Non-Gaussian Estimation of Nonlinear Continuous-discrete Models: Application of Ensemble Kalman Filter

Masaya Murata, Kaoru Hiramatsu
NTT Communication Science Laboratories
3-1, Morinosato Wakamiya, Atsugi-Shi, Kanagawa, 243-0198 Japan
E-mail: murata.masaya@lab.ntt.co.jp

Abstract
Ensemble Kalman filter, representing a sub-optimal non-Gaussian filter, for nonlinear continuous-discrete models is investigated. We formulate the filtering algorithm based on multiple distribution estimation and a bank of extended Kalman filters. The simulation study on satellite re-entry is also provided.

1 Statement of Problem
In this paper, we consider the following nonlinear continuous-discrete model:

\[ dx(t) = f(x(t))dt + Gd\beta(t) \]
\[ y(t_k) = h(x(t_k)) + \omega(t_k) \]

Here, \( x(t) \) and \( y(t_k) \) are arbitrary dimensional state at time \( t \) and observation at discrete time \( t_k \). \( \beta(t) \) and \( \omega(t_k) \) are arbitrary dimensional Brownian motion with mean zero and covariance matrix \( H \) and white Gaussian observation noise with mean zero and covariance matrix \( R \). The elements of \( \beta(t) \) are independent from each other. \( \beta(t) \) and \( \omega(t_k) \) are assumed as independent from each other and also from the initial state \( x(0) \). Equation (1) is the Itô’s stochastic differential equation (SDE). The filtering problem is to estimate \( p(x(t_k)|Y(t_k)) \) where \( Y(t_k) = \{y(t_1), y(t_2), \ldots, y(t_k)\} \).

The paper is organized as follows. We explain optimal and sub-optimal non-Gaussian filters in Section 2. The application of ensemble Kalman filter (EnKF), which is a representative sub-optimal non-Gaussian filter, to the problem of interest is explained in Section 3 and the simulation study on satellite re-entry is shown in Section 4. Section 5 concludes this paper.

2 Non-Gaussian estimation
2.1 Approximation of Optimal Filter
The \( p(x(t_k)|Y(t_k)) \) for Eqs. (1) is (2) is generally non-Gaussian and calculated as follows:

\[ p(x(t_k)|Y(t_k)) = \frac{p(y(t_k)|x(t_k))p(x(t_k)|Y(t_k-1))}{\int p(y(t_k)|x(t_k))p(x(t_k)|Y(t_k-1))dx(t_k)} \]

Therefore, when the \( p(x(t_k)|Y(t_k-1)) \) is approximated as

\[ p(x(t_k)|Y(t_k-1)) \approx \frac{1}{N} \sum_{i=1}^{N} \delta(x(t_k) - x(t_k)^{(i)}) \]

here, \( \delta(\cdot) \) is the Dirac’s delta function and \( x(t_k)^{(i)} \) is the \( i \)th realization of \( x(t_k) \) given \( Y(t_k-1) \), the \( p(x(t_k)|Y(t_k)) \) in Eq. (3) can be also approximated by the mixture of \( N \) Dirac’s delta functions. This is the simplest algorithm of a particle filter (PF) called the bootstrap filter or Monte Carlo filter.

2.2 Approximation of Sub-optimal Filter
Another approach is based on the linear optimal filter (LOF) which provides the state estimate and the estimation error covariance matrix that are optimal in the family of linear filters. It is well-known that when the \( p(x(t_k)|Y(t_k-1)) \) and \( p(x(t_k)|Y(t_k)) \) are both Gaussian, that is, when Eqs. (1) and (2) are both linear in the state and \( x(0) \) is also Gaussian, the LOF becomes the optimal filter. Therefore, the LOF is sub-optimal for the problem of interest.

