I. Introduction And Summary

Physical cryptography, the use of physical effects in addition to purely mathematical artifice for fast reliable cryptographic functions, has received considerable recent attention. For key generation, variously called key distribution, key expansion, key exchange or key agreement, the use of classical noise was first proposed [1]-[3] while the use of information-disturbance tradeoff in BB84 type protocols are the well known quantum key distribution (QKD) schemes [4]-[5]. Other quantum schemes that dispense with intrusion level estimation have been developed on the basis of incompatible quantum measurements in the KCQ (keyed communication in quantum noise) approach [6]-[9]. The fundamental merit of these physical cryptographic schemes is that the so-called information theoretic (IT) security is possible, in contrast to the expansion of a master key to session keys or key agreement from public-key protocols with security based on computational complexity. It is the purpose of this paper to demonstrate in detail that, contrary to widespread perception and belief, (i) it is not clear how strong IT security can even be obtained in principle from QKD; (ii) the security guarantee that can be experimentally obtained thus far is quite inadequate. Some assessment and suggestion will be made on the current situation. Note that we are not at all concerned with appropriate system modeling or device imperfection issues, but rather just with the fundamental quantitatively achievable security in a concrete protocol with realistic parameters, assuming perfect devices and applicable system model.

There are two main reasons for this present unsatisfactory state of affairs: The problem of quantitative security criteria and the security of the final generated key $K$ during actual use in a cryptography scheme — the so-called compositability problem. We will explain them by comparing a perfect fresh key $K^\text{p}$ and a session key $K'$ generated from a pseudorandom number generator (PRNG) to the key $K$ generated from BB84 type protocols [10]. The comparison between $K'$ and $K$ is meaningful because a shared secret key between the users is also needed in KCQ, and in QKD for message authentication, during the key generation process. The keys obtainable from KCQ or classical noise-based protocols would share all or some of the problems associated with $K$.

A perfect key $K^\text{p}$ has a uniform probability distribution $p(K^\text{p}) = U$ to the attacker (Eve). For $K^\text{p}$ consisting of $n$ bits, there are $2^n$ possible values for $K^\text{p}$ each with a probability $U_i = 2^{-n}$. Furthermore, any piece of additional information Eve may obtain in the actual use of $K^\text{p}$ in any cryptographic scheme should not give Eve more information about $K^\text{p}$. For example, when $K^\text{p}$ is used in one-time pad encryption of some $n$ data bits and $m < n$ bits are revealed in a known-plaintext attack (KPA), Eve still would have a uniform distribution with respect to the remaining $n - m$ bits in $K^\text{p}$. We would call the pre-use quantitative security of $K$ its “raw security” and the in-use security its “composition security”. Composition security guarantee is needed even when the raw security is adequate. In this paper we would deal with composition security mainly on the problem of the extent any bit of $K$ remains secret when some other part of $K$
is known to Eve. For such problem whether \( K \) is perfect for a classical protocol reduces to the question whether \( p(K) = U \), where \( p(K) \) is Eve’s probability distribution on \( K \) obtained through her attack with possible further information gained during the actual use of \( K \). The situation is far more complicated in the quantum case due to the possibility of quantum memory. It is possible to obtain a perfect key in practice when a secure method (but perhaps clumsy such as hand delivery) is used to deliver a randomly picked key string to be shared between two users Alice and Bob. If one generates the key by a public-key technique such as RSA, its security is based entirely on (presumed) computational complexity, i.e., it is (presumably) practically impossible for Eve to determine \( K \) due to the lack of an efficient algorithm even though it is possible in principle. In contrast, \( K^\text{p} \) has IT security that is not changed by Eve’s computational power.

One often hears that the generated key \( K \) in QKD has IT security, by itself a rather misleading statement because the \( K \) obtained in physical cryptography is never perfect just in the sense of \( p(K) = U \). A mistaken claim is maintained in the literature[5],[11]-[12] that \( p(K) = U \) with a high probability if one uses the criterion \( d \) to be discussed later. It is one of the main purposes of this paper to dispel this misconception. In contrast to a QKD generated key \( K \), a fresh key when available can typically be taken to be perfect in standard cryptography as indicated above. A more precise security claim on \( K \) is that it is sufficiently close to perfect. But then the crucial issue of quantitative security criteria arises for measuring the closeness, and in particular why any specific achievable quantitative value is adequately secure in a given application. This problem does not arise in a standard fresh key for which \( p(K) = U \).

There is no fundamental guarantee of IT security from “randomness test” given Eve’s knowledge of the key generation process. The \textit{proper security criterion} on \( K \) is the set of probabilities \( p(K) \) of Eve’s correct estimates on all the possible subsets \( \tilde{K} \) of \( K \) which she could derive from her possible \( p(K) \). Common single-number criterion such as her mutual information on \( K \) has no empirical operational security significance in itself and merely expresses a constraint on Eve’s possible \( p(K) \). For operational guarantee one would need to translate such a criterion into guarantees on \( p(K) \) which would be carried out in this paper.

To avoid possible confusion one may distinguish the following three different logical situations:

(a) A proof of security has been obtained that works against all possible attacks with high probability.

(b) A specific attack has been found that breaches security with high probability.

(c) Security level unknown for various possible attacks.

In this paper we are not talking about case (b). Instead, it is pointed out that case (a) has not been established and case (c) is the current situation contrary to the claims in the QKD literature.

For comparison to the case of \( K \), we first describe the raw and composition security for the key \( K’ \) obtained by PRNG in standard (conventional) cryptography. In typical ”key expansion” scheme a master key \( K^\text{m} \) is fed through a PRNG to generate a key \( K’ \) with many more bits that \( K^\text{m}, |K^\text{m}| < |K’| \). Different segments of \( K’ \) are then used as different session keys in various uses in order to reduce the total number of perfect key bits otherwise needed. On the raw security of \( K’ \), the Shannon measure of information or Eve’s entropy on \( K’ \), \( H_E(K’), \) is often employed. The well known Shannon limit [3],[9],[13]-[14] says

\[
H_E(K’) \leq |K^\text{m}|
\]  

(1)

If \( K’ \) is any subset (subsequence) of \( K’ \), it is possible that \( H_E(K’) = |K’| \) similar to the case of a uniform \( K’ \) even under (1) when \( |K’| \leq |K^\text{m}| \), but no other \( K’ \) can have a uniform distribution since that would violate (1). Thus, with \( H_E(K’), \) taken as the measure of raw security, there is IT security for such session keys in standard cryptography also. It is just that their quantitative level may be far from perfect.

