Statistical Model for the Nucleon Structure Functions

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Abstract

A phenomenological model for the nucleon structure functions is presented. Visualising the nucleon as a cavity filled with parton gas in equilibrium and parametrizing the effects due to the finiteness of the nucleon volume, we obtain a good fit to the data on the structure function $F_2^p$. The model then successfully predicts other unpolarized structure function data.

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Recent experiments have revealed some remarkable features of the nucleon structure functions $F_2^{p,n}$. Data on deep inelastic scattering of muons off proton and deuteron targets show that the quark sea in the nucleon is not flavor-symmetric, $\bar{u}(x) \neq \bar{d}(x)$; the Gottfried sum $S_G \equiv \int (F_2^p - F_2^n) (dx/x)$, at $Q^2 = 4 \text{ GeV}^2$, has the value $0.235 \pm 0.026$ compared to the usual quark model prediction of $1/3$. This result has been confirmed by the observed asymmetry in Drell-Yan production of dileptons in $pp$ and $pn$ collisions. Most notably, the HERA electron-proton scattering data reveal a rapid rise of the proton structure function $F_2^p(x)$ as $x$ decreases. Indeed over a wide range of small $x$, data from the various groups, for fixed $Q^2$, are all well described by a single inverse power of $x$. Figure 1 is a log-log plot of the data on $F_2^p(x)/x$ (the combination that enters $S_G$) versus $x$. We see that, for fixed $Q^2$, the data fall on straight lines defined by

$$\frac{F_2^p(x)}{x} = \frac{c}{x^m}, \quad (0.0004 \lesssim x \lesssim 0.2). \tag{1}$$

For instance, at $Q^2 = 15 \text{ GeV}^2$, the best-fit parameters are $c = 0.229 \pm 0.005$ and $m = 1.22 \pm 0.01$. In Fig. 1, the straight-line extrapolations in the unexplored low-$x$ region are our predictions based on Eq. (1).

Global fits to the nucleon structure data involve parametrizing the various parton distributions at some low $Q^2$ and evolving them to higher $Q^2$ relevant to observations. The fits so obtained have very high precision but contain several (typically $\sim 15-20$) arbitrary parameters and provide little physical insight into the structure of the nucleon. On the other hand, phenomenological models could give us some valuable clues into the physics of parton distributions in the nucleon. From this point of view the parton gas models or the statistical models of the structure functions have been quite interesting due to their intuitive appeal and simplicity. Bickerstaff and Londergan have provided a strong justification for the general philosophy of the statistical models and have also discussed limitations of other

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1For $Q^2 = 35, 120 \text{ GeV}^2$ the power-law fits give, respectively, $c = 0.229 \pm 0.006, 0.163 \pm 0.037$ and $m = 1.26 \pm 0.01, 1.35 \pm 0.04$. 
We present here a phenomenological model for the unpolarized nucleon structure functions by using ideas from statistical mechanics. As a starting point, we assume that inside the nucleon, the valence quarks together with the sea quarks, antiquarks and gluons constitute a noninteracting gas in equilibrium. (It may be recalled that particles are treated as noninteracting even in the quark-parton model of Feynman.) This simple picture is then improved upon in two respects: (i) Finite-size (of the nucleon) corrections (FSC) to the statistical expression for the number of states per unit energy interval are taken into account, and (ii) the resulting structure functions are evolved to the experimental values of $Q^2$ by using the standard techniques in quantum chromodynamics (QCD). Each of these two effects is shown to play an essential role. To our knowledge, calculation of the structure functions taking into account FSC has never been reported in the literature. The model reproduces all unpolarized structure function data from $x \approx 1$ to $x \approx 10^{-4}$ quite well.

The Model

We picture the nucleon (mass $M$) to consist of a gas of massless partons (quarks, antiquarks and gluons) in equilibrium at temperature $T$ in a spherical volume $V$ with radius $R$. We consider two frames, the proton rest frame and the infinite-momentum frame (IMF) moving with velocity $-v (\approx -1)$ along the common $z$ axis. Our interest lies in the limit when the Lorentz factor $\gamma \equiv (1 - v^2)^{-1/2} \to \infty$. The invariant parton number density in phase space \[9\] is given by (quantities in the IMF are denoted by the index $i$)\[
\frac{dn^i}{d^3p^i d^3r^i} = \frac{dn}{d^3p d^3r} = \frac{gf(E)}{(2\pi)^3},
\]where $g$ is the degeneracy ($g = 16$ for gluons and $g = 6$ for $q$ or $\bar{q}$ of a given flavor), $(E, p)$ is the parton four-momentum and $f(E) = \{\exp[\beta(E - \mu)] \pm 1\}^{-1}$ is the usual Fermi or Bose distribution function with $\beta \equiv T^{-1}$. In order to obtain the number distribution $dn^i/dx$ in the Bjorken scaling variable $x = p^i_z/(Mv\gamma)$, we note that\[
d^3p^i d^3r^i = 2\pi p^i_x dp^i_x (Mv\gamma dx) d^3r/\gamma = 2\pi M^2 x dE \, dx \, d^3r,
\]
\[ d^3 p \ d^3 r = 4 \pi E^2 \ dE d^3 r, \]

