Quantum Monte Carlo Study of Weakly Coupled Spin Ladders

Y. J. Kim

Division of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts 02138
and Center for Materials Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

R. J. Birgeneau, M. A. Kastner, and Y. S. Lee

Department of Physics and Center for Materials Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Y. Endoh

Department of Physics, Tohoku University, Sendai 980-8578, Japan

G. Shirane

Department of Physics, Brookhaven National Laboratory, Upton, New York 11973

K. Yamada

Department of Physics, Kyoto University, Gokasho, Uji 610-0011, Japan

(March 24, 2022)

We report a quantum Monte Carlo study of the thermodynamic properties of arrays of spin ladders with various widths (n), coupled via a weak inter-ladder exchange coupling αJ, where J is the intraladder coupling both along and between the chains. This coupled ladder system serves as a simplified model for the magnetism of presumed ordered spin and charged stripes in the two-dimensional CuO planes of hole-doped copper oxides. Our results for n = 3 with weak inter-ladder coupling α = 0.05, estimated from the t − t′ − t″ − J model, show good agreement with the ordering temperature of the recently observed spin density wave condensation in La2CuO4+y. We show that there exists a quantum critical point at αc ≃ 0.07 for n = 4, and determine the phase diagram. Our data at this quantum critical point agree quantitatively with the universal scaling predicted by the quantum nonlinear σ model. We also report results on random mixtures of n = 2 and n = 3 ladders, which correspond to the doping region near but above 1/8. Our study on the magnetic static structure factor reveals a saturation of the incommensurability of the spin correlations around 1/8, while the incommensurability of the charge stripes grows linearly with hole concentration. The implications of this result for the interpretation of neutron scattering experiments on the dynamic spin fluctuations in La2−xSrxCuO4 are discussed.

I. INTRODUCTION

One of the most surprising early observations in the field of high temperature superconductivity is that the onset of superconductivity as a function of hole concentration coincides with a commensurate-incommensurate transition in the low-energy dynamical spin fluctuations. These incommensurate spin fluctuations have been studied extensively over the past decade. We note especially the recent detailed quantitative study by Yamada et al. A variety of theoretical explanations of the spin incommensurability have been offered, varying from nesting of the Fermi surface in a nearly free electron model to microscopic phase separation of the holes in a doped-Mott insulator model for the copper oxides. Specifically, in the latter model under certain circumstances the holes could organize themselves into stripes with the charged stripes acting as antiphase domain walls for the intervening antiferromagnetic regions. The latter correspond to spin ladders which have been recently studied quite extensively in a different context. None of the experiments on the spin dynamics has been able to select unambiguously between these various theoretical models. However, in the past two years, the experimental situation has changed significantly. First, rather dramatic elastic magnetic scattering effects have been observed at low temperatures in samples of La1.6−xNd0.4SrxCuO4 with x=0.12, 0.15, and 0.20. These materials are all superconductors with onset Tc’s of ≃ 4K, 11K and 15K, respectively. In each case, Tranquada and coworkers observe elastic incommensurate magnetic peaks with onset temperatures of ≃ 50K, 46K and 15K, respectively. Recent work has shown that in the x=0.12 sample the correlation length reaches its maximum value below ~ 30K. The incommensurabilities are essentially identical to those of the corresponding dynamical fluctuations in samples of La2−xSrxCuO4 with the same x. In their study of the spin dynamics in La2−xSrxCuO4, Yamada et al. observe that the momentum space width of the low-energy spin fluctuations is a minimum for hole concentrations near 1/8. Further, at these hole concentrations the spin fluctuations extend down to very low energies even in the superconducting state. This is in contrast to the situation in samples with hole concentrations at
and above the optimal doping level, \( x \approx 0.15 \), where a well-defined spin gap is observed in the superconducting state. This has led Suzuki et al.\(^{11}\) and more recently, Kimura et al.\(^{23}\) to search for elastic magnetic scattering effects in \( \text{La}_{1-x}\text{Sr}_{x}\text{CuO}_4 \). Remarkably, they observe a spin density wave transition at \( T_m \approx 31K \) which coincides to within the errors with the onset temperature for superconductivity in this sample; the incommensurability is \( \epsilon = 0.12 \) which equals the Sr-doping, \( x \). Kimura and coworkers also observe magnetic order in a sample of \( \text{La}_{1.90}\text{Sr}_{0.10}\text{CuO}_4 \) \( (T_c = 31K) \) with incommensurability \( \epsilon = 0.105 \) below \( T_m \approx 15K \).

Most recently, Lee and coworkers\(^{24}\) have searched for spin density wave order in a sample of \( \text{La}_2\text{CuO}_4 \) doped with oxygen, that is, \( \text{La}_2\text{CuO}_{4+y} \), which has a superconducting \( T_c \) (onset) of 42K and is predominantly stage-4. They indeed find a transition to long-range incommensurate magnetic order at \( T_m \approx 42K \) with typical Bardeen–Cooper–Schrieffer (BCS) mean field behavior of the order parameter below \( T_m \). Importantly, they find that both the spin ordering direction and the three-dimensional stacking arrangement coincide with those in pure \( \text{La}_2\text{CuO}_4 \). The magnetic structure can be modeled quite well with \( n = 3 \) stripes of the \( \text{La}_2\text{CuO}_4 \) in-plane structure separated by \( n = 1 \) non-magnetic antiphase domain walls; this gives an incommensurability \( \epsilon = 0.125 \), close to the measured value. The measured long-range ordered moment is \( \sim 0.15 \mu_B \) and the inter-layer magnetic correlation length is \( \sim 3 \text{CuO}_2 \) layers.

In our view, these observations, including especially this recent work on \( \text{La}_2\text{CuO}_{4+y} \), give credence to stripe models for the microscopic magnetic structure. The recent observation\(^{11}\) of similar incommensurabilities in the spin dynamics of underdoped \( \text{YBa}_2\text{Cu}_3\text{O}_{6.6} \) strengthens this argument. Of course, a great deal of work remains to be done to explain all of the experimental observations, especially the electronic properties in the normal and superconducting states. Kivelson et. al.\(^{21}\) consider the electronic degrees of freedom within the charge stripes and characterize these phases as electronic liquid crystals. Although the charge and spin degrees of freedom of the charged stripes themselves are clearly essential to the physics, it is nevertheless important as a first step to develop a deeper understanding of the simpler problem of the magnetism of idealized insulating stripe arrays as a function of stripe width or, equivalently, hole concentration. Accordingly, we adopt a naive model, in which for hole concentrations between \( \sim 0.05 \) and \( 0.15 \) there is approximately one hole per two copper\(^{22,23}\) along site-centered charge stripes running along the tetragonal Cu–O–Cu–O... axes with the holes confined to single chains. We assume further that the charge stripes are effectively non-magnetic and that the magnetic coupling across the stripes is, to first order, determined by the third-nearest-neighbor coupling within the \( \text{CuO}_2 \) plane. This is believed to be antiferromagnetic and to about 5% of the nearest neighbor coupling.\(^{23}\) We omit entirely any effects of the low energy charge and spin excitations of the charged stripes themselves. In this greatly simplified model, one may use standard quantum Monte Carlo (QMC) techniques to determine the magnetic properties of the stripe arrays.