There are other important measures of quantitative IT security: Eve’s maximum probability \( p_1(K) \) of determining the whole key \( K \), her maximum probabilities \( p_1(\tilde{K}) \) of determining various subsets \( \tilde{K} \) of \( K \), and these are the ones with operational significance. In addition to \( H_E \), there is another common single-number measure used in quantifying the randomness of a bit sequence, namely the statistical distance \( \delta(p(K),U) \equiv \delta_E(K) \) between Eve’s probability distribution of \( K \) and \( U \). Generally, from different attacks on a physical cryptosystem and from different measured ciphertexts, Eve would obtain \textit{different} probability distribution \( p(K) \) on \( K \), which would determine the above quantities and whatever other measure one may employ. The significant point and a main difficulty is that no single number, be it \( H_E, p_1 \) or \( \delta_E \), could capture the full security picture in physical key generation with even just one probability distribution \( p(K) \) for Eve.

For raw security, many PRNG including those given just by a (maximum length) linear feedback shift register (LFSR) with perfect seedkey \( K^\text{m} \) has the following behavior [15],

\[
p_1(K’) = 2^{-|K^\text{m}|}, \quad p_1(\tilde{K}) = 2^{-|\tilde{K}|}
\]  

(2)

where \( \tilde{K} \) is a subsequence of \( K’ \) up to \( |K^\text{m}| \) consecutive bits. This is very favorable compared to the \( K \) that can be generated by QKD as will be seen in the next paragraph. However, such \( K’ \) has no IT composition security. Specifically, if \( K’ \) is used in “one-time pad” form, i.e., the PRNG is used as an additive stream cipher with \( K’ \) as running key, then a KPA with \( |K^\text{m}| \) known consecutive
data bits would lead to a unique determination of \( K^m \) for the usual nondegenerate ciphers including LFSRs in their common cipher configurations. The knowledge of \( K^m \) would then allow the complete determination of \( K' \) [14]-[15]. The situation is similar when \( K^p \) is used in a conventional symmetric-key block cipher such as AES.

For \( K \) obtained from QKD, the most commonly used criterion for raw security is \( I_E[K] = 1 - H_E[K]/|K| \), Eve’s information per bit on \( K \) from her attack. Under the quantitative security \( I_E[K] \leq 2^{-l} \), it was shown [9, section IIIB] the possibility is not ruled out that Eve may obtain a correct estimate of \( K \) and \( m \)-bits subsets \( K \) of \( K \) with probabilities, for \( l < |K| \),

\[
p_1(K) \sim 2^{-l}, \quad p_1(\tilde{K}) \sim \frac{n}{m} 2^{-l} \tag{3}
\]

Such possibility arises because the bits in \( K \) are not statistically independent to Eve. In BB84 they have been correlated from error correction and privacy amplification as well as from Eve’s joint attack. For concrete protocols that can be experimentally developed thus far [16], \( l \sim 10 \) for \( |K| > 10^3 \) which means a disastrous breach of security with \( p_1(K) \sim 10^{-3} \) is not ruled out. In any event, even \( l \sim 50 \) is rather unfavorable compared to (2) where \( |K^m| > 100 \) is typical in most conventional ciphers.

Another useful criterion is the statistical distance \( \delta_E(K) \) between \( p(K) \) and \( U \). For \( \delta_E(K) = 2^{-l} \) with \( l \leq |K| \), the possibility remains that [9, App. B]

\[
p_1(K) \sim 2^{-l}, \quad p_1(\tilde{K}) \sim 2^{-l} \tag{4}
\]

It should be noted that a sufficiently small \( p_1(K) \) guarantee is evidently necessary for meaningful security guarantee, see Section IV. It is apparent from (3)-(4) that according to the criterion \( I_E[K] \leq \epsilon \) and \( \delta_E \leq \epsilon \), \( K \) is only nearly uniform when \( \epsilon = 2^{-l/|K|} \) with \( \lambda \sim 1 \), a condition that appears impossible to achieve by a realistic protocol with a significant key generation rate. For the criterion \( I_E[K] \), QKD does much better than PRNG which is responsible for the better composition security of \( K \) as will be shown in this paper.

On the composition security of \( K \) it has been shown [12] that when Eve retains her probe, for \( I_E[K] \sim 2^{-l} \) it may be possible for her to tell the \((l+1)\)th bit of \( K \) knowing \( l \) of them from, say, a KPA on the of use of \( K \) as one-time pad. To overcome this problem, the use of a different criterion \( d \),

\[
d \equiv \frac{1}{2} \| \rho_{KE} - \rho_U \otimes \rho_E \|_1 \leq \epsilon \tag{5}
\]

was suggested and developed [11], [17]. The claim is that under (5) the users would get a perfect key \( K \) with probability at least \( 1 - \epsilon \). This is an incorrect interpretation of (5) as has been mentioned [9, App. B]. Given this prevailing misconception we will give a detailed discussion in section IIIC and bring out the point instead that \( p(K) \) is actually never given by \( U \) for \( d = \epsilon > 0 \) and that \( d \leq \epsilon \) has no clear raw or composition security significance. In any event, theoretical estimates [18]-[19] give \( \epsilon = 10^{-5} \) for various large \( |K| \), corresponding at best to (4) with \( l \sim 17 \).

As will be detailed in the paper, the following problem situation obtains on the security guarantee of the generated key \( K \) in concrete QKD protocols:

(i) The raw security of \( K \) is worse than that of a LFSR for the probabilities \( p_1(K) \) of identify the whole \( K \) by an attacker and \( p_1(\tilde{K}) \) for many smaller subsets \( \tilde{K} \) of \( K \).

(ii) There is no composition security guarantee for \( K \); an exponential decrease of the accessible information may only lead to a linear decrease of the number of compromised key bits while the situation is unknown for \( \delta_E \) or \( d \).

