Dark energy parametrization motivated by scalar field dynamics

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Abstract
We propose a new dark energy (DE) parametrization motivated by the dynamics of a scalar field $\phi$. We use an equation of state $w$ parametrized in terms of two functions $L$ and $y$, closely related to the dynamics of scalar fields, which is exact and has no approximation. By choosing an appropriate ansatz for $L$ we obtain a wide class of behavior for the evolution of DE without the need to specify the scalar potential $V$. We parametrize $L$ and $y$ in terms of only four parameters, giving $w$ a rich structure and allowing for a wide class of DE dynamics. Our $w$ can either grow and later decrease, or it can happen the other way around; the steepness of the transition is not fixed and it contains the ansatz $w = w_0 + w_0 (1 - a)$. Our parametrization follows closely the dynamics of a scalar field, and the function $L$ allows us to connect it with the scalar potential $V(\phi)$. While the Universe is accelerating and the slow roll approximation is valid, we get $L \simeq (V'/V)^2$. To determine the dynamics of DE we also calculate the background evolution and its perturbations, since they are important to discriminate between different DE models.

Keywords: cosmology, dark energy parametrization, scalar field dynamics, cosmological observations

1. Introduction

In recent years the study of our Universe has received a great deal of attention since on the one hand fundamental theoretical questions remain unanswered and on the other hand we now have the opportunity to measure the cosmological parameters with extraordinary precision. Existing observational experiments involve measurement on cosmic microwave background (CMB) [1, 2] or large scale structure (LSS) [3] or supernovae SNIa [4], and new research proposals are being carried out [5].
Taking a flat Universe dominated at present time by matter and dark energy (DE), and using a constant equation of state \( w \) for DE, one finds \( \Omega_{\text{DE}} \approx 0.714 \pm 0.012 \), \( \Omega_m \approx 0.286 \pm 0.012 \) with \( w = -1.037^{+0.071}_{-0.070} \) from WMAP9 results \(^1\) and \( \Omega_{\text{DE}} \approx 0.691^{+0.019}_{-0.021}, \Omega_m \approx 0.208^{+0.019}_{-0.021} \) with \( w = -1.13^{+0.13}_{-0.14} \) from Planck \(^2\), Supernova Legacy Survey (SNLS) \(^4\) and baryon acoustic oscillations (BAO) \(^3\) measurements. The constraint on curvature is \(-0.0013 < \Omega_k < 0.0028 \) for WMAP9 \(^1\) and |\( \Omega_k \)| \( \leq 0.0005 \) for Planck \(^2\) using a \( \Lambda \)CDM model, i.e. \( w = -1 \) for DE. At present time, the equation of state (EoS) of DE depends on the priors, choice of parameters and on the data used for the analysis as can be seen from the results obtained by either WMAP or Planck collaboration groups \(^1,2\). A more precise determination of the EoS of DE will be carried out in \(^5\), which together with precise measurements of CMB such as \(^1,2\) will yield a better understanding of the dynamics of DE. With better data we should be able to study in more detail the nature of DE, a topic of major interest in the field \(^6\). Since the properties of DE are still under investigation, different DE parametrizations have been proposed to help discern the dynamics of DE \(^7-14\). Some of these DE parametrizations have the advantage of having a reduced number of parameters, but they may lack a physical motivation and may also be too restrictive. The evolution of DE background may not be enough to distinguish between different DE models and therefore the perturbations of DE may be fundamental to differentiating between them.

Perhaps the best physically motivated candidates for DE are scalar fields, which can interact only via gravity \(^12-14\) or interact weakly with other fluids, e.g. interacting dark energy (IDE) models \(^17,18\). In this work we will concentrate on canonically normalized scalar fields minimally coupled to gravity. Scalar fields have been widely studied in the literature \(^12-14\) and special interest was devoted to tracker fields \(^13\), since in this case the behavior of the scalar field \( \phi \) is weakly dependent on the initial conditions set at an early epoch, well before matter–radiation equality. In this class of models a fundamental question is why DE is relevant now, also called the coincidence problem, and this can be understand by the insensitivity of the late time dynamics on the initial conditions of \( \phi \). However, tracker fields may not give the correct phenomenology since they have a large value of \( w \) at present time. We are more interested at this stage to work from present day redshift \( z = 0 \) to larger values of \( z \) in the region where DE and its perturbations are most relevant. Interesting models for DE and dark matter (DM) have been proposed using gauge groups, similar to quantum chromodynamics (QCD) in particle physics, and have been studied to understand the nature of DE \(^15\) and also DM \(^16\).

Here we propose a new DE parametrization based on scalar field dynamics, but the parametrization of \( w \) can be used without the connection to scalar fields. This parametrization has a rich structure that allows \( w \) to have different evolutions: for example, it may grow and later decrease or the reverse can happen, and it can have steep or smooth transition from an initial value of \( w \) to the present day \( w_\text{obs} \) value. We also determine the perturbations of DE, which together with the evolution of the homogenous part can single out the nature of DE. With the underlying connection between the evolution of \( w \) and the dynamics of scalar field we could determine the potential \( V(\phi) \). The same motivation of parametrizing the evolution of scalar field was presented in an interesting paper \(^11\). We share the same motivation but we follow a different path. We have the same number of parameters but a richer structure and it is easier to obtain information of the scalar potential \( V(\phi) \).

We organize the work as follows: in section 1.1 we give a brief overview of our DE parametrization. In section 2 we present the scalar field and we define the variables used in this work. In section 3 we present the dynamics of a scalar field and the set up for our DE
parametrization given in section 5. We calculate the DE perturbations in section 6 and finally we conclude in section 7.

1.1. Overview

We present here an overview of our \( w \) parametrization. The EoS is

\[
w = \frac{p}{\rho} = \frac{x - 1}{x + 1},
\]

with \( x \equiv \frac{E_k}{V} \) the ratio of kinetic energy \( E_k = \dot{\phi}^2/2 \) and potential \( V(\phi) \). This corresponds to a canonically normalized scalar field. The equation of motion of the scalar field gives (c.f. equation (21)),

\[
x = \frac{\sqrt{1 + \frac{2L}{3(1+y)^2}} - 1}{2}(1 + y),
\]

where \( y \equiv \frac{\rho_m}{V} \) is the ratio of matter density and \( V, \ L \equiv \left(V'/V\right)^2 A \) with \( A \equiv (1 + q)^2, \ q \equiv \dot{\phi}/V' \). Equation (2) is an exact equation and is valid for any fluid evolution and/or for an arbitrary potential \( V(\phi) \). In terms of \( L \) and \( y \) the EoS is

\[
w = \frac{6 + L - 6\sqrt{(1+y)^2 + 2L/3}}{L + 6y} = -1 + \frac{6(1+y) - 6\sqrt{(1+y)^2 + 2L/3}}{L + 6y}
\]

which we consider our master equation. It is valid for any value of \( L \) and \( y \) and not only in the slow roll regime.

The aim of our proposed parametrization for \( L, y \) is to cover a wide range of DE behavior. Of course other interesting parametrizations are possible. The dynamics of scalar fields with canonical kinetic terms has an EoS constrained between \(-1 < w < 1\) and gives an accelerating Universe only if \( \lambda = -V'/V \to 0 \) or to a constant \( \lambda < 1 \) with \( w = -1 + \lambda^2/3 \) [14] (for an EoS \( w < -0.8 \) one needs \( \lambda < 0.8 \)). From the dynamics of scalar fields we know that the evolution of \( w \) close to present time is model dependent. For example, in the case of \( V = V_0 \phi^{2/3} \), used as a model of DE derived from gauge theory [15], the shape of \( w(z) \) close to present time depends on the initial conditions and it may grow or decrease as a function of redshift \( z \). We also know that tracker fields are attractor solutions but in most cases they do not give a negative enough \( w_o \) [13]. In this class of models the EoS, regardless of its initial value, goes to a period of kinetic domination where \( w \approx 1 \) and later has a steep transition to \( w \approx -1 \), which may be close to present time, and finally it grows to \( w_o \) in a very model and initial condition dependent way. Furthermore, if instead of having a single potential term we have two competing terms close to present time, the evolution of \( w(z) \) would be even more complicated. Therefore, instead of deriving the potential \( V \) from theoretical models as in [15], we propose to use an ansatz for the functions \( L \) and \( y \), which on the one hand should cover as wide a range of DE behavior with the least number of parameters without sacrificing generality and on the other hand, we would like to have this ansatz as close as possible to the known scalar field dynamics. We believe that using our model will greatly simplify the extraction of DE dynamics from future observational data. We propose therefore the ansatz (c.f. equation (53))

\[
L = L_o + L_1 y f(z)
\]
\[ f(z) = \frac{(z/z_t)^k}{1 + (z/z_t)^k} \]  

(5)

where \( y = \chi_0 a^{3w_0} \) and \( L_0, L_1 \) are free parameters giving \( w_0 \) and \( w_1 = w(\infty \gg z_t) \) at early times and \( f(z) \) is a function that goes from \( f(z = 0) = 0 \) at \( z = 0 \) to \( f(z \gg 1) = 1 \) at large \( z \). The parameter \( z_t \) sets the transition redshift between \( w_0 \) and \( w_1 \) (a subscript \( o \) represents present time quantities) while \( k \) is the steepness of the transition and \( \xi \) takes only two values \( \xi = 1 \) or \( \xi = 0 \) (see section 5).

