Absence of $\pi^2$ terms in physical anomalous dimensions in DIS: Verification and resulting predictions

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A B S T R A C T

We study the higher-order corrections to structure functions in inclusive deep-inelastic scattering (DIS) in massless perturbative QCD, in the context of the conjectured absence of even-$n$ values of the Riemann zeta-function $\zeta_n$, i.e., of powers of $\pi^2$, in Euclidean physical quantities. We provide substantial additional support for this conjecture by demonstrating that it holds, as far as it can be tested by the results of diagram computations, for the physical anomalous dimensions of structure functions at the fourth and fifth order in the strong coupling constant $\alpha_s$. The conjecture is then employed to predict hitherto unknown $\zeta_4$ and $\zeta_5$ contributions to the anomalous dimensions for parton distributions and to the coefficient functions for the longitudinal structure function $F_1$.

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As far as they are presently known, the anomalous dimensions – i.e., the even-$N$ or odd-$N$ Mellin moments of the splitting functions – for the scale dependence (evolution) of parton distributions can be expressed in terms of rational numbers and integer-$n$ values $\zeta_n$ of the Riemann $\zeta$-function. The same holds for the $N$-space coefficient functions for inclusive lepton–hadron deep-inelastic scattering (DIS) via the exchange of a boson with spacelike four-momentum $q$, i.e., $q^2 \equiv -Q^2 < 0$. Three-loop calculations at $N \leq 14$ of these quantities were performed in refs. [1–3]. The corresponding all-$N$ expressions were derived in refs. [4] for the anomalous dimensions (see also ref. [5] for the helicity-dependent case) and in refs. [6] for the most important structure functions in DIS.

It is an old observation that the above ‘spacelike’ quantities do not include terms linear in $\zeta_2 = \pi^2/6$. Terms with $\zeta_2^2$ or $\zeta_4 = 2/5 \zeta_2^2 = \pi^4/90$ do occur in the three-loop coefficient functions in DIS (with the exception of the longitudinal structure function $F_1$) [1–3] and the four-loop anomalous dimensions [9–11], which together define the next-to-next-to-leading order $\mathcal{O}(\alpha_s^5\alpha_s^N)$ approximation in renormalization-group improved perturbation theory. The corresponding $\mathcal{O}(\alpha_s^5\alpha_s^N)$ quantities, i.e., the four-loop coefficient functions and five-loop anomalous dimensions, include contributions with $\zeta_4$ and $\zeta_5 = 8/35 \zeta_2^2 = \pi^6/945$ [11–13].

Already about 20 years ago, the absence of also $\zeta_4$ in the perturbative expansion of spacelike (Euclidean) physical quantities was referred to as an empirical rule [2]. In the standard $\overline{\text{MS}}$ renormalization scheme, however, this rule is violated at $\mathcal{O}(\alpha_s^4\alpha_s^N)$ by the scalar quark and gluon correlators that enter the hadronic decays of the Higgs boson [14]. Very recently, it has been demonstrated in ref. [15] that the $\zeta_4$ terms in the above quantities vanish after transforming the coupling constant to the $C$-scheme introduced in ref. [16]. This highly non-trivial cancellation can occur since the five-loop beta function of QCD and its gauge-group generalizations include $\zeta_4$ with most colour factors [17]. It has thus been conjectured that all even-$n$ $\zeta$-values, i.e., all powers of $\pi^2$, are absent from all spacelike physical quantities in massless perturbative QCD in this scheme [15].

The factorization-scheme dependent anomalous dimensions and coefficient functions can be combined to form physical anomalous dimensions for structure functions in DIS. In this manner, the above ‘$\pi^2$ conjecture’ can be (a) supported (or falsified) by quantities not considered in this context so far, and (b) used to predict new results for higher-order coefficients in the perturbation series. The latter possibility was, in fact, already mentioned (but not followed up) in ref. [2]. In the present letter, we perform both steps at $\mathcal{O}(\alpha_s^5\alpha_s^N)$ and $\mathcal{O}(\alpha_s^5\alpha_s^N)$ for the non-singlet structure func-
tions $F_{2ns}$ and $F_3$ and for the flavour-singlet system $(F_2, F_3)$. Here
$F_\phi$ is the structure function for DIS via the exchange of a scalar
that (like the Higgs boson in the limit of a heavy top quark and
$\eta_t$ effectively massless flavours [18]) couples directly only to gluons [19]. We finally also address physical anomalous dimensions
for $F_3$.