The EnKF approximates the LOF[1]. The EnKF equations using \( N \) ensembles are

\[ \bar{x}(t_k) = 1/N \sum_{j=1}^{N} x(t_k)^{(j)} \]
\[ \bar{y}(t_k)^{(i)} = h(x(t_k)^{(i)}) \]
\[ \bar{y}(t_k)^{(i)} = y(t_k)^{(i)} + \omega(t_k)^{(i)} \]
\[ \bar{y}(t_k)^{(i)} = \frac{1}{N} \sum_{j=1}^{N} y(t_k)^{(j)} \]
\[ \bar{x}(t_k)^{(j)} = x(t_k)^{(j)} - \bar{x}(t_k) \]
\[ \bar{y}(t_k)^{(j)} = y(t_k)^{(j)} - \bar{y}(t_k) \]
\[ U(t_k) = \frac{1}{N-1} \sum_{j=1}^{N} \bar{x}(t_k)^{(j)}(\bar{y}(t_k)^{(j)})^T \]
\[ V(t_k) = \frac{1}{N-1} \sum_{j=1}^{N} \bar{y}(t_k)^{(j)}(\bar{y}(t_k)^{(j)})^T \]
\[ K(t_k) = U(t_k)V(t_k)^{-1} \]
\[ \bar{x}(t_k)^{(j)} = x(t_k)^{(j)} + K(t_k)(y(t_k) - y(t_k)^{(i)}) \]
here, \( x(t_k)^{(j)} \) is the \( j \)th realization of \( x(t_k)|\mathcal{Y}(t_{k-1}) \). Then the ensemble average of \( \hat{x}(t_k)^{(j)} \) is
\[
\frac{1}{N} \sum_{j=1}^{N} \hat{x}(t_k)^{(j)} = \bar{x}(t_k) + K(t_k)(y(t_k) - \hat{y}(t_k))
\] (6)
and as \( N \) becomes sufficiently large, Eq. (6) asymptotically approaches the state estimate of the LOF. The estimation error covariance matrix is
\[
\frac{1}{N-1} \sum_{j=1}^{N} (\hat{x}(t_k)^{(j)} - \bar{x}(t_k))(\hat{x}(t_k)^{(j)} - \bar{x}(t_k))^T = \frac{1}{N-1} \sum_{j=1}^{N} m(t_k)^{(j)}
\] (7)
where \( \bar{x}(t_k) = \frac{1}{N} \sum_{j=1}^{N} \hat{x}(t_k)^{(j)} \). The \( m(t_k)^{(j)} \) is
\[
(\bar{x}(t_k)^{(j)} - K(t_k)(y(t_k)^{(j)} - \hat{y}(t_k) + \omega(t_k)^{(j)})) \times
(\bar{x}(t_k)^{(j)} - K(t_k)(y(t_k)^{(j)} - \hat{y}(t_k) + \omega(t_k)^{(j)}))^T
\]
and there are four terms in the above equation. The first term approaches the covariance matrix for the \( x(t_k)|\mathcal{Y}(t_{k-1}) \) as the \( N \) becomes sufficiently large. Under the same condition, the second, third, and fourth terms approach the following matrices:
\[
-\bar{U}(t_k)K(t_k)^T, \quad -K(t_k)\bar{U}(t_k)^T, \quad \bar{U}(t_k)K(t_k)^T
\]
Therefore, Eq. (7) asymptotically approaches
\[
\bar{P}(t_k) = K(t_k)\bar{U}(t_k)^T
\] (8)
which is the state estimation error covariance matrix of the LOF. The filtered ensembles of the EnKF (Eq. (5)) thus approximate the state estimation results of the LOF by the ensemble averages (by the first two moments of the distribution for the filtered ensembles).

Therefore, under the condition of sufficiently large \( N \) and that the ensemble averages of the \( x(t_k)^{(j)}, (j = 1, 2, \cdots, N) \) well approximate the expectation and covariance matrix for the \( x(t_k)|\mathcal{Y}(t_{k-1}) \), the first two moments of the distribution for the filtered ensembles become almost identical to the state estimation results of the LOF.

