Abstract—In a space of arbitrary dimensions, the effect of an external magnetic field on the vacuum of a quantized charged scalar field is studied for the field configuration in the form of a singular vortex. The zeta-function technique is used to regularize ultraviolet divergences. The expression for the effective action is derived. It is shown that the energy density and current induced in the vacuum decrease exponentially at large distances from the vortex. The analytic properties of vacuum features as functions of the complex-valued space dimension are discussed.

1. INTRODUCTION

In contrast to what occurs in classical mechanics, the motion of charged particles in quantum theory is affected by an external electromagnetic field even if the region where the field strength is operative is inaccessible for the particles. For the first time, this was demonstrated for the example of quantum-mechanical scattering on a flux tube that is formed by the magnetic lines + of force of an external field and which is impenetrable for particles undergoing scattering: the scattering cross section proved to be a periodic function of the total magnetic flux through the tube [1]. This brings about the question of whether this configuration, which becomes a singular filament (vortex) when the transverse dimensions of the flux tube are disregarded, can change the properties of the vacuum in a second-quantized theory.

As a matter of fact, this problem has been studied for a long time. Although the first
steps along these lines were made in [2], only in [3] - [9] were reliable results obtained for various geometries of space, for induced vacuum quantum numbers, and for boundary conditions at the locus of a singular magnetic vortex. Those studies focused primarily on effects associated with the polarization of the vacuum of a quantized spinor field. Here, we aim at performing systematic and exhaustive inquiries into such effects for a quantized scalar field. This case differs from that of a spinor field in that some vacuum quantum numbers (charge and angular momentum) are not induced in the scalar case; as a result, the problem reduces to studying the vacuum energy and current, as well as quantities that are associated with them directly (such as the effective potential generated by the vacuum-energy density and the strength of the vacuum magnetic field generated by the vacuum current) and those that generalize them (zeta function). The second distinction between the two cases in question is that, in the scalar case, the regularity condition can be used for the boundary condition at the locus of the vortex. It is the condition that is used throughout this study. Finally, the third distinction, which follows from the second one, is that the results can be immediately generalized to spaces of arbitrary dimensions.

2. QUANTIZATION OF A SCALAR FIELD AGAINST THE BACKGROUND OF A SINGULAR MAGNETIC VORTEX

The operator of a second-quantized complex scalar field can be represented in the form

\[ \Psi(x, t) = \sum_{\lambda} \frac{1}{\sqrt{2 E_\lambda}} \left[ e^{-iE_\lambda t} \psi_\lambda(x) a_\lambda + e^{iE_\lambda t} \psi_{-\lambda}(x) b_\lambda^{\dagger} \right], \tag{1} \]

where \( \lambda \) is the set of parameters (quantum numbers) that specify a state; \( E_\lambda = E_{-\lambda} > 0 \) is the energy of a given state; the symbol \( \sum_{\lambda} \) denotes summation over discrete and integration (with a definite measure) over continuous values of \( \lambda \); \( a_\lambda^{\dagger} \) and \( a_\lambda \) (\( b_\lambda^{\dagger} \) and \( b_\lambda \)) are, respectively, the creation and annihilation operators for scalar particles (antiparticles), the standard commutation relations being satisfied for these operators; and the functions \( \psi_\lambda(x) \) represent solutions to the time-independent Klein-Gordon equation

\[ \left( -\nabla^2 + m^2 \right) \psi_\lambda(x) = E_\lambda^2 \psi_\lambda(x), \tag{2} \]
Here, $\nabla$ is the operator of covariant differentiation in a static external (background) field. Proceeding in a standard way, we can show that the vacuum-energy density is given by

$$\varepsilon(x) = \sum_\lambda E_\lambda \psi_\lambda^*(x) \psi_\lambda(x)$$

(3)

and that the vacuum current has the form

$$j(x) = (2i)^{-1} \sum_\lambda E_\lambda^{-1} \left\{ \psi_\lambda^*(x) [\nabla \psi_\lambda(x)] - [\nabla \psi_\lambda(x)]^* \psi_\lambda(x) \right\}.$$  

(4)

If a magnetic field is chosen to represent the background field, the covariant derivative takes the form

$$\nabla = \partial - iV(x),$$

(5)

where $V(x)$ is the vector potential of the field in question. In $d$-dimensional space, the strength of the magnetic field is an antisymmetric tensor of rank $(d - 2)$ related to the above vector potential by the equation

$$B^\nu_{\nu_1 \cdots \nu_{d-2}}(x) = \left[ \partial_{\mu_1} V_{\mu_2}(x) \right] \epsilon^{\mu_1 \mu_2 \nu_{\nu_1} \cdots \nu_{d-2}},$$

(6)

where $\epsilon^{\mu_1 \cdots \mu_d}$ is the fully antisymmetric tensor normalized by the condition $\epsilon^{12 \cdots d} = 1$.