Since the Universe is accelerating at present time, we may take the slow roll approximation where \( |q| \ll 1, A \simeq 1 \) and \( L \simeq (V'/V)^2 \). In this case, one has \( w \simeq -1 + L/3(1 + y) \). However, the derivation of \( L \) in equation (2) is exact and has no approximation. We will show in section 5 that \( w \) can have a wide range of behaviors and in particular \( w \) can decrease and later increase as a function of redshift and vice versa, therefore the shape and steepness are not predetermined by the choice of our parametrization. Of course we could use other parametrization since the evolution of \( x \) and \( w \) in equations (1) and (2) are fully valid. There is also no need to have any reference to the underlying scalar field dynamics, i.e. our parametrization is not constrained to scalar field dynamics. However, it is when we interpret \( x, L \) and \( y \) as \( \dot{\phi}^2/2V, L = (V'/V)^2A \) and \( y = \rho_m/V \), that we connect the evolution of \( w \) to the scalar potential \( V(\phi) \).

Finally, DE perturbations are important in distinguishing between different DE models [21–25] and we will show that a steep transition of \( w \) has a bump in the adiabatic sound speed \( c_a^2 \) which could be detected in large scale structure [24, 25]. In an epoch where the Universe is dominated by DM and DE, the total perturbation \( \delta_T = \delta \rho_T/\rho_T = \Omega_{\text{DE}} \delta \rho_{\text{DE}} + \Omega_{\text{DM}} \delta \rho_{\text{DM}} \) and if \( \Omega_{\text{DE}} \) is not much smaller than \( \Omega_{\text{DM}}, \) i.e. for low redshift \( z \), then the evolution of \( \delta \rho_{\text{DE}} \) may have an important contribution to \( \delta_T \), as discussed in section 6.

2. Scalar field dynamics

We are interested in obtaining a new DE parametrization inferred from scalar fields [12–18]. Since it is derived from the dynamics of a scalar field \( \phi \), we can also determine its perturbations which are relevant in large scale structure formation. We start with a Friedmann Robertson Walker (FRW) metric with a line element

\[ ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \right) \]  

(6)

and a canonically normalized scalar field \( \phi(t, x) \) with a potential \( V(\phi) \), minimally coupled to gravity. The homogenous part of \( \phi \) has an equation of motion

\[ \dot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \]  

(7)

where \( V' \equiv dV/d\phi, \) \( H = \dot{a}/a \) is the Hubble constant, \( a(t) \) is the scale factor and a dot represents derivative with respect to time \( t \). Since we are interested in the epoch for small redshift \( z \), with \( a_o/a = 1 + z \), we only need to consider matter and DE and we have

\[ 3H^2 = \rho_m + \rho_{\text{DE}}, \]  

(8)

in natural units \( 8\pi G = 1, \ c = \hbar = 1 \). The energy density \( \rho \) and pressure \( p \) for the scalar field are
\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V, \quad \rho_\phi = \frac{1}{2} \dot{\phi}^2 - V \]  

(9)

We define the ratio of kinetic energy and potential energy as

\[ x \equiv \frac{\dot{\phi}^2}{2V} \]  

(10)

and the equation of state parameter EoS becomes

\[ w \equiv \frac{\rho_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V}{\dot{\phi}^2/2 + V} = \frac{x - 1}{x + 1} \]  

(11)

or

\[ w = -1 + \delta w, \quad \delta w \equiv \frac{2x}{1 + x}. \]  

(12)

The value of \( x \) determines \( w \) or inverting equation (11) we have \( x = (1 + w)/(1 - w) \). Since \( x \geq 0 \) the DE EoS \( w \) is in the range \(-1 \leq w \leq 1\). For growing \( x \) the EoS \( w \) becomes larger and at \( x \gg 1 \) one has \( w \approx 1 \) while a decreasing \( x \) has \( w \rightarrow -1 \) for \( x \rightarrow 0 \). In terms of \( x \) we have

\[ \rho_\phi = V(x + 1), \quad \rho_\phi = V(x - 1), \quad \rho_m = Vy \]  

(13)

where we defined the ratio of and scalar potential \( V \) as

\[ y \equiv \frac{\rho_m}{V}. \]  

(14)

Finally, we can express \( \Omega_\phi, \Omega_m \) and \( H \) in terms of \( x, y \) and \( V \) as

\[ \Omega_\phi = \frac{1 + x}{1 + x + y}, \quad \Omega_m = \frac{y}{1 + x + y}, \]  

(15)

\[ 3H^2 = \rho_m + \rho_\phi = V(1 + x + y). \]  

(16)

The equation of motion for \( \phi \) is

\[ \ddot{\phi} = -\frac{V' + \dot{\phi}}{3H} = -\frac{V'(1 + q)}{3H} \]  

(17)

and we can write

\[ x \equiv \frac{\dot{\phi}^2}{2V} = \frac{V}{3H^2} \frac{V'^2}{6V^2} (1 + q)^2 = \frac{V}{3H^2} \frac{L}{6} \]  

(18)

with

\[ L \equiv \left( \frac{V'}{V} \right)^2 (1 + q)^2 = \lambda^2 A, \]

\[ \lambda \equiv -\frac{V'}{V}, \quad q \equiv \frac{\dot{\phi}}{V'}, \quad A \equiv (1 + q)^2. \]  

(19)

Since the right-hand side of equation (18) still depends on \( x \) through \( H \), we use equations (16), and (18) then becomes
\[ x = \frac{L}{6(1 + x + y)}, \]  
\text{i.e. } L = 6x(1 + x + y), \text{ which has a simple solution}\n\[ x = \frac{\left(\sqrt{1 + \frac{2L}{3(1 + y)^2}} - 1\right)(1 + y)}{2} = \frac{\sqrt{(1 + y)^2 + 2L/3} - (1 + y)}{2}. \]  
Substituting equation (21) into equation (11) we obtain our DE parametrization as a function of \(L\) and \(y\)
\[ w = \frac{\sqrt{(1 + y)^2 + 2L/3} - (3 + y)}{\sqrt{(1 + y)^2 + 2L/3} + (1 + y)}. \]  
If we multiply in equation (22) the numerator and denominator by \(\sqrt{(1 + y)^2 + 2L/3} - (1 + y)\), we obtain an alternative and useful expression for the EoS
\[ w = \frac{6 + L - 6\sqrt{(1 + y)^2 + 2L/3}}{L + 6y} = -1 + \frac{6(1 + y) - 6\sqrt{(1 + y)^2 + 2L/3}}{L + 6y} \]  
and
\[ \delta w = w + 1 = \frac{6(1 + y) - 6\sqrt{(1 + y)^2 + 2L/3}}{L + 6y}. \]  
which we consider our master expression for \(w\). It is a relatively simple expression, but more important is that equations (22) and (23) are exact and therefore valid for any values of \(L\) and \(y\) and not only at a slow roll regime. We will see in the next section some physically motivated limits of equation (23). Finally, inverting the expression of \(w\) we can obtain \(L\) as a function of \(w\) and \(y\)
\[ L = \frac{12(1 + w) + 6y(1 - w^2)}{(1 - w^2)} \]  
and \(y\) as a function of \(w\) and \(L\)
\[ y = \frac{L(1 - w^2) - 12(1 + w)}{6(1 - w^2)}. \]  

2.1. Dynamics and limits for \(L, x, w\)

Using equation (20) we can express
\[ w = -1 + \delta w = -1 + \frac{2L}{L + 6(1 + x + y)}, \]  
and it is an exact equation, however \(x\) depends on \(y, L\) through equation (21). If we take \(x < (1 + y)\), valid in the slow roll approximation, one has for arbitrary values of \(y\) and \(L\)
\[ x = \frac{L}{6(1 + y)}, \quad w = -1 + \frac{L}{3(1 + y)}. \]
and as \( L \to 0 \) we have \( x \to 0 \) and \( w \to -1 \). Notice that
\[
y = \frac{\rho_m}{V} = \frac{\rho_m}{\rho_0} \left( 1 - w \right) = \frac{\Omega_m}{\Omega_0} \left( 1 - w \right)
\]  
and we then expect that \( y \) increases as a function of redshift \( z \) when matter dominates over DE and with \( y = y(a_0) \approx 0.25 \) at present time. We expect then that \( x < (1 + y) \) will be satisfied beyond the slow roll approximation and equation (28) is valid in a wide range of the parameter’s values.

If we take \( L \gg 1 \) with \( y \) constant in equation (23), we get the limit
\[
x = \sqrt{\frac{L}{6}}, \quad w = 1 - 2 \sqrt{\frac{6}{L}}
\]  
and for \( L/y \to L_1 \) constant with \( y \gg 1 \), \( L_1 \to 1 \) finds a constant \( x \) and \( w \) with
\[
x = \frac{L_1}{6}, \quad w = \frac{L_1 - 6}{L_1 + 6} = -1 + \frac{2L_1}{6 + L_1}.
\]  
Clearly, depending on the value of \( L \) we can have a decreasing or increasing \( x, w \) as a function of redshift. For example, for \( w = 0 \) one has \( L_1 = 6 \) while for \( w = 1/3 \) one requires \( L_1 = 12 \).

