I had originally intended to give a talk on homological reduction of first class constrained Hamiltonian systems, as in my joint work with Henneaux, Fisch and Teitelboim [FHST]. Since the organizers have given me the "honor" of opening the conference, I will attempt to set that work in a larger context, namely that of ghost techniques in mathematical physics.

What are 'ghosts' and what are they doing in physics? The name reflects the fact that they are new, auxiliary variables that are NOT physical, but are added to the system for computational reasons. An analogy familiar to many mathematicians is that of a resolution in homological algebra - the generators added to construct the resolution are the analogs of ghosts. Indeed it is more than an analogy in some cases; I first became seriously interested in the subject when I read a preprint of Browning and McMullan [BM] in which certain 'anti-ghosts' were clearly identified as generators of the Koszul resolution of an appropriate ideal.

But I am getting ahead of my story, both conceptually and historically. My intention this morning is to set the stage for a set of techniques and results which can be grouped under the rubric of 'cohomological physics', particularly BRST cohomology.

It will be easier to illustrate why BRST cohomology is of interest than it will be to say what it is. The acronym BRST has come to be applied in mathematical physics very widely; at times one gets the feeling it could be applied any time one has an operator of square zero (called in physics 'nilpotence'), but I will try to hold back the sea and restrict the term somewhat.

First, BRS refers to Becchi, Rouet and Stora who in 1975 [BRS] called attention to the "so-called Slavnov identities which express an invariance of the Fadeev-Popov Lagrangian". The T refers to Tyutin who, at about the same time [Ty] had a preprint on the same subject - the symmetry revealed is that of gauge transformations.

In pursuit of quantization of certain problems in gauge field theory, Fadeev and Popov had modified certain Lagrangians by introducing new non-physical variables
which they called ghosts. Becchi, Rouet and Stora and Tyutin discovered a transformation \( s \) which had a striking behavior on one of the Fadeev-Popov ghosts \( c \):

\[
sc = 1/2[c, c].
\]

Stora, among others, soon recognized the resemblance to the Maurer-Cartan form on a Lie group. Those were the days when the fibre bundle setting for gauge field theories was just becoming accepted (though implicit in the work of Dirac in 1931 (!) \([D2]\), full recognition occurred around 1975 in the interaction of Yang and Simons \([Y]\)).

The setting is this: we have a principal bundle

\[
P \downarrow \quad M
\]

Here we invoke the now common dictionary between physics and mathematics:

- vector potential = connection \( A \)
- field strength = curvature \( F = dA + 1/2[A, A] \)
- matter field = section of an associated vector bundle.

From a Lagrangian field theory point of view, the connection \( A \) is treated as a variable, so the action \( S \) is a function on \( A \), the space of connections on \( P \). The Yang-Mills functional, for example,

\[
YM(A) = \int_M |F|^2 dvol : A \to \mathbb{R}
\]

is in fact constant under changes in \( A \), known as gauge equivalences. It is hard to recall now, but in those days it was not so clear what was "the group of gauge transformations".

By 1982, the gauge transformation group had been identified clearly as the group \( G \) of vertical automorphisms of \( P \) as a principal bundle. In a seminal paper, Bonora and Cotta-Ramusino \([BCR]\) identified the BRS transformation with the standard Cartan-Chevelley-Eilenberg coboundary for the Lie algebra cohomology of the gauge algebra \( \text{Lie} \; G \) with appropriate coefficients. In that setting, the Fadeev-Popov ghosts can be identified with elements of a weak dual of the Lie algebra \( \text{Lie} \; G \) of \( G \), namely the space of sections of the bundle with fibre \( g = \text{Lie} \; G \) associated to \( P \) via the adjoint action. (If \( P = M \times G \) is the trivial bundle, then \( \text{Lie} \; G \) can be identified with \( \text{Map}(M, g^*) \).)

Meanwhile, also around 1975, Fradkin and Vilkovisky \([FV]\) initiated a different approach to quantization in the Hamiltonian setting. They started with a phase space (= symplectic manifold \( W \), e.g. the cotangent bundle \( T^*A \)) and constraints \( \phi_\alpha : W \to \mathbb{R} \). Via the symplectic structure, these corresponded to Hamiltonian vector fields which were assumed tangent to and foliating the constraint ‘surface’...
the zero locus of the constraints. They proceeded by adjoining ‘ghosts’, i.e. Grassmann algebra generators, to the Poisson algebra of smooth functions on $W$ and defining an operator $D$ on the extended algebra such that $D^2 = 0$. The operator $D$ contained a piece which was the Chevalley-Eilenberg differential $d$ used in defining Lie algebra cohomology. At least in nice cases, the resulting cohomology in degree zero gave the algebra of functions on the reduced phase space.

