Étale monodromy and rational equivalence for 1-cycles on cubic hypersurfaces in $\mathbb{P}^5$

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Abstract. Let $k$ be an uncountable algebraically closed field of characteristic 0, and let $X$ be a smooth projective connected variety of dimension $2p$, embedded into $\mathbb{P}^m$ over $k$. Let $Y$ be a hyperplane section of $X$, and let $A^p(Y)$ and $A^{p+1}(X)$ be the groups of algebraically trivial algebraic cycles of codimension $p$ and $p+1$ modulo rational equivalence on $Y$ and $X$, respectively. Assume that, whenever $Y$ is smooth, the group $A^p(Y)$ is regularly parametrized by an abelian variety $A$ and coincides with the subgroup of degree 0 classes in the Chow group $\text{CH}^p(Y)$. We prove that the kernel of the push-forward homomorphism from $A^p(Y)$ to $A^{p+1}(X)$ is the union of a countable collection of shifts of a certain abelian subvariety $A_0$ inside $A$. For a very general hyperplane section $Y$ either $A_0 = 0$ or $A_0$ coincides with an abelian subvariety $A_1$ in $A$ whose tangent space is the group of vanishing cycles $H^{2p-1}_{\text{van}}(Y)$. Then we apply these general results to sections of a smooth cubic fourfold in $\mathbb{P}^5$.

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§ 1. Introduction

Let $X$ be a smooth projective variety over an algebraically closed field of characteristic 0. The Picard-Lefschetz theory yields that the monodromy action on the $(n-1)$st vanishing cohomology of a smooth section of the variety $X$ is irreducible. The proof of this fact in terms of étale cohomology groups was given by Katz in his second article (Exposé XVIII) in [12], and by Deligne in [7]. It is also explained in terms of singular cohomology in §7.3 of Lamotke’s paper [14].

The irreducibility of the monodromy action plays an important role in the Hodge-theoretic study of algebraic varieties over $\mathbb{C}$, and it was amply utilized in the work by Voisin (see [27]–[29], [32] and [33], for example). It also affects algebraic cycles through Hodge theory (see Ch. III, §10, in the second volume of the book [30] and Proposition 2.4 in [31]).
To explain the latter idea, let $X$ be a smooth projective complex surface, embedded into a projective space, let $Y$ be a general hyperplane section of $X$ with the Jacobian $A = \text{Jac}(Y)$, and let $A_0(X)$ be the Chow group of 0-cycles of degree 0 on $X$. Furthermore, let $A_1$ be an abelian subvariety in $A$ which corresponds, via Hodge theory, to the vanishing cohomology in $H^1(Y)$. Then the kernel of the push-forward homomorphism from $A$ to $A_0(X)$ is a countable union of shifts of a certain abelian subvariety $A_0$ inside $A_1$. The monodromy argument implies that, for a general $Y$, either $A_0 = A_1$ or $A_0 = 0$ (see Ch. III, §10, in the second volume of [30]). Clearly, this alternative for $A_0$ must play an important role in the study of 0-cycles on surfaces, especially in the context of S. Bloch’s conjecture.

The aim (and novelty) of our paper is twofold. First, we will extend the monodromy argument from 0-cycles on surfaces to algebraic cycles of dimension $p-1$ on a smooth projective variety $X$ of even dimension $2p$, embedded into a projective space such that, if $Y$ is a smooth hyperplane section of $X$, the ‘continuous’ Chow group $A^p(Y)$ is weakly representable in the sense of Bloch (see Definition 1.1 in [2] or Definition 3.3 in [4]). The main case for us will be the case of 1-cycles on a 4-fold $X$ with representable 1-cycles on its section $Y$ (for example, take a general cubic hypersurface $X$ in $\mathbb{P}^5$). Second, we will develop the monodromy argument working in terms of étale cohomology groups over an abstract field of definition. This is of high importance because the nature of the abelian variety $A_0$, appearing also in the general $(p-1)$-dimensional case, has actually nothing to do with complex-analytic arguments. Though we were not able to avoid the uncountability and 0 characteristic of the ground field in the present paper, we strongly believe that these two requirements can effectively be omitted in a more subtle arithmetic theory, in which the intrinsic nature of the abelian variety $A_0$ will be revealed.

Let us now describe the results of the paper. Let $k$ be an uncountable algebraically closed field of characteristic 0, and let $r : Y \to X$ be a codimension $e$ closed embedding of smooth projective varieties over $k$. The morphism $r$ induces a push-forward homomorphism

$$r_* : A^p(Y) \to A^{p+e}(X)$$

on the Chow groups of algebraically trivial algebraic cycles, and our aim is to study the kernel

$$K = \ker(r_*)$$

under the assumption that the group $A^p(Y)$ is weakly representable or, equivalently, it can be regularly parametrized by an abelian variety $A$ over $k$.

Fix an embedding of the variety $X$ into a projective space $\mathbb{P}^m$ such that $X$ is not contained in a hyperplane. For simplicity, and as it is sufficient for the applications we have in mind, we will also assume that the group $\text{CH}^p(Y)_{\deg=0}$, defined by the embedding of $Y$ in $\mathbb{P}^m$, coincides with the group $A^p(Y)$. Clearly, this assumption is satisfied if $p = 1$ and the Néron-Severi group is of rank 1, or if $p$ is the dimension of $Y$ (that is, we study 0-cycles). It is also satisfied when $Y$ is a Fano threefold of Picard number 1 inside a four-dimensional $X$ (the case $p = 2$), for example, when $Y$ is a smooth section of a smooth cubic fourfold in $\mathbb{P}^5$. 
Our first result is a generalization of the Mumford-Roitman countability lemma for 0-cycles (see [16] and [18]) to cycles of positive dimension (Theorem 13).

Under the assumptions above, there exists an abelian subvariety $A_0$ in $A$ and a countable set $\Xi$ of closed points on $A$, such that

$$K = \bigcup_{x \in \Xi} (x + A_0)$$

inside the abelian variety $A$.

The presentation of the group $A^p(Y)$ by the abelian variety $A$ provides a homomorphism from $H^1(A)$ to $H^{2p-1}(Y)$, in terms of $l$-adic cohomology groups. Assuming that this homomorphism is an isomorphism, which is known to be true for $p \leq 2$, and also using the Tate conjecture for abelian varieties proved by Faltings, we can construct an abelian subvariety $A_1$ in $A$ whose tangent space is the kernel of the Gysin homomorphism from $H^{2p-1}(Y)$ to $H^{2(p+e)-1}(X)$. We prove in the paper that the abelian variety $A_0$ is a subvariety in $A_1$.

Now let $S$ be an integral algebraic scheme over $k$, let $\overline{\eta}$ be the geometric generic point of $S$, and choose a $c$-open subset $U$ of $S$ such that for any closed point $P$ in $U$ there is a scheme-theoretic isomorphism between $\overline{\eta}$ and $P$ over $\mathbb{Q}$. Consider a closed embedding $\mathcal{Y} \to \mathcal{X}$ of smooth families over $S$. The scheme-theoretic isomorphisms $\overline{\eta} \simeq P$ induce scheme-theoretic isomorphisms $\kappa_P$ between $\mathcal{Y}_P$ and $\mathcal{Y}_{\overline{\eta}}$ over $\mathbb{Q}$. Assume that $A^p(\mathcal{Y}_{\overline{\eta}})$ is presented by an abelian variety $A_{\overline{\eta}}$ and $A^p(\mathcal{Y}_P)$ is presented by an abelian variety $A_P$, for every closed point $P$ in $U$. Then the isomorphisms $\kappa_P$ induce isomorphisms $\kappa_P$ between $A_P$ and $A_{\overline{\eta}}$, which are compatible with the isomorphisms on Chow groups induced by the isomorphisms $\kappa_P$. In the paper we show that $\kappa_P(A_{P,0}) = A_{\overline{\eta},0}$ and $\kappa_P(A_{P,1}) = A_{\overline{\eta},1}$ for every $k$-point $P$ in $U$. In other words, we can study the varieties $A_0$ in a family either working at the geometric generic point or at a very general closed point on the base scheme $S$. The $c$-open set $U$ is not unique, of course, and all remains the same over any of them. What happens to $A_0$ beyond the union of such $c$-open sets is a big and important question which deserves a separate research programme.

Within this paper we are interested in the case where the family in question is a family of smooth hyperplane sections, so that we can enhance the study of the abelian variety $A_0$ by the monodromy action. So, let $X$ be a smooth projective variety of even dimension $2p$ over the ground field $k$, embedded into a projective space $\mathbb{P}^m$, let $\mathcal{X} = X \times \mathbb{P}^{m \vee}$, where $\mathbb{P}^{m \vee}$ is the dual projective space, and let $\mathcal{Y}$ be the intersection of $\mathcal{X}$ with the universal hyperplane inside $\mathbb{P}^m \times \mathbb{P}^{m \vee}$. Also let $T$ be the complement to the discriminant locus inside the dual projective space, and consider the family $\mathcal{Y}_T \to T$ of smooth hyperplane sections of the variety $X$ over $k$. Clearly, $\mathcal{Y}_T$ is embedded into $\mathcal{X}_T$ over $T$. Furthermore, let $\xi$ be the generic point of $T$, let $\overline{\xi}$ be the corresponding geometric generic point, and choose a $c$-open subset $U$ in $T$ so that the $k$-points $P$ in $U$ are scheme-theoretically isomorphic to $\overline{\xi}$. Now again, making our standard assumptions for the fibres $Y_{\overline{\xi}}$ and $Y_P$ and applying the $l$-adic monodromy argument in Lefschetz pencils passing through $U$, in conjunction with Theorem 13, we obtain the following result (Theorem 23).

In the terms above we have the following alternative: either $A_{\overline{\xi},0} = 0$ or $A_{\overline{\xi},0} = A_{\overline{\xi},1}$.

Respectively, either $A_{P,0} = 0$ or $A_{P,0} = A_{P,1}$ for any closed point $P$ in $U$. 
Notice that the assumptions of Theorem 23 are satisfied, for example, for all smooth projective fourfolds $X$ whose very general hyperplane sections are Fano varieties of Picard number 1. In all such cases Theorem 23 brings new information about the rational equivalence of algebraic 1-cycles on the fourfold $X$.

Now let $\tilde{Y}_P$ be the resolution of singularities on $Y_P$, and set $\tilde{Y}_P = Y_P$ whenever the section $Y_P$ is smooth. Assume that $p \leq 2$ and the group $A^p(\tilde{Y}_P)$ is weakly representable for every section $Y_P$ having at worst one ordinary double point. The next theorem generalizes Voisin’s result in Ch. III, §10, in Volume II of [30] (Theorem 26).

If the group $A^{p+1}(X)$ is notrationally weakly representable, it follows that the kernel of the push-forward homomorphism from $A^p(Y_P)$ to $A^{p+1}(X)$ is countable, for avery general hyperplane section $Y_P$.

The practical meaning of Theorem 26 is as follows. Suppose we want to prove that $A^{p+1}(X)$ is weakly representable. Then, by Theorem 26, ‘all we need’ is to find a positive-dimensional variety in the kernel of the homomorphism from $A^p(Y)$ to $A^{p+1}(X)$, for a very general ample section $Y$ on $X$.

Our original motivation for proving Theorem 26 was to understand more about the structure of the huge Chow group $\text{CH}^3(X)$ for a general cubic hypersurface $X$ in $\mathbb{P}^5$, hence the title of the paper. Recall that $\text{CH}^3(X)$ is generated by lines (see Theorem 1.1 in [23]), and $A^3(X)$ is not weakly representable by Theorem 0.5 in [20]. Then, as a particular case of Theorem 26, we obtain the following result (Corollary 27).

Let $X$ be a smooth cubic hypersurface in $\mathbb{P}^5$, and let $Y$ be a very general hyperplane section of $X$. Then the kernel of the push-forward homomorphism from $A^2(Y)$ to $A^3(X)$ is countable.

Intuitively, this corollary tells us that one can think of $A^3(X)$ as a collection of Prymians of smooth hyperplane sections modulo countable kernels generated by 1-cycles rationally equivalent to 0 on $X$. We expect that these countable kernels are of a deep arithmetical nature, relevant to the famous non-rationality conjecture.

§ 2. The setting of the problem and standard assumptions

The purpose of § 2 is to set up the main problem and justify three basic assumptions which we will keep throughout this paper.

Let $k$ be an algebraically closed field of characteristic 0, and let $Y$ be an algebraic variety over $k$. Denote the Chow group of codimension $p$ algebraic cycles modulo rational equivalence on $Y$ by $\text{CH}^p(Y)$, and let $A^p(Y)$ be the subgroup in $\text{CH}^p(Y)$ generated by algebraically trivial algebraic cycles on $Y$. Furthermore, let $V$ be another algebraic variety over $k$, and let $Z = \sum m_i Z_i$ be an algebraic cycle on the product $Y \times V$ such that the composition $g_i: Z_i \to V$ of the closed embedding $Z_i \to Y \times V$ and the projection $Y \times V \to V$ is surjective for each $i$. If $P$ is a closed point on $V$, the scheme-theoretic preimage $g_i^{-1}(P)$ is also a closed subscheme in the fibre $Y \times \{P\} = Y$. Let $Z_i(P)$ be the corresponding fundamental cycle, and define $Z(P)$ to be the cycle $\sum m_i Z_i(P)$ on $Y$. If the relative dimension of $Z$ over $V$ is $n$, then $Z(P)$ is acodimension $p = \dim(Y) - n$ algebraic cycle on the variety $Y$, for each closed point $P$ on $V$. Fix a closed point $P_0$ on $V$. The cycle

$$Z(P) - Z(P_0)$$
is algebraically trivial, and we obtain a map

\[ V(k) \to A^p(Y), \]

which sends \( P \) to the class of the cycle \( Z(P) - Z(P_0) \) on \( Y \). This map can be considered as a family of codimension \( p \) algebraically trivial cycle classes on \( Y \), induced by the cycle \( Z \) on \( Y \times V \) and the fixed point \( P_0 \) on \( V \).

In terms of relative cycles (see [13] or [24]) the same can be expressed by saying that a relative cycle \( Z \) of relative dimension \( n \) on \( Y \times V \) over \( V \), and a closed point \( P_0 \) on \( V \), induce a family of algebraically trivial cycles of codimension \( p + e - n \) on \( Y \).

The next definition appeared in Murre’s paper [17], and is important for what follows.

**Definition 1.** Let \( A \) be an abelian variety over \( k \). A group homomorphism \( A^p(Y) \to A(k) \) is said to be **regular** if its pre-composition with any family of algebraic cycles \( V(k) \to A^p(Y) \), in the above sense, is a regular morphism of varieties over \( k \). A regular homomorphism \( \psi: A^p(Y) \to A(k) \) to an abelian variety \( A \) over \( k \) is said to be **universal** if, having another regular homomorphism \( A^p(Y) \to B(k) \), there exists a unique homomorphism of abelian varieties \( A \to B \) such that the obvious diagram commutes, see [17], p. 981. Clearly, if \( \psi \) exists, then it is an epimorphism in the category of abelian groups.

Let \( r: Y \to X \)

be a closed embedding of smooth projective connected varieties over \( k \) of codimension \( e \), let

\[ r_*: A^p(Y) \to A^{p+e}(X) \]

be the push-forward homomorphism induced by the proper morphism \( r \), and let \( K \) be the kernel of \( r_* \). Of course, it is difficult to study \( K \) in general, and therefore we need to impose some reasonable assumptions on the Chow group \( A^p(Y) \).

**Assumption 1.** Our first assumption is that the universal regular epimorphism

\[ \psi: A^p(Y) \to A(k) \]

exists, and that the group \( A^p(Y) \) is weakly representable in the sense of Bloch, in which case the universal regular homomorphism \( \psi \) is an isomorphism of abelian groups, so that we can identify \( A^p(Y) \) and \( A(k) \) by means of \( \psi \); see [2] or [4].

