Abstract. Rigid tree honeycombs were introduced by Knutson, Tao, and Woodward \cite{9} and they were shown in \cite{2} to be sums of extreme rigid honeycombs, with uniquely determined summands up to permutations. Two extreme rigid honeycombs are essentially the same if they have proportional exit multiplicities and, up to this identification, there are countably many equivalence classes of such honeycombs. We describe two ways to approach the enumeration of these equivalence classes. The first method produces a (finite) list of all rigid tree honeycombs of fixed weight by looking at the locking patterns that can be obtained from a certain quadratic Diophantine equation. The second method constructs arbitrary rigid tree honeycombs from rigid overlays of two rigid tree honeycombs with strictly smaller weights. This allows, in principle, for an inductive construction of all rigid tree honeycombs starting with those of unit weight. We also show that some rigid overlays of two rigid tree honeycombs give rise to an infinite sequence of rigid tree honeycombs of increasing complexity but with a fixed number of nonzero exit multiplicities. This last result involves a new inflation/deflation construction that also produces other infinite sequences of rigid tree honeycombs.

1. Introduction

The notion of a rigid honeycomb was introduced in \cite{9} in relation to the Horn problem. It also played a central role in proving that the Horn inequalities hold in an arbitrary factor of type \( II_1 \) in \cite{2}, where it was shown that every rigid honeycomb arises as a special kind of overlay (more general than the clockwise overlays of \cite{9}) of extreme rigid honeycombs. Extreme rigid honeycombs are scalar multiples of the rigid tree measures (which we now call rigid tree honeycombs) studied in \cite{3}.

In retrospect, the results of \cite{2} depended on the study of these extreme rigid honeycombs, and one of the initial difficulties was the lack of examples of sufficiently complicated such honeycombs, which in turn hampered effective experimentation. (Producing just six such honeycombs of weight four required an extensive search; an intersection problem generated by one of these six examples is fully analyzed at the end of \cite{2} Section 6.) Subsequently, several infinite families of extreme rigid honeycombs were constructed in \cite{3}.

Our purpose is to address the following two questions:

(a) Can one effectively enumerate all the rigid tree honeycombs of a given weight \( \omega \)?

(b) How can one generate rigid tree honeycombs of high weight from rigid tree honeycombs of smaller weight?

In answer to (a), we offer a method based on the study of certain combinatorial objects called locking patterns. This makes the enumeration of (types of) rigid ...
tree honeycombs possible, though still tedious for large weights. For (b), we show that every rigid tree honeycomb of weight at least two can be constructed from an overlay of honeycombs with strictly lower weights. In principle, this allows one to use the enumeration of rigid tree honeycombs of weight less than a fixed \(k \in \mathbb{N}\) to construct all possible such honeycombs of weight \(k\). In practice, this is difficult to achieve because there are usually many overlays of two honeycombs of given type and these overlays are hard to construct directly. However, once an overlay of two honeycombs, or even a single honeycomb, is known, we show how to construct honeycombs of much greater complexity and weight. These constructions recover, for instance, some of the infinite families constructed in \([3]\) and produce many additional infinite families of rigid tree honeycombs. The honeycombs constructed this way should be instrumental in the experimental study of related conjectures, such as those of \([5]\).

The paper is organized as follows. Basic definitions about honeycombs, puzzles, and duality are covered in Sections 2 and 3. Locking patterns and their connection to the enumeration of rigid tree honeycombs are also covered in Section 3. (The term locking pattern was suggested by Ken Dykema before we really understood what these overlays are, and this is why the notion is not mentioned in \([2]\).) In Section 4, we describe two constructions that help calculate the exit multiplicities of honeycombs supported in the edges of the puzzle of a given rigid honeycomb. One of these results yields the basic inductive construction of rigid tree honeycombs from other rigid tree honeycombs of smaller weights. Section 5 is dedicated to the study of degenerations of a rigid honeycomb, particularly simple degenerations. These simple degenerations, applied to a rigid tree honeycomb, produce either an extreme rigid honeycomb or an overlay of two extreme rigid honeycombs. Necessary conditions are derived on such overlays and Section 6 is dedicated to showing that these necessary conditions are sufficient as well. The arguments are based on a new construction that starts with a puzzle and a compatible honeycomb and produces another puzzle and compatible honeycomb using an inflation and a partial deflation. The constructions produce infinite sequences of rigid tree honeycombs whose weights increase rapidly (like the powers of a number \(>1\)) but with a fixed number of nonzero exit multiplicities. Some (but not all) of the infinite families produced in \([3]\) are shown to arise from this construction and some new infinite families are produced.

2. Honeycombs and rigidity

We review the definition of a honeycomb as presented in \([8]\). Fix vectors \(u_1, u_2, u_3\) of length \(\sqrt{3}/3\) in the plane, arranged in counterclockwise order such that \(u_1 + u_2 + u_3 = 0\) (see Figure 2.1), and define unit vectors

\[
    w_1 = u_3 - u_2, \quad w_2 = u_1 - u_3, \quad w_3 = u_2 - u_1.
\]

Every point \(X \in \mathbb{R}^2\) can be written uniquely as \(X = x_1u_1 + x_2u_2 + x_3u_3\) such that \(x_1, x_2, x_3 \in \mathbb{R}\) and \(x_1 + x_2 + x_3 = 0\). We view the numbers \(x_j\) as the coordinates of \(x\). Given a fixed \(c \in \mathbb{R}\), the equation \(x_j = c\) represents a line parallel to \(w_j\).

A honeycomb \(\mu\) is described by two elements, namely,

1. the support of \(\mu\). This consists of a finite union \(I_1 \cup \cdots \cup I_k \subset \mathbb{R}^2\) where
   a. each \(I_j\) is a closed segment, a closed ray, or a line,
   b. each \(I_j\) is parallel to one of the vectors \(w_1, w_2, w_3\), and
Figure 2.1. The vectors $u_j$ and $w_j$

Figure 2.2. Some honeycombs

(c) if $i \neq j$ then $I_i$ and $I_j$ have at most one point in common that is an endpoint of both $I_i$ and $I_j$.

(2) the multiplicities of the honeycomb. These are (not necessarily integer) numbers $\mu(I_1), \ldots, \mu(I_k) \in (0, +\infty)$.

It is convenient to consider that a honeycomb $\mu$ assigns zero multiplicity to segments parallel to some $w_i$ that intersect the support of $\mu$ in finitely many points, and that $\mu$ assigns multiplicity $\mu(I_j)$ to all subintervals of $I_j$. A branch point of a honeycomb is a point where at least three of the sets $I_j$ meet. The segments $I_j$ are called edges of $\mu$. The multiplicities of a honeycomb are subject to the following balance condition: Suppose that that $J_1, J_2, J_3, J_4, J_5, J_6$ are six segments, each parallel to some $w_i$, containing no branch points in their interior, having a common endpoint $X$ and arranged clockwise around $X$. Then

$$(2.1) \quad \mu(J_1) - \mu(J_4) = \mu(J_5) - \mu(J_2) = \mu(J_3) - \mu(J_6).$$

It may be easier to think of a honeycomb as a Borel measure in the plane that assigns $\mu(I_j) \times \text{length}(A)$ to each Borel subset $A \subset I_j$. This allows us to add honeycombs, thus giving the collection of all honeycombs the structure of a convex cone.

In this work, we do not distinguish between two honeycombs that are simply translates of each other, with the translation preserving multiplicities.

Suppose that $\mu$ is a honeycomb with support $I_1 \cup \cdots \cup I_k$. It is easy to see that any endpoint of some $I_j$ must also be an endpoint of another $I_i$. The support of a honeycomb $\mu$ must contain lines or rays. Each ray points in the direction of one of the vectors $\pm w_j$, $j = 1, 2, 3$. Figure 2.2 shows a few examples of honeycombs. All edge multiplicities, with one exception indicated by a thicker line, are equal to 1.

We denote by $M$ (respectively, $M_*$) the collection of all honeycombs with the property that the rays in the support of $\mu$ point in the direction of $w_1, w_2$, or $w_3$ (respectively $-w_1, -w_2$, or $-w_3$). The second and third examples in Figure 2.2 represent a honeycomb in $M$ and its mirror image (relative to a vertical line) in $M_*$. Suppose that $\mu$ is a honeycomb in $M$, $c \in \mathbb{R}$, and $j \in \{1, 2, 3\}$. If the support
of $\mu$ contains a ray $I \subset \{x_j = c\}$, we define
\begin{equation}
(2.2) \quad \mu^{(j)}(c) = \mu(I).
\end{equation}
Otherwise, we set $\mu^{(j)}(c) = 0$. The numbers $\mu^{(j)}(c)$ are called the exit multiplicities of $\mu$. (The ordered collection of the nonzero exit multiplicities will later be called the exit pattern of $\mu$.) Of course, $\mu$ only has finitely many nonzero exit multiplicities and the balance condition (2.1) is easily seen (see, for instance, [8]) to imply the identity
\begin{equation}
(2.3) \quad \sum_{c \in \mathbb{R}} \mu^{(1)}(c) = \sum_{c \in \mathbb{R}} \mu^{(2)}(c) = \sum_{c \in \mathbb{R}} \mu^{(3)}(c).
\end{equation}
This common value is called the weight of $\mu$ and is denoted
\begin{equation}
(2.4) \quad \omega(\mu) = \sum_{c \in \mathbb{R}} \mu^{(1)}(c).
\end{equation}
The exit multiplicities also satisfy the equality
\begin{equation}
(2.5) \quad \sum_{c \in \mathbb{R}} c(\mu^{(1)}(c) + \mu^{(2)}(c) + \mu^{(3)}(c)) = 0,
\end{equation}
sometimes known as the trace identity on account of its connection with linear algebra.

A honeycomb $\mu \in \mathcal{M}$ (or $\mu \in \mathcal{M}_e$) is said to be rigid if there is no other honeycomb that has the same exit multiplicities as $\mu$. It is easy to see that an arbitrary honeycomb is not rigid if it has a branch point adjacent to six edges in its support; see Figure 2.3. In fact, one can determine whether a honeycomb $\mu$ is rigid just by looking at its support. Let $\mu$ be a honeycomb and let $A, B, C$ be branch points of $\mu$ such that the segments $AB$ and $BC$ are edges of $\mu$. We say that $ABC$ is an evil turn [2, p. 1592] if one of the following situations occurs:

(1) $C = A$, and the support of $\mu$ contains edges $BX, BY, BZ$ that are $120^\circ$, $180^\circ$, and $240^\circ$ clockwise from $AB$.
(2) $BC$ is $120^\circ$ clockwise from $AB$.
(3) $C \neq A$ and $A, B, C$ are collinear.
(4) $BC$ is $120^\circ$ counterclockwise from $AB$ and the support of $\mu$ contains an edge $BX$ that is $120^\circ$ clockwise from $AB$.
(5) $BC$ is $60^\circ$ counterclockwise from $AB$ and the support of $\mu$ contains edges $BX, BY$ which are $120^\circ$ and $180^\circ$ clockwise from $AB$.

The possible evil turns are illustrated in Figure 2.4 where the edges that must be in the support of $\mu$ are labeled, but the support may contain all six edges incident to $B$. A sequence $A_1, A_2, \ldots, A_n, A_{n+1} = A_1$ of branch points of $\mu$ is called an evil loop if $A_{j-1}A_jA_{j+1}$ is an evil turn for every $j = 1, \ldots, n$. Each of the
first three configurations in Figure 2.5 contains exactly one evil loop (up to cyclic permutations) but the fourth one contains none.

The following result is proved as in [2, Proposition 2.2]. (The proof uses a characterization of rigidity via puzzles; see [9, Theorems 4 and 7].)

**Proposition 2.1.** A honeycomb $\mu \in \mathcal{M}$ is rigid if and only if its support contains none of the following configurations:

1. Six edges meeting in one branch point.
2. An evil loop.

Suppose that $e = AB$ and $f = BC$ are edges of a rigid honeycomb $\mu$. As in [2], we write $e \rightarrow_{\mu} f$ (or $e \rightarrow f$ when $\mu$ is understood; see Figure 2.6) if one of the following two situations arises:

1. $\triangle ABC = 120^\circ$ and the segment $BX$ opposite $AB$ is not in the support of $\mu$.
2. $e$ and $f$ are opposite and there exists a segment $BX$ not in the support of $\mu$ such that $\angle CBX = 60^\circ$.

The balance condition shows that $\mu(e) \leq \mu(f)$ whenever $e \rightarrow f$. In fact, $\mu(f)$ equals $\mu(e)$ if $e \rightarrow f$ and $f \rightarrow e$, and it equals $\mu(e) + \mu(e')$ when $e \rightarrow f$, $e' \rightarrow f$, $e' \rightarrow f$. 

![Figure 2.4. Evil turns](image1)

![Figure 2.5. Some loops](image2)

![Figure 2.6. The relation $\rightarrow$](image3)
and \( e \neq e' \). The transitive closure \( \Rightarrow \) of the relation \( \rightarrow \) was called descendant in \([2]\). Thus, \( e \Rightarrow f \) if there exists a descendant path, that is, a sequence \( e_1 \ldots e_n \) of edges such that \( e_1 = e, e_n = f \), and \( e_j \rightarrow e_{j+1} \), \( j = 1, \ldots, n - 1 \). Given such a path, \( e \) is called an ancestor of \( f \) and \( f \) is called a descendant of \( e \). We write \( e \Leftrightarrow f \), and we say that \( e \) and \( f \) are equivalent, if either \( e = f \) or both \( e \Rightarrow f \) and \( f \Rightarrow e \) hold. An edge \( e \) of \( \mu \) is said to be a root edge if the relation \( f \Rightarrow e \) implies \( e \Leftrightarrow f \). The above observations show that the multiplicity of an arbitrary edge of \( \mu \) can be written as a linear combination with positive integer coefficients of the multiplicities of root edges. More precisely, suppose that \( \mu \) is a rigid honeycomb and that \( e_1, \ldots, e_n \) is a maximal sequence of pairwise inequivalent root edges. Then an arbitrary edge \( f \) of \( \mu \) satisfies

\[
\mu(f) = \sum_{j=1}^{n} d_j \mu(e_j),
\]

where \( d_j \geq 0 \) is the number of distinct descendant paths from \( e_j \) to \( f \). Another way of formulating this is to say that

\[
\mu = \sum_{j=1}^{n} \mu(e_j) \mu_j,
\]

where each \( \mu_j \) is itself a rigid honeycomb (see \([2]\) Section 3). The summands \( \mu_j \) belong to extreme rays in the cone \( \mathcal{M} \). Extreme rigid honeycombs are characterized by the fact that they have a unique equivalence class of root edges. An extreme rigid honeycomb is a tree honeycomb precisely when the multiplicity assigned to its root edges is equal to 1. Alternatively, rigid tree honeycombs are obtained as immersions of a special kind of tree that we now define.

The trees relevant to the construction of rigid honeycombs are finite trees whose edges are labeled 1, 2, or 3, and the following conditions are satisfied:

1. each vertex has order 1 or 3, and
2. the three edges adjacent to a vertex of order 3 have distinct labels.

An immersion of a tree is a map \( f \) that associates to each vertex \( v \) of order 3 of a tree a point \( f(v) \) in the plane such that the following conditions are satisfied:

(i) if \( a \) and \( b \) are joined by an edge labeled \( j \), then \( f(a) \neq f(b) \) and the segment \( f(a)f(b) \) is parallel to \( w_j \), and

(ii) if \( xa, xb, \) and \( xc \) are three edges adjacent to a vertex \( x \) then the angle between any two of the segments \( f(x)f(a), f(x)f(b), \) and \( f(x)f(c) \) equals 120°.

Given a tree \( T \) that has at least two vertices of order 3, and an immersion \( f \) of \( T \), we define a honeycomb \( \mu_f \) as follows. Let \( ab \) be an edge of \( T \). If \( a \) and \( b \) have order 3, we add the segment \( f(a)f(b) \) to the support of \( \mu_f \) and we add 1 to the multiplicity of this segment. If \( a \) is of order 1, \( ab \) is labeled \( j \), and \( c, d \) are the other vertices connected to \( b \), we add to the support of \( \mu_f \) a ray \( h \) with endpoint \( f(b) \) and parallel to \( w_j \) and we add 1 to the multiplicity of this ray. The ray is chosen such that the angles between any two of \( f(h)f(c), \) and \( f(b)f(d) \) equal 120°. The segments \( f(a)f(b) \) and the rays constructed above can be viewed as images under the immersion \( f \) of the edges of \( T \). The honeycomb \( \mu_f \) is simply determined by adding all of the multiplicities just defined. If \( T \) has exactly one vertex of order 3, there are two possible honeycombs associated to an immersion \( f \), one in \( \mathcal{M} \) and
the other in $M_\ast$. The honeycombs obtained from immersions of trees are called tree honeycombs.

Not every tree honeycomb is rigid. Figure 2.7 represents a tree $T$ and two immersions of $T$, the second of which yields a rigid honeycomb.

