Abstract

The aim of this paper is twofold: on one side we review the classical concept of musical mode from the viewpoint of modern music, reading it as a superimposition of a base-chord (seventh chord) and a tension-chord (triad). We associate to each modal scale an oriented plane graph whose homotopy properties give a measure of the complexity of the base-chord associated to a certain mode. Using these graphs we prove the existence of special modes which are not deducible in the standard way.

On the other side we give a more deep musical insight by developing a braid theoretical interpretation of some cadential harmonic progressions in modern music and we use braid theory in order to represent them and voice leadings among them.

A striking application is provided by the analysis of an harmonic fragment from Peru by Tribal Tech. We approximate the octatonic scale used in the improvisation by Scott Henderson, through the special mode myxolydian ♭2♯4 and we finally associate a braid representation to the fragment we analysed.

Keywords: Musical modes, voice leading, harmonic progressions, graph theory, homotopy theory, braid theory.

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Introduction

Classical harmony has been largely studied, however musicians often are hard put to understand the real nature of modes and the concept of sonority in modern music. In Western music theory, the word “mode” (from Latin modus, “measure, way, size, method”) generally refers to a type of scale, coupled with a set of characteristic melodic behaviours. More precisely the concept of mode incorporates the idea of the diatonic scale, but differs from it involving an element of melody type. This concerns particular repertories of short musical figures or groups of tones within a certain scale so that, depending on the point of view, mode takes the meaning of either a “particularized scale” or a “generalized tune”.

The goal of this paper is to construct modes from a genuine harmonic viewpoint; more precisely, a mode is not treated only like a scale (and hence an ordered sequence of pitches) but also as a superimposition of two chords: a base-chord and a tension chord. From a technical musical viewpoint, the base-chord is a seventh chord and the tension one is a triad. Of course as one can easily figure out not every seventh chord is admissible as base-chord as well as not every triad is compatible with a fixed seventh chord. Specifically, the base-chords have to be into 7 classes corresponding to the types of chord we can deduce harmonizing each degree of the major, harmonic and melodic minor scales. Each base-chord and the fact that we deduce modes using three scales “force” the superimposition of some specific triads which are chosen decomposing the classical modal scales; namely, not every triad can be used to produce a mode, as we shall see later.

Starting from a fixed base-chord we shall consider only the triads that give us the classical modes and we shall associate in a quite elementary way an oriented planar connected graph $G$ (the order being established by the scale’s degrees) and hence a topological measure of complexity given by the $1 - \chi(G)$ where $\chi(\cdot)$ denotes the Euler characteristic. It is a well-known fact that the fundamental group of $G$, i.e. $\pi_1(G)$ is a free group over $1 - \chi(G)$ generators. Moreover $\chi(\cdot)$ is a topological invariant. Each mode is a subgraph of the associated graph such that each vertex (except the first and the last one) has valence at least 2 (i.e. at least two edges connect the vertex in the graph).

Our first result is that out of the classical 21 modes which we shall refer as standard modes, other 12 “exceptional” modes arise. We shall refer to them as admissible special modes (See proposition 2.18). The most important fact is that one of these modes which is called Myxolidian♭2♯4, allows us to well “approximate” an octatonic half step/whole step scale used in a famous fusion piece, with a 7 notes scale, conserving entirely the notes of the chord played in the harmonic structure (a dominant chord in the case we examine). This fact allows either the composer or the improviser to constantly see the notes of resolutions which are the base-chord and to clearly distinguish them from the tension notes which are collected in the tension-triad. Moreover in the Myxolidian♭2♯4 we have the presence of the tritone which makes the dominant chord recognizable, as the 92 and the augmented fourth which greatly contribute to the tension of the mode.

We would like to stress the fact that modes attracted the attention of many music theorists over the years. However, to the knowledge of the authors, the analysis carried over is mainly of quantitative type and in fact it is based on algebra and combinatorics. We shall refer to the papers of [Agm89, Nol09, Row00, Maz02] and references therein. Our approach is in the opposite direction being somehow qualitative in the essence and for this reason the suitable mathematical tools are mainly based on the use of (elementary) algebraic topology.

In contrast to the first part which is “static” in the essence, in the second part of the paper we analyse some harmonic progressions. More precisely we take into account some well-known cadential chord progressions in modern music and in the special case in jazz music which are based on perfect, deceptive, plagal and secondary dominant progressions. Our main mathematical device in order to carry over our investigation is based on the geometrical interpretation through braids which give us a suggestive and net interpretation of the mathematical structure behind this corner of the musical world. We analyse the same harmonic fragment we used to test the special modes, which put on evidence a sort of “musical complexity” through its representation with braids.

This is the starting point of a forthcoming paper in which the braids structure of some cadences are used in order to construct suitable topological spaces having highly non-trivial homotopic properties. These properties allow us to define in a very clear and rigorous way a sort of musical complexity which could be used in order to classify some musical pieces, as well as to distinguish
the degree of complexity of a composition or improvisation. (See section 5 for further details and future projects).

1 Standard modes and superimposition of chords

The aim of this section is to recall some well-known facts and definitions about the classical construction of modes in modern music. A definition of mode [Lev95] is that it is a seven-note scale created by starting on any of the seven note of a major or a melodic minor scale. The section is structured as follows: in the first paragraph we deduce the modes from the major, melodic minor and harmonic minor scale. In the second paragraph a mathematical way to represent chords and modes is shown. We conclude the section showing a non-standard way to think about modes as a superimposition of two chords.

1.1 Deducing the standard modes

Following the definition given in [Lev95], we assume that a mode is an heptatonic scale, however we include the harmonic minor in the set of scales we use to deduce the standard modes. The reason which lead us to consider this scale is that it allows us to define naturally some really used and interesting modes like the phrygian dominant. The classical way to deduce modes from scales is to consider each degree of the scale as the root of the modal scale. In table 1 we list the 21 modes given by the major, melodic minor and harmonic minor scale, examples have been built on the C major (C, D, E, F, G, A, B), melodic minor (C, D, E♭, F, G, A, B) and harmonic minor (C, D, E♭, F, G, A♭, B) scale respectively.

| Scale         | Degree | Modes          | Example                  |
|---------------|--------|----------------|--------------------------|
| Major         | I      | Ionian         | C - D - E - F - G - A - B |
|               | II     | Dorian         | D - E - F - G - A - B - C |
|               | III    | Phrygian       | E - F - G - A - B - C - D |
|               | IV     | Lydian         | F - G - A - B - C - D - E |
|               | V      | Mixolydian     | G - A - B - C - D - E - F |
|               | VI     | Eolian         | A - B - C - D - E - F - G |
|               | VII    | Locrian        | B - C - D - E - F - G - A |
| Melodic Minor | I      | Hypoionian     | C - D - E♭ - F - G - A - B |
|               | II     | Dorian ♭        | D - E♭ - F - G - A - B - C |
|               | III    | Lydian Augmented| E♭ - F - G - A - B - C - D |
|               | IV     | Lydian Dominant | F - G - A - B - C - D - E♭ |
|               | V      | Mixolydian ♭♭♭ | G - A - B - C - D - E♭ - F |
|               | VI     | Locrian ♭♭♭     | A - B - C - D - E♭ - F - G |
|               | VII    | Super locrian  | B - C - D - E♭ - F - G - A |
| Harmonic Minor| I      | Hypoionian     | C - D - E♭ - F - G - A♭ - B |
|               | II     | Locrian ♭♭♭     | D - E♭ - F - G - A♭ - B - C |
|               | III    | Ionian augmented| E♭ - F - G - A♭ - B - C - D |
|               | IV     | Dorian ♭♭♭       | F - G - A♭ - B - C - D - E♭ |
|               | V      | Phrygian dominant| G - A♭ - B - C - D - E♭ - F |
|               | VI     | Lydian ♭♭♭      | A♭ - B - C - D - E♭ - F - G |
|               | VII    | Ultra locrian    | B - C - D - E♭ - F - G - A♭ |

Table 1: The 21 modes derived from the major, melodic minor and harmonic minor scale.

