RELATIVE NON-COMMUTING GRAPH OF A FINITE RING

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Abstract: Let $S$ be a subring of a finite ring $R$ and $C_R(S) = \{r \in R : rs = sr \forall s \in S\}$. The relative non-commuting graph of the subring $S$ in $R$, denoted by $\Gamma_{S,R}$, is a simple undirected graph whose vertex set is $R \setminus C_R(S)$ and two distinct vertices $a, b$ are adjacent if and only if $a$ or $b \in S$ and $ab \neq ba$. In this paper, we discuss some properties of $\Gamma_{S,R}$, determine diameter, girth, some dominating sets and chromatic index for $\Gamma_{S,R}$. Also, we derive some connections between $\Gamma_{S,R}$ and the relative commuting probability of $S$ in $R$. Finally, we show that the relative non-commuting graphs of two relative $Z$-isoclinic pairs of rings are isomorphic under some conditions.

Key words: Non-commuting graph, Commuting probability, $Z$-isoclinism.

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1. INTRODUCTION

Let $R$ be a finite ring with subring $S$. Let $C_R(S) = \{r \in R : rs = sr \forall s \in S\}$. The relative non-commuting graph of the subring $S$ in $R$, denoted by $\Gamma_{S,R}$, is defined as a simple undirected graph whose vertex set is $R \setminus C_R(S)$ and two distinct vertices $a, b$ are adjacent if and only if $a$ or $b \in S$ and $ab \neq ba$. For $S = R$, we have $\Gamma_{S,R} = \Gamma_R$, the non-commuting graph of $R$. The notion of non-commuting graph of a finite ring was introduced by Erfanian et al. [8] in the year 2015. The study of algebraic structures by means of graph theoretical properties became more popular during the last decade (see [1, 2, 3, 4, 11] etc.). Motivated by the works of Erfanian et al. [12], in this paper, we obtain some graphs that are not isomorphic to $\Gamma_{S,R}$ for any ring $R$ with subring $S$. We also determine diameter, girth, some dominating sets and chromatic index for $\Gamma_{S,R}$ and derive some connections between $\Gamma_{S,R}$ and the relative commuting probability of $S$ in $R$. Recall that the

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relative commuting probability of a subring $S$ in a finite ring $R$, denoted by $\Pr(S, R)$, is the probability that a randomly chosen pair of elements, one from $S$ and the other from $R$ commute. That is

$$\Pr(S, R) = \frac{|\{(s, r) \in S \times R : sr = rs\}|}{|S||R|}.$$ 

This notion was introduced and studied in [7]. Note that $\Pr(R, R)$ is the commuting probability of $R$, a notion introduced by MacHale [10].

In the last section, we show that the relative non-commuting graphs of two relative $\mathbb{Z}$-isoclinic pairs of rings are isomorphic under some conditions.

For a graph $G$, we write $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of $G$ respectively. We write $\text{deg}(v)$ to denote the degree of a vertex $v$, which is the number of edges incident on $v$. Let $\text{diam}(G)$ and $\text{girth}(G)$ be the diameter and girth of a graph $G$ respectively. Recall that $\text{diam}(G) = \max\{d(x, y) : x, y \in V(G)\}$, where $d(x, y)$ is the length of the shortest path from $x$ to $y$; and $\text{girth}(G)$ is the length of the shortest cycle obtained in $G$. A graph $G$ is called connected if there is a path between every pair of vertices. A star graph is a tree on $n$ vertices in which one vertex has degree $n - 1$ and the others have degree 1. A bipartite graph is a graph whose vertex set can be partitioned into two disjoint parts in such a way that the two end vertices of every edge lie in different parts. A complete bipartite graph is a bipartite graph such that two vertices are adjacent if and only if they lie in different parts. A complete graph is a graph in which every pair of distinct vertices is adjacent. Throughout the paper $R$ denotes a finite non-commutative ring.

## 2. Some properties of $\Gamma_{S, R}$

Let $S$ be a subring of a ring $R$, $r \in R$ and $A \subseteq R$. We write $C_R(r) := \{x \in R : xr = rx\}$, $C_S(r) := C_R(r) \cap S$ and $C_R(A) := \{x \in R : xa = ax \forall a \in A\}$. Note that $C_R(r)$ and $C_S(r)$ are subrings of $R$. Also $\bigcap_{r \in R} C_R(r) := Z(R)$ is the center of $R$. We begin this section with the following useful result.

