the function \( g_N : Y \to D_N \) by setting

\[
g_N(x) = \frac{2t + 1}{2^{N+1}} \quad \text{if} \ x \in \left[\frac{t}{2^N}, \frac{t + 1}{2^N}\right), \; 0 \leq t < 2^N
\]
Clearly, $|g_N - g| < 1/2^n$. Observe that $g_N$ assumes values in the finite set $D_N$ so that we may regard $g_N$ as a partition of $Y$.

Given a function $g : Y \rightarrow [0, 1)$ and positive integers $m$ and $n$, recall from §1.3 that we have the partition $g_m^\tau : Y \rightarrow (D_m)^\tau$ by mapping a point $y$ to its $g_m$-name and the partition $g_m^\nu : Y \rightarrow (D_m)^\nu$ by mapping a point $y$ to the equivalence class in $(D_m)^\nu$ containing $g_m^\tau(y)$. Let us also define the partition $g_m^+ : Y \rightarrow (D_m)^n$ by mapping a point $y$ to its $g_m$-name. Observe that $g_m^\tau n \leq g_m^{\tau n}$, $g_m^\nu n \leq g_m^{\nu n}$ and $g_m^n \leq g_m^{m^n}$ if $n' \leq m'$ and $n \leq m$.

The following proposition shows that closeness in tree name distributions implies closeness in forward name distributions.

**Proposition 2.4.5.** Let $Y = (Y, \mathcal{C}, \nu, S) \in \text{End}(p)$. For $n$, $m \in \mathbb{N}$ and any functions $g, h : Y \rightarrow [0, 1)$, if

$$\text{dist}(g_m^n) \sim \text{dist}(h_m^n),$$

then

$$\text{dist}(g_m^n) \sim \text{dist}(h_m^n).$$

**Proof:** For each element $\zeta \in (D_n)^m$ and $\alpha \in (D_n)^{m\nu}$, and for any representative $\beta \in \alpha$, let $w(\zeta, \alpha)$ be the total weights of nodes $v$ of length $m$ such that $(\beta(v), \beta(\sigma v), \ldots, \beta(\sigma^{m-1}(v)) = \zeta$. (This is clearly independent of the representative chosen.) Then

$$\sum_{\zeta \in (D_n)^m} |\text{dist}(g_m^n)(\zeta) - \text{dist}(h_m^n)(\zeta)| \leq \sum_{\zeta \in (D_n)^m} \sum_{\alpha \in (D_n)^{m\nu}} w(\zeta, \alpha) |\text{dist}(g_m^n)(\alpha) - \text{dist}(h_m^n)(\alpha)|$$

$$= \sum_{\alpha \in (D_n)^{m\nu}} \sum_{\zeta \in (D_n)^m} w(\zeta, \alpha) |\text{dist}(g_m^n)(\alpha) - \text{dist}(h_m^n)(\alpha)|$$

$$= \sum_{\alpha \in (D_n)^{m\nu}} |\text{dist}(g_m^n)(\alpha) - \text{dist}(h_m^n)(\alpha)| < \varepsilon.$$

Let $X = (X, \mathcal{B}, \mu, T)$ and $Y = (Y, \mathcal{C}, \nu, S)$ be $p$-endomorphisms. Suppose $g : X \rightarrow [0, 1)$ and $h : Y \rightarrow [0, 1)$, we wish to define a $\bar{t}$ distance between the processes $(X, g)$ and $(Y, h)$ analogous to the $\bar{d}$ distance in Ornstein’s Theory. To do this, we first let

$$\bar{t}_n((X, g), (Y, h)) = \int \bar{t}_n(\tau^g_x, \tau^h_y) d\mu(x) d\nu(y).$$

Then define

$$\bar{t}((X, g), (Y, h)) = \liminf \bar{t}_n((X, g), (Y, h)).$$
We remark that this is not the definition in [H,R] in which they define $\bar{t}_n((X,g),(Y,h))$ as an infimum of the integral of $\bar{t}_n(\tau^g_x,\tau^h_y)$ over the set of one-sided couplings of $X$ and $Y$ (see Chapter 3). Here, we only define it with respect to the product measure. Note that closeness in $\bar{t}$ essentially means that for some large $n$, $\bar{t}_n(\tau^g_x,\tau^h_y)$ is small for any $x$ in a large set in $X$ and for any $y$ in a large set in $Y$.

The following proposition shows that if $X$ is tvwB, then $\bar{t}((X,g),(Y,h))$ is small provided that $\text{dist}(g^N_N)$ is sufficiently close to $\text{dist}(h^N_N)$ for some $n$. The argument follows along the same lines as Lemma 4.4 in [H,R], though the notations found there need to be modified for our situation, since we are considering $p$-endomorphisms so that the weights of the nodes within a given level are not uniform.

**Proposition 2.4.6.** Suppose $X$ is tvwB and $g : X \to [0,1)$. For all $\varepsilon > 0$, there exist $\delta > 0$ and $N \in \mathbb{N}$ with the following property: for any $Y \in \text{End}(p)$ and any function $h : Y \to [0,1)$, if $\text{dist}(g^N_N) \sim \text{dist}(h^N_N)$, then $\bar{t}((X,g),(Y,h)) < \varepsilon$.

**Proof:** Let $\eta > 0$, to be specified later. Since $X$ is tvwB, we may choose $N$ large enough such that there exists a set $G$ of measure $> 1 - \eta$ such that $\bar{t}_N(\tau^g_{x'},\tau^h_y) < \eta$ whenever $x'$ and $y$ are in $G$. Clearly, we may assume that $1/2^N < \eta$. Fix $x' \in G$. If $\text{dist}(g^N_N) \sim \text{dist}(h^N_N)$, then we have a set $H$ in $Y$ of measure at least $1 - 2\eta$ such that if $y \in H$, then

$$\bar{t}_N(\tau^g_{x'},\tau^h_y) < \eta + (1/2^N) < 2\eta.$$

Let $\tau' = \tau^g_{x'}$. Create a tree name $\tau : T' \to [0,1)$ by tiling with copies of $\tau'$ as follows: If $0 < |v| \leq N$ and $|v'| = kN$ for some integer $k \geq 0$, we define $\tau$ by setting $\tau(vv') = \tau'(v)$.

We would like to show that $\tau$ is close to $\tau^g_x$ and $\tau^h_y$ for large sets in $X$ and in $Y$ in $\bar{t}_{kN}$ for all $k \in \mathbb{N}$. Roughly speaking, the idea is to think of each tree of height $kN$ as consisting of subtrees of height $N$ and to realize that the tvwB condition guarantees that the tree names of these subtrees are close to $\tau'$ in $\bar{t}_N$ on average.

We will now define inductively, $N$-levels at a time, a tree automorphism $A$ which makes $\bar{t}_{kN}(\tau^g_x,\tau)$ small on a large set in $X$ for all $k$. For $x \in X$, let $A_x$ be a tree automorphism which realizes the minimum in the definition of $\bar{t}_N(\tau^g_x,\tau)$. For $0 < |v| \leq N$, set $A(v) = A_x(v)$.

Inductively, for each $v \in T$ such that $jN < |v| \leq (j+1)N$, $j \geq 1$, write $v = v'u$ for unique nodes $v'$ and $u$ such that $|u| = jN$ and $0 < |v'| \leq N$, and define $A(v) = A_{x,v}(v')A(u)$.

For $j \geq 0$ and $x \in X$, let $S_j(x) = \{v \in T \mid |v| = jN \text{ and } T_vx \in G\}$. Let $w(S_j(x))$ denote the sum $\sum_{v \in S_j(x)} w_v$. Then by Lemma 2.4.2, for all $k \geq 0$, there exists a set $G_k \subseteq X$ of
measure $\geq 1 - \sqrt{\eta}$ such that
\[
\frac{1}{k} \sum_{j=0}^{k-1} w(S_j(x)) \geq 1 - \sqrt{\eta}
\]
for each $x \in G_k$. By our construction of the tree automorphism $A$, $\tau_y^g$ and $\tau'$ are matched to within $\eta$ in $\bar{t}_N$ whenever $y = T_v x$ for $v \in S_j(x)$. For each $x \in G_k$, by calculating $\bar{t}_{kN}(\tau_x^g, \tau)$ as an average of $\bar{t}_{N}(\tau_{T_v x}^g, \tau')$ over all $y \in T^{-jN}x, 0 \leq j \leq k - 1$, we have
\[
\bar{t}_{kN}(\tau_x^g, \tau) = \frac{1}{kN} \sum_{j=0}^{k-1} \sum_{v \in S_j(x)} w_v N\bar{t}_N(\tau_{T_v x}^g, \tau')
\]
\[
= \frac{1}{kN} \sum_{j=0}^{k-1} \left( \sum_{v \in S_j(x)} w_v N\bar{t}_N(\tau_{T_v x}^g, \tau') + \sum_{v \notin S_j(x)} w_v N\bar{t}_N(\tau_{T_v x}^g, \tau') \right)
\]
\[
\leq \frac{1}{kN} \sum_{j=0}^{k-1} N\eta w(S_j(x)) + N(1 - w(S_j(x)))
\]
\[
\leq 1 - \frac{1}{k} \sum_{j=0}^{k-1} w(S_j(x))(1 - \eta)
\]
\[
\leq 1 - (1 - \sqrt{\eta})(1 - \eta).
\]
Hence for all but a set of measure $2\sqrt{\eta}$ in $X$, we have $\bar{t}_{kN}(\tau_x^g, \tau) < 2\sqrt{\eta}$.

By the same argument, for all but a set of measure $4\sqrt{\eta}$ in $Y$, we have $\bar{t}_{kN}(\tau_y^h, \tau) < 4\sqrt{\eta}$.

Hence, by choosing $\eta$ sufficiently small, we have
\[
\int \bar{t}_{kN}(\tau_x^g, \tau_y^h) d\mu(x)d\nu(y) \leq \int \bar{t}_{kN}(\tau_x^g, \tau) d\mu(x)d\nu(y) + \int \bar{t}_{kN}(\tau, \tau_y^h) d\mu(x)d\nu(y)
\]
\[
\leq 4\sqrt{\eta} + 8\sqrt{\eta} < \epsilon
\]
Since $k$ is arbitrary, the result now follows. 

Let $X = (X, B, \mu, T)$ and consider a partition $P : X \to C$ for some finite set $C$. Given $M \in N$ and $x \in X$, recall from §2.2 that we defined a measure $\theta_{x,M,P}$ on $C$ by
\[
\theta_{x,M,P}(c) = \frac{1}{M} \sum_{v \in T_M | P(T_v x) = c} w_v
\]
for each $c \in C$. Also recall that $x \in X$ is $\varepsilon,M$-generic for $P$ if
\[
|\theta_{x,M,P} - \text{dist}(P)| < \varepsilon.
\]

Note that if $X$ is ergodic, the Tree Ergodic Theorem implies that for all sufficiently large $M$, there is a large set $G \subseteq X$ such that all points $x \in G$ are $\varepsilon,M$-generic for $P$. 

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Combined with proposition 2.4.6, the next proposition (the perturbation lemma), shows that if $X$ is twwB and $\hat{t}((X, f), (Y, g)) < \varepsilon$, then we only need to perturb $g$ slightly (depending on $\varepsilon$) to get a function $h$ on $Y$ such that the processes $(X, f)$ and $(Y, h)$ are as close in $\hat{t}$ as we like. This will be the key observation in the proof of Sinai’s Theorem (Proposition 2.4.9).

For the proof of the perturbation lemma (Proposition 2.4.7) and the copying lemma (Proposition 2.4.12), we will be constructing a function on a chosen tree Rokhlin tower $M$ using one or more chosen tree names in a way analogous to “painting” columns of a Rokhlin tower with a name as in the proof of Ornstein’s theorem. Since this construction is central to the proofs, let us now define it explicitly. Suppose $Y = (Y, C, \nu, S) \in \text{End}(p)$, $N \in \mathbb{N}$ and $A \in \mathcal{A}_N$. Given a tree name $\tau : T_N \to R$ and a point $y \in Y$, we may define a function $h : \{S^{-t}y \mid 1 \leq t \leq N\} \to R$ such that $h(S_v y) = \tau(Av)$ for all $v \in T_N'$. We will refer to $h$ as a laying of $\tau$ on $\{S^{-t}y \mid 1 \leq t \leq N\}$ via $A$. Doing this for all $y$ in the base $B$ of a tree Rokhlin tower $M$ of height $N + 1$ for fixed $A \in \mathcal{A}_N$ and tree name $\tau$ defines a function on $\bigcup M \setminus B$.

**Proposition 2.4.7 (Perturbation Lemma).** Let $X = (X, \mathcal{B}, \mu, T)$ and $Y = (Y, C, \nu, S)$ be ergodic. Suppose $\delta > 0$ and $f : X \to [0, 1]$. For any $\varepsilon > 0$, $N \in \mathbb{N}$ and for any function $g : Y \to [0, 1]$ such that $\hat{t}((X, f), (Y, g)) < \delta^4$, there exists some function $h : Y \to [0, 1]$ such that $|g - h| < 8\delta$ and $\text{dist}(h_N^{N'}) \sim \text{dist}(f_N^{N'})$.

**Proof:** There is no loss of generality in assuming that $N$ is large enough such that $1/2^N \leq \delta^4$. Let $\eta < \delta$ be specified later. Choose $M \in \mathbb{N}$ and a set $G \subseteq X$ of measure $> 1 - \eta$ such that the following conditions hold:

a) $N/M < \eta/2$

b) $x$ is $\eta,(M - N)$-generic for $f_N^{N'}$ for all $x \in G$

c) $\int \hat{t}_M(\tau_x^f, \tau_y^g) \, d\mu(x)d\nu(y) < \delta^4$.

The Tree Ergodic theorem and the hypothesis show that such an integer $M$ and set $G$ exist.

From c), we have

$$\int \hat{t}_M(\tau_x^f, \tau_y^g) \, d\nu(y) < \delta^2$$

except on a set $H \subseteq X$ of measure $< \delta^2$. Choose some point $x' \in G \setminus H$. It follows that on a set $G' \subseteq Y$ of measure $\geq 1 - \delta$, we have $\hat{t}_M(\tau_{x'}^f, \tau_y^g) \leq \delta$ for $y \in G'$. By the Strong Tree Rokhlin Lemma, we may build a $\eta/2$-tree Rokhlin tower $M$ of height $M + 1$ in $Y$ with base $B$ such that $\nu_B(G') \geq 1 - \delta$.

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We now construct the required function $h$. For each $y \in Y$, let $A_y \in \mathcal{A}_M$ be a tree automorphism which realizes $t_M(\tau^g_y, \tau^f_x)$. For each atom $\alpha$ in the partition $g_M^{M^r}|B$, we pick a representative point $y(\alpha) \in \alpha$. We define $h$ on $\cup \mathcal{M}|B$ by laying $\tau^f_x$ on $\{S^{-t}y \mid 1 \leq t \leq N\}$ via $A_y(\alpha)$ for each point $y \in \alpha$ and then for each atom $\alpha$. Note that for a fixed node $v$ and a fixed atom $\alpha$ in $g_M^{M^r}|B$, $h$ is constant on $S_v\alpha$, thus $h$ defines a measurable function on $\cup \mathcal{M}|B$. We then extend $h$ measurably to the rest of the space in any way we desire.

For each atom $\alpha \in g_M^{M^r}|B$, let $C_\alpha$ be the column of the tower over $\alpha$. Let us say that $C_\alpha$ is a **good column** if $\alpha$ contains a point in $G'$. Now, if $y \in G' \cap B$ and $y \in \alpha$, then since $1/2^M < \delta$, we have

$$
\bar{t}_M(\tau^g_{y(\alpha)}, \tau^f_x) \leq \bar{t}_M(\tau^g_y, \tau^f_x) + \delta \leq \bar{t}_M(\tau^g_y, \tau^f_x) + 2\delta < 3\delta. \tag{1}
$$

For each good column $C_\alpha$, a simple calculation using (1) and our definition of $h$ shows that

$$
\int_{C_\alpha} |g_M(z) - h(z)| \, d\nu(z) < 3\delta \nu(C_\alpha).
$$

Since $\nu_B(G') \geq 1 - \delta$, conditionally on $B$, the bases of the good columns is a set of measure $\geq 1 - \delta$. Thus,

$$
\nu(\{\cup C_\alpha \mid C_\alpha \text{ is good}\}) \geq (1 - \delta)\nu(\cup_{i=1}^M S_{i-B}) > 1 - 2\delta.
$$

Hence,

$$
|g - h| \leq |g - g_M| + |g_M - h| \\
\leq 1/2^M + 5\delta < 8\delta.
$$

For the second part of the conclusion, fix an atom $\alpha \in h_N^{M^r}|B$. Then there exists some tree automorphism $A$ such that $h_N(S_y, v) = f^M_N(T_{Av}x')$ for all $y \in \alpha$ and for all $v \in T_M$. Hence, for each $\xi \in (D_N)^{NV}$, if $f^{NV}_N(T_vx') = \xi$ for $v \in T^\prime_{M-N}$, then $h^{NV}_N(S_{A^{-1}v}y) = \xi$ for all $y \in \alpha$. Let $\check{\xi} = \{v \in T^\prime_{M-N} \mid f^{NV}_N(T_vx') = \xi\}$. Since tree automorphisms preserve weights of nodes, we have

$$
dist(h^{NV}_N | \cup_{0<j\leq M-N} S^{-j}\alpha)(\xi) = \sum_{\nu \in \check{\xi}} \frac{\nu(S_{A^{-1}v}\alpha)}{(M-N)\nu(\alpha)} \\
= \frac{1}{M-N} \sum_{\nu \in \check{\xi}} w_v \\
= \theta_{x', M-N, f^{NV}_N}(\xi).
$$

Thus,

$$
dist(h^{NV}_N | \cup_{0<j\leq M-N} S^{-j}\alpha) = \theta_{x', M-N, f^{NV}_N}.
$$
Obviously, this is true for each $\alpha \in h^M|B$. Hence, (b) implies that

$$\text{dist}(h^N| \bigcup_{0 \leq j \leq M-N} S^{-j}B) \sim \text{dist}(f^N).$$

Using (a) and the fact that the measure of $\cup M$ is large, we also have

$$\text{dist}(h^N) \sim \text{dist}(f^N)$$

for $\eta$ small enough. This completes the proof. ■

By imitating the proof of the previous proposition, but without the need to make $h$ close to a predefined $g$, the following proposition is immediate.

**Proposition 2.4.8.** Suppose $X = (X, B, \mu, T)$ and $Y = (Y, C, \nu, S)$ are both ergodic p-endomorphisms. For any function $f : X \to [0, 1]$, and for all $\varepsilon$ and $N$, we have some function $h : Y \to [0, 1]$ such that $\text{dist}(h^N) \sim \text{dist}(f^N)$.

For $Y = (Y, C, \nu, S)$ and a function $g : Y \to [0, 1)$, we have a map $g^N : Y \to [0, 1)^N$ defined by $g^N(y) = (g(y), g(Sy), \ldots)$. The measure $\nu$ pushes forward via $g^N$ to the shift invariant measure $\text{dist}(Y, g) = \nu \circ (g^N)^{-1}$ on $[0, 1)^N$. We will refer to the map $g^N$ as the g-name map.

We will now prove the analogue of Sinai’s theorem in Ornstein’s theory for tvwB p-endomorphisms. In fact, we will need a slightly stronger version of it.

**Proposition 2.4.9 (Strong Sinai’s Theorem).** Suppose $X = (X, B, \mu, T)$ is tvwB. Let $f : X \to [0, 1)$ be a generating function. Given $\varepsilon > 0$, there exist $\delta$ and $N$ such that if $Y = (Y, C, \nu, S)$ is ergodic and if $g : Y \to [0, 1)$ satisfies $\text{dist}(g^N) \sim \text{dist}(f^N)$, then there exists a tree-adapted function $h : Y \to [0, 1)$ such that $\text{dist}(Y, h) = \text{dist}(X, f)$ and $|h - g| < \varepsilon$.

**Proof:** By Proposition 2.2.1, $X$ is ergodic. The strategy is to apply Propositions 2.4.6 and 2.4.7 repeatedly. Precisely, choose $N_k \not\to \infty$ and $\varepsilon_k \not\to 0$ such that for each $k$, $N_k$ and $\varepsilon_k$ correspond to $(\varepsilon/2^{k+3})^4$ in Proposition 2.4.6 applied to $(X, f)$. We show that if $N = N_1$ and $\delta = \varepsilon_1$, the result holds.

To see this, notice that if

$$\text{dist}(g^N) \sim \text{dist}(f^N),$$

we have by Proposition 2.4.6,

$$\bar{t}((X, f), (Y, g)) < (\varepsilon/16)^4.$$
Hence, by Proposition 2.4.7, we have some function $g^2$ on $Y$ such that $|g - g^2| < \varepsilon/2$ and
\[
\text{dist}((g^2)_{N_{g}}^{N_{g}})^{\varepsilon_{2}} \sim \text{dist}(f_{N_{g}}^{N_{g}}).
\]
Inductively, we obtain a sequence of functions $g^j$'s such that $|g^j - g^{j+1}| < \varepsilon/2^j$ and
\[
\text{dist}((g^j)_{N_{g}}^{N_{g}})^{\varepsilon_{j}} \sim \text{dist}(f_{N_{g}}^{N_{g}}).
\]
Hence, the $g^j$ approach some function $h$ pointwise a.e. Note that $|g - h| < \varepsilon$.

