The Generalized Rank Invariant: Möbius invertibility, Discriminating Power, and Connection to Other Invariants

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Abstract

In addition to inherent computational challenges, the absence of a canonical method for quantifying ‘persistence’ in multi-parameter persistent homology remains a hurdle in its application.

One of the best known quantifications of persistence for multi-parameter persistent homology is the rank invariant, which has recently evolved into the generalized rank invariant (GRI) by naturally extending its domain. This extension enables us to quantify persistence across a broader range of regions in the indexing poset compared to the rank invariant. However, the size of the domain of the GRI is generally formidable, making it desirable to restrict its domain to a more manageable subset for computational purposes. The foremost questions regarding such a restriction of the domain are: (1) How to restrict, if possible, the domain of the GRI without any loss of information? (2) When can we more compactly encode the GRI as a ‘persistence diagram’? (3) What is the trade-off between computational efficiency and the discriminating power of the GRI as the amount of the restriction on the domain varies? (4) What proxies exist for persistence diagrams in the multi-parameter setting that can be derived from the GRI? To address the first three questions, we generalize and axiomatize the classic fundamental lemma of persistent homology via the notion of Möbius invertibility of the GRI which we propose. This extension also contextualizes known results regarding the (generalized) rank invariant within the classical theory of Möbius inversion. We conduct a comprehensive comparison between Möbius invertibility and other existing concepts related to the structural simplicity of persistence modules, such as Miller’s notion of tameness. During this investigation, we identify an example of a persistence module whose GRI is not Möbius invertible.

We address the fourth question through the notion of motivic invariants. We demonstrate that many invariants from the literature can be both derived from the GRI and recast as motivic invariants.

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1 Introduction

One-parameter persistent homology is a central concept in topological data analysis (TDA) with a wide range of applications [10, 17, 23, 28, 31, 35, 49, 57, 79]. Under fairly general assumptions, a one-parameter persistence module $M : \mathbb{R} \to \text{vec}$ is completely characterized by its so-called persistence diagram, which provides a canonical, compact and interpretable summary of $M$.

However, one-parameter persistent homology has well-known struggles when dealing with noise and outliers, motivating a surge in research in recent years on the more robust multi-parameter persistent homology, i.e. the setting in which we contend with multi-parameter persistence modules $M : \mathbb{R}^d \to \text{vec}$. Unfortunately, in contrast with
The case of one-parameter persistence modules (where persistence diagrams are full invariants thereof), there is no complete and simultaneously discrete invariant for multi-parameter persistence modules [24]. Accordingly, many discrete and necessarily incomplete invariants have been studied, e.g. [4, 5, 9, 14, 26, 50, 56, 59, 69, 74]. The central hurdle preventing widespread use of many of these invariants in practice is their computational complexity.

In this work, we focus on a particular class of invariants for multi-parameter persistence modules, or more generally for $P$-modules $M: P \rightarrow \text{vec}$, where $P$ is an arbitrary poset. For these invariants, we study the tension that exists between their computational complexity and their inherent discriminating power.

One of these invariants that is intimately tied to this work is the classical rank invariant (RI), which encodes the rank of all linear maps present in a given $P$-module; see [24, 76]. For one-parameter persistence modules, the RI contains equivalent information to the classical persistence diagram [1, 24], making it a natural notion to import into the multi-parameter setting. The RI allows us to quantify persistence over segments in the indexing poset (a segment is the set of points lying between a given pair of comparable points), and it is a lossless invariant when applied to so-called rectangle-decomposable modules [13]. As a downside, the RI only allows us to quantify persistence over segments, and in the multi-parameter setting there typically are many natural non-segment like subsets of the indexing poset $P$ over which we desire to quantify persistence.

Motivated by this, the RI has recently evolved into the generalized rank invariant (GRI) by extending the domain of the RI from the set $\text{Seg}(P)$ of segments of the indexing poset $P$ to the set $\text{Int}(P)$ of intervals of $P$ or to the even larger set $\text{Con}(P)$ of connected subposets of $P$ [52]. By recording the rank of the canonical limit-to-colimit map of the diagram over each element of $\text{Int}(P)$ (or of $\text{Con}(P)$), the GRI quantifies persistence across a broader range of regions in the indexing poset compared to the RI and, therefore, the GRI acquires more discriminating power than the RI.

However, the size of $\text{Int}(P)$, let alone that of the larger $\text{Con}(P)$, are generally formidable, causing a bottleneck for computing the GRI over the entire $\text{Int}(P)$ or $\text{Con}(P)$. For example, if $P$ is the 2-dimensional 10-by-10 grid (which serves as a much simplified setup for 2-parameter persistent homology), then $\text{Int}(P)$ comprises 1,497,925,315 elements [3, Theorem 31]. Therefore, restricting the domain of the GRI to a more manageable subset is desirable, if not downright necessary, for computational purposes.

Questions regarding the magnitude and consequent effect of this restriction can be grouped into two paradigms:

- **In the lossless paradigm**, we ask when we can, and if so how to, restrict the GRI’s domain without losing information and efficiently represent the information.

- **In the lossy paradigm**, we ask how much information a restricted GRI retains.

Within these two categories, the foremost questions regarding the restriction of the GRI’s domain are:

(☞) **Question 1.** How to restrict, if possible, the domain of the GRI without any loss of information?

(☞) **Question 2.** Under what conditions can we more compactly encode the GRI as a ‘persistence diagram,’ even when the indexing poset $P$ is not discrete?

\[1\text{I.e. when we ‘compress’}.\]
Question 3. In the lossy regime, what is the trade-off between computational efficiency and the discriminating power of the GRI as the amount of the restriction varies?

Finally, we move beyond considering the GRI itself and inquire as to what other invariants exist for quantifying persistence which are downstream from the GRI:

Question 4. What proxies exist for persistence diagrams in the multi-parameter setting that can be derived from the GRI?

Our work embarks on addressing these questions.

Contributions.

In order to tackle Question 1 and Question 2, we introduce the concept of Möbius invertibility of the GRI which is obtained by axiomatizing the classical fundamental lemma of persistent homology [42, Section VII]. For persistence modules over finite posets, Möbius invertibility of the GRI is always guaranteed, and its Möbius inverse is known as the generalized persistence diagram [52] (see also [5]) — a compact encoding of the GRI.\(^2\)

In addition, Möbius invertibility is useful to contextualize known theorems regarding the RI or GRI within the framework of classical Möbius inversion theory, which also sometimes enables us to simplify existing proofs or strengthen statements of existing theorems. Furthermore, Möbius invertibility is closely linked to several invariants of multi-parameter persistence [9, 14, 47, 59], as we elucidate in this work. We also observe that Möbius invertibility of the GRI of a P-module M is connected to the structural simplicity of M in a sense akin to Miller’s notion of tameness [69]. In relation to this, we conduct a thorough comparison between tameness and Möbius invertibility.

To tackle Question 3, we study the completeness properties of the GRI. Namely, fixing any collection of intervals \(I\) in the indexing poset \(P\), we characterize the collection of \(P\)-modules on which the GRI restricted to \(I\) is a complete invariant. Further, if the GRI restricted to \(I\) is not a complete invariant on an arbitrary fixed collection of modules, we suitably quantify this lack of completeness.

We tackle Question 4 by considering the concept of motivic invariants, which is based on the idea of ‘probing’ the indexing poset via a simpler poset called a motif. We demonstrate that many invariants from the literature can be derived from the GRI, and showcase how they fit into the framework of motivic invariants: see Table 1, Examples 2.32, 2.33, and Remark 2.36. One specific motivic invariant we focus on is the Zigzag-path-Indexed Barcode (ZIB, Definition 2.35), which consists of the restrictions of a given persistence module along all zigzag paths in its indexing poset.

A recent finding that the GRI of a \(\mathbb{Z}^2\)-module \(M\) is determined by the collection of all zigzag persistence modules that arise from restricting \(M\) to certain paths in \(\mathbb{Z}^2\) [40] indicates that identifying the minimal set of paths over which zigzag barcodes should be computed to determine the (restricted) GRI is important in order to be able to:

(i) exploit existing efficient zigzag persistence algorithms [19, 39, 70] for computing the (restricted) GRI of \(\mathbb{Z}^2\)-modules as well as for computing other related invariants [2, 4, 9, 14, 26].

\(^2\)In fact, the generalized persistence diagram is well-defined for persistence modules over a slightly more general class of indexing posets; see [52, Section 3].
(ii) examine how much is gained in terms of efficiency when computing the restricted GRI of a $\mathbb{Z}^2$-module as opposed to computing the entire GRI. Also, from a different perspective, by investigating the extent to which the ZIB can be recovered from the (restricted) GRI, we can ascertain the discriminating power of the (restricted) GRI.

Therefore, clarifying the connection between the GRI and the ZIB for $\mathbb{Z}^2$-modules also aids in addressing Question 3 for $\mathbb{Z}^2$-modules.

A priori, there is the possibility that the class of zigzag paths in $\mathbb{Z}^2$ composing the domain of the ZIB can be further restricted compared to the class considered in [40], while still being able to recover the entire GRI. One natural candidate for such a class is the collection of simple paths, i.e. paths with no repeated vertices (Definition 4.14). Motivated by this, we elucidate the precise relationship between the ZIB over simple paths and the GRI over $\text{Int}(\mathbb{Z}^2)$, the set of all intervals of $\mathbb{Z}^2$.

Details about Questions 1, 2 and 3.

(I) Our results related to Question 1 and Question 2 are the following.

(i) The notion of Möbius invertibility of the GRI encompasses generalized persistence diagrams [52] and rank decompositions [14], and properly adapts Möbius inversion of a constructible function [47]: see Section 3.1.

(ii) We show that if a GRI admits a rank decomposition, then even without assuming that $\text{Int}(P)$ is locally finite, we have that the minimal rank decomposition of the GRI is obtained from Möbius inversion over a specific locally finite $\mathcal{I} \subset \text{Int}(P)$: see Theorem A.

(iii) We clarify the relationship among the concepts pertaining to structural simplicity of persistence modules, including Möbius invertibility of the GRI, tameness [69], constructibility [47, 75], interval decomposability [12], finite presentability: see Figure 1.

(iv) We identify a 2-parameter persistence module whose GRI over intervals is not Möbius invertible (which implies that the GRI admits no rank decomposition). To the best of our knowledge, this is the first example of its kind in the existing literature: see Theorem B and Remark 3.10.

(v) We establish a number of sufficient conditions for Möbius invertibility of the GRI. For instance, our results imply that the GRI over intervals of any finitely presentable $\mathbb{R}^d$-module is Möbius invertible. Furthermore, in this scenario, we construct a finite poset of intervals to which the GRI over intervals can be restricted without any loss of information. That is, the Möbius inversion of the GRI over this finite poset encodes the entire GRI over the uncountable set of intervals in $\mathbb{R}^d$: see Theorem C and Proposition 3.17. In another instance, we establish the Möbius invertibility of $P$-modules satisfying certain assumptions akin to tameness [69]: see Theorem D.

All these results shed light on computational aspects of the GRI and also on its compact encoding as a ‘persistence diagram’, which provides a concrete answer to Question 1.

(II) We address Question 3 as follows.
Let $I \subset \text{Int}(P)$ be any collection of intervals. We use the Möbius inversion formula to prove that the GRI over $I$ is a complete invariant for $P$-modules whose indecomposable summands are interval modules supported on elements in $I$: see Theorem E (note: the statement itself was already known; see Remark 4.6).

We show that Theorem E is, in a certain precise sense, optimal: see Theorem F and Corollary 4.7.

We go one step further and describe the equivalence classes of $P$-modules that have the same GRI over any $I \subset \text{Int}(P)$. Interestingly, we show this via exploiting the Möbius inversion of functions which do not arise as GRIs of $P$-modules: see Theorem G and Corollary 4.9.

In what follows, we give a special attention to the discriminating power and computational cost of the (restricted) GRI of 2-parameter persistence modules. Let $\text{int}(\mathbb{Z}^2)$ be the collection of all finite intervals of $\mathbb{Z}^2$.

We show that the ZIB over simple paths and the GRI over $\text{int}(\mathbb{Z}^2)$ do not determine each other. As a corollary to this result and [40, Theorem 24], it follows that the ZIB over all paths is a strictly finer invariant than both the ZIB over simple paths and the GRI over $\text{int}(\mathbb{Z}^2)$: see Examples 4.16, 4.17 and Figure 5.

Using Möbius inversion, we show the ZIB over simple paths and the GRI over $\text{int}(\mathbb{Z}^2)$ estimate each other: see Remark 4.20 and Proposition 4.22.

We show that the ZIB (or equivalently GRI) over zigzag paths of the form $\bullet \rightarrow \bullet \leftarrow \bullet$ and $\bullet \leftarrow \bullet \rightarrow \bullet$ is a strictly stronger invariant than the bigraded Betti numbers: see Proposition 4.26.

A few subsidiary contributions follow: We establish a stability theorem for GRIs and their restrictions – a property that was not addressed in [52]: see Theorem H. A suitable reinterpretation of this theorem implies stability of ZIBs: see Theorem I. Also, we analyze the trade-off between computational complexity of the erosion distance (Definition 5.2) between restricted GRIs and the discriminating power of restricted GRIs as the domain of the restriction grows; Remark 5.3.
Figure 1: Implications and non-implications among the concepts pertaining to the structural simplicity of a persistence module $M$ over $\mathbb{R}^d$, as detailed in Sections 3.3 and 3.4:

(1) Proposition 3.5,  (4) Theorem C (ii),  (7) Theorem B,
(2) Pf. of Thm. C (iii),  (5) By definition,  (8) Remark 3.4 (i),
(3) Theorem C (i),  (6) Proposition 3.7,  (9) Remark 3.10 (i).
Table 1: Comparison between the GRI of a $P$-module $M$, denoted by $\text{rk}_M$, and other invariants of $M$. Different choices of the indexing poset $P$, and domain $\mathcal{J} \subset \text{Int}(P)$ of $\text{rk}_M$ are made. $\partial \text{rk}_M^\mathcal{J}$ denotes the M"obius inversion of either $\text{rk}_M^\mathcal{J}$ or an appropriate restriction of $\text{rk}_M^\mathcal{J}$, which is essentially unique (Proposition 3.2). Section 4.4 is dedicated to providing detailed explanations of this table.

| # | P | Domain $\mathcal{J}$ of $\text{rk}_M^\mathcal{J}$ | Name or Comparison | $\partial \text{rk}_M^\mathcal{J}$ | $\partial \text{rk}_M^\mathcal{J}$ always exists? |
|---|---|---|---|---|---|
| 1 | Any | $(\{p\} : p \in P)$ | $\leftrightarrow$ Dimension function | Dimension function | Yes |
| 2 | Any | $\text{Seg}(P)$ | $\leftrightarrow$ Rank Invariant | Signed barcode [14] | ? |
| 3 | $\mathbb{R}^d$ | $\{\text{Seg}(\ell) : \ell$ is a monotone line in $\mathbb{R}^d\}$ | $\leftrightarrow$ Rank invariant | Fibered barcode [59] | Yes |
| 4 | Any | $\text{Int}(P)$ | Int-GRI | GPD over $\text{Int}(P)$ | Rank decomposition | No |
| 5 | Any | $\text{Con}(P)$ | Con-GRI | GPD over $\text{Con}(P)$ | No |
| 6 | $[m] \times [n]$ | $\text{Int}(P)$ | tot-Compressed multiplicity ($\leftrightarrow$ Int-GRI) | $\delta_{\text{tot}}^M$ | Yes |
| 7 | $[m] \times [n]$ | $\{\min(I) \cup \max(I) : I \in \text{Int}(P)\}$ | ss-Compressed multiplicity ($\leftrightarrow$ Int-GRI) | $\delta_{ss}^M$ | Yes |
| 8 | $\mathbb{Z}^2$ | (Zigzag paths of the forms $\bullet \rightarrow \bullet \leftarrow \bullet$ and $\bullet \leftarrow \bullet \rightarrow \bullet$ in $\mathbb{Z}^2$) | $\Rightarrow$ Bigraded Betti numbers | Barcodes over zigzag paths of length 3 | Yes |
| 9 | $\mathbb{Z}^2$ | (Zigzag paths in $\mathbb{Z}^2$) | $\Rightarrow$ Int-GRI | Barcodes over zigzag paths | Yes |
| 10 | $\mathbb{Z}^2$ | (Simple zigzag paths in $\mathbb{Z}^2$) | $\nRightarrow$ Int-GRI | Barcodes over simple zigzag paths | Yes |
Related work.

Patel first noted that in the one-parameter setting the persistent diagram can be defined as the Möbius inversion of the RI and thereby introduced the generalized persistence diagram \[75\] of a constructible $\mathbb{R}$-indexed functor whose target can be different from the category of vector spaces. Patel’s work became a motivation for the work by Kim and Mémoli \[52\] and the work by McCleary and Patel \[65, 66\]. In particular, in \[52\], the generalized persistence diagram of a $P$-module is defined as the Möbius inversion of the GRI over \(\text{Con}(P)\), which can be viewed as a multiset of signed elements of \(\text{Con}(P)\) (assuming that \(\text{Con}(P)\) is locally finite).

The following are also related to our work.

- Asashiba et al. also invoke Möbius inversion to devise methods for approximating a persistence module $M$ over a finite 2d-grid by an interval-decomposable module \[5\]. One of their approximation methods yields an invariant that possesses the same amount of information as (the Möbius inversion of) the GRI of $M$ over the intervals in the grid \[54, \text{Remark 2.19}\]. These two equivalent invariants naturally encode the bigraded Betti numbers of $M$ \[54\]. See also \[6, 51\] for further developments of these ideas.

- Amiot et al. take a systematic approach to the invariant obtained by restricting the indexing poset of a given persistence module $M$ to a subposet $X \subset P$ of a finite representation type \[2\]. Their work is related to ours in the sense that invariants from both works consider restrictions of the indexing poset of a given module. Indeed, as the GRI is a complete invariant for interval-decomposable modules, whenever a choice of $X$ ensures that $M|_X$ is interval-decomposable, their work can be seen as studying the restriction of the GRI to $X$.

- Recently, connections between relative homological algebra and the RI/GRI have been explored \[4, 9, 14, 15\].

- In recent years, Möbius inversion has been utilized alongside other invariants of persistence modules, including the graded rank function \[8\], the birth-death function \[47, 66, 72\], the meta-rank \[32\], the Hilbert function \[74\], the multi-rank \[80\], the persistent cup length and persistent cup product \[68\], and the Grassmanian persistence diagram \[48\].

- Very recently, through an extension of the main result from \[40\], in \[41\] Dey et al. also propose to utilize zigzag persistence for computing the rank of the limit-to-colimit map \[52\] for persistence modules over posets more general than $\mathbb{Z}^2$.

Organization.

In Section 2 we review the requisite concepts about persistence modules, their decompositions, and introduce the concept of a motivic invariant, relating it to several existing invariants. In Sections 3, 4, and 5, we establish the results outlined in Contributions (I), (II), and (III) respectively. In Section 6, we discuss open questions.

A table with the main nomenclature used in this paper is given below.
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Nomenclature

Collections of subposets

\([n]\)  The totally ordered set \(\{1 < \ldots < n\}\).
\(P\)  A poset \(P = (P, \leq)\), regarded as a category.
\(\text{Con}(P)\)  Poset of connected subposets of \(P\), ordered by containment (pg. 16).
\(\text{Seg}(P)\)  Poset of segments of the poset \(P\), ordered by containment (Definition 2.1).
\(\text{Int}(P)\)  Poset of intervals of the poset \(P\), ordered by containment (Definition 2.1).
\(\text{int}(P)\)  Poset of finite intervals of the poset \(P\), ordered by containment (pg. 41).
\(\text{Int}_{m,n}(P)\)  Poset of intervals of \(P\) with at most \(m\) minimal points and \(n\) maximal points (pg. 17).
\(\hat{I}\)  The limit completion of \(I \subset \text{Int}(P)\) (Equation (4.1)).

Invariants

\(\text{RI}\)  Rank invariant (Definition 2.18).
\(\text{GRI}\)  Generalized rank invariant (Definition 2.18).
\(\text{rk}(M)\)  The (generalized) rank of \(M\) (pg. 16).
\(\text{rk}_{\mathcal{I}}(M)\)  The generalized rank invariant of \(M\) over \(\mathcal{I} \subset \text{Con}(P)\) (Definition 2.18).
\(\text{rk}_{\text{Int}}(M)\)  The generalized rank invariant of \(M\) over \(\text{Int}(P)\) (Definition 2.18).
\(\text{dgm}_{\mathcal{I}}(M)\)  The generalized persistence diagram of \(M\) over \(\mathcal{I}\) (Definition 2.21).
\(\text{ZIB}\)  The zigzag-path-indexed-barcode (Definition 2.35).
\(\text{MSZZ}\)  The Zigzag-path-indexed-barcode over simple paths of \(M : P \to \text{vec}\) (Definition 4.14).

Functions

\(I(Q, k)\)  The incidence algebra of \(Q\) over \(k\) (pg. 13).
\(\delta_Q\)  The Dirac delta function (Equation (2.4)).
\(\zeta_Q\)  The zeta function (Equation (2.5)).
\(\mu_Q\)  The Möbius function (Equation (2.6)).

Other

\(\text{barc}(M)\)  The barcode of \(M\) (pg. 11).
\(\Gamma\)  A path in \(\mathbb{Z}^2\) (pg. 23).
\(k^Q\)  The vector space of functions \(f : Q \to k\) (pg. 13).
\(k_1\)  Interval module for the interval \(I\) (Equation (2.1)).
\(p^+\)  The set of elements in \(P\) less than or equal to \(p\) (pg. 13).
\(p^{\dagger}\)  The set of elements in \(P\) greater than or equal to \(p\) (pg. 12).

2 Preliminaries

This section is organized as follows: in Section 2.1, we review the concepts of persistence modules and their structures. In Section 2.2, we review the notion of the incidence al-
gebra as well as the Möbius inversion formula. In Section 2.3, we recall the notions of the rank invariant, the generalized rank invariant, and their properties. In Section 2.4, we recall the notion of the generalized persistence diagram. In Section 2.5, we recall the notion of the rank decomposition and its connection with the notion of the generalized persistence diagram. In Section 2.6, we introduce the notion of a motivic invariant, provide historical context and motivation for the definition, and exemplify how certain invariants we discuss in this paper fit into this framework. In Section 2.7, we introduce a new motivic invariant called the zigzag-path-indexed barcode, which is based on slicing a P-module over zigzag paths in P.

2.1 Persistence modules

Throughout this paper, P = (P, ≤) is a poset, regarded as the category with objects the elements p ∈ P, and a unique morphism p → q if and only if p ≤ q ∈ P. All vector spaces in this paper are over a fixed field k. Let vec denote the category of finite-dimensional vector spaces and linear maps over k.

Persistence modules and their decompositions. A persistence module over P is a functor M : P → vec.\(^3\) We refer to M simply as a P-module. For any p ∈ P, we denote the vector space \(M_p := M(p)\), and for any \(p ≤ q ∈ P\), we denote the linear map \(φ_M(p, q) := M(p ≤ q)\). Given any P-modules M and N, the direct sum \(M ⊕ N\) is defined pointwisely at each \(p ∈ P\). We say that a nontrivial P-module M is decomposable if M is isomorphic to \(N_1 ⊕ N_2\) for some non-trivial P-modules \(N_1\) and \(N_2\), which we denote by \(M ≅ N_1 ⊕ N_2\). Otherwise, we say that M is indecomposable.

By the Azumaya-Krull-Remak-Schmidt Theorem [7, Theorem 1], every P-module is isomorphic to a direct sum of indecomposable P-modules; see also [11, Theorem 1.1]. This direct sum decomposition is unique up to isomorphism and permutations of summands. The multiset of isomorphism classes of indecomposable summands of M is called the barcode of M, denoted by barc(M).

Definition 2.1 ([12]). An interval of a poset P is any non-empty subset I ⊂ P such that

(i) (convexity) If \(p, r ∈ I\) and \(q ∈ P\) with \(p ≤ q ≤ r\), then \(q ∈ I\),

(ii) (connectivity) For any \(p, q ∈ I\), there is a sequence \(p = r_0, r_1, \ldots, r_n = q\) of elements of I, where \(r_i\) and \(r_{i+1}\) are comparable for \(0 ≤ i ≤ n − 1\).

Given an interval I of P, the interval module \(k_I\) is the P-module, with

\[
(k_I)_p := \begin{cases} 
    k & \text{if } p ∈ I \\
    0 & \text{otherwise}
\end{cases}, \quad φ_{k_I}(p, q) := \begin{cases} 
    \text{id}_k & \text{if } p ≤ q ∈ I \\
    0 & \text{otherwise}
\end{cases} \quad (2.1)
\]

Every interval module is indecomposable [12, Proposition 2.2]. A P-module M is interval-decomposable if it is isomorphic to a direct sum of interval modules. If the equivalence class of the interval module \(k_I\) belongs to \(\text{barc}(M)\), then we simply say that I belongs to \(\text{barc}(M)\) and write \(I ∈ \text{barc}(M)\). A zigzag poset of n points is \(\bullet_1 ↔ \bullet_2 ↔ \cdots ↔ \bullet_{n-1} ↔ \bullet_n\) where \(↔\) stands for either \(<\) or \(>\). A functor from a zigzag poset (of n points) to vec is called a zigzag module [18].

\(^3\)In the literature, M is often referred to as a pointwisely finite-dimensional persistence module.
**Theorem 2.2 ([18, 43]).** Zigzag modules are interval-decomposable.

