MOVING PLANES FOR NONLINEAR FRACTIONAL LAPLACIAN EQUATION WITH NEGATIVE POWERS

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Abstract. In this paper, we study symmetry properties of positive solutions to the fractional Laplace equation with negative powers on the whole space. We can use the direct method of moving planes introduced by Jarohs-Weth-Chen-Li-Li to prove one particular result below. If \( u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_\alpha \) satisfies

\[
(-\Delta)^{\alpha/2} u(x) + u^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n,
\]

with the growth/decay property

\[
u(x) = a|x|^m + o(1), \quad \text{as} \quad |x| \to \infty,
\]

where \( \frac{\alpha}{\alpha+1} < m < 1, \ a > 0 \) is a constant, then the positive solution \( u(x) \) must be radially symmetric about some point in \( \mathbb{R}^n \). Similar result is also true for Hénon type nonlinear fractional Laplace equation with negative powers.

1. Introduction. Analytical problems with negative powers arise naturally in the thin film equations and electrostatic micro-electromechanical system (MEMS) device \[1\] \[2\] \[8\] \[16\] \[17\] \[13\] \[9\] \[11\] \[21\] \[26\], singular minimal surface equations \[25\], Lichnerowicz equations in general relativity \[20\] \[23\] \[21\] \[22\] \[28\], and prescribed curvature equations in conformal geometry \[29\]. In the recent study, there are a few very interesting results obtained in various semilinear equations/system with fractional Laplacian \[5\] \[7\] \[24\] \[19\]. Both research directions lead us to study semilinear fractional Laplacian equations with negative powers on the whole space. Our main concern in this paper is to study the symmetry properties of positive solutions to the nonlinear fractional Laplace equation

\[
(-\Delta)^{\alpha/2} u(x) + u^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n,
\]

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with $\beta > 0$ and $\alpha \in (0, 2)$. Here $(-\Delta)^{\alpha/2}u(x)$ is the fractional Laplacian of a regular function $u(x)$, which will be defined below. We now point out one important special case which will guide us to propose suitable conditions for the symmetry properties of non-negative solutions to (1). When $\alpha = 2$, $\beta = 1$, the equation (1) becomes $-\Delta u + u^{-1} = 0$ and this model arises in differential geometry [25] and one can easily check that $u(x) = C|x|$ (with $C = 1/\sqrt{n - 1}$) is a solution and $|x|$ is only in the class $C^1(R^n)$.

Recall that the fractional Laplacian $(-\Delta)^{\alpha/2}u(x)$ ($\alpha \in (0, 2)$) is well-defined for $u \in C^{1,1}_{loc}(R^n) \cap L_\alpha$. Here

$$L_\alpha = \{u \in L^1_{loc}(R^n)| \int_{R^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}}dx < \infty \}.$$  

Note that for $u \in C^{1,1}_{loc} \cap L_\alpha$, the fractional Laplacian is a nonlocal operator taking the form

$$(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha}PV \int_{R^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}}dy,$$

where $0 < \alpha < 2$, $C_{n,\alpha} > 0$ is a uniform constant, and PV stands for the Cauchy principal value. The difficulties in the study of nonlinear fractional Laplacian equation (1) are the non-locality of the fractional Laplacian and the negative powers. To circumvent the non-locality trouble, Caffarelli and Silvestre [4] introduced the extension method that reduced a nonlocal problem into a local one in higher dimensions. Their extension method can be stated as below. For a function $u : R^n \rightarrow R$, define the extension $U : R^n \times [0, \infty) \rightarrow R$ that satisfies

$$\begin{cases} 
\text{div}(y^{1-\alpha}\nabla U) = 0, & (x, y) \in R^n \times [0, \infty), \\
U(x, 0) = u(x). 
\end{cases}$$

They showed that

$$(-\Delta)^{\alpha/2}u(x) = -C_{n,\alpha} \lim_{y \rightarrow 0^+} y^{1-\alpha} \frac{\partial U}{\partial y}, \ x \in R^n.$$  

We may use the above extension method to reduce the nonlocal problem into a local one for $U(x, y)$ in one higher dimensional half space $R^n \times [0, \infty)$ and apply the method of moving planes to the local problem

$$\begin{cases} 
\text{div}(y^{1-\alpha}\nabla U) = 0, & (x, y) \in R^n \times [0, \infty), \\
-\lim_{y \rightarrow 0^+} y^{1-\alpha} \frac{\partial U}{\partial y} = -U^{-\beta}(x, 0), & x \in R^n,
\end{cases} \ (2)$$

as in [3]. Thanks to the works [5] and [15] (also [12], [27], and [10]), we can use the direct method of moving planes to study symmetry properties of non-negative solutions to the equation (1). This will be the first part of this paper.

