SEMI-ABELIAN SPECTRAL DATA FOR SINGULAR FIBERS OF THE $\text{SL}(2, \mathbb{C})$-HITCHIN SYSTEM

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Abstract. We describe spectral data for singular fibers of the $\text{SL}(2, \mathbb{C})$-Hitchin fibration with irreducible and reduced spectral curve. Using higher Hecke transformations we give a stratification of these singular spaces by fiber bundles over Prym varieties. By analysing the parameter spaces of higher Hecke transformations this describes the singular Hitchin fibers as compactifications of semi-abelian varieties. We prove that a large class of singular fibers are themselves fiber bundles over Prym varieties. As applications we study irreducible components of singular Hitchin fibers and give a description of $\text{SL}(2, \mathbb{R})$-Higgs bundles in terms of these semi-abelian spectral data.

1. Introduction

Since the introduction of Higgs bundles in the late 80’s by Hitchin [Hit87a] and Simpson [Sim88], the geometry of Higgs bundle moduli spaces is an extensively studied research topic at the interface of algebraic geometry, differential geometry and mathematical physics. From the algebro-geometric perspective the moduli space of Higgs bundles is a non-compact analogue of the moduli space of holomorphic vector bundles. In addition, it can be interpreted as the moduli space of solutions to a differential geometric PDE, the Hitchin equation. Moreover, it enables the study of moduli spaces of representations of the fundamental group of an orientable surface by the so-called Non-Abelian Hodge Correspondence. Thereby, Higgs bundles are an important tool to study components of these moduli spaces of representations in the realm of higher Teichmüller theory [Hit92, BGG06, AC19, Apa+19].

In the last ten years, there is renewed interest in Higgs bundles motivated by physics. Higgs bundle moduli spaces are hyper-Kähler and hence subject to mirror symmetry conjectures. Emerging from a duality between electric and magnetic charges in physics, mirror symmetry was interpreted as a duality of torus fibrations [SYZ01] and connected to the geometric Langlands program [KW07].

The moduli space of $G$-Higgs bundles associated to a complex reductive Lie group $G$ contains a dense subset fibered by Lagrangian tori, the so-called Hitchin fibration. Mirror symmetry reincarnates as a duality between the Hitchin fibrations of the moduli space of $G$-Higgs bundles and the moduli space of $G^L$-Higgs bundles, where $G^L$ denotes the Langlands dual group of $G$. This Langlands duality was established for classical Lie groups by Hausel, Thaddeus [HT03] and Hitchin [Hit07] and in general by Donagi,
In this paper, we study singular Hitchin fibers - degenerations of the Lagrangian tori forming the Hitchin fibration. Motivated by Langlands duality, singular Hitchin fibers were more closely studied in [HP12] [GO13] [Bra18] [Fra+18] [BBS19]. We give a complex-geometric description of these singular spaces emphasizing what is left of the rich geometry of the regular fibers. Indeed, singular fibers naturally contain abelian subvarieties and one can describe their geometry building on this observation. These complex subtori played an important role in the recent work [Hit19], where Hitchin studied subintegrable systems on the singular locus of the Hitchin fibration.

Let $X$ be a Riemann surface of genus $g \geq 2$. Let $M$ denote the moduli space of polystable $SL(2,\mathbb{C})$-Higgs bundles on $X$. In this case, the Hitchin map is given by

$$\text{Hit} : M \to H^0(X, K_X^3), \quad (E, \Phi) \mapsto \det(\Phi).$$

It was shown in [Hit87a] [Hit87b], that this defines an algebraically completely integrable system over the subset of quadratic differentials with simple zeroes. This defines the fibration by Lagrangian tori, we referred to as the Hitchin fibration. We will call the subset of quadratic differentials with simple zeroes the regular locus and its complement the singular locus.

To every quadratic differential $q_2 \in H^0(X, K_X^3)$, we associate a complex curve $\Sigma \subset \text{Tot}(K_X)$, the so-called spectral curve. For a quadratic differential in the regular locus it is smooth and defines a branched covering of Riemann surfaces $\pi : \Sigma \to X$. In this case, $\text{Hit}^{-1}(q_2)$ can be identified with the Prym variety of the spectral covering $\pi$. Hence, the complex tori forming the integrable system are abelian torsors.

The singular Hitchin fibers are divided in 3 types, depending on the regularity of the spectral curve. The most singular fiber is the nilpotent cone $\text{Hit}^{-1}(0)$. It has many irreducible components, which intersect in a complicated way [Hit87a]. A more detailed study can be found in the recent work [ALS20]. Secondly, there are singular fibers, where the spectral curve is reduced and reducible. In explicit, there exists an abelian differential $\lambda \in H^0(X, K_X)$, such that $\lambda^2 = q_2$. These fibers are still quite complicated. It was shown in [GO13], that they are connected. We study the case of Hitchin fibers with reduced and irreducible spectral curve. This is equivalent to $q_2$ having no global square root on $X$. By a classical theorem [BNR89], these singular Hitchin fibers can be identified with certain moduli spaces of rank 1 torsion-free sheaves on $\Sigma$. These moduli spaces were studied in [GO13], who again proved connectedness.

We take a different approach. The normalisation naturally associates to the singular curve $\Sigma$ a smooth curve $\tilde{\Sigma}$. The Prym variety $\text{Prym}(\Sigma)$ associated to the covering $\tilde{\pi} : \tilde{\Sigma} \to X$ is naturally embedded in the singular Hitchin fiber through the pushforward along $\tilde{\pi}$. We describe the singular Hitchin fiber by gluing together holomorphic fiber bundles over the Prym varieties $\text{Prym}(\Sigma)$.
Theorem 1.1 (Thm. 5.3, Thm. 6.2). Let \( q_2 \in H^0(X, K_X^2) \) a quadratic differential with at least one zero of odd order. There is a stratification

\[ \text{Hit}^{-1}(q_2) = \bigcup_{i \in I} S_i \]

by finitely many locally closed analytic subsets \( S_i \). Each stratum \( S_i \) is a holomorphic \((\mathbb{C}^*)^{r_i} \times \mathbb{C}^{s_i}\)-bundle over \( \text{Prym}(\bar{\Sigma}) \) with \( r_i, s_i \in \mathbb{N}_0 \), in particular, a semi-abelian complex space.

We prove a more general version of this result in Theorem 6.2 covering all Hitchin fibers with irreducible and reduced spectral curve. The stratification is indexed by the local shape of the Higgs field at the higher order zeroes of \( q_2 \). There is a unique open stratum \( S_0 \), which is dense in the singular Hitchin fiber and is compactified by the lower-dimensional strata.

The \((\mathbb{C}^*)^{r_i} \times \mathbb{C}^{s_i}\)-fibers - the non-abelian part of the spectral data - parametrize higher Hecke transformations. A Hecke transformation of a holomorphic vector bundle is the generalization to higher rank of twisting a line bundle by a divisor. Hecke transformations are parametrized by Hecke parameters determining the direction, in which the holomorphic vector bundle is twisted. Hwang-Ramanan [HR04] studied Hecke transformations of order one. Here the moduli space of Hecke parameters is one-dimensional and Hecke transformations define so-called Hecke curves in the moduli space of holomorphic vector bundles. They showed that the space of tangents to Hecke curves is dual to the space of cotangent Higgs bundles lying over the singular locus. This result indicated a strong relationship between Hecke transformations and the geometry of singular Hitchin fibers.

In this work, we introduce a generalization of this concept by allowing higher order twists, which yields to higher dimensional moduli spaces of Hecke parameters. We can study how the different strata are glued together by compactifying the space of Hecke parameters of the open stratum \( S_0 \). These compact moduli spaces of Hecke parameters are given in terms of a quotient by a non-reductive group action. We can build a model for these quotient spaces by explicitly computing invariant functions (Theorem 7.5). For quadratic differentials with only zeroes of odd order, the fiber bundle structure of \( S_0 \) is preserved under the degeneration to lower-dimensional strata. We obtain the following theorem.

Theorem 1.2 (Thm. 7.12). Let \( q_2 \in H^0(X, K_X^2) \) be a quadratic differential, such that all zeroes have odd order. Then \( \text{Hit}^{-1}(q_2) \) fibers holomorphically over \( \text{Prym}(\bar{\Sigma}) \) with fibers given by the compact moduli of Hecke parameters.

For quadratic differentials with zeroes of even order, we can still understand how the strata are glued together by analysing the compact moduli of Hecke parameters (see Section 8). However, we observe that there can not be a global fibering of \( \text{Hit}^{-1}(q_2) \) over \( \text{Prym}(\bar{\Sigma}) \) in this case. Nevertheless, we can study the irreducible components of the singular Hitchin fibers.

Corollary 1.3 (Cor. 8.5, Thm. 8.7). Let \( q_2 \in H^0(X, K_X^2) \) with at least one zero of odd order, then \( \text{Hit}^{-1}(q_2) \) is an irreducible complex space. If all zeroes of \( q_2 \) are of even order, then \( \text{Hit}^{-1}(q_2) \) is connected and has two irreducible components.
These results will be generalized to a description of singular Hitchin fibers of the $\text{Sp}(2n, \mathbb{C})$-Hitchin system in a subsequent paper.

To emphasize the analogies to the regular fibers, we generalize the correspondence between $\text{SL}(2, \mathbb{R})$-Higgs bundles in regular Hitchin fibers and two-torsion points of $\text{Prym}(\Sigma)$ \cite{Sch13b} to singular fibers.

**Theorem 1.4** (Thm. 9.1). Let $q_2 \in H^0(K_X^2)$ be a quadratic differential, such that all zeroes have odd order. For every stratum $\mathcal{S}_i$, the $\text{SL}(2, \mathbb{R})$-Higgs bundles in $\mathcal{S}_i$ correspond to the two-torsion points of the Prym variety $\text{Prym}(\tilde{\Sigma})$.

Counting the two-torsion points of $\text{Prym}(\tilde{\Sigma})$, this describes the branching of the moduli spaces of $\text{SL}(2, \mathbb{R})$-Higgs bundles as a covering over the Hitchin base (cf. \cite{Sch13a}). A similar result for quadratic differentials with even zeroes is proven in Theorem 9.5.

**Structure** The paper is structured as follows: After introducing the framework (2), we identify the singular Hitchin fibers with irreducible spectral curve with a certain moduli spaces of Higgs bundles on $\Sigma$ (3). In Section 4 we introduce the concept of higher Hecke transformations and prove basic properties. We elaborate the stratification result in Section 5 and 6. In Section 7 we analyse the compact moduli of Hecke parameters for singular fibers over quadratic differentials, such that all zeroes are of odd order, and prove Theorem 1.2. Hereafter, we give an explicit description of the first degenerations (7.5). In Section 8 we do the same analysis for general Hitchin fibers with irreducible and reduced spectral curve. In the final section, we parametrize the real points in the singular Hitchin fibers by semi-abelian spectral data.

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2. Preliminaries

Let $X$ be a Riemann surface of genus $g \geq 2$. Let $p_K : K = K_X \to X$ denote the line bundle of holomorphic 1-forms. Denote by $\mathcal{M} = \mathcal{M}(X)$ the moduli space of polystable $\text{SL}(2, \mathbb{C})$-Higgs bundles on $X$. These are pairs $(E, \Phi)$ of a holomorphic vector bundle $E$ of rank 2 with $\det(E) = \mathcal{O}_X$ and a holomorphic section $\Phi$ of $\text{End}(E) \otimes K$, such that $\text{tr}(\Phi) = 0$. The smooth locus of $\mathcal{M}$, the moduli space of stable $\text{SL}(2, \mathbb{C})$-Higgs bundles, carries a holomorphic symplectic structure extending the holomorphic symplectic structure of the cotangent bundle to the moduli space of stable holomorphic
vector bundles. The Hitchin map is defined by

$$\text{Hit} : \mathcal{M} \to H^0(X, K^2), \quad (E, \Phi) \mapsto \det(\Phi).$$

This map is proper, surjective and defines an algebraically completely integrable system on the dense subset of quadratic differentials with simple zeroes ([Hit87a, Hit87b]). The torus fibers can be identified algebraically as follows. The Hitchin map computes the coefficients of the characteristic polynomial of $(E, \Phi) \in \text{Hit}^{-1}(q_2)$

$$\eta^2 + q_2.$$

Here $\eta$ can be interpreted as the tautological section $\eta : K \to p_K^* K$. The pointwise eigenvalues of the Higgs field form the complex analytic curve

$$\Sigma := Z_K(\eta^2 + p_K^* q_2) \subset \text{Tot} K.$$

This is the so-called spectral curve. The projection $p_K$ restricts to a two-sheeted branched covering $\pi : \Sigma \to X$ with branch points at the zeroes of $q_2$. This curve is smooth besides the ramification points. It is smooth at a ramification point if and only if the corresponding zero of the quadratic differential $q_2$ is of order 1. Due to the specific type of characteristic equation the spectral curve comes with an involutive automorphism $\sigma : \Sigma \to \Sigma$ interchanging the sheets.

The subset of quadratic differentials with simple zeroes is an open and dense subset of $H^0(K^2)$, which we refer to as the regular locus. Its compliment will be referred to as the singular locus and the fibers over the singular locus as singular fibers. The Hitchin fibers over the regular locus form a dense subset of $\mathcal{M}$ fibered by tori.

**Theorem 2.1** (Abelian Spectral Data [Hit87a]). Let $q_2 \in H^0(X, K^2)$ be a quadratic differential with simple zeroes, then $\text{Hit}^{-1}(q_2)$ is a torsor over the Prym variety

$$\text{Prym}_0(\Sigma) := \{ L \in \text{Jac}(\Sigma) \mid L \otimes \sigma^* L = O_X \}.$$

**Proof.** This will be a special case of the description of Hitchin fibers with irreducible spectral curve given below. We want to sketch the classical construction for context. Let $\lambda = \eta|_{\Sigma}$ and $\Lambda = \text{div}(\lambda)$. Let $(E, \Phi) \in \text{Hit}^{-1}(q_2)$, then

$$L = \ker(\pi^* \Phi - \lambda \text{id}_{\pi^* E}) \in \text{Prym}_\Lambda(\Sigma).$$

(see Theorem 5.3). By Proposition 5.6 $\text{Prym}_\Lambda(\Sigma)$ is a torsor over $\text{Prym}_0(\Sigma)$. The eigenline bundle uniquely determines the Higgs bundle by the algebraic pushforward $(E, \Phi) = \pi_* (L \otimes \pi^* K, \lambda)$. $\square$

Prym varieties are abelian varieties and were intensively studied in [Mum74].

In this work, we study Hitchin fibers with singular, but irreducible spectral curve. The spectral curve is irreducible if and only if $q_2$ has no global square root on $X$, i.e. there exists no $\lambda \in H^0(X, K)$, such that $\lambda^2 = q_2$. In this case, there is a covering of Riemann surfaces associated to the characteristic equation. It is the unique two-sheeted branched covering of Riemann surfaces $\tilde{\pi} : \tilde{\Sigma} \to X$, such that there exists $\lambda \in H^0(\tilde{\Sigma}, \tilde{\pi}^* K)$ solving

$$\lambda^2 + \tilde{\pi}^* q_2 = 0.$$
From a algebro-geometric perspective $\tilde{\Sigma}$ is the normalisation of $\Sigma$ and we will refer to $\tilde{\Sigma}$ as the normalised spectral curve. The geometry of this covering can be easily understood. The restriction

$$\pi : \Sigma \setminus \pi^{-1}(Z(q_2)) \to X \setminus Z(q_2)$$

is a unbranched covering of Riemann surfaces and there is a unique way to extend it in a smooth way. Whenever the local polynomial equation for $\Sigma$ in a neighbourhood of $p \in \pi^{-1}(Z(q_2))$ is irreducible, or equivalently the corresponding zero of $q_2$ is of odd order, we glue in a disc, such that the covering map locally extends to $\tilde{\pi} : \tilde{z} \mapsto z^2$. If instead the local polynomial is reducible, or equivalently the zero of $q_2$ is of even order, we glue in two discs separating the two sheets. Hence, the branch points of $\tilde{\pi} : \tilde{\Sigma} \to X$ are the zeroes of $q_2$ of odd order and by Riemann-Hurwitz the genus of $\tilde{\Sigma}$ is

$$g(\tilde{\Sigma}) = 2g - 1 + \frac{n_{\text{odd}}}{2},$$

where $n_{\text{odd}}$ denotes the number of odd zeroes of $q_2$ (without multiplicity). Hitchin fibers with irreducible spectral curve have the following useful property:

**Lemma 2.2.** Let $q_2$ be a quadratic differential with no global square root and $(E, \Phi) \in \text{Hit}^{-1}(q_2)$. Then $(E, \Phi)$ is stable.

