A FRACTIONAL DIRICHLET-TO-NEUMANN OPERATOR ON BOUNDED LIPSCHITZ DOMAINS

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary $\partial \Omega$. We define a fractional Dirichlet-to-Neumann operator and prove that it generates a strongly continuous analytic and compact semigroup on $L^2(\partial \Omega)$ which can also be ultracontractive. We also use the fractional Dirichlet-to-Neumann operator to compare the eigenvalues of a realization in $L^2(\Omega)$ of the fractional Laplace operator with Dirichlet boundary condition and the regional fractional Laplacian with the fractional Neumann boundary conditions.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial \Omega$. Given $g \in L^2(\partial \Omega)$, let $u \in W^{1,2}(\Omega)$ be the solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega. \quad (1.1)$$

The operator $\mathbb{D}_{1,0}$ defined on $L^2(\partial \Omega)$ by

$$\begin{cases} D(\mathbb{D}_{1,0}) = \{ g \in L^2(\partial \Omega), \exists u \in W^{1,2}(\Omega) \text{ solution of } (1.1) \text{ and } \partial_\nu u \text{ exists in } L^2(\partial \Omega) \}, \\ \mathbb{D}_{1,0} g = \partial_\nu u \end{cases} \quad (1.2)$$

is called the Dirichlet-to-Neumann operator. Here, $\partial_\nu u$ denotes the normal derivative of the function $u$ in direction of the outer normal vector $\nu$. The Dirichlet-to-Neumann operator is well-known and has been studied by several authors (see e.g. [3, 4, 5, 12, 16, 26] and their references). Some properties of the operator $\mathbb{D}_{1,0}$ have been used in [3] to give another proof on Lipschitz domains of the Friedlander’s result [16] (initially proved on smooth domains) that is, $\lambda_{n+1}^N \leq \lambda_n^D$ for all $n \in \mathbb{N}$, where $\lambda_{n+1}^N$ is the $(n+1)^{th}$-eigenvalue of the Neumann Laplace operator and $\lambda_n^D$ is the $n^{th}$-eigenvalue of the Dirichlet Laplacian. The Dirichlet-to-Neumann operator has also been defined on very rough domains in [4] by using the method of bilinear forms, where the authors have shown that $-\mathbb{D}_{1,0}$ generates a strongly continuous Markov semigroup on $L^2(\partial \Omega)$ and the asymptotic behavior of this semigroup is related to properties of the trace of functions in the first order Sobolev space $W^{1,2}(\Omega)$. Other generation of semigroups results are contained in [3, 12, 17, 26].
More precisely, heat kernel estimates for the classical Dirichlet-to-Neumann operator and fractional Laplacians (but not the fractional Dirichlet-to-Neumann) have been obtained in [17, 26].

The main concerns of the present paper are the following:

- Find a right definition of a fractional Dirichlet-to-Neumann type operator associated with the fractional Laplace operator, study its spectral properties and some generation of semigroup results.
- Use the spectral properties of the fractional Dirichlet-to-Neumann operator to obtain the Friedlander’s type result for the fractional Laplace operator on bounded domains in \( \mathbb{R}^N \) with Lipschitz continuous boundary. That is, for \( 0 < s < 1 \), we would like to show that \( \lambda^{N+1}_{n+1,s} \leq \lambda^D_n \) for all \( n \in \mathbb{N} \), where \( \lambda^{N+1}_{n+1,s} \) is the \((n+1)^{th}\) eigenvalue of the realization in \( L^2(\Omega) \) of the fractional Laplace operator with fractional Neumann type boundary conditions and \( \lambda^D_n \) denotes the \( n^{th}\)-eigenvalue of a realization in \( L^2(\Omega) \) of the fractional Laplace operator with the Dirichlet boundary condition.

For the convenience of the reader and in order to make the paper as self-contained as possible, we start by introducing the fractional Laplace operator which is not so familiar as the well-known Laplace operator. For \( 0 < s < 1 \) and \( \Omega \subset \mathbb{R}^N \) an arbitrary open set, we let

\[ L^1(\Omega) := \{ u : \Omega \rightarrow \mathbb{R} \text{ measurable}, \int_{\Omega} \frac{|u(x)|}{(1 + |x|)^{N+2s}} \, dx < \infty \} . \]

For \( u \in L^1(\mathbb{R}^N) \), \( x \in \mathbb{R}^N \) and \( \varepsilon > 0 \), we write

\[ (-\Delta)^s u(x) = C_{N,s} \int_{\{y \in \mathbb{R}^N, |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy, \]

with the normalized constant \( C_{N,s} \) given by

\[ C_{N,s} := \frac{s^{2s} \Gamma \left( \frac{N+2s}{2} \right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}, \tag{1.3} \]

where \( \Gamma \) denotes the usual Gamma function. The fractional Laplacian \((-\Delta)^s u\) of the function \( u \) is defined by the formula,

\[ (-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy = \lim_{\varepsilon \downarrow 0} (-\Delta)^s_\varepsilon u(x), \quad x \in \mathbb{R}^N, \tag{1.4} \]

provided that the limit exists.

Most recently, Caffarelli et al. [8, 9] have deeply studied the operator \((-\Delta)^s\) on \( \mathbb{R}^N \) and on subsets of \( \mathbb{R}^N \) with the Dirichlet boundary conditions. Some fundamental and beautiful results have been obtained. Some semi-linear problems involving \((-\Delta)^s\) with Dirichlet boundary condition have been also investigated in [8, 9, 23] and the references therein. The fractional Laplacian and fractional derivative operators are commonly used to model anomalous diffusion. Physical phenomena exhibiting this property cannot be modeled accurately by the usual advection-dispersion equation; among others, we mention turbulent flows and chaotic dynamics of classical conservative systems. The operator \((-\Delta)^s\) is the most typical non-local operator. Further properties and applications of the fractional Laplace operator and more general non-local operators are contained in [15] and the references therein.

As we have mentioned above, the aim of the present paper is to find a right definition of a fractional Dirichlet-to-Neumann type operator. We have the following
situation. Let $g \in C(\mathbb{R}^N)$ be a given function and $\Omega \subset \mathbb{R}^N$ a bounded domain with Lipschitz continuous boundary $\partial \Omega$. It is well known (see e.g. [20] and their references) that the following Dirichlet type problem

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega,$$

(1.5)
is not well-posed. The well-posed Dirichlet problem for the fractional Laplace operator is given by

$$(-\Delta)^s u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \mathbb{R}^N \setminus \Omega.$$

Since the problem (1.5) is not well-posed, compare with (1.1), the operator $(-\Delta)^s$ is not suitable to obtain a fractional Dirichlet-to-Neumann operator. We have to consider an operator that depends on the domain $\Omega$. We proceed as follows. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary open set. We restrict the integral kernel in (1.4) to the open set $\Omega$. For $u \in L^1(\Omega)$, $x \in \Omega$ and $\varepsilon > 0$, we let

$$A^s_{\Omega, \varepsilon} u(x) = C_{N,s} \frac{1}{\varepsilon} \int_{\{y \in \Omega : |y-x| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy,$$

and we define the operator $A^s_{\Omega}$ as follows:

$$A^s_{\Omega} u(x) = C_{N,s} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy = \lim_{\varepsilon \downarrow 0} A^s_{\Omega, \varepsilon} u(x), \quad x \in \Omega,$$

provided that the limit exists. In [18, 19] the operator $A^s_{\Omega}$ has been called the **regional fractional Laplacian**. We have the following. Let $u \in D(\Omega)$. Since $u = 0$ on $\mathbb{R}^N \setminus \Omega$, a simple calculation gives,

$$A^s_{\Omega} u := C_{N,s} \text{P.V.} \int_{\Omega} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy$$

$$- C_{N,s} \int_{\mathbb{R}^N \setminus \Omega} \frac{u(x)}{|x-y|^{N+2s}} \, dy = (-\Delta)^s u(x) - V_{\Omega}(x) u(x),$$

(1.6)

where the potential $V_{\Omega}$ is given for every $x \in \Omega$ by

$$V_{\Omega}(x) := C_{N,s} \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{|x-y|^{N+2s}} \, dy.$$

More details and the relation between $A^s_{\Omega}$ and $(-\Delta)^s$, and some useful properties of the potential $V_{\Omega}$ can be found in [6, 14] and their references.

Now, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial \Omega$. Given a function $g \in C(\partial \Omega)$, it has been shown in [19] that the Dirichlet problem for the regional fractional Laplacian,

$$A^s_{\Omega} u = 0 \quad \text{in} \quad \Omega, \quad u = g \quad \text{on} \quad \partial \Omega,$$

(1.7)
is well-posed. Therefore, to define a fractional Dirichlet-to-Neumann type operator, it makes sense to consider the regional fractional Laplacian. Let $A_D$ and $A_N$ be the realization in $L^2(\Omega)$ of the operator $A^s_{\Omega}$ with Dirichlet and fractional Neumann boundary conditions, respectively, that is,

$$D(A_D) = \{ u \in W^{s,2}_0(\Omega), \ A^s_{\Omega} u \in L^2(\Omega) \}, \quad A_D u = A^s_{\Omega} u,$$

and

$$D(A_N) = \{ u \in W^{s,2}(\Omega), \ A^s_{\Omega} u \in L^2(\Omega), \ A^{2-2s} u = 0 \in L^2(\partial \Omega) \},$$

$$A_N u = A^s_{\Omega} u,$$
where $N^{2-2s}u$ denotes the fractional normal derivative of the function $u$ (see (2.11) and (2.13) below). We refer to Section 2.2 below for more details. We notice that if $0 < s \leq \frac{1}{2}$, then the operators $A_D$ and $A_N$ coincide. That is, if $0 < s \leq \frac{1}{2}$, then $D(A_D) = D(A_N)$ and $A_Du = A_Nu$ for every $u \in D(A_D) = D(A_N)$. Therefore without any restriction we may assume that $\frac{1}{2} < s < 1$. Let $\sigma(A_D)$ denote the spectrum (which is discrete) of $A_D$. Let $\lambda \in \mathbb{R} \setminus \sigma(A_D)$. Then using [19], we have that given $g \in L^2(\partial \Omega)$, there exists $u \in W^{s,2}(\Omega)$ solution of the Dirichlet problem

\begin{equation}
A_D^s u = \lambda u \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega.
\end{equation}

Next, let $\frac{1}{2} < s < 1$ and $\lambda \in \mathbb{R} \setminus \sigma(A_D)$. As in (1.2) for the Laplace operator, the fractional Dirichlet-to-Neumann operator $D_{s,\lambda}$ is defined on $L^2(\partial \Omega)$ by

\begin{align*}
D(D_{s,\lambda}) &= \{g \in L^2(\partial \Omega), \exists u \in W^{s,2}(\Omega) \text{ solution of (1.8) and } N^{2-2s}u \text{ exists in } L^2(\partial \Omega)\}, \\
D_{s,\lambda}u &= C_s N^{2-2s}u,
\end{align*}

where $C_s$ is an explicit normalized constant (see (2.8) below). We show in Section 3 below that $D_{s,\lambda}$ is well defined, is associated with a bilinear symmetric, continuous and elliptic form and has also a compact resolvent. Further spectral properties of the operator $D_{s,\lambda}$ and some generation of semigroup results have been investigated. More precisely, we obtain that for every $\lambda \in \mathbb{R} \setminus \sigma(A_D)$, the operator $-D_{s,\lambda}$ generates a strongly continuous analytic and compact semigroup on $L^2(\partial \Omega)$ satisfying the following:

- If $\lambda < \lambda^D_{1,s}$, then the semigroup is positive.
- If $\lambda \leq 0$, then the semigroup is also submarkovian and ultracontractive.
- If $\lambda = 0$, then the semigroup is in addition Markovian.

Using the spectral properties of $D_{s,\lambda}$, we prove in Section 4 the Friedlander type result for the regional fractional Laplace operator, that is, $\lambda^N_{n+1,s} \leq \lambda^D_{n,s}$ for all $n \in \mathbb{N}$.

The rest of the paper is organized as follows. In Section 2 we introduce the function spaces needed to investigate our problem and we prove some intermediate results on the regional fractional Laplace operator $A_D^s$ as they are needed throughout the paper. The fractional Dirichlet-to-Neumann operator is introduced in Section 3 where we also show that it has a compact resolvent and generates a strongly continuous analytic semigroup which can also be ultracontractive. Finally in Section 4 we use the fractional Dirichlet-to-Neumann operator to compare the eigenvalues of the regional fractional Laplacian with Dirichlet boundary condition and the regional fractional Laplace operator with fractional Neumann boundary conditions.