The physical anomalous dimensions $K$ can be obtained by
considering the scaling violations $dF/d\ln Q^2$ of the (vector of)
$N$-space structure functions $F$, using the evolution equations for
the parton distributions $q$ and then expressing these in terms of the
structure functions, viz

$$
\frac{dF}{d\ln Q^2} = \frac{d}{d\ln Q^2}(Cq) = \frac{dC}{d\ln Q^2} q - C'\gamma q
\left(\beta \frac{dC}{dq} - C'\gamma\right) K \equiv K F .
$$

(1)

Here $C$ represents the coefficient functions, $\gamma$ the anomalous
dimensions (we use the standard convention $\gamma = -\lambda$ for their
relation to the moments of the splitting functions), $a_\alpha \approx \alpha_s/4\pi$ the
reduced strong coupling constant and $\beta$ the beta function of QCD
and its generalizations. We have suppressed here, as in many instances
below, the dependence of $C$, $\gamma$ and $K$ on $N$ for brevity.

Our notation (and normalization) for the expansion coefficients of
these quantities is

$$
C = 1 + \sum_{l=1}^{\infty} a^l \epsilon^{(l)}, \quad \{\gamma, K\} = \sum_{l=1}^{\infty} a^l \{\gamma, K\}^{(l-1)} .
$$

(2)

For the non-singlet cases, eq. (1) is simplified by $C\gamma C^{-1} = \gamma$,
thus only the $N^{3}$LO anomalous dimensions $\gamma^{(n)}$ (together with the
coefficient functions at this and all previous orders) enter the physi-
cal anomalous dimensions at $N^{3}$LO. In this case one has $K = -\gamma$ for
those colour factors, such as $C^a_4$ at order $a^4_1$, that cannot be
generated by multiplying the beta function and powers of the co-
efficient functions. For the system $(F_2, F_3)$, the quantities $F$, $C$, $\gamma$
and $K$ represent the matrices

$$
F = \begin{pmatrix} F_{2ns} \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2ns} \\ C_{\phi, q} \\ C_{\phi, g} \end{pmatrix},
$$

$$
K = \begin{pmatrix} K_{2ns} & K_{\phi, q} & K_{\phi, g} \\ K_{\phi, q} & K_{\phi, g} & K_{\phi, \phi} \end{pmatrix}, \quad \gamma' = \begin{pmatrix} \gamma_{2ns} \\ \gamma_{\phi, q} \\ \gamma_{\phi, g} \end{pmatrix} .
$$

(3)

The non-singlet expansion coefficients of $K$ have been written
down to order $a^4_1$, the maximal order of the present study, in eqs.
(2.7) and (2.8) of ref. [20] for the choice $\mu^2 = Q^2$ of the renor-
malization scale already employed in eq. (1). The generalization
to $\mu^2 \neq Q^2$, see eq. (2.9) of ref. [20], does not provide new
information. Corresponding matrix notation results for systems like
$(F_2, F_3)$ and $(F_2, F_2)$ have been given to $a^4_1$ in eq. (2.25) of ref.
[21]; their extension to the $a^2_2$ terms is straightforward.

Only a small part of the complete expressions is required for
the $c_4$ and $c_6$ contributions. Consequently, the $c_4$ parts of the
$a^2_2$ ($N^{3}$LO) physical anomalous dimensions considered for now take
the simple form

$$
\tilde{K}^{(3)}_{2ns} = -\tilde{\gamma}^{(3)}_{2ns} - 3\beta_0 \tilde{c}^{(3)}_{2ns}, \quad \tilde{K}^{(3)}_3 = -\tilde{\gamma}^{(3)}_3 - 3\beta_0 \tilde{c}^{(3)}_3
$$

