Weak universality for a class of 3d stochastic reaction–diffusion models

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Abstract

We establish the large scale convergence of a class of stochastic weakly nonlinear reaction–diffusion models on a three dimensional periodic domain to the dynamic $\Phi^4_3$ model within the framework of paracontrolled distributions. Our work extends previous results of Hairer and Xu to nonlinearities with a finite amount of smoothness (in particular $C^3$ is enough). We use the Malliavin calculus to perform a partial chaos expansion of the stochastic terms and control their $L^p$ norms in terms of the graphs of the standard $\Phi^4_3$ stochastic terms.

Keywords: weak universality, paracontrolled distributions, stochastic quantisation equation, Malliavin calculus, partial chaos expansion.

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1 Introduction

Consider a family of stochastic reaction–diffusion equation in a weakly nonlinear regime:
\[ \mathscr{L} u(t, x) = -\varepsilon^\alpha F_\varepsilon(u(t, x)) + \eta(t, x), \quad (t, x) \in [0, T/\varepsilon^2] \times (\mathbb{T}/\varepsilon)^3 \]  

(1.1)

with \( \varepsilon \in (0, 1) \), \( T > 0 \), initial condition \( \tilde{u}_{0, \varepsilon} : (\mathbb{T}/\varepsilon)^3 \to \mathbb{R} \), \( F_\varepsilon \in C^0(\mathbb{R}) \) with exponential growth at infinity, \( \alpha > 0 \) and \( \mathscr{L} := (\partial_t - \Delta) \) the heat flow operator and \( \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \). Here \( \eta \) denotes a family of centered Gaussian noises on \([0, T/\varepsilon^2] \times (\mathbb{T}/\varepsilon)^3 \) with stationary covariance

\[ \mathbb{E}(\eta(t, x)\eta(s, y)) = \tilde{C}^\varepsilon(t - s, x, y) \]

such that \( \tilde{C}^\varepsilon(t - s, x, y) = \Sigma(t - s, y - x) \) if \( \text{dist}(x, y) \leq 1 \) and 0 otherwise where \( \Sigma : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^+ \) is a smooth, positive function compactly supported in \([0, 1] \times B_{\mathbb{R}^3}(0, 1) \). We assume also that there exists a compactly supported function \( \psi \) such that \( \psi * \psi = \Sigma \) (this is true e.g. when \( \eta \) is obtained by space-time convolution of the white noise with \( \psi \)).

We look for a large scale description of the solution to eq. (1.1) and we introduce the “mesoscopic” scale variable \( u_\varepsilon(t, x) = \varepsilon^{-\beta}u(t/\varepsilon^2, x/\varepsilon) \) where \( \beta > 0 \). Substituting \( u_\varepsilon \) into (1.1) we get

\[ \mathscr{L} u_\varepsilon(t, x) = -\varepsilon^{\alpha-2-\beta} F_\varepsilon(u_\varepsilon^\beta(t, x)) + \varepsilon^{-2-\beta} \eta \left( \frac{t}{\varepsilon^2}, \frac{x}{\varepsilon} \right). \]

(1.2)

In order for the term \( \varepsilon^{-2-\beta}\eta(t/\varepsilon^2, x/\varepsilon) \) to converge to a non-trivial random limit we need that \( \beta = 1/2 \). Indeed the Gaussian field \( \eta_\varepsilon(t, x) := \varepsilon^{-5/2}\eta(t/\varepsilon^2, x/\varepsilon) \) has covariance \( \varepsilon^{-5}\tilde{C}^\varepsilon(t/\varepsilon^2, x/\varepsilon) \) and converges in distribution to the space-time white noise on \( \mathbb{R} \times \mathbb{T}^3 \). For large values of \( \varepsilon \) the non-linearit will be negligible with respect to the additive noise term. Heuristically, we can attempt an expansion of the reaction term around the stationary solution \( Y_\varepsilon \) to the linear equation

\[ \mathscr{L} Y_\varepsilon = -Y_\varepsilon + \eta_\varepsilon, \]

i.e. \( Y_\varepsilon(t, x) = \int_{-\infty}^t \tilde{P}(t - s, x - y)\eta_\varepsilon(s, y)dsdy \) with \( \tilde{P}(t, x) = \frac{1}{(4\pi t)^{3/2}}e^{-|x|^2/4t}e^{-t}1_{t>0} \).

Let us denote with \( C_\varepsilon \) the covariance of \( Y_\varepsilon \). We approximate the reaction term as

\[ \varepsilon^{\alpha-5/2} F_\varepsilon(u_\varepsilon^2) \sim \varepsilon^{\alpha-5/2} F_\varepsilon(u_\varepsilon^2 Y_\varepsilon(t, x)). \]

The Gaussian r.v. \( \varepsilon^{1/2} Y_\varepsilon(t, x) \) has variance \( \sigma_\varepsilon^2 = \varepsilon \mathbb{E}[(Y_\varepsilon(t, x))^2] = \varepsilon \mathbb{E}[(Y_\varepsilon(0, 0))^2] = \varepsilon C_\varepsilon(0, 0) \) independent of \( (t, x) \). Although \( \sigma_\varepsilon^2 \) depends on \( \varepsilon \), it can be bounded from above and below by two positive constants uniformly on \( \varepsilon \in (0, 1] \). We can expand the random variable \( F_\varepsilon(u_\varepsilon^2 Y_\varepsilon) \) according to the chaos decomposition relative to \( \varepsilon^{1/2} Y_\varepsilon(t, x) \) and obtain

\[ F_\varepsilon(u_\varepsilon^2 Y_\varepsilon(t, x)) = \sum_{n \geq 0} f_{n, \varepsilon} H_n(u_\varepsilon^2 Y_\varepsilon(t, x), \sigma_\varepsilon^2), \]

(1.4)

where \( H_n(x, \sigma_\varepsilon^2) \) are standard Hermite polynomials with variance \( \sigma_\varepsilon^2 \) and highest-order term normalized to 1. Note also that the coefficients \( (f_{n, \varepsilon})_{n \geq 0} \) do not depend on \( (t, x) \) by stationarity of the law of \( \varepsilon^{1/2} Y_\varepsilon(t, x) \) since they are given by the formula

\[ f_{n, \varepsilon} = \frac{1}{n!\sigma_\varepsilon^{2n}} \mathbb{E}[F_\varepsilon(u_\varepsilon^2 Y_\varepsilon(t, x))H_n(u_\varepsilon^2 Y_\varepsilon(t, x), \sigma_\varepsilon^2)] = n! \mathbb{E}[F_\varepsilon(\varepsilon G)H_n(\varepsilon G, \sigma_\varepsilon^2)]. \]
where $G$ is a standard Gaussian variable of unit variance.

Let $X$ be the stationary solution to the equation

$$\mathcal{L} X = -X + \xi,$$

with $\xi$ the space–time white noise on $\mathbb{R} \times \mathbb{T}^3$ and denote by $[X^N]$ the generalized random fields given by the $N$-th Wick power of $X$, which are well defined as random elements of $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^3)$ as long as $N \leq 4$. Denote with $C_X$ the covariance of $X$. The Gaussian analysis which we set up in this paper shows in particular that if \( \varepsilon^{(n-N)/2} f_{n, \varepsilon} \to g_n \) as $\varepsilon \to 0$ for every $0 \leq n \leq N$, $N \leq 4$, and $(F_{\varepsilon})_\varepsilon \subseteq C^{N+1}(\mathbb{R})$ with exponential growth, then the family of random fields

$$\mathbb{P}_\varepsilon : (t, x) \mapsto \varepsilon^{-N/2} F_{\varepsilon}(\varepsilon^{1/2} X_\varepsilon(t, x)), \quad (t, x) \in \mathbb{R} \times \mathbb{T}^3,$$

converges in law in $\mathcal{S}'(\mathbb{R} \times \mathbb{T}^3)$ as $\varepsilon \to 0$ to $\sum_{n=0}^N g_n [X^n]$. Consider the smallest $n$ such that $f_{n, \varepsilon}$ converges to a finite limit as $\varepsilon \to 0$. Since $H_n(\varepsilon^{1/2} Y_\varepsilon, \sigma_2^2) = \varepsilon^{n/2}[Y_\varepsilon^n]$, the $n$-th term in the expansion (1.4) is $f_{n, \varepsilon} \varepsilon^{n+(n-5)/2}[Y_\varepsilon^n]$. Therefore, the equation yields a non-trivial limit only if $\alpha = (5 - \varepsilon)/2$. We are interested mainly in the case $n = 3 \Rightarrow \alpha = 1$ and $n = 1 \Rightarrow \alpha = 2$. The case $\alpha = 2$ gives rise to a Gaussian limit and its analysis is not very difficult. In the following we will concentrate in the analysis of the $\alpha = 1$ case where the limiting behaviour of the model is the most interesting and given by the $\Phi^3$ family of singular SPDEs. In this case we obtain the family of models

$$\mathcal{L} u_\varepsilon(t, x) = -\varepsilon^{-2} F_{\varepsilon}(\varepsilon^{1/2} u_\varepsilon(t, x)) + \eta_\varepsilon(t, x) \quad (1.5)$$

with initial condition $u_{0, \varepsilon}(\cdot) := \varepsilon^{-2} \tilde{u}_{0, \varepsilon}(\varepsilon^{-1} \cdot)$ where $\tilde{u}_{0, \varepsilon}$ is the initial condition of the microscopic model (1.4).

In order to state our main result, Theorem 1.1 below, let us introduce some notations and specify our assumptions. Let $\tilde{F}_{\varepsilon}$ be the centering (up to the third Wiener chaos relative to $\varepsilon^{-1/2} Y_\varepsilon(t, x)$) of the function $F_{\varepsilon}$, i.e.

$$\tilde{F}_{\varepsilon}(x) := F_{\varepsilon}(x) - f_{0, \varepsilon} - f_{1, \varepsilon} x - f_{2, \varepsilon} H_2(x, \sigma_2^2) = \sum_{n \geq 3} f_{n, \varepsilon} H_n(x, \sigma_2^2). \quad (1.6)$$

The decomposition of $\tilde{F}_{\varepsilon}$ is obviously the same as in (1.4) except for the fact that we have discarded the orders 0, 1, 2. Let $F_{\varepsilon}^{(m)}$ be the $m$-th derivative of the function $\tilde{F}_{\varepsilon}$ for $0 \leq m \leq 3$ and define the following $\varepsilon$-dependent constants:

$$d_{\varepsilon} := \frac{\varepsilon^{-2}}{6} \int_{s, x} P_s(x) \mathbb{E}[\tilde{F}_{\varepsilon}^{(1)}(\varepsilon^{1/2} Y_\varepsilon(s, x))\tilde{F}_{\varepsilon}^{(1)}(\varepsilon^{1/2} Y_\varepsilon(0, 0))],$$

$$d_{\varepsilon} := 2\varepsilon^{-1/2} f_{3, \varepsilon} f_{2, \varepsilon} \int_{s, x} P_s(x) (C_\varepsilon(s, x))^2,$$

$$d_{\varepsilon} := 2\varepsilon^{-2} \int_{s, x} P_s(x) \mathbb{E}[\tilde{F}_{\varepsilon}^{(0)}(\varepsilon^{1/2} Y_\varepsilon(s, x))\tilde{F}_{\varepsilon}^{(2)}(\varepsilon^{1/2} Y_\varepsilon(0, 0))],$$

$$d_{\varepsilon} := \frac{\varepsilon^{-5/2}}{3} \int_{s, x} P_s(x) \mathbb{E}[\tilde{F}_{\varepsilon}^{(0)}(\varepsilon^{1/2} Y_\varepsilon(s, x))\tilde{F}_{\varepsilon}^{(1)}(\varepsilon^{1/2} Y_\varepsilon(0, 0))],$$

where $P_s(x)$ is the heat kernel and $\int_{s, x}$ denotes integration on $\mathbb{R}^+ \times \mathbb{T}^3$.

**Assumption 1** All along the paper we enforce the following assumptions:

a) $(u_{0, \varepsilon})_\varepsilon$ converges in law to a limit $u_0$ in $\mathcal{C}^{-1/2-\kappa}$ and is independent of $\eta$;

b) $(\tilde{u}_{0, \varepsilon})_\varepsilon$ is uniformly bounded in $L^\infty$ in probability, i.e. $\exists C > 0$ such that $\forall \varepsilon \in (0, 1] \| \tilde{u}_{0, \varepsilon} \|_{L^\infty((\mathbb{T}/\varepsilon)^3)} \leq C$;
c) \((F_\varepsilon)_\varepsilon \subseteq C^0(\mathbb{R})\) and there exists constants \(c,C > 0\) such that
\[
\sup_{\varepsilon, x} \sum_{k=0}^{9} |\partial_x^k F_\varepsilon(x)| \leq C \varepsilon^{c|\varepsilon|},
\]
(1.8)
d) the family of vectors \(\lambda_\varepsilon = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4\) given by
\[
\begin{align*}
\lambda_3, & \quad \lambda_1 := f_{3, \varepsilon}, \\
\lambda_2, & \quad \lambda_0 := \varepsilon^{-1} f_{1, \varepsilon} - 9d_\varepsilon - 6d_\varepsilon d_\varepsilon - \Psi - 3d_\varepsilon - 3\bar{d}_\varepsilon
\end{align*}
\]
(1.9)
has a finite limit \(\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4\) as \(\varepsilon \to 0\).

**Theorem 1.1 (Convergence of the solution)**
Under Assumption 1 the family of random fields \((u_\varepsilon)_\varepsilon\) given by the solution to eq. (1.9) converges in law and locally in time to a limiting random field \(u(\lambda)\) in the space \(C_T C_\varepsilon^{-\alpha}(\mathbb{T}^3)\) for every \(1/2 < \alpha < 2/3\). The law of \(u(\lambda)\) depends only on the value of \(\lambda\) and not on the other details of the nonlinearity or on the covariance of the noise term. We call this limit the dynamic \(\Phi^4\) model with parameter vector \(\lambda \in \mathbb{R}^4\).

Here \(C_T C_\varepsilon^{-\alpha}(\mathbb{T}^3)\) denotes the space of continuous functions from \([0,T]\) to the Besov space \(C_\varepsilon^{-\alpha}(\mathbb{T}^3) = B_{\infty, \infty}^{-\alpha}(\mathbb{T}^3)\) (see Appendix 1A for the notation on Besov spaces and paraproducts). Theorem 1.1 is actually just a corollary of the more precise result Theorem 2.2 in which we identify the paracontrolled equation satisfied by the limit random field \(u(\lambda)\).

**Remark 1.2** We are interested only in local-in-time convergence of \(u_\varepsilon\), as a way to show the potential of our method for controlling stochastic terms with infinite chaos decomposition (developed in Section 3). Nevertheless, we expect it to be possible to obtain global-in-time convergence of the solution with more stringent assumptions on \(F_\varepsilon\), although we do not treat this problem here.

**Remark 1.3** As a special case we can take
\[
F_\varepsilon(x) = \lambda_3 H_3(x, \sigma^2_\varepsilon) + \varepsilon^{1/2} \lambda_2 H_2(x, \sigma^2_\varepsilon) + \varepsilon (\lambda_1 + \gamma_{1, \varepsilon}) H_1(x, \sigma^2_\varepsilon) + \varepsilon^{3/2}(\lambda_0 + \gamma_{0, \varepsilon})
\]
so that
\[
\begin{align*}
f_{3, \varepsilon} = \lambda_3, & \quad \varepsilon^{-1/2} f_{2, \varepsilon} = \lambda_2, & \quad \varepsilon^{-1} f_{1, \varepsilon} + \gamma_{1, \varepsilon} & = \lambda_1 + \gamma_{1, \varepsilon}, & \quad \varepsilon^{-3/2} f_{0, \varepsilon} = 3 \lambda_0 + \gamma_{0, \varepsilon},
\end{align*}
\]
and
\[
\begin{align*}
d_\varepsilon \Psi & = (\lambda_3)^2 L_\varepsilon, & \bar{d}_\varepsilon \Psi & = \lambda_3 \lambda_2 L_\varepsilon, & \bar{d}_\varepsilon \bar{d}_\varepsilon & = d_\varepsilon = 0,
\end{align*}
\]
where \(L_\varepsilon := 2 \int_{s, \varepsilon} P_\varepsilon(s)(C_{Y, \varepsilon}(s, x))^2\). Choosing
\[
\gamma_{1, \varepsilon} := 9d_\varepsilon \Psi = 9(\lambda_3)^2 L_\varepsilon, & \quad \gamma_{0, \varepsilon} := 3\bar{d}_\varepsilon \Psi = 3 \lambda_3 \lambda_2 L_\varepsilon,
\]
we obtain \(\lambda_\varepsilon \to (\lambda_0, \lambda_1, \lambda_2, \lambda_3)\). This shows that all the possible limits \(\lambda \in \mathbb{R}^4\) are attainable. In this case (1.10) takes the form
\[
\mathcal{L} u_\varepsilon = -\lambda_3 u_\varepsilon^3 - \lambda_2 u_\varepsilon^2 - [\lambda_1 - 3\lambda_3 \varepsilon^{-1} \sigma^2_\varepsilon + 9(\lambda_3)^2 L_\varepsilon] u_\varepsilon - \lambda_0 + \lambda_2 \sigma^2_\varepsilon - 3 \lambda_3 \lambda_2 L_\varepsilon + \eta_\varepsilon.
\]
(1.10)
The name dynamic \(\Phi^4\) equation (or stochastic quantisation equation) derives from the fact that the simplest class of models which approximate the limiting random field \(u(\lambda)\) is precisely obtained by choosing a cubic polynomial like in (1.10) as non-linear term (which is the gradient of a fourth order polynomial playing the role of local potential).
In two dimensions, this model has been subject of various studies since more than thirty years \cite{12, 11, 5}. For the three dimensional case, the kind of convergence results described above are originally due to Hairer \cite{8, 9} and constitute one of the first groundbreaking applications of his theory of regularity structures. Similar results were later obtained by Catellier and Chouk \cite{4} using the paracontrolled approach of Gubinelli, Imkeller and Perkowski \cite{6}. Kupiainen \cite{13} described a third approach using renormalization group ideas.

Weak universality is the observation that the same limiting object describes the large scale behaviour of the solutions of more general equations, in particular that of the many parameters present in a general model, only a finite number of their combinations survive in the limit to describe the limiting object. The adjective “weak” is related to the fact that in order to control the large scale limit the non-linearity has to be very small in the microscopic scale. This sets up a perturbative regime which is well suited to the analysis via regularity structures or paracontrolled distributions.

The first result of weak universality for a singular stochastic PDE has been given by Hairer and Quastel \cite{10} in the context the Kardar–Parisi–Zhang equation. Using the machinery developed there Hairer and Wu \cite{11} proved a weak universality result for three dimensional reaction–diffusion equations driven by non Gaussian noise and a polynomial non–linearity, within the context of regularity structures. Weak universality for reaction–diffusion equations driven by non Gaussian noise is analysed in Shen and Wu \cite{22}. Recently, important results concerning the stochastic quantisation equation we obtained by Mourrat and Weber. In particular the convergence to the dynamic $\Phi^4_3$ model for a class of Markovian dynamics of discrete spin systems \cite{15} and also the global wellposedness of $\Phi^4_3$ in space and time \cite{16} and in time for $\Phi^4_3$ \cite{17}. The recent preprint \cite{21} analyzes an hyperbolic version of the stochastic quantisation equation in two dimensions, including the associated universality in the small noise regime.

The present work is the first which considers in detail the weak universality problem in the context of paracontrolled distributions, showing that on the analytic side the apriori estimates can be obtained via standard arguments and that the major difficulty is related to showing the convergence of a finite number of random fields to universal limiting objects. The main novelty of our work is our use of the Malliavin calculus \cite{20, 19} to perform the analysis of these stochastic terms without requiring polynomial non–linearity as in the previous works cited above. In particular we were inspired by the computations in \cite{25} and in general by the use of the Malliavin calculus to establish normal approximations \cite{19}. The main technical results of our paper, Theorem 3.1 below, is not particularly linked to paracontrolled distributions. A similar analysis is conceivable for the stochastic models in regularity structures. Moreover the same tools can also allow to prove similar non-polynomial weak universality statements for the KPZ along the lines of the present analysis. This is the subject of ongoing work.

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2 Analysis of the mesoscopic model

The goal of this section is to obtain a paracontrolled structure for equation \cite{15} analogous to that introduced in \cite{4} for the cubic polynomial case and use it to set up the limiting procedure. Convergence of the stochastic terms and some apriori estimates will be the subject of the following sections. Definitions and a reminder of the basic results of paradifferential calculus needed here can be found in Appendix A.
\section{Paracontrolled structure}

Let us start our analysis by centering the reaction term $F_\varepsilon(\varepsilon^{1/2}u_\varepsilon)$ in \eqref{1.5} using decomposition \eqref{1.6} to obtain:

\[
\mathcal{L} u_\varepsilon = -\varepsilon^{-\frac{3}{2}} \tilde{F}_\varepsilon(\varepsilon^{\frac{1}{2}}u_\varepsilon(t, x)) + \eta_\varepsilon - \varepsilon^{-3/2} f_{0, \varepsilon} - \varepsilon^{-1} f_{1, \varepsilon} u_\varepsilon - \varepsilon^{-3/2} f_{2, \varepsilon} H_2 \left( \varepsilon^{\frac{1}{2}} u_\varepsilon, \sigma_\varepsilon^2 \right).
\]

We write $u_\varepsilon = Y_\varepsilon + v_\varepsilon$ with $Y_\varepsilon$ as in \eqref{1.3}, and perform a Taylor expansion of $\tilde{F}_\varepsilon(\varepsilon^{1/2}Y_\varepsilon + \varepsilon^{1/2}v_\varepsilon)$ around $\varepsilon^{1/2}Y_\varepsilon$ up to the third order to get

\[
\mathcal{L} u_\varepsilon = \eta_\varepsilon - \varepsilon^{-\frac{3}{2}} \tilde{F}_\varepsilon(\varepsilon^{\frac{1}{2}}Y_\varepsilon) - \varepsilon^{-1} \tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) v_\varepsilon - \frac{1}{2} \varepsilon^{-\frac{3}{2}} \tilde{F}_\varepsilon^{(2)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) v_\varepsilon^2 - \frac{1}{6} \varepsilon^{-\frac{3}{2}} \tilde{F}_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) v_\varepsilon^3
\]

\[-\varepsilon^{-3/2} f_{0, \varepsilon} - \varepsilon^{-1} f_{1, \varepsilon} (Y_\varepsilon + v_\varepsilon) - \varepsilon^{-3/2} f_{2, \varepsilon} (Y_\varepsilon^2 + 2 v_\varepsilon Y_\varepsilon + v_\varepsilon^2) - R_\varepsilon(v_\varepsilon),
\]

where $R_\varepsilon(v_\varepsilon)$ is the remainder of the Taylor series and we used the fact that $H_2(\varepsilon^{1/2}Y_\varepsilon, \sigma_\varepsilon^2) = \varepsilon \llbracket Y_\varepsilon^2 \rrbracket$. Notice that we stopped the Taylor expansion at the first term for which $\varepsilon$ does not appear anymore with a negative exponent (that is $\tilde{F}_\varepsilon^{(3)}(\varepsilon^{1/2}Y_\varepsilon)$). One can then expect the remainder $R_\varepsilon(v_\varepsilon)$ to converge to zero in some sense. On the other hand, all the other terms except $\tilde{F}_\varepsilon^{(3)}(\varepsilon^{1/2}Y_\varepsilon)$ and $R_\varepsilon(v_\varepsilon)$ appear to diverge in the limit $\varepsilon \to 0$, but in analogy with well-known renormalization methods for random fields, we try to find a combination of them that can be made to converge in some function space. Define the following random fields:

\[
\begin{align*}
\mathcal{L} Y_\varepsilon & := -Y_\varepsilon + \eta_\varepsilon \\
\tilde{Y}_\varepsilon \tilde{Y}_\varepsilon & := \varepsilon^{-1/2} f_{2, \varepsilon} \llbracket Y_\varepsilon^2 \rrbracket \\
Y_\varepsilon \tilde{Y}_\varepsilon & := \varepsilon^{-\frac{3}{2}} \tilde{F}_\varepsilon(\varepsilon^{\frac{1}{2}}Y_\varepsilon) \\
Y_\varepsilon Y_\varepsilon & := \varepsilon^{-1} \tilde{F}_\varepsilon^{(1)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) Y_\varepsilon \\
Y_\varepsilon Y_\varepsilon & := \frac{1}{6} \varepsilon^{-\frac{3}{2}} \tilde{F}_\varepsilon^{(3)}(\varepsilon^{\frac{1}{2}}Y_\varepsilon) \\
\tilde{Y}_\varepsilon \tilde{Y}_\varepsilon & := \tilde{Y}_\varepsilon \circ Y_\varepsilon \tilde{Y}_\varepsilon - d_\varepsilon \tilde{Y}_\varepsilon Y_\varepsilon \\
Y_\varepsilon \tilde{Y}_\varepsilon & := Y_\varepsilon \circ Y_\varepsilon \tilde{Y}_\varepsilon - d_\varepsilon Y_\varepsilon Y_\varepsilon \\
Y_\varepsilon \circ Y_\varepsilon - d_\varepsilon & := Y_\varepsilon \circ Y_\varepsilon - d_\varepsilon Y_\varepsilon - Y_\varepsilon - Y_\varepsilon - Y_\varepsilon
\end{align*}
\]

with $Y_\varepsilon$ stationary solution, while $Y_\varepsilon, Y_\varepsilon, Y_\varepsilon, Y_\varepsilon$ have 0 initial condition in $t = 0$. The last four trees $\tilde{Y}_\varepsilon \tilde{Y}_\varepsilon, Y_\varepsilon \tilde{Y}_\varepsilon, Y_\varepsilon \tilde{Y}_\varepsilon$ are obtained from the others via the resonant Bony’s paraproduct $\circ$ recalled in Appendix \ref{appendix} and $d_\varepsilon, d_\varepsilon, d_\varepsilon, d_\varepsilon$ are just $\varepsilon$-dependent constants whose exact value will matter only in Section \ref{section3}. Indeed, in the scope of this section we only need the following relation to be verified:

\[
d_\varepsilon Y_\varepsilon = 2d_\varepsilon Y_\varepsilon + 3d_\varepsilon Y_\varepsilon.
\]

The notation $\tilde{Y}_\varepsilon Y_\varepsilon$ denotes that this tree has finite chaos expansion and can be treated with the well-known techniques of \ref{Bony} or \ref{Hara} (we put a bar on $\tilde{Y}_\varepsilon Y_\varepsilon$ just because is the only tree obtained from $\tilde{Y}_\varepsilon Y_\varepsilon$)

With the definitions \eqref{2.2}, equation \eqref{2.1} takes the form

\[
\mathcal{L} v_\varepsilon = Y_\varepsilon - \tilde{Y}_\varepsilon Y_\varepsilon - 2 Y_\varepsilon Y_\varepsilon - 3 Y_\varepsilon Y_\varepsilon - 3 Y_\varepsilon Y_\varepsilon - 3 Y_\varepsilon Y_\varepsilon - Y_\varepsilon Y_\varepsilon - Y_\varepsilon Y_\varepsilon - Y_\varepsilon Y_\varepsilon
\]

\[-\varepsilon^{-3/2} f_{0, \varepsilon} - \varepsilon^{-1} f_{1, \varepsilon} (Y_\varepsilon + v_\varepsilon) - \varepsilon^{-1/2} f_{2, \varepsilon} (2 Y_\varepsilon v_\varepsilon + v_\varepsilon^2) - R_\varepsilon(v_\varepsilon).
\]
At this point it is worth noting that the trivial case $\hat{F}_e(x) = H_3(x, \sigma^2_e)$ yields $Y_e^{\Psi^\gamma} = [Y_e^{\Psi^\gamma}], Y_e^{\Psi^\gamma} = [Y_e^{\Psi^\gamma}], Y_e^{\Psi^\gamma} = Y_e$, $Y_e^{\Psi^\gamma} = 1$. By comparing these random fields to the ones defined in [1] we can *guess* that $Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}$ can be controlled respectively in $\mathcal{C}^{1/2-\kappa}, \mathcal{C}^{1-\kappa}, \mathcal{C}^{1/2-\kappa}, \mathcal{C}^{\kappa}$ for any $F_e$ satisfying Assumption [1] and carry on the paracontrolled analysis of (2.3) as if it were the case. Clearly, the paracontrolled structure is robust and does not depend on how the terms $Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}$ are defined as long as they have the desired regularity.

