The equivalence between local inertial frames and electromagnetic gauge in
Einstein-Maxwell theories

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We are going to prove that locally the inertial frames and gauge states of the electromagnetic field are equivalent. This proof will be valid for Einstein-Maxwell theories in four-dimensional Lorentzian spacetimes. Use will be made of theorems proved in a previous manuscript. These theorems state that locally the group of electromagnetic gauge transformations is isomorphic to the local Lorentz transformations of a special set of tetrad vectors. The tetrad that locally and covariantly diagonalizes any non-null electromagnetic stress-energy tensor. Two isomorphisms, one for each plane defined locally by two separate sets of two vectors each. In particular, we are going to use the plane defined by the timelike and one spacelike vector, plane or blade one. These results will be extended to any tetrad that results in a local Lorentz transformation of the special tetrad that locally and covariantly diagonalizes the stress-energy tensor.
I. INTRODUCTION

In manuscript\textsuperscript{1} a covariant method for the local diagonalization of the $U(1)$ electromagnetic stress-energy tensor was presented. At every point in a curved four dimensional Lorentzian spacetime a new tetrad was introduced for non-null electromagnetic fields such that this tetrad locally and covariantly diagonalizes the stress-energy tensor. At every point the timelike and one spacelike vectors generate a plane that we called blade one\textsuperscript{1,2}. The other two spacelike vectors generate a plane that we called blade two. These vectors are constructed with the local extremal field\textsuperscript{2}, its dual, the very metric tensor and a pair of vector fields that represent a generic choice as long as the tetrad vectors do not become trivial. Let us display for the Abelian case the explicit expression for these vectors,

$$U^\alpha = \xi^{\alpha\lambda} \xi_{\rho\lambda} X^\rho / (\sqrt{-Q/2} \sqrt{X_\mu \xi^{\mu\sigma} \xi_{\nu\sigma} X^\nu}) \quad (1)$$

$$V^\alpha = \xi^{\alpha\lambda} X_\lambda / (\sqrt{X_\mu \xi^{\mu\sigma} \xi_{\nu\sigma} X^\nu}) \quad (2)$$

$$Z^\alpha = \ast \xi^{\alpha\lambda} Y_\lambda / (\sqrt{Y_\mu \ast \xi^{\mu\sigma} \ast \xi_{\nu\sigma} Y^\nu}) \quad (3)$$

$$W^\alpha = \ast \xi^{\alpha\lambda} \ast \xi_{\rho\lambda} Y^\rho / (\sqrt{-Q/2} \sqrt{Y_\mu \ast \xi^{\mu\sigma} \ast \xi_{\nu\sigma} Y^\nu}) \quad (4)$$

We start by stating that at every point in spacetime there is a duality rotation by an angle $-\alpha$ that transforms a non-null electromagnetic field into an extremal field,

$$\xi_{\mu\nu} = e^{-*\alpha} f_{\mu\nu} = \cos(\alpha) f_{\mu\nu} - \sin(\alpha) \ast f_{\mu\nu}. \quad (5)$$

where $\ast f_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\sigma\tau} f^{\sigma\tau}$ is the dual tensor of $f_{\mu\nu}$. The local scalar $\alpha$ is known as the complexion of the electromagnetic field. It is a local gauge invariant quantity. Extremal fields are essentially electric fields and they satisfy,

$$\xi_{\mu\nu} \ast \xi^{\mu\nu} = 0. \quad (6)$$

Equation (6) is a condition imposed on (5) and then the explicit expression for the complexion emerges $\tan(2\alpha) = -f_{\mu\nu} \ast f^{\mu\nu} / f_{\lambda\rho} f^{\lambda\rho}$. As antisymmetric fields in a four dimensional Lorentzian spacetime, the extremal fields also verify the identity,

$$\xi_{\mu\alpha} \xi^{\nu\alpha} - \ast \xi_{\mu\alpha} \ast \xi^{\nu\alpha} = \frac{1}{2} \delta_{\nu}^{\mu} Q. \quad (7)$$
where \( Q = \xi_{\mu\nu} \xi^{\mu\nu} = -\sqrt{T_{\mu\nu} T^{\mu\nu}} \) according to equations (39) in\(^3\). \( Q \) is assumed not to be zero, because we are dealing with non-null electromagnetic fields. It can be proved that condition (6) and through the use of the general identity,