Suppose that $\mu \in M$ is a rigid honeycomb with a unique equivalence class of root edges, all of which have multiplicity 1. We construct a tree $T_0$ as follows. Fix a root edge $e_0 = XY$ of $\mu$. The vertices of $T_0$ are $x, y$, and the descendance paths $e_0 \ldots e_n$. Given such a descendance path, the vertices $v = e_0 \ldots e_n$ and $v' = e_0 \ldots e_n$ are joined by an edge of $T_0$ labeled $j$ if $e_n$ is parallel to $w_j$. In addition, $x$ is joined to $y$ by an edge labeled $j$ if $XY$ is parallel to $w_j$, $x$ is joined to $e_0 e_1$ if $X$ is an endpoint of $e_1$, and $y$ is joined to $e_0 e_1$ if $Y$ an endpoint of $e_1$. The same labeling rule applies to the last two kinds of edges. The tree $T_0$ obtained this way may have some vertices of order 2. To obtain a tree $T$ satisfying condition (1) we simply remove the vertices of order 2 as follows: suppose that $v_0 v_1 \ldots v_n$ is a path in $T_0$ such that $v_0$ and $v_n$ have order 1 or 3 and $v_1, \ldots, v_{n-1}$ have order 2. Then $v_1, \ldots, v_{n-1}$ are removed from $T_0$ and an edge joining $v_0$ and $v_n$, carrying the label of $v_0 v_1$, is added instead. Clearly, we have $\mu = \mu_f$ for some immersion $f$ of $T$. The construction of $T_0$ and $T$ is illustrated in Figure 2.8 for a particular rigid tree honeycomb.

Suppose that $T$ is a tree, that $f$ is an immersion of $T$, and that $\mu = \mu_f$ belongs to $M$. Suppose also that there exists an edge $xy$ of $T$ such that $\mu$ assigns unit multiplicity to part of $f(x)f(y)$. Orient the edges of $T$ other than $xy$ away from $xy$. We say that the immersion $f$ is coherent if the following condition is satisfied: if a segment $I$ is contained in the images under $f$ of two different edges of $T$, then
The following result is [4, Theorem 4.1].

**Theorem 2.2.** Suppose that \( T \) is a tree, that \( f \) is an immersion of \( T \), and that \( \mu = \mu_f \in \mathcal{M} \). Then \( \mu \) is rigid if and only if the following three conditions are satisfied.

1. There exists an edge \( xy \) of \( T \) such that \( \mu \) assigns multiplicity 1 to part of \( f(x)f(y) \).
2. The immersion \( f \) is coherent.
3. There is no branch point \( X \) of \( \mu \) such that four segment \(XA_1,XA_2, XB_1\), and \(XB_2\) have nonzero multiplicity, the points \( A_1A_2 \) are collinear, and the points \( B_1XB_2 \) are collinear; see Figure 2.9.

Having obtained a description of rigid tree honeycombs, it is important to determine which linear combinations \( c_1\mu_1 + \cdots + c_n\mu_n \), \( c_1, \ldots, c_n > 0 \), of rigid tree honeycombs are rigid. Such combinations are called *rigid overlays* and their description involves a bilinear map [3] defined on honeycombs in \( \mathcal{M} \) as follows: given \( \mu, \nu \in \mathcal{M} \), we set

\[
(2.6) \quad \Sigma_\nu(\mu) = \sum_{c<d} [\mu^{(1)}(c)\nu^{(1)}(d) + \mu^{(2)}(c)\nu^{(2)}(d) + \mu^{(3)}(c)\nu^{(3)}(d)] - \omega(\mu)\omega(\nu),
\]

where \( \mu^{(j)}(c) \) is as defined above (2.2). It was seen in [3, end of Section 2] that

\[
(2.7) \quad \Sigma_\mu(\mu) = \frac{1}{2} \left[ \omega(\mu)^2 - \sum_{c\in\mathbb{R}} [\mu^{(1)}(c)^2 + \mu^{(2)}(c)^2 + \mu^{(3)}(c)^2] \right].
\]

The following result is [3, Theorem 4.2].

**Theorem 2.3.** Let \( \mu \) and \( \nu \) be two distinct rigid tree honeycombs in \( \mathcal{M} \). Then \( \Sigma_\nu(\mu) \geq 0 \) and \( \Sigma_\mu(\nu) = -1 \).

It is possible that \( \Sigma_\nu(\mu) > 0 \) and \( \Sigma_\mu(\nu) > 0 \) for two rigid tree honeycombs \( \mu \) and \( \nu \), see Figure 2.10. It follows from (2.7) that the second assertion in Theorem 2.3 can be written as

\[
(2.8) \quad \sum_{c\in\mathbb{R}} [\mu^{(1)}(c)^2 + \mu^{(2)}(c)^2 + \mu^{(3)}(c)^2] = \omega(\mu)^2 + 2.
\]

**Remark 2.4.** Suppose that \( \mu \) and \( \nu \) are distinct rigid honeycombs in \( \mathcal{M} \) obtained from immersions \( f \) of \( T \) and \( g \) of \( S \), respectively. Then [3, Theorem 5.1] provides a method for the calculation of \( \Sigma_\nu(\mu) \). Fix an edge \( e \) in \( S \) that is mapped by \( g \).
onto a root edge of $\nu$ that is not contained in the support of $\mu$. Orient all the other edges of $S$ away from $e$. Then $\Sigma_\nu(\mu)$ is the total number of occurrences of events of the four types indicated in the Figure 2.11. In these diagrams, the dashed lines represent the images under $g$ of edges of $S$, the dotted lines represent the images under $f$ of edges of $T$, and the solid lines represent common edges of $\mu$ and $\nu$. (Note that, in counting these events we consider edges of the trees rather than edges of the honeycombs themselves. Thus, when looking at edges of the honeycombs, one must take into account their multiplicities.)

The following characterization of rigid overlays is [3, Corollary 4.5] (see also [2, Theorem 3.8]).

**Theorem 2.5.** Let $\mu_1, \ldots, \mu_n$ be distinct rigid tree honeycombs and let $c_1, \ldots, c_n$ be positive numbers. Then the following are equivalent:

1. The honeycomb $\sum_{j=1}^n c_j \mu_j$ is rigid.
There exists a permutation $\nu_1, \ldots, \nu_n$ of $\mu_1, \ldots, \mu_n$ such that $\sum_{\nu_i}(\nu_j) = 0$ for $i > j$.

3. Puzzles and duality

Suppose that $\mu \in \mathcal{M}$. The puzzle of $\mu$ is obtained by a process of inflation that we describe next. Cut the plane along the edges of $\mu$ to obtain a collection of white puzzle pieces and translate these pieces away from each other in such a way that, given an edge $AB$ of $\mu$, the parallelogram formed by the two translates of $AB$ has two sides of length $\mu(AB)$ that are $60^\circ$ clockwise from $AB$. We call this parallelogram the inflation of $AB$ and we color it dark gray. The balance condition for $\mu$ implies that these white and dark gray pieces fit together and leave a space corresponding to each branch point of $\mu$. These extra spaces are colored light gray.

The light gray piece associated to a branch point $X$ of $\mu$ is a convex polygon with as many sides as there are edges incident to $X$. One can recover the honeycomb $\mu$, up to a translation, from its puzzle by deflation. This process consists simply of moving the white pieces together and replacing each dark gray parallelogram (or half-infinite strip) by a segment parallel to its white sides and with multiplicity equal to the length of its light gray side. There are similar processes of $\ast$-inflation and $\ast$-deflation which are the same as inflation and deflation except that ‘clockwise’ is replaced by ‘counterclockwise’ in the construction of the $\ast$-inflation.

Starting with a nonzero honeycomb $\mu \in \mathcal{M}$, one can construct the puzzle of $\mu$ and apply $\ast$-deflation to this puzzle. The only difficulty is that the segments arising from dark gray infinite strips would be assigned infinite multiplicity. To overcome this difficulty, choose a closed equilateral triangle $\Delta$ containing all the branch points of $\mu$ whose sides are $60^\circ$ clockwise from those rays in the support of $\mu$ that they intersect. Apply now the process of inflation only to the part of the support of $\mu$ that is contained in $\Delta$ and apply $\ast$-deflation to the resulting puzzle. The result of these operations is the part of a honeycomb $\mu^*_\Delta \in \mathcal{M}_\ast$ contained in an equilateral triangle $\Delta_\ast$ with sides equal to $\omega(\mu)$. (A simple example is illustrated in Figure 3.1.) Choosing a different triangle $\Delta$ changes the multiplicities of $\mu^*_\Delta$ only on the boundary of $\Delta$. Denote by $\Delta_0$ the smallest triangle satisfying the requirements above. We define $\mu^*$ to be equal to $\mu^*_\Delta$. The honeycomb $\mu^*$ has the property that each side of $\Delta_\ast$ contains a segment with zero multiplicity.

In general, we observe that the sides of $\Delta_\ast$ are divided by the rays in the support of $\mu^*$ into segments with lengths equal to the exit multiplicities of $\mu$. (The triangle $\Delta_\ast$ is partitioned by the support of $\mu^*$ into light gray pieces that form the degeneracy graph of $\mu$ introduced in [8].) If $\mu$ has only one branch point, we define $\mu^* = 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.1}
\caption{A honeycomb $\mu$ in $\Delta$, its puzzle, the $\ast$-deflation, and the honeycombs $\mu^*_\Delta$ and $\mu^*$}
\end{figure}
Remark 3.1. As just observed, the nonzero exit multiplicities of a honeycomb $\mu$ are precisely the lengths of the segments on $\partial \Delta$, between consecutive exit rays of $\mu^*$. For instance, suppose that the support of $\mu^*$ is contained in the support of $\nu^*$. If $\text{exit}(\mu) = \text{exit}(\nu)$, then $\mu^*$ and $\nu^*$ share their exit rays, and therefore $\mu$ and $\nu$ have precisely the same nonzero exit multiplicities. On the other hand, if $\text{exit}(\mu) = \text{exit}(\nu) - 1$, then $\mu^*$ has one fewer exit ray, and thus $\mu$ has the same nonzero exit multiplicities, except that the two nonzero exit multiplicities of $\nu^*$ corresponding to segments separated by the extra exit ray are replaced by their sum. (If $\text{exit}(\mu)$ is even smaller, more of the multiplicities of $\nu^*$ are aggregated in this manner into multiplicities of $\mu$.)

It is easily seen that a honeycomb $\mu \in \mathcal{M}$ is rigid if and only if $\mu^* \in \mathcal{M}_*$ is rigid (see [2, p. 1591]). Given a rigid honeycomb $\mu \in \mathcal{M}$, we denote by $\text{exit}(\mu)$ the number of nonzero exit multiplicities of $\mu$, and we denote by $\text{root}(\mu)$ the number of equivalence classes of root edges in the support of $\mu$. As noted earlier, $\mu$ is a linear combination with positive coefficients of $\text{root}(\mu)$ distinct rigid tree honeycombs. The following result is equivalent to [1, Theorem 1.1]. (To see the equivalence, observe that the number of attachment points of $\mu$ relative to the minimal triangle $\Delta_0$ considered above is equal to $\text{exit}(\mu) - 3$.)

**Theorem 3.2.** For every rigid honeycomb $\mu \in \mathcal{M}$ we have

$$\text{root}(\mu) + \text{root}(\mu^*) = \text{exit}(\mu) - 2.$$ 

In order to study the exit multiplicities of a rigid tree honeycomb, we require an auxiliary result of convex analysis.

**Lemma 3.3.** Let $K_1$ and $K_2$ be two convex sets such that $K_2$ is finite-dimensional, and let $f : K_1 \to K_2$ be a surjective affine map. Suppose that there exists an interior point $b_0 \in K_2$ such that $f^{-1}(b_0)$ is a singleton. Then $f$ is injective.

**Proof.** Let $a_0 \in K_1$ satisfy $f(a_0) = b_0$ and suppose that $a_1', a_1'' \in K_1$ are such that $f(a_1') = f(a_1'') = b_1$. Since $b_0$ is an interior point of $K_2$, there exist $b_2 \in K_2$ and $t \in (0, 1)$ such that $b_0 = t b_1 + (1 - t) b_2$. Choose $a_2 \in f^{-1}(b_2)$. Since $f$ is affine, we deduce that

$$f(t a_1' + (1 - t) a_2) = t b_1 + (1 - t) b_2 = b_0,$$

and thus

$$t a_1' + (1 - t) a_2 = t a_1'' + (1 - t) a_2 = a_0.$$ 

It follows immediately that $a_1' = a_1''$. $\square$

Fix a positive number $r$ and consider an equilateral triangle $\Delta = ABC$ with side length $r$ and sides parallel to $w_1, w_2$, and $w_3$. For simplicity, suppose that $A, B$, and $C$ are arranged counterclockwise around $\Delta$ and $AB$ is parallel to $w_1$. A pattern consists of a sequence $A_0 = A, A_1, \ldots, A_{k_1 - 1}, A_{k_1} = B = B_0, B_1, \ldots, B_{k_2 - 1}, B_{k_2} = C = C_0, C_1, \ldots, C_{k_3 - 1}, C_{k_3} = A$ of points arranged counterclockwise on the boundary of $\Delta$; see Figure 3.2. Alternatively, a pattern is determined by the distances $\alpha_1^{(1)}, \ldots, \alpha_{k_1}^{(1)}, \alpha_1^{(2)}, \ldots, \alpha_{k_2}^{(2)}, \alpha_1^{(3)}, \ldots, \alpha_{k_3}^{(3)}$ defined by

$$\alpha_j^{(1)} = \text{length}(A_{j-1}, A_j), \quad \alpha_j^{(2)} = \text{length}(B_{j-1}, B_j), \quad \alpha_j^{(3)} = \text{length}(C_{j-1}, C_j).$$
These positive numbers are subject to
\[ \sum_{j=1}^{k_1} \alpha_j^{(1)} = \sum_{j=1}^{k_2} \alpha_j^{(2)} = \sum_{j=1}^{k_3} \alpha_j^{(3)} = r. \]

**Definition 3.4.** We say that a pattern is a **locking pattern** if the following condition is satisfied: Suppose that \( \mu \in \mathcal{M} \) is such that the branch points of \( \mu \) are contained in \( \Delta \) and that the boundary of \( \Delta \) only intersects the rays in the support of \( \mu \) in a subset of \( \{ A_j \}_{j=0}^{k_1} \cup \{ B_j \}_{j=0}^{k_2} \cup \{ C_j \}_{j=0}^{k_3} \). Then \( \mu \) is rigid.

It is easy to see that \( \mathcal{M} \) can be replaced by \( \mathcal{M}_* \) in the above definition. Moreover, the above condition need only be satisfied for honeycombs \( \mu \in \mathcal{M} \) with rays intersecting \( \partial \Delta \) in a subset of \( \{ A_j \}_{j=0}^{k_1-1} \cup \{ B_j \}_{j=0}^{k_2-1} \cup \{ C_j \}_{j=0}^{k_3-1} \); such honeycombs are allowed to have a ray meeting \( \partial \Delta \) in \( A_0 (= C_{k_3}) \) that is parallel to the other rays passing through some \( A_j \) with \( j \neq k_3 \) but not such a ray that is parallel to the ones passing through some \( C_j \) for \( j \neq k_3 \). A similar observation (with \( A_0, B_0, C_0 \) in place of \( A_{k_1}, B_{k_2}, C_{k_3} \)) applies to honeycombs in \( \mathcal{M}_* \). It is also obvious that a pattern that has a locking refinement must itself be locking.

Suppose now that \( \mu \in \mathcal{M} \) and the only nonzero exit multiplicities of \( \mu \) are \( \alpha_j^{(1)} = \mu_j^{(1)}(c_j^{(1)}), \alpha_j^{(2)} = \mu_j^{(2)}(c_j^{(2)}), \) and \( \alpha_j^{(3)} = \mu_j^{(3)}(c_j^{(3)}) \), where \( c_j^{(1)} < \cdots < c_j^{(k_1)}, \) \( c_j^{(2)} < \cdots < c_j^{(k_2)}, \) and \( c_j^{(3)} < \cdots < c_j^{(k_3)} \). These numbers define a pattern with \( r = \omega(\mu) \) that we call the **exit pattern** of \( \mu \).

**Proposition 3.5.** Suppose that \( \mu \in \mathcal{M} \) is a rigid tree honeycomb. Then the exit pattern of \( \mu \) is a locking pattern.

**Proof.** Choose rigid tree honeycombs \( \rho_1, \ldots, \rho_n \in \mathcal{M}_* \) and \( c_1, \ldots, c_n > 0 \) such that \( \mu^* = \sum_{j=1}^{n} c_j \rho_j \). Theorem 3.2 and the hypothesis imply that \( n = \text{exit}(\mu) - 3 \). Recall that all the branch points of \( \mu^* \) are contained in a triangle \( \Delta_* \) with side lengths \( \omega(\mu) \) and that the rays of \( \mu^* \) determine precisely the exit pattern of \( \mu \) on the boundary of \( \Delta_* \). Denote by \( K_1 \) the collection of all honeycombs in \( \mathcal{M}_* \) whose branch points are contained in \( \Delta_* \) and whose rays intersect the boundary of \( \Delta_* \) in a subset of \( \{ A_j \}_{j=1}^{k_1} \cup \{ B_j \}_{j=1}^{k_2} \cup \{ C_j \}_{j=1}^{k_3} \). Observe that \( k_1 + k_2 + k_3 = \text{exit}(\mu) \). Given \( \nu \in K_1 \), denote by \( f(\nu) \in \mathbb{R}^{\text{exit}(\mu)} \) the vector \((a_1, \ldots, a_{k_1}, b_1, \ldots, b_{k_2}, c_1, \ldots, c_{k_3})\), where \( a_j \) is the exit multiplicity that \( \nu \) assigns to the ray passing through \( A_j \), and \( b_j, c_j \) are defined similarly. Denote by \( K_2 \subset \mathbb{R}^{\text{exit}(\mu)} \) the range of \( f \). In order to apply Lemma 3.3, we verify that \( f(\mu^*) \) is an interior point of \( K_2 \). First, we note that the dimension of \( K_2 \) is at most \( \text{exit}(\mu) - 3 \) because every honeycomb must satisfy (2.3) and (2.5).
On the other hand, $f$ is injective on the set $S = \{\sum_{j=1}^{n} t_j \rho_j : t_1, \ldots, t_n > 0\}$ and thus $f(K_1)$ has dimension at least $\text{exit}(\mu) - 3$. It follows that $K_2$ has dimension exactly $\text{exit}(\mu) - 3$ and that each point in $f(S)$ is an interior point of $K_2$. The proposition follows from Lemma 3.3.\hfill \Box

The multiplicity pattern of a rigid tree honeycomb has an additional property that helps determine whether a given pattern arises from such a honeycomb. Suppose that an arbitrary pattern $A_0, \ldots, A_k, B_0, \ldots, B_k, C_0, \ldots, C_k$ is given on a triangle $\Delta$ with sides parallel to $w_1, w_2$, and $w_3$. A convex region $\Omega \subset \Delta$ is said to be flat relative to this pattern if every honeycomb $\mu \in \mathcal{M}$, whose rays intersect $\partial \Delta$ in a subset of $\{A_j, B_j, C_j\}$, assigns zero multiplicity to all segments contained in the interior of $\Omega$.

