Wave functions in abelian Chern-Simons theory on the torus as modular forms of weight two

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Abstract

It is argued that in the context of geometric quantization holomorphic part of zero-mode wave functions in abelian Chern-Simons theory on the torus can naturally be considered as modular forms of weight 2. In the geometric quantization the zero-mode wave function can be defined as $\Psi[a, \bar{a}] = e^{-\frac{K(a, \bar{a})}{2}} f(a)$ where $K(a, \bar{a})$ denotes a Kähler potential for the zero-mode variable $a \in \mathbb{C}$ on the torus. We first review that $f(a)$ can be described in terms of Jacobi theta functions by imposing gauge invariance on $\Psi[a, \bar{a}]$ where gauge transformations are induced by doubly periodic translations of $a$. We discuss that $f(a)$ is quantum theoretically characterized by (i) an operative relation in the $a$-space representation and (ii) an inner product of $\Psi[a, \bar{a}]$’s including ambiguities in the choice of $K(a, \bar{a})$. We then consider a similar gauge invariance condition on $\Psi[a, \bar{a}]$ associated with modular transformations of $a$. We choose the modular parameter $\tau = i\tau_2$ of the torus such that the number density of the zero modes is preserved under the modular transformations of $a$. The results illustrate that we can quantum theoretically identify $f(a)$ as a modular form of weight 2. We confirm this statement by use of a specific representation of $f(a)$ in terms of the Jacobi theta function. We also indicate that the wave-function inner product gives rise to a quantum version of the Petersson inner product for the modular forms of weight 2.
1 Introduction

In this paper we present an argument that holomorphic part of zero-mode wave functions in abelian Chern-Simons (CS) theory on the torus (or, more precisely, on $S^1 \times S^1 \times \mathbb{R}$) can be interpreted as modular forms of weight two. The argument is made in the context of geometric quantization of the theory studied by Bos and Nair [1, 2, 3, 4]. (In the early papers [1, 2] the relevant quantization is called the coherent-state quantization but it is essentially the same as the geometric one; the latter provides more mathematical foundations of the quantization program.) For a standard mathematical literature on the geometric quantization of CS theory, see [5]. A review on physical aspects of CS theory in general, see, e.g., [6].

It has been known for a long time that the holomorphic zero-mode wave functions in abelian CS theory on the torus can be expressed in terms of a Jacobi theta function in geometric quantization [1, 3, 4]; similar results were also obtained in the cases of the non-abelian CS theory [2, 7]. Since any complex functions on the torus can, by definition, be expressed in terms of the elliptic functions, this result sounds natural. But, as far as the author notices, modular transformations of the zero-mode wave functions and their quantum properties have not yet been fully investigated in the literature. An initial motivation for this paper has been to carry out such a study partly in search for an application to a problem in number theory.

The idea of interpreting modular forms as quantum wave functions is not new; the author first learns such an idea from Rajeev's seminar presentation based on [8]. It is intriguing to understand such correspondences for the application of quantum theory to number theory. For example, such a correspondence, if realized, would open up a new perspective on Hecke operators acting on the modular forms and associated $L$-functions. For introduction to the modular forms and related subjects such as elliptic curves and $L$-functions, see, e.g., [9, 10, 11, 12].

The organization of this paper is as follows. In the next section, following [1, 3, 4], we briefly review geometric quantization of abelian CS theory on the torus. We indicate that crucial ingredients of the quantization (such as a Kähler potential, a symplectic potential,
and a wave function) are all derived from a Kähler form for the zero-mode variables defined on the torus of interest. We emphasize that there exist ambiguities in the choice of the Kähler potential. At a prequantum level the zero-mode wave function can be interpreted as a wave function of a complex scalar field that couples to the symplectic potential. The ordinary quantum wave function is obtained by imposing a polarization condition on the prequantum wave function. We also present an explicit form of an inner product for the (quantum) zero-mode wave functions.

In section 3 we consider gauge transformations of the symplectic potential induced by the doubly periodic translations of the zero-mode variable. We then impose gauge invariance on the zero-mode wave function and see its consequences. We find that holomorphic part of the zero-mode wave function can be described in terms of a Jacobi theta function, reproducing the results in [1, 3, 4]. We also find an alternative representation of the holomorphic zero-mode wave function in terms of another version of the Jacobi theta functions. Along the way, we discuss that the holomorphic zero-mode wave functions are characterized by an operative relation (in the holomorphic-coordinate-space representation) and the inner product of the zero-mode wave functions (which includes ambiguities in the choice of the Kähler potential).

In section 4 we apply the same analysis, i.e., imposition of the gauge invariance on the zero-mode wave function, to the cases of modular transformations. We consider the modular transformations of the zero-mode variable, rather than those of the modular parameter of the torus. We also assume that the real part of the modular parameter is zero throughout the paper. The relevant modular transformations are thus different from the conventional ones used in string theory and conformal field theory. Since the modular $T$-transformations are included in the doubly periodic translations, we focus on the modular $S$-transformations. We choose the modular parameter of the torus such that the number density of the zero modes is preserved under the modular transformations of the zero mode variable.

In section 5 we first review basic facts on the modular forms. We then utilize the results in the previous sections to indicate that the holomorphic zero-mode wave functions in abelian CS theory on the torus, as characterized in section 3, can be considered as modular forms of weight 2. Applying a specific form of the holomorphic wave function in terms of the Jacobi theta function, we confirm this statement in the unit of $\hbar = 1$ where $\hbar$ corresponds the Planck constant or a quantum parameter of the physical system of interest to us. We also show that the inner product of the zero-mode wave functions is quantum theoretically invariant under the modular transformation. We indicate that at the classical limit $\hbar \to 0$ the wave-function inner product reduces to the Petersson inner product for the modular forms of weight 2. Lastly, in section 6 we present brief conclusions.

2 Zero-mode wave functions in abelian CS theory on the torus

We first briefly review how to construct zero-mode wave functions of abelian Chern-Simons (CS) theory on the torus in the context of geometric quantization, following [1, 3, 4].
Basics on the torus parametrization

The torus can be described in terms of two real coordinates \( \xi_1, \xi_2 \), satisfying the periodicity condition \( \xi_r \to \xi_r + \text{integer} \) where \( r = 1, 2 \). In other words, \( \xi_r \) take real values in \( 0 \leq \xi_r \leq 1 \), with the boundary values 0, 1 being identical. Complex coordinates of the torus can be parametrized as \( z = \xi_1 + \tau \xi_2 \) where \( \tau \in \mathbb{C} \) is the modular parameter of the torus.

By definition, we can impose the doubly periodic condition on \( z \). Namely, functions of \( z \) are invariant under the doubly periodic translations

\[
z \to z + m + n\tau
\]

where \( m \) and \( n \) are integers. Notice that we can absorb the real part of \( \tau \) into \( \xi_1 \) without losing generality. In the following, we then assume \( \text{Re}\tau = 0 \), i.e.,

\[
\tau = \text{Re}\tau + i\text{Im}\tau = i\text{Im}\tau := i\tau_2
\]

with \( \tau_2 > 0 \).