To see that $\text{dist}(Y, h) = \text{dist}(X, f)$, note that for any $N \in \mathbb{N}$ and any $\theta > 0$, we have by proposition 2.4.5,
\[
\text{dist}((g^j)_{N}^{1})^{\theta} \sim \text{dist}(f_{N}^{1})
\]
for all large $j$. Thus, for any given $\gamma > 0$, we have
\[
\text{dist}(Y, g^j) \sim \text{dist}(X, f)
\]
in the $w^*$-topology for all large $j$. Now, as $g^j \to h$ a.e., we also have for large $j$,
\[
\text{dist}(Y, h) \sim \text{dist}(Y, g^j) \sim \text{dist}(X, f).
\]
Since $\gamma$ is arbitrary, we have $\text{dist}(Y, h) = \text{dist}(X, f)$.

It remains to show that $h$ is tree-adapted. By construction, for any $N \in \mathbb{N}$ and $\gamma > 0$, $\text{dist}((g^j)_{N}^{1}) \sim \text{dist}(f_{N}^{1})$ for all large $j$. As $g^j \to h$, it immediately follows that for any $M \in \mathbb{N}$, $\text{dist}(h_{M}^{1}) = \text{dist}(f_{M}^{1})$. Since $f$ is generating and hence tree-adapted, so is $h$. ■

The following is an immediate consequence of propositions 2.4.8 and 2.4.9.

**Corollary 2.4.10 (Sinai’s Theorem).** Suppose $X = (X, \mathcal{B}, \mu, T)$ is tvwB. Let $f : X \to [0,1)$ be a generating function. If $Y = (Y, \mathcal{C}, \nu, S)$ is ergodic, then there exists a tree-adapted function $h : Y \to [0,1)$ such that $\text{dist}(Y, h) = \text{dist}(X, f)$. ■

For a function $g$ on $Y$, consider the $g$-name map $g^{N} : Y \to [0,1)^{N}$ defined previously. Similarly, if $f$ is a function on $X$, consider the $f$-name map $f^{N} : X \to [0,1)^{N}$. If $\text{dist}(X, f) = \text{dist}(Y, g)$ for a generating $f$ on $X$, then we have a factor map $\pi : Y \to X$ defined by $\pi = (f^{N})^{-1} \circ g^{N}$. Note that $\pi$ is tree adapted if $g$ is a tree-adapted function on $Y$. Thus, Sinai’s Theorem implies that if $X$ is tvwB, then it is a tree-adapted factor of $Y$.

Before we establish the copying lemma, Proposition 2.4.12, we need the following result which shows that if $\text{dist}(Y, g) = \text{dist}(X, f)$ for a generating $f$ and a tree-adapted $g$ as in Sinai’s Theorem, the tree name distributions induced by $g$ and $f$ are the same.
**Proposition 2.4.11.** Suppose $X$ and $Y$ are in $\text{End}(p)$. Let $f : X \to [0, 1)$ be a generating function. Suppose $g : Y \to [0, 1)$ is a tree-adapted function such that $\text{dist}(Y, g) = \text{dist}(X, f)$. Then for all $M \in \mathbb{N}$,
\[
\text{dist}(g^M_N) = \text{dist}(f^M_N).
\]

**Proof:** This follows directly from proposition 2.1.4 applied to the tree-adapted factor map $\pi : Y \to X$ defined by $\pi = (f\mathbb{N}^{-1})g\mathbb{N}$ and on noticing that $g_M = f_M \circ \pi$. \hfill \blacksquare

If $h, h' : Y \to [0, 1)$, let $h \triangleright h'$ be the joined function defined by $h \triangleright h'(y) = (h(y), h'(y))$. By analogy with our definitions of $h^M_N$ and $h^M_N$, we can construct the partitions $(h \triangleright h')^M_N : Y \to (D_M \times D_M)^M_N$ and $(h \triangleright h')^M_N : Y \to (D_M \times D_M)^M_N$ by discretizing $h \triangleright h'$ as $h_M \triangleright h'_M$.

**Proposition 2.4.12 (Copying Lemma).** Let $Y = (Y, C, \nu, S) \in \text{End}(p)$ and suppose $X = (X, B, \mu, T) \in \text{End}(p)$ is ergodic. Let $f : X \to [0, 1)$ and $g : Y \to [0, 1)$ be generating. Suppose $f'$ is a tree-adapted function on $Y$ such that $\text{dist}(X, f) = \text{dist}(Y, f')$. For all $\varepsilon$ and $N$, there exists a function $g'$ on $X$ such that
\[
\text{dist}(f_N \triangleright g'_N) \sim \text{dist}(f'_N \triangleright g_N).
\]

**Proof:** By proposition 2.4.11, we have
\[
\text{dist}(f^M_N) = \text{dist}(f^M_N).
\]
Choose $M$ such that $N/M < \varepsilon/2$. Construct a $\varepsilon/2$-tree Rokhlin tower $M$ of height $M + 1$ in $X$. Let $B$ be the base of $M$. By the Strong Tree Rokhlin Lemma, we may assume that $f^M_N$ is independent of $B$. Hence, $\text{dist}(f^M_N | B) = \text{dist}(f^M_N)$. Let $\pi : Y \to X$ denote the tree adapted factor map defined by $\pi = (f\mathbb{N})^{-1} \circ (f')\mathbb{N}$.

Fix an atom $q$ of $f^M_N$. Define a partition $P_q : q \cap B \to (D_M \times D_M)^M_N$ such that
\[
\text{dist}(P_q) = \text{dist}((f^M_M \triangleright g^M_M)^M_N | \pi^{-1}(q)).
\]
This gives a bijection $\varphi$ of the atoms of $P_q$ and those of $(f^M_M \triangleright g^M_M)^M_N | \pi^{-1}(q)$ such that corresponding atoms have the same conditional measures and are mapped to the same element in $(D_M \times D_M)^M_N$.

For each atom $\alpha \in P_q$, choose any representative point $y \in \varphi(\alpha)$. Note that as $f_M = f_M' \circ \pi$ and $\pi(y) \in q$, by proposition 2.1.4, $f^M_M(z) = f^M_M(y)$ for all $z \in \alpha$. For each $z \in \alpha$, choose a tree automorphism $A \in A_M$ such that $f_M(T_v z) = f_M(S_A v y)$ for each $v \in T_M'$. We then define
choosing the same tree automorphism for all \( z \in \alpha \) which are in the same atom of \( f_M^{M^\tau} \).

This defines the function \( g' \) on the column over \( \alpha \).

Clearly we have for every \( z \in \alpha \) and \( v \in T'_M \),

\[
 f_M \vee g'_M(T_vz) = f'_M \vee g_M(S_{Av}y),
\]

so that

\[
 (f_M \vee g'_M)^{M^\tau}(z) = (f'_M \vee g_M)^{M^\tau}(y) = P_q(z).
\]

Repeat the above procedure for each \( \alpha \in P_q \). We then have \( (f_M \vee g'_M)^{M^\tau} = P_q \) on \( q \cap B \). We thus have

\[
 dist((f_M \vee g'_M)^{M^\tau}|q \cap B) = dist((f'_M \vee g_M)^{M^\tau}|\pi^{-1}(q))
\]

We then extend \( g' \) to the rest of the tower by following the above procedure for each atom \( q \) of \( f_M^{M^\tau} \). We thus have a function \( g' \) defined on \( \bigcup M \backslash B \). Extend \( g' \) measurably to the rest of the space in any way we like.

We wish to prove that \( dist(f_N \vee g'_N)^{N^\tau} \overset{\sim}{=} dist(f'_N \vee g_N)^{N^\tau} \). First, let us observe that by our construction and by (1), we have

\[
 dist((f_M \vee g'_M)^{M^\tau}|B) = \sum_{q \in f_M^{M^\tau}} \mu(q \cap B|B)dist((f_M \vee g'_M)^{M^\tau}|q \cap B)
\]

\[
 = \sum_{q \in f_M^{M^\tau}} \mu(q)dist((f'_M \vee g_M)^{M^\tau}|\pi^{-1}(q))
\]

\[
 = \sum_{\pi^{-1}(q) \in f_M^{M^\tau}} \nu(\pi^{-1}(q))dist((f'_M \vee g_M)^{M^\tau}|\pi^{-1}(q))
\]

\[
 = dist(f'_M \vee g_M)^{M^\tau}.
\]

We claim that for each \( 0 \leq j \leq M - N \),

\[
 dist((f_N \vee g'_N)^{N^\tau}|T^{-j}B) = dist(f'_N \vee g_N)^{N^\tau}.
\]

To see this, note that if \( \beta \) and \( \beta' \) are corresponding atoms of \( (f_M \vee g'_M)^{M^\tau} \) and \( (f'_M \vee g_M)^{M^\tau} \), then

\[
 dist((f_N \vee g'_N)^{N^\tau}|T^{-j}\beta \cap T^{-j}B) = dist((f'_N \vee g_N)^{N^\tau}|S^{-j}\beta').
\]

Moreover, by (2), we have

\[
 \mu(T^{-j}\beta|T^{-j}B) = \mu(\beta|B) = \nu(\beta') = \nu(S^{-j}\beta').
\]
Thus (3) follows from (4) and (5) since the distributions in (3) are just weighted averages of the distributions equated in (4). As \( N/M < \varepsilon/2 \) and the rest of the space is a set of measure \( < \varepsilon/2 \), we see that

\[
\text{dist}(f_N \lor g'_N)^{N\nu} \sim \text{dist}(f'_N \lor g_N)^{N\nu}.
\]

This finishes the result. ■

For \( Y = (Y, \mathcal{C}, \nu, S) \in \text{End}(p) \) and partitions \( P, R : Y \to C \) for a finite set \( C \), we write \( P \sim R \) if \( \{ y \in Y \mid P(y) \neq R(y) \} \) has measure \( < \varepsilon \). For two partitions \( P : Y \to C \) and \( Q : Y \to C' \), we write \( P \preceq Q \) if there exists a partition \( R : Y \to C \) with \( R \leq Q \) such that \( P \sim R \).

Given a function \( g : Y \to [0,1) \), let \( \{g_i\} \) denote the sub-sigma-algebra of \( \mathcal{C} \) generated by \( g \), i.e. \( \{g_i\} = \bigvee_{i=0}^{\infty} S^{-i}(g^{-1}(D)) \) for the Borel sigma-algebra \( D \) of \([0,1)\). If \( g \) and \( h \) are functions on \( Y \), we write \( h \preceq g \) if there exists some integer \( M \) such that \( h_N \preceq g_M \). We write \( h \subset \{g\} \) if \( h \) is \( \{g\}\)-measurable. Clearly, \( h \subset \{g\} \) if and only if \( h \preceq \{g\} \) for all \( \varepsilon \) and \( N \).

For the remainder of this section, let \( X = (X, \mathcal{B}, \mu, T) \) and \( Y = (Y, \mathcal{C}, \nu, S) \) be tvwB \( p \)-endomorphisms. Moreover, we fix generating functions \( f : X \to [0,1) \) and \( g : Y \to [0,1) \). Before delving into details, we briefly describe the strategy to proving Theorem 2.4.1 from this point on. The idea is to construct a Cauchy sequence of functions \( \{g_j\} \) on \( Y \) such that \( \text{dist}(Y, g_j) = \text{dist}(X, f) \) for each \( j \), and which converge to a function \( \bar{g} \) on \( Y \) such that \( g \subset \{\bar{g}\} \). Thus \( \text{dist}(Y, \bar{g}) = \text{dist}(X, f) \) and \( \{\bar{g}\} = \{g\} = \mathcal{C} \). Now, the sigma-algebra \( \mathcal{B} \) pulls backs to \( \{g\} \) via the factor map \( \pi = (f^N)^{-1} \circ \bar{g}^N : Y \to X \) as constructed prior to Proposition 2.4.11. It thus follows that \( \pi \) is an isomorphism as \( \pi^{-1} \) gives an isomorphism of \( \sigma \)-algebras of Lebesgue spaces. As we have previously remarked, \( B^+(p) \) is tvwB. Theorem 2.4.1 thus follows.

For technical reasons, we will choose the generators \( f \) and \( g \) such that \( \{x \in X \mid f(x) = q\} \) is a null set for each rational \( q \), and similarly for the sets \( \{y \in Y \mid g(x) = q\} \). Observe that we can always do this as \( X \) and \( Y \) are nonatomic Lebesgue spaces. (Just choose any point isomorphism \( f : X \to [0,1) \) and \( g : Y \to [0,1) \).) This assumption is needed to ensure that for the functions we construct via the copying lemma, a large enough set of points assume values sufficiently bounded away from the boundary points in the dyadic intervals of length \( 1/2^N \) in \( P_N \). For any such function \( h \), if \( |\bar{h} - h| \) is sufficiently small, then \( h_N(x) = \bar{h}_N(x) \) for all \( x \) in a set of large measure.
Proposition 2.4.13. Let \( f' : Y \to [0, 1) \) be a tree-adapted function such that \( \text{dist}(Y, f') = \text{dist}(X, f) \). Then for any \( \eta > 0, \varepsilon > 0 \) and \( M \in \mathbb{N} \), there exists a tree-adapted function \( \tilde{f} : Y \to [0, 1) \) such that

a) \( g \in \mathbb{M} \setminus \sum(\tilde{f}) \)

b) \( |\tilde{f} - f'| < \eta \)

c) \( \text{dist}(Y, \tilde{f}) = \text{dist}(X, f) \).

Proof: Let \( N \in \mathbb{N} \) be specified later. By Proposition 2.4.12, for any \( \delta \) and \( L \), we have some function \( g' \) on \( X \) such that

\[
\text{dist}(f_L \vee g'_L)^{L \vee} \sim \text{dist}(f'_L \vee g_L)^{L \vee}. \tag{1}
\]

As \( f' \subset \sum(g) \), for each \( \theta > 0 \), we can choose some \( k \) such that

\[
f'_N \subset g^k. \tag{2}
\]

Now, by our choice of \( g \), \( \{ y \in Y \mid g(y) = q \} \) is a null set for each rational \( q \). Thus, for any \( \beta > 0 \), there exist open intervals containing the rationals \( \{ t/2^k \mid t = 1, \ldots, 2^k - 1 \} \) such that \( g(y) \) assumes a value in one of these intervals on a set of measure \( < \beta \). Thus, (1) also implies that \( g'(x) \) assumes a value in one of these intervals on a set of measure \( < 2\beta \) for all sufficiently small \( \delta \) and \( 1/L \). If \( \beta > 0 \) is small enough, we may choose \( \rho > 0 \) such that for any function \( \tilde{g} \) with \( |g' - \tilde{g}| < \rho \), \( g'_{k}(x) = \tilde{g}_k(x) \) on a set of sufficiently large measure to give

\[
\text{dist}(f'_k \vee g'_{k^\vee}) \sim \text{dist}(f_k \vee \tilde{g}_k)^{k \vee}. \tag{3}
\]

From (1), \( \text{dist}(g'_L)^{L \vee} \sim \text{dist}(g_L)^{L \vee} \). By the Strong Sinai’s Theorem, if \( \delta \) and \( 1/L \) are chosen small enough, then we may choose \( \tilde{g} \) with \( |g' - \tilde{g}| < \rho \), \( \text{dist}(X, \tilde{g}) = \text{dist}(Y, g) \), and (2) holds.

Once more by proposition 2.4.12, for any \( L' \) and \( \delta' \), we have some function \( \tilde{f} \) on \( Y \) such that

\[
\text{dist}(f'_{L'} \vee \tilde{g}_{L'})^{L' \vee} \sim \text{dist}(f_{L'} \vee \tilde{g}_{L'}^{L'}). \tag{4}
\]

Using the Strong Sinai’s Theorem again, for any \( 0 < \rho' < \theta \), we may choose \( \delta' \) and \( L' \) to give a tree-adapted function \( \tilde{f} \) on \( Y \) such that \( |\tilde{f} - \tilde{f}'| < \rho' < \theta \) with \( \text{dist}(Y, \tilde{f}) = \text{dist}(X, f) \).

This gives (c) of the proposition.

Choose \( k' \in \mathbb{N} \) such that \( \tilde{g}_{k'} \subset f'^{k'} \). By (3) and decreasing \( \delta' \) and \( 1/L' \) if needed, we have \( g_{k'} \subset f'^{k'} \). Once again, since \( \{ x \in X \mid f(x) = q \} \) is a null set for each rational \( q \), we have \( g_{k'} \subset f'^{k'} \) by choosing \( \rho' \), \( \delta' \) and \( 1/L' \) small enough. This gives (a) of the proposition.
It remains to prove (b). Now,
\begin{align*}
|f' - \bar{f}| & \leq |f' - f_N| + |f_N - \bar{f}_N| + |\bar{f}_N - \bar{f}| + |\bar{f} - \bar{f}| \\
& < 1/2^N + |f_N - \bar{f}_N| + 1/2^N + \theta.
\end{align*}
Using (1), (2) and (3) and ensuring \(\min(L, L') \geq k\), and \(\delta, \delta'\) small enough we have
\[
\text{dist}(f'^\theta_k \lor g_k)^{k^\nu} \sim \text{dist}(\bar{f}_k \lor g_k)^{k^\nu}.
\]
Thus,
\[
\text{dist}(f'^\theta_k \lor g_k)^{k^\nu} \sim \text{dist}(\bar{f}_k \lor g_k)^{k^\nu}.
\]
Since \(f'^\theta_N \subset g_k^{k^\nu}\), we may also conclude that \(|f'^\theta_N - \bar{f}_N| < 3\theta\). We may now conclude b) by choosing \(\theta\) and \(N\) in the beginning to satisfy \(4\theta + 1/2^{N-1} < \eta\).

We are now ready to prove Theorem 2.4.1.

**Proof (Theorem 2.4.1):** The idea is to use Sinai’s Theorem and Proposition 2.4.13 repeatedly to construct a Cauchy sequence of functions \(\{g_j\}\) on \(Y\) converging to some function \(\bar{g}\) pointwise a.e., and \(\text{dist}(Y, g_j) = \text{dist}(X, f)\) for each \(j\), from which it follows that \(\text{dist}(Y, \bar{g}) = \text{dist}(X, f)\). We now need to ensure that the functions \(\{g_j\}\) are chosen in such a way that \(g \subset \sum(\bar{g})\). From this, we may conclude that \(\sum(\bar{g}) = \sum(g)\), thus concluding the proof of Theorem 2.4.1.

To carry out the above plan, choose a sequence of reals \(\varepsilon_j \downarrow 0\) and a sequence of integers \(M_j \nearrow \infty\). Let \(\eta_j \downarrow 0\) be a sequence of reals, to be specified later. Using Sinai’s Theorem, we begin by choosing a tree-adapted function \(g^1\) on \(Y\) such that \(\text{dist}(Y, g^1) = \text{dist}(X, f)\). Using proposition 2.4.13, we construct a sequence of tree-adapted functions \(\{g^j\}\) on \(Y\), \(j > 1\), such that
i) \(\text{dist}(Y, g^j) = \text{dist}(X, f)\), for \(j \geq 1\)
ii) \(|g^j - g^{j+1}| < \eta_j\), for \(j \geq 1\)
iii) \(g \subset \sum(\bar{g}^j)\), for \(j > 1\).

By ii), we may obviously arrange the \(\eta_j\)'s such that the \(g^j\)'s converge to a function \(\bar{g}\) pointwise a.e. Thus \(\text{dist}(Y, \bar{g}) = \text{dist}(X, f)\). Since \(g \subset \sum(\bar{g}^j)\), for \(j > 1\), we have some integer \(k_j\) such that
\[
g_M^{\varepsilon_j} \subset (g^{k_j})^{k_j^+}.
\]
Now as \(\text{dist}(Y, g^j) = \text{dist}(X, f)\) for each \(j\), the fact that \(\{x \in X \mid f(x) = q\}\) is a null set for each rational \(q\) implies \(\{y \in Y \mid g^j(x) = q\}\) is also. Hence, we have some \(\theta_n > 0\) for each
$n > 1$ such that whenever

$$\sum_{j=n}^{\infty} |g^j - g^{j+1}| < \theta_n,$$

we have

$$g_{M_n}^{2\varepsilon_n} \subseteq g_{k_n}^{k_n}.$$ 

For any $m \in \mathbb{N}$ and $\varepsilon > 0$, it follows that for all large $n$,

$$g^{\varepsilon,m} \subseteq \sum(g).$$

We thus have $\sum(g) = \sum(\bar{g})$, which gives Theorem 2.4.1.

Hence we are done if we can choose the $g^j$'s to satisfy the inequalities specified in (1) for all $n > 1$. To see that this is possible, note that they are chosen in the order $g^1 \to g^2 \to g^3 \ldots$. In view of this, when $\theta_n$ is determined, only $g^1, \ldots, g^n$ have been chosen and $g^{n+1}$ has not been chosen yet. Now, $\eta_n$ only need to be chosen when we choose $g^{n+1}$. As a result, $\eta_n$ has not been declared at the time we specify $\theta_n$.

Thus, we may choose $\eta_n < \min(\frac{\theta_1}{2}, \ldots, \frac{\theta_n}{2})$, it follows that

$$\sum_{j=n}^{\infty} |g^j - g^{j+1}| < \frac{\theta_n}{2} + \frac{\theta_n}{2^2} + \ldots = \theta_n.$$ 

Thus the inequalities in (1) can be satisfied. This completes the proof of Theorem 2.4.1. ■

### 2.5. An Elementary Proof of a Special Case of Theorem 2.4.1

In this section, we give a direct proof of Theorem 2.4.1 in the case that $\mathbf{p} = (p_1, \ldots, p_s)$ is a probability vector such that $p_1 > \ldots > p_s$. A simplified proof exists in this special case because the group $A$ of tree automorphisms is trivial.