**Proposition 2.1.** Let $S$ be a non-commutative subring of a ring $R$. Then

(a) $\text{deg}(r) = |R| - |C_R(r)|$ if $r \in V(\Gamma_{S, R}) \cap S$.
(b) $\text{deg}(r) = |S| - |C_S(r)|$ if $r \in V(\Gamma_{S, R}) \cap (R \setminus S)$.
(c) $\Gamma_{S, R}$ is connected.
(d) $\Gamma_{S, R}$ is empty graph if and only if $S$ is commutative.
Proof. The proof of part (a), (b) and (d) follow from the definition of $\Gamma_{S,R}$. For part (c), suppose $\Gamma_{S,R}$ has an isolated vertex, namely $v$. Then $\deg(v) = |R| - |C_R(v)| = 0$ or $|S| - |C_S(v)| = 0$ for $v \in S$ or $v \in R \setminus S$. Thus, in both cases $v \in C_R(S)$, a contradiction. \qed

In the following theorems we shall show that if $G$ is a star graph or an $n$-regular graph, where $n$ is a square free odd positive integer, then $G$ can not be realized by $\Gamma_{S,R}$ for any subring $S$ of a ring $R$. Also, $\Gamma_{S,R}$ is not a bipartite graph, for any proper subring $S$ of a ring $R$.

**Theorem 2.2.** Let $S$ be a non-commutative subring of a ring $R$. Then $\Gamma_{S,R}$ is not a star graph.

Proof. Suppose, $\Gamma_{S,R}$ is a star graph, where $S$ is a non-commutative subring of $R$. Then all but one vertices of $\Gamma_{S,R}$ have degree 1. Let $v$ be a vertex of $\Gamma_{S,R}$ having degree 1. Then, by Proposition 2.1, we have $[R : C_R(v)] = |R|/(|R| - 1)$ or $[S : C_S(v)] = |S|/(|S| - 1)$ according as $v \in S$ or $v \in R \setminus S$; which is absurd. Hence the result follows. \qed

**Theorem 2.3.** Let $S$ be a proper non-commutative subring of a ring $R$. Then $\Gamma_{S,R}$ is not bipartite.

Proof. Let $\Gamma_{S,R}$ be a bipartite graph. Then, there exist two disjoint subsets $S_1$ and $S_2$ of $V(\Gamma_{S,R})$ such that $|S_1| + |S_2| = |R| - |C_R(S)|$. Therefore, $S \cap S_1 = \emptyset$ or $S \cap S_2 = \emptyset$. So, $S \subseteq S_2$ or $S \subseteq S_1$. Without loss of generality we may assume that $S \subseteq S_1$. Then, for $v \in S_1$ we have $uv = sv$ for all $s \in S \setminus C_R(S)$. Thus, $v \in Z(S) \subseteq C_R(S)$, a contradiction. Hence, the theorem follows. \qed

**Theorem 2.4.** Let $S$ be a non-commutative subring of a ring $R$. Then $\Gamma_{S,R}$ is not an $n$-regular graph for any square free odd positive integer $n$.

Proof. Let $\Gamma_{S,R}$ be an $n$-regular graph. Suppose, $n = p_1p_2\ldots p_m$, where $p_i$’s are distinct odd primes. If $v \in V(\Gamma_{S,R}) \cap S$ then, by Proposition 2.1, we have

$$n = \deg(v) = |R| - |C_R(v)| = |C_R(v)|([R : C_R(v)] - 1).$$

Here, $|C_R(v)| \neq 1$, as $0, v \in C_R(v)$. Thus $|C_R(v)| = \prod_{p_i \in Q} p_i$ and $[R : C_R(v)] - 1 = \prod_{p_j \in P \setminus Q} p_j$, where $Q \subseteq \{p_1, p_2, \ldots, p_m\} = P$. So, $|R| = \prod_{p_i \in Q} p_i (\prod_{p_j \in P \setminus Q} p_j + 1)$. If $r \in R \setminus S$ then, using similar argument, we have $|S| = \prod_{p_i \in T} p_i (\prod_{p_j \in P \setminus T} p_j + 1)$, where $T \subseteq P$. So, $\prod_{p_i \in T ^{(T \cap Q)}} p_i (\prod_{p_j \in P \setminus T ^{(T \cap Q)}} p_j + 1)$.
1) divides $\prod_{p_j \in P \setminus Q} p_j + 1$, which is not possible. Hence, the theorem follows.

