Macrossopic quantum tunneling of two coupled particles in the presence of a transverse magnetic field.

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ABSTRACT

Two coupled particles of identical masses but opposite charges, with a constant transverse external magnetic field and an external potential, interacting with a bath of harmonic oscillators are studied. We show that the problem cannot be mapped to a one-dimensional problem like the one in Ref. [1], it strictly remains two-dimensional. We calculate the effective action both for the case of linear coupling to the bath and without a linear coupling using imaginary time path integral at finite temperature. At zero temperature we use Leggett’s prescription to derive the effective action. In the limit of zero magnetic field we recover a two dimensional version of the result derived in Ref. [1] for the case of two identical particles. We find that in the limit of strong dissipation, the effective action reduces to a two dimensional version of the Caldeira-Leggett form in terms of the reduced mass and the magnetic field. The case of Ohmic dissipation with the motion of the two particles damped by the Ohmic frictional constant $\eta$ is studied in detail.

INTRODUCTION

Macroscopic quantum tunneling with dissipation has become the subject of interest in quantum statistical mechanics and condensed matter physics for many years [1, 2, 3]. This mainly involves the influence of the environment (thermal bath of harmonic oscillators) on the tunneling of a macroscopic particle with variable, say $q$, out of an external potential $V(q)$, which is assumed to have a metastable minimum. In most cases of physical interest, it is assumed that $q$ interacts linearly with the environmental coordinate say $x_\alpha$ ($\alpha = 1, 2, \cdots$) at a certain temperature $T$. The breakthrough in this subject was made by Caldeira and Leggett [3]. They considered a Euclidean Lagrangian of the form

$$\mathcal{L}_E = \frac{1}{2} Mq^2 + V(q) + \sum_\alpha \frac{1}{2} m_\alpha (x_\alpha^2 + \omega_\alpha^2 q^2) + q \sum_\alpha c_\alpha x_\alpha,$$

where the parameters $m_\alpha, \omega_\alpha, c_\alpha$ need not to be known in detail. The partition function is given by

$$K(q, x_\alpha; \tau) = \int Dq(\tau) \prod_\alpha Dx_\alpha(\tau) \exp (-S_E),$$

(2)

where

$$S_E = \int_0^\tau d\tau \mathcal{L}_E.$$  

(3)

Performing the functional integral over $x_\alpha$ in the limit $\tau \to \infty$ gives

$$K(q; \tau) = \int Dq(\tau) \exp (-S^{\text{eff}}_E),$$

(4)

where the effective action is given by

$$S^{\text{eff}}_E = \int_0^\tau d\tau \left[ \frac{1}{2} Mq^2 + V(q) \right] + \eta \int_{-\infty}^\infty d\tau' \int_0^\tau d\tau \frac{(q(\tau) - q(\tau'))^2}{(\tau - \tau')^2},$$

(5)

and $\eta$ is the frictional constant. Chudnovsky [4] generalized this formalism by considering two macroscopic particles that interact with each other via a nonlinear potential $V(|x_1 - x_2|)$ with the coordinate $x_2$ linearly coupled to the environment. The Euclidean Lagrangian is of the form

$$\mathcal{L}_E = \frac{1}{2} M_1 \dot{x}_1^2 + \frac{1}{2} M_2 \dot{x}_2^2 + V(|x_1 - x_2|) + \frac{1}{2} \sum_\alpha m_\alpha \dot{x}_\alpha^2$$

$$+ \frac{1}{2} \sum_\alpha m_\alpha \omega_\alpha^2 (x_\alpha - x_2)^2,$$

(6)

Integrating out the environmental degree of freedom and using the new coordinates

$$q = x_1 - x_2,$$

$$r = \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2},$$

(7)

he found that in the limit $M_1 \to \infty$, the effective action reduces to the form of Caldeira and Leggett:

$$S^{\text{eff}}_E = \int_0^{h/T} d\tau \left[ \frac{1}{2} M_2 \dot{q}_E^2 + V(q) \right]$$

$$+ \frac{1}{2} \int_{-\infty}^\infty d\tau' \int_0^{h/T} d\tau \alpha(\tau - \tau') (q(\tau) - q(\tau'))^2,$$

