Tame integrals of motion and 0-minimal structures

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1 Introduction

An integrability of a Hamiltonian flow presumes that we can describe the flow in a simple way and, in particular, the decomposition of its phase space into invariant Liouville tori and the singular locus looks very simple geometrically. In this case it is said that integrals of motion are tame. Usually this is the case when the flow in integrable in terms of (real-) analytic functions as it was already shown for different examples. Here we would like to expose some ideas from mathematical logics, i.e. from the theory of o-minimal structures, which clarify an analytic notion of a tame integral of motion, and demonstrate this approach for the integrability problem of geodesic flows.

2 Different meanings of integrability

Let $M^{2n}$ be a symplectic manifold with a symplectic form

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j.$$ 

Let us denote by $\omega^{ij}$ the inverse matrix to a skew-symmetric matrix $\omega_{ij}$. Then any smooth function $H$ on this manifold define a Hamiltonian flow by the equation which describes the evolution of any smooth function $f$ along the trajectories of the flow:

$$\frac{df}{dt} = \{f, H\} = \omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial H}{\partial x^j}.$$ 

We recall that the function $H$ is called the Hamiltonian function of the flow (or just the Hamiltonian) and the skew-symmetric operation $\{f, g\}$ on smooth function is called the Poisson brackets.

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It is said that a function \( f \) is a first integral, or an integral of motion, of the flow if it is preserved by the flow:

\[
\{f, H\} = 0.
\]

The Liouville or complete integrability of a Hamiltonian flow is defined as follows:

- a flow on an \( n \)-dimensional symplectic manifold \( M^{2n} \) is called integrable if it admits a family of first integrals \( I_1, \ldots, I_n = H \) such that these integrals are in involution:
  \[
  \{I_j, I_k\} = 0, \quad 1 \leq j, k \leq n,
  \]
  and they are functionally independent almost everywhere, i.e. outside some nowhere dense set \( \Sigma \) which is called the singular locus.

The mapping

\[
\Phi : M^{2n} \to \mathbb{R}^n, \quad \Phi(q) = (I_1(q), \ldots, I_n(q)),
\]

is called the momentum map.

The condition that the first integrals are in involution means that Hamiltonian flows generated by the functions \( I_1, \ldots, I_n \) as Hamiltonians commute everywhere. If \( X \) is a compact component of the level surface \( \Phi = c \) on which the integrals are functionally independent then this component is diffeomorphic to a torus on which the Hamiltonian flows corresponding to \( I_1, \ldots, I_n \) are linearized and moreover such a linearization can be extended to a neighborhood of \( X \).

The proofs of these statements on integrable flows are exposed in [1].

But we see that there is some freedom in the definition when we say about functional independence of first integrals almost everywhere. It could be that

- they are functionally independent on an open dense set;

- given a smooth measure on \( M^{2n} \) such that the measure of \( M^{2n} \) is finite, the first integrals are functionally independent on the full measure subset.

Moreover in classical mechanics the most popular situation is when \( M^{2n} \) and \( H \) are (real-)analytic and

- first integrals \( I_1, \ldots, I_n \) are analytic.

In this case it is said that the flow is analytically integrable.

Another reasonable treatment of what means “almost every functional independence of integrals of motion” for a compact phase space is as follows

- there is a finite smooth (or even analytic) simplicial decomposition of the phase space \( M^{2n} \) such that a singular locus \( \Sigma \) forms a subcomplex of this decomposition and the complement to it is cutted by another subcomplex of positive codimension to a union of finitely many sets \( U_\alpha \) which are foliated by invariant tori over their images under the momentum map.
We considered such a notion in [13] and called it a geometric simplicity.

It is reasonable to say that integrals of motion are tame if they lead to a geometrically simple behaviour of the flow.

Some important examples of Hamiltonian flows do not have a compact phase space. This is, for instance, the geodesic flow of a Riemannian manifold $M^n$ which is a Hamiltonian flow on a cotangent bundle $T^* M^n$ to this manifold. The symplectic structure on $T^* M^n$ is given by a form

$$\omega = \sum_{j=1}^{n} dx^j \wedge dp_j$$

where $p_j = g_{jk} \dot{x}^k$ and $g_{jk} dx^j dx^k$ is the Riemannian metric. The Hamiltonian of the geodesic flow is homogeneous in momenta:

$$H(x, p) = g^{jk}(x) p_j p_k$$

where $g^{jk} g_{kl} = \delta^j_l$ and therefore the restrictions of the geodesic flow onto different non-zero level surfaces of $H$ are smoothly trajectory equivalent, moreover they are related by reparametrization of trajectories by a constant. Therefore it is reasonable to call the geodesic flow on $M^n$ integrable if it satisfies a weaker condition (see [13]) which is

- there are $(n-1)$ additional integrals of motion $I_1, \ldots, I_{n-1}$ which are in involution and almost everywhere independent on the unit covector bundle $SM^n = \{ H = 1 \}$.