Our understanding of two-dimensional quantum magnetism has grown enormously since the discovery of high temperature superconductivity. Theoretical efforts to understand the underlying antiferromagnetism in the parent compounds as well as experimental efforts to synthesize and study new cuprate materials have both contributed significantly. In particular, the two-dimensional quantum Heisenberg antiferromagnet (2DQHA) has been studied extensively, since the pioneering work by Chakravarty, Halperin, and Nelson\(^{26}\). They have mapped the long-wavelength, low-temperature behavior of the 2DQHA to a quantum nonlinear σ model (QNLSM) in \((2+1)\) dimensions and have obtained a phase diagram with three regimes: quantum disordered (QD), quantum critical (QC), and renormalized classical (RC). They have argued that the 2DQHA on the square lattice for \( S \geq 1/2 \) with nearest neighbor interactions is in the RC regime, and consequently has a correlation length diverging exponentially in \( 1/T \) as temperature is lowered to \( T = 0 \), implying the existence of a long-range ordered ground state. The temperature dependence of the correlation length in the RC regime, which has been solved exactly to three loop order by Hasenfratz and Niedermayer (HN)\(^{27}\) agrees quantitatively with the results of neutron scattering experiments on both \( \text{Sr}_2\text{CuO}_2\text{Cl}_2 \)\(^{28}\) and \( \text{La}_2\text{CuO}_4 \)\(^{29}\).

One-dimensional (1D) quantum magnetism has also drawn much interest in the last decade, mainly due to a conjecture made by Haldane regarding the different ground state properties between spin chains with integer and half-integer spin quantum number.\(^{30}\) In his seminal work in 1983\(^{26}\), Haldane mapped the 1D Heisenberg model onto the QNLσM and showed that the half-integer spin chain has an additional topological term, which generates gapless low-energy excitations together with algebraically decaying spin correlation functions. Integer spin chains, however, can be described by the standard QNLSM in \((1+1)\) dimensions, in which the ground state is disordered due to quantum fluctuations and the excitation spectrum acquires a gap – the so-called Haldane gap. This conjecture has been subsequently confirmed both numerically\(^{31,32}\) and experimentally\(^{33}\). Spin ladder have been studied mostly in the same context when even numbers of \( S = 1/2 \) chains are coupled to form a ladder (even-width), they show the same universal behavior as integer spin chains, while odd-width \( S = 1/2 \) ladders behave essentially like a single \( S = 1/2 \) chain at low temperatures and long wavelengths.

In this paper we report a detailed QMC study of ladder arrays as a function of the width of the spin ladders \((n)\), the strength of the coupling across the non-magnetic line \((J' = \alpha J)\), and temperature \((T)\). We have carried out such calculations for \( n = 4 \) ladder arrays, \( n = 3 \) ladder arrays, and for random mixtures of \( n = 3 \) and \( n = 2 \) lad-
ders. These various arrays are illustrated in Fig. 1. In the
limit of $\alpha \to 0$, the model corresponds to a set of isolated
ladders, and in the limit $\alpha \to 1$ we recover the isotropic
$s = 1/2$ square lattice QHA. One can easily see that a
quantum critical point as a function of $\alpha$ should exist for
non-zero $\alpha$ for coupled even-width ladders. Indeed, the
recent study by Tworzydlo et al. shows that $\alpha_c \approx 0.3$
for coupled $n = 2$ ladders. We show that arrays of $n = 4$
ladders exhibit a quantum critical point around $\alpha \approx 0.07$.
We discuss our QMC results in the context of available
experimental information; we also suggest future measure-
ments which should determine whether or not these
calculations are in fact relevant to the real monolayer
CuO$_2$ superconductors.

The format of this paper is as follows: In Sec. II we
discuss our QMC techniques. Section III contains our
results for weakly coupled three-leg ladders ($n = 3$).
Results for weakly coupled four-leg ladders ($n = 4$) includ-
ing especially the quantum critical behavior are given in
Sec. IV. In Sec. V, we discuss the results for random
mixtures of two-leg and three-leg ladders. We have also
studied strongly coupled $n = 3$ ladders and present the
results in Sec. VI. Finally, a discussion, summary, and
conclusions are given in Sec. VII.

![Schematic diagram of the model.](image)

**FIG. 1.** Schematic diagram of the model. (a) Weakly
coupled three leg ladder array with a periodicity of $8a$. $J'$
is the inter-ladder coupling $\alpha J$. (b) Weakly coupled four
leg ladder array. (c) Random mixture of two and three leg
ladders.

### II. QUANTUM MONTE CARLO

We have carried out quantum Monte Carlo simulations
on large lattices utilizing the loop cluster algorithm. The
same algorithm previously employed to study spin
ladders and chains is used with minor modifications.

We obtain the temperature dependence of the uniform
susceptibility, $\chi_u(T)$, the spin-spin correlation length
in the $x$-direction, $\xi_x(T)$, and the $y$-direction, $\xi_y(T)$,
the staggered susceptibility, $\chi_s(T)$, and the static structure
factor at the antiferromagnetic wave vector $Q \equiv (\pi, \pi)$,
$S_Q(T)$. The Hamiltonian for our model is based on the
square lattice with a coupling $J$, but every $n$-th bond in
the $x$-direction is replaced by a bond with exchange
coupling $\alpha J$:

$$
\mathcal{H} = J \sum_{i,j} S_{i,j} \cdot S_{i,j+1} + \sum_{i \neq l, j} S_{i,j} \cdot S_{l+1,j} + \alpha \sum_{i = l, j} S_{i,j} \cdot S_{i+1,j},
$$

where $i$ and $j$ run over the $x$ and $y$ directions, respect-
atively. $n$ is the width of the ladder and $l$ counts the
ladders. We use units in which $\hbar = k_{\text{B}} = g \mu_B = 1$. We
also set the lattice constant $a = 1$. Periodic boundary
conditions are used in both directions.

