On the $L^p$-distortion of finite quotients of amenable groups

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Abstract We study the $L^p$-distortion of finite quotients of amenable groups. In particular, for every $2 \leq p < \infty$, we prove that the $\ell^p$-distortions of the groups $C_2 \wr C_n$ and $C_{2^n} \rtimes C_n$ are in $\Theta((\log n)^{1/p})$, and that the $\ell^p$-distortion of $C_{2^n} \rtimes_A \mathbb{Z}$, where $A$ is the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ is in $\Theta((\log \log n)^{1/p})$.

Keywords Distortion of Bilipschitz embeddings · Finite metric spaces · Isoperimetry · Isometric group actions · $L^p$-spaces

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1 The main results

1.1 Distortion

Let us first recall some basic definitions.

Definition 1.1 Let $0 < R \leq \infty$. The distortion at scale $\leq R$ of an injection between two discrete metric spaces $F : (X, d) \to (Z, d)$ is the number (possibly infinite)
\[
\text{dist}_R(F) = \sup_{0 < d(x, y) \leq R} \frac{d(f(x), f(y))}{d(x, y)} \cdot \sup_{0 < d(x, y) \leq R} \frac{d(x, y)}{d(f(x), f(y))}.
\]

If \( R = \infty \), we just denote \( \text{dist}(F) \) and call it the distortion of \( F \).

- The \( \ell^p \)-distortion \( c_p(X) \) of a finite metric space \( X \) is the infimum of all \( \text{dist}_F \) over all possible injections \( F \) from \( X \) to \( \ell^p \).

Let \( G \) be a finitely generated group. Let \( S \) be a symmetric finite generating subset of \( G \). We equip \( G \) with the left-invariant word metric associated to \( S \): \( d_S(g, h) = |g^{-1}h|_S = \min\{n \in \mathbb{N}, g^{-1}h \in S^n\} \). Let \( (G, S) \) denote the associated Cayley graph of \( G \): the set of vertices is \( G \) and two vertices \( g \) and \( h \) are joined by an edge if there is \( s \in S \) such that \( g = hs \). Note that the graph metric on the set of vertices on \((G, S)\) coincides with the word metric \( d_S \).

Let \( \lambda_{G, p} \) denote the regular representation of \( G \) on \( \ell^p(G) \) for every \( 1 \leq p \leq \infty \) (i.e. \( \lambda_{G, p}(g)f(x) = f(g^{-1}x) \)). The \( \ell^p \)-direct sum of \( n \) copies of \( \lambda_{G, p} \) will be denoted by \( n\lambda_{G, p} \).

Our main results are the following theorems.

**Theorem 1** Let \( m \) be an integer \( \geq 2 \). For all \( n \in \mathbb{N} \), consider the finite lamp-lighter group \( C_m \downarrow C_n = (C_m)^{C_n} \rtimes C_n \) equipped with the generating set \( S = ((\pm 1, 0), (0, \pm 1)) \), where \( 1_0 \in (C_m)^{C_n} \) is the characteristic function of the singleton \( \{0\} \). For every \( 2 \leq p < \infty \), there exists \( C = C(p, m) < \infty \) such that

\[
C^{-1}(\log n)^{1/p} \leq c_p(C_m \downarrow C_n, S) \leq C(\log n)^{1/p}.
\]

A different proof for \( p = 2 \) has been given very recently by Austin, Naor, and Valette [1], using certain irreducible representations of the lamplighter group. On the other hand, the lower bound was known (see [5], or Sect. 2).

**Theorem 2** Let \( m \) be an integer \( \geq 2 \). For all \( n \in \mathbb{N} \), consider the group \( BS_{m,n} = C_m^{C_n} \rtimes C_n \), where the element \( 1 \in C_n \) acts by multiplication by \( m \) on \( C_m^{C_n} \). We consider the Cayley graph of this group associated with the generating set \( S = \{ (\pm 1, 0), (0, \pm 1) \} \). For every \( 2 \leq p < \infty \), there exists \( C = C(p, m) < \infty \) such that

\[
C^{-1}(\log n)^{1/p} \leq c_p(BS_{m,n}, S) \leq C(\log n)^{1/p}.
\]

**Theorem 3** For all \( n \in \mathbb{N} \), consider the group \( SOL_n = C_{n} \rtimes_A C_{o(A,n)} \), where \( A \) is a matrix of \( SL_2(\mathbb{Z}) \) with eigenvalues of modulus different from 1, e.g. the matrix \( \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \), and where \( o(A, n) \) denotes the order of \( A \) in \( SL_2(C_n) \). Equip this group with the generating set \( S = \{ (\pm 1, 0), (0, \pm 1) \} \). For every \( 2 \leq p < \infty \), there exists \( C = C(p) < \infty \) such that

\[
C^{-1}(\log \log n)^{1/p} \leq c_p(SOL_n, S) \leq C(\log \log n)^{1/p}.
\]
1.2 About the constructions

