Global Attractors for the Higher-Order Evolution Equation

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Abstract

In this paper, we obtain the existence of a global attractor for the higher-order evolution type equation. Moreover, we discuss the asymptotic behavior of global solution.

Keywords: Global attractor, existence, asymptotic behavior.

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1 Introduction

We consider the following nonlinear evolution equation

\[
\begin{cases}
    u_{tt} + (-\Delta)^m u + (-\Delta)^m u_t + g(x,u) = f(x), & (x,t) \in \Omega \times (0,\infty), \\
    u(x,0) = u_0(x), & u_t(x,0) = u_1(x), \\
    \frac{\partial u(x,t)}{\partial \nu} = 0, & i = 1,2,\ldots, m-1, \\
    (x,t) \in \partial \Omega \times [0,\infty),
\end{cases}
\] (1.1)

where in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial \Omega$, the assumption on $f$, $g$, $u_0$ and $u_1$ will be made below.

When $m = 1$, the equation (1.1) is following form

\[
u_{tt} - \Delta u - \Delta u_t + g(x,u) = f(x).
\] (1.2)

Chen and Wang [19] proved the existence of global attractor for the problem (1.2). Lately, Xie and Zhong in [8] studied the existence of global attractor of solution for the problem (1.1) with $f = 0$. Also, there are some authors studied the existence and nonexistence, asymptotic behavior of global solution for (1.2) (see [2–7] for more details). Nakao and Yang in [9] showed the global attractor of the Kirchhoff type wave equation.

In this paper, we improve our result by adopting and modifying the method of [19], we studied more general form of the equation.
This paper is organized as follows: In section 2, we give some assumptions and state the main results. In section 3, we prove the global existence of solution using the Faedo-Galerkin method. Also, we write some important estimates for the solution. In section 4, the existence of the global attractor is proved. In Section 5, the proof of decay property for solution is showed.

2 Preliminaries and main results

We write the Sobolev space \( H^k(\Omega) = W^{k,2}(\Omega) \), \( H^0_0(\Omega) = W_0^{k,2}(\Omega) \). Furthermore, we show by \((\ldots)\) the inner product of \( L^2(\Omega) \), by \( \|\|_p \) the norm of \( L^p(\Omega) \), \( p \geq 1 \) and by \( \|\|_E \) the norm of any other Banach space \( E \). As usual, we give \( u(t) \) instead of \( u(x,t) \), and \( u'(t) \) for \( u_i(t) \) and so on.

We write the following assumptions on \( f \) and \( g \).

\( (A_1) \) Assume \( f(x) \in L^2(\Omega) \) and show \( F = \|f\|_2^2 \).

\( (A_2) \) Suppose \( g(x,u) \in C^1(\Omega \times R^1) \) and \( \exists k_1, k_2 > 0, h_1(x) \in L^2(\Omega), h_2(x) \in L^2(\Omega) \cap L^{n/2}(\Omega) \) such that

\[
g(x,u)u + h_1(x)u \geq k_1(G(x,u) + h_1(x)|u|) \geq 0, (x,u) \in \Omega \times R^1
\]

and the growth condition in \( u \)

\[
|g(x,u)| \leq k_2(|u|^{\alpha} + h_2(x)), |g_u(x,u)| \leq k_2 \left( |u|^{\alpha - 1} + h_2(x) \right), (x,u) \in \Omega \times R^1
\]

with \( \alpha \geq 1, (n = 1, 2) \), and \( 1 \leq \alpha \leq \frac{n+2}{n-2}, (n \geq 3) \), \( G(x,u) = \int_0^u g(x,s)ds \).

Later, we assume \( H_1 = \|h_1\|_2, H_2 = \max \left\{ \|h_2\|_2, \|h_2\|_{n/2} \right\} \).

Clearly, the function \( g(x,u) = a(x)|u|^{\alpha - 1} - b(x)|u|^{\beta - 1}u (1 \leq \beta < \alpha) \) supplies \((2.1)\) and \((2.2)\) for some \( a(x), b(x) \).

Next, we show the definition and lemmas relating to the global attractor, (see \([9, 11, 12]\)).

**Definition 1.** Suppose that \( E \) is Banach space and \( \{S(t)\}_{t \geq 0} \) a semigroup on \( E \). A set \( A \subset E \) is said a \((E,E)-\)global attractor if and only if

1. \( A \) is never changing (invariant), namely, \( S(t)A = A \) for whole \( t \geq 0 \);

2. \( A \) is compact in \( E \);

3. \( A \) is a bounded set in \( E \) and absorbs all bounded subset \( B \) in \( E \) relating with \( E \) topology, that is, for whichever bounded subset \( B \subset E \),

\[
\text{dist}_E(S(t)B,A^c) = \sup_{y \in B} \inf_{x \in A} \|S(t)y - x\|_E \to 0 \text{ as } t \to \infty.
\]

**Lemma 2.** Assume \( E \) is Banach space and \( \{S(t)\}_{t \geq 0} \) is a semigroup of continuous operators on \( E \). Then, there exists \((E,E)-\)global attractor \( A \) if the following conditions are supplied:

1. \( T \) exists a bounded absorbing set \( B_0 \) in \( E \), that is, for whichever bounded subset \( B \subset E \), there is a \( T = T(B) \) such that \( S(t)B \subset B_0 \) for any \( t \geq T \).

2. \( \{S(t)\}_{t \geq 0} \) as asymptotically compact in \( E \), that is, for any bounded sequence \( \{y_n\} \) in \( E \) and \( t_n \to \infty \) as \( n \to \infty \), \( \{S(t_n)y_n\}_{n=1}^\infty \) has a convergent subsequence relating to \( E \) topology.

We show the basic results now.

**Theorem 3.** Suppose \((A_1)-(A_2)\) satisfy and \((u_0, u_1) \in X \). Then, the problem \((1.1)\) admits a unique weak solution \( u(t) \) in the class

\[
C^1([0,\infty);H^m_0) \cap C([0,\infty);H^{2m} \cap H^{m}_0) \cap W^{2,\infty}(\{0,\infty\);H^m_0) \cap W^{1,\infty}(\{0,\infty\);H^{2m})
\]
holds.
\[ \left\| P^1 u(t) \right\|^2 + \left\| P^2 u(t) \right\|^2 \leq C_1 e^{-\lambda_1 t} + C_2, \quad t \geq 0 \] (2.5)
\[ \left\| u_t(t) \right\|^2 + \left\| P^1 u_t(t) \right\|^2 + \left\| P^2 u_t(t) \right\|^2 \leq C_3 e^{-\lambda_2 t} + C_4, \quad t \geq 0 \] (2.6)
and
\[ \left\| P^3 u(t) \right\|^2 + \left\| P^4 u(t) \right\|^2 + \left\| Pu(t) \right\|^2 \leq C_5 e^{-\lambda_3 t} + C_4, \quad t \geq 0 \] (2.7)
with some \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \). In this theorem \( C_1 = C_1 \left( \left\| P^1 u_0 \right\|_2, \left\| P^2 u_1 \right\|_2 \right), C_2 = C_2 (F, H_1), \; C_3 = C_3 \left( \left\| P^1 u_0 \right\|_2, \left\| P^2 u_1 \right\|_2, F, H_1, H_2 \right), C_4 = C_4 (F, H_1, H_2), C_5 = C_5 \left( \left\| Pu_0 \right\|_2, \left\| Pu_1 \right\|_2, F, H_1, H_2 \right). \)

Show the solution in Theorem 1 by \( S(t)(u_0, u_1) = (u(t), u_t(t)) \). We are now in a position to prove some continuity of \( S(t) \) relating to the initial data \( (u_0, u_1) \), which will be needed for the proof of the existence of global attractor.

