A free boundary problem with non local interaction

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Abstract

We prove local existence for classical solutions of a free boundary problem which arises in one of the biological selection models proposed by Brunet and Derrida, [2] and Durrett and Remenik, [15]. The problem we consider describes the limit evolution of branching brownian particles on the line with death of the leftmost particle at each creation time as studied in [13]. We use extensively results in [5] and [16].

1 Introduction

In [2] Brunet and Derrida have proposed several models to study selection mechanisms in biological systems which give rise to very interesting questions not only in the applications to biology but also in the areas of stochastic particle systems and of PDE’s with free boundaries. This paper concerns mostly the last issue but it is worth, we think, to give first a more general overview.

In the line of the Brunet-Derrida’s proposal Durrett and Remenik in [15] have introduced and studied a model of particles on \( \mathbb{R} \) which independently at rate 1 creates a new particle whose position is chosen randomly with probability \( p(x, y)dy, p(x, y) = p(0, y - x) \), if \( x \) is the position of the generating particle. Instantaneously after the creation the leftmost particle is deleted so that the total number of particles is constant.

The biological interpretation is that the position of a particle is “its degree of fitness”, the larger the position the higher the fitness. The removal of the leftmost (and hence less fitted) particle gives rise to an improvement of the general fitness of the population and in fact Durrett and Remenik have proved the existence of traveling fronts moving with positive velocity.

The main difficulty in the analysis of the model is the apparently simple deleting mechanism of killing the leftmost particle. In fact the notion of leftmost particle is highly non local: one needs to know the positions of all the particles to determine which is the leftmost one. This is therefore a “topological” interaction which cannot be treated with the usual methods of interacting particle systems, it is the analogue in PDE’s of free boundary problems when the domain where the PDE’s are defined is itself one of the unknowns, see

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for instance the survey by Carinci, De Masi, Giardinì and Presutti, [7], on topological interactions and their relation in the “hydrodynamic limit” with free boundary problems.

In the biological applications the size of the population is very large and therefore the main interest is in the analysis of the asymptotic behavior of the particle system in the continuum limit when $N$, (i.e. the total number of particles) diverges. Under suitable assumptions on the initial datum Durrett and Remenik have proved that as $N \to \infty$ a limit density exists and it satisfies:

$$
\frac{\partial}{\partial t} \rho(x,t) = \int_{X_t}^{\infty} dy \ p(y,x) \rho(y,t)dy, \quad \rho(x,0) = \rho_0(x) \quad (1.1)
$$

where $X_t = \inf \{ r : \rho(r,t) > 0 \}$. Notice that the domain of integration on the right hand side of (1.1) is also an unknown since one needs to know the whole function $\rho(x,t)$ to determine the value $X_t$ of “the edge”.

As it stands (1.1) does not select $\rho(x,t)$ because we can give “arbitrarily” $X_t$ and still solve (1.1). To get uniqueness we would need to know a priori $X_t$ which should be the limit position (as $N \to \infty$) of the leftmost particle in the system. This is in itself an interesting issue but apparently very difficult to carry out. Durrett and Remenik have circumvented the difficulty by using the other information coming from the particle system, namely that the total number of particles is conserved. In the continuum limit where $N \to \infty$ the above is reflected into the condition that

$$
\int_{X_t}^{\infty} dx \ \rho(x,t) = 1, \text{ for all } t \geq 0 \quad (1.2)
$$

The pair (1.1)–(1.2) is a “free boundary problem” but not in its more usual formulation where (1.1) is usually replaced by a parabolic diffusion equation and instead of (1.2) there is a condition relating the velocity of the edge to the spatial derivative of the solution at the edge. This is indeed what happens in the classical Stefan problem, see for instance the survey by Fasano, [16].

Under suitable assumptions on the initial datum $\rho_0$ and on the probability kernel $p(x,y)$ Durrett and Remenik have been able to prove that the pair (1.1)–(1.2) has a unique solution which is the limit density of the particles system.

An important ingredient in the proof is that $X_t$ is monotonically non decreasing, a feature is clear at the particles level where in fact the position of the leftmost particle if it moves can only increase: it stays put when the new particle is created to its left (because then this is the one which is deleted) while, in the other case, the previous second leftmost particle becomes the leftmost one. Such a simplifying effect is not present in the next models we are going to discuss.

In [12] De Masi, Ferrari, Presutti and Soprano-Loto (in the sequel DFPS for brevity), have studied the so called N-BBM model, which is an acronym for $N$ branching Brownian motions. The selection mechanism in the N-BBM model is similar to the Durrett-Remenik one: once a new particle is created the leftmost one is deleted. There are however two main differences: the particles move as independent Brownian motions and the new particle is created at exactly the same position of the generating one (the previous kernel $p(x,y)$ becomes a Dirac delta, $\delta(x - y)$). Biologically this means that the individual fitness changes randomly in time and the duplicating processes are exact, the fitness of the son is exactly equal to that of the father.
Believing that the Durrett-Remenik arguments extend to this case one would conjecture that the limit density \( \rho(x, t) \) satisfies the equation

\[
\frac{\partial}{\partial t} \rho(x, t) = 1 - \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho(x, t) + \rho(x, t), \quad x > X_t, \quad \rho(x, 0) = \rho_0(x)
\]  

(1.3)

(where again \( X_t = \inf\{x : \rho(x, t) > 0\} \)). (1.3) is in fact obtained from (1.1) by adding on the right hand side the Laplacian which takes into account the Brownian motion of the particles while the last term \( \rho(x, t) \) is the right hand side of (1.1) when \( p(x, y) = \delta(x - y) \).

The free boundary problem (1.3)–(1.2) is “incomplete” because even if \( X_t \) is known yet (1.3) does not have a unique solution: we must also give the value of \( \rho(x, t) \) at the edge \( X_t \).

The natural choice would be to derive it as the limit particles density at the edge which is still not at all easy due to the poor control of the position of the leftmost particle. However taking into account the regularizing effect of the heat diffusion one may suppose that

\[
\rho(X_t, t) = 0 \quad \text{at all times } t \geq 0
\]  

(1.4)

(notice that in the Durrett-Remenik model (1.4) does not hold, recall however that in (1.1) there is no Laplacian!).

DFPS have proved (under suitable assumptions on the initial datum) that in the limit \( N \to \infty \) the particle density has a limit \( \rho(x, t) \) for any \( t \geq 0 \). It is also proved that \( \rho(x, t) \) satisfies (1.3)–(1.2)–(1.4) if this has a “regular” solution. As far as we know there is only a “local” existence theorem under suitable assumptions on the initial datum (as discussed in the next section) which therefore coincides with the limit density of the N-BBM system.

From [12] we know that \( \rho(x, t) \) is well defined at all times, but it is not clear if at times larger than for local existence it is still a solution of (1.3)–(1.2)–(1.4) at least in a “weak sense”.

Notice that uniqueness in the local existence theorem follows from [12] as DFPS have shown that any “smooth solution” is necessarily equal to the limit density of the particles system and hence unique.

The question of traveling fronts is of great interest: in [1] Berestycki, Brunet and Derrida have considered (1.2) complemented by conditions on the values of the solution and its derivative at the edge. They were mainly interested in the precise asymptotics of the velocity of the front underlying connections with the Fisher-KPP type fronts, see also [19] where Groisman and Jonckheere discuss front propagation and quasi-stationary distributions.