(4)

and

$$
\tilde{K}^{(3)}_{22} = -\tilde{\gamma}^{(3)}_{22} - 3\beta_0 \tilde{c}^{(3)}_{22} - \gamma_{2qs} \tilde{c}^{(3)}_{2g} + \gamma_{gs} \tilde{c}^{(3)}_{g, q},
$$

$$
\tilde{K}^{(3)}_{2\phi} = -\tilde{\gamma}^{(3)}_{2\phi} - 3\beta_0 \tilde{c}^{(3)}_{2\phi} - \gamma_{2qs} \tilde{c}^{(3)}_{g, q} + \gamma_{gs} \tilde{c}^{(3)}_{g, q}
$$

$$
- \tilde{c}^{(3)}_{2 \phi} \left(\tilde{\gamma}^{(3)}_{qs} - \tilde{\gamma}^{(3)}_{gs}\right) .
$$

(5)

In terms of the four-loop anomalous dimensions and three-loop
coefficient functions. The latter are completely known [6,21]. Here
and below a tilde above a quantity indicates the coefficient of $c_4$. If
the no-$\pi^2$ conjecture is correct, then the left-hand sides of eqs. (4)
and eqs. (5) vanish for all $N$.

Diagram calculations of the $N^{3}$LO non-singlet anomalous
dimensions up to $N = 16$ (and all-$N$ expressions in the limit of
a large number of colours $n_c$) have been presented in ref. [10].
These results are sufficient for determining the all-$N$ expressions
for $\tilde{\gamma}^{(4)}_{2ns}(N)$, which are found to read

$$
\tilde{\gamma}_{2ns}^{(4)} = 8C_F (C_A - C_F) \beta_0
\times \left[9 (C_A - 2C_F) \left(\frac{5}{4} - (\eta + \eta^2)(\pm 1)^N + 2S_{-2}\right)
- 3n_\gamma \left(\frac{3}{2} + \eta - 2S_{-1}\right)\right]
$$

(6)

with $\eta = D_0 - D_1 = 1/N - 1/(N + 1)$. Here and below the argument
$N$ of all harmonic sums $S_{2n}(N)$ [22] is suppressed for brevity.
For $\tilde{\gamma}_3^{(4)}$, the anomalous dimension for flavour differences of quark-
 antiquark sums, eq. (6) is valid at even $N$. For its quark–antiquark
difference counterpart $\tilde{\gamma}_3^{(4)}$ it provides the odd-$N$ values with, of
course, $\tilde{\gamma}_3^{(4)}(N = 1) = 0$ as required by fermion number conser-
vation. Using eq. (6) in eq. (4) we indeed find

$$
\tilde{K}^{(3)}_{3, ns} = \tilde{K}^{(3)}_3 = 0 \quad \text{for all even/odd } N .
$$

(7)

The diagram calculations of four-loop singlet splitting functions,
performed along the lines of refs. [1–3] with the FORCER pro-
gram [23] for massless four-loop self-energy integrals, have been com-
mpleted so far only at $N = 2$ and $N = 4$. The result of the hardest
of these calculations, $\tilde{\gamma}_{22}^{(4)}(N = 4)$, has been given in eq. (2.1) of
ref. [9]; the other results will be presented in ref. [11]. Inserting
these results into eq. (5), we find

$$
\tilde{K}^{(3)}_{22} = \tilde{K}^{(3)}_{2\phi} = \tilde{K}^{(3)}_{2q} = \tilde{K}^{(3)}_{2g} = 0
$$

(8)

for $N = 2$ and $N = 4$, thus verifying the no-$\pi^2$ conjecture also in
the singlet sector. Imposing eq. (8) for all even $N$, we can now pre-
dict the complete results for all four quantities $\tilde{\gamma}_3^{(4)}(N)$ in eq. (3):

$$
\tilde{\gamma}_{22}^{(3)} = 16C_F (C_A - C_F) \left[\frac{n_f^2}{2} \left(15\eta + 10\eta^2 - 20\eta^3\right)
+ C_F n_\gamma \left(138\nu - 72\nu^2 - 117\nu^3 - 87\nu^4 - 18\nu^5\right)
+ (C_A - C_F)n_\gamma \left(114\nu - 72\nu^2 - 105\frac{\eta}{2} - 69\eta^2 - 12\eta^3\right.
+ \left[24\nu - 18\eta - 12\eta^2\right]S_1\right]\right] ,
$$

