“Triviality” and the Perturbative Expansion in $\lambda\Phi^4$ Theory

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Abstract:

The “triviality” of $\lambda\Phi^4$ quantum field theory means that the renormalized coupling $\lambda_R$ vanishes for infinite cutoff. That result inherently conflicts with the usual perturbative approach, which begins by postulating a non-zero, cutoff-independent $\lambda_R$. We show how a “trivial” solution $\lambda_R = 0$ can be compatible with the known structure of perturbation theory to arbitrarily high orders, by a simple re-arrangement of the expansion. The “trivial” solution reproduces the result obtained by non-perturbative renormalization of the effective potential. The physical mass is finite, while the renormalized coupling strength vanishes: the two are not proportional. The classically scale-invariant $\lambda\Phi^4$ theory coupled to the Standard Model predicts a 2.2 TeV Higgs, but does not imply strong interactions in the scalar sector.
Suppose we accept that the 4-dimensional $\lambda \Phi^4$ theory is indeed “trivial” [1], meaning that it has no observable particle interactions; what is the theory’s effective potential?

Since there are no interactions the effective potential can only be the classical potential plus the zero-point energy of the free-field fluctuations. This is the crucial insight of Ref. [2]:— for a “trivial” theory the one-loop effective potential is effectively exact. (A recent lattice calculation provides striking confirmation of this fact [3].)

The usual perturbative renormalization [4] is then not appropriate because it would spoil this exactness — it does not properly absorb the infinities, but merely pushes them into “higher-order terms” which are then neglected. However, it is simple to renormalize the one-loop effective potential in an exact way [5, 6, 7, 2]. (This was first discovered in the context of the Gaussian effective potential [8, 9].) The constant background field $\phi$, the argument of $V_{\text{eff}}$, requires an infinite re-scaling, but the fluctuation field $h(x) \equiv \Phi(x) - \phi$ (i.e., the $p_\mu \neq 0$ projection of the field) is not re-scaled [2]. The particle mass $m_h$ is related to the cutoff $\Lambda$ and the bare coupling constant $\lambda = \lambda(\Lambda)$ by

$$m_h^2 = \Lambda^2 \exp \left(- \frac{32\pi^2}{3\lambda} \right). \quad (1)$$

Thus, for $m_h$ to remain finite $\lambda$ must vanish like $1/\ln(\Lambda/m_h)$ in the continuum limit ($\Lambda \to \infty$). As a consequence one finds that the connected $n$-point functions at non-zero momentum vanish for $n > 2$, implying no particle interactions; i.e., “triviality”. In particular, the connected 4-point function, from which one might have hoped to define a renormalized coupling constant $\lambda_R$, vanishes.

The usual perturbative approach, by contrast, is based on an attempt to generate a cutoff-independent and non-vanishing $\lambda_R$. No meaningful continuum limit is possible in perturbation theory. In fact, as discussed by Shirkov [10], perturbative calculations of the $\beta$ function up to 5 loops [11] provide the following results: In odd orders, $\beta_{\text{pert}}^{1-\text{loop}}$, $\beta_{\text{pert}}^{3-\text{loop}}$, $\beta_{\text{pert}}^{5-\text{loop}}$ are positive and monotonically increasing. In even orders $\beta_{\text{pert}}^{2-\text{loop}}$, $\beta_{\text{pert}}^{4-\text{loop}}$ each have an ultraviolet fixed point, which would imply a finite bare coupling constant, in contradiction with the rigorous results of Ref. [1]. The magnitude of this spurious fixed point at even orders appears to decrease to zero with increasing perturbative order. A Borel re-summation procedure [10, 11] yields a positive, monotonically increasing $\beta$ function, as in odd orders. That does not allow a continuum limit because the renormalized coupling will have an unphysical Landau pole.

The moral is that only by abandoning, at the start, the vain attempt to define a non-zero renormalized 4-point function can one obtain a continuum limit. In the effective potential analysis [2] one actually starts from an approximation scheme (one-loop or
Gaussian) in which, by definition, the shifted field \( h(x) \) is non-interacting. The resulting effective potential exhibits spontaneous symmetry breaking (SSB) and allows a continuum limit in which “dimensional transmutation” occurs, with massive particles arising from a scale-invariant bare action. The renormalization never introduces a “\( \lambda_R \)” but simply requires the particle mass and \( V_{\text{eff}} \) to be finite. One finds, as a consequence, that this renormalization implies “triviality” — thereby revealing that the original “approximation” was effectively exact.