(8)

where

$$\alpha(\tau) = \frac{1}{4} \sum_\alpha m_\alpha \omega_\alpha^2 \exp(-\omega_\alpha |\tau|).$$

(9)
In this paper, we will generalize Chudnovsky’s idea by considering two coupled macroscopic particles, in the presence of a constant transverse external magnetic field and an external potential. Due the presence of an external magnetic field, this problem is at least two dimensional.

**MODEL**

In the presence of a magnetic field \( \mathbf{B} \) derivable from a vector potential \( \mathbf{A} = \nabla \times \mathbf{A} \), the Euclidean Lagrangian we will first consider is of the form

\[
\mathcal{L}_E = \frac{m_1}{2} |\dot{x}_1|^2 + \frac{m_2}{2} |\dot{x}_2|^2 + ie (\dot{x}_1 \cdot \mathbf{A}_1 - \dot{x}_2 \cdot \mathbf{A}_2) + \frac{1}{2} m_1 \omega_1^2 x_1^2 + \frac{1}{2} m_2 \omega_2^2 x_2^2 + V(|x_1 - x_2|). \tag{10}
\]

Here the vectors have two components given by \( x_1 = (x, y) \) and \( x_2 = (X, Y) \), where \( \mathbf{A}_1 \) and \( \mathbf{A}_2 \) are the vector potentials of particle 1 and 2 respectively. Notice that the third term in Eq. (10) is completely imaginary. This comes from the fact that it is first order in time derivative, hence analytically continuing to imaginary time \( t \to e^{-i\theta} \), \( \theta = \pi/2 \) makes it completely imaginary. The results obtained here are not restricted only to two dimensions. It is completely general and can be extended to \( n \)-dimensional Euclidean space.

By choosing the symmetric gauge vector potential for the two particles \( A_i = \frac{1}{2} \mathbf{B} \times \mathbf{x}_i, \quad i = 1, 2 \), where \( \mathbf{B} = B_\perp \hat{z} \), the Euclidean Lagrangian can be written as

\[
\mathcal{L}_E = \frac{m_1}{2} |\dot{x}_1|^2 + \frac{m_2}{2} |\dot{x}_2|^2 + \frac{e B_\perp}{2} (\dot{x}_1 \times \dot{x}_1 - \dot{x}_2 \times \dot{x}_2) + \frac{1}{2} m_1 \omega_1^2 x_1^2 + \frac{1}{2} m_2 \omega_2^2 x_2^2 + V(|x_1 - x_2|). \tag{11}
\]

One can show that the results are independent of the choice of gauge. The Lagrangian in Eq. (11) describes motion of two coupled particles of opposite charges in the plane, in the presence of a constant transverse external magnetic field and an external potential. These two particles interact with each other by a nonlinear potential \( V(|x_1 - x_2|) \) which has a metastable minimum. The magnetic field breaks the time reversal symmetry of the Lagrangian and the weak harmonic oscillator potentials break the spatial translation symmetry of the Lagrangian. However, we can restore spatial translation invariance up to a total derivative in the limit \( \omega_1 = \omega_2 = 0 \). This will be the case at the end of the calculation in this paper. Hence, the total linear momentum is conserved and the dynamics of the system cannot, therefore, change the position of the center of mass [4]. Notice that the presence of the magnetic field makes the Lagrangian strictly two-dimensional.

