A Lyapunov-Based ISS Small-Gain Theorem for Infinite Networks of Nonlinear Systems

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Abstract—In this article, we show that an infinite network of input-to-state stable (ISS) subsystems, admitting ISS Lyapunov functions, itself admits an ISS Lyapunov function, provided that the couplings between the subsystems are sufficiently weak. The strength of the couplings is described in terms of the properties of an infinite-dimensional nonlinear positive operator, built from the interconnection gains. If this operator induces a uniformly globally asymptotically stable (UGAS) system, a Lyapunov function for the infinite network can be constructed. We analyze necessary and sufficient conditions for UGAS and relate them to small-gain conditions used in the stability analysis of finite networks.

Index Terms—Infinite-dimensional systems, input-to-state stability, large-scale systems, Lyapunov methods, nonlinear systems, small-gain theorems.

I. INTRODUCTION

W

ERE surrounded by networks: social networks, power grids, transportation and manufacturing networks, etc. These networks grow in size every year, and emerging technologies, such as cloud computing and fifth-generation (5G) communication, make this trend even more apparent. As stability properties of networks may deteriorate with the increase in the number of participating agents [1], it is natural to study infinite networks that overapproximate large-scale networks as a worst-case scenario.

A prominent place in these investigations is occupied by the theory of linear spatially invariant systems [2]–[4]. In such networks, infinitely many subsystems are coupled via the same pattern. This nice geometrical structure together with the linearity of subsystems allowed researchers to develop powerful criteria for the stability of such networks.

On the other hand, in the stability analysis of finite networks with nonlinear components, groundbreaking results have been obtained within the framework of input-to-state stability (ISS) [5]. According to the ISS small-gain approach, the influence of any subsystem on other subsystems in a network is characterized by so-called gain functions. The gain operator constructed from these functions characterizes the interconnection structure of the network. The small-gain theorems for finite networks of ISS systems described by ordinary differential equations (ODEs) [6]–[9] show that if the gains are small enough [which is expressed in terms of a so-called small-gain condition (SGC) of the gain operator], the network is ISS. These results have numerous applications in systems theory [10], [11], and play a major role in a large part of modern nonlinear control [12], [13].

Recently, significant advances have been achieved in an infinite-dimensional ISS theory, see [14], [15] for a comprehensive overview of the topic, and [16] for an overview of the linear theory.

This progress motivated the development of the ISS small-gain framework for the stability analysis of infinite interconnections of nonlinear systems without any spatial invariance assumption. This research was initiated in [17], where nonlinear Lyapunov-based small-gain theorems have been obtained under the very strong assumption that all gains are uniformly less than the identity. In [17], the authors also apply their small-gain theorem for stabilization of infinite nonlinear networks by revisiting a backstepping method. In [18], tight Lyapunov-based small-gain theorems have been obtained for networks of exponentially ISS systems with linear gains, and these results have been applied to distributed observer design and cooperative control of infinite networks in [19].

Nonlinear trajectory-based small-gain theorems for infinite networks have been developed in [20], where it was shown that an infinite network of ISS systems is ISS if the corresponding nonlinear gain operator satisfies the so-called monotone limit property. The monotone limit property implies the uniform SGC [20], which is equivalent to the monotone bounded invertibility property. The latter played a key role in the derivation of the ISS small-gain theorem for finite networks in [8, Lem. 13].

This article is strongly motivated by [21], where the robust strong SGC was introduced and a method to construct paths of strict decay, based on the concept of the strong transitive
The gain operator, was proposed. For finite networks, this method was used in [22, Prop. 2.7, Rem. 2.8]; see also [23] for more details on the importance of this concept in small-gain theory. Based on these results, in [21], a small-gain theorem for infinite networks and the construction of associated ISS Lyapunov functions were proposed under the assumption that a linear path of strict decay for the gain operator exists. Although there are examples of nonlinear infinite networks with nonlinear gains, whose gain operators admit a linear path of strict decay, in general, this requirement is quite restrictive. In fact, the Lyapunov-based small-gain theorems for finite networks developed in [9] do not require the linearity of the path of strict decay.

A. Contribution

We consider infinite networks of ISS control systems described by ODEs, admitting ISS Lyapunov functions with a corresponding gain operator, characterizing the influence of the subsystems on each other. In our main first result (see Theorem III.1), we show that the existence of a (possibly nonlinear) path of strict decay for the gain operator \( \Gamma \) (together with some uniformity conditions) implies that the whole network is ISS and a corresponding ISS Lyapunov function for the network can be constructed. Our result extends (up to some extra uniformity condition that we require from the path of strict decay) the nonlinear Lyapunov-based small-gain theorem for finite networks (in maximum formulation) shown in [9] to the setting of infinite networks, recovers the Lyapunov-based small-gain theorem for infinite networks from [21], and partially recovers the main result in [17]; see Section VIII for a detailed discussion.

Next, we introduce the concept of a max-robust SGC, which is less conservative than the robust SGC from [21], but better compatible with max-type gain operators, and can be fully characterized in terms of the asymptotic properties of the discrete-time system induced by the gain operator.

In our second main result (see Theorem VI.1), we show that the uniform global asymptotic stability (UGAS) of the system induced by a scaled gain operator guarantees the existence of a path of strict decay. Furthermore, we explicitly construct this path via the concept of the strong transitive closure of the gain operator, adopted from [21]. Finally, we characterize the UGAS property of the induced system in terms of SGCs and give useful sufficient conditions for it.

B. New vistas

Our nonlinear ISS small-gain theorem (see Theorem III.1) has been developed for the case of the maximum formulation of the ISS property for the subsystems. However, our proof technique can be used for other formulations of the ISS property (semi-maximum, summation, etc.), leading to very flexible small-gain results. In contrast to that, our method for the construction of the paths of strict decay for the gain operator relies strongly on the maximum formulation of the ISS property for subsystems. If the internal gains are all linear, and the ISS property for subsystems is given in a sum formulation, then the gain operator is linear and a path of strict decay with respect to such a gain operator exists (and can be explicitly constructed) if and only if the spectral radius of \( \Gamma \) is less than one. Numerous characterizations of this condition have been developed in [24]. Some of these characterizations, and in particular, the method for the construction of paths of strict decay, have been extended in [25] to the case of homogeneous and subadditive gain operators. The general nonlinear case remains, however, a challenging open problem.

C. Notation

We write \( \mathbb{R} \) for the reals and \( \mathbb{Z} \) for the integers. The sets of nonnegative reals and integers are denoted by \( \mathbb{R}_+ \) and \( \mathbb{Z}_+ \), respectively. By \( C^0(X, Y) \), we denote the set of all continuous mappings from a space \( X \) to a space \( Y \). In any metric space, we write \( B_1(x) \) for the open ball of radius \( 1 \) centered at \( x \), and \( \text{int}(A) \) for the interior of a subset \( A \subset X \). We use the following classes of comparison functions:

\[
P := \{ \gamma \in C^0(\mathbb{R}_+, \mathbb{R}_+) : \gamma(0) = 0, \gamma(r) > 0 \quad \forall r > 0 \}\]

\[
K := \{ \gamma \in P : \gamma \text{ strictly increasing} \}
\]

\[
K_\infty := \{ \gamma \in K : \gamma \text{ is unbounded} \}
\]

\[
L := \{ \gamma \in C^0(\mathbb{R}_+, \mathbb{R}_+) : \gamma \text{ is strictly decreasing with} \lim_{t \to \infty} \gamma(t) = 0 \}
\]

\[
KL := \{ \beta \in C^0(\mathbb{R}_+^2, \mathbb{R}_+) : \beta(t, \cdot) \in K \quad \forall t \geq 0 \}
\]

\[
\beta(t, \cdot) \in L \quad \forall r > 0 \}\}
\]

We write \( \ell^\infty \) for the space of bounded real sequences \( s = (s_i)_{i \in \mathbb{N}} \), which is a Banach space with the norm \( ||s||_{\ell^\infty} := \sup_{i \in \mathbb{N}} |s_i| \). The positive cone in \( \ell^\infty \) is given by \( \ell^\infty_+ := \{s \in \ell^\infty : s_i \geq 0 \quad \forall i \in \mathbb{N} \} \). We define \( \mathbb{I} := \{(1, 1, 1, \ldots) \in \ell^\infty_+ \). By \( e_i, e_i \in \mathbb{N} \), we denote the \( i \)th unit vector in \( \ell^\infty_+ \). Given \( s^1, s^2 \in \ell^\infty_+ \), we write \( s^1 \oplus s^2 \) for the vector given by the componentwise maximum of \( s^1 \) and \( s^2 \). If \( \mathbb{N} \) is replaced by another index set \( I \), we also write \( \ell^\infty(I) \) for the corresponding Banach space and \( \ell^\infty_+(I) \) for its positive cone. The notation \( L^\infty(\mathbb{R}_+, U) \) is used for the Banach space of essentially bounded strongly measurable functions from \( \mathbb{R}_+ \) into a Banach space \( U \).

A function \( \lambda : \mathbb{R}_+ \to X \) into some space \( X \) is called piecewise right-continuous if there is a partition of \( \mathbb{R}_+ \) into disjoint subintervals, \( \mathbb{R}_+ = [0, t_1) \cup [t_1, t_2) \cup [t_2, t_3) \cup \ldots \), such that the restriction of \( \lambda \) to each of the subintervals is continuous.

II. Technical Setup

A. Interconnections

Consider a family of control systems of the form

\[
\Sigma_i : \quad \dot{x}_i = f_i(x_i, \bar{x}_i, u_i), \quad i \in \mathbb{N}.
\]  

(1)

This family comes with sequences \((n_i)_{i \in \mathbb{N}}\) and \((m_i)_{i \in \mathbb{N}}\) of positive integers as well as finite (possibly empty) sets \( I_i \subset \mathbb{N} \), \( i \notin I_i \), such that the following assumptions are satisfied.

1) The state vector \( x_i \) is an element of \( \mathbb{R}^{n_i} \).
2) The internal input vector \( \bar{x}_i \) is composed of the state vectors \( x_j, j \in I_i \), and thus is an element of \( \mathbb{R}^{n_i} \), where \( N_i := \sum_{j \in I_i} n_j \).
3) The external input vector \( u_i \) is an element of \( \mathbb{R}^{m_i} \).
4) The right-hand side \( f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i} \) is a continuous function.

5) For every initial state \( x_{i0} \in \mathbb{R}^{n_i} \) and all essentially bounded inputs \( \bar{x}_i(\cdot) \) and \( u_i(\cdot) \), there is a unique solution of \( \Sigma_i \), which we denote by \( \phi_i(t, x_{i0}, \bar{x}_i, u_i) \) (it may be defined only on a bounded time interval).

For each \( i \in \mathbb{N} \), we fix norms on the spaces \( \mathbb{R}^{n_i} \) and \( \mathbb{R}^{m_i} \), respectively (these norms can be chosen arbitrarily). For brevity in notation, we avoid adding an index to these norms, indicating to which space they belong, and simply write \( \| \cdot \| \) for each of them. The interconnection of the systems \( \Sigma_i, i \in \mathbb{N} \), is defined on the state space \( X := \ell_\infty(\mathbb{N}, (n_i)) \), where

\[
\ell_\infty(\mathbb{N}, (n_i)) := \left\{ x = (x_i)_{i \in \mathbb{N}} : x_i \in \mathbb{R}^{n_i}, \sup_{i \in \mathbb{N}} |x_i| < \infty \right\}.
\]

This space is a Banach space with the \( \ell_\infty \)-type norm

\[
\|x\|_X := \sup_{i \in \mathbb{N}} |x_i|.
\]

The space of admissible external input values is likewise defined as the Banach space

\[
U := \ell_\infty(\mathbb{N}, (m_i)), \quad \|u\|_U := \sup_{i \in \mathbb{N}} |u_i|.
\]

We choose the class of admissible external input functions as

\[
\mathcal{U} := \{ u \in \ell_\infty(\mathbb{R}_+, U) : u \text{ is piecewise right-continuous} \}
\]

which will be equipped with the \( \ell_\infty \)-norm

\[
\|u\|_U := \text{ess sup}_{t \in \mathbb{R}_+} |u(t)|/U.
\]

We define the right-hand side of the interconnected system by

\[
f : X \times U \to \prod_{i \in \mathbb{N}} \mathbb{R}^{n_i}, \quad f(x, u) := (f_i(x_i, \bar{x}_i, u_i))_{i \in \mathbb{N}}.
\]

Hence, the interconnected system can formally be written as the differential equation

\[
\Sigma : \quad \dot{x} = f(x, u).
\]

To make sense of this equation, we need to define an appropriate notion of solution. For fixed \((u, x^0) \in \mathcal{U} \times X\), a function \( \lambda : J \to X \), where \( J \subset \mathbb{R} \) is an interval of the form \([0, T]\) with \( 0 < T \leq \infty \), is called a solution of the Cauchy problem

\[
\dot{x} = f(x, u), \quad x(0) = x^0
\]

provided that \( s \to f(\lambda(s), u(s)) \) is a locally integrable X-valued function (in the Bochner integral sense) and

\[
\lambda(t) = x^0 + \int_0^t f(\lambda(s), u(s)) \, ds \quad \text{for all } t \in J.
\]

We say that the system \( \Sigma \) is well-posed if for every initial state \( x^0 \in X \) and every external input \( u \in \mathcal{U} \) there exists a unique maximal solution. We denote this solution by \( \phi(\cdot, x^0, u) : [0, t_{\max}(x^0, u)) \to X \), where \( 0 < t_{\max}(x^0, u) \leq \infty \).

