Research article

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Gradient estimate of a variable power for nonlinear elliptic equations with Orlicz growth

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Abstract: In this paper, we prove a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for a variable power of the gradients to the zero-Dirichlet problem of general nonlinear elliptic equations with the nonlinearities satisfying Orlicz growth. It is mainly assumed that the variable exponents \( p(x) \) satisfy the log-Hölder continuity, while the nonlinearity and underlying domain \((A, \Omega)\) is \((\delta, R_0)\)-vanishing in \( x \in \Omega \).

Keywords: Nonlinear elliptic equations; Orlicz growth; Lorentz estimate of the variable power; log-Hölder continuity; \((\delta, R_0)\)-vanishing of \((A, \Omega)\)

MSC: 35J60; 35B65

1 Introduction

Throughout this paper, let \( \Omega \subset \mathbb{R}^n \) for \( n \geq 2 \) be a given bounded domain with its rough boundary specified later. Given a vectorial valued function \( f = (f^1, f^2, \cdots, f^n) : \Omega \rightarrow \mathbb{R}^n \). The aim of this present article is to study a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for a variable power of the gradients of weak solutions to the zero-Dirichlet problem of general nonlinear elliptic equations

\[
\begin{aligned}
\text{div} A(x, Du) &= \text{div} G(x, f) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where the nonlinearity \( A = A(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is measurable for each \( \xi \in \mathbb{R}^n \) and differentiable for almost every \( x \in \mathbb{R}^n \), and there exist constants \( 0 < \nu \leq 1 \leq \Lambda < \infty \) such that for all \( x, \xi, \eta \in \mathbb{R}^n \),

\[
\begin{aligned}
D_\xi A(x, \xi) \eta \cdot \eta &\geq \nu \varphi^\prime(|\xi|) |\eta|^2, \\
|A(x, \xi)| + |\xi||D_\xi A(x, \xi)| &\leq \Lambda \varphi(|\xi|).
\end{aligned}
\]

Here \( D_\xi \) denotes the differentiation in \( \xi \), and \( \varphi(s) : [0, +\infty) \rightarrow [0, +\infty) \) has the following properties

\[
\begin{aligned}
\varphi(0) &= 0 \text{ if and only if } s = 0, \\
\varphi &\in C^1(\mathbb{R}^+), \\
\sigma_\varphi &\leq \frac{\varphi(s)}{s^2} \leq \tau_\varphi \text{ for every } s > 0, \text{ where } 0 < \sigma_\varphi \leq 1 \leq \tau_\varphi < \infty.
\end{aligned}
\]

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Moreover, the nonhomogeneous term $G(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|G(x, \xi)| \leq A\varphi(|\xi|) \quad \text{for any } x, \xi \in \mathbb{R}^n. \quad (1.4)$$

We also define

$$\Phi = \Phi(s) = \int_0^s \varphi(r) \, dr \quad \text{for } s \geq 0. \quad (1.5)$$

The weak solution of (1.1) is understood in the following usual sense, if for $u \in W^{1,\Phi}(\Omega)$ it holds

$$\int_{\Omega} A(x, Du) D\phi dx = \int_{\Omega} G(x, f) D\phi dx \quad \forall \phi \in W^{1,\Phi}_0(\Omega),$$

where $W^{1,\Phi}$ is an Orlicz-Sobolev space defined in the following.

**Definition 1.1.** An Orlicz space $L^\Phi(\Omega)$ is the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\int_{\Omega} \Phi(|f|) dx < \infty.$$ 

Therefore, $L^\Phi(\Omega)$ is a Banach space equipped with the Luxemburg norm

$$\|f\|_{L^\Phi(\Omega)} := \inf \left\{ \mu > 0 : \int_{\Omega} \Phi\left(\frac{|f|}{\mu}\right) dx \leq 1 \right\}.$$ 

A Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$ is the set of all measurable functions $f \in L^\Phi(\Omega)$ for which $Df \in L^\Phi(\Omega, \mathbb{R}^n)$ with the norm $\|f\|_{W^{1,\Phi}(\Omega)} = \|f\|_{L^\Phi(\Omega)} + \|Df\|_{L^\Phi(\Omega, \mathbb{R}^n)}$. For more details about Orlicz spaces, we refer to [14, 15, 28, 29].

The equation of generalized $p$-Laplacian type is arising in the fields of fluid dynamics, magnetism and mechanics. Particularly, if $\varphi(s) = s^{p-1}$ for $1 < p < \infty$, our problem becomes the $p$-Laplace equation. For this case, there is a great deal of literature concerning the regularity of weak solutions, for instance, see [4, 9, 11, 13, 24]. While $\varphi(s) = s^{p-1} + as^{q-1}$, where $1 < p < q$ and $a$ is a positive constant, Filippis and Mingione [18] obtained a global estimate in the setting of Lebesgue spaces, which is covered by our problem (1.1). We also refer to [8] for the existence of solutions to the $(p, q)$-Laplace equations.

In recent years, there have been significant advances of this type of problem in the regularity theory, see for example [12, 14, 34, 35] since starting with a seminal paper of Lieberman [23]. In 2011, Verde [34] studied the elliptic system

$$-\text{div } A(x, Du) = -\text{div } G(x, f),$$

where $A(x, \xi) = G(x, \xi) = \frac{\varphi(|\xi|)}{|\xi|^q} \xi$ with $\varphi(s)$ defined as in (1.3). He proved the global Calderón-Zygmund estimates over whole domain $\mathbb{R}^n$ for the weak solutions. Very recently, Yao and Zhou [35] obtained local $L^q$-estimates for weak solutions of (1.1), which implies the fact that

$$\Phi(|f|) \in L^q_{\text{loc}}(\Omega) \Rightarrow \Phi(|Du|) \in L^q_{\text{loc}}(\Omega), \quad \text{for any } q \geq 1.$$ 

Also, Byun and Cho [12] further extended it to a global gradient estimate in the setting of Orlicz spaces under the assumption that the boundary of underlying domain is Reifenberg flat. We would like to point out that Cho [14] also considered the zero-Dirichlet problem of (1.1) for $A$ defined as in (1.2) and $G(x, \xi) = \frac{\varphi(|\xi|)}{|\xi|^q} \xi$, who obtained a global Calderón-Zygmund estimate in the setting of Orlicz spaces. We also refer to [6, 7, 17, 36] for a further study of these problems with Orlicz growth. In particular, Baroni and Lindfors [7], and Yao [36] studied parabolic problems with Orlicz growth.

In this paper we are to prove a global Calderón-Zygmund type estimate in the Lorentz spaces for general nonlinear elliptic equations (1.1). As we know, there are mainly three kinds of different arguments to handle the Calderón-Zygmund theory for elliptic and parabolic problems with VMO or small BMO discontinuous coefficients except for a classical technique by using singular integral operators and their commutators. The first one is the so-called geometric method originally traced from Byun and Wang’s work in [11], which is based on the weak compactness, the Hardy-Littlewood maximal operators and the modified Vitali covering. Here,
the so-called modified Vitali covering actually refers to the argument as “crawling of ink spots” as in the early papers by Safonov and Krylov [20, 30]. Indeed, this is also a development from Caffarelli and Peral’s paper in [13]. Secondly, Kim and Krylov [19, 21] gave a unified approach of studying $L^p$ solvability for elliptic and parabolic problems due to the Fefferman-Stein theorem, which is mainly based on the sharp functions. In this present paper, we have to highlight the third technique being called large-$M$-inequality principle originating from Acerbi and Mingione’s work [1, 2], which is directly based on arguing on certain Calderón-Zygmund-type covering instead of the boundedness of a maximal function operator and the so-called good-$\lambda$-inequality used by Byun and Wang [11] and Kim and Krylov’s papers [19, 21].

Here we are revising the so-called large-$M$-inequality principle and geometric method to get an estimate in the Lorentz spaces for the variable power of the gradients to (1.1). Regarding what we consider, we would like to point out that Byun, Ok and Wang [10] studied the zero-Dirichlet problem of linear elliptic systems by a geometrical argument. Adimurthi and Phuc [3] proved the global Lorentz and Lorentz-Morrey weighted Lorentz estimate to quasilinear equations below the natural exponent. Meanwhile, Baroni [4, 5] obtained interior estimates for fully nonlinear parabolic $p$-Laplacian equations with small BMO nonlinearities. Indeed, this is also a development from Caffarelli and Peral’s paper [26].

For examples, the first two-parameter scale of the Lebesgue space obtained by refining it in the fashion of a second index, and there are a lot of research activities on Lorentz regularity for partial differential equations. For example, Byun and Wang [22] gave a unified approach of studying $L^p$ solvability for evolutionary estimates for quasilinear equations below the natural exponent. Meanwhile, Baroni [4, 5] obtained interior estimates for evolutionary $p$-Laplacian systems and obstacle parabolic $p$-Laplacian with the given obstacle function $D\psi \in L^{p,q}$ locally in $\Omega_T$, respectively, which means that

$$F, D\psi \in L_{\text{loc}}^{p,q}(\Omega_T) \Rightarrow Du \in L_{\text{loc}}^{p,q}(\Omega_T)$$

for $y > p$ and $q \in (0, \infty)$, where he just used the so-called large-$M$-inequality principle. Very recently, Tian and Zheng [31] showed a global weighted Lorentz estimate to linear elliptic equations with lower order items under assumptions of partially BMO coefficients in Reifenberg flat domain. Zhang and Zheng [36, 37] also proved Hessian Lorentz estimates for fully nonlinear parabolic and elliptic equations with small BMO nonlinearities, and weighted Hessian Lorentz estimates of strong solutions for nondivergence linear elliptic equations with partially BMO coefficients, respectively.

To this end, we introduce some related notation and basic facts being useful in the article. The Lorentz space $L^{t,q}(U)$ for open subset $U \subset \mathbb{R}^n$ with parameters $1 \leq t < \infty$ and $0 < q < \infty$, is the set of all measurable functions $g : U \rightarrow \mathbb{R}$ requiring

$$\|g\|_{L^{t,q}(U)}^q := t \int_0^\infty \left( \mu^t\{\xi \in U : |g(\xi)| > \mu\} \right)^{\frac{q}{t}} d\mu < \infty;$$

while the Lorentz space $L^{t,\infty}$ for $1 \leq t < \infty$ and $q = \infty$ is defined by the Marcinkiewicz space $M^{t}(U)$ as usual, which is the set of all measurable functions $g$ with

$$\|g\|_{L^{t,\infty}} = \|g\|_{M^{t}(U)} := \sup_{\mu > 0} \left( \mu^t\{\xi \in U : |g(\xi)| > \mu\} \right)^{\frac{1}{t}} < \infty.$$
which implies $L'(U) = L'^{(i)}(U)$, see also [4, 5, 24, 37].

