Is Minisuperspace Quantum Gravity Reliable?

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Abstract

We study a minisuperspace quantum cosmology for a 2+1 dimensional de Sitter universe and find the wave function both exactly and in WKB approximation. Then we extend the model to a canonically quantized field theory for quantum gravity, i.e., a midisuperspace, and obtain the wave functional of the resulting field theory in the saddle point approximation. It is shown that these two approaches yield different results.

1 Introduction

Quantum cosmology (QC) was initiated by B.S. Dewitt in his seminal paper on Hamiltonian quantization of gravity [1]. Up to advent of reasonable boundary conditions to pick out a unique solution to the Wheeler-Dewitt equation (WDW), QC was out of attention. In the early eighties, proposals on possible boundary conditions revitalized QC. [2,3,4]. Equivalent to Hamiltonian quantization of 4-dimensional gravity, one can path integrate over all metric configurations and sum over all possible 4-topologies which have a given $\partial M$ as their boundary [5]. The space of all configurations for 4-metric is called superspace (SS) where diffeomorphically equivalent configurations are factored out. Mathematically, dealing with the full SS is too difficult, if not impossible. Inevitably, we must invoke to some approximate schemes. A commonly used approximate model is called ”minisuperspace” model [1]. In this approximation, one confines his attention to a restricted region of SS. In practice, one freezes or suspends many infinite degrees of freedom of gravitational field on a time constant slice of 3-geometry and retains a few of them alive.

Despite the problems surrounding the interpretation of the wave function of the universe, by using some generally accepted interpretational rules [4], many attempts have been made to present explanations for some observed features of the universe in the context of ”minisuperspace” models, e.g., cosmological constant problem [7] or lacking the existence of primordial black holes at the present state of the universe [8].

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There are some other and more realistic approaches. Inhomogeneous perturbations and midisuperspace models are examples of more realistic approximations to the full quantized theory. Inhomogeneous perturbations approach to quantum cosmology is a perturbative way to pass beyond the minisuperspace models [9]. In a typical midisuperspace model, one retains some dependence of metric configurations on spatial coordinates. This means that the less symmetric configurations are considered as the space of path integration. The spherically symmetric midisuperspaces, the BCMN models [10], have been used as models for quantized Schwarzschild black holes. In the light of these models some of expected features of quantum black holes have been derived successfully [11]. Also a Hamiltonian quantization of false-vacuum bubbles within the framework of a spherically symmetric midisuperspace has been presented [12]. There is no support in favor of this claim that the minisuperspace configurations have the dominant contribution to the partition function of quantum gravity. Moreover minisuperspace is not known to be a part of a systematic approximation to the full theory. Therefore, it is possible that some of derived predictions from these minisuperspace calculations to be artifact of this approximation. In this way, a comparison between amplitudes derived from a minisuperspace calculation and of midisuperspace one, will be useful.

Through this paper, it is shown that a minisuperspace wave function can differ from midisuperspace one. The organization of the paper is as follows:

In section 2, we take a minisuperspace for QC in 2+1 dimensions and obtain the "wave function" in saddle point approximation. Through Section 3, we will give the exact solutions to the Wheeler-Dewitt equation. Section 4 has been devoted to extension of this minisuperspace model to a midisuperspace one. We solve the resulting field theory in the saddle point approximation and obtain the wave functional of the theory.

Finally, we present a comparison between the obtained wave functions and wave functionals. Paper will end up with a conclusion.

2 Minisuperspace Saddle Point Wave Functions

In this section, we review the quantum cosmology of a 2+1 dimensional de Sitter universe. Take the following 2+1 dimensional FRW line element as a minisuperspace:

$$ds^2 = -N(t)^2 + a(t)^2 \left( \frac{dr^2}{1 - r^2} + r^2 d\theta^2 \right).$$

Spatial section of the 3-manifold taken to be $S^2$, i.e., $k=1$, to avoid getting infinite action due to infinite extension of a spatially flat $k=0$ or hyperbolic $k=-1$ universe. Our minisuperspace obviously contains classical de Sitter (dS3) universes.

