Approximate Parametrization of Plane Algebraic Curves by Linear Systems of Curves*

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Abstract

It is well known that an irreducible algebraic curve is rational (i.e. parametric) if and only if its genus is zero. In this paper, given a tolerance $\epsilon > 0$ and an $\epsilon$-irreducible algebraic affine plane curve $\mathcal{C}$ of proper degree $d$, we introduce the notion of $\epsilon$-rationality, and we provide an algorithm to parametrize approximately affine $\epsilon$-rational plane curves, without exact singularities at infinity, by means of linear systems of $(d - 2)$-degree curves. The algorithm outputs a rational parametrization of a rational curve $\overline{\mathcal{C}}$ of degree at most $d$ which has the same points at infinity as $\mathcal{C}$. Moreover, although we do not provide a theoretical analysis, our empirical analysis shows that $\overline{\mathcal{C}}$ and $\mathcal{C}$ are close in practice.

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Introduction

Let $\mathcal{O}^*$ be an algebraic or geometric object that satisfies a property $\mathcal{P}$ that implies the existence of certain associated objects $\mathcal{O}^*_i$; for instance, $\mathcal{O}^*$ might be a polynomial, $\mathcal{P}$ the fact of being reducible and $\mathcal{O}^*_i$ the irreducible factors. Computer algebra techniques provide, for a wide class of situations, algorithms to check $\mathcal{P}$, and to compute exactly the associated objects $\mathcal{O}^*_i$. However, in many practical applications, we receive a perturbation $\mathcal{O}$ of $\mathcal{O}^*$, where $\mathcal{P}$ does not hold anymore neither the associated objects $\mathcal{O}^*_i$ exist. The problem, then, consists in computing a new object $\mathcal{O}$, close to $\mathcal{O}$ and satisfying $\mathcal{P}$, as well as the associated objects $\mathcal{O}^*_i$ to $\mathcal{O}$. We call approximate to an algorithm solving a problem of the above type. Here, the notion of “closeness” depends in general on the particular problem that one is solving.

One can find in the literature approximate algorithms for computing gcds (see [3], [7], [16]), factoring polynomials (see [4], [10], [15], [21]), etc. For algebraic varieties there also exist approximate solutions: see [5], [6] for the implicitization problem, in [8] the numerical condition of implicitly given algebraic curves and surfaces has been analyzed, and see [2], [9], [12], [17], [18] where the parametrization questions are treated.

In this paper we consider the approximate parametrization problem for affine plane algebraic curves. That is, with the above terminology, $\mathcal{O}^*$ is an affine plane curve, $\mathcal{P}$ is the fact of being rational, and $\mathcal{O}^*_i$ is a rational parametrization of $\mathcal{O}^*$. So, the problem is stated as follows: we are given an affine curve (say that it is a perturbation of a rational curve) and we want to compute a rational parametrization of a rational affine curve near it; where we use the notion of “vecinity” introduced in [17].

In [17] and [18] the approximate parametrization problem is solved for the special case of affine plane curves and affine surfaces being a perturbation of a monomial curve and surface, respectively. In both papers, the basic tool is the use of $\epsilon$-points (see also [19]). More precisely, given a tolerance $\epsilon > 0$, in [17], the parametrization problem is solved for the case of affine plane curves having an $\epsilon$-singularity of maximum multiplicity, and in [18] the problem is solved for affine surfaces having also an $\epsilon$-singularity of maximum multiplicity. The basic idea was to use a pencil of lines through the $\epsilon$-singularity and, hence, it was solved working as in the exact case for monomial varieties.

In this paper, we generalize the ideas in [17] to the case of affine plane curves without singularities at infinity. For this purpose, the first obstacle is to associate suitably the different $\epsilon$-singularities. This leads to the notion of cluster. Then, we introduce the notion of (affine) $\epsilon$-rationality, and we provide an algorithm to parametrize approximately $\epsilon$-rational curves without exact singularities at infinity. The idea of the algorithm is to work with linear systems of curves of degree $d - 2$, where $d$ is the degree of the input curve. This system plays the role of the linear system of adjoint curves in the exact parametrization algorithm. In addition, we prove that the degree of the
output rational curve is bounded by the degree of the input one, and that both curves have the same points at infinity. Differently to [17] we do not provide a theoretical analysis of the error (i.e. on the closeness of input and output). However, our empirical analysis shows that the curves are in practice near, and it allows us to think about a theoretical treatment of this fact as a future project.

The paper is structured as follows. In Section 1 we recall the main notions and properties on $\varepsilon$-singularities. Section 2 is devoted to recall the main ideas of the exact parametrization algorithm for curves. In Section 3 we develop the idea of cluster and we introduce the notion of $\varepsilon$-rationality. In Section 4 we derive the approximate algorithm, as well as the main properties of the output curve. In Section 5 we illustrate the algorithm by some example, and in Section 6 we analyze empirically the error.

Throughout this paper, we use the following terminology. $\| \cdot \|$ and $\| \cdot \|_2$ denote the polynomial $\infty$–norm and the usual unitary norm in $\mathbb{C}^2$, respectively. $| \cdot |$ denotes the module in $\mathbb{C}$. The partial derivatives of a polynomial $g \in \mathbb{C}[x, y]$ are denoted by $g^{\vec{v}} := \frac{\partial^{i+j} g}{\partial x^i \partial y^j}$ where $\vec{v} = (i, j) \in \mathbb{N}^2$; we assume that $g^{\vec{v}} = g$. Moreover, for $\vec{v} = (i, j) \in \mathbb{N}^2$, $| \vec{v} | = i + j$. Also, $\vec{e}_1 = (1, 0)$, and $\vec{e}_2 = (0, 1)$.

In addition, we use the following general assumptions. A tolerance $\varepsilon$ is fixed such that $0 < \varepsilon < 1$. $\mathcal{C}$ is an affine real plane algebraic curve over $\mathbb{C}$ of proper degree $d > 0$ (see Def. 1.1), without (exact) singularities at infinity, not passing through $(1:0:0), (0:1:0)$, and defined by an $\varepsilon$-irreducible polynomial $f(x, y) \in \mathbb{R}[x, y]$; that is $f$ can not be expressed as $f(x, y) = g(x, y)h(x, y) + \mathcal{E}(x, y)$ where $h, g, \mathcal{E} \in \mathbb{C}[x, y]$ and $\|\mathcal{E}(x, y)\| < \varepsilon\|f(x, y)\|$ (see [3], [14]). We denote by $\mathcal{C}^h$ the projective closure of $\mathcal{C}$.

Let us mention that, although we require that $\mathcal{C}$ is real, the results in this paper are also valid for non-real plane algebraic curves. In addition, the condition $(1 : 0 : 0), (0 : 1 : 0) \not\in \mathcal{C}^h$ can be avoided by performing a suitable affine orthogonal linear change of coordinates. The requirement on the smoothness of $\mathcal{C}^h$ at infinity, might be avoided by performing a suitable projective linear change of coordinates. However, differently to affine orthogonal linear changes, in general, projective changes of coordinates do not preserve properly the closeness between the input and output curves.

1 Preliminaries on $\varepsilon$-points

Our fundamental technique to deal with the approximate parametrization problem is the use of $\varepsilon$-points. The notion of $\varepsilon$–point of an algebraic variety was introduced by the authors (see [17], [18], [19]) as a generalization of the notion of approximate root of a univariate polynomial. In this section, we briefly summarize some previous notions introduced in [17] and [18], and geometric properties obtained in [19]. We start with the notion of proper degree.

Definition 1.1. We say that a polynomial $g \in \mathbb{C}[x, y]$ has proper degree $\ell$ if the total degree of $g$ is $\ell$, and $\exists \, \vec{v} \in \mathbb{N}^2$, with $| \vec{v} | = \ell$, such that $| g^{\vec{v}} | > \varepsilon\|g\|$. 

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We say that an algebraic plane curve has proper degree \( \ell \) if its defining polynomial has proper degree \( \ell \).

The notion of \( \varepsilon \)-point is as follows.

**Definition 1.2.** \( P \in \mathbb{C}^2 \) is an \( \varepsilon \)-(affine) point of \( C \) if \( |f(P)| < \varepsilon \|f\| \).

In this situation, we introduce the notion of \( \varepsilon \)-singularity, pure \( \varepsilon \)-singularity, and \( \varepsilon \)-ramification point.

**Definition 1.3.** Let \( P \in \mathbb{C}^2 \) be an \( \varepsilon \)-point of \( C \).

(i) The \( \varepsilon \)-multiplicity of \( P \) on \( C \) (we denote it by \( \text{mult}_\varepsilon(P,C) \)) is the smallest natural number \( r \in \mathbb{N} \) satisfying that

1. \( \forall \, \vec{v} \in \mathbb{N}^2, \text{ such that } 0 \leq |\vec{v}| \leq r - 1, \text{ it holds that } |f^{\vec{v}}(P)| < \varepsilon \|f\|, \)
2. \( \exists \, \vec{v} \in \mathbb{N}^2, \text{ with } |\vec{v}| = r, \text{ such that } |f^{\vec{v}}(P)| \geq \varepsilon \|f\|. \)

(ii) \( P \) is an \( \varepsilon \)-(affine) simple point of \( C \) if \( \text{mult}_\varepsilon(P,C) = 1 \); otherwise, \( P \) is an \( \varepsilon \)-(affine) singularity of \( C \).

(iii) \( P \) is a \( k \)-pure \( \varepsilon \)-singularity of \( C \), with \( k \in \{1,2\} \), if \( \text{mult}_\varepsilon(P,C) > 1 \) and \( |f^{\text{mult}_\varepsilon(P,C)-\varepsilon^k}(P)| \geq \varepsilon \|f\| \).

(iv) \( P \) is an \( \varepsilon \)-(affine) ramification point of \( C \) if \( \text{mult}_\varepsilon(P,C) = 1 \), and either \( |f^{\varepsilon^1}(P)| < \varepsilon \|f\| \) or \( |f^{\varepsilon^2}(P)| < \varepsilon \|f\| \).

Note that, since \( C \) has proper degree, \( 0 \leq \text{mult}(P,C) \leq \text{mult}_\varepsilon(P,C) \leq \deg(C) \), where \( \text{mult}(P,C) \) denotes the “exact” multiplicity of \( P \) on \( C \). For instance, the origin has exact multiplicity 1, and \( \varepsilon \)-multiplicity 2, on the curve defined by \( \frac{\varepsilon}{2}x + x^3 + y^2 \).

In the exact case, if \( C \) is irreducible, \( \text{mult}(P,C) < \deg(C) \). Thus one may expect that in the approximate case, if \( C \) is \( \varepsilon \)-irreducible, then \( \text{mult}_\varepsilon(P,C) < \deg(C) \). Although this is the case in all the examples we have tried, we have not been able to prove it. So in this paper, when computing \( \varepsilon \)-multiplicities, we also consider the possibility \( \text{mult}_\varepsilon(P,C) = \deg(C) \).

The following lemma is a direct generalization of Lemma 3 in [17].

**Lemma 1.4.** Let \( P \in \mathbb{C}^2 \) be an \( \varepsilon \)-point of \( C \). There exists \( \delta > 0 \) such that every \( Q \in \mathbb{C}^2 \), satisfying that \( \|P - Q\|_2 < \delta \), is an \( \varepsilon \)-point of \( C \) with \( \text{mult}_\varepsilon(Q,C) \geq \text{mult}_\varepsilon(P,C) \).