By 1977, BFV (Batalin, Fradkin and Vilkovisky) saw their work as a variant of BRST. Since then the term ‘BRST’ has been applied to an ever increasing range of situations - primarily quantum but also classical - in which there is an operator $Q$ of square zero and hence ‘BRST cohomology’ $H = \ker Q / \im Q$. I would like to restrict the term to situations in which $Q$ contains a piece which is specifically of Cartan-Chevalley-Eilenberg type. This still includes one of the most important variants in the infinite dimensional case - the semi-infinite cohomology of Feigin [F].

Although many of the variants of BRST cohomology have been developed for use in quantum field theory, they do have classical analogs which are of considerable interest in their own rite and help to reveal the relevance of cohomology, both classical and quantum. I will focus on one particular example, the Batalin-Fradkin-Vilkovisky approach to first class constrained Hamiltonian systems. These systems are particularly appropriate for this conference, their structure as mathematics has become particularly clear and I can present them as more than a spectator. I will present only the classical aspects. Since these can be expressed entirely in terms of differential algebra, although of a new kind, algebraic techniques for quantization apply quite well; see in particular work of Huebschman [Hu] and also of Figueroa-O’Farrell and Kimura [FK][Ki] for the latest results and excellent expositions of the subject. After presenting the classical aspects in some detail, I will sketch several other related areas of what Witten has recently blessed with the name ‘cohomological field theory’ [Wi].

As mentioned, the Hamiltonian setting refers to functions on a symplectic manifold, proto-typically a cotangent bundle. For present purposes, however, it is sufficient to consider a Poisson algebra, the formal algebraic object modeled on the algebra of smooth functions on a symplectic manifold. In light of the mixed audience today, let me present a tri-lingual dictionary:
| PHYSICS | GEOMETRY | ALGEBRA |
|---------|----------|---------|
| Hamiltonian system | Differential system on symplectic $W$ e.g. $T^*M$ | |
| Fields with Poisson bracket | Functions on $W$ with Poisson bracket | Poisson algebra $P$ |
| Constraints $\{\varphi_\alpha\}$ $\varphi_\alpha \in \mathcal{C}^\infty(W)$ | $\phi : W \to \mathbb{R}^N$ | Ideal $I \subset P$ |
| Constraint surface | $V = \phi^{-1}(0) \subset W$ | $P/I$ |
| $f \approx g$ (weakly equal) | $f\vert V = g\vert V$ | $f \equiv g \mod I$ |
| Symmetries $\{\varphi_\alpha,\}$ | Hamiltonian v.f. $X_{\varphi_\alpha}$ | ad action of $I$ |
| $1^{st}$ Class: | $X_{\varphi_\alpha}$ tangent to $V$ | $I$ closed under $\{,\}$ |
| structure functions | and foliating | Lie algebra over $\mathbb{R}$ |
| true degrees of freedom | reduced phase space $V/\mathfrak{g}$ | $(P/I)^{1-\text{invariant}}$ |

**Definition.** A Poisson algebra $P$ over a field $k$ is a vector space $P$ over $k$ with two operations, denoted respectively by $f, g \to fg$ and $f, g \to \{f, g\}$, satisfying the following three conditions:

1. the product $fg$ makes $P$ an associative algebra;
2. the bracket $\{f, g\}$ makes $P$ a Lie algebra;
3. the two are related by a Leibnitz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$
   (otherwise said: $\{f, \}$ acts as a derivation of the associative algebra $P$.)

It is common to assume that $P$ is in fact commutative, i.e. $fg = gf$, and we shall in fact do so, but the additional generality as stated is appropriate in light of both quantization and current interest in non-commutative geometry.

