Analysis of Submarines Motion Stability based on Homotopy Continuation Theory

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Abstract. In this paper the motion stability of submarine was analyzed by using the homotopy continuation algorithm tool MATCONT, the Lyapunov motion stability theory and the bifurcation theory. Then the numerical simulation was used to verify the analysis results and method. The paper proposed an easy and effective method for analysis of submarines motion stability.

Introduction

As one of the main combat forces of the modern Navy, Submarine plays an important role in safeguarding national marine rights and interests. However, Submarine accidents are common occurrences since the Holland which is the originator of modern submarine appeared in 1897. From 1954 to 2005, Nuclear submarine accidents reached 285 in the United States, Russia (the Soviet Union), France and Britain. More than 600 submarine sailors died of submarine sunk accidents. Therefore, it has been an important research subject of Navy that how to ensure the safety of submarine. The modern submarine has higher speed and better maneuverability than ever. The mathematical equations of submarine motion are high dimensional and nonlinear. So, submarines motion stability is no longer judged by traditional stability index of slow and approximatively linear motion. The forces acting on the submarine moving underwater are very complex and it is easy to appear nonlinear phenomenon such as bifurcation and so on. Submarines can be very dangerous due to bad motion stability.

Numerical Analysis Method for Submarine Motion Stability

The motion stability theory is an important branch of mathematics which was founded by the famous Russian mathematician Lyapunov. The Lyapunov motion stability theory was used to study the influence of a small perturbation on system movement and two solutions for these problems were proposed, namely the first approximate theory and V-function method. The key to the success of the V-function method lies in constructing suitable V-function. But there are not Common regulations for constructing V-function of nonlinear system. Furthermore, it’s difficult to judge the sign of V-function. The basic idea of the first approximate theory is that a linearized equation is obtained from the nonlinear equation by neglecting high-order nonlinear terms and then the stability of the linear approximate equation is studied. The zero solution of the original system is stable if the real part of all the eigenvalues of the linear approximation equation is less than zero. The zero solution of the original system is instable if the real part of all the eigenvalues of the linear approximation equation is greater than zero. The stability of zero solution is in a critical state if the real part of all the eigenvalues of the linear approximation equation is equal to zero, the first approximate theory is failure in this time, and this paper analyze it by using the bifurcation theory.
In addition, the maximum real part of all eigenvalues $\lambda_i (i = 1, 2, \ldots, n)$ of the linear approximation system is defined as the system stability degree (Degree of stability namely DS) \((DS \neq 0)\), i.e. \(DS = \max \{R_i(\lambda_i)\}\). According to the first Lyapunovs approximate theory, the system is stable when \(R_i(\lambda_i) < 0\), and the larger \(\left|R_i(\lambda_i)\right|\) is, the faster the decay rate of system disturbance movement is, the better the original system stability is; the system is unstable if there is an eigenvalue $\lambda_i$ whose real part \(R_i(\lambda_i) > 0\), and the larger \(\left|R_i(\lambda_i)\right|\) is, the faster the divergence rate of system disturbance is, the worse the original system stability is. Consequently, the more positive DS is, the worse the system stability is. Contrariwise, the more negative DS is, the better the system stability is.

Nonlinear characteristic analysis methods for the system mainly include analytical method and numerical method. It is difficult to obtain the analytical solution of the high dimensional nonlinear system. Recently, with the development of the computing technology and the system hardware, numerical method has developed rapidly and become a valid means which used for analysis of nonlinear system. Homotopy continuation algorithm is a valid numerical method for the nonlinear equations in the last few years. Compare to traditional numerical method such as Newton iteration, Homotopy continuation algorithm is no strict demand for initial value, for example when using traditional methods solve nonlinear problem, these methods result in that the iterative process is not convergence, the equation can’t get all solutions because of the Singularities of systems Jacobian Matrix, an complete and smoothing equilibrium solution manifold can be obtained, therefore it has a board application. Using Homotopy continuation algorithms tool MATCONT, this paper study the motion stability of submarine, the specific process is shown in Fig.1: First is to model MATCONT0 on Matlab platform, and obtain state variable by numerical analysis when submarine is stationary motion. Then the control parameter is specified as bifurcation parameter, when the state variable as the starting point, and computing Continuation Homotopy to obtain the system equilibrium solution manifold. Next judge equilibrium points types of bifurcation by bifurcation theory and search all kinds of bifurcation points. Finally, Analyzing the result by the first approximate Lyapunovs law and gain the motion stability of submarine.

