NEW RESULTS ON HEAVY QUARKS NEAR THRESHOLD

M. BENEKE

Theory Division, CERN, CH-1211 Geneva, Switzerland

We review in brief the threshold expansion, a method to perform the expansion of Feynman integrals near the heavy quark-antiquark threshold, and its relation to the construction of two effective theories, non-relativistic QCD (NRQCD) and potential-NRQCD. We then summarize recent next-to-next-to-leading order results on the decay $J/\Psi \rightarrow l^+l^-$, the bottom quark mass from $\Upsilon$ sum rules and the top-anti-top production cross section near threshold in $e^+e^-$ collisions.

1 Motivation

In this talk we discuss systems of the form $QQ + X$, where $Q$ is a heavy quark of mass $m$ and $X$ a collection of massless particles, in the kinematic region, where the total invariant mass $q^2$ of the system is close to $4m^2$. The physics of such systems is characterized by the fact that in the reference frame with $\vec{q} = 0$ the heavy quark velocities are small, i.e. $v \ll 1$.

The threshold region is evidently sensitive to the mass of the quark and physical quantities that probe the threshold region can therefore be used to determine the bottom and top quark mass. In case of bottom quarks the threshold region is populated by narrow Upsilon resonances and the cross section $e^+e^- \rightarrow b\bar{b} + X$ cannot be predicted locally. However, dispersion relations equate an average over Upsilon resonances and the continuum to the derivatives of $b$-quark current correlation function at $q^2 = 0$, which can be calculated in perturbation theory\cite{1,2,3}. In case of top quarks the rapid decay of the top prohibits the formation of long-lived resonances\cite{4,5,6}. For $m_t = 175$ GeV a resonance-like structure remains visible in the energy dependence of the cross section $e^+ e^- \rightarrow t\bar{t} + X$\cite{7,8,9}. Perturbation theory can still be applied in the vicinity of the ‘pseudo-resonance’, but not arbitrarily close to threshold. The shape of the $t\bar{t}$ cross section near threshold, to be measured at a Next Linear Collider, is believed to provide us with the most accurate determination of the top quark mass, if the strong interaction corrections are indeed well understood.

In addition to quark masses as parameters of the standard model, there are more intrinsically QCD-related problems involving heavy quarks near threshold. Heavy quarkonia are non-relativistic and their production and decay can be treated as an expansion in $v$. One can also consider heavy quark production in hadron-hadron collisions. In this case one is mainly concerned with the resummation of logarithms of $v$ that arise as a consequence of soft and collinear gluon emission, predominantly from the massless initial state particles. This last application will not be discussed here (see, for instance, Refs.\cite{10}). Incidentally, we note that at

\begin{itemize}
  \item Talk presented at the XXXIIIrd Rencontres de Moriond ‘Electroweak Interactions and Unified Theories’, 14-21 March 1998, Les Arcs, France. Compared to the presentation at the meeting, this write-up has been updated to account for results published after the date of the conference.
\end{itemize}
Tevatron centre-of-mass energies the average velocity squared of top quarks is \( \langle v^2 \rangle \sim 1/2 \) and the non-relativistic dynamics discussed subsequently is probably not important there.

A non-relativistic system involves more than one momentum scale, related to the small parameter \( v \). The first part of the problem consists of organizing the calculation in a systematic expansion in \( v \) and has been addressed in several papers recently \cite{10-17}, especially in the context of dimensional regularization. These works clarify the definition of non-relativistic QCD (NRQCD)\cite{18-20} with a matching prescription based on dimensional regularization and show that one can proceed to a second effective theory by integrating out more scales. This development is summarized in the first part of the talk.

The second part summarizes recent next-to-next-to-leading order (NNLO) results on the phenomenological applications mentioned above. ‘NNLO’ in this context refers either to fixed-order calculations in \( \alpha_s \)\cite{21-24}, required for matching effective theories, or the resummation of Coulomb-enhanced corrections to all orders \cite{25-31}, required for the solution of the low-energy problem.

## 2 Complications near the threshold

The need for resummation arises, because even when \( \alpha_s \) is small the effective interaction between a heavy quark and anti-quark becomes strong at small relative velocity. The exchange of a Coulomb gluon – the 00-component of the propagator for a gluon with momentum of order \( mv \) and energy of order \( mv^2 \) – leads to an effective coupling of order \( \pi \alpha_s (mv^2)/v \). Resummation leads to weakly coupled bound states (\( E_{\text{bind}} \sim mv^2 \ll m \)) – the Coulomb bound states analogous to hydrogen and positronium. In first approximation the physics is indeed exactly as in QED. However, due to massless quarks and gluons, the coupling runs below the scale \( m \) in QCD. This leads to complications, and differences in the power counting, when \( mv^2 \sim \Lambda_{\text{QCD}} \).