One of our main results is below

**Theorem 1.1.** Let $\beta > 0$. Assume that a positive function $u \in C^{1,1}_{loc}(R^n) \cap L_\alpha$ satisfies the equation (1) with the growth/decay property

$$u(x) = a|x|^m + o(1), \ \text{as} \ |x| \rightarrow \infty,$$

where $\frac{\alpha}{\beta+1} < m < 1$, $a > 0$ is a constant. Then the positive solution $u(x)$ must be radially symmetric and monotone increasing about some point in $R^n$.

**Remark 1.** If $u(x) = a|x|^m + o(1)$ near infinity, then by the condition $u \in L_\alpha$, $m$ must be smaller than $\alpha$. Hence when $0 < \alpha \leq 1$ our assumption on $m$ automatically becomes $\frac{\alpha}{\beta+1} < m < \alpha$. It seems to us that it is difficult to use the direct method
of moving planes without the growth/decay assumption about the solution \( u(x) \). However, at this moment, we don’t know any example of such solutions with the growth/decay assumption to the fractional problem (1). Similar result is true for the positive functions \( u \in C^{1,1}_{loc}(\mathbb{R}^n) \cap L_\alpha \) satisfy

\[
(-\Delta)^{\alpha/2}u(x) + (1 + u(x))^{-\beta} = 0, \quad x \in \mathbb{R}^n,
\]

with the growth/decay property.

**Remark 2.** To a first look, one may want to weaken the growth/decay property (3) in Theorem 1.1. The conclusion of Theorem 1.1 is still true if we replace growth/decay property (3) by the weak growth/decay property that \( u(x) = a|x|^m + o(1) \) for \( |x| \) large, where \( a \) is a positive constant, \( 0 < m < 1 \) and \( m > \frac{\alpha}{\beta+1} \), and \( u \) is monotone increasing in \( |x| \) about the origin. See Theorem 1.2 below. How to make a sharp condition to replace the growth/decay property (3) is an open question.

We now compare Theorem 1.1 above with Theorem 1.1 in [14] when \( n = 3 \). In [14], the \( \alpha \) there is \( \frac{2}{\nu+1} \), which corresponds to \( \frac{\alpha}{\beta+1} \) in our case with \( \nu = \beta \). Guo and Wei imposed the condition \( \nu \leq 3 \) in [14] and we did not require this in Theorem 1.1 above. Though the method used by Guo and Wei may not be useful to our case with the fractional Laplacian, the results in [14] for standard Laplacian operator are sharp in the sense that the function \( u(x) = C|x|^{-\frac{n}{\nu}} \) is the exact radially symmetric solution to their equation (I). In the spirit of Theorem 1.1 in [14], we expect Theorem 1.1 above is also true for \( \frac{\alpha}{\beta+1} = m < 1 \) and \( 0 < \beta \leq \frac{n}{\max(2\alpha-n,0)} \) and we leave this as an open question.

In the second part of the paper, we study the symmetry properties of non-negative solutions to Hénon type fractional Laplace equation

\[
(-\Delta)^{\alpha/2}u(x) + |x|^\sigma u^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{4}
\]

where \( \sigma < 0 \) and \( \beta > 0 \) are constants. Note that when we use the Kelvin transform to the solution \( u(x) \) to (1), we are lead to the nonlinear problem above. In fact, let \( x^0 \) be a point in \( \mathbb{R}^n \), and

\[
\tilde{u}(x) = \frac{1}{|x - x^0|^{n-\alpha}} u \left( \frac{x - x^0}{|x - x^0|^\tau} + x^0 \right), \quad x \in \mathbb{R}^n \setminus \{x^0\}
\]

be the Kelvin transform of \( u \) centered at \( x^0 \). Then as in (55) of the paper [5], we have

\[
(-\Delta)^{\alpha/2}\tilde{u}(x) = -\frac{\tilde{u}^{-\beta}(x)}{|x - x^0|^{n-\alpha}}, \quad x \in \mathbb{R}^n \setminus \{x^0\}, \tag{5}
\]

with \( \tau = n + \alpha + \beta(n - \alpha) \). Obviously, when \( x^0 = 0 \), the equation (5) reduces to the equation (4) above with \( \sigma = -\tau \). Similar to the classical Hénon equation in the whole space, one may consider the Hénon type equation with negative powers

\[
(-\Delta)^{\alpha/2}u(x) + |x|^\sigma u^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{6}
\]

where \( \beta > 0 \) and \( 0 < \sigma_1 < n + \alpha + \beta(n - \alpha) \) are constants. Let

\[
\bar{u}(x) = \frac{1}{|x|^{n-\alpha}} u \left( \frac{x}{|x|^\tau} \right), \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