Let us denote

$$B_r(y) = \left\{ x \in \mathbb{R}^n : |x - y| < r \right\}$$

for $y \in \Omega$ and radius $r > 0$, and

$$\Omega_r(y) = B_r(y) \cap \Omega, \quad B^*(r) = B_r(y) \cap \{ x_n > 0 \}, \quad T_r(y) = B_r(y) \cap \{ x_n = 0 \}$$

with briefly $B_r = B_r(0)$, $\Omega_r = \Omega_r(0)$ and $T_r = T_r(0)$. For a bounded open set $U \subset \mathbb{R}^n$, we write the integral average of $g(x)$ over $U$ of a locally integrable function $g$ in $\mathbb{R}^n$ by

$$g_U = \frac{1}{|U|} \int_U g(x) dx.$$

As usual, it is a necessary assumption that the variable exponent $p(\cdot)$ is log-Hölder continuous, which ensures that the mollification, the singular integrals and the Hardy-Littlewood maximal operator are all bounded within the framework of generalized Lebesgue space. For this, we are recalling the definition that $p(x)$ is log-Hölder continuous denoted it by $p(x) \in LH(\Omega)$, if there exist positive constants $c_0$ and $\delta$ such that for all $x, y \in \Omega$ with $|x - y| < \delta$ it holds

$$|p(x) - p(y)| \leq \frac{c_0}{\log(|x - y|)}.$$

In the following context, we assume that $p(x) : \Omega \to \mathbb{R}$ is a log-Hölder continuous function and there exist positive constants $y_1$ and $y_2$ such that

$$1 < y_1 \leq p(x) \leq y_2 < \infty, \text{ for all } x \in \Omega. \quad (1.6)$$

Without loss of generality, let

$$|p(x) - p(y)| \leq \omega(|x - y|), \text{ for all } x, y \in \Omega, \quad (1.7)$$

where $\omega : [0, \infty) \to [0, \infty)$ is a modulus of continuity of $p(x)$ such that $\omega$ is a nondecreasing continuous function with $\omega(0) = 0$ and $\limsup_{r \to 0} \omega(r) \log \left( \frac{1}{r} \right) < \infty$. With the above assumptions in hand, it is clear that $p(x) \in LH(\Omega)$ yields that there exists a positive number $K_0$ such that

$$\omega(r) \log \left( \frac{1}{r} \right) \leq K_0 \iff r^{-\omega(r)} \leq e^{K_0} \text{ for any } r \in (0, 1). \quad (1.8)$$

To obtain a global Calderón-Zygmund type estimate for general nonlinear elliptic problem (1.1), it is also necessary to impose some regular assumptions on the nonlinearity $A = A(x, \xi)$ and the rough boundary of underlying domain $\Omega$. Let us set

$$\theta(A, B_r(y))(x) := \sup_{\xi \in \mathbb{R}^n \setminus \{ 0 \}} \frac{|A(x, \xi) - \hat{A}_{B_r(y)}(\xi)|}{\varphi(\xi)}$$

with

$$\hat{A}_{B_r(y)}(\xi) = \int_{B_r(y)} A(x, \xi) dx.$$

**Assumption 1.2.** Let $R_0 > 0$, we say that $(A, \Omega)$ is $(\delta, R_0)$-vanishing if

(i) \hspace{1cm}$$\sup_{0 < r < R_0} \sup_{y \in \mathbb{R}^n} \int_{B_r(y)} \theta(A, B_r(y))(x) dx \leq \delta. \quad (1.9)$$

(ii) For any $y \in \partial \Omega$ and for every number $r \in (0, R_0]$, there exists a coordinate system depending only on $y$ and $r$, such that in this new coordinate system, $y$ is the origin and

$$B_r \cap \{ x_n > \delta r \} \subset B_r \cap \Omega \subset B_r \cap \{ x_n > -\delta r \}. \quad (1.10)$$
Remark 1.3. In this article, we always assume that $\delta$ is a small positive constant with $0 < \delta < \frac{1}{8}$ by a scaling transformation. Note that the rough boundary with the so-called $(\delta, R_0)$-Reifenberg flatness (1.10) yields that the boundary might be locally very rough between two hyperplanes, which may go beyond the boundaries with $C^1$-smooth or the Lipschitz category with a small Lipschitz constant. However, this is still an $A$-type domain with the following measure density condition

$$\sup_{0<r<R_0} \sup_{y \in \Omega} |B_r(y) \cap \Omega| \leq \left( \frac{2}{1-\delta} \right)^n \leq \left( \frac{16}{7} \right)^n,$$

(1.11)

ensuring some natural properties in geometric analysis to hold, such as Sobolev embedding theorem and Sobolev extension theorem, see [11, 22, 25, 33].

Finally, we state the main result of this article.

**Theorem 1.4.** Let $u \in W_0^{1,\Phi}(\Omega)$ be a solution of (1.1) under the assumptions (1.2), (1.3) and (1.4) with the following higher integrability data

$$\Phi([f])^{p(t)} \in L^{1,q}(\Omega) \quad \text{for} \quad t > 1, \quad q \in (0, +\infty].$$

If $p(\cdot) \in LH(\Omega)$ with (1.6) and (1.7), and there exists a small constant $\delta = \delta(n, v, \Lambda, y_1, y_2, t, q, \sigma_{\phi}, \tau_{\phi}, R_0, K_0, |\Omega|) > 0$ such that $(A, \Omega)$ is $(\delta, R_0)$-vanishing with Assumption 1.2. Then we have $\Phi([Du])^{p(t)} \in L^{1,q}(\Omega)$ with the estimate

$$\| (\Phi([Du])^{p(t)})^p \|_{L^{1,q}(\Omega)} \leq c \left( \| (\Phi([f])^{p(t)})^p \|_{L^{1,q}(\Omega)} + 1 \right)^{\frac{q}{q-1}},$$

(1.12)

where the constant $c$ depends only on $n, v, \Lambda, y_1, y_2, t, q, \sigma_{\phi}, \tau_{\phi}, R_0, K_0, \omega(\cdot)$ and $|\Omega|$ (except in the case $q = \infty$, where $c$ depends on $n, v, \Lambda, y_1, y_2, t, \sigma_{\phi}, \tau_{\phi}, R_0, K_0, \omega(\cdot)$ and $|\Omega|$).

This article is devoted to a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for a variable power of the gradients of weak solutions to the zero-Dirichlet problem of general nonlinear elliptic equations (1.1) over Reifenberg domains. As already mentioned, our problem and proof are inspired by work of Acerbi and Mingione [1, 2] and Baroni [4, 5], and recent work from Cho [14]. The key ingredient is to make use of the so-called large-$M$-inequality principle, Calderón-Zygmund-type covering, approximate estimate and an iteration argument to attain the variable Lorentz estimate (1.12).

The rest of the paper is organized as follows. In Section 2, we introduce notation and some useful lemmas. In Section 3, we focus on proving the main theorem.

## 2 Technical tools

In this section, we introduce some useful lemmas. From now on, we denote $c$ to mean a universal constant that can be computed in terms of given data such as $n, v, \Lambda, y_1, y_2, t, q, \sigma_{\phi}, \tau_{\phi}, R_0, K_0$, $\omega(\cdot)$ and $|\Omega|$. First of all, we recall the existence and energy estimate of weak solution to general nonlinear elliptic problem (1.1).

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and $f \in L^\Phi(\Omega)$. Then there exists a unique solution $u \in W_0^{1,\Phi}(\Omega)$ to (1.1) such that the following energy estimate is valid:

$$\int_\Omega \Phi([Du])dx \leq c \int_\Omega \Phi([f])dx,$$

(2.1)

where $c = c(n, v, \Lambda, \sigma_{\phi}, \tau_{\phi})$.

**Proof.** The existence and uniqueness of weak solution to (1.1) has been proved in [14]. Next, we take a test function $\phi = u \in W_0^{1,\Phi}(\Omega)$ and use (1.2) and (1.4) to get the energy estimate (2.1). \qed

In what follows, let us give the scaling invariant property of (1.1), see Lemma 3.1 in [14].
Lemma 2.2. Let $u \in W^{1,p}_0(\Omega)$ be the weak solution of (1.1) under the assumptions (1.2), (1.3) and (1.4). For fixed $x_0 \in \Omega$, $\rho > 0$ and $K > 0$, we define

$$\tilde{A}(x, \xi) := A(\rho x, K\xi), \quad \tilde{G}(x, \xi) := G(\rho x, K\xi),$$

$\tilde{u}(x) := \frac{u(x_0 + \rho x)}{K}, \quad \tilde{f}(x) := \frac{f(x_0 + \rho x)}{K}, \quad \tilde{\varphi}(t) := \varphi(Kt)$

and the set $\tilde{\Omega} = \{ \frac{z-x_0}{\rho} : z \in \Omega \}$, where $\xi \in \mathbb{R}^n$ and $t \geq 0$. It leads to the following conclusions:

(I) $\tilde{u} \in W^{1,p}_0(\tilde{\Omega})$ is a weak solution to

$$\begin{cases}
\text{div} \tilde{A}(x, D\tilde{u}) = \text{div} \tilde{G}(x, \tilde{f}) & \text{in } \tilde{\Omega}, \\
\tilde{u} = 0 & \text{on } \partial \tilde{\Omega},
\end{cases}$$

where

$$\tilde{\Phi}(s) = \int_0^s \tilde{\varphi}(r) \, dr = \frac{1}{K} \Phi(ks).$$

(II) $\tilde{A}$ and $\tilde{G}$ satisfy assumptions (1.2) and (1.4), respectively, with the same constants $v$ and $L$.

(III) $\tilde{\varphi}$ satisfies (1.3).

(IV) If $(\tilde{A}, \tilde{\Omega})$ is $(\delta, R_0)$-vanishing, then $(\tilde{A}, \tilde{\Omega})$ is $(\delta, \frac{R_0}{\rho})$-vanishing.

Next, we recall a reverse Hölder inequality for $\Phi(|\xi|)$, see Theorem 9 in [16].

Lemma 2.3. Let $u \in W^{1,p}_0(\Omega)$ be the weak solution of (1.1) under the assumptions (1.2), (1.3) and (1.4). Suppose that $(\Phi(|f|))^{p(x)} \in L^1$ for $p(x) > y_1 > 1$ and $t > 1$. Then there exists $\sigma_0 = \sigma_0(n, v, \Lambda, \sigma_\varphi, \tau_\varphi) \in (0, ty_1 - 1)$ such that for $B_r$ with $B_{2r} \subset \subset \Omega$ and any $\sigma \in (0, \sigma_0)$, it holds

$$\left( \int_{B_r} (\Phi(|Du|))^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \leq c \int_{B_{2r}} \Phi(|Du|) \, dx + c \left( \int_{B_{2r}} (\Phi(|f|))^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}},$$

where $c = c(n, v, \Lambda, \sigma_\varphi, \tau_\varphi) > 0$.