Let $\mathcal{L}$ be any collection of indecomposable $P$-modules. A $P$-module $M$ is called \textit{$\mathcal{L}$-decomposable}, if every indecomposable summand of $M$ is isomorphic to an element in $\mathcal{L}$.\footnote{This terminology was used in [3].} If $\mathcal{L}$ consists solely of interval modules supported on intervals in a collection $I$ of intervals of $P$, we simply say that $M$ is $I$-decomposable.

**Tame, Finitely presentable, and Constructible $P$-modules.** Aside from interval decomposability, there are other notions of structural simplicity for persistence modules. One of them is \textit{q-tameness}. An $R$-module $M$ is considered $q$-tame if, for any $p < q$ in $R$, the rank of the map $\varphi_M(p, q)$ is finite \cite{29, 30}. This definition can be directly generalized to $P$-modules for arbitrary posets $P$. Since this work only deals with pointwise finite-dimensional persistence modules, we readily assumed $q$-tameness throughout. Another is \textit{constructibility} \cite{36, 47, 75}:

**Definition 2.5.** An order-preserving map $c : P \to P$ such that $c(p) \leq p$ for all $p \in P$, $c \circ c = c$ is called a \textit{co-closure}. A $P$-module $M$ (resp. any function $f$ from $P$) is \textit{constructible} if there is a co-closure $c : P \to P$ with finite image, and $M \circ c = M$ (resp. $f \circ c = f$). Specifically, if the image of $c$ is $S \subset P$, then $M$ (resp. $f$) is called $S$-\textit{constructible}.

### 2.2 The Möbius inversion formula

We review the notions of incidence algebra and Möbius inversion \cite{77, 78}. Throughout this section, let $Q$ denote a \textit{locally finite} poset, i.e. for all $p, q \in Q$ with $p \leq q$, the segment

$$[p, q] := \{x \in Q : p \leq x \leq q\}$$

is finite. Let

$$\text{Seg}(Q) := \{[p, q] : p \leq q \text{ in } Q\}. \quad (2.2)$$

\footnote{In [69], it is also assumed that for all $q \in Q$, the vector space $N_q$ has finite dimension. However, in this paper, that must readily be the case since $M_p$ is assumed to be finite-dimensional for all $p \in P$.}
Given any function $\alpha : \text{Seg}(Q) \to k$, we write $\alpha(p, q)$ for $\alpha([p, q])$. The incidence algebra $I(Q, k)$ of $Q$ over $k$ is the $k$-algebra of all functions $\text{Seg}(Q) \to k$ with the usual structure of a vector space over $k$, where multiplication is given by convolution:

$$(\alpha \beta)(p, r) := \sum_{q \in [p, r]} \alpha(p, q) \cdot \beta(q, r). \quad (2.3)$$

Since $Q$ is locally finite, the above sum is finite and hence $\alpha \beta$ is well-defined. This multiplication is associative and thus $I(Q, k)$ is an associative algebra. The Dirac delta function $\delta_Q \in I(Q, k)$ is given by

$$\delta_Q(p, q) := \begin{cases} 1, & p = q \\ 0, & \text{else,} \end{cases} \quad (2.4)$$

which is the two-sided multiplicative identity.

**Remark 2.6 ([78])**. An element $\alpha \in I(Q, k)$ admits a multiplicative inverse if and only if $\alpha(q, q) \neq 0$ for all $q \in Q$.

Another important element of $I(Q, k)$ is the zeta function:

$$\zeta_Q(p, q) := \begin{cases} 1, & p \leq q \\ 0, & \text{else}. \end{cases} \quad (2.5)$$

By Remark 2.6, the zeta function $\zeta_Q$ admits a multiplicative inverse, which is called the Möbius function $\mu_Q \in I(Q, k)$. The Möbius function can be computed recursively as

$$\mu_Q(p, q) = \begin{cases} 1, & p = q, \\ - \sum_{p < r < q} \mu_Q(p, r), & p < q, \\ 0, & \text{otherwise}. \end{cases} \quad (2.6)$$

Let $k^Q$ denote the vector space of all functions $Q \to k$. Also, for $q \in Q$, let

$$q^\uparrow := \{p \in Q : p \leq q\},$$

called a principal ideal. Assuming that $q^\uparrow$ is finite for each $q \in Q$, every element in $I(Q, k)$ acts on $k^Q$ by right multiplication: for any $f \in k^Q$ and for any $\alpha \in I(Q, k)$, we have

$$(f \ast \alpha)(q) := \sum_{p \leq q} f(p) \alpha(p, q). \quad (2.7)$$

In fact, even when not every principal ideal in a poset $Q$ is finite, Equation (2.7) still specifies a well-defined multiplication between $f$ and $\alpha$ under the weaker assumption that

$$\text{for every } q \in Q, f(r) = 0 \text{ for all but finitely many } r \in q^\uparrow. \quad (2.8)$$

**Definition 2.7.** Given a locally finite poset $Q$, we call a function $f : Q \to k$ **convolvable** (over $Q$) if $f$ satisfies Condition (2.8).
For a function $f \in k^Q$, the support of $f$ is the set $\{q \in Q : f(q) \neq 0\}$.

**Remark 2.8 (About convolvability).** Let $Q$ be a locally finite poset.

(i) Any function $Q \to k$ with finite support is convolvable. Hence, if $Q$ is finite, any function $Q \to k$ is convolvable.

(ii) If every principal ideal in $Q$ is finite, then every $f \in k^Q$ is convolvable.

(iii) The collection of all convolvable functions $Q \to k$ is a linear subspace of $k^Q$.

(iv) If $f : Q \to k$ is convolvable over $Q$, then for any $P \subset Q$, the restriction $f|_P$ is convolvable.

(v) Let $P_1, P_2 \subset Q$. Assume that, for each $i = 1, 2$, $f_i : P_i \to k$ is convolvable over $P_i$, and that $f_1 = f_2$ on $P_1 \cap P_2$. Then $f_1 \cup f_2 : P_1 \cup P_2 \to k$ is convolvable over $P_1 \cup P_2$.

**Remark 2.9.** Let $Q$ be a locally finite poset. Let $k^Q_c$ be the space of convolvable functions $Q \to k$, which is a subspace of $k^Q$. It follows that:

(i) By Remark 2.8 (i), if $Q$ is finite, then $k^Q_c = k^Q$.

(ii) Let $\alpha \in I(Q, k)$. The right multiplication map $\ast \alpha : k^Q_c \to k^Q_c$ given by $f \mapsto f \ast \alpha$ is an automorphism if and only if $\alpha$ is invertible.

(iii) By Remark 2.6 and the previous item, the right multiplication map $\ast \zeta_Q$ by the zeta function is an automorphism on $k^Q_c$ with inverse $\ast \mu_Q$.

The Möbius inversion formula is a powerful tool that has found widespread applications in combinatorics despite the fact that it has its origins in number theory. It will be a central tool for establishing our main results.

**Theorem 2.10 (Möbius Inversion formula).** Let $Q$ be a locally finite poset. For any pair of convolvable functions $f, g : Q \to k$,

$$g(q) = \sum_{r \leq q} f(r) \text{ for all } q \in Q$$  \hspace{1cm} (2.9)

if and only if

$$f(q) = \sum_{r \leq q} g(r) \cdot \mu_Q(r, q) \text{ for all } q \in Q.$$  \hspace{1cm} (2.10)

**Proof.** Equation (2.9) can be represented as $g = f \ast \zeta_Q$. By multiplying both sides by $\zeta_Q^{-1} = \mu_Q$ on the right, we have $g \ast \mu_Q = f$, which is precisely Equation (2.10).

**Definition 2.11.** The function $f = g \ast \mu_Q$ is referred to as the Möbius inversion of $g$ (over $Q$).

Möbius inversion serves as a discrete analogue of the concept of a derivative in calculus, as illustrated by the following example.
Example 2.12. Let \( q \in \mathbb{Q} \) and define the two functions \( 1_q, 1_{\geq q} : \mathbb{Q} \to k \) to be

\[
1_q(p) := \begin{cases} 
1, & p = q \\
0, & \text{otherwise}.
\end{cases} \quad 1_{\geq q}(p) := \begin{cases} 
1, & p \geq q \\
0, & \text{otherwise}.
\end{cases}
\]

Then, both functions are convolvable. Indeed, Remark 2.8 (i) implies that \( 1_q \) is convolvable and the local finiteness of \( \mathbb{Q} \) guarantees that \( 1_{\geq q} \) is convolvable. Notice that the Möbius inversion of \( 1_{\geq q} \) is equal to \( 1_q \) and that \( 1_q \) captures where the function values of \( 1_{\geq q} \) change.

The following proposition will be useful for some proofs in the sequel.

**Proposition 2.13.** Let \( g : \mathbb{Q} \to \mathbb{R} \) be non-decreasing and convolvable over \( \mathbb{Q} \). Then, the Möbius inversion of \( g \) is convolvable over \( \mathbb{Q} \).

**Proof.** Let \( f \) be the Möbius inversion of \( g \). Since \( g \) is non-decreasing, if \( g(r) = 0 \), then \( g(r') = 0 \) for all \( r' \leq r \) and thus \( f(r) = \sum_{r' \leq r} g(r') \cdot \mu(r', r) = 0 \). Furthermore, since \( g \) is convolvable, \( g(r) = 0 \) for all but finitely many \( r \in \mathbb{Q} \), for every \( q \in \mathbb{Q} \). Hence, we have \( f(r) = 0 \) for all but finitely many \( r \in \mathbb{Q} \), Desired.

A constructible function (Definition 2.5) admits Möbius inversion after its proper restriction:

**Proposition 2.14** ([47, Proposition 3.4]). Let \( S \) be a nonempty finite subset of a given poset \( P \). Let \( m : P \to \mathbb{Z} \) be an \( S \)-constructible function. Then, for all \( q \in P \),

\[
m(q) = \sum_{\substack{p \leq q \\ p \in S}} (m|_S * \mu_S)(p).
\]

**A matrix algebra perspective on the incidence algebra** Let \( m \in \mathbb{N} \) and let \( \mathbb{Q} \) be a poset with \( m \) elements. By the order-extension principle, we can extend the order on \( \mathbb{Q} \) to a total order. Thus, we fix \( \mathbb{Q} := \{q_1, \ldots, q_m\} \), where \( q_i < q_j \) implies \( i < j \). Then, each element \( \alpha \) in the incidence algebra \( I(Q,k) \) is canonically identified with the \((m \times m)\)-upper-triangular matrix \( (\alpha_{ij}) \) whose \((i, j)\)-entry is

\[
\alpha_{ij} := \begin{cases} 
\alpha(q_i, q_j) & \text{if } q_i < q_j \\
0, & \text{else}.
\end{cases}
\]

Then, for \( \beta \in I(Q,k) \), the product \( \alpha \beta \) in Equation (2.3) can be expressed as the multiplication of the upper-triangular matrices \( (\alpha_{ij}) \) and \( (\beta_{ij}) \), where

\[
(\alpha \beta)_{ij} = \text{(the } i\text{-th row of } \alpha) \cdot \text{(the } j\text{-th column of } \beta)\]

Now let us identify each \( f \in k^Q \) with the \( m \)-dimensional row vector \( (f_i) \) where \( f_i = f(q_i) \) for \( i = 1, \ldots, m \). Then, the multiplication \( f * \alpha \) in Equation (2.7) can be expressed as the multiplication of the \((1 \times m)\)-matrix \( (f_i) \) and the \((m \times m)\)-matrix \( (\alpha_{ij}) \).

**Remark 2.15.** Recall that an upper-triangular matrix is invertible if and only if all of its diagonal entries are nonzero. Remark 2.6 can be seen as a straightforward from this fact.
2.3 Generalized rank invariant

In this section, we recall the definitions of the rank invariant \([24, 76]\) and the generalized rank invariant \([52]\).

Let \(M\) be any \(P\)-module. Then, \(M\) admits both a \textit{limit} and a \textit{colimit} \([64, \text{Chapter V}]\): A limit of \(M\), denoted by \(\lim M\), consists of a vector space \(L\) together with a collection of linear maps \(\{\pi_p : L \to M\}_{p \in P}\) such that
\[
\varphi_M(p, q) \circ \pi_p = \pi_q \text{ for every } p \preceq q \text{ in } P. \tag{2.11}
\]
A colimit of \(M\), denoted by \(\lim M\), consists of a vector space \(C\) together with a collection of linear maps \(\{i_p : M_p \to C\}_{p \in P}\) such that
\[
i_q \circ \varphi_M(p, q) = i_p \text{ for every } p \preceq q \text{ in } P. \tag{2.12}
\]
Both \(\lim M\) and \(\lim M\) satisfy certain universal properties, making them unique up to isomorphism.

Let us assume that \(P\) is connected (Definition 2.1 (ii)). The connectedness of \(P\) alongside the equalities given in Equations (2.11) and (2.12) imply that \(i_p \circ \pi_p = i_q \circ \pi_q : L \to C\) for any \(p, q \in P\). This fact ensures that the \textit{canonical limit-to-colimit map}
\[
\psi_M : \lim M \longrightarrow \lim M
\]
given by \(i_p \circ \pi_p\) for any \(p \in P\) is well-defined. The \textit{(generalized) rank} of \(M\) is defined to be:\footnote{This construction was considered in the study of quiver representations \([55]\).}
\[
\text{rk}(M) := \text{rk}(\psi_M). \tag{2.13}
\]

\textbf{Remark 2.16.} \(\text{rk}(M)\) is finite as \(\text{rk}(M) = \text{rk}(i_p \circ \pi_p) \leq \dim(M_p) < \infty\) for any \(p \in P\).

The rank of \(M\) is a count of the ‘persistent features’ in \(M\) that span the entire indexing poset \(P\). Such ‘persistent features’ appear as summands of \(M\) in the form of \(k_p\):

\textbf{Theorem 2.17 ([27, Lemma 3.1])}. Let \(P\) be a connected poset. Assume that a \(P\)-module \(M\) is isomorphic to a direct sum \(\bigoplus_{a \in A} M_a\) for some indexing set \(A\) where each \(M_a\) is indecomposable. Then, the rank of \(M\) is equal to the cardinality of the set \(\{a \in A : M_a \cong k_p\}\).

We refine the rank of a \(P\)-module, which is a single integer, into an integer-valued function. Possible domain options for the function include the following sets, in addition to \(\text{Seg}(P)\) as shown in Equation (2.2):
\[
\text{Con}(P) := \{I \subset P : I \text{ is connected}\},
\]
\[
\text{Int}(P) := \{I \subset P : I \text{ is an interval}\}.
\]
Note the inclusions \(\text{Seg}(P) \subset \text{Int}(P) \subset \text{Con}(P)\).

\textbf{Definition 2.18 ([5, 24, 40, 52])}. The \textit{generalized rank invariant (GRI)} of a \(P\)-module \(M\) is the map
\[
\text{rk}_M : \text{Con}(P) \to \mathbb{Z}_{\geq 0}
\]
\[
I \mapsto \text{rk}(M|_I)
\]
where \(M|_I\) is the restriction of \(M\) to \(I\). The restriction of \(\text{rk}_M\) to \(I \subset \text{Con}(P)\) is denoted by \(\text{rk}_M^I\) and is called the \textit{GRI of \(M\) over \(I\)}). When \(I = \text{Int}(P)\), the GRI over \(I\) is simply called the \textit{Int-GRI} and denoted by \(\text{rk}_M^\text{Int}\). When \(I = \text{Seg}(P)\), the GRI of \(M\) over \(I\) is the \textit{rank invariant (RI)} of \(M\), introduced in \([24]\).
Remark 2.19. The following are useful properties of the GRI.

(i) (Monotonicity) If \( I \subset J \) in \( \text{Con}(P) \), then \( \text{rk}_M(I) \geq \text{rk}_M(J) \). This is because the canonical limit-to-colimit map over \( I \) is a factor of the canonical limit-to-colimit map over \( J \) [52, Proposition 3.8].

(ii) (Additivity) If \( M \cong \bigoplus_{i=1}^n M_i \), and \( I \in \text{Con}(P) \), then \( \text{rk}_M(I) = \sum_{i=1}^n \text{rk}_{M_i}(I) \).

(iii) (The GRI of an interval module) Let \( J \in \text{Int}(P) \). For the interval module \( k_J \) and any \( I \in \text{Con}(P) \), we have

\[
\text{rk}_{k_J}(I) = \begin{cases} 1, & J \supset I \vspace{1mm} \\ 0, & \text{else.} \end{cases}
\]

Remark 2.20 (Domain of the GRI). (i) For 2-parameter persistence modules [54], the Con-GRI was proved to be strictly stronger than the Int-GRI.

(ii) The Int-GRI is strictly stronger than the bigraded Betti numbers, which follows from [54] together with Proposition 4.26.

(iii) Some invariants of multi-parameter persistence modules determine and/or are determined by the GRI over a collection that includes non-intervals (e.g. fibered barcode, ss-compressed multiplicity [5], and zigzag-path-indexed barcode (Definition 4.14)).

We discuss the GRI’s properties and discriminating power in Section 4.

We close this section by describing a parameterized family of subsets of \( \text{Int}(P) \) that will be useful later. Given a poset \( P \), let \( a \in P \) and let \( B \subset P \) be an antichain. We write \( a \leq B \) (resp. \( a \geq B \)) if there exists \( b \in B \) such that \( a \leq b \) (resp. \( a \geq b \)). When two antichains \( A, B \subset P \) are given, we write \( A \leq B \) if \( \forall a \in A, \forall b \in B, a \leq b \) and \( A \leq B \). This defines a partial order on the set of antichains in \( P \) [9, Section 2.1]. When \( A \leq B \), define

\[
[A, B) := \{ x \in P : \exists a \in A, \exists b \in B, a \leq x < b \},
\]

\[
[A, B] := \{ x \in \mathbb{R}^2 : \exists a \in A, \exists b \in B, a \leq x \leq b \}.
\]

\([A, B)\) is either empty (when \( A = B \)) or an interval of \( P \) (when \( A < B \)), whereas \([A, B]\) is always an interval of \( P \). For \( m, n \in \mathbb{N} \), let

\[
\text{Int}_{m,n}(P) := \{ [A, B) : A, B \subset P \text{ are antichains s.t. } A \leq B, |A| \leq m, |B| \leq n \}.
\]

(2.14)

See for an illustration corresponding to the case \( P = \mathbb{R}^2 \).

\[
\begin{array}{c|c|c|c}
\text{Int}_{2,2}(\mathbb{R}^2) & \text{Int}_{1,1}(\mathbb{R}^2) & \text{Int}_{1,2}(\mathbb{R}^2) & \text{Int}_{1,3}(\mathbb{R}^2) \\
\hline
\end{array}
\]

Similarly, define

\[
\text{Int}_{m,n}^{cc}(P) := \{ [A, B] : A, B \subset P \text{ are antichains s.t. } A \leq B, |A| \leq m, |B| \leq n \}.
\]

(2.15)
2.4 Generalized persistence diagrams

In this section we review the notion of the generalized persistence diagram (GPD) introduced in [52], but we modify the original formulation by adding more flexibility in choosing the domain of the GPD. This additional flexibility is useful when clarifying the relationship between the GRI/GPD and other invariants of persistence modules such as graded Betti numbers [54] and rank decompositions (Section 2.5).

Let $\text{Con}(P)$ and any subcollection of $\text{Con}(P)$ be ordered by containment $\supset$. Let $M$ be a $P$-module. If $\mathcal{I} \subset \text{Con}(P)$ is locally finite and $\text{rk}^\mathcal{I}_M$ is convolvable over $\mathcal{I}$, we say that the GRI of $M$ is convolvable over $\mathcal{I}$, cf. Definition 2.7.

**Definition 2.21.** Let $M$ be a $P$-module and let $\mathcal{I} \subset \text{Con}(P)$. Assume that the GRI of $M$ is convolvable over $\mathcal{I}$. The generalized persistence diagram (GPD) of $M$ over $\mathcal{I}$ is the Möbius inversion of $\text{rk}^\mathcal{I}_M$ over the poset $(\mathcal{I}, \supset)$, i.e. the function $dgm^\mathcal{I}_M : \mathcal{I} \to \mathbb{Z}$ given by:

$$dgm^\mathcal{I}_M := \text{rk}^\mathcal{I}_M \ast \mu_\mathcal{I}. \quad (2.16)$$

When $\mathcal{I} = \text{Int}(P)$, the GPD of $M$ over $\mathcal{I}$ will be referred to as simply the Int-GPD of $M$, and will be denoted by $dgm^\mathcal{I}_M$. As will be shown later, $\mathcal{I} = \text{Int}(P)$ is the minimal subset of $\text{Con}(P)$ which allows the generalized rank invariant over $\mathcal{I}$ to be at least as strong an invariant as the barcode of interval-decomposable $P$-modules (Theorem F).

**Remark 2.22.** When the GRIs of $P$-modules $M$ and $N$ are both convolvable over $\mathcal{I} \subset \text{Con}(P)$, by additivity of the GRI (cf. Remark 2.19 (i)), we have $dgm^\mathcal{I}_{M \oplus N} = dgm^\mathcal{I}_M + dgm^\mathcal{I}_N$.

We establish a few other desirable properties of the GPD.

**Proposition 2.23.** If the GRI of a $P$-module $M$ is convolvable over a locally finite $\mathcal{I} \subset \text{Con}(P)$, then:

(i) The support of $dgm^\mathcal{I}_M$ is contained in the support of $\text{rk}^\mathcal{I}_M$. In other words, for any $I \in \mathcal{I}$, if $\text{rk}^\mathcal{I}_M(I) = 0$, then $dgm^\mathcal{I}_M(I) = 0$.

(ii) $dgm^\mathcal{I}_M$ is convolvable over $\mathcal{I}$.

(iii) The unique function $d : \mathcal{I} \to \mathbb{R}$ satisfying the following equality is $dgm^\mathcal{I}_M$:

$$\text{rk}_M(I) = \sum_{J \supset I} d(J) \text{ for all } I \in \mathcal{I}.$$ 

In other words, Equation (2.16) implies $\text{rk}^\mathcal{I}_M = dgm^\mathcal{I}_M \ast \zeta_\mathcal{I}$.

**Proof.** Item (i) follows from Remark 2.19 (i) and Proposition 2.13. Item (ii) is an immediate consequence of item (i). Item (iii) follows from Theorem 2.10 and item (ii). \qed

Item (i) together with the fact that $dgm^\mathcal{I}_M$ determines $\text{rk}^\mathcal{I}_M$ suggests that $dgm^\mathcal{I}_M$ is a concise encoding of $\text{rk}^\mathcal{I}_M$. 

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18
2.5 Rank decompositions

In this section, we provide a review of the notion of rank decomposition [14], with a strong focus on its connection with the Möbius inversion formula.

For any multiset $\mathcal{R}$ of intervals in a poset $P$, let $\text{mult}_{\mathcal{R}} : \text{Int}(P) \to \mathbb{Z}_{\geq 0}$ be defined by $R \mapsto (\text{the multiplicity of } R \text{ in } \mathcal{R})$. The multiset $\mathcal{R}$ is said to be pointwisely finite if for all $p \in P$, the sum $\sum_{R \ni p} \text{mult}_{\mathcal{R}}(R)$ is finite.

Remark 2.24. It is not difficult to check that the following are equivalent:

(i) A multiset $\mathcal{R}$ of intervals in $P$ is pointwisely finite.

(ii) The direct sum $k_{\mathcal{R}} := \bigoplus_{R \in \mathcal{R}} k_R$ is pointwisely finite-dimensional, i.e. $\dim((k_{\mathcal{R}})_p) < \infty$ for all $p \in P$.

(iii) For all $I \in \text{Int}(P)$, $\text{rk}_{k_{\mathcal{R}}}(I) < \infty$ (cf. Remark 2.19 (i)).

(iv) Every principal ideal of the subposet $\{ R \in \text{Int}(P) : \text{mult}_{\mathcal{R}}(R) \neq 0 \} \subset \text{Int}(P)$ is finite.

Definition 2.25. Let $M$ be a $P$-module. Whenever they exist, any pair $(\mathcal{R}, S)$ of pointwisely finite multisets of elements in $\text{Int}(P)$ such that

$$\text{rk}_M^{\text{Int}} = \text{rk}_{k_{\mathcal{R}}}^{\text{Int}} - \text{rk}_{k_S}^{\text{Int}}$$

is called a rank decomposition of $\text{rk}_M^{\text{Int}}$.

If $\mathcal{R}$ and $S$ are disjoint, then the rank decomposition is called minimal and we write $\mathcal{R}_M$ and $S_M$ (the uniqueness of the pair $(\mathcal{R}_M, S_M)$ follows from the next proposition).

Proposition 2.26 ([14, Corollary 2.12]). Let $M$ be a $P$-module such that $\text{rk}_M^{\text{Int}}$ admits a rank decomposition $(\mathcal{R}, S)$. Then, the unique minimal rank decomposition of $\text{rk}_M^{\text{Int}}$ is given by $(\mathcal{R}_M, S_M)$, where $\mathcal{R}_M := \mathcal{R} - (\mathcal{R} \cap S)$ and $S_M := S - (\mathcal{R} \cap S)$.

Proposition 2.27 ([14, Proposition 3.3]). Assume that $\text{Int}(P)$ is locally finite. If the Int-GRI of a $P$-module $M$ is convolable over $\text{Int}(P)$, then the minimal rank decomposition of $\text{rk}_M^{\text{Int}}$ can be obtained as

$$\mathcal{R}_M = \{ d_1 \cdot I : \text{dgm}_M(I) > 0 \} \text{ and } S_M = \{ d_1 \cdot I : \text{dgm}_M(I) < 0 \},$$

where $d_1 := |\text{dgm}_M(I)|$ and $d_1 \cdot I$ stands for $d_1$ copies of $I$.

We include the proof from [14].

