Collapses and revivals of matter waves

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Abstract

Quantum collapses and revivals are fascinating manifestations of interference. Of particular interest in recent years are macroscopic quantum interference effects in Bose-Einstein condensates. In this letter such effects will be studied for the two site Bose-Hubbard model that is a standard model for exploration of Bose-Einstein condensates. An analytic expression that is valid in the weak coupling limit for the difference in the occupation of the two sites is developed and tested numerically. It describes correctly the collapses and revivals. Moreover, it is demonstrated that a calculation to the first order in the interparticle interaction is required for the prediction of the collapse and revival times while the second order is required for the evaluation of the shape of the revival peaks. We believe that the result is relevant for a variety of situations where collapses and revivals are found.
Collapses and revivals are fascinating wave phenomena. By a collapse one means that a pattern or an expectation value that is initially pronounced, practically vanishes after some time, and by revival we mean that at a latter time the pattern nearly returns to its initial value. In optics such an effect is the Talbot effect discovered nearly 200 years ago [1, 2] (see also [3]). A related phenomenon is the “Quantum carpet” [3, 4]. For the Jaynes-Cummings (JC) model [5] that is central in quantum optics, an expression exhibiting collapses and revivals was found analytically [6] and it is of the form reminiscent of the one found in the present work for a model of interacting bosons in a regime of parameters that is of experimental relevance. Collapses and revivals can be found in many situations. A generic picture is outlined in [7, 8].

In recent years, matter waves such as for Bose-Einstein condensates (BECs) and other systems involving cold atoms were extensively studied [8–10]. Collapses and revivals were observed in experiments where a BEC was confined to a lattice. The interference pattern of the matter wave field originating in various lattice sites showed collapses and revivals as a function of time [11, 12]. These were found also experimentally for other condensates [13, 14]. Collapses were found in numerical calculations in the framework of a theoretical model similar to the model we use here [15]. These studies are related to the double well problem. This is a system defined by a potential with two minima of equal depth separated by a barrier. The potential is infinite at infinity. In the context of the present paper, a large number of bosons $N$ is trapped in this potential. It was studied experimentally [16–19]. In particular, the Bosonic Josephson effect attracted much interest [16, 17], since it is a clear manifestation of macroscopic quantum coherence. It encourages theoretical exploration of this and related systems [15, 20–25]. The double well was explored in detail [26–28]. In particular, it was shown that collapses take place [27] and agreement with the result of the Bose-Hubbard (BH) model was found for inter-particle interactions that are very weak [26]. Collapses and revivals were found theoretically for interacting bosons in a harmonic well [29–33], for wave packets in harmonic wells with small nonlinearities [7, 8, 34], for dynamics of atoms on optical lattices [35, 36] and in experiments on Rydberg atoms [37, 38].

The BH model where bosons can occupy only two sites [39] was studied extensively numerically and analytically [15, 20, 26, 27, 40]. It is an approximation of the double well model in the limit where the inter-particle interaction is sufficiently weak so that only the two lowest levels of the double well are occupied. For the double well, collapses were found
in exact numerical calculations \cite{27} and it was demonstrated that some of the results are similar to those found for the BH model. The static properties of weakly interacting BECs are often described by the Gross-Pitaevskii Equation (GPE). This is not the case for the dynamics as was demonstrated for the double well potential \cite{21}. In particular, it does not reproduce neither collapses nor revivals.

Analytical results on the time-evolution of interacting many-particle systems are relatively rare, and to the best of our knowledge no analytical formula for a collapsing and reviving quantity for a specific model of interacting bosons was derived in a controlled way. The main result of this letter is such an expression for the difference in occupation of the two sites of the BH model for weak interaction \( u \) between the particles and large number of particles \( N \). It is reminiscent of the one found for the JC model where the physics is completely different \cite{6}. In particular, we find that in the first order in \( \frac{1}{N} \) revival and collapse times are found correctly using just the first order in \( u \) while the shape of the reviving peaks requires the second order in \( u \). A detailed version will be published \cite{41}. It is of experimental relevance as demonstrated by various experiments and of conceptual importance for understanding of coherence. The reason is that for the collapse we discuss no information is lost and after some time a revival takes place. It should be distinguished from collapses that often take place in experiments where information is lost.