for massless partons. For fixed \( x \) the parton energy \( E \) varies between the kinematic limits \((xM/2)\) and \((M/2)\), where the lower limit is attained when \( p_T = 0 \) and the upper limit follows simply from energy-momentum considerations. (The kinematics will be described in detail in [10].) Hence \( dn^i/dx \) in the IMF is related to \( dn/dE \) and \( f(E) \) in the proton rest frame as follows

\[
\frac{dn^i}{dx} = \frac{M^2 x}{2} \int_{xM/2}^{M/2} \frac{dE}{E^2} \frac{dn}{dE} = g V M^2 x \int_{xM/2}^{M/2} E f(E). \tag{3}
\]

The structure function \( F_2(x) \) is given by

\[
F_2(x) = x \sum_{q} e_q^2 \left\{ \left( \frac{dn^i}{dx} \right)_q + \left( \frac{dn^i}{dx} \right)_{\bar{q}} \right\}.
\]

The number distribution \( dn^i/dx \) in Eq. (3) vanishes linearly as \( x \to 0 \) (and also as \( x \to 1 \)) and leads to the behavior of the structure function \( F_2(x) \sim x^2 \) at small \( x \), which disagrees with the observations noted in Eq. (1).

In an attempt to obtain the rise of \( F_2^p(x) \) at small \( x \), we now explore the effects arising from the finiteness of the nucleon volume \( V \). We note from Eq. (2) that \( dn/dE = gf(E)VE^2/2\pi^2 \), which is strictly valid in the large-volume limit, i.e., when the surface and curvature terms are negligible. We shall modify the model by incorporating these subleading terms. Various studies [11] of finite-size corrections (FSC) show that they depend on the particular equation of motion that is employed and are sensitive to the precise shape and size of the enclosure, the type of boundary conditions imposed on the wave function, and to the details such as whether the particles are strictly massless. Moreover, these studies invariably involve some simplifying assumptions and thus a blind use of their results cannot be justified in the present context. We feel more work needs to be done to fully understand the finite-size effects in a QCD bound state such as the nucleon.

In keeping with the phenomenological nature of the model, we have incorporated the FSC by rewriting \( dn/dE \) in Eq. (3) as in [11]:

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\[ \frac{dn}{dE} = gf(E)(VE^2/2\pi^2 + aR^2E + bR), \] (4)

and treating the numerical coefficients \( a \) and \( b \) in the surface and curvature terms as free parameters for reasons stated in the previous paragraph.

The model described above is assumed to hold at a certain input momentum scale \( Q_0^2 \), and if necessary can be evolved to higher \( Q^2 \) by means of the standard techniques in QCD. To complete the statement of the model, we require the parton distributions to obey the following three constraints at the input scale. The constraints on the net quark numbers in the proton are \( n_u - n_{\bar{u}} = 2 \) and \( n_d - n_{\bar{d}} = 1 \), i.e.,

\[
\frac{M^2}{2} \int_0^1 dx \int_{xM/2}^{M/2} dE \left\{ \left( \frac{dn}{dE} \right)_\alpha - \left( \frac{dn}{dE} \right)_{\bar{\alpha}} \right\} = n_\alpha - n_{\bar{\alpha}}. \quad (\alpha = u, d) \tag{5}
\]

Obviously, chemical potentials for heavy flavors are necessarily zero. As regards the third constraint, we assume that the longitudinal momentum fractions in the \( u, d \) flavors and the gluons add up to unity:

\[
\frac{M^2}{2} \int_0^1 dx \int_{xM/2}^{M/2} dE \left\{ \left( \frac{dn}{dE} \right)_u + \left( \frac{dn}{dE} \right)_{\bar{u}} + \left( \frac{dn}{dE} \right)_d + \left( \frac{dn}{dE} \right)_{\bar{d}} + \left( \frac{dn}{dE} \right)_g \right\} = 1. \tag{6}
\]

The quark flavors \( s \) and \( c \) which are not introduced in Eq. (6) show up at higher \( Q^2 \) as a result of QCD evolution.