Proof. Let $1_{\mathcal{J}} : \text{Int}(P) \to \{0, 1\}$ be defined by $1_{\mathcal{J}}(I) = 1$ if $\mathcal{J} \supset I$ and $1_{\mathcal{J}}(I) = 0$, otherwise. For every $I \in \text{Int}(P)$, we have

$$\text{rk}_M^{\text{Int}}(I) = \sum_{J : I \supset J} \text{dgm}_M(J)$$

$$\quad = \sum_{d_1 \cdot I \supset J : \text{dgm}_M(I) > 0} \text{dgm}_M(J) + \sum_{d_1 \cdot I \supset J : \text{dgm}_M(I) < 0} \text{dgm}_M(J)$$

$$\quad = \sum_{d_1 \cdot I \supset J : \text{dgm}_M(I) > 0} \text{dgm}_M(J) \cdot 1_{\mathcal{J}}(I) + \sum_{d_1 \cdot I \supset J : \text{dgm}_M(I) < 0} \text{dgm}_M(J) \cdot 1_{\mathcal{J}}(I).$$

By Remark 2.19 (ii) and (iii), the right-hand side of the equation above equals $\text{rk}_{k_{\mathcal{R}_M}}^{\text{Int}}(I) - \text{rk}_{k_{S_M}}^{\text{Int}}(I)$ with $\mathcal{R}_M \cap S_M = \emptyset$, as desired.
Under the assumption that \( \text{Int}(P) \) is locally finite and \( \text{rk}_M \) is convolvable over \( \text{Int}(P) \), Proposition 2.27 implies that:

(i) the existence of a rank decomposition is implied by the forward direction of Theorem 2.10.

(ii) the uniqueness of a minimal rank decomposition is implied by the backward direction of Theorem 2.10.

In Definition 2.25 and Propositions 2.26 and 2.27, the collection \( \text{Int}(P) \) and the function \( \text{rk}_M : \text{Int}(P) \to \mathbb{Z} \) can be replaced by any \( I \subseteq \text{Int}(P) \) and any map \( r : I \to \mathbb{R} \), respectively:

**Theorem 2.28** (Restatement of [14, Theorem 2.5]). Let \( I \subseteq \text{Int}(P) \) be locally finite and let \( r : I \to \mathbb{Z} \) be convolvable. Then, there exists \( a_I \in \mathbb{Z} \) for each \( I \in \text{Int}(P) \) such that for \( M := \bigoplus_{a_I > 0} (k_I)^{a_I} \) and \( N := \bigoplus_{a_I < 0} (k_I)^{-a_I} \), we have \( r = \text{rk}_M - \text{rk}_N \).\(^7\)

This theorem is proved by simply replacing \( \text{Int}(P) \), \( \text{rk}_M \), \( \text{dgm}_M \) in the proof of Proposition 2.27 with \( I \), \( r \), and \( * \mu_I \), respectively. When \( I \) is finite, the theorem above can also be shown via elementary linear algebra as detailed below. This approach is distinct from that of [14].

*Another proof of Theorem 2.28 when \( I \) is finite.* In this proof, we view the GPD and GRI as rational number valued functions in order to utilize the fact that the set \( \mathbb{Q} \) of rational numbers is a field. For the interval module \( k_I \), note that \( \text{dgm}_{k_I}^I : I \to \mathbb{Q} \) is identical to the indicator function \( 1_I : I \to \mathbb{Q} \) with support \( \{I\} \). Therefore, the canonical basis \( \{1_I : I \in I\} \) of the vector space \( \mathbb{Q}^I \) coincides with \( \{\text{dgm}_{k_I}^I : I \in I\} \). By Remark 2.8 (i), \( 1_I \) is convolvable over \( I \). Now, as \( \mathbb{Q}^I \) is finite-dimensional, the image of \( \{\text{dgm}_{k_I}^I : I \in I\} \) under the automorphism \( * \zeta_I \) on \( \mathbb{Q}^I \) forms another basis for \( \mathbb{Q}^I \) (cf. Remark 2.9 (i) and (iii)). This implies that, since

\[
\text{dgm}_{k_I}^I * \zeta_I = \text{rk}_{k_I},
\]

any function \( r : I \to \mathbb{Q} \) can be uniquely expressed as a linear combination \( r = \sum_{1 \in I} a_1 \cdot \text{rk}_{k_I}, a_I \in \mathbb{Q} \). It remains to show that \( a_I \) is an integer for every \( I \in I \). On one hand, since \( r \) is a \( \mathbb{Z} \)-valued function, its M"obius inverse \( * \mu_I \) over \( I \) takes values in \( \mathbb{Z} \). On the other hand,

\[
* \mu_I = \left( \sum_{1 \in I} a_1 \cdot \text{rk}_{k_I} \right) * \mu_I = \sum_{1 \in I} a_1 \cdot (\text{rk}_{k_I} * \mu_I) = \sum_{1 \in I} a_1 \cdot \text{dgm}_{k_I} = \sum_{1 \in I} a_1 \cdot 1_I.
\]

Since \( a_I = r(1) \in \mathbb{Z} \) for every \( I \in \mathcal{I}, \) we are done. (It is also noteworthy that for each \( I \in \mathcal{I}, \) the coefficient \( a_I \) of \( \text{rk}_{k_I} \) is equal to the M"obius inversion of \( r \) evaluated at \( I \), i.e. \( r * \mu_I \).)

\(^7\)Equivalently,

\[
r(j) = \sum_{I \in \mathcal{I}} a_{1r(k)_I}^I (j) \quad (2.17)
\]

for every \( J \in \mathcal{I} \) where the RHS contains only finitely many nonzero terms.
2.6 Motivic invariants

Throughout this section, P and Q will denote posets.

In this section we introduce a parametric notion of invariant of persistence modules over posets that helps conceptualize several existing invariants at the same time that it permits designing new ones (as we do in Section 2.7). We call these motivic invariants since they are, roughly speaking, specified by the choice of a given fixed poset (or a fixed collection thereof), the motif, which one uses to probe a given Q-module.

The underlying idea of defining invariants through the process of studying all substructures of a given type (where this type is specified through the choice of motif) present in an object is manifested in different fields:

- In category theory through the pervasive notion of representable functor [64].

- In metric geometry, motivated by the notion of curvature sets considered by Gromov in [46, Chapter 3]. For a given compact metric space X, Gromov considers $K_n(X)$, the n-th curvature set of X, as the set containing all $n \times n$ matrices produced by $n$-tuples of points in X. The collection of all curvature sets completely characterize compact metric spaces up to isometry.

- In Lovasz’s study of graphs and graphons [61, Chapter 5] whereby the general notion of homomorphism number $\text{hom}(F, G)$ is considered as a count of the number of times a given graph F appears as substructure (subgraph) of a given larger graph G. The numbers $\text{hom}(F, G)$ (or closely related concepts) are then used to characterize graphs and graphons and to define distances between them.
Invariants inspired by these notions have been considered in contexts closely related to persistence. In [44, 45], a persistence-like invariant of metric spaces was constructed via curvature sets. In [20] the authors consider (hierarchical) clustering methods that are induced by a given motif or by a collection of motives (giving rise to representable clustering functors). This line was further explored in [21, 22, 67] in the context of clustering directed graphs and networks and, in [63], by directly exploiting homomorphism densities, for the analysis of network data [62].

We now provide a formal definition of motivic invariants, and then discuss how the invariants mentioned in Table 1 can be seen as particular instantiations of this definition.

**Definition 2.29.** Denote the set of all order-preserving maps from $P$ to $Q$ as $\text{Hom}(P, Q)$. For $P$ a set of posets, let $\text{Hom}(P, Q)$ denote the set

$$\text{Hom}(P, Q) := \bigcup_{P \in P} \text{Hom}(P, Q).$$

We denote the collection of all $P$-modules as mod $P$. For a $Q$-module $M$ and $\varphi \in \text{Hom}(P, Q)$, recall the definition of the pullback $\varphi^*M$ (Definition 2.4).

**Definition 2.30.** Let $P$ be a set of posets, for which we call $P \in P$ a motif. Let $\Phi \subset \text{Hom}(P, Q)$, and let $F$ be an invariant for $P$-modules with codomain a category $D$. A **motivic invariant of a $Q$-module** $M$ defined by the triple $(P, \Phi, F)$ is the map $\Psi_F(M) : \Phi \to D$, defined by:

$$\Psi_F(M)(\varphi) := F(\varphi^* M).$$

The motivic invariant defined by the triple $(P, \Phi, F)$ is the assignment $M \mapsto \Psi_F(M)$. We also say that an invariant $F$ for $Q$-modules is a motivic invariant if there exists a parametrization $(P, \Phi, F)$ such that for all $Q$-modules $M$, $F(M)$ can be computed from $\Psi_F(M)$ and vice versa. See Figure 2 for an illustration.

**Remark 2.31.** In fact, all invariants for $P$-modules are motivic invariants via a trivial parametrization, i.e. a parametrization $(P, \Phi, F)$ where $|P| = |\Phi| = 1$. Namely, if $F$ is an invariant and $M$ is a $Q$-module, then by letting $P = \{Q\}$ and $\Phi = \{id_Q\}$, the map $\Psi_F$ in Definition 2.30 parametrized by $(P, \Phi, F)$ is precisely $\Psi_F(M)(id_Q) = F(M)$.

Nonetheless, as we see in the coming examples, numerous invariants from the literature are motivic invariants with nontrivial parametrizations.

**Example 2.32.** Let $P = \text{Con}(Q)$, $\Psi \subset \text{Hom}(P, Q)$ consist of the canonical embeddings $\iota_I : I \hookrightarrow Q$ of connected subsets $I \in \text{Con}(Q)$ into $Q$, and $F = \text{rk}$, the generalized rank (Equation (2.13)). This parametrization yields a motivic invariant $\Psi_{\text{rk}}$, such that for a $Q$-module $M$ we have $\Psi_{\text{rk}}(M) : \Phi \to \mathbb{Z}$ is given by

$$\Psi_{\text{rk}}(M)(\iota_I) = \text{rk}(M|_I).$$

This demonstrates that the GRI of a $Q$-module $M$ is a motivic invariant of $M$. In a similar fashion, a restriction of the GRI to any $I \subset \text{Con}(P)$ is a motivic invariant.

**Example 2.33** (Connection to Table 1). The invariants $\text{rk}^I$ for each $I$ in column 3 of Table 1 are motivic invariants. For one example, letting $P = \{[*]\}$, $\Phi = \text{Hom}(\{[*]\}, Q)$, and $F = \text{rk}$ defines a parametrization for the motivic invariant $\Psi_{\text{rk}}$, which is equivalent to the dimension function in row 1 of Table 1.
Note that a motivic invariant does not necessarily have a unique parametrization. For instance, if we set \( \mathcal{P} := \{ \{ p \} : p \in P \} \), then the motivic invariant parametrized by \( \mathcal{P} \), \( \Phi := \{ p \in \mathcal{P} \} \), \( F = \text{rk} \) is again the dimension function.

**Remark 2.34.** Amiot et al. [2] consider a notion of invariants for \( P \)-modules based on embeddings of posets, resulting in invariants directly related to the notion of motivic invariant we introduce. The invariants they study require all posets \( P \in \mathcal{P} \) to be of finite representation type, and they require that for all \( \varphi \in \text{Hom}(P,Q) \) and \( p, p' \in P \), \( p \leq p' \) \( \iff \) \( \psi(p) \leq \psi(p') \). These added conditions allow them to prove strong results about the invariant they introduce. This motivates a study of motivic invariants by focusing on the properties of a motivic invariant ensured by its parametrization(s).

### 2.7 Zigzag-path-indexed barcodes

We introduce a motivic invariant of \( P \)-modules based on the idea of considering embeddings of zigzag posets. We begin with preliminary terminology.

Given a poset \( P \), a path in \( P \) is a nonempty finite sequence \( \Gamma : p_1, p_2, \ldots, p_n \) in \( P \) such that \( p_i \leq p_{i+1} \) or \( p_{i+1} \leq p_i \) for each \( i = 1, \ldots, n - 1 \). By inheriting the order on \( P \), a path \( \Gamma \) can be viewed as a zigzag poset \( p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n \), where \( \leftrightarrow \) stands for either \( \leq \) or \( \geq \). For a path \( \Gamma \) in \( P \), with canonical inclusion \( \iota_{\Gamma} : \Gamma \hookrightarrow P \), and a \( P \)-module \( M \), we denote by \( M_{\Gamma} \) the zigzag module \( M_{\Gamma} := M \circ \iota_{\Gamma} : \Gamma \to \text{vec} \), i.e. \( M_{\Gamma} = \iota_{\Gamma}^* M \).

**Definition 2.35.** Given a \( P \)-module \( M \), we define the zigzag-path-indexed barcode (ZIB) of \( M \) as the map sending each path \( \Gamma : p_1, p_2, \ldots, p_n \) in \( P \) to \( \text{barc}(M_{\Gamma}) \), a multiset of intervals in the zigzag poset \( p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n \).

**Remark 2.36.** Let \( \mathcal{P} \) be the set of all finite zigzag posets given by paths \( \Gamma \) in \( P \), and let \( \Phi \) consist of the canonical embeddings \( \iota_{\Gamma} : \Gamma \hookrightarrow P \). The ZIB of a \( \mathcal{P} \)-module \( M \) is a motivic invariant with parametrization \( ( P, \Phi, \text{barc}) \).

Similarly, the fibered barcode introduced by Cerri et al. [25] and further studied by Lesnick and Wright [59], as well as the pathwise persistence barcode introduce by Neumann et al. [73] are all motivic invariants.

A zigzag poset is of finite representation type, but the ZIB does not fit into the framework of Amiot et al. [2], as the canonical inclusion maps \( \varphi_{\Gamma} : \Gamma \hookrightarrow P \) do not satisfy \( p \leq p' \iff \varphi_{\Gamma}(p) \leq \varphi_{\Gamma}(p') \). Hence, the ZIB example justifies the level of generality in our definition of motivic invariant.

In Section 4.3.1, we will restrict our focus to the ZIB for \( \mathbb{Z}^2 \)-modules, and compare its discriminating power with that of the GRI.

### 3 Möbius invertibility of the generalized rank invariant

In Section 3.1, we eliminate redundancy in the existing assumptions in the literature for defining the GPD and thereby further generalize the notion of a GPD. This yields the notion of Möbius invertibility of the GRI, which is useful to compare various invariants of multi-parameter persistence modules in a unified viewpoint (cf. Table 1). In Section 3.2, we demonstrate that any rank decomposition of the GRI can always be achieved through Möbius inversion, regardless of the local finiteness of the domain of the GRI. In Section 3.3, we compare Möbius invertibility of the GRI with other concepts regarding
the structural simplicity of persistence modules. In Section 3.4, we identify some sufficient conditions that guarantee the Möbius invertibility of the GRI. As a consequence, the Int-GRI of finitely presentable multi-parameter persistence modules is proved to be Möbius invertible.

3.1 Axiomatizing the fundamental lemma of persistent homology

In this section, we generalize the notion of GPD (Definition 2.21) by (i) dropping the local finiteness assumption on the domain of the GRI (thus also the convolvability assumption on the GRI), and (ii) dissociating the domain of the GRI from that of the GPD.

Definition 3.1. Let \( M \) be a \( P \)-module and let \( J \subset \text{Con}(P) \). Then \( \text{rk}_J^M \) is said to be Möbius invertible (over \( I \)) if there exist \( I \subset J \) and a map \( d_M : I \to \mathbb{Z} \) such that

\[
\text{rk}_J^M(I) = \sum_{J \ni I \cap J \in I} d_M(J) \quad \text{for every } I \in J,
\]

where the RHS contains only finitely many nonzero summands. In particular, if the collection \( I \) can be taken as a subset of \( \text{Int}(P) \), then \( \text{rk}_J^M \) is said to be Möbius invertible over intervals.

When \( P \) is a finite totally ordered set, \( I = J = \text{Int}(P) \), and \( M \) arises from applying the homology functor to a filtration over \( P \) of a given simplicial complex, \( \text{rk}_I^M \) given in Equation (3.2) coincides with the classical persistence diagram. In particular, in that setting, Equation (3.2) is known as the fundamental lemma of persistent homology [42, Chapter VII]. Hence, Definition 3.1 axiomatizes the fundamental lemma of persistent homology without any assumptions on \( P, \text{Con}(P) \) (or \( \text{Int}(P) \)), \( M \), or \( \text{rk}_J^M \). In this viewpoint, Möbius invertibility of \( \text{rk}_I^M \) stands for availability of (a generalized version of) the fundamental lemma of persistent homology for \( \text{rk}_J^M \).

Next we show that \( d_M^I \) is acquired via Möbius inversion, providing justification for the term Möbius invertibility.

Proposition 3.2. If \( \text{rk}_I^M \) is Möbius invertible, then there exists a unique minimal locally finite \( I \subset J \) over which \( \text{rk}_I^M \) is convolvable and

\[
\text{rk}_I^M(I) = \sum_{J \ni I \cap J \in I} \text{dgm}_M^I(J) \quad \text{for all } I \in J.
\]

When Equation (3.2) holds, we call \( \text{dgm}_M^I \) the Möbius inversion of \( \text{rk}_I^M \) (over \( I \)). From another viewpoint, Möbius invertibility of the GRI is a relaxation of the interval decomposability of persistence modules, as evidenced by Theorem C (i). Note also that, when assuming \( I = J \) and \( \text{rk}_M^I \) is convolvable, Definition 3.1 reduces to Definition 2.21.

Proof of Proposition 3.2. Let \( I_1, I_2 \subset J \) be locally finite, and let \( d_M^{I_1} : I_1 \to \mathbb{Z} \) and \( d_M^{I_2} : I_2 \to \mathbb{Z} \) be any two functions such that

\[
\forall I \in J, \quad \text{rk}_I^M(I) = \sum_{J \ni I \cap J \in I_1} d_M^{I_1}(J) = \sum_{J \ni I \cap J \in I_2} d_M^{I_2}(J),
\]

24
and such that \( \text{rk}_M^\mathcal{I} \) and \( \text{rk}_M^\mathcal{J} \) are convolvable. Clearly, \( \mathcal{I}_1 \cup \mathcal{I}_2 \) is also locally finite, and by Remark 2.8 (v), \( \text{rk}_M^\mathcal{I}_1 \cup \mathcal{I}_2 \) is convolvable. Now, we regard both \( d_M^1 \) and \( d_M^2 \) as functions \( \mathcal{J} \to \mathbb{Z} \) that vanish outside \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), respectively. The above equation gives:

\[
\forall I \in \mathcal{J}, \quad 0 = \sum_{J \in \mathcal{I}_1 \cup \mathcal{I}_2} (d_M^1 - d_M^2)(J),
\]

which implies that

\[
\forall I \in \mathcal{I}_1 \cup \mathcal{I}_2, \quad 0 = \sum_{J \in \mathcal{I}_1 \cup \mathcal{I}_2} (d_M^1 - d_M^2)(J).
\]

This means that the map \( d_M^1 - d_M^2 \) is the Möbius inversion of the zero function on \( \mathcal{I}_1 \cup \mathcal{I}_2 \). Since the zero function itself is the unique Möbius inversion of the zero function, we have that \( d_M^1 - d_M^2 \) is the zero function on \( \mathcal{I}_1 \cup \mathcal{I}_2 \). Also, as both \( d_M^1 \) and \( d_M^2 \) vanish outside of \( \mathcal{I}_1 \cup \mathcal{I}_2 \), the map \( d_M^1 - d_M^2 \) is the zero function on \( \mathcal{J} \). Now, the unique minimal \( \mathcal{I} \subset \mathcal{J} \) in the statement is the support of \( d_M^1 (= d_M^2) \), completing the proof.

\[\square\]

**Remark 3.3 (Comparing Definition 3.1 with ideas from [14, Section 8]).**

(i) When the sets \( \mathcal{I} \) and \( \mathcal{J} \) in Definition 3.1 consists solely of intervals of \( P \) and one contains the other, then saying that \( \text{rk}_M^\mathcal{J} \) is Möbius invertible over \( \mathcal{I} \) is equivalent to saying that the GRI of \( M \) with test set \( \mathcal{J} \) admits a rank decomposition with dictionary \( I \) [12, Section 8], i.e. there exist \( I \)-decomposable \( P \)-modules \( N_1 \) and \( N_2 \) such that

\[
\text{rk}_M^\mathcal{J} = \text{rk}_M^{\mathcal{J}_1} - \text{rk}_M^{\mathcal{J}_2}.
\]

This can be proved in a similar way as Proposition 2.27. However, Definition 3.1 generalizes ideas in [14, Section 8] in that neither the test set nor the dictionary set needs to be a set of intervals. This level of generality is useful in building a connection between invariants of persistence modules or clarifying the discriminating power of the GRI: Remarks 3.13, 4.15, Rows 3, 9, 10 of Table 1.

(ii) In contrast to [14, Section 8], we consider the case where \( \mathcal{I} \subsetneq \mathcal{J} \) to be important both in theory and in practice: Since \( \text{dg}_M^\mathcal{J} \) and \( \text{rk}_M^\mathcal{J} \) determine each other (cf. Proposition 2.23 (iii)), the Möbius invertibility of \( \text{rk}_M^\mathcal{J} \) over \( \mathcal{I} \) implies that \( \text{rk}_M^\mathcal{J} \) can be restricted to the smaller domain \( \mathcal{I} \) without losing any information. In particular, significant discrepancy between \( \mathcal{I} \) and \( \mathcal{J} \) indicates \( \text{rk}_M^\mathcal{J} \) is mostly constant. Identifying such an \( \mathcal{I} \), ideally the minimal one, is crucial to minimize the computational burden as well as facilitating vectorization of the information for usage in machine learning pipeline [60, 81]. Consider, for instance, a finitely presentable \( \mathbb{R}^d \)-module \( M \). While the poset \( \mathcal{J} := (\text{Int}(\mathbb{R}^d), \subset) \) is not locally finite, \( \text{rk}_M^\mathcal{J} \) is completely encoded into \( \text{dg}_M^\mathcal{J} \) (Theorem C (iii)).

**Remark 3.4.** Further remarks on Definition 3.1 follow.

(i) (Monotonicity) Let \( \mathcal{J}' \subset \mathcal{J} \) and \( \mathcal{I} \subset \mathcal{I}' \). If \( \text{rk}_M^\mathcal{J} \) is Möbius invertible over \( \mathcal{I} \), then \( \text{rk}_M^\mathcal{J}' \) is Möbius invertible over \( \mathcal{I}' \).

(ii) Assume \( \mathcal{I} \subset \mathcal{J} \) and \( \mathcal{I} \) is finite. If \( \text{rk}_M^\mathcal{J} \) is \( \mathcal{I} \)-constructible, then by Proposition 2.14, \( \text{rk}_M^\mathcal{J} \) is Möbius invertible over \( \mathcal{I} \). The converse does not hold.
3.2 Rank decomposition can always be obtained via Möbius inversion

In this section, we demonstrate that any rank decomposition of the GRI can always be achieved through Möbius inversion, regardless of the local finiteness of the domain of the GRI. This stands in contrast to the method presented in [14], where the authors establish the uniqueness of the minimal rank decomposition (if it exists) through two different means, one of which is contingent upon the local finiteness of the domain of the GRI.

We assume $\mathcal{J} = \text{Int}(P)$ for ease of notation, bearing Remark 3.4 (i) in mind.

**Theorem A.** Given any $P$-module $M$, regardless of the locally finiteness of $\text{Int}(P)$, the following are equivalent.

(i) $\text{rk}^{\text{Int}} M$ is Möbius invertible.

(ii) $\text{rk}^{\text{Int}} M$ admits a rank decomposition.

(iii) $\text{rk}^{\text{Int}} M$ admits a unique minimal rank decomposition.

(iv) There exists a unique minimal $I \subset \text{Int}(P)$ over which $\text{rk}^{\text{Int}} M$ is Möbius invertible.

(v) There exists a unique function $d_M: \text{Int}(P) \to \mathbb{Z}$ such that for all $I \in \text{Int}(P)$, the set $\{ J \supset I : J \in \text{Int}(P), \ d_M(J) \neq 0 \}$ is finite and

$$\text{rk}^{\text{Int}} M(I) = \sum_{J \supset I} d_M(J). \quad (3.3)$$

**Proof of Theorem A.** The implications (i) ⇔ (iv) ⇔ (v) are direct from Proposition 3.2. (i) ⇒ (ii) follows from Proposition 2.27 and Theorem 2.28. (ii) ⇒ (i): Let $(\mathcal{R}, \mathcal{S})$ be a rank decomposition of the Int-GRI of $M$. Then, by Remark 2.24 (iv), every principal ideal of the poset $\mathcal{I} := \{ I \in \text{Int}(P) : I$ belongs to either $\mathcal{R}$ or $\mathcal{S} \}$ is finite. Define $d_M^R: \mathcal{I} \to \mathbb{Z}$ by $I \mapsto \text{mult}_\mathcal{R}(I) - \text{mult}_\mathcal{S}(I)$. Then, for all $I \in \text{Int}(P)$:

$$\text{rk} M(I) = (\text{rk}_{k_R} - \text{rk}_{k_S})(I) = \sum_{J \supset I \in \mathcal{I}} d_M^R(J).$$

By Definition 3.1, $\text{rk}^{\text{Int}} M$ is Möbius invertible.

(ii) ⇔ (iii) follows from Proposition 2.26.

(iii) ⇔ (iv) can be proved using a similar argument as in the proof of (i) ⇔ (ii). \qed

3.3 Möbius invertibility vs. other structural simplicity measures for persistence modules

In this section, we elucidate the relationship among various concepts pertaining to the structural simplicity of persistence modules, as depicted in Figure 1. Along the way, we find a persistence module over $\mathbb{Z}^2$ whose Int-GRI is not Möbius invertible (which implies that its Int-GRI admits no rank decomposition).