We conclude this section showing that a complete graph cannot be realized by $\Gamma_{S,R}$ for a subring $S$ of a ring $R$ with unity.

**Theorem 2.5.** Let $R$ be a ring with unity and $S$ a subring of $R$. Then $\Gamma_{S,R}$ is not complete.

**Proof.** Suppose that there exists a subring $S$ of $R$ with unity such that $\Gamma_{S,R}$ is complete. Then, for any $s \in V(\Gamma_{S,R}) \cap S$ we have

$$\deg(s) = |V(\Gamma_{S,R})| - 1 = |R| - |C_R(S)| - 1.$$ 

By Proposition 2.1, we have $|R| - |C_R(S)| = |R| - |C_R(S)| - 1$. This gives $|C_R(S)| = 1$ and $|C_R(s)| = 2$, which is not possible, since $R$ is a ring with unity. Hence, the result follows.

3. **Diameter, girth, dominating set and chromatic index**

In this section, we obtain diameter, girth, some dominating sets and chromatic index of the graph $\Gamma_{S,R}$.

**Theorem 3.1.** Let $S$ be a non-commutative subring of a ring $R$. If $Z(S) = \{0\}$ then $\text{diam}(\Gamma_{S,R}) = 2$ and $\text{girth}(\Gamma_{S,R}) = 3$.

**Proof.** Suppose, $v_1$ and $v_2$ are two vertices of $\Gamma_{S,R}$ such that they are not adjacent. So, there exist vertices $s_1, s_2 \in S$ such that $v_1 s_1 \neq s_1 v_1$ and $v_2 s_2 \neq s_2 v_2$. If $v_2$ is adjacent to $s_1$ or $v_1$ is adjacent to $s_2$, then $d(v_1, v_2) = 2$. Suppose that both are not adjacent, that is $v_1 s_2 = s_2 v_1$ and $v_2 s_1 = s_1 v_2$. Then $s_1 + s_2$ is adjacent to $v_1$ and $v_2$, which give $d(v_1, v_2) = 2$. Therefore, $\text{diam}(\Gamma_{S,R}) = 2$.

In order to determine $\text{girth}(\Gamma_{S,R})$, suppose that $v, s \in V(\Gamma_{S,R})$ where $s \in S$ and $v, s$ are adjacent. So, there exist $v_1, v_2 \in V(\Gamma_{S,R})$ such that $v$ and $s$ are adjacent to $v_1$ and $v_2$ respectively. If $v, v_2$ or $s, v_1$ are adjacent then $\{v, s, v_2\}$ or $\{v, s, v_1\}$ is a cycle of length 3 in $\Gamma_{S,R}$. If both are not adjacent then $v_1 + v_2$ is adjacent to $v$ and $s$. Therefore, $\{v, s, v_1 + v_2\}$ is a cycle of length 3 in $\Gamma_{S,R}$. Hence, $\text{girth}(\Gamma_{S,R}) = 3$.

Let $G$ be a graph and $D$ a subset of $V(G)$ such that every vertex not in $D$ is adjacent to at least one member of $D$ then $D$ is called the dominating set for $G$. It is obvious that $V(G)$ is a dominating set for $G$. Again, it is easy to see that for any non-commutative subring $S$ of $R$, the set $S \setminus Z(S)$ is a dominating set for $\Gamma_{S,R}$. Let $A$ and $B$ be two subsets of $R$. We define $A + B := \{a + b : a \in A, b \in B\}$. Then it can be seen that $(S + C_R(S)) \setminus C_R(S)$ is a dominating set for $\Gamma_{S,R}$ if $S$ is a
Proposition 3.2. Let $S$ be a subring of a ring $R$ and $A \subseteq V(\Gamma_{S,R})$. Then $A$ is a dominating set for $\Gamma_{S,R}$ if and only if $C_R(A) \subseteq A \cup C_R(S)$.