Musically speaking, from the classification given in table 1 it is really hard to understand what a mode is: a non-trained listener could say that the dorian mode is simply a C major scale played from the second degree, but every musician knows that this is a great reduction. However suffice to do a simple experiment to understand that modes are really distinguishable when they are played on the seventh chord built on the degree of the scale associated to the modal scale. (See table 2). For a non-trained listener a D-dorian scale played on a Cmaj7 chord is simply a C major scale.
played on its second degree, but the same scale played on a $D - 7$ chord sounds is perceived as a new scale, that results completely different from the $C$-major scale.

Getting to the point, we focus our attention on this strong relation between the modal scale and the chord we associate to the scale, see table 1 and 2.

| Scale     | Degree | 7th Chord | Arpeggio (example) |
|-----------|--------|-----------|--------------------|
| Major     | I      | maj7      | C - E - G - B      |
|           | II     | -7        | D - F - A - C      |
|           | III    | -7        | E - G - B - D      |
|           | IV     | maj7      | F - A - C - E      |
|           | V      | 7         | G - B - D - F      |
|           | VI     | -7        | A - C - E - G      |
|           | VII    | -7²5      | B - D - F - A      |
| Melodic Minor | I     | -maj7     | C - E♭ - G - B    |
|            | II     | -7        | D - F - A - C      |
|            | III    | maj7²5    | E♭ - G - B - D     |
|            | IV     | 7         | F - A - C - E♭      |
|            | V      | 7         | G - B - D - F      |
|            | VI     | -7²5      | A - C - E♭ - G     |
|            | VII    | -7²5      | B - D - F - A      |
| Harmonic Minor | I     | -maj7     | C - E♭ - G - B    |
|              | II     | -7²5      | D - F - A♭ - C    |
|              | III    | maj7²5    | E♭ - G - B - D     |
|              | IV     | -7        | F - A♭ - C - E♭    |
|              | V      | 7         | G - B - D - F      |
|              | VI     | maj7      | A♭ - C - E♭ - G    |
|              | VII    | -7²7      | B - D - F - A♭     |

Table 2: Seventh chord harmonization on the major, melodic minor and harmonic minor scale.

Chords notation: $Rmaj7$ is given by the root $R$, its major third ($M3$), perfect fifth ($P5$) and major seventh ($M7$); we denote with $R - 7$ the chord given by $R - m3 - P5 - m7$ where $m3$ and $m7$ are respectively the minor third and the minor seventh in respect to the root note. $R - maj7$ is $R - m3 - P5 - m7$, $R - 7²5$ is $R - m3 - dim5 - m7$, $Rmaj7²5$ is given by $R - M3 - aug5 - M7$ and $R²7$ is $R - m3 - dim5 - dim7$.

1.2 Modes as a superimposition of chords

In this paragraph we want to stress the importance of the harmonic choice which lies behind the modal scale. Since either the chord and the modal scale is associated to a certain degree of the scale, the notes of the chord have to be part of the modal scale, more precisely they are the first, third, fifth and seventh degree of the modal scale.

Example 1.1 Consider $F$ lydian as a modal scale. The chord associated to this mode is $Fmaj7$, then we have

$$F \text{ lydian scale } \quad F - G - A - B - C - D - E$$

$$Fmaj7 \text{ arpeggio } \quad F - A - C - E$$

Every mode is associated to the proper seventh chord built on the root of the modal scale, this chord is called for brevity the base-chord. We list them in table.
Table 3: The twenty-one modes derived from the major, melodic minor and harmonic minor scale and their base-chords.

| Mode          | Example - Base-chord | Example - Scale |
|---------------|-----------------------|-----------------|
| Ionian        | Cmaj7                 | C - D - E - F - G - A - B |
| Dorian        | D - 7                 | D - E - F - G - A - B - C |
| Phrygian      | E - 7                 | E - F - G - A - B - C - D |
| Lydian        | Fmaj7                 | F - G - A - B - C - D - E |
| Mixolydian    | G7                    | G - A - B - C - D - E - F |
| Eolian        | A - 7                 | A - B - C - D - E - F - G |
| Locrian       | B - 75\textsuperscript{5} | B - C - D - E - F - G - A |

| Hypoionian    | C - maj7              | C - D - Es - F - G - A - B |
| Dorian 2\textsuperscript{b} | D - 7                 | D - Es - F - G - A - B - C |
| Lydian Augmented | E\textsuperscript{b}maj75\textsuperscript{5} | Es - F - G - A - B - C - D |
| Lydian Dominant | F7                    | F - G - A - B - C - D - Es |
| Mixolydian 2\textsuperscript{13} | G7                    | G - A - B - C - D - Es - F |
| Locrian 2\textsuperscript{2} | A - 75\textsuperscript{5} | A - B - C - D - Es - F - G |
| Super locrian | B - 75\textsuperscript{5} | B - C - D - Es - F - G - A |

| Hypoionian 2\textsuperscript{6} | C - maj7              | C - D - Es - F - G - A - B |
| Locrian 3\textsuperscript{6} | D - 75\textsuperscript{5} | D - Es - F - G - A - B - C |
| Ionian augmented | E\textsuperscript{b}maj75\textsuperscript{5} | Es - F - G - A - B - C - D |
| Dorian 2\textsuperscript{4} | C - 7                 | F - G - A - B - C - D - Es |
| Phrygian dominant | G7                    | G - Ab - B - C - D - Es - F |
| Lydian 2\textsuperscript{2} | A\textsuperscript{b}maj7 | A\textsuperscript{b} - B - C - D - Es - F - G |
| Ultra locrian | B5\textsuperscript{7} | B - C - D - Es - F - G - A\textsuperscript{b} |

From either example 1.1 and table 3 one can also deduce that deleting the base-chord from a modal scale a triad remains, we call that triad the tension-triad which is composed by the second, the fourth and the sixth degree of the modal scale.