By interchanging the order of \( x \) and \( E \) integrations in Eqs. (5-6) and performing the \( x \)-integration analytically, we see that in order to keep the integrals finite, large powers of \( 1/E \) are not allowed in the integrand. This means that the three terms displayed in Eq. (4) are the only ones that are allowed. \textit{Thus the model effectively has only two free parameters} \( a \) and \( b \).

The temperature \((T)\) and two chemical potentials \((\mu_u, \mu_d)\) are not free parameters; we determine them by solving the three coupled nonlinear equations (5-6) by the Davidenko-Broyden method \([12]\). The resulting values of \( T, \mu_u \) and \( \mu_d \) are such that the left and right hand sides of these equations agree with each other to typically one part in \( 10^6 \). The parton distributions were evolved by means of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations \([13]\) in leading order, taking the input scale \( Q_0^2 = M^2 \) and \( \Lambda_{QCD} = 0.3 \).
GeV. Finally, the root-mean-square (rms) radius of the parton distribution was taken to be the same as the charge rms radius (\(\rho\)) of the proton; since \(\rho \simeq 0.862\) fm \([14]\), this yields \(R = \sqrt{5/3}\ \rho = 1.11\) fm.

Results and Discussion

Since the two arbitrary constants \(a\) and \(b\) in Eq. (4) are not known, we have determined them by fitting the deep inelastic scattering data on \(F_2^p(x)\) at \(Q^2 = 15\) GeV\(^2\) \([4,5]\). The results of our fit incorporating QCD evolution and FSC are shown by the solid curve in Fig. 2. Also shown for comparison in Fig. 2 are: (a) the (dot-dashed) curve labeled ‘GAS’ giving the prediction of the (unmodified) parton gas model which has no free parameters by virtue of the constraints, (b) the (dashed) curve labeled ‘QCD’ showing the effect of QCD evolution on the gas model, and (c) the (dotted) curve labeled ‘FSC’ showing a fit to the data when only the FSC are introduced in the gas model. For the solid curve in Fig. 2, the fitted values of the two parameters are \(a = -0.400\) and \(b = 0.475\), and the corresponding temperature and chemical potentials are \(T = 72\) MeV, \(\mu_u = 162\) MeV and \(\mu_d = 81\) MeV.

As a test of the model, we show in Fig. 3 the prediction (solid curve) for the difference \([F_2^p(x) - F_2^n(x)]\). Also shown for comparison is the result (dashed curve) based on the parametrization of Glück et al. \([6]\). The agreement with the NMC data is reasonable.

As for the Gottfried sum \(S_G\), we have

\[
S_G = \frac{1}{3} - \frac{2}{3} \frac{M^2}{2} \int_0^1 dx \int_{x M/2}^{M/2} dE E^2 \left\{ \left( \frac{dn}{dE} \right)_\bar{d} - \left( \frac{dn}{dE} \right)_\bar{u} \right\}.
\]

The inequality \(S_G < \frac{1}{3}\) is thus a result of having in the proton, more valence \(u\) quarks than valence \(d\) quarks, \((n_u - n_{\bar{u}}) > (n_d - n_{\bar{d}})\), implying that \(\mu_u > \mu_d\) (see Eq. (5)) and hence the integral in \(S_G\) is positive. Our model predicts at \(Q^2 = 4\) GeV\(^2\), the value \(S_G = 0.22\) which

\(^2\)It is amusing to note that the values of \(a\) and \(b\) determined by us are close to the values \(a = -1/2\) and \(b = 3/(2\pi)\) which follow from one of the expressions for \(dn/dE\) given by Morse and Ingard \([11]\); substitute \(c = 1, \nu = E/(2\pi), \Lambda = 4\pi R^2\) and \(\ell_x = \ell_y = \ell_z = 2R\) in \(dN_{ob}\) in their Eq. (9.5.12).
is consistent with the experimental value $S_G = 0.235 \pm 0.026$. The natural explanation of the violation of the Gottfried sum rule is an attractive feature of our general framework.