Proof. Suppose, $A$ is a dominating set for $\Gamma_{S,R}$ and $v \in V(\Gamma_{S,R})$ such that $v \in C_R(A)$. If $v \notin A$ then there exists an element $a \in A$ such that $va \neq av$, a contradiction.

Conversely, we suppose that $C_R(A) \subseteq A \cup C_R(S)$. Let $v \in V(\Gamma_{S,R})$ such that $va = av$ for all $a \in A$. Then $v \in C_R(A)$ and so $v \in A \cup C_R(S)$. Thus, $v \in A$, a contradiction. Hence, $A$ is a dominating set for $\Gamma_{S,R}$. \hfill $\Box$

Proposition 3.3. Let $R$ be a ring with unity and $S$ a subring of $R$. If $L = \{s_1, s_2, \ldots, s_n\}$ is a generating set for $S$ and $L \cap C_R(S) = \{s_{m+1}, \ldots, s_n\}$ then $K = \{s_1, s_2, \ldots, s_m\} \cup \{s_1+s_{m+1}, s_1+s_{m+2}, \ldots, s_1+s_n\}$ is a dominating set for $\Gamma_{S,R}$.

Proof. Clearly, $K \subseteq V(\Gamma_{S,R})$. Let $v \in V(\Gamma_{S,R})$ such that $v \notin L$. If $v \in S$ then there exists an element $s = \beta_1 s_1^{\alpha_{11}} s_2^{\alpha_{12}} \ldots s_d^{\alpha_{1d}}$, where $\beta_i \in \mathbb{Z}$, $\alpha_{ij} \in \mathbb{N} \cup \{0\}$ and $s_j \in L$ such that $vs \neq sv$. Therefore, $vs_i \neq si v$ for some $1 \leq i \leq m$ and so, $v$ is adjacent to $s_i$.

If $v \in R \setminus S$ then there exists an element $u = \gamma_i s_1^{\alpha_{i1}} s_2^{\alpha_{i2}} \ldots s_p^{\alpha_{ip}}$, where $\gamma_i \in \mathbb{Z}$, $\alpha_{ii} \in \mathbb{N} \cup \{0\}$ and $s_i \in L$ such that $vu \neq uv$. If $vs_i \neq si v$ for some $1 \leq i \leq m$ then $v$ is adjacent to $s_i$. Otherwise, $vs_i = si v$ for all $1 \leq i \leq m$. So, there exists an element $s_l$ for some $m+1 \leq l \leq n$ such that $vs_l \neq si v$. Therefore, $v$ is adjacent to $s_1 + s_l$. Hence, the proposition. \hfill $\Box$

An edge coloring of a graph $G$ is an assignment of “colors” to the edges of the graph so that no two adjacent edges have the same color. The chromatic index of a graph denoted by $\chi'(G)$ and is defined as the minimum number of colours needed for a colouring of $G$. Let $\Delta$ be the maximum vertex degree of $G$, then Vizing’s theorem [6] gives $\chi'(G) = \Delta$ or $\Delta + 1$. Thus, Vizing’s theorem divides the graphs into two classes according to their chromatic index. Graphs satisfying $\chi'(G) = \Delta$ are called graphs of class 1 and those with $\chi'(G) = \Delta + 1$ are called graphs of class 2. Following theorem shows that $\Gamma_{R,R}$ is of class 2.

Theorem 3.4. Let $R$ be a ring. Then the non-commuting graph $\Gamma_{R,R}$ is of class 2.
We conclude this section with some consequences of Theorem 4.1.

Proof. Clearly, $\Delta \leq |R| - |Z(R)| - 1$. If $\chi'(\Gamma_{S,R}) = \Delta$ then $\chi'(\Gamma_{S,R}) \leq |R| - |Z(R)| - 1$, which is not true for the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_2 \right\}$. Hence, $\Gamma_{R,R}$ is class of 2.

We conclude this section with the following conjecture.

**Conjecture 3.5.** Let $S$ be a proper non-commutative subring of $R$. Then the relative non-commuting graph $\Gamma_{S,R}$ is of class 1.

4. RELATIVE NON-COMMUTING GRAPHS AND $Pr(S,R)$

In this section, we give some connections between $\Gamma_{S,R}$ and $Pr(S,R)$, where $S$ is a subring of a finite ring $R$. We start with the following result.