**EFFECTIVE ACTION**

We proceed to the effect of an external transverse magnetic field on the tunneling of the particles out of a metastable state by following the method of Caldeira and Leggett[1]. The partition function is given by

\[
Z = \int dx_1 dx_2 K(x_1, x_2, \beta), \tag{12}
\]

where

\[
K(x_1, x_2, \beta) = \int D\dot{x}_1 \int D\dot{x}_2 \exp (-S_E), \tag{13}
\]

and

\[
S_E = \int_0^\beta d\tau \mathcal{L}_E, \tag{14}
\]

\( \beta = 1/T \) is the inverse temperature. The tunneling rate is proportional to \( \exp(-S_E^c) \), where the Euclidean classical action \( S_E^c \) is determined from the bounce solution of the equation \( \delta S_E = 0 \), in which the periodic boundary condition \( x_1(0) = x_1(\beta) \) and \( x_2(0) = x_2(\beta) \) are required.

We will set \( h = 1 \) throughout the calculation in this paper. Let us simplify the problem by taking \( m_1 = m_2 = m \) and \( \omega_1 = \omega_2 = \omega' \) and now introduce the following change of variables

\[
\mathbf{q} = x_1 - x_2, \quad \mathbf{r} = \frac{x_1 + x_2}{2} \tag{15}
\]

where \( \mathbf{q} \) is the position of particle 1 relative to particle 2 and \( \mathbf{r} \) is the position vector of the center of mass of particles 1 and 2. The Lagrangian in the new coordinate system is of the form

\[
\mathcal{L}_E = \frac{1}{2} \dot{m} \left( \dot{q}_i^2 + \omega^2 q_i^2 \right) + \mathcal{V}(|\mathbf{q}|) + \frac{M}{2} \left( \dot{r}_i^2 + \omega^2 r_i^2 \right) + ieB_\perp \epsilon_{ij} \dot{r}_i q_j + \frac{ieB_\perp}{2} \frac{d}{d\tau} (\epsilon_{ij} q_i r_j), \tag{16}
\]

where subscript \( i = 1, 2 \), \( \dot{m} = \frac{1}{2} m \) is the reduced mass and \( M = 2m \) is the total mass. The last term in Eq. (16) is a total derivative and thus has no contribution to the classical equation of motion. However, this term cannot in general be ignored when computing the quantum transition amplitude because it can generate phase terms in the Euclidean action that may, in principle, produce oscillations of the tunnelling amplitude on the applied field, but since we impose periodic boundary condition on the coordinates \( q_i(\beta) = q_i(0) \) and \( r_i(\beta) = r_i(0) \), the total derivative term integrates out from the action. Thus, it can be ignored from Eq. (16).

The density matrix becomes

\[
K(q_i, r_i, \beta) = \int_{q_i(0) = q_{i0}}^{q_i(\beta) = q_{i0}} Dq_i \int_{r_i(0) = r_{i0}}^{r_i(\beta) = r_{i0}} Dr_i \exp (-S_E). \tag{17}
\]
Exploiting the periodic boundary conditions on \( q_i \) and \( r_i \), one can expand these coordinates in terms of Fourier series \( [5, 6] \):
\[
r_i(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{n=\infty} r_{in} e^{i\omega_n \tau}, \quad \text{etc,}
\]
where \( r_{in} = r_{in}^* \) and \( \omega_n = -\omega_n = 2\pi n/\beta \) is the bosonic Matsubara frequency. The classical equation of motion for \( r_i \) is
\[
M\ddot{r}_i + i eB\perp \epsilon_{ij} \dot{q}_j - M\omega_i^2 r_i = 0.
\]

