Quantifying lawlessness in finitely generated groups

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Abstract

We introduce a quantitative notion of lawlessness for finitely generated groups, encoded by the lawlessness growth function $\mathcal{A}_\Gamma : \mathbb{N} \to \mathbb{N}$. We show that $\mathcal{A}_\Gamma$ is bounded iff $\Gamma$ has a nonabelian free subgroup. By contrast we construct, for any nondecreasing unbounded function $f : \mathbb{N} \to \mathbb{N}$, an elementary amenable lawless groups for which $\mathcal{A}_\Gamma$ grows more slowly than $f$. We produce torsion lawless groups for which $\mathcal{A}_\Gamma$ is at least linear using Golod-Shafarevich theory, and give some upper bounds on $\mathcal{A}_\Gamma$ for Grigorchuk’s group and Thompson’s group $F$. We note some connections between $\mathcal{A}_\Gamma$ and quantitative versions of residual finiteness. Finally, we also describe a function $\mathcal{M}_\Gamma$ quantifying the property of $\Gamma$ having no mixed identities, and give bounds for nonabelian free groups. By contrast with $\mathcal{A}_\Gamma$, there are no groups for which $\mathcal{M}_\Gamma$ is bounded: we prove a universal lower bound on $\mathcal{M}_\Gamma(n)$ of the order of $\log(n)$.

1 Introduction

A law for a group $\Gamma$ is a non-trivial word-map which vanishes identically on $\Gamma$. $\Gamma$ is lawless if it has no laws. The goal of this article is to introduce and study a quantitative notion of lawlessness.

1.1 Statement of results

Throughout, $\Gamma$ is a group generated by a finite set $S$. Given a nontrivial reduced word $w$ which is not a law for $\Gamma$, the complexity of $w$ in $\Gamma$ is the minimal length of a tuple in $\Gamma$ not evaluating to the identity under $w$ (where the length of a tuple of elements of $\Gamma$ is the sum of the lengths of those elements in the word-metric induced by $S$ on $\Gamma$). The lawlessness growth function $\mathcal{A}_\Gamma^S : \mathbb{N} \to \mathbb{N}$ of $\Gamma$ then sends $n \in \mathbb{N}$ to the maximal complexity occurring among the words of length at most $n$. It is easy to see that if $\Gamma$ has a nonabelian free subgroup, then $\Gamma$ is lawless and $\mathcal{A}_\Gamma^S$ is a bounded function. Our first observation is that this is the only way that bounded lawlessness growth can arise.

Theorem 1.1. Suppose $\Gamma$ is lawless. Then $\Gamma$ has a nonabelian free subgroup iff $\mathcal{A}_\Gamma^S$ is bounded.
Theorem 1.1 immediately begs the question of how slowly $\mathcal{A}_\Gamma^S$ can grow for a lawless finitely generated group which contains no non-abelian free subgroup. We give a satisfying answer to this question, constructing examples where the growth can be as slow as desired.

Theorem 1.2. Let $f : \mathbb{N} \to \mathbb{N}$ be an unbounded nondecreasing function with $f(1) \geq 2$. There exists an elementary amenable lawless group $\Gamma$, generated by a finite set $S$ such that for all $n \in \mathbb{N}$, $\mathcal{A}_\Gamma^S(n) \leq f(n)$.

The groups $\Gamma$ can all be built out of a fixed lawless elementary amenable group, using a wreath product construction. Being amenable, $\Gamma$ contains no non-abelian free subgroup, so that by Theorem 1.1, $\mathcal{A}_\Gamma^S$ is unbounded. Next we turn to groups of fast lawlessness growth.

Theorem 1.3. There exists a lawless finitely generated group $\Gamma$ with $\mathcal{A}_\Gamma^S(n) \gg n$.

The group $\Gamma$ is constructed using Golod-Shafarevich theory: it has long been known that G-S theory enables the construction of infinite (indeed lawless) finitely generated $p$-torsion groups. We run this argument in an effective manner: ensuring that the orders of elements grow as slowly as possible as a function of word-length.

We then turn to the problem of estimating $\mathcal{A}_\Gamma^S$ for specific well-known examples of lawless groups without non-abelian free subgroups. The first example we investigate is Grigorchuk’s (first) group of automorphisms of the binary rooted tree. This is a torsion group (so has no non-abelian free subgroup) and has every finite $2$-group as a subgroup (so has no laws).

Theorem 1.4. Let $\Gamma$ be the first Grigorchuk group. Then there exists $C > 0$ such that:

$$n^{2/3} \ll \mathcal{A}_\Gamma^S(n) \ll \exp(Cn).$$

The upper bound in Theorem 1.4 is based on embedding iterated wreath products of $C_2$ into $\Gamma$, close to the identity, and bounds on the lengths of the shortest laws for these groups. The lower bound derives from a bound on the rate of growth of element-orders in $\Gamma$, due to Bartholdi and Šunič. Being as it is based on complexity-bounds for words of a very specific type (power words), we expect that our lower bound in Theorem 1.4 is far from best-possible; indeed, we conjecture that Grigorchuk’s group has super-polynomial lawlessness growth.

Our second example is Thompson’s group $F$ of homeomorphisms of the interval. Brin and Squier [5] prove that $F$ has no non-abelian free subgroup and satisfies no law. Making their arguments effective, we obtain the following conclusion.

Theorem 1.5. Thompson’s group $F$ satisfies $\mathcal{A}_F^S(n) \ll n$.

If the word $w$ does not vanish under evaluation at the tuple $g$, then whenever $\pi : \Gamma \to Q$ is a homomorphism in whose kernel $\omega(g)$ does not lie, $\omega$ is not a law for $Q$. For residually finite lawless groups $\Gamma$, this observation draws a link between the behaviour of $\mathcal{A}_\Gamma^S$, the lengths of the shortest laws holding in finite groups, and the residual finiteness growth function $\mathcal{F}_\Gamma^S : \mathbb{N} \to \mathbb{N}$ due to Bou-Rabee [2]. For instance, in [3] the author and A. Thom proved the following result.
Theorem 1.6. Suppose \( \Gamma \) contains a nonabelian free subgroup and let \( \delta > 0 \). There exists \( C > 0 \) such that:

\[
F^S_\Gamma(n) \geq Cn^{3/2}/\log(n)^{9/2+\delta}.
\]

By the preceding paragraph, we deduce the following conclusion valid for all lawless groups, of which Theorem 1.6 is a special case (since by Theorem 1.1 groups satisfying the hypothesis of Theorem 1.6 are precisely those for which \( A_1^S \) is bounded).

Theorem 1.7. Let \( \Gamma \) be a finitely generated lawless group and let \( \delta > 0 \). There exists \( C > 0 \) such that:

\[
F^S_\Gamma(nA_1^S(n)) \geq Cn^{3/2}/\log(n)^{9/2+\delta}.
\]

We also have an analogous bound for the residual \( p \)-finiteness growth. Finally we note a connection between \( A_1^S \) and decision problems for recursively presented groups.

Theorem 1.8. If \( \Gamma \) is a finitely generated recursively presented lawless group with decidable word problem, then \( A_1^S \) is computable.

If, instead of word-maps, we consider word-maps with coefficients, then we have the class of MIF groups (instead of lawless groups) and we can define an MIF growth function \( M_\Gamma \). By contrast with Theorem 1.1 we have a nonconstant universal lower bound.

Theorem 1.9. There are no groups of bounded MIF growth. Indeed, for any finitely generated group \( \Gamma \), \( M_\Gamma(l) \gg \log(l) \).

We also have an effective version of the fact (due to G. Baumslag) that nonabelian free groups are MIF.

Theorem 1.10. Let \( \Gamma \) be a nonabelian free group of finite rank. Then \( M_\Gamma(n) \ll n\log(n) \).

1.2 Background and structure of the paper

There are some important similarities between the class of groups satisfying laws and the class of amenable groups: both classes are closed under taking subgroups; quotients and extensions; both contain all finite and abelian groups, and neither contains \( F_2 \). An important difference between the two classes is that an ascending union of amenable groups is amenable, whereas an ascending union of groups with laws may be lawless. Our Theorem 1.2 shows how stark a distinction is opened up by this property: by taking (an extension of) an ascending union of finite groups, we may construct amenable groups which are nonetheless “very” lawless.

In the other direction, there are known examples of nonamenable groups satisfying laws: for instance Adian showed that free Burnside groups of sufficiently large odd exponent are nonamenable. Such examples are, however, quite exotic. It is, for example, an open question whether every nonamenable residually finite group is lawless \([6]\). Our Theorem 1.7 shows that there is a tension between residual finiteness and lawlessness, in a quantitative sense: a group cannot be simultaneously “very” residually finite and “very” lawless.
There has also been great interest in recent years in estimating the lengths of the shortest laws for finite groups. One may view the following problem as the “finitary” version of the problem of estimating $A_\Gamma$ for a fixed lawless group: given a sequence $(G_n)$ of finite groups which do not share a common law, can we estimate the asymptotic behaviour of the length of the shortest law $w_n \in F_2$ for $G_n$ as $n \to \infty$? This latter problem has been explored with $G_n$ taken to be the symmetric group $\text{Sym}(n)$ [11]; a finite simple group of Lie type of bounded rank [4], or the direct product of all groups of order at most $n$ (see Theorem 8.2 below). We exploit analogies between the finite and infinite settings several times in this paper, most notably in Theorems 1.9 and 1.10.

The paper is structured as follows. In Section 2 we define $A_\Gamma^S$ and establish its basic properties. We also prove Theorem 1.8 in this Section. Sections 3-7 are devoted, respectively, to the proofs of Theorems 1.1-1.5. In Section 8 we establish connections between $A_\Gamma^S$ and quantitative versions of residual finiteness. In Section 9 we define $A_\Gamma^S$ and prove Theorems 1.9 and 1.10. We conclude with a selection of open problems, speculations and directions for future research.