The following theorem provides sufficient conditions for well-posedness of \( \Sigma \), see [18, Cor. III.3].

**Theorem II.1:** Consider the coupled system \( \Sigma \), composed of the subsystems \( \Sigma_i \), and let the following assumptions hold.

1) \( f(x, u) \in X \) for all \((x, u) \in X \times U\).

2) \( f(\cdot, u) : X \to X \) is continuous for each \( u \in U \).

3) \( f(x, \cdot) : U \to X \) is continuous for each \( x \in X \).

4) For each \( u \in \mathcal{U} \) and \( x^0 \in X \), there are \( \delta > 0 \) and locally integrable functions \( \ell, \ell_0 : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\|f(x^1, u(t)) - f(x^2, u(t))\|_X \leq \ell(t)\|x^1 - x^2\|_X
\]

\[
\|f(x^0, u(t))\|_X \leq \ell_0(t)
\]

for all \( x^1, x^2 \in B_\delta(x^0) \) and almost all \( t \in \mathbb{R}_+ \).

Then, \( \Sigma \) is well-posed.

If \( \Sigma \) is well-posed, one has

\[
\pi_i(\phi(t, x^0, u)) = \phi_i(t, x^0_i, \bar{x}_i, u_i)
\]

for all \( t \in [0, t_{\max}(x^0, u)) \) and \( i \in \mathbb{N} \), where \( \pi_i : X \to \mathbb{R}^{n_i} \) denotes the canonical projection onto the \( i \)-th component, \( \bar{x}_i(\cdot) = (\pi_j(\phi(\cdot, x^0, u)))_{j \neq i} \), and \( x^0_i, u_i \) denote the \( i \)-th components of \( x^0 \) and \( u \), respectively, see [18, Sec. 3].

In the rest of the article, we assume that the following holds.

**Assumption II.2:** The system \( \Sigma \) is well-posed, and all of its uniformly bounded maximal solutions \( \phi(\cdot, x, u) \) are global, i.e., they exist on \( \mathbb{R}_+ \) (this latter property is also called boundedness-implies-continuation (BIC) property).

**Remark II.3:** If the function \( f \) is uniformly bounded on bounded balls, and Lipschitz continuous on bounded balls with respect to the first argument, then \( \Sigma \) is well-posed, and for any \( R > 0 \) there is \( \tau_R > 0 \) such that for all \( x \in \mathbb{R}^n \) with \( \|x\|_X \leq R \) and \( u \) with \( \|u\|_U \leq R \) the solution \( \phi(\cdot, x, u) \) exists at least on \([0, \tau_R]\), which easily implies the BIC property; see [26, Th. 4.3.4] for a related result for systems without inputs.

### B. Input-to-State Stability

We now recall the definition of input-to-state stability.

**Definition II.4:** A well-posed system \( \Sigma \) is called (uniformly) input-to-state stable (ISS) if it is forward complete and there exist \( \beta \in \mathcal{K} \) and \( \gamma \in \mathcal{K}_\infty \) such that

\[
\|\phi(t, x, u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_U)
\]

for all \((t, x, u) \in \mathbb{R}_+ \times X \times U\).

Input-to-state stability is most often verified via the construction of an ISS Lyapunov function that is defined as follows.

**Definition II.5:** A function \( V : X \to \mathbb{R}_+ \) is called an ISS Lyapunov function (in an implication form) for \( \Sigma \) if it satisfies the following properties.

1) \( V \) is continuous.

2) There exist \( \psi_1, \psi_2 \in \mathcal{K}_\infty \) such that

\[
\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) \quad \text{for all } x \in X.
\]

3) There exist \( \gamma \in \mathcal{K} \) and \( \alpha \in \mathcal{P} \) such that for all \( x \in X \) and \( u \in \mathcal{U} \) the following implication holds:

\[
V(x) > \gamma(\|u\|_U) \Rightarrow D^+V_u(x) \leq -\alpha(V(x))
\]

where \( D^+V_u(x) \) denotes the right upper Dini orbital derivative, defined as

\[
D^+V_u(x) := \limsup_{t \to 0^+} \frac{V(\phi(t, x, u)) - V(x)}{t}.
\]
The importance of ISS Lyapunov functions is due to the following result (cf., [15, Th. 2.17]).

**Proposition II.6:** If Σ admits an ISS Lyapunov function, then it is ISS.

The construction of an ISS Lyapunov function is a complex problem, which becomes especially challenging if the system is nonlinear and of large size. In this article, we assume that all components Σi of an infinite network are ISS with corresponding ISS Lyapunov functions Vi. To find an ISS Lyapunov function V for Σ, we exploit the interconnection structure and construct V from Vi. Hence, we make the following assumption.

**Assumption II.7:** For each i ∈ N, there exists a continuous function Vi : Rn→ R+ that is continuously differentiable outside of xi = 0 and satisfies the following properties.

1) There exist ψ1, ψ2 ∈ K̄∞ such that

$$ψ_1(|x_i|) ≤ V_i(x_i) ≤ ψ_2(|x_i|) \quad \text{for all } x_i ∈ R^n.$$  \hspace{1cm} (5)

2) There exist γij ∈ K ∪ {0}, where γij = 0 for all j ∈ N \ {i}, and γiu ∈ K as well as αi ∈ P such that for all x = (xj)j∈N ∈ X and u = (ui)j∈U ∈ U the following implication holds:

$$V_i(x_i) > \max \left\{ \sup_{j \in I_i} γ_{ij}(V_j(x_j)), γ_{iu}(|u_i|) \right\} \Rightarrow \nabla V_i(x_i)f_i(x_i, x, u_i) ≤ -α_i(V_i(x_i)).$$ \hspace{1cm} (6)

The function Vi is called an ISS Lyapunov function for Σi. The functions γij and γiu are called internal gains and external gains, respectively.

Using the internal gains γij from Assumption II.7, we define the gain operator Γ : ℓ̄+∞ → ℓ̄+∞ by

$$Γ(s) := \left( sup_{j \in N} γ_{ij}(s_j) \right)_{i \in N}. \hspace{1cm} (7)$$

In general, Γ might be neither well-defined nor continuous. The following assumption guarantees both, see [21, Lem. 2.1] and [20, Prop. 2].

**Assumption II.8:** The family \{γij : i, j ∈ N\} is pointwise equicontinuous. That is, for every r ≥ 0 and ε > 0, there exists δ = δ(ε, r) > 0 such that |r − r̃| ≤ δ, r̃ ∈ R+, implies |γij(r) − γij(r̃)| ≤ ε for all i, j ∈ N.

Additionally, we make the following assumption on the external gains.

**Assumption II.9:** There is γ̄u = sup ∈ K such that γiu ≤ γ̄u for all i ∈ N.

We now introduce the concept of a path of strict decay (for the gain operator Γ) that is of crucial importance in the construction of an ISS Lyapunov function for the interconnected system.

**Definition II.10:** A mapping γ : R+ → ℓ̄+∞ is called a path of strict decay (for Γ), if the following properties hold.

1) There exists a function ρ ∈ K̄∞ such that

$$Γ(σ(r)) ≤ (id + ρ)^{-1} o σ(r) \quad \text{for all } r ≥ 0$$

where (id + ρ)^{-1} is applied componentwise.

2) There exist σmin, σmax ∈ K̄∞ satisfying

$$σ_{min} ≤ σ_i ≤ σ_{max} \quad \text{for all } i ∈ N.$$
Step 2: We prove that $V$ is continuous and locally Lipschitz continuous outside of $x = 0$. Continuity at $x = 0$ follows from coercivity as shown in Step 1. Hence, let $0 \neq x \in X$. Define

$$\delta = \delta(x) := \frac{1}{3} \varphi^{-1}_2 \circ \sigma_{\min} \circ \sigma^{-1}_{\max} \circ \psi_1 \left( \frac{\|x\|}{4} \right) \leq \frac{\|x\|}{12}.$$  

For all $y \in B_\delta(x)$, it holds that $\|y\| \geq \frac{1}{2} \|x\|$, implying

$$V(y) \geq \sigma^{-1}_{\min} \circ \psi_1 \left( \frac{\|y\|}{2} \right) \geq \frac{1}{2} \sigma^{-1}_{\max} \circ \psi_1 \left( \frac{\|x\|}{4} \right).$$  

(11)

Define $I_\delta := \{ i \in \mathbb{N} : |x_i| \geq 2\delta \}$. For any $y \in B_\delta(x)$ and $i \in I_\delta$, we have

$$|y_i| \leq |x_i| + |x_i - y_i| \leq 3\delta = \varphi^{-1}_2 \circ \sigma_{\min} \circ \sigma^{-1}_{\max} \circ \psi_1 \left( \frac{\|x\|}{4} \right).$$

This implies

$$\sigma^{-1}_i(V_i(y_i)) \leq \sigma^{-1}_2(\|y_i\|) \leq \ker \sigma^{-1}_\min \circ \psi_1 \left( \frac{\|x\|}{4} \right).$$

In view of (11), we see that

$$V(y) = \sup_{i \in I_\delta} \sigma^{-1}_i(V_i(y_i))$$

for all $y \in B_\delta(x)$. It is then easy to see that

$$|V(y^1) - V(y^2)| \leq \sup_{i \in I_\delta} |\sigma^{-1}_i(V_i(y^1_i)) - \sigma^{-1}_i(V_i(y^1_i))|$$

for all $y^1, y^2 \in B_\delta(x)$. Since $V_i(y^1_i)$ and $V_i(y^2_i)$ on the right-hand side of this inequality are contained in the compact interval $[\psi_2(\delta), \psi_2(\|x\| + \delta)] \subset (0, \infty)$, the definition of a path of strict decay together with Assumption 5) of Theorem III.1 implies the existence of a constant $L > 0$ such that for all $y^1, y^2 \in B_\delta(x)$,

$$|V(y^1) - V(y^2)| \leq \sup_{i \in I_\delta} L|y^1_i - y^2_i| \leq L\|y^1 - y^2\|_X.$$

This proves Lipschitz continuity of $V$ on $B_\delta(x)$.

Step 3: We prove an auxiliary result needed for the proof of implication (4). Let $\rho \in \mathcal{K}_\infty$ satisfy

$$\Gamma(\sigma(r)) \leq (\id + \rho)^{-1} \circ \sigma(r) \quad \text{for all } r \in \mathbb{R}_+$$

(12)

as required in the definition of a path of strict decay. Then, we fix a function $\mu \in \mathcal{K}_\infty$ such that $\mu(r) \leq \rho(r)$ for all $r > 0$ and introduce for every $0 \neq x \in X$ the set

$$I(x) := \{ i \in \mathbb{N} : V(x) \leq \sigma^{-1}_i((\id + \mu)(V_i(x_i))) \}.$$  

(13)

Now we prove the following claim:

Assume toward a contradiction that the claim is false. Then, we can find sequences $y^n \to x$ and $i_n \in \mathbb{N} \setminus I(x)$ such that

$$V(y^n) \leq \sigma^{-1}_i(V_i(y^n_i)) + \frac{1}{n} \quad \text{for all } n \in \mathbb{N}. \quad (15)$$

At the same time, $i_n \notin I(x)$ implies

$$V(x) > \sigma^{-1}_i((\id + \mu)(V_i(x_i))) \quad \text{for all } n \in \mathbb{N}.$$  

Combining these two inequalities, we obtain

$$V(x) - V(y^n) > \sigma^{-1}_i((\id + \mu)(V_i(x_i))) - \frac{1}{n}.$$

We can find a compact interval $K \subset (0, \infty)$ such that for sufficiently large $n$ we have $(\id + \mu)(V_i(x_i)), V_i(y^n_i) \in K$. Indeed, this follows from the estimates

1) $(\id + \mu)(V_i(x_i)) \leq (\id + \mu) \circ \psi_1(\|x\|)\chi$;
2) $V_i(y^n_i) \leq \psi_2(\|y^n\|) \chi \leq \psi_2(2\|x\|)\chi$ for all $n$ large enough;
3) $V_i(y^n_i) \geq \sigma^{-1}_i(V_i(y^n_i)) - \frac{1}{n} \geq \ker \sigma^{-1}_\min(\psi_1(\frac{\|x\|}{4}) \chi) - 1/n > 0$ for all $n$ large enough;
4) $(\id + \mu)(V_i(x_i)) \geq (\id + \mu) \circ \psi_1(\|x_i\| \chi) \geq (\id + \mu) \circ \psi_1(\|x_i - y^n_i\| \chi) \geq (\id + \mu) \circ \psi_1(\frac{\|x_i\|}{2} \chi)$ for all $n$ large enough; this can be lower bounded by using the previous estimates.