Substituting the above ansatz into the Einstein-Hilbert action leads us to:

$$S = \frac{m_p^2}{4\pi} \int \left( \frac{\dot{a}^2}{N^2} - N + \Lambda Na^2 \right) dt.$$  

Varying with respect to $a(t)$ and $N$ yields an equation of motion

$$\frac{d}{dt} \left( \frac{\dot{a}}{N} \right) = \Lambda Na,$$
and the Hamiltonian constraint:

\[ H = \frac{1}{4} \pi_a^2 + 1 - \Lambda a^2 = 0, \] (4)

where \( \pi_a \) is momentum conjugate of \( a \). There is a solution to equations (3) and (4) for \( \Lambda > 0 \):

\[ a(t) = \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda} t + \beta), \] (5)

where \( \beta \) is an integration constant. This solution describes classical dynamics of a dS3 universe.

Euclideanized version of the action, resulting equation of motion and the Hamiltonian constraint read:

\[ S_E = -\frac{m_p^2}{4\pi} \int \left( a^2 \frac{N^2}{N^2} + N - \Lambda Na^2 \right) d\tau, \] (6)

\[ \frac{d}{d\tau} \left( \frac{\dot{a}}{N} \right) = -\Lambda Na, \] (7)

\[ H_E = \frac{1}{4} \pi_a^2 - 1 + \Lambda a^2 = 0. \] (8)

A solution to these set of equations which meets the requirements of a no boundary (NB) instanton [8] is:

\[ a(\tau) = \frac{1}{\sqrt{\Lambda}} \sin(\sqrt{\Lambda} \tau). \] (9)

Note that the condition \( \frac{da}{dt} = 1 \), necessary for regular closing off of four geometry at singularity \( a(0) = 0 \), is automatically satisfied by the Hamiltonian constraint. For \( a_{\partial M} < \frac{1}{\sqrt{\Lambda}} \), there is a "real" instanton which is a portion of a \( S^3 \) sphere [8].

For \( a_{\partial M} > \frac{1}{\sqrt{\Lambda}} \). In complex plane of \( \tau \), we should choose a path along \( \tau_{Re} \) axis to \( \tau_{max} = \frac{\pi}{2\sqrt{\Lambda}} \) which determines the maximum radius of such a compact instanton. This part of instanton describes one half of a \( S^3 \) sphere. Choosing the path to continue parallel to the \( \tau_{Im} \) axis to a given \( a > \frac{1}{\sqrt{\Lambda}} \) on the boundary, \( a(\tau) \) still remains real:

\[ a(\frac{\pi}{2\sqrt{\Lambda}} + i\tau_{Im}) = \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda} \tau_{Im}). \] (10)

This part of the instanton describes half of a Lorentzian dS3 universe. There is also another instanton which satisfies NBP conditions and contributes to the saddle point approximation [6].

Resulting NB wave function, for \( a < \frac{1}{\sqrt{\Lambda}} \) will be:

\[ \psi_{NB} \sim \exp \left( -2a \sqrt{1 - \Lambda a^2} - \frac{2}{\sqrt{\Lambda}} \sin^{-1} \sqrt{\Lambda a} \right) \] (11)

and for \( a > \frac{1}{\sqrt{\Lambda}} \):

\[ \psi_{NB} \sim e^{-\frac{\pi}{\sqrt{\Lambda}}} \cos(2a \sqrt{1 - \Lambda a^2} - \frac{2}{\sqrt{\Lambda}} \cosh^{-1} \sqrt{\Lambda a} - \frac{\pi}{4}). \] (12)
Vilenkin wave function in WKB approximation [3,4] will have the following form for \( a < \frac{1}{\sqrt{\Lambda}} \):

\[
\psi_V \sim \exp \left( 2a \sqrt{1 - \Lambda a^2} + \frac{2}{\sqrt{\Lambda}} \sin^{-1} \sqrt{\Lambda}a \right) \quad (13)
\]

and for \( a > \frac{1}{\sqrt{\Lambda}} \):

\[
\psi_V \sim \exp \left( \frac{\pi}{\sqrt{\Lambda}} \right) \exp \left[ -i(2a \sqrt{1 - \Lambda a^2} + \frac{2}{\sqrt{\Lambda}} \cosh^{-1} \sqrt{\Lambda}a) \right] \quad (14)
\]