**Remark 2.** Clearly, the universal homomorphism \( \psi \) exists and is an isomorphism if \( p = 1 \). The main result in [17] asserts that \( \psi \) exists in case \( p = 2 \). Therefore, Assumption 1 is satisfied whenever \( p = 2 \) and \( A^2(Y) \) is weakly representable in the sense of Bloch, see [2] and [4]. Notice that if the group \( A^3(Y) \) of 0-cycles on \( Y \) is weakly representable, then so is the group \( A^2(Y) \) (see Lemma 3.1 in [11]). Therefore, Assumption 1 is satisfied whenever \( A^3(Y) \) is representable. In particular, it is satisfied when \( Y \) is rationally connected. This is so, for example, if \( Y \) is a Fano threefold inside a smooth projective variety \( X \) over \( k \).
Assumption 2. Fix an embedding of the variety $X$ into a projective space $\mathbb{P}^m$. Since $Y$ is a subvariety in $X$, it induces an embedding of $Y$ into the same space $\mathbb{P}^m$. Clearly, $A^p(Y) \subset CH^p(Y)_{\text{deg}=0}$. We shall assume that

$$A^p(Y) = CH^p(Y)_{\text{deg}=0}.$$ 

Remark 3. If $p = 1$, this assumption is obviously satisfied. If $p = 2$, Assumption 2 is satisfied, for example, for all Fano threefolds $Y$ inside a smooth projective fourfold $X$ over $k$, such that the Picard number of $Y$ is equal to 1. In particular, Assumption 2 is satisfied for smooth sections $Y$ of a smooth cubic fourfold in $\mathbb{P}^5$.

Assumption 3. Since $\psi$ is an isomorphism, and hence the group $A^p(Y)$ is weakly representable, there exists a smooth projective curve $\Gamma$, a cycle $Z$ of codimension $p$ on $\Gamma \times Y$, and an algebraic subgroup $G \subset J_\Gamma$ in the Jacobian variety $J_\Gamma$ of the curve $\Gamma$, such that the induced homomorphism

$$z_*: J_\Gamma = A^1(\Gamma) \to A^p(Y) \simeq A$$

is surjective, and its kernel is $G$. Here $z$ is the cycle class of $Z$ in the Chow group $CH^p(\Gamma \times Y)$. The class $z$ gives us a morphism

$$z: M(\Gamma) \otimes \mathbb{L}^{p-1} \to M(Y),$$

where $M(\cdot)$ is the functor from smooth projective varieties over $k$ to (contravariant) Chow motives over $k$, $\mathbb{L}$ is the Lefschetz motive and $\mathbb{L}^n$ is the $n$-fold tensor power of $\mathbb{L}$. Fix a point on $\Gamma$ and consider the induced embedding

$$i_\Gamma: \Gamma \to J_\Gamma.$$ 

Also let

$$\alpha: J_\Gamma \to A$$

be the projection from the Jacobian $J_\Gamma$ onto the abelian variety $A = J_\Gamma/G$. Define $w$ to be the composition

$$z \circ (M(\alpha \circ i_\Gamma) \otimes id_{\mathbb{L}^{p-1}})$$

in the category of Chow motives with coefficients in $\mathbb{Z}$. Then $w$ is a morphism from the motive $M(A) \otimes \mathbb{L}^{p-1}$ to $M(Y)$, which induces a homomorphism

$$w_*: H^1_{\text{ét}}(A, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l) \quad (1)$$

(see [11]). Our third assumption is that $w_*$ is an isomorphism of cohomology groups.

Remark 4. If $p = 1$ and $\dim(Y) = 1$, then $w_*$ is an isomorphism by the standard argument. If $p = 2$ and $\dim(Y) = 3$, by Lemma 4.3 in [11] Assumption 3 is satisfied. Conjecturally, $w_*$ is an isomorphism for any $p > 2$ as well, but we do not prove that here. The reason for that is that Lemma 4.3 in [11] uses a result due to Merkurjev and Suslin on the injectivity of Bloch’s map $\lambda^2_p$; see [15]. Though we believe that the Bloch-Kato conjecture, which is now a theorem due to Voevodsky and Rost, can help us to prove that $w_*$ is an isomorphism for an arbitrary $p$, this may well be quite a big piece of work, deserving a separate paper to work it out.

Of course, if $k = \mathbb{C}$, then the isomorphism between $H^1(A, \mathbb{C})$ and $H^{2p-1}(Y, \mathbb{C})$ can easily be achieved using Hodge-theoretic methods, and Assumption 3 is always satisfied.
Consider the Gysin homomorphism
\[ H_{\text{ét}}^{2p-1}(Y, \mathbb{Q}_l) \xrightarrow{r_*} H_{\text{ét}}^{2(p+e)-1}(X, \mathbb{Q}_l) \tag{2} \]
induced by the closed embedding \( r \) on the \( l \)-adic cohomology groups. Assumption 3 gives us the advantage that we can now describe the kernel of the homomorphism (2) in terms of an abelian subvariety in \( A \), carrying Hodge-theoretical arguments over to \( l \)-adic representations.

Indeed, let
\[ A_{l^n} = \ker(A \xrightarrow{l^n} A) \]
be the \( l^n \)-torsion subgroup in the abelian variety \( A \) over \( k \), let
\[ T_l(A) = \lim A_{l^n} \]
be the Tate module of \( A \) and let
\[ V_l(A) = T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \]

Let \( G \) be the image in the group \( H_{\text{ét}}^1(A, \mathbb{Q}_l(1-p)) \) of the kernel of the Gysin homomorphism (2) under the isomorphism \( \omega_*^{-1} \) inverse to (1). Then \( G \) induces a \( \mathbb{Q}_l \)-vector subspace in \( H_{\text{ét}}^1(A, \mathbb{Q}_l) \). But the group \( H_{\text{ét}}^1(A, \mathbb{Q}_l) \) is isomorphic to the dual vector space \( \text{Hom}_{\mathbb{Q}_l}(V_l(A), \mathbb{Q}_l) \). Since the space \( V_l(A) \) is finite-dimensional, the dual space \( \text{Hom}_{\mathbb{Q}_l}(V_l(A), \mathbb{Q}_l) \) is isomorphic to \( V_l(A) \), and therefore \( G' \) induces a vector subspace in \( V_l(A) \). The latter vector subspace determines an idempotent \( e_G \) in the associative ring \( \text{End}_{\mathbb{Q}_l}(V_l(A)) \).

Now, without loss of generality we can temporarily assume that \( k \) is the algebraic closure of a field which is finitely generated over \( \mathbb{Q} \). In such a case, the Tate conjecture for abelian varieties, proved by Faltings, tells us that the canonical \( l \)-adic representation
\[ \text{End}(A) \otimes \mathbb{Q}_l \rightarrow \text{End}(V_l(A)) \]
is an isomorphism; see the article [9] (or p. 72 in [25], or p. 74 in [21]). Therefore, the idempotent \( e_G \) induces an idempotent \( e_{G'} \) in the associative ring \( \text{End}(A) \otimes \mathbb{Q}_l \). This idempotent determines a unique, up to an isogeny, abelian subvariety
\[ A_1 \subset A \]
in the abelian variety \( A \), such that the image of the injective homomorphism
\[ H_{\text{ét}}^1(A_1, \mathbb{Q}_l)(1-p) \rightarrow H_{\text{ét}}^1(A, \mathbb{Q}_l)(1-p), \]
induced by the inclusion \( A_1 \subset A \), coincides with the kernel of the composition of the isomorphism \( w_* \) with the Gysin homomorphism \( r_* \) from \( H_{\text{ét}}^{2p-1}(Y, \mathbb{Q}_l) \) to \( H_{\text{ét}}^{2p+1}(X, \mathbb{Q}_l) \).

Remark 5. If \( p = 1 \), \( \dim(Y) = 1 \) and \( \dim(X) = 2 \), then \( A_1 \) is the connected component of the kernel of the induced homomorphism from the Jacobian \( A \) of the curve \( Y \) to the Albanese variety \( \text{Alb}(X) \) of the surface \( X \).
Fixing an isomorphism between a hyperplane section of an embedded into a projective space, the dimension of $X$ is $2p$ and $Y$ is a smooth hypersurface in $\mathbb{P}^{2p+1}$. If the group $H^p_{\text{et}}(X, \mathbb{Q}_l)$ does not vanish, the abelian variety $A_1$ can be smaller than $A$.

Remark 6. In the applications below we will be dealing with the case when $X$ is embedded into a projective space, the dimension of $X$ is $2p$ and $Y$ is a smooth hypersurface in $\mathbb{P}^{2p+1}$, in which case $A_1 = A$. This is so, for example, when $X$ is a smooth hypersurface in $\mathbb{P}^{2p+1}$. If the group $H^p_{\text{et}}(X, \mathbb{Q}_l)$ does not vanish, the abelian variety $A_1$ can be smaller than $A$.

Remark 7. If, moreover, $k = \mathbb{C}$, then $A_1$ can be described Hodge-theoretically. Indeed, for any algebraic variety $V$ over $\mathbb{C}$ and any nonnegative integer $n$ the étale cohomology group $H^n_{\text{et}}(V, \mathbb{Q}_l)$ is isomorphic to the singular cohomology group $H^n(V(\mathbb{C}), \mathbb{Q}_l) = H^n(V(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_l$. The étale cohomology groups with coefficients in $\mathbb{Q}_l$ can be tensored further with the algebraic closure $\overline{\mathbb{Q}}_l$ of the $l$-adic field over $\mathbb{Q}_l$. Fixing an isomorphism between $\overline{\mathbb{Q}}_l$ and $\mathbb{C}$, the étale cohomology groups $H^n_{\text{et}}(\cdot, \overline{\mathbb{Q}}_l)$ are isomorphic to the singular cohomology groups $H^n(\cdot, \mathbb{C})$. The Gysin homomorphism $r_*$ from $H^{2p-1}(Y, \mathbb{C})$ to $H^{2p+1}(X, \mathbb{C})$ is a morphism of Hodge structures, so that its kernel $H_1$ is a Hodge substructure in $H^{2p-1}(Y, \mathbb{C})$. Suppose $p = 2$. By Remark 4 the group $H^3(Y, \mathbb{C})$ is isomorphic to $H^1(A, \mathbb{C})$ via the homomorphism $w_*$, and $w_*$ is obviously a morphism of Hodge structures too. It follows that $H_1$ is of weight 1. This gives the abelian subvariety $A_1$ in $A$, where $A = J^2(Y)_{\text{alg}}$ is the intermediate Jacobian of the threefold $Y$ (see [17]).

§ 3. The generalization of the Mumford-Roitman theorem

In this section we generalize a certain result due to Mumford and Roitman, which appeared first in [16] and then in [18], to algebraic cycles of positive dimension. To do this, we shall use the theory of relative cycles developed by Suslin and Voevodsky in [24] and, independently, by Kollár in [13].

So, let $N$ be the category of locally Noetherian schemes over $k$. Let $X$ be a scheme of finite type over $k$ and consider the presheaf of monoids $\mathcal{C}_r(X)$ on $N$, where for any scheme $S$ in $N$ the value $\mathcal{C}_r(X)(S)$ is the monoid $\mathcal{C}_r(X \times S/S)$ freely generated by relative cycles on $X \times S$ of relative dimension $r$ over $S$, in the sense of Suslin and Voevodsky, and the pullbacks are the pullbacks as constructed in § 3 in [24]. To understand why the monoids $\mathcal{C}_r(X)(S)$ are free, see Corollary 3.1.6, or Corollaries 3.4.5 and 3.4.6, in loc. cit. The presheaf $\mathcal{C}_r(X)$ is actually a sheaf in $\text{cdh}$, and hence in the Nisnevich topology on $N$ (see Theorem 4.2.9 in [24]). If $X$ is equidimensional, we will also write $\mathcal{C}^p(X)(S)$, or $\mathcal{C}^p(X \times S/S)$, for the same monoids of relative cycles of relative dimension $p$, where $p = \dim(X) - r$. If $X$ is projective over $k$, we fix a closed embedding of $X$ into a projective space $\mathbb{P}^m$ over $k$ and consider the subsheaf $\mathcal{C}^d_p(X)$ of relative cycles of degree $d$ in $\mathcal{C}^p(X)$, whose sections $\mathcal{C}^d_p(X \times S/S)$ are generated by relative cycles of relative codimension $p$ and degree $d$, where the degree is understood with regard to the closed embedding of $X$ into $\mathbb{P}^m$.

From now on we will always assume that $X$ is equidimensional and projective over $k$, and that the embedding of $X$ into $\mathbb{P}^m$ is fixed. Corollary 4.4.13 in [24] says then that the sheafification of the presheaf $\mathcal{C}^d_p(X)$ in the $h$-topology on the

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1 All monoids in this paper will be commutative monoids.
category $\mathbb{N}$ is representable by a Chow scheme $C^p_d(X)$, projective over $k$. Let

$$C^p(X) = \coprod_{d \geq 0} C^p_d(X)$$

be the total Chow scheme, where the coproduct is taken in the category $\mathbb{N}$.

Now let $\mathbb{N}_0$ be the full subcategory of seminormal schemes in $\mathbb{N}$. Since $k$ is of characteristic $0$, the above $h$-representability can be replaced by usual representability if we restrict the presheaves to the category $\mathbb{N}_0$ (see Theorem 3.21 in [13]), and notice that the Suslin-Voevodsky pullbacks of relative cycles coincide with Kollár’s ones in our case. Thus, for each seminormal $S$, there is a bijection

$$\Upsilon^p_d(X, S) : \mathcal{C}^p_d(X \times S/S) \sim \rightarrow \text{Hom}(S, C^p_d(X)),$$

which is functorial in $S$. Moreover, these bijections are also functorial in $X$ by Corollary 3.6.3 in [24].

Notice that if $d = 0$, then, by convention, $C^p_0(X) = \text{Spec}(k)$ and the unique $k$-point of $\text{Spec}(k)$ is the neutral element $0$ of the free monoid $\mathcal{C}^p(X)$. Since $0$ can also be considered as the empty codimension-$p$ relative cycle over $\text{Spec}(k)$, we in fact identify the unique $k$-point of $\text{Spec}(k)$ with the empty relative cycle.

It is trivial but worth noticing that if $k'$ is another field and $\alpha : k \sim \rightarrow k'$ is an isomorphism of fields, the scheme $C_r(X')$ is the pull-back of the scheme $C_r(X)$ with respect to the morphism $\text{Spec}(\alpha)$, where $X'$ is the pull-back of $X$, and the corresponding morphism from $C^p(X')$ to $C^p(X)$ is an isomorphism of schemes. The bijections $\Upsilon^p_d(X', S)$ and $\Upsilon^p(X, S)$ commute through the obvious isomorphisms on monoids and Hom-sets, induced by the isomorphism $\alpha$.

Next, given a commutative (additive) monoid $M$, its completion $M^+$ can be constructed as the quotient of $M \oplus M$ by the image of the diagonal embedding. Let $\tau : M \oplus M \rightarrow M^+$ be the corresponding quotient homomorphism, and let

$$\nu : M \rightarrow M^+$$

be the composition of the embedding of $M$ as one of the two direct summands and the homomorphism $\tau$. Then $\nu$ possesses the obvious universal property, and for any $(a, b)$ in $M \oplus M$ the value $\tau(a, b)$ is the difference $\nu(a) - \nu(b)$. If $M$ is a cancellation monoid then $\nu$ is injective, and we can identify $M$ with its image in $M^+$. Modulo this identification, $\tau(a, b) = a - b$.

In particular, we can consider the presheaf $\mathcal{Z}^p(X)$ of abelian groups on $\mathbb{N}$ such that for each $S$ the group of sections

$$\mathcal{Z}^p(X \times S/S) = \mathcal{C}^p(X \times S/S)^+$$

is the group completion of the monoid $\mathcal{C}^p(X \times S/S)$.

Identifying schemes with representable presheaves, we identify the presheaf $\mathcal{C}^p_d(X)$ with the Chow scheme $C^p_d(X)$. Looking at $\mathcal{C}^p(X) = \coprod_d \mathcal{C}^p_d(X)$ as a monoid
object in the category of presheaves, we can also interpret the presheaf $Z_p^p(X)$ as the group completion $C_p^p(X)^+$ of this monoid in the category of presheaves.