The analysis of the front before the limit \( N \to \infty \) is also particularly interesting, see for instance the work of Maillard, [22] on its large fluctuations.

We are mainly interested here on the existence of solutions for a free boundary problem introduced in [13]. The particles system is an extension of the N-BBM model obtained by making the branching mechanism non local as in the case considered by Durrett and Remenik. In [13] the conjectured evolution equation is in fact

\[
\frac{\partial}{\partial t} \rho(x, t) = 1 - \frac{1}{2} \frac{\partial^2}{\partial x^2} \rho(x, t) + \int_{X_t}^{\infty} dy \, p(y, x) \rho(y, t) dy, \quad x > X_t, \quad \rho(x, 0) = \rho_0(x)
\]  

(1.5)

which is a combination of (1.1) (for the branching) and (1.3) for the Brownian diffusion. The results for the N-BBM model have been extended in [13] to this case, a limiting
density $\rho(x,t)$ exists and it is uniquely defined, moreover if there is a smooth solution of the free boundary problem (1.5)–(1.2)–(1.4) then this is the limit particles density of the model. As mentioned the proof of local existence of smooth solutions for (1.5)–(1.2)–(1.4) is the main result in this paper, the precise statement is as follows.

Assumptions.

- **On the initial datum.** We suppose that: $\rho_0(x) = 0$ for $x \leq 0$, it is in $C^3$ for $x \geq 0$ and it has compact support. Moreover
  \[
  \frac{d}{dx} \rho_0(x) \bigg|_{x=0^+} = 2 \int_0^\infty \int_0^\infty \rho_0(y)p(y,x)dydx > 0 \quad (1.6)
  \]
- **On the kernel $p(x,y)$.** We suppose that: $p(x,y) = p(0,y-x)$, $p(0,x)$ is non negative with compact support, it is in $C^1$ and its integral is equal to 1 (i.e. $p(x,y)$ is a transition probability kernel).

Remarks. By the first assumption $X_0 = 0$: by translation invariance there is no loss in generality by fixing the edge initially at 0. The regularity assumption on $\rho_0$ comes from the necessity of controlling the velocity of the edge which involves, as we will see, the second derivative of $\rho(x,t)$ with respect to $x$. Finally the “strange condition (1.6)” is required to avoid initial layer problems as discussed in the next section.

**Theorem 1** Under the above assumptions there are $T > 0$, $X_t, t \in [0,T]$, and $\rho(x,t)$, $x \geq X_t, t \in [0,T]$, such that:

- $X_0 = 0$, $X_t$ is differentiable and its derivative $V_t$ is Hölder continuous with exponent $1/2$.
- $\rho(x,t)$ is $C^{3,1}$ (three derivatives in $x$ and one in $t$) in the domain $x > X_t, t > 0$.
- The pair $(X_t, \rho(x,t))$, $x \geq X_t, t \in [0,T]$, solves the free boundary problem (1.5)–(1.2)–(1.4).

In the next section we outline the strategy of the proof and discuss what known in the literature. In Section 5 we give the proof of Theorem 1 and in a last section we discuss the extension to the other free boundary problems mentioned in this introduction.

## 2 Strategy of proof

We suppose tacitly hereafter that the initial position of the edge is $X_0 = 0$, then a simple version of the classical Stefan free boundary problem is

\[
\rho_t = \frac{1}{2} \rho_{xx}, \quad x \geq X_t, t \geq 0 \quad (2.7)
\]
\begin{align}
\rho(x,0) &= \rho_0(x), \quad \rho(X_t,t) = 0 \quad (2.8) \\
\frac{dX_t}{dt} &= -\rho_x(X_t,t) \quad (2.9)
\end{align}

where $X_t$ and $\rho(x,t)$ are the unknowns, to simplify notation space and time derivatives are denoted hereafter by adding suffices.

The classical strategy for solving such a free boundary problem is to fix a curve $X_t$, solve (2.7)–(2.8), call $\rho(x,t)$ such a solution. Define then a new curve $\tilde{X}_t$ starting at 0 by setting its velocity $\tilde{V}_t = -\rho_x(X_t,t)$: this defines a map $\psi$: $X_t \rightarrow \tilde{X}_t$ and we look for a fixed point of $\psi$. In the above case the best is to work on the compact space of uniformly Lipschitz curves $X_t$: one can then prove that for small $t$ the map $\psi$ is a contraction and then the existence of a fixed point $X_t$ follows. To conclude one must then show that the solution of (2.7)–(2.8) with $X_t$ the fixed point satisfies (2.9) as well. See for instance [16].

The N-BBM problem looks similar. The differences are: (i) in (2.7) there is an additional term on the right hand side, (ii) there is the additional constraint that the mass is conserved, (1.2); (iii) we miss the condition (2.9). (i) can be dealt with by changing variables: $\rho(x,t) \rightarrow w(x,t) := e^{-t}\rho(x,t)$. Conservation of mass can be written in differential form and a relation for the velocity of the front can be obtained by differentiating with respect to time (1.4). One then obtains the system of equations:

\begin{align}
w_t &= \frac{1}{2}w_{xx}, \quad x \geq X_t, t \geq 0 \quad (2.10) \\
w(x,0) &= \rho_0(x), \quad w(X_t,t) = 0 \quad (2.11) \\
w_x(X_t,t) &= 2e^{-t} \quad (2.12) \\
\frac{dX_t}{dt} &= -\frac{1}{4}e^t w_{xx}(X_t,t) \quad (2.13)
\end{align}

Define next $u(x,t) := w_x(x,t)$ and ignore (2.11), we then get

\begin{align}
u_t &= \frac{1}{2}u_{xx}, \quad x \geq X_t, t \geq 0 \quad (2.14) \\
u(X_t,t) &= 2e^{-t} \quad (2.15) \\
\frac{dX_t}{dt} &= -\frac{1}{4}e^t u_x(X_t,t) \quad (2.16)
\end{align}

The free boundary problem for $(X_t,u(x,t))$ looks like the classical Stefan problem (2.7)–(2.8)–(2.9): it requires an additional analysis which is contained in [17] (for a more general system of equations), see also [20] where the above case is treated explicitly.

In the case we are mainly interested here there is a non local term and this prevents us to use at least directly the above approach. An alternative way to study the free boundary problems is to look at the evolution from the edge. This is often done at the particles level to study the shape and structure of the traveling waves independently of their location. A derivation of the hydrodynamic equations for the N-BBM model and its non local variant is however missing. The advantage of studying the free boundary problem in the frame where the edge is always at the origin is that by its very definition the spatial domain is fixed, it is no longer one of unknowns. The difficulties however have not disappeared as in the evolution equations appears a drift term which depends on the velocity of the edge. The natural setting from the problem requires now that the motion of the edge is
(we will also require that the derivative \( V_t \) is Hölder continuous with exponent 1/2). More precisely we call \( u(x,t) = \rho_x(x,t) \) and then change variables: \( \rho(x,t) \to \rho(x - X_t, t) \), \( u(x,t) \to u(x - X_t, t) \). By an abuse of notation we denote by the same symbols \( \rho \) and \( u \) the new functions and we get the following system of equations:

\[
\rho_t(x,t) = \frac{1}{2} \rho_{xx}(x,t) + V_t \rho_x(x,t) + \int_0^\infty dy p(y,x) \rho(y,t) dy, \quad x > 0 \tag{2.17}
\]

\[
\rho(x,0) = \rho_0(x), \quad x \geq 0, \quad \rho(0,t) = 0, \quad t \geq 0 \tag{2.18}
\]

\[
u_t(x,t) = \frac{1}{2} u_{xx}(x,t) + V_t u_x(x,t) + \int_0^\infty dy p(y,x) u(y,t) dy, \quad x > 0 \tag{2.19}
\]

\[
u(x,0) = u_0(x) := \frac{d\rho_0(x)}{dx} \quad \text{for} \quad x \geq 0 \tag{2.20}
\]

\[
u(0,t) = 2 \int_0^\infty \int_0^\infty dy \rho(y,t) p(y,x) dx \tag{2.21}
\]

We also have:

\[
u(0,t) V_t = -\frac{1}{2} \nu_x(0,t) + \int_0^\infty \int_0^\infty dy \nu(y,t) p(y,x) dy dx \tag{2.22}
\]

which is obtained by differentiating (1.4) with respect to time.