3 Exact Solutions

According to Dirac prescription for quantization [13], the wave function of a constraint system should be annihilated by the operator version of classical constraints. Replacing \( \pi \) by \( -i \hbar \frac{\partial}{\partial a} \) in the Hamiltonian constraint, we will find the following Schrödinger like equation, a Wheeler-Dewitt equation [1], for the wave function of a dS3 universe:

\[
- \frac{1}{4a^P} \frac{\partial}{\partial a} \left( a^P \frac{\partial}{\partial a} \psi \right) + (1 - \Lambda a^2) \psi = 0, \quad (15)
\]

where \( P \) carries some part of factor ordering ambiguity due to indefiniteness of measure of path integral or equivalently quadratic form of Hamiltonian in \( \pi_a \). We will set it to 0.

There are exact solutions to the differential equation (15) in terms of Whittaker functions of type W and M [14]:

\[
\psi(a, \Lambda) = \frac{\zeta_m}{\sqrt{a}} WM\left(-\frac{1}{4\sqrt{-\Lambda}}, \frac{1}{4}, \sqrt{-\Lambda}a^2\right) + \frac{\zeta_w}{\sqrt{a}} WW\left(-\frac{1}{4\sqrt{-\Lambda}}, \frac{1}{4}, \sqrt{-\Lambda}a^2\right). \quad (16)
\]

Constant coefficients \( \zeta_m \) and \( \zeta_w \) should be determined by considering appropriate boundary conditions.

Asymptotic expansion of \( \psi(a, \Lambda) \) can be obtained easily [9]:

\[
\psi(a, \Lambda) \sim \frac{1}{\sqrt{a}} \Lambda^{-\frac{1}{4} + \frac{\sqrt{\pi}}{\nu \Lambda}} \left( \frac{i - 1}{\sqrt{\frac{1}{4} + \frac{\sqrt{\pi}}{\nu \Lambda}}} \right) e^{\frac{i}{\nu \Lambda} \zeta_m} a^{\frac{1}{4} + \frac{\sqrt{\pi}}{\nu \Lambda}} e^{-\frac{i}{\nu \Lambda} \sqrt{a^2}}
\]

\[
+ \frac{1}{\sqrt{a}} \left( \Lambda^{-\frac{1}{4} + \frac{\sqrt{\pi}}{\nu \Lambda}} \right) \zeta_m a^{\frac{1}{4} + \frac{\sqrt{\pi}}{\nu \Lambda}} e^{-\frac{i}{\nu \Lambda} \sqrt{a^2}}. \quad (17)
\]

\( a \to \infty \)

Vilenkin wave function, has only outgoing sector at very large values of \( a \), therefore:

\[
\zeta_w = -\frac{(i - 1)\sqrt{2\pi}}{4\Gamma\left(\frac{3}{4} + \frac{i}{\nu \Lambda}\right)} e^{\frac{i}{\nu \Lambda} \zeta_m}. \quad (18)
\]

We obtain the Vilenkin wave function up to a constant \( \gamma \):

\[
\psi_V(a, \Lambda) = \gamma \left\{ \frac{1}{\sqrt{a}} WM\left(-\frac{1}{4\sqrt{-\Lambda}}, \frac{1}{4}, \sqrt{-\Lambda}a^2\right) - \frac{1}{\sqrt{a}} \frac{(i - 1)\sqrt{2\pi}}{\Gamma\left(\frac{3}{4} + \frac{i}{\nu \Lambda}\right)} e^{\frac{i}{\nu \Lambda}} WW\left(-\frac{1}{4\sqrt{-\Lambda}}, \frac{1}{4}, \sqrt{-\Lambda}a^2\right) \right\}. \quad (19)
\]
To find the Hartle-Hawking state, we note that the ingoing and outgoing sectors of \( \psi(a, \Lambda) \) should have the same amplitudes for \( a \to \infty \). This implies:

\[
\zeta_w = \left( \frac{\sqrt{\pi}}{2\Gamma(\frac{3}{4} - i\frac{1}{4} \sqrt{\Lambda})} - \frac{\sqrt{2\pi}(i - 1)}{4\Gamma(\frac{3}{4} + i\frac{1}{4} \sqrt{\Lambda})} e^{i\sqrt{\Lambda}} \right) \zeta_m. \tag{20}
\]