Let $\mathbf{X} = (X, \mathcal{B}, \mu, T) \in \text{End}(\mathbf{p})$. Let $I = \{1, \ldots, s\}$. Consider the one-sided Bernoulli shift $B^+(\mathbf{p})$ represented as the shift space $(I^{\mathbb{N}}, \mathcal{C}, \nu, S)$ such that the $j$-th symbol has weight $p_j$. We then have a canonical factor map $\phi : \mathbf{X} \to B^+(\mathbf{p})$ defined by setting $\phi(x)_k = j$ if $p_{\mathbf{x}}(T^k x) = p_j$, for $k \geq 0$.

Recall that a node of the $\mathbf{p}$-tree $\mathcal{T}$ is a finite sequence of integers in $\{1, \ldots, s\}$. For each node $v$ and a point $z$ in $I^\mathbb{N}$, let $vz$ be the point in $I^\mathbb{N}$ obtained by concatenating $v$ to the left of $z$. For each $z$ in $I^\mathbb{N}$, let $z[0, m]$ be the cylinder set

$$\{z' \in I^\mathbb{N} \mid z'_j = z_j \text{ for } 0 \leq j \leq m\}.$$
Our goal is to prove the following special case of Theorem 2.4.1.

**Theorem 2.5.1.** Let \( p \) be a probability vector with pairwise distinct components. Let \( X = (X, \mathcal{B}, \mu, T) \in \text{End}(p) \) be tvwB, then the canonical factor map \( \phi : X \to B^+(p) \) is an isomorphism.

To prove this, we need a preliminary lemma. Recall that for \( X = (X, \mathcal{B}, \mu, T) \) and \( Y = (Y, \mathcal{C}, \nu, S) \), if \( \psi : X \to Y \) is a factor map, then we have fiber measures \( \mu_y \) supported on \( \psi^{-1}(y) \) with the property that \( \mu = \int \mu_y \, d\nu(y) \).

**Lemma 2.5.2.** Consider the one-sided Bernoulli shift \( B^+(p) = (I^N, \mathcal{C}, \nu, S) \). Let \( X = (X, \mathcal{B}, \mu, T) \in \text{End}(p) \). If \( D \subseteq X \) and \( v \in \mathcal{T} \), then for a.a. \( z \) in \( I^N \), \( \mu_z(T_v D) = \mu_z(D) \) (i.e. \( T^v : (X, \mu_z) \to (X, \mu_z) \) is measure preserving).

**Proof:** This follows easily from the fact that

\[
\mu(T_v B | T_v C) = \mu(B | C)
\]

for positive measurable sets \( B \subseteq X \) and \( C \subseteq X \). \( \blacksquare \)

**Proof (of Theorem 2.5.1):** Let \( f : X \to [0,1] \) be generating. By assumption, \( (X, f) \) is tvwB. It suffices to prove that for a.a. \( z \) in \( I^N \), there exists some \( r \in [0,1] \) such that for \( \mu_z \)-a.a. \( x, f(x) = r \) (i.e. \( f \) is constant on fibres). Indeed, this implies that \( f \) is \( \phi^{-1}(\mathcal{C}) \)-measurable. Since \( f \) is generating, we thus have \( \mathcal{B} = \phi^{-1}(\mathcal{C}) \) and so \( \phi \) is an isomorphism.

We proceed by contradiction. Hence, we suppose that there exist \( \eta > 0 \), a set \( Z \subseteq I^N \) of positive measure and two disjoint intervals \( J \) and \( J' \) in \([0,1] \) separated by a distance of at least \( \eta \) such that for each \( z \in Z, \mu_z(x \in X \mid f(x) \in J) \) and \( \mu_z(x \in X \mid f(x) \in J') \) are both at least \( \eta \). We may clearly assume \( \eta < \nu(Z) \). By the Tree Ergodic Theorem, there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \), we have a set \( K \subseteq I^N \) of measure at least \( \eta \) such that for each \( z \in K \),

\[
\frac{1}{n} \sum_{0 < |v| \leq n, |v| \in Z} w_v \geq \frac{\nu(Z)}{2} > \frac{\eta}{2}.
\]

(1)

We now show that the integral

\[
\int t_n(\tau_x^f, \tau_y^f) \, d\mu_z(x) d\mu_z(y)
\]

is bounded away from zero for \( z \in K, n \geq N \). To see this, for \( v \in \mathcal{T} \), by Lemma 2.5.2,

\[
\int |\tau_x^f(v) - \tau_y^f(v)| \, d\mu_z(x) d\mu_z(y) = \int |f(T_v x) - f(T_v y)| \, d\mu_z(x) d\mu_z(y)
\]

\[
= \int |f(x) - f(y)| \, d\mu_{vz}(x) d\mu_{vz}(y).
\]

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Hence, if \( z \in K \) for \( n \geq N \), then from (1) and the fact that \( A \) consists of only the identity automorphism,

\[
\int \tilde{t}_n(\tau^I_x, \tau^I_y) \, d\mu_z(x) d\mu_z(y) = \frac{1}{n} \sum_{0 < |v| \leq n} w_v \int |\tau^I_x(v) - \tau^I_y(v)| \, d\mu_z(x) d\mu_z(y)
\]

\[
\geq \frac{1}{n} \sum_{0 < |v| \leq n \, |v_z \in Z} w_v \int |f(T_v x) - f(T_v y)| \, d\mu_z(x) d\mu_z(y)
\]

\[
= \frac{1}{n} \sum_{0 < |v| \leq n \, |v_z \in Z} w_v \int |f(x) - f(y)| \, d\mu_z(x) d\mu_z(y)
\]

\[
\geq \frac{1}{n} \sum_{0 < |v| \leq n \, |v_z \in Z} w_v \eta \geq \eta^4 / 2,
\]

where the second last inequality follows from the fact for each \( vz \) in \( Z \), we have two disjoint sets \( U \) and \( U' \) with \( \mu_{vz} \) measures at least \( \eta \) and \( |f(x) - f(y)| \geq \eta \) whenever \( x \in U \) and \( y \in U' \). Hence,

\[
\int \tilde{t}_n(\tau^I_x, \tau^I_y) \, d\mu_z(x) d\mu_z(y) \geq \eta^4 / 2
\]

for all \( z \in K, n \geq N \).

We will now use the fact that \((X, f)\) is tree very weak Bernoulli to arrive at a contradiction. Let \( 0 < \varepsilon < 1 \) be specified later. Choose \( n \) corresponding to \( \varepsilon \) in the definition of tvwB for \((X, f)\). We may assume that \( n \geq N \) (for the \( N \) chosen in the last paragraph). By definition, we have a set \( G \) of measure \( > 1 - \varepsilon \) such that for all \( x \) and \( y \) in \( G \), \( \tilde{t}_n(\tau^I_x, \tau^I_y) \leq \varepsilon \).

Now, we have

\[
\mu(G^c) = \int \mu_z(G^c) \, d\nu(z) \leq \varepsilon.
\]

Hence,

\[
\mu_z(G^c) \leq \sqrt{\varepsilon}
\]

for all \( z \) in a set \( G' \subseteq \mathbb{I}^\mathbb{N} \) of measure \( \geq 1 - \sqrt{\varepsilon} \). Consequently, we see that for each \( z \) in \( G' \),

\[
\int \tilde{t}_n(\tau^I_x, \tau^I_y) \, d\mu_z(x) d\mu_z(y) \leq \varepsilon + 2\sqrt{\varepsilon}.
\]

Choosing \( \varepsilon > 0 \) sufficiently small ensures that \( \nu(K \cap G') > 0 \). Moreover, choose \( \varepsilon \) such that \( \varepsilon + 2\sqrt{\varepsilon} < \eta^4 / 2 \). Now, if \( z \in K \cap G' \), we have on the one hand,

\[
\int \tilde{t}_n(\tau^I_x, \tau^I_y) \, d\mu_z(x) d\mu_z(y) < \eta^4 / 2
\]

as \( z \in G' \), while

\[
\int \tilde{t}_n(\tau^I_x, \tau^I_y) \, d\mu_z(x) d\mu_z(y) \geq \eta^4 / 2
\]

as \( z \in K \). This is a contradiction and thus completes the proof of the theorem. ■
Chapter 3:

A Joinings Proof of the Isomorphism Theorem

The goal of this chapter is to present a joinings proof of the isomorphism theorem, Theorem 2.4.1, in the previous chapter. The proof is a modification of Hoffman and Rudolph’s ([H,R]) proof of the isomorphism theorem in the case of the uniform probability vector. While the joinings proof is more technical in certain aspects than the proof presented in the previous chapter, it has the advantage of showing that there are uncountably many automorphisms of $B^+(p)$, unless the components of the probability vector $p$ are pairwise distinct (see Proposition 3.4.2). Most of the definitions and theorems below are modelled after Hoffman and Rudolph. There are, however, two modifications that we will make to their proof which will allow us to extend their arguments to the general probability vector. First, we will need to extend the definition of one-sided joinings introduced in [H,R]. In particular, we need to impose an additional condition in the definition of a one-sided joining which is trivial when $p$ is uniform. Second, our statement of the copying lemma will differ from that in Hoffman and Rudolph ([H,R]’s Lemma 5.4). In our presentation, we will need to copy tree distributions induced by the functions on the $p$-endomorphisms under consideration. This approach will save us from tackling the technical issue of whether dist and tdist generate the same topology on tree-adapted functions on $p$-endomorphisms for a general $p$ ([H,R]’s
Lemma 3.6).

This chapter is organized into four sections. In §3.1, we define the notion of a one-sided joining of two \( \mathbf{p} \)-endomorphisms and discuss some topological properties of such joinings. §3.2 is devoted to the proof of the copying lemma, which is similar in form to Proposition 2.4.12 but it is more involved. §3.3 examines the \( \bar{t} \) distance between two processes, as defined in [H,R]. §3.4 contains the proof of Theorem 2.4.1, using the machinery of one-sided joinings developed in §3.1 to §3.3. The idea is to show that for a tvwB \( \mathbf{p} \)-endomorphism \( \mathbf{X} = (X, \mathcal{B}, \mu, T) \) and for \( B^+(\mathbf{p}) = (Y, \mathcal{C}, \nu, S) \), the set of one-sided joinings \( \lambda \) such that \( B^\lambda \mathcal{C} \) (the isomorphic joinings) is a dense \( G_\delta \) in the space of one-sided joinings, which is a compact metric space in the \( w^* \)-topology. The Baire Category Theorem then implies that the set of isomorphic joinings is non-empty, provided that the space of one-sided joinings is non-empty (which will indeed be the case). However, each such joining gives an isomorphism and so \( \mathbf{X} \cong B^+(\mathbf{p}) \).

### 3.1. One-sided Joinings

Let \( \mathbf{p} = (p_1, \ldots, p_s) \) be a fixed probability vector such that \( p_1 \geq \ldots \geq p_s \). Let \( p_0 = 1 \) and assume that \( 0 = s_0 < s_1 < \ldots < s_r = s \) are chosen such that for all \( 0 \leq i \leq r - 1 \), \( p_{s_i+1} = \ldots = p_{s_{i+1}} \) and \( p_{s_i} > p_{s_{i+1}} \). Define the probability vector \( \bar{\mathbf{p}} \) by summing the identical components of \( \mathbf{p} \), i.e.

\[
\bar{\mathbf{p}} = \left( \sum_{i=1}^{s_1} p_i, \ldots, \sum_{i=s_{r-1}+1}^{s_r} p_i \right)
\]

Let \( I(\bar{\mathbf{p}}) = \{1, \ldots, r\}^N \) and assign \( j \) with weight equal to the \( j \)-th component of \( \bar{\mathbf{p}} \). Construct the one-sided Bernoulli shift \( B^+(\bar{\mathbf{p}}) = (I(\bar{\mathbf{p}}), \mathbf{m}, \sigma) \) where \( \mathbf{m} \) is the product measure and \( \sigma \) is the shift defined in the usual way. Given \( \mathbf{X} = (X, \mathcal{B}, \mu, T) \in End(\mathbf{p}) \), define a map \( \psi_\mathbf{X} : X \to I(\bar{\mathbf{p}}) \) by setting \( \psi_\mathbf{X}(x)_i = j \) if \( p_\mathbf{X}(T^i x) = p_{s_j} \). Note that if \( \mathbf{p} \) is a uniform probability vector, \( \psi_\mathbf{X} \) is a constant map into a single point system. For two m.p.s. \( \mathbf{X} = (X, \mathcal{B}, \mu, T) \) and \( \mathbf{Y} = (Y, \mathcal{C}, \nu, S) \), a coupling of \( \mathbf{X} \) and \( \mathbf{Y} \) is a measure \( \lambda \) on the product space \( (X \times Y, \mathcal{B} \times \mathcal{C}) \) such that \( \lambda(B \times Y) = \mu(B) \) for all \( B \subseteq X \) and \( \lambda(X \times C) = \nu(C) \) for all \( C \subseteq Y \) (i.e. \( \lambda \) has marginals \( \mu \) and \( \nu \)). A joining of \( \mathbf{X} \) and \( \mathbf{Y} \) is a coupling \( \lambda \) which is also \( T \times S \)-invariant. For brevity, for a set \( B \subseteq X \), we will also use \( B \) to denote the subset \( B \times Y \) in the product space \( X \times Y \), with similar convention for a set \( C \subseteq Y \). (It will always be clear from the
context whether $B \subseteq X$ refers to $B$ or $B \times Y$.

Two examples of joinings are worth mentioning at this stage. First, the product measure

$\mu \times \nu$ is always a joining of $X$ and $Y$. Second, the diagonal measure $\chi_\Delta$ on $X \times X$ defined

by $\chi_\Delta(B \times C) = \mu(B \cap C)$ for any $B \subseteq X$ and $C \subseteq X$ is a self-joining of $X$.

**Definition 3.1.1.** For $X = (X, B, \mu, T)$ and $Y = (Y, C, \nu, S)$ in $\text{End}(p)$, a **one-sided coupling (joining)** $\lambda$ of $X$ and $Y$ is a coupling (joining) of $X$ and $Y$ such that

i) for any pair of compact-valued generating functions $f : X \to R$ and $g : Y \to U$, if $f^i = f \circ T^i$ and $g^i = g \circ S^i$, then for each $j \geq 0$, we have

$$\text{Dist}_\lambda(\{f^i\}_{i \leq j} \mid \{f^j\}_{j < i}) = \text{Dist}_\lambda(\{g^i\}_{i \leq j} \mid \{g^j\}_{j < i})$$

and

$$\text{Dist}_\lambda(\{g^i\}_{i \leq j} \mid \{f^j\}_{j < i}) = \text{Dist}_\lambda(\{g^i\}_{i \leq j} \mid \{g^j\}_{j < i})$$

ii) $\lambda$ projects to the diagonal measure on $I(p) \times I(p)$ via the map $\psi_X \times \psi_Y : X \times Y \to I(p) \times I(p)$.

Some remarks on definition 3.1.1 are in order. First, condition i) holds if the equations

hold for some pair of generating functions. Indeed, note that the equalities in i) hold if and

only if

$$\text{Dist}_\lambda(\{f^i\}_{i \geq j} \mid \{f^j\}_{j < i}) = \text{Dist}_\lambda(\{f^i\}_{i \geq j} \mid \{f^j\}_{j < i})$$

and

$$\text{Dist}_\lambda(\{g^i\}_{i \geq j} \mid \{f^j\}_{j < i}) = \text{Dist}_\lambda(\{g^i\}_{i \geq j} \mid \{g^j\}_{j < i}).$$

Since the sigma-algebras generated by $\{f^i\}_{i \geq j}$ and $\{g^i\}_{i \geq j}$ for any $j \geq 0$ do not depend on

the choice of the generators, condition i) is just the Hoffman and Rudolph definition ([H,R]'s

Definition 3.2). In the case that $\lambda$ is $T \times S$-invariant, condition i) essentially says that for

$\lambda$-a.a $(x, y)$, if $x'$ is a preimage of $x$, then the conditional probability of $x'$ given $(x, y)$ is just

the conditional probability of $x'$ given only $x$. Condition ii) essentially says that for $\lambda$-a.a

$(x, y)$, the $p_X$-name of $x = p_Y$-name of $y$. Thus if $p$ is a uniform probability vector, condition

ii) always holds and Definition 3.1.1 is just the Hoffman and Rudolph definition.

For brevity, given $X, Y \in \text{End}(p)$, let $C(X, Y)$ and $C^+(X, Y)$ denote the set of couplings

and one-sided couplings of $X$ and $Y$. Similarly, let $J(X, Y)$ and $J^+(X, Y)$ denote the set

of joinings and one-sided joinings of $X$ and $Y$. By viewing the set of probability measures
on \((X \times Y, \mathcal{B} \times \mathcal{C})\) as a subset of bounded linear functionals on the continuous functions on \(X \times Y\), we have a metrizable topology, the \(w^*\)-topology, on the set of couplings and joinings. It is standard that \(C(X, Y)\) and \(J(X, Y)\) are \(w^*\)-compact.

We will now establish some basic facts of one-sided couplings and joinings which will be used in subsequent sections.

**Proposition 3.1.2.** \(C^+(X, Y)\) and \(J^+(X, Y)\) are \(w^*\)-closed convex subsets of \(C(X, Y)\) and \(J(X, Y)\) respectively.

**Proof:** The statement that \(J^+(X, Y)\) is a \(w^*\)-closed convex subset of \(J(X, Y)\) easily follows from the corresponding statement for \(C^+(X, Y)\) and \(C(X, Y)\) and the fact that the set of joinings is a closed convex subset of all couplings. Hence, it is enough to prove that \(C^+(X, Y)\) is a \(w^*\)-closed convex subset of \(C(X, Y)\).

For convexity, notice that condition i) of Definition 3.1.1 says that for each \(t \geq 0\), each one-sided coupling couples \(B\) and \(S^{-t}C\) independently when conditioned on \(T^{-t}B\). As each one-sided coupling projects to \(\mu\) on \(B\), this implies that conditionally on \(T^{-t}B\), each one-sided coupling can be viewed as a product measure of the form \(\mu \times \nu_i\) on \(B \times S^{-t}C\). It follows that any convex combinations of one-sided couplings is still a product measure of this form on \(B \times S^{-t}C\) conditionally on \(T^{-t}B\). Thus any convex combinations couples \(B\) and \(S^{-t}C\) independently over \(T^{-t}B\). By symmetry, \(C\) and \(T^{-t}B\) are also coupled independently when conditioned on \(S^{-t}C\). Thus, condition i) holds for convex combinations. The fact that condition ii) of Definition 3.1.1 also holds for convex combinations of one-sided couplings is obvious since any convex combination of measures projecting to the diagonal measure on \(I(p) \times I(p)\) also projects to the diagonal measure. Hence, one-sided couplings are convex.

We prove closure. Fix \(t \in \mathbb{N}\). Let \(\lambda_i \in C^+(X, Y)\) be a sequence of one-sided couplings such that \(\lambda_i \to \lambda\) in \(w^*\). Clearly, \(\lambda \in C(X, Y)\). Suppose \(P\) is a finite partition of \(X\), and \(Q\) is a finite partition of \(Y\) which is \(S^{-t}C\)-measurable. Choose finite partitions \(R_j\) of \(X\) such that \(R_j \uparrow T^{-t}B\). For condition i) of definition 3.1.1, it suffices to show that for each atom \(\alpha \in P\),

\[
E_\lambda(\alpha | T^{-t}B \vee Q) = E_\lambda(\alpha | T^{-t}B) \quad \text{a.e.}
\]

Indeed, this implies conditionally on \(T^{-t}B\), \(\lambda\) couples \(P\) and \(Q\) independently. Since \(Q\) and \(P\) are arbitrary, condition i) of Definition 3.1.1 follows.

Note that for each \(s \in \mathbb{N}\) and \(\alpha \in P\),
and
\[ E_{\lambda}(\alpha|\frac{s}{1} R_j \vee Q) \to E_{\lambda}(\alpha|\frac{s}{1} R_j \vee Q) \quad \text{a.e.} \]

and
\[ E_{\lambda}(\alpha|\frac{s}{1} R_j \vee Q) \geq E_{\lambda}(\alpha|T^{-i}\mathcal{B} \vee Q) = E_{\mu}(\alpha|T^{-i}\mathcal{B}) \quad \text{a.e.} \]

as \( \lambda_i \) is one-sided. Thus, we have \( E_{\lambda}(\alpha|T^{-i}\mathcal{B} \vee Q) \geq E_{\mu}(\alpha|T^{-i}\mathcal{B}) \) a.e. by the Martingale Convergence Theorem. Since \( T^{-i}\mathcal{B} \) is a sub-\( \sigma \)-algebra of \( T^{-i}\mathcal{B} \vee Q \), the reverse inequality also holds so that condition i) of Definition 3.1.1 is proved. For condition ii), note that for each cylinder set \( c \subseteq I(\tilde{\mathbf{p}}) \), \( (\psi_\mathbf{x} \times \psi_\mathbf{y})^{-1}(c \times c) \) is a set of the form \( B \times C \), where \( B \subseteq X \) and \( C \subseteq Y \). Since \( \lambda_i \to \lambda \) in \( w^* \), we have \( \lambda_i(B \times C) \to \lambda(B \times C) \). Hence, as
\[ \lambda_i((\psi_\mathbf{x} \times \psi_\mathbf{y})^{-1}(c \times c)) = \mathbf{m}(c), \]

the same holds for \( \lambda \) so that \( \lambda \) also projects to the diagonal measure on \( I(\tilde{\mathbf{p}}) \times I(\tilde{\mathbf{p}}) \). ■

The following proposition, which extends Lemma 3.7 in [H,R] to \( \mathbf{p} \)-endomorphisms, gives an example of a one-sided joining. Given a factor map \( \phi : X \to Y \), \( \phi \) yields a probability measure \( \lambda_\phi \) on \( X \times Y \) defined by \( \lambda_\phi(B \times C) = \mu(B \cap \phi^{-1}C) \) for \( B \in \mathcal{B} \) and \( C \in \mathcal{C} \). It is easy to check that \( \lambda_\phi \in J(X,Y) \). We say that \( \lambda_\phi \) is the graphical joining induced by \( \phi \).