Since \( w \) is only a function of \( x \), we have \( dw = w_1 dx \) or as a function of redshift
\[
w_z = w_1 x_z
\]  
with
\[
w_1 = \frac{2}{(1 + x)^2} \geq 0.
\]  
The sign of \( w_z \) then depends only on the sign of \( x_z \) given by
\[
x_z = x_y y_z + x_L L_z
\]  
with
\[
x_y = -\frac{x}{1 + 2x + y}, \quad x_L = \frac{1}{6(1 + 2x + y)}.
\]  
Clearly, \( x_L \) is positive definite while \( x_y \leq 0 \). In general we can assume that DE redshifts more slowly than matter, at least for small \( z \), since DE has \( w < 0 \) and matter \( w_m = 0 \), so \( y = \rho_m/V \) is a growing function of \( z \), i.e. \( y_0 > 0 \) and \( x_y y_z \leq 0 \). Therefore, if \( L_z \) is negative we have \( x_y L_z < 0 \) and a decreasing \( x \) and \( w \) as a function of redshift. However, if \( L_z \) is positive, then the sign of \( x_y L_z \) is positive and \( w \) may grow or decrease depending on the magnitude of \( |x_y y_z| \) compared to \( |x_L L_z| \).

### 2.2. Slow roll approximation

The evolution of \( x \) as a function of \( L \) and \( y \) is given in equation (21) and here we will show some phenomenological interesting limits. Since the field \( \phi \) should be responsible for accelerating the Universe we know that \( w \) must be close to \(-1\) at present time, and the field \( \phi \) must satisfy the slow roll approximation. In the slow roll approximation one has \( |\dot{\phi}| \ll |3H\dot{\phi}| \approx |V| \) and we then have
with $A \simeq 1$ and the function $L$ becomes

$$L \simeq \left(\frac{V'}{V}\right)^2 = \lambda^2. \quad (37)$$

Let us now take different limits for $x$ and $w$ as a function of $L$ and $y$. We name a full slow roll approximation when $\dot{\phi} = 0$ (i.e. $3H\dot{\phi} = -V'$ and $q = 0$). This full slow roll approximation is more suitable in inflation where one has a long period of inflation; however DE has dominated only recently and so we do not expect this approximation to hold for a long period of time and it should not be taken as a working hypothesis for DE parametrization.

### 2.3. Late time attractor solution

The evolution of scalar fields has been studied in [14] and a late time attractor for an accelerating Universe requires $w < -1/3$, $\lambda^2 < 2$ with $\Omega_\phi \to 1$. In the limit $\lambda^2 < 3$ with $\rho_m \ll \rho_\phi$, one has

$$\frac{\dot{\phi}^2}{6H^2} = \frac{\lambda^2}{6}, \quad \frac{V}{3H^2} = 1 - \frac{\lambda^2}{6} \quad (38)$$

giving

$$x \equiv \frac{\dot{\phi}^2}{2V} = \frac{\lambda^2}{6 - \lambda^2}, \quad w = -1 + \frac{\lambda^2}{3} \quad (39)$$

Notice that equation (28) reduces to equation (39) in the limit $y \ll 1$ with $L \simeq \lambda^2$, and therefore equation (28) generalizes equation (39). For large $z$ we expect $y$ to increase and equation (39) would no longer be valid, so we should take instead equation (28).

### 3. Dynamical evolution

Differentiating $x$ and $y$ with respect to time or equivalently as a function $N = \ln[|a|]$, where $a$ is the scale factor, we get the evolution of $x$ and $y$. Using the definition of $x$ in equation (18), $\rho_m = -3H\rho_\phi, \dot{\phi} = -3H\dot{\phi} = V'$ and $f_N \equiv df/dN = \dot{f}/H$ for any function $f(N(t))$, we get

$$\dot{x} = \frac{\dot{\phi}}{V}(\phi - xV') = \frac{\dot{\phi}V'}{V}(q - x) = 6Hx\left(\frac{x - q}{1 + q}\right) \quad (40)$$

$$x_N = \frac{\dot{x}}{H} = 6x\left(\frac{x - q}{1 + q}\right) = 6x\left(\frac{1 + x}{1 + q} - 1\right) \quad (41)$$

and $\ddot{y} = H\dot{y}_N$ with

$$y_N = \frac{\ddot{y}}{H} = -3y - \frac{yV'\dot{\phi}}{HV} = -3y\left(1 - \frac{2x}{1 + q}\right). \quad (42)$$

We see that equations (41) and (42) are uniquely determined by a single function $q \equiv \ddot{\phi}/V'$. The critical points for $\dot{x} = 0$, i.e. $\dot{\phi} = 0$, have $x = 0$ or $x = q$ while $\ddot{y} = 0$ is satisfied for $y = 0$ and $2x = 1 + q$. The case $\dot{\phi} = x = 0$ has $w = -1$ and $\ddot{y} = -3Hy$, which gives a solution $y = y_i(a/a_i)^{-3}$ and a constant $V(\phi)$ with $\Omega_\phi \to 1, y \to 0$ and $\Omega_m \to 0$. 
In the case \( q = x \) the EoS becomes \( w = (q - 1)/(q + 1) \) and it will take different constant values. Setting \( q = x \) constant in equation (42), we get \( \dot{y} = 3H y w \) with \( w \) constant one has a solution
\[
y = \left( \frac{a}{a_0} \right)^{3w}.
\] (43)

For \( q = x < 1 \) we have \( w < 0, y \to 0 \) and \( \Omega_0 \to 1 \). Finally, we can satisfy \( \dot{x} = \ddot{y} = 0 \) for \( x = y = 0 \) or \( x = q = 1 \) giving \( w = 0 \) and \( y \) constant (c.f. equation (43)).

Therefore, having an increasing or decreasing \( x \) depends on the sign of \( \dot{x} = \dot{V} (q - x)/V \) and it can vary as a function on time depending on the values of \( x \) and \( q \), i.e. on the choice of the potential \( V(\phi) \). If the field is rolling down the potential, then \( V < 0 \) and the sign of \( x \) is given by \( (q - x) \) with \( \dot{x} > 0 \) for \( x > q \) and \( \dot{x} < 0 \) for \( x < q \). If we take the full slow roll approximation, defined by \( \ddot{\phi} = q = 0 \) and \( 3H \dot{\phi} = -V' \), then equation (41) becomes
\[
\dot{x} = 6H x^2
\] (44)

which is positive definite, i.e. \( \dot{x} \geq 0 \). Using \( H = \dot{a}/a \) we can express equation (44) giving a solution
\[
x = \frac{x_0}{1 + 6x_o \text{Ln}(a_o/a)} = \frac{x_0}{1 + 6x_o \text{Ln}(1 + z)}.
\] (45)

Therefore, if the condition \( q = 0 \) or \( |q| \equiv |\ddot{\phi}/V'| \ll x \) is satisfied, equation (45) gives a decreasing function for \( x \) as a function of redshift \( z \) and therefore \( w(z) \) also decreases. However, we do not expect the Universe to be in a full slow roll regime and when \( x \) is small, e.g. \( w < -0.9 \) and one has \( x < 0.05 \), the slow roll condition \( |\ddot{\phi}| < |V'| \) does not imply that \( q \ll x \). Therefore, the sign of \( \dot{x} \) can be positive or negative depending on the sign and size of \( q = \ddot{\phi}/V' \) compared to \( x \), and \( x \) can either grow or decrease. In the region where \( q = \ddot{\phi}/V' < x \) we have \( \dot{x} > 0 \) while for \( q = \ddot{\phi}/V' > x \) we have \( \dot{x} < 0 \). The value of \( q \) then parametrizes the amount of slow roll of the potential and a full slow roll has \( q = 0 \), but we expect to be only in an approximate slow roll regime with \( |q| < 1 \) and \( L \simeq \lambda^2 \). We discuss the dynamics of \( q \) in appendix C. In the present work we do not want to study the critical points of the dynamical equations but the evolution of \( x \) close to present time when the Universe is accelerating with \( x \) close to zero (\( w \) close to \(-1 \)) but not exactly zero with \( \ddot{\phi} \not\equiv 0 \) and \( \ddot{x} \equiv x V' \).

Finally, for completeness, we derive in appendix A an alternative set of the dynamical evolution for \( x \) and \( y \) as a function of a single quantity \( \zeta \equiv -V_0/V \), which determines the evolution of \( x \) and \( y \). We also present in appendix B the dynamical equations for a scalar field in terms of the variables \( \tilde{x} \equiv \ddot{\phi}/\sqrt{6H}, \tilde{y} \equiv \sqrt{V/3H} \), as widely used in the literature [6, 14], which are determined by a single quantity \( \lambda \equiv -V'/V \).