In this study, we consider the magnetic-field configuration in the form of a singular vortex represented by a point for $d = 2$, a line for $d = 3$, and a $(d - 2)$-dimensional hypersurface for $d > 3$; that is, we have

$$V_1(x) = -\frac{x^2}{(x^1)^2 + (x^2)^2}, \quad V_2(x) = \frac{x^1}{(x^1)^2 + (x^2)^2}, \quad V_\nu(x) = 0, \quad \nu = 3, d,$$

(7)

$$B^{3 \cdots d}(x) = 2\pi \Phi \delta(x^1) \delta(x^2),$$

(8)

where $\Phi$ is the flux of the vortex in question (in $2\pi$ units).

Let us construct the complete set of solutions to equation (2) in the field of the vortex that satisfy the regularity condition on the vortex hypersurface. We can easily obtain

$$\psi_{knp}(x) = (2\pi)^{\frac{1-d}{2}} J_{n-\Phi_1(kr)} e^{i\nu p \varphi} e^{ipx_{d-2}},$$

(9)

where

$$0 < k < \infty, \quad n \in \mathbb{Z}, \quad -\infty < p_\nu < \infty, \quad \nu = 3, d,$$

(10)
$J_\omega(u)$ is a Bessel function of order $\omega$, $r = \sqrt{(x^1)^2 + (x^2)^2}$, $\varphi = \arctg(x^2/x^1)$, $x_{d-2} = (0, 0, x^3, ..., x^d)$ and $\mathbb{Z}$ is the set of integers. Since the solutions given by (9) correspond to a continuous spectrum ($E_{knp} = \sqrt{p^2 + k^2 + m^2} > m$), they are normalized to a delta function:

$$\int d^d x \psi_{knp}^*(x)\psi_{kn'p'}(x) = \frac{\delta(k - k')}{k} \delta_{nn'}\delta(p - p').$$

(11)

Taking all the above into account, we obtain

$$\varepsilon(x) = (2\pi)^{1-d} \int d^{d-2} p \int_0^\infty dk k \left(p^2 + k^2 + m^2\right)^{\frac{d}{2}} \sum_{n \in \mathbb{Z}} J^2_{|n-\Phi|}(kr)$$

(12)

$$j_\varphi(x) \equiv r^{-1} \left[x^1 j_2(x) - x^2 j_1(x)\right] =$$

$$= (2\pi)^{1-d} r^{-1} \int d^{d-2} p \int_0^\infty dk k \left(p^2 + k^2 + m^2\right)^{-\frac{d}{2}} \sum_{n \in \mathbb{Z}} (n - \Phi) J^2_{|n-\Phi|}(kr)$$

(13)

(the remaining components of the vacuum current vanish identically). It is not quite correct to define the vacuum-energy density via (12) because the integral involved diverges for $E_{knp} \to \infty$. All problems associated with regularization of ultraviolet divergences and with renormalization can be conveniently solved with the aid of a zeta function, which therefore comes to be of paramount importance for problems of the type considered here.

3. ZETA FUNCTION

We define the density of the zeta function as [compare with (3)]

$$\zeta_x(s) = \sum_\lambda E_\lambda^{-2s} \psi^*_\lambda(x)\psi_\lambda(x).$$

(14)

For an external field in the form of a singular magnetic vortex, expression (14) takes the form

$$\zeta_x(s) = (2\pi)^{1-d} \int d^{d-2} p \int_0^\infty dk k \left(p^2 + k^2 + m^2\right)^{-s} \sum_{n \in \mathbb{Z}} J^2_{|n-\Phi|}(kr).$$

(15)

The last expression is well defined only for $\text{Re} \ s > \frac{d}{2}$. 

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Representing the flux of the vortex as the sum of the integral \((n')\) and fractional \((F)\) parts,
\[
\Phi = n' + F, \quad n' \in \mathbb{Z}, \quad 0 \leq F < 1, \tag{16}
\]
we find that the zeta-function density \((15)\) depends periodically on 0 and that the case of integral flux values \((F = 0)\) is equivalent to the flux-free case \((\Phi = 0)\). Defining the zeta-function density in a free theory (that is, in the absence of vortices) as
\[
\zeta^{(0)}_{x}(s) = \zeta_{x}(s)|_{F=0}, \tag{17}
\]
we can easily find that, under the condition \(\text{Re } s > \frac{d}{2}\) it can be recast into the form
\[
\zeta^{(0)}_{x}(s) = (2\pi)^{1-d} \int d^{d-2}p \int_{0}^{\infty} dk \kappa (p^2 + k^2 + m^2)^{-s} = \frac{m^{d-2s}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)}, \tag{18}
\]
where \(\Gamma(z)\) is the Euler gamma function.