**Proposition 3.1.3.** Suppose \( X = (X, \mathcal{B}, \mu, T) \) and \( Y = (Y, \mathcal{C}, \nu, S) \) are ergodic \( \mathbf{p} \)-endomorphisms. Let \( \phi : X \to Y \) be a tree-adapted factor map. Then \( \lambda_\phi \in J^+(X,Y) \).

**Proof:** Note that for any \( t \in \mathbb{N} \) and any node \( u \) of length \( t \), the conditional probability of \( T_u x \) given \( x \) and \( \phi(x) \) equals that of \( T_u x \) given \( x \), as \( \lambda_\phi \) is the graphical joining arising from the factor map \( \phi : X \to Y \). On the other hand, note that as \( \phi \) is tree-adapted, for any \( x \in \phi^{-1}(y) \), there is a unique node \( v \) of length \( t \) such that \( \phi(T_v x) = S_v y \). Clearly, the conditional probability of \( S_v y \) given \( x \) and \( y \) equals that of \( T_v x \) given \( x \). By Proposition 2.1.3, since \( \phi \) is a tree adapted factor map, \( w_u = w_v \). Thus, the conditional probability of \( S_v y \) given \( x \) and \( y \) equals that of \( S_v y \) given \( y \) (in fact, both are equal to \( w_u \)). Since \( \lambda_\phi \) is stationary, it follows that condition i) of Definition 3.1.1 of one-sided joinings holds for \( \lambda_\phi \).

By Proposition 2.1.3, for any cylinder set \( c \) in \( I(\tilde{\mathbf{p}}) \), \( \phi^{-1}(\psi_\mathbf{y}^{-1}(c)) = \psi_\mathbf{x}^{-1}(c) \). Thus,
\[ \lambda_\phi(\psi_\mathbf{x}^{-1}(c) \times \psi_\mathbf{y}^{-1}(c)) = \mu(\psi_\mathbf{x}^{-1}(c)) = C(c), \]

This implies condition ii) of Definition 3.1.1. Hence \( \lambda_\phi \in J^+(X,Y) \). ■
Note that Proposition 3.1.3 implies that if $X = (X, \mathcal{B}, \mu, T)$ is a $p$-endomorphisms, then $J^+(X, B^+(p))$ is non-empty. Indeed, we can easily construct a tree-adapted factor map $\phi_1 : X \to B^+(p)$ by choosing a tree partition $K_X$ of $X$ and mapping a point $x$ to its $K_X$-name. By Proposition 3.1.3, the graphical joining arising from this factor map gives a one-sided joining.

3.2. The Copying Lemma

In this section, we will establish the copying lemma for one-sided joinings which will be the key ingredient in the proof of Theorem 2.4.1.

Let $X$ and $Y$ be $p$-endomorphisms and let $\lambda \in J^+(X, Y)$. For $(x, y) \in X \times Y$, and a pair of nodes $(v, u)$ of the same length $j$, let $p_{x,y}(v, u)$ be the conditional mass of $(T_v x, S_u y)$ given $(x, y)$. By condition i) of Definition 3.1.1, we have $\sum_{|u|=j} p_{x,y}(v, u) = w_v$ and $\sum_{|v|=j} p_{x,y}(v, u) = w_u$ for $\lambda$-a.a. $(x, y)$. Moreover, if $w_v \neq w_u$, then $p_{x,y}(v, u) = 0$ for $\lambda$-a.a. $(x, y)$ by condition ii) of Definition 3.1.1. For a tree automorphism $A$ and two nodes $v$ and $u$, let $A(v, u) = w_v$ if $u = Av$, and $A(v, u) = 0$ otherwise.

Before we state the next proposition, we recall the following well-known theorem which states that any doubly stochastic matrix (i.e. a square matrix such that the entries in each row and column sum to one) is expressible as an average of permutation matrices (i.e. square matrices such that each row and column consists of precisely a single entry of 1 with the rest 0’s). The proof is based on Hall’s Marriage Lemma and can be found in many standard combinatorial texts (eg. [Ryd]).

**Proposition 3.2.1.** Let $M$ be a doubly stochastic matrix of order $n$. Then $M$ can be expressed as a convex combination of permutation matrices, i.e. there exist nonnegative reals $c_1, \ldots, c_t$ which sum to one, along with permutation matrices $P_1, \ldots, P_t$ such that

$$M = c_1 P_1 + \cdots + c_t P_t.$$ 

**Remark:** If in Proposition 3.2.1, $M$ is a matrix such that the entries in each row and column sum to some fixed number $\alpha$, then the conclusion still holds with $P_1, \ldots, P_t$ replaced by “permutation” matrices where each nonzero entry is $\alpha$.

**Proposition 3.2.2.** Let $\lambda \in J^+(X, Y)$ and $N \in \mathbb{N}$. Then for $\lambda$-a.a. $(x, y) \in X \times Y$, we
have a probability measure $m_{x,y}$ on $A_N$ such that for any $1 \leq j \leq N$, if $|v| = |u| = j$, then

$$p_{x,y}(v, u) = \int A(v, u) \, dm_{x,y}(A).$$

**Proof:** Fix $(x, y) \in X \times Y$. We prove the result by induction on $N$. First, suppose $N = 1$. Let $V$ be the set of nodes of length 1. Define a measure $\rho$ on $V \times V$ by setting

$$\rho(U) = \sum_{(v, u) \in U} p_{x,y}(v, u)$$

for $U \subseteq V \times V$. We wish to apply Proposition 3.2.1 to construct a measure on $A_1$ from $\rho$. To do this, recall that each node of length 1 is an integer in $\{1, \ldots, s\}$. We can represent the measure $\rho$ as a $s \times s$ matrix $M$ whose columns are indexed by $1, \ldots, s$ such that $M_{vu} = \rho(v, u)$. By the one-sidedness of $\lambda$, note that $M$ is a block diagonal matrix such that for each block, the entries in each row and column have the same sum (in fact, the sum for the $j$-th block is just $p_{s_j}$). For each of the blocks $M_j$, we can apply the remark following Proposition 3.2.1 and express it as a convex combination of permutation matrices. Doing this block by block, gives a decomposition of $M$ in the form

$$M = \sum_{n=1}^{t} a_n Q_n,$$

where the $a_n$’s sum to 1, $Q_n$’s are distinct block diagonal matrices such that for each $Q_n$, the $j$-th block is a permutation matrix with each nonzero entry being $p_{s_j}$, and the sum is taken over all such possible matrices. We may then use this decomposition on $M$ to define a measure on $A_1$ as follows. For $A \in A_1$, choose the unique matrix $Q_n$ such that $(Q_n)_{vu} = A(v, u)$ for all $v, u \in \{1, \ldots, s\}$, and set $m(A) = a_n$. Then (1) immediately implies that

$$p_{x,y}(v, u) = \rho(v, u) = M_{vu} = \int A(v, u) \, dm(A).$$

Next, assume that the result holds for $N = t$. We wish to build a measure on $A_{t+1}$ such that the asserted equality in the statement of the proposition continues to hold for nodes of length $\leq t + 1$. Notice that each tree automorphism in $A_{t+1}$ is defined by a tree automorphism in $A_t$ combined with a collection of tree automorphisms in $A_1$ indexed by the nodes of length $t$. To define the required measure $m_{t+1}$ on $A_{t+1}$, we proceed as follows. Fix $A' \in A_{t+1}$ and let $A$ denote the restriction of $A'$ to $T_i$. For each node $|v| = t$, consider the
pair of points \((T_v, x, S_{Av}, y)\). Using the basis case, we have a measure \(m_v\) on \(A_1\) such that for any \(|u| = |u'| = 1\),

\[ p_{T_v, x, S_{Av}, y}(u, u') = \int A(u, u') dm_v(A). \]

We then define

\[ m(A') = \prod_{|v| = t} m_v(B_v) m(A), \]

where \(B_v\) is the tree automorphism in \(A_1\) induced by \(A'\) on the trees of height one rooted at \(v\) and \(Av\). A simple calculation using the basis case and the induction hypothesis when \(N = t\) shows that for every pair of nodes \((v, u)\) of common length \(\leq t + 1\)

\[ p_{x, y}(v, u) = \int_{A' \in A_{t+1}} A'(v, u) dm(A'). \]

\[ \square \]

For a finite set \(C\) and partitions \(P : X \to C\) and \(Q : Y \to C\), define the joined partition \(P \otimes Q : X \times Y \to C \times C\) by \(P \otimes Q(x, y) = (P(x), Q(y))\). If \(\lambda \in J(X, Y)\), then \(\lambda\) induces a stationary measure \(\lambda_{P \otimes Q}\) on the shift space \((C \times C)^\mathbb{N}\) via the map \((x, y) \to (P \otimes Q(T^t x, S^t y))_{t \geq 0}\). Clearly, we can extend \(\lambda_{P \otimes Q}\) to a stationary measure on \((C \times C)^\mathbb{Z}\) and then restrict it to a measure on \((C \times C)^{-N}\).

For any \(N \in \mathbb{N}\) and \(j \geq N\), and a pair of elements \((\alpha, \beta)\) in \(C^N\), let \((\alpha \times \beta)^{-j}\) denote the cylinder set

\[ \{ z \in (C \times C)^{-N} \mid z_t = (\alpha_{t+j+1}, \beta_{t+j+1}), -j \leq t \leq -j + N - 1 \}. \]

Given \(A \in \mathcal{A}\) and tree names \(h, h' : T' \to C\), we then define a measure \(\lambda_{(h, h', A)}\) on the cylinder sets by setting \(\lambda_{(h, h', A)}((\alpha \times \beta)^{-j})\) to be the total weights of all nodes \(|v| = j\) such that

\[ (h(v), h(\sigma(v)), \ldots, h(\sigma^{N-1}(v))) = \alpha \]

and

\[ (h'(Av), h'(\sigma(Av)), \ldots, h'(\sigma^{N-1}(Av))) = \beta. \]

The following proposition states that we can represent \(\lambda_{P \otimes Q}\) as an average of measures of the form \(\lambda_{(\tau^x, \tau^y, A)}\). This will be used to prove the copying lemma.
Proposition 3.2.3. With the notations above, for \( \lambda \in J^+(X, Y) \) and partitions \( P : X \to C \) and \( Q : Y \to C \) for a finite set \( C \), we have for each \( N \in \mathbb{N} \), a family of probability measures \( m_{x,y} \) on \( A_N \) such that

\[
\lambda_{P \otimes Q}((\alpha \times \beta)^{-N}) = \int \int \lambda_{(\tau^P_x, \tau^Q_y, A)}((\alpha \times \beta)^{-N}) \, dm_{x,y}(A) \, d\lambda(x, y)
\]

for each pair of elements \((\alpha, \beta)\) in \( C^N \).

Proof: Abbreviate the cylinder set \((\alpha \times \beta)^{-N}\) as \((\alpha \times \beta)\). Notice that as \( \lambda \) is \( T \times S \)-invariant, we have

\[
\lambda_{P \otimes Q}(\alpha \times \beta) = \int \sum_{|v|=|u|=N} p_{x,y}(v, u) \, d\lambda(x, y)
\]

By Proposition 3.2.2, we have a measure \( m_{x,y} \) on \( A_N \) such that

\[
p_{x,y}(v, u) = \int A(v, u) \, dm_{x,y}(A)
\]

for all pairs of nodes \( v \) and \( u \) of length \( N \). We then have

\[
\lambda_{P \otimes Q}(\alpha \times \beta) = \int \sum_{|v|=|u|=N} p_{x,y}(v, u) \, d\lambda(x, y)
\]

\[
= \int \sum_{|v|=|u|=N} \int A(v, u) \, dm_{x,y}(A) \, d\lambda(x, y)
\]

\[
= \int \int \sum_{|v|=|u|=N} A(v, u) \, dm_{x,y}(A) \, d\lambda(x, y)
\]

\[
= \int \int \lambda_{(\tau^P_x, \tau^Q_y, A)}((\alpha \times \beta)) \, dm_{x,y}(A) \, d\lambda(x, y).
\]

\[ \blacksquare \]

We will now prove the copying lemma for one-sided joinings. Note that while it appears to be more general than the copying lemma in [H,R] (Proposition 5.4) in that we also copy distribution of tree names, the proof is essentially the same. Following the convention in Chapter 2, we will work with functions taking values in \([0,1)\).

Proposition 3.2.4 (Copying Lemma). Suppose \( X = (X, \mathcal{B}, \mu, T) \) and \( Y = (Y, \mathcal{C}, \nu, S) \) are ergodic \( p \)-endomorphisms. Let \( \lambda \in J^+(X, Y) \). If \( g : X \to [0,1) \) and \( h : Y \to [0,1) \), then for all \( \varepsilon \) and \( N \), there exists a function \( \tilde{g} \) on \( Y \) such that

\[
|\text{dist}(g_N^{\mathcal{C}}) - \text{dist}(\tilde{g}_N^{\mathcal{C}})| < \varepsilon
\]

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and
\[ |\text{dist}(g_N^N \otimes h_N^N) - \text{dist}(\tilde{g}_N^N \vee h_N^N)| < \varepsilon. \]

**Proof:** Let \( \tilde{g} = g_N^{N^\vee} \vee g_N \) and \( \tilde{h} = h_N^{N^\vee} \vee h_N \) and let \( \tilde{D} \) denote \((D_N)^{N^\vee} \times D_N\). (Recall that \( D_N \) is the set of midpoints of the dyadic intervals \([t/2^N, (t+1)/2^N)\).) Choose \( M > N \) such that \( 2N/M < \varepsilon/2 \). Construct a \( \varepsilon/2 \)-tree Rokhlin tower \( M \) of height \( M+1 \) in \( Y \). By Proposition 3.2.3, we have measures \( m_{x,y} \) on \( A_M \) such that for each \( N \leq j \leq M \) and for each pair of elements \((\alpha, \beta)\) in \( \tilde{D}^N \),
\[
\lambda_{\tilde{g} \otimes \tilde{h}}((\alpha \times \beta)^{-j}) = \int \int \lambda_{(\tau_1^j \times \tau_2^j, A)}((\alpha \times \beta)^{-j}) dm_{x,y}(A)d\lambda(x, y). \tag{1}
\]

Consider the partitions \( \tilde{g}^{M^\tau} : X \to \tilde{D}^{M^\tau} \) and \( \tilde{h}^{M^\tau} : Y \to \tilde{D}^{M^\tau} \), we can assign a measure \( \hat{\lambda} \) to each atom of the partition \( \tilde{g}^{M^\tau} \times \tilde{h}^{M^\tau} \times A_M \) of \( X \times Y \times A_M \) by setting
\[
\hat{\lambda}(p \times q \times A) = \int_{px} m_{x,y}(A) d\lambda(x, y). \tag{2}
\]

By the Strong Tree Rokhlin Lemma, we may assume that the base \( C \) of \( M \) is chosen such that \( \text{dist}(\tilde{h}^{M^\tau} \mid C) = \text{dist}(\hat{h}^{M^\tau}) \). For each atom \( \gamma \in \hat{h}^{M^\tau} \), we define a partition
\[
P_\gamma : \gamma \cap C \to \tilde{D}^{M^\tau} \times \tilde{D}^{M^\tau} \times A_M
\]
such that
\[
\text{dist}(P_\gamma) = \hat{\lambda}(\tilde{g}^{M^\tau} \times \tilde{h}^{M^\tau} \times A_M \mid \gamma).
\]
The partitions \( P_\gamma \) over all \( \gamma \) collectively define a partition \( P \) of \( C \) such that
\[
\text{dist}(P) = \text{dist}(\tilde{g}^{M^\tau} \times \tilde{h}^{M^\tau} \times A_M).
\]

This gives a bijective correspondence \( \rho \) of the atoms of \( P \) and those of \( \tilde{g}^{M^\tau} \times \tilde{h}^{M^\tau} \times A_M \) such that if \( \alpha \in P \) and \( \rho(\alpha) = (\beta, \gamma, A) \), then \( \alpha \subseteq \gamma \), and \( \nu(\alpha \mid C) = \hat{\lambda}(\rho(\alpha)) \).

We will now construct the required function \( \tilde{g} \) on the tower. Fix an atom \( \alpha \in P \) such that \( \rho(\alpha) = (\beta, \gamma, A) \). Choose any point \( x \in \beta \). We define the function \( \tilde{g} \) on \( \cup_{i=1}^M S^{-i} \alpha \) by setting \( \tilde{g}(z) = g(T_{A^{-1} \beta} x) \) for \( z \in S^{-i} \alpha \). By repeating this procedure for each atom \( \alpha \in P \), we can extend \( \tilde{g} \) to a function on \( \cup M \setminus B \). Extend the function \( \tilde{g} \) to the rest of \( Y \) in any way we like.

Set \( \tilde{g} = g_N^{N^\vee} \vee g_N \). For an integer \( j \) such that \( N \leq j \leq M - N \), we claim that
\[
\text{dist}(\tilde{g}^{N^+} \otimes \tilde{h}^{N^+}) = \text{dist}(\tilde{g}^{N^+} \vee \tilde{h}^{N^+} | S^{-j} C_{\phi}) \tag{3}
\]
If $\alpha \in P$ and $\rho(\alpha) = (\beta, \gamma, A)$, then by our construction, notice that

$$(\tilde{g}_N(S_v), h_N(S_v)) = (g_N(T_{A^{-1}}v), h_N(S_v))$$

for any $y \in \alpha$ and $x \in \beta$ and $v \in T'_M$. We thus have

$$(\tilde{g}(S_v y), \tilde{h}(S_v y)) = (\tilde{g}(T_{A^{-1}}v), \tilde{h}(S_v y))$$

(4)

for $v \in T'_{M-N}$.

Let $\zeta_1 \vee \zeta_2 = \{ y \in Y \mid \tilde{g}^{N^+}(y) = \zeta_1, \tilde{h}^{N^+}(y) = \zeta_2 \}$. Then (4) implies that for any $x \in \beta$ and $y \in \gamma$,

$$\nu(\zeta_1 \vee \zeta_2 | S^{-j}C) = \lambda_{(\tau^g_x, \tau^\lambda \gamma, A)}((\zeta_1 \times \zeta_2)^{-j}).$$

(5)

Note that (1) and (2) implies that for $x(\beta) \in \beta$ and $y(\gamma) \in \gamma$,

$$\lambda_{\tilde{g} \circ \tilde{h}}((\zeta_1 \times \zeta_2)^{-j}) = \sum_{(\beta, \gamma, A) \in \tilde{g} \circ \tilde{h}} \lambda_{(\tau^g_x, \tau^\lambda \gamma, A)}((\zeta_1 \times \zeta_2)^{-j})\lambda(\beta, \gamma, A).$$

(6)

Hence, since $\nu(S^{-j}C) = \nu(A|C) = \lambda(\beta, \gamma, A)$, (5) and (6) imply that

$$\lambda_{\tilde{g} \circ \tilde{h}}((\zeta_1 \times \zeta_2)^{-j}) = \sum_{\alpha \in P} \nu(\zeta_1 \vee \zeta_2 | S^{-j}C)\nu(S^{-j}C)$$

$$= \nu(\zeta_1 \vee \zeta_2 | S^{-j}C).$$

It follows that for $N \leq j \leq M-N$,

$$dist(\tilde{g}^{N^+} \otimes \tilde{h}^{N^+}) = dist(\tilde{g}^{N^+} \vee \tilde{h}^{N^+}|S^{-j}C),$$

which is (3).