The following proposition is rather straightforward (cf. Definitions 2.3, 2.4, and 2.5):
Proposition 3.5. Constructible persistence modules are finitely presentable.

Proof. Let \( M : P \to \text{vec} \) be an S-constructible module with \( S \subset P \). Then, there exist an S-constructible P-module \( F_0 = \bigoplus_{i=1}^{n} k_{p_i} \) and a natural transformation \( \Psi^0 : F_0 \Rightarrow M \) such that each component is surjective, i.e. \( \Psi^0(a) : F_a \to M_a \) is surjective for each \( a \in P \) [47, Proposition 4.13]. It is easy to see that the kernel of \( \Psi^0 \) is also S-constructible and thus there exists another S-constructible P-module \( F_1 = \bigoplus_{i=1}^{m} k_{q_i} \) with a natural transformation \( \Psi^1 : F_1 \Rightarrow F_0 \) such that \( \text{im}\Psi^1 = \ker\Psi^0 \). Now we have that \( M \) is the cokernel of \( \Psi^1 \), as desired. \( \square \)

Proposition 3.6 ([69, Theorem 6.12, Remark 6.15]). Finitely presentable persistence modules are tame.

Proposition 3.7. There exists a non-tame P-module whose Int-GRI is Möbius invertible.

Proof. Let \( P \) be an infinite poset, and consider any interval-decomposable P-module \( M \) that has infinitely many summands, e.g. \( M = \bigoplus_{p \in P} k_{i(p)} \). Then, observe that \( M \) cannot be finitely encoded and thus \( M \) is not tame. However, by Theorem C (ii), its GRI is Möbius invertible. \( \square \)

We consider the dual statement:

Theorem B. There exists a tame P-module \( M \) whose Int-GRI is not Möbius invertible.

Before proving this, we remark the following: (i) By Remark 3.4 (i), this theorem implies that the Con-GRI of a tame persistence module is not necessarily Möbius invertible. (ii) Theorems A and B imply that the existence of a finite encoding of \( M \) does not imply the rank decomposability of the Int-GRI of \( M \).

In proving Theorem B, we will utilize the following well-known concrete formulation for the limit and colimit of any \( M : P \to \text{vec} \) (see, for instance, [52, Appendix E]):

Convention 3.8. (i) The limit of \( M \) is the pair \( (W, (\pi_p)_{p \in P}) \) described as:

\[
W := \left\{ (\ell_p)_{p \in P} \in \prod_{p \in P} M_p : \forall p \leq q \in P, \varphi_M(p, q)(\ell_p) = \ell_q \right\}
\]

where for each \( p \in P \), the map \( \pi_p : W \to M_p \) is the canonical projection. Elements of \( W \) are called sections of \( M \).

(ii) The colimit of \( M \) is isomorphic to a pair \( (C, (\iota_p)_{p \in P}) \) that is described as follows. For \( p \in P \), let \( \iota_p : M_p \to \bigoplus_{p \in P} M_p \) be the canonical injection. \( C \) is the quotient space \( \left( \bigoplus_{p \in P} M_p \right) / V \), where \( V \) is generated by the vectors \( \iota_p(v_p) - \iota_{p'}(v_{p'}) \) over all \( p \leq p' \) in \( P \), with \( v_p \in M_p, v_{p'} \in M_{p'} \). Letting \( f \) be the quotient map from \( \bigoplus_{p \in P} M_p \) to \( C \), for \( p \in P \), \( \iota_p : M_p \to C \) is the composition \( f \circ \iota_p \).

Proof of Theorem B. We prove the statement by constructing a \( Z^2 \)-module that is tame while its Int-GRI is not Möbius invertible. Let \( N \) be the persistence module defined in Figure 3 (B), over the 6-point poset which we denote \( Q \). Define the order-preserving map \( \pi : Z^2 \to Q \) by mapping like colors, i.e. in Figure 3 red points in (A) map to red points in (B), blue points in (A) map to blue points in (B), etc. Define \( M := \pi^* N \), which is visualized in Figure 3 (A). It is immediate by this definition that \( M \) is tame.
Now we show that \( \text{rk}_{M}^{\text{int}} \) is not Möbius invertible. Define the interval \( I_0 \) as follows:

\[
I_0 := \{ (x, y) : y \leq -x \} \cup \{ (-2i - 1, 2i + 2) \}_{i=0}^{\infty} \cup \{ (2i + 2, -2i - 1) \}_{i=0}^{\infty}.
\]

Then for \( a \in \mathbb{Z} \), define the interval \( I_a \) as the shift of \( I_0 \) by \((a, -a)\). \( I_a \) contains the part of \( \mathbb{Z}^2 \) at or below \( y = -x \), and skips every other point along the line \( y = -x + 1 \) except for skipping two consecutive points at \((a, -a + 1)\) and \((a + 1, -a)\). See Figure 3 (C) for a visualization.

**Claim 1.** For each \( a \in \mathbb{Z} \), \( \text{rk}_{M}^{\text{int}}(I_a) = 1 \).

**Proof.** Following Convention 3.8, we first describe a nonzero section of \( M|_{I_a} \). In fact, a basis for the only section of \( M|_{I_a} \) which is nonzero everywhere consists of \( 1 \in k \) for every copy of \( k \) appearing in \( M|_{I_a} \), and \((1, 1) \in k^2 \) for every \( k^2 \) appearing in \( M|_{I_a} \). It is immediate to see this satisfies the definition (Convention 3.8) to be a section. It is straightforward to check that the image of this section in the colimit is nonzero, and as a result \( \text{rk}_{M}^{\text{int}}(I_a) \geq 1 \).

Lastly, as \((−1, −1) \in I_a \) and \( M|_{−1,−1} = k \) has dimension 1, we know \( \text{rk}_{M}^{\text{int}}(I_a) \leq 1 \), and thus \( \text{rk}_{M}^{\text{int}}(I_a) = 1 \).

**Claim 2.** For \( a \in \mathbb{Z} \) and \( J \in \text{Con}(P) \) such that \( J \supseteq I_a \), \( \text{rk}_{M}^{\text{con}}(J) = 0 \).

**Proof.** Let \( J \in \text{Con}(P) \) such that \( J \supseteq I_a \) for some fixed \( a \in \mathbb{Z} \). Then there are two possibilities. First, suppose \( J \) contains a point \((x, y)\) with \( y > -x + 1 \). In this case, it is immediate \( \text{rk}_{M}^{\text{con}}(J) = 0 \) as \( M(x, y) = 0 \).

In the second case, suppose \( J \) contains a point \((x, y) \notin I_a \) with \( y = -x + 1 \). By definition of \( I_a \), we have that at least one of \((x + 1, y − 1)\) or \((x − 1, y + 1)\) must be in \( J \) as well. This creates a sub-diagram of \( M|_{J} \) of the form \( k \xleftarrow{[0 \ 1]} k^2 \xrightarrow{[1 \ 0]} k \). The rank over this diagram is 0, and thus it follows by Remark 2.19 (i) that \( \text{rk}_{M}^{\text{con}}(J) = 0 \).

Assume by way of contradiction that \( \text{rk}_{M}^{\text{int}} \) is Möbius invertible, i.e. that there exists a map \( d_M : \text{Int}(\mathbb{Z}^2) \to \mathbb{Z} \) as stated in Theorem A (v).

We claim that for any \( a \in \mathbb{Z} \) there is no \( J \in \text{Int}(\mathbb{Z}^2) \) such that \( I_a \nsubseteq J \) and \( d_M(J) \neq 0 \). Suppose not, i.e. there exists a fixed \( a \in \mathbb{Z} \) and \( J_1 \in \mathcal{I} \) with \( I_a \nsubseteq J_1 \) and \( d_M(J_1) \neq 0 \).
By Claim 2, we know \( \text{rk}_M(J_1) = 0 \), but as \( d_M(J_1) \neq 0 \), this alongside Equation (3.3) implies there must exist \( J_2 \in I \) with \( J_1 \subsetneq J_2 \), and \( d_M(J_2) \neq 0 \). Again, Claim 2 alongside Equation (3.3) imply the existence of a \( J_3 \in I \) with \( J_2 \subsetneq J_3 \) and \( d_M(J_3) \neq 0 \). Repeating this procedure, it follows that there exists an infinite ascending chain of intervals \( I_1 \subsetneq I_2 \subsetneq \ldots \), such that \( d_M(J_i) \neq 0 \) for all \( i \geq 1 \). This implies that the set \( \{ J \supset I_a : J \in \text{Int}(P), \ d_M(J) \neq 0 \} \) is infinite, contradicting Theorem A (v).

Therefore, for any \( a \in Z \) there is no \( J \in I \) such that \( J \supsetneq I_a \) and \( d_M(J) \neq 0 \). This fact alongside Claim 1 implies that \( d_M(I_a) = 1 \).

Now, observe that for all \( a \in Z \), the interval \( I_a \) contains the point \((-1, -1) \in Z^2 \). This implies that for the singleton interval \( I = \{ (-1, -1) \} \), the set \( \{ J \supset I_a : J \in \text{Int}(P), \ d_M(J) \neq 0 \} \) is infinite, contradicting Theorem A (v) and completing the proof. \( \square \)

Remark 3.9. The module \( M : Z^2 \to \text{vec} \) from the previous gives rise to infinitely many modules which have non-Möbius invertible Int-GRI’s. For example, let \( P \) be an any poset into which \( Z^2 \) can be embedded via some \( f : Z^2 \hookrightarrow P \) such that \( f(Z^2) \in \text{Int}(P) \) (e.g. \( P = Z^d \) for \( d > 2 \)). Define a \( P \)-module \( N \) by setting \( N|_{f(Z^2)} \) as the push-forward of \( M \) along \( f \), and setting \( N := 0 \) outside of \( f(Z^2) \). Then \( N \) has a non-Möbius invertible Int-GRI.

As another example, we can naturally extend \( M \) to an \( \mathbb{R}^2 \)-module \( M' \) where \( M'_p := M_{(p)Z^2} \) for all \( p \in Z^2 \). Then, the Int-GRI of \( M \) is not Möbius invertible.

Remark 3.10. For the \( Z^2 \)-module \( M \) considered in the previous proof, we observe the following.

(i) We have actually showed that the Int-GRI of \( M \) is not Möbius invertible over any subcollection of \( \text{Con}(P) \) containing \( \text{Int}(P) \).

(ii) By Item (i) and Remark 3.4 (i), the Con-GRI of \( M \) is also not Möbius invertible.

(iii) The map \( \pi \) is not ‘rank-preserving’, e.g. \( \text{rk}_M(I_a) = 1 \neq 0 = \text{rk}_N(\pi(I_a)) \) for each \( a \in Z \).

It is somewhat surprising that even for a persistence module \( M \) that is simple enough to admit a finite encoding, both the Con-GRI and Int-GRI of \( M \) might not be Möbius invertible. Motivated by Remark 3.10 (iii), in the next section, we adapt the notion of finite encoding, aimed at ensuring the Möbius invertibility of the GRI.

### 3.4 Sufficient conditions for Möbius invertibility of the GRI

In this section, we explore several conditions on persistence modules \( M \) that ensure the Möbius invertibility of the Con- or Int-GRI of \( M \).

Let \( \pi : P \to Q \) be an order-preserving map. If for all \( I \in \text{Int}(P) \), the image \( \pi(I) \) belongs to \( \text{Int}(Q) \), then we say \( \pi \) is interval-preserving. Let \( M \) be a \( P \)-module and \( N \) be a \( Q \)-module. We say that \( \pi \) is Int-rank-preserving (from \( M \) to \( N \)), if \( \forall I \in \text{Int}(P), \ \text{rk}_M(I) = \text{rk}_N(\pi(I)) \).

**Theorem C.** Each of the following assumptions implies that the Int-GRI of a given \( P \)-module \( M \) is Möbius invertible over intervals.

(i) \( M \) is interval-decomposable.

(ii) There exist a finite poset \( Q \), an order-preserving map \( \pi : P \to Q \), and a \( Q \)-module \( N \) such that \( \pi \) is interval-preserving and Int-rank-preserving.

(iii) \( P = \mathbb{R}^d \) and \( M \) is finitely presentable (we refine this statement in Proposition 3.17).
Remark 3.11. Although the condition on the map \( \pi \) given in Item (ii) seems weaker than that of a finite encoding, it is actually not: see Remark 3.10 (iii). Conversely, \( \pi \) described in Item (ii) is also not necessarily a finite encoding: Consider any pair of persistence modules \( M, N \) over the same finite poset \( P \) that are not isomorphic, but have the same Int-GRI. Then the identity map \( \text{id}_P \) on \( P \) is interval-preserving and Int-rank-preserving, but \( \text{id}_P \) is not a finite encoding.

Nevertheless, natural candidates for such \( \pi \) are finite encodings: see Example 3.16.

Theorem C (i) and (ii) respectively correspond to implications (3) and (4) of Figure 1. Within the proof of Theorem C (iii), Implication (2) of Figure 1 is established.

Now we prove Theorem C. Item (i) is straightforward: Define \( d_M : \text{Int}(P) \to \mathbb{Z} \) by sending each interval \( I \) to the multiplicity of \( I \) in the barcode of \( M \). Then, the function \( d_M \) satisfies the condition given in item (v) of Theorem A, completing the proof. In proving Item (ii), the proposition below is useful.

**Proposition 3.12.** Let \( \pi : P \to Q \) be an order-preserving map and let \( J \) be an interval of \( Q \). Then \( \pi^{-1}(J) \) is a disjoint union of intervals of \( P \).

**Proof.** It suffices to show that \( \pi^{-1}(J) \) is convex (Definition 2.1 (i)). Let \( p, r \in \pi^{-1}(J) \) and \( q \in P \) with \( q \in [p, r] \). Since \( \pi \) is order-preserving, we have that \( \pi(q) \in [\pi(p), \pi(r)] \subset J \). Hence, \( q \) belongs to \( \pi^{-1}(J) \), as desired. \( \square \)

**Proof of Theorem C (ii).** Since \( Q \) is finite, the set \( \pi^{-1}\text{Int}(Q) := \{ \pi^{-1}(J) : J \in \text{Int}(Q) \} \) is finite. For \( J \in \text{Int}(Q) \), let \( C(\pi^{-1}(J)) \) be the set of connected components of \( \pi^{-1}(J) \). This implies that any pair of elements in \( C(\pi^{-1}(J)) \) are disjoint. Also, by Proposition 3.12, every element of \( C(\pi^{-1}(J)) \) is in \( \text{Int}(P) \).

Let \( \mathcal{I} := \bigcup_{J \in \text{Int}(Q)} C(\pi^{-1}(J)) \). For each \( I \in \text{Int}(P) \), the set \( \mathcal{I}_{\geq 1} := \{ J \in \mathcal{I} : J \supset I \} \) is finite, because for each \( J \in \text{Int}(Q) \), the intersection \( \mathcal{I}_{\geq 1} \cap C(\pi^{-1}(J)) \) contains at most one element. This proves that \( \text{rk}_M^{\mathcal{I}} \) is convolvable.

Next, for \( I \in \text{Int}(P) \), let \( I^\mathcal{I} \in \mathcal{I} \) be the connected component of \( \pi^{-1}(\pi(I)) \) containing \( I \). Since \( \pi(I^\mathcal{I}) = \pi(I) \) and \( \pi \) is Int-rank-preserving, we have \( \text{rk}_M(I) = \text{rk}_M(I^\mathcal{I}) \). Also, observe that for any \( J \in \mathcal{I} \), we have \( I \subset J \) if and only if \( I^\mathcal{I} \subset J \). Therefore,

\[
\text{rk}_M(I) = \text{rk}_M(I^\mathcal{I}) = \sum_{J \in \mathcal{I} : I \subset J} \text{dgm}_M(J) = \sum_{J \in \mathcal{I} : I^\mathcal{I} \subset J} \text{dgm}_M(J),
\]

i.e. \( \text{rk}_M^{\mathcal{I}} \) is Möbius invertible over \( \mathcal{I} \subset \text{Int}(P) \), as desired. \( \square \)

We now prove Theorem C (iii), building on Theorem C (ii). When a given \( \mathbb{R}^d \)-module \( M \) is finitely presentable, a natural finite encoding of \( M \) exists. Proving that the finite encoding is Int-rank-preserving demands careful scrutiny.

**Proof of Theorem C (iii).** Let \( M \) be a finitely presentable \( \mathbb{R}^d \)-module. By assumption, \( M \) is the cokernel of a morphism \( \bigoplus_{a \in A} k_{a!} \to \bigoplus_{b \in B} k_{b!} \) where \( A \) and \( B \) are some finite multisets of elements from \( \mathbb{R}^d \). Let \( G_1, \ldots, G_d \) be finite subsets of \( \mathbb{R} \) such that \( Q' = \Pi_{i=1}^d G_i \subset \mathbb{R}^d \) is the smallest grid including all the elements in \( A \) and \( B \). Let \( q_i \) be the unique minimal element of \( Q' \). Let \( Q := Q' \cup \{ -\infty \} \), where we declare that \( -\infty < q \) for all \( q \in Q' \). We define the \( Q \)-module \( N \) by \( N_{Q'} := M_{|Q'} \) and \( N_{-\infty} := 0 \).

\[8\] It is noteworthy that \( M \) is the left-Kan extension of \( N \) along the canonical inclusion \( Q' \hookrightarrow \mathbb{R}^d \).
Let \([-\cdot]_Q : \mathbb{R}^d \to Q\) be the map sending each \(p \in \mathbb{R}^d\) to the maximal \(q \in Q\) such that \(q \leq p\) (we declare that \(-\infty < p\) for all \(p \in \mathbb{R}^d\)). By Theorem C (ii), it suffices to show that \([-\cdot]_Q\) is interval-preserving, and \(\text{Int}\)-rank-preserving with respect to \(M\) and \(N\). It is not difficult to see that \([-\cdot]_Q\) is interval-preserving, and thus we only show that \([-\cdot]_Q\) is \(\text{Int}\)-rank-preserving. First, we make the following observations regarding the map \([-\cdot]_Q\).

1. For any \(p \leq p' \in \mathbb{R}^d\), if \([p]_Q = [p']_Q\), then \(\varphi_M(p, p') : M_p \to M_{p'}\) is the identity.

2. \(\mathbb{R}^d\) is partitioned into the preimages \(B_q\) of \(q \in Q\) under \([-\cdot]_Q\). In particular,

\[
B_q = \begin{cases} 
\mathbb{R}^d \setminus q_i^\uparrow, & \text{if } q = -\infty \\
\prod_{i=1}^d [a_i, b_i), & \text{if } q \neq -\infty,
\end{cases}
\]

for some intervals \([a_i, b_i) \subset \mathbb{R}\). Each \(B_q\) will be called a block.

Now it remains to prove:

\[
\text{rk}_M(I) = \text{rk}_N([I]_Q) = 0. 
\]

for all intervals \(I \subset q_i^\uparrow\), \(\text{rk}_M(I) = \text{rk}_N([I]_Q)\).

Let \(I \in \text{Int}(\mathbb{R}^d)\) with \(I \subset q_i^\uparrow\). Let \(\Psi_M\) be the canonical limit-to-colimit map over \(M|_I\) and \(\Psi_N\) be the canonical limit-to-colimit map over \(N|_{[I]_Q}\). By the rank-nullity theorem, it suffices to prove that \(\varprojlim M|_I \cong \varprojlim N|_{[I]_Q}\) and \(\ker(\Psi_M) \cong \ker(\Psi_N)\).

Consider the map \(\Phi : \varprojlim M|_I \to \varprojlim N|_{[I]_Q}\) by, for each \((\ell_p)_{p \in I} \in \varprojlim M|_I\) (cf. Convention 3.8),

\[
\Phi((\ell_p)_{p \in I}) := (j_q)_{q \in [I]_Q},
\]

where \(j_q := \ell_p\) for some \(p \in I\) such that \([p]_Q = q\) (see Figure 4 (A)). It suffices to prove the following subclaims:

**Subclaim 1.** \(\Phi\) is well-defined.

**Subclaim 2.** \(\Phi\) is a linear isomorphism.

**Subclaim 3.** The restriction of \(\Phi\) to \(\ker(\Psi_M)\) is a bijection with \(\ker(\Psi_N)\).

**Proof of Subclaim 1.** Obs.1 implies that \(j_q\) does not depend on the choice of \(p\). Next, we show that for \((\ell_p)_{p \in I} \in \varprojlim M|_I\), \(\Phi((\ell_p)_{p \in I}) \in \varprojlim N|_{[I]_Q}\). To this end, it suffices to show that if \(q \leq q' \in [I]_Q\), then \(\varphi_N(q, q')(j_q) = j_{q'}\). Let \(p, p' \in I\) such that \([p]_Q = q\) and \([p']_Q = q'\). If \(p \leq p'\), then \(\varphi_N(q, q')(j_q) = j_{q'}\) immediately follows from the fact that \(\varphi_M(p, p') (\ell_p) = \ell_{p'}\).

Thus, suppose that there is no comparable pair \(p, p' \in I\) with \([p]_Q = q\) and \([p']_Q = q'\). For an example of this scenario, see Figure 4 (A). Since \([p]_Q < [p']_Q\), we must have that \(p \in B_1\) and \(p' \in B_n\) are in different blocks \(B_1 \neq B_n\). We say two blocks \(B \neq B'\) are adjacent if \(B = \prod_{i=1}^d [a_i, b_i)\) and \(B' = \prod_{i=1}^d [a_i', b_i')\), with \(a_i = a_i'\) and \(b_i = b_i'\) for all but one \(1 \leq i \leq d\), for which \(b_i = a_i'\). If \(B\) and \(B'\) are adjacent, with \(a_i < a_i'\) for some \(1 \leq i \leq d\), then we write \(B \leq B'\). As \(I\) is an interval, there exists a finite sequence of adjacent blocks \(B_1 \leq B_2 \leq \ldots \leq B_{n-1} \leq B_n\) that each intersects \(I\).

Observe that the convexity of \(I\) implies if \(B\) and \(B'\) are adjacent blocks, there must exist \(p \in B \cap I\) and \(p' \in B' \cap I\) such that \(p \leq p'\). Thus, for \(2 \leq i \leq n - 1\), we can find pairs of points \(p, p' \in B_i \cap I\) such that \(p_{i-1} \leq p_i\) for \(2 \leq i \leq n\), further having the property that \(p_i = p\) and \(p_{n-1} = p'\). As noted before, since \(p, p'\) are in the same block, we have...
is well-defined, we demonstrated the existence of a (non-unique) path in \( I \) connecting \( p \) and \( q \). For an example of such a path, see Figure 4 (A).

Let \( q_i := [p_i]_Q \) for \( 1 \leq i \leq n \). It is clear that we have \( q = q_1 \leq q_2 \leq \ldots \leq q_{n-1} \leq q_n = q' \). By the fact that \((\ell_p)_p\in I \) is a section, we have \( \varphi_M(p_{i-1}, p_i)(l_{p_{i-1}}) = l_p \), for \( 2 \leq i \leq n \), which implies \( \varphi_N(q_{i-1}, q_i)(j_{q_{i-1}}) = j_{q_i} \), for \( 2 \leq i \leq n \). By composing the internal morphisms in \( N \), this gives us \( \varphi_N(q, q')(j_q) = \varphi_N(q, q_n)(j_{q_i}) = j_{q_n} = j_q' \). Hence, \( \Phi((\ell_p)_p) = (j_q)_q\in I_Q \in \lim N|_{I_Q} \). For an example of \( \Phi \), see Figure 4 (B).

**Proof of Subclaim 2.** We construct the inverse \( \Phi^{-1} : \lim N|_{I_Q} \to \lim M|_{I} \). For \( (j_q)_q\in I_Q \) a section in \( \lim N|_{I_Q} \), define \( \Phi^{-1}((j_q)_q) := (\ell_p)_p \), where \( \ell_p := j_q \) for \( q = [p]_Q \).

It is straightforward to see that \( \Phi^{-1} \) maps sections to sections as if \( p \leq p' \in I \) then \( [p]_Q \leq [p']_Q \). Lastly, it is immediate to see by the definitions that \( \Phi \circ \Phi^{-1} \) is the identity on \( \lim N|_{I_Q} \) and \( \Phi^{-1} \circ \Phi \) is the identity on \( \lim M|_{I} \). \( \square \)

**Proof of Subclaim 3.** We have already demonstrated that \( \Phi \) injective, so it remains to show that \( \Phi(\ker \Psi_M) = \ker \Psi_N \). To see this, suppose \( (\ell_p)_p \in \ker \Psi_M \). By Lemma B.2, this means that there exists \( p^* \in I \) and a path \( \Gamma \) of comparable elements in \( I \) connecting \( p^* \) to some \( p' \in I \), and a section \((m_p)_p\in I \) in \( \lim M|_{I} \) with \( m_{p'} = \ell_{p'} \) and \( m_{p^*} = 0 \). Define \( (j_q)_q\in I_Q := \Phi((\ell_p)_p) \) and \( (h_q)_q\in I_Q := \Phi((m_p)_p) \). Then if we denote \( q^* := [p^*]_Q \), \( q' := [p']_Q \) and \( \Gamma_Q := [\Gamma]_Q \), we see that \( h_{q^*} = \ell_{p^*} \), \( h_{q'} = 0 \), and \( \Gamma_Q \) is a path of comparable elements in \( I_Q \) connecting \( q^* \) to \( q' \), so \( \Psi_N((j_q)_q) = 0 \). Hence, \( \Phi((\ell_p)_p) \in \ker \Psi_N \), meaning \( \Phi(\ker \Psi_M) \subset \ker \Psi_N \).