In the way the above equations have been derived \( u \) is the spatial derivative of \( \rho \), but we will regard the system (2.17) to (2.21) without imposing such relation. Namely we fix a function \( V_t \) which is Hölder continuous with exponent 1/2; we then solve (2.17)-(2.18) and find \( \rho(x,t) \). We then solve (2.19)-(2.21) with \( \rho(x,t) \) as determined above and thus get \( u(x,t) \). With such \( \rho(x,t) \) and \( u(x,t) \) we determine a new speed \( V_t \) via (2.22) and thus get an iterative scheme. We will prove that all this can be done and the iterative scheme has a fixed point. For such fixed point we re-establish the identity that \( u \) is the spatial derivative of \( \rho \) and then get a proof of Theorem 1.

The change of variables which fixes the position of the edge has been used in [18]. Our approach is similar but we have extra difficulties for the presence of the non local term. Moreover, [18] relies on the result of [21] which does not include our case since its initial and boundary conditions are stronger. As one of our analysis we give an improved estimate in Lemma 3 than (4.24) of [18].

3 Main results

Let us denote by \( K \) a Gaussian function and \( G \) a Green function as

\[
K(x,t;\xi,\tau) = \frac{1}{\sqrt{2\pi(t-\tau)}} \exp \left\{ -\frac{|x-\xi|^2}{2(t-\tau)} \right\}, \quad G(x,t;\xi,\tau) = K(x,t;\xi,\tau) - K(x,t;-\xi,\tau).
\]

From now on, we write independent positive constants as \( \{c_i\}_{i \geq 1} \) and use the following facts extensively

\[
\int_{-\infty}^{\infty} K(x,t;\xi,\tau) dx = 1, \quad \int_{-\infty}^{\infty} |K(x,t;\xi,\tau)| dx = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sqrt{t-\tau}}.
\]
Theorem 2

Let \( X \in C^1([0,T]) \) where \( T > 0 \) and \( X_0 = 0 \). There is a unique solution \( \rho \in C \bar{D_T} \) which satisfies (2.17)-(2.18) where \( D_T = \{(x,t) : 0 < x, 0 < t \leq T\} \) with \( \rho_x \in C \bar{D_T} \) and \( \|\rho\|_\infty + \|\rho_x\|_\infty < \infty \).

Proof

Similarly as Theorem 20.3.1 of [5], let us define a mapping \( F : \mathcal{B}_\eta \rightarrow \mathcal{B}_\eta \), where \( \mathcal{B}_\eta = \{\rho(x,t) \in C((0,\infty) \times [0,\eta]) : \rho_x \in C((0,\infty) \times [0,\eta]), \|\rho\|_\infty + \|\rho_x\|_\infty < \infty\} \) as

\[
F \rho(x,t) := \int_0^\infty G(x,t;\xi,0)\rho_0(\xi)d\xi + \int_0^t \int_0^\infty G(x,t;\xi,\tau) \left[ V_\tau \rho(\xi,\tau) + \int_0^\infty dy p(y,\tau)p(y,\xi) \right] d\xi d\tau.
\]

Then \( \mathcal{B}_\eta \) is a Banach space with the norm \( \|\cdot\|_\infty + \|\partial_x(\cdot)\|_\infty \) and we have

\[
\|F \rho_1 - F \rho_2\|_\infty \leq 2\eta \|V\|_\infty \|\rho_1 - \rho_2\|_\infty + \|\rho_1 - \rho_2\|_\infty, \tag{3.25}
\]

and

\[
\left\| \frac{\partial F \rho_1}{\partial x} - \frac{\partial F \rho_2}{\partial x} \right\|_\infty \leq c_1 \sqrt{\eta} \|V\|_\infty \|\rho_1 - \rho_2\|_\infty + \|\rho_1 - \rho_2\|_\infty. \tag{3.26}
\]

Thus for all sufficiently small \( \eta > 0 \), we get \( F \) is a contraction mapping so that there is a unique fixed point \( \rho^0 \). To extend \( \eta \) to \( T \), if we define \( \mathcal{H} \) as

\[
\mathcal{H} \rho(x,t) := \int_0^\infty G(x,t;\xi,0)\rho_0(\xi)d\xi + \int_\eta^T \int_0^\infty G(x,t;\xi,\tau) \left[ V_\tau \rho(\xi,\tau) + \int_0^\infty dy p(y,\tau)p(y,\xi) \right] d\xi d\tau.
\]

Then we have

\[
\|\mathcal{H} \rho_1 - \mathcal{H} \rho_2\|_\infty \leq 2(T-\eta) \|V\|_\infty \|\rho_1 - \rho_2\|_\infty + \|\rho_1 - \rho_2\|_\infty, \tag{3.26}
\]

and

\[
\left\| \frac{\partial \mathcal{H} \rho_1}{\partial x} - \frac{\partial \mathcal{H} \rho_2}{\partial x} \right\|_\infty \leq c_1 \sqrt{t-\eta} \|V\|_\infty \|\rho_1 - \rho_2\|_\infty + \|\rho_1 - \rho_2\|_\infty, \tag{3.27}
\]

such that we have \( \mathcal{H} \) is a contraction mapping from \( \{\rho(x,t) \in C([0,\infty) \times [\eta,2\eta]) : \rho_x \in C((0,\infty) \times [\eta,2\eta]), \|\rho\|_\infty + \|\rho_x\|_\infty < \infty\} \) to itself. Thus we can extend \( \eta \) to \( 2\eta \), inductively also to \( T \) and this completes the proof.

We will prove the existence of a classical solution of the FBP (2.19)-(2.22) in \([0,T]\) in the next sections:

Theorem 2

There is \( T > 0 \) and a pair \((X,u)\) which satisfies the FBP (2.19)-(2.22) in \([0,T]\) with \( X \in C^1([0,T]) \), \( X_0 = 0 \), and \( u \in C(D_T) \cap C^2(D_T) \), where \( D_T = \{(x,t) : 0 < x, 0 < t \leq T\} \).

We did not find a proof of Theorem 2 in the existing literature, see for instance [4], [9] and references therein. Our proof exploits the one dimensionality of the problem and uses extensively the Cannon estimates, [5], following the strategy proposed by Fasano in [6]. After then we will prove Theorem 1 as a corollary of Theorem 2.
4 Proof of Theorem 2

Theorem 2 is proved at the end of the section. The idea is to reduce the analysis of the FBP (2.19)-(2.22) to a fixed point problem:

- Take a curve \( V_t, t \in [0,T] \), let \( X_t = \int_0^t V_x d\tau \), and find \( u \) such that \((X, u)\) solves (2.19)-(2.21).
- Construct a new curve \( Q[V](t) = \frac{-\frac{1}{2}u_x(0,t) + \int_0^\infty \int_0^\infty u(y, t) p(y,x) dydx}{u(0,t)} \) for \( 0 \leq t \leq T \).
- Find \( V \) so that \( Q[V] = V \) and prove that the corresponding pair \((X, u)\) solves the FBP (2.19)-(2.22).