From these observations, we do not expect to be able to control the products $Y_e^{\Psi^\gamma}v_e, Y_e^{\Psi^\gamma}v_e^2$ and $Y_e^{\Psi^\gamma}v_e$ in eq. (2.4) uniformly in $\varepsilon > 0$. In order to proceed with the analysis we make the Ansatz:

$$\begin{align*}
  v_e &= Y_e + v_e, \\
v_e &= -Y_e^{\Psi^\gamma} - Y_e^{\Psi^\gamma} - 3v_e < Y_e^{\Psi^\gamma} + v_e^2
\end{align*}$$

(5.5)

and proceed to decompose the ill-defined products using the paracontrolled techniques recalled in Appendix A. We start by writing $v_e Y_e^{\Psi^\gamma} = v_e < Y_e^{\Psi^\gamma} + v_e > Y_e^{\Psi^\gamma} + v_e \circ Y_e^{\Psi^\gamma}$. The resonant term, together with Ansatz (2.5), yields:

$$v_e \circ Y_e^{\Psi^\gamma} = -Y_e^{\Psi^\gamma} \circ Y_e^{\Psi^\gamma} - Y_e^{\Psi^\gamma} \circ Y_e^{\Psi^\gamma} - 3v_e(Y_e^{\Psi^\gamma} \circ Y_e^{\Psi^\gamma}) - 3\text{com}(v_e, Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}) + v_e^2 \circ Y_e^{\Psi^\gamma},$$

with the definition and bounds of $\text{com}_2(\cdot, \cdot, \cdot)$ given in Lemma A.7. Then we define

$$Y_e^{\Psi^\gamma} v_e := v_e Y_e^{\Psi^\gamma} - v_e < Y_e^{\Psi^\gamma} + (3v_e Y_e^{\Psi^\gamma} + d_e Y_e^{\Psi^\gamma} + d_e Y_e^{\Psi^\gamma})$$

$$= v_e > Y_e^{\Psi^\gamma} - Y_e^{\Psi^\gamma} - Y_e^{\Psi^\gamma} - 3v_e Y_e^{\Psi^\gamma} + v_e^2 \circ Y_e^{\Psi^\gamma} - 3\text{com}(v_e, Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}).$$

Moreover we have for $v_e Y_e$:

$$v_e Y_e = \varphi_e Y_e - Y_e^{\Psi^\gamma} \prec Y_e - Y_e^{\Psi^\gamma} \succ Y_e - Y_e^{\Psi^\gamma} \circ Y_e,$$

where we introduced the shorthand $\varphi_e := v_e + Y_e^{\Psi^\gamma}$. So we let

$$v_e \circ Y_e := v_e Y_e + d_e Y_e^{\Psi^\gamma} = \varphi_e Y_e - Y_e^{\Psi^\gamma} \prec Y_e - Y_e^{\Psi^\gamma} \succ Y_e - Y_e^{\Psi^\gamma}.$$  

Finally to analyse the product $Y_e^{\Psi^\gamma} v_e^2$ we write

$$Y_e^{\Psi^\gamma} v_e^2 = Y_e^{\Psi^\gamma} (Y_e^{\Psi^\gamma})^2 - 2Y_e^{\Psi^\gamma} \varphi_e + Y_e^{\Psi^\gamma} \varphi_e^2,$$

and consider the products involving only $Y^\tau$ factors: first

$$Y_e^{\Psi^\gamma} Y_e^{\Psi^\gamma} = Y_e^{\Psi^\gamma} \succ Y_e^{\Psi^\gamma} + Y_e^{\Psi^\gamma} \prec Y_e^{\Psi^\gamma} + Y_e^{\Psi^\gamma} \circ Y_e^{\Psi^\gamma} + d_e Y_e^{\Psi^\gamma} := Y_e^{\Psi^\gamma} \circ Y_e^{\Psi^\gamma} + d_e Y_e^{\Psi^\gamma},$$

and then we define the term $Y_e^{\Psi^\gamma} \circ (Y_e^{\Psi^\gamma})^2$ as follows:

$$Y_e^{\Psi^\gamma} \circ (Y_e^{\Psi^\gamma})^2 := Y_e^{\Psi^\gamma} (Y_e^{\Psi^\gamma})^2 - 2d_e Y_e^{\Psi^\gamma} Y_e^{\Psi^\gamma}$$

$$= Y_e^{\Psi^\gamma} \prec (Y_e^{\Psi^\gamma})^2 + Y_e^{\Psi^\gamma} \succ (Y_e^{\Psi^\gamma})^2 + Y_e^{\Psi^\gamma} \circ (Y_e^{\Psi^\gamma} Y_e^{\Psi^\gamma}) + 2\text{com}_1(Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}, Y_e^{\Psi^\gamma}) + 2Y_e^{\Psi^\gamma} Y_e^{\Psi^\gamma} Y_e^{\Psi^\gamma}.$$
so that
\[ Y^I_e \circ v^2_e := Y^I_e \circ v^2_e + 2d_e \circ v_e = Y^I_e \circ (Y^I_e)^2 - 2(Y^I_e \circ Y^I_e) \varphi_e + Y^I_e \varphi^2_e. \]

We note also that
\[ \mathcal{L} v_e = -\mathcal{L} Y^I_e - \mathcal{L} \bar{Y}_e^I + \mathcal{L} v^2_e - 3v_e < \mathcal{L} Y^I_e - 3 \text{com}_3(v_e, Y^I_e) - 3 \text{com}_2(v_e, Y_e^\vee), \]
with \( \text{com}_2(\cdot, \cdot) \) and \( \text{com}_3(\cdot, \cdot) \) specified in Lemma A.7. Substituting these renormalized products into (2.4) and recalling the definition (1.9) for \( \lambda_e = (\lambda_{0,e}, \lambda_{1,e}, \lambda_{2,e}, \lambda_{3,e}) \), we obtain the following equation for \( v^2_e \):
\[
\begin{align*}
\mathcal{L} v^2_e &= 3 \text{com}_3(v_e, Y^I_e) + 3 \text{com}_2(v_e, Y_e^\vee) \\
&\quad - Y^\sigma_e v^3_e - 3 Y^I_e \circ v^2_e - 3 Y^\vee_e \phi v_e \\
&\quad + Y_e - \lambda_{2,e} (2v_e \circ Y_e + v^2_e) \\
&\quad - \lambda_{1,e} (Y_e + v_e + |9 d_e^V| + 6 d_e^V + 3d_e^V \phi v_e - \lambda_{0,e} - R_e(v_e),
\end{align*}
\]
where we can use the constraint (2.3) to remove the term proportional to \( v_e \). Summarizing, we obtain the following equation, together with Ansatz (2.5):
\[
\begin{aligned}
\left\{ \begin{array}{l}
v_e = -Y^I_e - 3v_e < Y^I_e + v^2_e \\
\mathcal{L} v^2_e = U(\lambda_e, Y^I_e; v_e, v^2_e) - R_e(v_e)
\end{array} \right.
\end{aligned}
\]
with initial condition \( v_{e,0} = u_{0,e} - Y_e(0) \) and \( U \) given by
\[
U(\lambda_e, Y^I_e; v_e, v^2_e) := 3 \text{com}_3(v_e, Y^I_e) + 3 \text{com}_2(v_e, Y_e^\vee) - Y^\sigma_e v^3_e \\
-3 Y^I_e \circ v^2_e - 3 Y^\vee_e \phi v_e + Y_e - \lambda_{2,e} (2v_e \circ Y_e + v^2_e) \\
- \lambda_{1,e} (Y_e + v_e) - \lambda_{0,e} - R_e(v_e).
\]

The enhanced noise vector \( \Psi_e \) is defined by
\[
\Psi_e := (Y^\sigma_e, Y^I_e, Y_v^\vee, \bar{Y}_v^\vee, Y^\vee_e, \bar{Y}_v^\vee, \bar{Y}_e^\vee, \bar{Y}_v^\vee, \bar{Y}_e^\vee)
\]
\[
\mathcal{X}_e := C_T e^{-\kappa} \times C_T e^{-\frac{\kappa}{2}} \times (C_T e^{-\kappa})^2 \times \mathcal{L}^{1/2 - \kappa} \times (C_T e^{-\kappa})^3 \times C_T e^{-\frac{\kappa}{2}}
\]
for every \( \kappa > 0, T > 0 \). We use the notation \( \|\Psi_e\|_{\mathcal{X}_e} = \sum_\tau \|\Psi^\tau_e\|_{\mathcal{X}_e} \) for the associated norm where \( Y^\tau_e \) is a generic tree in \( \Psi_e \). The homogeneities \( \tau \in \mathbb{R} \) are given by
\[
\begin{align*}
\begin{array}{cccccccc}
Y^\tau_e & = & Y^\sigma_e & Y^I_e & Y_v^\vee & \bar{Y}_v^\vee & Y^\vee_e & \bar{Y}_v^\vee & \bar{Y}_e^\vee \\
|\tau| & = & 0 & -1/2 & -1 & -1 & 1/2 & 0 & 0 & -1/2
\end{array}
\end{align*}
\]
Note that for every \( \varepsilon > 0 \) eq. (2.6) is equivalent to eq. (1.6).
Remark 2.1 The paracorrelated structure we developed in this section is the same as in the work of Catellier and Chouk [1], plus an additive source term (which is $R_e(v_e)$ in equation (2.6)). Therefore, there exists $T = T (\|Y_e\|_{L^p}, \|u_e, 0\|_{E^{-1/2-\varepsilon}}, |\lambda_e|)$ such that we can define for $\alpha \in (1/2, 2/3), p \in [4, \infty)$, $\gamma > 1/4 + 3/2\varepsilon$ a solution map

$$
\Gamma : \mathcal{C}^{-1/2-\varepsilon} \times X^{\tau} \times \mathbb{R}^4 \times \mathcal{M}\frac{\gamma}{2}L^p(\mathbb{T}^3) \to \mathcal{C}^{\frac{\varepsilon}{2}}C^{-\alpha}(\mathbb{T}^3)
$$

so that $u_e = \Gamma(u_e, 0, Y_e, \lambda_e, R)$ with $u_e = Y_e + v_e$ and $v_e$ that solves (2.6), with the remainder $R_e(v_e)$ replaced by $R$. The space $\mathcal{M}\frac{\gamma}{2}L^p(\mathbb{T}^3)$ is specified in Appendix A. Indeed, we can use Lemma A.3 and Lemma A.3 to control $IR$ as

$$
\|IR\|_{\mathcal{C}^{1/2, 1+2\varepsilon}} \lesssim T^\delta \|R\|_{\mathcal{M}\frac{\gamma}{2}L^p}
$$

for $\delta > 0$ small enough, and thanks to this bound it is easy to see that the the fixed point procedure of Section 3 of [1] still holds with a fixed additive source term $R$. In the same way, the continuity of the solution map $\Gamma$ follows easily as in Theorem 1.2 of [1].

### 2.2 Identification of the limit

In order to identify interesting limits for equation (1.5), we introduce $\forall \lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4$ the enhanced noise $\mathcal{Y}(\lambda)$ which is constructed from universal noises $X^{\tau}$ as

$$
\mathcal{Y}(\lambda) := (Y^{\tau}(\lambda), Y^{\tau}(\lambda), Y^{\tau}(\lambda), Y^{\tau}(\lambda), Y^{\tau}(\lambda), Y^{\tau}(\lambda), Y^{\tau}(\lambda))
$$

where $X$ is the stationary solution to the linear equation $\mathcal{L} X = -X + \xi$ and $\xi$ is the time-space white noise on $\mathbb{R} \times \mathbb{T}^3$. We will sometimes use the shorter notation $\mathcal{Y}(\lambda) = (Y^{\tau}(\lambda))_X$ for (2.9).

We define the universal fields $X^{\tau}$ through their Littlewood-Paley decomposition $\forall (t, \bar{x}) \in \mathbb{R}^+ \times \mathbb{T}^3$ as:

$$
\begin{align*}
X^{\mathcal{Y}} &:= [X^{\mathcal{Y}}], \quad \mathcal{L} X^{\mathcal{Y}} = X^{\mathcal{Y}}, \quad \text{with } X^{\mathcal{Y}}(t = 0) = 0, \\
X^{\mathcal{Y}} &:= [X^{\mathcal{Y}}], \\
\Delta_{q} X^{\mathcal{Y}}(t, \bar{x}) &:= \Delta_{q}(1 - J_0)(X^{\mathcal{Y}} \circ X^{\mathcal{Y}})(t, \bar{x}) = \int_{\zeta_1, \zeta_2} (1 - J_0)([X^2(\zeta_1)][X^2(\zeta_2)]) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_{q} X^{\mathcal{Y}}(t, \bar{x}) &:= \Delta_{q}(X^{\mathcal{Y}} \circ X)(t, \bar{x}) = \int_{\zeta_1, \zeta_2} [X^2(\zeta_1)][X(\zeta_2)] \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_{q} X^{\mathcal{Y}}(t, \bar{x}) &:= \int_{\zeta_1, \zeta_2} (1 - J_1)([X^3(\zeta_1)][X^2(\zeta_2)]) \mu_{q, \zeta_1, \zeta_2} + 6 \int_{s, \bar{x}} [\Delta_{q} X(t + s, \bar{x} - x) - \Delta_{q} X(t, \bar{x})] P_s(x)[C(s, x)]^2,
\end{align*}
$$

where as before $[\ ]$ stands for the Wick product, $\zeta_i = (x_i, s_i) \in \mathbb{R} \times \mathbb{T}^3$, $C(\cdot, \cdot)$ is the covariance of $X$ and $\mu_{q, \zeta_1, \zeta_2}$ is the measure

$$
\mu_{q, \zeta_1, \zeta_2} := \delta(t - s_2) d\zeta_1 d\zeta_2 \int_{x, y} K_{q, \bar{x}}(x) \sum_{i, j} K_{i, x}(y) K_{j, x}(y) P_{t-s_1}(y - x)
$$

with the usual heat kernel $P_t(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{x^2}{4t}} 1_{t > 0}$. We commit an abuse of notation by writing $X(\zeta)$ since $X$ is actually a distribution in space: the integrals in (2.10) should obviously be intended as functionals.
Standard computations (see e.g. [1] or [18]) show that, \( \forall \lambda \in \mathbb{R}^d \) and for any \( T > 0, 0 < \kappa < \kappa' \)
\[
\mathbb{Y}(\lambda) \in C_T^\kappa \mathcal{C}_{-\frac{1}{2} - 2\kappa'} \times \left( C_T^{\frac{\kappa}{2}} \mathcal{C}_{-\frac{1}{2} - 2\kappa'} \right)^2 \times C_T^{\kappa} \mathcal{C}_{-\frac{1}{2} - 2\kappa'},
\]
almost surely.

Using the paracontrolled structure we developed in Section 2.1 we can identify the limiting solution \( u(\lambda) \) introduced in Theorem 1.1.

**Theorem 2.2**

The family of random fields \( u_\varepsilon \) given by the solutions of eq. (1.5) converges in law and locally in time to a limiting random field \( u(\lambda) \) in the space \( C_T^\kappa \mathcal{C}_{-\alpha} (\mathbb{T}^3) \) for every \( 1/2 < \alpha < 2/3 \). The limiting random field \( u(\lambda) \) solves the paracontrolled equation

\[
\begin{align*}
 u(\lambda) &= X + v(\lambda), \\
 v(\lambda) &= -\lambda_2 X Y - \lambda_3 Y X - 3 \lambda_3 v(\lambda) \quad \in \mathcal{L}^\infty (\mathbb{T}^3), \\
 v^\kappa(\lambda) &\approx \mathcal{L}^\infty (\mathbb{T}^3), \\
 v^\beta(\lambda)(t = 0) &= v_0 + \lambda_3 X Y (t = 0) + \lambda_2 X Y (t = 0) + 3 \lambda_3 v_{\ast,0} X Y (t = 0)
\end{align*}
\]

with \( U \) defined in (2.7) and \( v_0 = u_0 - X (t = 0) \).

**Proof** Fix \( T > 0 \). Let \( u_\varepsilon = Y_\varepsilon + v_\varepsilon \) be the solution of eq. (1.5) for fixed \( \varepsilon > 0 \), which is seen to be unique in the \((\varepsilon\)-dependent) time interval \([0, T_\varepsilon]\) by a usual fixed-point argument on the original equation (without resorting to the paracontrolled decomposition). Let \( u_\varepsilon = \Gamma(u_{\varepsilon,0}, Y_\varepsilon, \lambda_\varepsilon, R_\varepsilon(v_\varepsilon)) \) on \([0, T_\varepsilon]\) with \( \Gamma \) defined as in Remark 2.1 and \( R_\varepsilon(v_\varepsilon) \) seen as an exogenous source term. We know from the a-priori estimations of Section 4.2 that there exists a time \( T_* = T_* (\| Y_\varepsilon \|_{\mathcal{X}_T}, \| u_{\varepsilon,0} \|_{\mathcal{C}_{-1/2 - \kappa}}, |\lambda_\varepsilon|) \) and a family of events \( (\mathcal{E}_\varepsilon)_{\varepsilon > 0} \) such that \( \mathbb{P}(\mathcal{E}_\varepsilon) \to 1 \) for \( \varepsilon \to 0 \) and we can control \( \| v_\varepsilon(t) \|_{\mathcal{M}_{1/4 + 3\kappa/2} L^\infty} \). Thus, we can control the \( L^\infty \) norm of \( v_\varepsilon(t) \) in \([T_\varepsilon, T_\varepsilon/2, T_\varepsilon] \) and extend the solution \( v_\varepsilon \) on \([0, T_*]\) for every \( \varepsilon \). Denote by \( u_\varepsilon^* = \Gamma^*(u_{\varepsilon,0}, Y_\varepsilon, \lambda_\varepsilon, R_\varepsilon(v_\varepsilon)) \) the process \( u_\varepsilon \) stopped at time \( T_* \) and \( \Gamma^* \) the corresponding stopped solution map.

Note that \( u(\lambda) \) solves the same equation as \( u_\varepsilon^* \) with \( Y_\varepsilon \) replaced by \( \mathbb{Y}(\lambda) \), \( u_{\varepsilon,0} \) replaced by \( u_0 \), \( \lambda_\varepsilon \) replaced by \( \lambda \) and \( R_\varepsilon(v_\varepsilon) \) equals 0. So \( u(\lambda) = u_\varepsilon^* = \Gamma^*(u_0, \mathbb{Y}(\lambda), \lambda, 0) \) up to time \( T_* \). Let us introduce the random field \( \bar{u}_\varepsilon^* = \Gamma^*(u_{\varepsilon,0}, Y_\varepsilon, \lambda_\varepsilon, 0) \) with \( \bar{u}_\varepsilon^* = \bar{u}_\varepsilon^* - Y_\varepsilon \) that solves the paracontrolled equation (2.6) but with remainder \( R_\varepsilon(v_\varepsilon) = 0 \).

Consider the n-tuple of random variables \((u_{\varepsilon,0}, Y_\varepsilon, u_\varepsilon^*, \bar{u}_\varepsilon^*)\) and let \( \mu_\varepsilon \) be its law on \( \mathcal{Z} = \mathcal{C}_{-\alpha} \times \mathcal{X}_T \times (C_T^{\kappa} \mathcal{C}_{-\alpha})^2 \) conditionally on \( \mathcal{E}_\varepsilon \). Observe that \( \Gamma^* \) is continuous as discussed in Section 2.1 and this gives that \( \forall \delta > 0, \mu_\varepsilon(\| u_\varepsilon^* - \bar{u}_\varepsilon^* \|_{C_T^{\kappa} \mathcal{C}_{-\alpha}} > \delta) \to 0 \) as \( \varepsilon \to 0 \). Indeed \( R_\varepsilon(v_\varepsilon) \to 0 \) in probability in the space \( \mathcal{M}_{1/4 + 3\kappa/2} L^\infty \) by Lemma 4.5. This shows that \( \mu_\varepsilon \) concentrates on \( \mathcal{C}_{-\alpha} \times \mathcal{X}_T \times \{(z, z) \in (C_T^{\kappa} \mathcal{C}_{-\alpha})^2 \} \). Let \( \mu \) any accumulation point of \( (\mu_\varepsilon)_\varepsilon \). Then \( \mu \left( \mathcal{C}_{-\alpha} \times \mathcal{X}_T \times \{(z, z) \in (C_T^{\kappa} \mathcal{C}_{-\alpha})^2 \} \right) = 1 \). The apriori estimations of Section 4.2 yield the tightness of \( \mu_\varepsilon \) and from the concentration of \( \mu_\varepsilon \) on the diagonal we know that there exists a subsequence such that for any test function \( \varphi \),

\[
\int_Z \varphi(x, y, z, t) \mathrm{d}\mu_\varepsilon(x, y, z, t) \to \int_Z \varphi(x, y, z, t) \mathrm{d}\mu(x, y, z, t) = \int_Z \varphi(x, y, t, t) \mathrm{d}\mu(x, y, z, t). \tag{2.12}
\]

Moreover, still along subsequences we have that for any bounded continuous function \( \varphi \)

\[
\mathbb{E}(\varphi(u_{\varepsilon,0}, Y_\varepsilon, \bar{u}_\varepsilon^*)) = \mathbb{E}(\varphi(u_{\varepsilon,0}, Y_\varepsilon, \Gamma^*(u_{\varepsilon,0}, Y_\varepsilon, \lambda_\varepsilon, 0))) \to \mathbb{E}(\varphi(u_0, \mathbb{Y}(\lambda), \Gamma^*(u_0, \mathbb{Y}(\lambda), \lambda, 0))).
\]
since by Theorem 3.1 the vector $\mathcal{Y}_\varepsilon$ converges in law to $\mathcal{Y}(\lambda)$, $u_{\varepsilon,0}$ to $u_0$, and $\Gamma_\ast$ is a continuous function as discussed in Remark 2.1. This shows that

$$\int_{\mathcal{Z}} \varphi(x, y, t, t) d\mu_{\varepsilon}(x, y, z, t) \to \int_{\mathcal{Z}} \varphi(x, y, \Gamma_\ast(x, y,\lambda,0), \Gamma_\ast(x, y,\lambda,0)) d\mu(x, y, z, t)$$

(2.13)

and then by comparing (2.12) and (2.13) we can conclude that there exists a subsequence such that

$$\int_{\mathcal{Z}} \varphi(x, y, z, t) d\mu_{\varepsilon}(x, y, z, t) \to \int_{\mathcal{Z}} \varphi(x, y, \Gamma_\ast(x, y,\lambda,0), \Gamma_\ast(x, y,\lambda,0)) d\mu(x, y, z, t).$$

We can identify the limit distribution $\mu$ by noting that since $\mathbb{P}(\mathcal{E}_\varepsilon) \to 1$ we have

$$\mathbb{E}[\psi(u_{\varepsilon,0}, \mathcal{Y}_\varepsilon)|\mathcal{E}_\varepsilon] = \frac{\mathbb{E}[\psi(u_{\varepsilon,0}, \mathcal{Y}_\varepsilon)|\mathbb{P}(\mathcal{E}_\varepsilon)}{\mathbb{P}(\mathcal{E}_\varepsilon)} \to \mathbb{E}[\psi(u_0, \mathcal{Y}(\lambda))]$$

for any test function $\psi$. So the first two marginals of $\mu$ have the law of $(u_0, \mathcal{Y}(\lambda))$ and they are independent since $(u_{\varepsilon,0}, \mathcal{Y}_\varepsilon)$ are independent for any $\varepsilon$. Calling $\nu$ the law of $(u_0, \mathcal{Y}(\lambda))$ we have that

$$\int_{\mathcal{Z}} \varphi(x, y, z, t) d\mu(x, y, z, t) = \int_{\mathcal{Z}} \varphi(x, y, \Gamma_\ast(x, y,\lambda,0), \Gamma_\ast(x, y,\lambda,0)) d\nu(x, y)$$

which implies that $\mu$ is unique and that the whole family $(\mu_{\varepsilon})_\varepsilon$ converges to $\mu$. We can conclude that $u_{\varepsilon,0}^\ast \to u^\ast$ in law with $u^\ast = u(\lambda)$ up to the time $T_\ast(\|\mathcal{Y}(\lambda)\|_{\mathcal{X}_\varepsilon}, \|u_0\|_{\mathcal{E}_{\varepsilon,-1/2-\varepsilon}}, |\lambda|)$ since the function $T^\ast$ is lower semicontinuous (as obtained from the a-priori estimates).

\section{Convergence of the enhanced noise}

This is the central section of the paper, in which we present a new method to estimate certain random fields that do not have a finite chaos decomposition, and we apply it to the treatment of the random fields $\mathcal{Y}_\varepsilon$ of (2.2).

\subsection{An example of convergence}

We choose to give first a complete example (the convergence of the tree $Y_\varepsilon^{\mathcal{N}}$ to $Y^{\mathcal{N}}(\lambda)$) in order to put in evidence the main idea in the proof of Theorem 3.1. Recall its definition (2.2):

$$Y_\varepsilon^{\mathcal{N}} = \frac{\varepsilon^{-1}}{3} \tilde{F}_\varepsilon^{(1)}(\varepsilon^{1/2} Y_\varepsilon),$$

with $\tilde{F}_\varepsilon^{(1)}$ being the first derivative of the centered function $\tilde{F}_\varepsilon$ defined in (1.6). Since $\frac{d}{dx}H_n(x,\sigma^2_\varepsilon) = nH_{n-1}(x,\sigma^2_\varepsilon)$ the Wiener chaos decomposition of $Y_\varepsilon^{\mathcal{N}}$ reads:

$$\frac{\varepsilon^{-1}}{3} \tilde{F}_\varepsilon^{(1)}(\varepsilon^{1/2} Y_\varepsilon) = \frac{\varepsilon^{-1}}{3} \sum_{n \geq 3} n f_n H_{n-1}(\varepsilon^{1/2} Y_\varepsilon, \sigma^2_\varepsilon)$$

$$= \frac{\varepsilon^{-1}}{3} f_3 H_2(\varepsilon^{1/2} Y_\varepsilon, \sigma^2_\varepsilon) + \frac{\varepsilon^{-1}}{3} \sum_{n \geq 4} n f_n H_{n-1}(\varepsilon^{1/2} Y_\varepsilon, \sigma^2_\varepsilon)$$

$$= f_3 \mathbb{E}[Y^2_\varepsilon] + \frac{\varepsilon^{-1}}{3} \sum_{n \geq 4} n f_n H_{n-1}(\varepsilon^{1/2} Y_\varepsilon, \sigma^2_\varepsilon).$$

(3.1)
where \([\ldots]\) is the Wiener product and we used the fact that \(\varepsilon^{-\Delta}H_t(\varepsilon^{1/2} Y_t, \sigma_x^2) = [Y^n_t]\). Now one can use hypercontractivity (as done in \([10, 11]\) to control the \(L^p\) norm of each chaos order by its \(L^2\) norms. However this strategy does not give useful bounds for the infinite series in the second term of (3.1). Instead, we just observe that
\[
\sum_{n \geq 4} n f_{n,\varepsilon} H_{n-1}(\varepsilon^{1/2} Y_{\varepsilon}, \sigma_x^2) = (\id - J_0 - \ldots - J_2) \hat{F}^{(1)}_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon}),
\]
where \(J_i\) is the projection of on the \(i\)-th chaos, and look for a different way to write this remainder. One of the main insights of this paper is that we can write it as:
\[
(\id - J_0 - \ldots - J_2) \hat{F}^{(1)}_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon}) = \delta^3 G^{[3]}_{[1]} D^3 \hat{F}^{(1)}_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon})
\]
where \(D, \delta\) are the Malliavin derivative and divergence operators, and \(G^{[3]}_{[1]} = (1 - L)^{-1}(2 - L)^{-1}(3 - L)^{-1}\) with \(L\) the Ornstein-Uhlenbeck operator. This is proven in Lemma \([13]\).

