Quantification of uncertainty in metallic elements subjected to fatigue

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Abstract. This investigation analyzes the propagation of the uncertainty of the number of life cycles to fatigue failure, taking into account the strain life method, considering as random variables some parameters of the material and loading in a steel plate under fatigue, with constant and variable loading through multidimensional Hermite polynomials. The application of series of multidimensional Hermite polynomials allowed the prediction of the randomness of the output vector. This research has shown that a multidimensional Hermite polynomial adequately estimates the propagation of the uncertainty of the input random variables. The results showed that variations in material and loading parameters can generate important failure probabilities and allowed quantifying the variability in the number of life cycles for fatigue failure steel parts.

1. Introduction

The quantification of uncertainty is important in mechanical processes. There are two commonly used methods to establish the propagation of uncertainty in the parameters of the model: (i) the analytical method, (ii) the Monte Carlo method. In the analytical method [1,2] the uncertainty in the output is explicitly represented as uncertainty functions in the input variables. The Monte Carlo method [3, 4] involves a sufficiently large number of simulations model with a sample of the input random variables, estimated by the probability density functions, \( f_X(x) \), with great computational cost and time. It is necessary to use other more efficient computational methods to estimate the uncertainty in complex models that use a significantly reduced number of solutions.

Recently, an approximate new and cheaper technique called multidimensional Hermite polynomials or polynomial chaos expansions (PCE) [5, 6] has emerged. PCE can be classified into two approaches: (i) intrusive formulation, (ii) non-intrusive formulation; in the intrusive formulation the uncertainty is expressed explicitly within the analysis of the system under investigation (uses the Galerkin method [7]). In the non-intrusive formulation the PCE are used to create response surfaces (called Stochastic response surface [8]) without interfering with the system analysis [9].

The literature does not report any case studies that quantify the uncertainty of steel elements subject to fatigue using PCE. This research examines the randomness of materials, in life cycle determination, \( N_f \), under the strain life method using PCE in a non-intrusive approach. The polynomial coefficients were determined using the probabilistic collocation method [10, 11].
main contribution of this research was the quantification of the propagation of the uncertainty of the material through the determination of $f_X(x)$.

2. Polynomial chaos expansion

The random variable $u(\theta)$ is defined in probabilistic space $(\Theta, F, P)$ expanded on a multivariate polynomial basis that follows a probability distribution function. In which, the probabilistic space is defined as a triple $(\Theta, F, P)$ formed by a set $\Theta$ (called sample space), a $\sigma$-algebra $F$ (called events) in $\Theta$ and a positive measure $p$ in this $\sigma$-algebra such that the probability $P(\Theta) = 1$. The multivariate polynomial can be written as a random process $u(\theta)$, see Equation (1),

$$u(\theta) = \sum_{i=0}^{\infty} a_i \Psi_i [\theta]$$  \hspace{1cm} (1)

where $a_i$ are the unknown deterministic coefficients, $\xi_i(\theta)$ is a set of normal random variables standardized and identically distributed and independent, and $\Psi_i (\xi_1, \xi_2, \xi_3, \xi_4, \ldots, \xi_m)$ are Hermite polynomials that form an orthogonal base $L^2(\Theta, F, P)$ in which is a Hilbert space of random variables with finite variance.

The multivariate polynomial expansion is commonly called chaos homogeneous of degree $p$, the dimensions $i$ and $\theta$ represent the results in the space of possible random event results. Another way of representing the multivariate orthogonal polynomials according to [1] is shown in Equation (2),

$$u(\theta) = a_0 \Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1} \Gamma_1 (\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} a_{i_1 i_2} \Gamma_2 (\xi_{i_1}(\theta), \xi_{i_2}(\theta)) + \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{i_3=1}^{\infty} a_{i_1 i_2 i_3} \Gamma_3 (\xi_{i_1}(\theta), \xi_{i_2}(\theta), \xi_{i_3}(\theta)) + \ldots$$  \hspace{1cm} (2)

where $\Gamma_p (\xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \ldots, \xi_{ip})$ denotes the $p$-order Hermite polynomials in terms of the independent normal multivariate random variables $\xi = [\xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \ldots, \xi_{ip}]$.

