Higher Spin Currents 
in the Orthogonal Coset Theory

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Abstract

In the coset model \((D_N^{(1)} \oplus D_N^{(1)}, D_N^{(1)})\) at levels \((k_1, k_2)\), the higher spin 4 current that contains the quartic WZW currents contracted with completely symmetric \(SO(2N)\) invariant \(d\) tensor of rank 4 is obtained. The three-point functions with two scalars are obtained for any finite \(N\) and \(k_2\) with \(k_1 = 1\). They are determined also in the large \(N\) \'{t} Hooft limit. When one of the levels is the dual Coxeter number of \(SO(2N)\), \(k_1 = 2N - 2\), the higher spin \(\frac{7}{2}\) current, which contains the septic adjoint fermions contracted with the above \(d\) tensor and the triple product of structure constants, is obtained from the operator product expansion (OPE) between the spin \(\frac{3}{2}\) current living in the \(\mathcal{N} = 1\) superconformal algebra and the above higher spin 4 current. The OPEs between the higher spin \(\frac{7}{2}, 4\) currents are described. For \(k_1 = k_2 = 2N - 2\) where both levels are equal to the dual Coxeter number of \(SO(2N)\), the higher spin 3 current of \(U(1)\) charge \(\frac{4}{7}\), which contains the six product of spin \(\frac{1}{2}\) (two) adjoint fermions contracted with the product of \(d\) tensor and two structure constants, is obtained. The corresponding \(\mathcal{N} = 2\) higher spin multiplet is determined by calculating the remaining higher spin \(\frac{7}{2}, \frac{7}{2}, 4\) currents with the help of two spin \(\frac{3}{2}\) currents in the \(\mathcal{N} = 2\) superconformal algebra. The other \(\mathcal{N} = 2\) higher spin multiplet, whose \(U(1)\) charge is opposite to the one of above \(\mathcal{N} = 2\) higher spin multiplet, is obtained. The OPE between these two \(\mathcal{N} = 2\) higher spin multiplets is also discussed.
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1 Introduction

The Gaberdiel and Gopakumar proposal [1], the duality between the higher spin gauge theory on $AdS_3$ space [2] and the large $N$ ’t Hooft limit of a family of $W_N (\equiv WA_{N-1})$ minimal models is the natural analogue of the Klebanov and Polyakov duality [3] relating the $O(N)$ vector model in three-dimensions to a higher spin theory on $AdS_4$ space. Then the obvious generalization of [1] is to consider the Klebanov and Polyakov duality in one dimension lower. By replacing the $SU(N)$ group by $SO(2N)$, the relevant most general coset model is described as [4, 5]

$$\frac{SO(2N)_{k_1} \oplus SO(2N)_{k_2}}{SO(2N)_{k_1 + k_2}}$$

(1.1)
One can also consider the case where the $SU(N)$ group by $SO(2N+1)$ but this is not described in this paper. It is known that the conformal weight (or spin) of the primary state is equal to the quadratic Casimir eigenvalue divided by the sum of the level and the dual Coxeter number of the finite Lie algebra $[6,7]$. For example, for $SO(2N)$, the quadratic Casimir eigenvalue for the adjoint representation is given by $2N - 2$ while the dual Coxeter number is $2N - 2$. Then we are left with the adjoint fermion of spin $\frac{1}{2}$ at the critical level which is equal to the dual Coxeter number. One can apply this critical behavior to the two numerator factors in (1.1) simultaneously. In the description of these adjoint free fermions, the central charge grows like $N^2$ in the large $N$ 't Hooft limit: so-called stringy coset model $[8]$. See also the relevant works in $[9,10,11,12,13]$.

Although some constructions on the higher spin currents in $[14]$ have been done, there are two unknown coefficients in the expression of higher spin 4 current. Moreover, the spin 1 currents in the numerators of (1.1) are described with the double index notation. Each index is a vector representation of $SO(2N)$ and because of antisymmetric property of these spin 1 currents, the number of independent fields is given by $\frac{1}{2} [(2N)^2 - 2N] = N(2N - 1)$. In order to obtain the description of above free adjoint fermions, one should write down the spin 1 currents with a single adjoint index. It is known that the real free fermions transforming in the adjoint representation of $SO(2N)$ realize the affine Kac Moody algebra for the critical level. It is equivalent to the theory of $\frac{1}{2} 2N(2N - 1) = N(2N - 1)$ free fermions $[7]$.

Before one considers the adjoint free fermion description, one should obtain the higher spin 4 current from the spin 1 currents living in the numerator factors of (1.1) and having a single adjoint index. The higher spin 4 current is $SO(2N)$ singlet field $[6]$. Then one should have a quantity contracted with the quartic terms in the above spin 1 currents. This is known as $d$ symbol which is completely symmetric $SO(2N)$ invariant tensor of rank 4. In calculation of any OPE between the higher spin currents, one should use various contraction identities between the above $d$ symbol and the structure constant $f$. Recall that in the defining OPE between the spin 1 currents, the structure constant $f$ symbol appears. As far as I know, there are no known identities between $f$ symbol and $d$ symbol except of $ff$ contraction in the literature. This is one of the reasons why the double index notation in $[14]$ is used.

In this paper, one starts with the definition of $d$ symbol which is given by one half times the trace over six quartic terms in the $SO(2N)$ generators. When one meets the relevant contraction identities in the calculation of any OPE, one can try to obtain the tensorial structure in the right-hand sides of these identities. Of course in each term, there should be present $N$ dependence coefficients explicitly. The tensorial structure in terms of multiple product of $f$ symbol, $d$ symbol and the symmetric $SO(2N)$ invariant tensor $\delta$ of rank 2 occurs.
naturally during the explicit calculation of OPE. As one applies for \( N = 2, 3, 4 \) and 5 cases in the \( SO(2N) \) generators, one can determine the \( N \) dependence coefficients explicitly.

It turns out that the higher spin 4 current is obtained completely except of overall normalization factor. The eigenvalue equations of zero mode of the higher spin 4 current acting on several primary states can be determined explicitly. The corresponding three-point functions can be obtained. By choosing the overall factor correctly, one observes the standard three-point functions in the large \( N \) 't Hooft limit from the asymptotic symmetry algebra in the \( AdS_3 \) bulk theory. From the description of adjoint fermions living in the first factor in the numerator of (1.1), one obtains the well known \( \mathcal{N} = 1 \) superconformal algebra generated by the spin 2 stress energy tensor and its superpartner, spin \( \frac{3}{2} \) current. It turns out that the higher spin \( \frac{7}{2} \) current consists of septic, quintic, cubic and linear terms in the adjoint fermions with appropriate derivative terms. The \( \mathcal{N} = 2 \) superconformal algebra is realized by two adjoint fermions living in the two numerator factors in (1.1). In this case, the higher spin 3 current with \( U(1) \) charge \( \frac{4}{3} \) is given by the multiple product of two fermions contracted with \( dff \) or \( ff \) tensor without any derivative terms. Moreover, its three partners, higher spin \( \frac{7}{2}, \frac{7}{2}, \frac{7}{2} \), and 4 currents are determined.

In section 2, the higher spin 4 current is obtained, the three-point functions are given and the OPE between the higher spin 4 current and itself is described under some constraints.

In section 3, the higher spin \( \frac{7}{2} \) current is obtained, and the three OPEs between this higher spin \( \frac{7}{2} \) current and the higher spin 4 current are described using the Jacobi identities.

In section 4, the lowest higher spin 3 current is obtained, and its three other higher spin \( \frac{7}{2}, \frac{7}{2}, \frac{7}{2} \) and 4 currents are obtained which can be denoted as \( \mathcal{N} = 2 \) lowest higher spin multiplet with definite \( U(1) \) charge \( \frac{4}{3} \). Furthermore, another \( \mathcal{N} = 2 \) lowest higher spin multiplet with definite \( U(1) \) charge \( -\frac{4}{3} \) is obtained. The OPE between these higher spin multiplets in \( \mathcal{N} = 2 \) superspace are given using the Jacobi identities.

In section 5, we list some future directions related to this work.

In Appendices A-L, the technical details appearing in sections 2, 3 and 4 are given.

2 The coset model with arbitrary two levels \((k_1, k_2)\)

From the spin 1 currents of the coset model, one constructs the spin 2 stress energy tensor. By generalizing the Sugawara construction, the higher spin 4 current is obtained from the quartic terms in the spin 1 currents with the \( SO(2N) \) invariant tensors of ranks 4, 2. The corresponding three-point functions of zero mode of the higher spin 4 current with two scalars are described. The OPE between the higher spin 4 current and itself for particular \( k_1 \) and \( N \)
is obtained.

2.1 Spin 2 current and Virasoro algebra

The standard stress energy tensor satisfies the following OPE [6]

\[ T(z) T(w) = \frac{1}{(z-w)^4} c + \frac{1}{(z-w)^2} 2T(w) + \frac{1}{(z-w)} \partial T(w) + \cdots. \] (2.1)

For the coset model in (1.1), the above stress energy tensor can be obtained by usual Sugawara construction [6]

\[ T(z) = -\frac{1}{2(k_1 + 2N - 2)} J^a J^a(z) - \frac{1}{2(k_2 + 2N - 2)} K^a K^a(z) \]
\[ + \frac{1}{2(k_1 + k_2 + 2N - 2)} (J^a + K^a)(J^a + K^a)(z). \] (2.2)

The affine Kac-Moody algebra \( \hat{SO}(2N)_{k_1} \oplus \hat{SO}(2N)_{k_2} \) in (1.1) is described by the following OPEs [6]

\[ J^a(z) J^b(w) = -\frac{1}{(z-w)^2} k_1 \delta^{ab} + \frac{1}{(z-w)} f^{abc} J^c(w) + \cdots, \]
\[ K^a(z) K^b(w) = -\frac{1}{(z-w)^2} k_2 \delta^{ab} + \frac{1}{(z-w)} f^{abc} K^c(w) + \cdots. \] (2.3)

The adjoint indices \( a, b, \cdots \) corresponding to \( SO(2N) \) group run over \( a, b = 1, 2, \cdots, \frac{1}{2} 2N(2N-1) \). The Kronecker delta \( \delta^{ab} \) appearing in (2.3) is the second rank \( SO(2N) \) symmetric invariant tensor. The structure constant \( f^{abc} \) is antisymmetric as usual. The diagonal affine Kac-Moody algebra \( \hat{SO}(2N)_{k_1+k_2} \) in (1.1) can be obtained by adding the above two spin 1 currents, \( J^a(z) \) and \( K^a(z) \). Of course, we have \( J^a(z) K^b(w) = + \cdots \).

The central charge appearing the above OPE (2.1) is given by [6]

\[ c(k_1, k_2, N) = \frac{1}{2} 2N(2N-1) \left[ \frac{k_1}{(k_1 + 2N - 2)} + \frac{k_2}{(k_2 + 2N - 2)} - \frac{(k_1 + k_2)}{(k_1 + k_2 + 2N - 2)} \right]. \] (2.4)

Note that the dual Coxeter number of \( SO(2N) \) is equal to \( (2N-2) \) and the dimension of \( SO(2N) \) is given by \( \frac{1}{2} 2N(2N-1) \).

Then the Virasoro algebra realized in the coset model (1.1) [5, 15] is summarized by (2.1) together with (2.2) and (2.4).
2.2 Higher spin 4 current

The 28 SO(8) generators $T^a$ are given in Appendix (A.1). Then the structure constant introduced in the above is given by

$$f^{abc} = -\frac{i}{2} \text{Tr} \left[ T^c T^a T^b - T^c T^b T^a \right].$$

(2.5)

Then one obtains $[T^a, T^b] = i f^{abc} T^c$.

The totally symmetric SO(2N) invariant tensor of rank 4 is defined as $[16, 17]$

$$T^a T^b T^c + T^a T^c T^b + T^b T^a T^c + T^b T^c T^a + T^c T^b T^a = d^{abcd} T^d.$$ 

(2.6)

That is, one can express the $d$ tensor as

$$d^{abcd} = \frac{1}{2} \text{Tr} \left[ T^d T^a T^b T^c + T^d T^a T^c T^b + T^d T^c T^a T^b + T^d T^b T^a T^c + T^d T^b T^c T^a + T^d T^c T^b T^a \right].$$

(2.7)

Note that one uses $\text{Tr}(T^a T^b) = 2 \delta^{ab}$.

One obtains the product of the structure constants

$$f^{abc} f^{abd} = 2 (2N - 2) \delta^{cd},$$

(2.8)

and the triple product leads to

$$f^{adf} f^{bce} f^{cfb} = -(2N - 2) f^{def}.$$ 

(2.9)

Furthermore, one obtains the following nontrivial triple product between $d$ tensor (2.7) and $f$ tensor (2.5)

$$d^{adeb} f^{bfc} f^{cg} = -\frac{4}{3} (N - 1) d^{defg} + 4 \delta^{df} \delta^{eg} + 4 \delta^{dg} \delta^{ef} - 8 \delta^{de} \delta^{fg} - \frac{1}{3} (2N - 5) f^{dgh} f^{hef}.$$ 

(2.10)

By multiplying $f^{dfh}$ into (2.10) and rearranging the indices, one obtains with (2.9)

$$d^{abef} f^{agd} f^{bde} f^{cgh} = 2 (2N^2 - 7N + 11) f^{fg}. $$

(2.11)

For the index condition $f = d$ in (2.10) together with the identity (2.8), one obtains

$$d^{abc} = 2 (4N - 1) \delta^{bc}.$$ 

(2.12)

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1 One can consider the rank 3 tensor as $d^{abc} = \frac{1}{2} \text{Tr} \left[ T^a T^b T^c + T^b T^a T^c \right]$ which is identically zero.
Note that this behavior is different from the one of unitary case where the trivial result \( d^{aabc} = 0 \) arises [18]. One also has
\[
d^{abcd}d^{abce} = 12 \left[ N(2N - 1) + 2 \right] \delta^{de}. \quad (2.13)
\]

Let us describe how one can obtain the higher spin current with the help of \( d \) tensor we introduced. For the second rank \( SO(2N) \) invariant symmetric tensor \( \delta^{ab} \), one describes the stress energy tensor in (2.2). According to the observation of footnote[1] there is no nontrivial third rank \( SO(2N) \) invariant symmetric tensor \( d^{abc} \). Then the next nontrivial higher spin current can be constructed from the fourth rank \( SO(2N) \) invariant symmetric tensor \( d^{abcd} \) (2.6).

Let us consider the following higher spin 4 current, along the line of [18, 19, 6],
\[
W^{(4)}(z) = d^{abcd}\left[ A_1 J^a J^b J^c J^d + A_2 J^a J^b J^c K^d + A_3 J^a J^b K^c K^d \right. \\
+ A_4 J^a K^b K^c K^d + A_5 K^a K^b K^c K^d \left( z \right) + \left. \left[ A_6 \partial J^a \partial J^a + A_7 \partial^2 J^a J^a \right. \right. \\
+ A_8 \partial K^a \partial K^a + A_9 \partial^2 K^a K^a + A_{10} \partial J^a \partial K^a + A_{11} \partial^2 J^a K^a \\
+ A_{12} J^a \partial^2 K^a + A_{13} f^{abc} J^a \partial J^b K^c + A_{14} f^{abc} J^a K^b \partial K^c + A_{15} J^a J^b J^b \\
+ A_{16} K^a K^a K^b K^b + A_{17} J^a J^a K^b K^b + A_{18} J^a J^a J^b K^b + A_{19} J^a K^a K^b K^b \\
+ A_{20} J^a J^b K^a K^b \left. \right] (z). \quad (2.14)
\]

One should obtain the twenty relative \((k_1, k_2, N)\)-dependent coefficients. The first five quartic terms in (2.14) can be easily understood in the sense that they are the only possible terms from each spin 1 current, \( J^a(z) \) and \( K^a(z) \) using the \( d^{abcd} \) tensor. The next seven derivative terms in (2.14) can be seen from the second derivative of stress energy tensor \( \partial^2 T(z) \). The remaining eight terms can arise in \( TT(z) \).

First of all, the higher spin 4 current should have the regular terms with the diagonal spin 1 current in the coset model as follows [18, 19, 6]:
\[
J^a(z) W^{(4)}(w) = + \cdots, \quad J^a(z) \equiv (J^a + K^a)(z). \quad (2.15)
\]

Let us calculate the OPEs between the diagonal spin 1 current and the twenty terms in (2.14) in order to use the condition (2.15). One can perform the various OPEs by following the procedures done in the unitary case [20]. Let us focus on the \( A_1 \) term in (2.14) which has the regular OPE with \( K^a(z) \). Then the equations (2.22), (2.23) and (2.24) of [20] can be used. For example, the equation (2.24) of [20] provides the information of the OPE between
the $J^a(z)$ and the above $A_1$ term. Using the relations (2.11) and (2.8), one can simplify the fourth-order pole in (2.24) of [20] which was given by $f_{abc} f_{feci} d_{bcde} f_{geh} f_{idg} J^b(w)$.

It turns out that we are left with $J^a(w)$ with $N$-dependent $SO(2N)$ group theoretical factor. The third-order pole,

$$f_{abcd} (f_{jdh} f_{fej} J^d J^e + f_{heg} f_{fej} J^d J^g + f_{heg} f_{fej} J^e J^g)(w) + d_{bcde} f_{geh} f_{idg} J^b J^c J^d J^e,$$

(2.16)

can be simplified with the help of (2.11). We are left with $f_{abc} J^b J^c(w)$ in (2.16) with $N$ dependent coefficient factor which is proportional to $\partial J^a(w)$. Finally the second-order pole,

$$-4k_1 d_{abcd} J^b J^c J^d(w)$$

$$ + d_{bcde} (f_{abc} f_{feg} J^d J^e + f_{abc} f_{fde} J^b J^d) + f_{abc} f_{fde} J^b J^d (w),$$

(2.17)

can be simplified further together with (2.11).

Then we obtain the final OPE as follows:

$$J^a(z) d_{bcde} J^b J^c J^d J^e(w) = \frac{1}{(z - w)^2} 2(2N - 2)(4N^2 - 14N + 22) J^a(w)$$

$$- \frac{1}{(z - w)^3} 2(4N^2 - 14N + 22) f_{abc} J^b J^c(w)$$

$$+ \frac{1}{(z - w)^2} \left[ - (4k_1 + 8(N - 1)) d_{abcd} J^b J^c J^d - (12 + (2N - 2)(2N - 5)) f_{abc} \partial J^b J^c$$

$$+(12 + (2N - 2)(2N - 5)) f_{abc} J^b J^c \partial J^c \right] (w) + \cdots.$$  

(2.18)

There is no first order pole in the above (2.18). One can check the second order pole in (2.18) from (2.17).

Let us consider the $A_2$ term in (2.14) where there exists $K^d(z)$ dependence. Starting from the (2.23) and (2.21) of [20] with the relations (2.11) and (2.10), one can simplify the third order pole, $f_{abc} f_{dcd} f_{geh} f_{idg} J^b K^b(w)$, as $f_{abc} J^b J^c(w)$ with $N$ dependent factor. Similarly, the second order pole,

$$-3k_1 d_{abcd} J^b J^c J^d K^b(w) - k_2 d_{abcd} J^b J^c J^d(w)$$

$$+ d_{bcde} (f_{ac} f_{fde} J^d J^e + f_{ac} f_{fde} J^b J^d) + f_{ac} f_{fde} J^b J^d (w),$$

(2.19)

can be simplified in terms of several independent terms. It turns out that in this case also there are no first order poles.

Therefore, one obtains the following OPE corresponding to $A_2$ term

$$J^a(z) d_{bcde} J^b J^c J^d K^e(w) = \frac{1}{(z - w)^3} (4N^2 - 14N + 22) f_{abc} J^b K^c(w)$$
It is useful to realize that this OPE remains the same after the exchange of tensor is totally symmetric. The twelve terms in the second order pole can be divided into two groups and each of them has their own counterpart.

One can see the second order pole in (2.20) from (2.19). For example, the second order pole, spin 4 current. Therefore, one should have the following condition \([18, 19, 6]\) OPE between the stress energy tensor in the numerator of the coset model and the higher \(J\) tensor (2.2). According to the previous regular condition (2.15), the diagonal spin 1 current \(B\) being 17 OPEs in Appendix (B.2). Then there are no singular terms in the OPE and Appendix (B.1).

It turns out that the relevant OPE coming from (2.21) can be summarized as

\[
J^a(z) \, d^{bcde} \, J^b \, J^c K^d K^e (w) = \frac{1}{(z-w)^2} \left[-(2k_1 + 4(N-1)) d^{bcde} \, J^b \, J^c K^d K^e \right. \\
- (2k_2 + \frac{4}{3}(N-1)) d^{bcde} \, J^b K^d + 8J^b K^a K^b + 4f^{abc} \partial J^b K^c \\
- \frac{1}{3}(2N-5) f^{abc} f^{cde} J^d K^e K^b \\
- \frac{1}{3}(2N-5) f^{abc} f^{cde} J^d K^e K^b - \frac{1}{3}(2N-5) f^{abc} f^{cde} J^d K^e K^d \right] (w) + \cdots. \tag{2.22}
\]

It is useful to realize that this OPE remains the same after the exchange of \(J^a(w)\) and \(K^a(w)\) together with \(k_1 \leftrightarrow k_2\). The left hand side is invariant under this transformation because the \(d\) tensor is totally symmetric. The twelve terms in the second order pole can be divided into two groups and each of them has their own counterpart.

It is straightforward to complete this calculation step by step. We summarize the remaining 17 OPEs in Appendix B. Then we have the complete expressions in (2.18), (2.20), (2.22), and Appendix (B.1).

The higher spin 4 current should transform as a primary field under the stress energy tensor (2.2). According to the previous regular condition (2.15), the diagonal spin 1 current \(J^a(z)\) does not have any singular terms in the OPE with the higher spin 4 current \(W^{(4)}(w)\) after we use the results of Appendix (B.2). Then there are no singular terms in the OPE between the stress energy tensor in the denominator of the coset model (11) and the higher spin 4 current because the former is given by \(J^a, J^a(z)\). The singular terms can arise from the OPE between the stress energy tensor in the numerator of the coset model and the higher spin 4 current. Therefore, one should have the following condition \([18, 19, 6]\)

\[
\hat{T}(z) \, W^{(4)}(w) \left|_{\frac{1}{(z-w)^n}, \, n=3,4,5,6} \right. = 0. \tag{2.23}
\]
Here the stress energy tensor in the numerator is described by

\[ \hat{T}(z) \equiv -\frac{1}{2(k_1 + 2N - 2)} J^a(z) - \frac{1}{2(k_2 + 2N - 2)} K^a(z). \]  \tag{2.24}

Of course, the higher spin 4 current has the standard OPE (the second and first order poles) with stress energy tensor (2.2) as usual.

Let us calculate the OPE between the stress energy tensor (2.24) and the \( A_1 \) term in (2.14). First of all, because the \( A_1 \) term does not contain the \( K^a(w) \) spin 1 current, one can consider the OPE between the first term of (2.24) and the \( A_1 \) term. It is known that the spin 1 current \( J^a(w) \) transforms as a primary field under the first term of (2.24) (i.e., stress energy tensor in the first factor of the numerator). Then one should obtain the OPE \( J^b(z) d^{bcde} J^c J^d J^e(w) \) and this turns out that there exists a nontrivial second order pole given by \(-3k_1(8N - 2)J^c J^c(w)\) where the identity (2.12) is used. Note that the structure constant term vanishes due to the presence of \( d^{bcde} \). Furthermore, one should calculate the OPE between the above stress energy tensor and the previous expression \( d^{bcde} J^c J^d J^e(w) \) where the order of the singular terms is greater than 2. Then we are left with \(-3k_1(8N - 2)J^c J^c(w)\) by combing the contribution \(-2k_1(8N - 2)J^c J^c(w)\) from the contraction between the stress energy tensor and \( J^c(w) \) and the contribution \(-k_1(8N - 2)J^c J^c(w)\) from the OPE between the stress energy tensor and \( d^{bcde} J^d J^e(w) \). Therefore, the final total contribution is summarized by \(-6k_1(8N - 2)J^c J^c(w)\) and we present this OPE as follows:

\[ \hat{T}(z) d^{bcde} J^b J^c J^d J^e(w) = -\frac{1}{(z-w)^4} 12k_1(4N - 1)J^a J^a(w) + O\left(\frac{1}{(z-w)^2}\right). \]  \tag{2.25}

This result in (2.25) is different behavior from the corresponding OPE in the unitary case because in the latter, there is no contribution from the fourth order pole because the above \( d^{abce} \) tensor for the \( SU(N) \) group vanishes [18].

Let us move on the \( A_2 \) term in (2.14). In this case, the spin 1 current \( K^d(w) \) is present. However, the contribution in the higher singular terms of the stress energy tensor coming from the second term of (2.24) vanishes. Then one can calculate the OPE between the stress energy tensor in the first factor of the numerator and the \( A_2 \) term. By using the previous procedure one can obtain the contribution \(-2k_1(8N - 2)J^c K^c(w)\) from the contraction with \( J^b(w) \) current and the contribution \(-k_1(8N - 2)J^c K^c(w)\) from the OPE between other remaining factor \( d^{bcde} J^c J^d K^e(w) \). By adding these two, one obtains the following OPE

\[ \hat{T}(z) d^{bcde} J^b J^c J^d K^e(w) = -\frac{1}{(z-w)^4} 6k_1(4N - 1)J^a K^a(w) + O\left(\frac{1}{(z-w)^2}\right). \]  \tag{2.26}

Now let us describe the contribution from \( A_3 \) term in (2.14) where the quadratic \( K^c K^d(w) \) appears. In this case, one should also calculate the contribution from the stress energy tensor
in the second term in (2.24). As done before, the contribution from the contraction with \( J_b(w) \) spin 1 current is given by \(-k_1(8N - 2)K^aK^c(w)\). Similarly the contribution from the contraction with the remaining factor is given by \(-k_2(8N - 2)J^cJ^c(w)\). Then we are left with

\[
\hat{T}(z) d^{bcde} J^b J^c K^d K^e(w) = -\frac{1}{(z-w)^4} \left[ 2k_2(4N-1)J^aJ^a + 2k_1(4N-1)K^aK^a \right](w) + O\left(\frac{1}{(z-w)^2}\right),
\]

(2.27)

One also sees the symmetry under the transformation \( J^a(z) \leftrightarrow K^a(z) \) and \( k_1 \leftrightarrow k_2 \).

It is straightforward to determine other remaining calculation step by step. We summarize the remaining 17 OPEs in Appendix C. Then we are left with (2.25), (2.26), (2.27), and Appendix (C.1).

Now one can determine the undetermined coefficient functions \( A_1, A_2, \cdots, A_{20} \) appearing in the higher spin 4 current in (2.14). The twenty three linear equations are given in Appendix (B.2) explicitly. The eight linear equations are given in Appendix (C.2). By solving them, one obtains the final expressions in Appendix D. They depend on \( k_1, k_2 \) and \( N \). The corresponding coefficients for \( k_1 = 1 \) are presented in Appendix E. Appendix F corresponds to the case where \( k_1 = 2N - 2 \).

### 2.3 Three-point functions [21] with two scalars where \( k_1 = 1 \)

The zero modes of the current satisfy the commutation relations of the underlying finite dimensional Lie algebra \( SO(2N) \). For the state \(|(v;0)\rangle\), \( T^a \) corresponds to \( iK^a_0 \) and for the state \(|(0;v)\rangle\), \( T^a \) corresponds to \( iJ^a_0 \) as follows:

\[
|(v;0)\rangle: \quad T^a \leftrightarrow iK^a_0, \quad |(0;v)\rangle: \quad T^a \leftrightarrow iJ^a_0.
\]

(2.28)

Note that from the defining equation of the OPEs (2.3), one obtains

\[
[J^a_m, J^b_n] = -k_1m\delta^{ab}\delta_{m+n,0} + f^{abc}J^c_{m+n}, \quad [K^a_m, K^b_n] = -k_2m\delta^{ab}\delta_{m+n,0} + f^{abc}K^c_{m+n},
\]

(2.29)

In (2.29), the central terms for the zero modes vanish. Recall that our generators for the \( SO(2N) \) satisfy \([T^a, T^b] = if^{abc}T^c\) [6].

The large \( N \) ’t Hooft limit is described as [22, 23]

\[
N, k_2 \to \infty, \quad \lambda \equiv \frac{2N}{2N - 2 + k_2} \quad \text{fixed}.
\]

(2.30)

The presence of numerical value \(-2\) in the denominator of (2.30) is not important under the large \( N \) ’t Hooft limit [24].
Compared to the large $\mathcal{N} = 4$ holography in [10, 25, 26] where one can obtain the eigenvalue equations from the several low $N$ values inside the package of [27], one should analyze both the coefficients and zero modes of the twenty terms in higher spin 4 current in order to obtain the corresponding eigenvalue equations.

### 2.3.1 Eigenvalue equation of the zero mode of the higher spin 4 current acting on the state $|(0; v)\rangle$

Let us consider the eigenvalue equation of the zero mode of the $A_1$ term of the higher spin 4 current in (2.14) acting on the primary state $(0; v)$

$$d^{abcd}(J^a J^b J^c J^d)|0; v\rangle >. \tag{2.31}$$

Using the fact that the zero mode is nothing but the product of each zero mode but the ordering is reversed [18, 19], the above expression (2.31) becomes

$$d^{abcd}(J^d J^c J^b J^a)|0; v\rangle >. \tag{2.32}$$

Note that the ground state transforms as a vector representation with respect to $J^a_0$ while the zero mode $K^a_0$ has vanishing eigenvalue equation [21]

$$K^a_0 |(0; v)\rangle = 0. \tag{2.33}$$

Then the above expression (2.32) becomes

$$\frac{1}{2N} d^{abcd}(-i)^4 \text{Tr}(T^d T^c T^b T^a)|0; v\rangle >. \tag{2.34}$$

In order to use the previous identity in (2.6), one can express the above $A_1$ term as follows:

$$\frac{1}{6} d^{abcd}(J^a J^b J^c J^d + J^b J^c J^a J^d + J^c J^d J^a J^b + J^d J^a J^b J^c + J^a J^b J^c J^d)|0; v\rangle >. \tag{2.35}$$

due to the symmetric property of $d$ tensor. Then the equivalent expression corresponding to (2.31) with (2.35) can be written in terms of

$$\frac{1}{2N} \frac{1}{6} d^{abcd} \text{Tr}(T^d T^c T^b T^a + T^d T^a T^c T^b + T^d T^c T^a T^b + T^d T^c T^a T^c) \times |(0; v)\rangle >. \tag{2.36}$$

The reason why there exists the extra $\frac{1}{2N}$ is that one should have the eigenvalue not the trace. Using the identity (2.6), one can reexpress (2.36) as

$$\frac{1}{2N} \frac{1}{3} d^{abcd} d^{abcd} = \frac{1}{2N} \frac{1}{3} 12 [N(2N - 1) + 2] \delta^{aa} = \frac{1}{2N} \frac{1}{3} 12 [N(2N - 1) + 2] \frac{1}{2} 2N(2N - 1) \rightarrow 8N^3. \tag{2.37}$$
Here the identity (2.13) is used and we take the large $N$ limit at the last result in (2.37).

One can analyze the other 19 terms in (2.14). Among them, the 16 terms which have the $K^a(z)$ spin 1 current do not contribute to the eigenvalue equation because one can take the zero mode and change the ordering of the zero modes as in (2.32). Then one can move the rightmost zero mode $K_a^0$ to the right and use the previous condition (2.33). On the other hand, the remaining $A_6, A_7$ and $A_{15}$ terms can contribute to the eigenvalue equation.

The zero mode of the $A_6$ term of the higher spin 4 current acting on the primary state $(0; \nu)$ is

$$(\partial J^a \partial J^a)_{0}(0; \nu) > = (\partial J^a)_{0}(\partial J^a)_{0}(0; \nu) > = (J^a_0)(-J^a_0)(0; \nu) > = J^a_0 J^a_0(0; \nu) > , \quad (2.38)$$

where the zero mode of $\partial J^a$ in (2.38) can be obtained from the usual mode expansion and is given by the zero mode of $-J^a$. Now using the correspondence (2.28), the above expression leads to

$$\frac{1}{2N} \text{Tr}(i T^a i T^a) |(0; \nu) > = -\frac{1}{2N} 2\delta^{aa} = -\frac{1}{2N} 2 \frac{1}{2} 2N(2N - 1) \rightarrow -2N, \quad (2.39)$$

where the extra factor $\frac{1}{2N}$ is considered as in (2.36) and the large $N$ limit is taken.

Now the final contribution from the zero mode of the $A_7$ term of the higher spin 4 current acting on the primary state $(0; \nu)$ is given by

$$(\partial^2 J^a J^a)_{0}(0; \nu) > = J^a_0 (\partial^2 J^a)_{0}(0; \nu) > = J^a_0 J^a_0(0; \nu) > , \quad (2.40)$$

where the zero mode of $\partial^2 J^a$ in (2.40) can be obtained from the usual mode expansion also and is given by the zero mode of $2J^a$. Therefore, one can follow the previous description. It turns out that

$$2 \frac{1}{2N} \text{Tr}(i T^a i T^a) |(0; \nu) > = -2 \frac{1}{2N} 2 \frac{1}{2} 2N(2N - 1) \rightarrow -4N. \quad (2.41)$$

For the $A_{15}$ term, one has the eigenvalue equation

$$(J^a J^a J^b J^b)_{0}(0; \nu) > = \delta^{ab} \delta^{cd}(J^a J^b J^c J^d)_{0}(0; \nu) > . \quad (2.42)$$

By following the procedure in the $A_1$ term, one obtains that the above (2.42) can be written as

$$\frac{1}{2N} \frac{1}{3} \delta^{ab} \delta^{cd} d^{abcd} = \frac{1}{2N} \frac{1}{3} (4N - 1) \frac{1}{2} \frac{1}{2} 2N(2N - 1) \rightarrow \frac{8}{3} N^2, \quad (2.43)$$

where the identity (2.12) is used in (2.43). Furthermore, the $A_{15}$ term itself behaves as $N^0$ in Appendix (E.3). Then there is no contribution at the leading order approximation.
By combining (2.37), (2.39) and (2.41) with the corresponding coefficients in the large $N$ limit of Appendix (E.3), the zero mode eigenvalue equation leads to

$$W_0^{(4)}(0; v) = \left[ 8N^3A_1 + (-2N)\left( \frac{N^2 12(2\lambda - 9)}{5(2\lambda - 3)} A_1 \right) + (-4N)\left( -N^2 \frac{8(2\lambda - 9)}{5(2\lambda - 3)} A_1 \right) \right] \times |(0; v) > = N^3 \left[ \frac{96(\lambda - 2)}{5(2\lambda - 3)} \right] A_1 |(0; v) > . \tag{2.44}$$

One can also calculate the same eigenvalue equation at finite $N$ and $k_2$ corresponding to (2.44) which will appear later.

2.3.2 Eigenvalue equation of the zero mode of the higher spin 4 current acting on the state $|v; 0\rangle$

Let us describe the eigenvalue equation of the zero mode of the $A_1$ term of the higher spin 4 current in (2.14) acting on the primary state $|v; 0\rangle$

$$d^{abcd}(J^a J^b J^c J^d)|v; 0\rangle = d^{abcd}J_0^a J_0^b J_0^c J_0^d |v; 0\rangle > . \tag{2.45}$$

Note that the ground state transforms as a vector representation with respect to $K^a_0$ and the singlet condition for the primary state $|v; 0\rangle$ can be described as

$$(J^a_0 + K^a_0)|v; 0\rangle = 0. \tag{2.46}$$

Then the above expression (2.45) is equivalent to

$$-d^{abcd}J_0^a J_0^b J_0^c K^d_0 |v; 0\rangle = -d^{abcd}K^a_0 J_0^a J_0^b J_0^d |v; 0\rangle > , \tag{2.47}$$

where the relation (2.46) is used and the zero mode $K^a_0$ is moved to the left. Now the singlet condition is applied to the rightmost $J^d_0$ and we are left with

$$d^{abcd}K^a_0 J_0^a J_0^b K^d_0 |v; 0\rangle = d^{abcd}K^a_0 K^b_0 J_0^a J_0^b |v; 0\rangle > . \tag{2.48}$$

One can further take the previous steps and obtains

$$d^{abcd}K^a_0 K^b_0 K^c_0 K^d_0 |v; 0\rangle > . \tag{2.49}$$

Then using the correspondence (2.28), the above expression (2.49) becomes

$$\frac{1}{2N} d^{abcd}(-i)^4 \text{Tr}(T^a T^b T^c T^d) |v; 0\rangle > , \tag{2.50}$$
which leads to the previous eigenvalue in (2.37)\footnote{The eigenvalue equation of the zero mode of the $A_2$ term of the higher spin 4 current in (2.14) acting on the primary state $(v; 0)$ can be written as}

What happens for $A_5$ term of higher spin 4 current in (2.14)? According to the large $N$ behavior of the coefficient $A_5$, this coefficient behaves as $\frac{1}{N}$ in Appendix (E.3) and moreover the analysis of eigenvalue equation leads to $N^3$ behavior. Therefore, the total power for the large $N$ behavior is given by $N^2$ and can be ignored in this approximation.