To compute the Malliavin derivative of \(\hat{F}^{(1)}_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon})\) we observe that for every \(\varepsilon > 0\) \((t, x) \in \mathbb{R} \times \mathbb{T}^3\) there exists \(h_{(t, x)} \in L^2(\mathbb{R} \times \mathbb{T}^3)\) such that the Gaussian random variable \(Y_{\varepsilon, (t, x)} := Y_{\varepsilon}(t, x)\) can be written as
\[
Y_{\varepsilon, (t, x)} \xrightarrow{\text{law}} \langle \xi, h_{(t, x)} \rangle.
\]
Here \(\xi\) is the Gaussian white noise on \(\mathbb{R} \times \mathbb{T}^3\), which can be seen as a Gaussian Hilbert space \(\langle \xi, h \rangle_{H} = \{W(h)\}_{h \in H}\) indexed by the Hilbert space \(H := L^2(\mathbb{R} \times \mathbb{T}^3)\). This is the framework in which we apply the Malliavin calculus results of Appendix \([3]\). Notice that by construction
\[
\langle h_{(t, x)}, h_{(t', x')} \rangle = C_\varepsilon(t - t', x - x') := \mathbb{E}[Y_{\varepsilon, (t, x)} Y_{\varepsilon, (t', x')}].
\]
The function \(h_{(t, x)}\) can actually be written as the space-time convolution
\[
h_{(t, x)} = \hat{P} \ast \psi_\varepsilon(t, x),
\]
with \(\hat{P}(t, x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} \mathbb{1}_{t \geq 0}\) and \(\psi_\varepsilon\) such that \(\eta_\varepsilon = \psi_\varepsilon \ast \xi, \psi_\varepsilon(t, x) = \varepsilon^{-5/2} \psi(\varepsilon^{-1} t, \varepsilon^{-1} x)\). We omit the dependence on \(\varepsilon\) of \(h_{(t, x)}\) not to burden the notation.

Going back to the calculations, we obtain from \([5, 2]\) that \(D \hat{F}^{(1)}_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon}) = \varepsilon^{1/2} \hat{F}^{(2)}(\varepsilon^{1/2} Y_{\varepsilon}) h\). Then (noting that \(\hat{F}^{(4)}_\varepsilon = F^{(4)}_\varepsilon\)):
\[
Y_{\varepsilon} \overset{\mathcal{Y}}{=} \int f_{3, \varepsilon} Y_\varepsilon^{[2]} + \frac{1}{2} \delta^3 G^{[3]}_{[1]} F^{(4)}(\varepsilon^{1/2} Y_{\varepsilon}) h \overset{\mathcal{Y}}{=} f_{3, \varepsilon} Y_\varepsilon^{[2]} + \hat{Y}_\varepsilon \overset{\mathcal{Y}}{=}
\]
It can be easily seen from \([5, 2]\) that \(Y_\varepsilon\) has the same law of a time-space mollification of \(X\) by convolution (with \(X\) defined in Section \([2, 2]\)). Then the convergence in law of \(f_{3, \varepsilon} Y_\varepsilon^{[2]}\) to \(Y \overset{\mathcal{Y}}{=} \langle \lambda \rangle\) can be easily established by standard techniques (see \([3, 8]\)). We are only left to show that \(\hat{Y}_\varepsilon \overset{\mathcal{Y}}{=} \text{in (3.4)}\) converges to zero in \(C_T\varepsilon^{-1-\kappa}\).

It is well known (see Section \([5, 2]\) for details) that in order to control the norm of \(\hat{Y}_\varepsilon \overset{\mathcal{Y}}{=}(t, \cdot)\) for \(t \in (0, T]\) in the Besov space \(\mathcal{C}^{\alpha-\kappa}\) \(\forall \kappa > 0\) and in probability, it is enough to have suitable estimates for
\[
\sup_{x \in \mathbb{T}^3} \left\| \Delta_\eta \hat{Y}_\varepsilon \overset{\mathcal{Y}}{=}(t, \cdot) \right\|_{L^p(\Omega)} = \left\| \Delta_\eta \hat{Y}_\varepsilon \overset{\mathcal{Y}}{=}(t, \cdot) \right\|_{L^p(\Omega)} ,
\]
for any \(\bar{x} \in \mathbb{T}^3\) since \(\hat{Y}_\varepsilon \overset{\mathcal{Y}}{=}\) is stationary in space. We then proceed to compute:
\[
\left\| \Delta_\eta \hat{Y}_\varepsilon \overset{\mathcal{Y}}{=}(t, \bar{x}) \right\|_{L^p(\Omega)} = \frac{1}{3} \left\| \delta^3 G^{[3]}_{[1]} \int K_\eta, \bar{x}(x) F^{(4)}(\varepsilon^{1/2} Y_{\varepsilon, (t, x)}) h \overset{\mathcal{Y}}{=} 3 dx \right\|_{L^p(\Omega)}
\]
where $K_{q,\tilde{x}}(x)$ is the kernel associated to the Littlewood-Paley block $\Delta_q$. Observe that $\|F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x)\|_{L^p}^p = \int_{\mathbb{R}} |F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x)|^p \gamma(dx)$ where $\gamma(dx)$ is the density of a centered Gaussian with variance $\sigma_\varepsilon^2$. This norm is then finite by the bound (1.8) of Assumption 1.

Another fundamental idea of this work is that we can “estimate out” the bounded term $\|F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x)\|_{L^p}$, which has an infinite chaos decomposition, and obtain a standard $\Phi^4_3$ diagram that can be treated with well understood techniques. Using Lemma B.2 and Corollary B.6, one has

$$\left\| \delta^3 G^4_{1} \int K_{q,\tilde{x}}(x) F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x) h^{\otimes 3}_{(t,x)} dx \right\|_{L^p(\Omega)} \leq \sum_{k=0}^{3} \left\| D^k G^4_{1} \int K_{q,\tilde{x}}(x) F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x) h^{\otimes 3}_{(t,x)} dx \right\|_{L^p(\Omega, H^{\otimes 3+k})} \leq \left\| \int K_{q,\tilde{x}}(x) F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x) h^{\otimes 3}_{(t,x)} dx \right\|_{L^p(\Omega, H^{\otimes 3})}.$$

Then note that we can decompose the norm $\|u\|_{L^p(\Omega, H^{\otimes 3})} = \|u\|_{H^{\otimes 3}}^{1/2}$ for $u \in L^p(\Omega, H^{\otimes 3})$ and since the norm of the Hilbert space $H^{\otimes 3}$ is given by the scalar product $\|h^{\otimes 3}\|_{H^{\otimes 3}}^2 = \langle h, h \rangle$ we obtain

$$\left\| \Delta_q Y^{\vee}_{\varepsilon}(t, \tilde{x}) \right\|_{L^p(\Omega)} \leq \varepsilon^{1/2} \left\| \int K_{q,\tilde{x}}(x) F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x) h^{\otimes 3}_{(t,x)} dx \right\|_{H^{\otimes 3}}^{1/2} \leq \varepsilon \left( \varepsilon \int |K_{q,\tilde{x}}(x)K_{q,\tilde{x}}(x')| \left\| F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x) F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x') \right\|_{L^p(\Omega)} \right)^{1/2} + \varepsilon |\langle h(t,x), h(t,x') \rangle|^3 dx dx'.$$

The norm containing $F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x)$ can then be easily estimated using Hölder’s inequality as

$$\left\| F^4(\varepsilon^{1/2}t\gamma,\varepsilon,t,x) \right\|_{L^p(\Omega)} \leq \varepsilon \left( \varepsilon \int |K_{q,\tilde{x}}(x)K_{q,\tilde{x}}(x')| |\langle h(t,x), h(t,x') \rangle|^3 dx dx' \right)^{1/2} \leq 1.$$

This yields

$$\left\| \Delta_q Y^{\vee}_{\varepsilon}(t, \tilde{x}) \right\|_{L^p(\Omega)} \leq \left( \varepsilon \int |K_{q,\tilde{x}}(x)K_{q,\tilde{x}}(x')| |\langle h(t,x), h(t,x') \rangle|^3 dx dx' \right)^{1/2}$$

which is a standard $\Phi^4_3$ diagram that can be analysed with the techniques of [8] (recalled in Appendix A.3). We just remark that $\langle h(t,x), h(t,x') \rangle = C_\varepsilon(0, x-x')$ and the bound $\varepsilon |C_\varepsilon(t,x)| \leq 1$ of Lemma A.12 yields $\forall \delta (0, 1)$:

$$\left\| \Delta_q Y^{\vee}_{\varepsilon}(t, \tilde{x}) \right\|_{L^p(\Omega)} \leq \varepsilon^{1/2} \left[ \varepsilon \int |K_{q,\tilde{x}}(x)K_{q,\tilde{x}}(x')| |C_\varepsilon(0, x-x')|^2 \right]$$

and then $Y^{\vee}_{\varepsilon}(t, \cdot)$ converges to zero in probability in the space $C^{-1-\delta/2}_{T}$ $\forall \delta (0, 1)$, as $\varepsilon \to 0$. The time regularity of $Y^{\vee}_{\varepsilon}(t, \cdot)$ needed to obtain the convergence in $C_T \varepsilon^{-1-\delta/2}$ does not need new ideas, and it is done in Section 3.3.

The method shown in this section is valid verbatim for the trees $Y^{\vee}_{\varepsilon}, Y^I_{\varepsilon}, Y^Y_{\varepsilon}$, while for the composite trees in [22] (namely $Y^Y_{\varepsilon}, Y^I_{\varepsilon}, Y^Y_{\varepsilon}, Y^Y_{\varepsilon}$) that are obtained via paraproducts of simple trees, one has to be able to
write the remainder $\hat{Y}_T^\tau$ as an iterated Skorohod integral $\delta^n(\ldots)$ in order to exploit the boundedness of this operator. Moreover, these trees require a second renormalization (on top of the Wick ordering) which is not easy to control for infinite chaos decompositions. We deal with both these difficulties introducing the product formula (3.6), which allows to write products of iterated Skorohod integrals as combinations of iterated Skorohod integrals. The details and calculations for composite trees can be found in Section 3.4.

3.2 Main theorem and overview of the proof

**Theorem 3.1**

*Under Assumption 1 there exists $C > 0$ such that for any $p \in [2, \infty)$ we have $\|Y_\varepsilon\|_{L^p} < C$ in $L^p(\Omega)$. Moreover, $Y_\varepsilon \rightharpoonup Y(\lambda) \in \mathcal{X}_T$ and $Y_\varepsilon \rightarrow X \in C_T\mathcal{C}^{-1/2-\kappa}$ in law.*

The rest of Section 3 is dedicated to the proof of Theorem 3.1.

From the definition $Y(\lambda) = (Y^\tau(\lambda))_\tau$ of (2.4) it is clear that we only need to prove that $Y_\varepsilon^\tau \rightarrow Y^\tau(\lambda)$ for every $\tau$. Note that we can write each tree $Y^\tau(\lambda)$ as $Y^\tau(\lambda) = f_r(\lambda)K^\tau(X)$ for a measurable function $K^\tau$ of the Gaussian process $X \in C_T\mathcal{C}^{-1/2-\kappa}$ defined (2.10), and a suitable deterministic function $f_r(\lambda)$ of $\lambda$. For example, we can write $Y^\lambda(\lambda) = \lambda_3[|X|^2]$ with $K^\lambda(X) = X^\lambda = [X^2]$ and $f_r(\lambda) = \lambda_3$.

We will show (eqs. (3.13) and (3.21)) that every random field $Y_\varepsilon^\tau$ defined in (2.2) can be decomposed with the same functions $f_r(\cdot)$ and $K^\tau$ as

$$Y_\varepsilon^\tau = f_r(\varepsilon)K^\tau(Y_\varepsilon) + \hat{Y}_\varepsilon^\tau$$

where $\hat{Y}_\varepsilon^\tau$ are suitable remainder terms. For all $p \geq 2$ it is well-known (see [4], [8]) that the term $f_r(\varepsilon)K^\tau(Y_\varepsilon)$ is uniformly bounded in $L^p(\Omega; \mathcal{X}^\tau)$ (with $\mathcal{X}^\tau$ given by (2.8)). Thus, we will prove that $\hat{Y}_\varepsilon^\tau$ converges to zero in $L^p(\Omega; \mathcal{X}^\tau)$. This can be done by showing that, by Besov embedding, for $p \in [2, \infty)$ and $\forall \alpha < |\tau|$ we have

$$\mathbb{E}(\|\hat{Y}_\varepsilon^\tau(t)\|_{B_{p,q}^{\alpha-\delta/p}}^p) \lesssim \mathbb{E}(\|\hat{Y}_\varepsilon^\tau(t)\|_{B_{p,q}^{\alpha}}^p) \lesssim \sum_q 2^{\alpha pq} \int_{\mathbb{T}^3} \|\Delta_q \hat{Y}_\varepsilon^\tau(t,x)\|_{L^p(\Omega)}^p dx \rightarrow 0$$

(3.6)

thanks to the stationarity of the process $Y(t,x)$. In order to prove the bound (3.6) it suffices to show that $\forall t \in [0,T]$ and $\varepsilon \rightarrow 0$,

$$\sum_q 2^{\alpha pq} \sup_x \|\Delta_q \hat{Y}_\varepsilon^\tau(t,x)\|_{L^p(\Omega)}^p \rightarrow 0$$

(3.7)

which is one of the key estimation of this paper and will be performed in Sections 3.3 and 3.4.2.

In order to obtain uniform convergence for $t \in [0,T]$ it suffices to show that $\forall \sigma \in [0,1/2]$, $q \geq -1$:

$$\sup_x \|\Delta_q \hat{Y}_\varepsilon^\tau(t,x) - \Delta_q \hat{Y}_\varepsilon^\tau(s,x)\|_{L^p(\Omega)}^p \leq C_\varepsilon |t-s|^{\sigma 2^{q(\alpha-2\sigma-\delta)/pq}}$$

(3.8)

Indeed, by the Garsia-Rodemich-Rumsey inequality we obtain for $\delta > 0$ small enough and $p$ large enough

$$\sup_{\varepsilon} \mathbb{E}(\|\hat{Y}_\varepsilon^\tau\|_{C_T^{\alpha-2\sigma\delta/p}B_{p,q}^{\alpha-2\delta/p}}^p) \leq T^2 \sum_q 2^{(\alpha-2\sigma-\delta)/pq} \sup_{s \in [0,T]} \sup_x \|\Delta_q \hat{Y}_\varepsilon^\tau(t,x) - \Delta_q \hat{Y}_\varepsilon^\tau(s,x)\|_{L^p(\Omega)}^p$$

$$\leq C_\varepsilon T^2 \sum_q 2^{-\delta pq}$$

which by Besov embedding yields an estimation on $\mathbb{E}(\|Y_\varepsilon^\tau\|_{C_T^{\alpha-2\sigma\delta/p}B_{p,q}^{\alpha-2\delta/p}}^p)$ for $\kappa > 0$ small enough. This gives us the necessary tightness to claim that $Y_\varepsilon$ has weak limits along subsequences.
The only thing left after proving (3.5), (3.7) and (3.8) is that for each $\tau$ we have $K^\tau(Y_\varepsilon) \to K^\tau(X)$ in law. However this is clear and already well-known, since by hypothesis we can introduce a space-time convolution regularisation of $X$ (let’s call it $X_\varepsilon$) which has the same law of $X$ for any $\varepsilon > 0$. This yields immediately the convergence $Y_\varepsilon \to X$ in law. At this point an approximation argument gives that $K^\tau(Y_\varepsilon)$ has the same law of $K^\tau(X_\varepsilon)$. Transposing the regularisation to the kernels of the chaos expansion we can write $K^\tau(X_\varepsilon) = K^\tau_0(X)$ and now it is easy to check that $K^\tau_0(X) \to K^\tau(X)$ in probability (as done systematically in [3], [13]). We can then conclude that $K^\tau(Y_\varepsilon) \to K^\tau(X)$ and therefore $Y_\varepsilon \tau \to Y^{\tau}(\lambda)$ in law for every $\tau$, since from Assumption [H] we have immediately $f_\varepsilon(\lambda_\varepsilon) \to f_\varepsilon(\lambda)$.

Let us give some more details on how to prove the decomposition (3.3) and the bounds (3.7) and (3.8). As seen in Section 3.1 we have $Y_{\varepsilon, \xi} = (\xi, h_\xi)$ in law for $\xi = (t,x) \in \mathbb{R} \times \mathbb{T}^3$ and this gives
\[
D^n F_{\varepsilon}(\varepsilon^{1/2} Y_{\varepsilon, \xi}) = \tilde{F}_{\varepsilon}(m+n)(\varepsilon^{1/2} Y_{\varepsilon, \xi}) h_{\xi}^{\otimes n}.
\]
We define for $m \in \mathbb{N}$, $\xi \in \mathbb{R} \times \mathbb{T}^3$:
\[
\Phi[m]_{\xi} := \varepsilon^{m/2} F_{\varepsilon}(m^{1/2} Y_{\varepsilon, \xi})
\]
(3.9)
Note that the term $\Phi[m]$ above is not the $m$-th derivative of some function $\Phi$ (we use the square parenthesis notation to emphasize this fact). It easy to see from (3.9) that $D^k \Phi[m]_{\xi} = \Phi[m+k]_{\xi} h_{\xi}^k$. Therefore, the partial chaos expansion (3.3) takes a more explicit form when applied to $\Phi[m]_{\xi}$:
\[
\Phi[m]_{\xi} = \sum_k e^{(m-k+3)/2} (m+k)! f_{m+k, \varepsilon} Y_{\varepsilon, \xi}
\]
(3.10)
with $G_{[1]}^{m}$ defined in (2.1). Here we used the fact that $\delta^n(h_{\xi}^{\otimes n}) = [Y_{\varepsilon, \xi}^{n}]$ (see Remark B.1) and that by the definition of $\Phi[m]_{\xi}$ we obtain $\forall \xi \in \mathbb{R} \times \mathbb{T}^3$:
\[
\mathbb{E} (\Phi[m]_{\xi}^{m+k}) = e^{(m+k-3)/2} (m+k)! f_{m+k, \varepsilon}
\]
with $f_{n, \varepsilon}$ the coefficients in the decomposition $\tilde{F}_{\varepsilon}(\varepsilon^{1/2} Y_{\varepsilon}) := \sum_{n \geq 0} f_{n, \varepsilon} H_n(\varepsilon^{1/2} Y_{\varepsilon}, \sigma_{n}^2)$. Choosing $n = 4 - m$ in eq. (3.10) we obtain
\[
\Phi[m]_{\xi} = \frac{3!}{(3-m)!} f_{3, \varepsilon} [Y_{\varepsilon, \xi}^{3-m}] + \hat{\Phi}[m]_{\xi}
\]
(3.11)
and a remainder with an infinite chaos decomposition strictly greater that $3 - m$:
\[
\hat{\Phi}[m]_{\xi} = \delta^{3-m} (G_{[4]}^{4-m} \Phi[4]_{\xi} h_{\xi}^{4-m}).
\]
(3.12)

This is a key step in the proof of Theorem 3.1. Indeed, it suffices to substitute (3.11) into definition (2.2) to identify the remainder $\tilde{Y}_{\varepsilon}^{\tau}$ in decomposition (3.5) that has to converge to zero, and see that it always contains the term $\hat{\Phi}[m]_{\xi}$. Moreover, the structure (3.12) of $\Phi[m]_{\xi}$ makes it possible to bound its $L^p$ norm and obtain (3.7), (3.8) in the same way as done in Section 3.1 for $Y_{\varepsilon}^{\tau}$. We will consider separately simple trees (namely $Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}$) which are linear functions of $\Phi[m]_{\xi}$ in Section 3.3, and composite trees (namely $Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}, Y_{\varepsilon}^{\tau}$) which are quadratic in simple trees and need to be further renormalized in order to converge to some limit as $\varepsilon \to 0$. We will show the decomposition (3.7) for composite trees in Section 3.4.1 and the bounds (3.7), (3.8) in Section 3.4.2.
Remark 3.2 We can easily estimate terms of the form $\varepsilon^{-(m-3)/2}\Phi_\xi^{[m]}$ for $3 \leq m \leq 9$ and every $p \in [2, \infty)$. We have (as already observed for $F^{[1]}_\varepsilon$): 

$$\| \varepsilon^{m-3}\Phi_\xi^{[m]} \|_{L^p}^p = \| F^{(m)}_\varepsilon(\varepsilon^{1/2} Y_{\varepsilon, \xi}) \|_{L^p}^p = \int_\mathbb{R} |F^{(m)}_\varepsilon(x)|^p \gamma(dx)$$

where $\gamma(dx)$ is the density of a centered Gaussian with variance $\sigma_\varepsilon^2$. The integral is finite by Assumption [4] in particular we only need to assume that the first $m$ derivatives of $F_\varepsilon$ have exponential growth (actually, it is easy to see that one can require even weaker growth conditions).

3.3 Analysis of simple trees

First of all note that the term $\hat{Y}_\varepsilon^\vee$ has no remainder, and then it can be shown to converge in law to $\lambda^{(2)}Y^\vee$ by usual techniques (see [4]). In this section we show the convergence of the trees $Y^\varnothing, Y^\varepsilon, Y^\vee, Y^{\vee\vee}$. We obtain easily from (3.11):

$$\Delta_q Y^\tau_\varepsilon(t, x) := \frac{(3-m)!}{3!} \int_\mathbb{R} \Phi_\xi^{[m]} \mu_{q, \zeta} \left[ Y_{\varepsilon, \xi}^{(3-m)} \right] \mu_\zeta + \frac{(3-m)!}{3!} \int_\mathbb{R} \Phi_\xi^{[m]} \mu_{q, \zeta} \left[ Y_{\varepsilon, \xi}^{(3-m)} \right] \mu_\zeta = f_\varepsilon(\lambda_\varepsilon) \Delta_q K^\tau_\varepsilon(t, x) + \Delta_q \hat{Y}^\tau_\varepsilon(t, x),$$

with $\zeta = (s, y)$ and either

$$\mu_{q, \zeta} = \delta(t-s)K_{q, x}(y)\text{d}s\text{d}y, \quad \text{for } \Delta_q Y^\varnothing, \Delta_q Y^\varepsilon, \Delta_q Y^\vee,$$

$$\mu_{q, \zeta} = \text{d}s\text{d}y \int K_{q, x}(z)P_{t-s}(z-y)\text{d}z, \quad \text{for } \Delta_q Y^{\vee\vee},$$

where $K_{q, x}(y)$ is the kernel associated to the Littlewood-Paley block $\Delta_q$ and $P_t(x)$ is the heat kernel.

As said before, $f_\varepsilon \int_\mathbb{R} \left[ Y_{\varepsilon, \xi}^{(3-m)} \right] \mu_{q, \zeta}$ converges in law in $L^p$ for every $2 \leq p < +\infty$ to $\lambda_3 \int_\mathbb{R} \left[ Y_{\varepsilon, \xi}^{(3-m)} \right] \mu_{q, \zeta}$ since $f_\varepsilon \rightarrow \lambda_3$ by Assumption [4]. We can bound the remainder term $\int_\mathbb{R} \Phi_\xi^{[m]} \mu_{q, \zeta}$ in $L^p(\Omega)$ using Lemma [B.2] and the definition of the norm $\| \cdot \|_{D^{1-m, p}(H^{4-m})}$ to obtain:

$$\left\| \int_\mathbb{R} \Phi_\xi^{[m]} \mu_{q, \zeta} \right\|_{L^p(\Omega)} \leq \delta^{4-m} \left\| C_{[1]}^{[4-m]} \int_\mathbb{R} \Phi_\xi^{[4]} h^{4-m}_{\zeta} \mu_{q, \zeta} \right\|_{L^p(\Omega)} \leq \sum_{k=0}^{4-m} \delta^{4-m} \left\| C_{[1]}^{[4-m]} \int_\mathbb{R} \Phi_\xi^{[4]} h^{4-m}_{\zeta} \mu_{q, \zeta} \right\|_{L^p(\Omega, H^{4-m+k})}.$$ 

From Corollary [B.6] we know that $(j-L)^{-1}$ and $D(j-L)^{-1}$ are bounded in $L^p$ for every $p \in [2, \infty)$ and every $j \geq 1$. Applying repeatedly these estimations we obtain:

$$\left\| D^k C_{[1]}^{[4-m]} \int_\mathbb{R} \Phi_\xi^{[4]} h^{4-m}_{\zeta} \mu_{q, \zeta} \right\|_{L^p(\Omega, H^{4-m+k})} \leq \left\| \int_\mathbb{R} \Phi_\xi^{[4]} h^{4-m}_{\zeta} \mu_{q, \zeta} \right\|_{L^p(\Omega, H^{4-m})}.$$
Now we can proceed to implement the idea we already described in Section 3.1, i.e. estimating out the term \( \|e^{-\frac{\Phi}{2}}\|_{L^p(\Omega)} \) (which is bounded by Remark 3.2 but with infinite chaos decomposition) and considering the finite-chaos term that is left. We do this by decomposing the \( L^p(\Omega, H^{\otimes 4-m}) \) norm as norms on \( H^{\otimes 4-m} \) and \( L^{p/2}(\Omega) \) as follows:

\[
\left\| \int_{\zeta} \Phi^{[4]} \zeta_h^{4-m} \mu_{q,\zeta} \right\|_{L^p(\Omega, H^{\otimes 4-m})} \lesssim \left\| \int_{\zeta} \Phi^{[4]} \zeta_h^{4-m} \mu_{q,\zeta} \right\|_{L^{p/2}(\Omega)}^{1/2} \lesssim \left\| \int_{\zeta} \Phi^{[4]} \zeta_h^{4-m} \mu_{q,\zeta} \right\|_{L^{p/2}(\Omega)}^{1/2} \lesssim \left[ \int_{\zeta'} \|\Phi^{[4]} \zeta_h^{4-m} \mu_{q,\zeta'}\|_{L^{p/2}(\Omega)} \left| \langle h_{\zeta}, h_{\zeta'} \rangle \right| \right]^{1/2}.
\]

Finally, putting the estimations together and using Hölder’s inequality, together with the bound \( \varepsilon|\langle h_{\zeta}, h_{\zeta'} \rangle| = \varepsilon|C_\varepsilon(\zeta - \zeta')| \lesssim 1 \) of Lemma A.12, we obtain for every \( \delta \in (0,1) \):

\[
\left\| \int_{\zeta} \Phi^{[m]} \mu_{q,\zeta} \right\|_{L^p(\Omega)} \lesssim \left[ \int_{\zeta,\zeta'} \left| \langle h_{\zeta}, h_{\zeta'} \rangle \right|^{3-m+\delta} |\mu_{q,\zeta}| \right]^{1/2}.
\]

Now using Remark 3.2 (note that to bound \( e^{-\frac{\Phi}{2}} \Phi^{[4]} \) we only need to control the first 4 derivatives of \( F_\varepsilon \)) and the fact that \( \langle h_{\zeta}, h_{\zeta'} \rangle_H = C_\varepsilon(\zeta - \zeta') \) we obtain as a final estimation

\[
\left\| \int_{\zeta} \Phi^{[m]} \mu_{q,\zeta} \right\|_{L^p(\Omega)} \lesssim \varepsilon^{\frac{\delta}{2}} \left[ \int_{\zeta,\zeta'} \left| C_\varepsilon(\zeta - \zeta') \right|^{3-m+\delta} |\mu_{q,\zeta}| \right]^{1/2}.
\]