A typical Hamiltonian system consists of differential equations of the form $\{f, H\} = \frac{df}{dt}$ where $H$ is a fixed function on the manifold $W$. In some physical problems, solutions are sought which are constrained to lie on a sub-manifold $V \subset W$. As in algebraic geometry, we can think of $V$ as the zero set of some functions $\phi_\alpha : W \to \mathbb{R}$.
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Stasheff called constraints. The algebra of functions $C^\infty$-in-the-sense-of-Whitney on $V$ can be identified with $C^\infty(W)/I$ where $I$ is the ideal of functions which vanish on $V$. We restrict attention to situations in which the $\phi_\alpha$ generate $I$. Example: Zero angular momentum.

$$W = \mathbb{R}^2 \times \mathbb{R}^2$$

$$(P, Q) = (q^1, q^2) \times (p_1, p_2)$$

$$\phi = P \otimes Q = p_1 q^2 - p_2 q^1.$$ 

Now if $W$ is symplectic (or just given a Poisson bracket on $C^\infty(W)$), Dirac [D1] calls the constraints first class if $I$ is closed under the Poisson bracket. (If the $\mathbb{R}$-linear span $\Phi$ of the $\phi_\alpha$ is closed under bracket, physicists say the $\phi_\alpha$ close on a Lie algebra; this is a very nice case, but the more general FIRST CLASS case in which the ideal $I$ is closed but $\Phi$ is not is where homological techniques are really important.) When the constraints are first class, we have that the Hamiltonian vector fields $X_{\phi_\alpha}$ determined by the constraints are tangent to $V$ (where $V$ is smooth) and give a foliation $\mathcal{F}$ of $V$. Similarly, $C^\infty(W)/I$ is an $I$-module with respect to the bracket. (In symplectic geometry, the corresponding variety is called coisotropic [We].)

An example of the special case that is particularly relevant to this conference is that of an equivariant moment map [AGJ]. Here we are given a Lie group $G$ acting on $W$ by symplectic diffeomorphisms (symplectomorphisms) and an equivariant moment map

$$J = \Phi : W \to g^*,$$

equivariant with respect to the coadjoint action of $G$ on $g^*$, the dual of $g$, the Lie algebra of $G$. In the physics literature, it is common to choose a basis $\{T^\alpha\}$ and to write $\phi(w) = \phi_\alpha T^\alpha$.

In many cases of interest, $I$ does not arise as the Lie algebra of some Lie group of transformations of $W$ or even $V$, but the corresponding Hamiltonian vector fields $X_{\phi_\alpha}$ are still referred to as (infinitesimal) symmetries. In the nicest case, e.g. when the foliation $\mathcal{F}$ is given by a principal $G$-bundle structure on a smooth $V$, the algebra $C^\infty(V/\mathcal{F})$ can be identified with the $I$-invariant sub-algebra of $C^\infty(W)/I$. In great (if not complete) generality, this $I$-invariant sub-algebra represents the true observables of the constrained system. Sniatycki and Weinstein [SW] have defined an algebraic reduction in the context of group actions and momentum maps which is guaranteed to produced a reduced Poisson algebra but not necessarily a reduced space of states. The S-W (Sniatycki and Weinstein) reduced Poisson algebra is $(C^\infty(W)/I)^G$ where $V = J^{-1}(0)$ for some equivariant moment map $J : W \to g^*$. (If $G$ is compact, $(C^\infty(W)/I)^G$ is isomorphic to the Dirac reduction $C^\infty(W)^G/I^G$. ) With hindsight, the generalization of S-W reduction to a general FIRST CLASS constraint ideal $I$ is obvious. The issue of its suitability is not one of geometry necessarily, but rather one of physics.

Now - where are the ghosts? Instead of considering just the “observable” functions, one can consider the deRham complex of longitudinal or vertical forms of the foliation $\mathcal{F}$, that is, the complex $\Omega(V, \mathcal{F})$ consisting of forms on vertical vector...
fields. In local coordinates \((x^1, ..., x^{r+s})\) with \((x^1, ..., x^r)\) being coordinates on a leaf, a typical longitudinal form is

\[ f_I(x)dx^J \quad \text{where} \quad J = (j_1, ..., j_q) \quad \text{with} \quad 1 \leq j_1 < ... j_q \leq r, \quad \text{the leaf dimension}. \]

Another description of \(\Omega(V, F)\) is in terms of alternating functions of vertical vector fields which are multi-linear with respect to \(C^\infty(V)\). To become more fully algebraic, consider \(P\), an arbitrary Poisson algebra with an ideal \(I\) which is closed under the Poisson bracket. Reduction is then achieved by passing to the \(I\)-invariant subalgebra of \(P/I\). Note that a class \([g]\) is \(I\)-invariant if \([I, g] \subset I\), equivalently, if \(\{\phi, g\} \approx 0\) for all constraints \(\phi \in I\). This subalgebra inherits a Poisson bracket even though \(P/I\) does not: For \(f, g \in P\) and \(\phi \in I\), \(\{f + \phi, g\} = \{f, g\} + \{\phi, g\}\) where \(\{\phi, g\}\) need not belong to \(I\), but will if the class of \(g\) is \(I\)-invariant.