The torus has a holomorphic one-form \( \omega = \omega(z)dz \), satisfying

\[
\int_\alpha \omega = 1, \quad \int_\beta \omega = \tau = i\tau_2
\]

where the integrations are made along two non-contractible cycles on the torus, which are conventionally labeled as \( \alpha \) and \( \beta \) cycles. The one-form \( \omega \) is a zero mode of the anti-holomorphic derivative \( \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} \). We can assume \( \omega(z) = 1 \). In terms of \( \omega \) a gauge potential of abelian CS theory on the torus can be parametrized as

\[
A_z = \partial_{\bar{z}}\theta + \frac{\pi \bar{\omega}}{\tau_2}a
\]

where \( \theta \) is a complex function \( \theta(z, \bar{z}) \) and \( a \) is a complex number corresponding to the value of \( A_z \) along the zero mode of \( \partial_{\bar{z}} \). The abelian gauge transformations can be represented by

\[
\theta \to \theta + \chi
\]

where \( \chi \) is a complex constant or a phase factor of the \( U(1) \) theory. With a suitable choice of \( \chi \) we can parametrize the gauge potential solely by the zero-mode contributions, \( a \) and its complex conjugate \( \bar{a} \):

\[
A_z = \frac{\pi \omega}{\tau_2}a, \quad A_{\bar{z}} = \frac{\pi \bar{\omega}}{\tau_2}a.
\]

Since the complex variable \( a \) is defined on the torus it is natural to require that physical observables of the zero modes are also invariant under the doubly periodic translations

\[
a \to a + m + in\tau_2
\]

where \( m \) and \( n \) correspond to the winding numbers along the \( \alpha \) and \( \beta \) cycles, respectively. The physical configuration space of the zero mode is given by

\[
\mathcal{C} = \mathbb{C} / \mathbb{Z} + i\tau_2\mathbb{Z}.
\]
This is nothing but a complex torus, with the modular parameter being $i\tau_2$. Mathematically, this space is known as an abelian variety over the field of complex numbers. It is also known that the complex torus can be embedded into a complex projective space, that is, the zero-mode variable $a$ may satisfy the scale invariance

$$a \sim \lambda a$$

where $\lambda$ is a complex constant.

**Key ingredients in geometric quantization**

Geometric quantization provides a powerful quantization scheme when a phase space of a physical system is given by a Kähler manifold [3, 4]. The Kähler manifold has both symplectic and complex structures. The symplectic structure takes origin from the classical physics (to be quantized) while the complex structure makes it automatic to realize irreducibility (or polarization) of operators. The torus $S^1 \times S^1$ is a Kähler manifold with the simplest nontrivial topology. A physical system on the torus can be described in terms of an inherent zero-mode variable. As discussed above, this is particularly true in the case of abelian Chern-Simons theory on the torus where the contributions from ordinary non-zero-mode part of the abelian gauge potential can be gauged away.

The zero-mode dynamics can then be encoded by a Kähler form for the zero-mode part of the CS gauge potentials (2.6), i.e.,

$$\Omega^{(\tau_2)} = \frac{l}{2\pi} da \wedge d\bar{a} + \tau_2 \int_{z,\bar{z}} \frac{\pi \omega}{\tau_2} \wedge \frac{\pi \bar{\omega}}{\tau_2} = \frac{i\pi l}{\tau_2} da \wedge d\bar{a}$$

where the integral is taken over $dzd\bar{z}$ and $l$ is the level number associated to the abelian Chern-Simons theory. We here use the normalization of $\omega$ and $\bar{\omega}$ given by

$$\int_{z,\bar{z}} \bar{\omega} \wedge \omega = i2\tau_2 .$$

A Kähler potential $K(a, \bar{a})$ associated with the zero-mode Kähler form $\Omega^{(\tau_2)}$ is defined as

$$\Omega^{(\tau_2)} = i\partial\bar{\partial}K(a, \bar{a})$$

where $\partial, \bar{\partial}$ denote the Dolbeault operators. This definition leads to

$$K(a, \bar{a}) = \frac{\pi l}{\tau_2} a\bar{a} + u(a) + v(\bar{a})$$

where $u(a)$ and $v(\bar{a})$ are purely holomorphic and anti-holomorphic functions, respectively. These functions represent ambiguities in the choice of $K(a, \bar{a})$.

A symplectic potential (or a canonical one-form) $A^{(\tau_2)}$ corresponding to the Kähler form $\Omega^{(\tau_2)}$ is defined as

$$\Omega^{(\tau_2)} = dA^{(\tau_2)}.$$
Under a canonical transformation the Kähler form $\Omega^{(\tau_2)}$ does not change but the symplectic potential transforms as

$$A^{(\tau_2)} \rightarrow A^{(\tau_2)} + d\Lambda$$  \hspace{1cm} (2.15)

where $\Lambda$ is a function of $(a, \bar{a})$. In other words, the symplectic potential $A^{(\tau_2)}$ undergoes a $U(1)$ gauge transformation.

In the program of geometric quantization a quantum wave function arises from a prequantum wave function $\Psi[A_z]$ which is a function of $(a, \bar{a})$ in the present case. Mathematically we can state that the prequantum wave function is a section of line bundle on the torus with curvature $\Omega^{(\tau_2)}$. This means that $\Psi[A_z]$ transform as

$$\Psi[A_z] \rightarrow \Psi'[A_z] = e^{i\Lambda}\Psi[A_z]$$ \hspace{1cm} (2.16)

under the $U(1)$ gauge transformation (2.15). At the prequantum level $\Psi[A_z]$ can be interpreted as a wave function of a complex scalar field that couples to the symplectic potential $A^{(\tau_2)}$.

In order to obtain a quantum wave function we need to impose the so-called polarization condition on $\Psi[A_z]$, i.e.,

$$\left(\partial_{\bar{a}} + \frac{1}{2}\partial_a K\right)\Psi[A_z] = 0$$ \hspace{1cm} (2.17)

where $K = K(a, \bar{a})$ is the zero-mode Kähler potential in (2.13). The polarization condition leads to the specific form

$$\Psi[A_z] = e^{-\frac{K}{2}}\psi[A_z]$$ \hspace{1cm} (2.18)

where $\psi[A_z]$ is a holomorphic function of $A_z$. In the present case the physical variables are given by $(a, \bar{a})$ so that the wave function can be expressed as

$$\Psi[A_z] := \Psi[a, \bar{a}] = e^{-\frac{K(a, \bar{a})}{2}}f(a)$$ \hspace{1cm} (2.19)

where $f(a)$ is a function of $a$. We call $f(a)$ a holomorphic zero-mode wave function. Notice that we here define $\Psi[a, \bar{a}]$ with $K(a, \bar{a})$ including the above-mentioned ambiguities in its choice.

Let us fix the Kähler potential by $K_0 := \frac{\omega}{\tau_2}a\bar{a}$. The symplectic potential can also be chosen as

$$\mathcal{A}^{(\tau_2)} = \frac{l}{4\pi} \int_{\tau_2} \left(\frac{\pi \omega a}{\tau_2} \wedge \frac{\pi \omega \bar{a}}{\tau_2} d\bar{a} - \frac{\pi \omega \bar{a}}{\tau_2} \wedge \frac{\pi \omega}{\tau_2} da\right) = \frac{i\pi l}{2\tau_2} (a d\bar{a} + \bar{a} da)$$ \hspace{1cm} (2.20)

In these choices the polarization condition (2.17) can be expressed as $D_a \Psi = 0$ where $D_a := \partial_a + \frac{1}{2}\partial_a K_0 = \partial_a - iA_0^{(\tau_2)}$ is given in a form of a covariant derivative. As mentioned before, under a canonical transformation $\mathcal{A}^{(\tau_2)}$ transforms as $\mathcal{A}^{(\tau_2)} \rightarrow \mathcal{A}^{(\tau_2)} + d\Lambda$ while $\Omega^{(\tau_2)}$ remains the same. Thus, from the definition (2.12), a change of $K(a, \bar{a})$ under the
canonical transformation should be absorbed in the terms of $u(a) + v(\bar{a})$ in (2.13). In other words, the form of Kähler potential does not change under the $U(1)$ gauge transformations of $\mathcal{A}^{(\tau_2)}$. Thus, it is appropriate to define the polarization condition in terms of $K(a, \bar{a})$ as in (2.17) rather than in terms of $\mathcal{A}_8^{(\tau_2)}$.

This allows us to choose $\mathcal{A}^{(\tau_2)}$ independently of the polarization condition. For example, we can define $\mathcal{A}^{(\tau_2)}$ as

$$\mathcal{A}^{(\tau_2)} = i\frac{\pi l}{\tau_2} da.$$ (2.21)

This is a suitable choice in the $a$-representation space. In fact, we can regard this as an irreducible representation of $\mathcal{A}^{(\tau_2)}$ in the $a$-space since, as a general feature in quantum theories, irreducibility of an operator can be realized by a (holomorphic) polarization condition.

