1. Introduction

A simplicial poset \( P \) (also called a boolean poset and a poset of boolean type) is a finite poset with a smallest element \( \hat{0} \) such that every interval \([\hat{0}, y]\) for \( y \in P \) is a boolean algebra, i.e., \([\hat{0}, y]\) is isomorphic to the set of all subsets of a finite set, ordered by inclusion. The set of all faces of a (finite) simplicial complex with empty set added forms a simplicial poset ordered by inclusion, where the empty set is the smallest element. Such a simplicial poset is called the face poset of a simplicial complex, and two simplicial complexes are isomorphic if and only if their face posets are isomorphic. Therefore, a simplicial poset can be thought of as a generalization of a simplicial complex.

Although a simplicial poset \( P \) is not necessarily the face poset of a simplicial complex, it is always the face poset of a CW-complex \( \Gamma(P) \). In fact, to each \( y \in P \setminus \{\hat{0}\} = \overline{P} \), we assign a (geometrical) simplex whose face poset is \([\hat{0}, y]\) and glue those geometrical simplices according to the order relation in \( P \). Then we get the CW-complex \( \Gamma(P) \) such that all the attaching maps are inclusions. For instance, if two simplices of a same dimension are identified on their boundaries via the identity map, then it is not a simplicial complex but a CW-complex obtained from a simplicial poset. The CW-complex \( \Gamma(P) \) has a well-defined barycentric subdivision which is isomorphic to the order complex \( \Delta(\overline{P}) \) of the poset \( \overline{P} \). Here \( \Delta(\overline{P}) \) is a simplicial complex on the vertex set \( \overline{P} \) whose faces are the chains of \( \overline{P} \).

We say that \( y \in P \) has rank \( i \) if the interval \([\hat{0}, y]\) is isomorphic to the boolean algebra of rank \( i \) (in other words, the face poset of an \((i-1)\)-simplex), and the rank of \( P \) is defined to be the maximum of ranks of all elements in \( P \). Let \( d = \text{rank} \ P \).

In exact analogy to simplicial complexes, the \( f \)-vector of the simplicial poset \( P \), \((f_0, f_1, \ldots, f_{d-1})\), is defined by

\[
f_i = f_i(P) = \# \{ y \in P \mid \text{rank} \ y = i + 1 \}
\]

and the \( h \)-vector of \( P \), \((h_0, h_1, \ldots, h_d)\), is defined by the following identity:

\[
\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i},
\]
where \( f_{-1} = 1 \), so \( h_0 = 1 \). Note that the number of facets of \( P \), that is \( f_{d-1} \), is related to \( h \)-vectors as follows:

\[
(1.1) \quad f_{d-1} = \sum_{i=0}^{d} h_i.
\]

When \( P \) is the face poset of a simplicial complex \( \Sigma \), the \( f \)- and \( h \)-vector of \( P \) coincide with the classical \( f \)- and \( h \)-vector of the simplicial complex \( \Sigma \) respectively.

\( f \)-vectors and \( h \)-vectors have equivalent information, but \( h \)-vectors are often easier than \( f \)-vectors. In [8], R. Stanley discussed characterization of \( h \)-vectors for certain classes of simplicial posets. For example, he proved that a vector \((h_0, h_1, \ldots, h_d)\) of integers with \( h_0 = 1 \) is the \( h \)-vector of a Cohen-Macaulay simplicial poset of rank \( d \) if and only if \( h_i \geq 0 \) for all \( i \). Gorenstein* simplicial posets are more special than Cohen-Macaulay simplicial posets. If the CW-complex \( \Gamma(P) \) is homeomorphic to a sphere of dimension \( d-1 \), then the simplicial poset \( P \) of rank \( d \) is Gorenstein* (see Section 5 for more details). It is known that \( h \)-vectors of Gorenstein* simplicial posets satisfy Dehn-Sommerville equations \( h_i = h_{d-i} \) for all \( i \), in addition to the non-negativity conditions \( h_i \geq 0 \). In this paper we will prove that \( h \)-vectors of Gorenstein* simplicial posets must satisfy one more subtle condition conjectured by Stanley in [8], see [1], [5], [8] for partial results.

**Theorem 1.1.** If \( P \) is a Gorenstein* simplicial poset of rank \( d \) and \( h_i(P) = 0 \) for some \( i \) between 0 and \( d \), then \( \sum_{i=0}^{d} h_i(P) \), that is the number of facets of \( P \) by (1.1), is even.

Combining this with Theorem 4.3 in [8], one completes characterization of \( h \)-vectors of Gorenstein* simplicial posets.