Let us move on the $A_6$ term. The eigenvalue equation leads to

\[
(\partial J^a \partial J^a)_0|\hspace{1mm} (v; 0) >= J^a_0 J^a_0 |\hspace{1mm} (v; 0) >= -J^a_0 K^a_0 |\hspace{1mm} (v; 0) >= K^a_0 K^a_0 |\hspace{1mm} (v; 0) >, \tag{2.54}
\]

where the singlet condition (2.46) is used. After using the correspondence (2.28), this becomes the previous result in (2.39).

Similarly, the $A_7$ term eigenvalue equation gives

\[
(\partial^2 J^a J^a)_0|\hspace{1mm} (v; 0) >= J^a_0 2J^a_0 |\hspace{1mm} (v; 0) >= -2 J^a_0 K^a_0 |\hspace{1mm} (v; 0) >= 2 K^a_0 K^a_0 |\hspace{1mm} (v; 0) >, \tag{2.55}
\]

which leads to (2.41).

For the $A_8$ and $A_9$ terms of the higher spin 4 current, these coefficients behave as $N$ from Appendix (E.3) in the large $N$ limit and the corresponding eigenvalues behave as $N$. Then the total power of the large $N$ behavior is given by 2 and these terms can be ignored at the leading order calculation.

\footnote{Let us describe the next $A_{10}$ term of the higher spin 4 current in (2.14). One obtains}

\[
(\partial J^a \partial K^a)_0|\hspace{1mm} (v; 0) >= K^a_0 J^a_0 |\hspace{1mm} (v; 0) >= -K^a_0 K^a_0 |\hspace{1mm} (v; 0) >, \tag{2.56}
\]

where this (2.56) is equivalent to the previous relation (2.54) with an extra minus sign. We can also calculate the eigenvalue equation for the $A_{11}$ term

\[
(\partial^2 J^a K^a)_0|\hspace{1mm} (v; 0) >= K^a_0 2J^a_0 |\hspace{1mm} (v; 0) >= -2 K^a_0 K^a_0 |\hspace{1mm} (v; 0) >. \tag{2.57}
\]
Let us consider $A_{13}$ term of the higher spin 4 current in (2.14). One can easily see that there exists the relation

$$ f^{abc} J^a \partial J^b K^c(z) = J^a J^b J^a K^b(z) - J^a J^b J^b K^a(z), \quad (2.59) $$

by writing the derivative term as the commutator of normal ordered product. Then the zero mode of this expression (2.59) is given by

$$ (K^b_0 J^a_0 - K^a_0 J^b_0)\langle v; 0 \rangle = -(K^b_0 K^a_0 K^b_0 K^a_0 - K^a_0 K^a_0 K^b_0 K^b_0)\langle v; 0 \rangle. \quad (2.60) $$

Then this (2.60) becomes

$$ -\frac{1}{2N} (-i)^4 Tr(T^b T^a T^b T^a - T^a T^b T^a T^b)\langle v; 0 \rangle. \quad (2.61) $$

Furthermore, this (2.61) will reduce to

$$ -\frac{1}{2N} (-i)^4 f^{bac} Tr(T^b T^a T^c - T^b T^c T^a)\langle v; 0 \rangle. \quad (2.62) $$

One can use the identity (2.5) and obtains, together with (2.8),

$$ -\frac{1}{2N} (-i)^4 f^{bac} \frac{1}{2} 2i f^{bac} = \frac{1}{2N} 2(2N-2)\frac{1}{2} 2N(2N-1) \to 4N^2. \quad (2.63) $$

Let us focus on the $A_{14}$ term. One has the relation

$$ f^{abc} J^a K^b \partial K^c(z) = J^a K^b K^a K^b(z) - J^a K^b K^b K^a(z). \quad (2.64) $$

The zero mode of (2.64) can be described as

$$ (K^b_0 K^a_0 K^b_0 J^a_0 - K^a_0 K^b_0 K^b_0 J^a_0)\langle v; 0 \rangle = -(K^b_0 K^a_0 K^b_0 K^a_0 - K^a_0 K^a_0 K^b_0 K^b_0)\langle v; 0 \rangle. \quad (2.65) $$

Then this (2.65) becomes

$$ -\frac{1}{2N} (-i)^4 Tr(T^b T^a T^b T^a - T^a T^b T^b T^a)\langle v; 0 \rangle. \quad (2.66) $$

Furthermore, this (2.66) will reduce to

$$ -\frac{1}{2N} (-i)^4 f^{bac} Tr(T^c T^b T^a)\langle v; 0 \rangle, \quad (2.67) $$

This (2.67) is equivalent to (2.55) with an extra minus sign.

One can continue to calculate the eigenvalue equation corresponding to $A_{12}$ term as follows:

$$ (J^a \partial^2 K^a)\langle v; 0 \rangle = 2K^a_0 J^a_0\langle v; 0 \rangle = -2K^a_0 K^a_0\langle v; 0 \rangle. \quad (2.58) $$

Then this (2.58) is the same contribution from $A_{11}$ term.
by combining the first two generators. This (2.67) is equivalent to (2.62) and (2.63).

Are there any contributions from the $A_{15}$-$A_{20}$ terms? These coefficients behave as $N^0$, $\frac{1}{N^2}$, $\frac{1}{N}$, $N^0$, $\frac{1}{N}$ and $N^0$ respectively from Appendix (E.3). There are no contributions. Then one obtains the final eigenvalue equation as follows:

$$W_0^{(4)}|(v; 0) > = \left[ 8N^3 A_1 - 8N^3 \left( \frac{4\lambda}{\lambda - 1} \right) A_1 + 8N^3 \left( \frac{12\lambda^2}{(\lambda - 1)(2\lambda - 3)} \right) A_1 \right. \\
- 8N^3 \left( \frac{8\lambda^3}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} \right) A_1 - 2N \left( \frac{N^2 12(2\lambda - 9)}{5(2\lambda - 3)} \right) A_1 \\
- 4N \left( -N^2 \frac{8(2\lambda - 9)}{5(2\lambda - 3)} \right) A_1 + 2N \left( \frac{N^2 48(\lambda - 2)\lambda}{5(\lambda - 1)(2\lambda - 3)} \right) A_1 \\
+ 4N \left( -N^2 \frac{16(\lambda - 12)\lambda}{5(\lambda - 1)(2\lambda - 3)} \right) A_1 + 4N \left( -N^2 \frac{16\lambda^2 + 15\lambda + 6}{5(\lambda - 3)(\lambda - 1)(2\lambda - 3)} \right) A_1 \\
+ 4N^2 \left( -N \frac{24\lambda}{(\lambda - 1)(2\lambda - 3)} \right) A_1 + 4N^2 \left( N \frac{48\lambda^2}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} \right) A_1 \\
\times \left. |(v; 0) > = -N^3 \left[ \frac{96(\lambda + 1)(\lambda + 2)(\lambda + 3)}{5(\lambda - 3)(\lambda - 1)(2\lambda - 3)} \right] A_1 |(v; 0) > \right]. \hspace{1cm} (2.68)$$

The eigenvalue has a simple factorized form.

With the following normalization,

$$A_1 = -\frac{5}{96N^3} (\lambda - 3)(\lambda - 1)(2\lambda - 3), \hspace{1cm} (2.69)$$

the two eigenvalue equations, (2.44) and (2.68), lead to

$$W_0^{(4)}|(v; 0) > = (1 + \lambda)(2 + \lambda)(3 + \lambda)|/(v; 0) >, \hspace{1cm} (2.70)$$

If one takes the overall normalization factor for the $W^{(4)}(z)$ as $A_4$ rather than $A_1$ as in (2.69), then $A_4$ becomes $A_4 = -\frac{5}{12N^3}\lambda^3$. In principle, one can calculate the OPE between $W^{(4)}(z)$ and $W^{(4)}(w)$ from the explicit twenty terms in (2.14) although the complete computation of the eighth order singular terms is rather involved for general $(k_2, N)$ manually. Then one expects that the central term, the eighth order pole of the above OPE, is given by $A_4^2 f(\lambda, N)$ where $f(\lambda, N)$ is a (fractional) function of $\lambda$ and $N$ (after the large $N$ limit is taken). That is, our normalization is given by the central term of the OPE between the higher spin 4 current and itself which behaves as $\frac{25}{144N^6}\lambda^6 f(\lambda, N)$ where $f(\lambda, N)$ is not known at the moment.

The above eigenvalues are also observed in [23] by following the descriptions in [28] where the unitary case is analyzed.
One of the primaries is given by \((v; 0) \otimes (v; 0)\) and the other primary is given by \((0; v) \otimes (0; v)\) by pairing up identical representations on the holomorphic and antiholomorphic sectors in the context of diagonal modular invariant [29]. Let us denote them as follows:

\[
\mathcal{O}_+ = (v; 0) \otimes (v; 0), \quad \mathcal{O}_- = (0; v) \otimes (0; v).
\] (2.71)

The ratio of the three-point functions, from (2.70), is given by

\[
\frac{\langle \mathcal{O}_+ \mathcal{O}_+ W^{(4)} \rangle}{\langle \mathcal{O}_- \mathcal{O}_- W^{(4)} \rangle} = \frac{(1 + \lambda)(2 + \lambda)(3 + \lambda)}{(1 - \lambda)(2 - \lambda)(3 - \lambda)},
\] (2.72)

in the notation of (2.71). This is the same form for the unitary case [20, 30]. In the corresponding unitary bulk calculation of [29], for \(\lambda = \frac{1}{2}\), this ratio for generic spin is given by \((-1)^s(2s - 1)\) with spin \(s\). One expects that the orthogonal bulk computation will give rise to the behavior of (2.72).

### 2.3.3 Eigenvalue equation of the zero mode of the higher spin 4 current acting on the state \(|(v; v)\rangle\)

For the primary \((v; v)\) with the condition \(J^a_0|(v; v)\rangle = 0\), one can calculate the eigenvalue equation [8]. The nontrivial contributions arise from the \(A_5, A_8,\) and \(A_9\) terms. It turns out that

\[
W_0^{(4)}|(v; v)\rangle = -N^2 \frac{48\lambda^2(\lambda^2 + 1)}{5(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1|(v; v)\rangle.
\] (2.73)

In (2.73), Appendix (E.3) is used.

### 2.3.4 Further eigenvalue equations

One also presents the eigenvalue equations [30] at finite \(N\) and \(k_2\), by using Appendix (E.1) and Appendix (E.2), as follows:

\[
W_0^{(4)}|(0; v)\rangle = \frac{6A_1}{(3k_2 + 2N - 2)d(1, k_2, N)} \times (32k_2^3N^4 - 64k_2^2N^2 + 96k_2^2N - 55k_2 + 160k_2^2N^5 - 168k_2^2N^4 - 268k_2^2N^3 + 690k_2^2N^2 - 617k_2^2N + 203k_2^2 + 128k_2N^6 - 64k_2N^5 - 680k_2N^4 + 1548k_2N^3 - 1726k_2N^2 + 987k_2N - 220k_2 + 256N^6 - 1024N^5 + 1584N^4 - 1384N^3 + 900N^2 - 382N + 68)|\langle 0; v\rangle|,
\]

\[
W_0^{(4)}|(v; 0)\rangle = \frac{6(k_2 + 2N - 1)(k_2 + 4N - 3)(3k_2 + 8N - 5)A_1}{k_2(k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(1, k_2, N)} \times (32k_2^3N^4 - 64k_2^2N^2 + 96k_2^2N - 55k_2 + 224k_2^2N^5 - 120k_2^2N^4 - 500k_2^2N^3 + \ldots)
\]

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+ 1038k_2^2N^2 - 907k_2^2N + 292k_2^2 + 384k_2N^6 - 64k_2N^5 - 1752k_2N^4
+ 3636k_2N^3 - 3930k_2N^2 + 2213k_2N - 487k_2 + 1024N^6 - 3328N^5
+ 5184N^4 - 5576N^3 + 3976N^2 - 1566N + 250)|v; 0⟩. \quad (2.74)

Of course, these eigenvalue equations (2.74) becomes (2.44) and (2.68) respectively under the large N ’t Hooft limit. Compared to the unitary case in [30], the above eigenvalues do not have simple factorized form. This is because of the fact that the identities between f and d symbols contain rather complicated functions of \(N\).

For convenience, we also present the eigenvalue equations for the spin 2 stress energy tensor (2.2) with \(k_1 = 1\)

\[
T_0|0; v⟩ = \left(\frac{k_2}{2(k_2 + 2N - 1)}\right)|0; v⟩ \rightarrow (1 - \lambda) \left(\frac{2}{2}\right)|0; v⟩,
\]

\[
T_0|v; 0⟩ = \left(\frac{(k_2 + 4N - 3)}{2(k_2 + 2N - 2)}\right)|v; 0⟩ \rightarrow (1 + \lambda) \left(\frac{2}{2}\right)|v; 0⟩. \quad (2.75)
\]

Note that the conformal dimension of \((0; v)\) can be obtained from the formula \([4, 31, 32, 22, 8]\)

\[
h(0; v) = \frac{1}{2}(2N - 1) \left[\frac{1}{1 + (2N - 2)} - \frac{1}{1 + k_2 + (2N - 2)}\right] = \left(\frac{k_2}{2(k_2 + 2N - 1)}\right), \quad (2.76)
\]

where the overall factor \(\frac{1}{2}(2N - 1)\) is the quadratic Casimir of \(SO(2N)\) vector representation. Similarly, the conformal dimension of \((v; 0)\) can be obtained

\[
h(v; 0) = \frac{1}{2}(2N - 1) \left[\frac{1}{1 + (2N - 2)} + \frac{1}{k_2 + (2N - 2)}\right] = \left(\frac{k_2 + 4N - 3}{2(k_2 + 2N - 2)}\right). \quad (2.77)
\]

Then the two results \((2.76)\) and \((2.77)\) are coincident with the ones in \((2.75)\).

Furthermore, one can write down the eigenvalue equation for the state \(|v; v⟩\)

\[
T_0|v; v⟩ = \left(\frac{2N - 1}{2(k_2 + 2N - 2)(k_2 + 2N - 1)}\right)|v; v⟩ \rightarrow \lambda^2 \left(\frac{4N}{4N}\right)|v; v⟩. \quad (2.78)
\]

Note that under the large N ’t Hooft limit the eigenvalue \((2.78)\) reduces to zero.

The conformal dimension of \((v; v)\) can be obtained as follows:

\[
h(v; v) = \frac{1}{2}(2N - 1) \left[\frac{1}{k_2 + (2N - 2)} - \frac{1}{1 + k_2 + (2N - 2)}\right]
= \frac{(2N - 1)}{2(k_2 + 2N - 2)(k_2 + 2N - 1)}. \quad (2.79)
\]

This looks similar to the unitary case [28]: the overall factor is again the quadratic Casimir of \(SO(2N)\) in vector representation. In the denominator one has \((k_2 + 2N - 2)\) and this quantity plus one. There exists a relation together with \((2.76), (2.77)\) and \((2.79)\),

\[
h(v; v) = h(0; v) + h(v; 0) - 1, \quad (2.80)
\]

which was also observed in [22]. The identity in \((2.80)\) is checked from \((2.75)\) and \((2.78)\).
2.4 The OPE between the higher spin 4 current and itself where \( k_1 = 1, N = 4 \) and \( k_2 \) is arbitrary

Let us describe the OPE between the higher spin 4 current and itself. Because it is rather involved to calculate this OPE manually, one fixes the value of \( N \) and then one can compute this OPE inside the package of [27]. For fixed \( N = 4 \) which is the lowest value one can consider nontrivially, one obtains the following fourth order pole of this OPE, by realizing that the right structure constants should behave according to the known results [33], as follows:

\[
\left. W^{(4)}(z) W^{(4)}(w) \right|_{(z-w)^4} = \frac{3}{10} \partial^2 T(w) + \frac{42}{(5c + 22)} \left( T^2 - \frac{3}{10} \partial^2 T \right)(w) + \sqrt{\frac{18(c + 24)}{(5c + 22)}} W^{(4)}(w) + W^{(4')}(w).
\] (2.81)

Here the central charge reduces to

\[
c(k_1 = 1, k_2, N = 4) = \frac{4k_2(k_2 + 13)}{(k_2 + 6)(k_2 + 7)},
\] (2.82)

which can be obtained from (2.4) by substituting the two values of \( k_1 = 1 \) and \( N = 4 \). The overall factor can be fixed as

\[
A_1(k_1 = 1, k_2, N = 4) = \frac{k_2}{2520(k_2 + 7)} \sqrt{\frac{(k_2 + 2)(k_2 + 4)}{3(k_2 + 9)(k_2 + 11)}},
\] (2.83)

by comparing the coefficient of the first term in the right hand side of (2.81).

Let us emphasize that there exists a new primary field in (2.81) which is given by

\[
\left. W^{(4')}(z) \right|_{k_1 = 1, k_2, N = 4} = \frac{1}{(k_2 + 2)(k_2 + 11)} \left[ -\frac{1}{9} d^{abcd} J^a J^b K^c K^d + \frac{2}{35} (k_2 - 1) k_2 \partial J^a \partial J^a - \frac{4}{105} (k_2 - 1) k_2 \partial^2 J^a J^a + \frac{28}{15} \partial J^a \partial K^a + \frac{64}{105} (k_2 - 1) \partial^2 J^a K^a - \frac{4}{35} (k_2 - 1) f^{abc} J^a \partial J^b K^c - \frac{2}{735} (k_2 - 1) k_2 J^a J^a J^b J^b - \frac{68}{315} J^a J^a K^b K^b + \frac{8}{105} (k_2 - 1) J^a J^a J^b K^b - \frac{28}{45} J^a J^b K^a K^b + \frac{1}{90} d^{abcdef} d^{gfh} J^a J^b K^c K^d \right](z). \] (2.84)

In other words, there exists a nonzero expression by combining the fourth order pole with the first line of (2.81) with minus sign. Furthermore, one can express the various nonzero terms as the one in (2.84). One can easily see that the ten operators except the last operator appear in the previous higher spin 4 current in (2.14). It is straightforward to analyze the description appearing in Appendix B and Appendix C for the last operator in (2.84).

Let us further restrict to the simplest case where one can see the full structure of the corresponding OPE without losing any terms in the right hand side. In other words, in this
particular limit where \( k_2 \to \infty \) corresponding to \( c = 4 \), the structure constants do not vanish. That is, there is no \((c - 4)\) factor in the right hand side of the OPE.

Then the higher spin 4 current can be written in terms of

\[
W^{(4)}(z) \bigg|_{k_1=1,k_2 \to \infty,N=4} = \frac{1}{2520\sqrt{3}} \left(d^{abcd} J^a J^b J^c J^d + 18 \partial J^a \partial J^a - 12 \partial^2 J^a J^a - 3 J^a J^b J^b \right)(z),
\]

(2.85)

by substituting \( N = 4 \) and \( k_2 \to \infty \) limit in Appendix E. The normalization factor is consistent with the general form in (2.83). The field contents in (2.85) are given in terms of the numerator spin 1 current (having the level \( k_1 = 1 \)) of the coset model. Of course, the stress energy tensor contains only the first term with \( k_1 = 1 \) in (2.2) in this limit.

Then one can obtain the corresponding higher spin 4' current from (2.84) by taking \( k_2 \to \infty \) limit and it turns out that

\[
W^{(4)'}(z) \bigg|_{k_1=1,k_2 \to \infty,N=4} = \frac{2}{35} \left( \partial J^a \partial J^a - \frac{2}{3} \partial^2 J^a J^a - \frac{1}{21} J^a J^a J^b J^b \right)(z).
\]

(2.86)

In (2.86), there is no \( d \) symbol.

Now we can calculate the OPE between the higher spin 4 current (2.85) and itself as follows:

\[
\begin{align*}
W^{(4)}(z) W^{(4)}(w) & = \frac{1}{(z-w)^8} \cdot \frac{c}{4} + \frac{1}{(z-w)^6} 2T(w) + \frac{1}{(z-w)^5} \frac{1}{2} \partial T(w) \\
+ \frac{1}{(z-w)^4} \left[ \frac{3}{20} \partial^2 T + \frac{42}{(5c+22)} \left( T^2 - \frac{3}{10} \partial^2 T \right) + C_{44}^4 W^{(4)} \right](w) \\
+ \frac{1}{(z-w)^3} \left[ \frac{1}{30} \partial^2 T + \frac{1}{2} \frac{42}{(5c+22)} \partial \left( T^2 - \frac{3}{10} \partial^2 T \right) + \frac{1}{2} C_{44}^4 \partial W^{(4)} \right](w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{1}{168} \partial^2 T + \frac{5}{36} \frac{42}{(5c+22)} \partial^2 \left( T^2 - \frac{3}{10} \partial^2 T \right) + \frac{5}{36} C_{44}^4 \partial^2 W^{(4)} \right](w) \\
+ \frac{24(72c+13)}{\left(5c+22\right)(2c-1)(7c+68)} \left( T(T^2 - \frac{3}{10} \partial^2 T) + \frac{1}{70} \partial^4 T \right) \\
- \frac{95c^2 + 1254c - 10904}{6\left(5c+22\right)(2c-1)(7c+68)} \left( \frac{1}{2} \partial^2(T^2 - \frac{3}{10} \partial^2 T) - \frac{9}{5} \partial^2 TT + \frac{3}{70} \partial^4 T \right) \\
+ \frac{28}{3\left(c+24\right)} C_{44}^4 \left( TW^{(4)} - \frac{1}{6} \partial^2 W^{(4)} \right) + C_{44}^6 W^{(6)} \right](w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{1120} \partial^2 T + \frac{1}{36} \frac{42}{(5c+22)} \partial^3 \left( T^2 - \frac{3}{10} \partial^2 T \right) + \frac{1}{36} C_{44}^4 \partial^3 W^{(4)} \right](w) \\
+ \frac{1}{2} \frac{24(72c+13)}{(5c+22)(2c-1)(7c+68)} \partial \left( T(T^2 - \frac{3}{10} \partial^2 T) - \frac{3}{5} \partial^2 TT + \frac{1}{70} \partial^4 T \right)
\end{align*}
\]
\[-\frac{1}{2} \frac{(95c^2 + 1254c - 10904)}{6(5c + 22)(2c - 1)(7c + 68)} \partial \left( \frac{1}{2} \partial^2 (T^2 - \frac{3}{10} \partial^2 T) - \frac{9}{5} \partial^2 TT + \frac{3}{70} \partial^4 T \right) + \frac{1}{2} \frac{28}{3(c + 24)} C^4_{44} \partial \left( TW^{(4)} - \frac{1}{6} \partial^2 W^{(4)} \right) + \frac{1}{2} C^6_{44} \partial W^{(6)} \right] (w)
\]
\[+ \frac{1}{(z-w)^3} W^{(4')}(w) + \frac{1}{(z-w)^3} \frac{1}{2} \partial W^{(4')}(w) + \frac{1}{(z-w)^3} \frac{1}{3} \left( TW^{(4')} - \frac{1}{6} \partial^2 W^{(4')} \right) (w)
\]
\[+ \frac{1}{(z-w)^3} \frac{1}{2} \partial \left( TW^{(4')} - \frac{1}{6} \partial^2 W^{(4')} \right) (w) + \ldots. \quad (2.87)\]

Here the central charge coming from (2.82) is given by

\[c(k_1 = 1, k_2 \to \infty, N = 4) = 4, \quad (2.88)\]

from (2.4) by substituting the right numbers. Moreover the two structure constants are given by

\[C^4_{44} = \sqrt{\frac{18(c + 24)}{(5c + 22)}}, \quad C^6_{44} = \sqrt{\frac{12(c - 1)(11c + 656)}{(2c - 1)(7c + 68)}}, \quad (2.89)\]

together with (2.88). Note that there are extra two last lines in (2.87) associated with the new primary higher spin 4' current, compared to the previous result in [33]. The expression in (2.89) already appeared in [34, 35, 33].

### 2.5 Next higher spin currents

In the second order pole of (2.87), there exists a primary higher spin 6 current. One can imagine the six product of spin 1 current with correct contractions of \(SO(2N)\) indices. Let us consider the higher spin 4 current \(W^{(4)}(z)\) which contains \(d^{abcd} J^a J^b J^c J^d(z)\) and the same higher spin 4 current which contains \(d^{defg} J^d J^e J^f J^g(z)\). Then one has the second order pole of this OPE, \(d^{abcd} d^{defg} \delta^{dd'} J^a J^b J^c J^d J^e J^f J^g(w)\), by considering the singular term between \(J^d(z)\) and \(J^{d'}(w)\). This gives rise to the term of \(d^{abcd} d^{defg} J^a J^b J^c J^d J^e J^f J^g(w)\). Then one expects that the higher spin 6 current contains this term and is given by \(W^{(6)}(z) = d^{abcd} d^{defg} J^a J^b J^c J^d J^e J^f J^g(z) + \ldots.\) According to the description of [16, 17], the tensorial structure of \(SO(2N)\) symmetric invariant tensor of rank 6 can be determined by the product of two rank 4 \(d\) symbols. Therefore the above expression can be rewritten in terms of \(d\) tensor of rank 6 and one should have \(W^{(6)}(z) = d^{abcde} J^a J^b J^c J^d J^e J^f J^g(z) + \ldots.\) It would be interesting to observe the full expression for the higher spin 6 current.
3 Higher spin currents with $\mathcal{N} = 1$ supersymmetry in the stringy coset model with two levels $(2N - 2, k_2)$

In the presence of adjoint fermions coming from the equality of one of the levels and the dual Coxeter number of $SO(2N)$, one can construct the higher spin $\frac{7}{2}$ current which is the superpartner of the previous higher spin $4$ current. In doing this, the role of spin $\frac{3}{2}$ current living in the $\mathcal{N} = 1$ superconformal algebra is crucial. The OPE between this $\mathcal{N} = 1$ lowest higher spin multiplet denoted by $(\frac{7}{2}, 4)$ is described using the Jacobi identities.

3.1 Spin $\frac{3}{2}, 2$ currents and $\mathcal{N} = 1$ superconformal algebra

The spin $\frac{3}{2}$ current can be obtained from the spin $\frac{1}{2}$ current and spin $1$ current as follows [15, 6]:

$$G(z) = \sqrt{\frac{4(N - 1)}{(2N - 2 + k_2)(4N - 4 + k_2)}} \left( \frac{k_2}{6(N - 1)} \psi^a J^a - \psi^a K^a \right)(z).$$  \hspace{1cm} (3.1)

Here one has

$$\psi^a(z) \psi^b(w) = -\frac{1}{(z-w)^{\frac{3}{2}}} \delta^{ab} + \cdots.$$  \hspace{1cm} (3.2)

Furthermore, we can express the spin $1$ current from the above spin $\frac{1}{2}$ current satisfying (3.2) as

$$J^a(z) \equiv f^{abc} \psi^b \psi^c(z).$$  \hspace{1cm} (3.3)

It is easy to check this spin $1$ current satisfies the first equation of (2.3) with $k_1 = (2N - 2)$. Then it is easy to see that there are only two terms in (3.1) and the relative coefficients can be fixed by using the above spin $\frac{3}{2}$ current should transform as a primary field under the stress energy tensor (2.2) with $k_1 = (2N - 2)$ as follows:

$$T(z) G(w) = \frac{1}{(z-w)^2} \frac{3}{2} G(w) + \frac{1}{(z-w)} \partial G(w) + \cdots.$$  \hspace{1cm} (3.4)

In other words, the condition (3.4) determines the relative coefficients of (3.1).

The overall factor in (3.1) can be determined by the following OPE between the spin $\frac{3}{2}$ current and itself

$$G(z) G(w) = \frac{1}{(z-w)^3} \frac{2c}{3} + \frac{1}{(z-w)} 2T(w) + \cdots.$$  \hspace{1cm} (3.5)

Here the central charge in (3.5) is given by (2.4) with the condition $k_1 = (2N - 2)$. 

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It is useful to write down the following OPEs which will be used in later calculations

\[
\hat{G}(z) \psi^a(w) = \frac{1}{(z-w)^2} k_2 \psi^a(w) + \frac{1}{(z-w)} \left[ -k_2 \partial \psi^a - f^{abc} \psi^b K^c \right](w) + \cdots,
\]

\[
\hat{G}(z) J^a(w) = \frac{1}{(z-w)} k_2 \psi^a(w) + \frac{1}{(z-w)} \left[ k_2 \partial \psi^a + f^{abc} \psi^b K^c \right](w) + \cdots,
\]

\[
\hat{G}(z) K^a(w) = \frac{1}{(z-w)^2} k_2 \psi^a(w) + \frac{1}{(z-w)} \left[ k_2 \partial \psi^a + f^{abc} \psi^b K^c \right](w) + \cdots,
\]

where we introduce the following quantity

\[
\hat{G}(z) \equiv \left( \frac{k_2}{6(N-1)} \psi^a J^a - \psi^a K^a \right)(z).
\]

Compared to the unitary case \([36, 37, 38]\), the behavior of relative coefficient, which is equal to one over three times the level divided by dual Coxeter number, occurs in (3.7). See also \([6, 15]\).

### 3.2 Eigenvalue equation of the zero mode of the higher spin 4 current

We also present the eigenvalue equations for the spin 2 stress energy tensor (2.2) with \(k_1 = (2N - 2)\)

\[
T_0(0; v) > = \frac{k_2(2N-1)}{8(N-1)(k_2 + 4N - 4)} |(0; v) > \rightarrow \frac{(1 - \lambda)}{4(1 + \lambda)} |(0; v) >, \\
T_0(v; 0) > = \frac{(2N-1)(k_2 + 6N - 6)}{8(N-1)(k_2 + 2N - 2)} |(v; 0) > \rightarrow \frac{(1 + 2\lambda)}{4} |(v; 0) >.
\]

In (3.8), the large \(N\) ’t Hooft limit is taken at the final stage. Note that the conformal dimension of \((0; v)\) can be obtained from the formula

\[
h(0; v) = \frac{1}{2}(2N - 1) \left[ \frac{1}{(2N - 2) + (2N - 2)} - \frac{1}{(2N - 2) + k_2 + (2N - 2)} \right] \\
= \frac{k_2(2N-1)}{8(N-1)(k_2 + 4N - 4)},
\]

where the overall factor \(\frac{1}{2}(2N - 1)\) is the quadratic Casimir of \(SO(2N)\) vector representation. Similarly, the conformal dimension of \((v; 0)\) can be obtained

\[
h(v; 0) = \frac{1}{2}(2N - 1) \left[ \frac{1}{(2N - 2) + (2N - 2) + \frac{1}{k_2 + (2N - 2)}} \right]
\]

Moreover, the eigenvalue equation for the state \(|(v; v) >\) can be obtained as follows:

\[
T_0(v; v) > = \frac{(N-1)(2N-1)}{(k_2 + 2N - 2)(k_2 + 4N - 4)} |(v; v) > \rightarrow \frac{\lambda^2}{2(\lambda + 1)} |(v; v) >.
\]
\[
\frac{(2N - 1)(k_2 + 6N - 6)}{8(N - 1)(k_2 + 2N - 2)}.
\]  

(3.13)

As done in section 2, one obtains the following eigenvalue equations

\[
W^{(4)}_0|0; v\rangle = -N^3 \left[ \frac{48}{(2\lambda - 3)} \right] A_1|0; v\rangle,
\]

\[
W^{(4)}_0|(v; 0)\rangle = -N^3 \left[ \frac{48(\lambda + 1)^2(2\lambda + 1)(4\lambda + 3)}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} \right] A_1|(v; 0)\rangle,
\]

\[
W^{(4)}_0|(v; v)\rangle = -N^3 \left[ \frac{96\lambda^2(4\lambda^2 + \lambda + 1)}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} \right] A_1|(v; v)\rangle.
\]

(3.14)

Using these relations (3.14), one can obtain the several three-point functions. The relevant coefficients are given in Appendix (F.3).

### 3.3 Higher spin \( \frac{7}{2}, 4 \) currents

One way to determine the higher spin \( \frac{7}{2}, 4 \) current is to use the OPE between the spin \( \frac{3}{2} \) current and the higher spin 4 current in previous section. Note that the corresponding coefficients at the critical level \( k_1 = (2N - 2) \) are given in Appendix F. In other words, from the \( \mathcal{N} = 1 \) super primary condition \cite{36,37}, one should have

\[
\hat{G}(z) W^{(4)}(w) \bigg|_{(z-w)^2} = \frac{1}{(z-w)^2} \sqrt{\frac{(2N - 2 + k_2)(4N - 4 + k_2)}{4(N - 1)}} W^{(4)}(w)
\]

\[
+ \mathcal{O}\left(\frac{1}{(z-w)}\right).
\]

(3.15)

In order to calculate the second order pole of (3.15), one can use the three OPEs in (3.6). The explicit results are given in Appendix G. Of course, this will give us the final higher spin \( \frac{7}{2} \) current but it is rather nontrivial to simplify in simple form. Therefore, after we identify

The conformal dimension of \( (v; v) \) in (3.10) can be also obtained as follows:

\[
h(v; v) = \frac{1}{2} (2N - 1) \left[ \frac{1}{k_2 + (2N - 2)} - \frac{1}{(2N - 2) + k_2 + (2N - 2)} \right]
\]

\[
= \frac{(N - 1)(2N - 1)}{(k_2 + 2N - 2)(k_2 + 4N - 4)}.
\]

(3.11)

There exists a relation together with (3.9), (3.13) and (3.11),

\[
h(v; v) = h(0; v) + h(v; 0) - \frac{(2N - 1)}{4(N - 1)}.
\]

(3.12)

Here the last term in (3.12) is the ratio of quadratic Casimir for the vector representation and dual Coxeter number of \( SO(2N) \).
the correct field contents for fixed $N = 4$, introduce the undetermined coefficients and fix them using the previous methods we used in previous section. That is, the higher spin $\frac{7}{2}$ current should not have any singular terms with the diagonal spin 1 current and transform as a primary higher spin current under the stress energy tensor.

