NEGATIVE CURVES ON VERY GENERAL BLOW-UPS OF $\mathbb{P}^2$  
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Abstract. This note contains new evidence to a conjecture, related to the Nagata conjecture and the Segre-Harbourne-Gimigliano-Hirschowitz conjecture, on the cone of effective curves of blow-ups of $\mathbb{P}^2$ at very general points.  

Introduction  

The solution to the problem of determining the dimension of every linear system of curves in $\mathbb{P}^2$ with assigned multiplicities at general points is predicted by the equivalent conjectures of Segre, Harbourne, Gimigliano and Hirschowitz [Se, Ha1, Gi, Hi], hereafter “SHGH conjecture”. In this paper we are interested in the following weaker form of the conjecture:  

Conjecture 0.1. Suppose that $C$ is an integral curve with negative self-intersection on the blow-up $Y$ of $\mathbb{P}^2$ at a set of points in very general position. Then $C$ is a $(-1)$-curve of $Y$ (that is, a smooth rational curve with self-intersection $-1$).  

If one also includes the conjecture that $H^1(Y, \mathcal{O}_Y(D)) = 0$ for every effective nef divisor $D$ on any such $Y$, then this becomes one of the formulations of the SHGH conjecture [Ha2]. There is an interesting relationship between Conjecture 0.1 which of course contains Nagata’s conjecture [Na1], and the symplectic packing problem in dimension four [MP, Bi]. In the first section of this paper, we will review how the above conjecture characterizes the geometry of the cone of effective curves of $Y$.  

It is not difficult to see that the conjecture is satisfied by all rational curves (see Proposition 2.4 below). There are several results nowadays, giving evidence to the SHGH conjecture, that are valid under suitable assumptions on the multiplicities assigned to the centers of the blow-up. These include [AH, Ha3, CM1, CM2, Mi, Ya]. In the same spirit, we prove the following result.  

Theorem 0.2. Conjecture 0.1 is satisfied by all curves whose image on $\mathbb{P}^2$ is a curve with a singularity of multiplicity 2 at one of the centers of the blow-up.  

The proof of this theorem is based on a local study of dynamic self-intersection, very much inspired to the methods in [EL] and [Xu]. The novelty here with respect to [EL, Xu] is to consider deformations with two-parameters families: it is indeed by computing the Kodaira-Spencer map from different directions of deformation that we produce a linear system on $C$ giving an isomorphism to $\mathbb{P}^1$ and showing that $C^2 = -1$. A similar result is proven to hold for an arbitrary smooth projective surface in place of $\mathbb{P}^2$ (see Theorem 2.5 below).  

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An analogous result on linear systems on smooth projective surfaces was given in [dVL, Theorem 4.1] under the assumption that the “specialty” of the system (namely, the gap between its dimension and its expected dimension) increases by imposing a double point. We remark that this hypothesis does not translate well to the context of determining negative curves on the blow-up. To the best of our knowledge, there are no other results in which the assumptions only involve one of the multiplicities.

The precise notion of “very general position” adopted in this paper is given in Definition 2.1 below. Throughout this paper we work over the complex numbers.

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1. Geometry of the cone of curves

In this section we discuss the implication that Conjecture 0.1 has on the geometry of the cone of effective curves of the blow-up of $\mathbb{P}^2$. This section is mostly of an expository nature, as the material here contained is probably well known to the specialist.

We start by fixing some notation. Let $X$ be a smooth projective surface. Let $N := \text{Pic}(X)/\equiv \otimes \mathbb{R}$ and $\rho := \text{rk Pic}(X)$. By the Hodge index theorem, we can identify $N$ with $\mathbb{R}^\rho$ in such a way that the intersection product is given by the matrix diag$(1, -1, \ldots, -1)$. For a divisor $D$ on $X$ we denote by $D^{\leq 0}$ (resp. $D^{= 0}$, $D^{\geq 0}$, $D^\perp$) the subset of $N(X)$ defined by $[D] \cdot x < 0$ (resp. $[D] \cdot x \leq 0$, $[D] \cdot x \geq 0$, $[D] \cdot x = 0$). Let

