Minimally doubled fermions at one loop

Stefano Capitani, Johannes Weber and Hartmut Wittig

Institut für Kernphysik, Becher Weg 45, University of Mainz, D-55099 Mainz, Germany

Abstract

Minimally doubled fermions have been proposed as a cost-effective realization of chiral symmetry at non-zero lattice spacing. Using lattice perturbation theory at one loop, we study their renormalization properties. Specifically, we investigate the consequences of the breaking of hyper-cubic symmetry, which is a typical feature of this class of fermionic discretizations. Our results for the quark self-energy indicate that the four-momentum undergoes a renormalization which is linearly divergent. We also compute renormalization factors for quark bilinears, construct the conserved vector and axial-vector currents and verify that at one loop the renormalization factors of the latter are equal to one.
1. The past ten years have witnessed two major breakthroughs in lattice QCD. The first concerns the significant acceleration of simulation algorithms for dynamical quarks, in particular as the light quark masses are tuned towards the chiral regime. Owing to these developments, simulations with pion masses close to the physical value have become routine. The second achievement was the solution to the long-standing problem of constructing discretizations of the quark action which preserve chiral symmetry, and the realization of the rôle of the Ginsparg-Wilson relation [1, 2]. However, the well-known discretizations based on the perfect action formalism [3], or, alternatively, the domain wall [4, 5] and overlap constructions [6] all involve non-local interactions. As a result, their implementation is numerically much more expensive than conventional Wilson [7] or staggered [8] fermions.

Minimally doubled fermions [9–13] share the desirable features of strict locality with traditional discretizations, whilst preserving exact chiral symmetry for a degenerate doublet of quark fields. The question whether or not they are suitable for the determination of hadronic properties in practical simulations has not been thoroughly investigated so far. Before embarking on extensive numerical studies of minimally doubled fermions, it is useful to examine some of their properties in perturbation theory. In this letter, we present our results for the quark self-energy and the renormalization properties of quark bilinears at one loop in lattice perturbation theory. In particular, we shall elucidate the consequences of the breaking of hyper-cubic symmetry, which is a typical feature of this class of lattice actions. Our main findings indicate that hyper-cubic symmetry breaking generates a renormalization of the quark’s four-momentum.

After fixing our notation in section 2, we list expression for the quark propagator and the vertices in section 3. The perturbative calculation of the quark self-energy at one loop in lattice perturbation theory is presented in section 4. Sections 5 and 6 discuss the renormalization properties of quark bilinears and provide expressions for the conserved vector and axial-vector currents. In section 7 we present our conclusions and discuss the consequences of our results for numerical simulations.

2. Following the works of Boriçi [12] and Creutz [13] we employ the particular construction of minimally doubled fermions of ref. [13]. The corresponding lattice Dirac operator respects chiral symmetry, and is also $O(a)$-improved. For massless quarks, the general expression reads

\[ D = D + \mathcal{D} - 2i\Gamma. \]

In momentum space the terms $D$ and $\mathcal{D}$ are given by

\[ D(p) = i \sum_\mu (\gamma_\mu \sin p_\mu), \quad \mathcal{D}(p) = i \sum_\mu (\gamma'_\mu \cos p_\mu). \]

The matrices $\Gamma$ and $\gamma'_\mu$ are defined by

\[ \Gamma = \frac{1}{2} \sum_\mu \gamma_\mu, \quad \gamma'_\mu = \Gamma \gamma_\mu \Gamma = \Gamma - \gamma_\mu, \]

with $\Gamma^2 = 1$. Other useful relations are

\[ \sum_\mu \gamma_\mu = \sum_\mu \gamma'_\mu = 2\Gamma, \quad \{\Gamma, \gamma_\mu\} = 1, \quad \{\Gamma, \gamma'_\mu\} = 1. \]

The construction of the Dirac operator $D$ involves a particular linear combination of two (physically equivalent) naïve fermion actions, corresponding to $D$ and $\mathcal{D}$ in eq. (1). The first
term, as is widely known, has 16 zeros in the first Brillouin zone, when any component of \( p \) is equal to 0 or \( \pi \). The term \( \overline{D} \) has also 16 zeros, which are, however, positioned at the points where \( p_\mu = \pm \pi/2 \).