The fact that \(I\) is a sub-Lie algebra of \(P\) but is not a Lie algebra over \(P\) (the bracket is \(R\)-linear but not \(P\)-linear) is a significant subtlety. The pair \((P, I)\) is, however, a Rinehart algebra \([R]\) over \(R:\)

\(P\) is a commutative algebra over \(R,\)

\(I\) is a Lie algebra over \(R\) and a \(P\)-module,

\(\{\phi, \psi\}\) gives a representation \(\rho : I \to \text{Der } P\), the Lie algebra of derivations of \(P\), such that \(\{\phi, f\psi\} = (\rho(\phi)f)\psi + f\{\phi, \psi\}\) for \(\phi, \psi \in I, f \in P\).

Hence we can consider the Rinehart complex \(\text{Alt}_P(I, M)\) where \(M\) is a \(P\)-module with a representation \(\pi\) as a Lie module over \(I\) such that

\[ \pi(\phi)(fm) = f\pi(\phi)m + \pi(f\phi)m, \quad f \in P, \phi \in I, m \in M. \]

The underlying vector space of \(\text{Alt}_P(I, M)\) consists of the alternating \(P\)-multi-linear functions from \(I\) to \(M\). The differential \(d\) given by Rinehart is an obvious generalization of that of Cartan-Chevalley-Eilenberg:

\[ (dh)(\phi_0, ..., \phi_q) = \sum_{i<j}(-1)^{i+j}h(\{\phi_i, \phi_j\}, ..., \hat{\phi}_i, ..., \hat{\phi}_j, ...) + \sum_i(-1)^{i}\pi(\phi_i)h(..., \hat{\phi}_i, ...). \]

(In case, \(M = P = C^\infty(W)\) and \(I\) corresponds to vector fields on \(W\), the Rinehart complex is the de Rham complex of \(W\).)

When \(I\) is a subalgebra of FIRST CLASS constraints, \(P\) is not a \((P, I)\)-module via the adjoint action:

\[ \{\phi, fg\} = \{\phi, f\}g + f\{\phi, g\} \neq f\{\phi, g\} + \{f(\phi, g)\}, \]

but \(P/I\) is a \((P, I)\)-module via the adjoint action since \([f\phi, g] \equiv f\{\phi, g\} \mod I\). As remarked by Stephen Halperin, the Rinehart complex \(\text{Alt}_P(I, P/I)\) is, in this case, the complex \(\Omega^*(V, F)\) of longitudinal forms. (See [Hu] for further applications of Rinehart’s complex to Poisson algebras.)

In some special cases, what the physicists [BFV], [He], [BM] did was to construct a homological “model” for \(\Omega(V, F)\) in roughly the sense of rational homotopy theory [Su]. That is to say, a differential graded commutative algebra with a map to \(\Omega(V, F)\) giving an isomorphism in cohomology. The model was itself crucially a
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Poisson algebra extension of the Poisson algebra $P = C^\infty(W)$ and its differential contained a piece which reinvented the Koszul complex for the ideal $I$. The differential also contained a piece which looked like the Cartan-Chevalley-Eilenberg differential.

But still - where are the ghosts? The ghosts are easiest to describe and their meaning clearest in the case in which the ideal is regular. (At one time, regular ideals were known as Borel ideals.) This is an algebraic condition, but implied by $I$ being the defining ideal in $C^\infty(W)$ for $V = \phi^{-1}(0)$ when 0 is a regular value of $\phi : W \to \mathbb{R}^N$.