Proposition 3.6. Let $\mu \in \mathcal{M}$ be a rigid tree honeycomb. Then the light gray pieces determined in $\Delta_\ast$ by the support of $\mu^\ast$ are flat relative to the exit pattern of $\mu$.

Proof. We denote by $\mathcal{N} \subset \mathcal{M}_\ast$ the collection of those honeycombs that have branch points contained in $\Delta_\ast$ and exit points among $A_0, \ldots, A_k, B_0, \ldots, B_k, C_0, \ldots, C_k$. Suppose that $I \subset \Delta_\ast$ is a segment parallel to $w_j$ for some $j = 1, 2, 3$. We say that $I$ is flat if the following condition is satisfied for every $\nu \in \mathcal{N}$:

\begin{enumerate}
  \item There is no edge $J$ of $\nu$ such that $J \cap I$ is a single point in the interior of $I$.
\end{enumerate}

If $AB$ and $AC$ are two flat edges, each parallel to $w_1, w_2$, or $w_3$, and $\angle BAC = 60^\circ$, then the smallest convex polygon with sides parallel to $w_1, w_2$, or $w_3$ containing $A, B$, and $C$ is flat relative to the exit pattern of $\mu$. This polygon is either an equilateral triangle or a trapezoid. It suffices to show that all the edges of $\mu^\ast$ in $\Delta_\ast$ are flat. These edges are of the form $e^\ast$, with $e$ an edge of $\mu$, where $e^\ast$ is a segment of length $\mu(e)$ that is $60^\circ$ clockwise from $e$. Denote by $S$ the set of all edges of $\mu$ and by $S_1 \subset S$ the collection of those edges $e$ with the property that $e^\ast$ is flat. It is clear that the rays of $\mu$ belong to $S_1$. We show that the set $S_1$ has the following properties (see Figure 3.3):

\begin{enumerate}
  \item If $e_1, e_2$, and $e_3$ are the only three edges meeting at a branch point, and if $e_1, e_2 \in S_1$, then $e_3 \in S_1$.
  \item If $e_1, e_2, e_3$, and $e_4$ are four edges meeting at a branch point of $\mu$ such that $\mu(e_1) = \mu(e_2) + \mu(e_4)$, and if $e_1, e_2 \in S_1$, then $e_3, e_4 \in S_1$.
  \item If $e_1, e_2, e_3$, and $e_4$ are four edges meeting at a branch point of $\mu$ such that $\mu(e_1) = \mu(e_2) + \mu(e_4)$ and $e_1$ is collinear with $e_4$, and if $e_1 \in S_1$, then $e_4 \in S_1$.
\end{enumerate}
Supposing for the moment that properties (1)–(3) have been proved, we argue
by contradiction that \( S_1 = S \). If \( S_1 \neq S \), choose an edge \( e \in S \setminus S_1 \) with the
property that the length of a descendance path from some root edge to \( e \) is the
largest possible. All the descendants of \( e \) have longer descendance paths from the
same root edge and hence they belong to \( S_1 \). Since \( e \) is not a ray, it must have
descendants. There are three possibilities:

(a) \( e \) has two descendants \( e_2 \) and \( e_3 \) and there is no other edge adjacent to
their common endpoint. This possibility is excluded by property (1) above
because \( e_2, e_3 \in S_1 \).

(b) \( e \) has two descendants \( e_1, e_3 \), and there is a fourth edge \( e_4 \) adjacent to their
common endpoint. This possibility is excluded by (2) for similar reasons.

(c) \( e \) has only one descendant \( e_1 \), in which case there must exist \( e_2 \) and \( e_3 \)
adjacent to their common point such that \( \mu(e_1) = \mu(e) + \mu(e_2) \). This is
excluded by (3).

All three possibilities being excluded, we conclude that \( S_1 = S \).

Finally, we verify properties (1)–(3). In case (1), the edges \( e_1^*, e_2 \), and \( e_3^* \) form an
equilateral triangle. Suppose that \( \nu \in \mathcal{N} \) has an edge that intersects \( e_1^* \) in a single
interior point. It is easy to see that in this case there must also be some edge of \( \nu \)
that intersects either \( e_1^* \) or \( e_2 \) in its interior, thus contradicting the flatness of these
segments.

In cases (2) and (3), the edges \( e_1 \) and \( e_3^* \) are the two parallel edges of an isosceles
trapezoid and \( e_2 \) and \( e_3^* \) are the other two edges. Suppose that \( \nu \in \mathcal{N} \) has an edge
that intersects \( e_3 \) in a single interior point. Then \( \nu \) must also have an edge that
intersects either \( e_1^* \) or \( e_2 \) in a single interior point, so \( e_3 \in S_1 \) if \( e_1, e_2 \in S_1 \). Finally,
suppose that \( \nu \in \mathcal{N} \) has an edge that intersects \( e_4 \) in a single interior point. Then
\( \nu \) must also have an edge that intersects \( e_1^* \) in a single interior point, so \( e_4 \in S_1 \) if
\( e_1 \in S_1 \). This concludes the proof.

The last two results indicate an algorithm for checking whether a given pattern
arises from the exit multiplicities of a rigid tree honeycomb. Thus, starting with a
pattern, we construct flat segments and flat regions in stages, as follows. At stage 0,
we have the collection of flat segments determined by the pattern on the boundary
of the triangle. Suppose that the flat segments and flat regions of stage \( k \) have
been constructed, and let \( AB \) and \( CD \) be two nonparallel flat edges that appear
in this configuration. Suppose that \( AB \) and \( CD \) lie on different sides of a 60°
angle, and that the interiors of two equilateral triangles \( \Delta_1 \) and \( \Delta_2 \), based on \( AB \)
and \( CD \) respectively, intersect. Then we construct segments \( A'B' \subset \Delta_1 \) parallel
to \( AB \) with \( A', B' \in \partial \Delta_1 \) and \( C'D' \subset \Delta_2 \) parallel to \( CD \) with \( C', D' \in \partial \Delta_2 \)
(see Figure 3.4) that have one endpoint in common. Denote by \( \Omega \) the smallest
trapezoid (or equilateral triangle) with edges parallel to \( w_1, w_2 \), or \( w_3 \) that contains
\( A'B' \) and \( C'D' \). Then, in stage \( k + 1 \) we declare that \( \Omega \) and all of its sides are
flat. If the pattern is in fact the exit pattern of a rigid tree honeycomb, then the
preceding result shows that this process ends in a finite number of stages and that
the flat regions it creates (which are only equilateral triangles and trapezoids) tile
the original triangle. Moreover, the shortest flat segments, corresponding to root
edges of the rigid tree honeycomb, must have unit length.

In order to verify that a given rigid pattern, for which the preceding process
yields a collection of flat triangles and trapezoids that tile the entire triangle \( \Delta \),
really arises as the exit pattern of some rigid tree honeycomb $\mu$, we must construct $\mu$ or some honeycomb homologous to it. One way to do this is to find a honeycomb $\nu$ in $\mathcal{M}_*$ whose support consists of the flat segments created in the process, construct $\mu$ such that $\mu^* = \nu$, and verify that it is in fact a rigid tree honeycomb. This may be a laborious process. Fortunately, there exists a simpler process that yields such as honeycomb $\mu$: each flat region being a triangle or a trapezoid, it has a circumcenter. The segment joining the circumcenters of two adjacent flat regions is part of the perpendicular bisector of the common edge. Add to these segments rays contained in the perpendicular bisector of each flat segment in $\partial \Delta$ terminating at the circumcenter of the adjacent flat region. Assign to each bisecting segment a multiplicity equal to the length of the original flat segment. This way, we obtain a honeycomb, rotated counterclockwise by 30°. It is fairly easy to check whether this is a rigid tree honeycomb since we know what its root edges are.

Figure 3.5 illustrates the two above processes for the pattern 3, 1|2, 1, 1|2, 1, 1. The flat regions are determined in two stages, they tile $\Delta$, and the shortest flat edges have unit length. However, the (red) honeycomb constructed using circumcenters is rigid but not extreme because not all the root edges are equivalent. For the pattern 1, 2, 1|2, 1, 1|1, 2, 1 shown in Figure 3.6, all the flat regions are determined in stage 1 and they do not cover $\Delta$, so this is not the exit pattern of a rigid tree honeycomb. On the other hand, for the pattern 4, 2|2, 2, 2|2, 2, 2 the flat regions do cover $\Delta$ but the shortest flat edge has length 2. In this case, the circumcenter construction yields twice a rigid tree honeycomb. There are locking patterns for which the flat regions created by the method described above overlap and thereby create greater flat regions that may not by triangles or trapezoids. Such an example is 3, 2|1, 3, 2|2, 2, 1, illustrated in Figure 3.7.
Suppose that $\mu \in \mathcal{M}$ is a rigid tree honeycomb and its dual is written as in the above proof $\mu^* = \sum_{j=1}^{n} c_j \rho_j$. If $t_1, \ldots, t_n > 0$ and $\nu \in \mathcal{M}$ is such that $\nu^* = \sum_{j=1}^{n} t_j \rho_j$ then $\mu$ and $\nu$ are homologous in the sense defined in [3]. In other words, up to a translation, $\nu$ can be obtained by stretching or shrinking the edges of $\mu$ while preserving their multiplicities (see Figure 3.8). Conversely, if $\nu$ is an arbitrary rigid tree honeycomb of weight $\omega(\mu)$ that determines the same pattern on $\Delta_*$, then $\nu$ must satisfy $\nu^* = \sum_{j=1}^{n} t_j \rho_j$ for some $t_1, \ldots, t_n > 0$. Thus, in order to classify the rigid tree honeycombs of a given weight $\omega$, it suffices to find the locking patterns on a triangle $\Delta$ with sides equal to $\omega$ that also correspond to rigid tree honeycombs. There is only a finite number of such patterns because the lengths of the segments in the boundary of $\Delta$ determined by the relevant patterns are exit multiplicities of a tree honeycomb and are therefore integers.

Let $\mu \in \mathcal{M}$ be a rigid tree honeycomb of weight $\omega$. According to [2.8], the integers $\mu^{(j)}(c)$, $j = 1, 2, 3, c \in \mathbb{R}$, are subject to the following requirements:
If we arrange the nonzero exit multiplicities of \( \mu \) in a nonincreasing list \( \alpha_1, \ldots, \alpha_N \), we have

\[
\sum_{j=1}^{N} \alpha_j = 3\omega, \quad \sum_{j=1}^{N} \alpha_j^2 = \omega^2 + 2.
\]

This system of Diophantine equations has finitely many solutions that are fairly easy to list. We observe that subtracting the first equation above from the second we obtain

\[
\sum_{j=1}^{N} \alpha_j (\alpha_j - 1) = \frac{(\omega - 1)(\omega - 2)}{2}.
\]

In other words, we need to write the triangular number \( (\omega - 1)(\omega - 2)/2 \) as a sum of triangular numbers, and this sum will determine precisely the values of those \( \alpha_j \) that are greater than 1. Once numbers \( \alpha_j \) have been determined, there may be several ways that they can be arranged to make a pattern on a triangle with sides \( \omega \). For each of these arrangements, we can use the algorithm outlined above to determine whether it is the exit pattern of a rigid tree honeycomb. We illustrate this for small values of \( \omega \). For \( \omega \leq 2 \) we have \( (\omega - 1)(\omega - 2)/2 = 0 \) and thus \( \alpha_j = 1 \) for all \( j \). The corresponding types of rigid tree honeycombs are pictured in Figure 3.9(A) and (B). For \( \omega = 3 \) we have \( (\omega - 1)(\omega - 2)/2 = 1 \) so the only numbers \( \alpha_j \) satisfying (3.1) are 2, 1, 1, 1, 1, 1, 1, 1. These lengths can be arranged around a triangle \( \Delta \) in six different ways but all these arrangements can be obtained from one of them by rotations or reflections. It is seen immediately that these arrangements are in fact the exit patterns of rigid tree honeycombs. Thus, there are exactly six types of rigid tree honeycombs of weight 3, all of which can be obtained by rotations and reflections from the one pictured in Figure 3.9(C) (the thicker edge has multiplicity 2).
For $\omega = 4$, we have $(\omega - 1)(\omega - 2)/2 = 3$, and this can be written as $3 = 3$ and $3 = 1 + 1 + 1$. The corresponding solutions $\alpha_j$ are

$$3, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1.$$  

We find that there are 44 distinct types of rigid tree honeycombs, all of which can be obtained from one of the eight depicted in Figure 3.10 by rotations and reflections. Each of the types pictured, except for the first one, yields six different types via rotations and reflections. The first one is invariant under rotations by 120°. For larger values of $\omega$, the number of types of rigid tree honeycombs grows fairly rapidly. For instance, we get 272 such types for $\omega = 5$ and more than 1000 for $\omega = 6$. Rigid tree honeycombs with a rotational symmetry exist for every weight $\omega$ that is not a multiple of 3. Some examples of such honeycombs are provided in [3] but for large $\omega$ there are other examples. To find them we denote by $\beta_1 \geq \cdots \geq \beta_k$ the possible exit multiplicities of such a honeycomb in the direction of $w_1$, and note that these numbers satisfy

$$\sum_{j=1}^{k} \beta_j = \omega, \quad \sum_{j=1}^{k} \beta_j^2 = \frac{\omega^2 + 2}{3}.$$  

These equations combine to yield

$$\sum_{j=1}^{k} \beta_j(\beta_j - 1) = \frac{(\omega - 1)(\omega - 2)}{6}.$$  

For instance, if $\omega = 8$, we must write $(\omega - 1)(\omega - 2)/6 = 7$ as a sum of triangular numbers. The acceptable possibilities are

$$6 + 1, 3 + 3 + 1,$$

and they correspond to the following sequences of $\beta_j$:

$$4, 2, 1, 1$$

$$3, 3, 2.$$
(The representation $7 = 1+1+1+1+1+1+1$ requires that $\beta_1 = \cdots = \beta_7 = 2$ but $7 \times 2 > 8$. Similarly, $7 = 3+1+1+1+1$ requires $\beta_1 = 3$ and $\beta_2 = \cdots = \beta_5 = 2$.)

We obtain, up to symmetry, the 4 distinct rigid tree honeycomb types depicted in Figure 3.11.

4. Honeycombs compatible with a puzzle

Suppose that we have the puzzle of a honeycomb $\nu \in \mathcal{M}$ and, in addition, a honeycomb $\mu \in \mathcal{M}$ superposed on it. We say that $\mu$ is compatible with the puzzle of $\nu$ if the dark gray parallelogram contain no branch points of $\mu$ in their interior and the edges of $\mu$ only cross such a parallelogram along segments parallel to an edge of the parallelogram. This concept appeared in [9] where it was used to explain saturated Horn inequalities. It was also employed in [3, Section 7] to give a different argument for some factorizations of Littlewood-Richardson coefficients first found in [7, 6]. Here we are mostly interested in the existence and construction of such compatible honeycombs $\mu$, particularly rigid tree honeycombs. It is useful to observe that a tree honeycomb that is compatible with the puzzle of $\nu$ crosses the gray parallelograms along segments that are either all parallel to the white edges or all parallel to the light gray edges (see Figure 4.1). Deflating the puzzle of $\nu$ yields, in addition to the original honeycomb $\nu$, a honeycomb $\mu_{\nu}$ constructed as follows:

1. the parts of the support of $\mu$ inside the white puzzle pieces are translated along with those white pieces,
2. the parts of the support of $\mu$ that cross dark gray parallelograms between white puzzle pieces are discarded, and
Figure 4.2. Evil turns in a rigid puzzle

(3) if $AB$ is an edge of $\nu$ and the support of $\mu$ crosses the inflation of $AB$ on segments $I$ parallel to $AB$, then $AB$ is also contained in the support of $\mu_{\nu}$ and its multiplicity is the sum of the multiplicities of all such segments $I$.

