ABSTRACT

How can we tell when accounts are fake or real in a social network? And how can we tell which accounts belong to liberal, conservative or centrist users? Often, we can answer such questions and label the class of a node in a network based on its neighbors and appropriate assumptions of homophily (“birds of a feather flock together”) or heterophily (“opposites attract”). One of the most widely used methods for this kind of reasoning is Belief Propagation (BP) which iteratively propagates the information from a few nodes with explicit beliefs throughout the network until it converges. However, one main problem with BP is that there are no guarantees of convergence in general graphs with loops. This paper introduces Linearized Belief Propagation (LPB), a linearization of BP that allows a closed-form solution via intuitive matrix calculations and, thus, comes with convergence guarantees. It handles homophily, heterophily, and more general cases that arise in multi-class settings. The paper also introduces Turbo Belief Propagation (TBP), a “localized” version of LBP for which the final class assignments depend only on the nearest labeled neighbors. TBP (in contrast to standard BP and LBP) allows fast incremental updates in case of new explicit labels or new edges in the graph. We show an intuitive connection between LBP and TBP by proving that the labeling assignments for both are identical in the limit of decreasing coupling strengths between nodes in the graph. Importantly, the linearized matrix equations of both new methods allow compact implementations in SQL. Finally, our runtime experiments show that both new methods are orders of magnitude faster than standard BP while leading to almost identical node labels.

1. INTRODUCTION

Network effects are powerful and often appear in terms of homophily (“birds of a feather flock together”). For example, if we know the political leanings of most of Alice’s friends on Facebook, then we have a good estimate of her leaning as well. Occasionally, the reverse is true, also called heterophily (“opposites attract”). For example, in an online dating site, we may observe that talkative people prefer to date silent ones, and vice versa. In general, we can have k different classes, and k × k affinities or coupling strengths between the classes. Those affinities can be organized in a “coupling matrix” as shown in Figure 1 with three examples. Figure 1a represents homophily for the case of k=2 classes. It indicates that a connection between people with similar political orientations is more likely than between people with different orientations.

In contrast to homophily, the coupling matrix in Figure 1b captures heterophily, as, e.g., required for our online dating example.

In this work, we not only cover these two popular cases, but also capture a more general setting with any symmetric, doubly stochastic coupling matrix and multiple (k > 2) classes. This scenario is illustrated by the example in Figure 1c, which is taken from an on-line auction settings like eBay [36]. Here, we observe k=3 classes of people: Fraudster (F), Accomplice (A) and Honest (H). Honest people buy and sell from other honest people, as well as accomplices; accomplices establish a good reputation (thanks to multiple interactions with honest people), they never interact with other accomplices (waste of effort and money), but they do interact with fraudsters, forming near-bipartite cores with them. Fraudsters primarily interact with accomplices (to build reputation). The interaction with honest people, to

\[ \begin{pmatrix}
D & 0.8 & 0.2 \\
R & 0.2 & 0.8 \\
\end{pmatrix} \]

\[ \begin{pmatrix}
T & 0.3 & 0.7 \\
S & 0.7 & 0.3 \\
\end{pmatrix} \]

\[ \begin{pmatrix}
H & 0.6 & 0.3 & 0.1 \\
A & 0.3 & 0.0 & 0.7 \\
F & 0.1 & 0.7 & 0.2 \\
\end{pmatrix} \]

(a) homophily (b) heterophily (c) general case

Figure 1: Three types of network effects with example coupling matrices. Shading intensity corresponds to the affinities or coupling strengths between classes of neighboring nodes. (a): D: Democrats, R: Republicans. (b): T: Talkative, S: Silent. (c): H: Honest, A: Accomplice, F: Fraudster.
defraud them, happens in last few days shortly before the fraudster’s account is shut down.

In all of the above scenarios, we are interested in the most likely “beliefs” (or labels) for all nodes in the graph. The underlying problem is then: How can we assign class labels, when we know who-contacts-whom and the a priori (“explicit”) labels for some of the nodes in the network?\(^2\)

One of the most widely used methods for this kind of semi-supervised reasoning in networked data is Belief Propagation (BP), which has been successfully applied in scenarios, such as fraud detection \(^{30, 36}\) and malware detection \(^{5}\). BP helps to propagate the information from a few explicitly labeled nodes throughout the network by iteratively propagating information between neighboring nodes. However, when applied to loopy graphs, BP does not always converge and, while there is a lot of work \(^{8, 33}\), exact criteria for convergence are not known (see Sec. 22 for a detailed discussion). In addition, whenever we get additional explicit labels (e.g., we identify more fraudsters in the online auction setting), we need to re-run BP from start. These issues raise fundamental theoretical questions of practical importance: How can we ensure convergence of the algorithm? And how can we support fast incremental updates?

Contributions. In this paper, we introduce two new formulations of BP. Unlike standard BP, these come with convergence guarantees, allow closed form solutions, give a clear intuition about the algorithms, can be implemented on top of standard SQL, and one can even be updated incrementally. We see our three main contributions as follows:

1. **LBP.** Section 3 gives a new matrix formulation for multi-class BP called Linearized Belief Propagation (LBP). Section 4 proves LBP to be the result of applying a certain linearization process to the update equations of BP. Section 4.4 goes one step further and shows that the solution to LBP can be obtained in closed form by the inversion of an appropriate Kronecker product. Section 5.1 shows that this new closed form provides us with convergence guarantees (even on graphs with loops) and a clear intuition about the reasons for convergence/non-convergence. Finally, experiments in Sect. 5 show that a main-memory implementation of LBP takes 4 sec for a graph for which standard BP takes 40 min, while giving almost identical classifications.

2. **TBP.** Section 6 gives a novel semantics for “local” transductive reasoning called Turbo Belief Propagation (TBP). TBP borrows many ideas from relational learners and extends their concepts to heterophily and even more general couplings between classes in a sound and intuitive fashion. In particular, the final labels depend only on the nearest neighbors with explicit labels. We show a deeper connection between LBP and TBP by proving that the labeling assignments for both are guaranteed to be identical in the case of decreasing affinities between nodes in a graph. Importantly, TBP (in contrast to standard BP and LBP) allows fast incremental maintenance for the predicated labels if there are updated or newly found explicit labels. Finally, experiments in Sect. 8 show that a disk-bound implementation of TBP is even faster than LBP by one order of magnitude as every edge in the graph is visited only once.

3. **SQL implementation.** Section 7 shows that both our new formalism allow compact implementations in SQL. For LBP (Sect. 7.1), we translate our linearized matrix formulation into a simple implementation with standard joins and groupings. For TBP (Sect. 7.2), we maintain an intuitive indexing structure based on shortest paths to explicitly labeled nodes that allows incremental updates.

Outline. Section 2 provides necessary background on BP. Sect. 3 introduces the LBP matrix formulation. Sect. 4 gives its derivation. Sect. 5 provides convergence guarantees and extends LBP to weighted graphs. Sect. 6 introduces the TBP semantics. Sect. 7 gives SQL implementations of LBP and TBP. Sect. 8 gives experiments. Sect. 9 contrasts related work, and Sect. 10 concludes. The appendix contains all proofs together with an additional algorithm for updating TBP when incrementally adding edges to the graph. The actual SQL implementation of LBP and TBP can be found on the authors’ home pages.

2. BELIEF PROPAGATION

Belief Propagation (BP), also called the sum-product algorithm, is an exact inference method for graphical models with tree structure \(^{37}\). The idea behind BP is that all nodes receive messages from their neighbors in parallel, then they update their belief states, and finally they send new messages back out to their neighbors. In other words, at iteration \(i\) of the algorithm, the posterior belief of a node is conditioned on the evidence that is \(i\) steps away in the underlying network. This process repeats until convergence and is well-understood on trees.

When applied to loopy graphs, however, BP is not guaranteed to converge to the marginal probability distribution. Indeed, Judea Pearl, who invented BP, cautioned about the indiscriminate use of BP in loopy networks, but advocated to use it as an approximation scheme \(^{37}\). More importantly, loopy BP is not even guaranteed to converge at all. Despite this lack of exact criteria for convergence, many papers have since shown that “loopy BP” gives very accurate results in practice \(^{43}\), and it is thus widely used today in various applications, such as error-correcting codes \(^{26}\) or stereo imaging in computer vision \(^{9}\). Our practical interest in BP comes from the fact that it is not just an efficient inference algorithm on probabilistic graphical models, but it has also been successfully used for semi-supervised learning.

In these learning scenarios, we are interested in the most likely beliefs (or classes) for all nodes in a network. BP helps to iteratively propagate the information from a few nodes with explicit beliefs throughout the network. More formally, consider a graph of \(n\) nodes (or keys) and \(k\) possible classes (or values). Each node maintains a \(k\)-dimensional belief vector where each element \(i\) represents a weight proportional to the belief that this node belongs to class \(i\). We denote by \(e_i\), the vector of prior (or explicit) beliefs and \(b_i\), the vector of posterior (or implicit or final) beliefs at node \(s\), and require that \(e_i\) and \(b_i\) are normalized to 1, i.e. \(\sum_i e_i(i) = \sum_i b_i(i) = 1\). Using \(m_{st}\) for the \(k\)-dimensional

\(^2\)This learning scenario, where we reason from observed training cases directly to test cases, is also called transductive learning. Contrast this with inductive learning where we reason from observed training cases to general rules, which are only then applied to the test cases.

\(^3\)Notice that here and in the rest of this paper, we write \(\sum_i\) as short form for \(\sum_{i \in [k]}\) whenever \(k\) is clear from the context.
message that node \( s \) sends to node \( t \), we can write the BP update formulas [3] for the belief vector of each node and the messages it sends w.r.t. class \( i \) as:

\[
b_s(i) \leftarrow \frac{1}{Z_s} e_s(i) \prod_{u \in N(s)} m_{us}(i) \quad (1)
\]

\[
m_{st}(i) \leftarrow \sum_j e_s(j) H_{st}(j, i) \prod_{u \in N(s) \setminus t} m_{us}(j) \quad (2)
\]

Here, we write \( Z_s \) for a normalizer that makes the elements of \( b_s \) sum up to 1, and \( H_{st}(j, i) \) for a proportional "coupling weight" that indicates the relative influence of class \( j \) of node \( s \) on class \( i \) of node \( t \) (cf. Fig. 1). We assume that the relative coupling between classes is the same in the whole graph, i.e., \( H(i, j) \) is identical for all edges in the graph. We further assume this coupling matrix \( H \) to be doubly stochastic and symmetric. In BP, the above update formulas are repeatedly computed for each node until the values (hopingly) converge to the final beliefs.