The calculations in this paper will be preformed in the framework of the two site BH model. It is defined by the Hamiltonian

\[
H_{\text{BH}} = -J \left( a_L^\dagger a_R + a_R^\dagger a_L \right) + U \left[ n_L(n_L - 1) + n_R(n_R - 1) \right]. \tag{1}
\]

The sites are denoted by \( L \) (Left) and \( R \) (Right). The creation and annihilation operators on the sites are \( a_L^\dagger, a_R^\dagger \) and \( a_L, a_R \). The number operators for the two sites are \( n_L = a_L^\dagger a_L \) and \( n_R = a_R^\dagger a_R \). The commutation relations are \( [a_L, a_L^\dagger] = 1, [a_R, a_R^\dagger] = 1 \), and the units are such that \( \hbar = \frac{1}{N} \). It is assumed that the site energies on the two sites are identical. The total number of particles \( n_L + n_R = N \) is conserved. The first term in (1) represents the hopping between the two sites while the second one is the energy of the interparticle interaction. The BH Hamiltonian (1) can be written up to multiplicative and additive constants as

\[
H = -S_x + uS_x^2 \tag{2}
\]

where \( u \equiv \frac{UN}{J} \) and \( \vec{S} = (S_x, S_y, S_z) \). The components of \( \vec{S} \) are

\[
S_x = \frac{1}{2N} \left( a_R^\dagger a_L + a_L^\dagger a_R \right),
\]

\[
S_y = \frac{1}{2N} \left( a_R^\dagger a_L - a_L^\dagger a_R \right),
\]

\[
S_z = \frac{1}{2N} \left( n_L - n_R \right).
\]
\( S_y = \frac{i}{2N} \left( a_R^\dagger a_L - a_L^\dagger a_R \right) \), \( S_z = \frac{1}{2N} \left( a_R^\dagger a_L - a_L^\dagger a_R \right) + \frac{1}{2N} (n_L - n_R) \). They satisfy the standard commutation relations of the angular momentum, with \( \hbar \) replaced by \( \frac{1}{N} \). The relation between \( H_{BH} \) and \( H \) is
\[
H_{BH} = 2JNH + \frac{1}{2} UN^2 - NU. \tag{3}
\]
For large \( N \) the Semiclassical analysis of the dynamics generated by \( H \) is useful \[15, 20, 42, 43\]. The Josephson regime \( 1 < u < N^2 \) was extensively studied. Here we confine ourselves to the Rabi regime \( u < 1 \). First, we note that \( S^2 = \frac{1}{2N} \left( \frac{N}{2} + 1 \right) \) is a constant of motion. The classical dynamics of the vector \( \vec{S} \) is the motion on the Bloch sphere of radius \( \frac{1}{2} \). For \( u < 1 \) it is just a motion with an angle \( \varphi \) around the \( x \) axis, that is \( \vec{S} = (S_x, S_\perp \cos \varphi, S_\perp \sin \varphi) \) with \( S_\perp = \sqrt{1 - S_x^2} \). The Hamiltonian (2) takes the form
\[
H = -S_x + u \left( \frac{1}{4} - S_x^2 \right) \sin^2 \varphi, \tag{4}
\]
where \( S_x \) and \( \varphi \) are conjugate variables. For \( u = 0 \) the angular frequency is \( \dot{\varphi} = \frac{\partial H}{\partial S_x} = -1 \), therefore, in these units the normalized difference in the occupation between the Left and Right sites is proportional to \( S_z \) and to \( \sin t \).

As a result of the contribution of the second term in (4), \( \dot{\varphi} \) is not a constant but exhibits small variations. Classically, the difference between the occupation of the Left and Right sites oscillates with constant amplitude and period. Since \( \varphi \) and \( S_x \) are not angle-action variables, \( \dot{\varphi} \) and \( S_x \) vary with time. The standard transformation to angle-action variables \((I, \bar{\varphi})\) is \( I = \frac{1}{2\pi} \int_0^{2\pi} S_x d\varphi \) and \( \dot{\bar{\varphi}} = \frac{\partial H}{\partial I} \). To order \( u \), the Hamiltonian is
\[
H \approx -I + \frac{1}{8} u - \frac{1}{2} uI^2. \tag{5}
\]
The action variable is quantized \[44\] so that
\[
I_n = \frac{n}{N} \tag{6}
\]
where \( n = -\frac{N}{2}, ..., \frac{N}{2} \) are integers. Note that \( \dot{\varphi} = -\frac{\partial H}{\partial S_x} \approx -1 \) for small \( u \) and therefore \( \dot{\varphi} \) never vanishes. Consequently the Maslov index vanishes. Hence, the spectrum of the Hamiltonian (2) is
\[
E_n^{(1)} \approx -\frac{n}{N} + \frac{1}{8} u - \frac{1}{2N^2} un^2. \tag{7}
\]
The corresponding spectrum of the BH Hamiltonian is
\[
E_n^{(BH)} = 2JNE_n^{(1)} + \frac{1}{2} UN^2 - NU \approx 2J \left( -n + \frac{3}{8} uN - \frac{1}{2} u - \frac{1}{2N} un^2 \right). \tag{8}
\]
The calculation was extended to the second order in $u$, resulting in (for details see [41])