An inner product of the zero-mode wave functions $\Psi[a, \bar{a}]$ in (2.19) can be expressed as

$$\langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a}) \overline{\Psi[A_{\bar{a}}]} \Psi'[A_{\bar{a}}] = \int d\mu(a, \bar{a}) e^{-K(a, \bar{a})} \overline{f(a)} f'(a)$$ (2.22)

where $\overline{f(a)}$ is the complex conjugate of $f(a)$. The integral is taken over the complex plane $\mathbb{C}$ although, strictly speaking, the zero-mode variable is defined on $\mathbb{C} = \mathbb{C} + i\pi \mathbb{Z}$ as discussed in (2.8). In the above expression we then need to impose the doubly periodic condition on $a$ by hand. In this sense, the integral over $d\mu(a, \bar{a}) = d\bar{a} da$ is the same as the integral over $dzd\bar{z}$ in (2.10, 2.11). Lastly, we notice that, as a consequence of the geometric quantization [3, 4], an action of the derivative $\frac{\partial}{\partial a}$ on $f(a)$ leads to the factor of $\frac{i\pi l}{\tau_2}$.

## 3 Gauge transformations induced by doubly periodic translations

As mentioned in the introduction, any complex functions on the torus may obey the double periodicity condition. Thus, at a classical level, we can naturally expect that the holomorphic zero-mode wave function $f(a)$ is invariant under the doubly periodic translations

$$a \rightarrow a + m + in\tau_2.$$ (3.1)

Quantum theoretically, however, this invariance is not necessarily guaranteed. In this section we consider this problem by use of the quantum wave function $\Psi[a, \bar{a}]$ in (2.19). Our strategy is to impose a gauge invariance of the zero-mode wave function $\Psi[a, \bar{a}] = \Psi'[a, \bar{a}] = e^{i\Lambda} \Psi[a, \bar{a}]$ under the gauge transformation $\mathcal{A}^{(\tau_2)} \rightarrow \mathcal{A}^{(\tau_2)} + d\Lambda$ which is induced by the doubly periodic translation of $a$.

**Connection to the Jacobi theta functions**
Inspired by the results in [1, 3, 4], we now chose the symplectic potential and the Kähler potential by

\[ A_{1}^{(\tau_2)} = -\frac{i\pi l}{2\tau_2} (\bar{a} - a) d(\bar{a} + a), \]

\[ K_1 = -\frac{\pi l}{2\tau_2} (\bar{a} - a)^2 = K_0 - \frac{\pi l}{2\tau_2} (\bar{a}^2 + a^2) \]

where, as before, \( K_0 = \frac{\pi l}{\tau_2} a \bar{a} \). The gauge invariance condition for the zero-mode wave function

\[ \Psi_1[a, \bar{a}] = e^{-\frac{K_1}{2}} f_1(a) \]

is then expressed as

\[ e^{i\Lambda} e^{-\frac{K_1'}{2}} f_1(a) = e^{-\frac{K_1'}{2}} f_1(a + m + in\tau_2) \]

where \( \Lambda = -\pi \ln(\bar{a} + a) \) and \( K_1' = -\frac{\pi l}{2\tau_2} (\bar{a} - a - i2n\tau_2)^2 \). We here label the holomorphic function by \( f_1(a) \) to indicate that it associates with the choice of the Kähler potential \( K_1 \).

The condition \( (3.5) \) simplifies as

\[ f_1(a + m + in\tau_2) = e^{-i2\pi lan + \pi \ln^2 \tau_2} f_1(a). \]

This is reminiscent of a Jacobi theta function. Indeed, for \( l = 1 \) we can identify \( f(a) \) as one of the Jacobi theta functions:

\[ \vartheta_3(\tau, a) := \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{4}} y^n \]

where \( q := e^{i2\pi \tau} \) and \( y := e^{i2\pi a} \). Under the doubly periodic translations \( \vartheta_3(\tau, a) \) transforms as

\[ \vartheta_3(\tau, a + m + n\tau) = q^{-\frac{n^2}{2}} y^{-n} \vartheta_3(\tau, a). \]

In the case of \( \tau = i\tau_2 \), this can be expressed as

\[ \vartheta_3(i\tau_2, a + m + in\tau_2) = e^{-2\pi ian + \pi \tau^2 n^2} \vartheta_3(i\tau_2, a). \]

Thus \( f_1(a) \) with \( l = 1 \) in \( (3.6) \) can be identified with \( \vartheta_3(i\tau_2, a) \). Apart from our setting \( \tau = i\tau_2 \), this result agree with the literature [1, 3, 4].

At this stage one may wonder weather the rest of the Jacobi theta functions can also be described in the same context. The other Jacobi theta functions \( \vartheta_j(\tau, a) \) (\( j = 1, 2, 4 \)) are defined as

\[ \vartheta_1(\tau, a) := \sum_{n=-\infty}^{\infty} i(-1)^n q^{\frac{n^2}{2}} (n+\frac{1}{2})^2 y^{n-\frac{1}{2}} \]

\[ \vartheta_2(\tau, a) := \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}} (n+\frac{1}{2})^2 y^{n-\frac{1}{2}} \]

\[ \vartheta_4(\tau, a) := \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n^2}{2}} y^n \]
Note that there are different conventions in the definition of the Jacobi theta functions; we here follow those in a recent textbook [13]. These definitions are conventional in physics. Analogs of (3.9) are given by

\[
\begin{align*}
\vartheta_1(i\tau_2, a + m + in\tau_2) &= (-1)^{n+m}e^{-2\pi i an+\pi \tau_2 n^2} \vartheta_1(i\tau_2, a) \tag{3.13} \\
\vartheta_2(i\tau_2, a + m + in\tau_2) &= (-1)^m e^{-2\pi i an+\pi \tau_2 n^2} \vartheta_2(i\tau_2, a) \tag{3.14} \\
\vartheta_4(i\tau_2, a + m + in\tau_2) &= (-1)^n e^{-2\pi i an+\pi \tau_2 n^2} \vartheta_4(i\tau_2, a) \tag{3.15}
\end{align*}
\]

Changing the variable \(a \to a + \frac{1}{2}\) in (3.6), we can easily find

\[
f_1(a + \frac{1}{2} + m + in\tau_2) = (-1)^{ln}e^{-i2\pi ln a+\pi ln^2\tau_2} f_1(a + \frac{1}{2}). \tag{3.16}
\]

Namely, we have \(f_1(a + \frac{1}{2}) = \vartheta_4(i\tau_2, a)\) for \(l = 1\). This agrees with the relation

\[
\vartheta_3(\tau, a + \frac{1}{2}) = \vartheta_4(\tau, a). \tag{3.17}
\]

Similarly, we can check other theta-function formulae

\[
\begin{align*}
\vartheta_3(\tau, a + \frac{\tau}{2}) &= q^{-\frac{1}{8}} y^{-\frac{1}{2}} \vartheta_2(\tau, a), \tag{3.18} \\
\vartheta_3(\tau, a + \frac{\tau}{2} + \frac{1}{2}) &= i q^{-\frac{1}{8}} y^{-\frac{1}{2}} \vartheta_1(\tau, a) \tag{3.19}
\end{align*}
\]

\((q = e^{i2\pi \tau}, y = e^{i2\pi a})\) as follows. Let \(t_2(a) = \exp(-\frac{\pi \tau}{4} + i\pi ln a) f_1(a + \frac{i\tau}{2})\). Then from (3.6) we find \(t_2(a + m + i\tau_2) = (-1)^m \exp(-i2\pi ln + \pi ln^2\tau_2) t_2(a)\). With (3.14) this leads to \(t_2(a) = \vartheta_2(a)\) for \(l = 1\), which is consistent with the formula (3.18). The formula (3.19) can also be checked by showing \(t_1(a) = \vartheta_1(a)\) \((l = 1)\) where \(t_1(a)\) is defined as \(t_1(a) = \exp(-\frac{\pi \tau}{4} + i\pi ln a) f_1(a + \frac{1}{2} + \frac{i\tau}{2})\). Note that from (3.13, 3.14) we can also find rather trivial relations, \(f_1(a + \frac{m}{a}) = \vartheta_2(i\tau_2, a)\) and \(f_1(a + \frac{m+n}{a}) = \vartheta_1(i\tau_2, a)\) for \(l = 1\).