**Theorem 4.** Suppose whole conditions in Theorem 3. Assume \( S(t)(u_{0k}, u_{1k}) \) and \( S(t)(u_0, u_1) \) are the solutions of the problem (1.1) with the initial data \( (u_{0k}, u_{1k}) \) and \( (u_0, u_1) \). If \( (u_{0k}, u_{1k}) \rightarrow (u_0, u_1) \) in \( X \) as \( k \rightarrow \infty \), then \( S(t)(u_{0k}, u_{1k}) \rightarrow S(t)(u_0, u_1) \) in \( X \) as \( k \rightarrow \infty \).

Theorem 4 denotes that the semigroup \( S(t) : X \rightarrow X \) is continuous on \( X \).

**Theorem 5.** Assume every assumptions in Theorem 3 be provided. Then, the semigroup \( \{S(t)\}_{t \geq 0} \) related with the solution of the problem (1.1) accepts a \( (X, X) \)-global attractor \( A \).

For the decay property of solution \( u(t) \) for the problem (1.1), we get

**Theorem 6.** Suppose \( u \) is a weak solution in Theorem 3 with \( f = 0 \) and \( g(x, u) = g(u) \). Besides, suppose \( 0 \leq 2G(u) \leq u g(u) \). Then, for whichever \( q > 0 \), there is \( C_1 = C_1 \left( \left\| P^1 u_0 \right\|_2, \left\| P^2 u_1 \right\|_2 \right) \) such that
\[ E(t) = \frac{1}{2} \left( \left\| u(t) \right\|^2 + \left\| P^1 u(t) \right\|^2 + \left\| P^2 u_t(t) \right\|^2 \right) + \int_{\Omega} G(u(t)) \, dx \leq C_1 (1 + t)^{-1/q}. \] (2.8)

### 3 The Proof of Theorem 3

In this section, we suppose that all assumptions in Theorem 3 are supplied. Firstly, we establish the global existence of a solution to problem (1.1) with Fadeo-Galerkin method as in [16, 17].

Assume \( \omega_j(x) \ (j = 1, 2, \ldots) \) is the complete set of properly normalized eigenfunctions for the operator \( (-\Delta)^m \) in \( H_0^m(\Omega) \). Then, the family \( \{\omega_1, \omega_2, \ldots, \omega_k, \ldots\} \) holds an orthogonal basis for both \( H_0^m(\Omega) \) and \( L^2(\Omega) \), see [16, 17]. For each positive integer \( k \), show \( V_k = \text{span} \{\omega_1, \omega_2, \ldots, \omega_k, \ldots\} \). We search for an approximation solution \( u_k(t) \) to the problem (1.1) in the form
\[ u_k(t) = \sum_{j=1}^{k} d_{jk}(t) \omega_j \]
where \( d_{jk}(t) \) are the solution of the nonlinear ordinary differential equation (ODE) system in the variant \( t \):
\[ (u_k''(t), \omega_j) - (P u_k(t), \omega_j) - (P u_k'(t), \omega_j) - (P u_k^m(t), \omega_j) + (g, \omega_j) = (f, \omega_j), \quad j = 1, 2, \ldots, k, \] (3.1)
with the initial conditions
\[ d_{jk}(0) = (u_{0k}, \omega_j), \quad d_{jk}'(0) = (u_{1k}, \omega_j) \] (3.2)

where \( u_{0k} \) and \( u_{1k} \) are chosen in \( V_k \) so that
\[ u_{0k} \to u_0, \quad u_{1k} \to u_1 \text{ in } H^{2m}(\Omega) \cap H_0^m(\Omega) \text{ as } k \to \infty. \] (3.3)

Here \((.,.)\) shows the inner product in \( L^2(\Omega) \). Then, Sobolev imbedding theorem means that \( \exists c_0 > 0 \), such that
\[ \| u_k(0) \|_{H_0^m}^2 \leq c_0 \| P^j u_k \|_2^2, \quad \| u_k'(0) \|_{H_0^m}^2 \leq c_0 \| P^j u_1 \|_2^2 \quad \forall k = 1, 2, ..., \] (3.4)

and (3.1) shows that for any \( v \in V_k \),
\[ (u_k', v) - (Pu_k, v) - (Pu_k', v) + (g, v) = (f, v), \quad \forall v \in V_k. \] (3.5)

We know, the system (3.1) and (3.2) accept a unique solution \( u_k(t) \) on the interval \([0, T]\) for any \( T > 0 \). Such a solution can be expanded to the overall interval \([0, \infty)\). We show by \( C_1, C_2, \ldots \) the constants that are independent of \( k \) and \( t \geq 0 \), by \( C_0 \) the constant depending on \( k_1, k_2 \) in (A2) and Sobolev imbedding constant \( c_0 \) in (3.4). These constants may be different from line to line.

Multiplying (3.1) by \( d_{jk}(t) \) and summing the resulting equations over \( j \), we obtain
\[ E_1(t) + \| P^j u_k(t) \|_2^2 = 0, \quad \forall t \geq 0 \] (3.6)

where
\[ E_1(t) = \frac{1}{2} \left( \| u_k(t) \|_2^2 + \| P^j u_k(t) \|_2^2 + \| P^j u_k'(t) \|_2^2 \right) + \int_\Omega G(x, u_k(t)) \, dx - \int_\Omega f(x) u_k(t) \, dx. \] (3.7)

Also, multiplying (3.1) by \( d_{jk}'(t) \), we get
\[ E_2(t) + \| P^j u_k(t) \|_2^2 + \int_\Omega g(x, u_k) u_k(t) \, dx = \| u_k'(t) \|_2^2 + \| P^j u_k'(t) \|_2^2 + \int_\Omega f(x) u_k(t) \, dx \] (3.8)

where
\[ E_2(t) = \frac{1}{2} \| P^j u_k(t) \|_2^2 + \int_\Omega u_k(t) u_k'(t) \, dx + \int_\Omega P^j u_k(t) P^j u_k'(t) \, dx. \] (3.9)

If we take sufficient large \( k_1 > 0 \) and use the assumption (A2), we get
\[ \psi'_k(t) + \lambda_1 \psi_k(t) \leq C_0 \left( F^2 + H_1^{2m} \right), \quad \psi_k(t) = k_1 E_1(t) + E_2(t) \] (3.10)

with some positive \( \lambda_1 \), relating to the indicated constants in (A2).