As $\nu(\bigcup_{i=M-N}^{M-N} S^{-j}C) > 1 - \varepsilon$, we then have

$$\left| dist(\tilde{g}^{N^+} \otimes \tilde{h}^{N^+}) - dist(\tilde{g}^{N^+} \vee \tilde{h}^{N^+}) \right| < \varepsilon,$$

from which the conclusion easily follows, since $\tilde{g}$ refines the partitions $g_N$ and $g_N^{N^+}$ (and similarly for $\tilde{g}$ and $\tilde{h}$). $\blacksquare$

### 3.3. The $\bar{t}$ Distance

Suppose $X = (X, B, \mu, T)$ and $Y = (Y, C, \nu, S)$ are $p$-endomorphisms. Following [H,R], for a fixed $m \in N$ and functions $g : X \to [0,1]$ and $h : Y \to [0,1]$ respectively, define the $\bar{t}_m$ distance between the pair of processes $(X,g)$ and $(Y,h)$ by

$$\bar{t}_m((X,g),(Y,h)) = \inf_{\lambda \in C^+(X,Y)} \frac{1}{\lambda} \int_{\sigma \in C^+(X,Y)} \bar{t}_m(\tau^{g}_{x}, \tau^{h}_{\gamma}) d\lambda.$$
Let
\[ \bar{t}(X, g), (Y, h) = \liminf \bar{t}_m((X, g), (Y, h)). \]

Note that this definition of \( \bar{t}_m((X, g), (Y, h)) \) differs from the definition in §2.4 in which we consider only the product joining \( \lambda \). Nonetheless, note that Proposition 2.4.6 (which will be used in §3.4) will also hold with the present definition. In fact, for a tvwB \( X \) and \( \varepsilon > 0 \), if \( \text{dist}(g^N, h^N) \) is sufficiently close to \( \text{dist}(h^N, g^N) \) for some \( N \), the proof of Proposition 2.4.6 shows that \( \int \bar{t}_m(T_x^i, S_y^i) d\lambda < \varepsilon \) for all large \( m \) for any coupling \( \lambda \) (not just one-sided).

Our goal in this section is to prove the following proposition which will be combined with the copying lemma in the previous section to give a joinings proof of Theorem 2.4.1.

**Proposition 3.3.1.** Suppose \( \bar{t}((X, g), (Y, h)) < \varepsilon \), then there exists \( \lambda \in J^+(X, Y) \) such that \( \int |g(x) - h(y)| \ d\lambda < \varepsilon \).

The proof of Proposition 3.3.1 will depend on following proposition whose proof will be postponed until the end of this section.

**Proposition 3.3.2.** For each \( m \in \mathbb{N} \), there exists a \( \lambda \in C^+(X, Y) \) such that
\[
\frac{1}{m} \int \sum_{i=0}^{m-1} |g(T^i x) - h(S^i y)| \ d\lambda(x, y) \leq \bar{t}_m((X, g), (Y, h)) + 1/2^{m-2}.
\]

**Proof (of Proposition 3.3.1):** By Proposition 3.3.2, for each \( m \in \mathbb{N} \), we have some \( \lambda_m \in C^+(X, Y) \) such that
\[
\frac{1}{m} \int \sum_{i=0}^{m-1} |g(T^i x) - h(S^i y)| \ d\lambda_m(x, y) \leq \bar{t}_m((X, g), (Y, h)) + 1/2^{m-2}.
\]

Choose an increasing sequence of integers \( n_1, n_2, \ldots \) such that
\[
\bar{t}_{n_m}((X, g), (Y, h)) \to \bar{t}((X, g), (Y, h)),
\]
so that by passing to a further subsequence if necessary, we have some \( L < \varepsilon \) such that
\[
\frac{1}{n_m} \int \sum_{i=0}^{n_m-1} |g(T^i x) - h(S^i y)| \ d\lambda_{n_m}(x, y) \to L. \tag{1}
\]

For \( i \geq 0 \) and any measure \( \lambda \) on \( X \times Y \), let \( (T \times S)^i \lambda \) denote the measure defined by \( (T \times S)^i \lambda(D) = \lambda((T \times S)^{-i} D) \) for \( D \subseteq X \times Y \). Define
\[
\tilde{\lambda}_m = \frac{1}{n_m} \sum_{i=0}^{n_m-1} (T \times S)^i \lambda_{n_m}.
\]
Since the set of one-sided couplings is convex, \( \tilde{\lambda}_m \in C^+(X, Y) \). For any measure \( \lambda \) on \( X \times Y \) and \( i \geq 0 \), let
\[
\lambda(g^i \Delta h^i) = \int |g(T^i x) - h(S^i y)| \, d\lambda.
\]
Then,
\[
\tilde{\lambda}_m(g^0 \Delta h^0) = \frac{1}{n_m} \sum_{i=0}^{n_m-1} (T \times S)^i \lambda_m(g^0 \Delta h^0)
\]
\[
= \frac{1}{n_m} \sum_{i=0}^{n_m-1} \lambda_m(g^i \Delta h^i).
\]
Let \( \lambda^* \) be any \( w^* \)-limit point of the \( \tilde{\lambda}_m \)'s. Since \( C^+(X, Y) \) is \( w^* \)-closed in \( C(X, Y) \), \( \lambda^* \in C^+(X, Y) \). It is clear that \( \lambda^* \) is stationary so \( \lambda^* \in J^+(X, Y) \). By (1) and (2), given \( \delta > 0 \), we have for all sufficiently large \( m \),
\[
\lambda^*(g^0 \Delta h^0) \sim \tilde{\lambda}_m(g^0 \Delta h^0) \sim \delta.
\]
Hence, \( \lambda^*(g^0 \Delta h^0) = L < \varepsilon \), since \( \delta \) is arbitrary. 

We now turn our attention to the proof of Proposition 3.3.2. The proof will require the construction of a one-sided coupling of \( X = (X, B, \mu, T) \) and \( Y = (Y, C, \nu, S) \), which we now describe. Given functions \( g : X \to [0, 1) \) and \( h : Y \to [0, 1) \), and \( m \in \mathbb{N} \), we have the joined partition \( g^m \otimes h^m \) of \( X \times Y \) defined by \( g^m \otimes h^m(x, y) = (g^m(x), h^m(y)) \). For an atom \( \beta \in g^m \otimes h^m \), let \( A_{\beta} \) denote a tree automorphism in \( A_m \) which realizes \( \tilde{t}_m(\tau_x^m, \tau_y^m) \) for any \( (x, y) \in \beta \). Set \( A_{x,y} = A_{\beta} \) for any \( (x, y) \in \beta \).

For each \( (x, y) \in X \times Y \), we define a measure supported on \( (T \times S)^{-m}(x, y) \), which we will denote as \( A^m_{x,y} \), in the following way. For any measurable set \( D \subseteq X \times Y \), let \( A^m_{x,y}(D) \) be the total weights of the nodes \( v \) of length \( m \) such that \( (T_v x, S_{A_{x,y}} v y) \in D \). Given \( \lambda \in C^+(X, Y) \), define the measure \( \tilde{\lambda} \) on \( (X \times Y, B \times C) \) by setting
\[
\tilde{\lambda}(D) = \int A^m_{x,y}(D) d\lambda(x, y)
\]
for \( D \subseteq X \times Y \). It is straightforward to check that \( \tilde{\lambda} \) defines a probability measure on \( X \times Y \).

We now show:

**Lemma 3.3.3.** \( \tilde{\lambda} \in C^+(X, Y) \).

**Proof:** First, we need to see that \( \tilde{\lambda} \) is a coupling. Let \( \{\mu_x\} \) be a disintegration of \( \mu \) induced by the factor map \( T^m : X \to X \). If \( B \subseteq X \), then \( A^m_{x,y}(B) = \mu_x(B) \). Thus,
\[
\tilde{\lambda}(B) = \int A^m_{x,y}(B) d\lambda = \int \mu_x(B) d\mu = \mu(B)
\]
for all measurable \( B \subseteq X \times Y \).
and so $\lambda$ projects to $\mu$ on $X$. By symmetry, $\lambda$ projects to $\nu$ on $Y$. Hence $\lambda \in C(X, Y)$.

We now need to check that $\lambda$ is one-sided. We first check condition i) of Definition 3.1.1. For convenience, let us represent $X$ and $Y$ as one-sided shift spaces on $[0,1]^\mathbb{N}$ (i.e. $X = Y = [0,1]^\mathbb{N}$, and $T = S$ is the shift map). Given integers $0 \leq i \leq j$, and a point $z \in [0,1]^\mathbb{N}$, we let

$$z[i, j] = \{y \in [0,1]^\mathbb{N} \mid y_t = z_t, i \leq t \leq j\}$$

and

$$z[i, \infty) = \{y \in [0,1]^\mathbb{N} \mid y_t = z_t, t \geq i\}.$$

By symmetry, it clearly suffices to prove that for each integer $t > 0$ and for $\lambda$-a.a. $(x', y')$ in $X \times Y$,

$$E(x'[0, t-1]|x'[t, \infty) \times y'[t, \infty)) = E(x'[0, t-1]|x'[t, \infty)).$$  \hspace{1cm} (1)

We prove (1) by considering three cases: $t = m$, $t < m$ and $t > m$:

**Case I: $t = m$**

Using the definition of $\lambda$, if $x = T^m x'$ and $y = S^m y'$, we have

$$E(x'[0, m-1]|x'[m, \infty) \times y'[m, \infty)) = E(x'[0, m-1]|T^{-m}x \times S^{-m}y)$$

$$= A_{x,y}^m(x'[0, m-1])$$

$$= E(x'[0, m-1]|x'[m, \infty)).$$

**Case II: $t < m$**

Observe that for $\lambda$-a.a. $(x', y')$, if $x = T^m x'$ and $y = S^m y'$, then $A_{x,y}^m(x', y') > 0$. For each such $(x', y')$, we have

$$A_{x,y}^m(x'[0, m-1] \times y'[0, m-1]) = E(x'[0, m-1]|x'[m, \infty)).$$

We may then use Case I) to show that

$$E(x'[0, t-1]|x'[t, \infty) \times y'[t, \infty)) = \frac{E(x'[0, m-1] \times y'[t, m-1]|x'[m, \infty) \times y'[m, \infty))}{E(x'[t, m-1]|x'[m, \infty)) \times y'[t, m-1]|x'[m, \infty))}$$

$$= \frac{A_{x,y}^m(x'[0, m-1] \times y'[t, m-1])}{A_{x,y}^m(x'[t, m-1] \times y'[t, m-1])}$$

$$= \frac{E(x'[0, m-1]|x'[m, \infty))}{E(x'[t, m-1]|x'[m, \infty))}$$

$$= E(x'[0, t-1]|x'[t, \infty)).$$

**Case III: $t > m$**

This follows from a direct calculation using Case I) and the fact that $\lambda$ is one-sided.
We now prove condition ii) of Definition 3.1.1. Recall the Bernoulli shift $B^+(\mathfrak{p}) = (I(\mathfrak{p}) = \{1, \ldots, r\}^N, \mathfrak{m}, \sigma)$ as defined in §3.1. Consider the cylinder set

$$C = \{ z \in I(\mathfrak{p}) \mid z_0 = r_0, \ldots, z_{m-1} = r_{m-1} \},$$

where $r_j \in \{1, \ldots, r\}$. By the definition of the maps $\psi_X$ and $\psi_Y$, we have a set $V_C$ of nodes of length $m$ whose total weights is $\mathfrak{m}(C)$ such that

$$\psi_X^{-1}(C) = \bigcup_{v \in V_C} T_vX \text{ and } \psi_Y^{-1}(C) = \bigcup_{v \in V_C} S_vY.$$ 

Hence,

$$A^m_{x,y}(\psi_X^{-1}(C) \times \psi_Y^{-1}(C)) = \sum_{v,u \in V_C} A_{x,y}(v, u) = \mathfrak{m}(C)$$

Condition ii) now follows immediately from the definition of $\bar{\lambda}$ and the one-sidedness of $\lambda$.

We are now ready to prove Proposition 3.3.2.

\textbf{Proof (of Proposition 3.3.2):} Consider the partition $g^m_m \otimes h^m_m$ on $X \times Y$. Let $\lambda \in C^+(X, Y)$, construct $\bar{\lambda}$ as in Lemma 3.3.3. For each atom $\beta \in g^m_m \otimes h^m_m$ and a pair of nodes $(v, u)$ of length $m$, note that

$$\bar{\lambda}(T_v \times S_u) \beta = A_{\beta}(v, u) \lambda(\beta). \quad (1)$$

Choose a point $(x_\beta, y_\beta) \in \beta$ for each atom $\beta$. Note that by (1) and our choice of $A_\beta$,

$$\frac{1}{m} \sum_{i=0}^{m-1} \int_{(T \times S)^{-m, \beta}} |g_m(T^ix) - h_m(S^iy)| \, d\bar{\lambda}$$

$$= \sum_{|v|=|u|=m} \int_{T_v \times S_u(\beta)} \frac{1}{m} \sum_{i=0}^{m-1} |g_m(T^ix) - h_m(S^iy)| \, d\bar{\lambda}$$

$$= \sum_{|v|=|u|=m} \frac{1}{m} \sum_{i=0}^{m-1} |g_m(T^ix, x_\beta) - h_m(S^iy, y_\beta)| \, A_{\beta}(v, u) \lambda(\beta)$$

$$= \bar{\lambda}(\tau_{x_\beta} g^m_m, \tau_{y_\beta} h^m_m) \lambda(\beta).$$

Thus,

$$\frac{1}{m} \sum_{i=0}^{m-1} \int |g_m(T^ix) - h_m(S^iy)| \, d\bar{\lambda} = \frac{1}{m} \sum_{i=0}^{m-1} \sum_{\beta \in g^m_m \otimes h^m_m} \int_{(T \times S)^{-m, \beta}} |g(T^ix) - h(S^iy)| \, d\bar{\lambda}$$

$$= \sum_{\beta \in g^m_m \otimes h^m_m} \bar{\lambda}(\tau_{x_\beta} g^m_m, \tau_{y_\beta} h^m_m) \lambda(\beta)$$

$$= \int \bar{\lambda}(\tau_{x} g^m_m, \tau_{y} h^m_m) \, d\lambda.$$
Since $|g - g_m| \leq 1/2^m$, we have
\[
\frac{1}{m} \sum_{i=0}^{m-1} \int |g(T_i x) - h(S_i y)| \, d\lambda \leq \int \tilde{t}_m(\tau_x^y, \tau_y^x) \, d\lambda + 1/2^{m-1}.
\]
As this argument holds true for all $\lambda \in C^+(X, Y)$, the result follows. ■

3.4. A Joinings Proof of Theorem 2.4.1

In this section, we prove Theorem 2.4.1 using the machinery of one-sided joinings developed in §3.1 to §3.3. In fact, we will prove a stronger result. We say that $\lambda \in J^+(X, Y)$ is an isomorphic joining if $B^\lambda = C$. Note that each isomorphic joining is the graphical joining of an isomorphism from $X$ to $Y$. We now prove

Theorem 3.4.1. Let $p$ be a probability vector. If $X = (X, B, \mu, T) \in \text{End}(p)$ is tvwB, the set of isomorphic joinings is a dense $w^*-G_\delta$ of $J^+(X, B^+(p))$.

Proof: Let $Y = B^+(p)$. Choose a pair of generating functions $f : X \rightarrow [0, 1)$ and $g : Y \rightarrow [0, 1)$. Fix $N \in \mathbb{N}$ and $\delta > 0$.

Let $\lambda \in J^+(X, Y)$. For each $N' \in \mathbb{N}$ and $\delta' > 0$, by the copying lemma, we have a function $\bar{f}$ on $Y$ such that
\[
|\text{dist}(f_{N'}^{-\nabla}) - \text{dist}(\bar{f}_{N'}^{-\nabla})| < \delta'
\]
and
\[
|\text{dist}_\lambda(f_{N'}^{\nabla} \otimes g_{N'}^{\nabla}) - \text{dist}(\bar{f}_{N'}^{\nabla} \vee g_{N'}^{\nabla})| < \delta'.
\]
For $\varepsilon > 0$, if $1/N'$ and $\delta'$ are sufficiently small then by (1) and the proof of Proposition 2.4.6,
\[
\tilde{t}((X, f), (Y, \bar{f})) < \varepsilon.
\]
By Proposition 3.3.1, we have some $\lambda' \in J^+(X, Y)$ such that
\[
\int |f(x) - \bar{f}(y)| \, d\lambda'(x, y) < \varepsilon.
\]
Then provided that $\varepsilon$, $1/N'$, $\delta'$ are small enough, we may use (2) to conclude that
\[
|\text{dist}_\lambda(f_N^{\nabla} \otimes g_N^{\nabla}) - \text{dist}_\lambda(f_N^{\nabla} \otimes g_N^{\nabla})| < \delta.
\]
As in [H,R], for $\lambda \in J^+(X, Y)$, we say that $f \triangleleft Y$ if there exists some function $h$ on $Y$ such that $\int |f(x) - h(y)| \, d\lambda < \varepsilon$. Let $O_\varepsilon = \{\lambda \in J^+(X, Y) \mid f \triangleleft Y\}$. Then $O_\varepsilon$ is $w^*$-open in
distinct, there are uncountably many automorphisms of the one-sided Bernoulli shift and $X$ joinings. Note that by Proposition 3.1.3, the set of isomorphic joinings is non-empty. Thus, all $\epsilon > 0$ (and hence in general between any two $tvwB$ isomorphic joining. ■

Now, by Baire’s Theorem, $\bigcap_{n=1}^{\infty} O_{1/n}$ is a dense $w^*-G_\delta$ in $J^+(X,Y)$. By symmetry, if $O'_\epsilon = \{ \lambda \in J^+(X,Y), g \subseteq \lambda \} = \bigcap_{n=1}^{\infty} O'_{1/n}$, we see that the set $\bigcap_{n=1}^{\infty} O_{1/n}$ is also dense in $J^+(X,Y)$. Hence, the intersection $\bigcap_{n=1}^{\infty} O_{1/n} \cap O'_{1/n}$ is a dense $w^*-G_\delta$ in $J^+(X,Y)$ so that it is a residual set. This concludes the proof as any $\lambda$ in the intersection satisfies $f \subseteq \lambda Y$ and $g \subseteq \lambda X$ so that $\lambda$ is an isomorphic joining.

This concludes the proof of the isomorphism theorem, using the machinery of one-sided joinings. Note that by Proposition 3.1.3, the set of isomorphic joinings is non-empty. Thus, $X$ and $B^+(p)$ are isomorphic.

One corollary of the joinings proof is that unless the components of $p$ are pairwise distinct, there are uncountably many automorphisms of the one-sided Bernoulli shift $B^+(p)$ (and hence in general between any two $tvwB$ $p$-endomorphisms). To prove this, we first need the lemma below, which is just a special case of Proposition 3.1.3. In the following, we suppose $B^+(p)$ is represented as the one-sided shift space $(\{1,\ldots,s\}^N, \mu, T)$ for the Bernoulli measure $\mu = \{p_1,\ldots,p_s\}^N$ and the shift map $T$ (so that $j$ has weight $p_j$ for $1 \leq j \leq s$).

**Lemma 3.4.2.** Suppose $h : \{1,\ldots,s\}^N \to \{1,\ldots,s\}$ is tree-adapted and for $a.a.x$ in $\{1,\ldots,s\}^N$, $x_0$ and $h(x)$ have the same weight. Then the $h$-name map $\theta : B^+(p) \to B^+(p)$ defined by $\theta(x) = (h(x), h(Tx), \ldots)$ is a tree-adapted factor map. Moreover, the graphical self-joining $\lambda_n$ of $B^+(p)$ derived from $\theta$ is one-sided. ■

**Proposition 3.4.3.** For a probability vector $p$ with at least two identical components, there are uncountably many automorphisms of $B^+(p)$.

**Proof:** Without loss of generality, suppose $p_1 = p_2$. It suffices to show that $J^+(B^+(p), B^+(p))$ has no isolated points. Indeed, from the proof of the isomorphism theorem, the isomorphic joinings is a dense $w^*-G_\delta$ in $J^+(B^+(p), B^+(p))$. However, by the Baire Category Theorem, a dense $w^*-G_\delta$ in a complete metric space with no isolated points is necessarily uncountable.

To this end, notice that there are at least two elements in $J^+(B^+(p), B^+(p))$. Indeed, the graphical joining derived from the identity automorphism certainly is one. Another one is the graphical joining derived from the $h$-name map for the function $h : \{1,\ldots,s\}^N \to \{1,\ldots,s\}$.
defined by

\[ h(x) = \begin{cases} 
2, & \text{if } x_0 = 1; \\
1, & \text{if } x_0 = 2; \\
x_0, & \text{otherwise.}
\end{cases} \]

(This is one-sided by Lemma 3.4.2).

Let \( \lambda \in J^+(B^+(p), B^+(p)) \) and \( \varepsilon > 0 \). It suffices to construct \( \lambda' \in J^+(B^+(p), B^+(p)) \) such that \( \lambda' \neq \lambda \) and \( \lambda' \sim_* \lambda \) in \( w^* \). To see this, since there are at least two elements in \( J^+(B^+(p), B^+(p)) \), just choose some one-sided joining \( \lambda_1 \) distinct from \( \lambda \). By Proposition 3.1.2, convex combinations of \( \lambda \) and \( \lambda_1 \) remain one-sided. Let \( \lambda_\delta = (1 - \delta)\lambda + \delta \lambda_1 \). If \( \delta \) is small enough, then \( \lambda_\delta \) will be sufficiently close to \( \lambda \) in \( w^* \), and we are done. ■
Chapter 4:
Examples of TvwB p-Endomorphisms

In this chapter, we will present two classes of examples of tvwB p-endomorphisms. It follows from the isomorphism theorem, Theorem 2.4.1, that all of these are isomorphic to $B^+(p)$.

4.1. One-Sided Markov Shifts

Besides the one-sided Bernoulli shift $B^+(p)$, the simplest examples of p-endomorphisms can be found among the one-sided Markov shifts, which we will define below.