2 Preliminaries

Let $|\cdot|_S : \Gamma \to \mathbb{N}$ be the word-length function induced on $\Gamma$ by $S$. Denote by $B_\Gamma(l)$ the set of elements of $\Gamma$ of length at most $l$. For $F_k$ the free group of rank $k$, we denote by $|\cdot|$ the word length on $F_k$ induced by a fixed free basis $X_k = \{x_1, \ldots, x_k\}$, and may write $B(l)$ for $B_{X_k}(l)$. The word $w \in F_k$ is a law for $\Gamma$ if it is non-trivial in $F_k$ and lies in the kernel of every homomorphism $F_k \to \Gamma$. The set of non-laws for $\Gamma$ in $F_k$ is:

$$N_k(\Gamma) = F_k \setminus \bigcap_{\pi} \ker(\pi)$$

where the intersection is taken over all homomorphisms $\pi : F_k \to \Gamma$. For $w \in N_k(\Gamma)$, we define the complexity of $w$ in $\Gamma$ (with respect to $S$) to be:

$$\chi^S_\Gamma(w) = \min \left\{ \sum_{i=1}^k |g_i|_S : g \in \Gamma^k, w(g) \neq 1 \right\}.$$

For $W \subseteq N_k(\Gamma)$, the $W$-lawlessness growth function (or anarchy growth; pandemonium growth etc.) of $\Gamma$ with respect to $S$ is defined to be:

$$A^S_{\Gamma,W}(l) = \max \left\{ \chi^S_\Gamma(w) : w \in W, |w| \leq l \right\}.$$

If $N_k(\Gamma) = F_k \setminus \{1\}$, we denote $A^S_{\Gamma,F_k\setminus\{1\}}$ by $A^S_{\Gamma}$ and refer to it simply as the lawlessness growth of $\Gamma$ (with respect to $S$).

**Lemma 2.1.** If $W' \subseteq W \subseteq N_k(\Gamma)$, then $A^S_{\Gamma,W'}(l) \geq A^S_{\Gamma,W}(l)$ for all $l \in \mathbb{N}$.

**Notation 2.2.** For nondecreasing functions $F_1, F_2 : \mathbb{N} \to \mathbb{N}$ we write $F_1 \preceq F_2$ if there exists $K \in \mathbb{N}$ such that $F_1(l) \leq K F_2(Kl)$ for all $l \in \mathbb{N}$, and $F_1 \approx F_2$ if $F_1 \preceq F_2$ and $F_2 \preceq F_1$. It is clear that $\approx$ is an equivalence relation.
Remark 2.3. In our notation $\mathcal{A}_s^S$ for the lawlessness growth function, we leave implicit the rank $k$ of the free group $F_k$ in which our words lie. We shall assume $k$ to be an arbitrary (but fixed) integer at least 2. It turns out that very little is lost by making this assumption. First suppose that $\Gamma$ has no law in $F_k$ and $1 \leq k' \leq k$. Embed $F_{k'}$ into $F_k$ by extending the set of free variables (with the inclusion map). Then for $w \in F_{k'}$ and $g_1, \ldots, g_{k'} \in \Gamma$, 
\[ w(g_1, \ldots, g_{k'}) = i(w)(g_1, \ldots, g_{k'}, e, \ldots, e), \text{ so } \chi^S_\Gamma (w) = \chi^S_\Gamma (i(w)). \] (2)
We apply Lemma 2.1 with $W' = \langle F_{k'} \rangle \setminus \{1\}$ and $W = F_k \setminus \{1\}$ so that by (2),
\[ \mathcal{A}_{\Gamma, F_k \setminus \{1\}}^S (l) \geq \mathcal{A}_{\Gamma, F_{k'} \setminus \{1\}}^S (l) \]
for all $l \in \mathbb{N}$. Conversely suppose that $2 \leq k' \leq k$ and that $\Gamma$ has no laws in $F_{k'}$. Let $\phi : F_k \hookrightarrow F_{k'}$ be an embedding, and for $1 \leq i \leq k$ write $\phi(x_i) = v_i \in F_{k'}$. For $w \in F_k$ non-trivial, $\phi(w) = w(v_1, \ldots, v_k)$ is not a law for $\Gamma$, so there are $g \in \Gamma^{k'}$ with $\phi(w)(g) = \phi(v_1(g), \ldots, v_k(g)) \neq e$. Hence the $k$-tuple $(v_1(g), \ldots, v_k(g))$ witnesses that $w$ is not a law for $\Gamma$, and that $\chi^S_\Gamma (w) \leq C \chi^S_{\Gamma, \Gamma'} (\phi(w))$, where $C = \max \{|v_i| : 1 \leq i \leq k\}$. Since $|\phi(w)| \leq C|w|$ and $|v_i(g)| \leq C(|g_1|_S + \ldots + |g_k|_S)$,
\[ \mathcal{A}_{\Gamma, F_{k'} \setminus \{1\}}^S (l) \leq Ck \mathcal{A}_{\Gamma, \Gamma'}^S (\langle \phi(F_k) \setminus \{1\} \rangle) \leq Ck \mathcal{A}_{\Gamma, F_{k'} \setminus \{1\}}^S (Cl) \]
(3) for all $l \in \mathbb{N}$. Thus $\mathcal{A}_{\Gamma, F_{k'} \setminus \{1\}}^S \approx \mathcal{A}_{\Gamma, F_k \setminus \{1\}}^S$.

Lemma 2.4. Let $\Delta \leq \Gamma$ be a subgroup generated by a finite set $T$. Then there exists $C > 0$ such that for all $1 \neq w \in F_k$, if $w$ is not a law for $\Delta$, then:
\[ \chi^T_\Delta (w) \geq C \chi^S_\Gamma (w). \] (4)
Thus, if $W \subseteq N_k(\Delta)$, then $\mathcal{A}_{\Delta, W}^T (l) \geq C \mathcal{A}_{\Gamma, W}^S (l)$ for all $l \in \mathbb{N}$.

Proof. There exists $C > 0$ such that for all $g \in \Gamma$, $|g|_T \geq C |g|_S$. (4) follows immediately. \qed

In view of Remark 2.3 and Lemma 2.4 we study $\mathcal{A}_\Gamma^S$ up to $\approx$.

Corollary 2.5. Let $S_1$ and $S_2$ be finite generating sets for $\Gamma$. Then $\mathcal{A}_{\Gamma, S_1}^S \approx \mathcal{A}_{\Gamma, S_2}^S$.

Lemma 2.6. Let $\pi : \Gamma \rightarrow Q$ be an epimorphism of groups. Then $N_k(Q) \subseteq N_k(\Gamma)$ and, for any $w \in N_k(Q)$,
\[ \chi^S_\Gamma (w) \leq \chi^{\pi(S)}_Q (w). \]
(5)
Thus, if $W \subseteq N_k(Q)$, then $\mathcal{A}_{\Gamma, W}^S (l) \leq \mathcal{A}_{\pi(W)}^S (l)$ for all $l$.

Example 2.7. Suppose that $\Gamma$ is a torsion group. The torsion growth function of $\Gamma$ (with respect to $S$) is defined to be:
\[ \pi^S_\Gamma (n) = \max \{ o(g) : |g|_S \leq n \} \]
and was introduced in [9]. In case \( \Gamma \) is a \( p \)-group of infinite exponent, there is an intimate connection between torsion growth and lawlessness growth: let \( \mathcal{W} = \{ x^{p^k} : k \in \mathbb{N} \} \subset F_2 \) be the set of \( p \)-power words. Then for \( m, n \in \mathbb{N} \),
\[
A_{\Gamma, \mathcal{W}}^S(p^m) \geq n + 1 \Leftrightarrow x_{\Gamma}^S(n) \leq p^m.
\]
Known finitely generated torsion \( p \)-groups of infinite exponent include various branch groups, for which the torsion growth has been estimated. We discuss this further in Section 6.

Our proof of Theorems 1.2 and 1.9 uses the following Proposition, which is Lemma 2.2 from [11]. For \( w \in F_k \) and \( \Gamma \) any group, let:
\[
Z(w, \Gamma) = \{ g \in \Gamma^k : w(g) = e \}
\]
be the vanishing set of \( w \) in \( \Gamma \).

**Proposition 2.8.** Let \( k \geq 2 \) and let \( w_1, \ldots, w_m \in F_k \) be nontrivial. There exists \( w \in F_k \) nontrivial such that, for any group \( \Gamma \),
\[
Z(w, \Gamma) \supseteq Z(w_1, \Gamma) \cup \cdots \cup Z(w_m, \Gamma)
\]
and \( |w| \leq 16m^2 \max |w_i| \).

**Corollary 2.9.** Let \( \Gamma \) be a lawless group and let \( k \geq 2 \). Then for all \( l \geq 1 \), there exist \( g_1, \ldots, g_k \in \Gamma \) such that, for all \( v \in F_k \) nontrivial with \( |v| \leq l \), \( v(g_1, \ldots, g_k) \neq e \).

**Proof.** Suppose not. Let \( w_1, \ldots, w_m \in F_k \) be a list of all nontrivial reduced words of length at most \( l \), and let \( w \) be as in the conclusion of Proposition 2.8. Then for every \( g_1, \ldots, g_k \in \Gamma \), there exists \( 1 \leq i \leq m \) such that \( (g_1, \ldots, g_k) \in Z(w_i, \Gamma) \subseteq Z(w, \Gamma) \), so \( w(g_1, \ldots, g_k) \), and therefore \( w \) is a law for \( \Gamma \).

**Proof of Theorem 1.8.** Let \( S \) be a finite generating set for \( \Gamma \). We are given an algorithm \textsc{WordProblem} which takes as input an element of \( F(S) \) and determines whether or not it evaluates to the identity element of \( \Gamma \). We describe an algorithm \textsc{Complex} which takes as input a word \( w \in F_k \) and returns \( \chi_{\Gamma}^S(w) \). Applying \textsc{Complex} to all \( w \) of length \( \leq n \), we compute \( A_{\Gamma, \mathcal{W}}^S(n) \).