**Proof.** Simply observe that the reasoning of Lemma 3 in [17] is also valid over \( \mathbb{C} \).

The following example shows that, in Lemma 1.4, the \( \varepsilon \)-multiplicity of \( Q \) can be strictly bigger than \( \text{mult}_\varepsilon(P,C) \).
Example 1.5. Let $C$ be defined by $f(x, y) = x^3y + y^3x + x^3 + \frac{8}{2}x^2 + ey + \frac{1}{2};$ note that $\|f\| = 1$. For $P = (0, 0)$, one has

$$f(P) = \frac{\varepsilon}{2}, \quad f^{(1)}(P) = 0, \quad f^{(2)}(P) = \varepsilon.$$ 

So, $\text{mult}_{\varepsilon}(P, C) = 1$. Now, we consider the sequence of points $\{P_n = (-\frac{1}{n}, 0)\}_{n \geq 1}$. Then,

$$f(P_n) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2n^2} - \frac{1}{n^3}, \quad f^{(1)}(P_n) = \frac{3}{n^2} - \frac{\varepsilon}{n}, \quad f^{(2)}(P_n) = \varepsilon - \frac{1}{n^3},$$

$$f^{(2,0)}(P_n) = \varepsilon - \frac{6}{n}, \quad f^{(1,1)}(P_n) = \frac{3}{n^2}, \quad f^{(0,2)}(P_n) = 0, \quad f^{(3,0)}(P_n) = 6.$$ 

So, for $n$ sufficiently large, $\text{mult}_{\varepsilon}(P_n, C) = 3$. 

Definition 1.6. Let $P$ be an $\varepsilon$-point of $C$ and $r = \text{mult}_{\varepsilon}(P, C)$. If $P$ is $k$-pure, with $k \in \{1, 2\}$, we define the $k$-weight of $P$ as

$$\text{weight}_k(P) = \max_{i=0, \ldots, r-1} \left\{ \frac{r! \cdot f^{i} \cdot \text{det}(P)}{i! \cdot f^{r-i} \cdot \text{det}(P)} \right\}.$$ 

We define the weight of $P$, denoted by $\text{weight}(P)$, as $\max\{\text{weight}_1(P), \text{weight}_2(P)\}$; if $P$ is pure in both directions, and as the corresponding $k$-weight otherwise. 

The following two rational functions were introduced in [22], and will play an important role in this development:

$$R_{\text{in}}(x) = 2x \left( \frac{1}{1 + 3x} + \frac{16x}{(1 + 3x)^3} \right), \quad R_{\text{out}}(x) = \frac{1}{2} - \frac{x(1 - 9x)}{2(1 + 3x)} - \frac{32x^2}{(1 + 3x)^3}.$$ 

Furthermore, these two rational functions give information on how close an $\varepsilon$-point is to an exact point of the curve $C$ (see Theorem 2 and Corollary 3 in [19]).

2 Preliminaries on Symbolic Parametrization

In this section, we briefly recall the symbolic parametrization algorithm for rational plane algebraic curves of degree $d > 2$ (note that lines and conics can be trivially parametrized by lines) based on $(d-2)$ adjoint curves; for further details see [24]. For this purpose, throughout this section we assume that $C$ is rational (i.e. its genus is zero). In addition, taking into account our requirements in Section 4 and for simplicity sake, we assume in this section that all singularities are affine and ordinary. Again, for a complete description see [24].

The idea is to use a linear system of curves such that for almost every curve in this system, all its intersections with $C^h$, except one, are predetermined; recall that $C^h$ is the
projective closure of \( C \). Moreover, the set of all these intersection points is the same one for every curve in the system, and the points in this set are called the “base points”. Thus, if one computes the intersection points of \( C^h \) with a generic representative of the system, the expression of the unknown intersection point gives the parametrization of the curve in terms of the parameter defining the linear system.

More precisely, let \( \mathcal{H}_{d-2} \) be the linear system of adjoint curves to \( C^h \) of degree \( d - 2 \). That is, \( \mathcal{H}_{d-2} \) is the linear system of curves of degree \( d - 2 \) having each \( r \)-fold of \( C^h \) as a base point of multiplicity \( r - 1 \); i.e. as a point of multiplicity at least \( r - 1 \). In particular it implies that the multiplicity of intersection of a curve in \( \mathcal{H}_{d-2} \) and \( C^h \) at a base point of multiplicity \( r - 1 \) is at least \( r(r - 1) \). Thus, using that the genus of \( C \) is zero, and taking into account Bézout’s Theorem, one deduces that \( d - 2 \) intersections of \( C^h \) and a generic element in \( \mathcal{H}_{d-2} \) are not predetermined. In this situation, one may take \((d - 3)\) simple points on \( C^h \), and determine the 1-dimensional linear subsystem \( \mathcal{H}^*_{d-2} \) of \( \mathcal{H}_{d-2} \) obtained when these simple points are required to be base points of multiplicity 1. In this way, the number of predetermined intersections (counted with multiplicity) is \((d - 1)(d - 2) + (d - 3)\), i.e. only one intersection point is missing. Thus, computing this free intersection one finds a rational parametrization of \( C^h \). Summarizing these ideas one has the following process:

1. Compute the singularities of \( C^h \) as well as their multiplicities (recall that we have assumed that all singularities are affine and ordinary).
2. Determine the linear system \( \mathcal{H}_{d-2} \) of adjoint curves of degree \((d - 2)\) to \( C^h \).
3. Compute \( d - 3 \) different simple points on \( C^h \).
4. Determine the linear subsystem \( \mathcal{H}^*_{d-2} \) of \( \mathcal{H}_{d-2} \) by requiring that every simple points in Step 3 is a base point of multiplicity one.
5. Compute the free intersection point of \( \mathcal{H}^*_{d-2} \) and \( C^h \).

Let us make a comment on how to computationally perform the steps in the above process. Step (1) can be performed, for instance, using resultants. In Step (2), one considers a homogeneous polynomial \( H(x, y, z) \) of degree \((d - 2)\) with undetermined coefficients. Now, for each singular point \( P \) of multiplicity \( r \) one requires that \( H \) and all its partial derivatives till order \((r - 1)\) vanish at \( P \). This generates a linear system of equations in the undetermined coefficients of \( H \). Solving it, and substituting in \( H \), we get the defining polynomial of \( \mathcal{H}_{d-2} \); let us call it again \( H \). Step (3) may be performed by intersecting \( C^h \) with lines (see [21] for advanced approaches); although it is not necessary, looking for the parallelism with the reasoning in Section [4] we take affine simple points. Step (4) can be approached as Step (2), i.e. requiring that \( H \) vanishes at each simple point, solving the provided linear system and substituting the solution in \( H \); let \( H^*(t, x, y, z) \) be the defining polynomial of \( \mathcal{H}^*_{d-2} \) (note that \( \dim(\mathcal{H}^*_{d-2}) = 1 \)).
Finally, let us deal with Step (5). For this purpose, let \( \{ Q_i := (q_{1,i} : q_{2,i} : 1) \}_{i=1}^{s} \) be the singularities and \( r_i \) the multiplicity of \( Q_i \). Also, let \( \{ P_i := (p_{1,i} : p_{2,i} : 1) \}_{i=1}^{d-3} \) be the simple points determined in Step (3). Then, the free intersection point is obtained by computing the primitive part, w.r.t. \( t \), of the resultants of \( H^*(t, x, y, 1) \) and \( f(x, y) \) with respect to \( x \) and \( y \), respectively. Indeed, it holds that (see [20])

\[
S_1(x, t) = \text{Res}_y(H^*(t, x, y, 1), f(x, y)) = \prod_{i=1}^{s}(x - q_{1,i})^{r_i (r_i - 1)} \prod_{i=1}^{d-3}(x - p_{1,i})M_1(x, t),
\]

\[
S_2(y, t) = \text{Res}_x(H^*(t, x, y, 1), f(x, y)) = \prod_{i=1}^{s}(y - q_{2,i})^{r_i (r_i - 1)} \prod_{i=1}^{d-3}(y - p_{2,i})M_2(y, t),
\]

where \( \deg_x(M_1) = \deg_y(M_2) = 1 \). Therefore, the parametrization is the solution in \( \{x, y\} \) of \( \{M_1(x, t) = 0, M_2(y, t) = 0\} \).

3 \( \epsilon \)-Rational Curves

In this section we introduce the notion of \( \epsilon \)-rationality of a plane algebraic curve. This notion plays the corresponding role in the approximate frame that the rationality does for exact algebraic curves. We will have two main difficulties. On one hand, computing the \( \epsilon \)-multiplicity and on the other, differently to the exact case, we will have in general more \( \epsilon \)-singularities than expected, and we will need to associate them; we will solve this last difficulty introducing a suitable concept of cluster.

We first need to determine the \( \epsilon \)-singularities. To check the existence and perform the actual computation of the \( \epsilon \)-singularities, one has to solve the system of algebraic equations

\[
\{ f^0(x, y) = 0, \quad f^{\epsilon_1}(x, y) = 0, \quad f^{\epsilon_2}(x, y) = 0 \},
\]

under fixed precision \( \epsilon \| f \| \). This can be done, for instance, by applying root finding techniques (see [3], [11], [13]). Note that since \( f \) is \( \epsilon \)-irreducible then it is irreducible, and hence the above system has finitely many solutions. Let \( S_1 \) be the set of solutions. One may accelerate the computation by working (if possible) with two co-prime polynomials, instead of three, to get a finite super-set of the set of solutions, from where the \( \epsilon \)-singularities are detected afterwards.

Now, for \( P \in S_1 \), we want to compute \( \text{mult}_\epsilon(P, C) \). This can be obviously done by substituting \( P \) at the corresponding partial derivatives and checking the conditions in Def. 1.3 (1). Seemingly, there is no difficulty on that. However, in Lemma 1.4 we have seen that for each \( \epsilon \)-point \( P \) of \( \epsilon \)-multiplicity \( r \) there exists an open disk \( U \) centered at \( P \) such that if \( Q \in U \), then \( Q \) is an \( \epsilon \)-point of \( \epsilon \)-multiplicity at least \( r \). So, an small perturbation of \( P \) may produce an incorrect answer for the \( \epsilon \)-multiplicity; see, for instance, Example 1.5. We are indeed interested in assigning the maximum possible \( \epsilon \)-multiplicity to the \( \epsilon \)-point. The proof of Lemma 3 in [17], and hence of Lemma 1.4 shows how to detect the radius of one of these open disks, so one may try to estimate the maximum \( \epsilon \)-multiplicity at the disk. Nevertheless, in practice, this is
unfeasible. Instead, we propose a different strategy that, although it does not ensure the achievement of the maximum, in practical examples turns to work efficiently.

More precisely, for each \( k \in \{2, \ldots, d-1\} \), we take \( \bar{u}_1, \ldots, \bar{u}_s, u \in \mathbb{N}^2 \), with \( 2 \leq s \leq k+1 \) (in practice \( s = 2 \)) such that for all \( i \), \( |\bar{u}_i| = k \) and \( \gcd(f^{\bar{u}_1}, \ldots, f^{\bar{u}_s}) = 1 \), and we solve \( \{ f^{\bar{u}_1} = 0, \ldots, f^{\bar{u}_s} = 0 \} \), under fixed precision \( \epsilon \). Let \( A_k \) be the set of solutions. Then, for \( k \in \{2, \ldots, d-1\} \) we consider the set (note that \( S_1 \) is defined above)

\[
S_k = \{ P \in A_k / |f^u(P)| < \epsilon \|f\| \land \bar{u} \in \mathbb{N}^2 \text{ with } |\bar{u}| \leq k \}.
\]

If for a given \( k \) and for all \( s \) it holds that \( \gcd(f^{\bar{u}_1}, \ldots, f^{\bar{u}_s}) \neq 1 \), we take \( S_k = \emptyset \). Finally we consider the set

\[
S = \bigcup_{k=1}^{d-1} S_k.
\]

It is clear that in general we introduce additional \( \epsilon \)-singularities, and we will have to generate a process (the cluster construction) to identify them. Nevertheless, each new \( \epsilon \)-singularity, after identification, will increase the \( \epsilon \)-multiplicity of the original one.

**Definition 3.1.** The set \( S \), introduced above, is called the \( \epsilon \)-(affine)-singular locus of \( C \). We denote it by \( \text{Sing}_\epsilon(C) \).