For \(0 < F < 1\), the result obtained for the zeta-function density by performing integration with respect to \(p\) and summation over \(n\) in \((15)\) is
\[
\zeta_{x}(s) = \frac{2}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2} + 1)}{\Gamma(s)} \int_{0}^{\infty} dk \kappa (k^2 + m^2)^{\frac{d}{2} - s - 1} \times \int_{0}^{\infty} dy \left[ J_{F}(y) J_{1+F}(y) + J_{1-F}(y) J_{-F}(y) \right]. \tag{19}
\]
This expression can be recast into the form
\[
\zeta_{x}(s) = \frac{r}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} \times \int_{0}^{\infty} dk \left( k^2 + m^2 \right)^{\frac{d}{2} - s} \left[ J_{F}(kr) J_{1+F}(kr) + J_{1-F}(kr) J_{-F}(kr) \right]. \tag{20}
\]
By using the relation
\[
\alpha^{-z} = \frac{1}{\Gamma(z)} \int_{0}^{\infty} dy y^{z-1} e^{-\alpha y}, \quad \text{Re } z > 0, \tag{21}
\]
we can then reduce expression \((20)\) to the final form
\[
\zeta_{x}(s) = \frac{4 \sin(F\pi)}{(4\pi)^{\frac{d}{2} + 1}} \frac{m^{d-2s}}{\Gamma(s)} \int_{0}^{\infty} du e^{-u} \left[ K_{F}(u) + K_{1-F}(u) \right] \gamma \left( s - \frac{d}{2}, \frac{m^2 r^2}{2u} \right), \tag{22}
\]
where
\[ \gamma(z, w) = \int_0^w dy y^{z-1} e^{-y} \] (23)
is the incomplete gamma function, and \( K_\omega(z) \) is the Macdonald function of order \( \omega \).

With the aid of the inverse Mellin transformation, we find that the kernel of the heat equation is given by
\[ \langle x \mid \exp \left[ -t \left( -\nabla^2 + m^2 \right) \right] \mid x \rangle = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{e^{-st}}{\Gamma(s)} \zeta_x(s), \quad c > \frac{d}{2}. \] (24)

Although expression (24) was obtained under the condition \( \text{Re} \, s > \frac{d}{2} \), we can construct an analytic continuation of this expression to the region \( \text{Re} \, s < \frac{d}{2} \) by using repeatedly the recursion relation
\[ \gamma(z, w) = \frac{1}{z} \left[ \gamma(z+1, w) + w e^{-w} \right], \] (25)
As a result, we arrive at the zeta-function density expressed in terms of a meromorphic function on the \( s \) plane, the products \( \Gamma(s)\zeta_x(s) \) having only simple poles on the real axis. The residues at these poles are given by
\[ \text{Res}_{s=\frac{d}{2}-N+1} \Gamma(s)\zeta_x(s) = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{(-m^2)^{N-1}}{\Gamma(N)}, \quad N = 1, 2, \ldots \] (26)

Shifting the contour of integration in (24) to the left, closing it by the corresponding part of a large circle, and summing the contributions of the residues, we obtain
\[ \langle x \mid \exp \left[ -t \left( -\nabla^2 + m^2 \right) \right] \mid x \rangle = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-m^2 t} \left\{ 1 + O \left[ \frac{t}{r} \exp \left( -\frac{r^2}{t} \right) \right] \right\}. \] (27)
The same expression \( [O \left( \frac{t}{r} \exp \left( -\frac{r^2}{t} \right) \right) \text{ term apart}] \) is obtained for the kernel of the heat equation in free theory as well if the function \( \zeta^{(0)}_x(s) \) (18), which can be treated as an analytic continuation to the entire complex \( s \) plane, is substituted for the function \( \zeta_x(s) \) (23). Thus, the functions \( \zeta_x(s) \) and \( \zeta^{(0)}_x(s) \) have the same structure of singularities. Hence, the difference
\[ \zeta^{\text{ren}}_x(s) = \zeta_x(s) - \zeta^{(0)}_x(s) \] (28)
which represents the renormalized zeta-function density, is a holomorphic function on the complex $s$ plane, and so is the product $\Gamma(s)\zeta_x^{ren}(s)$ . We can easily obtain

$$
\zeta_x^{ren}(s) = -\frac{4\sin(F\pi)}{(4\pi)^{\frac{d-2s}{2}+1}} \frac{m^{d-2s}}{\Gamma(s)} \int_0^\infty du e^{-u} [K_F(u) + K_{1-F}(u)] \Gamma\left(s - \frac{d}{2}, \frac{m^2r^2}{2u}\right),
$$

(29)

where

$$
\Gamma(z, w) = \Gamma(z) - \gamma(z, w) = \int_w^\infty dy y^{z-1} e^{-y}.
$$

(30)

Considering that the Macdonald function has the integral representation

$$
K_\omega(u) = \int_0^\infty dy \chi(\omega y) \exp(-u \chi y),
$$

(31)

we eventually obtain

$$
\zeta_x^{ren}(s) = -\frac{16 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \frac{r^{s-\frac{d}{2}}}{m^{s-d}} \times
\int_1^\infty \frac{du}{\sqrt{u^2 - 1}} \chi[(2F - 1)\text{Arch}] u^{s-\frac{d}{2}-1} K_{s-\frac{d}{2}}(2mr\nu).
$$

(32)