**Example 1.2** Considering the same setting of example 1.1, we have

\[ F \text{ lydian scale} \quad F - G - A - B - C - D - E \]
Base-chord arpeggio \[ F - A - C - E \]
Tension-triad arpeggio \[ G - B - D \]

**Thus it is possible to think about the** F lydian mode as the superimposition of a Fmaj7 chord and a G major triad.

Every modal scale can be decomposed uniquely in a seventh chord built on its root note and a triad built on its second degree. As we said before the modal scale is recognizable if it is played on its base-chord, so one can consider the base-chord as the set of stable notes and the tension-triad as the collection of unstable notes for a certain mode, see table 4 for a complete description of modes in terms of base-chords and tension-triads.
Table 4: Modes as a superimposition of two chords. Notation $\sharp$ means that the triad is augmented ($R - M3 - aug5$), while $\flat$ means that it is diminished ($R - m3 - dim5$).

This decomposition which arise naturally from the standard way of deducing modes from the classical scales, is the reason why we want to introduce in the next paragraph the space of 4 and 3 note-chords.

1.3 The space of chords: some notations

The aim of this section is to recall some well-known facts about the space of chords and to fix our notations. Our basic references are [CQT08, Tym11, Tym06, Tym09] and references therein.

From a mathematical point of view, we recall that pitches are modeled by real numbers. A pitch class (modulus octave) is modeled by a point in the 1-dimensional torus $T_1 := \mathbb{R}/(12\mathbb{Z})$. In general the space $T^n := \mathbb{R}^n/(12\mathbb{Z})^n$ represents the space of ordered pairs of pitch classes. (Cfr. [Tym06], Section 2-5, for further details). However this space is bigger than the space of $n$-notes chords which corresponds to the unordered pairs of pitch class. Thus by passing to the quotient of $T^n$ by the group of all permutations over the set of $n$ elements, namely $S(n)$, we get the space of $\mathcal{C}_n$ of $n$-notes chords. More precisely:

$$\mathcal{C}_n := T^n/S(n) = \mathbb{R}^n/(12\mathbb{Z})^n \rtimes S(n)$$

where $\rtimes$ denotes the semi-direct product. We observe that $\mathcal{C}_3$ is the Möbius strip which is a not orientable surface. Also for dimension higher than two the quotient is an orbifold. In the following we mainly consider the space of seventh chords (i.e. 4-notes chords) and triads (3-notes chords) that corresponds respectively to the orbifold constructed gluing with a twist two opposite top and bottom faces of a 4-dimensional and 3-dimensional prism. (Cfr. [Tym06] Fig. S5-S6). However we do not use the geometric structure of the space of chords except as an ambient space.

**Definition 1.3** Given two chords $[A] \in \mathcal{C}_n$ and $[C] \in \mathcal{C}_m$, we define the chord intersection of both as the chord $[D] \in \mathcal{C}_k$ for $k \leq \min\{m, n\}$ defined by the maximal dimensional subset of notes appearing both in $[A]$ and $[C]$.

These structures allow us to represent a mode as the superimposition of a 4-notes chord and a 3-notes chord. More precisely:
Definition 1.4 We set: \( \mathcal{M} := \mathcal{C}_4 \times \mathcal{C}_3 \).

A musical mode or simply a mode is a point \( m = ([B], [T]) \in \mathcal{M} \) such that \( [B] \cap [T] = \emptyset \).

As we will see later on, not every point in \( \mathcal{M} \) is a mode. The decomposition has been formalized, thus we can give another classification of modal scales fixing the base-chord \( B \) and varying the tension-triad to associate every possible modal choice to a certain seventh chord (available tension-triads are the ones we derived in paragraph 1.2, table 4). See table 5 for a list of all possible modal choices on a fixed type of seventh chord [BG12].

Base-chord \( \text{maj7} \) | Modes | Example (root C)
---|---|---
T-M III-P V-M VII | Ionian | C - D - E - F - G - A - B
Lydian | C - D - E - F\# - G - A - B
Lydian \#2 | C - D\# - E - F\# - G - A - B

Base-chord \( \text{maj7}^{\flat 5} \) | Modes | Example (root C)
---|---|---
T-M III-aug V-M VII | Lydian augmented | C - D - E - F\# - G - A - B
Ionian augmented | C - D - E - F\# - G - A - B

Base-chord 7 | Modes | Example (root C)
---|---|---
T-III M-V P-VII m | Mixolydian | C - D - E - F - G\# - A - B\#
Mixolydian \#13 | C - D - E - F - G\# - A\# - B\#
Phrygian dominant | C - D\# - E - F - G - A\# - B\#
Lydian dominant | C - D - E - F - G\#\# - A - B\#

Base chord \( \text{−7} \) | Modes | Example (root C)
---|---|---
T-m III-P V-m VII | Dorian | C - D - E\# - F - G - A - B\#
Phrygian | C - D\# - E - F - G - A\# - B\#
Eolian | C - D - E\# - F - G - A\# - B\#
Dorian \#2 | C - D\# - E\# - F - G - A - B\#
Dorian \#4 | C - D - E\# - F - G\# - A - B\#

Base chord \( \text{−7}^{\flat 5} \) | Modes | Example (root C)
---|---|---
T-m III-dim V-m VII | Locrian | C - D\# - E\# - F - G\# - A - B\#
Locrian \#2 | C - D\# - E\# - F - G\# - A\# - B\#
Superlocrian | C - D\# - E\# - F - G\#\# - A\# - B\#
Locrian \#6 | C - D\# - E\# - F - G\# - A - B\#

Base-chord \( \text{−maj7} \) | Modes | Example (root C)
---|---|---
T-m III-P V-M VII | Hypoionian | C - D - E\# - F - G - A - B\#
Hypoionian \#6 | C - D - E\# - F - G\# - A\# - B\#

Base-chord \( \text{−57} \) | Modes | Example (root C)
---|---|---
T-m III-dim V-dim VII | Ultralocrian | C - D\# - E\# - F - G\# - A\# - B\#

Table 5: Modal scales associated to a fixed base-chord

For instance, choose \([B]\) as a major seven chord, then it is possible to associate to \([B]\) three different classes of tension-triads \([T]\). Considering rows associated to the \( \text{maj7} \) chord type in table 5 we have that the modes listed in the table are representable as the following points in \( \mathcal{M} \):

1. Ionian: \( i := ([C\text{maj7}], [D-]) \);
2. Lydian: \( l := ([C\text{maj7}], [D]) \);
3. Lydian \#2: \( l_2 := ([C\text{maj7}], [D_2\#-\flat 5]) \)

Every couple \(([B], [T])\) identifies a point in \( \mathcal{M} \) which can be associated uniquely to a modal scale. The scale is given by the set of notes \( \{b_1, b_2, b_3, b_4, t_1, t_2, t_3\} \) where \( B = \{b_1, \ldots, b_4\} \) and \( T = \{t_1, t_2, t_3\} \). Following [Pis59, Chapter 10] for the analysis of nonharmonic tones in classical harmony we are entitled to define
Definition 1.5. Let $B = \{b_1, \ldots, b_4\}$ and $T = \{t_1, t_2, t_3\}$. Non chord tones are notes which do not belong to the base-chord; i.e. $t_i \in T$ such that $t_i \notin B$.