**Proof.** If there is a $\Phi$-invariant subbundle $M \subseteq E$, then it is an eigen line bundle. Hence, the eigensections of $\Phi$ must exists on $X$ as global sections of $K$. This contradicts the assumption on $q_2$. □

In the following, we will also need to look at $M$-twisted $\text{SL}(2, \mathbb{C})$-Higgs bundles. These are pairs $(E, \Phi)$, where $E$ is a holomorphic vector bundle of rank 2, such that $\det(E) = O_X$ and $\Phi \in H^0(X, \text{End}(E) \otimes M)$ with $\text{tr}(\Phi) = 0$. Nitsure [Nit91] constructed the moduli space of polystable $M$-twisted Higgs bundles in the algebraic category. It will be denoted by $\mathcal{M}(X, M)$. The Hitchin map is given by

$$\text{Hit}_M : \mathcal{M}(X, M) \to H^0(X, M^2), \quad (E, \Phi) \mapsto \det(\Phi).$$

As above, the generic fibers can be identified with torsors over Prym varieties of dimension $g - 1 + \deg(M)$. However, the smooth locus of $\mathcal{M}(X, M)$ is no longer a holomorphic symplectic manifold and hence this torus fibration is no longer defined as a completely integrable system. If there exists a section $s \in H^0(X, MK^{-1})$, the smooth locus of $\mathcal{M}(X, M)$ has the structure of a Poisson manifold (see [Mar94]).

### 3. $\sigma$-invariant Higgs bundles on the normalised spectral curve

#### 3.1. The Pullback.

Let $p : Y \to X$ be a branched 2-sheeted covering of Riemann surfaces and $\sigma$ the involutive biholomorphism changing the sheets.

**Definition 3.1.** A $\sigma$-invariant holomorphic vector bundle $(E, \hat{\sigma})$ on $Y$ is holomorphic vector bundle $E$ on $Y$ with a lift

$$\begin{array}{ccc}
E & \xrightarrow{\hat{\sigma}} & E \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\sigma} & Y
\end{array}$$
such that

i) \( \hat{\sigma}^2 = \text{id}_E \), and

ii) \( \hat{\sigma}|_y = \text{id}_{E_y} \) for all ramification points \( y \in Y \).

Let \( (M, \hat{\sigma}_M) \) be \( \sigma \)-invariant holomorphic line bundle on \( Y \). A \( \sigma \)-invariant \( (M, \hat{\sigma}_M) \)-twisted Higgs bundle \( (E, \Phi, \hat{\sigma}_E) \) on \( Y \) is a \( M \)-twisted Higgs bundle \( (E, \Phi) \) on \( Y \), such that \( (E, \hat{\sigma}_E) \) is \( \sigma \)-invariant holomorphic vector bundle and

iii) \( (\hat{\sigma}_E \otimes \hat{\sigma}_M) \circ \Phi = \Phi \circ \hat{\sigma}_E \).

Lemma 3.2. Let \( (E, \Phi, \hat{\sigma}_E) \) be a \( \sigma \)-invariant \( (M, \hat{\sigma}_M) \)-twisted Higgs bundle, \( g \in A^0(\text{SL}(E)) \) an element of the gauge group, then \( (gE, g\Phi g^{-1}, g \circ \hat{\sigma} \circ g^{-1}) \) is a \( \sigma \)-invariant \( (M, \hat{\sigma}_M) \)-twisted Higgs bundle.

Define

\[
\mathcal{M}_\sigma(Y, M, \hat{\sigma}_M) = \left\{ (E, \Phi) \in \mathcal{M}(Y, M) \mid \exists \hat{\sigma} : (E, \Phi, \hat{\sigma}) \ \sigma \text{-invariant } (M, \hat{\sigma}_M) \text{-twisted} \right\}.
\]

Proposition 3.3.

i) Let \( E \) be a holomorphic vector bundle on \( X \). Then \( p^*E \) has a induced lift \( \hat{\sigma}_{p^*E} \), such that \( (p^*E, \hat{\sigma}_{p^*E}) \) is a \( \sigma \)-invariant holomorphic vector bundle.

ii) We have a natural map \( p^* : \mathcal{M}(X, M) \to \mathcal{M}_\sigma(Y, p^*M, \hat{\sigma}_{p^*M}) \).

Proof. i) Let \( U \subset X \) open, such that \( E|_U \cong U \times \mathbb{C}^r \). The trivialisation induces a trivialisation \( p^*E|_{p^{-1}(U)} \cong p^{-1}U \times \mathbb{C}^r \). If \( x \in U \) is not a branch point, i.e. \( p^{-1}(x) = \{ y, \sigma(y) \} \), such trivialisation induces a identification of the fibers \( p^*E_y \cong p^*E_{\sigma(y)} \). This defines a lift \( \hat{\sigma}_{p^*E} : p^*E \to p^*E \) away from the ramification points. This lift extends over the ramification points by the identity. Therefore, \( (p^*E, \hat{\sigma}_{p^*E}) \) is a \( \sigma \)-invariant holomorphic vector bundle.

ii) Clearly, \( (p^*E, p^*\Phi) \in \mathcal{M}(Y, p^*M) \) and by i) \( (p^*E, \hat{\sigma}_{p^*E}) \) is a \( \sigma \)-invariant holomorphic vector bundle. Property iii) of Definition 3.1 becomes clear in a trivialisation as in the proof of i).

\[\square\]

So a pullback bundle comes with a induced lift \( \hat{\sigma} \). In the following, we will omit the lift of the pullback in the notation.

3.2. The \( \sigma \)-invariant Pushforward.

Definition 3.4. Let \( \xi \) be an analytic sheaf on \( Y \). A lift \( \hat{\sigma} : \xi \to \xi \) of \( \sigma \) is a family of involutive homomorphisms of abelian groups

\[
\hat{\sigma}_V : H^0(V, \xi) \to H^0(\sigma(V), \xi)
\]

commuting with restriction maps, such that for all \( f \in \mathcal{O}_V \) and \( s \in H^0(V, \xi) \)

\[
\hat{\sigma}(fs) = (\sigma^*f)\hat{\sigma}(s).
\]

The pair \( (\xi, \hat{\sigma}) \) is called a \( \sigma \)-invariant analytic sheaf.
Definition 3.5. Let \((\xi, \hat{\sigma})\) be a \(\sigma\)-invariant analytic sheaf on \(Y\), then the \(\sigma\)-invariant pushforward \(p_*(\xi, \hat{\sigma})\) is the analytic sheaf on \(X\) defined through

\[
H^0(U, p_*(\xi, \hat{\sigma})) = H^0(p^{-1}U, \xi)^{\hat{\sigma}}
\]

for open sets \(U \subset X\). Here \(H^0(p^{-1}U, \xi)^{\hat{\sigma}}\) denotes the \(\hat{\sigma}\)-invariant sections of \((\xi, \hat{\sigma})\).

Lemma 3.6.  

i) Let \((\xi, \hat{\sigma})\) be a locally free \(\sigma\)-invariant sheaf of rank \(r\) on \(Y\), such that for every ramification point \(y \in Y\) there exists an open, \(\sigma\)-invariant neighbourhood \(V \subset Y\) of \(y\) and an isomorphism

\[
H^0(V, \xi) \cong \mathcal{O}_V^r,
\]

such that

\[
\hat{\sigma} |_V : \mathcal{O}_V^r \to \mathcal{O}_V^r, \quad f \mapsto f \circ \sigma.
\]

Then \(p_*(\xi, \hat{\sigma})\) is locally free of rank \(r\).

ii) Let \((E, \hat{\sigma})\) be a \(\sigma\)-invariant holomorphic vector bundle of rank \(r\), then \((\mathcal{O}(E), \hat{\sigma})\) satisfies the assumption in i). In particular, the pushforward \(p_*(\mathcal{O}(E), \hat{\sigma})\) is locally free of rank \(r\).

Proof.  

i) Let \(U \subset X\) an open subset trivializing the covering. Let \(p^{-1}(U) = U_1 \cup U_2\). A section in \(H^0(p^{-1}U, \xi)^{\hat{\sigma}}\) is fixed by its values on \(U_1\). Hence \(H^0(p^{-1}U, \xi)^{\hat{\sigma}} \cong \mathcal{O}_{U_1}^r \cong \mathcal{O}_U^r\). Let \(x \in X\) a branch point. By assumption there exists a neighbourhood \(U \subset X\), such that

\[
H^0(p^{-1}U, \xi)^{\hat{\sigma}} = \{ f \in \mathcal{O}_{p^{-1}U}^r \mid f = \sigma^* f \} \cong p^{-1}\mathcal{O}_U^r \cong \mathcal{O}_U^r.
\]

ii) Clearly a lift \(\hat{\sigma}\) induces a lift on the sheaf of sections \(\hat{\sigma} : \mathcal{O}(E) \to \mathcal{O}(E)\) satisfying Definition 3.3. To check the extra assumption in i), let \(y \in Y\) be a ramification point. Assumption ii) of Definition 3.1 guarantees the existence of a local frame of \(\sigma\)-invariant sections in a \(\sigma\)-invariant neighbourhood \(V\) of \(y\). Take a local basis for \(E_y\) and extend it to a holomorphic frame \(s_1, \ldots, s_r\) of \(E_V\). Then a \(\sigma\)-invariant frame is given by \(s_1 + \hat{\sigma}s_1, \ldots, s_r + \hat{\sigma}s_r\) for a small enough neighbourhood \(V\) of \(y\). A \(\sigma\)-invariant frame induces an isomorphism \(\mathcal{O}_V(E) \cong \mathcal{O}_V^r\) such that \(\hat{\sigma}|_V\) has the desired form.

\[\square\]

Definition 3.7. Let \((E, \hat{\sigma})\) be a \(\sigma\)-invariant vector bundle. We define the \(\sigma\)-invariant pushforward \(p_*(E, \hat{\sigma})\) to be the vector bundle corresponding to the locally free sheaf \(p_*(\mathcal{O}(E), \hat{\sigma})\).

Lemma 3.8. Let \(E\) be a holomorphic vector bundle on \(X\) and \((p^*E, \hat{\sigma}_{p^*E})\) the corresponding \(\sigma\)-invariant holomorphic vector bundle on \(Y\), then

\[
p_*(p^*E, \hat{\sigma}_{p^*E}) = E.
\]

Example 3.9. Let \(p : Y \to X\) be a unbranched 2-covering of Riemann surfaces. Let \(L\) be a line bundle on \(X\) and \((p^*L, \hat{\sigma})\) the induced \(\sigma\)-invariant line bundle on \(Y\). Then \(\hat{\sigma}\) is another lift of \(\sigma\) on \(L\). However, \(p_*(p^*L, -\hat{\sigma}) \not\cong L\).

We have

\[
p_*(p^*L, -\hat{\sigma}) \cong L \otimes I,
\]

where \(I = p_*(\mathcal{O}_Y, \text{id}_{\mathcal{O}_Y})\) is the unique non trivial line bundle on \(X\), which pulls back to the trivial bundle on \(Y\) (compare [Mum74] Section 3). \(p^*(I^2) \cong \mathcal{O}_Y\) and the induced lift \(\hat{\sigma}_{p^*(I^2)}\) is the identity. Hence, \(I^2 = \mathcal{O}_X\).
3.3. Pullback and Pushforward of singular Hitchin fibers. Let \( q_2 \in H^0(K^2_X) \) be a quadratic differential with no global square root on \( X \). Let \( \tilde{\pi} : \tilde{\Sigma} \to X \) be the covering by the normalized spectral curve and \( \sigma : \Sigma \to \tilde{\Sigma} \) the involution changing the sheets. We want to parametrize the singular fibers by parametrizing their pullback to \( \Sigma \). However, the pullback

\[
\pi^* : \text{Hit}^{-1}(q_2) \to M^\sigma(\Sigma, \pi^*K)
\]

is not injective in general, because there can be multiple lifts of \( \sigma \).

**Example 3.10.** Let \( q_2 \in H^0(K^2) \) be a quadratic differential with only double zeroes, which has no global square root on \( X \). Then \( \Sigma \to X \) is a 2-sheeted unbranched covering of Riemann surfaces. In this case, there exists a non-trivial line bundle \( I \) with \( \tilde{\pi}^*(I) \cong O_\Sigma \) and \( I^2 = O_X \) (see Example 3.9). For \( (E, \Phi) \in \text{Hit}^{-1}(q) \), also \( (E \otimes I, \Phi) \in \text{Hit}^{-1}(q) \). But we clearly have

\[
\tilde{\pi}^*(E, \Phi) \cong \tilde{\pi}^*(E \otimes I, \Phi).
\]

**Proposition 3.11.** Let \( q_2 \in H^0(X, K^2) \) be a quadratic differential with no global square root. Let

\[
(E, \Phi) \in \tilde{\pi}^*\text{Hit}^{-1}(q_2) \subset M^\sigma(\tilde{\Sigma}, \tilde{\pi}^*K).
\]

i) If \( q_2 \) has at least one zero of odd order there is a unique lift \( \tilde{\sigma} \) such that \((E, \Phi, \tilde{\sigma})\) is a \( \sigma \)-invariant Higgs bundle.

ii) If \( q_2 \) has only zeroes of even order there are two such lifts \( \pm \tilde{\sigma} \).

**Proof.** Let \( (E, \Phi) \in \tilde{\pi}^*\text{Hit}^{-1}(q) \), such that there are two lifts \( \tilde{\sigma}_1, \tilde{\sigma}_2 \) such that \((E, \Phi, \tilde{\sigma}_i)\) is a \( \sigma \)-invariant Higgs bundle. Then \( \tilde{\sigma}_1 \circ \tilde{\sigma}_2 \in \text{Aut}(E, \Phi) \). If \((E, \Phi)\) is stable, this implies that \( \tilde{\sigma}_1 = \pm \tilde{\sigma}_2 \). If in addition, \( q \) has only even zeroes, the spectral covering \( \tilde{\pi} \) is unbranched and this gives the two possible lifts. If \((E, \Phi)\) is stable, and \( q \) has at least on zero of odd order, \( \tilde{\pi} : \tilde{\Sigma} \to X \) has at least one ramification point \( p \in Y \). In particular, \((\tilde{\sigma}_1)_p = (\tilde{\sigma}_2)_p = \text{id}_{E_p} \), and therefore \( \tilde{\sigma}_1 = \tilde{\sigma}_2 \). \((E, \Phi) \in \tilde{\pi}^*\text{Hit}^{-1}(q_2) \) is strictly polystable if and only if

\[
(E, \Phi) = \left( M \oplus M^{-1}, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right).
\]

with \( \deg(M) = 0 \). Hence, \( q_2 \) has only even zeroes. Then \( \tilde{\sigma}_1 = g\tilde{\sigma}_2 \) with

\[
g \in \text{Aut}(E, \Phi) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mid t \in \mathbb{C}^* \right\},
\]

such that \( g^2 = \text{id}_E \). Hence again, \( g = \pm \text{id}_E \). \( \Box \)

**Proposition 3.12.** Let \( q_2 \in H^0(X, K^2) \) be a quadratic differential with no global square root. The pullback

\[
\tilde{\pi}^* : \text{Hit}^{-1}(q_2) \to M^\sigma(\tilde{\Sigma}, \tilde{\pi}^*K)
\]

i) is injective, if \( q_2 \) has at least one zero of odd order, and

ii) is generically two-to-one, if \( q_2 \) has only even zeroes.

Let \( I \) be the unique non-trivial line bundle with \( \tilde{\pi}^*I = O_Y \). The non-injectivity in ii is due to fact that \((E, \Phi)\) and \((E \otimes I, \Phi)\) are identified via pullback.
Proof. We already saw in Lemma 3.11 that in the first case there is a unique lift \( \hat{\sigma} \). Hence the injectivity follows from Lemma 3.8. In the second case, we saw that there are two possible lifts \( \pm \hat{\sigma} \). From Example 3.9 this implies 
\[
\pi_*(E, \hat{\sigma}) = (\pi_*(E, -\hat{\sigma})) \otimes I.
\]
Together with Lemma 3.8, this gives the result in case ii. \( \square \)

Example 3.13. In case ii branching exists. Consider 
\[
E = I^\frac{1}{2} \oplus I^{-\frac{1}{2}} = I^\frac{1}{2} \oplus I^{\frac{3}{2}}, \quad \Phi = \left( \begin{array}{c} 0 \\ \alpha \\ \alpha \\ 0 \end{array} \right)
\]
with \( \alpha \in H^0(KI) \), such that \( \alpha^2 \) has no global square root. Such sections exist by Riemann-Roch and the resulting Higgs bundle is indeed stable.

4. Higher Hecke transformations

In Section 5, we will stratify singular Hitchin fibers by fiber bundles over Prym varieties. The Prym varieties will parametrize the eigenline bundles, whereas the fibers of these bundles decode some extra data. These extra data will parametrize the manipulation of Higgs bundles by higher Hecke transformations. In this section, we recall the definition of Hecke transformation (see [HR04]) and introduce higher Hecke transformations. For simplicity, we will only treat the case of rank 2 bundles.

Let us first recall the rank 1 analogue. A Hecke transformation of a line bundle \( L \) is the same as the twist \( L(-p) \) at a point \( p \in X \). We have an exact sequence 
\[
0 \to O(L(-p)) \xrightarrow{s_p} O(L) \to \mathcal{T}_X(p) \to 0,
\]
where \( s_p \) is a canonical section of \( O(p) \) and \( \mathcal{T}_X(p) \) is the torsion sheaf of length 1 at \( p \).