For any path, we write
\[
\dot{r}_i(\tau) = \dot{r}_i(\tau) + y_i(\tau),
\]
so \( y_i(0) = y_i(\beta) = 0 \). The Fourier transform of the center of mass coordinate \( r_i(\tau) \) action becomes
\[
S^r_{eff} = \frac{1}{\beta} \sum_{n=-\infty}^{n=\infty} \frac{1}{2} M(\omega_n^2 + \omega_i^2)|y_{in}|^2
+ \frac{1}{2\beta} \sum_{n=-\infty}^{n=\infty} \frac{1}{2} (eB\perp \omega_n)^2 |\dot{q}_{in}|^2.
\]
Notice that the linear term in \( y_{in} \) vanishes by means of the equation of motion. \( Dr_i = Dr_{in} = D\dot{y}_{in} \), the Gaussian integral over \( y_{in} \) in Eqn. (17) is easily done, we finally obtain the effective action:
\[
S^r_{eff} = \int_0^\beta d\tau \left\{ \frac{m}{2} \dot{q}_i^2 + V(|q_i|) \right\}
- \frac{1}{2\beta} \sum_{n=-\infty}^{n=\infty} A_n |\dot{q}_{in}|^2,
\]
where
\[
A_n = -\left( \frac{m\omega_n^2}{2} + \frac{(eB\perp \omega_n)^2}{M(\omega_n^2 + \omega_i^2)} \right).
\]

The equivalent form of the Caldeira Leggett effective action \( [11] \) one can obtain from (23) is
\[
S^\alpha_{eff} = \int_0^\beta d\tau \left\{ \frac{m}{2} \dot{q}_i^2 + V(|q_i|) \right\}
+ \frac{1}{4} \int_0^\beta d\tau \int_0^\beta d\tau' A(\tilde{\tau}) |q_i(\tau) - q_i(\tau')|^2.
\]
where \( \tilde{\tau} = \tau - \tau' \) and
\[
A(\tau) = -\frac{M\omega_i^2}{\beta} \sum_{n=-\infty}^{n=\infty} \frac{\omega_n^2 e^{i\omega_n \tau}}{(\omega_n^2 + \omega_i^2)}.
\]
\( \omega_c = eB\perp /M \) is the cyclotron frequency. The first term in (24) is independent of \( \omega_n \) and thus gives an unnecessary delta function contribution to (26) which does not have any contribution to the effective action and hence can be neglected. Further simplification of (26) yields
\[
A(\tau) \approx \frac{M\omega_i^2}{\beta} \sum_{n=-\infty}^{n=\infty} \frac{\omega_n^2}{(\omega_n^2 + \omega_i^2)} e^{i\omega_n \tau}.
\]

The above expression can now be summed easily by means of Residue theorem or the summation formula \( [5, 6] \), it is given by
\[
A(\tau) = \frac{M\omega_i^2}{2} \cosh \left[ \frac{\omega' (\beta/2 - |\tau|)}{\sinh(\beta\omega'/2)} \right].
\]

The effective action then becomes
\[
S^\alpha_{eff} = \int_0^\beta d\tau \left\{ \frac{m}{2} \dot{q}_i^2 + V(|q_i|) \right\}
+ \frac{M\omega_i^2}{4} \int_0^\beta d\tau' \int_0^\beta d\tau \frac{\omega' \cosh(\omega'(\beta/2 - |\tau|))}{\sinh(\beta\omega'/2)}
\times |q_i(\tau) - q_i(\tau')|^2.
\]
In the limit \( \omega' \to 0 \), Eqn. (28) simplifies to
\[
A(\tau) = \frac{M\omega_i^2}{\beta},
\]
and the action is thus
\[
S^\alpha_{eff} = \int_0^\beta d\tau \left\{ \frac{m}{2} \dot{q}_i^2 + V(|q_i|) \right\}
+ \frac{M\omega_i^2}{4\beta} \int_0^\beta d\tau' \int_0^\beta d\tau \frac{\omega' \cosh\left( \frac{\omega'(\beta/2 - |\tau|)}{\sinh(\beta\omega'/2)} \right)}{\sinh(\beta\omega'/2)}
\times |q_i(\tau) - q_i(\tau')|^2.
\]

The effective actions in Eqn’s (29) and (31) are strictly of the form (20) and version of Eqn. (20) and Eqn. (13) in Ref. [2] for the case of finite pinning, no dissipation and no pinning, no dissipation respectively. However, in the present case the effective action depends on the relative coordinate of the two particles.