Defining \( v = (1 - 4m^2/q^2)^{1/2} \), the cross section \( e^+e^- \rightarrow Q\bar{Q} + X \) can be expanded in a double series in \( \alpha_s \) and \( v \):

\[
R_{Q\bar{Q}} = v \sum_{k=0}^{\infty} \sum_{l=-k}^{k} c_{kl} \alpha_s^k v^l \times \text{logs of } v. \tag{1}
\]

A NNLO calculation in the kinematic region where \( \alpha_s \sim v \) has to account for all terms with \( k + l \leq 2 \). Powers of \( v \) arise from ratios of momentum scales. We have to disentangle the contributions from the different scales in order to be sure that for a high-order loop graph, which cannot be calculated exactly, we have taken into account all terms with \( k + l \leq 2 \).

Eventually we will be led to calculating diagrams with Coulomb Green functions rather than free Green functions. There are, however, two 2-loop calculations in conventional perturbation theory that enter at NNLO: (a) The 2-loop correction to the Coulomb potential, i.e. \( a_2 \) in

\[
V(\vec{q}) = -\frac{4\pi\alpha_s}{q^2} \sum_k a_k \alpha_s^k. \tag{2}
\]

It has been calculated in Ref.\cite{21}. (b) The 2-loop correction to the matching of the vector heavy quark current in QCD to its non-relativistic analogue, i.e. \( c_2 \) in

\[
\bar{Q}\gamma_i Q = \left( \sum_k c_k \alpha_s^k \right) \psi^i \sigma_i \chi + \ldots, \tag{3}
\]

where \( \psi \) and \( \chi \) are 2-spinors. Only the spatial component is needed at NNLO. The coefficient \( c_2 \) has been calculated in Refs.\cite{23,24}. Note that the series expansion of the Coulomb potential...
is uniquely specified (at least to order $\alpha_s^2$) by choosing a scheme for coupling renormalization. But the non-relativistic current is not conserved and one has to choose a factorization scheme to define the $c_k$. In Refs. 23, 24 the $\overline{\text{MS}}$ scheme is used.

3 Scales and their Separation

We now discuss how to construct the expansion (1), first at a given loop order $k$, then including resummation.

3.1 Threshold expansion of Feynman integrals

Take a loop integral that contributes to the heavy quark cross section. To be precise, we consider the two-point function of the current (3) and take the imaginary part at the end. This avoids calculating the threshold expansion of phase space integrals. The threshold expansion is a method to calculate the expansion in $v$ without calculating the full integral. Inspection of the denominators of the Feynman integrand shows that there are four essential regions of loop momentum:

- hard (h): $l_0 \sim m, \vec{l} \sim m$,
- soft (s): $l_0 \sim mv, \vec{l} \sim mv$,
- potential (p): $l_0 \sim mv^2, \vec{l} \sim mv$,
- ultrasoft (us): $l_0 \sim mv^2, \vec{l} \sim mv^2$.

The threshold expansion is constructed by writing the diagram as a sum of terms that follow by dividing each loop momentum integration into these four regions. More precisely, one has to account for the possibility that for example $l_1$ and $l_2$ are hard, but their sum is not. That is, one should sum over one-particle-irreducible subgraphs. Note that the division is done implicitly, through the expansions of the propagators. No explicit cut-offs are needed. The soft region has often been omitted in the discussion of non-relativistic dynamics, probably because it gives rise only to instantaneous potentials as we discuss below. However, it also gives rise to the (standard) evolution of the strong coupling between the scales $m$ and $mv$, as can be seen from the fact that a light quark loop inserted into a potential gluon line can only be hard or soft.