2. Intermediate results. In this section we introduce the function spaces needed to investigate our problem and we prove some intermediate results that will be used to obtain our main results.

2.1. The functional setup. Let $\Omega \subset \mathbb{R}^N$ be an arbitrary bounded open set. For $s \in (0,1)$, we denote by

\[ W^{s,2}(\Omega) := \{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy < \infty \} \]
the fractional order Sobolev space endowed with the norm
\[
\|u\|_{W^{s,2}(\Omega)} := \left( \int_\Omega |u|^2 \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}.
\]
We let
\[
W^{s,2}_0(\Omega) = D(\Omega)^{W^{s,2}(\Omega)}.
\]
By definition, \(W^{s,2}_0(\Omega)\) is the smallest closed subspace of \(W^{s,2}(\Omega)\) containing \(D(\Omega)\) and it can be characterized as follows [1, Theorem 10.1.1]:
\[
W^{s,2}_0(\Omega) = \{ u \in W^{s,2}(\mathbb{R}^N) : \tilde{u} = 0 \text{ on } \mathbb{R}^N \setminus \Omega \},
\]
where \(\tilde{u}\) is the quasi-continuous version (with respect to the capacity defined with the space \(W^{s,2}(\mathbb{R}^N)\)) of the function \(u\). Using the characterization (2.1), we have that \(W^{s,2}_0(\Omega)\) is continuously embedded into \(W^{s,2}(\mathbb{R}^N)\) (see e.g. [14]). This shows that for \(u \in W^{s,2}_0(\Omega)\),
\[
\|u\|_{W^{s,2}_0(\Omega)} = \left( \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy \right)^{\frac{1}{2}}
\]
defines an equivalent norm on \(W^{s,2}_0(\Omega)\). Moreover, it is well-known (see e.g. [14]) that the continuous injection \(W^{s,2}_0(\Omega) \hookrightarrow L^2(\Omega)\) is also compact. Another characterization of \(W^{s,2}_0(\Omega)\) has been given in [27, Theorem 4.5] (see also [6]). More precisely, let
\[
\tilde{W}^{s,2}(\Omega) = W^{s,2}(\Omega) \cap C(\overline{\Omega})^{W^{s,2}(\Omega)}.
\]
Then
\[
W^{s,2}_0(\Omega) = \{ u \in \tilde{W}^{s,2}(\Omega) : \tilde{u} = 0 \text{ quasi-everywhere on } \partial \Omega \},
\]
where here, \(\tilde{u}\) denotes the quasi-continuous version of \(u\) with respect to the capacity defined on subsets of \(\overline{\Omega}\) with the space \(\tilde{W}^{s,2}(\Omega)\). If \(\Omega\) has a Lipschitz continuous boundary, then \(\tilde{W}^{s,2}(\Omega) = W^{s,2}(\Omega)\), that is, \(W^{s,2}(\Omega) \cap C(\overline{\Omega})\) is dense in \(W^{s,2}(\Omega)\).

We have the following result taken from [6, Corollary 2.8 and Remark 2.3].

**Theorem 2.1.** Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain with a Lipschitz continuous boundary. Then for every \(0 < s \leq \frac{1}{2}\), the spaces \(W^{s,2}(\Omega)\) and \(W^{s,2}_0(\Omega)\) coincide with equivalent norms.

In view of Theorem 2.1, the characterization (2.2) and [27], we have that if \(0 < s \leq \frac{1}{2}\) and \(\Omega\) has a Lipschitz continuous boundary, then every \(u \in W^{s,2}(\Omega)\) is zero quasi-everywhere and \(\sigma\)-a.e. on \(\partial \Omega\). Therefore, to talk about traces (not necessarily zero) of functions in \(W^{s,2}(\Omega)\), it is not a restriction to assume that \(\frac{1}{2} < s < 1\). We have the following result taken from [10, 14].

**Theorem 2.2.** Let \(\frac{1}{2} < s < 1\) and \(\Omega \subset \mathbb{R}^N\) a bounded domain with a Lipschitz continuous boundary \(\partial \Omega\). Then the following assertions hold.
(a) The continuous injection \(W^{s,2}(\Omega) \hookrightarrow L^q(\partial \Omega)\) is also compact.
(b) There exists a linear continuous trace operator
\[
\text{Tr} : W^{s,2}(\Omega) \hookrightarrow L^q(\partial \Omega), \quad \forall q \in [1, 2^*] \text{ if } N > 2s, \forall q \in [1, \infty) \text{ if } N = 2s,
\]
such that \(\text{Tr}(u) = u\) on \(\partial \Omega\) for every \(u \in W^{s,2}(\Omega) \cap C(\overline{\Omega})\), where the constant
\[
2^* := \frac{2(N-1)}{N-2s} > 2.
\]
Moreover, the continuous embedding \(W^{s,2}(\Omega) \hookrightarrow L^2(\partial \Omega)\) is also compact.

The following theorem gives another equivalent norm for the space \(W^{s,2}(\Omega)\).
Theorem 2.3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz continuous boundary $\partial \Omega$ and $\frac{1}{2} < s < 1$. Then there exists a constant $C = C(\Omega, N, s) > 0$ such that for every $u \in W^{s, 2}(\Omega)$,

$$
\int_{\Omega} |u|^2 \, dx \leq C \left( \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\partial \Omega} |u|^2 \, d\sigma \right). \tag{2.4}
$$

Proof. Let $\frac{1}{2} < s < 1$. It suffices to show that

$$
\|u\| := \left( \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\partial \Omega} |u|^2 \, d\sigma \right)^{1/2} \tag{2.5}
$$
defines an equivalent norm on $W^{s, 2}(\Omega)$. Since $2^* := \frac{2(N-1)}{N-2s} > 2$ and $\sigma(\partial \Omega) < \infty$, it follows from (2.3) that there exists a constant $C > 0$ such that for every $u \in W^{s, 2}(\Omega)$,

$$
\int_{\partial \Omega} |u|^2 \, d\sigma \leq C \left( \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\Omega} |u|^2 \, dx \right). \tag{2.6}
$$

To prove the converse inequality, we proceed by contradiction. Assume that for every $n \in \mathbb{N}$, there exists a sequence $u_n \in W^{s, 2}(\Omega)$ such that

$$
\int_{\Omega} |u_n|^2 \, dx > n \left( \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \int_{\partial \Omega} |u_n|^2 \, d\sigma \right). \tag{2.7}
$$

By possibly dividing (2.6) by $\|u_n\|_{L^2(\Omega)}^2$ we may assume that $\|u_n\|_{L^2(\Omega)}^2 = 1$ for any $n$. Hence, by (2.6), $u_n$ is a bounded sequence in the Hilbert space $W^{s, 2}(\Omega)$. Therefore, after passing to a subsequence, if necessary, we may assume that $u_n$ converges weakly to some $u \in W^{s, 2}(\Omega)$ and strongly to $u$ in $L^2(\Omega)$ (since the embedding $W^{s, 2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact by Theorem 2.2). Moreover, $u_n|_{\partial \Omega}$ converges strongly to $u|_{\partial \Omega}$ in $L^2(\partial \Omega)$ (since the embedding $W^{s, 2}(\Omega) \hookrightarrow L^2(\partial \Omega)$ is compact by Theorem 2.2). It follows from (2.6) that

$$
\lim_{n \to \infty} \int_{\Omega} \int_{\Omega} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy = 0 \quad \text{and} \quad \lim_{n \to \infty} \int_{\partial \Omega} |u_n|^2 \, d\sigma = 0.
$$

This implies that $u_n|_{\partial \Omega}$ converges strongly to $0$ in $L^2(\partial \Omega)$ and (after passing to a subsequence, if necessary)

$$
\lim_{n \to \infty} |u_n(x) - u_n(y)| = 0 \quad \text{for a.e. } (x, y) \in \Omega \times \Omega. \tag{2.7}
$$

Using (2.7) and the fact that (after passing to a subsequence, if necessary) $u_n$ converges a.e. to $u$ in $\Omega$, we get that $\lim_{n \to \infty} u_n(x) = \lim_{n \to \infty} u_n(y)$ for a.e. $x, y \in \Omega$. Hence, $u_n$ converges almost everywhere to some constant function $c$. The uniqueness of the limit and the uniqueness of the trace (since $u_n|_{\partial \Omega}$ converges to $0$ on $\partial \Omega$ after passing to a subsequence, if necessary) imply that $c = u = 0$ a.e. on $\Omega$. On the other hand, we have $\|u\|_{L^2(\Omega)}^2 = \lim_{n \to \infty} \|u_n\|_{L^2(\Omega)}^2 = 1$, and this is a contradiction. The proof is finished. \qed

For more information on the fractional order Sobolev spaces we refer to [1, 10, 14, 27] and their references.
2.2. The regional fractional Laplacian with various boundary conditions.

Before we introduce various realizations of the regional fractional Laplace operator, we give a Green type formula. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,1}$ with boundary $\partial \Omega$. Let $\frac{1}{2} < s < 1$ and the constant

$$C_s := \frac{C_{1,s}}{2s(2s - 1)} \int_0^\infty \frac{\lvert t - 1 \rvert^{1 - 2s} - (t\lor 1)^{1 - 2s}}{t^{2 - 2s}} \, dt,$$

(2.8)

where $C_{1,s}$ is given by (1.3) with $N = 1$. Let the constant $B_{N,s}$ be such that

$$\frac{C_{1,s}}{C_{N,s}} B_{N,s} := \begin{cases} C_s, & \text{if } N = 1, \\ \frac{2^{\frac{N-2}{2}} C_s}{\Gamma\left(\frac{N-1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos^{2s}(\theta) \sin^{N-2}(\theta) \, d\theta, & \text{if } N \geq 2. \end{cases}$$

(2.9)

Remark 1. We notice that a simple calculation gives that

$$\int_0^{\frac{\pi}{2}} \cos^{2s}(\theta) \sin^{N-2}(\theta) \, d\theta = \frac{1}{2}\mathbb{B}\left(\frac{2s + 1}{2}, \frac{N - 1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{2s + 1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N+2s}{2}\right)},$$

(2.10)

where $\mathbb{B}$ denotes the usual beta function. Replacing (2.10) into (2.9), we get that $B_{N,s} = C_s$, so that it is independent of $N$ and depends on $s$ only.

The following integration by parts formula has been recently obtained by Guan [18, Theorem 3.3].

Theorem 2.4. Let $\frac{1}{2} < s < 1$ and

$$C_{2s}(\Omega) := \{ u: u(x) = f(x)\rho(x)^{2s-1} + g(x), \forall x \in \Omega, \text{ for some } f, g \in C^2(\overline{\Omega}) \},$$

where $\rho(x) := \text{dist}(x, \partial \Omega)$, $x \in \Omega$. For $u \in C_{2s}(\Omega)$ and $z \in \partial \Omega$, we define the fractional normal derivative $N^{2-2s}u$ of the function $u$ by

$$N^{2-2s}u(z) = -\lim_{t \downarrow 0} \frac{du(z - \rho(z)t)}{dt} t^{2-2s}, \quad z \in \partial \Omega.$$ 

(2.11)

Then for every $u \in C_{2s}^2(\Omega)$ and $v \in W^{s,2}(\Omega)$, one has $A_{\Omega}^s u \in L^2(\Omega)$, $N^{2-2s}u \in L^2(\partial \Omega)$ and

$$\int_{\Omega} v A_{\Omega}^s u \, dx = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy$$

$$- C_s \int_{\partial \Omega} v N^{2-2s}u \, d\sigma,$$

(2.12)

where $C_s$ is the constant given in (2.8).

Comparing (2.12) and the classical Green formula for the Laplace operator, we get that the function $C_s N^{2-2s}u$ plays the role (for the regional fractional Laplace operator) that the normal derivative $\partial_\nu u$ does for the Laplace operator.

Next, we introduce a weak formulation on non-smooth domains of a fractional normal derivative.

Definition 2.5. Let $\frac{1}{2} < s < 1$ and $\Omega \subset \mathbb{R}^N$ a bounded domain with Lipschitz continuous boundary $\partial \Omega$.