In addition, the following reverse Hölder inequality on the boundary version is also a self-improving result due to the Reifenberg flatness domain belonging to $A$-type condition as in (1.11).

Lemma 2.4. Let $u \in W^{1,p}_0(\Omega)$ be the weak solution of (1.1) under the assumptions (1.2), (1.3) and (1.4). Suppose that $(\Phi(|f|))^{p(x)} \in L^1$ for $p(x) > y_1 > 1$ and $t > 1$, and $(A, \Omega)$ satisfies $(\delta, R_0)$-vanishing with Assumption 1.2. Then there exists $\sigma_0 = \sigma_0(n, v, \Lambda, \sigma_\varphi, \tau_\varphi) \in (0, ty_1 - 1)$ such that for $\Omega_r = \Omega \cap B_r(x_0)$ with $x_0 \in \partial \Omega$ and any $\sigma \in (0, \sigma_0)$, it holds

$$\left( \int_{\Omega_r} (\Phi(|Du|))^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}} \leq c \int_{\Omega_{2r}} \Phi(|Du|) \, dx + c \left( \int_{\Omega_{2r}} (\Phi(|f|))^{1+\sigma} \, dx \right)^{\frac{1}{1+\sigma}},$$

where $c = c(n, v, \Lambda, \sigma_\varphi, \tau_\varphi) > 0$.

Proof. By a similar procedure to Theorem 9 in [16] and using the measure density property (1.11) for $\Omega$, and a zero-extension of $u$ in $B_{2r}$, the conclusion is clearly true. \qed

Note that $\Phi(s)$ is an $N$-function from the definition of $\Phi(s)$ in (1.5). Besides, we deduce that

$$\min \left\{ \alpha^{\sigma+1}, \alpha^{\sigma+1} \right\} \Phi(s) \leq \Phi(\alpha s) \leq \max \left\{ \alpha^{\sigma+1}, \alpha^{\sigma+1} \right\} \Phi(s), \quad \text{for any } s, \alpha \geq 0, \quad (2.2)$$

and

$$\Phi(s_1 + s_2) \leq \frac{1}{2} \Phi(2s_1) + \frac{1}{2} \Phi(2s_2), \quad \text{for } s_1, s_2 \geq 0. \quad (2.3)$$
We define an auxiliary vector field \( V : \mathbb{R}^n \to \mathbb{R}^n \) given by

\[
V(\xi) := \left( \frac{\varphi(\xi)}{\|\xi\|} \right)^{1/2} \xi, \quad \text{for each } \xi \in \mathbb{R}^n.
\]

Then there is the following relation between \( V \) and \( \Phi \), see [14]:

\[
\Phi(\xi - \eta) \leq m_0 (|V(\xi) - V(\eta)|^2 - \Phi(\eta)),
\]

for all \( \xi, \eta \in \mathbb{R}^n \) and \( m_0 > 0 \) depending only on \( n, \nu, \Lambda, \sigma_\varphi, \tau_\varphi \).

Now, we give the comparison estimates with a limiting problem, and show its Lipschitz regularity by following from Proposition 5.5 and 5.11 in [14]. We state the required comparison estimates in the following lemmas.

For the boundary case, let \( \Omega_5 \) satisfy that

\[
B_5^t \subset \Omega_5 \subset B_5 \cap \{ x \in \mathbb{R}^n, x_n > -5\delta \},
\]

where \( B_5^t = \int_{B_5} A \right| \right| dx \), such that we have

\[
\int_{B_1} |V(Du) - V(Dv)|^2 dx \leq \epsilon
\]

and

\[
\|\Phi(|Dv|)\|_{L^\infty(B_1)} \leq c_1,
\]

where \( c_1 = c_1(n, \nu, \Lambda, \sigma_\varphi, \tau_\varphi) > 1 \).

For the boundary case, let \( \Omega_5 \) satisfy that

\[
\int_{B_5} \theta(A, B_5^t) dx \leq \delta,
\]

and

\[
\int_{B_5} \Phi(|Du|) dx \leq 1 \quad \text{and} \quad \int_{B_5} (\Phi(|f|))^\nu dx \leq \delta^{\frac{1}{\nu} \eta}.
\]

**Lemma 2.6.** Let \( u \in W^{1, \Phi}(\Omega) \) be a weak solution of (1.1) under the assumptions (1.2), (1.3) and (1.4). If for any \( 0 < \epsilon < 1 \), there exists a constant \( \delta = \delta(n, \epsilon, \nu, \Lambda, \sigma_\varphi, \tau_\varphi) > 0 \) such that (2.5) and (2.6) hold. Then there exists a weak solution \( v \in W^{1, \Phi}(B_1) \) of

\[
\begin{cases}
\text{div} \bar{A}_{B_1}(Dv) = 0 & \text{in } B_1, \\
v = u & \text{on } \partial B_1,
\end{cases}
\]

where \( \bar{A}_{B_1} = \int_{B_1} A(\xi, \eta) dx \), such that we have

\[
\int_{B_1} |V(Du) - V(Dv)|^2 dx \leq \epsilon
\]

and

\[
\|\Phi(|Dv|)\|_{L^\infty(B_1)} \leq c_1,
\]

where \( c_1 = c_1(n, \nu, \Lambda, \sigma_\varphi, \tau_\varphi) > 1 \).
where \( \mathcal{A}_{B_2} = \int_{B_2} A(x, \xi) dx \), and we have
\[
\int_{\Omega_1} |V(Du) - V(D\bar{v})|^2 dx \leq \epsilon
\]
and
\[
||\Phi(\bar{v})||_{L^\infty(\Omega_1)} \leq c_2,
\]
where \( c_2 = c_2(n, \nu, \lambda, \sigma_\phi, \tau_\phi) > 1 \) and \( \bar{v} \) is zero extension of \( v \) from \( B_1^* \) to \( \Omega_1 \).

Let us now collect some preliminary results concerning the so-called embedding relations involved in the Lorentz spaces, which will be used in the sequel.

**Proposition 2.7.** Let \( U \) be a bounded measurable subset of \( \mathbb{R}^n \), then the following relations hold:

(I) If \( 0 < q \leq \infty \) and \( 1 \leq t_1 < t_2 < \infty \), then \( L^{t_2, q}(U) \subset L^{t_1, q}(U) \) with the estimate
\[
||g||_{L^{t_2, q}(U)} \leq c ||g||_{L^{t_1, q}(U)}.
\] (2.10)

(II) If \( 1 \leq t < \infty \), and \( 0 < q_1 < q_2 \leq \infty \), then \( L^{t, q_1}(U) \subset L^{t, q_2}(U) \subset L^{\infty}(U) \) with the estimate
\[
||g||_{L^{t, q_2}(U)} \leq c(t, q_1, \bar{q}_2)||g||_{L^{t, q_1}(U)}.
\] (2.11)

(III) If \( ||g|| \in L^{t, q}(U) \) for some \( 0 < q \leq \infty \), then \( g \in L^{t, \infty}(U) \) with the estimate
\[
||g||_{L^{t, \infty}(U)} = ||g||_{L^{t, q}(U)}.
\] (2.12)

The following two lemmas will play important roles in our main proof, which are the variants of classical Hardy’s inequality and a reverse Hölder inequality, respectively, see Lemma 3.4 and 3.5 in [4].

**Lemma 2.8.** Let \( \psi : [0, +\infty) \rightarrow [0, +\infty) \) be a measurable function such that
\[
\int_0^\infty \psi(\lambda) d\lambda < \infty.
\] (2.13)

Then for any \( \alpha \geq 1 \) and \( r > 0 \), there holds
\[
\int_0^\infty \lambda^{-r} \left( \int_\lambda^\infty \psi(\mu) d\mu \right)^{\alpha} \frac{d\lambda}{\alpha} \leq \left( \frac{\alpha}{r} \right)^{\alpha} \int_0^\infty \lambda^{-\alpha} (\lambda \psi(\lambda))^{\alpha} d\lambda.
\]

**Lemma 2.9.** Let \( h : [0, +\infty) \rightarrow [0, +\infty) \) be a nonincreasing, measurable function. For \( \alpha_1 \leq \alpha_2 \leq \infty \) and \( r > 0 \), if \( \alpha_2 < \infty \), then we have
\[
\left( \int_\lambda^\infty (\mu^r h(\mu))^{\alpha_2} \frac{d\mu}{\mu} \right)^{\frac{1}{\alpha_2}} \leq \varepsilon \lambda^r h(\lambda) + \frac{c}{\varepsilon^{\alpha_2/(\alpha_2 - 1)}} \left( \int_\lambda^\infty (\mu^r h(\mu))^{\alpha_1} \frac{d\mu}{\mu} \right)^{\frac{1}{\alpha_1}}
\] (2.14)

with every \( \varepsilon \in (0, 1] \) and \( \lambda \geq 0 \). If \( \alpha_2 = \infty \), then it holds
\[
\sup_{\mu > \lambda} (\mu^r h(\mu)) \leq c \lambda^r h(\lambda) + c \left( \int_\lambda^\infty (\mu^r h(\mu))^{\alpha_1} \frac{d\mu}{\mu} \right)^{\frac{1}{\alpha_1}},
\] (2.15)

where the constant \( c \) depends only on \( \alpha_1, \alpha_2 \) and \( r \); except in the case \( \alpha_2 = \infty \) with \( c \equiv c(\alpha_1, r) \).

### 3 Proof of Theorem 1.4

This section is mainly devoted to proving Theorem 1.4. Our proof consists of 6 steps. In step 1, for given \( \lambda_0 \) in (3.7), we show the Calderón-Zygmund type covering on the super-level set \( E(\lambda, \Omega_t, (x_0)) \), and establish a
decay estimate of $\Omega_r(y)$. In step 2, we give various comparison estimates to the reference problems. In step 3, we employ the so-called “crawling of ink spots” approach to show an estimate for the super-level set. In steps 4, we get the estimate of $\|\Phi(\{Du\})^{p(x)}\|_{L^{q_2}(\Omega_r(y)))}$ for $q < \infty$. In steps 5 and 6, we deduce our conclusions in the cases of $q < \infty$ and $q = \infty$, respectively, under a priori assumption $\|\Phi(\{Du\})^{p(x)}\|_{L^{q_2}(\Omega_r(y)))} < \infty$ which is proved in step 5.

**Proof.** Let $\sigma_0$ be the same as in Lemma 2.4, and let $\sigma_2 = \min\{\sigma_0, y_1 - 1\} > 0$ due to $y_1 > 1$. For a fixed point $y \in \Omega$, we take $r_y < \frac{R}{2\gamma}$ with

$$R \leq \min\left\{\frac{R_0}{2}, \frac{R_0}{c'}, 1\right\} \text{ and } \omega(4R) < y_1 \sigma_2,$$

where $c' = c'(n, y_1, y_2, \nu, A, \sigma_\phi, r_\omega, \omega(\cdot), |\Omega|) \geq |\Omega| + 1$ determined later.