From definition 1.4 we know that for a certain mode $m \in \mathcal{M}$, $m = ([B], [T])$ it has to be $[B] \cap [T] = \emptyset$. So every note belonging to a tension-triad associated to a base-chord is a non chord tone. Every point in $\mathcal{M}$ has different musical meanings, here follows a list of information we can read for each point $m \in \mathcal{M}$:

1. $m$ identifies a unique mode;
2. chord tones and non chord tones are splitted into two components, respectively $[B]$ and $[T]$;
3. considering the notes belonging to $[B]$ and $[T]$ we deduce the modal scale associated to $m$, that can be re-ordered in a 7-uple in which the degrees of the scale are displayed from the root, to the seventh note.

Thus fixed a base-chord we can write down an ordered modal scale associated to the chord for every available choice of tension-triad.

Example 1.6. Fix a seventh chord, for instance a Cmaj7. The idea is to split stable and unstable notes of an heptatonic scale belonging to the base-chord, as follows

$\begin{align*}
C & \rightarrow \square \rightarrow E \rightarrow \square \rightarrow G \rightarrow \square \rightarrow B \\
C & \rightarrow D \rightarrow E \rightarrow F \rightarrow G \rightarrow A \rightarrow B \\
C & \rightarrow D\# \rightarrow E \rightarrow F\# \rightarrow G \rightarrow A \rightarrow B
\end{align*}$

White squares are placeholders for the note of a suitable triad. As we showed in the previous paragraph, choosing a $D$ minor triad one can find the $C$ ionian scale, considering a $D$ major triad we have a $C$ lydian and with a $D\#$ diminished triad we obtain the $C$ lydian $\sharp 2$ scale.

2 A geometrical representation of modes through graphs

This section ideally splits into two parts. In the first part, for the sake of the reader as well as for fixing our notations, we shall recall some well-known facts about graphs, maximal trees and fundamental group. For further details we shall refer to [Gib10] (or any elementary textbook in algebraic topology) and references therein.

In the second part which is the core of the first part of the paper, we start defining the associated planar oriented graph to each base-chord. Since every mode is an heptatonic scale, the orientation of the graph is induced by the natural sequence of the degrees of the scale.

2.1 Some mathematical preliminaries

Definition 2.1. An abstract unoriented graph is a pair $(V, E)$ where $V$ is a finite set and $E$ is a set of unordered pairs of different elements of $V$. Thus an element of $E$ is of the form $\{v, w\}$ where $v$ and $w$ belong to $V$ and $v \neq w$. We call vertices the elements of $V$ and edges the elements $\{v, w\}$ of $E$ connecting $v$ and $w$ (or $w$ and $v$).

Definition 2.2. Let $(V, E)$ be an abstract graph. A realization of $(V, E)$ is a set of points in $\mathbb{R}^N$, one point for each vertex and segments joining precisely those pairs of points which correspond to edges. The points are the vertices and the segments are the edges; the realization is termed a graph. We require that the following two intersection conditions hold:

1. two edges meet either in a common end-point or at all;
2. no vertex lies on an edge except at one of its ends.

We denote by \((vw)\) the edge joining \(v\) and \(w\).

**Definition 2.3** Two abstract (unoriented) graphs \((V, E)\) and \((V', E')\) are isomorphic if there exists a bijective map \(f : V \rightarrow V'\) such that

\[
\{v, w\} \in E \iff \{f(v), f(w)\} \in E'.
\]

**Remark 2.4** Analogous definitions for oriented graphs are obtained by replacing unordered pairs \(
\{\cdot, \cdot\} \) by ordered pairs \((\cdot, \cdot)\).

**Definition 2.5** For \(n \geq 1\), a path on a graph \(G\) from \(v_1\) to \(v_{n+1}\) is a sequence of vertices and edges

\[
v_1 e_1 v_2 e_2 \ldots v_n e_n v_{n+1}
\]

where \(e_1 = (v_1 v_2), e_2 = (v_2 v_3), \ldots, e_n = (v_n v_{n+1})\).

If \(G\) is oriented, we only require that \(e_i = (v_i v_{i+1})\) or \(e_i = (v_{i+1} v_i)\) for \(i = 1, \ldots, n\); that is, the edges along the path are oriented in the opposite way.

**Definition 2.6** The path is simple if \(e_1, \ldots, e_n\) are all distinct, and \(v_1, \ldots, v_{n+1}\) are all distincts except that possibly \(v_1 = v_{n+1}\). If the simple path has \(v_1 = v_{n+1}\) and \(n > 0\) is called a loop.

A graph \(G\) is said to be connected if, given any two vertices \(v\) and \(w\) of \(G\) there is a path on \(G\) from \(v\) to \(w\). A graph which is connected and without loops is called a tree.

![Diagram](a) An example of a planar graph  
(b) A maximal tree

**Definition 2.7** Given a graph \(G\), a graph \(H\) is called a subgraph of \(G\) is the vertices of \(H\) are vertices of \(G\) and the edges of \(H\) are edges of \(G\). Also \(H\) is called a proper subgraph of \(G\) if \(H \neq G\).

The following definition is central in the sequel of the paper.

**Definition 2.8** Let \(G\) be a graph, \(H\) be any maximal tree in \(G\), \(S\) be a subset of the vertices set and \(k\) is an integer. An admissible path \(\gamma\) in \(G\) with respect to \(S\) of length \(k\) is any proper subgraph of \(H\) satisfying the two conditions:

1. each vertex \(v \in S\) lies in \(\gamma\);
2. the total number of vertices in \(\gamma\) is \(k\).

We also observe that any graph \(G\) as a subgraph which is a tree (e.g. the empty subgraph is a tree) so that the set \(\mathcal{F}\) of subgraphs of \(G\) which are trees will have maximal elements. That is, there exists at least one \(T \in \mathcal{F}\) such that \(T\) is not a proper subgraph of any \(T' \in \mathcal{F}\).

**Lemma 2.9** Let \(G\) be a connected graph. A subgraph \(T\) of \(G\) is a maximal tree for \(G\) if and only if \(T\) is a tree containing all the vertices of \(G\).

**Proof.** Cfr. [Gib10, Proposition 1.11, pag.18].

For a connected graph \(G\) there is a standard way to compute the homotopy group. In fact the following result holds:
Proposition 2.10 For a connected graph $G$ with maximal tree $T$, $\pi_1(G)$ is a free group with basis the classes $[f_e]$ corresponding to the edges $e$ of $X \setminus T$.

Proof. Cfr. [Hat02, Proposition 1A.2 pag.84].

Given a finite connected graph $G$, we denote by $\chi(G)$ the Euler characteristic defined as the number of vertices minus the number of edges. From proposition 2.10 we immediately get:

Corollary 2.11 Let $G$ be a finite connected graph. Then $\pi_1(G)$ is a free group over $1 - \chi(G)$ generators. More precisely the Euler characteristic of a finite connected graph is a topological invariant.