**Leggett’s Prescription at Zero Temperature**

Let us now derive the zero temperature (\( \beta \to \infty \)) version of Eqn. (31) by applying the Leggett’s prescription \( [10] \). This prescription simply tells us that if the Fourier transform of the real time classical equation of motion of is of the form
\[
K(\omega)q_i(\omega) = -\left( \frac{dV}{dq_i} \right)(\omega),
\]
then the formula for the tunneling rate can be obtained from the effective action
\[
S^\alpha_{eff} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2} K(-i|\omega|)|\dot{q}_i(\omega)|^2 d\omega + S_0(\dot{q}_i(\omega)).
\]
where
\[ S_v(\dot{q}_i(\omega)) = \int_{-\infty}^{\infty} d\tau V(q_i(\tau)), \] (34)
and \( \dot{q}_i(\omega) \) is the Fourier transform of the imaginary-time trajectory. Now the real time classical equations of motion from [16] are
\[ M(\ddot{r}_i + \omega^2 r_i) - eB_\perp \epsilon_{ij} \dot{q}_j = 0. \] (35)
\[ \ddot{q}_i + \omega^2 q_i - eB_\perp \epsilon_{ij} \dot{r}_j = -\frac{dV}{dq_i}. \] (36)

Next, we perform the real time Fourier transform Eqn.’s [35] and [36] at zero temperature (see Appendix) and solve for \( q_i(\omega) \). The result is of the form
\[ K(\omega)q_i(\omega) = -\left(\frac{dV}{dq_i}\right)(\omega), \] (37)
where
\[ K(\omega) = -\dot{m}(\omega^2 - \omega^2) + \frac{(eB_\perp \omega)^2}{M(\omega^2 - \omega^2)}. \] (38)

Plugging (38) into (33), we obtain
\[ S_{\text{eff}}^f = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{1}{2} \left[ \dot{m}(\omega^2 + \omega^2) + \frac{(eB_\perp \omega)^2}{M(\omega^2 + \omega^2)} \right] |\dot{q}_i(\omega)|^2 + S_v(\dot{q}_i(\omega)). \] (39)

One can simply derive this equation from (33) as \( \beta \to \infty \) by Fourier transforming the kinetic term in (33), and replacing \( \omega_n \) by \( \omega \) and the summation over \( n \) by integration over \( \omega \) with a normalization factor of \( 1/2\pi \). Therefore, we see that the results are consistent.

**DISSIPATIVE ENVIRONMENT**

In this section we shall consider the coupling of the Lagrangian in (10) to a thermal harmonic oscillators in two dimensions. Therefore classical dynamics of the system will be dissipative. The Euclidean Lagrangian we will consider is of the form
\[ L_E = \frac{m_1}{2} |\dot{x}_1|^2 + \frac{m_2}{2} |\dot{x}_2|^2 + \frac{eB_\perp}{2} (\dot{x}_1 \times \dot{x}_1 - \dot{x}_2 \times \dot{x}_2)
+ \frac{1}{2} m_1 \omega_1^2 |\dot{x}_1|^2 + \frac{1}{2} m_2 \omega_2^2 |\dot{x}_2|^2 + V(|x_1 - x_2|) \] (40)
\[ + \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \left[ \dot{x}_\alpha^2 + \omega_\alpha^2 (x_\alpha - x)^2 \right], \]
where \( x = (x_1, x_2). \)

In the absence of a transverse magnetic field and weak harmonic oscillator potentials, the Lagrangian corresponds to a two dimensional version of the one considered in Ref. [4] except for the coupling in \( x \) instead of \( x_2. \)

The Lagrangian is also similar to the one studied in Ref. [5] where it was shown that the action can be mapped to a one-dimensional problem. However, this is not the case in Eqn. (40). We have assumed that both particles are coupled to a large environmental harmonic oscillators. Notice that the Lagrangian is still translational invariant up to a total derivative when \( \omega_1 = \omega_2 = 0. \) Therefore translational invariance of the system will be restored by taking the limit \( \omega_1 = \omega_2 = 0 \) at the end of the calculation.