From the definition 3.14 of the measure \( \mu_{q,\zeta} \), the l.h.s of (3.15) can be estimated in a standard way using Lemma A.14 to obtain for every \( x \in \mathbb{T}^d, q > 0 \):

\[
\left\| \Delta_{q} \dot{Y}_\varepsilon(t,x) \right\|_{L^p(\Omega)} \lesssim \varepsilon^{\frac{\delta}{2} - \frac{1+q}{2}} \quad \left\| \Delta_{q} \check{\dot{Y}}_\varepsilon(t,x) \right\|_{L^p(\Omega)} \lesssim \varepsilon^{-\frac{\delta}{2} + \frac{1+q}{2}}.
\]

**Time regularity of trees**

We want to show 3.3. In order to do that, we compute in the same way as before:

\[
\left\| \int_{\zeta} \Phi^{[m]} \mu_{q,\zeta} \right\|_{L^p(\Omega)} \lesssim \left\| \int_{\zeta} \Phi^{[4]} \zeta_h^{4-m} \mu_{q,\zeta} \right\|_{L^{p/2}(\Omega)}^{1/2} \lesssim \left\| \int_{\zeta} \Phi^{[4]} \zeta_h^{4-m} \mu_{q,\zeta} \right\|_{L^{p/2}(\Omega)}^{1/2}.
\]

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We focus on the first term above to obtain that it is bounded by

\[
\left\| \int_{\zeta} \left( \Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]} \right) \left( h_{s,x}^{\otimes 4-m}, h_{s,x'}^{\otimes 4-m} \right) H_{\otimes 4-m} \mu_{q,\zeta} \mu_{q',\zeta'} \right\|_{L^{p}(\Omega)}^{1/2}.
\]

Now note that

\[
\varepsilon \int_{\zeta} \left\| \left( \Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]} \right) \left( h_{s,x}^{\otimes 4-m}, h_{s,x'}^{\otimes 4-m} \right) \right\|_{L^{p/2}(\Omega)} \left\| \left( h_{s,x}, h_{s,x'} \right) \right\|^{4-m} \mu_{q,\zeta} \mu_{q',\zeta'} \right\|^{1/2}.
\]

and we can estimate \( \left\| \varepsilon^{-\frac{1}{2}} \left( \Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]} \right) \right\|_{L^{p}(\Omega)} \) by hypercontractivity and using Lemma A.15 as

\[
\lesssim_{p} \varepsilon^{1/2} \left\| \int_{0}^{1} F(\varepsilon^{\frac{1}{2}} Y_{\varepsilon}(t, s)) \right\|_{L^{2p}(\Omega)} \left\| Y_{\varepsilon}(t, x) - Y_{\varepsilon}(s, x) \right\|_{L^{2}(\Omega)}
\]

for any \( \sigma \in [0, 1/2] \). The other term can be estimated more easily by

\[
\left\| \left( \Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]} \right) \left( h_{s,x}, h_{s,x'} \right) \right\|^{3-m+\delta} \right\|_{L^{p}(\Omega)}^{1/2}.
\]

and finally obtain

\[
\left\| \int_{\zeta} \left( \Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]} \right) \mu_{q,\zeta} \mu_{q',\zeta'} \right\|_{L^{p}(\Omega)}^{1/2} \lesssim \varepsilon^{\frac{1}{2}} \left\| \left( \Phi_{t,x}^{[4]} - \Phi_{s,x}^{[4]} \right) \mu_{q,\zeta} \mu_{q',\zeta'} \right\|_{L^{p}(\Omega)}^{1/2}.
\]

Which yields estimation (3.8) by applying Lemma A.14 as before. This concludes the treatment of simple trees. Notice that in this section we only needed \( F_{\varepsilon} \in C^{\infty}(\mathbb{R}) \) with the first 5 derivatives having exponential growth: indeed we need to take 4 derivatives to bound \( \varepsilon^{1/2} \Phi_{t,x}^{[4]} \) as of Remark 3.2, plus one more derivative for the time regularity of \( \varepsilon^{1/2} \Phi_{t,x}^{[4]} \).
3.4 Analysis of composite trees

In this section we show the decomposition (3.5) and the bound (3.7) for the trees $Y_\varepsilon$, $Y_\varepsilon$, $\bar{Y}_\varepsilon$, $\bar{Y}_\varepsilon$ and $Y_\varepsilon$. The time regularity (3.8) of $\bar{Y}_\varepsilon$ can be obtained with the same technique as in the previous section, assuming that we can control one more derivative of $F_\varepsilon$ than what is needed to prove the boundedness of $Y_\varepsilon$ (thus we will need $F_\varepsilon \in C^9(\mathbb{R})$ with exponential growth, as discussed in Remark 3.7). Looking at the definitions in (2.22) it is clear that we can write the $q$-th Littlewood-Paley blocks of $Y_\varepsilon$, $Y_\varepsilon$, $\bar{Y}_\varepsilon$, $\bar{Y}_\varepsilon$ and $Y_\varepsilon$ for $\varepsilon > 0$ as:

$$
\Delta_q Y_{\varepsilon}(\bar{\zeta}) = \frac{1}{\varepsilon} \int_{\zeta_1, \zeta_2} \Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]} \mu_{q, \zeta_1, \zeta_2} - d_\varepsilon \Delta_q Q(\bar{\zeta}),
$$

(3.16)

for any time-space point $\bar{\zeta} = (t, \bar{x})$ that we keep fixed throughout this section. In order to keep the notation shorter we defined

$$
\Phi_{\zeta_1}^{[1]} := \varepsilon^{-1/2} f_{2,\varepsilon}[Y_\varepsilon^2(\zeta_1)],
$$

which can be thought of as a finite-chaos equivalent of $\Phi_{\zeta}^{[1]}$ (modulo a constant $f_{2,\varepsilon}/f_{3,\varepsilon}$) in the same way as $\bar{Y}_\varepsilon$ is a finite-chaos equivalent of $Y_\varepsilon$. The measure $\mu_{q, \zeta_1, \zeta_2}$ on $(\mathbb{R} \times \mathbb{T}^3)^2$ is given by

$$
\mu_{q, \zeta_1, \zeta_2} := \left[ \int_{x,y} K_q(x) \sum_{i,j} K_{i,x}(y) K_{j,x}(y) P_{t-s_1}(y-x) \right] \delta(t-s_2) d\zeta_1 d\zeta_2,
$$

with $\zeta_i = (s_i, x_i)$ for $i = 1, 2$, $K$ being the kernel associated to the Littlewood-Paley decomposition and $P$ being the heat kernel. The first step for decomposing (3.16) is to expand them using the partial chaos expansion (B.3) to obtain

$$
\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]} = \mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}] + \delta G_1 \mathbb{D}(\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}),
$$

(3.17)

$$
\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]} = \mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}] + \delta G_1 \mathbb{D}(\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}),
$$

$$
\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]} = \mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}] + \delta G_1 \mathbb{D}(\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[1]}),
$$

$$
\Phi_{\zeta_1}^{[2]} = \mathbb{E}[\Phi_{\zeta_1}^{[2]}] + \delta G_1 \mathbb{D}(\Phi_{\zeta_1}^{[2]}).
$$

Like the trees appearing in the $\Phi_3$ model, we expect composite trees to require a further renormalisation, on top of the Wick ordering. We developed (3.17) to the smallest order that allows us to see the effect of renormalization.

3.4.1 Renormalisation of composite trees

In this section we show how to renormalize (3.16) by estimating terms of the type $\mathbb{E}[\Phi_{\zeta_1}^{[m]} \Phi_{\zeta_2}^{[n]}]$ in expansion (3.17). This poses an additional difficulty, as in principle we would need to compute an infinite number of contractions between $\Phi_{\zeta_1}^{[m]}$ and $\Phi_{\zeta_2}^{[n]}$. However, we can again decompose $\Phi_{\zeta_1}^{[m]}$ as in (3.11), and then the product formula (B.6) ensures that we only need to control a finite number of contractions. This is another important step in the proof and will be carried out in Lemma 3.5. First we need some preparatory results:
Lemma 3.3

We have

$$\int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_1) \mathbb{E}[\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2}] \mu_{q, \zeta_1, \zeta_2} = \int_{s, x} \Delta_q Y_\varepsilon(s, \bar{x} - x) G(t - s, x).$$

and

$$\int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_2) \mathbb{E}[\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2}] \mu_{q, \zeta_1, \zeta_2} = \int_{x} \Delta_q Y_\varepsilon(t, \bar{x} - x) H(t, x),$$

where we introduced the kernels:

$$G(t - s, x) := \int_{x_1', x_2} \sum_{i \sim j} K_{i,x}(x_1') K_{j,x}(x_2) P_{t-s}(x_1') \mathbb{E}[\Phi^{[1]}_0 \Phi^{[1]}_{(t-s, x_2)}],$$

$$H(t, x) := \int_{s, x_1, x_1'} \sum_{i \sim j} K_{i,x}(x_1') K_{j,x}(0) P_{t-s}(x_1' - x_1) \mathbb{E}[\Phi^{[0]}_0 \Phi^{[2]}_{(t-s, -x_1)}].$$

Remark 3.4 Some caveat on the notation: although we use the same letter for the kernel $G(\cdot, \cdot)$ and the Green operator $G^{[n]}_{(\cdot, \cdot)}$, those two are not related in any possible way. It is always clear which one the notation refers to.

Proof We have

$$\int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_1) \mathbb{E}[\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2}] \mu_{q, \zeta_1, \zeta_2} = \int_{s_1, x_1, x_2, x_1'} K_{q, \bar{x}}(x + x_1) Y(s_1, x_1) \int_{x_1', x_2} \sum_{i \sim j} K_{i,x}(x_1') K_{j,x}(x_2) P_{t-s_1}(x_1' - x_1) Y(s_1, x_1) \mathbb{E}[\Phi^{[1]}_0 \Phi^{[1]}_{(t-s_1, x_2-x_1)}]$$

and by change of variables, exploiting the translation invariance of the problem we obtain:

$$= \int_{s_1, x_1, x} \Delta_q Y_\varepsilon(s_1, \bar{x} - x) \int_{x_1', x_2} \sum_{i \sim j} K_{i,x}(x_1') K_{j,x}(x_2) P_{t-s_1}(x_1') \mathbb{E}[\Phi^{[1]}_0 \Phi^{[1]}_{(t-s_1, x_2)}].$$

Using the definition of $K_q$ we have

$$= \int_{s_1, x} \Delta_q Y_\varepsilon(s_1, \bar{x} - x) \int_{x_1', x_2} \sum_{i \sim j} K_{i,x}(x_1') K_{j,x}(x_2) P_{t-s_1}(x_1') \mathbb{E}[\Phi^{[1]}_0 \Phi^{[1]}_{(t-s_1, x_2)}].$$

Finally we can write

$$\int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_1) \mathbb{E}[\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2}] \mu_{q, \zeta_1, \zeta_2} = \int_{s_1, x} \Delta_q Y_\varepsilon(s_1, \bar{x} - x) G(t - s_1, x).$$
Similar computations holds for the other term, indeed

\[
\int_{\zeta_1, \zeta_2} Y_\varepsilon(\zeta_2) \mathbb{E}[\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2}]_{\mu_q, \zeta_1, \zeta_2} = \int_{s_1, x_1, x_2, x_1'} K_{q, \varepsilon}(x) \sum_{i \sim j} K_{i, x}(x_1') K_{j, x}(x_2) P_{t-s_1}(x_1' - x_1) Y_\varepsilon(t, x_2) \mathbb{E}[\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2}]_{(t-s_1, x_2-x_1)} \\
= \int_{x_2} K_{q, \varepsilon}(x + x_2) Y_\varepsilon(t, x_2) \int_{s_1, x_1, x_1'} \sum_{i \sim j} K_{i, x}(x_1') K_{j, x}(0) P_{t-s_1}(x_1' - x_1) \mathbb{E}[\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2}]_{(t-s_1, -s_1)} \\
= \int_x \Delta_q Y_\varepsilon(t, \bar{x} - x) \int_{s_1, x_1, x_1'} \sum_{i \sim j} K_{i, x}(x_1') K_{j, x}(0) P_{t-s_1}(x_1' - x_1) \mathbb{E}[\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2}]_{(t-s_1, -s_1)} \\
= \int_x \Delta_q Y_\varepsilon(t, \bar{x} - x) H(t, x)
\]

Substituting the lemma above and (3.17) in the expressions (3.16), we can write them as:

\[
\Delta_q Y_\varepsilon(\bar{\zeta}) = \frac{1}{9} \int_{\zeta_1, \zeta_2} \delta G_1 D(\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2})_{\mu_q, \zeta_1, \zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[ \frac{1}{5} \int_{s, x} G(t-s, x) - d_\varepsilon \Phi \right] \\
\Delta_q Y_\varepsilon(\bar{\zeta}) = \frac{1}{3} \int_{\zeta_1, \zeta_2} \delta G_1 D(\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2})_{\mu_q, \zeta_1, \zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[ \frac{1}{3} \int_{s, x} \bar{G}(t-s, x) - d_\varepsilon \Phi \right] \\
\Delta_q Y_\varepsilon(\bar{\zeta}) = \frac{1}{6} \int_{\zeta_1, \zeta_2} \delta G_1 D(\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2})_{\mu_q, \zeta_1, \zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[ \frac{1}{6} \int_x H(t, x) - d_\varepsilon \Phi \right] \\
\Delta_q Y_\varepsilon(\bar{\zeta}) = \frac{1}{3} \int_{\zeta_1, \zeta_2} \delta^2 G_1^2 D^2(\Phi^{[0]}_{\zeta_1} \Phi^{[1]}_{\zeta_2})_{\mu_q, \zeta_1, \zeta_2} + \Delta_q(1)(\bar{\zeta}) \left[ \frac{1}{3} \int_{s, x} \mathbb{E}[\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2}]_{\mu_q, \zeta_1, \zeta_2} - d_\varepsilon \Phi \right] \\
+ \Delta_q Y_\varepsilon(\bar{\zeta}) \left[ \frac{1}{3} \int_{s, x} G(t-s, x) + \frac{1}{3} \int_x H(t, x) - d_\varepsilon \Phi \right] \\
+ \frac{1}{3} \Delta_q R_\varepsilon(\bar{\zeta}) + \frac{1}{3} \Delta_q R_\varepsilon(\bar{\zeta})
\]

with the additional definitions

\[
\bar{G}(t-s, x) := \int_{x_1, x_1} \sum_{i \sim j} K_{i, x}(x_1') K_{j, x}(0) P_{t-s_1}(x_1' - x_1) \mathbb{E}[\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2}]_{(t-s_1, x_2-x_1)} \\
\Delta_q R_\varepsilon(\bar{\zeta}) := \int_{s, x} [\Delta_q Y_\varepsilon(s, \bar{x} - x) - \Delta_q Y_\varepsilon(t, \bar{x})] G(t-s, x) \\
\Delta_q R_\varepsilon(\bar{\zeta}) := \int_x [\Delta_q Y_\varepsilon(t, \bar{x} - x) - \Delta_q Y_\varepsilon(t, \bar{x})] H(t, x)
\]

Now we can characterise the local behaviour of \( \mathbb{E}[\Phi^{[m]}_{\zeta_1} \Phi^{[n]}_{\zeta_2}] \) appearing in the integrals above. Decomposing separately
\( \Phi^{[m]}_{\zeta_{1}} \) and \( \Phi^{[n]}_{\zeta_{2}} \) as in (3.11) we obtain:

\[
\mathbb{E}[\Phi^{[m]}_{\zeta_{1}} \Phi^{[n]}_{\zeta_{2}}] = \frac{3!^2}{(3-m)!(3-n)!} (f_{\varepsilon \zeta})^2 \mathbb{E}[[Y_{\varepsilon \zeta_{1}}^{3-m}][Y_{\varepsilon \zeta_{2}}^{3-n}]] + \frac{3!}{(3-m)!} f_{\varepsilon \zeta} \mathbb{E}[[Y_{\varepsilon \zeta_{1}}^{3-m}] \Phi^{[n]}_{\zeta_{2}}] \\
+ \frac{3!}{(3-n)!} f_{\varepsilon \zeta} \mathbb{E}[[Y_{\varepsilon \zeta_{2}}^{3-n}] \Phi^{[m]}_{\zeta_{1}}] + \mathbb{E}[\Phi^{[m]}_{\zeta_{1}} \Phi^{[n]}_{\zeta_{2}}],
\]

where \( \mathbb{E}[[Y_{\varepsilon \zeta_{1}}^{3-m}][Y_{\varepsilon \zeta_{2}}^{3-n}]] = (3-m)! \delta(3-m,3-n)C_{\varepsilon}(\zeta_{1} - \zeta_{2})^{3-n} \) and to bound all other terms we introduce the following result.

Lemma 3.5

Under Assumption \( \mathbb{F} \) (in particular if \( F \in C^8(\mathbb{R}) \) with exponentially growing derivatives) we have, for every 0 ≤ \( m, n \leq 3 \) and \( m \neq n \):

\[
|\mathbb{E}[\Phi^{[m]}_{\zeta_{1}} \Phi^{[n]}_{\zeta_{2}}]| \lesssim \sum_{i=0}^{4-n} \varepsilon^{1+\frac{m+n+3}{2}} |\langle \zeta_{1}, \zeta_{2} \rangle|^{4-m+i} \lesssim \varepsilon^{\delta} |\langle \zeta_{1}, \zeta_{2} \rangle|^{3-\frac{m+n}{2}+\delta}, \quad \forall \delta \in [0, 1].
\]

Moreover for every 0 ≤ \( m, n \leq 3 \),

\[
|\mathbb{E}[[Y_{\varepsilon \zeta_{1}}^{m} \Phi^{[n]}_{\zeta_{2}}]| \lesssim \varepsilon^{\frac{m+n-3}{2}} |\langle \zeta_{1}, \zeta_{2} \rangle|^m \quad \text{if} \quad m \geq 4-n,
\]

\[
\mathbb{E}[[Y_{\varepsilon \zeta_{1}}^{m} \Phi^{[n]}_{\zeta_{2}}]] = 0 \quad \text{if} \quad m < 4-n.
\]

Proof Using formula (B.3) we decompose

\[
\mathbb{E}[\Phi^{[m]}_{\zeta_{1}} \Phi^{[n]}_{\zeta_{2}}] = \mathbb{E}[\delta^{4-m}(G_{[1]}^{[4-m]} \Phi^{[4]}_{\zeta_{1}} h_{\zeta_{1}}^{4-m}) \delta^{4-n}(G_{[1]}^{[4-n]} \Phi^{[4]}_{\zeta_{2}} h_{\zeta_{2}}^{4-n})]
\]

\[
= \sum_{i=0}^{4-n} \binom{4-n}{i} \binom{4-m}{i} \mathbb{E}(G_{[5-n-i]}^{[8-m-n-i]} \Phi^{[8-m-n-i]}_{\zeta_{1}} G_{[5-m-i]}^{[8-m-n-i]} \Phi^{[8-m-n-i]}_{\zeta_{2}} \langle \zeta_{1}, \zeta_{2} \rangle)^{(8-m-n-i)}.
\]

We can bound the term

\[
\varepsilon^{\frac{m+n-3}{2}} \mathbb{E}(G_{[5-n-i]}^{[8-m-n-i]} \Phi^{[8-m-n-i]}_{\zeta_{1}} G_{[5-m-i]}^{[8-m-n-i]} \Phi^{[8-m-n-i]}_{\zeta_{2}} \langle \zeta_{1}, \zeta_{2} \rangle)^{(8-m-n-i)} \lesssim \varepsilon^{\frac{m+n-5}{2}} \phi^{[8-m-n-i]}_{\zeta_{1}} \mathbb{E} \Phi^{[8-m-n-i]}_{\zeta_{2}} \mathbb{E} \Phi^{[8-m-n-i]}_{\zeta_{2}} \mathbb{E}
\]

knowing 8 - n - i ≤ 8 derivatives of \( F_{\varepsilon} \) (see Remark B.2) and using the bound \( \varepsilon|C_{\varepsilon}(\zeta_{1} - \zeta_{2})| \lesssim 1 \) of Lemma A.12 with \( |\langle \zeta_{1}, \zeta_{2} \rangle| = |C_{\varepsilon}(\zeta_{1} - \zeta_{2})| \) we have:

\[
|\mathbb{E}[\Phi^{[m]}_{\zeta_{1}} \Phi^{[n]}_{\zeta_{2}}]| \lesssim \sum_{i=0}^{4-n} \varepsilon^{1+\frac{m+n+3}{2}} |\langle \zeta_{1}, \zeta_{2} \rangle|^{4-m+i} \lesssim \varepsilon^{\delta} |\langle \zeta_{1}, \zeta_{2} \rangle|^{3-\frac{m+n}{2}+\delta}.
\]

For the second bound we recall that \( [Y_{\varepsilon \zeta_{1}}^{m}] = \delta^{m}(\hat{h}_{\zeta_{1}}^{m}) \) (Remark B.12) and compute

\[
\mathbb{E}[[Y_{\varepsilon \zeta_{1}}^{m} \Phi^{[n]}_{\zeta_{2}}] = \mathbb{E}[\delta^{m}(\hat{h}_{\zeta_{1}}^{m}) \delta^{4-n}(G_{[1]}^{[4-n]} \Phi^{[4]}_{\zeta_{2}} h_{\zeta_{2}}^{4-n})]
\]

\[
= \sum_{i=0}^{m+4-n} \binom{m}{i} \mathbb{E}(\delta^{4-n-i}(\hat{h}_{\zeta_{1}}^{m+i}) G_{[m+1-i]}^{[m+4-n-i]} \Phi^{[m+4-n-i]}_{\zeta_{2}} h_{\zeta_{2}}^{m+4-n-i}) \mathbb{E}
\]

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Since \( D_h^{\otimes m} = 0 \) we obtain \( \mathbb{E}[\|Y_{\varepsilon, h_{\zeta_1}}^{m} \| \Phi_{\zeta_2}^{[m]}] = 0 \) if \( m < 4 - n \) and
\[
|\mathbb{E}[\|Y_{\varepsilon, h_{\zeta_1}}^{m} \| \Phi_{\zeta_2}^{[m]}]| \lesssim \varepsilon^{\frac{m-n}{2}} \mathbb{E}[\varepsilon^{\frac{3m-n}{2}} G_{m+n-3}^{[m]} \Phi_{\zeta_2}^{[m+n]}]|(h_{\zeta_1}, h_{\zeta_2})|^m
\]
if \( m \geq 4 - n \), with
\[
\mathbb{E}[\varepsilon^{\frac{3m-n}{2}} G_{m+n-3}^{[m]} \Phi_{\zeta_2}^{[m+n]}] \lesssim 1.
\]

Using Lemma 3.5 we obtain
\[
\mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}] = 9 \mathbb{E}[(f_{3, \varepsilon}[Y_{\varepsilon, \zeta_1}^{2}] + \Phi_{\zeta_1}^{[1]})(f_{3, \varepsilon}[Y_{\varepsilon, \zeta_2}^{2}] + \Phi_{\zeta_2}^{[1]})] = 18(f_{3, \varepsilon})^2[C_{\varepsilon}(\zeta_1 - \zeta_2)^2] + \mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}]
\]
and thus \( G(t-s, x) = 18(f_{3, \varepsilon})^2 \int_{x_1, x_2} K_{i,x}(x_1)K_{j,x}(x_2)P_{t-s}(x_1)[C_{\varepsilon}(\zeta_1 - \zeta_2)^2] + \hat{G}(t-s, x) \) with the remainder term defined by
\[
\hat{G}(t-s, x) := \int_{x_1, x_2} \sum_{i\neq j} K_{i,x}(x_1)K_{j,x}(x_2)P_{t-s}(x_1)\mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}(t-s, x_2)].
\]
We have the estimation
\[
|\mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}]| \lesssim \varepsilon^\delta C_{\varepsilon}(\zeta_1 - \zeta_2)^{2+\delta}.
\]
Similarly
\[
\mathbb{E}[\Phi_{\zeta_1}^{[1]} \Phi_{\zeta_2}^{[1]}] = 3\varepsilon^{-1/2}f_{3, \varepsilon}\mathbb{E}[\|Y_{\varepsilon, \zeta_1}^{2}\| (f_{3, \varepsilon}[Y_{\varepsilon, \zeta_2}^{2} + \Phi_{\zeta_2}^{[1]}]) = 6\varepsilon^{-1/2}f_{3, \varepsilon}f_{3, \varepsilon}[C_{\varepsilon}(\zeta_1 - \zeta_2)^2],
\]
and
\[
|\mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}]| = |\mathbb{E}[(f_{3, \varepsilon}[Y_{\varepsilon, \zeta_1}^{3}] + \Phi_{\zeta_1}^{[0]})(6f_{3, \varepsilon}[Y_{\varepsilon, \zeta_2}^{2} + \Phi_{\zeta_2}^{[2]}])| \lesssim |f_{3, \varepsilon}||\mathbb{E}[\|Y_{\varepsilon, \zeta_1}^{3}\| \Phi_{\zeta_2}^{[2]}]| + |\mathbb{E}[\Phi_{\zeta_1}^{[0]} \Phi_{\zeta_2}^{[2]}]|
\]
\[
\lesssim \varepsilon^\delta (f_{3, \varepsilon} + 1)C_{\varepsilon}(\zeta_1 - \zeta_2)^{2+\delta},
\]
and
\[
|\mathbb{E}[\Phi_{\zeta_2}^{[0]} \Phi_{\zeta_2}^{[1]}]| = |\mathbb{E}[\Phi_{\zeta_2}^{[0]}(3f_{3, \varepsilon}[Y_{\varepsilon, \zeta_2}^{2} + \Phi_{\zeta_2}^{[2]}])| \lesssim |f_{3, \varepsilon}||\mathbb{E}[\|Y_{\varepsilon, \zeta_2}^{3}\| \Phi_{\zeta_2}^{[1]}]| + |\mathbb{E}[\Phi_{\zeta_2}^{[0]} \Phi_{\zeta_2}^{[1]}]|
\]
\[
\lesssim \varepsilon^{1/2} (f_{3, \varepsilon} + 1)C_{\varepsilon}(\zeta_1 - \zeta_2)^{3}.
\]
We have by Lemma A.16 that for all \( \delta \in (0, 1) \), \( |\hat{G}(t-s, x)| \lesssim \varepsilon^\delta (|t-s|^{1/2} + |x|)^{5-\delta} \). Using estimate (3.19) together with Lemma A.10 we have that for all \( \delta \in (0, 1) \), \( \delta' \in (0, \delta) \) that \( |H(t, x)| \lesssim \varepsilon^{\delta'} (|t-s|^{1/2} + |x|)^{-\delta} \). Furthermore, we have
\[
\frac{1}{3} \Delta_q \hat{Y}_x^\Psi = 6(f_{3, \varepsilon})^2 \int_{s,x} |\Delta_q Y_x(t+s, \bar{x}-x) - \Delta_q Y_x(t, \bar{x})|P_s(x)[C_{\varepsilon}(s, x)]^2 + \frac{1}{3} \Delta_q \hat{Y}_x^\Psi
\]
with the remainder term \( \Delta_q \hat{Y}_x^\Psi \) given by
\[
\Delta_q \hat{Y}_x^\Psi = \int_{s,x} |\Delta_q Y_x(t, \bar{x}-x) - \Delta_q Y_x(t, \bar{x})|\hat{G}(t-s, x).
\]
The term
\[
6(f_{3, \varepsilon})^2 \int_{s,x} |\Delta_q Y_x(t+s, \bar{x}-x) - \Delta_q Y_x(t, \bar{x})|P_s(x)[C_{\varepsilon}(s, x)]^2
\]
can be shown to converge in law to

\[6(\lambda^2) \int_{s,x} [\Delta_q Y(t + s, \bar{x} - x) - \Delta_q Y(t, \bar{x})] P_s(x) \mathbb{E}(Y(0, 0)Y(s, x))^2\]

in \(C^\infty_{\mathcal{F}} -1/2 - 2\kappa\) with the standard techniques used in the analysis of the \(\Phi^4_3\) model. On the other hand, for all \(\delta > 0\) sufficiently small we have the bounds

\[
\|\Delta_q Y_{\varepsilon}\|_{L^\infty} + \|\Delta_q \tilde{Y}_{\varepsilon}\|_{L^\infty} \lesssim \varepsilon^5 \|Y_{\varepsilon}\|_{C^\infty_{\mathcal{F}} -1/2 - 2\kappa} 2^{q(1/2 + 2\kappa + 2\delta)} \int_{s,x} (|x| + |t - s|^{1/2})^{\delta - 5}
\]

\[
\lesssim \varepsilon^5 \|Y_{\varepsilon}\|_{C^\infty_{\mathcal{F}} -1/2 - 2\kappa} 2^{q(1/2 + 2\kappa + 2\delta)},
\]

which shows that these remainders go to zero in \(C^\infty -1/2 - 2\kappa\) as \(\varepsilon \to 0\), since \(\|Y_{\varepsilon}\|_{C^\infty_{\mathcal{F}} -1/2 - 2\kappa}\) is bounded in \(L^p(\Omega)\).