In the following we will flesh out these points and discuss their implications on the development of physical cryptography. In section II we will discuss the raw security of the generated key \( K \). In section III the composition security of \( K \) will be treated which shows the lack of such guarantee thus far for both specific attack scenarios and in general. In Section IV we discuss the relevance of rigorous security proofs and the importance of actual numerical values. Some suggestions on possible future development of physical cryptography is outlined in section V.

II. RAW SECURITY OF THE GENERATED KEY

We will first review the raw security of a running key \( K' \) generated from a master key in standard cryptography for later comparison with the key generated from QKD. Recall that by the raw security of \( K \) we mean the quantitative security level of \( K \) against attacks during the key generation process without any additional information that Eve may obtain during its actual cryptographic use. For a key obtained from public-key technique such as RSA, there is no IT security at all since the key can be determined by Eve if she has sufficient computational power. Thus, we would not further discuss public-key schemes in this paper which deals only with IT security. For the usual symmetric-key ciphers there is IT security for \( K' \) and it is better than that of concrete realistic QKD schemes. As we shall see, this arises as a result of the detailed quantitative behavior of the security measures that have been adopted to describe the quantitative security. We will begin with a discussion of the security measures.

IIA. Security Criterion and Security Parameter

The attacker’s optimal probabilities \( p_1(\hat{K}) \) of correctly estimating \( \hat{K} \), any subset of \( K \), constitute the operationally meaningful criterion on the security of \( K \). In conventional cryptography the closeness of \( p_1(\hat{K}) \) with
p_1(\hat{K}^p) for all \hat{K} is called semantic security [20], whether polynomial complexity is included [21] on an attack algorithm or not [22]. A security parameter \( \epsilon \) may be introduced to measure the deviation of such semantic security from a perfect key \( K^p \), \( \epsilon = 0 \) corresponding to the perfect case.

Such semantic security guarantee is difficult to obtain and to prove. Simpler criterion is often employed instead. However, theoretical constructs such as the attacker's mutual information \( I_E(K) \) on \( K \) has no operational meaning by itself. In the context of communications, these information theoretic quantities derive their empirical significance via the Shannon Coding Theorems through which they are related to operational quantities [23]. In the context of cryptography, operational significance could only be obtained from the security guarantee on the various \( p_1(\hat{K}) \) implied by such single-number criterion. This is what we would spell out in this paper for the common criteria \( I_E(K)/|K| \) and \( \delta_E \).

IIB. Conventional Key Expansion Raw IT Security

Consider the running key \( K' \) generated from a PRNG with master key \( K^m \), with number of bits \( |K'| > |K^m| \). Segments of \( |K'| \) could be used as session keys in different applications. It can also be used as a running key, i.e., in one-time pad form as an additive stream cipher. Let \( X_n \) be the \( n \)-bit data random variable, we use lower case \( x_n \) to denote a specific \( n \)-sequence from \( X_n \). An additive stream cipher would have output \( Y_n \) with \( y_n = x_n \oplus k_n \), where \( k_n \) is a specific value of the \( n \)-bit \( K_n' \). (We add the subscript \( n \) on occasion for clarity.) The general Shannon limit for a standard cipher is given by (1). It holds for any cipher that satisfies unique description \( H(X_n;Y_n,K^m) = 0 \), thus covers also 'random ciphers' with randomized encryption [13]-[14].

To derive (1) the so-called Kerekhoff's assumption [14] is sometimes invoked which says Eve knows everything about the cipher except the seedkey \( K^m \). Such an assumption is not actually needed so long as all the information Alice and Bob need to share but which Eve does not have can be quantified by a proper key \( K^m \). The ciphertext \( y_n \) is also assumed to be available to Eve, evidently a most common situation in reality.

We will discuss several quantitative criteria of security. First Eve has only one probability distribution on the possible \( K_n' \) for nonrandom nondegenerate ciphers. All standard ciphers are nonrandom, i.e., one for which the ciphertext \( y_n \) is uniquely determined by \( x_n \) and the specific \( K^m \) used. From her knowledge on the structure of a nonrandom cipher Eve could generate the at most \( 2^{|K^m|} \) possible sequences of \( K_n' \) for any \( n \) and determine its probability distribution. For \( p(K^m) = U \), each of these \( K_n' \) - sequence would have the same probability \( 2^{-|K^m|} \) under the normal nondegeneracy assumption of exactly \( 2^{|K^m|} \) number of \( K' \) sequences. The case of random ciphers is covered in section IIC that treats classical noise and QKD generated keys.

From \( p(K_n') \) one can determine Eve’s maximum probability \( p_1(K_n') \) of getting the whole \( K_n' \) correctly, which is just \( 2^{-|K^m|} \) in the above typical situation. From (1) Eve’s mutual information per bit on \( K_n' \) is at least

\[
I_E(K_n')/n \geq 1 - H(K_n')/n \quad (6)
\]

The \( \delta_E(K_n) \) is also large for typical \( n > |K^m| \) similar to \( I_E(K_n)/n \). For a maximum length LFSR or more generally nondegenerate ciphers, the probability of various consecutive subsequences \( K' \) of \( K_n' \) with \( |K_n'| < |K^m| \) is given by \( p_1(K') = 2^{-|K'|} \) since such \( K' \) is uniformly distributed. Thus we have arrived at (2) in section I. However, different subsequences of \( K' \) are correlated through \( K^m \) and no overall joint probability on any subset of \( K' \) can be smaller than \( 2^{-|K'|} \).

It is clear from this description what the mechanism of the Shannon limit (1) is: however long \( X_n \) is for \( n > |K^m| \), from \( Y_n \) there are at most \( 2^{|K^m|} \) possible \( X_n \) sequences from \( x_n = y_n \oplus k_n \) or any injective encryption map. Such a cipher is considered adequate in standard cryptography for the protection of long data \( X_n \) although it is far from semantically secure, perhaps partly because there is no alternative that does better information theoretically than what the key size \( |K^m| \) allows which is always far less than \( n = |X_n| \). Perhaps it is partly due to (2), and partly due to the practical complexity of \( p(K') \) evaluation of more general subsets \( K' \). But see [20, section 5.5.3].