There is a relationship between $\Gamma_p (\xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \ldots, \xi_{ip})$ and $\Psi_i(\xi)$ one by one. The sum in Equation (1) is performed according to the Hermite polynomial order, while Equation (2) a recount is made, starting with the minor order polynomials. It is possible to construct the Hermite polynomials of any size and degree using the standard normal probability density function or an algorithm.

2.1. Construction of polynomial chaos expansions

The Hermite polynomials can be generated from the standardized normal probability density function, see Equation (3) for each degree and dimension according to [1],

$$\Gamma_p (\xi) = (-1)^p e^{\frac{1}{2} \xi^\top \xi} \frac{\partial^p}{\partial \xi_{i_1}, \ldots, \partial \xi_{i_p}} e^{-\frac{1}{2} \xi^\top \xi}$$  \hspace{1cm} (3)

where $\Gamma_p (\xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \ldots, \xi_{ip})$ denotes the Hermite polynomials of order $p$ in terms of the multidimensional independent normal random variables $\xi = (\xi_{i_1}, \xi_{i_2}, \xi_{i_3}, \ldots, \xi_{ip})$.

In the literature it is possible to find algorithms that allow the construction of Hermite polynomials of order $n + 1$ in terms of the orders of the previous polynomials, see Equation (4),

$$H_{n+1} (\xi(\theta)) = \xi H_n (\xi(\theta)) - n H_{n-1} (\xi(\theta))$$  \hspace{1cm} (4)
The series of Equation (2) can be truncated in a finite number of terms $N$ and $\hat{u}$ is an approximate answer, see Equation (5) which will depend on the number of truncation terms $N$,

$$u(\theta) \approx \hat{u}(\theta) = \sum_{i=0}^{N-1} a_i \Psi_i[\xi(\theta)] \tag{5}$$

where the number of truncation terms $N$ in an order expansion $p$ involving $n$ random variables is given by Equation (6).

$$N = \frac{(n + p)!}{n! p!} \tag{6}$$

In order to determine the coefficients $a_\alpha$ there are several methods that can be used depending on the type of formulation used in the mathematical model.

2.2. Determination of coefficients of polynomial chaos expansions

Two classes of methods are distinguished for the determination of the coefficients; the intrusive and non-intrusive method. The propagation of uncertainty in the model’s computational simulation via an intrusive approach, where all the dependent variables and random parameters in the governing equation are replaced by their polynomial chaos expansions, generates a system of deterministic equations to be solved. This procedure needs to make modifications to the existing (deterministic) computational code. The non-intrusive method is simpler in theory, but a non-intrusive formulation for complex problems can be difficult to implement and a high computational cost. Among the non-intrusive methods, we highlight: projection method [12–14], stochastic collocation method [15, 16], non-intrusive regression method [8, 17–19]. The non-intrusive regression method or stochastic regression method developed by [8], was used in this research. The quantity and quality of the collocation points for each standardized random variable will define the quality of the approach.

2.3. Selection of collocation points

The selection of the collocation points of PCE of dimension $n$ and order $p$ correspond to the roots of the following polynomial, that is, $p+1$, of each of the input variables [10, 20]. The set of collocation points are chosen from all possible combinations $(\xi_1, \xi_2, \xi_3, \cdots, \xi_n)$, $(\xi_1, \xi_2, \xi_3, \cdots, \xi_n)$, $(\xi_1, \xi_2, \xi_3, \cdots, \xi_n)$, with the highest probability. The number of collocation points selected should be the double of the number of unknown coefficients to be estimated in the chaos polynomials, to obtain a robust coefficient estimator [8].

PCE is a spectral method, [1, 6, 8, 21, 22] which allows the propagation of the uncertainty of the input random variables through the construction of non-linear functionals dependent on the solution [1, 23] to the random output variable. This facilitates the quantification of the uncertainty of the output through $f_X(x)$ [10]. The convergence of the approximate solution is measured by comparing the PCE of different degrees according to [8] or between PCE and Monte Carlo simulation (SMC) [11, 23], ensuring accuracy of the results. The spectral convergence will depend on the high differentiability of the Hermite functions contained in each standard normal random variable.