2.2 Graphs and base-chords

Definition 2.12 Given a base-chord $[B] \in C_4$ the associated graph $\mathcal{G}([B])$ is the realization of the abstract graph whose vertex set is given by the set of all notes forming $[B]$ and of every compatible tension-triad and the oriented edge set is represented by all possible oriented connections between each vertex according to the order of the degrees of the scale; i.e. from the root to the seventh. (See figure 1).

Figure 1: A graph built assuming (without loss of generality) that the modal choices on a base-chord $B = \{b_1, b_2, b_3, b_4\}$ are given by two tension-triads $T = \{t_1, t_2, t_3\}$ and $\bar{T} = \{\bar{t}_1, \bar{t}_2, \bar{t}_3\}$

Figure 2: A dominant chord graph (M:=major, m:=minor, P:=perfect and a:=augmented).

Remark 2.13 A priori the degrees can be arranged in the plane in infinitely many ways, giving rise to completely different unoriented graphs. However considering the orientation induced by the degrees of the scale all the oriented graphs are homeomorphic. Since homeomorphisms induce isomorphisms in homotopy, all of the homotopic classification as well as the topological measure of complexity, is not affected by the convention given in definition 2.12.

We also observe that on the second, fourth and sixth degrees we have at maximum two choices. This is a straightforward consequence of the constructions of the modes from the major, melodic minor and harmonic minor scales. (Read table 5 for further details).

Example 2.14 Labelling the previous graph using $C$ as root note for each base-chord, we obtain the following oriented planar graph.
Following definition 2.12 we can build a graph for each type of chord $G^7$, $\Gamma_{G^7}$.

1. **Diminished seven**: $\Gamma_{07}$. This kind of chord appears only in the harmonization of the seventh degree of the harmonic minor scale, therefore we have no choice except for the ultralocrian mode. So, the graph associated to this seventh chord is

   ![Graph associated to diminished seventh chords, $\Gamma_{07}$](image)

2. **Major seven**: $\Gamma_{maj7^{5}}$. Fixing a $maj7^{5}$ chord as base of the mode, we have two different possibilities: either the ionian sharp five or the lydian sharp five modal scale. In terms of graph, we have

   ![Graph associated to diminished seventh chords, $\Gamma_{maj7^{5}}$](image)

---

These types of chord are the one which are deduced from the harmonization of the major, harmonic minor and melodic minor scale.
3. **Minor major seven**: $\Gamma_{-maj7}$. In this case we can choose between two different modes: hypoionian and hypoionian $b6$. The graph is

![Figure 8: The graph associated to minor major seventh chords, $\Gamma_{-maj7}$.](image)

4. **Major seven**: $\Gamma_{maj7}$. This is certainly a more common chord than the previous ones. We expect to have more possibilities, in fact a well known and simple base-chord surely will bear more tension-triads than a naturally dissonant one.

![Figure 10: The graph associated to major seven chords, $\Gamma_{maj7}$.](image)
5. Dominant: $\Gamma_7$. Dominant chords are largely used in blues and traditional jazz thanks to their capability of bearing tensions. Thus we have:

![Diagram of dominant chords]

6. Minor seven: $\Gamma_{-7}$. For a minor seventh chord the only forbidden notes are the augmented second and the diminished fourth. So we have a graph isomorphic to $\Gamma_7$.

![Diagram of minor seven chords]

Figure 11: A maximal tree associated to $\Gamma_{maj7}$.

Figure 12: The graph associated to dominant chords, $\Gamma_7$.

Figure 13: A maximal tree of $\Gamma_7$.

Figure 14: The graph associated to minor seven chords, $\Gamma_{-7}$.
7. **Minor seven** \(\bar{5}\): \(\Gamma_{-7}\). In this case the *base-chord* makes the difference: the root note and the diminished fifth form a tritone interval which gives a *stable* sense of dissonance to the half-diminished seventh chord, that is emphasized by the minor second which is natural in three of the four modal solutions we find on this type of chord, i.e. locrian, superlocrian and locrian \(\sharp 6\) scales.

**Example 2.15** In this example we display three of the graphs investigated above, labelling the vertices with notes instead of the degrees of the modal scale.
(a) Minor major seven modes - Given a $-maj7$ chord, available modes are ipoionic and ipoionic $♭6$.

(b) Major seven modes - This graph represent modes on a major seven chords, Cmaj7 in this case.

(c) Ultra locrian - In this example tension-triad’s notes are double circled. The base-chord is C$\#7$.

**Remark 2.16** It is clear by previous discussion that even if the graphs associated to $\Gamma_7$, $\Gamma_-, \Gamma_-, \Gamma_-$ are isomorphic, they are built on different notes and hence they are quite different in the essence so the homotopy is not suitable to distinguish among them.

All of these graphs show in a clear and net way how to construct new modes from the existing ones. Given a graph $G$, let us consider any proper tree $H$ (not maximal, in general!) contained in $G$ and having 7 vertices. By taking into account definition 2.8, we give the following:

**Definition 2.17** Let $[B]$ a base-chord and $\mathcal{G}([B])$ be the associated base-chord graph. An admissible mode is any admissible connected subgraph (or path in the graph) $\gamma([B])$ in $\mathcal{G}$ with respect to $[B]$ of length 7. If $\gamma([B])$ is not a mode constructed above, we refer to as admissible special mode.

Summing up, the previous discussion, we get:

**Proposition 2.18** Given the base-chord $[B] \in \mathcal{E}_4$ the following modes are the only admissible special modes:

1. if $[\Gamma_B] = [\Gamma_{maj7}]$ then $\gamma([B])$ is the path $\gamma_{Ion2} := \{I, aII, MIII, PIV, PV, MV1, MVII\}$

2. if $[\Gamma_B] = [\Gamma_7]$ then $\gamma([B])$ are the paths $\gamma_{Mix92} := \{I, mII, MIII, PIV, PV, MV1, MVII\}$ $\gamma_{Mix244} := \{I, mII, MIII, aIV, PV, MV1, mVII\}$ $\gamma_{Mix2496} := \{I, MII, MIII, aIV, PV, MV1, mVII\}$ $\gamma_{Mix2496} := \{I, mII, MIII, aIV, PV, MV1, mVII\}$

3. if $[\Gamma_B] = [\Gamma_-]$ then $\gamma([B])$ are the paths $\gamma_{Loc2} := \{I, mII, mIII, PIV, PV, MV1, MVII\}$ $\gamma_{Loc44} := \{I, MIII, mIII, aIV, PV, MV1, MVII\}$ $\gamma_{Phr44} := \{I, mII, mIII, aIV, PV, MV1, mVII\}$

4. if $[\Gamma_B] = [\Gamma_-♭5]$ then $\gamma([B])$ are the paths $\gamma_{Loc2} := \{I, mII, mIII, PIV, dV, MV1, MVII\}$ $\gamma_{Loc44} := \{I, MIII, mIII, dIV, dV, MV1, MVII\}$ $\gamma_{Loc44} := \{I, mII, mIII, dIV, dV, MV1, MVII\}$ $\gamma_{Loc44} := \{I, MIII, mIII, dIV, dV, MV1, MVII\}$
Proof. The result readily follows by the previous graph classification. Let us consider every class of chord to prove the existence of special modes.