\[
A_\mu A^\nu - *B_\mu *A^\nu = \frac{1}{2} \delta_\mu^\nu A_\alpha B^\alpha \beta ,
\]

which is valid for every pair of antisymmetric tensors in a four-dimensional Lorentzian spacetime\(^3\), when applied to the case \( A_\mu = \xi_\mu \) and \( B^\nu = *\xi^\nu \) yields the equivalent condition,

\[
\xi_\alpha \xi^{\mu\nu} = 0 ,
\]

which is equation (64) in\(^3\). The duality rotation given by equation (59) in\(^3\),

\[
f_{\mu\nu} = \xi_{\mu\nu} \cos \alpha + *\xi_{\mu\nu} \sin \alpha ,
\]

allows us to express the stress-energy tensor in terms of the extremal field,

\[
T_{\mu\nu} = \xi_{\mu\lambda} \xi^\nu \lambda + *\xi_{\mu\lambda} *\xi^\nu \lambda .
\]

With all these elements it becomes trivial to prove that the tetrad\(^1\) (1-4) is orthonormal and diagonalizes the stress-energy tensor (11). We notice then that we still have to define the vectors \( X^\mu \) and \( Y^\mu \). Let us introduce some names. The tetrad vectors have two essential components. For instance in vector \( U^\alpha \) there are two main structures. First, the skeleton, in this case \( \xi^{\alpha\lambda} \xi_{\rho\lambda} \), and second, the gauge vector \( X^\mu \). These do not include the normalization factor \( 1/ ( \sqrt{-Q/2} \sqrt{X_\mu \xi^{\mu\sigma} \xi_{\nu\sigma} X^\nu } ) \). The gauge vectors it was proved in manuscript\(^1\) could be anything that does not make the tetrad vectors trivial. That is, the tetrad (1-4) diagonalizes the stress-energy tensor for any non-trivial gauge vectors \( X^\mu \) and \( Y^\mu \). It was therefore proved that we can make different choices for \( X^\mu \) and \( Y^\mu \). In geometrodynamics, the Maxwell equations,

\[
f_{\mu\nu} = 0, \quad *f_{\mu\nu} = 0 ,
\]

\[\text{3}\]
are telling us that two potential vector fields $A_\nu$ and $*A_\nu$ exist,

\[
f_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \quad \quad *f_{\mu\nu} = *A_{\nu;\mu} - *A_{\mu;\nu} .
\]

(13)

The symbol "\(\cdot\)" stands for covariant derivative with respect to the metric tensor $g_{\mu\nu}$. We can define then, a tetrad,

\[U^\alpha = \xi^\alpha{}_{\lambda} \xi_{\rho\lambda} A^\rho / (\sqrt{-Q/2} \sqrt{A_\mu \xi_{\mu\sigma} \xi_{\nu\sigma} A^\nu}) \quad \quad (14)\]

\[V^\alpha = \xi^\alpha{}_{\lambda} A_\lambda / (\sqrt{A_\mu \xi_{\mu\sigma} \xi_{\nu\sigma} A^\nu}) \quad \quad (15)\]

\[Z^\alpha = *\xi^\alpha{}_{\lambda} *A_\lambda / (\sqrt{*A_\mu *\xi_{\mu\sigma} *\xi_{\nu\sigma} *A^\nu}) \quad \quad (16)\]

\[W^\alpha = *\xi^\alpha{}_{\lambda} \xi_{\rho\lambda} *A^\rho / (\sqrt{-Q/2} \sqrt{*A_\mu *\xi_{\mu\sigma} *\xi_{\nu\sigma} *A^\nu}) . \quad \quad (17)\]

The four vectors (14-17) have the following algebraic properties,

\[- U^\alpha U_\alpha = V^\alpha V_\alpha = Z^\alpha Z_\alpha = W^\alpha W_\alpha = 1 . \quad \quad (18)\]

Using the equations (7-9) it is simple to prove that (14-17) are orthogonal. When we make the transformation,

\[A_\alpha \to A_\alpha + \Lambda_\alpha , \quad \quad (19)\]