**Example 3.2.** Let us take \( \epsilon = 0.001 \) in Example 1.5. The \( \epsilon \)-singular locus of this curve is \( \text{Sing}_\epsilon(C) = S_1 \cup S_2 \cup S_3 \) where

\[
S_1 = \{ P_1 = (0.02131893405 + 0.009609927603i, 0.02442855631 + 0.1171004584i),
P_2 = (0.04713033954 + 0.02355323617i, -0.07491796596 - 0.0903219938i),
P_3 = (-0.01424770212 + 0.01818884517i, 0.1084633939 + 0.05315246871i),
P_4 = (-0.02443272919, -0.1159479025),
P_5 = (-0.01424770212 - 0.01818884517i, 0.1084633939 - 0.05315246871i),
P_6 = (0.004713033954 - 0.02355323617i, -0.07491796596 + 0.0903219938i),
P_7 = (0.02131893405 - 0.009609927603i, 0.02442855631 - 0.1171004584i) \},
S_2 = \{ P_8 = (-0.0001666666667, 0) \},
S_3 = \emptyset.
\]

Moreover, \( \text{mult}_\epsilon(P_1) = \cdots = \text{mult}_\epsilon(P_7) = 2 \) but \( \text{mult}_\epsilon(P_8) = 3 \). Note that considering only \( S_1 \) we would have not found a point with \( \epsilon \)-multiplicity 3.

As we could check in the previous example, the difficulty appears when observing that we may have two (in general more than two) \( \epsilon \)-singularities \( P \) and \( Q \) that are very “close”, and somehow we need to identify them. To approach this, we introduce the notion of cluster of \( \epsilon \)-singularities. Intuitively, two \( \epsilon \)-singularities \( P \) and \( Q \) of \( C \) are in the same cluster, if the disks centered at \( P \) and \( Q \) (of certain radius) are a small vibration of each other. The radius and the vibration are measured by means of the value of the function \( R_{\text{out}} \) at the weight and the tolerance, respectively (see Section 4). Since the notion of weight requires that the \( \epsilon \)-singularities are pure, for non-pure
\(\epsilon\)-singularities we will take radius zero. More precisely, we introduce the following definition.

**Definition 3.3.** Let \( P \) be an \( \epsilon \)-point of \( \mathcal{C} \). We define its **radius**, and we denote it by \( \text{radius}(P) \), as \( \mathcal{R}_{\text{out}}(\text{weight}(P)) \) if \( P \) is pure and zero otherwise.

**Definition 3.4.** Let \( \mathcal{A} \) be a finite set of \( \epsilon \)-points of \( \mathcal{C} \). For \( P \in \mathcal{A} \) we define the **cluster of \( P \) w.r.t. \( \mathcal{A} \)** as the set of all points \( Q \in \mathcal{A} \) such that at least one of the following conditions is verified:

\[
\begin{align*}
(1) \quad & \|P - Q\|_2 + |\text{radius}(P) - \text{radius}(Q)| < \mathcal{R}_{\text{out}}(\epsilon), \\
(2) \quad & \text{there exists } P' \in \mathcal{A} \text{ such that } \|P' - P\|_2 + |\text{radius}(P') - \text{radius}(P)| < \mathcal{R}_{\text{out}}(\epsilon) \text{ and } \|P' - Q\|_2 + |\text{radius}(P') - \text{radius}(Q)| < \mathcal{R}_{\text{out}}(\epsilon).
\end{align*}
\]

We say that \( R \) is a **candidate to be the representative of a cluster**, if \( R \) is a point of the cluster of maximum \( \epsilon \)-multiplicity. We say that \( R \) is a **representative of a cluster** if it is a candidate and \( |f(R)| \leq |f(Q)| \) for all the other candidates \( Q \). We define the **\( \epsilon \)-multiplicity of the cluster** as the \( \epsilon \)-multiplicity of any of its representatives.

We denote a cluster by \( \mathcal{C}_{\text{Cluster}}(R, \mathcal{A}) \), where \( r \) is the \( \epsilon \)-multiplicity and \( R \) a representative, and by \( \mathcal{C}_{\text{Cluster}}(R) \) when \( \mathcal{A} = \text{Sing}_\epsilon(\mathcal{C}) \).

Now, we are ready to introduce the notion of \( \epsilon \)-rationality.

**Definition 3.5.** If \( \{\mathcal{C}_{\text{Cluster}}(P_i)\}_{i=1,...,s} \) is the cluster decomposition of \( \text{Sing}_\epsilon(\mathcal{C}) \), we say that \( \mathcal{C} \) is **\( \epsilon \)**-(affine) rational if \( (d - 1)(d - 2) - \sum_{i=1}^{s} r_i(r_i - 1) = 0 \).

**Remark 3.6.** Note that in the previous theoretical development we have not considered singularities (neither \( \epsilon \)-singularities) at infinity. We leave this extension of the concept of \( \epsilon \)-rationality for further research.

If we apply the previous ideas to Example 3.2 (see also Example 1.5), with \( \epsilon = 0.001 \) we get that the 8 points of \( \text{Sing}_\epsilon(\mathcal{C}) \) belong to the same cluster. So, the cluster decomposition is \( \{\mathcal{C}_{\text{Cluster}}(P_8) = \{P_1, \ldots, P_8\} \} \). Therefore, \( \mathcal{C} \) is \( \epsilon \)-rational; indeed, it is \( \epsilon \)-monomial, and thus parametrizable with the techniques in [17]. We finish the section with a more general example.

**Example 3.7.** Let us consider \( \epsilon = 0.005 \) and the curve \( \mathcal{C} \) of proper degree 5 defined by the polynomial (see Fig.1):

\[
f(x, y) = -2.199771784x^2 - 0.2197717843x^4y - 0.9016804979x^3y^2 + 1.858817427x^3 - 1.891680498y^4 + 0.9899999999x^3y^2 + 0.989999999x^2y^3 + 0.340954356xy^2 + 0.9899999999x^4 + 0.9899999999y^4 + 0.9899999999y^3 - 0.1869087137x^5 + 5.235497925xy^2 - 1.770497925x^2y^2 + 1.45213693x^3y - 0.1440456432xy - 0.52786307y^5 + 0.01.
\]
The $\epsilon$-singular locus is $\text{Sing}_\epsilon(C) = S_1 \cup S_2 \cup S_3$, where

\[
S_1 = \{ P_1 = (-0.9956027274 + 0.0004067223817i, 0.001447687187 + 0.9982777543i), \\
P_2 = (1.011706789 - 0.1320874194i, -1.008532436 + 0.06832949372i), \\
P_3 = (1.007458642, -1.044045331), P_4 = (0.9909273695, -0.9540334161), \\
P_5 = (1.011706789 + 0.1320874194i, -1.008532436 - 0.06832949372i), \\
P_6 = (-0.9956027274 - 0.0004067223817i, 0.001447687187 - 0.9982777543i), \\
P_7 = (0, 0), \\
P_8 = (-0.9956027274 - 0.0004067223817i, 0.001447687187 - 0.9982777543i), \\
P_9 = (1.000000001, -1.1), \\
P_{10} = (0, 0). \\
S_2 = \{ P_8 = (1.000000001, -1.1) \}, \\
S_3 = \emptyset.
\]

Moreover, $\text{mult}_\epsilon(P_1) = \text{mult}_\epsilon(P_2) = \text{mult}_\epsilon(P_7) = 2$, and $\text{mult}_\epsilon(P_3) = \text{mult}_\epsilon(P_4) = \text{mult}_\epsilon(P_5) = \text{mult}_\epsilon(P_6) = \text{mult}_\epsilon(P_8) = 3$. Furthermore, the cluster decomposition is (see Fig. 1):

\[
\text{Cluster}_2(P_1) = \{ P_1 \}, \\
\text{Cluster}_2(P_2) = \{ P_2 \}, \\
\text{Cluster}_2(P_7) = \{ P_7 \}, \\
\text{Cluster}_3(P_8) = \{ P_3, P_4, P_5, P_6, P_8 \}.
\]

Thus, $C$ is $\epsilon$-rational.

![Figure 1: Left: Clusters. Right: Curve $C$](image)

4 Approximate Parametrization Algorithm

In this section, we present our approximate parametrization algorithm. For this purpose, we assume that $C$ is $\epsilon$-rational of proper degree $d > 2$ (note that for $d = 1$ the problem is trivial, and for $d = 2$ one can apply the algorithm in [17]), and that

\[
\{ \text{Cluster}_{r_i}(Q_i) \}_{i=1,...,s}, \text{ where } Q_i := (q_{i,1} : q_{i,2} : 1),
\]

is the cluster decomposition of $\text{Sing}_\epsilon(C)$. Furthermore, if possible, i.e. when there exists a real representative of the cluster, we take $Q_i$ real.
In this situation, we adapt the algorithm in Section 2 as follows. Let $C^h$ be the projective closure of $C$. We consider the linear system of curves $\mathcal{H}_{d-2}$ of degree $(d - 2)$ given by the divisor $\sum_{i=1}^s r_i Q_i$. That is, $Q_i$ is a base point of (exact) multiplicity $r_i - 1$ of the linear system. Afterwards, one computes $(d - 3)$ $\epsilon$–simple affine points on $C^h$ (see below for details), and determines the linear subsystem $\mathcal{H}_{d-2}^*$ of $\mathcal{H}_{d-2}$ obtained by intersecting $\mathcal{H}_{d-2}$ with the linear system of $(d - 2)$-degree curves generated by the divisor $\sum_{i=1}^{d-3} P_i$; say that $P_i := (p_{i,1} : p_{i,2} : 1)$. If $P_i$, $Q_j$ would be exact points and singularities, respectively, of $C^h$, then $\dim(\mathcal{H}_{d-2}^*) = 1$ (see Chap. 4 in [24]). However, in our case, since we are working with $\epsilon$-points we can only ensure that $\dim(\mathcal{H}_{d-2}^*) \geq 1$ (see Theorem 2.56 in [24]). If this dimension is strictly bigger than 1, we can either take more $\epsilon$-simple points till dimension 1 is reached, or we can take a small perturbation of the $\epsilon$-points such that the effective divisor $\sum_{i=1}^s r_i Q_i + \sum_{i=1}^{d-3} P_i$ is in general position (see page 49 in [24]), and hence the dimension is 1. So, we can assume w.l.o.g. that $\dim(\mathcal{H}_{d-2}^*) = 1$. Let, then, $\mathcal{H}_{d-2}(t, x, y, z)$ be the defining homogeneous polynomial of $\mathcal{H}_{d-2}$.

At this point, if $P_i$, $Q_j$ would be exact points and singularities, respectively, of $C$, the symbolic algorithm presented in Section 2 would output the parametrization $\mathcal{P}(t) = (\frac{p_1(t)}{q_1(t)}, \frac{p_2(t)}{q_2(t)})$, where

$$q_1(t)x - p_1(t) = \frac{\text{Res}_y(\mathcal{H}_{d-2}(t, x, y, 1), f(x, y))}{\prod_{i=1}^s (x - q_{i,1})^{r_i s + 1} \prod_{i=1}^{d-3} (x - p_{i,1})},$$

$$q_2(t)y - p_2(t) = \frac{\text{Res}_x(\mathcal{H}_{d-2}(t, x, y, 1), f(x, y))}{\prod_{i=1}^s (y - q_{i,2})^{r_i s + 1} \prod_{i=1}^{d-3} (y - p_{i,2})}.$$ 

However, in our case, $P_i$, $Q_j$ are not exact points, but $\epsilon$–points. So these rational functions are not, in general, polynomials. Nevertheless, considering if necessary a small perturbation of $\mathcal{H}_{d-2}$, the quotient of the division of each numerator by its denominator is linear as polynomial in either $x$ or $y$. Then, the idea is to determine the parametrization from these linear quotients. For this purpose, we will consider (if necessary) two perturbations, both affecting $\mathcal{H}_{d-2}$. The first one will ensure that the degree in the resultants is the expected one, namely $d(d - 2)$. The second will guarantee that the output is indeed a parametrization; i.e. that not both components are constants. Note that, in the exact case, these two facts are provided by the theory.