The goal in our paper is to find the top beliefs for each node in the network, and to assign these beliefs to the respective nodes. That is, for each node \( s \) we are interested in determining the classes with the highest values in \( b_s \).

**Problem 1 (Top belief assignment). Given:** (i) an undirected graph with \( n \) nodes and adjacency matrix \( A \), where \( A(s, t) \neq 0 \) if the edge \( s-t \) exists, (ii) a symmetric, doubly stochastic coupling matrix \( H \) representing \( k \) classes, where \( H(j, i) \) indicates the relative influence of class \( j \) of a node on class \( i \) of its neighbor, and (iii) a matrix of explicit beliefs \( E \), where \( E(s, i) \neq 0 \) is the strength of belief in class \( i \) by node \( s \). The goal of top belief assignment is to infer for each node the set of classes with highest final belief.

In other words, we are looking for a mapping from nodes to sets of classes in order to allow for ties. This is the main problem that this paper is addressing in a principled way.

### 3. Linearized Belief Propagation

In this section, we derive Linearized Belief Propagation (LBP), which is a closed form description for the final beliefs after convergence of BP under mild restrictions of our parameters. The main idea is to center values around default values (similar to expansions in Taylor series) and to then restrict our parameters to small deviations from these defaults. The resulting equations replace multiplication with addition and can thus be put into a matrix framework with a closed form solution. This allows us to later give exact convergence criteria based on the chosen parameters.

We chose the symbol \( H \) for the coupling weights as reminder of our motivating concepts of homophily and heterophily. Concretely, if \( H(i, i) > H(i, j) \) for \( j \neq i \), we say homophily is present, otherwise heterophily or a mix between the two.

Double stochasticity is essential for our mathematical derivations. Symmetry is not essential for the proofs, but justified by the undirected edges and thus symmetry in both directions.

\[
\text{Logarithm: } \ln(1 + c) = (1 + c)/(1 + e) \approx \frac{1}{2} c + \frac{1}{2} e^2
\]

\[
\text{Division: } (\frac{a}{b}) = \frac{a + c}{b + e} \approx \frac{c + e}{a + b}
\]

**Figure 2:** Table of the linearizing approximations.

**Figure 3:** LBP equation (Eq. 4): Notice our matrix conventions: \( H(j, i) \) indicates the relative influence of class \( j \) of a node on class \( i \) of its neighbor, \( A(s, t) = A(t, s) \neq 0 \) if the edge \( s - t \) exists, and \( B(s, i) \) is the belief in class \( i \) by node \( s \).

**Definition 2 (Centering).** We call a vector or matrix \( x \) "centered around \( c \)" if all its entries are close to \( c \) and their average is exactly \( c \).

**Definition 3 (Residual vector/matrix).** If a vector \( x \) is centered around \( c \), then the residual vector around \( c \) is defined as \( x = [x_1 - c, x_2 - c, \ldots] \). Accordingly, we denote a matrix \( X \) as a residual matrix if each column and row vector corresponds to a residual vector.

For example, we call the vector \( x = [1.01, 1.02, 0.97] \) centered around \( c = 1 \). The residuals from \( c \) will form the residual vector \( \tilde{x} = [0.01, 0.02, -0.03] \). Note that the entries in a residual vector always sum up to 0, by construction.

The main ideas in our proofs are as follows: (1) the \( k \)-dimensional message vectors \( m \) are centered around 1; (2) all the other \( k \)-dimensional vectors are probability vectors, they have to sum up to 1, and thus they are centered around 1/k. This holds for the belief vectors \( b \), \( e \), and for all the entries of matrix \( H \); and (3) we make heavy use of the linearizing approximations shown in Fig. 2.

According to aspect (1) of the previous paragraph, we require that the messages sent are normalized so that the average value of the elements of a message vector is 1 or, equivalently, that the elements sum up to \( k \):

\[
m_{st}(i) \leftarrow \frac{1}{Z_{st}} \sum_j e_s(j) H_{st}(j, i) \prod_{u \in N(s) \setminus t} m_{us}(j) \quad (3)
\]

Here, we write \( Z_{st} \) as a normalizer that makes the elements of \( m_{st} \) sum up to \( k \). Scaling all elements of a message vector by the same constant does not affect the resulting beliefs since the normalizer in Eq. 1 makes sure that the beliefs are always normalized to 1, independent of the scaling of the messages. Thus, scaling messages still preserves the exact solution, yet it will be essential for our derivation.

**Theorem 4 (Linearized BP (LBP)).** Let \( \tilde{B} \) and \( \tilde{E} \) be the residual matrices of final and explicit beliefs centered around 1/k, \( \hat{H} \) the residual coupling matrix centered around 1/k, \( A \) the adjacency matrix, and \( D = \text{diag}(d) \) the diagonal degree matrix. Then, the final belief assignment from belief propagation is approximated by:

\[
\tilde{B} = \tilde{E} + \tilde{A} \tilde{B} \hat{H} - \hat{D} \tilde{B} \hat{H}^2
\]

\[\footnote{Unless otherwise stated, all vectors in this paper are assumed to be column vectors even if written as row vectors.}
Figure 3 illustrates Eq. 4 and shows our matrix conventions. We refer to the term DBH^2 as “echo cancellation”. For increasingly small residuals, the echo cancellation becomes increasingly negligible, and by further ignoring it, Eq. 4 can be further simplified to
\[ B = E + ABH \]  
(5)

We will refer to Eq. 4 (with echo cancellation) as LBP and Eq. 5 (without echo cancellation) as LBP*.

Iterative updates. Note that while these equations give an implicit definition of the final beliefs after convergence, they can also be used as update equations, allowing an iterative calculation of the final solution. Starting with an arbitrary initialization of \( B \) (e.g., all values zero), we repeatedly compute the right hand side of the equations and update the values of \( B \) until the process converges.

\[ \hat{B}(t) \leftarrow E + A\hat{B}(t-1)\hat{H} - DB\hat{B}(t-1)\hat{H}^2 \]  
(6)

Thus, the final beliefs of each node can be computed via elegant matrix operations and optimized solvers, while the closed form gives us guarantees for the convergence of this process, as explained in Sect. 4.4.

4. DERIVATION OF LBP

This section proves our first technical contribution: Section 4.1 linearizes the update equations (Lemma 9) by centering around appropriate defaults and using the approximations from Fig. 2. Sect. 4.2 expresses the steady state messages in terms of beliefs (Lemma 6). Sect. 4.3 gives the steady state beliefs and, thus, proves Theorem 4. Sect. 4.4 gives an explicit expression for the steady-state beliefs (Eq. 15) using Roth’s column lemma.

4.1 Centering Belief Propagation

In the following derivation, we will center the elements of the coupling matrix and all message and belief vectors around their natural default values, i.e., the elements of \( m \) around 1, and the elements of \( H, e, \) and \( b \) around \( \frac{1}{k} \). We are interested in the residual values given by: \( m(i) = 1 + \hat{m}(i) \), \( H(j,i) = \frac{1}{k} + \hat{H}(j,i) \), \( e(i) = \frac{1}{k} + \hat{e}(i) \), and \( b(i) = \frac{1}{k} + \hat{b}(i) \). As a consequence, \( \hat{H} \in \mathbb{R}^{k \times k} \) is a coupling matrix that makes explicit the relative attraction and repulsion: the sign of \( \hat{H}(j,i) \) tells us if the class \( j \) attracts or repels class \( i \) in a neighbor, and the magnitude of \( \hat{H}(j,i) \) indicates the extent. Subsequently, this centering allows us to rewrite belief propagation in terms of the residuals.

\[ \hat{b}(i) \leftarrow \hat{e}(i) + \left( \frac{1}{k} \sum_{u \in N(i)} \hat{m}_{us} \right) \]  
(7)

\[ \hat{m}_{st}(i) \leftarrow k \sum_{j} \hat{H}(j,i)\hat{b}_s(j) - \sum_{j} \hat{H}(j,i)\hat{m}_{st}(j) \]  
(8)

The proof is given in the appendix.

4.2 Messages in Steady State

We next derive a closed-form description of the steady-state of belief propagation.

Lemma 6 (Steady state messages). For small deltas of all values from their expected values, and after convergence of belief propagation, message propagation can be expressed in terms of the steady beliefs as:
\[ \hat{m}_{st} = k(I_k - \hat{H}^2)^{-1}\hat{H}(b_s - \hat{H}b_e) \]  
(9)

where \( I_k \) is the identity matrix of size \( k \).

Proof Lemma 6. Using Eq. 8 and plugging for \( \hat{m}_{ts}(j) \) back into the equation for \( \hat{m}_{st}(j) \), we get:
\[ \hat{m}_{st}(i) \leftarrow k \sum_{j} \hat{H}(j,i)\hat{b}_s(j) - \sum_{j} \hat{H}(j,i)\hat{m}_{st}(j) \]
\[ \left( \frac{k}{2} \sum_{j} \hat{H}(g,j)\hat{b}_s(g) - \sum_{j} \hat{H}(g,j)\hat{m}_{st}(g) \right) \]

Now, for the case of convergence, both \( \hat{m}_{st}(g) \) on the left and right side of the equation need to be equivalent. We can, therefore, group related terms together and replace the update symbol with equality:
\[ \hat{m}_{st}(i) = \frac{k}{2} \sum_{j} \hat{H}(j,i)\hat{b}_s(j) - \sum_{j} \hat{H}(j,i)\hat{m}_{st}(j) \]
\[ = k \sum_{j} \hat{H}(j,i)\hat{b}_s(j) - k \sum_{j} \hat{H}(j,i)\hat{m}_{st}(j) \]
\[ = k \sum_{j} \hat{H}(g,j)\hat{b}_s(g) - k \sum_{j} \hat{H}(g,j)\hat{m}_{st}(g) \]
\[ = k \sum_{j} \hat{H}(g,j)\hat{m}_{st}(g) \]
\[ = k \hat{H}\hat{b}_s - k\hat{H}^2\hat{b}_s \]  
(10)

This equation can then be written in matrix notation as:
\[ (I_k - \hat{H}^2)\hat{m}_{st} = k\hat{H}\hat{b}_s - k\hat{H}^2\hat{b}_s \]  
(11)

which leads to Eq. 9 given that all entries of \( \hat{H} << \frac{1}{k} \) and thus the inverse of \( (I_k - \hat{H}^2) \) always exists.

4.3 Beliefs in Steady State

We are now ready to prove Theorem 4.