$f_{\mu\nu}$ remains invariant, and the transformation,

\[*A_\alpha \to *A_\alpha + *\Lambda_\alpha , \quad \quad (20)\]

leaves $*f_{\mu\nu}$ invariant, as long as the functions $\Lambda$ and $*\Lambda$ are scalars. Schouten defined what he called, a two-bladed structure in a spacetime. These blades are the planes determined by the pairs $(U^\alpha, V^\alpha)$ and $(Z^\alpha, W^\alpha)$. It was proved in (19) that the transformation (19) generates a “rotation” of the tetrad vectors $(U^\alpha, V^\alpha)$ into $(\tilde{U}^\alpha, \tilde{V}^\alpha)$ such that these “rotated” vectors $(\tilde{U}^\alpha, \tilde{V}^\alpha)$ remain in the plane or blade one generated by $(U^\alpha, V^\alpha)$. It was also proved in (1) that
the transformation \((20)\) generates a “rotation” of the tetrad vectors \((Z^\alpha, W^\alpha)\) into \((\tilde{Z}^\alpha, \tilde{W}^\alpha)\) such that these “rotated” vectors \((\tilde{Z}^\alpha, \tilde{W}^\alpha)\) remain in the plane or blade two generated by \((Z^\alpha, W^\alpha)\). For the sake of simplicity we are going to assume that the transformation of the two vectors \((U^\alpha, V^\alpha)\) on blade one, given in \((14-15)\), by the “angle” \(\phi\) is a proper transformation, that is, a boost. For discrete improper transformations the result follows the same lines\(^1\). Therefore we can write the transformation generated by \((19)\) as,

\[
U^\alpha_{(\phi)} = \cosh(\phi) U^\alpha + \sinh(\phi) V^\alpha
\]

\[
V^\alpha_{(\phi)} = \sinh(\phi) U^\alpha + \cosh(\phi) V^\alpha.
\]

The transformation generated by \((20)\) of the two tetrad vectors \((Z^\alpha, W^\alpha)\) on blade two, given in \((16,17)\), by the “angle” \(\varphi\), can be expressed as,

\[
Z^\alpha_{(\varphi)} = \cos(\varphi) Z^\alpha - \sin(\varphi) W^\alpha
\]

\[
W^\alpha_{(\varphi)} = \sin(\varphi) Z^\alpha + \cos(\varphi) W^\alpha.
\]

It is a simple exercise in algebra to see that the equalities \(U^\alpha_{(\phi)} V^\beta_{(\phi)} = U^\alpha V^\beta\) and \(Z^\alpha_{(\varphi)} W^\beta_{(\varphi)} = Z^\alpha W^\beta\) are true. These equalities are telling us that these antisymmetric tetrad objects are gauge invariant. We remind ourselves that it was proved in manuscript\(^1\) that the group of local electromagnetic gauge transformations is isomorphic to the group LB1 of boosts plus discrete transformations on blade one, and independently to LB2, the group of spatial rotations on blade two. Equations \((21,22)\) represent a local electromagnetic gauge transformation of the vectors \((U^\alpha, V^\alpha)\). Equations \((23,24)\) represent a local electromagnetic gauge transformation of the vectors \((Z^\alpha, W^\alpha)\). Written in terms of these tetrad vectors, the electromagnetic field is,

\[
f_{\alpha\beta} = -2 \sqrt{-Q/2} \cos \alpha \ U_{[\alpha} V_{\beta]} + 2 \sqrt{-Q/2} \sin \alpha \ Z_{[\alpha} W_{\beta]} .
\]

Equation \((25)\) represents maximum simplification in the expression of the electromagnetic field. The true degrees of freedom are the local scalars \(\sqrt{-Q/2}\) and \(\alpha\). Local gauge invariance is manifested explicitly through the possibility of “rotating” through a scalar angle \(\phi\) on blade one by a local gauge transformation \((21,22)\) the tetrad vectors \(U^\alpha\) and \(V^\alpha\), such that \(U_{[\alpha} V_{\beta]}\)
remains invariant. Analogous for discrete transformations on blade one. Similar analysis on blade two. A spatial “rotation” of the tetrad vectors $Z^{\alpha}$ and $W^{\alpha}$ through an “angle” $\varphi$ as in (23-24), such that $Z_{[\alpha} W_{\beta]}$ remains invariant. All this formalism clearly provides a technique to maximally simplify the expression for the electromagnetic field. The new expression for the metric tensor is,

$$g_{\alpha\beta} = -U_\alpha U_\beta + V_\alpha V_\beta + Z_\alpha Z_\beta + W_\alpha W_\beta.$$  (26)