Then by (6), \( \bar{u}(x) \) satisfies

\[
(-\Delta)^{\alpha/2}\bar{u}(x) + |x|^\sigma \bar{u}^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{7}
\]
where \( \sigma = \sigma_1 - [n + \alpha + \beta(n - \alpha)] < 0 \). The equation (7) belongs to the class of (4). In general, one may consider the Hénon type equation

\[
(-\Delta)^{\alpha/2} u(x) + |x|^\gamma u^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n \setminus \{0\},
\]

where \( \beta > 0 \) and \(-\infty < \sigma_1 < n + \alpha + \beta(n - \alpha) \) are constants. Let

\[
\tilde{u}(x) = \frac{1}{|x|^{n-\alpha}} u \left( \frac{x}{|x|^2} \right), \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

Then by (8), \( \tilde{u}(x) \) satisfies

\[
(-\Delta)^{\alpha/2} \tilde{u}(x) + |x|^\gamma \tilde{u}^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n \setminus \{0\},
\]

where \( \sigma = \sigma_1 - [n + \alpha + \beta(n - \alpha)] < 0 \). The equation (9) belongs to the class of (4).

To our best knowledge, there is almost no symmetry result for non-negative solutions to equation (4). There is one more difficulty in the study of equation (4), which is that the equation (4) has a singularity at \( x = 0 \).

We have the following result about (4).

**Theorem 1.2.** Assume \( \beta > 0 \) and \( \sigma < 0 \) are constants. Assume that \( u \in C_{\text{loc}}^{1,1}(\mathbb{R}^n \setminus \{0\}) \cap L_\alpha \) is a positive solution of the equation (4) with the growth/decay property that \( u(x) = a|x|^{\alpha} + o(1) \) for \( |x| \) large, where \( a \) is a positive constant, \( 0 < m < 1 \) and \( m > \frac{\alpha+n}{n+1} \), and \( u \) is monotone increasing in \( |x| \) about the origin. Then \( u \) must be radially symmetric about origin.

The direct method of moving planes has some advantages. In [5], the authors first developed the direct method of moving planes for the semilinear fractional Laplacian with subcritical and critical Sobolev exponent on the whole space (see [15] for the bounded domain case), which overcome the necessary of imposing extra assumptions on the solutions when using the extension method or the equivalent integral equation method. One typical beautiful result in [5] is below. Consider the non-trivial non-negative solutions to the problem

\[
(-\Delta)^{\alpha/2} u = u^p(x), \quad x \in \mathbb{R}^n,
\]

where \( 1 < p \leq \frac{n+\alpha}{n-\alpha} \). Then for any \( u \in L_\alpha \cap C_{\text{loc}}^{1,1} \) being a non-negative solution of equation (10), one has that (i) in the critical case \( p = \frac{n+\alpha}{n-\alpha} \), \( u \) is radially symmetric and monotone decreasing about some point and (ii) in the subcritical case \( 1 < p < \frac{n+\alpha}{n-\alpha} \), \( u \equiv 0 \).

The key ingredients in the direct method of Chen-Li-Li [5] are maximum principles of decay at infinity and narrow region for the anti-symmetric functions, for the convenience, we put them in the appendix (see lemmata 4.1 and 4.2). One may see [19] for the system case. Here is the key idea. Choose any direction to be the \( x_1 \) direction. For any \( x \in \mathbb{R}^n, \lambda \in \mathbb{R} \), let

\[
x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n), \quad T_\lambda = \{x \in \mathbb{R}^n | x_1 = \lambda\},
\]

\[
\Sigma_\lambda = \{x \in \mathbb{R}^n | x_1 < \lambda\},
\]

and

\[
u_\lambda(x) = u(x^\lambda).
\]

Let

\[
w_\lambda(x) = u(x) - u_\lambda(x)
\]

and

\[
\Sigma^-_\lambda = \{x \in \Sigma_\lambda | w_\lambda(x) < 0\}.
\]
We want to show $\Sigma^\lambda = \emptyset$ for $\lambda < 0$. To this end we consider the inequality or equation for $w_{\lambda}$. In our case, for $x \in \Sigma^\lambda$,

$$(-\Delta)^{\alpha/2} w_{\lambda}(x) = u_{\lambda}^{-\beta}(x) - u^{-\beta}(x) = \beta \xi_{\lambda}^{-\beta-1} w_{\lambda}(x) \geq \beta u^{-\beta-1} w_{\lambda}(x),$$