We note that
\[ \psi_k(t) \leq c_0 (k_1 + c_0) \left( \| P^j u_k'(t) \|_2^2 + \| P^j u_k(t) \|_2^2 \right) \]
\[ + k_1^2 (F^2 + H_1^{2m}) + k_1 \int_\Omega (G + h_1 |u_k(t)|) \, dx \] (3.11)

and
\[ 2 \psi_k(t) \geq (k_1 - 1) \left( \| u_k'(t) \|_2^2 + \| P^j u_k'(t) \|_2^2 \right) + (k_1 - 5c_0) \| P^j u_k(t) \|_2^2 \]
\[ + k_1 \int_\Omega (G + h_1 |u_k(t)|) \, dx - k_1^2 (F^2 + H_1^{2m}) \] (3.12)
with \( k_1 \geq \max \{3, 2 + 5c_0 \} \).

The application of Gronwall lemma to (3.10) holds
\[
\|P^2 u_k(t)\|_2^2 + \|P^2 u'_k(t)\|_2^2 + \int_{\Omega} (G(x, u_k(t)) + h_1(x)|u_k(t)|) \, dx \leq C_1 e^{-\lambda_1 t} + C_2, \; t \geq 0
\]
which shows
\[
\|P^2 u_k(t)\|_2^2 + \|P^2 u'_k(t)\|_2^2 \leq C_1 e^{-\lambda_1 t} + C_2, \; t \geq 0
\]
(3.13)

where \( C_1 = C_1 \left( \|P^2 u_0\|_2, \|P^2 u_1\|_2 \right), \; C_2 = C_2 (F, H_1) \).

Also, we differentiate (3.1) with respect to \( t \) and get
\[
(u''_k, \omega_j) - (Pu'_k, \omega_j) - (Pu''_k, \omega_j) + (g_u u'_k, \omega_j) = 0, \; j = 1, 2, \ldots, k.
\]
(3.14)

Multiplying (3.14) by \( d'_{jk} (t) \) and summing the resulting equations over \( j \), we obtain
\[
E'_3 (t) + \|P^2 u''_k (t)\|_2^2 + \int_{\Omega} g_u u'_k u''_k \, dx = 0
\]
(3.15)

with
\[
E_3 (t) = \frac{1}{2} \left( \|u''_k(t)\|_2^2 + \|P^2 u'_k(t)\|_2^2 + \|P^2 u''_k(t)\|_2^2 \right)
\]
\[
\leq C_0 \left( \|P^2 u''_k(t)\|_2^2 + \|P^2 u'_k(t)\|_2^2 \right), \; t \geq 0
\]
(3.16)
in which the Sobolev embedding theorem has been used.

Furthermore, the growth condition (2.2) and the Hölder inequality mean that
\[
\int_{\Omega} |g_u u'_k| \, dx \leq k_2 \int_{\Omega} \left( |h_2| \|u'_k\|_2 + |u_k|^{\alpha-1} \|u'_k\|_2 \right) \, dx
\]
\[
\leq C_0 \left( |h_2|_{n/2} + \|P^2 u_m\|_{2}^{\alpha-1} \right) \|P^2 u'_k\|_2 \|P^2 u''_k\|_2.
\]

Therefore, we get
\[
\int_{\Omega} |g_u u'_k u''_k| \, dx \leq \frac{1}{2} \|P^2 u''_k(t)\|_2^2 + C_0 \|P^2 u'_k(t)\|_2^2 \left( \|P^2 u_k(t)\|_2^{2(\alpha-1)} + H_2^{2m} \right)
\]
(3.17)

and
\[
E'_3 (t) + \frac{1}{2} \|P^2 u''_k(t)\|_2^2 \leq C_0 \|P^2 u'_k(t)\|_2^2 \left( \|P^2 u_k(t)\|_2^{2(\alpha-1)} + H_2^{2m} \right).
\]
(3.18)

Then, the applications of the estimates (3.13) and (3.15)-(3.18) give that \( \exists \lambda_1 \geq \lambda_2 > 0 \), depending on \( C_0 \), such that
\[
E'_3 (t) + \lambda_2 E_3 (t) \leq C_0 \|P^2 u'_k(t)\|_2^2 \left( 1 + \|P^2 u_k(t)\|_2^{2(\alpha-1)} + H_2^2 \right)
\]
\[
\leq C_3 e^{-\lambda_1 t} + C_4.
\]
(3.19)

Here, assume \( C_3 = C_3 \left( \|P^2 u_0\|_2, \|P^2 u_1\|_2, F, H_1, H_2 \right), \; C_4 = C_4 (F, H_1, H_2) \). Then (3.19) means that
\[
E_3 (t) \leq E_3 (0) e^{-\lambda_2 t} + C_3 e^{-\lambda_1 t} + \lambda_2^{-1} C_4, \; t \geq 0.
\]
(3.20)
We show that $E_3(0)$ is uniformly bounded for $k$ under the conditions in Theorem 3 now. It follows by (3.1) that
\[
(u_k''(t) - Pu_k'(t) - Pu_k''(t), u_k''(t)) = (f, u_k''(t)) - (g, u_k''(t)).
\]
(3.21) Especially, suppose $t = 0$, we get
\[
\|u_k''(0)\|_2^2 + \|P^1 u_k''(0)\|_2^2 + \int_\Omega P^2 u_k''(0).\left( P^1 u_k(0) + P^1 u_k'(0) \right)^2 dx = \int f(x) u_k''(0) dx - \int (g, u_k(0)) u_k'(0) dx.
\]
(3.22) By Young inequality with $\varepsilon$,
\[
\int_\Omega |P^1 u_k'(0).P^1 u_k(0)| dx \leq \varepsilon \left( \|P^1 u_k'(0)\|_2^2 + C_\varepsilon \|P^1 u_k(0)\|_2^2 \right),
\]
\[
\int_\Omega |P^1 u_k'(0).P^2 u_k''(0)| dx \leq \varepsilon \left( \|P^1 u_k''(0)\|_2^2 + C_\varepsilon \|P^2 u_k'(0)\|_2^2 \right),
\]
\[
\int_\Omega |g(x, u_k(0)) u_k''(0) dx| \leq \|u_k''(0)\|_2 \frac{\varepsilon}{\varepsilon - 1} \|g\|_{\mu_1} \leq \varepsilon \left( \|P^2 u_k''(0)\|_2^2 + C_\varepsilon \|g\|_{\mu_1}^2 \right),
\]
(3.23) and
\[
\int_\Omega |f(x) u_k''(0)| dx \leq \varepsilon \left( \|P^2 u_k'(0)\|_2^2 + C_\varepsilon \|f\|_2^2 \right)
\]
with $\mu_1 = 2n/(n+2)$. Since $\mu_1 \alpha = 2n\alpha/(n+2) \leq 2n/(n-2)$, we obtain by (2.2) that
\[
\int_\Omega |g|^{\mu_1} dx \leq C_0 \int_\Omega (|u_0(0)|^{\mu_1} + |h_2|^{\mu_1}) dx \leq C_0 \left( \|P^1 u_0\|_2^{\mu_1} + \|h_2\|_2^{\mu_1} \right).
\]
(3.24) Suppose $0 < \varepsilon \leq 1/6$. Then, from (3.22) to (3.24) that
\[
E_3(0) \leq \left( \|P^1 u_k''(0)\|_2^2 + \|u_k''(0)\|_2^2 + \|P^2 u_k''(0)\|_2^2 \right) - C_0 \left( \|P^1 u_k'(0)\|_2^2 + \|P^2 u_k'(0)\|_2^2 + F^2 + \|g\|_{\mu_1} \right) \leq C_0 \left( \|P^1 u_1\|_2^2 + \|P^1 u_0\|_2^2 + F^2 + \|P^1 u_0\|_2^{2\alpha} + \|h_2\|_2^2 \right) \equiv C_3.
\]
(3.25) Therefore, the inequality (3.20) shows
\[
\|P^1 u_k''(t)\|_2^2 + \|P^1 u_k'(t)\|_2^2 + \|P^2 u_k''(t)\|_2^2 \leq C_3 e^{-\lambda_2 t} + \lambda_2^{-1} C_4, t \geq 0
\]
(3.26) and the estimates (3.13) and (3.26) give that
\[
\begin{cases}
\|u_k(t)\| & \text{is bounded in } L^\infty([0, \infty); H^m_0(\Omega)), \\
\|u_k'(t)\| & \text{is bounded in } L^\infty([0, \infty); H^m_0(\Omega)), \\
\|u_k''(t)\| & \text{is bounded in } L^\infty([0, \infty); H^m_0(\Omega)).
\end{cases}
\]
(3.27) So, there exists a subsequences in $\{u_k\}$ (still showed by $\{u_k\}$) such that
\[
\begin{cases}
u_k \to u \text{ weakly star in } L^\infty([0, \infty); H^m_0(\Omega)), \\
u_k' \to u' \text{ weakly star in } L^\infty([0, \infty); L^2(\Omega)), \\
u_k'' \to u'' \text{ weakly star in } L^2([0, \infty); H^m_0(\Omega)).
\end{cases}
\]
(3.28)
From applying the fact that $L^\infty([0, \infty) ; H^m_0(\Omega)) \hookrightarrow L^2([0, \infty) ; H^m_0(\Omega))$ and the Lions-Aubin compactness Lemma in [20], we obtain from (3.27) and (3.28) that