Thus, the product $\Gamma(s)\zeta_x^{ren}(s)$ is a nonpositive function of $F$, is symmetric with respect to the substitution $F \rightarrow 1 - F$, has a minimum at $F = \frac{1}{2}$, and vanishes at $F = 0$. In contrast to the functions $\zeta_x^{(0)}(s)$ [18] and $\zeta_x(s)$ [22], the function $\zeta_x^{ren}(s)$ [32] decreases exponentially at large distances:

$$
\zeta_x^{ren}(s) = -\frac{\sin(F\pi)}{(4\pi)^{\frac{d}{2}} \Gamma(s)} e^{-2mr} m^{\frac{d}{2} - s - 1} r^{s - \frac{d}{2} - 1} \left\{1 + O\left[\left(\frac{mr}{r}\right)^{-1}\right]\right\}, \quad mr \gg 1.
$$

(33)

Instead of (21), we can use the relation

$$
\alpha^{-z} = \frac{2 \sin(z\pi)}{\pi} \int_0^\infty dy \frac{y^{1-2z}}{y^2 + \alpha}, \quad 0 < \text{Re } z < 1,
$$

(34)

whence it follows that the function $\zeta_x^{ren}(s)$ can alternatively be represented as

$$
\zeta_x^{ren}(s) = -\frac{16 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \frac{r^{2s-d}}{\Gamma(s)\Gamma(1 - s + \frac{d}{2})} \int_{mr}^\infty dw \left( w^2 - m^2r^2 \right)^{\frac{d}{2} - s} K_F(w)K_{1-F}(w).
$$

(35)

In this representation, the product $\Gamma(s)\zeta_x^{ren}(s)$ is a holomorphic function in the region $\text{Re } s < \frac{d}{2} + 1$, but it is undefined for $\text{Re } s > \frac{d}{2} + 1$. 

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In the absence of a vortex, integration of the zeta-function density with respect to any of the space coordinates obviously leads to a divergence because \( \zeta_x^{(0)}(s) \) is independent of \( x \) and because the space being considered is not compact. In the presence of a vortex, a similar situation arises if we integrate the zeta-function density \( \zeta_x(s) \), which is uniform only in the direction parallel to the vortex. The situation is different if integration is performed with the renormalized zeta-function density \( \zeta_x^{\text{ren}}(s) \), which decreases exponentially in the plane orthogonal to the vortex. It is because of this property of the renormalized zeta-function density that there exist \( s \) values at which the following integral is finite:

\[
\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \zeta_x^{\text{ren}}(s) = -\frac{m^{d-2s-2}}{2(4\pi)^{d/2-1}} \frac{\Gamma(s - \frac{d}{2} + 1)}{\Gamma(s)} F(1 - F). \tag{36}
\]

Here, integration is performed for \( \text{Re} \ s > \frac{d}{2} - 1 \) in the case of representation (32) and for \( \frac{d}{2} - 1 < \text{Re} \ s < \frac{d}{2} + 1 \) in the case of representation (35). Obviously, the result can be continued analytically over the entire complex \( s \) plane.

4. EFFECTIVE ACTION AND VACUUM ENERGY

Going over to the imaginary time \( t = -i\tau \), we define the effective action in \((d + 1)\)-dimensional Euclidean spacetime as

\[
S_{\text{eff}}[V(x)] = \int d\tau d^d x \Omega(x, \tau) = \ln \det \left(-\partial_t^2 - \nabla^2 + m^2\right) M^{-2}, \tag{37}
\]

where \( M \) is a parameter that has dimensions of mass, and \( \Omega \) is the effective-action density. In \((d + 1)\)-dimensional space, the effective-action density is expressed in terms of the zeta-function density as

\[
\Omega(x, \tau) = -\left[\frac{d}{ds} \zeta_x^{\text{ren}}(s)\right]_{s=0} - \zeta_x^{\text{ren}}(0) \ln M^2. \tag{38}
\]

Taking into account (18), (22), (28), and (35), we find that, in the presence of a singular magnetic vortex, expression (35) can be recast into the form

\[
\Omega(x, \tau) = -\frac{4 \sin(F\pi)}{(4\pi)^{N+\frac{1}{2}}} m^{2N+1} \times \\
\times \int_0^{\infty} du \, e^{-u} [K_F(u) + K_{1-F}(u)] \gamma \left(-N - \frac{1}{2}, \frac{m^2 r^2}{2u}\right), \; d = 2N, \tag{39}
\]
\[
\Omega(x, \tau) = \frac{m^{2(N+1)}}{(4\pi)^{N+1}} \left\{ \frac{(-1)^N}{\Gamma(N+2)} \left[ \ln \frac{M^2}{m^2} + \psi(N+2) + \gamma \right] + \right.
\]
\[
+ \frac{\sin(F\pi)}{\pi} \int_0^\infty du \ e^{-u} \left[ K_F(u) + K_{1-F}(u) \right] \Gamma \left( -N - 1, \frac{m^2r^2}{2u} \right) \biggr\}, \ d = 2N + 1,
\]
(40)