Let $S$ and $Y$ be two Noetherian schemes over $k$. Consider a functor $\mathcal{Hom}(S,Y)$ on $\mathbb{N}$ sending a scheme $T$ to the set $\mathcal{Hom}(S,Y)(T)$ of morphisms from $S \times T$ to $Y \times T$ over $T$. The graphs of such morphisms give us an embedding of $\mathcal{Hom}(S,Y)$ into the Hilbert functor $\mathcal{H}ilb(S \times Y)$. If $S$ and $Y$ are projective over $k$, the latter functor is representable by the projective Hilbert scheme $\mathcal{H}ilb(S \times Y)$ and $\mathcal{Hom}(S,Y)$ is representable by an open subscheme $\text{Hom}(S,Y)$ in $\mathcal{H}ilb(S \times Y)$. By the universal property of fibred products over $k$ one has the natural bijection between $\mathcal{Hom}(S,Y)(T)$ and $\text{Hom}(T \times S,Y)$. This yields the adjunction

$$\text{Hom}(T \times S,Y) \simeq \text{Hom}(T,\text{Hom}(S,Y))$$

and the corresponding regular evaluation morphism

$$e_{S,Y}: \text{Hom}(S,Y) \times S \to Y.$$ 

In this paper we are interested in the case when $S = \mathbb{P}^1$. Fixing a very ample sheaf $\mathcal{O}(1)$ on $Y$ allows us also to define an appropriate functor $\mathcal{Hom}^d(\mathbb{P}^1,Y)$ for any nonnegative integer $d$, and a quasi-projective scheme $\text{Hom}^d(\mathbb{P}^1,Y)$ parametrizing morphisms from $\mathbb{P}^1$ to $Y$ whose graphs are of degree $d$ with regard to the corresponding projective embedding of $\mathbb{P}^1 \times Y$. Then we obtain a regular evaluation morphism of quasi-projective schemes

$$e_{\mathbb{P}^1,Y}: \text{Hom}^d(\mathbb{P}^1,Y) \times \mathbb{P}^1 \to Y,$$

for each non-negative integer $d$. If $P$ is a closed point of $\mathbb{P}^1$, then we have the evaluation-at-$P$ morphism

$$e_P: \text{Hom}^d(\mathbb{P}^1,Y) \to Y,$$

sending $f$ to $f(P)$. More details about schemes of morphisms can be found in Kollár’s book [13], or in [6].

Now let $X$ be a smooth projective variety embedded into $\mathbb{P}^n$ over $k$. Let $A$ and $A'$ be two algebraic cycles of codimension $p$ on $X$. Then $A$ is rationally equivalent to $A'$ if and only if there exists an effective codimension $p$ algebraic cycle $Z$ on $X \times \mathbb{P}^1$ and an effective codimension $p$ algebraic cycle $B$ on $X$ such that

$$Z(0) = A + B \quad \text{and} \quad Z(\infty) = A' + B.$$ 

Assume that $A$ is rationally equivalent to $A'$, and let

$$f_Z = \Upsilon(Z)$$

and

$$f_{B \times \mathbb{P}^1} = \Upsilon(B \times \mathbb{P}^1)$$

be two regular morphisms from $\mathbb{P}^1$ to $\mathcal{C}^p(X)$, where

$$\Upsilon = \mathcal{U}^p(X,\mathbb{P}^1): \mathcal{C}^p(X \times \mathbb{P}^1/\mathbb{P}^1) \to \text{Hom}(\mathbb{P}^1,\mathcal{C}^p(X))$$
be the evaluation morphism sending. Also let

\[ f = f_Z \oplus f_{B \times \mathbb{P}^1} : \mathbb{P}^1 \to C^p(X) \oplus C^p(X) \]

be the morphism generated by \( f_Z \) and \( f_{B \times \mathbb{P}^1} \). Since \( C^p(X) \) is a cancellation monoid, for any two elements \( a, b \in C^p(X) \) the value \( \tau(a, b) \) in \( C^p(X)^+ \) is \( a - b \), after the identification of \( C^p(X) \) with its image in \( C^p(X)^+ \) under the injective homomorphism \( \nu \) from \( C^p(X) \) to \( C^p(X)^+ \). Then

\[ \tau f(0) = \tau(f_Z(0), f_{B \times \mathbb{P}^1}) = f_Z(0) - f_{B \times \mathbb{P}^1}(0) = Z(0) - B = A \]

and

\[ \tau f(\infty) = \tau(f_Z(\infty), f_{B \times \mathbb{P}^1}) = f_Z(0) - f_{B \times \mathbb{P}^1}(\infty) = Z(\infty) - B = A'. \]

Conversely, suppose there is a regular morphism \( f = f_1 \oplus f_2 \) from \( \mathbb{P}^1 \) to the direct sum \( C^p(X) \oplus C^p(X) \) such that \( \tau f(0) = A \) and \( \tau f(\infty) = A' \). Let \( Z_1 \) and \( Z_2 \) be two algebraic cycles in \( C^p(X \times \mathbb{P}^1/\mathbb{P}^1) \) such that \( \Upsilon(Z_i) = f_i \) for \( i = 1, 2 \), and let \( Z = Z_1 - Z_2 \). Then \( Z(0) = A \) and \( Z(\infty) = A' \), which means that \( A \) is rationally equivalent to \( A' \).

For any nonnegative integers \( d_1, \ldots, d_s \) let

\[ C^p_{d_1, \ldots, d_s}(X) = C^p_{d_1}(X) \times \cdots \times C^p_{d_s}(X) \]

be the fibred product over the ground field \( k \). For any degree \( d \geq 0 \) let

\[ W_d = \{(A, B) \in C^p_{d,d}(X) | A \sim B \} \]

be the Zariski closed subset in \( C^p_{d,d}(X) \) determined by ordered pairs \((A, B)\) of closed points in \( C^p_d(X) \) such that the cycle \( A \) is rationally equivalent to the cycle \( B \) on \( X \). For any nonnegative \( u \) and positive \( v \) also let

\[ W_d^{u,v} = \{(A, B) \in C^p_{d,d}(X) | \exists f \in \text{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) : \tau f(0) = A, \tau f(\infty) = B \}. \]

Then

\[ W_d^{u,v} \subset W_d \quad \text{and} \quad W_d = \bigcup_{u,v} W_d^{u,v}. \]

Also, let \( \overline{W}_d^{u,v} \) be the Zariski closure of the set \( W_d^{u,v} \) in the scheme \( C^p_{d,d}(X) \).

**Proposition 8.** For any \( d, u \) and \( v \), the set \( W_d^{u,v} \) is a quasi-projective subscheme in \( C^p_{d,d}(X) \) whose Zariski closure \( \overline{W}_d^{u,v} \) is contained in \( W_d \).

**Proof.** To prove the proposition all we need is to extend the arguments in [18] from zero-cycles and symmetric powers to codimension \( p \) cycles and Chow schemes.

Let \( e: \text{Hom}^v(\mathbb{P}^1, C^p_{d+u,u}(X)) \to C^p_{d+u,u,d+u,u}(X) \) be the evaluation morphism sending \( f \) to the ordered pair \((f(0), f(\infty))\), and let

\[ s: C^p_{d,u,d,u}(X) \to C^p_{d+u,u,d+u,u}(X) \]
be the regular morphism given by the formula
\[ s(A, C, B, D) = (A + C, C, B + D, D). \]

The two morphisms \( e \) and \( s \) allow us to take the fibred product
\[ V = \text{Hom}^v(\mathbb{P}^1, C_{d+u,u}^p(X)) \times_{C_{d+u,u,d+u,u}^p(X)} C_{d,u,d,u}^p(X), \]
which is a closed subvariety in the product
\[ \text{Hom}^v(\mathbb{P}^1, C_{d+u,u}^p(X)) \times C_{d,u,d,u}^p(X) \]
over \( \text{Spec}(k) \) consisting of quintuples \((f, A, C, B, D)\) with \( e(f) = s(A, C, B, D) \), that is,
\[ (f(0), f(\infty)) = (A + C, C, B + D, D). \]

The latter equality gives us that \( \text{pr}_{2,3}(V) \subset W_{d,u,v}^u \),
where \( \text{pr}_{2,3} \) is the projection of \( \text{Hom}^v(\mathbb{P}^1, C_{d+u,u}^p(X)) \times C_{d,u,d,u}^p(X) \) onto \( C_{d,d,u}^p(X) \).

Conversely, if \((A, B)\) is a closed point of \( W_{d,u,v}^u \), there exists a regular morphism
\[ f \in \text{Hom}^v(\mathbb{P}^1, C_{d+u,u}^p(X)) \]
with
\[ \tau f(0) = A \quad \text{and} \quad \tau f(\infty) = B. \]

Let \( f(0) = (C', C) \) and \( f(\infty) = (D', D) \). Then
\[ \tau f(0) = C' - C = A \]
and
\[ \tau f(\infty) = D' - D = B \]
in the completed monoid \( Z^p(X) = C^p(X)^+ \). This means that there exist effective codimension \( p \) algebraic cycles \( M \) and \( N \) on \( X \) such that
\[ C' + M = C + A + M \quad \text{and} \quad D' + N = D + B + N \]
in \( C^p(X) \). Since \( C^p(X) \) is a free monoid, it possesses the cancellation property. Therefore, the last two equalities imply the equalities
\[ C' = C + A \quad \text{and} \quad D' = D + B \]
respectively. This yields
\[ e(f) = s(A, C, B, D), \]
and hence
\[ (f, A, C, B, D) \in V. \]

This means that \((A, B)\) is in \( \text{pr}_{2,3}(V) \).
Thus, 
\[ \text{pr}_{2,3}(V) = W^{u,v}_d. \]
As it is the image of a quasi-projective variety under the projection \( \text{pr}_{2,3} \), the set \( W^{u,v}_d \) is itself a quasi-projective variety.

Let 
\[ \tilde{s} : C^p_{d, d, u, u}(X) \to C^p_{d+u, d+u, u}(X) \]
be the morphism obtained by composing and precomposing \( s \) with the transposition of the second and third factors in the domain and codomain of \( s \). Then
\[ W_d = \text{pr}_{1,2}(\tilde{s}^{-1}(W_{d+u} \times W_u)). \]

Let \((A, B, C, D)\) be a closed point in \( C^p_{d, d, u, u}(X) \) such that the value
\[ e_s(A, B, C, D) = (A + C, B + D, C, D) \]
is in \( W^{0,v}_d \times W^{0,v}_u \). This means that there exist two regular morphisms
\[ g \in \text{Hom}^v(\mathbb{P}^1, C^{p}_{d+u}(X)) \quad \text{and} \quad h \in \text{Hom}^v(\mathbb{P}^1, C^{p}_{u}(X)) \]
with
\[ g(0) = A + C, \quad g(\infty) = B + D \quad \text{and} \quad h(0) = C, \quad h(\infty) = D. \]

Then
\[ f = g \times h \in \text{Hom}^v(\mathbb{P}^1, C^{p}_{d+u, u}(X)), \]
\[ f(0) = (A + C, C) \quad \text{and} \quad f(\infty) = (B + D, D). \]

Hence, \( \tau f(0) = A \) and \( \tau f(\infty) = B \). It means that \( (A, B) \in W^{u,v}_d \). We have shown that
\[ \text{pr}_{1,2}(\tilde{s}^{-1}(W^{0,v}_{d+u} \times W^{0,v}_u)) \subset W^{u,v}_d. \]

Conversely, suppose \((A, B) \in W^{u,v}_d\), and let \( f \) be a morphism from \( \mathbb{P}^1 \) to \( C^{p}_{d+u, u}(X) \) with \( \tau f(0) = A \) and \( \tau f(\infty) = B \). Composing \( f \) with the projections of \( C^{p}_{d+u, u}(X) \) onto \( C^{p}_{d+u}(X) \) and \( C^{p}_{u}(X) \) we can easily show that
\[ W^{u,v}_d \subset \text{pr}_{1,2}(\tilde{s}^{-1}(W^{0,v}_{d+u} \times W^{0,v}_u)). \]

Thus,
\[ W^{u,v}_d = \text{pr}_{1,2}(\tilde{s}^{-1}(W^{0,v}_{d+u} \times W^{0,v}_u)). \]

Since \( \tilde{s} \) is continuous and \( \text{pr}_{1,2} \) is proper, we obtain
\[ \overline{W}^{u,v}_d = \text{pr}_{1,2}(\tilde{s}^{-1}(\overline{W}^{0,v}_{d+u} \times \overline{W}^{0,v}_u)). \]

This means that in order to prove the second assertion of the proposition it suffices to show that \( \overline{W}^{0,v}_d \) is contained in \( W_d \).

Let \((A, B)\) be a closed point of \( \overline{W}^{0,v}_d \). If \((A, B)\) is in \( W^{0,v}_d \), then it is also in \( W_d \). Suppose that \((A, B)\) is in \( \overline{W}^{0,v}_d \setminus W^{0,v}_d \). Let \( W \) be an irreducible component of the quasi-projective variety \( W^{0,v}_d \) whose Zariski closure \( \overline{W} \) contains the point \((A, B)\).
Let $U$ be an affine neighbourhood of $(A, B)$ in $\overline{W}$. Since $(A, B)$ is in the closure of $W$ the set $U \cap W$ is non-empty. Let $C$ be an irreducible curve passing through $(A, B)$ in $U$ and let $\overline{C}$ be the Zariski closure of $C$ in $\overline{W}$. The evaluation regular morphisms

$$e_0 : \text{Hom}^v(\mathbb{P}^1, C_d^p(X)) \to C_d^p(X) \quad \text{and} \quad e_\infty : \text{Hom}^v(\mathbb{P}^1, C_d^p(X)) \to C_d^p(X)$$

give us the regular morphism

$$e_{0, \infty} : \text{Hom}^v(\mathbb{P}^1, C_d^p(X)) \to C_{d,d}^p(X).$$

Then $W_{d,0}^v$ is exactly the image of the regular morphism $e_{0, \infty}$, and we can choose a quasi-projective curve $T$ in $\text{Hom}^v(\mathbb{P}^1, C_d^p(X))$ such that the closure of the image $e_{0, \infty}(T)$ is $\overline{C}$. Since $\text{Hom}^v(\mathbb{P}^1, C_d^p(X))$ is a quasi-projective scheme, we can embed it in some projective space $\mathbb{P}^m$. Let $\overline{T}$ be the closure of $T$ in $\mathbb{P}^m$, let $T$ be the normalization of $\overline{T}$ and let $\overline{T}_0$ be the preimage of $T$ in $\overline{T}$. Consider the composition

$$f_0 : \overline{T}_0 \times \mathbb{P}^1 \to T \times \mathbb{P}^1 \subset \text{Hom}^v(\mathbb{P}^1, C_d^p(X)) \times \mathbb{P}^1 \to C_d^p(X),$$

where $e$ is the evaluation morphism $e_{\mathbb{P}^1, C_d^p(X)}$. The regular morphism $f_0$ defines a rational map

$$f : \tilde{T} \times \mathbb{P}^1 \dashrightarrow C_d^p(X).$$

Since $\tilde{T}$ is a smooth projective curve, the product $\tilde{T} \times \mathbb{P}^1$ is a smooth projective surface over the ground field. Under this condition there exists a finite chain of $\sigma$-processes $(\tilde{T} \times \mathbb{P}^1)’ \to \tilde{T} \times \mathbb{P}^1$ resolving indeterminacy of $f$ and giving a regular morphism

$$f' : (\tilde{T} \times \mathbb{P}^1)’ \to C_d^p(X).$$

The regular morphism $\tilde{T}_0 \to T \to \overline{C}$ extends to a regular morphism $\tilde{T} \to \overline{C}$. Let $P$ be a point in the fibre of this morphism at $(A, B)$. For any closed point $Q$ on $\mathbb{P}^1$ the restriction $f|_{\tilde{T} \times \{Q\}}$ of the rational map $f$ to $\tilde{T} \times \{Q\}$ is regular on the whole curve $\tilde{T}$, because $\tilde{T}$ is smooth. Then

$$(f|_{\tilde{T} \times \{Q\}})(P) = A$$

and

$$(f|_{\tilde{T} \times \{\infty\}})(P) = B.$$

This means that the points $A$ and $B$ are connected by a finite collection of curves which are the images of rational curves on $(\tilde{T} \times \mathbb{P}^1)’$ under the regular morphism $f’$. In turn, it follows that $A$ is rationally equivalent to $B$, so that

$$(A, B) \in W_d.$$ 

Proposition 8 is proved.

In what follows, for any equi-dimensional algebraic scheme $V$ over $k$, let $\text{CH}^p(V)$ be the Chow group, with coefficients in $\mathbb{Z}$, of codimension-$p$ algebraic cycles modulo rational equivalence on $V$. If a closed embedding $V \subset \mathbb{P}^m$ is fixed, let $\text{CH}^p(V)_{\deg=0}$
be the subgroup generated by classes of cycles of degree 0 in $\text{CH}^p(V)$. Then for a nonnegative integer $d$ we have a map

$$\theta^p_d : C^p_{d,d}(X) \rightarrow \text{CH}^p(X)_{\text{deg}=0}$$

sending an ordered pair $(A, B)$ of closed points on the Chow scheme $C^p_d(X)$ to the class of the difference $Z_A - Z_B$, where $Z_A$ and $Z_B$ are codimension-$p$ algebraic cycles on $X$ corresponding to the points $A$ and $B$ respectively.