IIC. QKD Key Raw Security

The case of classical noise key generation will be described first. In such cryptosystem where noise is involved including randomized encryption systems, there is no longer a fixed observation random variable \( Y_n^E \) for Eve in contrast to the case of the previous section on standard ciphers. In any key generation scheme, the user Alice picks a random bit sequence \( Z_{n'} \) of length \( n' \) and transmits it to Bob via modulated physical channel inputs, who could extract an error free (with high probability) bit sequence \( W_{n''} \) of length \( n'' \leq n' \) from his observation. For example, this can be done with an openly known error correcting code (ECC) so that both Alice and Bob know what \( W_{n''} \) is. Then a “privacy amplification” (PA) function \( f_{PA} \) is applied by both to obtain the \( n \) bit generated key \( K_n = f_{PA}(W_{n''}(Z_{n'})), n < n'' \). One can combine the ECC and PA, to write \( K_n = F(Z_{n''}) \) for an openly known function \( F \). Eve learns about \( K_n \) through an attack with some observed random variable \( Y_{n'}^E \) that depends on \( Z_{n'} \) via \( p(Y_{n'}^E|Z_{n'}) \), her probability of getting various \( Y_{n'}^E \) given the possible \( z_{n'} \). With \( p(Z_{n'}) = U \) and the known ECC and PA, Eve then obtains \( p(K_n|Y_{n'}^E) \), to be called Eve’s conditional probability distribution (CPD) on her estimate of \( K_n \). Note that a single number security criterion is just a constraint on Eve’s possible CPD.
The generation rate $r$ is usually taken to be $r = n/n'$, which is further reduced in BB84 from basis matching.

There is typically one complete $Y_n^E$ that Eve may observe in classical-noise key generation and $Y_n^E$ can be drawn from a continuous alphabet, for example when $p(Y_n^E|Z_n)$ is given by that of an additive noise channel. On the other hand, in QKD and KCQ different incompatible $p(Y_n^E)$ may be obtained from different incompatible quantum measurements, i.e., $p(Y_n^E)$ depends on the specific attack. In all cases the conditioning on different possible observations $y_n^E$ in $p(K_n|Y_n^E)$ is an added complication that does not exist in standard ciphers. We first assume that $y_n^E$ is fixed and order the resulting $p(k_n|y_n^E) = p_i$ for $i \in \{1, \ldots, N = 2^n\} \equiv 1\!\!\!N$, $p_1 \geq p_2 \ldots \geq p_N$.

The most common quantitative security criterion in classical-noise and QKD key generation is Eve's mutual information per bit $I_E(K_n)/n$ must be small, which cannot be satisfied in standard cipher key expansion from (6). The question is: how small is small enough for what security? Note that this question does not arise for the fresh keys in standard cryptography where the IT security can be presumed perfect. It is clear that Eve's maximum probability of getting the whole key $K_n$ must be sufficiently small. With the distribution $p_1, p_2 = \ldots = p_n = (1 - p_1)/(N - 1)$, it has been shown that [9, Section IIIB]

Theorem 1:
For $I_E/n \leq 2^{-l}$, there are CPD that give $p_1 \geq 2^{-l} - \frac{1}{n^2}$. Furthermore, there are CPD [9] that gives $p(\tilde{K}_m) \sim \frac{m}{n} p_1$ when $p_1 > 1/N$ for $m$-bit subsets $\tilde{K}_m$ of $K, m < n$. Together these give the quantitative results (3), which gives the operational significance of a $I_E/n$ guarantee.

In many QKD security proofs Eve’s possible $I_E/n$ is bounded for any possible attack she may launch. Whatever one’s notion of “secure enough” may be in a realistic application, the experimental results thus far [16] that yields at best $l \sim 10$ for $n > 1,000$ is very inadequate as a security guarantee, because it does not rule out the large chance $10^{-3}$ of identifying the whole 1,000 bit key with high probability. See section IVA for a discussion on the role of security proof in this connection. Thus, bounding $I_E/n$ insufficiently just leaves open the possibility of a disastrous breach of security. This situation cannot be expected to improve significantly in concrete realistic protocols because from Theorem 1, decreasing the security parameter $I_E/n$ exponentially only leads to linear increase in the effective number of random bits given by $l$. In this connection, it may be observed that it is misleading to consider “exponentially small” as quantitatively adequate in key generation. See section IV.

In the above $p_1$ for a given CPD we have suppressed the $y_n^E$ dependence. The common criteria $I_E/n$ and $\delta_E$ are averages over all possible $y_n^E$ for any given attack. As guarantee for each $y_n^E$, the Markov inequality [24] for a negative-valued random variable $X$ would need to be employed,

$$Pr[X \geq \epsilon] \leq E[X]/\epsilon$$

where $E[X]$ is the average value of $X$. Generally this would lead to the more stringent requirement from an $E[X] < \epsilon$ to $E[X] < \epsilon^2$ to guarantee $X < \epsilon$ with probability $> 1 - \epsilon$, the latter probability requirement is especially appropriate when $X$ is essentially a probability itself. Thus, by taking $I_E/n \sim p_1$ from (2) the actual full guarantee would reduce the exponent $l$ by $1/2$, making it so much more difficult to achieve a good value in practice.

The criterion of statistical distance ($L_1$-distance, variational distance, Kolmogorov distance) $\delta_E \equiv \delta(P, U)$ between Eve’s CPD $P = \{p_i\}$ and the uniform $U$ can be used,

$$\delta(P, U) = \frac{1}{2} \sum_{i=1}^{N} |p_i - \frac{1}{N}|$$

It is a direct consequence of the definition (8) that [24, p.299]

$$\delta_E \leq \epsilon \implies p(\tilde{K}_m) \leq \epsilon + \frac{1}{2m}$$

where $\tilde{K}_m$ is any $m$-bit subset of $K_n$. As a numerical measure, $\delta_E$ suffers exactly the same $p_1$ problem as $I_E/n$. From the same distribution that gives Theorem 1 for $p_1 = 2^{-l}$, one obtains [9, App B].

Theorem 2:
For $\delta_E = 2^{-l}$, there are CPD that give $p_1 = 2^{-l} + \frac{1}{N}$.