The first task is to prove existence and smoothness of \( u \), that we do in this section using the lemmas below, see [5]. Let \( A > 0 \) and

\[
\Sigma(A, T) := \{ V \in C([0,T]) : V_0 = \frac{-\frac{1}{2} \rho_0''(0) + \int_0^\infty \int_0^\infty dy \rho_0'(y) p(y,x) dx}{\rho_0'(0)} \text{, } |V|_1^1 \leq A \},
\]

where \(| \cdot |_1^1\) is the H"older seminorm with exponent \( \frac{1}{2} \).

**Proposition 2** Let \( X \in C^1([0,T]) \) where \( T > 0 \) and \( X_0 = 0 \). There is a unique solution \( u \in C(D_T) \) which satisfies (2.19)-(2.22) where \( D_T = \{(x,t) : 0 < x, 0 < t \leq T \} \) with \( u_x \in C(D_T) \) and \( \|u\|_{\infty} + \|u_x\|_{\infty} < \infty \).

**Proof** Let \( \rho \) as in Proposition 1 and let us define a mapping \( \mathcal{F} : \mathcal{B}_\eta \to \mathcal{B}_\eta \) where \( \mathcal{B}_\eta = \{ u(x,t) \in C([0,\infty) \times [0,\eta]) : u_x \in C([0,\infty) \times [0,\eta]) \text{, } \|u\|_{\infty} + \|u_x\|_{\infty} < \infty \} \) as

\[
\mathcal{F}u(x, t) := - \int_0^t K_x(x, t; \tau) g(\tau) d\tau + \int_0^\infty G(x, t; \xi, 0) \varphi(\xi) d\xi + \int_0^t \int_0^\infty G(x, t; \xi, \tau) \left[ V_x u_x (\xi, \tau) + \int_0^\infty dy u(y, \tau) p(y, \xi) \right] d\xi d\tau,
\]

where

\[
u(0, t) = 2 \int_0^\infty \int_0^\infty dy \rho(y, t) p(y, x) dx =: g(t), \ \varphi(\xi) := \rho_0'(\xi).
\]

Then we can show that \( \mathcal{F} \) is a contraction mapping for all sufficiently small \( \eta > 0 \) and extend \( \eta \) to \( T \) as same as the proof of Proposition 1 so that \( \mathcal{F} \) has a unique fixed point \( u \in C(D_T) \). This completes the proof.

Thus we can write \( u \) as

\[
u(x, t) = - \int_0^t K_x(x, t; \tau) g(\tau) d\tau + \int_0^\infty G(x, t; \xi, 0) \varphi(\xi) d\xi + \int_0^t \int_0^\infty G(x, t; \xi, \tau) \left[ V_x u_x (\xi, \tau) + \int_0^\infty dy u(y, \tau) p(y, \xi) \right] d\xi d\tau \tag{4.28}
\]
where
\[ u(0, t) = 2 \int_0^\infty \int_0^\infty dy \rho(y, t)p(y, x)dx =: g(t), \quad \varphi(\xi) := \rho'_0(\xi). \]

In addition, by differentiating \(4.28\) and integration by parts, we have
\[
\begin{align*}
    u_x(x, t) &= -2 \int_0^t K(x, t; 0, \tau)g'(\tau)d\tau + \int_0^\infty [K(x, t; \xi, 0) + K(x, t; -\xi, 0)]\varphi'(\xi)d\xi \\
    &\quad + \int_0^t \int_0^\infty G_x(x, t; \xi, \tau) \left[V_\tau u_\xi(\xi, \tau) + \int_0^\infty dyu(\tau, \tau)p(y, \xi)\right]d\xi d\tau. \quad (4.29)
\end{align*}
\]

We introduce the following lemma from \([5]\) which plays an essential role in our analysis.

**Lemma 1** Let \(\phi(t)\) satisfy
\[
0 \leq \phi(t) \leq \psi(t) + C_1 \int_0^t \frac{\phi(\tau)}{\sqrt{T - \tau}}d\tau, \quad 0 \leq t \leq T, \quad (4.30)
\]
where \(C_1 \geq 0\) and \(\psi(t)\) is nonnegative and nondecreasing. Then
\[
0 \leq \phi(t) \leq [1 + 2C_1\sqrt{T}]\psi(t)\exp\{\pi C_1^2 t\} \leq C_2\psi(t) \quad (4.31)
\]
with
\[
C_2 = [1 + 2C_1\sqrt{T}]\exp\{\pi C_1^2 T\}. \quad (4.32)
\]

**Proof** See Lemma 17.7.1 of \([5]\). \(\bullet\)

Let us denote by \(\mathcal{S}\) a collection of continuous functions \(C : [0, \infty) \times [0, \infty) \to [0, \infty)\) such that for each \(x \in [0, \infty)\), \(C(x, \cdot)\) is increasing with respect to the second variable and \(C(x, 0) > 0\) is independent of the first variable.

**Lemma 2** Let \(V \in \Sigma(A,T)\) and let \(X_t = \int_0^t V_\tau d\tau\). For \((X, \rho)\) as in Proposition \([7]\) and \((X, u)\) as in Proposition \([2]\) then \(\|\rho\|_\infty \leq C_1(A, T), \|\rho_x\|_\infty \leq C_2(A, T), \sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^1} \leq C_3(A, T), \|u\|_\infty \leq C_4(A, T), \|u_x\|_\infty \leq C_5(A, T), \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^1} \leq C_6(A, T)\), where \(C_i \in \mathcal{S}\) for all \(1 \leq i \leq 6\).