**Corollary 1.2.** Let \((h_0, h_1, \ldots, h_d)\) be a vector of non-negative integers with \( h_i = h_{d-i} \) for all \( i \) and \( h_0 = 1 \). There is a Gorenstein* simplicial poset \( P \) of rank \( d \) with \( h_i(P) = h_i \) for all \( i \) if and only if either \( h_i > 0 \) for all \( i \), or else \( \sum_{i=0}^{d} h_i \) is even.

Our proof of Theorem 1.1 is purely algebraic but the idea stems from topology, so we will explain how our proof is related to topology in Section 2. A main tool to study the \( h \)-vector of a simplicial poset \( P \) is a (generalized) face ring \( A_P \) introduced in [8] of the poset \( P \). In Section 3 we discuss restriction maps from \( A_P \) to polynomial rings. In Section 4 we construct a map called an index map from \( A_P \) to a polynomial ring. Theorem 1.1 is proven in Section 5.

I am grateful to Takayuki Hibi for informing me of the above problem and for his interest. I am also grateful to Akio Hattori and Taras Panov for the successful collaborations ([4], [5]) from which the idea used in this paper originates. Finally I am grateful to Ezra Miller for his comments on an earlier version of the paper, which were very helpful to improve the paper.
2. Relation to topology

In the toric geometry, simplicial convex polytopes are closely related to toric manifolds or orbifolds (see [2]). Similarly to this, Gorenstein* simplicial posets, which contain the boundary complexes of simplicial polytopes as examples, are closely related to objects (in topology) called torus manifolds or orbifolds (see [4], [5]), and the proof of Theorem 1.1 is motivated by a topological observation described in this section. Here a torus manifold (resp. orbifold) means a closed smooth manifold (resp. orbifold) of dimension $2d$ with an effective smooth action of a $d$-dimensional torus group having at least one fixed point.

We shall illustrate relations between combinatorics and topology with simple examples. In the following, $T$ will denote the product of $d$ copies of the circle group consisting of complex numbers with unit length, i.e., $T$ is a $d$-dimensional torus group.

Example 2.1. A complex projective space $\mathbb{C}P^d$ has a $T$-action defined in the homogeneous coordinates by

$$(t_1, \ldots, t_d) \cdot (z_0 : z_1 : \cdots : z_d) = (z_0 : t_1 z_1 : \cdots : t_d z_d).$$

The orbit space $\mathbb{C}P^d / T$ has a natural face structure. Its facets are the images of (real) codimension two submanifolds $z_i = 0$ ($i = 0, 1, \ldots, d$) under the quotient map $\mathbb{C}P^d \rightarrow \mathbb{C}P^d / T$. The map (called a moment map)

$$(z_0 : z_1 : \cdots : z_d) \mapsto \frac{1}{\sum_{i=0}^{d} |z_i|^2} (|z_1|^2, \ldots, |z_d|^2)$$

induces a face preserving homeomorphism from the orbit space $\mathbb{C}P^d / T$ to a standard $d$-simplex. The face poset of $\mathbb{C}P^d / T$ ordered by reverse inclusion (so $\mathbb{C}P^d / T$ itself is the smallest element) is the face poset of a simplicial complex of dimension $d - 1$ and Gorenstein*.

Similarly, the product of $d$ copies of $\mathbb{C}P^1$ admits a $T$-action, the orbit space $(\mathbb{C}P^1)^d / T$ is homeomorphic to a $d$-cube, and the face poset of $(\mathbb{C}P^1)^d / T$ ordered by reverse inclusion is also the face poset of a simplicial complex of dimension $d - 1$ and Gorenstein*.

In any case, the orbit space is a simple convex polytope and its polar is a simplicial convex polytope. The Gorenstein* simplicial complex is the boundary complex of the simplicial convex polytope.

Example 2.2. Let $S^{2d}$ be the $2d$-sphere identified with the following subset in $\mathbb{C}^d \times \mathbb{R}$:

$$\{(z_1, \ldots, z_d, y) \in \mathbb{C}^d \times \mathbb{R} \mid |z_1|^2 + \cdots + |z_d|^2 + y^2 = 1\},$$

and define a $T$-action on $S^{2d}$ by

$$(t_1, \ldots, t_d) \cdot (z_1, \ldots, z_d, y) = (t_1 z_1, \ldots, t_d z_d, y).$$
The facets in the orbit space \( S^{2d}/T \) are the images of codimension two submanifolds \( z_i = 0 \) \((i = 1, \ldots, d)\) under the quotient map \( S^{2d} \to S^{2d}/T \), and the map

\[
(z_1, \ldots, z_d, y) \to (|z_1|, \ldots, |z_d|, y)
\]

induces a face preserving homeomorphism from \( S^{2d}/T \) to the following subset of the \( d \)-sphere:

\[
\{(x_1, \ldots, x_d, y) \in \mathbb{R}^{d+1} \mid x_1^2 + \cdots + x_d^2 + y^2 = 1, \ x_1 \geq 0, \ldots, x_d \geq 0\}.
\]