In order to construct a model of $\text{Alt}_P(I, P/I)$, we reverse the procedure of BFV and first provide a model for $P/I$ as a $P$-module. This model is a differential graded commutative algebra $(P \otimes \Lambda \Psi, \delta)$ where $\Psi$ is a graded vector space (in fact, negatively graded) and $\Lambda \Psi$ denotes the free graded commutative algebra on the graded vector space $\Psi$. (This grading is the opposite of the usual convention in homological algebra, but is chosen to correspond to the (anti-) ghost grading in the physics literature.) This model is constructed as follows in terms of a set of constraints (a more invariant description is given in HRCPA [S]): Let $\{\phi_\alpha\}$ be a regular sequence of constraints (physics: irreducible set), i.e., there are no relations of the form $f^i \phi_1 + \cdots + f^i \phi_i = 0$ for non-zero $f^i$ in $P$. Adjoin Grassmann variables, ghosts $\eta^\alpha$ and anti-ghosts $P_\alpha$ in 1-1 correspondence with the constraints. That is, form the graded commutative algebra $\Lambda \eta^\alpha \otimes P \otimes \Lambda P_\alpha$. Extend the Poisson bracket on $P$ to this new algebra by decreeing

$$\{\eta^\alpha, P_\beta\} = \delta^\alpha_\beta$$

and then apply the Leibnitz rule to determine the Poisson bracket on general monomials in the ghosts and anti-ghosts. Notice that we can interpret this bracket as follows: The span of the $\phi_\alpha$ is isomorphic to $\Phi$ and the span of the $\eta^\alpha$ is isomorphic to the dual, $\Phi^*$, so the bracket formula above is the usual symplectic structure on $\Phi^* \oplus \Phi$. The ghost degree is defined to be 1 for $\eta^\alpha$ and $-1$ for $P_\alpha$ and is extended to monomials ‘additively’, i.e. denoting ghost degree by $gh$, we have $gh(\omega_1 \omega_2) = gh \omega_1 + gh \omega_2$.

**Theorem.** (Batalin-Fradkin-Vilkovisky): There exists $Q \in \Lambda \eta^\alpha \otimes P \otimes \Lambda P_\alpha$ of ghost degree 0 such that $\{Q, Q\} = 0$. The operator $D = \{Q, \}$ satisfies $D^2 = 0$ and in ghost degree 0, we have $\text{Ker}D \approx (P/I)^{I\text{-invariant}}$.

Many years later, after reinterpretation by Henneaux [He] and then Browning and McMullan [BM], I was able to reinterpret the existence of $Q$ in terms of Homological Perturbation Theory.

First consider just the anti-ghost complex, $P \otimes \Lambda P_\alpha$ with the derivation $\delta$ defined by $\delta P_\alpha = \phi_\alpha$. Browning and McMullan recognized this as the Koszul complex for the ideal $I$ in the commutative algebra $P$ [K], [Bo]. The condition that the ideal $I$ is regular is equivalent to the Koszul complex being a model for $P/I$ or, in algebracists’ terms, a resolution of $P/I$. (For more general ideals, this fails, i.e., $H^i(P \otimes \Lambda P_\alpha, \delta) \neq 0$ for some $i \neq 0$. The Tate resolution [Ta] kills this homology by systematically enlarging the set of anti-ghosts, cf. [EHST] and [S3]).
On the other hand, the ghost complex $\Lambda \eta^\alpha \otimes P$ with the Chevalley-Eilenberg differential $d$ computes $H_{\text{Lie}}(I, P)$ so that $H^0(I, P) = I$-invariants of $P$. Similarly $\Lambda \eta^\alpha \otimes P/I, d$ computes the $I$-invariants of $P/I$ in degree 0.

The BFV differential $D$ looks like $d + \delta + \text{terms of higher order}$. To construct it as an inner derivation $D = \{Q, \cdot \}$ with $Q = Q_0 + Q_1 + \text{terms of higher order}$, start with $Q_0 = \eta^\alpha \phi_\alpha$. We then obtain the following formulas for the action of $Q_0$ on $P$ and on the ghost generators:

$$\{Q_0, f\} = \eta^\alpha \{\phi_\alpha, f\} = df$$
$$\{Q_0, \eta^\beta\} = 0$$
$$\{Q_0, \mathcal{P}_\beta\} = \delta_\beta^\gamma \phi_\alpha = \delta \mathcal{P}_\beta.$$