In this Letter we shall follow a different route, considering the 4-point function of the already massive theory. At the leading-log level, because of the Landau-pole problem, we shall see that the only possibility for defining a continuum limit of the regularized theory corresponds to \( \lambda_R = 0 \). This yields the same relation \( (1) \) as above. We then show that this solution is compatible with all orders of sub-leading logarithms.

2. Let us start by defining \( \lambda_R \) as the 4-point function in the limit of zero external momenta (which for massive particles is not an exceptional point.) We calculate this in terms of the bare, cutoff-dependent coupling \( \lambda = \lambda(\Lambda) \), taking into account the basic one-loop bubble of particles with mass \( m_h \). This gives:

\[
\lambda_R = \lambda - b_0 \lambda^2 t, \tag{2}
\]

where

\[
b_0 \equiv \frac{3}{16\pi^2}, \tag{3}
\]

\[
t \equiv \ln(\Lambda/m_h). \tag{4}
\]

It is evident that the actual expansion is not in powers of \( \lambda \) but rather in powers of \( \lambda \) and \( t \). However, one can define a (perturbative) \( \beta \)-function that depends on \( \lambda \) alone:

\[
\beta_{\text{pert}} \equiv \Lambda \frac{\partial \lambda}{\partial \Lambda} = \frac{\partial \lambda}{\partial t} = b_0 \lambda^2 + b_1 \lambda^3 + \ldots. \tag{5}
\]

[Note that we are defining the \( \beta \) function in terms of the cutoff dependence of the bare coupling constant. In the conventional perturbative context this is completely equivalent to the more usual definition as the renormalization-point dependence of the renormalized coupling constant. Since we want to consider the case where \( \lambda_R \) vanishes identically the above definition is obviously preferable.] Formally, by integrating the \( \beta \) function one resums large logarithms in the series for \( \lambda_R/\lambda \): The first term takes into account all leading-log terms, \( (b_0 \lambda t)^n \); the second term accounts for the sub-leading logarithms \( \lambda(b_0 \lambda t)^n \), etc.. Using \( \beta \) seemingly allows one to relax the requirement \( b_0 \lambda t \ll 1 \) to just \( \lambda \ll 1 \).
This is powerful magic, and very familiar, but one should be aware of the hidden assumptions behind it. The $\beta_{\text{pert}}$ function is extracted from an RG equation that is satisfied only in a perturbative sense, neglecting higher-order terms. The statement that the leading term is $b_0\lambda^2$ is equivalent to assuming that the leading-log series converges and so can be summed. That is, one is assuming $|b_0\lambda t| < 1$. If the theory were perturbatively asymptotically free this would create no difficulty, but here $b_0$ is positive and one has the “Landau-pole” problem. Explicitly, the solution to $d\lambda/dt = b_0\lambda^2$, in terms of the boundary condition at $t = 0$, is
\[
\lambda(t) = \frac{\lambda(0)}{1 - b_0\lambda(0)t}.
\]
One is forced to identify $\lambda_R = \lambda(0)$ for consistency with the original equation (2), which is seen as the first two terms in the infinite expansion of
\[
\lambda_R = \lambda(0) = \frac{\lambda(t)}{1 + b_0\lambda(t)t}.
\]
One wants to take $\Lambda$, and hence $t$, to infinity, but as $t$ is increased from zero $\lambda(t)$ grows without bound; indeed it becomes infinite at $t = 1/(b_0\lambda_R)$. Thus, the condition $|b_0\lambda t| < 1$ is inevitably violated. No sensible $\Lambda \to \infty$ limit is possible. This pushes the problem of the continuum limit to the next-to-leading level. There, since $b_1 < 0$, one finds an ultraviolet fixed point; but this conflicts with the rigorous results of Ref [1], and in any case it disappears at next-to-next-to-leading order. These results actually signal the inconsistency, in the $\lambda\Phi^4$ case, of assuming that the leading-log series can be naively re-summed.