3. Numerical implementation

A numerical model was developed to predict the number of cycles to fatigue failure ($N_f$) using PCE. The results were validated using SMC in notched bodies subjected to fatigue with constant and variable load.
3.1. Failure modeling in notched bodies under constant amplitude loading

A probabilistic analysis of fatigue life was performed by the strain life method in the scenario of components loaded under constant amplitude loading using PCE and SMC as a way of evaluating the results obtained from the PCE simulation. In the strain life method implemented, the relationship of [24] was used to estimate the stresses and deformations at the root of the notch and the ratio of [25] to calculate the number of cycles for failure. The study was carried out on a plate with two semi-circular lateral notches subjected to axial force of constant amplitude [26,27]. Figure 1(a) shows the plate configuration, where $K_t$ corresponds to the elastic stress concentration factor elastic, $K_f$ is the fatigue stress concentration factor, and $B$ is the plate thickness. Figure 1(b) shows a general scheme of the constant amplitude force. The plate was fabricated from an AISI 4340 steel whose cyclic mechanical properties are listed in Table 1, where $H'$ and $n'$ are the coefficient and the cyclic strain exponent, $b$ and $\sigma_f'$ are the exponent and coefficient of resistance to fatigue. As well as, $c$ and $\varepsilon_f'$ are the exponent and the fatigue ductility coefficient, $\sigma_0$ is the yield stress, and $\sigma_u$ is the tensile strength limit.

![Diagram](a)

**Figure 1.** Uniaxial fatigue. (a) Steel plate with two semi-circular lateral notches, (b) constant amplitude loading scheme.

The parameters used in the quantification of uncertainty are given in Table 1 for deterministic and probabilistic values of the random variables ($VA$), such as the average value ($\mu$), the coefficient of variation ($V$) and the $f_X(x)$ if it is applicable. The amplitudes of the force applied and the number of cycles for fatigue failure observed in the laboratory are listed in Table 2.

| $VA$          | $\mu$     | $V$(%) | $f_X(x)$   |
|--------------|-----------|--------|------------|
| $H'$ (MPa)   | 11.62E+02 | 15     | Normal     |
| $n'$ (-)     | 1.23E-01  | 15     | Normal     |
| $\sigma_f'$ (MPa) | 11.65E+02 | 15     | Normal     |
| $\varepsilon_f'$ (-) | 11.42E-01 | 15     | Normal     |
| $b$ (-)      | -81.00E-02| 0      | ‡          |
| $c$ (-)      | -67.00E-02| 0      | ‡          |
| $\sigma_0$ (MPa) | 6.48E+02  | 0      | ‡          |
| $\sigma_u$ (MPa) | 7.86E+02  | 0      | ‡          |

(-) Dimensionless; ‡ Deterministic, without $f_X(x)$.
The experimental study carried out by [26, 27] determined the number of life-cycles for the crack initiation of 0.51 mm by fatigue ($N_{f_{obs}}$), applying different constant amplitude loads ($F_a$), are given in Table 2.

| $F_a$ (kN) | $N_{f_{obs}}$ (cycles) |
|------------|------------------------|
| 88.96      | 62                     |
| 71.17      | 635                    |
| 62.28      | 1300                   |

The coefficients of PEC ($y$), were obtain using the regression-based collocation method, in which the number of collocation points used by PEC ($N_{pc}$), and the sample space was generated using Sobol sequence. The sample space generated using Sobol sequence is transformed to the standard normal space using isoprobabilistic transformations [28]. Table 3 shows the $p$, the degree of the polynomial; $N_{PEC}$, number of coefficients of the PEC; $N_{pc}$, number of collocation points; $N_{SMC}$ the number of SMC used to establish $f_{N_f}(N_f)$ and $N_{SMC/PEC}$ the number of SMC used in the PEC to determine the $f_{N_f}(N_f)$. The $p$ which generated the $f_{N_f}(N_f)$, with minor errors with respect to $f_{N_f}(N_f)$, were grades $p = 3, 4$ and 4 for $F_a = 88.96$ kN; 53.38 kN and 31.14 kN, respectively, and the $f_{N_f}(N_f)$ and $f_{N_f}(N_f)$ together with the errors found for the three analyzed loading, are presented in the Figure 2(a) to Figure 2(c). For the $F_a$ used, a sample space was define with $N_{pc} = 35, 70$ and 70, to the $p = 3, 4$ and 4. In addition, the $N_{PEC} = N_{pc}$, because the $N$ collocation points were used, according to Equation (6) for a non-optimized PCE.

| $F_a$ (kN) | $N_{pc}$ (-) | $N_{SMC}$ (-) | $N_{SMC/PEC}$ (-) |
|------------|--------------|---------------|-------------------|
| 88.96      | 35           | 1.00E+06      | 5.00E+04          |
| 53.38      | 70           | 1.00E+06      | 5.00E+04          |
| 31.14      | 70           | 1.00E+06      | 7.00E+04          |

(-) Dimensionless.