Since $I$ is closed under Poisson bracket,

$$\{\phi_\alpha, \phi_\beta\} = C^\gamma_{\alpha \beta} \phi_\gamma$$

where the $C^\gamma_{\alpha \beta}$ may be functions. Let $Q_1 = 1/2 \eta^\alpha \eta^\beta C^\gamma_{\alpha \beta} \phi_\gamma \phi_\gamma$ which then acts according to the formulas:

$$\{Q_1, f\} = 1/2 \eta^\alpha \eta^\beta \{C^\gamma_{\alpha \beta}, f\} \mathcal{P}_\gamma$$
$$\{Q_1, \eta^\gamma\} = 1/2 \eta^\alpha \eta^\beta C^\gamma_{\alpha \beta} = d \eta^\gamma$$
$$\{Q_1, \mathcal{P}_\epsilon\} = \eta^\alpha C^\gamma_{\alpha \epsilon} \mathcal{P}_\gamma \delta \mathcal{P}_\beta.$$

Thus $\{Q_0 + Q_1, \cdot\} = d + \delta + \text{stuff}$ where ‘stuff’ stands for the terms above which do not appear in $d + \delta$. Because of these extra terms, e.g. $\{Q_1, f\}$, we have $\{Q_0 + Q_1, Q_0 + Q_1\} \neq 0$. How can we add “terms of higher order” $Q_i$ so as to achieve $D^2 = 0$?

Here is the inductive step. Note $Q_0$ has one ghost and no anti-ghosts while $Q_1$ has two ghosts and one anti-ghost. Assume we have defined $Q_i$ with $i + 1$ ghosts and $i$ anti-ghosts and that

$$R_n := \sum_0^n Q_i$$

is such that $\{R_n, R_n\}$ is a sum of terms, each of which has at least $n + 1$ anti-ghosts ($\mathcal{P}'s$) and that $\delta \{R_n, R_n\}$ has terms with at least one more. Because $\delta$ is acyclic, there is a suitable $Q_{n+1}$. In fact, there is a contracting homotopy for $\delta$, i.e. a linear map $h : P \otimes \Lambda P \to P \otimes \Lambda P$ which raises the number of anti-ghosts by 1 such that $\delta h + h \delta = Id - \bar{\pi}$ where $\bar{\pi} : P \otimes \Lambda \Psi \to P \to P/I \hookrightarrow P \otimes \Lambda \Psi$ is given by $\pi$ composed with an $\mathbf{R}$-linear splitting $P/I \to P$. Having made one such choice, we can then systematically choose

$$Q_{n+1} = -1/2 h \{R_n, R_n\}.$$
When the ideal $I$ is not regular or the chosen set of constraints is reducible even though the ideal is regular, the Koszul complex is not a resolution of $P/I$. It can however be extended to the (Koszul)-Tate resolution $\mathbf{[Ta]}$, by adjoining alternately new even and odd variables, called in physics “anti-ghosts of anti-ghosts” etc. The Tate resolution again admits a contracting homotopy, so the construction of $Q$ proceeds as above. If the set of constraints is reducible but the ideal is regular (for example if the corresponding vector fields define the action of a Lie group $G$ but the orbits are in fact homogeneous spaces $G/H$), the result is again a model for the deRham complex of forms along the leaves. In essence, the extra constraints have led to a model containing a factor which is acyclic and so makes no contribution to the cohomology. When the ideal is NOT regular, the cohomology in degree zero is still isomorphic to the $I$-invariants of $P/I$, but the interpretation of the other groups is less clear. Even though the ideal is not regular, the zero locus $V$ of the constraints may still be a sub-manifold of $W$ and the corresponding Hamiltonian vector fields may give a true foliation (without singularities). That is the case considered in $\mathbf{[FHST]}$ where we show we again have a model for the deRham complex of forms along the leaves. Should the BFV complex have cohomology different from that of the deRham complex of forms along the leaves, it is an interesting question as to which cohomology is physically significant.

**Ghosts in the Lagrangian setting.** Problems in classical field theory are, if anything, more familiar in the Lagrangian setting. We start with an “action” $S_0 = S_0(\phi)$ where $\phi$ denotes one or several ‘fields’. Nowadays the word seems to indicate a function or section of some bundle $p : E \to M$. We seek solutions of a variational equation or system of equations

$$\delta S_0 := \delta_{\delta \phi} S_0 = 0.$$  

The point of view relevant to cohomological (ghost) techniques considers $\Sigma$, the space of all solutions as a subspace of the space $\mathcal{S} = \text{Sections} E$ of all fields. The following discussion of these techniques is essentially just an introduction to Marc Henneaux’s talk, which will provide a more thorough treatment.