- 7. It is not possible to have path associated to special modes on the graph $\Gamma_7$ (figure 4), since there is only one path available which represent the ultra locrian mode.

- $maj7^\#$ and $-maj7$. In both $\Gamma_{maj7^\#}$ (figure 6) and $\Gamma_{-maj7}$ (figure 8) The only available choice is on the fourth and the sixth degree of the modal scale, respectively. This fact implies that only two admissible modes can be built on such graph and they differs exactly for one note. So we can choose among two paths on the graph which are exactly the two admissible modes we used to build the graph.

- $maj7$. In $\Gamma_{maj7}$ (figure 10) there are $2^2$ available choices. The modes which generate this graph are three, so there is a special mode which represent the admissible path on $\Gamma_{maj7}$ which is different from the paths representing the Ionian, Lydian and Lydian $2^\#$ scales. The only possible, admissible path is

\[
\{I, aII, MIII, PIV, PV, MV1, MVII\}.
\]

- 7. Four admissible non special modes generate $\Gamma_7$ (see table 5 and figure 12). The total number of admissible modes in this graph is $2^3 = 8$. We expect to find 4 special modes:

\[
\begin{align*}
&\{I, mII, MIII, PIV, PV, MV1, mVII\} \\
&\{I, mII, MIII, aIV, PV, MV1, mVII\} \\
&\{I, MIII, MIII, aIV, PV, mVI, mVII\} \\
&\{I, mII, MIII, aIV, PV, mVI, mVII\}
\end{align*}
\]

- $-7$ and $-7^\#$. This cases are similar to the previous one. $\Gamma_{-7}$ (figure 14) is generated by 5 admissible, non special modes (table 5), so we have 3 special modes which are

\[
\begin{align*}
&\{I, mII, mIII, PIV, PV, mVI, mVII\} \\
&\{I, MIII, mIII, aIV, PV, mVI, mVII\} \\
&\{I, mII, mIII, aIV, PV, mVI, mVII\}
\end{align*}
\]

$\Gamma_{-7^\#}$ (figure 16) is generated by four admissible non special modes (table 5), we have the following four special modes:

\[
\begin{align*}
&\{I, mII, mIII, PIV, dV, mVI, mVII\} \\
&\{I, MIII, mIII, dIV, dV, mVI, mVII\} \\
&\{I, mII, mIII, dIV, dV, MV1, mVII\} \\
&\{I, MIII, mIII, dIV, dV, MV1, mVII\}
\end{align*}
\]

\[\square\]

### 2.3 Topological complexity measure of a base-chord

The aim of this section is to associate a measure of complexity to each base chord.

**Definition 2.19** Given a base-chord $[B]$, let $\mathcal{G}([B])$ be the associated base-chord graph. We call topological complexity measure (t.c.m., for brevity) of $[B]$, i.e. $\tau([B])$, the number of generators of the fundamental group of $\mathcal{G}([B])$.

**Lemma 2.20** Let $[B]$ be a base-chord. The integer $\tau([B])$ is well-defined.
Proof. It is enough to observe that given any base-chord \([B]\), the associated integer \(\tau([B])\) is uniquely defined. In fact by the classification given in section 2, at each base-chord \([B]\) we can uniquely associate a planar connected graph \(\mathcal{G}([B])\). As direct consequence of proposition 2.10, the fundamental group of \(\pi_1(\mathcal{G}([B]))\) is a free group having \(\tau([B])\) generators.

Proposition 2.21 Let \([B]\) be a base-chord, \(\mathcal{G}([B])\) be a planar graph. Then the fundamental group \(\pi_1(\mathcal{G}([B]))\) and the topological measure of complexity are given below:

| Base-chord \([B]\) | \(\pi_1([B])\) | \(\tau([B])\) |
|-------------------|--------------|-----------|
| \(\sigma^7\)      | \(\{1\}\)   | 0         |
| \(\text{maj}^{75}\) | \(\mathbb{Z}\) | 1         |
| \(-\text{maj}^7\) | \(\mathbb{Z}\) | 1         |
| \(\text{maj}^7\)  | \(\mathbb{Z}^{*^2}\) | 2         |
| \(7\)             | \(\mathbb{Z}^{*^3}\) | 3         |
| \(-7\)            | \(\mathbb{Z}^{*^3}\) | 3         |
| \(-7^5\)          | \(\mathbb{Z}^{*^3}\) | 3         |

Proof. The proof follows from the classification given in section 2 and proposition 2.10. By recalling a classical result from algebraic topology, namely that collapsing a contractible subcomplex is a homotopy equivalence, it is possible to provide another proof of the proposition 2.10. In fact, the following holds:

Lemma 2.22 If the pair \((X,A)\) satisfies the homotopy extension property and \(A\) is contractible, then the quotient map \(q : X \to X/A\) is a homotopy equivalence.

For the proof see, for instance [Hat02, Propoisition 0.17, pag.15]. Applying lemma 2.22 to the pair \((\mathcal{G}([B]), T([B]))\) where \(T([B])\) is a maximal tree contained in the graph \(\mathcal{G}([B])\), the first homotopy group of the graph \(\mathcal{G}([B])\) can be easily computed (by using the Seifert Van-Kampen theorem) by observing that the projection to the quotient:

\[
\Pi : \mathcal{G}([B]) \to \mathcal{G}([B])/T([B])
\]

is an homotopy equivalence and that the quotient is homotopic to the wedge of \(\tau([B])\) times \(S^1\) we get the result.

3 Harmonic progressions and braids

Here we start recalling some definitions from braid theory, then we give a braids theoretical interpretation of modes constructed over a fixed base-chord and by using the concatenation property for braids we analyse some chord progressions. Our basic reference is [Han89].

3.1 Braids and links

Definition 3.1 A braid \(\beta\) on \(n\) strands is a collection of embeddings

\[
\mathcal{B} := \{\beta^\alpha; \beta^\alpha : [0,1] \to \mathbb{R}^3, \alpha = 1,\ldots,n\}
\]

with disjoint images such that:

- \(\beta^\alpha(0) = (0,\alpha,0)\);
- \(\beta^\alpha(1) = (1,\tau(\alpha),0)\) for some permutation \(\tau\);
• the images of each $\beta^\alpha$ is transverse to all planes $\{x = \text{const}\}$, where $\mathbb{R}^3$ is equipped with the $(x, y, z)$ coordinates.

![Diagram of transverse images]

Figure 18: The action of generators $\{\sigma_i\}_{i=1}^n$ on the $i^{th}$ strand.

**Definition 3.2** Two such braids are said to lie in the same topological braid class if they are homotopic in the sense of braids: one can deform one braid to the other without any intersection among the strands.

There is a natural group structure on the space of topological braids with $n$ strands, $B_n$, given by concatenation. Using generators $\sigma_i$ which interchanges the $i$th and $(i + 1)$th strands with a positive crossing yields the presentation for $B_n$.