Therefore, the resulting Hartle-Hawking state will be:

\[
\psi_{H-H}(a, \Lambda) = \frac{\zeta_m}{\sqrt{a}} W\left( -\frac{1}{4\sqrt{-\Lambda}}, \frac{1}{4}, \sqrt{-\Lambda} a^2 \right)
+ \frac{1}{\sqrt{a}} \left( \frac{\sqrt{\pi}}{2\Gamma(\frac{3}{4} - i\frac{1}{4} \sqrt{\Lambda})} - \frac{\sqrt{2\pi}(i - 1)}{4\Gamma(\frac{3}{4} + i\frac{1}{4} \sqrt{\Lambda})} e^{i\sqrt{\Lambda}} \right) \zeta_m W\left( -\frac{1}{4\sqrt{-\Lambda}}, \frac{1}{4}, \sqrt{-\Lambda} a^2 \right). \tag{21}
\]

4 Midisuperspace Saddle Point Wave Functionals

Is there any dramatic discrepancy between using a minisuperspace instead of full theory or even a midisuperspace? A way to answer to this question is to compare a minisuperspace amplitude with a midi one. Here we present and, approximately, solve a "midi" superspace with axial symmetry for quantum cosmology of dS3 universes which is described by the following line element:

\[
ds^2 = -N(t,r)^2 dt^2 + \phi(t,r) dr^2 + \psi(t,r) d\theta^2. \tag{22}
\]

By axial symmetry we mean that the metric components are considered to be \( \theta \) independent.

Substituting the above metric ansatz, Lagrangian density reads:

\[
\mathcal{L} = N(\psi\phi)^\frac{3}{2}\left[ \frac{1}{2N^2\phi}\left( -\dot{\phi} + 2N_r - \frac{\phi^\prime}{\phi} N_r \right) \left( -\dot{\psi} + \frac{\psi^\prime}{\psi} N_r \right) \right] + \frac{1}{2} R + 2\Lambda \] \tag{23}

, where \( \frac{1}{2} R \) is the scalar curvature of spatial sector of three geometry \( \frac{1}{2} R \) is given by:

\[
\frac{1}{2} R = \frac{2\psi'' \phi - \psi^2 \phi^\prime - \phi \psi^\prime \psi^\prime}{2\psi^2 \phi^2}. \tag{24}
\]

Momentum conjugates to the field variables are simply:

\[
\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2N(\phi)} \left( \dot{\phi} - \frac{\phi^\prime}{\phi} N_r \right), \tag{25}
\]

\[
\pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{1}{2N(\phi)} \left( \dot{\psi} + \frac{\psi^\prime}{\psi} N_r - 2N_r \right). \tag{26}
\]

A Legendre transformation will result in the Hamiltonian constraint density:

\[
H = \frac{1}{2N(\phi)} \left[ \dot{\phi} \left( \psi - \frac{\psi^\prime}{\psi} N_r \right) + \dot{\psi} \left( \phi + \frac{\phi^\prime}{\phi} N_r - 2N_r \right) \right] \mathcal{L} - \mathcal{L}, \tag{27}
\]
\[ H = \frac{1}{2N(\phi \psi)^{1/2}} \left( \psi \phi' + N \phi \psi' \right) - 2 N \phi' \psi' - 2 \phi \psi' - 2N(\phi \psi)^{1/2} \Lambda. \]  
(28)

Now, we should express the time derivatives of the field variables in terms of momenta, fields and their spatial derivatives:

\[ \dot{\phi} = 2N(\phi \psi)^{1/2} \pi \phi + 2N r - \frac{\phi'}{\phi} N r, \]  
(29)

\[ \dot{\psi} = 2N(\phi \psi)^{1/2} \pi \psi + \frac{\psi'}{\phi} N r. \]  
(30)

Integration over a spacelike hypersurface \( \Sigma \) leads to the Hamiltonian constraint:

\[ H = 2\pi \int_{\Sigma} \{ N \left[ 2(\phi \psi)^{1/2} \pi \phi \pi \psi - (\phi \psi)^{1/2} 2\Lambda(\phi \psi)^{1/2} - 2\phi^\prime \phi^\prime - \frac{\phi^\prime}{\phi} \pi \phi - 2\pi \phi \right] \} d\tau. \]  
(31)