As in the Hamiltonian setting, we begin with an algebra of functions, e.g. $C^\infty \mathcal{S}$, although this time just a commutative algebra, not a Poisson algebra. We ignore all difficulties associated with the infinite dimensional nature of $\mathcal{S}$ and proceed with an algebraic formalism. We approach the subspace $\Sigma$ via a Koszul complex.

Let $\phi^i$ be a (minimal) set of solutions of the variational equation or rather let $\phi^i$ be functions on $\mathcal{S}$ corresponding to a minimal set of solutions generating all solutions. Introduce a new set of Grassmann variables $\phi^*_i$ of ghost degree $-1$, called “anti-fields”, to generate the Koszul complex:

$$C^\infty \mathcal{S} \otimes \Lambda \phi^*_i.$$  

Define a Poisson “anti-bracket”, denoted $(\ ,\ )$, by declaring $(\phi^i, \phi^*_j) = \delta^i_j$ and extending according to a (slightly mis)graded Leibnitz rule. Notice that the above formula holds in ghost degree 0 but is applied to terms of ghost degree $-1$ and 0 respectively; this Interrupt (equivalent ways) of describing the Leibnitz rule:

(1) the anti-bracket itself is an operation of ghost degree 1, i.e. (denoting total
ghost degree by “gh”) $gh(A, B) = ghA + ghB + 1$, so the Leibnitz rule is:

$$(A, BC) = (-1)^c(A, B)C + (-1)^{b(a+1)}B(A, C);$$

(2) the anti-bracket is a graded Poisson bracket with respect to degree where degree $A = ghA + 1$.

The variational system or, more importantly, the space of solutions $\Sigma$ may have Noether symmetries, i.e. may support the vector field action of a Lie algebra $\mathfrak{g}$. Choose a basis $C^\alpha_\alpha$ for $\mathfrak{g}$ declared to have ghost degree -2 and a dual basis $C^\alpha_\beta$ declared to have ghost degree 1 for the dual of $\mathfrak{g}^*$. Define the anti-bracket of these new generators to be

$$(C^\alpha_\alpha, C^\beta_\beta) = \delta^\alpha_\beta$$

and extend by the above Leibnitz rule. This notation is a mild modification of that in the physics literature: $C^\alpha_\alpha$ is more traditionally denote $\phi^*_\alpha$.

What’s going on here! As Henneaux will explain more fully, the anti-fields $\phi^*_\alpha$ can be interpreted as vector fields on $\Sigma$ and $(\ , \ )$ as the Schouten bracket. Here is a very preliminary attempt to interpret the curious regrading by degree.

Following Henneaux and others, interpret $S$ as a space of histories, which I take to mean sections of the bundle $E \to M$ which is of the form $D \times I \to N \times I$. Thus sections can be interpreted as 1-parameter families of sections of $D \to N$ and $S$ as

$$\text{Map } (I, \text{Sections } (D \to N)).$$

There are homological models for such path spaces in the mathematical literature; hopefully one of them can be related straightforwardly to this anti-bracket model.

The major result of Batalin and Vilkovisky in this Lagrangian setting is that the appropriate action $S$ to be quantized is of the form $S = S_0 + \text{“ghost terms”}$ where the ghost terms each have ghost degree 0 as does the $S_0$ with which we began the discussion. It should come as no surprise that the existence of the ghost terms follows from the acyclicity of the Koszul complex (assuming appropriate regularity conditions) or of the Koszul-Tate resolution.

**Cohomological field theory.**

Witten has recently [Wi] introduced the term “cohomological field theory” for yet another situation in which techniques of homological algebra play a significant role. I would suggest that the term be applied to the Hamiltonian and Lagrangian methods above as well.

Witten’s paper is concerned in passing with “equivariant cohomology”, apparently inspired by discussions with Scott Axelrod. My remarks will provide an introduction to a small part of Axelrod’s talk later this week.

Equivariant cohomology refers to cohomology in the setting of transformation groups, i.e. a topological group $G$ acting on a space $X$ without assumptions on the action $G \times X \to X$ other than its continuity or smoothness (and the algebraic conditions that the unit of $G$ act as the identity on $X$ and that $g(ha) = (gh)a$ for
$g,h \in G, x \in X$). Without further restriction on the action, the quotient space (= orbit space) $X/G$ may fail to be a manifold or even Hausdorff. The projection $X \to X/G$ need not be anything like a principal bundle. Thus the cohomology of $X/G$ may be very difficult to relate to that of $X$ and $G$.