$$\overline{\text{NE}}(X) \subset N(X)$$

be the closure of the cone spanned by the classes of effective curves on $X$, and

$$\text{Nef}(X) \subset N(X)$$

be the closure of the cone spanned by the classes of ample divisors on $X$. By Kleiman’s criterion, these two cones are put in duality by the intersection product in $N(X)$. Fix an ample class $H$ on $X$, let

$$\text{Pos}(X) := \{ \alpha \in N(X) \mid \alpha^2 \geq 0, \alpha \cdot H \geq 0 \},$$

and let $\text{Nul}(X)$ denote the boundary of $\text{Pos}(X)$. Then $\text{Nul}(X)$ is supported by the quadratic equation $x_1^2 = x_2^2 + \cdots + x_\rho^2$. Note that

$$\text{Nef}(X) \subseteq \text{Pos}(X) \subseteq \overline{\text{NE}}(X).$$

It follows by a result of Campana and Peternell [CP] that $\partial \overline{\text{NE}}(X)$ is supported by $\text{Nul}(X)$ and countably many hyperplanes, so we can write

$$\partial \overline{\text{NE}}(X) = \mathcal{B}_1 \sqcup \mathcal{B}_2,$$

where $\mathcal{B}_1$ is the closure (in $\partial \overline{\text{NE}}(X)$) of the union of the facets of $\partial \overline{\text{NE}}(X)$, and $\mathcal{B}_2$ is supported by $\text{Nul}(X)$.

The following is a more precise formulation of Conjecture 0.1.
Conjecture 1.1. Let $Y$ be the blow-up of $\mathbb{P}^2$ at a set of $r$ points in very general position. Then, writing $\partial \overline{\text{NE}}(Y) = \mathcal{B}_1 \sqcup \mathcal{B}_2$ as above (setting $X = Y$), the extremal rays of $\mathcal{B}_1$ that do not lie on $\text{Nul}(Y)$ are spanned by classes of $(-1)$-curves, and we have

$$\mathcal{B}_1 = \partial \overline{\text{NE}}(Y) \cap K_Y^{>0} \quad \text{and} \quad \mathcal{B}_2 = \partial \overline{\text{NE}}(Y) \cap K_Y^{\geq 0}.$$  

In particular,

$$\overline{\text{NE}}(Y) \cap K_Y^{\geq 0} = \text{Pos}(Y) \cap K_Y^{\geq 0}.$$  

In the following, we review the equivalence of the two conjectures. We begin with a general consideration on the clustering of extremal rays. As at the beginning of the section, let $X$ be a smooth projective surface. We consider the metric on the set of rays in $N(X)$ given by the angular distance: for any two rays $R_1$ and $R_2$ in $N(X)$, we set $d(R_1, R_2)$ to be the angle between them. For an extremal ray $R$ of $\overline{\text{NE}}(X)$, we set

$$d(R) = \inf \{ d(R, R') \mid R' \text{ is an extremal ray of } \overline{\text{NE}}(X) \text{ different from } R \}.$$  

Lemma 1.2. If $R$ is an extremal ray of $\overline{\text{NE}}(X)$, then $d(R, \text{Nul}(X)) \leq d(R)$; in particular, every Cauchy sequence of extremal rays of $\overline{\text{NE}}(X)$ converges to a ray on $\text{Nul}(X)$. Moreover, if $\rho \geq 4$ and $F$ is a facet of $\overline{\text{NE}}(X)$ such that $\partial F \cap \text{Nul}(X) \neq \emptyset$, then $F$ has infinitely many extremal rays forming a Cauchy sequence converging to $\partial F \cap \text{Nul}(X)$.