We let

- $p_1 : \sigma_i \sigma_j = \sigma_j \sigma_i; \quad |i - j| > 1$
- $p_2 : \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}; \quad i < n - 1$.

From a graphical point of view, the first property can be visualized as follows:

![Diagram of $p_1$ property]

(a) $p_1$ property: $\sigma_i \sigma_j$
(b) $p_1$ property: $\sigma_j \sigma_i$

The second properties can be displayed as follows:

![Diagram of $p_2$ properties]
Then the presentation of $B_n$ is as follows:

$$B_n := \langle \sigma_1, \ldots, \sigma_{n-1} : p_1 \text{ and } p_2 \text{ hold} \rangle.$$ 

Now we consider $n$ distinct points in the plane, i.e. $(x_j)_{j=1}^n \subset \mathbb{R}^2$. Let

$$\Delta_{ij} = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^{2n} : x_i = x_j \}, \quad i \neq j$$

and borrowing the terminology of celestial mechanics, we can define the \textit{collision set} as:

$$\Delta = \bigcup_{i,j=1}^n \Delta_{ij}.$$ 

Its complement $\hat{\chi}_n(\mathbb{R}^2) := \mathbb{R}^{2n} \setminus \Delta$ is called the \textit{(collision-free) configuration space}. It can be thought as the configuration space for a set of $n$ points without collisions. We equip $\hat{\chi}_n(\mathbb{R}^2)$ with the topology induced from the topology of the Euclidean space. Since the configuration space is the complement of a finite union of two dimensional linear subspaces, by dimensional arguments, readily follows that $\hat{\chi}_n(\mathbb{R}^2)$ is connected.

There is a natural right action of $\mathfrak{S}(n)$ on the configuration space $\hat{\chi}_n(\mathbb{R}^2)$:

$$\mu : \hat{\chi}_n(\mathbb{R}^2) \times \mathfrak{S}(n) \to \hat{\chi}_n(\mathbb{R}^2)$$

deﬁned by permutation of coordinates, i.e.

$$\mu((x_1, x_n), \sigma) = (x_1, \ldots, x_n) \cdot \sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

It is to prove that $\mu$ is indeed a right and free action. Denote the orbit space for the free action $\mu$ by $C_n(\mathbb{R}^2)$, or in standard notation,

$$C_n(\mathbb{R}^2) = \hat{\chi}_n(\mathbb{R}^2) / \mathfrak{S}(n).$$

**Proposition 3.3** The (Artin) braid group $B_n$ can be canonically identiﬁed with the fundamental group

$$\pi_1(C_n(\mathbb{R}^2), c_0).$$
Consider and arrange the voices of each chord, such that \[ \text{We refer to [CQT08, Tym06, Tym09, Tym11, Pis59, Pis47, Pis55].} \]

Set \( B \) voice-leading among them as \[ \text{through braids.} \]

By using the concatenation product in the braid group, we shall represent harmonic progressions \[ \text{3.2 A representation of cadential chord progression through braids} \]

\[ \text{σ section 1.} \]

Every loop in the graphs turns out to be the represented by \( \sigma_i \), see figure \[ \text{[18]} \]

\[ \text{Figure 20: From graph to braids. These braids can be associated to the graphs we represented in section [1].} \]

\[ \text{Proof. Cfr. [Han89, Theorem 3.1, pag 19].} \]

We close the paragraph showing a braid theoretical representation of the planar oriented graphs associated to the base-chords of section \[ \text{II} \]

\[ \text{Figure 20: From graph to braids. These braids can be associated to the graphs we represented in section [1]. Every loop in the graphs turns out to be the represented by } \sigma_i \text{, see figure [18].} \]

\[ \text{3.2 A representation of cadential chord progression through braids} \]

By using the concatenation product in the braid group, we shall represent harmonic progressions \[ \text{[Pis59] Chapter 12]} \text{ through braids.} \]

Given two base-chords \( [B_1] \) and \( [B_2] \), we can treat them as multisets of pitches, we denote the voice-leading among them as \( [B_1] \rightarrow [B_2] \) and we assume the voice leading to be crossing free.

We refer to [CQT108, Tym06, Tym09, Tym11, Pis59, Pis57, Pis58]. Set \( [B_1] = [(p_1, p_2, p_3, p_4)] \) and \( [B_2] = [(q_1, q_2, q_3, q_4)] \), then we can write every note of each base-chord as an element of \( \mathbb{R}/12\mathbb{Z} \), and arrange the voices of each chord, such that

\[ p_i > p_j \Rightarrow q_i \geq q_j \text{ for all } i, j \in \{1, \ldots, 4\} \]

\[ \text{Example 3.4 Consider } [\text{Cmaj7}] = [(C, E, G, B)] \text{ and } [\text{Gmaj7}] = [(G, B, D, F\sharp)] \text{ thus:} \]

\[ [\text{Cmaj7}] = [(0, 4, 7, 11)], \quad [\text{Gmaj7}] = [(7, 11, 2, 6)]. \]

To avoid voice crossings in \( (0, 4, 7, 11) \rightarrow (7, 11, 2, 6) \) we have that

\[ 0 \mapsto 2, \quad 4 \mapsto 5, \quad 7 \mapsto 7, \quad 11 \mapsto 11. \]

\[ \text{which fulfills the condition [2].} \]
A single chord can be represented as a trivial braid, we denote by $B(\cdot)$ the braid associated to the chord $\cdot$. See diagrams (a) and (b) of figure 21 for a braid representation of $[Cmaj7]$ and $[Gmaj7]$. The voice leading $Gmaj7 \rightarrow Cmaj7$ generates a non trivial braid. For example, to move from $[C] = [0] \in [Cmaj7]$ to $[D] = [2] \in [Gmaj7]$ (see figure 21 (c)) we use the generators $\sigma_1 \sigma_2$ (see section 3). In this way it is possible to represent any chord progressions by braids.

$\begin{align*}
\text{(a) } & B(Cmaj7) \\
\text{(b) } & B(Gmaj7) \\
\text{(c) A voice moving on a braid}
\end{align*}$

Figure 21: The braid diagram associated to the base-chords

3.3 Some jazz harmonic progressions

In the following list we give a braid theoretic representation of some of the most relevant jazz harmonic progressions.

- $II - V - I$ and $II - V - VI$. The braid in figure 22 (a), represents the harmonic progression

  $D - 7 \rightarrow G7 \rightarrow Cmaj7$.

  where the second degree prepares the authentic cadence $V - I$.

In figure 22 (b) is represented the harmonic progression

  $D - 7 \rightarrow G7 \rightarrow A - 7$

  where the second degree $D - 7$ prepares the deceptive cadence $V - VI$

$\begin{align*}
\text{(a) } & B(D - 7) \rightarrow B(G7) \rightarrow B(Cmaj7) \\
\text{(b) } & B(D - 7) \rightarrow B(G7) \rightarrow B(A - 7)
\end{align*}$

Figure 22: Authentic and deceptive cadences
\* IV – I and Secondary Dominant. The braid in figure 23 (a) represents the plagal cadence $F\text{maj}7 \rightarrow C\text{maj}7$.