At the \( m \)th step, \textsc{Complex} either verifies that \( \chi_{\Gamma}^S(w) \leq m \) and terminates, or verifies that \( \chi_{\Gamma}^S(w) > m \) and proceeds to the \((m + 1)\)th step. We thus establish the exact value of \( \chi_{\Gamma}^S(w) \).

The \( m \)th step of \textsc{Complex} runs as follows. Let \( B \) be a list of all elements of \( F(S) \) of length at most \( m \) (a finite set). For each ordered \( k \)-tuple \( \nu \) of (not necessarily distinct) elements of \( B \), apply \textsc{WordProblem} to \( w(\nu) \). Note that \( \chi_{\Gamma}^S(w) \leq m \) iff for some \( \nu, w(\nu) \) is non-trivial in \( \Gamma \).

**3 Bounded lawlessness growth**

We recall two basic facts about free groups. Let \( F \) be a free group of finite rank.
Lemma 3.1. Let $1 \neq N \lhd F$.

(i) If $|F : N| < \infty$, then $\text{rk}(N) - 1 = |F : N| \cdot (\text{rk}(F) - 1)$;

(ii) If $|F : N| = \infty$ then $\text{rk}(N) = \infty$.

In particular, if $F$ is nonabelian then $N$ is not cyclic.

Lemma 3.2. The centralizer of a non-trivial element of $F$ is a cyclic subgroup.

Proof. Suppose $a, b \in \Gamma$ with $(a, b) \cong F_2$. Let $C > 0$ with $|a|_S + |b|_S \leq C$. Then for all $1 \neq w \in F_2$, $w(a, b) \neq 1$, so $\chi^S_1(w) \leq C$. Thus $A^S_1$ is bounded.

Conversely suppose (for a contradiction) that $G^S_1$ is bounded but that $\Gamma$ has no subgroup isomorphic to $F_2$. Let $r \in \mathbb{N}$ be minimal such that there exist $(g_1, h_1), \ldots, (g_r, h_r) \in \Gamma \times \Gamma$ such that for all nontrivial $w(x, y) \in F_2$, there exists $1 \leq i \leq r$ such that $w(g_i, h_i) \neq 1$. Note that $r \geq 2$ (else $(g_1, h_1) \cong F_2$). By minimality of $r$, there exists $1 \neq w_r \in F_2$ such that:

$$w_r(g_1, h_1), \ldots, w_r(g_{r-1}, h_{r-1}) = 1.$$ 

Let $N = \{w \in F_2 : w(g_1, h_1) = 1\}$. Then $N \lhd F_2$ (it is the kernel of the homomorphism sending an ordered basis of $F_2$ to $g_1, h_1$) and $1 \neq N$ (for instance, $w_r \in N$). But $N \leq C_{F_2}(w_r)$, as otherwise any nontrivial word of the form $[w_r, v]$, for $v \in N$, would be a law for $\Gamma$ (contradicting hypothesis). By Lemma 3.2, $N$ is cyclic, contradicting Lemma 3.1. 

4 Amenable groups of slow lawlessness growth

In this Section we prove Theorem 1.1. We start with an elementary combinatorial fact.

Lemma 4.1. There exist increasing functions $p, q : \mathbb{N} \to \mathbb{N} \cup \{0\}$ such that $p(1) = q(1) = 0$ and, for all $i, j, k, l \in \mathbb{N}$,

(i) For $r = p$ or $q$, if $r(j) - r(k) = r(l) - r(i)$ then either (a) $i = k$ and $j = l$, or (b) $j = k$ and $i = l$;

(ii) If $q(j) - p(k) = q(l) - p(i)$ then $i = k$ and $j = l$.

Proof. We construct $p$ and $q$ via a recursive process. We have $p(1) = q(1) = 0$ and set $p(2) = 1, q(2) = 2$, so that (i) and (ii) hold for $i, j, k, l \in \{1, 2\}$. Supposing we have constructed $p(1) \leq q(1) < p(2) \leq q(2) < \ldots < p(n) \leq q(n)$ such that (i) and (ii) hold for all $1 \leq i, j, k, l \leq n$, we define $p(n+1)$ such that:

$$p(n+1) \geq p(n) + q(n) + 1 \tag{6}$$

and then define $q(n+1)$ by:

$$q(n+1) \geq p(n+1) + q(n) + 1 \tag{7}$$

so that $p(n) < q(n) < p(n+1) < q(n+1)$. We check that (i) and (ii) hold for all $1 \leq i, j, k, l \leq n + 1$. 

For (i), consider the equation \( p(j) - p(k) = p(l) - p(i) \). We may assume that \( l \geq j \geq k \). Since \( p \) is increasing, if \( j = k \) then \( i = l \) and if \( l = j \) then \( i = k \), so we may assume \( l > j > k \). We claim that \( p(j) - p(k) < p(l) - p(i) \). Subject to our assumptions, the maximal value of \( p(j) - p(k) \) is attained for \( j = l - 1 \) and \( k = 1 \), while the minimal value of \( p(l) - p(i) \) is attained for \( i = l - 1 \), and \( p(l - 1) - p(1) < p(l) - p(l - 1) \) by construction. The argument for \( q \) is exactly the same.

For (ii), consider the equation \( q(j) - p(k) = q(l) - p(i) \). We may assume \( l > j \) (as before, WLOG \( l \geq j \) and if \( l = j \) then \( i = k \) as \( p \) is increasing). We distinguish two cases. In the case that \( l \geq i \), we have \( j \geq k \) (since the quantity in the equation is non-negative) so that:

\[
q(j) - p(k) \leq q(l - 1) < q(l) - p(l) \leq q(l) - p(i).
\]

In the second case, \( l < i \) so \( j < k < i \) and:

\[
q(l) - q(j) \leq q(l) - p(i) - p(i - 1) \leq p(i) - p(k)
\]

so that \( q(l) - p(i) < q(j) - p(k) \). \( \square \)

**Remark 4.2.** Though we shall not need it in the sequel, it is not difficult to see that a minimal solution to the inequalities (6) and (7) yields functions \( p \) and \( q \) growing at most exponentially.

Fix a lawless elementary amenable group \( \Delta \) (not necessarily finitely generated). For example, we may take \( \Delta \) to be the direct sum of any sequence of finite groups which do not have a common law, such as the sequence of all finite symmetric groups \([11]\). By Corollary (2.29) for each \( l \geq 1 \) we have \( g_l, h_l \in \Delta \) such that for all \( v \in F_2 \) nontrivial, if \(|v| \leq l\) then \( v(g_l, h_l) \neq e \). Let \( L : \mathbb{N} \to \mathbb{N} \) be an increasing function to be determined.

Let \( \hat{g}, \hat{h} : \mathbb{Z} \to \Delta \) be defined as follows:

(i) For each \( n \in \mathbb{N} \), \( \hat{g}(p(n)) = g_{L(n)} \) and \( \hat{h}(q(n)) = h_{L(n)} \);

(ii) \( \hat{g}(m) = e \) for \( m \not\in \text{im}(p) \);

(iii) \( \hat{h}(m) = e \) for \( m \not\in \text{im}(q) \),

where \( p \) and \( q \) are as in Lemma (4.1). Let \( G = \Delta \text{ Wr } \mathbb{Z} = \Delta^Z \rtimes \mathbb{Z} \) be the unrestricted wreath product of \( \Delta \) and \( \mathbb{Z} \), with the \( \mathbb{Z} \)-factor being generated by \( t \). Let \( S = S(L) = \{ \hat{g}, \hat{h}, t \} \) and define \( \Gamma = \Gamma(L) = \langle S(L) \rangle \leq G \). We claim that, for an appropriate choice of the function \( L \), the group \( \Gamma \) satisfies the conclusion of Theorem (1.2), the proof of which we divide between the next two results.

**Theorem 4.3.** For any \( L, \Gamma \) is elementary amenable.

*Proof.* Let \( N = \Gamma \cap \Delta^Z \). Then \( N \lhd \Gamma \) with \( \Gamma / N \cong \mathbb{Z} \), so it suffices to check that \( N \) is elementary amenable. \( N / [N, N] \) is a countable abelian group, so it suffices to check that \([N, N]\) is elementary amenable.

The group \( N \) is generated by \( X = \{ \hat{g}^n, \hat{h}^n : n \in \mathbb{Z} \} \), so \([N, N] \) is normally generated in \( N \) by \( Y = \{ [\hat{f}_1, \hat{f}_2] : \hat{f}_1, \hat{f}_2 \in X \} \). If \( Y \subseteq \bigoplus \Delta \), then \([N, N] \leq \bigoplus \Delta \)
(since $\bigoplus \Delta \ll \Delta^\mathbb{Z}$, so that $[N, N]$ is elementary amenable (being a subgroup of the elementary amenable group $\bigoplus \Delta$).

We therefore claim that $Y \subseteq \bigoplus \Delta$. Recall that the support of $\hat{f} \in \Delta^\mathbb{Z}$ is $\operatorname{supp}(\hat{f}) = \{ n \in \mathbb{Z} : \hat{f}(n) \neq e \}$, so that $\bigoplus \Delta$ is precisely the group of finite-support elements of $\Delta^\mathbb{Z}$. We assert that:

(a) for every $\hat{f}_1, \hat{f}_2 \in \Delta^\mathbb{Z}$, $\operatorname{supp}(\hat{f}_1 \hat{f}_2) \subseteq \operatorname{supp}(\hat{f}_1) \cap \operatorname{supp}(\hat{f}_2)$;

(b) For all pairs of distinct elements $\hat{f}_1, \hat{f}_2 \in X$, $|\operatorname{supp}(\hat{f}_1) \cap \operatorname{supp}(\hat{f}_2)| \leq 1$,

whence the desired claim. Observation (a) is clear, and (b) follows from Lemma 4.1.

Proof. Let $\hat{f} \in \Delta^\mathbb{Z}$ be arbitrary (but fixed) prime number. We follow the treatment of Golod-Shafarevich.