**Proof** We can write \(\rho\) as
\[
\begin{align*}
    \rho(x, t) &= \int_0^\infty G(x, t; \xi, 0)\rho_0(\xi)d\xi \\
    &\quad + \int_0^t \int_0^\infty G(x, t; \xi, \tau) \left[V_\tau \rho_\xi(\xi, \tau) + \int_0^\infty dy(p(y, \tau)p(y, \xi)\right]d\xi d\tau \\
    &= \int_0^\infty G(x, t; \xi, 0)\rho_0(\xi)d\xi - \int_0^t \int_0^\infty G_\xi(x, t; \xi, \tau)V_\tau \rho(\xi, \tau)d\xi d\tau \\
    &\quad - \int_0^t \int_0^\infty G(x, t; \xi, \tau) \int_0^\infty dy(p(y, \tau)p(y, \xi)\right]d\xi d\tau. \quad (4.33)
\end{align*}
\]
Then we have
\[
\sup_{x \geq 0} |\rho(x, t)| \leq 2\|\rho_0\|_\infty + c_1\|V\|_\infty \int_0^t \frac{\sup_{\xi \geq 0} |\rho(\xi, \tau)|}{\sqrt{t - \tau}} d\tau \\
+ c_2 \int_0^t \sup_{y \geq 0} |\rho(y, \tau)| d\tau. \quad (4.34)
\]
Applying Lemma 1 on (4.34), we have
\[
\sup_{x \geq 0} |\rho(x, t)| \leq \left(1 + 2c_1\|V\|_\infty \sqrt{T}\right) \exp \left\{\pi c_1^2\|V\|_\infty^2 T\right\} \left(2\|\rho_0\|_\infty + c_2 \int_0^t \sup_{y \geq 0} |\rho(y, \tau)| d\tau\right). 
\]
By Gronwall’s lemma and \(\|V\|_\infty \leq |V_0| + A\sqrt{T}\), we obtain for some \(C_1 \in S\),
\[
\|\rho\|_\infty \leq C_1(A, T). 
\]
Also we can write \(\rho_x\) as
\[
\rho_x(x, t) = \int_0^\infty [K(x, t; \xi, 0) + K(x, t; -\xi, 0)]\varphi(\xi)d\xi \\
+ \int_0^t \int_0^\infty G_x(x, t; \xi, \tau) \left[V_\tau \rho_\xi(\xi, \tau) + \int_0^\infty dy \rho(\xi, \tau) p(\xi, y)\right] d\xi d\tau. \quad (4.36)
\]
Then we have
\[
\sup_{x \geq 0} |\rho_x(x, t)| \leq 2\|\varphi\|_\infty + c_3\|V\|_\infty \int_0^t \frac{\sup_{\xi \geq 0} |\rho_\xi(\xi, \tau)|}{\sqrt{t - \tau}} d\tau + c_4\sqrt{T}\|\rho\|_\infty. \quad (4.37)
\]
Using (4.35) and Lemma 1, we also obtain for some \(C_2 \in S\),
\[
\|\rho_x\|_\infty \leq C_2(A, T). \quad (4.38)
\]
Taking absolute value on both sides of (4.33) and integrating them with respect to \(x\), we have
\[
\|\rho(\cdot, t)\|_{L^1} \leq 2 + c_5\|V\|_\infty \int_0^t \frac{\|\rho(\cdot, \tau)\|_{L^1}}{\sqrt{t - \tau}} d\tau + 2 \int_0^t \|\rho(\cdot, \tau)\|_{L^1} d\tau. \quad (4.39)
\]
Similarly, we get for some \(C_3 \in S\),
\[
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^1} \leq C_3(A, T). \quad (4.40)
\]
Using (4.28) and integration by parts, we have
\[
\sup_{x \geq 0} |u(x, t)| \leq \|g\|_\infty + 2\|\varphi\|_\infty + c_6\sqrt{T}\|V\|_\infty \|g\|_\infty \\
+ c_7\|V\|_\infty \int_0^t \frac{\sup_{\xi \geq 0} |u(\xi, \tau)|}{\sqrt{t - \tau}} d\tau + c_8 \int_0^t \sup_{y \geq 0} |u(y, \tau)| d\tau. \quad (4.41)
\]
Using
\[
\|g\|_\infty \leq 2 \sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^1} \leq 2C_3(A, T) \quad (4.42)
\]
and we apply both Gronwall’s lemma and Lemma 1 on (4.41), we obtain for some \( C_4 \in \mathcal{S} \),
\[
\|u\|_{\infty} \leq C_4(A, T). \tag{4.43}
\]
By integration by parts, we have
\[
|g'(t)| \leq c_0|\rho_x(0, t)| + (c_{10}|V_t| + c_{11})\|\rho(\cdot, t)\|_{L^1}. \tag{4.44}
\]
Using (4.44), (1.29) and similar arguments as above, we finally get for some \( C_5, C_6 \in \mathcal{S} \),
\[
\|u_x\|_{\infty} \leq C_5(A, T), \quad \sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^1} \leq C_6(A, T). \tag{4.45}
\]

**Lemma 3** Let \( V \in \Sigma(A, T) \) and let \( X_t = \int_0^t V_\tau d\tau \). For \((X, u)\) as in Proposition 4 then there exists \( u_{xx} \in C(\overline{D_T} \setminus 0) \) such that \( \|u_{xx}\|_{\infty} \leq C_7(A, T) \) for some \( C_7 \in \mathcal{S} \).

**Proof** By differentiating (1.29) with respect to spatial variable, let us define a mapping \( \mathcal{F} : \mathcal{B}_\eta \rightarrow \mathcal{B}_\eta \) where \( \mathcal{B}_\eta = \{v(x, t) \in C([0, \infty) \times [0, \eta] \setminus 0) : \|v\|_{\infty} < \infty \} \) as
- if \( x \neq 0, t \neq 0 \),
  \[
  \mathcal{F}v(x, t) := -2 \int_0^t K_x(x, t, \rho(\cdot, \tau)) \left[ g'(\tau) - V_\tau u_x(0, \tau) - \int_0^\infty dyu(y, \tau)p(y, 0) \right] d\tau + \int_0^\infty [K_x(x, \xi, 0) + K_x(x, \xi, -\eta, 0)]\varphi'(\xi)d\xi
  + \int_0^t \int_0^\infty [K_x(x, \xi, \tau) + K_x(x, \xi, -\eta, \tau)]V_\tau v(\xi, \tau)d\xi d\tau
  - \int_0^t \int_0^\infty [K_x(x, \xi, \tau) + K_x(x, \xi, -\eta, \tau)] \int_0^\infty dyu(y, \tau)p(y, 0)d\xi d\tau. \tag{4.46}
  \]
- if \( x = 0, t \neq 0 \), \( \mathcal{F}v(0, t) := 2 \left[ g'(t) - V_t u_x(0, t) - \int_0^\infty dyu(y, t)p(y, 0) \right], \)
- if \( x \neq 0, t = 0 \), \( \mathcal{F}v(x, 0) := \varphi''(x). \)

Then for \( v_1, v_2 \in \mathcal{B}_\eta \), we have the following estimate:
\[
\|\mathcal{F}v_1 - \mathcal{F}v_2\|_{\infty} \leq c_1 \sqrt{\eta} \|V\|_{\infty} \|v_1 - v_2\|_{\infty}.
\]
Thus for all sufficiently small \( \eta > 0 \), \( \mathcal{F} \) is a contraction mapping so that there is a unique fixed point \( v^\eta \). We can extend \( \eta \) to \( T \) by a similar way of the proof of Proposition 4. Let us say a unique \( v \in C(\overline{D_T} \setminus 0) \) such that for \( x \neq 0, t \neq 0 \),
\[
v(x, t) := -2 \int_0^t K_x(x, t, 0, 0) \left[ g'(\tau) - V_\tau u_x(0, \tau) - \int_0^\infty dyu(y, \tau)p(y, 0) \right] d\tau + \int_0^\infty [K_x(x, \xi, 0) + K_x(x, \xi, -\eta, 0)]\varphi'(\xi)d\xi
  + \int_0^t \int_0^\infty [K_x(x, \xi, \tau) + K_x(x, \xi, -\eta, \tau)]V_\tau v(\xi, \tau)d\xi d\tau
  - \int_0^t \int_0^\infty [K_x(x, \xi, \tau) + K_x(x, \xi, -\eta, \tau)] \int_0^\infty dyu(y, \tau)p(y, 0)d\xi d\tau. \tag{4.47}
\]
By integrating (4.47) with respect to spatial variable on both sides and using integration by parts, we obtain for \( x \geq 0, t > 0, \)