As was shown in ref. [13], the presence of the term \(-2i\Gamma\) in eq. (1) guarantees that \( D(p) \) exhibits only two Fermi points, located at \( p = (0,0,0,0) \) and \( p = (\pi/2,\pi/2,\pi/2,\pi/2) \), which represent two degenerate fermion species of opposite chirality. Here we simply note that the extra zeros of \( D \) at the corners of the Brillouin zone are lifted by the presence of \( \overline{D} - 2i\Gamma \). An entirely similar statement applies to the zeros of \( \overline{D} \).

The matrix \( \Gamma \) is not unique: There are altogether 16 definitions of \( \Gamma \), in which the coefficient of a particular gamma-matrix can be chosen as 1 or \(-1\). Each choice selects different zeros of \( D \), which correspond to the physical degrees of freedom, but otherwise all such definitions yield an equivalent theory.

The inclusion of the term proportional to \( \Gamma \) implies that the action is no longer symmetric under the full hyper-cubic group. Depending on its definition, the matrix \( \Gamma \) selects a particular direction in Euclidean space. The action of minimally doubled fermions is only symmetric with respect to a subgroup of the full hyper-cubic group, which preserves this fixed direction (up to a sign). For the action considered in this paper, which corresponds to the definition of \( \Gamma \) in eq. (3), this is the positive major diagonal. For other minimally doubled actions, such as those considered in refs. [9,10], it is the temporal axis.

The breaking of hyper-cubic symmetry implies the possibility of mixing with operators of different dimensionality, such as \( \psi \Gamma \Box \psi \). In refs. [14–16] it was argued that mixing with dimension-3 operators cannot be avoided. In any case, the lack of hyper-cubic symmetry generates not only mixing with dimension-3 operators, but also mixing with marginal operators of dimension 4. In the next two sections we will show that hyper-cubic symmetry breaking generates a linearly divergent additive renormalization of the quark’s momentum.

3. By inverting the Dirac operator of eq. (1), and restoring the proper factors of \( a \), we obtain the fermion propagator

\[
S(p) = a - i \sum_\mu \left[ \gamma_\mu (\sin ap_\mu - \cos ap_\mu) - i\Gamma(\sum_\mu \cos ap_\mu - 2) + am_0 \right]
\sin ap_\mu \sum_\nu \cos ap_\nu - 2 \sin ap_\mu (\cos ap_\mu + 1) - 2 \cos ap_\mu \] + 8 + (am_0)^2.
\]

Unlike many standard fermionic discretizations one finds that the denominator of this propagator cannot be cast into a form which possesses a definite behaviour under parity transformation in each single coordinate \((p_i \to -p_i)\). This is not surprising in view of the fact that the definition of \( \Gamma \) singles out an intrinsic Euclidean direction.

By using the identities \( \{ \gamma_\mu, \gamma_\nu \} = \{ \gamma_\mu', \gamma_\nu' \} = 2\delta_{\mu\nu} \) and \( \{ \gamma_\mu, \gamma_\nu' \} = 1 - 2\delta_{\mu\nu} \), the above propagator can also be written in a form which is more convenient for lattice perturbation theory, i.e.

\[
S(p) = a \frac{-i \sum_\mu \left[ \gamma_\mu \sin ap_\mu - 2 \gamma_\mu' \sin^2 ap_\mu/2 \right] + am_0}{4 \sum_\mu \left[ \sin^2 ap_\mu/2 + \sin ap_\mu \left( \sin^2 ap_\mu/2 - \frac{1}{2} \sum_\nu \sin^2 ap_\nu/2 \right) \right] + (am_0)^2},
\]

where the limit of small \( p \) (i.e. the continuum limit) becomes more transparent. In particular, one can use the standard methods for the treatment of lattice divergences, and calculate for example the so-called \( J \) and \( I - J \) parts along the lines of [17].
via the substitution \( a p_\mu \to \pi/2 + a p_\mu \) one obtains the propagator for the fermionic mode associated with the Fermi point at \( a p_\mu = \pi/2 \):

\[
S'(p) = e^{-i \sum_\mu \left[- \gamma'_\mu \sin a p_\mu - 2 \gamma_\mu \sin^2 a p_\mu/2 \right] + am_0}
\]

\[
\frac{-i}{4 \sum_\mu \left[ \sin^2 a p_\mu/2 - \sin a p_\mu \left( \sin^2 a p_\mu/2 - \frac{1}{2} \sum_\nu \sin^2 a p_\nu/2 \right) \right] + (am_0)^2}.
\]

By changing the direction of the four-momentum \( p_\mu \) and exchanging \( \gamma_\mu \) with \( \gamma'_\mu \), one recovers the propagator of eq. (4). Since \( \gamma'_5 = -\gamma_5 \) this implies that the modes corresponding to the two Fermi points have indeed opposite chirality.