**Definition 4.1** ([HR04]). Let \( E \) be a holomorphic vector bundle of rank 2 on a Riemann surface \( X \). Let \( p \in X \) and \( \alpha \in E_p^\vee \setminus \{0\} \), the dual fiber at \( p \). The Hecke transformation \( \hat{E}^{(p,\alpha)} \) of \( E \) is defined through the exact sequence of coherent sheaves 
\[
0 \to O(\hat{E}^{(p,\alpha)}) \to O(E) \xrightarrow{\alpha} \mathcal{T}_X(p) \to 0.
\]
To get a more concrete description of such transformation, we want to describe it on the level of transition functions.

Let \( \mathcal{U} = \{U_i\}_{i=1}^m \) a covering of \( X \) by contractible open sets, such that \( p \in U_i \) if and only if \( i = 1 \). Let \( \{\psi_{ij}\} \in \tilde{H}^1(\mathcal{U}, \text{GL}(2, \mathbb{C})) \) transition functions for \( E \). Choose a holomorphic frame \( s_1, s_2 \) of \( E|_{U_1} \), such that \( \alpha = (s_2)^\vee_p \). Define a new covering \( \mathcal{V} = \{V_i\}_{i=0}^m \) by \( V_0 = U_1, V_1 = U_1 \setminus \{p\} \) and \( V_i = U_i \) for \( i \geq 2 \). Define new transition functions \( \{\hat{\psi}_{ij}\} \in \tilde{H}^1(\mathcal{V}, \text{GL}(2, \mathbb{C})) \) by 
\[
\hat{\psi}_{0j} : V_0 \cap V_1 \times \mathbb{C}^2 \to V_0 \cap V_1 \times \mathbb{C}^2, \\
(z, x_1, x_2) \mapsto (z, x_1, zx_2)
\]
(1) respective the frame \( s_1, s_2 \),
(2) \( \hat{\psi}_{0j} = \psi_{1j} \circ \hat{\psi}_{01}, \quad \hat{\psi}_{j0} = \hat{\psi}_{0j}^{-1} \) for \( j \geq 1 \), and \( \hat{\psi}_{ij} = \psi_{ij} \) for \( i, j \geq 1 \).
Lemma 4.2. The holomorphic vector bundle associated to the transition functions \( \{\psi_{ij}\} \in \check{H}^1(V, GL(2, \mathbb{C})) \) is the Hecke transformation \( \check{E}^{p,\alpha} \) of \( E \).

Proof. By definition of the transition function \( \check{\psi}_{01} \), the associated vector bundle fits in the exact sequence from Definition 4.1. \( \square \)

We generalize this concept by allowing higher order twists. Let \( \text{Div}^+(X) \) denote the set of effective divisors on \( X \). Let \( D \in \text{Div}^+(X) \) and \( E \) a holomorphic vector bundle on \( X \). Higher Hecke transformations will be parametrised by polynomial germs on \( D \). Define

\[
H^0(D, E) := \bigoplus_{p \in \text{supp} D} \mathcal{O}(E)_p / \sim,
\]

where \([s_1] \sim [s_2]\) if and only if \( \text{ord}_p([s_1] - [s_2]) \geq D_p \), for all \( p \in \text{supp} D \). Furthermore, denote by \( H^0(D, E)^* \subset H^0(D, E) \) the equivalence classes of germs, such that for all \( p \in \text{supp} D \) the evaluation at \( p \) is non-zero.

**Definition 4.3.** Let \( E \) be a holomorphic vector bundle of rank 2. Let \( D \in \text{Div}^+(X) \) and \( \alpha \in H^0(D, E^\vee)^* \). Then the Hecke transformation \( \hat{E}^{(D,\alpha)} \) of \( E \) at \( D \) in direction \( \alpha \) is defined by the exact sequence of locally free sheaves

\[
0 \to \mathcal{O}(\hat{E}^{(D,\alpha)}) \to \mathcal{O}(E) \xrightarrow{\alpha} T_X(D) \to 0,
\]

where \( T_X(D) \) is the torsion sheaf of length \( D_p \) at \( p \in \text{supp} D \).

**Lemma 4.4.** Let \( D \in \text{Div}^+(X) \) and \( \alpha \in H^0(D, E^\vee)^* \), then \( \det(\hat{E}^{(D,\alpha)}) = \det(E)(-D) \).

For our purpose, it will be more convenient to use the dual version of this concept.

**Definition 4.5.** Let \( D \in \text{Div}^+(X) \) and \( \alpha \in H^0(D, E)^* \). Then the (dual) Hecke transformations \( \check{E}^{(D,\alpha)} \) of \( E \) at \( D \) in direction \( \alpha \) is defined by the exact sequence of locally free sheaves

\[
0 \to \mathcal{O}((\check{E}^{(D,\alpha)})^\vee) \to \mathcal{O}(E^\vee) \xrightarrow{\alpha} T_X(D) \to 0.
\]

**Lemma 4.6.** Let \( \{\psi_{ij}\} \in \check{H}^1(U, GL(2, \mathbb{C})) \) transition functions of \( E \) as above. For \( p \in X, l \in \mathbb{N}, \) let \( D := lp \in \text{Div}^+(X) \). Let further \( \alpha \in H^0(D, E)^* \). The Hecke transformations \( E^{(D,\alpha)} \) is given by the transition functions \( \{\check{\psi}_{ij}\} \in \check{H}^1(V, GL(2, \mathbb{C})) \) defined as in \([1],[3]\), where the frame \( s_1, s_2 \) is chosen, such that

\[
[(s_2)_p] = \alpha \in H^0(D, E)
\]

and

\[
\check{\psi}_{01} : V_0 \cap V_1 \times \mathbb{C}^2 \to V_0 \cap V_1 \times \mathbb{C}^2
\]

\[
(z, x_1, x_2) \mapsto (z, x_1, z^{-l}x_2).
\]

For general \( D \in \text{Div}^+(X) \) and \( \alpha \in H^0(D, E)^* \), we have to introduce a new transition functions for all \( p \in \text{supp}(D) \).

**Lemma 4.7.** Let \( D \in \text{Div}^+(X) \) and \( \alpha \in H^0(D, E)^* \), then \( \det(\check{E}^{(D,\alpha)}) = \det(E)(D) \).
4.1. Parameter spaces of higher Hecke transformations.

Lemma 4.8. Let \( D \in \text{Div}^+(X) \), \( \alpha \in H^0(D, E)^* \) and \( \phi \in H^0(D, \mathcal{O}_X)^* \). Then
\[
E^{(D, \alpha)} \cong E^{(D, \phi \alpha)}.
\]

An equivalence class in the quotient \( H^0(D, E)/H^0(D, \mathcal{O}_X)^* \) is referred to as a Hecke parameter.

Proposition 4.9. \( H^0(D, \mathcal{O}_X)^* \) is a complex solvable Lie group respective multiplication of germs of non-vanishing holomorphic functions. Let \( D = lp \) with \( l \in \mathbb{N} \) and \( p \in X \). Then
\[
H^0(D, \mathcal{O}_X)^* \cong \left\{ \begin{pmatrix} x_0 & x_1 & \cdots & x_{l-1} \\ \vdots & \ddots & \vdots & \vdots \\ x_0 & x_1 & \cdots & x_0 \\ x_0 \\ x_1 \\ \vdots \\ x_{l-1} \end{pmatrix} \mid x_0 \in \mathbb{C}^*, x_i \in \mathbb{C} \right\} \cong \mathbb{C}^* \times \mathbb{C}^{l-1}.
\]
In general, \( H^0(D, \mathcal{O}_X)^* \) is isomorphic to the Cartesian product of such groups.

4.2. Leading example. As a leading example, we show how the algebraic pushforward of a line bundle along a 2-covering of Riemann surfaces can be recovered using Hecke transformations and the \( \sigma \)-invariant pushforward defined in Section 3.

Let \( p : Y \to X \) be a covering of Riemann surfaces and \( \sigma : Y \to Y \) the holomorphic involution changing the sheets. Denote by \( B \subset X \) the branch divisor. Let \( L \in \text{Jac}^l(Y) \), then \( E = L \oplus \sigma^*L \) has a natural lift \( \hat{\sigma} : E \to E \) induced by pullback along \( \sigma \). Restricted to the fiber over a ramification point, \( \hat{\sigma} \) is locally given by
\[
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Hence, \( E \) is not yet a \( \sigma \)-invariant holomorphic vector bundle (cf. Definition 3.1). This can be corrected by a Hecke transformation. Choose a neighbourhood \( U \) of \( \text{Fix}(\sigma) \) separating all ramification points and a frame \( s \in H^0(U, L) \). Then
\[
s_1 = s \oplus \sigma^*s, \quad s_2 = s \oplus -\sigma^*s
\]
is a frame of \( E \) diagonalizing \( \hat{\sigma} \). Let \( D = \frac{1}{2}p^*B \) and
\[
\alpha = [s_2]^\vee \in H^0(D, E^\vee)^*.
\]
Then the Hecke transformation \( E^{(D, \alpha)} \) can be constructed form \( E \) by introducing new transition functions of the form
\[
\hat{\psi}_{01} : V_0 \cap V_1 \times \mathbb{C}^2 \to V_0 \cap V_1 \times \mathbb{C}^2
\]
(at every point \( y \in \text{supp}(D) = \text{Fix}(\sigma) \) (where we choose a coordinate in the neighbourhood of \( y \) such that \( \sigma : z \mapsto -z \)). \( \hat{\sigma} \) induces a lift of \( \sigma \) on \( \hat{E}^{(D, \alpha)} \), that we keep calling \( \hat{\sigma} \). The frame \( s_1, zs_2 \) extends to a \( \hat{\sigma} \)-invariant frame \( s_1^\sigma, s_2^\sigma \) of \( \hat{E}^{(D, \alpha)} \). Hence, \( (\hat{E}^{(D, \alpha)}, \hat{\sigma}) \) is a \( \sigma \)-invariant holomorphic vector bundle. In particular, \( p_*(\hat{E}^{(D, \alpha)}, \hat{\sigma}) \) is a locally free sheaf of rank 2 on \( X \).
**Lemma 4.10.** \( p_*(\hat{E}^{(D,\alpha)}, \hat{\sigma}) = p_! L. \)

**Proof.** Let \( U_1 \subset X \) be open, contractible subset trivializing the covering \( p \), i.e. \( p^{-1}U_1 = \hat{U}_1^+ \sqcup \hat{U}_1^- \). \( p_* L \) is a free module of rank 2 over \( O_{U_1} \). This is apparent from decomposing
\[
H^0(U_1, p_* L) = H^0(p^{-1}U_1, L) = H^0(\hat{U}_1^+, L) \oplus H^0(\hat{U}_1^-, L).
\]

Hence, we have a natural isomorphism
\[
H^0(U_1, p_* L) \cong H^0(U_1, p_*(\hat{E}^{(D,\alpha)}, \hat{\sigma})) = H^0(p^{-1}U_1, L \oplus \sigma^* L) \hat{\sigma}.
\]

Let \( U_2 \subset X \) be open, contractible neighbourhood of a branch point \( x \in X \). Choose a coordinate on \( p^{-1}(U_2) \), such that \( \sigma \big|_{p^{-1}(U_2)} : z \mapsto -z \). After the choice of a local frame \( s \in H^0(p^{-1}U_2, L) \), we write a section \( \phi \in H^0(p^{-1}U_2, L) \) as
\[
\phi(z) = \phi_1(z^2)s + \phi_2(z^2)zs.
\]

Then \( p_* L \big|_{p^{-1}(U_2)} \) is free over \( O_{U_2} \) of rank 2 with generators \( s, zs \). Let \( s_1, s_2 \) be the \( \sigma \)-invariant frame of \( \hat{E}^{(D,\alpha)} \) defined above, then we define an isomorphism
\[
H^0(p^{-1}U_2, L) \to H^0(p^{-1}U_2, \hat{E}^{(D,\alpha)}) \hat{\sigma}, \quad \phi \mapsto \phi_1 s_1 + \phi_2 s_2.
\]

The isomorphisms \( \hat{\sigma} \) and \( \hat{\sigma} \) define an isomorphism of locally free sheaves, i.e. they commute with the restriction functions. Let \( U_1, U_2 \subset X \) as above and \( U_1 \subset U_2 \) and \( \phi \in H^0(U_2, p_! L) = H^0(p^{-1}U_2, L) \) as in \( \hat{\sigma} \)
\[
\phi \big|_{\hat{U}_1^+} = \phi_1 s_1 \big|_{\hat{U}_1^+} + \phi_2 z s_2 \big|_{\hat{U}_1^+},
\]
\[
\phi \big|_{\hat{U}_1^-} = \phi_1 s_1 \big|_{\hat{U}_1^-} + \phi_2 z s_2 \big|_{\hat{U}_1^-} = \phi_1 s_1 \circ \sigma - \phi_2 z s_2 \big|_{\hat{U}_1^+} \circ \sigma.
\]

So the restriction map is given by
\[
\begin{pmatrix}
1 & z \\
1 & -z
\end{pmatrix}.
\]

This agrees with the restriction map of \( p_*(\hat{E}^{(D,\alpha)}, \hat{\sigma}) \) by construction. \( \square \)

**Corollary 4.11.** Consider a two-sheeted covering of Riemann surfaces \( p : Y \to X \) with at least one branch point. Then
\[
p_* O_Y = O_X \oplus O_X(-B)^{\hat{\sigma}},
\]
where \( B \in \text{Div}^+(X) \) is the branch divisor of \( p \).

**Proof.** Let \( L = O_Y \) in the construction above. So \( E = O_Y \oplus O_Y \) and
\[
\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

The diagonalizing frame for \( \hat{\sigma} \) gives a global splitting
\[
E = O_Y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus O_Y \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

Then
\[
\hat{E}^{(B,\alpha)} = O_Y \oplus O_Y(-\frac{1}{2}p^* B) = p^* O_X \oplus p^* (O_X(-B)^{\hat{\sigma}}).
\]
5. MODULI OF $\sigma$-INARIANT HIGGS BUNDLES

After identifying the Hitchin fibers with certain moduli spaces of $\sigma$-invariant Higgs bundles on the normalised spectral curve in Section 3, we will now prove the stratification result for these moduli spaces in a more general setting. Thereafter, we will identify these strata as fiber bundles over Prym varieties. We return to the case of singular $\text{SL}(2,\mathbb{C})$-Hitchin fibers in the following section.

5.1. The Stratification. Let $Y$ be a Riemann surface of genus $g(Y) \geq 2$ with an involutive biholomorphism $\sigma : Y \to Y$. Denote by $Y/\sigma$ the unique Riemann surface such that there exists a ramified two-sheeted covering of Riemann surfaces $p : Y \to Y/\sigma$ factoring through $\sigma$. In particular, the genus of $Y/\sigma$ is given by

$$g(Y/\sigma) = \frac{1}{2} g(Y) + 1 - \frac{1}{2} \# \text{Fix}(\sigma).$$

Depending on $g(Y)$, this restricts the number of fix points of the involution $\sigma$. Let $M$ be a line bundle on $Y/\sigma$ with a non-zero section $\lambda : Y \to p^*M$, such that $\hat{\sigma}\lambda = -\lambda$. Here $p^*M$ is regarded as a $\sigma$-invariant holomorphic line bundle with the lift $\hat{\sigma}$ induced by pullback (cf. Proposition 3.3). In particular, $\lambda$ has a zero of odd order at every branch point. Let $\Lambda = \text{div}(\lambda)$.

In this section, we parametrize $M_{\sigma,\lambda} = M_{\sigma}(Y, p^*M, \lambda) := M_{\sigma}(Y, p^*M) \cap \text{Hit}^{-1}_{p^*M}(-\lambda^2)$, the polystable $\sigma$-invariant $p^*M$-twisted $\text{SL}(2,\mathbb{C})$-Higgs bundles on $Y$ with characteristic equation $(X + \lambda)(X - \lambda)$.

By assumption, $-\lambda^2$ is a $\sigma$-invariant section of $p^*M^2$ and hence descends to $a \in H^0(X, M^2)$. $M_{\lambda}^\sigma$ is identified with the image of $p^* : \text{Hit}^{-1}_M(a) \to M(Y, p^*M)$ by Proposition 3.3 and is therefore an analytic subset.