It appears that an analytic integrability looks the strongest assumption which implies, in particular, geometric simplicity as it was shown in [13].

Some recent examples of integrable geodesic flows of analytic metrics show that even using for integration of geodesic flows such mildly non-analytic $C^\infty$ functions as, for instance,

$$f(x, p) = \exp(-Q(p)^{-2}) \sin(\mu \log|p_x - \tau p_y|)$$

where $\mu, \tau$ are constants and $Q(p)$ is a polynomial in momenta $p_x, p_y$ divided by $(p_x - \tau p_y)$, leads to geometrically non-simple flows [2,3].

Therefore

for considering topological properties of integrable flows by means of topology of finite CW-complexes or tame subsets in $\mathbb{R}^n$ we have to restrict ourselves to geometrically simple flows or to tame integrals of motion.

In the next chapter we expose some background from mathematical logics which leads to the most clarified analytic approach to understanding what it means that an integral of motion is tame.

Before we do that we would like to notice that an analogue of the Morse theory for integrable systems on four-dimensional symplectic manifolds developed by Fomenko and his collaborators also needs some analytic condition fulfilled. It reads that an additional (to the Hamiltonian) integral of motion has to be of
the Bott type, i.e. its restrictions onto normal planes to critical level surfaces would be locally a Morse function. This is another kind of tameness condition which was generalized in [11] to a geometric condition under which this theory works.

3 O-minimal structures and analytic-geometric categories

3.1 O-minimal structures

By definition, a family $\mathcal{S}$ of subsets of the Euclidean spaces $\mathbb{R}^n$ is called an o-minimal structure (on $(\mathbb{R}, +, \cdot, <)$) if being graded by the dimensions of ambient Euclidean spaces:

$$\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}_n,$$

it meets the following conditions:

1) $\mathcal{S}_n$ is a Boolean algebra of some subsets of $\mathbb{R}^n$ with a standard union operation (in particular, this means that this algebra is closed with respect to complements and finite unions and intersections):

2) if $X \in \mathcal{S}_n$ and $Y \in \mathcal{S}_k$, then $X \times Y \in \mathcal{S}_{n+k}$;

3) let $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a projection $(x^1, \ldots, x^n, x^{n+1}) \to (x^1, \ldots, x^n)$, then $X \in \mathcal{S}_{n+1}$ implies $\pi(X) \in \mathcal{S}_n$;

4) $\mathcal{S}_n$ contains all algebraic sets in $\mathbb{R}^n$, i.e. if $P(x^1, \ldots, x^n)$ is a polynomial then its zero set $\{P = 0\}$ belongs to $\mathcal{S}$;

5) $\mathcal{S}_1$ consists exactly of all finite unions of open intervals and points.

Let us present the main known examples of o-minimal structures:

- $\mathbb{R}_{\text{alg}}$: semialgebraic sets. It consists exactly of all semialgebraic sets, i.e. determined by finitely many equations $P_1 = \ldots = P_k = 0$, and inequalities $Q_1 > 0, \ldots, Q_l > 0$ with $P_1, \ldots, P_k, Q_1, \ldots, Q_l$ polynomials. Such sets form an o-minimal structure by the Tarski–Seidenberg theorem.

- $\mathbb{R}_{\text{an}}$: finite subanalytic sets. It consists of intersections of subanalytic sets in $\mathbb{R}^n$ with cubes $[-D, D]^n$ and their projections. Notice that the family formed only by intersections of subanalytic sets with cubes is not closed under projections. It is the theorem of Gabrielov [7] which implies that $\mathbb{R}_{\text{an}}$ is an o-minimal structure.