Then one can express the higher spin $\frac{7}{2}$ current as follows [18, 19, 6]:

$$W^{(2)}(z) = B_1 d^{abcd} \psi^a J^b J^c J^d(z) + B_2 d^{abcd} f^{aef} f^{beg} J^f K^g(z) + B_3 d^{abcd} J^a K^b \psi^c K^d(z)$$

$$+ B_4 d^{abcd} f^{aef} f^{beg} K^c K^d \psi^f K^g(z) + B_5 J^a \psi^a J^b(z) + B_6 K^a \psi^b K^b(z)$$

$$+ B_7 J^a J^a \psi^b K^b(z) + B_8 J^a \psi^b J^b(z) + B_9 \psi^a K^a K^b(z)$$

$$+ B_{10} f^{abc} f^{cde} K^a K^c \psi^b K^d(z) + B_{11} J^a \psi^a K^b K^b(z) + B_{12} \psi^a J^b K^a K^b(z)$$

$$+ B_{13} J^a J^b \psi^a K^b(z) + B_{14} f^{abc} f^{cde} J^a J^b \psi^c K^d(z) + B_{15} J^a J^b K^a \psi^b(z)$$

$$+ X(k_2, N)(GT - \frac{1}{8} \partial^2 G)(z).$$

(3.16)

The $B_7$ term can be written as $(\psi^a J^b J^b K^a - 2 f^{abc} \psi^a \partial J^b K^c - (2N - 2) \partial^2 \psi^a K^a)(z)$ by moving the field $\psi^b$ to the left. Similarly, the $B_8$ term can be described as $(\psi^a J^b J^b + (2N - 2) \partial^2 \psi^a J^a(2N - 2) \partial^2 \psi^a \partial J^a)(z)$. For the $B_{13}$ term one obtains $(\psi^a J^b J^a K^b + (2N - 2) \partial^2 \psi^a K^a)(z)$. For the $B_{14}$ term one can write down $(3(2N - 2) f^{abc} \psi^a \partial J^b K^c + (2N - 2) \partial^2 \psi^a K^a)(z)$. For the $B_{15}$ term, one has $(\psi^a J^b J^a K^b + f^{abc} \psi^a \partial J^b K^c)(z)$ by moving $\psi^b$ to the left. Furthermore, the $B_2$ and the $B_4$ term can be simplified using the identity (2.10). For the remaining other terms, the fermion $\psi^a$ can be moved to the leftmost without any extra terms because of the properties of $f$ and $d$ symbols. The $B_5, B_6, B_7, B_{11}, B_{12},$ and $B_{13}$ terms can be seen from $GT(z)$. The $B_8, B_9,$ and $B_{15}$ terms are written in terms of $B_5, B_6,$ and $B_{13}$ terms plus derivative terms respectively.

Note that the last term in (3.16) is a quasiprimary field in the sense that the OPE between the stress energy tensor and this field does not contain the third order pole. We realize that this term does not appear for the particular $N = 4$ case.

We would like to determine the undetermined coefficients $B_1 - B_{15}$ and $X(k_2, N)$ in (3.16). As done in (2.15), one should have the regular condition as follows:

$$J^a(z) W^{(2)}(w) = + \cdots.$$  

(3.17)

In Appendix $H$, we present the OPEs between the diagonal spin 1 current and the 15 fields in (3.16). Moreover, the higher spin $\frac{7}{2}$ current transforms as a primary field under the stress energy tensor. In other words, one has

$$\hat{T}(z) W^{(2)}(w) \bigg|_{\frac{1}{(z-w)^{n+1}}, n=3,4,5} = 0,$$  

(3.18)
as in \((2.23)\). One obtains the corresponding OPEs in Appendix \(I\).

By solving the various linear equations on the coefficients satisfying the above requirements \((3.17)\) and \((3.18)\), one obtains the final coefficients in Appendix \(J\). There are in Appendix \((J.1)\), Appendix \((J.2)\) and Appendix \((J.3)\).

For consistency check, one can calculate the OPE between \(\hat{G}(z)\) and \(W^{(\frac{7}{2})}(w)\). The first order pole should behave as
\[
\sqrt{\frac{(2N-2+k_2)(4N-4+k_2)}{4(N-1)}} W^{(4)}(w).
\]
In doing this, the OPEs in \((3.6)\) are crucial. In order to see the presence of higher spin 4, the rearrangement of the normal ordered product should be taken because the above first order pole terms contain unwanted terms. Of course, we do not have to worry about the extra contractions in the OPEs because we are interested in the first order pole as described above.

### 3.4 The OPEs between the higher spin \(\frac{7}{2}, 4\) currents

It is natural to ask how the OPEs between the higher spin \(\frac{7}{2}\) current and the higher spin 4 current arise. They have rather long expressions for \(N = 4\) case.

Therefore, one tries to obtain the corresponding OPEs from the Jacobi identities for the above higher spin currents and other relevant higher spin currents. We will consider only the three OPEs, \(W^{(\frac{7}{2})}(z) W^{(\frac{7}{2})}(w)\), \(W^{(\frac{7}{2})}(z) W^{(4)}(w)\) and \(W^{(4)}(z) W^{(4)}(w)\). What kind of new primary higher spin currents are present in the right hand side of OPEs? From the OPEs of \(W^{(\frac{7}{2})}(z) W^{(4)}(w)\) or \(W^{(4)}(z) W^{(\frac{7}{2})}(w)\), one can think of the presence of new higher spin \(\frac{13}{2}\) current at the first order pole. Furthermore, from the OPE \(W^{(4)}(z) W^{(4)}(w)\), the new higher spin 6 current can appear in the second order pole of this OPE. Note that there is no new higher spin 7 current in the first order pole. The reason is as follows. One can calculate the OPE \(W^{(4)}(w) W^{(4)}(z)\) in the presence of the new higher spin 7 current at the first order pole, use the symmetry \(z \leftrightarrow w\) and end up with the OPE \(W^{(4)}(z) W^{(4)}(w)\). By focusing on the new higher spin 7 current, one realizes that there exists an extra minus sign. Therefore, the new higher spin 7 current should vanish.

Then one can assign the above two higher spin currents as one single \(\mathcal{N} = 1\) higher spin current, denoted by \((6', \frac{13}{2})\) where the numbers stand for each spin. From the OPE in \(W^{(4)}(z) W^{(4)}(w)\), the second order pole provides a new higher spin 6 current. Then one can think of \(\mathcal{N} = 1\) higher spin current denoted by \((\frac{11}{2}, 6)\). Furthermore, from the bosonic higher spin 4' current in previous section, one can introduce its superpartner whose spin is given by \(\frac{9}{2}\). The corresponding \(\mathcal{N} = 1\) higher spin current is characterized by \((4', \frac{9}{2})\) via above notation.

For \(N = 4\), one can calculate the OPE in \(W^{(\frac{7}{2})}(z) W^{(\frac{7}{2})}(w)\). By requiring that the seventh
order pole should be equal to $2^{\frac{2c}{7}}$, one can determine the coefficient $A_1$ as

$$A_1(k_1 = 6, k_2, N = 4) = \frac{k_2}{5040(k_2 + 12)} \sqrt{\frac{(k_2 + 2)(k_2 + 4)}{6(k_2 + 6)(k_2 + 12)(k_2 + 14)(k_2 + 16)}}.$$  (3.19)

The fifth order pole gives $2T(w)$ and the fourth order pole gives $\partial T(w)$. Similar behaviors arise in (2.87). Let us describe the third order pole. One can easily check that the following quantity together with (3.19),

$$\frac{1}{(4c + 21)(10c - 7)} \left[ 8(37c + 3)TT + 3(2c - 117)\partial GG - \frac{3}{10}(302c - 327)\partial^2 T \right](w),$$  (3.20)

is a quasiprimary field. The third order pole subtracted by both (3.20) and $\frac{2}{10}\partial^2 T(w)$ (which is a descendant field) is a primary field. However this is not written in terms of the previous higher spin $4'$ current. This implies that there exists a new primary higher spin $4'$ current. The structure constants appearing in (3.20) are obtained from the Jacobi identities. Because we are dealing with the extensions of $\mathcal{N} = 1$ superconformal algebra, the $\partial GG(w)$ term appears in addition to $TT(w)$ and $\partial^2 T(w)$.

By assuming that the $\mathcal{N} = 1$ OPE between the $\mathcal{N} = 1$ higher spin $\frac{7}{2}$ multiplet contains the $\mathcal{N} = 1$ higher spin $4'$, $\frac{11}{2}$, $6'$ multiplets, one obtains the complete structure of these OPEs in component approach (and $\mathcal{N} = 1$ superspace). They are given in Appendix K in terms of the central charge and some undetermined structure constants. It would be interesting to see whether there exist other additional higher spin currents or not. See also the work in [39] where the Jacobi identities are used.

### 3.5 The OPE in the $\mathcal{N} = 1$ superspace

From the three OPEs in component approaches described in Appendix K, one summarizes its $\mathcal{N} = 1$ superspace in simple notation as follows:

$$[W^{(\frac{7}{2})} \cdot W^{(\frac{7}{2})}] = [I] + [W^{(4')} + W^{(\frac{11}{2})} + W^{(6')}] + [W^{(\frac{11}{2})}] + [W^{(6')}],$$  (3.21)

where $[I]$ appearing in (3.21) is the $\mathcal{N} = 1$ superconformal family of the identity operator. According to the field contents in [40] where $k_2$ is fixed as $k_2 = 1$, the above OPE should not contain the $\mathcal{N} = 1$ higher spin integer multiplets. See also [41]. The right-hand side should contain the first, the second and the fourth terms. It would be interesting to observe this behavior explicitly. First of all, the single higher spin $4$ current should exist by combining the previous two kinds of higher spin $4$ currents under the constraint $k_2 = 1$.28
4 Higher spin currents with $\mathcal{N} = 2$ supersymmetry in the stringy coset model with two levels $(2N - 2, 2N - 2)$

The additional adjoint fermions allow us to construct the spin $1, \frac{3}{2}$ currents in the $\mathcal{N} = 2$ superconformal algebra. Furthermore, the additional higher spin $3, \frac{7}{2}$ currents can be found explicitly along the line of [42]. The lowest higher spin 3 current of $U(1)$ charge $\frac{4}{3}$ is obtained and it can be written in terms of two adjoint fermions. There exists another $\mathcal{N} = 2$ higher spin multiplet which consists of the above same spin contents, $(3, \frac{7}{2}; \frac{7}{2}, 4)$ with different $U(1)$ charges. Finally, the OPE between these two $\mathcal{N} = 2$ higher spin multiplets is described.

4.1 Spin $1, \frac{3}{2}, \frac{3}{2}, 2$ currents and $\mathcal{N} = 2$ superconformal algebra

Let us introduce the second adjoint fermions which satisfy the following OPE

$$\chi^a(z) \chi^b(w) = -\frac{1}{(z - w)} \frac{1}{2} \delta^{ab} + \ldots. \tag{4.1}$$

It is easy to see that one can express the spin 1 current from the above spin $\frac{1}{2}$ current with (4.1) as

$$K^a(z) \equiv f^{abc} \chi^b(z). \tag{4.2}$$

This spin 1 current satisfies the second equation of (2.3) with $k_2 = (2N - 2)$.

Then it is straightforward to construct the four generating currents, denoted by $(1, \frac{3}{2}, \frac{3}{2}, 2)$, corresponding to the $\mathcal{N} = 2$ superconformal algebra as follows [43]:

$$J(z) = \frac{2}{3} i \psi^a \chi^a(z),$$

$$G^+(z) = -\frac{1}{6 \sqrt{3(2N - 2)}} \left[ \psi^a J^a - 3 \psi^a K^a - i \chi^a K^a + 3 i \chi^a J^a \right](z),$$

$$G^-(z) = -\frac{1}{6 \sqrt{3(2N - 2)}} \left[ \psi^a J^a - 3 \psi^a K^a + i \chi^a K^a - 3 i \chi^a J^a \right](z), \tag{4.3}$$

$$T(z) = -\frac{1}{4(2N - 2)} J^a J^a(z) - \frac{1}{4(2N - 2)} K^a K^a(z) + \frac{1}{6(2N - 2)} (J^a + K^a)(J^a + K^a)(z).$$

By realizing that the difference between $G^+(z)$ and $G^-(z)$ occurs in the third and fourth terms, under the $\chi^a(z) \rightarrow -\chi^a(z)$, one sees the relation $G^+(z) \leftrightarrow G^-(z)$.

Let us introduce the following spin 1 current by taking the product of two adjoint fermions

$$L^a \equiv f^{abc} \psi^b \chi^c. \tag{4.4}$$

The central charge can be reduced to

$$c = \frac{1}{3} N(2N - 1), \tag{4.5}$$
which can be obtained from \([2,4]\) by substituting the corresponding two levels. In order to construct the higher spin currents, let us introduce the following intermediate spin 2 current

\[
M^a_1 \equiv \epsilon^{abcd} \psi^b \chi^c J^d, \\
M^a_2 \equiv \epsilon^{abcd} \psi^b \chi^c K^d, \\
M^a_3 \equiv \epsilon^{abcd} \psi^b \chi^c L^d, \quad (4.6)
\]

together with \((3.3), (4.2)\) and \((4.4)\). Compared to the unitary case in \([44]\), the contracted indices appear in the two different adjoint fermions (because of the symmetric \(d\) tensor) as well as the spin 1 currents.

### 4.2 Higher spin 3, \(\frac{7}{2}, \frac{7}{2}, 4\) currents

From the experience of section 2 and section 3, there exist the higher spin 4 current and the \(\mathcal{N} = 1\) higher spin \(\frac{7}{2}\) current denoted by \((\frac{7}{2}, 4)\), then there are two choices where the above \(\mathcal{N} = 1\) higher spin \(\frac{7}{2}\) multiplet can arise from the lower two component currents or higher two component currents. Let us try to find the higher spin currents by taking the second choice.

By writing the possible candidate terms for the higher spin 3 current, one can think of the product of spin 1 currents \((3.3), (4.2)\) or \((4.4)\) and the intermediate spin 2 currents in \((4.6)\). Furthermore, one can think of the product of each component field in the spin \(\frac{7}{2}\) currents living in the \(\mathcal{N} = 2\) superconformal algebra. Of course, one should consider the possible derivative terms. Therefore, one can consider the following higher spin 3 current \([0]\)

\[
W^{(3)}_{\frac{7}{2}}(z) = a_1 J^a M^a_1(z) + a_2 K^a M^a_1(z) + a_3 L^a M^a_1(z) + a_4 J^a M^a_2(z) + a_5 K^a M^a_2(z) + a_6 L^a M^a_2(z) \\
+ a_7 J^a M^a_3(z) + a_8 K^a M^a_3(z) + a_9 L^a M^a_3(z) + a_{10} J^a \partial J^a(z) + a_{11} J^a \partial K^a(z) \\
+ a_{12} J^a \partial L^a(z) + a_{13} J^a K^a(z) + a_{14} K^a \partial K^a(z) + a_{15} K^a \partial L^a(z) + a_{16} J^a \partial L^a(z) \\
+ a_{17} \partial K^a L^a(z) + a_{18} L^a \partial L^a(z) + a_{19} (\psi^a J^a)(\psi^b J^b)(z) + a_{20} (\psi^a J^a)(\psi^b K^b)(z) \\
+ a_{21} (\psi^a J^a)(\chi^b J^b)(z) + a_{22} (\psi^a J^a)(\chi^b K^b)(z) + a_{23} (\psi^a K^a)(\psi^b K^b)(z) \\
+ a_{24} (\psi^a K^a)(\chi^b J^b)(z) + a_{25} (\psi^a K^a)(\chi^b K^b)(z) + a_{26} (\chi^a J^a)(\chi^b J^b)(z) \\
+ a_{27} (\chi^a J^a)(\chi^b K^b)(z) + a_{28} (\chi^a K^a)(\chi^b K^b)(z). \quad (4.7)
\]

The \(U(1)\) charge \(\frac{4}{3}\) will be determined later.

As done in previous sections, one can use two requirements in order to fix the above coefficients. One of them is the regularity with the diagonal spin 1 current as follows:

\[
J^a(z) W^{(3)}_{\frac{7}{2}}(w) = + \cdots. \quad (4.8)
\]

Here the diagonal spin 1 current in \((4.8)\) is the sum of \((3.3)\) and \((4.2)\). The other is given by the primary condition, which can be described as follows together with \((4.7)\):

\[
\hat{T}(z) W^{(3)}_{\frac{7}{2}}(w) \bigg|_{\frac{1}{(z-w)^n}, n = 3, 4, 5} = 0. \quad (4.9)
\]
Here the stress energy tensor is given by (2.24) substituted by (3.3) and (1.2).

It turns out, from (4.8) and (1.9), that the above higher spin 3 current with the explicit coefficients is given by

\begin{align*}
W^{(3)}_{\frac{3}{4}}(z) &= \left[ -\frac{i}{4}(a_7 - a_8) J^a M^a_1 + \frac{i}{2}(a_7 - a_8) K^a M^a_1 - a_8 L^a M^a_1 - \frac{i}{4}(a_7 - a_8) K^a M^a_2 \\
&- a_7 L^a M^a_2 + a_7 J^a M^a_3 + a_8 K^a M^a_3 + i(a_7 - a_8) L^a M^a_3 \\
&+ \frac{3i}{4}(a_7 - a_8) J^a \partial L^a + \frac{3i}{4}(a_7 - a_8) K^a \partial L^a + \frac{3i}{4}(a_7 - a_8) \partial J^a L^a \\
&+ \frac{3i}{4}(a_7 - a_8) \partial K^a L^a + (-a_7 + a_8) (\psi^a J^b)(\psi^b K^a) + \frac{i}{2}(a_7 - a_8) (\psi^a J^a)(\chi^b J^b) \\
&- \frac{i}{2}(a_7 - a_8) (\psi^a J^a)(\chi^b K^b) + \frac{3i}{2}(a_7 - a_8) (\psi^a K^a)(\chi^b J^b) \\
&+ \frac{i}{2}(a_7 - a_8) (\psi^a K^a)(\chi^b K^b) + (a_7 - a_8) (\chi^a J^a)(\chi^b K^b) \right](z). \quad (4.10)
\end{align*}

Note that there exist also $a_{1,4,11,14,20,24}$ and $a_{28}$ dependent terms (other coefficients depend on these six coefficients and $a_7$ and $a_8$ after the above two conditions are used) but they are identically zero respectively. From the definitions of (4.6), the first eight terms in (4.10) contain the rank 4 $d$ symbol. One can see the common nonderivative expression in the third term and sixth term and then one can combine them with coefficient $(a_7 - a_8)$. Similarly, the fifth term and the seventh term share the common nonderivative term with the coefficient $-a_7 - a_8$. Furthermore, the composite fields appearing in (4.10) contain the various derivative terms (it is obvious that the ninth-twelfth terms do have the derivative terms and also they can appear from the ordering for the composite fields) but the precise coefficients will lead to the vanishing of these derivative terms.

For the extended $\mathcal{N} = 2$ superconformal algebra, there is one additional condition for the higher spin current which is the $U(1)$ charge (i.e., the coefficient of the first order pole of the OPE with the spin 1 current). That is $[44]$.

\begin{equation}
J(z) W^{(3)}_q(w) = \frac{1}{(z - w)} q W^{(3)}_q(w) + \cdots. \quad (4.11)
\end{equation}

It turns out that the $U(1)$ charge is fixed and for $q = \frac{4}{3}$, there is a relation $a_{12} = \frac{3i}{4}(a_7 - a_8)$. This relation is used in (4.10). For $q = -\frac{4}{3}$, there is a relation $a_{12} = -\frac{3i}{4}(a_7 - a_8)$. It is useful to express the above higher spin 3 current in manifestly $U(1)$ charge symmetric way. Let us focus on the first term in (4.10). If one substitutes the definition of $M^a_i$ in (4.6), one has $f^{abcdef} \psi^b \psi^e \psi^d \chi^e J^f(w)$ where $J^a$ is replaced by the fermions. One substitutes for the $J^f$ using the relation (3.3) and obtains $f^{abcdef} f^{fg} \psi^b \psi^c \psi^d \chi^e \psi^g \psi^h(z)$. Now move the composite field $\psi^d \chi^e$ to the right. One obtains $f^{abcdef} f^{fg} \psi^b \psi^c \psi^d \psi^g \psi^h \psi^d \chi^e(z)$ which can
be written in terms of \( \frac{i}{2} f^{abc} d^{adef} f^{fg} h^{b} \psi^{c} \psi^{d} \psi^{e} (\psi^{f} + i \chi^{d}) (\psi^{e} - i \chi^{e}) (z) \) from the symmetric property of \( d^{adef} \). Then the overall factor is given by \( \frac{1}{8} (a_{7} - a_{8}) \). Let us describe the eighth term which is the last term which contains the \( d \) symbol. One has \( f^{abc} d^{adef} f^{fg} h^{b} \chi^{c} \psi^{d} \psi^{e} \psi^{f}(z) \) which can be identified with \(-\frac{i}{2} f^{abc} d^{adef} f^{fg} h^{b} i \chi^{c} \psi^{d} \psi^{e} \psi^{f}(z) \). By multiplying the overall factor \( i(a_{7} - a_{8}) \), one obtains \( \frac{1}{8} (a_{7} - a_{8}) f^{abc} d^{adef} f^{fg} h^{b} i \chi^{c} \psi^{d} \psi^{e} \psi^{f}(z) \). This can be further rewritten in terms of \( \frac{1}{8} (a_{7} - a_{8}) f^{abc} d^{adef} f^{fg} h^{b} i \chi^{c} \psi^{d} \psi^{e} \psi^{f}(z) \) where we use the fact that there exists a minus sign when the first three factors \( \psi^{a} i \chi^{c} \chi^{d} \) move to the right. Therefore, there should be an overall factor \( \frac{1}{4} \). One can analyze the other terms up to the seventh term.

Now we are considering the last six terms in (4.10). The first term is given by \( -(a_{7} - a_{8}) f^{abc} d^{adef} f^{fg} h^{b} \chi^{c} \chi^{d}(z) \). This can be rewritten as \(-\frac{4}{3} (a_{7} - a_{8}) f^{abc} d^{adef} f^{fg} h^{b} \chi^{c} \chi^{d}(z) + \frac{3}{4} f^{abc} d^{adef} f^{fg} h^{b} \chi^{c} \chi^{d}(z) \) where we use the fact that there exists a minus sign when the first three factors \( \psi^{a} i \chi^{c} \chi^{d} \) move to the right. Therefore, there should be an overall factor \( \frac{1}{4} \). One can analyze the other four terms. Finally, one can summarize the last six terms in (4.10) are given by \( \frac{1}{3} (a_{7} - a_{8}) f^{abc} d^{adef} f^{fg} h^{b} \chi^{c} \chi^{d}(z) \).

By putting \( a_{7} - a_{8} = 1 \) one obtains the following higher spin 3 current with \( U(1) \) charge \( \frac{1}{3} \) as follows:

\[
W^{(3)}(z) = \frac{1}{8} f^{abcd} f^{adef} f^{bg} \chi^{c} \psi^{d} \psi^{e} \chi^{f} \psi^{g}(z) = \frac{1}{8} f^{abcd} f^{adef} f^{bg} \chi^{c} \psi^{d} \psi^{e} \chi^{f} \psi^{g}(z) \]

One can calculate the \( U(1) \) charges for the adjoint fermions with (4.13) as follows:

\[
J(z)(\psi^{a} + i \chi^{a})(w) = \frac{1}{(z - w)} \frac{1}{3} (\psi^{a} + i \chi^{a})(w) + \cdots ,
\]

\[
J(z)(\psi^{a} - i \chi^{a})(w) = \frac{1}{(z - w)} (-1) \frac{1}{3} (\psi^{a} - i \chi^{a})(w) + \cdots . \quad (14.14)
\]
Then it is obvious that the above higher spin 3 current (4.13) has $U(1)$ charge $\frac{4}{3}$: there exist five factors with $U(1)$ charge $\frac{1}{3}$ and one factor with $U(1)$ charge $-\frac{1}{3}$ according to (4.14).

For the unitary case [44], one sees the factor $f^{aef}f^{bgh}(\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g)(\psi^h + i\chi^h)$ and the other factor is given by $d^{abc}f^{chi}(\psi^h - i\chi^h)(\psi^i - i\chi^i)$ of $U(1)$ charge $-\frac{2}{3}$ in the nonderivative terms. However, the orthogonal case contains the different factor $d^{abcd}(\psi^c + i\chi^c)(\psi^d - i\chi^d)$ of $U(1)$ charge 0 in (4.13).

In order to obtain the other higher spin currents, it is useful to calculate the following OPEs,

\[
G^+(z) (\psi^a + i\chi^a)(w) = + \cdots,
\]

\[
G^+(z) (\psi^a - i\chi^a)(w) = \frac{1}{(z-w)} \frac{1}{2\sqrt{3}(2N-2)} f^{abc}(\psi^b + i\chi^b)(\psi^c + i\chi^c)(w) + \cdots,
\]

\[
G^-(z) (\psi^a - i\chi^a)(w) = + \cdots,
\]

\[
G^-(z) (\psi^a + i\chi^a)(w) = \frac{1}{(z-w)} \frac{1}{2\sqrt{3}(2N-2)} f^{abc}(\psi^b - i\chi^b)(\psi^c - i\chi^c)(w) + \cdots. \tag{4.15}
\]

We will use this property to calculate the OPEs for the particular singular terms. One sees the $U(1)$ charge conservation in (4.15).

How does one determine other higher spin currents related to the lowest one? Let us recall that the following OPE [44, 45, 46]

\[
G^+(z) W^{(3)}_{\frac{4}{3}}(w) = -\frac{1}{(z-w)} W^{(3)}_{\frac{4}{3}}(w) + \cdots. \tag{4.16}
\]

Here the higher spin current appears in the first order pole. Once we have obtained the first order pole in the above OPE, then we obtain the higher spin current. See also the relevant work in [47]. Because the lowest higher spin 3 current is written in terms of adjoint fermions, it is better to calculate the OPE between $G^+(z)$ and fermions appearing in (4.13). According to the observations of (4.15), the spin $\frac{3}{2}$ current $G^+(z)$ has nontrivial OPE with spin $\frac{1}{2}$ current of $U(1)$ charge $-\frac{1}{3}$ while the spin $\frac{3}{2}$ current $G^-(z)$ has nontrivial OPE with spin $\frac{1}{2}$ current of $U(1)$ charge $\frac{1}{3}$. Then it is obvious that when one calculates the left hand side of (4.16), the only nontrivial singular terms appear at the location of the last factors, $(\psi^d - i\chi^d)(w)$ and $(\psi^f - i\chi^f)(w)$ in (4.13). This leads to the following higher spin $\frac{7}{2}$ current of $U(1)$ charge $\frac{7}{3}$

\[
W^{(3)}_{\frac{7}{3}}(z) = \frac{1}{2\sqrt{3}(2N-2)} \left[ \frac{1}{8} d^{abcd} f^{aef} f^{bgh} f^{di} j \right.
\]

\[
\times (\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^i + i\chi^i)(\psi^j + i\chi^j)
\]

\[
- \frac{1}{4} f^{abc} f^{def} f^{fgh}
\]

\[
\times (\psi^a + i\chi^a)(\psi^b + i\chi^b)(\psi^c + i\chi^c)(\psi^d + i\chi^d)(\psi^e + i\chi^e)(\psi^g + i\chi^g)(\psi^h + i\chi^h) \left. \right] (z). \tag{4.17}
\]

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In (4.17), the $N$ dependence appears in the overall factor rather than the relative coefficients. One easily sees that the above two expressions preserve $U(1)$ charge by counting the $U(1)$ charge at each factor. In other words, each factor has $U(1)$ charge of $\frac{1}{3}$.

From the OPE \[W\]

\[G^{-}(z) W^{(3)}_{\frac{\tau}{2}}(w) = \frac{1}{(z-w)} W^{(\frac{7}{2})}_{\frac{\tau}{2}}(w) + \cdots, \] (4.18)

one can obtain the other higher spin $\frac{7}{2}$ current of $U(1)$ charge $\frac{1}{3}$. It turns out, from the first order pole of (4.18), that

\[W^{(\frac{7}{2})}_{\frac{\tau}{2}}(z) = \frac{1}{2\sqrt{3(2N-2)}} \frac{1}{8} d_{abcd} f^{ae} f^{bgh} \left[ \right. \]

\[+ f^{eij} \left( (\psi^j - i\chi^j)(\psi^j - i\chi^j) \right) (\psi^j + i\chi^j)(\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^c + i\chi^c)(\psi^d - i\chi^d) \]

\[- f^{fij} (\psi^e + i\chi^e) \left( (\psi^e - i\chi^e)(\psi^e - i\chi^e) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^c + i\chi^c)(\psi^d - i\chi^d) \]

\[+ f^{gij} (\psi^e + i\chi^e)(\psi^f + i\chi^f) \left( (\psi^f - i\chi^f) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^c + i\chi^c)(\psi^d - i\chi^d) \]

\[- f^{hij} (\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g) \left( (\psi^f - i\chi^f) \right) (\psi^e + i\chi^e)(\psi^d - i\chi^d) \]

\[+ f^{eij} (\psi^e + i\chi^e)(\psi^f + i\chi^f)(\psi^g + i\chi^g)(\psi^h + i\chi^h) \left( \right) (\psi^d - i\chi^d) \] (z)

\[- \frac{1}{2\sqrt{3(2N-2)}} \frac{1}{4} f^{abc} f^{def} \left[ \right. \]

\[+ f^{aij} \left( (\psi^j - i\chi^j)(\psi^j - i\chi^j) \right) (\psi^b + i\chi^b)(\psi^c + i\chi^c)(\psi^d + i\chi^d)(\psi^e + i\chi^e)(\psi^f - i\chi^f) \]

\[- f^{bij} (\psi^a + i\chi^a) \left( (\psi^j - i\chi^j)(\psi^j - i\chi^j) \right) (\psi^e + i\chi^e)(\psi^f - i\chi^f) \]

\[+ f^{cij} (\psi^a + i\chi^a)(\psi^b + i\chi^b) \left( (\psi^j - i\chi^j)(\psi^j - i\chi^j) \right) (\psi^d + i\chi^d)(\psi^e + i\chi^e)(\psi^f - i\chi^f) \]

\[- f^{dij} (\psi^a + i\chi^a)(\psi^b + i\chi^b)(\psi^c + i\chi^c) \left( (\psi^j - i\chi^j)(\psi^j - i\chi^j) \right) (\psi^e + i\chi^e)(\psi^f - i\chi^f) \]

\[+ f^{eij} (\psi^a + i\chi^a)(\psi^b + i\chi^b)(\psi^c + i\chi^c)(\psi^d + i\chi^d) \left( (\psi^j - i\chi^j)(\psi^j - i\chi^j) \right) (\psi^f - i\chi^f) \] (z).

From (4.15), the OPE between $G^{-}(z)$ and $(\psi^a - i\chi^a)(w)$ does not have any singular terms and the contribution from this OPE in (4.19) vanishes. Note that the big bracket stands for the normal ordered product \[18 19\]. Of course, one can move those factors to the right in order to simplify further. Each term has the $U(1)$ charge $\frac{1}{3}$ because there are four factors for the $U(1)$ charge $\frac{1}{3}$ and three factors for the $U(1)$ charge $-\frac{1}{3}$. Totally one has $\frac{1}{3} U(1)$ charge.

From the relation \[14\],

\[G^{-}(z) W^{(\frac{7}{2})}_{\frac{\tau}{2}}(w) = \frac{1}{(z-w)^2} \left(-1\right)^\frac{7}{3} W^{(3)}_{\frac{\tau}{2}}(w) + \frac{1}{(z-w)} \left[ W^{(4)}_{\frac{\tau}{2}} - \frac{1}{2} \partial W^{(3)}_{\frac{\tau}{2}} \right] (w) + \cdots, \] (4.20)

34
one obtains, by calculating the left hand side of (4.20) with (4.3) and (4.17) and reading off the first order pole,

\[
(W^{(4)}_{\frac{1}{2}} - \frac{1}{2} \partial W^{(3)}_{\frac{1}{2}})(z) = \frac{1}{2\sqrt{3(2N - 2)}} \frac{1}{8} \epsilon^{abcd} f^{aef} f^{fgih} f^{dij} \left[ \frac{1}{4} f^{ijkl} \left( (\psi^k - i\chi^k)(\psi^j - i\chi^j) \right) (\psi^l + i\chi^l)(\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^e + i\chi^e)(\psi^i + i\chi^i)(\psi^d + i\chi^d) - \frac{1}{2} \frac{1}{4} \epsilon^{abcd} f^{aef} f^{fgih} f^{dij} \right] (\psi^e + i\chi^e) \left( (\psi^k - i\chi^k)(\psi^j - i\chi^j) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^i + i\chi^i)(\psi^d + i\chi^d) + \frac{1}{2} \frac{1}{4} \epsilon^{abcd} f^{aef} f^{fgih} f^{dij} \right] (\psi^e + i\chi^e) \left( (\psi^k - i\chi^k)(\psi^j - i\chi^j) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^i + i\chi^i)(\psi^d + i\chi^d) - \frac{1}{2} \frac{1}{4} \epsilon^{abcd} f^{aef} f^{fgih} f^{dij} \right] (\psi^e + i\chi^e) \left( (\psi^k - i\chi^k)(\psi^j - i\chi^j) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^i + i\chi^i)(\psi^d + i\chi^d) + \frac{1}{2} \frac{1}{4} \epsilon^{abcd} f^{aef} f^{fgih} f^{dij} \right] (\psi^e + i\chi^e) \left( (\psi^k - i\chi^k)(\psi^j - i\chi^j) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^i + i\chi^i)(\psi^d + i\chi^d) - \frac{1}{2} \frac{1}{4} \epsilon^{abcd} f^{aef} f^{fgih} f^{dij} \right] (\psi^e + i\chi^e) \left( (\psi^k - i\chi^k)(\psi^j - i\chi^j) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^i + i\chi^i)(\psi^d + i\chi^d) + \frac{1}{2} \frac{1}{4} \epsilon^{abcd} f^{aef} f^{fgih} f^{dij} \right] (\psi^e + i\chi^e) \left( (\psi^k - i\chi^k)(\psi^j - i\chi^j) \right) (\psi^g + i\chi^g)(\psi^h + i\chi^h)(\psi^i + i\chi^i)(\psi^d + i\chi^d)
\]

\[(4.21)\]

The properties in (4.15) are used. One can check that the U(1) charge of each term is equal to $\frac{1}{3}$ where there are six positive ones and two negative ones. In order to obtain the primary current, one should consider \((W^{(4)}_{\frac{1}{2}} - \frac{1}{2} \partial W^{(3)}_{\frac{1}{2}})(z)\) [44] which can be obtained from (4.21) and (4.13).

Then, the higher spin 3, $\frac{7}{2}$, and 4 currents are summarized by (4.13), (4.17), (4.19) and (4.21) with addition of the derivative of (4.13).

### 4.3 Other higher spin 3, $\frac{7}{2}$, $\frac{7}{2}$, 4 currents

In the description of (4.11), for the opposite U(1) charge, there exists also other solution for the higher spin 3 current. One obtains the higher spin 3 current of U(1) charge $-\frac{1}{3}$ as follows:

\[
W^{(3)}_{-\frac{1}{3}}(w) = W^{(3)}_{\frac{1}{2}}(w) \Big|_{\chi^a \rightarrow -\chi^a}.
\]

(4.22)
More explicitly, one can read off the explicit expression which can be obtained from (4.13) by replacing the second adjoint fermions with those together with minus sign. It is obvious to see that the $U(1)$ charge $-\frac{4}{3}$ of this higher spin current can be seen during this process: five factors of $U(1)$ charge $-\frac{5}{3}$ and one factor with $U(1)$ charge $\frac{1}{3}$.

Let us calculate other higher spin currents. From the known OPE [44]

$$G^+(z) W^{(3)}_{-\frac{4}{3}}(w) = -\frac{1}{(z-w)} W^{(7)}_{-\frac{4}{3}}(w) + \cdots, \quad (4.23)$$

one obtains the higher spin $\frac{7}{2}$ current together with (4.22) and (4.23) as follows:

$$W^{(7)}_{-\frac{4}{3}}(w) = -W^{(7)}_{-\frac{4}{3}}(w) \bigg|_{\chi^a \to -\chi^a}. \quad (4.24)$$

Note that under the change of $\chi^a \to -\chi^a$, the original $U(1)$ charge is changed into the negative one. More explicitly, one can take this operation in (4.18). Then the left hand side of (4.18) leads to the left hand side of (4.23) with the help of (4.22) and the right hand side of (4.18) becomes $W^{(7)}_{\frac{4}{3}}(w) \bigg|_{\chi^a \to -\chi^a}$. By realizing that the first order pole from (4.23), then we are left with (4.24).

Similarly, the OPE [44] with (4.22)

$$G^-(z) W^{(3)}_{-\frac{4}{3}}(w) = \frac{1}{(z-w)} W^{(7)}_{-\frac{4}{3}}(w) + \cdots, \quad (4.25)$$

provides the following result for the higher spin current, by considering the relation (4.16) where the operation $\chi^a \to -\chi^a$ is taken and the relation (4.22),

$$W^{(7)}_{-\frac{4}{3}}(w) = -W^{(7)}_{\frac{4}{3}}(w) \bigg|_{\chi^a \to -\chi^a}. \quad (4.26)$$

In other words, the left-hand side of (4.25) is equal to the left hand side of (4.16) with the additional operation $\chi^a \to -\chi^a$. We also use the previous relation (4.22). Then the right-hand side of (4.25) can be read off from this relation and we arrive at (4.26).