The results shown in Figures 2(a) to Figure 2(c) for the PEC and SMC methodologies resulted in a log-normal probability density function using the maximum likelihood estimator. The probability density function for SMC was used to find the mean and standard deviation using 1.00E + 06 simulations, whereas using the PCE methodology only 35, 70 and 70 simulations were required for the loading conditions $F_a = 88.96$ kN, 71.17 kN and 62.28 kN respectively.
The results obtained from the $N_f$ simulation using SMC and PCE are listed in Table 4 and show that PEC can be used without loss of accuracy. Where $N_f$ is the number of cycles for fatigue failure using SMC, $\hat{N}_f$ is the number of cycles for fatigue failure using PCE, the value of the mean ($\mu$), the standard deviation ($\sigma$) and the coefficient of variation ($V$) of $N_f$ are $\mu_{N_f}$, $\sigma_{N_f}$ and $V_{N_f}$, similarly to $\hat{N}_f$. The behavior of $f_{N_f}(\hat{N}_f)$ and $f_{N_f}(N_f)$ are similar, as are the values of $\mu_{\hat{N}_f}$ and $\mu_{N_f}$, in the same way as the $V$, whose values are presented in Table 4.

The prediction of $\mu_{\hat{N}_f}$, $\sigma_{\hat{N}_f}$ and $V_{\hat{N}_f}$, using PEC, compared to those found using SMC, $\mu_{N_f}$, $\sigma_{N_f}$ and $V_{N_f}$ are similar, since the $\text{err}_{\mu_{\hat{N}_f}}$ and $\text{err}_{\sigma_{\hat{N}_f}}$ are smaller than 4.00%.

**Table 4.** Statistical comparison of results $\hat{N}_f$ and $N_f$ using PCE and SMC in the plate with two semicircular and different notches $F_a$ (kN).

| $F_a$ | $\mu_{\hat{N}_f}$ | $\sigma_{\hat{N}_f}$ | $V_{\hat{N}_f}$ | $\hat{f}_{\hat{N}_f}(\hat{N}_f)$ | $\mu_{N_f}$ | $\sigma_{N_f}$ | $V_{N_f}$ | $f_{N_f}(N_f)$ | $\text{err}_{\mu_{\hat{N}_f}}$ | $\text{err}_{\sigma_{\hat{N}_f}}$ |
|-------|------------------|------------------|----------------|--------------------------------|----------------|----------------|----------------|----------------|------------------|----------------|
| 88.96 | 338.24           | 120.46           | 36             | LN                            | 338.36         | 121.13         | 36             | LN             | 0                | 1              |
| 53.38 | 1959.83          | 832.85           | 42             | LN                            | 1963.51        | 840.83         | 43             | LN             | 0                | 1              |
| 31.14 | 30654.40         | 23177.51         | 76             | LN                            | 29856.30       | 22320.73       | 75             | LN             | 4                | 4              |

Note: $F_a$; $\mu_{\hat{N}_f}$, $\sigma_{\hat{N}_f}$, $V_{\hat{N}_f}$ (cycles); LN = Lognormal; $v_{\hat{N}_f}$, $\text{err}_{\mu_{\hat{N}_f}}$, $\text{err}_{\sigma_{\hat{N}_f}}$ (%).

The values $\hat{N}_f$ and $N_f$ obtained for $F_a = 88.96$ kN, 53.38 kN and 31.14 kN are classified as low cycle fatigue, besides that the $V$ has a high variation.

**4. Conclusion**

Using the strain life method to estimate cycle numbers for fatigue failure and implementing the PCE methodology, it is possible to obtain good $\hat{N}_f$ predictions compared to the $N_f$ predictions using SMC, with an error smaller than 4%, indicating that with a smaller number of simulations for PCE is possible to obtain $N_f$ with the same accuracy as using SMC.
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