Alternative choices of the symplectic potential and the Kähler potential

One of the reasons why we can relate \(f_1(a)\) to \(\vartheta_3(i\tau_2, a)\) is that the exponent in (3.6) is independent of \(m\) while we are considering the transformation \(a \to a + m + in\tau_2\). This happens because of our particular choices of \(A^{(\tau_2)}\) and \(K_1\) in (3.2, 3.3). In terms of (Rea, Ima) these can be expressed as \(A^{(\tau_2)} \sim (\text{Ima})d(\text{Rea})\) and \(K_1 \sim (\text{Ima})^2\), respectively. Motivated by this thought, we now choose

\[
\begin{align*}
A_2^{(\tau_2)} &= -\frac{i\pi \tau}{2\tau_2} (\bar{a} + a) d(\bar{a} - a), \tag{3.20} \\
K_2 &= \frac{\pi \tau}{2\tau_2} (\bar{a} + a)^2 = K_0 + \frac{\pi \tau}{2\tau_2} (\bar{a}^2 + a^2) \tag{3.21}
\end{align*}
\]

and consider the gauge invariance of the zero-mode wave function

\[
\Psi_2[a, \bar{a}] = e^{-\frac{K_2}{\tau}} f_2(a) \tag{3.22}
\]
where, for a reason to be clarified in a moment, we define the wave function in terms of the anti-holomorphic function $f_2(a)$. The gauge invariance condition is expressed as
\[ e^{i\Lambda}e^{-\frac{K_2}{2}f_2(a)} = e^{-\frac{K_2}{2}f_2(a + m + in\tau_2)} \quad (3.23) \]
where $\Lambda = -i\frac{\pi l}{\tau_2}m(\bar{a} - a)$ and $K_2' = -\frac{\pi}{2\tau_2}(\bar{a} + a + 2m)^2$. The condition (3.23) then simplifies as
\[ f_2(a + m + in\tau_2) = e^{\frac{\pi l}{\tau_2}m(2a + m)}f_2(a). \quad (3.24) \]
The exponent does not depend on $a$ but on $\bar{a}$; this is why we consider the anti-holomorphic function in the definition of the wave function in (3.22). The exponent is also independent of $n$, contrary to the previous case in (3.6).

Now let us introduce an anti-holomorphic function of the form
\[ f_3(a) := e^{-\frac{\pi l}{\tau_2}a^2}f_2(a). \quad (3.25) \]
From (3.24) we find
\[ f_3(a + m + in\tau_2) = e^{2\pi l m a + \pi l n^2\tau_2}f_3(a). \quad (3.26) \]
Namely, we have
\[ f_3(a) = e^{-\frac{\pi l}{\tau_2}a^2}f_2(a) = f_1(a). \quad (3.27) \]
Thus $f_3(a)$ is also equivalent to $\bar{v}_3(i\tau_2, a)$ for $l = 1$.

As mentioned previously, there are ambiguities in the choice of Kähler potential up to addition of holomorphic and anti-holomorphic functions. Including these ambiguities, we may define a zero-mode wave function in a twofold way:
\[
\Psi[a, \bar{a}] = e^{-\frac{1}{2}(u_1(a) + v_1(\bar{a}))}\Psi_1[a, \bar{a}] \quad (3.28)
\]
\[
\bar{\Psi}[a, \bar{a}] = e^{-\frac{1}{2}(u_2(a) + v_2(\bar{a}))}\bar{\Psi}_2[a, \bar{a}] \quad (3.29)
\]
where $u_i(a)$ and $v_i(\bar{a})$ ($i = 1, 2$) are holomorphic and anti-holomorphic functions, respectively; we denote these by $u_i$ and $\bar{v}_i$ in the following. Using (3.28), we can express the inner product of the zero-mode wave function as
\[ \langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a})e^{-(K_0 - \frac{\pi l}{\tau_2}(\bar{a}^2 + a^2) + u_1 + \bar{v}_1)}f_1(a)f'_1(a) \quad (3.30) \]
where $f'_1(a)$ denotes an alternative solution of (3.6) with possibly different choices of $(m, n)$ from those of the other solution $f_1(a)$. On the other hand, using (3.29) and (3.27), we can similarly express the inner product as
\[ \langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a})e^{-(K_0 + \frac{\pi l}{\tau_2}(\bar{a}^2 + a^2) + u_2 + \bar{v}_2)}f_2(a)f'_2(a) \]
\[ = \int d\mu(a, \bar{a})e^{-(K_0 - \frac{\pi l}{\tau_2}(\bar{a}^2 + a^2) + u_2 + \bar{v}_2)}f_1(a)f'_1(a). \quad (3.31) \]
These two inner products differ by the holomorphic and anti-holomorphic functions $(u_i, \bar{v}_i)$ in the exponents. In other words, the expressions (3.30, 3.31) give rise to a concrete realization
of the fact that there exist ambiguities in the choice of the Kähler potential up to the addition of holomorphic and anti-holomorphic functions. The relation (3.27) can be seen as a reflection of this fact.

Characterization of the holomorphic zero-mode wave function

We have shown that \( f_1(a) \) and \( f_2(a) \) are described by the Jacobi theta functions. Under the doubly periodic translations these transform as \( f_1(a + m + in\tau_2) = e^{-i2\pi ln a + i2\pi^2 m\tau_2} f_1(a) \) and \( f_2(a + m + in\tau_2) = e^{i\pi m(2a + m)} f_2(a) \), respectively. Thus, both of them do not satisfy the double periodicity condition to be satisfied by holomorphic functions on the torus in general. Notice, however, that for any \( l \in \mathbb{Z} \) the former relation becomes \( f_1(a + m) = f_1(a) \) for \( n = 0 \) while the latter becomes \( f_2(a + in\tau_2) = f_2(a) \) for \( m = 0 \). This means that each of \( f_1(a) \) and \( f_2(a) \) forms a subset of holomorphic functions \( f(a) \) that satisfies the double periodicity condition.

We can then naturally expect the double periodicity for the quantum holomorphic wave function \( f(a) \). Indeed, as shown in [14, 15], we can argue that such an expectation is true for \( l \in 2\mathbb{Z} \). To be more specific, a similar gauge invariance condition on a zero-mode wave function with certain choices of \( \mathcal{A}(\tau_2) \) and \( K \) leads to the relation \( f(a + m + in\tau_2) = e^{i\pi l mn} f(a) \). We do not make any reviews on this relation here since we shall not utilize it in the rest of this paper. We will, however, present a careful review and a detailed derivation of this relation in a forthcoming paper [16].

The holomorphic wave functions \( f_1(a) \) and \( f_2(a) \) are distinct to each other but this does not mean that they represent distinct physical states to each other. As a matter of fact, quantum theoretically, these are both representing the same physical state and reflect the ambiguities in the choice of the Kähler potential. This is a quintessential feature in geometric quantization. **What characterizes the holomorphic zero-mode wave function** \( f(a) \) **is the operative relation**

\[
\frac{\partial}{\partial a} f(a) = \frac{\pi l}{\tau_2} \bar{a} f(a)
\]

and the inner product of the zero-mode wave functions

\[
\langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a}) e^{-K(a, \bar{a})} \overline{f(a)} f'(a).
\]

The operative relation (3.32) is guaranteed as long as the Kähler potential is given in the form of \( K(a, \bar{a}) = K_0 + u(a) + v(\bar{a}) \) where \( K_0 = \frac{\pi l}{\tau_2} a \bar{a} \). In the present paper we have always dealt with such potentials; see (2.22), (3.30) and (3.31). The inner product is conjugate symmetric, that is, we have \( \langle \Psi | \Psi' \rangle = \overline{\langle \Psi' | \Psi \rangle} \). This means that there is a freedom to choose either \( f(a) \) or \( \overline{f(a)} \) as the “holomorphic” zero-mode wave function. Namely, at a level of definition we have a dual relation between \( f(a) \) and \( \overline{f(a)} \), i.e.,

\[
f(a) \leftrightarrow \overline{f(a)}.
\]