One should note that, although this Hamiltonian cap-
tures the essential magnetic interactions of coupled lad-
ers, if Eq. (1) is to describe the stripe arrays shown in
Fig. 1, the lattice constant in the $x$-direction is $\frac{2\pi}{n+1}$
instead of one. This affects only two quantities in our com-
putation: $\xi_x(T)$ and $S_Q(T)$. The correction for $\xi_x(T)$ is
trivial, since $\xi_x(T)$ is measured in units of the lattice
constant. Unless noted otherwise, this $\frac{2\pi}{n+1}$ correction is
incorporated in the $\xi_x$ data presented in this paper. $S_Q(T)$,
however, cannot be converted in a simple manner for the
stripe model. Therefore, in the following sections, $S_Q(T)$
should be understood as the structure factor at $(\pi, \pi)$
for the model Hamiltonian, Eq. (1), rather than for the
stripe model as shown in Fig. 1. The incommensurate
structure factor is discussed only in Sec. V, where it is
separately computed as a function of $q_x$, $S(q_x, \pi)$.

In order to deduce the spin-spin correlation lengths,$\xi_x(T)$
and $\xi_y(T)$, the instantaneous spin-spin correlation function,
$C(x, y)$, is computed and fitted to the asymptotic form

$$
C(x, 0) \sim x^{-\lambda} e^{-x/\xi_x} + (L_x - x)^{-\lambda} e^{-(L_x-x)/\xi_x},
$$

$$
C(0, y) \sim y^{-\lambda} e^{-y/\xi_y} + (L_y - y)^{-\lambda} e^{-(L_y-y)/\xi_y},
$$

which is a symmetrized 2D Ornstein-Zernike form ($\lambda = 0.5$).
Only data with $x \gtrsim 3\xi_x$ and $y \gtrsim 3\xi_y$ are included in
the fits to ensure that the asymptotic behavior is probed.
Technically, due to the non-uniformity of the exchange
couplings in the $x$-direction, $C(x, 0)$ can be better de-
scribed by the function $\sim x^{-1/2} e^{-x/\xi_x} \sin(x/n)$ rather
than Eq. (2a); however, the values for $\xi_x$ obtained with
both methods agree within the error bars.

The lengths and Trotter numbers of the simulated lat-
tices are chosen so as to minimize any finite-size and
lattice-spacing effects. The linear sizes of the lattice, \(L_x\) and \(L_y\), are kept at least 10 times larger than the respective correlation lengths. Spin states are updated about \(2 \times 10^4\) times to reach equilibrium and then measured \(\sim 5 \times 10^4\) times. For the random mixtures of \(n = 2\) and \(n = 3\) ladders, we generate each configuration in the following way: Beginning from the first column \((i = 1)\), we assign a spin or hole for each column; the number of spin columns between the hole columns represents the width of the spin stripe. In order to make sure that the width is either two or three, we put restrictions such that a spin column must be assigned after each hole column, while a hole column must be assigned after three consecutive spin columns. The width of each ladder is determined by generating a random number \(r\) \((0 < r < 1)\) and comparing \(r\) with \(p\), where \(p\) is the ratio of the three leg ladder in the mixture. After two spin columns are assigned, the spin column is assigned only if \(r < p\), otherwise the hole column is assigned and the width becomes two. For example, for an equal mixture of two and three leg ladders, \(p = 0.5\). For each \(T\) and \(p\), typically 10 to 20 different configurations are generated and averaged over. Averaging over more configurations does not alter our results.

### III. WEAKLY COUPLED THREE LEG LADDERS

We present our QMC data for the \(n = 3\) case, Fig. 1(a), in this section. From detailed studies of isolated ladders, it is known that at low temperatures and long length scales three-leg ladders exhibit the same behavior as a single chain \((S = 1/2)\). It is well known that any non-zero inter-ladder coupling eventually enhances the correlations across the non-magnetic stripe and drives the system towards the 2DQHA-QNL\(\sigma\)M RC fixed point. Thus, we expect to see qualitatively different behavior for an array of weakly coupled \(n = 3\) ladders from that of isolated ladders. The correlation lengths for arrays of \(n = 3\) ladders are shown in Fig. 2 as a function of inverse temperature for various inter-ladder couplings \((\alpha)\). Note that \(\alpha = 0\) corresponds to an isolated three leg ladder; these data are taken from Ref. 11. Not surprisingly, one can see clearly \(\sim \exp(1/T)\) behaviors except for \(\alpha = 0\), as discussed above. In the inset of Fig. 2, we also show the ratio \(\xi_y/\xi_x\). At low temperatures, \(\xi_y/\xi_x\) approaches the mean field prediction, \(\sqrt{\alpha}\), shown here as a solid line for each \(\alpha\).

Our correlation length data are fitted to the crossover form given by Castro Neto and Hone, which interpolates between the HN result at low temperatures and \(\xi \sim T^{-1}\) at high temperatures:

\[
\xi = A \exp\left(\frac{2\pi \rho_s}{T}\right) \frac{1}{1 + \frac{1}{2} \frac{T}{2 \pi \rho_s}},
\]

where \(\rho_s\) is the spin stiffness. Our fitting results are presented in Table I. We note first that the values deduced for \(2\pi \rho_s\) separately from \(\xi_y\) and \(\xi_x\) are the same within the error bars, which means that there is only one temperature scale for the low-temperature behavior of this model, despite the fact that the correlation lengths themselves are highly anisotropic.

| \(\alpha\) | \(2\pi \rho_s/J\) | \(\xi_y\) | \(\xi_x\) | \(\xi_y\) |
|---|---|---|---|---|
| 0.05 | 0.10(1) | 0.11(1) | 0.26(2) | 1.14(5) |
| 0.1 | 0.17(2) | 0.19(1) | 0.52(5) | 1.4(1) |
| 0.3 | 0.55(2) | 0.58(1) | 0.83(5) | 1.22(5) |
| 1.0* | 1.1310(3) | 0.4978(8) |

*Ref. 36
we have assumed perfectly ordered charge stripes. We note that \( \alpha = 0.1 \) gives \( T_m \approx 75K \). We should also note that these estimates of \( T_m \)'s are not very sensitive to the explicit value of \( \gamma_{eff} \). Even if \( \gamma_{eff} \) changes by a factor of two, \( T_m \) remains within 10% of the value obtained here. However, \( T_m \) depends sensitively on the choice of the inter-ladder exchange coupling \( \alpha J \). The low values for \( T_m \) found experimentally clearly constrain the inter-ladder coupling to rather small values (or, as discussed in Sec. VI, very large values), at least within the simplified model considered here.