Then, in each term, the propagators can be simplified. The propagator of a heavy quark with momentum $(q/2 + l_0, \vec{p} + \vec{l})$ is

$$\frac{1}{l_0^2 - \vec{l}^2 - 2\vec{p} \cdot \vec{l} - \vec{p}^2 + ql_0 + y + i\epsilon},$$

and we will assume that $\vec{p}$ scales as $mv$ and $y = q^2/4 - m^2$ as $mv^2$. When $l$ is hard, we expand the terms involving $\vec{p}$ and $y$ and the leading term in the expansion scales as $v^0$. When $l$ is soft, the term $ql_0$ is largest and the remaining ones are expanded. The propagator becomes static and scales as $v^{-1}$. When $l$ is potential, the propagator takes its usual form, when the gluon line is soft and ultrasoft and scales as $v^{-2}$ and $v^{-4}$, respectively. If the gluon momentum is potential, one can expand $l_0^2$ and the interaction becomes instantaneous. If we add the scaling rules for the loop integration measure, $d^4l \sim 1(h), v^4(s), v^5(p), v^6(\text{us})$, we can immediately estimate the size of the leading term from a given region. Because all terms that are small in a given region are expanded in the Feynman integrand, each term in the resulting sum contributes only to a single power of $v$. 
(multiplied, in general, by powers of logarithms of $v$). It is important that the power behaviour can be determined before calculating. It is also important that the integrals that appear in each term of the expansion are much easier to calculate than the original integral, because they are homogeneous in $v$, that is, contain only a single scale. In particular, any term in the expansion of massive 2-loop 3-point integrals is calculable \cite{[15,24]}

The diagrammatic rules for the expansion of propagators (and vertices) can be considered as following from an effective Lagrangian. Because different loop momentum regions do not overlap, one can introduce separate fields for hard, potential, etc. quarks and gluons. (For the coupling of potential quarks to ultrasoft gluons, the threshold expansion entails the multipole expansion.) This has been done in Ref.\cite{[17]} and, in part, in Refs.\cite{[11,13]}. One can then think of integrating out first the hard fields, then soft fields and potential gluons.

One can use Coulomb gauge or a covariant gauge. (In covariant gauge the ghost fields are treated like other massless fields. The can be hard, soft, potential and ultrasoft.) The hard region is very inconvenient to calculate in Coulomb gauge. But Coulomb gauge makes gauge cancellations manifest that occur in the coupling of soft and ultrasoft gluons to heavy quarks. In fact, one can use different gauges for different regions with one exception: The distinction between soft and potential gluons is not gauge-invariant. This should be no surprise. When one integrates out soft gluons alone, one has to compute graphs with external potential gluons. But potential gluons are off their mass shell. On the other hand, when one integrates out soft and potential gluons together, only on-shell graphs have to be considered.

### 3.2 NRQCD

First we integrate out the hard region. This means that we assume that all loop momenta are hard and the external momenta soft, potential or ultrasoft. That is, we Taylor-expand the integrand in the external momenta of the graph. The result is obviously polynomial in the external momenta and can therefore be written as a local operator. This yields the non-relativistic QCD (NRQCD) Lagrangian \cite{[18,20]}

$$
\mathcal{L}_{\text{NRQCD}} = \psi^\dagger \left( i D^0 + \frac{D^2}{2m} \right) \psi + \frac{1}{8m^2} \psi^\dagger D^\dagger \psi - \frac{g_s}{2m} \psi^\dagger \bar{\sigma} \cdot \vec{B} \psi \\
- \frac{g_s}{8m^2} \psi^\dagger \left( \vec{D} \cdot \vec{E} - \vec{E} \cdot \vec{D} \right) \psi - \frac{ig_s}{8m^2} \psi^\dagger \bar{\sigma} \cdot \left( \vec{D} \times \vec{E} - \vec{E} \times \vec{D} \right) \psi \\
+ \text{antiquark terms} + \mathcal{L}_{\text{light}}
$$

and an expansion of QCD operators (such as the vector current) in terms of non-relativistic fields. We have written down only those terms of the Lagrangian that are needed for the NNLO calculations described below, provided one treats the difference between $\alpha_s(m)$ and $\alpha_s(m v^2)$ as small. Otherwise one should use the renormalization group to sum up the leading logarithms sensitive to the scale $m v^2$ in the coefficient functions.

Note that the threshold expansion provides a matching prescription, in which one does not have to calculate NRQCD diagrams explicitly. The QCD diagram is already broken up into the contributions from different scales and the hard regions are exactly the relativistic effects which are contained in the coefficient functions of the effective Lagrangian. The matching prescription is very simple. Despite the fact that the QCD diagrams are divergent, when the relative momentum of the heavy quarks goes to zero, the matching coefficients are given by the Taylor expansion around zero relative momentum. For the single-heavy quark sector, this matching prescription coincides with that of Ref.\cite{[22]}. 