Proof. We can assume that $R \not\subset \text{Nul}(X)$. By [Ko] Lemma II.4.2, $R$ is spanned by the class of a curve $C$ with $C^2 < 0$, and every other extremal ray of $\overline{\text{NE}}(X)$ is contained in the half space $C^{\geq 0}$ [Laz Exercise 1.4.33(ii)]. Thus the inequality $d(R, \text{Nul}(X)) \leq d(R)$ will follow once we show that the orthogonal projection (in the euclidian metric) of $R$ onto $C^{\perp}$ is contained in $\text{Pos}(X)$. This can be easily checked as follows. Let $(\ , \ )$ denote the standard inner product in $\mathbb{R}^\rho$, and let $( \cdot , \cdot )$ denote the intersection product defined by the diagonal matrix $\text{diag}(1, -1, \ldots, -1)$. Let $h = (1, 0, \ldots, 0) \in \mathbb{R}^\rho$, fix a vector $c \in \mathbb{R}^\rho$ such that $(c \cdot h) > 0$ and $(c \cdot c) < 0$, and consider the orthogonal projection

$$\pi : \mathbb{R}^\rho \to \{ x \in \mathbb{R}^\rho \mid (c \cdot x) = 0 \}.$$  

It is an exercise to check that

$$\frac{(\pi(c) \cdot \pi(c))}{(c \cdot c)} = \frac{(c \cdot c)^2}{(c \cdot c)^2} - 1 \quad \text{and} \quad (\pi(c) \cdot h) = (c \cdot h) - \frac{(c \cdot c)}{(c \cdot c)} \cdot (c, h).$$  

Since $(c \cdot c)^2 \leq (c, c)^2$, the first equation gives $(\pi(c) \cdot \pi(c)) \geq 0$. The second equation gives $(\pi(c) \cdot h) > 0$. Therefore the projection of $R$ onto $C^{\perp}$ is contained in $\text{Pos}(X)$.

To prove the second part of the lemma, we note that $F$ is supported by the hyperplane $H$ that is tangent to $\text{Nul}(X)$ along the ray $\partial F \cap \text{Nul}(X)$; this follows by the convexity of $\overline{\text{NE}}(X)$ and the inclusion $\text{Pos}(X) \subseteq \overline{\text{NE}}(X)$. If $F$ has only finitely many rays, then we find another facet $F'$ containing $\partial F \cap \text{Nul}(X)$ that, for the same reason, is also supported by $H$. Since this is impossible, $F$ must have infinitely many extremal rays, and these cluster to $\partial F \cap \text{Nul}(X)$ by what proved in the first part of the lemma. \hfill \square

Now we can go back to the question on the equivalence between the two conjectures. Recall that we are denoting by $Y$ the blow-up of $\mathbb{P}^2$ at a set of $r$ points in very general position. Clearly both conjectures are certainly true for $r \leq 2$, so we can assume that $r \geq 3$, hence $\text{rk} \text{Pic}(Y) \geq 4$. Since one direction is obvious, let us assume that the only integral curves $C \subset Y$ with $C^2 < 0$ are the $(-1)$-curves. Write $\partial \overline{\text{NE}}(Y) = \mathcal{B}_1 \sqcup \mathcal{B}_2$ as
described above. The extremal rays of $\mathcal{B}_1$ not lying on $\text{Nul}(Y)$, which are also extremal rays of $\text{NE}(Y)$, are spanned by classes of $(-1)$-curves. In particular, by adjunction formula and Lemma 1.2, we have $\mathcal{B}_1 \subset K_Y^{\leq 0}$. On the other hand, the Cone Theorem implies that $\mathcal{B}_2 \subset K_Y^{>0}$. In fact, observing that $\partial \text{NE}(Y) \cap K_Y^{>0} \neq \emptyset$ and that $\mathcal{B}_2$ is open in $\partial \text{NE}(Y)$, we actually have $\mathcal{B}_2 \subset K_Y^{>0}$. Therefore we conclude that

$$\mathcal{B}_1 = \partial \text{NE}(Y) \cap K_Y^{\leq 0} \quad \text{and} \quad \mathcal{B}_2 = \partial \text{NE}(Y) \cap K_Y^{>0}.$$ 

The last assertion follows by continuity.