The quark-quark-gluon vertex is derived as

\[
V_1(p_1, p_2) = -i g_0 \left( \gamma_\mu \cos \frac{a(p_1 + p_2)_\mu}{2} - \gamma'_\mu \sin \frac{a(p_1 + p_2)_\mu}{2} \right),
\]

and the quark-quark-gluon-gluon vertex is

\[
V_2(p_1, p_2) = \frac{i}{2} a g_0^2 \left( \gamma_\mu \sin \frac{a(p_1 + p_2)_\mu}{2} + \gamma'_\mu \cos \frac{a(p_1 + p_2)_\mu}{2} \right),
\]

where \( p_1 \) and \( p_2 \) are the incoming and outgoing quark momenta at the vertex. Following ref. [18], the expressions for the vertices can be easily derived by comparing the Dirac operator of minimally doubled fermions, eq. (1), with the Wilson-Dirac operator

\[
D_w(p) = \frac{1}{a} \sum_\mu \left\{ i \gamma_\mu \sin a p_\mu + r (1 - \cos a p_\mu) \right\} + m_0,
\]

and noting that the hopping terms of these two actions are related by the replacement \( r \to -i \gamma'_\mu \). Indeed, since the terms \( e^{iap_\mu} \) and \( e^{-iap_\mu} \) are coupled to the Fourier transforms of \( U_\mu(x) \) and \( U_\mu(x-a\tilde{\mu}) \) respectively, it is sufficient to substitute \( r \to -i \gamma'_\mu \), in order to obtain the vertices for minimally doubled fermions from those of the Wilson case.

4. We are now going to describe the calculation of the quark self-energy at one loop. Figure 1 lists all diagrams which are relevant for the perturbative calculation in this letter.

Using the expression for the vertex \( V_2(p, p) \), the tadpole contribution to the self energy (diagrams (g) and (h) in Fig. 1) is easily computed. In a general covariant gauge, where \( \partial_\mu A_\mu = 0 \), the expression is

\[
\frac{1}{a^2} \cdot \frac{Z_0}{2} \left( 1 - \frac{1}{4}(1 - \alpha) \right) \cdot i a g_0^2 C_F \sum_\mu \left( \gamma_\mu a p_\mu + (\Gamma - \gamma_\mu)(1 + O(a^2)) \right)
\]

\[
= a g_0^2 C_F \frac{Z_0}{2} \left( 1 - \frac{1}{4}(1 - \alpha) \right) \left( i \psi + \frac{1}{a} \sum_\mu (\Gamma - \gamma_\mu) \right) + O(a),
\]

where \( Z_0 \) is given by [19–21]

\[
Z_0 = \int_{-\pi/a}^{\pi/a} d^4 p \frac{1}{(2\pi)^4 \vec{p}^2} = 0.1549333 \ldots = 24.466100 \frac{1}{16\pi^2}, \quad \vec{p}^2 = \frac{4}{a^2} \sum_\mu \sin^2 \left( \frac{ap_\mu}{2} \right).
\]

Terms of \( O(a) \) and higher are not important here. Since \( \sum_\mu \gamma_\mu = 2\Gamma \), the result of the one-loop tadpole is

\[
g_0^2 C_F \frac{Z_0}{2} \left( 1 - \frac{1}{4}(1 - \alpha) \right) \left( i \psi + \frac{2i\Gamma}{a} \right).
\]
already noted in [14], would imply a power-divergent mixing of order 1
three operator
the contribution of the sunset diagrams to the self-energy (diagrams (e) and (f) in Fig. 1). In
The term proportional to ...
W e have computed the sunset diagram using special computer codes written in FORM
Figure 1: The diagrams needed for the one-loop renormalization of the lattice operators.

The term proportional to $i\bar{\psi}\gamma^a\psi$ is the same as for Wilson fermions, while the other term, as already noted in [14], would imply a power-divergent mixing of order $1/a$ with the dimension-three operator $\bar{\psi}\Gamma\psi$, provided that there is no cancellation by an analogous term coming from the contribution of the sunset diagrams to the self-energy (diagrams (e) and (f) in Fig. 1). In this section we show that there is no such compensation.