Similar constructions give (9) with $\leq$ replaced by $=$, thus yielding (4) above. Note that (9) gives the operational significance of a $\delta_E$ guarantee. From Theorems 1 and 2 it follows that $K$ is only nearly uniform when $l \sim |K|$. It appears there is no experimental protocol that has been quantified with $\delta_E$ or $d$. Theoretical estimates thus far concentrate on $\epsilon = 10^{-5}$ for $d \leq \epsilon$ with various large $n$ [18]-[19]. Even without using (7) for individual guarantee such $p_1$ is inadequate for many purposes. In particular, it is questionable to say a $10^5$ bit key $K$ has IT security with just $l \sim 17$ as compared to the $l \sim 10^5$ for a really perfect key.

The raw security significance of $d \leq \epsilon$ is unknown as shown in the next subsection. Thus, while QKD may in principle provide better $p_1(K)$ guarantee than conventional key expansion it is much worse in practice and unlikely to significantly improve. This is because it is difficult to capture the possible probability distribution behavior with a mere single parameter value. A good $p_1$ guarantee, by itself or through $I_E/n$ or $\delta_E$, is inadequate unless $p_1 = 2^{-l}$ with $l \sim n$. This exponential-linear problem appears to be a major quantitative stumbling block to physical cryptography.
III. Composition security of the generated key

In this section we will show that composition security has not been guaranteed at all in QKD under $I_E/n$ or $\delta_E$ and their quantum counterparts. Recall that by the composition security of $K$ we mean its quantitative security level against attacks that utilize information obtained both during the key generation process and during the actual cryptographic use of $K$. We will discuss only a specific composition security scenario of partial key leakage (PKL) —whether it is possible for Eve to predict a future bit exactly if part of $K$ is known to her.

III A. Conventional Cipher Composition Security

We first review the case of a standard nonrandom cipher, for which the raw security of $K'$ was discussed in section IIB. For an additive stream cipher $y_n = x_n \oplus k_n$ where $k_n$ is a specific value of $K'$ with distribution $p(K')$ on $\{−N, \ldots, N\}$, Eve would learn $m$ bits of $k_n$ from a KPA where she knows $m < n$ bits of $x_n$ and the openly observed value of $y_n$. From these $m$ bits of $k_n$ Eve could try to determine the seedkey $K^m$ to whatever degree possible and with such knowledge on $K^m$ she could then get information on the other $n − m$ bits in $x_n$ through $y_n$. Such KPA can often be launched in real world situations. It is convenient to assume the $m$ known bits of $x_n$ are its first $m$ bits. For the situation where Eve knows nothing about $x_n$, i.e., $p(X_n) = U$, the key $K'$ and hence $K^m$ is totally hidden from observation of $Y_n$ alone since $p(k_n|y_n)$ becomes a memoryless binary symmetric channel with crossover probability $1/2$. That does not mean, of course, that the security of $X_n$ is good enough as section IIB shows how it is limited by (1). It is possible that Eve knows something about $X_n$ so that to her $p(X_n) ≠ U$ but she does not know any bit in $x_n$ for sure. We will not discuss such scenario of “statistical attack” [13], [14] in this paper and focus on just KPA.

For nondegenerate nonrandom ciphers [13], [14] there is a one-to-one mapping between $K^m$ and $(\{X_{m'}, Y_{m'}\} | m' \in \{−N, \ldots, N\})$, the pairs of $|K^m|$ consecutive data bits in $X_n$ and the corresponding output bits in $Y_n$. This includes block ciphers and their stream cipher modes of operation. Thus, in a KPA with $m = |K^m|$ the key $K^m$ can be uniquely determined and the rest $n − m$ bits in $X_n$ are totally compromised information theoretically. This is the situation in conventional symmetric-key ciphers such as AES where security depends exclusively on the complexity of finding $K^m$ from $(\{X_{m'}, Y_{m'}\})$. Indeed, such weak composition security is a manifestation of the weak $I_E/n$ raw security from the Shannon Limit (6). As will be seen in the next subsection IIB, it is removed (classically) by a strong IT guarantee on the raw security. Thus, the composition security situation of classical key expansion is similar to both the raw and composition security of asymmetrical key ciphers such as RSA whose security depends on the complexity of factoring large integers.

Generally, we have the PKL problem of the extent to which knowledge on one part of the key $K$ would reveal about another part, all through the probability distribution $p(K)$ itself without any further information as in the case of the above KPA. This problem does not exist for a key $K$ with $p(K) = U$, not to mention a perfect key $K^p$. The security against PKL cannot be guaranteed by $p_1$ alone, but it was thought that “exponentially small” $I_E/n$ and $\delta_E$ would be sufficient even in the context of QKD with Eve holding onto her probe with quantum memory. In the rest of this section III we will show that is the case in a purely classical situation but the presence of quantum effect, while allowing key generation with small $I_E/n$ and $\delta_E$, also takes away the composition security guarantee that obtains in a purely classical scenario.

Before proceeding, it may be mentioned that the possibility of IT security on the key against KPA is not ruled out for degenerate ciphers, meaningful versions of which can be developed for classical ciphers with randomized encryption [13]. That is the subject for future detailed treatment.

III B. Composition Security Under $I_E$ and $\delta_E$

In this subsection we develop the composition security significance of an $I_E$ and an $\delta_E$ guarantee on a key $K$ with distribution $p(K)$ to Eve. This applies to any classical protocol directly and also to a quantum protocol through reduction of the quantum security criterion guarantee. The quantum case would be handled in the next subsection.

For a symmetric-key conventional cipher such as AES under KPA or statistical attack, Eve would obtain in general information on $p(K)$ distributed through the whole $K$. We consider the specific case where an $m$-bit subsequence $K_m$ of an $n$-bit $K$ is known exactly to Eve, for example obtained from an $m$-bit KPA for $K$ used as one-time pad, and ask whether any other bit in $K$ would be revealed with a significant probability from such knowledge. It is clear that a $p_1(K)$ guarantee does nothing for this problem unless it is very close to $2^{-n}$, because it applies to just one $x_n$ and it says little about correlations among the $n$ bits of $K$ which is the matter of concern here. On the other hand, since $I_E$ and $\delta_E$ are themselves already constraints on the whole $p(K)$, it may be expected a sufficiently small value would lead to good composition security in this case. The question is how small.