**4. Scalar field dynamics and \( L, \zeta \) and \( \lambda \)**

As seen in section 3 and in the appendix A and B, the dynamics of a scalar field are given by a single function which can be either \( L, \lambda \) or \( \zeta \) given in equations (19) and (A2), respectively. Both \( \zeta \) and \( \lambda \) determine uniquely the set of differential equations (A1) and (B1). The solution to these equations then depends on the functional form of \( \zeta \) or \( \lambda \) as a function of \( N \) and on the initial conditions \( x_i, y_i \) and the three quantities \( L, \lambda \) or \( \zeta \) are related by equations (19) and (A3).
and in the slow roll approximation $|q| \ll 1$ (with $x \ll 1, y \ll 1$) they coincide since $L \simeq \lambda^2 \simeq \zeta \simeq 6x$. Therefore, all three quantities are equivalent in the full slow roll approximation and differ slightly once the evolution of $\phi$ does not obey it any more. All of them can be used to determine the evolution of $\phi$ uniquely.

If we determine $\lambda$ and $y = \rho_\phi/V$ as a function of the scale factor $a$, then we can extract $\phi$ as a function of the scale factor using $dV/V = \lambda \phi_N dN$ and $\phi_N = \sqrt{V}$. We get

$$V = V_0 e^{\lambda \phi_N dN}, \quad \phi = \int \sqrt{\lambda} \rho_\phi dN$$

with $V(a) = \rho_\phi(a)/y(a)$. From equation (47) we have $V(\phi)$ and $\phi$ as a function of the scale factor and we can then determine $V(\phi)$ as a function of $\phi$.

We have studied the critical points for $x$, $y$ or $\hat{x}$, $\hat{y}$ for constant $\zeta$ or $\lambda$, respectively, which give the asymptotic behavior of the system. However, we do not expect in general to have $\zeta$ (or $\lambda$) constant and we would need to obtain the dynamical equation of motion for $\zeta(\lambda)$. This can be easily done by taking, for example, the derivative $\zeta_N$, i.e. $\zeta_N = V_{NN}/V - V_N V^2 = \zeta^2 (V_{NN}/V_N^2 - 1)$, and we can use the equation of motion of the scalar field to determine the r.h.s. of the equation of $\zeta_N$. This is not a closed system and we could take further derivatives of $\zeta$ and have an infinite series of equations in which the functions $d^m \zeta/d^m N$ are given in terms of $d^m \zeta/d^m N$ with $m < n$. In order to have an exact solution for the scalar field we would need to solve the equation of motion of $\phi$ given a potential $V(\phi)$, but then we would have an exact solution only for the specific potential used.

In principle, the quantity $\zeta_N = \lambda \phi_N$ may have many free parameters and may be a complicated function since it depends on the potential $V$, $V'$ via $\lambda$ and the kinetic term $\phi$. Instead of using $\zeta$ as our free function, we prefer to work with $\lambda$ since it is only a function of the potential $V(\phi)$ and not of $\phi_N$. If we take the derivative of $\lambda$, we get

$$\lambda_N = \left( \lambda^2 - \frac{V''}{V} \right) \phi_N$$

and we will distinguish two different cases for $\lambda_N$. If we have a rolling scalar field in such a way that $V'$ only vanishes at the infinite value of $\phi$, i.e. there is no local minimum, then the term

$$\Gamma = \frac{V''V}{V'^2}$$

can be treated in a good approximation as constant, at least for tracker potentials with $\Gamma > 1$ [13], and equation (48) can be written as

$$\lambda_N = \lambda^2 (1 - \Gamma) \phi_N,$$
attractor. But what figures 2(b) and (c) show is that the model may not have reached the attractor solution by the time one has $W = f_0 = 0.74$, and the evolution of $w$ and $L$ as a function of $z$ depends strongly not only on the value of $G$ but also on the initial conditions $x, \gamma, \hat{x}$ and $L_g$. We have used a $\Gamma = 1.1 + 0.5k$ with $k = 1, 2, 3, 4$ (green, red, yellow and blue, respectively and $L_i = 1 + 10j$ with $j = 0, 1, 3, 5, 7$.

5. Scalar field DE parametrization

The aim here is to test a wide class of DE models in order to constrain the dynamics of DE from the observational data. We could parametrize $w$ or $x$ with a single function in terms of the scale factor (c.f. section 5.3) but as we have seen in equation (21) we get a better understanding of the evolution of $\phi$ if we parametrize the functions $L$ and $y$.

In order to have the evolution of DE, we need to either choose a potential $V(\phi)$ and solve equation (A1) or parametrize the EoS $w$ or $x$ as a function of the scale factor $a$. If we choose to parametrize $\zeta$ or $\lambda$, then we need to solve the differential equations equation (A1) or (B1). On the other hand, $L$ gives the evolution of $x$ and $w$ without the need to solve any differential equations and $L$ reduces to $\lambda$ (or $\zeta$) in the slow roll approximation. Therefore, we propose here to use the exact equation for $w$ given in equation (23) and parametrize the function $L$.

A priori it is impossible to know how many free parameters have the potential $V(\phi)$; and since the evolution of different scalar field models requires one to solve the equation (A1), it is difficult to test a wide range of potentials $V$. The potential $V$ may involve a single term, as in a runaway potential like $V = V_0 \phi^{-\alpha}$ or $V = V_0 e^{-\phi^2}$, or it may have different terms in the potential with the same order of magnitude, as for example $V = A\phi^{-1} + B e^{\phi^2}$ or $V = A\phi^{-1} + m^2 \phi^2$, that may lead to a local minimum with $V = 0$. 

![Figure 1](https://example.com/figure1.png)

**Figure 1.** In figures 1(a)–(c) we show the evolution of $w$ for a potential $V = V_0 \phi^{-3/2}$, $V = V_0 e^{-\phi^2/2}$ and $V = V_0 \phi^{-1/2} e^{\phi^2/2}$, respectively, for different initial conditions.
In all these cases the evolution of $w$ would be different but still constrained to $-1 \leq w \leq 1$, and in the slow roll approximation ($L \ll 1, x \ll 1$) one has $w \simeq -1 + L/[3(1 + y)]$ given by equation (28). However, we do not know if there was a steep descent in $w$ or $x$ close to present time in such a way that for smaller $a < a_0$, the value of $w$ was much larger, e.g. $w$ close to 0, as matter, or even positive $w \simeq 1/3$, as radiation, with $L(a < a_0) \gg L(a_0)$, and the slow roll approximation may not be valid anymore. In this work, we would like to include a parametrization which also includes this kind of steep transition.

Figure 2. In figure 2(a) we show the evolution of $\dot{x}, \dot{y}$ for different initial conditions. We also show in figures 2(b) and (c) $w$ and the corresponding $L$ as a function of $z$ for different values of $\Gamma$ and $L_{\omega}, \dot{\omega}, \dot{\gamma}$.

In all these cases the evolution of $w$ would be different but still constrained to $-1 \leq w \leq 1$, and in the slow roll approximation ($L \ll 1, x \ll 1$) one has $w \simeq -1 + L/[3(1 + y)]$ given by equation (28). However, we do not know if there was a steep descent in $w$ or $x$ close to present time in such a way that for smaller $a < a_0$, the value of $w$ was much larger, e.g. $w$ close to 0, as matter, or even positive $w \simeq 1/3$, as radiation, with $L(a < a_0) \gg L(a_0)$, and the slow roll approximation may not be valid anymore. In this work, we would like to include a parametrization which also includes this kind of steep transition.

How many parameters should we use? We would say that we should use the minimum number of parameters as long as we still track the behavior of scalar fields and it is generic enough to have different behavior for $w$ and allow the cosmological data to fix these parameters.
5.1. Ansatz

We will propose an ansatz for $L$ and $y$ that covers the generic behavior of scalar field leading to an accelerating Universe. If we want to have a constant EoS for DE at early times $z \gg 1$, as for example matter $w = 0$ or radiation $w = 1/3$, which are reasonable behaviors for particles, we should choose $L$ proportional to $y$ for large $y$ with $L/y \rightarrow L_1$; or if we want $w \rightarrow -1$ at a large redshift the limit $L/y \rightarrow 0$ must be satisfied. We then propose to take

$$L = L_0 + L_1 y^{k} f(a)$$

(51)

with two constant free parameters $L_0$ and $L_1$, and a transition function $f(a)$ constrained between $0 \leq f(a) \leq 1$. The quantity $\xi$ takes the values $\xi = 1$ and $\xi = 0$ only and we do not consider it as a free parameter but more as two different ansätze for $L$, depending on whether we want $L$ to grow proportional to $y$ (i.e. $\xi = 1$) or to a constant value, in which case we take $\xi = 0$. Within the limit of small $a \ll 1$ we have $y \gg 1$ and $L/y$ goes in equation (51) to a constant value $L_1$ or zero for $\xi = 1, 0$, respectively. This limit is independent of the functional form of $y$, since at large $y$ the EoS $w$ depends on $L/y \rightarrow L_1$ and the dependence on $y$ cancels out giving a constant value of $w$ (c.f. equation (31)). We therefore choose to take $y$ in equation (51) as in equation (A9) with $w \approx w_o$, i.e.

