On the intersection graph of ideals of a commutative ring

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Abstract

Let $R$ be a commutative ring and $M$ be an $R$-module, and let $I(R)^*$ be the set of all non-trivial ideals of $R$. The $M$-intersection graph of ideals of $R$, denoted by $G_M(R)$, is a graph with the vertex set $I(R)^*$, and two distinct vertices $I$ and $J$ are adjacent if and only if $IM \cap JM \neq 0$. For every multiplication $R$-module $M$, the diameter and the girth of $G_M(R)$ are determined. Among other results, we prove that if $M$ is a faithful $R$-module and the clique number of $G_M(R)$ is finite, then $R$ is a semilocal ring. We denote the $Z_n$-intersection graph of ideals of the ring $Z_m$ by $G_n(Z_m)$, where $n, m \geq 2$ are integers and $Z_n$ is a $Z_m$-module. We determine the values of $n$ and $m$ for which $G_n(Z_m)$ is perfect. Furthermore, we derive a sufficient condition for $G_n(Z_m)$ to be weakly perfect.

1 Introduction

Let $R$ be a commutative ring, and $I(R)^*$ be the set of all non-trivial ideals of $R$. There are many papers on assigning a graph to a ring $R$, for instance see [1–4]. Also the intersection graphs of some algebraic structures such as groups, rings and modules have been studied by several authors, see [3, 6, 8]. In [6], the intersection graph of ideals of $R$, denoted by $G(R)$, was introduced as the graph with vertices $I(R)^*$ and for distinct $I, J \in I(R)^*$, the vertices $I$ and $J$ are adjacent if and only if $I \cap J \neq 0$. Also in [3], the intersection graph of submodules of an $R$-module $M$, denoted by $G(M)$, is defined to be the graph whose vertices are the non-trivial submodules of $M$ and two distinct vertices are adjacent if and only if they have non-zero intersection. In this paper, we generalize $G(R)$ to $G_M(R)$, the

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**M-intersection graph of ideals** of $R$, where $M$ is an $R$-module.

Throughout the paper, all rings are commutative with non-zero identity and all modules are unitary. A module is called a *uniform* module if the intersection of any two non-zero submodules is non-zero. An $R$-module $M$ is said to be a *multiplication* module if every submodule of $M$ is of the form $IM$, for some ideal $I$ of $R$. The *annihilator* of $M$ is denoted by $\text{ann}(M)$. The module $M$ is called a *faithful* $R$-module if $\text{ann}(M) = 0$. By a non-trivial submodule of $M$, we mean a non-zero proper submodule of $M$. Also, $J(R)$ denotes the Jacobson radical of $R$ and $\text{Nil}(R)$ denotes the ideal of all nilpotent elements of $R$. By $\text{Max}(R)$, we denote the set of all maximal ideals of $R$. A ring having only finitely many maximal ideals is said to be a *semilocal* ring. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and the integers modulo $n$, respectively.

A graph in which any two distinct vertices are adjacent is called a *complete graph*. We denote the complete graph on $n$ vertices by $K_n$. A *null graph* is a graph containing no edges. Let $G$ be a graph. The *complement* of $G$ is denoted by $\overline{G}$. The set of vertices and the set of edges of $G$ are denoted by $V(G)$ and $E(G)$, respectively. A subgraph $H$ of $G$ is said to be an *induced subgraph* of $G$ if it has exactly the edges that appear in $G$ over $V(H)$. Also, a subgraph $H$ of $G$ is called a *spanning subgraph* if $V(H) = V(G)$. Suppose that $x, y \in V(G)$. We denote by $\text{deg}(x)$ the degree of a vertex $x$ in $G$. A *regular graph* is a graph where each vertex has the same degree. We recall that a *walk* between $x$ and $y$ is a sequence $x = v_0 - v_1 - \cdots - v_k = y$ of vertices of $G$ such that for every $i$ with $1 \leq i \leq k$, the vertices $v_{i-1}$ and $v_i$ are adjacent. A *path* between $x$ and $y$ is a walk between $x$ and $y$ without repeated vertices. We say that $G$ is *connected* if there is a path between any two distinct vertices of $G$. For vertices $x$ and $y$ of $G$, let $d(x, y)$ be the length of a shortest path from $x$ to $y$ ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path between $x$ and $y$). The *diameter* of $G$, $\text{diam}(G)$, is the supremum of the set $\{d(x, y) : x$ and $y$ are vertices of $G\}$. The *girth* of $G$, denoted by $\text{gr}(G)$, is the length of a shortest cycle in $G$ ($\text{gr}(G) = \infty$ if $G$ contains no cycles). A *clique* in $G$ is a set of pairwise adjacent vertices and the number of vertices in the largest clique of $G$, denoted by $\omega(G)$, is called the *clique number* of $G$. The *chromatic number* of $G$, $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. A graph $G$ is *perfect* if for every induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$. Also, $G$ is called *weakly perfect* if $\chi(G) = \omega(G)$.