3. Let us re-examine the $\beta$-function approach, relying on just two key ingredients; (i) a basic equation from which one obtains the $\Lambda$ dependence of $\lambda$, and (ii) the necessity of achieving a continuum limit $\Lambda \to \infty$. Our basic equation is Eq. (4) and we attempt to keep $\lambda_R$ and the physical mass $m_h$ fixed (i.e. $\Lambda$ independent) while taking the continuum limit $\Lambda \to \infty$. That is, we demand
\[
\frac{d\lambda_R}{dt} = 0,
\]
which yields
\[
\frac{d\lambda(t)}{dt} - b_0\lambda^2(t) - 2b_0t\lambda(t)\frac{d\lambda(t)}{dt} = 0.
\]
In the usual perturbative analysis one would neglect the third term on the left-hand side of the above equation and arrive at
\[
\frac{d\lambda(t)}{dt} = b_0\lambda^2(t).
\]

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Seemingly, the neglected term is then $O(\lambda(t)^3)$, justifying the procedure, *a posteriori*. However, one cannot obtain a continuum limit in this way, as just explained.

If, instead, we *do* keep the third term in Eq. (9) we obtain

$$\frac{d\lambda(t)}{dt} = -b_0\lambda^3(t) \frac{1}{\lambda(t) - 2\lambda_R}. \quad (11)$$

Assuming that $\lambda(t)$ and $\lambda_R$ are both non-negative we find that Eq. (11) has to be studied separately for $\lambda(t) - 2\lambda_R > 0$ and for $\lambda(t) - 2\lambda_R < 0$ to preserve the uniqueness of the solution. In neither case, however, is a limit $t \to \infty$ possible if $\lambda_R > 0$. The only possibility is associated with the case $\lambda_R = 0$, which gives:

$$\frac{d\lambda(t)}{dt} = -b_0\lambda^2(t), \quad (12)$$

$$\lambda(t) = \frac{1}{b_0 t}. \quad (13)$$

Thus, now we find a negative $\beta$ function, giving a bare coupling constant that tends to zero in the continuum limit. Eq. (13) is precisely the relation (11), obtained from the effective-potential analysis of the massless theory [2]. [The above explicitly answers the objection of Ref. [12]: our $\beta$ function is not, of course, $\beta_{\text{pert}} + (\text{non-perturbative corrections})$; it is simply the right $\beta$ function for achieving a continuum limit.]

4. To discuss higher orders it is convenient to introduce the variable

$$x = b_0\lambda(t)t. \quad (14)$$

The basic one-loop correction, Eq. (2), then has the form $\lambda_R^{(0)} = \lambda(t)(1 - x)$. Explicit calculation of the higher-order leading-logarithmic corrections to this formula would of course give $\lambda_R = \lambda(t)(1 - x + x^2 - x^3 + \ldots)$, in agreement with a formal expansion of $\lambda_R = \lambda(t)/(1 + x)$ (Eq. (7)). However, that expression represents a re-summation of the geometric series that is only valid if $|x| < 1$. Our solution, Eq. (13), is $x = 1$ with $\lambda_R = 0$. It is easy to see that this can be a solution to arbitrarily high order if we rearrange the perturbative expansion suitably. We can view the higher-order diagrams as modifying, and multiplicatively renormalizing, $\lambda_R^{(0)}$ rather than $\lambda(t)$. In a sense, this makes the effective expansion parameter $x^n(1-x)$ rather than $x^n$ itself. For $x \ll 1$ this would make essentially no difference, of course. It produces a sequence of approximations (for $N = 0, 1, 2, \ldots$) of the form

$$\lambda_R^{(N)} = \lambda(t)(1 - x)(1 + x^2 + x^4 + \ldots x^{2N}) = \lambda(t)\frac{1 - (x^2)^{N+1}}{1 + x} \quad (15)$$
which, for any \( N \), gives
\[
\lambda_R^{(N)} \big|_{x=1} = 0. \tag{16}
\]
Note that the limits \( x \to 1 \) and \( N \to \infty \) do not commute. Indeed, for any \( x < 1 \) one has
\[
\lambda_R = \lim_{N \to \infty} \lambda_R^{(N)}(x) = \frac{\lambda(t)}{1+x}, \tag{17}
\]
whose \( x \to 1 \) limit is \( \lambda_R = \frac{1}{2} \lambda(t) \), whereas we have
\[
\lambda_R = \lim_{N \to \infty} \lim_{x \to 1} \lambda_R^{(N)}(x) = 0, \tag{18}
\]
yielding again Eqs. (12, 13).