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Note also that perturbative calculations with the NRQCD Lagrangian have up to now been mainly done with a cutoff regularization, either in QED or lattice QCD, while we assume dimensional regularization here. The change is not quite trivial. For, if we calculated a NRQCD integral according to the Feynman rules of its Lagrangian, we would obtain an incorrect result, because dimensional regularization treats the UV cutoff of NRQCD as larger than \( m \). We can use dimensional regularization provided we expand the integrand before integration. The precise prescription is again specified by the threshold expansion. The NRQCD integral is given by expanding the integrand according to the rules for the soft, potential and ultrasoft region. Indeed, since the hard region is accounted for by the coefficient functions, this reproduces the original QCD diagram.

The interaction terms in NRQCD do not have a definite scaling in \( v \), because a gluon field can be soft, potential or ultrasoft, and a quark field can be soft or potential and different scaling rules apply to each of these.

### 3.3 Potential NRQCD

If the scale \( mv \) is already non-perturbative, we stop with NRQCD. If it is not, the threshold expansion suggests that one can integrate out the soft region together with potential gluons to arrive at another effective Lagrangian. We call the effective theory potential NRQCD (PNRQCD), following Refs. 14, 16, where the corresponding tree-level Lagrangian was considered first. We cannot integrate out potential quarks, because the Green functions relevant to our applications have external potential quark lines.

To integrate out the soft contributions we consider (NRQCD) graphs in which all momenta are soft and the external momenta potential or ultrasoft. The graph is Taylor-expanded in the external ultrasoft momenta and in the zero components of external potential momenta. But it cannot be expanded in the spatial components of potential external momenta, because they are not small compared to the spatial components of the loop momenta. Hence the result is non-polynomial in the spatial components of the potential momenta. This is an instantaneous, but spatially non-local interaction. We call such interactions potentials. Potential gluons have propagators with \( l_0^2 \) expanded. They also give rise to potentials.

PNRQCD contains only potential quark fields and ultrasoft gluon (massless quark, ghost) fields. NRQCD is matched on PNRQCD order by order in \( \alpha_s \). Consider quark-anti-quark scattering at small relative momentum. At leading order in \( \alpha_s \) the quark and antiquark interact by the exchange of a potential gluon. The leading term in \( v \) yields the Coulomb potential at order \( \alpha_s \). The corresponding non-local operator is

\[
\int d^3 \vec{r} \left[ \psi^\dagger T^A \psi \right] (x + \vec{r}) \left( -\frac{\alpha_s}{r} \right) \left[ \chi^\dagger T^A \chi \right] (x). \tag{7}
\]

The corrections of order \( v^2 \) are known as the Breit potential. At order \( \alpha_s^2 \) one has to compute the soft and potential contributions to the 1-loop NRQCD graphs and subtract the PNRQCD graphs constructed from the order-\( \alpha_s \) potentials in the PNRQCD Lagrangian. The soft contributions to NRQCD graphs have no analogue in PNRQCD and renormalize the PNRQCD interactions. The 1-loop correction to the Coulomb potential is generated in this way together with a potential of form \( \alpha_s^2/(mr^2) \) and higher order terms in \( v \). For the potential contributions to NRQCD graphs it is necessary to perform an explicit matching to avoid double counting. As mentioned above, the contributions from soft and potential gluons may look different in different gauges. But their sum and hence the PNRQCD Lagrangian is gauge-invariant.

In general the PNRQCD Lagrangian can be written as

\[
\mathcal{L}_{\text{PNRQCD}} = \mathcal{L}'_{\text{NRQCD}} + \mathcal{L}_{\text{non-local}}, \tag{8}
\]
where $\mathcal{L}_{\text{non-local}}$ collects all non-local interactions. The local interactions are exactly those of NRQCD, but the interpretation is different, because only ultrasoft gluons are left over. In loop graphs constructed from $\mathcal{L}'_{\text{NRQCD}}$ the gluon propagators are always expanded according to their ultrasoft scaling rule, while in loop graphs constructed from $\mathcal{L}_{\text{NRQCD}}$ gluons are also soft and potential. The prime reminds us of this difference.