The stress-energy tensor can be written,

$$T_{\alpha\beta} = (Q/2) \left[-U_\alpha U_\beta + V_\alpha V_\beta - Z_\alpha Z_\beta - W_\alpha W_\beta\right].$$  (27)

In section II we are going to prove the equivalence between the local inertial frames and local gauge states of the electromagnetic field for the tetrad that locally and covariantly diagonalizes the stress-energy tensor. In section III we are going to generalize the proof to any locally Lorentz transformed tetrad. Throughout the paper we use the conventions of manuscript. In particular we use a metric with sign conventions -++. The only difference in notation with will be that we will call our geometrized electromagnetic potential $A^{\alpha}$, where $f_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the geometrized electromagnetic field $f_{\mu\nu} = (G^{1/2}/c^2) F_{\mu\nu}$.

II. EQUIVALENCE FOR THE TETRAD THAT DIAGONALIZES THE STRESS-ENERGY TENSOR

The theorem proved in manuscript for blade one states that there is an isomorphism between the local electromagnetic gauge group of transformations and the local group LB1, essentially the local boosts on blade one and two kinds of discrete transformations, see reference. Therefore, to each local gauge state of the electromagnetic field corresponds either a local boost of the two local tetrad vectors that span plane one, that is vectors, or a discrete transformation of them. These all means that locally, to each absolute value of a velocity corresponds a unique electromagnetic gauge. For local Lorentz boosts on the plane one.
III. EQUIVALENCE FOR LORENTZ TRANSFORMED TETRADS

However, the point remains to be proved that there is a similar relationship for a locally Lorentz transformed tetrad, such that the new plane or blade one will be Lorentz transformed with respect to the one that diagonalizes the stress-energy tensor. We proceed then to call generically the tetrad set \( \{1, 2, 3, 4\} \) by the standard name \( E_{\alpha}^\mu \). For the second electromagnetic tetrad we are going to need a local Lorentz transformation. Let us analyze the expression 

\[ \tilde{E}_\delta^\rho = \Lambda_\delta^\rho E_\alpha^\rho. \]

This is going to be a Lorentz transformed electromagnetic tetrad vector. Then, keeping the same notation as in \(^1\), we call,

\[ \tilde{\xi}^{\mu \nu} = -2 \sqrt{-Q/2} \Lambda_\delta^\rho \Lambda_\gamma^\lambda E_\delta^{[\mu} E_\gamma^\rho]. \]  
(28)

\[ \ast \tilde{\xi}^{\mu \nu} = 2 \sqrt{-Q/2} \Lambda_\delta^\rho \Lambda_\gamma^\lambda E_\delta^{[\mu} E_\gamma^\rho]_\ast. \]  
(29)