We will say that map $F : G \rightarrow E$ from a group $G$ to a Banach space is equivariant if it is the orbit of 0 of an isometric affine action of $G$ on $E$. Let $\sigma$ be such an action. The equivariance of $F(g) = \sigma(g)0$ implies that $\|F(g) - F(h)\| = \|F(g^{-1}h)\|$. Hence the distortion at scale $\leq R$ of $F$ is just given by

$$dist_R(F) = \sup_{0 < |g| \leq R} \frac{|g|}{\|F(g)\|} \cdot \sup_{0 < |g| \leq R} \frac{\|F(g)\|}{|g|}.$$

Let us introduce some basic notation. If $E_1$ and $E_2$ are two normed spaces, we denote by $E = E_1 \oplus \ell^p E_2$ the direct sum of the two vector spaces $E_1$ and $E_2$ equipped with the norm $\|(x_1, x_2)\| = (\|x_1\|^p + \|x_2\|^p)^{1/p}$. If $F_1$ and $F_2$ respectively map a set $X$ to $E_1$ and $E_2$, then the direct sum of these maps from $X$ to $E$ will be denoted by $F_1 \oplus \ell^p F_2$.

All the groups involved in the main theorems are of the form $G = N \rtimes A$ where $A$ is a finite cyclic group. To prove an upper bound on $c_p(G)$, our general approach is to construct an embedding $F = F_1 \oplus \ell^p F_2$, where $F_1$ is the orbit of 0 of an affine action $\sigma_1$ of $G$, whose linear part is $K\lambda_{G,p}$ (for some $K \in \mathbb{N}$), and such that for $R = \text{Diam}(N)$, we have

$$\text{dist}_R(F_1) \approx (\log R)^{1/p}.$$ 

More precisely, for $F_{m,n}$ and $BS_{m,n}$ (resp. for $\text{SOL}_{A,n}$), we will need $K \approx \log(mn)$ (resp. $K \approx \log \log n$) copies of $\lambda_{G,p}$.

For $G = F_{m,n}$ or $BS_{m,n}$, we can take $F = F_1$ since $\text{Diam}(N) \approx \text{Diam}(G) \approx n$ (see Proposition 3.1). But, for $G = \text{SOL}_{A,n}$, we have $\text{Diam}(N) \approx \log n$, which can be much less than $\text{Diam}(G) \approx o(A, n)$. Hence, the solution in this case is to add some map $F_2 : G/N \approx C_\sigma(A,n) \rightarrow \ell^p$ with a bounded distortion (for instance, take the orbit of 0 under the action of $C_\sigma(A,n)$ on $\mathbb{R}^2$ such that 1 acts by rotation of center $o(A, n)$, 0) and angle $2\pi/o(A, n)$).

Note that Theorem 3 also holds for the group $C_n \rtimes_A \mathbb{Z}$, in which case we can take an action of $\mathbb{Z}$ by translations on $\mathbb{R}$ to embed the quotient with bounded distortion (i.e. for $F_2$).

2 Upper bounds on the distortion

Let $1 \leq p \leq \infty$. Recall [9] that the left-$\ell^p$-isoperimetric profile in balls of $(G, S)$ is defined by

$$J_{G,S,p}(n) = \sup_{\text{Supp}(f) \subset B(1,n)} \frac{\|f\|_p}{\sup_{s \in S} \|\lambda(s)f - f\|_p},$$

where $B(1, n)$ denotes the open ball of radius $n$ and center 1 in $(G, S)$. 
In [9], we provided a general construction of metrically proper affine isometric actions of an amenable group $G$ on $\ell^p(G)$, whose compressions are related to the isoperimetric profile. Here, we will use the isoperimetric profile to produce upper bounds on the $\ell^p$-distortion of finite groups.

On the other hand, as explained in [10], if $X = (G, d_S)$ is a Cayley graph, then the inequality $J_{p,S,G} \geq J$ for some non-decreasing function $J : \mathbb{R}_+ \to \mathbb{R}_+$ implies Property A(J,p) (see [10, Definition 4.1]) for the space $X$ (if the group $G$ is amenable, a standard average argument actually shows that this is an equivalence). So in a large extent, the results of the present paper are easy consequences of the method explained in [10].

A crucial remark is that $\mu_{G,S,p}$ is a local quantity, and hence behaves well under quotients. Namely, we recall the following easy fact.