(9)

$$
\tilde{\gamma}_{2g}^{(3)} = 16(C_A - C_F) \left[(C_A - C_F) n_\gamma^2 \left(22D_0 - \frac{16}{3}D_{-1} - 4D_0^2
+ 6D_3 - 50D_1 + 10D_2 - 12D_3 + 109\frac{3}{2}D_2 + 16D_2^2
+ [4D_0 - 8D_1 + 8D_2]S_1\right)
+ (C_A - C_F) n_\gamma \left(\frac{122}{3}D_{-1} - 8D_2^2 - \frac{91}{2}D_0 - 21D_2^2
- 18D_3^2 + 208D_1 - 96D_2^2 + 36D_3^2 - 659\frac{3}{2}D_2 - 114D_2^2\right)\right] .
$$


\[ \begin{align*}
-24D_0^2 + [12D_1 - 13D_0 - 24D_0^2 + 46D_1 - 48D_0^2 \\
- 91D_0 - 36D_1]S_1 + [24D_1 - 12D_0 - 24D_2]S_{1,1} \\
+ [6D_0 - 12D_1 + 12D_2]S_2 + CF\eta_f^2 \left( \frac{49}{3} D_0 - 8D_1 - D_0^2 \right) \\
+ 6D_0^2 - 70D_1 + 20D_1^2 - 12D_1^3 + 55D_2 + 24D_2^2 \\
+ [2D_0 - 4D_1 + 4D_2]S_1 \right) + CF(C_A - CF)\eta_f^2 \left( \frac{239}{2} \eta + 69\eta^2 + 12\eta^3 - 180v + 72\eta^2 \\
+ [22 - 24v + 18\eta + 12\eta^2]S_1 \right). \end{align*} \] (12)

The abbreviations \( \eta \) and \( D_0 \) have been defined below eq. (6); in addition \( v \equiv D_1 - D_2 \) is used in eqs. (9) and (12). \( \gamma^{(4)}_{\text{KL}} \) is obtained by adding \( \gamma^{(3)}_{\text{KL}} \) in eq. (6) to the pure-singlet contribution (9) which we were able to check at \( N = 6 \). At least at this value of \( N \), we expect further checks in the near future. A complete determination of the all-\( N \) expressions in eqs. (9)-(12) from diagram calculations, on the other hand, would be a very (currently: too) challenging task.

It is interesting to note that all \( \zeta_4 \) terms of \( \gamma^{(3)}_{N\pi} \) vanish for \( C_F = C_A \), which is part of the colour-factor choice leading to an \( \mathcal{N} = 1 \) supersymmetric theory [24]. The same behaviour has been found (to all orders) before for the double-logarithmic large-\( N \) contributions to the off-diagonal anomalous dimensions \( \gamma_{N\pi} \) and \( \gamma_{N\pi}^{(3)} \) [25]. We also note that the ‘diagonal’ quantities (6), (9) and (12) are reciprocity respecting, see ref. [10] and references therein.

All flavour-singlet quantities in eqs. (9)-(12) include terms with \( 1/(N - 1)^2 \) which correspond to \( \zeta_4 \)-\( \text{N} \text{LO} \) contributions in the small-\( x \) expansion of the \( \text{N} \text{LO} \) splitting functions \( b^{(3)}_{N\pi} \). Contributions of this x-space form are also generated, however, by terms without \( \zeta_4 \) in \( N\pi \)-space. Therefore the above results are not sufficient to obtain the \( \zeta_4 \) coefficients of the next-to-next-to-leading logarithms of \( b^{(3)}_{N\pi} \) in the small-\( x \) (high-energy) limit.

We now turn to the \( \zeta_6 \) (N\text{4LO}) contributions to the physical anomalous dimensions. Their \( \zeta_6 \) terms, which we denote by \( \tilde{R}^{(4)} \), are given by simple modifications of eqs. (4) and (5): on the right-hand-sides replace \( f^{(3)} \) everywhere by \( f^{(4)} \) for \( f = \gamma, C \), and replace \(-3\beta_0 \) everywhere by \(-4\beta_0 \). For the non-singlet cases, the \( \zeta_4 \) terms at N\text{4LO} are given by

\[ \tilde{R}^{(4)}_{N\pi} = -\gamma^{(4)}_{N\pi} - 3\beta_1 c^{(3)}_4 - 4\beta_0 \left( c^{(4)}_4 - c^{(1)}_4 \right) c^{(3)}_4 \] (13)

with \( \sigma = + \) for \( a = 2, \text{ns} \) and \( \sigma = \) - for \( a = 3 \). At this order, the scheme transformation of ref. [16] includes the \( \zeta_4 \) term of the five-loop beta function, see eq. (4) of ref. [15] (where this coefficient, \( \beta_4 \) in our notation, is denoted by \( \beta_5 \)). Consequently, the conjecture of ref. [15] implies

\[ \rho_0 K^{(4)}_0 = 0. \] (14)