A similar construction, using the $\ast$-deflation of the puzzle of $\nu$, yields a honeycomb $\mu_{\nu^\ast}$ with $\omega(\mu_{\nu^\ast}) = \omega(\mu_{\nu}) = \omega(\mu)$. Moreover, it was shown in [3] that

$$\mu^\ast = (\mu_{\nu})^\ast + (\mu_{\nu^\ast})^\ast.$$  

Our inductive procedure for constructing honeycombs from rigid overlays depends on the construction of honeycombs that are compatible with the puzzle of a rigid honeycomb. We begin with compatible honeycombs whose support does not intersect the interior of any puzzle piece. Such a support must be contained in the edges of the dark gray parallelograms. Thus, suppose that $\nu \in \mathcal{M}$ is a rigid honeycomb and regard the union of the edges of the dark gray parallelograms in its puzzle as a prospective support for a honeycomb. For simplicity, we simply call puzzle edges the edges of the dark gray parallelogram. These edges are either white or light gray, according to the color of the other neighboring puzzle piece.

**Lemma 4.1.** Let $\nu$ be a honeycomb in $\mathcal{M}$. Every evil loop contained in the puzzle edges of $\nu$ is a gentle loop, that is, consecutive edges in the loop form $120^\circ$ angles.

**Proof.** Since all the branch points in this support are ‘rakes’ (as in the second part of Figure 3.3), it follows that every evil turn of an evil loop is either a gentle turn (of $120^\circ$) or a $180^\circ$ turn along the ‘handle’ of a rake. Suppose that an evil loop $e_1, \ldots, e_n = e_1$ contains at least one $180^\circ$ turn. We may assume that $e_1 = AB$ and $e_2 = BA$, in which case it follows that the edges $e_2, \ldots, e_k$ point to the acute angles of the adjacent parallelograms until we meet again a $180^\circ$ turn $e_k e_{k+1}$ (see Figure 4.2). Continuing this way around the loop, we conclude that $e_1$ also points to the acute angle of the adjacent parallelogram, an obvious contradiction. □

According to [9, Theorems 4 and 7], the puzzle of a rigid honeycomb $\nu$ contains no gentle (and, by Lemma 4.1, no evil) loops. It follows that any honeycomb supported by the puzzle edges of $\nu$ is necessarily rigid. Suppose for a moment that there exists a honeycomb $\mu$ whose support is the union of all the edges of the puzzle of $\nu$. The root edges of such a honeycomb are easily identified. For each ray $e$ in the support of $\nu$ there is one equivalence class of root edges containing the ray in the inflation of $e$ incident to the acute angle of that inflation (see Figure 4.3 for an example with these root edges colored red). We now orient all edges towards the acute angles of the parallelogram to which they belong (or away from the obtuse angles in the case of some of the rays). (Note that the opposite orientation was
used in [9, Section 4] for different purposes.) Thus, if \( f \) is a ray in the support of \( \nu \), the boundary of its inflation has two edges that are rays: an incoming ray pointing to the acute angle of the inflation, and an outgoing ray. This orientation is useful because, except for the root edges, it is precisely the orientation of the descendant relation, that is, each edge (with the exception of the root edges) is oriented from an ancestor to a descendant. The resulting oriented graph contains no loops and, in addition, for each edge \( e \) there is at least one path from one of the root edges to \( e \). It is easy now to see that there actually exists a rigid tree honeycomb rooted at each of the root edges of the hypothetical honeycomb \( \mu \). Figure 4.3 shows (in red) the support of one of the honeycombs constructed in the manner just described.

**Theorem 4.2.** Let \( \nu \in \mathcal{M} \) be a rigid honeycomb. Then there are exactly \( \text{exit}(\nu) \) rigid tree honeycombs whose support is contained in the union of the puzzle edges of \( \nu \). The root of the honeycomb corresponding to a ray \( e \) of \( \nu \) is the incoming ray in the inflation of \( e \). Every edge of the puzzle of \( \nu \) is contained in the support of one of these tree honeycombs. If \( \mu_1 \) and \( \mu_2 \) are two of these tree honeycombs, then \( \Sigma_{\mu_1}(\mu_2) = 0 \). The following observation about arbitrary compatible honeycombs is useful in Section 6.

**Lemma 4.3.** Let \( \nu \in \mathcal{M} \) be a rigid honeycomb and let \( \mu \in \mathcal{M} \) be a honeycomb that is compatible with the puzzle of \( \nu \). Suppose that the support of \( \mu \) contains an edge \( e \) of the puzzle. Then the support of \( \mu \) contains every descendant of \( e \) relative to the orientation of the puzzle edges of \( \nu \).

**Proof.** It suffices to show that, given two edges \( e_1 \) and \( e_2 \) in the puzzle of \( \nu \) such that \( e_1 e_2 \) is a gentle path and \( e_1 \) is contained in the support of \( \mu \), it follows that \( e_2 \) is also contained in the support of \( \mu \). The three possible configurations, up to rotations and reflections, are shown in Figure 4.4. We observe first that at least part of \( e_2 \) must be contained in the support of \( \mu \) because otherwise the support of \( \mu \) would contain one edge crossing a dark gray parallelogram in a direction that is not allowed by compatibility. Similarly, if \( I \subset e_2 \) is the largest segment in the support of \( \mu \) that contains \( e_1 \cap e_2 \), then \( I \) must equal \( e_2 \) because otherwise the endpoint of \( I \) (other than \( e_1 \cap e_2 \)) would need to be a branch point for \( \mu \) with one edge of \( \mu \) crossing the neighboring dark gray parallelogram in the direction forbidden by compatibility. □
It is of interest to calculate the exit pattern of the tree honeycombs described in Theorem 4.2, and we do eventually provide an answer amounting to an inductive procedure depending on the number of extreme summands of the honeycomb $\nu$ (see Remark 4.15). We start with the special case of an extreme rigid honeycomb. The direction of the rays corresponding to the multiplicities $\alpha_j$ are not specified in the following statement, but the multiplicity $\alpha_1 \alpha_j$ corresponds to the outgoing ray in the inflation of the ray with multiplicity $\alpha_j$, while the multiplicities $\alpha_1^2 - 1$ and 1 correspond to outgoing and incoming rays, respectively, in the inflation of the ray with multiplicity $\alpha_1$.

**Proposition 4.4.** Suppose that $\nu$ is a rigid tree honeycomb and that the nonzero exit multiplicities of $\nu$ are, in counterclockwise order, $\alpha_1, \ldots, \alpha_n$. Let $\mu$ be the rigid tree honeycomb, supported in the union of the puzzle edges of $\nu$, and rooted in the incoming ray in the inflation of the ray with multiplicity $\alpha_1$. Then the nonzero exit multiplicities of $\mu$ are, in counterclockwise order,

$$\alpha_1^2 - 1, 1, \alpha_1 \alpha_2, \ldots, \alpha_1 \alpha_n,$$

where the first term is omitted if $\alpha_1 = 1$. In particular, $\omega(\mu) = \alpha_1 \omega(\nu)$,

$$(4.2)$$

$$\operatorname{exit}(\mu) = \operatorname{exit}(\nu) + 1$$

if $\alpha_1 > 1$, and $\mu$ is homologous to $\nu$ if $\alpha_1 = 1$.

**Proof.** Suppose that the exit multiplicities of $\mu$ are $\beta_1, 1, \beta_2, \ldots, \beta_n$, where $\beta_j$ is the multiplicity of the outgoing ray corresponding to $\alpha_j$. The presence of 1 is explained by the choice of root edge which happens to be an incoming ray. Deflate now the puzzle of $\nu$ and obtain thereby a honeycomb $\mu_\nu$ with exit multiplicities $\beta_1 + 1, \beta_2, \ldots, \beta_n$. The honeycomb $\mu_\nu$ has support contained in that of $\nu$ and is therefore of the form $k\nu$ for some positive integer $k$. We deduce that the exit multiplicities of $\mu$ are $k\alpha_1 - 1, 1, k\alpha_2, \ldots, k\alpha_n$ and $\omega(\mu) = k\omega(\nu)$. Thus,

$$\sum_{j=1}^n \alpha_j^2 = \omega(\nu)^2 + 2,$$

$$(k\alpha_1 - 1)^2 + 1 + \sum_{j=2}^n (k\alpha_j)^2 = (k\omega(\nu))^2 + 2,$$

and subtracting $k^2$ times the first equation from the second we obtain

$$-2k\alpha_1 + 2 = -2k^2 + 2.$$
The only positive solution of this equation is $k = \alpha_1$. 

**Corollary 4.5.** Suppose that $\nu$ is a rigid tree honeycomb and that the nonzero exit multiplicities of $\nu$ are, in counterclockwise order, $\alpha_1, \ldots, \alpha_n$. There exists a rigid tree honeycomb $\mu$ whose nonzero exit multiplicities are, in counterclockwise order,

$$1, \alpha_1^2 - 1, \alpha_1 \alpha_2, \ldots, \alpha_1 \alpha_n,$$

where the second term is omitted if $\alpha_1 = 1$. In particular, $\omega(\mu) = \alpha_1 \omega(\nu)$.

**Proof.** Let $\nu'$ be a honeycomb obtained by reflecting $\nu$ in a line parallel to $w_1$. Thus the nonzero exit multiplicities of $\nu'$ can be listed, in counterclockwise order, as $\alpha_1, \alpha_n, \alpha_{n-1}, \ldots, \alpha_2$. Proposition 4.4, applied to $\nu'$, shows that there exists a rigid tree honeycomb $\mu'$ whose nonzero exit multiplicities are, in counterclockwise order,

$$\alpha_1^2 - 1, 1, \alpha_1 \alpha_n, \alpha_1 \alpha_{n-1}, \ldots, \alpha_1 \alpha_2.$$

The honeycomb $\mu$ is now obtained by reflecting $\mu'$ in a line parallel to $w_1$. □

**Remark 4.6.** An alternate proof of the preceding corollary uses the $*$-inflation of $\mu$.

We next construct compatible honeycombs starting with a rigid overlay of two rigid tree honeycombs. The following result was proved in [3, Theorem 5.2].

**Theorem 4.7.** Suppose that $\nu_1$ and $\nu_2$ are rigid tree honeycombs and $\Sigma_{\nu_2}(\nu_1) = 0$. Then there exists a honeycomb $\nu_2'$, homologous to $\nu_2$ and compatible with the puzzle of $\nu_1$, such that the parts of the support of $\nu_2'$ contained in the white puzzle pieces are simply translates of the corresponding parts of the support of $\nu_2$.

We recall briefly the construction of $\nu_2'$. Choose a root edge $e$ of $\nu_2$ that is not contained in the support of $\nu_1$ and orient all the edges of $\nu_2$ away from $e$ (that is, in the direction of the descendence paths from $e$. If a portion $f$ of an edge of $\nu_2$ is contained in an edge of $\nu_1$, attach $f$ to the white puzzle piece on its right relative to this orientation. This way we obtain the part of the support of $\nu_2$ that is contained in the boundary of a white puzzle piece. The part of the support of $\nu_2'$ in the interior of the white pieces is obtained simply by translation, as in the statement above. Finally, one reconnects the edges of $\nu_2'$ that are transversal to the support of $\nu_1$ and this is done by adding a number of segments that cross dark gray parallelograms and are parallel to their light gray sides. The requirement that $\Sigma_{\nu_2}(\nu_1) = 0$ makes this construction possible. This process is illustrated in Figure 4.5 where the supports of $\nu_2$ and $\nu_2'$ are drawn with dashed lines. (An interesting feature of the overlay in Figure 4.5 is that $\nu_1$ and $\nu_2$ assign nonzero multiplicity to precisely the same rays.)

**Proposition 4.8.** Suppose that $\nu_1$ and $\nu_2$ are rigid tree honeycombs such that $\Sigma_{\nu_2}(\nu_1) = 0$, and let $f$ be a ray of $\nu_1$ to which $\nu_1$ assigns exit multiplicity $\alpha_1$. Let $\alpha_2 \geq 0$ be the exit multiplicity that $\nu_2$ assigns to the same ray. Denote by $\nu_2'$ the rigid tree honeycomb whose support is contained in the union of the puzzle edges of $\nu_1$ and with one root edge equal to the incoming ray of the inflation of $f$. Finally, let $\nu_2''$ be the rigid tree honeycomb, homologous to $\nu_2$, constructed in Theorem 4.7. Then $\Sigma_{\nu_2'}(\nu_1) = 0$ and $\Sigma_{\nu_2''}(\nu_2) = \alpha_2 + \alpha_1 \Sigma_{\nu_1}(\nu_2)$.

**Proof.** Suppose that $\nu_2^{(j)}(c) \nu_1^{(j)}(d) \neq 0$ for some $j = 1, 2, 3$ and some $c < d$. Unless the ray $f$ is on the line $x_j = d$, there exist numbers $c' < d'$ such that $\nu_2^{(j)}(c') =$...
all the pairs arising from the ray case, we obtain inflation of $f$. If the ray $f$ is on the line $x_j = d$, we find $c' < d' < d''$ such that $\nu_2^{(j)}(c') = \nu_1^{(j)}(c), \nu_1^{(j)}(d') = \alpha_3^2 - 1$, and $\nu_2^{(j)}(d'') = 1$. This describes all the pairs $c' < d'$ (or $c' < d''$) such that $\nu_2^{(j)}(c')\nu_1^{(j)}(d') \neq 0$ with one exception arising from the ray $f$ that corresponds to $\nu_1^{(j)}(c) = \alpha_1$ and $\nu_2^{(j)}(c) = \alpha_2$. In this case, we obtain $d' < d''$ (corresponding to the outgoing and incoming rays in the inflation of $f$) such that $\nu_2^{(j)}(d') = \alpha_2$ and $\nu_1^{(j)}(d'') = 1$. We conclude that

$$\sum_{c' < d'} \nu_2^{(j)}(c')\nu_1^{(j)}(d') = \alpha_2 + \alpha_1 \sum_{c < d} \nu_2^{(j)}(c)\nu_1^{(j)}(d)$$

and thus

$$\Sigma_{\nu_1}(\tilde{\nu}_2) = \alpha_2 + \alpha_1 \omega(\nu_1) + \alpha_1 \omega(\nu_1)\omega(\nu_2)$$

$$= \alpha_2 + \alpha_1 \Sigma_{\nu_2}(\nu_1)$$

because $\omega(\tilde{\nu}_1) = \alpha_1 \omega(\nu_1)$ and $\omega(\tilde{\nu}_2) = \omega(\nu_2)$. The calculation of $\Sigma_{\nu_1}(\tilde{\nu}_1)$ is somewhat simpler because there is no extra pair $c' > d'$ such that $\nu_1^{(j)}(c)\nu_1^{(j)}(d') \neq 0$.

**Remark 4.9.** The preceding proposition can also be proved using the calculation of $\Sigma$ described in Remark 4.4, though the extra $\alpha_2$ crossings needed for $\Sigma_{\nu_1}(\tilde{\nu}_2)$ may be difficult to locate.

**Remark 4.10.** Suppose that $\nu_1, \nu_2$, and $\nu_3$ are three rigid tree honeycombs such that $\Sigma_{\nu_2}(\nu_1) = \Sigma_{\nu_3}(\nu_1) = 0$. Let $\tilde{\nu}_1$ and $\tilde{\nu}_2$ be as in Proposition 4.8 and construct a third honeycomb $\tilde{\nu}_3$ that is homologous to $\nu_3$ and compatible with the puzzle of $\nu_1$ (the same way that $\tilde{\nu}_2$ was constructed). Then the entire overlay $\tilde{\nu}_2 + \tilde{\nu}_3$ is homologous to $\nu_2 + \nu_3$, and therefore

$$\Sigma_{\nu_2}(\tilde{\nu}_3) = \Sigma_{\nu_2}(\nu_3)$$

and

$$\Sigma_{\nu_3}(\tilde{\nu}_2) = \Sigma_{\nu_3}(\nu_2).$$

There is a second way to construct a rigid tree honeycomb from a rigid overlay of two tree honeycombs. We show in Theorem 5.1 that every rigid tree honeycomb with weight at least 2 can be obtained from this construction applied to an overlay of honeycombs with smaller weights.

**Theorem 4.11.** Suppose that $\nu_1$ and $\nu_2$ are rigid tree honeycombs, $\Sigma_{\nu_1}(\nu_2) = \sigma > 0$ and $\Sigma_{\nu_2}(\nu_1) = 0$. Then there exists a rigid tree honeycomb $\tilde{\nu}_1$ compatible with the puzzle of $\nu_2$, such that the parts of the support of $\tilde{\nu}_1$ contained in the white
puzzle pieces are simply translates of the corresponding parts of the support of $\nu_1$. Moreover, the exit pattern of $\hat{\nu}_1$ is the same as the exit pattern of $\nu_1 + \sigma \nu_2$.

Proof. Fix a root edge $e$ of $\nu_1$ that is disjoint from the support of $\nu_2$ and orient all the other edges of $\mu_1$ away from $e$, that is, in the direction given by descendance paths. We construct a collection $C$ of segments and rays containing:

1. the edges of the puzzle of $\nu_2$,
2. the segments obtained as translates of segments in the support of $\nu_1$ that are contained in the interior of a white puzzle piece, and
3. the inflation of every branch point of $\nu_1 + \nu_2$ that is in the interior of a side of a white puzzle piece. That is, if $Z$ is a branch point in the interior of an edge $AB$ of $\nu_2$, and if $A'B', A''B''$ are the two white edges of the inflation of $AB$, then we include in our collection the segment $Z'Z''$ joining the points on $A'B'$ and $A''B''$ that are the translates of $Z$.