$$u_k \to u, \ u'_k \to u' \text{ strongly in } L^2([0, \infty) ; L^2(\Omega)) \quad (3.29)$$

and then $u_k \to u \text{ a.e in } \Omega \times [0, \infty)$.

Using the growth condition (2.2), for any $T > 0$, the integral

$$\int_0^T \int_\Omega |g(x,u_k(x,t))| \frac{a(x)}{\alpha} \, dx \, dt$$

is bounded. Accordingly, by Lemma 2 in Chap. 1 [17], we conclude

$$g(x,u_k) \to g(x,u) \text{ weakly in } L^\frac{a(x)}{\alpha}([0,T] ; L^\frac{a(x)}{\alpha}(\Omega)) \quad (3.30)$$

with these convergences, by using the limit in the approximate equation (3.5), we get

$$(u''(t),v) - (Pu,v) - (Pu',v) - (Pu'',v) + (g(x,u),v) = (f,v), \ \forall v \in H^m_0(\Omega), \quad (3.31)$$

So, $u(t)$ is a weak solution of (1.1) and supplies (2.5) and (2.6), and the proof of existence for the solution $u(t)$ of (1.1) is completed.

We derive the estimates for $\|Pu(t)\|_2$ and $\|Pu(t)\|_1$ now. Also, we write $u$ instead of $u_k$ for convenience and view the estimates for $u$ as a limit of $u_k$. Supposing $v = -Pu$ in (3.31), we obtain

$$E'_4(t) + \|Pu(t)\|_2^2 \leq \|P^2u(t)\|_2^2 + \|Pu(t)\|_2^2 + C_0 \left( F^2 + \|g\|_2^2 \right) \quad (3.32)$$

with some $C_0 > 0$ and

$$E_4(t) = \frac{1}{2} \|Pu(t)\|_2^2 + \int_\Omega P^2u(t) P^2u(t) \, dx + \int_\Omega Pu(t) Pu(t) \, dx. \quad (3.33)$$

Also, assuming $v = -Pu$, in (3.31), we get

$$\int_\Omega Pu(-u_{tt} + Pu + Pu_{tt}) \, dx + \|Pu\|_2^2 = \int_\Omega gP^2u \, dx - \int_\Omega fPu \, dx \leq \frac{1}{2} \|Pu\|_2^2 + C_0 \left( F^2 + \|g\|_2^2 \right). \quad (3.34)$$

This means that

$$E'_5(t) + \frac{1}{2} \|Pu(t)\|_2^2 \leq C_0 \left( F^2 + \|g\|_2^2 \right) \quad (3.35)$$

with

$$E_5(t) = \frac{1}{2} \left( \|P^2u(t)\|_2^2 + \|Pu(t)\|_2^2 + \|Pu(t)\|_2^2 \right). \quad (3.36)$$

We note that

$$\|u\|_{2\alpha}^2 \leq C_0 \|P^2u\|_{2\alpha}^{2\alpha} + \|Pu\|_{2\alpha}^{2\alpha(1-\theta)} \leq \eta \|Pu\|_2^2 + C_\eta \|P^2u\|_2^{2\beta} \quad (3.37)$$

with small $\eta > 0$ and $2\alpha \theta = (n - 2) \alpha - n < 2$, $\beta = \alpha (1-\theta) / (1-\alpha \theta) > 0$. Then, (3.37) shows