where $\xi_{\lambda}(x)$ between $u(x)$ and $u_{\lambda}(x)$, that is,

$$(-\Delta)^{\alpha/2} w_{\lambda}(x) + c(x) w_{\lambda}(x) \geq 0,$$

with 

$$c(x) = -\beta u^{-\beta-1}(x).$$

Every time when applying the method of moving planes to our problems (1) and (4), we need to check that the coefficient $c(x)$ satisfies the conditions in those two maximum principles, that is to check $\lim_{|x| \to \infty} |x|^\alpha c(x) \geq 0$ and $c(x)$ bounded from below on the target region. Our assumptions in Theorems 1.1 and 1.2 will play roles in these steps. The verifications of these steps are really non-trivial works in the problems above with negative powers. Related symmetry results for fractional Laplacian equation or system with negative powers have been studied by integral method of moving planes ([6]) in [21] [18].

The plan of this paper is below. In section 2, we present the proof of Theorem 1.1. In the proof, we need to set up a key lemma. In section 3, we give the proof of Theorem 1.2.

2. Fractional Laplace equation with negative powers. In this section, we consider the fractional Laplace equation with negative powers (1), that is

$$(-\Delta)^{\alpha/2} u(x) + u^{-\beta}(x) = 0, \quad x \in \mathbb{R}^n, \quad (11)$$

where $\beta > 0$. The growth/decay property (3) about the non-negative $u(x)$ is important for us to use the direct method of moving planes. If one use the extension method and apply the method of moving planes to the extended local problem, one may need to assume the growth/decay property about the extended solution, which is not natural.

Proof. (of Theorem 1.1) We may take $a = 1$. We may choose any direction to be the $x_1$ direction. For any $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, let

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n), u_{\lambda}(x) = u(x^\lambda),$$

$$T_{\lambda} = \{x \in \mathbb{R}^n | x_1 = \lambda\}, \Sigma_{\lambda} = \{x \in \mathbb{R}^n | x_1 < \lambda\},$$

and

$$v_{\lambda}(x) = u(x) - u_{\lambda}(x),$$

$$\Sigma_{\lambda}^- = \{x \in \Sigma_{\lambda} | v_{\lambda}(x) < 0\}.$$

Then for $x \in \Sigma_{\lambda}^-$,

$$(-\Delta)^{\alpha/2} v_{\lambda}(x) = u_{\lambda}^{-\beta}(x) - u^{-\beta}(x) = \beta \xi_{\lambda}^{-\beta-1} v_{\lambda}(x) \geq \beta u^{-\beta-1} v_{\lambda}(x),$$

where $\xi_{\lambda}(x)$ between $u(x)$ and $u_{\lambda}(x)$, that is,

$$(-\Delta)^{\alpha/2} v_{\lambda}(x) + c(x) v_{\lambda}(x) \geq 0,$$

(12)
with
\[ c(x) = -\beta u^{-\beta - 1}(x). \]

Since \( u(x) = |x|^m + o(1) \) near infinity, it holds that
\[ u^{-\beta - 1}(x) = \frac{1}{|x|^{m(\beta + 1)}} + o\left(\frac{1}{|x|^{m(\beta + 1)}}\right) \]
near infinity. Hence \( c(x) \) satisfies
\[ \lim_{|x| \to \infty} |x|^\alpha c(x) \geq 0, \]
which is needed when using the maximum principle of decay at infinity [5] (see also Lemma 4.1). Furthermore \( c(x) \) is clearly bounded from below and thus satisfies the condition in the maximum principle of narrow region [5] (see also Lemma 4.2). To applying those maximum principles we only need to show that for \( x \in \Sigma \),
\[ \lim_{|x| \to \infty} v_\lambda(x) \geq 0. \]

Notice that, for \( x \in \Sigma, |x| \geq |x^\lambda| \) when \( \lambda \leq 0 \). We can show that for \( x \in \Sigma \),
\[ \lim_{|x| \to \infty} v_\lambda(x) \geq 0. \] (14)

To this end, we need the following lemma.

**Lemma 2.1.** For any \( \sigma > 1 \) and \( a < b \), there exists \( N = N(\sigma) > 0 \) such that
\[ a^\sigma - b^\sigma \leq (a - b)N(\sigma)(a^{\sigma - 1} + b^{\sigma - 1}). \]

The proof of this elementary fact is given in the appendix.