\[
\int_{x}^{\infty} v(y, t) dy = 2 \int_{0}^{t} K(x, t; 0, \tau) \left[ g'(\tau) - V_\tau u_x(0, \tau) - \int_{0}^{\infty} dyu(y, \tau)p(y, 0) \right] d\tau \\
+ \int_{0}^{\infty} [-K(x, t; \xi, 0) - K(x, t; -\xi, 0)] \varphi'(\xi) d\xi \\
+ \int_{0}^{t} \int_{0}^{\infty} [-K(x, t; \xi, \tau) - K(x, t; -\xi, \tau)] V_\tau v(\xi, \tau) d\xi d\tau \\
+ \int_{0}^{t} \int_{0}^{\infty} [K(x, t; \xi, \tau) + K(x, t; -\xi, \tau)] \int_{0}^{\infty} dyu(y, \tau)p(y, \xi) d\xi d\tau \\
= 2 \int_{0}^{t} K(x, t; 0, \tau) \left[ g'(\tau) - V_\tau u_x(0, \tau) \right] d\tau \\
+ \int_{0}^{\infty} [-K(x, t; \xi, 0) - K(x, t; -\xi, 0)] \varphi'(\xi) d\xi \\
+ \int_{0}^{t} \int_{0}^{\infty} G_x(x, t; \xi, \tau)V_\tau \int_{0}^{\infty} v(y, \tau) dyd\xi d\tau - 2 \int_{0}^{t} K(x, t; 0, \tau)V_\tau \int_{0}^{\infty} v(y, \tau) dyd\tau \\
- \int_{0}^{t} \int_{0}^{\infty} G_x(x, t; \xi, \tau) \int_{0}^{\infty} dyu(y, \tau)p(y, \xi) d\xi d\tau.
\]

Adding (4.29) to (4.48), we get

\[
u(x, t) + \int_{x}^{\infty} v(y, t) dy = 2 \int_{0}^{t} K(x, t; 0, \tau)V_\tau \left\{ -u_x(0, \tau) - \int_{0}^{\infty} v(y, \tau) dy \right\} d\tau \\
+ \int_{0}^{t} \int_{0}^{\infty} G_x(x, t; \xi, \tau)V_\tau \left\{ u_\xi(\xi, \tau) + \int_{\xi}^{\infty} v(y, \tau) dy \right\} d\xi d\tau.
\]

Adding \((4.29)\) to \((4.48)\), we get

\[
u(x, t) = -2 \int_{0}^{t} K(x, t; 0, \tau)V_\tau f(0, \tau) d\tau + \int_{0}^{t} \int_{0}^{\infty} G_x(x, t; \xi, \tau)V_\tau f(\xi, \tau) d\xi d\tau.
\]

By the uniqueness of the contraction mapping argument, \( f \) should be identically 0. Thus we conclude \( v = u_{xx} \in C(D_T) \), \(4.49\) becomes

\[
u(x, t) = -2 \int_{0}^{t} K(x, t; 0, \tau)V_\tau f(0, \tau) d\tau + \int_{0}^{t} \int_{0}^{\infty} G_x(x, t; \xi, \tau)V_\tau u_x(\xi, \tau) d\xi d\tau.
\]

By the uniqueness of the contraction mapping argument, \( f \) should be identically 0. Thus we conclude \( v = u_{xx} \in C(D_T \setminus 0) \) and \((4.47)\) becomes

\[
u(x, t) = -2 \int_{0}^{t} K(x, t; 0, \tau) \left[ g'(\tau) - V_\tau u_x(0, \tau) - \int_{0}^{\infty} dyu(y, \tau)p(y, 0) \right] d\tau \\
+ \int_{0}^{\infty} G(x, t; \xi, 0) \varphi''(\xi) d\xi \\
+ \int_{0}^{t} \int_{0}^{\infty} [K_x(x, t; \xi, \tau) + K_x(x, t; -\xi, \tau)] V_\tau u_{xx}(\xi, \tau) d\xi d\tau \\
- \int_{0}^{t} \int_{0}^{\infty} [K_x(x, t; \xi, \tau) + K_x(x, t; -\xi, \tau)] \int_{0}^{\infty} dyu(y, \tau)p_y(y, \xi) d\xi d\tau.
\]

Then we have

\[
\sup_{x \geq 0} |u_{xx}(x, t)| \leq 2(\|g'\|_\infty + \|V\|_\infty \|u_x\|_\infty + \|u\|_\infty + 2\|\varphi''\|_\infty + c_1\|V\|_\infty \int_{0}^{t} \frac{\sup_{\xi \geq 0} |u_{xx}(\xi, \tau)|}{\sqrt{t-\tau}} d\tau + c_2\sqrt{T} \|u\|_\infty.
\]

(4.50)
By applying Lemma 1 and Lemma 2 on (1.50), we deduce that for some $C \in S$,
\[
\|u_{x\tau}\|_{\infty} \leq C(A, T).
\]

**Lemma 4** Let $V \in \Sigma(A, T)$ and let $X_t = \int_0^t V_r \, d\tau$. For $(X, u)$ as in Proposition A then $u_x(0, t)$ is Hölder continuous with exponent $\frac{1}{2}$ with $|u_x(0, \cdot)|_{\frac{1}{2}} \leq C_8(A, T)$ for some $C_8 \in S$.

**Proof**
\[
u_x(0, t) = -2 \int_0^t \frac{g'(\tau)}{\sqrt{2\pi(t-\tau)}} \, d\tau + 2 \int_0^\infty \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{\xi^2}{2t}\right\} \varphi'(\xi) \, d\xi
\]
\[
+ \int_0^t \int_0^\infty \frac{1}{\sqrt{2\pi(t-\tau)}} \frac{2\xi}{t-\tau} \exp\left\{-\frac{\xi^2}{2(t-\tau)}\right\} \left[V_r u_x(z, \tau) + \int_0^\infty du(y, \tau)p(y, \xi)\right] \, d\xi \, d\tau
\]
By change of variable, $w = \frac{\xi}{\sqrt{t-\tau}}$ and $z = \frac{\xi}{\sqrt{t-\tau}}$, we have
\[
u_x(0, t) = -2 \int_0^t \frac{g'(\tau)}{\sqrt{2\pi(t-\tau)}} \, d\tau + 2 \int_0^\infty \frac{1}{\sqrt{2\pi t}} \exp\left\{-\frac{w^2}{2}\right\} \varphi'(w\sqrt{t}) \, dw
\]
\[
+ \int_0^t \frac{1}{\sqrt{t-\tau}} \int_0^\infty \frac{2z}{\sqrt{2\pi t-\tau}} \exp\left\{-\frac{z^2}{2}\right\} \left[V_r u_x(z\sqrt{t-\tau}, \tau) + \int_0^\infty du(y, \tau)p(y, z\sqrt{t-\tau})\right] \, dz \, d\tau
\]
Lemma 5 There is $A > 0$ such that for all sufficiently small $T > 0$,
\[
Q[V](t) = \frac{-\frac{1}{2}u_x(0,t) + \int_0^\infty \int_0^\infty u(y,t)p(y,x)dydx}{u(0,t)}, \quad 0 \leq t \leq T,
\]
maps $Q : \Sigma(A,T) \longrightarrow \Sigma(A,T)$.