More precisely, let $\mathcal{H}_{d-2}(t, x, y, z) = H_1(x, y, z) + t H_2(x, y, z)$, and let $\mathcal{D}_i$ be the projective curve defined by $H_i$, $i = 1, 2$. We recall that $(1 : 0 : 0), (0 : 1 : 0) \notin C^h$. Now, we need to ensure that either $C^h, \mathcal{D}_1$ or $C^h, \mathcal{D}_2$ do not have common points at infinity. If this is not the case, let $\{R_1, \ldots, R_m\}$ be the points of $C$ at infinity and $K(\rho_1, \rho_2, x, y, z) = \rho_1 x^{d-2} + \rho_2 y^{d-2}$, where $\rho_i$ are parameters. Then, we consider in $\mathbb{C}^2$ the union $\mathcal{L}$ of the affine lines defined by $H_2(R_i) + K(\rho_1, \rho_2, R_i) = 0$, for $i = 1, \ldots, m$. Note that, since $R_i$ are points at infinity, the polynomials $H_2(R_i) + K(\rho_1, \rho_2, R_i) \in \mathbb{C}^2$.
Lemma 1, it is proved that for $\varepsilon$ small real numbers we consider an small perturbation that ensures that the above requirement is satisfied.

Thus, in what follows we assume that $D_2$ and $C^h$ do not have common points at infinity. Therefore, if $F$ is the homogenization of $f$, by Lemma 3.1 in [1], one has that

$$\deg_x(\text{Res}_y(\overline{H}_{d-2}^*, F)) = \deg_y(\text{Res}_x(\overline{H}_{d-2}^*, F)) = d(d - 2).$$

Moreover, since $\overline{H}_{d-2}^*$ and $C^h$ do not have common points at infinity, it holds that

$$\deg_x(\text{Res}_y(\overline{H}_{d-2}(t, x, y, 1), f)) = \deg_y(\text{Res}_x(\overline{H}_{d-2}(t, x, y, 1), f)) = d(d - 2).$$

Now, we consider the polynomials

$$A_1(x) = \prod_{i=1}^s (x - q_{i1})^{r_i(r_i - 1)} \prod_{i=1}^{d-3} (x - p_{i1}), \quad A_2(y) = \prod_{i=1}^s (y - q_{i2})^{r_i(r_i - 1)} \prod_{i=1}^{d-3} (y - p_{i2}).$$

Since $C$ is $\varepsilon$-rational, it holds that

$$\deg_x(A_1(x)) = \deg_y(A_2(y)) = d(d - 2) - 1.$$

Let $B_1(x, t) := \overline{q}_1(t)x - \overline{p}_1(t)$ be the quotient of $S_1(x, t) := \text{Res}_y(\overline{H}_{d-2}(t, x, y, 1), f(x, y))$ and $A_1(x)$. Similarly let $B_2(y, t) := \overline{q}_2(t)x - \overline{p}_2(t)$ be the quotient of $S_2(y, t) := \text{Res}_x(\overline{H}_{d-2}(t, x, y, 1), f(x, y))$ and $A_2(y)$. Then, we output

$$\overline{\mathcal{P}}(t) = \left(\frac{\overline{p}_1(t)}{\overline{q}_1(t)} \cdot \frac{\overline{p}_2(t)}{\overline{q}_2(t)}\right)$$

as approximate parametrization of $C$.

Intuitively one sees that, in practice, $\overline{\mathcal{P}}(t)$ will be always a parametrization. In order to prove this claim, we repeat the reasoning but introducing a new perturbation of $\overline{H}_{d-2}^*$. More precisely, let $\Delta = (\delta_1, \ldots, \delta_6)$ be a family of perturbing parameters and let

$$G(\Delta, x, y, z) = \delta_1 y^{d-2} + \delta_2 y^{d-3} z + \delta_3 x^{d-2} + \delta_4 x^{d-3} z + \delta_5 x^{d-3} y + \delta_6 x y^{d-3}.$$

If $d = 3$ we take $\Delta = (\delta_1, \delta_2, \delta_3)$ and $G = \delta_1 y + \delta_2 z + \delta_3 x$. Observe also that in [17], Lemma 1, it is proved that for $\varepsilon$-monomial curves, and hence for $d = 3$, $\overline{\mathcal{P}}(t)$ is always a parametrization. Then we consider $\overline{H}^{**}(\Delta, t, x, y, z) = \overline{H}_{d-2}^*(t, x, y, z) + G(\Delta, x, y, z)$; that is

$$\overline{H}^{**}(\Delta, t, x, y, z) = H_1(x, y, z) + tH_2(x, y, z) + G(\Delta, x, y, z).$$

Note that we are perturbing $H_1$ and hence $H_2$ keeps the required conditions on the point at infinity of $C$.
In this situation, repeating the above process with $H^*$ and $F$, instead of with $H^d_{d-2}$ and $F$, we introduce $S^\Delta_1$, $S^\Delta_2$, $B^\Delta_1$, $B^\Delta_2$, $R^\Delta_1$, $R^\Delta_2$ and $\overline{P}^\Delta(\Delta, t)$. So

$$S^\Delta_1(\Delta, x, t) = \text{Res}_y(H^*(\Delta, t, x, y, 1), f), S^\Delta_2(\Delta, y, t) = \text{Res}_x(H^*(\Delta, t, x, y, 1), f),$$

and $B^\Delta_1(\Delta, x, t), R^\Delta_1(\Delta, x, t)$ are the quotient and the remainder of the division of $S^\Delta_1$ by $A_1(x)$, respectively. Similarly, for $B^\Delta_2(\Delta, y, t), R^\Delta_2(\Delta, y, t)$ using $S^\Delta_2$ and $A_2(y)$. Finally, the components of $\overline{P}^\Delta(\Delta, t)$ are the roots of $B^\Delta_1(\Delta, x, t)$ and $B^\Delta_2(\Delta, y, t)$ as univariate polynomials over $\mathbb{C}[\Delta, t]$.

We start with some lemmas.

**Lemma 4.1.** The leading coefficient w.r.t. $x$ of $B^\Delta_1(\Delta, x, t)$ and the leading coefficient of $B^\Delta_2(\Delta, y, t)$ w.r.t. $y$, as polynomials in $\mathbb{C}(\Delta)[t]$, are the same up to multiplication by non-zero constants in $\mathbb{C}$.

Furthermore, the roots are

$$\left\{ \frac{-H_1(a, b, 0) + G(\Delta, a, b, 0)}{H_2(a, b, 0)} \right\}_{(a:b) \in \mathbb{C}^h}.$$

**Proof.** Let $B^\Delta_1(\Delta, x, t) = q_1(\Delta, t)x - p_1(\Delta, t)$, and $B^\Delta_2(\Delta, y, t) = q_2(\Delta, t)y - p_2(\Delta, t)$. By hypothesis $F(1, 0, 0) \neq 0, F(0, 1, 0) \neq 0$. So, the leading coefficient of $F$ w.r.t. $y$ is a non-zero constant; similarly w.r.t. $x$. Thus, by well known properties on resultants (see, e.g. Lemma 4.3.1. in [25]), it holds that up to multiplication by a non-zero element in $\mathbb{C}$:

$$\text{Res}_y(H^*(\Delta, t, x, y, 0), F(x, y, 0)) = (S^\Delta_1)^H(\Delta, x, 0, t),$$

$$\text{Res}_x(H^*(\Delta, t, x, y, 0), F(x, y, 0)) = (S^\Delta_2)^H(\Delta, y, 0, t),$$

where $(S^\Delta_i)^H$ denotes the homogenization of $S^\Delta_i$ as polynomials in $\mathbb{C}(\Delta, t)[x, y]$. Now, observe that

$$(S^\Delta_1)^H(\Delta, x, 0, t) = q_1(\Delta, t)x^{d-2}, (S^\Delta_2)^H(\Delta, y, 0, t) = q_2(\Delta, t)y^{d-2}.$$ 

Moreover, let $F(x, y, 0)$ factor as

$$F(x, y, 0) = \prod_{i=1}^{d}(\beta_i x - \alpha_i y).$$

Since $F(0, 1, 0) \neq 0$ then $\alpha_i \neq 0$ for all $i$. Hence, up to multiplication by non-zero constants

$$\text{Res}_y(H^*(\Delta, t, x, y, 0), F(x, y, 0)) = \prod_{i=1}^{d} \text{Res}_y(H^*(\Delta, t, x, y, 0), \beta_i x - \alpha_i y) =$$
\[= (−1)^{d(d−2)}x^{d(d−2)}\prod_{i=1}^{d} \overline{\mathcal{H}}^{**}(\Delta, t, \alpha_i, \beta_i, 0).\]

Analogously,

\[\text{Res}_x(\overline{\mathcal{H}}^{**}(\Delta, t, x, y, 0), F(x, y, 0)) = (−1)^{d(d−2)}y^{d(d−2)}\prod_{i=1}^{d} \overline{\mathcal{H}}^{**}(\Delta, t, \alpha_i, \beta_i, 0).\]

So, up to multiplication by non-zero constants

\[q_1(\Delta, t) = q_2(\Delta, t) = \prod_{i=1}^{d} \overline{\mathcal{H}}^{**}(\Delta, t, \alpha_i, \beta_i, 0) = \prod_{i=1}^{d} (H_1(\alpha_i, \beta_i, 0) + G(\Delta, \alpha_i, \beta_i, 0) + tH_2(\alpha_i, \beta_i, 0)).\]

**Lemma 4.2.** For all \(\Delta_0 \in \mathbb{C}^d\), \(\deg_x(B_1^\Delta(\Delta_0, x, t)) = d\) and \(\deg_x(B_2^\Delta(\Delta_0, y, t)) = d\).

**Proof.** First note that \(\deg_x(B_1^\Delta) \leq d\) and \(\deg_y(B_2^\Delta) \leq d\). The equality follows from the last equality in the proof of Lemma 4.1, and using that \(H_2(\alpha_i, \beta_i, 0) \neq 0\) for all \(i\). \(\square\)

**Lemma 4.3.** There exists a non-empty Zariski open subset \(\Omega\) of \(\mathbb{C}^d\) such that if \(\Delta_0 \in \Omega\) then \(B_1^\Delta(\Delta_0, x, t)\) and \(B_2^\Delta(\Delta_0, y, t)\) are primitive w.r.t. \(x\) and \(y\), respectively.

**Proof.** We assume that \(d > 3\); if \(d = 3\) the reasoning is analogous. Let us assume that

\[B_1^\Delta(\Delta, x, t) = D(\Delta, t)\Lambda(\Delta, x, t),\]

with \(\deg_x(D) > 0\). Then,

\[S_1^\Delta(\Delta, x, t) = D(\Delta, t)\Lambda(\Delta, x, t)A_1(x) + R_1^\Delta(\Delta, x, t).\]

By Lemma 4.1 we know how the roots of \(D(\Delta, t) \in \mathbb{C}(\Delta)[t]\) are. Now for each root \(t_0\) of \(D\) (say that \(t_0\) is defined by \(P := (a : b : 0) \in \mathbb{C}^h\) ), \(\deg_x(S_1^\Delta(\Delta, x, t_0)) = \deg_x(R_1^\Delta(\Delta, x, t_0)) \leq d(d−2)−2\). Let \(\mathcal{D}(t_0)\) be the projective curve defined by \(\overline{\mathcal{H}}^{**}(\Delta, t_0, x, y, z)\) over the algebraic closure \(\mathbb{F}\) of \(\mathbb{C}(\Delta)\). Then, \(\mathcal{D}(t_0)\) and \(\mathcal{H}\) intersect at infinity at an additional point different from \(P\), or the multiplicity of intersection of both curves at \(P\) is at least two. We analyze each case. But first we introduce some additional notation. We express \(F, H_1\), and \(H_2\) as

\[F(x, y, z) = f_0(x, y) + f_1(x, y)z + \cdots + f_d(x, y)z^d,\]

\[H_i(x, y, z) = h_{i,0}(x, y) + h_{i,1}(x, y)z + \cdots + h_{i,d−2}(x, y)z^{d−2},\]

where \(f_j, h_{i,j}\) are homogeneous of degree \(d−j\) and \((d−2)−j\), respectively. Moreover, we denote by \(F^x, H_i^x, \overline{\mathcal{H}}^{**x}, f_j^x, h_{i,j}^x\) the corresponding partial derivative w.r.t. \(x\); similarly w.r.t. \(y\) and \(z\).