From the above lemma, when \( \lambda \leq 0 \), to \( a = |x^\lambda|^m, b = |x|^m, \sigma = \frac{1}{m} \), we get that
\[ \frac{|x^\lambda|^2 - |x|^2}{|x|^m - |x|^{\lambda m}} = \frac{a^{2\sigma} - b^{2\sigma}}{a - b} \geq \frac{(a^\sigma + b^\sigma)(a^\sigma - b^\sigma)}{a - b} \geq N(\sigma)(a^{\sigma - 1} + b^{\sigma - 1})(a^\sigma + b^\sigma). \]
Then
\[ 0 \leq |x|^m - |x^\lambda|^m = b - a \leq \frac{b^{2\sigma} - a^{2\sigma}}{N(\sigma)(a^{\sigma - 1} + b^{\sigma - 1})(a^\sigma + b^\sigma)} \]
\[ = \frac{b^\sigma - a^\sigma}{N(\sigma)} \cdot \frac{1}{a^{\sigma - 1} + b^{\sigma - 1}} \]
\[ = \frac{|x^\lambda|^2 - |x|^2}{|x|^m} \cdot \frac{1}{|x^\lambda| + |x|} \cdot \frac{1}{N(\sigma)(a^{\sigma - 1} + b^{\sigma - 1})} \]
\[ = \frac{4\lambda^2 - 4\lambda x_1}{|x^\lambda| + |x|} \cdot \frac{1}{N(\sigma)(a^{\sigma - 1} + b^{\sigma - 1})} \to 0, \] (15)
as \( |x| \to \infty \) since \( \frac{4\lambda^2 - 4\lambda x_1}{|x^\lambda| + |x|} \) is bounded. Recall that
\[ v_\lambda(x) = |x|^m - |x^\lambda|^m + o(1). \]

Hence \( \lim_{|x| \to \infty} v_\lambda(x) = 0 \) for \( \lambda \leq 0 \). Thus (14) holds. Similarly, \( \lim_{|x| \to \infty} -v_\lambda(x) = 0 \) for \( \lambda \geq 0 \).

This indicates that if \( v_\lambda \) is negative somewhere in \( \Sigma \), then the negative minima of \( v_\lambda \) were attained in the interior of \( \Sigma \).

**Step 1.** (Move the plane along \( x_1 \) direction from near \( -\infty \))

In this step, we show that for \( \lambda \) sufficiently negative,
\[ v_\lambda(x) \geq 0, x \in \Sigma. \] (16)
Since \( v_\lambda(x) \) satisfies the following
\[
\begin{cases}
-\Delta^{\alpha/2} v_\lambda(x) + c(x)v_\lambda(x) \geq 0, & x \in \Sigma^\lambda_0, \\
v_\lambda(x) \geq 0, & x \in \Sigma_\lambda \setminus \Sigma^\lambda_0, \\
v_\lambda(x^\lambda) = -v_\lambda(x), & x \in \Sigma_\lambda,
\end{cases}
\]
with \( c(x) \sim \frac{1}{|x|^{n+\alpha}} \) for \( |x| \) large, applying the maximum principle of decay at infinity to \( v_\lambda(x) \), we conclude that, there exists a \( R_0 > 0 \) (independent of \( \lambda \) as in line 1 of page 422 in [5]), such that if \( \tilde{x} \) is a negative minimum of \( v_\lambda(x) \) in \( \Sigma_\lambda \), then
\[
|x| \leq R_0.
\]
That is, for \( \lambda \) sufficiently negative, (16) holds.

**Step 2.** (Move the plane to the limiting position which gives the symmetry about the function \( u \))

(16) provides a starting point to move the plane, now move the plane \( T_\lambda \) as long as (16) holds. Define
\[
\lambda_0 = \sup\{\lambda|w_\mu(x) \geq 0, \forall x \in \Sigma_\mu, \mu \leq \lambda\}.
\]
Then \( \lambda_0 < \infty \) since we can also move the plane from \( \lambda \) sufficiently positive.

In this part, we show that
\[
v_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}.
\]
Assume not. Then there exists some \( \hat{x} \in \Sigma_{\lambda_0} \) such that
\[
v_{\lambda_0}(\hat{x}) > 0.
\]
Since we already have \( v_{\lambda_0}(x) \geq 0, \quad x \in \Sigma_{\lambda_0} \), by strong maximum principle, we have
\[
v_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}.
\]
In fact, if there exists some \( \hat{x} \) such that
\[
v_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} v_{\lambda_0}(x) = 0,
\]
then
\[
(-\Delta)^{\alpha/2} v_{\lambda_0}(\hat{x}) = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-v_{\lambda_0}(y)}{|\hat{x} - y|^{n+\alpha}} dy
\]
\[
= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-v_{\lambda_0}(y)}{|\hat{x} - y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_0}} \frac{-v_{\lambda_0}(y)}{|\hat{x} - y|^{n+\alpha}} dy
\]
\[
= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-v_{\lambda_0}(y)}{|\hat{x} - y|^{n+\alpha}} dy + \int_{\Sigma_{\lambda_0}} \frac{v_{\lambda_0}(y)}{|\hat{x} - y|^{n+\alpha}} dy
\]
\[
= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|\hat{x} - y|^{n+\alpha}} - \frac{1}{|\hat{x} - y|^{n+\alpha}} v_{\lambda_0}(y) \right) dy
\]
< 0,
which contradicts
\[
(-\Delta)^{\alpha/2} v_{\lambda_0}(\hat{x}) = u_{\lambda_0}^{-\beta}(\hat{x}) - u^{-\beta}(\hat{x}) = 0.
\]
It follows from (19) that there exists a small \( \delta \) and a constant \( a_0 > 0 \), such that
\[
v_{\lambda_0}(x) \geq a_0, \quad x \in \Sigma_{\lambda_0} - \delta \cap B_{R_0}(0).
\]
Since \( v_\lambda(x) \) depends on \( \lambda \) continuously, there exists \( \epsilon > 0 \) and \( \epsilon < \delta \) such that
\[
v_\lambda(x) \geq 0, \quad x \in \Sigma_{\lambda_0} - \delta \cap B_{R_0}(0),
\]
for all \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \).