**Proposition 2.1** (for a proof, see [11, Proposition 4.5]) Let $\pi : G \to Q$ be a surjective homomorphism between two finitely generated groups and let $S$ be a symmetric generating subset of $G$. Then

$$J_{G,S,p} \leq J_{Q,\pi(S),p}.$$

Our main technical tool is the following proposition, which is an analogue of [10, Proposition 4.5]. For the convenience of the reader, we give its relatively short proof in Sect. 4.

**Theorem 4** Let $X = (G, S)$ be a finite Cayley graph such that $J_{G,S,p}(r) \geq J(r)$ when $r \leq R$, for some $R \leq \text{Diam}(G)/2$. Then, there exists an affine isometric action $\sigma$ of $G$ on such that

- the linear part of $\sigma$ is the $\ell^p$-direct sum of $K = [\log R]$ regular representations of $G$ in $\ell^p(G)$.
- The orbit of $0$ induces an injection $F : G \to \bigoplus_{k=0}^{K-1} \ell^p(G)$ such that

$$\text{dist}_R(F) \leq 2 \left( \frac{R}{2} \right)^{1/p} \left( \frac{t}{J(t)} \right)^p \int_0^t \frac{dt}{t} \right)^{1/p}.$$

In particular, if $J(t) = t/C$, then

$$\text{dist}_R(F) \leq 2C \left( 2 \log(R/2) \right)^{1/p}.$$

**Corollary 2.2** Assume that $G_n$ has diameter $n$ and that $J_{G,p}(t) \geq t/C$, then, $c_p(G_n) \leq 2C \left( 2 \log(n/4) \right)^{1/p}$.

On the other hand, we have proved in [9] that the following finitely generated groups satisfy $J_p(t) \geq t/C$ for some $C < \infty$ and for all $1 \leq p < \infty$. 

• the lamplighter group $L_m = C_m \rtimes \mathbb{Z}$;
• solvable Baumslag-Solitar groups $BS_m = \mathbb{Z}[1/m] \rtimes \mathbb{Z}$ for all $m \in \mathbb{N}$, where $n \in \mathbb{Z}$ acts by multiplication by $m^n$;
• polycyclic groups. Here, we will focus on the following example: $SOL_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where $A$ is a matrix of $SL_2(\mathbb{Z})$ with eigenvalues of modulus different from 1, e.g. the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Note that respectively $L_{m,n}$, $BS_{m,n}$ and $SOL_{A,n}$ are quotients of $L_m$, $BS_m$ and $SOL_A$.

3 Proofs of the main theorems

3.1 Upper bounds

Thanks to Corollary 2.2, the upper bounds in Theorems 1, 2 and 3 follow from the following upper bounds on the diameters of the groups $L_{m,n}$, $BS_{m,n}$ and $SOL_{A,n}$ (for the latter, see the discussion in Sect. 1.2).

Proposition 3.1 We have

(i) $\text{Diam}(L_{m,n}) \leq (m + 3)n$;
(ii) $\text{Diam}(BS_{m,n}) \leq (m + 1)n$;
(iii) Let $N_n \simeq C_2^n$ be the kernel of $SOL_A \to C_{o(A,n)}$. Then, with the distance on $N_n$ induced by the word distance on $SOL_A$, we have $\text{Diam}(N_n) \leq c \log n$ for some $c = c(A) > 0$.

Proof For (i), see [8]. For (ii), note that every element of $C_{m^n}$ can be written as

$$\sum_{i=0}^{n-1} a_i m^i = a_0 + m(a_1 + m(a_2 + \cdots \cdots)),$$

where $0 \leq a_i \leq m - 1$. Finally, (iii) follows from the following well-known lemma. □

Lemma 3.2 Let $N \simeq \mathbb{Z}^2$ be the kernel of $SOL_A \to \mathbb{Z}$. For all $r \geq 1$, denote by $B_{N,SOL_A}(r)$ (resp. $B_N(r)$), the ball of radius $r$ for the metric on $N$ induced by the word length on $SOL_A$ (resp. for the usual metric on $\mathbb{Z}^2$). There exists some $\alpha = \alpha(A) < \infty$ such that

$$B_N(1, e^{r/\alpha}) \subset B_{N,SOL_A}(r) \subset B_N(1, e^{\alpha r}).$$

Proof Note that $SOL_A$ embeds as a co-compact lattice in the connected solvable Lie group $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$, such that $N$ maps on a co-compact lattice of $\tilde{N} = \mathbb{R}^2$. Recall that the exponential radical $\text{Exp}(G)$ of a connected solvable Lie group $G$ is a closed connected nilpotent normal subgroup satisfying, for all $r > 0$

$$B_{\text{Exp}(G)}(1, e^{r/\beta}) \subset B_G(r) \leq B_{\text{Exp}(G)}(1, e^{\beta r}),$$
for some $\beta > 0$ depending on a choice of left-invariant riemannian metrics on $G$ and $\text{Exp}(G)$. The lemma follows from the observation [4,7] that $\tilde{N}$ is the exponential radical of $G$ ([4] was the first one to introduce and to study the exponential radical of a connected solvable Lie group, without actually naming it, and this was rediscovered by Osin [7]). 