From this relation we can also understand the definition of \( f_2(a) \) in (3.22).
4 Gauge transformations induced by modular transformations

In the previous section we consider gauge invariance of the zero-mode wave function in abelian CS theory on the torus where the gauge transformations of the symplectic potential are induced by the doubly periodic translations $a \rightarrow a + m + in\tau_2$ of the zero-mode variable. In this section we continue to carry out the same analysis for modular transformations of the zero-mode variable. The modular transformations are generated by combinations of the so-called modular $S$- and $T$-transformations

$$S : a \rightarrow -\frac{1}{a}, \quad T : a \rightarrow a + 1. \quad (4.1)$$

Basics of modular forms and the modular transformations are reviewed in section 5. Notice that the modular $T$-transformations are obtained from the doubly periodic translations $a \rightarrow a + m + in\tau_2$ simply by setting $(m, n) = (1, 0)$. From (3.6) we then find that the holomorphic zero-mode wave function $f(a)$ is invariant under the $T$-transformation regardless the choice of the level number $l \in \mathbb{Z}$. It is therefore an intriguing question whether one can derive any conditions for $f(a)$ from the gauge invariance of the zero-mode wave function under the gauge transformations induced by the modular $S$ transformation. This is exactly what we consider in this section.

Since $\tau_2$ originally represents the modular parameter of the torus to start with, we have a freedom to change it up to the modular transformation of $\tau = i\tau_2$ ($\tau_2 > 0$). Under the modular $S$-transformation of $a$ we can then properly assume that the parameter $\tau_2$ varies as

$$\tau_2 \rightarrow \tau'_2 := \frac{\tau_2}{\alpha} \quad (4.2)$$

where $\alpha$ is some constant or a function of $(a, \bar{a})$. The modular transformations of our interest are thus described by

$$S : (a, \tau_2) \rightarrow \left( -\frac{1}{a} \frac{\tau_2}{\alpha} \right),$$

$$T : (a, \tau_2) \rightarrow (a + 1, \tau_2).$$

Notice that these are qualitatively different from the conventional $S$- and $T$-transformations used in string theory and conformal field theory, $S : (\tau, z) \rightarrow (-\frac{1}{\tau}, \frac{z}{\tau})$, $T : (\tau, z) \rightarrow (\tau + 1, z)$ where $\tau$ and $z$ are the modular parameter and the physical variable on the torus, respectively. In the present case we do not focus on functions of $\tau$. Instead, we are interested in the modular transformations of the zero-mode complex coordinate $a$, while $\frac{\pi l}{\tau_2}$ serves as the Planck constant $\hbar$ in the zero-mode dynamics.

To carry out our analyses, we choose the symplectic potential $A^{(\tau_2)}$ in (2.21) and the Kähler potential $K_0$ in (3.3), i.e.,

$$A^{(\tau_2)} = i\frac{\pi l}{\tau_2} \bar{a} da, \quad (4.3)$$

$$K_0 = \frac{\pi l}{\tau_2} a \bar{a} \quad (4.4)$$
and consider the gauge invariance of the zero-mode wave function

$$\Psi_0[a, \bar{a}] = e^{-\frac{K_0}{2}} f(a).$$  \hspace{1cm} (4.5)$$

The gauge transformations of $A^{(\tau_2)}$ are induced by the $S$-transformation $a \rightarrow -\frac{1}{a}$ with (4.2). Under the $S$-transformation we have

$$K'_0 = \frac{\alpha}{|a|^2} K_0,$$

(4.6)

$$\delta A^{(\tau_2)} = -\frac{i\pi l}{\tau_2} \left( \frac{\alpha}{|a|^2} + |a|^2 \right) \frac{da}{a} := d\Lambda.$$  \hspace{1cm} (4.7)

In terms of $\Lambda$ the gauge transformations of $\Psi_0[a, \bar{a}]$ in (4.5) is given by $\Psi'_0 \rightarrow e^{i\Lambda} \Psi_0$. Thus the gauge invariance of $\Psi_0$ is implemented by

$$e^{i\Lambda} e^{-\frac{K_0}{2}} f(a) = e^{-\frac{K'_0}{2}} f\left( -\frac{1}{a} \right).$$  \hspace{1cm} (4.8)

We now impose the gauge invariance of $K_0$, i.e.,

$$\alpha = |a|^4.$$  \hspace{1cm} (4.9)

Then $\Lambda$ becomes $\Lambda = -i2K_0 \log a$ and the gauge invariance condition (4.8) simplifies as

$$f\left( -\frac{1}{a} \right) = e^{K_0 \log a^2} f(a).$$  \hspace{1cm} (4.10)

Imposition of the invariance for $K_0$ in (4.9) is an appropriate physical condition. Since the quantization scheme begins with the choice of $\Omega^{(\tau_2)}$, the Kähler form $\Omega^{(\tau_2)}$ should physically be preserved under the transformations of $(a, \bar{a})$. In the case of the doubly periodic translations or the modular $T$-transformations, $K_0$ is not invariant but its variation can be split into purely holomorphic or antiholomorphic functions. Thus the corresponding Kähler form $\Omega^{(\tau_2)}$ preserves. In the present case the transformations are simply given by $a \rightarrow -\frac{1}{a}$, which means that the mixture of holomorphic and antiholomorphic parts does not come in under the $S$-transformations. Thus the preservation of $\Omega^{(\tau_2)}$ can be encoded by the invariance of $K_0$.

What means by the invariance of $K_0$

As discussed in (3.32), the factor of $\frac{\pi l}{\tau_2} a$ arises from the action of the derivative $\frac{\partial}{\partial a}$ on $f(a)$ in the $a$-space representation. Thus in the $a$-space $K_0$ as an operator can be expressed as either (a) $\hat{K}_0 = a \frac{\partial}{\partial a}$ or (b) $\hat{K}_0 = \frac{\partial}{\partial a} a$. In the former case (a), the action of $\hat{K}_0$ on $a^p$ ($p \in \mathbb{Z}$) can be calculated as $\hat{K}_0 a^p = pa^p$. Similarly, the action of $\hat{K}'_0 = -\frac{1}{a} \frac{\partial}{\partial (-\frac{1}{a})}$ on $a^p = (-1)^p(-\frac{1}{a})^{-p}$ becomes $\hat{K}'_0 a^p = -pa^p$. These results contradict with the invariance of $K_0$ we have discussed above. On the other hand, the latter case (b) simply shows that $\hat{K}_0$
in the $a$-space can be evaluated as the identity operator. This is a suitable choice in our context. Thus, in the $a$-space the condition (4.10) can be represented as

$$f\left(-\frac{1}{a}\right) = a^2 f(a).$$  \hspace{1cm} (4.11)

We should emphasize here that this relation is obtained in the $a$-space representation.

We have chosen the modular parameter $\tau = i\tau_2$ of the torus such that the Kähler potential $K_0 = \frac{\pi l}{\tau_2} |a|^2$ is preserved under the modular $S$-transformation of $a$. This means that $\frac{d a d \bar{a}}{\tau_2}$ is preserved under the $S$-transformation. Since $\tau_2$ denotes the area of the torus $\frac{d a d \bar{a}}{\tau_2}$ represents the number density of the zero modes per unit area or the degeneracy. This feature applies to the case of the $T$-transformation as well. Thus, from a physical perspective, our choice (4.9) guarantees that the degeneracy of the zero modes is preserved under the modular transformations of $a$. The feature of degeneracy preservation is in accord with a semiclassical picture of the zero-mode dynamics.