We orient all the segments in $C$, with the exception of the translate of the root edge $e$, as follows:

1. the edges of every dark gray parallelogram point to the acute angles of the parallelogram,
2. the translate of a (part of) an edge $f$ of $\nu_1$ other than $e$ is pointing in the same direction as $f$,
3. the segments $Z'Z''$ described above are oriented as in Figure 4.6 where the six possible configurations (up to rotations) are shown. (Here, the horizontal edges are in the support of $\nu_2$ and are represented by dashed lines. The support of $\nu_1$ is represented by dotted lines and the solid lines represent common parts of the two supports. No other configurations are possible because $\Sigma_{\nu_2}(\nu_1) = 0$; see Remark 2.4.)

We treat $C$ as the prospective support of a honeycomb and, accordingly, we say that a point where three or more of these segments meet is a branch point. Denote by $e' = xy$ the translate of $e$ and construct (as in Section 3) a tree $T_0$ by following the descendance paths in $C$. Thus, the vertices of $T_0$ are $x, y$, and the gentle paths $e'e_1\ldots e_n$, where $e_j \in C$, $e_1$ points away from $x$ or from $y$, and $e_j$ points away from $e_{j-1}$ for $j = 2, \ldots, n$. If $e'e_1\ldots e_n$ is such a path, then it is joined to $e'e_1\ldots e_{n-1}$ by an edge labeled $j$ if $e_n$ is parallel to $w_j$. In addition, $x$ is joined to $y$ by an edge labeled $j$ if $e'$ is parallel to $w_j$. Finally, either $x$ or $y$ is joined to $e_1$ by an edge labeled using the same rule.

We first argue that the tree $T_0$ is finite. In the contrary case, there would exist an infinite gentle oriented path. Because $C$ is finite, some edge of $C$ must appear
twice in this path, and we conclude that there exists an oriented gentle loop $e_1 \ldots e_n$ formed by edges in $C$. Some of the edges $e_j$, but no two consecutive ones, may be of the form $Z'Z''$ as in the case (3) above. Each other $e_j$ is the translate of some edge $f_j$ of $\nu_1 + \nu_2$. These edges $f_j$ form a loop in the support of $\nu_1 + \nu_2$. We claim that this loop is evil, thus contradicting the rigidity of $\nu_1 + \nu_2$. This is verified by considering any two, three, or four consecutive edges that form an oriented gentle path in $C$. Such a sequence of edges arises from a branch point of $\nu_1 + \nu_2$ and the relevant evil turns are observed by examining the situations depicted in Figures 4.6 and 4.7. All other possibilities for the branch points of $\nu_1 + \nu_2$ are precluded by the condition $\Sigma_{\nu_2}(\nu_1) = 0$.

We also note that ends of the tree $T_0$ are precisely those paths $e'e_1 \ldots e_n$ such that $e_n$ is either an outgoing ray in the puzzle of $\nu_2$ or the translate of a ray of $\nu_1$. We define now multiplicities $\tilde{\nu}_1(e)$ for each $e \in C$ by setting $\tilde{\nu}_1(e') = 1$ and, for other edges, $\tilde{\nu}_1(e)$ is the number of gentle descendance paths $e'e_1 \ldots e_n$ satisfying $e_n = e$.

Define now a honeycomb $\tilde{\nu}_1$ by setting $\tilde{\nu}_1(e') = 1$ and, given an arbitrary edge $f$ in the collection $C$, $\tilde{\nu}_1(f)$ is the number of distinct oriented descendance paths from $e'$ to $f$. The balance condition at each branch point is checked by looking at the various cases from Figures 4.6 and 4.7. It is clear that $\tilde{\nu}_1$ is a tree honeycomb, and Theorem 2.2 implies that it is rigid. Since the edges in the support of $\tilde{\nu}_1$ are either translates of the original edges in the support of $\nu_1 + \nu_2$ or segments of the form $Z'Z''$, it follows that the support of the honeycomb $(\tilde{\nu}_1)_{\nu_2}$ (obtained by deflating the puzzle of $\nu_2$) is contained in the support of $\nu_1 + \nu_2$. Thus $(\tilde{\nu}_1)_{\nu_2}$ is a rigid honeycomb of the form $c_1\nu_1 + c_2\nu_2$ for some $c_1, c_2 \geq 0$. Clearly, $(\tilde{\nu}_1)_{\nu_2}$ assigns unit mass to $e'$, so $c_1 = 1$. To determine $c_2$, we note that, since $(\tilde{\nu}_1)_{\nu_2}$ and $\nu_1 + c_2\nu_2$
have same exit pattern, relation (2.7) shows that
\[-1 = \Sigma_{(\hat{\nu}_1)_{\nu_2}}((\hat{\nu}_1)_{\nu_2}) = \Sigma_{\nu_1 + c_2\nu_2}(\nu_1 + c_2\nu_2)\]
\[= \Sigma_{\nu_1}(\nu_1) + c_2\Sigma_{\nu_1}(\nu_2) + c_2\Sigma_{\nu_2}(\nu_1) + c_2^2\Sigma_{\nu_2}(\nu_2)\]
\[= -1 + 0 + c_2\sigma - c_2^2.\]
This equality is only satisfied by \(c_2 = 0\) and \(c_2 = \sigma\). If \(\sigma > 0\), the alternative \(c_2 = 0\) can be dismissed because in this case one of the first two configurations depicted in Figure 4.6 occurs, thus ensuring that the honeycomb \(\hat{\nu}_1\) assigns positive multiplicity to some white edges of the puzzle of \(\nu_2\), and so \(c_2 > 0\).

**Corollary 4.12.** Suppose that \(\nu_1\) and \(\nu_2\) are rigid tree honeycombs such that \(\Sigma_{\nu_2}(\nu_1) = 0\), and set \(\sigma = \Sigma_{\nu_2}(\nu_2)\). Then there exist a rigid tree honeycomb \(\mu_1\) (respectively, \(\mu_2\)) that has the same exit pattern as \(\nu_1 + \sigma\nu_2\) (respectively, \(\sigma\nu_1 + \nu_2\)).

**Proof.** The honeycomb \(\mu_1 = \hat{\nu}_1\) satisfies the requirement. To prove the existence of \(\mu_2\), we argue as in Corollary 4.5. Thus, we consider the reflections \(\nu_1'\) and \(\nu_2'\) of \(\nu_1\) and \(\nu_2\) in a line parallel to \(w_1\). These rigid tree honeycombs satisfy \(\Sigma_{\nu_2'}(\nu_2') = 0\) and \(\Sigma_{\nu_2'}(\nu_1') = \sigma\), so there exists a rigid tree honeycomb \(\mu_2'\) with the same exit pattern as \(\sigma\nu_1' + \nu_2'\). We obtain \(\mu_2\) as the reflection of \(\mu_2'\) in a line parallel to \(w_1\). \(\square\)

Once we know that a pattern comes from a rigid tree honeycomb \(\mu\), one can find a honeycomb homologous to \(\mu\) using the flat region construction from Section 3. This allows for a fairly efficient construction of measures \(\mu_1\) and \(\mu_2\) with the exit patterns prescribed by Corollary 4.12.

**Remark 4.13.** With the notation of Theorem 4.11, relation (4.1) applies to the honeycomb \(\hat{\nu}_1\) to yield
\[(\hat{\nu}_1)^* = ((\hat{\nu}_1)_{\nu_2})^* + ((\hat{\nu}_1)_{\nu_2'})^* = (\nu_1 + \sigma\nu_2)^* + ((\hat{\nu}_1)_{\nu_2'})^*.\]

The support of \((\hat{\nu}_1)_{\nu_2'}\) inside the triangle \(\Delta_\ast\) is always contained in the support of \(\nu_2'.\) In many cases, the support of \((\hat{\nu}_1)_{\nu_2'}\) contains the entire support of \(\nu_2'\) (inside \(\Delta_\ast\)). When this occurs, the honeycomb \(((\hat{\nu}_1)_{\nu_2'})^*\) is in fact a tree honeycomb with the same exit pattern as \(\nu_2',\) so \(((\hat{\nu}_1)_{\nu_2'})^*\) is homologous (after a 60° rotation) to \(\nu_2'.\) Here is one particular way to form overlays with this special property. Given an arbitrary rigid tree honeycomb \(\nu_2,\) find another rigid tree honeycomb \(\nu_1\) such that \(\omega(\nu_1) = 3,\) \(\nu_2\) is clockwise from \(\nu_1,\) and every exit ray of \(\nu_2\) crosses an edge of \(\nu_1.\) (See Figure 4.8 for a particular case. The honeycomb \(\nu_2\) is pictured in black and \(\nu_2\) in red. The second part of the figure shows the puzzle of \(\nu_2\) and the support of \(\hat{\nu}_1.)\) Since the support of \(\hat{\nu}_1\) contains a part of each incoming ray of the puzzle of \(\nu_2,\) this support must also contain the descendants of these incoming rays, and these include all the parallelogram edges other than the incoming rays. It is clear that the support of \((\hat{\nu}_1)_{\nu_2'}\) contains all the edges of the light gray puzzle pieces. We record this fact as follows.

**Proposition 4.14.** Let \(\mu \in \mathcal{M}\) be an arbitrary rigid tree honeycomb. There exist rigid tree honeycombs \(\nu \in \mathcal{M}\) and \(\rho \in \mathcal{M}_\ast\) such that \(\rho \leq \nu^*\) and \(\rho\) is homologous to a 60° rotation of \(\mu.\)
Remark 4.15. At this point, we have enough information to calculate the exit pattern of any of the rigid tree honeycombs whose support is contained in the union of the boundaries of the dark gray parallelograms in a rigid puzzle. Thus, suppose that $\nu$ is a rigid honeycomb, and write it as $\nu = \sum_{j=1}^{n} c_j \nu_j$, where each $\nu_j$ is a rigid tree honeycomb and $\Sigma_{\nu_i} (\nu_j) = 0$ for $i > j$. Replacing $c_j$ by 1 does not alter the structure of the puzzle of $\nu$, and let $\alpha_j \geq 0$ be the corresponding exit multiplicity of $\nu_j$. The main observation is that the rigid honeycomb $\mu$, supported by the edges of parallelograms in the puzzle of $\nu$ and rooted in the incoming ray in the inflation of $f$, can be obtained as a result of a sequence of operations that we now describe.

Suppose for the moment that $\alpha_1 > 0$.

1. Construct rigid tree honeycombs $\nu_1(1), \ldots, \nu_n(1)$, compatible with the puzzle of $\nu_1$ as follows: $\nu_1(1)$ is supported by the puzzle edges and is rooted in the inflation of $f$ while, for $j \geq 2$, $\nu_j(1)$ is obtained by translating the support of $\nu_j$ along with the white pieces as in Theorem 4.7. As seen above (Proposition 4.8 and Remark 4.10), these new tree honeycombs form again a rigid overlay.

2. Construct rigid tree honeycombs $\nu_2(2), \ldots, \nu_n(2)$, compatible with the puzzle of $\nu_2(1)$, as follows: $\nu_2(2)$ is obtained by translating the support of $\nu_1(1)$ as in Theorem 4.11 and, for $j \geq 3$, $\nu_j(2)$ is obtained using the procedure of Theorem 4.7. These tree honeycombs also form a rigid overlay.

3. For $k = 3, \ldots, n$, construct rigid tree honeycombs $\nu_k(k), \nu_{k+1}(k), \ldots, \nu_n(k)$.

The construction is done by induction and the inductive step is the procedure described in (2). Thus, supposing that these honeycombs are constructed for some $k < n$, we construct the puzzle of $\nu_{k+1}(k)$ and we construct $\nu_{k+1}(k+1)$ by an application of Theorem 4.11 to $\nu_k(k)$ and we construct $\nu_{j}(j)$ by an application of Theorem 4.7 to $\nu_k(j)$, $j = k+2, \ldots, n$.

4. The rigid tree honeycomb $\nu_n(n)$ is precisely the desired honeycomb $\mu$.

One can calculate the exit patterns of all the honeycombs constructed above, in particular, the exit pattern of $\mu$.

1. After the first operation described above, the honeycomb $\nu_j(1)$ is homologous to $\nu_j$ for $j \geq 2$ and the exit pattern of $\nu_1(1)$ is obtained by multiplying the exit pattern of $\nu_1$ by $\alpha_1$, except for the ray $f$ that gives...
rise to two exit multiplicities equal to $\alpha_j^2 - 1$ and 1. Moreover, we have $\Sigma_{\nu_1(i)}(\nu_1(j)) = \Sigma_{\nu_1}(\nu_j)$ for $i, j \geq 2$, and $\Sigma_{\nu_1(1)}(\nu_1(j)) = \alpha_j + \alpha_1 \Sigma_{\nu_1}(\nu_j)$ for $j \geq 2$ (see Proposition 4.8).

(2) After the second operation, the honeycomb $\nu_2(j)$ is homologous to $\nu_j$ for $j \geq 3$ and the exit pattern of $\nu_2(2)$ is the same as the one for $\nu_1(1) + \sigma \nu_1(2)$, where $\sigma = \Sigma_{\nu_1(1)}(\nu_1(2))$ (see Theorem 4.11). Moreover, we have $\Sigma_{\nu_2(i)}(\nu_2(j)) = \Sigma_{\nu_1}(\nu_j)$ for $i, j \geq 3$, and an easy calculation shows that

$$\Sigma_{\nu_2(2)}(\nu_2(j)) = \Sigma_{\nu_1(1)}(\nu_1(j)) + \sigma \Sigma_{\nu_1(2)}(\nu_1(j)), \quad j \geq 3.$$

Further operations follow this model and they eventually yield the exit pattern of $\mu$.

In case $\alpha_1 = 0$, we look for the first nonzero $\alpha_j$ and we can simply discard the honeycombs $\nu_1, \ldots, \nu_{j-1}$ because they do not contribute to the exit pattern of $\mu$.

**Example 4.16.** The special case of a rigid overlay of two tree honeycombs is used in Section 6. Thus, let $\nu_1$ and $\nu_2$ be rigid tree honeycombs such that $\Sigma_{\nu_2}(\nu_1) = 0$ and $\Sigma_{\nu_1}(\nu_2) = \sigma > 0$. Let $f$ be a ray to which $\nu_j$ assigns multiplicity $\alpha_j$, $j = 1, 2$. The first operation in the preceding remark provides $\nu_1(1)$ and $\nu_1(2)$ such that $\Sigma_{\nu_1(2)}(\nu_1(1)) = 0$ and $\Sigma_{\nu_1(1)}(\nu_1(2)) = \alpha_2 + \sigma \alpha_1$. The exit multiplicities assigned by $\nu_1(1)$ to the outgoing and incoming rays in the inflation of $f$ are $\alpha_1^2 - 1$ and 1, respectively, while $\nu_1(2)$ assigns multiplicity $\alpha_2$ to the outgoing ray. If $e$ is another ray such that $\nu_j(e) = \beta_j$, $j = 1, 2$, then the corresponding (outgoing) ray is assigned multiplicities $\alpha_1 \beta_1$ and $\beta_2$, respectively, by $\nu_1(1)$ and $\nu_1(2)$. After the second operation, we obtain a honeycomb $\mu = \nu_2(2)$ with typical exit multiplicities $\alpha_1 \beta_1 + (\alpha_2 + \sigma \alpha_1) \beta_2$, and with exit multiplicities $\alpha_1^2 - 1 + (\alpha_2 + \sigma \alpha_1) \alpha_2$ and 1 assigned to the outgoing and incoming rays in the inflation of $f$ (in the puzzle of $\nu_1 + \nu_2$). This process is illustrated in Figure 4.9, where $\nu_1$ is drawn in black, $\nu_2$ in red, and the numbers represent exit multiplicities. For this example, $\sigma = 2$, $\alpha_1 = 2$, and $\alpha_2 = 1$. For comparison, we show in Figure 4.10 the same final honeycomb constructed directly on the puzzle of $\nu_1 + \nu_2$.

5. Degeneration of a rigid tree honeycomb

Consider an arbitrary rigid honeycomb $\nu \in \mathcal{M}$ and write its dual as

$$\nu^* = c_1 \rho_1 + \cdots + c_n \rho_n,$$

where $n = \text{root}(\nu^*) = \text{exit}(\nu) - \text{root}(\nu) - 2$, $c_1, \ldots, c_n > 0$, and each $\rho_j$ is a rigid tree honeycomb in $\mathcal{M}_+$ (see Theorem 3.2). As we pointed out before, the honeycombs
Figure 4.10. The same honeycomb constructed using the puzzle of $\nu_1 + \nu_2$

$\mu \in \mathcal{M}$ that are homologous to $\nu$ are precisely the ones that satisfy

\begin{equation}
\mu^* = t_1\rho_1 + \cdots + t_n\rho_n
\end{equation}

for some $t_1, \ldots, t_n > 0$. Decreasing one of the coefficients $t_j$ amounts to decreasing the lengths of some of the edges of $\mu$. If we replace some of the coefficients $t_j$ by 0, the honeycomb $\mu$ satisfying (5.1) is no longer homologous to $\nu$. We call it a degeneration of $\nu$. A simple degeneration of $\mu$ is obtained by replacing exactly one of the $t_j$ by 0. All the degenerations of $\nu$ satisfy $\omega(\mu) = \omega(\nu)$. Moreover, the exit multiplicities of $\mu$ are sums of one or several (consecutive) exit multiplicities of $\nu$ (see Remark 3.1). For instance, if all the coefficients $t_j$ are replaced by 0, we obtain the degeneration $\mu = \omega(\nu)\nu_0$, where $\nu_0$ is a rigid tree honeycomb of unit weight.