The following result shows that a linear leak of information is possible under $I_E$ in the absence of any quantum effect.

Theorem 3:

With $I_E/n = \epsilon$ for any $0 < \epsilon ≤ 1$ there are $p(K)$ for which Eve knows one additional bit from knowing $\lfloor 1/\epsilon \rfloor$ number of them in an $n$-bit $K$. Equivalently, for such
The proof can be obtained from the following simple construction. Let \( P(k_n) = P(k_1, \ldots, k_n) = 2^{-n} \) for \( n \) bits of an \((n+1)\)-bit \( K \), and let \( k_{n+1} = f(k_n) \) where \( f \) is a known deterministic Boolean function of \( k_n \). Then \( P(k_{n+1}) = P(k_{n+1} | k_n)P(k_n) = 2^{-n} \) from which it follows that \( I_E/(n+1) = 1/(n+1) \). Either by extension to \( m > 1 \) other bits determined by \( k_n \) or by forming a product distribution, the theorem follows.

Thus under a KPA with \( I_E/n \leq 2^{-l} \), one bit may be leaked for every \( 2^l \) known bits. This is perhaps tolerable in some applications when \( l \geq 10 \), although there is still the issue of the distribution of such leaked bits in \( K \). Unfortunately, the quantum situation is much worse as discussed in subsection IIIC with respect to the corresponding accessible information.

The situation for \( \delta_E \) is much more favorable than \( I_E \). It is easy to show by simple counting that no deterministic bit of \( K \) can be leaked this way when \( \delta_E < 1/2 \). It can be shown that for \( \delta_E \leq \epsilon \) Eve could not achieve even a probability of knowing a bit better than \( \epsilon + 1/2 \) compared to 1/2 from pure guessing. We will not dwell on the composition security of a \( \delta_E \) classical guarantee with no quantum probe that has been held in quantum memory, as it appears entirely adequate. On the other hand, as will be discussed in the next subsection IIIC there is no known guarantee in the case when quantum memory is available for a related quantum criterion \( d \). A contrary claim has been repeatedly made in the literature [25] including a recent broad review of QKD security [5]. The error in such a claim can be traced to an incorrect inference on the meaning of the classical statistical distance to be presently discussed.

It was suggested that between two distributions \( P, Q \) for two random variables \( X \) and \( X' \) over the same range \( \chi \) of \( N \) elements, the statistical distance

\[
\delta(P, Q) = \frac{1}{2} \sum_{x \in \chi} |P(x) - Q(x)| \tag{10}
\]

"can be interpreted as the probability that two random experiments described by \( P \) and \( Q \) respectively, are different" [11],[17], an interpretation repeated in refs. [12], [28]. The justification for the interpretation is given by lemma 1 in refs [11], [28] which states that for any two distributions \( P \) and \( Q \) for \( X \) and \( X' \) there exists a joint distribution \( P_{XX'} \) that gives \( P, Q \) as marginals with

\[
Pr[X \neq X'] = \delta(P, Q). \tag{11}
\]

However, to the extent it makes sense to talk about such a joint distribution, the interpretation would obtain only if "there exists" is replaced by "for every". This is because since there is no knowledge on such joint distribution, one cannot assume the most favorable case via "there exists" for security guarantee or for general interpretation. Indeed, it is not clear at all what realistic meaning can be given or claimed for the realization of such a joint distribution, other than the independent case \( P_{XX'} = P \cdot Q \). This independent case is the appropriate one to consider since one is just comparing two distributions \( P \) and \( Q \) with \( \delta(P, Q) \). In such case, even if both \( P \) and \( Q \) are the same uniform distribution so that \( \delta(P, Q) = 0 \), we have \( Pr[X \neq X'] = 1 - \frac{1}{2^n} \) and the two sides of (11) are almost as far apart as it could be since both must be between 0 and 1. This is also a counter-example to the interpretation. As a matter of fact, whether (11) holds is irrelevant to how close \( P \) and \( Q \) are according to \( \delta(P, Q) \).

Furthermore, instead of (11) the following equation (12) is a consequence of the interpretation,

\[
P(x) = (1 - \epsilon)Q(x) + \epsilon P'(x) \tag{12}
\]

where \( P' \) is a probability distribution on \( \chi \). Indeed (12) may be taken as the mathematical representation of the interpretation, apart from possible “partition ensemble fallacy” which we would not discuss since there is no need. However, (12) cannot be true when \( \delta(P, Q) = \epsilon \) because that occurs if and only if \( \delta(P', Q) = 1 \) which in turn holds if and only if \( P' \) and \( Q \) are never both nonzero on the same \( x \). The latter never occurs when \( \chi \) is taken to be the common range of \( P \) and \( Q \) as indicated. Thus, not only the interpretation is not proven, it has nothing to do with (11) and cannot even be true. One can also see this from the immediate fact that \( P \neq Q \) for sure whenever \( \delta(P, Q) = \epsilon > 0 \).

IIIC. Composition Security in QKD and the Criterion d

In the quantum case Holevo’s bound is often used to bound the accessible information \( I_E \) Eve may obtain via some quantum measurement in the key generation process. In the presence of enduring quantum memory, however, Eve may utilize the bit knowledge she obtained on parts of \( K \) via KPA in conjunction with the quantum probe still in her possession to get at the rest of \( K \). In this quantum case there is the additional issue of lockable information [12], that a random variable side information \( S \) may reveal to Eve more than \( H(S) \) bits of information on \( K \) which is impossible classically. Without the quantum probe there would be at most \( H(S) \) bits of classical information which is already reflected in \( H_E(K) \) for the PKL problem. In [29], it was suggested that if Eve’s optimal mutual information on \( K \) from a quantum measurement, called the accessible information \( I_{sec} \), is exponentially small in \( n \) for large \( n \), then the \( n \)-bit \( K \) is composition secure according to their quantitative definition. While the mathematical result in [29] is correct, it was pointed out in [12] via a counter-example with one-time pad use of \( K \) that the result does not have the interpretation given in [29] to guarantee their composition security. In particular, it was shown that for \( I_E/n \sim 2^{-l} \), each \( l \) bits of knowledge on \( K \) may yield another bit. For \( I_E/n \leq \epsilon \) the fraction of bit leakage
may thus go up from $\epsilon$ to $\log_2$, an exponential increase from the case where no quantum memory is available. For $l = 10$, the possible leak thus increases from 0.1% to 10%.