In Fig. 3 Monte Carlo results for \( \chi_u \), \( \chi_s \), and the inverse of \( S_Q \) are plotted as a function of \( T \), for different \( \alpha \)'s. The reason \( S_Q^{-1} \) is plotted is to emphasize the effect of the inter-ladder coupling. One of the most significant results of Ref. 31 is that the divergence of \( S_Q \) is only logarithmic in 1D; thus \( S_Q^{-1} \) does not extrapolate to zero at zero temperature. As shown in Fig. 3(c), coupled \( n = 3 \) ladders behave completely differently from the isolated \( n = 3 \) ladder; namely, the \( S_Q \) of coupled ladders exhibits a clear crossover from the weak divergence in \( T^{-1} \) of an isolated ladder to the strong divergence in \( T^{-1} \) of a 2D spin system.

Since the 2D Heisenberg model cannot have long-range order at any non-zero temperature, either a small anisotropy in the spin Hamiltonian or a non-zero 3D coupling is necessary to explain the Néel transition in real materials. In pure \( \text{La}_2\text{CuO}_4 \), one can estimate the Néel transition temperature reliably as the temperature at which the correlation length squared becomes of order of the inverse of the effective anisotropy \( \gamma' \) which, in the case of \( \text{La}_2\text{CuO}_4 \), is about \( \gamma' \approx 10^{-4} \). We can apply the same heuristic formula to determine the spin density wave ordering temperature, \( T_m \), for our presumed stripe array model for doped \( \text{La}_2\text{CuO}_4 \). For example, \( \text{La}_2\text{CuO}_{4+y} \) has the same spin ordering direction and stacking scheme as in pure \( \text{La}_2\text{CuO}_4 \) except that the inter-layer order extends over only about three planes; thus, we take the same effective anisotropy and find the temperature satisfying \( \xi_x(T_m)\xi_y(T_m) \approx \gamma'_{eff} \). In our simplified model the inter-ladder coupling is determined by the third-nearest-neighbor exchange coupling in the \( \text{CuO}_2 \) plane; from current values for \( t', t'', J \) in the extended \( t-J \) model we estimate \( \alpha \approx (t''/t) \approx 0.05 \). By extrapolating our correlation length data to lower temperature for \( \alpha = 0.05 \), we obtain \( T_m \approx 0.029J \), or \( T_m \approx 44K \) for \( J = 1500K \), which agrees remarkably well with the experimental result. This precise agreement is a coincidence since \( \alpha = 0.05 \) is only a crude estimate and...
an ordered ground state. In QNLσM language, there is a quantum critical point dividing the QD ground state and the RC ground state at $\alpha_c$. We show the phase diagram of arrays of $n = 4$ ladders in Fig. 4. In this figure, we take $2\pi \rho_s$ as the temperature scale where crossover from the QC to the RC regime occurs, and $\Delta$ as the crossover temperature from the QC to the QD regime. $2\pi \rho_s$ is determined in the manner explained in Sec. III; $\Delta$ can be obtained by fitting to $\chi_u(T) \sim \exp(-\Delta/T)$ at low temperatures ($T \ll \Delta$). One should note that the thick shaded lines in Fig. 4 represent crossover lines rather than true phase boundaries. The phase diagram shown here is basically that of the QNLσM, with $\alpha$ playing the role of the coupling constant $g$. As shown in Fig. 4, a quantum critical point occurs at $\alpha_c = 0.07(1)$.

**Fig. 4.** Uniform susceptibility, staggered susceptibility, and inverse static structure factor peak intensity for arrays of $n = 4$ ladders with different inter-ladder couplings $\alpha$.

Monte Carlo results for $\chi_u$, $\chi_s$, and $S_Q^{-1}$ are plotted as a function of $T$, for different $\alpha$'s, in Fig. 5. The correlation length obtained from the simulation is plotted in Fig. 6 as a function of $T^{-1}$. Figures 5 and 6 are plotted in such a way as to contrast their behaviors with those of arrays of $n = 3$ ladders. The difference between the behaviors in the QD and the RC regime is evident in these figures; namely, $\xi$, $\chi_s$, and $S_Q$ all diverge exponentially at low temperatures in the RC regime (e.g., $\alpha = 0.3$), while they all saturate at finite values as $T \to 0$ in the QD regime (e.g., $\alpha = 0.05$).

To illustrate the quantum critical behavior more dramatically, we plot in Fig. 7(a), the dimensionless ratio $S_Q/T\chi_s$. At very high temperatures, this ratio shows classical behavior, $S_Q/T\chi_s = 1$, while the same behavior shows up again at very low temperatures for $\alpha > \alpha_c$, that is, in the RC regime. This is as expected, since RC behavior is closely similar to classical behavior, if the spin wave velocity and spin stiffness renormalizations are accounted for. In the QD regime, that is, $\alpha < \alpha_c$, $S_Q$ and $\chi_s$ are constant; therefore $S_Q/T\chi_s$ should be linear in $T^{-1}$. According to the quantum critical scaling predicted for the QNLσM, this ratio should show universal behavior in the QC regime with the specific value: $S_Q/T\chi_s = 1.10(2)$. As may be seen in Fig. 7(b), the $\alpha = 0.07$ data for $S_Q/T\chi_s$ indeed are constant $\approx 1.12$ at low temperatures, in quantitative agreement with the QC theory. This strongly supports the above claim that $\alpha_c = 0.07$ for the $n = 4$ stripe array system is a quantum critical point. Another quantity plotted in Fig. 7(b) is $\xi_yT$. Although $\xi_yT$ is not shown here, it shows essentially the same temperature dependence. Again at very high temperatures, this quantity shows classical behavior, $\xi_yT = J$, while the low-temperature behavior can...
distinguish the QC regime from the RC and QD regimes. In the QC regime, the QNL $\sigma_3$ predicts $\xi T = c/1.04$, where $c$ is the spin wave velocity. As may be seen in Fig. 7(b), $\xi y T$ is indeed a constant at low temperatures (high $1/T$) for $\alpha = 0.07$; the explicit value $\xi y T \approx 1.4$, corresponds to $c_y \approx 1.46 J$, close to, but somewhat less than, the value $c = 1.657 J$ for the square lattice.