Suppose that $\mu$ is an arbitrary degeneration of $\nu$. An application of Theorem 3.2 to $\mu$ and to $\nu$ yields

\begin{align*}
\text{root}(\mu) &= \text{exit}(\nu) - \text{exit}(\mu) - 1, \\
\text{root}(\nu) &= \text{exit}(\nu) - \text{exit}(\mu) - 1,
\end{align*}

and subtracting these equalities we obtain the equation

\begin{equation}
\text{root}(\mu) - \text{root}(\nu) = [\text{exit}(\nu) - \text{exit}(\mu)] - [\text{exit}(\nu) - \text{exit}(\mu)],
\end{equation}

where $\text{root}(\nu^*) = \text{root}(\mu^*) > 0$ and $\text{exit}(\nu) - \text{exit}(\mu) \geq 0$.

Suppose now that $\nu$ is a rigid tree honeycomb and that $\mu$ is a simple degeneration of $\nu$. Then we have $\text{root}(\nu^*) = \text{root}(\mu^*) = 1$ and (5.2) allows for two possibilities:

1. $\text{root}(\mu) = 2$ and $\text{exit}(\mu) = \text{exit}(\nu)$, or
2. $\text{root}(\mu) = 1$ and $\text{exit}(\mu) = \text{exit}(\nu) - 1$.

We show that the first situation arises for at least one of the simple degenerations of $\nu$.

**Theorem 5.1.** Suppose that $\nu \in \mathcal{M}$ is a rigid tree honeycomb such that $\omega(\nu) \geq 3$. Then there exist rigid tree honeycombs $\nu_1, \nu_2 \in \mathcal{M}$ satisfying $\Sigma_{\nu_1}(\nu_2) = 0$ and such that the $\nu$ has the same exit pattern as either $\nu_1 + \sigma \nu_2$ or $\sigma \nu_1 + \nu_2$, where $\sigma = \Sigma_{\nu_2}(\nu_1) \neq 0$.

**Proof.** The support of $\nu^* = c_1\rho_1 + \cdots + c_n\rho_n$ inside the triangle $\Delta_\nu$ of size $\omega(\nu)$ has at least one flat segment $e$ of unit length, dual to a root edge of $\nu$. We can select $j, k, \ell \in \{1, \ldots, n\}$ such that $e$ is also an edge of $\rho_j + \rho_k + \rho_\ell$ (we may actually
need three such honeycombs, one whose support contains \( e \) and two more whose supports contain just the endpoints of \( e \).) Observe that \( n \geq 5 \) because \( \omega(\nu) \geq 3 \), and thus we can choose \( i \in \{1, \ldots, n\}\backslash\{j, k, \ell\} \). Consider the simple degeneration \( \mu \) of \( \nu \) defined by \( \mu^* = \nu^* - c_i \rho_i \). The choice of \( i \) ensures that \( e \) is an edge of \( \mu^* \), so \( \mu \) has some edges with multiplicity 1.

If \( \text{root}(\mu) = 1 \), it follows that \( \mu \), having an edge of multiplicity 1, must itself be a rigid tree honeycomb and we are in the second situation described above, that is, \( \text{exit}(\mu) = \text{exit}(\nu) - 1 \). It follows that the nonzero exit multiplicities of \( \nu \) can be arranged in counterclockwise order \( \alpha_1, \ldots, \alpha_{n+3} \) (see Theorem 3.2) so the nonzero exit multiplicities of \( \mu \) are \( \alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_{n+3} \). We see then that

\[
\omega(\mu)^2 = -2 + (\alpha_1 + \alpha_2)^2 + \sum_{j=3}^{n+3} \alpha_j^2
\]

\[
= 2\alpha_1\alpha_2 - 2 + \sum_{j=1}^{n+3} \alpha_j^2
\]

\[
= 2\alpha_1\alpha_2 + \omega(\nu)^2,
\]

and this is impossible because \( \omega(\mu) = \omega(\nu) \) and \( \alpha_1\alpha_2 \neq 0 \). We conclude that we must be in the first situation described above, that is, \( \text{exit}(\mu) = \text{exit}(\nu) \) and \( \mu \) is a sum of two extreme honeycombs. One of these summands is a tree honeycomb because an edge of \( \mu \) has unit multiplicity, so \( \mu = \nu_1 + \sigma\nu_2 \) for some rigid tree honeycombs \( \nu_1 \) and \( \nu_2 \), and \( \sigma > 0 \). Since \( \mu \) is rigid, one of the numbers \( \Sigma_{\nu_1}(\nu_2) \) and \( \Sigma_{\nu_2}(\nu_1) \) is equal to zero. Finally, the equality \( \text{exit}(\mu) = \text{exit}(\nu) \) implies that \( \mu \) and \( \nu \) have the same exit pattern (see Remark 3.1), in particular \( \Sigma(\mu) = \Sigma(\nu) = -1 \). Thus,

\[
-1 + \sigma \Sigma_{\nu_2}(\nu_1) + \sigma \Sigma_{\nu_1}(\nu_2) - \sigma^2 = -1,
\]

so \( \sigma \) is equal to either \( \Sigma_{\nu_2}(\nu_1) \) or \( \Sigma_{\nu_1}(\nu_2) \). The theorem follows.

It may happen that all the simple degenerations of \( \nu \) yield honeycombs \( \nu_1 \) and \( \nu_2 \) as in the preceding result. For instance, this is the case for all rigid tree honeycombs \( \nu \) of weight at most 5. This however is not true for all rigid tree honeycombs. The smallest examples correspond to the patterns 2, 3, 2, 2, 3, 2, 3, 2 and 2, 3, 1, 2, 2, 2, 2, 2. They are pictured in Figure 5.1 along with their duals and one particular extreme rigid summand in the dual (drawn in red) that yields a simple degeneration unlike the ones envisaged in the proof of Theorem 5.1. One of the following situations arises in these examples:
(A) A simple degeneration $\mu$ of $\nu$ has $\text{exit}(\mu) = \text{exit}(\nu)$ and the decomposition $\mu = m_1\nu_1 + m_2\nu_2$, where $\nu_1$ and $\nu_2$ are rigid tree honeycombs, has integer coefficients $m_1, m_2 \geq 2$.

(B) A simple degeneration $\mu$ of $\nu$ has $\text{exit}(\mu) = \text{exit}(\nu) - 1$. In this case, $\mu = c\nu_0$ for some rigid tree honeycomb $\nu_0$ and some integer $c \geq 2$.

In both cases, all the edges of the dual honeycomb $\mu^*$ have length greater than 1. Later, we describe all the rigid tree honeycombs $\nu$ that have a degeneration of one of these two types (see Theorems 6.2 and 6.8). Here we find necessary conditions for such honeycombs. We start with case (A). In this situation, $\mu$ has the same exit pattern as $\nu$ and so $\Sigma_{\nu^2}(\mu) = \Sigma_{\nu}(\nu) = -1$. Suppose without loss of generality that $\Sigma_{\nu^2}(\nu) = 0$ and calculate $\Sigma_{\mu}(\mu) = -m_1^2 - m_2^2 + \sigma m_1 m_2$, where $\sigma = \Sigma_{\nu^1}(\nu)$.

Thus, the pair $(m_1, m_2)$ is a solution of the quadratic Diophantine equation

\[ m_1^2 + m_2^2 - \sigma m_1 m_2 = 1. \tag{5.3} \]

In case (B), we arrange the nonzero exit multiplicities of $\nu$ in a counterclockwise sequence $\alpha_1, \ldots, \alpha_m$ such that the nonzero exit multiplicities of $\mu$ are $\alpha_1 + \alpha_2, \alpha_3, \ldots, \alpha_m$. Note that the exit multiplicities of $\nu_0$ are $\beta_2 = (\alpha_1 + \alpha_2)/c, \beta_3 = \alpha_3/c, \ldots, \beta_m = \alpha_m/c$. We have $\Sigma_{\mu}(\mu) = \Sigma_{c\nu_0}(c\nu_0) = -c^2$ and an application of (2.7) to $\nu$ and to $\mu$ yields

\[ \sum_{j=1}^{m} \alpha_j^2 = \omega(\nu)^2 + 2, \quad (\alpha_1 + \alpha_2)^2 + \sum_{j=3}^{m} \alpha_j^2 = \omega(\mu)^2 + 2c^2. \]

Recalling that $\omega(\mu) = \omega(\nu)$, subtract the two equalities to obtain

\[ \alpha_1 \alpha_2 = c^2 - 1. \tag{5.4} \]

Now substitute $\beta_2 c - \alpha_1$ for $\alpha_2$ to obtain

\[ c^2 + \alpha_1^2 - \beta_2 \alpha_1 c = 1, \tag{5.5} \]

and similarly,

\[ c^2 + \alpha_2^2 - \beta_2 \alpha_2 c = 1. \tag{5.6} \]

These equations are of the same kind as (5.3) with $\beta_2$ in place of $\sigma$.

We discuss briefly the structure of the nonnegative integer solutions $(p, q)$ of the equation

\[ p^2 + q^2 - \sigma pq = 1 \tag{5.6} \]

for a given integer $\sigma \geq 1$. If $\sigma = 1$, the equation can be rewritten as

\[ (p - q)^2 + pq = 1, \]

and one immediately deduces that the only nonnegative integer solutions are $(0, 1)$, $(1, 0)$, and $(1, 1)$.

If $\sigma = 2$, the equation becomes $(p - q)^2 = 1$, so the nonnegative integer solutions are $(n, n + 1)$ and $(n + 1, n)$ for $n = 0, 1, \ldots$. If $\sigma \geq 3$, a simple substitution transforms (5.6) into a standard Pell equation (see [10, Chapter 8]). It is easier however to treat the equation directly. Denote by
We need to study the multiplicative group $G = \mathbb{Z}/N$ and we define the multiplicative function (sometimes called norm) $N : \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$N(p - \xi q) = (p - \xi q)(p - \xi' q) = p^2 + q^2 - \sigma pq, \quad p - \xi q \in \mathbb{Z}.$$

We need to study the multiplicative group $G = \{ z \in \mathbb{Z} : N(z) = 1 \}$. This group is generated, for instance, by $-1$ and $\xi'$. Define a sequence $\{ p_n : n = 0, 1, \ldots \}$ of nonnegative integers by setting

$$p_0 = 0, p_1 = 1, p_{n+1} = \sigma p_n - p_{n-1}, \quad n \in \mathbb{N}.$$

An induction argument shows that

$$\xi^n = p_{n+1} - p_n \xi \quad \text{and} \quad \xi^{-n} = -(p_{n-1} - p_n \xi), \quad n \in \mathbb{N}.$$

We deduce that the nonnegative integer solutions of (5.6) are precisely the pairs $(p_n, p_{n+1})$ and $(p_{n+1}, p_n)$ for $n = 0, 1, \ldots$.

We note for further use some identities that the sequence $\{ p_n \}_{n=0}^{\infty}$ satisfies.

First, consider the column vectors $v_n = \begin{bmatrix} p_{n+1} \\ p_n \end{bmatrix}$ that satisfy $v_{n+1} = \sigma v_n - v_{n-1}$ for $n > 0$. We deduce that

$$\det[v_{n+1}, v_{n+2}] = \sigma \det[v_{n+1}, v_{n+1}] - \det[v_{n+1}, v_n] = \det[v_n, v_{n+1}],$$

and thus $\det[v_n, v_{n+1}]$ does not depend on $n$. Evaluating this determinant for $n = 0$ we see that

$$p_{n+1}^2 - p_n p_{n+2} = \det[v_n, v_{n+1}] = 1, \quad n \geq 0.$$}

Equivalently,

$$p_{n+1}^2 - 1 = p_n p_{n+2}, \quad n \geq 0.$$

The inductive argument above is easily seen to yield the more general equality

$$\det[v_n, v_{n+k}] = p_k, \quad k, n \geq 0,$$

or, equivalently,

$$p_{n+1} p_{n+k} - p_n p_{n+k+1} = p_k, \quad k, n \geq 0.$$

In Section 6 we need the special case $k = 2$. Since $p_2 = \sigma$, this can be rewritten as

$$p_n p_{n+1} - \sigma = p_{n-1} p_{n+2}, \quad n \geq 1.$$

These identities show, for instance, that $p_n$ and $p_{n+1}$ are relatively prime and that the greatest common divisor of $p_n$ and $p_{n+2}$ is $\sigma$ if $n$ is even.

The first few terms of the sequence $p_n$ are

$$0, 1, \sigma, \sigma^2 - 1, \sigma^3 - 2\sigma, \sigma^4 - 3\sigma^2 + 1, \sigma^5 - 4\sigma^3 + 3\sigma,$$

and a closed formula for these numbers is

$$p_n = \frac{\xi^n - \xi'^n}{\xi - \xi'}, \quad n \geq 0.$$
Remark 5.2. If \( \sigma = 2 \) and \( p_n \) is defined inductively by (5.7), then \( p_n = n \). Thus, even in this case, the nonnegative solutions of (5.6) are the pairs \((p_n, p_{n+1})\) and \((p_n, p_n)\). If \( \sigma = 1 \), then the sequence \( p_n \) defined inductively by (5.7) is periodic and the nonnegative solutions of (5.6) are the pairs \((p_n, p_{n+1})\) with \( n = 0, 1, 2 \).

Returning now to the discussion of the degenerations of a rigid tree honeycomb, we have the following result.

**Proposition 5.3.** Let \( \nu \) be a rigid tree honeycomb and let \( \mu \) be a simple degeneration of \( \nu \). Then one of the following cases occurs:

1. There exist rigid tree honeycombs \( \nu_1 \) and \( \nu_2 \) such that \( \Sigma_{\nu_1}(\nu_2) = 1 \), \( \Sigma_{\nu_2}(\nu_1) = 0 \), and \( \mu = \nu_1 + \nu_2 \).
2. There exist rigid tree honeycombs \( \nu_1 \) and \( \nu_2 \) such that \( \Sigma_{\nu_2}(\nu_1) = 0 \), \( \Sigma_{\nu_1}(\nu_2) = \sigma \) for some \( \sigma > 1 \), and \( \mu = m_1\nu_1 + m_2\nu_2 \), where \( m_1 \) and \( m_2 \) are consecutive terms in the sequence \( \{p_n\}_{n=1}^{\infty} \) defined by (5.7).
3. There exists a rigid tree honeycomb \( \nu_0 \) that has an exit multiplicity \( \sigma > 1 \) and there exist three consecutive terms \( p_n, p_{n+1}, p_{n+2} \) in the sequence \( \{p_n\}_{n=1}^{\infty} \) defined by (5.7), such that the exit multiplicities \( \mu \) are obtained by replacing each exit multiplicity \( \beta \) of \( \nu_0 \) by \( p_{n+1}\beta \) except for the multiplicity \( \sigma \) which is replaced by two consecutive exit multiplicities equal to \( p_n \) and \( p_{n+2} \), respectively.

**Proof.** Parts (1) and (2) follow immediately from the above discussion of the equation (5.6). For (3), set \( \sigma = \beta_2 \) in equations (5.4) and (5.5) to see that \( c, \alpha_1 \) and \( c, \alpha_2 \) must be consecutive pairs of elements in the sequence \( p_n \). Moreover, \( \alpha_2 = \sigma c - \alpha_1 \), showing that \( \alpha_1, c, \alpha_2 \) are three consecutive terms of this sequence (in either increasing or decreasing order).

The proof of Theorem 5.1 shows that all but at most three of the simple degenerations of a rigid tree honeycomb fall under either case (1) or case (2) above with \( \min\{m_1, m_2\} = 1 \). Case (A) above corresponds to (2) with \( \min\{m_1, m_2\} > 1 \) and (B) corresponds to (3). The following result shows that at most two degenerations are as in (B).

**Theorem 5.4.** Suppose that \( \nu \) is a rigid tree honeycomb. Then:

1. If \( \nu \) has two simple degenerations that are extreme honeycombs, then \( \nu \) has no simple degeneration of the form \( m_1\nu_1 + m_2\nu_2 \) with \( \nu_1, \nu_2 \) rigid tree honeycombs and \( m_1, m_2 \geq 2 \).
2. At most two of the simple degenerations of \( \nu \) are extreme honeycombs.
3. If two of the simple degenerations of \( \nu \) are extreme honeycombs, then there exists a rigid tree honeycomb \( \nu' \), with exit multiplicities \( \delta_1, \ldots, \delta_k \), listed in counterclockwise order, such that:
   (a) \( \delta_1 > 1 \).
   (b) If the sequence \( \{p_n\}_{n=0}^{\infty} \) is defined by (5.7) with \( \sigma = \delta_1 \), then there exists \( n > 1 \) such that \( \omega(\mu) = p_n p_{n+1} \omega(\nu') \), and the exit multiplicities of \( \mu \), arranged in counterclockwise order, are either
   \[ p_n^2 - 1, 1, p_{n+1}^2 - 1, p_n p_{n+1} \delta_2, \ldots, p_n p_{n+1} \delta_k, \]
   or
   \[ p_{n+1}^2 - 1, 1, p_n^2 - 1, p_n p_{n+1} \delta_2, \ldots, p_n p_{n+1} \delta_k, \]
   where the first three exit multiplicities correspond to parallel rays.
Proof. Part (1) follows immediately from part (3) because the nonzero exit multiplicities of \( m_1 \nu_1 + m_2 \nu_2 \), and hence those of \( \nu \), are at least 2. As observed already, the exit multiplicities of a simple degeneration that is an extreme honeycomb are obtained simply by replacing two neighboring exit multiplicities of \( \nu \) by their sum and leaving the others multiplicities unchanged.