(a) Let $u \in W^{s,2}(\Omega)$. We say that $A_{\Omega}^s u \in L^2(\Omega)$ if there exists $w \in L^2(\Omega)$ such that

$$\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_{\Omega} uv \, dx$$

$$- C_s \int_{\partial \Omega} v N^{2-2s}u \, d\sigma, \quad \forall v \in W^{s,2}(\Omega).$$
for all \( v \in \mathcal{D}(\Omega) \) and hence, for all \( v \in W^{s,2}_0(\Omega) \) by density. In that case we write \( A^s_{\Omega}u = w \).

(b) Let \( u \in W^{s,2}(\Omega) \) such that \( A^s_{\Omega}u \in L^2(\Omega) \). We say that \( u \) has a fractional normal derivative in \( L^2(\partial \Omega) \) if there exists \( g \in L^2(\partial \Omega) \) such that

\[
\int_\Omega (A^s_{\Omega}u)v \, dx = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x-y|^{N+2s}} \, dxdy - \int_{\partial \Omega} gv \, d\sigma \tag{2.13}
\]

for all \( v \in W^{s,2}(\Omega) \cap C(\overline{\Omega}) \), hence for all \( v \in W^{s,2}(\Omega) \) by density and by using (2.3). In that case, the function \( g \) is uniquely determined by (2.13), we write \( C_{s,N}N^{2-2s}u = g \) and call \( g \) the fractional normal derivative of \( u \).

**Remark 2.** It follows from Definition 2.5 that the Green’s type formula

\[
\int_\Omega vA^s_{\Omega}u \, dx = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x-y|^{N+2s}} \, dxdy - C_s \int_{\partial \Omega} vN^{2-2s}u \, d\sigma, \tag{2.14}
\]

holds for all \( v \in W^{s,2}(\Omega) \) whenever \( u \in W^{s,2}(\Omega) \), \( A^s_{\Omega}u \in L^2(\Omega) \) and \( N^{2-2s}u \) exists in \( L^2(\partial \Omega) \).

If \( \Omega \) is a bounded open set of class \( C^{1,1} \) and \( u \in C^2_{\mathbb{Z}}(\overline{\Omega}) \), then \( N^{2-2s}u \) coincides with the function introduced in (2.11). Moreover, by \( \cite{18,27} \), \( N^{2-2s}u \in L^2(\partial \Omega) \) and \( A^s_{\Omega}u \in L^2(\Omega) \).

Throughout the remainder of the article without any mention, \( \Omega \subset \mathbb{R}^N \) denotes a bounded domain with Lipschitz continuous boundary \( \partial \Omega \). Let \( \frac{1}{2} < s < 1 \) and \( \mu \in \mathbb{R} \). We define the bilinear symmetric form \( \mathcal{E}_\mu \) with domain \( W^{s,2}(\Omega) \) by

\[
\mathcal{E}_\mu(u,v) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy - \mu \int_{\partial \Omega} uv \, d\sigma.
\]

We have the following result.

**Proposition 2.1.** Let \( \mu \in \mathbb{R} \). Then the bilinear symmetric form \( \mathcal{E}_\mu \) is continuous and elliptic.

**Proof.** Let \( \mu \in \mathbb{R} \). It is easy to see that the form \( \mathcal{E}_\mu \) is continuous. We show that it is elliptic. We notice that (2.3) together with the fact the embedding \( W^{s,2}(\Omega) \hookrightarrow L^2(\Omega) \) is compact, imply that there exists a constant \( C > 0 \) such that for every \( u \in W^{s,2}(\Omega) \),

\[
\mu \int_{\partial \Omega} |u|^2 \, d\sigma \leq \frac{1}{2} \|u\|^2_{W^{s,2}(\Omega)} + C \int_\Omega |u|^2 \, dx
\]

\[
= \frac{1}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dxdy + \left( C + \frac{1}{2} \right) \int_\Omega |u|^2 \, dx.
\]
Therefore, for every $u \in W^{s,2}(\Omega)$,
\[
\mathcal{E}_\mu(u, u) + (C + 1) \int_\Omega |u|^2 \, dx = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy \\
- \mu \int_{\partial \Omega} |u|^2 \, d\sigma + (C + 1) \int_\Omega |u|^2 \, dx \\
\geq \frac{1}{2} \left( \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dxdy + \int_\Omega |u|^2 \, dx \right) \\
= \frac{1}{2} \|u\|^2_{W^{s,2}(\Omega)}.
\]
We have shown that $\mathcal{E}_\mu$ is elliptic and the proof is finished. \hfill \Box

Next, let $A_\mu$ be the closed linear self-adjoint operator on $L^2(\Omega)$ associated with the form $\mathcal{E}_\mu$ in the sense that,
\[
\begin{aligned}
D(A_\mu) &= \{ u \in W^{s,2}(\Omega), \exists w \in L^2(\Omega), \\
& \quad \mathcal{E}_\mu(u, v) = (w, v)_{L^2(\Omega)} \forall v \in W^{s,2}(\Omega) \}, \\
A_\mu u &= w.
\end{aligned}
\tag{2.15}
\]

The operator $A_\mu$ is the realization in $L^2(\Omega)$ of the regional fractional Laplace operator $A^{s}_\mu$ with the fractional Robin type boundary conditions. More precisely, we have the following characterization.

**Proposition 2.2.** Let $A_\mu$ be the operator defined in (2.15). Then
\[
\begin{aligned}
D(A_\mu) &= \{ u \in W^{s,2}(\Omega), A^{s}_\mu u \in L^2(\Omega), \quad \mathcal{N}^{2-2s} u \text{ exists in } L^2(\partial \Omega) \\
& \quad \text{and } C_s \mathcal{N}^{2-2s} u = \mu u \}, \\
A_\mu u &= A^{s}_\mu u,
\end{aligned}
\tag{2.16}
\]
where $\mathcal{N}^{2-2s} u$ is to be understood in the sense of (2.13) in Definition 2.5.

**Proof.** Let $A_\mu$ be the operator on $L^2(\Omega)$ defined in (2.15). Let $D(A_\mu)$ be given by (2.15) and set
\[
\begin{aligned}
D := \{ u \in W^{s,2}(\Omega), A^{s}_\mu u \in L^2(\Omega), \quad \mathcal{N}^{2-2s} u \text{ exists in } L^2(\partial \Omega) \\
& \quad \text{and } C_s \mathcal{N}^{2-2s} u = \mu u \}.
\end{aligned}
\]

Let $u \in D(A_\mu)$. Then by definition, $u \in W^{s,2}(\Omega)$ and there exists $w \in L^2(\Omega)$ such that $\mathcal{E}_\mu(u, v) = \int_\Omega uv \, dx$ for all $v \in W^{s,2}(\Omega)$. That is, for every $v \in W^{s,2}(\Omega)$, we have
\[
\int_\Omega uv \, dx = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dxdy - \mu \int_{\partial \Omega} uv \, d\sigma. \tag{2.17}
\]
It follows from (2.17) that in particular, for every $v \in D(\Omega)$,
\[
\int_\Omega uv \, dx = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dxdy.
\]
By Definition 2.5, this implies that $A^{s}_\mu u \in L^2(\Omega)$ and $w = A^{s}_\mu u$. It also follows from (2.17), the fact that $A^{s}_\mu u = w$ and Definition 2.5 that $u$ has a fractional normal derivative and $C_s \mathcal{N}^{2-2s} u = \mu u$ in $L^2(\partial \Omega)$. Hence, $u \in D$ and $A_\mu u = A^{s}_\mu u$. 

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To prove the converse inclusion, let \( u \in D \). Then \( u \in W^{s,2}(\Omega) \), \( w := A^\omega_D u \in L^2(\Omega) \) and \( C_s\mathcal{N}^{2-2s}u = \mu u \in L^2(\partial\Omega) \). It follows from the identity (2.14) that for every \( v \in W^{s,2}(\Omega) \),
\[
\int_{\Omega} v A^\omega_D u \, dx = \int_{\Omega} w v \, dx = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy - \mu \int_{\partial\Omega} uv \, d\sigma = \mathcal{E}_\mu(u,v).
\]
Hence, \( u \in D(A_\mu) \) and \( A_\mu u = A^\omega_D u \). We have shown (2.16) and the proof is finished.

Since the embedding \( W^{s,2}(\Omega) \hookrightarrow L^2(\Omega) \) is compact, we have that \( A_{\mu} \) has a compact resolvent, hence, it has a discrete spectrum. For \( \mu \in \mathbb{R} \) and \( n \in \mathbb{N} \), we denote by \( \lambda_{n,s}(\mu) \) the \( n^{th} \)-eigenvalue of \( A_\mu \).

**Remark 3.** We have the following situation.

(a) If \( \mu = 0 \), then we denote the operator \( A_0 \) by \( A_N \) which is given by
\[
\begin{cases}
D(A_N) = \{ u \in W^{s,2}(\Omega), \ A^\omega_D u \in L^2(\Omega), \mathcal{N}^{2-2s}u \text{ exists in } L^2(\partial\Omega) \\
\quad \text{ and } \mathcal{N}^{2-2s}u = 0 \}, \\
A_N u = A^\omega_D u.
\end{cases}
\]

The operator \( A_N \) is the realization in \( L^2(\Omega) \) of the regional fractional Laplace operator \( A^\omega_D \) with fractional Neumann boundary condition. The spectrum of \( A_N \) is an increasing sequence \( 0 = \lambda_{1,s}^N < \lambda_{2,s}^N < \cdots < \lambda_{n,s}^N < \cdots \) of real numbers such that \( \lim_{n \to \infty} \lambda_{n,s}^N = \infty \).

(b) We extend \( \mathcal{E}_\mu \) to \( \mu = -\infty \). If \( \mu = -\infty \), since \( \infty \cdot \int_{\partial\Omega} |u|^2 \, d\sigma \) must be finite for every \( u \in D(\mathcal{E}_{-\infty}) \), then \( u = 0 \) a.e. on \( \partial\Omega \) and hence, the form \( \mathcal{E}_{-\infty} \) has domain \( W^{s,2}_0(\Omega) \) and is given for \( u, v \in W^{s,2}_0(\Omega) \) by
\[
\mathcal{E}_{-\infty}(u,v) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(v(x) - v(y))(u(x) - u(y))}{|x - y|^{N+2s}} \, dx \, dy.
\]

We denote the associated operator by \( A_D \) which is the realization in \( L^2(\Omega) \) of \( A^\omega_D \) with the Dirichlet boundary condition. By [27], we have that
\[
D(A_D) = \{ u \in W^{s,2}_0(\Omega), \ A^\omega_D u \in L^2(\Omega) \}, \ A_D u = A^\omega_D u. \tag{2.19}
\]

Since the embedding \( W^{s,2}_0(\Omega) \hookrightarrow L^2(\Omega) \) is compact, then \( A_D \) also has a compact resolvent. Its spectrum is an increasing sequence \( 0 < \lambda_{1,s}^D < \lambda_{2,s}^D < \cdots < \lambda_{n,s}^D < \cdots \) of real numbers and \( \lim_{n \to \infty} \lambda_{n,s}^D = \infty \). We have the following Poincaré type inequality. For every \( u \in W^{s,2}_0(\Omega) \),
\[
\lambda_{1,s}^D \int_{\Omega} |u|^2 \, dx \leq \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy. \tag{2.20}
\]

(c) It is important not to confuse the operator \( A_D \) and the realization \( (-\Delta)^s_D \) in \( L^2(\Omega) \) of the fractional Laplace operator with the Dirichlet boundary condition given by
\[
D((-\Delta)^s_D) = \{ u \in W^{s,2}_0(\Omega), \ (-\Delta)^s u \in L^2(\Omega) \}, \ (-\Delta)^s_D u = (-\Delta)^s u. \tag{2.21}
\]
In fact, \((-\Delta)^s\Omega\) is the self-adjoint operator on \(L^2(\Omega)\) associated with the closed bilinear symmetric form

\[
\mathcal{E}(u,v) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy, \quad u,v \in W^{s,2}_0(\Omega).
\]

The two operators are different, but if \(u \in D(A_D) \cap D((-\Delta)^s\Omega)\), then \((-\Delta)^s\Omega u = (-\Delta)^s u = A^s_D u + V_\Omega(x)u\) by (1.6). They coincide if and only if \(\mathbb{R}^N \setminus \Omega\) has zero capacity and this cannot happen if \(\Omega\) is bounded. The operator \((-\Delta)^s\Omega\) also has a compact resolvent. Its spectrum is an increasing sequence \(0 < \lambda_{1,s,\Omega} < \lambda_{2,s,\Omega} < \cdots < \lambda_{n,s,\Omega} < \cdots\) of real numbers and \(\lim_{n \to \infty} \lambda_{n,s,\Omega} = \infty\).