Since \( v_\lambda(x) \) satisfies

\[
\begin{cases}
\ (-\Delta)^{\alpha/2} v_\lambda(x) + c(x)v_\lambda(x) \geq 0, & x \in \Sigma^-_\lambda \setminus \Sigma_{\lambda_0 - \delta}, \\
\ v_\lambda(x) \geq 0, & x \in \Sigma_{\lambda_0 - \delta}, \\
\ v_\lambda(x^\lambda) = -v_\lambda(x), & x \in \Sigma_\lambda,
\end{cases}
\]

where \( \Sigma^-_\lambda \setminus \Sigma_{\lambda_0 - \delta} \) is a narrow region and \( c(x) \) bounded from below, applying narrow region principle, we obtain,

\[
v_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}.
\] (21)

Combining (19), (20), and (21), we conclude that for all \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \),

\[
v_\lambda(x) \geq 0, \ \forall x \in \Sigma_\lambda,
\]

which is a contradiction to the definition of \( \lambda_0 \). Therefore,

\[
v_{\lambda_0}(x) \equiv 0, \ \forall x \in \Sigma_{\lambda_0}.
\]

Thus the positive solutions must be radially symmetric about some point in \( \mathbb{R}^n \). The monotonicity is a consequence of the fact that (16) holds for all \(-\infty < \lambda \leq \lambda_0 \).

This completes the proof of Theorem 1.1. \( \square \)

3. \textbf{Hénon type nonlinear fractional Laplace equation with negative powers.} In this section we consider positive solutions to the Hénon type nonlinear fractional Laplace equation with negative powers (4), which we recall here,

\[
(-\Delta)^{\alpha/2} u(x) + |x|^{-\beta}u(x) = 0, \ \forall x \in \mathbb{R}^n \setminus \{0\},
\] (22)

where \( \sigma < 0 \) and \( \beta > 0 \) are constants. As we pointed out before, this equation has a singularity at the origin.

\textbf{Proof.} (of Theorem 1.2). We may take \( a = 1 \) in the growth/decay condition about \( u(x) \). We use the notations \( x^\lambda, u_\lambda, v_\lambda, T_\lambda, \Sigma_\lambda \) and \( \Sigma^-_\lambda \) as in the section 2. Then for \( \lambda < 0 \) and \( x \in \Sigma^-_\lambda \setminus \{0^\lambda\} \), \( v_\lambda \) satisfies

\[
\begin{align*}
\ (-\Delta)^{\alpha/2} v_\lambda(x) &= |x|^\sigma u^-_\lambda(x) - |x|^{-\beta}u(x) \\
&= [\ |x|^\alpha - |x|^\sigma] u^-_\lambda(x) + |x|^{\sigma}[u^-_\lambda(x) - u^{-\beta}(x)] \\
&\geq \beta |x|^{\sigma} \xi_\lambda^{-\beta}v_\lambda(x) \\
&\geq \beta |x|^\sigma u^{-\beta-1}v_\lambda(x),
\end{align*}
\] (23)

where \( \xi_\lambda(x) \) between \( u(x) \) and \( u_\lambda(x) \). This implies,

\[
\ (-\Delta)^{\alpha/2} v_\lambda(x) + c(x)v_\lambda(x) \geq 0,
\] (24)

with

\[
\begin{align*}
c(x) &= -\beta |x|^{\sigma} u^{-\beta-1}(x).
\end{align*}
\] (25)