Checking this prediction and its \( \zeta_6 \) counterpart \( \tilde{K}^{(4)}_0 = 0 \) requires the four-loop coefficient functions for \( F_{2,m} \) and \( F_3 \) and the corresponding five-loop splitting functions. The former quantities have been computed so far at \( N \leq 6 \) [9,11]. The latter have been obtained, very recently, at \( N = 2 \) and \( N = 3 \) [12,13] by using the \( R^* \)-operation [26] as extended to generic numerators in ref. [27] and implemented using the latest version [28] of \textsc{Form} [29], together with the \textsc{Forcer} program [23]. The leading large-\( \eta \) contributions to \( \gamma^{(4)}_{N\pi} \) and \( c^{(4)}_{2,\text{ms}} \), both of which include \( \zeta_4 \) terms, have been determined at all \( N \) in refs. [30].

We now show, by explicitly writing down all contributions to eqs. (13) and (14), that the latter relation is fulfilled for \( K_{2,ns} \) at \( N = 2 \). The corresponding verification for \( K_3 \) at \( N = 3 \) is completely analogous, but will be suppressed for brevity. For the same reason we do not show the (less critical, since the scheme transformation of ref. [16] is not required) verification for the \( \zeta_6 \) parts of \( K_{2,ns} \) and \( K_3 \) at these values of \( N \), nor the all-\( N \) verification for the \( \zeta_4 \) part of \( K_{2,ns} \) in the large-\( \eta \) limit. The recent result for \( \gamma^{(4)}_{\text{KL}}(N = 2) \) [12,13] is given by
Here \( T_f = 1/2 \) has been inserted; the power of \( T_f \) for each term can be readily reconstructed. The result for QCD is obtained for \( C_A = n_f = 3, n_l = 8, C_F = 4/3, \frac{\alpha^b A}{\alpha^b A / n_A} = 135/8, \frac{d_{abcd}^A}{d_{abcd}^A / n_A} = 5/2 \) and \( d_{abcd}^A / n_A = 5/36 \). The \( \xi \) parts of the four-loop coefficient function \( c_{2\text{ns}}^4 = \frac{1}{2} \) [9,11] and of the five-loop beta function \( \beta_\xi \) [17] read

\[
\begin{align*}
\tilde{c}_{2\text{ns}}^4 &= \frac{248}{3} \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{128}{3} \frac{\alpha^b A}{\alpha^b A / n_A} + 16 \frac{C_F n_f^3}{C_F} - 16 \frac{C_F^2 n_f^2}{C_F^2} \\
- \frac{220}{27} C_F n_f + \frac{1552}{27} C_A C_F n_f^2 + 16 C_A C_F^2 n_f \\
- \frac{3592}{27} C_A^2 C_F^3 - \frac{505}{3} C_A^2 C_F n_f - 354 C_A^2 C_F^2 n_f + 4367 C_A^2 C_F 
\end{align*}
\]  

and

\[
\begin{align*}
\tilde{\beta}_4 &= 176 n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} - 416 n_f \frac{\alpha^b A}{\alpha^b A / n_A} + 128 n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} - 44 \frac{C_F n_f^3}{C_F^2} - 968 C_A \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + 2288 C_A \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
- 704 C_A n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{28}{3} C_A C_F n_f^2 + \frac{286}{3} C_A C_F^2 n_f^2 + 14 \frac{C_A n_f^3}{C_F^2} \\
- 236 \frac{C_A^2 C_F n_f}{C_F} - 242 \frac{C_A^2 C_F n_f^2}{C_F^2} - 26 \frac{C_A^2 C_F n_f^3}{C_F^3} + 451 \frac{C_A^3 C_F n_f}{C_F^3} \\
- 583 \frac{C_A^3 C_F n_f}{C_F^3} + 121 \frac{C_A^4 C_F}{C_F^4} 
\end{align*}
\]  