For every unbounded nondecreasing function $f : \mathbb{N} \to \mathbb{N}$ with $f(1) \geq 2$, there exists $L$ such that for all $n \in \mathbb{N}$, $\mathcal{A}^S_{\Gamma(L)}(n) \leq f(n)$.

Proof. Let $m \in \mathbb{N}$ and let $w \in F_2$ be nontrivial with $|w| \leq L(m)$. Then:

$$ w(g^{p(m)-p(m)}, \hat{h})(q(m)) = w(g_{L(m)}, h_{L(m)}) \neq e $$

so that $w$ is not a law for $\Gamma(L)$ and:

$$ \mathcal{A}^S_{\Gamma(L)}(w) \leq \left| g^{p(m)-p(m)} \right|_{S(L)} + \left| \hat{h} \right|_{S(L)} \leq 2(q(m) - p(m) + 1). \quad (8) $$

We require $L(m) \in \mathbb{N}$ sufficiently large that $f(L(m)) \geq 2(q(m + 1) - p(m + 1) + 1)$ (possible since $f$ is unbounded).

Now let $n \in \mathbb{N}$. Suppose first that $n \geq L(1)$. Let $m \in \mathbb{N}$ with $L(m) \leq n \leq L(m + 1)$. Then:

$$ \mathcal{A}^S_{\Gamma(L)}(n) \leq \mathcal{A}^S_{\Gamma(L)}(n) \leq \mathcal{A}^S_{\Gamma(L)}(L(m + 1)) $$

$$ \leq 2(q(m + 1) - p(m + 1) + 1) \quad \text{(by (8))} $$

$$ \leq f(L(m)) $$

$$ \leq f(n). $$

On the other hand, if $n \leq L(1)$ then for any $w \in F_2$ nontrivial with $|w| \leq n$, $w(\hat{g}, \hat{h}) \neq e$, so $\mathcal{A}^S_{\Gamma(L)}(w) \leq 2$ and $\mathcal{A}^S_{\Gamma(L)}(n) \leq 2$. \hfill $\square$

5 Golod-Shafarevich groups of linear lawlessness growth

The goal of this Section is to prove Theorem 1.3. Throughout this Section $p$ is an arbitrary (but fixed) prime number. We follow the treatment of Golod-Shafarevich.
groups from Chapter 3 of [8]. For \( \Gamma \) an abstract group, denote by \( \hat{\Gamma}_{(p)} \) the pro-\( p \) completion of \( \Gamma \). Let \( \mathbb{F}_p\langle U_k \rangle \) be the algebra of power-series in the non-commuting variables \( U_k = \{u_1, \ldots, u_k\} \) over \( \mathbb{F}_p \). For \( f \in \mathbb{F}_p\langle U_k \rangle \), the degree \( \text{deg}(f) \) of \( f \) is the minimal length of a monomial occurring in \( f \) with nonzero coefficient. Let \( \hat{F}_k \) be a free pro-\( p \) group on the finite set \( X_k = \{x_1, \ldots, x_k\} \). There is a continuous monomorphism (the Magnus embedding) \( \mu : \hat{F}_k \hookrightarrow \mathbb{F}_p\langle U_k \rangle^\times \) extending \( x_i \mapsto 1 + u_i \) [8].

**Definition 5.1.** (i) A pro-\( p \) group \( G \) is Golod-Shafarevich if it admits a pro-\( p \) presentation \( G = \langle X_k \mid R \rangle \) such that there exists \( \tau \in (0, 1) \) with:

\[
1 - k\tau + \sum_{r \in R} r^{D(r)} < 0,
\]

where for \( r \in \hat{F} \), \( D(r) = \text{deg}(\mu(r) - 1) \) is the degree of \( r \);

(ii) An abstract group \( \Gamma \) is Golod-Shafarevich if \( \hat{\Gamma}_{(p)} \) is a Golod-Shafarevich pro-\( p \) group.

All the relations in the presentations we construct will be \( p \)-powers, and estimating the degrees of these is easily done, direct from the definition of the function \( D \).

**Lemma 5.2.** For all \( w \in \hat{F}_k \), \( D(p^r) = pD(w) \).

The next, basic Lemma immediately implies that if \( G = \langle X_k \mid R \rangle \) is an abstract group presentation satisfying (9), then \( G \) is an abstract Golod-Shafarevich group.

**Lemma 5.3.** Let \( \Gamma \) be an abstract group, and suppose \( \langle S \mid R \rangle \) is an abstract group presentation for \( \Gamma \). Then \( \langle S \mid R \rangle \) is a pro-\( p \) presentation for \( \hat{\Gamma}_{(p)} \).

Finally, we need a guarantee that the groups we construct are indeed lawless (see [14] p.224).

**Theorem 5.4.** Let \( G \) be a Golod-Shafarevich pro-\( p \) group. Then \( G \) has a non-abelian free subgroup.

**Corollary 5.5.** Suppose \( \Gamma \) is an (abstract) Golod-Shafarevich group. Then \( \Gamma \) is lawless.

**Proof.** Let \( 1 \neq w \in F_2 \). Let \( \mathcal{N} \) be the inverse system of normal subgroups of \( p \)-power index in \( \Gamma \), and let \( G = \lim_{\Gamma \in \mathcal{N}} \Gamma / N \) be the pro-\( p \) completion of \( \Gamma \). By Theorem 5.4 there exist \( g, h \in G \) freely generating a rank-2 free subgroup of \( G \). Then there exists \( N \in \mathcal{N} \) such that the image of \( w(g, h) \) is non-trivial in \( \Gamma / N \). In particular, \( w \) is not a law for \( \Gamma / N \). But \( \Gamma / N \) is also a quotient of \( \Gamma \), so \( w \) is not a law for \( \Gamma \) either.

Theorem 5.3 is immediate from Corollary 5.5 and the next result. Recall that \( \pi_1^S \) is the torsion growth function of \( \Gamma \) (see Example 2.7 above).

**Proposition 5.6.** For all \( k \geq 2 \) there is a torsion Golod-Shafarevich pro-\( p \) group \( \Gamma = \langle X_k \mid R' \rangle \), and a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \),

\[
\pi_1^\infty(X_k)(n) \leq Cn.
\]
Proof of Theorem 1.3. Let \( v \) be as in Proposition 5.6. Let \( W = \{ x^k : k \in \mathbb{N} \} \). Then for all \( n \in \mathbb{N} \),
\[
A_k^X(Cn) \geq A_k^{X, W}(Cn) \geq n + 1
\]
(by Lemma 2.1 and Example 2.7).

Proof of Proposition 5.6. Let \( w_1, w_2, w_3 \ldots \) be an enumeration of the non-trivial elements of \( F(X_k) \), ordered such that \( |w_n| \) is non-decreasing. Recall that there exists \( C > 0 \) such that \( |B_{X_k}(i)| \leq C(2k - 1)^i \). Choose \( q > 1 \) and let \( 0 < c < \log(q)/\log(2k - 1) \), so that:
\[
a_m := Cq^m \geq \left| B_{X_k}(c^m) \right|.
\]
for \( m \geq 1 \) (with \( a_0 := 0 \)). Choose \( m_0 \in \mathbb{N} \) and set \( r_n = w_n^{m+m_0} \) for \( a_{m-1} + 1 \leq n \leq a_m \) and \( R' = \{ r_n : n \in \mathbb{N} \} \). Let \( \Gamma = \langle X_k | R' \rangle \) and let \( g \in \Gamma \). If \( m \in \mathbb{N} \) is such that \( c^m \leq g |X_k| \leq c^{m+1} \), then there exists \( 1 \leq n \leq a_m \) such that \( g = w_n \) in \( \Gamma \), so that the order of \( g \) in \( \Gamma \) divides \( p^{m+m_0} \leq p^{m_0+1} |g|_{X_k}/c \), so \( \tau^X_{\Gamma} \) grows at most linearly.

It therefore suffices to check that we can choose \( q \) and \( m_0 \) such that \( \Gamma \) is Golod-Shafarevich. By Lemma 5.2, \( D(r_n) \geq p^{m+1} \) for \( n > a_m \), hence for \( \tau \in (0, 1) \),
\[
\sum_{r \in R'} \tau^D(r) \leq \sum_{m=1}^{\infty} a_m^m \tau^{m+m_0} = C \sum_{m=1}^{\infty} \left( q \tau^{p^m} \right)^m = C \sum_{m=1}^{\infty} h^m
\]
where \( h = q \tau^{p^m} \). It is clear that the right-hand side of (12) can be made arbitrarily small by making \( h \) arbitrarily small. If we take \( q = 2 \) and \( \tau = 3/4 \), then we have (9) provided \( h \) is sufficiently small that the right-hand side of (12) is \( < (3k - 4)/4 \). This can be achieved for \( m_0 \) larger than an absolute constant.

Remark 5.7. It is not difficult to strengthen the proof of Proposition 5.6 to show that if \( \Gamma \) is Golod-Shafarevich, then there is a torsion Golod-Shafarevich \( p \)-group \( \Gamma \), a surjective homomorphism \( \pi : \Gamma \to \Gamma \) and a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \),
\[
\pi^\pi(S)(n) \leq Cn.
\]
For, the pro-\( p \) completion of \( \Gamma \) already has a presentation satisfying (9). Adding in the relations \( R' \) from the proof of Proposition 5.6 yields a presentation for the pro-\( p \) completion of \( \Gamma \), which has at most linear torsion growth. But we can make the contribution of \( R' \) to the left-hand side of (9) arbitrarily small, so that the inequality (9) still holds, and \( \Gamma \) is still Golod-Shafarevich.