In order to obtain the effective action, we expand \( x_\alpha \) and \( x \) in a Fourier series:
\[ x_\alpha(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{n=\infty} x_{\alpha n} e^{i\omega_n \tau}, \] (41)
\[ x(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{n=\infty} x_n e^{i\omega_n \tau}, \] etc.

Performing the Gaussian integration over \( x_{\alpha n} \) we obtain
\[ K(x_1, x_2, \beta) = \int Dx_1(\tau) \int Dx_2(\tau) \exp (-S_E), \] (42)
where
\[ S_E = \int_0^\beta d\tau \left[ \frac{m_1}{2} (\dot{x}_1^2 + \dot{x}_2^2) + \frac{eB_\perp}{2} (\dot{x}_1 \times \dot{x}_1 - \dot{x}_2 \times \dot{x}_2) + \frac{1}{2} (m_1 \omega_1^2 + \sum_{\alpha=1}^{N} m_\alpha \omega_\alpha^2 |x|^2 + V(|x_1 - x_2|) \right]
- \frac{1}{\beta} \sum_{\alpha=1}^{N} \frac{m_\alpha \omega_\alpha^4}{2(\omega_\alpha^2 + \omega_n^2)} |x_n|^2. \] (43)

We have set \( m_1 = m_2 = m \) and \( \omega_1 = \omega_2 = \omega \) to arrive at this result. Next, we rewrite (43) in terms of \( q_i \) and \( r_i \) using [15], Fourier transform using (18) and use the fact that the Fourier coefficient \( |x_n|^2 = |x_{1 n}|^2 + |x_{2 n}|^2 \) where
\[ x_{1 n} = r_n + \frac{1}{2} q_n, \quad \text{and} \quad x_{2 n} = r_n - \frac{1}{2} q_n. \] (44)

Then the Gaussian integration over \( r_n \) can be easily done, and we obtain
\[ K(q_i; \beta) = \int Dq_i(\tau) \exp (-S_{\text{eff}}^f) \], (45)
where
\[ S_{\text{eff}}^f = \int_0^\beta d\tau \left[ \frac{\dot{m}}{2} \dot{q}_i^2 + V(|q_i|) \right]
+ \frac{1}{4} \int_0^\beta d\tau \int_0^\beta d\tau' (A(\tilde{\tau}) + B(\tilde{\tau})) [q_i(\tau) - q_i(\tau')]^2 \] (46)
The coefficients are
\[ A(\tau) = \frac{(eB_\perp)^2}{M\beta} \sum_n \frac{M\omega_n^2 + 2\lambda_n\omega_n^2}{M(\omega_n^2 + \omega^2) + 2\lambda_n\omega_n^2} e^{i\omega_n\tau} \]  
(47)
\[ B(\tau) = -\frac{1}{2\beta} \sum_n \omega_n^2 \lambda_ne^{i\omega_n\tau} \]  
(48)
\[ \lambda_n(\beta) = \sum_\alpha \frac{m_\alpha \omega_\alpha^2}{\omega_\alpha^2 + \omega_n^2}. \]  
(49)

Constant terms independent of \(\omega_n\) have been dropped since they give no contribution to (46). In general, Eqn.(47) is difficult to sum unless one considers some limiting cases.

**OMHIC DISSIPATION**

In line with Ref.[3], we will assume that the effect of the oscillators on the motion of the particles result in the force of friction \(\eta x\). This requires that the spectral density should be defined as
\[ J(\tilde{\omega}) = \frac{\eta}{2} \sum_\alpha m_\alpha \omega_\alpha^2 \delta(\tilde{\omega} - \omega_n). \]  
(50)