Proof Theorem 4 (BP in closed form). For steady-state, we can write Eq. 7 in vector form as:
\[ \hat{b}_s = \hat{e}_s + \left( \frac{1}{k} \sum_{u \in N(s)} \hat{m}_{us} \right) \]
and by substituting \( \hat{H} \) for \((I_k - \hat{H}^2)^{-1}\hat{H}\), we write Eq. 9 as
\[ \hat{m}_{us} = k\hat{H}\hat{H}(b_s - \hat{H}b_e) \]
Combining the last two equations, we get

$$\mathbf{b}_s = \mathbf{e}_s + \hat{\mathbf{H}} \sum_{u \in N(s)} \mathbf{b}_u - d_s \mathbf{H} \hat{\mathbf{H}} \mathbf{b}_s$$

(12)

where $d_s$ is the degree or number of bi-directional neighbors for node $s$, i.e. neighbors that are connected to $s$ with edges in both directions (see Sect. 5.2 for a discussion of the implication). By using $\hat{\mathbf{B}}$ and $\hat{\mathbf{E}}$ as $n \times k$ matrices of final and initial beliefs, $\mathbf{D}$ as the diagonal degree matrix, and $\mathbf{A}$ as the adjacency matrix, Eq. 12 can be written in matrix form

$$\hat{\mathbf{B}} = \hat{\mathbf{E}} + \mathbf{A} \hat{\mathbf{H}} \mathbf{b}_s - \mathbf{D} \hat{\mathbf{H}} \hat{\mathbf{H}} \mathbf{b}_s$$

(13)

By approximating $(\mathbf{I}_k - \hat{\mathbf{H}}^2) \approx \mathbf{I}_k$ (recall that all entries of $\hat{\mathbf{H}} << \frac{1}{k}$), we can simplify to

$$\hat{\mathbf{B}} = \hat{\mathbf{E}} + \mathbf{A} \hat{\mathbf{H}} \mathbf{b}_s - \mathbf{D} \hat{\mathbf{H}}^2 \mathbf{b}_s$$

And by further ignoring the second term with residual terms of third order, we can further simplify to get Eq. 5.

### 4.4 Closed form solution for LBP

In practice, we will solve Eq. 4 and Eq. 5 via an iterative computation (see end of Sect. 3). However, we will next give a closed form solution, which will allow us later to study the convergence of the iterative updates.

We need to introduce two new notions and one lemma: Let $\mathbf{X}$ and $\mathbf{Y}$ be matrices of order $m \times n$ and $p \times q$, respectively, and let $x_{ij}$ denote the $j$-th column of matrix $\mathbf{X}$, i.e., $\mathbf{X} = \{x_{ij}\} = [x_1 \ldots x_n]$. The vectorization of matrix $\mathbf{X}$ stacks the columns of a matrix one underneath the other to form a single column vector, i.e.

$$\text{vec}(\mathbf{X}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The Kronecker product of $\mathbf{X}$ and $\mathbf{Y}$ is the $mp \times nq$ matrix defined by

$$\mathbf{X} \otimes \mathbf{Y} = \begin{bmatrix} x_{11} \mathbf{Y} & x_{12} \mathbf{Y} & \cdots & x_{1n} \mathbf{Y} \\ x_{21} \mathbf{Y} & x_{22} \mathbf{Y} & \cdots & x_{2n} \mathbf{Y} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} \mathbf{Y} & x_{m2} \mathbf{Y} & \cdots & x_{mn} \mathbf{Y} \end{bmatrix}$$

Finally, Roth’s column lemma [18] states that

$$\text{vec}(\mathbf{XYZ}) = (\mathbf{Z}^\top \otimes \mathbf{X}) \text{vec}(\mathbf{Y})$$

With $\hat{\mathbf{H}}^2 = \hat{\mathbf{H}}$, we can thus write Eq. 4 as

$$\text{vec}(\mathbf{B}) = \text{vec}(\hat{\mathbf{E}}) + (\mathbf{H} \otimes \mathbf{A}) \text{vec}(\mathbf{B}) - (\hat{\mathbf{H}}^2 \otimes \mathbf{D}) \text{vec}(\mathbf{B})$$

$$= \text{vec}(\hat{\mathbf{E}}) + (\mathbf{H} \otimes \mathbf{A} - \hat{\mathbf{H}}^2 \otimes \mathbf{D}) \text{vec}(\mathbf{B})$$

(14)

which can be solved for $\text{vec}(\mathbf{B})$ as

$$\text{vec}(\mathbf{B}) = (\mathbf{I}_{nk} - \mathbf{H} \otimes \mathbf{A} + \hat{\mathbf{H}}^2 \otimes \mathbf{D})^{-1} \text{vec}(\hat{\mathbf{E}})$$

(15)

Similarly to before, by ignoring the echo cancellation $\hat{\mathbf{H}}^2 \otimes \mathbf{D}$, we can further simplify to get:

$$\text{vec}(\mathbf{B}) = (\mathbf{I}_{nk} - \mathbf{H} \otimes \mathbf{A})^{-1} \text{vec}(\hat{\mathbf{E}})$$

(16)

Overall, using Eq. 15 and Eq. 16 we are able to compute the final beliefs in a closed form as long as the inverse of the matrix exists. Likewise, the update equation Eq. 6 is equivalent to

$$\text{vec}(\mathbf{B}_{(l)}^{(l-1)}) = \text{vec}(\hat{\mathbf{E}}) + (\mathbf{H} \otimes \mathbf{A} - \hat{\mathbf{H}}^2 \otimes \mathbf{D}) \text{vec}(\mathbf{B}_{(l-1)})$$

(17)

These updates are of the form $\mathbf{y}_{(l)}^{(l)} \leftarrow \mathbf{y} + \mathbf{M}\mathbf{y}_{(l-1)}^{(l-1)}$, which are known to converge for any choice of initial values for $\mathbf{y}_{(0)}$, as long as $\mathbf{M}$ has a spectral radius $\rho(\mathbf{M}) < 1$. We will use this fact to derive exact convergence guarantees next.

### 5. ADDITIONAL BENEFITS OF LBP

In this section we give convergence criteria for LBP and show that our formalism generalizes to weighted graphs.

#### 5.1 Convergence guarantees

Equation 15 and Eq. 16 give us a closed form for the final beliefs after convergence. Before we give our convergence criteria, we start with a few general properties of matrices, eigenvalues, norms and the Kronecker product [27, 31, 38].

Let $\rho(\cdot)$ be the spectral radius (i.e. the supremum among the absolute values of the eigenvalues) of the enclosed matrix, and $\mathbf{X}$ and $\mathbf{Y}$ be appropriate matrices. Then, $\rho(\mathbf{X} \otimes \mathbf{Y}) = \rho(\mathbf{X})\rho(\mathbf{Y})$. Furthermore, the Neumann series $\sum_{k=0}^{\infty} \mathbf{X}^k$ converges if and only if $\rho(\mathbf{X}) < 1$. In other words, if this condition holds, then $(\mathbf{I} - \mathbf{X})$ is invertible and we have $(\mathbf{I} - \mathbf{X})^{-1} = \sum_{k=0}^{\infty} \mathbf{X}^k$. Based on these two properties, we can give a sufficient criterion for convergence of the iterative approaches for LBP and a sufficient and necessary criterion for LBP*.

**Lemma 7 (Eigenvalue Convergence).** LBP and LBP* have the following sufficient, respectively, sufficient and necessary criteria for convergence:

LBP converges $\iff \rho(\hat{\mathbf{H}}) < \frac{\sqrt{\rho(\mathbf{A})^2 + 4\rho(\mathbf{D})} - \rho(\mathbf{A})}{2\rho(\mathbf{D})}$

LBP* converges $\iff \rho(\hat{\mathbf{H}}) < \frac{1}{\rho(\mathbf{A})}$

In practice, computation of the largest eigenvalues can be expensive. Instead, we can exploit certain properties of sub-multiplicative norms that give upper bounds to the spectral radius of a matrix to establish sufficient and easier-to-compute conditions for convergence.

**Lemma 8 (Norm Convergence).** Let $|| \cdot ||$ stand for any sub-multiplicative norm of the enclosed matrix. Further, let $M$ be a set of such norms and $f_M(\mathbf{X}) := \min_{\| \cdot \| \in M} || \mathbf{X} ||$, be the function returning the smallest value when applying all norms in $M$ on $\mathbf{X}$. Then, the two criteria from Lemma 7 still hold as sufficient but not necessary conditions of convergence after replacing any norm $\rho(\cdot)$ of a matrix with any

\footnote{For all practical purposes, due to the size of the matrix $\mathbf{H} \otimes \mathbf{A}$, the iterative approach of Eq. 6 is preferable over Eq. 17.}
such norm \( ||\cdot|| \in M \). In particular, by replacing \( \rho(\cdot) \) with \( f_M(\cdot) \), we get better bounds.

Vector/Elementwise \( p \)-norms for \( p \in [1, 2] \) (e.g., the Frobenius norm) and all induced \( p \)-norms are sub-multiplicative.\(^\text{10}\) Furthermore, vector \( p \)-norms are monotonically decreasing for increasing \( p \), and thus: \( \rho(X) \leq ||X|| \leq ||X||^p \). We thus suggest to use the set \( M \) of the three norms (i) Frobenius norm, (ii) induced-1 norm, and (iii) induced-\( \infty \) norm which are all fast to calculate.

5.2 Weighted graphs

Note that Theorem 4 can be generalized to allow weighted graphs by simply using a weighted adjacency matrix \( A \) with elements \( A(i,j) = w > 0 \) if the edge \( j - i \) exists with weight \( w \), and \( A(i,j) = 0 \) otherwise. Our derivation remains the same, we only need to make sure that the degree \( d_i \) of a node \( s \) is the sum of the squared weights to its neighbors (squared for going back and forth). The weight on an edge simply scales the coupling strengths between two neighbors, and we have to add up parallel paths. In other words, Theorem 4 can be applied for weighted graphs as well.

6. TURBO BELIEF PROPAGATION

Our ultimate goal with belief propagation is to assign the most likely classes to each unlabeled node, i.e. every node without explicit beliefs. In this section, we define a semantics for top belief assignment that is strongly related to BP and LBP (it gives the same classiﬁcation for increasingly small coupling weights), but that has two strong algorithmic advantages: (i) calculating the final beliefs requires to visit every node only once, and (ii) the beliefs can be incrementally maintained when adding explicit beliefs or edges to the graph.

6.1 Scaling Beliefs

We start with a simple deﬁnition that helps us separate the relative strength of beliefs from their absolute values.

