The one-dimensional Oslo model is studied under self-organized criticality (SOC) conditions and under absorbing state (AS) conditions. While the activity signals the phase transition under AS conditions by a sudden increase, this is not the case under SOC conditions. The scaling parameters of the activity are found to be identical under SOC and AS conditions, but in SOC the activity lacks a pickup.

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INTRODUCTION

Self-organized criticality (SOC) \[1\] is a form of non-equilibrium critical behavior \[2\] where, in contrast to traditional critical phenomena, only a separation of timescales between external driving and internal relaxation is needed for the system to develop into a scale invariant state. The description of SOC as the result of feedback between an order parameter and a tuning parameter was put forward by Tang and Bak \[2\]. Later Dickman et al. \[3\] linked SOC to models with AS phase transitions, where a tuning parameter, the particle density \(\xi\), determines whether the system is in an active phase where it changes in time or in an inactive phase where the system is stuck in one configuration. The order parameter of these transitions is the density of sites about to topple, called the activity. Dickman et al. suggested that the critical behavior of SOC systems is numerically identical to that of these well understood \[1\] non-equilibrium critical phenomena.

The intuition that the two types of critical phenomena are related can be understood as follows: conservative SOC sandpile models can be turned into AS systems by making the open boundaries periodic, and ceasing to drive the systems. Thereby, for each SOC system a corresponding AS system can be constructed. Activity in SOC systems transports particles in a random fashion, which can lead to dissipation at the open boundaries. The activity stops whenever the system gets stuck in an inactive (absorbing) state. It is then re-activated by the external drive. Thus, the SOC system is repeatedly driven through the corresponding absorbing-state phase transition. A key assumption in the explanation of SOC in terms of this “AS mechanism” is that the SOC activity behaves just like in AS with a sudden pickup: “Slow driving pins \(\xi\) at its critical value: if it exceeds \(\xi_c\), activity is generated, and thereby dissipation” \[5\].

The main result of the present study is that this assumption is not valid. The critical properties of the AS transition are reflected in the finite-size scaling of the SOC activity. But as a function of the particle density, the activity shows no sign of a phase transition under SOC conditions.

We note that finding the critical parameters, exponents, moment ratios etc. is incomparably easier in an SOC system than under AS conditions. For instance, the interesting long-time limit is difficult to access in simulations of AS transitions as the number of active systems in an ensemble shrinks exponentially with time \[6, 7\]. As a numerical tool for studying non-equilibrium phase transitions, SOC could be extremely useful if it was fundamentally understood (as opposed to numerically determined) which properties measured in SOC are universal.

THE OSLO MODEL

We study the Oslo model \[8\] in one dimension. This model is believed to be in the C-DP universality class (directed percolation coupled to a non-diffusive conserved field), as is the Manna model \[9\]. In addition to the results shown here, all findings were confirmed for the two-dimensional Oslo model and the one- and two-dimensional Manna models. Further details will appear elsewhere.

Initialization in the AS version: \(\xi_L \in \mathbb{N}\) particles for a fixed, given density \(\xi\) are added at uniformly randomly chosen sites in a lattice of linear size \(L\) with periodic boundary conditions (topology of a ring). Every site is assigned a threshold particle number, either 1 or 2, randomly, independently and with equal probabilities. Sites
exceeding this local threshold are called “active” and subject to the relaxation process described under Toppling below.

**Initialization in the SOC version:** The system starts out empty and is filled only by the External drive (below). Every site is assigned a random threshold as in AS.

**Toppling:** Every active site is selected randomly with equal probability. One particle is moved to each of the nearest neighbors, and a new random threshold, 1 or 2, is assigned to the originating site.

**Avalanches:** The toppling is repeated until no active site is left. Avalanches are defined as a sequence of topplings induced by a single driving step (below).