V. RANDOM MIXTURE OF WEAKLY COUPLED TWO AND THREE LEG LADDERS

In Fig. 8(a), we show representative static structure factor data at low temperatures for random mixtures of $n = 2$ and $n = 3$ weakly coupled ladders, obtained as described in Sec. II. Here, $p$ is the fraction of $n = 3$ ladders in the mixture; therefore, $p = 1.0$ corresponds to the pure $n = 3$ case. We choose $\alpha = 0.05$ for all of the simulations discussed in this section. We show data at low temperatures ($T \sim 0.05 J$), since the magnetic structure factor develops well defined peaks only at these low temperatures. At high temperatures, due to the short correlation lengths, the structure factor exhibits only broad peaks, making it difficult to extract meaningful values for the incommensurability. Nevertheless, we can fit the data with two identical Lorentzians, split symmetrically about the antiferromagnetic wave vector $Q \equiv (\pi, \pi)$, together with a broad temperature-independent background term centered around $(\pi, \pi)$:

$$S(q_x, \pi) = \frac{S_Q}{4} \left( \frac{1}{1 + q_x^2/\kappa_x^2} + \frac{1}{1 + q_x^2/\kappa_y^2} \right) + \frac{B}{1 + (q_x - \pi)^2/\kappa_y^2}. \quad (4)$$

Here $q_\pm = q_x - (\pi \pm \epsilon)$ and the last term is a temperature-independent background. The width of this background term corresponds to $\sim 1$ lattice constant. There are only two temperature dependent fitting parameters, $\epsilon$ and $\kappa_x$, since the peak intensity $S_Q/4$ is calculated separately in our QMC study. The peak position corresponds to the incommensurability $\epsilon$, and the peak width corresponds to the inverse correlation length $\kappa_x$. These data are presented in Fig. 9. In Fig. 8(b), we also show $\kappa_y = 1/\xi_y$; $\xi_y$ is computed directly in the way explained in Sec. II.

FIG. 7. (a) The ratio $S_Q/T \chi_s$ as a function of inverse temperature as described in the text. (b) The correlation length in the $y$-direction of arrays of $n = 4$ ladders multiplied by temperature is plotted to contrast the different low-temperature behaviors in the QD and RC phases. The solid lines in both (a) and (b) are the predictions for quantum critical scaling.

FIG. 8. (a) Static Structure factor near the antiferromagnetic wave vector for the $n = 2$ and $n = 3$ mixed spin ladders. Representative data at $T = 0.05$ for $p = 0.5$ and $p = 0.25$, and at $T = 0.08$ for $p = 1.0$ are shown; $\alpha = 0.05$ for all data. The solid lines are the results of fits to two Lorentzians [Eq. (4)]. (b) Corresponding charge structure factor near the nuclear zone center, calculated as described in the text. We show $p = 0.95$ data instead of $p = 1.0$ for graphical purpose.
The most prominent feature in Fig. 8(a) is that the incommensurability $\epsilon$ of the magnetic structure factor does not change significantly as $p$ is varied. This is verified quantitatively in Fig. 9(a), where it may be seen that the values for $\epsilon$ for both $p = 0.5$ and $p = 0.25$ cluster around $1/8$, the exact value for $p = 1.0$. Note that the high-temperature data have large error bars, since the peaks are very broad. This behavior of the magnetic structure factor may be contrasted with that of the charge structure factor. We show in Fig. 8(b) the charge structure factor for the charge stripes in the random mixtures. The calculation is made for the simplest charge distribution: we assume that the scattering is unity from the charge stripes (anti-phase domain walls) and zero from the spin ladders. Although greatly oversimplified, as an illustration this calculation nevertheless provides useful intuitive guidance. The most important qualitative result is that the incommensurability of the charge stripes is not equal to twice the magnetic incommensurability in the random mixtures. For example, the $p = 0.25$ case has a charge incommensurability of $0.31$, which is quite different from $2\epsilon = 0.25(2)$, where $\epsilon$ is the magnetic incommensurability.

**FIG. 9.** (a) The incommensurability obtained from the fitting static structure factor to two Lorentzians together with a fixed background term [Eq. (4)]. Even if there is a substantial fraction of $n = 2$ ladders, the incommensurability remains fixed near 1/8. (b) Inverse correlation length as a function of $T$. $\kappa_x$ is obtained from the fit. $\kappa_y$ is determined from the correlation function Eq. (2b) as described in Sec. II.

This saturation of $\epsilon$ of the magnetic structure factor around 1/8 can be understood in the following way: At low temperatures ($T \ll \Delta, \Delta \approx 0.41J$) the spins on the two-leg ladder will form a spin singlet ground state with a large spin gap in the excitation spectrum. These spin singlets have effective spin zero, and therefore do not contribute in first order to the magnetic structure. Therefore, the magnetic structure factor originates predominantly from the $n = 3$ components of the mixture. In their systematic study of the *dynamic* incommensurate spin fluctuations in La$_{2-x}$Sr$_x$CuO$_4$, Yamada and coworkers showed that the incommensurability is linear in hole concentration $x$ with $\epsilon \approx x$ for $0.06 \lesssim x \lesssim 0.12$ and saturates around 1/8 on further doping. In our simplified model of stripes, the range $0.12 \lesssim x \lesssim 0.16$ corresponds to mixtures of $n = 2$ and $n = 3$ ladders. Although we have calculated the static structure factor, we believe that the saturation in the dynamic fluctuation incommensurability in La$_{2-x}$Sr$_x$CuO$_4$ may have the same origin. We also have computed $\chi_u$, $\chi_s$, and $S_{Q^{-1}}$, which are plotted in Fig. 10 as functions of $T$, for different $p$’s, all with $\alpha = 0.05$. Note the differing low-temperature behaviors of the $p = 0$ arrays from those with $p \neq 0$; it is evident that the $n = 3$ physics indeed dominates in the magnetism of the random mixture.

**FIG. 10.** Uniform susceptibility, staggered susceptibility, and inverse static structure factor peak intensity for arrays of $n = 2$ and $n = 3$ mixed ladders with different $p$.

Finally, we should note that in our model, in order to
explain the fact that the spin incommensurability saturates near 1/8 even for very high dopings, we must hypothesize that for doping above the optimal value \( x \approx 0.15 \), the charge per stripe increases progressively with increasing \( x \) above \( \sim 0.15 \). That is, we assume that as the doping increases beyond \( x = 0.15 \) charge stripes cannot become closer due to the Coulomb repulsion; instead, additional charges go into already existing charge stripes.

VI. STRONGLY COUPLED THREE LEG LADDERS

The previous calculations have all been done in the weak inter-ladder coupling limit; indeed, in order to obtain reasonable magnetic ordering temperatures in our model, the reduced inter-ladder coupling must be of order \( \alpha \sim 0.05 \). However, as pointed out to us by Kivelson \( ^{[37]} \), a similar situation also should obtain in the limit of large \( \alpha \). In this case, physically one would imagine the carriers along the charge stripes mediating a large effective antiferromagnetic exchange between the bounding spin chains; these neighboring chains would then form an array of two-leg ladders coupled to single chains [Fig. 1(a)].