In the next section, we introduce the $M$-intersection graph of ideals of $R$, denoted by $G_M(R)$, where $R$ is a commutative ring and $M$ is a non-zero $R$-module. It is shown that for every multiplication $R$-module $M$, $\text{diam}(G_M(R)) \in \{0, 1, 2, \infty\}$ and $\text{gr}(G_M(R)) \in \{3, \infty\}$. 

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Among other results, we prove that if $M$ is a faithful $R$-module and $\omega(G_M(R))$ is finite, then $|\text{Max}(R)| \leq \omega(G_M(R)) + 1$ and $J(R) = \text{Nil}(R)$. In the last section, we consider the $Z_n$-intersection graph of ideals of $Z_m$, denoted by $G_n(Z_m)$, where $n, m \geq 2$ are integers and $Z_n$ is a $Z_m$-module. We show that $G_n(Z_m)$ is a perfect graph if and only if $n$ has at most four distinct prime divisors. Furthermore, we derive a sufficient condition for $G_n(Z_m)$ to be weakly perfect. As a corollary, it is shown that the intersection graph of ideals of $Z_m$ is weakly perfect, for every integer $m \geq 2$.

2 The $M$-intersection graph of ideals of $R$

In this section, we introduce the $M$-intersection graph of ideals of $R$ and study its basic properties.

**Definition.** Let $R$ be a commutative ring and $M$ be a non-zero $R$-module. The $M$-intersection graph of ideals of $R$, denoted by $G_M(R)$, is the graph with vertices $I(R)^*$ and two distinct vertices $I$ and $J$ are adjacent if and only if $IM \cap JM \neq 0$.

Clearly, if $R$ is regarded as a module over itself, that is, $M = R$, then the $M$-intersection graph of ideals of $R$ is exactly the same as the intersection graph of ideals of $R$. Also, if $M$ and $N$ are two isomorphic $R$-modules, then $G_M(R)$ is the same as $G_N(R)$.

**Example 1.** Let $R = Z_{12}$. Then we have the following graphs.

![Graphs](image)

**Example 2.** Let $n \geq 2$ be an integer. If $[m_1, m_2]$ is the least common multiple of two distinct integers $m_1, m_2 \geq 2$, then $m_1Z_n \cap m_2Z_n = m_1Z_n \cap m_2Z_n = [m_1, m_2]Z_n$. Thus $m_1Z$ and $m_2Z$ are adjacent in $G_{Z_n}(Z)$ if and only if $n$ does not divide $[m_1, m_2]$. 

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Example 3. Let \( p \) be a prime number and \( n, m \) be two positive integers. If \( p^n \) divides \( m \), then \( m \mathbb{Z} \) is an isolated vertex of \( G_{\mathbb{Z}_{p^n}}(\mathbb{Z}) \). Therefore, since \( \mathbb{Z}_{p^n} \) is a uniform \( \mathbb{Z} \)-module, so \( G_{\mathbb{Z}_{p^n}}(\mathbb{Z}) \) is a disjoint union of an infinite complete graph and its complement. Also, \( \mathbb{Z}_{p^\infty} \) (the quasi-cyclic \( p \)-group), is a uniform \( \mathbb{Z} \)-module and \( \text{ann}(\mathbb{Z}_{p^\infty}) = 0 \). Hence \( G_{\mathbb{Z}_{p^\infty}}(\mathbb{Z}) \) is an infinite complete graph.

Remark 1. Obviously, if \( M \) is a faithful multiplication \( R \)-module, then \( G_M(R) \) is a complete graph if and only if \( M \) is a uniform \( R \)-module.

Remark 2. Let \( R \) be a commutative ring and let \( M \) be a non-zero \( R \)-module.

(1) If \( M \) is a faithful \( R \)-module, then \( G(R) \) is a spanning subgraph of \( G_M(R) \). To see this, suppose that \( I \) and \( J \) are adjacent vertices of \( G(R) \). Then \( I \cap J \neq 0 \) implies that \( (I \cap J)M \neq 0 \) and so \( IM \cap JM \neq 0 \). Therefore \( I \) is adjacent to \( J \) in \( G_M(R) \).