The variational characterization and some basic properties of \(\lambda_{n,s,\Omega}\) have been investigated in [24, Proposition 9 and Appendix A]. It follows directly from the min-max variational formula for the eigenvalues that

\[
0 < \lambda^D_{n,s} \leq \lambda_{n,s,\Omega}, \quad \forall \ n \in \mathbb{N}.
\] (2.22)

In [25] the authors have shown that \(\lambda_{1,s,\Omega} < (\lambda^D)^s\), where we recall that \(\lambda^D_1\) denotes the first eigenvalue of the Dirichlet Laplace operator.

For more details on this topics we refer to [14, 24, 25, 27] and their references.

3. The fractional Dirichlet-to-Neumann operator. Let \(\sigma(A_D)\) denote the spectrum of the operator \(A_D\) introduced in (2.19). Recall that \(\sigma(A_D)\) is a discrete set and its elements form an increasing sequence \(0 < \lambda^D_1 < \lambda^D_2 < \cdots < \lambda^D_n < \cdots\) of real numbers such that \(\lim_{n \to \infty} \lambda^D_n = \infty\).

Lemma 3.1. Let \(\frac{1}{2} < s < 1\) and \(\lambda \in \mathbb{R} \setminus \sigma(A_D)\). Then

\[
W^{s,2}(\Omega) = W^{s,2}_0(\Omega) \oplus \mathcal{H}^{s,\lambda}(\Omega),
\] (3.1)

where \(\mathcal{H}^{s,\lambda}(\Omega) := \{u \in W^{s,2}(\Omega) : A^s_D u = \lambda u\}\) and by \(A^s_D u = \lambda u\) we mean that

\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \lambda \int_{\Omega} uv \, dx
\] (3.2)

for all \(v \in \mathcal{D}(\Omega)\), and hence, for all \(v \in W^{s,2}_0(\Omega)\) by density.

Proof. Let \(\frac{1}{2} < s < 1\) and \((W^{s,2}_0(\Omega))^*\) the dual of the Hilbert space \(W^{s,2}_0(\Omega)\).

Consider the operator \(B : W^{s,2}_0(\Omega) \to (W^{s,2}_0(\Omega))^*\) given by

\[
\langle Bu, v \rangle = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy,
\]

where \(\langle \cdot, \cdot \rangle\) is the duality map between \((W^{s,2}_0(\Omega))^*\) and \(W^{s,2}_0(\Omega)\). Note that \(B\) is a bounded operator. First, we claim that the operator \(B\) is bijective. In fact, if one defines \(\mathcal{E}_B\) with domain \(W^{s,2}_0(\Omega)\) by

\[
\mathcal{E}_B(u, \varphi) := \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \, dx \, dy,
\]

we have that \(\mathcal{E}_B\) is a bilinear, symmetric and continuous form. Moreover, using the Poincaré inequality (2.20), we have that there exists a constant \(C > 0\) such that \(\mathcal{E}_B(u, u) \geq C\|u\|^{2}_{W^{s,2}_0(\Omega)}\) for every \(u \in W^{s,2}_0(\Omega)\). Hence, the form \(\mathcal{E}_B\) is also coercive. Now it follows from the classical Lax-Milgram theorem (see e.g. [7, Corollary 5.8]) that for every \(v \in (W^{s,2}_0(\Omega))^*\) there exists a unique \(u \in W^{s,2}_0(\Omega)\) such that \(\langle Bu, \varphi \rangle = \mathcal{E}_B(u, \varphi) = \langle v, \varphi \rangle\) for every \(\varphi \in W^{s,2}_0(\Omega)\) and this proves the claim.
Next, since \( B \) is bijective, then it is invertible and \( 0 \in \mathbb{R} \setminus \sigma(B) \). Considering \( L^2(\Omega) \) as a subspace of \((W^{s,2}_0(\Omega))^\ast\), that is, \( L^2(\Omega) \hookrightarrow (W^{s,2}_0(\Omega))^\ast \) by letting
\[
(f, v) = \int_{\Omega} f v \, dx, \quad f \in L^2(\Omega), \ v \in W^{s,2}_0(\Omega),
\]
it is easy to see that \( B^{-1}(L^2(\Omega)) \subset W^{s,2}_0(\Omega) \). Since the operator \( A_D \) (given in (2.19)) is also invertible (hence, \( 0 \in \rho(A_D) \)) and \( B^{-1} u = A_D^{-1} u \) for every \( u \in L^2(\Omega) \), it follows from [2, Lemma 3.10.2], that \( A_D \) is the part of \( B \) in \( L^2(\Omega) \), that is, \( D(A_D) = \{ u \in D(B) = W^{s,2}_0(\Omega) : B u \in L^2(\Omega) \} \) and \( A_D u = B u \) for every \( u \in D(A_D) \). We recall that \( B \) as an operator from \( W^{s,2}_0(\Omega) \) into \((W^{s,2}_0(\Omega))^\ast\) is bounded, but we mention that its part \( A_D \) in \( L^2(\Omega) \) which is an operator from \( D(A_D) \) into \( L^2(\Omega) \) is unbounded. Note that the \((W^{s,2}_0(\Omega))^\ast\)-norm is smaller than the \( L^2(\Omega)\)-norm.

Now, since \( 0 \in \rho(B) \), \( B^{-1}(L^2(\Omega)) \subset L^2(\Omega) \) and \( D(B) = W^{s,2}_0(\Omega) \subset L^2(\Omega) \), it follows from the abstract result in [2, Proposition 3.10.3] that \( \sigma(B) = \sigma(A_D) \). Let \( \lambda \in \mathbb{R} \setminus \sigma(A_D) = \mathbb{R} \setminus \sigma(B) \), \( u \in W^{s,2}(\Omega) \) and \( F \in (W^{s,2}_0(\Omega))^\ast \) given for every \( v \in W^{s,2}_0(\Omega) \) by
\[
F(v) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy - \lambda \int_{\Omega} u v \, dx.
\]
Since the bounded operator \( \lambda - B : W^{s,2}_0(\Omega) \to (W^{s,2}_0(\Omega))^\ast \) is bijective, it follows that there exists \( u_0 \in W^{s,2}_0(\Omega) \) such that \( B(u_0) = \lambda u_0 = F \). Let \( u_1 := u - u_0 \). We claim that \( u_1 \in \mathcal{H}^{s,\lambda}(\Omega) \). It is clear that \( u_1 \in W^{s,2}_0(\Omega) \). Let \( v \in D(\Omega) \). From the equality \( B(u_0) - \lambda u_0 = F \) we get that \( B(u_0), v - (\lambda u_0, v = F(v), \) that is,
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{u_0(x) - u_0(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy - \lambda \int_{\Omega} u_0 v \, dx = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy - \lambda \int_{\Omega} u_0 v \, dx.
\]
The preceding identity implies that for all \( v \in D(\Omega) \),
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy - \lambda \int_{\Omega} (u - u_0) v \, dx = 0.
\]
That is, for all \( v \in D(\Omega) \),
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u_1(x) - u_1(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx \, dy - \lambda \int_{\Omega} u_1 v \, dx = 0.
\]
Therefore \( A_D^* u_1 = \lambda u_1 \) in the sense of (3.2). Hence, \( u = u_0 + u_1 \in W^{s,2}_0(\Omega) + \mathcal{H}^{s,\lambda}(\Omega) \) and we have shown that \( W^{s,2}(\Omega) = W^{s,2}_0(\Omega) + \mathcal{H}^{s,\lambda}(\Omega) \). Since \( \lambda \notin \sigma(A_D) \) we have that \( W^{s,2}_0(\Omega) \cap \mathcal{H}^{s,\lambda}(\Omega) = \{0\} \). We have shown (3.1) and the proof is finished.

**Remark 4.** Let \( \frac{1}{2} < s < 1 \) and \( \mathcal{V}^s(\partial \Omega) := \{ u|_{\partial \Omega} : u \in W^{s,2}(\Omega) \} \) the trace space. Then \( \mathcal{V}^s(\partial \Omega) \) is a subspace of \( L^2(\partial \Omega) \). If \( \lambda \in \mathbb{R} \setminus \sigma(A_D) \), then it follows from Lemma 3.1 that the trace operator restricted to \( \mathcal{H}^{s,\lambda}(\Omega) \), i.e., the mapping \( u \in \mathcal{H}^{s,\lambda}(\Omega) \mapsto u|_{\partial \Omega} \in \mathcal{V}^s(\partial \Omega) \) is linear and bijective. Letting \( \| u\|_{\mathcal{V}^s(\partial \Omega)} := \| u\|_{\mathcal{H}^{s,\lambda}(\Omega)} \), then \( \mathcal{V}^s(\partial \Omega) \) becomes a Hilbert space. By the closed graph theorem, different choice of \( \lambda \in \mathbb{R} \setminus \sigma(A_D) \) leads to an equivalent norm on \( \mathcal{V}^s(\partial \Omega) \). Moreover, the embedding \( \mathcal{V}^s(\partial \Omega) \hookrightarrow L^2(\partial \Omega) \) is compact and \( \mathcal{V}^s(\partial \Omega) \) is dense in \( L^2(\partial \Omega) \).

The following result will be useful.
Lemma 3.2. Let \( \frac{1}{2} < s < 1 \), \( 0 \leq \varphi \in \mathcal{V}^s(\partial \Omega) \), \( \lambda \leq 0 \) and \( u \in \mathcal{H}^{s, \lambda}(\Omega) \) such that \( u|_{\partial \Omega} = \varphi \). Then \( u \geq 0 \) on \( \Omega \).

Proof. Let \( \lambda \leq 0 \). Notice that \( \lambda \not\in \sigma(A_\Omega) \). Let \( 0 \leq \varphi \in \mathcal{V}^s(\partial \Omega) \) and \( u \in \mathcal{H}^{s, \lambda}(\Omega) \) such that \( u|_{\partial \Omega} = \varphi \). We mention that, by [27, Lemma 2.6], \( u^+, u^- \in W^{s, 2}(\Omega) \). By definition of \( u \in \mathcal{H}^{s, \lambda}(\Omega) \), we have that for all \( v \in W^{s, 2}_0(\Omega) \),

\[
\frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N + 2s}} \, dx dy - \lambda \int_\Omega u v \, dx = 0. \tag{3.3}
\]

Since \( u = \varphi \geq 0 \) on \( \partial \Omega \) we have that \( u^- = 0 \) on \( \partial \Omega \). Since \( u^- \in W^{s, 2}(\Omega) \) and \( u^- = 0 \) on \( \partial \Omega \), it follows from (2.2) that \( u^- \in W^{s, 2}(\Omega) \). Choosing \( v = u^- \) in (3.3), we get that

\[
\frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(u^-(x) - u^-(y))}{|x - y|^{N + 2s}} \, dx dy - \lambda \int_\Omega u^- \, dx = 0. \tag{3.4}
\]

Since \( u = u^+ - u^- \), it follows from (3.4) that

\[
0 = -\frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u^-(x) - u^-(y)|^2}{|x - y|^{N + 2s}} \, dx dy + \lambda \int_\Omega |u^-|^2 \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x - y|^{N + 2s}} \, dx dy
\]

and this implies that (recall that \(- \lambda \geq 0\))

\[
0 \leq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u^-(x) - u^-(y)|^2}{|x - y|^{N + 2s}} \, dx dy - \lambda \int_\Omega |u^-|^2 \, dx = -\frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x - y|^{N + 2s}} \, dx dy \leq 0.
\]

Therefore, we have the following two cases.

- If \( \lambda < 0 \), then

\[
\min\{1, \lambda\} \|u^-\|_{W^{s, 2}(\Omega)}^2 \leq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u^-(x) - u^-(y)|^2}{|x - y|^{N + 2s}} \, dx dy - \lambda \int_\Omega |u^-|^2 \, dx = 0.
\]

Hence, \( u^- = 0 \) in \( \Omega \).

- If \( \lambda = 0 \), then using also the Poincaré inequality (2.20), we get that

\[
C \int_\Omega |u^-|^2 \, dx \leq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u^-(x) - u^-(y)|^2}{|x - y|^{N + 2s}} \, dx dy = 0.
\]

Hence, \( u^- = 0 \) in \( \Omega \).

We have shown that in any case, \( u^- = 0 \) in \( \Omega \) and this implies that \( u = u^+ \geq 0 \) on \( \Omega \). The proof is finished.