The following remedy was suggested [11],[17] by using the criterion (5). Let $\rho_E^k$ be the state in Eve’s possession conditioned on a generated key value $k$, and let

$$\rho_U \equiv \frac{1}{|K|} \sum_k |k\rangle\langle k|$$  \hspace{1cm} (13)

be the completely mixed uniform state on the $|K| = 2^n$ orthonormal $|k\rangle$’s. It is assumed that the a priori probability of $K$ to Eve before she measures on her probe is uniform. Then the security criterion $d$ is the trace distance

$$d \equiv \frac{1}{2} \| \rho_{KE} - \rho_U \otimes \rho_E \|_1$$  \hspace{1cm} (14)

where

$$\rho_{KE} \equiv \frac{1}{|K|} \sum_k |k\rangle\langle k| \otimes \rho_E^k$$  \hspace{1cm} (15)

and

$$\rho_E \equiv \frac{1}{|K|} \sum_k \rho_E^k.$$  \hspace{1cm} (16)

The trace distance $\| \rho - \sigma \|_1$ between two states is related to the classical statistical distance $\delta(P, Q)$ between two probability distributions as follows [11]. For any POVM or von Neumann measurement made on $\rho$ and $\sigma$ with resulting distribution $P$ and $Q$,

$$\| \rho - \sigma \|_1 \leq \epsilon \implies \delta(P, Q) \leq \epsilon$$  \hspace{1cm} (17)

Using (17) and (11) it is concluded [11, p. 414], [17, Prop 2.1.1] that when $d \leq \epsilon$ the key is $\epsilon$-secure: with probability $p \geq 1 - \epsilon$ the real and the ideal situation of perfect security can be considered identical, where the ideal situation is one where $K$ is replaced by a uniformly distributed random variable $u$ which is independent of $\rho_E^k$. This statement is repeatedly made [25] and provides the following two very desirable consequences to supply both raw and composition security significance. Under $d \leq \epsilon$, with probability $p \geq 1 - \epsilon$ the key $K$ is universally composable (or at least so for partial key leakage) and it is the same as the uniform $K$ to Eve for the raw security apart from composition. Note that similar to the invalidity of (12), $d = \epsilon$ does not imply

$$\rho_{KE} = (1 - \epsilon)\rho_U \otimes \rho_E + \epsilon\sigma_{KE}$$  \hspace{1cm} (18)

for some joint density operator $\sigma_{KE}$. Equation (18) may lend itself to the above incorrect interpretation. Specific counter examples and further discussion of (18) can be found in ref [30].

To see how (5) does not give an $\epsilon$-secure key and how $\delta_E$ enters, we now trace the steps of the above incorrect derivation [11]. Let $P^k_Y = P(y|k)$ be Eve’s probability distribution on her measurement result $y$ conditioned on an actual generated $k$, i.e., through $\rho_E^k$, with $P_Y = P(y)$ the distribution obtained through $\rho_E$. For $d \leq \epsilon$, (17) and (14) imply, with Eve’s a priori distribution on $k$ given by $U$,

$$\delta(P^k_Y U_k, P_Y U_k) \leq \epsilon$$  \hspace{1cm} (19)

Under the incorrect interpretation, this implies $P^k_Y = P_Y$ independent of $k$ with probability $\geq 1 - \epsilon$, thus Eve’s corresponding CPD $P(k|y)$ is equal to $U$ also with probability $\geq 1 - \epsilon$. Not only this conclusion does not follow unless (11) is true, one can see that $P(k|y) \neq U$ for sure as long as $\rho_E^k$ or $P_Y^k$ carries any $k$-dependence, i.e., when there exists $k_1 \neq k_2$ with $P^k_Y \neq P^k_Y$ for a given $y$. This is because $P^k_Y$ then depends on $k$ and thus Eve cannot have $U$ as her CPD for the given $y$.

To derive the raw security meaning of $d \leq \epsilon$, we first observe that since Bob and Eve perform their “local” operations separately, the criterion $d$ is exactly equivalent to

$$d = E_k[\| \rho_E^k - \rho_E \|_1]$$  \hspace{1cm} (20)

and (5) is just a condition on the $\rho_E^k$. With $E_k$, the average over the $2^n$ possible values of $K$, equality of the right hand sides of (14) and (20) follows from lemma 2 of ref. [11] directly. The right-hand side of (20) is actually one of the criteria proposed in [29].

Apart from an increase in $\epsilon$ from a Markov inequality guarantee for individual $k$, (20) implies

$$\| \rho_E^k - \rho_E \|_1 \leq \epsilon$$  \hspace{1cm} (21)

From (21), (17), and (10) it follows that given $n'$-bit $y_n^E$, Eve’s CPD $P(k|y)$ satisfies

$$|P(k|y) - U_k| \leq \epsilon \cdot U_k/P_Y.$$  \hspace{1cm} (22)

The best guarantee from (22) on the smallest $P(k|y)$ over all possible $n'$-bit $y_n$ is $P_{Y_n'} = U_{Y_n'}$, i.e., the minimax of $P_Y$ over $y$ and $P_Y$ is obtained by $P_Y = U_Y$. Exactly similar results are obtained for subsequences $y_{n''}$ of $y_n$ corresponding to (9). Thus, (22) would reduce to $\delta(P, U) = \delta_E \leq \epsilon$ when the number of possible $y$’s is $N$. With key sifting, error correction and privacy amplification, the number of such possible $y$’s is much larger and so $U_k/P_Y$ is a very large number that would render the guarantee (22) useless. Thus, (21),(22) does not turn into a useful $\delta_E \leq \epsilon$ guarantee on Eve’s CPD. This also means (5) implies no composition security guarantee through $\delta_E$ for the case of no quantum memory.