**Definition 9 (Standardization).** Given a vector \( x = [x_1, x_2, \ldots, x_k] \), we deﬁne its standardization as the new vector \( x' = \zeta(x) \) with \( x'_i = \frac{x_i - \mu(x)}{\sigma(x)} \), where \( \mu(x) \) and \( \sigma(x) \) are the mean and the standard deviation of the elements of \( x \) respectively if \( \sigma \neq 0 \), and with \( x'_i = 0 \) if \( \sigma = 0 \).\(^\text{11}\)

For example, \( \zeta([1, 0]) = [1, -1] \), \( \zeta([1, 1, 1]) = [0, 0, 0] \), and \( \zeta([1, 0, 0, 0]) = [2, -0.5, -0.5, -0.5, -0.5] \). The standardized belief assignment \( \hat{b}'_s \) for a node \( s \) is then the standardization of the final beliefs after convergence of belief propagation: \( \hat{b}'_s = \zeta(\hat{b}_s) \). For example, assume two nodes \( s \) and \( t \) with final beliefs \( \hat{b}_s = [4, -1, -1, -1, -1] \) and \( \hat{b}_t = [40, -10, -10, -10, -10] \), respectively. The standardized belief assignment is then the same for both nodes: \( \hat{b}'_s = \hat{b}'_t = [2, -0.5, -0.5, -0.5, -0.5] \) whereas the standard deviations maintain the original scaling: \( \sigma(\hat{b}_s') = 2, \sigma(\hat{b}_t') = 20 \).

**Lemma 10 (Scaling \( \hat{B} \)).** Scaling the explicit beliefs with a constant factor \( \lambda \) leads to scaled final beliefs by \( \lambda \). In other words, \( \forall \lambda \in \mathbb{R} : (\hat{B} \leftarrow \lambda \cdot \hat{B}) \Rightarrow (\hat{B} \leftarrow \lambda \cdot \hat{B}) \).

**Proof.** Follows immediately from Eq. 19. \( \square \)

**Corollary 11 (Scaling \( \hat{E} \)).** Scaling \( \hat{E} \) with a constant factor does not change the standardized belief assignment \( \hat{B}' \).

The last corollary implies that scaling the explicit beliefs has no effect on the top belief assignment, and thus the ultimate classiﬁcation of LBP.\(^\text{12}\)

6.2 Scaling Coupling Strengths

While scaling \( \hat{E} \) has no effect on the standardized beliefs, the scale of the coupling matrix \( \hat{H} \) is important. To separate (i) the relative difference among beliefs from (ii) their absolute scale, we introduce a parameter \( \epsilon_{BH} \) and deﬁne with \( \hat{H}_s \) the unscaled (“original”) coupling matrix implicitly by: \( \hat{H} = \epsilon_{BH} \hat{H}_s \). This separation allows us to keep the relative scaling ﬁxed as \( \hat{H}_s \), and to thus analyze the inﬂuence of the absolute scaling on the standardized belief assignment (and thereby the top belief assignment) by varying \( \epsilon_{BH} \) only.

It was previously observed in experiments \(^\text{24}\) that the top belief assignment is the same for a large range of small \( \epsilon_{BH} \) in belief propagation with binary classes, but that it deviates for very small \( \epsilon_{BH} \). Here we show that the standardized belief assignment for LBP converges for \( \epsilon_{BH} \rightarrow 0 \), and that any deviations are only due to limited computational precision. We also give a new closed form for the predictions of LBP in the limit and name this semantics Turbo Belief Propagation (TBP). TBP has several advantages: (i) it is faster to calculate (therefore the choice of name), (ii) it can be maintained incrementally, and (iii) it provides a simple intuition about its behavior and an interesting connection to relational learners. For that, we need one more notion:

**Definition 12 (Geodesic number \( g_t \)).** The geodesic number \( g_t \) of a node \( t \) is the length of the shortest path to \( t \) from among all nodes with explicit beliefs.

Notice that any node with explicit beliefs has geodesic number 0. For the following deﬁnition, let the weight \( w \) of a path \( p \) be the product of the weights of its edges (if the graph is unweighted, the weights are simply 1).

**Definition 13 (Turbo BP (TBP)).** Given a node \( t \) with geodesic number \( k \), let \( P^k_t \) be the set of all paths with length \( k \) from a node with explicit beliefs to \( t \). For any such path \( p \in P^k_t \), let \( w_p \) be its weight, and \( \hat{e}_p \) the explicit beliefs of the node at the start of path \( p \). The final belief assignment \( \hat{b}_t \) for turbo belief propagation (TBP) is deﬁned by

\[
\hat{b}_t = \hat{H}^k \sum_{p \in P^k_t} w_p \hat{e}_p \tag{18}\]

\(^\text{12}\) Notice that their absolute magnitude still needs to be small compared to \( \frac{1}{\lambda} \) in order to fulﬁll our original centering assumption.
The simple intuition behind TBP is that nodes with increasing distance have an increasingly negligible influence:
By order of magnitude, nodes at distance $k$ have an influence that is $\epsilon^k$ times the influence of nodes at distance $k-1$.
Thus in the limit of $\epsilon_H \to 0$, the nearest neighbors with explicit beliefs will dominate the influence of any other node.
Since linear scaling does not change the standardization of a vector, $\zeta(\epsilon \mathbf{x}) = \zeta(\mathbf{x})$, scaling $\mathbf{H}$ has no effect on the standardized and thus also top belief assignments for TBP.
In other words, the standardized belief assignment of TBP is independent of $\epsilon_H$ (as long as $\epsilon_H > 0$), and WLOG, we can therefore use the unsealed coupling matrix $\mathbf{H}_\epsilon$ ($\epsilon_H = 1$).

**Example 14 (TBP illustration).** Consider the undirected and unweighted graph of Fig. 5a. Node $v_0$ has geodesic number 2 since the closest nodes with explicit beliefs are $v_1$ and $v_6$ two hops away. There are three highlighted shortest paths to those beliefs. The TBP standardized belief assignment is then $\mathbf{b}_{v_0}^H = \zeta(\mathbf{H}_0^H(2\mathbf{e}_{v_1} + 4\mathbf{e}_{v_6}))$. Notice that the factor 2 for $\mathbf{e}_{v_1}$ arises from the 2 shortest paths from $v_1$ to $v_0$ (both paths with the same weight of 1).

The following theorem gives the connection between LBP and TBP and is the main result of this section.

**Theorem 15 (Limit of LBP).** For $\epsilon_H \to 0$, the standardized belief assignment for LBP converges towards the standardized belief assignment for TBP.

**Example 16 (Detailed example).** Consider the unweighted and undirected torus graph shown in Fig. 5b and assume explicit beliefs $\mathbf{e}_{v_1} = [2,-1,-1]$, $\mathbf{e}_{v_2} = [-1,2,-1]$, $\mathbf{e}_{v_3} = [-1,-1,2]$, plus the unsealed coupling matrix from Fig. 1c: $\mathbf{H}_0 = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.3 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix} \approx \begin{bmatrix} 1 \end{bmatrix}_{3 \times 3}$. We focus on node $v_4$ and compare the standardized belief assignment $\mathbf{b}_{v_4}^H$ and the standard deviation $\sigma(\mathbf{b}_{v_4}^H)$ between BP, LBP, $\rho LBP$, and TBP for the limit of $\epsilon_H \to 0$. TBP predicts the standardized beliefs to be the consequence of the two shortest paths, $v_4 - v_8 - v_5 - v_1$ and $v_4 - v_8 - v_7 - v_3$, and thus $\mathbf{b}_{v_4}^H = \zeta(\mathbf{H}_0^H(\mathbf{e}_{v_1} + \mathbf{e}_{v_3})) \approx [-0.069,1.258,-1.189]$. For the standard deviation, we get $\sigma(\mathbf{b}_{v_4}^H) = \sigma(\mathbf{H}_0^H(\mathbf{e}_{v_1} + \mathbf{e}_{v_3})) = \epsilon_H^3 \sigma(\mathbf{H}_0^H(\mathbf{e}_{v_1} + \mathbf{e}_{v_3})) = \epsilon_H^3 \rho(\mathbf{H}_0^H(\mathbf{e}_{v_1} + \mathbf{e}_{v_3})) \approx 0.332$. LBP is guaranteed to converge for $\epsilon_H < \frac{\sqrt{\rho(\mathbf{A})^2 + 4\rho(\mathbf{D})} - \rho(\mathbf{A})}{2\rho(\mathbf{H}_0^H(\mathbf{D}))}$. Given $\rho(\mathbf{A}) \approx 2.414$, $\rho(\mathbf{H}_0) \approx 0.629$, and $\rho(\mathbf{D}) = 3$, we thus get $\epsilon_H \lesssim 0.479$. LBP converges if and only if $\epsilon_H < \frac{1}{\rho(\mathbf{H}_0)}$ and thus for $\epsilon_H \lesssim 0.658$. Using the norm approximations instead of the spectral radii, we get $\epsilon_H \lesssim 0.360$ for LBP, and $\epsilon_H \lesssim 0.455$ for $\rho LBP$. Figure 4c and 4d illustrate that the spectral radius bound for $\rho LBP$ captures the convergences exactly.

7. IMPLEMENTATIONS IN SQL

In this section, we give translations of our two semantics into the relational model, which allows any standard DBMS to perform this kind of semi-supervised learning.

7.1 LBP in SQL

In the following, we use Datalog notation extended with aggregates in the tradition of [7]. Such an aggregate query has the form $Q(\bar{x}, \alpha(\bar{y})) = C(\bar{z})$ with $C$ being a conjunction of non-negated relational atoms and comparisons, and $\alpha(\bar{y})$ being the aggregate term $\bar{y}$. When translating into SQL, the head of the query ($\bar{x}, \alpha(\bar{y})$) defines the SELECT clause. In addition, the variables $\bar{z}$ also appear in the GROUP BY clause.

Note that in a slight abuse of notation (and for the sake of conciseness), we use variables to express both attribute names and join variables in Datalog notation.
create table H2 as
select H1.c1, H2.c2,
sum(H1.h*H2.h) as h
from H H1, H H2
where H1.c2 = H2.c1
group by H1.c1, H2.c2
(a) Equation 19

Algorithm 1: (LBP) Returns the final beliefs $B$ with LBP for a weighted network $A$ with explicit beliefs $E$, coupling strengths $H$, and calculated tables $D$ and $H_2$.