Note that the Gaussianity is not imposed on the \( x(t_k)|\mathcal{Y}(t_{k-1}) \) and the distribution of \( x(t_k)^{(j)}, (j = 1, 2, \cdots, N) \) is generally asymmetric, that is, non-Gaussian. Then, the distribution of \( \hat{x}(t_k)^{(j)}, (j = 1, 2, \cdots, N) \) also becomes non-Gaussian and only the first two moments are guaranteed in the aforementioned LOF sense. The advantage of the EnKF is that the algorithm is completely liberated from the particle degeneracy problem that terribly hurts the filtering accuracy of the PF family. The EnKF is also generally much faster than the PF family since a computationally heavy particle re-sampling is not necessary.

3 Design of EnKF for Eqs. (1) and (2)

3.1 Standard Approach

We apply the EnKF to the problem of interest. As explained in the previous section, the EnKF first requires \( x(t_k)^{(j)}, (j = 1, 2, \cdots, N) \). It can be obtained from the following equation:
\[
x(t_k)^{(j)} = x(t_{k-1})^{(j)} + \left( \int_{t_{k-1}}^{t_k} f(x(t)) dt \right)^{(j)}
\]
+ \( G \left( \int_{t_{k-1}}^{t_k} d\beta(t) \right)^{(j)} \) (9)

Here, \( j \) denotes the \( j \)th realization of the random variable. Although the third term is easy to calculate (it is \( G\Delta\beta \sim N(0, \Delta GG^T) \) where \( \Delta = t_k - t_{k-1} \)), we need to discretize the second term in the stochastic manner.

Neglecting all of the derivative terms in the Itô-Taylor series expansion of the \( f(x(t)) \) around \( x(t_{k-1}) \) yields \( f(x(t_{k-1})) \) and using this approximation makes the second term \( \Delta f(x(t_{k-1})) \). The resulting discretized equation is equivalent to the EM: Euler-Maruyama method. The EM method is simple but obviously the approximation accuracy is very limited. Leaving the first-order derivatives in the Itô-Taylor series makes the integrand of the second term as follows:
\[
f(x(t_{k-1})) + \int_{t_{k-1}}^{t_k} \frac{\partial f(x(t))}{\partial x(t)} \bigg|_{x(t) = \bar{x}(s)} ds + \int_{t_{k-1}}^{t_k} \frac{\partial f(x(t))}{\partial x(t)} \bigg|_{x(t) = \bar{x}(s)} G d\beta(s)
\] (10)

Although using Eq. (10) is theoretically more accurate than the EM, it is generally much more complicated and we prefer to use the EM as the stochastic discretization method.

3.2 Proposed Approach

The \( p(x(t_k)|\mathcal{Y}(t_{k-1})) \) is calculated as follows:
\[
p(x(t_k)|\mathcal{Y}(t_{k-1})) = \int p(x(t_k), x(t_{k-1})|\mathcal{Y}(t_{k-1})) dx(t_{k-1})
\]
\[
= \int p(x(t_k)|x(t_{k-1})) p(x(t_{k-1})|\mathcal{Y}(t_{k-1})) dx(t_{k-1})
\]
\[
\approx \frac{1}{N} \sum_{j=1}^{N} p(x(t_k)|x(t_{k-1})^{(j)})
\] (11)

Equation (11) shows the multiple distribution estimation for the \( p(x(t_k)|\mathcal{Y}(t_{k-1})) \). When we assume that the \( p(x(t_k)|x(t_{k-1})^{(j)}), (j = 1, 2, \cdots, N) \) is Gaussian, each distribution can be calculated by extended Kalman filter (EKF) with \( N \) different initial states \( x(t_{k-1})^{(j)}, (j = 1, 2, \cdots, N) \). Moreover, since the EKF is the fastest Gaussian filter for the problem of interest, we prefer to use the EKF for the estimation of each distribution.
After obtaining the mixture of the N Gaussian distributions estimated by the EKFs, we sample one state realization from each distribution and regard that the total of N realizations approximate the \( p(x(t_k)|Y(t_{k-1})) \) as shown below.

\[
p(x(t_k)|Y(t_{k-1})) \approx \frac{1}{N} \sum_{j=1}^{N} \delta(x(t_k) - x(t_k)^{(j)}),
\]

where \( x(t_k)^{(j)} \sim N(\bar{x}(t_k)^{(j)}, P(t_k)^{(j)}) \).