By the definition of a path of strict decay, we thus have constants $0 < c \leq C < \infty$ such that

$$|\sigma^{-1}_i((\id + \mu)(V_i(x_i))) - \sigma^{-1}_i(V_i(y^n_i))| = c_n \|((\id + \mu)(V_i(x_i)) - V_i(y^n_i))\|$$

for some $c_n \in [c, C]$, when $n$ is large enough, which by monotonicity of $\sigma^{-1}_i$ yields

$$V(x) - V(y^n) > c_n ((\id + \mu)(V_i(x_i)) - V_i(y^n_i)) - \frac{1}{n}.$$

From (9), it follows that for some $L > 0$ (independent of $n$),

$$V_i(y^n_i) = V_i(x_i) + V_i(y^n_i) - V_i(x_i) \leq L\|y^n - x\|_X.$$  

Putting $\delta_n := \|y^n - x\|_X$, we thus obtain

$$V(x) - V(y^n) > c_n ((\id + \mu)(V_i(x_i)) - [V_i(x_i) + L\delta_n]) - \frac{1}{n}$$

$$= c_n \mu V_i(x_i) - c_n L\delta_n - \frac{1}{n}$$

$$\geq c_n \mu \psi_1(\|x_i\| \chi) - c_n L\delta_n - \frac{1}{n}.$$

We can also write this as

$$0 \leq \mu \circ \psi_1(\|x_i\| \chi) < b_n$$

$$b_n := c_n^{-1} \frac{[V(x) - V(y^n)] + c_n L\delta_n + \frac{1}{n}}{\chi}.$$  

\]
Note that $b_n \to 0$ as $n \to \infty$, which implies $|x_{in}| \to 0$ as $n \to \infty$, and since $y^n \to x$, also $|y^n_{in}| \to 0$. Then, (15) yields

$$0 \leq V(y^n) \leq \sigma^{-1}_{\min} \circ \psi_2(|y^n_{in}|) + \frac{1}{n} \to 0.$$ 

Since $V(y^n) \to V(x)$, we obtain $V(x) = 0$, and hence, $x = 0$, a contradiction.

**Step 4:** We define the $\mathcal{K}$-function $\gamma$ by

$$\gamma(r) := \sigma^{-1}_{\min} \circ (id + \rho) \circ \gamma_{\max}^u(r) \quad \text{for all } r \geq 0,$$

$$\gamma(r) := \sigma^{-1}_{\min} \circ (id + \rho) \circ \gamma_{\max}^u(r) \quad \text{for all } r \geq 0$$

and prove the following claim.

The inequality $V(x) > \gamma(\|u\|_U)$ for some $x \in X$ and $u \in U$ implies the existence of $T > 0$ such that

$$\nabla V_i(\phi(t)) f_i(\phi(t), \tilde{\phi}(t), u_i(t)) \leq -\tilde{\alpha}(V_i(\phi(t))) \quad (16)$$

for all $i \in I(x)$ and $t \in [0, T]$, where $\phi(t) := (\phi(t, x, u), \tilde{\phi}(t))$ is the $j$th component of $\phi(t)$ (for every $j \in \mathbb{N}$) and $\phi(t) = (\phi(t))_{j \in I_t}$.

Let us fix $x$ and $u$ as in the claim and note that $x \neq 0$. As $V$ and $\phi(\cdot)$ are continuous, $V(x) > \gamma(\|u\|_U)$ implies

$$V(\phi(t)) > \sigma^{-1}_{\min} \circ (id + \rho) \circ \gamma_{\max}^u(\|u\|_U)$$

$$\geq \sigma^{-1}_{\min} \circ (id + \rho) \circ \gamma_{\min}(u_i(t))$$

(17)

for all $i \in \mathbb{N}$ and $t \in [0, T]$, where $T > 0$ is chosen sufficiently small. If $T$ is chosen further small enough, then

$$V(\phi(t)) < \sigma^{-1}_i \circ (id + \rho) \circ V_i(\phi_i(t)) \quad \forall i \in I(x), t \in [0, T]$$

(18)

We prove this by contradiction. Assume that there are sequences $0 < t_n \to 0$ and $x_n \in I(x)$ such that

$$V(\phi(t_n)) \geq \sigma^{-1}_i \circ (id + \rho) \circ V_i(\phi_i(t_n)).$$

The left-hand side of this inequality converges to $V(x)$ as $n \to \infty$. Hence, we find a sequence $0 < \varepsilon_n \to 0$ such that

$$\sigma^{-1}_i \circ (id + \rho) \circ V_i(\phi_i(t_n)) \geq V(x) + \varepsilon_n$$

$$\leq \sigma^{-1}_i \circ (id + \mu) \circ V_i(x_{in}) + \varepsilon_n.$$

We can further estimate

$$V_i(\phi_i(t_n)) \geq V_i(x_{in}) - |V_i(x_{in}) - V_i(\phi_i(t_n))|$$

$$\geq V_i(x_{in}) - L|x_{in} - \phi_i(t_n)|$$

$$\geq V_i(x_{in}) - L|x - \phi(t_n)||x|$$

where we use that $|\phi_i(t_n)| \leq |\phi_i(t_n) - x_{in}| + |x_{in}| \leq \|\phi(t_n) - x||x| + \|x||x|$ (implying the existence of $L > 0$ by Assumption 5 of Theorem III.1.). Hence, we obtain

$$\sigma^{-1}_i \circ (id + \rho)(V_i(x_{in}) - L|x - \phi(t_n)||x)$$

$$\leq \sigma^{-1}_i \circ (id + \mu)(V_i(x_{in})) + \varepsilon_n.$$

We write this inequality as

$$\varepsilon_n \geq \sigma^{-1}_i \circ (id + \rho)(V_i(x_{in}) - L|x - \phi(t_n)||x)$$

$$- \sigma^{-1}_i \circ (id + \mu)(V_i(x_{in})).$$

With a similar reasoning as used before, we can show that the arguments of $\sigma^{-1}_{\min}$ are contained in a compact subset of $(0, \infty)$ for all sufficiently large $n$. Hence, there are constants $c_n \in [c, C]$ such that

$$\varepsilon_n \geq c_n \left[ \rho(V_i(x_{in}) - L|x - \phi(t_n)||x) \right.$$

$$\left. - \mu(V_i(x_{in})) - L|x - \phi(t_n)||x \right].$$

Using that $V_i(x_{in})$ is contained in a compact interval for all $n$ and $\rho$ is uniformly continuous on this interval, we find a sequence $0 < \delta_n \to 0$ such that

$$\rho(V_i(x_{in}) - L|x - \phi(t_n)||x) \geq \rho(V_i(x_{in})) - \delta_n \quad \forall n \in \mathbb{N}.$$ 

Also using that $\rho - \mu > 0$ (on $(0, \infty)$) and $V_i(x_{in}) \geq (id + \mu)^{-1} \circ \sigma_{\min}(V(x))$, this implies

$$\varepsilon_n \geq c_n \left[ (\rho - \mu)((id + \mu)^{-1} \circ \sigma_{\min}(V(x))) \right.$$

$$\left. - \delta_n - L|x - \phi(t_n)||x| \right].$$

Hence, as $n \to \infty$, we obtain the contradiction $V(x) = 0$, as $V(x) > \gamma(\|u\|_U) \geq 0$. This proves (18).

From (18), it then follows that for all $i \in I(x)$ and $t \in [0, T]$:

$$V_i(\phi_i(t)) > (id + \rho)^{-1} \circ \sigma_i(\phi_i(t)) \quad (12)$$

$$\sup_{j \in I_t} \gamma_{ij}(\sigma_i(V_i(\phi(t)))) \geq \sup_{j \in I_t} \gamma_{ij}(V_j(\phi_j(t))).$$

At the same time, (17) together with (18) implies

$$V_i(\phi_i(t)) > (id + \rho)^{-1} \circ \sigma_i \circ \sigma^{-1}_{\min}(id + \mu) \circ \nu_i(\|u_i(t)\|)$$

$$\geq \gamma_{ij}(\|u_i(t)\|).$$

Putting both estimates together, we obtain

$$V_i(\phi_i(t)) > \max \left\{ \sup_{j \in I_t} \gamma_{ij}(V_j(\phi_j(t))), \nu_i(\|u_i(t)\|) \right\}.$$ 

By (6) together with Assumption 6 of Theorem III.1, this implies (16), which proves the claim.

**Step 5:** We show the implication (4). First observe that the case $x = 0$ does not occur, since $V(0) = 0 > \gamma(\|u\|_U)$ is never satisfied. Hence, let us fix $0 \neq x \in X$ and $u \in U$ satisfying $V(x) > \gamma(\|u\|_U)$. By Step 3, there is $\delta > 0$ such that

$$V(y) = \sup_{i \in I(x)} \sigma^{-1}_i(V_i(y_i)) \quad \text{for all } y \in B_\delta(x).$$

Step 4 shows that for all sufficiently small $t \geq 0$, we have

$$\nabla V_i(\phi_i(t)) f_i(\phi_i(t), \tilde{\phi}(t), u_i(t)) \leq -\tilde{\alpha}(V_i(\phi_i(t))) \quad \forall i \in I(x).$$

Now let us introduce the Cauchy problem

$$\dot{v}(t) = -\tilde{\alpha}(v(t)), \quad v(0) = v_0 \in \mathbb{R}_+.$$ 

By [20, Lem. 6], we can assume that $\tilde{\alpha}$ is globally Lipschitz. Then, the Cauchy problem has a globally defined unique solution that we denote by $\mathcal{V}(t, v_0)$. For every $i \in I(x)$ and all sufficiently small $t \geq 0$, Lemma IX.1 guarantees that

$$V_i(\phi_i(t)) \leq \mathcal{V}(t, V_i(x_i)).$$

It thus follows that

$$\sup_{i \in I(x)} \sigma^{-1}_i(V_i(\phi_i(t))) - V(x) \leq 0.$$
\[
\begin{align*}
&= \frac{1}{t} \left[ \sup_{t \in I(x)} \sigma_i^{-1}(V_i(\phi_i(t))) - \sup_{t \in I(x)} \sigma_i^{-1}(V_i(x_i)) \right] \\
&\leq \frac{1}{t} \sup_{t \in I(x)} \left[ \sigma_i^{-1}(V_i(\phi_i(t))) - \sigma_i^{-1}(V_i(x_i)) \right] \\
&\leq \frac{1}{t} \sup_{t \in I(x)} \left[ \sigma_i^{-1}(\mathcal{V}(t, V_i(x_i))) - \sigma_i^{-1}(V_i(x_i)) \right] \\
&= \frac{1}{t} \sup_{t \in I(x)} \left[ \sigma_i^{-1}(\mathcal{V}(t, V_i(x_i))) - \sigma_i^{-1}(V_i(x_i)) \right].
\end{align*}
\]

For all \( t \geq 0 \), we have
\[
\mathcal{V}(t, V_i(x_i)) \leq V_i(x_i) \leq \sigma_i \circ V(x) \leq \sigma_{\text{max}}(V(x))
\]
and
\[
V_i(x_i) \geq (i \sigma + \mu)^{-1} \circ \sigma_i \circ V(x) \geq (i \sigma + \mu)^{-1} \circ \sigma_{\text{min}} \circ V(x).
\]
With \( t^* > 0 \) chosen small enough, for all \( t \in (0, t^*) \), we have
\[
\mathcal{V}(t, V_i(x_i)) \geq \mathcal{V}(t, (i \sigma + \mu)^{-1} \circ \sigma_{\text{min}} \circ V(x)) \geq \frac{1}{2} (i \sigma + \mu)^{-1} \circ \sigma_{\text{min}} \circ V(x).
\]

Now, define
\[
K(r) := \left[ \frac{1}{2} (i \sigma + \mu)^{-1} \circ \sigma_{\text{min}}(r), \sigma_{\text{max}}(r) \right]
\]
and let \( c = c(K(r)) > 0 \) be the maximal constant such that
\[
|\sigma_i^{-1}(r_1) - \sigma_i^{-1}(r_2)| \geq c |r_1 - r_2| \quad \forall r_1, r_2 \in K(r).
\]

For all \( t \in (0, t^*) \), we obtain
\[
\frac{1}{t} (\mathcal{V}(\phi(t)) - V(x)) \leq -c(K(V(x))) \inf_{t \in I(x)} \frac{1}{t} \mathcal{V}(V_i(x_i)) - \mathcal{V}(t, V_i(x_i))
\]
\[
= -c(K(V(x))) \inf_{t \in I(x)} \frac{1}{t} \int_0^t \mathcal{\dot{\alpha}}(\mathcal{V}(s, V_i(x_i))) \, ds
\]
\[
\leq -c(K(V(x))) \frac{1}{t} \int_0^t \min_{s \in K(V(x))} \mathcal{\dot{\alpha}}(\mathcal{V}(s, \mathcal{g})) \, ds.
\]

Observe that the function
\[
s \mapsto \min_{s \in K(V(x))} \mathcal{\dot{\alpha}}(\mathcal{V}(s, \mathcal{g}))
\]
is continuous as \( \mathcal{V}(\cdot, \cdot) \) is continuous, and thus uniformly continuous on compact sets. Hence,
\[
\mathcal{D}^+ V_u(x) \leq -c(K(V(x))) \min_{s \in K(V(x))} \mathcal{\dot{\alpha}}(\mathcal{g}).
\]