Proof First of all, let $A > 0$ and $T > 0$ be arbitrary numbers. For $V \in \Sigma(A,T)$, let $(X,u)$ as in Proposition 2. Since $u(0,t) = g(t)$ and $|f|^\frac{1}{2} \leq \sqrt{T}||f'||\infty$ for all $f \in C^1([0,T])$, so we have
\[
\frac{1}{2} |u_x(0,\cdot)|^\frac{1}{2} + \sqrt{T} \sup_{0 \leq t \leq T} |\int_0^\infty \int_0^\infty u_t(y,t)p(y,x)dydx| \leq \frac{1}{(\inf_{[0,T]} |g|)^2} \left( \frac{\sup_{0 \leq t \leq T} ||u(\cdot,t)||_{L^1}}{T} \right)
\]
\[+ \frac{\sqrt{T}||g'||\infty (\frac{1}{2} |u_x(0,\cdot)|^\infty + \sup_{0 \leq t \leq T} ||u(\cdot,t)||_{L^1})}{\inf_{[0,T]} |g|)^2} \]
We also have
\[
\int_0^\infty \int_0^\infty u_t(y,t)p(y,x)dydx \leq c_1 |u_x(0,t)| + (c_2 |V_\epsilon| + c_3) |u(0,t)| + (c_4 |V_\epsilon| + c_5) ||u(\cdot,t)||_{L^1}
\]
and
\[
\inf_{[0,T]} |g| \geq |\varphi(0)| - T ||g'||\infty \geq |\varphi(0)| - TC(A,T)
\]
for some $C \in S$ by (4.41). Then by previous lemmas, for some $\tilde{C} \in S$,
\[
\frac{1}{2} \left| \frac{-\frac{1}{2}u_x(0,t) + \int_0^\infty \int_0^\infty u(y,t)p(y,x)dydx}{u(0,t)} \right| \leq \frac{\tilde{C}(A,T)}{(|\varphi(0)| - TC(A,T))^2}.
\]
Let us choose $A > \frac{\tilde{C}(\cdot,0)}{|\varphi(0)|^2}$ and then for all sufficiently small $T > 0$, $Q : \Sigma(A,T) \longrightarrow \Sigma(A,T)$ is well-defined.

Lemma 6 The map $Q$ defined in Lemma 5 is continuous on $\Sigma(A,T)$ with sup norm.

Proof Let two pairs $(V,X,\rho,\tilde{\rho})$, $(\tilde{V},\tilde{X},\tilde{\rho},\tilde{\tilde{\rho}})$ as Proposition 1 and 2. Using abuse of notation, for each $C_i \in S$, we will write simply $C_i$ instead of $C_i(A,T)$.
Using (3.23) and taking a difference between $\rho$ and $\tilde{\rho}$, we have
\[
\sup_{x \geq 0} |\rho(x,t) - \tilde{\rho}(x,t)| \leq c_1 \sqrt{T} ||\rho||\infty ||V - \tilde{V}||\infty + c_2 ||\tilde{V}||\infty \int_0^t \sup_{\xi \geq 0} |\rho(\xi,\tau) - \tilde{\rho}(\xi,\tau)| \sqrt{T - \tau} d\tau
\]
\[+ c_3 \int_0^t \sup_{y \geq 0} |\rho(y,\tau) - \tilde{\rho}(y,\tau)| d\tau.
\]
Then similarly as before, by applying Gronwall's lemma and Lemma 1, we obtain

\[\|\rho - \tilde{\rho}\|_\infty \leq C_1\|V - \tilde{V}\|_\infty.\]

In addition,

\[\|\rho(\cdot, t) - \tilde{\rho}(\cdot, t)\|_{L^1} \leq c_4\sqrt{T}\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^1} \|V - \tilde{V}\|_{\infty} + c_5\tilde{V}\|_{\infty} \int_0^t \frac{\|\rho(\cdot, \tau) - \tilde{\rho}(\cdot, \tau)\|_{L^1}}{\sqrt{t - \tau}} d\tau + c_6 \int_0^t \|\rho(\cdot, \tau) - \tilde{\rho}(\cdot, \tau)\|_{L^1} d\tau\]

so that

\[\sup_{0 \leq t \leq T} \|\rho(\cdot, t) - \tilde{\rho}(\cdot, t)\|_{L^1} \leq C_2\|V - \tilde{V}\|_{\infty}.\]

Moreover, we also get

\[\sup_{t \geq 0} |\rho_x(x, t) - \tilde{\rho}_x(x, t)| \leq c_7\sqrt{T}\|\rho_x\|_{L^\infty} \|V - \tilde{V}\|_{\infty} + c_8\tilde{V}\|_{\infty} \int_0^t \sup_{\xi \geq 0} |\rho(\xi, \tau) - \tilde{\rho}(\xi, \tau)| \frac{d\tau}{\sqrt{t - \tau}}\]

so that

\[\sup_{t \geq 0} |\rho_x(x, t) - \tilde{\rho}_x(x, t)| \leq C_3\|V - \tilde{V}\|_{\infty}.\]

By taking the difference of \(u\) and \(\tilde{u}\) written as (4.28), we have

\[\sup_{t \geq 0} |u(x, t) - \tilde{u}(x, t)| \leq \|g - \tilde{g}\|_{\infty} + c_{10}\sqrt{T}\|\tilde{V}\|_{\infty} \|g - \tilde{g}\|_{\infty} + c_{11}\sqrt{T}\|g\|_{\infty} \|V - \tilde{V}\|_{\infty} + c_{12}\sqrt{T}\|u\|_{L^\infty} \|V - \tilde{V}\|_{\infty} + c_{13}\tilde{V}\|_{\infty} \int_0^t \sup_{\xi \geq 0} |u(\xi, \tau) - \tilde{u}(\xi, \tau)| \frac{d\tau}{\sqrt{t - \tau}}\]

so that

\[\sup_{t \geq 0} |u(x, t) - \tilde{u}(x, t)| \leq C_4\|V - \tilde{V}\|_{\infty}.\]

Taking the difference between \(g'\) and \(\tilde{g}'\), we also have

\[|g'(t) - \tilde{g}'(t)| \leq c_{15}|\rho_x(0, t) - \tilde{\rho}_x(0, t)| + c_{16}|\nu_1||\rho(\cdot, t) - \tilde{\rho}(\cdot, t)||_{L^1} + c_{17}|\rho(\cdot, t)||_{L^1} |\nu_1 - \tilde{\nu}_1| + c_{18}|\rho(\cdot, t) - \tilde{\rho}(\cdot, t)||_{L^1} .\]

so that

\[\|g' - \tilde{g}'\|_{\infty} \leq C_5\|V - \tilde{V}\|_{\infty}.\]

Taking the difference between \(u_x\) and \(\tilde{u}_x\) and using previous results, we deduce

\[\sup_{t \geq 0} |u_x(x, t) - \tilde{u}_x(x, t)| \leq c_{19}\sqrt{T}\|g' - \tilde{g}'\|_{\infty} + c_{20}\sqrt{T}\|u_x\|_{\infty} \|V - \tilde{V}\|_{\infty} + c_{21}\tilde{V}\|_{\infty} \int_0^t \sup_{\xi \geq 0} |u(\xi, \tau) - \tilde{u}(\xi, \tau)| \frac{d\tau}{\sqrt{t - \tau}} + c_{22}\sqrt{T}\|u - \tilde{u}\|_{\infty}\]
so that
\[ \|u_x - \bar{u}_x\|_\infty \leq C_6 \|V - \bar{V}\|_\infty. \]

In a similar way as before and using the estimates above, we obtain
\[
\|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1} \leq c_{23} \sqrt{T} \|g - \bar{g}\|_\infty + c_{24} T \|V - \bar{V}\|_\infty + c_{25} T \|\bar{V}\|_\infty \|g - \bar{g}\|_\infty \\
+ c_{26} \sqrt{T} \|u\|_{L^1} \|V - \bar{V}\|_\infty + c_{27} \|\bar{V}\|_\infty \int_0^t \|u(\cdot, \tau) - \bar{u}(\cdot, \tau)\|_{L^1} \sqrt{t - \tau} d\tau \\
+ c_{28} \int_0^t \|u(\cdot, \tau) - \bar{u}(\cdot, \tau)\|_{L^1} d\tau,
\]

so that
\[
\sup_{0 \leq t \leq T} \|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^1} \leq C_7 \|V - \bar{V}\|_\infty.
\]

Finally we conclude that
\[
\|Q[V] - Q[\bar{V}]\|_\infty \leq C_8 \|V - \bar{V}\|_\infty = C_8 \|V - \bar{V}\|_\infty.
\]

**Proof of Theorem 2**

\(Q\) is continuous, then, since \(\Sigma(A, T)\) is convex and compact we can apply the Schauder fixed point theorem to conclude that \(Q\) has a fixed point. This completes the proof.