Let us assume that \(Q \in \mathcal{D}(t_0)\), with \(Q = (n : m : 0) \neq P\). This is equivalent to

\[\delta_1C_1 + \delta_3C_3 + \delta_5C_5 + \delta_6C_6 = C_0,\]
where

\[ C_1 = b^{d-2}h_{2,0}(Q) - m^{d-2}h_{2,0}(P), \quad C_2 = a^{d-3}bh_{2,0}(Q) - n^{d-3}mh_{2,0}(P), \]
\[ C_3 = a^{d-2}h_{2,0}(Q) - n^{d-2}h_{2,0}(P), \quad C_4 = ab^{d-3}h_{2,0}(Q) - nm^{d-3}h_{2,0}(P), \]
\[ C_5 = a^{d-3}bh_{2,0}(Q) - n^{d-3}mh_{2,0}(P), \quad C_6 = ab^{d-3}h_{2,0}(Q) - nm^{d-3}h_{2,0}(P), \]
\[ C_0 = h_{1,0}(P)h_{2,0}(Q) - h_{2,0}(P)h_{1,0}(Q). \]

Observe that \( h_{2,0}(Q) \neq 0, h_{2,0}(P) \neq 0 \). Let us see that all \( C_i, i > 0 \), can not vanish simultaneously. Let \( C_1 = C_3 = C_5 = C_6 = 0 \). We assume that \( a \neq 0 \). If \( a = 0 \) then \( b \neq 0 \), and the reasoning is similar. From \( C_3 = 0 \) one has that \( n \neq 0 \). So

\[ P = (a : b : 0) = (a^{d-2} : ba^{d-3} : 0) = (a^{d-2}h_{2,0}(Q) : ba^{d-3}h_{2,0}(Q) : 0) = \]
\[ = (n^{d-2}h_{2,0}(P) : n^{d-3}mh_{2,0}(P) : 0) = (n : m : 0) = Q, \]
which is a contradiction. Therefore, if \( V_1 \) is the hyperplane in \( \mathbb{C}^6 \) defined by \( \delta_1C_1 + \delta_3C_3 + \delta_5C_5 + \delta_6C_6 = C_0 \), for all \( \Delta_0 \) in \( \mathbb{C}^6 \setminus V_1 \) this case does not happen.

Let us assume that the multiplicity of intersection of \( \mathcal{D}(t_0) \) and \( \mathcal{C}^h \) at \( P \) is at least two. Since \( \mathcal{C}^h \) does not have singularities at infinity, this implies that both curves have the same tangent at \( P \). This is equivalent to demand

\[ (F^x(P) : F^y(P) : F^z(P)) = (H^{**}_x(\Delta, t_0, P) : H^{**}_y(\Delta, t_0, P) : H^{**}_z(\Delta, t_0, P)). \]

By hypothesis \( ab \neq 0 \). So, by Euler’s formula and taking into account that \( P \) is at infinity, the condition is equivalent to

\[ F^x(P)H^{**}_z(\Delta, t_0, P) = F^z(P)H^{**}_x(\Delta, t_0, P). \]

That is equivalent to

\[ \delta_1C_1 + \delta_2C_2 + \delta_3C_3 + \delta_4C_4 + \delta_5C_5 + \delta_6C_6 = C_0, \]

where

\[ C_1 = b^{d-2}(f_1(P)h_{2,0}^z(P) - f_0^x(P)h_{2,1}(P)), \quad C_2 = b^{d-3}f_0^x(P)h_{2,0}(P), \]
\[ C_3 = a^{d-3}(af_1(P)h_{2,0}^z(P) - af_0^x(P)h_{2,1}(P) - (d - 2)f_1(P)h_{2,0}(P)), \]
\[ C_4 = a^{d-3}f_0^x(P)h_{2,0}(P), \quad C_5 = a^{d-4}b(-f_0^x(P)h_{2,1}(P) + f_1(P)h_{2,0}^z(P) - (d - 3)f_1(P)h_{2,0}(P)), \]
\[ C_6 = b^{d-4}(f_1(P)h_{2,0}^z(P) - f_1(P)h_{2,0}(P) - f_0^x(P)h_{2,1}(P) + f_1(P)h_{2,0}^z(P) - h_{2,0}^z(P)h_{1,0}(P)). \]

Let us see that \( C_i, i > 0 \), cannot vanish simultaneously. Let \( C_1 = \cdots = C_6 = 0 \). Since \( h_{2,0}(P) \neq 0 \), and \( ab \neq 0 \), one has that

\[ C_1 = 0 \Rightarrow f_1(P)h_{2,0}^z(P) = f_0^x(P)h_{2,1}(P), \quad C_2 = 0 \Rightarrow f_0^x(P) = 0, \]
\[ C_6 = 0 \Rightarrow f_1(P)(h_{2,0}^z(P) - h_{2,0}(P)) = 0. \]
Note that \( f_1(P) \neq 0 \), since otherwise it would imply that \( F^s(P) = 0 \) and using that \( F^{s\prime}(P) = f_0^{\prime \prime}(P) = 0 \) and that (by Euler’s formula) \( F^y(P) = 0 \), one would deduce that \( P \) is a singularity of \( \mathbb{C}^6 \) which is excluded by hypothesis. So, the first and second equalities imply that \( h_{2,0}(P) = 0 \) and this yields to (using the last equality) \( h_{2,0}(P) = 0 \) which is a contradiction. Therefore, out of the hyperplane defined in \( \mathbb{C}^6 \) by \( \sum_{i=1}^{6} C_i \delta_i = C_0 \), this case cannot happen.

For each point of \( \mathcal{C} \) at infinity we generate the hyperplanes described above and corresponding to each one of the two cases. Let \( V \) be the union of all of them, and let \( \Omega_1 = \mathbb{C}^6 \setminus V \). Repeating the same reasoning with \( B_{2}^{\Delta} \) (note that \( G \) is symmetric in terms of \( x \) and \( y \)), we get \( \Omega_2 \). Finally, let \( \Omega = \Omega_1 \cap \Omega_2 \).

Now, the next theorem follows directly.

**Theorem 4.4.** There exists a non-empty Zariski open subset \( \Omega \) of \( \mathbb{C}^6 \) such that if \( \Delta_0 \in \Omega \) then \( \overline{\mathcal{P}}(\Delta_0, t) \) is a rational parametrization of a rational curve of degree at most \( d \).

**Proof.** Taking \( \Omega \) as in Lemma 4.3 we ensure that \( \overline{\mathcal{P}}(\Delta_0, t) \) is a rational parametrization. By Lemmas 4.1 and 4.2 we get that the degree of the curve is at most \( d \). \( \square \)

**Remark 4.5.** Let \( \overline{\mathcal{H}}_{\Delta_0}^{\ast} \) be the linear system of \((d - 2)\)-degree curves defined by \( \overline{\mathcal{H}}^{\ast}(\Delta_0, t, x, y, z) \). If no perturbation is needed, i.e. \( \Delta = \emptyset \), then \( \overline{\mathcal{H}}_{\emptyset}^{\ast} = \overline{\mathcal{H}}_{d-2}^{\ast} \), and hence it is generated by the effective (exact) divisor \( \mathcal{Q}_1 \). Now, if we identify (as usual) \( \mathcal{C}^2 \) with \( \mathbb{R}^4 \) and we consider the perturbing parameters \( \delta_i \) as real variables, it holds that for each \( Q_i \) (similarly of \( P_i \)) there exists \( \rho(Q_i) > 0 \) such that for almost every element \( \langle Q_i, \Delta_0 \rangle \) in the open Euclidean disk of \( \mathbb{R}^{10} \), of center \( (Q_i, \emptyset) \) and radius \( \rho(Q_i) \), \( \overline{\mathcal{P}}(\Delta_0, t) \) is a parametrization and \( Q_i^{\ast} \) is an \( \epsilon \)-point of \( \epsilon \)-multiplicity (at least) \( r_i \) of the curves defined by \( H_1(x, y, z) + G(\Delta_0, x, y, z) \) and \( H_2(x, y, z) \) (recall that \( \overline{\mathcal{H}}^{\ast} = H_1 + G + tH_2 \)); i.e. of the generating curves of \( \overline{\mathcal{H}}_{\Delta_0}^{\ast} \). This can be seen by applying Theorem 4.1, taking \( \delta_i \) small enough to ensure that \( \|H_1\| = \|H_1 + G(\Delta_0, x, y, z)\| \), and noting that if \( M \) is any of the derivatives of \( H_1 + G \) and \( H_2 \) involved in the \( \epsilon \)-multiplicity, then \( \|M\|^2_2 \) is a continuous function that vanishes at \( (Q_i, \emptyset) \).

Finally, and before outlining the algorithm, we briefly describe how to proceed with the selection and computation of the (affine simple) \( \epsilon \)-points \( P_i \). We first observe that, in general, an \( \epsilon \)-point can be computed by solving \( \{ f(x, y) = 0, \alpha x + \beta y = \rho \} \), where \( \alpha, \beta, \rho \in \mathbb{C} \), under fixed precision \( \epsilon \|f\| \). However, we are intersected in working with either real \( \epsilon \)-points or pairs of conjugate complex points. We can always compute all points, but at most one, in pairs of conjugate complex points. For choosing real points one can always analyze the roots of the discriminant of \( f \) (see Theorem 7.7 in [24]). On the other hand we have observed, in our examples, that taking (when possible) the simple \( \epsilon \)-points as (affine) \( \epsilon \)-ramification points (see Def. 1.3) the error distance between the original curve and the output curve decreases. So we tend to use first such points. Finally, one has to take care of the fact that a chosen \( \epsilon \)-point can be too close
(i.e. in the same cluster) to an \( \epsilon \)-singularity or to a previously computed \( \epsilon \)-point, and hence identifiable with it. To avoid this, whenever a new simple \( \epsilon \)-point is computed we check whether it belongs to the cluster of the others points.

The above process provides the following approximate parametrization algorithm for deciding whether a real \( \epsilon \)-irreducible (with proper degree) plane algebraic curve \( C \) is \( \epsilon \)-rational, and in the affirmative case, compute an approximate parametrization. Recall that we assume that \( C \) does not have exact singularities at infinity, and that \((0:1:0), (1:0:0) \notin C^h\). If this last condition fails, one may consider an affine orthogonal change of coordinates to achieve the requirement.

**Approximate Parametrization Algorithm**

- **Given** a tolerance \( \epsilon > 0 \) and an \( \epsilon \)-irreducible polynomial \( f(x, y) \in \mathbb{R}[x, y] \), of proper degree \( d > 2 \) (for \( d = 1 \) it is trivial, if \( d = 2 \) apply [17]), without exact singularities at infinity, not passing through \((0:1:0), (1:0:0)\), and defining a real plane algebraic curve \( C \); let \( F(x, y, z) \) be the homogenization of \( f \).

- **Decide** whether \( C \) is \( \epsilon \)-rational and in the affirmative case

- **Compute** a rational parametrization \( \overline{\mathbf{P}}(t) \) of a curve \( \overline{C} \) close to \( C \).