The orbit space \( S^{2d}/T \) is not (isomorphic to) a simple convex polytope because the intersection of \( d \) facets consists of two points, but it is a manifold with corners and every face (even \( S^{2d}/T \) itself) is acyclic. The face poset of \( S^{2d}/T \) ordered by reverse inclusion is not the face poset of a simplicial complex when \( d \geq 2 \). However, it is a simplicial poset and Gorenstein*. The geometric realization of the face poset of \( S^{2d}/T \) is formed from two \((d-1)\)-simplices by gluing their boundaries via the identity map.

More generally, it is proved in [5] that if a torus manifold \( M \) has vanishing odd degree cohomology, then the orbit space \( M/T \) is a manifold with corners and every face (even \( M/T \) itself) is acyclic; so the face poset of \( M/T \) ordered by reverse inclusion is a Gorenstein* simplicial poset, say \( P \). Moreover, \( h_i(P) \) agrees with the 2\(i\)-th betti number \( b_{2i}(M) \) of \( M \) and the equivariant cohomology ring \( H^*_T(M; \mathbb{Z}) \) of \( M \) is isomorphic to the face ring \( A_P \) of \( P \) (defined over \( \mathbb{Z} \)). Here \( H^*_T(M; k) \) for a ring \( k \) is defined as

\[
H^*_T(M; k) := H^*(ET \times_T M; k)
\]

where \( ET \) is the total space of the universal principal \( T \)-bundle (on which \( T \) acts freely) and \( ET \times_T M \) is the orbit space of the product \( ET \times M \) by the diagonal \( T \)-action.

A projective toric orbifold is related to a simplicial convex polytope as in Example 2.1 and the \( h \)-vector of the simplicial convex polytope agrees with the (even degree) betti numbers of the toric orbifold. Noting this fact, Stanley [7] deduced constraints on the \( h \)-vector by applying the hard Lefschetz theorem to the toric orbifold and completed the characterization of \( h \)-vectors of simplicial convex polytopes. In some sense our proof of Theorem 1.1 is on this line. The topological argument developed below in this section is not complete but would be helpful for the reader to understand what is done in subsequent sections.

Let \( P \) be a Gorenstein* simplicial poset of dimension \( d - 1 \). Looking at the result in [5] mentioned above, it is likely that there exists a torus orbifold \( M \) which have the following properties:

Properties.

1. \( H^{odd}(M; \mathbb{Q}) = 0 \),
2. \( h_i(P) = b_{2i}(M) \),
3. \( H^*_T(M; \mathbb{Q}) \) is isomorphic to \( A_P \) (defined over \( \mathbb{Q} \)).
What we will use to deduce the necessity in Theorem 1.1 is the index map (or evaluation map) in equivariant cohomology:

\[ \text{Ind}_T: H^*_T(M; \mathbb{Q}) \to H^{*-2d}(pt; \mathbb{Q}) = H^{*-2d}(BT; \mathbb{Q}), \]

where \( BT = ET/T \) is the classifying space of principal \( T \)-bundles. The index map is nothing but the Gysin homomorphism in equivariant cohomology induced from the collapsing map \( \pi: M \to pt \). As is well-known, \( BT \) is the product of \( d \) copies of \( \mathbb{C}P^\infty \) (up to homotopy) and \( H^*(BT; \mathbb{Q}) \) is a polynomial ring in \( d \) variables of degree two. The index map \( \text{Ind}_T \) decreases cohomological degrees by \( 2d \) because the dimension of \( M \) is \( 2d \). Moreover, \( H^*_T(M; \mathbb{Q}) \) is a module over \( H^*(BT; \mathbb{Q}) \) through \( \pi^*: H^*(BT; \mathbb{Q}) = H^*_T(pt; \mathbb{Q}) \to H^*_T(M; \mathbb{Q}) \) and \( \text{Ind}_T \) is an \( H^*(BT; \mathbb{Q}) \)-module map. Since \( H^{	ext{odd}}(M; \mathbb{Q}) = 0 \) and \( H^*(BT; \mathbb{Q}) \) is a polynomial ring in \( d \) variables, say \( t_1, \ldots, t_d \), the quotient ring of \( H^*_T(M; \mathbb{Q}) \) by the ideal generated by \( \pi^*(t_1), \ldots, \pi^*(t_d) \) agrees with the ordinary cohomology \( H^*(M; \mathbb{Q}) \). Similarly, the quotient ring of \( H^*_T(pt; \mathbb{Q}) = H^*(BT; \mathbb{Q}) \) by the ideal generated by \( t_1, \ldots, t_d \) agrees with \( H^*(pt; \mathbb{Q}) \). Therefore the index map in equivariant cohomology induces the index map in ordinary cohomology:

\[ \text{Ind}: H^*(M; \mathbb{Q}) \to H^{*-2d}(pt; \mathbb{Q}). \]