$$\|g\|_2^2 \leq C_0 \left( \|u\|_{2\alpha}^{2\alpha} + H_2^{2m} \right) \leq \eta \|Pu\|_2^2 + C_\eta \|P^2u\|_2^{2\beta} + C_0 H_2^{2m}. \quad (3.38)$$
Then, by (2.5), (3.35) and (3.38) that
\[
E_5^t(t) + \frac{1}{2} \| Pu_t(t) \|^2 \leq \eta \| Pu(t) \|^2 + \| Pu \|^{2\beta} + C_0 (F^2 + H^{2m}_2)
\]
\[
\leq C_1 e^{-\lambda_1 \beta t} + \eta \| Pu(t) \|^2 + C_2.
\]
Assume \( \phi (t) = k_1 E_5(t) + E_4(t) \). We get from (3.32) and (3.39) that
\[
\phi'(t) + \frac{k_1 - 1}{2} \| Pu(t) \|^2 + (1 - (1 + k_1/2) \eta) \| Pu(t) \|^2 \leq C_1 e^{-\lambda_1 \beta t} + C_2.
\]
Suppose \( k_1 \geq 3 \) and \( \eta \) is small that \( 1 - \eta (1 + k_1/2) \geq 4/5 \). Then, (3.40) shows
\[
\phi'(t) + \| Pu(t) \|^2 + \frac{1}{2} \| Pu(t) \|^2 \leq C_1 e^{-\lambda_1 \beta t} + C_2.
\]
We note that
\[
E_4(t) \leq \frac{3}{5} \| Pu \|^2 + 3 \| Pu_t \|^2 + \frac{1}{2} \left( \| P^2 u \|^2 + \| P^2 u_t \|^2 \right)
\]
and
\[
\phi(t) \leq \left( \frac{3}{5} + \frac{k_1}{2} \right) \| Pu \|^2 + \left( 3 + \frac{k_1}{2} \right) \| Pu_t \|^2 + \frac{1}{2} \left( \| P^2 u \|^2 + \| P^2 u_t \|^2 \right)
\]
\[
\leq C_0 \left( \| Pu \|^2 + \| Pu_t \|^2 \right) + C_1 e^{-\lambda_1 \beta t} + C_2.
\]
Also (3.41) and (3.43) give that \( \exists \lambda_1 \beta \geq \lambda_3 > 0 \), depending on \( C_0 \), such that
\[
\phi'(t) + \lambda_3 \phi(t) \leq C_1 e^{-\lambda_1 \beta t} + C_2, \ t \geq 0.
\]
So,
\[
\phi(t) \leq \phi(0) e^{-\lambda_3 t} + C_1 e^{-\lambda_1 \beta t} + C_2 \lambda_3^{-1}, \ t \geq 0.
\]
Otherwise, we get
\[
\phi(t) = k_1 E_4(t) + E_3(t) \geq \frac{k_1}{2} \left( \| P^2 u \|^2 + \| Pu \|^2 \right)
\]
\[
\geq \frac{k_1 - 1}{2} \| P^2 u \|^2 + \left( \frac{k_1}{2} - 1 \right) \| Pu_t \|^2 + \frac{k_1 - c_0}{2} \| Pu \|^2
\]
\[
\geq \| P^2 u_t \|^2 + \| Pu \|^2 + \| Pu_t \|^2,
\]
where the facts \( k_1 \geq \{4, 2 + c_0\} \) and Sobolev imbedding theorem (see [17])
\[
\| P^2 u \|_2^2 \leq c_0 \| Pu \|_2^2 \ \forall u \in H^{2m} (\Omega) \cap H_0^m (\Omega)
\]
have been used. So, by the estimates (3.45) and (3.46) that
\[
\| P^2 u_t (t) \|^2 + \| Pu(t) \|^2 + \| Pu_t(t) \|^2 \leq C_5 e^{-\lambda_3 t} + C_4 \lambda_3^{-1}, \ t \geq 0
\]
with \( C_4 = C_4 (F, H_1, H_2), C_5 = C_5 (\| Pu_0 \|_2, \| Pu \|_2, F, H_1, H_2) \).
To establish the uniqueness, we suppose that \( u(t) \) and \( v(t) \) are two solutions of (1.1), which supply the estimates (2.5)-(2.7) and \( u(0) = v(0), u'(0) = v'(0) \). Taking \( U(t) = u(t), V(t) = v(t) \) and \( W(t) = U(t) - V(t) \), then we see from (1.1) that
\[
W_t - PW - PW_t - P(u - v) = g(x, v) - g(x, u), \quad x \in \Omega, \ t \geq 0.
\]

(3.48)

Multiplying (3.48) by \( W \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|W(t)\|_2^2 + \|P^1_t W(t)\|_2^2 \right) + \|P^2_t W(t)\|_2^2 + \int_\Omega P^2_t (u - v) P^1_t W dx = \int_\Omega (g(x, v) - g(x, u)) W dx
\]
and
\[
\|W(t)\|_2^2 + \|P^1_t W(t)\|_2^2 + 2 \int_0^t \|P^1_s W(s)\|_2^2 ds + 2 \int_0^t \int_\Omega P^2_s (u(s) - v(s)) P^1_s W(s) dx ds = 2 \int_0^t \int_\Omega (g(x, v(s)) - g(x, u(s))) W(s) dx ds.
\]

(3.50)

Since
\[
\left\|P^1_s (u(s) - v(s))\right\| \leq \int_0^t \left\|P^1_\tau (u(\tau) - v(\tau))\right\| d\tau = \int_0^t \left\|P^1_\tau W(\tau)\right\| d\tau
\]
then
\[
\left\|P^1_s (u(s) - v(s))\right\|_2 \leq s^{1/2} \left( \int_0^t \left\|P^1_\tau W(\tau)\right\|_2^2 d\tau \right)^{1/2}
\]
and
\[
\int_0^t \int_\Omega \left|P^2_s (u(s) - v(s)) P^1_s W(s)\right| dx ds \leq \int_0^t \int_\Omega \left|P^2_s W(s)\right| \left\|P^1_\tau W(\tau)\right\|_2 dx d\tau ds \leq \int_0^t \int_\Omega \left\|P^2_s W(s)\right\|_2 \left\|P^1_\tau W(\tau)\right\|_2 d\tau ds \leq t \int_0^t \left\|P^2_s W(s)\right\|_2^2 ds.
\]

(3.51)

Now, taking \( U_\varepsilon (s) = \varepsilon u(s) + (1 - \varepsilon) v(s), 0 \leq \varepsilon \leq 1 \), we get
\[
G = \int_0^t \int_\Omega \left|g(x,u(s)) - g(x,v(s))\right| |W(s)| dx ds = \int_0^t \int_\Omega \left|\frac{d}{d\varepsilon} g(x,U_\varepsilon)\right| |W(s)| dx d\varepsilon ds \leq \int_0^t \int_\Omega \int_0^1 \left|g_u(x,U_\varepsilon)\right| |u(s) - v(s)| |W(s)| dx d\varepsilon ds \leq k_2 \int_0^t \int_\Omega \left( \left\|u\right\|_{\alpha - 1}^\alpha + \left\|v\right\|_{\alpha - 1}^\alpha + h_2(x) \right) |u(s) - v(s)| |W(s)| dx ds \leq c_0 \int_0^t \left( \|u(s)\|_{\sigma_1}^{\sigma_1} + \|v(s)\|_{\sigma_1}^{\sigma_1} + h_2\|\sigma_2\|\sigma_2^{\sigma_2} \right) \left\|P^1_s (u(s) - v(s))\right\|_2 \left\|P^2_s W(s)\right\|_2 dx ds
\]
where \( \sigma_1 = n(\alpha - 1)/2 \leq 2n/(n - 2), \sigma_2 = n/2 \).