1. **Compute** the cluster decomposition \( \{\text{Cluster}_{r_i}(Q_i)\}_{i=1,...,s} \) of \( \text{Sing}_{\epsilon}(C) \); say \( Q_i = (q_{i,1} : q_{i,2} : 1) \).

2. If \( \sum_{i=1}^{s} r_i(r_i - 1) \neq (d - 1)(d - 2) \), RETURN “\( C \) is not (affine) \( \epsilon \)-rational”. If \( s = 1 \) one may apply the algorithm in [17].

3. **Determine** the linear system \( \overline{H}_{d-2} \) of degree \( d - 2 \) given by the divisor \( \sum_{i=1}^{s} r_i Q_i \).

4. **Compute** \( (d - 3) \) \( \epsilon \)-ramification points \( \{P_j\}_{1 \leq j \leq d-3} \) of \( C \); if there are not enough \( \epsilon \)-ramification points, complete with simple \( \epsilon \)-point. Take the points over \( \mathbb{R} \), or as conjugate complex points. After each point computation check that it is not in the cluster of the others (including the clusters of \( Q_i \)); if this fails take a new one. Say \( P_i = (p_{i,1} : p_{i,2} : 1) \).

5. **Determine** the linear subsystem \( \overline{H}_{d-2}^* \) of \( \overline{H}_{d-2} \) given by the divisor \( \sum_{i=1}^{d-3} P_i \). Let \( H^*(t, x, y, z) = H_1(x, y, z) + tH_2(x, y, z) \) be its defining polynomial.

6. If \( [\gcd(F(x, y, 0), H_1(x, y, 0)) \neq 1] \) and \( [\gcd(F(x, y, 0), H_2(x, y, 0)) \neq 1] \) replace \( H_2 \) by \( H_2 + \rho_1 x^{d-2} + \rho_2 y^{d-2} \), where \( \rho_1, \rho_2 \) are real and strictly smaller than \( \epsilon \). Say that \( \gcd(F(x, y, 0), H_2(x, y, 0)) = 1 \); similarly in the other case.

7. Set \( \delta_1 = \cdots = \delta_6 = 0 \).

8. If \( d > 3 \) then \( g := \delta_1 y^{d-2} + \delta_2 y^{d-3} + \delta_3 x^{d-2} + \delta_4 x^{d-3} + \delta_5 x^{d-3} y + \delta_6 x y^{d-3} \) else \( g := \delta_1 y + \delta_2 + \delta_3 x \).
(9) $S_1(x, t) = \text{Res}_y(H^*(x, y, 1) + g, f)$ and $S_2(y, t) = \text{Res}_x(H^*(x, y, 1) + g, f)$.

(10) $A_1(x) = \prod_{i=1}^r(x - q_i, 1)^{(r_i - 1)} \prod_{i=1}^{d-3}(x - p_i, 1)$,

$A_2(y) = \prod_{i=1}^r(y - q_i, 2)^{(r_i - 1)} \prod_{i=1}^{d-3}(y - p_i, 2)$.

(11) For $i = 1, 2$ compute the quotient $B_i$ of $S_i$ by $A_i$ w.r.t. either $x$ or $y$.

(12) If the content of $B_1$ w.r.t $x$ or the content of $B_2$ w.r.t. $y$ does depend on $t$, take \{δ_1, ..., δ_t\} as small real numbers (strictly smaller than ε) and go to Step 8.

(13) Determine the root $\mathcal{P}_1(t)$ of $B_1$ as a polynomial in $x$ and the root $\mathcal{P}_2(t)$ of $B_2$ as a polynomial in $y$.

(14) RETURN $\mathcal{P}(t) = (\mathcal{P}_1(t), \mathcal{P}_2(t))$.

The next theorem states the main properties of the curve output by the algorithm. But first, we need the following technical lemma.

**Lemma 4.6.** Let $\mathcal{L}$ be the algebraic closure of $\mathbb{C}(t)$, and $\mathcal{C}_1, \mathcal{C}_2$ two plane projective curves over $\mathcal{L}$ with defining polynomials $G_1(x, y, z), G_2(x, y, z) \in \mathbb{C}[t][x, y, z]$, respectively. If there exist $K, W, L \in \mathbb{C}[t][x, y, z]$ such that $KG_1 + WG_2 = zL$, and

1. $G_1(x, y, 0)G_2(x, y, 0) \neq 0$,
2. $\gcd(G_1(x, y, 0), G_2(x, y, 0)) = 1$,

then either $z$ divides $K$ and $W$ or there exist $U_1, U_2, U_3 \in \mathbb{C}[t][x, y, z]$ such that

$L = U_1G_1(x, y, 0) + U_2G_2(x, y, 0) + zU_3$.

**Proof.** If $z$ divides $K$, then $z$ divides $WG_2$, and by (2) $z$ divides $W$. So let us assume that $z$ does not divides $K$, and let us denote by $G_i^0$ the polynomial $G_i(x, y, 0)$; similarly with $K^0, W^0$. Then, $K^0G_1^0 + W^0G_2^0 = 0$. Since $G_i^0 \neq 0$ and $\gcd(G_1^0, G_2^0) = 1$, then $G_1^0$ divides $W^0$ and $G_2^0$ divides $K^0$. Let $K^0 = \Delta_1G_2^0, W^0 = \Delta_2G_1^0$. So $(\Delta_1 + \Delta_2)G_1^0G_2^0 = 0$, and since $G_i^0 \neq 0$, one gets $\Delta_1 + \Delta_2 = 0$. Now, we write

$K = K^0 + z\overline{K}, W = W^0 + z\overline{W}, G_i = G_i^0 + z\overline{G_i}$,

where $\overline{K}, \overline{W}, \overline{G_i} \in \mathbb{C}[t][x, y, z]$. Then, $KW_1 + WG_2 = z(G_1^0\overline{K} + G_2^0\overline{W} + z(\overline{K}\overline{G_1} + \overline{W}\overline{G_2}))$. □

**Theorem 4.7.** The rational curve $\overline{C}$, output by the algorithm, and $\mathcal{C}$ have the same points at infinity, and $\deg(\overline{C}) \leq \deg(\mathcal{C})$. 

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Proof. The fact on the degree follows from Theorem 4.4. For the reasoning of the rest of the proof, we can assume w.l.o.g. that no perturbation $\Delta$ is required (i.e. $\Delta = 0$) in the execution of the algorithm. Let $\overline{H}_{d-2}(t, x, y, z, F(x, y, z), S_1(x, t), S_2(y, t), A_1(x), A_2(y), B_1(x, t), B_2(y, t), R_1(x, t),$ and $R_2(y, t)$ be defined as above. Let $B_1 := q_1(t)x - p_1(t), B_2 := q_2(t)y - p_2(t),$ and recall that $R_i$ is the remainder of the division of $S_i$ by $A_i.$ Furthermore, by Lemma 4.1, $q_1(t) = \lambda q_2(t),$ with $\lambda \in \mathbb{C}^*.$ By Lemma 4.2, $\deg(A_1) = \deg(B_2) = \deg(F) = d,$ and, by Lemma 4.3 $\gcd(q_1, p_1) = \gcd(q_2, p_2) = 1.$ So,

$$\overline{F}^H(t) := (\lambda^{-1}p_1(t) : p_2(t) : q_2(t))$$

parametrizes the projective closure of $\overline{C}.$ Furthermore, since $\deg(p_i) \leq \deg(q_2),$ then all points of $\overline{C}$ at infinity are reachable by $\overline{F}^H(t)$ (see [23]). In addition, we note that

$$\deg_{(x,y,z)}(\overline{H}_{d-2}) = d - 2, \deg(A_j) = d(d - 2) - 1, \deg_{(x,y)}(R_j) \leq d(d - 2) - 2.$$}

Moreover, if $m^H(x, y, z, w)$ denotes the homogenization of $m(x, y, w)$ as a polynomial in $\mathbb{C}[w][x, y],$ we have that

$$\overline{S}_1^H(x, z, t) = \text{Res}_y(\overline{H}_{d-2}(t, x, y, z), F(x, y, z)) = B_1^H(x, z, t)A_1^H(x, z) + R_1^H(x, z, t)z^{n_1},$$

$$\overline{S}_2^H(y, z, t) = \text{Res}_x(\overline{H}_{d-2}(t, x, y, z), F(x, y, z)) = B_2^H(y, z, t)A_2^H(y, z) + R_2^H(y, z, t)z^{n_2},$$

where $n_j + \deg(R_j^H) = d(d - 2), j = 1, 2.$ So $n_j \geq 2.$ Also, we denote by $\overline{C}_\infty$ and $\mathbb{C}_\infty$ the set of points at infinity $\overline{C}$ and $\mathbb{C}$ respectively. By resultant properties, there exist polynomials $M_1, N_1, M_2, N_2 \in \mathbb{C}[t, x, y, z]$ such that

$$M_i\overline{H}_{d-2} + N_iF = \overline{S}_i^H, \ i = 1, 2.$$}

So,

$$yA_2^H S_1^H - \lambda x A_1^H S_2^H = z A_1^H A_2^H (\lambda x p_2 - yp_1) + z^{n_3} R_3,$$

where $n_3 \geq 2$ and $R_3$ a polynomial; namely $z^{n_3} R_3 = y A_2 z^{n_1} R_1^H - \lambda x A_1 z^{n_2} B_2^H.$ On the other hand, if $K = yA_2^H M_1 - \lambda x A_1^H M_2$ and, $W = yA_2^H N_1 - \lambda x A_1^H N_2,$ then

$$yA_2^H S_1^H - \lambda x A_1^H S_2^H = K(x, y, z, t)(H_1 + tH_2) + W(x, y, z, t)F.$$}

Therefore, $z$ divides the right hand side of the above equation. We now check that $H_1 + tH_2$ and $F$ satisfy the hypothesis of Lemma 4.6. Since $F$ is irreducible and non-linear, $F(x, y, 0) \neq 0.$ Moreover, if $H_1(x, y, 0) + tH_2(x, y, 0) = 0$ then $H_2(x, y, 0) = 0$ and this implies that $\mathcal{D}_2$ contains all the points at infinity of $\mathbb{C}^h,$ which is a contradiction. Finally, if $\gcd(H_1(x, y, 0) + tH_2(x, y, 0), F(x, y, 0)) \neq 1,$ then $\gcd(H_2(x, y, 0), F(x, y, 0)) \neq 1,$ and this implies that $\mathcal{D}_2$ and $\mathbb{C}^h$ share points at infinity. Therefore, applying Lemma 4.6 one deduces that either there exist polynomials $M_3, N_3 \in \mathbb{C}[t][x, y, z]$ such that

$$M_3\overline{H}_{d-2} + N_3 F = A_1^H A_2^H (\lambda x p_2 - yp_1) + z^{n_4} R_3,$$
where \( n_4 \geq 1 \), or there exist polynomials \( U_1, U_2, U_3 \in \mathbb{C}[t][x, y, z] \) such that
\[
U_1 H_{d-2}(t, x, y, 0) + U_2 F(x, y, 0) + z U_3 = A_1^H A_2^H (\lambda x p_2 - y p_1) + z^{n_4} R_3.
\]

In this situation, using \( \mathcal{C}_\infty \subset \mathcal{P}^H(\mathbb{C}) \), we first observe that \( \text{Card}(\mathcal{C}_\infty) \) is less or equal to the number of different roots of \( q_2(t) \) and, by Lemma 4.1, this number is less or equal to \( \text{Card}(\mathcal{C}_\infty) \). So, \( \text{Card}(\mathcal{C}_\infty) \leq \text{Card}(\mathcal{C}_\infty) \). Now, we prove that \( \mathcal{C}_\infty \subset \mathcal{C}_\infty \), from where one concludes the proof. Let \( P = (x_0 : y_0 : 0) \in \mathcal{C}_\infty \), and let \( t_0 \) be the root of \( q_2 \) generated by \( P \) (see Lemma 4.1). So, \( \overline{H}_{d-2}(t_0, x_0, y_0, 0) = F(x_0, y_0, 0) = 0 \). Applying the corresponding equality above, and using that \( n_4 \geq 1 \), we get
\[
A_1^H (x_0, 0) A_2^H (y_0, 0) (\lambda x_0 p_2 (t_0) - y_0 p_1 (t_0)) = 0.
\]
Moreover, since \((1 : 0 : 0), (0 : 1 : 0) \notin \mathcal{C}^h \) then \( x_0 y_0 \neq 0 \), and hence \( A_1^H (x_0, 0) A_2^H (y_0, 0) \neq 0 \). So, \( \lambda x_0 p_2 (t_0) = y_0 p_1 (t_0) \). In addition, \( p_1 (t_0) p_2 (t_0) \neq 0 \) because \( \gcd(q_2, p_1) = 1 = \gcd(q_2, p_2) \). Therefore,
\[
\overline{P}^H (t_0) = (\lambda^{-1} p_1 (t_0) : p_2 (t_0) : 0) = (y_0 \lambda^{-1} p_1 (t_0) : y_0 p_2 (t_0) : 0) = (x_0 p_2 (t_0) : y_0 p_2 (t_0) : 0) = (x_0 : y_0 : 0) = P. \]

\( \square \)

## 5 Displaying Examples.

In this section we present several examples (the degrees are 5, 6, 7) to illustrate the algorithm. These examples have been computed in Maple.