If we write \( \nu^* = \sum_{j=1}^n c_j \rho_j \) as above, this situation corresponds to the fact that there exists a ray \( I \) such that \( \rho_{j_0}(I) > 0 \) but \( \rho_j(I) = 0 \) for every \( j \neq j_0 \). Suppose that \( \nu \) has at least two simple degenerations of this type. Reordering the honeycombs \( \rho_j \), we may assume that these degenerations \( \mu_1 \) and \( \mu_2 \) satisfy \( \mu_1^* = \nu^* - c_1 \rho_1 \) and \( \mu_2^* = \nu^* - c_2 \rho_2 \). Consider also the degeneration \( \mu_0 \) such that \( \mu_0^* = \nu^* - c_1 \rho_1 - c_2 \rho_2 \).

Since there are two distinct rays \( I_1, I_2 \) to which \( \nu^* \) assigns positive multiplicity such \( \mu_i^* \) assigns zero multiplicity to \( \mu_i, i = 1, 2 \), we see that \( \text{exit}(\mu_i^*) \leq \text{exit}(\mu) - 2 \). By Theorem 3.2, we have \( \text{exit}(\mu) = n + 3 \) and

\[
\text{root}(\mu_0^*) + \text{root}(\mu_0) = \text{exit}(\mu_0) - 2 \leq \text{exit}(\mu) - 4 = n - 1.
\]

Since \( \text{root}(\mu_0) = n - 2 \), we see that \( \text{root}(\mu_0) \leq 1 \), and thus \( \mu_0 \) is an extreme rigid honeycomb. Thus, there are rigid tree honeycombs \( \nu_0, \nu_1, \nu_2 \) and integers \( k_0, k_1, k_2 > 1 \) such that \( \mu_1 = k_j \nu_j \) for \( j = 0, 1, 2 \). Moreover, since \( \mu_0 \) is also a simple degeneration of \( \mu_1, k_0 > k_1 \); similarly, \( k_0 > k_2 \) and, in fact, \( k_0 \) is a multiple of both \( k_1 \) and \( k_2 \). The exit pattern of \( \mu_1 \) is the same as that of \( \nu \), except that two consecutive exit multiplicities \( \alpha_1, \alpha_2 \) of \( \nu \) are replaced by \( \alpha_1 + \alpha_2 \). We next exclude the possibility that the exit pattern of \( \mu_2 \) is the same as that of \( \nu \), except that two consecutive exit multiplicities \( \alpha_3, \alpha_4 \), other than \( \alpha_1, \alpha_2 \), are replaced by \( \alpha_3 + \alpha_4 \). Suppose that this possibility does arise. In this case, the exit multiplicities of the honeycombs involved can be listed as follows.

| \( \nu \) | \( \alpha_1, \alpha_2 \) | \( \alpha_3, \alpha_4 \) | \( \alpha_5, \ldots \) |
|---|---|---|---|
| \( \mu_1 \) | \( \alpha_1 + \alpha_2 = k_1 \beta_1 \) | \( \alpha_3 = k_1 \beta_3, \alpha_4 = k_1 \beta_4 \) | \( \alpha_5 = k_1 \beta_5, \ldots \) |
| \( \mu_2 \) | \( \alpha_1 = k_2 \gamma_1, \alpha_2 = k_2 \gamma_2 \) | \( \alpha_3 + \alpha_4 = k_2 \delta_3 \) | \( \alpha_5 = k_2 \delta_5, \ldots \) |
| \( \mu_0 \) | \( \alpha_1 + \alpha_2 = k_0 \delta_1 \) | \( \alpha_3 + \alpha_4 = k_0 \delta_3 \) | \( \alpha_5 = k_0 \delta_5, \ldots \) |

Here, \( \beta_j, \gamma_j, \delta_j \) represent the nonzero exit multiplicities of \( \nu_1, \nu_2, \nu_0 \), respectively. Note that \( \beta_2, \gamma_4, \delta_2, \) and \( \delta_4 \) are not listed because \( \mu_1, \mu_2, \) and \( \mu_0 \) have fewer nonzero exit multiplicities than \( \nu \). We now apply (2.8) to the four rigid tree honeycombs involved to obtain

\[
\alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2 + \sum_{j \geq 5} \alpha_j^2 = \omega(\nu)^2 + 2,
\]

\[
(\alpha_1 + \alpha_2)^2 + \alpha_3^2 + \alpha_4^2 + \sum_{j \geq 5} \alpha_j^2 = k_1^2(\omega(\nu_1)^2 + 2) = \omega(\nu)^2 + 2k_1^2,
\]

\[
\alpha_1^2 + \alpha_2^2 + (\alpha_3 + \alpha_4)^2 + \sum_{j \geq 5} \alpha_j^2 = \omega(\nu)^2 + 2k_2^2,
\]

\[
(\alpha_1 + \alpha_2)^2 + (\alpha_3 + \alpha_4)^2 + \sum_{j \geq 5} \alpha_j^2 = \omega(\nu)^2 + 2k_0^2.
\]

Subtract now the first equality from the other three to see that

\[
\alpha_1 \alpha_2 = k_1^2 - 1, \quad \alpha_3 \alpha_4 = k_2^2 - 1, \quad \alpha_1 \alpha_2 + \alpha_3 \alpha_4 = k_0^2 - 1,
\]

and thus

\[
k_0^2 + 1 = k_1^2 + k_2^2.
\]
Since \(k_0\) is a common multiple of \(k_1\) and \(k_2\), it follows that \(k_1\) and \(k_2\) are relatively prime, so \(k_0 = mk_1k_2\) for some \(m \in \mathbb{N}\). Therefore

\[
k_1^2 + k_2^2 = m^2k_1^2k_2^2 + 1 \geq k_1^2k_2^2 + 1,
\]
hence \((k_1^2 - 1)(k_2^2 - 1) \leq 0\), and this is impossible because \(k_j > 1\).

We see right away that it is not possible to have three degenerations of \(\nu\) of this type because two of them would have to involve disjoint sets of exit multiplicities. This proves (2).

Let now \(\mu_1, \mu_2, \mu_0, \nu_1, \nu_2, \nu_0\), and \(k_0, k_1, k_2\) be as above. The preceding argument shows that the exit multiplicities of these honeycombs can be arranged as follows.

| \(\nu\) | \(\alpha_1, \alpha_2, \alpha_3\) | \(\alpha_4, \ldots\) |
|-------|------------------|------------------|
| \(\mu_1\) | \(\alpha_1 + \alpha_2 = k_1\beta_1, \alpha_3 = k_1\beta_3\) | \(\alpha_4 = k_1\beta_4, \ldots\) |
| \(\mu_2\) | \(\alpha_1 = k_2\gamma_1, \alpha_2 + \alpha_3 = k_2\gamma_2\) | \(\alpha_4 = k_2\gamma_4, \ldots\) |
| \(\mu_0\) | \(\alpha_1 + \alpha_2 + \alpha_3 = k_0\delta_1\) | \(\alpha_4 = k_0\delta_4, \ldots\) |

Of course, \(\alpha_1, \alpha_2\), and \(\alpha_3\) must be consecutive exit multiplicities corresponding to parallel rays. As in the situation discussed above, the numbers \(k_1\) and \(k_2\) must be relatively prime. Indeed, a common factor \(d\) of these integers must divide \(\omega(\nu)\), \(\alpha_1, \alpha_3\), and \(\alpha_j\) for \(j \geq 4\), and therefore \(d\) also divides \(\alpha_2 = 3\omega(\nu) - \sum_{j \neq 2} \alpha_j\).

Now, the relation \(\sum_{j \geq 1} \alpha_j^2 = \omega(\nu)^2 + 2\) shows that \(d^2\) divides 2, so \(d = 1\). Thus \(k_0 = mk_1k_2\) for some \(m \in \mathbb{N}\). We apply again (2.8) to obtain

\[
a_1^2 + a_2^2 + a_3^2 + \sum_{j \geq 4} a_j^2 = \omega(\nu)^2 + 2,
\]

\[
(a_1 + a_2)^2 + a_3^2 + \sum_{j \geq 4} a_j^2 = \omega(\nu)^2 + 2k_1^2,
\]

\[
a_1^2 + (a_2 + a_3)^2 + \sum_{j \geq 4} a_j^2 = \omega(\nu)^2 + 2k_2^2,
\]

\[
(a_1 + a_2 + a_3)^2 + \sum_{j \geq 4} a_j^2 = \omega(\nu)^2 + 2k_0^2,
\]

and we subtract the first equality from the others to see that

\[
a_1a_2 = k_1^2 - 1, \quad a_2a_3 = k_2^2 - 1, \quad a_1a_2 + a_2a_3 + a_1a_3 = k_0^2 - 1.
\]

Thus

\[
a_1a_3 = m^2k_1^2k_2^2 - k_1^2 - k_2^2 + 1 \geq (k_1^2 - 1)(k_2^2 - 1),
\]

and since \(a_1 = (k_1^2 - 1)/a_2, a_3 = (k_2^2 - 1)/a_2\), these relations are only possible when \(m = a_2 = 1, a_1 = k_1^2 - 1\), and \(a_3 = k_2^2 - 1\). Now set \(\sigma = \delta_1\) and let \(\{p_n\}_{n=0}^{\infty}\) be defined by (5.7). Since \(\mu_0\) is a simple degeneration of \(\mu_1\), it follows from Proposition 5.3 that \(\beta_1, k_2 = k_0/k_1\), and \(\beta_3\) are consecutive terms of this sequence. Finally, \(\beta_1 = (a_1 + a_2)/k_1 = k_1\), and therefore \(k_1\) and \(k_2\) are consecutive terms in the sequence \(\{p_n\}_{n=0}^{\infty}\). This concludes the proof of (3). \qed

6. Regeneration

In Section 5, we found necessary conditions for a honeycomb to be one of the simple degenerations of a rigid tree honeycomb. Our purpose in this section is to show that these necessary conditions are sufficient as well. The argument requires a basic construction of rigid tree honeycombs that involves a combination of an inflation with a partial deflation.
Construction. We start with the following data:

(1) a rigid honeycomb $\nu$.

(2) a rigid honeycomb $\mu$ that is compatible with the puzzle of $\nu$. We set $\nu' = \mu_\nu$, where $\mu_\nu$ is defined in Section 4. It follows from [6, Theorem 7.14] and [7, Theorem 1.4] (see also the discussion following [3, Theorem 7.2]) that $\nu'$ is also rigid.

(3) a ray $f$ in the support of $\mu$.

We proceed in three steps.

(i) Construct the puzzle of $\mu$ and preserve the coloring of the ‘white’ pieces. Color the remainder of the pieces as follows:

(a) The pieces corresponding to branch points of $\mu$ are colored light red.
(b) The inflation of (part of) an edge is colored dark red if it is contained in the closure of a white piece of the puzzle of $\nu$ or if it crosses a dark gray parallelogram and is parallel to its white sides. Two of the sides of a dark red parallelograms are white and the other two are light red.
(c) The inflation of (part of) an edge is colored black if it is contained in the closure of a light gray piece of the puzzle of $\nu$ or if it crosses a dark gray parallelogram and is parallel to its light gray sides.

(ii) Construct the measure $\tilde{\mu}$ with support in the edges of the puzzle of $\mu$ and rooted in the incoming ray corresponding to $f$ (Theorem 4.7).

(iii) Remove all the black, dark gray, and light gray areas of the puzzle constructed in (i), translate the remaining pieces to tile the plane, and preserve the multiplicities assigned by $\tilde{\mu}$ for those (parts of) edges that are contained in the closure of a white piece or in a dark gray parallelogram and are parallel to its white sides. Add multiplicities in case several such edges are translated to the same segment. Call $\mu'$ the collection of multiplicities obtained this way.

**Proposition 6.1.** The construction above yields:

(1) The puzzle of $\nu'$, with the color gray replaced by red, and

(2) A rigid honeycomb $\mu'$ that is compatible with the puzzle of $\nu'$.

**Proof.** The white areas in the puzzle of $\nu$ are divided by the support of $\mu$ into smaller regions, and it is these regions that are the white puzzle pieces that remain after the operation described in (iii) above. Suppose that two such areas $G_1$ and $G_2$
are separated by an edge $e$ of $\mu$. Then their counterparts after operation (iii) are separated by a dark red parallelogram with two edges parallel to $e$ and two edges of length $\mu(e)$ that are $60^\circ$ clockwise from $e$. Of course, $e$ is the translate of some edge $e'$ of $\nu'$ with multiplicity $\nu(e') = \mu(e)$, and thus the dark red parallelogram is the one that normally appears in the construction of the puzzle of $\nu'$. Another possibility is that $G_1$ and $G_2$ are separated by a dark gray parallelogram in the puzzle of $\nu$. Suppose that $AB$ and $A'B'$ are the two white sides of that parallelogram, and that $I_1, \ldots, I_m$ are the edges of $\mu$ that cross the parallelogram and are parallel to its white sides. Then, in the picture produced by (iii), $G_1$ and $G_2$ are separated by a dark red parallelogram that is decomposed into $m + 2$ parallelograms with light red sides of lengths $\mu(AB), \mu(I_1), \ldots, \mu(I_m), \mu(A'B')$. The sum of these lengths is precisely $\nu'(e')$, where $e'$ is the edge of $\nu$ with inflation $ABA'B'$. Similarly, the light red pieces assemble in step (iii) to complete the puzzle of $\nu'$. This process is illustrated in Figure 6.1 for white pieces in the puzzle of $\nu$, in Figure 6.3 for light gray puzzle pieces, and in Figure 6.2 for dark gray puzzle pieces. These figures are sufficiently general (they deal with branch points of order five contained in the interior, on the boundary, or in a corner of a puzzle piece) to demonstrate that the white, dark red, and light red pieces do actually fit together to form the puzzle of $\nu'$. The multiplicities chosen for these figures are the smallest positive integers that satisfy the balance condition on the (red) edges of $\mu$. Choosing different multiplicities (possibly zero) produce essentially the same picture (possibly without some of the dark red parallelograms).

In order to verify that $\mu'$ also satisfies the balance condition we observe that $\mu'$ is obtained from $\tilde{\mu}$ by decreasing the lengths of its edges without changing their
multiplicities, except in those cases in which two edges are translated to the same segment and their multiplicities are added. In other words, $\mu'$ is a degeneration of $\tilde{\mu}$ and is therefore a rigid honeycomb. \hfill \Box

**Theorem 6.2.** Suppose that $\nu_1$ and $\nu_2$ are rigid tree honeycombs such that $\Sigma_{\nu_1}(\nu_2) = \sigma \geq 1$ and $\Sigma_{\nu_2}(\nu_1) = 0$. Let $\{p_n\}_{n=0}^{\infty}$ be the sequence defined by (5.7). Then, for every $n \geq 1$, there exists a rigid tree honeycomb $\mu_n$ such that:

1. $\mu_n$ and $p_n\nu_1 + p_{n+1}\nu_2$ have the same exit pattern, and
2. $\mu_n$ is compatible with the puzzle of $p_{n-1}\nu_1 + p_n\nu_2$.