Here, we only consider the boundary case, saying $B_{25\ell_1}(y) \subset \Omega$. For this, we can find a boundary point $\tilde{y} \in B_{25\ell_1}(y) \cap \partial \Omega$ and for $r_y \in (0, \frac{R}{2\gamma})$, there exists a coordinate system depending only on $\tilde{y}$ and $r_y$, such that in this new coordinate system it holds

$$\left\{\begin{array}{l}
z = y, \\
\tilde{y} + 125\delta r_y(0, \cdots, 0, 1) \text{ is the origin,}
\end{array}\right.$$

We select $0 < \delta < \frac{1}{\tilde{y}}$ such that $\Omega_{5r_y}(z) \subset \Omega_{50r_y}(0)$, which implies the fact that

$$\Omega_{125r_y}(0) \subset \Omega_{170r_y}(z).$$

Then we also have

$$\int_{B_{125r_y}(0)} \theta(A, B_{125r_y}(0)) (x) dx \leq 4 \int_{B_{125r_y}(0)} \theta(A, B_{125r_y}(0)) (x) dx \leq 4 \delta.$$

For a fixed point $x_0 \in \Omega$, we set

$$p^- := \inf_{\Omega_{24}(x_0)} p(x), \quad p^+ := \sup_{\Omega_{24}(x_0)} p(x);$$

$$p^- := \inf_{\Omega_{125r_y}(x_0)} p(x), \quad p^+ := \sup_{\Omega_{125r_y}(x_0)} p(x).$$

For the weak solution $u$ of original problem, by a scaling argument to the nonhomogeneous terms $f$ and the $N$-functions $\phi$, $\Phi$, then we write

$$\tilde{u} = \frac{u}{\left(\|\Phi(f)\|_{L^{q_2}(\Omega)} + 1\right)^{1/y_1}}, \quad \tilde{f} = \frac{f}{\left(\|\Phi(f)\|_{L^{q_2}(\Omega)} + 1\right)^{1/y_1}},$$

$$\tilde{\phi}(s) = \phi\left(\left(\|\Phi(f)\|_{L^{q_2}(\Omega)} + 1\right)^{1/y_1} s\right), \quad \tilde{\Phi}(s) = \Phi\left(\left(\|\Phi(f)\|_{L^{q_2}(\Omega)} + 1\right)^{1/y_1} s\right).$$

By the assumption $\Phi(f)_{L_{\infty}(\Omega)} \in L_{\infty}(\Omega)$, then there holds

$$\|\Phi(f)\|_{L_{\infty}(\Omega)} = \left(\left(\|\Phi(f)\|_{L^{q_2}(\Omega)} + 1\right)^{1/y_1} \right)^{p(x)} \leq \left(\left(\|\Phi(f)\|_{L^{q_2}(\Omega)} + 1\right)^{1/y_1} \right)^{p(x)} \leq 1.$$

Hereafter, for the sake of simplicity, we still use $u, f, \phi$ and $\Phi$ replacing $\tilde{u}, \tilde{f}, \tilde{\phi}$ and $\tilde{\Phi}$ in the following.

**Step 1.** In this step, we give a Calderón-Zygmund type covering on the super-level set $E(\lambda, \Omega_r(y_0))$ as below. Let $u$ be the weak solution of (1.1), we define the quantity for any $r_1$ and $r_2$ with $R \leq r_1 \leq r_2 \leq 2R$

$$\lambda := \int_{\Omega_r(y_0)} (\Phi(\{Du\}))^{\frac{p(x)}{\phi(x)}} dx + \frac{1}{\delta} \int_{\Omega_r(y_0)} (\Phi(f))^{\frac{p(x)}{\phi(x)}} dx + 1,$$
where \( \delta > 0 \) and \( \eta > 1 \) will be specified later. We treat the super-level set
\[
E(\lambda, \Omega_r(x_0)) := \{ x \in \Omega_r(x_0), (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} > \lambda \}
\]
for any \( \lambda > M\Lambda \geq 1 \) with \( M = \left( \frac{2320r}{7(r_2^2 - r_1^2)} \right)^n \). For \( y \in E(\lambda, \Omega_r(x_0)) \) and radii \( 0 < r \leq r_2 - r_1 \), we let
\[
CZ(\Omega_r(y)) := \int_{D_r(y)} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} dx + \frac{1}{\delta} \left( \int_{D_r(y)} (\Phi(f))^\frac{\rho(\delta)}{\rho} \eta \right)^\frac{1}{\eta}.
\]
If \( \frac{r_2^2 - r_1^2}{105} \leq r \leq r_2 - r_1 \), then we discover that
\[
CZ(\Omega_r(y)) \leq \left( \frac{|\Omega_r(x_0)|}{|\Omega_r(y)|} \right) \int_{D_{r_2}(x_0)} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} dx + \left( \frac{|\Omega_r(x_0)|}{|\Omega_r(y)|} \right)^\frac{1}{\delta} \left( \int_{D_{r_2}(x_0)} (\Phi(f))^\frac{\rho(\delta)}{\rho} \eta \right)^\frac{1}{\eta} \]
\[
\leq \left( \frac{|B_{r_2}(x_0)|}{|B_r(y)|} \right)^\frac{n}{|\Omega_r(y)|} \left( \frac{1}{\sigma_r} \int_{D_{r_2}(x_0)} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} dx + \frac{1}{\delta} \left( \int_{D_{r_2}(x_0)} (\Phi(f))^\frac{\rho(\delta)}{\rho} \eta \right)^\frac{1}{\eta} \right) \]
\[
\leq \left( \frac{r_2^2 - r_1^2}{105} \right)^n \lambda_0 < \lambda,
\]
which means that while \( \frac{r_2^2 - r_1^2}{105} \leq r \leq r_2 - r_1 \) one has \( CZ(\Omega_r(y)) < \lambda \). At the same time, by Lebesgue’s differentiation theorem we find that \( CZ(\Omega_r(y)) > \lambda \) for \( 0 < r \ll 1 \). Therefore, by absolute continuity of the integral with respect to the domain we can pick the maximal radius \( r_y \) such that
\[
CZ(\Omega_y(y)) = \int_{D_{r_y}(y)} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} dx + \frac{1}{\delta} \left( \int_{D_{r_y}(y)} (\Phi(f))^\frac{\rho(\delta)}{\rho} \eta \right)^\frac{1}{\eta} = \lambda
\]
for each point \( y \in E(\lambda, \Omega_r(x_0)) \). Moreover, one has
\[
CZ(\Omega_y(y)) < \lambda, \quad \text{for any } r \in (r_y, r_2 - r_1).
\]
From (3.9), we conclude the following alternatives:
\[
\frac{\lambda}{2} \leq \int_{D_{r_y}(y)} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} dx \quad \text{or} \quad \left( \frac{\delta_\lambda}{2} \right)^{\eta} \leq \int_{D_{r_y}(y)} (\Phi(f))^\frac{\rho(\delta)}{\rho} \eta dx.
\]
First, we suppose that the first case of (3.11) is valid to have
\[
\int_{D_{r_y}(y)} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} dx
\[
= \frac{|\Omega_r(y) \backslash E(\frac{1}{4}, \Omega_r(x_0))|}{|\Omega_r(y)|} \int_{D_{r_y}(y) \backslash E(\frac{1}{4}, \Omega_r(x_0))} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} dx
\[
+ \frac{1}{|\Omega_r(y)|} \int_{D_{r_y}(y) \backslash E(\frac{1}{4}, \Omega_r(x_0))} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} \eta dx
\[
\leq \frac{\lambda}{4} + c \left( \frac{|\Omega_r(y) \cap E(\frac{1}{4}, \Omega_r(x_0))|}{|\Omega_r(y)|} \right)^{\frac{1}{\pi_1}} \left( \int_{D_{r_y}(y)} (\Phi(|Du|))^\frac{\rho(\delta)}{\rho} \eta dx \right)^{\frac{1}{\eta}},
\]
where \( \sigma_1 > 0 \) is determined later.
Considering that \( \omega(4R) < y_1 \sigma_2 \) in (3.1), it leads to \( \frac{y_1(1 + \sigma_2)}{y_1 + \omega(4R)} - 1 > 0 \). Now let us take

\[
0 < \sigma_1 \leq \frac{y_1(1 + \sigma_2)}{y_1 + \omega(4R)} - 1,
\]

which yields that

\[
\frac{p^*_y}{p^*} \left( 1 + \sigma_1 \right) = \left( 1 + \frac{p^*_y - p^*}{p^*} \right) \left( 1 + \sigma_1 \right) \leq \left( 1 + \frac{\omega(4R)}{p^*} \right) \left( 1 + \sigma_1 \right) \leq \left( 1 + \sigma_2 \right),
\]

where \( p^*_y := \sup_{\Omega_{12}\sigma_2} p(x) \) with \( \Omega_{12}\sigma_2 \subseteq \Omega_{2R}(x_0) \), and \( \omega(\cdot) \) is the modulus of continuity for \( p(x) \). Then we use the reverse Hölder inequality shown in Lemma 2.4 to obtain that

\[
\left( \int_{\Omega_{\sigma_2}(y)} (\Phi(|Du|))^{\frac{p^*_y}{p^*} (1 + \sigma_1)} \right)^{\frac{1}{1 + \sigma_1}} \leq \left( \int_{\Omega_{\sigma_2}(y)} (\Phi(|Du|))^{\frac{p^*_y}{p^*} (1 + \sigma_1)} \right)^{\frac{1}{1 + \sigma_1}} + c
\]

Taking into account (3.10), we have \( CZ(\Omega_{2\sigma_2}(y)) < \lambda \), then by a similar proof of (3.19) in **Step 2**, we get that

\[
\begin{cases}
\int_{\Omega_{\sigma_2}(y)} (\Phi(|Du|)) \leq c \lambda^{\frac{p^*_y}{p^*}}, \\
\int_{\Omega_{\sigma_2}(y)} (\Phi(|f|))^\eta \leq c \lambda^{\frac{p^*_y}{p^*} \delta \eta \eta}.
\end{cases}
\]

Thus, by taking \( \eta = 1 + \sigma_2 \) we have

\[
\left( \int_{\Omega_{\sigma_2}(y)} (\Phi(|Du|))^{\frac{p^*_y}{p^*} (1 + \sigma_1)} \right)^{\frac{1}{1 + \sigma_1}} \leq c \left( \lambda + \lambda \delta \eta \frac{p^*_y}{p^*} + 1 \right) \leq c \lambda.
\]

Therefore, we combine (3.11), (3.12) and (3.13) to have

\[
\frac{\lambda}{\eta} \leq c \left( \frac{|\Omega_{\sigma_2}(y) \cap E(\frac{\lambda}{\eta}, \Omega_{\tau}(x_0))|}{|\Omega_{\sigma_2}(y)|} \right)^{1 - \frac{1}{1 + \sigma_1}} \lambda.
\]

which implies

\[
|\Omega_{\sigma_2}(y)| \leq c |\Omega_{\sigma_2}(y) \cap E(\frac{\lambda}{\eta}, \Omega_{\tau}(x_0))|
\]

with the positive constant \( c \) depending only on \( n, \nu, \lambda, y_2, y_2, \sigma, \tau, R_0, K_0 \), and \( |\Omega| \).