**Lemma 5.1.** Let $(E, \Phi) \in M_{\lambda}^\sigma$. Let $y \in Y$ and $m \in H^0(U, \pi^*M)$ be a local frame in an open neighbourhood $U$ of $y$. There exists a local frame of $E|_U$, such that the Higgs field is given by

$$\Phi = z^D \begin{pmatrix} 0 & 1 \\ z^{2\Lambda_y - 2D_y} & 0 \end{pmatrix} \otimes m.$$

**Proof.** Choose a coordinate disc $(U, z)$ centred at $y$, such that the determinant $\det(\Phi) = -z^{2\Lambda_y} m^2$. There exists a non vanishing section $\phi \in H^0(U, \text{End}(E))$ such that

$$\Phi(z) = z^D \phi(z) m.$$

There are two possible Jordan forms of $\phi$ at $y$. If $D_y < \Lambda_y$ there is one Jordan block of size 2, if $D_y = \Lambda_y$, $\phi$ is diagonalizable with eigenvalues $\pm 1$. Thus, after a constant gauge transformation we can assume

$$\phi(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & -a(z) \end{pmatrix} \quad \text{with} \quad \phi(0) = \begin{pmatrix} 0 & 1 \\ * & 0 \end{pmatrix}.$$
Hence,
\[ g = \frac{1}{\sqrt{b(z)}} \begin{pmatrix} b(z) & 0 \\ -a(z) & 1 \end{pmatrix} \in \text{SL}(E|\nu), \]
is a well-defined gauge, such that
\[ g^{-1}\phi g = \begin{pmatrix} 0 & 1 \\ -\det(\phi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^{2\Lambda_y - 2D_y} & 0 \end{pmatrix} \]
is a well-defined gauge, such that
\[ \frac{g^{-1}\phi g}{\det(\phi)} = \begin{pmatrix} 0 & 1 \\ -\det(\phi) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^{2\Lambda_y - 2D_y} & 0 \end{pmatrix} \]

For \((E, \Phi) \in M^\sigma\), we denote by \(\text{div}(\Phi)\) the vanishing divisor of \(\Phi\). The properties of such divisors are summarized in the following definition.

**Definition 5.2.** An effective divisor \(D \in \text{Div}(Y)\) is called \(\sigma\)-Higgs divisor on \((Y, \sigma, \lambda)\) if \(0 \leq D \leq \Lambda\), \(\sigma^*D = D\) and
\[ D_y \equiv 0 \mod 2 \]
for all \(y \in \text{Fix}(\sigma) \subset \text{supp} \Lambda\).

**Theorem 5.3.** There exists a stratification
\[ M^\sigma(Y, p^*M, \lambda) = \bigcup_D S_D \]
by locally closed analytic subsets
\[ S_D = \{(E, \Phi) \in M^\sigma(Y, p^*M, \lambda) \mid \text{div}(\Phi) = D\} \]
indexed by \(\sigma\)-Higgs divisors \(D \in \text{Div}(Y)\).

**Proof.** First, it is easy to see that for \((E, \Phi) \in M^\sigma(Y, p^*M, \lambda)\) the vanishing divisor \(\text{div}(\Phi)\) is a \(\sigma\)-Higgs divisor. These divisors form a lower semi-continuous invariant on \(M^\sigma_{\lambda}\) (cf. Lemma 5.1). In particular, for fixed \(\sigma\)-Higgs divisor \(D\)
\[ \bigcup_{D' \geq D} S_{D'} \]
is closed and \( \bigcup_{D' \leq D} S_{D'} \) is open.

Hence, \(S_D\) is locally closed. To see that the closed subset is an analytic subset, we need to identify it as the Hitchin fiber of \(\sigma\)-invariant \(\text{SL}(2, \mathbb{C})\)-Higgs bundles with a different twist. Fix a \(\sigma\)-Higgs divisor \(D\) and let \(s_D\) be the canonical section of \(O(D)\), which is \(\sigma\)-invariant. Then \((p^*M)(-D) = p^*(M(-\frac{3}{2}\text{Nm}D))\) and hence it is a pullback. Moreover \(\frac{\Lambda_y}{s_D} \in H^0(p^*M(-D))\) satisfies \(\sigma^*(\lambda/s_D) = -\lambda/s_D\). So
\[ M^\sigma(Y, p^*M(-D), \lambda/s_D) \]
is the subspace of \(\sigma\)-invariant Higgs bundles \(p^*M(-D)\)-twisted \(\text{SL}(2, \mathbb{C})\)-Higgs bundles on \(Y\) with determinant \(-\lambda^2/s_D^2\). This is the pullback of a Hitchin fiber in the moduli space of \(M(-\frac{3}{2}\text{Nm}D)\)-twisted \(\text{SL}(2, \mathbb{C})\)-Higgs bundles on \(Y/\sigma\) and hence it is an analytic subspace of \(M(Y, p^*M(-D))\).

There is a holomorphic bijective map
\[ M^\sigma(Y, p^*M(-D), \lambda/s_D) \to \bigcup_{D' \geq D} S_{D'} \subset M^\sigma(Y, p^*M, \lambda), \]
\[ (E, \Phi) \quad \mapsto \quad (E, s_D \Phi), \]
Therefore, its image is an analytic subspace of \(M^\sigma(Y, p^*M, \lambda)\) (see [Gra+94] 1.10.13).
5.2. Prym varieties.

**Definition 5.4.** Let \( D \) be an effective divisor on \( Y \), such that \( \sigma^*D = D \). Then the \( D \)-twisted Prym variety of \((Y, \sigma)\) is defined by
\[
Prym_D = \{ L \in \text{Pic}(Y) \mid L \otimes \sigma^*L = \mathcal{O}_X(D)^{-1} \}.
\]

**Theorem 5.5.** Consider \( \mathcal{M}^*_\lambda \) as above. For every stratum \( S_D \), there exists a holomorphic map
\[
\text{Eig}_D : S_D \to Prym_{\Lambda-D}, \quad (E, \Phi) \mapsto \ker(\Phi - \lambda \text{id}_E).
\]

**Proof.** Let \( (E, \Phi) \in S_D \) and let \( \mathcal{O}(L) = \ker(\Phi - \lambda \text{id}_E) \) the sheaf-theoretical kernel. Then \( \mathcal{O}(\sigma^*L) = \ker(\Phi + \lambda \text{id}_E) \). The inclusions \( \mathcal{O}(L) \to \mathcal{O}(E), \mathcal{O}(\sigma^*L) \to \mathcal{O}(E) \) define an exact sequence of coherent analytic sheaves
\[
0 \to \mathcal{O}(L) \oplus \mathcal{O}(\sigma^*L) \to \mathcal{O}(E) \to \mathcal{T} \to 0,
\]
where \( \mathcal{T} \) is a torsion sheaf supported at \( Z(\lambda) \). \( \mathcal{T} \) can be explicitly constructed using the local description of \( \Phi \) in a neighbourhood of \( p \in Z(\lambda) \) given in Lemma 5.1. In particular,
\[
\mathcal{O}_Y = \det(E) = L \otimes \sigma^*L \otimes \det(T) = L \otimes \sigma^*L \otimes \mathcal{O}(\Lambda - D).
\]

\( \square \)

**Proposition 5.6.** Let \( D \) be a \( \sigma \)-Higgs divisor on \((Y, \sigma, \lambda)\). The twisted Prym variety \( Prym_{\Lambda-D} \) is an abelian torsor over \( Prym_0 \) of dimension
\[
dim Prym_0 = g(Y) - g(Y/\sigma) = \frac{1}{2} \left( g(Y) - 1 + \frac{1}{2} \# \text{Fix}(\sigma) \right).
\]

**Proof.** Let \( N = \Lambda - D \). We need to show that \( Prym_N \) is non-empty. If there exists \( L \in Prym_N \), the group action
\[
Prym_0 \to Prym_N, \quad M \mapsto L \otimes M
\]
is simply transitive. Due to the map \( \text{Eig}_D : S_D \to Prym_N \), it is enough to show that \( S_D \) is non-empty for \( \sigma \)-Higgs divisors \( D \). After identifying \( \mathcal{M}^*_\lambda \) with the \( \text{SL}(2, \mathbb{C}) \)-Higgs bundles on \( Y/\sigma \) in the corresponding Hitchin fiber it will be easy to construct examples with all possible \( \sigma \)-Higgs divisors by looking at \( \text{SL}(2, \mathbb{R}) \)-Higgs bundles (see Proposition 9.5). \( \square \)

5.3. Hecke parameters. The fibers of each stratum \( S_D \) over \( Prym_{\Lambda-D}(Y) \) can be identified as Hecke parameters (cf. Section 3). Consider \( \mathcal{M}^*_\lambda \) as above, \( D \) a \( \sigma \)-Higgs divisor and \( L \in Prym_{\Lambda-D}(Y) \). Let
\[
(E_L, \Phi_L) = \left( L \oplus \sigma^*L, \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \right).
\]

\( E_L \) has a natural lift \( \hat{\sigma} : E_L \to E_L \) of \( \sigma \) induced by the pullback. Choose a local frame \( s \) of \( L \) at a branch point \( p \in Y \), then
\[
\hat{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
respectively the frame \( s, \sigma^*s \) of \( E_L \). Fix the diagonalizing frame
\[
s_+ = s + \sigma^*s, \quad s_- = s - \sigma^*s.
\]

**Lemma 5.7.** Let \( \alpha \in H^0(\Lambda - D, E_L)^* \). Then \( E_L^{(\Lambda-D, \alpha)} \) is a \( \sigma \)-invariant holomorphic vector bundle if
Applying Let's (with notation as in 1, 2). Because $\Lambda - C$ is a respective frame, we can explicitly parametrize $\sigma$ with respect to which factors through the action of $\hat{\sigma}$.

Let $\Lambda = \{ \alpha \in H^0(D, E_L) \mid \alpha \text{ satisfies i) and ii) } \}$

$H^0(D, (E_L, \hat{\sigma}))^* := \{ \alpha \in H^0(D, E_L)^* \mid \alpha \text{ satisfies i) and ii) } \}$

$G_D := \{ \phi \in H^0(D, O_Y)^* \mid \sigma^* \phi = \phi \}$.

Proposition 5.8. Let $\Lambda = \text{Fix}(\sigma), D$ a $\sigma$-Higgs divisor and $N := \Lambda - D$.

Let $(U, z)$ be a union of coordinate neighbourhoods $(U_p, z_p)$ around $p \in \text{Fix}(\sigma)$ disconnecting $\text{Fix}(\sigma)$, such that $\sigma : z_p \mapsto -z_p$. Let $s \in H^0(U, L)$ a frame. Then there is a holomorphic surjective map

$$u_s : H^0(N, (E_L, \sigma))^* \to \bigoplus_{p \in \text{Fix}(\sigma)} \mathbb{C} z_p + \mathbb{C} z_p^3 + \cdots + \mathbb{C} z_p^{-2}$$

which factors through the action of $G_N$ and induces a bijection on the quotient. Let $s' = f s$ with $f = f_e + f_o \in H^0(U, O_Y f)$ and $f_e, f_o$ the even and odd part respective $\sigma$, then

$$u_{s'} = f_e u_s - f_o u_s \mod z^N.$$

Proof. A choice of $s$ induces a frame $s_+ = s + \sigma^* s, s_- = s - \sigma^* s$ of $E_L$. Respective such frame, we can explicitly parametrize

$$H^0(N, (E_L, \hat{\sigma}))^* = \{ (as_+ + bs_-) \mid a, b \in H^0(N, O_Y) \mid \sigma^* a = -a, \sigma^* b = b \}.$$

Let $N = lp$ with $l \in \mathbb{N}, p \in Y$. We define the isomorphism

$$u_s : H^0(N, (E_L, \hat{\sigma}))^* \to \{ u \in H^0(N, O_Y) \mid \sigma^* u = -u \}$$

$$[as_+ + bs_-] \mapsto \frac{a}{b} \mod z^{Nl}.$$

This map clearly factors through $G_N$ and separates the orbits. The right side is a $\mathbb{C}$ vector space over the basis $z, z^3, \ldots, z^{Np-2}$. Let now $s' = (f_e + f_o)s$, then

$$s'_+ = f_e s_+ + f_o s_-, \quad s'_- = f_e s_- + f_o s_+$$

and

$$\alpha = \frac{1}{f_e^2 - f_o^2} \left( (a f_e - b f_o) s'_+ + (-a f_o + b f_e) s'_- \right).$$

Applying $u_{s'}$ gives the result.
Proposition 5.9. Let $\text{supp}\Lambda = \text{Fix}(\Sigma)$, $D \in \text{Div}^+(Y)$ a $\sigma$-Higgs divisor and $N = \Lambda - D$. Then

$$F_D = \{H^0(N, (E_L, \hat{\sigma}))^*/G_N \mid L \in \text{Prym}_N(Y)\} \to \text{Prym}_N(Y)$$

is a holomorphic vector bundle of rank

$$r = \sum_{y \in Z(\lambda)} \frac{1}{2}(\Lambda_y - D_y - 1) = \frac{1}{2} (\deg(\Lambda) - \deg(D) - \#\text{Fix}(\sigma)).$$

Proof. There exists a universal line bundle $L \to Y \times \text{Prym}_N$. Let $U$ be a disconnecting neighbourhood of $\text{Fix}(\sigma)$ as above. A local trivialization of $L$ over $U \times V \subset Y \times \text{Prym}_N$ is equivalent to choosing a local frame $s \in H^0(U, L)$ over $V \in \text{Prym}_N$ in a coherent way. It defines a $u$-coordinate of $F_D$ over $V$, in other words, a local trivialisation $F_D \cong V \times \mathbb{C}^r$. Changing the trivialisation corresponds to choosing a different holomorphic frame $s' \in H^0(U, L)$. The corresponding transformation of the $u$-coordinate is holomorphic by the previous proposition. \hfill \Box

Theorem 5.10. Let $\text{supp}\Lambda = \text{Fix}(\sigma)$ and $D$ a $\sigma$-Higgs divisor. Then there is an isomorphism of $F_D \to \mathcal{S}_D$ making the following diagram commute:

$$\begin{align*}
F_D & \to \mathcal{S}_D \\
\text{Prym}_{\Lambda-D} & \to \text{Prym}_{\Lambda-D} \\
\text{Eig}_D & \\
\text{id} & \\
\end{align*}$$

In particular,

$$\dim \mathcal{S}_D = \deg(M) - \frac{1}{2} \deg(D) + \frac{1}{2} (g(Y) - 1) - \frac{1}{4} \#\text{Fix}(\sigma).$$

Proof. Let $N = \Lambda - D$. Let $L \in \text{Prym}_N(Y)$. Let $(U, z)$ be a union of coordinate neighbourhood of $\text{Fix}(\sigma)$ disconnecting $\text{Fix}(\sigma)$, such that $\sigma : z \mapsto -z$. Let $t \in H^0(U, p^*M)$ and $s \in H^0(U, L)$ local frames. We will show in i) how to produce Higgs bundles in $\mathcal{S}_D$ by applying Hecke transformation to $(E_L, \Phi_L)$. This defines the map $F_D \to \mathcal{S}_D$. To see that it is an isomorphism, we will show in ii) how to recover the $u$-coordinate from $(E, \Phi) \in \mathcal{S}_D$.

i) Let $\alpha \in H^0(N, (E_L, \hat{\sigma}))^*$, we saw in Lemma 5.7 that $\hat{E}^{(N, \alpha)}$ with the induced lift $\hat{\sigma}$ is a $\sigma$-invariant holomorphic vector bundle. Furthermore,

$$\text{det}(\hat{E}^{(N, \alpha)}) = \text{det}(E_L)(N) = L \otimes \sigma^*L \otimes \mathcal{O}(\Lambda - D) = \mathcal{O}_Y.$$

The Higgs field $\Phi_L$ induces a Higgs field on $\hat{E}^{(N, \alpha)}$. From Lemma 4.6 it is easy to see that the Hecke transformation of $E_L$ in direction $\alpha = us_+ + s_-$ is given by introducing a new transition function

$$\hat{\psi}_{01} = \begin{pmatrix} 1 & -u z^{-N} \\ 0 & z^{-N} \end{pmatrix}$$
Corollary 5.11. Consider \( \mathcal{M}^\sigma(Y, p^*L, \lambda) \), such that \( Z(\lambda) = \text{Fix}(\sigma) \) and \( \lambda \) has only simple zeroes. Then

\[ \mathcal{M}^\sigma(Y, p^*L, \lambda) = \text{Prym}_\Lambda. \]
Consider parameters in Proposition 8.1. To use extension classes. We will give an interpretation in terms of Hecke of the complex of locally free sheaves \( E \), we need to parametrise the possible \( \sigma \)-Higgs divisor \( D \) and \( L \in \text{Prym}_{\Lambda - D} \). Then

\[
\text{Eig}^{-1}_D(L) \cong \{ [c] \in H^0(\Lambda y, L^2 \pi^* M) \mid \text{ord}_y[c] = D_y \} \cong \mathbb{C}^* \times \mathbb{C}^{\wedge - D_y - 1}.
\]

The last isomorphism is determined by the choice of a local coordinate \((U, z)\) centred at \( y \) and the choice of a local frame \( s \in H^0(U, L^2 \pi^* M) \).