**Input:** $A(s, t, w), E(v, c, b), H(c_1, c_2, h), D(v, d), H_2(c_1, c_2, h)$

**Output:** $B(v, c, b)$

1. Initialize final beliefs for nodes with explicit beliefs:
   $B(s, c, b) := E(s, c, b)$
2. for $i \leftarrow 1$ to $i$ do
   3. Create two temporary views:
      $V_1(t, c_2, \sum(w \cdot b \cdot h)) := A(s, t, w), B(s, c_1, b), H(c_1, c_2, h)$
      $V_2(s, c_2, \sum(d \cdot b \cdot h)) := D(s, d), B(s, c_1, b), H_2(c_1, c_2, h)$
4. Update final beliefs:
   $B(v, c, b_1 + b_2 - b_3) := E(v, c, b_1), V_1(v, c, b_2), V_2(v, c, b_3)$
return $B(v, c, b)$
(b) Top belief assignment.

corollary 17 (LBP in SQL). The iterative updates for LBP can be expressed in standard SQL with loops.

14Remember from Sect. 5.2 that the degree of a node in a weighted graph is the sum of the squares of the weights to all neighbors.
15In practice, we use union all, followed by a grouping on $v, c$.

Figure 6: SQL translation for example Datalog statements.

Algorithm 2: (TBP) Returns the final beliefs $B$ and geodesic numbers $G$ with TBP for a weighted network $A$ with explicit beliefs $E$, and coupling scores $H$.

**Input:** $A(s, t, w), E(v, c, b), H(c_1, c_2, h)$

**Output:** $B(v, c, b), G(v, g)$

1. Initialize geodesic $n$, and beliefs for nodes with explicit beliefs:
   $G(v, 0') := E(v, c, b)$
2. $i \leftarrow 1$
3. repeat
   4. Find next nodes to calculate:
      $G(t, i) := G(s, i - 1), A(s, t, \_), \neg G(t, \_)$
   5. Calculate beliefs for new nodes:
      $B(t, c_2, \sum(w \cdot b \cdot h)) := G(t, i), A(s, t, w), B(s, c_1, b), G(s, i - 1), H(c_1, c_2, h)$
   6. $i \leftarrow i + 1$
until no more inserts into $G$
return $B$ and $G$
(d) Equation 20

Finally, we return the top beliefs for each node with the query from Fig. 6b.

7.2 TBP in SQL

The TBP semantics may assign beliefs to a node that depend on an exponential number of paths (exponential in the geodesic number of a node). However, TBP actually allows a simple algorithm in SQL that visits every edge only once. We achieve this by adding a table $G(v, g)$ to the schema that stores the geodesic number $g$ for each node $v$. This table $G$, in turn, also supports efficient updates. In the following, we give two algorithms for (1) the initial assignments of beliefs and (2) addition of explicit beliefs. The Appendix also includes an algorithm for (3) addition of edges to the graph.

(1) Initial belief assignment. Algorithm 2 shows the initial calculation of all final beliefs: We start with the node's beliefs of explicit beliefs, i.e. geodesic number 0. At each subsequent iteration, we then determine nodes with increasing geodesic number by following edges from previously inserted nodes (i.e. those with geodesic number smaller by 1), but ignoring nodes that have already been visited (i.e. those that are already in $G$). Note that in a slight abuse of Datalog notation (and for the sake of conciseness), we allow negation on relational atoms with anonymous variables implying a nested not exist query. Figure 6b shows the query from line 4 in SQL. The beliefs of the new nodes are then calculated by following all edges from nodes that
Algorithm 3: (TBP.newExplicitBeliefs) Updates \( B \) and \( G \), given new explicit beliefs \( E_n \) and weighted network \( A \).

**Input**: \( E_n(v,c,b), A(s,t,u) \)

**Output**: Updated \( B(v,c,b) \) and \( G(v,g) \)

1. Initialize geodesic numbers for new nodes with explicit beliefs:
   \[ G_v(v', 0') := E_n(v, w) \]
   \[ !G_v(v', 0') := {\overline{G_v(v, g)}} \]
2. Initialize beliefs for new nodes:
   \[ B_n(v, c, b) := E_n(v, c, b) \]
   \[ !B_n(v, c, b) := {\overline{B_n(v, c, b)}} \]
3. \( i \leftarrow 1 \)
4. **repeat**
   5. Find next nodes to update:
      \[ G_v(t, i) := G_v(s, i - 1), A(s, t, u), \neg(G_v(t, g), g < i) \]
      \[ !G_v(t, i) := {\overline{G_v(s, i)}}, \neg({\overline{G_v(t, g)}}, g < i) \]
6. Calculate new beliefs for these nodes:
   \[ B_n(t, c, b, \sum(w \cdot b \cdot h)) := G_v(t, i), A(s, t, u), B(s, c, b), \]
   \[ G_v(s, i - 1), H(c, g, h) \]
7. \( !B_n(v, c, b) := {\overline{B_n(v, c, b)}} \)
8. \( i \leftarrow i + 1 \)
9. **until** no more inserts into \( G_v \)
10. **return** \( B \) and \( G \)

> have just been assigned their beliefs in the previous step.

**Proposition 18** (Algorithm 2). Algorithm 2 terminates in finite number of iterations and returns a sound and complete enumeration of final beliefs according to TBP.

(2) Addition of explicit beliefs. We assume the set of changed or additional explicit beliefs to be available in table \( E_n(v,c,b) \) and use tables \( G_v(v,g) \) and \( B_n(v,c,b) \) to store temporary information for nodes that get updated. We will further use an exclamation mark left of a Data- log query to imply that the respective data record is either inserted or an existing one updated. To illustrate, Fig. 6d shows the SQL equivalent for the following update:

\[ !B(v,c,b) := B_n(v,c,b) \]  

Algorithm 3 shows the SQL translation for batch updates of explicit beliefs: Line 1 and line 2 initialize tables \( G_v \) and \( B_n \) for all new explicit nodes. At each subsequent iteration \( i \) line 3, we then determine all nodes \( t \) that need to be updated with new geodesic number \( g_i = i \) by following edges from previously updated nodes \( s \) with geodesic number \( g_s = i \) and ignoring those that already have a smaller geodesic number \( g_s < i \). Line 5. For these nodes \( t \), the updated beliefs are then calculated by only following edges that start at nodes \( s \) with geodesic number \( g_s = i \) independent of whether those were updated or not. Line 6. The algorithm terminates when there are no more inserts in table \( G_v \). Line 8.

**Proposition 19** (Algorithm 3). Algorithm 3 terminates in finite number of iterations and returns a sound and complete enumeration of updated beliefs.

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8. **Experiments**

In this section, we experimentally verify how well our new methods LBP and TBP scale, and how close the top belief classification of both methods matches that of standard BP.

**Experimental setup.** We implemented main memory-based versions of BP and LBP in JAVA, and disk-bound versions of LBP and TBP in PostgreSQL. The JAVA implementation uses optimized libraries for sparse matrix operations. When timing our memory-based algorithms, we focus on the running times for computations only and ignore the time for loading data and initializing matrices. For the SQL implementation, we report the times from start to finish. We are mainly interested in relative performance within a platform (LBP vs. BP in JAVA, and TBP vs. LBP in SQL) and scalability with graph sizes. Both implementations run on a 2.5 Ghz Intel Core i5 with 16G of main memory and a 1TB SSD hard drive. To allow comparability across implementations, we limit evaluation to one processor. For timing results, we run BP and LBP for 5 iterations.

**Synthetic Data.** We assume a scenario with \( k = 3 \) classes and the matrix \( H \) from Fig. 7a as the unscaled coupling matrix. We study the convergence of our algorithms by scaling \( H \) with a varying parameter \( \varepsilon \). We created 9 “Kronecker graphs” of varying sizes (see Fig. 7a) which are known to share many properties with real world graphs. To generate initial class labels (explicit beliefs), we pick 5% of the nodes in each graph and assign to them two random numbers from \((-0.1, -0.09, \ldots, 0.09, 0.1\) as centered beliefs for two classes (the belief in the third class is then their negative sum due to centering). For timing of incremental updates for TBP (shown as \( \Delta \)TBP), we similarly created updates for 2% of the nodes with explicit beliefs (corresponding to \( 1\% = 0.1\% \) of all nodes in a graph).

**Measuring classification quality**. We take the top beliefs returned by BP as “ground truth” and are interested in how close the classification returned by LBP and TBP comes for varying scaling of \( H \). We measure quality of our methods by using precision and recall as follows: Given a set of top beliefs \( B_{GT} \) for a ground truth labeling method (GT) and a set of top beliefs \( B \) of another method (O), let \( B \)

\[\text{Figure 7: Synthetic data used for our experiments.}\]

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Notes:

1. **Note that edges \( s \rightarrow t \) with \( g_s \geq g_t \) cannot contain a geodesic path in that direction and are thus ignored. Also note that, again for the sake of conciseness, we write \( G(t,g), g < i \) to indicate that nodes \( t \) with \( g_t < i \) are not updated. In SQL, we used an except clause.**

18.Parallel Colt: [http://sourceforge.net/projects/parallelcolt/](http://sourceforge.net/projects/parallelcolt/)

19.**Note that we count as the number of edges the number of entries in \( A \); thus each edge is counted twice (\( s \rightarrow t \) equal to \( s \rightarrow t \) and \( t \rightarrow s \)).**

20.**This approach is justified as BP has previously been shown to work well in real-life classification scenarios. Our goal in this paper is to replace BP with a faster and simpler semantics that works as well.**
be the set of shared beliefs: \( B_C \cap B_O \). Then, recall \( r \) measures the portion of GT beliefs that are returned by \( \Omega : r = |B_C|/|B_C| \), and precision \( p \) the portion of “correct” beliefs among \( B_C \): \( p = |B_C|/|B_C| \). Notice that this method also naturally handles ties. For example, assume that the GT assigns classes \( c_1, c_2, c_3 \) as top beliefs to 3 nodes \( v_1, v_2, v_3 \), respectively: \( v_1 \rightarrow c_1, v_2 \rightarrow c_2, v_3 \rightarrow c_3 \), whereas the comparison method assigns 4 beliefs: \( \{v_1 \rightarrow \{c_1, c_2\}, v_2 \rightarrow c_2, v_3 \rightarrow c_2\} \). Then \( r = 2/3 \) and \( p = 2/4 \).

**QUESTION 1.** How fast and scalable are our LBP and TBP as compared to BP in both implementations?

**Result 1.** The main memory implementation of LBP is up to 600 times faster than BP, and the SQL implementation of TBP is more than 10 times faster than LBP.