This procedure can be extended to include all orders of sub-leading logarithms. The essential point is that any sub-leading-log term \( A \) appearing at some order in \( \lambda \) will itself be modified in subsequent orders by a series of leading-log corrections, \( A(1-x+x^2-...) \), and so is multiplied by a \( \lambda_R^{(N)} \) factor. For instance, the sequence of approximations
\[
\lambda_R^{(N,M+1)} = \frac{\lambda_R^{(N)}}{1 - c \lambda_R^{(N)} \ln \left( \frac{\lambda_R^{(N,M)}(1+c\lambda(t))}{\lambda(t)(1+c\lambda_R^{(N,M)})} \right)}, \tag{19}
\]
with \( c \equiv b_1/b_0 \), contains, in the limit \( N \to \infty, M \to \infty \), all the leading and next-to-leading corrections to the zero-momentum coupling, \( \lambda_R \). One can see this as follows. For \( |x| < 1 \) the above sequence corresponds to an iterative solution of the implicit equation
\[
\lambda_R = \frac{\lambda_{ll}}{1 - c \lambda_{ll} \ln \left( \frac{\lambda_R^{(N,M)}(1+c\lambda(t))}{\lambda(t)(1+c\lambda_R^{(N,M)})} \right)}, \tag{20}
\]
where \( \lambda_{ll} = \lim_{N \to \infty} \lambda_R^{(N)} \) is the leading-log solution, which is \( \lambda_{ll} = \lambda(t)/(1+x) \) for \( |x| < 1 \). It is then straightforward to check that for \( \lambda_R \) to be cutoff independent one requires \( \lambda(t) \) to satisfy
\[
\frac{d\lambda(t)}{dt} = b_0 \lambda(t)^2 (1 + c\lambda(t)), \tag{21}
\]
which is the two-loop perturbative \( \beta \) function. However, for \( x = 1 \) the sequence \( \lambda_R^{(N,M+1)} \) gives identically
\[
\lambda_R = \lim_{N \to \infty} \lim_{M \to \infty} \lim_{x \to 1} \lambda_R^{(N,M)}(x) = 0. \tag{22}
\]
and the associated relations \( \lambda_R^{(N,M)} \). [Note that the re-summations producing the logarithmic term in the denominator of Eq. (21) can be performed consistently even when \( x \to 1 \) as \( t \to \infty \) since both \( \lambda_R^{(N)} \) and \( \lambda(t) \) vanish in that limit.]

In other words, we have exploited the fact that the structure of the sub-leading logarithms can be inferred from the usual perturbative \( \beta \) function, which just represents a
formal re-summation of those terms. However, that re-summation is valid only for \( |x| < 1 \). The sub-leading logarithmic structure itself, though, when examined at \( x = 1 \), is consistent to all orders with the ‘trivial’ solution \( \lambda_R = 0 \). The point is that all sub-leading corrections are themselves multiplied by a \( \lambda_R^{(N)} \) factor.

Of course all of the above is open to the objection that we are merely re-arranging the terms of a divergent series. There is no defence to this charge. Our point, though, is that the conventional procedure, re-summing leading logs to all orders, then sub-leading logs, etc., is itself a re-arrangement of a divergent series. Moreover, because of the Landau pole, one is forced into a region with \( x \geq 1 \) where this re-arrangement is highly dubious because the sub-series being re-summed are themselves divergent.

5. To further illustrate our point we give a concrete example. This is not meant to represent how things actually work in \( \lambda \Phi^4 \) theory, but merely to reinforce the point that the conventional procedure, although sanctified by time and custom, can in fact give the wrong answer. Consider the mathematical example in which \( \lambda_R \) and the bare \( \lambda \) are related by:

\[
\lambda_R = \lambda (1 - x) \sum_{n=0}^{\infty} g_n(\delta) x^{2n},
\]

where \( \delta \) is a parameter that vanishes in the infinite-cutoff limit (say, as \( 1/\Lambda \)). If the coefficients \( g_n(\delta) \) all become unity in the infinite-cutoff limit (i.e., as \( \delta \to 0 \)), then this reproduces the leading-log series \( \lambda_R = \lambda (1 - x + x^2 - \ldots) \). However, suppose that in the double limit \( \delta \to 0 \) and \( n \to \infty \)

\[
g_n(\delta) \to \begin{cases} 
1 & \text{if } n\delta \ll 1, \\
0 & \text{if } n\delta \gg 1.
\end{cases}
\]