6 Grigorchuk’s group

For the duration of this Section, \( \Gamma \) will denote the first Grigorchuk group. For background on this group and automorphisms of rooted trees more generally, we refer to [10]. The group \( \Gamma \) is defined as a group of automorphisms of the binary rooted tree \( T : V(T) = \{ 0, 1 \}^* \) is the set of finite formal words in the alphabet \( \{ 0, 1 \} \), and for each \( v \in \{ 0, 1 \}^* \), \( \epsilon \in \{ 0, 1 \} \), there is an edge joining \( v \) to \( ve \). For each \( v \in V(T) \), let \( T_v \) be
the subtree rooted at $v$, that is the induced subgraph on $\{vw : w \in V(\mathcal{T})\}$. Note that $\mathcal{T}_v \cong \mathcal{T}$ via $vw \mapsto w$. Let:

$$\text{top}(v) = \{g \in \text{Aut}(\mathcal{T}) : w^g = w \text{ for all } w \in V(\mathcal{T}) \setminus V(\mathcal{T}_v)\}$$

be the restricted stabilizer at $v$. Then for every $v$, the above isomorphism $\mathcal{T}_v \cong \mathcal{T}$ induces an isomorphism $\text{top}(v) \to \text{Aut}(\mathcal{T})$; we write $g \mid_{\mathcal{T}_v} \in \text{Aut}(\mathcal{T})$ for the image under this isomorphism of $g \in \text{top}(v)$.

Let $V_n = \{0, 1\}^n$ be the set of words of length $n$ (geometrically, the set of vertices at distance $n$ from the root of the tree). Let $\text{Stab}(n) \leq \text{Aut}(\mathcal{T})$ be the pointwise stabilizer of $V_n$. Then for $g \in \text{Stab}(n)$, there exist unique $g_v \in \text{top}(v)$, for $v \in V_n$, such that:

$$g = \prod_{v \in V_n} g_v$$

(note that the $g_v$, being disjointly supported, commute). Extending our notation above, we write $g \mid_{\mathcal{T}_v} = g_v \mid_{\mathcal{T}_v}$ for $g \in \text{Stab}(n)$, $v \in V_n$ and $g_v \in \text{top}(v)$ as in (14). The decomposition (14) yields an isomorphism $\text{Aut}(\mathcal{T}) \cong \text{Stab}(n)$, and for $K \leq \text{Aut}(\mathcal{T})$ we shall identify $K^{V_n}$ with its image in $\text{Aut}(\mathcal{T})$ under this isomorphism. Likewise, when $g$ and $g_v$ are as in (14) we shall write $g = (g_v)_{v \in V_n}$.

The automorphisms $a, b, c$ and $d$ of $\mathcal{T}$ are defined as follows:

\begin{align*}
(0w)^a &= 1w; & (1w)^a &= 0w; \\
(0w)^b &= 0(0w); & (1w)^b &= 1(w^c); \\
(0w)^c &= 0(0w); & (1w)^c &= 1(w^d); \\
(0w)^d &= 0w; & (1w)^d &= 1(w^b)
\end{align*}

for any $w \in \{0, 1\}^*$. In other words, $a$ swaps the two subtrees $\mathcal{T}_0$ and $\mathcal{T}_1$, and $b, c, d \in \text{Stab}(1)$ are given by:

$b = (a, c), c = (a, d), d = (1, b)$.

Grigorchuk’s group $\Gamma$ is defined to be the subgroup of $\text{Aut}(\mathcal{T})$ generated by $S = \{a, b, c, d\}$. Let $x = abab \in \Gamma$ and let $K = \langle x \rangle^\Gamma \leq \Gamma$ be the normal closure of $x$ in $\Gamma$.

**Proposition 6.1** (10, Proposition 8). We have $K^{V_1} \leq K$ (and hence $K^{V_n} \leq K$ for all $n$).

**Proposition 6.2.** Let $y_n \in K^{V_n}$ be given by:

$$y_n \mid_{\mathcal{T}_v} = \begin{cases} x & v = 0^n \\ e & \text{otherwise} \end{cases}$$

for $v \in V_n$. Then there exists $C > 0$ such that for all $n$, $|y_n|_S \leq C(1 + \sqrt{3})^n$.

**Proof.** We have $y_0 = x$ and $y_1 = y = (x, 1)$. As noted in the proof of (10, Proposition 9), $[x, y] = (x^{-1}, 1, 1, 1) = y_2^{-1}$, so $y_2 = [y_1, y_0]$. Since for $n \geq 1$ we have $y_n = (y_{n-1}, 1)$, so by induction, for $n \geq 3$ we have:

$$y_n = (y_{n-1}, 1) = ([y_{n-2}, y_{n-3}], 1) = (y_{n-2}, 1), (y_{n-3}, 1) = [y_{n-1}, y_{n-2}].$$

(15)
Now there exists $C_0 > 0$ such that $|y_{0}|_S, |y_{1}|_S ≤ C_0$ (the latter since $y_{1} ∈ K^{V_1} ≤ K ≤ Γ$), and by (15) we have $|y_{n}|_S ≤ 2|y_{n-1}|_S + |y_{n-2}|_S$ for $n ≥ 2$. Solving the corresponding recurrence yields the required bound.

Let $W_n$ be the $(n+1)$-fold iterated regular wreath product of $C_2$; that is $W_0 = C_2$ and $W_{n+1} = W_n \wr C_2$. Alternatively we may view $W_n$ as a subgroup of $\text{Aut}(T)$, as follows: let $a ∈ \text{Aut}(T)$ be as above, and identify $W_0 = C_2$ with $\langle a \rangle ≤ \text{Aut}(T)$. Having defined $W_n ≤ \text{Aut}(T)$, a general element of $W_{n+1}$ is $(g_0, g_1)α ∈ W_n \wr C_2$ (with $g_0, g_1 ∈ W_n$ and $c ∈ \{0, 1\}$). We identify this with the unique $g ∈ \text{Aut}(T)$ satisfying:

$$gα ∈ \text{Stab}(1) \text{ and } (gα)|_t = g_v \text{ for } v ∈ V_1.$$ 

It is easily seen that this identification yields an embedding of $W_{n+1}$ as a subgroup of $\text{Aut}(T)$. Moreover, the action of $W_n ≤ \text{Aut}(T)$ on $V_{n+1}$ is faithful and yields an isomorphism of permutation groups from $W_{n+1}$ to the imprimitive permutational wreath product $\langle a \rangle \wr_{n+1} W_n$ (since $V_{n+2} = V_{n+1} × V_1$ via the identification of $v c$ with $(v, c)$).

Define, for $n ∈ \mathbb{N}$, $a_n ∈ \text{Aut}(T)$ by: $a_0 = a$ and $a_{n+1} ∈ \text{top}((0^{n+1})$ by:

$$a_{n+1}|_{t^{n+1}} = a$$

so that $S_n = \{a_0, a_1, \ldots, a_n\}$ generates $W_n$.

**Lemma 6.3.** For any $v ∈ V_{n+1}$, there exists $h ∈ W_n$ with $|h|_{S_n} ≤ n+1$ and $(0^{n+1})^h = v$.

**Proof.** If $v = ε_0ε_1 \cdots ε_n$, then $h = a_n^{ε_n} a_{n-1}^{ε_{n-1}} \cdots a_1^{ε_1} a_0^{ε_0}$ works.

**Proposition 6.4.** $W_n$ has no law in $F_k$ of length at most $n + 1$. In other words, $B(n + 1) \setminus \{1\} ⊆ N_k(W_n)$. More precisely, for any $1 ≠ w ∈ F_k$ with $|w| ≤ n + 1$, there exist $g_1, \ldots, g_k ∈ W_n$ with:

$$w(g_1, \ldots, g_k) ≠ e \text{ and } ∑_{i=1}^{k} |g_i|_{S_n} ≤ (n + 1)^2. \tag{16}$$

**Proof.** We proceed by induction on $n$. In fact, we make a stronger claim. For $w ∈ F_k$ and $0 ≤ m ≤ |w|$ define $w^{(m)}$ to be the $m$-prefix of $w$, that is, $w^{(m)} ∈ F_k$ is the unique element satisfying (i) $|w^{(m)}| = m$ and (ii) there exists $w^{(m)} ∈ F_k$ such that $|w^{(m)}| = |w| - m$ and $w = w^{(m)}w^{(m)}$ (so that $w^{(0)}$ is the empty word and $w^{(|w|)} = w$). Our claim is that for all $w ∈ F_k$ with $|w| ≤ n + 1$, there exists $g = (g_1, \ldots, g_k) ∈ W_n^k$ with:

$$∑_{i=1}^{k} |g_i|_{S_n} ≤ (n + 1)^2$$

such that the points:

$$v_0 = 0^{n+1}, v_1 = t_0^{w(0)}, v_2 = t_0^{w(1)}(g), \ldots, v_0^{w(|w|)}(g) = t_0^{v(g)}$$
are all distinct. In particular, \(v_0 \neq v_0^{w(g)}\), so \(w(g) \neq e\), and (16) follows. The claim clearly holds for \(n = 0\).