Now, with these fields, the \( \tilde{\xi}_{\mu \nu} \), and its dual \( \ast \tilde{\xi}_{\mu \nu} \), we can repeat the procedure followed in \(^1\), and the transformed tetrads \( \tilde{E}_\alpha^\rho \), can be rewritten completely in terms of these “new” extremal fields. It is straightforward to prove that \( \tilde{\xi}^{\mu \lambda} \ast \tilde{\xi}_{\mu \nu} = 0 \). It is also evident that \( \tilde{E}_0^{\mu} \ast \tilde{\xi}_{\mu \nu} = 0 = \tilde{E}_1^{\mu} \ast \tilde{\xi}_{\mu \nu} \). Therefore \( \tilde{E}_0^{\mu} \) and \( \tilde{E}_1^{\mu} \) belong to the plane generated by the normalized version of vectors like \( \tilde{\xi}^{\mu \nu} \tilde{\xi}_{\lambda \nu} \tilde{X}^\lambda \) and \( \tilde{\xi}^{\mu \nu} \tilde{X}_\nu^\lambda \). Then, for instance we are going to be able to write the timelike \( \tilde{E}_0^{\mu} \) as the the normalized version of the timelike \( \tilde{\xi}^{\mu \nu} \tilde{\xi}_{\lambda \nu} \tilde{X}^\lambda \) for some vector field \( \tilde{X}^\lambda \). We remind ourselves that the relation between the normalized versions of the two vectors that locally determine blade one, \( \tilde{\xi}^{\mu \nu} \tilde{\xi}_{\lambda \nu} X^\lambda \) and \( \tilde{\xi}^{\mu \nu} X_{\nu} \), on one hand, and \( \tilde{\xi}^{\mu \nu} \tilde{\xi}_{\lambda \nu} \tilde{X}^\lambda \) on the other hand, is established through a LB1 gauge transformation on the vector \( X^\lambda \rightarrow X^\lambda + \Lambda^\lambda \). Analogous analysis for \( \tilde{E}_2^{\mu} \) and \( \tilde{E}_3^{\mu} \) on blade two. Gauge transformations of the electromagnetic tetrads we remind ourselves are nothing but a special kind of tetrad transformations that belong either to the groups LB1 or LB2. This method essentially says that the local Lorentz transformation of the electromagnetic tetrads is structure invariant, or construction invariant. This means that after a Lorentz transformation we can manage to rewrite the new transformed tetrads using skeletons and gauge vectors following the same pattern as for the original tetrad before the Lorentz transformation. We are going to call this property tetrad structure covariance. Therefore, we next proceed to write the four orthonormal vectors \( \tilde{E}_\delta^\rho \),

\[ \tilde{\xi}^{\mu \nu} = -2 \sqrt{-Q/2} \Lambda_\delta^\rho \Lambda_\gamma^\lambda E_\delta^{[\mu} E_\gamma^\rho], \]  
(28)

\[ \ast \tilde{\xi}^{\mu \nu} = 2 \sqrt{-Q/2} \Lambda_\delta^\rho \Lambda_\gamma^\lambda E_\delta^{[\mu} E_\gamma^\rho]_\ast. \]  
(29)
\[
\begin{align*}
\tilde{U}^\alpha &= \tilde{\xi}^{\alpha\lambda} \tilde{X}^\lambda / (\sqrt{-\tilde{Q}}/2 \sqrt{\tilde{X}_\mu \tilde{\xi}^{\mu\sigma} \tilde{\xi}^{\nu\sigma} \tilde{X}^\nu}) \\
\tilde{V}^\alpha &= \tilde{\xi}^{\alpha\lambda} \tilde{X}_\lambda / (\sqrt{\tilde{X}_\mu \tilde{\xi}^{\mu\sigma} \tilde{\xi}^{\nu\sigma} \tilde{X}^\nu}) \\
\tilde{Z}^\alpha &= \star \tilde{\xi}^{\alpha\lambda} \star \tilde{Y}_\lambda / (\sqrt{\star \tilde{Y}_\mu \star \tilde{\xi}^{\mu\sigma} \star \tilde{\xi}^{\nu\sigma} \star \tilde{Y}^\nu}) \\
\tilde{W}^\alpha &= \star \tilde{\xi}^{\alpha\lambda} \star \tilde{\xi}^{\rho\lambda} \star \tilde{Y}^\rho / (\sqrt{-\tilde{Q}}/2 \sqrt{\star \tilde{Y}_\mu \star \tilde{\xi}^{\mu\sigma} \star \tilde{\xi}^{\nu\sigma} \star \tilde{Y}^\nu}).
\end{align*}
\]