We have computed the sunset diagram using special computer codes written in FORM [22,23] and Mathematica, and also checked it against calculations by hand. The result of this diagram is

$$
\Sigma^{\text{sunset}}(p, m_0) = i\bar{\psi}\gamma^a \frac{g_0^2}{16\pi^2} C_F \left[ \log a^2 p^2 - 5.42642 + (1 - \alpha) \left( - \log a^2 p^2 + 7.850272 \right) \right]
$$

$$
+ m_0 \cdot \frac{g_0^2}{16\pi^2} C_F \left[ 4 \log a^2 p^2 - 29.48729 + (1 - \alpha) \left( - \log a^2 p^2 + 5.792010 \right) \right]
$$

$$
+ 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F \cdot i \Gamma \sum_{\mu} p_\mu + (5.07558 + 6.11653 (1 - \alpha)) \cdot \frac{g_0^2}{16\pi^2} C_F \cdot i \frac{\Gamma}{a}.
$$

(14)
Note that gauge invariance forces the terms proportional to \((1 - \alpha)\) to be the same as, for example, in the case of Wilson or overlap fermions. This is an important check of the correctness of our calculations.

The total contribution of the one-loop diagrams to the quark self-energy is then

\[
\Sigma(p, m_0) = \frac{g_0^2}{16\pi^2} C_F \left[ \log a^2 p^2 + 6.80663 \right. \\
+ \left. (1 - \alpha) \left( -\log a^2 p^2 + 4.792010 \right) \right],
\]

where

\[
\Sigma_1(p) = \frac{g_0^2}{16\pi^2} C_F \left[ 4 \log a^2 p^2 - 29.48729 \right. \\
+ \left. (1 - \alpha) \left( -\log a^2 p^2 + 5.792010 \right) \right],
\]

\[
c_1(g_0^2) = 1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F,
\]

\[
c_2(g_0^2) = 29.54170 \cdot \frac{g_0^2}{16\pi^2} C_F.
\]

As indicated above, the two terms proportional to \(\Gamma/a\) arising from the tadpole and the sunset diagrams do not cancel — they actually reinforce each other. Note, however, that the parts proportional to \((1 - \alpha)\) cancel exactly, as required by gauge invariance.

The full inverse propagator at one loop can be written as

\[
\Sigma^{-1}(p, m_0) = \left( 1 - \Sigma_1 - \frac{c_1}{2} \right) \cdot \left\{ i\not{\! p} + m_0 \left( 1 - \Sigma_2 + \Sigma_1 + \frac{c_1}{2} \right) - \frac{ic_1}{2} \sum_{\mu \neq \nu} \gamma_\mu p_\nu - \frac{ic_2}{a} \Gamma \right\},
\]

where we have collected all terms proportional to \(i\not{\! p}\) in the wave-function renormalization, which then contains, in addition to the standard term \(\Sigma_1\), also \(c_1\). By contrast, the linear divergence (i.e. the term proportional to \(c_2\)) must be absorbed into a redefinition of the four-momentum, which amounts to a uniform additive shift,

\[
p'_\mu = p_\mu - \frac{c_2(g_0^2)}{2a}.
\]

After replacing \(p\) by \(p'\) and neglecting terms of \(O(g_0^4)\), we obtain

\[
\Sigma(p', m_0) = \frac{Z_2}{i\not{\! p}' + Z_m m_0 - \frac{ic_1}{2} \sum_{\mu \neq \nu} \gamma_\mu p_\nu},
\]

where the wave-function renormalization at one loop is given by

\[
Z_2 = \left( 1 - \Sigma_1 - \frac{c_1}{2} \right)^{-1},
\]

while

\[
Z_m = 1 - \left( \Sigma_2 - \Sigma_1 - \frac{c_1}{2} \right).
\]
is the result for the quark mass renormalization factor.

Our results demonstrate that the self-energy generates a power-divergent mixing with an operator of the form $\Gamma/a$. As can be seen from the Dirac structure, this mixing is not a renormalization of the mass. Indeed, chiral symmetry protects the quark mass against an additive renormalization like in the Wilson case. Rather, the power-divergent mixing implies that all components of the four-momentum $p_\mu$ are shifted under renormalization by an equal amount, i.e. $p_\mu \rightarrow p'_\mu = p_\mu + \text{const.}/a$. The constant can be determined either order by order in perturbation theory, or at the non-perturbative level in a Monte Carlo simulation. For instance, in our perturbative calculation it is given by $-c_2(g_0^2)/2$.