On the flip side, suppose we have a section \((j_q)_q\in I_Q \in \ker \Psi_N \). Again, this means that there exists \( q^* \in [I]_Q \) a path of comparable elements in \( [I]_Q \) connecting \( q^* \) to some \( q' \in [I]_Q \), and a section \((h_q)_q\in I_Q \) such that \( h_{q^*} = j_{q^*} \) and \( h_{q'} = 0 \). We can define \( (\ell_p)_p := \Phi^{-1}((j_q)_q) \) and \( (m_p)_p := \Phi^{-1}((h_q)_q) \). Fix \( p^*, p' \in I \) such that \( [p^*]_Q = q^* \) and \( [p']_Q = q' \). Then we have \( \ell_{p^*} = m_{p^*} \) and \( m_{p'} = 0 \). When we showed \( \Phi \) is well-defined, we demonstrated the existence of a (non-unique) path \( \Gamma \) in \( I \) connecting
\( p^* \) and \( p' \), and so \((\ell_p)_{p \in I} \in \ker \Psi_M \). As \( \Phi^{-1} \) is the inverse of \( \Phi \), this implies \( \Phi(\ker \Psi_M) \supset \ker \Psi_M \), and so \( \ker \Psi_M \cong \ker \Psi_N \). \( \square \)

**Remark 3.13.** By Remark 3.4 (i) and Theorem C (iii), the GRI over any \( J \subset \text{Int}(\mathbb{R}^d) \) of any finitely presentable \( \mathbb{R}^d \)-module \( M \) is Möbius invertible over intervals (equivalently, by Remark 3.3 (i), \( \text{rk}_J \) admits a rank decomposition). This generalizes the fact that the RI of any finitely presentable \( \mathbb{R}^d \)-module admits a rank decomposition [14, Corollary 5.6].

For the Con-GRI, the statement analogous to Theorem C (ii) also holds:

**Proposition 3.14.** Let \( M \) be a \( P \)-module. If there exists a finite poset \( Q \), a poset morphism \( \pi : P \to Q \) and \( Q \)-module \( N \) such that \( \pi \) is rank-preserving, i.e. for all \( I \in \text{Con}(P) \), \( \text{rk}_M(I) = \text{rk}_N(\pi(I)) \), then \( \text{rk}_M^{\text{Con}} \) is Möbius invertible.

**Proof.** For any \( J \subset Q \), let \( C(\pi^{-1}(J)) \) be the set of connected components of \( \pi^{-1}(J) \). This implies \( C(\pi^{-1}(J)) \subset \text{Con}(P) \). Define \( I \subset \text{Con}(P) \) as \( I := \bigcup_{J \in \text{Con}(Q)} C(\pi^{-1}(J)) \). From here, the argument that \( \text{rk}_M^{\text{Con}} \) is Möbius invertible over \( I \) follows the same as the proof of Theorem C (ii) upon replacing all instances of \( \text{Int}(-) \) with \( \text{Con}(-) \). \( \square \)

Interestingly, the statement analogous to Theorem C (iii) for the Con-GRI is false:

**Remark 3.15.** There exist finitely presentable \( \mathbb{R}^d \)-modules whose Con-GRIs are not Möbius invertible. The construction of such an example is parallel to the one described in the proof of Theorem B, and thus we omit it. In particular, when constructing such an example, the map \([-1]_Q \) considered in the proof of Theorem C (iii) is not Con-rank-preserving, and thus we cannot utilize Proposition 3.14 in that case.

**Pulling-back of the GPD.** Theorem C (ii) and Proposition 3.14 show that the presence of a rank-preserving order-preserving map \( \pi : P \to Q \) with \( |Q| < \infty \) ensures the Möbius invertibility of the GRI. Next, we will demonstrate that by imposing an additional assumption on \( \pi \), the GPD of \( M \) can be fully determined by the GPD of \( N \).

An order-preserving map \( \pi : P \to Q \) is called a covering if \( \pi \) is surjective and for each \( J \in \text{Con}(Q) \), \( \pi^{-1}(J) \) is a disjoint union of connected subposets \( I \subset P \) such that \( \pi(I) = J \). We say that \( \pi \) is Con-rank-preserving if for all \( I \in \text{Con}(P) \), \( \text{rk}_M(I) = \text{rk}_M(\pi(I)) \).

**Theorem D.** Let \( M \) be a \( P \)-module and let \( N \) be a \( Q \)-module with a covering \( \pi : P \to Q \) that is Con-rank-preserving. Then, the Con-GRI of \( M \) is Möbius invertible, and for \( I \in \text{Con}(P) \),

\[
\text{dgm}_M(I) = \begin{cases} 
\text{dgm}_N(\pi(I)), & \text{if } I \text{ is a connected component of } \pi^{-1}(\pi(I)) \\
0, & \text{otherwise}
\end{cases}
\]

Also, the analogous statement obtained by replacing every Con by Int holds, under the extra assumption that \( \pi \) is interval-preserving.

**Proof.** We only prove the statements for the Con-GRI. Because \( \pi \) is a covering and rank-preserving, we can extend the domain \( \text{Con}(P) \) of \( \text{rk}_M \) to \( \text{Con}(P) \cup \pi^{-1}(\text{Con}(Q)) \) by:

\[
I \mapsto \begin{cases} 
\text{rk}_M(I), & I \in \text{Con}(P) \\
\text{rk}_M(I'), & I \in \pi^{-1}(\text{Con}(Q)) \text{ and } I' \text{ is any connected component of } I.
\end{cases}
\]

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Indeed, this map is well-defined even on the intersection $\text{Con}(P) \cap \pi^{-1}\text{Con}(Q)$. For the containment relation $\supset$ on $\text{Con}(P) \cup \pi^{-1}\text{Con}(Q)$, we claim that $\text{rk}_M$ is $\pi^{-1}\text{Con}(Q)$-constructible. Indeed, the rank-preserving property of $\pi$ and the definition of $\text{rk}_M$ on $\pi^{-1}\text{Con}(Q)$ imply that for any $I \in \text{Con}(P)$,

$$\text{rk}_M(I) = \text{rk}_N(\pi(I)) = \text{rk}_M(\pi^{-1}\pi(I)),$$

showing that the map $\pi^{-1}\pi$ on $\text{Con}(P) \cup \pi^{-1}\text{Con}(Q)$ is the co-closure with image $\pi^{-1}\text{Con}(Q)$. Since $Q$ is finite, $\pi^{-1}\text{Con}(Q)$ is finite. Therefore, by Proposition 2.14, the Con-GRI of $M$ is Möbius invertible over $\pi^{-1}\text{Con}(Q)$, i.e. there exists a function $d_M : \pi^{-1}\text{Con}(Q) \to \mathbb{Z}$ such that

$$\text{rk}_M(I) = \sum_{L \supset I, \pi^{-1}\text{Con}(Q)} d_M(L), \ \forall I \in \text{Con}(P). \tag{3.5}$$

Now, consider the subcollection of $\text{Con}(P)$ given by

$$\mathcal{I} := \{J : J \text{ is a connected component of an element in } \pi^{-1}\text{Con}(Q)\}.$$ 

Since $\pi^{-1}\text{Con}(Q)$ is finite, for each $I \in \text{Con}(P)$, we have that

$$\mathcal{I}_{\supset I} := \{J \supset I : J \text{ is a connected component of an element in } \pi^{-1}\text{Con}(Q)\}$$

is finite. Now let us define $\bar{d}_M : \text{Con}(P) \to \mathbb{Z}$ by

$$I \mapsto \begin{cases} d_M(\pi^{-1}\pi(I)), & I \in \mathcal{I} \\ 0, & \text{otherwise.} \end{cases}$$

We claim that

$$\text{rk}_M(I) = \sum_{J \supset I, \mathcal{I}_{\supset I}} \bar{d}_M(J), \ \forall I \in \text{Con}(P). \tag{3.6}$$

Fix any $I \in \text{Con}(P)$. The RHS contains only finite many nonzero summands since $\mathcal{I}_{\supset I}$ is finite. For any $L$ containing $I$ with $L \in \pi^{-1}\text{Con}(Q)$, there exists a unique connected component $J \in \text{Con}(P)$ of $L$ containing $I$. Furthermore, by construction, we have $\bar{d}_M(J) = d_M(L)$. This proves that the sums given in Equations (3.5) and (3.6) coincide.

Note that, since $\pi$ is surjective, the two maps

$$\pi : \pi^{-1}\text{Con}(Q) \supseteq \text{Con}(Q) : \pi^{-1}$$

are inverse to each other and order isomorphisms. Therefore, $d_M(\pi^{-1}(I)) = \text{dgm}_N(I)$ for all $I \in \text{Con}(Q)$. This implies that

$$\bar{d}_M(I) = \begin{cases} \text{dgm}_N(\pi(I)), & I \in \mathcal{I} \\ 0, & \text{otherwise,} \end{cases}$$

completing the proof. \hfill \Box

**Example 3.16.** Consider the $\mathbb{Z}^2$-module $M$ defined as follows:

$$M_{(x,y)} = \begin{cases} 0, & y < -x \\ k^2, & y = -x \\ k, & y > -x, \end{cases}$$

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where $M$ is constant on $y > -x$ and each map $k^2 \to k$ in $M$ is the projection $p_1$ to the first factor. Although $M$ is not finitely presentable, $M$ is tame: Let $N$ be the $\{1 < 2 < 3\}$-module given by $0 \to k^2 \xrightarrow{1} k$. There exists the obvious order-preserving map $\pi : \mathbb{Z}^2 \to \{1, 2, 3\}$ such that $M = \pi^* N$. This $\pi$ is a covering and Int-rank-preserving. Thus, by the previous theorem, not only $dgm_M$ exists, but also $dgm_M$ is directly obtained from $dgm_N$.

Another way to show that the GRI of $M$ is Möbius invertible is to prove that $M$ is interval-decomposable, and invoke Theorem C (i).

By exploiting Theorem D, we refine Theorem C (iii) by specifying the precise types of intervals that appear in the support of the Int-GPD of a finitely presentable $\mathbb{R}^d$-module.

Proposition 3.17. The Int-GRI of any finitely presentable $\mathbb{R}^d$-module is Möbius invertible over a finite subset of $\text{Int}_{m,n}(\mathbb{R}^d)$ (cf. Equation (2.14)) for large enough $m, n \in \mathbb{N}$.

In plain words, the Int-GRI of any finitely presentable $\mathbb{R}^d$-module admits a compact encoding as a ‘persistence diagram’ consisting of elements from $\text{Int}_{m,n}(\mathbb{R}^d)$. This result sheds light on computational aspects of the Int-GRI and also on its compact encoding.

Proof. Let $\pi : \mathbb{R}^d \to Q$ be the order-preserving map from the proof of Theorem C (iii), where $Q = Q' \cup \{-\infty\}$. The map $\pi$ is a covering that is both interval-preserving and Int-rank-preserving. Also, observe that there exist $m, n \in \mathbb{N}$ (which depend on the size of the grid $Q'$) such that for all $I \in \text{Int}(\mathbb{R}^d)$, $\pi^{-1}(1)$ belongs to $\text{Int}_{m,n}(\mathbb{R}^d)$. Now, the statement immediately follows from Theorem D.

\section{Discriminating power of the generalized rank invariant}

In this section, we investigate the discriminating power of the GRI from various perspectives: Section 4.1 recalls known results, and Section 4.2 presents new results. In Section 4.3, we restrict our attention to the setting of 2-parameter persistence modules. In Section 4.4, we compare the GRI on various domains with other invariants of persistence modules.

\subsection{Known results}

In this section, we review known results regarding the discriminating power of the GRI.

Proposition 4.1. Let $M$ be any interval-decomposable $P$-module. Then, for each $I \in \text{Con}(P)$, $\text{rk}_M(I)$ is equal to the total multiplicity of intervals $J \in \text{barc}(M)$ such that $J \supset I$.

Proof. When $P$ is finite, Remark 2.19 (ii) and (iii) directly imply the claim. The claim still holds even without these assumptions, see [52, Proposition 3.17] and [14, Proposition 2.1]. \hfill \Box

Theorem 4.2. Assume that the GRI of a P-module $M$ is convolvable over $I \subset \text{Int}(P)$. If $M$ is $I$-decomposable, then for all $I \in I$, $dgm_M(I)$ is equal to the multiplicity of $I$ in $\text{barc}(M)$.

The proof of Theorem 4.2 below is along the same lines as that of [40, Theorem 9].
Proof. For \( J \in \mathcal{I} \), let \( m_J \) be the multiplicity of \( J \) in \( \text{barc}(M) \). We have \( M \cong \bigoplus_{J \in \mathcal{I}} (k_J)^{m_J} \) where \( (k_J)^{m_J} \) denotes the direct sum of \( m_J \) copies of \( k_J \). Then, by Proposition 4.1, we have that
\[
\text{rk}_M(I) = \sum_{J \supseteq I} m_J \quad \forall I \in \mathcal{I}.
\]
By the uniqueness of \( \text{dgm}_M^\mathcal{I} \) (Theorem 2.10 and Definition 2.21), we have that \( \text{dgm}_M^\mathcal{I}(I) = m_I \) for all \( I \in \mathcal{I} \).

**Corollary 4.3.** Let \( \mathcal{I} \subset \text{Int}(P) \) be such that every principal ideal of \( \mathcal{I} \) is finite (and thus \( \mathcal{I} \) is locally finite). Then, the GRI over \( \mathcal{I} \) is a complete invariant of \( \mathcal{I} \)-decomposable \( P \)-modules.

We emphasize that this corollary is a direct consequence of the Möbius inversion formula. Depending on the poset structures of \( \text{Int}(P) \) and \( \mathcal{I} \subset \text{Int}(P) \), the GRI over \( \mathcal{I} \) can actually be a complete invariant over an even larger collection than the collection of all \( \mathcal{I} \)-decomposable modules. Let \( \widehat{\mathcal{I}} \) be the **limit completion** of \( \mathcal{I} \), i.e. the collection of all unions of nested families of intervals in \( \mathcal{I} \). In other words,
\[
\widehat{\mathcal{I}} := \left\{ \bigcup_{x \in X} I_x : X \text{ is totally ordered, } I_x \in \mathcal{I} \text{ and } I_x \subset I_y \text{ for all } x \leq y \text{ in } X \right\}, \quad (4.1)
\]
which is a subcollection of \( \text{Int}(P) \) containing \( \mathcal{I} \). For example, if \( P = \mathbb{R} \) and \( \mathcal{I} := \{[0, a] \in \text{Int}(\mathbb{R}) : a \in [0, 1]\} \), then \( \widehat{\mathcal{I}} = \mathcal{I} \cup \{[0, b) : b \in (0, 1]\} \).

**Proposition 4.4** (Restatement of [14, Proposition 2.10]). Let \( \mathcal{I} \subset \text{Int}(P) \). Then, the GRI over \( \mathcal{I} \) is a complete invariant on the collection of \( P \)-modules \( M \) that are \( \widehat{\mathcal{I}} \)-decomposable.\(^9\)

### 4.2 Optimality of Completeness, and Extent of Incompleteness

This section presents three key results concerning the discriminating power of the GRI:

- **Theorem E**, which establishes the completeness of the GRI for a specific class of persistence modules,
- **Theorem F**, which demonstrates the optimality of the previous completeness result in a suitable sense, and
- **Theorem G** and Corollary 4.9, which quantify the failure of the GRI to be complete on a collection of persistence modules.

We emphasize that the Möbius inversion formula plays a central role in the proofs of all of these theorems, even without the assumption of local finiteness on the domain of the GRI.

First, we generalize Theorem 4.2 and Corollary 4.3 by dispensing with redundant assumptions on \( \mathcal{I} \in \text{Int}(P) \).

**Theorem E** (Completeness). Let \( P \) be any poset. Then,

\(^9\)The original statement includes the assumption \( \text{rk}_M(I) < \infty \) for all \( I \in \mathcal{I} \). This is automatically guaranteed by our assumption that \( M \) is pointwisely finite-dimensional.
(i) The GRI over any $\mathcal{I} \subset \text{Int}(P)$ is a complete invariant on the collection of all $\mathcal{I}$-decomposable $P$-modules.

(ii) The direct sum decomposition of any interval-decomposable $P$-module $M$ can be obtained via Möbius inversion of $\text{rk}_M$ over the subposet

$$\mathcal{I}_M := \{ I \in \text{Int}(P) : I \in \text{barc}(M) \} \subset \text{Int}(P).$$

The statement given in item (i) is weaker than Proposition 4.4. However, the proof of item (i) given below is not only simpler than that of Proposition 4.4, but also simultaneously proves item (ii), that is not implied by Proposition 4.4.

Proof of Theorem E. Consider the function $\text{mult}^\mathcal{I}_M : \mathcal{I}_M \to \mathbb{Z}_{\geq 0}$ sending each $I \in \mathcal{I}$ to the multiplicity of $I$ in $\text{barc}(M)$. Proposition 4.1 implies

$$\text{rk}_M^\mathcal{I}(I) = \sum_{J \supseteq I} \text{mult}^\mathcal{I}_M(J) \text{ for all } I \in \mathcal{I}.$$

By Definition 3.1, $\text{rk}_M^\mathcal{I}$ is Möbius invertible over $\mathcal{I}_M$. By Proposition 3.2, the function $\text{mult}^\mathcal{I}_M$ equals $\text{dgm}^\mathcal{I}_M$, the Möbius inversion of $\text{rk}_M^\mathcal{I}$. Since $\text{rk}_M^\mathcal{I}$ is the restriction of $\text{rk}_M$ to $\mathcal{I}_M$, we have proved that $\text{rk}_M^\mathcal{I}$ uniquely determines $\text{mult}^\mathcal{I}_M$, Item (i), as well as Item (ii). □

We now generalize the well-known result that the RI is a complete invariant on the collection of rectangle-decomposable $\mathbb{Z}^2$-modules [13, Theorem 2.1] to the case of $\mathbb{Z}^d$- and $\mathbb{R}^d$-modules for $d \geq 2$.

Corollary 4.5. The RI is a complete invariant for a rectangle-decomposable $\mathbb{R}^d$- or $\mathbb{Z}^d$-module $M$ for any dimension $d$. Furthermore, the multiplicity of each segment $I$ in the barcode of $M$ is the Möbius inversion of $\text{rk}_M$ (over some $I \subset \text{Int}(P)$ with $I \in \mathcal{I}$) evaluated at $1$.

Proof. Theorem E (i) (or Proposition 4.4) implies that the RI is a complete invariant for rectangle-decomposable $\mathbb{R}^d$- or $\mathbb{Z}^d$-modules for any dimension $d$. Theorem E (ii) implies that the multiplicity of each $I \in \text{barc}(M)$ can be obtained via Möbius inversion of $\text{rk}_M$ over

$$\{ I \in \text{Int}(P) : I \in \text{barc}(M) \}.$$ □

Remark 4.6. We clarify the relationship between Proposition 4.4 and Theorem E.

(i) If every principal ideal of $\mathcal{I}$ is finite, then the limit completion $\hat{\mathcal{I}}$ of $\mathcal{I}$ (Equation (4.1)) is the same as $\mathcal{I}$, and thus Proposition 4.4 reduces to Item (i) of Theorem E.

(ii) If $\mathcal{I} \subsetneq \hat{\mathcal{I}}$, by Remark 2.24, $\text{mult}^\mathcal{I}_M$ has the support $\mathcal{I}' \subsetneq \hat{\mathcal{I}}$ for which every principal ideal is finite (and thus locally finite). Then, the GRI over $\mathcal{I}'$ is a complete invariant of $M$ (note: $\mathcal{I}'$ is not necessarily contained in $\mathcal{I}$), and the barcode of $M$ can be obtained from Möbius inversion of the GRI over $\mathcal{I}'$.\[\text{When } P = \mathbb{R}^d \text{ or } \mathbb{Z}^d \text{ and } \mathcal{I} = \text{Seg}(P), \mathcal{I} \text{-decomposable } P \text{-modules are often called rectangle-decomposable.}\]
The statement of Corollary 4.3 is optimal in the following sense.

**Theorem F** (Optimality of Corollary 4.3). Let $\mathcal{I} \subset \text{Int}(\mathcal{P})$ where every principal ideal is finite. Let $\mathcal{L}$ be any collection of indecomposable $\mathcal{P}$-modules properly containing $\{k_1 : I \in \mathcal{I}\}$. Then, the GRI over $\mathcal{I}$ is not a complete invariant on the collection of $\mathcal{L}$-decomposable modules.

**Proof.** It suffices to find a non-isomorphic pair $N, N'$ of $\mathcal{L}$-decomposable $\mathcal{P}$-modules that have the same GRI over $\mathcal{I}$. Let $M \in \mathcal{L}$ such that it is not isomorphic to $k_1$ for any $I \in \mathcal{I}$. Consider $\text{dgm}^\mathcal{I}_M$, the GPD of $M$ over $\mathcal{I}$, which exists by the assumption that every principal ideal of $\mathcal{I}$ is finite. Now consider the two $\mathcal{P}$-modules

$$N := \bigoplus_{I \in \mathcal{I}} (k_1)^{\text{dgm}^\mathcal{I}_M(1)_{>0}}$$

and

$$N' := M \bigoplus \left( \bigoplus_{I \in \mathcal{I}} (k_1)^{-\text{dgm}^\mathcal{I}_M(1)_{<0}} \right),$$

where $(k_1)^n$ stands for the direct sum of $n$ copies of $k_1$. While $N$ is $\mathcal{I}$-decomposable, $N'$ is not $\mathcal{I}$-decomposable as it has $M$ as a summand, and thus $N \not\cong N'$. By additivity of $\text{dgm}^\mathcal{I}_M$ (cf. Remark 2.22), the GPDs of $N$ and $N'$, $\text{dgm}^\mathcal{I}_N, \text{dgm}^\mathcal{I}_{N'} : \mathcal{I} \to \mathbb{Z}$ are both equal as functions to the map $(\text{dgm}^\mathcal{I}_M)_+ : \mathcal{I} \to \mathbb{Z}$ given by $I \mapsto \max(\text{dgm}^\mathcal{I}_M(I), 0)$. Therefore, $\text{rk}_N$ coincides with $\text{rk}_{N'}$, both of which are equal to $(\text{dgm}^\mathcal{I}_M)_+ \ast \zeta_\mathbb{Z}$. \hfill \qed

**Corollary 4.7.** Let $\mathcal{I} \subset \text{Int}(\mathcal{P})$ be such that every principal ideal is finite. Then, for any $\mathcal{J} \subset \text{Int}(\mathcal{P})$ with $\mathcal{I} \subset \subset \mathcal{J}$, the GRI over $\mathcal{I}$ is not a complete invariant on the collection of $\mathcal{J}$-decomposable $\mathcal{P}$-modules.

Here is an application of Corollary 4.7: We claim that as $m, n \in \mathbb{N}$ increase, the discriminating power of the GRI over $\text{Int}_{m,n}(\mathbb{R}^d)$ (cf. Equation (2.14)) on $\mathbb{R}^d$-modules strictly increases:

**Corollary 4.8.** If $(m, n) < (m', n')$ in $\mathbb{N}^2$, then there is a pair of $\mathbb{R}^d$-modules that are distinguished by their GRIs over $\text{Int}_{m',n'}(\mathbb{R}^d)$, but are confounded by their GRIs over $\text{Int}_{m,n}(\mathbb{R}^d)$.

For any natural number $n$, let $[n] := \{1, \ldots, n\}$ be equipped with the canonical order.

**Proof.** Let $\ell \in \mathbb{N}$ with $\ell > m', n'$. By Corollary 4.7, there exists a pair of $[\ell]^d$-modules $M$ and $N$ that are distinguished by their GRIs over $\text{Int}^\mathcal{cc}_{m',n'}([\ell]^d)$ but not by $\text{Int}^\mathcal{cc}_{m,n}([\ell]^d)$. Let $\iota : [\ell]^d \to \mathbb{R}^d$ be the canonical inclusion and let $[- -] : \mathbb{R}^d \to [\ell]^d \cup \{-\infty\}$ send each $p \in \mathbb{R}^d$ to the maximal $q \in [\ell]^d \cup \{-\infty\}$ such that $q \leq p$ (where we declare that $-\infty < p$ for all $p \in \mathbb{R}^d$). Let $M'$ and $N'$ respectively be the $\mathbb{R}^d$-modules given as the left Kan extensions of $M$ and $N$ along $\iota$, i.e. for all $p \leq q$ in $\mathbb{R}^d$,

$$M'_p = M_{[p]}, \quad \varphi_{M'}(p, q) = \varphi_M([p] \lor [q]),$$

where $M_{-\infty}$ is defined to be 0. Define $N'$ similarly. Observe that the GRIs of $M'$ and $N'$ over $\text{Int}_{m,n}(\mathbb{R}^d)$ coincide as, for every $I \in \text{Int}_{m,n}(\mathbb{R}^d)$, we have $[I]_i \in \text{Int}^\mathcal{cc}_{m,n}([\ell]^d \cup \{-\infty\})$ so that

$$\text{rk}_{M'}(I) = \text{rk}_M([I]_i) = \text{rk}_N([I]_i) = \text{rk}_{N'}(I).$$

On the other hand, we claim that the GRIs of $M'$ and $N'$ over $\text{Int}^\mathcal{cc}_{m',n'}(\mathbb{R}^d)$ do not coincide. To see this, pick any $K \in \text{Int}^\mathcal{cc}_{m,n}([\ell]^d)$ such that $\text{rk}_M(K) \neq \text{rk}_N(K)$. Then we pick any element $J \in \text{Int}^\mathcal{cc}_{m',n'}(\mathbb{R}^d)$ such that $[J]_i = K$, which implies $\text{rk}_{M'}(J) \neq \text{rk}_{N'}(J)$. \hfill \qed

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11The collection $\mathcal{L}$ may include non-interval modules.
Next, we investigate the extent to which the GRI over \( \mathcal{I} \) fails to be complete on the collection of \( \mathcal{J} \)-decomposable modules. We will see that the failure of completeness of the GRI over \( \mathcal{I} \) can be quantified via the dimension of the kernel of a linear map, which equals the cardinality of \( \mathcal{J} \setminus \mathcal{I} \).