**Corollary 9.** $(\theta^p_d)^{-1}(0)$ is a countable union of irreducible Zariski closed subsets in the Chow scheme $C^p_{d,d}(X)$.

**Proof.** Proposition 8 means that $W_d$ is the countable union of Zariski closed sets $W_{d,u,v}$ over $u$ and $v$. This completes the proof.

### §4. Countability lemmas and the abelian variety $A_0$

The purpose of this section is to prove Theorem 13, given in the introduction, which generalizes the argument from §10.1.2 in the second volume of the book [30] (see also pp. 304, 305 there). Theorem 13 introduces the abelian variety $A_0$ which is of key importance to the whole approach. We shall also prove that $A_0$ is a subvariety of the abelian variety $A_1$ introduced in §2. Now and throughout the rest of the paper we shall assume that the ground field $k$ is uncountable.

**Lemma 10.** Let $V$ be an irreducible quasi-projective algebraic variety over $k$. Then $V$ cannot be written as a countable union of its Zariski closed subsets none of which is the whole of $V$.

**Proof.** Since $V$ is supposed to be irreducible, without loss of generality we may assume that $V$ is affine. Let $d$ be the dimension of $V$ and suppose $V = \bigcup_{n \in \mathbb{N}} V_n$ is the union of closed subsets $V_n$ in $V$ such that $V_n \neq V$ for each $n$. By E. Noether’s lemma, there exists a finite surjective morphism $f : V \rightarrow \mathbb{A}^d$ over $k$. Let $W_n$ be the image of $V_n$ under $f$. Since $f$ is finite, it is proper. Therefore, the $W_n$ are closed in $\mathbb{A}^d$, so that the affine space $\mathbb{A}^d$ is the union of the $W_n$. Since the ground field $k$ is uncountable, the set of all hyperplanes in $\mathbb{A}^d$ is uncountable. Therefore, there exists a hyperplane $H$ such that $W_n \not\subset H$ for any index $n$. Induction reduces the assertion of the lemma to the case when $d = 1$.

The lemma is proved.

A countable union $V = \bigcup_{n \in \mathbb{N}} V_n$ of algebraic varieties will be called **irredundant** if $V_n$ is irreducible for each $n$ and $V_m \not\subset V_n$ for $m \neq n$. In an irredundant decomposition, the sets $V_n$ will be called **c-components**\(^2\) of $V$.

**Lemma 11.** Let $V$ be a countable union of algebraic varieties over an uncountable algebraically closed ground field. Then $V$ admits an irredundant decomposition, and such an irredundant decomposition is unique.

**Proof.** Let $V = \bigcup_{n \in \mathbb{N}} V'_n$ be a countable union of algebraic varieties over $k$. For each $n$ let $V'_n = V'_{n,1} \cup \cdots \cup V'_{n,r_n}$ be the irreducible components of $V'_n$. Ignoring all components $V'_{m,i}$ with $V'_{m,i} \subset V'_{n,j}$ for some $n$ and $j$, we end up with an irredundant

\(^2\)Here, and in what follows, ‘c’ is from ‘countable’.

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Étale monodromy

be the subgroup generated by classes of cycles of degree 0 in $\text{CH}^p(V)$. Then for a nonnegative integer $d$ we have a map

$$\theta^p_d : C^p_{d,d}(X) \rightarrow \text{CH}^p(X)_{\text{deg}=0}$$

sending an ordered pair $(A, B)$ of closed points on the Chow scheme $C^p_d(X)$ to the class of the difference $Z_A - Z_B$, where $Z_A$ and $Z_B$ are codimension-$p$ algebraic cycles on $X$ corresponding to the points $A$ and $B$ respectively.

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\(^2\)Here, and in what follows, ‘c’ is from ‘countable’.
decomposition. Having two irredundant decompositions $V = \bigcup_{n \in \mathbb{N}} V_n$ and $V = \bigcup_{n \in \mathbb{N}} W_n$, suppose there exists $V_m$ such that $V_m$ is not contained in $W_n$ for any $n$. Then $V_m$ is the union of the closed subsets $V_m \cap W_n$, each of which is not $V_m$. This contradicts Lemma 10. Therefore, any $V_m$ is contained in some $W_n$. By symmetry, any $W_n$ is in $V_l$ for some $l$. Then $V_m \subset V_l$. By irredundancy, $l = m$ and $V_m = W_n$.

The lemma is proved.

**Lemma 12.** Let $A$ be an abelian variety over $k$, and let $K$ be a subgroup which can be represented as a countable union of Zariski closed subsets in $A$. Then the irredundant decomposition of $K$ contains a unique irreducible component passing through 0, and this component is an abelian subvariety in $A$.

**Proof.** Let $K = \bigcup_{n \in \mathbb{N}} K_n$ be the irredundant decomposition of $K$ that exists by Lemma 11. Since $0 \in K$, there exists at least one component in the irredundant decomposition that contains 0. Suppose there are $s$ components $K_1, \ldots, K_s$ containing 0, and that $s > 1$. Summation in $K$ gives us a regular morphism from the product $K_1 \times \cdots \times K_s$ to $A$, whose image is the irreducible Zariski closed subset $K_1 + \cdots + K_s$ in $A$. By Lemma 10, there exists $n \in \mathbb{N}$ such that $K_1 + \cdots + K_s \subset K_n$ and so $K_i \subset K_n$ for each $1 \leq i \leq n$. By irredundancy, $s = 1$, which contradicts the assumption that $s > 1$.

Renumbering the components, we may assume that $0 \in K_0$. If $K_0 = \{0\}$, then $K_0$ is an abelian variety. Suppose $K_0 \neq \{0\}$ and take a nontrivial element $x$ of $K_0$. Since $-x + K_0$ is irreducible, it must be in some $K_n$ by Lemma 10. As $0 \in -x + K_0$ it follows that $0 \in K_n$ and so $n = 0$. It follows that $-x \in K_0$. Similarly, since $K_0 + K_0$ is irreducible and contains $K_0$, we see that $K_0 + K_0 = K_0$. Being a Zariski closed abelian subgroup in $A$, the set $K_0$ is an abelian subvariety in $A$.

The lemma is proved.

Now we are ready to prove Theorem 13, which is the first result we stated in the introduction.

**Theorem 13.** In the terms above, there exists an abelian subvariety $A_0$ in $A$ and a countable subset $\Xi$ of closed points in $A$ such that

$$K = \bigcup_{x \in \Xi} (x + A_0)$$

inside the abelian variety $A$.

**Proof.** Let $n$ be the dimension of $Y$. Since $\mathcal{C}_{n-p}$ is a functor on Noetherian schemes over $k$, the closed immersion $r$ induces a morphism from $\mathcal{C}_{n-p}(Y)$ to $\mathcal{C}_{n-p}(X)$ in the category of presheaves on seminormal schemes. In upper indices, this is a morphism from $\mathcal{C}^p(Y)$ to $\mathcal{C}^{p+e}(X)$. Passing to Chow schemes we obtain a regular morphism

$$r_* : C^p(Y) \to C^{p+e}(X)$$

of projective schemes over $k$. Since $X$ is embedded into $\mathbb{P}^m$, and as $Y$ is a closed subvariety in $X$, $r_*$ induces morphisms

$$r_* : C^p_d(Y) \to C^{p+e}_d(X),$$
one for each degree $d$. Taking Assumption 2 into account we obtain the obvious commutative diagram

$$
\begin{array}{c}
\prod_d C_{d,d}^p(Y) \\
\downarrow \prod_d \theta_d^p \\
A^p(Y) \\
\end{array} \quad \begin{array}{c}
\xrightarrow{r_*} \\
\downarrow \prod_d \theta_d^{p+e} \\
\xrightarrow{r_*} \\
\end{array} \quad \begin{array}{c}
\prod_d C_{d,d}^{p+e}(X) \\
\downarrow \\
\prod_d \theta_d^{p+e} \\
\end{array} \quad \begin{array}{c}
\xrightarrow{\deg=0} \\
\downarrow \\
\end{array} \quad \begin{array}{c}
\prod_d CH^{p+e}(X) \\
\end{array}
$$

Since $A^p(Y)$ is weakly representable, there exists a smooth projective curve $\Gamma$ over $k$, and an algebraic cycle $Z$ of codimension $p$ on $\Gamma \times Y$ such that the induced homomorphism

$$Z_*: A^1(\Gamma) \to A^p(Y)$$

is surjective. On the other hand, since $A^1(\Gamma)$ is representable by the Jacobian of $\Gamma$, the map

$$\theta_d^1: C_{d,d}^1(\Gamma) \to A^1(\Gamma)$$

is surjective for big enough $d$. Using these two facts we can show that the right-hand vertical arrow in the above commutative diagram is surjective. Then the kernel $K$ of the bottom horizontal homomorphism is the image under the map $\prod \theta_d^p$ of the preimage of 0 under the composition of the maps $\prod_d r_*$ and $\prod \theta_d^{p+e}$. Corollary 9 implies that the latter preimage is the coproduct of countable unions of Zariski closed subsets in the schemes $C_{d,d}^p(Y)$.

Now consider the composition

$$\psi \circ \theta_d^p: C_{d,d}^p(Y) \to A^p(Y) \to A$$

for each number $d$. By the definition of the regularity of $\psi$, this composition is a regular morphism of schemes. Since these schemes are projective, the composition is proper. It follows that the subgroup $\psi(K)$ is a countable union of Zariski closed subsets in the abelian variety $A$.

For simplicity of notation we identify $\psi(K)$ and $K$. By Lemma 11, the set $K$ admits a unique irredundant decomposition inside the abelian variety $A$, say

$$K = \bigcup_{n \in \mathbb{N}} K_n.$$ 

Let $A_0$ be the unique component of that decomposition that passes through 0, which is an abelian subvariety in $A$ by Lemma 12. For any $x$ in $K$ the set $x + A_0$ is an irreducible Zariski closed subset in $K$. Since $K$ coincides with $\bigcup_{x \in K} (x + A_0)$, ignoring each set $x + A_0$ that is a subset of $y + A_0$ for some $y \in K$, we can find a subset $\Xi$ of $K$ such that

$$K = \bigcup_{x \in \Xi} (x + A_0)$$

and for any two elements $x, y \in \Xi$ the irreducible sets $x + A_0$ and $y + A_0$ are not contained one in another. Since $x + A_0$ is irreducible, it is contained in $K_n$ for
some \( n \) by Lemma 10. Then \( A_0 \subset -x + K_n \). Similarly, \(-x + K_n \subset K_l \) for some \( l \), so that \( K_l = A_0 \) by Lemma 12. This yields
\[
x + A_0 = K_n
\]
for each \( x \in \Xi \). This means that the set \( \Xi \) is countable.

The theorem is proved.

Let us also prove that the abelian variety \( A_0 \), provided by Theorem 13, is contained in the abelian variety \( A_1 \) introduced in §2. Choose an ample line bundle \( L \) on the abelian variety \( A \). Let \( i: A_0 \to A \) be the closed embedding of \( A_0 \) in \( A \), and let \( L_0 \) be the pull-back of \( L \) to \( A_0 \) under the embedding \( i \). Define a homomorphism \( \zeta \) on divisors via the commutative diagram

\[
\begin{array}{ccc}
A^1(A_0) & \xrightarrow{\zeta} & A^1(A) \\
\downarrow{\lambda_{L_0}^*} & & \uparrow{\lambda_L^*} \\
A^1(A_0^\vee) & \xrightarrow{i^\vee*} & A^1(A^\vee)
\end{array}
\]

Similarly, we define a homomorphism \( \zeta_{Z_l} \) on cohomology by means of the commutative diagram

\[
\begin{array}{ccc}
H^1_{\text{ét}}(A_0, Z_l) & \xrightarrow{\zeta_{Z_l}} & H^1_{\text{ét}}(A, Z_l) \\
\downarrow{\lambda_{L_0}^*} & & \uparrow{\lambda_L^*} \\
H^1_{\text{ét}}(A_0^\vee, Z_l) & \xrightarrow{i^\vee*} & H^1_{\text{ét}}(A^\vee, Z_l)
\end{array}
\]

and analogously for the homomorphism
\[
\zeta_{Q_l/Z_l}: H^1_{\text{ét}}(A_0, Q_l/Z_l) \to H^1_{\text{ét}}(A_0, Q_l/Z_l).
\]

The homomorphism \( \zeta_{Z_l} \) induces the injective homomorphisms
\[
\zeta_{Q_l}: H^1_{\text{ét}}(A_0, Q_l) = H^1_{\text{ét}}(A_0, Z_l) \otimes Q_l \to H^1_{\text{ét}}(A_0, Z_l) = H^1_{\text{ét}}(A, Z_l) \otimes Q_l
\]
and
\[
\zeta_{Z_l} \otimes Q_l/Z_l: H^1_{\text{ét}}(A_0, Z_l) \otimes Q_l/Z_l \to H^1_{\text{ét}}(A_0, Z_l) \otimes Q_l/Z_l.
\]

**Proposition 14.** The image of the composition
\[
H^1_{\text{ét}}(A_0, Q_l(1-p)) \xrightarrow{\zeta_{Q_l}} H^1_{\text{ét}}(A, Q_l(1-p)) \xrightarrow{w_*} H^{2p-1}_{\text{ét}}(Y, Q_l)
\]
is contained in the kernel of the Gysin homomorphism
\[
H^{2p-1}_{\text{ét}}(Y, Q_l) \xrightarrow{r_*} H^{2(p+e)-1}_{\text{ét}}(X, Q_l).
\]
Proof. To prove this, we will use Bloch’s $l$-adic Abel-Jacobi maps. For any abelian group $A$, a prime $l$ and positive integer $n$ let $A_{l^n}$ be the kernel of the endomorphism of $A$ and let $A(l)$ be the $l$-primary part of $A$, that is, the union of the groups $A_{l^n}$ for all $n$. For any smooth projective variety $V$ over $k$, there is a canonical homomorphism

$$\lambda^p_l(V) : \text{CH}^p(V)(l) \to H^{2p-1}_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(p)),$$

which was constructed by Bloch in [3]. The homomorphisms $\lambda^p_l(V)$ are functorial with respect to the action of correspondences between smooth projective varieties over $k$. Moreover, all the homomorphisms

$$\lambda^1_l(V) : \text{CH}^1(V)(l) \to H^1_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(1))$$

are isomorphisms (see [3]).

Since $A_0$ and $A$ are abelian varieties, their Néron-Severi groups are torsion free. It follows that $\text{CH}^1(A_0)(l) = A^1(A_0)(l)$ and $\text{CH}^1(A)(l) = A^1(A)(l)$, so that we actually have the isomorphism $\lambda^1_l(A_0)$ between $A^1(A_0)(l)$ and $H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1))$, and the isomorphism $\lambda^1_l(A)$ between $A^1(A)(l)$ and $H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1))$. Similarly, one has the isomorphisms $\lambda^1_l(A_0^\vee)$ and $\lambda^1_l(A^\vee)$ for the dual abelian varieties.

The functorial properties of Bloch’s maps $\lambda^1_l$ give us that the diagram

$$\begin{array}{ccc}
A^1(A_0)(l) & \xrightarrow{\zeta} & A^1(A)(l) \\
\lambda^1_l(A_0) \sim & & \lambda^1_l(A) \sim \\
H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\zeta_{\mathbb{Q}_l/\mathbb{Z}_l}} & H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1))
\end{array}$$

(3)

is commutative.

For a smooth projective $V$ over $k$ we have homomorphisms

$$\varrho^{i,j}_l(V) : H^i_{\text{ét}}(V, \mathbb{Z}_l(j)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^i_{\text{ét}}(V, \mathbb{Q}_l/\mathbb{Z}_l(j))$$

with finite kernels and cokernels, which were used in [11] and much earlier in [4]. In particular, we have the commutative diagram

$$\begin{array}{ccc}
H^1_{\text{ét}}(A_0, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{\zeta_{\mathbb{Q}_l/\mathbb{Z}_l}} & H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \\
\varrho^{1,-1}_l(A_0) & & \varrho^{1,1}_l(A) \\
H^1_{\text{ét}}(A_0, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l & \xrightarrow{\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l} & H^1_{\text{ét}}(A, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l
\end{array}$$

(4)

Let $\sigma : A_0 \xrightarrow{\sim} A^1(A_0^\vee)$ and $\sigma : A \xrightarrow{\sim} A^1(A^\vee)$ be the autoduality isomorphisms. The morphism of motives

$$w : M(A) \otimes \mathbb{L}^{p-1} \to M(Y)$$
induces a homomorphism $w_*: A^1(A) \to A^p(Y)$ on Chow groups. A straightforward verification shows that the diagram

\[
\begin{array}{ccc}
A^1(A)(l) & \xrightarrow{w_*} & A^p(Y)(l) \\
\uparrow{\lambda_l^*} & & \uparrow{(\psi_Y^p)^{-1}} \\
A^1(A)(l) & \xleftarrow{\sigma} & A(l)
\end{array}
\]

is commutative.