This map agrees with the Gysin homomorphism in ordinary cohomology induced from the collapsing map \( \pi \), so it is the evaluation map on a fundamental class of \( M \). Thus, we have a commutative diagram:

\[
\begin{array}{ccc}
H^{2d}_T(M; \mathbb{Q}) & \xrightarrow{\text{Ind}_T} & H^0_T(pt; \mathbb{Q}) = H^0(BT; \mathbb{Q}) = \mathbb{Q} \\
\downarrow & & \downarrow \\
H^{2d}(M; \mathbb{Q}) & \xrightarrow{\text{Ind}} & H^0(pt; \mathbb{Q}) = \mathbb{Q},
\end{array}
\]

where the right vertical map is the identity.

A key thing is to find an element \( \omega_T \) in \( H^{2d}_T(M; \mathbb{Q}) \) such that

(i) \( \omega_T \) is a polynomial in elements of \( H^{2d}_T(M; \mathbb{Q}), \)
(ii) \( \text{Ind}_T(\omega_T) \) is an integer and \( \text{Ind}_T(\omega_T) \equiv \chi(M) \pmod{2} \), where \( \chi(M) \) is the Euler characteristic of \( M \).

We may think of \( \omega_T \) as a “lifting” of the equivariant top Stiefel-Whitney class \( w^{2d}_T(M) \in H^{2d}_T(M; \mathbb{Z}/2) \) of \( M \). If we find such an element \( \omega_T \), then it follows from the commutativity of the above diagram that

\[ \text{Ind}_T(\omega_T) = \text{Ind}(\omega) \]

where \( \omega \) is the image of \( \omega_T \) under the left vertical map in the above diagram.

Now suppose \( h_i(P) = 0 \) for some \( 1 \leq i \leq d - 1 \). Then the \( 2i \)-th betti number \( b_{2i}(M) \) of \( M \) is zero by property (2) and the element \( \omega \) vanishes because it is a polynomial in degree two elements by (i) above, so the right hand side of (2.1) is
zero and \( \chi(M) \) is even by (ii) above. On the other hand, it follows from properties (1) and (2) that

\[
\chi(M) = \sum_{i=0}^{d} b_{2i}(M) = \sum_{i=0}^{d} h_{i}(P).
\]

These prove that \( \sum_{i=0}^{d} h_{i}(P) \) is even.

It turns out that the argument developed above works without assuming the existence of the torus orbifold \( M \). In fact, the face ring \( A_{P} \) takes the place of \( H^{*}_{T}(M; \mathbb{Q}) \) by property (3) and an l.s.o.p. for \( A_{P} \) plays the role of \( \pi^{*}(t_{1}), \ldots, \pi^{*}(t_{d}) \) so that the polynomial ring generated by the l.s.o.p. corresponds to the polynomial ring \( \pi^{*}(H^{*}(BT; \mathbb{Q})) \) (or \( H^{*}(BT; \mathbb{Q}) \) since \( \pi^{*} \) is injective). The index map \( \text{Ind}_{T} \) has an expression (so-called Lefschetz fixed point formula) in terms of local data around \( T \)-fixed points of \( M \), and since the formula is purely algebraic, one can use it to define an “index map” from \( A_{P} \). To carry out this idea, we need to study restriction maps from \( A_{P} \) to polynomial rings because restriction maps to \( T \)-fixed points in equivariant cohomology are involved in the Lefschetz fixed point formula. We will discuss such restriction maps in Section 3 and construct the index map from \( A_{P} \) in Section 4.

3. Restriction maps

In this and next sections, we consider rings over \( \mathbb{Q} \). A main tool to study the \( h \)-vector of a (finite) simplicial poset \( P \) is the face ring \( A_{P} \) of the poset \( P \) introduced by Stanley in [8]. We recall it first.