Here, \( N(\bar{x}(t_k)^{(j)}, P(t_k)^{(j)}) \) is the estimated Gaussian distribution by the jth EKF and \( \sim \) indicates the sampling. The advantage of this approach is the utilization of the EKFs. Since the EKF equations are deterministic ordinary differential equations (ODEs), the integration of the EKFs is straightforward. Considering the tradeoff between accuracy and complexity, we prefer to use the fourth-order Runge-Kutta method (RK) to integrate the EKFs (ODEs) by discretizing using the EM scheme with 100 steps between each observation to generate the simulation data.

### 4 Simulation Study

#### 4.1 Model and Data

The satellite re-entry model is often used as a benchmark simulation problem for nonlinear filters[2][3]. The system model is formally expressed as follows:

\[
\begin{align*}
\dot{x}_1 &= x_4 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= Dx_3 + Gx_1 + \omega_1 \\
\dot{x}_4 &= Dx_4 + Gx_2 + \omega_2 \\
\dot{x}_5 &= \omega_3 \\
D &= b \cdot \exp \left( \frac{r_0 - r}{b_0} \right) v \\
b &= b_0 \exp(x_5) \\
r &= \sqrt{x_1^2 + x_2^2} \\
v &= \sqrt{x_3^2 + x_4^2} \\
G &= -\frac{\mu}{r^3} \\
r_0 &= 6374, \quad b_0 = -0.59783 \\
b_0 &= 13.406, \quad \mu = 3.9860 \times 10^5
\end{align*}
\]

The units are kilometers, kilograms, and seconds. Here, \( D \) is the drag force, \( G \) is the gravitational force from the earth (\( \mu \) is the gravitational parameter of the earth), and \( \omega_i, (i = 1, 2, 3) \) are zero-mean, delta-correlated Gaussian processes with joint spectral density \( Q = \text{diag}(2.4064 \times 10^{-4}, 2.4064 \times 10^{-4}, 0) \). \( x_5 \) is the aerodynamic property of the satellite. The true initial state is Gaussian whose expectation and covariance matrix are \( x(0) = [6500.4, 349.14, -1.8093, -6.7967, 0.6932]^T \) and \( P_0 = \text{diag}(10^{-6}, 10^{-6}, 10^{-6}, 10^{-6}, 0) \).

The radar is located at \((r_0, 0)\) and the observation model is

\[
\begin{align*}
r(t_k) &= \sqrt{(x_1(t_k) - r_0)^2 + x_2(t_k)^2} + \omega(t_k)_1 \\
\theta(t_k) &= \tan^{-1} \frac{x_2(t_k)}{x_1(t_k) - r_0} + \omega(t_k)_2
\end{align*}
\]

Here, \( \omega(t_k)_j, (j = 1, 2) \) are zero-mean, white Gaussian noises with joint covariance matrix \( R(t_k) = \text{diag}(10^{-6}, (0.01\pi/180)^2) \). \( t_k, (k = 1, 2, \cdots) \) is set to \( 0.1k(s) \) and the simulation duration is 250 seconds. The system model is re-expressed by the Itô’s SDE and discretized using the EM scheme with 100 steps between each observation to generate the simulation data.