From the relation (44),

$$G^+(z) W^{(7)}_{-\frac{4}{3}}(w) = \frac{1}{(z-w)^2} \frac{7}{3} W^{(3)}_{-\frac{4}{3}}(w) + \frac{1}{(z-w)} \left[ W^{(4)}_{-\frac{4}{3}} + \frac{1}{2} \partial W^{(3)}_{-\frac{4}{3}} \right] (w) + \cdots, \quad (4.27)$$

one obtains that the first order pole of (4.27) leads to

$$(W^{(4)}_{-\frac{4}{3}} + \frac{1}{2} \partial W^{(3)}_{-\frac{4}{3}})(w) = -(W^{(4)}_{\frac{4}{3}} - \frac{1}{2} \partial W^{(3)}_{\frac{4}{3}})(w) \bigg|_{\chi^a \to -\chi^a}, \quad (4.28)$$
where the previous relation (4.20) together with the operation $\chi^a \rightarrow -\chi^a$ is used. Moreover, the previous relation (4.26) is used also. As described before, the field (4.28) is not a primary under the stress energy tensor. The primary current is given by $(-W_{\frac{4}{3}}^{(4)} + \frac{1}{9} \partial W_{\frac{4}{3}}^{(3)})(w)$ which can be obtained from $(-W_{\frac{4}{3}}^{(4)} + \frac{1}{9} \partial W_{\frac{4}{3}}^{(3)})(w)$ by changing of $\chi^a(w) \rightarrow -\chi^a(w)$.

Therefore, the higher spin 3, 7, 2, 4 currents are summarized by (4.22), (4.24), (4.26), and (4.28). They are obtained from the higher spin currents appearing in previous subsection by simple change of the adjoint fermions $\chi^a(z)$ up to signs.

4.4 The OPE between the two lowest higher spin currents in $\mathcal{N} = 2$ superspace

Because the coset with the critical levels has the $\mathcal{N} = 2$ supersymmetry, one can describe the OPE between the two lowest higher spin multiplets in the $\mathcal{N} = 2$ superspace. Let us consider the OPE between the two $\mathcal{N} = 2$ lowest higher spin 3 multiplets where they have two opposite $U(1)$ charges. That is,

$$W_{\frac{3}{3}}^{(3)}(Z_1) W_{-\frac{3}{3}}^{(3)}(Z_2),$$

(4.29)

where each four component current, which obtained in previous subsection, is given by

$$W_{\frac{4}{3}}^{(3)} \equiv \left( W_{\frac{4}{3}}^{(3)}, W_{\frac{7}{3}}^{(7)}, W_{\frac{1}{3}}^{(7)}, W_{\frac{4}{3}}^{(4)} \right),$$

$$W_{-\frac{4}{3}}^{(3)} \equiv \left( W_{-\frac{4}{3}}^{(3)}, W_{-\frac{7}{3}}^{(7)}, W_{-\frac{1}{3}}^{(7)}, W_{-\frac{4}{3}}^{(4)} \right).$$

(4.30)

In principle, in order to obtain the explicit OPE in (4.29), one should calculate only the four OPEs between the four component currents living in $W_{\frac{4}{3}}^{(3)}(Z_1)$ in (4.30) and the lowest component current in $W_{-\frac{4}{3}}^{(3)}(Z_2)$ in (4.30), due to the $\mathcal{N} = 2$ supersymmetry. See also the relevant works in [48, 49, 50] where the various $\mathcal{N} = 2$ multiplets in different coset model are studied. From the four OPEs, one can realize that the right hand sides of these OPEs should have $U(1)$ charges, 0, 1, or $-1$ by adding the $U(1)$ charges. Recall that the four currents characterized by the $\mathcal{N} = 2$ stress energy tensor $T \equiv (J, G^+, G^-, T)$ of $\mathcal{N} = 2$ superconformal algebra have 0, 1, $-1$, and 0 respectively. It is natural to consider the right hand side of (4.29) in terms of $\mathcal{N} = 2$ stress energy tensor $T(Z_2)$ with its various descendant fields in minimal way.

Inside of the package of [51], one can introduce the OPEs, $T(Z_1) T(Z_2)$ which is the standard OPE corresponding to the $\mathcal{N} = 2$ superconformal algebra, $T(Z_1) W_{\frac{4}{3}}^{(3)}(Z_2)$, which is the $\mathcal{N} = 2$ primary condition with $U(1)$ charge $\frac{4}{3}$ and $T(Z_1) W_{\frac{-4}{3}}^{(3)}(Z_2)$, which is the $\mathcal{N} = 2$ primary condition with $U(1)$ charge $\frac{-4}{3}$. The explicit calculation of these OPEs is left for future work.
primary condition with $U(1)$ charge $-\frac{4}{3}$. All the coefficients appearing in these OPEs are constants except the central charge $c$ which is a function of $N$ in (4.5). Then one can write down the right hand side of OPE (4.29) with arbitrary coefficients which depend on $c$ or $N$. After using the Jacobi identities, one summarizes the structure constants in Appendix L explicitly. See also the relevant work in [52].

One expects that there should be present other higher spin multiples in the various OPEs. For example, $W^{(3)}_{\frac{4}{3}}(Z_1) W^{(3)}_{\frac{4}{3}}(Z_2)$ or $W^{(3)}_{-\frac{4}{3}}(Z_1) W^{(3)}_{-\frac{4}{3}}(Z_2)$ as in the unitary case [44]. It would be interesting to obtain these higher spin multiplets explicitly further.

5 Conclusions and outlook

In the coset model (1.1), we have constructed the higher spin 4 current for general levels. For $k_1 = 1$ with arbitrary $N$ and $k_2$, the eigenvalue equations of the zero mode of the higher spin 4 current acting on the states are obtained. The corresponding three-point functions are also determined. The $\mathcal{N} = 1$ higher spin multiplet characterized by $(\frac{7}{2}, 4)$ for $k_1 = 2N - 2$ in terms of adjoint fermions and spin 1 current is obtained. The two $\mathcal{N} = 2$ higher spin multiplets denoted by $(3, \frac{7}{2}, \frac{7}{2}, 4)$ for $k_1 = k_2 = 2N - 2$ in terms of two adjoint fermions are determined. Some of the OPEs in $\mathcal{N} = 1$ or $\mathcal{N} = 2$ coset models are given explicitly.

We consider the possible related open problems as follows:

• One can also try to obtain the higher spin currents in the following coset model

$$\tilde{SO}(2N + 1)_{k_1} \oplus \tilde{SO}(2N + 1)_{k_2}$$

It seems that the minimum value of $N$ for the nontrivial existence of $d$ symbol (and corresponding higher spin 4 current) is given by $N = 2$. In the present paper, the minimum value of $N$ is given by $N = 4$ and the number of independent fields in the higher spin currents is rather big implying that it is rather nontrivial to extract the corresponding OPEs. In the coset model (5.1), for $N = 2$ or $N = 3$ case, one expects that one can analyze the OPEs further and observe more structures in the right-hand sides of the OPEs.

• Further algebraic structures

In order to observe the algebraic structures living in the bosonic, $\mathcal{N} = 1$ or $\mathcal{N} = 2$ higher spin multiplets for generic $N$ (and generic $k_2$), one should calculate the various OPEs between them manually. In practice, this is rather involved because for example, the higher spin 4 current in the bosonic coset model consists of twenty terms and the number of OPEs is greater than two hundreds. In [27], one can try to obtain the various OPEs for the fixed low
\( N \) values (for example, \( N = 4, 5, 6, 7, \ldots \)) and expect the \( N \) dependence of structure constants appearing in the right-hand side of the OPEs.

- \( \mathcal{N} = 2 \) enhancement of [24]
  
  One considers the critical level condition in [24, 53]. It would be interesting to observe any \( \mathcal{N} = 2 \) enhancement or not. One can easily see the breaking of adjoint representation in \( SO(2N + 1) \) into the adjoint representation of \( SO(2N) \) plus the vector representation of \( SO(2N) \). The first step is to construct the \( \mathcal{N} = 2 \) superconformal algebra realization.

- The additional numerator factors
  
  For example, one considers the following coset model where the extra numerator factor exists in the coset

\[
\frac{SO(2N)_{2N-2} \oplus \tilde{SO}(2N)_{2N-2} \oplus \tilde{SO}(2N)_{2N-2}}{SO(2N)_{6N-6}}. \tag{5.2}
\]

It is an open problem to see whether one constructs the \( \mathcal{N} = 3 \) superconformal algebra [54] from the three kinds of adjoint fermions or not. It is nontrivial to obtain the three spin \( \frac{3}{2} \) currents satisfying the standard OPEs between them. Then one can try to obtain the higher spin currents living in the above coset model (5.2). Furthermore, one can describe another coset model where the additional numerical factor occurs. It is an open problem to construct the linear (or nonlinear) \( \mathcal{N} = 4 \) superconformal algebra from the four kinds of adjoint fermions.

- Further identities between \( f \) and \( d \) tensors of \( SO(2N) \)
  
  One can analyze the various identities involving \( f \) and \( d \) tensors by following the description of [16, 17]. They will be useful in order to calculate the OPEs between the higher spin currents in the context of section 3 and 4.

- Zero mode eigenvalue equations in other representations
  
  There exists an adjoint representation of \( SO(2N) \). It is an open problem to describe the eigenvalue equations for the zero mode of the higher spin 4 current acting on the states associated with the adjoint representation. For the \( SO(8) \) generators in the adjoint representation, one has \( 28 \times 28 \) matrices whose elements are given by the structure constant.

- Marginal operator
  
  One of the motivations in section 4 is based on the presence of perturbing marginal operator [55] which breaks the higher spin symmetry but preserving the \( \mathcal{N} = 2 \) supersymmetry. It would be interesting to obtain this operator and calculate the mass terms with the explicit eigenvalues along the lines of [56, 57, 58, 59]. Under the large \( c \) limit, the right hand side of the OPE has the simple linear terms.

- \( \mathcal{N} = 2 \) superspace description for the adjoint fermions
We obtained the two $\mathcal{N} = 2$ higher spin multiplets. It is an open problem to see whether one can write down the two adjoint fermions in $\mathcal{N} = 2$ superspace. This will allow us to write down the $\mathcal{N} = 2$ higher spin multiplets in $\mathcal{N} = 2$ superspace.

- Asymptotic quantum symmetry algebra

We have obtained the eigenvalue equations and three-point functions at finite $N$ and $k_2$ in section 2. Along the line of [9], it is an open problem to study the asymptotic quantum symmetry algebra of the higher spin theory on the $AdS_3$ space. See also [23] where the brief sketch for the large $N$ 't Hooft limit is given.

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A The $f$ and $d$ tensors of $SO(8)$

The 28 generators of $SO(8)$ are given by

\[
T^1 = \begin{pmatrix}
0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad T^2 = \begin{pmatrix}
0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
T^3 = \begin{pmatrix}
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad T^4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
T^5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad T^6 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
T^7 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad T^8 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
T^9 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad T^{10} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]
\[ T^{11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ T^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T^{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ T^{15} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T^{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ T^{17} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T^{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ T^{19} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, T^{20} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]
Using this explicit form of Appendix [A.1], the structure constants (2.5) and $d$ tensor (2.6) can be written explicitly. Inside of mathematica [61], one can write down the $f$ and $d$ symbols as follows:

$$f = \text{Table}[	ext{Simplify}[\frac{1}{2}(-i)\text{Tr}[T[c1].(T[a1].T[b1] - T[b1].T[a1])]],
\{a1, 1, 28\}, \{b1, 1, 28\}, \{c1, 1, 28\}];$$

$$d = \text{Table}[	ext{Simplify}[\frac{1}{2}\text{Tr}[T[d1].(T[a1].T[b1].T[c1] + T[a1].T[c1].T[b1] + T[c1].T[a1].T[b1] + T[b1].T[a1].T[c1] + T[b1].T[c1].T[a1] + T[c1].T[b1].T[a1])]],
\{a1, 1, 28\}, \{b1, 1, 28\}, \{c1, 1, 28\}, \{d1, 1, 28\}].$$
B The OPEs between the diagonal spin 1 current and various spin 4 currents

As done in the section 2 (the equations (2.18), (2.20) and (2.22)), the remaining 17 OPEs are described by

\[ J^a(z) \epsilon^{bcde} J^b K^c K^d K^e(w) = \frac{1}{(z - w)^3} (4N^2 - 14N + 22) f^{abc} K^b J^c(w) \]
\[ + \frac{1}{(z - w)^2} \left[ - (3k_2 + 4(N - 1)) d^{abcd} K^b K^c J^d + (2N - 5) f^{abc} f^{cde} K^b K^c J^d \right. \]
\[ \left. - k_1 d^{abcd} K^b K^c K^d + 12K^b J^a - 12K^a K^b J^a - 12 f^{abc} \partial K^b J^c \right] (w) + \cdots, \]
\[ J^a(z) \epsilon^{bcde} K^b K^c K^d K^e(w) = \frac{1}{(z - w)^3} 2(2N - 2)(4N^2 - 14N + 22) K^a(w) \]
\[ - \frac{1}{(z - w)^3} 2(4N^2 - 14N + 22) f^{abc} K^b K^c K^d \]
\[ + \frac{1}{(z - w)^2} \left[ - (4k_2 + 8(N - 1)) d^{abcd} K^b K^c K^d \right. \]
\[ \left. - (12 + (2N - 2)(2N - 5)) f^{abc} \partial K^b K^c + (12 + (2N - 2)(2N - 5)) f^{abc} K^b \partial K^c \right] (w) + \cdots, \]
\[ J^a(z) \partial J^b \partial J^b(w) = - \frac{1}{(z - w)^3} 2(2N - 2) J^a(w) - \frac{1}{(z - w)^3} 2((2N - 2) + 2k_1) \partial J^a(w) \]
\[ + \frac{1}{(z - w)^2} \left[ f^{abc} \partial J^b J^c - f^{abc} J^b \partial J^c \right] (w) + \cdots, \]
\[ J^a(z) \partial^2 J^b(w) = - \frac{1}{(z - w)^3} 2(2N - 2) + 3k_1) J^a(w) - \frac{1}{(z - w)^3} 2(2N - 2) \partial J^a(w) \]
\[ - \frac{1}{(z - w)^2} \left[ \frac{k_1}{(2N - 2)} f^{abc} \partial J^c + (2 + \frac{k_1}{(2N - 2)}) f^{abc} \partial J^c \right] (w) + \cdots, \]
\[ J^a(z) \partial K^b \partial K^b(w) = - \frac{1}{(z - w)^3} 2(2N - 2) K^a(w) - \frac{1}{(z - w)^3} 2((2N - 2) + 2k_2) \partial K^a(w) \]
\[ + \frac{1}{(z - w)^2} \left[ f^{abc} \partial K^b K^c - f^{abc} K^b \partial K^c \right] (w) + \cdots, \]
\[ J^a(z) \partial^2 K^b K^b(w) = - \frac{1}{(z - w)^3} 2(2N - 2) + 3k_2) K^a(w) - \frac{1}{(z - w)^3} 2(2N - 2) \partial K^a(w) \]
\[ -\frac{1}{(z - w)^2} \left[ \frac{k_2}{(2N - 2)} f^{abc} K^b \partial K^c + (2 + \frac{k_2}{(2N - 2)}) f^{abc} \partial K^b K^c \right] (w) + \cdots, \]

\[ J^a(z) \partial J^b \partial K^c(w) = -\frac{1}{(z - w)^3} 2(k_1 \partial K^a + k_2 \partial J^a)(w) \]

\[ + \frac{1}{(z - w)^2} \left[ f^{abc} \partial J^b K^c - f^{abc} J^b \partial K^c \right] (w) + \cdots, \]

\[ J^a(z) \partial^2 J^b K^c(w) = -\frac{1}{(z - w)^4} 6k_1 K^a(w) - \frac{1}{(z - w)^3} 2 f^{abc} J^b K^c(w) \]

\[ - \frac{1}{(z - w)^2} \left[ \frac{k_2}{(2N - 2)} f^{abc} J^b + \frac{k_2}{(2N - 2)} f^{abc} J^b \partial f^{abc} J^c + 2 f^{abc} \partial J^b K^c \right] (w) + \cdots, \]

\[ J^a(z) J^b \partial^2 K^c(w) = -\frac{1}{(z - w)^4} 6k_2 J^a(w) - \frac{1}{(z - w)^3} 2 f^{abc} J^b K^c(w) \]

\[ - \frac{1}{(z - w)^2} \left[ \frac{k_1}{(2N - 2)} f^{abc} \partial K^b K^c + \frac{k_1}{(2N - 2)} f^{abc} K^b \partial K^c + 2 f^{abc} \partial K^b J^c \right] (w) + \cdots, \]

\[ J^a(z) f^{bcd} J^b \partial J^c K^d(w) = -\frac{1}{(z - w)^4} 8k_1 (N - 1) K^a(w) \]

\[ - \frac{1}{(z - w)^3} 2(N - 1 - k_1) f^{abc} J^b K^c(w) \]

\[ + \frac{1}{(z - w)^2} \left[ - (2N - 1 + k_1) f^{abc} \partial J^b K^c + f^{abc} f^{cde} J^d J^b K^e - k_2 f^{abc} J^b \partial J^c \right] (w) + \cdots, \]

\[ J^a(z) f^{bcd} J^b K^c \partial K^d(w) = -\frac{1}{(z - w)^4} 8k_2 (N - 1) J^a(w) \]

\[ - \frac{1}{(z - w)^3} 2(N - 1 - k_2) f^{abc} K^b J^c(w) \]

\[ + \frac{1}{(z - w)^2} \left[ - (2N - 1 + k_2) f^{abc} \partial K^b J^c + f^{abc} f^{cde} K^d K^b J^e - k_1 f^{abc} K^b \partial K^c \right] (w) + \cdots, \]

\[ J^a(z) J^b J^c J^d(w) = \frac{1}{(z - w)^4} 4(2N - 2)(2N - 2 + k_1) J^a(w) \]

\[ - \frac{1}{(z - w)^3} 4(2N - 2)(2N - 2 + k_1) \partial J^a(w) \]

\[ + \frac{1}{(z - w)^2} \left[ - 4(2N - 1 + k_1) J^a J^b J^c - 2(2N - 2 + k_1) f^{abc} \partial J^b J^c \right. \]

\[ + 2(2N - 2 + k_1) f^{abc} J^b \partial J^c \left] (w) + \cdots, \right. \]

\[ J^a(z) K^b K^c K^d(w) = \frac{1}{(z - w)^4} 4(2N - 2)(2N - 2 + k_2) K^a(w) \]

\[ - \frac{1}{(z - w)^3} 4(2N - 2)(2N - 2 + k_2) \partial K^a(w) + \frac{1}{(z - w)^2} \left[ - 4(2N - 1 + k_2) K^a K^b K^c \right. \]

\[ - 2(2N - 2 + k_2) f^{abc} \partial K^b K^c + 2(2N - 2 + k_2) f^{abc} K^b \partial K^c \left] (w) + \cdots, \right. \]
\[ J^a(z) J^b J^c K^c(w) = -\frac{1}{(z-w)^2} \left[ 2(2(N-1) + k_1) J^a K^b K^b + 2(2N - 2 + k_2) J^b J^b K^a \right] (w) + \ldots, \]
\[ \frac{1}{(z-w)^2} \left[ -k_2 J^a J^b J^c + k_2 f^{abc} J^b J^c K^c - k_1 J^b J^b K^a \right] (w) + \ldots, \]
\[ \frac{1}{(z-w)^2} \left[ -k_1 K^a K^b K^b - k_2 J^a J^b K^b \right] (w) + \ldots, \]
\[ \frac{1}{(z-w)^2} \left[ f^{abc} f^{cde} J^b J^c K^d + f^{abc} f^{cde} J^b J^d K^e \right] (w) + \ldots. \] (B.1)

One observes that there exists an invariance under the change of \( J^a(z) \leftrightarrow K^a(z) \) and \( k_1 \leftrightarrow k_2 \). For example, the first equation of Appendix (B.1) can be obtained from (2.20) by this change.

We collect each independent field with its coefficients in (2.14) appearing at various poles in Appendix (B.1) where the rearrangement \([18]\) of the normal ordered product is used and the coefficients should vanish in order to satisfy the regular condition (2.15).

**pole-4:** \[ 2(2N - 2)(4N^2 - 14N + 22)A_1 + 2(2N - 2)A_6 - 2(2N - 2) + 3k_1 \]
\[-6k_2 A_{12} - k_2 4(2N - 2)A_{14} + 4(2N - 2)(k_1 + 2N - 2)A_{15} \] \( J^a(w) = 0, \)

**pole-4:** \[ 2(2N - 2)(4N^2 - 14N + 22)A_5 + 2(2N - 2)A_8 - 2(2N - 2) + 3k_2 \]
\[-6k_1 A_{11} - k_1 4(2N - 2)A_{13} + 4(2N - 2)(k_2 + 2N - 2)A_{16} \]
\[ + 2(2N - 2)k_1 A_{18} \] \( K^a(w) = 0, \)

**pole-3:** \[ -2(4N^2 - 14N + 22)(2N - 2)A_1 - 2((2N - 2) + 2k_1)A_6 - 2(2N - 2)A_7 \]

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-2k_2A_{10} - 4(2N - 2)(k_1 + 2N - 2)A_{15} - (2N - 2)k_2A_{20} \partial J^a(w) = 0,

pole-3 : \left[- 2(2N - 2)(4N^2 - 14N + 22)A_5 - 2((2N - 2) + 2k_2)A_8 - 2(2N - 2)A_9 - 2k_1A_{10} - 4(2N - 2)(k_2 + 2N - 2)A_{16} - (2N - 2)k_1A_{20} \right] \partial K^a(w) = 0,

pole-3 : \left[(4N^2 - 14N + 22)A_2 - (4N^2 - 14N + 22)A_4 - 2A_{11} + 2A_{12} - ((2N - 2) - 2k_1)A_{13} + ((2N - 2) - 2k_2)A_{14} + 2(-k_1 + 2N - 2)A_{18} - 2(k_2 + 2N - 2)A_{19} \right] f^{abc}J^bK^c(w) = 0,

pole-2 : \left[\left(\frac{k_1}{2N - 2} A_7 - \frac{k_2}{2N - 2} A_{11} - k_2A_{13} + 2(k_1 + 2N - 2)A_{15} + k_2A_{18} \right) f^{abc}J^b \partial J^c(w) = 0,\right.

pole-2 : \left[- ((2N - 5)(2N - 2) + 12)A_1 - A_6 - \left(\frac{k_1}{2N - 2} + 2\right)A_7 - \frac{k_2}{2N - 2} A_{11} - 2(k_1 + 2N - 2)A_{15} - k_2A_{18} \right] f^{abc} \partial J^bJ^c(w) = 0,

pole-2 : \left[\left(\frac{k_2}{2N - 2} A_8 - \frac{k_1}{2N - 2} A_{12} - k_1A_{14} + 2(k_2 + 2N - 2)A_{16} \right) f^{abc}K^b \partial K^c(w) = 0,\right.

pole-2 : \left[- ((2N - 5)(2N - 2) + 12)A_5 - A_8 - \left(\frac{k_2}{2N - 2} + 2\right)A_9 + A_8 - \frac{k_1}{2N - 2} A_{12} - 2(k_2 + 2N - 2)A_{16} \right] f^{abc} \partial K^bK^c(w) = 0,

pole-2 : \left[- 4(k_1 + 2N - 2)A_{15} - k_2A_{18} \right] J^aJ^bJ^b(w) = 0,

pole-2 : \left[12A_2 - 8A_3 - 2(k_2 + 2N - 2)A_{17} - k_1A_{18} \right] J^bJ^bK^a(w) = 0,

pole-2 : \left[- 12A_2 + 8A_3 - 2(k_1 + 2N - 2)A_{18} - 2k_2A_{20} \right] J^aJ^bK^b(w) = 0,

pole-2 : \left[8A_3 - 12A_4 - 2(k_2 + 2N - 2)A_{19} - 2k_1A_{20} \right] J^bK^aK^b(w) = 0,

pole-2 : \left[- 8A_3 + 12A_4 - 2(k_1 + 2N - 2)A_{17} - k_2A_{19} \right] J^aK^bK^b(w) = 0,
pole-2 : \( -4(k_2 + 2N - 2)A_{16} - k_1 A_{19} \) \( K^a K^b K^c(w) = 0, \)
pole-2 : \( -(8(N - 1) + 4k_1)A_1 - k_2 A_2 \) \( d^{abcd} J^b J^c J^d(w) = 0, \)
pole-2 : \( -(4(N - 1) + 3k_1)A_2 - \left( \frac{4(N - 1)}{3} + 2k_2 \right) A_3 \) \( d^{abcd} J^b J^c K^d(w) = 0, \)
pole-2 : \( -(4(N - 1) - 3k_1)A_3 - (4(N - 1) + 3k_2)A_4 \) \( d^{abcd} J^b K^c K^d(w) = 0, \)
pole-2 : \( -k_1 A_4 - (8(N - 1) + 4k_2)A_5 \) \( d^{abcd} K^b K^c K^d(w) = 0, \)
pole-2 : \( (k_2 + (2N - 2))A_{14} + 2(k_2 + 2N - 2)A_{19} - \left( \frac{1}{3} (2N - 2)(2N - 5) + 4 \right) A_3 \)
\( + (12 + (2N - 5)(2N - 2))A_4 - (k_1 + 2N - 2)A_{20} - A_{10} + 2A_{12} \) \( f^{abc} J^b \partial K^c(w) = 0, \)
pole-2 : \( -(k_1 + (2N - 2))A_{13} - k_2 A_{20} - \frac{1}{3} (2N - 2)(2N - 5)A_3 - (2N - 2)A_{13} \)
\( -12A_2 + 4A_3 + A_{10} - 2A_{11} \) \( f^{abc} \partial J^b K^c(w) = 0, \)
pole-2 : \( (2N - 5)A_2 - \frac{2}{3} (2N - 5)A_3 - A_{13} + 2A_{18} - A_{20} \) \( f^{abc} f^{cde} J^b J^e K^d(w) = 0, \)
pole-2 : \( -\frac{2}{3} (2N - 5)A_3 + (2N - 5)A_4 - A_{14} - A_{20} \) \( f^{abc} f^{cde} J^d K^e K^b(w) = 0. \) (B.2)

The regular condition provides the vanishing of these coefficients appearing in Appendix (B.2) for the independent composite fields.

Let us emphasize that one can understand the identity (2.10) in different point of view. In Appendix (B.2), the second order poles having \( A_2 \) term with a single \( K^a \) are given by \( J^b J^c K^a(w), J^a J^b K^b(w), d^{abcd} J^b J^c K^d(w), f^{abc} \partial J^b K^c(w), \) and \( f^{abc} f^{cde} J^b J^e K^d(w). \) Of course, these are obtained from (2.19) with the help of (2.10). Although one does not know the tensorial structure in the right hand side of (2.10), one can figure out it from the above field contents. In other words, one can determine the tensorial structure from the above requirement (regular condition). Then one can fix the coefficients appearing in (2.10) by applying for several lower \( N \) values.
The OPEs between the numerator spin 2 current and various spin 4 currents

As done in the section 2 (the equations (2.25), (2.26) and (2.27)), one obtains the remaining 17 OPEs described by

\[
\begin{align*}
\hat{T}(z) d^{bcde} J^b K^c K^d K^e(w) &= -\frac{1}{(z-w)^4} 6k_2(4N-1)J^a K^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) d^{bcde} K^b K^c K^d K^e(w) &= -\frac{1}{(z-w)^4} 12k_2(4N-1)K^a K^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) \partial J^a \partial J^a(w) &= -\frac{1}{(z-w)^6} 4k_1 N(2N-1) + \frac{1}{(z-w)^3} 2\partial(J^a J^a)(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) \partial^2 J^a J^a(w) &= -\frac{1}{(z-w)^6} 6k_1 N(2N-1) + \frac{1}{(z-w)^4} 6J^a J^a(w) \\
&+ \frac{1}{(z-w)^3} 6\partial J^a J^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) \partial K^a \partial K^a(w) &= -\frac{1}{(z-w)^6} 4k_2 N(2N-1) + \frac{1}{(z-w)^3} 2\partial(K^a K^a)(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) \partial^2 K^a K^a(w) &= -\frac{1}{(z-w)^6} 6k_2 N(2N-1) + \frac{1}{(z-w)^4} 6K^a K^a(w) \\
&+ \frac{1}{(z-w)^3} 6\partial K^a K^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) \partial J^a \partial K^a(w) &= \frac{1}{(z-w)^3} 2\partial(J^a K^a)(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) \partial^2 J^a K^a(w) &= \frac{1}{(z-w)^4} 6J^a K^a(w) + \frac{1}{(z-w)^3} 6\partial J^a K^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) J^a \partial^2 K^a(w) &= \frac{1}{(z-w)^4} 6J^a K^a(w) + \frac{1}{(z-w)^3} 6J^a \partial K^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) f^{abc} J^a \partial J^b K^c(w) &= \frac{1}{(z-w)^4} 2(2N-2) J^a K^a(w) \\
&+ \frac{1}{(z-w)^3} 4(2N-2) \partial J^a K^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) f^{abc} J^a K^b \partial K^c(w) &= \frac{1}{(z-w)^4} 2(2N-2) J^a K^a(w) \\
&+ \frac{1}{(z-w)^3} 4(2N-2) J^a \partial K^a(w) + O\left(\frac{1}{(z-w)^2}\right), \\
\hat{T}(z) J^a J^b J^c(w) &= -\frac{1}{(z-w)^4} 2 \left[ 2(2N-2 + k_1) + k_1 N(2N-1) \right] J^a J^a(w) \\
&+ O\left(\frac{1}{(z-w)^2}\right),
\end{align*}
\]
It is obvious that there exists an invariance under the change of $J^a$ energy tensor (2.2) in Appendix (C.1) and the coefficients should vanish in order to be a primary under the stress pole-4:

$$\hat{T}(z) K^a K^b K^c(w) = -\frac{1}{(z-w)^4} 2 \left[ 2(2N-2+k_2) + k_2 N(2N-1) \right] K^a K^c(w)$$

$$\hat{T}(z) J^a J^b K^c(w) = -\frac{1}{(z-w)^4} N(2N-1) \left[ k_2 J^a J^b + k_1 K^a K^c \right] (w) + O\left(\frac{1}{(z-w)^2}\right),$$

$$\hat{T}(z) J^a J^b J^c(w) = -\frac{1}{(z-w)^4} \left[ k_1 (N(2N-1)+2) \right] J^a K^c(w)$$

$$-\frac{1}{(z-w)^4} (2N-2) \partial J^a K^c(w) + O\left(\frac{1}{(z-w)^2}\right),$$

$$\hat{T}(z) J^a K^a K^c(w) = -\frac{1}{(z-w)^4} \left[ 2(2N-2+k_2) + N(2N-1)k_2 \right] J^a K^c(w)$$

$$+ O\left(\frac{1}{(z-w)^2}\right),$$

$$\hat{T}(z) J^a J^b K^a(w) = -\frac{1}{(z-w)^4} \left[ k_2 J^a J^b + k_1 K^a K^a \right] (w) + O\left(\frac{1}{(z-w)^2}\right).$$

(C.1)

It is obvious that there exists an invariance under the change of $J^a(z) \leftrightarrow K^a(z)$ and $k_1 \leftrightarrow k_2$. For example, the first equation of Appendix (C.1) can be obtained from (2.26) by this change.

We collect each independent field with its coefficients in (2.14) appearing at various poles in Appendix (C.1) and the coefficients should vanish in order to be a primary under the stress energy tensor (2.2):

pole-6: \[
\begin{bmatrix}
-4(2N-1)k_1 NA_6 - 6k_1 N(2N-1)A_7 - 4k_2 N(2N-1)A_8 \\
-6k_2 N(2N-1)A_9
\end{bmatrix} = 0,
\]

pole-4: \[
\begin{bmatrix}
-6(8N-2)k_1 A_1 - k_2 (8N-2)A_3 + 6A_7 \\
-2(2k_1 + 2N - 2) + (2N-1)k_1 N)A_15 - k_2 N(2N-1)A_17 - k_2 A_20
\end{bmatrix} J^a J^a(w) = 0,
\]

pole-4: \[
\begin{bmatrix}
-3(8N-2)k_1 A_2 - 3k_2 (8N-2)A_4 + 6A_{11} + 6A_{12} + 2(2N-2)A_{13} \\
+2(2N-2)A_{14} - k_1 ((2N-1)N + 2)A_{18} \\
-(2k_2 + 2N - 2) + (2N-1)k_2 N)A_{19}
\end{bmatrix} J^a K^a(w) = 0,
\]

pole-4: \[
\begin{bmatrix}
(8N-2)(-k_1)A_3 - N(2N-1)A_{17}k_1 - 6k_2 (8N-2)A_5
\end{bmatrix}
\]

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\[ +6A_9 - 2(2k_2 + 2N - 2) + (2N - 1)k_2 N)A_{16} - k_1 A_{20}\] \(K^aK^a(w) = 0,\)

\begin{align*}
pole-3 : & \quad \left[4A_6 + 6A_7\right] \partial J^a J^a(w) = 0, \\
pole-3 : & \quad \left[4A_8 + 6A_9\right] \partial K^a K^a(w) = 0, \\
pole-3 : & \quad \left[2A_{10} + 6A_{11} + 4(2N - 2)A_{13} - 2(2N - 2)A_{18} + (2N - 2)A_{20}\right] \partial J^a K^a(w) = 0, \\
pole-3 : & \quad \left[2A_{10} + 6A_{12} + 4(2N - 2)A_{14} + (2N - 2)A_{20}\right] J^a \partial K^a(w) = 0. \tag{C.2}
\end{align*}

The primary condition (2.23) leads to the vanishing of these coefficients appearing in Appendix (C.2) for the independent composite fields.