**Example 5.1.** Let \( \epsilon = 0.01 \) and \( \mathcal{C} \) the curve of proper degree 5 defined by the polynomial (see Fig.4):
\[
f(x, y) = 0.006521014507 x^4 + 0.0006521014507 x^3 y^2 - 0.3174429862 x^3 + 0.006521014507 y^4 + 0.03536521618 y^4 + 0.008903520149 x^2 y^3 - 0.1541837293 y^3 - 0.3561209555 x^2 - 0.251465855 y^2 - 0.1517989182 x y^4 + 0.006177658243 y^5 + 0.006521014507 x^5 - 0.6503293396 x y - 0.969591291 x y^2 + 0.1751383118 x^2 y^2 + 0.1487535027 x y^3 - 1. x y + 0.00065868834.
\]
First we compute the \( \epsilon \)-singularities of \( \mathcal{C} \), obtaining the \( \epsilon \)-singular locus \( \text{Sing}_\epsilon (\mathcal{C}) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \):
\[
\mathcal{S}_1 = \{ Q_1 = (-3.999854219, 2.000094837), Q_2 = (0, 0, \cdot), Q_3 = (0.9998153818, -2.999388343), Q_4 = (-2.001190360 + 0.05414244305i, 3.001898191 - 0.08039416354i), Q_5 = (-1.980207988, 3.002780607), Q_6 = (-2.019931003, 2.997118979), Q_7 = (-2.001190360 - 0.05414244305i, 3.001898191 + 0.08039416354i), Q_8 = (-2.000000001, 3.000000001) \},
\mathcal{S}_2 = \emptyset.
\]
Moreover, the cluster decomposition of the singular locus is (see Fig. 2, Left):

\[ \text{Cluster}_2(Q_1) = \{Q_1\}, \text{Cluster}_2(Q_2) = \{Q_2\}, \text{Cluster}_2(Q_3) = \{Q_3\} \]

and

\[ \text{Cluster}_3(Q_8) = \{Q_1, Q_5, Q_6, Q_7, Q_8\} \]

We observe that \( \mathcal{C} \) is \( \epsilon \)-rational. Following Step 4 in the algorithm we obtain two \( \epsilon \)-ramification points, namely \( P_1 = (3.437938023, 4.260660564), P_2 = (7.712891931, 1.573609575) \). We note that these points are not in the cluster of each other and they are not in the clusters of the cluster decomposition of the singular locus (see Fig. 2, Right).

Finally, the algorithm outputs the parametrization \( \overline{\mathcal{P}}(t) = (\overline{\mathcal{P}}_1(t), \overline{\mathcal{P}}_2(t)) \) where (see Fig. 3 to compare the input and the output curves):

\[
\begin{align*}
\overline{\mathcal{P}}_1(t) &= -0.5918689071 \times 10^{-20} (0.7256428750 \times 10^{579} t + 0.1009796140 \times 10^{581} t^4 + 0.4757134093 \times 10^{580} t^5 + 0.3531628351 \times 10^{580} t^2 + 0.8491037424 \times 10^{580} t^3 + 0.5883163866 \times 10^{578}) \\
\overline{\mathcal{P}}_2(t) &= 0.3851669500 \times 10^{-31} (0.1621127956 \times 10^{583} t^3 + 0.1491645111 \times 10^{582} t + 0.6997748561 \times 10^{582} t^2 + 0.8444468165 \times 10^{582} t^5 + 0.1252710479 \times 10^{581} + 0.1858263849 \times 10^{583} t^4) \\
\overline{T}(t) &= 0.1265532998 \times 10^{551} t^3 + 0.1372217100 \times 10^{551} t^4 + 0.1260572385 \times 10^{549} + 0.1356321818 \times 10^{550} t + 0.5851539780 \times 10^{550} t^2 + 0.5967959572 \times 10^{550} t^5
\end{align*}
\]

We note that the algorithm did no require perturbing \( \overline{H}_{d-2}^* \).

\[\Box\]

**Example 5.2.** Let \( \epsilon = 0.004 \) and \( \mathcal{C} \) the curve of proper degree 6 defined by the polynomial (see Fig. 5):
We get the $\epsilon$-singular locus $\text{Sing}_\epsilon(\mathcal{C}) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ where

$$\mathcal{S}_1 = \{Q_1 = (-1.994232333, 1.00504304), Q_2 = (-2.000005299 + 0.005645280797i, -1.000026945 - 0.0002822677587i), Q_3 = (-2.000014217 + 0.004619269427i, 1.000004775 - 0.003559494332i), Q_4 = (-2.003547061, -1.006293429), Q_5 = (-2.005740475, 0.994897497), Q_6 = (-1.996418580, -0.9936748962), Q_7 = (-2.000014217 .004619269427i, 1.000004775 + 0.003559494332i), Q_8 = (-2.000005299 - 0.005645280797i, -1.000026945 + 0.002822677587i), Q_9 = (1.000036272 + 0.008596901071i, 2.000017052 - 0.00305992359i), Q_{10} = (5.999999669, -2.999998564), Q_{11} = (1.000036272 - 0.008596901071i, 2.000017052 + 0.00305992359i), Q_{12} = (0.9978910941, 1.994329680), Q_{13} = (1.002094534, 2.005650021), \}

$$

$$\mathcal{S}_2 = \{Q_{14} = (-2.0000000001, 1.), Q_{15} = (-2., -1.000000005), Q_{16} = (1., 2.)\}, \mathcal{S}_3 = \emptyset.$$

The singular cluster decomposition is (see Fig. 4 Left):
$$\text{Cluster}_2(Q_{10}) = \{Q_{10}\}, \text{Cluster}_3(Q_{14}) = \{Q_1, Q_3, Q_5, Q_7, Q_{11}\}, \text{Cluster}_3(Q_{15}) = \{Q_2, Q_4, Q_6, Q_8, Q_{15}\}, \text{ and Cluster}_3(Q_{16}) = \{Q_9, Q_{11}, Q_{12}, Q_{13}, Q_{16}\}.$$ We observe that $\mathcal{C}$ is $\epsilon$-rational. In Step 4 we obtain three $\epsilon$-ramification points: $P_1 = (-1.330235522, 0.9268173641), P_2 = (-1.979908167, 0.02661222172),$ and $P_3 = ...
We note that these points are not in the cluster of each other and they are not in the clusters of the $\epsilon$-singularities (see Fig. 4 Right).

The algorithm outputs the parametrization $\overline{P}(t) = (\overline{p}_1(t), \overline{p}_2(t))$ where (see Fig. 5 to compare the input and the output curves):

\[
\overline{p}_1(t) = -0.2992985374 \times 10^{-13} (-0.4498780650 \times 10^{665} t^2 - 0.1104625259 \times 10^{666} t + 0.3823432112 \times 10^{665} t^3 - 0.4349945664 \times 10^{664} t^4 - 0.8977530140 \times 10^{665} t^5 + 0.2445532363 \times 10^{662} t^6 + 0.1487140379 \times 10^{664} t^7)
\]

\[
\overline{p}_2(t) = -0.5410017657 \times 10^{-14} (0.4254697372 \times 10^{662} t^5 - 0.4450231957 \times 10^{665} t^4 - 0.8325944623 \times 10^{665} t^2 + 0.3137371087 \times 10^{665} t - 0.1036628422 \times 10^{663} t^5 - 0.7148994612 \times 10^{664} t^4 + 0.1592853294 \times 10^{666})
\]
$$\overline{t}(t) = 0.3388387927 \cdot 10^{649} t^5 + 0.1633128101 \cdot 10^{651} t^2 + 0.3082510569 \cdot 10^{648} t^6 + 0.4492917291 \cdot 10^{651} t + 0.1270749205 \cdot 10^{650} t^4 + 0.3531547270 \cdot 10^{650} t^3 + 0.4139801407 \cdot 10^{651}.$$  

We note that the algorithm did no require perturbing $$\overline{t}^{*}_{d-2}$$. \hfill \Box

**Example 5.3.** Let us consider $$\epsilon = 0.001$$ and the curve $$C$$ of proper degree 7 defined by the polynomial (see Fig[5]):

$$f(x, y) = 0.005242164122x + 0.0000006109092905 + 0.4234041949y - 0.05219720755xy^4 - 0.1626221914x^2y^2 - 0.006150324474x^5y^2 - 0.009115378696y^3 + 0.01468749412x^3 - 0.17265929757y^4 - 0.0517878117x^5 + 0.00061029838126y^6 + 0.7056394692y^2 + 0.007271579029x^4 - 0.00904345878x^3y^4 + 0.02810421594y^3 + 0.01517536020x^4y^3 - 0.0333531981xy^3 + 0.07030423460x^3y^3 + 0.999999999y^3x^2 + 0.023960264475x^3y + 0.0635929287x^2y^3 + 0.0006102983812xy^5 + 0.06037915453x^4y - 0.05961614786x^2y^2 + 0.1735938027x^2y^2 + 0.00967336920xy - 0.1183989159x^3y - 0.3997312415x^2y - 0.0150464143x^2y^4 - 0.0002034043985x^7 + 0.007152781730x^6y + 0.009647777670x^6y^6 - 0.01996027092x^2 - 0.001858780227x^6 - 0.008636725103x^2y^5 - 0.002554427076y^7.$$  