This could happen in many ways; e.g. \( g_n(\delta) = (1 - \delta)^n \) or \( g_n(\delta) = 1/(1 + n!\delta^n) \). While all \( g_n \)’s become unity as \( \delta \to 0 \) for any finite \( n \), we must be careful because our series involves infinitely large \( n \). For any finite \( \delta \), no matter how small, the \( g_n \) coefficients at very large \( n \) (\( n > 1/\delta \)) become much less than unity. Thus, for \( \delta \to 0 \) we have

\[
\lambda_R \sim \lambda (1 - x) \left( \frac{1}{1-x^2/\delta} + \sum_{n=1/\delta}^{\infty} g_n(\delta) x^{2n} \right)
\]

\[
\sim \lambda (1 - x) \left( \frac{1 - x^{2/\delta}}{1 - x^2} + R(x) \right)
\]

\[
\sim \lambda \frac{(1 - x^{2/\delta})}{(1 + x)} + \lambda (1 - x) R(x),
\]
The remainder term $R(x)$ is a series beginning at order $x^{2/\delta}$ with a radius of convergence greater than unity, and so is non-singular at $x = 1$.

For $|x| < 1$ one has $x^{2/\delta} \to x^\infty \to 0$, and $R(x) \to 0$, so that:

$$\lambda_R \sim \frac{\lambda}{(1 + x)}$$

which is the usual perturbative relationship, at the leading-log level. In this case the subtlety about the $\delta \to 0$ limit of the $g_n$’s is irrelevant.

However, for $x = 1$ the last equation is not valid. One then has $x^{2/\delta} = 1^\infty = 1$ in (25), and so $\lambda_R = 0$. This is obvious from the $(1 - x)$ factor in the original equation, (23). The point is that the $g_n x^{2n}$ series does not generate a $1/(1 - x)$ factor to cancel it. Thus, this example admits the “triviality” solution, $\lambda_R = 0$ and $x = 1$, associated with the negative $\beta$ function of Eq. (12).

6. In conclusion, we have presented a simple rearrangement procedure which reproduces the full perturbative expansion at arbitrarily high orders and is valid in the full range $x \leq 1$ ($x \equiv b_0 \lambda t$). It is based on the simple remark that $x^n(1 - x)$ is a more suitable expansion parameter than $x^n$ itself. The assumption $x < 1$ allows one to re-sum the various sub-series and leads to the conventional results. However, one cannot then obtain any consistent continuum limit, and moreover one cannot avoid being dragged into a region with $x \geq 1$, invalidating the original assumption. However, if the continuum limit is governed by $x \to 1$, then the condition $\lambda_R = 0$ holds to all orders in this modified expansion. This solution is entirely consistent with the “triviality” found in mathematically rigorous analyses [1]. It is also entirely consistent with the effective-potential analysis [2], which is based on the very physical consideration that the effective potential of a “trivial” theory is just the classical potential plus the zero-point energy of the free-field fluctuations.

Our results here prove nothing, since we start from an inherently divergent Feynman-diagram expansion. However, they do provide a way to understand how “triviality” can be consistent with a seemingly highly non-trivial perturbative structure.

The consequences of our picture are substantial, and are discussed in more detail in Ref. [2]. Although the $\lambda \Phi^4$ theory is “trivial” (i.e., has non-interacting particles), it has SSB. When coupled to the Standard Model — and the gauge and Yukawa interactions may be treated as small perturbations — it leads to the Higgs-Kibble mechanism in the usual way. In the theoretically most attractive classically-scale-invariant case, one finds [3, 4] the relation $m_h^2 = 8\pi^2 v^2$, where $v$ is the renormalized expectation value of the scalar field, known from the Fermi constant to be $246$ GeV. Thus, one predicts a $2.2$ TeV Higgs boson
In the perturbative picture $m_A$ would be proportional to $\lambda_R$, but in the “trivial” solution $m_h$ and $\lambda_R$ are quite distinct quantities: the former remains finite while the latter vanishes [13]. Thus, in spite of the large Higgs mass, the Higgs/longitudinal-$W, Z$ sector in our picture is not strongly interacting. Indeed, the interactions in this sector are of electroweak strength, and would vanish if the gauge and Yukawa couplings were turned off.

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