Since \( u(x) = |x|^m + o(1) \) near infinity, it holds that \( c(x) \sim \frac{1}{|x|^{m(\beta+1)-\sigma}} + o\left(\frac{1}{|x|^{m(\beta+1)-\sigma}}\right) \). By our assumption \( m > \frac{\alpha+\sigma}{\beta+1} \), we see

\[
c(x) \sim o\left(\frac{1}{|x|^\sigma}\right),
\]

and it follows that \( c(x) \) is clearly bounded from below on \( \Sigma_\lambda \) since the singular point \( 0 \notin \Sigma_\lambda \).
To applying maximum principles of decay at infinity and narrow region, we need pay more attention near the singular point, that is, we need to show the following two points:

(i) \[ \lim_{|x| \to \infty} v_\lambda(x) \geq 0. \]

(ii) \[ \lim_{x \to 0^\lambda} v_\lambda(x) > 0. \]

Note that (i) is true by our monotone assumption about \( u(x) \) in \( |x| \).

**Step 1.** (Move the plane along \( x_1 \) direction from near \(-\infty\))

In this step, we show that for \( \lambda \) sufficiently negative, \( v_\lambda(x) \geq 0 \), \( x \in \Sigma_\lambda \setminus \{0^\lambda\} \).

Near the singular point \( 0^\lambda \) of \( v_\lambda \), we claim that \( \Sigma_\lambda^- \) has no intersection with \( B_{\epsilon}(0^\lambda) \) for some \( \epsilon > 0 \) small. To see this, we just notice that for \( \lambda \) sufficiently negative, \( 0^\lambda \) is also sufficiently negative and

\[ \lim_{x \to 0^\lambda} v_\lambda(x) = \lim_{x \to 0^\lambda} (u(x) - u_\lambda(x)) = \lim_{x \to 0^\lambda} u(x) - u(0) > 0 \]

by the growth/decaying assumption.

Since \( c(x) \sim o\left(\frac{1}{|x|^{1+\alpha}}\right) \), we can then obtain by decay at infinity that there exists a \( R_0 > 0 \) (which is independent of \( \lambda \)), such that if \( \bar{x} \) is a negative minimum of \( w_\lambda(x) \) in \( \Sigma_\lambda \), then

\[ |\bar{x}| \leq R_0. \] (27)

That is, for \( \lambda \) sufficiently negative, (26) holds.

**Step 2.** (Move the plane to the limiting position which gives the symmetry about the function \( u \))

From the relation (23) we can move the plane \( T_\lambda \) to the right as long as (26) holds to its limiting position. Define

\[ \lambda_0 = \sup\{\lambda < 0 | v_\mu(x) \geq 0, \forall x \in \Sigma_\mu \setminus \{0^\mu\}, \mu \leq \lambda\}. \]

Claim that

\[ \lambda_0 = 0, \ v_{\lambda_0}(x) \equiv 0, \ x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}. \] (28)

The idea to prove this claim is below. Suppose \( \lambda_0 < 0 \), we can show that the plane \( T_\lambda \) can be moved further right to cause a contradiction to the definition of \( \lambda_0 \) via the use of maximum principle of decay at infinity and the narrow region principle.

Note that by our monotonicity assumption on \( u \),

\[ \lim_{x \to 0^{\lambda_0}} v_{\lambda_0}(x) = \lim_{x \to 0^{\lambda_0}} (u(x) - u_{\lambda_0}(x)) = \lim_{x \to 0^{\lambda_0}} u(x) - u(0) > 0. \]

That is, there exists \( \epsilon > 0 \) small and \( c_0 > 0 \) such that

\[ v_{\lambda_0}(x) \geq c_0, \ \forall x \in B_{\epsilon}(0^{\lambda_0}) \setminus \{0^{\lambda_0}\}. \] (29)

The relation (27) tells us that the negative minimum of \( v_{\lambda_0}(x) \) cannot attained in \( B_{R_0}(0) \). We want to show that it can not be attained inside of \( B_{R_0}(0) \). That is, we show that for \( \lambda \) sufficiently close to \( \lambda_0 \),

\[ v_\lambda(x) \geq 0, \ x \in (\Sigma_\lambda \cap B_{R_0}(0)) \setminus \{0^\lambda\}. \] (30)
When \( \lambda_0 < 0 \), we already have \( w_{\lambda_0}(x) \geq 0, \ x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\} \). In fact, there holds
\[
v_{\lambda_0}(x) > 0, \ x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}.
\] (31)

Assume not. By (29), there exists some point \( \hat{x} \) such that
\[
v_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}} v_{\lambda_0}(x) = 0,
\]
and
\[
(-\Delta)^{\alpha/2} v_{\lambda_0}(\hat{x}) < 0,
\]
while
\[
(-\Delta)^{\alpha/2} v_{\lambda}(\hat{x}) = |\hat{x}^\lambda|^\sigma u_{\alpha}^{-\beta}(\hat{x}) - |\hat{x}|^\sigma u^{-\beta}(\hat{x})
= \left[|\hat{x}^\lambda|^\sigma - |\hat{x}|^\sigma\right] u_{\alpha}^{-\beta}(\hat{x})
> 0.
\]