Due to eqs. (4) and (7), the three-loop contribution \( c_{2\text{ns}}^4 = \frac{1}{2} \) [2] can be read off from Eq. (6). For the convenience of the reader, we also recall the required one-loop quantities in the normalization used in this letter. \( K_{2\text{ns}}(N_f = 2) = \gamma_{2\text{ns}}(N_f = 2) = 8/3 C_F, c_{2\text{ns}}^4(N_f = 2) = 1/3 C_F \) and \( \rho_0 = 11/3 C_A - 2/3 n_f \). Assembling these contributions, we arrive at eq. (14). This and the other verifications mentioned above provide substantial and highly non-trivial extra evidence for the no-\( n^2 \) conjecture in the form presented in ref. [15]. The \( \xi \) coefficient of \( \gamma_{2\text{ns}} \) at \( N = 2 \) reads

\[
\begin{align*}
\tilde{\gamma}_{2\text{ns}}^{(4)} &= \frac{800}{27} \rho_0 \left( 36 C_A^4 - 36 C_F C_A^3 - 42 C_A^2 C_F^2 + 38 C_A^3 C_F \right) \\
- \frac{48}{9} \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + 18 n_f C_A^3 - 3 n_f C_A C_F^2 - 14 n_f C_A^2 C_F \\
+ 24 n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \right). 
\end{align*}
\]  

The \( N = 4 \) expressions corresponding to eqs. (15) and (18) represent the first new results for the N\( ^4 \)LO non-singlet anomalous dimensions obtained from this conjecture. They are given by

\[
\begin{align*}
\tilde{\gamma}_{2\text{ns}}^{(4)} &= -\frac{7977620}{50625} A C F^4 + \frac{6688679}{162000} A C F^3 + \frac{60495779}{33750} A C F^2 \\
- \frac{202467481}{162000} A C F + \frac{6908}{15} A C F \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
- \frac{349624}{45} A C F \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{20332714}{50625} n_f C_A^4 \\
+ \frac{41674913}{81000} n_f C_A C_F^3 + \frac{33077541}{67500} n_f C_A^2 C_F^2 \\
+ \frac{428850767}{81000} n_f C_A C_F^2 + \frac{124036}{75} n_f A C F \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
+ \frac{39076}{45} n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{1704086}{10125} n_f^2 C_A^3 + \frac{8505499}{6750} n_f C_A C_F^2 \\
- \frac{4882673}{3375} n_f^2 C_A C_F + \frac{11704}{25} n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} - \frac{2146}{25} n_f^3 C_A C_F \\
+ \frac{139286}{2025} n_f^3 C_A C_F + \frac{1256}{405} n_f^4 C_A C_F . 
\end{align*}
\]  

It is also possible to predict the \( \xi_4 \) and \( \xi_6 \) coefficients \( \tilde{\gamma}_{2\text{ns}}^{(4)} \) and \( \tilde{\gamma}_{2\text{ns}}^{(4)} \) of the N\( ^4 \)LO singlet anomalous dimensions at \( N = 2 \) and \( N = 4 \). The prediction at \( N = 2 \) includes a further check, since the four results must be pairwise equal due to the momentum sum rule. By evaluating the elements of \( K \) in eq. (3) as given by eq. (1), we obtain

\[
\begin{align*}
\tilde{\gamma}_{2\text{ns}}^{(4)}(N_f = 2) &= -\tilde{\gamma}_{2\text{ns}}^{(4)}(N_f = 2) \\
= \frac{2048}{27} \frac{d_{abcd}^A}{d_{abcd}^A / n_A} - \frac{512}{9} \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{704}{3} \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{640}{81} C_F n_f^3 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
+ \frac{10496}{81} C_F^2 n_f^3 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{2896}{27} C_F^3 n_f^2 + \frac{1024}{3} C_F n_f^3 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} - \frac{10912}{9} C_A \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
- \frac{7040}{9} C_A n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{6178}{81} C_A C_F n_f^2 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
- \frac{142996}{81} C_A^2 C_F n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{6596}{81} C_A C_F^2 n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} - \frac{59840}{81} C_A C_F^4 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
- \frac{48958}{27} C_F^2 n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + \frac{196028}{81} C_A C_F^2 n_f \frac{d_{abcd}^A}{d_{abcd}^A / n_A} - \frac{220472}{81} C_A C_F^4 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
+ \frac{6254 C_A C_F n_f^3 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} + 170968}{27} C_A^3 C_F^2 \frac{d_{abcd}^A}{d_{abcd}^A / n_A} \\
- \frac{221920}{81} C_A^4 C_F \frac{d_{abcd}^A}{d_{abcd}^A / n_A} . 
\end{align*}
\]  