(2) If \( M \) is a multiplication \( R \)-module, then \( G(M) \) is an induced subgraph of \( G_M(R) \). Note that for each non-trivial submodule \( N \) of \( M \), there is a non-trivial ideal \( I \) of \( R \), such that \( N = IM \) and so we can assign \( N \) to \( I \). Also, \( N = IM \) is adjacent to \( K = JM \) in \( G(M) \) if and only if \( IM \cap JM \neq 0 \), that is, if and only if \( I \) is adjacent to \( J \) in \( G_M(R) \).

Theorem 1. Let \( R \) be a commutative ring and let \( M \) be a faithful \( R \)-module. If \( G_M(R) \) is not connected, then \( M \) is a direct sum of two \( R \)-modules.

Proof. Suppose that \( C_1 \) and \( C_2 \) are two distinct components of \( G_M(R) \). Let \( I \in C_1 \) and \( J \in C_2 \). Since \( M \) is a faithful \( R \)-module, so \( IM \cap JM = 0 \) implies that \( I \not\subseteq J \) and \( J \not\subseteq I \). Now if \( I + J \neq R \), then \( I - I + J - J \) is a path between \( I \) and \( J \), a contradiction. Thus \( I + J = R \) and so \( M = IM \oplus JM \).

The next theorem shows that for every multiplication \( R \)-module \( M \), the diameter of \( G_M(R) \) has 4 possibilities.

Theorem 2. Let \( R \) be a commutative ring and \( M \) be a multiplication \( R \)-module. Then \( \text{diam}(G_M(R)) \in \{0, 1, 2, \infty\} \).

Proof. Assume that \( G_M(R) \) is a connected graph with at least two vertices. So \( M \) is a faithful module. If there is a non-trivial ideal \( I \) of \( R \) such that \( IM = M \), then \( I \) is
adjacent to all other vertices. Hence $\text{diam}(G_M(R)) \leq 2$. Otherwise, we claim that $G(M)$ is connected. Let $N$ and $K$ be two distinct vertices of $G(M)$. Since $M$ is a multiplication module, so $N = IM$ and $K = JM$, for some non-trivial ideals $I$ and $J$ of $R$. Suppose that $I = I_1 - I_2 - \cdots - I_n = J$ is a path between $I$ and $J$ in $G_M(R)$. Therefore, $N = I_2M - \cdots - I_{n-1}M - K$ is a walk between $N$ and $K$. Thus, we conclude that there is also a path between $N$ and $K$ in $G(M)$. The claim is proved. So by [3, Theorem 2.4], $\text{diam}(G(M)) \leq 2$. Now, suppose that $I_1$ and $I_2$ are two distinct vertices of $G_M(R)$. If $I_1M \cap I_2M = 0$, then $I_1M$ and $I_2M$ are two distinct vertices of $G(M)$. Hence there exists a non-trivial submodule $N$ of $M$ which is adjacent to both $I_1M$ and $I_2M$ in $G(M)$. Since $M$ is a multiplication module, so $N = JM$, for some non-trivial ideal $J$ of $R$. Thus $J$ is adjacent to both $I_1$ and $I_2$ in $G_M(R)$. Therefore $\text{diam}(G_M(R)) \leq 2$. 

\textbf{Theorem 3.} Let $R$ be a commutative ring and $M$ be a multiplication $R$-module. If $G_M(R)$ is a connected regular graph of finite degree, then $G_M(R)$ is a complete graph.

\textbf{Proof.} Suppose that $G_M(R)$ is a connected regular graph of finite degree. If $\text{ann}(M) \neq 0$, then $G_M(R) = K_1$. So assume that $\text{ann}(M) = 0$. We claim that $M$ is an Artinian module. Suppose to the contrary that $M$ is not an Artinian module. Then there is a descending chain $I_1M \supset I_2M \supset \cdots \supset I_nM \supset \cdots$ of submodules of $M$, where $I_i$’s are non-trivial ideals of $R$. This implies that $\text{deg}(I_1)$ is infinite, a contradiction. The claim is proved. Therefore $M$ has at least one minimal submodule. To complete the proof, it suffices to show that $M$ contains a unique minimal submodule. By contrary, suppose that $N_1$ and $N_2$ are two distinct minimal submodules of $M$. Hence $N_1 = I_1M$ and $N_2 = I_2M$, where $I_1$ and $I_2$ are two non-trivial ideals of $R$. Since $N_1 \cap N_2 = 0$, so $I_1$ and $I_2$ are not adjacent. By Theorem 2 there is a vertex $J$ which is adjacent to both $I_1$ and $I_2$. So both $I_1M$ and $I_2M$ are contained in $JM$. Thus each vertex adjacent to $I_1$ is adjacent to $J$ too. This implies that $\text{deg}(J) > \text{deg}(I_1)$, a contradiction.