**D The coefficients appearing in (2.14) which depend on \(k_1, k_2\) and \(N\)**

By solving the linear equations in Appendix (B.2) and Appendix (C.2), we determine the various coefficients which are the functions of \(k_1, k_2\) and \(N\) as follows:

\begin{align*}
A_2 &= -\frac{4(k_1 + 2N - 2)}{k_2} A_1, \\
A_3 &= \frac{6(k_1 + 2N - 2)(3k_1 + 4N - 4)}{k_2(3k_2 + 2N - 2)} A_1, \\
A_4 &= -\frac{4(k_1 + 2N - 2)(3k_1 + 2N - 2)(3k_1 + 4N - 4)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)} A_1, \\
A_5 &= \frac{k_1(k_1 + 2N - 2)(3k_1 + 4N - 4)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)} A_1, \\
A_6 &= \frac{6}{(3k_2 + 2N - 2)D(k_1, k_2, N)} A_1 \\
x \quad (224k_1k_2N^6 + 64N^6 + 200k_1k_2^2N^5 - 128k_1N^5 + 248k_1^2k_2N^5 - 1344k_1k_2N^5 \\
+ 352k_2N^5 - 480N^5 + 36k_1k_2^2N^4 - 224k_1^2N^4 + 132k_2^2N^4 - 1000k_2^2N^4 + 304k_2^3N^4 \\
+ 832k_1N^4 + 96k_1^3k_2N^4 - 1240k_1^3k_2N^4 + 2296k_1k_2N^4 - 2288k_2N^4 + 2208N^4 - 96k_1^3N^3 \\
+ 12k_1^3k_2N^3 - 144k_1k_2^2N^3 + 72k_2^2N^3 + 1232k_1^2N^3 + 24k_1^2k_2N^3 - 528k_1^2k_2N^3 + 1298k_1k_2^2N^3 \\
- 1672k_2^2N^3 - 416k_1N^3 + 12k_1k_2N^3 - 384k_1^3k_2N^3 + 830k_1^2k_2N^3 - 1688k_1k_2N^3 + 6688k_2N^3 \\
- 5312N^3 - 12k_1^2N^2 + 432k_1^2N^2 - 36k_1k_2^2N^2 + 105k_1k_2N^2 - 324k_2^2N^2 - 1080k_2^3N^2 \\
- 72k_1k_2N^2 + 21k_1k_2N^2 - 956k_1k_2N^2 + 3672k_2N^2 - 2464k_1N^2 - 36k_1^3k_2N^2 - 96k_1^2k_2N^2 \\
+ 940k_1^3k_2N^2 + 2928k_2^2N^2 - 9856k_2N^2 + 6528N^2 + 42k_1^2N - 144k_1^3N - 45k_2^2k_2N \\
- 93k_1k_2N + 504k_2N - 640k_1^2N - 90k_1k_2^2N + 213k_1k_2^2N + 152k_1k_2^2N - 3544k_2N \\
+ 3488k_1N - 45k_1k_2N + 348k_1k_2N + 70k_1k_2N - 4600k_1k_2N + 7040k_2N - 3936N + 24k_1^4 \\
\quad \times \quad (224k_1k_2N^6 + 64N^6 + 200k_1k_2^2N^5 - 128k_1N^5 + 248k_1^2k_2N^5 - 1344k_1k_2N^5 \\
+ 352k_2N^5 - 480N^5 + 36k_1k_2^2N^4 - 224k_1^2N^4 + 132k_2^2N^4 - 1000k_2^2N^4 + 304k_2^3N^4 \\
+ 832k_1N^4 + 96k_1^3k_2N^4 - 1240k_1^3k_2N^4 + 2296k_1k_2N^4 - 2288k_2N^4 + 2208N^4 - 96k_1^3N^3 \\
+ 12k_1^3k_2N^3 - 144k_1k_2^2N^3 + 72k_2^2N^3 + 1232k_1^2N^3 + 24k_1^2k_2N^3 - 528k_1^2k_2N^3 + 1298k_1k_2^2N^3 \\
- 1672k_2^2N^3 - 416k_1N^3 + 12k_1k_2N^3 - 384k_1^3k_2N^3 + 830k_1^2k_2N^3 - 1688k_1k_2N^3 + 6688k_2N^3 \\
- 5312N^3 - 12k_1^2N^2 + 432k_1^2N^2 - 36k_1k_2^2N^2 + 105k_1k_2N^2 - 324k_2^2N^2 - 1080k_2^3N^2 \\
- 72k_1k_2N^2 + 21k_1k_2N^2 - 956k_1k_2N^2 + 3672k_2N^2 - 2464k_1N^2 - 36k_1^3k_2N^2 - 96k_1^2k_2N^2 \\
+ 940k_1^3k_2N^2 + 2928k_2^2N^2 - 9856k_2N^2 + 6528N^2 + 42k_1^2N - 144k_1^3N - 45k_2^2k_2N \\
- 93k_1k_2N + 504k_2N - 640k_1^2N - 90k_1k_2^2N + 213k_1k_2^2N + 152k_1k_2^2N - 3544k_2N \\
+ 3488k_1N - 45k_1k_2N + 348k_1k_2N + 70k_1k_2N - 4600k_1k_2N + 7040k_2N - 3936N + 24k_1^4 \\
\end{align*}
\[-192k_1^4 - 12k_1^2 k_2^3 + 150k_1 k_2^3 - 252k_2^3 + 712k_1^2 - 24k_1^4 k_2^2 + 270k_1^2 k_2^3 - 1064k_1 k_2^2 + 1240k_2^2 \]
\[-1312k_1 - 12k_1^4 k_2 + 144k_1^3 k_2 - 848k_1^2 k_2 + 2184k_1 k_2 - 1936k_2 + 928),\]

\[A_7 = \frac{4}{(3k_2 + 2N - 2)(D(k_1, k_2, N) A_1 \times (12k_1 k_2 N^3 - 36k_1^3 k_2 N^2 - 45k_1^4 k_2 N - 12k_1^4 k_2 - 12k_1^4 N^2 + 42k_1^4 N + 24k_1^4 + 24k_1^3 k_2 N^3 - 72k_1^3 k_2 N^2 - 90k_1^4 k_2^2 - 24k_1^4 k_2^2 + 96k_1^4 k_2 N^3 - 384k_1^4 k_2 N^2 - 96k_1^4 k_2 N^2 + 348k_1^3 k_2 N)} + 144k_1^3 k_2 N^3 + 432k_1^3 k_2 N^2 - 144k_1 k_2 N - 192k_1^3 + 12k_1^2 k_2 N^3 - 36k_1^2 k_2 N^2 - 45k_1^2 k_2 N^2 - 12k_1^2 k_2 N^2 - 72k_1^2 k_2 N^2 - 90k_1^2 k_2 N^3 - 24k_1^4 k_2^3 + 132k_1^2 k_2 N^3 + 528k_1^2 k_2 N^2 - 252k_2^5 + 504k_2^3 N - 1672k_2 N^3 + 3672k_2 N^2 - 354k_2^2 N + 1240k_2 k_2 N - 270k_2^2 k_2 N + 200k_2^2 k_2 N^5 - 1000k_2 k_2 N^3 + 1298k_2 k_2 N - 956k_2 N^2 - 1522k_2 k_2 N^2 - 1064k_2 k_2 N^2 - 1672k_2 N^2 + 3672k_2 N^2 - 354k_2^2 N + 1240k_2 N^2 - 12k_1 k_2^4 N^3 - 36k_1 k_2^3 N^2 - 45k_1 k_2^2 N^2 - 12k_1 k_2 N^2 - 45k_1 k_2^3 N^3 - 96k_1 k_2^3 N^2 - 96k_1 k_2^3 N^2 - 232k_1 k_2^3 N^2 + 12k_1 k_2^3 N^2 - 1240k_1 k_2^4 + 144k_1 k_2^3 + 248k_1 k_2^3 N^5 - 1240k_1 k_2^4 N^2 + 830k_1 k_2^3 N^2 + 940k_1 k_2^3 N^2 - 780k_1 k_2^3 N^2 - 848k_1 k_2^3 N^2 \]

\[A_8 = \frac{6}{k_2(2 + N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)D(k_1, k_2, N) A_1 \times k_1 k_2 + 2184k_2 + 12k_1 k_2 + 150k_1 k_2 - 24k_1^4 k_2 + 270k_1^2 k_2^3 - 1064k_1 k_2^2 + 1240k_2^2 \]

\[A_9 = \frac{4}{(3k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)D(k_1, k_2, N) A_1 \times k_1(k_1 + 2N - 2)(3k_1 + 4N - 4)(12k_1^2 k_2 N^3 - 36k_1^3 k_2 N^2 - 45k_1^4 k_2 N - 12k_1^4 k_2 - 12k_1^4 N^2 + 42k_1^4 N + 24k_1^4 + 24k_1^3 k_2 N^3 - 72k_1^3 k_2 N^2 - 90k_1^4 k_2^2 - 24k_1^4 k_2^2 + 96k_1^4 k_2 N^3 - 384k_1^4 k_2 N^2 - 96k_1^4 k_2 N^2 + 348k_1^3 k_2 N)} + 144k_1^3 k_2 N^3 + 432k_1^3 k_2 N^2 - 144k_1 k_2 N - 192k_1^3 + 12k_1^2 k_2 N^3 - 36k_1^2 k_2 N^2 - 45k_1^2 k_2 N^2 - 12k_1^2 k_2 N^2 - 72k_1^2 k_2 N^2 - 90k_1^2 k_2 N^3 - 24k_1^4 k_2^3 + 132k_1^2 k_2 N^3 + 528k_1^2 k_2 N^2 - 252k_2^5 + 504k_2^3 N - 1672k_2 N^3 + 3672k_2 N^2 - 354k_2^2 N + 1240k_2 k_2 N - 270k_2^2 k_2 N + 200k_2^2 k_2 N^5 - 1000k_2 k_2 N^3 + 1298k_2 k_2 N - 956k_2 N^2 - 1522k_2 k_2 N^2 - 1064k_2 k_2 N^2 - 1672k_2 N^2 + 3672k_2 N^2 - 354k_2^2 N + 1240k_2 N^2 - 12k_1 k_2^4 N^3 - 36k_1 k_2^3 N^2 - 45k_1 k_2^2 N^2 - 12k_1 k_2 N^2 - 45k_1 k_2^3 N^3 - 96k_1 k_2^3 N^2 - 96k_1 k_2^3 N^2 - 232k_1 k_2^3 N^2 + 12k_1 k_2^3 N^2 - 1240k_1 k_2^4 + 144k_1 k_2^3 + 248k_1 k_2^3 N^5 - 1240k_1 k_2^4 N^2 + 830k_1 k_2^3 N^2 + 940k_1 k_2^3 N^2 - 780k_1 k_2^3 N^2 - 848k_1 k_2^3 N^2 \]
\[
A_{10} = -\frac{12}{k_2(3k_2 + 2N - 2)D(k_1, k_2, N)}A_1
\times (k_1 + 2N - 2)(3k_1 + 4N - 4)(4k_1^3k_2N^3 - 12k_1^3k_2N^2 - 15k_1^3k_2N - 4k_1^3k_2 - 4k_1^3N^2
+ 14k_1^3k_2^2N^2 + 8k_1^3k_2^2N^3 - 24k_1^3k_2^2N^2 - 30k_1^3k_2^2N - 8k_1^3k_2^2 + 20k_1^3k_2N - 80k_1^3k_2N^3
- 71k_1^3k_2N^2 + 161k_1^3k_2N + 6k_1^3k_2 - 56k_1^3k_2N^3 + 252k_1^3k_2N^2 - 216k_1^3k_2N + 20k_1^3k_2N^3
- 12k_1^3k_2N^2 - 15k_1^3k_2^2N - 4k_1^3k_2^2 - 20k_1^3k_2N^2 - 80k_1^3k_2N^3 - 71k_1^3k_2N^2 + 161k_1^3k_2N
+ 6k_1^3k_2 + 16k_1^3k_2N^5 - 80k_1^3k_2N^4 - 212k_1^3k_2N^3 + 884k_1^3k_2N^2 - 700k_1k_2N^2 + 92k_1k_2
\]
\]

\[
A_{11} = \frac{4}{k_2(3k_2 + 2N - 2)D(k_1, k_2, N)}A_1
\times (k_1 + 2N - 2)(12k_1^3k_2N^3 - 36k_1^3k_2N^2 - 45k_1^3k_2N - 12k_1^3k_2 - 12k_1^3N^2 + 42k_1^3N + 24k_1^3N
+ 24k_1^3k_2^2N^3 + 72k_1^3k_2^2N^2 + 90k_1^3k_2^2N - 24k_1^3k_2^2N + 156k_1^3k_2N^3 - 624k_1^3k_2N^2
+ 513k_1^3k_2N + 204k_1^3k_2 - 156k_1^3k_2N^3 + 702k_1^3k_2N^2 - 234k_1^3k_2N - 312k_1^3k_2N^3 - 12k_1^3k_2N^3
- 45k_1^3k_2N^2 - 12k_1^3k_2^2 + 252k_1^3k_2N^2 - 1008k_1^3k_2N^3 - 501k_1^3k_2N^2 + 1245k_1^3k_2N + 120k_1^3k_2^2
+ 608k_1^3k_2N^5 - 3040k_1^3k_2N^4 + 1376k_1^3k_2N^3 + 4468k_1^3k_2N^2 - 2900k_1^3k_2N^2 - 512k_1^3k_2
+ 584k_1^3N^4 + 3212k_1^3N^3 - 4032k_1^3N^2 + 764k_1^3N + 640k_1^3N^2 + 96k_1^3N^2 - 384k_1^3k_2N^3
- 372k_1^3k_2N^2 + 774k_1^3k_2N - 60k_1^3k_2 + 560k_1^3k_2^2N^5 - 2800k_1^3k_2N^4 + 224k_1^3k_2N^3 + 6622k_1^3k_2N^2
- 5174k_1^3k_2N + 568k_1^3k_2 + 70k_1^3k_2N^6 - 422k_1k_2N^5 + 3928k_1k_2N^4 + 9076k_1k_2N^3 - 17808k_1k_2N^2
+ 9572k_1k_2N - 1240k_1k_2 - 704k_1k_2N^5 - 4576k_1k_2N^4 + 10208k_1k_2N^3 + 4576k_1k_2N^2 + 96k_2^3N^3
+ 432k_2^3N^2 + 936k_2^3N + 600k_2^3N^2 + 3080k_2^3N^3 - 7056k_2^3N^2 + 7112k_2^3N^3 - 2576k_2^3N^2 - 800k_2^3N^5
+ 5200k_2^3N^4 - 14480k_2^3N^3 + 20240k_2^3N^2 - 13840k_2^3N^2 + 3680k_2^2 - 128N^6 + 960N^5
- 4416N^4 + 10624N^3 - 13056N^2 + 7872N - 1856),
A_{12} = \frac{4}{k_2(3k_2 + 2N - 2)(3k_1 + 4N - 4)D(k_1, k_2, N)}A_1
\times (k_1 + 2N - 2)(3k_1 + 4N - 4)(12k_1^3k_2N^3 - 36k_1^3k_2N^2 - 45k_1^3k_2N - 12k_1^3k_2 - 46k_1^3k_2N^4
- 384k_1^3k_2N^3 + 492k_1^3k_2N^2 + 9360k_1^3k_2N - 45k_1^3k_2N - 12k_1^3k_2 + 96k_1^3k_2N^4
- 912k_1^3 + 24k_1^3k_2N^3 + 24k_1^3k_2N^3 + 24k_1^3k_2N^3 + 24k_1^3k_2N^3 + 24k_1^3k_2N^3 + 24k_1^3k_2N^3 + 24k_1^3k_2N^3 + 24k_1^3k_2N^3
+ 363k_1^3k_2N^2 - 159k_1^3k_2N^2 + 660k_1^3k_2N^2 + 560k_1^3k_2N^2 - 2800k_1^3k_2N^2 + 4688k_1^3k_2N^3
- 5906k_1^3k_2N^2 + 6922k_1^3k_2N - 3464k_1^3k_2 + 1456k_1^3k_2N^4 - 8008k_1^3k_2N^3 + 16128k_1^3k_2N^3
- 14056k_1^3N^2 + 4480k_1^3N^2 + 12k_1k_2N^3 - 36k_1k_2N^2 - 45k_1k_2N^2 - 12k_1k_2 + 156k_1k_2N^4
- 624k_1k_2N^3 - 141k_1k_2N^2 + 513k_1k_2N + 204k_1k_2 + 608k_1k_2N^5 - 3040k_1k_2N^4
- 128k_1N^6 - 320k_1N^5 - 208k_1N^4 + 80k_1N^3 + 56k_1N^2 + 28k_1N + 4),
\]
\]
\]

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\[ A_{13} = \frac{-12}{k_2(3k_2 + 2N - 2)}D(k_1, k_2, N)A_1 \]
\[ = \frac{-12}{k_2(3k_2 + 2N - 2)}D(k_1, k_2, N)A_1 \]
\[ \times (k_1 + 2N - 2)(20k_1^2k_2N^3 - 60k_1^3k_2N^2 - 75k_1^3k_2N - 20k_1^3k_2 - 20k_1^3N^2 + 70k_1^3N + 40k_1^3) \]
\[ + 40k_1^2k_2N^3 + 120k_1^2k_2N^2 - 294k_1^2k_2N + 50k_1^2k_2^2 + 120k_1^2k_2N^2 - 480k_1^2k_2N^3 - 298k_1^2k_2^2 \]
\[ + 88k_1^2k_2N^3 - 120k_1^2k_2N^2 - 120k_1^2k_2N + 540k_1^2k_2^2N - 444k_1^2k_2^2 + 24k_1^2 + 20k_1^2N^3 - 60k_1^2N^2 \]
\[ - 219k_1^2N + 70k_1k_2^2 + 120k_1k_2^2N^4 - 480k_1k_2^2N^3 - 838k_1k_2^2N^2 + 1688k_1k_2^2N - 544k_1k_2^2 \]
\[ + 160k_1k_2N^5 - 800k_1k_2N^4 - 256k_1k_2N^3 + 332k_1k_2N^2 - 358k_1k_2N + 114k_1k_2 \]
\[ - 192k_1N^4 + 1056k_1N^3 - 2208k_1N^2 + 2016k_1N - 672k_1 - 656k_1N^2 + 196k_1N^3 - 284k_1^2 \]
\[ - 288k_2N^5 + 1296k_2N^4 + 2280k_2N^3 + 172k_2N^2 - 384k_2N^3 + 2112k_2N^2 - 494k_2N^2 \]
\[ + 508k_2N - 1872k_2 - 64N^3 + 416N^4 - 1792N^3 + 3520N^2 - 3008N + 928 \],

\[ A_{14} = \frac{-12}{(3k_2 + 2N - 2)(3k_2 + 4N - 4)}D(k_1, k_2, N)A_1 \]
\[ \times (k_1 + 2N - 2)(20k_1^2k_2N^3 - 60k_1^3k_2N^2 - 75k_1^3k_2N - 20k_1^3k_2 - 20k_1^3N^2 + 70k_1^3N + 40k_1^3) \]
\[ + 88k_1^2k_2N^3 - 308k_1^2N^2 + 220k_1^2N + 40k_1^2k_2^2N - 120k_1^2k_2N^2 - 130k_1^2k_2N + 120k_1^2k_2N^2 \]
\[ - 480k_1^2k_2N^3 + 650k_1^2k_2N^2 + 1000k_1^2k_2N + 800k_1^2k_2 + 384k_1^2k_2^2N - 1728k_1^2k_2^2 \]
\[ + 242k_1^2N^2 \]
\[ - 1080k_1^2N^3 + 20k_1k_2^3N^3 - 60k_1k_2^3N^2 - 75k_1k_2^3N - 20k_1k_2^3N + 120k_1k_2^3N^4 - 480k_1k_2^3N^2 \]
\[ + 38k_1k_2^3N^2 + 62k_1k_2^3N + 512k_1k_2^3 + 160k_1k_2N^5 - 800k_1k_2N^4 + 1088k_1k_2N^3 \]
\[ - 1276k_1k_2N^2 + 2708k_1k_2N - 1880k_1k_2 + 384k_1N^4 - 2112k_1N^3 + 494k_1N^2 - 5088k_1N \]
\[ + 1872k_1 - 20k_2N^2 + 70k_2N^2 + 40k_2^3 - 120k_2N^2 + 540k_2N^2 - 84k_2N^2 - 50k_2 + 192k_2N^4 \]
\[ + 1056k_2N^3 - 96k_2N^2 - 2208k_2N + 1440k_2 - 64N^5 + 416N^4 + 320N^3 - 2816N^2 \]
\[ + 332N - 1184 \],

\[ A_{15} = \frac{-18}{(3k_2 + 2N - 2)}D(k_1, k_2, N)A_1 \]
\[ \times (24k_1^2k_2^2N - 15k_1^2k_2^2N^2 - 68k_1^2k_2N + 52k_1^2k_2 + 44k_1^2N - 44k_1^2 + 24k_1k_2^3N \]
\[ - 15k_1^2 + 124k_1k_2N^2 - 224k_1k_2^2N + 112k_1k_2N^2 - 384k_1k_2N^2 + 52k_1k_2^2N \]
\[ - 252k_1k_2 + 176k_1N^2 - 352k_1N + 176k_1 + 12k_1N^2 - 42k_1N^3 - 42k_1N^3 + 56k_1N^3 - 252k_1N^2 \]
\[ + 392k_2N - 196k_2^2 + 64k_2N^4 - 352k_2N^3 + 824k_2N^2 - 848k_2N + 312k_2 + 176N^3 - 528N^2 \]
\[ + 528N - 176 \],

\[ A_{16} = \frac{-18}{k_2(k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)}D(k_1, k_2, N)A_1 \]
\[ \times k_1(1 + 2N - 2)(3k_1 + 4N - 4)(24k_1^2k_2N - 15k_1^2k_2^2N^2 - 42k_1N + 42k_1^3) \]
\[ + 24k_1^2k_2^2N - 15k_1^2k_2^2N + 124k_1^2k_2N^2 - 224k_1^2k_2^2N + 112k_1^2k_2 + 56k_1^2N^3 - 252k_1^2N^2 + 392k_1^2N \]
\[ - 196k_1^2 + 28k_1k_2^2N^2 - 68k_1k_2^2N + 52k_1k_2^2 + 112k_1k_2N^3 - 384k_1k_2N^2 + 524k_1k_2N \]
In this paper, we consider the two cases where $k_1 = 1$ and $k_1 = 2N - 2$ with arbitrary $k_2$, although the most general case is an interesting subject.

Here one introduces the common factor which appears in the denominator in Appendix (D.1) as

$$D(k_1, k_2, N) \equiv (40k_1k_2N^3 + 176N^3 + 10k_1k_2^2N^2 + 176k_1N^2 + 10k_2^2k_2N^2 - 60k_1k_2N^2$$
$$- 528N^2 + 44k_2^2N - 5k_1k_2^2N + 44k_2^2N - 352k_1N - 5k_2^2k_2N + 152k_1k_2N$$
$$- 352k_2N + 528N - 44k_1^2 + 22k_1k_2^2 - 44k_2^2 + 176k_1 + 22k_1k_2 - 132k_1k_2$$
$$- 176 + 176k_2N^2 + 176k_2).$$

(D.2)
E The coefficients appearing in (2.14), which depend on \( k_2 \) and \( N \) when \( k_1 = 1 \)

The above coefficients in Appendix (D.1) and Appendix (D.2), for the critical level \( k_1 = 1 \), are given by

\[
A_2 = -\frac{4(2N - 1)}{k_2} A_1, \\
A_3 = \frac{6(2N - 1)(4N - 1)}{k_2(3k_2 + 2N - 2)} A_1, \\
A_4 = -\frac{4(2N - 1)(2N + 1)(4N - 1)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)} A_1, \\
A_5 = \frac{(2N - 1)(2N + 1)(4N - 1)}{k_2(2N + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)} A_1, \\
A_6 = \frac{6}{(3k_2 + 2N - 2)d(1, k_2, N)} A_1 \left( 18k_2^3 N^3 - 21k_2^2 N^2 - 138k_2^3 N + 114k_2^3 + 100k_2^2 N^4 - 232k_2^3 N^3 - 555k_2^2 N^2 + 1055k_2^2 N - 422k_2^2 + 112k_2 N^5 - 316k_2 N^4 - 726k_2 N^3 + 2366k_2 N^2 - 1877k_2 N + 468k_2 + 32N^5 - 288N^4 + 1264N^3 - 1664N^2 + 870N - 160) \right), \\
A_7 = -\frac{4}{(3k_2 + 2N - 2)d(1, k_2, N)} A_1 \left( 18k_2^3 N^3 - 21k_2^3 N^2 - 138k_2^3 N + 114k_2^3 + 100k_2^3 N^4 - 232k_2^3 N^3 - 555k_2^2 N^2 + 1055k_2^2 N - 422k_2^2 + 112k_2 N^5 - 316k_2 N^4 - 726k_2 N^3 + 2366k_2 N^2 - 1877k_2 N + 468k_2 + 32N^5 - 288N^4 + 1264N^3 - 1664N^2 + 870N - 160) \right), \\
A_8 = \frac{6(2N - 1)(2N + 1)(4N - 1)}{k_2(k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(1, k_2, N)} A_1 \times (3k_2^4 N - 12k_2^4 + 24k_2^3 N^2 - 114k_2^3 N + 72k_2^3 + 62k_2^2 N^3 - 333k_2^2 N^2 + 402k_2^2 N - 122k_2^2 + 56k_2 N^4 - 318k_2 N^3 + 555k_2 N^2 - 317k_2 N + 42k_2 + 16N^4 - 32N^3 + 60N^2 - 64N + 20), \\
A_9 = -\frac{4(2N - 1)(2N + 1)(4N - 1)}{k_2(k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(1, k_2, N)} A_1 \times (3k_2^4 N - 12k_2^4 + 24k_2^3 N^2 - 114k_2^3 N + 72k_2^3 + 62k_2^3 N^3 - 333k_2^2 N^2 + 402k_2^2 N - 122k_2^2 + 56k_2 N^4 - 318k_2 N^3 + 555k_2 N^2 - 317k_2 N + 42k_2 + 16N^4 - 32N^3 + 60N^2 - 64N + 20), \\
A_{10} = -\frac{12(2N - 1)(4N - 1)}{k_2(3k_2 + 2N - 2)d(1, k_2, N)} A_1 \times (2k_2^3 N^2 - 7k_2^3 N - 4k_2^3 + 10k_2^3 N^3 - 59k_2^2 N^2 + 49k_2^2 N)
\[ A_{11} = \frac{4}{k_2(3k_2 + 2N - 2)d(1, k_2, N)} \cdot \left( 96k_2^3N^4 - 468k_2^3N + 24k_2^2N^2 \right) \]
\[ \quad - 207k_2^3N + 528k_2^3N - 3108k_2^2N^4 + 2320k_2^2N^3 \]
\[ \quad - 1007k_2^2N^2 + 3093k_2^2N - 1912k_2^2 + 704k_2N^6 - 4416k_2N^5 \]
\[ \quad + 624k_2N^4 - 4640k_2N^3 + 672k_2N^2 - 6700k_2N + 2112k_2 \]
\[ \quad - 128N^6 + 256N^5 - 424N^4 + 3472N^3 - 6190N^2 + 3868N - 800, \]
\[ \frac{1}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(1, k_2, N)} \cdot A_1 \]
\[ \times (3k_2^2N - 12k_2^4 + 39k_2^2N^2 - 189k_2^2N + 132k_2^4) \]
\[ + 152k_2^3N^3 - 843k_2^3N^2 - 1188k_2^3N - 488k_2^3 + 176k_2N^4 \]
\[ - 1092k_2N^3 + 2502k_2N^2 - 2360k_2N + 792k_2 - 32N^4 \]
\[ + 616N^3 - 1608N^2 + 1520N - 496, \]
\[ A_{13} = \frac{12}{k_2(3k_2 + 2N - 2)d(1, k_2, N)} \cdot A_1 \]
\[ \times (20k_2^3N^3 - 116k_2^3N^2 - 23k_2^3N - 214k_2^3 + 120k_2^2N^4 \]
\[ - 728k_2^3N^3 + 338k_2^3N^2 - 886k_2N^2 + 778k_2^2 + 160k_2N^5 \]
\[ - 1064k_2^3N^3 + 1396k_2N^3 - 1970k_2N^2 + 2311k_2N - 860k_2 \]
\[ - 64N^5 + 224N^4 - 856N^3 + 1832N^2 - 1366N + 320, \]
\[ A_{14} = \frac{12}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(1, k_2, N)} \cdot A_1 \]
\[ \times (5k_2^3N - 20k_2^3 + 30k_2^2N^2 - 140k_2^2N + 122k_2^2 + 40k_2N^3 \]
\[ - 218k_2N^2 + 431k_2N - 250k_2 - 16N^3 + 200N^2 - 356N + 172, \]
\[ A_{15} = \frac{18}{(2N - 1)(3k_2 + 2N - 2)d(1, k_2, N)} \cdot A_1(12k_2^3N^2 - 18k_2^3N + 27k_2^3 \]
\[ + 56k_2^3N^3 - 128k_2^3N^2 + 192k_2^2N^2 - 99k_2^2 + 64k_2N^4 \]
\[ - 240k_2N^3 + 468k_2N^2 - 392k_2N + 112k_2 + 176N^3 \]
\[ - 352N^2 + 220N - 44, \]
\[ A_{16} = \frac{18}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(1, k_2, N)} \cdot A_1 \]
\[ \times (7k_2^2 + 28k_2N - 21k_2 + 16N^2 - 30N + 18), \]
\[ A_{17} = \frac{36(4N - 1)}{k_2(3k_2 + 2N - 2)d(1, k_2, N)} \cdot A_1 \]
\[ \times (8k_2^2N + 17k_2^2 + 28k_2N^2 + 58k_2N - 43k_2 + 88N^2 - 88N + 22), \]

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$A_{18} = \frac{72}{k_2(3k_2 + 2N - 2)d(1, k_2, N)} A_1(12k_2^3N^2 - 18k_2^3N + 27k_2^3$
$+ 56k_2^2N^3 - 128k_2^2N^2 + 192k_2^2N - 99k_2^2 + 64k_2N^4$
$- 240k_2N^3 + 468k_2N^2 - 392k_2N + 112k_2 + 176N^3$
$- 352N^2 + 220N - 44),$
$A_{19} = \frac{72(2N - 1)(2N + 1)(4N - 1)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(1, k_2, N)} A_1$
$\times (7k_2^2 + 28k_2N - 21k_2 + 16N^2 - 30N + 18),$
$A_{20} = -\frac{72(2N - 1)(4N - 1)}{k_2(3k_2 + 2N - 2)d(1, k_2, N)} A_1$
$\times (3k_2^2N - 5k_2^2 + 14k_2N^2 - 36k_2N + 11k_2 + 16N^3 - 66N^2 + 47N - 2).$  \hspace{1cm} (E.1)

The simplified notation is used

d(1, k_2, N) = (5k_2^2N + 22k_2^2 + 20k_2N^2 + 73k_2N - 66k_2 + 88N^2 - 132N + 44). \hspace{1cm} (E.2)

For the computation of the three-point functions with finite $N$ and $k_2$ corresponding to (2.74),
the above expression is needed.

Under the large ’t Hooft limit (2.30), the coefficients in Appendix (E.1) with (E.2) become

$A_2 = \frac{4\lambda}{(\lambda - 1)} A_1, \quad A_3 = \frac{12\lambda^2}{(\lambda - 1)(2\lambda - 3)} A_1,$

$A_4 = \frac{8\lambda^3}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_5 = -\frac{\lambda^4}{N (\lambda - 3)(2\lambda - 3)(\lambda - 1)} A_1,$

$A_6 = N^2 \frac{12(2\lambda - 9)}{5(2\lambda - 3)} A_1, \quad A_7 = -N^2 \frac{8(2\lambda - 9)}{5(2\lambda - 3)} A_1,$

$A_8 = -N^2 \frac{12\lambda^2(\lambda^2 + 6)}{5(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_9 = -N \frac{12\lambda^2(\lambda^2 + 6)}{5(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1,$

$A_{10} = N^2 \frac{48(\lambda - 2)\lambda}{5(\lambda - 1)(2\lambda - 3)} A_1, \quad A_{11} = -N^2 \frac{16(\lambda - 12)\lambda}{5(\lambda - 1)(2\lambda - 3)} A_1,$

$A_{12} = -N^2 \frac{16\lambda(\lambda^2 + 15\lambda + 6)}{5(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_{13} = -N \frac{24\lambda}{(\lambda - 1)(2\lambda - 3)} A_1,$

$A_{14} = \frac{N}{N^2 \frac{48\lambda^2}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_{15} = \frac{36(\lambda + 3)}{5(\lambda + 1)(2\lambda - 3)} A_1,$

$A_{16} = \frac{1}{N^2 \frac{18\lambda^4(3\lambda^2 - 7)}{5(\lambda - 3)(\lambda + 1)(2\lambda - 3)(\lambda - 1)^2} A_1,$

$A_{17} = -\frac{1}{N \frac{72\lambda^2(3\lambda + 4)}{5(\lambda + 1)(2\lambda - 3)(\lambda - 1)} A_1, \quad A_{18} = \frac{144\lambda(\lambda + 3)}{5(\lambda - 1)(\lambda + 1)(2\lambda - 3)} A_1,$

$A_{19} = -\frac{1}{N \frac{144\lambda^3(3\lambda^2 - 7)}{5(\lambda - 3)(\lambda - 1)^2(\lambda + 1)(2\lambda - 3)} A_1,$

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\[ A_{20} = \frac{144\lambda^3(\lambda + 3)}{5(\lambda - 1)^2(\lambda + 1)(2\lambda - 3)} A_1. \] (E.3)

The nineteen coefficients are given by \( A_1 \) coefficient. By recognizing the explicit \( N \) behavior of the above coefficients, one can extract the leading behavior of the zero mode of the higher spin 4 current by considering the \( N \) behavior of the various zero modes consisting of the higher spin 4 current. For example, \((2.44)\) and \((2.68)\).