(21)
\[
\begin{align*}
\tilde{\gamma}_{qq}^{(4)}(N = 2) &= \tilde{\gamma}_{qq}^{(4)}(N = 2) \\
&= \frac{25600}{27} \cdot \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_A} \cdot \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_R} \\
&= \left( \frac{800}{9} C^2_{qq} \eta_f^2 - \frac{2}{3} C^4_{qq} \eta_f + \frac{14080}{27} C_A \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_R} \right) \\
&+ \frac{139600}{27} C_A C^2_{qq} \eta_f + \frac{35200}{9} C_A C^4_{qq} + \frac{56800}{27} C^2_{qq} \eta_f^2 \\
&+ \frac{10600}{27} \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_A} - \frac{12200}{27} C^2_{qq} + \frac{324400}{27} C_A C^4_{qq} \\
&= \tilde{\gamma}_{qq}^{(4)}(N = 2) \\
&= \frac{12800}{27} \cdot \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_A} \\
&= \left( \frac{12800}{27} \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_A} - \frac{800}{9} C^2_{qq} \eta_f^2 + \frac{3200}{9} C^4_{qq} \eta_f + \frac{70400}{27} C_A \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_R} \right) \\
&+ \frac{70400}{27} C_A C^2_{qq} \eta_f + \frac{4000 C^2_{qq} \eta_f^2}{n_A} \\
&+ \frac{41200}{27} C_A C^4_{qq} + \frac{800}{81} C^2_{qq} \eta_f^2 + \frac{1400}{9} C_A C^4_{qq} \eta_f \\
&+ \frac{16400}{27} C_A C^4_{qq} - \frac{7400}{27} \frac{\tilde{a}^{(4)}_{\tilde{q} \tilde{q}}^{abcd,abcd}}{n_R} - \frac{12100}{27} C^2_{qq} \eta_f^2 \\
&+ \frac{97900}{81} C^4_{qq} \\
&\text{Hence the no-}\pi^2\text{ conjecture also passes this further five-loop check. We note that this check succeeds only due to the (}\xi_c\text{ part of the) scheme transformation of ref. [16]. This is not due to the (}\hat{F}_2, \hat{F}_3\text{) analogues of the }\tilde{b}_A\text{ shift [14], as the resulting contributions to the momentum sum rule cancel. Instead it arises from the need to refer to a renormalization-group invariant current [15], which is not }G^{uv}G_{\mu
u}\text{ but }\beta(a_0)G^{uv}G_{\mu
u}\text{ for the structure function }F_3.\text{ The resulting overall factor of }\tilde{a}_0\text{ induces a scheme shift }\tilde{b}_A/b_0 c_{(0)}^{(0)}\text{ of the }N^4\text{LO coefficient }\tilde{c}_{n,q}^{(0)}\text{ with }c_{(0)}^{(0)} = 1.}
\end{align*}
\]
the $c_4$ contributions to $\beta_s$ occurring in $\overline{\text{MS}}$ [17] – these checks include four ‘all-$N$’ relations at $N^3$LO, three for even $N$ and one for odd $N$.

Based on the evidence presented in ref. [15] and in this letter, this conjecture can be employed to predict new $\pi^2$ contributions to higher-order anomalous dimensions and coefficient functions. At $N^3$LO we have presented the $c_4$ contributions to the flavour-singlet splitting functions and to the coefficient functions for the longitudinal structure function $F_L$ at all even $N$. Based on present four-loop FORCER [23] computations of DIS [9,11], it is possible to predict hitherto unknown $c_4$ and $c_6$ parts of $N^4$LO anomalous dimensions at $N \leq 6$. Here we have shown, for brevity, only the $N = 2$ ($N = 4$) results for the singlet (non-singlet) case. These predictions, and the no-$\pi^2$ conjecture in general, will serve as useful partial checks for very complicated future high-order computations. They may also provide input for future studies of the structure of perturbative quantum field theory.

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