**Microscopic time:** Active sites topple at unit rate, which defines the microscopic time unit. In AS, observables are clearly time-dependent, as the system relaxes from the initial unstable state to complete quiescence, generating a single avalanche. The average over initial states in AS is denoted $\langle \cdot \rangle_{\text{AS}}$. Where the time $t$ is dropped as a parameter from the observables, time averaging with appropriate weighting has been applied. In SOC, averaged observables are denoted $\langle \cdot \rangle_{\text{SOC}}$. This ensemble can be conditioned on the density of particles $\zeta$.

**Conditional activity:** The activity is the density of sites exceeding the local particle threshold. It is the order parameter in the sense that a non-zero asymptotic value of $\lim_{t \to \infty} \lim_{L \to \infty} \langle \rho_a(\zeta; L, t) \rangle_{\text{AS}}$ indicates the active phase. Only non-zero measurements of the instantaneous activity contribute to estimates of the averages – in AS this is done by explicit conditioning (only active systems are part of the AS ensemble), while SOC systems are re-activated immediately whenever they fall into an absorbing state. Nonetheless, it is important to distinguish between

1) the AS activity $\langle \rho_a(\zeta; L) \rangle_{\text{AS}}$ measured at constant (conserved) externally set $\zeta$ and
2) the SOC activity $\langle \rho_a(\zeta; L) \rangle_{\text{SOC}}$, conditioned on the value $\zeta$ of the fluctuating density, $\zeta(t)$, in the SOC ensemble.

**Boundary conditions:** In the AS version, boundaries are periodically closed and no dissipation of particles takes place. In the SOC version, boundaries are open, i.e. toppling across a boundary removes particles from the system.

**External drive:** The AS models are not driven. Time series (see below) terminate when all activity has ceased. This happens, in a finite system, with a rate that is bounded away from zero, i.e. in any finite system the avalanche eventually comes to a halt (below a trivial density limit). While the model in the AS version is restarted by resetting it to a new initial condition, the SOC version is driven externally to compensate the loss of particles at the boundary. The driving occurs only at quiescence, thereby completely separating the timescales of toppling and driving.

**Time series in the AS version:** An ensemble of systems is prepared, and observables are averaged over the members of the ensemble (of systems that are still active) at equal times, producing time series of fluctuating observables. We focus here on moments of the activity in the quasi-stationary state, $\langle \rho_a^k(\zeta) \rangle_{\text{AS}}$, that is, moments of the activity after a transient but conditional to activity $\langle \cdot \rangle_{\text{AS}}$.

**Time series in the SOC version:** Because the SOC systems are driven externally, they do not need to be reset to the initial state and restarted. After a sufficient transient, which is dominated by the time it takes the external drive to fill the system to a density near $\zeta_c^{\text{SOC}}$, the observables fluctuate about an average value.

Typical SOC observables include the avalanche size, defined as the number of topplings, and the duration, defined as the number of microscopic time units that elapse during an avalanche.

Studies of AS systems typically focus on stationary and dynamic properties more commonly studied in critical phenomena, such as the finite-size scaling of the order parameter and its fluctuations, survival probabilities, and spreading of activity. These observables will be referred to as AS observables.

**SIMILARITIES BETWEEN AS AND SOC**

Given the claim that corresponding observables scale identically under SOC and AS conditions, surprisingly few direct comparisons exist. Specifically, we are not aware of any studies of critical scaling (in the tuning parameter) of observables under SOC conditions. However, it has been reported that

- the asymptotic ($L \to \infty$) particle density in an SOC system coincides with the critical density of the corresponding AS system [11][12][13][14].

- the avalanche sizes measured in an AS system, that is, the response of a finite quiescent AS system at the critical particle density to the addition of a particle, obey the same finite-size scaling as avalanche sizes in bulk-driven SOC systems [11][14].

- scaling relations between avalanche exponents describing SOC observables and exponents characterising AS systems [11][13][15][16] are apparently valid. This implies that $\langle \rho_a(L) \rangle_{\text{SOC}}$ follows the same scaling as $\langle \rho_a(\zeta_c; L) \rangle_{\text{AS}}$.