Proof. By assumption, we can trivialize the covering in a neighbourhood of \( p(y) \in Y/\sigma \). Let \( p^{-1}(U) = U_y \cup U_{\sigma y} \), such that \( y \in U_y \). Let \( (E, \Phi) \in M^\lambda_y \). By the \( \sigma \)-invariance \( (E, \Phi)^{p^{-1}(U)} \) is uniquely determined by \( (E, \Phi) |_{U_y} \). So we need to parametrise the possible \( (E, \Phi) |_{U_y} \) with eigenvalues \( \pm \lambda |_{U_{\sigma y}} \). Regarding \( (E, \Phi) |_{U_y} \) as a SL(2, \( \mathbb{C} \))-Higgs bundle with reducible spectral curve, we will use the description of SL(2, \( \mathbb{C} \))-Hitchin fibers with this property given in [GO13] Section 7. Write \( (E, \Phi) \) as an extension

\[
0 \to (L, \lambda) \to (E, \Phi) \to (L^*, -\lambda) \to 0.
\]

These extensions are parametrised by the hypercohomology group \( H^1(L^2, 2\lambda) \) of the complex of locally free sheaves

\[
O_Y(M^2) \xrightarrow{2\lambda} O_Y(L^2 \pi^* M).
\]

The 5-term exact sequence of (one of) the associated spectral sequences reveals that

\[
H^1(L^2, 2\lambda) \cong H^0(\Lambda, L^2 \pi^* M) := \bigoplus_{y \in \text{supp}(\Lambda)} O(L^2 \pi^* M)_y / \sim,
\]

where for \( v, v' \in \bigoplus_{y \in \text{supp}(\Lambda)} O(L^2 \pi^* M)_y \)

\[
v \sim v' \iff v = v' + f \lambda \quad \text{with} \quad f \in \bigoplus_{y \in \text{supp}(\Lambda)} O_y.
\]

By Theorem 5.10, the extension data at the simple zeroes in \( \text{Fix}(\sigma) \) is uniquely determined by \( \sigma \)-invariance. Hence, the fibers of \( \text{Eig}_D \) are parametrised by

\[
H^0(\Lambda_y y, L^2 \pi^* M),
\]

where we consider \( \Lambda_y y \) as a divisor supported at the point \( y \). Furthermore, one can explicitly construct a Higgs bundle

\[
\left( E = L \oplus C \cong L^{-1}, \partial_E = \begin{pmatrix} \partial_L & b \\ 0 & \partial_{L^{-1}} \end{pmatrix}, \Phi = \begin{pmatrix} \lambda & c \\ 0 & -\lambda \end{pmatrix} \right)
\]

from the extension data \([c] \in H^0(\Lambda, L^2 \pi^* M)\) by extending \([c]\) to a smooth section \( c \in A^0(Y, L^2 \pi^* M) \) and solving the equation

\[
\bar{\partial} c = 2b\lambda.
\]
for \( b \in \mathcal{A}^{(0,1)}(Y, L^2) \). In this way, we see that \( \text{div}(\Phi)_y = D_y \) if and only if \( D_y = \text{ord}_c \). So \((E, \Phi) \in S_D\) are parametrized by the polynomial germs \([c] \in H^0(\Lambda_y y, L^2 \pi^* M)\) with \( \text{ord}_y([c]) = D_y \).

**Theorem 5.13.** Fix a moduli space \( \mathcal{M}^\sigma(Y, p^* M, \lambda) \) and a compatible \( \sigma \)-Higgs divisor \( D \). Then the stratum indexed by \( D \) is a holomorphic fiber bundle

\[
(C^\times)^{r_1} \times \mathbb{C}^{r_2} \to S_D \to \text{Prym}_{\Lambda - D}.
\]

with

\[
r_1 = \frac{1}{2} (\#Z(\lambda) - \#\text{Fix}(\sigma)) \quad r_2 = \frac{1}{2} (\deg(\Lambda) - \deg(D) - \#Z(\lambda)).
\]

In particular, the dimension of the stratum is given by

\[
\frac{1}{2} (\deg(M) - \deg(D) + g(Y) - 1 - \frac{1}{2} \#\text{Fix}(\sigma)).
\]

**Proof.** All the extension data depends only on the structure of the Higgs bundle at \( Z(\lambda) \). So if we have more than one higher order zero in \( \lambda \) the fiber of \( \text{Eig}_p \) is a Cartesian product of the fibers described in Theorem 5.10 and Proposition 5.12. The coordinates in Proposition 5.12 depend holomorphically on the choice of a local frame \( s \in H^0(U, L^2 \pi^* M) \). Hence, the argument given in the proof of Theorem 5.13 establishes the structure of a holomorphic fiber bundle on \( S_D \). \( \square \)

### 6. Stratification of singular Hitchin fibers

In this section, we specialize the results of the previous section to a description of singular Hitchin fibers of the moduli space of \( K \)-twisted Higgs bundles on \( X \).

**Definition 6.1.** Let \( q_2 \in H^0(X, K^2) \). A Higgs divisor \( D \in \text{Div}(X) \) is a divisor such that for all \( p \in Z(q_2) \)

\[
0 \leq D_p \leq \lfloor \frac{1}{2} \text{ord}_p(q_2) \rfloor,
\]

where \( \lfloor \cdot \rfloor \) denotes the floor function.

For \( q_2 \in H^0(K^2) \) let

\[
\begin{align*}
n_{\text{even}} &= \# \{ p \in Z(q_2) \mid p \text{ zero of even order} \} \\
n_{\text{odd}} &= \# \{ p \in Z(q_2) \mid p \text{ zero of odd order} \}
\end{align*}
\]

**Theorem 6.2.** Let \( q_2 \in H^0(K^2) \) be a quadratic differential with no global square root on \( X \). Then there is a stratification of the Hitchin fiber

\[
\text{Hit}^{-1}(q_2) = \bigcup_D S_D
\]

indexed by Higgs divisors \( D \). If \( n_{\text{odd}} \geq 1 \), each stratum has the structure of a holomorphic fiber bundle

\[
(C^\times)^{r_1} \times \mathbb{C}^{r_2} \to S_D \to \text{Prym}_{\Lambda - D}(\tilde{\Sigma}),
\]
where

\[ r_1 = n_{\text{even}}, \quad \text{and} \quad r_2 = 2g - 2 - \deg(D) - n_{\text{even}} - \frac{1}{2} n_{\text{odd}}. \]

If \( n_{\text{odd}} = 0 \), \( S_D \) is a branched two-to-one covering over a holomorphic fiber bundle like this. In general, the dimension of a stratum \( S_D \) is given by

\[ 3g - 3 - \deg(D). \]

**Proof.** The stratification by Higgs divisors is obtained in the same way as in Theorem 5.3. We analysed the map \( p^* : \operatorname{Hit}^{-1}(q_2) \to \mathcal{M}^\sigma(\Sigma, \tilde{\pi}^*K, \lambda) \) in Section 3. It is bijective, if there is at least one zero of odd order, and generically two-to-one, if \( n_{\text{odd}} = 0 \). We showed above how \( \mathcal{M}^\sigma_{\lambda} \) is stratified by \( \sigma \)-Higgs divisors (Theorem 5.3). Here a \( \sigma \)-Higgs divisors on \( (\Sigma, \sigma, \lambda) \) is the pullback \( \tilde{\pi}^*D \) of a Higgs divisor \( D \in \operatorname{Div}(X) \) associated to \( q_2 \). The structure of the strata was described in Theorem 5.13. We have \( \#\operatorname{Fix}(\sigma) = n_{\text{odd}}, \) \( \#Z(\lambda) = n_{\text{odd}} + 2n_{\text{even}}, \) and

\[ g(\Sigma) = 2g - 1 + \frac{n_{\text{odd}}}{2}. \]

Hence, the dimension of the stratum \( S_D \) is given by

\[ \frac{1}{2}(\deg(L) - \deg(D) + g(Y) - 1 - \frac{1}{2}\#\operatorname{Fix}(\sigma)) = 3g - 3 - \deg(D). \]

\[ \square \]

6.1. **Algebraic and Geometric Interpretations.** The torus structure is not completely lost, once we degenerate to the singular locus. Starting with a torus of codimension 1 in the first degeneration, the deeper we go into the singular locus the lower the dimension of the subtorus becomes. All singular fibers with irreducible and reduced spectral curve have a subtorus of at least dimension \( g - 1 \). Furthermore, these subtori are torsors of the Prym variety of the normalised spectral cover.

The highest dimensional stratum \( S_0 \) corresponds to Higgs bundles \( (E, \Phi) \in \operatorname{Hit}^{-1}(q_2) \) with non-vanishing Higgs field. One can show, that this stratum corresponds to the locally free sheaves on the singular spectral curve by the Beauville-Narasimhan-Ramanan correspondence [BNR89]. The lowest dimensional stratum \( S_{D_{\text{max}}} \) contains the Higgs bundles \( (E, \Phi) \in \operatorname{Hit}^{-1}(q_2) \) with maximal vanishing order. At a zero of odd order \( 2m + 1 \), they can be locally written as

\[ \Phi(z) = z^m \begin{pmatrix} 0 & 1 \\ \bar{z} & 0 \end{pmatrix} \, dz \]

and at a zero of order \( 2m \), they are diagonalizable with local eigensections \( \pm z^m \, dz \). If \( B \in \operatorname{Div}(X) \) is the branch divisor of the covering by the normalised spectral curve we have

\[ D_{\text{max}} = \Lambda - \frac{1}{2} \tilde{\pi}^*B. \]

This stratum \( S_{D_{\text{max}}} \) has no Hecke parameters or extension data. It is obtained as the algebraic pushforward along \( \tilde{\pi} : \Sigma \to X \) of the line bundles \( L(\frac{1}{2} \tilde{\pi}^*B) \) for all

\[ L \in \operatorname{Prym}_{\frac{1}{2} \tilde{\pi}^*B}(\Sigma) \]
(compare Lemma [4.10]). Restricted to certain subsets of the quadratic differentials one recovers the subintegrable system described by Hitchin [Hit19].

7. Singular fibers with locally irreducible spectral curve

In this section, we start analyzing how the strata glue together to form the singular Hitchin fiber. We will consider the case where the spectral curve is locally irreducible, i.e., the quadratic differential has only zeroes of odd order. To do so, we need to compactify the moduli of Hecke parameters of the highest stratum. We show that these singular Hitchin fibers are themselves holomorphic fiber bundles over Prym varieties with fibers given by the compactified moduli of higher Hecke parameters. This allows a very explicit description of the fibers for the first degenerations.

7.1. Higher Hecke transformations - revisited. We will again work in the setting introduced in Section [5]. So \((Y, \sigma)\) is Riemann surface with an involution \(\sigma, p : Y \to Y/\sigma\) the associated two-covering, \(M\) a line bundle on \(Y/\sigma\) and \(\lambda : Y \to p^*M\) a section with the properties described there. We consider \(M^\alpha = M^\theta(Y, p^*M, \lambda)\), such that \(Z(\lambda) = \text{Fix}(\sigma)\). Under identifying \(M^\alpha\) with the singular Hitchin fiber by pullback, this extra condition means that the spectral curve is locally irreducible. We saw in Theorem 5.10 that for fix \(L \in \text{Prym}_\Lambda(Y)\) the Higgs bundles \((E, \Phi) \in S_0\), which project to \(L\), are parametrized by

\[
\text{Eig}^{-1}(L) = H^0(\Lambda, (E, \hat{\sigma}))^*/G_\Lambda.
\]

After choices of frames of \(L\) at \(\text{supp}\Lambda\), the Hecke parameters are decoded in the polynomial germs \(u\). We reconstructed the \(\sigma^*\)-invariant Higgs bundle from the spectral data \((L, u)\) as the Hecke transformation of \((E_L, \Phi_L)\) at \(\Lambda\) in direction of \(\alpha = us_+ + s_-\) by introducing the new transition function

\[
\psi_{01} = \begin{pmatrix} 1 & -uz^{-\Lambda_p} \\ 0 & z^{-\Lambda_p} \end{pmatrix}
\]

(see Theorem 5.10). To compactify the moduli of Hecke parameters, we need to allow Hecke parameters \(\alpha \in H^0(\Lambda, (E, \hat{\sigma}))\), which possibly vanish on \(\text{supp}\Lambda\) (cf. Lemma 5.7). Fix \(L \in \text{Prym}_\Lambda(Y)\) and a frame \(s \in H^0(U, L)\) in a neighbourhood \(U\) of \(\text{supp}\Lambda\).

**Definition 7.1.** Let \(\alpha \in H^0(\Lambda, (E, \hat{\sigma}))\setminus \{0\}\). Define \((\tilde{E}_L^{(\Lambda, \alpha)}, \tilde{\Phi}_L^{(\Lambda, \alpha)})\) by introducing a new transition function

\[
\tilde{\psi}_{01} = \begin{pmatrix} b^{-1} & -az^{-\Lambda_p} \\ 0 & bz^{-\Lambda_p} \end{pmatrix}
\]

respective \(s_+, s_-\) for all \(y \in \text{supp}\Lambda\), where \(a, b\) are defined through \(\alpha = as_+ + bs_-\) (cf. [6]).

Recall that the Hecke transformation is invariant under the group action of

\[G_\Lambda = \{ \phi \in H^0(\Lambda, \mathcal{O}_Y) \mid \sigma^* \phi = \phi \}\]

on \(H^0(\Lambda, (E, \hat{\sigma}))^*\). There is an extra equivalence on Hecke parameters, when we allow them to vanish.
Lemma 7.2. \( \text{i)} \) Let \( \alpha \in H^0(\Lambda, (E_L, \hat{\sigma})) \) and \( \phi \in G_\Lambda \), then \( \hat{E}_L^{(\Lambda, \alpha)} \cong \hat{E}_L^{(\Lambda, \phi \alpha)} \). In particular, Definition 7.1 and Definition 4.5 agree for \( \alpha \in H^0(\Lambda, (E_L, \hat{\sigma}))^* \).

\( \text{ii)} \) Let \( \alpha, \alpha' \in H^0(\Lambda, (E_L, \hat{\sigma})) \), such that \( \text{div}(\alpha) = \text{div}(\alpha') = D \). Then \( \hat{E}_L^{(\Lambda, \alpha)} \cong \hat{E}_L^{(\Lambda, \alpha')}, \) whenever the projections to \( H^0(\Lambda - D, (E_L, \hat{\sigma})) \) agree.

Proof. In i), the new transition function of the Hecke transformation in direction of \( \phi \alpha \) is given by

\[
\begin{pmatrix}
\phi^{-1} b^{-1} & -\phi a z^{-\Lambda_p} \\
0 & \phi b z^{-\Lambda_p}
\end{pmatrix} = \begin{pmatrix}
b^{-1} & -a z^{-\Lambda_p} \\
0 & b z^{-\Lambda_p}
\end{pmatrix} \begin{pmatrix}
\phi^{-1} & 0 \\
0 & \phi
\end{pmatrix}
\]

Hence, the transition functions define isomorphic vector bundles. For ii), let \( \hat{\sigma}^* \phi = \phi \), then the Hecke transformations respective \( \alpha = as_+ + bs_- \) and \( \alpha' = (a + \phi(b))s_+ + bs_- \) are isomorphic. The isomorphism is given by the gauge transformation

\[
\begin{pmatrix}
b^{-1} & -a z^{-\Lambda_p} \\
0 & b z^{-\Lambda_p}
\end{pmatrix} \begin{pmatrix}1 & \phi \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
b^{-1} & (\phi z^{-\Lambda_p} - a) z^{-\Lambda_p} \\
0 & b z^{-\Lambda_p}
\end{pmatrix}.
\]

This provides the equivalence in the a-coordinate. Using the \( G_\Lambda \) action one obtains equivalence ii). \( \square \)

7.2. Weighted projective spaces. We will obtain a topological model for the compact moduli of Hecke parameters by gluing subsets of weighted projective spaces. Let us recall some basic facts about weighted projective spaces.

A weight vector \( (i_0, \ldots, i_n) \in \mathbb{N}^n \) defines a \( \mathbb{C}^* \)-action on \( \mathbb{C}^{n+1} \) by

\[
\mathbb{C}^* \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, \quad (\lambda, x_0, \ldots, x_n) \mapsto (\lambda^{i_0} x_0, \ldots, \lambda^{i_n} x_n).
\]

The weighted projective space \( \mathbb{P}(i_0, \ldots, i_n) \) is defined as the quotient of \( \mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\} \) by this action. We will denote the equivalence class of \( (x_0, \ldots, x_n) \) by \( (x_0 : \cdots : x_n) \). Weighted projective spaces are complex orbifolds. We obtain orbifold charts in the same way one defines affine charts of projective space \( \mathbb{P}^n \). In explicit, on points of the form \( (1, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \) the action of \( \mathbb{C}^* \) restricts to an action of \( \mathbb{Z}_{i_0} \) given by

\[
(1, x_1, \ldots, x_n) \mapsto (1, \xi_{i_0}^{i_1} x_1, \ldots, \xi_{i_0}^{i_n} x_n),
\]

where \( \xi_{i_0} \) is a primitive \( i_0 \)-th root of unity.

Weighted projective spaces are normal toric complex spaces. In an orbifold chart, the torus action is given by

\[
(1 : x_1 : \cdots : x_n) \mapsto (1 : \lambda_1^{i_1} x_1 : \cdots : \lambda_n^{i_n} x_n)
\]

for \( (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \). This extends to an analytic action on \( \mathbb{P}(i_0, \ldots, i_n) \).