Accordingly, we have carried out a limited number of calculations on arrays of \( n = 3 \) ladders coupled with strong exchange couplings; that is, in the \( \alpha > 1 \) limit. Specifically, we have repeated our simulations of Sec. \( ^{[11]} \) with \( \alpha = 4.0 \) and \( \alpha = 10.0 \). The correlation length data so-obtained are shown in Fig. 11(a). To facilitate a comparison, we have also plotted the \( \alpha = 0.05 \) data from Fig. 3. At low temperatures, one can see the RC behavior clearly, similar to that of the weak coupling data. The solid lines are the results of fits to Eq. (3). It is evident, however, that the uniform susceptibility data for strongly coupled \( n = 3 \) ladders shown in Fig. 11(b) are quite different from those of weakly coupled \( (\alpha = 0.05) \) \( n = 3 \) ladders. In this limit, since \( J' \) is the primary coupling, as discussed above, two-leg ladders are effectively formed across the charge stripes; therefore, we should consider this system as a mixture of \( S = 1/2 \) chains and \( n = 2 \) ladders rather than as an array of \( n = 3 \) ladders. Specifically, \( S = 1/2 \) chains and \( n = 2 \) ladders alternate in the \( x \)-direction. The spin gap of these \( n = 2 \) ladders is large (of order \( 0.5J' \)) and the contribution from these to \( \chi_u \) is essentially zero. Thus one might surmise that \( \chi_u \) would be entirely due to the \( S = 1/2 \) chains. That this idea is correct may be seen from Fig. 11(b), where we have plotted the uniform susceptibility per spin of the \( S = 1/2 \) chain from Ref. \( ^{[22]} \) divided by three. Both the \( \alpha = 4.0 \) and the \( \alpha = 10.0 \) data agree with the spin chain results for \( T \gtrsim 0.4J \). At low temperatures, the correlations between the chain and the ladder become important and we observe behavior similar to that of mixtures of ladders [Fig. 11(a)].

The correlation length for \( \alpha = 4.0 \) grows as a function of \( T^{-1} \) much faster than that of \( \alpha = 0.05 \), such that the prediction for \( T_m \) in this case is \( T_m \approx 144K \) for \( J = 1500K \), which is more than a factor of three larger than the experimental result. In order to obtain reasonable values for \( T_m \) in the large \( \alpha \) limit, one needs to have \( \alpha \) close to 10 as shown in Fig. 11. In such a case, for \( \alpha = 10.0 \), we obtain \( T_m \approx 55K \). Such large values for \( J' \approx 1.3eV \) seem to us to be unlikely. For this model (\( J' \gg J \)) to be relevant one would need to invoke some other mechanism such as extensive structural disorder or effects from charge and spin fluctuations of the charged stripes themselves to reduce \( T_m \) down to the observed values of 30K to 40K.

![FIG. 11. (a) Correlation lengths for arrays of \( n = 3 \) ladders with strong inter-ladder coupling (\( \alpha > 1 \)) plotted as a function of inverse temperature. The correlation lengths in the \( x \)-direction are shown as solid symbols, while those in the \( y \)-direction are shown as empty symbols. (b) Uniform susceptibility per spin as a function of \( T \).]

VII. DISCUSSION

A few comments on the applicability of our model are in order. First and foremost, our model is clearly an extremely simplified version of any actual charge and spin ordering in the doped copper oxide planes. We assume that (1) the charge stripes are perfectly ordered...
with the charges confined to Cu–O–Cu–O chains and (2)
the charge and spin degrees of freedom along the stripes
can be ignored. We know that dynamic transverse fluct-
uations of the charge stripes could be very large, espe-
cially for incommensurate charge stripe spacings. How-
ever, this effect may be less significant if the charge or-
dering occurs at a much higher temperature than the
spin ordering, so that the charge fluctuations are signifi-
cantly reduced near the spin ordering temperature. Se-
veral neutron scattering experiments on La$_{2-x}$Sr$_x$NiO$_4$
and La$_{1.6-x}$Nd$_{0.4}$Sr$_x$CuO$_4$ indeed report a charge or-
dering temperature that is higher than the spin order-
ting temperature. The charge degrees of freedom in the
metallic stripes, which we have ignored, clearly are essen-
tial for the transport and superconducting properties,
and we have nothing to contribute on this aspect of the
problem. Second, microscopic phase separation of holes
and spins can occur in other geometries; for example, it
is possible to have grid-like spin-rich regions separated
by hole-rich domain walls as well as diagonal stripes.
In both cases, the incommensurate spin density wave peaks
would be rotated by 45° with respect to those for the
stripe model discussed here. In a recent experiment on an
$x = 0.05$ sample [3] which notably is an insulator rather
than a superconductor at low temperatures, 45° rotated
magnetic peaks are indeed observed; in that case it is
believed that this rotation is due to the establishment of
diagonal stripes.

Despite the success of the current model in explain-
ing a number of experimental observations for incom-
mensurabilities near 1/8, many points remain to be un-
derstood. As is obvious in Figs. 2 and 3, $\xi_x$ and $\xi_y$ are very anisotropic, which would, in turn, predict
anisotropic widths in the quasielastic and dynamic neu-
tron scattering measurements. Currently available exper-
imental data are not complete enough to test this predic-
tion. In recent work, Tranquada et al. [15] calculated the
elastic structure factor of $n = 4$ and $n = 3$ mixtures for
sinusoidally varying charge and spin density waves;
they hypothesized that the stripe disorder is caused by
pinning by the random Sr dopants. They also argued
that these defects would explain the absence of any peak
width anisotropy in their data.

An obvious major deficiency of the current model is
that it does not naturally explain the special role which
hole dopings $x$ and concomitantly, incommensurabilities
$\epsilon$ near 1/8 play in the La$_2$CuO$_4$-based superconductors.
It is evident that $T_m$ will be a maximum locally around
$x \approx \epsilon \approx 1/8$, since for small $\alpha$, admixtures of $n = 4$
($x < 1/8$) or $n = 2$ ($x > 1/8$) ladders will decrease
the magnetic correlation lengths and hence will lower $T_m$. However, there is no obvious reason why other odd-
width ladder arrays would not also be favorable. We
note that for $n = 5$, 0.05 = 1/20 the spin correlations are
commensurated [6] so that the uniform stripe model does
not apply. Thus, the other relevant odd-width ladder ar-
rays are $n = 7$, corresponding to $x = \epsilon = 0.0625$, and
$n = 5$, corresponding to $x = \epsilon = 0.083$. We have carried
out some limited calculations for $n = 5$ and, as one would
intuit from the results of Greven et al. [16] for isolated lad-
ders, at a given temperature the correlation lengths for
arrays of $n = 5$ ladders are somewhat larger than those for
$n = 3$ arrays. For perfectly ordered charge stripes, $T_m$
should be correspondingly higher. In fact, the opposite
seems to be true, although more experimental data are
required to document this completely. At the minimum,
from the inelastic neutron scattering studies of Yamada
et al. [17] we know that the low-energy dynamical coherence
length at low temperatures for $x \approx \epsilon \approx 1/12$ is shorter
than that for $x \approx \epsilon \approx 1/8$.