If the case of the second estimate in (3.11) is valid, by taking \( \zeta = \frac{\lambda}{\eta} \) and Fubini’s theorem, we get

\[
\left( \frac{\lambda + \delta \eta \frac{p^*_y}{p^*}}{\lambda} \right)^\eta \leq \int_{\Omega_{\sigma_2}(y)} (\Phi(|f|))^{\frac{p^*_y}{p^*} \eta} \mu
\]

\[
= \frac{\eta}{|\Omega_{\sigma_2}(y)|} \int_0^{\mu \eta} \eta \mu |\{ x \in \Omega_{\sigma_2}(y) : (\Phi(|f|))^{\frac{p^*_y}{p^*} \eta} > \mu \}| \frac{d\mu}{\mu}
\]

\[
= \frac{\eta}{|\Omega_{\sigma_2}(y)|} \int_0^{\lambda \mu} \eta \mu |\{ x \in \Omega_{\sigma_2}(y) : (\Phi(|f|))^{\frac{p^*_y}{p^*} \eta} > \mu \}| \frac{d\mu}{\mu}
\]

\[
+ \frac{\eta}{|\Omega_{\sigma_2}(y)|} \int_{\lambda \mu}^{\eta \mu} \eta \mu |\{ x \in \Omega_{\sigma_2}(y) : (\Phi(|f|))^{\frac{p^*_y}{p^*} \eta} > \mu \}| \frac{d\mu}{\mu}
\]
Let $\delta = 4\zeta$, we derive that

$$\left|\Omega_{r_1}(y)\right| \leq \frac{\eta}{(\zeta\Lambda)^n} \int_{\Omega} \mu^n \left\{ x \in \Omega_{r_1}(y) : (\Phi(f))^{\frac{p(x)}{p'}} > \mu \right\} \frac{d\mu}{\mu}. \quad (3.15)$$

Now we put (3.14) and (3.15) together to get that

$$\left|\Omega_{r_1}(y)\right| \leq c_{\text{ref}} \left|\Omega_{r_1}(y) \cap E\left(\frac{\Lambda}{\Lambda}, \Omega_{r_1}(x_0)\right)\right| + \frac{c\eta}{(\zeta\Lambda)^n} \int_{\Omega} \mu^n \left\{ x \in \Omega_{r_1}(y) : (\Phi(f))^{\frac{p(x)}{p'}} > \mu \right\} \frac{d\mu}{\mu}. \quad (3.16)$$

**Step 2.** This step is devoted to various comparison estimates with the reference problems and the limiting one. Note that $\Omega$ is $\delta, R_0$-Reifenberg flat, then it follows from (3.3) and (3.10) that

$$\begin{aligned}
\int_{\Omega_{125r_y}} \left(\Phi(|Du(z')|)\right)^{\frac{p(x)}{p'}} dz' + \frac{1}{\delta} \left(\int_{\Omega_{170r_y}} \left(\Phi(|Du(z')|)\right)^{\frac{p(x)}{p'} \eta} dz'\right)^{\frac{1}{\eta}} \\
\leq 2 \cdot \left(\frac{34}{25}\right)^{n} \left(\int_{\Omega_{170r_y}} \left(\Phi(|Du(z')|)\right)^{\frac{p(x)}{p'} \eta} dz'\right)^{\frac{1}{\eta}}
\end{aligned} \quad (3.17)$$

which implies that

$$\begin{aligned}
\left\{ \int_{\Omega_{125r_y}} \Phi(|Du|) \frac{p}{p'} dx \leq c\Lambda, \\
\left(\int_{\Omega_{125r_y}} \left(\Phi(|Du(z')|)\right)^{\frac{p(x)}{p'} \eta} dz'\right)^{\frac{1}{\eta}} \leq c\Lambda \delta, \right. \end{aligned} \quad (3.18)$$

where we still use variable $x$ for simplicity. Therefore, it suffices to show that

$$\begin{aligned}
\left\{ \int_{\Omega_{125r_y}} \Phi(|Du|) dx \leq c_3 \Lambda^{\frac{\eta}{p'}} , \\
\int_{\Omega_{125r_y}} \left(\Phi(|Du|)\right)^{\frac{p}{p'}} dx \leq c_3 \Lambda^{\frac{\eta}{p'}} \delta^{\frac{1}{\eta}} \right. \end{aligned} \quad (3.19)$$

for a constant $c_3 \geq 1$. We first claim that

$$\begin{aligned}
\left(\int_{\Omega_{125r_y}} \Phi(|Du|) dx\right)^{p_y - p'_y} \leq c_4 \\
\end{aligned} \quad (3.20)$$

with $c_4 \geq 1$ a universal constant. In fact, since $(\Phi(f))^{p(x)} \in L^{1,q}(\Omega)$ for $t > 1$ and $0 < q < \infty$, it deduces that

$$\int_{\Omega} \Phi(f) dx \leq \int_{\Omega} (\Phi(f) + 1)^{p(x)} dx \leq c \left\| (\Phi(f))^{p(x)} \right\|_{L^{1,q}(\Omega)} + c |\Omega| \leq c \left(1 + |\Omega| \right),$$

where we have used (3.6). By the energy estimate of Lemma 2.1, it leads to that

$$\int_{\Omega} \Phi(|Du|) dx \leq c \left(1 + |\Omega| \right). \quad (3.21)$$

By considering $p_y - p'_y \leq \omega(250r_y)$, we have

$$\begin{aligned}
\left(\int_{\Omega_{125r_y}} \Phi(|Du|) dx\right)^{p_y - p'_y} \\
= \left(\frac{1}{|\Omega_{125r_y}|}\right)^{p_y - p'_y} \left(\int_{\Omega_{125r_y}} \Phi(|Du|) dx\right)^{p_y - p'_y}
\end{aligned} \quad (3.22)$$
Similarly, recalling (3.21) and $\frac{1}{250r_\gamma} \geq \frac{1}{R} \geq \frac{1}{|\Omega|} + 1$ with (3.1) to find

$$\int_{\Omega_{125r_\gamma}} \Phi(Du) \, dx \leq c\left( \int_{\Omega} \Phi(Du) \, dx \right)^{p_\gamma-p_r} \left( \int_{\Omega_{125r_\gamma}} \Phi(Du) \, dx \right)^{p_r-p_\gamma} \leq c(\Omega + 1)^{p_\gamma-p_r} \leq c \left( \frac{1}{250r_\gamma} \right)^{1/2} \leq c,$$

where we have used the so-called log-Hölder continuity (1.8) for $p(x)$ in the last inequality, which yields (3.20). Recalling $y_1 \leq p_\gamma$ and (3.20) with $\lambda > 1$, we obtain

$$\int_{\Omega_{125r_\gamma}} \Phi(Du) \, dx \leq c \left( \int_{\Omega} \Phi(Du) \, dx \right)^{p_\gamma-p_r} \left( \int_{\Omega_{125r_\gamma}} \Phi(Du) \, dx \right)^{p_r-p_\gamma} \leq c \left( \int_{\Omega_{125r_\gamma}} \Phi(Du) \, dx \right)^{p_\gamma-p_r} \leq c \left( \int_{\Omega_{125r_\gamma}} \Phi(Du) \, dx \right)^{p_r-p_\gamma} \leq c \int_{\Omega_{125r_\gamma}} \Phi(Du) \, dx \leq c \lambda^{\frac{1}{p_\gamma}}.$$

Note that $(\Phi(f))^p(x) \in L^{1,q}(\Omega)$ for $1 < y_1 \leq p(x) \leq y_2 < \infty$, and $1 < q \leq 1 + \sigma$ with $\sigma$ as the same of Lemma 2.4, it holds

$$\int_{\Omega} (\Phi(f))^q \, dx \leq \left( \int_{\Omega} (\Phi(f))^{1+\sigma} \, dx \right)^{1/q} + \int_{\Omega} \left( (\Phi(f))^q \right)^{1/q} \, dx \leq c \left( 1 + |\Omega| \right).$$

Similarly, recalling $\delta \lambda_0 \geq 1$ and $\lambda \geq M \lambda_0$ we find

$$\int_{\Omega_{125r_\gamma}} (\Phi(f))^q \, dx \leq c \left( \int_{\Omega_{125r_\gamma}} (\Phi(f))^{p^{(a)}} \, dx \right)^{1/q} \leq c \left( \delta \lambda \right)^{\frac{p^{(a)}}{q}} \leq c \left( \delta \lambda \right)^{\frac{p^{(a)}}{q}} \leq c \left( \delta \lambda + \delta \lambda_0 \right)^{\frac{p^{(a)}}{q}} \leq c \lambda^{\frac{p^{(a)}}{q}} \delta^{\frac{1}{q^2}}.$$ 

Next, we define

$$\tilde{A}(x, \xi) := A(25r_\gamma, x, \lambda^{\frac{p}{q}} \xi), \quad \tilde{G}(x, \xi) := G(25r_\gamma, x, \lambda^{\frac{p}{q}} \xi),$$
\[ \tilde{u}(x) := \frac{u(25r_\delta x)}{25r_\delta \lambda^{\frac{1}{p}}} , \quad \tilde{f}(x) := \frac{f(25r_\delta x)}{\lambda^{\frac{1}{p}}} , \quad \tilde{\phi}(s) := \phi\left(\lambda^{-\frac{1}{p}}s\right) , \quad \tilde{\Phi}(s) := \frac{\Phi\left(\lambda^{-\frac{1}{p}}s\right)}{\lambda^{\frac{1}{p}}} . \]

In light of (3.2), (3.4), (3.19) and Lemma 2.2, we find that the hypothesis of Lemma 2.6 is valid, which implies that
\[ \int_{D_{2r_\delta}} |V(Du) - V(D\tilde{v})|^2 \, dx \leq e\lambda^{\frac{1}{p}} , \]
and
\[ \|\tilde{\Phi}(D\tilde{v})\|_{L^\infty(\Omega_{2r_\delta})} \leq c_0 \lambda^{\frac{1}{p}} , \]
where \( e > 0 \) is small and \( c_0 = \max\{c_1, c_2\} \geq 1 \).