Because only potential quarks and ultrasoft gluons are left in PNRQCD, the interaction terms have definite scaling rules. They agree with those given in Refs. [4–13]. Note that the NRQCD scaling rules of Ref. [2] are really those of PNRQCD. A potential quark propagator in coordinate space scales as $v^3$, so a quark field in NRQCD scales as $v^{3/2}$. Comparing the scaling of $\psi^\dagger \partial^0 \psi$ with the scaling of $[\bar{\psi}, \chi]$, we find that the former scales as $v^5$ and the latter scales as $\alpha_s v^4$. As is well-known, the Coulomb interaction cannot be treated as a perturbation when $v \sim \alpha_s(mv)$. Note, however, that the matching on PNRQCD can be done by treating the Coulomb interaction as a perturbation. The unperturbed PNRQCD Lagrangian is

$$\mathcal{L}^\dagger_{\text{NRQCD}} = \psi^\dagger \left( i\partial^0 + \frac{\vec{\partial}^2}{2m} \right) \psi + \chi^\dagger \left( i\partial^0 - \frac{\vec{\partial}^2}{2m} \right) \chi$$

$$+ \int d^3\vec{r} \left[ \psi^\dagger T^A \psi \right] (\vec{r}) \left(-\frac{\alpha_s}{r} \right) \left[ \chi^\dagger T^A \chi \right] (0) + \mathcal{L}_{\text{light}}^\dagger. \quad (9)$$

One can rewrite this in terms of a ‘tensor field’ $[\psi \otimes \chi^\dagger](\vec{R}, \vec{r})$ that depends on the cms and relative coordinates. The unperturbed Lagrangian describes free propagation (with mass $2m$) in the cms coordinate. The propagation of $[\psi \otimes \chi^\dagger](\vec{R}, \vec{r})$ in its relative coordinate is given by the Coulomb Green function of a particle with reduced mass $m/2$. In calculating diagrams with Coulomb Green functions, one sums corrections of order $(\alpha_s/v)^n$ to all orders. The remaining terms can be treated as perturbations in $v$ and $\alpha_s$ around the unperturbed Lagrangian. These are calculations familiar from QED bound state problems. What is new, in our opinion, is that we understand how to perform such calculations systematically in dimensional regularization, without double counting, and, if necessary, including retardation effects (graphs with ultrasoft lines).

When an ultrasoft gluon line with momentum $l$ connects to a quark line with loop momentum $k-l/2$ for the incoming and $k+l/2$ for the outgoing quark line, the threshold expansion instructs us to expand the quark-gluon vertex and quark propagator in $\vec{l}/\vec{k} \sim v$. All gluon interaction terms in $\mathcal{L}'_{\text{NRQCD}}$ should be understood as multipole-expanded, for instance

$$[\psi^\dagger \vec{A} \cdot \vec{\partial} \psi](x) \equiv \psi^\dagger(t, \vec{x}) A(t, 0) \cdot \vec{\partial} \psi(t, \vec{x}) + \psi^\dagger(t, \vec{x}) (\vec{x} \cdot \vec{\partial}) A(t, 0) \cdot \vec{\partial} \psi(t, \vec{x}) + \ldots, \quad (10)$$

and likewise for all other interactions.

Up to this point we have neglected the fact that in QCD – contrary to QED – the coupling constant evolves below the scale $m$. When $mv^2 \ll m$, but $\alpha_s(mv^2) \ll 1$, one can sum up logarithms of $v$, but otherwise the power counting remains unaffected. In particular, only the Coulomb interaction has to be treated non-perturbatively. When $mv^2 \sim \Lambda_{\text{QCD}}$ the situation changes, because ultrasoft gluons couple with unit strength, since $\alpha_s(mv^2) \sim 1$. The coupling to heavy quarks is still small, of order $v$ at least, but the self-coupling of gluons is unsuppressed. An ultrasoft gluon propagator in coordinate space scales as $v^4$, hence the gluon kinetic term scales as $v^8$. The three-gluon and four-gluon vertices also scale as $v^8$. When $\alpha_s(mv^2) \sim 1$, ultrasoft gluons enter the calculation of $R_{QQ}$ as a non-perturbative contribution of relative order $v^2$ and ‘retardation effects’ cannot be neglected at NNLO. A perturbative treatment of the problem cannot be extended beyond NLO, because the unperturbed PNRQCD Lagrangian must be modified to contain $\mathcal{L}_{\text{light}}$ rather than $\mathcal{L}_{\text{light}}^\dagger$. The unperturbed problem is then no longer
exactly solvable. This is why the energy spectra of charmonia and bottomonia are not exactly Coulomb-like.