It will be convenient for us to define a one-sided Markov shifts over left-infinite shift spaces of the form $C^{-N}$ for a finite set $C$. We say that a square matrix $A$ is stochastic if each entry is nonnegative and the sum of the entries in each row equals one. Let $|A|$ denote the number of rows (or columns) of $A$. We may obviously index the rows and columns of $A$ by the integers from 1 to $|A|$ and denote the entries in $A$ by $A_{ij}$, where $1 \leq i, j \leq |A|$. A stochastic matrix is irreducible if for each pair $(i, j)$, there exists $k \in \mathbb{N}$ such that $(A^k)_{ij}$ is nonzero. It is well known that for an irreducible stochastic matrix, there exists a unique row probability vector $q$ with all components positive such that $qA = q$. We say that $q$ is a left fixed probability vector of $A$. Let $q_j$ denote the $j$-th component of $q$. Using $q$ and $A$, we may define a measure $\mu$ on the cylinder sets of $\{1, \ldots, |A|\}^{-N}$ by
\[\mu(x \in \{1, \ldots, |A|\}^{-N} | x_{-n} = a_{-n}, \ldots, x_{-1} = a_{-1}) = q_{a_{-n}}A_{a_{-n}a_{-(n-1)}} \cdots A_{a_{-2}a_{-1}}.\]

It is easily seen that \(\mu\) extends to a measure on the Borel sigma-algebra \(\mathcal{B}\) of \(\{1, \ldots, |A|\}^{-N}\). If \(T\) is the shift map on \(\{1, \ldots, |A|\}^{-N}\), then note that \(\mu = \mu T^{-1}\) (i.e. \(\mu\) is shift invariant). We define the **one-sided Markov shift over** \(A\), denoted \(X_A^-\), to be the m.p.s. \((\{1, \ldots, |A|\}^{-N}, \mathcal{B}, \mu, T)\). We will refer to the integers \(\{1, \ldots, |A|\}\) as the **states** of \(X_A^-\). We say that \(X_A^-\) is an **irreducible Markov shift over** \(A\) if \(A\) is irreducible. Note that \(X_A^-\in End(p)\) if and only if the entries in each row of \(A\) are the components of \(p\), after deleting all zero entries.

For each stochastic matrix \(A\), consider the function \(f_A : \{1, \ldots, |A|\}^{-N} \to \{1, \ldots, |A|\}\) defined by \(f_A(x) = x_{-1}\). By placing the discrete metric \(d\) on \(\{1, \ldots, |A|\}\), it is clear that \(f_A\) is generating. Note that for the one-sided Markov shift \(X_A^-\), the \((-1)^{st}\) coordinates and the conditional probabilities of the inverse images of any point \(x\) are completely determined by \(x_{-1}\). If \(X_A^-\in End(p)\), we can thus choose a tree partition \(K_{X_A^-}\) of \(X_A^-\) with the following property: whenever \(x_{-1} = y_{-1}\), then \(x\) and \(y\) generate the same \(f_A\)-tree name if the tree names are defined with respect to that partition. We will henceforth assume that for each \(X_A^-\in End(p)\), we choose \(K_{X_A^-}\) with this property. If \(I\) is a state of \(X_A^-\), let \(\tau_{f_A}^I\) be the common \(f_A\)-tree name generated by all points \(y\) with \(f_A(y) = I\).

We now wish to give examples of one-sided Markov shifts that are tvwB. Our first example is motivated by the well-known fact (Ornstein and Friedman) that a strongly mixing two-sided Markov shift is two-sided Bernoulli. Unfortunately, the one-sided analogue of this fact requires considerably more restrictions. Indeed, consider the one-sided Markov shift induced by the stochastic matrix:

\[
\begin{pmatrix}
2/3 & 1/3 \\
1/3 & 2/3
\end{pmatrix}
\]

It is not difficult to see that \(X_A^-\in End(\frac{2}{3}, \frac{2}{3})\). However, as \(X_A^-\) is not tvwB (since the standard generator yields two tree names whose \(t_m\) distance is 1 for all \(m\in \mathbb{N}\), \(X_A^-\) is not isomorphic to \(B^+(\frac{2}{3}, \frac{2}{3})\) and hence cannot be one-sided Bernoulli. (Note that, however, the **two-sided** Markov shift over \(A\) is isomorphic to the **two-sided** Bernoulli shift \(B(\frac{2}{3}, \frac{2}{3})\).) Nonetheless, as the following shows, for one-sided Markov shifts which are uniformly \(p\)-to-1 endomorphisms, strong mixing does imply one-sided Bernoulli.
Proposition 4.1.1. Let \( p \in \mathbb{N} \) and \( \mathbf{p} = (\frac{1}{p}, \ldots, \frac{1}{p}) \). Suppose \( A \) is a \( N \times N \) stochastic matrix such that the one-sided Markov shift \( X_A^- \) is in \( \text{End}(\mathbf{p}) \) and is strongly mixing. Then \( X_A^- \) is \( tvwB \).

Some observations will be helpful before we prove Proposition 4.1.1. Note that for every stochastic matrix \( A \), we may associate a directed weighted graph to it. Specifically, we define \( G(A) \) to be the graph with \( |A| \) vertices identified by the integers \( 1, \ldots, |A| \) with a directed edge from \( I \) to \( J \) labeled with weight \( w \) if \( A_{I,J} = w \) if \( w > 0 \). Note that if \( X_A^- \in \text{End}(\mathbf{p}) \), the set of weights of the edges extending out from any vertex in \( G(A) \) is the same. The critical observation which will be of use to us in the proofs of the ensuing propositions is the following: for any states \( I \) and \( J \), there exists a node \( v \) of length \( n \) such that \( \tau_I^f(v) = J \) if and only if we have a path in \( G(A) \) of length \( n \) from vertices \( I \) to \( J \).

Proof (Proposition 4.1.1): For brevity, we let \( f \) denote \( f_A \) throughout the proof. Since \( X_A^- \) is strongly mixing, there exists some integer \( n \) such that \( (A^n)_{ij} > 0 \) for all \( 1 \leq i, j \leq N \). We thus have some path in \( G(A) \) of length \( n \) between any two vertices in \( G(A) \). By the observation made just prior to this proof, this implies in particular that there exists some state \( J \) such that for all state \( I \), \( \tau_I^f(v) = J \) for some node \( v \) of length \( n \). Hence, for all \( x \in X_A^- \), \( \tau_x^f(v) = J \) for some node \( v \) of length \( n \).

For any pair of points \((x', y')\), let \( B_{x', y'} \) be any tree automorphism in \( A_n \) such that \( \tau_{x'}^f(v) = \tau_{y'}^f(B_{x', y'}(v)) = J \) for some node \( v \) of length \( n \). Given a fixed pair of points \((x, y)\) in \( X_A^- \), we now build a tree automorphism \( B \) \( n \)-levels at a time which makes \( \overline{t}_m(\tau_{x'}^f, \tau_{y'}^f) \) small for all large \( m \). For \( 0 < |v| \leq n \), let \( B(v) = B_{x,y}(v) \). Inductively, assume that \( B(v) \) is defined for all \( |v| \leq sn \). For \( sn < |v| \leq (s + 1)n \), let \( v = uv' \), where \( |v'| = sn \) and \( 0 < |u| \leq n \).

We then define \( B(v) = uB(v') \) if \( f(T_vx) = f(T_{B(v')}y) \) and define \( B(v) = B(T_vx, T_{B(v')}y)(u)B(v') \) otherwise.

Now, we have some node \( u \) of length \( n \) such that \( f(T_ux) = f(T_{B(u)}y) \). Since the \( f \)-tree name of \( x \) depends only on \( x(-1) \) and we are extending by the identity automorphism in the subtree rooted at \( u \), it follows that if \( n \leq t < 2n \), the weights of the nodes \( |v| = t \) such that \( f(T_ux) = f(T_{B(v)}y) \) total to at least \( 1/p^n \). For the nodes \( |u| = n \) such that \( f(T_ux) \neq f(T_{B(u)}y) \), \( B \) is defined in such a way that we have some node \( |v| = n \) such that \( f(T_{vu}x) = f(T_{B(vu)}y) \). Thus, if \( 2n \leq t < 3n \), the weights of the nodes \( |v| = t \) with \( f(T_vx) = f(T_{B(v)}y) \) sum to at least
\[
\frac{1}{p^n}(1 - \frac{1}{p^n}) + \frac{1}{p^n}.
\]

Inductively, we see that in general, if \(sn \leq t < (s + 1)n\), the weights of the nodes \(|v| = t\) with \(f(T_vx) = f(T_{B(v)}y)\) sum to at least
\[
\sum_{i=0}^{s-1} \frac{1}{p^n}(1 - \frac{1}{p^n})^i.
\]

This sum approaches 1 as \(s \to \infty\), independent of \(x\) and \(y\). Hence it follows that \(X^-_A\) is tree v.w.B. ■

The next proposition gives an additional class of tvwB Markov shift.

**Proposition 4.1.2.** Let \(p\) be a probability vector (not necessarily uniform). Suppose that \(A\) is an irreducible \(N \times N\) matrix such that \(X^-_A \in End(p)\) and for every pair of rows in \(A\), we can find two identical nonzero entries in the same column, then \(X^-_A\) is tvwB.

**Proof:** Once again, consider the graph \(G(A)\) associated to \(A\). Note that it suffices to find \(n \in \mathbb{N}\) along with paths of length \(n\) from each vertex ending at a common vertex such that the corresponding edges in the paths have equal weights. Indeed, this will allow us to conclude that for any \(x\) and \(y\) in \(X^-_A\), there exists a node \(v\) of length \(n\) such that \(\tau^f_{x}(v) = \tau^f_{y}(B(v))\) for some tree automorphism \(B\), and we may argue as before to reach the conclusion.

To construct the required paths, note that by assumption, there are edges extending from vertices 1 and 2 with the same weight \(w_1\) ending at a common vertex, say \(J_1\), in \(G(A)\). We then choose any edge of weight \(w_1\) extending from vertex 3. If this edge ends at vertex \(J_2\), then again by assumption we have edges of equal weights from \(J_1\) and \(J_2\) which end at a common vertex. This allows us to extend the paths from each of vertices 1, 2 and 3 such that they all end at the same vertex and the corresponding edges in the paths have the same weight. A simple inductive argument enables us to construct the desired paths. ■

We end this section with a proposition which shows that we can decide whether a one-sided Markov shift is tvwB simply by checking tree names of a finite height (depending on the dimension of the stochastic matrix \(A\)). For a general \(p\)-endomorphism, deciding whether it is tree v.w.B is obviously a more difficult problem.

Let \(X^-_A \in End(p)\) and consider its associated graph \(G(A)\). Note that every path in \(G(A)\) “sees” a sequence of weights by reading the weights attached to the edges from the start to the end of the path. We now prove:
Proposition 4.1.3. Suppose that $A$ is an irreducible $N \times N$ matrix such that the Markov shift $X_A^- \in \text{End}(p)$. Then $X_A^-$ is tvwB if and only if there exist paths of common length $\leq N^{3N}$ from each vertex in $G(A)$ which see a common sequence of weights and end in the same vertex.

Proof: We note that every edge $e$ in the directed graph $G(A)$ can be represented by the triple $(s(e), t(e), w(e))$, where $s(e)$, $t(e)$ and $w(e)$ are the starting vertex, terminal vertex and weight of $e$. Since the number of different nonzero weights cannot exceed the number of vertices, we have at most $N^3$ different types of edges under this representation.

Now, if $(X_A^-, f_A)$ is tvwB then we have paths $v_J$ from each vertex $J$ in $G(A)$ which end at a common vertex and see the same sequence of weights. Indeed, by the tvwB condition, for any two vertices $I$ and $I'$, we must have paths $v_I$ and $v_{I'}$ with the desired property. If $I''$ is another vertex, then construct any path $v_{I''}$ from $I''$ which see the same sequence of weights as $v_I$. The three paths just constructed end in at most two distinct vertices so we may extend these three paths such that they end at a common vertex. An inductive argument gives us the required paths.

Assume that the paths chosen have a common length $> N^{3N}$. For a path $u$, let $u(j)$ be the $j$-th edge of $u$. Then there are at most $N^{3N}$ possible ordered $N$-tuple of edges $(v_1(j), \ldots, v_N(j))$ for each $j$. Now, if the paths $v_1, \ldots, v_N$ have more than $N^{3N}$ edges, then there exist integers $j < j'$ such that

$$(v_1(j), \ldots, v_N(j)) = (v_1(j'), \ldots, v_N(j')).$$

Hence, we may shorten the path $v_I$ by deleting the edges $v_I(k)$ for $j \leq k < j'$, for each vertex $I$. Clearly, the new paths still have the desired property but they have a shorter length. Hence, we may continue to shorten the paths to have a common length $\leq N^{3N}$.

Conversely, if the asserted property holds, then for every pair of states $I$ and $J$ of $X_A^-$, there is a node $u$ of length $\leq N^{3N}$ such that there exists a tree automorphism $B$ with $\tau_I^{f_A}(u) = \tau_J^{f_A}(Bu)$. The same argument in the proof of Proposition 4.1.1 shows that $X_A^-$ is tvwB. ■

Since the tree names $\tau_I^{f_A}$ restricted to $\mathcal{T}_{N^{3N}}$ over all states $I$ of $X_A^-$ determine all paths of length at most $N^{3N}$ in $G(A)$, Proposition 4.1.4 shows that it suffices to look at tree names of that height to determine whether $X_A^-$ is tvwB. It is worth mentioning that Ashley, Marcus
and Tuncel [A,M,T] developed a general, though necessarily more complicated, algorithm for deciding whether any two one-sided Markov shifts are isomorphic.

4.2. A Generalization of \([T, Id]\)

In this section, we shall consider a case of the well-known \([T, Id]\) transformations in the context of a general probability vector \(p = (p_1, \ldots, p_n)\) and characterize those that are one-sided Bernoulli.

The \([T, Id]\) transformation can be described as follows. We consider the 2-shift \(B^+(p) = (\{0,1\}^\mathbb{N}, \mu, \sigma)\) with \(p = (\frac{1}{2}, \frac{1}{2})\), and a Lebesgue space \((Y, C, \nu)\). Suppose \(T\) is an automorphism of \(Y\), define the map \([T, Id]\) on the product space \(\{0,1\}^\mathbb{N} \times Y\) with product measure \(\mu \times \nu\) by

\([T, Id](x, g) = (\sigma x, T^x(0)g)\).

It is not difficult to show that \([T, Id]\) is measure preserving and the resulting m.p.s. \((\{0,1\}^\mathbb{N} \times Y, \mu \times \nu, [T, Id]\) is a \(p\)-endomorphism.

A special case of \([T, Id]\) occurs when \(Y\) is just the circle represented as \([0,1)\) and \(T = R_\alpha\) is a rotation on \(Y\) by an irrational \(\alpha\). Hoffman and Rudolph [H,R] showed that this particular \([T, Id]\) system (along with other isometric extensions of the uniformly \(p\)-to-1 endomorphisms with certain properties) are all tvwB and hence one-sided Bernoulli.

Let us now extend the \([T, Id]\) system to the situation when \(p\) is an arbitrary finite probability vector. Consider a compact abelian metrizable group \(G\) with a translation invariant metric \(d'\) and Haar measure \(\nu\) (i.e. \(d'(hg, h'g) = d'(h, h')\) for all \(h, h'\) and \(g \in G\)). Let \(d\) be the discrete metric on \(\{1, \ldots, n\}\). Define a metric \(D\) on the set \(R = \{1, \ldots, n\} \times G\) by

\[D((x_1, g_1), (x_2, g_2)) = \frac{1}{2}d(x_1, x_2) + \frac{1}{2}d'(g_1, g_2).\]

Let \(p = (p_1, \ldots, p_n)\) and let \(B = \{1, \ldots, n\}^\mathbb{N}\). As usual, let \(B^+(p) = (B, \mu, \sigma)\) be the one-sided Bernoulli shift such that state \(j\) has weight \(p_j\), for \(1 \leq j \leq n\). For each state \(j\), we associate to it some element \(g_j \in G\). Consider the transformation \(S\) on \(B \times G\) defined by \(S(x, g) = (\sigma x, gg_j)\) if \(x_0 = j\). Let \(\lambda\) be the product measure \(\mu \times \nu\), then note that \((B \times G, \lambda, S) \in \text{End}(p)\).

Consider the function \(f : B \times G \rightarrow R\) defined by \(f(x, g) = (x_0, g)\). Note that \(f\) is generating. Let \(f_1\) and \(f_2\) denote the component functions of \(f\), i.e. \(f_1(x, g) = x_0\) and
Clearly \( f_1 : B \times G \to \{1, \ldots, n\} \) defines a tree partition of \((B \times G, \lambda, S)\). We may thus use \( f_1 \) to define a set of partial inverses \( S_v \) for each node \( v \) of the \( p \)-tree \( T \) as described in Chapter 1. Clearly, we have \( f_1(S_v(x, g)) = f_1(S_v(x', g')) \) for all nodes \( v \) of length \( \geq 1 \) and for all \((x, g)\) and \((x', g')\) in \( B \times G \).

**Proposition 4.2.1.** \((B \times G, \lambda, S)\) is tvwB if there exist \( i \neq j \) with \( p_i = p_j \) such that \( g_j g_i^{-1} \) has dense orbit in \( G \).

**Proof:** We first assume the stated condition and prove that \(((B \times G, \lambda, S), f)\) is tvwB. With no loss of generality, we may assume that \( p_1 = p_2 \) and \( g_2 g_1^{-1} \) has dense orbit. Given \( \varepsilon > 0 \), we note that there exists some \( s \in \mathbb{N} \) such that for any \( h \) and \( h' \) in \( G \), we can find an integer \( 0 < r < s \) such that \( d'((h g_1^r, h' g_2^{-r})) < \varepsilon \). To see this, partition \( G \) into sets of diameter \(< \varepsilon/3 \). On each set, pick an arbitrary element in it. Say that \( y_1, \ldots, y_k \) are the elements picked. Since \( g_2 g_1^{-1} \) has dense orbits and \( G \) is abelian, for any pair \((y_i, y_j)\) in \( G \), there exists an integer \( r > 0 \) such that \( d'(y_i g_1^{-r}, y_j g_2^{-r}) < \varepsilon/3 \). We may then choose \( s \) to be larger than all these \( r \)'s. Thus, for any two elements \( h \) and \( h' \) in \( G \), we have some integer \( 0 < r < s \) such that \( d'(h g_1^{-r}, h' g_2^{-r}) < \varepsilon \).

For any pair of points \( x \) and \( y \) in \( B \times G \), by the above paragraph, there exists some tree automorphism \( \bar{A} \) and some node \( v \) of length \( r < s \) such that

\[
d'(f_2(S_v x), f_2(S_{\bar{A}v} y)) < \varepsilon.
\]

(In fact, the inequality can be met by considering the node \( v = (1, \ldots, 1) \) of length \( r \) and letting \( \bar{A} \) be any tree automorphism such that \( \bar{A}v = (2, \ldots, 2) \).) Moreover, for any node \( v' \), we have

\[
d'(f_2(S_{v'} x), f_2(S_{v'} \bar{A}v y)) < \varepsilon.
\]

If \( v' \) is not the root node, then as we have observed,

\[
d(f_1(S_{v'} x), f_1(S_{v'} \bar{A}v y)) = 0.
\]

Hence, we have some node \( u \) of length \( s \) and a tree automorphism \( A \) such that

\[
D(f(S_u x), f(S_{Au} y)) < \varepsilon.
\]

If \( p_n \) is the smallest component in the probability vector \( p \), we can imitate the proof of Proposition 4.1.1 to construct the required tree automorphism \( A \) \( s \)-levels at a time such that
whenever $sN \leq t < s(N + 1)$, the nodes $v$ of length $t$ for which $D(f(S_vx), f(S_Ay)) < \varepsilon$ have total weights at least

$$\sum_{i=0}^{N-1} (p_n)^i(1 - (p_n)^i).$$

From this, we see that $(B \times G, \lambda, S)$ is tvwB. □

In the case when $G$ is the circle $[0,1)$ with Lebesgue measure, we can improve Proposition 4.2.1 as follows:

**Proposition 4.2.2.** $(B \times [0,1), \lambda, S)$ is tvwB if and only if there exist $i \neq j$ with $p_i = p_j$ such that $g_jg_i^{-1}$ is irrational.

**Proof:** Since the irrationals have dense orbits, the fact that $(B \times [0,1), \lambda, S)$ is tvwB given the stated condition is just a special case of Proposition 4.2.1. Conversely, suppose the stated condition is false but $(B \times [0,1), f)$ is tvwB. We then have some positive integer $N \geq 2$ such that $g_jg_i^{-1}$ is some integral multiple of $\frac{1}{N}$ whenever $p_j = p_i$. Suppose $x$ and $y$ are points in $B \times [0,1)$ with the property that $f_2(x) = t$ and $f_2(y) = t + \frac{1}{2N}$ for some $t \in [0,1)$. Then by our choice of the rotation factors $g_j$, for every tree automorphism $A$ and node $v$, if $f_2(S_vx) = t'$, then $f_2(S_Ay) = t' + \frac{1}{2N} + \frac{k}{N}$ for some integer $k$. Hence, $d'(f_2(S_vx), f_2(S_Ay))$ is bounded away from zero and so there exists some $\beta > 0$ such that for all $n$,

$$\bar{t}_n(\tau_x', \tau_y') \geq \beta$$

for all pair of points $(x, y)$ such that $f_2(x) = t$ and $f_2(y) = t + \frac{1}{2N}$ for some $t \in [0,1)$. Now, as any large set must contain a pair of such points, this contradicts the tvwB condition and completes the proof. □
Chapter 5:

Finite Group Extensions of One-Sided Bernoulli Shifts

In this chapter, we consider Lebesgue spaces with certain semigroup actions defined on them. The main result of this chapter is that for any two such spaces with semigroup actions \( \{ T_g \}_{g \in G} \) and \( \{ S_g \}_{g \in G} \), there is an isomorphism \( \phi \) such that \( \phi \circ T_g = S_g \circ \phi \) for all \( g \in G \).