It follows from (31) that there exists a constant \( a_0 > 0 \), such that
\[
v_{\lambda_0}(x) \geq a_0, \ x \in (\Sigma_{\lambda_0} - \delta \cap B_R(0)) \setminus \{0^{\lambda_0}\}.
\] (32)

Since \( c(x) \) depends on \( \lambda \) continuously, there exists \( \epsilon > 0 \) and \( \epsilon < \delta \), such that for all \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \), we have
\[
v_{\lambda}(x) \geq 0, \ x \in (\Sigma_{\lambda_0} - \delta \cap B_R(0)) \setminus \{0^{\lambda_0}\}.
\] (33)

Combining these results, we get that for all \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \),
\[
v_{\lambda}(x) \geq 0, \ x \in \Sigma_{\lambda} \setminus \{0^{\lambda}\}.
\]

This contradicts the definition of \( \lambda_0 \). Hence, we must have
\[
\lambda_0 = 0, \ v_{\lambda_0}(x) = 0, \ x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}.
\]

This completes the proof of Theorem 1.2. \( \Box \)

4. Appendix. We now give the proof of Lemma 2.1.

Proof. Recall that \( \sigma > 1 \) in this lemma. We may assume \( a = 1, \ b > 1 \).

Since
\[
\lim_{b \to \infty} \frac{b^\sigma - 1}{(b - 1)(1 + b^\sigma - 1)} = 1,
\]
and
\[
\lim_{b \to 1^+} \frac{b^\sigma - 1}{(b - 1)(1 + b^\sigma - 1)} = \frac{\sigma}{2} > \frac{1}{2},
\]
so there exists \( N = N(\sigma) > 0 \) satisfies
\[
\frac{b^\sigma - 1}{b - 1} \geq N(\sigma)(1 + b^\sigma - 1).
\]

This completes the proof of Lemma 2.1. \( \Box \)
The following two lemmas are the key ingredients needed in applying the standard direct method of moving planes, which have been explained in [5] with detailed proof.

Let $T$ be a hyperplane in $\mathbb{R}^n$. Without loss of generality, we may assume that

$$T = \{ x \in \mathbb{R}^n | x_1 = \lambda, \lambda \in \mathbb{R} \}.$$

Let

$$\tilde{x} = (2\lambda - x_1, x_2, \cdots, x_n)$$

be the reflection of $x$ about the plane $T$. Denote

$$H = \{ x \in \mathbb{R}^n | x_1 < \lambda \}, \quad \tilde{H} = \{ x|\tilde{x} \in H \}.$$

**Lemma 4.1. (Decay at Infinity)**

Let $\Omega$ be a unbounded region in $H$. Assume

$$\begin{cases}
(-\Delta)^{\alpha/2}u(x) + c(x)u(x) \geq 0, & x \in \Omega, \\
u(x) \geq 0, & x \in H \setminus \Omega, \\
u(\tilde{x}) = -u(x), & x \in H,
\end{cases}$$

with

$$\lim_{|x| \to \infty} |x|^\alpha c(x) \geq 0.$$  \hfill (35)

There exists a constant $R_0 > 0$ such that if

$$u(x^0) = \min_\Omega u(x) < 0,$$

then

$$|x^0| \leq R_0.$$

**Remark 3.** As pointed out in Theorem 3 in [5], $R_0$ is depending on $c(x)$, but independent of $u$.

**Lemma 4.2. (Narrow Region Principle)**

Let $\Omega$ be a bounded narrow region in $H$ such that it is contained in

$$\{ x | \lambda - l < x_1 < \lambda \}$$

with small $l > 0$. Suppose that $u \in C^{1,1}_{loc}(\Omega) \cap L_\alpha$ and is lower semi-continuous on $\Omega$. If $c(x)$ is bounded from below in $\Omega$ and

$$\begin{cases}
(-\Delta)^{\alpha/2}u(x) + c(x)u(x) \geq 0, & x \in \Omega, \\
u(x) \geq 0, & x \in H \setminus \Omega, \\
u(\tilde{x}) = -u(x), & x \in H,
\end{cases}$$

then

$$u(x) \geq 0, \quad x \in \Omega.$$  \hfill (36)

This conclusion holds for unbounded region $\Omega$ if we further assume that

$$\lim_{|x| \to \infty} u(x) \geq 0.$$  

Similar lemmata can be obtained for half spaces (see [5]).

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