Therefore, we have proved the implication
\[
V(x) > \gamma(\|u\|_d) \quad \Rightarrow \quad \mathcal{D}^+ V_u(x) \leq -\mathcal{\dot{\alpha}}(V(x))
\]
for all \( x \in X \) with
\[
\mathcal{\dot{\alpha}}(r) := c(K(r)) \min_{s \in K(r)} \mathcal{\dot{\alpha}}(\mathcal{g}) \quad \forall r > 0.
\]

**Step 6:** It remains to lower bound \( \mathcal{\dot{\alpha}} \) by a positive definite function. For each \( r > 0 \), define
\[
\mathcal{K}_2(r) := \bigcup_{q \in K(r)} \text{str}(1, q),
\]
where \( \text{str}(1, q) \) equals \( [1, q] \) for \( q \geq 1 \) and \( \text{str}(q, 1) \) for \( q < 1 \). Clearly, \( \mathcal{K}_2(r) \) is a compact subset of \((0, \infty)\). Furthermore, we introduce
\[
\mathcal{\dot{\alpha}}_2(r) := c(K(r)) \min_{q \in \mathcal{K}_2(r)} \mathcal{\dot{\alpha}}(r) \quad \forall r > 0.
\]

As \( K(r) \subset \mathcal{K}_2(r) \) for any \( r > 0 \), \( \mathcal{\dot{\alpha}}(r) \geq \mathcal{\dot{\alpha}}_2(r) \) for all \( r > 0 \). Furthermore, there is \( r_{\text{min}} \) such that \( \mathcal{K}_2(r_1) \supset \mathcal{K}_2(r_2) \) for all \( r_1, r_2 \in (0, r_{\text{min}}) \) with \( r_1 < r_2 \). This implies that \( \mathcal{\dot{\alpha}}_2 \) is a nondecreasing positive function on \((0, r_{\text{min}})\), and \( \lim_{r \to 0} \mathcal{\dot{\alpha}}_2(r) = 0 \). Moreover, for all \( r \in (0, r_{\text{min}}) \), we have
\[
\mathcal{\dot{\alpha}}_2(r) = \frac{2}{r} \int_{r/2}^{r} \mathcal{\dot{\alpha}}_2(s) \, ds \quad \forall r > 0.
\]

where \( \mathcal{\dot{\alpha}}_2 \) is integrable on \((0, r_{\text{min}})\) as it is monotone on this interval. Hence, \( \mathcal{\dot{\alpha}}_2 \) and thus \( \mathcal{\dot{\alpha}}_2 \) can be lower bounded by a continuous function on \((0, r_{\text{min}})\). Similarly, there is \( r_{\text{max}} \) such that \( \mathcal{K}_2(r_1) \supset \mathcal{K}_2(r_2) \) for all \( r_1, r_2 \in (r_{\text{max}}, \infty) \) with \( r_1 > r_2 \). This implies that \( \mathcal{\dot{\alpha}}_2 \) is a nonincreasing positive function on \((r_{\text{max}}, \infty)\). Consequently, for \( r \in (r_{\text{max}}, \infty) \), we have
\[
\mathcal{\dot{\alpha}}_2(r) = \frac{2}{r} \int_{r/2}^{r} \mathcal{\dot{\alpha}}_2(s) \, ds \quad \forall r > 0.
\]

Hence, \( \mathcal{\dot{\alpha}}_2 \), and thus \( \mathcal{\dot{\alpha}}_2 \), can be lower bounded by a continuous function on \((r_{\text{max}}, \infty)\). As \( \mathcal{\dot{\alpha}}_2 \) assumes positive values and is bounded away from zero on every compact interval in \((0, \infty)\), \( \mathcal{\dot{\alpha}}_2 \) can be lower bounded by a positive definite function, which we denote by \( \alpha \), on \( \mathbb{R}^+ \). This implies
\[
V(x) > \gamma(\|u\|_d) \quad \Rightarrow \quad \mathcal{D}^+ V_u(x) \leq -\alpha(V(x)) \quad \forall x \in X.
\]

Thus, \( V \) is an ISS Lyapunov function for \( \Sigma \). Proposition II.6 implies that \( \Sigma \) is ISS. The proof is complete.

**IV. Gain Operators and Their Properties**

A crucial assumption in our small-gain theorem is the existence of a path of strict decay for the operator \( \Gamma \). Our next goal is to understand under which conditions such a path exists, and to provide an explicit expression for it. We base our analysis on the properties of the gain operator, derived in this section.

From now on, we always assume that the family \( \{\gamma_{ij}\} \) is pointwise equicontinuous (see Assumption II.8), implying that \( \Gamma \) is well-defined and continuous. We assume throughout this section that for each \( i \in \mathbb{N} \), \( \gamma_{ij} \neq 0 \) only for finitely many \( j \in \mathbb{N} \), though many of the following results hold without this assumption.

The most important property of \( \Gamma \) is its monotonicity: for all \( s^1, s^2 \in \ell^+_{\infty} \), we have the implication
\[
s^1 \leq s^2 \quad \Rightarrow \quad \Gamma(s^1) \leq \Gamma(s^2).
\]

Moreover, we note that \( \Gamma(0) = 0 \) and that \( \Gamma \) is a max-preserving operator, i.e.,
\[
\Gamma(s^1 + s^2) = \Gamma(s^1) \oplus \Gamma(s^2) \quad \text{for all } s^1, s^2 \in \ell^+_{\infty}.
\]

Now, we recall the important robust and robust strong SGCs, introduced in [21], which are closely related to the stability properties of the discrete-time system induced by \( \Gamma \). We modify
these properties to make them more compatible with max-type gain operators.

**Definition IV.1:** We say that the operator $\Gamma$ satisfies the following.

1. The SGC if
   \[ \Gamma(s) \not\geq s \quad \text{for all } s \in \ell^+_{\infty} \setminus \{0\}. \]  
   (20)

2. The strong SGC if there is $\rho \in K_{\infty}$ with
   \[ D_\rho \circ \Gamma(s) \not\geq s \quad \text{for all } s \in \ell^+_{\infty} \setminus \{0\} \]  
   for the operator $D_\rho : \ell^+_{\infty} \to \ell^+_{\infty}$, defined by
   \[ D_\rho(s) := (\text{id} + \rho)(s_i) \quad i \in \mathbb{N}. \]  
   (21)

3. The max-robust SGC if there is $\omega \in K_{\infty}$ with $\omega < \text{id}$ such that for all $i, j \in \mathbb{N}$ the operator
   \[ \Gamma_{ij}(s) := \Gamma(s) \oplus \omega(s_j)e_i \quad \text{for all } s \in \ell^+_{\infty} \]  
   satisfies the SGC.

4. The max-robust strong SGC if there are $\omega \in K_{\infty}$ with $\omega < \text{id}$ and $\rho \in K_{\infty}$ such that for all $i, j \in \mathbb{N}$ the operator $\Gamma_{ij}$, defined in (22), satisfies the strong SGC with the same $\rho$ for all $i, j$.

In the following lemma, we introduce the so-called strong transitive closure (or Kleene star operator) $Q$ of the gain operator $\Gamma$, which provides the crucial tool for the construction of a path of strict decay. This result was first shown in [21, Lem. 4.3], strengthened in [20, Lem. B.5], and is now even more strengthened, since the robust SGC is replaced by the less restrictive max-robust SGC.

**Lemma IV.2:** Assume that $\Gamma$ satisfies the max-robust SGC with some $\omega \in K_{\infty}$. Then, the operator
\[ Q(s) := \bigoplus_{k \in \mathbb{Z}^+} \Gamma^k(s) \quad \text{for all } s \in \ell^+_{\infty} \]  
(23)

is well-defined and has the following properties:
\[ s \leq Q(s) \leq \omega^{-1}(\|s\|_{\ell_{\infty}})1 \quad \text{for all } s \in \ell^+_{\infty} \]  
(24)
\[ \Gamma(Q(s)) \leq Q(s) \quad \text{for all } s \in \ell^+_{\infty} \]  
(25)

**Proof:** The proof is only a slight variation of the proof of [20, Lem. B.5], Here, it suffices to show that the assumption $\sup_{k \in \mathbb{Z}^+} \Gamma^k(s) > \omega^{-1}(\|s\|_{\ell_{\infty}})1$ for some $i \in \mathbb{N}$ and $s \in \ell^+_{\infty}$ leads to a contradiction. From this assumption, the existence of $j \in \mathbb{N}$ and $j_1, \ldots, j_{k-1} \in \mathbb{N}$ follows such that
\[ \gamma_{i_{j_1}} \circ \gamma_{i_{j_2}} \circ \cdots \circ \gamma_{i_{j_{k-1}}} (s_j) > \omega^{-1}(\|s\|_{\ell_{\infty}}). \]

Now, consider the operator $\Gamma_{ji}$, as defined in (22)
\[ \Gamma_{ji}(s) = \left( \max_{k \in \mathbb{N}} \{ \sup_{k \in \mathbb{N}} \gamma_{i_{j_k}}(s_k), \omega(s_i)\delta_{ji} \} \right)_{i \in \mathbb{N}} \]
\[ = \left( \sup_{k \in \mathbb{N}} \gamma_{i_{j_k}}(s_k), \sup_{k \in \mathbb{N}} \omega(s_k)\delta_{ji}\delta_{ik} \right)_{i \in \mathbb{N}} \]
\[ = \left( \sup_{k \in \mathbb{N}} \gamma_{i_{j_k}}(s_k), \omega(s_k)\delta_{ji}\delta_{ik} \right)_{i \in \mathbb{N}} \].

Hence, $\Gamma_{ji}$ is a gain operator induced by the gains
\[ \hat{\gamma}_{i_{j_k}}(r) := \max_{k \in \mathbb{N}} \{ \gamma_{i_{j_k}}(r), \delta_{ji}\delta_{ik}\omega(r) \}. \]

Since $\Gamma_{ji}$ satisfies the SGC by assumption, by [20, Lem. B.3], all cycles built from the gains $\hat{\gamma}_{i_{j_k}}$ are contractions. This implies a contradiction, namely
\[ s_j > \hat{\gamma}_{i_{j_1}} \circ \hat{\gamma}_{i_{j_2}} \circ \cdots \circ \hat{\gamma}_{i_{j_{k-1}}} (s_j) \]
\[ > \gamma_{i_{j_1}} \circ \gamma_{i_{j_2}} \circ \cdots \circ \gamma_{i_{j_{k-1}}} (s_j) \]
\[ > 1 \quad \text{im} \quad Q = \{ s \in \ell^+_{\infty} : \Gamma(s) \leq s \} \]  
The rest of the proof is the same as that of [20, Lem. B.5]. □

Some further simple properties of the operator $Q$ are summarized in the following proposition.

**Proposition IV.3:** Assume that $\Gamma : \ell^+_{\infty} \to \ell^+_{\infty}$ is well-defined, continuous, and satisfies the max-robust SGC. Then, the operator $Q$ defined in (23) has the following properties.

1. $Q(0) = 0$ and $Q$ is a monotone operator.
2. The image of $Q$ is the set of all points of decay for $\Gamma$:
   \[ \text{im} \quad Q = \{ s \in \ell^+_{\infty} : \Gamma(s) \leq s \} \]
   This set is closed, contains $s = 0$, is cofinal (i.e., for any $x \in \ell^+_{\infty}$, there is $s \in \text{im} \quad Q$ with $x \leq s$) and forward-invariant with respect to $\Gamma$, i.e., $\Gamma(\text{im} \quad Q) \subset \text{im} \quad Q$.

3. $Q \circ Q = Q$.

**Proof:**

1. This immediately follows from the corresponding properties of $\Gamma$ and the definition of $Q$.

2. Lemma IV.2 implies that $\text{im} \quad Q \subset \{ s \in \ell^+_{\infty} : \Gamma(s) \leq s \}$. Conversely, $\Gamma(s) \leq s$ implies $\Gamma^k(s) \leq s$ for all $k \geq 0$, and hence, $Q(s) = s$ implying $s \in \text{im} \quad Q$. Since $\Gamma$ is continuous, it follows that $\text{im} \quad Q$ is closed. Since for each $s \in \ell^+_{\infty}$ it holds that $s \leq Q(s) \in \text{im} \quad Q$, the set $\text{im} \quad Q$ is cofinal. Since for any $s \in \text{im} \quad Q$ we have $\Gamma(s) \leq s$, by monotonicity of $\Gamma$ it follows that $\Gamma(\text{im} \quad Q) \subset \text{im} \quad Q$, showing forward-invariance of $\text{im} \quad Q$.

3. This follows immediately from the proof of property 2).□

For the gain operator $\Gamma$ and any $\theta \in K_{\infty}$, we define the operator $\Gamma_{\theta} : \ell^+_{\infty} \to \ell^+_{\infty}$ by
\[ \Gamma_{\theta}(s) := (\text{id} + \theta) \circ \Gamma(s) \quad \text{for all } s \in \ell^+_{\infty}. \]  
(26)

Here, we apply the function $\text{id} + \theta$ componentwise, i.e.,
\[ \Gamma_{\theta}(s) = \left( \sup_{i \in \mathbb{N}} (\text{id} + \theta) \circ \gamma_{i_{j}}(s_j) \right)_{i \in \mathbb{N}}. \]

Hence, the operator $\Gamma_{\theta}$ is structurally the same as $\Gamma$, but with scaled gain functions.