The $$\epsilon$$-singular locus is Sing$$\epsilon(C) = S_1 \cup S_2 \cup S_3$$ where

$$S_1 = \{ Q_1 = (4.998181206 + 0.0004639222080i, 6.997094116 - 0.0003357295061i), \quad Q_2 = (4.998181206 - 0.0004639222080i, 6.997094116 + 0.0003357295061i), \quad Q_3 = (5.001816967 + 0.000435470406i, 7.002902744 - 0.0003352187676i), \quad Q_4 = (0.9999998537, -3.000000118), \quad Q_5 = (5.001816967 - 0.0004635470406i, 7.002902744 + 0.0003352187676i), \quad Q_6 = (-2.000211362 + 0.00008683312445i, -0.001218550314 - 0.9986341029i), \quad Q_7 = (-2.000211362 - 0.00008683312445i, -0.001218550314 + 0.9986341029i), \quad Q_8 = (1.998594026 + 0.000145301485i, -0.0005005279912 - 0.9994646423i), \quad Q_9 = (-1.998594026 - 0.000145301485i, -0.0005005279912 + 0.9994646423i), \quad Q_{10} = (1.000000133, -0.000000553958512), \quad Q_{11} = (-2.001405450 - 0.0001446569643i, 0.0050000190394 - 1.000535463i), \quad Q_{12} = (-2.001405450 + 0.0001446569643i, 0.0050000190394 + 1.000535463i), \quad Q_{13} = (-1.999787395 - 0.00008416274464i, 0.001216340837 - 1.001365249i), \quad Q_{14} = (-1.999787395 + 0.00008416274464i, 0.001216340837 + 1.001365249i), \quad Q_{15} = (4.997608917 - 0.001980994691i, 1.999734804 + 0.002346469993), \quad Q_{16} = (4.997608917 + 0.001980994691i, 1.999734804 - 0.002346469993), \quad Q_{17} = (-3.999997183, 1.999998082), \quad Q_{18} = (5.002393988 - 0.001973712849i, 2.000267815 - 0.002341789270i), \quad Q_{19} = (5.002393988 + 0.001973712849i, 2.000267815 + 0.002341789270i), \quad S_2 = \{ Q_{20} = (-2.000000398 - 0.0000003109941563i, -0.0000005243819124 - 0.9999997083i), \quad Q_{21} = (-2.000000398 + 0.0000003109941563i, -0.0000005243819124 + 0.9999997083i), \quad Q_{22} = (5.000000495, 2.000000179), Q_{23} = (4.999999480, 6.999999337) \}, \quad S_3 = \emptyset.$$
The cluster decomposition of the singular locus is (see Fig. 4, Left):

\[ \text{Cluster}_2(Q_4) = \{Q_4\}, \text{Cluster}_2(Q_{10}) = \{Q_{10}\}, \text{Cluster}_2(Q_{17}) = \{Q_{17}\}, \]

\[ \text{Cluster}_3(Q_{23}) = \{Q_1, Q_2, Q_3, Q_5, Q_{23}\}, \text{Cluster}_3(Q_{20}) = \{Q_6, Q_8, Q_{11}, Q_{18}, Q_{20}\}, \]

\[ \text{Cluster}_3(Q_{21}) = \{Q_7, Q_9, Q_{12}, Q_{14}, Q_{21}\}, \text{Cluster}_3(Q_{22}) = \{Q_{15}, Q_{16}, Q_{18}, Q_{19}, Q_{22}\}. \]

We observe that \( C \) is \( \epsilon \)-rational. Now, we obtain four \( \epsilon \)-ramification points:

\[ P_1 = (-2.972405737, -7.933174980), P_2 = (23.79950366, 17.84891277), P_3 = (-10.06218879, 1.300686562) \] and \( P_4 = (24.47385001, 17.37936091) \). We note that these points are not in the cluster of each other and they are not in the clusters of the \( \epsilon \)-singularities. Finally, the algorithm outputs a parametrization (for space limitation we do not include it here). In Fig. 6 we plot the input and the input curve.

\[ \text{Figure 6: Left: Input (in dots) and output curve in Example 5.3. Right: A zoom at } (-4, 2) \]

### 5.1 Empirical Analysis of Error

The aim of this section is to analyze empirically the performance of the algorithm proposed. A good performance would mean to obtain an output curve \( \overline{C} \) close to the input curve \( C \) and by close we mean that \( C \) is contained in the offset region of \( \overline{C} \) at a small distance and viceversa (see [17]). To estimate the distance, between the curves \( C \) and \( \overline{C} \), we designed the next method (see Fig. 7 Left):

1. We randomly generate a set \( E \) of (affine) \( \epsilon \)-points on the input curve \( C \) as follows. Fix real numbers \( a, b \), and take \( n \) random integer values \( \alpha_i \in [a, b] \), \( i = 1, \ldots, n \). Let \( E_1 \) be the set of intersection points of the curve \( C \) with the lines \( x = \alpha_i \). We also take \( n \) random integer values \( \beta_i \in [a, b] \) and we intersect the curve \( C \) with the lines \( y = \alpha_i \) to obtain the set of points \( E_2 \). We set \( E = E_1 \cup E_2 \).

2. Let \( r \) be a positive integer and \( \Theta_r = \{ k \pi r^{-1} \mid k = 1, \ldots, r \} \). For each \( \epsilon \)-point \( P \) in \( E \) and each \( \theta \in \Theta_r \) let \( L_{P, \theta} \) be the line through \( P \) in the direction of \( (\cos(\theta), \sin(\theta)) \).
3. For each $P \in \mathcal{E}$ we compute $d_{P,\overline{C}} = \min\{\|P - Q\|_2 \mid Q \in \bigcup_{\theta \in \Theta_s} (L_{P,\theta} \cap \overline{C})\}$.

4. Let $\mathcal{D} = \{d_{P,\overline{C}} \mid P \in \mathcal{E}\}$. We compute the mean value $\mu$ of the elements of $\mathcal{D}$ as well as the statistical standard error $\rho$. We can say that the estimated distance is, in average, in the interval $[\mu - 1.96 \rho, \mu + 1.96 \rho]$.

Given the curve $\mathcal{C}$ of degree $d = 4$ defined by the irreducible polynomial

$$f(x, y) = 1.000065y^2 + 1.00000028y^3 + y^4 + 1.000065xy - 11.49999972xy^2$$
$$+xy^3 + 0.760065x^2 + 5.74000028x^2y + 3.69x^2y^2 - 0.75999972x^3 - 3.12x^3y$$
$$+0.19x^4 + 0.01x + 0.01y. \quad (1)$$

For $\epsilon = 0.01$ the algorithm concluded that $\mathcal{C}$ is $\epsilon$-rational and returned a rational parametrization $\mathcal{P}(t) = (\overline{p}_1(t), \overline{p}_2(t))$ which corresponds to the rational curve $\overline{\mathcal{C}}$ with implicit equation

$$\overline{f}(x, y) = 0.01642553x^4 + 0.06494377x^2 + 0.08804654y^2 - 0.06552116x^3$$
$$+0.08169391y^3 + 0.091025077xy + 0.49976135x^2y - 0.99999999xy^2 + 0.08645018y^4$$
$$+0.31900118x^2y^2 - 0.26972458x^3y + 0.08645019xy^3 - 0.00001398 + 0.00078781x$$
$$+0.0007408y.$$
Until now we have empirically measured the distance in the examples used in paper. We describe next a different experiment. We randomly generate a set of curves and for each curve we estimate its distance to the output curve given by our algorithm. Our experiments are satisfactory and allow us to think about a theoretical treatment of this fact as a future project. We explain next how the family of curves was constructed.

We fix three points \( P_1 = (2 : 0 : 1), P_2 = (0 : 0 : 1) \) and \( P_3 = (1 : 1 : 1) \) in \( \mathbb{P}^2(\mathbb{C}) \). We consider the linear system of curves of degree 4 defined by the divisor \( 2P_1 + 2P_2 + 2P_3 \). Its defining polynomial is

\[
G(x, y, z) = u_2 y^2 z^2 + u_3 y^3 z + u_4 x y^4 + u_5 x y z^2 + (-2 u_2 - 3 u_3) x y z + (u_6 x y^2 + 1/2 u_5 - 2 u_6) x y^2 z + u_6 x y^3 + u_1 x^2 z^2 + (-3/2 u_5 + 2 u_3) x^2 y z + (u_2 + u_5 + 1/2 u_5 + 1/4 u_1 + u_4) x^2 y^2 \\
- u_1 x^3 z + (1/2 u_5 - 3 u_4 - u_6 + 1/2 u_1) x^3 y + 1/4 u_1 x^4
\]

(2)

For \( j = 1, \ldots, 6 \) and \( i = 1, \ldots, 10 \) let \( r_{ij} \) be a random integer number in the interval \([0, 100]\). We obtain 60 different polynomials \( G_{ij}(x, y, z), j = 1, \ldots, 6, i = 1, \ldots, 10 \) setting

\[
u_k = \begin{cases} 
\frac{r_{ij}}{100} & \text{if } k = j \\
1 & \text{if } k \neq j
\end{cases} \quad k = 1, \ldots, 6
\]

in equation \( (2) \). Given \( i \in \{1, \ldots, 6\} \) and \( j \in \{1, \ldots, 10\} \) we obtain a random perturbation \( g_{ij}(x, y) \in \mathbb{R}[x, y] \) of \( G_{ij}(x, y, 1) \) as follows

\[
g_{ij}(x, y) = G_{ij}(x, y, 1) + \epsilon \frac{r_1}{100}(x + y) + \epsilon^2 \frac{r_2}{100}(x^2 + 2 x y + y^2) + \epsilon^3 \frac{r_3}{100}(x^3 + x^2 y + xy^2 + y^3)
\]

where \( r_1, r_2, r_3 \) are integer numbers taken randomly in the interval \([0, 100]\) and \( \epsilon = 0.01 \). The polynomials \( g_{ij}(x, y), j = 1, \ldots, 6, i = 1, \ldots, 10 \) have proper degree 4 and define 60 curves \( C_{ij} \) verifying \((1 : 0 : 0), (0 : 1 : 0) \notin C_{ij} \) and without (exact) singularities at infinity. Using our algorithm we conclude that 28 of the 60 curves are \( \epsilon \)-rational. We show those curves in Fig. 7 Right. The implicit equation of the curve \( C_{11} \) is equation \( (1) \).

With the notation used in the method described above to compute the experimental distance, we take \([a, b] = [-100, 100]\) and \( n = 15 \) so that the number of points used to compute the distance is expected to be \(|E| = 120\). Set the number of lines going through each point equal \( r = 10 \). We compute \( \mu, \rho \) and \( I_{\mu, \rho} = [\mu - 1.96 \rho, \mu + 1.96 \rho] \) for each one of the 28 \( \epsilon \)-rational curves to obtain the next table.
Figure 7: Left: Illustration of the method. Right: The 28 $\epsilon$-rational curves $C_{ij}$ randomly generated.

| $\mu$   | $\rho$  | $I_{\mu,\rho}$         | $\mu$   | $\rho$  | $I_{\mu,\rho}$         |
|---------|---------|-------------------------|---------|---------|-------------------------|
| 0.007541| 0.000855| [0.005866, 0.009217]    | 0.006807| 0.000385| [0.006051, 0.007563]    |
| 0.006977| 0.001184| [0.004656, 0.009299]    | 0.100902| 0.013253| [0.074926, 0.126879]    |
| 0.006977| 0.001184| [0.004656, 0.009299]    | 0.003049| 0.000254| [0.002551, 0.000355]    |
| 0.003577| 0.000503| [0.002592, 0.004563]    | 0.003924| 0.000212| [0.003508, 0.004339]    |
| 0.004011| 0.000553| [0.002928, 0.005094]    | 0.003995| 0.000549| [0.002919, 0.005072]    |
| 0.006007| 0.000808| [0.004423, 0.007590]    | 0.008330| 0.000806| [0.006749, 0.009911]    |
| 0.004239| 0.000844| [0.002584, 0.005894]    | 0.005638| 0.000536| [0.004586, 0.006690]    |
| 0.005758| 0.000585| [0.004610, 0.006905]    | 0.003020| 0.000316| [0.002399, 0.003639]    |
| 0.002882| 0.000224| [0.002442, 0.003322]    | 0.000854| 0.000091| [0.000677, 0.001032]    |
| 0.005477| 0.000756| [0.003996, 0.006958]    | 0.004077| 0.000027| [0.003540, 0.004614]    |
| 0.003123| 0.000437| [0.002266, 0.003979]    | 0.004077| 0.000274| [0.003540, 0.004614]    |
| 0.004752| 0.000359| [0.004049, 0.005455]    | 0.035130| 0.000520| [0.024898, 0.045361]    |
| 0.001453| 0.000148| [0.001163, 0.001744]    | 0.006209| 0.000619| [0.004996, 0.007423]    |
| 0.004956| 0.007123| [0.035599, 0.063522]    | 0.013406| 0.001179| [0.011094, 0.015718]    |
| 0.001049| 0.000113| [0.000827, 0.001272]    | 0.009037| 0.000687| [0.007691, 0.010385]    |

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