3.2 Lower bounds

To obtain the lower bound on the distortion, we will need the following notion of relative girth.

**Definition 3.3** Let $\pi : G \to Q$ be a surjective homomorphism between two finitely generated groups and let $S$ be a symmetric generating subset of $G$. Denote by $X = (G, S)$ and $Y = (Q, \pi(S))$. The relative girth $g(Y, X)$ of $Y$ with respect to $X$ is the maximum integer $n \in \mathbb{N}$ such that a ball of radius $n$ in $Y$ is isometric to a ball of radius $n$ in $X$.

Recall [2] that the rooted binary tree $T_n$ of depth $n$ satisfies $c_p(T_n) \geq c(\log n)^{1/p}$ for all $2 \leq p < \infty$ and for some constant $c > 0$.

**Proposition 3.4** We keep the notation of the previous definition. Assume that $X$ contains a bi-Lipschitz embedded 3-regular tree. Then there exists some $c > 0$ such that $c_p(Y) \geq c(\log g(X, Y))^{1/p}$.

**Proof** The assumption implies that for every $n$, $T_n$ embeds into $X$ with uniform bi-Lipschitz constants. In particular, there is a constant $C$ such that $T_n$ maps into a ball of radius $Cn$. Suppose that $n$ satisfies $Cn \leq g(X, Y)$, which means that the balls of radius $Cn$ in $X$ and $Y$ are isometric. The proposition now clearly follows from Bourgain’s result. 

On the other hand, the groups $L_m$, $BS_m$ and $SOLA$ are solvable non-virtually nilpotent. Hence by [3], they admit a bi-Lipschitz embedded 3-regular tree (for the lamplighter, see also [6]). So to prove the lower bounds of Theorems 1, 2 and 3, we just need to find convenient lower bounds for the relative girths, which is done by the following proposition.

**Proposition 3.5** We have

(i) $g(L_{m,n}, L_m) \geq n$;
(ii) $g(BS_{m,n}, BS_m) \geq n$;
(iii) $g(SOLA_{A,n}, SOLA_A) \geq c \log n$ for some $c = c(A) > 0$.

**Proof** The only non-trivial case, (iii), follows from Lemma 3.2.

4 Proof of Theorem 4

Let $f_0$ be the dirac at 1, and for every integer $1 \leq k \leq K$, choose a function $f_k \in \ell^p(G)$ such that
the support of $f_k$ is contained in the ball $B(1, 2^k)$,
\( \| f_k \|_p \geq J(2^k) \)
\( \sup_{s \in S} \| \lambda(s) f_k - f_k \|_p \leq 1 \)

For all $v = (v_k)_{1 \leq k \leq K} \in K \ell^P(G)$ and all $g \in G$, define

$$
\sigma(g)v = \bigoplus_k (\lambda(g)v_k + F_k)
$$

where

$$
F_k(g) = \left( \frac{2^k}{J(2^k)} \right) (f_k - \lambda(g) f_k).
$$

Now consider the map $F = \bigoplus \ell^P F_k : G \to K \ell^P(G)$. For all $g \in G$, we have

$$
\| F(g) \|_p \leq \left( \sum_{k=0}^{K} \left( \frac{2^k}{J(2^k)} \right)^p \| \lambda(g) f_k - f_k \|_p^p \right)^{1/p}
\leq \left( \sum_{k=0}^{K} \left( \frac{2^k}{J(2^k)} \right)^p \right)^{1/p}
\leq |g|_S \left( \int_1^{Diam(G)/2} \left( \frac{t}{J(t/2)} \right)^p \frac{dt}{t} \right)^{1/p}
= 2^{2/p} |g|_S \left( \int_1^{Diam(X)/4} \left( \frac{t}{J(t)} \right)^p \frac{dt}{t} \right)^{1/p}.
$$

On the other hand, since $f_k$ is supported in $B(1, 2^k)$, if $|g|_S \geq 2.2^k$, then the supports of $f_k$ and $\lambda(g) f_k$ are disjoint. Thus,

$$
\| F(g) \|_p \geq \| F_k(g) \|_p
= 2^{1/p} \frac{2^k}{J(2^k)} \| f_k \|_p
\geq 2^{1/p} 2^k,
$$

whenever $d_S(x, y) \geq 2.2^k$. To conclude, we have to consider the case when $g \in S \setminus \{1\}$. But as $f_0$ is a dirac at 1, $\| F(g) \|_p \geq 1$. So we are done. \( \square \)
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