**Definition.** Let \( P \) be a simplicial poset of rank \( d \) with elements \( \hat{0}, y_{1}, \ldots, y_{p} \). Let \( \bar{A} = \mathbb{Q}[y_{1}, \ldots, y_{p}] \) be the polynomial ring over \( \mathbb{Q} \) in the variables \( y_{i} \) and define \( \mathcal{I}_{P} \) to be the ideal of \( \bar{A} \) generated by the following elements:

\[
y_{i}y_{j} - (y_{i} \wedge y_{j})(\sum z),
\]

where \( y_{i} \wedge y_{j} \) is the greatest lower bound of \( y_{i} \) and \( y_{j} \), \( z \) ranges over all minimal upper bounds of \( y_{i} \) and \( y_{j} \), and we understand \( \sum z = 0 \) if \( y_{i} \) and \( y_{j} \) have no common upper bound. Then the face ring \( A_{P} \) of the simplicial poset \( P \) is defined as the quotient ring \( \bar{A}/\mathcal{I}_{P} \) and made graded

\[
A_{P} = (A_{P})_{0} \oplus (A_{P})_{1} \oplus \ldots
\]

by defining \( \deg y_{i} = \text{rank} y_{i} \). The ring \( A_{P} \) reduces to a classical Stanley-Reisner face ring when \( P \) is the face poset of a simplicial complex.

We denote by \( P_{s} \) the subset of \( P \) consisting of elements of rank \( s \). Elements in \( P_{1} \) will be denoted by \( x_{1}, \ldots, x_{n} \) and called **atoms** in \( P \). The set \( \{x_{1}, \ldots, x_{n}\} \) is a basis of \( (A_{P})_{1} \).
Suppose that \( y \) is an element of \( P_d \). Then the interval \([0, y]\) is a boolean algebra of rank \( d \) and \( A_{[0,y]} \) is a polynomial ring in \( d \) variables. Sending all elements in \( P \) which are not lower than \( y \) to zero, we obtain an epimorphism

\[ t_y : A_P \to A_{[0,y]} \]

Since \( \mathbb{Q} \) is a field with infinitely many elements, \( A_P \) admits an l.s.o.p. \( \theta_1, \ldots, \theta_d \) (see the proof of Theorem 3.10 in [8]). In the following we fix the l.s.o.p. and denote by \( \Theta \) the vector space of dimension \( d \) spanned by \( \theta_1, \ldots, \theta_d \) over \( \mathbb{Q} \), and by \( \mathbb{Q}[\Theta] \) the polynomial ring generated by \( \theta_1, \ldots, \theta_d \). Note that \( \Theta \) is a vector subspace of \( (A_P)_1 \) and \( \mathbb{Q}[\Theta] \) is a subring of \( A_P \).

**Lemma 3.1.** The restriction of \( t_y \) to \( \mathbb{Q}[\Theta] \) is an isomorphism onto \( A_{[0,y]} \).

**Proof.** Since \( A_P \) is finitely generated as a \( \mathbb{Q}[\Theta] \)-module, so is \( A_{[0,y]} \). This implies that \( t_y \) maps the vector space \( \Theta \) isomorphically onto the vector space spanned by \( d \) elements of degree one generating the polynomial ring \( A_{[0,y]} \); thus the lemma follows. \( \square \)

Henceforth, we identify \( A_{[0,y]} \) with \( \mathbb{Q}[\Theta] \) via \( t_y \), and think of \( t_y \) as a map to \( \mathbb{Q}[\Theta] \), i.e.,

\[ t_y : A_P \to \mathbb{Q}[\Theta] \]

Note that \( t_y \) is the identity on the subring \( \mathbb{Q}[\Theta] \) and a \( \mathbb{Q}[\Theta] \)-module map.

For \( w \in P_s \), we set

\[ \mathcal{A}(w) := \{ i \in \{1, \ldots, n\} | x_i \text{ is an atom lower than } w \} \]

The cardinality of \( \mathcal{A}(w) \) is \( s \). Let \( y \in P_d \). By definition of \( t_y \),

\[ t_y(x_i) = 0 \text{ whenever } i \notin \mathcal{A}(y). \quad (3.1) \]

We set

\[ \theta_i(y) := t_y(x_i) \quad \text{for } i \in \mathcal{A}(y). \quad (3.2) \]

Since \( t_y : (A_P)_1 \to \Theta \) is surjective and the cardinality of \( \mathcal{A}(y) \), that is \( d \), agrees with the dimension of \( \Theta \), the set \( \{ \theta_i(y) | i \in \mathcal{A}(y) \} \) is a basis of \( \Theta \).