5.1. An Isomorphism Theorem on TvwB G-extensions

Let \( G' \) be a metrizable semigroup. For an arbitrary nonatomic Lebesgue space \((X, B, \mu)\), we define a \( G' \)-action on \((X, B, \mu)\) as a collection of measure preserving endomorphisms \( \{ T_g \} \) with the property that \( T_g T_{g'} = T_{gg'} \) for all \( g, g' \in G' \), and the map \( \pi : G' \times X \to X \) defined by \( \pi(g, x) = T_g x \) is measurable. If \((Y, C, \nu)\) is another nonatomic Lebesgue space with a \( G' \)-action \( \{ S_g \} \) defined on it, then we say that \((X, B, \mu), \{ T_g \}\) and \((Y, C, \nu), \{ S_g \}\) are \( G' \)-isomorphic if there exists a measure preserving bijection \( \phi : X \to Y \) such that \( \phi T_g = S_g \phi \) for all \( g \in G' \).

Consider a finite group \( G \). Note that \( \mathbb{N}^* \times G \) is a semigroup with operation defined by \( (n, g) \cdot (n', g') = (n + n', g'g) \). For a probability vector \( p = (p_1, \ldots, p_s) \), we say that a \( \mathbb{N}^* \times G \)-action \( \{ T_{(n,g)} \} \) on a nonatomic Lebesgue space \((X, B, \mu)\) is a \( (\mathbb{N}^* \times G, p) \)-action if the m.p.s. \((X, B, \mu, T_{(1,e)})\) is a \( p \)-endomorphism and \( T_{(0,g)} \) is an automorphism of \((X, B, \mu)\) with no fixed...
measure preserving map obtained by composing the maps \( \phi \) and \( \mu \), and \( \nu \) be two nonatomic Lebesgue spaces with \( (N^* \times G, p) \)-actions \( \{T_{n,g}\} \) and \( \{S_{n,g}\} \) defined on them. If the systems \( X = (X, \mathcal{B}, \mu, T_{(1,e)}) \) and \( Y = (Y, \mathcal{C}, \nu, S_{(1,e)}) \) are both tvwB, then \((X, \mathcal{B}, \mu, \{T_{n,g}\}) \) and \((Y, \mathcal{C}, \nu, \{S_{n,g}\}) \) are \( N^* \times G \)-isomorphic.

A natural example of a space with a \( (N^* \times G, p) \)-action is given by a G-extension of \( B^+(p) \), which we will now define. Let \( \nu \) be the uniform probability measure on the finite group \( G \) equipped with the discrete topology. Let \( I \) denote the set \( \{1, \ldots, s\} \). As we have seen, by assigning weight \( p_j \) to \( j \), we may represent the one-sided Bernoulli shift \( B^+(p) \) as \( (\mathbb{N}^*, \mathcal{B}, \mu, T) \), where \( T \) is the shift map and \( \mu \) is the product measure \( \{p_1, \ldots, p_s\}^{\mathbb{N}^*} \) on the Borel sigma-algebra \( \mathcal{B} \). Let \( P : I^{\mathbb{N}^*} \rightarrow I \) be the time zero partition defined by \( P(x) = x_0 \).

Consider the product space \( I^{\mathbb{N}^*} \times G \) with the product measure \( \mu \times \nu \) on its product sigma-algebra \( \mathcal{P} \). For a measurable partition \( \varphi : I^{\mathbb{N}^*} \rightarrow G \), define the map \( T^\varphi : I^{\mathbb{N}^*} \times G \rightarrow I^{\mathbb{N}^*} \times G \) by \( T^\varphi(x, g) = (Tx, \varphi(x)g) \). We will refer to such partitions \( \varphi \) as a cocycle. It is easy to verify that \( T^\varphi \) is measure preserving. We define the \( (G, \varphi) \)-extension of \( B^+(p) \), denoted as \( B^+(p)^\varphi \), to be the m.p.s. \( (I^{\mathbb{N}^*} \times G, \mathcal{P}, \mu \times \nu, T^\varphi) \). More generally, we say that a m.p.s. is a \( G \)-extension of \( B^+(p) \) if it is the \( (G, \varphi) \)-extension of \( B^+(p) \) for some cocycle \( \varphi : I^{\mathbb{N}^*} \rightarrow G \). It is clear that any \( G \)-extension of \( B^+(p) \) is a \( p \)-endomorphism. Define the partition \( P' : I^{\mathbb{N}^*} \times G \rightarrow I \times G \) by \( P'(x, g) = (P(x), g) \). Let \( d \) be the discrete metric on \( I \times G \). Clearly, \( (I \times G, d) \) is a compact metric space and \( P' \) is generating.

Given a \( G \)-extension \( B^+(p)^\varphi \), note that each element \((n, g) \in \mathbb{N}^* \times G \) corresponds to a measure preserving map obtained by composing the maps \( \varphi_{(1,e)} \) and \( \varphi_{(0,g')} \) given by

\[
\varphi_{(1,e)}(x, g) = (Tx, \varphi(x)g)
\]

and

\[
\varphi_{(0,g')}(x, g) = (x, gg').
\]

It is easy to verify that the maps \( \varphi_{(n,g)} \) define a semigroup action, namely a \( \mathbb{N}^* \times G \)-action, on \( I^{\mathbb{N}^*} \times G \). (This is the reason why we defined \((n, g) \cdot (n', g') \) as \((n + n', g'g) \) instead of \((n + n', gg') \).) We say that the \( G \)-extensions \( B^+(p)^\varphi \) and \( B^+(p)^\phi \) are \( \mathbb{N}^* \times G \)-isomorphic if there is a measure preserving bijection \( \rho : I^{\mathbb{N}^*} \times G \rightarrow I^{\mathbb{N}^*} \times G \) such that \( \rho \varphi_{(n,g)} = \phi_{(n,g)} \rho \) a.e. for all \((n, g) \in \mathbb{N}^* \times G \). We say that \( \rho \) is a \( \mathbb{N}^* \times G \)-isomorphism from \( B^+(p)^\varphi \) to \( B^+(p)^\phi \).
As $\rho \circ T^x = T^\phi \circ \rho$, the $N^* \times G$-isomorphism $\rho$ is also an isomorphism from $B^+(p)^\varphi$ to $B^+(p)^\phi$ in the sense of our original definition in Chapter 1.

We begin the proof of Theorem 5.1.1 with some reductions. First, note that if $(X, \mathcal{B}, \mu)$ has a $(N^* \times G, p)$-action on it such that $X = (X, \mathcal{B}, \mu, T_{(1,e)})$ is tvwB then we can represent it as a $G$-extension of $B^+(p)$. In fact, consider the space $X/G$ of $G$-orbits (so each point in $X/G$ is a set of the form $T_{(0,g)}x$, $g \in G$). $T_{(1,e)}$ projects to a measure preserving transformation $\bar{T}$ on $X/G$ such that $(X/G, \bar{T})$ is a $p$-endomorphism. Since $T_{(0,g)}$ has no fixed points unless $g = e$, the canonical projection is $\Pi_G : X \to X/G$ is a tree-adapted factor map. Thus $(X/G, \bar{T})$ is isomorphic to $B^+(p)$ by Theorem 2.4.1 and Proposition 2.4.2. Representing the space $X$ as $X/G \times G$, it is immediate that $(X, \{T_{(n,g)}\})$ is $N^* \times G$-isomorphic to $(N^* \times G, \{\phi_{(n,g)}\})$ for some cocycle $\phi : N^* \to G$. Thus we may rephrase Theorem 5.1.1 as follows.

**Theorem 5.1.1’.** If $B^+(p)^\varphi$ and $B^+(p)^\phi$ are tvwB, then they are $N^* \times G$-isomorphic.

Note that by our definition, all $G$-extensions of $B^+(p)$ are defined on the space $N^* \times G$, although different $G$-extensions have different dynamics. To emphasize the difference in dynamics, for any cocycle $\psi : I^{N^*} \to G$, we will henceforth use $I_\psi$ to denote the product space $I^{N^*} \times G$ of the m.p.s. $B^+(p)^\psi$ and we will denote the generating partition $P'$ of $I_\psi$ by $P^\psi$.

Let $\Pi : I^{N^*} \times G \to I^{N^*}$ be the canonical projection. For $x \in I^{N^*}$ and a subset $A \subseteq I^{N^*}$, let the fiber over $x$ denote the set $\Pi^{-1}(x)$ and the fiber over $A$ denote the set $\Pi^{-1}(A)$. Note that for any $N^* \times G$-isomorphism $\rho : I_\varphi \to I_\phi$, since $\rho \phi_{(0,g)} = \phi_{(0,g)} \rho$ for all $g \in G$, $\rho$ maps fibres to fibres, i.e. for $x \in I^{N^*}$, $\rho(\Pi^{-1}(x)) = \Pi^{-1}(y)$ for some $y \in I^{N^*}$. Moreover, it is easily seen that the fibres are mapped to one another by a group rotation in the sense that for each $x \in I^{N^*}$, there is $g \in G$ and $x' \in I^{N^*}$ such that $\rho(x, g') = (x', gg')$ for all $g' \in G$.

Conversely, given measurable partitions $\varphi : I^{N^*} \to G$ and $\phi : I^{N^*} \to G$, an isomorphism $\rho_1 : B^+(p) \to B^+(p)$, and a measurable partition $\theta : I^{N^*} \to G$ such that

$$\theta(Tx) \varphi(x) = \phi(\rho_1(x)) \theta(x),$$

then it is direct that the map $\rho : I_\varphi \to I_\phi$ defined by $\rho(x, g) = (\rho_1(x), \theta(x)g)$ defines a $N^* \times G$-isomorphism from $B^+(p)^\varphi$ to $B^+(p)^\phi$. Thus, proving that $B^+(p)^\varphi$ and $B^+(p)^\phi$ are $N^* \times G$-isomorphic amounts to constructing the maps $\rho_1$ and $\theta$.

The basic approach to the proof of Theorem 5.1.1’ is the same as that of Theorem 2.4.1, with some necessary additions. We will focus primarily on these additions and refer the
reader to the various propositions in Chapter 2 when the proofs are exactly the same.

Before we embark on the proof, some remarks on notations are in order. Rather than considering functions taking values in the compact metric space \([0,1]\) as in Chapter 2, we will assume in this chapter that all the functions take values in the finite set \(I \times G\) (i.e. are partitions), with the discrete metric \(d\), unless otherwise specified. To emphasize this fact, we will denote the functions by capital letters such as \(Q\) and \(R\) instead of small letters \(g\) and \(h\). Among the functions which range in \(I \times G\), we shall primarily be interested in those functions \(Q : I^{N_r} \times G \rightarrow I \times G\) for which there exists a pair of measurable partitions \(Q_1 : I^{N_r} \rightarrow I\) and \(\xi : I^{N_r} \rightarrow G\) such that

\[
Q(x, g) = (Q_1(x), \xi(x)g)
\]

for all \((x, g) \in I^{N_r} \times G\) and \(Q_1\) is tree-adapted. We say that \(Q\) is a \(G\)-map if it has the above property. We can thus think of each \(G\)-map \(Q\) as being defined by an ordered pair of functions \((Q_1, \xi)\) of the above form. We will write \(Q = (Q_1, \xi)\) if \(Q\) is defined by \((Q_1, \xi)\).

Note that the generating partition \(P'\) is a \(G\)-map. If \(Q\) is a \(G\)-map, then for each \(x \in I^{N_r}\), \(Q\) assigns the same first coordinate to any two distinct points in the fiber over \(x\) but distinct group coordinates. Thus, a \(G\)-map takes every fiber onto a set of the form \(\{j\} \times G\) for some \(j \in I\).

For the purpose of constructing tree names, it will be convenient to choose a “canonical” set of partial inverses for \(G\)-extensions as follows. Recall that each node \(v\) of the \(p\)-ary tree \(T\) is a finite sequence of integers in \(I = \{1, \ldots, s\}\). Consider any \(G\)-extension \(B^+(p)^{\bar{r}}\). For each \(z \in I^{N_r}\), let \(vz\) be the point in \(I^{N_r}\) obtained by concatenating \(v\) to the left of \(z\). If \(|v| = t\), then for each \((x, g)\) in \(I_x\), we define the partial inverse \(T_v^x\) by setting \(T_v^x(x, g)\) to be the unique element in \((T^r)^{-t}(x, g)\) whose first coordinate is \(vx\).

Given a function \(Q\) on \(B^+(p)^{\bar{r}}\), we may then define the partitions \(Q^{N_r} : I_{\bar{r}} \rightarrow (I \times G)^{N_r}\), \(Q^{N_{\bar{r}}} : I_{\bar{r}} \rightarrow (I \times G)^{N_{\bar{r}}}\) and \(Q^{N} : I_{\bar{r}} \rightarrow (I \times G)^{N}\) exactly as before. Moreover, we may define the \(\bar{t}\) distance between any two processes \((B^+(p)^{\bar{r}}, Q)\) and \((B^+(p)^{\bar{r}}, R)\) for functions \(Q\) and \(R\) on the respective spaces as described in Chapter 2. Proposition 2.4.6 translates into the following.

**Proposition 5.1.2.** Let \(B^+(p)^{\bar{r}}\) be \(tvwB\). Suppose \(Q\) is a function on \(I_{\bar{r}}\). For all \(\varepsilon > 0\), there exist \(\delta\) and \(N\) with the following property: for any \(G\)-extension \(B^+(p)^{\bar{r}}\) and function \(R\) on \(I_{\bar{r}}\), if \(R^{N_{\bar{r}}} \stackrel{\delta}{\sim} Q^{N_{\bar{r}}},\) then \(\bar{t}((B^+(p)^{\bar{r}}, Q), (B^+(p)^{\bar{r}}, R)) < \varepsilon\). ■
Proposition 5.1.3. Consider $G$-extensions $B^+(p)^\varphi$ and $B^+(p)^\phi$, along with $G$-maps $Q$ on $I_\varphi$ and $R$ on $I_\phi$. Then for all $g \in G$ and all $v \in T'$, 
\[ d(Q(T_v^\varphi(x, g)), R(T_v^\varphi(x', g))) = d(Q(T_v^\varphi(x, e)), R(T_v^\varphi(x', e))). \]

Proof: This follows immediately from the fact that as $Q$ is a $G$-map, if $Q(T_v^\varphi(x, e)) = (j, g')$, then $Q(T_v^\varphi(x, g)) = (j, g'g)$, and likewise for the $G$-map $R$. \[ \blacksquare \]

We are now ready to prove the analogue of the perturbation lemma, Proposition 2.4.7. For $(j, g) \in I \times G$ and $\bar{g} \in G$, set $(j, g) \cdot \bar{g} = (j, g\bar{g})$. For any $G$-extension $B^+(p)^\varphi$, a measurable set $B$ in $I_\varphi$ and $g \in G$, let $B_g$ denote the set $B \cap (I^{N'} \times \{g\})$ (so $B_g$ is the subset of $B$ with group coordinate $g$). In the following proposition and its proof, we will use the notation $O_i(\delta), i = 1, 2, \ldots$ to denote real-valued functions of $\delta$ such that $\lim_{\delta \to 0} O_i(\delta) = 0$.

Proposition 5.1.4. Consider an ergodic $G$-extension $B^+(p)^\varphi$. Then there exists some $O_1(\delta)$ with the following property: For any ergodic $G$-extension $B^+(p)^\phi$ and any $G$-map $Q$ on $I_\phi$, if $\tilde{t}((B^+(p)^\varphi, P^\varphi), (B^+(p)^\phi, Q)) < \delta$, then for all $\varepsilon$ and $N$, there exists a $G$-map $Q'$ on $I_\phi$ such that $|Q - Q'| < O_1(\delta)$ and $(P^\varphi)^{N\varepsilon} \bowtie Q'^{N\varepsilon}$.

Proof: We proceed along the lines of Proposition 2.4.7 except that we cannot use only a single tree name to define a function on the chosen tree Rokhlin tower as we did in Chapter 2. For $\eta > 0$ to be specified later, choose $M \in \mathbb{N}$ such that

a) $N/M < \eta/2$

b) there exists a set $S \subseteq I^{N'}$ with $\mu(S) > 1 - \eta$ such that whenever $x \in S$, $(x, g)$ is $\eta_i(M - N)$-generic for $(P^\varphi)^{N\varepsilon}$ for all $g \in G$

c) $\int \tilde{t}_M(\tau_{x,g}^P, \tau_{y,h}^Q) d(\mu \times \nu)(x, g)d(\mu \times \nu)(y, h) < \delta$.

From b) and c), if $\eta$ is sufficiently small, we have some $x \in S$ such that there exists a set $S' \subseteq I^{N'}$ with $\mu(S') > 1 - O_2(\delta)$ and $\tilde{t}_M(\tau_{x,e}^P, \tau_{z,e}^Q) < O_2(\delta)$ whenever $z \in S'$. Using the Strong Tree Rokhlin Lemma, build a $\eta/2$-tree Rokhlin tower $M$ of height $M + 1$ in $B^+(p)$ such that at least $1 - O_2(\delta)$ fraction of the base $B$ is in $S'$. Note that $M' = \Pi^{-1}(M)$ is also a tree Rokhlin tower in $B^+(p)^\phi$ with base $B' = \Pi^{-1}(B)$. For each $(y, e) \in B'_e$, we define $Q'$ by laying the tree name $\tau_{y,e}^P$ on $\{(T^\phi)^{-i}(y, e) \mid 1 \leq i \leq M\}$ via a suitable tree automorphism $A$ which optimizes $\tilde{t}_M(\tau_{x,e}^P, \tau_{y,e}^Q)$ as we did in Proposition 2.4.7. This defines $Q'$ on the column over $B'_e$. We then extend $Q'$ to points in the column over $B'_g$ for each $g \in G$ by defining $Q'(T_v^\varphi(y, g)) = Q'(T_v^\phi(y, e)) \cdot g$. Note that restricted to $\cup M \setminus B'$, $Q'$ is a $G$-map. By our construction, $\cup M \setminus B'$ is a union of complete fibers, i.e. $\cup M \setminus B' = \Pi^{-1}(E)$ for some set $E$. 

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Thus, since $\mu I$ in $(\delta, \rho)$ exceed as in Chapter 2 to construct a tree-adapted factor map $Q$, Proposition 5.1.6 (Strong Sinai’s Theorem).

Consider ergodic Proposition 5.1.5. The next proposition is the Strong Sinai’s Theorem which is formally identical to Proposition 2.4.7 shows that the required tree distributions differ by less than $\varepsilon$.

The following is an immediate consequence of the proof of Proposition 5.1.4.

**Proposition 5.1.5.** Consider ergodic $G$-extensions $B^+(p)\circ$ and $B^+(p)\phi$. For every $\varepsilon$ and $N$, there exists a $G$-map $Q$ on $I_\phi$ such that $(P^\circ)^{N\nu} \simeq Q^{N\nu}$. ■

The next proposition is the Strong Sinai’s Theorem which is formally identical to Proposition 2.4.9.

**Proposition 5.1.6 (Strong Sinai’s Theorem).** Suppose $B^+(p)\circ$ is tvwB. Given $\varepsilon > 0$, there exist $\delta$ and $N$ with the following property: If $B^+(p)\phi$ is an ergodic $G$-extension, $Q$ is a $G$-map on $B^+(p)\phi$ with $(P^\circ)^{N\nu} \simeq Q^{N\nu}$, then there exists a $G$-map $Q'$ on $B^+(p)\phi$ such that $\text{dist}(B^+(p)\phi, Q') = \text{dist}(B^+(p)\circ, P^\circ)$ and $|Q - Q'| < \varepsilon$.

**Proof:** Use the approach in Proposition 2.4.9 to choose $\delta$ and $1/N$ small enough such that we have a Cauchy sequence of $G$-maps $Q^j$ on $I_\phi$ converging to some map $Q'$ such that $|Q - Q'| < \varepsilon$ with $\text{dist}(B^+(p)\phi, Q') = \text{dist}(B^+(p)\circ, P^\circ)$. It is easily checked that $Q'$ is also a $G$-map.

Given the $G$-map $Q = (Q_1, \zeta)$ such that $\text{dist}(B^+(p)\phi, Q) = \text{dist}(B^+(p)\circ, P^\circ)$, we can proceed as in Chapter 2 to construct a tree-adapted factor map $\rho : B^+(p)\phi \rightarrow B^+(p)\circ$ defined in $I^{N}$. We may thus extend $Q'$ to the rest of the space such that it remains a $G$-map by defining $Q'(x, g) = (P(x), g)$ on $\Omega M \setminus B'$.

By Proposition 5.1.3 and our definition of $Q'$, whenever $x' \in S' \cap B$, we have for each $g \in G$,

$$\frac{1}{M} \sum_{0 < |v| \leq M} w_v d(Q(T_v^\circ(x', g)), Q'(T_v^\circ(x', g))) = \frac{1}{M} \sum_{0 < |v| \leq M} w_v d(Q(T_v^\circ(x', e)), Q'(T_v^\circ(x', e))) = \bar{t}_M(\tau_{x,e}^\circ, \tau_{x,e}^Q) < O_2(\delta)$$

Thus, since $\mu (S') > 1 - O_2(\delta)$, we have

$$\int_{\Omega M \setminus B'} d(\mu(z, g), Q'(z, g)) d\mu(z) d\nu(g) < O_3(\delta).$$

Since $\mu \times \nu(\Omega M \setminus B') > 1 - \delta$ for sufficiently small $\eta$, we have $|Q - Q'| < O_1(\delta)$.