Moreover, it is easy to see that \(\|\Delta_q Y_{\varepsilon} - \Delta_q \tilde{Y}_{\varepsilon}\|_{L^\infty(\Omega)} \sim O_{L^\infty}(2^{q(1/2 + 2\kappa + 2\delta)}).\)

Note that

\[
\int_{s,x} G(t - s, x) = \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[1]} \Phi_0^{[1]}(s, x)] = 18(f_{2,3}\varepsilon)^2 \int_{s,x} P_s(x) |C_s(s, x)|^2 + \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[1]} \Phi_0^{[1]}(s, x)],
\]

\[
\int_{s,x} H(t, x) = \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \Phi_0^{[2]}(s, x)] = \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \Phi_0^{[2]}(s, x)].
\]

Here we used the fact that

\[
\int_{x} \sum_{i,j} K_{i,x}(x_1') K_{j,z}(0) = \int_{x} \sum_{i,j} K_{i,x}(x_1') K_{j,z}(0) = \delta(x_1'),
\]

since \(\int_{x} K_{i,x}(x_1') K_{j,z}(0) = 0\), where \(|i - j| > 1\). This is readily seen in Fourier space taking into account the support properties of the Littlewood-Paley blocks. Now,

\[
\int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[1]} \Phi_0^{[1]}(s, x)], \quad \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \Phi_0^{[2]}(s, x)],
\]

converge to finite constants due to the bounds \(3.18\) and \(3.19\) and by Lemma \(A.13\) \(\int_{s,x} P_s(x) C_s(s, x)^2 \lesssim |\log \varepsilon|\).

Finally, from \(3.20\) we have

\[
\int_{s,x} \mathbb{E}[\Phi_0^{[0]} \Phi_0^{[1]}] \mu_{q, \xi_1, \xi_2} = \int_{s,x} P_s(x) \mathbb{E}[\Phi_0^{[0]} \Phi_0^{[1]}(s, x)] = O(\varepsilon^{-1/2}).
\]

Indeed Lemma \(A.13\) again yields \(\varepsilon \int_{s,x} P_s(x) C_s(s, x)^3 \lesssim 1\).

Thus \(\int_{\xi_1, \xi_2} \mathbb{E}[\Phi_0^{[0]} \Phi_0^{[1]}] \mu_{q, \xi_1, \xi_2}\) gives a diverging constant which depends on all the \((f_{n,\varepsilon})_n\). Making the choice to define the renormalisation constants \(d'\) as in eq. \(1.7\) we cancel exactly these contributions which are either \((F_{\varepsilon})_\varepsilon\) dependent and/or diverging. In particular we verify that we can satisfy the constraint \(2.3\).
Let us summarize our results so far. We have shown that \( \Delta \bar{Y}_\varepsilon (\bar{\zeta}) \) (as seen in (3.16)), we can write the trees of \( \Delta \bar{Y}_\varepsilon (\bar{\zeta}) \) as

\[
\Delta \bar{Y}_\varepsilon (\bar{\zeta}) = \begin{cases} 
(3.21) 
\end{cases}
\]

Let us summarize our results so far. We have shown that \( \Delta \bar{Y}_\varepsilon (\bar{\zeta}) \) and \( \Delta \bar{R}_\varepsilon (\bar{\zeta}) \) are \( O_{L^\infty} (\varepsilon^2 (1/2 + 2\kappa + 2\delta)) \) in \( L^p (\Omega) \) and then these terms converge to 0 in the right topology as \( \varepsilon \to 0 \). As already mentioned, the convergence in law of

\[
(f_{3,\varepsilon})^2 \int_{\zeta, \zeta} (1 - J_0) \mathbb{E} \left[ \left( Y^2_{\varepsilon, \zeta} \right)^2 \right] \mu_{q, \zeta, \zeta} \to (\lambda_3)^2 \Delta \bar{Y}_\varepsilon (\bar{\zeta})
\]

is easy to establish with standard techniques (as done in [4], [18]) assuming the convergence of \( \lambda_3 \) as in (1.9) to \( \lambda \). Then, comparing (3.21) with the canonical trees in (2.10) we can identify the remainder terms \( \Delta \bar{Y}_\varepsilon (\bar{\zeta}) \) that still need to be bounded, that are precisely those in which \( \Phi_{\zeta}^{[n]} \) appears. Estimating these terms is the content of next section.

### 3.4.2 Estimation of renormalised composite trees

In this section we prove the bound (3.7) for composite trees. The difficulty we encounter here is that the remainder \( \bar{Y}_\varepsilon \) cannot be written as an iterated Skorohod integral as in Section 3.1 but instead as a product of iterated Skorohod integrals. We will then use the product formula (B.6) to write the remainder in the desired form. We can
write (3.21) in a much shorter way as:

\[
\begin{align*}
\Delta_1 Y_{\epsilon} (\hat{\zeta}) & = \frac{1}{3} \delta G_1 \int_{\zeta_1, \zeta_2} D(\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_2 Y_{\epsilon} (\hat{\zeta}) & = \frac{1}{3} \delta G_1 \int_{\zeta_1, \zeta_2} D(\Phi^{[1]}_{\zeta_1} \Phi^{[1]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_3 Y_{\epsilon} (\hat{\zeta}) & = \frac{1}{6} \delta G_1 \int_{\zeta_1, \zeta_2} D(\Phi^{[0]}_{\zeta_1} \Phi^{[2]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2}, \\
\Delta_4 Y_{\epsilon} (\hat{\zeta}) & = \frac{1}{3} \delta G_1 \int_{\zeta_1, \zeta_2} D(\Phi^{[2]}_{\zeta_1} \Phi^{[0]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2}, \\
\end{align*}
\]

just substituting again \( \delta G_1 D = (1 - J_0) \) and \( \delta^2 G^{[2]}_{[1]} D^2 = (1 - J_0 - J_1) \). In order to treat all trees at the same time, we can write the first terms in the r.h.s. above (modulo a constant that we discard) as:

\[
\delta^r G^{[r]}_{[1]} \int_{\zeta_1, \zeta_2} D^r(\Phi^{[i]}_{\zeta_1} \Phi^{[j]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2} \quad \text{with } r = 1, i + j = 2 \text{ or } r = 2, i + j = 1.
\]

First notice that by Lemma B.2 we have

\[
\left| \delta^r G^{[r]}_{[1]} \int_{\zeta_1, \zeta_2} D^r(\Phi^{[i]}_{\zeta_1} \Phi^{[j]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2} \right|_{L^p(\Omega)} \lesssim \sum_{\theta = 0}^r \left| D^\theta G^{[\theta]}_{[1]} \int_{\zeta_1, \zeta_2} D^\theta(\Phi^{[i]}_{\zeta_1} \Phi^{[j]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2} \right|_{L^p(\Omega)_{H^{\theta+r}}},
\]

and from the boundedness of the operator \( D^\theta G^{[\theta]}_{[1]} \) given by Corollary B.6 we obtain:

\[
\left| \delta^r G^{[r]}_{[1]} \int_{\zeta_1, \zeta_2} D^r(\Phi^{[i]}_{\zeta_1} \Phi^{[j]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2} \right|_{L^p(\Omega)} \lesssim \left| \int_{\zeta_1, \zeta_2} D^r(\Phi^{[i]}_{\zeta_1} \Phi^{[j]}_{\zeta_2}) \mu_{q, \zeta_1, \zeta_2} \right|_{L^p(\Omega)_{H^{\theta+r}}},
\]

Computing the \( r \)-th derivative of the integrand we obtain

\[
\Phi^{[4-m]}_{\zeta_1} \Phi^{[4-n]}_{\zeta_2} h^{\otimes k}_G \otimes h^{\otimes \ell}_G = \left[ \frac{3! f_{3, \epsilon}}{(3 - m)!} [Y^{m-1} + \Phi^{[4-m]}_{\zeta_1}] \right] \left[ \frac{3! f_{3, \epsilon}}{(3 - n)!} [Y^{n-1} + \Phi^{[4-n]}_{\zeta_2}] \right] h^{\otimes k}_G \otimes h^{\otimes \ell}_G \tag{3.22}
\]

for \( m + n = 5 \) and \( 0 \leq k + \ell \leq 2 \). The constraints on \( m, n, k, \ell \) are related to the number of branches in the graphical notation of the trees: each tree has \( m + k - 1 \) leaves with height 2 and \( n + \ell - 1 \) leaves with height 1, as follows

\[
\begin{align*}
Y_{\epsilon} & \iff m + k = 3, \ n + \ell = 3, \\
\bar{Y}_{\epsilon} & \iff m = 3, k = 0, n = 2, \ell = 1, \\
Y_{\epsilon} & \iff m + k = 4, \ n + \ell = 2, \\
\bar{Y}_{\epsilon} & \iff m + k = 4, \ n + \ell = 3.
\end{align*}
\tag{3.23}
\]

In (3.22), the terms which do not contain \( \hat{\Phi}^{[n]}_{\zeta} \) will generate finite contributions in the limit, as seen in Section 3.4.1 by writing the decomposition (3.21). We just consider the terms proportional to \( \hat{\Phi}^{[4-m]}_{\zeta_1} \hat{\Phi}^{[4-n]}_{\zeta_2} \), because
all the other similar terms featuring at least one remainder $\Phi^{[m]}_\zeta$ can be estimated with exactly the same technique, and are easily shown to be vanishing in the appropriate topology.

We can use one of the key observations of this paper, the product formula (3.30), to rewrite products of Skorohod integrals in the form $\delta^n(u)\delta^m(v)$ as a sum of iterated Skorohod integrals $\delta^l(w)$, which are bounded in $L^p$ by Lemma B.2. We obtain

\[
\Phi^{[4-n]}_{\zeta_1} \Phi^{[4-n]}_{\zeta_2} = \sum_{(q,r,i) \in I} C_{q,r,i} \delta^m(G^{[m]}_{\zeta_1} \Phi^{[4]}_{\zeta_1} h_{\zeta_1}^{m+n-q-r} (h_{\zeta_1}^{m+n-q-r} - h_{\zeta_2}^{m+n-q-r}))
\]

with $I = \{(q,r,i) \in \mathbb{N}^3 : 0 \leq q \leq m, 0 \leq r \leq n, 0 \leq i \leq q \land r \}$ and the notation shortcut:

\[
\Theta^{[j]}_{[i]}(\zeta) := e^{-\frac{1}{2}G^{[j]}_{[i]} \Phi^{[3+i]}_\zeta}.
\]

By Remark B.3 for every $n,m \geq 1$ and $\Psi \in \text{Dom} \delta^n$ we can write $\delta^n(\Psi) h^{\otimes m} = \delta^n(\Psi \otimes h^{\otimes m})$, and therefore

\[
\sum_{(q,r,i) \in I} C_{q,r,i} e^{\frac{1}{2}m-n-q-r} \delta^m H^{n+q-r} (h_{\zeta_1}^{m+q-i} (h_{\zeta_1}^{m+q-i} - h_{\zeta_2}^{m+q-i})) =
\]

With the term to estimate in this form, we can proceed as in Section 3.1 to estimate separately the terms $\Theta^{[j]}_{[i]}(\zeta) = e^{-\frac{1}{2}G^{[j]}_{[i]} \Phi^{[3+i]}_\zeta}$ in $L^p(\Omega)$, which are bounded as discussed in Remark 3.2.

**Lemma 3.6**

*Under Assumption 2 (in particular if $F_\varepsilon \in C^8(\mathbb{R})$ and the first 8 derivatives have exponential growth) we have the bound:*

\[
||\delta^{m+n-q-r} \Theta^{[m+q-i]}_{1+q-i} (h_{\zeta_1}^{m+q-i} (h_{\zeta_1}^{m+q-i} - h_{\zeta_2}^{m+q-i})) ||_{L^p(\mathbb{H}^k)}^2 \lesssim \sum_{j=0, h \in \mathbb{N}} \||D^h \Theta^{[m+q-i]}_{1+q-i} (h_{\zeta_1}^{m+q-i} (h_{\zeta_1}^{m+q-i} - h_{\zeta_2}^{m+q-i})) ||_{L^p(V)}^2
\]

*Proof* Thanks to Lemma B.3 the integral can be estimated with

\[
\sum_{j=0, h \in \mathbb{N}} \||D^h \Theta^{[m+q-i]}_{1+q-i} (h_{\zeta_1}^{m+q-i} (h_{\zeta_1}^{m+q-i} - h_{\zeta_2}^{m+q-i})) ||_{L^p(V)}^2
\]

with $V = H^{\otimes m+n+q-i+j}$. We have that $\| \cdot \|_{L^p(\mathbb{H}^k)} = \| \cdot \|_{H^{\otimes k}}^{1/2} \| \cdot \|_{L^p/2}$ and therefore we can bound each term in the sum above as

\[
\lesssim \left( \sum_{j=0, h \in \mathbb{N}} \||D^h \Theta^{[m+q-i]}_{1+q-i} (h_{\zeta_1}^{m+q-i} (h_{\zeta_1}^{m+q-i} - h_{\zeta_2}^{m+q-i})) ||_{L^p(\mathbb{H}^k)} \right)^{1/2} \times \left( \sum_{j=0, h \in \mathbb{N}} \||D^h \Theta^{[m+q-i]}_{1+q-i} (h_{\zeta_1}^{m+q-i} (h_{\zeta_1}^{m+q-i} - h_{\zeta_2}^{m+q-i})) ||_{L^p(V)} \right)^{1/2}
\]

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Using Hölder’s inequality we get the estimate
\[
\|\langle D^h \Theta_{[1+a]}^{[m+a]} (\zeta), D^h \Theta_{[1+a]}^{[m+a]} (\zeta') \rangle\|_{L^p} \lesssim \|\langle D^h \Theta_{[1+a]}^{[m+a]} (\zeta), D^h \Theta_{[1+a]}^{[m+a]} (\zeta') \rangle\|_{L^{p'}} \lesssim \|D^h G_{[1+a]}^{[m+a]} (\zeta)\|_{L^p} \|D^h G_{[1+a]}^{[m+a]} (\zeta')\|_{L^p}.
\]

Now to bound \(\|\langle D^h \Theta_{[1+a]}^{[m+a]} (\zeta), D^h \Theta_{[1+a]}^{[m+a]} (\zeta') \rangle\|_{L^p}\) (with \(h \leq j \leq m + n - q - r \) and \(a = r - i\)) we can use the boundedness of the operator \(D^h G_{[1+a]}^{[m+a]}\) for \(h \leq 2m\) given by Corollary \(3.6\). Consider the two regions \(h \leq 2m\) and \(h > 2m\). In the first region we just use Corollary \(3.6\) to obtain:
\[
\|\langle D^h \Theta_{[1+a]}^{[m+a]} (\zeta), D^h \Theta_{[1+a]}^{[m+a]} (\zeta') \rangle\|_{L^p} \lesssim \|D^h G_{[1+a]}^{[m+a]} (\zeta)\|_{L^p} \|D^h G_{[1+a]}^{[m+a]} (\zeta')\|_{L^p}.
\]

If \(h > 2m\) we first use the bound \(D^{2m} G_{[1+a]}^{[m+a]}\) and then take the remaining \(h - 2m\) derivatives on \(e^{-\frac{1+i\alpha}{\bar{\nu}}\Phi_{\zeta}}\):
\[
\|\langle D^h \Theta_{[1+a]}^{[m+a]} (\zeta), D^h \Theta_{[1+a]}^{[m+a]} (\zeta') \rangle\|_{L^p} \lesssim \|D^{2m} G_{[1+a]}^{[m+a]} (\zeta)\|_{L^p} \|D^{2m} G_{[1+a]}^{[m+a]} (\zeta')\|_{L^p}.
\]

From Remark \(3.2\) we see that this last term is bounded by a constant if \(F \in C^{4+a+h-2m}(\mathbb{R})\) with the first \(4 + a + h - 2m\) derivatives having an exponential growth, with \(a = r - i\) and \(h \leq m + n - q - r\).

Applying the same reasoning to \(\|\langle D^{[m+r-i]} G_{[1+q-i]}^{[n-q-i]} (\zeta), D^{[m+r-i]} G_{[1+q-i]}^{[n-q-i]} (\zeta') \rangle\|_{L^p}\) we conclude that we need to control \(4 + n + 4 + m\) derivatives of \(F\) in order to perform the estimates of this Lemma. From the constraints (3.23) we see that \(4 + n + 4 + m \leq 8\).

From Lemma \(3.6\) we obtain \(\forall \delta \in [0, 1/2]: \varepsilon^{\frac{2+a+h-2m}{2}} \delta^{m+q-r} \int \Theta_{[1+q-i]}^{[n-q-i]} (\zeta) \Theta_{[1+r-i]}^{[m+r-i]} (\zeta) \mu_{\zeta, \zeta}^{m+q-r} \mu_{\zeta, \zeta}^{n+q-r} \mu_{\zeta, \zeta}^{q+r-i} \mu_{\zeta, \zeta}^{q+r-i} \|C_{\zeta, \zeta}^{1/2}\|_{L^p} \lesssim \varepsilon^\delta (\varepsilon^{2+a+h-2m} \delta)^{\frac{1}{2}}.
\]

Our aim now is to estimate the quantity \(J\). The idea is to use the bound \(\varepsilon \|h_{c, h_{c'}}\| = \varepsilon^2 (\zeta - \zeta') \lesssim 1\) of Lemma \(A.12\) to cancel strategically some of the covariances \(\|h_{c, h_{c'}}\|\). We will consider three regions:

If \(q + r \leq 2\) we use the bounds
\[
\varepsilon^{q+r-2i} \|h_{c, h_{c'}}\|^q \|h_{c', h_{c'_1}}\|^q \|h_{c'_2, h_{c'_2}}\|^q \lesssim \varepsilon^2 \|h_{c_1, h_{c'_2}}\|^q \|h_{c'_1, h_{c'_2}}\|^q
\]
and then (suppose \(r < 2\))
\[
\varepsilon^{q+r-2i} \|h_{c, h_{c'}}\|^q \|h_{c'_2, h_{c'_2}}\|^q \lesssim \|h_{c_2, h_{c'_2}}\|^q \|h_{c'_2, h_{c'_2}}\|^q
\]
to obtain
\[ I \lesssim \epsilon^{r-\delta} \int \langle \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{m+k-q} \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{n+\ell-r} \langle \langle h_{\zeta_1}, h_{\zeta_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle. \]  
(3.24)

(IF vice-versa \( q < 2 \) it suffices to put \( \delta \) on the term \( \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{q+\delta} \).

Notice that in this case \( m+k-q > 0 \).

In the case \( q+r = 3 \) if \( m+k-q \geq 2 \) we estimate like before to obtain
\[ I \lesssim \epsilon^{2-\delta} \int \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{m+k-q} \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{n+\ell-r} \langle \langle h_{\zeta_1}, h_{\zeta_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle. \]  
(3.25)

Note that \( m+k-q+\delta-1 > 0 \) and \( m+k-q+2\delta-3 > -1 \) here. If \( m+k-q = 1 \) we bound
\[ I \lesssim \epsilon^{3-2\delta} \int \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{m+k-q} \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{n+\ell-r-2} \langle \langle h_{\zeta_1}, h_{\zeta_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle. \]  
(3.26)

and note that \( m+k-q-1/2+\delta/2 > 0, m+k-q-1+\delta > 0, n+\ell-r-2 \geq 0 \). Finally if \( m+k-q = 0 \) we can only have \( m+k = 3, q = 3, r = 0, i = 0 \) and thus
\[ I \lesssim \epsilon^{3-2\delta} \int \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{m+k-q} \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{n+\ell-r} \langle \langle h_{\zeta_1}, h_{\zeta_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle \]  
(3.27)

If \( q+r \geq 4 \) we bound first
\[ \epsilon^{2q+2r-2i+\delta-4} \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{q+r-i} | \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{q+r-1} \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{q+r-i} \lesssim | \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{2-\frac{q}{2}} | \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{2-\frac{q}{2}}. \]

(note that \( 2q+2r-2i+\delta-4 \geq \delta \)) to obtain:
\[ I \lesssim \epsilon^{6-q-\delta} \int \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{m+k-q} \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{n+\ell-r} \langle \langle h_{\zeta_1}, h_{\zeta_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle. \]

Now in the cases \( m+k = 3, n+\ell = 3 \) and \( m+k = 4, n+\ell = 2 \) we can just write \( \epsilon^{6-q-\delta} = \epsilon^{m+k-q} \epsilon^{6-m-k-r-\delta} \) and cancel the corresponding number of covariances to obtain
\[ I \lesssim \int | \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{\delta} | \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle. \]  
(3.28)

while for the case \( m+k = 4, n+\ell = 3 \) we have either \( \ell \geq 1 \) or \( k \geq 1 \) and therefore with one of the following bounds
\[ \epsilon^{m+k-1-q} \epsilon^{n+\ell-r-\delta} | \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{m+k-q} | \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{n+\ell-r} \lesssim | \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle | \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{\delta} \]
\[ \epsilon^{m+k-q} \epsilon^{n+\ell-1-r-\delta} | \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle^{m+k-q} | \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{n+\ell-r} \lesssim | \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{1+\delta}. \]

we obtain the estimates
\[ I \lesssim \int | \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{1+\delta} | \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle. \]  
(3.29)
\[ I \lesssim \int | \langle \langle h_{\zeta_1}, h_{\zeta'_1} \rangle \rangle | \langle \langle h_{\zeta_2}, h_{\zeta'_2} \rangle \rangle^{\delta} | \langle \langle h_{\zeta_1}, h_{\zeta'_2} \rangle \rangle^{2-\frac{q}{2}} | \mu_{q,\zeta_1,\zeta_2} \rangle | \mu_{q,\zeta_1,\zeta'_2} \rangle | \mu_{q,\zeta_1,\zeta'_1} \rangle. \]  
(3.30)
We can use directly Lemma A.17 to obtain a final estimate of (3.24), (3.25), (3.26), (3.29). For (3.27), (3.28) and (3.30) notice that the integral over $\zeta_1, \zeta_2$ is finite and thus the whole quantity is proportional to $|\langle h_{\zeta_1}, h_{\zeta_2} \rangle|^n$. Globally, we have

$$\mathfrak{I} \lesssim 2^{(m+k+n+t-6)q}$$

as needed to prove (3.7).

**Remark 3.7** Finally, by controlling one more derivative of $F_{\varepsilon}$ as done in Section 3.3 we can show (3.8) for \( Y_\tau = Y_{\varepsilon} \), \( \dot{Y}_\tau \), \( \ddot{Y}_\tau \), \( \dddot{Y}_\tau \), \( \ddddot{Y}_\tau \), thus proving that $\hat{Y} \rightarrow 0$ in $C_T^{\alpha/2}\mathcal{G}^{\alpha-\kappa}$ in probability $\forall \alpha < |\tau|$. From the proof of Lemma 3.6 together with this observation, we conclude that we need to control the derivatives of $F_{\varepsilon}$ up to order 9 to be able to show the convergence for composite trees.

### 4 Convergence of the remainder and a-priori bounds

In this section we prove the convergence of the remainder (Lemma 4.5), as well as some technical results on the norm of the solution, needed in the proof of Theorem 2.2. In order to prove Lemma 4.1 we need first to prove Lemma 4.3 and Lemma 4.4 in this order.

#### 4.1 Boundedness of the remainder

We show that the remainder $R_{\varepsilon}(v_{\varepsilon})$ that appears in equation (2.10) can be controlled by a stochastic term $M_{\varepsilon, \delta}$ that converges to zero in probability, times a function of the solution $v_{\varepsilon}$. Let.

$$M_{\varepsilon, \delta}(Y_{\varepsilon}, u_{0,\varepsilon}) := \varepsilon^{\delta/2} ||e^{c\varepsilon^{1/2} |Y_{\varepsilon}| + c\varepsilon^{1/2} |P(u_{0,\varepsilon} - Y_{\varepsilon}(0))||}_{L^p[0, T]} |_{L^q(T^3)},$$

for $p \geq 1$, $\delta \in [0, 1]$ and define

$$v_{\varepsilon}^\varnothing := v_{\varepsilon} - v_{\varepsilon}^0 \quad \text{with} \quad v_{\varepsilon}^0 : t \mapsto P_1(u_{0,\varepsilon} - Y_{\varepsilon}(0)).$$

**Lemma 4.1 (Boundedness of remainder)**

For every $\gamma \in (0, 1)$, $\delta \in [0, 1]$ we have

$$||R_{\varepsilon}(v_{\varepsilon}, v_{\varepsilon}^0, v_{\varepsilon}^0)(t, x)||_{\mathcal{M}_T^{\gamma/(3+\delta), p} L^p(T^3)} \lesssim M_{\varepsilon}(Y_{\varepsilon}, u_{0,\varepsilon}) ||v_{\varepsilon}^0||_{\mathcal{M}_T^{\gamma/(3+\delta)} L^\infty(T^3)} e^{c\varepsilon^{1/2} ||v_{\varepsilon}^0||_{C_T L^\infty}}$$

with $M_{\varepsilon}$ as in (4.1), $v_{\varepsilon}^0$ as in (4.2) and $\mathcal{M}_T^{\gamma/(3+\delta), p} L^p(T^3)$, $\mathcal{M}_T^{\gamma/(3+\delta)} L^\infty(T^3)$ defined in (A.1).

**Proof** We can write the remainder in two ways:

$$R_{\varepsilon}(v_{\varepsilon}) = v_{\varepsilon}^3 \int_0^1 [F_{\varepsilon}^{(3)}(\varepsilon^{\frac{1}{2}} Y_{\varepsilon} + \tau \varepsilon^{\frac{1}{2}} v_{\varepsilon}) - F_{\varepsilon}^{(3)}(\varepsilon^{\frac{1}{2}} Y_{\varepsilon})] \frac{(1 - \tau)^2}{2!} \, d\tau$$

$$= \varepsilon^{\frac{3}{2}} v_{\varepsilon}^3 \int_0^1 F_{\varepsilon}^{(4)}(\varepsilon^{\frac{1}{2}} Y_{\varepsilon} + \tau \varepsilon^{\frac{1}{2}} v_{\varepsilon}) \frac{(1 - \tau)^3}{3!} \, d\tau.$$
and we estimate, \(\forall \gamma \in [0,1]\),

\[
\|t \mapsto t^\gamma R_\epsilon(v_\epsilon)(t,x)\|_{L^p((0,T],L^p(\mathbb{T}^3))} \lesssim \|t^\gamma v_\epsilon(t)\|_{C^\gamma_T L^\infty}^{3+\delta} + \|v_\epsilon\|_{C^\gamma_T L^\infty} \|Y_\epsilon(t,x) + c_\epsilon^\frac{1}{2}|v_\epsilon|^2(t,x)\|_{L^p[0,T]L^p(\mathbb{T}^3)}.
\]

\(\square\)

We can also verify that \(M_{\epsilon,\delta} \to 0\) in probability for every \(\delta > 0\):

**Lemma 4.2 (Convergence of the stochastic term)**

Under Assumption 7 the random variable \(M_{\epsilon,\delta}(Y_\epsilon, u_{0,\epsilon})\) defined in [4.7] converges to zero in probability \(\forall \delta \in (0,1]\).