**DIFFERENCES BETWEEN AS AND SOC**

These similarities are remarkable because the ensemble sampled by SOC systems is very different from that sampled by AS systems.
Our main finding is illustrated in Fig. 1. Appropriate rescaling with \( L \) of \( \langle \rho_a(\zeta; L) \rangle_{\text{AS}} \) and \( \zeta \) collapses the curves \( \langle \rho_a(\zeta; L) \rangle_{\text{AS}} \). The same scaling parameters produce a collapse of the curves \( \langle \rho_a(\zeta; L) \rangle_{\text{SOC}} \), but the two collapsed curves differ. The AS activity hence follows

\[
\langle \rho_a(\zeta; L) \rangle_{\text{AS}} = a^{\text{AS}}_\rho L^{-\beta/\nu_L} G^{\text{AS}}(a^{\text{AS}}_\zeta (\zeta - \zeta_c) L^{1/\nu_L})
\]  

(1)

where \( a^{\text{AS}}_\rho \) denotes metric factors \[17\], \( G^{\text{AS}} \) a scaling function and \( \nu_L \) and \( \beta \) are the usual \[4\] critical exponents. The SOC activity follows

\[
\langle \rho_a(\zeta; L) \rangle_{\text{SOC}} = a^{\text{SOC}}_\rho L^{-\beta/\nu_L} G^{\text{SOC}}(a^{\text{SOC}}_\zeta (\zeta - \zeta_c) L^{1/\nu_L})
\]  

(2)

with numerically identical constants as in eq. (2). Crucially, however, the scaling functions \( G^{\text{SOC}} \) and \( G^{\text{AS}} \) are fundamentally different, with \( \langle \rho_a(\zeta; L) \rangle_{\text{SOC}} \) having no powerlaw pickup. In other words, the ensemble probed in the SOC regime lacks the one key feature that motivates the AS mechanism.

Since the scaling parameters are identical under SOC and AS conditions, our findings do not contradict the many observations of valid scaling relations between SOC and AS exponents. But our findings call into question the physically appealing narrative of the AS mechanism, where an SOC system is pushed through an AS phase transition, into the active phase by a slow drive and strongly repelled from there as a sudden increase in activity above \( \zeta_c \) leads to dissipation. This sudden increase does not exist in SOC. Our critique \[18\] was based on the disbelief that a linear feedback loop like the equation of motion eq. [3] could drive a nonlinear system like the Oslo model to a (highly non-linear) phase transition. Fig. 1 specifies the discord between SOC and AS, invalidating the key proposition \( \langle \rho_a \rangle_{\text{SOC}}(\zeta; L) = \langle \rho_a \rangle_{\text{AS}}(\zeta; L) \) underlying the AS mechanism.

Neither the order parameter nor its variance (not shown) have special features near the critical particle density \( \zeta_c \). Whereas a non-linear \( \zeta \)-dependence is clearly seen under AS conditions, \( \langle \rho_a(\zeta; L) \rangle_{\text{AS}} \), over the range of \( \zeta \) sampled by the SOC system, \( \langle \rho_a(\zeta; L) \rangle_{\text{SOC}} \) increases approximately linearly with \( \zeta \).

The distribution of the SOC system’s tuning parameter \( P(\zeta_{\text{SOC}}(t); L) \) is well described by a Gaussian. Fig. 2 shows its width \( \sigma_{\text{SOC}}(\zeta) \), which is a measure for the apparent critical region in the SOC model. It is found to shrink as \( L^{-1/\nu_L} \), which is the minimum rate required for universal finite size scaling in the light of the AS mechanism \[18\].

**DISCUSSION**

The activity \( \langle \rho_a(\zeta) \rangle_{\text{SOC}} \) in the SOC model does not signal the onset of a phase transition around \( \zeta_c \) by a kink.