#### 4.2 Filter Setting and Execution

The initial filtered state PDF is Gaussian, having \( \hat{x}(0) = [6500.4, 349.14, -1.8093, -6.7967, 0]^T \) and \( \hat{P}_0 = \text{diag}(10^{-6}, 10^{-6}, 10^{-6}, 10^{-6}, 1) \). For the filter execution, the \( \omega_i, (i = 1, 2, 3) \) is set to having \( Q = \text{diag}(2.4064 \times 10^{-4}, 2.4064 \times 10^{-4}, 10^{-5}) \) to simultaneously estimate the satellite state and the unknown aerodynamic property. The observation noise filtered is exactly the same as that used for simulating the data.

Non-Gaussian filters compared are the EnKF1 in Section 3.1 and the EnKF2 in Section 3.2. The EKFs for the EnKF2 are integrated by the fourth-order RK using 10 steps between each observation, while the SDEs for the EnKF1 are integrated by the EM with 10 steps between each observation.

#### 4.3 Results

Table 1 shows the following root mean square errors (RMSEs) evaluated by 100 Monte Carlo runs:

| \( \text{RMSE1}(t_k) = \sqrt{\mathbb{E}[(x_1(t_k) - \hat{x}_1(t_k))^2 + (x_2(t_k) - \hat{x}_2(t_k))^2]} \) |
| \( \text{RMSE2}(t_k) = \sqrt{\mathbb{E}[(x_3(t_k) - \hat{x}_3(t_k))^2]} \) |

Here, \( \hat{x}_s(t_k) (s = 1, 2, 5) \) is the corresponding element of the filtered state estimate at discrete time \( t_k \). The RMSEs further averaged over the simulation duration are shown in the table. The ensemble number \( N \) was varied from 1000 to 5000 with 1000 intervals to investigate the improvement in the filtering accuracy.

| \( \) \( 500 \) | \( 1000 \) | \( 3000 \) | \( 5000 \) | \( 10000 \) |
| --- | --- | --- | --- | --- |
| \( \text{EnKF1} \) | 0.0057 | 0.0077 | 0.0074 | 0.0075 | 0.0071 |
| \( \text{EnKF2} \) | 0.0077 | 0.0077 | 0.0074 | 0.0077 | 0.0071 |

Table 1: Simulation results. The first row is \( N \). The second and third rows denote the averaged RMSE1 and RMSE2 for EnKF1. The third and fourth rows denote those for EnKF2.
From this table, the averaged RMSE$_1$s and RMSE$_2$s are almost the same for all of the filter settings (approximately 7 meter error for the estimation of satellite position and 0.07 error for the aerodynamic property). The values improve as $N$ increases for both filters. We also evaluated the EnKF1 and EnKF2 using 100 ensembles and the RMSE$_2$s were considerably worse than those shown in table 1. The EnKF1 and EnKF2 using 10 ensembles did not work well due to a numerical problem. Therefore, for the simulation problem, we observe that the EnKF1 and EnKF2 both show satisfactory filtering results using the relatively small number of ensembles such as $N > 3000$.

5 Conclusion

We investigated the EnKF for nonlinear continuous-discrete problems and proposed the algorithm based on the multiple distribution estimation. Since the proposed approach is a bank of EKFs, high-order deterministic integration techniques are applicable. For the simulation problem, the filtering results for the standard and proposed EnKF algorithms were almost identical and the use of relatively small ensemble numbers guaranteed the reasonable state estimates.

References

[1] M. Murata, K. Hiramatsu, “On ensemble Kalman filter, particle filter, and Gaussian particle filter,” in Transactions of the Institute of Systems, Control and Information Engineers, vol.29, no.10, pp.448-462, 2016.

[2] S. Sarkka, “On unscented Kalman filtering for state estimation of continuous-time nonlinear systems.” IEEE Transactions on Automatic Control, vol.52, no.9, pp.1631-1641, 2007.

[3] K. Berntorp, “Feedback particle filter: application and evaluation,” in Proc. the 18th International Conference on Information Fusion, Washington, D.C., U.S.A., July 6-9, pp.1633-1640, 2015.