4  $J/\psi \rightarrow l^+l^-$

The two-loop short-distance correction to the leptonic decay of an $S$-wave quarkonium state such as $J/\psi$, $\psi'$ and $\Upsilon(nS)$ has been analyzed in Ref.\textsuperscript{[23]}. The decay rate can be expressed as

$$\Gamma(J/\psi \rightarrow l^+l^-) = \left( \sum_{k=0} c_k(\mu) \left( \frac{\alpha_s(m_c)}{\pi} \right)^k \right)^2 \frac{4\pi e_c^2 \alpha_s^2}{3M_{J/\psi}} \frac{12 |\Psi(0)|^2(\mu)}{M_{J/\psi}},$$

neglecting relativistic corrections which can be added systematically\textsuperscript{[7]} at the expense of further non-perturbative parameters in addition to the wave-function at the origin. The coefficients $c_k$ are those of (3) (up to a normalization) and $\Psi(0)$ is related to the vacuum-to-$J/\psi$ matrix element of the non-relativistic current. We have $c_0 = 1$, $c_1 = -\frac{2}{3} C_F^3\frac{3}{14}$, and the 2-loop coefficient in the \text{MS} factorization scheme reads\textsuperscript{[23]}

$$c_2(\mu) = C_F^2 \left\{ \pi^2 \left[ \frac{1}{6} \ln \left( \frac{m_Q^2}{\mu^2} \right) - \frac{79}{36} + \ln 2 \right] + \frac{23}{8} - \frac{\zeta(3)}{2} \right\}$$

$$+ C_F C_A \left\{ \pi^2 \left[ \frac{1}{4} \ln \left( \frac{m_Q^2}{\mu^2} \right) + \frac{89}{144} - \frac{5}{6} \ln 2 \right] - \frac{151}{72} - \frac{13\zeta(3)}{4} \right\} + \frac{11}{18} C_F T_F n_f,$$

The calculation amounts to picking up the hard contributions only of the 2-loop three-point integrals contributing to $c + \bar{c} \rightarrow \gamma\gamma$. Numerically

$$\sum_{k=0} c_k(\mu) \left( \frac{\alpha_s(m_c)}{\pi} \right)^k = 1 - \frac{8\alpha_s(m_c)}{3\pi} - \left[ 44.55 - 0.41n_f - 25.59 \ln \frac{m_c}{\mu} \right] \left( \frac{\alpha_s}{\pi} \right)^2 + \ldots$$

Take $\mu = m_c$ for the factorization scale. Before squaring the coefficient function, the 1-loop correction is $-25\%$, but the 2-loop correction amounts to $-50\%$. Even for bottomonium, the 2-loop correction is as large as the 1-loop correction.

At two loops the wavefunction at the origin becomes factorization scale and scheme-dependent. The anomalous dimension is very large. The leptonic width is an important observable with respect to tuning the parameters of potential models. The large scheme-dependence is a problem for potential models, because it is not clear which scheme the wavefunction at the origin in potential models corresponds to.

The above result suggests that the perturbative expansion is not reliable, so that perturbative factorization would not work quantitatively. But the large coefficients could also be the consequence of a ‘bad’ factorization scheme. We will know only once a second quarkonium decay such as $\eta_c \rightarrow \gamma\gamma$ is computed to second order. With present analytic methods this seems to be a challenging piece of work.

5  $t\bar{t}$ Production

5.1 The total cross section

The total cross section for $e^+e^- \rightarrow t\bar{t} + X$ has been calculated at NNLO in Refs.\textsuperscript{[26,27]} for the vector coupling of the $t\bar{t}$ pair. (The axial-vector contribution is a NNLO effect but has not been
considered in Refs. \cite{26,27}. This can be done by combining the hard matching coefficient \cite{12} with the integrals over Coulomb Green functions in dimensional regularization. In practice, Ref. \cite{26,27} used a different factorization scheme by comparing the resummed cross section with the cross section at the threshold at fixed order $\alpha_s^2$. The final result is independent of this choice.