$$y = \frac{m}{V} = y_o \left( \frac{a}{a_o} \right)^{w_0} = y_o (1 + z)^{-3w_0}.$$  

(52)

However, taking $y$ as in equation (A9) does not mean that $\rho_0 \propto (a_o/a)^{3(1+w_0)}$ since the kinetic energy $\dot{\phi}^2/2$, or equivalently $x$, may grow faster or more slowly than $V$. For $\xi = 1$ equation (51) allows a wide class of behaviors for $w$. If we want $w$ to increase to $w = 0, 1/3$, we would take $L_1 = 6, 12$, respectively; and since in many scalar field models the evolution of $w$ goes from $w_o$ to a region dominated by the kinetic energy density with $w = 1$, in this case we would should take $L_1 \gg 1$. Of course a $w(z \gg 1) = 1$ would only be valid for a limited period since $\Omega_y$ should not dominate the Universe at early times. We have included in equation (51) the case $\xi = 0$ because we want to allow $w$ to increase from $w_o$ at small $z$ and later go to $w \rightarrow -1$ (c.f. pink-dashed line in figure 5), since this is the behavior of potentials used as a model of DE, as for example $V = V_o \phi^n$, $n = 2/3$ derived from gauge group dynamics [15] where the behavior of $w(z)$ close to present time depends on the initial conditions.

We propose to take the simple ansatz for the transition $f$ as a function of redshift $z$ as

$$f(z) = \frac{(z/z_t)^k}{1 + (z/z_t)^k}$$

(53)

with $f_o \equiv f(z = 0) = 0, f(z = z_t) = 1/2$ and $f_1 \equiv f(z \gg 1) = 1$. The transition function $f(z)$ given in equation (53) has four free parameters $L_0, L_1, z_t$ and $k$. The parameters $L_0$ and $L_1$ give the EoS $w$ at present time and at large redshift $z \gg z_t$, while the transition epoch from $w_o$ to $w_1 \equiv w(z \gg z_t)$ is given by $z_t$, and $k$ sets the steepness of this transition. This parametrization has a simple expression in terms of $z$ and it reduces to Chevallier-Polarski-Linder parametrization (CPL) [9], i.e. $w = w_o + w_0(1 - a)$, in the full slow roll approximation, with $\xi = 0, z_t = 1, k = 1$ giving $f(z) = z(1 + z) = 1 - a$ and $L = L_0 + L_1(1 - a)$. Using equation (39) we have $w \approx -1 + L/3 = -1 + (L_0 + L_1 f(z))/3$ and therefore we identify $w_0 = -1 + L_0/3$ and $w_1 = L_1/3$. However, our parametrization given in equations (51) and (53) goes well beyond the CPL parametrization, and the analysis will be presented in full in [22].
5.2. Initial conditions and free parameters

Let us summarize the parameters and initial conditions of our parametrization. The EoS $w$ is only a function of $x$, and $x$ is a function of $L$ and $y$. From equation (26) we see that $y$ depends on two parameters $w_0$ and $\Omega_{\text{DE}}$ (or equivalently on $\chi_0$, $w_0$), since we are assuming a flat Universe with DE and matter and $\Omega_{\text{mo}} = 1 - \Omega_{\text{DE}}$.

From equation (51) we have the conditions at present time as

$$\chi_0 \equiv y(a_0 = 1) = \frac{\rho_{\text{DE}}}{V_0} = \frac{2}{(1 - w_0) \Omega_{\text{DE}}}$$

and from equation (26) we express $L_0$ as a function of $w_0$ and $\chi_0$ and we have

$$L(z = 0) = L_0 = \frac{12(1 + w_0) + 6\chi_0(1 - w_0^2)}{(1 - w_0)^2}$$

since $f(z = 0) = 0$. While the value is at an early time we get

$$\gamma_1 \equiv y(\alpha \ll 1) = \frac{2}{(1 - w_1) \Omega_{\text{DE}}}$$

and

$$L_1 \equiv L(z \gg 1) = L_0 + L_4 \gamma^2 f_i = \frac{12(1 + w_i) + 6\chi_i(1 - w_i^2)}{(1 - w_i)^2} \simeq \frac{6\gamma_i(1 + w_i)}{(1 - w_i)}$$

with $f_i = f(z \gg 1) = 1$. Since we expect to have negligible DE at early times we have $\gamma_i \gg 1$ and we keep in the right-hand side of equation (57) only the term proportional to $\gamma_i$. We see from this equation that for $\xi = 1$ the value of $L_1$ is determined by the early time EoS $w_i$ with (c.f. equation (31)),

$$L_1 = \frac{6(1 + w_i)}{1 - w_i}$$

giving for example $L_1 = 6$ for $w_i = 0$, $L_1 = 12$ for $w_i = 1/3$ while $L_1 = 0$ requires $w_i \to -1$ and $L_1 \gg 1$ has $w_i \to 1$. However, when $\xi = 0$ we have the limit $L/y \to 0$ with $L_i = L_0 + L_4$ a finite constant, so equation (57) requires $w_i = -1$ independently of the value of $L_i$.

From equation (51) we have that $L$ depends on $y$ and $L_0$, $L_1$, $z$, $k$ and $w_i$, $\Omega_{\text{DE}}$. However, not all parameters are independent, since $L_0$ is a function of $w_0$, $\Omega_{\text{DE}}$, $L_4$, $z$, $q$ and we are left with $\Omega_{\text{DE}}$ and four parameters in $w$. To conclude, the free parameters are $\Omega_{\text{DE}}$ and for the EoS we can take $w_0$, $w_i$, the transition redshift $z_t$ and the steepness of the transition $k$.

5.3. Other parametrizations

We present here some widely used parametrizations and we compare them with our DE parametrizations in this work. We begin with a simple DE parametrization [9] given in terms of only two parameters and widely used in most data analysis projects:

$$w(a) = w_i + w_0(1 - a) = w_0 + w_i \frac{z}{1 + z}$$
with the derivative
\[
\frac{dw}{da} = -w, \quad \frac{dw}{dz} = \frac{da}{dz} \frac{dw}{da} = (1 + z)^{-2}w_i
\]  

Clearly \( w \) in equation (59) is convenient since it is a simple EoS and it has only two parameters. However, it may be too restrictive and we do not see a clear connection between the value of \( w \) at small \( a \) and its derivative at present time \( w_i \). It has only three parameters, \( \Omega_c, w_0, w_i \), two less than our model but our model has a much richer structure.

Another interesting parametrization was presented in [10], with four free parameters. It is given by
\[
w = w_0 + (w_i - w_0)G, \quad G \equiv 1 + e^{a_d/d} \frac{1 - e^{1-a}/d}{1 + e^{a_d - a}/d}
\]  

where \( w_0, w_i, a_d, d \) are constant parameters. The function \( G \) is constrained between \( 0 \leq G \leq 1 \) with \( G = 0 \) for \( a \gg a_d \) and \( G = 1 \) for \( a \ll a_d \). Therefore \( a_d \) is the scale factor where the transition of the EoS \( w \) goes from \( w_0 \) to \( w_i \). The parameter \( d \) gives the width between the transition, for small \( d \) the transition between \( w_0 \) and \( w_i \) is steep. Even though \( w \) in equation (61) gives a large variety of DE behavior [10], since the sign of the slope is fixed our parametrization in equations (21) and (51) has a richer structure with the same number of parameters.

In [11] the researchers’ motivation was similar to ours in the present work. They presented a DE parametrization motivated by the dynamics of a scalar field. Their

\[ \text{Figure 3.} \text{ We show the evolution of } \Omega_c, w \text{ and } c_s^2 \text{ for different models. We have taken } \xi = 1, k = 2, L = 6 \text{ with } z = 0.1, 1, 2, 10, 100 \text{ (red, dark blue, light blue, green and yellow, respectively). In black we have } w, \Omega_c \text{ using } w \text{ in equation (59) and } \Omega_c \text{ for a cosmological constant (black dot-dashed). We take in all cases } w_0 = -0.9, \Omega_c = 0.74 \text{ and } L = 6 \text{ giving } w_i = w(z \gg z_i) = 0 \text{ for large } z. \]
parametrizations have either two or three parameters in equations (25) and (28), respectively, in the paper [11] (they do not take $a_{eq}$ as a free parameter but we do think that it is an extra parameter). The two parameters involved determine the quantities $w_i(a \gg a_0)$, which gives the EoS at an early time, and $\lambda(a_{eq}) = -V'/V|_{a_{eq}}$ at DM–DE equality (i.e. $\Omega_m = \Omega_{de}$). In the second case, the parametrization also involves a term $\zeta_i$ (equation (23) in [11]) which depends on a second derivative of $V$ and on the value of $\dot{\phi}/H$ at DM–DE equality. Since the functional form of the evolution of the EoS $w(a/a_{eq})$ is fixed in their parametrization, the value of $w_i$ at present time is determined if we know the value of $a_0/a_{eq}$. Therefore, the quantity $a_{eq}$ must also be assumed as a free parameter. As in our present work, the system of equations does not close without the knowledge of the complete $V$ as a function of $\phi$. However, since we are both interested in extracting information from the observational data to determine the scalar potential, the parametrization, given in equations (25) and (28) in [11], is an interesting proposal to study a wide range of potentials $V$. Here we have taken a different parametrization which has a closer connection to the scalar potential $V(\phi)$ given by equations (10) and (11).