Also, the following theorem shows that for every multiplication $R$-module $M$, the girth of $G_M(R)$ has 2 possibilities.

\textbf{Theorem 4.} Let $R$ be a commutative ring and $M$ be a multiplication $R$-module. Then $\text{gr}(G_M(R)) \in \{3, \infty\}$.

\textbf{Proof.} Suppose that $I_1 - I_2 - \cdots - I_n - I_1$ is a cycle of length $n$ in $G_M(R)$. If $n = 3$, we are done. Thus assume that $n \geq 4$. Since $I_1M \cap I_2M \neq 0$ and $M$ is a multiplication
module, we have $I_1M \cap I_2M = JM$, where $J$ is a non-zero ideal of $R$. If $J$ is a proper ideal of $R$ and $J \neq I_1, I_2$, then $I_1 - J - I_2 - I_1$ is a triangle in $G_M(R)$. Otherwise, we conclude that $I_1M \subseteq I_2M$ or $I_2M \subseteq I_1M$. Similarly, we can assume that $I_iM \subseteq I_{i+1}M$ or $I_{i+1}M \subseteq I_iM$, for every $i$, $1 < i < n$. Without loss of generality suppose that $I_1M \subseteq I_2M$.

Now, if $I_2M \subseteq I_3M$, then $I_1 - I_2 - I_3 - I_1$ is a cycle of length 3 in $G_M(R)$. Therefore assume that $I_3M \subseteq I_2M$. Since $I_3M \subseteq I_4M$ or $I_4M \subseteq I_3M$, so $I_2 - I_3 - I_4 - I_2$ is a triangle in $G_M(R)$. Hence if $G_M(R)$ contains a cycle, then $gr(G_M(R)) = 3$.  

**Lemma 1.** Let $R$ be a commutative ring and $M$ be a non-zero $R$-module. If $I$ is an isolated vertex of $G_M(R)$, then the following hold:

1. $I$ is a maximal ideal of $R$ or $I \subseteq \text{ann}(M)$.
2. If $I \not\subseteq \text{ann}(M)$, then $I = Ra$, for every $a \in I \setminus \text{ann}(M)$.

**Proof.** (1) There is a maximal ideal $m$ of $R$ such that $I \subseteq m$. Assume that $I \neq m$. Then we have $IM = IM \cap mM = 0$, since $I$ is an isolated vertex. So $I \subseteq \text{ann}(M)$.

(2) Suppose that $a \in I \setminus \text{ann}(M)$ and $I \neq Ra$. Since $I$ is an isolated vertex, we have $RaM = IM \cap RaM = 0$ and so $a \in \text{ann}(M)$, a contradiction. Thus $I = Ra$.  

**Theorem 5.** Let $R$ be a commutative ring and $M$ be a faithful $R$-module. If $G_M(R)$ is a null graph, then it has at most two vertices and $R$ is isomorphic to one of the following rings:

1. $F_1 \times F_2$, where $F_1$ and $F_2$ are fields;
2. $F[x]/(x^2)$, where $F$ is a field;
3. $L$, where $L$ is a coefficient ring of characteristic $p^2$, for some prime number $p$.

**Proof.** By Lemma 1, every non-trivial ideal of $R$ is maximal and so by [10, Theorem 1.1], $R$ cannot have more than two different non-trivial ideals. Thus $G_M(R)$ has at most two vertices. Also, by [11, Theorem 4], $R$ is isomorphic to one of the mentioned rings.  

In the next theorem we show that if $M$ is a faithful $R$-module and $\omega(G_M(R)) < \infty$, then $R$ is a semilocal ring.
Theorem 6. Let $R$ be a commutative ring and $M$ be a faithful $R$-module. If $\omega(G_M(R))$ is finite then $|\text{Max}(R)| \leq \omega(G_M(R)) + 1$ and $J(R) = \text{Nil}(R)$.