**FIG. 1:** Rescaled order parameter versus rescaled reduced density (circles: \( L = 256 \), squares: \( L = 1024 \), triangles: \( L = 4096 \)). The dependence on \( \zeta \) of the SOC order parameter \( \langle \rho_a \rangle_{\text{SOC}}(\zeta; L) \) (open symbols), looks very different from the AS order parameter \( \langle \rho_a \rangle_{\text{AS}}(\zeta; L) \) (filled symbols). The same scaling parameters, \( \beta = 0.26, \nu_L = 1.33 \) \[19\], and \( \zeta_c = 1.73260 \) \[14\] are used for SOC systems and for AS systems. The active phase can be seen in AS mode as a power-law pick-up in the scaling function but is invisible in SOC mode. For the extreme values of \( \zeta \) reached by the SOC system, some error bars are curiously small, which is due to small sample sizes at extreme parameter values.

**FIG. 2:** The width of the distribution \( P(\zeta_{\text{SOC}}(t); L) \) as a function of system size \( L \) shrinks as \( L^{-1/\nu_L} \), enabling the observation of universal (AS-) exponents in the SOC ensemble. The value \( \nu_L = 1.33 \) was used for the rescaling in this figure. The re-scaled quantity varies by less than 4% between \( L = 256 \) and \( L = 65,536 \).

This seems to contradict the widely accepted explanation of how the SOC system finds the critical point.

The comparatively suppressed activity for \( \zeta > \zeta_c \) under SOC conditions confirms that the difference between the ensembles is significant: In order for the SOC system to advance deep into the high-\( \zeta \) region, low dissipation and, presumably, low activity are needed. The particle density
\( \zeta \) under SOC conditions thus has a different effect from \( \zeta \) under AS conditions.

Scaling laws relating SOC and AS exponents are based on the assumption that moments \( \langle \rho_a^T(L) \rangle_{SOC} \) and \( \langle \rho_a^T(\zeta; L) \rangle_{AS} \) follow the same scaling. The validity of such scaling laws has been confirmed repeatedly in the literature [14][20]. We have added to these studies the direct observation of identical scaling of \( \langle \rho_a(\zeta; L) \rangle_{SOC} \) and \( \langle \rho_a(\zeta; L) \rangle_{AS} \), i.e., over a range of \( \zeta \), but we have also shown where the equivalence between corresponding AS and SOC models breaks down.

Given the clear difference between the AS and SOC behavior of such a defining observable as the order parameter near criticality—why do we observe identical finite-size scaling and, even more puzzlingly, why identical critical densities? The ideal theory answering this question includes a recipe for turning any phase transition into SOC—i.e., it is only possible for some “special” AS transitions (maybe there is only one such universality class), the theory has to explain this restriction, and clarify how much of our understanding of phase transitions applies to AS transitions at all.

The AS mechanism of self-organization, as formulated by Dickman et al. can be paraphrased in a mean-field equation of motion for the tuning parameter [3],

\[
\frac{\partial \tilde{\zeta}}{\partial t} = h(t, L) - \epsilon(t, L)\rho^a_{SOC}(t, \tilde{\zeta}; L),
\]

where \( h(t; L) \) is the rate of increase in \( \zeta \), measured on the microscopic time scale, due to the addition of particles, and \( \epsilon(t; L)\rho^a_{SOC}(t, \tilde{\zeta}; L) \) is the loss rate. Even if one accepts that such a simple feedback loop can drive a system into a phase transition, the models we investigated (Oslo and Manna 1-d and 2-d) do not seem to be governed by its supposed equation of motion, eq. [3], because the detailed behavior of \( \rho^a_{SOC} \) in this equation cannot be equated to that of \( \rho^a_{AS} \).

Because it is so widely accepted in the literature that SOC probes a corresponding AS phase transition [5][11][12][21], it is often difficult to see which results are based on this assumption and which support it independently. We emphasize that the scaling relations between AS and SOC are not an explanation of the equivalence between AS and SOC; their apparent validity constitutes the observation of this equivalence that still needs to be understood.

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