To prove that $(P^\circ)^{N\nu} \simeq Q^{N\nu}$, note that by our construction of $Q'$, we have $Q^{\nu}(x', g) = (P^\circ)^{\nu}(x, g)$ for all $(x', g) \in B'$. By a) and b), since $x \in S$, the same reasoning in Proposition 2.4.7 shows that the required tree distributions differ by less than $\varepsilon$. ■

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by \(\rho(x,g) = (Q_1^N(x), \zeta(Q_1^N(x))g)\). Moreover, for any \(G\)-extension \(B^+(p)^\phi\), the canonical projection \(\Pi : I_\phi \to I^N\) is clearly a factor map from \(B^+(p)^\phi\) to \(B^+(p)\). Combining Propositions 5.1.5 and 5.1.6, along with the fact that the partition \(Q'\) constructed in Proposition 5.1.6 is a \(G\)-map, the following corollary is immediate.

**Corollary 5.1.7 (Sinai’s Theorem).** Suppose \(B^+(p)^\varphi\) is tvwB and \(B^+(p)^\phi\) is ergodic. Then there exists a \(G\)-map \(Q\) on \(B^+(p)^\phi\) such that \(dist(B^+(p)^\phi, Q) = dist(B^+(p)^\varphi, P^\varphi)\). Moreover, the factor map \(\rho : B^+(p)^\varphi \to B^+(p)^\varphi\) as constructed above projects to a factor map \(\rho_\Pi : B^+(p) \to B^+(p)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
B^+(p)^\phi & \xrightarrow{\rho} & B^+(p)^\varphi \\
\Pi & \downarrow & \downarrow \Pi \\
B^+(p) & \xrightarrow{\rho_\Pi} & B^+(p).
\end{array}
\]

Our next goal is the copying lemma. Recall that for \(B \subseteq I^N \times G\) and \(g \in G\), we defined the set \(B_g\) to be the subset of \(B\) with group coordinate \(g\). For \(g \in G\) and a set \(S \subseteq I^N \times G\), let

\[
S \cdot g = \{(x, g'g) \in I^N \times G \mid (x, g') \in S\}.
\]

**Proposition 5.1.8 (Copying Lemma).** For ergodic \(G\)-extensions \(B^+(p)^\phi\) and \(B^+(p)^\varphi\), let \(Q\) be a \(G\)-map on \(I_\phi\) such that \(dist(B^+(p)^\phi, Q) = dist(B^+(p)^\varphi, P^\varphi)\). Then for all \(\varepsilon\) and \(N\), we have a \(G\)-map \(Q'\) on \(I_\varphi\) such that

\[
dist(Q' \vee P^\varphi)^N \sim dist(P^\phi \vee Q)^N.
\]

**Proof:** Let \(M\) be chosen such that \(N/M < \varepsilon/2\). Define a partition \(\tilde{\varphi} : I^N \to I \times G\) by \(\tilde{\varphi}(x) = (P(x), \varphi(x))\). Construct a \(\varepsilon/2\)-tree Rokhlin tower \(M\) of height \(M + 1\) in \(B^+(p)\) with base \(B\) such that \(dist(\tilde{\varphi}^M) = dist(\tilde{\varphi}^M|B)\). Using the canonical projection \(\Pi : I_\varphi \to I^N\), we may lift the partition \(\tilde{\varphi}^M\) and the tree Rokhlin tower \(M\) so that we may regard \(\tilde{\varphi}^M\) as being a partition of \(I_\varphi\) and the tree Rokhlin tower \(M\) with base \(B\) as being in \(B^+(p)^\varphi\). Clearly, we still have \(dist(\tilde{\varphi}^M) = dist(\tilde{\varphi}^M|B)\). For any atom \(\alpha \in \tilde{\varphi}^M\) and \(g \in G\), note that all points in \(\alpha_g\) have the same \(P^\varphi\)-\(M\)-tree name. Thus, \(dist((P^\varphi)^M) = dist((P^\varphi)^M|B)\).

Consider the tree-adapted factor map \(\rho : B^+(p)^\phi \to B^+(p)^\varphi\) as constructed in Sinai’s Theorem. For any atom \(\alpha \in \tilde{\varphi}^M\), note that if \((x, g) \in \alpha\), then \((P^\varphi)^M(x, g) = Q^M(y, g')\) for any \((y, g') \in \rho^{-1}(x, g)\) by Proposition 2.1.4. Thus \((P^\varphi)^M(\alpha_g) = Q^M(\rho^{-1}(\alpha_g))\).
For each atom $\alpha \in \varphi^{\mathcal{M}^\nabla}$, construct a partition $U^e_\alpha$ of $\alpha \cap B$ such that
\[
\text{dist}(U^e_\alpha) = \text{dist}((P^\phi \lor Q)^{\mathcal{M}^\nabla} | \rho^{-1}(\alpha))
\] (1)
We then define the required partition $Q'$ on the column of the tower over $B_e$ as follows. Fix an atom $\alpha \in \varphi^{\mathcal{M}^\nabla}$. By (1), we can define a bijective correspondence $\Lambda$ of the atoms of $U^e_\alpha$ and those of $(P^\phi \lor Q)^{\mathcal{M}^\nabla}$ with the same conditional measures. Consider an atom $\beta \in U^e_\alpha$. As we have already noted, $(P^\phi)^{\mathcal{M}^\nabla}(x, e) = Q^{\mathcal{M}^\nabla}(x', g')$ for any $(x, e) \in \beta$ and $(x', g') \in \Lambda(\beta)$. We may then define $Q'$ on the tower over $\beta$ by laying the $P^\phi$-tree name of some point $(x', g')$ in $\Lambda(\beta)$ via suitable tree automorphisms such that $(Q' \lor P^\phi)^{\mathcal{M}^\nabla}(x, e) = (P^\phi \lor Q)^{\mathcal{M}^\nabla}(x', g')$ for any $(x, e) \in \beta$. Repeating for each $\beta \in U^e_\alpha$ and then for each $\alpha \in \varphi^{\mathcal{M}^\nabla}$, we can then define $Q'$ on the tower over $B_e$.

We now extend $Q'$ on the column of the tower over $B_g$ for all $g \in G$. To do this, note that if $(x, g'g)$ is in the column over $B_e$, then $(x, g'g)$ is in the column over $B_g$. Define $Q'(x, g'g) = Q'(x, g') \cdot g$. This defines $Q'$ on $\cup \mathcal{M} \setminus B$. It is clear that $Q'$ is a $G$-map restricted to $\cup \mathcal{M} \setminus B$. Since $\cup \mathcal{M} \setminus B$ is $\mathcal{B}$-measurable, we may then extend $Q'$ so that it is a $G$-map on the full space by setting $Q'(x, g) = (P(x), g)$.

We now prove that $\text{dist}(Q' \lor P^\phi)^{\mathcal{M}^\nabla} \approx \text{dist}(P^\phi \lor Q)^{\mathcal{M}^\nabla}$. If $\kappa \in (I \times G)^{\mathcal{M}^\nabla}$ and $g \in G$, let $\kappa \cdot g \in (I \times G)^{\mathcal{M}^\nabla}$ be defined by $(\kappa \cdot g)(v) = \kappa(v) \cdot g$. If $\bar{\kappa} \in (I \times G)^{\mathcal{M}^\nabla}$ and $\kappa$ is a representative in $\bar{\kappa}$, define $\bar{\kappa} \cdot g \in (I \times G)^{\mathcal{M}^\nabla}$ to be the equivalence class containing $\kappa \cdot g$. For any atom $\alpha \in \varphi^{\mathcal{M}^\nabla}$ and $\beta_\alpha \in U^e_\alpha$, note that if $(x, e) \in \beta_\alpha$ and $(x', g') \in \Lambda(\beta_\alpha)$, then for each $g \in G$,
\[
(Q' \lor P^\phi)^{\mathcal{M}^\nabla}(x, g) = ((Q' \lor P^\phi)^{\mathcal{M}^\nabla}(x, e)) \cdot g
\]
\[
= ((P^\phi \lor Q)^{\mathcal{M}^\nabla}(x', g')) \cdot g
\]
\[
= (P^\phi \lor Q)^{\mathcal{M}^\nabla}(x', g'g).
\]
This implies that $\beta_\alpha \cdot g$ and $\Lambda(\beta_\alpha) \cdot g$ have the same image under $(Q' \lor P^\phi)^{\mathcal{M}^\nabla}$ and $(P^\phi \lor Q)^{\mathcal{M}^\nabla}$ respectively. Moreover, since $Q$ is a $G$-map, $\rho^{-1}(\alpha_g) = \rho^{-1}(\alpha_e) \cdot g$ and so
\[
\mu \times \nu(\Lambda(\beta_\alpha) \cdot g \cap \rho^{-1}(\alpha_g)) = \mu \times \nu(\Lambda(\beta_\alpha) \cdot g | \rho^{-1}(\alpha_g)) \mu \times \nu(\rho^{-1}(\alpha_g))
\]
\[
= \mu \times \nu(\Lambda(\beta_\alpha) | \rho^{-1}(\alpha_e)) \mu \times \nu(\rho^{-1}(\alpha_e))
\]
\[
= \mu \times \nu(\Lambda(\beta_\alpha) | \rho^{-1}(\alpha_e)) \mu \times \nu(\alpha_e)
\]
\[
= \mu \times \nu(\beta_\alpha | \alpha_e \cap B) \mu \times \nu(\alpha_e | B)
\]
\[
= \mu \times \nu(\beta_\alpha | B) = \mu \times \nu(\beta_\alpha \cdot g | B)
\]
As this holds for all $g \in G$, $\alpha \in \varphi^{\mathcal{M}^\nabla}$, and $\beta_\alpha \in U^e_\alpha$, we have
\[
\text{dist}(P^\phi \lor Q)^{\mathcal{M}^\nabla} = \text{dist}((Q' \lor P^\phi)^{\mathcal{M}^\nabla} | B)
\]
Hence, by the same reasoning as in the proof of Proposition 2.4.12, we see that the distribution of \((Q' \lor P^e)^{N^\mathcal{V}}\) on each level of the tower except the top \(N\) levels equals \(\text{dist}(P^e \lor Q)^{N^\mathcal{V}}\). Since \(N/M < \varepsilon/2\), the result follows. ■

The rest of the proof now follows along the same lines as Theorem 2.4.1. To begin, we have the following proposition which is formally identical to Proposition 2.1.13.

**Proposition 5.1.9.** Consider \(tvwB \mathcal{G}\)-extensions \(B^+(p)^\phi\) and \(B^+(p)^\varphi\). Suppose \(Q\) is a \(\mathcal{G}\)-map on \(I^\phi\) such that \(\text{dist}(B^+(p)^\phi, Q) = \text{dist}(B^+(p)^\varphi, P^\varphi)\). Then for every \(\eta > 0, \varepsilon > 0\), there exists a \(\mathcal{G}\)-map \(Q'\) on \(I^\phi\) such that

i) \(P^\phi \prec \sum(Q')\)

ii) \(\text{dist}(B^+(p)^\phi, Q') = \text{dist}(B^+(p)^\varphi, P^\varphi)\)

iii) \(|Q - Q'| < \eta\).

■

**Proof (Theorem 5.1.1’):** Imitating the proof of Theorem 2.4.1, we choose, via proposition 5.1.9, a Cauchy sequence of \(\mathcal{G}\)-maps \(\{U^j\}\) on \(I^\phi\) converging to a \(\mathcal{G}\)-map \(U\) on \(I^\phi\) such that \(\text{dist}(B^+(p)^\phi, U) = \text{dist}(B^+(p)^\varphi, P^\varphi)\) and \(\sum(U) = \sum(P^\phi)\).

Let \(U = (U_1, \xi)\). Since \(\sum(U) = \sum(P^\phi)\), the factor map \(\rho : B^+(p)^\phi \rightarrow B^+(p)^\varphi\) defined by \(\rho(x, g) = (\bar{U}_1(x), \xi(x)g)\) is an isomorphism, where \(\bar{U}_1(x)\) is the \(U_1\)-name of \(x\). Since \(\rho T^\phi = T^\varphi \rho\), it follows that

\[
\varphi(\bar{U}_1(x))\xi(x) = \xi(Tx)\phi(x)
\]

Obviously, \(\bar{U}_1\) defines an automorphism of \(B^+(p)^\phi\). Hence, \(\rho\) defines a \(N^\ast \times \mathcal{G}\)-isomorphism from \(B^+(p)^\varphi\) to \(B^+(p)^\phi\). ■

### 5.2. Some Applications of Theorem 5.1.1

**Example 5.2.1.** As an application of Theorem 5.1.1, consider a finite group \(G\) of order \(\geq 3\) with two distinct generators \(h\) and \(h'\) (thus \(G \cong \mathbb{Z}/n\mathbb{Z}\), for \(n \geq 3\)). Fix a probability vector \(p = (p_1, \ldots, p_s)\) with \(p_1 = p_2\). Define a map \(\varphi : I^{N^\ast} \rightarrow G\) by

\[
\varphi(z) = \begin{cases} 
h & \text{if } z_0 = 2; \\
e & \text{otherwise.}
\end{cases}
\]

Note that the \(P^\varphi\)-tree name of \((x, g)\) is completely determined by \(P^\varphi(x, g) = (x_0, g)\). Let \(\tau_{(j, g)}\) be the common \(P^\varphi\)-tree name of all points \((x, g)\) such that \(x_0 = j\). Using the
ideas in §4.1, to prove that $B^+(p)^e$ is tvwB, it suffices to show that for any two elements $(j, g)$ and $(j', g')$ in $I \times G$, we have some node $v$ and some tree automorphism $A$ such that $\tau_{(j, g)}(v) = \tau_{(j', g')}(Av)$. In fact, it is enough to show this in the case when $j = j' = 1$.

Choose $k \in \mathbb{N}$ such that $g' = h^kg$. Then for the node $v = (1, 1, 1, \ldots, 1)$ of length $k + 1$, $\tau_{(j, g)}(v) = (1, g)$; for the node $u = (1, 2, 2, \ldots, 2)$ of length $k + 1$, $\tau_{(j', g')}(u) = (1, g)$. Since these two nodes can be matched by a tree automorphism, we have $\tau_{(j, g)}(v) = \tau_{(j', g')}(Av)$ for some tree automorphism $A$. Thus, $B^+(p)^e$ is tvwB.

In the same way, for the map $\psi : I^{N^*} \rightarrow G$ defined by
\[
\psi(z) = \begin{cases} 
    h' & \text{if } z_0 = 2; \\
    e & \text{otherwise},
\end{cases}
\]
the same argument shows that $B^+(p)^\psi$ is tvwB. Thus Theorem 5.1.1 shows that $B^+(p)^e$ and $B^+(p)^\psi$ are $\mathbb{N}^* \times G$-isomorphic.

Note that the two cocycles $\phi$ and $\psi$ are not cohomologous provided that $hh'^{-1}$ is a generator (if they were cohomologous, then a $\mathbb{N}^* \times G$-isomorphism between the corresponding extensions exists trivially). Indeed, if they were cohomologous, the $G$-extension obtained from the map $\phi - \psi$ defined by $(\phi - \psi)(x) = \phi(x)\psi(x)^{-1}$ would not even be ergodic, contradicting the above argument which shows that it must be tvwB and hence ergodic. ■

The following proposition, as observed by del Junco, gives another interesting application of Theorem 5.1.1. We say that a tree-adapted factor map $\phi : X \rightarrow Y$ is uniformly p-to-1 if the fiber over $a.a.y$ in $Y$ contains $p$ points each with conditional probability $1/p$. One example of such a factor map is if $X = Y$ is the one-sided $(1/2, 1/2)$-Bernoulli shift and $\phi$ is the addition map defined by $\phi(x)_i = x_i + x_{i+1}$. The following proposition essentially says that in the case when $p = (1/2, 1/2)$, then this is (up to automorphism) the only example.

**Proposition 5.2.2.** Suppose $X = (X, \mathcal{B}, \mu, T) \in \text{End}(p)$ is tvwB. Let $Y = (Y, \mathcal{C}, \nu, S)$ be a tree adapted factor of $X$. For any pair of uniformly 2-to-1 tree-adapted factor maps $\Pi_1 : X \rightarrow Y$ and $\Pi_2 : X \rightarrow Y$, there exist automorphisms $\rho : X \rightarrow X$ and $\varphi : Y \rightarrow Y$ such that $\Pi_2 \circ \rho = \varphi \circ \Pi_1$.

**Proof:** We define two $(\mathbb{N}^* \times \mathbb{Z}_2, p)$-actions on $(X, \mathcal{B}, \mu)$ as follows. Let $\mathbb{Z}_2 = \{0, 1\}$. Define $T_{(1,0)} = T$, and $T_{(0,0)}$ to be the identity map. Define $T_{(0,1)} : X \rightarrow X$ by sending $x$ to the unique point $x'$ distinct from $x$ such that $\Pi_1(x') = \Pi_1(x)$. The fact that $\Pi_1$ is uniformly 2-to-1 implies that $T_{(0,1)}$ is an automorphism of $X$. Using $T_{(1,0)}$, $T_{(0,0)}$ and $T_{(0,1)}$, we can define $T_{(n,g)}$ by group
composition. By tree-adaptedness, it is easy to check that \( \{T_{(n,g)}\} \) defines a \((\mathbb{N}^* \times \mathbb{Z}_2, p)\)-action on \( X \). In a similar way, using \( \Pi_2 \), we can define a second \((\mathbb{N}^* \times \mathbb{Z}_2, p)\)-action \( \{S_{(n, g)}\} \) on \( X \).

By Theorem 5.2.2, we have a \( \mathbb{N}^* \times \mathbb{Z}_2 \)-isomorphism \( \rho : (X, \{T_{(n,g)}\}) \to (X, \{S_{(n,g)}\}) \). In particular, \( \rho \) defines an automorphism of \( X \). Note that as \( S_{(0,1)} \circ \rho = \rho \circ T_{(0,1)} \), \( \rho \) maps fibres of \( \Pi_1 \) to those of \( \Pi_2 \) and so the map \( \varphi : Y \to Y \) defined by \( \varphi = (\Pi_2) \circ \rho \circ (\Pi_1)^{-1} \) is well-defined and is clearly an isomorphism with the required property.

**Remark 5.2.3.** It is reasonable to believe that Theorem 5.1.1 should also hold in the case when \( G \) is a compact metrizable group instead of just a finite group. We were unable to prove this, although it seems that the same ideas should apply and is likely a technical extension. One possible approach would be to define a partition \( Q_N \) on \( G \) by dividing \( G \) into subsets of diameter \( \leq 1/2^N \) for each \( N \) and define tree distributions by a suitable discretization.

Second, it would be interesting to give a joinings proof of this result, though the definition of one-sided joinings might need to be modified to achieve the required isomorphism.
Chapter 6:
Some Open Problems

This thesis has extended the investigation of the isomorphism problem of non-invertible endomorphisms initiated by Hoffman and Rudolph. Obviously, much is left unaddressed. In this chapter, we state some problems in this area which warrant further investigation.

Problem 1. Consider a Lebesgue probability space \((X, \mathcal{B}, \mu)\) with two commuting endomorphisms \(T\) and \(S\) on it, i.e. \(TS = ST\) a.e. The basic model is the two-dimensional lattice \(\{0,1\}^{(\mathbb{N}^*)^2}\) with product measure \((1/2, 1/2)^{((\mathbb{N}^*)^2)}\), and with \(T\) and \(S\) being the shift maps in each direction. Given another Lebesgue space \((Y, \mathcal{C}, \nu)\) with two commuting endomorphisms \(T'\) and \(S'\) on it with the same property as \(T\) and \(S\), is there a reasonable way to decide when \((X, \mathcal{B}, \mu)\) and \((Y, \mathcal{C}, \nu)\) are \(\mathbb{N}^* \times \mathbb{N}^*\)-isomorphic, i.e. when can we find a measure preserving bijection \(\phi : X \to Y\) such that \(\phi T = T' \phi\) and \(\phi S = S' \phi\) a.e.?

Problem 2. So far, all endomorphisms \(X = (X, \mathcal{B}, \mu, T)\) considered are homogeneous in the sense that for a.a. \(x\), \(x\) has the same number of inverse images with the same set of conditional probabilities. Obviously, most endomorphisms do not share this property. Can one give a reasonable classification for a subset of these? If we have a fixed one-sided Markov shift which is not homogeneous, is there some reasonable criterion to ensure that a given endomorphism is isomorphic to it?
Problem 3. Proposition 5.2.2 in particular implies that there is essentially only one tree-adapted uniformly 2-to-1 endomorphisms of $B^+(p)$. It would be interesting to classify tree-adapted uniformly 3-to-1 endomorphisms (and in general $p$-to-1 endomorphisms) of $B^+(p)$. The proof of Proposition 5.2.2 cannot be used since even for tree-adapted uniformly 3-to-1 maps of $B^+(p)$, there is no canonical way of constructing a $\mathbb{N}^* \times \mathbb{Z}_3$-action on the Bernoulli shift space.
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