From (2.5) and Sobolev imbedding theorem, there is \( C_3 > 0 \) such that
\[
\|u(s)\|_{\sigma_1}^{\sigma_1} + \|v(s)\|_{\sigma_1}^{\sigma_1} + h_2\|\sigma_2\|\sigma_2^{\sigma_2} \leq C_0 \left( \left\|P^1_s u(s)\right\|_2^{\sigma_1} + \left\|P^1_s v(s)\right\|_2^{\sigma_1} + h_2\|\sigma_2\|\sigma_2^{\sigma_2} \right) \leq C_3 \forall s \geq 0.
\]
Then,
\[ G \leq C_3 \int_0^t s^{1/2} \left( \int_0^s \left\| \frac{\partial u}{\partial t} \left( \frac{\partial^2 u}{\partial \tau^2} \right) \right\|_2^2 \, d\tau \right)^{1/2} \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2 \, ds \leq C_3 t \int_0^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2 \, d\tau. \] (3.52)

Then, the estimates (3.50)-(3.52) indicate that
\[ \left\| W(t) \right\|_2^2 + \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2^2 + 2 \int_0^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2 \, ds \leq (C_3 + 1) t \int_0^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2 \, ds. \] (3.53)

The integral inequality (3.53) represents that there exists \( T_1 > 0 \), such that \( W(t) = 0 \) in \([0, T_1]\). As a result, \( u(t) - v(t) = u(0) - v(0) = 0 \) in \([0, T_1]\).

Then, we conclude that \( u(t) = v(t) \) on \([T_1, 2T_1] \), \([2T_1, 3T_1] \), ..., and \( u(t) = v(t) \) on \([0, \infty) \). This shows the proof of uniqueness.

Now, we establish \( u \in C([0, \infty) ; H^m_0(\Omega)) \). Assume \( t > s \geq 0 \). Then,
\[ \left\| \frac{\partial^2 (u(t) - u(s))}{\partial \tau^2} \right\|_2^2 = \int_\Omega \int_s^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2^2 \, d\tau \, dx \leq (t-s) \int_s^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2^2 \, d\tau \rightarrow 0 \text{ as } t \rightarrow s. \] (3.54)

This indicates \( u(t) \in C([0, \infty) ; H^m_0(\Omega)) \). Also, we get
\[ \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2^2 = \int_\Omega \int_s^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2^2 \, d\tau \, dx \leq (t-s) \int_s^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2^2 \, d\tau \rightarrow 0 \text{ as } t \rightarrow s. \] (3.55)

and \( u(t) \in C([0, \infty) ; H^m_0(\Omega) \cap H^m_0(\Omega)) \).

Moreover, we get
\[ \left\| \frac{\partial^2 (u(t) - u(s))}{\partial \tau^2} \right\|_2^2 \leq (t-s) \int_s^t \left\| \frac{\partial^2 u}{\partial \tau^2} \right\|_2^2 \, d\tau \rightarrow 0 \text{ as } t \rightarrow s. \] (3.56)

This shows that \( u(t) \in C^1([0, \infty) ; H^m_0) \) and the proof of Theorem 3 is completed.

4 Global attractor for the problem (1)

By Theorem 3, we see that the solution operator \( S(t)(u_0, u_1) = (u(t), u_t(t)) \), \( t \geq 0 \) of the problem (1.1) creates a semigroup on \( X = (H^2(\Omega)) \times (H^m_0(\Omega)) \), which supplies these properties:
(1) \( S(t) : X \rightarrow X \) for all \( t \geq 0 \);
(2) \( S(t+s) = S(t) S(s) \) for \( t, s \geq 0 \);
(3) \( S(t)(u_0, u_1) \rightarrow S(s)(u_0, u_1) \) in \( X \) as \( t \rightarrow s \) for any \( (u_0, u_1) \in X \).

For establishing the existence of the \((X, X)\)-global attractor for the problem (1.1), firstly, we show the continuity of \( S(t) \) relating to the initial data \((u_0, u_1)\).

The proof of Theorem 4

Suppose \( u_k(t) \), \( u(t) \) is corresponding solution of the problem (1.1) with the initial data \((u_{0k}, u_{1k}) \) and \((u_0, u_1) \) respectively, \( k = 1, 2, \ldots \).

Since \( (u_{0k}, u_{1k}) \rightarrow (u_0, u_1) \) in \( X \), \( \{ (u_{0k}, u_{1k}) \} \) is bounded in \( X \). Set \( w_k(t) = u_k(t) - u(t) \). Then, \( w_k \) holds
\[
\begin{cases}
    w_k'' - Pw_k - Pw_k' - Pw_k'' = g(x, u) - g(x, u_k) = G_k, & (x, t) \in \Omega \times (0, \infty), \\
    w_k(x, 0) = u_{0k}(x) - u_0(x), & x \in \Omega, \\
    w_k(x, t) = 0, & (x, t) \in \partial \Omega \times [0, \infty). 
\end{cases}
\] (4.1)
Multiplying the equation in (4.1) by \( w_k' \), \(-Pw_k \) and \(-Pw_k' \), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \|w_k\|_2^2 + \|P^2 w_k\|_2^2 + \|P^2 w_k'\|_2^2 \right) + (1 - \eta) \|P^2 w_k'\|_2^2 \leq C_\eta \|G_k\|_2^2
\]  
(4.2)

and
\[
\frac{d}{dt} \left( \frac{1}{2} \|Pw_k\|_2^2 + \int \Omega (Pw_k' Pw_k + P^2 w_k P^2 w_k') \, dx \right) + (1 - \eta) \|Pw_k\|_2^2 \leq \|Pw_k'\|_2^2 + C_\eta \|G_k\|_2^2
\]  
(4.3)

and
\[
\frac{1}{2} \frac{d}{dt} \left( \|P^2 w_k\|_2^2 + \|Pw_k\|_2^2 + \|Pw_k'\|_2^2 \right) + (1 - \eta) \|Pw_k'\|_2^2 \leq C_\eta \|G_k\|_2^2
\]  
(4.4)

with small \( \eta > 0 \). Then, by (4.2) and (4.4) we obtain
\[
y_k'(t) + (k (1 - \eta) - c_0) \|Pw_k\|_2^2 + (1 - \eta) \|P^2 w_k'\|_2^2 \leq kC_\eta \|G_k\|_2^2
\]  
(4.5)

where
\[
y_k(t) = \frac{k_1 + 1}{2} \left( \|Pw_k(t)\|_2^2 + \|P^2 w_k(t)\|_2^2 \right)
+ \frac{1}{2} \left( \|P^2 w_k(t)\|_2^2 + \|w_k(t)\|_2^2 \right) + \frac{k_1}{2} \|P^2 w_k(t)\|_2^2
+ \int \Omega (Pw_k'(t) Pw_k(t) + P^2 w_k(t) P^2 w_k'(t)) \, dx
\leq \frac{k_1 + 2}{2} \left( \|Pw_k(t)\|_2^2 + \|P^2 w_k(t)\|_2^2 \right)
+ \frac{k_1 + 1}{2} \|Pw_k'(t)\|_2^2 + \|w_k(t)\|_2^2 + \|P^2 w_k(t)\|_2^2
\leq C_0 \left( \|Pw_k'(t)\|_2^2 + \|Pw_k(t)\|_2^2 + \|P^2 w_k(t)\|_2^2 \right).
\]  
(4.6)