We call a analytic subspace \( Y \subset \mathbb{P}(i_1, \ldots, i_n) \) toric, if it is preserved by the torus action.
7.3. Compact Moduli of Hecke parameters. In this section, we want to study the compact moduli of Hecke parameters. To do so we will restrict our attention to the Hecke parameters at a single higher order zero. Let \((U, z)\) be holomorphic disc centred at zero and \(\sigma : z \mapsto -z\). Let \(d \in \mathbb{N}\) a odd number and \(D\) the divisor with coefficient \(d\) at zero. Define

\[
\text{Heck}_d = \left\{ \left( \frac{a}{b} \right) \in H^0(D, \mathcal{O}_U^2) \mid \sigma^*a = -a, \; \sigma^*b = b \right\} / \sim,
\]

where \((a, b) \sim (a', b') \iff \text{ord}_0(a) = \text{ord}_0(b) =: n\) and \((a, b) = (a', b') \pmod{z^{d-n}}\).

These are the equivalence classes of relation ii) in Lemma 7.2. For \(0 \leq n < \frac{d}{2}\) let

\[
V_n := \{ \alpha \in \text{Heck}_d \mid \text{ord}_0(\alpha) = n \}.
\]

We can understand the quotient of \(\text{Heck}_d\) by \(G_D\) by gluing subsets, on which we find explicit invariant polynomials. By Proposition 4.9, \(G_D = \mathbb{C}^* \times H_D\), where

\[
H_D = \{ \phi = 1 + \phi_2 z^2 + \cdots \in H^0(D, \mathcal{O}_U) \}
\]

is the maximal unipotent normal subgroup. We will first factor through \(H_D\) as orbits of unipotent group actions on affine spaces are closed. The resulting intermediate quotient can be factored through \(\mathbb{C}^*\). The subsets \(V_n\) will correspond to the strata of the stratification 7.3.3.

**Lemma 7.3.** Let \(0 \leq n \leq \frac{d-3}{2}\). There is a holomorphic map \(u_n : V_n \to \mathbb{P}^\frac{1}{2}(d-2n-1)\) invariant under the \(G_D\) action, which separates orbits. Its image is an affine chart of \(\mathbb{P}^\frac{1}{2}(d-2n-1)\). For \(n = \frac{d-1}{2}\) the \(G_D\) action identifies \(V_n\) to a point.

**Proof.** Let's assume \(n \leq \frac{d-3}{2}\) is even. Every \(\alpha \in V_n\) has a unique representative of the form

\[
\left( \frac{a_n z^{n+1} + \cdots + a_d z^{d-2}}{1 + b_n z^2 + \cdots + b_2 z^2} \right) \mod z^{d-n}
\]

with \(b_n \neq 0\) in its \(H_D\)-orbit. In particular, \(b_n\) and the \((n+1)\)-th, \ldots, \((d-n-2)\)-th derivatives of the fraction in the first coordinate define \(\frac{1}{2}(d-2n+1)\) holomorphic functions invariant under the \(H_D\)-action. This defines a map

\[
V_n \to \mathbb{C}^* \times \mathbb{C}_{\frac{1}{2}}^\frac{1}{2}(d-2n-1)
\]

The \(\mathbb{C}^*\)-action acts with weight 1 on every coordinate. By factoring through \(\mathbb{C}^*\) we obtain the desired map to an affine chart of \(\mathbb{P}^\frac{1}{2}(d-2n-1)\). For \(n\) odd, every \(\alpha \in V_n\) has a unique representative

\[
\left( \frac{a_n z^n}{1 + b_n z^2 + \cdots + b_2 z^2} \right) \mod z^{d-n}
\]

By recording \(a_n\) and the \((n+1)\)-th, \ldots, \((d-n-2)\)-th derivative of the second coordinate and again factoring through the \(\mathbb{C}^*\)-action we obtain invariant map

\[
V_n \to \mathbb{P}^\frac{1}{2}(d-2n+1).
\]
As \( a_n \neq 0 \), the image is an affine chart. If \( n = \frac{d-1}{2} \) is odd, the only \( H_D \)-invariant function on \( V_n \) is \( a_n \neq 0 \). Hence, \( V_n \) is identified to a point by the \( C^* \)-action. Similarly, for \( n = \frac{d-1}{2} \) even.

It seems impossible to find enough invariant functions to define the global quotient \( \text{Heck}_d/G_D \). However, we obtain a topological model by gluing the quotients of subsets, which are easier to understand.

**Proposition 7.4.** There exist finitely many locally closed connected subsets \( N_i \subset \text{Heck}_d \), \( i \in I \), such that

i) for every \( n < l \leq \frac{d-1}{2} \) and \( \alpha \in V_i \) there exist \( i \in I \), such that \( \alpha \in N_i \) and \( N_i \cap V_n \neq 0 \),

ii) there exist algebraic maps \( N_i \to \mathbb{P}(1,1,2,3,\ldots,m_i) \) invariant by the action of \( G_D \), which separate the \( G_D \)-orbits. Their images are toric subspaces and contain no singular points.

**Proof.** For \( n \leq l \leq \frac{d-1}{2} \) let

\[
N_i^n := H_D \cdot \{ a = x_1z + x_3z^3 + \ldots, b = x_0z + x_2z^2 + \ldots \mod z^{d-n} | \ x_0 = \cdots = x_{n-1} = x_{n+1} = \cdots = x_{l-1} = 0, x_l \neq 0 \}.
\]

Let \( \alpha = (a,b) \in N_i^n \). If \( x_n \neq 0 \), we have \( \text{ord}_0(\alpha) = n \), hence \( \alpha \in V_n \). If \( x_n = 0 \), we have \( \alpha \in V_l \). So \( N_i^n \) describes a locally closed subset of \( V_n \) containing \( V_l \) in its closure. We first want to find invariant polynomials by the \( H_D \)-action and then take the quotient by \( C^* \). Let \( l \) be odd and \( n \) be even, then

\[
a = x_lz^l + \cdots + x_{d-n-2}z^{d-n-2},
b = x_nz^n + x_{l+1}z^{l+1} + \cdots + x_{d-n-1}z^{d-n-1}
\]

Every orbit in \( N_i^n \) has a representative of the form

\[
(9) \quad \left( \frac{x_lz^l}{x_nz^n + x_{l+1}z^{l+1} + \cdots + x_{d-n-1}z^{d-n-1}} \right) \mod z^{d-n}.
\]

The \( n \)-th, \( n+2 \)-th, \ldots, \((d-l-2)\)-th derivative give (after multiplying with their common divisor)

\[
\frac{1}{2}(d-l-n)
\]

homogeneous polynomials of degree

\[
1, 2, \ldots, \frac{1}{2}(d-l-n).
\]

The representative in (9) is not quite unique because if we act by \( (1 + z^{d-n-l}a) \in H_D \) the \( a \)-coordinate stays unchanged modulo \( z^{d-n} \). However, as we only record up to the \( d-l-2 \)-th derivative of the \( b \)-coordinate these homogeneous polynomials are invariant under the \( H_D \)-action. Furthermore, it is easy to see that they are independent elements of the algebra of \( H_D \)-invariant polynomials on \( N_i^n \) because the \( n+2k \)-th derivative is the first on to contain \( x_{l+2k} \). By recording \( x_l \) in addition, we have \( \frac{1}{2}(d-l-n+2) \) independent homogeneous polynomials. This defines a map

\[
N_i^n \to \mathbb{C}^{\frac{1}{2}(d-l-n+2)}
\]
invariant by the $H_D$-action. Factoring through the $\mathbb{C}^*$-action we obtain the desired algebraic map

$$N^n_l \rightarrow \mathbb{P}(1, 1, 2, 3, \ldots, \frac{d-l-n}{2}).$$

To show that it separates orbits we first consider $N^n_l \cap V_l$, i.e. $x_n = 0$. Here every element has a unique representative $b \over a$ mod $z^{d-l}$. Those are determined by the invariant polynomials induced from the derivatives $l+1$ till $d-l-2$. Instead on $N^n_l \cap V_n$ we can uniquely represent each element by a $u$-coordinate, see Lemma [7.3]. This $u$-coordinate can be recovered from the invariant polynomials. The $(n+2)$-th derivative decodes $x_{l+2}$, the $(n+4)$-th $x_{l+4}$ etc. and the $(d-l-2)$-th decodes $x_{d-n-2}$. So the map separates orbits.

As $x_l \neq 0$ we see that the image is contained in

$$\{(y_0 : \cdots : y_{\frac{1}{2}(d-l-n)}) \in \mathbb{P}(1, 1, \ldots, \frac{1}{2}(d-l-n)) \mid y_0 \neq 0\}.$$ 

This subset contains no singularity of the weighted projective space. Furthermore, by the explicit description of the homogeneous polynomials it is easy to verify that the image is closed under the torus action of $(\mathbb{C}^*)^{\frac{1}{2}(d-l-n)}$. Now lets consider the case of even $n$ and even $l$. Here we have to take a finer decomposition. Let $k > l$ a odd number then

$$k N^n_l := H_D \{ \begin{array}{l}
\frac{a}{b} = x_1 z + x_3 z^3 + \cdots + x_{d-n-2} z^{d-n-2} \\
b = x_0 z + x_2 z^2 + \cdots + x_{d-n-1} z^{d-n-1} \\
x_0 = \cdots = x_{n-1} = x_{n+1} = \cdots = x_{l-1} = 0, x_l \neq 0,
\end{array} \}.$$ 

Clearly

$$\bigcup_{odd \ k \geq l} k N^n_l = N^n_l.$$ 

So with these subsets we still satisfy property i). For fixed $k$ we proceed as before by computing the $n$-th, $\ldots$, $(d-k-2)$-th derivative of

$$\frac{x_n z^n + x_{l+1} z^{l+1} + \cdots + x_{d-1} z^{d-1}}{1 + \frac{x_{k+2}}{x_k} z^2 + \cdots + \frac{x_{d-2}}{x_k} z^{d-2-l}}.$$ 

They define a map

$$k N^n_l \rightarrow \mathbb{P}(1, 1, 2, 3, \ldots, \frac{1}{2}(d-k-n)).$$ 

invariant by the action of $G_D$. For $x_n \neq 0$ we can recover the $u$-coordinate of $k N^n_l \cap V_n$ as above. If $x_n = 0$ the $u$-coordinate of the lower stratum is now given by $a \over b$ mod $z^{d-l}$. We recover $a_{k+2}$ from the $l+2$-th derivative, $a_{k+4}$ from the $l+4$-derivative till $a_{d-l-2}$ from the $d-k-2$-th derivative. These uniquely defines the $u$-coordinate on $V_l$. With the same argument as above the image contains no singular points and is closed under the torus action.

When $n$ is odd we can obtain the same results by changing the role of $a$ and $b.$

**Theorem 7.5.** The quotient of $\text{Heck}_d$ by the action of $G_D$ is made up from toric subspaces of weighted projective spaces glued algebraically along torus orbits.
Figure 1. Schematic image of the compact moduli of Hecke parameters for $d = 5$: $S_0$ Beige, $S_1$ Blue, $S_2$ Red, $N_2^0$ Green.

Proof. Most of the work was already done in the previous lemma by introducing the sets $N_i \subset \Heck_d$ and the $G_D$-invariant, orbit-separating maps $N_i \to \mathbb{P}(1, 1, 2, 3, \ldots, m_i)$. These maps identify the quotients $N_i/G_D$ with toric subspaces of weighted projective spaces. We can build a model for the quotient $\Heck_d/G_D$ by gluing together this subsets $N_i/G_D$ along their intersection. We are left to show that this happens algebraically along torus orbits. It is enough to show that for all $i \in I$ and $0 \leq l \leq \frac{d-3}{2}$ the intersection $N_i \cap V_l$ is mapped onto a toric subspace under the two maps to weighted projective spaces and that the coordinate change is polynomial.

We will show this for $N_i^n \cap V_n$ with $n < l \leq \frac{d-3}{2}$, $n$ even and $l$ odd. For the other cases, it works in the same way. Denote by $u_n : V_n \to \mathbb{P}^{\frac{1}{2}(d-2n-1)}$, $f_l : N_i^n \to \mathbb{P}(1, 1, 2, \ldots, \frac{d-l-n}{2})$ the $G_D$-invariant maps defined in Lemma 7.3 and Proposition 7.4. Let $\alpha \in N_i^n \cap V_n$. We can choose a representative of the form

$$\left( x_l z^l + x_{l+2} z^{l+2} + \cdots + x_{d-n-2} z^{d-n-2} \right).$$

The image under $u_n$ is given by

$$(1 : 0 : \cdots : 0 : x_l : x_{l+2} : x_{d-n-2}) \in \mathbb{P}^{\frac{1}{2}(d-2n-1)}.$$

So $u_n(N_i^n \cap V_n)$ is clearly a union of torus orbits. On the other hand, we can explicitly compute the values of the invariant polynomials defining $f_l$.
and obtain

\[
\left( x_1 : 1 : x_{l+2} : x_{l+4}x_l + x_{l+2}^2 : \ldots : x_{d-n-2}(x_l)^{\frac{1}{2}(d-l-2-n)} + \ldots \right) \\
\in \mathbb{P}(1, 1, \ldots, \frac{d-l-n}{2}).
\]

This is again a union of torus orbits. It is easy to check that the gluing maps are polynomial \((x_l \neq 0)\). □

### 7.4. Global fibering over Prym Varieties

We will show that the singular fibers with locally irreducible spectral curve fiber over the Prym varieties with fibers given by compact moduli of Hecke parameters. As a first step we identify the Prym varieties of the different strata.

**Definition 7.6.** Let \( \mathcal{M}_\Lambda^\sigma \), such that \( \text{Fix}(\sigma) = Z(\lambda) \). Define

\[
\text{Eig}_{tw}: \mathcal{M}_\Lambda^\sigma = \bigcup_D S_D \to \text{Prym}_\Lambda(Y), \quad (E, \Phi) \mapsto \text{Eig}_D(E, \Phi)(-\frac{1}{2}D).
\]

**Remark 7.7.** This is well-defined, because \( D \) has only even coefficients. If we allow \( q_2 \) to have even zeroes, there is no canonical way to identify the twisted Prym varieties of the different strata. See Section 8 for more details.

We defined two kinds of \( u \)-coordinates: First in Proposition 5.8 when parametrising the strata and second in Lemma 7.3 when parametrising \( V_n \subset \text{Heck}_d \). They are equivalent in the following way.

**Proposition 7.8.** Let \( p \in Y \) and \( \Lambda = d \cdot p \). Let \( 0 \leq n \leq \frac{d-3}{2} \) and \((a, b) \in V_n \subset \text{Heck}_d \). Let \( L \in \text{Prym}_\Lambda(Y) \), choose a frame \( s \) of \( L \) in neighbourhood of \( p \) and let \( \alpha = as_+ + bs_- \in H^0(\Lambda, (E, \hat{\sigma})) \). Then

\[
(\hat{E}_L(\Lambda, \alpha), \hat{\Phi}_L(\Lambda, \alpha)) \in S_{2np},
\]

its image under \( \text{Eig}_{2np} \) is \( L(np) \) and the \( u \)-coordinate defined in Proposition 5.8 is given by

\[
u_n(\alpha) \in \mathbb{C}_{\frac{1}{2}}^{(d-2n-1)} \subset \mathbb{P}_{\frac{1}{2}}^{(d-2n-1)},
\]

where \( \nu_n \) was defined in Lemma 7.3.

**Proof.** The Higgs field of the Hecke transformation at \( p \) is given by

\[
\hat{\Phi}_L = \hat{\Phi}_L(\Lambda, \alpha) = \psi_{01}^{-1} \Phi_L \psi_{01} = \begin{pmatrix} \frac{a}{b} & \frac{1}{2}b^2 - a^2 \\ \frac{3}{2} \sqrt{a^2 + b^2} & -\frac{a}{b} \end{pmatrix} \text{d}z
\]

respective the induced frame on \( \hat{E}_L(\Lambda, \alpha) \). A section of the eigen bundle \( L \) at \( p \) is given by

\[
s = \begin{pmatrix} b + a \\ z^d b^{-1} \end{pmatrix}.
\]

Let \( s = z^{\text{ord}_{\hat{\Phi}_L}(a+b)} \hat{s} \), then \( \hat{s} \) defines a non-vanishing section of the eigen line bundle \( \hat{L} = \ker(\hat{\Phi}_L - \lambda \text{Id}_{\hat{E}_L}) \). In particular, \( \hat{L} = L(np) \) and

\[
(\hat{E}_L(\Lambda, \alpha), \hat{\Phi}_L(\Lambda, \alpha)) \in S_{2np}.
\]
To compute the \( u \)-coordinate at \( p \), let us first assume that \( n \) is even, i.e. \( \text{ord}_p(a + b) = \text{ord}_p(b) \). Then

\[
\tilde{s} = \left( \frac{\tilde{b} + \tilde{a}}{z^{d-2n} \tilde{b}^{n-1}} \right)
\]

with \( \tilde{a} \) an odd polynomial of degree \( d - n - 2 \) and \( \tilde{b} \) a non-vanishing even polynomial of degree \( d - n - 1 \). The sections \( s_\pm \) are given by

\[
s_+ = \tilde{s} + \sigma^* \tilde{s} = \left( \frac{\tilde{b}}{0} \right), \quad s_- = \tilde{s} - \sigma^* \tilde{s} = \left( \frac{\tilde{a}}{z^{d-n} \tilde{b}^{n-1}} \right).
\]

Hence, the \( u \)-coordinate as defined in Proposition 5.8 is given by \( u = \frac{\tilde{a}}{\tilde{b}} \mod z^{d-2n} \). Respective the basis \( z, z^3, \ldots, z^{d-2n-2} \), this gives exactly the coordinates \( u_n \) defined in Lemma 7.3. When \( n \) is odd, a similar consideration gives the result. \( \square \)

Let \( D \in \text{Div}^+(X) \) be a \( \sigma \)-Higgs divisor associated to \( M^\sigma_{\lambda} \). Define

\[
\text{Heck}_D := \prod_{p \in \text{supp}(D)} \text{Heck}_{D_p}.
\]

**Proposition 7.9.** Consider \( M^\sigma_{\lambda} \), such that \( Z(\lambda) = \text{Fix}(\sigma) \). Then the map \( \text{Eig}_{\text{tw}}: M^\sigma_{\lambda} \to \text{Prym}(Y) \) is a topological fiber bundle with fibers given by the compact moduli of Hecke parameters \( \text{Heck}_D/G_D \).