Proof. First we prove that $|\text{Max}(R)| \leq \omega(G_M(R)) + 1$. Let $\omega = \omega(G_M(R))$. By contradiction, assume that $m_1, \ldots, m_{\omega+2}$ are distinct maximal ideals of $R$. We know that $m_1 \cdot m_i \neq 0$, for every $i$, $1 \leq i \leq \omega + 1$. Otherwise, $m_1 \cdot m_j = 0$, for some $j$, $1 \leq j \leq \omega + 1$. So $m_1 \cdot m_j \subseteq m_{j+1}$ and hence by Prime Avoidance Theorem [5, Proposition 1.11], we have $m_t \subseteq m_{j+1}$, for some $t, 1 \leq t \leq j$, which is impossible. This implies that $\{m_1, m_1m_2, \ldots, m_1 \cdots m_{\omega+1}\}$ is a clique in $G_M(R)$, a contradiction. Thus $|\text{Max}(R)| \leq \omega + 1$.

Now, we prove that $J(R) = \text{Nil}(R)$. By contrary, suppose that $a \in J(R) \setminus \text{Nil}(R)$. Since $Ra^t M \cap Ra^j M \neq 0$, for every $i, j, i < j$ and $\omega(G_M(R))$ is finite, we conclude that $Ra^t = Ra^s$, for some integers $t < s$. Hence $a^t(1 - ra^{s-t}) = 0$, for some $r \in R$. Since $a \in J(R)$, so $1 - ra^{s-t}$ is a unit. This yields that $a^t = 0$, a contradiction. The proof is complete. $\Box$

3 The $\mathbb{Z}_n$-intersection graph of ideals of $\mathbb{Z}_m$

Let $n, m \geq 2$ be two integers and $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. In this section we study the $\mathbb{Z}_n$-intersection graph of ideals of the ring $\mathbb{Z}_m$. Also, we generalize some results given in [9]. For abbreviation, we denote $G_{\mathbb{Z}_n} (\mathbb{Z}_m)$ by $G_n(\mathbb{Z}_m)$. Clearly, $\mathbb{Z}_n$ is a $\mathbb{Z}_m$-module if and only if $n$ divides $m$.

Throughout this section, without loss of generality, we assume that $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ and $n = p_1^{\beta_1} \cdots p_s^{\beta_s}$, where $p_i$'s are distinct primes, $\alpha_i$'s are positive integers, $\beta_i$'s are non-negative integers, and $0 \leq \beta_i \leq \alpha_i$ for $i = 1, \ldots, s$. Let $S = \{1, \ldots, s\}$ and $S' = \{i \in S : \beta_i \neq 0\}$. The cardinality of $S'$ is denoted by $s'$. For two integers $a$ and $b$, we write $a \mid b$ ($a \nmid b$) if $a$ divides $b$ ($a$ does not divide $b$).

First we have the following remarks.

Remark 3. It is easy to see that $I(\mathbb{Z}_m) = \{d\mathbb{Z}_m : d \text{ divides } m\}$ and $|I(\mathbb{Z}_m)^*| = \prod_{i=1}^{s}(\alpha_i + 1) - 2$. Let $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. If $n \mid d$, then $d\mathbb{Z}_m$ is an isolated vertex of $G_n(\mathbb{Z}_m)$. Obviously, $d_1\mathbb{Z}_m$ and $d_2\mathbb{Z}_m$ are adjacent if and only if $n \nmid [d_1, d_2]$. This implies that $G_n(\mathbb{Z}_m)$ is a subgraph of $G(\mathbb{Z}_m)$.

Remark 4. Let $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module and $d = p_1^{\gamma_1} \cdots p_s^{\gamma_s}(\neq 1, m)$ be a divisor of $m$. We set $D_d = \{i \in S : r_i < \beta_i\}$. Clearly, $D_d \subseteq S'$. Suppose that $W$ is a clique of $G_n(\mathbb{Z}_m)$.
Then $\Gamma_W = \{ D_d : d\mathbb{Z}_m \in W \}$ is an intersecting family of subsets of $S'$. (A family of sets is intersecting if any two of its sets have a non-empty intersection.) Also, if $\Gamma$ is an intersecting family of subsets of $S'$ and $W_\Gamma = \{ d\mathbb{Z}_m : d \neq 1, m, d|m, D_d \in \Gamma \}$ is non-empty, then $W_\Gamma$ is a clique of $G_n(\mathbb{Z}_m)$. (If $D$ is a non-empty subset of $S'$ and $\Gamma = \{ D \}$, then we will denote $W_\Gamma$ by $W_D$.) Thus we have

$$\omega(G_n(\mathbb{Z}_m)) = \max \{ |W_\Gamma| : \Gamma \text{ is an intersecting family of subsets of } S' \}.$$ 

Now, we provide a lower bound for the clique number of $G_n(\mathbb{Z}_m)$.