We may understand the situation in the other way around. Namely, we start with the condition

$$K_0 = \hbar |a|^2 = 1$$  \hspace{1cm} (4.12)

while the modular parameter $\tau = i\tau_2$ of the torus being fixed. Since the Planck constant (or a quantum parameter) is defined as

$$\hbar = \frac{\pi l}{\tau_2},$$  \hspace{1cm} (4.13)

the fixation of $\tau_2 > 0$ provides a more desirable description of the physical system for the zero modes. In the unit of $\hbar = 1$ the condition (4.12) becomes $|a|^2 = 1$, that is, the zero-mode variable $a$ is defined on the unit circle in the complex plane. In this case we can literally realize the fixation of $\tau_2$, i.e., $\tau_2' = \frac{\tau_2}{|a|^4} = \tau_2$. This suggests that under the modular $S$-transformation we have two equivalent pictures of the holomorphic zero-mode wave function:

$$f \left(-\frac{1}{a}, \frac{\tau_2}{|a|^4}\right) \leftrightarrow f \left(-\frac{1}{a}, \tau_2\right) \bigg|_{|a|^2=1}$$  \hspace{1cm} (4.14)

in the unit of $\hbar = 1$. We shall make use of this relation in the next section.

Revisit to the modular $T$-transformation

Although we have essentially thought of the modular $T$-transformation of the holomorphic zero-mode wave function $f(a)$ in the previous section, we now reconsider it with the choices of (4.3)-(4.5). The gauge invariance condition for $\Psi_0$ in (4.5) under the gauge transformation of $A^{(\tau_2)}$, induced by the modular $T$-transformation $a \rightarrow a + 1$, can be expressed as

$$e^{i \Lambda} e^{-\frac{K_0}{\tau_2}} f(a) = e^{-\frac{K_0'}{\tau_2}} f(a + 1)$$  \hspace{1cm} (4.15)

where $\Lambda = i \frac{\pi l}{\tau_2} a$, $K_0 = \frac{\pi l}{\tau_2} a \bar{a}$ and $K_0' = \frac{\pi l}{\tau_2} (a + 1)(\bar{a} + 1)$. This simplifies as

$$f(a + 1) = e^{\frac{\pi l}{\tau_2} (a-\bar{a}+1)} f(a)$$  \hspace{1cm} (4.16)
where the factor of $\frac{\pi l}{\tau_2} \bar{\alpha}$ can be replaced by $\frac{\partial}{\partial a}$ in the $a$-space representation.

Similarly, for the gauge transformation induced by $a \rightarrow a + m$ ($m \in \mathbb{Z}$), the gauge invariance of $\Psi_0$ leads to the relation $f(a + m) = e^{\frac{\pi l m}{\tau_2} (\bar{a} - a + m)} f(a)$. On the other hand, by direct use of (4.16), we have $f(a + m) = e^{\frac{\pi l m}{\tau_2} (\bar{a} - a + 1)} f(a)$. These relations contradict each other. In other words, for $m > 1$ the gauge invariance condition on $\Psi_0$ makes sense only at the classical limit

$$\hbar = \frac{\pi l}{\tau_2} \rightarrow 0. \tag{4.17}$$

The classical limit can be realized by either $\tau_2 \rightarrow \infty$ or $l = 0$. The latter case means the vanishing of the level number in the CS theory; so it is unphysical. The former case, on the other hand, implies that the zero-mode configuration space is not a torus but a very "long" cylinder. Thus it contradicts the existence of two winding numbers. In either case we can not properly construct a quantum theory of interest. Therefore we can not apply the above data (4.3)-(4.5) to our discussion of the modular $T$-transformation and the construction of the holomorphic wave function $f(a)$.

One may wonder what happens if we use a different $A^{(\tau_2)}$. For the choice of $A^{(\tau_2)} = i\frac{\pi l}{2\tau_2} (ad\bar{a} + \bar{a}da)$ in (2.20), an analog of the gauge invariance condition (4.15) becomes

$$f(a + 1) = e^{\frac{\pi l}{2\tau_2}} f(a). \tag{4.18}$$

We can similarly derive discrepancies of this relation, unless at the classical limit (4.17), from computations of $f(a + m)$; repetitive use of (4.18) leads $f(a + m) = e^{\frac{\pi l m}{\tau_2}} f(a)$ while the gauge invariance condition under the $T$-transformation becomes $f(a + m) = e^{\frac{\pi l m^2}{\tau_2}} f(a)$. Thus for both choices of $A^{(\tau_2)}$'s we find the same results. This is as expected since these symplectic potentials (2.20) and (2.21) give rise to the identical Kähler form (2.10).

Part of the reasons why we can not properly describe the modular $T$-transformation in terms of $(a, \bar{a})$ is that we have ignored the scale invariance of the zero-mode variable $a \sim \lambda a$ where $\lambda$ is a complex number. As mentioned in (2.9), the scale invariance arises from the fact that the complex torus, as an abelian variety, can be embedded into a complex projective space. Given the scale invariance, $a \rightarrow a + \lambda^{-1}$ can also be regarded as the modular $T$-transformation $a \rightarrow a + 1$. In this sense, the doubly periodic translations $a \rightarrow a + m + in\tau_2$ provides a larger class of the the modular $T$-transformation. In other words, using the scale invariance of $a \sim \lambda a$ (with $\text{Re}\lambda \in \mathbb{Z}$), we see that the modular $T$-transformation $a \rightarrow a + 1$ can be extended to the doubly periodic translations $a \rightarrow a + m + in\tau_2$. This is why we could derive the modular $T$-invariance $f(a + m) = f(a)$ for the holomorphic zero-mode wave function from the doubly periodic translations in the previous section. We can then attribute the above reasons to the scale invariance of the zero-mode variable. Note that this issue does not arise in the modular $S$-transformations.
5 Holomorphic wave functions as modular forms of weight 2

Elements of modular forms

We first review some basics of modular forms, following mathematical textbooks [9, 10, 12]. In general, a modular form \( f(z) \) of weight \( k \) is defined by

\[
f \left( \frac{\alpha z + \beta}{\gamma z + \delta} \right) = (\gamma z + \delta)^k f(z) \tag{5.1}
\]

where \( \alpha, \beta, \gamma, \delta \) are matrix elements of the modular group

\[
SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \big| \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \alpha \delta - \beta \gamma = 1 \right\} \tag{5.2}
\]

The modular forms are defined on the upper-half plane \( \mathbb{H} = \{ z \in \mathbb{C} | \text{Im } z > 0 \} \). Accordingly, to be rigorous, the modular group can be defined as \( PSL(2, \mathbb{Z}) := SL(2, \mathbb{Z})/\{ \pm I \} \), with \( I \) the identity matrix. The fundamental domain \( \mathcal{F} \) for the action of \( SL(2, \mathbb{Z}) \) generators on \( \mathbb{H} \) corresponds to the space of \( \mathbb{H}/PSL(2, \mathbb{Z}) \). This can be specified by

\[
\mathcal{F} = \left\{ z \in \mathbb{C} \big| \text{Im } z > 0, \ |z| \geq 1, \ |\text{Re } z| \leq \frac{1}{2} \right\} . \tag{5.3}
\]

This region is shown in Figure 1. Any point \( z \in \mathbb{H} \) can be obtained by a linear fractional transformation of \( z_0 \in \mathcal{F} \), i.e., \( z = \frac{\alpha z_0 + \beta}{\gamma z_0 + \delta} \). In this sense we can properly consider \( \Gamma = SL(2, \mathbb{Z}) \) in (5.2) as the modular group.