![Flow Chart of Submarines Motion Stability Analysis](image)

**Figure 1. The Flow Chart of Submarines Motion Stability Analysis.**

**Analysis of Underwater Submarine Turning Motion Stability and Simulation Experiment**

Course-altering maneuver is one of the most important and basic three kinds of submarine sports, this paper study the motion stability of submarine based on Submarine Underwater Turning Motion. When submarine in underwater turning motion, its hull bears complex forces and its coupling factors...
have a great influence, submarine prone to happen Nonlinear motion phenomena such as Bifurcation and Mutation, the system motion stability goes worse.

**Motion Equations of Submarine**

Foreign scholars have done lots of studies on submarine motion equation. DTNSRDC published standard submarine motion equation based on lots of ship model tests and experiment trial voyage in 19670, which is adopted by ITTC and cited by majority of researchers because of its high degree of. People regard submarine as a rigid body in standard motion equation and study the submarine motion by the parameter of submarine barycenter, they can accurately predict the movement track of submarine motion in most cases. However, Feldman pointed that the prediction of the standard motion equation is inaccuracy when submarine in strong maneuver such as in high velocity or large helm motion statement, this method should be further improvement. This is because of the structure characteristics of slender type, for example when submarine in underwater turning motion, the submarine hull receives the complicated stress in the direction of the hull length, the coupling effect in different directions and the nonlinear phenomenon of submarine motion are apparent, the standard motion equation cannot apply to these conditions. DTNSRDC publish a correction submarine standard motion equation in 19790, this article is based on the submarine motion equation, it refer to the equation of document0 and adapted the stability condition of submarine underwater turning motion:

\[
\dot{u} = \dot{v} = \dot{w} = \dot{p} = \dot{q} = \dot{r} = \dot{\phi} = \dot{\theta} = 0; \quad \psi = \text{const}
\]  

To transform the submarine underwater stationary motion equations to the ordinary differential equations:

\[
f(x, \mu) = 0
\]

Due to limitation on space, we dont specify any equations. The state variable is \(x=[u, v, w, p, q, r, \ldots]\), bifurcation parameter is \(=[r, b, s, B, x_{GB}, y_{GB}, z_{GB}]\), is heeling angle, is trim angle, center of gravity position is \((x_G, y_G, z_G)\), center of floating position is \((x_B, y_B, z_B)\), \(x_{GB} = x_G - x_B\), \(y_{GB} = y_G - y_B\), \(z_{GB} = z_G - z_B\), other meaning of characters refer to the equation of document0.

**Stability Analysis**

In submarine motion, The main safety submarine parameters are heeling angle and trim angle changes of submarine with bifurcation parameter \(=[r, b, s, B, x_{GB}, y_{GB}, z_{GB}]\) in underwater turning motion, and then study rules of motion stability to provide evidence for submarine safety maneuver. The submarine structure parameter and hydrodynamic coefficients this paper adopted refer to the equation of documents [17], the length of submarine is 5.3m, the width of submarine is 1m, the high of submarine is 0.6m, the range of \(z_{GB}\) is between -0.339m~0.461m (approximate -55%~75% the high of submarine).

**Error! Reference source not found.** Due to limitation on space, this section just analyzes the stability of submarine motion whose speed is \(U=18\)kn, its bifurcation parameter is \(z_{GB}\) and the other one is fixed, the value of these parameter; \(r=20, b, s, B, x_{GB}, y_{GB}, z_{GB} = 0\). the starting point is \(u= 9.118 \text{ m/s}, v=1.633 \text{ m/s}, w=-0.414 \text{ m/s}, p=14.427/\text{s}, q=10.845/\text{s}, r=-36.938/\text{s}, =-16.363, =20.548\) by MATCONT numerical integration. Taking \(z_{GB}\) as the bifurcation parameter, the system equilibrium solution manifold and the bifurcation point is found as shown in Fig.3~ Fig.4. The variation of the real part \(\text{Re}()\) with the \(z_{GB}\) is shown in Fig.5:

\[
\begin{aligned}
R_x(\lambda) > 0 & \implies z_{GB} \in (-0.339m, 0) \\
R_x(\lambda) = 0 & \implies z_{GB} = 0 \\
R_x(\lambda) < 0 & \implies z_{GB} \in (0, 0.461m)
\end{aligned}
\]
According to The First Approximate Lyapunovs Law, For heeling angle $\alpha$ and trim angle $\beta$: If $z_{GB} \in (-0.339m, 0)$, Submarine is instability, the best negative instability $D_{S_{\text{min}}} = -0.273(z_{GB}=0.461m$, Submarine motion is stability, as show in Fig.5); If $z_{GB} \in (-0.339m, 0)$, Submarine is instability, the best positive instability $D_{S_{\text{max}}} = 0.162 (z_{GB}=-0.339m$, submarine motion is stability, as show in Fig.5). If $z_{GB} = 0$, It will lead to bifurcating, Submarine will produce a strong shock, and it lost stability on the basis of bifurcating theory.

This is consistent with the actual movement of the submarine, when trim angle is not zero, submarine has a couple moment (righting moment) $M$, which is formed by Gravity and buoyancy to help right submarine, its size is:

$$M_{\theta} = -m_0 g \cdot z_{GB} \cdot \sin \theta$$ (10)

In this equation, $m_0$ is submarine quality of underwater total displacement, - means the effect of righting moment to restrain submarine trim. Submarine is generally a stern trim shape or 0 when it is underwater high speed and large rudder angle rotary motion. When $z_{GB}=z_G-z_B=0$, The center of gravity of the submarine under the floating heart, at this time the righting moment $M_0$ and restrain the further increase of the rake angle of submarine, submarine motion is stability; When $z_{GB} \neq 0$, The center of gravity of the submarine above the floating heart, at this time the righting moment $M_0$ and cant restrain but to promote the further increase of the rake angle of submarine, submarine motion is instability; When $z_{GB}=0$ and $M=0$, at this time, the influence of external disturbance is obvious, the submarine motion is complex and the system is easy to be oscillation instability.

**Numerical Simulation**

In order to verify the above results, this section will use the same submarine motion equations, structural parameters and hydrodynamic coefficients to carry out the computer simulation test. The 4 and 5 order Runge-Kutta algorithm (ode45) is used to analyze the above stable points and the unstable points (including bifurcation points) as the initial value of the simulation calculation and carry out the simulation test in the Matlab software platform.

(1) Stable point: From the above section know system in the range of motion is stable, which $z_{GB}=0.2m$ submarine of the state and bifurcation parameter value as the initial value substitution test system. The results is shown in Fig.6 is shown in Fig.7 after simulation.

So heeling angle and pitch angle is oscillation and stable between -15.3 and 13.7 after a short time, submarine motion is stability which is consist with the above results of stability analysis.
(2) Hopf bifurcation point (unstable point): In the above section, system is in Hopf bifurcation. When $z_{GB} = 0$, Put the Submarine condition and bifurcation parameter values in this point as initial value into the test system, the results as shown in Fig. 8 and Fig. 9 through simulation and calculation.

So heeling angle and pitch angle is constant amplitude oscillation with time $t$, the submarine motion is instability. This is consistent with the results of the stability analysis in the above section.

(3) Instability Point: In the above section, the system in the range of $z_{GB} \in (-0.339m,0)$ movement is not stable, when $z_{GB}=-0.2m$, Put the Submarine condition and bifurcation parameter values in this point as initial value into the test system, the results as shown in Fig.10 and Fig.11 through simulation and calculation.

So, heeling angle and pitch angle has a large mutation or oscillation around $t=160s$, the hull has a large lurching and roll motion, the submarine motion is instability. This is consistent with the results of the stability analysis in the above section.

In totally, the results of the numerical simulation are very verified by the above results. It shows that it is correct and effective to analyze the motion stability of submarine by using Homotopy continuation algorithm.

Conclusions

Modern submarine has a great maneuvering performance. Because of complexity of stress and oblivious nonlinear phenomenon, its motion stability has drawn wide attention. This paper adopts Homotopy continuation method to analyze the submarine motion stability according to the Lyapunov motion stability theory and bifurcation theory. As a sample, the study of submarine underwater turning motion focused on the analysis of the heeling angle and pitch angle changing with the bifurcation parameter $z_{GB}$, then the correctness of analysis results was verified by means of numerical simulation. It demonstrates that the application of the Homotopy continuation algorithm which used
to analyze the submarine motion stability is correct and effective. This paper provides a feasible and effective research method for submarine motion stability analysis.

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