**F. The coefficients appearing in (2.14), which depend on \( k_2 \) and \( N \) when \( k_1 = 2N - 2 \)**

The above coefficients in Appendix (D.1) and Appendix (D.2), for the critical level \( k_1 = 2N - 2 \), are given by

\[
\begin{align*}
A_2 &= -\frac{16(N - 1)}{k_2} A_1, \\
A_3 &= \frac{240(N - 1)^2}{k_2(3k_2 + 2N - 2)} A_1, \\
A_4 &= -\frac{1280(N - 1)^3}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)} A_1, \\
A_5 &= \frac{640(N - 1)^4}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)} A_1, \\
A_6 &= \frac{6}{(3k_2 + 2N - 2)d(k_2, N)} A_1 \\
&\times (60k_2^2 N^4 - 240k_2^3 N^3 + 123k_2^2 N^2 - 153k_2 N + 300k_2^3 N + 560k_2^2 N^5 \\
&- 2800k_2^2 N^4 + 2860k_2 N^3 - 920k_2^2 N^2 + 2620k_2 N - 2320k_2^2 + 1200k_2 N^6 \\
&- 7200k_2 N^5 + 10476k_2 N^4 - 1788k_2 N^3 + 468k_2 N^2 - 8676k_2 N + 5520k_2 \\
&- 1024N^5 + 6656N^4 - 9664N^3 - 704N^2 + 8896N - 4160), \\
A_7 &= \frac{4}{(3k_2 + 2N - 2)d(k_2, N)} A_1 (60k_2^3 N^4 - 240k_2^3 N^3 + 123k_2^2 N^2 - 153k_2 N \\
&+ 300k_2^3 + 560k_2^2 N^5 - 2800k_2^2 N^4 + 2860k_2 N^3 - 920k_2^2 N^2 \\
&+ 2620k_2 N - 2320k_2^2 + 1200k_2 N^6 - 7200k_2 N^5 + 10476k_2 N^4 \\
&- 1788k_2 N^3 + 468k_2 N^2 - 8676k_2 N + 5520k_2 - 1024N^5 + 6656N^4 \\
&- 9664N^3 - 704N^2 + 8896N - 4160), \\
A_8 &= \frac{480}{k_2(3k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(k_2, N)} A_1 \\
&\times (N - 1)^2 (12k_2^2 N^4 - 48k_2^3 N^3 - 15k_2^2 N^2 \\
&+ 54k_2 N + 24k_2^4 + 144k_2^3 N^5 - 720k_2^2 N^4 + 396k_2^3 N^3 \\
&+ 828k_2^2 N^2 - 360k_2 N - 288k_2^3 + 560k_2^2 N^6 - 3360k_2^2 N^5 + 4772k_2^2 N^4 \\
&+ 24k_2^2 N^3 - 1932k_2 N^2 - 1856k_2^2 N + 1792k_2 + 768k_2 N^7 \\
&- 5376k_2 N^6 + 13152k_2 N^5 - 16176k_2 N^4 + 17424k_2 N^3 - 22224k_2 N^2)
\end{align*}
\]
\[ + 18000k_2N - 5568k_2 + 1280N^6 - 9600N^5 + 31488N^4 - 55552N^3 \\
+ 54528N^2 - 28032N + 5888), \\
A_9 = -\frac{320(N-1)^2}{k_2(2k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(k_2, N)}A_1 \\
\times (12k_2^2N^4 - 48k_2^3N^3 - 15k_2^4N^2 + 54k_2^3N + 24k_2^4 + 144k_2^5N^5 \\
- 720k_2^3N^4 + 396k_2^4N^3 + 828k_2^3N^2 - 360k_2^4N - 288k_2^2 \\
+ 560k_2^5N^6 - 3360k_2^3N^5 + 4772k_2^3N^4 + 24k_2^5N^3 - 1932k_2^3N^2 \\
- 1856k_2^2N + 1792k_2 + 768k_2N^7 - 5376k_2N^6 + 13152k_2N^5 \\
- 16176k_2N^4 + 17424k_2N^3 - 22224k_2N^2 + 18000k_2N - 5568k_2 \\
+ 1280N^6 - 9600N^5 + 31488N^4 - 55552N^3 + 54528N^2 - 28032N + 5888), \\
A_{10} = -\frac{480(N-1)}{k_2(3k_2 + 2N - 2)d(k_2, N)}A_1(4k_2^3N^4 - 16k_2^3N^3 - 5k_2^3N^2 \\
+ 18k_2^3N + 8k_2^3 + 36k_2N^5 - 180k_2^2N^4 + 33k_2^3N^3 \\
+ 414k_2^2N^2 - 291k_2^2N - 12k_2^2 + 72k_2N^6 - 432k_2N^5 \\
+ 130k_2N^4 + 2030k_2N^3 - 3234k_2N^2 + 1586k_2N - 152k_2 \\
- 384N^5 + 2496N^4 - 5472N^3 + 5280N^2 - 2208N + 288), \\
A_{11} = \frac{32(N-1)}{k_2(3k_2 + 2N - 2)d(k_2, N)}A_1(60k_2^3N^4 - 240k_2^3N^3 - 219k_2^3N^2 \\
+ 504k_2^3N - 168k_2^3 + 580k_2^3N^5 - 2900k_2^2N^4 + 635k_2^3N^3 \\
+ 5855k_2^2N^2 - 4930k_2^2N + 760k_2^2 + 1320k_2N^6 - 7920k_2N^5 \\
+ 9102k_2N^4 + 9084k_2N^3 - 20970k_2N^2 + 9960k_2N - 576k_2 \\
- 1328N^5 + 8632N^4 - 17384N^3 + 13640N^2 - 3016N - 544), \\
A_{12} = \frac{160(N-1)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(k_2, N)}A_1(12k_2^4N^4 - 48k_2^4N^3 - 15k_2^4N^2 \\
+ 54k_2^4N + 24k_2^4 + 204k_2^4N^5 - 1020k_2^3N^4 + 561k_2^3N^3 \\
+ 1173k_2^3N^2 - 510k_2^3N - 408k_2^3 + 1160k_2^2N^6 - 6960k_2^2N^5 \\
+ 10790k_2N^4 - 3540k_2^2N^3 + 1614k_2^2N^2 - 7928k_2N^3 + 4864k_2^3 \\
+ 2208k_2N^7 - 15456k_2N^6 + 41256k_2N^5 - 64032k_2N^4 + 85488k_2N^3 \\
- 99360k_2N^2 + 69384k_2N - 19488k_2 + 5696N^6 - 42720N^5 \\
+ 136320N^4 - 232000N^3 + 219840N^2 - 109536N + 22400), \\
A_{13} = -\frac{48}{k_2(3k_2 + 2N - 2)d(k_2, N)}A_1(20k_2^3N^4 - 80k_2^3N^3 - 187k_2^3N^2 \\
+ 387k_2^3N - 212k_2^3 + 200k_2^3N^5 - 1000k_2^2N^4 - 530k_2^3N^3 \\
+ 4210k_2^2N^2 - 4160k_2^2N + 1280k_2^2 + 480k_2N^6 - 2880k_2N^5 \\
+ 2576k_2N^4 + 6652k_2N^3 - 14136k_2N^2 + 9532k_2N - 2224k_2 \\
- 544N^5 + 3536N^4 - 8368N^3 + 9328N^2 - 4976N + 1024), \\
A_{14} = -\frac{480(N-1)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(k_2, N)}A_1(20k_2^3N^4 - 80k_2^3N^3 \\
- 25k_2^3N^2 + 90k_2^3N + 40k_2^3 + 200k_2^3N^5 - 1000k_2^3N^4
A_{15} = \frac{36}{(N-1)(3k_2 + 2N - 2)d(k_2, N)} A_1(15k_2^3N^2 - 30k_2^3N + 18k_2^3) \\
+ 100k_2^3N^3 - 300k_2^3N^2 + 320k_2^3N - 120k_2^3 + 100k_2N^4 \\
- 460k_2N^3 + 876k_2N^2 - 772k_2N + 256k_2 + 176N^3 - 528N^2 + 528N - 176), \\
A_{16} = -\frac{1}{k_2(3k_2 + 2N - 2)(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(k_2, N)} A_1 \\
\times (38k_2^2N^2 - 73k_2^2N + 52k_2^2 + 228k_2N^3 - 666k_2N^2 \\
+ 750k_2N - 312k_2 + 112N^4 - 616N^3 + 1376N^2 - 1352N + 480), \\
A_{17} = -\frac{1440(N - 1)}{k_2(3k_2 + 2N - 2)d(k_2, N)} A_1(8k_2^2N^2 - 13k_2^2N + 16N^2) \\
+ 44k_2N^3 - 108k_2N^2 + 168k_2N - 104k_2 + 176N^2 - 352N + 176), \\
A_{18} = \frac{144(N + 2)}{k_2(3k_2 + 2N - 2)} A_1 \\
\times \frac{1}{(30k_2^2N^2 + 7k_2^2N + 44k_2^2 + 120k_2N^3 + 88k_2N^2 + 190k_2N + 88k_2 + 88N^3 + 264N^2 - 352)} \\
\times (6k_2^2N^2 + 51k_2^2N - 24k_2^2 + 28k_2N^3 + 246k_2N^2 - 44k_2^2N \\
- 32k_2^2 + 32k_2N^3 + 160k_2N^2 - 236k_2N^2 - 76k_2N + 336k_2 + 88N^3 + 264N^2 - 352), \\
A_{19} = \frac{5760(N - 1)^2}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)d(k_2, N)} A_1(38k_2^2N^2 - 73k_2^2N + 52k_2^2 \\
+ 228k_2N^3 - 666k_2N^2 + 750k_2N - 312k_2 + 112N^4 \\
- 616N^3 + 1376N^2 - 1352N + 480), \\
A_{20} = -\frac{1440(N - 1)}{k_2(3k_2 + 2N - 2)d(k_2, N)} A_1(22k_2^2N^2 - 47k_2^2N + 20k_2^2 + 140k_2N^3 \\
- 450k_2N^2 + 414k_2N - 104k_2 + 112N^4 - 616N^3 + 1024N^2 - 648N + 128), \quad (F.1)

where the simplified notation
\[ d(k_2, N) = \frac{1}{(10k_2^2N^2 - 5k_2^2N + 44k_2^2 + 60k_2N^3 - 90k_2N^2 + 294k_2N - 264k_2 + 352N^2 \\
- 704N + 352), \quad (F.2) \]

is used. Also one can use these coefficients for the finite N behavior in the N = 1 coset model corresponding to \((3.14)\).

The coefficients in Appendix \([F.1]\) and Appendix \([F.2]\) under the large ‘t Hooft limit \([2.30]\) lead to
\[ A_2 = \frac{8\lambda}{(\lambda - 1)} A_1, \quad A_3 = \frac{60\lambda^2}{(\lambda - 1)(2\lambda - 3)} A_1, \]

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\[ A_4 = \frac{160 \lambda^3}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_5 = -\frac{40 \lambda^4}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \]
\[ A_6 = -N^2 \frac{12(2\lambda + 3)}{(2\lambda - 3)} A_1, \quad A_7 = N^2 \frac{8(2\lambda + 3)}{(2\lambda - 3)} A_1, \]
\[ A_8 = -N^2 \frac{48\lambda^2 (2\lambda^2 + 3\lambda + 3)}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_9 = N^2 \frac{32\lambda^2 (2\lambda^2 + 3\lambda + 3)}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \]
\[ A_{10} = -N^2 \frac{48\lambda (\lambda + 2)}{(\lambda - 1)(2\lambda - 3)} A_1, \quad A_{11} = N^2 \frac{16\lambda(5\lambda + 6)}{(\lambda - 1)(2\lambda - 3)} A_1, \]
\[ A_{12} = -N^2 \frac{16\lambda (19\lambda^2 + 21\lambda + 6)}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_{13} = -N \frac{48\lambda (\lambda + 1)}{(\lambda - 1)(2\lambda - 3)} A_1, \]
\[ A_{14} = N \frac{240\lambda^2 (\lambda + 1)}{(\lambda - 3)(\lambda - 1)(2\lambda - 3)} A_1, \quad A_{15} = -\frac{1}{N} \frac{18 (2\lambda^2 - 4\lambda - 3)}{(\lambda - 3)(2\lambda - 3)(2\lambda + 1)} A_1, \]
\[ A_{16} = \frac{1}{N} \frac{36\lambda^4 (24\lambda^2 - 19\lambda - 19)}{(\lambda - 3)(\lambda - 1)^2(2\lambda - 3)(2\lambda + 1)} A_1, \]
\[ A_{17} = -\frac{1}{N} \frac{72\lambda^5 (\lambda + 4)}{(2\lambda - 3)(2\lambda + 1)(\lambda - 1)} A_1, \quad A_{18} = -\frac{1}{N} \frac{144\lambda (2\lambda^2 - 4\lambda - 3)}{(\lambda - 3)(2\lambda + 1)(\lambda - 1)} A_1, \]
\[ A_{19} = -\frac{1}{N} \frac{144\lambda^3 (24\lambda^2 - 19\lambda - 19)}{(\lambda - 3)(\lambda - 1)^2(2\lambda - 3)(2\lambda + 1)} A_1, \]
\[ A_{20} = -\frac{1}{N} \frac{72\lambda^2 (10\lambda^2 - 13\lambda - 11)}{(\lambda - 1)^2(2\lambda - 3)(2\lambda + 1)} A_1. \]

The nineteen coefficients are given by \( A_1 \) coefficient. One can analyze the \( N \) behavior on these coefficients explicitly as done before and they will contribute to the various eigenvalue equations (and three-point functions) in (3.14).

### G. The OPEs between the spin \( \frac{3}{2} \) current and various spin 4 currents

One presents the various OPEs between the spin \( \frac{3}{2} \) current in (3.7) and various higher spin 4 currents in (2.14)

\[ \hat{G}(z) a^{abcd} J^a J^b J^c J^d (w) \bigg|_{\frac{1}{(z-w)^2}} = a^{abcd} \left( -k_{2d} \psi^a J^b J^c J^d + f^{bb'd'} f^{ab'c'} (\psi^d K^c') (J^c J^d) \right. \]
\[ + f^{cb'd'} f^{ab'c'} J^b (\psi^d K^c') J^d \bigg|_{\frac{1}{(z-w)^2}} = \frac{1}{(z-w)^2} \]
\[ \hat{G}(z) a^{bcde} J^b J^c J^d K^e (w) \bigg|_{\frac{1}{(z-w)^2}} = a^{bcde} \left( k_{2a} \psi^b J^c J^d + f^{ade'd'} f^{bc'f'} (\psi^d K^f') (J^b J^c) \right. \]
\[ - f^{bc'd'} f^{ade'} J^a (\psi^d K^f') J^c \bigg|_{\frac{1}{(z-w)^2}} = \frac{1}{(z-w)^2}. \]
$\nabla^a f^{abcde} J^b J^c J^d J^e K^f$ $- 2k_2 K^d J^a J^b J^c + f^{bde} f^{cde} K^d J^a J^b J^c$, 

$\hat{G}(z) \frac{d^{bcde} J^b J^c K^d K^e}{(z-w)^2} = d^{abcd} \left( - k_2 J^a J^b K^c K^d + f^{aef} f^{bfg} J^a J^b J^c \right)$ 

$+ f^{c} g f^{aef} J^b (\psi^e K^g) K^d + f^{aef} f^{d} g f^{bfg} K^b J^c \psi^e K^g = k_2 J^a J^b K^c K^d + d^{abcd} f^{aef} f^{bfg} J^a J^b J^c K^e K^d$ 

$+ f^{bfg} J^b J^c \psi^e K^a + 2k_2 J^a J^b \psi^e K^d - f^{c} g f^{d} g J^a J^b J^c \psi^e K^g$, 

$\hat{G}(z) \frac{d^{bcde} K^b K^c K^d K^e}{(z-w)^2} = d^{abcd} \left( k_2 J^a J^b K^c K^d + f^{aef} f^{bfg} (\psi^e K^g) K^c K^d \right)$ 

$- f^{aef} f^{c} g f^{bfg} (\psi^e K^g) K^d + f^{aef} f^{d} g f^{bfg} K^b K^c \psi^e K^g = k_2 J^a J^b K^c K^d + d^{abcd} f^{aef} f^{bfg} (\psi^e K^g) K^c K^d$ 

$- f^{c} g f^{d} g J^a J^b \psi^e K^c + 2k_2 J^a J^b \psi^e K^d - f^{c} g f^{d} g J^a J^b \psi^e K^g$, 

$\hat{G}(z) \frac{\partial J^a \partial J^a}{(z-w)^2} = -2k_2 \partial^a J^a \partial J^a - f^{abc} (\psi^b K^c) \partial J^a - 2k_2 \partial J^a \partial \psi^a - f^{abc} \partial J^a \psi^b K^c$, 

$\hat{G}(z) \frac{\partial J^a J^a}{(z-w)^2} = -3k_2 \partial^2 J^a J^a - 2f^{abc} (\psi^b K^c) J^a - k_2 \partial^2 J^a J^a$, 

$\hat{G}(z) \frac{\partial K^a \partial K^a}{(z-w)^2} = 2k_2 \partial \psi^a \partial K^a + f^{abc} (\psi^b K^c) \partial K^a + 2k_2 \partial K^a \partial \psi^a + f^{abc} \partial K^a \psi^b K^c$, 

$\hat{G}(z) \frac{\partial^2 K^a K^a}{(z-w)^2} = 3k_2 \partial^2 \psi^a K^a + 2f^{abc} (\psi^b K^c) K^a + k_2 \partial^2 K^a \psi^a$, 

$\hat{G}(z) \frac{\partial J^a \partial K^a}{(z-w)^2} = -2k_2 \partial \psi^a \partial K^a - f^{abc} (\psi^b K^c) \partial K^a + 2k_2 \partial J^a \partial \psi^a + f^{abc} \partial J^a \psi^b K^c$, 

$\hat{G}(z) \frac{\partial^2 J^a K^a}{(z-w)^2} = -3k_2 \partial^2 \psi^a K^a - 2f^{abc} (\psi^b K^c) K^a + k_2 \partial^2 J^a \psi^a$, 

$\hat{G}(z) \frac{J^a \partial^2 K^a}{(z-w)^2} = -k_2 \partial^2 \psi^a K^a - 2(2N-2) \partial^2 (\psi^a K^a) + 3k_2 J^a \partial^2 \psi^a$ 

$+ 2f^{abc} J^a \partial (\psi^b K^c)$,
\[
\hat{G}(z) f^{abc} J^a \partial J^b K^c(w) = -k_2 f^{abc} \psi^a \partial J^b K^c - (2N - 2) f^{ef} \partial (\psi^f K^e) K^c
\]

\[
+ (2N - 2) f^{dj} \partial J^b \psi^d K^f
\]

\[
- 2k_2 f^{abc} J^a \partial J^b \psi^c K^c - f^{bde} \psi^d f^{abc} J^a (\psi^f K^e) K^c + k_2 f^{abc} J^a \partial J^b \psi^c + 2k_2 (2N - 2) \partial J^a \partial \psi^a,
\]

\[
\hat{G}(z) f^{abc} J^a K^b \partial K^c(w) = -k_2 f^{abc} \psi^a K^b \partial K^c - (2N - 2) f^{abc} (\psi^a K^b) \partial K^c
\]

\[
+ (2N - 2) f^{abc} K^a \partial \psi^b K^c + k_2 f^{abc} J^a \partial \psi^b K^c + 2k_2 (2N - 2) J^a \partial^2 \psi^a + 2k_2 (2N - 2) K^a \partial^2 \psi^a
\]

\[
+ (2N - 2) f^{abc} J^a \partial \psi^b K^c + 2k_2 f^{abc} J^a K^b \partial \psi^c + f^{abc} f^{cde} J^a K^b \psi^d K^e - 2k_2 (2N - 2) \partial \psi^a \partial K^a,
\]

\[
\hat{G}(z) J^a J^b J^c(w) = -k_2 \psi^a J^b J^c + 2(2N - 2)(\psi^a K^a)(J^b J^c) + 2(2N - 2) J^a J^b \psi^b K^b
\]

\[
+ 2f^{acd} f^{bce} J^a (\psi^d K^d) J^b J^c K^d - k_2 J^a \psi^a J^b J^c - k_2 J^a J^b \psi^a J^b - k_2 J^a J^b \psi^b J^b - k_2 J^a J^a J^b \psi^b,
\]

\[
\hat{G}(z) K^a K^b K^c(w) = k_2 \psi^a K^a K^b K^c + 2(2N - 2)(\psi^a K^a)(K^b K^c)
\]

\[
- 4k_2 f^{abc} K^a \partial \psi^b K^b - f^{fde} f^{cde} K^a (\psi^e K^a) K^b - 2f^{abc} f^{cde} K^a \psi^b K^d + k_2 K^a \psi^a K^b K^b
\]

\[
- f^{abc} f^{cde} K^a (\psi^d K^d) K^c + 2(2N - 2) K^a \partial \psi^a K^b,
\]

\[
\hat{G}(z) J^a J^b K^c(w) = -2k_2 \psi^a J^b K^c + 2(2N - 2)(\psi^a K^a)(K^b K^c)
\]

\[
+ 4k_2 f^{abc} J^a \partial \psi^b K^c + 2f^{acd} f^{bce} J^a (\psi^c K^c) J^b + 2f^{acd} f^{bde} J^a K^b \psi^c K^e
\]

\[
+ 2(k_2 + 2N - 2) J^a J^a \psi^b K^b,
\]

\[
\hat{G}(z) J^a J^b K^b(w) = -2k_2 \psi^a J^b K^b + 2(2N - 2)(\psi^a K^a)(J^b K^b)
\]

\[
+ 2f^{acd} f^{bce} J^a (\psi^d K^d) K^b + 2k_2 f^{abc} J^a \partial \psi^c + 2f^{acd} f^{bde} J^a J^b \psi^c K^e - k_2 J^a J^b \psi^b K^b
\]

\[
- 2(2N - 2) J^a J^a \psi^b K^b + k_2 J^a J^a J^b \psi^b,
\]

\[
\hat{G}(z) J^a K^b K^a(w) = -2k_2 \psi^a K^a K^b K^b - 2(2N - 2)(\psi^a K^a)(K^b K^b)
\]

\[
+ k_2 f^{abc} K^a \partial \psi^c K^b + f^{acd} f^{bde} K^a (\psi^e K^e) K^b + k_2 f^{abc} K^a \partial \psi^c + f^{acd} f^{bde} K^a K^b \psi^e K^e
\]

\[
+ k_2 J^a \psi^a K^b K^b - k_2 f^{abc} J^a \partial \psi^b K^c - f^{acd} f^{bde} J^a (\psi^c K^c) K^b + k_2 f^{abc} J^a K^b \partial \psi^c
\]

\[
- f^{acd} f^{bde} J^a K^b \psi^c K^e + 2(k_2 + 2N - 2) J^a \psi^b K^b,
\]

\[
\hat{G}(z) J^a J^b K^a K^b(w) = -2k_2 \psi^a J^b K^a K^b + f^{acd} f^{bce} (\psi^d K^d)(K^a K^b)
\]

\[
- 2(2N - 2) J^a (\psi^b K^b) K^a + 2k_2 f^{abc} J^a \partial \psi^b K^c + f^{acd} f^{bde} J^a J^b K^a \psi^c K^e - k_2 J^a \psi^b K^a K^b
\]

\[
+ f^{acd} f^{bde} J^a (\psi^c K^c) K^b - 2(2N - 2) J^a K^a \psi^b K^b
\]

\[
+ k_2 J^a J^b \psi^a K^b - k_2 f^{abc} J^a J^b \partial \psi^c - f^{acd} f^{bde} J^a J^b \psi^c K^e + k_2 J^a J^b K^a \psi^b.
\]
Note that the field $d^{abcd} \psi^a J^b J^c J^d w$ has the coefficient $-4(k_2 A_1 + k_2 A_2)$ which becomes $-4(-4 + k_2 + 4N)A_1$ using Appendix [E.1]. Further arrangement for the normal ordered product is needed to express singular terms in simple form. In principle, one can obtain the higher spin $\frac{7}{2}$ current (3.15) by simplifying the above results in Appendix [G.1].

H  The OPEs between the diagonal spin 1 current and various spin $\frac{7}{2}$ currents

The OPE between diagonal spin 1 current (3.3) and the various higher spin $\frac{7}{2}$ currents in (3.16) can be written as

$$J^a(z) d^{bcde} \psi^b J^c J^d J^e(w) = \frac{1}{(z-w)^2} 2(4N^2 - 14N + 22)(2N - 2)\psi^a(w)$$

$$- \frac{1}{(z-w)^2} 4(4N^2 - 14N + 22)(2N - 2)\partial \psi^a(w) + \frac{1}{(z-w)^2} \left[ 2(2N - 2)(2N^2 - 7N + 11)\partial^2 \psi^a \right. $$

$$- 14(N - 1)d^{abcd} \psi^b J^c J^d - 12\psi^a J^b J^d + 12\psi^b J^b J^c + 6(2N^2 - 7N + 13)f^{abc} \psi^b \partial J^c \right] (w) + \cdots,$$

$$J^a(z) d^{bcde} f^{fgh} f^{fgh} J^d J^e J^h K^h(w) = - \frac{1}{(z-w)^2} 4(N - 1)(2N^2 - 7N + 11)f^{abc} \psi^b K^c(w)$$

$$+ \frac{1}{(z-w)^2} \left[ \frac{8}{3}(2N - 2)(N - 1)d^{abcd} \psi^b J^d \right. $$

$$- \frac{4}{3}(N - 1)d^{cdef} f^{fgh} f^{fgh} J^e K^d + \frac{4}{3}k_2(N - 1)d^{abc} \psi^b J^c J^d \left[ 2(2N^2 - 7N + 11)\partial^2 \psi^a \right. $$

$$+ 4f^{abc} f^{cdef} \psi^c J^b K^d - \frac{44}{3}(N - 1)(2N^2 - 7N + 11)f^{abc} \partial \psi^b K^c \right] (w) + \cdots,$$

$$J^a(z) d^{bcde} J^b K^c \psi^d K^e(w) = \frac{1}{(z-w)^2} \left[ d^{bcde} f^{fgh} f^{fgh} \psi^g K^d K^e \right. $$

$$- (2N - 2)d^{abcd} \psi^b K^c K^d + d^{bcde} f^{fgh} f^{fgh} \psi^b J^c K^g - 2k_2d^{abcd} \psi^b J^c K^d \right] (w) + \cdots,$$

$$J^a(z) d^{bcde} f^{fgh} f^{fgh} K^d K^e \psi^g K^h(w) = - \frac{1}{(z-w)^2} 8k_2(N - 1)(2N^2 - 7N + 11)\psi^a$$

$$+ \frac{1}{(z-w)^3} 4(N - k_2 - 1)(2N^2 - 7N + 11)f^{abc} \psi^b K^c$$

$$+ \frac{1}{(z-w)^2} \left[ 4k_2(N - 1)d^{abcd} \psi^b K^c K^d - \frac{4}{3}(N - 1)d^{bcd} f^{fgh} f^{fgh} \psi^g K^b K^g K^e \right. $$

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\[\frac{8}{3}(N-1)d^{bcde} f^{adef} f^{feg\psi^b K^c K^d - 4(2k_2 + 2(2N - 2))\psi^b K^c K^d} \]

\[+ (4(k_2 + 2N - 2) + (2N - 5)(2N - 2)(k_2 + 2N - 2) - \frac{2}{3}(2N - 5)(2N - 2)^2) f^{abc\psi^b \partial K^c} \]

\[+ (4 + (2N - 2)(2N - 5)) f^{cde\psi^e K^d K^c} + 4(2k_2 + 2(2N - 2))\psi^b K^a K^b \]

\[− 8 f^{abc f^{cde\psi^d K^b K^c}}(w) + \cdots, \]

\[J^a(z) J^b J^c J^e(w) = \frac{1}{(z - w)^2} 4(2N - 2)^2 \psi^a(w) - \frac{1}{(z - w)^3} 8(2N - 2)^2 \partial \psi^a(w) \]

\[+ \frac{1}{(z - w)^2} \left[ - 3(2N - 2)\psi^a J^b J^b + 2(2N - 2) f^{abc\psi^b \partial J^c} \right](w) + \cdots, \]

\[J^a(z) J^b J^c J^c(w) = \frac{1}{(z - w)^2} 2k_2(2N - 2)\psi^a(w) \]

\[+ \frac{1}{(z - w)^3} (2k_2 - 2(2N - 2)) f^{abc\psi^b K^c}(w) \]

\[+ \frac{1}{(z - w)^2} \left[ - (2k_2 + 2(2N - 2)) \psi^b K^a K^b + 2 f^{abc f^{cde\psi^d K^b K^c} - k_2 \psi^a K^b K^b} \right](w) + \cdots, \]

\[J^a(z) J^b J^c K^c(w) = \frac{1}{(z - w)^3} 2(2N - 2) f^{abc\psi^b K^c}(w) \]

\[+ \frac{1}{(z - w)^2} \left[ - 4(2N - 2)\psi^a J^b J^b - k_2 \psi^a J^b J^b + k_2(2N - 2)\partial^2 \psi^a \right] \]

\[+ 2(2N - 2) f^{abc \partial \psi^b K^c} - 2 f^{abc f^{cde\psi^d J^b K^c} + 2k_2 f^{abc\psi^b \partial J^c} \right](w) + \cdots, \]

\[J^a(z) J^b J^c J^e(w) = \frac{1}{(z - w)^2} 6(2N - 2)^2 \psi^a(w) - \frac{1}{(z - w)^3} 12(2N - 2)^2 \partial \psi^a(w) \]

\[+ \frac{1}{(z - w)^2} \left[ - 3(2N - 2)\psi^a J^b J^b + 3(2N - 2)^2 \partial^2 \psi^a \right] \]

\[- 4(2N - 2)\psi^b J^a J^b - 2(2N - 2) f^{abc\psi^b \partial J^c} \right](w) + \cdots, \]

\[J^a(z) \psi^b K^c K^c(w) = - \frac{1}{(z - w)^3} (2k_2 + 2(2N - 2)) f^{abc\psi^b K^c}(w) \]

\[+ \frac{1}{(z - w)^2} \left[ - k_2 \psi^a K^b K^b - (2k_2 + 2(2N - 2)) \psi^b K^b K^a \right](w) + \cdots, \]

\[J^a(z) f^{bcdef} K^b K^f \psi^e K^e(w) = - \frac{1}{(z - w)^4} 4(2N - 2)^2 k_2 \psi^a(w) \]

\[+ \frac{1}{(z - w)^3} (2N - 2)(-2k_2 + 2(2N - 2)) f^{abc\psi^b K^c}(w) \]

\[+ \frac{1}{(z - w)^2} \left[ (2N - 2)(k_2 + (2N - 2)) f^{abc\psi^b \partial K^c} + (2N - 2) f^{abc f^{cde\psi^d K^d K^b} \right](w) + \cdots, \]
\[ J^a(z) J^b \psi^b K^c K^c(w) = \frac{1}{(z-w)^2} \left[ -3(2N-2)\psi^a K^b K^b - (2k_2 + 2(2N-2))\psi^b J^b K^a \right](w) + \cdots , \]
\[ J^a(z) \psi^b J^c K^b K^c(w) = \frac{1}{(z-w)^2} \left[ f^{abc} f^{cde} \psi^b K^b K^d - (2N-2)\psi^b K^b K^a \right] + \cdots , \]
\[ J^a(z) J^b J^c \psi^b K^c K^c(w) = \frac{1}{(z-w)^2} \left[ -k_2(2N-2)\partial^2 \psi^a + f^{abc} f^{cde} \psi^b J^b K^d \right] + \cdots , \]
\[ J^a(z) f^{bcd} f^{def} J^b J^c \psi^c K^e(w) = \frac{1}{(z-w)^2} (2N-2)^2 f^{abc} \psi^b K^c K^e(w) + \cdots , \]
\[ J^a(z) f^{bcd} f^{def} J^b J^c \psi^c K^e(w) = \frac{1}{(z-w)^2} \left[ -5(2N-2)^2 f^{abc} \partial \psi^b K^c + 3(2N-2) f^{abc} f^{cde} \partial \psi^c J^b K^e \right] + \cdots , \]
\[ J^a(z) J^b J^c \psi^b K^e(w) = \frac{1}{(z-w)^2} 3(2N-2) f^{abc} \psi^b K^c K^e(w) + \cdots , \]
\[ J^a(z) J^b J^c \psi^b K^c(w) = \frac{1}{(z-w)^2} \left[ -k_2 f^{abc} \partial J^c - 3(2N-2)\psi^a J^b K^b - (2N-2)\psi^b J^b K^a \right] + \cdots . \]

It is nontrivial to express these OPEs in terms of independent terms with the rearrangement of the normal ordered product \[18 \sqbrack{19,62}. \]

We describe each independent terms with their coefficients in (3.16) appearing in the various poles of Appendix (H.1) where the rearrangement \[18 \sqbrack{18,62} \] of the normal ordered product is used as follows:

pole-4: \[ -4B_1 k_2 (2N-2)^2 + 2B_5 k_2 (2N-2) - B_4 8k_2 (N-1)(2N^2 - 7N + 11) + 2B_1 (4N^2 - 14N + 22)(2N-2) + 4B_5 (2N-2)^2 + 6B_6 (2N-2)^2 \] \( \psi^a(w) = 0, \)
pole-3: \[ -4B_1 (4N^2 - 14N + 22)(2N-2) - 8B_5 (2N-2)^2 - 12B_6 (2N-2)^2 \] \( \partial \psi^a(w) = 0, \)
pole-3: \[ B_4 (N-k_2 - 1)(2N^2 - 7N + 11) + B_1 (2N-2)(-2k_2 + 2N-2) \]
\[ +B_6(2k_2 - 2(2N - 2)) - B_9(2k_2 + 2(2N - 2)) - B_{24}(N - 1)(2N^2 - 7N + 11) \\
+ B_{14}(2N - 2)^2 + 2B_7(2N - 2) + 3B_{15}(2N - 2) \left[ f^{abcd}\psi^b K^c(w) = 0, \right. \\
\text{pole-2 :} \quad \left[ \frac{4}{3}B_2k_2(N - 1) - 14B_1(N - 1) \right] d^{abcd}\psi^b J^d = 0, \\
\text{pole-2 :} \quad \left[ 8B_2k_2 - B_7k_2 - 3B_5(2N - 2) - 3B_8(2N - 2) - 12B_1 \right] \psi^a J^b J^c(w) = 0, \\
\text{pole-2 :} \quad \left[ -8B_2k_2 - B_{13}k_2 - B_{15}k_2 - 4B_5(2N - 2) - 4B_8(2N - 2) \\
\quad + 12B_1 \right] \psi^b J^b J^d(w) = 0, \\
\text{pole-2 :} \quad \left[ -B_{14}k_2(2N - 2)^2 + B_7k_2(2N - 2) - B_{13}k_2(2N - 2) \\
\quad - B_24k_2(N - 1)(2N^2 - 7N + 11) + 2B_5(2N - 2)^2 + 3B_8(2N - 2)^2 \\
\quad + B_12(2N^2 - 7N + 11)(2N - 2) \right] \partial^2 \psi^a(w) = 0, \\
\text{pole-2 :} \quad \left[ -B_2\frac{10}{3}k_2(2N^2 - 7N + 11) - 3B_{14}k_2(2N - 2) + 2B_7k_2 - B_{13}k_2 \\
\quad - 2B_{15}k_2 + B_1(3(2N - 5)(2N - 2) + 48) + 2B_5(2N - 2) \\
\quad - 2B_8(2N - 2) \right] f^{abcd}\psi^b \partial J^c(w) = 0, \\
\text{pole-2 :} \quad \left[ -B_6k_2 - B_9k_2 - \frac{1}{3}32B_4(N - 1) - \frac{16}{3}B_4(N - 1) - 3B_{11}(2N - 2) \\
\quad - 8B_3 \right] \psi^a K^b \psi^b K^c(w) = 0, \\
\text{pole-2 :} \quad \left[ B_4(4k_2(N - 1) + \frac{16}{3}(N - 1)^2) - \frac{1}{3}4B_3(N - 1) \\
\quad - B_3(2N - 2) \right] d^{abcd}\psi^b K^c K^d = 0, \\
\text{pole-2 :} \quad \left[ -4B_4(2k_2 + 2(2N - 2)) - 4B_4(-2k_2 - 2(2N - 2)) + B_6(-2k_2 - 2(2N - 2)) \\
\quad - B_9(2k_2 + 2(2N - 2)) + \frac{32}{3}B_4(N - 1) - \frac{32}{3}B_4(N - 1) + \frac{4}{3}16B_4(N - 1) \\
\quad - \frac{16}{3}B_4(N - 1) - B_{12}(2N - 2) + 4B_3 + 4B_3 \right] \psi^b K^b \psi^c K^d = 0, \\
\text{pole-2 :} \quad \left[ -\frac{8}{9}B_4(N - 1)(2N - 5) - \frac{4}{9}B_4(N - 1)(2N - 5) - B_4((2N - 5)(2N - 2) + 4) \\
\quad - \frac{16}{9}B_4(N - 1)(2N - 5) - B_4((2N - 5)(2N - 2) + 4) \right] \psi^b K^b \psi^c K^d = 0, \]
\[-B_{10}(2N - 2) - 8B_4 + 2B_6 - B_{12} - \frac{2}{3}(2N - 5)B_3 \left[ f^{abc} f^{cde} \psi^d K^b K^e \right] = 0,\]

\[\text{pole-2 : } \left[ -2B_3k_2 + \frac{1}{3}16B_2(N - 1)^2 - \frac{4}{3}B_3(N - 1) \right.\]
\[\quad + \frac{8}{3}B_2(2N - 2)(N - 1) \left. \right] d^{abcd}_{\psi^b J^c K^d}(w) = 0,\]

\[\text{pole-2 : } \left. \left[ -B_{12}k_2 + \frac{2}{3}32B_2(N - 1) - \frac{16}{3}B_2(N - 1) - 24B_2(2N - 2) \right.\]
\[\quad - 3B_{13}(2N - 2) - 3B_{15}(2N - 2) + 4B_3 \left. \right] \psi^a J^b K^b(w) = 0,\]

\[\text{pole-2 : } \left[ -B_{11}(2k_2 + 2(2N - 2)) - 8B_2(2N - 2) - 16B_2(N - 1) - B_{13}(2N - 2) \right.\]
\[\quad - B_{15}(2N - 2) - 8B_3 \left. \right] \psi^b J^b K^a(w) = 0,\]

\[\text{pole-2 : } \left[ -\frac{1}{3}B_3(2N - 5) - \frac{4}{9}B_2(N - 1)(2N - 5) - \frac{4}{9}B_2(N - 1)(2N - 5) \right.\]
\[\quad - \frac{4}{9}B_2(N - 1)(2N - 5) + \frac{1}{3}B_3(-2N - 5) - B_2(\frac{5}{3}(2N - 5)(2N - 2) + 16) \]
\[\quad - 3B_{14}(2N - 2) + 4B_2 + 2B_7 - B_{12} + B_{13} \left. \right] f^{abc} f^{cde} \psi^e J^b K^d(w) = 0,\]

\[\text{pole-2 : } \left[ -\frac{4}{3}(-1)B_2(N - 1)(4N^2 - 14N + 22) - 5B_{14}(2N - 2)^2 + 2B_7(2N - 2) \right.\]
\[\quad - 4B_{13}(2N - 2) - 3B_{15}(2N - 2) \]
\[\quad + B_2(-44(2N - 2) - \frac{11}{3}(2N - 5)(2N - 2)^2) \right] f^{abc} \psi^b K^c(w) = 0,\]

\[\text{pole-2 : } \left. \left[ B_{10}(2N - 2)(k_2 + 2N - 2) + B_4((2N - 5)(2N - 2)(k_2 + 2N - 2) \right.\]
\[\quad + 4(k_2 + 2N - 2) - \frac{2}{3}(2N - 5)(2N - 2)^2) - 4B_4(-2k_2 - 2(2N - 2)) \]
\[\quad + B_6(-2k_2 - 2(2N - 2)) - \frac{1}{9}8B_4(N - 1)(2N - 5)(2N - 2) \]
\[\quad + \frac{1}{3}B_3(-(2N - 5))(2N - 2) - 8B_4(2N - 2) + 2B_6(2N - 2) \]
\[\quad - B_{12}(2N - 2) - \frac{32}{3}B_4(N - 1) + \frac{1}{9}B_4(-4(N - 1)(2N - 5)(2N - 2)) \]
\[\quad + \frac{32}{3}B_4(N - 1) + 4B_3 \right] f^{abc} \psi^b \partial K^c(w) = 0.\] (H.2)
Here one should obtain the various independent terms from Appendix (H.1).