**Figure 8a** and **Fig. 8b** show our timing experiments for both JAVA and SQL, respectively. **Fig. 9** compares the times for the 5 largest graphs. Notice that all implementations except BP show approximate linear scaling behavior in the number of edges (as reference, both **Fig.8a** and **Fig.8b** show a dashed grey line that represents an exact linear scalability of 100k edges per second). The main-memory implementation of LBP is 600 times faster than that of BP for the largest graph. The SQL implementation of TBP is 10 times faster than the one for LBP. Not surprisingly, the main-memory JAVA implementation of LBP is faster than the disk-bound LBP implementation in SQL. On the one hand, we don’t measure time for loading the data in JAVA. More importantly, however, is the fact that our matrix formulation of LBP enables us to use well-established libraries for matrix operations. These optimized operations lead to a highly efficient algorithm. It is worth mentioning that even though our SQL implementation did not exploit special libraries, the disk-bound TBP implementation is faster than the main-memory implementation of BP (!).

**QUESTION 2.** How big is the benefit of incrementally updating TBP instead of recalculating from scratch?

**Result 2.** For few changes, incremental updates are faster.

**Figure 9** shows that updating 1% of the nodes in a graph that previously had 5% nodes with explicit beliefs is faster than re-running TBP from the start. The relative speed-up is around 2.5 for the larger graphs. We observed there to be a natural switching point above which it is faster to re-calculate from scratch than to incrementally update. We suppose that these results occur due to the increased number of required random disk-accesses.

**QUESTION 3.** How do the top belief assignments of LBP, LBP* and TBP compare to that of BP?

**Result 3.** BP, LBP, LBP*, and TBP give almost identical top belief assignments.

**Figure 8c** shows recall \( r \) and precision \( p \) of LBP with BP as GT (“LBP w.r.t. BP”) on graph #5 (we observed similar results for all remaining graphs). The vertical grey line shows \( \epsilon_H = 0.0002 \) for which our Lemma 8 guarantees convergence of LBP, and the vertical drop in \( r \) and \( p \) to the right corresponds to the choice of \( \epsilon_H \) for which LBP stops converging, in practice. We observe that LBP matches the top belief assignment of BP over the whole range of guaranteed convergence. **Figure 8d** shows that the predictions of LBP and LBP* match very well as long as both algorithms converge (LBP and LBP* return different but unique top belief assignments; therefore, \( r \) and \( p \) turned out to be equivalent). We thus conclude that LBP and LBP* lead to the same classifications as long as \( \epsilon_H \) is small enough (and thus that the additional summand in LBP does not add to increased precision). We also validated that the results of LBP in Java and SQL are identical (not shown in our graphs).

**Figure 8e** also validates that TBP closely matches LBP (and thus BP). The averaged recall of TBP vs. LBP between
\(10^{-9} < \epsilon_H < 0.0002\) is 0.9992 and the averaged precision 0.9968. The visible oscillations and the observation that TBP’s precision values are generally lower than its recall values are dominantly due to tied top beliefs for TBP. That means that TBP returns two top beliefs, while LBP only one. For example, we observed the following final beliefs which lead to a drop in precision (due to TBP’s tie):

TBP: \([1, 1, -2] \cdot 10^{-2}\)
LBP: \([1.00000000014, 1.0000000002, -2.0000000016] \cdot 10^{-2}\)

And the following more rare belief assignments, which are due to numerical rounding errors, led to a drop in both precision and recall (two different top beliefs returned):

TBP: \([7.6, 7.59999999999999, -15.2] \cdot 10^{-11}\)
LBP: \([7.60009, 7.60047, -15.20056] \cdot 10^{-11}\)

Minimizing the possibility of ties, e.g., by choosing initial explicit beliefs with additional digits (e.g., 0.0503 instead of 0.05) removed these oscillations. In summary, TBP and LBP capture the classification of BP very well. Any misclassification is commonly due to closely tied top beliefs, in which case returning both tied beliefs as done by TBP would arguably be the preferable alternative in the first place.

9. RELATED WORK

Transductive inference. In this paper, we propose two solutions to the transductive inference problem, which in its generality appears in a number of applied scenarios in the database and machine learning literature. In its generality, it can be described as follows: Consider a set of keys \(X = \{x_1, \ldots, x_n\}\), a domain of values \(Y = \{y_1, \ldots, y_k\}\), a partial labeling function \(l\) that maps a subset of the keys to values \(l : X_L \rightarrow Y\) with \(X_L \subseteq X\), a weighted mapping \(w : (X_1, X_2) \rightarrow \mathbb{R}\), with \((X_1, X_2) \subseteq X \times X\), and a local condition that needs to hold for a solution to be accepted: \(f_i(X, w, x_i, l_i)\). The three problems are then to find an appropriate semantics that determines the solutions, an efficient algorithm that implements this semantics, and efficient ways to update algorithms in case either the partial labeling function \(l\), or the the weighted mapping \(w\) changes.

The two main philosophies for transductive inference are logical approaches and connectionist approaches (see Fig. 10).

\textbf{Logical approaches} determine the solution based on hard rules, and are most common in the database literature. Examples are trust mappings, preference-based updates, stable model semantics, but also tuple-generating dependencies, inconsistency-resolution, database repairs, community databases. Example applications are peer-data management and collaborative data sharing systems that have to deal with conflicting data and lack of consensus about which data is correct during integration, update exchange, and that have adopted some form of conflict handling or trust mappings in order to facilitate data sharing among users \([3, 11, 12, 15, 17, 16, 21, 22, 20]\). Commonly, those inconsistencies are expressed with key violations \([10]\) and resolved at query time through database repairs \([1]\).

\textbf{Connectionist approaches} determine the solution based on soft rules. The related work comprises guilt-by-association approaches, which use limited prior knowledge and network effects in order to derive new knowledge. The main alternatives are semi-supervised learning (SSL), random walks with restarts (RWR), and label or belief propagation (BP).

SSL methods can be divided into low-density separation methods, graph-based methods, methods for changing the representation, and co-training methods (see \([44, 29]\) for overviews). A multi-class approach has been introduced in \([19]\). RWR methods are used to compute mainly node relevance, e.g., original and personalized PageRank \([1, 17]\), lazy random walks \([32]\), and fast approximations \([35, 41]\).

\textbf{Belief Propagation} (or min-sum or product-sum algorithm) is an iterative message-passing algorithm that is a very expressive formalism for assigning classes to unlabeled nodes and has been used successfully in multiple settings for solving inference problems, such as error-correcting codes \([26]\) or stereo imaging in computer vision \([9]\), fraud detection \([30, 36]\), malware detection \([5]\), graph similarity \([25, 26]\), structure identification \([29]\), and pattern mining and anomaly detection \([20]\). BP solves the inference problem approximately: it is known that when the factor graph has a tree structure, it reaches a stationary point (convergence to the true marginals) after a finite number of iterations. Although in loopy factor graphs, convergence to the correct marginals is not guaranteed, the true marginals may still be achieved in \textit{locally tree-like} structures. As consequence, approaches in the database community that rely on BP-type of inference also commonly lack convergence guarantees \([39]\).

There exist various works that speed up BP by: (i) exploiting the graph structure \([6, 30]\), (ii) changing the order of message propagation \([8, 13, 33]\), or (iii) using the MapReduce framework \([20]\). Here, we derive a linearized formulation of the standard belief propagation algorithm FABP \([24]\) from binary to multiple labels for classification handled by any symmetric, doubly stochastic matrix. In addition, we provide translations into SQL and a new faster semantics that captures the underlying intuition and provides efficient incremental updates.

10. CONCLUSIONS

This paper showed that the widely used belief propagation algorithm for multiple classes can be approximated by a linear system that replaces multiplication with addition. This allows us to give a fast and compact implementation in standard SQL. The linear system also allows a closed form solution with the help of the inverse of an appropriate matrix. We can thus explain when the system will converge, and what the limit value is when the neighbor-to-neighbor influence tends to zero. For the latter case, we show that the scores depend on the “nearest labeled neighbor”, which leads to a new super-fast algorithm that also incremental updates (something that was not possible before for BP).
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B. APPENDIX FOR TECHNICAL REPORT

B.1 Proofs for Section 4

Lemma 5: Centered BP.

Proof: Substituting the expansions into the belief updates Equ. 1, leads to

\[ \frac{1}{k} + \hat{b}_i(i) = \frac{1}{\tilde{Z}_s} \left( \frac{1}{k} + \hat{c}_i(i) \right) \prod_{u \in N(s)} \left( 1 + \hat{m}_{us}(i) \right) \]

\[ \ln \left( 1 + k b_u(i) \right) = - \ln \tilde{Z}_s - \ln \left( 1 + k \hat{c}_i(i) \right) + \sum_{u \in N(s)} \hat{m}_{us}(i) \]

\[ k \hat{b}_i(i) = - \ln \tilde{Z}_s + k \hat{c}_i(i) + \sum_{u \in N(s)} \hat{m}_{us}(i) \quad (21) \]

For the last step, we use the approximation \( \ln(1 + \epsilon) \approx \epsilon \) for small \( \epsilon \). Summing both sides over \( i \) gives us:

\[ k \sum_i \hat{b}_i(i) = - k \ln \tilde{Z}_s + k \sum_i \hat{c}_i(i) + \sum_{i, u \in N(s)} \hat{m}_{us}(i) \]

Hence, we see that \( \ln \tilde{Z}_s \) needs to be 0, and therefore our normalizer is actually a normalization constant and independent for all nodes \( \tilde{Z}_s = 1 \). Plugging \( \tilde{Z}_s = 1 \) back into Eq. 21 leads to Eq. 7:

\[ \hat{b}_i(i) = \hat{c}_i(i) + \frac{1}{k} \sum_{u \in N(s)} \hat{m}_{us}(i) \]

Equation 8 We first write Eq. 3 as:

\[ m_{st}(i) = \frac{1}{\tilde{Z}_s} \sum_j \hat{c}_s(j) H(j, i) \prod_{u \in N(s) \setminus t} m_{us}(j) \]

\[ = \frac{Z_s}{\tilde{Z}_s} \sum_j H(j, i) \frac{1}{\tilde{Z}_s} \hat{c}_s(j) \prod_{u \in N(s) \setminus t} m_{us}(j) \]

\[ = \frac{Z_s}{\tilde{Z}_s} \sum_j H(j, i) \frac{b_j(j)}{m_{ts}(j)} \]

(23)