Let \(n \geq 1\), let \(1 \neq u \in F_k \) with \(2 \leq |u| \leq n+1\) (if \(|u| = 1\) the claim is trivial) and suppose the claim fails for \(u\) and \(W_{n+1}\). Let \(u = u^{(|u|-1)}\), WLOG (by permuting and inverting the variables \(x_i\)) we may assume that \(u = u x_1\). Since \(1 \neq u, |u| \leq n\), we may assume by induction that there exists \(g' = (g'_1, \ldots, g'_k) \in W^k_{n-1}\) witnessing the truth of the claim for \(u\) and \(W_{n-1}\). Let \(v'_0 = (0^{|u|})^{w(u)}(g') \in V_n\) for \(0 \leq i \leq |u|\) (a sequence of \(|u| + 1\) distinct points). Now consider the \(g_j'\) as elements of \(W_n = \langle a \rangle \gamma v' \| W_{n-1}\), acting naturally on \(V_{n+1}\), and consider the points \(\tilde{v}_i = (0^{i+1})^{w(u)}(g') \) for \(0 \leq i \leq |u|\) (note \(w(u)\) is a reduced word so the final letter of \(u\) is not \(x_i^{-1}\)) these points are distinct. But then:

\[
\tilde{v}_0 = \tilde{v}_0' = 0^{n+1}.
\]

Let \(h \in W_{n-1}\) with \(|h|_{S_{n-1}} \leq n\) and \((0^{|u|}) h = v'_a\): such exists by Lemma 6.3. Then:

\[
(h^{-1} a h)^{v'_0} = a, \text{ and } |h^{-1} a h|_{S_n} \leq 2n + 1.
\]

Set \(g_1 = h^{-1} a h g_1'\) and \(g_j = g_j'\) for \(2 \leq j \leq k\). Consider \(v_i = (0^{i+1})^{w(u)}(g)\). We have \(v_i = \tilde{v}_i\) for \(0 \leq i \leq |u|-1\) since \(v'_0 \neq v'_a\) for \(i \leq |u| - 1\) (note \(w(u)\) is a reduced word so the final letter of \(u\) is not \(x_i^{-1}\)) these points are distinct. But then:

\[
v_i |_{v'_0} = (v'_0)^{w(u)}(h^{-1} a h)^{v'_0} = (v'_0)^{w(u)}(h^{-1} a h)^{v'_0} = 0^{n+1}
\]

differs in its final letter from all \(v_i = v'_0\) for \(i \leq |u|\). Moreover,

\[
\sum_{i=1}^k |g_i|_{S_n} \leq (2n + 1) + \sum_{i=1}^k |g_i'|_{S_n}.
\]

The claim follows. \(\square\)

The next Proposition is essentially proved in the course of Proposition 10 of [10]. We include a proof for the reader’s convenience.

**Proposition 6.5.** For each \(n \in \mathbb{N}\) there is an injective homomorphism \(\Phi_n : W_n \rightarrow \Gamma\) sending \(a_i\) to \(k_{i+1} = x_{3i}^4\) for \(0 \leq i \leq n\).

**Proof.** We follow the proof of Proposition 4 of [10], specialized to our setting. We have \(k_{i+1} \in \text{top}(0^5)\), with \(k_{i+1} |_{a} = x^4\), and \(x^4 \in \text{Stab}(4)\) is given, for \(v \in V_3\), by:

\[
x^4 |_{v_0} = a \text{ and } x^4 |_{v_i} = c;
\]

in particular, every \(k_{i+1}\) has order 2. We prove the conclusion by induction on \(n\). Certainly \(W_0 = C_2 \cong \langle k_1 \rangle \leq \Gamma\), so the conclusion holds for \(n = 0\). Let \(Q_n = \langle k_1, \ldots, k_n \rangle \leq \Gamma\); we suppose by induction that there is an isomorphism \(\Phi_n : W_{n-1} \rightarrow Q_n\) sending \(a_i\) to \(k_{i+1}\) for \(0 \leq i \leq n-1\). Note that the natural isomorphism \(\text{Aut}(T) \cong \text{top}(0^5)\) (induced by the isomorphism of trees \(T \cong T_{(0)}\) sends \(k_i\) to \(k_{i+1}\): the restriction of this map to \(Q_n\) yields an isomorphism from \(Q_n\) to \(P_n = \langle k_2, \ldots, k_{n+1} \rangle\). Similarly the natural isomorphism \(\text{Aut}(T) \cong \text{top}(0)\) sends \(a_i\) to \(a_{i+1}\), so restricts to an isomorphism from \(W_{n-1}\) to
Theorem 6.7. For the upper bound, let 

\[ \Psi_n : P_n \rightarrow W_n \]

sending \( k_{i+1} \) to \( a_i \) for \( 1 \leq i \leq n \). We shall extend \( \Psi_n \) from \( P_n \) to \( Q_{n+1} \).

Since \( P_n \leq \text{top}(0^5) \) and \( k_1 |_{F_{12}} = a \), we have \( P_n^{k_1} \leq \text{top}(0^4) \), so \( P_n, P_n^{k_1} \leq \Gamma \) generate their direct product, and \( k_1 \) (being of order 2) acts by conjugation by swapping the two factors. Thus \( Q_{n+1} = \langle k_1, P_n \rangle \cong P_n \wr C_2 \), with the \( C_2 \)-factor generated by \( k_1 \). To finish, we compose with the isomorphism \( P_n \wr C_2 \cong W_{n-1} \wr C_2 = W_{n+1} \) induced by \( P_n \cong W_{n-1} \).

One of the original motivations for introducing \( \Gamma \) was the following, now-famous result.

**Theorem 6.6.** For all \( g \in \Gamma \), there exists \( k \in \mathbb{N} \) such that \( g^{2^k} = 1 \).

As we have seen (Example 2.7 above) slow torsion growth yields fast lawlessness growth for \( p \)-groups. The following is a consequence of Theorem 7.7 of [1] (see also the bullet-points at the end of Section 1 of that paper).

**Theorem 6.7.** There exists \( C > 0 \) such that for all \( g \in \Gamma \), \( o(g) \leq C|g|^{3/2} \).

**Proof of Theorem 6.4** For the upper bound, let \( n \in \mathbb{N} \) and let \( 1 \neq w \in F_k \) with \( |w| \leq n + 1 \). Let \( g_1, \ldots, g_k \in W_n \) be as in Proposition 6.4 and let \( \Phi_n : W_n \rightarrow \Gamma \) be as in Proposition 6.5. We have:

\[ e \neq \Phi_n(w(g_1, \ldots, g_k)) = w(\Phi_n(g_1), \ldots, \Phi_n(g_k)) \]

so that:

\[ \chi^S_i(w) \leq \sum_{i=1}^k |\Phi_n(g_i)| \leq 4C(1 + \sqrt{3})^5n \]

\[ \sum_{i=1}^k |g_i| \leq 4C(n + 1)^2(1 + \sqrt{3})^5n \]

(the second inequality being by Proposition 6.2). Thus:

\[ A^S_i(n) \leq 4C(n + 1)^2(1 + \sqrt{3})^5n \leq \exp(n) \]

For the lower bound, let \( w_m(x, y) = x^{2^m} \in F_2 \). Suppose that \( g, h \in \Gamma \) satisfies \( |g|, |h| \leq l \). Then by Theorems 6.6 and 6.7 \( w_m(g, h) = e \) for all \( m \leq \log_2 C + (3/2) \log_2 l \). Thus:

\[ A^S_i(2^m) \geq \chi^S_i(w_m) \geq l + 1 \geq 2^{m/3}/C. \]

\[ \square \]

### 7 Thompson’s group \( \text{F} \)

In this Section we prove Theorem 1.5. We adopt the following model for Thompson’s group \( \text{F} \).
Definition 7.1. **F** is the subgroup of Homeo(\(\mathbb{R}\)) generated by the homeomorphisms \(A\) and \(B\), where:

\[
A(x) = x + 1 \text{ and } B(x) = \begin{cases} 
    x & x \leq 0 \\
    2x & x \in (0, 1] \\
    x + 1 & x > 1 
\end{cases}
\]

Proposition 7.2. **F** is the group of orientation-preserving piecewise-linear homeomorphisms \(H\) of \(\mathbb{R}\) which are differentiable except at finitely many dyadic rational numbers; all of whose slopes are powers of 2, and such that there exist integers \(K\) and \(L\) such that \(H(x) = x + K\) for all \(x \in \mathbb{R}\) sufficiently large and \(H(x) = x + L\) for all \(x \in \mathbb{R}\) sufficiently small.

Our proof of Theorem 1.5 is based closely on [5], where the following was proved.

Theorem 7.3 (Brin–Squier). The group \(\mathbb{F}\) is lawless.

The argument presented in [5] is constructive, and we extract from it a linear upper bound on \(A_\mathbb{F}\), as follows. Define \(T : \mathbb{R} \to \mathbb{R}\) on each interval \([8n, 8(n+1)]\) by:

\[
T(x) = \begin{cases} 
    2(x - 4n) & x \in [8n, 8n + 1] \\
    x + 1 & x \in [8n + 1, 8n + 2] \\
    (x + (8n + 4))/2 & x \in [8n + 2, 8n + 6] \\
    x - 1 & x \in [8n + 6, 8n + 7] \\
    2(x - 4(n + 1)) & x \in [8n + 7, 8(n + 1)] 
\end{cases}
\]

(so that \(T\) restricts to an orientation-preserving homeomorphism of \([8n, 8(n+1)]\)) and set \(U = T^2\). For \(n \in \mathbb{N}\) define \(U_n : \mathbb{R} \to \mathbb{R}\) by:

\[
U_n(x) = \begin{cases} 
    U(x) & 0 \leq x \leq 8(n + 1) \\
    x & \text{otherwise} 
\end{cases}
\]

and \(V_n = A^2U_nA^{-2}\), so that \(U_n\) and \(V_n \in \mathbb{F}\).

Lemma 7.4. There exists \(C > 0\) such that for all \(n \in \mathbb{N}\),

\[
\left|U_n\right|_{S}\left|V_n\right|_{S} \leq C(n + 1).
\]

Proof. By Proposition 7.2 \(U_0 \in \mathbb{F}\). Let \(M = |U_0|_{S} \in \mathbb{N}\). Then \(U_{n+1} = A^8U_nA^{-8}U_0\), so \(|U_n|_{S} \leq (M + 16)n + M\) and \(|V_n|_{S} \leq (M + 16)n + M + 4\).

Proposition 7.5. Let \(w \in F_2\) be non-trivial, with \(|w| \leq n\). Then \(w(U_n, V_n) \neq e\).

Proof. This is essentially the content of [5] Section 4; we give a sketch. In [5] there are defined homeomorphisms \(f_0, f_1\) of \(S^1\), such that \(f_0^2, f_1^2\) freely generate a rank-2 free subgroup of Homeo(\(S^1\)). This is shown using a ping-pong argument. Identifying \(S^1\) with \(\mathbb{R}/2\mathbb{Z}\) and letting \(\pi : \mathbb{R} \to S^1\) be the induced covering map, it follows that for any continuous lifts \(\tilde{f}_0, \tilde{f}_1\) of \(f_0, f_1\) to this cover, \(\tilde{f}_0^2, \tilde{f}_1^2\) freely generate a rank-2 free
subgroup of Homeo(ℝ). As per the description of f₀ given in [5], our map T is such a lift of f₀ to ℝ fixing 0.