The homomorphism $w_*: A^1(A) \to A^p(Y)$ on Chow groups and the homomorphism $w_*: H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p))$ induced by $w$ on the cohomology fit into the commutative diagram

\[
\begin{array}{ccc}
A^1(A)(l) & \xrightarrow{w_*} & A^p(Y)(l) \\
\uparrow{\lambda_l^*} & & \uparrow{\lambda_Y^p} \\
H^1_{\text{ét}}(A, \mathbb{Q}_l/\mathbb{Z}_l(1)) & \xrightarrow{w_*} & H^{2p-1}_{\text{ét}}(Y, \mathbb{Q}_l/\mathbb{Z}_l(p))
\end{array}
\]

The commutativity of the diagrams (3)–(6), the definition of the abelian variety $A_0$ and an easy diagram chase over the obvious commutative diagrams with Gysin mappings show that the image of the triple composition

$r_* \circ w_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l) : H^1_{\text{ét}}(A_0, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^{2(p+\varepsilon)-1}_{\text{ét}}(X, \mathbb{Z}_l(p + \varepsilon)) \otimes \mathbb{Q}_l/\mathbb{Z}_l$

is contained in the kernel of the homomorphism

$\varrho_{l}^{2(p+\varepsilon)-1,p+\varepsilon}(X) : H^{2(p+\varepsilon)-1}_{\text{ét}}(X, \mathbb{Z}_l(p + \varepsilon)) \otimes \mathbb{Q}_l/\mathbb{Z}_l \to H^{2(p+\varepsilon)-1}_{\text{ét}}(X, \mathbb{Q}_l/\mathbb{Z}_l(p + \varepsilon))$.

Since the latter kernel is finite, the image of the composition $r_* \circ w_* \circ (\zeta_{\mathbb{Z}_l} \otimes \mathbb{Q}_l/\mathbb{Z}_l)$ is finite too. Since the étale cohomology groups of smooth projective varieties with $\mathbb{Z}_l$-coefficients are finitely generated $\mathbb{Z}_l$-modules, it follows that the image of the triple composition

$r_* \circ w_* \circ \zeta_{\mathbb{Q}_l} : H^1_{\text{ét}}(A_0, \mathbb{Q}_l(1)) \to H^{2(p+\varepsilon)-1}_{\text{ét}}(X, \mathbb{Q}_l(p + \varepsilon))$

is zero, which finishes the proof of the proposition.

**Corollary 15.** The abelian variety $A_0$ is contained in the abelian variety $A_1$.

**Proof.** Since the triple composition $r_* \circ w_* \circ \zeta_{\mathbb{Q}_l}$ is 0 by Proposition 14, the homomorphism $\zeta_{\mathbb{Q}_l}$ factorizes through the group $H^1_{\text{ét}}(A_1, \mathbb{Q}_l(1 - p))$.

The corollary is proved.
§ 5. Geometric generic versus very general version of $A_0$

Since the ground field $k$ is uncountable and algebraically closed, its transcendental degree over the primary subfield is infinite. We will be using the following terminology. If $S$ is an integral algebraic scheme, a (Zariski) $c$-closed subset in $S$ is a union of a countable collection of Zariski closed irreducible subsets in $S$. A (Zariski) $c$-open subset of $S$ is the complement to a $c$-closed subset in $S$. A property $P$ of points in $S$ holds for a very general point on $S$ if there exists a $c$-open subset $U$ in $S$ such that $P$ holds for each closed point in $U$.

The purpose of this section is to convince the reader that, given a flat family $\mathcal{X} \to S$ over an integral base $S$ over $k$, there exists a natural $c$-open subset $U$ in $S$ such that for all closed $k$-points $P \in U$ the fibres $\mathcal{X}_P$ are isomorphic to the geometric generic fibre $\mathcal{X}_{\eta}$, as schemes over $\text{Spec}(\mathbb{Q})$, and these isomorphisms preserve algebraic and rational equivalence of algebraic cycles. This is certainly part of the folklore (see [26], for example), but we give all the proofs here for clarity.

Let $\mathcal{X} \to S$ be an integral affine scheme of finite type over $k$, let $k(S)$ be the function field of $S$, and let $k(\mathcal{X})$ be the algebraic closure of the field $k(S)$. Let $I(S)$ be the ideal of $S$ and let $f_1, \ldots, f_n$ be a set of generators of $I(S)$. The polynomials $f_i$ have a finite number of coefficients. Then we can choose a countable algebraically closed subfield $k_0$ in $k$ such that there exists an integral affine scheme $S_0$ over $k_0$ with $S = S_0 \times_{\text{Spec}(k_0)} \text{Spec}(k)$.

Let $Z$ be a closed subscheme of $S_0$, and let $i_Z : Z \subset S_0$ be the corresponding closed embedding. Then $Z$ is defined by an ideal $\mathfrak{a}$ in $k_0[S_0]$. Since the field $k_0$ is countable and $\mathfrak{a}$ is finitely generated, there exist only countably many closed subschemes $Z$ in $S_0$. For each $Z$, let $U_Z$ be the complement $S_0 \setminus \text{im}(i_Z)$, $Z_k = Z \times_{k_0} k$ and $(U_Z)_k = U_Z \times_{k_0} k$. If $(i_Z)_k : Z_k \to S$ is the pull-back of $i_Z$ with respect to the extension $k/k_0$, then $(U_Z)_k$ is the complement $S_k \setminus \text{im}((i_Z)_k)$. Furthermore, let

$$U = S \setminus \bigcup_Z \text{im}((i_Z)_k) = \bigcap_Z (U_Z)_k,$$

where the union is taken over closed subschemes $Z$ such that $\text{im}((i_Z)_k) \neq S$. Notice that the last condition is equivalent to $\text{im}(i_Z) \neq S_0$. The set $U$ is $c$-open by its construction (see also the proof of Lemma 2.1 in [26]).

Lemma 16. For any closed $k$-point $P$ in $U$, one can construct a field isomorphism between $k(\mathcal{X})$ and $k$ whose value at $f \in k_0[S_0]$ is $f(P)$.

Proof. Let $P$ be a closed $k$-point in the set $U$ defined by the corresponding morphism $f_P : \text{Spec}(k) \to S$. Its image under the projection $\pi : S \to S_0$ belongs to $U_Z$ for each $Z$ such that $\text{im}(i_Z) \neq S_0$. This means that this image is nothing but the generic point $\eta_0 = \text{Spec}(k_0(S_0))$ of the scheme $S_0$. In other words, there exists a morphism $h_P : \text{Spec}(k) \to \text{Spec}(k_0(S_0)) = \eta_0$ such that $\pi \circ f_P = g_0 \circ h_P$, where $g_0$ is a morphism from the generic point $\eta_0$ to $S_0$. In terms of commutative rings, this means the following. If $e_{\nu_P} : k[S] \to k$ is the evaluation at $P$, that is, the morphism inducing $f_P$ on spectra, there exists a homomorphism of fields $e_P$ making
the diagram

\[ \begin{array}{ccc}
  k[S] & \xrightarrow{ev_P} & k \\
  \uparrow & & \uparrow \\
  k_0[S_0] & \longrightarrow & k_0(S_0)
\end{array} \]

which can be extended to an isomorphism.

As the square (7) is commutative, \( e_P(f) = f(P) \) for each \( f \) in \( k_0[S_0] \).

The lemma is proved.
Remark 17. We must note that the above isomorphism \( e_P \) is non-canonical, as it depends on the choice of the transcendental basis \( B \) containing the quantities \( b_1, \ldots, b_d \).

Now let \( f: \mathcal{X} \to S \) be a smooth morphism of schemes over \( k \). Extending \( k_0 \) if necessary, we may assume that there exists a morphism of schemes \( f_0: \mathcal{X}_0 \to S_0 \) over \( k_0 \) such that \( f \) is the pull-back of \( f_0 \) under the field extension from \( k_0 \) to \( k \). Let \( \eta_0 = \text{Spec}(k_0(S_0)) \) be the generic point of the scheme \( S_0 \), \( \eta = \text{Spec}(k(S)) \) the generic point of the scheme \( S \), and \( \bar{\eta} = \text{Spec}(k(S)) \) the geometric generic point of \( S \). Then we also have the corresponding fibres \( \mathcal{X}_{0,\eta_0}, \mathcal{X}_\eta \) and \( \mathcal{X}_{\bar{\eta}} \).

Pulling back the scheme-theoretic isomorphism \( \text{Spec}(e_P) \) to the fibres of the family \( f \), we obtain the Cartesian square

\[
\begin{array}{ccc}
\mathcal{X}_P & \xrightarrow{\kappa_P} & \text{Spec}(k) \\
\downarrow & & \downarrow \\
\mathcal{X}_{\bar{\eta}} & \xrightarrow{\kappa_{\bar{\eta}}} & \bar{\eta}
\end{array}
\]

Since \( \text{Spec}(e_P) \) is an isomorphism of schemes over \( \eta_0 \), the morphism \( \kappa_P \) is an isomorphism of schemes over \( \mathcal{X}_{0,\eta_0} \).

Given a field \( F \), a scheme \( Y \) over \( F \) and an automorphism \( \sigma \) of \( F \), let \( Y_\sigma \) be the fibred product of \( Y \) and \( \text{Spec}(F) \) over \( \text{Spec}(F) \), with regard to the automorphism \( \text{Spec}(\sigma) \), and let \( w_\sigma: Y_\sigma \simeq Y \) be the corresponding isomorphism of schemes over \( \text{Spec}(F^\sigma) \), where \( F^\sigma \) is the subfield of \( \sigma \)-invariant elements in \( F \).

Let \( L \) be a field subextension of \( k/k_0 \). The projection \( \mathcal{X} \to \mathcal{X}_0 \) factorizes through \( \mathcal{X}_{0L} = \mathcal{X}_0 \times_{\text{Spec}(k_0)} \text{Spec}(L) \). Composing the embedding of the fibre \( \mathcal{X}_P \) into the total scheme \( \mathcal{X} \) with the morphism \( \mathcal{X} \to \mathcal{X}_{0L} \) we can consider \( \mathcal{X}_P \) as a scheme over \( \mathcal{X}_{0L} \).

Now if \( P' \) is another closed \( k \)-point in \( U \), let \( \sigma_{PP'} = e_{P'} \circ e_P^{-1} \) be the automorphism of the field \( k \), and let \( \kappa_{PP'} = \kappa_{P'}^{-1} \circ \kappa_P \) be the induced isomorphism of the fibres as schemes over \( \text{Spec}(k^{\sigma_{PP'}}) \). In these terms, \( (\mathcal{X}_P)_{\sigma_{PP'}} = \mathcal{X}_{PP'} \), the isomorphism \( w_{\sigma_{PP'}}: \mathcal{X}_{PP'} \simeq \mathcal{X}_P \) is over \( \mathcal{X}_0 \times_{\text{Spec}(k_0)} \text{Spec}(k^{\sigma_{PP'}}) \), and \( w_{\sigma_{PP'}} = \kappa_{PP'} \). To see this we just need to use Lemma 16 and pull-back the scheme-theoretic isomorphisms between points on \( S \) to isomorphisms between the corresponding fibres of the morphism \( f: \mathcal{X} \to S \).

Remark 18. The assumption that \( S \) is affine is not essential, of course. We can always cover \( S \) by open affine subschemes, construct the system of isomorphisms \( \kappa \) in each affine chart and then construct ‘transition isomorphisms’ between very general fibres in a smooth family over an arbitrary integral base \( S \) of finite type over \( k \).
Now let $S$ be an integral scheme of finite type over $k$, let $X$ and $Y$ be two connected schemes, both smooth and projective over $S$, and let

![Diagram](image)

be a closed embedding of schemes over the base $S$. Extending $k_0$ appropriately we may assume that there exist models $f_0, g_0$ and $r_0$ over $k_0$ of the morphisms $f, g$ and $r$, respectively, such that $g_0 \circ r_0 = f_0$. Then, for any closed $k$-point $P$ in $U$, the diagram

![Diagram](image)

is commutative, where $r_P$ and $r_\pi$ are the obvious morphisms on fibres induced by the morphism $r$. Then, of course, the isomorphisms $\kappa_{P'}$ commute with the morphisms $r_P$ and $r_{P'}$, for any two closed $k$-points $P$ and $P'$ in $U$. Cutting out more Zariski closed subsets from $U$ we may assume that the fibres of the families $f$ and $g$ over the points from $U$ are smooth.

**Lemma 19.** The scheme-theoretic isomorphisms $\kappa_P$ preserve the algebraic and rational equivalence of algebraic cycles.

**Proof.** As we already explained in §3, if $\alpha : k \cong k'$ is an isomorphism of fields, the functorial bijections $\Upsilon$ from the representation of Chow monoids by Chow schemes commute through the isomorphisms of monoids and Hom-sets induced by the isomorphism $\text{Spec}(\alpha)$. In particular, if $k' = \overline{k(S)}$ and $\alpha = e_P^{-1}$, the bijections $\Upsilon(\mathcal{X}_\pi) \cong \overline{k(S)}$ commute with the bijections $\Upsilon(\mathcal{X}_P)$ over $k$, and the same for $\mathcal{Y}$. The commutativity for sections of the corresponding pre-sheaves on an algebraic curve $C$ over $k$ and its pull-back $C'$ over $k'$ gives us the first assertion of the lemma. If $C = \mathbb{P}^1$, we obtain the second one.

The lemma is proved.

Assume now that Assumptions 1–3 are satisfied for the geometric generic fibre $\mathcal{Y}_\pi$, and for the fibre $\mathcal{Y}_P$ for each closed point $P$ in $U$. Also let

$$\psi_\pi : A^p(\mathcal{Y}_\pi) \cong A_\pi$$

and

$$\psi_P : A^p(\mathcal{Y}_P) \cong A_P$$

be the corresponding regular parametrizations. By Lemma 19 the isomorphism $\kappa_P$ induces the push-forward isomorphism of abelian groups

$$\kappa_P^* : A^p(\mathcal{Y}_P) \to A^p(\mathcal{Y}_\pi).$$
Let
\[ \kappa_P : A_P \to A_{\pi} \]
be the composition given by the commutative diagram
\[
\begin{array}{ccc}
A_P & \xrightarrow{\psi_P^{-1}} & A^p(Y_P) \\
\downarrow \kappa_P & & \downarrow \varphi_P \\
A_{\pi} & \xleftarrow{\psi_{\pi}} & A^p(Y_{\pi})
\end{array}
\]
(9)

Consider the obvious commutative diagram
\[
\begin{array}{ccc}
A_P & \xleftarrow{\psi_P} & A^p(Y_P) & \xleftarrow{\theta_d} & C_{d,d}^p(Y_P) \\
\downarrow \kappa_P & & \downarrow \varphi_P & & \downarrow \theta_d \\
A_{\pi} & \xleftarrow{\psi_{\pi}} & A^p(Y_{\pi}) & \xleftarrow{\theta_d} & C_{d,d}^p(Y_{\pi})
\end{array}
\]
(10)

The top and bottom horizontal compositions in this diagram are regular morphisms of schemes over \( k \) and \( \bar{k}(S) \), respectively, and the vertical morphism from the right-hand side is a regular morphism of schemes over \( \mathbb{Q} \). It follows that the homomorphism \( \kappa_P : A_P \to A_{\pi} \) is a regular morphism of schemes over \( \mathbb{Q} \) too.

Now, the commutative diagram (8) gives us the commutative diagram
\[
\begin{array}{ccc}
A^p(Y_P) & \xrightarrow{r_P} & A^{p+e}(Y_P) \\
\downarrow \varphi_P & & \downarrow \varphi_P \\
A^p(Y_{\pi}) & \xrightarrow{r_{\pi}} & A^{p+e}(Y_{\pi})
\end{array}
\]
(11)

where \( e \) is the codimension of \( Y_{\pi} \) in \( Y_P \). Let \( A_{P,1} \) and \( A_{\pi,1} \) be the abelian subvarieties in \( A_P \) and \( A_{\pi} \), respectively, constructed in § 2. Furthermore, let \( A_{P,0} \) and \( A_{\pi,0} \) be the abelian subvarieties in \( A_{P,1} \) and \( A_{\pi,1} \), respectively, provided by Theorem 13 and Corollary 15.