Remark 1.3. As it is well known, an interest in Conjecture 1.1 comes from the search for examples of irrational Seshadri constants. Indeed, if true, the conjecture would allow us to construct such examples. This is easy to see: let $Y$ be the blow-up of $\mathbb{P}^2$ at $r \geq 9$ points in very general position, and denote by $H$ the pull-back on $Y$ of the hyperplane class of $\mathbb{P}^2$ and by $E$ the exceptional divisor of $Y \to \mathbb{P}^2$. Then, for a very general point $q \in Y$ and for every rational $0 < a < 1/\sqrt{7}$, we would have $\epsilon_q(H-aE) = \sqrt{1-ra^2}$. Note that, for most (rational) values of $a$, this is an irrational number.

2. Negative curves on $\mathbb{P}^2$ blown up at very general points

The necessity to assume very general position in the conjecture is clear in the case of more than 9 points. Indeed the blow-up of $\mathbb{P}^2$ at 9 points will generally carry infinitely $(-1)$-curves [Na2 Theorem 4a], and certainly one cannot allow to blow up points lying on these curves. In this paper we adopt the following notion of very general position.

Definition 2.1. We say that a set $\{x_1, \ldots, x_r\}$ of $r \geq 1$ distinct points on a smooth projective surface $X$ is in very general position if the following condition is fulfilled. For every integral curve $C \subset X$, the pair $(C, (x_1, \ldots, x_r))$ belongs to an irreducible algebraic family

$$\{(C_t, (x_{1,t}, \ldots, x_{r,t})) \mid t \in T\},$$

where $C_t \subset X$ is an integral curve and $(x_{1,t}, \ldots, x_{r,t}) \in X^r$ for every $t \in T$, satisfying the following properties:

(i) $p_q(C_t) = p_q(C)$ and $\text{mult}_{x_i,t} C_t = \text{mult}_{x_i} C$ for every $t \in T$ and $i = 1, \ldots, r$;

(ii) the morphism $\psi : T \to X^r$ given by $\psi(t) = (x_{1,t}, \ldots, x_{r,t})$ is an isomorphism to an open subset of $X^r$.

We say that a point $x \in X$ is a very general point if the set $\{x\}$ is in very general position.

Remark 2.2. For every family of integral curves on $X$, the set of $r$-ples $(x_1, \ldots, x_r) \in X^r$ for which the condition of the definition is not satisfied by some curve of the family is contained in a proper closed subvariety of $X^r$. We conclude that the complement of the locus in $X^r$ of points corresponding to subsets of $X$ in very general position is a countable union of proper closed subvarieties. Any subset of a set of points in very general position is a set of points in very general position. Given a set of points in very general position, the image of any of the points of the set on the blowing-up of the surface at the residual points of the set is a very general point.

The blow-up of $\mathbb{P}^2$ at at most 8 points in general position is a Del Pezzo surface. This surface contains finitely many $(-1)$-curves, and these are the only integral curves with negative self-intersection. The picture in the case of 9 points is well known too [Na2 Proposition 12]; a proof of the following statement will also be given below.
Proposition 2.3. Every integral curve with negative self-intersection on the blow-up of \( \mathbb{P}^2 \) at a set of 9 points in very general position is a \((-1)\)-curve.

Very little is known on the cone of effective curves of the blow-up of \( \mathbb{P}^2 \) at more than 9 points. If we restrict our attention to rational curves, then it is easy to get the desired statement. The following proposition, which is well known to the experts, can be viewed as the first step with respect to another formulation of the SHGH conjecture, due to Harbourne [Ha4], which says that if \( C \) is any irreducible and reduced curve on the blow-up of \( \mathbb{P}^2 \) at very general points, then \( C^2 \geq p_a(C) - 1 \), where \( p_a(C) \) is the arithmetic genus of \( C \).

Proposition 2.4. Let \( Y \) be the blow-up of \( \mathbb{P}^2 \) at a set of points in very general position, and assume that \( C \) is a integral rational curve on \( Y \) with \( C^2 < 0 \). Then \( C \) is a \((-1)\)-curve.