**Theorem 7.** Let $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. Then

$$\omega(G_n(\mathbb{Z}_m)) \geq \max \left\{ \beta_j \prod_{i \neq j} (\alpha_i + 1) - 1 : \beta_j \neq 0 \right\}.$$ 

**Proof.** Suppose that $\beta_j \neq 0$. With the notations of the previous remark, let $\Gamma = \{ D \subseteq S' : j \in D \}$. Then $\Gamma$ is an intersecting family of subsets of $S'$ and so $W_\Gamma$ is a clique of $G_n(\mathbb{Z}_m)$. Clearly, $|W_\Gamma| = \beta_j \prod_{i \neq j} (\alpha_i + 1) - 1$. Therefore $\omega(G_n(\mathbb{Z}_m)) \geq \beta_j \prod_{i \neq j} (\alpha_i + 1) - 1$ and hence the result holds. $\square$

Clearly, if $n = p_1^{\beta_1}$ ($\beta_1 > 1$), then equality holds in the previous theorem. Also, if $n$ has only two distinct prime divisors, that is, $s' = 2$, then again equality holds. So the lower bound is sharp.

**Example 4.** Let $m = n = p_1^2 p_2^2 p_3^2$, where $p_1, p_2, p_3$ are distinct primes. Thus $S' = S = \{1, 2, 3\}$ and $G_n(\mathbb{Z}_m) = G(\mathbb{Z}_m)$. It is easy to see that $|W_{\{1\}}| = |W_{\{2\}}| = |W_{\{3\}}| = 2$ and $|W_{\{1,2\}}| = |W_{\{1,3\}}| = |W_{\{2,3\}}| = 4$. Also, $|W_{\{1,2,3\}}| = 7$. Let $\Gamma_j = \{ D \subseteq S' : j \in D \}$, for $j = 1, 2, 3$. Hence $|W_{\Gamma_j}| = 17$, for $j = 1, 2, 3$. If $\Gamma = \{ \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$, then $|W_\Gamma| = 19$. Therefore $\omega(G(\mathbb{Z}_m)) = 19$.

By the strong perfect graph theorem, we determine the values of $n$ and $m$ for which $G_n(\mathbb{Z}_m)$ is a perfect graph.

**Theorem A.** (The Strong Perfect Graph Theorem [7]) A finite graph $G$ is perfect if and only if neither $G$ nor $\overline{G}$ contains an induced odd cycle of length at least 5.

**Theorem 8.** Let $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. Then $G_n(\mathbb{Z}_m)$ is perfect if and only if $n$ has at most four distinct prime divisors.
Proof. First suppose that \( s' \geq 5 \) and \( n = p_1^{\beta_1} \cdots p_{s'}^{\beta_{s'}} \), where \( p_i \)'s are distinct primes and \( \beta_i \)'s are positive integers. Let \( D_1 = \{p_1, p_5\} \), \( D_2 = \{p_1, p_2\} \), \( D_3 = \{p_2, p_3\} \), \( D_4 = \{p_3, p_4\} \), and \( D_5 = \{p_4, p_5\} \). Now, assume that \( d_i \mathbb{Z}_m \in W_{D_i} \), for \( i = 1, \ldots, 5 \). Hence \( d_1 \mathbb{Z}_m - d_2 \mathbb{Z}_m - d_3 \mathbb{Z}_m - d_4 \mathbb{Z}_m - d_5 \mathbb{Z}_m - d_1 \mathbb{Z}_m \) is an induced cycle of length 5 in \( G_n(\mathbb{Z}_m) \). So by Theorem \( A \), \( G_n(\mathbb{Z}_m) \) is not a perfect graph.

Conversely, suppose that \( G_n(\mathbb{Z}_m) \) is not a perfect graph. Then by Theorem \( A \) we have the following cases:

Case 1. \( d_1 \mathbb{Z}_m - d_2 \mathbb{Z}_m - d_3 \mathbb{Z}_m - d_4 \mathbb{Z}_m - d_5 \mathbb{Z}_m \) is an induced cycle of length 5 in \( G_n(\mathbb{Z}_m) \). Let \( D_i = D_{d_i} \), for \( i = 1, \ldots, 5 \). So \( D_5 \cap D_1 \not= \emptyset \) and \( D_i \cap D_{i+1} \not= \emptyset \), for \( i = 1, \ldots, 4 \). Let \( p_5 \in D_5 \cap D_1 \) and \( p_i \in D_i \cap D_{i+1} \), for \( i = 1, \ldots, 4 \). Clearly, \( p_1, \ldots, p_5 \) are distinct and thus \( s' \geq 5 \).