We close the section with a simple lemma on gain operators satisfying the max-robust strong SGC.

**Lemma IV.4:** Assume that $\Gamma$ satisfies the max-robust strong SGC with some $\rho, \omega \in K_{\infty}$. Then, $\Gamma_{\rho}$ satisfies the max-robust SGC with the same $\omega \in K_{\infty}$. Furthermore, there is $\theta \in K_{\infty}$ such that $\Gamma_{\theta}$ also satisfies the max-robust strong SGC.

**Proof:** Let $\Gamma$ satisfy the max-robust strong SGC with certain $\omega \in K_{\infty}$ and $\rho \in K_{\infty}$. By [27, Lem. 1.1.3], we can find $\rho_1, \rho_2 \in K_{\infty}$ such that $\text{id} + \rho = (\text{id} + \rho_1) \circ (\text{id} + \rho_2)$. Note that for any
\( s \in \ell^\infty_+ \) and \( i, j \in \mathbb{N} \), it holds that
\[
(id + \rho) \circ \Gamma_{ij}(s) = (id + \rho_1) \circ (id + \rho_2) \circ (\Gamma(s) \oplus \omega(s_j)e_i) \\
\geq (id + \rho_1) \circ ((id + \rho_2) \circ (\Gamma(s) \oplus \omega(s_j)e_i)).
\]
Since \((id + \rho) \circ \Gamma_{ij}(s) \geq s\) for all \(s \neq 0\), it also holds that
\[
(id + \rho_1) \circ (\Gamma_{ij}(s) \oplus \omega(s_j)e_i)(s) \geq s \quad \forall s \in \ell^\infty_+ \setminus \{0\}
\]
showing that \(\Gamma_{ij}\) satisfies the max-robust strong SGC. Similar arguments show that \(\Gamma_{ij}\) satisfies the max-robust SGC. \(\square\)

V. DISCRETE-TIMe SYSTEM INDUCED BY THE GAIN OPERATOR

In this section, we relate the properties of the gain operator \(\Gamma\) and its strong transitive closure \(Q\) to the stability properties of the discrete-time system induced by the gain operator \(\Gamma\):
\[
s(k + 1) = \Gamma(s(k)), \quad k \in \mathbb{Z}_+.
\]

As we will see, the stability properties of system (27) play an important role in the construction of paths of strict decay for \(\Gamma\), and hence for the construction of ISS Lyapunov functions for interconnected systems via the small-gain approach.

The following proposition characterizes the max-robust SGC in terms of the stability properties of the system (27).

Proposition VI (Criterion for max-robust SGC): Let \(\Gamma\) be well-defined and continuous. Then, \(\Gamma\) satisfies the max-robust SGC if and only if the following two properties hold.
1) The system (27) is uniformly globally stable (UGS), i.e., there is \(\sigma \in \mathcal{K}_\infty\) such that for any initial state \(s \in \ell^\infty_+\), the solution of (27) satisfies
\[
\|\Gamma^k(s)\|_{\infty} \leq \sigma(\|s\|_{\infty}) \quad \forall k \in \mathbb{Z}_+.
\]
(28)

2) Each trajectory of (27) converges to zero componentwise, i.e., \(\pi_i \circ \Gamma^k(s) \to 0 \) as \(k \to \infty\) for every \(s \in \ell^\infty_+\) and \(i \in \mathbb{N}\).

Proof: \("\Rightarrow\":\) We first show that our assumption implies the existence of \(\varphi \in \mathcal{K}_\infty\) satisfying
\[
s \leq \Gamma(s) \oplus b \quad \Rightarrow \quad \|s\|_{\infty} \leq \varphi(\|b\|_{\infty})
\]
(29)
for any \(s, b \in \ell^\infty_+\). To this end, assume that \(s \leq \Gamma(s) \oplus b\). Using that \(\Gamma\) is a max-preserving operator, inductively we obtain
\[
s \leq \Gamma^k(s) \oplus \bigoplus_{l=0}^{k-1} \Gamma^l(b) \quad \text{for all } k \geq 1.
\]
(30)

Looking at this inequality componentwise and letting \(k \to \infty\) yields \(s \leq \bigoplus_{l=0}^{\infty} \Gamma^l(b)\). Consequently, by the assumption of uniform global stability, there is \(\varphi \in \mathcal{K}_\infty\) such that
\[
\|s\|_{\infty} \leq \sup_{l \in \mathbb{Z}_+} \|\Gamma^l(b)\|_{\infty} \leq \varphi(\|b\|_{\infty}).
\]
Hence, the implication (29) holds. To complete the proof, assume that \(\Gamma(s) \oplus \omega(s_j)e_i \geq s\) for some \(i, j \in \mathbb{N}\), \(s \in \ell^\infty_+ \setminus \{0\}\) and \(\omega < \varphi^{-1}\). Then, \(\|s\|_{\infty} \leq \varphi(\omega(s_j)) < s_j \leq \|s\|_{\infty}\), a contradiction.

\("\Leftarrow\":\) By definition of \(Q\), we have \(\Gamma^k(s) \leq Q(s)\) for all \(s \in \ell^\infty_+\), and by monotonicity of the norm and using Lemma IV.2, we obtain that \(\|\Gamma^k(s)\|_{\infty} \leq \|Q(s)\|_{\infty} \leq \omega^{-1}(\|s\|_{\infty})\) for all \(s, k\). This shows the UGS property.

To show 2), consider the operator \(Q\) induced by \(\Gamma\) and let \(s \in \text{im} Q\). Then, \(\Gamma(s) \leq s\) implying \(\Gamma^{k+1}(s) \leq \Gamma^k(s)\) for all \(k \in \mathbb{Z}_+\). Hence, each of the sequences \((\Gamma_t^k(s))_{t \in \mathbb{N}}\), \(i \in \mathbb{N}\), is monotonically decreasing and bounded below by zero. So the following limits exist:
\[
s_i^* := \lim_{k \to \infty} \Gamma^k(s), \quad i \in \mathbb{N}.
\]
The vector \(s^* := (s_i^*)_{i \in \mathbb{N}}\) is an element of \(\ell^\infty_+\), since \(0 \leq s^* \leq s\). We claim that \(s^* = 0\). To prove this, let \(I_i\) be the finite set of \(j \in \mathbb{N}\) with \(\gamma_{ij} \neq 0\). Then, observe that for each \(i \in \mathbb{N}\)
\[
\Gamma_i(s^*) = \sup_{j \in \mathbb{N}} \gamma_{ij}(s^*_j) = \sup_{j \in \mathbb{N}} \gamma_{ij}(\lim_{k \to \infty} \Gamma^k(s))
\]
\[
(a) := \sup_{j \in \mathbb{N}} \lim_{k \to \infty} \gamma_{ij}(\Gamma^k_j(s)) = \max_{j \in J_i, k \to \infty} \gamma_{ij}(\Gamma^k_j(s))
\]
\[
(b) := \lim_{k \to \infty} \max_{j \in J_i} \gamma_{ij}(\Gamma^k_j(s)) = \lim_{k \to \infty} \Gamma_i(\Gamma^k(s))
\]
\[
= \lim_{k \to \infty} \Gamma^k+1(s) = s^*_i.
\]

The identity (a) holds because \(\gamma_{ij}\) is continuous and (b) holds because the maximum (over finitely many quantities) commutes with the limit operation. Hence, the pointwise limit \(s^*\) of the trajectory \((\Gamma^k(s))_{k \in \mathbb{Z}_+}\) is a fixed point of \(\Gamma\). Since \(\Gamma\) satisfies the SGC, this implies \(s^* = 0\). We thus completed the proof for the case that \(s \in \text{im} Q\). For any other \(s\), we have \(s \leq Q(s)\), and hence, \(0 \leq \Gamma(s) \leq \Gamma^k(Q(s))\) for all \(k \in \mathbb{Z}_+\), implying \(\Gamma^k(s) \to 0\) componentwise.

In the following proposition, we define several basic stability properties of (27) and show their equivalence.

Proposition V.2: Assume that \(\Gamma\) satisfies the max-robust SGC.

The following statements are equivalent.
1) System (27) is uniformly globally asymptotically stable (UGAS), i.e., there is \(\beta \in \mathcal{KL}\) such that for any initial condition \(s \in \ell^\infty\), the solution of (27) satisfies
\[
\|\Gamma^k(s)\|_{\infty} \leq \beta(\|s\|_{\infty}, k) \quad \forall k \in \mathbb{Z}_+.\n\]
(31)

2) System (27) is globally attractive, i.e., for all \(s \in \ell^\infty_+\), it holds that \(\Gamma(s) \to 0 \) as \(k \to \infty\).

3) System (27) is globally weakly attractive on im \(Q\), i.e., \(\inf_{k \geq 0} \|\Gamma^k(s)\|_{\infty} = 0 \quad \forall s \in \text{im} Q\).

Proof: Clearly, \(1 \Rightarrow 2 \Rightarrow 3\) holds. Let us show the implication \(3 \Rightarrow 1\).

Let (27) be globally weakly attractive on im \(Q\). For any \(r > 0\) and any \(s \in B_r(0)\), it holds that \(s \leq r \mathbb{I} \leq Q(r \mathbb{I})\) in im \(Q\). By monotonicity of \(\Gamma\), it holds that \(\Gamma^k(s) \leq \Gamma^k(Q(r \mathbb{I}))\), and thus, \(\|\Gamma^k(s)\|_{\infty} \leq \|\Gamma^k(Q(r \mathbb{I}))\|_{\infty} \quad \forall k \in \mathbb{Z}_+\). Hence, \(\inf_{k \geq 0} \sup_{s \in B_r(0)} \|\Gamma^k(s)\|_{\infty} = 0\) [the so-called uniform global weak attractivity of (27)]. Together with the UGS property, this implies UGAS (for continuous-time systems one can find this result, e.g., in [28, Th. 4.2]; the proof of the discrete-time version is completely analogous). \(\square\)

The next example shows that even if the gains \(\gamma_{ij}\) are all linear and \(\Gamma\) satisfies the strong as well as the max-robust SGC, the system induced by \(\Gamma\) is not necessarily UGAS.
Example V.3: Consider linear gains defined by
\[
\gamma_{k+1,k} := \delta_k \frac{k}{k+1} \quad \text{if } k \in \{2^s : s \in \mathbb{N}\}
\]
and \(\gamma_{ij} = 0\) whenever \(i \neq j + 1\). Since \(\gamma_{ij} \leq 1\) for all \(k\), the gain operator \(\Gamma\) is well-defined. Furthermore, we have
\[
\left( \Gamma^{2^k-1} (I) \right)_{2^k} = \gamma_{2^k,2^k-1} \cdots \gamma_{2^k+2,2^k+1} \geq \frac{2^{k-1} + 1}{2^k} \geq \frac{1}{2}
\]
showing that \(\Gamma^k(I) \not\to 0\) as \(k \to \infty\), and thus, the discrete-time system (27) induced by \(\Gamma\) is not UGAS. Now, [25, Prop. 8] implies, in particular, that \(\Gamma\) does not satisfy the so-called robust strong SGC with linear \(\rho\) and \(\omega\). On the other hand,

1) \(\Gamma\) satisfies the strong SGC. Assume to the contrary that there is \(s \in \ell_\infty^2 \setminus \{0\}\) such that \(\Gamma r \geq (1-\varepsilon)s\) for a fixed but arbitrary \(\varepsilon \in (0,1)\). Then, we get \(Rs \geq s \geq (1-\varepsilon)s\), where \(R\) is the right-shift operator from [24, Ex. 3.15]. However, in view of [24, Ex. 3.15], \(R\) satisfies the strong SGC with any \(\varepsilon \in (0,1)\), and we come to a contradiction.

2) \(\Gamma\) satisfies the max-robust SGC. Pick any \(i,j \in \mathbb{N}\) and perturb the \(ij\)-component of \(\Gamma\) by the linear function \(\omega(r) = \frac{r}{2}\). The resulting operator \(\Gamma_{ij}\) is a block-diagonal operator with finite-dimensional blocks, and all its finite cycles are contractions. Thus, by [20, Prop. B.4], \(\Gamma_{ij}\) satisfies the SGC (here, we use the fact that the robust SGC used in [20] implies the max-robust SGC).

We thus arrive at the following relations.

\[
\begin{align*}
\text{UGS} \land \text{global attractivity (GATT)} & \iff \text{UGAS} \\
\text{UGS} \land \text{componentwise GATT} & \iff \text{max-robust SGC}
\end{align*}
\]

Since we need UGAS for the construction of a path of strict decay, we need to understand what is required in addition to the max-robust SGC to obtain UGAS.

It is well-known that for finite networks the max-preserving gain operator \(\Gamma\) induces a UGAS discrete-time system if and only if all cycles composed of gains are contractions, see, e.g., [29, Th. 6.4]. In the case of infinite networks, UGAS of the induced system can be characterized in terms of sufficiently long chains of gains, as shown in the following proposition.