Clearly, therefore, if this stripe model is to describe
the real La$_2$CuO$_4$-based superconductors, then some fur-
ther physics is required. At least three different factors
could act to reduce the charge and concomitantly the
spin correlation lengths at lower doping. First, as evi-
cenced by the commensurate spin structure for $x \lesssim 0.05$,
pinning of the stripes becomes much more important at
lower dopings. Second, as the charge stripe separation
grows, the stripe-stripe interaction energy will decrease
and, correspondingly, stripe positional fluctuations will
grow. These would in turn inhibit the development of the
spin correlations. Third, as discussed above, it is
believed that at $x = 0.05$ the stripes switch from being
approximately along the CuO–Cu–O direction to being
along the diagonal direction that is from (1 0) to (1 1).
Presumably, as the doping decreases towards 0.05, fluc-
tuations into the diagonal stripe phase could occur and
thus would shorten the spin correlation length. These
ideas are, of course, purely heuristic. Some real theo-
retical calculations plus further experiments are clearly
required to put the model on a firmer basis.

We have assumed that the charge stripes are site-
centered rather than bond-centered [14]. The basic reason
for this is that, at least within our simplified model, we
are unable to match at all the observed experimental
trends with a bond-centered model. This may be eas-
ily seen as follows. First, if the spins across the charged
stripe bonds are coupled antiferromagnetically, then the
basic commensurate antiferromagnetic situation is unau-
terful. Second, if, as originally argued by Aharony et al.
[15] holes on the oxygens induce a strong ferromagnetic
coupling between the two neighboring copper spins, then
one has effective $S = 1$ chains. Thus, for the specific
case of $\epsilon = 1/8$, the system breaks up into alternating
spin-1 chains and two-leg ladders. Both the spin-1 chain
($\Delta \approx 0.4J$) and the two-leg ladder ($\Delta \approx 0.5J$) have large
spin gaps and short correlation lengths, so that this sys-
tem only orders at extremely low temperatures, if at all.
On the other hand, for $x = 1/10$ or $x = 1/6$, one has
single chains or three-leg ladders respectively in between
the spin-1 chains. Hence the correlation length along the
chain diverges at low temperature, making a spin density
wave ordering possible. The difficulty for a bond-centered
model then is that one has facile ordering at $x = 1/10$
and $x = 1/6$, but not at 1/8, which is the opposite of the
general trend observed experimentally. Therefore,
we are required within the context of our model to adopt site-centered stripes in order to obtain sensible behavior around \( x = 1/8 \) doping regime. It would clearly be interesting to see if this same difficulty occurs for the \( t-J \) model as discussed by White and Scalapino.

Manifestly, more experiments are needed to address many unresolved features of the stripe model. First, one needs to understand in detail the doping dependence of the spin density wave ordering in the \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) and related systems. Second, better characterization of the charge stripes themselves, including both the incommensurabilities and the correlation lengths is crucial. Third, unambiguous determination of the charge ordering temperatures is essential, since it is still not certain as to whether charge ordering drives the spin ordering, or vice versa. Since neutrons scatter from the small nuclear displacements induced by the modulated charge density, the charge ordering peaks observed via neutron scattering are extremely weak. Electron diffraction is, of course, especially sensitive to the charge modulation, but it is only useful for surfaces or very thin films which may differ from their bulk counterparts. X-ray scattering seems to be a logical choice to study the charge modulation in these materials. In their study of \( \text{La}_{1.48}\text{Nd}_{0.4}\text{Sr}_{0.12}\text{CuO}_4 \), Zimmermann et al.\[12\] used high-energy x-ray (100 keV) to investigate charge stripes, confirming results from neutron scattering experiments. They were able to determine peak widths at various temperatures with somewhat better precision than that of neutron scattering; however, a much higher resolution x-ray study is necessary to compare quantitatively the correlation length data with various theoretical predictions. Generally, it is important to determine the nature of the charge-charge stripe correlations in both the normal and superconducting states.

In summary, we have studied the magnetism arising from the charge and spin stripe order in monolayer cuprate superconductors. Quantum Monte Carlo simulations have been carried out on a simplified model, which is essentially weakly coupled insulating spin ladders. Our calculations are consistent with recent results of spin density wave ordering and dynamical spin fluctuations in \( \text{La}_{2-x}\text{Sr}_x\text{CuO}_4 \) and \( \text{La}_2\text{CuO}_4+y \) for doping and incommensurabilities near 1/8. We show that the periodicity of the incommensurate spin order is not necessarily twice that of the charge order. The behavior at lower doping remains problematic. We have also studied in detail the quantum critical behavior in coupled four-leg ladders.

**ACKNOWLEDGMENTS**

We would like to thank V. J. Emery, M. Greven, T. Imai, S. A. Kivelson, and J. M. Tranquada for invaluable discussions. We thank S. A. Kivelson especially for detailed comments on this manuscript. The present work was supported by the US-Japan Cooperative Research Program on Neutron Scattering. The work at Tohoku has been supported by a Grant-in-Aid for Scientific Research of Monbusho and the Core Research for Evolutional Science and Technology (CREST) Project sponsored by the Japan Science and Technology Corporation. The work at MIT was supported by the NSF under Grant No. DMR97-04532 and by the MRSEC Program of the National Science Foundation under Award No. DMR98-08941. The work at Brookhaven National Laboratory was carried out under Contract No. DE-AC02-98CH10886, Division of Materials Science, U. S. Department of Energy.