Now, we are ready to introduce the fractional Dirichlet-to-Neumann operator.

**Definition 3.3.** Let \( \frac{1}{2} < s < 1 \) and \( \lambda \in \mathbb{R} \setminus \sigma(A_\Omega) \). The fractional Dirichlet-to-Neumann operator \( D_{s, \lambda} \) is defined on \( L^2(\partial \Omega) \) by

\[
\begin{cases}
D(D_{s, \lambda}) = \{ g \in L^2(\partial \Omega) : \exists u \in W^{s, 2}(\Omega), \ u|_{\partial \Omega} = g, \ A_\Omega^s u = \lambda u \text{ and } \mathcal{N}^{2-2s} u \text{ exists in } L^2(\partial \Omega) \}, \\
D_{s, \lambda} g = C_s \mathcal{N}^{2-2s} u,
\end{cases}
\tag{3.5}
\]
Proposition 3.1. Using (3.6) we get that for all $\lambda \in \mathbb{R} \setminus \sigma(A_D)$, we define the bilinear symmetric form $F_\lambda$ with domain $V^s(\partial \Omega)$ by

$$F_\lambda(\varphi, \psi) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dy \, dx - \lambda \int_\Omega uv \, dx$$

where $u, v \in H^{s,\lambda}(\Omega)$ with $\varphi = u|_{\partial \Omega}$ and $\psi = v|_{\partial \Omega}$, that is, $\varphi, \psi \in V^s(\partial \Omega)$.

We have the following result.

**Proposition 3.1.** The bilinear symmetric form $F_\lambda$ is continuous, elliptic and the operator $D_{s,\lambda}$ defined in (3.5) is the closed linear self-adjoint operator on $L^2(\partial \Omega)$ associated with $F_\lambda$. Moreover, $D_{s,\lambda}$ has a compact resolvent and $-D_{s,\lambda}$ generates a strongly continuous analytic semigroup $(e^{-tD_{s,\lambda}})_{t \geq 0}$ on $L^2(\partial \Omega)$.

**Proof.** Let $\frac{1}{2} < s < 1$ and $\lambda \in \mathbb{R} \setminus \sigma(A_D)$. We prove the proposition in several steps.

**Step 1.** Using the Hölder inequality and Remark 4 we get that there exists a constant $C > 0$ such that for all $u, v \in H^{s,\lambda}(\Omega)$ with $\varphi = u|_{\partial \Omega}$ and $\psi = v|_{\partial \Omega}$, we have

$$|F_\lambda(\varphi, \psi)| \leq C\|u\|_{H^{s,\lambda}(\Omega)}\|v\|_{H^{s,\lambda}(\Omega)}$$

$$\leq C\|u\|_{\partial \Omega}\|v\|_{\partial \Omega} = C\|\varphi\|_{V_0(\partial \Omega)}\|\psi\|_{V_0(\partial \Omega)} = C\|\varphi\|_{V_0(\partial \Omega)}\|\psi\|_{V_0(\partial \Omega)}.$$

Hence, the form $F_\lambda$ is continuous.

**Step 2.** Since the embedding $W^{s,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact and $H^{s,\lambda}(\Omega)$ is continuously embedded into $L^2(\partial \Omega)$, then using (2.4), we have that, given $\varepsilon > 0$, there exists a constant $C_1 > 0$ such that

$$\int_\Omega |u|^2 \, dx \leq \varepsilon \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \, dx + C_1 \int_{\partial \Omega} |u|^2 \, d\sigma$$

(3.6)

for all $u \in H^{s,\lambda}(\Omega)$. Let $\varepsilon > 0$ be such that $\varepsilon(|\lambda| + \frac{1}{2}) = \frac{1}{2}$ and let $\omega := C_1(|\lambda| + \frac{1}{2})$. Using (3.6) we get that for all $u \in H^{s,\lambda}(\Omega)$,

$$F_\lambda(u|_{\partial \Omega}, u|_{\partial \Omega}) + \omega \int_{\partial \Omega} |u|^2 \, d\sigma$$

$$= \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \, dx - \lambda \int_\Omega |u|^2 \, dx + \omega \int_{\partial \Omega} |u|^2 \, d\sigma$$

$$\geq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \, dx + \frac{1}{2} \int_\Omega |u|^2 \, dx$$

$$\geq \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \, dx + \frac{1}{2} \int_\Omega |u|^2 \, dx$$

$$- \varepsilon(|\lambda| + \frac{1}{2}) C_{N,s} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \, dx$$

$$- \varepsilon(|\lambda| + \frac{1}{2}) \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dy \, dx$$

$$= \frac{1}{2}\|u\|_{H^{s,\lambda}(\Omega)}^2 = \frac{1}{2}\|u|_{\partial \Omega}\|_{V_0(\partial \Omega)}^2.$$

We have shown that $F_\lambda$ is elliptic.
Step 3: Let \( B \) be the closed linear self-adjoint operator on \( L^2(\partial \Omega) \) associated with the symmetric, continuous and elliptic form \( \mathcal{F}_\lambda \) in the sense that,
\[
\begin{align*}
D(B) &= \{ \varphi \in \mathcal{V}^s(\partial \Omega), \ \exists g \in L^2(\partial \Omega), \ \mathcal{F}_\lambda(\varphi, \psi) = (g, \psi)_{L^2(\partial \Omega)} \ \forall \psi \in \mathcal{V}^s(\partial \Omega) \}, \\
B\varphi &= g.
\end{align*}
\]
We claim that \( B = \mathbb{D}_{s,\lambda} \). Indeed, let \( u \in \mathcal{H}^{s,\lambda}(\Omega) \) and \( w \in L^2(\partial \Omega) \). Assume that \( u|_{\partial \Omega} \in D(B) \) and \( B(u|_{\partial \Omega}) = w \). Then by definition, for all \( v \in \mathcal{H}^{s,\lambda}(\Omega) \),
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} uv \, dx = \int_{\partial \Omega} uv \, d\sigma. \tag{3.7}
\]
We notice that for \( u \in \mathcal{H}^{s,\lambda}(\Omega) \) one has that
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} uv \, dx = 0 = \int_{\partial \Omega} uv \, d\sigma
\]
for all \( v \in W^{s,2}_0(\Omega) \). Since \( W^{s,2}_0(\Omega) \oplus \mathcal{H}^{s,\lambda}(\Omega) = W^{s,2}(\Omega) \) (by (3.1)), we have that (3.7) holds for all \( v \in W^{s,2}(\Omega) \). Now replacing \( \lambda u \) by \( A_{\Omega}^s u \) in (3.7) (since \( u \in \mathcal{H}^{s,\lambda}(\Omega) \)) we get that for all \( v \in W^{s,2}(\Omega) \),
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\partial \Omega} v A_{\Omega}^s u \, dx = \int_{\partial \Omega} uv \, d\sigma.
\]
Hence, \( u \) has a fractional normal derivative in \( L^2(\partial \Omega) \) and \( C_{s}N^{2-2s}u = w \) in the sense of Definition 2.5. We have shown that \( u|_{\partial \Omega} \in D(\mathbb{D}_{s,\lambda}) \) and \( \mathbb{D}_{s,\lambda}(u|_{\partial \Omega}) = w = B(u|_{\partial \Omega}) \).

Conversely, let \( \varphi \in D(\mathbb{D}_{s,\lambda}) \) and \( w := \mathbb{D}_{s,\lambda} \varphi \). Then there exists \( u \in \mathcal{H}^{s,\lambda}(\Omega) \) such that \( u|_{\partial \Omega} = \varphi \), the fractional normal derivative \( N^{2-2s}u \) exists in \( L^2(\partial \Omega) \) and \( C_{s}N^{2-2s}u = w \). By definition, this implies that (by using (2.14))
\[
\begin{align*}
\frac{C_{N,s}}{2} &\int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} uv \, dx \\
&= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \int_{\partial \Omega} v A_{\Omega}^s u \, dx \\
&= C_s \int_{\partial \Omega} v N^{2-2s}u \, d\sigma = \int_{\partial \Omega} uv \, d\sigma
\end{align*}
\]
for all \( v \in W^{s,2}(\Omega) \). We have shown that \( \varphi \in D(B) \), \( B\varphi = w \) and this completes the proof of the claim.

Step 4: The generation result of the strongly continuous analytic semigroup on \( L^2(\partial \Omega) \) by the operator \(-\mathbb{D}_{s,\lambda}\) follows from the fact that \( \mathcal{F}_\lambda \) is continuous and elliptic, and that \( \mathcal{V}^s(\partial \Omega) \) is dense in \( L^2(\partial \Omega) \). Since the embedding \( \mathcal{V}^s(\partial \Omega) \hookrightarrow L^2(\partial \Omega) \) is compact, we have that \( \mathbb{D}_{s,\lambda} \) has a compact resolvent and hence, has a discrete spectrum. The proof is finished. \( \square \)

Remark 5. Let \( \frac{1}{2} < s < 1 \), \( \lambda \in \mathbb{R} \setminus \sigma(A_D) \), \( u \in W^{s,2}(\Omega) \) and \( w \in L^2(\partial \Omega) \). It follows from the proof of Proposition 3.1 that \( u|_{\partial \Omega} \in D(\mathbb{D}_{s,\lambda}) \) and \( \mathbb{D}_{s,\lambda}(u|_{\partial \Omega}) = w \) if and only if for all \( v \in W^{s,2}(\Omega) \),
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} uv \, dx = \int_{\partial \Omega} uv \, d\sigma. \tag{3.8}
\]

The following theorem is the main result of this section.

Theorem 3.4. Let \( \frac{1}{2} < s < 1 \) and \( \lambda \in \mathbb{R} \setminus \sigma(A_D) \).
(a) If \( \lambda < \lambda_D^{s,2} \), then the semigroup \((e^{-tD^{s,2}_\lambda})_t\) is positive.
(b) If \( \lambda \leq 0 \), then the semigroup \((e^{-tD^{s,2}_\lambda})_t\) is Markovian, that is, \((e^{-tD^{s,2}_\lambda})_t\) is positive and
\[
0 \leq e^{-tD^{s,2}_\lambda} \varphi \leq 1, \quad \forall \ t \geq 0 \quad \text{whenever} \quad \varphi \in L^2(\partial \Omega), \ 0 \leq \varphi \leq 1.
\]
(c) If \( \lambda = 0 \), then \((e^{-tD^{s,2}_\lambda})_t\) is Markovian, that is, \((e^{-tD^{s,2}_\lambda})_t\) is positive and one has
\[
e^{-tD^{s,2}_\lambda} 1|_{\partial \Omega} = 1|_{\partial \Omega} \quad \text{for all} \ t \geq 0.
\]
(d) If \( \lambda_1 \leq \lambda_2 \leq 0 \), then
\[
0 \leq e^{-tD^{s,2}_{\lambda_1}} \varphi \leq e^{-tD^{s,2}_{\lambda_2}} \varphi
\]
pointwise for all \( t > 0 \) and \( 0 \leq \varphi \in L^2(\partial \Omega) \).