- $\mathbb{R}_{\text{exp}}$. Let us correspond to any polynomial $P(x^1, \ldots, x^{2k})$ an exponential polynomial $Q(x^1, \ldots, x^k) = P(x^1, \ldots, x^k, e^{x^1}, \ldots, e^{x^k})$ and denote by $\mathbb{R}_{\text{exp}}$ the family of all sets generated by the zero sets of such exponential polynomials under projections $\mathbb{R}^{n+1} \to \mathbb{R}^n$. Wilkie proved that this family is closed under complements and forms an o-minimal structure [15].

- $\mathbb{R}_f$, where $f$ is a Pfaffian function. This is a generalization of $\mathbb{R}_{\text{exp}}$. It is said that a chain of $C^1$-functions $f_1, \ldots, f_k: \mathbb{R}^n \to \mathbb{R}$ is a Pfaffian chain if for each $j = 1, \ldots, k$ the first derivatives of $f_j$ with respect to $x^1, \ldots, x^n$ are
polynomials in $x^1, \ldots, x^n, f_1, \ldots, f_j$. In this event the function $f = f_k$ is called a Pfaffian function. The similarity between the zero sets of Pfaffian functions and algebraic sets was first pointed out by Khovanskii [8]. It was proved by Wilkie that if we replace $\exp$ by a Pfaffian function $f$ in the definition of $R_{\exp}$ and close this family with respect to Boolean operations, projections and products then we get an o-minimal structure which is denoted by $R_f$ [14].

- $R_{\text{an,exp}}$. It was proved by van den Dries, Macintyre, and Marker [6] by using the results of Wilkie [15] that sets from $R_{\text{an}}$ and $R_{\exp}$ generate by Boolean operations and projections an o-minimal structure which is denoted by $R_{\text{an,exp}}$.

These o-minimal structures are related by the following evident inclusions:

$R_{\text{alg}} \subseteq R_{\text{an}} \subseteq R_{\text{an,exp}}$.

Given an o-minimal structure $S$, we say that

- a subset $X \in \mathbb{R}^n$ is definable if $X \in S$ [1];
- a mapping $f : X \to \mathbb{R}^k$ with $X \subseteq \mathbb{R}^n$ is definable if its graph $\{(x, f(x))\}$ is a definable set, i.e. belongs to $S_{n+k}$;
- a set $X$ is $S$-triangulable if $X \in S$ and there is a definable mapping $f : X \to \mathbb{R}^n$ which maps $X$ homeomorphically onto a union of open simplices of finite simplicial complex $K \subseteq \mathbb{R}^n$. In this event we say that $f$ defines an $S$-triangulation of $X$.

By this definition, any $S$-triangulable set is definable. The converse is also true:

**Theorem 1 (Triangulation Theorem)** Every definable set $X \in S$ is $S$-triangulable.

This theorem is proved by a general method for all o-minimal structures [3]. The proof follows by induction on the dimension of a definable set and starts with an evident statement that all sets from $S_1$ are $S$-triangulable. We sketched such a proof for sets from $R_{\text{an}}$ in [13].

Let us also notice that it follows from the definition of an o-minimal structure that images and preimages of definable sets under definable proper mappings are definable. Here we recall that a mapping is called proper if preimages of compact sets are compact.

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1 This terminology originates in mathematical logics and reflects the fact that definable sets are exactly sets which are defined by the first order logics formulas involving the summation “+”, the multiplication “·” and the linear ordering “<” plus some additional functions which lead to extensions of the smallest o-minimal structure on $(\mathbb{R}, +, \cdot, <)$, i.e. the subalgebraic sets. These are analytic functions restricted to cubes $[-1, 1]^n$ for $R_{\text{an}}$, the exponent function $\exp$ for $R_{\text{an}}$ and etc. If we drop the multiplication from the signature of our language (in the sense of mathematical logics) we have to replace the 4th condition by another which reads that the graphs of some functions coming into the signature are definable. For instance, the smallest o-minimal structure which includes $+, <$ and the multiplications $r$ by all real numbers $r \in \mathbb{R}$ is formed by all semilinear sets.
3.2 Geometric and analytic-geometric categories

For using the theory of o-minimal structures in topology and geometry it needs to develop its analog for subsets in manifolds and it was done in [4].