It is important to realize that the term proportional to $c_1(g_0^2)$ cannot be absorbed into a redefinition of $p_\mu$, since otherwise the conserved vector and axial-vector currents do not have unit normalization. We address this issue in more detail in section 6. Thus, the renormalized quark propagator, eq. (22), contains also a term $\sum_{\mu\neq\nu} \gamma_\mu p_\nu$. This should not come as a surprise, since the present formalism is no longer isotropic. Note that the presence of this term does not move the pole of the propagator at $p = 0$.

Since the mass is protected from an additive renormalization, the redefinition of the four-momentum amounts to a renormalization of the velocity. Noting that $p_\mu$ is a fermionic momentum (it is the external momentum of the fermionic self-energy), and that no such phenomenon can occur for the gluonic self-energy, we can interpret this mixing as a renormalization of the quark velocity. These findings support Creutz’s conjecture [11, 13] that “interactions at finite lattice spacing can result in the gluons and fermions not having the same speed of light.”

We remark that the power-divergent mixing proportional to $c_2$, as well as the one of the same dimensionality (proportional to $c_1$), occur among operators which are not invariant under the hyper-cubic group. Such mixings are lattice artefacts which are peculiar to minimally doubled fermions.

Once the above subtraction has been made, and $p_\mu$ replaced with $p'_\mu$ in all renormalized quantities, the power divergence disappears. Although we have no proof, it is not unreasonable to expect that this can be done consistently at every order of perturbation theory, similar to the subtraction of the $1/a$ divergence in the self-energy of Wilson fermions, which is consistently removed from the theory by the replacement $m_0 \rightarrow m_q = m_0 - m_{cr}$.

One may wonder how the redefinition of the four-momentum affects numerical simulations and how it could be determined non-perturbatively. We postpone this discussion to our conclusions in section 7.

5. We have also computed the renormalization factors of local bilinears, and we list the results for the vertex diagrams below. The complete renormalization factors are obtained after including wave-function renormalization, which is achieved by adding the contributions of $\Sigma_1$ and $c_1$, eqs. (16) and (15), of the self-energy. For the scalar density the result for the vertex (diagram (a) in Fig. 1) is

$$\frac{g_0^2}{16\pi^2} C_F \left[ -4 \log a^2 p^2 + 29.48729 + (1 - \alpha) \left( \log a^2 p^2 - 5.792010 \right) \right]. \quad (25)$$

Here there is no mixing term arising from the breaking of hyper-cubic invariance. The only such mixing occurs after adding the wave-function contribution, which includes the term.
proportional to \( c_1 (g_0^2) \).

For the vector current the vertex diagram yields

\[
\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[ -\log a^2 p^2 + 9.54612 + (1 - \alpha) \left( \log a^2 p^2 - 4.792010 \right) \right] + c_{vtx}^v (g_0^2) \cdot \Gamma \tag{26}
\]

where the coefficient of the mixing is given by

\[
c_{vtx}^v (g_0^2) = -0.10037 \cdot \frac{g_0^2}{16\pi^2} C_F. \tag{27}
\]

This is a mixing with an operator of the same dimension, which is not invariant under the hyper-cubic group. Note that there can be no power-divergent mixing here (and in all the other bilinears), as one can see by simple dimensional counting.

As a consequence of chiral symmetry, the vertex corrections are identical for the vector and axial-vector currents, and the same is true also for the scalar and pseudo-scalar densities. We have verified this in the course of our calculations. After taking the renormalization of the wave-function into account, the renormalization factors \( Z_V \) and \( Z_A \) of the local vector and axial-vector currents are not equal to one. In order to identify the conserved currents, which are protected against renormalization, one has to consider the chiral Ward identities. We postpone this discussion to the next section.

Finally, for the tensor current we obtain the result for the vertex diagram as

\[
\frac{g_0^2}{16\pi^2} C_F \sigma_{\mu\nu} \left[ 2.16548 + (1 - \alpha) \left( \log a^2 p^2 - 3.792010 \right) \right]. \tag{28}
\]

Again, the breaking of hyper-cubic invariance does not generate any extra mixing, apart from the one arising from the self-energy.