For \( I \in \text{Int}(P) \), let \( 1_I : \text{Int}(P) \to \{0,1\} \) be the indicator supported on \( I \). Then, by Remark 2.8 (i), \( 1_I \) is convolvable over any locally finite \( \mathcal{T} \subset \text{Int}(P) \). For simplicity, we denote the Möbius inversion of the restriction \( 1_I \big|_{\mathcal{T}} \) over \( \mathcal{T} \) as \( 1_I \circ \mu_\mathcal{T} \).

**Theorem G.** Let \( \mathcal{I} \subset \mathcal{J} \subset \text{Int}(P) \). Assume that the GRIs of \( P \)-modules \( M \) and \( N \) are convolvable over \( \mathcal{J} \). Then, \( \text{rk}_M^\mathcal{J} \) and \( \text{rk}_N^\mathcal{J} \) coincide on \( \mathcal{I} \) if and only if \( \text{dgm}_M^\mathcal{J} - \text{dgm}_N^\mathcal{J} \) is a linear combination of the Möbius inverses over \( \mathcal{J} \) of the indicators \( 1_I \) for \( I \in \mathcal{J} \setminus \mathcal{I} \).

We defer the proof to the end of this section.

Any pair of non-isomorphic interval-decomposable \( P \)-modules \( M \) and \( N \) that have the same GRI over \( \mathcal{I} \subset \text{Int}(P) \) is called \( \mathcal{I} \)-minimal if there are no proper nonzero summands \( M',N' \) of \( M,N \) respectively such that \( M' \) and \( N' \) have the same GRI over \( \mathcal{I} \).

**Corollary 4.9.** Let \( \mathcal{I} \subset \mathcal{J} \subset \text{Int}(P) \) and assume that every principal ideal of \( \mathcal{J} \) is finite. Then, there exist \( |\mathcal{J} \setminus \mathcal{I}| \) distinct \( \mathcal{I} \)-minimal non-isomorphic pairs of \( \mathcal{J} \)-decomposable \( P \)-modules whose GRIs coincide on \( \mathcal{I} \).

We remark that when \( \mathcal{I} = \mathcal{J} \), the corollary above reduces to Theorem E.

**Proof.** Let \( I \in \mathcal{J} \setminus \mathcal{I} \) and consider the Möbius inversion \( 1_I \circ \mu_\mathcal{J} \) of the indicator \( 1_I : \mathcal{J} \to \{0,1\} \). For \( J \in \mathcal{J} \), let \( d_J := (1_I \circ \mu_\mathcal{J})(J) \). Clearly, the two \( \mathcal{J} \)-decomposable \( P \)-modules
\[
M_I^+ := \bigoplus_{d_J > 0} (k_J)^{d_J} \quad \text{and} \quad M_I^- := \bigoplus_{d_J < 0} (k_J)^{-d_J}
\]
for any \( I \in \mathcal{J} \setminus \mathcal{I} \) with \( I \neq I' \). \hfill \Box

Theorem G and Corollary 4.9 offer a theoretical foundation for well-known examples of non-isomorphic interval-decomposable multi-parameter persistence modules that share the same rank invariant:

**Example 4.10.** Let \( P := [4] \) and consider the subposets \( \mathcal{I} := \{[2,3],[1,3],[2,4]\} \) and \( \mathcal{J} := \mathcal{I} \cup \{[1,4]\} \) of \( \text{Int}(P) \). Using the recursion in Equation (2.6), it is not hard to verify that \( 1_{[1,4]} \circ \mu_\mathcal{J} = 1_{[1,4]} - 1_{[1,3]} - 1_{[2,4]} + 1_{[2,3]} \). Since \( |\mathcal{J} \setminus \mathcal{I}| = 1 \), Corollary 4.9 implies that there is a single minimal non-isomorphic pair of \( \mathcal{J} \)-decomposable \( P \)-modules whose GRIs coincide on \( \mathcal{I} \). As \( |\mathcal{J} \setminus \mathcal{I}| = 1 \), we let \( I = [1,4] \) and compute the \( \mathcal{I} \)-minimal pair \( M_I^+ \) and \( M_I^- \) by Equation (4.2). This yields \( M_I^+ = k_{[1,4]} \oplus k_{[2,3]} \), and \( M_I^- = k_{[1,3]} \oplus k_{[2,4]} \).

**Example 4.11.** For the intervals \( I,J_1,J_2,J_3 \) in the poset \( [2]^2 \) depicted below, let \( M,N : [2]^2 \to \text{vec} \) be given by
\[
M = k_I \oplus k_{J_3} \quad \text{and} \quad N = k_{J_1} \oplus k_{J_2}.
\]
Note that \( M \) and \( N \) have the same GRI over \( I := \{I_1, I_2, I_3\} \), but not over \( J := I \cup \{I\} \). By Theorem 4.2, we have \( \dim^J_M = 1 + 1_3 \) and \( \dim^J_N = 1_1 + 1_2 \). Using the recursion in Equation (2.6), one can verify that \( 1_1 \ast \mu_J = 1_1 - 1_1 - 1_1 + 1_1 = \dim^J_M - \dim^J_N \).

Since \(|J \setminus I| = 1\), Corollary 4.9 implies that \( M \) and \( N \) form the unique minimal pair of \( J \)-decomposable non-isomorphic \([2]^2\)-modules.

Before proving Theorem G, we set up notation. Let \( Q \) denote the set of rational numbers. For any \( J \subset \operatorname{Int}(P) \), let \( Q^J_c \) be the vector space of convolvable maps \( J \to Q \) (cf. Remark 2.8 (iii)). Assume that \( I \subset J \subset \operatorname{Int}(P) \). For the zeta function \( \zeta_J \) of the poset \((J, \supset)\), the map \(*\zeta_J\) is an automorphism on \( Q^J_c \) with inverse \( *\mu_J \) (cf. Remark 2.9 (iii)). Let \( \pi_J : Q^J_c \to Q^I_c \) be the restriction \( f \mapsto f|_I \). We have the following diagram:

\[
\begin{array}{ccc}
Q^J_c & \xleftarrow{\ast \zeta_J} & Q^I_c \\
\downarrow{\pi_J} & & \downarrow{\pi_I} \\
Q^I_c & \xrightarrow{\ast \mu_J} & Q^J_c
\end{array}
\]  
(4.3)

Given any \( P \)-module \( M \) whose GRI over \( J \) is convolvable, the functions \( \dim^J_M, \rk^J_M \) and \( \rk^I_M \) are mapped to each other via the maps given in Equation (4.3):

\[
\begin{array}{ccc}
\dim^J_M & \xleftarrow{\ast \zeta_J} & \rk^J_M \\
\downarrow{\pi_J} & & \downarrow{\pi_I} \\
\rk^I_M & \xrightarrow{\ast \mu_J} & \dim^I_M
\end{array}
\]  
(4.4)

Remark 4.12. The kernel of \( \pi_J \) consists of those functions \( f : J \to \mathbb{Q} \) such that \( f(1) = 0 \) for all \( I \in I \). Equivalently, \( \ker(\pi_J) = \operatorname{Span}(1_I : I \in J \setminus I) \).

Proof of Theorem G. From the diagram given in Equation (4.3), we have:

\[
\begin{align*}
\rk^J_M &= \rk^I_M \\
\iff \rk^J_M - \rk^I_M &= 0 \\
\iff \pi_J(\rk^J_M - \rk^I_M) &= 0 \\
\iff \rk^I_M - \rk^J_M &\in \operatorname{Span}(1_I : I \in J \setminus I) \quad \text{by Remark 4.12} \\
\iff (\rk^J_M - \rk^I_M) * \mu_J &\in \operatorname{Span}(1_I : I \in J \setminus I) \quad \text{by Remark 2.9 (iii)} \\
\iff \dim^J_M - \dim^I_M &\in \operatorname{Span}(1_I : I \in J \setminus I) \quad \text{by Definition 2.21.} \quad \Box
\end{align*}
\]

Remark 4.13 (Another proof of Corollary 4.7 for the case when \( I, J \) are finite). When \( I \subset J \subset \operatorname{Int}(P) \) are finite, we can prove Corollary 4.7 using the rank-nullity theorem. Note that, in Equation (4.3), (i) \( Q^I_c = Q^J_c \) and \( Q^J_c = Q^J_c \), (ii) \( \{1_I : I \in I\} \) and \( \{1_I : I \in J\} \) are bases for \( Q^I_c \) and \( Q^J_c \) respectively, and (iii) Since \( *\zeta_J \) is an automorphism with the inverse \( *\mu_J \), the kernel of the composition \( \pi_J \circ (*\zeta_J) \) coincides with the image of \( \ker(\pi_J) \) via \( *\mu_J \), which is \(|J \setminus I|-\text{dimensional space by the rank-nullity theorem. We omit further details.}\)

### 4.3 Results for 2-parameter persistence modules

In this section, we focus on clarifying the discriminating power of the GRI of \( \mathbb{Z}^2 \)-modules. Specifically, we do this by comparing the GRI with the ZIB (Section 4.3.1), and with the bigraded Betti numbers (Section 4.3.2).
4.3.1 Comparison with ZIB over simple paths

Let \text{con}(P) and \text{int}(P) denote the sets of finite connected subsets and finite intervals of a given poset \( P \), respectively. We use the term \text{int-GRI} to refer to the GRI over \text{int}(P). We already know that the ZIB of a \( \mathbb{Z}^2 \)-module \( M \) determines the \text{int-GRI} of \( M \), i.e. if two \( \mathbb{Z}^2 \)-modules have the same ZIB, then they have the same \text{int-GRI}; this is implied by [40, Theorem 24] stating that for \( I \in \text{int}(\mathbb{Z}^2) \), \( \text{rk}_M(I) \) equals the rank of a zigzag module arising as the restriction of \( M \) to a certain path \( \partial I \) along the boundary of \( I \). A priori, the path \( \partial I \) can contain repeated points in \( \mathbb{Z}^2 \). Hence, a natural question for efficient computation of the \text{int-GRI} is whether the ZIB over only simple paths (i.e. paths without repeated points) determines the \text{int-GRI}.

In this section, we show that the answer is negative. Furthermore, we show that the \text{int-GRI} also does not determine the ZIB over simple paths. Hence, the discriminating power of the \text{int-GRI} and of the ZIB over simple paths are not well-ordered. It then follows that the ZIB over all paths is a strictly finer invariant than both the ZIB over simple paths and the \text{int-GRI}: see Examples 4.16, 4.17 and Figure 5. We take one step further and investigate how much the \text{int-GRI} can be recovered from the ZIB over simple paths, and the other way around: see Remark 4.20 and Proposition 4.22.

**Comparison between the ZIB and the GRI.** For \( a, b \in \mathbb{Z}^2 \), we write \( a \triangleleft b \) if and only if \( a < b \) and \( [a, b] = \{a, b\} \). Let \( \Gamma : p_1, p_2, \ldots, p_n \) be a path in \( \mathbb{Z}^2 \). We call \( \Gamma \) **faithful** if \( p_i \triangleleft p_{i+1} \) or \( p_{i+1} \triangleleft p_i \) for each \( i = 1, \ldots, n-1 \). In what follows, we consider only faithful paths; this is because for any \( \mathbb{Z}^2 \)-module \( M \) and for any non-faithful path \( \Gamma \), there is a faithful path \( \Gamma' \) containing \( \Gamma \) as a subsequence, and thus \( \text{barc}(M_\Gamma) \) can be read off from \( \text{barc}(M_{\Gamma'}) \).

We call \( \Gamma \) **simple** if all of the points \( p_1, p_2, \ldots, p_n \) are distinct from each other. There are two special types of simple paths: We call \( \Gamma \) a **monotone path** or **positive path**, if \( p_i \triangleleft p_{i+1} \) for each \( i \). We call \( \Gamma \) a **negative path** if \( \Gamma \) is obtained from the reflection of a monotone path with respect to the \( y \)-axis.

**Definition 4.14.** The restriction of the domain of the ZIB of any \( P \)-module to the set of all simple paths of \( P \) called the **ZIB over simple paths** and is denoted by \( M_{\text{SZ}} \).

Let \( F \) and \( G \) be two invariants of \( \mathbb{Z}^2 \)-modules. If \( F \) determines \( G \), then we write \( F \Rightarrow G \), which defines a transitive relation on the class of invariants of \( \mathbb{Z}^2 \)-modules. For example, \( \text{ZIB} \Rightarrow \text{ZIB over simple paths} \). If \( F \) determines \( G \) and vice versa, then we write \( F \Leftrightarrow G \). We say that \( F \) **strictly determines** \( G \) if \( F \Rightarrow G \) and \( G \not\Rightarrow F \). Now, we compare the ZIB and the GRI of \( \mathbb{Z}^2 \)-modules over various domains:

**Remark 4.15.** In Figure 5 (B):

(i) If \( F \) and \( G \) are in the same column and \( F \) is at a higher level than \( G \), then \( F \Rightarrow G \) is clear by definition.

(ii) \( \text{GRI over con}(\mathbb{Z}^2) \Rightarrow \text{ZIB over simple paths} \) follows from the definition of GRI over con(\( P \)).

(iii) (RI \( \Leftrightarrow \text{ZIB over monotone paths} \)) was noted in [59, Proposition 1.2].

(iv) \( \text{ZIB over all paths} \Rightarrow \text{GRI over int}(\mathbb{Z}^2) \) is a direct corollary of [40, Theorem 24].

(v) Theorem E and Corollary 4.7 demonstrate an interpolation between the \text{int-GRI} and RI of \( \mathbb{Z}^2 \)-modules. Namely, these results show that if \( \text{Seg}(\mathbb{Z}^2) \subset I \subseteq J \subset \text{Int}(\mathbb{Z}^2) \), and every principal ideal of \( I \) is finite, then \( \text{rk}^J \Rightarrow \text{rk}^I \) but \( \text{rk}^I \not\Rightarrow \text{rk}^J \).\[^{12}\]

\[^{12}\]Theorem E and Corollary 4.7 claim more general statements as they consider \( P \)-modules not just \( \mathbb{Z}^2 \)-modules.
Figure 5: (A) Barcodes of a $\mathbb{Z}^2$-module over zigzag paths. (B) Hierarchy of invariants for $\mathbb{Z}^2$-modules. The hierarchy of the GRI (left column) is comparable to the hierarchy of barcodes over zigzag paths (right column). In this figure, an invariant $F$ strictly determines another invariant $G$ at a lower height, regardless of which columns $F$ and $G$ belong to. The notations $\leftarrow$ and $\rightarrow$ indicate that one invariant does not determine another, but rather can be used to estimate the other. See Remark 4.15 for a full explanation.

(vi) Overall, by transitivity of the relation $\Rightarrow$, $(F \Rightarrow G)$ holds if $F$ is at a higher level than $G$ regardless of the columns they belong to.

In the following two examples, we will see that $(\text{int-GRI} \not\Rightarrow \text{ZIB over simple paths})$ and $(\text{ZIB over simple paths} \not\Rightarrow \text{int-GRI})$.

**Example 4.16** $(\text{int-GRI} \not\Rightarrow \text{ZIB over simple paths})$. Let $M, N$ be $\mathbb{Z}^2$-modules defined as below whose supports are contained in $[3]^2 \subset \mathbb{Z}^2$. Also, let $\Gamma : (1, 2), (1, 3), (2, 3), (3, 3), (3, 2), (3, 1), (2, 1)$ (cf. Figure 6 (C)). It is not difficult to check that $M_{\Gamma} \neq N_{\Gamma}$, whereas $\text{rk}_{M}(I) = \text{rk}_{N}(I)$ for all $I \in \text{Int}(3^2)$ [54, Example A.2]. This shows that the GRI over $\text{int}(\mathbb{Z}^2)$ cannot fully recover the ZIB over simple paths, the ZIB over all paths, nor the GRI over $\text{con}(\mathbb{Z}^2)$.

**Example 4.16** $(\text{int-GRI} \not\Rightarrow \text{ZIB over simple paths})$. Let $M, N$ be $\mathbb{Z}^2$-modules defined as below whose supports are contained in $[3]^2 \subset \mathbb{Z}^2$. Also, let $\Gamma : (1, 2), (1, 3), (2, 3), (3, 3), (3, 2), (3, 1), (2, 1)$ (cf. Figure 6 (C)). It is not difficult to check that $M_{\Gamma} \neq N_{\Gamma}$, whereas $\text{rk}_{M}(I) = \text{rk}_{N}(I)$ for all $I \in \text{Int}(3^2)$ [54, Example A.2]. This shows that the GRI over $\text{int}(\mathbb{Z}^2)$ cannot fully recover the ZIB over simple paths, the ZIB over all paths, nor the GRI over $\text{con}(\mathbb{Z}^2)$.}

\[
M := \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & k & 1
\end{pmatrix}
\quad
N := \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & k & 1
\end{pmatrix}
\]

\[
\begin{align*}
&\quad \begin{array}{cccc}
1 & k & 1 & k \\
1 & 0 & 1 & k \\
1 & 1 & 0 & k \\
0 & k & 0 & k
\end{array} \\
&\quad \begin{array}{cccc}
1 & k & 1 & k \\
1 & 0 & 1 & k \\
1 & 1 & 0 & k \\
0 & k & 0 & k
\end{array} \\
&\quad \begin{array}{cccc}
1 & k & 1 & k \\
1 & 0 & 1 & k \\
1 & 1 & 0 & k \\
0 & k & 0 & k
\end{array}
\end{align*}
\]
Example 4.17 (ZIB over simple paths \( \not\Rightarrow \) int-GRI). Let \( I \in \text{int}(\mathbb{Z}^2) \) with the following directed Hasse diagram.

\[
\begin{array}{c}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\uparrow & & \uparrow & & \uparrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet
\end{array}
\]

Let us define the following \( \mathbb{Z}^2 \)-modules \( M \) and \( N \) supported on \( I \):

\[
M := k \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus k \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

\[
N := k \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus k \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \oplus k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

Since each summand of \( M \) and \( N \) is indecomposable, by Theorem 2.17, we have \( \text{rk}_M(I) = 1 \) and \( \text{rk}_N(I) = 0 \). Now we claim that \( M_{SZZ} = N_{SZZ} \). Since \( M \) and \( N \) are supported on \( I \), it suffices to show \( M_{\Gamma} \cong N_{\Gamma} \) for all maximal simple paths \( \Gamma \) in \( I \). Indeed, it is not difficult to check that \( M_{\Gamma} \cong N_{\Gamma} \) for all of the six maximal simple paths \( \Gamma \) in \( I \):

\[
\begin{array}{c}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\uparrow & & \uparrow & & \uparrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet
\end{array}
\]

Therefore, the ZIB over simple paths cannot determine the int-GRI in general. In addition, invoking the fact that the ZIB determines the int-GRI, this example proves that the ZIB strictly determines the ZIB over simple paths.

Remark 4.18. Let \( d \geq 3 \). Since \( \mathbb{Z}^2 \) can be embedded into \( \mathbb{Z}^d \), for \( \mathbb{Z}^d \)-modules, neither the int-GRI nor the ZIB over simple paths determine the other.

**ZIB over simple paths and the int-GRI estimate each other.** Although the int-GRI and the ZIB over simple paths fail to determine each other, we can estimate one from the other. We describe how this is done in Remark 4.20 and Proposition 4.22 below.

Let \( \Gamma \) be a path in \( \mathbb{Z}^2 \). Let \( I_{\Gamma} \) be the smallest interval of \( \mathbb{Z}^2 \) that contains \( \Gamma \), i.e.

\[
I_{\Gamma} := \{ q \in \mathbb{Z}^2 : \exists p, r \in \Gamma, \ p \leq q \leq r \}.
\]
Figure 6: (A) An interval $I$ in $\mathbb{Z}^2$ with points in $\min_{\mathbb{Z}^2}(I)$ and $\max_{\mathbb{Z}^2}(I)$ labeled. (B) Paths $\Gamma_i$ for $i = 1, 2, 3, 4$ in $\mathbb{Z}^2$ (directions are not specified) and their interval-hulls. Whereas $\Gamma_i$ is a tame path for $i = 1, 2, 4$, $\Gamma_3$ is not a tame path. (C) An example showing that the rank over the non-tame path $\Gamma$ does not coincide with the rank over the interval-hull $I_\Gamma$. (D) An interval in $\mathbb{Z}^2$ which is not solid nor thin. The interval in (A) is solid. The interval $I_{\Gamma_3}$ in (B) is thin. (E) Let $\Gamma$ be the path $r_1, \ldots, r_5$ and $\Gamma'$ be the subpath $r_2, r_3, r_4$. Then $\Gamma^{-}$, $\Gamma'^{+}$, $\Gamma'^{\pm}$ are the paths $r_1, r_2, r_3, r_4$ and $r_2, r_3, r_4$, and $r_1, r_2, r_3, r_4, r_5$, respectively.
We call \( I_\Gamma \) the interval-hull of \( \Gamma \). See Figure 6 (B) for illustrative examples.

Given any \( I \in \text{int}(Z^2) \), the set \( \text{min}(I) \) (resp. \( \text{max}(I) \)) of minimal (resp. maximal) points of \( I \) forms an antichain, i.e. any two different points in \( \text{min}(I) \) (resp. \( \text{max}(I) \)) are not comparable. Hence, we can list the elements of \( \text{min}(I) \) in ascending order of their \( x \)-coordinates, i.e. \( \text{min}(I) := \{ p_0, \ldots, p_k \} \) and such that for each \( i = 0, \ldots, k \), the \( x \)-coordinate of \( p_i \) is less than that of \( p_{i+1} \). Similarly, let \( \text{max}(I) := \{ q_0, \ldots, q_\ell \} \) be ordered in ascending order of their \( x \)-coordinates. We have that \( p_0 \leq q_0 \) (cf. Figure 6 (A)). Let \( \min_{ZZ}(I) \) be the shortest faithful path in \( Z^2 \) that contains the following sequence:

\[
p_0 < (p_0 \lor p_1) > p_1 < (p_1 \lor p_2) > \cdots < (p_{k-1} \lor p_k) > p_k.
\]

Similarly, let \( \max_{ZZ}(I) \) be the shortest faithful path in \( Z^2 \) containing

\[
q_0 > (q_0 \land q_1) < q_1 > (q_1 \land q_2) < \cdots > (q_{\ell-1} \land q_\ell) > q_\ell.
\]

Let \( \Gamma : p_1, \ldots, p_n \) be a path in \( Z^2 \). By \( \Gamma^{-1} \) we will denote its reverse path \( p_n, p_{n-1}, \ldots, p_1 \). We call \( \Gamma \) tame if:

\[
\left[ \min_{ZZ}(I_\Gamma) \leq \Gamma \text{ or } \min_{ZZ}(I_\Gamma)^{-1} \leq \Gamma \right] \text{ and } \left[ \max_{ZZ}(I_\Gamma) \leq \Gamma \text{ or } \max_{ZZ}(I_\Gamma)^{-1} \leq \Gamma \right].
\]

For example, in Figure 6 (B), \( \Gamma_1, \Gamma_2 \), and \( \Gamma_3 \) are tame but \( \Gamma_3 \) is not. Given a finite interval \( I \subset Z^2 \), the concatenation of the paths \( \min_{ZZ}(I)^{-1} \) and \( \max_{ZZ}(I) \) is called the boundary cap of \( I \) \([40]\), which is an instance of a tame path. The boundary cap of \( I \) is a simple path iff \( \min_{ZZ}(I) \) and \( \max_{ZZ}(I) \) do not intersect.

Let us fix a \( Z^2 \)-module \( M \).

**Proposition 4.19.** Given any tame path \( \Gamma \) in \( Z^2 \), it holds that \( \text{rk}(M_\Gamma) = \text{rk}_M(I_\Gamma) \).

This proposition is a generalization of \([40, \text{Theorem 24}]\) and its proof, which we defer to the appendix, is similar to that of \([40, \text{Theorem 24}]\).

Now, we describe how to estimate the int-GRI via the ZIB over simple paths. An \( I \in \text{int}(Z^2) \) is called solid if \( \min_{ZZ}(I) \) and \( \max_{ZZ}(I) \) do not intersect (e.g. the interval given in Figure 6 (A)). On the contrary, if \( I \) equals \( I_\Gamma \) for some negative path \( \Gamma \), then \( I \) is called thin (e.g. \( \Gamma_2 \) in Figure 6 (B)).

**Remark 4.20 (Estimating the int-GRI via the ZIB over simple paths).** Let \( I \in \text{int}(Z^2) \).

(i) If \( I \) is a thin interval, then \( I \) is covered by a simple path \( \Gamma \). Thus, by Theorem 2.17 and Proposition 4.19, \( \text{rk}_M(I) \) equals the multiplicity of the ‘full bar’ \( \Gamma \) in \( \text{barc}(M_\Gamma) \).

(ii) If \( I \) is solid, then the boundary cap \( \Gamma \) of \( I \) is a simple tame path such that \( I = I_\Gamma \). Hence, \( \text{rk}_M(I) \) equals the multiplicity of the full bar in \( \text{barc}(M_\Gamma) \).