$$E^{(BH2)}_n \approx 2J \left( -n + \frac{3}{8}uN - \frac{1}{2}u - \frac{1}{2N}un^2 - \frac{1}{16}u^2n + \frac{1}{4N^2}u^2n^3 \right). \tag{9}$$

We verified that the expression (9) can be obtained in the framework of standard second order perturbation theory in $u$, where in each term only the leading contribution in $\frac{1}{N}$ was kept. It is important to notice that our result is the contribution up to the order $u^2$ in the semiclassical approximation. It holds for $u < 1$, while the standard perturbation theory requires $uN < 1$. Finally we compared the perturbative results for the spectrum with the ones obtained from direct diagonalization of (1) and found excellent agreement even for $u = 2$. Such results were encountered also in other situations [45].

We turn now to calculate the time dependence of the normalized difference in population between the left and right sites (denoted by $\Delta (t)$). We start from a state where all particles are on the Left site, in this state $\langle S_z \rangle = \frac{1}{2}$. Since $u$ is small, it is convenient to expand this state in terms of eigenstates of $S_x$, and then calculate the correction of second order in $u$ [41]. Such states are

$$|n\rangle \equiv \frac{1}{\sqrt{\left(\frac{N}{2} + n\right)!\left(\frac{N}{2} - n\right)!}} \left(a^+_+^{N+n} a^-_-^{N-n}\right)|0\rangle. \tag{10}$$

where $a^\pm = \frac{1}{\sqrt{2}} \left(a^\pm_L \pm a^\pm_R\right)$ and $[a_+, a_-] = 0$, $[a_+, a^+_+] = 1$, $[a_-, a^+_+] = 1$. The reason for (10) is that $S_x = \frac{1}{2N} \left(a^+_+ a_- - a^+_+ a_-\right)$.

The initial state is

$$|\psi (t = 0)\rangle = \frac{1}{\sqrt{N!}} \left(a^+_L\right)^N |0\rangle = \frac{1}{2^{N/2}N!} \left(a^+_+ + a^+_-\right)^N |0\rangle. \tag{11}$$

It is useful to expand $|\psi (t = 0)\rangle$ in the basis of (10),

$$|\psi (t = 0)\rangle = \sum_{n=-N/2}^{N/2} c_n |n\rangle, \tag{12}$$

and for large $N$

$$c_n \approx \left(\frac{2}{\pi N}\right)^{\frac{1}{4}} e^{-\frac{n^2}{N}}. \tag{13}$$

We note that the normalized difference between the occupation of the two sites is

$$\Delta (t) = \langle \psi \left| S_z \right| \psi \rangle = \frac{1}{N} \text{Re} \left\langle \psi \left| \bar{S}_+ \right| \psi \right\rangle. \tag{14}$$
where \( \tilde{S}_+ \equiv N(S_z - iS_y) = \frac{1}{2} (a_L^+ + a_R^+) (a_L - a_R) = a_L^+ a_R \). In the basis \( \{ |n \rangle \} \), \( \tilde{S}_+ \) is a raising operator, therefore \( \tilde{S}_+ |n \rangle \propto |n+1 \rangle \). For large \( N \),

\[
\left\langle \psi(t) | \tilde{S}_+ | \psi(t) \right\rangle = \frac{N}{2} \sum_{n=-N/2}^{N/2} c_n c_{n+1} e^{-i(E_n^{BH}) - E_{n+1}^{BH})} t. \tag{15}
\]

Substitution of the energies (9) results for large \( N \) in

\[
\left\langle \psi(t) | \tilde{S}_+ | \psi(t) \right\rangle = \tilde{S} e^{-i\phi t} \tag{16}
\]

where the phase \( \phi \) will be specified at a later stage and

\[
\tilde{S} = \frac{\sqrt{N}}{\sqrt{2\pi}} \sum_{n=-N/2}^{N/2} e^{-\frac{2n^2+2n+1}{4N}} e^{-\frac{i}{N} (2un \pi^2/2N)^2} e^{iJn/T_R} \tag{17}
\]