**Proof.** Let \( \frac{1}{2} < s < 1 \) and \( \lambda \in \mathbb{R} \setminus \sigma(A_D) \).
(a) Suppose that \( \lambda < \lambda_D^{s,2} \). Let \( \varphi \in V^s(\partial \Omega) \). Then \( \varphi^+, \varphi^- \in V^s(\partial \Omega) \). Indeed, let \( u \in W^{s,2}(\Omega) \) such that \( u|_{\partial \Omega} = \varphi \). We recall that by [27, Lemma 2.6], \( u^+, u^- \in W^{s,2}(\Omega) \). Moreover, \( u^+|_{\partial \Omega} = \varphi^+ \) and \( u^-|_{\partial \Omega} = \varphi^- \). Now, let \( \varphi \in V^s(\partial \Omega) \), \( u|_{\partial \Omega} = \varphi \) where \( u \in H^{s,\lambda}(\Omega) \). We write \( u^+ = u_0 + u_1 \in W^{s,2}_0(\Omega) \oplus H^{s,\lambda}(\Omega) \) and \( u^- = u_0 - u_2 \in W^{s,2}_0(\Omega) \oplus H^{s,\lambda}(\Omega) \). Since \( u = u^+ - u^- = (u_0 - \bar{u}_0) + (u_1 - u_2) \in W^{s,2}_0(\Omega) \oplus H^{s,\lambda}(\Omega) \), we have that \( u_0 - \bar{u}_0 = 0 \). Hence \( u_0 = \bar{u}_0 \). Since \( u_1, u_2 \in H^{s,\lambda}(\Omega) \) and \( u_0 \in W^{s,2}_0(\Omega) \), it follows that
\[
\frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_j(x) - u_j(y))(u_0(x) - u_0(y))}{|x - y|^{N+2s}} \ dxdy = \lambda \int_\Omega u_j u_0 \ dx, \quad j = 1, 2. \quad (3.9)
\]
Using (3.9) and Definition 2.5 we get that
\[
\begin{align*}
F_{\lambda}(\varphi^+, \varphi^-) &= \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} \ dxdy - \lambda \int_\Omega u_1 u_2 \ dx \\
&= \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))u_0(x) - u_0(y)}{|x - y|^{N+2s}} \ dxdy \\
&\quad - \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_2(x) - u_2(y))u_0(x) - u_0(y)}{|x - y|^{N+2s}} \ dxdy \\
&\quad - \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \ dxdy \\
&\quad - \lambda \int_\Omega (u_1 + u_0)(u_2 + u_0) \ dx + \lambda \int_\Omega u_1 u_0 \ dx + \lambda \int_\Omega u_0 u_2 \ dx + \lambda \int_\Omega |u_0|^2 \ dx \\
&= \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} \ dxdy - \lambda \int_\Omega u_1 u_2 \ dx \\
&\quad - \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \ dxdy + \lambda \int_\Omega |u_0|^2 \ dx \\
&= \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} \ dxdy - \lambda \int_\Omega u_1 u_2 \ dx \\
&\quad - \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \ dxdy + \lambda \int_\Omega |u_0|^2 \ dx \\
&= \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} \ dxdy - \lambda \int_\Omega u_1 u_2 \ dx \\
&\quad - \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \ dxdy + \lambda \int_\Omega |u_0|^2 \ dx \\
&= \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{(u_1(x) - u_1(y))(u_2(x) - u_2(y))}{|x - y|^{N+2s}} \ dxdy - \lambda \int_\Omega u_1 u_2 \ dx \\
&\quad - \frac{C_{N.s}}{2} \int_\Omega \int_\Omega \frac{|u_0(x) - u_0(y)|^2}{|x - y|^{N+2s}} \ dxdy + \lambda \int_\Omega |u_0|^2 \ dx. \quad (3.10)
\end{align*}
\]
Since \( u^+(x)u^-(y) + u^-(x)u^+(y) \geq 0 \) for almost every \((x, y) \in \Omega \times \Omega\), we have that
\[
-\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{u^+(x)u^-(y) + u^-(x)u^+(y)}{|x-y|^{N+2s}} \, dx \, dy \leq 0. \tag{3.11}
\]
Since \( \lambda < \lambda^D_{1,s} \) and \( u_0 \in W^{s,2}_0(\Omega) \), it follows from the Poincaré inequality (2.20) that
\[
-\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \lambda \int_{\Omega} |u_0|^2 \, dx \leq 0. \tag{3.12}
\]
Combining (3.10), (3.11) and (3.12) we get that
\[
\mathcal{F}_\lambda(\varphi^+, \varphi^-) \leq 0. \tag{3.13}
\]
By [13, Theorem 1.3.2], (3.13) implies that the semigroup \((e^{-tD_{s,\lambda}})_{t \geq 0}\) is positive.

(b) Let \( \lambda \leq 0 \). It follows from part (a) that \((e^{-tD_{s,\lambda}})_{t \geq 0}\) is positive. Let
\( 0 \leq \varphi \in V^s(\partial \Omega) \). Then \( \varphi \wedge 1 \in V^s(\partial \Omega) \). In fact, let \( 0 \leq u \in W^{s,2}(\Omega) \) such that \( u|_{\partial \Omega} = \varphi \). It follows from [27, Lemma 2.7] that \( u \wedge 1 \in W^{s,2}(\Omega) \) and
\[
|u \wedge 1|^2_{W^{s,2}(\Omega)} \leq |u|^2_{W^{s,2}(\Omega)}. \tag{3.14}
\]
Moreover, \((u \wedge 1)|_{\partial \Omega} = \varphi \wedge 1 \). Now, let \( 0 \leq \varphi \in V^s(\partial \Omega) \), \( u|_{\partial \Omega} = \varphi \) where \( 0 \leq u \in H^{s,\lambda}(\Omega) \). We write \( u \wedge 1 = u_0 + u_1 \in W^{s,2}_0(\Omega) \oplus H^{s,\lambda}(\Omega) \). Since \( u_1 \in H^{s,\lambda}(\Omega) \) and \( u_0 \in W^{s,2}_0(\Omega) \), we have that
\[
\frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u_1(x) - u_1(y))(u_0(x) - u_0(y))}{|x-y|^{N+2s}} \, dx \, dy = \lambda \int_{\Omega} u_0 u_1 \, dx. \tag{3.15}
\]
Using (3.15), Definition 2.5, the fact that \( \lambda \leq 0 < \lambda^D_{1,s} \), the Poincaré inequality (2.20) and (3.14), we get that
\[
\mathcal{F}_\lambda(\varphi \wedge 1, \varphi \wedge 1) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_1(x) - u_1(y)|^2}{|x-y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} |u_1|^2 \, dx
\]
\[
= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_1 + u_0(x) - (u_1 + u_0(y))|^2}{|x-y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} |u_1 + u_0|^2 \, dx
\]
\[
- \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_1(x) - u_1(y))(u_0(x) - u_0(y))}{|x-y|^{N+2s}} \, dx \, dy
\]
\[
- \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + 2\lambda \int_{\Omega} u_0 u_1 \, dx + \lambda \int_{\Omega} |u_0|^2 \, dx
\]
\[
= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u \wedge 1(x) - (u \wedge 1(y)|^2}{|x-y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} |u \wedge 1|^2 \, dx
\]
\[
- \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \lambda \int_{\Omega} |u_0|^2 \, dx
\]
\[
\leq \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u \wedge 1(x) - (u \wedge 1(y)|^2}{|x-y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} |u \wedge 1|^2 \, dx
\]
\[
- \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u_0(x) - u_0(y)|^2}{|x-y|^{N+2s}} \, dx \, dy + \lambda^D_{1,s} \int_{\Omega} |u_0|^2 \, dx
\]
\[
\leq \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} |u|^2 \, dx = \mathcal{F}_\lambda(\varphi, \varphi). \tag{3.16}
\]
By [13, Theorem 1.3.3], the estimate (3.16) implies that \((e^{-tD_{s,\lambda}})_{t \geq 0}\) is submarkovian.
(c) Now assume that \( \lambda = 0 \). By part (b), the semigroup \( (e^{-tD_s, \lambda})_{t \geq 0} \) is submarkovian. Since the constant function \( 1_{|\partial \Omega} \in D(D_{s, 0}) \) and \( D_{s, 0}(1_{|\partial \Omega}) = 0 \) (by Remark 5), it follows that \( e^{-tD_s, \lambda}1_{|\partial \Omega} = 1_{|\partial \Omega} \) for all \( t \geq 0 \).

(d) Finally let \( \lambda_1 \leq \lambda_2 \leq 0 \). It follows from part (b) that \( (e^{-tD_s, \lambda_1})_t \geq 0 \) and \( (e^{-tD_s, \lambda_2})_t \geq 0 \) are both submarkovian. Since \( F_{\lambda_1}(\varphi, \varphi) \geq 0, j = 1, 2, \) for all \( \varphi \in V^s(\partial \Omega) \), then using the domination criteria of semigroups associated with symmetric forms (see e.g. [22, Theorem 2.24]), it suffices to show that for all \( 0 \leq \varphi, \psi \in V^s(\partial \Omega) \),

\[
F_{\lambda_2}(\varphi, \psi) \leq F_{\lambda_1}(\varphi, \psi).
\]

Indeed, let \( 0 \leq \varphi, \psi \in V_s(\partial \Omega) \). There exist \( u_1, v_1 \in \mathcal{H}^{s, \lambda_1}(\Omega) \) and \( u_2, v_2 \in \mathcal{H}^{s, \lambda_2}(\Omega) \) such that \( u_1|_{\partial \Omega} = u_2|_{\partial \Omega} = \varphi \) and \( v_1|_{\partial \Omega} = v_2|_{\partial \Omega} = \psi \). Since \( u_2 - u_1 \in W^s_0(\Omega) \) and \( v_2 \in \mathcal{H}^{s, \lambda_2}(\Omega) \), we have

\[
F_{\lambda_2}(\varphi, \psi) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u_2(x) - u_1(x))(v_2(x) - v_1(x))}{|x - y|^{N+2s}} \, dx \, dy - \lambda_2 \int_{\Omega} u_1 v_2 \, dx
\]

\[
= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u_1(x) - u_1(y))(v_2(x) - v_2(y))}{|x - y|^{N+2s}} \, dx \, dy - \lambda_1 \int_{\Omega} u_1 v_1 \, dx
\]

\[
+ (\lambda_1 - \lambda_2) \int_{\Omega} u_1 v_2 \, dx
\]

\[
= \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{(u_1(x) - u_1(y))(v_1(x) - v_1(y))}{|x - y|^{N+2s}} \, dx \, dy - \lambda_1 \int_{\Omega} u_1 v_1 \, dx
\]

\[
+ (\lambda_1 - \lambda_2) \int_{\Omega} u_1 v_2 \, dx
\]

\[
= F_{\lambda_1}(\varphi, \psi) + (\lambda_1 - \lambda_2) \int_{\Omega} u_1 v_2 \, dx.
\]

Since \( \lambda_1 - \lambda_2 \leq 0 \) and \( u_1 \geq 0, \, v_2 \geq 0 \) (by Lemma 3.2), it follows that \( (\lambda_1 - \lambda_2) \int_{\Omega} u_1 v_2 \, dx \leq 0 \). Hence, the estimate (3.17) follows from (3.18). The proof is finished.

The following result shows that the semigroup can also be ultracontractive.

**Proposition 3.2.** Let \( \frac{1}{2} < s < 1 \), \( \lambda \leq 0 \) and suppose that \( N > 2s \). Let \( \mu_{1,s}(\lambda) \) be the first eigenvalue of \( D_{s, \lambda} \). Then for all \( 1 \leq p \leq q \leq \infty \) and \( t > 0 \), the operator \( e^{-tD_s, \lambda} \) is bounded from \( L^p(\partial \Omega) \) into \( L^q(\partial \Omega) \). More precisely, there is a constant \( C > 0 \) such that

\[
\|e^{-tD_s, \lambda}\|_{L^q(\partial \Omega) \rightarrow L^p(\partial \Omega)} \leq C(t \wedge 1)^{-\frac{(N-1)}{2s} \left( \frac{1}{p} - \frac{1}{q} \right)} e^{-\mu_{1,s}(\lambda) \frac{1}{2} - \frac{1}{s} t},
\]

(3.19) for all \( t > 0 \) and \( p, q \in [1, \infty] \) with \( p \leq q \).