Proof. To prove that \( C \) is a \((-1)\)-curve, it suffices to show that \( K_Y \cdot C < 0 \), as the conclusion then follows by adjunction. Let \( B \subset \mathbb{P}^2 \) be the image of \( C \). We can assume that \( B \) is a curve, otherwise \( C \) would be exceptional, hence a \((-1)\)-curve. Let \( p_1, \ldots, p_r \in \mathbb{P}^2 \) be the centers of the blow-up, and let \( m_i = \text{mult}_{p_i} B \). We can assume that \( m_1 > 0 \).

For short, let \( p = (p_1, \ldots, p_r) \). By the definition of very general position, the pair \((B, p)\) belongs to an irreducible algebraic family
\[
\{(B_t, p_t) \mid t \in T\},
\]
where \( p_t = (p_1, \ldots, p_r, t) \in (\mathbb{P}^2)^r \), \( B_t \subset X \) is an integral rational curve with \( \text{mult}_{p_t,t} B_t = m_t \) for every \( t \in T \), and the morphism \( \psi : T \to (\mathbb{P}^2)^r \) given by \( \psi(t) = p_t \) is an isomorphism to an open subset of \((\mathbb{P}^2)^r\).

Fix \( r \) points \( q_1, \ldots, q_r \) in general position on a smooth cubic \( \Gamma \subset \mathbb{P}^2 \). Let \( Z \subset (\mathbb{P}^2)^r \) be a smooth, irreducible curve passing through \( p \) and \( q = (q_1, \ldots, q_r) \), and consider the open set \( U = Z \cap \psi(T) \) of \( Z \). Note that \( U \) is not empty, and that \( q \) is in the closure of \( U \). The family \( \{(B_t, p_t) \mid t \in \psi^{-1}(U)\} \) determines an effective divisor
\[
\mathcal{B} \subset \mathbb{P}^2 \times U,
\]
whose restriction to \( \mathbb{P}^2 \times \{p_t\} \) (for every \( t \in \psi^{-1}(U) \)) is the divisor \( B_t \). Viewing \( \mathcal{B} \) as a scheme, we take its flat closure \( \overline{\mathcal{B}} \) inside \( \mathbb{P}^2 \times Z \), and let \( B_0 \) be the restriction of \( \overline{\mathcal{B}} \) to \( \mathbb{P}^2 \times \{q\} \). Then \( B_0 \) is an effective divisor on \( \mathbb{P}^2 \) (see for instance [Har Example III.9.8.5]). Since \( \text{mult}_{p_t,t} B_t = m_t \) for every \( p_t \in U \), we have \( \text{mult}_{q_i} B_0 \geq m_i \) by the semi-continuity of the multiplicity. Moreover, since \( B_t \) is a rational curve for every \( t \in \psi^{-1}(U) \), so is every irreducible component of \( B_0 \), hence \( \Gamma \) is not contained in the support of \( B_0 \). Thus
\[
-K_{\mathbb{P}^2} \cdot B = \Gamma \cdot B_0 \geq \sum m_i.
\]

If this inequality is strict, then we have
\[
K_Y \cdot C = K_{\mathbb{P}^2} \cdot B + \sum m_i < 0.
\]

Therefore, to conclude the proof, it is enough to show that the inequality in (2.1) is strict. Suppose this is not the case. Then \( \text{mult}_{q_i} B_0 = m_i \) and \( \mathcal{O}_{\mathbb{P}^2}(B_0)|_\Gamma = \mathcal{O}_\Gamma(\sum m_i q_i) \). Moreover, by the way we chose the \( q_i \), we can fix a different deformation in which \( q_1 \) is replaced by another general point \( q_1' \) of \( \Gamma \) while the other \( q_i \) are kept the same, obtaining
in this way another curve $B_0'$. Since the previous arguments also apply to $B_0'$, we have $\text{mult}_{q_i}(B_0') = m_1$ and $\text{mult}_{q_i}(B_0') = m_i$ for $i \geq 2$, and moreover

$$\mathcal{O}_\Gamma \left( m_1 q_1 + \sum_{i=2}^r m_i q_i \right) = \mathcal{O}_{\mathbb{P}^2}(B_0)|_\Gamma = \mathcal{O}_{\mathbb{P}^2}(B_0')|_\Gamma = \mathcal{O}_\Gamma \left( m_1 q_1' + \sum_{i=2}^r m_i q_i \right).$$