It is then shown that similarly, if f₀ is a continuous lift of f₀ to ℝ fixing 0; and g₀ : ℝ → ℝ is given by:

\[ g₀(t) = \begin{cases} f₀²(t) & t \in [-4(n + 1), 4(n + 1)] \\ \text{otherwise} & \end{cases} \]

(so that g₀ ∈ Homeo(ℝ), and g₁ = A²gA⁻², then w(g₀, g₁) ≠ e in Homeo(ℝ) (T₂ is the notation of [5] is A in our notation and actions in [5] are on the right while ours are on the left). Taking f₀ = T as above, we have U₀ = A⁴(n+1)g₀A⁻⁴(n+1) and V₀ = A^(k(n+1))g₁A⁻^(k(n+1)), so w(U₀, V₀) = A^(k(n+1))w(g₀, g₁)A⁻^(k(n+1)) ≠ e also. ☐

Proof of Theorem[7,5] By Lemma[7,4] and Proposition[7,5] any non-trivial w ∈ F₂, with |w| ≤ n, satisfies χ_{F}[A,B](w) ≤ 2C(n+1).

\[ \text{Proposition 8.1.} \]

Let C be a class of finite groups. Recall that Γ is residually C if, for every 1 ≠ g ∈ Γ, there exists Q ∈ C and a surjective homomorphism π : Γ → Q such that π(g) ≠ 1. In this case we denote by DΓ,C(g) the minimal value of |Q| among Q ∈ C admitting such a homomorphism π. If S is a finite generating set for Γ, then the residual C-finiteness growth function of Γ (with respect to S) is:

\[ \mathcal{F}^{S}_{C,G}(n) = \max \{ D_{Γ,C}(g) : |g|^S ≤ n \}. \]

Residual finiteness growth was introduced by Bou-Rabee [2] and has been extensively studied for a wide variety of residually finite groups. Two classes C which are of particular interest are the class of all finite groups, and the class of all p-groups (for p a fixed prime). In these cases we denote the function \( \mathcal{F}^{S}_{C,G} \) by \( \mathcal{F}^{S}_{G} \) and \( \mathcal{F}^{S}_{p,G} \), respectively.

\[ \text{Proposition 8.1. Let} \ W \subseteq N₂(Γ). \ Let f : ℕ → ℕ \ and \ suppose \ that \ for \ all \ l, \ there \ exists \ w_l \in W \ of \ length \ at \ most \ f(l) \ which \ is \ a \ law \ for \ every \ member \ of \ C \ of \ order \ at \ most \ l. \ Then: } \]

\[ \mathcal{F}^{S}_{C,G}(f(l) · (A^{S}_{Γ,W} ◦ f)(l)) > l. \]  \hspace{1cm} (17)

Proof. Let \( w_l \in W \) be as in the statement. There exist \( g, h \in Γ \) such that:

\[ \max \{|g|^S, |h|^S\} ≤ |g|^S + |h|^S ≤ A^{S}_{Γ,W}(f(l)) \]

and \( w_l(g, h) ≠ e \), while by construction, \( D_{Γ,C}(w_l(g, h)) > l \) and:

\[ |w_l(g, h)|^S ≤ |w_l| · \max \{ |g|^S, |h|^S \} ≤ f(l) · A^{S}_{Γ,W}(f(l)). \]

Proposition 8.1 is useful in the presence of good upper bounds on the function f. The best currently known bounds for the classes of finite groups and finite p-groups are as follows, taken from [3] and [7], respectively.
Theorem 8.2. For each $n \in \mathbb{N}$ there exists $1 \neq w_n \in F_2$ such that:

(i) For all finite groups $G$ of order at most $n$, $w_n$ is a law for $G$;
(ii) For all $\delta > 0$, $|w_n| = \mathcal{O}(n^{2/3} \log(n)^{3+\delta})$.

Theorem 8.3. For each $m \in \mathbb{N}$ there exists $1 \neq w'_m \in F_2$ such that:

(i) For all nilpotent groups $G$ of class at most $m$, $w'_m$ is a law for $G$ (in particular, $w'_m$ is a law for every finite $p$-group of order at most $p^m$);
(ii) $|w'_m| = \mathcal{O}(m^a)$, where $a = \log(2)/(\log(1 + \sqrt{5}) - \log(2)) \approx 1.440$.

Corollary 8.4. Let $\Gamma$ be a finitely generated lawless group and let $\delta > 0$. Then:

$$\mathcal{F}_\Gamma(lA_\Gamma(l)) \geq l^{3/2}/\log(l)^{9/2+\delta}.$$  

Proof. Let $w_n$ be as in Theorem 8.2 and set $W = \{w_n\}_{n \in \mathbb{N}}$. Apply Proposition 8.1 with $C$ the class of all finite groups.

Corollary 8.5. Let $\Gamma$ be a lawless group and let $p$ be a prime. Then:

$$\mathcal{F}_{\Gamma,p}(lA_\Gamma(l)) \geq \exp \left(\log(p)^{1/a}\right),$$

where $a$ is as in Theorem 8.3 (so that $1/a \approx 0.694$).

Proof. Let $w'_m$ be as in Theorem 8.3 and set $W = \{w'_m\}_{m \in \mathbb{N}}$. Apply Proposition 8.1 with $C$ the class of all finite $p$-groups.

9 Equations with coefficients

Laws are generalized by mixed identities. Given a group $\Gamma$, a non-trivial element $w$ of the free product $\Gamma * F_k$, lying in the kernel of every homomorphism $\Gamma * F_k \rightarrow \Gamma$ which restricts to the identity on $\Gamma$. A mixed identity is therefore a law for $\Gamma$ precisely when it lies in $F_k$. Mixed-identity-free (MIF) groups are rather rarer than lawless groups: any group decomposing as a non-trivial direct product, or with a non-trivial finite conjugacy class satisfies a mixed identity, as does any group possessing a non-trivial normal subgroup with a mixed identity.

One may define an analogue of lawlessness growth for mixed identities. For $\Gamma$ a finitely generated group, and $w \in \Gamma * F_k$ non-trivial, a mixed identity for $\Gamma$, the complexity of $w$ in $\Gamma$ is defined exactly as for elements of $F_k$: it is the minimal word-length of a $k$-tuple $(g_i)$ in $\Gamma$ such that $w$ does not lie in the kernel of the homomorphism $\Gamma * F_k \rightarrow \Gamma$ restricting to the identity on $\Gamma$ and sending $x_i$ to $g_i$. To define the induced MIF growth function $\mathcal{M}_\Gamma$ of $\Gamma$, we take the maximal complexity over elements $w \in \Gamma * F_k$ of word-length at most $l$. As with lawlessness growth, it is easy to see that the equivalence class of the function $\mathcal{M}_\Gamma$ does not depend on a choice of finite generating set for $\Gamma$ or $\Gamma * F_k$. Moreover, since $\Gamma * F_k$ embeds into $F_k * \mathbb{Z}$ for every $k$, we may assume $k = 1$. Henceforth we take $\mathbb{Z} = \langle x \rangle$ and fix a finite generating set $\mathcal{S}$ for $\Gamma$. 

Remark 9.1. It might alternatively occur to one to assign length one to all coefficients from $\Gamma$ appearing in $w$. The distinction between such a length function and the word metric on $\Gamma \ast F_k$ coming from a finite generating set is roughly the same as that between the degree and “height” of a polynomial with integer coefficients. Importantly though, if we were to assign length one to every element of $\Gamma$, then for $\Gamma$ infinite, there would be infinitely many words of bounded length, so it would not be clear that the associated MIF growth function would take finite values, even for $\Gamma$ MIF-free.

We recall the following basic fact about subgroups of free groups, the proof of which is an easy consequence of the uniqueness of reduced-word representatives for elements of free products.

Lemma 9.2. Let $\Gamma$ and $\Delta$ be nontrivial groups, and let $e \neq h \in \Delta$. Then $\{[g, h] : e \neq g \in \Gamma\}$ freely generate a free subgroup of $\Gamma \ast \Delta$.

Proof of Theorem 1.9 Let $g_1, \ldots, g_m$ be an enumeration of the nontrivial elements of $B_S(l)$, so that for some $C > 0$ (independent of $l$) we have $m \leq \exp(Cl)$. For $1 \leq i \leq m$ let $w_i = [g_i, x]$. By Lemma 9.2 the $w_i$ freely generate a free subgroup $F$ of $\Gamma \ast \mathbb{Z}$ of rank $m$. Applying the construction of Proposition 2.8 to the $w_i$, we obtain a nontrivial word $w$ in the $w_i$ of reduced length at most $16m^2$, with the property that, whenever $\pi : F \to \Gamma$ is a homomorphism whose kernel contains some $w_i$, $w \in \ker(\pi)$ also.

Since every $y_j$ has word-length at most $2(l + 1)$ in $S \cup \{x\}$, $w$ has word-length at most $32(l + 1)m^2 \leq \exp(l)$ when viewed as an element of $\Gamma \ast \mathbb{Z}$. Meanwhile, $w$ has complexity at least $l + 1$ in $\Gamma \ast \mathbb{Z}$. For every homomorphism $\Gamma \ast \mathbb{Z} \to \Gamma$ restricting to the identity on $\Gamma$ restricts to a homomorphism $F \to \Gamma$. If such a homomorphism sends $x$ to some $g_i$, then $w_i$ lies in its kernel, and by Proposition 2.8 so does $w$. \qed

Now we turn to the proof of Theorem 1.10. Henceforth $\Gamma$ is a finite-rank nonabelian free group with free basis $S$.