**Proof.** Let \( \frac{1}{2} < s < 1 \) and \( \lambda \leq 0 \). Recall that the semigroup is submarkovian. Hence, for every \( p \in [1, \infty] \) and \( t \geq 0 \), we have that

\[
\|e^{-tD_s, \lambda}\|_{L^p(\partial \Omega)} \leq 1.
\]

(3.20) Moreover there exist consistent semigroups of contractions on \( L^p(\partial \Omega) \), \( p \in [1, \infty] \) and each of such semigroup is strongly continuous if \( p \in [1, \infty] \). Since the semigroup is also analytic on \( L^2(\partial \Omega) \), we have that \( e^{-tD_{s, \lambda} \phi} \in D(\mathcal{D}_{s, \lambda}) \subset V^s(\partial \Omega) \) for every
\( \phi \in L^2(\partial \Omega) \) and \( t > 0 \). Moreover, there exists a constant \( C_1 > 0 \) such that for every \( \phi \in L^2(\partial \Omega) \) and \( t > 0 \),

\[
\| D_{s,\lambda} e^{-tD_{s,\lambda}} \phi \|_{L^2(\partial \Omega)} \leq \frac{C_1}{t} \| \phi \|_{L^2(\partial \Omega)}. \tag{3.21}
\]

Next, suppose that \( N > 2s \). It follows from (2.3) that there is a constant \( C > 0 \) such that for every \( \phi \in \mathcal{V}^s(\partial \Omega) \),

\[
\| \phi \|_{L^2^*(\partial \Omega)} \leq C \left( \mathcal{F}_\lambda(\phi, \phi) + \| \phi \|_{L^2(\partial \Omega)}^2 \right), \quad 2^* := \frac{2(N-1)}{N-2s}. \tag{3.22}
\]

Using the estimates (3.20), (3.21) and (3.22) with \( \phi = e^{-tD_{s,\lambda}} \varphi \), we get that

\[
\| e^{-tD_{s,\lambda}} \varphi \|_{L^2^*(\partial \Omega)} \leq C \left( \mathcal{F}_\lambda(e^{-tD_{s,\lambda}} \varphi, e^{-tD_{s,\lambda}} \varphi) + \| e^{-tD_{s,\lambda}} \varphi \|_{L^2(\partial \Omega)}^2 \right)
\]

\[
= C \left( \| D_{s,\lambda} e^{-tD_{s,\lambda}} \varphi, e^{-tD_{s,\lambda}} \varphi \|_{L^2(\partial \Omega)} + \| e^{-tD_{s,\lambda}} \varphi \|_{L^2(\partial \Omega)}^2 \right)
\]

\[
\leq C \left( \| D_{s,\lambda} e^{-tD_{s,\lambda}} \varphi \|_{L^2(\partial \Omega)} + \| \varphi \|_{L^2(\partial \Omega)}^2 \right)
\]

for all \( t > 0 \) and \( \varphi \in L^2(\partial \Omega) \). Therefore, \( e^{-tD_{s,\lambda}} \) maps \( L^2(\partial \Omega) \) into \( L^2^*(\partial \Omega) \) with

\[
\| e^{-tD_{s,\lambda}} \|_{L^2^*(\partial \Omega), L^2^*(\partial \Omega)} \leq Ct^{-\frac{N-1}{2s}} e^{\omega_1 t}. \tag{3.23}
\]

Note that \( 2^* := \frac{2(N-1)}{N-2s} \) with \( d = \frac{2(N-1)}{2s-1} \). We claim that the estimate (3.23) extrapolates and gives the estimate

\[
\| e^{-tD_{s,\lambda}} \|_{L^p(\partial \Omega), L^{p^*}(\partial \Omega)} \leq C_1 t^{-\frac{N-1}{2s}} e^{\omega_1 t} \tag{3.24}
\]

for some constant \( C_1 > 0 \), \( \omega_1 = \frac{2(N-1)}{2s-1} \) and uniformly for all \( t > 0 \). We proceed as in the proof of [22, Lemma 6.1]. By the Riesz-Thorin interpolation theorem [13, Section 1.1.5], we get from (3.23) that for every \( p \in [2, \infty) \),

\[
\| e^{-tD_{s,\lambda}} \|_{L^p(\partial \Omega), L^{\frac{2p}{p-1}}(\partial \Omega)} \leq C_1 t^{-\frac{N-1}{2s}} e^{\omega_1 t}, \quad \forall t > 0. \tag{3.25}
\]

Let \( t_k := \frac{2^* - 1}{2^*}(2^*)^{-k} \) and \( p_k := 2 \left( \frac{2^*}{2} \right)^k \) for \( k \geq 0 \). Then

\[
\sum_{k=0}^\infty t_k = 1 \quad \text{and} \quad \sum_{k=0}^\infty \frac{1}{p_k} = \frac{2^*}{2(2^* - 2)} = \frac{N-1}{2(2s-1)}.
\]

Applying the estimate (3.25) with \( p = p_k \) yields

\[
\| e^{-tD_{s,\lambda}} \|_{L^2(\partial \Omega), L^{\infty}(\partial \Omega)} \leq \prod_{k=0}^\infty \| e^{-t_k D_{s,\lambda}} \|_{L^p_k(\partial \Omega), L^{p_k+1}(\partial \Omega)}
\]

\[
\leq \prod_{k=0}^\infty C_1 t^{-\frac{N-1}{2s}} e^{\omega_1 t} = C_1 t^{-\frac{N-1}{2s-1}} e^{\omega_1 t}, \tag{3.26}
\]

with \( \omega := \frac{N-1}{2s-1} \). By duality, we get from (3.26) that

\[
\| e^{-tD_{s,\lambda}} \|_{L^1(\partial \Omega), L^2(\partial \Omega)} \leq C_1 t^{-\frac{N-1}{2s-1}} e^{\omega_1 t}. \tag{3.27}
\]
Combining (3.26) and (3.27) we get the estimate (3.24) and the claim is proved. Next, proceeding as in [22, Lemma 6.5] by using the estimate
\[ \|e^{-tD_{s,\lambda}}\|_{L^p(\partial\Omega)} \leq C e^{-\mu_{1,s}(\lambda)t}, \quad \text{for some } C > 0, \forall t \geq 0, \tag{3.28} \]
we get that the estimate (3.24) improves to
\[ \|e^{-tD_{s,\lambda}}\|_{L^1(\partial\Omega), L^\infty(\partial\Omega)} \leq C_1 t^{-\frac{N-1}{2\sigma-1}} e^{-\mu_{1,s}(\lambda)t}. \quad \tag{3.29} \]

Next, let \( \frac{1}{p} = \frac{\alpha}{2} + \frac{1}{\infty}, \) i.e., \( \alpha = \frac{1}{p}. \) Using the Riesz-Thorin interpolation theorem again, we get that
\[ \|e^{-tD_{s,\lambda}}\|_{L^p(\partial\Omega)}, L^\infty(\partial\Omega)} \leq \|e^{-tD_{s,\lambda}}\|_{L^1(\partial\Omega), L^\infty(\partial\Omega)} \|e^{-tD_{s,\lambda}}\|_{L^\infty(\partial\Omega)} \]
\[ \leq C_1 t^{-\frac{N-1}{2\sigma-1}} e^{-\mu_{1,s}(\lambda)t} (1 + t)^{-\frac{\alpha}{p}}. \quad \tag{3.30} \]

Finally, let \( \frac{1}{q} = \frac{\beta}{p} + \frac{1}{\infty}, \) i.e., \( \beta = \frac{p}{q}. \) It follows from the Riesz-Thorin interpolation theorem and (3.30) that
\[ \|e^{-tD_{s,\lambda}}\|_{L^p(\partial\Omega), L^\infty(\partial\Omega)} \leq \|e^{-tD_{s,\lambda}}\|_{L^p(\partial\Omega), L^\infty(\partial\Omega)} \|e^{-tD_{s,\lambda}}\|_{L^\infty(\partial\Omega)} \]
\[ \leq \left( C_1 t^{-\frac{N-1}{2\sigma-1}} e^{-\mu_{1,s}(\lambda)t} (1 + t)^{-\frac{\beta}{q}} \right)^{1-\frac{\beta}{p}} \]
\[ = C_1 \frac{1}{t} t^{-\frac{N-1}{2\sigma-1}} e^{-\mu_{1,s}(\lambda)(1 + t)^{-\frac{\beta}{q}}}. \]

The preceding estimate implies (3.19). The proof is finished.

We conclude the section by giving the sign of the first eigenvalue of \( D_{s,\lambda}. \)

**Proposition 3.3.** Let \( \lambda \in \mathbb{R} \setminus \sigma(A_D) \) and \( \mu_{1,s}(\lambda) \) the first eigenvalue of \( D_{s,\lambda}. \) If \( \lambda < 0 \) then \( \mu_{1,s}(\lambda) > 0, \) if \( \lambda > 0 \) then \( \mu_{1,s}(\lambda) < 0 \) and \( \mu_{1,s}(0) = 0. \)

We postpone the proof of the proposition to the next section. We also notice that using (3.28), it follows from Proposition 3.3 that if \( \lambda < 0, \) then the semigroup \( (e^{-tD_{s,\lambda}})_{t \geq 0} \) is exponentially stable and if \( \lambda > 0, \) then the semigroup is unstable. This is consistent with the corresponding result for the semigroup generated by the classical Dirichlet-to-Neumann operator contained in [12, Proposition 2.1].

4. **Comparison of the eigenvalues.** In the present section we use the fractional Dirichlet-to-Neumann operator to compare the eigenvalues of the operators \( A_D \) and \( A_N \) defined in (2.19) and (2.18), respectively.

**Theorem 4.1.** Let \( \frac{1}{2} < s < 1, \lambda \in \mathbb{R} \setminus \sigma(A_D) \) and \( \mu \in \mathbb{R}. \) Then the following assertions hold.

(a) \( \mu \in \sigma(D_{s,\lambda}) \) if and only if \( \lambda \in \sigma(A_{\mu}), \) and

(b) \( \dim \ker(\mu - D_{s,\lambda}) = \dim \ker(\lambda - A_{\mu}). \)

**Proof.** For a linear closed operator \( A \) with domain \( D(A), \) we denote by \( \ker(A) := \{ u \in D(A), \ Au = 0 \}. \) Let \( \frac{1}{2} < s < 1, \lambda \in \mathbb{R} \setminus \sigma(A_D) \) and \( \mu \in \mathbb{R}. \) To prove the two assertions, it suffices to show that the mapping
\[ T : \ker(\lambda - A_{\mu}) \to \ker(\mu - D_{s,\lambda}), \ u \mapsto u|_{\partial\Omega}, \]
is an isomorphism. Indeed, let \( u \in \ker(\lambda - A_{\mu}). \) Then \( E_{\mu}(u, v) = \lambda \int_{\Omega} uv \, dx \) for all \( v \in W^{s,2}(\Omega), \) that is,
\[ \frac{C_N}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dxdy - \lambda \int_{\Omega} uv \, dx = \mu \int_{\Omega} uv \, d\sigma. \quad \tag{4.1} \]
Theorem 4.2. Let $v \in W^{s,2}(\Omega)$. By Remark 5, this implies that $u|_{\partial \Omega} \in D(D_{s,\lambda})$ and $D_{s,\lambda}(u|_{\partial \Omega}) = \mu u|_{\partial \Omega}$. If $u|_{\partial \Omega} = 0$, then $u \in W^{s,2}_0(\Omega) \cap D(D_{s,\lambda}) \subset W^{s,2}_0(\Omega) \cap H^{s,\lambda}(\Omega) = \{0\}$. We have shown that $T$ defines a one-to-one mapping from $\ker(\lambda - A_\mu)$ into $\ker(\mu - D_{s,\lambda})$. Now we show the surjectivity. Let $\varphi \in \ker(\mu - D_{s,\lambda})$. Then by Remark 5, there exists $u \in H^{s,\lambda}(\Omega)$ such that $\varphi = u|_{\partial \Omega}$ and (4.1) holds for all $v \in W^{s,2}(\Omega)$. That is, $\mathcal{E}_\mu(u, v) = \lambda \int_{\Omega} u v \, dx$ for all $v \in W^{s,2}(\Omega)$. Therefore, $u \in D(A_\mu)$ and $A_\mu u = \lambda u$. The proof is finished.

We have the following result.

Theorem 4.2. Let $\frac{1}{2} < s < 1$ and $n \in \mathbb{N}$. Then the function $\lambda_{n,s}(\cdot) : [-\infty, \infty) \to \mathbb{R}$ is continuous and decreasing. In particular, we have that

$$\lim_{\mu \to -\infty} \lambda_{n,s}(\mu) = \lambda_{n,s}^D.$$  

(4.2)

To prove the theorem, we need the following result taken from [21].

Lemma 4.3. Let $B_n, B$ be self-adjoint operators with compact resolvent on a separable Hilbert space $H$. Assume that there exists a constant $\omega \in \mathbb{R}$ such that $(B_n x, x)_H \geq \omega (x, x)_H$ for all $x \in D(B_n)$, $n \in \mathbb{N}$ and that $(\lambda + B_n)^{-1} \to (\lambda + B)^{-1}$ in $\mathcal{L}(H)$ as $n \to \infty$ for all $\lambda > \omega$. Denote by $\lambda_{n,k}$ the $k$th eigenvalue of $B_n$ and by $\lambda_k$ the $k$th eigenvalue of $B$ repeating according to their multiplicity. Then $\lim_{n \to \infty} \lambda_{n,k} = \lambda_k$.