In figure 23 (b) it is possible to see the harmonic progression $F7 \rightarrow C7 \rightarrow G7$.

![Diagram showing harmonic progressions](image)

Figure 23: Plagal cadence and secondary dominant

4 An application: Mixolydian $b2\#4$ in Peru

We introduced in paragraph 3.2 the representation through braids of chords progressions. Let us analyse a modal harmonic structure excerpted from Peru by Tribal Tech (see figure 24).

![An harmonic fragment from Peru by Tribal Tech.](image)

Figure 24: An harmonic fragment from Peru by Tribal Tech.

This choice is due to the clear harmonic and modal analysis of this track which is provided by Scott Henderson in [Hen]. First of all we give a representation through braids of the harmonic progression. The algorithm we used to draw the braids which represent voice-leadings between chords is exactly the one we described in 3.2. See figure 25 for such representation.
The braid 25 (a) is generated by
\[ \sigma_9 \sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1 \sigma_2 \sigma_4 \]

While, braids in figure 25 (b) and (c) have generators
\[ \sigma_{10} \sigma_{11} \sigma_6 \sigma_7 \sigma_5 \sigma_6 \sigma_3 \]
and
\[ \sigma_{11} \sigma_{10} \sigma_7 \sigma_6 \sigma_5 \sigma_4 \sigma_3 \sigma_2 \sigma_1 \sigma_2 \]
respectively. Either an improviser or a composer can see each chord of the structure in figure 24 as a base-chord to build a modal melody. In [Hen] the following modal choices has been used to build a solo:

| F−9 | D−9 | B13♭9 | C/6/9 |
|-----|-----|--------|-------|
| F dorian | D dorian | B octatonic (half step-whole step) | C lydian |

Table 6: Example of modal choices on a given harmonic structure

We do not considered any octatonic scale, however, thanks to the special modes we introduced in proposition 2.18 we can give an heptatonic approximation of the half step, whole step octatonic scale (hs/ws scale for brevity). See table 7 for the notes of the octatonic scale with root note B.

\[
\begin{array}{cccccccc}
B & C & D & D_2 & F & F_2 & G_2 & A \\
\end{array}
\]

Table 7: Octatonic scale. Bold notes belong to a B7 chord
Since the base-chord for the hs-ws scale is a dominant seventh chord, it is reasonable to look for a good approximation of this scale among the modes which are built on a B7:

- B mixolydian: \((B, C^\sharp, D^\sharp, E, F^\sharp, G^\sharp, A)\)
- B lydian dominant: \((B, C^\sharp, D^\sharp, E^\sharp, F^\sharp, G^\sharp, A)\)
- B mixolydian \(\flat\)13: \((B, C^\sharp, D^\sharp, E, F^\sharp, G, A)\)
- B mixolydian \(\flat\)2\(\flat\)13: \((B, C, D^\sharp, E, F^\sharp, G, A)\)
- B mixolydian \(\sharp\)4: \((B, C^\sharp, D^\sharp, E^\sharp, F^\sharp, G^\sharp, A)\)
- B mixolydian \(\flat\)2\(\sharp\)4\(\flat\)13: \((B, C, D^\sharp, E^\sharp, F^\sharp, G, A)\)

where special modes are emphasized. The choice is almost trivial, to find the best approximation of the hs-ws scale it suffices to consider the nearest special mode to the chord \(B^{13}\flat\flat\), which shares as many notes as possible with the hs-ws scale, i.e. the mixolydian \(\flat\)2\(\sharp\)4. The comparison between the two scale is shown in Table 8, where the black square represents the note missing in the approximation of the 8-note scale with a 7-note one. The dropped note respect to the root of the base-chord is the minor third, which is a huge tension for a dominant chord, however the approximation provided by the special mode is a good one, since we are not giving up the tensions that are explicit in the chord \(B^{13}\flat\flat\) and at the same time the modal scale contains the arpeggio of the B7 chord.

\[
\begin{array}{cccccccc}
\text{hs-ws scale} & B & C & D & D^\sharp & F & F^\sharp & G & A \\
\text{mix\(\flat\)2\(\sharp\)4 scale} & B & C & \square & D^\sharp & E^\sharp = F & F^\sharp & G & A \\
\end{array}
\]

Table 8: Comparing the hs-ws scale and the most suitable special mode.

5 Conclusion and future projects

We start this section by putting on evidence the major achievements of this paper. We starting by revising an heptatonic modal scale as a superimposition of two chords, playing two completely different roles. A base-chord \([B]\) which is represented by a seventh chord and a tension-triad constructed above it. Thus we can summarise it by as: “7 = 4 + 3.” From a genuinely musical viewpoint this introduce a striking difference since magnify the harmonic and melodic characteristic of the modal scale.

The second achievement is to associate a 2-dimensional oriented planar graph to each base-chord reflecting its freedom to support tensions. By using this graph we are able to pull out from the same tonal system, or even better on the classical harminization, some special modes and hence some new (heptatonic) modal scales.

The third, from our point of view, interesting achievement is the topological measure of complexity. This is given in terms of the Euler characteristic of a graph, and hence it is stable under homeomorphism of graphs. This is an important property making our construction independent on the choices of the positions of the degrees of the scale. From a mathematical point of view the Euler characteristic is an integer between \(-2\) and \(1\) included. Bigger this invariant, less degrees of freedom a base-chord has to support tension. From a musical point of view this reflects a major degree of freedom in the improvisation as well as composing.

Although this qualitative analysis give us a new insight on the modal scales, it is still poor in order to classify some modern tracks as well as to analyse some popular modern chord progressions and cadences. For this reason in section 4 we introduce a braids theoretical interpretation of the above introduced graphs. This in particular allows us to associate to each chord progression, a braid in a quite effective and explicitly way by showing the voice-leading of the progression.

We finally applied our results in order to better understand Peru. More precisely, by using the graph theoretic results and in particular the new special mode \(\flat\)2\(\flat\)13 we are able to give a “nice”
approximation of the octatonic scale by using an heptatonic one. Of course we have to give up to
one note, but as we showed in the previous section, the note dropped out does not contribute to the
tensions required explicitly by the chord. In fact the modal scale used as approximation includes
the tensions which characterize the octatonic scale and the stable notes needed to be as melodic as
possible on a such altered chord, making our result very reasonable also from the musical point of
view. We point out this approximation is good with respect to any chosen metric (intervallic, etc),
simply because it shares 7 notes with the octatonic scale chosen by Scott Henderson.

Using the braid theoretical approach to analyse Peru, we are able to figure out how complicate
is the chord progression used by Scott Henderson in comparison to the most standard and popular
chord progressions.

In a forthcoming paper we shall try to distinguish about different melodic lines, constructed
over the same base-chord progressions. From a mathematical viewpoint we shall construct some
different topological spaces reflecting the crossings between the strand associated to a melodic line
with respect to the underground braid. This will give us a sort of measure of complexity associated
to a melodic phrase or to an improvisation and even to a classification of jazz and modern standard
tracks.

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