**Proposition 20.** For any closed point \( P \) in \( U \),
\[
\kappa_P(A_{P,1}) = A_{\pi,1}
\]
and
\[
\kappa_P(A_{P,0}) = A_{\pi,0}.
\]


\textbf{Proof.} The first claim is actually true for any closed point \( P \) on \( S \), not only on \( U \), and can easily be deduced using specialization isomorphisms on étale cohomology groups. Let us prove the second claim. Let \( \Xi_P \) be the countable subset in \( A_P \) and \( \Xi_\eta \) the countable subset in \( A_\eta \) such that we have the presentations kernels

\[ K_P = \bigcup_{x \in \Xi_P} (s + A_{P,0}) \quad \text{and} \quad K_\eta = \bigcup_{x \in \Xi_\eta} (x + A_{\eta,0}) \]

in \( A_P \) and \( A_\eta \), respectively, as provided by Theorem 13. Then

\[ \kappa_P(K_P) = \kappa_P \left( \bigcup_{x \in \Xi_P} (x + A_{P,0}) \right) = \bigcup_{x \in \Xi_P} (\kappa_P(x) + \kappa_P(A_{P,0})). \]

The definition of \( \kappa_P \) and the commutative diagram (11) give

\[ \kappa_P(K_P) = K_\eta. \]

Therefore,

\[ \bigcup_{x \in \Xi_P} (\kappa_P(x) + \kappa_P(A_{P,0})) = \bigcup_{x \in \Xi_\eta} (x + A_{\eta,0}) \]

inside the abelian variety \( A_\eta \). Since the group isomorphisms \( \kappa_P \) are regular morphisms of schemes over \( \text{Spec}(\mathbb{Q}) \), we obtain that \( \kappa_P(A_{P,0}) \) is a Zariski closed subset in \( A_\eta \). Since \( \kappa_P(A_{P,0}) \) is a subgroup in \( A_\eta \), it is an abelian subvariety in \( A_\eta \). Lemmas 11 and 12 finish the proof.

The proposition is proved.

\textbf{Remark 21.} Of course, the set \( U \) is not uniquely defined, and the same argument works over the union of all such \( c \)-open subsets in the integral scheme \( S \). The behaviour of \( A_0 \) outside the union of the sets \( U \) is an open question of particular importance and deserves a separate research programme.

\section{Étale monodromy argument for cycles of dimension \( p - 1 \)}

This is the main section of the paper, in which we apply Theorem 13 to the family of hyperplane sections of a projective variety embedded into a projective space. In such a case we can enhance the study of \( A_0 \) by the monodromy argument in terms of étale \( l \)-adic cohomology over \( k \).

Let \( d = 2p \) and let \( X \) be a smooth \( d \)-dimensional projective variety over the ground field \( k \). Fix a closed embedding \( X \subset \mathbb{P}^m \) such that \( X \) is not contained in a smaller linear subspace in \( \mathbb{P}^m \). Let

\[ \mathcal{H} = \{(P, H) \in \mathbb{P}^m \times \mathbb{P}^m{}^\vee \mid P \in H\} \]

be the universal hyperplane, and let \( p_1 \) and \( p_2 \) be the projections of \( \mathcal{H} \) on \( \mathbb{P}^m \) and \( \mathbb{P}^m{}^\vee \), respectively. Let

\[ \mathcal{X} = X \times \mathbb{P}^m{}^\vee, \]

and let

\[ \mathcal{Y} = \mathcal{X} \cap \mathcal{H} \]

inside \( \mathbb{P}^m \times \mathbb{P}^m{}^\vee \).
Let
\[ f : Y \to \mathbb{P}^m \]
be the composition of the closed embedding \( Y \subset \mathcal{H} \) with the projection \( p_2 \), let
\[ g : X \to \mathbb{P}^m \]
be the composition of the closed embedding of \( X \) in \( \mathbb{P}^m \times \mathbb{P}^m \) with the projection onto \( \mathbb{P}^m \), and let
\[ Y_f, r \quad \xymatrix{ Y \ar[rr]^r & & X \ar[dl]^g \ar[ld]_f \ar@{.>}[dl] \ar@{.>}[d]^{p_2} \ar@{.>}[dl]^{p_1} } \]
be the obvious closed embedding over the dual projective space.

For any scheme \( S \) and any morphism of schemes
\[ S \to \mathbb{P}^m \]
let \( \mathcal{H}_S \to S \) be the pull-back of \( p_2 \) with respect to the morphism \( S \to \mathbb{P}^m \), let \( \mathcal{Y}_S \) be the fibred product of \( \mathcal{Y} \) and \( \mathcal{H}_S \) over the universal hyperplane \( \mathcal{H} \), and let
\[ f_S : \mathcal{Y}_S \to S \]
be the induced projection, that is, the composition of the closed embedding of \( \mathcal{Y}_S \) into \( \mathcal{H}_S \) and the morphism \( \mathcal{H}_S \to S \). Also let \( \mathcal{X}_S \to S \) be the pull-back of trivial family \( \mathcal{X} \to \mathbb{P}^m \) with respect to the morphism \( S \to \mathbb{P}^m \). Then we obtain the closed embedding
\[ \mathcal{Y}_S \xleftarrow{r_S} \mathcal{X}_S \]
over \( S \).

Assume that the scheme \( S \) is integral. Let \( k(S) \) be the function field of \( S \),
\[ \eta = \text{Spec}(k(S)) \]
be the generic point of \( S \), \( \overline{k(S)} \) be the algebraic closure of \( k(S) \) and
\[ \overline{\eta} = \text{Spec}(\overline{k(S)}) \]
be the geometric generic point of \( S \). Then we also have the closed embeddings \( r_\eta \) and \( r_\pi \) over \( \eta \) and \( \overline{\eta} \) respectively.

As in the previous section, choose an appropriate \( \epsilon \)-open subset \( U \) in \( S \) such that the point \( \overline{\eta} \) is scheme-theoretically isomorphic to each closed point \( P \) in \( U \), and assume that Assumptions 1–3 are satisfied for the geometric generic fibre
\[ Y_\pi = \mathcal{Y}_\pi \]
and for the fibre
\[ Y_P = \mathcal{Y}_P \]
for each closed point \( P \) in \( U \). Also let
\[
\psi_\eta : A^P(Y_\eta) \sim \to A_\eta
\]
and
\[
\psi_P : A^P(Y_P) \sim \to A_P
\]
be the corresponding regular parametrizations. Then we have the abelian subvarieties
\[ A_{\eta,0} \subset A_{\eta,1} \subset A_\eta \]
and
\[ A_{P,0} \subset A_{P,1} \subset A_P \]
for each closed point \( P \) in \( U \).

First let \( S = D \) be a projective line inside the dual space \( \mathbb{P}^m^\vee \) such that the morphism \( f_D \) is a Lefschetz pencil for the variety \( X \). Let \( L \) be the minimal subextension of \( k(D) \) in \( k(D) \) such that the abelian varieties \( A_{\eta,0}, A_{\eta,1} \) and \( A_\eta \) are defined over \( L \). Then \( L \) is finitely generated and algebraic of finite degree \( n \) over \( k(D) \). Let \( D' \) be a smooth projective curve such that \( L = k(D') \) and the embedding of \( k(D) \) in \( k(D') \) is induced by a generically of degree \( n \) morphism from \( D' \) onto \( D \). Since the closed embeddings of \( A_{\eta,0} \) in \( A_{\eta,1} \) and \( A_{\eta,1} \) in \( A_\eta \) are now defined over \( L \), there exist a Zariski open subset \( U' \) in \( D' \), spreads \( \mathcal{A}_{\eta,0}, \mathcal{A}_{\eta,1} \) and \( \mathcal{A}_{\eta} \) of \( A_{\eta,0}, A_{\eta,1} \) and \( A_\eta \), respectively, over \( U' \), and morphisms
\[
\mathcal{A}_{\eta,0} \to \mathcal{A}_{\eta,1}
\]
and
\[
\mathcal{A}_{\eta,1} \to \mathcal{A}_{\eta}
\]
over \( U' \) such that, when passing to the fibres at the geometric generic point \( \eta \), we obtain the closed embeddings
\[
A_{\eta,0} \to A_{\eta,1}
\]
and
\[
A_{\eta,1} \to A_\eta
\]
over \( k(D) \).

Let \( \alpha \) be the morphism from \( \mathcal{A} \) onto \( U' \), and let \( \alpha_0 \) and \( \alpha_1 \) be the morphisms from \( \mathcal{A}_0 \) and, respectively, \( \mathcal{A}_1 \) onto \( U' \). Since \( \mathcal{A} \) is a spread of \( A_\eta \) over \( U' \) and \( A_\eta \) is a projective variety over \( L = k(D') \), the morphism \( \alpha \) is locally projective and therefore proper. Similarly, the morphisms \( \alpha_0 \) and \( \alpha_1 \) are proper. Cutting more points from \( D' \) we may assume that all the morphisms \( \alpha, \alpha_0 \) and \( \alpha_1 \) are smooth over \( U' \).

Let \( \eta' \) be the generic point of \( D' \), let \( \eta' = \eta \) be the geometric generic point of \( D' \), let \( \pi_1(U', \eta) \) be the étale fundamental group of \( D' \) pointed at \( \eta \), and let \( \pi^{\text{tame}}_1(U', \eta) \) be the corresponding tame fundamental group. For any scheme \( V \) and nonnegative integer \( n \) let \( (\mathbb{Z}/l^n)_V \) be the constant sheaf on \( V \) associated to the group \( \mathbb{Z}/l^n \).

Since the morphisms \( \alpha_0, \alpha_1 \) and \( \alpha \) are smooth and proper, the higher direct images
\[
R^1\alpha_{0*}(\mathbb{Z}/l^n)_{\mathcal{A}_0}, \quad R^1\alpha_{1*}(\mathbb{Z}/l^n)_{\mathcal{A}_1}, \quad \text{and} \quad R^1\alpha_*\left(\mathbb{Z}/l^n\right)_{\mathcal{A}}
\]
are locally constant (see [10], Ch. I, Theorem 8.9). Then the fibres of these sheaves at the geometric generic point $\eta$ are finite continuous $\pi_1(U', \eta)$-modules (see [10], Proposition A I.7). The proper base change (see, for example, [10], Ch. I, Theorem 6.1') gives that

$$(R^1\alpha_{0*}(\mathbb{Z}/l^n))_{\mathfrak{A}_0} = H^1_{\text{ét}}(\mathfrak{A}_{0\eta}, \mathbb{Z}/l^n),$$

$$(R^1\alpha_{1*}(\mathbb{Z}/l^n))_{\mathfrak{A}_1} = H^1_{\text{ét}}(\mathfrak{A}_{1\eta}, \mathbb{Z}/l^n)$$

and

$$(R^1\alpha_{*}(\mathbb{Z}/l^n))_{\mathfrak{A}} = H^1_{\text{ét}}(\mathfrak{A}_{\eta}, \mathbb{Z}/l^n).$$

Then we see that $\pi_1(U', \eta)$ acts continuously on

$$H^1_{\text{ét}}(\mathfrak{A}_{0\eta}, \mathbb{Z}/l^n),$$

$$H^1_{\text{ét}}(\mathfrak{A}_{1\eta}, \mathbb{Z}/l^n)$$

and

$$H^1_{\text{ét}}(\mathfrak{A}_{\eta}, \mathbb{Z}/l^n).$$

Passing to the limits on $n$ and then tensoring with $\mathbb{Q}_l$ we find that $\pi_1(U', \eta)$ acts continuously on

$$H^1_{\text{ét}}(\mathfrak{A}_{0\eta}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{0\eta}, \mathbb{Q}_l),$$

$$H^1_{\text{ét}}(\mathfrak{A}_{1\eta}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{1\eta}, \mathbb{Q}_l)$$

and

$$H^1_{\text{ét}}(\mathfrak{A}_{\eta}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{\eta}, \mathbb{Q}_l).$$

The homomorphism

$$\zeta_{\mathfrak{A}_l} : H^1_{\text{ét}}(A_{0\eta}, \mathbb{Q}_l) \rightarrow H^1_{\text{ét}}(A_{\eta}, \mathbb{Q}_l)$$

is the composition of the obvious homomorphisms

$$\zeta'_{\mathfrak{A}_l} : H^1_{\text{ét}}(A_{0\eta}, \mathbb{Q}_l) \rightarrow H^1_{\text{ét}}(A_{1\eta}, \mathbb{Q}_l)$$

and

$$\zeta''_{\mathfrak{A}_l} : H^1_{\text{ét}}(A_{1\eta}, \mathbb{Q}_l) \rightarrow H^1_{\text{ét}}(A_{\eta}, \mathbb{Q}_l).$$

The action of $\pi_1(U', \eta)$ commutes naturally with both $\zeta'_{\mathfrak{A}_l}$ and $\zeta''_{\mathfrak{A}_l}$.

Without loss of generality, we may assume that $U'$ is the pre-image of a Zariski open subset $U$ in $D$ and all the fibres of the Lefschetz pencil

$$f_D : \mathfrak{Y}_D \rightarrow D$$

over closed points of $U$ are smooth. Let

$$f_{D'} : \mathfrak{Y}_{D'} \rightarrow D'$$

be the pull-back of the pencil $f_D$ with respect to the morphism $D' \rightarrow D$, and let

$$f_{U'} : \mathfrak{Y}_{U'} \rightarrow U'$$

the fibres of these sheaves at the geometric generic point $\eta$ are finite continuous $\pi_1(U', \eta)$-modules (see [10], Proposition A I.7). The proper base change (see, for example, [10], Ch. I, Theorem 6.1') gives that

$$(R^1\alpha_{0*}(\mathbb{Z}/l^n))_{\mathfrak{A}_0} = H^1_{\text{ét}}(\mathfrak{A}_{0\eta}, \mathbb{Z}/l^n),$$

$$(R^1\alpha_{1*}(\mathbb{Z}/l^n))_{\mathfrak{A}_1} = H^1_{\text{ét}}(\mathfrak{A}_{1\eta}, \mathbb{Z}/l^n)$$

and

$$(R^1\alpha_{*}(\mathbb{Z}/l^n))_{\mathfrak{A}} = H^1_{\text{ét}}(\mathfrak{A}_{\eta}, \mathbb{Z}/l^n).$$

Then we see that $\pi_1(U', \eta)$ acts continuously on

$$H^1_{\text{ét}}(\mathfrak{A}_{0\eta}, \mathbb{Z}/l^n),$$

$$H^1_{\text{ét}}(\mathfrak{A}_{1\eta}, \mathbb{Z}/l^n)$$

and

$$H^1_{\text{ét}}(\mathfrak{A}_{\eta}, \mathbb{Z}/l^n).$$

Passing to the limits on $n$ and then tensoring with $\mathbb{Q}_l$ we find that $\pi_1(U', \eta)$ acts continuously on

$$H^1_{\text{ét}}(\mathfrak{A}_{0\eta}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{0\eta}, \mathbb{Q}_l),$$

$$H^1_{\text{ét}}(\mathfrak{A}_{1\eta}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{1\eta}, \mathbb{Q}_l)$$

and

$$H^1_{\text{ét}}(\mathfrak{A}_{\eta}, \mathbb{Q}_l) = H^1_{\text{ét}}(A_{\eta}, \mathbb{Q}_l).$$

The homomorphism

$$\zeta_{\mathfrak{A}_l} : H^1_{\text{ét}}(A_{0\eta}, \mathbb{Q}_l) \rightarrow H^1_{\text{ét}}(A_{\eta}, \mathbb{Q}_l)$$

is the composition of the obvious homomorphisms

$$\zeta'_{\mathfrak{A}_l} : H^1_{\text{ét}}(A_{0\eta}, \mathbb{Q}_l) \rightarrow H^1_{\text{ét}}(A_{1\eta}, \mathbb{Q}_l)$$

and

$$\zeta''_{\mathfrak{A}_l} : H^1_{\text{ét}}(A_{1\eta}, \mathbb{Q}_l) \rightarrow H^1_{\text{ét}}(A_{\eta}, \mathbb{Q}_l).$$

The action of $\pi_1(U', \eta)$ commutes naturally with both $\zeta'_{\mathfrak{A}_l}$ and $\zeta''_{\mathfrak{A}_l}$.