Given an o-minimal structure \( S \), we distinguish a class of \( S \)-manifolds. We say that a smooth manifold \( M^n \) is a \( S \)-manifold if it admits a finite \( S \)-atlas \( \{U_\alpha\} \), i.e. such an atlas that

- every coordinate mapping \( \varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n \) homeomorphically maps a chart \( U_\alpha \) onto a definable set \( V_\alpha \);
- for any intersection \( U_\alpha \cap U_\beta \) the transition mapping

  \[
  \varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)
  \]

  is definable.

Now we say that

- a subset \( X \in M^n \) is definable if for any chart \( U_\alpha \) the set \( \varphi_\alpha(X \cap U_\alpha) \subset \mathbb{R}^n \) is definable;
- for definable sets \( X \in M^n \) and \( Y \in N^k \) a mapping \( f : X \to Y \) is definable if it is continuous and its graph \( \{(x, f(x))\} \) is definable in \( M^n \times N^k \).

Notice that these definitions are independent on choosing \( S \)-atlases for \( M^n \) and \( N^k \).

Thereby an o-minimal structure \( S \) defines a “geometric category” of \( S \)-manifolds and their definable subsets as objects and definable maps between them as morphisms. For instance, if \( S = \mathbb{R}_{\text{alg}} \) we have a category of so-called Nash manifolds [12].

To an o-minimal structure \( S \) which contains \( \mathbb{R}_{\text{an}} \) there corresponds an “analytic-geometric” category which is defined as follows

- given an analytic manifold \( M^n \) a subset \( X \subset M^n \) is called definable if for any point \( x \in X \) there are an open neighborhood \( U \) of this point and an analytic isomorphism \( \varphi : U \to V \in \mathbb{R}^n \) such that \( \varphi(X \cap U) \) is definable;
- a mapping \( f : X \to Y \) of two definable sets \( X \subset M^n \) and \( Y \subset N^k \) is called definable if its graph is definable in \( M^n \times N^k \).

Notice that by this definition an image of a definable set under a proper analytic mapping is a definable set.

The analytic-geometric category corresponding to \( S \) has definable sets as objects and definable mappings as morphisms.

Introduction of these categories allows us to use machinery developed for definable sets in \( \mathbb{R}^n \), for instance Triangulation Theorem and many other facts (see [4, 5]), for subsets in manifolds.
4 On obstructions to integrability

The first obstruction to integrability of geodesic flows was found by Kozlov who proved that if the geodesic flow on a compact oriented analytic Riemannian two-manifold admits an additional analytic first integral then the surface is diffeomorphic to a sphere or to a torus [10].

The analyticity condition was strongly used in the proof and it is still unknown is the theorem is valid under only $C^\infty$ assumptions for a manifold and first integrals. Using the theory of modular forms Kolokoltsov extended the Kozlov theorem assuming that an additional first integral is $C^\infty$ but polynomial in momenta [9].

High-dimensional obstructions were obtained by us in [13] in two steps:
1) there were found some obstructions to a nice geometric behaviour of the geodesic flow on a manifold, i.e. obstructions to its geometric simplicity.
2) some analytic properties of first integrals which imply geometric simplicity were established.

We shall remark here that the condition of geometric simplicity can be weakened and by using the language of o-minimal structures the analytic condition can be clarified and slightly strengthened.

For realizing the first step we proved the following

**Theorem 2** If the geodesic flow on a compact analytic manifold $M^n$ is geometrically simple then there is an invariant torus $T^n \subset SM^n$ such that the natural projection $\pi : SM^n \to M^n$ induced a homomorphism

$$\pi_* : \pi_1(T^n) \to \pi_1(M^n)$$

those image is a subgroup of finite index in $\pi_1(M^n)$. 

It implies

**Corollary 1** If a geodesic flow on a compact manifold $M^n$ is geometrically simple then

1) the fundamental group $\pi_1(M^n)$ of $M^n$ is almost commutative, i.e., contains a commutative subgroup of finite index;
2) the real cohomology ring $H^\ast(M^n;\mathbb{R})$ of $M^n$ contains a subring $A$ which is isomorphic to the real cohomology ring $H^\ast(T^k;\mathbb{R})$ of the $k$-dimensional torus where $k$ is the first Betti number of $M^n$: $b_1 = \dim H^1(M^n;\mathbb{R}) = k$;
3) moreover if the first Betti number of $M^n$ equals its dimension: $b_1 = n$ then the ring $H^\ast(M^n;\mathbb{R})$ is isomorphic to $H^\ast(T^n;\mathbb{R})$.