Variation with respect to \( N \) and \( N r \) leads into the Hamiltonian and the momentum constraints:

\[ H_0 = 2(\phi \psi)^{1/2} \pi \phi \pi \psi - 2\Lambda(\phi \psi)^{1/2} - \frac{2\phi^\prime \phi^\prime - \phi^\prime}{2\phi^\prime \phi^\prime} = 0, \]  
(32)

\[ H_r = \frac{\psi'}{\phi} \pi \phi - \frac{\phi'}{\phi} \pi \phi - 2\pi \phi = 0. \]  
(33)

Hamiltonian constraint carries time reparametrization invariance of the classical theory and momentum constraint is the generator of infinitesimal coordinate transformations within spacelike hypersurfaces.

The number of field variables and constraints are equal, then it is tempting to solve \( \pi \phi \) and \( \pi \psi \) in terms of field variables and their spatial derivatives by constructing a superposition of constraints to eliminate \( \pi \psi \).

Note that \( ^2 R \) can be rewritten as follows:

\[ ^2 R = \frac{1}{2\psi} \partial_r \left( \frac{\psi^2}{\phi \psi} \right). \]  
(34)

Now, consider the below superposition:

\[ 2(\phi \psi)^{1/2} \frac{\phi'}{\psi} \pi \phi H_r - H_0 = \frac{(\phi \psi)^{1/2}}{\psi'} \left( -2\phi^\prime \pi \phi' - 4\phi \pi \phi \pi \phi' + 2 R \phi' + 2\Lambda \right) = 0. \]  
(35)

By considering the eqs.(34) and (35), we obtain:

\[ \partial_r \left( -2\phi^\prime \pi \phi^2 + \frac{\psi^2}{2\phi \psi} + 2\Lambda \right) = 0 \]  
(36)

which simply results in:

\[ \pi \phi = \sqrt{\frac{1}{2\phi} \left( \frac{\psi^2}{2\phi \psi} + 2\Lambda \psi - 2 \right)}. \]  
(37)
Regarding eq.(32), \( \pi_\psi \) will become:

\[
\pi_\psi = \frac{1}{2\psi} (\frac{\psi'}{2\psi})' + 2\Lambda \quad \frac{1}{2\sqrt{\frac{\psi'^2}{2\psi^2} + 2\Lambda\psi - 2}}.
\]  

(38)

The leading term in a semiclassical expansion of the "wave functional" of this field theory will be \( e^{\pm iS_{cl}} \), where \( S_{cl} \) is the action evaluated for classical path:

\[
\delta S = \int \left( \pi_\phi \delta \phi + \pi_\psi \delta \psi \right).
\]

(39)

We can choose a specific contour of integration to simplify integration process. First of all, we hold \( \psi \) constant and then integrate over \( \phi \), to a configuration of \( \phi \) such that:

\[
\frac{\psi'^2}{2\psi \phi} + 2\Lambda\psi - 2 = 0.
\]

(40)

This configurations are the "boundary" between classically allowed and forbidden regions. After that, we will hold \( \phi \) constant such that eq.(40) holds and integrate over \( \psi \) to a given configuration.

Before going ahead, we note that:

\[
\pi_\psi = \frac{1}{2\psi} (\frac{\psi'}{2\psi})' + 2\Lambda \quad \frac{1}{2\sqrt{\frac{\psi'^2}{2\psi^2} + 2\Lambda\psi - 2}} \quad \frac{1}{\psi'} \sqrt{\frac{1}{2\phi}},
\]

(41)

then this part will not contribute to the action. Therefore, the action reads:

\[
S = \int \sqrt{\frac{1}{2\phi} (\frac{\psi'^2}{2\psi \phi} + 2\Lambda\psi - 2) d[\phi] dr}.
\]

(42)

Integration is straightforward and results in:

\[
S = \int \sqrt{\frac{\psi'^2}{4\psi} - (1 - \Lambda\psi)\phi - 2\sqrt{\frac{\psi'^2}{4\psi} \frac{1}{Arctanh(\sqrt{\frac{\psi'^2}{4\psi} - (1 - \Lambda\psi)\phi})}}) dr}.
\]

(43)

Wave functional for classically forbidden region will be a superposition of exponentially decaying and growing forms and in the allowed region is a linear superposition of oscillating exponentials of the action.