5.4. Results

We have plotted $w$ for different sets of the parameters in figures 3–5 to show how $w$ depends on $L_{\phi}, L_1, L_2$ and $k$. We notice that our parametrization in equation (51) has a very rich structure allowing for $w$ to grow and/or decrease at different redshifts. We have also plotted $\Omega_{\phi}$ and the adiabatic sound speed $c_a^2$ defined in equation (69) for each model. We are showing
some extreme cases which we do not expect to be observationally valid but we want to show the full extent of our parametrization. We have also included a cosmological constant (CC) (black dot-dashed) in the figures of $W_f$ and since $\omega_{-1}^w = c_0^2$ and $c_0^2 = 0$ we do not include them in the graphs for $w_c$, $a_2$. We also plotted $W_f$, $w_c$, $a_2$ (in black) for the parametrization in equation (59) for comparison. We take in all cases $w_0 = -0.9$, $\Omega_0 = 0.74$ and $L_4 = 6$ giving $w(z > z_t) = 0$ for large $z$.

In figure 3 we show the evolution of $\Omega_0$, $w$ and $c_0^2$ for different models. We have taken $\xi = 1$, $z_t = 2$ fixed and $(k, L_4) = (2, 6), (20, 12), (2, 100), (20, 1000), (20, 0)$ (red, dark blue, light blue, green and yellow, respectively) and $\xi = 0$ with $z_t = 0.1$, $k = 10$, $L1 = 12$ in the pink-dashed line. In black we have $w$, $\Omega_w$ using $w$ in equation (59) and $\Omega_0$, for a cosmological constant (black dot-dashed). We take in all cases $w_0 = -0.9$, $\Omega_0 = 0.74$ and $L_4 = 6$ giving $w(z > z_t) = 0$ for large $z$.

In figure 4 we show the evolution of $\Omega_0$, $w$ and $c_0^2$ for different models. In this case we have taken $\xi = 1$, $z_t = 2$, $L_4 = 6$ fixed with $w_0 = 0$ and we vary $k = 1/2, 2, 5, 10, 20$ (red, dark blue, light blue, green and yellow, respectively). We clearly see in the evolution $w$ how the steepness of the transition depends on $k$ and that $c_0^2$ has a bump at $z_t$ and it is more prominent for steeper transition. This is generic behavior and we could expect to see a signature of the transition in large scale structure as discussed in section 6.
In figure 5 we show the evolution of $W_f$, $w$, and $c^2_a$ and we take $\xi = 1$, $z_t = 2$ fixed and $(k, L_1) = (2, 6), (20, 12), (20, 100), (20, 1000), (20, 0)$ (red, dark blue, light blue, green and yellow, respectively) and $\xi = 0$ with $z_t = 0.1$, $q = 10$, $L_1 = 12$ (pink-dashed line). In this case we vary $L_1$ and we see that for large $L_1$ the EoS $w$ becomes bigger and it may approach $w \approx 1$ (e.g. green line). Of course this case is not phenomenologically viable but we plot it to show the distinctive cases of our $w$ parametrization. Once again, a steep transition gives a bump in $c^2_a$. The pink-dashed line shows how $w$ can increase at low $z$ and then approach $w = -1$.

We have seen that our parametrization gives a wide class of $w$ behavior, with increasing and decreasing $w$. From the observational data we should be able to fix the parameters of $L$ in equation (51) and we could then have a much better understanding on the underlying potential $V(\phi)$.

6. Perturbations

Besides the evolution of the homogenous part of DE $\phi(t, x)$, its perturbations $\delta \phi(t, x)$ are also an essential ingredient in determining the nature of DE. We work in the synchronous gauge and the linear perturbations with a line element $ds^2 = a^2(-dt^2 + (\delta_{ij} + h_{ij})dx^idx^j)$, where $h$ is the trace of the metric perturbations [19, 21]. Here in section 6 a dot represents derivative with respect to conformal time $\tau$ and $H = \dot{a}/a = (da/d\tau)/a$ is the Hubble constant w.r.t. $\tau$, while $\dot{H} = (da/dt)/a$.

6.1. Scalar field perturbations

For a DE given in terms of a scalar field, the evolution requires the knowledge of $V$ and $V'$, while the evolution of $\delta \phi(t, x)$ depends also on $V''$ through [19, 21]

$$\delta \ddot{\phi} + 2H\delta \dot{\phi} + [k^2 + a^2V'']\delta \phi = -\frac{1}{2}h\phi. \quad (62)$$

Equation (62) can be expressed as a function of $a$ with $\ddot{Y} = a\dot{H}_a$ for $Y = \delta \phi, \phi, H, h$ and the subscript $a$ means derivative w.r.t. $a$ (i.e. $Y_a \equiv dY/da$), giving

$$\delta \phi_{aa} + \left(\frac{3}{a} + \frac{H_a}{H}\right)\delta \phi_a + \left[\frac{k^2}{a^2H^2} + \frac{V''}{H^2}\right]\delta \phi = -\frac{1}{2}h_a\phi_a. \quad (63)$$

In the slow roll approximation we have

$$\left|\frac{V''}{3H^2}\right| = \Gamma x < 3 \quad (64)$$

where we have used equation (21) and $\Gamma \equiv \frac{V'V}{V'^2}$ given in equation (49).

Equation (64) implies that an EoS of DE between $-1 \leq w \leq -1/3$, $0, 1/3$, with $0 \leq x \leq 1/2, 1, 2$, requires $\Gamma < 3/x = 6, 3, 3/2$, respectively. For a scalar field $\phi$ to be in the tracking regime one requires $\Gamma$ to be approximately constant with $\Gamma > 1$ [13]. Therefore the regime $1 < \Gamma < 3/x$ allows a tracking behavior satisfying also the slow roll approximation. Here we are more interested in the late time evolution of DE and the tracking regime is not required; and in fact we expect deviations from it. However, if $\Gamma$ is nearly constant, the evolution of the perturbations in equation (63) are then given only in terms of $x$ and we can use our DE parametrization in equations (51)–(53) to calculate them.
We can express the slow roll parameter \( \epsilon \), \( \eta \) in terms of \( \Gamma \) and \( L \) in the limit \( q \ll 1 \) as

\[
\epsilon \equiv \frac{1}{2} \left( \frac{V''}{V} \right)^2 = \frac{\lambda^2}{2}, \quad \eta \equiv \frac{V''}{V} = \Gamma \lambda^2.
\]  

(65)

We have decided to use \( L, \eta \) instead of \( \epsilon, \eta \) in order not to confuse the reader with the inflation parameters and the DE ones.

### 6.2. Fluid perturbations

The evolution of the energy density perturbation \( \delta = \delta \rho/\rho, \theta = k (1 + w) v \), with \( v \) as the velocity perturbation, is [19–21, 23]

\[
\dot{\delta} = -(1 + w)(k^2 + 9H^2) [c_s^2 - c_a^2] \frac{\theta}{k^2} - \frac{\dot{h}}{2} - 3H(c_s^2 - w) \frac{\delta}{1 + w}
\]

(66)

\[
\dot{\theta} = -H(1 - 3c_s^2) \theta + c_s^2 k^2 \frac{\delta}{1 + w},
\]

(67)

and we do not consider an anisotropic stress. The evolution of the perturbations depends on three quantities [21, 23]

\[
w = \frac{p}{\rho}
\]

(68)

\[
c_{a}^2 = \frac{\dot{p}}{\dot{\rho}} = w + \frac{\dot{\rho}}{\dot{\rho}}
\]

(69)

\[
c_{s}^2 = \frac{\dot{\rho} - \dot{\rho}}{3H(1 + w)} = w + \frac{\dot{\rho} - \dot{\rho}}{3H(1 + w)}
\]

(70)

where \( w \) is the EoS, \( H \) the Hubble constant in conformal time, \( c_a^2 \) is the adiabatic sound speed and \( c_s^2 \) is the sound speed in the rest frame of the fluid [20, 21]. For a perfect fluid one has \( c_s^2 = c_a^2 \) but scalar fields are not perfect fluids. The entropy perturbation \( G_i \) for a fluid \( \rho_i \) with \( \hat{\delta}_i = \delta \rho_i/\rho_i \) is

\[
w_i G_i \equiv (c_{a}^2 - c_{a}^2) \delta_i = \frac{\dot{p}_i}{\dot{p}_i} \left( \frac{\delta \rho_i}{\rho_i} - \frac{\delta \rho_i}{\rho_i} \right)
\]

(71)

where the quantities \( G_i \) and \( c_{a}^2 \) are scale independent and gauge invariant but \( c_s^2 \) can be neither [19, 23]. In its rest frame a scalar field \( \phi \) with a canonical kinetic term has \( c_s^2 = \delta p/\delta \phi = 1 \) [20, 21]. One can relate the rest frame \( \delta, \theta \) to an arbitrary frame \( \hat{\delta}, \hat{\theta} \) by [23]

\[
\hat{\delta} = \delta + 3H(1 + w) \frac{\theta}{k^2}
\]

(72)

and

\[
\delta \rho = \dot{c}_s^2 \delta \rho + (\dot{c}_s^2 - \dot{c}_s^2) 3H(1 + w) \rho_i \frac{\theta}{k^2}.
\]

(73)
As we see from equations (66) and (67) the evolution of $\dot{c}_s^2$ depends on $c_s^2$, $c_t^2$ and $w$. Using equation (68) and since $w$ is a function of $x(a)$ we have

$$c_s^2 = w + \frac{x_w w_t}{3a(1 + w)}$$

(74)

with $w_t/(1 + w_t) = 2/[x(1 + x)]$. From equations (23) and (21) we can express $c_s^2$ as a function of the parameters of $x$.