Let \( z \in P_{d-1} \). Let \( y \) be an element in \( P_d \) above \( z \) and define \( \ell \in \{1, \ldots, n\} \) by

\[ \mathcal{A}(y) \setminus \mathcal{A}(z) = \{ \ell \}. \]

The canonical map \( A_{[0,y]} = \mathbb{Q}[\Theta] \to A_{[0,z]} \) is surjective and \( A_{[0,z]} \) can canonically be identified with \( \mathbb{Q}[\Theta]/(\theta_\ell(y)) \). Let \( y' \) be another element in \( P_d \) above \( z \) and define \( \ell' \in \{1, \ldots, n\} \) similarly to \( \ell \). It may happen that \( \ell = \ell' \). Since

\[ \mathbb{Q}[\Theta]/(\theta_\ell(y)) = A_{[0,z]} = \mathbb{Q}[\Theta]/(\theta_{\ell'}(y')), \]

\( \theta_i(y) \) and \( \theta_i(y') \) are same up to a non-zero scalar multiple; so the following lemma makes sense.

**Lemma 3.2.** \( t_y(\alpha) \equiv t_{y'}(\alpha) \mod \theta_\ell(y) \) for any \( \alpha \in A_P \). In particular, \( \theta_i(y) \equiv \theta_i(y') \mod \theta_\ell(y) \) for \( i \in \mathcal{A}(z) \) (\( = \mathcal{A}(y) \setminus \{ \ell \} = \mathcal{A}(y') \setminus \{ \ell' \} \)).
Proof. We have canonical surjections $A_P \to A_{[0,y]} \to A_{[0,z]}$ and $A_P \to A_{[0,y']} \to A_{[0,z]}$, whose composite surjections $A_P \to A_{[0,z]}$ are the same. Therefore the lemma follows from (3.3). □

4. INDEX MAPS

In this section, we define an “index map” from $A_P$ to the polynomial ring $\mathbb{Q}[\Theta]$, which corresponds to the index map $\text{Ind}_\tau$ in Section 2. It is a $\mathbb{Q}[\Theta]$-module map, so it induces a homomorphism from the quotient $A_P/(\Theta)$ modulo the linear system of parameters $\theta_1, \ldots, \theta_d$ to $\mathbb{Q}$. This induced map corresponds to the index map $\text{Ind}$ in Section 2.

We shall make some observations needed later before we define the index map. Let $z \in P_{d-1}$ and let $y, y' \in P_d$ lie above $z$ as before. Give an orientation on $\Theta$ determined by an ordered basis $(\theta_1, \ldots, \theta_d)$ and choose an order of the basis $\{\theta_i(y) \mid i \in A(y)\}$ whose induced orientation on $\Theta$ agrees with the given orientation. We then define $m(y)$ to be the determinant of a matrix sending the ordered basis $\{\theta_i(y) \mid i \in A(y)\}$ to the ordered basis $(\theta_1, \ldots, \theta_d)$. Note that $m(y)$ is positive. It follows from the latter statement in Lemma 3.2 that

$$m(y)\theta_\ell(y) = \lambda(y, y')m(y')\theta_\ell'(y'),$$

where $\lambda(y, y') = \pm 1$. If $A(y) = A(y')$, then $\ell = \ell'$ and both $\theta_\ell(y)$ and $\theta_\ell'(y')$ restrict to the element $x_\ell \in A_{[0,x_\ell]}$. Therefore

$$m(y) = m(y') \quad \text{(and $\lambda(y, y') = 1$) if $A(y) = A(y')$.}$$

The order of the basis $\{\theta_i(y) \mid i \in A(y)\}$ determines an order of atoms $x_i (i \in A(y))$ and then determines an orientation on the $(d-1)$-simplex with those atoms as vertices. The oriented $(d-1)$-simplex obtained in this way is denoted by $\langle y \rangle$. Then the boundaries $\partial \langle y \rangle$ and $\partial \langle y' \rangle$ of $\langle y \rangle$ and $\langle y' \rangle$ have opposite orientations on the $(d-2)$-simplex $[z]$ corresponding to $z$ (in other words, $[z]$ does not appear in $\partial \langle y \rangle + \partial \langle y' \rangle$) if and only if $\lambda(y, y') = -1$.

Now we pose the following assumption, which we shall see in Section 5 is satisfied by all Gorenstein* simplicial posets.

Assumption.

1. For any $z \in P_{d-1}$, there are exactly two elements in $P_d$ above $z$.
2. One can assign a sign $\epsilon(y) \in \{\pm 1\}$ to each $y \in P_d$ so that $\sum_{y \in P_d} \epsilon(y)\langle y \rangle$ is a cycle (hence defines a fundamental class in $H_{d-1}(\Gamma(P); \mathbb{Z})$ where $\Gamma(P)$ denotes the CW-complex explained in the Introduction).