**5 Proof of Theorem 1**

Let \((X, \rho, u)\) in Theorem 2 and let us define \(\tilde{\rho}(x, t) := -\int_x^\infty u(y, t)dy\). Since \(u\) satisfies
\[
\frac{d}{dt} \left( \int_0^\infty u(y, t)dy \right) = \frac{1}{2} u_x(0, t) - V_t u(0, t) + \int_0^\infty \int_0^\infty dz u(z, t)p(z, y)dy = 0,
\]
thus we have \(\tilde{\rho}(0, t) = -\int_0^\infty u(y, t)dy = -\int_0^\infty \rho_0(y)dy = 0\).
Then \(\tilde{\rho}\) satisfies
\[
\tilde{\rho}_t(x, t) = -\int_x^\infty u_t(y, t)dy = \frac{1}{2} u_x(x, t) + V_t u(x, t) - \int_x^\infty \int_0^\infty dz \tilde{\rho}_x(z, t)p(z, y)dy \\
= \frac{1}{2} \tilde{\rho}_{xx}(x, t) + V_t \tilde{\rho}_x(x, t) + \int_x^\infty \int_0^\infty dz \tilde{\rho}(z, t)p(z, y)dy \\
= \frac{1}{2} \tilde{\rho}_{xx}(x, t) + V_t \tilde{\rho}_x(x, t) - \int_0^\infty dz \tilde{\rho}(z, t) \int_x^\infty p_y(z, y)dy \\
= \frac{1}{2} \tilde{\rho}_{xx}(x, t) + V_t \tilde{\rho}_x(x, t) + \int_0^\infty dz \tilde{\rho}(z, t)p(z, x).
\]
Since $\tilde{\rho}(x, 0) = \rho_0(x)$, by the uniqueness of Proposition [1], we also get $\rho = \tilde{\rho}$ such that
\[
\tilde{\rho}_x(0, t) = u(0, t) = 2 \int_0^\infty \int_0^\infty dy \rho(y, t)p(y, x)dx = 2 \int_0^\infty dy \tilde{\rho}(y, t)p(y, x)dx. \tag{5.51}
\]
By (5.51) and $\tilde{\rho}(0, t) = 0$, we have $\frac{d}{dt} \left( \int_0^\infty \tilde{\rho}(y, t)dy \right) = 0$ and finally deduce that
\[
\int_0^\infty \tilde{\rho}(x, t)dx = \int_0^\infty \rho_0(x)dx = 1. \tag{5.52}
\]
This completes the proof.

6 Further results

Let us try to apply our $C^1$-argument to the FBP of [1] as follows:

\[
(\star) \begin{cases}
\rho_t(x, t) = \frac{1}{2} \rho_{xx}(x, t) + \rho, & \text{if } X_t < x, \ t > 0, \\
\rho(X_t, t) = \alpha, & \text{if } t \geq 0, \\
\rho_x(X_t, t) = \beta, & \text{if } t > 0. \\
\rho(x, 0) = \rho_0(x), & \text{if } 0 \leq x,
\end{cases}
\]

where $\rho_0$ is specified later.

If $\alpha \neq 0$ and $\beta = 0$, then $\frac{d}{dt} \rho(X_t, t) = V_t \rho_x(X_t, t) + \rho_t(X_t, t) = \frac{1}{2} \rho_{xx}(X_t, t) + \alpha = 0$ so that, by change of variable $u := \rho_x$, $(\star)$ becomes

\[
(\star \star) \begin{cases}
u_t(x, t) = \frac{1}{2} v_{xx}(x, t) + v, & \text{if } X_t < x, \ t > 0, \\
u(X_t, t) = 0, & \text{if } t \geq 0, \\
u_x(X_t, t) = -2\alpha, & \text{if } t > 0. \\
u(x, 0) = \rho_0'(x), & \text{if } 0 \leq x,
\end{cases}
\]

Again by change of variable $v := -\frac{1}{2\alpha} u_x$, $(\star \star)$ becomes $(\star \star \star)$

\[
(\star \star \star) \begin{cases}
\nu_t(x, t) = \frac{1}{2} v_{xx}(x, t) + v, & \text{if } X_t < x, \ t > 0, \\
\nu(X_t, t) = 1, & \text{if } t \geq 0, \\
\nu(x, 0) = -\frac{1}{2\alpha} \rho_0''(x), & \text{if } 0 \leq x, \\
V_t = -\frac{1}{2} \nu_x(X_t, t), & \text{if } t > 0.
\end{cases}
\]

To make each step valid, it needs that the value of the initial datum at 0 is same as the boundary value which is $\rho_0(0) = \alpha$, $\rho_0'(0) = 0$, $\rho_0''(0) = -2\alpha$ and $\rho_0$ should be $C^4_c([0, \infty))$ to have $v_{xx}$ of $(\star \star \star)$ as in Lemma 3.

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Similarly as before, we can shift the boundary $X$ to 0 so that we have the following equivalent FBP:

$$
(\star\star\star')
\begin{cases}
    v_t(x,t) = \frac{1}{2} v_{xx}(x,t) + V_t v_x(x,t) + v(x,t) & \text{if } 0 < x, \ t > 0, \\
v(0,t) = 1, & \text{if } t \geq 0, \\
v(x,0) = -\frac{1}{2\alpha} \rho_0''(x), & \text{if } 0 \leq x, \\
V_t = -\frac{1}{2} v_x(0,t), & \text{if } t > 0.
\end{cases}
$$

By writing $v$ as

$$
v(x,t) = -\int_0^t K_x(x,t;0,\tau)d\tau + \int_0^\infty G(x,t;\xi,0)\psi(\xi)d\xi + \int_0^t \int_0^\infty G(x,t;\xi,\tau) [V_\tau v_\xi(\xi,\tau) + v(\xi,\tau)] d\xi d\tau,
\tag{6.53}
$$

where $\psi(\xi) = -\frac{1}{2\alpha} \rho_0''(\xi)$, we can repeat the same argument as previous sections such that there is a pair $(X,v)$ which satisfies $(\star\star\star')$.

If $\alpha = 0$, $\beta \neq 0$, it can be done similarly as $(\alpha \neq 0, \beta = 0)$-case with the initial condition $\rho_0 \in C^3_c([0,\infty))$ such that $\rho_0(0) = 0$, $\rho_0'(0) = \beta$.

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