We end this section with a brief comment on the renormalization of the quark mass. Chiral symmetry protects the bare quark mass \( m_0 \) from undergoing an additive renormalization. The relation between the bare and renormalized quark masses, \( m_0 \) and \( m_R \), respectively, is then obtained via

\[
m_R = Z_m m_0, \tag{29}
\]

where \( Z_m \) is given in eq. \((24)\). The full expression for the renormalization factors of the scalar and pseudo-scalar densities in perturbation theory at one loop is

\[
Z_S = Z_P = 1 - \left( \Lambda_S + \Sigma_1 + \frac{c_1}{2} \right), \tag{30}
\]

where the results for the self-energy contributions \( \Sigma_1 \), and \( c_1 \) are given in eqs. \((16)\) and \((18)\). Here \( \Lambda_S \) is the result for the one-loop vertex diagram of the scalar density, given in eq. \((25)\), which is exactly equal to the \( O(g_0^2) \)-contribution to the quark self-energy \( \Sigma_2 \), but comes with an opposite sign. Thus, when we compare with eq. \((24)\), we see that the renormalization factors \( Z_S \) and \( Z_P \) of the scalar and pseudo-scalar densities satisfy

\[
1/Z_m = Z_S = Z_P, \tag{31}
\]

where the last equality is a consequence of chiral symmetry. We have thus verified at one loop in perturbation theory, that the renormalization of the quark mass for minimally doubled fermions has the same form as, say, in the case of overlap fermions.
It is important to recall that the Dirac spinor in this expression describes a degenerate doublet of quarks. The action in the massless case is invariant under an axial U(1) transformation, and it is then clear that the chiral Ward identities associated with this exact symmetry yield the isospin-singlet currents of the theory.

If one applies the usual vector and axial transformations, i.e.

\[
\delta_V \psi = i \alpha_V \psi, \quad \delta_V \overline{\psi} = -i \overline{\psi} \alpha_V,
\]
\[
\delta_A \psi = i \alpha_A \gamma_5 \psi, \quad \delta_A \overline{\psi} = i \overline{\psi} \alpha_A \gamma_5,
\]

one identifies the conserved vector current as

\[
V^c_\mu(x) = \frac{1}{2} \left( \overline{\psi}(x) (\gamma_\mu + i \gamma'_\mu) U_\mu(x) \psi(x + a\hat{\mu}) + \overline{\psi}(x + a\hat{\mu}) (\gamma_\mu - i \gamma'_\mu) U_\mu^\dagger(x) \psi(x) \right),
\]

while the axial-vector current (which is conserved in the massless case) is given by

\[
A^a_\mu(x) = \frac{1}{2} \left( \overline{\psi}(x) (\gamma_\mu + i \gamma'_\mu) \gamma_5 U_\mu(x) \psi(x + a\hat{\mu}) + \overline{\psi}(x + a\hat{\mu}) (\gamma_\mu - i \gamma'_\mu) \gamma_5 U_\mu^\dagger(x) \psi(x) \right).
\]

Below we list the results for the individual diagrams of the conserved vector current. Due to chiral symmetry, the corresponding expressions for the conserved axial current are trivially obtained by replacing \( \gamma_\mu \) with \( \gamma_\mu \gamma_5 \), and \( \Gamma \) with \( \Gamma \gamma_5 \). The vertex (diagram (a) in Fig. 1) gives the result

\[
\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[ - \log a^2 p^2 + 0.61800 + (1 - \alpha) \left( \log a^2 p^2 - 1.73375 \right) \right] + c_{\text{vc}}^{\text{vtx}}(g_0^2) \cdot \Gamma,
\]

where the mixing coefficient \( c_{\text{vc}}^{\text{vtx}} \) is given by

\[
c_{\text{vc}}^{\text{vtx}}(g_0^2) = -0.43749 \cdot \frac{g_0^2}{16\pi^2} C_F.
\]

The result for the sails (diagrams (b) and (c) in Fig. 1) is

\[
\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[ 4.80841 - 6.11653 (1 - \alpha) \right] + c_{\text{vc}}^{\text{sls}}(g_0^2) \cdot \Gamma,
\]

where \( c_{\text{vc}}^{\text{sls}} \) is obtained as

\[
c_{\text{vc}}^{\text{sls}}(g_0^2) = -1.09017 \cdot \frac{g_0^2}{16\pi^2} C_F.
\]