**Theorem 4.1.** Let $S$ be a subring of a ring $R$. Then the number of edges of $\Gamma_{S,R}$ is

$$|E(\Gamma_{S,R})| = |S||R|(1 - Pr(S,R)) - \frac{|S|^2}{2}(1 - Pr(S)).$$

Proof. Let $I = \{(r_1, r_2) \in S \times R : r_1 r_2 \neq r_2 r_1\}$ and $J = \{(r_1, r_2) \in R \times S : r_1 r_2 \neq r_2 r_1\}$. Therefore, we have $|I| = |S||R| - |\{(r_1, r_2) \in S \times R : r_1 r_2 = r_2 r_1\}| = |S||R| - |S||R| Pr(S,R) = |J|$ and so $|I \cap J| = |\{(a, b) \in S \times S : ab \neq ba\}| = |S|^2 - |S|^2 Pr(S)$. Thus, the result follows from the fact that $|E(\Gamma_{S,R})| = \frac{1}{2}|I \cup J|$.

The above theorem shows that lower or upper bounds for $Pr(S)$ and $Pr(S, R)$ will give lower or upper bounds for $|E(\Gamma_{S,R})|$ and vice-versa. More bounds for $|E(\Gamma_{S,R})|$ are obtained in the next few results.

**Proposition 4.2.** Let $S$ be a subring of a ring $R$. Then

$$|E(\Gamma_{S,R})| \geq \frac{1}{2}|S||R| - \frac{1}{4}|S|^2 - \frac{1}{4}|Z(S)||R| - \frac{1}{4}|S||C_R(S)| + \frac{1}{4}|Z(S)||S|.$$

Proof. Let $A = V(\Gamma_{S,R}) \cap S$ and $B = V(\Gamma_{S,R}) \cap (R \setminus S)$. Therefore, $|A| = |S| - |Z(S)|$ and $|B| = |R| - |S| - |C_R(S)| + |Z(S)|$. So, we have

$$2|E(\Gamma_{S,R})| = \sum_{v \in V(\Gamma_{S,R})} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v)
= \sum_{v \in A} (|R| - |C_R(v)|) + \sum_{v \in B} (|S| - |C_S(v)|)
\geq |A||R| - \frac{|A||R|}{2} - |B||S| - \frac{|S||B|}{2}.$$

Thus, putting the values of $|A|$ and $|B|$, we get the required result. □

We conclude this section with some consequences of Theorem 4.1.
Proposition 4.3. Let $S$ be a non-commutative subring of a ring $R$ and $p$ the smallest prime dividing $|R|$. Then

$$|E(\Gamma_{S,R})| \leq |S|(|R| - \frac{3|S|}{16} - p) - |Z(R) \cap S|(|R| - p)$$

Proof. By [7, Theorem 2.5], we have

$$\frac{|Z(R) \cap S|}{|S|} + \frac{p(|S| - |Z(R) \cap S|)}{|S||R|} \leq \text{Pr}(S,R). \quad (1)$$

Now, using (1) and the fact that $\text{Pr}(S) \leq \frac{5}{8}$ in Theorem 4.1 we get the required result. □

Proposition 4.4. Let $S$ be a non-commutative subring of a ring $R$. Then

$$|E(\Gamma_{S,R})| \geq -\frac{3|S|^2}{16} + \frac{3|S||R|}{8}.$$

Proof. Using [7, Theorem 2.2], we have that $\text{Pr}(S,R) \leq \text{Pr}(S) \leq \frac{5}{8}$. Therefore, $1 - \text{Pr}(S,R) \geq 1 - \text{Pr}(S) \geq \frac{2}{5}$. Hence, putting these results in Theorem 4.1, we get the required proposition. □

Proposition 4.5. Let $S$ be a non-commutative subring of a ring $R$. If $|C_R(S)| = 1$ then

$$2|R|\text{Pr}(S,R) - |S|\text{Pr}(S) \neq -2\frac{|R|}{|S|} + \frac{4}{|S|} + 2|R| - |S|. $$

Proof. Suppose there exists a finite ring $R$ with non-commutative subring $S$ such that $|C_R(S)| = 1$ and

$$2|R|\text{Pr}(S,R) - |S|\text{Pr}(S) = -2\frac{|R|}{|S|} + \frac{4}{|S|} + 2|R| - |S|. $$