**Proof** We can use Young’s inequality to estimate \(M_{\epsilon,\delta}(Y_\epsilon, u_{0,\epsilon})\) for some \(c' > 0\) as

\[
M_{\epsilon,\delta}(Y_\epsilon, u_{0,\epsilon}) \lesssim \epsilon^{\delta/2}\|e^{c'\epsilon^{1/2}}Y_\epsilon\|_{L^p[0,T]L^p(\mathbb{T}^3)} + \epsilon^{\delta/2}\|e^{c'\epsilon^{1/2}}|P Y_\epsilon(0)|\|_{L^p[0,T]L^p(\mathbb{T}^3)}
\]

Under Assumptions \(\|e^{1/2}u_{0,\epsilon}\|_{L^\infty(\mathbb{T}^3)}\) is uniformly bounded, so the third term above converges to zero in probability. Note that \(\epsilon^{1/2}Y_\epsilon(t,x)\) and \(P e^{1/2}Y_\epsilon(t = 0)\) are centered Gaussian random variables, and then both \(\mathbb{E}\|e^{c'\epsilon^{1/2}|Y_\epsilon|\|_{L^p[0,T]L^p(\mathbb{T}^3)}\) and \(\mathbb{E}\|e^{c'\epsilon^{1/2}|P Y_\epsilon(0)|\|_{L^p[0,T]L^p(\mathbb{T}^3)}\) are uniformly bounded in \(\epsilon > 0\) for every \(p \in [1,\infty)\). This yields the convergence in probability of \(M_{\epsilon,\delta}(Y_\epsilon, u_{0,\epsilon})\). \(\square\)

In order to show that \(\|R_\epsilon(v_\epsilon)\|_{M^{\gamma'}_{\epsilon,T} L^p} \to 0\) in probability for \(\gamma' > 1 + \frac{3}{2} \kappa\) as needed in the proof of Theorem 2.2, we still need to control the norms \(\|v_\epsilon\|_{M^{\gamma'/3+\gamma+\delta}_{\epsilon,T} L^\infty}\) and \(\|v_\epsilon^0\|_{C_T L^\infty}\) that appear in Lemma 4.1. This is done in next section.

### 4.2 Apriori bounds on the solution

**Lemma 4.3 (Apriori bound on the solution)**

Fix \(T > 0\). There exists \(\kappa > 0\), \(T_* = T_*(\|\mathbb{Y}_\epsilon\|_{X_T}, \|u_{\epsilon,0}\|_{\mathbb{D}^{-1/2-\kappa}}, |\lambda_\epsilon|) \in (0,T)\) a lower semicontinuous function depending only on \((\|\mathbb{Y}_\epsilon\|_{X_T}, \|u_{\epsilon,0}\|_{\mathbb{D}^{-1/2-\kappa}}, |\lambda_\epsilon|)\) and a collection of events \(\mathcal{E}_\epsilon\rangle_\epsilon > 0\) such that

\[
\mathbb{P}(\mathcal{E}_\epsilon) \to 1 \quad \text{as } \epsilon \to 0
\]

and conditionally on \(\mathcal{E}_\epsilon\) there exists a universal constant \(C > 0\) such that:

\[
\|v_\epsilon\|_{C_{T_\epsilon} \mathbb{D}^{-1/2-\kappa}} + \|v_\epsilon\|_{M_{T_\epsilon}^{\frac{3}{2}} L^\infty} \leq C(1 + |\lambda_\epsilon|)(1 + \|\mathbb{Y}_\epsilon\|_{X_T})^3 (1 + \|u_{\epsilon,0}\|_{\mathbb{D}^{-1/2-\kappa}})^3
\]

for any \(v_\epsilon\) that solves equation (2.7). Moreover, still conditionally on \(\mathcal{E}_\epsilon\) we have

\[
\|v_\epsilon^0\|_{C_{T_\epsilon} L^\infty} \leq C(1 + |\lambda_\epsilon|)(1 + \|\mathbb{Y}_\epsilon\|_{X_T})^3 (1 + \|u_{\epsilon,0}\|_{\mathbb{D}^{-1/2-\kappa}})^3
\]

with \(v_\epsilon^0\) as in [4.2].

**Proof** We know from Lemma 4.1 that the bounds above on \(\|v_\epsilon\|_{C_{T_\epsilon} \mathbb{D}^{-1/2-\kappa}} + \|v_\epsilon\|_{M_{T_\epsilon}^{\frac{3}{2}} L^\infty} \) and \(\|v_\epsilon^0\|_{C_{T_\epsilon} L^\infty}\) hold whenever \(M_{\epsilon,\delta} \leq T_*^{\gamma/2}\). The event \(\mathcal{E}_\epsilon = \{M_{\epsilon,\delta} \leq T_*^{\gamma/2}\}\) has \(\mathbb{P}(\mathcal{E}_\epsilon) \to 1\) by Lemma 4.2 and this proves the result. \(\square\)
The only thing left to prove is Lemma 4.4 which just a standard application of some well-known bounds on paraproducts, that are recalled in Appendices A.2 and A.2.

First observe that for $\varepsilon > 0$ a pair $(u_\varepsilon, \nabla_\varepsilon)$ solves the paracontrolled equation (2.6) if and only if $v_\varepsilon = v_\varepsilon^0 + v_\varepsilon^1$ and $(u_\varepsilon, \nabla_\varepsilon)$ solves:

\[
\begin{align*}
\mathcal{L} v_\varepsilon^0 & = -\nabla Y_\varepsilon - Y_\varepsilon \nabla Y_\varepsilon - 3\varepsilon Y_\varepsilon \nabla Y_\varepsilon + v_\varepsilon^0 + v_\varepsilon^1 \\
\mathcal{L} v_\varepsilon^1 & = U(\lambda_\varepsilon, \nabla Y_\varepsilon; u_\varepsilon, v_\varepsilon^0 + v_\varepsilon^1) - R_\varepsilon(v_\varepsilon^1) \\
v_\varepsilon^0(0) & = Y_\varepsilon(0) + \hat{Y}_\varepsilon(0) + 3\varepsilon R_\varepsilon(v_\varepsilon^1) - Y_\varepsilon(0) 
\end{align*}
\]

(4.3)

Here $U$ is the same as in (2.6). The initial condition of (4.3) is given by $v_{\varepsilon,0} := u_{0,\varepsilon} - Y_\varepsilon(0)$. The a-priori bounds of Lemma 4.4 come from being able to find closed estimates for (4.3).

Let us specify now all the notations we are going to use in the rest of this section. We consider the spaces

\[
\mathcal{V}^b_T := \mathcal{L}^{2\kappa}_T \cap \mathcal{L}^{1/4,1/2+2\kappa}_T \cap \mathcal{L}^{1/2,1+2\kappa}_T, \quad \mathcal{V}_T := \mathcal{L}^{1/2,1/2-\kappa}_T \cap \mathcal{L}^{1/4+3\kappa/2,2\kappa}_T,
\]

with the corresponding norms

\[
\|v_\varepsilon^0\|_{\mathcal{V}^b_T} := \|v_\varepsilon^0\|_{\mathcal{L}^{2\kappa}_T} + \|u_\varepsilon^0\|_{\mathcal{L}^{1/4,1/2+2\kappa}_T} + \|v_\varepsilon^0\|_{\mathcal{L}^{1/2,1+2\kappa}_T},
\]

\[
\|v_\varepsilon\|_{\mathcal{V}_T} := \|v_\varepsilon\|_{\mathcal{L}^{1/2,1/2-\kappa}_T} + \|v_\varepsilon\|_{\mathcal{L}^{1/4+3\kappa/2,2\kappa}_T}.
\]

(4.4) (4.5)

We refer to Appendix A.4 for the definition of the parabolic spaces $\mathcal{L}^{\gamma;\alpha}_T$. We let

\[
v_\varepsilon^0 := v_\varepsilon - v^\# = -\nabla Y_\varepsilon - Y_\varepsilon \nabla Y_\varepsilon - 3(\nabla^0 + \nabla^1) \nabla Y_\varepsilon + v_\varepsilon^1,
\]

\[
v_\varepsilon^1 := v_\varepsilon^0 + Y_\varepsilon \nabla Y_\varepsilon = -\nabla Y_\varepsilon - 3(\nabla^0 + \nabla^1) \nabla Y_\varepsilon + v_\varepsilon^1,
\]

and $v_\varepsilon^1(t = 0) = v_\varepsilon^0(t = 0) = 0$. We define also the norm

\[
\|v_\varepsilon^0\|_{\mathcal{Y}_T} := \|v_\varepsilon^0\|_{\mathcal{L}^{2\kappa}_T} + \|v_\varepsilon^0\|_{\mathcal{M}^{1/4+3\kappa/2,2\kappa}_T},
\]

with $\mathcal{M}^{1/4+3\kappa/2,2\kappa}_T$ given in Appendix A.3. In order not to get lost in these definitions the reader can keep in mind the following:

- $v_\varepsilon$ is the solution without the linear term;
- $v_\varepsilon^0$ is the contribution of the initial condition, which give origin to some explosive norm (near the initial time);
- $v_\varepsilon^1$ is the regular part of the solution;
- $v_\varepsilon^0, v_\varepsilon^1$ enter in the estimation of the remainder, they are just convenient shortcuts for certain contributions appearing in $v_\varepsilon$.

**Lemma 4.4**

There exists $T_* = T_*$ ($\|Y_\varepsilon\|_{\mathcal{X}_T}, \|u_{0,\varepsilon}\|_{\mathcal{Y}_T}, |\lambda_\varepsilon| \in (0, T]$) a lower semicontinuous function depending only on $\|Y_\varepsilon\|_{\mathcal{X}_T}, \|u_{0,\varepsilon}\|_{\mathcal{Y}_T}$ and $|\lambda_\varepsilon|$, a constant $M_{\varepsilon,\delta} = M_{\varepsilon,\delta}(Y_\varepsilon, u_{0,\varepsilon}) > 0$ defined by (4.4), and a universal constant $C > 0$ such that, whenever $M_{\varepsilon,\delta} \leq T_*^{1/2}$ we have

\[
\|v_\varepsilon\|_{\mathcal{V}_T} \leq C(1 + |\lambda_\varepsilon|)(1 + \|Y_\varepsilon\|_{\mathcal{X}_T})^3(1 + \|u_{0,\varepsilon}\|_{\mathcal{Y}_T})^3,
\]

\[
\|v_\varepsilon\|_{\mathcal{V}_T} \leq C\left(\|Y_\varepsilon\|_{\mathcal{X}_T} + \|u_{0,\varepsilon}\|_{\mathcal{Y}_T} + \|v_\varepsilon^0\|_{\mathcal{Y}_T}\right).
\]

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Proof. Using the well-known Schauder estimates of Lemma A.2 (and the fact that \( \|f\|_{L^q_T} \lesssim T^\kappa \|f\|_{L^q_T} \)) we obtain for \( \kappa, \theta > 0 \) small enough

\[
\|If\|_{L^{\theta+2\kappa,2\kappa}_T} + \|If\|_{L^{1/2-\theta+2\kappa,1/2+2\kappa}_T} + \|If\|_{L^{1/2-\theta+2\kappa,1/2+2\kappa}_T} \lesssim T^{\frac{\kappa}{\theta}} \left( \|f\|_{M^{1-\theta,q'-\kappa}_T} + \|f\|_{M^{1/2+2\kappa,q'-1/2-2\kappa}_T} \right). \tag{4.6}
\]

We choose \( \theta > 2\kappa \) small enough so that

\[
L^{\theta+3\kappa/2,2\kappa}_T \cap L^{1/4-\theta+3\kappa,2/2+2\kappa}_T \cap L^{1/2-\theta+3\kappa/2,2\kappa}_T \subset V_T.
\]

Now

\[
\|v^\circ \|_{V_T} \lesssim \|Y_v \|_{V_T} + \|v^\circ \|_{C_T L^\infty} (\|Y_v \|_{C_T L^{1-\kappa}} + \|Y_v \|_{C_T L^{-\kappa}}) + \left( \|v^\circ \|_{C_T L^{-1/2-\kappa}} + \|v^\circ \|_{M_T^{1/4,q'-\kappa}} \right) (\|Y_v \|_{C_T L^{1-\kappa}} + \|Y_v \|_{C_T L^{-\kappa}}) + \|v^\circ \|_{V_T} \lesssim \|Y_v \|_{V_T} + \|v_0 \|_{L^{1/2-\kappa}} + \|v^\circ \|_{V_T},
\]

where we used that \( v^\circ (0) = 0 \) and as a consequence that \( \|v^\circ \|_{C_T L^\infty} \lesssim T^\kappa \|v^\circ \|_{C_T L^\infty} \leq T^\kappa \|v^\circ \|_{V_T} \) to gain a small power of \( T \). So provided \( T \) is small enough (depending only on \( Y_v \)) this yields the following a-priori estimation on \( v^\circ \):

\[
\|v^\circ \|_{C_T L^\infty} \lesssim \|v^\circ \|_{V_T} \lesssim \|Y_v \|_{V_T} + \|v_0 \|_{L^{1/2-\kappa}} + \|v^\circ \|_{V_T}.
\]

Therefore we have an estimation on \( v_e \):

\[
\|v_e \|_{V_T} \leq \|v^\circ \|_{V_T} + \|v_0 \|_{L^{1/2-\kappa}} + \|v^\circ \|_{V_T} \lesssim \|Y_v \|_{V_T} + \|v_0 \|_{L^{1/2-\kappa}} + \|v^\circ \|_{V_T}.
\]

In order to estimate terms in \( U(\lambda_e, Y_v; v_e, v^\circ_e + v^\circ) \) we decompose the renormalised products as

\[
\begin{align*}
Y_v \circ v_e &= v_e \circ Y_v \circ Y_v - Y_v \circ Y_v - 3v_e Y_v \circ Y_v + v^\circ_e \circ Y_v \circ Y_v + v^\circ \circ Y_v \circ Y_v - 3\text{com}(v_e, Y_v \circ Y_v, Y_v) \\
v_e \circ Y_v &= -Y_v \circ Y_v - 3(v_e \circ Y_v) Y_v + Y_v \lesssim (v^\circ_e + v^\circ) + Y_v \circ (v^\circ_e + v^\circ) \\
Y_e \circ v^2 &= Y_e \circ (Y_v \circ Y_v) + 2(Y_e \circ Y_v)(Y_v + 3v_e - Y_v) - 2(Y_e \circ Y_v) \lesssim (v^\circ_e + v^\circ) + 2(Y_e \circ Y_v) \circ (v^\circ_e + v^\circ) + Y_e \circ (v^\circ_e + v^\circ)^2 + Y_e \circ (v^\circ_e + v^\circ)^2.
\end{align*}
\]
We decompose $U(\lambda, Y; v, v_0, v_1, v_2) + Q_0(\lambda, Y, v_0, v, v_1, v_2) + Q_{\lambda, Y}$.

$Q_{-1/2} = -3[v \rightharpoonup Y^* - 3 \text{com}(v, Y)] + (v_0^2 + v_1^2)
-6[(Y \circ Y) + (Y \circ Y)] + (v_0^2 + v_1^2)\]
$3\lambda_2\,(3v - v_0^2) + 2\lambda_2\,(3v - Y^* - v_0^2) + 3 \text{com}_3(v, Y^*) + 3 \text{com}_2(v, Y)
$Q_0 = 3[v Y^* - v_0^2 \circ Y + v_1^2 \circ Y^* + 2(Y \circ Y) - Y^* - Y^* - 2(Y \circ Y)]
$-Y^\delta_{-2} - 2[Y \circ Y^* - 2(Y \circ Y)] + (Y \circ Y^*)]
$2\lambda_2\,(v - Y^* - v_0^2) - Y^* + Y^* + Y^* + Y^*.

Here $Q_{\lambda, Y}$ does not depend on the solution but only on $\lambda, Y$ (as the notation suggests) and we have grouped the other terms which we expect to have regularity $C^{-1/2-2\kappa}$ in $Q_{-1/2}$, (and the same for $Q_0$ and regularity $C^{-k}$).

With the same technique we used above for $v_0^2$, we obtain the following estimate on $v_0^2$:

$$\|v_0^2\|_{\mathcal{L}^{3/2+\kappa/2,1+\kappa}} \leq \|Y\|_{\mathcal{X}} + \|v_0,0\|_{\mathcal{C}^{-1/2-\kappa}} + \|v_0^2\|_{\mathcal{X}}.$$

and this yields

$$\|v_0^2\|^2_{\mathcal{L}^{3/2+\kappa/2,1+\kappa}} \leq \left(\|Y\|_{\mathcal{X}} + \|v_0,0\|_{\mathcal{C}^{-1/2-\kappa}} + \|v_0^2\|_{\mathcal{X}}\right)^2.$$

Then we are ready to bound $Q_{-1/2}, Q_0, Q_{\lambda, Y}$ using the standard paraproduct estimations recalled in Appendix A.2

$$\|Q_{-1/2}\|_{\mathcal{M}_{3/2+\kappa/2,1+\kappa}} \leq (1 + \|\lambda\|) (1 + \|Y\|_{\mathcal{X}})^3 (1 + \|v_0,0\|_{\mathcal{C}^{-1/2-\kappa}} + \|v_0^2\|_{\mathcal{X}})^3.$$

In order to control the estimation of $\|v_0^2\|_{\mathcal{X}}$ we have to control $\|IR_\epsilon(v)\|_{\mathcal{X}}$. This is achieved easily by using the results of Section 4.1. Thanks to Lemma 1.1, $\forall \delta \in (0,1) \forall \theta > 0$ such that $\frac{\theta}{\epsilon^2} > \frac{1}{\delta} + \frac{2}{\delta}$ (note that it is possible to choose $\theta > 2\kappa$) that satisfies this property as long as $\kappa$ and $\delta$ are small enough) we have:

$$\|R_\epsilon(v)\|_{\mathcal{M}_{3/2+\kappa/2,1+\kappa}} \leq M_{\delta}(Y, u_0, \epsilon) \|v_0\|_{\mathcal{X}}^{3+\delta} \epsilon^{1/2} \|v_0^2\|_{\mathcal{X}}.$$

By Lemma A.3 together with A.2 we obtain then

$$\|IR_\epsilon(v)\|_{\mathcal{X}} \leq M_{\delta}(Y, u_0, \epsilon) \|v_0\|_{\mathcal{X}}^{3+\delta} \epsilon^{1/2} \|v_0^2\|_{\mathcal{X}}.$$

Using that

$$\|P_\epsilon(v_0)\|_{\mathcal{X}} \leq \|v_0^2\|_{\mathcal{X}} \|v_0,0\|_{\mathcal{C}^{-1/2-\kappa}} \leq (1 + \|v_0,0\|_{\mathcal{C}^{-1/2-\kappa}}) \|Y\|_{\mathcal{X}}.$$
we obtain that \( \exists C' > 0 \) such that
\[
\|v_{\varepsilon_n}^p\|_{V_T^p} \leq C'(1 + |\lambda_{\varepsilon_n}|)(1 + \|Y_{\varepsilon_n}\|_{X_T})^3 (1 + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})^3 + C'T_{\varepsilon,\delta}^{5/2}(1 + |\lambda_{\varepsilon_n}|)(1 + \|Y_{\varepsilon_n}\|_{X_T})^3 \|v_{\varepsilon}^p\|_{V_T^p}^3 + C'M_{\varepsilon,\delta}(Y_{\varepsilon_n}, u_{\varepsilon,0}) e^{c\varepsilon^{1/2}/(\|v_{\varepsilon}\|_{X_T} + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})} e^{c\|v_{\varepsilon}\|_{V_T^p}} \|v_{\varepsilon}^p\|_{V_T^p}^{3+\delta} \\
+ C'M_{\varepsilon,\delta}(Y_{\varepsilon_n}, u_{\varepsilon,0}) e^{c\varepsilon^{1/2}/(\|v_{\varepsilon}\|_{X_T} + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})} e^{c\|v_{\varepsilon}\|_{V_T^p}} \|v_{\varepsilon}^p\|_{V_T^p}^{3+\delta} \\
\leq D + C M_{\varepsilon,\delta}(Y_{\varepsilon_n}, u_{\varepsilon,0}) e^{c\varepsilon^{1/2}/(\|v_{\varepsilon}\|_{X_T} + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})} e^{c\|v_{\varepsilon}\|_{V_T^p}} + C M_{\varepsilon,\delta}(Y_{\varepsilon_n}, u_{\varepsilon,0}) e^{c\varepsilon^{1/2}/(\|v_{\varepsilon}\|_{X_T} + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})} e^{c\|v_{\varepsilon}\|_{V_T^p}} \|v_{\varepsilon}^p\|_{V_T^p}^{3+\delta}
\]
with
\[
C := C'(1 + |\lambda_{\varepsilon_n}|)(1 + \|Y_{\varepsilon_n}\|_{X_T})^3 e^{c\varepsilon^{1/2}/(\|v_{\varepsilon}\|_{X_T} + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})} (1 + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})^3 + \epsilon.
\]
and
\[
D := C'(1 + |\lambda_{\varepsilon_n}|)(1 + \|Y_{\varepsilon_n}\|_{X_T})^3 (1 + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})^3.
\]
Let \( T_* \in (0, T] \) such that:
\[
C T_*^{5/2}(5C)^2 + e^{c\varepsilon^{1/2}/(5C)} (5C)^2 + D \leq \frac{1}{2}.
\]
Assume that \( M_{\varepsilon,\delta} \leq T_*^{5/2} \). Define a closed interval \([0, S] = \{ t \in [0, T_*] : \|v_{\varepsilon_n}^p\|_{V_T^p} \leq 4D \} \subseteq [0, T_*] \). This interval is well defined and non-empty since \( t \rightarrow \|v_{\varepsilon_n}^p\|_{V_T^p} \) is continuous and nondecreasing and \( \|v_{\varepsilon_n}^p\|_{V_T^p} \leq 4D \). Let us assume that \( S < T_* \), then we can take \( \epsilon > 0 \) small enough such that \( S + \epsilon < T_* \) and by continuity \( \|v_{\varepsilon_n}^p\|_{V_{S^*}^p} \leq 5C \), then
\[
\|v_{\varepsilon_n}^p\|_{V_{S^*}^p} \leq D + C M_{\varepsilon,\delta}(Y_{\varepsilon_n}, u_{\varepsilon,0}) e^{c\varepsilon^{1/2}/(\|v_{\varepsilon}\|_{X_T} + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})} e^{c\|v_{\varepsilon}\|_{V_T^p}} + C M_{\varepsilon,\delta}(Y_{\varepsilon_n}, u_{\varepsilon,0}) e^{c\varepsilon^{1/2}/(\|v_{\varepsilon}\|_{X_T} + \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}})} e^{c\|v_{\varepsilon}\|_{V_T^p}} \|v_{\varepsilon}^p\|_{V_T^p}^{3+\delta} \\
\leq D + C M_{\varepsilon,\delta}(Y_{\varepsilon_n}, u_{\varepsilon,0}) e^{c\varepsilon^{1/2}/(5C)} + C T_*^{5/2}(5C)^2 \|v_{\varepsilon}^p\|_{V_{S^*}^p} + C T_*^{5/2}(5C)^2 e^{c\varepsilon^{1/2}/(5C)} (5C)^2 + D \leq \frac{1}{2} \|v_{\varepsilon}^p\|_{V_{S^*}^p} \]
which gives \( \|v_{\varepsilon_n}^p\|_{V_{S^*}^p} \leq 4D \). This implies \( S = T_* \) (by contradiction). From the construction of \( T_* \) it is easy to see that \( T_* (\|Y_{\varepsilon_n}\|_{X_T}, \|v_{\varepsilon,0}\|_{\mathcal{E}^{-1/2-\varepsilon}}, |\lambda_{\varepsilon_n}|) \) is lower semicontinuous. \( \square \)

4.3 Convergence of the remainder

It suffices to put together the results obtained in Sections 4.1 and 4.2 to obtain the convergence of \( R_\varepsilon(v_{\varepsilon}) \):

Lemma 4.5
The remainder \( R_\varepsilon(v_{\varepsilon}) \) that appears in equation (2.9) converges in probability to 0 as \( \varepsilon \to 0 \) in the space \( \mathcal{M}_{\varepsilon_{T_*}}^{p}L^p(\mathbb{T}^d) \).

Proof From the estimation on \( R_\varepsilon(v_{\varepsilon}) \) of Lemma 4.1 together with the fact that \( M_{\varepsilon,\delta} \to 0 \) in probability (Lemma 4.2) and the bounds on \( \|v_{\varepsilon}\|_{\mathcal{M}_{\varepsilon_{T_*}}^{p}L^\infty} \) and \( \|v_{\varepsilon}\|_{C_T L^\infty} \) of Lemma 4.3 we see immediately that
\[
\|R_\varepsilon(v_{\varepsilon})\|_{\mathcal{M}_{\varepsilon_{T_*}}^{p}L^p} \to 0 \quad \text{in probability}.
\]
\( \square \)
## A Paracontral controlled analysis and kernel estimations

In this section we first recall the basic results of paracontral calculus first introduced in [6], without proofs. For more details on Besov spaces, Littlewood-Paley theory, and Bony’s paraproduct the reader can refer to the monograph [2]. We then proceed to give some results on the convolution of functions with known singularity and the estimation of finite-chaos Gaussian trees. We refer to Section 10 of [8] and to the nice pedagogic exposition [13] for further details.

### A.1 Notation

Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$, independent of the variables under consideration, such that $a \leq c \cdot b$. If we want to emphasize the dependence of $c$ on the variable $x$, then we write $a(x) \lesssim_x b(x)$. If $f$ is a map from $A \subset \mathbb{R}$ to the linear space $Y$, then we write $f_{x,t} = f(t) - f(s)$. For $f \in L^p(\mathbb{T}^d)$ we write $\|f(x)\|_{L^p(\mathbb{T}^d)}^p := \int_{\mathbb{T}^d} |f(x)|^p dx$.