Proof of Theorem 4.2. Recall that $\lambda_{n,s}(\mu)$, $n \in \mathbb{N}$, are the eigenvalues of the operator $A_\mu$ associated with the bilinear form $\mathcal{E}_\mu$ given for $u, v \in D(\mathcal{E}_\mu) = W^{s,2}(\Omega)$, $-\infty < \mu < \infty$, by

$$\mathcal{E}_\mu(u, v) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy - \mu \int_\Omega uv \, dx,$$

and for $u, v \in D(\mathcal{E}_{-\infty}) = W^{s,2}_0(\Omega)$ by

$$\mathcal{E}_{-\infty}u(u, v) = \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy.$$

Since $\mathcal{E}_\mu(u, u) \geq 0$ for every $u \in D(\mathcal{E}_\mu)$ and $-\infty \leq \mu \leq 0$, it follows from the min-max formula for the eigenvalues that $\lambda_{n,s}(\mu) \geq 0$ for all $n \in \mathbb{N}$ if $-\infty \leq \mu \leq 0$. Note also that $\lambda_{1,s}(0) = \lambda_{1,s}^N = 0$. We prove the theorem in three steps.

Step 1: Let $-\infty \leq \mu_0 < \infty$ be fixed. We claim that for $\lambda \in \mathbb{R}$ large enough,

$$\lim_{\mu \to \mu_0} (\lambda + A_\mu)^{-1} = (\lambda + A_{\mu_0})^{-1} \text{ in } \mathcal{L}(L^2(\Omega)).$$  

(4.3)

Indeed, let $\lambda \in \mathbb{R}$ be large enough, $f \in L^2(\Omega)$ and $f_n \in L^2(\Omega)$ a sequence that converges weakly to $f$ in $L^2(\Omega)$. Let $\mu_n \in (-\infty, \infty)$ be a sequence that converges to $\mu_0 \in [-\infty, \infty)$ and set $u_n := (\lambda + A_{\mu_n})^{-1} f_n \in W^{s,2}(\Omega)$. We have to show that after passing to a subsequence, if necessary, $u_n$ converges to $(\lambda + A_{\mu_0})^{-1} f$ in $L^2(\Omega)$. By definition, we have that for all $v \in W^{s,2}(\Omega)$,

$$\lambda \int_\Omega u_n v \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u_n(x) - u_n(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy$$

$$- \mu_n \int_{\partial \Omega} u_n v \, d\sigma = \int_\Omega f_n v \, dx.$$  

(4.4)
In particular, taking \( v = u_n \) in (4.4), we get that
\[
\lambda \int_\Omega |u_n|^2 \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \mu_n \int_{\partial \Omega} |u_n|^2 \, d\sigma
= \int_\Omega f_n u_n \, dx.
\]
(4.5)
Since \( \lambda > 0 \) is large enough and the bilinear form \( E_{\mu_n} \) is elliptic (by Proposition 2.1), we have that
\[
\lambda \int_\Omega |u_n|^2 \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \mu_n \int_{\partial \Omega} |u_n|^2 \, d\sigma
\geq \alpha \|u_n\|_{W^{s,2}(\Omega)}^2
\]
for some constant \( \alpha > 0 \) (independent of \( n \)) and for all \( n \in \mathbb{N} \). Since
\[
\int_\Omega f_n u_n \, dx \leq \|f_n\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega)},
\]
it follows from (4.5) and (4.6) that \( u_n \) is a bounded sequence in \( W^{s,2}(\Omega) \). Therefore (after passing to a subsequence, if necessary), \( u_n \) converges weakly to some \( u \) in \( W^{s,2}(\Omega) \). Since the embedding \( W^{s,2}(\Omega) \hookrightarrow L^2(\Omega) \) and \( W^{s,2}(\Omega) \hookrightarrow L^2(\partial \Omega) \) are compact, we have that \( u_n \) converges strongly to \( u \in L^2(\Omega) \) and \( u_n|_{\partial \Omega} \) converges strongly to \( u|_{\partial \Omega} \in L^2(\partial \Omega) \).

- If \( \mu_0 \neq -\infty \), then taking the limit of (4.4) as \( n \to \infty \), we get that
\[
\lambda \int_\Omega uv \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy
- \mu_0 \int_{\partial \Omega} uv \, d\sigma = \int_\Omega fv \, dx
\]
for all \( v \in W^{s,2}(\Omega) \). Hence, \( u = (\lambda + A_{\mu_0})^{-1} f \). Taking the limit of (4.5) as \( n \to \infty \), we get that
\[
\lim_{n \to \infty} \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} \, dx \, dy = \int_\Omega fu \, dx - \lambda \int_\Omega u^2 \, dx + \mu_0 \int_{\partial \Omega} |u|^2 \, d\sigma
= \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.
\]
We have shown that \( u_n \to u \) weakly in \( W^{s,2}(\Omega) \) and \( \lim_{n \to \infty} \|u_n\|_{W^{s,2}(\Omega)} = \|u\|_{W^{s,2}(\Omega)} \). This implies that \( u_n \to u \) strongly in \( W^{s,2}(\Omega) \).

- If \( \mu_0 = -\infty \), then \( \lim_{n \to \infty} (-\mu_n) = \infty \). Therefore, we get from (4.5) that
\[
\lim_{n \to \infty} \int_{\partial \Omega} |u_n|^2 \, d\sigma = 0.
\]
Hence, \( u|_{\partial \Omega} = 0 \) and by (2.2), \( u \in W^{s,2}_0(\Omega) \). Taking the limit of (4.4) as \( n \to \infty \), we get that for all \( v \in W^{s,2}_0(\Omega) \),
\[
\lambda \int_\Omega uv \, dx + \frac{C_{N,s}}{2} \int_\Omega \int_\Omega \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy = \int_\Omega fv \, dx.
\]
Hence, \( u = (\lambda + A_D)^{-1} f \). Since \( u_n \) converges strongly to \( u \in L^2(\Omega) \), it follows from [11, Appendix B] that \( (\lambda + A_{\mu_n})^{-1} \to (\lambda + A_D)^{-1} \) in \( L(L^2(\Omega)) \) as \( n \to \infty \) and we have shown the claim (4.3).
Step 2: Since the operators $A_{\mu_n}$ and $A_{\mu_0}$ are self-adjoint with compact resolvent and there exists a constant $\omega \in \mathbb{R}$ such that for all $u \in D(A_{\mu_n})$,
\[
\int_{\Omega} u A_{\mu_n} u \, dx \geq \omega \int_{\Omega} |u|^2 \, dx,
\]
and since $(\lambda + A_{\mu_n})^{-1} \rightarrow (\lambda + A_{\mu_0})^{-1}$ in $L(L^2(\Omega))$ as $n \to \infty$, then (4.2) follows from Lemma 4.3.

Step 3: It remains to show that $\lambda_{n,s}(\cdot)$ is decreasing and continuous. Indeed, it follows from the definition of $E_\mu$ and the min-max definition of the eigenvalues that $\lambda_{n,s}(\cdot)$ is at least non-increasing. Suppose that $\lambda_{n,s}(\mu_1) = \lambda_{n,s}(\mu_2)$ for some $\mu_1 < \mu_2$. Let $\lambda := \lambda_{n,s}(\mu_1)$. It follows from Theorem 4.1 that $\mu \in \sigma(\mathbb{D}_{s,\lambda})$ for all $\mu \in [\mu_1, \mu_2]$. But this is impossible since $\sigma(\mathbb{D}_{s,\lambda})$ is discrete (recall that $\mathbb{D}_{s,\lambda}$ has a compact resolvent by Proposition 3.1). Hence, $\lambda_{n,s}(\cdot)$ is decreasing. Finally a standard eigenvalue perturbation theory shows that $\lambda_{n,s}(\cdot)$ is continuous (see e.g. [21]). The proof is finished.

Proof of Proposition 3.3. Let $\lambda \in \mathbb{R} \setminus \sigma(A_D)$ and $\mu_{1,s}(\lambda)$ the first eigenvalue of $\mathbb{D}_{s,\lambda}$ which is given by the infimum of the Rayleigh coefficient, that is,
\[
\mu_{1,s}(\lambda) = \inf_{\varphi \in V^s(\partial \Omega), \varphi \neq 0} \frac{\mathcal{F}_\lambda(\varphi, \varphi)}{\|\varphi\|_{L^2(\partial \Omega)}^2}.
\]
Fix, $\varphi \in V^s(\partial \Omega)$ and let $u \in W^{s,2}(\Omega)$ be the solution of the Dirichlet problem
\[
A_D^s u = \lambda u \text{ in } \Omega, \quad u = \varphi \text{ on } \partial \Omega.
\]
The existence of solution follows from [19]. If $\lambda < 0$, then
\[
\mathcal{F}_\lambda(\varphi, \varphi) = \frac{C_{N,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \lambda \int_{\Omega} |u|^2 \, dx \geq 0,
\]
so that $\mu_{1,s}(\lambda) \geq 0$. If $\lambda = 0$ and $\varphi = c$ is a constant, then $u = c$ and $\mu_{1,s}(0) = 0$.
Recall that, since the embedding $V^s(\partial \Omega) \hookrightarrow L^2(\partial \Omega)$ is compact, we have that the spectrum of $\mathbb{D}_{s,\lambda}$ is discrete. We also recall that $\mu \in \sigma(\mathbb{D}_{s,\lambda})$ if and only if $\lambda \in \sigma(A_D)$ (by Theorem 4.1). Let $\mu_{k,s}(\lambda)$, $k \in \mathbb{N}$, be the eigenvalues of $\mathbb{D}_{s,\lambda}$. Then the functions $\mu_{k,s}(\cdot)$ are the inverses of the functions $\lambda_{k,s}(\cdot)$, $k \in \mathbb{N}$ (the existence of the inverse follows from the fact that $\lambda_{k,s}(\cdot)$ is decreasing by Theorem 4.2). Since $\lambda_{k,s}(\mu_2) < \lambda < \lambda_{k,s}(\mu_1)$, $k \in \mathbb{N}$, is decreasing and continuous in $\mu$ (by Theorem 4.2), we have that $\mu_{k,s}(\lambda)$, $k \in \mathbb{N}$, is decreasing and continuous in $\lambda$. We are interested to the case $k = 1$. Since $\mu_{1,s}(0) = 0$, we have that if $\lambda > 0$, then $\mu_{1,s}(\lambda) < \mu_{1,s}(0) = 0$. We have shown that $\mu_{1,s}(\lambda) > 0$ if $\lambda < 0$ and $\mu_{1,s}(\lambda) < 0$ if $\lambda > 0$. The proof is finished.

Now we are ready to state and prove the main result of this section.

Theorem 4.4. Let $\frac{1}{2} < s < 1$. Then
\[
\lambda_{n+1,s}^N \leq \lambda_{n,s}^D, \quad n \in \mathbb{N}.
\]
Proof. Let $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$. Recall that by Theorem 4.2, $\lambda_{n,s}(\mu)$ is decreasing in $\mu$. Moreover, $\lambda_{n,s}(0) = \lambda_{n,s}^N$ and $\lim_{\mu \to -\infty} \lambda_{n,s}(\mu) = \lambda_{n,s}^N$. Assume that there exists $k \in \mathbb{N}$ such that $\lambda_{k,s}^D < \lambda_{k+1,s}^N$. Choose $\lambda_{k,s}^D < \lambda < \lambda_{k+1,s}^N$. Then for all $\mu \leq 0$ we have
\[
\begin{cases}
\lambda_{n,s}(\mu) \leq \lambda_{k,s}(\mu) < \lambda_{k,s}^D < \lambda, & \forall \ n \leq k, \\
\lambda_{n,s}(\mu) \geq \lambda_{k+1,s}(\mu) \geq \lambda_{k+1,s}(0) = \lambda_{k+1,s}^N > \lambda, & \forall \ n \geq k + 1.
\end{cases}
\]
Hence, \( \lambda \neq \lambda_{n,s}(\mu) \) for all \( \mu \leq 0, \ n \in \mathbb{N} \), that is, \( \lambda \not\in \sigma(A_\mu) \) whenever \( \mu \leq 0 \). By Theorem 4.1 this implies that

\[
\sigma(D_{s,\lambda}) \cap (-\infty, 0] = \emptyset. \tag{4.8}
\]

On the other hand, since \( \lambda > 0 \), it follows from Proposition 3.3 that the first eigenvalue of \( D_{s,\lambda} \) is negative, that is, \( \sigma(D_{s,\lambda}) \cap (-\infty, 0] \neq \emptyset \). This is a contradiction to (4.8) and the proof is finished.

To conclude the paper, we notice that it follows from (2.22) and (4.7) that

\[
0 < \lambda_{n+1,s}^N \leq \lambda_{n,s}^D \leq \lambda_{n,s,\Omega}, \quad n \in \mathbb{N}.
\]

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