This implies that $\mathcal{O}_\Gamma \left( m_1 (q_1 - q_1') \right) = \mathcal{O}_\Gamma$. But $\Gamma$ is an elliptic curve, and after fixing $q_1$ as zero, we know that there are finitely many $m_1$-torsion points on $\Gamma$. A contradiction. □

**Proof of Proposition 2.3.** Let $Y$ be the blow-up of $\mathbb{P}^2$ at a set of 9 points in very general position, and suppose that $C \subset Y$ is an integral curve with $C^2 < 0$. Since $-K_Y$ is nef, we have $K_Y \cdot C \leq 0$. By adjunction, we conclude that $C$ is rational, hence it is a $(-1)$-curve by Proposition 2.4. □

The following is the main result of this paper. Choosing $S = \mathbb{P}^2$ in the statement gives the theorem stated in the introduction.

**Theorem 2.5.** Let $S$ be a smooth projective surface, and let $f : Y \to S$ be the blowing up of $S$ at a set $\Sigma$ of points in very general position. Let $C \subset Y$ be an integral curve with negative self-intersection, and assume that $f(C)$ is a curve with a singularity of multiplicity 2 at one of the points of $\Sigma$. Then $C$ is a $(-1)$-curve of $Y$.

**Proof.** Let $p \in \Sigma$ be the double point of $f(C)$ whose existence we are assuming, and let $X$ be the blow-up of $S$ at $\Sigma \setminus \{p\}$. Then $f$ factors through the blow-up $g : Y \to X$ of the image $x \in X$ of $p$. Let $D = g(C)$. Note that $\text{mult}_x D = 2$. Since $x$ is a very general point of $X$, there exists an irreducible algebraic family of integral curves with marked points

$$\{(D_t, x_t) \mid t \in T\} \subset X \times X$$

with $D = D_{t^*}$ and $x = x_{t^*}$ for some $t^* \in T$, such that the morphism $\psi : T \to X$ given by $\psi(t) = x_t$ is an isomorphism to an open subset of $X$ and, moreover, $\text{mult}_x D_t = 2$ for all $t \in T$. The Kodaira-Spencer map induced by any 1-dimensional degeneration $t \to t^*$ inside $T$ defines a section of the normal bundle $N := N_{D/X}$ of $D$ in $X$. Bearing in mind that we are dealing with a family of irreducible and reduced curves with marked singularities, these sections are non-zero whenever the degeneration $t \to t^*$ is performed along a curve of $T$ that is smooth at $t^*$.

As in the proof of [PL, Lemma 1.1], we reduce to a local computation in some open set $\Omega$ in $\mathbb{C}^2$. We fix local coordinates $u = (u_1, u_2)$ in $\Omega$. Let $f = f(u)$ be the holomorphic function locally defining $D$. Let us start considering the case in which $p$ is an ordinary node. Writing $f$ as a power series centered at $(0,0)$, we can assume that the coordinates are chosen so that

$$f(u) = u_1 u_2 + (\text{higher degree terms}).$$

We can reduce to the case when $T$ is a small disk in $\mathbb{C}^2$, with $t^* = (0,0)$, and fix coordinates $t = (t_1, t_2)$ in $T$ such that $t_i = u_i \psi$. The total space of the deformation is defined in $\Omega \times T$ by a power series $F = F(u, t)$. The deformation determines a Kodaira-Spencer map

$$\rho : T_{t^*}(T) \to H^0(D, N),$$

by Proposition 2.4.
which is non-trivial by our previous assumptions. This map is locally defined by
\[ p \left( \lambda_1 \frac{\partial}{\partial t_1} + \lambda_2 \frac{\partial}{\partial t_2} \right) = \left( \lambda_1 \frac{\partial F}{\partial t_1}(u, 0) + \lambda_2 \frac{\partial F}{\partial t_2}(u, 0) \right) \bigg|_C =: \tau_\lambda. \]