Finally, the operator tadpole (diagram (d) in Fig. 1) gives the same result as for Wilson fermions:

\[
- g_0^2 C_F \gamma_\mu \frac{Z_0}{2} \left( 1 - \frac{1}{4}(1 - \alpha) \right).
\]
Summing up all contributions gives
\[
\frac{g_0^2}{16\pi^2} C_F \gamma_\mu \left[ - \log a^2 p^2 - 6.80664 + (1 - \alpha) \left( \log a^2 p^2 - 4.79202 \right) \right] + c_{vc}(g_0^2) \cdot \Gamma, \quad (41)
\]
where the total mixing coefficient is given by
\[
c_{vc}(g_0^2) = -1.52766 \cdot \frac{g_0^2}{16\pi^2} C_F. \quad (42)
\]
These numbers exactly compensate the contributions of \(\Sigma_1(p)\) and \(c_1\) of eqs. (16) and (18) of the quark self-energy. Although we are not yet able to give an algebraic proof that the term \(i c_1 \sum_{\mu \neq \nu} \gamma_\mu p_\nu\) in the denominator of the self-energy, eq. (22), cancels the unwanted contribution of the conserved currents proportional to \(\Gamma\), we know that this has to happen, for otherwise the renormalization factors of the conserved currents would be different from one. We have explicitly derived these currents using chiral Ward identities, corresponding to transformations which leave the Lagrangian invariant, and this proves that these currents are conserved. Thus, it is certain that such cancellation must occur, and the one-loop result \(c_{vc} = -c_1\) (which holds to all significant digits that we have achieved) lends support to this statement.

It is for the above reasons why the term proportional to \(c_1\) cannot be absorbed into the redefinition of the four-momentum, as this would spoil the correct renormalization of the conserved currents. Our one-loop calculation has thus confirmed within our numerical precision that the renormalization factors of these currents is unity, as expected. It is remarkable that the use of the conserved currents exactly cancels not only the self-energy terms that contribute to the multiplicative renormalization, but also the mixing with dimension-four operators, coming from hyper-cubic symmetry breaking. Of course, radiative corrections to quark bilinears cannot generate any terms proportional to \(1/a\), and so an uncancelled, power-divergent factor arising from the self-energy remains in the total renormalization constant. However, as already stated, the latter can be absorbed into a redefinition of the renormalized four-momentum. By contrast, for the local vector and axial-vector currents any mixing coming from the hyper-cubic breaking remains uncancelled after the self-energy has been added to the vertex diagram.

7. In this article we have presented the first perturbative study of a particular realization of minimally doubled fermions at one-loop order. Our analysis has shown that minimally doubled fermions are described at one loop by a fully consistent quantum field theory. We have elucidated the consequences of the breaking of hyper-cubic symmetry. In particular, we found that in the Borici-Creutz construction [12, 13], all components of the four-momentum \(p_\mu\) undergo a subtraction under renormalization, which consists of a uniform shift.

Furthermore, local vector and axial-vector currents can mix with other operators of the same dimension which are not invariant under the hyper-cubic group. By contrast, no such mixing occurs for the scalar density and the tensor current. We have derived expressions for the conserved isospin-singlet vector and axial currents, which involve only nearest-neighbour points. They do not undergo any mixing, and we have verified that their renormalization constants are one. In fact, apart from the staggered formulation, minimally doubled fermions are the only discretization which yields a simple expression for a conserved (point-split) axial-vector current.

It remains to discuss the implications of our findings for practical simulations. Since there is one exact chiral \(U(1) \otimes U(1)\) symmetry, there must be exactly one Goldstone boson as
the quark mass is tuned to zero. It is natural to associate the neutral pion with this particle. The charged pions, by contrast, will retain a finite mass in the chiral limit at non-zero lattice spacing. An interesting observation is that the influence of disconnected diagrams, which are required for the determination of the $\pi^0$ mass, must become weaker as the lattice spacing goes to zero, since the exact masslessness of the charged pions is recovered in the continuum limit.\footnote{We thank Mike Creutz for clarifying this issue.}

Even though the neutral pion in the chiral limit is exactly massless in this theory, the renormalization of the four-momentum will modify the rate of the exponential fall-off of its two-point correlation function. Then, the extracted energies will be different for every hadron from the ones given by the dispersion relations, and in particular the rest energy of the neutral pion will not be zero (in the chiral limit).

We can infer from our perturbative calculations, eq. (21), that this renormalization is governed by one parameter, $c_2(g_0^2)$, with a further explicit dependence on $a$. It remains to be investigated whether this functional form is preserved at higher loop order and also non-perturbatively, and what will be the practical prescriptions that one has to infer from it. What is clear is that in numerical simulations the subtraction would depend on $\beta$ and $a$ in a different way from what the Callan-Symanzik renormalization group equations dictate in the scaling region. This derives from the further explicit dependence on the lattice spacing of the term multiplying $c_2$.