Then the above equation, in view of Theorem 4.1, gives

$$|E(\Gamma_{S,R})| = |R| - |C_R(S)| - 1 = |V(\Gamma_{S,R})| - 1.$$

This shows that there is a finite non-commutative ring $R$ with non commutative subring $S$ such that $\Gamma_{S,R}$ is a star graph, which is not possible (by Theorem 2.2). Hence, the proposition follows. □

5. Relative non-commuting graph and relative \(Z\)-isoclinism

In 1940, Hall [9] introduced the notion of isoclinism between two groups. Following Hall, Buckley et al. [5] introduced the concept of \(Z\)-isoclinism between two rings. Recently, Dutta et al. [7] introduced the concept of relative \(Z\)-isoclinism between two pairs of rings. For a subring $S$ of $R$, $[S, R]$ is the subgroup of $(R, +)$ generated by all
commutators \([s, r], s \in S, r \in R\). Let \(S_1\) and \(S_2\) be two subrings of the rings \(R_1\) and \(R_2\) respectively. Recall that a pair of rings \((S_1, R_1)\) is said to be relative \(\mathbb{Z}\)-isoclinic to a pair of rings \((S_2, R_2)\) if there exist additive group isomorphisms \(\phi : \frac{R_1}{Z(R_1) \cap S_1} \to \frac{R_2}{Z(R_2) \cap S_2}\) such that \(\phi \left( \frac{S_1}{Z(R_1) \cap S_1} \right) = \frac{S_2}{Z(R_2) \cap S_2}\), and \(\psi : [S_1, R_1] \to [S_2, R_2]\) such that \(\psi([s_1, r_1]) = [s_2, r_2]\) whenever \(\phi(s_1 + (Z(R_1) \cap S_1)) = s_2 + (Z(R_2) \cap S_2)\) and \(\phi(r_1 + (Z(R_1) \cap S_1)) = r_2 + (Z(R_2) \cap S_2)\) whenever \(s_1 \in S_1, s_2 \in S_2, r_1 \in R_1, r_2 \in R_2\). Such pair of mappings \((\phi, \psi)\) is called a relative \(\mathbb{Z}\)-isoclinism from \((S_1, R_1)\) to \((S_2, R_2)\). In this section, we have the following main result.

**Theorem 5.1.** Let \(S_1\) and \(S_2\) be two subrings of the finite rings \(R_1\) and \(R_2\) respectively. Let the pairs \((S_1, R_1)\) and \((S_2, R_2)\) are relative \(\mathbb{Z}\)-isoclinic. Then \(\Gamma_{S_1, R_1} \cong \Gamma_{S_2, R_2}\) if \(|Z(R_1) \cap S_1| = |Z(R_2) \cap S_2|\) and \(|Z(R_1)| = |Z(R_2)|\).

**Proof.** Suppose \((\phi, \psi)\) is a relative \(\mathbb{Z}\)-isoclinism between \((S_1, R_1)\) and \((S_2, R_2)\). If \(|Z(R_1) \cap S_1| = |Z(R_2) \cap S_2|\) and \(|Z(R_1)| = |Z(R_2)|\), then \(|S_1| = |S_2|\), \(\frac{R_1}{Z(R_1)} \cong \frac{R_2}{Z(R_2)}\), \(Z(R_1) \cap S_1 = Z(R_2) \cap S_2\) and \(|S_1 \setminus Z(R_1)| = |S_2 \setminus Z(R_2)|\). Now, by second isomorphism theorem (of groups), we have \(\frac{S_1 + Z(R_1)}{Z(R_1)} \cong \frac{S_2 + Z(R_2)}{Z(R_2)}\). Let \(\{s_1, s_2, \ldots, s_m\}\) be a transversal for \(\frac{S_1 + Z(R_1)}{Z(R_1)}\). So, the set \(\{s_1, s_2, \ldots, s_m\}\) can be extended to a transversal for \(\frac{R_1}{Z(R_1)}\). Suppose, \(\{s_1, s_2, \ldots, s_m, r_{m+1}, \ldots, r_k\}\) is a transversal for \(\frac{R_1}{Z(R_1)}\). Similarly, we can find a transversal \(\{s'_1, s'_2, \ldots, s'_m, r'_{m+1}, \ldots, r'_k\}\) for \(\frac{R_2}{Z(R_2)}\) such that \(\{s'_1, s'_2, \ldots, s'_m\}\) is a transversal for \(\frac{S_2 + Z(R_2)}{Z(R_2)}\).