In the alternative representation, we have
\[
\Omega(x, \tau) = \frac{1}{(4\pi)^{N+1} \Gamma(N + \frac{3}{2})} \left\{ (-1)^N \pi m^{2N+1} + \right.
\]
\[
+ \frac{4 \sin(F\pi)}{\pi r^{2N+1}} \int_m^\infty dw \ (w^2 - m^2 r^2)^{N+\frac{1}{2}} K_F(w) K_{1-F}(w) \biggr\}, \ d = 2N,
\]
(41)
\[
\Omega(x, \tau) = \frac{1}{(4\pi)^{N+1} \Gamma(N + 2)} \left\{ (-1)^N \left[ \ln \frac{M^2}{m^2} + \psi(N+2) + \gamma \right] m^{2(N+1)} + \right.
\]
\[
+ \frac{4 \sin(F\pi)}{\pi r^{2(N+1)}} \int_m^\infty dw \ (w^2 - m^2 r^2)^{N+1} K_F(w) K_{1-F}(w) \biggr\}, \ d = 2N + 1;
\]
(42)

where \(\psi(z) = \frac{d}{dz} \ln \Gamma(z)\) is the digamma function, and \(\gamma = -\psi(1)\) is the Euler constant.

The effective potential that is equivalent to the renormalized effective-action density is given by
\[
\Omega_{\text{ren}}(x, \tau) = - \left. \left[ \frac{d}{ds} \zeta_{\text{ren}}^{\text{en}}(x, \tau) \right] \right|_{s=0},
\]
(43)

Equation (43) can be rewritten in an integral form as
\[
\ln \left\{ \frac{\det \left( -\partial_\tau^2 - \nabla^2 + m^2 \right)}{\det \left( -\partial_\tau^2 - \partial^2 + m^2 \right)} \right\} = \int d\tau d^d x \Omega_{\text{ren}}(x, \tau);
\]
(44)

By considering that the product \(\Gamma(s)\zeta_{x,\tau}^{\text{ren}}(s)\) is a holomorphic function, it can be shown that the relation
\[
\zeta_{x,\tau}^{\text{ren}}(0) = 0,
\]
(45)

must hold. Taking into account the explicit form of the renormalized zeta-function density as given by (32) or (35), we can easily obtain
\[
- \left. \left[ \frac{d}{ds} \zeta_{x,\tau}^{\text{ren}}(s) \right] \right|_{s=0} = \zeta_{x,\tau}^{\text{ren}} \left( -\frac{1}{2} \right).
\]
(46)
By comparing equations (3) and (14), we can therefore naturally define the vacuum-energy density in terms of the effective potential as

$$\varepsilon^{\text{ren}}(x) = \Omega^{\text{ren}}(x, \tau). \quad (47)$$

As a result, we obtain

$$\varepsilon^{\text{ren}}(x) = \frac{16 \sin(F\pi)}{(4\pi)^{\frac{d+3}{2}}} \left( \frac{m}{r} \right)^{\frac{d+1}{2}} \int_1^\infty \frac{dv}{\sqrt{v^2 - 1}} \text{ch} [(2F - 1)Achuv] v^{-\frac{d+3}{2}} K_{d+1}(2mrv), \quad (48)$$

In the alternative representation, we have

$$\varepsilon^{\text{ren}}(x) = \frac{16 \sin(F\pi) \Gamma(-d-1)}{(4\pi)^{\frac{d+3}{2}} \Gamma\left(\frac{d+3}{2}\right)} \int_{mr}^\infty dw (w^2 - m^2 r^2)^{\frac{d+1}{2}} K_F(w) K_{1-F}(w). \quad (49)$$

By using the relations

$$\Gamma\left(-N - \frac{1}{2}, w\right) = \frac{(-1)^N}{\Gamma(N + \frac{3}{2})} \left[ -\pi \text{erfc}(\sqrt{w}) + e^{-w} \sum_{j=0}^{N} (-1)^j \Gamma\left(j + \frac{1}{2}\right) w^{-j+\frac{1}{2}} \right], \quad (50)$$

$$\Gamma(-N - 1, w) = \frac{(-1)^N}{\Gamma(N + 2)} \left[ -E_1(w) + e^{-w} \sum_{j=0}^{N} (-1)^j \Gamma(j + 1) w^{-j+1} \right], \quad (51)$$

where

$$\text{erfc}(w) = 2 \sqrt{\frac{2}{\pi}} \int_w^\infty du e^{-u^2}$$

is the complementary error function, and

$$E_1(w) = \int_w^\infty \frac{du}{u} e^{-u}$$

1) Generally, it may turn out that, at odd values of $d$, the product $\Gamma(s)\zeta^{\text{ren}}(s)$ has a simple pole at $s = 0$. In this case, the result corresponding to defining the renormalized vacuum-energy density in terms of the effective potential would be different from that for the renormalized vacuum-energy density defined in terms of the function $\zeta^{\text{ren}}(\frac{1-1}{2})$. It is the situation that arises when the vacuum energy is induced by an external magnetic field having a regular configuration (see [10]).
is the exponential integral (see, for example, [11]), we find that the maximum value of the renormalized vacuum-energy density as a function of $F$ is given by