By taking \( k_1 \geq 3 \)
\[
y_k(t) = \frac{k_1 + 1}{2} \left( \|Pw_k\|_2^2 + \|P^2 w_k\|_2^2 \right)
+ \frac{1}{2} \left( \|P^2 w_k\|_2^2 + \|w_k'\|_2^2 \right) + \frac{k_1}{2} \|Pw_k\|_2^2
- \frac{1}{2} \left( \|Pw_k\|_2^2 + \|Pw_k'\|_2^2 \right) - \frac{1}{2} \left( \|P^2 w_k\|_2^2 + \|P^2 w_k'\|_2^2 \right)
\geq \|Pw_k(t)\|_2^2 + \|P^2 w_k'(t)\|_2^2 + \|Pw_k(t)\|_2^2, \quad t \geq 0.
\]  
(4.7)

Otherwise, we obtain from assumption (A2),
\[
\|G_k\|_2^2 = \int \Omega |g(x,u_k) - g(x,u)|^2 \, dx = \int \Omega g_0^2 w_k^2 \, dx
\leq c_0 \int \Omega \left( |u_k|^{2(\alpha - 1)} + |u|^{2(\alpha - 1)} + h_2^2 \right) w_k^2 \, dx.
\]  
(4.8)
The application of Sobolev imbedding theorem and the estimate (2.7) gives

$$\int_{\Omega} |u_k|^{2(\alpha-1)} w_k^2 dx \leq \|w_k\|_{2\mu_2}^2 \|u_k\|_{2(\alpha-1)\mu_3}^2 \leq C_3 \|w_k\|_{2\mu_2}^2 \leq C_4 \|w_k\|_2^2 \tag{4.9}$$

with $\mu_2 = n/ (n-4)^+$ and $\mu_3 = \mu_2 / (\mu_2 - 1)$.

Similarly,

$$\int_{\Omega} |u|^2(\alpha-1) w_k^2 dx \leq \|w_k\|_{2\mu_2}^2 \|u\|_{2(\alpha-1)\mu_3}^2 \leq C_3 \|w_k\|_{2\mu_2}^2 \leq C_4 \|w_k\|_2^2 \tag{4.10}$$

and

$$\int_{\Omega} h_2^2 w_k^2 dx \leq \|w_k\|_{2\mu_2}^2 \|h_2\|_{\mathcal{H}/2}^2 \leq C_3 \|w_k\|_{2\mu_2}^2 \leq C_4 \|w_k\|_2^2 \tag{4.11}$$

Then, we get from (4.5) to (4.11) that $\lambda_4 > 0$, such that

$$y_k(t) + \lambda_4 y_k(t) \leq C_3 \|G_k\|_2^2 \leq C_3 \|w_k\|_{2\mu_2}^2 \leq C_3 \|w_k\|_{2\mu_2}^2 \leq C_3 y_k(t) \tag{4.12}$$

where $C_3$ is as in (2.6), independent of $k$. The differential inequality (4.12) means

$$y_k(t) \leq y_k(0) e^{(C_3-\lambda_4)t}, \quad t \geq 0. \tag{4.13}$$

Then, from (4.6) and (4.7), we obtain

$$y_k(0) \leq C_0 \left( \left\| P_\frac{1}{2} (u_{1k} - u) \right\|_2^2 + \left\| P(u_{0k} - u_0) \right\|_2^2 + \left\| P(u_{1k} - u_1) \right\|_2^2 \right) \to 0 \text{ as } k \to \infty \tag{4.14}$$

and

$$\left\| Pw_k(t) \right\|_2^2 + \left\| P^2 w_k(t) \right\|_2^2 + \left\| Pw_k(t) \right\|_2^2 \leq \lambda_4(t) \leq \lambda_4(0) e^{(C_3-\lambda_4)t} \to 0 \text{ as } k \to \infty. \tag{4.15}$$

This indicates that $S(t) : X \to X$ is continuous. Now we show that $\{S(t)\}_{t \geq 0}$ is asymptotically compact in $X$ from the method in [9].

Assume $\{(u_{0k}, u_{1k})\}$ is a bounded sequence and $\{u_k(t)\}$ be the corresponding solutions of the problem (1.1) in $C([0,\infty); H^{2m}(\Omega) \cap H^m_0(\Omega))$. We suppose $t_k \to \infty$ as $k \to \infty$. For any $T > 0$, assume $t_n, t_k > T$. Then, the application of (4.12) to $w_{kn}(t) = u_n(t + t_n - T) - u_k(t + t_n - T)$, we get

$$Y_{kn}(t) \leq Y_{kn}(0) e^{-\lambda_4 t} + C_3 \int_0^t e^{-\lambda_4(t-s)} \|w_{kn}(s)\|_{2\mu_2} ds, \quad t \geq 0 \tag{4.16}$$

with

$$Y_{kn}(t) = \left\| Pw_{kn}(t) \right\|_2^2 + \left\| P^2 w_{kn}(t) \right\|_2^2 + \left\| Pw_{kn}(t) \right\|_2^2. \tag{4.17}$$

Especially, we take $t = T$ and obtain

$$\left\| P(u_n(t_n) - u_k(t_k)) \right\|_2^2 + \left\| P^2 (u_n(t_n) - u_k(t_k)) \right\|_2^2 + \left\| P(u_n(t_n) - u_k(t_k)) \right\|_2^2 \leq Y_{kn}(0) e^{-\lambda_4 T} + C_3 \sup_{0 \leq s \leq T} \|u_k(t_k - T + s) - u_n(t_n - T + s)\|_{2\mu_2} \tag{4.18}$$

Since the embedding $(H^{2m}(\Omega) \cap H^m_0(\Omega)) \hookrightarrow L^{2\Omega_2}(\Omega)$ is compact, we can remove a subsequence $\{u_{k_i} \left(t_{k_i} - T + s \right)\}$ which converges in $L^{2\mu_2}(\Omega)$. Therefore, for any $\varepsilon > 0$, firstly we fix $T > 0$, such that

$$Y_{kn}(0) e^{-\lambda_4 T} < \frac{\varepsilon}{2}. \tag{4.19}$$
Supposing \( n_0 > 0 \) and \( k_1, j > n_0 \), we get
\[
C_3 \sup_{0 \leq s \leq T} \left\| u_{k_i} \left( t_{k_i} - T + s \right) - u_j \left( t_{k_j} - T + s \right) \right\|_{2\mu_2}^2 < \frac{\varepsilon}{2}.
\] (4.20)

Then, it follows by (4.18) to (4.20) that \( \{ u_{k_i} \} \) is a Cauchy sequence in \( X \) and we finalize that \( \{ S(t) \}_{t \geq 0} \) is asymptotically compact on \( X \) and now Theorem 4 is established.