It is well known that the modular group can be generated by \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). The definition of the modular form in (5.1) is then obtained from the conditions

\[
f(z + 1) = f(z), \tag{5.4}
\]

\[
f \left( \frac{-1}{z} \right) = z^k f(z). \tag{5.5}
\]

The first condition simply means that \( f(z) \) can be expressed in a form of the Fourier expansion

\[
f(z) = \sum_{n=0}^{\infty} a_n q^n \tag{5.6}
\]

where \( q = e^{i2\pi z} \) and \( a_n \) is the Fourier coefficient. If \( a_0 = 0 \), the modular form \( f(z) \) is called the cusp form. The vector space formed by the cusp forms of weight \( k \) is denoted by \( S_k(\Gamma) \), i.e.,

\[
S_k(\Gamma) := \left\{ f : \mathbb{H} \to \mathbb{C} \big| f \left( \frac{-1}{z} \right) = z^k f(z), \ f(z) = \sum_{n=1}^{\infty} a_n q^n \right\} . \tag{5.7}
\]
Let \( f(z), g(z) \in S_k(\Gamma) \), then the so-called \textit{Petersson inner product} is defined as

\[
\langle f, g \rangle = \frac{1}{\text{vol} F} \int_F f(z) \overline{g(z)} y^k \frac{dxdy}{y^2} \quad (5.8)
\]

where \( z = x + iy \). Notice that the integral measure \( \frac{dxdy}{y^2} \) is invariant under the modular transformations. Invariance under the \( T \)-transformation is obvious. Under the \( S \)-transformation we have \( y' = \frac{y}{|z|^2} \) and the unit area is changes as \( dx'dy' = \frac{dxdy}{|z|^4} \). Thus the measure \( \frac{dxdy}{y^2} \) is modular invariant. This measure is called the \textit{hyperbolic measure} on \( \mathbb{H} \).

The volume of \( F \) is evaluated with the hyperbolic measure and becomes finite:

\[
\text{vol} F = \int_{F} \frac{dxdy}{y^2} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \int_{-\frac{1}{2}}^{\infty} \frac{dy}{y^2} \right) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{3}. \quad (5.9)
\]

Notice also that the integrand of the Petersson inner product, \( f(z) \overline{g(z)} y^k \), is invariant under both the \( S \)- and \( T \)-transformations. Hence, the inner product (5.8) represents a manifestly modular invariant integral.

\underline{Holomorphic wave functions as modular forms of weight 2}

Having reviewed some basics of the modular forms, we now reconsider the holomorphic zero-mode wave function \( f(a) \) in abelian CS theory on the torus. The upshot of the previous sections is that under the modular \( S \)- and \( T \)-transformations \( f(a) \) varies as

\[
S : f \left( -\frac{1}{a} \right) = a^2 f(a), \quad (5.10)
\]

\[
T : f(a+1) = f(a), \quad (5.11)
\]
given that \( f(a) \) is quantum theoretically characterized by the operative relation (3.32) and the inner product (3.33). The holomorphic zero-mode wave function \( f(a) \) is also a function of the modular parameter \( \tau := i\tau_2 \) \((\tau_2 > 0)\) of the torus to start with. Transformations of \( \tau_2 \) under under the modular \( S\) - and \( T\)-transformations of \( a \) have been determined in section 4. The modular transformations of our interest are generated by

\[
S : (a, \tau_2) \rightarrow \left( -\frac{1}{a}, \frac{\tau_2}{|a|^2} \right),
\]

(5.12)

\[
T : (a, \tau_2) \rightarrow (a + 1, \tau_2).
\]

(5.13)

Since \( \tau_2 \) corresponds to the area of the torus, from the argument below (5.8) we find that (5.12) and (5.13) guarantee that the number density of the zero modes is preserved under the modular transformations. Note that under the \( S\)-transformation the area element \( \text{d}a\bar{a} \) changes as \( |\text{d}a\bar{a}| \rightarrow \frac{|\text{d}a\bar{a}|}{|a|^2} \). Thus \( \tau_2 \rightarrow \frac{\tau_2}{|a|^2} \) in (5.12) means that the number density per unit area \( \frac{|\text{d}a\bar{a}|}{\tau_2} \) is preserved.

From (5.4, 5.5) and (5.10, 5.11) we find that the holomorphic zero-mode wave function \( f(a) \) in the \( a\)-space representation behaves as a modular form of weight 2. We now check this statement by use of a specific form of the the holomorphic zero-mode wave function

\[
f_2(a; \tau_2) = e^{\frac{\pi \tau_2}{2} a^2} \vartheta_3(i\tau_2, a) \tag{5.14}
\]

which corresponds to the holomorphic function \( f_2(a) \) defined in (3.27) with \( l = 1 \). Using the formula (with Re \( \alpha > 0 \))

\[
\sum_{m \in \mathbb{Z}} e^{-\pi \alpha m^2 + 2\pi i \beta m} = \frac{1}{\sqrt{\alpha}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi}{\alpha}(n-\beta)^2},
\]

(5.15)

we can rewrite \( f_2(a; \tau_2) \) as

\[
f_2(a; \tau_2) = \frac{1}{\sqrt{\tau_2}} \vartheta_3 \left( -\frac{1}{i\tau_2}, \frac{a}{i\tau_2} \right).
\]

(5.16)

From (3.9) we find

\[
\vartheta_3 \left( -\frac{1}{i\tau_2}, \frac{a + 1}{i\tau_2} \right) = \vartheta_3 \left( -\frac{1}{i\tau_2}, \frac{a}{i\tau_2} \right).
\]

(5.17)

Thus \( f_2(a; \tau_2) \) satisfies the modular \( T\)-invariance (5.11).

Under the modular \( S\)-transformation \( f_2(a; \tau_2) \) becomes

\[
f_2 \left( -\frac{1}{a}; \tau_2' \right) = e^{\frac{\pi \tau_2}{2} a^2} \vartheta_3 \left( i\tau_2', \frac{-1}{a} \right) = \frac{1}{\sqrt{\tau_2}} \vartheta_3 \left( -\frac{1}{i\tau_2'}, \frac{-1}{i\tau_2} a \right) \tag{5.18}
\]

where \( \tau_2' = \frac{\tau_2}{|a|^2} \). Using the operative relation (3.32), we can express \( i\bar{a} f_2(a; \tau_2) \) as

\[
i\bar{a} f_2(a; \tau_2) = \frac{i}{\pi} \vartheta_3 \left( \frac{\tau_2}{\sqrt{\tau_2}} \right) \sum_{m \in \mathbb{Z}} e^{-\frac{\pi \tau_2 m^2 + 2\pi i \tau_2 ma}{\tau_2}} \]

\[
= \frac{1}{\sqrt{\tau_2}} \sum_{m \in \mathbb{Z}} (i2m) e^{-\frac{\pi \tau_2 m^2 + 2\pi i \tau_2 ma}{\tau_2}} \tag{5.19}
\]
where we keep fixing \( l = 1 \). On the other hand, using the conjugate symmetry of the holomorphic zero-mode wave function (3.34), we have the dual relation

\[
\begin{align*}
    i\bar{a} f_2 \left( -\frac{1}{a}; \tau' \right) & \leftrightarrow -ia f_2 \left( -\frac{1}{a}; \tau' \right) \\
    & = -i\frac{\tau'}{\pi} \partial_a \frac{1}{\sqrt{\tau_2}} \sum_{m \in \mathbb{Z}} e^{-\frac{\tau_2}{\tau_2} m^2 - \frac{2\tau_2}{\tau_2} \frac{1}{a^2}} \\
    & = i\frac{1}{\sqrt{\tau_2}} \sum_{m \in \mathbb{Z}} \left( 2m \frac{1}{a^2} \right) e^{-\frac{\tau_2}{2} m^2 + \frac{\tau_2}{2} m \frac{1}{a^2}} 
\end{align*}
\]

(5.20)

where in the second line we use the operative relation that the factor of \( a \) is given by the action of \( \frac{\tau_2}{\pi} \partial_a \) on an antiholomorphic function in the \( \bar{a} \)-space representation.

We now apply the relation (4.14) to the above expression:

\[
\begin{align*}
    i\bar{a} f_2 \left( -\frac{1}{a}; \tau' \right) & \leftrightarrow -ia f_2 \left( -\frac{1}{a}; \tau_2 \right) |_{|a|^2 = 1} \\
    & = i\frac{1}{\sqrt{\tau_2}} \sum_{m \in \mathbb{Z}} 2ma^2 e^{-\frac{\tau_2}{2} m^2 + \frac{\tau_2}{2} ma} \\
    & = i a^2 \bar{a} f(a; \tau_2) 
\end{align*}
\]

(5.21)

Thus we can obtain the relation \( f_2(-\frac{1}{a}; \tau'_2) = a^2 f_2(a; \tau_2) \) in the unit of \( \hbar = \frac{\pi}{\tau_2} = 1 \). This provides a concrete realization of (5.10) in terms of \( f_2(a; \tau_2) \). We can therefore interpret the holomorphic zero-mode wave function \( f_2(a; \tau_2) \) as a modular form of weight 2. The holomorphic wave function \( f_2(a; \tau_2) \) is quantum theoretically characterized by the operative relation (3.32) and the inner product (3.33). The dual relation used in (5.20) arises from the conjugate symmetry of the inner product as discussed in (3.34).