I  The OPEs between the numerator spin 2 current and various spin $\frac{7}{2}$ currents

The primary condition (3.18) for the higher spin $\frac{7}{2}$ current in (3.16) can be described as

$$
\hat{T}(z) \epsilon^{abcd} \psi^a J^b J^c J^d(w) = -\frac{1}{(z-w)^4} 3(2N-2)(8N-2) \psi^a J^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) \epsilon^{abef} \epsilon^{beg} \epsilon^{jcf} J^d \psi^f K^g(w) = \frac{1}{(z-w)^4} 16(N-1)^2(4N-1) \psi^a K^a(w)
$$

$$
+ \frac{1}{(z-w)^3} 16(N-1)(2N^2 - 7N + 11) \psi^a \partial K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) \epsilon^{abcd} J^a K^b \psi^c K^d(w) = -\frac{1}{(z-w)^4} k_2(8N-2) \psi^a J^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) \epsilon^{abef} \epsilon^{beg} \epsilon^{jcf} K^d \psi^f K^g(w) = \frac{1}{(z-w)^4} 8k_2(N-1)(4N-1) \psi^a K^a(w)
$$

$$
+ \frac{1}{(z-w)^3} 8(N-1)(2N^2 - 7N + 11) \psi^a \partial K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) J^a \psi^a J^b J^c(w) = -\frac{1}{(z-w)^4} (3 + N(2N-1))(2N-2) \psi^a J^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) K^a K^b \psi^b J^c(w) = -\frac{1}{(z-w)^4} k_2(N(2N-1) + 2) \psi^a K^a(w)
$$

$$
- \frac{1}{(z-w)^3} 2(2N-2) \psi^a \partial K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) J^a \psi^b K^b(w) = -\frac{1}{(z-w)^4} (2N-2)N(2N-1) \psi^a K^a(w)
$$

$$
- \frac{1}{(z-w)^3} 4(2N-2) \partial \psi^a K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) J^a \psi^b J^b(w) = -\frac{1}{(z-w)^4} (6 + N(2N-1))(2N-2) \psi^a J^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) \psi^a K^b K^b(w) = -\frac{1}{(z-w)^4} (2(k_2 + 2N-2) + k_2 N(2N-1)) \psi^a K^a(w)
$$

$$
+ \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) \epsilon^{abc} \epsilon^{cde} \psi^b K^c \psi^d K^d(w) = \frac{1}{(z-w)^4} 2(2N-2)^2 \psi^a K^a(w)
$$

$$
+ \frac{1}{(z-w)^3} 4(2N-2)^2 \psi^a \partial K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

$$
\hat{T}(z) J^a \psi^b K^b(w) = -\frac{1}{(z-w)^4} k_2 N(2N-1) \psi^a J^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}),
$$

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One should have vanishing coefficients in order to have primary higher spin ing to the fourth order pole in (I.2). The coefficient of fourth order pole is given by

\[ \psi^a J^b K^a K^b(w) = -\frac{1}{(z-w)^4}k_2 \psi^a J^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}), \]

\[ \hat{T}(z) J^a J^b \psi^a K^b(w) = \frac{1}{(z-w)^4} (2N-2) \psi^a K^a(w) + \frac{1}{(z-w)^3} 4(2N-2) \psi^a K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}), \]

\[ \hat{T}(z) f^{abc} f^{cde} J^a J^e \psi^b K^d(w) = \frac{1}{(z-w)^4} 4(2N-2) \psi^a K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}); \]

\[ \hat{T}(z) J^a J^b K^a \psi^b(w) = -\frac{1}{(z-w)^4} 3(2N-2) \psi^a K^a(w) + \mathcal{O}(\frac{1}{(z-w)^2}). \] (I.1)

One also uses the various identities between \( f \) and \( d \) symbols as in section 2.

Now one presents each independent terms with coefficients in (3.16) in Appendix (I.1)

\[ \text{pole-4 : } \left[-B_3k_2(8N - 2) - B_{11}k_2N(2N - 1) - B_{12}k_2 - 3B_1(2N - 2)(8N - 2)
\right. \]

\[ -B_5((2N - 1)N(2N - 2) + 3(2N - 2)) - B_8((2N - 1)N(2N - 2) + 6(2N - 2)) \psi^a J^a(w) = 0, \]

\[ \text{pole-4 : } \begin{bmatrix} B_48k_2(N - 1)(4N - 1) + B_6(-k_2)((2N - 1)N + 2) \\
-B_9(2k_2 + 2N - 2) + k_2(2N - 1)N + 2B_{10}(2N - 2)^2 \\
-B_7N(2N - 1)(2N - 2) + B_{13}(2N - 2) - 3B_{15}(2N - 2) \\
+B_216(N - 1)^2(4N - 1) \end{bmatrix} \psi^a K^a(w) = 0, \]

\[ \text{pole-3 : } \begin{bmatrix} 4B_{10}(2N - 2)^2 - 2B_6(2N - 2) \\
+B_48(N - 1)(2N^2 - 7N + 11) \end{bmatrix} \psi^a \partial K^a(w) = 0, \]

\[ \text{pole-3 : } \begin{bmatrix} 4B_{14}(2N - 2)^2 - 4B_7(2N - 2) + 4B_{13}(2N - 2) \\
+B_216(N - 1)(2N^2 - 7N + 11) \end{bmatrix} \partial \psi^a K^a = 0. \] (I.2)

One should have vanishing coefficients in order to have primary higher spin \( \frac{7}{2} \) current. Without \( X(K_2, N) \) term in (3.16), the coefficients \( B_1-B_{15} \) do not satisfy the two equations corresponding to the fourth order pole in (I.2). The coefficient of fourth order pole is given by

\[ \frac{48(N-4)(N-1)(N+1)(2N-3)(2N+1)(4N-4+k_2)(6N-6+k_2)}{(2N-2+3k_2)(4N-4+3k_2)} \hat{G}(w). \] (I.3)
This implies that the higher spin $\frac{7}{2}$ current without $X(K_2, N)$ term in (3.16) is a quasiprimary field. Therefore, we should add the other quasiprimary field of spin $\frac{7}{2}$ with arbitrary coefficient $X(k_2, N)$ which will appear later.

**J The coefficients appearing in (3.16) which depend on $k_2$ and $N$ when $k_1 = 2N - 2$**

By solving the linear equations Appendix (H.2) and Appendix (I.2), one has

\[
B_2 = \frac{21}{2k_2}B_1, \\
B_3 = \frac{168(N - 1)^2}{k_2(3k_2 + 2N - 2)}B_1, \\
B_4 = \frac{420(N - 1)^2}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_5 = \frac{-4(6k_2^2N - 15k_2^2 + 12k_2N^2 - 42k_2N + 30k_2 - 4N^3 + 18N^2 + 18N - 32)}{(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_6 = \frac{504(N - 1)^2(2N^2 - 7N + 11)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_7 = \frac{42(2k_2^2N^2 - 7k_2^2N + 23k_2^2 + 4k_2N^3 - 18k_2N^2 + 60k_2N - 46k_2 + 24N^2 - 48N + 24)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_8 = \frac{-1}{(N - 1)(3k_2 + 2N - 2)(3k_2 + 4N - 4)}(-10k_2^2N^2 + 35k_2^2N - 7k_2^2 - 20k_2N^3 + 90k_2N^2 - 84k_2N + 14k_2 + 16N^4 - 88N^3 + 72N^2 + 56N - 56)B_1, \\
B_9 = \frac{-168(N - 1)^2(2N^2 - 7N + 11)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_{10} = \frac{-84(N - 1)(2N^2 - 7N + 11)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_{11} = \frac{-56(N - 1)(2k_2N^2 - 7k_2N + 23k_2 + 36N - 36)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_{12} = \frac{-336(N - 1)^2(2k_2N - 5k_2 + 4N^2 - 14N + 4)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_{13} = \frac{-21}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}(2k_2^2N^2 - 7k_2^2N + 23k_2^2 + 4k_2N^3 - 18k_2N^2 + 60k_2N - 46k_2 + 8N^4 - 44N^3 + 72N^2 - 44N + 8)B_1, \\
B_{14} = \frac{-21(6k_2^2N - 15k_2^2 + 12k_2N^2 - 42k_2N + 30k_2 + 4N^3 - 18N^2 + 30N - 16)}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}B_1, \\
B_{15} = \frac{7}{k_2(3k_2 + 2N - 2)(3k_2 + 4N - 4)}(22k_2^2N^2 - 77k_2^2N + 37k_2^2 + 44k_2N^3 - 198k_2N^2 + 228k_2N - 74k_2 + 24N^4 - 132N^3 + 216N^2 - 132N + 24)B_1.
\]
From Appendix $G$, one determines the overall coefficient $B_1$ in terms of $A_1$ (the description of Appendix $G$) appearing in the higher spin 4 current with (3.15) as follows:

$$B_1 = -4(k_2 + 4N - 4)\frac{1}{7}\left[\frac{4(N - 1)}{(2N - 2 + k_2)(4N - 4 + k_2)}\right] A_1.$$  

(J.2)

The quantity $A_1$ in (J.2) appears in Appendix $F$. Furthermore, the quantity $X$ appearing in (3.16) can be determined by

$$X = \frac{384(N - 4)(N - 1)(2N - 3)(2N + 1)(k_2 + 2N - 2)(k_2 + 4N - 4)^2(k_2 + 6N - 6)}{(3k_2 + 2N - 2)(3k_2 + 4N - 4)}$$

$$\times \frac{A_1}{(4k_2^3N^2 - 2k_2^3N + 21k_2^2 + 36k_2N^2 + 138k_2N - 126k_2 + 168N^2 - 336N + 168)}.$$  

(J.3)

One can check that the fourth order pole of the OPE $T(z) (GT - \frac{i}{8}\partial^2 G)(w)$ is given by $\frac{1}{8}(4c + 21)G(w)$. In other words, the nonzero term in (J.3) is canceled by the above $X$ term. Note that there exists $(N - 4)$ factor in Appendix (J.3). In other words, for $N = 4$, there is no last term in (3.16).

It is straightforward to calculate the coefficients under the large $N$ 't Hooft limit as done before. Then they will be used in the eigenvalue equations involving the higher spin $\frac{7}{2}$ current.

K \hspace{1cm} The OPEs between the higher spin $\frac{7}{2}$ current and the higher spin 4 current and its $\mathcal{N} = 1$ superspace description

The OPE between the lowest higher spin $\frac{7}{2}$ current and itself in section 3 is summarized as

$$W^{(\frac{7}{2})}(z) W^{(\frac{7}{2})}(w) = \frac{1}{(z-w)^2} \frac{2c}{7} + \frac{1}{(z-w)^5} 2T(w) + \frac{1}{(z-w)^4} \partial T(w)$$

$$+ \frac{1}{(z-w)^3} \left[ C_7^{\frac{7}{2}} W^{(4)} + C_7^{\frac{3}{2}} W^{(4')} + \frac{1}{(4c + 21)(10c - 7)} (8(37c + 3)) TT$$

$$+ 3(2c - 117) \partial GG + 6(2c^2 - 6c + 9) \partial^2 T \right](w)$$

$$+ \frac{1}{(z-w)^2} \left[ \frac{1}{2} C_7^{\frac{7}{2}} \partial W^{(4)} + \frac{1}{2} C_7^{\frac{3}{2}} \partial W^{(4')} + \frac{1}{(4c + 21)(10c - 7)} (8(37c + 3)) \partial TT$$

$$+ \frac{3}{2} (2c - 117) \partial^2 GG + \frac{4}{3} (2c^2 - 6c + 9) \partial^3 T \right](w)$$

$$+ \frac{1}{(z-w)} \left[ C_7^{\frac{7}{2}} W^{(6)} + C_7^{\frac{3}{2}} W^{(6')} + \frac{1}{(2c + 61)(3c + 20)} C_7^{\frac{3}{2}} \left( \frac{1}{2}(4c - 229) GW^{(\frac{7}{2})} ight)$$

$$+ (50c + 589) T W^{(4')} + \frac{1}{12} (2c + 9)(5c + 112) \partial^2 W^{(4')} \right]$$
\[\begin{align*}
&\frac{1}{(c-1)(2c+53)}C^4_{\frac{1}{2}7} \left( -\frac{1}{2} (c-101) G \partial W^{(\frac{7}{2})} + \frac{1}{3} 50(c+1) T W^{(4)} \right) \\
&+ \frac{7}{6} (c-101) \partial G W^{(\frac{7}{2})} + \frac{1}{36} \left( 10c^2 + 157c - 567 \right) \partial^2 W^{(4)} \\
&+ \frac{1}{(c+11)(4c+21)(10c-7)(14c+11)} \left( 24(450c^2 + 1199c + 1) T T T \right) \\
&+ 36(18c^2 - 1037c + 169) T \partial G G + 2(520c^3 + 5946c^2 + 10067c + 867) \partial T \partial T \\
&+ (10c^3 - 303c^2 - 14968c + 9111) \partial^2 G G \\
&+ 2(620c^3 + 2580c^2 - 5459c - 891) \partial^2 T T \\
&+ \frac{3}{2} (8c^3 - 482c^2 + 619c - 6745) \partial^3 G G \\
&+ \left( \frac{1}{3} (20c^4 + 100c^3 - 2477c^2 + 2159c - 1302) \partial^4 T \right)(w) + \ldots.
\end{align*}\]

(K.1)

One can check that all the nonlinear terms vanish under the large \( c \to \infty \).

The OPE between the lowest higher spin \( \frac{7}{2} \) current and the higher spin 4 current is described as

\[W^{(\frac{7}{2})}(z) W^{(4)}(w) = \frac{1}{(z-w)^6} G(w) + \frac{1}{(z-w)^5} \partial G(w)\]

\[+ \frac{1}{(z-w)^4} \left[ 7C^4_{\frac{1}{2}7} W^{(\frac{7}{2})} + \frac{(2c-57)}{2(4c+21)} \partial^2 G + \frac{90}{(4c+21)} T G \right](w)\]

\[+ \frac{1}{(z-w)^3} \left[ - \frac{1}{2} C^4_{\frac{3}{2}7} W^{(\frac{3}{2})} + 3C^4_{\frac{1}{2}7} \partial \partial W^{(\frac{7}{2})} + \frac{1}{(4c+21)(10c-7)} \left( 2(154c - 339) T \partial G \right) \right.\]

\[+ 12(37c + 3) \partial T G + \frac{1}{2} (2c - 117)(2c - 3) \partial^3 G \left] \right](w)\]

\[+ \frac{1}{(z-w)^2} \left[ 11C^6_{\frac{1}{2}7} W^{(\frac{11}{2})} + \frac{1}{(3c+20)} C^6_{\frac{3}{2}7} \left( \frac{59}{2} G W^{(4)} - \frac{1}{3} (2c + 33) \partial W^{(\frac{7}{2})} \right) \right.\]

\[+ \frac{1}{(2c+53)} C^4_{\frac{1}{2}7} \left( \frac{51}{2} G W^{(4)} + 119 T W^{(\frac{7}{2})} + \frac{3}{4} (2c + 19) \partial^2 W^{(\frac{7}{2})} \right) \]

\[+ \frac{1}{(c+11)(4c+21)(10c-7)} \left( 36(99c + 31) T T G + 3(26c^2 - 667c - 2019) T \partial^2 G \right.\]

\[+ 2(19c + 71)(4c + 21) \partial T \partial G + 3(11c - 17)(4c + 21) \partial^2 T G \]

\[+ \frac{1}{24} (4c + 21)(2c^2 - 167c + 1233) \partial^4 G \left] \right](w)\]

\[+ \frac{1}{(z-w)} \left[ - \frac{1}{2} C^6_{\frac{3}{2}7} W^{(\frac{11}{2})} + 5C^6_{\frac{1}{2}7} \partial W^{(\frac{11}{2})} \right.\]

\[+ \frac{1}{(c-1)(2c+53)} C^4_{\frac{1}{2}7} \left( \frac{25}{2} (c+1) G \partial W^{(4)} + \frac{5}{3} (31c - 71) T \partial W^{(\frac{7}{2})} + \frac{5}{6} (11c - 91) \partial G W^{(4)} \right.\]

\[+ \frac{175}{3} (c+1) \partial T W^{(\frac{7}{2})} + \frac{5}{36} (2c^2 - 31c + 69) \partial^3 W^{(\frac{7}{2})} \right)\]

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\[
\frac{1}{(2c + 61)(3c + 20)} C_{z}^{\mu_{1} \nu_{1} \nu_{2}} \left( \frac{1}{2} (61c + 1685) G \partial W^{(4')} - 9(3c + 20) T W^{(\frac{7}{2})} \right) + \frac{1}{2} (34c + 1505) \partial G W^{(4')} - \frac{1}{3} (c^2 + 64c + 675) \partial^2 W^{(\frac{7}{2})} \right) \\
+ \frac{1}{(c + 11)(4c + 21)(10c - 7)(14c + 11)} \left( 36(486c^2 - 875c + 339) TT \partial G \\
+ 5(44c^3 - 2484c^2 - 3191c - 6069) T \partial^3 G + 144(342c^2 + 640c + 43) \partial TT G \\
+ 6(90c^3 - 2269c^2 - 4405c + 2034) \partial T \partial^2 G \\
+ (632c^3 + 1266c^2 + 31333c + 1119) \partial^2 T \partial G \\
+ 3(136c^3 + 698c^2 - 2631c - 1153) \partial^3 T G \\
+ \frac{1}{24} (16c^4 - 1692c^3 + 27416c^2 + 86847c + 103113) \partial^5 G \right) \right] (w) + \cdots \\
\] (K.2)

In this case, all the nonlinear terms become zero when the large \( c \to \infty \) is taken. In principle, one can obtain the OPE \( W^{(4)}(z) W^{(\frac{7}{2})}(w) \) from Appendix (K.2). In the \( \mathcal{N} = 1 \) description, the latter is more useful to the former.

The OPE between the higher spin 4 current and itself is given by

\[
W^{(4)}(z) W^{(4)}(w) = \frac{1}{(z - w)^8} 2c + \frac{1}{(z - w)^6} 16T(w) + \frac{1}{(z - w)^3} 8 \partial T(w) \\
+ \frac{1}{(z - w)^4} \left[ 10 C_{z}^{\mu_{1} \nu_{1}} W^{(4')} + 3 C_{z}^{\mu_{1} \nu_{1} \nu_{2}} W^{(4')} + \frac{1}{(4c + 21)(10c - 7)} \right] \\
+ 36(13c - 38) \partial G G + 6(c - 1)(16c - 69) \partial^2 T + 12(224c - 99) T T \\
+ \frac{1}{(z - w)^2} \left[ 5 C_{z}^{\mu_{1} \nu_{1} \nu_{2}} \partial W^{(4')} + \frac{3}{2} C_{z}^{\mu_{1} \nu_{1} \nu_{2}} \partial W^{(4')} \\
+ \frac{1}{(4c + 21)(10c - 7)} \left( 18(13c - 38) \partial^2 G G + 12(224c - 99) \partial T T \\
+ \frac{4}{3} (c - 1)(16c - 69) \partial^3 T \right) \right] (w) \\
+ \frac{1}{(z - w)^2} \left[ 12 C_{z}^{\mu_{1} \nu_{1} \nu_{2} \nu_{3}} W^{(6)} + C_{z}^{\mu_{1} \nu_{1} \nu_{2} \nu_{3} \nu_{4}} W^{(6')} + \frac{1}{(2c + 61)(3c + 20)} C_{z}^{\mu_{1} \nu_{1} \nu_{2} \nu_{3} \nu_{4}} \left( 12(14c + 349) T W^{(4')} \\
+ \frac{1}{4} (10c^2 + 285c + 92) \partial^2 W^{(4')} - 3(19c + 638) G W^{(\frac{7}{2})} \right) \\
+ \frac{1}{(c - 1)(2c + 53)} C_{z}^{\mu_{1} \nu_{1} \nu_{2} \nu_{3} \nu_{4}} \left( \frac{20}{3} (28c - 23) T W^{(4')} + \frac{14}{3} (13c - 38) G W^{(\frac{7}{2})} \\
- 2(13c - 38) G W^{(\frac{7}{2})} + \frac{1}{18} (50c^2 + 767c - 1017) \partial^2 W^{(4')} \right) \\
+ \frac{1}{(c + 11)(4c + 21)(10c - 7)(14c + 11)} \right] (w) \\
\]
\[+3(190c^3 + 7237c^2 + 12743c + 12530)\partial^2 G G + 3(326c^3 - 1591c^2 - 13109c - 12926)\partial^3 G G + 2(4720c^3 + 51330c^2 + 86729c + 35121)\partial T \partial T + 2(5632c^3 + 8310c^2 - 101995c - 77847)\partial^2 T T + \frac{4}{3}(40c^4 + 70c^3 - 8911c^2 - 8443c - 2256)\partial^4 T + 192(576c^2 + 721c + 128)T T T \right) \right) (w)
\]
\[+ \frac{1}{(z - w)} \left( 6C_{77}^6 \partial W^{(6)} + \frac{1}{2} C_{77}^6' \partial W^{(6)} \right)\]
\[+ \frac{1}{(c - 1)(2c + 53)} C_{77}^4 \left( \frac{4}{3} (13c - 38) \partial G \partial W^{(\frac{7}{2})} + \frac{7}{3} (13c - 38) \partial^2 W^{(\frac{7}{2})} \right) - (13c - 38)G \partial^2 W^{(\frac{7}{2})} + \frac{10}{3} (28c - 23) T \partial W^{(4)} + \frac{10}{3} (28c - 23) \partial T W^{(4)} + \frac{1}{18} (10c^2 + c - 111) \partial^3 W^{(4)} \right)\]
\[+ \frac{1}{(2c + 61)(3c + 20)} C_{77}^4' \left( - \frac{3}{2} (19c + 638) \partial G \partial W^{(\frac{7}{2})} - \frac{3}{2} (19c + 638) \partial G W^{(\frac{7}{2})} + 6(14c + 349) T \partial W^{(4)} + 6(14c + 349) \partial T W^{(4)} + \frac{1}{4} (2c^2 + 31c - 564) \partial^3 W^{(4)} \right)\]
\[+ \frac{1}{(c + 11)(4c + 21)(10c - 7)(14c + 11)} \times \left( \frac{1}{5} (40c^4 + 70c^3 - 8911c^2 - 8443c - 2256) \partial^5 T + 108(234c^2 + 81c + 85) T \partial^2 G G + 108(234c^2 + 81c + 85) \partial T \partial G G + 288(576c^2 + 721c + 128) \partial T T T + 18(12c^3 - 267c^2 - 701c - 694) \partial^4 G G + 12(472c^3 + 354c^2 + 3263c + 2361) \partial^2 T \partial T + 6(c + 11)(38c^2 + 187c + 200) \partial^3 G \partial G + 6(416c^3 + 766c^2 - 12783c - 9699) \partial^3 T T \right) \right) (w) + \cdots. \quad (K.3)
\]
All the nonlinear terms in Appendix (K.3) vanish under the large \(c\) limit.

By introducing the \(\mathcal{N} = 1\) stress energy tensor [63],
\[T = \frac{1}{2} G(z) + \theta T(z), \quad (K.4)\]
and representing the \(\mathcal{N} = 1\) lowest higher spin multiplet
\[W^{(\frac{7}{2})}(Z) = W^{(\frac{1}{2})}(z) + \theta W^{(4)}(z), \quad (K.5)\]
the above three OPEs, Appendix (K.1), Appendix (K.2) and Appendix (K.3), can be written as
\[W^{(\frac{7}{2})}(Z_1) W^{(\frac{7}{2})}(Z_2) = \frac{1}{2} \frac{2c}{7} + \frac{\theta_{12}}{\frac{6}{12}} 6T(Z_2) + \frac{1}{\frac{5}{12}} 2DT(Z_2) + \frac{\theta_{12}}{\frac{5}{12}} 4\partial T(Z_2)\]
\[
+ \frac{1}{z_{12}^4} \partial D T (Z_2) + \left[ \theta_{12} \right] \left[ 7 C_{\frac{67}{27}}^4 W(\frac{\hat{z}}{7}) + \frac{3(c - 6)}{(4c + 21)} 2 \partial^2 T + \frac{90}{(4c + 21)} 2 D T T \right] (Z_2) \\
+ \left[ \frac{1}{z_{12}^4} \left[ C_{\frac{67}{27}}^4 D W(\frac{\hat{z}}{7}) + C_{\frac{67}{27}}^6 W(4) \right] + \frac{1}{(4c + 21)(10c - 7)} \left( 8(37c + 3) DT DT + 3(2c - 117) \partial T T + 6(2c^2 - 6c + 9) \partial^2 D T \right) \right] (Z_2) \\
+ \left[ \frac{\theta_{12}}{z_{12}^4} \left[ \frac{1}{2} C_{\frac{67}{27}}^4 D W(\frac{\hat{z}}{7}) + \frac{1}{2} C_{\frac{67}{27}}^6 \partial W(\frac{\hat{z}}{7}) + \frac{1}{(4c + 21)(10c - 7)} \left( 8(37c + 3) \partial T D T + \frac{3}{2}(2c - 117) 4 \partial^2 T T + \frac{4}{3}(2c^2 - 6c + 9) \partial^3 D T \right) \right] (Z_2) \\
+ \left[ \frac{\theta_{12}}{z_{12}^4} \left[ \frac{1}{2} C_{\frac{67}{27}}^4 D W(\frac{\hat{z}}{7}) + \frac{1}{2} C_{\frac{67}{27}}^6 \partial W(\frac{\hat{z}}{7}) + \frac{1}{(4c + 21)(10c - 7)} \left( 8(37c + 3) \partial T D T + \frac{3}{2}(2c - 117) 4 \partial^2 T T + \frac{4}{3}(2c^2 - 6c + 9) \partial^3 D T \right) \right] (Z_2) \\
+ \left[ \frac{\theta_{12}}{z_{12}^4} \left[ \frac{1}{2} C_{\frac{67}{27}}^4 D W(\frac{\hat{z}}{7}) + \frac{1}{2} C_{\frac{67}{27}}^6 \partial W(\frac{\hat{z}}{7}) + \frac{1}{(4c + 21)(10c - 7)} \left( 8(37c + 3) \partial T D T + \frac{3}{2}(2c - 117) 4 \partial^2 T T + \frac{4}{3}(2c^2 - 6c + 9) \partial^3 D T \right) \right] (Z_2) \\
+ \frac{1}{(c + 1)(4c + 21)(10c - 7)} \left( 36(99c + 31) 2 D T D T T + 6(50c^2 + 501c + 519) 2 \partial D T D T + 6(23c + 34c - 822) 2 \partial^2 D T T + \frac{1}{3}(5c^3 - 135c^2 - 1139c + 939) 2 \partial^4 T + 4(55c^2 - 19c - 516) 2 D T \partial^2 T \right) (Z_2) \\
+ \left[ \frac{1}{z_{12}^4} \left[ C_{\frac{67}{27}}^6 D W(\frac{\hat{z}}{7}) + C_{\frac{67}{27}}^6 W(6) + \frac{1}{12} C_{\frac{67}{27}}^6 \left( - \frac{1}{2}(c + 101) 2 T \partial W(\frac{\hat{z}}{7}) + \frac{50}{3}(c + 1) D T D W(\frac{\hat{z}}{7}) + \frac{7}{6}(c - 101) 2 \partial T W(\frac{\hat{z}}{7}) + \frac{1}{36}(10c^2 + 157c - 567) \partial^2 D W(\frac{\hat{z}}{7}) \right) \right] (Z_2) \\
+ \frac{1}{(c + 1)(4c + 21)(10c - 7)(14c + 11)} \left( 24(450c^2 + 1199c + 1) D T D T D T + 36(18c^2 - 1037c + 169) 4 D T \partial T T \right) \right]
\]
\[ + \frac{1}{3}(20c^4 + 100c^3 - 2477c^2 + 2159c - 1302)\partial^1 DT \right] (Z_2) \\
+ \frac{\theta_{12}}{z_{12}} \left[ \frac{1}{2} C^{6'}_{\frac{3}{17}} DW^{(6')} + 6C^{6}_{\frac{3}{17}} \partial W^{(14')} + \frac{1}{(c-1)(2c+53)} C^4_{\frac{1}{17}} \left( \frac{14}{3}(13c - 38)\partial DTW^{(\frac{7}{2})} \right. \right. \]
\[ + (13c - 38)2T\partial DW^{(\frac{7}{2})} + \frac{1}{18}(10c^2 + c - 111)\partial^3 W^{(\frac{7}{2})} + \frac{2}{3}(101c - 1)DT\partial W^{(\frac{7}{2})} \]
\[ + \frac{1}{3}(49c + 151)2\partial TDW^{(\frac{7}{2})} \]
\[ + \frac{1}{(2c + 61)(3c + 20)} C^{4'}_{\frac{3}{17}} \left( \frac{3}{2}(19c + 638)2T\partial W^{(4')} + 9(3c + 20)DTDW^{(4')} \right) \]
\[ + 3(14c + 349)2\partial TW^{(4')} + \frac{1}{4}(2c^2 + 31c - 564)\partial^2 DW^{(4')} \right) \]
\[ + \frac{1}{(c + 11)(4c + 21)(10c - 7)(14c + 11)} \times \left( 72(450c^2 + 1199c + 1)2DTDT\partial T + 24(34c^3 - 505c^2 - 2197c - 332)2DT\partial^3 T \right. \]
\[ + 216(234c^2 + 81c + 85)2DTDT\partial T + 12(130c^3 - 93c^2 + 1970c - 357)2DT\partial^2 T \]
\[ + 6(212c^3 + 540c^2 - 677c + 3075)2\partial^2 DT\partial T + 9(48c^3 + 102c^2 - 2399c - 2351)2\partial^3 DTT \]
\[ + \frac{2}{5}(10c^4 - 437c^3 + 107c^2 + 11387c + 3183)2\partial^5 T \right] (Z_2) + \cdots \] (K.6)

Here in addition to Appendix (K.4) and Appendix (K.5), we also introduce the following three \( \mathcal{N} = 1 \) higher spin multiplets as follows [36, 37]:

\[
W^{(4')}(Z) = W^{(4')}(z) + \theta W^{(\frac{7}{2})}(z), \\
W^{(\frac{14}{3})}(Z) = W^{(\frac{14}{3})}(z) + \theta W^{(6)}(z), \\
W^{(6')}(Z) = W^{(6')}(z) + \theta W^{(\frac{14}{3})}(z). \] (K.7)

In the relation (K.21), the definitions in Appendix (K.7) is used. Note that the \( \theta_{12} \) independent terms in the right hand side of Appendix (K.6) are related to the OPE Appendix (K.1) while the \( \theta_{12} \) dependent terms in the right hand side of Appendix (K.6) are related to the OPE between \( W^{(4')}(z) W^{(\frac{7}{2})}(w) \). More explicitly, by the projection of \( \theta_1 = 0 = \theta_2 \) on the equation Appendix (K.6), one obtains the OPE in Appendix (K.1). By acting \( D_1 \) on the equation Appendix (K.6) and putting the condition \( \theta_1 = 0 = \theta_2 \), one obtains the OPE \( W^{(4')}(z) W^{(\frac{7}{2})}(w) \) which can be obtained from Appendix (K.2). The remaining component OPEs \( W^{(\frac{7}{2})}(z) W^{(4')}(w) \) and \( W^{(4')}(z) W^{(4)}(w) \) can be read off from Appendix (K.6): the result of \( \mathcal{N} = 1 \) supersymmetry.

The factor \( (4c+21)(10c-7) \) occurs in the OPE between the \( \mathcal{N} = 1 \) higher spin \( \frac{5}{2} \) current in the unitary case [37, 38, 38]. For the \( \mathcal{N} = 1 \) higher spin \( \frac{7}{2} \) current in the orthogonal case, the
extra factor $(c+11)(14c+11)$ occurs in Appendix (K.6). It would be interesting to see whether there are any null states at the particular values of the central charge, $c = -\frac{21}{4}, \frac{7}{10}, -11$ or $-\frac{11}{14}$.