For the case of interior, by Lemma 2.5 similarly we have
\[ \int_{B_{2r_\delta}} |V(Du) - V(D\tilde{v})|^2 \, dx \leq e\lambda^{\frac{1}{p}} , \]
and
\[ \|\tilde{\Phi}(D\tilde{v})\|_{L^\infty(B_{2r_\delta})} \leq c_0 \lambda^{\frac{1}{p}} . \]

**Step 3.** We are here to estimate the super level set \( E(\lambda, \Omega_r(x_0)) \). For any fixed point \( x \in \Omega \), we select a universal constant \( R \) with \( 0 < R \leq \min\left\{ \frac{\delta}{2}, \frac{R}{\delta(\epsilon)} , 1 \right\} \), and there exists a constant \( \delta = \delta(\epsilon) > 0 \) such that Lemma 2.5 and 2.6 hold. Let \( A = (2 r^{s+1} c_0 (m_0 + 1))^\frac{1}{p} \), for any \( x \in E(A\lambda, \Omega_r(x_0)) \) we consider the collection \( \mathcal{B}_A \) of all subset \( \Omega_r(y) \). By “crawling of ink spots” argument, we extract a countable subcollection \( \{\Omega_r(y_i)\} \in \mathcal{B}_A \), such that
\[ \Omega_r(y_i) \cap \Omega_r(y_j) = \emptyset , \quad \text{whenever } i \neq j , \quad \text{and } E(A\lambda, \Omega_r(x_0)) \subset \bigcup_{i \in \mathbb{N}} \Omega_r(y_i) \cup \mathcal{N}_A \]
with \( |\mathcal{N}_A| = 0 \). Let us denote \( p^*_i = p^*_j \), then we derive that
\[ \begin{align*}
|E(A\lambda, \Omega_r(x_0))| &= |E((2 r^{s+1} c_0 (m_0 + 1))^\frac{1}{p} \lambda, \Omega_r(x_0))| \\
&= \left| \{ x \in \Omega_r(x_0) : \Phi(|Du|) > (2 r^{s+1} c_0 (m_0 + 1))^\frac{1}{p} \lambda \} \right| \\
&\leq \sum_{i=1}^{\text{interior case}} \left| \{ x \in \Omega_r(y_i) : \Phi(|Du|) > 2 r^{s+1} c_0 (m_0 + 1)^\frac{1}{p} \lambda \} \right| \\
&+ \sum_{i=1}^{\text{boundary case}} \left| \{ x \in \Omega_r(y_i) : \Phi(|Du|) > 2 r^{s+1} c_0 (m_0 + 1)^\frac{1}{p} \lambda \} \right| .
\end{align*} \]

For the boundary case, we recall (2.2), (2.3) and (2.4) to find that
\[ \Phi(|Du|) \leq \Phi(|Du - D\tilde{v}| + |D\tilde{v}|) \leq \frac{1}{2} \left( \Phi(2|Du - D\tilde{v}|) + \Phi(2|D\tilde{v}|) \right) \leq 2^{s+1} \left( \Phi(|Du - D\tilde{v}|) + \Phi(|D\tilde{v}|) \right) \leq 2^{s+1} (m_0 + 1) \left( |V(Du) - V(D\tilde{v})|^2 + \Phi(|D\tilde{v}|) \right) , \]
which implies that
\[ \left| \{ x \in \Omega_r(y_i) : \Phi(|Du|) > 2 r^{s+1} c_0 (m_0 + 1)^\frac{1}{p} \lambda \} \right| . \]
\[ \begin{align*}
&\leq \left| \left\{ x \in \Omega_{5\alpha} : |V(Du) - V(D\bar{v})|^2 > c_0 \lambda^{\frac{\nu}{p'}} \right\} \right| \\
&\quad + \left| \left\{ x \in \Omega_{5\alpha} : \Phi(|D\bar{v}|) > c_0 \lambda^{\frac{\nu}{p'}} \right\} \right| \\
&\leq \frac{1}{c_0 \lambda^{\frac{\nu}{p'}}} \int_{\Omega_{5\alpha}} |V(Du) - V(D\bar{v})|^2 \, dx \\
&\leq ce|\Omega_{5\alpha}| \leq ce|B_{5\alpha}| = ce|B_r(y_i)| \leq ce\left| \frac{B_r(y_i)}{|\Omega_r(y_i)|} \right| \Omega_r(y_i) \\
&\leq ce\left( \frac{2}{1 - \delta} \right)^n \left| \Omega_r(y_i) \right| \leq ce|\Omega_r(y_i)|,
\end{align*} \]

where we used the following weak (1,1)-type estimate:
\[ \left| \left\{ x \in U : g(x) > \lambda \right\} \right| \leq \frac{1}{\lambda} \int_U g(x) \, dx. \]

Similarly, for the interior case, we discover that
\[ \left| \left\{ x \in \Omega_{2r} : \Phi(|D\bar{u}|) > 2^{r+1} c_0 (m_0 + 1) \lambda^{\frac{\nu}{p'}} \right\} \right| \]
\[ = \left| \left\{ x \in B_{2r} : \Phi(|D\bar{u}|) > 2^{r+1} c_0 (m_0 + 1) \lambda^{\frac{\nu}{p'}} \right\} \right| \]
\[ \leq ce|B_r(y_i)|. \]

Therefore, it follows from (3.22) that
\[ |E(A\lambda, \Omega_r(x_0))| \leq ce \sum_{i=1}^{\infty} |\Omega_r(y_i)|. \] (3.23)

Using "crawling of ink spots" argument again and (3.16), we conclude that
\[ \left| E(A\lambda, \Omega_r(x_0)) \right| \leq c e \sum_{i=1}^{\infty} \Omega_r(y_i) \cap E \left( \frac{\lambda}{q}, \Omega_r(x_0) \right) + c e \left( \frac{n}{\lambda^\mu} \right) \sum_{i=1}^{\infty} \int_{\Omega_r(y_i)} \mu^\eta \{ x \in \Omega_r(y_i) : (\Phi(f))^{\frac{\mu}{p'}} > \mu \} \, d\mu \]
\[ \leq ce \left| E \left( \frac{\lambda}{q}, \Omega_r(x_0) \right) \right| + c e \left( \frac{n}{\lambda^\mu} \right) \sum_{i=1}^{\infty} \int_{\Omega_r(y_i)} \mu^\eta \{ x \in \Omega_r(y_i) : (\Phi(f))^{\frac{\mu}{p'}} > \mu \} \, d\mu \]
\[ \leq ce \left| E \left( \frac{\lambda}{q}, \Omega_r(x_0) \right) \right| + \frac{c \eta}{\lambda^\mu} \int_{\Omega_r(y_i)} \mu^\eta \{ x \in \Omega_r(y_i) : (\Phi(f))^{\frac{\mu}{p'}} > \mu \} \, d\mu \] (3.24)

**Step 4.** This step is devoted to the estimate of \( \left\| (\Phi(|Du|))^{p(q)} \right\|_{L^p(\Omega_r(x_0))} \) for \( 0 < q < \infty \). Since \( t > 1 \), we multiply the inequality (3.24) by \( (p^q)^{\frac{t}{q}} (A\lambda)^p \), and integrate the resulting expression with a power \( \frac{t}{q} \) in the measure \( d\lambda \) from \( M\lambda_0 \) to \( \lambda \), which yields that
\[ \left| \frac{t}{q} \int_{M\lambda_0}^{\lambda} (A\lambda)^p \left\{ \{ x \in \Omega_r(x_0) : (\Phi(|Du|))^{\frac{\mu}{p'}} > \lambda \} \right\} \frac{d\lambda}{\lambda} \right| \]
\[ \leq c \cdot \left| \frac{t}{q} \right| \int_{\Omega_r(x_0)} (A\lambda)^p \left\{ \{ x \in \Omega_r(x_0) : (\Phi(|Du|))^{\frac{\mu}{p'}} > \lambda \} \right\} \frac{d\lambda}{\lambda} \\
+ c \cdot \left| \frac{t}{q} \right| \int_{\Omega_r(x_0)} (A\lambda)^p \left\{ \{ x \in \Omega_r(x_0) : (\Phi(|Du|))^{\frac{\mu}{p'}} > \lambda \} \right\} \frac{d\lambda}{\lambda} \]
\[ := ce \left( I_1 + I_2 \right), \] (3.25)

where \( c \) depends on \( n, v, \Lambda, \gamma_1, \gamma_2, t, q, \sigma_\rho, \tau_\rho, R_0, K_0, |\Omega|, \) and \( \omega(\cdot) \). Thanks to (2.12) in Proposition 2.7, we have
\[ \left\| (\Phi(|Du|))^{p(q)} \right\|_{L^p(\Omega_r(x_0))} \leq \left\| (\Phi(|Du|))^{\frac{\mu}{p'}} \right\|_{L^{p(q)}(\Omega_r(x_0))}. \]
By a simple change of variable and (3.26) it leads to that

$$I_1 = c(q) \left\| (\Phi(Du))^{p(x)} \right\|_{L^\lambda(\Omega_2(x_0))}^q.$$  

We are now to estimate $I_2.$ For this we part it in two cases.

**Case 1.** If $q \geq t,$ noticing that (2.13) is satisfied since $(\Phi(f))^{\frac{p(t)}{q}} \in L^q(\Omega_2).$ By making the change of variables

$$\lambda = \zeta \lambda \text{ and } \zeta = \frac{\lambda}{t},$$

then we employ Lemma 2.8 with $\psi(\mu) = \mu^{t-1} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \mu \} \right\}^\frac{1}{t}, a = \frac{t}{q} \geq 1,$ $r = q (\frac{1}{2} - \frac{t}{q}) > 0$ and (3.26) to infer

$$I_2 = c \cdot t p \int_0^\infty \lambda^{q(p-\frac{t}{q})} \left( \int_\lambda^\infty \mu^q \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \mu \} \right\}^\frac{1}{t} d\mu \right)^\frac{q}{t} d\lambda,$$

where $c = c(y_1, y_2, t, q).$

**Case 2.** If $0 < q < t,$ we use Lemma 2.9 with $h(\mu) = \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \mu \} \right\}^\frac{t}{q}, r = \frac{tq}{r}, a_1 = \frac{t}{q} < \frac{t}{q} = a_2$ and $\varepsilon = 1,$ which yields the following

$$\left( \int_\lambda^\infty \mu^q \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \mu \} \right\}^\frac{1}{t} d\mu \right)^\frac{q}{t} \leq \lambda^{\frac{t}{q}} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \lambda \} \right\}^\frac{q}{t} + t p \int_0^\infty \lambda^{q(p-\frac{t}{q})} \left( \int_\lambda^\infty \mu^q \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \mu \} \right\}^\frac{1}{t} d\mu \right)^\frac{q}{t} d\lambda.$$