**The wave-function inner product as a quantum version of the Petersson inner product**

So far we have argued that the holomorphic zero-mode wave function \( f(a) \) can be considered as a modular form of weight 2. Consequently, we may define the zero-mode variable \( a \) on the fundamental domain \( \mathcal{F} \) in Figure 1. The inner product of the zero-mode wave functions (3.33) is then written as

\[
\langle \Psi | \Psi' \rangle = \int d\mu(a, \bar{a}) e^{-K(a, \bar{a})} \overline{f(a)} f'(a)
\]

\[
\rightarrow \frac{1}{\text{vol} \mathcal{F}} \int_\mathcal{F} d(\text{Re} a) d(\text{Im} a) e^{-K(a, \bar{a})} \overline{f(a)} f'(a)
\]

\[
= \frac{3}{\pi} \int_\mathcal{F} d(\text{Re} a) d(\text{Im} a) \frac{e^{-K(a, \bar{a})}}{(\text{Im} a)^2} \overline{f(a)} f'(a)(\text{Im} a)^2
\]

(5.22)

where we express the integral measure \( d\mu(a, \bar{a}) \) as \( d(\text{Re} a) d(\text{Im} a) \) in the second line. The Kähler potential \( K(a, \bar{a}) \) can be expressed as

\[
K(a, \bar{a}) = \frac{\pi l}{\tau_2} \left[ a\bar{a} + u(a) + v(\bar{a}) \right]
\]

(5.23)
where $\pi \frac{i}{\tau_2} u(a)$ and $\pi \frac{i}{\tau_2} v(\bar{a})$ denote purely holomorphic and antiholomorphic functions, respectively. We extract the factor of $\pi \frac{i}{\tau_2}$ here as it serves as the Planck constant of the physical system of interest. Quantum effects in (5.22) are encoded in $K(a, \bar{a})$. In the classical limit $\tau_2 \to \infty$, these effects vanish and the inner product (5.22) reduces to the Petersson inner product (5.8) with $k = 2$. We can then consider the inner product of the zero-mode wave functions in abelian Chern-Simons theory on the torus as a quantum version of the Petersson inner product for modular forms of weight 2.

The modular invariance of the wave-function inner product (5.22) can easily be seen as follows. As in the case of the Petersson inner product, the classical part of the integrand $\overline{f(a)} f'(a) (\text{Im } a)^2$ and the hyperbolic measure $\frac{d(\text{Re } a) d(\text{Im } a)}{(\text{Im } a)^2}$ are both modular invariant. The quantum contribution comes from the factor $e^{-K(a, \bar{a})}$. As discussed in the previous section, under the modular $S$-transformation we make $K_0 = \pi \frac{i}{\tau_2} a \bar{a}$ invariant by suitably choosing the transformation of $\tau_2$, while under the $T$-transformation we have $K'_0 = K_0 + \pi \frac{i}{\tau_2} (a + \bar{a} + 1)$. Thus under the modular transformation the Kähler potential $K(a, \bar{a})$ is invariant up to the addition of the holomorphic and antiholomorphic functions $\pi \frac{i}{\tau_2} [u(a) + v(\bar{a})]$ in (5.23). In the context of geometric quantization such additional ambiguities are built in the definition of $K(a, \bar{a})$ in (2.12) as it leads to the identical Kähler form. In other words, the factor $e^{-K(a, \bar{a})}$ is quantum theoretically invariant under the modular transformation as well. Thus the wave-function inner product (5.22) can also be considered as a modular invariant integral.

6 Conclusion

In this paper we consider a wave function in abelian Chern-Simons theory on the torus in the context of geometric quantization. In a suitable gauge the wave function can be parametrized by a zero-mode variable $a$ on the torus, with a modular parameter of the torus chosen by $\tau = i \tau_2$ ($\tau_2 > 0$). All the key ingredients in the geometric quantization, such as a Kähler potential $K(a, \bar{a})$, a symplectic potential $\mathcal{A}(\tau_2)$ and a zero-mode wave function $\Psi[a, \bar{a}]$, can all be derived from the Kähler form $\Omega(\tau_2)$ for the zero-mode variable. In this sense the geometric quantization of our interest can be considered as a Kähler-form program. In general, the wave function satisfies a polarization condition. This allows us to express it as $\Psi[a, \bar{a}] = e^{-\frac{K(a, \bar{a})}{2}} f(a)$ where $f(a)$ is a holomorphic function which we call a holomorphic zero-mode wave function. In this paper we emphasize that there exist ambiguities in the choice of $K(a, \bar{a})$ up to an addition of holomorphic and antiholomorphic functions. We review these materials in section 2.

Basic ideas in sections 3 and 4 can be itemized as follows.

1. Since the zero-mode variable $a$ is defined on the torus, physical observables of the zero modes are expected to be invariant under the doubly periodic translation $a \to a + m + i n \tau_2$ ($m, n \in \mathbb{Z}$) and the modular $S$-transformation $a \to -\frac{1}{a}$. Note that the modular $T$-transformation $a \to a + 1$ is included in the former transformation.

2. The above $a$-transformations induce $U(1)$ gauge transformations of the symplectic
potential $\mathcal{A}^{(\tau_2)}$.

3. Accordingly, we can consider gauge transformations of the zero-mode wave function $\Psi[a, \bar{a}]$.

4. We impose gauge invariance on $\Psi[a, \bar{a}]$ under the gauge transformations of $\mathcal{A}^{(\tau_2)}$ induced by the transformations of the zero-mode variable. We can then express the gauge invariant condition in terms of the holomorphic zero-mode wave function $f(a)$.

In section 3 we consider the gauge invariance of $\Psi[a, \bar{a}]$ associated with the doubly periodic translations $a \rightarrow a + m + in\tau_2$ and find that $f(a)$ can be described in terms of the Jacobi theta functions. Along the way, we indicate that $f(a)$ is quantum theoretically determined by the operative relation (3.32) and the inner product of the zero-mode wave functions (3.33).

In section 4 we consider the gauge invariance of $\Psi[a, \bar{a}]$ in the cases of the modular $S$- and $T$-transformations of $a$ represented by $S: (a, \tau_2) \rightarrow (-\frac{1}{a}, \frac{\tau_2}{|a|^2})$ and $T: (a, \tau_2) \rightarrow (a + 1, \tau_2)$, respectively. We then find that the gauge invariance conditions on $\Psi[a, \bar{a}]$ lead to the relations $S: f(-\frac{1}{a}) = a^2 f(a)$ and $T: f(a + 1) = f(a)$, given the above mentioned characterization of $f(a)$.

The results in section 4 illustrate that we can quantum theoretically identify $f(a)$ as a modular form of weight 2. We confirm this statement by use of a specific representation of $f(a)$ in terms of the Jacobi theta function. We also show that the inner product of the zero-mode wave functions gives rise to a quantum version of the Petersson inner product for modular forms of weight 2.

Finally, we would like to mention a possible application of these results. In search for physical applications, it is natural to consider a hermitian operator acting on the holomorphic wave function $f(a)$. The quantum theoretic identification of $f(a)$ as a modular form of weight 2 suggests that one of such operators may be given by a Hecke operator acting on the modular form. Eigenvalues of the Hecke operator play an important role in number theory. Namely, an $L$-function associated with the modular form can be determined by these eigenvalues. Thus the results in this paper would open up a new avenue to connect quantum theory to number theory. We shall investigate these aspects in a forthcoming paper [16].

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