DE perturbations are important in distinguishing different DE models [21–25], and in the epoch where the Universe is dominated by DM and DE the total perturbation is $\delta T = \delta p_T/\rho_T = \Omega_{DM}\delta\rho_{DM} + \Omega_{DE}\delta\rho_{DE}$. If $\Omega_{DE}$ is not much smaller than $\Omega_{DM}$, i.e. at an epoch with low redshift $z$, then the perturbations of DE have an important contribution to $\delta T$, For a scalar field $c_s^2 = w$ and $c_t^2 = 1$ [20, 21] we see from equations (66) and (67) that a bump in $c_s^2$ may give a significant contribution to the evolution of $\delta\rho_{DE}$ depending on the value of $\Omega_{DE}$ (i.e. the redshift of the transition) and the steepness of the bump [24, 25].

Finally, we can relate $V''$ in terms of the adiabatic sound speed $c_s^2$ in equation (69) and its time derivative, using $c_s^2 = \rho/\rho_T = 1 + 2V/H(d\phi/dt)$, giving

$$\frac{dc_s^2}{dr} = (c_s^2 - 1)\left(\frac{V''}{V} - \frac{3H}{2}\left(\frac{2dH}{3H^2} - (c_s^2 + 1)\right)\right)$$

(75)

and for $c_s^2 = 1$ we can invert equation (75) to give

$$\frac{V''}{V'} = \frac{1}{(c_s^2 - 1)}\frac{dc_s^2}{dr} - \frac{3H}{2}(w_T + c_s^2 + 2),$$

(76)

where we have used $H = dH/dr = -(\rho_T + p_T)/2 = -3H^2(1 + w_T)/2$ with $\rho_T$, $p_T$, $w_T$ the total energy density, pressure and EoS, respectively. In our case we have $\rho_T = \rho_m + \rho_\phi$, $p_T = p_m + p_\phi = p_\phi$ and using $\rho_\phi = V(1 + x)$ and equations (11) and (16) we have

$$w_T \equiv \frac{p_T}{\rho_T} = w\Omega_\phi = \frac{w\rho_\phi}{3H^2} = \frac{w(1 + x)}{1 + x + y} = \frac{x - 1}{1 + x + y}.$$  

(77)

With equation (77) the left-hand side of equation (76) depends then only on $y$, $x$ and is fully determined by our parametrization. In the full slow roll approximation $\dot{\phi} = 0$ and one has $c_s^2 = -1$.

7. Conclusions

We have presented a new parametrization of DE motivated by the dynamics of a canonical normalized scalar field minimally coupled to gravity. Our parametrization has allows for $w(z)$ to have a wide class of behaviors in which it may grow and later decrease or decrease and later grow. The EoS $w(x(y, L))$ in equation (23) is given in terms of the functions $L$ and $y$ and it is an exact equation, valid also when the slow roll approximation is not satisfied. The EoS $w$ is constrained between $-1 \leq w \leq 1$ for any value of $x$, with $0 \leq x$ by definition. The parametrization proposed is given in equations (51)–(53), with $L = L_0 + L_1 y^2 f(z)$ and a transition function $f(z) = \frac{1}{1 + (c_t^2/2)}$. The EoS $w$ has four free parameters: $L_0$, $L_1$, $z_t$, $k$, which can be expressed in terms of $w_0$ and the EoS at an early time $w_i = w(z \gg z_t)$, given by $L_0$ and $L_1$, respectively (c.f. equation (55) and (57), while $z_t$ gives the transition redshift between $w_0$ and $w_i$ and $k$ sets the steepness of the transition. Besides studying the evolution of DE we also determined its perturbations from the adiabatic sound speed $c_s^2$ and $c_t^2$ given in equations (69)
and (70), which are functions of $x$ and its derivatives. We have seen that a steep transition has a bump in $c_2^a$ and this should be detectable in large scale structure if it takes place at late times.

We can use the parametrization of $x$ ($L$, $y$) in equations (35), (51) and (52) and $c_2^a$ and $c_2^s$ in equations (69) and (70) without any reference to the underlying physics, namely the dynamics of the scalar field $\phi$, and the parametrization is well defined. However, it is when we interpret $x = \dot{\phi}^2/2V$ and $L = (V'/V)^2A$ and $y = \rho_m/V$ that we are analyzing the evolution of a scalar field $\phi$ and we can connect the evolution of $w$ to the potential $V(\phi)$, once the free parameters are phenomenologically determined by the cosmological data. The slow roll approximation is when we take $|q| \ll 1, A \simeq 1$.

To conclude, we have proposed a new parametrization of DE which has a rich structure, and the determination of its parameters will help us to understand the dynamics of DE.

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Appendix A. Dynamical evolution of $x$, $y$ with $\zeta \equiv -\frac{V'}{V}$

The dynamical equations (41) and (42) can also be written in terms of the potential and its derivative $V_N/V$, $V_N \equiv dV/dN$ as

\[
x_N \equiv \frac{dx}{dN} = -\left(6x + (1 + x) \frac{V_N}{V}\right)
\]

\[
y_N \equiv \frac{dy}{dN} = -y\left(3 + \frac{V_N}{V}\right).
\]

From equations (A1) we clearly see that the behavior of $x$ and $y$ is completely determined by a single function $V_N/V$ and we define

\[
\zeta \equiv -\frac{V_N}{V} = -\frac{V'\phi_N}{V} = \lambda(\phi) \phi_N
\]

with $\lambda = -V'/V$. It is easy to see from equations (41), (42) and (A1) that the function $\zeta$ is given in terms of $x$ and $q$ as

\[
\zeta = -\frac{V_N}{V} = \frac{6x}{1 + q}.
\]

In the case where the scalar field rolls down to its minimum we will have $\dot{V} = V'\phi_N$ negative giving a $\zeta > 0$. In the simple case of having $\zeta$ constant, the solution to equations (A1) are simply

\[
x = x_0 e^{-6(\zeta(N-N_t))} + \left(\frac{\zeta}{6 - \zeta}\right)(1 - e^{-6(\zeta(N-N_t))})
\]

\[
y = y_0 e^{-(3-\zeta)(N-N_t)}
\]

with initial conditions $x = x_0$, $y = y_0$ at $N = N_t$. The asymptotic values of $x$ and $y$ depend on the value of $\zeta$. The critical points of the system in equations (A1) give for $x_N = 0$ a solution $q_1 x$ and
\[ \zeta = -\frac{V_N}{V} = \frac{6x}{1 + x} \]  
(A6)

and if we invert equation (A6) we simply get

\[ x = \frac{\zeta}{6 - \zeta} \]  
(A7)

and an EoS

\[ w = \frac{x - 1}{x + 1} = -1 + \frac{\zeta}{3}. \]  
(A8)

Using equation (A8) we have \( 3 - \zeta = -3w \) and equation (A5) then becomes

\[ y = y_i e^{3w(N - N_i)} = y_i \left( \frac{a_i}{a} \right)^{3w} \]  
(A9)

which is valid for \( \zeta \) constant. We clearly see that the value of \( \zeta \) determines the asymptotic behavior of \( x, y \) and of the scalar field energy density \( \rho_0 \). If \( \zeta < 3 \) the scalar field will dominate since \( w < 0 \) and we will have \( y \to 0, \Omega_\phi \to 1 \) and \( x = \zeta/(6 - \zeta) < 1 \), while for \( \zeta = 3 \) we have \( x = 1, w = 0 \) and \( y \) constant, given in equation (A14). Finally, for \( \zeta > 3 \) we get \( x > 1, w > 0 \) and \( y \to \infty \) with \( \Omega_\phi \to 0 \). Of course, an accelerated Universe requires \( w < 0, x < 1 \) and therefore \( \zeta < 3 \) close to present time. However, in general the value of \( \zeta \) will be time (or \( N \)) dependent.