Given a Banach space $X$ with norm $\|\cdot\|_X$ and $T > 0$, we note $C_T X = C([0,T],X)$ for the space of continuous functions from $[0,T]$ to $X$, equipped with the supremum norm $\|\cdot\|_{C_T X}$, and we set $C X = C(\mathbb{R}_+,X)$. For $\alpha \in (0,1)$ we also define $C^\alpha_T X$ as the space of $\alpha$-Hölder continuous functions from $[0,T]$ to $X$, endowed with the seminorm $\|f\|_{C^\alpha_T X} = \sup_{0 \leq s < t \leq T} \|f(t) - f(s)\|_X/|t - s|^{\alpha}$, and we write $C^\alpha_{loc} X$ for the space of locally $\alpha$-Hölder continuous functions from $\mathbb{R}_+$ to $X$. For $\gamma > 0$, $p \in [1, \infty)$, we define

$$
\mathcal{M}^\gamma_T X = \{v : L^p([0,T],X) : \|v\|_{\mathcal{M}^\gamma_T X} = \|t \mapsto t^{\gamma} v(t)\|_{L^p([0,T],X)} < \infty\},
$$

$$
\mathcal{M}^\gamma_T X = \{v : C((0,T],X) : \|v\|_{\mathcal{M}^\gamma_T X} = \|t \mapsto t^{\gamma} v(t)\|_{C_T X} < \infty\}. \tag{A.1}
$$

The space of distributions on the torus is denoted by $\mathcal{D}'(\mathbb{T}^3)$ or $\mathcal{D}'$. The Fourier transform is defined with the normalization

$$
\mathcal{F} u(k) = \hat{u}(k) = \int_{\mathbb{T}^d} e^{-i(k,x)} u(x) dx, \quad k \in \mathbb{Z}^3,
$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1} v(x) = (2\pi)^{-1} \sum_k e^{i(k,x)} v(k)$. Let $(\chi,\rho)$ denote a dyadic partition of unity such that $\text{supp}(\rho(2^{-i} \cdot)) \cap \text{supp}(\rho(2^{-j} \cdot)) = \emptyset$ for $|i - j| > 1$. The family of operators $(\Delta_j)_{j \geq -1}$ will denote the Littlewood-Paley projections associated to this partition of unity, that is $\Delta_{-1} u = \mathcal{F}^{-1} (\chi \mathcal{F} u)$ and $\Delta_j = \mathcal{F}^{-1} (\rho(2^{-j} \cdot) \mathcal{F} u)$ for $j \geq 0$. Let $S_j = \sum_{i < j} \Delta_i$, and $K_q$ be the kernel of $\Delta_q$ so that

$$
\Delta_q f(\bar{x}) = \int_{\mathbb{T}^d} K_{\bar{x},q}(x)f(x) dx.
$$

For the precise definition and properties of the Littlewood-Paley decomposition $f = \sum_{q \geq -1} \Delta_q f$ in $\mathcal{D}'(\mathbb{T}^3)$, see Chapter 2 of [2]. The Hölder-Besov space $B^\alpha_{p,q}(\mathbb{T}^3, \mathbb{R})$ for $\alpha \in \mathbb{R}$, $p, q \in [1, \infty]$ with $B^\alpha_{p,q}(\mathbb{T}^3, \mathbb{R}) := \mathcal{C}^\alpha$ is and equipped with the norm

$$
\|f\|_{B^\alpha_{p,q}} = \|2^{q\alpha} \|\Delta_i f\|_{L^p(\mathbb{T}^3)}\|_{L^q}.
$$

If $f$ is in $\mathcal{C}^{\alpha-\varepsilon}$ for all $\varepsilon > 0$, then we write $f \in \mathcal{C}^{\alpha-}$. For $\alpha \in (0,2)$, we define the space $\mathcal{L}^\alpha_T = C^{\alpha/2_T} L^\infty \cap C_T \mathcal{C}^{\alpha}$, equipped with the norm

$$
\|f\|_{\mathcal{L}^\alpha_T} = \max \left\{\|f\|_{C^{\alpha/2_T} L^\infty}, \|f\|_{C_T \mathcal{C}^{\alpha}}\right\}.
$$

The notation is chosen to be reminiscent of

$$
\mathcal{L} := \partial_{t,-}.
$$
by which we will always denote the heat operator with periodic boundary conditions on $\mathbb{T}^d$. We also write $\mathcal{L}^\alpha = C^\alpha_{\text{loc}} L^{\infty} \cap C^\alpha$. When working with irregular initial conditions, we will need to consider explosive spaces of parabolic type. For $\gamma > 0$, $\alpha \in (0, 1)$, and $T > 0$ we define the norm

$$\| f \|_{\mathcal{L}^{\gamma,\alpha}_T} = \max \left\{ \| t \mapsto t^\gamma f(t) \|_{C^\alpha_{\text{loc}} L^{\infty}}, \| f \|_{M^{\gamma,\alpha}_T} \right\},$$

and the space $\mathcal{L}^{\gamma,\alpha}_T = \left\{ f : [0, T] \rightarrow \mathbb{R} : \| f \|_{\mathcal{L}^{\gamma,\alpha}_T} < \infty \right\}$. In particular, we have $\mathcal{L}^{0,\alpha}_T = \mathcal{L}^\alpha_T$. We introduce the linear operator $I : C(\mathbb{R}_+, \mathcal{D}'(\mathbb{T})) \rightarrow C(\mathbb{R}_+, \mathcal{D}'(\mathbb{T}))$ given by

$$I(f(t) = \int_0^t P_{t-s} f(s) ds,$$

where $\{P_t\}_{t \geq 0}$ is the heat semigroup with kernel $P_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} 1_{t \geq 0}$.

Paraproducts are bilinear operations introduced by Bony [3] in order to linearize a class of non-linear PDE problems. They appear naturally in the analysis of the product of two Besov distributions. In terms of Littlewood–Paley blocks, the product $fg$ of two distributions $f$ and $g$ can be formally decomposed as

$$fg = f \prec g + f \succ g + f \circ g,$$

where

$$f \prec g = g \succ f := \sum_{j \geq -1} \sum_{i=-1}^{i-2} \Delta_i f \Delta_j g \quad \text{and} \quad f \circ g := \sum_{|i-j| \leq 1} \Delta_i f \Delta_j g.$$

This decomposition behaves nicely with respect to Littlewood–Paley theory. We call $f \prec g$ and $f \succ g$ paraproducts, and $f \circ g$ the resonant term. We use the notation $f \prec g = f \prec g + f \circ g$. The basic result about these bilinear operations is given by the following estimates, essentially due to Bony [3] and Meyer [14].

When dealing with paraproducts in the context of parabolic equations it would be natural to introduce parabolic Besov spaces and related paraproducts. But to keep a simpler setting, we choose to work with space–time distributions belonging to the scale of spaces $(C_T C^\alpha_{\gamma,\alpha})_{\alpha \in \mathbb{R}}$ for some $T > 0$. To do so efficiently, we will use a modified paraproduct which introduces some smoothing in the time variable that is tuned to the parabolic scaling. Let therefore $\varphi \in C^{\infty}(\mathbb{R}, \mathbb{R}_+)$ be nonnegative with compact support contained in $\mathbb{R}_+$ and with total mass 1, and define for all $i \geq -1$ the operator

$$Q_i : C^\beta \rightarrow C^\beta, \quad Q_i f(t) = \int_0^\infty 2^{-2i} \varphi(2^{2i}(t-s)) f(s) ds.$$

We will often apply $Q_i$ and other operators on $C^\beta$ to functions $f \in C_T C^\beta$ which we then simply extend from $[0, T]$ to $\mathbb{R}_+$ by considering $f(\cdot \wedge T)$. With the help of $Q_i$, we define a modified paraproduct

$$f \prec g := \sum_i (Q_i S_{i-1} f) \Delta_i g$$

for $f, g \in C(\mathbb{R}_+, \mathcal{D}'(\mathbb{T}))$. We define the commutators $\text{com}_1, \text{com}_2, \text{com}_3$ in Lemma [A.7] We write $\hat{P}(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}} e^{-t} 1_{t \geq 0}$ for a modified heat kernel which has the same bounds of the usual heat kernel $P(t, x)$. Let $Y_\varepsilon$ as in [1.3] and recall that $C_\varepsilon$ is the covariance of $Y_\varepsilon$, i.e. $C_\varepsilon(t, x) = \mathbb{E}(Y_\varepsilon(t, x) Y_\varepsilon(0, 0))$. We will sometimes write $Y_{\varepsilon, \zeta} := Y_\varepsilon(t, x)$ for $\zeta = (t, x) \in \mathbb{R} \times \mathbb{T}^3$. Let $\sigma^2_\varepsilon = \varepsilon \mathbb{E}[(Y_\varepsilon(0, 0))^2] = \varepsilon C_\varepsilon(0, 0)$. 37
A.2 Basic paracontrolled calculus results

First, let us recall some interpolation results on the parabolic time-weighted spaces $L_{T}^\gamma,\alpha$

**Lemma A.1**

For all $\alpha \in (0, 2)$, $\gamma \in [0, 1)$, $\varepsilon \in [0, \alpha \wedge 2\gamma)$, $T > 0$ and $f \in L_{T}^\gamma,\alpha$ with $f(0) = 0$ we have

$$\|f\|_{L_{T}^{\gamma-\varepsilon/2,\alpha-\varepsilon}} \lesssim \|f\|_{L_{T}^\gamma,\alpha}. \quad (A.2)$$

Let $\alpha \in (0, 2)$, $\gamma \in (0, 1)$, $T > 0$, and let $f \in L_{T}^\alpha$. Then for all $\delta \in (0, \alpha]$ we have

$$\|f\|_{L_{T}^\delta} \lesssim \|f(0)\|_{C_{T}^{\delta}} + T^{(\alpha-\delta)/2}\|f\|_{L_{T}^\gamma,\alpha}. \quad (A.3)$$

**Schauder estimates**

**Lemma A.2**

Let $\alpha \in (0, 2)$ and $\gamma \in [0, 1)$. Then

$$\|If\|_{L_{T}^{\gamma,\alpha}} \lesssim \|f\|_{M_{T}^{\gamma,\alpha}}. \quad (A.4)$$

for all $t > 0$. If further $\beta \geq -\alpha$, then

$$\|s \mapsto P_{s}u_{0}\|_{L_{T}^{\beta+\alpha/2,\alpha}} \lesssim \|u_{0}\|_{C_{T}^{\beta}}. \quad (A.5)$$

For all $\alpha \in \mathbb{R}$, $\gamma \in [0, 1)$, and $t > 0$ we have

$$\|If\|_{M_{T}^{\gamma,\alpha/2}} \lesssim \|f\|_{M_{T}^{\gamma,\alpha}}. \quad (A.6)$$

Proofs can be found e.g. in [7]. We need also some well known bounds for the solutions of the heat equation with sources in space–time Lebesgue spaces.

**Lemma A.3**

Let $\beta \in \mathbb{R}$ and $f \in L_{T}^{p}\mathcal{B}_{p,\infty}^{\beta}$, then for every $\kappa \in [0, 1]$ we have $If \in C_{T}^{\kappa/q,\beta+2(1-\kappa)-(2-2\kappa+d)/p}$ with

$$\|If\|_{C_{T}^{\kappa/q,\beta+2(1-\kappa)-(2-2\kappa+d)/p}} \lesssim T\|f\|_{L_{T}^{p}\mathcal{B}_{p,\infty}^{\beta}},$$

with $\frac{1}{q} + \frac{1}{p} = 1$. Moreover, for every $\gamma < \gamma' < 1 - 1/p$ and every $0 < \alpha < (2-5/p + \beta) \wedge 2$ we have

$$\|If\|_{L_{T}^{\gamma,\alpha}} \lesssim T\|f\|_{M_{T}^{\gamma,\alpha}\mathcal{B}_{p,\infty}^{\beta}}.$$ 

Proof We only show the second inequality as the first one is easier and obtained with similar techniques. Let $u = If$, we have

$$t^{\gamma}\|\Delta_{i}u(t)\|_{L_{\infty}} \leq t^{1/q}2^{d/2} \left[ \int_{0}^{1} s^{-\gamma q}e^{-cs2^{d/2}}(1-s)^{d}\mathrm{d}s \right]^{1/q} \left[ \int_{0}^{t} s^{\gamma p}\|\Delta_{i}f(s)\|_{L_{p}}^{p}\mathrm{d}s \right]^{1/p} \lesssim \gamma,q \cdot 2^{d/2} \left[ \int_{0}^{t} s^{\gamma p}\|\Delta_{i}f(s)\|_{L_{p}}^{p}\mathrm{d}s \right]^{1/p}.$$
which allows us to bound $\|If\|_{L^p}$. In order to estimate $\|t \rightarrow t^{r}If\|_{C^{1/2r}_{1/2}L^\infty}$ we write

$$\|t^{r} \Delta u(t) - s^{r} \Delta u(s)\|_{L^\infty} \lesssim \int_s^t \|\Delta u(v)\|_{L^\infty} dv + |t - s|^{2(d+2)/p} \|\Delta f\|_{M^{r-2}_{2}L^p(\mathbb{T}^3)}$$

$$+ \left\| \int_s^t v^{r-1} \Delta f(v) dv \right\|_{L^\infty}$$

We can estimate the first term as

$$\int_s^t \|\Delta u(v)\|_{L^\infty} dv \lesssim 2^{d+2/p} \|\Delta f\|_{M^{r-2}_{2}L^p(\mathbb{T}^3)} \int_s^t v^{r-1} dv.$$

For the third term we have

$$\left\| \int_s^t v^{r-1} \Delta f(v) dv \right\|_{L^\infty} \lesssim \left[ \int_s^t dv \right]^{1/q} \left[ \int_s^t v^{2(p-2)/p} \|\Delta f\|_{L^\infty} dv \right]^{1/p}$$

$$\lesssim 2^{d/p}|t - s|^{1/q} \|\Delta f\|_{M^{r-2}_{2}L^p(\mathbb{T}^3)}$$

We obtain then if $2^{2^i}|t - s| \leq 1$

$$\|t^{r} \Delta u(t) - s^{r} \Delta u(s)\|_{L^\infty} \lesssim 2^{d/p}|t - s|^{1/q} \|\Delta f\|_{M^{r-2}_{2}L^p(\mathbb{T}^3)}$$

and if $2^{2^i}|t - s| > 1$ we just use the trivial estimate

$$\|t^{r} \Delta u(t) - s^{r} \Delta u(s)\|_{L^\infty} \lesssim 2^{d/p}2^{-2i/q} \|\Delta f\|_{M^{r-2}_{2}L^p(\mathbb{T}^3)} \lesssim 2^{d/p}|t - s|^{1/q} \|\Delta f\|_{M^{r-2}_{2}L^p(\mathbb{T}^3)}.$$

Therefore, for every $\kappa \in [0, 1]$

$$\|t^{r} \Delta u(t) - s^{r} \Delta u(s)\|_{L^\infty} \lesssim 2^{(d/2 - 2)2^{2i/q}|t - s|^{\kappa/q}} \|\Delta f\|_{M^{r-2}_{2}L^p(\mathbb{T}^3)}.$$ 

Choosing $\kappa/q = \alpha/2$ we obtain the desired estimate.

\[\square\]

**Estimates on Bony’s paraproducts and commutators**

**Lemma A.4**

For any $\beta \in \mathbb{R}$ we have

$$\|f \prec g\|_{\mathcal{C}^0} \lesssim_{\beta} \|f\|_{L^\infty} \|g\|_{\mathcal{C}^\beta}, \quad (A.7)$$

and for $\alpha < 0$ furthermore

$$\|f \prec g\|_{\mathcal{C}^{\alpha+\beta}} \lesssim_{\alpha, \beta} \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}. \quad (A.8)$$

For $\alpha + \beta > 0$ we have

$$\|f \circ g\|_{\mathcal{C}^{\alpha+\beta}} \lesssim_{\alpha, \beta} \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\beta}. \quad (A.9)$$

A natural corollary is that the product $fg$ of two elements $f \in \mathcal{C}^\alpha$ and $g \in \mathcal{C}^\beta$ is well defined as soon as $\alpha + \beta > 0$, and that it belongs to $\mathcal{C}^{\gamma}$, where $\gamma = \min\{\alpha, \beta, \alpha + \beta\}$. We will also need a commutator estimation:

**Lemma A.5**

Let $\alpha > 0$, $\beta \in \mathbb{R}$, and let $f, g, h \in \mathcal{C}^\alpha$, and $h \in \mathcal{C}^\beta$. Then

$$\|f \prec (g \circ h) - (fg) \circ h\|_{\mathcal{C}^{\alpha+\beta}} \lesssim \|f\|_{\mathcal{C}^\alpha} \|g\|_{\mathcal{C}^\alpha} \|h\|_{\mathcal{C}^\beta}.$$
We collect in the following lemma various estimates for the modified paraproduct \( f \prec \prec g \), proofs are again in [7].

**Lemma A.6**

a) For any \( \beta \in \mathbb{R} \) and \( \gamma \in [0, 1) \) we have

\[
t^\gamma \| f \prec g(t) \|_{\mathcal{C}^\beta} \lesssim \| f \|_{\mathcal{M}_\gamma^1 L^\infty} \| g(t) \|_{\mathcal{C}^\beta}, \tag{A.10}
\]

for all \( t > 0 \), and for \( \alpha < 0 \) furthermore

\[
t^\gamma \| f \prec g(t) \|_{\mathcal{C}^{\alpha+\beta}} \lesssim \| f \|_{\mathcal{M}_\gamma^1 \mathcal{C}^{\alpha}} \| g(t) \|_{\mathcal{C}^\beta}. \tag{A.11}
\]

b) Let \( \alpha, \delta \in (0, 2), \gamma \in [0, 1), T > 0, \) and let \( f \in \mathcal{L}^{\gamma,\delta}_T, g \in \mathcal{C}_T^{\alpha}, \) and \( \mathcal{L} g \in \mathcal{C}_T^{\alpha-2}. \) Then

\[
\| f \prec g \|_{\mathcal{L}^{\gamma,\alpha}_T} \lesssim \| f \|_{\mathcal{L}^{\gamma,\alpha}_T} (\| g \|_{\mathcal{C}_T^{\alpha}} + \| \mathcal{L} g \|_{\mathcal{C}_T^{\alpha-2}}).	ag{A.12}
\]

We introduce various commutators which allow to control non-linear functions of paraproducts and also the interaction of the paraproducts with the heat kernel.

**Lemma A.7**

a) For \( \alpha, \beta, \gamma \in \mathbb{R} \) such that \( \alpha + \beta + \gamma > 0 \) and \( \alpha \in (0, 1) \) there exists bounded trilinear maps

\[
\text{com}_1, \overline{\text{com}}_1 : \mathcal{C}^\alpha \times \mathcal{C}^{\beta} \times \mathcal{C}^{\gamma} \to \mathcal{C}^{\alpha+\beta+\gamma},
\]

such that for smooth \( f, g, h \) they satisfy

\[
\text{com}_1(f, g, h) = (f \prec g) \circ h - f(g \circ h). \tag{A.13}
\]

\[
\overline{\text{com}}_1(f, g, h) = (f \prec g) \circ h - f(g \circ h). \tag{A.14}
\]

b) Let \( \alpha \in (0, 2), \beta \in \mathbb{R}, \) and \( \gamma \in [0, 1) \). Then the bilinear maps

\[
\text{com}_2(f, g) := f \prec g - f \prec g, \tag{A.15}
\]

\[
\text{com}_3(f, g) := [\mathcal{L}, f \prec ] g := \mathcal{L} (f \prec g) - f \prec \mathcal{L} g. \tag{A.16}
\]

have the bounds

\[
t^\gamma \| \text{com}_2(f, g)(t) \|_{\alpha+\beta} \lesssim \| f \|_{\mathcal{L}^{\gamma,\alpha}_T} \| g(t) \|_{\mathcal{C}^\beta}, \quad t > 0. \tag{A.17}
\]

as well as

\[
t^\gamma \| \text{com}_3(f, g)(t) \|_{\alpha+\beta-2} \lesssim \| f \|_{\mathcal{L}^{\gamma,\alpha}_T} \| g(t) \|_{\mathcal{C}^\beta}, \quad t > 0. \tag{A.18}
\]

Proofs can be found in [7].
A.3 Estimation of finite-chaos diagrams

In this section we recall a few well-known results on the estimation of finite chaos diagrams. For additional details see Chapter 10 of \cite{3} and \cite{18}. First of all we need to characterize the local behaviour of the heat kernel \( P_t(x) \) and of the covariance \( C_\varepsilon(t, x) \) of the Gaussian field \( Y_\varepsilon \).

Remark A.8 We sometimes use a slightly different version of the heat kernel, namely

\[
\tilde{P}_t(x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} e^{-t} \mathbb{1}_{t>0}
\]

in order to have that \( X_t(x, y) = \int_{-\infty}^t \frac{1}{2\pi} P_{t-s}(x-y)v(s, y)dsdy \) is the stationary solution to \( \Delta X = -X + v \). However, the bounds on the covariance \( C_\varepsilon \) remain trivially valid in this setting.

Lemma A.9

The heat kernel \( P(\zeta) := P(t, x) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{|x|^2}{4t}} \mathbb{1}_{t>0} \) has the bound

\[
|P(\zeta)| \lesssim (|t|^{1/2} + |x|)^{-3}.
\]

Let \( k \in \mathbb{N}^4 \) a multi-index with \( |k| = 2k_1 + 2k_2 + \cdots + 2k_4 \). Then for every multi-index \( |k| \leq 2 \) we have:

\[
|D^k P_t(x)| \lesssim (|t|^{1/2} + |x|)^{-3-|k|}.
\]

Proof

\[
|P_t(x)||t|^{1/2} + |x| \gtrsim 1 + (|x||t|^{-1/2})^3 e^{-\frac{|x|^2}{4t}} = (1 + |\alpha|^3) e^{-\frac{|\alpha|^2}{4t}} < +\infty
\]

In the same way we prove that \( |\partial_\alpha P_t(x)| \lesssim (|t|^{1/2} + |x|)^5, |\partial_{\alpha, \beta} P_t(x)| \lesssim (|t|^{1/2} + |x|)^4 \) and \( |\partial_\alpha, \partial_{\beta} P_t(x)| \lesssim (|t|^{1/2} + |x|)^5 \). \hfill \Box

We recall a special case of Lemma 10.14 of \cite{3}, which is enough for our purpose. We use the notation \( \|\zeta\| := (|t|^{1/2} + |x|) \) for \( \zeta = (t, x) \in \mathbb{R} \times \mathbb{T}^3 \).

Lemma A.10

Let \( f, g : \mathbb{R} \times \mathbb{T}^3 \setminus \{0\} \to \mathbb{R} \) smooth, integrable at infinity and such that \( |f(\zeta)| \lesssim \|\zeta\|^\alpha \) and \( |g(\zeta)| \lesssim \|\zeta\|^\beta \) in a ball \( B = \{ \zeta \in \mathbb{R} \times \mathbb{T}^3 : \|\zeta\| < 1, \zeta \neq 0 \} \). Then if \( \alpha, \beta \in (-5, 0) \) and \( \alpha + \beta + 5 < 0 \) we have

\[
|f \ast g(\zeta)| \lesssim \|\zeta\|^{\alpha + \beta + 5}
\]

in a ball centered in the origin.

Moreover, if \( \alpha, \beta \in (-5, 0) \) and \( 0 < \alpha + \beta + 5 < 1 \) and for every multi-index \( |k| \leq 2 \) we have \( |D^k f(\zeta)| \lesssim \|\zeta\|^{\alpha - |k|} \) and \( |D^k g(\zeta)| \lesssim \|\zeta\|^{\beta - |k|} \), then

\[
|f \ast g(\zeta) - f \ast g(0)| \lesssim \|\zeta\|^{\alpha + \beta + 5}
\]

in a ball centered in the origin.

Remark A.11 The covariance \( C_\varepsilon \) of \( Y_\varepsilon \) can be written as \( C_\varepsilon = \tilde{P} \ast \tilde{C}_\varepsilon \ast \tilde{P} \) with \( \tilde{C}_\varepsilon(t, x) := \mathbb{E}(\eta_\varepsilon(t, x)\eta_\varepsilon(0, 0)) \). Recall from the introduction that \( \tilde{C}_\varepsilon(t, x) = e^{-5C_\varepsilon(\varepsilon^{-2}t, \varepsilon^{-1}x)} \) where \( C_\varepsilon \) is the covariance of the Gaussian process \( \eta_\varepsilon \) defined on \( \mathbb{R} \times (\mathbb{T}/\varepsilon)^3 \), and \( C_\varepsilon(t-s, x-y) = \Sigma(t-s, x-y) \) if \( \text{dist}(x, y) \leq 1 \) and 0 otherwise (so that the family of functions \( C_\varepsilon \) is bounded uniformly on \( \varepsilon \) by a \( C^2 \) function). Then there exists a family of functions \( C_\varepsilon \) defined on \( \mathbb{R} \times (\mathbb{T}/\varepsilon)^3 \) such that \( C_\varepsilon(t, x) = \varepsilon^{-1}C_\varepsilon(\varepsilon^{-2}t, \varepsilon^{-1}x) \) and \( C_\varepsilon(t, x) = [\tilde{P} \ast \tilde{C}_\varepsilon \ast \tilde{P}](t, x) \).
Lemma A.12
The covariance \( C_\varepsilon \) has the bound, for every multi-index \(|k| \leq 2\):

\[
|D^k C_\varepsilon(t,x)| \lesssim (|t|^{1/2} + |x|)^{-1-|k|}.
\]

Moreover, we have

\[
\varepsilon^{|k|+1} |D^k C_\varepsilon(t,x)| \lesssim 1
\]

Proof  The first bound is obtained directly from Lemma A.9 and Lemma A.10. Indeed, since by hypothesis \( \tilde{C}_\varepsilon \) has compact support, it is easy to see that \( |\tilde{C}_\varepsilon(t,x)| \lesssim (|t|^{1/2} + |x|)^{-5} \). The second bound is obtained by a simple change of variables in the convolution defining \( C_\varepsilon \).

Lemma A.13
We have \( \int_{s,x} P_s(x)[C_\varepsilon(s,x)]^2 \lesssim |\log \varepsilon| \) and for every \( n \geq 3 \) \( \varepsilon^{n-2} \int_{s,x} P_s(x)[C_\varepsilon(s,x)]^n \lesssim 1 \).

Proof  From the fact that \( P_{\varepsilon^{2n}}(x) = \varepsilon^{-3} P_s(x) \) together with Remark A.11 we obtain

\[
\int_{\mathbb{R} \times \mathbb{T}^3} P_s(x)[C_\varepsilon(s,x)]^2 dsdx \lesssim \int_{B(0, \varepsilon^{-1})} P_s(x)[C_\varepsilon^2(s,x)]^2 dsdx \lesssim |\log \varepsilon|
\]

with \( B(0, R) = \{ \zeta \in \mathbb{R}^3 : ||\zeta|| < R, \zeta \neq 0 \} \) a “parabolic” ball centered in the origin. The second estimation is obtained in the same way.

Lemma A.14
For \( m \in (0, 3) \), \( n \in (3, 5) \), define for \( \zeta, \zeta' \in \mathbb{R} \times \mathbb{T}^3 \)

\[
I_m := \int |C_\varepsilon(\zeta - \zeta')|^m |\mu_{q,\zeta} \mu_{q,\zeta'}|, \quad \tilde{I}_n := \int |C_\varepsilon(\zeta - \zeta')|^n |\tilde{\mu}_{q,\zeta} \tilde{\mu}_{q,\zeta'}|
\]

with \( \mu_{q,\zeta} := K_{q,x}(s-t)\delta(t-s) d\zeta \) and \( \tilde{\mu}_{q,\zeta} := \left[ \int x K_{q,x}(x) P_{t-s}(x-y) \right] d\zeta \) for \( \zeta = (s, y) \). Then

\[
I_m \lesssim 2^{mq} \quad \text{and} \quad \tilde{I}_n \lesssim 2^{(n-1)q}.
\]

Proof  The estimation of \( I_m \) is easily obtained by Lemma A.12 and a change of variables. For \( \tilde{I}_n \) observe that for every \( q > 0 \)

\[
\tilde{\mu}_{q,\zeta} = \left[ \int x K_{q,x}(x) (P_{t-s}(x-y) - P_{t-s}(\bar{x}-y)) \right] d\zeta
\]

and then we can apply Lemma A.10 to obtain the result.

Lemma A.15
We have for every \( \sigma \in [0, 1] \)

\[
\sup_{x \in \mathbb{T}^3} |C_\varepsilon(t,x) - C_\varepsilon(0,x)| \lesssim \varepsilon^{-1-2\sigma} |t|^\sigma
\]

Proof  It is easy to obtain by interpolation knowing that \( |\partial_t C_\varepsilon(t,x)| \lesssim \varepsilon^{-3} \) from Lemma A.12.