Case 2. \( d_1 \mathbb{Z}_m - d_2 \mathbb{Z}_m - d_3 \mathbb{Z}_m - d_4 \mathbb{Z}_m - d_5 \mathbb{Z}_m - d_6 \mathbb{Z}_m \) is an induced path of length 5 in \( G_n(\mathbb{Z}_m) \). Let \( D_i = D_{d_i} \), for \( i = 1, \ldots, 6 \). So \( D_i \cap D_{i+1} \not= \emptyset \), for \( i = 1, \ldots, 5 \). Let \( p_i \in D_i \cap D_{i+1} \), for \( i = 1, \ldots, 5 \). Clearly, \( p_1, \ldots, p_5 \) are distinct and hence \( s' \geq 5 \).

Case 3. There is an induced cycle of length 5 in \( G_n(\mathbb{Z}_m) \). So \( G_n(\mathbb{Z}_m) \) contains an induced cycle of length 5 and by Case 1, we are done.

Case 4. \( d_1 \mathbb{Z}_m - d_2 \mathbb{Z}_m - d_3 \mathbb{Z}_m - d_4 \mathbb{Z}_m - d_5 \mathbb{Z}_m - d_6 \mathbb{Z}_m \) is an induced path of length 5 in \( G_n(\mathbb{Z}_m) \). Since \( D_{d_i} \cap D_{d_j} \not= \emptyset \), \( D_{d_i} \cap D_{d_j} \not= \emptyset \) and \( D_{d_3} \cap D_{d_4} = \emptyset \), we may assume that \( \{p_1, p_2\} \subseteq D_{d_1} \), where \( p_1 \in D_{d_3} \) and \( p_2 \in D_{d_4} \), for some distinct \( p_1, p_2 \in S' \). Similarly, we find that \( \{p_3, p_4\} \subseteq D_{d_2} \), for some distinct \( p_3, p_4 \in S' \setminus \{p_1, p_2\} \) and also \( |D_{d_3}| \geq 2 \). Now, since \( D_{d_3} \cap D_{d_2} = \emptyset \) and \( p_2 \not\in D_{d_3} \), we deduce that \( s' \geq 5 \). \( \square \)

Corollary 1. The graph \( G(\mathbb{Z}_m) \) is perfect if and only if \( m \) has at most four distinct prime divisors.

In the next theorem, we derive a sufficient condition for \( G_n(\mathbb{Z}_m) \) to be weakly perfect.

Theorem 9. Let \( \mathbb{Z}_m \) be a \( \mathbb{Z}_m \)-module. If \( \alpha_i \leq 2\beta_i - 1 \) for each \( i \in S' \), then \( G_n(\mathbb{Z}_m) \) is weakly perfect.

Proof. Let \( D \) be a non-empty subset of \( S' \) and \( \overline{D} = S' \setminus D \). As we mentioned in Remark 4 if \( W_{D} \) is non-empty, then \( W_{D} \) is a clique of \( G_n(\mathbb{Z}_m) \). Also, the vertices of \( W_{S'} \) (if \( S' \not= \emptyset \)) are adjacent to all non-isolated vertices. Suppose that \( D_1 \) and \( D_2 \) are two non-empty subsets of \( S' \) and \( D_1 \subseteq D_2 \). Since \( \alpha_i \leq 2\beta_i - 1 \) for each \( i \in S' \), so \( \Pi_{i \in D_1 \setminus D'_1} (\alpha_i - \beta_i + 1) \leq \Pi_{i \in D_1 \setminus D_1} (\alpha_i - \beta_i + 1) \leq \Pi_{i \in D_2 \setminus D_1} (\beta_i) \leq \Pi_{i \in D_1 \setminus D_1} (\alpha_i - \beta_i + 1) \leq \Pi_{i \in D_2 \setminus D_1} (\alpha_i - \beta_i + 1) \) and hence \( |W_{D_1}| \leq |W_{D_2}| \).
Let $\Gamma$ be an intersecting family of subsets of $S'$ and $\omega(G_n(Z_m)) = |W_\Gamma|$. Let $D \subseteq S'$. We show that $D \in \Gamma$ or $\overline{D} \in \Gamma$. Assume that $D \notin \Gamma$. So there is $D_1 \in \Gamma$ such that $D \cap D_1 = \emptyset$. Thus $D_1 \subseteq \overline{D}$ and hence $\overline{D} \in \Gamma$. We claim that $|W_\overline{D}| \leq |W_D|$, for each $D \in \Gamma$. Suppose to the contrary, $D \in \Gamma$ and $|W_\overline{D}| > |W_D|$. If $A \in \Gamma$ and $A \subseteq D$, then $\overline{D} \subseteq \overline{A}$. So we have $|W_A| \leq |W_D| < |W_\overline{D}| \leq |W_D|$. Let $\Phi = \Gamma \cup \{A: A \in \Gamma, A \subseteq D\} \{A \in \Gamma: A \subseteq D\}$. Then $\Phi$ is an intersecting family of subsets of $S'$ and $|W_\Gamma| < |W_\Phi|$, a contradiction. The claim is proved.