Then, using, \( \tilde{Z}_s = 1 \) and the expansions together with the approximation \( \frac{1 + \epsilon}{1 + \epsilon} \approx \frac{1 + \epsilon}{1 + \epsilon} \approx \frac{1}{k} + \epsilon \) for small \( \epsilon \), we get:

\[ 1 + \hat{m}_{st}(i) \approx \frac{1}{\tilde{Z}_s} \sum_j \left( \frac{1}{k} + \hat{H}(j, i) \right) \frac{1}{1 + \hat{m}_{ts}(j)} \]

\[ \approx \frac{1}{\tilde{Z}_s} \sum_j \left( \frac{1}{k} + \hat{H}(j, i) \right) \left( \frac{1}{k} + \hat{b}_j(j) - \frac{1}{k} \hat{m}_{ts}(j) \right) \]

\[ \approx \frac{1}{\tilde{Z}_s} \left( \frac{1}{k} + \frac{1}{k} \sum_j \hat{H}(j, i) + \frac{1}{k} \sum_j \hat{b}_j(j) + \sum_j \hat{H}(j, i) \hat{b}_j(j) \right) \]

\[ - \frac{1}{k^2} \sum_j \hat{m}_{ts}(j) - \frac{1}{k} \sum_j \hat{H}(j, i) \hat{m}_{ts}(j) \]

(24)

We can then determine the normalization factor \( \tilde{Z}_s \) to be a constant as well (\( \tilde{Z}_s = k^{-1} \)) by summing both sides of Eq. 24 over \( i \) and observing that \( \sum_j \hat{b}_j(j) \sum_i \hat{H}(j, i) = 0 \), since \( \sum_i \hat{H}(j, i) = 0 \):

\[ k + \sum_i \hat{m}_{st}(i) \approx \frac{1}{\tilde{Z}_s} \left( 1 + \sum_j \sum_i \hat{H}(j, i) \hat{b}_j(j) \right) \]

\[ - \sum_i \sum_j \hat{H}(j, i) \hat{m}_{ts}(j) \]

We get Eq. 8 from Eq. 24 and \( \frac{1}{\tilde{Z}_s} = k \).

\[ \hat{m}_{st}(i) \approx k \sum_j \hat{H}(j, i) \hat{b}_j(j) - \sum_j \hat{H}(j, i) \hat{m}_{ts}(j) \]

\[ \square \]

B.2 Proofs for Section 5

Lemma 7: Eigenvalue Convergence.

Proof: From the geometric series, we know that \( (I - \mathbf{X}) \) is invertible if \( \rho(\mathbf{X}) < 1 \). For Eq. 16 we have \( \mathbf{X} := \mathbf{H} \otimes \mathbf{A} \) and therefore \( \rho(\mathbf{H} \otimes \mathbf{A}) = \rho(\mathbf{H}) \rho(\mathbf{A}) < 1 \), which holds if and only if \( \rho(\mathbf{H}) < \frac{1}{\rho(\mathbf{A})} \).

For Eq. 15 we have \( \mathbf{X} := \mathbf{H} \otimes \mathbf{A} - \mathbf{H}^2 \otimes \mathbf{D} \) and therefore \( \rho(\mathbf{H} \otimes \mathbf{A} - \mathbf{H}^2 \otimes \mathbf{D}) \leq \rho(\mathbf{H} \otimes \mathbf{A}) + \rho(\mathbf{H}^2 \otimes \mathbf{D}) = \rho(\mathbf{H}) \rho(\mathbf{A}) + \rho(\mathbf{H}^2) \rho(\mathbf{D}) \leq \rho(\mathbf{H}) \rho(\mathbf{A}) + \rho(\mathbf{H}^2) \rho(\mathbf{D}) < 1 \), which holds if \( \rho(\mathbf{H}) \leq \frac{\sqrt{\rho(\mathbf{A})^2 + 4 \rho(\mathbf{D})}}{2 \rho(\mathbf{D})} \).

\[ \square \]

Lemma 8: \( p \)-norm Criteria.

Proof: Since \( \rho(\mathbf{X}) < \|\mathbf{X}\| \), it is sufficient to show that \( \|\mathbf{X}\| < 1 \). For Eq. 16 we have \( \rho(\mathbf{H} \otimes \mathbf{A}) = \rho(\mathbf{H}) \rho(\mathbf{A}) \leq \|\mathbf{H}\| \|\mathbf{A}\| < 1 \), which holds if \( \|\mathbf{H}\| < \frac{1}{\|\mathbf{A}\|} \).

Note that we can use different norms \( \|\cdot\|_1 \) and \( \|\cdot\|_2 \), and we get the best bounds for minimizing each norm individually.

For Eq. 15 we have \( \rho(\mathbf{H} \otimes \mathbf{A} - \mathbf{H}^2 \otimes \mathbf{D}) \leq \rho(\mathbf{H}) \rho(\mathbf{A}) + \rho(\mathbf{H}^2) \rho(\mathbf{D}) \leq \|\mathbf{H}\| \|\mathbf{A}\| + \|\mathbf{H}\|^2 \|\mathbf{D}\| \|\mathbf{H}\|_k < 1 \), which holds if \( \|\mathbf{H}\| \leq \frac{\sqrt{\|\mathbf{A}\|^2 + 4 \|\mathbf{D}\|_k}}{2 \|\mathbf{D}\|_k} \). Just as before, we can use different norms \( \|\cdot\|_1 \), \( \|\cdot\|_2 \), and \( \|\cdot\|_k \), and we get the best bounds for minimizing each norm individually.

We also give an additional, simpler bound for LBP.

Lemma 20 (Alternative \( p \)-norm Criteria). Let \( \|\cdot\| \) stand for the induced 1-norm or induced \( \infty \)-norm of the enclosed matrix. Then LBP converges if \( \|\mathbf{H}\| < \frac{1}{\|\mathbf{A}\|} \).

Proof: For the induced 1-norm or \( \infty \)-norm (which are the maximum absolute column or row sum of a matrix, respectively), we know from the definition of \( \mathbf{D} \), that \( \|\mathbf{D}\| \leq \|\mathbf{A}\| \). With \( \|\mathbf{H}\|^2 < \|\mathbf{H}\| < 1 \), we thus have \( \|\mathbf{H}\| \|\mathbf{A}\| + \|\mathbf{H}\|^2 \|\mathbf{D}\| < 2 \|\mathbf{H}\| \|\mathbf{A}\| < 1 \), from which \( \|\mathbf{H}\| < \frac{1}{\|\mathbf{A}\|} \).
B.3 Proofs for Section 6

**Theorem 15:** Convergence of LBP towards TBP.

**Proof Theorem 15** Given an unscaled coupling matrix $H_0$, Eq. 12 for LBP can be written as

$$\hat{b}_s = \hat{e}_s + \epsilon_H \sum_{t \in N(s)} w_{ts} \hat{b}_t$$

where $w_{us} = A(u, s)$ is the weight of the edge $u \rightarrow s$. Let $k$ be the geodesic number of node $s$.

If $k = 0$, i.e. node $s$ has explicit beliefs and thus $\hat{e}_s \neq 0$, \(\hat{b}_s \rightarrow \hat{e}_s\) for $\epsilon_H \rightarrow 0$ as $(\epsilon_H \hat{e}_s(j) \hat{b}_s(j)) / \hat{e}_s(j) \rightarrow 0$.

If $k > 0$, the final beliefs are $\hat{b}_s = \epsilon_H H_0 \sum_{t \in N(s)} w_{ts} \hat{b}_t$, i.e. the sum of all weighted neighbor’s beliefs, transformed by $\epsilon_H \hat{H}$. It follows that those final beliefs are by at least an order of $\epsilon_H$ smaller than those of nodes with explicit beliefs.

In turn, the propagated influence of this node to any other neighbor $u$ is $w_{us} \epsilon_H H_0 \sum_{t \in N(s)} w_{ts} \hat{b}_t$. It follows by induction that the influence of a node $g_1$ with geodesic number $k_1$ on its neighbors is of an order of $\epsilon_H^{k_1-k_2}$ smaller than a node with geodesic number $k_2$ and $k_1 > k_2$, assuming that weights in the order of 1. It further follows from induction that for a node $e$ with $k > 0$, $\hat{b}_e \rightarrow \epsilon_H \sum_{t \in N(s), w_{ts} \hat{b}_t}$ where $N_0(e)$ is the subset of neighbors with explicit beliefs.

Finally, all such paths leading to explicit beliefs have the same length $k$. Let $P^k_e$ stand for the set of such paths. Then,\(\hat{b}_s \rightarrow \epsilon_H H_0 \sum_{p \in P^k_e} w_{pe} \hat{e}_p\) (25)

Since standardization of $\hat{b}_s$ is equivalent to dividing it by its standard deviation, we see that $\hat{b}_s = \mathcal{N}(\hat{b}_s)$ converges towards the same assignment as TBP for $\epsilon_H \rightarrow 0$. \(\square\)

B.4 Proofs for Section 7

**Proposition 18** Initial assignment in TBP.

**Proof Proposition 18** There are two types of edges $(s \rightarrow t)$: “valid” edges where $g(t) = g(s) + 1$, i.e. edges that propagate beliefs from nodes $s$ to $t$, and “invalid” edges where $g(t) \leq g(s)$, i.e. directed edges where the target has a smaller or equal geodesic number and hence does not receive beliefs along this edge.

At iteration $i$, the algorithm calculates the implicit beliefs for all and only those nodes $v$ with $g(v) = i$. This follows from induction on the fact that in iteration $i$, tables $G$ and $B$ contain information for all nodes $v$ with $g(v) \leq i - 1$. Thus, by end of iteration $i$, all nodes with $g(v) \leq i$ are correctly calculated from valid edges only.

Finally, there cannot be any node $v$ with $g(v) = i$ if there is no other node $v'$ with $g(v') = i - 1$. Since the longest self-avoiding path in a graph is finite, the algorithm terminates in a finite number of iterations with the correct belief assignment. If there are nodes in the graph missing from $B$ at the end of the algorithm, then the graph must have distinct strongly connected components and those nodes are not connected from any node with explicit beliefs. \(\square\)

**Proposition 19** Update beliefs in TBP.

**Proof Proposition 19** We need to show that all nodes which are affected by inserted explicit beliefs are correctly updated when Algorithm 3 terminates. There are two types of updated nodes: (1) nodes that keep their geodesic number $g'(v) = g(v)$, and which receive their implicit beliefs from their original source, but now together with new sources; and (2) nodes whose number decreases $g'(v) < g(v)$, and whose beliefs are now received only by new nodes (the number can never increase as inserts of new nodes never remove an existing shortest path). Note that for both types, there need to be a new explicit node that was inserted at exactly $g'(v)$ hops away.