**Proposition V.4:** Assume that the gain operator \(\Gamma\) is well-defined, continuous, and satisfies the max-robust SGC. Then, the following are equivalent.

1) The induced system (27) is UGAS.

2) There exists \(\eta \in K\) with \(\eta < \text{id}\) such that for every \(r \geq 0\) there is \(n \in \mathbb{N}\) with
\[
\sup_{j_0,j_1,\ldots,j_n \in \mathbb{N}} \gamma_{j_0j_1} \cdots \gamma_{j_{n-1}j_n} (r) \leq \eta(r).
\]

3) There exist \(\eta \in K\) with \(\eta < \text{id}\) and \(i_0 \in \mathbb{N}\) such that for every \(r \geq 0\) there is \(n \in \mathbb{N}\) with
\[
\sup_{j_0,j_1,\ldots,j_n \in \mathbb{N}} \gamma_{j_0j_1} \cdots \gamma_{j_{n-1}j_n} (r) \leq \eta(r).
\]

**Proof:**

1) \(\Rightarrow\) 2: Assume that (27) is UGAS. Then, there exists \(\beta \in K\) such that \(\|\Gamma^r(s)\|_{\ell_\infty} \leq \beta(\|s\|_{\ell_\infty}, k)\) for all \(s \in \ell_\infty^2, k \in \mathbb{Z}_+\). We put \(\eta(r) := r/2\) for all \(r \in \mathbb{R}_+\). For a given \(r > 0\), choose \(n\) so large that \(\beta(r,n) \leq r/2\). This implies
\[
\sup_{j_0,j_1,\ldots,j_n \in \mathbb{N}} \gamma_{j_0j_1} \cdots \gamma_{j_{n-1}j_n} (r) = \sup_{j_0 \in \mathbb{N}} \Gamma^n_{j_0} (r \mathbb{I})
\]
\[
\leq \sup_{j_0 \in \mathbb{N}} \sup_{j_0,j_1,\ldots,j_n \in \mathbb{N}} \gamma_{j_0j_1} \cdots \gamma_{j_{n-1}j_n} (r) \leq \eta(r).
\]

Hence, (32) holds.

2) \(\Rightarrow\) 3: This is obvious.

3) \(\Rightarrow\) 1: By Proposition V.2, it suffices to prove global attractivity on im \(Q\). To this end, fix \(s \in \text{im } Q\) and put \(r_1 := \|s\|_{\ell_\infty}\). By assumption, there exists \(\tilde{n}_1 \in \mathbb{N}\) with
\[
\sup_{j_0 \in \mathbb{N}} \sup_{j_1,\ldots,j_n \in \mathbb{N}} \gamma_{j_0j_1} \cdots \gamma_{j_{n-1}j_n} (r_1) \leq \eta(r_1).
\]

This implies
\[
\Gamma_{i_1} \cdots \Gamma_{i_n} (s) \leq \eta(r_1) \quad \text{for all } i \geq i_0.
\]

By Proposition V.1, we further find \(\tilde{n}_1 \in \mathbb{N}\) with
\[
\Gamma_{i_1} \cdots \Gamma_{i_n} (s) \leq \eta(r_1) \quad \text{for } 1 \leq i < i_0.
\]

Now, put \(n_1 := \max\{\hat{n}_1, \tilde{n}_1\}\). As \(s \in \text{im } Q\), we obtain that
\[
\Gamma^{n_1}(s) \leq \eta(r_1) \quad \text{and } \Gamma^{n_1}(s) \leq \eta(r_1).
\]

Thus,
\[
\|\Gamma^{n_1}(s)\|_{\ell_\infty} = \sup_{i \in \mathbb{N}} \Gamma^{n_1}(s) = \max_{1 \leq i < i_0} \sup_{i \in \mathbb{N}} \Gamma^{n_1}(s) \leq \eta(r_1).
\]

Now, we put \(r_2 := \eta(r_1)\). Proceeding in the same way, we find \(n_2 \in \mathbb{N}\) with
\[
\|\Gamma^{n_1+n_2}(s)\|_{\ell_\infty} \leq \eta(r_2) = \eta^2(r_1).
\]

Inductively, we find a sequence \(N_k \to \infty\) such that
\[
\|\Gamma^{N_k}(s)\|_{\ell_\infty} \leq \eta^k(r_1) \quad \text{for all } k \geq 0.
\]

Since \(\eta < \text{id}\) and \(\eta \in K\), it follows that \(\eta^k(r_1) \to 0\) (the only fixed point of \(\eta\)) as \(k \to \infty\). This shows that 3) in Proposition V.2 is satisfied.

The UGAS property of (27) implies important approximation and continuity properties of the operator \(Q\).

**Proposition V.5:** Assume that \(\Gamma : \ell_\infty^2 \to \ell_\infty^2\) is well-defined and continuous, and system (27) is UGAS. Then, the following statements hold.
Using the simple representation of the iterates $\Gamma^k$ in terms of gains (see, e.g., [20, Lem. 12]), we have

$$
\|Q(s) - Q(\tilde{s})\|_{\ell_\infty} \leq \sup_{i \in \mathbb{N}} \left( \max_{k=0}^{m} \sup_{j_1, \ldots, j_k} \gamma_{ij_1} \circ \cdots \circ \gamma_{ij_k-1,j_k}(s_{j_k}) \right) - \sup_{k=0}^{m} \sup_{j_1, \ldots, j_k} \gamma_{ij_1} \circ \cdots \circ \gamma_{ij_k-1,j_k}(\tilde{s}_{j_k}) + \frac{\varepsilon}{2}.
$$

Since $\{\gamma_{ij} : i, j \in \mathbb{N}\}$ is pointwise equicontinuous by assumption, the same is true for the family $\{\gamma_{ij_1} \circ \cdots \circ \gamma_{ij_k-1,j_k} : i, j_1, \ldots, j_k \in \mathbb{N}, 0 \leq k \leq m\}$. Hence, this family is uniformly equicontinuous on compact intervals, and thus, there exists $\rho \in (0, \delta)$ such that $\|s - \tilde{s}\|_{\ell_\infty} \leq \rho$ implies

$$
\|\gamma_{ij_1} \circ \cdots \circ \gamma_{ij_k-1,j_k}(s_{j_k}) - \gamma_{ij_1} \circ \cdots \circ \gamma_{ij_k-1,j_k}(\tilde{s}_{j_k})\|_{\ell_\infty} \leq \frac{\varepsilon}{2}
$$

for all $i, j_1, \ldots, j_k \in \mathbb{N}$ and $0 \leq k \leq m$. This, in turn, implies $\|Q(s) - Q(\tilde{s})\|_{\ell_\infty} \leq \varepsilon$. We have thus proved continuity of $Q$ at $s$, and since any $s \in \ell_+^\infty$ satisfies $s \leq q$ for some $q \in \text{int}(\ell_+^\infty)$, the proof is complete. \hfill \Box

VI. CONSTRUCTION OF PATHS OF STRICT DECAY

We can finally present our main result on the existence and construction of paths of strict decay. It extends the first result of this kind in [21, Lem. 4.5], where properties (i)-(iii) of a path of strict decay have been shown under similar assumptions. See Section VIII for an extended discussion of this issue.

Theorem VI.1: Let the following assumptions hold.

1) There exists $\theta \in \mathcal{K}_\infty$ such that the system induced by

$$
\Gamma_\theta = (\id + \theta) \circ \Gamma
$$

is UGAS.

2) For each compact interval $K \subset (0, \infty)$, there are $0 < l \leq L < \infty$ with $l(r_2 - r_1) \leq \gamma_{ij}(r_2) - \gamma_{ij}(r_1) \leq L(r_2 - r_1)$ for all nonzero $\gamma_{ij}$ and $r_1 < r_2 \in K$.

Then, there exists a path of strict decay $\sigma : \mathbb{R}_+ \to \ell_+^\infty$ for $\Gamma$.

Proof: First, we fix $\theta \in \mathcal{K}_\infty$ such that the system induced by $\Gamma_\theta$ is UGAS. We also put $\gamma_{ij}^\theta := (\id + \theta) \circ \gamma_{ij}$ for all $i, j \in \mathbb{N}$, and define $\Gamma(\sigma(r)) := \Gamma_\theta(\sigma(r))$ for all $r \in \mathbb{R}_+$, where $\Gamma_\theta(\sigma) := \bigoplus_{k \in \mathbb{Z}_+} \Gamma^k_{\theta}(\sigma)$. Then, we can verify all properties of a path of strict decay for $\sigma$:

Since $\Gamma_\theta(\sigma(\mathbb{R}_+)) \leq \Gamma_\theta(\mathbb{R}_+)$ by Lemma IV.2, we obtain $\Gamma(\sigma(r)) \leq (\id + \theta)^{-1} \sigma(r)$ for all $r \in \mathbb{R}_+$. Hence, $\sigma$ satisfies property (i) of a path of strict decay.

Property (ii) of a path of strict decay holds with $\sigma_{\min} = \id$, since $\sigma(r) \geq r_1$, and $\sigma_{\max} = \omega^{-1}$ by Lemma IV.2.

From property (ii), we can conclude that $\sigma_t(0) = 0, \sigma_t(r) > 0$ for all $r > 0$, and $\sigma_t(r) \to \infty$ as $r \to \infty$. As $0$ is a globally attractive fixed point for $\Gamma_\theta, Q_\theta$ is continuous on $\ell_+^\infty$ by Proposition V.5. Thus, all $\sigma_t$ are continuous as well. Furthermore, for $r_1, r_2 \in (0, \infty)$ with $r_1 < r_2$, we obtain by Proposition V.52, that

$$
\sigma_t(r) = \max_{0 \leq k \leq r_0} \pi_t \circ \Gamma^k_{\theta}(r) \text{ for all } r \in [r_1, r_2].
$$

By our assumption that for each $i$ only finitely many $\gamma_{ij}$ are nonzero, the supremum in

$$
\sigma_t(r) = \max_{0 \leq k \leq r_0} \sup_{i} \gamma_{ij_1}^\theta \circ \cdots \circ \gamma_{ij_k-1,j_k}^\theta(r)
$$
is in fact a supremum over finitely many strictly increasing functions (since we can ignore all chains which contain a zero function). This implies that $\sigma_i$ is also strictly increasing on $[r_1, r_2]$, and hence everywhere. It follows that all $\sigma_i$ are $K_{\infty}$-functions [property (iii) of a path of strict decay].

We complete the proof by verifying property (iv) of a path of strict decay. It suffices to prove the statement for $\sigma_i$ in place of $\sigma_i^{-1}$. Indeed, assume that for every compact interval $L \subset (0, \infty)$, we have constants $\hat{c}, \hat{C} > 0$ satisfying

$$\hat{C}|r_1 - r_2| \leq |\sigma_i(r_1) - \sigma_i(r_2)| \leq \hat{C}|r_1 - r_2| \quad \forall r_1, r_2 \in L.$$

If $K = [a, b] \subset (0, \infty)$, then $\sigma_i^{-1}(K) = [\sigma_i^{-1}(a), \sigma_i^{-1}(b)]$ which is a subset of $[\sigma_{\max_i}(a), \sigma_{\min_i}(b)] =: L \subset (0, \infty)$. Hence, the above estimates imply

$$\frac{1}{\hat{C}}|r_1 - r_2| \leq |\sigma_i^{-1}(r_1) - \sigma_i^{-1}(r_2)| \leq \frac{1}{\hat{C}}|r_1 - r_2| \quad \forall r_1, r_2 \in K.$$

To verify the statement for $\sigma_i$, we first prove the following claim: If $K = [a, b] \subset (0, \infty)$ is a compact interval, then there exists another compact interval $[c, d] \subset (0, \infty)$ such that $\gamma_i^\theta(K) \subset [c, d]$ for all nonzero $\gamma_i$. By uniform equicontinuity of the functions $\{\gamma_i^\theta\}$ on compact intervals, we know that $\gamma_i^\theta(b)$ is uniformly bounded from above by some $d > 0$. Now pick $\rho > 0$ such that $a - \rho > 0$. Then, by assumption, there is $l > 0$ with $\gamma_i^\theta(a) - \gamma_i^\theta(a - \rho) \geq l \rho$ for all $i, j \in \mathbb{N}$ such that $\gamma_{ij}^\theta \neq 0$. Hence, $\gamma_i^\theta(a) \geq (1 + \theta)\rho$ whenever $\gamma_i^\theta \neq 0$. This implies $\gamma_i^\theta(K) \subset [c, d]$ with $c := (1 + \theta)\rho$.