To explain these results we recall that we say that a geodesic flow on $M^n$ is geometrically simple if the unit cotangent bundle $SM^n$ admits a decomposition

$$SM^n = \Gamma \cup \left( \bigcup_{\alpha=1}^{d} U_\alpha \right)$$

such that
• this decomposition is invariant under the flow;
• the set $\Gamma$ is closed and the complement to it is everywhere dense;
• for any point $q \in SM^n$ and every its neighborhood $V$ there is another neighborhood $W$ of $q$ such that $W \subset V$ and $W \cap (M^n \setminus \Gamma)$ has finitely many connected components;
• any component $U_\alpha$ is diffeomorphic to a product of an $n$-dimensional torus and an $(n-1)$-dimensional disc.

In fact we proved that omitting the fourth condition there is a component $U_\alpha$ such that the image of its fundamental group under the projection homomorphism $\pi_*(\pi_1(U_\alpha))$ has a finite index in $\pi_1(M^n)$.

Moreover in this formulation the proof of the theorem works for more general case when the flow is locally simple, i.e. there is a point $x \in M^n$ and its neighborhood $U$ such that

• the universal covering $\hat{M}^n \to M^n$ is trivial over $U$;
• the preimage of $U$ under the projection $\pi : SM^n \to M^n$ admits a decomposition

$$\pi^{-1}(U) = \hat{\Gamma} \cup \left( \bigcup_{\alpha=1}^{d} \hat{U}_\alpha \right)$$

where $\hat{\Gamma}$ is closed and the complement to it is dense, and each component $\hat{U}_\alpha$ is an intersection of $\pi^{-1}(U)$ with an invariant open set $U_\alpha$.

The second step was realized in [13] by the following

**Theorem 3** If a geodesic flow on a compact manifold is analytically integrable then it is geometrically simple.

In proving this theorem the basic point is to show that given analytic first integrals $I_1, \ldots, I_{n-1}$ (here we assume that $I_n$ is the Hamiltonian of the flow, $I_n = g^{ij}(x)p_ip_j$) the set $C$ of the critical values of the momentum map restricted onto $SM^n$

$$\Phi : q \to (I_1(q), \ldots, I_{n-1}(q)) \in \mathbb{R}^{n-1}$$

and its preimage in $SM^n$ are analytically-triangulable.

In the modern terminology of § 3, the proof of that consist in a remark that these sets $C$ and $\Phi^{-1}(C)$ are definable in $\mathbb{R}_{an}$-analytic-geometric category and therefore are $\mathbb{R}_{an}$-triangulable. We proved their analytic-triangulability directly by using the Gabrielov theorem [8]. We already mentioned that the proof of Triangulation Theorem for general o-minimal structures follows the same scheme as we used which probably originates in Hironaka’s proof of Triangulation Theorem for semialgebraic sets.

After proving that $C$ and $\Phi^{-1}(C)$ are analytically-triangulable we completed the set $C$ of $\Phi$ by adding some additional analytic $(n-2)$-dimensional simplices
to simplicial subcomplex $K$ those complement in $\Phi(SM^n)$ is a union of finitely many discs $V_\alpha$ and denote $\Phi^{-1}(V_\alpha)$ by $U_\alpha$ thus proving a geometric simplicity.

Now by using the general form triangulation Theorem we can generalize this theorem as follows:

**Theorem 4** Let $\mathcal{S}$ be an o-minimal structure. Let $M^n$ is a compact Riemannian $\mathcal{S}$-manifold and assume that the geodesic flow on $M^n$ is integrable in terms of $\mathcal{S}$-definable first integrals. Then this geodesic flow is geometrically simple.

For $\mathcal{S} = \mathbb{R}_{an}$ this theorem reduces to Theorem 3.

It was first shown by Butler that assuming only integrability in terms of $C^\infty$ first integrals we can not conclude that the fundamental group of the manifold is almost commutative. He did that by constructing a $C^\infty$ integrable geodesic flow on a three-dimensional nilmanifold. Later Bolsinov and the author even managed to construct a $C^\infty$ integrable geodesic flow on a solvmanifold those fundamental group has an exponential growth.

But as Theorem 4 shows we can derive topological conclusions of Corollary 1 by assuming that the flow is integrated in terms of $C^\infty$ first integrals which are definable in some analytic-geometric category. In this event the category corresponding to $\mathbb{R}_{an}$ is the smallest possible category.

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