We now show by induction that at iteration $i$, the algorithm calculates the implicit beliefs for all and only those nodes $v$ with $g'(v) = i$. We need to show two things: (1) that at iteration $i$, each node $v$ with $g'(v) = i$ is identified in table $G_i$, and (2) that the beliefs in table $B_i$ are correctly calculated. For (1) note that nodes to be updated at iteration $i$ are identified by following edges from nodes updated at iteration $i - 1$ and only ignoring those nodes $v$ that already have $g'(v) < i$. By induction, by the end of iteration $i - 1$ all such nodes are identified. For (2) note that calculating the updated beliefs, all parents $s$ with $g'(s) = i - 1$ are used together with their updated beliefs. By induction, all those beliefs were correctly calculated by the end of iteration $i - 1$. \(\square\)

B.5 TBP: Addition of edges

(3) Addition of edges. Algorithm 4 gives the SQL translation for batch inserts of additional edges assuming a new table $A_s(s, t, w)$ containing the set of new edges.

Note that this algorithm is more intricate than Algorithm 3 for the batch insert of explicit beliefs because edges and nodes can now be visited more than once. We provide here some intuition: In both Algorithm 2 and Algorithm 3 we started from original or new explicit beliefs, and were progressing the information in each iteration along paths of increasing geodesic numbers. This guaranteed that every node is updated only once as paths from later iterations to a node cannot contain shorter geodesic paths, and can thus ignored. In the case of added edges, however, we now have “seed nodes” from which updates need to be propagated throughout the immediate neighborhoods that have different geodesic numbers. Since these seed nodes can start propagating with different geodesic numbers in the first iteration, there may be later paths arriving at a node that actually have a shorter geodesic length than the previously arriving one. There are two cases: (1) if both nodes ending at a newly inserted edge have the same geodesic numbers before the update, then this edge cannot contain a geodesic path and can thus be ignored; (2) if one node $s$ has a smaller
Update beliefs for seed nodes:

6
7
Delete $G$ which the geodesic number of the source is smaller than the first set of nodes that need their beliefs updated; those geodesic number becomes $t$.

3
Update beliefs for seed nodes:

$B_n(t, c_2, \sum(w \cdot b \cdot h)) := G_n(t, g, A(s, t, w), B(s, c_1, b), G(s, g - 1), H(c_1, c_2, h))$

4
Repeat

5
Find next nodes to update:

$B_n(t, s, g)$

6
Calculate new beliefs for these nodes:

$B_n(t, s, g)$

7
Delete $G_n$ and rename $G_n'$ to $G_n$

8
Until no more inserts into $G_n'$

9
Return $B$ and $G$

geodesic number than the other node $t$ ($g_t < g_s$), then the direction $s \to t$ contains a new geodesic path, and $t$ becomes a new seed node. If $(2a)$ $g_t + 1 = g_s$ before the update, then the new edge contains only one additional geodesic path for $t$ does not change. If, however, $(2b)$ $g_t + 1 < g_s$, then the new edge contains all geodesic paths for $t$, whose new geodesic number becomes $g'_t = g_s + 1$.

Line 1 updates the adjacency matrix. Line 2 then finds the first set of nodes that need their beliefs updated; those are the ones that are the target of any new edge $s \to t$ for which the geodesic number of the source is smaller than the target ($g_s < g_t$), even before the update. Note that several edges may have been added to the same target, and we thus have to use the minimum of the newly found geodesic paths. Line 3 then updates beliefs of nodes $t$ by following only edges from source nodes $s'$ with geodesic number $g_{s'} = g_s - 1$. In each subsequent iteration (line 4), we now keep two tables, $G_n(s, g)$ for the nodes that were updated in the previous iteration, and $G_n(s, g)$ for the nodes to be updated during this iteration. Finding the new nodes to update $G_n'$ (line 5) and updating the beliefs of those nodes $B_n'$ (line 6) are similar as in the first iteration, except for the renaming of the tables (line 7). The algorithm stops when there are no more nodes to update (line 8).

Recall, that the geodesic paths and the beliefs of a node may be updated more than once. For this reason, some care needs to be taken when updating the result table $B_n$. For example, consider two new edges $s \to v$ and $v \to t$ where the adjacent nodes have original geodesic numbers $g_{v(0)} = 0$, $g_{v(0)} = 2$, and $g_{v(0)} = 4$. Then, both $v$ and $t$ become seed nodes for the first iteration with new geodesic numbers $g_{v(1)} = 1$, and $g_{t(1)} = 3$. Note since updates happen concurrently $t$ still inherits $g_{t(1)} = g_{t(1)} + 1$. Beliefs are not updated during this iteration. In the iteration 2, the updated information travels from $v$ to $t$ and we have $g_{t(2)} = 2$.

Using similar ideas, one can create a pathological example, in which updating of the graph takes $n^2$ time with our algorithm, where $n$ is the number of added edges.

We next sketch an idea that would avoid the quadratic time increase: sort the seed nodes by new updated geodesic numbers and let $i$ be the smallest updated geodesic number across all seed nodes. Start propagating beliefs, in each iteration $j$, from seed nodes with new geodesic number $i + j - 1$, unless this node was updated in a previous iteration. This simple change would avoid that a node gets updated its beliefs more than once. We have not implemented this idea and leave experimenting with it for future work.

**Proposition 21 (Algorithm 4).** Algorithm 4 terminates in finite number of iterations and returns a sound and complete enumeration of updated beliefs.

**Proof.** Proposition 21 works similar to the one for Proposition 19 with three differences: (1) the induction base are those nodes which were the target of a newly inserted edge and whose beliefs thus need to be updated ($g_s < g_t$); (2) we need to use a grouping and minimum when updating the geodesic number; and (3) the induction is now on the distance from the original “seed nodes.” Importantly, now the geodesic number and the beliefs of a node can change more than once, which is taken care of our algorithm by maintaining two separate tables $G_n'$ and $G_n$. □

### B.6 The binary case

Previous work [24] has given a linearization for belief propagation for the binary case ($k = 2$). Here we show that our more general results include the binary case as special case.

We start from Eq. 11 and use the normalization conditions for $k = 2$ to write $\hat{b} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$, $\hat{m} = \begin{bmatrix} 0 \\ -h \end{bmatrix}$, and $\hat{h} = \begin{bmatrix} k & -h \\ -h & k \end{bmatrix}$. We then get $\hat{H}^2 = 2 \begin{bmatrix} k^2 & -k^2 \\ -k^2 & k^2 \end{bmatrix}$. As the normalization $x(1) = -x(1)$ holds for all results, it suffices to only focus on one dimensions, which we choose WLOG to be the first. We get: $(k\hat{H}\hat{b})(1) = 4\hat{h}\hat{b}$, $(k\hat{H}\hat{b})(1) = 8\hat{h}^2$, $(\hat{I}_h - \hat{H})(1) = 1 - 4\hat{h}^2$, and finally:

$$\hat{m}_{s,t} = \frac{4\hat{h}}{1 - 4\hat{h}^2} \hat{b} - \frac{8\hat{h}^2}{1 - 4\hat{h}^2} \hat{b}$$

(26)

Note that our Eq. 26 differs from [24, Eq. 19] by a factor of 2 on the right side. This is result of the decision to center the messages around $\frac{1}{2}$ in [24], whereas this work decided to centered around 1: Centering messages around 1 allowed us to ignore incoming messages that have no residuals (i.e. $m_{s,t} = 1$) and was a crucial assumption in our derivations (see Sect. 4.1). This difference leads to a factor 2 in Eq. 8 for $k = 2$, and thus a factor 2 difference in Eq. 26. However, both alternative centering approaches ultimately lead to the same equation in the binary case:

$$\hat{b} = \left(\hat{I}_h - \frac{2\hat{h}}{1 - 4\hat{h}^2} A + \frac{4\hat{h}^2}{1 - 4\hat{h}^2} D\right)^{-1} e$$

where $\hat{b}$ and $e$ are the column vectors that contain the first dimension of the binary centered beliefs for each node. This
can be easily seen by writing the original, non-simplified version \[\text{Eq. 13}\] in the vectorized form of \[\text{Eq. 15}\].

Further note that experiments in \[\text{24}\] showed a decreasing quality of classification for decreasing \(\epsilon_H\) and, thus, recommended \(\epsilon_H\) to be chosen above a certain threshold. The conclusion from the present paper is that, in theory, the quality should converge quickly, and then remain constant for \(\epsilon_H \to 0\). Any deviations from this behavior must be the consequences of unavoidable roundoff errors due to limited precision of floating-point computations. Our new semantics TBP captures the exact theoretical behavior for \(\epsilon_H \to 0\) while avoiding the problem of roundoff errors.

### B.7 Additional Experiments

Here, we add two more experiments on our example Kronecker graphs to answer two more questions:

**Question 4.** How does the fraction of explicit beliefs in a fixed graph affect the time for LBP and TBP?

**Result 4.** LBP gets slightly slower, while TBP get slightly faster with an increasing fraction of explicit beliefs.

**Figure 11a** illustrates the result on our Kronecker graph #5 using 5 iterations for LBP. LBP gets slightly slower because the LBP update equations having an increased amount of calculations. In contrast, TBP gets slightly faster as the number of edges to propagate information along decreases with a denser fraction of explicit beliefs. However, both effects are minor.

**Question 5.** When is it faster to update a graph then to calculate all beliefs from scratch with TBP?

**Result 5.** In our experiments, it was faster to update TBP when less than \(\approx 60\%\) of the final explicit beliefs are new or less than \(\approx 3\%\) of the final edges are new.

**Figure 11b** and **Figure 11c** illustrate these results on our Kronecker graph #5. Both graphs assume that 10\% of the nodes have explicit beliefs after the update. **Figure 11b** assumes that, among these nodes with explicit beliefs, we have a certain fraction as new beliefs. For example, 20\% on the horizontal axis implies that we had 80\% of the explicit nodes (= 8\% of all nodes) known before the update, and are now adding 20\% of the explicit nodes (= 2\% of all nodes) with the incremental TBP \[\text{Algorithm 3}\] ("\(\Delta TBP\)"). For the alternative \[\text{Algorithm 2}\] ("TBP"), we recalculate the final beliefs starting from 100\% of the explicit nodes every time (therefore, shown with the constant horizontal line). **Figure 11c** keeps the 10\% of explicit beliefs fixed, and assumes that we have a certain fraction of edges as new edges. For example, the 5\% on the horizontal axis implies that we had known 95\% of all edges in the graph before the update, and are adding the remaining 5\% of edges with \[\text{Algorithm 4}\] ("\(\Delta TBP\)"). We show here only the first 10\%; beyond this, the updates need increasingly more time until up to \(\approx 16\) seconds (4 times the necessary time to update from scratch). Thus, both update algorithms (updating beliefs or edges) can speed up the calculations. However, the circumstances for which it is beneficial for incremental edges updates is smaller. This result seems to confirm our intuition that updating the graph structure is more complex than updating implicit beliefs. Future work is needed to see if the incremental edge updates can be made faster, in particular with the idea mentioned at the end of Sect. B.5.