Now fix a compact interval $K \subset (0, \infty)$, $k \subset \mathbb{N}$, and consider all chains of the form

$$c_{j_1, \ldots, j_k} := \gamma_{j_1, j_2} \circ \gamma_{j_2, j_3} \circ \cdots \circ \gamma_{j_k, j_1},$$

which are built from nonzero gains. From the claim and our assumptions, it then follows that

$$l_1 l_2 \cdots l_k |r_1 - r_2| \leq |c_{j_1, \ldots, j_k}(r_1) - c_{j_1, \ldots, j_k}(r_2)| \leq L_1 l_2 \cdots L_k |r_1 - r_2|$$

for certain positive numbers $l_i, L_i > 0$, $i = 1, \ldots, k$, and all $r_1, r_2 \in K$. The same Lipschitz condition then holds for the functions $r \mapsto \pi_i \circ \Gamma_i^\theta(r) = \sup_{j_1, \ldots, j_k} c_{j_1, \ldots, j_k}(r)$, where for the lower bound we need to require that at least one nonzero chain $c_{j_1, \ldots, j_k}$ exists. By what we have shown above, on every compact interval $K \subset (0, \infty)$, $\sigma_i$ can be written as the maximum over finitely many of such functions:

$$\sigma_i(r) = \max_{0 \leq k < k_0} c_{j_1, \ldots, j_k}(r) \quad \text{for all } r \in K. \quad (35)$$

With $C := \max\{1, L_1 L_2, \ldots, L_1 \cdots L_k\}$ (keeping in mind that $\pi_i \circ \Gamma_i^\theta(r) = \Gamma_i^\theta(r)$), the following holds:

$$|\sigma_i(r_1) - \sigma_i(r_2)| \leq C|r_1 - r_2| \quad \text{for all } r_1, r_2 \in K.$$

For the lower bound, we put $\hat{c} := \min\{1, l_1, l_1 l_2, \ldots, l_1 l_2 \cdots l_k\}$ and apply Lemma IX.2. Observe that in taking the supremum, we can ignore all functions that are identically zero. Considering $k = 0$, we see that at least one nonzero function is involved in taking the supremum, namely, the identity.

**Remark VI.2:** Properties (i)–(iii) of a path of strict decay actually hold for the mapping $\sigma(r) = Q_0(r) \mathbb{I}$ under the assumption that $\Gamma_0$ satisfies the max-robust SGC only, since this already implies that $\pi_i \circ \Gamma_i^\theta(\mathbb{I} r) \to 0$ as $k \to \infty$ for every $i \in \mathbb{N}$, which is enough to write that $\sigma_i$ locally as the maximum over finitely many continuous and strictly increasing functions. The stronger assumption of UGAS is only needed to verify the uniform local Lipschitz condition (iv), which requires that we can locally write $\sigma_i(r) = \max_{0 \leq k < k_0} \pi_i \circ \Gamma_i^\theta(\mathbb{I} r)$ with $k_0$ being independent of $i$.

**Remark VI.3:** Assumption 2) in Theorem VI.1 is, in general, unnecessarily strong. This can be seen best by looking at the case when all gains $\gamma_{ij}$ are linear functions. Then, the maximum in (35) is taken over linear functions, and since one of them is the identity, only those linear functions with slope $> 1$ need to be taken into account. However, then it is clear that no uniform lower bound on the slopes of the individual $\gamma_{ij}$ is necessary to obtain a uniform lower bound on the Lipschitz constants of the $\sigma_i$. In general, the situation is more complicated, and detailed information about the gains and their compositions is necessary to relax assumption 2).

## VII. SUFFICIENT CONDITIONS FOR UGAS

Since UGAS of the discrete-time system, induced by the scaled gain operator, is a key requirement for the existence of a path of strict decay (and thus, for the application of the small-gain theorem), in this section, we analyze sufficient conditions for UGAS of the system induced by the gain operator $\Gamma$. The following proposition describes a way of reducing the proof of UGAS of (27) to finitely many computations.

**Proposition VII.1:** Assume that there exists a positive integer $N$ and a map $p : \mathbb{N} \to \{1, \ldots, N\}$ as well as a family $\{\gamma_{ij} : i, j = 1, \ldots, N\} \subset K \cup \{0\}$ of virtual gains such that

$$\gamma_{ij} \leq \tilde{\gamma}_{ij}(p)(\mathbb{I} s) \quad \text{for all } i, j \in \mathbb{N}.$$

Let $\tilde{\Gamma} : \mathbb{R}_+^N \to \mathbb{R}_+^N, s \mapsto (\sup_{1 \leq i \leq N} \tilde{\gamma}_{ij}(p)(s))_{1 \leq i \leq N}$ be the associated virtual gain operator. If $\tilde{\Gamma}$ satisfies the SGC, $\tilde{\Gamma}(s) \geq s$ for all $s \in \mathbb{R}_+^N \setminus \{0\}$, the system (27) induced by $\Gamma$ is UGAS.

**Proof:** For any $s \in \mathbb{R}_+^N$ and $i \in \mathbb{N}$, we have with a convention $f_i := i:

$$\Gamma_i^k(s) = \sup_{j_1, \ldots, j_k \in \mathbb{N}} \gamma_{ij_1} \circ \cdots \circ \gamma_{j_{k-1}, j_k}(s_{j_k}) \leq \sup_{j_1, \ldots, j_k \in \mathbb{N}} \tilde{\gamma}_{ij_1}(p)(j_1) \circ \cdots \circ \tilde{\gamma}_{j_{k-1}, j_k}(p)(j_k)(\|s\|_{\mathbb{I} s}) \leq \sup_{1 \leq j_1, \ldots, j_k \leq N} \tilde{\gamma}_{ij_1}(p)(j_1) \circ \cdots \circ \tilde{\gamma}_{j_{k-1}, j_k}(p)(j_k)(\|s\|_{\mathbb{I} s}) = \Gamma_i^k(p)(\|s\|_{\mathbb{I} s}),$$

where $\mathbb{I} = (1, \ldots, 1) \in \mathbb{R}_+^N$ in the last line. By [29, Th. 6.4], the assumption that $\Gamma$ satisfies the SGC implies that the $L_\infty$-norm in $\mathbb{R}_+^N$ of $\Gamma_i^k(\|s\|_{\mathbb{I} s})$ is bounded by $\beta(\|s\|_{\mathbb{I} s}, k)$ for some $K\mathcal{L}$-function $\beta$ (depending on $\Gamma$ only). Hence, the system induced by $\Gamma$ is UGAS. □

Another method of checking UGAS of (27) via the introduction of virtual gains, based on a compactification of the index set $\mathbb{N}$, is described in the following proposition.
Proposition VII.2: Let $N^* := N \cup \{\infty\}$ and assume that there exist virtual gains $\overline{\gamma}_{ij} \in K \cup \{0\}$, $i, j \in N^*$ (where $\overline{\gamma}_{i\infty} \neq 0$ is allowed), satisfying the following assumptions.
1. $\overline{\gamma}_{ij} = \gamma_{ij}$ whenever $(i, j) \in N \times N$.
2. The virtual gain operator
   \[
   \Gamma : \ell^+_\infty(N^*) \to \ell^+_\infty(N^*), \quad s \mapsto \left(\sup_{j \in N^*} \overline{\gamma}_{ij}(s_j)\right)_{i \in N^*},
   \]
   is well-defined, continuous, and satisfies the max-robust SGC with some $\omega \in K_\infty$.
3. For each $i \in N^*$, $\overline{\gamma}_{ij} \neq 0$ only for finitely many $j \in N^*$.
4. There exists $k_0 \in N$ such that for all $r > 0$
   \[
   \limsup_{i \to \infty} \sup_{j_1, \ldots, j_{k_0} \in N^*} \overline{\gamma}_{ij_1} \circ \cdots \circ \overline{\gamma}_{j_{k_0-1}j_{k_0}} \circ \omega^{-1}(r) \leq \sup_{j_1, \ldots, j_{k_0} \in N^*} \overline{\gamma}_{i\infty j_1} \circ \cdots \circ \overline{\gamma}_{j_{k_0-1}j_{k_0}}(r). \tag{36}
   \]

Then, the system (27) induced by $\Gamma$ is UGAS.

Proof: First, we turn $N^*$ into a compact metrizable space by declaring $\infty$ to be an accumulation point and all other elements isolated points. Now, let $s = \overline{Q}(r \cdot \mathbb{I})$ for some $r > 0$, where $\overline{Q}(s) = \sup_{k \geq 0} \overline{\Gamma}^k(s)$. Then, by Lemma IV.2, $\overline{\Gamma}(s) \leq s$ and $r \mathbb{I} \leq s \leq \omega^{-1}(r) \mathbb{I}$. For every $k \in \mathbb{Z}_+$, we define
\[
 f_k(i) \defeq \overline{\Gamma}^k_i(s), \quad f_k : N_* \to \mathbb{R}._+
\]
By the choice of $s$, the functions $f_k$ form a decreasing sequence, i.e., $f_{k+1} \leq f_k$ for all $k \in \mathbb{Z}_+$. We claim that the function $f_k$ is upper semicontinuous. Since $\infty$ is the only accumulation point of $N_*$, this is equivalent to
\[
 \limsup_{i \to \infty} f_i^{k_0}(s) \leq f_i^{k_0}(s).
\]
In terms of the gains $\overline{\gamma}_{ij}$, this can be written as
\[
 \limsup_{i \to \infty} \sup_{j_1, \ldots, j_{k_0} \in N^*} \overline{\gamma}_{ij_1} \circ \cdots \circ \overline{\gamma}_{j_{k_0-1}j_{k_0}}(s_{j_{k_0}}) \leq \sup_{j_1, \ldots, j_{k_0} \in N^*} \overline{\gamma}_{i\infty j_1} \circ \cdots \circ \overline{\gamma}_{j_{k_0-1}j_{k_0}}(s_{j_{k_0}}).
\]
From $r \mathbb{I} \leq s \leq \omega^{-1}(r) \mathbb{I}$, it follows that this inequality is implied by our assumption (36). Hence, $f_k$ is upper semicontinuous. It now easily follows that also $f_{k_0}$ is upper semicontinuous for every $k \in \mathbb{N}$. Consequently, $(f_{k_0})_{k \in \mathbb{Z}_+}$ is a decreasing sequence of upper semicontinuous functions on a compact metric space. Hence, Lemma IX.3 implies
\[
 \lim_{k \to \infty} \|f_{k_0}(s)\|_{\infty} = \inf_{k \in \mathbb{Z}_+} \sup_{i \in \mathbb{N}_*} f_{k_0}(i) = \sup_{i \in \mathbb{N}_*} \inf_{k \in \mathbb{Z}_+} f_{k_0}(i) = \sup_{i \in \mathbb{N}_*} \inf_{k \to \infty} f_{k_0}(i) = 0
\]
where the last equality follows from Proposition V.1. This clearly implies $\overline{\Gamma}^k(s) \to 0$ for $k \to \infty$. Now, it is easy to see that the same property holds for the original gain operator $\Gamma$. The proof is completed by applying Proposition V.2.

Remark VII.3: To get a better understanding of condition (36), consider the case $k_0 = 1$:
\[
 \limsup_{i \to \infty} \sup_{j \in N^*} \overline{\gamma}_{ij} \circ \omega^{-1}(r) \leq \sup_{j \in N^*} \overline{\gamma}_{i\infty j}(r).
\]
This implies that for all sufficiently large $i$, up to some small error, one gets $\overline{\gamma}_{ij} \circ \omega^{-1} \leq \gamma_{i\infty j}$ for some $j^*$ and all $j \in N^*$. However, from the proof of Lemma IV.2, we know that $\gamma_{i\infty j} < \omega^{-1}$ as a consequence of the max-robust SGC. Hence,
\[
 \overline{\gamma}_{ij} \leq \gamma_{i\infty j} \circ \omega < \omega^{-1} \circ \omega = \text{id}.
\]
Thus, asymptotically, the gains become contractions. If $k_0 > 1$, this is true for all chains of $k_0$ gains.

Example VII.4: Consider a cascade network built from subsystems
\[
 \Sigma_1 : \quad \dot{x}_1 = f_1(x_1, u_1),
\]
\[
 \Sigma_i : \quad \dot{x}_i = f_i(x_i, x_{i-1}, u_i), \quad i \geq 2.
\]
We assume that ISS Lyapunov functions $V_i$ for $\Sigma_i$ exist with associated interconnection gains $\gamma_i(i-1)$, which form an equicontinuous family. Then, the gain operator $\Gamma$ is well-defined and continuous. Let us assume that $\Gamma$ satisfies the max-robust SGC with some $\omega \in K_\infty$, $\omega \leq \text{id}$. We add a virtual gain $\overline{\gamma}_{i\infty}(s) \in K \cup \{0\}$. It is then easy to see that with $\gamma_i(i-1) \defeq \gamma_i(i-1)$ for all $i \in N$, $i \geq 2$, the virtual gain operator $\Gamma$ satisfies the max-robust SGC with the same $\omega$ if and only if
\[
 \overline{\gamma}_{i\infty}(r) < r \quad \text{for all } r > 0.
\]
Additionally, the asymptotic condition (iv) in Proposition VII.2 for $k_0 = 1$ requires that
\[
 \limsup_{i \to \infty} \gamma_i(i-1) \circ \omega^{-1}(r) \leq \overline{\gamma}_{i\infty}(r) \quad \text{for all } r > 0.
\]
This means that the gains $\gamma_i(i-1)$ asymptotically have to become smaller than the composition of the two contractions $\overline{\gamma}_{i\infty}$ and $\omega$.

The following proposition characterizes an interesting special case of Proposition VII.2.