$$
\varepsilon_{\text{ren}}(x)|_{F=\frac{1}{2}} = \frac{2m^{2N+1}}{(4\pi)^{N+\frac{1}{2}}} \left( \frac{-1)^N}{\Gamma\left(N + \frac{3}{2}\right)} \right) \left\{ -\frac{\pi}{2} + \pi mr \left[ K_0(2mr)L_{-1}(2mr) + K_1(2mr)L_0(2mr) \right] + \frac{1}{\sqrt{\pi}} \sum_{l=0}^{N} (-1)^l \Gamma\left(l + \frac{1}{2}\right)(mr)^{-l}K_{l+1}(2mr) \right\},
$$

$$d = 2N,$$

(52)

$$
\varepsilon_{\text{ren}}(x)|_{F=\frac{1}{2}} = \frac{2m^{2(N+1)}}{(4\pi)^{N+1}} \left( \frac{-1)^N}{\Gamma\left(N + 2\right)} \right) \left\{ -E_1(2mr) + e^{-2mr} \sum_{l=0}^{N} (-1)^l \Gamma\left(l + 1\right) \sum_{n=0}^{l+1} \frac{\Gamma(l + n + 2)(mr)^{-l-n-1}}{2^{n+1} \Gamma(n + 1) \Gamma(l - n + 2)} \right\},
$$

$$d = 2N + 1,$$

(53)

where $L_\omega(u)$ is the modified Struve function of order $\omega$ [11]. In particular, we have

$$
\varepsilon_{\text{ren}}(x)|_{F=\frac{1}{2}} = \frac{m^3}{3\pi^2} \left\{ \frac{\pi}{2} - \pi mr \left[ K_0(2mr)L_{-1}(2mr) + K_1(2mr)L_0(2mr) \right] - K_1(2mr) + (2mr)^{-1}K_2(2mr) \right\},
$$

$$d = 2,$$

(54)

$$
\varepsilon_{\text{ren}}(x)|_{F=\frac{1}{2}} = \frac{m^4}{16\pi^2} \left\{ E_1(2mr) + \frac{e^{-2mr}}{2mr} \left[ -1 + \frac{1}{2mr} + \frac{3}{2(mr)^2} + \frac{3}{4(mr)^3} \right] \right\},
$$

$$d = 3.$$

(55)

To conclude this section, we present the asymptotic expressions for the renormalized vacuum-energy density at small and large distances from the vortex. We have

$$
\varepsilon_{\text{ren}}(x) = \frac{4\sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} \Gamma\left(\frac{d+1}{2} + F\right) \Gamma\left(\frac{d+1}{2} + 1 - F\right) \frac{r^{-d-1}}{(d+1)\Gamma\left(\frac{d+1}{2} + 1\right)} \left\{ 1 + O\left[(mr)^2\right] \right\},
$$

$$mr \ll 1,$$

(56)

$$
\varepsilon_{\text{ren}}(x) = \frac{\sin(F\pi)}{(4\pi)^{\frac{d+1}{2}}} e^{-2mr} m^{\frac{d+1}{2}} r^{-\frac{d+1}{2}} \left\{ 1 + O\left[(mr)^{-1}\right] \right\},
$$

$$mr \gg 1.$$

(57)
5. VACUUM CURRENT AND VACUUM MAGNETIC FIELD

Performing integration with respect to $p$ in (13) and using relation (21), we find that the vacuum current is given by

$$j_\phi(x) = \frac{4}{r \left(4\pi\right)^{d+1/2}} \int_0^\infty dy \, y^{1/2} \, e^{-m^2 y} \int_0^\infty dk \, k^2 e^{-k^2 y} \sum_{n \in \mathbb{Z}} (n - \Phi) J^2_{|n-\Phi|}(kr). \quad (58)$$

Performing summation over $n$ in the expression on the right-hand side of (58), we obtain

$$j_\phi(x) = \frac{2}{(4\pi)^{d+1/2}} \int_0^\infty dy \, y^{1/2} \, e^{-m^2 y} \int_0^\infty dk \, k^2 e^{-k^2 y} \left\{ kr \left[ J^2_{1-F}(kr) + J^2_{-F}(kr) - J^2_F(kr) - J^2_{-1+F}(kr) \right] + (2F - 1) \left[ J_{1-F}(kr) J_{-F}(kr) + J_F(kr) J_{-1+F}(kr) \right] \right\}. \quad (59)$$

Following integration with respect to $k$ and the substitution $r^2/2y = u$, we reduce the expression for $j_\phi(x)$ to the form

$$j_\phi(x) = \frac{2 \sin(F\pi)}{(2\pi)^{d+1/2}} r^{-d} \int_0^\infty du \, u^{d-1/2} \exp \left( -u - \frac{m^2 r^2}{2u} \right) \left[ K_F(u) - K_{1-F}(u) \right], \quad (60)$$