Since \( n \) is an integer, in first order in \( u \), the envelope of the sum (15) is a periodic function of \( t \) with period (revival time) of

\[
T_R = \frac{\pi N}{uJ}. \tag{18}
\]

Around the \( m \)-th revival, we write \( t = m \cdot T_R + \tau \) with \( -\frac{1}{2} T_R < \tau < \frac{1}{2} T_R \) and (17) takes the form

\[
\tilde{S}_m = \frac{\sqrt{N}}{\sqrt{2\pi}} \sum_{n=-N/2}^{N/2} e^{-\frac{(2n^2+2n+1)}{4N}} e^{iJn/T_R} e^{i\phi t}. \tag{19}
\]

Therefore, \( \tilde{S} = \sum_m \tilde{S}_m \). Around each revival the sum can be replaced by an integral since in the vicinity of a revival \( \frac{1}{N} u \tau + \frac{3}{16} u^2 n^2 \pi \) is small (while \( \frac{1}{N} u t \) is typically large). This will be discussed in what follows. Doing the integral over \( n \) for \( -\frac{1}{2} T_R < \tau < \frac{1}{2} T_R \), the final result is (for the detailed calculation see [41]),

\[
\Delta (t) = \left\langle \psi(t) | \tilde{S}_+ | \psi(t) \right\rangle = \frac{1}{2} \sum_m A_m \exp \left[ \frac{- \left( t - mT_R + \frac{3m \pi}{2J} \right)^2}{\left( \Delta t^m_R \right)^2} \right] \cos (\phi_1 - \phi t), \tag{20}
\]

where

\[
\Delta t^m_R \approx \sqrt{2N \left( 1 + \frac{9}{16} u^2 m^2 \pi^2 \right)} \frac{1}{Ju}, \tag{21}
\]

\[
A_m = \frac{e^{-\frac{u^2}{4}}}{\left[ 1 + \frac{9}{16} u^2 m^2 \pi^2 \right]^{1/4}} \exp \left[ \frac{2 + \frac{9}{2} u^2 m^2 \pi^2}{4N \left( 1 + \frac{9}{16} u^2 m^2 \pi^2 \right)} \right], \tag{22}
\]

\[
\phi \approx J \left( 2 + \frac{1}{8} u^2 + \frac{u}{N} \right). \tag{23}
\]
and

\[ \phi_1 \approx \frac{u^2}{8} \left( 2J\tau + \frac{3}{2} m\pi \right) + u \left( \frac{J\tau}{N} + \frac{3}{8} \left( m \cdot \pi + \frac{J}{N} u\tau \right) \right). \quad (24) \]

This is a sequence of Gaussians of width \( \Delta t_R^m \) of order \( \sqrt{N} \) with a separation \( T_R \) of order \( N \). For short times \((m = 0)\), the dynamics is described by

\[ \Delta (t) = \frac{1}{2} e^{-\frac{1}{N} J u^2 t^2} \cos (\phi t - \phi_1) \quad (25) \]

and the collapse time is given by

\[ T_c = \frac{\sqrt{2N}}{J u}. \quad (26) \]

It is important to note that for small \( m \) the width of the Gaussian peaks is of the order \( \sqrt{2N/J u} \), therefore, for such values of \( m \), \( \frac{J}{N} u\tau + \frac{3}{N^2} u^2 n (mT_R + \tau) \leq \sqrt{\frac{2}{N} + \frac{3}{N^2} u n \left( m\pi + \sqrt{\frac{2}{N}} \right)} \) that is much smaller than 1 \((\frac{J}{N} \) is small). Therefore the sum \((19)\) can be approximated by an integral leading to the relatively simple formula \((20)\).

The evolution of the expectation of the normalized difference in occupation of the two sites \( \Delta (t) \) is the main result of the present work. In Fig. 1 it is compared to exact results found by numerical diagonalization of the Hamiltonian \((1)\), for \( u = \frac{1}{2} \), \( N = 100 \) and \( J = 1 \). We note remarkable agreement of the envelope with the exact numerical result. The rapid oscillations, exhibit good agreement for short times (Fig. 1(b)) but it deteriorates for longer times (Fig. 1(c)).