Now, we show that $G_n(Z_m)$ has a proper $|W_\Gamma|$-vertex coloring. First we color all vertices of $W_\Gamma$ with different colors. Next we color each family $W_D$ of vertices out of $W_\Gamma$ with colors of vertices of $W_\overline{D}$. Note that if $D \notin \Gamma$, then $\overline{D} \in \Gamma$ and $|W_D| \leq |W_\overline{D}|$. Suppose that $d_1 Z_m$ and $d_2 Z_m$ are two adjacent vertices of $G_n(Z_m)$. Thus $D_{d_1} \cap D_{d_2} \neq \emptyset$. Without loss of generality, one can assume $D_{d_1} \neq D_{d_2}$. So we deduce that $\overline{D}_{d_1} \neq \overline{D}_{d_2}$ and $D_{d_1} \neq \overline{D}_{d_2}$. Therefore, $d_1 Z_m$ and $d_2 Z_m$ have different colors. Thus $\chi(G_n(Z_m)) \leq |W_\Gamma|$ and hence $\omega(G_n(Z_m)) = \chi(G_n(Z_m)) = |W_\Gamma|$. $\square$

As an immediate consequence of the previous theorem, we have the next result.

**Corollary 2.** The graph $G(Z_m)$ is weakly perfect, for every integer $m \geq 2$.

In the case that $\alpha_i = 2\beta_i - 1$ for each $i \in S'$, we determine the exact value of $\chi(G_n(Z_m))$. It is exactly the lower bound obtained in the Theorem [7].

**Theorem 10.** Let $Z_m$ be a $Z_m$-module. If $\alpha_i = 2\beta_i - 1$ for each $i \in S'$, then $\omega(G_n(Z_m)) = \chi(G_n(Z_m)) = 2^{s'-1} \prod_{i \in S'} \beta_i \prod_{i \in S \setminus S'} (\alpha_i + 1) - 1$.

**Proof.** Let $D \neq \emptyset$ be a proper subset of $S'$. Then $|W_D| = \prod_{i \in D} \beta_i \prod_{i \notin D} (\alpha_i - \beta_i + 1) = \prod_{i \in S'} \beta_i \prod_{i \in S \setminus S'} (\alpha_i + 1)$ and hence $|W_D| = |W_\overline{D}|$. Also, the vertices of $W_S'$ (if $W_S' \neq \emptyset$) are adjacent to all non-isolated vertices and $|W_S'| = \prod_{i \in S'} \beta_i \prod_{i \in S \setminus S'} (\alpha_i + 1) - 1$. Clearly if $\Gamma$ is an intersecting family of subsets of $S'$, then $|\Gamma| \leq 2^{s'-1}$. Moreover, if $\beta_j \neq 0$ and $\Gamma_j = \{D \subseteq S': j \in D\}$, then $|\Gamma_j| = 2^{s'-1}$. Thus by Theorem [9] $\omega(G_n(Z_m)) = \chi(G_n(Z_m)) = |W_{\Gamma_j}| = 2^{s'-1} \prod_{i \in S'} \beta_i \prod_{i \in S \setminus S'} (\alpha_i + 1) - 1$. $\square$

**Corollary 3.** Let $m = p_1 \cdots p_s$, where $p_i$'s are distinct primes. Then $\omega(G(Z_m)) = \chi(G(Z_m)) = 2^{s-1} - 1$.

We close this article by the following problem.
Problem. Let $\mathbb{Z}_n$ be a $\mathbb{Z}_m$-module. Then is it true that $G_n(\mathbb{Z}_m)$ is a weakly perfect graph?

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