Proposition VII.5: Assume that the virtual gains in Proposition VII.2 are chosen as $\gamma_{ij} := 0$ if $i = \infty$ or $j = \infty$. Furthermore, assume that $\Gamma$ is well-defined, continuous and satisfies the max-robust SGC. Then, the same is true for the virtual gain operator $\overline{\Gamma}$. Moreover, the following implications hold.
1. If (36) holds, then the operator $\overline{\Gamma}^{k_0}$ is compact, i.e., the image of any bounded set under $\overline{\Gamma}^{k_0}$ is relatively compact.
2. If the operator $\overline{\Gamma}^{k_0}$ is compact and every subsystem of $\Sigma$ can only influence finitely many other subsystems, then condition (36) is satisfied.

Proof: The proof is subdivided into three steps.

Step 1: The proof that $\Gamma$ is well-defined or continuous if $\Gamma$ has the corresponding property is trivial, and hence, we omit it. To verify that $\Gamma$ satisfies the max-robust SGC if $\Gamma$ does, we need to verify that for all $i, j \in N^*$, the operator $\Gamma_i(s) = \Gamma_i(s) \circ \omega(s_j)$ satisfies $\Gamma_i(s) \geq s$ for all $s \in \ell^\infty_\infty(N^*) \setminus \{0\}$. To this end, we fix $s = (s_i)_{i \in \mathbb{N}^*}$ and distinguish several cases.
1. $i, j < \infty$ and $s_k > 0$ for some $k \in N$. Then, with $s := (s_i)_{i \in \mathbb{N}}$, the claim follows directly from $\Gamma(s) \geq s$.
2. $i, j < \infty$ and $s_k = 0$ for all $k \in N$. Then, $s_{\infty} > 0$ and thus, $\overline{\Gamma}(s) \geq s$, showing that $\Gamma(s) \geq s$.
3. $i = \infty$ or $j = \infty$ and $s_k > 0$ for some $k \in N$. Then, with $s := (s_i)_{i \in \mathbb{N}}$, it follows that $\Gamma(s) \not\geq s$, and consequently, $\Gamma_i(s) \not\geq s$. 

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4) \((i = \infty, j < \infty)\) or \((i < \infty, j = \infty)\) and \(s_k = 0\) for all \(k \in \mathbb{N}\). In both cases, \(\varpi_\infty \circ \varpi_{ij}(s) = 0 < s_\infty\).

5) \(i, j = \infty\) and \(s_k = 0\) for all \(k \in \mathbb{N}\). Then, \(\omega < \text{id}\) implies

\[
\varpi_\infty(s) \oplus \omega(s_\infty) = \omega(s_\infty) < s_\infty.
\]

Step 2: We prove statement 1). Without loss of generality, we assume that \(k_0 = 1\). Condition (36) can then be reformulated as follows: for each \(r > 0\) and \(\varepsilon > 0\), there is \(i_0 = i_0(r, \varepsilon) \in \mathbb{N}\) such that for all \(i \geq i_0\) and \(j \in \mathbb{N}\) we have \(\gamma_{ij}(r) \leq \varepsilon\).\n
Now, let \(B \subset I^N\) be a bounded set, say \(||s||_\varepsilon \leq b\) for all \(s \in B\). We have to show that \(\varpi(B)\) is relatively compact. By [30, Th. 6, p. 260], this is equivalent to the following: \(\varpi(B)\) is bounded, and for every \(\varepsilon > 0\), there is a finite partition \(\mathbb{N} = J_1 \cup \ldots \cup J_N\) such that for every \(s \in \varpi(B)\) we have \(|s_i - s_j| \leq \varepsilon\) whenever \(i, j \in J_k\) for some \(k\). Boundedness of \(\varpi(B)\) is a consequence of the equicontinuity of \(\{\gamma_{ij}\}\). If we choose \(i_0 = i_0(b, \varepsilon)\) as above, we can put \(J_k := \{i\}\) for \(i = 1, \ldots, \ell - 1\) and \(J_0 := \{i \in \mathbb{N} : i \geq i_0\}\). For any \(s = \varpi(t) \in \varpi(B)\), we have

\[
s_{i+1} - s_i = \sup_{j \in \mathbb{N}} \gamma_{ij}(t_j) - \sup_{j \in \mathbb{N}} \gamma_{ij}(t_j) \leq \sup_{j \in \mathbb{N}} \gamma_{ij}(b) \leq \varepsilon
\]

whenever \(i_1 \geq i_0\). Interchanging the roles of \(i_1\) and \(i_2\) yields \(|s_i - s_{i+1}| \leq \varepsilon\) whenever \(i_1, i_2 \in J_k\). Hence, \(\varpi\) is compact.

Step 3: We prove statement 2) for \(k_0 = 1\). To this end, let us assume that condition (36) does not hold. Then, there exists \(r > 0\) such that

\[
\limsup_{i \to \infty} \sup_{j \in \mathbb{N}} \gamma_{ij}(r) > 0.
\]

Hence, there exist \(K_0 \in \mathbb{N}, \alpha > 0\) and sequences \(i_k \to \infty, j_k \in J_k\) such that \(\gamma_{ij_k}(r) \geq \alpha\) for all \(k \geq K_0\). Now, consider the bounded set \(C := \{r_{ij_k} : k \in \mathbb{N}\} \subset I^N\). We prove that \(\varpi(C)\) is not relatively compact, and thus, \(\varpi\) is not compact. First, observe that

\[
\gamma_{ij_k}(r) \geq \alpha \quad \text{for all} \quad k \geq K_0.
\]

Moreover, by our assumption that every subsystem can only influence finitely many other subsystems, we have \(\Gamma \gamma_{ij_k} = 0\) for almost all \(i \in \mathbb{N}\). Assume to the contrary that \(\mathbb{N} = J_1 \cup \ldots \cup J_N\) is a finite partition such that \(\Gamma \gamma_{ij_k} = \gamma_{ij_k}\) for some \(l \in \{1, \ldots, N\}\). Assuming that \(J_l\) contains infinitely many elements (which must be true for at least one of the index sets), we find that \(\Gamma \gamma_{ij_k} = \gamma_{ij_k}\) for almost all \(j \in J_l\), a contradiction.

\[\square\]

VIII. DISCUSSION OF THE OBTAINED RESULTS

A. Comparison to [17]

The first Lyapunov-based small-gain theorem for infinite networks can be found in [17, Th. 11]. It requires that all internal gains are identical \((\gamma_{ij} \equiv \gamma)\), and the SGC in [17] requires that \(\gamma < \text{id}\), which is in most cases much more conservative than the SGC employed in our article.

As \(\gamma_{ij} = \gamma < \text{id}\) for all \(i, j \in \mathbb{N}\), the gain operator in [17] has a simple representation

\[
\Gamma(s) = \left( \begin{array}{c}
\sup_{j \in \mathbb{N}, \ell} \gamma(s_j) \\
\end{array} \right)_{i \in \mathbb{N}} \leq \gamma(||s||_{\varepsilon}) I.
\]

Define \(\sigma : \mathbb{R}_+ \to \mathbb{R}_+\) by \(\sigma(r) := r I, r \geq 0\). Then,

\[
\Gamma(\sigma(r)) \leq \gamma(||r||_{\varepsilon}) I = \gamma(r) I = \gamma \circ \sigma(r) < \sigma(r).
\]

If \(\gamma < \text{id}\) satisfies

\[
id - \gamma \geq \eta \quad \text{with a certain} \quad \eta \in K_{\infty},\]

then \(\gamma = \text{id} - (\text{id} - \gamma) \leq \text{id} - \eta = (\text{id} + \rho)^{-1}\) for a suitable \(\rho \in K_{\infty}\) (this is easy to see, and was stated in [20, Lem. 8]). In this case, we obtain that \(\sigma\) is a path of strict decay as in Definition II.10, and the corresponding ISS Lyapunov function constructed by our small-gain theorem is \(V(s) = \sup_{i \in \mathbb{N}} V_i(s_i)\) for any \(s = (s_i)_{i \in \mathbb{N}} \in I_{\varepsilon}(\mathbb{N}, (n_i))\), which is precisely the Lyapunov function proposed in [17, Th. 1.1]. Thus, we obtain the main result of [17] as a special case of ours. Nevertheless, if the condition (38) does not hold, then \(\varpi\) decays at the points of the form \(\sigma(r)\), but not in a uniform way as we require in this work. In this case, the result in [17] is applicable, while our theorem is not.

B. Comparison to [21]

The setting of [21] is very similar to that of our article, in particular, the Lyapunov gains \(\gamma_{ij}\) may be distinct nonlinear functions, and the number of neighbors is also taken to be finite. The small-gain theorem [21, Th. 5.1] is shown under the requirement that there is a linear path of strict decay for the gain operator \(\varpi\), with a linear \(\rho\) (see Definition II.10), whereas we do not require linearity, which makes our result truly nonlinear.

In [21], it was proposed to use the strong transitive closure \(Q\) to construct a path of strict decay, and a robust SGC was introduced to describe under which conditions a path of strict decay exists. In [21, Lem. 4.5], it was shown that there is \(\sigma : \mathbb{R}_+ \to \mathbb{R}_+\), satisfying properties 1)-3) of Definition II.10 with a linear \(\rho\), provided that \(\Gamma\) satisfies the robust strong SGC with linear \(\rho, \omega\), and the discrete-time system (27) induced by \(\Gamma\) is USAS.

We improve these results in several directions. We introduce the concept of max-robust SGC, which is weaker than the robust SGC, and show that USAS of the discrete-time system induced by the gain operator implies the max-robust SGC (see Proposition V.1). Finally, we show in Theorem VI.1 that under certain regularity assumptions USAS of a system induced by a scaled gain operator is only needed to ensure the existence of a path of strict decay, which we also construct explicitly.

C. Finite networks

Finally, let us compare our results to the available results for finite networks. A nonlinear Lyapunov-based small-gain theorem for finite networks of ODE systems has been proposed in [9]. As in the first part of this section, we require a bit more uniformity in the definition of a path of strict decay, and thus,
we cannot fully recover the small-gain result in [9]. Also, since we formulate our result in a maximum formulation, the gain operator is a so-called max-preserving operator [22, 23], and it is known that the SGC (not even the strong one) is equivalent to UGAS of an induced discrete-time system and to the existence of a path of strict decay as defined in [9] (see [29, Th. 6.4]).

On the other hand, we know that for finite networks the strong SGC is equivalent to the robust strong SGC [20, Prop. 14], and implies UGAS of the discrete-time system [29, Th. 4.6]. Thus, for finite networks, our Theorem VI.1 states that the strong SGC for \( \Gamma \) implies the existence of a path of strict decay with a uniform decay rate, characterized by \( \rho \), which is not far from the sharp result for finite networks.

### IX. Conclusion

We have proved a fully nonlinear Lyapunov-based small-gain theorem for ISS of infinite networks. In our result, we use the maximum formulation and the implication form to describe the ISS property of the subsystems. A crucial assumption in our result is the existence of a (nonlinear) path of strict decay for the gain operator, which acts on the positive cone in the sequence space \( \mathcal{E}_\infty \). We have proved that such a path exists if the discrete-time system induced by a scaled gain operator is UGAS and, additionally, the interconnection gains satisfy a uniform local Lipschitz condition. While the second assumption is not hard to check in a concrete example, the first one is more delicate. We have characterized a weaker property in terms of the so-called max-robust SGC, which is again tractable. The difference between UGAS and this weaker property is that trajectories may only converge to zero componentwise and not necessarily uniformly, i.e., with respect to the \( \mathcal{E}_\infty \)-norm. Different types of sufficient conditions for the uniform convergence were also provided and their tractability was shown in an example.

One open question for future research is whether there exist characterizations of UGAS (of the gain-operator-induced system) that can be checked by a condition formulated in terms of single gains instead of their compositions. Another open problem is to extend the theory to different formulations of the ISS property for the subsystems, which lead to other types of gain operators.

### APPENDIX

#### Technical Lemmas

We omit the proofs of the following two simple lemmas, the first of which follows from [31, Lem. 3.4].

**Lemma IX.1:** Let \( s_1, s_2 : [0, T] \to \mathbb{R}_+ \) be differentiable functions such that \( s_1(0) = s_2(0), s_1(t) \leq -\alpha(s_1(t)), \) and \( s_2(t) = -\alpha(s_2(t)) \) for all \( t \in [0, T] \) and some locally Lipschitz \( \alpha \in \mathcal{P} \). Then, \( s_1(t) \leq s_2(t) \) for all \( t \in [0, T] \).

**Lemma IX.2:** Let \( f_i : [a, b] \to \mathbb{R}_+ \), \( a, b \in \mathbb{R}, a < b \). Further assume that \( |f_i(r_1) - f_i(r_2)| \geq l|r_1 - r_2| \) for all \( r_1, r_2 \in [a, b] \) and \( i \in \mathbb{N} \), where \( l > 0 \). Put \( f(r) := \max_{i=1,\ldots,n} f_i(r), f : [a, b] \to \mathbb{R}_+ \). Then, with \( l := \min\{l_1, \ldots, l_n\} \), we have \( |f(r_1) - f(r_2)| \geq l|r_1 - r_2| \) for all \( r_1, r_2 \in [a, b] \).

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