In addition, the model is in excellent agreement with data on the gluon distribution $g(x)$. It also reproduces well the ratio $F_2^p(x)/F_2^n(x)$, the ratio $\bar{u}(x)/\bar{d}(x)$ at $< x > = 0.18$ which has been deduced to be about 0.51 by the NA51 collaboration [3], the longitudinal momentum fraction carried by the charged partons, etc. These and other predictions of the model, on the quark and antiquark distributions $q(x), \bar{q}(x), q_{e}(x) = q(x) - \bar{q}(x)$ for various flavors, etc. will be discussed in detail elsewhere [10].

Now we briefly describe the salient features of some of the recent calculations of the nucleon structure functions, which use ideas from statistical mechanics. It has generally been difficult so far to model $F_2^p(x, Q^2)$ such that it is nonzero only in $x = [0,1]$, and is consistent with the two number constraints, the momentum constraint, the Gottfried sum and the observed low-$x$ rise. Mac and Ugaz [7a] calculated first-order QCD corrections to the statistical distributions and obtained a crude but reasonable agreement with $F_2^p(x)$ data for $x \gtrsim 0.2$. The momentum constraint was not imposed and the fitted value of the proton radius was 2.6 fm. Cleymans et al. [7b] used the framework of the finite temperature quantum field theory. They considered $O(\alpha_s)$ corrections to the statistical distributions and obtained a good fit to the $F_2^p(x)$ data for $x \geq 0.25$. They also calculated the ratio $\sigma_L/\sigma_T$ in this region; it was a factor of 6 above the experimental value. Bourrely et al. [7c] considered polarized as well as unpolarized structure functions and presented a statistical parametrization (with eight parameters) of parton distributions in the IMF. Their framework allowed chemical potential for quarks as well as for gluons. The number constraints were not satisfied very accurately. QCD effects were not considered. Parton distributions were nonzero for $x > 1$. $x\bar{q}(x)$ vanished as $x \rightarrow 0$ and so it was not possible to reproduce the fast increase of the antiquark distributions for $x < 0.1$. Bourrely and Soffer’s [7d] approach was similar to that in [7c]. By incorporating QCD evolution of parton distributions and allowing the antiquark chemical potential to depend on $x$, they were able to reproduce the HERA data on $F_2^p$. Recently Buccella et al. [7e] have introduced in this model a so-called liquid
term to reproduce the low-\(x\) behaviour of structure functions.

In conclusion, it is noteworthy that the application of ideas of statistical mechanics to the point constituents of the nucleon can provide a simple description of all the observed features of the unpolarized nucleon structure functions down to the lowest \(x\) values so far explored. The model presented here has two free parameters which arise from our treatment of the finite-size corrections. It is hoped that the success of the model would provide a stimulus to further studies of the finite-size effects in a QCD bound state such as the nucleon.

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FIGURES

FIG. 1. Log-log plot of the proton structure function data. Experimental data are from Refs. [4,5]; the error bars show statistical and systematic errors combined in quadrature. The straight lines are our fits described in Eq. (1), and are labeled by $Q^2 = 15, 35, \text{and } 120 \text{ GeV}^2$. Numbers have been scaled by the factors shown in parentheses for convenience in plotting.

FIG. 2. Proton structure function $F^p_2(x)$ at $Q^2 = 15 \text{ GeV}^2$. Data points are as in Fig. 1. Solid curve is our best fit to the data and includes both FSC and QCD. Also shown for comparison are: the (dot-dashed) curve labeled ‘GAS’ giving the (unmodified) gas model prediction, the (dashed) curve labeled ‘QCD’ showing the QCD-evolved gas model, and the (dotted) curve labeled ‘FSC’ which is a fit to the data when finite-size corrections are included in the gas model (without QCD).

FIG. 3. Difference $(F^p_2 - F^n_2)$ versus $x$, at $Q^2 = 4 \text{ GeV}^2$. Experimental data are from Ref. [1]; errors are statistical only. Solid curve is the prediction of our model. Dashed curve is based on the parametrization of Glück et al. [6].
Figure 1

$F_2^{p}/x$

$\chi$

$10^{-1}$

$10^{0}$

$10^{1}$

$10^{2}$

$10^{3}$

$10^{4}$

$10^{5}$

$10^{6}$

$10^{7}$

(100)

120

(10)

35

15
Figure 2

$Q^2 = 15 \text{ GeV}^2$

The figure shows the dependence of $F_2^p$ on $\chi$ for different models:
- **QCD**
- **FSC**
- **GAS**

The data points are represented with error bars and the curves represent theoretical predictions for each model.
$Q^2 = 4 \text{ GeV}^2$