Lemma A.16
We have for every \( \alpha < 3 \)

\[
\sum_{i,j} \left| \int K_i(x-y) P_i(y) dy \right| \int \frac{|K_j(x-y)|}{(|y| + t^{1/2})^{\alpha}} dy \lesssim \frac{1}{(|x| + t^{1/2})^{3+\alpha}}
\]
Proof. We will show that

\[ \left| \int K_i(x - y)P_t(y)dy \right| \lesssim 2^{-i}(|x| + t^{1/2} + 2^{-i})^{-4}, \]

(A.19)

and that

\[ \left| \int \frac{|K_i(x - y)|}{(|y| + t^{1/2})^\alpha} dy \right| \lesssim (|x| + t^{1/2} + 2^{-i})^{-\alpha}, \]

(A.20)

from which we deduce that

\[ \sum_{i \geq 1} \left| \int K_i(x - y)P_t(y)dy \int \frac{|K_i(x - y)|}{(|y| + t^{1/2})^\alpha} dy \right| \lesssim \sum_{i \geq 1} (|x| + t^{1/2} + 2^{-i})^{4+\alpha}. \]

Bounding the sum over \( i \) with an integral, we conclude

\[ \int_0^1 d\lambda \frac{\lambda}{(|x| + t^{1/2} + \lambda)^{4+\alpha}} = \frac{1}{(|x| + t^{1/2})^{3+\alpha}} \int_0^{1/(|x| + t^{1/2})} \frac{d\lambda}{(1 + \lambda)^{4+\alpha}} \lesssim \frac{1}{(|x| + t^{1/2})^{3+\alpha}}. \]

Let us show (A.19). We want to estimate

\[ I = \int K_i(x - y)P_t(y)dy = \int K_i(x - y)[P_t(y) - P_t(y)]dy \]

\[ = \int_0^1 d\tau \int K_i(x - y)\left[P_t(y) - P_t(y + \tau(y - x))\right]dy \]

\[ |I| \lesssim \int_0^1 d\tau \int |(y - x)K_i(x - y)||P_t'(y + \tau(y - x))|dy \lesssim 2^{-i} \int_0^1 d\tau \int |yK_i(y)||P_t'(y + 2^{-i}y)|dy \]

\[ \lesssim 2^{-i} \int_0^1 d\tau \int |yK_i(y)| \frac{e^{-c|y|^{2+i}}}{t^2} dy \]

where

\[ |P_t'(z)| = \frac{C e^{-c|z|^{2+i}}}{t^{1/2}} \lesssim C e^{-c|z|^{2+i}}. \]

When \( t^{1/2} \geq 2^{-i}, |x| \) we have

\[ |I| \lesssim 2^{-i}t^{-2} \lesssim 2^{-i}(|x| + t^{1/2} + 2^{-i})^{-4}. \]

When \( 2^{-i} \geq t^{1/2}, |x| \) we estimate simply

\[ |I| \lesssim \int |K_i(x - y)|P_t(y)dy \lesssim 2^{3i} \lesssim 2^{-i}(|x| + t^{1/2} + 2^{-i})^{-4}. \]

When \( |x| \geq 2^{-i}, t^{1/2} \) we have instead that either \( |x| \geq 2\tau 2^{-i}|y| \) or \( |x| < 2\tau 2^{-i}|y| \). In the first region \(|x + \tau 2^{-i}y| \geq c|x| \) so

\[ |I| \lesssim 2^{-i} \int_0^1 d\tau \int |yK_i(y)| \frac{e^{-c|y|^{2+i}}}{t^2} dy \lesssim 2^{-i} \frac{e^{-c|y|^{2+i}}}{t^2} \lesssim 2^{-i}|x|^{-4} \lesssim 2^{-i}(|x| + t^{1/2} + 2^{-i})^{-4}. \]
while in the second region $|y| \geq 2|\tau|/2\sigma$, then $|yK_1(y)| \leq |yK_1(y)|^{1/2} f(2^i|x|/2\sigma)$ where $f$ is a rapidly decreasing function and in this region

$$|I| \lesssim 2^{-i} \int_0^1 \frac{e^{-c|x+\tau 2^{-i}y|/t}}{t^2} \, dy$$

$$\lesssim 2^{-i} \int_0^1 \frac{e^{-c'|\tau 2^{-i}y|/t}}{t^{3/2}|x+2^{-i}y|} \, dy \lesssim 2^{-i} \int_0^1 \frac{e^{-c'|y|/t}}{t^{3/2}} \, dy$$

$$\lesssim \frac{2^{-i}}{|x|^3} \int_0^1 \frac{e^{-c'|y|/t}}{t^{3/2}} \, dy \lesssim \frac{2^{-i}}{|x|^3} \lesssim -2^{-i}(|x| + t^{1/2} + 2^{-i})^{-4}.$$  

So we conclude that (A.19) holds. Let us turn to (A.20). When $t^{1/2} \geq 2^{-i}, |x|$ we have

$$\int \frac{|K_i(x-y)|}{(|y| + t^{1/2})^\alpha} \, dy \lesssim \frac{1}{t^\alpha} \int \frac{|K_i(x-y)|}{(|y| + t^{1/2})^\alpha} \, dy \lesssim \frac{1}{t^\alpha} \lesssim (|x| + t^{1/2} + 2^{-i})^{-\alpha}.$$  

When $2^{-i} \geq t^{1/2}, |x|$ we estimate

$$\int \frac{|K_i(x-y)|}{(|y| + t^{1/2})^\alpha} \, dy \lesssim 2^{\alpha i} \int \frac{|K_i(y)|}{|x+y|^\alpha} \, dy \lesssim 2^{\alpha i} \sup_z \int \frac{|K_i(y)|}{|z+y|^\alpha} \, dy \lesssim 2^{\alpha i} \lesssim (|x| + t^{1/2} + 2^{-i})^{-\alpha},$$

and finally when $|x| \geq 2^{-i}, t^{1/2}$ we have either $|x| \geq 2^{-i} |y|$ or $|x| < 2^{-i} |y|$. In the first region $|x+2^{-i}y| \geq c|x|$ so

$$\int \frac{|K_i(x-y)|}{(|y| + t^{1/2})^\alpha} \, dy \lesssim \int \frac{|K_i(y)|}{|x+2^{-i}y|^\alpha} \, dy \lesssim |x|^{-\alpha} \lesssim (|x| + t^{1/2} + 2^{-i})^{-\alpha},$$

while in the second $|y| \geq 2^{i} |x|/2$, then $|K_i(y)| \lesssim |K_i(y)|^{1/2} f(2^i|x|/2)$ where $f$ is another rapidly decreasing function and in this region

$$\int \frac{|K_i(x-y)|}{(|y| + t^{1/2})^\alpha} \, dy \lesssim f(2^i|x|/2) \int \frac{|K_i(y)|^{1/2}}{|2^i y|^\alpha} \, dy \lesssim 2^{\alpha i} f(2^i|x|/2) \lesssim |x|^{-\alpha} \lesssim (|x| + t^{1/2} + 2^{-i})^{-\alpha},$$

concluding our argument. \(\square\)

**Lemma A.17**

For $m, n \in (0, 5)$, $k, \ell \in [0, 2]$ define

$$I_{k, m, n} := \int_{\xi_1, \zeta_2} C_\zeta (\zeta_1 - \zeta_2)^k C_\zeta (\zeta_1 - \zeta_2)^\ell C_\zeta (\zeta_1 - \zeta_2)^m C_\zeta (\zeta_2 - \zeta_2)^n |\mu_{q, \zeta_1, \zeta_2} - \|q, \zeta_1, \zeta_2\|_{\mu_{q, \zeta_1, \zeta_2}},$$

with $\mu_{q, \zeta_1, \zeta_2}$ for $\zeta = (t, x)$, $\zeta_i = (s_i, x_i)$ $i = 1, 2$ defined as

$$\mu_{q, \zeta_1, \zeta_2} := \int_{x, y} K_{q, x}(x) \sum_{i = 1} K_{i, x}(y) K_{j, x}(x_2) P_{t-s}(y-x_1) |(t-s_2)d\zeta_1 d\zeta_2.$$  

If $\ell = 0, 0 < m + k - 2 < 5, m + k - 2 \in (-1, 5)$ and $k + m - 4 \in (0, 5)$ we have the bound

$$I_{k, m, n} \lesssim 2^{(k+m+n-4)q}.$$  

If $(k + m - 2), (\ell + m - 2) \in (0, 5), k + m + \ell - 4 \in (0, 5)$ and $k + \ell + m + n - 4 \in (0, 5)$ we have the bound

$$I_{k, m, n} \lesssim 2^{(k+\ell+m+n-4)q}.$$
Proof Observe that

$$\mu_{q, \xi_1, \xi_2} = \left| \sum_{i,j} \int_{x,y} K_{q, \xi}(x_2) K_{i,x}(y) K_{j,x}(y_2)(P_{t-s}(y-x_1) - P_{t-s}(\bar{x} - x_1))\delta(t-s_2)d\zeta_1d\zeta_2 \right|$$

$$+ \left| \sum_{i,j} \int_{x,y} (K_{q, \xi}(x) - K_{q, \xi}(x_2)) K_{i,x}(y) K_{j,x}(y_2) P_{t-s}(y-x_1)\delta(t-s_2)d\zeta_1d\zeta_2 \right|$$

$$= K_{q, \xi}(x_2)[P_{t-s}(x_2 - x_1) - P_{t-s}(\bar{x} - x_1)]\delta(t-s_2)d\zeta_1d\zeta_2$$

$$+ \left| \sum_{i,j} \int_{x,y} [K_{q, \xi}(x) - K_{q, \xi}(x_2)] K_{i,x}(y) K_{j,x}(x_2) P_{t-s}(y-x_1)\delta(t-s_2)d\zeta_1d\zeta_2 \right|$$

$$= \tilde{\mu}_{q, \xi_1, \xi_2} + \mu_{q, \xi_1, \xi_2}$$

where in the first line we used \( \int_{x} K_{i,x}(y) = 0 \) and the fact that \( \int_{x} K_{i,x}(x') K_{j,x}(x_2) = 0 \) if \( |i-j| > 1 \) and \( \sum_{i,j} K_{i,x}(y) K_{j,x}(x_2) = \delta(x_2 - y)\delta(x_2 - x) \). Now the estimation of the term

$$\hat{I}_{k,m,n} := \int_{\xi_1, \xi_2} C_2(\zeta_1 - \zeta_2)^k C_2(\zeta_1' - \zeta_2') C_2(\zeta_1 - \zeta_2)^n|\tilde{\mu}_{q, \xi_1, \xi_2}|$$

with \( \tilde{\mu}_{q, \xi_1, \xi_2} = K_{q, \xi}(x_2)[P_{t-s}(x_2 - x_1) - P_{t-s}(\bar{x} - x_1)]\delta(t-s_2)d\zeta_1d\zeta_2 \) can be done with Lemma A.10 and gives the expected result. The integral

$$\hat{I}_{k,m,n} := \int_{\xi_1, \xi_2} C_2(\zeta_1 - \zeta_2)^k C_2(\zeta_1' - \zeta_2') C_2(\zeta_1 - \zeta_2)^n|\tilde{\mu}_{q, \xi_1, \xi_2}|$$

with \( \tilde{\mu}_{q, \xi_1, \xi_2} = \left| \sum_{i,j} \int_{x,y} [K_{q, \xi}(x) - K_{q, \xi}(x_2)] K_{i,x}(y) K_{j,x}(x_2) P_{t-s}(y-x_1)\delta(t-s_2)d\zeta_1d\zeta_2 \) can be estimated by multiple changes of variables. We have \( K_{q, \xi}(x) - K_{q, \xi}(x_2) = 2^2q(x_2 - x) \int_{0}^{1} K''(2^q(x_2 - x)\tau - 2^q(\bar{x} - x_2))d\tau, \) and by the scaling properties of \( C_2 \) and \( P_{t,s} \), namely \( C_2(2^{-2}s, 2^{-i}x) \lesssim 2^i C_2(s,x) \) and \( P_{2^{-2}s, 2^{-i}s}(2^{-i}x) \lesssim 2^{2i}P_{s}(x) \) given by Lemma A.9 and Lemma A.12 we obtain easily the bound on \( \hat{I}_{k,m,n} \) by rescaling the integral.}

## B Some Malliavin calculus

We recall here some tools from Malliavin calculus that are widely used in the rest of the paper. An introduction to Malliavin calculus and the proofs of some results of this Appendix can be found in [20, 19, 24]. Lemma B.7 was inspired by the calculations of [25].

### B.1 Notation

Let \( \{W(h)\}_{h \in H} \) be an isonormal Gaussian process indexed by a real separable Hilbert space \( H \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) a probability space with \( \mathcal{F} \) generated by the isonormal Gaussian process \( W \), we note \( L^2(\Omega) = \bigoplus_{n \geq 0} H_n \) the well-known Wiener chaos decomposition of \( L^2(\Omega) \). For any real separable Hilbert space \( V \) and \( k \in \mathbb{N} \), let \( D^k : L^2(\Omega; V) \to L^2(\Omega; H^\otimes k \otimes V) \) be the Malliavin derivative and \( \delta^k : L^2(\Omega; H^\otimes k \otimes V) \to L^2(\Omega; V) \) the divergence operator (also called Skorohod integral) defined as the adjoint of \( D^k \). For \( p \geq 1 \) we will write \( \mathbb{D}^{k,p}(V) \subset L^p(\Omega; V) \) for the closure of smooth random variables with respect to the norm

$$\|\Psi\|_{\mathbb{D}^{k,p}(V)} = \left[ \mathbb{E}(\|\Psi\|_{V}^p) + \sum_{j=1}^{k} \mathbb{E}(\|D^j\Psi\|_{H^\otimes j \otimes V}^p) \right]^{1/p}$$
with the notation $\mathbb{D}^{k,p} := \mathbb{D}^{k,p}(\mathbb{R})$. Let $\{P_t\}_{t \in \mathbb{R}^+}$ the Ornstein-Uhlenbeck semigroup and $L : L^2(\Omega) \to L^2(\Omega)$ its generator (i.e. $e^{tL} = P_t$). Following [24] we introduce the Green operator

$$G_j = (j - L)^{-1}$$

with the notation

$$G^{[m]}_{[j]} := \prod_{k=j}^m G_k \quad \text{for } 1 \leq j \leq m \quad \text{(B.1)}$$

so that $G^{[j]}_{[j]} = G_j$. To avoid confusion, it is worth stressing that $G^{[m]}_{[j]}$ is not the $m$-th power of the operator $G_j$ but just a shortcut for $\prod_{k=j}^m G_k$.

### B.2 Partial chaos expansion

Let $\Psi \in L^2(\Omega)$ which has the Wiener chaos decomposition $\Psi = \sum_{n \geq 0} J_n \Psi$. Then by Proposition 1.2.2 of [20] $DJ_n \Psi = J_{n-1} D \Psi$, and knowing that $LJ_n \Psi = -nJ_n \Psi$ we obtain the commutation property

$$D(j - L)^{-\alpha} \Psi = D \sum_{n \geq 0} \frac{1}{(j + n)^\alpha} J_n \Psi = \sum_{n \geq 1} \frac{1}{(j + n)^\alpha} J_{n-1} D \Psi = (j + 1 - L)^{-\alpha} D \Psi \quad \text{(B.2)}$$

for every $\alpha > 0$, $j > 0$. The above formula holds also for $j = 0$ if $\mathbb{E}(\Psi) = 0$.

The results we have recalled so far let us write an $n$th-order Wiener chaos expansion for a random variable in $\mathbb{D}^{n,2}$:

**Lemma B.1**

Let $\Psi \in \mathbb{D}^{n,2}$ and $G^{[n]}_{[1]}$ as in (B.1). Then for every $n \in \mathbb{N} \setminus \{0\}$ $G^{[n]}_{[1]} \mathbb{D}^n \Psi \in \text{Dom} \delta^n$, $J_0 D^k \Psi \in \text{Dom} \delta^k \forall 0 \leq k < n$ and

$$\delta^n G^{[n]}_{[1]} \mathbb{D}^n \Psi = (\text{id} - J_0 - \ldots - J_{n-1}) \Psi = \Psi - \sum_{k=0}^{n-1} \frac{1}{k!} \delta^k J_0 D^k \Psi. \quad \text{(B.3)}$$

**Proof** We have for any $\Psi \in L^2(\Omega)$, since $L = -\Delta$ (20), Proposition 1.4.3):

$$\Psi - \mathbb{E}(\Psi) = LL^{-1}(\Psi - J_0 \Psi) = -\delta DL^{-1}(\Psi - J_0 \Psi) = \delta (1 - L)^{-1} D \Psi$$

where we used (B.2), and the fact that $(1 - L)^{-1} D \Psi \in \text{Dom} \delta$ is obvious by construction. This yields the first order expansion $\Psi = \mathbb{E}(\Psi) + \delta (1 - L)^{-1} D \Psi$. Iterating the expansion up to order $n$ we obtain (B.3). It is clear that $J_0 D^k \Psi \in \text{Dom} \delta^k$ since $J_0 D^k \Psi$ is constant with values in $H^{\delta^k}$. The second equality comes from the fact that $\delta^k J_0 D^k \Psi \in \mathcal{H}_k \forall k \in \mathbb{N}$ and $\forall \Psi' \in \mathbb{D}^{k,2}$:

$$\mathbb{E}(\delta^k (J_0 D^k \Psi) J_k \Psi') = (J_0 D^k \Psi, J_k D^k \Psi')_{L^2(\Omega; H^{\delta^k})} = \mathbb{E}(\delta^k (J_0 D^k \Psi) \Psi')$$

\[\square\]

In order to obtain $L^p$ estimations of the remainder term $\delta^n G^{[n]}_{[1]} \mathbb{D}^n \Psi$ generated by expansion (B.3), we used the following lemmas:

**Lemma B.2** ([20], Prop. 1.5.7)

Let $V$ be a real separable Hilbert space. For every $p > 1$ and every $q \in \mathbb{N}$, $k \geq q$ and every $\Psi \in \mathbb{D}^{k,p}(H^q \otimes V)$ we have

$$\|\delta^q(\Psi)\|_{\mathbb{D}^{k-q,p}(V)} \lesssim_{k,p} \|\Psi\|_{\mathbb{D}^{k,p}(H^q \otimes V)}$$

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Remark B.3 Let $V$ be a real separable Hilbert space. For every $v \in V$ and every $\Psi \in \mathbb{D}^{q,2}(H^{\otimes q})$ with $q \in \mathbb{N}$ we have $\Psi \otimes v \in \text{Dom} \delta^q$ and
\[
\delta^q(\Psi)v = \delta^q(\Psi \otimes v).
\]
Indeed, notice that for every smooth $\Psi' \in \mathbb{D}^{q,2}(V)$ and every smooth $\Psi \in \mathbb{D}^{q,2}(H^{\otimes q})$ we have
\[
\mathbb{E}(\langle \delta^q(\Psi \otimes v), \Psi' \rangle_V) = \mathbb{E}(\langle \Psi \otimes v, D^q \Psi'_{H^{\otimes q} \otimes V} \rangle_V) = \mathbb{E}(\langle \delta^q(\Psi)v, \Psi' \rangle_V).
\]
Now since $D^q(\Psi \otimes v) = D^q \Psi \otimes v$ and $\Psi \in \mathbb{D}^{q,2}(H^{\otimes q})$, we have $\Psi \otimes v \in \mathbb{D}^{q,2}(H^{\otimes q} \otimes V)$. Lemma B.2 yields the bound $\|\delta^q(\Psi \otimes v)\|_{L^2(V)} \lesssim \|\Psi \otimes v\|_{\mathbb{D}^{q,2}(H^{\otimes q} \otimes V)}$ which allows to pass to the limit for $\Psi'$ and $\delta^q(\Psi \otimes v)$ in $L^2(V)$.

Lemma B.4 ([24], Prop. 4.3)
For every $j > 0$ the operator $(j - L)^{-1/2}$ is bounded in $L^p$ for every $1 \leq p < \infty$.

Lemma B.5
Let $j \in \mathbb{N}\setminus\{0\}$ and $V$ a real separable Hilbert space. There exists a finite constant $c_p$ such that for every $\Psi \in L^p(\Omega, V)$:
\[
\|D(j - L)^{-1/2}\Psi\|_{L^p(\Omega, H^{\otimes q} \otimes V)} \leq c_p \|\Psi\|_{L^p(\Omega, V)}
\]
(where the operator $D(j - L)^{-1/2}$ is defined on every $\Psi$ which is polynomial in $W(h_1), \ldots, W(h_n)$ and can be extended by density on $L^p$).

Proof First notice that we can suppose w.l.o.g. $\mathbb{E}(\Psi) = 0$ thanks to B.2. Therefore we can write $D(j - L)^{-\frac{1}{2}}$ as
\[
D(j - L)^{-1/2} = D(-C)^{-1}(j - L)^{-1/2}
\]
with $C = -\sqrt{-L}$. We decompose the second part as $-C(j - L)^{-1/2}\Psi = \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{1/2} J_n \Psi = T_\phi \Psi$, with $T_\phi \Psi := \sum_{n=0}^{\infty} \phi(n) J_n \Psi$. We apply Theorem 1.4.2 of [20] to show that $T_\phi$ is bounded in $L^p$, indeed $\phi(n) = h(1/n)$ and $h(x) = (jx + 1)^{-1/2}$ which is analytic in a neighbourhood of 0. Finally, we can apply Proposition 1.5.2 of [20] to show that $DC^{-1}$ is bounded in $L^p$, thus concluding the proof.

The two lemmas above give the following immediate corollary:

Corollary B.6
For every $1 \leq m \leq n$ the operator $G_{[m]} := \prod_{k=m}^{n} (j - L)^{-1}$ is bounded in $L^p$ for every $1 \leq p < \infty$.

Moreover, Let $j \in \mathbb{N}\setminus\{0\}$ and $V$ a real separable Hilbert space. Then for every $\Psi \in L^p(\Omega, V)$ we have:
\[
\|D(j - L)^{-1}\Psi\|_{L^p(\Omega, H^{\otimes q} \otimes V)} \lesssim \|\Psi\|_{L^p(\Omega, V)}.
\]

Moreover, for every $0 \leq k \leq 2m$, $i \geq 0$ we have
\[
\|D^k G_{[i]}^{[i+m]} \Psi\|_{L^p(\Omega, H^{\otimes k} \otimes V)} \lesssim \|\Psi\|_{L^p(\Omega, V)}.
\]

The next lemma is one of the most useful tools of this paper. It allows us to write products of decompositions of the type B.3 as sums of iterated Skorohod integrals. From now on we will note $\langle \cdot, \cdot \rangle_{H^{\otimes q}}$ the $r$-th contraction, which to avoid inconsistency has to be taken between symmetric tensors. We also note $h_{v_1, \ldots, v_n}^{\otimes n} := h_{v_1} \otimes \ldots \otimes h_{v_n}$ for $h_{v_1}, \ldots, h_{v_n} \in H$.
Lemma B.7
Let \( u = f(W(h_u))h_u^{\otimes m} \) and \( v = Fh_{v_1,\ldots,v_n} \) with \( f \in C^{m+n}(\mathbb{R}) \) and \( F \in \mathbb{D}^{m+n,2} \). Then
\[
\delta^m(u)\delta^n(v) = \sum_{(q,r,i)\in I_{m,n}} C_{m,n,q,r,i} \delta^{m+n-q-r}[f^{(r-i)}(W(h_u))\langle h_u^{\otimes m-i},D^{q-i}F \rangle_{H^{\otimes q+r-i}},\langle h_u^{\otimes r},h_{v_1,\ldots,v_n} \rangle_{H^{\otimes r}}] \quad (B.4)
\]
with \( C_{m,n,q,r,i} := \left(\begin{array}{c} q \\ r \end{array}\right) \left(\begin{array}{c} q \\ i \end{array}\right) \left(\begin{array}{c} r \\ i \end{array}\right) \cdot! \) and \( I_{m,n} := \{(q,r,i) \in \mathbb{N}^3 : 0 \leq q \leq m, 0 \leq r \leq n, 0 \leq i \leq q \wedge r\}. \)
A trivial change of variables gives also:
\[
\delta^m(u)\delta^n(v) = \sum_{(i,q,r)\in I'_{m,n}} C_{m,n,q,i,r} \delta^{m+n-q-r-2i}[f^{(r)}(W(h_u))\langle h_u^{\otimes m-i},D^{q}F \rangle_{H^{\otimes q+r+i}},\langle h_u^{\otimes r+i},h_{v_1,\ldots,v_n} \rangle_{H^{\otimes r+i}}] \quad (B.5)
\]
with \( I'_{m,n} := \{(i,q,r) \in \mathbb{N}^3 : 0 \leq i \leq m \wedge n, 0 \leq q \leq m - i, 0 \leq r \leq n - i\}. \)

Remark B.8 In the special case \( v = g(W(h_v))h_v^{\otimes m} \) eq. (B.4) takes the form
\[
\delta^m(u)\delta^n(v) = \sum_{(q,r,i)\in I_{m,n}} C_{m,n,q,r,i} \delta^{m+n-q-r}([D^{q-i}u,D^{r-i}v]_{H^{\otimes q+r-i}}) \quad (B.6)
\]
which is just a generalization to Skorohod integrals of the multiplication formula for multiple Wiener integrals \((23, 26)\). We can write the above formula more explicitly as
\[
\delta^m(u)\delta^n(v) = \sum_{(q,r,i)\in I_{m,n}} C_{m,n,q,r,i} \delta^{m+n-q-r}[f^{(r-i)}(W(h_u))g^{(q-i)}(W(h_v))h_u^{\otimes m-q} \otimes h_v^{\otimes n-r}],[h_u,h_v]_{H^{\otimes q+r-i}}.
\]

Remark B.9 Note that one can assume w.l.o.g. the argument of \( \delta^{m+n-q-r} \) in (B.4) to be symmetric, and this would allow to iterate Lemma [B.7]

Remark B.10 We can give the following intuition for the second formula in Lemma [B.7] The random variables \( u \) and \( v \) have an infinite chaos decomposition, and following the tree-like notation of \([6] \text{ or } [8]\) they could be thought of as having an infinite number of leaves which need to be contracted with each other.

It is apparent that the index \( i \) in the second equation denotes contractions between the already existing leaves of the trees \( u, v \). The indexes \( r \) and \( q \) count new leaves in each vertex that are created by the Malliavin derivatives, which are then contracted with other leaves from the other tree. There are then \( m + n - r - q - 2i \) overall unmatched leaves which are arguments to the iterated Skorokhod integral.

The more intuitive interpretation of the second equation in Lemma [B.7] is the reason why we gave two distinct expression for the same quantity. Nevertheless, the formula (B.4) is more practical in the calculations and is more widely used throughout the paper.

Proof [Lemma [B.7]] Using Cauchy-Schwarz inequality and Lemma [B.2] we can show that \( \langle D^r\delta^m(v),\delta^j(u) \rangle_{H^{\otimes r}} \in L^2(\Omega,H^{\otimes m-j-r}) \) for every \( 0 \leq r + j \leq m \). Then we apply Lemma [B.11] to get:
\[
\delta^m(u)\delta^n(v) = \sum_{r=0}^{n} \binom{n}{r} \delta^{n-r}(\langle D^r\delta^m(u),v \rangle_{H^{\otimes r}}).
\]
Using the commutation formula \(^{(B.7)}\) we rewrite the r.h.s. as

\[
\delta^m(u)\delta^n(v) = \sum_{r=0}^{\min(m,n)} \binom{n}{r} \binom{m}{i} i! \delta^{m-i}(D^{r-i}u, v)_{H^s}.
\]

We obtain

\[
\langle \delta^{m-i}(D^{r-i}u), v \rangle_{H^s} = \delta^{m-i}(f^{r-i}(W(h_u))h_u^{m-i}F(h_u, h_v))_{H^s}
\]

where we set \(h_u := h_u^{\otimes n} \otimes h_v\). From Lemma \(^{(B.11)}\) we have, since \(D h_u = h_u\):

\[
\delta^{n-i}(D^{r-i}u, v)_{H^s} = \delta^{n-i}(f^{r-i}(W(h_u))h_u^{m-i}F(h_u, h_v))_{H^s}
\]

Proof If \(j = 0, k = 1\) or \(k = 0, j = 1\) eq. \(^{(B.7)}\) is trivial. Let \(j = k = 1\) and \(u \in \mathbb{D}^{2,2}(H) \subset \mathbb{D}^{1,2}(H)\). We can apply Proposition 1.3.2 of \(^{(20)}\) to obtain \((D\delta)(u, h) = (u, h) + \delta(Du, h)\) \(\forall h \in H\). Since by hypothesis \(Du\) is symmetric we have \(\delta((Du, h)) = \langle Du, h \rangle\), and then \(D\delta(u) = u + \delta Du\). The proof by induction is easy noticing that \(D\delta = \delta D + j\delta^{j-1}\).\(\square\)
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