With the aid of the integral representation (31) for the Macdonald function, we then obtain

$$j_\phi(x) = \frac{32 \sin(F\pi)}{(4\pi)^{d+1/2}} m^{d+1/2} r^{-d-1} \int_1^\infty dv \, \text{sh} \left[ (2F - 1) \text{Arch} v \right] v^{d-1/2} \frac{K_{d+1}}{2} (2mr v). \quad (61)$$

If relation (34) is used instead of (21), we arrive at

$$j_\phi(x) = \frac{32 \sin(F\pi)}{(4\pi)^{d+1/2}} \frac{r^{-d}}{\Gamma(d-1/2)} \int_{mr}^\infty dw \, w^2 (w^2 - m^2 r^2)^{d-3/2} \times$$

$$\times \left\{ w \left[ K^2_{1-F}(w) - K^2_F(w) \right] + (2F - 1) K_F(w) K_{1-F}(w) \right\}; \quad (62)$$

2) As a matter of fact, an analytic continuation in the complex-valued variable $d$ is performed here from the region Re $d < 3$ to the region Re $d > 3$ (compare with the presentation in Section 3). We then arrive at the quantity expressed in (58) in terms of a function that is holomorphic over the entire complex $d$ plane.
In this representation, the vacuum current is a holomorphic function of \( d \) in the region \( \text{Re} \, d > 1 \), but it is not defined for \( \text{Re} \, d < 1 \).

In the case of spaces of lower dimensions \((d = 2, 3, 4, \text{ and } 5)\), it can be shown by transforming the integral in equation (60) that the corresponding expressions for the vacuum current are

\[
j_\varphi(x) = \frac{\sin(F\pi)}{4\pi^2 r^2} \left( F - \frac{1}{2} \right) \left\{ -4 \left[ (F - \frac{1}{2})^2 + m^2 r^2 \right] \int_{2mr}^{\infty} \frac{du}{u} K_{2F-1}(u) + 
+ mr \left[ K_{2F}(2mr) + K_{2(1-F)}(2mr) \right] \right\}, \quad d = 2, \tag{63}
\]

\[
j_\varphi(x) = \frac{\sin(F\pi) m}{6\pi^3} \left( F - \frac{1}{2} \right) \left\{ \left[ (F - \frac{1}{2})^2 + m^2 r^2 \right] mr K^2_F(mr) - 
- \left[ (1 - F)(\frac{1}{2} - F) + m^2 r^2 \right] mr K^2_{1-F}(mr) + 
+ 2 \left( F(1 - F) - m^2 r^2 \right) (F - \frac{1}{2}) K_F(mr) K_{1-F}(mr) \right\}, \quad d = 3, \tag{64}
\]

\[
j_\varphi(x) = \frac{\sin(F\pi) m}{32\pi^3 r^4} \left( F - \frac{1}{2} \right) \left( 4 \left\{ (F - \frac{1}{2})^2 \left[ (F - \frac{1}{2})^2 - 1 + 2m^2 r^2 \right] + m^4 r^4 \right\} \times 
\times \int_{2mr}^{\infty} \frac{du}{u} K_{2F-1}(u) + 2m^2 r^2 K_{2F-1}(2mr) - 
- \left[ (F - \frac{1}{2})^2 - 1 + m^2 r^2 \right] mr \left[ K_{2F}(2mr) + K_{2(1-F)}(2mr) \right] \right\}, \quad d = 4, \tag{65}
\]

\[
j_\varphi(x) = \frac{\sin(F\pi) m}{60\pi^4} \left( F - \frac{1}{2} \right) \left\{ \left[ (F - \frac{1}{2}) [(1 + F)(2 - F) - (2F - \frac{1}{2})m^2 r^2] - 
- m^4 r^4 \right] mr K^2_F(mr) - \left\{ (\frac{1}{2} - F)[(1 + F)(1 - F)(2 - F) - (\frac{3}{2} - 2F)m^2 r^2] - 
- m^4 r^4 \right\} mr K^2_{1-F}(mr) + 2 \left( (1 + F)(1 - F)(2 - F) + 
+ [1 - 2F(1 - F)] m^2 r^2 + m^4 r^4 \right) (F - \frac{1}{2}) K_F(mr) K_{1-F}(mr) \right\}, \quad d = 5, \tag{66}
\]
The expression on the right-hand side of (64) was first obtained in [2]. The integral representation (60) for the case of \(d = 3\) was also presented in [5].