The top quark is unstable on a scale comparable to the scale $m_t\alpha_s$ and finite width effects are essential in the threshold region. Following Ref. \cite{5}, the finite top quark width is taken into account by the substitution

$$E = \sqrt{s} - 2m_t \rightarrow E + i\Gamma_t$$

in the argument of the Coulomb Green function, where $E$ is the energy measured from the threshold, defined by twice the top quark pole mass. Note that $E$ depends on the renormalization convention for the top quark mass and is not a physical quantity, contrary to the cms energy $\sqrt{s}$. The prescription \cite{14} has been justified in Ref. \cite{5} at LO. It is probably not justified at NNLO. In general, one can expect that the finite width inhibits the radiation of ultrasoft gluons from top quarks but affects the potential interactions less strongly.

The result for the cross section in units of the point cross section as a function of $E$ is reproduced in Figure 1. We note that the NNLO calculation has an as large effect on the height of the peak as the NLO calculation. Furthermore both shift the location of the peak by somewhat less than a GeV. If the QCD corrections do not converge, this implies an uncertainty in $m_t$ of almost 500 MeV, with frustrating consequences for precision studies of the $t\bar{t}$ threshold at a Next Linear Collider.

5.2 Which mass?

There is reason to believe that the situation is not as bad. When one discusses uncertainties in quark masses in the range of a few hundred MeV, it is important to ask how sensitive a particular mass renormalization prescription is to long-distance QCD effects. One has to be particularly careful about this, if the physical process is intrinsically less long-distance sensitive than the mass renormalization convention one is about to use.
For the pole mass of a stable quark long-distance sensitivity causes an uncertainty of order $\Lambda_{\text{QCD}}^3$. The analysis of Refs. 35, 36 goes through unmodified for an unstable quark, when the pole mass is defined as the real part of the complex pole position in the top quark propagator. In perturbative calculations this long-distance sensitivity shows up as large radiative corrections, when the top quark pole mass as an input parameter is fixed as done above. It is advantageous to fix instead a mass renormalization scheme which is less sensitive to long-distances, provided one can show that the large corrections cancel then. (Effectively, this amounts to comparing a LO, NLO etc. calculation in terms of the pole mass at somewhat different input values of the pole mass. Because of the definition (14) of $E$, this leads to a shift in the horizontal scale of Figure 1, which compensates the shift in the LO, NLO and NNLO curves. It seems preferred to plot the cross section as a function of the physical parameter $\sqrt{s}$ rather than the scheme-dependent parameter $E$.)

Indeed the analysis of higher-order radiative corrections to the Coulomb potential reveals systematically large terms 38–40. In part, these large corrections are a consequence of long-distance sensitivity. The long-distance sensitive contributions to the Coulomb potential can be shown 41, 42 to cancel exactly against those in the pole mass renormalization. Contrary to intuition the threshold cross section is less sensitive to long distances than the pole mass and hence the threshold is not tied to twice the pole mass once we talk about accuracies of several hundred MeV. One can make this cancellation manifest 41 by subtracting the dangerously large terms from the potential and by adding them back to the pole mass. The result is a modified mass renormalization prescription, called the ‘potential subtracted mass’, which can be related to a more conventional definition like the $\overline{\text{MS}}$ mass by a reasonably well-behaved series, but which is at the same time not unphysically far away from the threshold like the $\overline{\text{MS}}$ mass (at the scale $m_t$) itself. It is of course possible to use other modified mass renormalization prescriptions, provided they also lead to a manifest cancellation of long-distance sensitive terms and can be computed to sufficient accuracy. One possible alternative is discussed in Ref. 43.

The discussion of the previous paragraph applies to a stable or unstable quark. Also, as mentioned, the top quark pole mass suffers from the same long-distance sensitivity as the bottom or charm quark pole mass despite the fact that the width of the top quark is significantly larger than $\Lambda_{\text{QCD}}$. Still, the width helps. The point is not that the top quark pole mass should be better behaved. The point is that contrary to bottom quarks, where one can find observables (such as the $B$ meson mass) which are as long-distance sensitive as the pole mass, there is no observable involving top quarks that would be as sensitive to long distances as the top quark pole mass. In this precise sense, the top quark pole mass is an irrelevant quantity. The top quark propagator is simply never on-shell. The propagator is given by $1/(p^2 - m_t^2 + i\Gamma_t/2)$, but the loop or external momentum $p$ can always be considered as real. The denominator of the propagator never gets smaller than about $m_t\Gamma_t$. We then find, for a quantity that would have a long-distance correction of (relative) order $\Lambda_{\text{QCD}}/m$ for a stable quark,

$$\frac{\Lambda_{\text{QCD}}}{m} \to \frac{\Lambda_{\text{QCD}}}{m} \cdot \frac{\Lambda_{\text{QCD}}}{\Gamma}. \quad (15)$$