(iii) If \( I \) is not solid nor thin, then there is no simple tame path spanning \( I \) (see e.g. Figure 6 (D)). However, by monotonicity of the GRI (Remark 2.19 (i)), we have the following upper and lower bounds for \( \text{rk}_M(I) \):

\[
\max_j \text{rk}_M(J) \leq \text{rk}_M(I) \leq \min_i \text{rk}(M_\Gamma)
\]

where the minimum is taken over all simple paths \( \Gamma \) in \( I \) and the maximum is taken over all solid intervals \( J \supset I \). The both bounds can be obtained from \( M_{SZZ} \): Clearly, \( \max_i \text{rk}(M_\Gamma) \) is determined by \( M_{SZZ} \). From item (ii), \( \max_j \text{rk}_M(J) \) can also be computed from \( M_{SZZ} \).
Conversely, we now describe how to estimate the ZIB over simple paths via the int-GRI. Let $\Gamma$ be a simple path in $\mathbb{Z}^2$ and consider the following nonnegative integers:

$$m_\Gamma := \text{rk}_M(I_\Gamma) \text{ and } \ell_\Gamma := \min(\text{rk}(M_{\Gamma'}) : \Gamma' \leq \Gamma, \text{ and } \Gamma' \text{ is tame}).$$

By monotonicity of $\text{rk}_M$, we have

$$m_\Gamma \leq \text{rk}(M_\Gamma) \leq \ell_\Gamma. \tag{4.7}$$

**Remark 4.21.** By Proposition 4.19,

(i) the int-GRI of $M$ can be used to compute $m_\Gamma$ and $\ell_\Gamma$.

(ii) if $\Gamma$ is tame, then $m_\Gamma = \ell_\Gamma = \text{rk}(M_\Gamma)$.

Let $\Gamma : r_1, r_2, \ldots, r_n$ be a path and let $\Gamma' : r_k, r_{k+1}, \ldots, r_\ell$ be a subpath of $\Gamma$. When $k \neq 1$, we consider the one-point extension $\Gamma'$ to the left, i.e. $\Gamma'^{-} : r_{k-1}, r_k, \ldots, r_\ell$. When $\ell \neq n$, we consider the one-point extension of $\Gamma'$ to the right, i.e. $\Gamma'^{+} : r_k, \ldots, r_\ell, r_{\ell+1}$. When $k \neq 1$ and $\ell \neq n$, we consider the two-point extension $\Gamma'^{\pm} : r_{k-1}, r_k, \ldots, r_\ell, r_{\ell+1}$ of $\Gamma'$ within $\Gamma$ (see Figure 6 (E)).

We obtain upper and lower bounds on the multiplicity of each subpath $\Gamma'$ of $\Gamma$ in $\text{barc}(M_\Gamma)$:

**Proposition 4.22** (Estimating the ZIB over simple paths via the int-GRI). Let $n_{\Gamma'}$ be the multiplicity of $\Gamma'$ in $\text{barc}(M_\Gamma)$. Then, we have

$$m_{\Gamma'} - \ell_{\Gamma'}^{-} - \ell_{\Gamma'}^{+} + m_{\Gamma'^{\pm}} \leq n_{\Gamma'}, \leq \ell_{\Gamma'}^{-} - m_{\Gamma'^{+}} - m_{\Gamma'^{-}} + \ell_{\Gamma'^{\pm}}, \tag{4.8}$$

where the undefined terms in the upper and lower bounds are set to be zero.\(^{13}\)

We remark that, by Remark 4.21 (ii), the upper and lower bounds match when (i) $\Gamma'^{\pm} \leq \Gamma$, and (ii) $\Gamma$ is either a monotone or negative path.

**Proof.** By the principle of inclusion and exclusion, we have that

$$n_{\Gamma'} = \text{rk}(M_{\Gamma'}) - \text{rk}(M_{\Gamma'^{+}}) - \text{rk}(M_{\Gamma'^{-}}) + \text{rk}(M_{\Gamma'^{\pm}}) \quad \text{(cf. } [52, \text{Section 3}]). \tag{4.9}$$

The claimed inequalities follow from the inequalities in Equation (4.7). \qed

**Remark 4.23.** By Theorem C (iii) and its proof, if $M : \mathbb{R}^2 \to \text{vec}$ is finitely presentable, then there is a discrete grid $G \subseteq \mathbb{R}^2$ such that the Int-GRI of $M$ can be computed from the Int-GRI of $M|_G$. The definitions and results in this section pertaining to the ZIB over $\mathbb{Z}^2$ naturally adapt to definitions and results for the setting of finite grids in $\mathbb{R}^2$, and so the ZIB is a useful tool for finitely presentable $\mathbb{R}^2$-modules as well.

**Remark 4.24.** Generalizing Remark 4.20 and Proposition 4.22 to the setting of $\mathbb{Z}^d$-modules for $d \geq 3$ seems difficult, as it is unclear how to define ‘tame paths’ in $\mathbb{Z}^d$ that can bridge between the GRI and the ZIB, as in Proposition 4.19. This uncertainty is supported by the fact that, for $d \geq 3$, there exists $P \in \text{int}(\mathbb{Z}^d)$ and a $P$-module $M$ with $\text{rk}(M) \neq \text{rk}(M_{\Gamma})$ for every path $\Gamma$ in $P$.\(^{14}\) For example, let $P := \{(x, y, z) \in \mathbb{Z}^3 : 0 \leq x, y, z \leq 1\}$ and consider the $P$-module:

---

\(^{13}\)For example, if $\Gamma$ and $\Gamma'$ have the same starting point but different end points, then $\Gamma'^{-}$ and $\Gamma'^{\pm}$ are undefined and thus Equation (4.8) reduces to $m_{\Gamma'} - \ell_{\Gamma'}^{-} - m_{\Gamma'^{-}} \leq n_{\Gamma'} \leq \ell_{\Gamma'} - m_{\Gamma'^{+}}$.

\(^{14}\)The recent work [41] utilizes zigzag persistence for computing the generalized rank of a persistence module over a general poset. This work is not directly applicable to our setting. In that work, given a finite poset $P$ and a $P$-module $M$, the authors consider a path $\Gamma$ that covers $P$ and for it they compute a special decomposition of the zigzag module $M_{\Gamma}$ with the goal of determining the generalized rank of $M$. This special decomposition contains information that is not detected by $\text{barc}(M_{\Gamma})$. 

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in which all arrows $k^2 \rightarrow k^2$ denote the identity map with the exception of the arrow labeled with $\phi_\lambda$ where $\phi_\lambda := \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$, $\lambda \neq 0$. Let $I := \{p \in P : M_p = k^2\}$, which is an interval of $P$. Then, it is not difficult to verify that for every path $\Gamma$ in $I$, we have $\text{rk}(M_\Gamma) = 2$, whereas $\text{rk}_M(I) = 0$.

### 4.3.2 Comparison with bigraded Betti numbers

In this section, we show that the bigraded Betti numbers of a $\mathbb{Z}^2$-module $M$ are a much weaker invariant than the int-GRI of $M$. This is in stark contrast with the fact that, for $\mathbb{Z}^d$-modules with $d \geq 3$, the int-GRI is not a stronger invariant than the $d$-graded Betti numbers; see [54, Theorem 4.1]. We do this by showing that the bigraded Betti numbers do not even determine the GRI over simple paths of length 3, while the reverse is already known to hold.

**Proposition 4.25.** The GRI of any $\mathbb{Z}^2$-module over the zigzag posets in Equation (4.10) below determines the bigraded Betti numbers of $M$.

**Proof.** The bigraded Betti numbers of $M$ is determined by the barcodes over the zigzag posets

$$(x, y+1) \leftrightarrow (x, y) \rightarrow (x+1, y) \land (x-1, y) \rightarrow (x, y-1) \text{ for all } x, y \in \mathbb{Z}^2;$$

(4.10) see [71, Theorem 2.1]. Since the zigzag modules are interval-decomposable (Theorem 2.2), and the Int-GRI of any interval-decomposable module determines its barcode (Corollary 4.3), the claim follows.

The reverse statement does not hold.

**Proposition 4.26.** The bigraded Betti numbers of a $\mathbb{Z}^2$-module do not determine its GRI over zigzag paths of length 3.

**Proof.** It suffices to construct a pair of $\mathbb{Z}^2$-modules that have the same bigraded Betti numbers but different barcodes over some zigzag path of length 3. Let $a = (0, 2)$, $b = (1, 1)$, $c = (2, 0)$ and $D = (2, 2)$. Let $M$ and $N$ be $\mathbb{Z}^2$-modules defined as

$$M := k_a \uparrow \cup c \uparrow \oplus k_b \uparrow \quad \text{and} \quad N := k_a \uparrow \oplus k_c \uparrow \oplus k_b \uparrow \setminus D \uparrow.$$ 

It is straightforward to compute that the zeroth Betti number for each module is 1 at $a$, $b$ and $c$, and that the first Betti number for each is 1 at $D$. For the both modules, the second and higher Betti numbers vanish.

Now, consider $\Gamma$, the zigzag path of length 3 given by $a \rightarrow D \leftarrow c$. Since there is an interval summand over the upset $a \uparrow \cup c \uparrow$ in $M$ but not $N$, the zigzag barcode of $M$ over $\Gamma$ includes a full bar, whereas the zigzag barcode of $N$ over $\Gamma$ does not include a full bar.
4.4 Generalized rank invariant compared to other invariants

This section is dedicated to providing detailed explanations of Table 1.

Row 1. The dimension function of \( M : P \to \text{vec} \), defined by the map \( p \mapsto \dim(M_p) \) for \( p \in P \), can be viewed as the GRI over \( \mathcal{J} := \{[p] : p \in P\} \). Clearly, \( \text{rk}_{\mathcal{J}}^P \) is Möbius invertible over \( \mathcal{J} \), and its Möbius inversion is identical to itself.

Row 3. We call a line \( \ell \) in \( \mathbb{R}^d \) monotone, if \( \ell \) forms a totally ordered set with the order inherited from \( \mathbb{R}^d \). In a fixed line \( \ell \), let Seg(\( \ell \)) be ordered by \( \preceq \). For different monotone lines, we declare that \( \ell \) and \( \ell' \), any pair of elements from Seg(\( \ell \)) and Seg(\( \ell' \)) is not comparable. For any fixed line \( \ell \), since \( M|_\ell \) is interval-decomposable \([34]\), \( \text{rk}_{\mathcal{J}}^\ell(M) \) is Möbius invertible (Theorem C (i)). Hence, the GRI of \( M \) is also Möbius invertible over the disjoint union \( \bigcup \{\text{Seg}(\ell) : \ell \text{ is monotone in } \mathbb{R}^d\} \).

Row 6-7. See \([5]\).

Row 8. ‘⇒ Bigraded Betti numbers’ follows from Propositions 4.25 and 4.26.

Row 9. ‘⇒ Int-GRI’ follows from \([40, \text{Theorem } 3.12]\).

Row 10 ‘∉ Int-GRI’ follows from Examples 4.16 and 4.17.

5 Stability of the generalized rank invariant

In Section 5.1, we prove that the GRI is stable in the erosion distance sense and thereby, in Section 5.2, we establish a stability result for ZIBs.

5.1 Stability of the generalized rank invariant

In this section, we prove that the GRI over \( I \) is stable in the erosion distance sense as long as \( I \) is closed under thickenings: see Definition 5.1 and Theorem H.

For ease of notation, in this section we focus on the setting of \( \mathbb{R}^d \) or \( \mathbb{Z}^d \)-modules with the usual interleaving distance. Appendix A deals with more general settings, specifically the stability of GRIs of \( P \)-modules under appropriate assumptions on \( P \).

We begin by reviewing the definition of interleaving distance between \( \mathbb{R}^d \) (or \( \mathbb{Z}^d \))-modules \([58]\). Let \( M : \mathbb{R}^d \to \text{vec} \) and \( \epsilon \in \mathbb{R}_{\geq 0} \). Denote \( \epsilon := \epsilon(1, \ldots, 1) \in \mathbb{R}^d \). Define the \( \epsilon \)-shift of \( M \), \( M^\epsilon : \mathbb{R}^d \to \text{vec} \) by \( M^\epsilon(a) := M(a) \) and \( \varphi_{M^\epsilon}(a, b) := M(a + \epsilon - b + \epsilon) \) for all \( a, b \in \mathbb{R}^d \). For a morphism of modules \( \alpha \), define \( \alpha(\epsilon)_a = \alpha + \epsilon \). Define the transition morphism \( \varphi_{M^\epsilon}^M : M \to M^\epsilon \) as the morphism whose restriction to \( M(a) \) is the linear map \( \varphi_{M^\epsilon}(a, a + \epsilon) \) for each \( a \in \mathbb{R}^d \). For \( \epsilon \geq 0 \), we say \( \mathbb{R}^d \)-modules \( M \) and \( N \) are \( \epsilon \)-interleaved if there exist morphisms \( \alpha : M \to N^\epsilon \) and \( \beta : N \to M^\epsilon \) such that \( \beta(\epsilon) \circ \alpha = \varphi_{M^\epsilon}^M \) and \( \alpha(\epsilon) \circ \beta = \varphi_{N^\epsilon}^N \). The interleaving distance is defined as:

\[
d_I(M, N) := \inf\{\epsilon \geq 0 : M \text{ and } N \text{ are } \epsilon - \text{interleaved}\},
\]

and \( d_I(M, N) := \infty \) if no such \( \epsilon \)-interleavings exist.

For \( I \in \text{Int}(\mathbb{R}^d) \), define the \( \epsilon \)-thickening of \( I \), \( I^\epsilon \), as the set

\[
I^\epsilon := \{p \in \mathbb{R}^d : \exists q \in I \text{ s.t. } \|p - q\|_\infty \leq \epsilon\}.
\]
Definition 5.1. Let $\mathcal{I} \subset \text{Int}(\mathbb{R}^d)$. We say that $\mathcal{I}$ is closed under thickenings if for all $\epsilon \geq 0$ and $I \in \mathcal{I}$, $I^\epsilon \in \mathcal{I}$.

For example, the set $\text{Int}_{m,n}(\mathbb{R}^d)$ given in Equation (2.14) is closed under thickenings.

We now adapt the definition of erosion distance from [75] to our setting:

Definition 5.2. Let $\mathcal{I} \subset \text{Int}(\mathbb{R}^d)$ be closed under thickenings, and $M, N : \mathbb{R}^d \to \text{vec}$. We say there is an $\epsilon$-erosion between $rk^\epsilon_M$ and $rk^\epsilon_N$ if for all $I \in \mathcal{I}$, we have

$$rk^\epsilon_M(I^\epsilon) \leq rk_N(I) \quad \text{and} \quad rk_N(I^\epsilon) \leq rk^\epsilon_M(I)$$

The erosion distance between $rk^\epsilon_M$ and $rk^\epsilon_N$ is defined as:

$$d_E(rk^\epsilon_M, rk^\epsilon_N) := \inf\{\epsilon \geq 0 : \exists \text{ an } \epsilon - \text{erosion between } rk^\epsilon_M \text{ and } rk^\epsilon_N\},$$

and $d_E(rk^\epsilon_M, rk^\epsilon_N) := \infty$ if no such erosion exists.

We remark that if $d_E(rk^\epsilon_M, rk^\epsilon_N) < \infty$, then $d_E(rk^\epsilon_M, rk^\epsilon_N)$ is a non-negative integer. We establish the following stability result:

Theorem H. Let $\mathcal{I} \subset \text{Int}(\mathbb{R}^d)$ be closed under thickenings. Then, for any $M, N : \mathbb{R}^d \to \text{vec}$,

$$d_E(rk^\epsilon_M, rk^\epsilon_N) \leq d_1(M, N). \tag{5.1}$$

Proof of Theorem H. If $d_1(M, N) = \infty$, there is nothing to prove. Let $\epsilon \geq 0$, and suppose that $\alpha : M \to N^\epsilon$ and $\beta : N \to M^\epsilon$ give an $\epsilon$-interleaving. Fix $I \in \mathcal{I}$. We will show that $rk_N(I^\epsilon) \leq rk^\epsilon_M(I)$. To this end, it suffices to show that there exist morphisms $\alpha'$ and $\beta'$ that make the following diagram commute:

$$\begin{array}{ccc}
\lim N|_{I^\epsilon} & \xrightarrow{\psi_{N|_{I^\epsilon}}} & \lim N|_{I^\epsilon} \\
\downarrow\alpha' & & \uparrow\beta' \\
\lim M|_I & \xrightarrow{\psi_{M|_I}} & \lim M|_I
\end{array} \tag{5.2}$$

Indeed, if such $\alpha'$ and $\beta'$ exist, then the limit-to-colimit map $\psi_{N|_{I^\epsilon}}$, whose rank is $rk_N(I^\epsilon)$, factors through the limit-to-colimit map $\psi_{M|_I}$, whose rank is $rk^\epsilon_M(I)$, implying the desired bound.

Define $\alpha'$ by $(\ell_p)_{p \in I} \mapsto (\alpha_p(\ell_p))_{p + \epsilon \in I}$. Then, from naturality of $\alpha$, and the fact that $p + \epsilon \in I$ implies $p \in I^\epsilon$, we have that $(\alpha_p(\ell_p))_{p + \epsilon \in I}$ is a section of $M|_I$. Define $\beta'$ by $[\nu_p] \mapsto [\beta_p(\nu_p)]$ for any $p \in I$ and $\nu_p \in M_p$. By naturality of $\beta$, and since $p + \epsilon \in I^\epsilon$, $\beta'$ is well-defined.

Fix any $p_0 \in I$. Then, both $p_0 + \epsilon$ and $p_0 - \epsilon$ belong to $I^\epsilon$. Let $(\ell_p)_{p \in I^\epsilon}$ be an element of $\lim N|_{I^\epsilon}$ and observe commutativity of the following diagram:

$$\begin{array}{ccc}
(\ell_p)_{p \in I^\epsilon} & \xrightarrow{\alpha'} & [(\ell_{p_0} - \epsilon)] = [\varphi_N(p_0 - \epsilon)(\ell_{p_0} - \epsilon)] \\
\downarrow\alpha' & & \uparrow\beta' \\
(\alpha_p(\ell_p))_{p + \epsilon \in I} & \xrightarrow{\beta'} & [\alpha_{p_0 - \epsilon}(\ell_{p_0} - \epsilon)]
\end{array}$$

Thus, $rk_N(I^\epsilon) \leq rk^\epsilon_M(I)$, and a symmetric argument gives $rk^\epsilon_M(I^\epsilon) \leq rk_N(I^\epsilon)$, so we obtain $d_E(rk^\epsilon_M, rk^\epsilon_N) \leq \epsilon$, as desired.\hfill \Box

\footnote{A similar theorem is provided in the arXiv version of [52], but not in the published version.
Remark 5.3 (Computational efficiency vs. discriminating power). Corollary 4.8 implies that, as \( m \) and \( n \) increase, the discriminating power of \( \text{rk}^{\text{Int}_{m,n}(\mathbb{R}^d)} \) strictly increases. Hence, if we let \( I = \text{Int}_{m,n}(\mathbb{R}^d) \) in Theorem H, we expect \( d_E(\text{rk}^Z_{M}, \text{rk}^Z_{N}) \) to give an increasingly better approximation to \( d_I(M, N) \) as \( m, n \) increase.

On the other hand, as \( m \) and \( n \) increase, the cost associated to computing \( d_E(\text{rk}^Z_{M}, \text{rk}^Z_{N}) \) is also expected to increase. To illustrate this, by adapting the binary-search based algorithm described in [53, Section 5],\(^{16}\) we can verify that the resulting computational cost of \( d_E \) between \( \mathbb{Z}^d \)-modules that are supported on the finite grid \( [\ell]^d \) is \( O(t^{d(m+n)} \log \ell) \) in time.

### 5.2 Stability of (restricted) ZIBs

In this section, we reinterpret Theorem H in terms of (restricted) ZIBs.

A path \( \Gamma \) in a poset \( P \) is called **rank-representing** if for every \( P \)-module \( M \), \( \text{rk}(M) = \text{rk}(M_\Gamma) \). A subset \( I \subset \text{Int}(P) \) is called **ZIB-rank-representing** if each \( I \in I \) admits at least one rank-representing path \( \Gamma \) in \( I \).

Let \( I \subset \text{Int}(\mathbb{R}^d) \) be closed under thickenings and ZIB-rank-representing. For example, when \( d = 2 \), the collection \( I \) can be \( \text{Int}^c_{m,n}(\mathbb{R}^2) \) (cf. Equation (2.15)). Let

\[
\text{ZZ}_I := \{ \Gamma : \Gamma \text{ is a rank-representing path for some } I \in I \}.
\]

For \( \Gamma \in \text{ZZ}_I \) and \( \epsilon \geq 0 \), let \( \Gamma^\epsilon \) be any path that is rank-representing of \( I^\epsilon \). We remark that \( \Gamma^\epsilon \) may not be unique but exists in \( \text{ZZ}_I \) by the assumption on \( I \). Let \( M_{\text{ZZ}_I} \) be the ZIB of \( M \) on \( \text{ZZ}_I \), i.e. the map sending each \( \Gamma \in \text{ZZ}_I \) to \( \text{barc}(M_\Gamma) \). Given another \( \mathbb{R}^d \)-module \( N \), we define the erosion distance \( d_E(M_{\text{ZZ}_I}, N_{\text{ZZ}_I}) \) as the infimum \( \epsilon \geq 0 \) for which, for any \( \Gamma \in \text{ZZ}_I \), we have \( \text{rk}(M_\Gamma) \leq \text{rk}(N_\Gamma) \) and \( \text{rk}(M_\Gamma^\epsilon) \leq \text{rk}(N_\Gamma^\epsilon) \). In light of Theorems 2.2, 2.17 and Proposition 4.19, the condition \( \text{rk}(M_\Gamma) \leq \text{rk}(N_\Gamma) \) can be read as:

\[
(\text{Multiplicity of the full bar in } \text{barc}(M_\Gamma)) \leq (\text{Multiplicity of the full bar in } \text{barc}(N_\Gamma)).
\]

**Theorem I** (Reinterpretation of Theorem H). Let \( I \subset \text{Int}(\mathbb{R}^d) \) be any collection of intervals closed under thickenings and ZIB-rank-representing. Then, for any \( \mathbb{R}^d \)-modules \( M \) and \( N \)

\[
d_E(M_{\text{ZZ}_I}, N_{\text{ZZ}_I}) \leq d_I(M, N). \tag{5.3}
\]

**Remark 5.4.** When \( d = 2 \), the collection \( I = \text{Int}^c_{m,n}(\mathbb{R}^2) \) for any \( m, n \in \mathbb{N} \) is a salient example for which this theorem is applicable. When \( d \geq 3 \), one such an example is \( I = \text{Seg}(\mathbb{R}^d) \). However, identifying some other such collections for \( d \geq 3 \) does not seem straightforward for a reason analogous to the one described in Remark 4.24.

### 6 Discussion

We have addressed the four driving questions

**Question 1.** How to restrict, if possible, the domain of the generalized rank invariant without any loss of information?

**Question 2.** Under what conditions can we more compactly encode the GRI as a ‘persistence diagram,’ even when the indexing poset \( P \) is not discrete?

\(^{16}\)This algorithm was implemented in [33].
**Question 3.** What is the trade-off between computational efficiency and the discriminating power of the GRI as the amount of the restriction varies?

**Question 4.** What proxies exist for persistence diagrams in the multi-parameter setting that can be derived from the GRI?

Below, we delve into a selection of future research directions.

- Given that in the case of $\mathbb{R}^2$ or $\mathbb{Z}^2$-modules the generalized rank invariant and the ZIB estimate each other (Section 4.3), it becomes natural to seek an efficient algorithm for computing or estimating the generalized persistence diagram of a finitely presentable $\mathbb{R}^2$-module by harnessing zigzag persistence update algorithms [38]. Such an algorithm could also potentially be useful for estimating other invariants that are closely related to the generalized rank invariant [2, 4, 9, 14, 26].

- A related direction is to study, for the case when $d \geq 3$, how to estimate the restricted or full GRI of an $\mathbb{R}^d$ or $\mathbb{Z}^d$-module via its ZIB over a set of zigzag paths of manageable size, and to utilize the results of this study for establishing a stability result for ZIB; see Remarks 4.24 and 5.4.

- Theorem I suggests the possibility of utilizing zigzag persistence for an efficient estimation of the interleaving distance.

- For the $\mathbb{Z}^2$-module $M$ given in the proof of Theorem B, the fact that $\text{rk}_{\text{Int}}^M$ is not Möbius invertible implies that there exists no finite exact sequence of interval-decomposable persistence modules $(M_i)_{i=0}^n$ and morphisms $(f_i)_{i=0}^n$ with
  \[
  0 \rightarrow M_n \xrightarrow{f_n} \cdots \rightarrow M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} M \rightarrow 0
  \]
  such that $\text{rk}_{\text{Int}}^M_{M_i} = \text{rk}_{\text{Int}}^\ker f_i + \text{rk}_{\text{Int}}^\ker f_{i-1}$ for each $i = 0, \ldots, n$. Indeed, the existence of such a sequence implies the rank decomposition $\text{rk}_{\text{Int}}^M = \text{rk}_{\text{Int}}^\ker M_{2n} - \text{rk}_{\text{Int}}^\ker M_{2n-1}$ and thus the Möbius invertibility of $\text{rk}_{\text{Int}}^M$. More generally, Remark 3.9 clarifies that there are $\mathbb{Z}^d$-modules, $d > 2$, that do not admit such finite sequences. This observation might have interesting implications in the perspective of recent studies at the intersection of representation theory and multi-parameter persistence [2, 4, 9, 14, 26].

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A Stability of the generalized rank invariants over general posets

In this section, we extend the erosion distance between the GRIs [75, 76] to the general setting of P-modules and generalize Theorem H to this general setting as well (Definition A.7 and Theorem J).