Without loss of generality, we may assume that $U'$ is the pre-image of a Zariski open subset $U$ in $D$ and all the fibres of the Lefschetz pencil

$$f_D : \mathfrak{Y}_D \rightarrow D$$

over closed points of $U$ are smooth. Let

$$f_{D'} : \mathfrak{Y}_{D'} \rightarrow D'$$

be the pull-back of the pencil $f_D$ with respect to the morphism $D' \rightarrow D$, and let

$$f_{U'} : \mathfrak{Y}_{U'} \rightarrow U'$$
be the pull-back of \( f_{D'} \) with respect to the open embedding of \( U' \) to \( D' \). Applying
the same arguments to the morphism \( f_{U'} \) we obtain a continuous action of the étale
fundamental group \( \pi_1(U', \overline{\eta}) \) on the cohomology group \( H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l) \), and it is well
known that this action is tame, in the sense that it factorizes through the surjective homomorphism from \( \pi_1(U', \overline{\eta}) \) onto \( \pi_1^{\text{tame}}(U', \overline{\eta}) \).

For each closed point \( s \) in the complement \( D \setminus U \) let
\[
\delta_s \in H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l)
\]
be the unique up to conjugation vanishing cycle corresponding to the point \( s \) in the
standard sense (see [10], Ch. III, Theorem 7.1), and let
\[
E \subset H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l)
\]
be the \( \mathbb{Q}_l \)-vector subspace generated by all the elements \( \delta_s \), \( s \in D \setminus U \). In other
words, \( E \) is the space of vanishing cycles in \( H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l) \). We can show that
\[
E = \ker(H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l) \to H^{2p+1}_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Q}_l)),
\]
where \( X_{\overline{\eta}} = X \times \overline{\eta} \) (see [8], §4.3).

In what follows we will be using the étale l-adic Picard-Lefschetz formula for the
monodromy action. For each \( s \in D \setminus U \) let
\[
\pi_{1,s} \subset \pi_1^{\text{tame}}(U, \overline{\eta})
\]
be the so-called tame fundamental group at \( s \), a subgroup uniquely determined
by the point \( s \) up to conjugation in \( \pi_1^{\text{tame}}(U, \overline{\eta}) \). In terms of [10], \( \pi_{1,s} \) is the image of the homomorphism
\[
\gamma_s : \hat{\mathbb{Z}}(1) \to \pi_1^{\text{tame}}(U, \overline{\eta}),
\]
where \( \hat{\mathbb{Z}}(1) \) is the limit of all groups \( \mu_n \) and \( \mu_n \) is the group of \( n \)-th roots of unity
in the algebraically closed field \( k(U) \) whose exponential characteristic is 1.

The tame fundamental group \( \pi_1^{\text{tame}}(U, \overline{\eta}) \) is generated by the subgroups \( \pi_{1,s} \). If \( u \) is an element in \( \hat{\mathbb{Z}}(1) \), let \( \overline{\pi} \) be the image of \( u \) in \( \mathbb{Z}_l(1) \). Now if \( v \) is an element in the \( \mathbb{Q}_l \)-vector space \( H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l) \), the Picard-Lefschetz formula says that
\[
\gamma_s(u)x = x \pm \overline{\pi}(x, \delta_s)\delta_s.
\]

**Proposition 22.** Under the assumptions above, either \( A_{\overline{\eta},0} = 0 \) or \( A_{\overline{\eta},0} = A_{\overline{\eta},1} \).

**Proof.** By Proposition 14 and the fact that the space \( E \) of vanishing cycles coincides with the kernel of the Gysin homomorphism
\[
r_{\overline{\eta}} : H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l) \to H^{2p+1}_{\text{ét}}(X_{\overline{\eta}}, \mathbb{Q}_l),
\]
we see that the image of the composition
\[
H^1_{\text{ét}}(A_{\overline{\eta},0}, \mathbb{Q}_l(1 - p)) \xrightarrow{\zeta_{\overline{\eta}}} H^1_{\text{ét}}(A_{\overline{\eta}, \mathbb{Q}_l(1 - p)) \xrightarrow{w_*} H^{2p-1}_{\text{ét}}(Y_{\overline{\eta}}, \mathbb{Q}_l)
\]
is contained in \( E \). The homomorphism \( \zeta_{\overline{\eta}} \) is injective and compatible with the action of \( \pi_1(U', \overline{\eta}) \). Since \( p \leq 2 \), the homomorphism \( w_* \) is bijective (see Remark 4). Then
\[
E \simeq H^1_{\text{ét}}(A_{\overline{\eta},1}, \mathbb{Q}_l(1 - p))
\]
via \( \zeta_{\overline{\eta}}^{\prime\prime} \) and \( w_* \).
Since the variety $Y_\eta$ satisfies Assumption 1, there exists a smooth projective curve $\Gamma$ and an algebraic cycle $Z$ on $\Gamma \times Y$ over $\eta$ such that the cycle class $z$ of $Z$ induces a surjective homomorphism

$$z_* : A^1(\Gamma) \to A^p(Y_\eta),$$

whose kernel is $G$. The homomorphism

$$w_* : H^1_{\text{ét}}(A_\eta, \mathbb{Q}_l(1-p)) \to H^{2p-1}_{\text{ét}}(Y_\eta, \mathbb{Q}_l)$$

is then induced by the composition of the embedding of the curve $\Gamma$ in its Jacobian $J_\Gamma$ over $\eta$, the quotient map from $J_\Gamma$ onto the abelian variety $A = J_\Gamma$, also over $\eta$, and the homomorphism induced by the correspondence $Z$ (see §2). Spreading out the morphisms $\Gamma \to J_\Gamma$ and $J_\Gamma \to A$, as well as the cycle $Z$, over a certain Zariski open subset in $D'$, we can ensure that the homomorphism $w_*$ is compatible with the action of the fundamental group $\pi_1(U', \eta)$.

Thus the composition $w_* \circ \zeta_{\eta l}$ is an injection of the $\pi_1(U', \eta)$-module $H^1_{\text{ét}}(A_\eta, \mathbb{Q}_l(1-p))$ into the $\pi_1(U', \eta)$-module of vanishing cycles $E$. Let

$$E_0 = \text{im}(w_* \circ \zeta_{\eta l})$$

be the image of this injection.

Since $U'$ is finite of degree $n$ over $U$, the group $\pi_1(U', \eta)$ is a subgroup of finite index $n$ in the étale fundamental group $\pi_1(U, \eta)$, which acts continuously on $E$ by the standard étale monodromy theory.

We are now going to use the Picard-Lefschetz formula in order to show that $E_0$ is a $\pi^\text{tame}_1(U, \eta)$-equivariant subspace in $E$. Obviously, it suffices to show that for each element $\gamma_s(u)$ in $\pi^\text{tame}_1(U, \eta)$ and any element $x$ in $E_0$ the element $\gamma_s(u)x$ is again in the space $E_0$.

Indeed, since $\langle \delta_s, \delta_s \rangle = 0$, the Picard-Lefschetz formula (12) and an easy induction give us

$$(\gamma_s(u))^m x = x \pm m \overline{u}(x, \delta_s) \delta_s$$

for a natural number $m$, so

$$\overline{u}(x, \delta_s) \delta_s = \frac{1}{m}((\gamma_s(u))^m x \pm x).$$

When $m$ is the index of $\pi_1(U', \eta)$ in $\pi_1(U, \eta)$, then $(\gamma_s(u))^m$ sits in the subgroup $\pi_1(U', \eta)$, so that the right-hand side of the latter formula is an element of $E_0$. Applying the Picard-Lefschetz formula again, we see that $\gamma_s(u)x$ is in $E_0$.

Thus, $E_0$ is a submodule in the $\pi^\text{tame}_1(U, \eta)$-module $E$. Since $E$ is known to be absolutely irreducible (see [10], Ch. III, Corollary 7.4, for example), we see that either $E_0 = 0$ or $E_0 = E$. In the first case $H^1_{\text{ét}}(A_{\eta,0}, \mathbb{Q}_l) = 0$, whence $A_{\eta,0} = 0$. In the second case

$$\zeta_{\eta l} : H^1_{\text{ét}}(A_{\eta,0}, \mathbb{Q}_l(1-p)) \to H^1_{\text{ét}}(A_{\eta,1}, \mathbb{Q}_l(1-p))$$

is an isomorphism, and so $A_{\eta,0} = A_{\eta,1}$.

The proposition is proved.
Now let $T$ be the complement to the discriminant locus of $X$ in $\mathbb{P}^m$. That is
to say, $T$ is the set of hyperplanes in $\mathbb{P}^m$ whose intersections with $X$ are smooth.
We want to consider the global case, when the base scheme $S$ is the scheme $T$.
Again, let $U$ be a $c$-open subset in $T$ constructed as in §5. In other words, we
define $U$ by removing the images of the pull-backs of all closed embeddings in the
model $T_0$ of $T$ defined over the minimal field of definition of $T$. Then $U$ is a $c$-open
subset in $T$ and in the dual projective space $(\mathbb{P}^m)^\vee$ such that, if $\xi$ is the generic point
of the projective space $\mathbb{P}^m$ and $\overline{\xi}$ is the corresponding geometric generic point, for
any closed point $P \in U$ we have an isomorphism $\mathcal{z}_P$ between $Y_P$ and $Y_{\overline{\xi}}$, and for any
two closed points $P$ and $P'$ in $U$ we have the scheme-theoretic isomorphism $\mathcal{z}_{P,P'}$
between $Y_P$ and $Y_{P'}$ constructed in §5. As above, we assume that Assumptions 1–3
are satisfied for the fibres at $\overline{\xi}$ and at every closed point $P$ of the set $U$.

The next theorem, Theorem 23, is the second result stated in the introduction.
It represents the main result in the paper.

**Theorem 23.** In the notation above, either $A_{\overline{\xi},0} = 0$, in which case $A_{P,0} = 0$, or
$A_{\overline{\xi},0} = A_{\overline{\xi},1}$, so that $A_{P,0} = A_{P,1}$ for any closed point $P$ in $U$.

**Proof.** For every closed point $P$ in $\mathbb{P}^m$ let $H_P$ be the corresponding hyperplane
in $\mathbb{P}^m$. Let $\Sigma$ be a Zariski closed subset in $\mathbb{P}^m$ such that for each point $P$ in
the complement to $\Sigma$ in $\mathbb{P}^m$ the hyperplane $H_P$ does not contain $X$ and the
scheme-theoretic intersection $X \cap H_P$ is either smooth or contains at most one
singular point, which is a double point (see Definition 1.4 and Proposition 1.5 in
Ch. III in [10], or read through Exposé XVII in [12]). Let $G$ be the Grassmannian
of lines in $\mathbb{P}^m$. There is a Zariski open subset $W$ in $G$ such that, for each line $D$
in $W$, the line $D$ does not intersect $\Sigma$ and the corresponding codimension 2 linear
subspace in $\mathbb{P}^m$ intersects $X$ transversally. In other words, any line $D$ from $W$ gives
rise to a Lefschetz pencil on $X$ (see Ch. III, §1, in [10] or Exposé XVII in [12]). Let $Z$
be the complement to the above $c$-open subset $U$ in $\mathbb{P}^m$. Then $Z$ is the union
of a countable collection of Zariski closed irreducible subsets in $\mathbb{P}^m$. In particular,$Z$ is $c$-closed. It follows that the condition for a line $D \in G$ not to be a subset
in $Z$ is it being $c$-open. By Lemma 10 the intersection of the corresponding $c$-open
subset in $G$ with $W$ is nonempty, so that we can choose a line $D$ such that $D$ gives
us a Lefschetz pencil $f_D : \mathcal{Y}_D \to D$ and $D \cap U \neq \emptyset$. Let $P_0$ be a point in $D \cap U$
and let $\overline{\eta}$ be the geometric generic point of $D$. By Proposition 22, either $A_{\overline{\eta},0} = 0$
or $A_{\overline{\eta},0} = A_{\overline{\eta},1}$. Suppose $A_{\overline{\eta},0} = 0$. Proposition 20 being applied to the pencil
$f_D$ gives us that $A_{P_0,0} = 0$. Applying the same proposition to the family $f_T$ we
obtain $A_{\xi,0} = 0$, and so for each closed point $P$ in $U$ the abelian variety $A_{P,0}$ is
zero. Similarly, if $A_{\overline{\eta},0} = A_{\overline{\eta},1}$ then, by Proposition 20 applied to $f_D$, we obtain
$A_{P_0,0} = A_{P_0,1}$. Applying Proposition 20 to the family $f$ we see that $A_{\xi,0} = A_{\xi,1}$
and $A_{P,0} = A_{P,1}$ for each closed point $P$ in $U$.

The theorem is proved.

§ 7. Applications to the study of 1-cycles on four-dimensional varieties

The purpose of this section is to apply Theorem 23 to the study of rational
equivalence of 1-dimensional algebraic cycles on 4-dimensional varieties, extending
Voisin’s idea in Ch. III, §10 in the second volume of [30]. We keep all the notation
and assumptions of the previous section. To enhance and, at the same time, simplify the exposition, we will assume that Assumptions 1–3 are satisfied not only for the fibres at closed points of the \( c \)-open subset \( U \subset T \), but rather for the fibres at all closed points of the set \( T \), that is, at all smooth sections of the variety \( X \) by hyperplanes in \( \mathbb{P}^m \). Assume, moreover, that \( p \leq 2 \) and that the group \( H_{\text{ét}}^{2p+1}(X, \mathbb{Q}_l) \) vanishes. The latter implies that \( A_{\eta,1} = A_{\eta}, A_{\xi,1} = A_\xi \) and \( A_{P,1} = A_P \) for each closed point \( P \) in \( T \).

For any closed point \( P \in \mathbb{P}^m \) let \( \tilde{Y}_P \) be the resolution of singularities of the section \( Y_P \). In addition to the assumptions above, we will also require that whenever the section \( Y_P \) has at worst one singular point and this point is an ordinary double point, the continuous group \( A_p(Y_P) \) is weakly representable.

Since the group \( A_p(Y_\xi) \) is weakly representable, we can choose a smooth projective curve \( C \) over \( _\xi \) and an appropriate algebraic cycle \( Z \) on \( C \times Y_\xi \) such that the induced homomorphism \( Z^* \) from \( A^1(C) \) to \( A^p(Y_\xi) \) is surjective. Then the homomorphism \( \theta^p_d \) from \( C^p_{d,d}(Y_\xi) \) to \( A^p(Y_\xi) \) is surjective for big enough \( d \) (see the proof of Theorem 13). Since the group \( A^p(Y_P) \) is weakly representable, whenever \( Y_P \) has at worst one singular point and this point is an ordinary double point, the homomorphism \( \theta^p_d \) from \( C^p_{d,d}(\tilde{Y}_P) \) to \( A^p(\tilde{Y}_P) \) is surjective as well.

It is important to stress that, as we are now assuming that Assumptions 1–3 are satisfied for the fibre at every closed point \( P \) of \( T \), accordingly the abelian varieties satisfy \( A_{P,0} \subset A_{P,1} \subset A_P \) for every closed point \( P \) of \( T \). However, it does not mean that we can extend the coherence provided by Proposition 20 from fibres at closed points of \( U \) to fibres at closed points of \( T \). Let \( T^\natural \) be the set of closed points in \( T \) such that \( P \in T^\natural \) if and only if \( A_{P,0} \) coincides with \( A_{P,1} \).