We end our conclusions with a discussion of a possible strategy to determine the renormalized momentum non-perturbatively. If we denote the unsubtracted momentum by $p_\mu$, then the $\pi^0$ correlation function at large Euclidean times will be

$$\sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \langle \pi^0(x) \pi^0(0) \rangle \propto A e^{-p_0 x_0}, \quad x_0 \rightarrow \infty,$$

(43)

where $p_0$ is a function of the bare quark mass, $m_0$, and the injected three-momentum $\vec{p}$. The relation to the temporal component of the renormalized momentum, $p_{\mu;\text{ren}}$, is then given by

$$p_0(\vec{p}; m_0) = p_{0;\text{ren}}(\vec{p}, p^c_\mu; m_0),$$

(44)

where $p^c_\mu$ denotes the value of the four-momentum for which the energies of the hadrons are restored to their physical values. Since the $\pi^0$ is exactly massless in the chiral limit, the above relation can serve as a renormalization condition to fix the value of $p^c_\mu$. To this end, one has to evaluate $p_0(\vec{p}; m_0)$ for several values of $m_0$ and $\vec{p}$ and determine $p^c_\mu$ by implicitly solving the relation

$$\lim_{m_0 \rightarrow 0} p_0(\vec{p}; m_0) \bigg|_{\vec{p} = \vec{p}^c} = p^c_0.$$

(45)

The energies in all hadronic channels can then be computed by setting the momenta to $p^c_0$.

One may wonder if other variants of minimally doubled fermions, such as the proposals by Karsten \cite{9} and Wilczek \cite{10} are easier to implement in simulations. We are currently investigating this issue, but leave a more thorough discussion to a future publication \cite{25}.

Acknowledgments We thank Mike Creutz for useful discussions and comments on the manuscript. This work was supported by Deutsche Forschungsgemeinschaft (SFB443) and Gesellschaft für Schwerionenforschung, GSI.
References

[1] P.H. Ginsparg and K.G. Wilson, Phys. Rev. D25 (1982) 2649.
[2] M. Lüscher, Phys. Lett. B428 (1998) 342, hep-lat/9802011.
[3] P. Hasenfratz, Nucl. Phys. B (Proc. Suppl.) 63 (1998) 53, hep-lat/9709110.
[4] D.B. Kaplan, Phys. Lett. B288 (1992) 342, hep-lat/9206013.
[5] V. Furman and Y. Shamir, Nucl. Phys. B439 (1995) 54, hep-lat/9405004.
[6] H. Neuberger, Phys. Lett. B417 (1998) 141, hep-lat/9707022; Phys. Lett. B427 (1998) 353, hep-lat/9801031.
[7] K.G. Wilson, Phys. Rev. D10 (1974) 2445.
[8] J.B. Kogut and L. Susskind, Phys. Rev. D11 (1975) 395; L. Susskind, Phys. Rev. D16 (1977) 3031.
[9] L.H. Karsten, Phys. Lett. B104 (1981) 315.
[10] F. Wilczek, Phys. Rev. Lett. 59 (1987) 2397.
[11] M. Creutz, JHEP 04 (2008) 017, arXiv:0712.1201.
[12] A. Boriçi, Phys. Rev. D78 (2008) 074504, arXiv:0712.4401.
[13] M. Creutz, (2008), arXiv:0808.0014.
[14] P.F. Bedaque, M.I. Buchoff, B.C. Tiburzi and A. Walker-Loud, Phys. Lett. B662 (2008) 449, arXiv:0801.3361.
[15] P.F. Bedaque, M.I. Buchoff, B.C. Tiburzi and A. Walker-Loud, Phys. Rev. D78 (2008) 017502, arXiv:0804.1145.
[16] M.I. Buchoff, (2008), arXiv:0809.3943.
[17] H. Kawai, R. Nakayama and K. Seo, Nucl. Phys. B189 (1981) 40.
[18] A. Boriçi, (2008), arXiv:0812.0092.
[19] A. González-Arroyo and C.P. Korthals-Altes, Nucl. Phys. B205 (1982) 46.
[20] R.K. Ellis and G. Martinelli, Nucl. Phys. B235 (1984) 93.
[21] S. Capitani, Phys. Rept. 382 (2003) 113, hep-lat/0211036.
[22] J.A.M. Vermaseren, (2000), math-ph/0010025.
[23] J.A.M. Vermaseren, Nucl. Phys. Proc. Suppl. 183 (2008) 19, arXiv:0806.4080.
[24] M. Bochicchio, L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, Nucl. Phys. B262 (1985) 331.
[25] S. Capitani, J. Weber and H. Wittig, in preparation.