We expect, at least, a suppression by $\Lambda_{\text{QCD}}/\Gamma_t$ due to the finite width. (For a related discussion of this point, see Refs. 43.)
Table 1: Bottom quark mass obtained from Υ sum rules in GeV. The $\overline{\text{MS}}$ mass is renormalized at the scale of its own value. The different values are not strictly comparable, because different values of $\alpha_s(m_Z)$ may be implied. Whenever available, we quote a result obtained from a fixed input rather than fitted $\alpha_s(m_Z)$. ‘$\alpha_s^2$’ means that no systematic resummation has been attempted.

6 Bottom quark mass from Upsilon sum rules

The derivatives of the bottom vector current two-point function at $q^2 = 0$ are related to the inclusive bottom cross section by

$$\frac{12\pi^2}{n!} \frac{d^n}{d(q^2)^n} \Pi(q^2) \bigg|_{q^2=0} = \int_0^\infty \frac{ds}{s^{n+1}} R_{\bar{b}b}(s)$$

up to a (small) correction due to $\bar{b}b$ radiation from light quarks. The left hand side can be computed in perturbation theory; the right hand side from data.

The parameters of the lowest $\Upsilon(nS)$ resonances are well-measured, but the continuum cross section near the threshold is not. Hence the experimental error of the right hand side decreases with increasing $n$, because higher moments weight lower $s$. But larger $n$ makes the theoretical calculation more difficult, because the expansion parameter is $\alpha_s \sqrt{n}$. The enhancement by a factor $\sqrt{n}$ is a consequence of the Coulomb enhanced terms in the theoretical calculation of $R$. A power of $v$ in $R$ corresponds to a power of $n^{-1/2}$ in the moment. When

$$\sqrt{\pi n} \alpha_s(m_b/\sqrt{n}) \ll 1$$

is no longer satisfied, a resummation by the methods discussed earlier in this talk is necessary. This is already the case for $n$ larger than 4. There is an intermediate $n$ where experimental and theoretical uncertainties are balanced.

Recently, there have been several calculations [28-30] which implemented this resummation at NNLO. Their results, together with results of previous (NLO or fixed-order $\alpha_s^2$) calculations [46-48] are compiled in Table 1. Note that as the accuracy of the calculation increases, the uncertainty in the result becomes larger. Instead of our own comments, we refer to Ref. [29] for a discussion of this intriguing point. The result of Ref. [47] is low, mainly because they do not include the sub-threshold poles of the Coulomb Green function into their calculation. But since to a large extent it is the average over Coulomb poles which is dual to the average of over the physical $\Upsilon$ resonances, their inclusion is necessary for moments which receive their largest contribution from the $\Upsilon$ resonances. Technically, this follows from the fact that the Coulomb poles contribute
and amount of order $\alpha_s(m_b/\sqrt{n})^3$ to the moments, to be compared with $1/n^{3/2}$, the tree graph contribution.

Note that because of the cancellation of long-distance contributions between the potential and the pole mass discussed in the previous section, it is advantageous to use a mass renormalization convention different from the pole mass beyond NLO. This strategy has been adopted in Refs. 30, 31.

We mentioned earlier that there is a non-perturbative contribution to the heavy quark cross section at NNLO when $m v^2 \sim \Lambda_{\text{QCD}}$. There is a contribution to the moments from the scale $m_b/n$. If we require for a perturbative treatment that all scales are larger than 0.5 GeV, this limits $n < 10$. The use of larger $n$ in most of the calculations quoted in Table 1 has been justified by the observation that even at $n = 20$ the gluon condensate contribution to the moments is very small. Hence one may be in the fortunate situation that the actual non-perturbative contribution is smaller than the power counting argument suggests. However, the condensate expansion converges parametrically only when $m_b/n \gg \Lambda_{\text{QCD}}$. If the expansion does not converge, the size of its first term is not very conclusive. (For the Coulomb energy levels, the contribution from dimension-6 operators has been considered in Ref. 49.)

7 Conclusion

The past year has seen significant progress in our understanding of perturbative quark-antiquark systems close to threshold, both methodical and in terms of explicit higher-order calculations. We have come nearer to the answer to the question how accurately the bottom and the top quark mass can be determined by purely perturbative means.

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