Borel in his celebrated Transformation Group Seminar constructed a principal $G$-bundle $\tilde{X} \to X_G$ where $\tilde{X}$ has the same homotopy type as $X$. In case the action of $G$ on $X$ does give a principal $G$-bundle $X \to X/G$, then $X_G$ has the homotopy type $X/G$ and hence the same cohomology. Otherwise, $X_G$ is a space with the cohomology that the orbit space should have. This cohomology is known as the **equivariant cohomology** of $X$, denoted $H_G(X) := H(X_G)$.

The construction of $X_G$ makes use of the *universal* principal $G$-bundle $EG \to BG$. The space $\tilde{X}$ is in fact just $EG \times X$ with the diagonal action of $G$ and $X_G$ is the orbit space $EG \times_G X$.

In the smooth setting, $G$ a Lie group acting smoothly on a manifold $M$, there is a model for the equivariant cohomology which uses the Weil algebra, a model for the differential forms on $EG$. The $G$-action on $M$ is reflected in two families of operators $\iota_X$ and $\theta_X$ on differential forms on $M$ for $X \in \mathfrak{g}$ (and similarly for the $G$-action on $EG$). For $X \in \mathfrak{g}$, $\iota_X$ is contraction with $X$ and $\theta_X$ is the infinitesimal action of the vector field corresponding to $X$. These operators obey the usual rules:

\[
\iota_{[X,Y]} = \iota_X \iota_Y - \iota_Y \iota_X
\]

\[
\theta_X = d\iota_X + \iota_X d.
\]

Now construct the **Weil algebra** $W(\mathfrak{g}) := \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{s}\mathfrak{g}^*)$, the free graded commutative algebra generated by a copy of the dual of $\mathfrak{g}^*$ in degree 1 and another copy $\mathfrak{s}\mathfrak{g}^*$ of $\mathfrak{g}^*$ in degree 2. (If we chose a basis $C^\alpha$ for $\mathfrak{g}^*$ in degree, the Weil algebra would be described as generated by the ghosts $C^\alpha$ with ghost degree 1 and by even generators $sC^\alpha$ with ghost degree 2.) This algebra $W(\mathfrak{g})$ is given a total differential $D$ of degree 1 which is the sum of two differentials $\delta$ and $s$ where $\delta$ is the Chevalley-Eilenberg differential for $\mathfrak{g}$ with coefficients in $S(\mathfrak{s}\mathfrak{g}^*)$ under the coadjoint action and $s$ is the differential determined on the odd generating copy of $\mathfrak{g}^*$ as an isomorphism with the even copy (in coordinates, $s : C^\alpha \to sC^\alpha$). Alternatively, $s$ can be regarded as isomorphic to the graded analog of the Koszul differential for the maximal ideal of $S(\mathfrak{s}\mathfrak{g}^*)$. Since this algebra is acyclic with respect to $s$, it is also acyclic with respect to $D = s + \delta$. As such, it is a model for (has the same cohomology as) $EG$. The principal $G$-action on $EG$ is reflected in $W(\mathfrak{g})$ by defining the operators $\iota_X$ and $\theta_X$ for $X \in \mathfrak{g}$ as follows:

\[
\theta_X(h) = -h([X, ]) \quad \text{for} \quad h \in \mathfrak{g}^*
\]

\[
\theta_X(sh) = s \coad(X) h
\]

where $\coad$ denotes the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ and

\[
\iota_X h = h(X) \quad \text{for} \quad h \in \mathfrak{g}^*
\]

while $\iota_X$ is 0 on $S(\mathfrak{s}\mathfrak{g}^*)$. These operators combine to give operators $\iota_X$ and $\theta_X$ on $W(\mathfrak{g}) \otimes \Omega^*(M)$. The cohomology of $M_G$ is then given by the subcomplex of “basic” forms: $\ker \iota_X \cap \ker \theta_X$. 
Since $\iota_X$ is 0 on $S(\mathfrak{g}^*)$ and non-degenerate on $\Lambda(\mathfrak{g}^*)$, the “basic” forms lie, in fact, in $S(\mathfrak{g}^*) \otimes \Omega^*(M)$. The algebra is reflecting the homotopy fibration $X \to X_G \to BG$ where $BG$ is the classifying space of the group $G$. Thus the ‘bosonic’ ghosts $sC_\alpha$ come ‘from below’. Axelrod will give some indication in his talk of how this construction is relevant to classical field theory (see also [A]).

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