L The OPE between the two lowest higher spin multiplets in $\mathcal{N} = 2$ superspace

Let us present the OPE in (4.29) here

\[
W^{(3)}_\frac{1}{3}(Z_1) W^{(3)}_{-\frac{1}{3}}(Z_2) = \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}} g_1 + \frac{c}{z_{12}^3} + \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}} g_2 T(Z_2) + \frac{1}{z_{12}^5} g_3 T(Z_2) \\
+ \frac{\theta_{12}}{z_{12}^5} g_4 DT(Z_2) + \frac{\theta_{12}}{z_{12}^5} g_5 \overline{T}(Z_2) + \frac{\theta_{12} \bar{\theta}_{12}}{z_{12}^5} \left[ g_6 [D, \overline{D}] T + g_7 T T + g_8 \partial T \right] (Z_2) \\
+ \frac{1}{z_{12}^4} \left[ g_9 [D, \overline{D}] T + g_{10} T T + g_{21} \partial T \right] (Z_2) + \frac{\theta_{12}}{z_{12}^4} \left[ g_{11} \partial DT + g_{12} TDT \right] (Z_2) \\
+ \frac{\theta_{12}}{z_{12}^4} \left[ g_{13} \partial D T + g_{14} TDT \right] (Z_2) + \frac{\theta_{12}}{z_{12}^4} \frac{1}{(c-1)(c+6)(2c-3)} \\
\times \left[ g_{15} \partial [D, \overline{D}] T + g_{16} T [D, \overline{D}] T + g_{17} TTT + g_{18} \overline{D} TDT + g_{19} \partial TT + g_{20} \partial^2 T \right] (Z_2) \\
+ \frac{1}{z_{12}^3} \frac{1}{(c-1)(c+6)(2c-3)} \\
\times \left[ g_{22} \partial [D, \overline{D}] T + g_{23} T [D, \overline{D}] T + g_{24} TTT + g_{25} \overline{D} TDT + g_{26} \partial TT + g_{27} \partial^2 T \right] (Z_2) \\
+ \frac{1}{z_{12}^3} \frac{1}{(c-1)(c+6)(2c-3)} \\
\times \left[ g_{27} \partial^2 DT + g_{28} TTT + g_{29} [D, \overline{D}] TDT + g_{30} \partial DTT + g_{31} \partial TDT \right] (Z_2) \\
+ \frac{1}{z_{12}^3} \frac{1}{(c-1)(c+6)(2c-3)} \\
\times \left[ g_{32} \partial^2 \overline{D} T + g_{33} TTT + g_{34} \overline{D} [D, \overline{D}] T + g_{35} \partial DTT + g_{36} \partial TDT \right] (Z_2) \\
+ \frac{1}{z_{12}^3} \frac{1}{(c-1)(c+1)(c+6)(2c-3)(5c-9)} \\
\times \left[ g_{37} \partial^2 [D, \overline{D}] T + g_{38} TTT + g_{39} TTT + g_{40} \overline{D} TDT + g_{41} \partial D TDT \\
+ g_{42} [D, \overline{D}] T [D, \overline{D}] T + g_{43} \partial [D, \overline{D}] T T + g_{44} \partial DTDT \right. \\
\left. + g_{45} \partial T [D, \overline{D}] T + g_{46} \partial TTT + g_{47} \partial T \partial T + g_{48} \partial^2 T + g_{49} \partial^3 T \right] (Z_2)
\]
\[
\frac{1}{z_{12}^2 (c-1)(c+6)(2c-3)(5c-9)} + \frac{1}{z_{12}^2 (c-1)(c+6)(2c-3)(5c-9)} \times \left[ g_{220} \partial^2 [D, \overline{D}]T + g_{51} TT [D, \overline{D}]T + g_{52} TTTT + g_{53} TTTDT \right. \\
+ g_{54} \partial \overline{D} \partial DT
\]

\[
+ g_{55} [D, \overline{D}]T + g_{56} \partial [D, \overline{D}]TT + g_{57} \partial DT \partial DT
\]

\[
+ g_{58} \partial T [D, \overline{D}]T + g_{59} \partial TTTT + g_{60} \partial T \partial T + g_{61} \partial^2 TT + g_{106} \partial^3 T \right] (Z_2)
\]

\[
\frac{1}{z_{12}^2 (c-1)(c+6)(2c-3)(5c-9)} \times \left[ g_{62} \partial^2 DT + g_{63} TTTDT + g_{64} T [D, \overline{D}] TDT + g_{65} \partial [D, \overline{D}] TDT + g_{66} \partial DT [D, \overline{D}] T
\]

\[
+ g_{67} \partial DT TT + g_{68} \partial DT \partial T + g_{69} \partial^2 DTT + g_{70} \partial TTTDT + g_{71} \partial^2 TDT \right] (Z_2)
\]

\[
\frac{1}{z_{12}^2 (c-1)(c+6)(2c-3)(5c-9)} \times \left[ g_{72} \partial^2 DT + g_{73} TTTDT + g_{74} TDT [D, \overline{D}]T + g_{75} \partial DT [D, \overline{D}]T + g_{76} \partial \overline{D} TTTT
\]

\[
+ g_{77} \partial DT \partial T + g_{78} \partial^2 DTT + g_{79} [D, \overline{D}] TDT + g_{80} \partial TTTDT + g_{81} \partial^2 TDT \right] (Z_2)
\]

\[
\frac{1}{z_{12}^2 (c-2)(c-1)(c+1)(c+6)(c+12)(2c-3)(5c-9)} \times \left[ g_{82} \partial^3 [D, \overline{D}]T + g_{83} TTTT [D, \overline{D}]T + g_{84} TTTTT + g_{85} TTDTDT
\]

\[
+ g_{86} T [D, \overline{D}] T [D, \overline{D}]T + g_{87} D T [D, \overline{D}] TDT + g_{88} \partial DT \partial DT + g_{89} \partial D TTT DT
\]

\[
+ g_{90} \partial \overline{D} DT DT + g_{91} \partial [D, \overline{D}] T [D, \overline{D}]T + g_{92} \partial [D, \overline{D}] TTT + g_{93} \partial [D, \overline{D}] T \partial T
\]

\[
+ g_{94} \partial^2 [D, \overline{D}] TTT + g_{95} \partial DT TTT DT + g_{96} \partial^2 DT DT DT + g_{97} \partial TTT [D, \overline{D}] T + g_{98} \partial \overline{TTTT TTT TTT + g_{99}} \partial T \overline{DT DT DT + g_{100}} \partial T \partial TT
\]

\[
+ g_{101} \partial^2 T [D, \overline{D}] T + g_{102} \partial^2 TTTT + g_{103} \partial^2 TJ \partial T + g_{104} \partial^3 TT + g_{105} \partial^4 T \right] (Z_2)
\]

\[
\frac{1}{z_{12} (c-2)(c-1)(c+1)(c+6)(c+12)(2c-3)(5c-9)} \times \left[ g_{107} \partial^3 [D, \overline{D}] T + g_{108} TTTT [D, \overline{D}] T + g_{109} TTTTT + g_{110} \overline{T D T DT
\]

\[
+ g_{111} T [D, \overline{D}] T [D, \overline{D}] T
\]

\[
+ g_{112} \overline{D} T [D, \overline{D}] T DT + g_{113} \partial \overline{D} T \partial DT + g_{114} \partial \overline{D} T T T DT + g_{115} \partial^2 \overline{D} T DT
\]

\[
+ g_{116} \partial [D, \overline{D}] T [D, \overline{D}] T + g_{117} \partial [D, \overline{D}] T T T + g_{118} \partial [D, \overline{D}] T \partial T + g_{119} \partial^2 [D, \overline{D}] T T
\]

\[
+ g_{120} \partial D T T T DT + g_{121} \partial^2 \overline{D} T D T DT + g_{122} \partial T T T [D, \overline{D}] T + g_{123} \partial T T T T + g_{124} \partial \overline{T D T DT T + g_{125}} \partial T \partial TT
\]

\[
+ g_{126} \partial^3 T [D, \overline{D}] T + g_{127} \partial^2 T T T + g_{128} \partial^2 T \partial T + g_{129} \partial^3 T T + g_{219} \partial^4 T \right] (Z_2)
\]
\[
\begin{aligned}
&\theta_{12}
\frac{1}{z_{12} (c-2)(c-1)(c+1)(c+6)(c+12)(2c-3)(5c-9)}
\times \left[ g_{130} \partial^4 DT + g_{131} TTTDT + g_{132} [D, \overline{D}] TDT + g_{133} [D, \overline{D}] T[D, \overline{D}] TDT \\
+ g_{134} \partial [D, \overline{D}] T DT + g_{135} \partial [D, \overline{D}] TTTDT + g_{136} \partial^2 [D, \overline{D}] TDT + g_{137} \partial DT TT [D, \overline{D}] T \\
+ g_{138} \partial DT TTT + g_{139} \partial DT TTDT + g_{140} \partial DT DT + g_{141} \partial DT DT DT + g_{142} \partial^2 DT TT \\
+ g_{143} \partial^2 DT DT + g_{144} \partial^2 DT TT + g_{145} \partial DT TDT + g_{146} \partial DT DT DT + g_{147} \partial DT DT DT + g_{148} \partial^2 DT DT + g_{149} \partial^2 DT TT + g_{150} \partial^3 DT \right] (Z_2)
\end{aligned}
\]

\[
\begin{aligned}
&\frac{1}{z_{12} (c-2)(c-1)(c+1)(c+6)(c+12)(2c-3)(5c-9)}
\times \left[ g_{151} \partial^4 DT + g_{152} TTTDT + g_{153} TDT [D, \overline{D}] T + g_{154} DT [D, \overline{D}] T [D, \overline{D}] T \\
+ g_{155} \partial DT \partial [D, \overline{D}] T + g_{156} \partial DT T [D, \overline{D}] T + g_{157} \partial DT TDT + g_{158} \partial DT DT DT \\
+ g_{159} \partial DT TTDT + g_{160} \partial DT DT T + g_{161} \partial DT TTDT + g_{162} \partial DT DT DT + g_{163} \partial^2 DT TDT \\
+ g_{164} \partial DT DTDT + g_{165} \partial [D, \overline{D}] TTTDT + g_{166} \partial [D, \overline{D}] T TDT + g_{167} \partial DT DTDT + g_{168} \partial DT TDT + g_{169} \partial^2 DT DT + g_{170} \partial^2 DT TT + g_{171} \partial^3 DT \right] (Z_2)
\end{aligned}
\]

\[
\begin{aligned}
&\frac{1}{z_{12} (c-2)(c-1)(c+1)(c+6)(c+12)(2c-3)(5c-9)(7c-15)}
\times \left[ g_{172} \partial^4 [D, \overline{D}] T + g_{173} TTTT [D, \overline{D}] T + g_{174} TTTTTT + g_{175} TTTTTTDT \\
+ g_{176} TTT [D, \overline{D}] T [D, \overline{D}] T + g_{177} TTT [D, \overline{D}] T DT + g_{178} \partial DT TTDT \\
+ g_{179} \partial DT [D, \overline{D}] TDT \\
+ g_{180} \partial DT \partial DTDT + g_{181} \partial DT \partial DTDT + g_{182} \partial DT \partial DTDT + g_{183} \partial DT TTDT \\
+ g_{184} \partial DT TTDT + g_{185} [D, \overline{D}] T [D, \overline{D}] T [D, \overline{D}] T + g_{186} \partial [D, \overline{D}] T [D, \overline{D}] T \\
+ g_{187} [D, \overline{D}] T TT [D, \overline{D}] T + g_{188} \partial [D, \overline{D}] TTTT + g_{189} \partial [D, \overline{D}] T TDT DT \\
+ g_{190} \partial [D, \overline{D}] T TDT TT \\
+ g_{191} \partial^2 [D, \overline{D}] T [D, \overline{D}] T + g_{192} \partial^2 [D, \overline{D}] TTT + g_{193} \partial^2 [D, \overline{D}] TDT + g_{194} \partial^3 [D, \overline{D}] TT \\
+ g_{195} \partial DT TTDT + g_{196} \partial DT TDT [D, \overline{D}] T + g_{197} \partial DT DT DT + g_{198} \partial^2 DT DT DT \\
+ g_{199} \partial^2 DT DT DT + g_{200} \partial^3 DT DT + g_{201} \partial DT TTT + g_{202} \partial DT TTTT \\
+ g_{203} \partial DT DT DT \\
+ g_{204} \partial DT [D, \overline{D}] T [D, \overline{D}] T + g_{205} \partial DT TT [D, \overline{D}] T + g_{206} \partial DT DT TT \\
+ g_{207} \partial DT DT TT + g_{208} \partial DT DT [D, \overline{D}] T + g_{209} \partial DT TT [D, \overline{D}] T + g_{210} \partial DT TTTT \\
+ g_{211} \partial DT TTTT + g_{212} \partial DT TT + g_{213} \partial DT TT + g_{214} \partial DT [D, \overline{D}] T + g_{215} \partial DT TTT + g_{216} \partial DT TTT + g_{217} \partial DT TTT + g_{218} \partial DT TTT \right] (Z_2) + \cdots.
\end{aligned}
\]
The structure constants in Appendix (L.1) are given by

\[
\begin{align*}
g_1 &= \frac{-2c}{9}, \quad g_2 = \frac{19}{9}, \quad g_3 = \frac{4}{3}, \quad g_4 = -\frac{7}{3}, \quad g_5 = \frac{11}{3}, \\
g_6 &= \frac{4(3c + 23)}{27(c - 1)}, \quad g_7 = \frac{104}{9(c - 1)}, \quad g_8 = \frac{23}{9}, \quad g_9 = \frac{(9c - 8)}{9(c - 1)}, \quad g_{10} = -\frac{1}{3(c - 1)} \\
g_{11} &= \frac{-7(2c - 7)}{9(c - 1)}, \quad g_{12} = \frac{35}{3(c - 1)}, \quad g_{13} = \frac{11(2c + 1)}{9(c - 1)}, \quad g_{14} = \frac{11}{(c - 1)} \\
g_{15} &= \frac{1}{108}(36c^3 - 859c^2 + 5814c - 12264), \quad g_{16} = \frac{1}{54}(780c^2 - 3641c - 7002), \\
g_{17} &= \frac{-2}{27}(c - 3571), \quad g_{18} = \frac{1}{9}(535c^2 - 1170c + 2336), \\
g_{19} &= \frac{155}{9}(c + 6)(2c - 3), \quad g_{20} = \frac{1}{54}(146c^3 - 24c^2 - 801c - 1022), \quad g_{21} = \frac{2}{3}, \\
g_{22} &= \frac{-1}{18}(18c^3 + 71c^2 - 558c + 336), \quad g_{23} = \frac{-2}{9}(36c^2 + 121c + 18), \\
g_{24} &= \frac{8}{9}(19c + 65), \quad g_{25} = \frac{4}{3}(c^2 - 54c + 32), \\
g_{26} &= \frac{-1}{3}(c + 6)(2c - 3), \quad g_{27} = \frac{-7}{6}(c^3 - 4c^2 + 24c - 84), \\
g_{28} &= \frac{-14}{3}(3c + 67), \quad g_{29} = \frac{7}{18}(39c^2 - 74c - 84), \\
g_{30} &= \frac{-7}{9}(19c^2 + 121c - 546), \quad g_{31} = \frac{-7}{6}(11c^2 + 2c + 8), \\
g_{32} &= \frac{11}{36}(6c^3 - 9c^2 + 10c - 84), \quad g_{33} = \frac{-22}{3}(5c - 19), \\
g_{34} &= \frac{-11}{18}(39c^2 - 62c - 12), \quad g_{35} = \frac{11}{9}(13c^2 - 13c + 42), \\
g_{36} &= \frac{11}{6}(5c^2 + 74c - 128), \quad g_{37} = \frac{1}{27}(9c^5 + 273c^4 - 2463c^3 + 5677c^2 - 4392c - 3150), \\
g_{38} &= \frac{-4}{27}(2334c^3 - 3862c^2 - 6821c - 5448), \quad g_{39} = \frac{-2}{27}(9572c^2 - 22722c - 17825), \\
g_{40} &= \frac{4}{9}(2819c^3 - 10177c^2 + 12112c + 5816), \\
g_{41} &= \frac{2}{27}(2694c^4 - 3327c^3 + 2937c^2 - 3134c + 7200), \\
g_{42} &= \frac{-1}{162}(576c^4 + 18801c^3 - 72025c^2 + 57786c - 6300), \\
g_{43} &= \frac{-1}{54}(2070c^4 + 463c^3 - 69242c^2 + 112335c + 64218), \\
g_{44} &= \frac{2}{27}(2676c^4 - 13749c^3 + 21621c^2 + 6938c - 50400), \\
g_{45} &= \frac{-2}{27}(c + 1)(5c - 9)(219c^2 - 220c - 1854), \quad g_{46} = \frac{4}{9}(c + 1)(5c - 9)(13c - 1147), \\
g_{47} &= \frac{1}{18}(1036c^4 + 3801c^3 - 22741c^2 + 26866c - 5504), \\
g_{48} &= \frac{1}{27}(2054c^4 + 1994c^3 - 1401c^2 - 53871c + 43930), 
\end{align*}
\]
\[ g_{49} = \frac{1}{81}(375c^5 - 1365c^4 - 2750c^3 + 33544c^2 - 50706c - 27384), \]
\[ g_{50} = \frac{2}{9}(2c^3 + 6c^2 + 27c - 14), \quad g_{51} = \frac{2}{9}(72c^3 - 1576c^2 - 2753c - 1560), \]
\[ g_{52} = -\frac{1}{9}(544c^2 + 6720c + 4811), \quad g_{53} = -\frac{2}{3}(13c^3 - 731c^2 - 2764c - 200), \]
\[ g_{54} = -\frac{1}{9}(3c^4 - 183c^3 - 3498c^2 - 916c + 576), \]
\[ g_{55} = \frac{1}{108}(1728c^4 - 2733c^3 - 10177c^2 + 9348c + 504), \]
\[ g_{56} = -\frac{1}{18}(360c^4 + 961c^3 - 3629c^2 - 10614c - 924), \]
\[ g_{57} = \frac{1}{9}(57c^4 - 3105c^3 + 7902c^2 + 5212c - 4032), \]
\[ g_{58} = -\frac{1}{9}(c + 1)(5c - 9)(36c^2 + 121c + 18), \quad g_{59} = -\frac{4}{3}(c + 1)(5c - 9)(19c + 65), \]
\[ g_{60} = -\frac{1}{12}(28c^4 - 609c^3 + 923c^2 + 3136c - 3884), \]
\[ g_{61} = -\frac{1}{18}(2c^4 + 746c^3 - 2817c^2 - 8691c + 3970), \]
\[ g_{62} = -\frac{14}{27}(3c^5 + 3c^4 - 95c^3 + 235c^2 - 342c - 126), \]
\[ g_{63} = \frac{14}{9}(404c^2 - 1722c - 1307), \quad g_{64} = \frac{7}{18}(981c^3 - 2009c^2 - 554c - 1932), \]
\[ g_{65} = \frac{7}{18}(99c^4 - 353c^3 + 720c^2 - 1096c - 84), \]
\[ g_{66} = \frac{7}{54}(384c^4 - 1673c^3 - 2771c^2 + 8946c + 5292), \]
\[ g_{67} = -\frac{7}{9}(37c^3 + 3087c^2 - 6190c - 6510), \quad g_{68} = -\frac{7}{18}(c + 6)(104c^3 - 461c^2 + 471c - 56), \]
\[ g_{69} = -\frac{7}{18}(69c^4 - 192c^3 + 3326c^2 - 8467c - 3318), \quad g_{70} = -\frac{7}{6}(83c^3 + 305c^2 + 386c - 928), \]
\[ g_{71} = -\frac{7}{18}(58c^4 - 191c^3 + 861c^2 - 1678c + 488), \]
\[ g_{72} = \frac{11}{108}(c + 1)(24c^4 - 192c^3 + 437c^2 - 1642c + 1212), \]
\[ g_{73} = -\frac{22}{9}(356c^2 + 126c + 43), \quad g_{74} = -\frac{11}{18}(579c^3 - 531c^2 + 142c - 204), \]
\[ g_{75} = -\frac{11}{54}(384c^4 - 65c^3 + 305c^2 - 810c - 108), \quad g_{76} = -\frac{11}{9}(83c^3 + 1013c^2 - 118c - 138), \]
\[ g_{77} = \frac{11}{18}(c + 6)(56c^3 - 75c^2 + 69c - 164), \quad g_{78} = \frac{11}{36}(2c - 1)(51c^3 - 177c^2 + 1804c + 576), \]
\[ g_{79} = -\frac{11}{18}(99c^4 - 305c^3 - 372c^2 + 844c + 84), \quad g_{80} = -\frac{11}{6}(117c^3 - 1221c^2 + 618c + 1592), \]
\[ g_{81} = \frac{11}{18}(22c^4 + 511c^3 - 1161c^2 - 742c + 2000), \]
\[ g_{82} = \frac{1}{972}(36c^7 - 2082c^6 - 114909c^5 + 190147c^4 + 979562c^3 - 2223812c^2 - 576408c + 1792224), \]
\[
g_{83} = \frac{1}{81}(8592c^4 - 690638c^3 + 492653c^2 + 301732c + 152436), \\
g_{84} = -\frac{1}{27}(14816c^3 + 458716c^2 - 638341c - 144486), \\
g_{85} = \frac{2}{9}(2320c^4 - 101889c^3 + 606844c^2 - 674264c + 142664), \\
g_{86} = \frac{1}{162}(21024c^5 - 155943c^4 - 477323c^3 + 2798086c^2 - 3446922c + 677124), \\
g_{87} = \frac{1}{54}(56463c^5 - 221675c^4 + 287658c^3 - 263908c^2 + 480360c - 170688), \\
g_{88} = -\frac{1}{81}(10764c^6 + 67305c^5 - 243325c^4 - 549414c^3 + 1592164c^2 + 1045896c - 1536192), \\
g_{89} = -\frac{1}{54}(46011c^5 + 233631c^4 - 1403284c^3 + 2121364c^2 - 3022480c + 322848), \\
g_{90} = -\frac{1}{108}(8091c^6 + 10017c^5 - 481476c^4 + 2033078c^3 - 3183264c^2 + 3023176c - 347232), \\
g_{91} = -\frac{1}{648}(3c - 10)(384c^5 - 19097c^4 + 354613c^3 - 952818c^2 + 808704c - 128016), \\
g_{92} = -\frac{1}{54}(9288c^5 + 56174c^4 - 230811c^3 - 1268420c^2 + 2402016c - 410424), \\
g_{93} = -\frac{1}{216}(90000c^6 + 92627c^5 - 178145c^4 - 3381908c^3 + 11292180c^2 - 6585920c - 5335680), \\
g_{94} = -\frac{1}{108}(1314c^6 - 7662c^5 - 147177c^4 + 893811c^3 - 1813360c^2 + 1082048c - 284424), \\
g_{95} = \frac{1}{54}(43257c^5 - 373377c^4 - 3279584c^3 + 8020988c^2 - 3334496c - 3284064), \\
g_{96} = \frac{1}{108}(8055c^6 - 12570c^5 - 9742c^4 - 511992c^3 + 4979588c^2 - 8095952c + 3884832), \\
g_{97} = -\frac{2}{27}(c - 2)(c + 12)(6954c^3 - 2861c^2 - 23299c - 25314), \\
g_{98} = -\frac{2}{27}(c - 2)(c + 12)(24202c^2 - 53445c - 42157), \\
g_{99} = -\frac{1}{9}(c - 2)(c + 12)(5899c^3 - 31814c^2 + 36521c + 26914), \\
g_{100} = -\frac{1}{18}(734c^5 - 32469c^4 - 211047c^3 + 1109960c^2 - 984776c - 211976), \\
g_{101} = -\frac{1}{81}(3405c^6 + 18707c^5 - 141714c^4 - 114585c^3 + 778432c^2 + 348368c - 1590744), \\
g_{102} = -\frac{1}{27}(c + 12)(296c^4 - 19977c^3 + 2186c^2 + 152690c - 196004), \\
g_{103} = \frac{2}{27}(c - 2)(c + 12)(644c^4 + 977c^3 - 6375c^2 + 7269c - 9683), \\
g_{104} = \frac{1}{81}(1923c^6 + 11271c^5 - 83708c^4 + 179760c^3 + 479801c^2 - 27546c - 1116048), \\
g_{105} = \frac{1}{1296}(1524c^7 + 192c^6 - 228911c^5 + 758549c^4 + 1510616c^3 \\
- 7264492c^2 + 3218632c + 5989872), \\
g_{106} = \frac{1}{54}(30c^5 - 294c^4 - 682c^3 + 2933c^2 + 10053c + 1302),
\[
g_{107} = -\frac{1}{81}(27c^7 + 171c^6 - 1578c^5 - 6019c^4 - 101c^3 + 55478c^2 + 36624c - 14112),
\]
\[
g_{108} = \frac{4}{27}(1872c^4 + 11665c^3 - 57505c^2 - 84080c - 57732),
\]
\[
g_{109} = \frac{4}{9}(640c^3 + 2234c^2 - 34793c - 10998),
\]
\[
g_{110} = \frac{8}{3}(31c^4 - 1488c^3 - 1757c^2 + 36922c + 32),
\]
\[
g_{111} = \frac{1}{27}(1728c^5 + 13656c^4 - 30415c^3 - 117361c^2 + 173622c - 74592),
\]
\[
g_{112} = \frac{2}{9}(9c - 7)(11c^4 - 591c^3 + 658c^2 + 2544c + 192),
\]
\[
g_{113} = -\frac{4}{27}(12c^6 - 525c^5 - 4828c^4 - 8937c^3 + 65674c^2 + 49488c - 12096),
\]
\[
g_{114} = \frac{1}{9}(63c^5 - 1713c^4 - 44560c^3 + 248500c^2 + 96752c + 7488),
\]
\[
g_{115} = \frac{1}{18}(9c^6 - 336c^5 - 3111c^4 + 77186c^3 - 239904c^2 - 91088c - 7104),
\]
\[
g_{116} = \frac{1}{108}(3c - 10)(576c^5 + 6967c^4 - 13895c^3 - 35142c^2 + 32184c + 2016),
\]
\[
g_{117} = \frac{1}{9}(72c^5 - 670c^4 - 29169c^3 - 1808c^2 + 272004c + 37632),
\]
\[
g_{118} = -\frac{1}{36}(360c^6 + 4549c^5 - 1243c^4 - 113704c^3 + 79500c^2 + 344432c - 172032),
\]
\[
g_{119} = -\frac{1}{9}(54c^6 + 318c^5 - 2403c^4 + 13806c^3 - 12505c^2 - 75262c + 22512),
\]
\[
g_{120} = \frac{1}{9}(141c^5 - 5319c^4 + 106876c^3 + 186724c^2 + 482768c + 36288),
\]
\[
g_{121} = -\frac{1}{18}(45c^6 - 2166c^5 - 8819c^4 + 30636c^3 - 75380c^2 + 316112c + 149184),
\]
\[
g_{122} = \frac{2}{9}(c - 2)(c + 12)(72c^3 - 1576c^2 - 2753c - 1560),
\]
\[
g_{123} = -\frac{2}{9}(c - 2)(c + 12)(54c^2 + 6720c + 4811),
\]
\[
g_{124} = -\frac{1}{3}(c - 2)(c + 12)(13c^3 - 731c^2 - 2764c - 200),
\]
\[
g_{125} = -\frac{1}{3}(208c^5 + 1926c^4 - 5559c^3 - 14291c^2 + 60518c - 38392),
\]
\[
g_{126} = -\frac{4}{27}(45c^6 + 542c^5 + 1065c^4 - 1899c^3 - 15641c^2 - 1786c + 25584),
\]
\[
g_{127} = -\frac{2}{9}(172c^4 + 744c^3 - 1277c^2 - 10355c + 7874),
\]
\[
g_{128} = -\frac{2}{9}(c - 2)(c + 12)(7c^4 - 149c^3 + 48c^2 + 93c - 1021),
\]
\[
g_{129} = \frac{1}{54}(12c^6 - 636c^5 - 10495c^4 - 53613c^3 + 5614c^2 + 814848c + 294816),
\]
\[
g_{130} = -\frac{7}{648}(30c^7 + 80c^6 - 5979c^5 + 16513c^4 + 55058c^3 - 248048c^2 + 179896c + 82992),
\]
\begin{align*}
g_{131} &= \frac{7}{9}(1376c^3 + 24184c^2 - 66847c - 18402), \\
g_{132} &= \frac{7}{9}(246c^4 + 11499c^3 - 27490c^2 + 10670c - 29484), \\
g_{133} &= -\frac{7}{108}(3c - 10)(675c^4 + 295c^3 - 7346c^2 + 9078c - 3780), \\
g_{134} &= \frac{7}{54}(3c - 10)(65c^5 + 499c^4 - 1428c^3 - 9172c^2 + 20508c - 504), \\
g_{135} &= \frac{7}{9}(249c^4 - 1212c^3 + 2929c^2 - 3716c + 812), \\
g_{136} &= \frac{7}{36}(60c^6 - 49c^5 - 2109c^4 + 16386c^3 - 38894c^2 + 31648c - 12264), \\
g_{137} &= \frac{7}{54}(1833c^5 + 20341c^4 - 149380c^3 + 41140c^2 + 280008c + 248976), \\
g_{138} &= \frac{14}{27}(814c^4 + 2355c^3 - 146337c^2 + 266620c + 131124), \\
g_{139} &= -\frac{14}{9}(10c^5 - 633c^4 + 8734c^3 - 43992c^2 + 60892c + 1512), \\
g_{140} &= -\frac{7}{18}(113c^5 + 6013c^4 + 23676c^3 - 166376c^2 + 131240c + 94080), \\
g_{141} &= \frac{7}{108}(285c^6 + 169c^5 - 670c^4 - 60784c^3 + 60116c^2 + 184872c - 38304), \\
g_{142} &= -\frac{7}{9}(9c^5 + 614c^4 + 1520c^3 + 62106c^2 - 172210c + 9660), \\
g_{143} &= -\frac{7}{36}(75c^6 + 331c^5 - 1806c^4 + 14276c^3 - 31932c^2 - 47136c + 106512), \\
g_{144} &= -\frac{7}{54}(54c^6 + 649c^5 + 2382c^4 - 11713c^3 + 34356c^2 - 98724c + 24864), \\
g_{145} &= \frac{7}{3}(3c - 10)(134c^3 + 1167c^2 - 904c - 2084), \\
g_{146} &= \frac{7}{9}(270c^5 + 868c^4 - 6159c^3 + 5271c^2 - 642c + 5712), \\
g_{147} &= -\frac{7}{12}(35c^5 + 705c^4 - 756c^3 - 11834c^2 + 37100c - 30752), \\
g_{148} &= -\frac{7}{54}(110c^6 + 915c^5 - 1745c^4 + 1154c^3 - 99200c^2 + 242168c - 82992), \\
g_{149} &= -\frac{7}{9}(58c^5 - 35c^4 - 1575c^3 + 27644c^2 - 71752c + 40536), \\
g_{150} &= -\frac{7}{108}(90c^6 - 539c^5 - 12299c^4 + 76248c^3 - 196162c^2 + 238668c - 67296), \\
g_{151} &= \frac{11}{2592}(120c^7 + 460c^6 - 8851c^5 + 52395c^4 - 231412c^3 + 528598c^2 + 4096c - 52008), \\
g_{152} &= \frac{11}{3}(96c^3 - 3776c^2 - 6907c - 2762), \\
g_{153} &= \frac{11}{9}(558c^4 - 4699c^3 - 5334c^2 + 2914c - 2196), \\
g_{154} &= \frac{11}{108}(2025c^5 - 4653c^4 - 15988c^3 + 41354c^2 - 19392c - 4536),
\end{align*}
\[ g_{155} = -\frac{11}{54}(c + 12)(195c^5 - 561c^4 + 84c^3 - 1896c^2 + 2164c + 168), \]
\[ g_{156} = -\frac{11}{54}(1239c^5 + 4231c^4 + 8424c^3 + 57092c^2 - 24552c - 13968), \]
\[ g_{157} = -\frac{22}{27}(742c^4 + 4873c^3 + 32681c^2 + 27908c - 5556), \]
\[ g_{158} = -\frac{22}{9}(20c^5 - 1087c^4 + 4654c^3 - 14524c^2 + 10180c + 1800), \]
\[ g_{159} = -\frac{11}{18}(199c^5 + 2455c^4 + 12816c^3 - 63256c^2 + 29608c + 4416), \]
\[ g_{160} = -\frac{11}{108}(285c^6 + 1058c^5 - 9411c^4 + 21276c^3 - 105890c^2 + 55464c + 19368), \]
\[ g_{161} = -\frac{11}{18}(54c^5 + 298c^4 + 1861c^3 - 62578c^2 - 48906c + 12468), \]
\[ g_{162} = \frac{11}{36}(c + 12)(45c^5 - 47c^4 + 1182c^3 - 4490c^2 + 2434c + 372), \]
\[ g_{163} = \frac{11}{54}(42c^6 + 151c^5 - 3066c^4 + 28255c^3 - 94028c^2 - 26656c + 13080), \]
\[ g_{164} = -\frac{77}{9}(3c - 10)(c + 12)(7c^3 - 2c^2 + c - 4), \]
\[ g_{165} = -\frac{11}{36}(60c^6 - 17c^5 - 3045c^4 + 5330c^3 + 14010c^2 - 21392c - 168), \]
\[ g_{166} = -\frac{11}{3}(346c^4 + 4327c^3 - 15410c^2 + 188c + 5656), \]
\[ g_{167} = -\frac{11}{9}(120c^5 + 2494c^4 - 8791c^3 + 4955c^2 - 450c + 5088), \]
\[ g_{168} = -\frac{11}{12}(85c^5 - 821c^4 - 14700c^3 + 52430c^2 - 18852c - 38848), \]
\[ g_{169} = \frac{11}{54}(50c^6 + 1085c^5 + 1845c^4 - 16450c^3 + 53796c^2 - 73808c + 42960), \]
\[ g_{170} = -\frac{11}{9}(54c^5 + 79c^4 - 9527c^3 + 20534c^2 + 3572c - 23784), \]
\[ g_{171} = \frac{11}{108}(30c^6 + 643c^5 + 2623c^4 - 18896c^3 + 1566c^2 + 134124c - 131136), \]
\[ g_{172} = \frac{77}{23328}(1440c^7 - 14997c^6 - 62076c^5 + 1287093c^4 - 4637476c^3 + 5140856c^2 + 3881352c - 551040), \]
\[ g_{173} = \frac{308}{81}(4368c^4 - 30343c^3 + 14437c^2 - 28765c - 14850), \]
\[ g_{174} = \frac{2156}{81}(c - 10)(20c - 13)(32c - 7), \]
\[ g_{175} = \frac{616}{27}(2912c^4 - 22307c^3 + 78585c^2 - 42856c + 33520), \]
\[ g_{176} = \frac{77}{81}(4032c^5 - 16497c^4 - 39946c^3 + 188221c^2 - 233572c + 9240), \]
\[ g_{177} = \frac{308}{27}(2688c^5 - 17111c^4 + 44127c^3 - 52462c^2 + 38522c + 4760), \]
\[ g_{178} = \frac{154}{27}(294c^5 - 2187c^4 + 34562c^3 - 82675c^2 + 308522c - 52240), \]
\[ g_{179} = \frac{77}{162} (10143c^6 - 33504c^5 - 9135c^4 + 114096c^3 - 78904c^2 + 157688c - 20160), \]
\[ g_{180} = -\frac{308}{81} (1008c^6 + 4230c^5 - 13346c^4 - 48561c^3 + 82163c^2 + 298018c - 38360), \]
\[ g_{181} = -\frac{77}{162} (6363c^6 + 14892c^5 - 323451c^4 + 764148c^3 - 451800c^2 - 5176c - 90880), \]
\[ g_{182} = -\frac{77}{972} (4410c^7 - 15369c^6 - 120012c^5 + 227883c^4 + 712944c^3 \]
\[ -275864c^2 - 6730616c + 981120), \]
\[ g_{183} = -\frac{154}{81} (1197c^6 - 285c^5 - 46635c^4 + 276678c^3 - 574875c^2 + 999112c - 164240), \]
\[ g_{184} = -\frac{77}{486} (882c^7 - 5664c^6 - 38073c^5 + 435645c^4 - 1390596c^3 \]
\[ + 1886018c^2 - 1948908c + 367360), \]
\[ g_{185} = \frac{77}{1458} (3c - 10)(9168c^4 - 41233c^3 + 63926c^2 - 32925c + 4284), \]
\[ g_{186} = \frac{77}{972} (3c - 10)(26195c^2 - 26718c^4 + 93996c^3 - 148037c^2 + 121372c - 29064), \]
\[ g_{187} = \frac{77}{162} (3c - 10)(672c^5 - 194c^4 - 32876c^3 + 96045c^2 - 103071c + 7728), \]
\[ g_{188} = \frac{154}{81} (252c^5 - 8779c^4 + 2830c^3 + 137012c^2 - 102195c + 169050), \]
\[ g_{189} = \frac{77}{27} (7c - 15)(189c^5 - 1045c^4 + 3357c^3 - 6704c^2 + 6324c - 672), \]
\[ g_{190} = -\frac{77}{162} (3780c^6 + 20748c^5 - 65190c^4 - 857481c^3 + 2953173c^2 - 2006758c - 516040), \]
\[ g_{191} = -\frac{77}{486} (2412c^6 - 14484c^5 - 43683c^4 + 380967c^3 - 687016c^2 + 234404c + 73920), \]
\[ g_{192} = -\frac{77}{81} (378c^6 - 900c^5 - 42774c^4 + 232857c^3 - 400095c^2 + 32236c - 8120), \]
\[ g_{193} = -\frac{77}{108} (7c - 15)(18c^6 - 27c^5 - 594c^4 + 4678c^3 - 14908c^2 + 12436c + 1176), \]
\[ g_{194} = -\frac{77}{972} (252c^7 - 309c^6 + 7686c^5 + 62992c^4 - 425287c^3 + 561346c^2 + 593504c + 146720), \]
\[ g_{195} = -\frac{154}{27} (546c^5 - 35523c^4 + 210850c^3 - 439679c^2 + 129322c - 145040), \]
\[ g_{196} = -\frac{154}{162} (10143c^6 - 76284c^5 + 127473c^4 + 181620c^3 - 549440c^2 + 8344c + 141120), \]
\[ g_{197} = \frac{77}{162} (6237c^6 + 28896c^5 - 74356c^4 + 3456888c^3 - 5422224c^2 + 843128c + 2656640), \]
\[ g_{198} = \frac{486}{77} (2205c^7 - 4947c^6 + 32040c^5 - 199680c^4 + 429834c^3 \]
\[ -290084c^2 + 1394512c + 809760), \]
\[ g_{199} = \frac{154}{81} (1071c^6 - 4551c^5 + 28176c^4 - 227049c^3 + 785763c^2 - 674102c + 633640), \]
\[ g_{200} = \frac{77}{486} (882c^7 + 1632c^6 - 76929c^5 + 207717c^4 + 408444c^3 \]
\[ -1785350c^2 + 1833900c + 240800), \]
Let us emphasize that the field contents in the right-hand side of Appendix (L.1) together with the nonlinear terms in Appendix (L.2) disappear. Note that in the large $c \to \infty$, all the nonlinear terms in Appendix (L.1) together with Appendix (L.2) disappear. Let us emphasize that the field contents in the right-hand side of
Appendix (L.1) can be read off from the $\mathcal{N} = 2$ higher spin $\frac{7}{2}$ current with $U(1)$ charge $\frac{1}{3}$ and the higher spin $\frac{3}{2}$ current with $U(1)$ charge $-\frac{1}{3}$ in the unitary case where the factor $(c - 2)(c - 1)(c + 1)(c + 6)(2c - 3)(5c - 9)$ occurs also. One can analyze the various null states at $c = 2, 1, -1, -6, -12, \frac{5}{2}$ or $\frac{2}{3}$. See also.

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