All the information concerning the effect of the environment on the dynamics of the particles is contained in \(J(\tilde{\omega})\). The spectral function is frequently assumed to be of the form [11][2]
\[ J(\tilde{\omega}) = \eta \tilde{\omega}^s \exp(-\tilde{\omega}/\omega_c), \]  
(51)
up to a frequency cutoff \(\omega_c\), where \(s > 1\) is the super-Ohmic case, \(s = 1\) is the Ohmic case, and \(0 \leq s < 1\) is the sub-Ohmic case. In this section, we will consider only the case of Ohmic dissipation with \(\omega_c \rightarrow \infty\). Using the definition of the spectral function (50) we have
\[ \lambda_n = \frac{2}{\pi} \int_0^\infty d\tilde{\omega} \frac{J(\tilde{\omega})}{\tilde{\omega}^2 + \omega_n^2} = \frac{\eta}{\omega_n}. \]  
(52)
The second equality follows from (51) for the Ohmic case. The effective action (46) in this case becomes
\[ S_E^{eff} = \int_0^\beta d\tau \left( \frac{1}{2} \tilde{m} \dot{q}_i \dot{q}_i + V(|q_i|) \right) + \frac{1}{4} \int_0^\beta d\tau \int_0^\beta d\tau' [A(\tilde{\tau}) + B(\tilde{\tau})] [q_i(\tau) - q_i(\tau')]^2 \]  
(53)
where
\[ A(\tau) = \frac{(eB_\perp)^2}{M\beta} \sum_n \frac{M\omega_n^2 + 2\eta|\omega_n|}{M(\omega_n^2 + \omega^2) + 2\eta|\omega_n|} e^{i\omega_n\tau}, \]  
(54)
\[ B(\tau) = -\frac{1}{2\beta} \sum_n \frac{\eta|\omega_n|}{2} e^{i\omega_n\tau}. \]  
(55)

Let us consider the limit of very strong dissipation \(\eta \gg M\), setting \(\omega = 0\) we have
\[ \frac{2\eta|\omega_n|}{M(\omega_n^2 + 2\eta|\omega_n|)} \approx 1 - \frac{M|\omega_n|}{2\eta}. \]  
(56)
Hence the effective action becomes
\[ S_E^{eff} = \int_0^\beta d\tau \left( \frac{1}{2} \tilde{m} \dot{q}_i \dot{q}_i + V(|q_i|) \right) + \frac{\eta_{eff}}{8\pi} \int_0^\beta d\tau \int_0^\beta d\tau' \frac{|q_i(\tau) - q_i(\tau')|^2}{(\beta/\pi)^2 \sin^2((\pi\tau/\beta)).} \]  
(57)
where
\[ \eta_{eff} = \frac{(eB_\perp)^2}{\eta} + \eta. \]  
(58)

In the limit \(B_\perp = 0\), the action corresponds to a two dimensional version of Eq. (44) in Ref.[4] for \(m_1 = m_2\).

At \(T = 0\), we apply the Leggett’s prescription outline above. The real time classical equations of motion for the case in which the motion of the two identical particles are damped by Ohmic friction with constant \(\eta\) are
\[ m\ddot{x}_1 + eB_\perp \epsilon_{ij} \dot{x}_1^i + \eta \dot{x}_1^i + m\omega_n^2 x_1^i = -\partial V / \partial x_1^i, \]  
(59)
\[ m\ddot{x}_2 + eB_\perp \epsilon_{ij} \dot{x}_2^i + \eta \dot{x}_2^i + m\omega_n^2 x_2^i = -\partial V / \partial x_2^i. \]  
(60)
Substituting \(x_1^i\) and \(x_2^i\) in terms of \(R_i\) and \(q_i\), using the inverse transformation of (15), we obtain, after adding and subtracting the resulting equations
\[ 2m\ddot{R}_i - eB_\perp \epsilon_{ij} \dot{q}_j + 2\eta \dot{q}_i + m\omega_n^2 \dot{R}_i = 0, \]  
(61)
\[ \frac{1}{2} \ddot{q}_i - eB_\perp \epsilon_{ij} \dot{R}_j + \frac{1}{2} \dot{\eta} + m\omega_n^2 q_i = -\partial V / \partial q_i. \]  
(62)
Fourier transforming (61) and solving for \(q_i(\omega)\) we obtain
\[ K(\omega)q_i(\omega) = -\partial V / \partial q_i(\omega), \]  
(63)
where \(K(\omega)\) in this case is given by
\[ K(\omega) = -\tilde{m}(\omega^2 - \omega'\omega) + \frac{(eB_\perp \omega)^2}{M(\omega^2 - \omega'^2) - 2i\eta\omega}. \]  
(64)
The effective action at \(T = 0\) is thus
\[ S_E^{eff} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \left[ \ddot{m} \omega^2 + \frac{\eta}{2} \right] + \frac{(eB_\perp \omega)^2}{M(\omega^2 + \omega'^2) + 2\eta |\omega|} \ddot{q}_i(\omega) + S_c(\ddot{q}_i(\omega)). \]  
(65)
Fourier transforming back to imaginary time domain (see Appendix) we obtain