**Lemma 24.** The set \( T^\natural \) is constructible.

**Proof.** Let

\[ \mathcal{V} = \{(Z, P) \in C_{d,d}^{p+1}(X) \times \mathbb{P}^m \mid Z \subset H_P\} \]

be the incidence subvariety, where \( Z \subset H_P \) means that the codimension \( p+1 \) algebraic cycle \( Z \) of degree \( d \) on \( X \) is supported on the hyperplane section \( X \cap H_P \) for a closed point \( P \) in \( \mathbb{P}^m \). Let

\[ v_T : \mathcal{V}_T \to T \]

be the corresponding pull-back of the projection to \( \mathbb{P}^m \) with respect to the inclusion of \( T \) into \( \mathbb{P}^m \), and let

\[ s_T : \mathcal{V}_T \to C_{d,d}^{p+1}(X) \]

be the obvious morphism from \( \mathcal{V}_T \) to \( C_{d,d}^{p+1}(X) \). Let

\[ \mathcal{V}^2_T = \mathcal{V}_T \times_T \mathcal{V}_T \]

be the 2-fold fibred product of \( \mathcal{V}_T \) over \( T \), and the consider the corresponding morphisms

\[ v_T^2 : \mathcal{V}^2_T \to T \]
and

\[ s^2_T : \mathcal{V}^2_T \rightarrow C^{p+1}_{d,d}(X). \]

By Corollary 9 we see that \((\theta^{p+1}_d)^{-1}(0)\) is the union of a countable collection of irreducible Zariski closed subsets in \(C^{p+1}_{d,d}(X)\), say

\[ (\theta^{p+1}_d)^{-1}(0) = \bigcup_{i \in I} Z_i. \]

Let

\[ W_i = (s^2_T)^{-1}(Z_i) \]

for each \(i \in I\). For any closed point \(P\) in \(T\) the pre-image \((v^2_T)^{-1}(P)\) is the 2-fold product \(\mathcal{V}^2_P\) of the fibre \(\mathcal{V}_P\) of the morphism \(v_T\) at \(P\) over \(\text{Spec}(k)\). Since the homomorphism \(\theta^d_P\) from \(C^p_{d,d}(Y_P)\) to \(A^p(Y_P)\) is surjective, the condition \(r^*_P = 0\) is equivalent to the condition that the fibre \(\mathcal{V}^2_P\) of the morphism \(v^2_T\) at \(P\) is a subset of the pre-image \(\bigcup_{i \in I} W_i\) of \(0\) under the composition \(\theta^{p+1}_d \circ s^2_T\). By Lemma 10, this is equivalent to saying that \(\mathcal{V}^2_P\) is a subset in

\[ W_{i_1} \cup \cdots \cup W_{i_n} \]

for a finite collection of indices \(i_1, \ldots, i_n\) in \(I\). It follows that the set \(T^3\) is constructible.

The lemma is proved.

We need one more easy lemma about \(c\)-open sets over an uncountable field.

**Lemma 25.** Let \(V\) be an irreducible quasi-projective variety over \(k\), and let \(U\) be a nonempty \(c\)-open subset in \(V\). Then the Zariski closure of \(U\) in \(V\) is \(V\).

**Proof.** Indeed, since \(U\) is \(c\)-open, there exists a countable union \(Z = \bigcup_{i \in I} Z_i\) of Zariski closed irreducible subsets in \(V\) such that \(U = V \setminus Z\). Then \(U\) is nothing but the complement to the interior \(\text{Int}(Z)\) of the set \(Z\) in \(V\). Assume that \(\text{Int}(Z)\) is nonempty. Then there exists a nonempty subset \(W\) in \(\text{Int}(Z)\) which is Zariski open in \(V\). By Lemma 10 there exists an index \(i_0 \in I\) such that \(W\) is contained in \(Z_{i_0}\). This shows us that \(\text{Int}(Z_{i_0})\) of the set \(Z_{i_0}\) is nonempty. This is not possible as \(Z_{i_0}\) is a closed proper subset in a Zariski topological space.

The lemma is proved.

Now, Bloch’s definition of weak representability in [2] (see also [4]) can also be given for Chow groups with coefficients in \(\mathbb{Z}\) and with coefficients in \(\mathbb{Q}\). In the latter case we will speak about rational weak representability. Keeping the assumptions made in the beginning of this section, we can now prove the following theorem.

**Theorem 26.** If the group \(A^{p+1}(X)\) is not rationally weakly representable, then the kernel of the push-forward homomorphism from \(A^p(Y_P)\) to \(A^{p+1}(X)\) is countable, for a very general hyperplane section \(\mathcal{V}_P\).

**Proof.** By Theorem 23, either \(A_{\xi,0} = 0\) or \(A_{\xi,0} = A_{\xi}\). Suppose the latter. By the same Theorem 23, \(A_{P,0} = A_P\) for each closed point \(P\) in the \(c\)-open subset \(U\) in \(T\). On the other hand, \(U\) is a subset in \(T^3\), and the set \(T^3\) is constructible.
by Lemma 24. We represent $U$ as the complement to a countable union $\bigcup_{i \in I} D_i$ of irreducible Zariski closed subsets $D_i$ in $T$ and $T^a$ as a countable union $\bigcup_{j \in J} T^a_j$, where $T^a_j$ is Zariski open in an irreducible Zariski closed subset $Z_j$ in $T$. Let $Z$ be the union $\bigcup_{j \in J} Z_j$ and let $W$ be the complement to $Z$ in $T$. Then $W$ is $c$-open in $T$ and $W \cap U = \emptyset$. The intersection of $W$ and $U$ is the complement to the union of all the $D_i$ and $Z_j$ in $T$, $i \in I$ and $j \in J$. As $U \neq \emptyset$, it follows that $D_i \neq T$ for each index $i$. Since $W \cap U = \emptyset$, by Lemma 10 there must exist an index $j_0 \in J$ such that $Z_{j_0} = T$. This gives $A_{P, 0} = A_P$, that is, $r_{P, *} = 0$, for each closed point $P$ in the nonempty Zariski open subset $T^a_{j_0}$ in $T$.

By Lemma 25 the intersection of $T^a$ with $U$ is nonempty. Let

$$f_D : \mathcal{Y}_D \to D$$

be a Lefschetz pencil for $X$ such that the set-theoretic intersection of the line $D = \mathbb{P}^1$ with the set $T^a \cap U$ is nonempty. Since the group $A^{p+1}(Y_P)$ is weakly representable for each closed point $P \in T$ and $D$ passes through $U$, it follows that the group $A^{p+1}(Y_\eta)$ is weakly representable too. Let $\Gamma_\eta$ be a smooth projective curve and $Z_\eta$ an algebraic cycle of codimension 1 on $\Gamma_\eta \times Y_\eta$ over $\eta$ implementing the weak representability of $A^{p+1}(Y_\eta)$. Let

$$D' \to D$$

be a finite extension of the curve $D$ such that both $\Gamma_\eta$ and $Z_\eta$ are defined over the function field $k(D')$. Spreading out the curve $\Gamma_\eta$ and the cycle $Z_\eta$ into a relative curve $\mathcal{G} \to V'$ and a relative cycle $\mathcal{Z}$ on $\mathcal{G} \times V, \mathcal{Y}_V$, over the preimage $V'$ of a certain Zariski open subset $V$ in $D$ under the map $D' \to D$, we obtain a homomorphism

$$\mathcal{Z}_* : A^1(\mathcal{G}) \to A^{p+1}(\mathcal{Y}_{V'}).$$

Compactifying and resolving singularities, we obtain a surface $\mathcal{G}'$, a codimension 1 algebraic cycle $\mathcal{Z}'$ on the variety $\mathcal{G}' \times_{D'} \mathcal{Y}_{D'}$ and a homomorphism

$$\mathcal{Z}'_* : A^1(\mathcal{G}') \to A^{p+1}(\mathcal{Y}_{D'}).$$

Consider an arbitrary element $\alpha$ in the group $A^{p+1}(\mathcal{Y}_{D'})$. Let $\alpha'$ be its image in $A^{p+1}(Y_{D'})$, and let $\bar{\alpha}$ be the image of $\alpha'$ in $A^{p+1}(Y_\eta)$. Take a cycle class $\bar{\beta} \in A^1(\Gamma_\eta)$ which goes to $\bar{\alpha}$ under the surjective homomorphism $Z_{\eta,*}$ from $A^1(\Gamma_\eta)$ to $A^{p+1}(Y_\eta)$, and consider a finite extension

$$D'' \to D'$$

such that $\bar{\beta}$ is defined over the function field $k(D'')$. 
Let $\eta'$ and $\eta''$ be the generic points of $D'$ and $D''$, respectively, and consider the following commutative diagram

\[
\begin{array}{ccc}
A^1(\Gamma_{\eta''}) & \xrightarrow{Z_{\eta''}} & A^{p+1}(Y_{\eta''}) \\
\uparrow & & \uparrow \\
A^1(\Gamma_{\eta'}) & \xrightarrow{Z_{\eta'}} & A^{p+1}(Y_{\eta'}) \\
\uparrow & & \uparrow \\
A^1(\mathcal{G}') & \xrightarrow{\beta'} & A^{p+1}(\mathcal{Y}_{D'}) \\
\end{array}
\]

which illustrates what is going on.

The cycle class $\overline{\beta}$ comes from a cycle class $\beta'' \in A^1(\Gamma_{\eta''})$ under the pullback from $A^1(\Gamma_{\eta''})$ to $A^1(\Gamma_{\eta'})$. Let $\beta'$ be the image of $\beta''$ under the pushforward homomorphism from $A^1(\Gamma_{\eta''})$ to $A^1(\Gamma_{\eta'})$, and let $\beta$ be a cycle class in $A^1(\mathcal{G}')$ going to $\beta'$ under the surjective homomorphism from $A^1(\mathcal{G}')$ to $A^1(\Gamma_{\eta'})$. Let $\gamma$ ($\gamma'$ and $\gamma''$) be the image of the cycle class $\beta$ ($\beta'$ and $\beta''$, respectively) under the homomorphism $Z_\ast$ ($Z_{\eta''\ast}$ and $Z_{\eta'\ast}$, respectively). Without loss of generality we can assume that $\eta''/\eta'$ is Galois. Let $N$ be the corresponding norm on the group $A^{p+1}(Y_{\eta''})$. Since the kernel of the homomorphism from $A^{p+1}(Y_{\eta''})$ to $A^{p+1}(Y_{\eta})$ is torsion, there exists a positive integer $m$ such that

\[m(\gamma'' - \alpha'') = 0.\]

Then

\[mN(\gamma'') = mN(\alpha'') = mn\alpha',\]

where

\[n = [\eta'': \eta'].\]

It follows that

\[m\gamma - mn\alpha\]

belongs to the kernel of the homomorphism from $A^{p+1}(\mathcal{Y}_{D'})$ to $A^{p+1}(Y_{\eta})$.

We see that if

\[B_1 = \text{im}(A^1(\mathcal{G}') \xrightarrow{\beta'} A^{p+1}(\mathcal{Y}_{D'}))\]
and

\[ B_2 = \ker(A^{p+1}(\mathcal{Y}_{D'}) \to A^{p+1}(Y_{1/2})), \]

then the \( \mathbb{Q} \)-vector space \( A^{p+1}(\mathcal{Y}_{D'}) \otimes \mathbb{Q} \) is generated by the vector subspaces \( B_1 \otimes \mathbb{Q} \) and \( B_2 \otimes \mathbb{Q} \).

Now, as \( A^{p+1}(Y_{1/2}) \) is the colimit of the groups \( A^{p+1}(\mathcal{Y}_{W'}) \), where \( W' \) runs through Zariski open subsets in \( D' \), the localization sequences for open embeddings \( \mathcal{Y}_{W'} \subset \mathcal{Y}_{D'} \) show that kernel \( B_2 \) is generated by the image of the homomorphism

\[ \bigoplus_{P' \in D'} A^p(Y_{P'}) \to A^{p+1}(\mathcal{Y}_{D'}), \]

induced by the proper push-forward homomorphisms \( r_{P'} \). If \( D^k \) is the intersection of \( T^n \) and \( D \), and \( D^k \) is the pre-image of \( D^k \) under the finite map from \( D' \) onto \( D \), then \( r_{P'} = 0 \) for each closed point \( P' \) in \( D^k \). It follows that \( B_2 \) is generated by the image of the homomorphism

\[ \bigoplus_{P' \in D' \setminus D^k} A^p(Y_{P'}) \to A^{p+1}(\mathcal{Y}_{D'}). \]

Notice that the complement \( D' \setminus D^k \) is finite.

Next, if \( Y_{P'} \) is smooth, then \( Y_{P'} = Y_P \), where \( P \) is the image of \( P' \) under the finite map from \( D' \) onto \( D \), and the group \( A^p(Y_{P'}) \) is isomorphic to the abelian variety \( A_P \) via the universal regular homomorphism \( \psi_P \). In particular, \( A^p(Y_{P'}) \) is weakly representable. If the section \( Y_{P'} = Y_P \) is singular, resolving the double point on it we obtain a nonsingular variety \( \tilde{Y}_P \) whose group \( A^p(\tilde{Y}_P) \) is weakly representable by our assumption.

Thus, \( B_2 \) is covered by the finite direct product of weakly representable groups \( A^p(Y_P) \) and \( A^p(\tilde{Y}_P) \). This means that \( B_2 \) itself is weakly representable. Then, of course, \( B_2 \otimes \mathbb{Q} \) is rationally weakly representable. The image \( B_1 \) of the homomorphism \( \mathcal{Z}' \otimes \mathbb{Q} \) is weakly representable because \( A^1(\mathcal{Y}') \) is parametrized by the Picard variety of the surface \( \mathcal{Y}' \). Since \( A^{p+1}(\mathcal{Y}_{D'}) \otimes \mathbb{Q} \) is generated by the rationally weakly representable \( \mathbb{Q} \)-vector subspaces \( B_1 \) and \( B_2 \), the whole group \( A^{p+1}(\mathcal{Y}_{D'}) \) is rationally weakly representable. This contradicts the assumption of the theorem.

Hence, \( A_{2,0} = 0 \), and applying Theorem 23 completes the proof.

§ 8. Application to hyperplane sections of cubic hypersurfaces in \( \mathbb{P}^5 \)

First let \( X \) be a \( K3 \)-surface embedded appropriately into \( \mathbb{P}^m \). Since smooth hyperplane sections of a projective surface are smooth projective curves, Assumptions 1–3 are satisfied. It is also well known that the group \( A^2(X) \) is divisible (see Lemma 0.1.1 in [1]). The third cohomology of a \( K3 \)-surface vanishes, so the Albanese variety is trivial. By Roitman’s theorem (see [19]) the group \( A^2(X) \) is uniquely divisible. Then there is no difference between rational and integral (weak) representability for this group. Moreover, we know from Mumford’s result in [16] that \( A^2(X) \) is not representable. By Theorem 26, for a very general hyperplane section \( Y_P \) of the surface \( X \) the kernel of the push-forward homomorphism \( r_{P*} \) from \( A^p(Y_P) \) to \( A^{p+1}(X) \) is countable. This is, of course, a particular case of Proposition 2.4 in [31]. Another application of Theorem 26 is the following.
Corollary 27. Let $X$ be a smooth cubic hypersurface in $\mathbb{P}^5$, and let $Y$ be a very general hyperplane section of $X$. Then the kernel of the push-forward homomorphism from $A^2(Y)$ to $A^3(X)$ is countable.

Proof. For the cubic $X$ we have that $H^5_{\text{et}}(X, \mathbb{Q}_l) = 0$ and so $A_1 = A$ for $Y_\xi$ and every smooth section $Y_P$ of the fourfold $X$. Any such section is a smooth cubic 3-fold in $H_P \simeq \mathbb{P}^4$, whose group $A^3(Y_P)$ is well known to be representable by the corresponding Prymian variety $\text{Prym}(Y_P)$ (see [1]). Since the Prym construction is of purely algebraic-geometric nature, we can do it over $\xi$ getting the Prym variety $\text{Prym}(Y_\xi)$ for the geometric generic fibre $Y_\xi$. In other words, all the Assumptions 1–3 are satisfied for $Y_\xi$, as well as for all smooth hyperplane sections $Y_P$.

If a hyperplane section $Y_P$ of the cubic fourfold $X$ has one singular point and this point is an ordinary double point, then the singular cubic $Y_P$ is rational, so that $\text{e}_{Y_P}$ is rational. It follows that the group $A^3(Y_P)$ is weakly representable. If $Y_P$ is smooth, then it is unirational and so rationally connected. Hence $A^3(Y_P)$ is trivial. The group $A^3(X)$ is not weakly representable by Theorem 0.5 in [20]. Since, moreover, it is uniquely divisible (see Theorem 4.7, (iii) in [22]), it is not rationally weakly representable either. Thus, all the assumptions of Theorem 26 are also satisfied.

By Theorem 26, for each closed point $P$ in the $c$-open subset $U$ of $\mathbb{P}^{m\lor}$ there exists a countable set $\Xi_P$ of closed points in the Prymian $\text{Prym}(Y_P)$ of the hyperplane section $Y_P$ such that the kernel of the homomorphism $r_P^*$ from $\text{Prym}(Y_P)$ to $A^3(X)$ is countable.

The corollary is proved.

In particular, if $\Sigma$ and $\Sigma'$ are two linear combinations of lines on $X$, supported on $Y_P$, then the cycle $\Sigma$ is rationally equivalent to the cycle $\Sigma'$ on $X$ if and only if the point on $\text{Prym}(Y_P)$ represented by the class of $\Sigma - \Sigma'$ occurs in the set $\Xi_P$.

Notice also that the group $A^3(Y_P)$ may be nonzero, but we know that it is a torsion group. Since $A^2(Y_P)$ is divisible, any cycle class in $A^3(X)$ is represented, up to torsion, by line configurations supported on hyperplane sections.

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