---

1. Y. Endoh, K. Yamada, M. Matsuda, K. Nakajima, K. Kuroda, Y. Hidaka, I. Tanaka, H. Kojima, R. J. Birgeneau, M. A. Kastner, B. Keimer, G. Shirane, and T. R. Thurston, in *Mechanisms of superconductivity: JJAP series*, 7, edited by Y. Muto (Publication Office, Japanese Journal of Applied Physics, Tokyo, Japan, 1992), pp. 174-177.
2. For a review, see M. A. Kastner, R. J. Birgeneau, G. Shirane, and Y. Endoh, Rev. Mod. Phys. **70**, 897 (1998).
3. K. Yamada, C. H. Lee, K. Kurahashi, J. Wada, S. Waki-moto, S. Ueki, H. Kimura, Y. Endoh, S. Hosoya, G. Shirane, R. J. Birgeneau, M. Greven, M. A. Kastner, and Y. J. Kim, Phys. Rev. B **57**, 6165 (1998).
4. N. Bulut, D. Hone, D. J. Scalapino, and N. E. Bickers, Phys. Rev. Lett. **64**, 2723 (1990).
5. Q. Si, Y. Zha, , K. Levin, and J. P. Lu, Phys. Rev. B **47**, 9055 (1993).
6. T. Tanamoto, H. Kohno, and H. Fukuyama, J. Phys. Soc. Jpn **63**, 2739 (1994).
7. J. Zaanen and O. Gunnarsson, Phys. Rev. B **40**, 7391 (1989); J. Zaanen, M. L. Horbach, and W. van Saarloos, *ibid.* **53**, 8671 (1996).
8. V. J. Emery and S. A. Kivelson, Physica C **66**, 763 (1994); S. A. Kivelson and V. J. Emery, in *Strongly Correlated Electronic Materials*, edited by K. S. Bedell, Z. Wang, D. E. Meltzer, A. V. Balatsky, and E. Abrahams (Addison-Wesley, Reading, Massachusetts, 1994), pp. 619-656; V. J. Emery, S. A. Kivelson, and O. Zachar, Phys. Rev. B **56**, 6120 (1997).
9. C. Nayak and F. Wilczek, Phys. Rev. Lett. **78**, 2465 (1997).
10. For a review, see E. Dagotto and T. M. Rice, Science **271**, 618 (1996).
11. M. Greven, R. J. Birgeneau, and U.-J. Wiese, Phys. Rev. Lett. **77**, 1865 (1996).
12. J. M. Tranquada, B. J. Sternlieb, J. D. Axe, Y. Nakamura, and S. Uchida, Nature **375**, 561 (1995); J. M. Tranquada, J. D. Axe, N. Ichikawa, Y. Nakamura, S. Uchida, and B. Nachumi, Phys. Rev. B **54**, 7489 (1996); J. M. Tranquada, J. D. Axe, N. Ichikawa, A. R. Moodenbaugh, Y. Nakamura, and S. Uchida, Phys. Rev. Lett. **78**, 338 (1997).
T. Suzuki, T. Goto, K. Chiba, T. Shinoda, T. Fukase, H. Kimura, K. Yamada, M. Ohashi, and Y. Yamaguchi, Phys. Rev. B 57, R3229 (1998).

H. Kimura, K. Hirotta, H. Matsushita, K. Yamada, Y. Endoh, S. H. Lee, C. F. Majkrzak, R. Erwin, G. Shirane, M. Greven, Y. S. Lee, M. A. Kastner, and R. J. Birgeneau, Phys. Rev. B (to be published).

Y. S. Lee, R. J. Birgeneau, M. A. Kastner, Y. Endoh, S. Wakimoto, K. Yamada, R. W. Erwin, S. H. Lee, and G. Shirane, cond-mat/9902157.

P. Dai, H. A. Mook, and F. Dogan, Phys. Rev. Lett. 80, 1738 (1998); H. A. Mook, P. Dai, S. M. Hayden, G. Aeppli, T. G. Perring, and F. Dogan, Nature 395, 580 (1998).

S. A. Kivelson, E. Fradkin, and V. J. Emery, Nature 393, 550 (1998); S. A. Kivelson and V. J. Emery, cond-mat/9809082.

P. Hasenfratz and F. Niedermayer, Phys. Lett. B 268, 231 (1991).

M. Greven, R. J. Birgeneau, Y. Endoh, M. A. Kastner, M. Matsuda, and G. Shirane, Z. Phys. B 96, 465 (1995).

R. J. Birgeneau, A. Aharony, N. R. Belk, F. C. Chou, Y. Endoh, M. Greven, S. Hosoya, M. A. Kastner, C. H. Lee, Y. Lee, G. Shirane, S. Wakimoto, B. O. Wells, and K. Yamada, J. Phys. Chem. Solids 56, 1913 (1995).

For a review, see I. Affleck, J. Phys. Condens. Matter 1, 3047 (1989).

F. D. M. Haldane, Phys. Lett. 93A, 464 (1983).

M. Nightingale and H. Blöte, Phys. Rev. B 33, 659 (1986).

J. Renard, M. Verdaguer, L. Regnault, W. Erkelens, J. Rossat-Mignod, and W. Stirling, Europhys. Lett. 3, 945 (1987).

J. Tworzydło, O. Y. Osman, C. N. A. van Duin, and J. Zaanen, Phys. Rev. B 59, 115 (1999).

For a review, see H. G. Evertz, in Numerical Methods for Lattice Many-Body Problems, edited by D. J. Scalapino (Addison Wesley Longman, Reading, Massachusetts, 1998), p. 6.

Y. J. Kim, M. Greven, U.-J. Wiese, and R. J. Birgeneau, Eur. Phys. J. B 4, 291 (1998).

A. H. Castro Neto and D. Hone, Phys. Rev. Lett. 76, 2165 (1996).

B. Keimer, N. Belk, R. J. Birgeneau, A. Cassanho, C. Y. Chen, M. A. Kastner, A. Aharony, Y. Endoh, R. W. Erwin, and G. Shirane, Phys. Rev. B 46, 14034 (1992).

A. V. Chubukov, S. Sachdev, and A. Sokol, Phys. Rev. B 49, 9052 (1994); A. V. Chubukov, S. Sachdev, and J. Ye, ibid. 49, 11919 (1994).

A. Sokol, R. L. Glenister, and R. R. P. Singh, Phys. Rev. Lett. 72, 1549 (1994).

B. B. Beard, R. J. Birgeneau, M. Greven, and U. J. Wiese, Phys. Rev. Lett. 80, 1742 (1998).

S. A. Kivelson (private communication).

For a review, see J. M. Tranquada, in Neutron Scattering in Layered Copper-Oxide Superconductors, edited by A. Furrer (Kluwer, Dordrecht, The Netherlands, 1998), pp. 225-260.

S. Wakimoto, R. J. Birgeneau, Y. Endoh, P. M. Gehring, K. Hirotta, M. A. Kastner, S. H. Lee, Y. S. Lee, G. Shirane, S. Ueki, and K. Yamada, cond-mat/9902201.

S. R. White and D. J. Scalapino, Phys. Rev. Let. 80, 1272 (1998).

A. Aharony, R. J. Birgeneau, A. Coniglio, M. A. Kastner, and H. E. Stanley, Phys. Rev. Let. 60, 1330 (1988).

O. Zachar, S. A. Kivelson, and V. J. Emery, Phys. Rev. B 57, 1422 (1998).

M. v. Zimmermann, A. Vigliante, T. Niemöller, N. Ichikawa, T. Frello, J. Madsen, P. Woehner, S. Uchida, N. H. Andersen, J. M. Tranquada, D. Gibbs, and J. R. Schneider, Europhys. Lett. 41, 629 (1998).