When $\langle y \rangle$ and $\langle y' \rangle$ share a $(d-2)$-simplex $[z]$, it follows from the above assumption that $[z]$ does not appear in $\partial(\epsilon(y)\langle y \rangle) + \partial(\epsilon(y')\langle y' \rangle)$. Therefore,

$$\lambda(y, y') \text{ and } \epsilon(y)\epsilon(y') \text{ have opposite signs}$$

by the remark mentioned above the Assumption.
Definition. For a simplicial poset $P$ which satisfies the Assumption above, we define the index map by

$$\text{Ind}_T(\alpha) := \sum_{y \in P_d} \frac{\epsilon(y) \iota_y(\alpha)}{m(y) \prod_{i \in A(y)} \theta_i(y)} \quad \text{for} \; \alpha \in A_P.$$  

(4.4) 

Apparently, $\text{Ind}_T(\alpha)$ lies in the quotient field of $\mathbb{Q}[\Theta]$, but we have

**Theorem 4.1.** $\text{Ind}_T(\alpha) \in \mathbb{Q}[\Theta]$ for any $\alpha \in A_P$.

Remark. The proof given below is essentially same as that of Theorem 2.2 in [3]. A similar result can be found in [4, Section 8].

Proof. The right hand side of (4.4) can be expressed as

$$g \prod_{j=1}^{N} \frac{f_j}{f_j}$$

with $g \in \mathbb{Q}[\Theta]$ and $f_j \in \Theta \subset \mathbb{Q}[\Theta]$ such that any two of $f_1, \ldots, f_N$ are linearly independent. It suffices to show that $f_1$ divides $g$.

Let $Q$ be the set of $y \in P_d$ such that $\theta_i(y)$ is not a scalar multiple of $f_1$ for every $i \in A(y)$, and let $Q^c$ be the complement of $Q$ in $P_d$. In (4.4), the sum of terms for elements in $Q$ reduces to

$$\sum_{y \in Q} \frac{\epsilon(y) \iota_y(\alpha)}{m(y) \prod_{i \in A(y)} \theta_i(y)} = \frac{g_1}{\prod_{j=2}^{N} f_j}$$

with $g_1 \in \mathbb{Q}[\Theta]$, so that $f_1$ does not appear in the denominator.

On the other hand, if $y \in Q^c$, then it follows from the definition of $Q$ that there is an element $\ell \in A(y)$ such that

$$\theta_\ell(y) = cf_1 \quad (0 \neq c \in \mathbb{Q}),$$

and there is a unique element $z \in P_{d-1}$ such that $z$ is lower than $y$ and $A(z) = A(y) \setminus \{\ell\}$. By assumption, there is a unique element in $P_d$ which lies above $z$ and is different from $y$. We denote it by $y'$. Now we are in the same situation as before. It follows from (4.4) and (4.7) that $y'$ is also an element in $Q^c$. Noting that $A(y) = A(z) \cup \{\ell\}$ and $A(y') = A(z) \cup \{\ell'\}$ and using (4.4), we combine the two terms in (4.4) for $y$ and $y'$ to get

$$\frac{\epsilon(y) \iota_y(\alpha)}{m(y) \prod_{i \in A(y)} \theta_i(y)} + \frac{\epsilon(y') \iota_{y'}(\alpha)}{m(y') \prod_{i \in A(y')} \theta_i(y')} = \frac{\epsilon(y) \iota_y(\alpha) \prod_{i \in A(z)} \theta_i(y') + \lambda(y, y') \epsilon(y') \iota_{y'}(\alpha) \prod_{i \in A(z)} \theta_i(y)}{m(y) \theta_\ell(y) \prod_{i \in A(z)} \theta_i(y) \prod_{i \in A(z)} \theta_i(y')}.$$ 

Here

$$\iota_y(\alpha) \prod_{i \in A(z)} \theta_i(y') \equiv \iota_{y'}(\alpha) \prod_{i \in A(z)} \theta_i(y) \mod \theta_\ell(y)$$

for any $\alpha \in A_P$. 


by Lemma 3.2 and
\[ \epsilon(y) + \lambda(y, y') \epsilon(y') = 0 \]
by (1.3), so the numerator of the right hand side of the identity (1.8) is divisible by \( \theta_1(y) = cf_1 \). This means that we can arrange the left hand side of (1.8) with a common denominator in which \( f_1 \) does not appear as a factor. Since elements in \( Q^c \) appear pairwise like this, one has

\[ \sum_{y \in Q^c} \frac{\epsilon(y) \iota_y(\alpha)}{m(y) \prod_{i \in A(y)} \theta_i(y)} = \frac{g_2}{\prod_{j=2}^{N} f_j} \]

with \( g_2 \in \mathbb{Q}[\Theta] \). This together with (1.6) implies that the numerator \( g \) in (4.9) is divisible by \( f_1 \).