After a change variable $\zeta \lambda \rightarrow \lambda,$ (3.26) and Fubini’s theorem we get

$$I_2 \leq c \cdot t p \int_0^\infty \lambda^{q(p-\frac{t}{q})} \lambda^{q(p-\frac{t}{q})} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \lambda \} \right\}^\frac{q}{t} d\lambda$$

$$+ c \cdot t p \int_0^\infty \lambda^{q(p-\frac{t}{q})} \int_\lambda^\infty \mu^{q-1} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \mu \} \right\}^\frac{q}{t} \frac{d\mu}{d\lambda}$$

$$\leq c \left\| (\Phi(f))^{p(x)} \right\|_{L^\lambda(\Omega_2(x_0))}^q + c \cdot t p \int_0^\infty \lambda^{q(p-\frac{t}{q})} \left( \int_\lambda^\infty \mu^{q-1} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(f))^{\frac{p(t)}{q}} > \mu \} \right\}^\frac{q}{t} d\mu \right)^\frac{q}{t} d\lambda \leq c \left\| (\Phi(f))^{p(x)} \right\|_{L^\lambda(\Omega_2(x_0))}^q,$$

where $c = c(y_1, y_2, t, q).$

We are now in a position to put the estimates of $I_1$ and $I_2$ into (3.25), and after simple manipulation, then for $t > 1$ it follows that

$$\left\| (\Phi(Du))^{p(x)} \right\|_{L^\lambda(\Omega_2(x_0))} \leq$$

$$c \left( t p \int_0^\infty \left( (AA)^{p(t)} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(Du))^{\frac{p(t)}{q}} > AA \} \right\} \right)^\frac{1}{t} d(AA) \right)^\frac{1}{t}$$

$$+ c \left( t p \int_0^\infty \left( (AA)^{p(t)} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(Du))^{\frac{p(t)}{q}} > AA \} \right\} \right)^\frac{1}{t} d(AA) \right)^\frac{1}{t}$$

$$\leq c \left( t p \int_0^\infty \left( (AA)^{p(t)} \left\{ \{ x \in \Omega_2(x_0) : (\Phi(Du))^{\frac{p(t)}{q}} > AA \} \right\} \right)^\frac{1}{t} d(AA) \right)^\frac{1}{t}$$

$$+ c \left( \left\| (\Phi(Du))^{p(x)} \right\|_{L^\lambda(\Omega_2(x_0))}^q + \left\| (\Phi(f))^{p(x)} \right\|_{L^\lambda(\Omega_2(x_0))} \right)^\frac{1}{t}.$$
\[
\leq c\lambda^\eta_0|\Omega_r(x_0)|^{\frac{d}{2}} + \tilde{c}e^{\frac{1}{2}} \left( \left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega_r(x_0))} + \left\| (\Phi(f))^{(p(x))} \right\|_{L^{\eta}(\Omega_r(x_0))} \right),
\]

(3.27)

where \( \tilde{c} = \tilde{c}(n, \nu, A, y_1, y_2, t, q, \sigma_\phi, \varphi, \tau, R_0, K_0, |\Omega|, \omega(\cdot)) > 0 \). Now we choose \( \epsilon > 0 \) small enough such that \( \tilde{c}e^{\frac{1}{2}} \leq \frac{1}{2} \), then we can find a corresponding positive constant \( \delta = \delta(n, \nu, A, y_1, y_2, t, q, \sigma_\phi, \varphi, \tau, R_0, K_0, |\Omega|) \) such that we deduce

\[
\left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega_r(x_0))} \leq c\lambda^\eta_0|\Omega_r(x_0)|^{\frac{d}{2}} + \frac{1}{2} \left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega_r(x_0))} + c \left\| (\Phi(f))^{(p(x))} \right\|_{L^{\eta}(\Omega_r(x_0))}.
\]

(3.28)

**Step 5.** This step is devoted to proving that \( \left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega_r(x_0))} < \infty \). To this end, we first refine the estimate of \( (\Phi(Du))^{(p(x))} \) in the scale of Lorentz spaces. Consider the following truncated function

\[
\left\| (\Phi(Du))^{(p(x))} \right\|_{k} = \min \left\{ (\Phi(Du))^{(p(x))}, k \right\}
\]

for \( x \in \Omega \) and \( k \in \mathbb{N} \cap [M\lambda_0, \infty) \).

By considering \( E_k(\lambda, \Omega_\rho) = \left\{ x \in \Omega_\rho : \left\| (\Phi(Du))^{(p(x))} \right\|_{k} > \lambda \} \) in line with (3.24), we discover that

\[
E_k(A_\lambda, \Omega_r(x_0)) = c\varepsilon E_k(\frac{\lambda}{\varepsilon}, \Omega_r(x_0)) + c\varepsilon \int_{\Omega} \mu_{\eta} \left\{ x \in \Omega_r(x_0) : \left\| (\Phi(f))^{(p(x))} \right\|_{\rho} > \mu \} \frac{d\mu}{\mu}.
\]

(3.29)

For \( 0 < k \leq A_\lambda \) we have \( E_k(A_\lambda, \Omega_r(x_0)) = \emptyset \), which implies that the above estimate holds trivially. For \( k > A_\lambda \), the estimate is also valid since

\[
E_k(A_\lambda, \Omega_r(x_0)) = E(A_\lambda, \Omega_r(x_0)) = \{ x \in \Omega_r(x_0), (\Phi(Du))^{(p(x))} > A_\lambda \} \text{ and } E_k(\frac{\lambda}{\varepsilon}, \Omega_r(x_0)) = E(\frac{\lambda}{\varepsilon}, \Omega_r(x_0)).
\]

Then working exactly as in Step 4, we get that (3.28) holds with \( \left\| (\Phi(Du))^{(p(x))} \right\|_{k} \) in place of \( (\Phi(Du))^{(p(x))} \).

Now we let \( L = c\lambda^\eta_0|\Omega_{2R}(x_0)|^{\frac{d}{2}} + c \left\| (\Phi(f))^{(p(x))} \right\|_{L^{\eta}(\Omega_{2R}(x_0))} \) for a fixed small radius \( R > 0 \), \( \Theta(\rho) = \left\| (\Phi(Du))^{(p(x))} \right\|_{k} \left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega_{\rho}(x_0))} \) for any \( R \leq \rho < 2R \), and take a small \( \epsilon > 0 \) such that \( \epsilon \varepsilon = \frac{1}{2} \) in (3.28). Then for any \( R \leq \rho_1 < \rho_2 < \rho_3 < \cdots < 2R \) with \( \rho_d \rightarrow 2R, d \rightarrow \infty \),

\[
\Theta(\rho_1) \leq \frac{1}{2} \Theta(\rho_2) + L, \quad \Theta(\rho_2) \leq \frac{1}{2} \Theta(\rho_3) + L, \quad \cdots
\]

by the iteration we get that

\[
\Theta(R) \leq \frac{1}{2^d} \Theta(\rho_d) + L \sum_{i=0}^{d-1} \frac{1}{2^i}.
\]

Note that

\[
\Theta(2R) = \left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega_{2R}(x_0))} < \infty,
\]

we let \( d \rightarrow \infty \) to get

\[
\left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega_{\rho}(x_0))} \leq c\lambda^\eta_0|\Omega_{2R}(x_0)|^{\frac{d}{2}} + c \left\| (\Phi(f))^{(p(x))} \right\|_{L^{\eta}(\Omega_{2R}(x_0))}.
\]

In what follows, we use a standard finite covering argument to realize our global estimate. Note that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and let us now take \( x_0 \) as every point in \( \Omega \). Then there exist \( N = N(n, |\Omega|) \in \mathbb{N} \) and \( x_j \in \Omega \) for \( j = 1, 2, \cdots, N \), where we replace the point \( x_0 \) by each \( x_j \), such that

\[
\overline{\Omega} \subset \bigcup_{j=1}^{N} \Omega_{R_k}(x_j).
\]

Therefore, we deduce that

\[
\left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega)} \leq \sum_{j=1}^{N} \left\| (\Phi(Du))^{(p(x))} \right\|_{k} \left\| (\Phi(Du))^{(p(x))} \right\|_{L^{\eta}(\Omega)}
\]
where we use the definition of $\lambda_0$. By (3.1), we notice that

$$\frac{p^+}{p^-} = 1 + \frac{p^+ - p^-}{p^-} \leq 1 + \frac{\omega(4R)}{y_1} \leq 1 + \sigma,$$

where $\sigma$ is the same as in Lemma 2.4. Then, it yields that

$$\int_{D_{2r}(x_j)} \left( \frac{\Phi(Du)}{p^-} \right)^{\frac{p^+}{p^-}} dx \leq \int_{D_{2r}(x_j)} \left( \frac{\Phi(Du)}{p^-} \right)^{\frac{p^+}{p^-}} dx + 1$$

$$\leq c \left( \int_{D_{2r}(x_j)} (\Phi(Du))^{\frac{p^+}{p^-}} dx + 1 \right) + c \int_{D_{2r}(x_j)} (\Phi(f))^{\frac{p^+}{p^-}} dx,$$

(3.31)

where we employed the reverse Hölder inequality of Lemma 2.4 in the last inequality. Using (2.1) and Hölder inequality, we obtain that

$$\left( \int_{D_{2r}(x_j)} \Phi(Du) dx \right)^{\frac{p^+}{p^-}}$$

$$\leq \left( \frac{1}{|D_{2r}(x_j)|} \right)^{\frac{p^+}{p^-}} \left( \int_{\Omega} (\Phi(Du)) dx \right)^{\frac{p^+}{p^-}}$$

$$\leq c \left( \frac{1}{|D_{2r}(x_j)|} \right)^{\frac{p^+}{p^-}} \left( \int_{\Omega} (\Phi(f)) dx \right)^{\frac{p^+}{p^-}}$$

$$\leq c \left( \frac{1}{|D_{2r}(x_j)|} \right)^{\frac{p^+}{p^-}} |\Omega|^{1 - \frac{p^+}{p^-}} \int_{\Omega} (\Phi(f))^{\frac{p^+}{p^-}} dx$$

$$\leq c \left( \frac{1}{|D_{2r}(x_j)|} \right)^{\frac{p^+}{p^-}} \int_{\Omega} \left( (\Phi(f))^{\frac{p^+}{p^-} \cdot \frac{p^+}{p^-}} + 1 \right) dx.$$  

(3.32)