**Proof.** By definition, it is clear that \( \text{Eig}_{\text{tw}} \) is continuous on each stratum and from Proposition 7.8 it is continuous under the degeneration from one stratum to another. Let \( U \) a union of neighbourhoods of \( Z(\lambda) \) and \( V \subset \text{Prym}_{\Lambda}(Y) \) open such that there exists a local frame of the universal bundle

\[
s : U \times V \to \mathcal{L}
\]

(cf. proof of Proposition 5.9). By applying Hecke transformations we obtain a commuting diagram

\[
\begin{array}{ccc}
\text{Heck}_D/G_D \times V & \xrightarrow{\text{Heck}} & \text{Eig}_{\text{tw}}^{-1}(V) \\
\text{pr}_2 \downarrow & & \downarrow \\
V & \xrightarrow{\text{Eig}_{\text{tw}}} & \text{Eig}_{\text{tw}}^{-1}(V)
\end{array}
\]

The identification of \( u \)-coordinates in the previous proposition shows that this map is bijective. \( \square \)

Following paragraph 2.49 in [KK83], an analytic subset of a complex space is called reducible, if it is the union of proper analytic subsets. Let \( X \) a complex space and \( \text{Sing}(X) \) the singular set, then a irreducible component \( Z \subset X \) is defined as the closure of a connected component \( X \setminus \text{Sing}(X) \). An irreducible component is an irreducible analytic subset.

**Corollary 7.10.** The space \( M^\sigma_{\lambda} \) with \( Z(\lambda) = \text{Fix}(\sigma) \) is an irreducible complex space, in particular connected.
Proof. The space of Hecke parameters of the highest stratum is connected. The Prym variety \( \text{Prym}_\Lambda(Y) \) is connected as long as there exists a branch point of \( p \) (see \[HP12\]). Furthermore the closure of the highest stratum is the whole singular Hitchin fiber by Theorem 7.5 and the previous proposition. In particular, the set of non-singular points is connected and hence \( \mathcal{M}_\Lambda^\sigma \) is irreducible. □

Remark 7.11. We want to point out that the connectedness was already shown in \[GO13\].

Theorem 7.12. The map \( \text{Eig}_{tw} : \mathcal{M}_\Lambda^\sigma \to \text{Prym}_\Lambda(Y) \) is holomorphic. In particular, the compact moduli of Hecke parameters \( \text{Heck}_\Lambda/G_\Lambda \) is a complex space.

Proof. We will use a version of the Riemann extension theorem for complex spaces to prove the theorem. To do so, we have to reduce the problem to codimension 2. Let’s again assume that there is only one higher order zero of \( \lambda \). We saw in Proposition 7.4 that an open neighbourhood \( N_1^0/G_\Lambda \) of the first stratum \( V_1 \) in the zeroth stratum \( V_0 \) can be identified with an open non-singular toric subspace of a weighted projective space \( \mathbb{P}(1,1,2,\ldots,n) \). Gluing this open subset to \( V_0 \), we obtain a complex manifold \( V = V_0 \cup V_1 \) of Hecke parameters of the zeroth and first stratum. We can build a holomorphic fibre bundle \( F_{01} \)

\[
V \to F_{01} \to \text{Prym}_\Lambda(Y)
\]

by choosing local frames of \( L \in \text{Prym}_\Lambda(Y) \) around \( Z(\lambda) \). Through Hecke transformations, we obtain an analytic map to \( S_0 \cup S_1 \), such that the following diagram commutes

\[
\begin{array}{ccc}
F_{01} & \xrightarrow{\text{Heck}} & S_0 \cup S_1 \\
\downarrow & & \downarrow \\
\text{Prym}_\Lambda(Y) & \xrightarrow{\text{Eig}_{tw}} & S_0 \cup S_1
\end{array}
\]

Hence \( \text{Eig}_{tw} \) is holomorphic on \( S_0 \cup S_1 \). To extend it we use the Riemann extension theorem (Thm. I.12.13 in \[Gra+94\]) for reduced locally pure dimensional complex spaces. By Theorem 7.10, \( \mathcal{M}_\Lambda^\sigma \) is an irreducible complex space. Furthermore, the Hitchin map is flat and therefore its fibres are locally pure dimensional (see Thm. II.2.13 in \[Gra+94\]). Let \( p \in \mathcal{M}_\Lambda^\sigma \setminus (S_0 \cup S_1) \). For a small neighbourhood \( U \subset \mathcal{M}_\Lambda^\sigma \) of \( p \) we can choose coordinates functions

\[
f : V \subset \text{Prym}_\Lambda(Y) \to \mathbb{C}^{\dim \text{Prym}_\Lambda},
\]

such that \( \text{Eig}_{tw}(U \cap (S_0 \cup S_1)) \subset V \). Then \( f \circ \text{Eig}_{tw} \) define holomorphic functions on \( U \) away from a analytic subset of codimension 2. By the extensions theorem they extend to \( U \) meromorphically. We already showed that \( \text{Eig}_{tw} \) as defined in Definition 7.6 is a continuous extension. Hence \( \text{Eig}_{tw} \) is holomorphic. □

In conclusion, we obtain the following description of singular \( SL(2,\mathbb{C}) \)-Hitchin fibers with locally irreducible spectral curve.
Theorem 7.13. Let $q_2$ be a quadratic differential with only zeroes of odd order. Then $\text{Hit}^{-1}(q_2)$ is holomorphic fiber bundle

$$\text{Heck}_A/G_A \to \text{Hit}^{-1}(q_2) \to \text{Prym}_A(\tilde{\Sigma}).$$

In particular, the singular Hitchin fiber is an irreducible complex space.

7.5. The first degenerations. Zeroes of order 3

Let $q_2$ be a quadratic differential with one zero of order 3, such that all other zeroes are simple. In this case, $G_A \cong \mathbb{C}^*$ is reductive and it is easy to see that the compact moduli of Hecke parameters is given by

$$\text{Heck}_3/\mathbb{C}^* \cong \mathbb{P}^1.$$

So as a direct consequence of Theorem 7.12, we obtain:

Corollary 7.14. Let $q_2$ be a quadratic differential with $k$ zeroes of order 3, such that all other zeroes are simple. Then the singular Hitchin fiber is a holomorphic fiber bundle

$$\left(\mathbb{P}^1\right)^k \to \text{Hit}^{-1}(q_2) \xrightarrow{\text{Eig}_\text{loc}} \text{Prym}_A(\tilde{\Sigma}).$$

Zeroes of order 5

Let us now consider quadratic differentials with zeroes of order 5.

Proposition 7.15. The normalisation of the compact moduli of Hecke parameters $\text{Heck}_5/G_D$ is given by $\mathbb{P}(1,1,2)$. In particular, it is a toric complex space.

Proof. In Proposition 7.4 we defined an isomorphism from the

$$N^0_1 \to \mathbb{P}(1,1,2) \setminus \{(y_0 : 0 : y_2)\}.$$ 

Its inverse is given by

$$\mathbb{P}(1,1,2) \setminus \{(y_0 : 0 : y_2)\} \to \text{Heck}_d/G_A, \quad (y_0 : y_1 : y_2) \mapsto \left(\frac{y_1^2 z}{y_0 y_1 + y_2 z^2}\right).$$

This map naturally extends to $(0 : 0 : 1)$ by mapping it onto $V_2$, which consists of a point. If $y_0 \neq 0$, the image lies in $V_0$ and

$$(u_0 \circ \psi)(y_0 : y_1 : y_2) = \frac{y_1}{y_0} z + \frac{y_2}{y_0^2} z^2.$$ 

Therefore, it extends holomorphically to $y_0 \neq 0, y_1 = 0$. The map is biholomorphic away from the point in the lowest stratum, which is a fixed point of the full-dimensional torus action on $\mathbb{P}(1,1,2)$. Hence, we can pushforward the torus action to the moduli of Hecke parameters. \hfill \Box

Corollary 7.16. Let $q_2 \in H^0(K^2)$ be a quadratic differential with $k$ zeroes of order 3, $l$ zeroes of order 5, such that all other zeroes are simple. Then, up the normalization, $\text{Hit}^{-1}(q_2)$ is a given by a holomorphic fiber bundle

$$\left(\mathbb{P}^1\right)^k \times \left(\mathbb{P}(1,1,2)\right)^l \to \text{Hit}^{-1}(q_2) \xrightarrow{\text{Eig}_\text{loc}} \text{Prym}_A(\tilde{\Sigma}).$$

In particular, $\text{Hit}^{-1}(q_2)$ is a toric variety.
8. Singular fibers with irreducible spectral curve

When we allow zeroes of even order, the singular Hitchin fibers do not fiber over Prym varieties. However, it is still true that we can describe the degeneration to lower strata using higher Hecke transformations. In Section 5, it was more convenient to parametrize the extra data at even zeroes with extension data. We will reinterpret these extra data as Hecke parameters now.

Fix \( M^\sigma_\lambda \), such that \( \{ y, \sigma(y) \} = Z(\lambda) \setminus \operatorname{Fix}(\sigma) \) and all other zeroes of \( \lambda \) have order 1. Let \( D \) be an associated \( \sigma \)-Higgs divisor. Let \( L \in \operatorname{Prym}_{\lambda-D} \) and \( (E, \Phi) \in M^\sigma_\lambda \) obtained from \( (E_L, \Phi_L) \) by applying the unique Hecke transformation at all simple zeroes. Choose frames \( s_1 \in H^0(U, L) \), \( s_2 \in H^0(U, \sigma^* L) \) for a neighbourhood \( U \) of \( y \) and let

\[
s_+ := s_1 \oplus s_2, \quad s_- := s_1 \oplus -s_2.
\]

the induced frame of \( E|_U = (L \oplus \sigma^* L)|_U \).

**Proposition 8.1.** Let \( l = (\Lambda - D)_y \) and \( \alpha = as_+ + bs_- \in H^0(ly, E)^* \), such that \( a(0) \neq \pm b(0) \). Then

\[
\left( \hat{E}(y+\sigma y, \alpha+\sigma^* \alpha), \hat{\Phi}(y+\sigma y, \alpha+\sigma^* \alpha) \right) \in S_D \subset M^\sigma_\lambda
\]

and the extension datum at \( y \) introduced in Proposition 5.12 is given by

\[
\left[ \frac{b+a}{b-a} s_1^2 \, dz \right] \in H^0(ly, L^2 K).
\]

**Proof.** This is a local computation from the description of the Higgs field given in [10]. \( \square \)

From this description, we see that for the Hecke parameters at even zeroes of the quadratic differential, there are two different ways to degenerate to lower strata:

i) By degenerating to lower strata in the moduli spaces of Hecke parameters, i.e. allowing \( \alpha \) to vanish. Here the eigenline bundle \( L \) is twisted by a divisor \( D + \sigma^* D \) invariant by \( \sigma \).

ii) By imposing

\[
a \equiv b \mod z^l \quad \text{or} \quad a \equiv -b \mod z^l
\]

for some \( l \leq \frac{1}{2} \text{ord}_{\alpha(y)}(q_2) \), while \( a(0), b(0) \neq 0 \). In this case, the eigenline bundle \( L \) is twisted by the divisor \( ly \) or \( l\sigma(y) \), respectively.

Consonant with the previous section, we can find a compactification of the Hecke parameters of the highest stratum by allowing Hecke parameters in \( \alpha \in H^0(\Lambda y, E) \). Denote

\[
\text{Heck}_{\Lambda y} := H^0(\Lambda y \cdot y, E)/\sim,
\]

where \( \sim \) denotes the analogue of relation ii) of Lemma 7.2. Along the lines of Section 7.3, we can study the quotient of \( \text{Heck}_{\Lambda y} \) by the non-reductive group action of \( H^0(\Lambda y, O_Y^\vee) \) and obtain a topological model by gluing toric subsets of weighted projective spaces. Following Section 7.4 one proves:
Theorem 8.2. Let \( q_2 \in H^0(K^2) \) be a quadratic differential with one zero \( x \) of order 2d, such that all other zeroes are simple. Let \( \tilde{\pi}^{-1}(x) = \{ y, \sigma y \} \) and \( L \in \text{Prym}_\Lambda(\Sigma) \). Let \( (E, \Phi) \in \tilde{\pi}^*\text{Hit}^{-1}(q_2) \) denote the Higgs bundles obtained by applying the unique Hecke transformation to \( (E_L, \Phi_L) \) at all simple zeroes. Applying Hecke transformations to \( (E, \Phi) \) at \( x \) defines a continuous injective map

\[
T_L : \text{Heck}_d/H^0(dy, O_y^*) \to \tilde{\pi}^*\text{Hit}^{-1}(q_2),
\]

whose image is the closure of \( \text{Eig}_0^{-1}(L) \subset S_0 \) and is given by

\[
\bigcup_{l_1 + l_2 \leq d} \text{Eig}_{D(l_1, l_2)}^{-1}(L(l_1y + l_2\sigma y))
\]

with \( D(l_1, l_2) = (l_1 + l_2)y + (l_1 + l_2)\sigma y \in \text{Div}^+(Y) \).

Define \( F_{q_2} \) as the topological fiber bundle over \( \text{Prym}_\Lambda \) with fibers given by the moduli of Hecke parameters (cf. Proposition 5.9). We can define a continuous map \( T : F_{q_2} \to \text{Hit}^{-1}(q_2) \) by applying \( T_L \) fiberwise. However, as we will see below, this map is not anymore injective. It has the property that it makes the following diagram commute

\[
\begin{array}{ccc}
T^{-1}(S_0) & \xrightarrow{T} & S_0 \\
\downarrow & & \downarrow \\
\text{Prym}_\Lambda & \xrightarrow{\text{Eig}_0} & \text{Eig}_0
\end{array}
\]

But there is no way to extend the fibering to the whole singular fiber. This was already encountered in [GO13] and [Hit19]. To illustrate why these two properties fail, we describe the case of zeroes of order 2.

Example 8.3 (Zeros of order 2). Let \( x \in Z(q_2) \) be a zero of order two and \( \{ y, \sigma y \} = \tilde{\pi}^{-1}(x) \). The moduli of Hecke parameters are easy to understand

\[
(H^0(y, E_L) \setminus \{ 0 \}) / H^0(y, O^*_y) = (\mathbb{C}^2 \setminus \{ 0 \}) / \mathbb{C}^* = \mathbb{P}^1.
\]

So in this case, the stratification of Hecke parameters by vanishing order is trivial. However, as we seen in Theorem 6.2 the stratification of \( \text{Hit}^{-1}(q_2) \) has two strata. One, where the Higgs field is non-vanishing and one, where it is diagonalizable and vanishes at \( x \) of order 1.

Let \( (L, \alpha) \in F_{q_2} \) and \( \alpha = as_+ + bs_- \) respective some choice of frame of \( L \) around \( y, \sigma y \). Then the Higgs field of \( T(L, \alpha) \) is given by (10). Hence,

\[
T(L, \alpha) \in S_0 \iff a_0 \neq \pm b_0.
\]

Furthermore, it is easy to check that for \( a_0 = b_0 \) the eigen line bundle has developed a zero at \( y \), whereas for \( a_0 = -b_0 \) it develops a zero at \( \sigma y \). We conclude that for given \( L \in \text{Prym}_\Lambda(\Sigma) \)

\[
T(L, s_+ + s_-) = T(L(y - \sigma y)), s_+ - s_-).
\]

In particular, \( T \) is not injective and the fibering can not be extended.
Remark 8.4. With our methods, we can not show that the compact moduli of Hecke parameters at even zeroes is a complex space. If this is the case, the map $T$ defined above is defining a one-sheeted analytic covering in the language of [Gra+94]. The analogue of a birational morphism in the analytic category.

Corollary 8.5. Let $q_2$ be quadratic differential with at least one zero of odd order, then the $\text{Hit}^{-1}(q_2)$ is irreducible.