Since \( \iota_y \) is a \( \mathbb{Q}[\Theta] \)-module map, so is \( \text{Ind}_T \). Therefore

\[ \text{Ind}_T : A_P \rightarrow \mathbb{Q}[\Theta] \]

induces a homomorphism

(4.9) \[ \text{Ind} : A_P/(\Theta) \rightarrow \mathbb{Q}. \]

This map decreases degrees by \( d \) because \( \text{Ind}_T \) does.

5. Gorenstein* simplicial posets

We shall prove Theorem 1.1 in this section. Let \( k \) be an arbitrary field. Suppose that a simplicial poset \( P \) is Gorenstein* over \( k \), i.e., the order complex \( \Delta(P) \) of \( P = P - \{ \hat{0} \} \), which is a simplicial complex, is Gorenstein* over \( k \). According to Theorem II.5.1 in [9], a simplicial complex \( \Delta \) of dimension \( d - 1 \) is Gorenstein* over \( k \) if and only if for all \( p \in |\Delta| \),

\[ \tilde{H}_q(|\Delta|, k) \cong H_q(|\Delta|, |\Delta| - p; k) \cong \begin{cases} k, & q = d - 1, \\ 0, & q < d - 1. \end{cases} \]

Therefore, it follows from the universal coefficient theorem ([6 Corollary 55.2]) that if a simplicial poset \( P \) is Gorenstein* over \( k \), then it is Gorenstein* over \( \mathbb{Q} \). In the sequel we may assume \( k = \mathbb{Q} \). According to Theorem II.5.1 in [9] again, \( \Delta(P) \) is an orientable pseudomanifold, so the assumption in Section 4 is satisfied for the Gorenstein* simplicial poset \( P \) because \( \Delta(P) \) is the barycentric subdivision of the CW-complex \( \Gamma(P) \).

Since a Gorenstein* simplicial poset is Cohen-Macaulay, \( h_i = h_i(P) \) agrees with the dimension of the homogeneous part of degree \( i \) in \( A_P/(\Theta) \), see the proof of Theorem 3.10 in [8]. Therefore, if \( h_i = 0 \) for some \( i \) \( (1 \leq i \leq d - 1) \), then a product of \( d \) elements in \( (A_P)_1 \) vanishes in \( A_P/(\Theta) \), in particular, the product is zero when evaluated by the index map in (4.9).
We take a subset \( I \) of \( \{1, \ldots, n\} \) with cardinality \( d \) such that \( I = A(y) \) for some \( y \in P_d \). If \( A(y) = A(y')(= I) \), then \( m(y) = m(y') \) by (4.2). Therefore we may write \( m(y) \) as \( m_I \). Since

\[
\ell_y(\prod_{i \in I} x_i) = \begin{cases} 
\prod_{i \in A(y)} \theta_i(y) & \text{if } A(y) = I, \\
0 & \text{otherwise}
\end{cases}
\]

by (3.1) and (3.2), we have

\[
\text{Ind}_T(m_I \prod_{i \in I} x_i) = \sum_{A(y) = I} \epsilon(y) \in \mathbb{Q}
\]

by \( 4.4 \). Hence, if we regard \( m_I \prod_{i \in I} x_i \) as an element in \( A_F/(\Theta) \), then we have

\[
\text{Ind}(m_I \prod_{i \in I} x_i) = \sum_{A(y) = I} \epsilon(y).
\]

Now suppose that \( h_i = 0 \) for some \( i \) \( (1 \leq i \leq d - 1) \). Then the left hand side of \( 5.1 \) is zero as remarked above. This means that (since \( \epsilon(y) = \pm 1 \)) there must be an even number of elements \( y \in P_d \) with \( A(y) = I \) at the right hand side of \( 5.1 \). Since \( I \) is arbitrary, we conclude that \( f_d-1 \) (the number of elements in \( P_d \)) is even. This together with \( 1.1 \) completes the proof of Theorem 1.1.

Remark. An element corresponding to \( \omega_T \) in Section 2 is \( \sum_I m_I \prod_{i \in I} x_i \), where \( I \) runs over all subsets of \( \{1, \ldots, n\} \) with cardinality \( d \) and \( m_I \) is understood to be zero if there is no \( y \in P_d \) such that \( I = A(y) \).

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