\[ S_{\text{eff}} \equiv \int_{-\infty}^{\infty} d\tau \left( \frac{1}{2} \hat{\mu} \dot{q}_i^2 + V(|q_i|) \right) + \eta \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau \frac{|q_i(\tau) - q_i(\tau')|^2}{|\tau - \tau'|} \]  \tag{65}

and the Fourier coefficient is

\[ G(\tau - \tau') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\omega) e^{i\omega(\tau - \tau')}, \]  \tag{66}

where

\[ G(\tau - \tau') = \frac{(eB_\perp)^2}{8\pi \eta} \frac{1}{|\tau - \tau'|^2}. \]  \tag{68}

Plugging this expression into Eq.(65) we recover a two-dimensional version of Eq.(18) in Ref. \[2\] for the case of no pinning and finite dissipation.

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\[ \text{APPENDIX} \]

The real time Fourier transform is defined as

\[ q_i(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega q_i(\omega)e^{i\omega t}, \text{ etc} \]  \tag{69}

and the imaginary time Fourier transform is defined as

\[ \tilde{q}_i(\omega) = \int_{-\infty}^{\infty} q_i(\tau)e^{-i\omega\tau} d\tau \]  \tag{70}

Fourier transforming Eq.(64) we have for the potential term

\[ S_v(\tilde{q}_i(\omega)) = \int_{-\infty}^{\infty} d\tau V(q_i(\tau)). \]  \tag{71}

The first term in Eq.(64) gives

\[ S_1 = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \frac{1}{2} \hat{\mu} \dot{q}_i(\tau)\dot{q}_i(\tau') \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(\tau - \tau')} \]  \tag{72}

Note the contribution from \( \omega' \) gives a delta function and hence a factor of \( q_i(\tau)^2 \) which can then be absorbed in the potential Eq.(71).

The second term in Eq.(64) gives

\[ S_2 = \frac{1}{8\pi} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\omega \eta|\omega|e^{i\omega(\tau - \tau')} q_i(\tau) q_i(\tau') \]  \tag{73}
Now we use the fact that
\[ \eta|\omega| = \frac{2\eta}{\pi} \int_{0}^{\infty} du \frac{\omega^2}{u^2 + \omega^2}. \] (74)

Plugging (74) into (73) and performing the contour integration over \( \omega \) and subsequently integration over \( u \) we obtain
\[ S_2 = -\frac{1}{4\pi} \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau \frac{q_i(\tau)q_i(\tau')}{|\tau - \tau'|^2} \] (75)
The last term in Eq. (64) is simply
\[ S_3 = 2 \int_{-\infty}^{\infty} d\tau' \int_{-\infty}^{\infty} d\tau \mathcal{G}(\tau - \tau')q_i(\tau)q_i(\tau') \] (76)

We can use the fact that
\[ q_i(\tau)q_i(\tau') = \frac{1}{2} \left[ q_i(\tau)^2 + q_i(\tau')^2 - (q_i(\tau) - q_i(\tau'))^2 \right] \]
to arrive at Eq. (65).

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