Proof. Theorems 7.9 and 8.2 show that $S_0 = \text{Hit}^{-1}(q_2)$. Furthermore, $S_0$ is irreducible as a $\mathbb{C}^* \times \mathbb{C}^n$-bundle over a Prym variety, which is connected as long as there exists a branch point of $\tilde{\pi} : \Sigma \to X$ (see [HP12]). In particular the smooth points of $\text{Hit}^{-1}(q_2)$ are connected and therefore it is irreducible. □

We already encountered above, that the case of quadratic differentials with only zeroes of even order is very special. As we saw in Proposition 3.12 the pullback of Higgs bundles to the normalized spectral curve is not injective. A second issue is that $\text{Prym}_0(\tilde{\Sigma})$ is not connected. A detailed discussion of this issue can be found in [Mum71] (and [HP12]). We give a short recall of Mumford’s approach here.

Lemma 8.6. Let $L \in \text{Prym}_0(\tilde{\Sigma})$, then there exists a line bundle $M$ on $\tilde{\Sigma}$ of degree 1 or 0, such that $L \sim M \otimes \sigma^*M^{-1}$.

The two connected components can be seen in two different ways: First they can be defined as $P_i = \{ M \otimes \sigma^*M^{-1} | M \in \text{Pic}^i(\tilde{\Sigma}) \}$ for $i = 0, 1$. To see that these define different components Mumford showed that the function $\text{Prym}_0 \to \mathbb{Z}_2$, $L \mapsto \dim H^0(L \otimes \tilde{\pi}^*K^{\frac{i}{2}}) \mod 2$ is locally constant and is equal to $i \mod 2$ restricted to $P_i$.

Secondly, one can see the different components in the following way: Let $L = M \otimes \sigma^*M^{-1}$ and $O(D) = M$. Let $D = D_+ - D_-$, where $D_\pm$ are effective divisors then $\deg(D_+) = \deg(D_-) + i$. We can also decompose $D - \sigma^*D = (D - \sigma^*D)_+ + (D - \sigma^*D)_-$ into effective divisors. Then $\deg((D - \sigma^*D)_+) = 2 \deg(D) - i \equiv i \mod 2$. This does only depend on the parity of $\deg(M)$ again. So if we represent a element $L \in \text{Prym}_0(\Sigma)$ with a divisor $C$ with $C + \sigma^*C = 0$, then $\deg(C_+) \mod 2$ tells us in which connected component it lies.

Theorem 8.7. The singular Hitchin fibers $\text{Hit}^{-1}(q_2)$ for a quadratic differential $q_2$, such that all zeroes have even order, have two irreducible components and are connected.

Proof. We showed in Proposition 3.12 that such singular Hitchin fiber is a branched two-to-one covering over the space $\mathcal{M}^\sigma(\Sigma, \pi^*K, \lambda)$. In particular, we can conclude connectedness of the singular fiber from the connectedness of $\mathcal{M}^\sigma(\Sigma, \pi^*K, \lambda)$. By the criterion above, we change the stratum, when we twist by a divisor of the form $p - \sigma(p)$ for $p \in \Sigma$. As we saw in Example
we can degenerate from a \( \text{Eig}_0^{-1}(L) \subset S_0 \) and \( \text{Eig}_0^{-1}(L(p - \sigma(p))) \subset S_0 \) to one and the same point in the lower stratum. This is true in general: By using degenerations of type ii), we can always connect two points over different components of the Prym variety by going to lower strata.

However, the fiber is not irreducible because \( S_0 \) is not. It decomposes into two connected components given by restricting the \( \mathbb{C}^* \times \mathbb{C}^n \)-bundle to the different components of \( \text{Prym}_\lambda \). In particular, the closures of the two connected components of \( S_0 \) are proper analytic subsets of \( \text{Hit}^{-1}(q_2) \), such that their union is the entire space. \( \square \)

9. Real points in singular Hitchin fibers

In this section, we are going to study real points in singular Hitchin fibers with irreducible and reduced spectral curve. We will show that they correspond to two-torsion points of the Prym variety and a discrete choice of Hecke parameters at the even zeroes of the quadratic differential.

\[ L \in \text{Prym}_0(\tilde{\Sigma}) \] is a two-torsion point, if \( L^2 \cong O_X \). Under the Prym condition, this is equivalent to \( \sigma^*L \cong L \). We call \( L \in \text{Prym}_D(\tilde{\Sigma}) \) \( \sigma \)-symmetric, if \( \sigma^*L \cong L \). Choosing a \( \sigma \)-symmetric base point for the simply transitive action of \( \text{Prym}_0(\tilde{\Sigma}) \) on \( \text{Prym}_D(\tilde{\Sigma}) \) the two-torsion points are bijectively mapped on the \( \sigma \)-symmetric points. Recall that our definition of \( \sigma \)-invariant holomorphic line bundle was stronger (cf. Definition 3.1).

**Theorem 9.1.** Let \( q_2 \in H^0(K_X^2) \) a quadratic differential, such that all zeroes have odd order and \( D \in \text{Div}(X) \) a associated Higgs divisor. Then the \( \text{SL}(2, \mathbb{R}) \)-Higgs bundles in \( S_D \subset \text{Hit}^{-1}(q_2) \) correspond to the \( \sigma \)-symmetric points of \( \text{Prym}_{\Lambda - \pi^* D}(\tilde{\Sigma}) \).

**Proof.** Let \( N := \Lambda - \tilde{\pi}^*D \) and \( L \in \text{Prym}_N(\tilde{\Sigma}) \) such that there exists an isomorphism \( \tilde{\phi} : \sigma^*L \to L \). \( \phi \) is unique up to \( \pm \text{id} \) and restricts to \( \pm 1 \) at each \( p \in \text{Fix}(\sigma) = \tilde{\pi}^{-1}(Z(q_2)) \). Choose a frame \( s \in H^0(U,L) \) at \( p \in \text{Fix}(\sigma) \), such that \( s = \pm \phi(\sigma^*s) \) for \( \phi_p = \pm 1 \) respectively. Such frame is uniquely defined up to multiplying by an \( \sigma \)-invariant holomorphic function and therefore defines a unique \( u \)-coordinate (cf. Proposition 5.8). The induced frame \( s_+, s_- \) is given by the global splitting

\[
(E_L, \Phi_L) = \left( L \oplus L, \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \right).
\]

We decompose \( N = N_+ + N_- \) such that \( N_\pm \) is supported at \( p \in \text{Fix}(\sigma) \) such that \( \phi_p = \pm 1 \) respectively. The Hecke transformation of \( (E_L, \Phi_L) \) at \( N \) in direction \( u = 0 \) is given by

\[
(\hat{E}_L, \hat{\Phi}_L) = \left( L(N_-) \oplus L(N_+), \begin{pmatrix} 0 & \frac{\eta_+}{\eta_-} \\ \frac{\eta_-}{\eta_+} & 0 \end{pmatrix} \right)
\]

with \( \eta_\pm \in H^0(O(N_\pm)) \) canonical. The induced lift of \( \sigma \) to \( L(N_\pm) \) is the identity over all branch points. Hence, this descends to a \( \text{SL}(2, \mathbb{R}) \)-Higgs bundle on \( X \). If we choose \( -\phi \) in the beginning the role of \( N_\pm \) are interchanged and we obtain the same \( \text{SL}(2, \mathbb{R}) \)-Higgs bundle (with interchanged order of
the splitting line bundles).

For the converse, consider a \( \text{SL}(2, \mathbb{R}) \)-Higgs bundle

\[
(E, \Phi) = \left( L \oplus L^{-1}, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right) \in S_D.
\]

There are divisors \( N_{\pm} \in \text{Div}(\tilde{\Sigma}) \), such that

\[
\tilde{\pi}^* \alpha = \frac{\lambda \eta_-}{\eta_+}, \quad \tilde{\pi}^* \beta = \frac{\lambda \eta_+}{\eta_-}
\]

for \( \eta_{\pm} \in H^0(O_{\tilde{\Sigma}}(N_{\pm})) \) canonical. The eigenline bundles are defined by

\[
(\tilde{\pi}^* L)(-N_{\pm}) \xrightarrow{(\eta_{\pm}, \eta_{\pm})} \tilde{\pi}^* L \oplus \tilde{\pi}^* L^{-1}.
\]

and correspond to a \( \sigma \)-symmetric point of \( \text{Prym}_N(\tilde{\Sigma}) \). Furthermore, the induced isomorphism

\[
\phi : \sigma^*(\tilde{\pi}^* L)(-N_{\pm}) \to (\tilde{\pi}^* L)(-N_{\pm})
\]

is \(-1\) at \( \text{supp} N_- \). So we recover \((E, \Phi)\) with the construction in the first part of the proof.

**Example 9.2.** For the highest stratum \( S_0 \) the pullback \( \pi^*(K^{-\frac{1}{2}}) \in \text{Prym}_\Lambda \) is \( \sigma \)-symmetric. The corresponding \( \text{SL}(2, \mathbb{R}) \)-Higgs bundle is the image of the Hitchin section

\[
\begin{pmatrix} K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}, & \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix} \end{pmatrix}.
\]

More generally, if \( \text{deg}(D) \equiv 0 \mod 2 \), there exist line bundles \( M \) on \( X \) such that \( M^2 \cong O_X(D) \). Then \( \tilde{\pi}^*(K^{-\frac{1}{2}} M) \in \text{Prym}_{\Lambda - \tilde{\pi}^* D} \) is \( \sigma \)-symmetric. The corresponding \( \text{SL}(2, \mathbb{R}) \)-Higgs bundle is of the form

\[
\begin{pmatrix} K^{-\frac{1}{2}} M \oplus K^{\frac{1}{2}} M^{-1}, & \begin{pmatrix} 0 & \eta \\ q_2 & 0 \end{pmatrix} \end{pmatrix}
\]

with \( \eta \in O_X(D) \) canonical. These are the only \( \text{SL}(2, \mathbb{R}) \)-Higgs coming from \( \sigma \)-invariant line bundles.

**Corollary 9.3.** Let \( q_2 \in H^0(K^2) \) be a quadratic differential, such that all zeroes have odd order. Then \( \text{Hit}^{-1}(q_2) \) contains

\[
2^{2g-2} \prod_{p \in Z(q_2)} (\text{ord}_p(q_2) + 1)
\]

\( \text{SL}(2, \mathbb{R}) \)-Higgs bundles.

**Proof.** By the previous theorem, every stratum contains \( 2^{2g-2-n} \) \( \text{SL}(2, \mathbb{R}) \)-points, where \( n \) is the number of zeroes. At a zero \( p \in Z(q_2) \), we have \( \frac{\text{ord}_p(q_2)}{2} + 1 \) possible values for \( D \) and hence there are

\[
\prod_{p \in Z(q_2)} \frac{1}{2}(\text{ord}_p(q_2) + 1)
\]

different strata. \( \square \)
The regular fibers contain $2^{6g-6}$ real points. If we have one triple zero and all other zeroes are simple, we have $2^{6g-8}$ of them. If we have $g - 1$ triple and $g - 1$ simple zeroes, the number is $2^{5g-5}$. For one zero of order $4g - 3$ and one simple zero, we have $(2g - 1)2^{2g}$ real points. In general, the moduli space of $\text{SL}(2, \mathbb{R})$-Higgs bundles branches at the singular fibers.

For quadratic differentials with zeroes of even order, there are two Hecke parameters in each stratum leading to $\text{SL}(2, \mathbb{R})$-Higgs bundles. We use the description of the extra data at even zeroes given in Proposition S.1

**Theorem 9.5.** Let $q_2 \in H^0(K^2)$ with at least one zero of odd order and $D \in \text{Div}^+(X)$ an associated Higgs divisor. The $\text{SL}(2, \mathbb{R})$-Higgs bundles in the stratum $S_D \subset \text{Hit}^{-1}(q_2)$ correspond to the $\sigma$-symmetric points of $\text{Prym}_{A-\pi^*D}(\Sigma)$ together with a choice of one of two possible Hecke parameters at every even zero $p$ of $q_2$, where $\frac{1}{2}\text{ord}_p(q_2) \neq D_p$. In particular, each stratum $S_N$ contains

$$2^{2g-2+n-n_0}$$

$\text{SL}(2, \mathbb{R})$-Higgs bundles, where $n$ is the number of zeroes of $q_2$ and $n_0$ is the number of zeroes $p$ of $q_2$, such that $\frac{1}{2}\text{ord}_p(q_2) = D_p$.

**Proof.** Let $L \in \text{Prym}_{A-\pi^*D}(\Sigma)$, such that there exists an isomorphism $\phi : \sigma^*L \rightarrow L$. Fix a choice of $\pm$ at every even zero such that $\frac{1}{2}\text{ord}_p(q_2) \neq \pi^*D_p$. Let $Z^c \subset Z(q_2)$ be the set of zeroes of even order and

$$N^{\text{even}} = (\Lambda - \pi^*D)|_{\tilde{\pi}^{-1}Z^c}.$$  

Let $N^{\text{even}} = N^{\text{even}}_+ + N^{\text{even}}_-$. Let $N^{\text{even}}_\pm$ be supported at the even zeroes assigned a $\pm$ respectively. As seen above, the isomorphism $\phi$ defines a unique Hecke transformation at $\tilde{\pi}^{-1}(p)$ for every odd zero $p$ of $q_2$. Performing these Hecke transformations we obtain a $\sigma$-invariant Higgs bundle

$$(E, \Phi) = \left( L_1 \oplus L_2, \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \right)$$

on $\Sigma$ with $L_1 \otimes L_2 \cong O_{\Sigma}(-N^{\text{even}})$, which is locally diagonalizable over all even zeroes of $q_2$. If $N^{\text{even}}_\pm = 0$, this descends to a $\text{SL}(2, \mathbb{R})$-Higgs bundle on $X$ and we are done. Let $N^{\text{even}}_\pm \neq 0$. Choose a frame $s \in H^0(U \cup \sigma(U), L)$ for a neighborhood $U$ of $\tilde{\rho}$, such that $\phi(\sigma^*s) = s$. Depending on the fixed choice of $\pm$, we define the Hecke parameter $\alpha = [s_\pm]$ respective the induced frame $s_+, s_-$. By Proposition S.1, this defines a $\sigma$-invariant Higgs bundle

$$\left( \tilde{E}(\tilde{\phi} + \sigma \tilde{\rho} \alpha + \sigma^* \alpha), \tilde{\Phi}(\tilde{\phi} + \sigma \tilde{\rho} \alpha + \sigma^* \alpha) \right).$$

Performing Hecke transformations like this over all even zeroes $p \in X$, such that $\frac{1}{2}\text{ord}_p(q_2) \neq \pi^*D_p$, we obtain

$$\left( L_1(N^{\text{even}}_-) \oplus L_2(N^{\text{even}}_+), \begin{pmatrix} 0 & \alpha \eta_- \\ \beta \eta_+ & 0 \end{pmatrix} \right).$$

Finally, this $\sigma$-invariant Higgs bundle descends to a $\text{SL}(2, \mathbb{R})$-Higgs bundle in the desired stratum.

For the converse, let

$$(E, \Phi) = \left( L \oplus L^{-1}, \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \right) \in S_D.$$
Then
\[ \tilde{\pi}^*(E, \Phi) = \left( \tilde{\pi}^*L \oplus \tilde{\pi}^*L^{-1}, \left( \begin{array}{cc} 0 & \lambda \eta_- \\ \eta_- & 0 \end{array} \right) \right) \]
for divisors \( N_{\pm} \in \text{Div}^+(\Sigma) \) with canonical sections \( \eta_{\pm} \). The eigenline bundles are defined by \( \sigma \) and define a \( \sigma \)-symmetric element of \( \text{Prym}_{\Lambda^{-1} \tilde{\pi}^*D} \). Moreover, this determines the decomposition \( N_{\text{even}} = N_{\text{even}}^+ + N_{\text{even}}^- \) and hence a choice of \( \pm \) for all \( p \in Z_e \), where \( N_{\text{even}}^p \neq 0 \), i.e. \( \frac{1}{2} \text{ord}_p(q_2) \neq D_p \). □

**Remark 9.6.** The choice of \( \pm \) in the previous theorem actually depends on choosing an isomorphism \( \phi : \sigma^*L \rightarrow L \). However this isomorphism is unique up to \( \pm \text{id}_\Sigma \). Choosing \( -\phi \) instead of \( \phi \) corresponds to switching all \( + \) to \( - \) and vice versa. For the \( SL(2,\mathbb{R}) \)-Higgs bundle, this corresponds to the gauge interchanging the splitting line bundles.

A general formula for the number of real points in a singular Hitchin fiber would be quite complicated. So let us finish by computing this number in some examples.

**Example 9.7.** Let \( q_2 \in H^0(K_X^2) \) be a quadratic differential with \( d < 2g - 2 \) double zeroes and \( 4g - 4 - 2d \) simple zeroes. Then the Hitchin fiber contains
\[ 2^{6g - 6 - 2d} \sum_{k=0}^{d} \binom{d}{k} 2^k \]
real points. Let \( q_2 \) be a quadratic differential with one zero of order \( 2d < 4g - 4 \) and \( 4g - 4 - 2d \) simple zeroes. Then this number is given by \( (4d - 3)2^{6g - 6 - 2d} \).

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