To compare P-modules, we consider the interleaving distance between P-modules, developed by Bubenik et al. [16] and expanded upon by de Silva et al. [37]. This interleaving distance is an extension of the classical interleaving distance between $\mathbb{R}^d$-modules [29, 58].

Let $P$ be a poset, viewed as a category. For two order-preserving maps $T_1, T_2 : P \to P$, we write $T_1 \leq T_2$ if for all $p \in P$, $T_1(p) \leq T_2(p)$. Let $I_P$ be the identity functor on $P$.

Definition A.1. A translation on $P$ is an endofunctor $T : P \to P$ together with a natural transformation $\eta : I_P \to T$ (i.e. $p \leq T(p)$ for all $p \in P$). A family of superlinear translations $\Omega = (\Omega_\epsilon)_{\epsilon \geq 0}$ is a family of translations $\Omega_\epsilon$ on $P$, for $\epsilon \geq 0$, such that $\Omega_0 = I_P$, and for $\epsilon, \zeta \geq 0$, $\Omega_\epsilon \Omega_\zeta \leq \Omega_{\epsilon + \zeta}$.

Throughout the following, $\Omega$ will refer to a family of superlinear translations on $P$. For $\Omega_\epsilon$ a superlinear translation on $P$ and $I \subset P$, we denote $\Omega_\epsilon(I) := \{\Omega_\epsilon(p) : p \in I\}$. For all $\epsilon \geq 0$, the translation $\Omega_\epsilon$ comes with a natural transformation $\eta_\epsilon : I_P \to \Omega_\epsilon$. For any $M : P \to \text{vec}$, this induces a natural transformation $M \eta_\epsilon : M \to M \Omega_\epsilon$. This is used to define:

Definition A.2. Two P-modules $M$ and $N$ are $\Omega_\epsilon$-interleaved if there exist a pair of natural transformations $\varphi : M \to N \Omega_\epsilon$ and $\psi : N \to M \Omega_\epsilon$ such that the diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{M \eta_\epsilon} & M \Omega_\epsilon \\
\downarrow{\psi} & & \downarrow{\psi \Omega_\epsilon} \\
N & \xrightarrow{N \eta_\epsilon} & N \Omega_\epsilon
\end{array}
\quad
\begin{array}{ccc}
M \Omega_\epsilon & \xrightarrow{M \eta_{\epsilon \Omega_\zeta}} & M \Omega_\epsilon \Omega_\zeta \\
\downarrow{\varphi \Omega_\epsilon} & & \downarrow{\varphi \Omega_\epsilon \Omega_\zeta} \\
N \Omega_\epsilon & \xrightarrow{N \eta_{\epsilon \Omega_\zeta}} & N \Omega_\epsilon \Omega_\zeta
\end{array}
$$

commutes. The pair $(\varphi, \psi)$ is said to be an $\Omega_\epsilon$-interleaving.

The interleaving distance with respect to $\Omega$ is

$$d^\Omega_I(M, N) := \inf\{\epsilon \geq 0 : M, N \text{ are } \Omega_\epsilon \text{-interleaved}\},$$

or $d^\Omega_I(M, N) := \infty$ if there is no $\Omega_\epsilon$-interleaving for any $\epsilon \geq 0$.

For example, let $P = \mathbb{Z}^n$ with the standard product order, and let $\Omega$ be the family with $\Omega_\epsilon$ the translation by $\{(\epsilon, 0, \ldots, 0), (0, \epsilon, \ldots, 0), \ldots, (0, 0, \ldots, \epsilon)\} \subset \mathbb{Z}^n$. Then $d_1^\Omega_I$ is the classical notion of interleaving used in Section 5.

We now extend the definition of erosion distance in a suitable way.

Definition A.3. Let $I \subset P$ be non-empty. For $\epsilon > 0$, we call $I^\epsilon$ the $\epsilon$-thickening of $I$, defined as:

$$I^\epsilon := \{r \in P : \exists p, p' \in I \text{ with } p \leq \Omega_\epsilon(r) \text{ and } r \leq \Omega_\epsilon(p')\}.$$

Clearly, $I \subset I^\epsilon$. For example, if $P = \mathbb{R}$, and $\Omega$ is the family with $\Omega_\epsilon$ the translation by $\epsilon$ for $\epsilon \geq 0$, then for an interval $I = [a, b]$, its $\epsilon$-thickening would be the interval $I^\epsilon = [a - \epsilon, b + \epsilon]$. Furthermore, the $\epsilon$-thickening of an interval is an interval:
**Proposition A.4.** Let $I \in \text{Int}(P)$. Then, $I^\varepsilon \in \text{Int}(P)$ for all $\varepsilon \geq 0$.

**Proof.** Let $\varepsilon \geq 0$. We need to show that $I^\varepsilon$ is non-empty, convex, and connected.

Since $I(\neq \emptyset) \subset I^\varepsilon$, we have that $I^\varepsilon$ is non-empty. Suppose $a, b \in I^\varepsilon$, and $a \leq c \leq b$. By definition, there exist $p, p_a, p_b, p_c \in I$ such that $p_a \leq \Omega_\varepsilon(a)$, $p_b \leq \Omega_\varepsilon(b)$, $a \leq \Omega_\varepsilon(p_a')$, and $b \leq \Omega_\varepsilon(p_b')$. Then, we have: $p_a \leq \Omega_\varepsilon(a) \leq \Omega_\varepsilon(c)$ and $c \leq b \leq \Omega_\varepsilon(p_b')$, and thus $c \in I^\varepsilon$, so $I^\varepsilon$ is convex.

We now establish connectivity. First note that for all $r \in I$, the point $\Omega_\varepsilon(r)$ belongs to $I^\varepsilon$, which can be seen by letting $p = p' = r$ in Definition A.3. Suppose $p, p' \in I^\varepsilon$. Then, we can find $r_p, r_{p'} \in I$ with $p \leq \Omega_\varepsilon(r_p)$ and $p' \leq \Omega_\varepsilon(r_{p'})$. As $r_p, r_{p'} \in I$, there is a chain $r_p = a_0, a_1, \ldots, a_n = r_{p'}$ of sequentially comparable elements of $I$. Then $p \leq \Omega_\varepsilon(r_p) \geq r_p = a_0, a_1, \ldots, a_n = r_{p'} \leq \Omega_\varepsilon(r_{p'}) \geq p'$ gives a chain of sequentially comparable elements of $I^\varepsilon$, so $I^\varepsilon$ is connected. 

**Definition A.5.** If $I \subset \text{Int}(P)$ and for all $I \in I$ and $\varepsilon \geq 0$, $I^\varepsilon \in I$, then we say $I$ is closed under $\Omega$-thickenings.

**Example A.6.** If $P = \mathbb{R}^2$ with the usual product order, and $\Omega$ is the family with $\Omega_\varepsilon$ the translation by $(\varepsilon, \varepsilon)$ for $\varepsilon \geq 0$, then for a rectangle $I = [a, b]$, its $\varepsilon$-thickening would be $I^\varepsilon = [a - (\varepsilon, \varepsilon), b + (\varepsilon, \varepsilon)]$. This is still a rectangle, so the collection of rectangles in $\mathbb{R}^2$ is closed under $\Omega$-thickenings.

Now we define the erosion distance for P-modules:

**Definition A.7.** Let $I \subset \text{Int}(P)$ be closed under $\Omega$-thickenings, and let $M, N$ be P-modules. We say there is an $\varepsilon$-erosion between $rk^T_M$ and $rk^T_N$ if for all $I \in I$, we have

$$rk_M(I^\varepsilon) \leq rk_N(I) \quad \text{and} \quad rk_N(I^\varepsilon) \leq rk_M(I).$$

Define the **erosion distance** (with respect to $\Omega$) between $rk^T_M$ and $rk^T_N$ as:

$$d^\Omega_{\varepsilon}(rk^T_M, rk^T_N) := \inf\{\varepsilon \geq 0 : \exists \text{ an } \varepsilon - \text{erosion between } rk^T_M \text{ and } rk^T_N\}$$

and $d^\Omega_{\varepsilon}(rk^T_M, rk^T_N) := \infty$ if $\varepsilon$-erosions do not exist for any $\varepsilon \geq 0$.

**Proposition A.8.** Fix a collection $I \subset \text{Int}(P)$ that is closed under $\Omega$-thickenings. Then, $d^\Omega_{\varepsilon}$ is an extended pseudometric on the collection $(rk^T_M : M : P \rightarrow \text{vec})$.

**Proof.** Since $\Omega_0 = I_P$, it is immediate that $d^\Omega_{\varepsilon}(rk^T_M, rk^T_M) = 0$. Symmetry is immediate from the definition.

It remains to show the triangle inequality. Note that $\Omega_\varepsilon \Omega_{\varepsilon'} \leq \Omega_{\varepsilon + \varepsilon'}$. This implies $(I^\varepsilon)^{\varepsilon'} \subset I^{\varepsilon + \varepsilon'}$. From this, if there is an $\varepsilon$-erosion between $rk^T_M$ and $rk^T_N$, and an $\varepsilon'$-erosion between $rk^T_N$ and $rk^T_L$, then for all $I \in I$:

$$rk_M(I) \geq rk_N(I^\varepsilon) \geq rk_L((I^\varepsilon)^{\varepsilon'}) \geq rk_L(I^{\varepsilon + \varepsilon'})$$

$$rk_L(I) \geq rk_N(I^{\varepsilon'}) \geq rk_M((I^{\varepsilon'})^\varepsilon) \geq rk_M(I^{\varepsilon + \varepsilon}),$$

hence there is an $(\varepsilon + \varepsilon')$-erosion between $rk^T_M$ and $rk^T_L$, as desired. 

To ensure stability of the generalized rank invariant for P-modules, we require the family $\Omega$ to satisfy two additional properties:

**Definition A.9.**

(i) **surjectivity** For all $\varepsilon \geq 0$, $\Omega_\varepsilon : P \rightarrow P$ must be surjective, and
(ii) **Order embedding** for all \( \epsilon \geq 0 \) and \( p, p' \in P, \ p \leq p' \iff \Omega_e(p) \leq \Omega_e(p') \). We call a family of superlinear translations \( \Omega \) a surjective order embedding if \( \Omega \) satisfies properties (i) and (ii).

**Remark A.10.** Note that property (ii) is a strengthening of the condition in Definition A.1, which only enforces the forward direction of the if and only if statement.

Further, note that property (ii) implies that \( \Omega_e \) is injective for all \( \epsilon \geq 0 \). This implies that if \( P \) is finite, properties (i) and (ii) imply each other. As demonstrated in the examples in **Remark A.11** below, the two properties are independent of each other when \( P \) is infinite.

For example, if \( \Omega \) is the shift by \( (|\epsilon|,|\epsilon|,\ldots,|\epsilon|) \) in \( \mathbb{Z}^n \), or more generally the shift by \( (\epsilon,\epsilon,\ldots,\epsilon) \) in \( \mathbb{R}^d \), then both properties are satisfied. In this case, we call the family \( \Omega \) a surjective order embedding.

**Theorem J.** Fix \( \Omega \) be a family of superlinear translations on \( P \) such that \( \Omega \) is a surjective order embedding. Let \( I \subseteq \text{Int}(P) \) be a collection of intervals closed under \( \Omega \)-thickenings. Then for any \( P \)-modules \( M \) and \( N \):

\[
d_{e}^{\Omega}(\text{rk}_{M}^{\Omega}, \text{rk}_{N}^{\Omega}) \leq d_{e}^{\Omega}(M, N). \tag{A.1}
\]

We omit most of the details of the proof of Theorem J as it follows the same steps as the proof of Theorem H, under adjustments to the general setting such as replacing \( p + \epsilon \) with \( \Omega_{e}(p) \). The assumption that \( \Omega \) is a surjective order embedding ensures that the map \( \alpha' : \lim N_{1} \rightarrow \lim M_{1} \) is well-defined, i.e. \( \alpha' \) sends sections to sections.

To see that \( \alpha' : \lim N_{1} \rightarrow \lim M_{1} \) sends a section to a section, let \( (\ell_{p})_{p \in I} \) be a section in \( \lim N_{1} \). \( \alpha' \) sends maps this section to the collection \( (\alpha_{\epsilon}(\ell_{p}))_{\epsilon \in \{p \in I \}} \). Since \( \Omega_{e} \) is surjective, we know this image \( (\alpha_{\epsilon}(\ell_{p})) \) contains an element for all \( p \in I \). Property (ii) alongside naturality of \( \alpha \) implies that for all \( p \leq p' \in I \) with \( \Omega_{e}(p) \leq \Omega_{e}(p') \in I \),

\[
\varphi_{M}(\Omega_{e}(p), \Omega_{e}(p'))(\alpha_{\epsilon}(\ell_{p})) = \alpha_{\epsilon'}(\ell_{p'}),
\]

demonstrating that \( (\alpha_{\epsilon}(\ell_{p}))_{\epsilon \in \{p \in I \}} \) is indeed a section.

**Remark A.11.** We show that if the family \( \Omega \) of Theorem J lacks either property (i) or property (ii), the inequality presented in Equation (A.1) cannot be guaranteed.

To see the necessity of property (i), let \( P = [0, \infty) \subseteq \mathbb{R} \) with the usual order. Let \( \Omega \) be the family of superlinear translations \( \Omega_{e} : P \rightarrow P \), where for \( \epsilon \geq 0 \) and \( \alpha \in P \), \( \Omega_{e}(\alpha) := \alpha + \epsilon \). Clearly, \( \Omega \) satisfies property (ii), but not (i).

Let \( M := k_{p} \) and \( N := k_{(0, \infty)} \). We have \( d_{\epsilon}^{\Omega}(M, N) = 0 \), as there is an \( \Omega_{e} \)-interleaving for all \( \epsilon > 0 \). However, we claim \( d_{\epsilon}^{\Omega}(\text{rk}_{M}^{\int}, \text{rk}_{N}^{\int}) = \infty \). To see this, let \( I = [0, 1] \subseteq P \). Then \( \text{rk}_{M}^{\int}(1) = 0 \) because \( N(0) = 0 \). However, for all \( \epsilon \geq 0 \), \( \text{rk}_{N}^{\int}(1) = 1 \), and so there is no \( \epsilon \geq 0 \) for which \( \text{rk}_{M}^{\int}(1) \leq \text{rk}_{N}^{\int}(1) \), implying the claim. Thus, if \( \Omega \) is not surjective, the inequality presented in Equation A.1 cannot be guaranteed.

To see the necessity of property (ii), let \( P \) be the \( x \)-axis in \( \mathbb{R}^2 \) union the negative part of the \( y \)-axis, \( \{0\} \times \langle -\infty, 0 \rangle \). Let the poset structure on \( P \) be induced by the standard order on \( \mathbb{R}^2 \). For \( \epsilon \geq 0 \), let \( \Omega_{e} \) be defined as \( \Omega_{e}((a, 0)) := (a + \epsilon, 0) \), and for \( b < 0 \), \( \Omega_{e}((b, 0)) := \begin{cases} (0, b + \epsilon), & \epsilon \leq -b \\ (\epsilon + b, 0), & \epsilon > -b \end{cases} \). It is straightforward to check that \( \Omega := (\Omega_{e})_{\epsilon \geq 0} \) forms a family of superlinear translations on \( P \). Further, it is straightforward to verify that \( \Omega_{e} \) is surjective for all \( \epsilon \geq 0 \), so property (i) holds. However, property (ii) fails. For one example, \( \Omega_{3}((-2, 0)) = (1, 0) \leq (2, 0) = \Omega_{3}((0, -1)) \), however \( -2, 0 \) and \((0, -1)\) are incomparable.
Let $M$ and $N$ be $P$-modules defined as follows. $M$ is the the direct sum of interval modules $k_P \oplus k_{(0,0)}$. Let $N$ consist of the same vector spaces as $M$ pointwise, but with maps given by, for any $a < 0$, $\varphi_N((a,0),(0,0)) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\varphi_N((0,a),(0,0)) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and for any $b > 0$, $\varphi_N((0,0),(b,0)) = [1 \ 1]$. All other admissible maps $\varphi_N(a,b) : k \to k$ are the identity. It is straightforward to check that $N$ is indecomposable, and not interval-decomposable as it has dimension 2 at $(0,0)$, so $M$ and $N$ are non-isomorphic.

We claim that $d^\Omega(M,N) = 0$. Let $\epsilon > 0$. We claim there exists interleaving maps $\psi_N : N \to M\Omega_\epsilon$ and $\psi_M : M \to N\Omega_\epsilon$. Define $\psi_N$ and $\psi_M$ pointwise as follows:

$$\psi_N|_{N(a)} := \begin{cases} \varphi_M(a,\Omega_\epsilon(a)) & a \neq (0,0) \\ \varphi_N(a,\Omega_\epsilon(a)) & a = (0,0) \end{cases}$$

$$\psi_M|_{M(a)} := \begin{cases} \varphi_N(a,\Omega_\epsilon(a)) & a \neq (0,0) \\ \varphi_M(a,\Omega_\epsilon(a)) & a = (0,0) \end{cases}$$

It is straightforward to check that $\psi_N$ and $\psi_M$ satisfy the all commutative diagrams for the interleaving condition (Definition A.2) are satisfied. Hence, there exists an $\epsilon$-interleaving between $M$ and $N$ for all $\epsilon > 0$, and so $d^\Omega(M,N) = 0$.

Lastly, we claim $d^\Omega(rk^\text{Int}_M, rk^\text{Int}_N) = \infty$. For any $I \in \text{Int}(P)$ such that $\{(0,0)\} \subsetneq I$, it is immediate that $rk^\text{Int}_M(I) = 1$, since $M$ has as a summand $k_P$. Let $I$ be the interval which is the convex hull (within $P$) of the points $(-1,0)$ and $(0,-1)$. We claim $rk^\text{Int}_N(I) = 0$. This follows immediately as $\lim N|_I = 0$. To see this limit is trivial, let $\ell := (\ell_p)_{p \in I}$ be a section of $N|_I$. Then $\exists a, b \in k$, with $\ell_{(-1,0)} = a$ and $\ell_{(0,-1)} = b$. Then by the definition of a section (Convention 3.8), we must have $\ell_{(0,0)} = \varphi_N((-1,0),(0,0))(a) = (a,0)$ and $\ell_{(0,0)} = \varphi_N((0,-1),(0,0))(b) = (0,b)$, which implies $a = b = 0$, and thus the only section $\ell$ over $N|_I$ has $\ell_p = 0$ for all $p \in I$.

Hence, for all $\epsilon \geq 0$, $rk^\text{Int}_N(I) = 0 < 1 = rk^\text{Int}_M(I^\epsilon)$, and so $d^\Omega(rk^\text{Int}_M, rk^\text{Int}_N) = \infty$ as claimed. Thus, if property (ii) does not hold, the inequality presented in Equation A.1 cannot be guaranteed.

\section*{B Proofs from Section 4.3}

\textbf{Proof of Proposition 4.19} To prove Proposition 4.19, we need the following definition and lemmas (which are also used in [40]). Recall the construction of the (co)limit from Convention 3.8.

Let $P$ be a poset and let $M$ be any $P$-module. Let $p, p' \in P$ and let $v_p \in M_p$ and $v_{p'} \in M_{p'}$. We write $v_p \sim v_{p'}$ if $p$ and $p'$ are comparable, and either $v_p$ is mapped to $v_{p'}$ via $\varphi_M(p,p')$ or $v_{p'}$ is mapped to $v_p$ via $\varphi_M(p',p)$.

\textbf{Definition B.1.} Let $\Gamma : p_0, \ldots, p_k$ be a path in $P$. A $(k+1)$-tuple $v \in \bigoplus_{i=0}^k M_{p_i}$ is called a section of $M$ along $\Gamma$ if $v_{p_i} \sim v_{p_{i+1}}$ for each $i$.

Note that $v$ is not necessarily a section of the restriction $M|_{[p_0,\ldots,p_k]}$ of $M$ to the subposet $\{p_0, \ldots, p_k\} \subset I$ [40, Example 21]. Furthermore, $\Gamma$ can contain multiple copies of the same point in $P$.
Lemma B.2. Let \( p, p' \in P \). For any vectors \( v_p \in M_p \) and \( v_{p'} \in M_{p'} \), it holds that \([v_p] = [v_{p'}]\) as elements of\(^{17}\) the colimit \( \lim M \) if and only if there exist a path \( \Gamma : p = p_0, p_1, \ldots, p_n = p' \) in \( P \) and a section \( \upsilon \) of \( M \) along \( \Gamma \) such that \( v_p = v_p \) and \( v_{p'} = v_{p'} \).

Lemma B.3. Let \( I \) be a finite interval of \( \mathbb{Z}^2 \). Let \( L := \min_{\mathbb{Z}^2}(I) \) and \( U := \max_{\mathbb{Z}^2}(I) \). Given any \( I \)-module \( M \), we have \( \lim M \cong \lim M|_L \) and \( \lim M \cong \lim M|_U \).

The isomorphism \( \lim M \cong \lim M|_L \) in Lemma B.3 is given by the canonical section extension \( e : \lim M|_L \to \lim M \). Namely,

\[
e : (v_p)_{p \in L} \mapsto (w_{p'})_{p' \in P}, \tag{B.1}
\]

where for any \( p' \in P \), the vector \( w_{p'} \) is defined as \( \varphi_M(p, p')(v_p) \) for any \( p \in L \cap p' \). The connectedness of \( L \cap p' \) guarantees that \( w_{p'} \) is well-defined. Also, if \( p \in L \), then \( w_{p'} = v_{p'} \). The inverse \( r := e^{-1} \) is the canonical section restriction. The other isomorphism \( \lim M \cong \lim M|_U \) in Lemma B.3 is given by the map \( i : \lim M|_U \to \lim M \) defined by \( [v_p] \mapsto [v_p] \) for any \( p \in U \) and any \( v_p \in M_p \); the fact that this map \( i \) is well-defined follows from Lemma B.2.

Proof of Proposition 4.19. Let \( L \) and \( U \) be as in Lemma B.3 above. Let us define \( \xi : \lim M|_L \to \lim M|_U \) by \( i^{-1} \circ \psi_M \circ e \). By construction, the following diagram commutes

\[
\begin{array}{ccc}
\lim M|_L & \xrightarrow{\xi} & \lim M|_U \\
\downarrow{=} & & \downarrow{i}\vspace{0.5cm}
\end{array}
\xrightarrow{=}
\begin{array}{ccc}
\lim M & \xrightarrow{\psi_M} & \lim M \\
\downarrow{=} & & \downarrow{=}\vspace{0.5cm}
\end{array}
\tag{B.2}
\]

where \( \psi_M \) is the canonical limit-to-colimit map of \( M \). Hence we have that \( \text{rk}(\psi_M) = \text{rk}(\xi) \). Now, it suffices to show:

\[
\text{rk}(\xi) = \text{rk}(\psi_{M_r} : \lim M_r \to \lim M_r).
\]

Let us recall the following: let \( \alpha : V_1 \to V_2 \) and \( \beta : V_2 \to V_3 \) be two linear maps. If \( \alpha \) is surjective, then \( \text{rk}(\beta \circ \alpha) = \text{rk}(\beta) \). If \( \beta \) is injective, then \( \text{rk}(\beta \circ \alpha) = \text{rk}(\alpha) \). Therefore, it suffices to show that there exist a surjective linear map \( f : \lim M_r \to \lim M_L \) and an injective linear map \( g : \lim M|_U \to \lim M_r \) such that \( \psi_{M_r} = g \circ \xi \circ f \). We define \( f \) as the canonical section \( \psi_q \), where \( q \in L \). We define \( g \) as the canonical map, i.e. \( [v_q] \mapsto [v_q] \) for any \( q \) in \( U \) and any \( v_q \in M_q \). By Lemma B.2 and by construction of \( M_r \), the map \( g \) is well-defined.

We now show that \( \psi_{M_r} = g \circ \xi \circ f \). Let \( v := (v_q)_{q \in L} \in \lim M_r \). Then, by definition of \( \psi_{M_r} \), the image of \( v \) via \( \psi_{M_r} \) is \( [v_{q_0} \in U] \) defined as in Equation (4.6). Also, we have

\[
v \xrightarrow{f} (v_q)_{q \in L} \xrightarrow{\xi} [v_{q_0}] \xrightarrow{g} [v_{q_0}] \in \lim M_r,
\]

which proves the equality \( \psi_{M_r} = g \circ \xi \circ f \).

We claim that \( f \) is surjective. Let \( r' : \lim M \to \lim M_r \) be the canonical section restriction map \( (v_q)_{q \in L} \mapsto (v_q)_{q \in L} \). Then, the restriction \( r : \lim M \to \lim M|_L \) can be seen as the composition of two restrictions \( r = f \circ r' \). Since \( r \) is the inverse of the isomorphism \( e \) in Equation (B.2), \( r \) is surjective and thus so is \( f \).

\(^{17}\)For simplicity, we write \([v_p]\) and \([v_{p'}]\) instead of \([j_p(v_p)]\) and \([j_q(v_{p'})]\) respectively where \( j_p : M_p \to \bigoplus r \in P M_r \) and \( j_{p'} : M_{p'} \to \bigoplus r \in P M_r \) are the canonical inclusion maps.
Next we claim that \( g \) is injective. Let \( i' : \prod M \to \prod M \) be defined by \( [v_q] \mapsto [v_q] \) for any \( q \in \Gamma \) and any \( v_q \in M_q \). By Lemma B.2 and by construction of \( M_\Gamma \), the map \( i' \) is well-defined. Then, for the isomorphism \( i \) in Equation (B.2), we have \( i = i' \circ g \). This implies that \( g \) is injective. \( \square \)