In Fig. 2 the evolution of the normalized difference in occupation between the two sites is presented for \( u = \frac{1}{20} \), \( N = 50 \) and \( J = 1 \). We note also the remarkable agreement between the analytical and numerical results found for the envelope. The prediction for the rapid oscillations agrees with the exact results for longer times and more revivals than in Fig. 1.
Figure 1: (Color online) The normalized difference between the occupation of the two sites $\Delta(t)$ for $J = 1$, $N = 100$ and $u = \frac{1}{2}$. The light gray line represents the numerical result, obtained by diagonalizing the Hamiltonian (1). The black line represents the envelope based on (20). (a) $\Delta(t)$ for the time regime $t < T_B$. The arrows show the time regimes which are presented in (c) and (d). The time $T_R$ of (18) is marked. (b) Long time blurring. The time $T_B = m_{\text{max}} T_R$ where the revivals mix (see Eq. (27)) is marked. (c) Short time dynamics. The red dashed-dot line is given by (20) where $\phi$ and $\phi_1$ are given by (23) and (24). The dashed black line presents oscillations with the unperturbed Rabi’s frequency $2J$ (that is approximating the phase $\phi t - \phi_1$ by $2Jt$) and $T_c$ of (26) is marked. (d) the same as (c) for a time interval near the revival $m = 1$, where the analytical result for the phase $\phi_1 - \phi t$ no longer agrees with the result of exact numerical calculation.
Figure 2: (Color online) Similar to Fig. 1 but for $J = 1$, $N = 50$ and $u = \frac{1}{20}$. (a) $\Delta (t)$ for a time regime $t < T_B$. The arrows show the time regimes which are presented in (b)-(d). (b) Short time dynamics. (c) the same as (b) for a time interval near the revival $m = 2$. (d) the same as (b) for a time interval near the revival $m = 3$, where the analytical result for the phase in (20) no longer agrees with the exact numerical calculation.

For small $m$, $\Delta t^m_R \ll T_R$. However, there is an $m_{\text{max}}$ where the width $\Delta t^m_R$ is comparable to $T_R$ and then the revivals mix and our calculations are not valid. Defining $m_{\text{max}}$ by $\Delta t^{m_{\text{max}}} = \frac{1}{2} T_R$, we estimate

$$m_{\text{max}} = \frac{\sqrt{2 (\pi^2 N - 8)}}{3 u \pi}.$$  \hfill (27)

We checked that indeed for $m > m_{\text{max}}$ the peaks mix and the picture presented in Figs. 1(a) and 2(a) deteriorates. The calculation can be extended to the case where initially both sites are occupied and in some situations a formula similar to (20) is found. If the initial occupation of the left site is given by $\cos^2 \alpha$, it exhibits revivals at times $T_R \left( 1 - \frac{3}{4} u \sin (2\alpha) \right)^{-1}$ where $T_R$ is given by (18) [41].

The main result of this paper is the analytic expression (20) for the normalized difference
between the two sites occupation $\Delta (t)$ of the Bose-Hubbard model defined by (1). It consists of a sequence of Gaussian peaks, superimposed on a rapid oscillation. Comparison between the approximate result and the exact numerical calculation demonstrates that the result obtained indeed requires the terms in order $u^2$ and $\frac{1}{N}$. The classical approximation (4) reproduces correctly the rapid oscillations for short times. Such a behavior is found also for the GPE in double well [15, 21]. Quantization is essential for the collapses and revivals. The collapse and revival times are predicted correctly by the first order in the interaction $u$, however for the width of the peaks the order $u^2$ is required. The population difference exhibits three time scales (superimposing the Rabi oscillations): The collapse time $T_c$ (26), the revival time $T_R$ (18) and $T_B = m_{\text{max}} T_R$ (27) where the revival picture is blurred. If initially both sites are occupied but the imbalance is large, a similar picture emerges but the time scales are different. We can see from (22) that the amplitude of the revival peaks decreases with time (see also Figs. 1 and 2). This decrease is completely coherent.

The result presented in this letter is a fascinating manifestation of macroscopic quantum coherence. It is of great importance for distinguishing collapses resulting of dephasing where quantum coherence is dumped (and consequently the revivals are dumped as well) from the situation presented in this work where quantum revivals are found. The knowledge of the function $\Delta (t)$, and in particular the decrease in the amplitude of the peaks that is completely coherent, can be used to measure the rate of destruction of coherence in experiments. The main result (20) can be used also for the comparison between the Bose-Hubbard model and the double well problem [27]. It is interesting to note that (20) is very similar to the result found in [6] for completely different physics. We believe that the method of the calculation used here can be applied to other physical situations as well, in particular in presence of interactions.

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