The asymptotic expressions for the vacuum current at small and large distances from the vortex are given by

\[
j_\Phi(x) = \frac{4 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \left( F - \frac{1}{2} \right) \Gamma\left(\frac{d-1}{2} + F\right) \Gamma\left(\frac{d-1}{2} + 1 - F\right) \frac{\Gamma\left(\frac{d}{2} + 1\right)}{\Gamma\left(\frac{d}{2} + 1\right)} r^{-d} \left\{ 1 + O\left([mr]^2\right) \right\}, \quad mr \ll 1, \quad (67)
\]

\[
j_\Phi(x) = \frac{2 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \left( F - \frac{1}{2} \right) e^{-2mr} m^{\frac{d-3}{2}} r^{-\frac{d+3}{2}} \left\{ 1 + O\left((mr)^{-1}\right) \right\}, \quad mr \gg 1. \quad (68)
\]

To conclude this section, we note that, according to the Maxwell equation

\[
\partial_r B_{(I)}^{3,d}(x) = -e^2 j_\Phi(x), \quad (69)
\]

where \(e\) is the coupling constant having dimensions of \(m^{\frac{d-3}{2}}\) in \((d+1)\)-dimensional spacetime, the magnetic field of strength

\[
B_{(I)}^{3,d}(x) = e^2 \int_0^\infty dr j_\Phi(x). \quad (70)
\]

is induced in the vacuum. Using relation (61) or (62) for \(j_\Phi(x)\), we obtain

\[
B_{(I)}^{3,d}(x) = \frac{16 e^2 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \left( \frac{m}{r} \right)^{\frac{d-1}{2}} \int_1^\infty dv \text{sh} [(2F - 1)\text{Arch}v] v^{-\frac{d+3}{2}} K_{d+1}(2mrv), \quad (71)
\]

In the alternative representation, we have

\[
B_{(I)}^{3,d}(x) = \frac{16 e^2 \sin(F\pi)}{(4\pi)^{\frac{d}{2}+1}} \frac{r^{1-d}}{\Gamma\left(\frac{d+1}{2}\right)} \int_{mr}^\infty dw \left\{ w \left[K_{1-F}(w) - K_F^2(w)\right] + (2F - 1) K_F(w) K_{1-F}(w) \right\}. \quad (72)
\]

The total flux of the vacuum magnetic field (in \(2\pi\) units),

\[
\Phi^{(I)} = \int_0^\infty dr r B_{(I)}^{3,d}(x) \quad (73)
\]

3) The results presented in [3] for the vacuum energy density at \(d = 3\) are self-contradictory and erroneous.
diverges for \( d \geq 3 \) and is finite in all other cases. Specifically, we have

\[
\Phi^{(I)} = \frac{2 e^2 m^{d-3}}{3(4\pi)^{\frac{d+1}{2}}} \Gamma \left( \frac{3-d}{2} \right) F (1-F) \left( F - \frac{1}{2} \right).
\]

(74)

It should be emphasized that the last expression can be continued analytically from the holomorphicity region \( \text{Re} d < 3 \) to the entire complex \( d \) plane. The result proves to be finite at even real values of \( d \).

6. CONCLUSION

Effects associated with boson-vacuum polarization by an external field in the form of a singular magnetic vortex have been studied comprehensively. Integral representations have been obtained for the zeta-function density [equation (32) or (33)] for the effective action density [equations (39) and (40) or (41) and (42)], for the vacuum-energy density [equation (48) or (49)], for the vacuum current [equation (61) or (62)], and for the strength of the vacuum magnetic field [equation (71) or (72)]. Remarkably, our results concerning the zeta function, energy, current, and magnetic-field strength admit analytic continuation in the space dimension \( d \), the representations given by (32), (48), (61), and (71) being implemented in terms of functions that are holomorphic in the entire complex \( d \) plane. The global features of the vacuum have also been determined [see equations (36) and (74)]; of these, only the total flux of the vacuum magnetic field is finite at the single value of \( d = 2 \), although expression (36) at \( s = -\frac{1}{2} \) and expression (74) treated in the sense of an analytic continuation are finite at all even values of \( d \).

Our results depend periodically, with a period equal to unity, on the flux \( \Phi \) of the vortex. At nonintegral values of \( \Phi \), the vacuum-energy density is a positive function (convex for \( \text{Re} d > -1 \)) of the fractional part \( F \) of the flux of the vortex, is symmetric with respect to the substitution \( F \to 1 - F \), and has a maximum at \( F = \frac{1}{2} \). At the same time, the vacuum current vanishes at \( F = \frac{1}{2} \), is a negative function (concave for \( \text{Re} d > 1 \)) in the interval \( 0 < F < \frac{1}{2} \), and is a positive function (convex for \( \text{Re} d > 1 \)) in the interval \( \frac{1}{2} < F < 1 \), the minimum and the maximum being located symmetrically with respect to the point \( F = \frac{1}{2} \). Accordingly, the vacuum-energy density is even under charge
conjugation, while the vacuum current is odd. It has been shown that the local features of the vacuum decrease exponentially at large distances from the vortex’ [see equations (33), (57) and (68)].

To conclude our brief summary of the results, we note that the formulas for the vacuum-energy density become especially simple in the case of a massless scalar field [see equations (56) and (57)].

ACKNOWLEDGEMENTS

This work was supported by the State Foundation for Basic Research of Ukraine (project no. 2.4/320) and by the Swiss National Science Foundation (grant no. CEEC/NIS/96-98/7 IP 051219).
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