Let \(\phi\) be defined as \(\phi(s_i + Z(R_1)) = s'_i + Z(R_2), \phi(r_j + Z(R_1)) = r'_j + Z(R_2)\) for \(1 \leq i \leq m, m + 1 \leq j \leq n\) and let the one-to-one correspondence \(\theta : Z(R_1) \to Z(R_2)\) maps elements of \(S_1\) to \(S_2\). Therefore, \(|C_{R_1}(S_1)| = |C_{R_2}(S_2)|\). Let us define a map \(\alpha : R_1 \to R_2\) such that \(\alpha(s_i + z) = s'_i + \theta(z), \alpha(r_j + z) = r'_j + \theta(z)\) for \(1 \leq i \leq m, m + 1 \leq j \leq n\) and \(z \in Z(R_1)\). Then \(\alpha\) is a bijection. This gives that \(\alpha\) is also a bijection from \(R_1 \setminus C_{R_1}(S_1)\) to \(R_2 \setminus C_{R_2}(S_2)\). Suppose \(u, v\) are adjacent in \(\Gamma_{S_1, R_1}\). Then \(u \in S_1\) or \(v \in S_1\), say \(u \in S_1\). So, \([u, v] \neq 0\), therefore \([s_i + z, r + z_1] \neq 0\), where \(u = s_i + z, v = r + z_1\) for some \(z, z_1 \in Z(R_1), r \in \{s_1, s_2, \ldots, s_m, r_{m+1}, \ldots, r_n\}\) and \(1 \leq i \leq m\). Thus \([s'_i + \theta(z), r + \theta(z_1)] \neq 0\), where \(\theta(z), \theta(z_1) \in Z(R_2)\) and so, \(\alpha(u)\) and \(\alpha(v)\) are adjacent. Hence, the theorem.

We conclude the paper with the following consequence of Theorem 5.1.
Corollary 5.2. Let \( R \) be a ring with subrings \( S \) and \( T \) such that \((S, R)\) is relative \( \mathbb{Z} \)-isoclinic to \((T, R)\). Then \( \Gamma_S \cong \Gamma_T \) if \( |Z(R) \cap S| = |Z(R) \cap T| \).

References

[1] Abdollahi, A. (2007) Engel graph associated with a group, J. Algebra, 318, 680–691.
[2] Abdollahi, A. (2008) Commuting graph of full matrix rings over finite fields, Linear Algebra Appl., 428, 2947–2954.
[3] Abdollahi, A., Akbari, S. and Maimani, H. R. (2006) Non-commuting graph of a group, J. Algebra, 298, 468–492.
[4] Beck, I. (1988) Coloring of commutative rings, J. Algebra, 116, 208–226.
[5] Buckley, S. M., Machale, D. and Ní Shé, A. Finite rings with many commuting pairs of elements, Preprint.
[6] Diestel, R. Graph Theory. Springer-Verlag, New York 1997, electronic edition 2000.
[7] Dutta, J., Basnet, D. K. and Nath, R. K. (2017) On commuting probability of finite rings, Indag. Math. (N.S.), 28(2), 372–382.
[8] Erfanian, A., Khashyarmanesh, K. and Nafar, Kh. (2015) Non-commuting graphs of rings, Discrete Math. Algorithms Appl., 7(3), 1550027-1–1550027-7.
[9] Hall, P. (1940) The classification of prime power groups, J. Reine Angew. Math., 182, 130–141.
[10] MacHale, D. (1976) Commutativity in finite rings, Amer. Math. Monthly, 83, 30–32.
[11] Omidi G. R. and Vatandoost E. (2011) On the commuting graph of rings, J. Algebra Appl., 10(3), 521–527.
[12] Tolue, B. and Erfanian, A. (2013) Relative non-commuting graph of a finite group, J. Algebra Appl., 12(2), 1250157-1–1250157-11.