Now, we are in the position to calculate the wave functional of a 2+1 dimensional FRW-like universe. It will be sufficient to consider a homogeneous and isotropic universe on the final hypersurface of simultaneity.

By substituting "FRW" form for the metric on the final hypersurface and regarding to eq.(43), we will reach the following saddle point "wave functionals":
Vilenkin wave functional in tunneling region will be:
\[ \Psi_V \sim e^{2a\sqrt{1-\Lambda a^2} - 2a \int_0^1 \tan^{-1}\left(\frac{r}{(1-r^2)^{\frac{1}{2}}}\right) \sqrt{1-\Lambda a^2} dr} \]  \hspace{1cm} (44)

and in classical region:
\[ \Psi_V \sim e^{-i(2a\sqrt{1+\Lambda a^2} - 2a \int_0^1 \tan^{-1}\left(\frac{r}{(1-r^2)^{\frac{1}{2}}}\right) \sqrt{-1+\Lambda a^2} dr}}. \]  \hspace{1cm} (45)

Hartle-Hawking wave functional in tunneling region will be:
\[ \Psi_{H-H} \sim e^{-2a\sqrt{-1-\Lambda a^2} + 2a \int_0^1 \tan^{-1}\left(\frac{r}{(1-r^2)^{\frac{1}{2}}}\right) \sqrt{-1-\Lambda a^2} dr} \]  \hspace{1cm} (46)

and in classical region:
\[ \Psi_{H-H} \sim \cos(2a\sqrt{-1+\Lambda a^2} - 2a \int_0^1 \tanh^{-1}\left(\frac{r}{(1-r^2)^{\frac{1}{2}}}\right) \sqrt{-1+\Lambda a^2} dr - \frac{\pi}{4}). \]  \hspace{1cm} (47)

As obviously can be seen from the resulting midisuperspace wave functionals, there is a discrepancy between minisuperspace wave functions and midisuperspace wave functionals.

5 Conclusion

Minisuperspace approximation to find the wave function of a 2+1 dimensional de Sitter universe, seems to be unreliable and a more realistic model such as a midisuperspace yields different result. Therefore, certain predictions derived from minisuperspace models may be wrong. Comparing predictions based on a canonically quantized field theory for quantum gravity (a midisuperspace) with minisuperspace one, e.g., tunneling amplitudes for creation of a dS3 universe or isotropy of a large universe are subjects of further investigations.

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7 References

1. B.S.Dewitt, Phys.Rev.160(1967)1113.
2. J.B.Hartle and S.W.Hawking, Phys.Rev.D28, 2960(1983).
3. A.Vilenkin, Phys.Rev.D33, 3560(1986).
4. A.Vilenkin, Phys.Rev.D37, 888(1988).
5. G.W. Gibbons, S.W. Hawking, Euclidean Quantum Gravity, World Scientific (1993).
6. P.D. D’eath, Supersymmetric Quantum Cosmology, Cambridge University Press (1995).
7. A. Strominger, Nucl. Phys. B319 (1989) 722-732.
8. R. Bousso and S. W. Hawking, Phys. Rev. D52, No. 10, 5659 (1995).
9. J.J. Halliwell, S.W. Hawking, Phys. Rev. D31, No. 8, 1777 (1985).
10. B.K. Berger, D.M. Chitre, V.E. Moncrief, and Y. Nutku, Phys. Rev. D5, 2467 (1972).
11. C. Vaz, Phys. Rev. D61 (2000), 064017.
12. W. Fischler, D. Morgan, and J. Polchinski, Phys. Rev. D42, No. 12 (1990).
13. P.A.M. Dirac, Lectures on quantum mechanics (New York, Belfer Graduate School of Science, Yeshiva University, 1964).
14. M. Abramowitz and I. Stegun, Hand Book of Mathematical Functions, Dover Publication, INC., New York, 1965.