Proposition 9.3. Let:

$$w(x) = a_1x^{k_1} \cdots a_Lx^{k_L} \in \Gamma \ast \mathbb{Z}$$ (18)

for some $e \neq a_i \in \Gamma$ and $k_i \in \mathbb{Z}$ with $k_i \neq 0$ for $i \leq L - 1$. Suppose $u \in \Gamma$ is not a proper power in $\Gamma$, and that $[u, a_i] \neq e$ for all $i$. Then for all $m \in \mathbb{N}$ sufficiently large that:

$$|u^m|_S \geq |u^2|_S l + \sum_{i=1}^{L} |a_i|_S,$$ (19)

$w(u^m)$ does not commute with $u$ (and in particular $w(u^m) \neq e$).

Proof. We proceed by induction on $l$. For $l = 1$ the claim is clear for any $m \geq 1$: $a_1u^{k_1}m$ commutes with $u$ iff $a_1$ does. Let $l \geq 2$ and suppose the claim holds for smaller $l$. If $w(u^m)$ commutes with $u$, then they are powers of a common word. Since $u$ is not a proper power, there exists $k_0 \in \mathbb{Z}$ such that $u^{k_0}a_1u^{k_1} \cdots a_Lu^{k_L} = e$.

Consider the Cayley graph $C$ of $\Gamma$ with respect to the free basis $S$. Since $C$ is a tree there is, for every $g, h \in \Gamma$, a unique reduced path $[g, h]$ from $g$ to $h$ in $C$ (of length equal to $|g^{-1}h|_S$). Writing $g_0 = e$, $h_0 = u^{k_0}$ and $g_i = h_{i-1}a_i$, $h_i = g_iu^{k_i}$ for $i \geq 1$ (so that $h_j = e$), the union of all the paths $[g_j, h_j]$ and $[h_j, g_{j+1}]$ is a closed loop in $C$.
and (using again the fact that $C$ is a tree), $[g_1, h_1]$ is contained in the union of the other intervals. Each $[h_{j-1}, g_j]$ has length $|a_j|_S$, so for $m$ as in the statement, there exists $i \neq 1$ such that $[g_1, h_1]$ overlaps with $[a_i, h_i]$ in an interval of length at least $|u|^2|_S$.

We may write $u$ uniquely as $y^{-1}zy$, for $y, z \in \Gamma$ reduced words in $S$, with $z$ cyclically reduced and $|u|_S = 2|y|_S + |z|_S$. There exist $M, N \in \mathbb{Z}$ with $g_1^{-1}h_1 = y^{-1}z^My$ and $g_1^{-1}h_1 = y^{-1}z^N y$. Further, $I_1 = [g_1y^{-1}, h_1y^{-1}] = [g_1y^{-1}, g_1y^{-1}z^M]$ overlaps with $I_2 = [g_iy^{-1}, h_1y^{-1}] = [g_iy^{-1}, g_iy^{-1}z^N]$ in an interval of length at least $|u|^2|_S - 2|y|_S \geq 2|z|_S$. Let $p$ and $q$ be the shortest prefixes of $z^M$ and $z^N$, respectively, such that $g_1y^{-1}p = g_1y^{-1}q$ (that is, the starting-points of common subinterval of $I_1$ and $I_2$).

We claim that $|p|_S \equiv |q|_S \mod |z|_S$, so that the initial common subinterval of $I_1$ and $I_2$ of length $|z|_S$ contains a point $g_1y^{-1}z^{k_1} = g_iy^{-1}z^{k_i}$ for some $k_1, k_i \in \mathbb{Z}$, so that:

$$u^{k_1-k_i-m} = g_1^{-1}g_i = a_2a_3^{k_2} \cdots a_i^{k_i-1}a_i$$

commutes with $u$, contradicting the inductive hypothesis.

Suppose the claim fails. First suppose $M$ and $N$ have the same sign. Comparing the initial common subintervals of $I_1$ and $I_2$ of length $2|z|_S$, we see that there exist $v, w \in \Gamma$ nontrivial reduced words with $z = vw = uv$, and $|z|_S = |v|_S + |u|_S$ (specifically, $|v|_S \equiv |p|_S - |q|_S \mod |z|_S$). Since $v$ and $w$ commute, they are powers of a common word, contradicting the fact that $u$ (and hence $z$) is not a proper power.

Similarly if $M$ and $N$ have opposite signs, we obtain $z = vw$ and $z^{-1} = uv = wzw^{-1}$, but no nontrivial element of a free group is conjugate to its inverse. We therefore have the desired claim.

**Proof of Theorem** 1.10. Let $e \neq w \in \Gamma \ast \mathbb{Z}$ with $|w| \leq n$. Conjugating, we may assume that $w(x)$ has the form given in (18). Moreover,

$$\sum_{i=1}^{|x|} |a_i| \leq n$$

so that by Proposition 9.3 $w$ has complexity at most $2|u|_S n + |u|_S + n$ for any $u \in \Gamma$ satisfying the conditions of Proposition 9.3. It therefore suffices to find such $u$ satisfying $|u| \ll \log n$.

There exists $C > 0$ such that $|B_{\Delta}(k)| \geq \exp(Ck)$ for all $k \in \mathbb{N}$. Since centralizers in free groups are cyclic, the union of the centralizers of the $a_i$ cover at most $(2k + 1)n$ points in $B_{\Delta}(k)$. Hence there exists $C' > 0$ such that for all $k \geq C' \log(n)$, $B_{\Delta}(k)$ contains an element $u$ such that $[u, a_i] \neq e$ for all $i$. Taking $u$ to be of minimal length among such elements, we may assume $u$ is not a proper power.

**Remark** 9.4. Unlike the class of lawless groups, the class of MIF groups is not closed under taking overgroups. For instance $\Gamma \times C_2$ is non-MIF, for any group $\Gamma$. There seems to be no straightforward relationship between the MIF growth of a group $\Delta$ and that of a finitely generated overgroup $\Gamma$ (in the spirit of Lemma 2.4), even if $\Gamma$ is itself MIF.
10 Open questions

There are many interesting questions one can ask about the spectrum of possible lawlessness growth functions. For instance one may wonder whether there is a universal upper bound on $A_{\Gamma}$ for lawless groups.

**Question 10.1.** Does there exist, for all non-decreasing functions $f : \mathbb{N} \to \mathbb{N}$, a finitely generated lawless group $\Gamma = \Gamma(f)$ satisfying $A_{\Gamma} \not\leq f$? Does there exist such a group satisfying $A_{\Gamma} \geq f$?

Our main source of strong lower bounds on $A_{\Gamma}$ comes from upper bounds on torsion growth for infinite finitely generated $p$-groups. Under this approach, there are viable methods for constructing finitely generated lawless $p$-groups which are unrelated to Golod-Shafarevich theory. For instance, there are many examples of branch $p$-groups beyond Grigorchuk’s group, and all (weakly) branch groups are lawless (by an argument similar to our proof of Theorem 1.4).

**Question 10.2.** How slow can the torsion growth of a finitely generated (weakly) branch $p$-group be? Can it be sublinear?

For example, by Theorem 7.8 of [1], Grigorchuk’s group has torsion growth at least linear. In a completely different direction, a result of Drutu-Sapir shows that any non-virtually-cyclic group which admits an asymptotic cone with a cut-point is lawless. By [13](Theorem 1.12) such groups include some finitely generated torsion groups (indeed $p$-groups, as is made clear in the proof). It is unclear to us at present how effective the construction in [13] is able to be made, and whether it could give rise to groups of fast lawlessness growth.

Our proof of the upper bound in Theorem [1.4] and the possibility of improving upon it, naturally connects to the following question about finite groups.

**Question 10.3.** Let $W_n$ be the $(n + 1)$-fold iterated wreath product of $C_2$, as in Section 6. What is the length of the shortest law in $F_k$ for $W_n$?

As far as the author is aware, there is no shorter law known for $W_n$ than the obvious power-word $x^{2^{n+1}}$.

Turning to MIF growth, we may ask for examples of groups for which $M_{\Gamma}$ grows slowly. Since MIF is a difficult property to satisfy, exhibiting such groups may be rather challenging. For instance, it is natural (if rather ambitious) to ask whether Theorem [1.9] is sharp.

**Question 10.4.** Does there exist $\Gamma$ with $M_{\Gamma}(n) \ll \log(n)$?

Gromov hyperbolic groups form an important class of MIF groups, properly containing the nonabelian finite-rank free groups. We may ask for a generalization of the bound from Theorem [1.10] to this class, and for insight into the features of the geometry of groups on which the bound depends.

**Problem 10.5.** Let $\Gamma$ be a torsion-free Gromov hyperbolic group. Give an upper bound on $M_{\Gamma}$. Does there exist, for each $\delta > 0$, an increasing function $f_\delta : \mathbb{N} \to \mathbb{N}$ such that, if $\Gamma$ admits a $\delta$-hyperbolic Cayley graph, then $M_{\Gamma} \leq f_\delta$?
Finally, since words in $F_k$ of a given length are a small subset of the elements of $\Gamma \ast F_k$, it is clear that for any MIF group $\Gamma$, $A_{\Gamma} \leq M_{\Gamma}$, and in general one would expect it would be a remarkable achievement to identify a group in which elements of $\Gamma \ast F_k$ of essentially maximal complexity for their length already occur in $F_k$.

**Question 10.6.** Does there exist a finitely generated MIF group $\Gamma$ with $A_{\Gamma} \approx M_{\Gamma}$?

By Theorems 1.1 and 1.9 a group providing a positive answer to Question 10.6 would need to have no $F_2$-subgroups. In a forthcoming work, we shall give a finite group analogue of a positive answer to Question 10.6: a sequence $(G_n)$ of finite groups having no common law, such that for each $n$, the length of the shortest law for $G_n$ is comparable to the length of the shortest mixed identity.

Following circulation of a preliminary version of the present article, J.M. Petschick has shared with the author a construction of a group answering the final part of Question 10.2 in the affirmative (as yet unpublished). His result therefore also strengthens the conclusion of our Theorem 1.3.

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