In view of the linearity of this map, the sections \( \tau_\lambda \) fill up a non-trivial linear subspace in \( H^0(D, N) \) as \( \lambda \) varies in \( \mathbb{C}^2 \). Mimicking [DL], we consider the function
\[ \Phi(u, t) := F(u + x(t), t), \]
where \( x(t) = (x_1(t), x_2(t)) \) are the coordinate of the marked point \( x_t \) of \( C_t \). Note that \( \Phi \in (u_1, u_2)^2 \) for all \( t \). We expand \( \Phi \) as a power series in \((t_1, t_2)\). The coefficients of the two terms of degree 1 are equal to
\[ \frac{\partial \Phi}{\partial t_i} = \frac{\partial f}{\partial u_1}(u) \cdot \frac{\partial x_1}{\partial t_i}(0) + \frac{\partial f}{\partial u_2}(u) \cdot \frac{\partial x_2}{\partial t_i}(0) + \frac{\partial F}{\partial t_i}(u, 0), \quad i = 1, 2. \]
These are functions of \( u \), and both are contained in \((u_1, u_2)^2\). Note that \( \partial x_j/\partial t_i(0) = \delta_{ij} \). Then, by combining (2.2), (2.3) and (2.4), we see that
\[ \partial \Phi / \partial t_i \bigg|_{C} + \tau_\lambda \in \mathfrak{m}_x^2, \]
where \( \mathfrak{m}_x \) is the maximal ideal of \( x \) in \( D \). Note that \( \left( \lambda_1 u_2 + \lambda_2 u_1 \right) \big|_C \in \mathfrak{m}_x \). We conclude that \( \tau_\lambda \in H^0(D, N \otimes \mathfrak{m}_x) \). Then \( \tau_\lambda \) gets pulled back by \( g|_C : C \to D \) to a section \( \sigma_\lambda \) of \((g|_C)^* N\) that vanishes at the two pre-images of \( x \). After suitably denoting these two pre-images by \( y_1 \) and \( y_2 \), we actually get sections
\[ \sigma_{(1,0)} \in H^0(C, (g|_C)^* N \otimes \mathfrak{m}_{y_1} \otimes \mathfrak{m}_{y_2}) \quad \text{and} \quad \sigma_{(0,1)} \in H^0(C, (g|_C)^* N \otimes \mathfrak{m}_{y_1} \otimes \mathfrak{m}_{y_2}^2) \]
when \( \lambda \in \{(1,0), (0,1)\} \). This implies that \( \deg(\text{Div}(\sigma_\lambda)) \geq 3 \) for every \( \lambda \neq 0 \). Since \( D^2 < 4 \), this yields \( \deg(\text{Div}(\sigma_\lambda)) = 3 \), that is \( D^2 = 3 \), and thus \( C^2 = -1 \). In fact, we observe that the linear system \( |\text{Div}(\sigma_\lambda)| \) contains a pencil parameterized by \( \lambda \). This pencil has base points at \( y_1 \) and \( y_2 \) and movable part of degree 1, which defines an isomorphism to \( \mathbb{P}^1 \).

It remains to discuss the case when \( x \) is not an ordinary node of \( D \). We now explain why this case cannot occur. Using analogous notation as in the previous discussion, we get the following local equation of \( D \):
\[ f = u_1^2 + (\text{higher degree terms}). \]
Arguing as before, we see this time that \( \tau_{(0,1)} \in H^0(C, N \otimes \mathfrak{m}_x^2) \). This implies that \( \deg(\text{Div}(\sigma_{(0,1)})) \geq 4 \), which is impossible.

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