A.1. \( \zeta \) as a function of \( x \) and \( y \)

The function \( \zeta \) is a function of \( \phi \) and \( \phi_N \), as we can see from equation (A2) and we can express \( \phi_N \) in terms of \( x, y \) as

\[ \phi_N = \text{sign}[\dot{\phi}] \sqrt{\frac{6x}{1 + x + y}} \]  
(A10)

(for a rolling scalar field \( \dot{\phi} > 0 \) but we have kept the term \( \text{sign}[\dot{\phi}] \) for completeness),

\[ \zeta = -\frac{V_N}{V} = \text{sign}[\dot{\phi}] \lambda \sqrt{\frac{6x}{1 + x + y}} \]  
(A11)

and the equations (A1) now read

\[ x_N = -6x + \text{sign}[\dot{\phi}] (1 + x) \lambda \sqrt{\frac{6x}{1 + x + y}} \]  
(A12)

\[ y_N = -3y + \text{sign}[\dot{\phi}] y \lambda \sqrt{\frac{6x}{1 + x + y}}. \]  
(A13)

The amount of \( \Omega_\phi \) can be determined at the critical point \( x_N = y_N = 0 \), giving \( x = 1, w = 0 \) (i.e. the scalar field redshifts as the barotropic fluid, which in our case is matter \( w_m = 0 \)) and

\[ y = \left( \frac{\lambda^2}{\zeta} - 1 \right) \left( \frac{6}{6 - \zeta} \right) = 2 \left( \frac{\lambda^2}{3} - 1 \right). \]  
(A14)

\[ \Omega_\phi = \frac{\zeta}{\lambda^2} = \frac{3}{\lambda^2}. \]  
(A15)
This solution is valid for $\lambda^2 > 3$ and has $\zeta = 3$. The critical point for $\lambda^2 < 3$ has
\[ x = \frac{\lambda^2}{6 - \lambda^2}, \quad y \to 0, \quad \Omega_\phi \to 1. \quad (A16) \]

**Appendix B. Dynamical system for $\ddot{x} \equiv \phi/6H$ and $\ddot{y} \equiv \sqrt{V/3H}$**

As a matter of completeness we also present here the dynamical evolution equations in terms of the variables $\dot{x} \equiv \phi/\sqrt{6H} = \phi_N/\sqrt{6}$ and $\dot{y} \equiv \sqrt{V/3H}$, which have been widely used in the literature for studying the dynamics of scalar fields [14]. In this case the dynamical equations are given by
\[ \dot{N} = -3 \dot{x} + \frac{\sqrt{3}}{2} \lambda \dot{y}^2 - \dot{x} \frac{H_N}{H}, \]
\[ \ddot{N} = -\frac{\sqrt{3}}{2} \lambda \ddot{x} - \dot{y} \dot{H}_N \frac{H}{H}, \]
\[ \frac{H_N}{H} = -\frac{3}{2} [1 + \dot{x}^2 - \dot{y}^2] \quad (B1) \]
valid for a flat Universe with a scalar field and matter. These equations depend only on the function $\lambda(\phi(N)) = -V'/V$. The variables $x, y$ in equations (10), (14) and (A2) are related to $\dot{x}, \dot{y}$ by
\[ x \equiv \frac{\dot{\phi}^2}{2V} = \frac{\dot{x}^2}{\dot{y}^2}, \quad y \equiv \frac{\rho_m}{V} = \frac{1 - \dot{x}^2 - \dot{y}^2}{\dot{y}^2}, \quad (B2) \]
and
\[ \zeta = -V_N/V = \lambda \sqrt{6} \dot{x}. \quad (B3) \]

We can write
\[ \ddot{x} \equiv \frac{\dot{\phi}}{\sqrt{6H}} = \text{sign}[\dot{\phi}] \sqrt{\frac{x}{1 + x + y}}, \quad (B4) \]
\[ \ddot{y} \equiv \sqrt{\frac{V}{3H}} = \sqrt{\frac{1}{1 + x + y}} \quad (B5) \]
with $\Omega_\phi = \dot{x}^2 + \dot{y}^2$. We can easily verify that $x_N = 2\lambda \left( \frac{\dot{x}}{x} - \frac{\dot{y}}{y} \right)$ in equation (41) is equivalent to equations (B1). The critical points of equations (B1) have been determined in [14] as a function of $\lambda$ giving for $\lambda^2 > 3$
\[ \ddot{x} = \frac{3}{2\sqrt{2\lambda^2}}, \quad \ddot{y} = \frac{3}{2\sqrt{2\lambda^2}}, \quad \Omega_\phi = \frac{3}{\lambda^2} \quad (B6) \]
and therefore $x = \dot{x}^2/\dot{y}^2 = 1, w = 0$ with $\zeta = \lambda \sqrt{6} \dot{x} = 3$ as in equation (A7). On the other hand, if $\lambda^2 < 3$, then one finds
\[ \ddot{x} = \sqrt{\frac{\lambda^2}{6}}, \quad \ddot{y} = \sqrt{1 - \frac{\lambda^2}{6}}, \quad \Omega_\phi = 1 \quad (B7) \]
giving

\[ x = \frac{\dot{x}^2}{y^2} = \frac{\lambda^2}{6 - \lambda^2}, \quad w = -1 + \frac{\lambda^2}{3}, \quad \zeta = \lambda^2 \]  

(B8)

with \( \lambda \sqrt{6} \dot{x} = \lambda^2 \) and we recover \( x \) as in equation (A7). A Universe with a late time acceleration has then \( \zeta = \lambda^2 < 3 \), which corresponds to the slow roll approximation.

### Appendix C. Dynamics of \( q \) and \( L \)

The parameter \( |q| = |\ddot{\phi}/V'| \) is clearly smaller than one in the slow roll regime \( (\ddot{\phi} < 3H \dot{\phi} \approx V') \). Let us now determine the dependence of \( q \) on the potential \( V(\phi) \) and its derivatives. The evolution of \( q \) is

\[ \dot{q} = HqN = \frac{\dddot{\phi}}{V'} - \frac{\ddot{\phi} \dot{\phi}}{V'} \]  

(C1)

\[ = 3H \left( -q + (1 + q) \frac{\dot{H}}{3H^2} + 2\dot{x} \right) \]  

(C2)

where we used equation (17),

\[ \frac{\ddot{\phi}}{V'} = \frac{V'' \dot{\phi}}{V'} - \frac{3H \ddot{\phi}}{V'} - \frac{3H^2 \dot{\phi}^2}{V'} \]

\[ = -3Hq + 3H (1 + q) \left( \frac{V''}{9H^2} + \frac{\dot{H}}{3H^2} \right). \]  

(C3)

and

\[ \frac{V''}{9H^2} = \frac{\Gamma x}{3}, \quad \Gamma \equiv \frac{V''V}{V'^2}. \]

We can estimate the value of \( q \) if we drop the term proportional to \( \ddot{\phi} \) in equation (C3) giving

\[ q \simeq \frac{\dddot{\phi}}{V'} + \frac{\ddot{\phi} \dot{\phi}}{V'^2} \cdot \frac{\dddot{\phi}}{V'} \]

(C6)

In a stable evolution of \( \phi \) we have a positive \( V'' \) and since \( \dot{H} \) is negative, both terms have opposite signs, but of course we do not expect a complete cancelation of these terms. However, both of them are smaller than one, since \( 0 \leq -\dot{H}/3H^2 < 1/2 \) for \( x < 1 \) and \( |V''/9H^2| = \Gamma x/3 < 1/3 \) in the slow roll approximation. A tracker behavior requires \( \Gamma > 1 \) [13] and \( x < 1/\Gamma < 1 \). Finally, the evolution of \( L \) is given by

\[ L = H L_N = 2\lambda \dot{\lambda}(1 + q)^2 + \lambda^2 \ddot{q}(1 + q) = 2L[\lambda/\lambda + \dot{q}(1 + q)], \]

\[ = \frac{12HLx(1 - \Gamma)}{(1 + q)} + \frac{2L\dot{q}}{1 + q} = 3HL \left( \frac{2x - q}{1 + q} + \frac{\dot{H}}{3H^2} \right) \]

\[ = 3HL \left( \frac{2(q - x)(1 + 2x + 2y) - y(1 + q)}{2(1 + q)(1 + x + y)} \right). \]  

(C7)

We see that at \(-1 < q < x \) we have \( L < 0 \) giving a decreasing \( L \) as a function of time or an increasing \( L \) as a function of \( \zeta \). For \((q - x)/(1 + q) > y/(1 + 2x + 2y) \) or equivalently for
\( q > (y + 2x(1 + 2x + 2y))/(2 + 4x + 3y) \) we have \( \dot{L} > 0 \) and a decreasing \( L \) as a function of \( z \).

Instead of choosing a DE parametrization as in equation (51) we could solve equations (C1) and (C7) for different potentials \( V(\phi) \) or by taking different approximated solutions or ansatze for \( \Gamma \). However, we choose to directly parametrize \( L \) as in equation (51). Still using \( L = aL_o = aHL_o \) and \( L_A^2 f = L - L_o \) we identify

\[
(L - L_o) \left(k f \left( \frac{a}{a_i} \right)^k - 3\xi w_o \right) = 3L \left( \frac{2x - a}{1 + q} + \frac{\dot{H}}{3H^2} \right)
\]

and the choices of \( \Gamma \) and \( q \) would fix the parameters \( L_o, k \) and \( a_i \).

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