**Proof.** We proceed by induction. The existence of $\mu_1 = \hat{\nu}_1$ is a consequence of Theorem 4.11. Suppose that $n \in \mathbb{N}$ and that $\mu_n$ has been constructed. Set $\nu = p_{n-1}\nu_1 + p_n\nu_2$ and $\nu' = (\mu_n)_{\nu} = p_n\nu_1 + p_{n+1}\nu_2$. The nonzero exit multiplicity of $\mu_n$ corresponding to a typical ray $e$ is of the form $p_n\alpha + p_{n+1}\beta$, where $\alpha$ and $\beta$ are exit multiplicities of $\nu_1$ and $\nu_2$, respectively, and $\alpha + \beta > 0$. Choose a particular ray $e_0$ of $\mu_n$ with exit multiplicity $p_n\alpha_0 + p_{n+1}\beta_0$ such that $\beta_0 > 0$. We apply the construction above to the measures $\nu, \mu = \mu_n$, and the incoming ray $f$ in the inflation of $e_0$. We obtain a honeycomb $\mu'$ compatible with the puzzle of $\nu'$. The honeycomb $\tilde{\mu}$ constructed in step (ii) has exit multiplicities $(p_n\alpha_0 + p_{n+1}\beta_0)^2 - 1$ and 1 corresponding to the outgoing and incoming rays of the puzzle of $\mu$ corresponding to $e_0$. The other nonzero exit multiplicities correspond to the remaining outgoing rays and they are of the form $(p_n\alpha_0 + p_{n+1}\beta_0)(p_n\alpha + p_{n+1}\beta)$. It follows, in particular, that

$$\mu'_\nu = (p_n\alpha_0 + p_{n+1}\beta_0)(p_n\nu_1 + p_{n+1}\nu_2).$$

Next, observe that $\mu'$ assigns unit multiplicity to the incoming ray in the inflation of $e_0$ in the puzzle of $\nu'$. Denote by $\rho$ the rigid tree honeycomb supported by the puzzle edges of $\nu'$ and rooted in the incoming ray corresponding to $e_0$. Lemma 4.3 implies that there exists a rigid honeycomb $\mu''$ such that $\mu' = \rho + \mu''$. Suppose that $f \neq f_0$ is an arbitrary ray of $\nu'$ such that $\nu_1(f) = \alpha$ and $\nu_2(f) = \beta$. Then, according to Example 4.16, the multiplicity that $\rho$ assigns to the exit ray corresponding to $e \neq e_0$ in the puzzle of $\nu'$ is $\alpha \alpha + (\beta_0 + \beta_0)\beta$. Therefore $\rho \neq \mu'$, so $\mu'' 
eq 0$ and thus $\rho_0(\mu') \geq 2$. Now, $\nu'$ is a degeneration of $\mu'$ and $\text{exit}(\mu') = \text{exit}(\nu')$, so (5.2) implies

$$2 - \text{root}(\mu') = \text{root}(\nu') = \text{root}(\mu') = \text{root}(\mu''') - \text{root}(\nu'') \geq 0.$$

We conclude that $\text{root}(\mu') = 2$, and thus $\mu''' = \gamma \mu''$ for some rigid tree measure $\mu'''$ and some $\gamma > 0$. We conclude the proof by showing that $\mu'''$ has the same exit pattern as $p_{n+1}\nu_1 + p_{n+2}\nu_2$ and thus $\mu_{n+1} = \mu'''$ satisfies the conclusion of the theorem with $n$ replaced by $n+1$. We start with a typical ray $e \neq e_0$ in the support of $\nu'$ and calculate the density that $\mu''$ assigns to the outgoing ray in the inflation of $\nu'$:

$$(p_n\alpha_0 + p_{n+1}\beta_0)(p_n\alpha + p_{n+1}\beta) - \alpha_0\alpha - (\beta_0 + \sigma\alpha_0)\beta = c\alpha + d\beta.$$
Figure 6.4. Rigid overlays where the nonzero exit multiplicities of $\nu_1$ and $\nu_2$ correspond to the same rays

\[ d = (p_n \alpha_0 + p_{n+1} \beta_0)p_{n+1} - \beta_0 - \sigma \alpha_0 \]
\[ = (p_n p_{n+1} - \sigma) \alpha_0 + (p_{n+1}^2 - 1) \beta_0 \]
\[ = p_{n-1} p_{n+2} \alpha_0 + p_n p_{n+2} \beta_0 = (p_{n-1} \alpha_0 + p_n \beta_0)p_{n+2}. \]

Similar calculations apply to the outgoing ray in the inflation of $e_0$, while the outgoing ray in that inflation is assigned multiplicity $1 - 1 = 0$ by $\mu''$. Thus, the exit pattern of $\mu''$ is the same as that of $(p_{n-1} \alpha_0 + p_n \beta_0)(p_{n+1} \nu_1 + p_{n+2} \nu_2)$, and the desired conclusion follows along with the equality $\gamma = p_{n-1} \alpha_0 + p_n \beta_0$. \[ \square \]

Remark 6.3. The proof above would be slightly simpler if one could choose $\alpha_0 = 0$ and $\beta_0 > 0$. This however is not always possible because there are rigid overlays $\nu_1 + \nu_2$ such that $\nu_1$ and $\nu_2$ assign positive multiplicity to precisely the same rays, as seen in the two examples in Figure 6.4 (the edges of $\nu_2$ outside the support of $\nu_1$ are drawn with dotted lines).

Example 6.4. We illustrate the entire process in the proof of Theorem 6.2 for the overlay pictured in Figure 6.5. In this example, $\Sigma_{\nu_1}(\nu_2) = 1$, and the red lines in the second part of the picture represent the support of the honeycomb $\mu_1 = \tilde{\nu}_1$, compatible with the puzzle of $\nu_2$. The inflation of $\mu_1$, colored as in the proof above
is shown in Figure 6.6. The support of $\mu'$ is colored blue and the black dot indicates the incoming ray $f$. The second part of the picture represents the support of $\mu_2$ (in black), and the parts of the support of $\rho$ not covered by the support of $\mu_2$ (in blue). Since $\sigma = 1$, the honeycomb $\mu_2$ has the exit pattern of $p_2\nu_1 + p_3\nu_2 = \nu_1$, so $\mu_2$ is homologous to $\nu_1$.

**Corollary 6.5.** With the notation of the Theorem 6.2, for every $n \in \mathbb{N}$ there exists a rigid tree honeycomb $\mu'_n \in \mathcal{R}$ with the same exit pattern as $p_{n+1}\nu_1 + p_n\nu_2$.

**Proof.** Reflection in a line parallel to $w_1$ changes the roles of $\nu_1$ and $\nu_2$. Apply Theorem 6.2 to these reflected honeycombs to get a honeycomb $\mu_n$. Finally, reflect $\mu_n$ in a line parallel to $w_1$ to obtain $\mu'_n$. $\square$

**Example 6.6.** The preceding corollary, applied to the first two overlays in Figure 6.7 shows that

$$n, n, n + 1|n + 1, n, n|n + 1, n, n
\begin{array}{l}
\end{array}$$

are the exit patterns of rigid tree honeycombs. These four families were already described in [3, Section 8]. Similarly, the third overlay yields the patterns

$$n, n + 1, n, n + 1|n + 1, n + 1, n + 1, n|2n + 2, 2n + 1
\begin{array}{l}
\end{array}$$

are the exit patterns of rigid tree honeycombs, where the sequence $\{p_n\}_{n=0}^\infty = \{0, 3, 8, 21, 55, 144, \ldots\}$ is defined by (5.7) with $\sigma = 3$. The overlays in Figure 6.8 satisfy $\Sigma\nu_1(\nu_2) = 3$. We deduce that

$$p_n, p_{n+1}, p_{n+1}, p_{n+1}|p_{n+1}, p_{n+1}, p_{n+1}, p_{n+1}|2p_{n+1}, p_{n+1}, p_{n+1}
\begin{array}{l}
\end{array}$$

are the exit patterns of rigid tree honeycombs, where the sequence $\{p_n\}_{n=0}^\infty = \{0, 3, 8, 21, 55, 144, \ldots\}$ is defined by (5.7) with $\sigma = 3$. 

---

**Figure 6.6.** The inflation of $\mu_1$, the support of $\mu'' = \tilde{\mu}_1$. Also, the support of $\mu''$ (black) and the edges of $\rho$ not contained in the support of $\mu' = \mu_2$.
Figure 6.7. Three rigid overlays with $\Sigma_{\nu_1}(\nu_2) = 2$

Figure 6.8. Two rigid overlays with $\Sigma_{\nu_1}(\nu_2) = 3$

Figure 6.9. $\Sigma_{\nu_j}(\nu_i) = 0$ for $i < j$ and $\Sigma_{\nu_j}(\nu_i) = 1$ for $i > j$

Remark 6.7. The preceding two results (combined with Theorem 4.11 for $\sigma = 1$) could be paraphrased as follows. Suppose that $\nu$ is a rigid honeycomb with $\text{root}(\nu) = 2$. If $\Sigma_\nu(\nu) = -1$, then there exists a rigid tree honeycomb $\mu$ that has the same exit pattern as $\nu$. One may wonder whether one can remove the restriction on $\text{root}(\nu)$. The answer is negative. If $\nu_1, \nu_2,$ and $\nu_3$ are three rigid honeycombs of weight 1 forming a clockwise overlay (see Figure 6.9), then $\nu = c_1\nu_1 + c_2\nu_2 + c_3\nu_3$, $c_1, c_2, c_3 \in \mathbb{N}$, satisfies $\Sigma_\nu(\nu) = -1$ precisely when:

$$c_1^2 + c_2^2 + c_3^2 - c_1c_2 - c_2c_3 - c_1c_3 = 1.$$  

Rewriting this equation as

$$(c_1 - c_2)^2 + (c_2 - c_3)^2 + (c_1 - c_3)^2 = 2,$$

we see that, for each positive solution $(c_1, c_2, c_3)$, two of the $c_j$ must be equal and differ from the third by 1. Among the resulting solutions, it seen that $n\nu_1 + (n + 1)$...
1) $\nu_2 + n\nu_3$ and $(n + 1)\nu_1 + n\nu_2 + (n + 1)\nu_3$ are not the exit patterns of rigid tree honeycombs for any $n \in \mathbb{N}$. The other solutions are the exit patterns of rigid tree honeycombs, as seen from Example 6.10 below.

**Theorem 6.8.** Suppose that $\mu$ is a rigid tree honeycomb and that its nonzero exit multiplicities, listed in counterclockwise order as $\alpha_1, \ldots, \alpha_k$ are such that $\nu_1 = \sigma > 1$. Let $\{p_n\}_{n=0}^{\infty}$ be the sequence defined by (5.7). Then, for every $n \geq 2$, there exists a rigid tree honeycomb $\mu_n$ such that:

1. The exit multiplicities of $\mu_n$, listed in counterclockwise order, are
   \[p_{n+1}, p_{n-1}, p_n^{\alpha_2}, \ldots, p_n^{\alpha_k}.\]
   In other words, $\mu_n$ has the same exit pattern as $p_n\mu$, except that $p_n\alpha_1$ is replaced by $p_{n+1}$ and $p_{n-1}$.

2. $\mu_n$ is compatible with the puzzle of $p_{n-1}\mu$.

**Proof.** Let $e_0$ be the ray in the support of $\mu$ corresponding to the multiplicity $\alpha_0$. We prove the existence of $\mu_n$ satisfying the following additional property: the rays in the support of $\mu_n$ are:

(i) the outgoing rays in the puzzle of $p_{n-1}\mu$, and

(ii) one other ray contained in inflation of $e_0$.

The existence of $\mu_2$ follows from Theorem 4.2. In this case, the additional ray in (ii) is the incoming ray in the inflation of $e_0$. Suppose that $\mu_n$ has been constructed for some $n \geq 2$. We apply the basic construction with $p_{n-1}\mu$ in place of $\nu$ and $\mu_n$ in place of $\mu$ and with the outgoing ray in the inflation of $e_0$ in place of $f$. The important observation is that (using notation from part (iii) of the construction, is that $\mu$ has consecutive rays with multiplicities $p_{n+1}^2 - 1, 1$, and $p_{n+1}p_{n-1}$, and that the rays with densities 1 and $p_{n-1}p_{n+1}$ are only separated by a dark gray strip (a translated part of the inflation of $e_0$). It follows that these three exit multiplicities collapse to two exit multiplicities $p_{n+1}^2 - 1$ and $p_{n-1}p_{n+1} + 1$ of $\mu'$. Since

\[
p_{n+1}^2 - 1 = p_n p_{n+2},
\]

\[
p_{n-1}p_{n+1} + 1 = p_n p_n,
\]

it suffices to show that $\mu'$ is extreme, in which case $\mu_{n+1} = \mu'/p_n$ satisfies the conclusion of the theorem with $n + 1$ in place of $n$. To see this, we note that $\nu' = p_n\mu$ and $\mu'_0 = p_n p_{n+1} \mu$, so $\text{root}(\mu'_0) = 1$ and $\text{exit}(\mu'_0) = k$. An application of (5.2) shows that

\[
1 - \text{root}(\mu') = \text{root}(\mu'_0) - \text{root}(\mu')
= [\text{root}(\mu'^*') - \text{root}(\mu'^*_0)] - [\text{exit}(\mu') - \text{exit}(\mu'_0)]
= [\text{root}(\mu'^*) - \text{root}(\mu'^*_0)] - 1.
\]

Since $\text{root}(\mu') \geq 1$ and $\text{root}(\mu'^*) - \text{root}(\mu'^*_0) > 0$, we conclude that $\text{root}(\mu') = 1$, as desired. □

A reflection argument, like the one used in the proof of Corollary 6.2 immediately yields the following result.

**Corollary 6.9.** Under the hypothesis of Theorem 6.8, for every $n \geq 2$ there exists a rigid tree honeycomb $\mu'_n$ such that the exit multiplicities of $\mu'_n$, listed in counterclockwise order, are

\[p_{n-1}, p_{n+1}, p_n^{\alpha_2}, \ldots, p_n^{\alpha_k}.\]
Example 6.10. An application of Theorem 6.8 and Corollary 6.9 to the rigid tree honeycomb with exit pattern $1, 1, 1|2, 1|1, 1, 1$ shows that

$$n, n|n + 1, n - 1, n|n, n$$

$$n, n|n - 1, n + 1, n|n, n$$

are the exit patterns of rigid tree honeycombs. These examples were first described in [3, Section 8]. Another rigid tree honeycomb has exit pattern $2, 2, 1|1, 3, 1|1, 1, 2, 1$. If we apply the results above with the first $2$ in the role of $\alpha_1$, we deduce that

$$n - 1, n + 1, 2n|n, 3n|n, n, 2n, n$$

$$n + 1, n - 1, 2n|n, 3n|n, n, 2n, n$$

are the exit patterns of rigid tree measures. The tree honeycombs in the first series are represented in Figure 6.10 for $n = 1, 2, 3$. It is easy to see how further honeycombs in this series are constructed. The results can also be applied with $3$ in place of $\alpha_1$ to generate the exit patterns

$$2p_n, 2p_n|p_n, p_{n+1}|p_{n-1}, p_n|p_n, 2p_n, p_n, 2p_n, p_n,$$

$$2p_n, 2p_n|p_n, p_{n+1}|p_{n-1}, p_n|p_n, 2p_n, p_n, 2p_n, p_n,$$

where the numbers $p_n$ come from the sequence $0, 3, 8, 21, \ldots$.

The following result is extracted from the proof of Theorem 6.8. The second part follows from the first using reflection.

**Corollary 6.11.** Under the hypothesis of Theorem 6.8, for every $n \geq 2$ there exists a rigid tree honeycomb $\tau_n$ such that the exit multiplicities of $\tau_n$, listed in counterclockwise order, are

$$p_{n+1}^2 - 1, 1, p_n^2 - 1, p_n p_{n+1} \alpha_2, \ldots, p_n p_{n+1} \alpha_k.$$  

Similarly, there exists a rigid tree measure $\tau'_n$ such that the exit multiplicities of $\tau'_n$, listed in counterclockwise order, are

$$p_n^2 - 1, 1, p_{n+1}^2 - 1, p_n \alpha_2, \ldots, p_n \alpha_k.$$  

The following result shows that all the possibilities described in Theorem 5.4 actually arise.

**Proposition 6.12.** Each of the honeycombs $\tau_n$ and $\tau'_n$ of Corollary 6.11 has two simple degenerations that are extreme honeycombs.
Proof. Fix \( n \geq 2 \) and consider the honeycomb \( \tau_n \). We know from the above results that there exist rigid tree honeycombs \( \mu_n \) and \( \mu_{n+1} \) with exit multiplicities
\[
p_{n+1}, p_{n-1}, p_n \alpha_2, \ldots, p_n \alpha_k
\]
and
\[
p_{n+2}, p_n, p_{n+1} \alpha_2, \ldots, p_{n+1} \alpha_k,
\]
respectively. Thus the extreme rigid honeycombs \( p_{n+1} \mu_n \) and \( p_n \mu_{n+1} \) have exit multiplicities
\[
p_{n+2}^2, p_{n+1} p_{n-1}, p_{n+1} p_n \alpha_2, \ldots, p_{n+1} p_n \alpha_k
\]
and
\[
p_n p_{n+2}, p_n^2, p_{n+1} p_n \alpha_2, \ldots, p_{n+1} p_n \alpha_k,
\]
respectively. Since
\[
p_{n+1}^2 = p_{n+1}^2 - 1 + 1, p_{n+1} p_{n-1} = p_{n+1}^2 - 1, p_n p_{n+2} = p_{n+1}^2 - 1,\]
and
\[
p_n^2 = p_n^2 - 1 + 1,\]
we see immediately that \( p_{n+1} \mu_n \) and \( p_n \mu_{n+1} \) are simple degenerations of \( \tau_n \). The case of \( \tau'_n \) is treated similarly. \( \square \)

Example 6.13. The smallest illustration of Proposition 6.12 arises from the rigid tree honeycomb \( \mu \) with exit pattern 1, 1, 1|2, 1|1, 1, 1. For \( n = 2 \), we obtain the rigid tree honeycomb with exit pattern 6, 6, 6|3, 1, 8, 6|6, 6, 6. Figure 6.11 shows the support of \( \tau_2^* \) along with the supports of the two extreme summands of \( \tau_2^* \) whose elimination yields the two extreme simple degenerations of \( \tau_2 \).

Remark 6.14. The simple degenerations \( \nu \) of a rigid tree honeycomb fall into three categories:

(G) \( \sigma \nu_1 + \nu_2 \), where \( \nu_1 \) and \( \nu_2 \) are distinct tree honeycombs and \( \sigma \) is a positive integer. According to Theorem 5.1, these simple degenerations are generic: at most three simple degenerations are not of this kind.

(A) \( c_1 \nu_1 + c_2 \nu_2 \), where \( \nu_1 \) and \( \nu_2 \) are distinct tree honeycombs and \( c_1, c_2 \geq 2 \) are integers.

(B) \( c \nu \), where \( \nu \) is a rigid tree measure and \( c \) is a positive integer.

Thus, in addition to generic simple degenerations, a rigid tree measure might have

\textbf{Figure 6.11.} A rigid tree honeycomb with exit pattern 6, 6, 6|3, 1, 8, 6|6, 6, 6 and two extreme summands of its dual that yield extreme simple degenerations.
(1) no other degenerations,
(2) one degeneration of type (B),
(3) one degeneration of type (A),
(4) two degenerations of type (B),
(5) two degenerations of type (A),
(6) one degeneration of type (B) and one of type (A),
(7) two degenerations of type (B) and one of type (A),
(8) two degenerations of type (A) and one of type (B),
(9) three degenerations of type (B),
(10) three degenerations of type (A).

We have seen examples in the first four categories, and Theorem 5.4 shows that (7) and (9) are impossible. Extensive experimentation has not produced any rigid tree honeycombs in categories (5–10) and we conjecture that no such examples exist.

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