**Proof of Theorem 5**

From Lemma 2, it is sufficient to indicate that there exists a continuous operator semigroup \( \{ S(t) \} \) on \( X \) such that \( S(t)(u_0, u_1) = (u(t), u_t(t)) \) for each \( t \geq 0 \). By the estimates (2.7), we accomplish that
\[
\beta_0 = \left\{ (u, v) \in X \mid \| P_2 v \|_2^2 + \| Pu \|_2^2 + \| Pv \|_2^2 \leq C_4 \right\}
\] (4.21)
is an absorbing set of \( \{ S(t) \}_{t \geq 0} \) and for any \( (u_0, u_1) \in X \),
\[
\text{dist}_X(S(t)(u_0, u_1), \beta_0) \leq C_5 e^{-\lambda s}, \ t \geq 0
\] (4.22)
where the constants \( C_4, C_5 \) are in (2.7). By Theorem 2, \( S(t) : X \rightarrow X \) is continuous and asymptotically compact on \( X \). From a general theory (see [1, 11]), we conclude that \( S(t) \) admits a global attractor \( A \) on \( X \) defined by
\[
A = \omega(\beta_0) = \bigcap_{\tau \geq 0} \left[ \bigcup_{t \geq \tau} S(t) \beta_0 \right]_X
\] (4.23)
where \( [D]_X \) is the closure of the set \( D \) in \( X \). Then we prove the Theorem 5.

**5 Decay property of solution for (1)**

In this section, we search the decay property of solution to (1.1) with \( f \equiv 0 \). Firstly, we present a well-known Lemma that will be needed.

**Lemma 7.** ([18]) Assume \( E : [0, \infty) \rightarrow [0, \infty) \) is a non-increasing function and suppose that there are constants \( q \geq 0 \) and \( \gamma > 0 \) such that
\[
\int_S \theta^{q+1}(t) \, dt \leq \gamma^{-1} E(0)^q E(s), \ \forall S \geq 0.
\] (5.1)

Then, we get
\[
E(t) \leq E(0) \left( \frac{1+q}{1+q\gamma} \right)^{1/q} \ \forall t \geq 0 \text{ if } q > 0
\] (5.2)
and
\[
E(t) \leq E(0) e^{1-q} \ \forall t \geq 0 \text{ if } q = 0.
\] (5.3)

**Proof of Theorem 7**

Suppose \( u(t) \) is a weak solution in Theorem 3 with \( f = 0 \). Show
\[
E(t) = \frac{1}{2} \left( \| u(t) \|_2^2 + \| P^2 u(t) \|_2^2 + \| P^2 u_t(t) \|_2^2 \right) + \int_\Omega G(u(t)) \, dx, \ t \geq 0.
\] (5.4)

Then, we obtain by (1.1) that
\[
E'(t) + \| P^2 u_t(t) \|_2^2 = 0, \ \forall t \geq 0.
\] (5.5)
This indicates that $E(t)$ is non-increasing in $[0, \infty)$.

Multiplying the equation in (1.1) by $E^q(t)u(t)$, $q > 0$, we obtain

$$
\int_S^T E^q(t) \int_\Omega u(u_t - Pu - Pu_{tt} + g(u)) \, dx \, dt = 0, \ \forall T > S \geq 0.
$$

We note that

$$
\int_S^T E^q(t)(u, u_t) \, dt = E^q(t)(u, u_t) \frac{\|\cdot\|_S^2}{S^2} - \int_S^T \left( qE(t)^{q-1} E'(t)(u, u_t) + E^q(t) \|u_t(t)\|^2 \right) \, dt;
$$

$$
- \int_S^T E^q(t)(u, Pu) \, dt = \int_S^T E^q(t) \|P^2 u\|_2^2
$$

and

$$
- \int_S^T E^q(t)(u, Pu_t) \, dt = - \int_S^T \left( qE(t)^{q-1} E'(t) \left( P^2 u, P^2 u_t \right) + E^q(t) \|P^2 u_t(t)\|^2 \right) \, dt
$$

$$
+ E^q(t) \left( P^2 u, P^2 u_t \right) \bigg|_S^T.
$$

Then, we get by (5.6) that

$$
2 \int_S^T E^{q+1}(t) \, dt = -E^q(t) \left[ (u, u_t) + \left( P^2 u, P^2 u_t \right) \right] \bigg|_S^T
$$

$$
+ q \int_S^T E(t)^{q-1} E'(t) \left[ (u, u_t) + \left( P^2 u, P^2 u_t \right) \right] \, dt
$$

$$
+ 2 \int_S^T E^q(t) \left( \|u_t(t)\|^2 + \|P^2 u_t(t)\|^2 \right) \, dt
$$

$$
+ \int_S^T E^q(t) \left( P^2 u, P^2 u_t \right) \, dt + \int_S^T E^q(t) (2G(u) - ug(u)) \, dt.
$$

Since $G(u) \geq 0$, $E(t) \geq 0$. Moreover, we get the following estimates from (5.5):

$$
\|P^2 u_t(t)\|_2 \leq (-E'(t))^{1/2}, \quad \|P^2 u\|_2^2 \leq 2 (E(t))^{1/2}, \quad \|P^2 u_t(t)\|_2 \leq 2 (E(t))^{1/2}, \ \forall t \geq 0,
$$

$$
E^q(t) \left( (u, u_t) + \left( P^2 u, P^2 u_t \right) \right) \leq C_0 E^q(t) \left[ P^2 u \right]_2 \left[ P^2 u_t \right]_2 \leq C_0 E^{q+1}(t),
$$

$$
\int_S^T \left( E(t)^{q-1} E'(t) \left[ (u, u_t) + \left( P^2 u, P^2 u_t \right) \right] \right) \, dt
$$

$$
\leq C_0 \int_S^T E(t)^{q-1} E'(t) \left[ P^2 u \right]_2 \left[ P^2 u_t \right]_2 \, dt \leq C_0 E^{q+1}(S),
$$

$$
2 \int_S^T E^q(t) \left( \|u_t(t)\|^2 + \|P^2 u_t(t)\|^2 \right) \, dt \leq C_0 \int_S^T E^q(t) (-E'(t))^{1/2} \leq C_0 E^{q+1}(S),
$$

$$
\int_S^T E^q(t) \left( P^2 u, P^2 u_t \right) \, dt \leq \int_S^T E^q(t) \left[ P^2 u \right]_2 \left[ P^2 u_t \right]_2
$$

$$
\leq \int_S^T E^{q+1}(t) \, dt + C_1 E^{q+1}(S).
$$
Then we obtain from (5.8) to (5.12) that
\[ \int_{S}^{T} E^{q+1}(t) \, dt \leq C_0 E^{q+1}(S) \leq C_0 E^{q}(0) E(S) \equiv \gamma^{-1} E^{q}(0) E(S). \]  
(5.13)

From Lemma 10, we get
\[ E(t) = \frac{1}{2} \left( \|u(t)\|_2^2 + \left\| \int_{t}^{2} \left( \int_{t}^{2} u(t) \left( \int_{t}^{2} u(t) \right)^2 \right) \right\|_2 + \int_{\Omega} G(u(t)) \, dx \right) \]
\[ \leq E(0) \left( \frac{1 + q}{1 + q\gamma} \right)^{1/q} \leq C_1 (1 + t)^{-1/q}. \]

This is the estimates (2.8) and the proof of Theorem 7 is completed.

**Conclusion 8.** In this paper, we obtained the global attractor and the asymptotic behavior of global solution for the higher-order evolution equation with damping term. This improves and extends many results in the literature such as (Xie and Zhong (2007); Chen et al. (2011)).

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