In order to prove the properties of the tetrad set (30-33) it is just necessary to transcribe many of the results introduced in section I. We are assuming that our choice for vectors \(\tilde{X}_\rho\) and \(\tilde{Y}_\rho\) are not making the tetrad trivial. Now, and this is the point of this section, if we choose \(\tilde{X}_\rho = A_\rho\) and \(\tilde{Y}_\rho = \star A_\rho\) and introduce local transformations \(A_\alpha \rightarrow A_\alpha + \Lambda_\alpha\) and \(\star A_\alpha \rightarrow \star A_\alpha + \star \Lambda_\alpha\) such that the new extremal fields \(\tilde{\xi}_{\mu\nu}\) and its dual \(\star \tilde{\xi}_{\mu\nu}\) remain invariant, then, many of the results of section II are reproduced once again. One might ask about the local choice of vectors \(X^\mu\) and \(Y^\mu\) hidden in the tetrads \(E^\mu_\alpha\) or equivalently the tetrad vectors (1-4). Because these tetrad vectors are hidden in \(\tilde{\xi}_{\mu\nu}\) and its dual \(\star \tilde{\xi}_{\mu\nu}\). How we manage to transform the gauge vectors \(A_\alpha \rightarrow A_\alpha + \Lambda_\alpha\) and \(\star A_\alpha \rightarrow \star A_\alpha + \star \Lambda_\alpha\) without affecting the new extremal fields \(\tilde{\xi}_{\mu\nu}\), and its dual \(\star \tilde{\xi}_{\mu\nu}\), that we claim will remain invariant. One simple local gauge choice for them would be for instance \(X^\rho = Y^\rho = \alpha_\rho \, g^{\rho\rho}\), where \(\alpha\) is the local complexion scalar defined in section I. It is a local gauge invariant, and this choice solves the problem with the local invariance of the new extremal fields. In fact, any local gauge invariant scalar would do the job like \(Q_{\rho\nu} \, g^{\nu\rho}\), for instance. Returning to our issue of the equivalence of the local inertial frames and gauge on a new plane one, which is the result of a local Lorentz transformation of the plane one that “diagonalizes” the stress-energy tensor, the same conclusions that were reached for the plane one that “diagonalizes” the stress-energy tensor, are reached for the new plane. Locally, to each absolute value of a velocity there corresponds a unique electromagnetic gauge. Since the local Lorentz transformation is generic, we conclude that locally, to each absolute value of any velocity (less that c, of course) in any direction there corresponds an electromagnetic gauge. If we pick any local plane one, the relationship between velocity absolute value and electromagnetic gauge is one to one.
IV. CONCLUSIONS

It is not only interesting but also surprising that the local inertial frames are even related locally to the electromagnetic gauge. Not only that, there are on each local plane one, a one to one correspondence. This result is not trivial. It goes to the heart of a unified structure involving local inertial frames and gauge. The new tetrads introduced in manuscript\footnote{1} make this relationship to become evident. Several properties of these tetrads are remarkable. For instance their skeleton-gauge vector structure. Their structure covariant or structure invariant nature under local Lorentz transformations. The fact that they allow to prove that the local electromagnetic gauge group is both isomorphic to the local groups LB1 and LB2, see reference\footnote{1}. A non-compact group like the boosts plus two kinds of discrete transformations, all on plane one, is isomorphic by transitivity to local spatial rotations on a plane two, a compact group. This tetrad introduces maximal simplification in the expression of the electromagnetic field. Automatically diagonalizes locally and covariantly the stress-energy tensor. It is truly outstanding. We quote from\footnote{7} “Here is not the place to write down the Lorentz transformations and to sketch how special relativity theory with its fixed causal and inertial structure gave way to general relativity where these structures have become flexible by their interaction with matter. I only want to point out that it is the inherent symmetry of the four-dimensional continuum of space and time that relativity deals with. We found that objectivity means invariance with respect to the group of automorphisms”. We also quote H. Weyl from\footnote{8} “By this new situation, which introduces an atomic radius into the field equations themselves -but not until this step- my principle of gauge-invariance, with which I had hoped to relate gravitation and electricity, is robbed of its support. But it is now very agreeable to see that this principle has an equivalent in the quantum-theoretical field equations which is exactly like it in formal respects; the laws are invariant under simultaneous replacement of $\psi$ by $\exp(i\hbar\lambda)\psi$, $\phi_{\alpha}$ by $\phi_{\alpha} - \frac{\partial\lambda}{\partial x_{\alpha}}$, where $\lambda$ is an arbitrary real function of position and time. Also the relation of this property of invariance to the law of conservation of electricity remains exactly as before ... the law of conservation of electricity $\frac{\partial\rho_{\alpha}}{\partial t} = 0$ follows from the material as well as from the electromagnetic equations. The principle of gauge-invariance has the character of general relativity since it contains an arbitrary function $\lambda$, and can certainly be understood in terms of it”.
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