We now combine (3.30), (3.31) and (3.32) to get that

$$\| (\Phi(Du))^{\frac{p^+}{p^-}} \|_{L^{\frac{4}{\gamma}}(\Omega)}$$

$$\leq c \sum_{j=1}^{N} \left( \frac{1}{|D_{2r}(x_j)|} \right)^{\frac{p^+}{p^-}} \int_{\Omega} \left( (\Phi(f))^{\frac{p^+}{p^-} \cdot \frac{p^+}{p^-}} + 1 \right) dx + \left( \frac{1}{|D_{2r}(x_j)|} \right)^{\frac{p^+}{p^-}} \left( \int_{\Omega} \left( (\Phi(f))^{\frac{p^+}{p^-} \cdot \frac{p^+}{p^-}} + 1 \right) dx \right)^{\frac{1}{p^-}}$$

$$+ c N \| (\Phi(f))^{\frac{p^+}{p^-}} \|_{L^{\frac{4}{\gamma}}(\Omega)}.$$  

(3.33)
Using a standard Hardy’s inequality in the Marcinkiewicz spaces (cf. Lemma 2.8 in [26]) and the reverse Hölder inequality of Lemma 2.9, we conclude that

\[
\int_{\Omega} (\Phi(f_j))^\frac{p_0}{p_0 - 1} \frac{p_0}{p_0 - 1} dx \\
\leq \frac{t(p)^2}{t(p)^2 - p^*} |\Omega|^{\frac{1}{p^*}} \| (\Phi(f_j))^\frac{p_0}{p_0} \|_{L^p(\Omega)}^{\frac{p_0}{p_0 - 1}} \\
= \frac{t(p)^2}{t(p)^2 - p^*} |\Omega|^{\frac{1}{p^*}} \left( \sup_{h > 0} \left( h^{p^*} \left\{ x \in \Omega : (\Phi(f_j))^\frac{p_0}{p_0} > h \right\} \right) \right)^{\frac{p_0}{p_0 - 1}} \\
\leq c |\Omega|^{\frac{1}{p^*}} \| (\Phi(f_j))^\frac{p_0}{p_0} \|_{L^{p^*}(\Omega)}^{\frac{p_0}{p_0 - 1}} \\
\leq c |\Omega|^{\frac{1}{p^*}} \| (\Phi(f_j))^p(x) \|_{L^p(\Omega)}^{\frac{1}{p_0}}.
\]

Similarly, we also show that

\[
\left( \int_\Omega (\Phi(f_j))^{\frac{p_0}{p_0 - q}} dx \right)^{\frac{1}{q}} \leq c(y_1, y_2, t, q) |\Omega| \left( \frac{1}{p^*} \| (\Phi(f_j))^\frac{p_0}{p_0} \|_{L^{p_0}(\Omega)} \right)^{\frac{1}{p}} \\
\leq c(y_1, y_2, t, q) |\Omega| \left( \frac{1}{p^*} \| (\Phi(f_j))^p(x) \|_{L^p(\Omega)} \right)^{\frac{1}{p_0}}.
\]

For the case of \( q < \infty \), from (3.33) we then infer the following relations

\[
\left\| (\Phi(|Df|))^{p(x)} \right\|_{L^{p(x)}(\Omega)} \\
\leq c \sum_{j=1}^N \left( \frac{1}{|\Omega_2(x)|} \right)^p \left( \left\| (\Phi(f_j))^{p(x)} \right\|_{L^{p(x)}(\Omega)} + 1 \right)^{\frac{p}{p(x)}} \\
+ c \sum_{j=1}^N \left( \frac{1}{|\Omega_2(x)|} \right)^p \left( \left\| (\Phi(f_j))^{p(x)} \right\|_{L^{p(x)}(\Omega)} + 1 \right)^{\frac{p}{p(x)}} \\
\leq c \sum_{j=1}^N \left( \frac{1}{|\Omega_2(x)|} \right)^p \left( \left\| (\Phi(f_j))^{p(x)} \right\|_{L^{p(x)}(\Omega)} + 1 \right)^{\frac{p}{p(x)}} \\
+ c \sum_{j=1}^N \left( \frac{1}{|\Omega_2(x)|} \right)^p \left( \left\| (\Phi(f_j))^{p(x)} \right\|_{L^{p(x)}(\Omega)} + 1 \right)^{\frac{p}{p(x)}} \\
\leq cN \left( \left\| (\Phi(f_j))^{p(x)} \right\|_{L^{p(x)}(\Omega)} + 1 \right)^{\frac{p}{p(x)}} \leq c,
\]

where we used the uniformly estimate (3.6). Now let us take \( k \to \infty \). By the lower semi-continuity of Lorentz quasi-norm we have

\[
\left\| (\Phi(|Df|))^{p(x)} \right\|_{L^{p(x)}(\Omega)} \leq c,
\]

where \( c \) depends only on \( n, v, \Lambda, y_1, y_2, t, q, \sigma, \tau, R_0, K_0, \omega(\cdot) \), and \( |\Omega| \). And recalling the definition in (3.5), we get the desired result (1.12)

**Step 6.** Finally, for the case of \( q = \infty \), we come back to the second inequality in (3.11) and split it into two parts with a small \( t > 0 \) determined later:
\[ \left( \frac{\lambda}{2} \right)^{\eta} \leq \frac{1}{\delta^n} \int_{\Omega_t(y)} \left( \Phi(|f|) \right)^{\frac{\rho(t)}{\lambda}} \eta \, dx \leq \frac{(\lambda)^{\eta}}{\delta^n} + \frac{1}{\delta^n |\Omega_t(y)|} \int_{\{x \in \Omega_t(y) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} > 1\}} (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} \eta \, dx. \]

We set
\[ \Psi(\mu, \Omega_t(y)) = \{ x \in \Omega_t(y) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} > \mu \}, \]
\[ \Psi(\mu, \Omega_t(y)) = \{ x \in \Omega_t(y) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} > \mu \}. \]

Similar to the estimate (3.34), by using Hölder inequality we get
\[ \left( \frac{\lambda}{2} \right)^{\eta} - \left( \frac{\lambda}{\delta} \right)^{\eta} \leq \frac{1}{\delta^n |\Omega_t(y)|} \int_{\{x \in \Omega_t(y) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} > 1\}} (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} \eta \, dx \]
\[ \leq \frac{t}{(t - \eta) \delta^n} \sup_{\mu > 1} \left\{ x \in \Psi(\lambda, \mu, \Omega_t(y)) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} \geq \mu \right\} \]
\[ \leq \frac{t |\Psi(\lambda, \Omega_t(y))|^{\frac{1 - \eta}{\eta}}}{(t - \eta) \delta^n |\Omega_t(y)|} (\lambda)^{\eta} |\Psi(\lambda, \Omega_t(y))|^{\frac{1 - \eta}{\eta}} \sup_{\mu > 1} \mu^{\eta} |\Psi(\mu, \Omega_t(y))|^{\frac{1 - \eta}{\eta}} \]
\[ = \frac{t}{(t - \eta) \delta^n} \left[ (\lambda)^{\eta} + \frac{\sup_{\mu > 1} \mu^{\eta} |\Psi(\mu, \Omega_t(y))|}{\sup_{\mu > 1} |\Psi(\mu, \Omega_t(y))|} \right]. \]

Now we choose \( \delta > 0 \) sufficiently small so as to satisfy
\[ \left( \frac{\lambda}{2} \right)^{\eta} - \left( \frac{\lambda}{\delta} \right)^{\eta} = \left( \frac{\lambda}{2} \right)^{\eta} - \left( \frac{\lambda}{\delta} \right)^{\eta} \leq \left( \frac{\lambda}{2} \right)^{\eta} \left( 1 + \frac{t}{(t - \eta) \delta^n} \right) \geq \left( \frac{\lambda}{2} \right)^{\eta}, \]
and there exists a positive constant \( c(t) \) depending only on \( t \) such that \( \delta \leq c(t) \delta^n \). Therefore, it follows that
\[ |\Omega_t(y)| \leq c(t) |\Omega_t(y)|^{\frac{1 - \eta}{\eta}} \left( \sup_{\mu > 1} \mu^{\eta} |\Psi(\mu, \Omega_t(y))| \right) \]
\[ \leq \frac{ct(t)^{\frac{1}{\eta}}}{(t - \eta) \delta^n} \left( (\lambda)^{\eta} |\Psi(\lambda, \Omega_t(y))| \right)^{\frac{1 - \eta}{\eta}} \sup_{\mu > 1} \mu^{\eta} |\Psi(\mu, \Omega_t(y))| \]
\[ \leq \frac{ct(t)^{\frac{1}{\eta}}}{(t - \eta) \delta^n} \sup_{\mu > 1} \mu^{\eta} |\Psi(\mu, \Omega_t(y))|. \quad (3.35) \]

Next we put the estimates of two cases in (3.11) together into (3.23), that is, we insert the formulas (3.14) and (3.35) into (3.23) to get
\[ |E(\Lambda, \Omega_t(x_0))| \leq c\ v_{\eta} E(\Lambda, \Omega_t(x_0)) + c\ u^{\eta} \sup_{\mu > \lambda} \left\{ \mu^{\eta} |\Psi(\mu, \Omega_t(x_0))| \right\}, \quad (3.36) \]
then we multiply (3.36) by \( (\Lambda)^{\eta} \), and take the supremum with respect to \( \lambda \) over \( (M\lambda_0, \infty) \) to show
\[ \sup_{\lambda > M\lambda_0} (\Lambda)^{\eta} \left\{ x \in \Omega_t(x_0) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} > \Lambda \right\} \]
\[ \leq c \left( \sup_{\lambda > M\lambda_0} (\Lambda)^{\eta} \left\{ x \in \Omega_t(x_0) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} > \frac{\lambda}{4} \right\} + \sup_{\lambda > M\lambda_0} \left( \sup_{\mu > \lambda} (\Lambda)^{\eta} \sup_{\mu > \lambda} \left\{ \mu^{\eta} |\Psi(\mu, \Omega_t(x_0))| \right\} \right) \]
\[ \leq c \left( \sup_{\lambda > M\lambda_0} (\Lambda)^{\eta} \left\{ x \in \Omega_t(x_0) : (\Phi(|f|))^{\frac{\rho(t)}{\lambda}} > \frac{\lambda}{4} \right\} + \sup_{\lambda > M\lambda_0} \left( \sup_{\mu > \lambda} (\Lambda)^{\eta} \sup_{\mu > \lambda} \left\{ \mu^{\eta} |\Psi(\mu, \Omega_t(x_0))| \right\} \right) \right). \]

Note that
In the remainder we use a similar way of the argument in [18] C. De Filippis and G. Mingione, A borderline case of Calderón-Zygmund estimates for non-uniformly elliptic problems. St.

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[19] If we take $\epsilon > 0$ so small that it ensures $c\epsilon^{\frac{1}{p}} \leq \frac{1}{2}$, it follows that

$$
\sup_{\lambda \in M_{A_0}} \sup_{\mu > \lambda} \mu^{\frac{p}{p^*}} \left| \Phi(\mu, \Omega_2(x_0)) \right| \leq \left\| \left( \Phi(|Du|) \right)^{p(x)} \right\|_{M^t(\Omega_2(x_0))}.
$$

In the remainder we use a similar way of the argument in Step 5, and it leads to the desired result for the case $q = \infty$.  

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