An $\epsilon$-secure key is evidently “universally composable” as concluded previously. Since $d \leq \epsilon$ does not imply
the key is $\epsilon$-secure, the compositability problem remains in the presence of quantum memory. It is yet not known what the level of PKL leakage may be under the $d \leq \epsilon$ guarantees, whether leakage similar to the accessible information case is ruled out.

Since $\delta(P, Q) > 0$ implies $P \neq Q$ for sure [31], we have the following situation

(A) Under $d = \epsilon$, Eve’s probability distribution $p(K)$ is not the uniform $U$ for sure.

which can be compared to the following claim in the literature [25]

(A’) Under $d = \epsilon$, Eve’s probability distribution $p(K)$ is $U$ with probability $1 - \epsilon$.

Note the huge difference between between (A) and (A’), the latter is currently used to justify the IT security guarantee of a QKD key. On the other hand, (A’) is actually impossible from (A), and the probabilistic or operational significance of $d \leq \epsilon$ for both the quantitative raw and composition security are unknown.

IV. Importance of Security Proof and Numerical Values

Physical cryptography raises new conceptual issues in addition to the already subtle ones in both conventional symmetric-key and asymmetric key cryptography. In order to fully assess the significance of the above results for actual security guarantee, we will discuss some such conceptual issues in this section.

IVA. Security Proof in Physical Cryptography

We first observe that, in contrast to almost all problems in physics and most in engineering, a guarantee of cryptographic security cannot be obtained by experiments which could show a task is well carried out but not something general is impossible. An experiment can implement a specific attack and show that it does not work, but one cannot implement all possible attacks.

All proofs are based on reasoning on specific givens, in this case the mathematical model of the physical cryptosystem must be valid for the actual situation if the proof is to mean what it says in a real application. In standard or conventional cryptography where purely mathematical relations constitute the entire security mechanism, there is already a problem on the realistic features of an operating cryptosystem that cannot be incorporated in a general mathematical representation and must be treated on an individual ad hoc basis, such as the case of the RSA timing attack. In a physical cryptosystem involving either classical noise sources or quantum effects, the actual mathematical representation is a major issue due to the presence of other interfering physical effects that may play a crucial role in the actual cryptographic security. In particular, quantum information is an unusual area in physics where very small disturbance can lead to major consequence. In BB84 type QKD there is a serious problem of system and device modeling at the time of use, see, e.g., ref [32]-[35], which arises from the single-photon nature of the signal. In particular, a device imperfection can entirely compromise the security of a BB84 protocol [34]-[35]. The issue here is not that the device imperfection cannot be removed, but rather how many such undiscovered loopholes there are in practice. However, in this paper we do not deal with these issues but only with the fundamental quantitative security assuming the physical model is exactly correct.

The excitement of physical cryptography and particularly QKD is mainly derived from the belief that unconditional information-theoretic security is possible for generating fresh keys which can be proved mathematically given a model. This is in sharp contrast to conventional cryptography, in which asymmetric key cryptosystem has only complexity based security the strength of which is further based on unproved though widely accepted assumptions on the difficulties of various mathematical problems. For symmetric-key conventional ciphers against KPA, the design is even more of “an art”, with security based on less widely shared beliefs in the problem complexity of various attack algorithms. In QKD, “unconditional security” means all possibilities of an attacker gaining more “information” than a designed level is ruled out except for a small probability which is itself a design parameter. That is, claim (a) in the Introduction of this paper is maintained.

In contrast to a perfect key $K^p$ the QKD generated key $K$ can never be perfect because Eve could always obtain some “information” by an attack during the physical key generation process. The crucial questions are then what operational meaning the various security criteria and proofs have. These questions are already subtle ones in conventional cryptography, see for example the dispute described in ref [36] on public-key systems and the complaint on lack of proper security foundation in symmetric-key ciphers [20]. In physical cryptography such questions are much more acute while similar security situation has not arisen previously in any real cipher. These problems do arise in a more restricted manner (no quantum memory) in classical-noise key generation which, however, has never found publicly known actual deployment, and the criterion of $I_E/n$ was employed without any discussion on its adequacy in cryptographic context [1]-[3].

The various probabilities one can obtain from a mathematical model have a clear empirical or operational meaning, in the same sense that probability in quantum physics or communication engineering has empirical meaning. However, various theoretical constructs such as $I_E$ and $\delta_E$ do not automatically have the meaning that would ensure whatever security we may desire in an application. They are really no more than mere constraints
on the possible distribution \( p(K) \) Eve may obtain in an attack and need to be transformed into operational guarantees as done in this paper. In particular, it is misleading to claim that the system is secure if \( \epsilon \) can be made exponentially small in \(|K|\), as the following shows.

For example, for a 1,000 bit \( K \), if \( I_E/n \lesssim 2^{-20} \) which is an “exponentially small” number to many, one may thus claim \( K \) is “secure”. From Theorem 1 it is not ruled out that Eve may identify \( K \) with a probability \( 2^{-20} \). Since \( I_E/n \) is obtained under average over all possible key values, from (7) and the surrounding discussion the final security guarantee then becomes: with a probability \( \geq 1 - 2^{-10} \), Eve has \( I_E/n \leq 2^{-10} \) and thus she may not be able to get the whole \( K \) from her measurement with probability more than \( 2^{-10} \). This is a weaker security guarantee than the simple statement that for sure Eve could not get \( K \) with probability more than \( p_1 = 2^{-10} \) from her measurement. Since \(|K| = 10^3 >> 10 \), such guarantee clearly does not rule out with a small enough overall probability of a disastrous breach of security that Eve determines the whole \( K \) with probability \( 2^{-10} \sim 10^{-3} \) from her measurement result alone without any further use context on \( K \). Even for \( p_1 = 2^{-100} \), one may ask in what sense such a \( K \) is near a perfect 1,000 bit \( K^p \), particularly in view of all the other subset breach probabilities given in (3).

It is the role of a security proof to rule out such disastrous breach of security —— here it is Eve being able to identify \( K \) at a probability \( p_1 \sim 10^{-3} \) from her measured result —— by making it very unlikely if not impossible, certainly not at a probability \( \sim 10^{-3} \). Security breach with probability \( \sim 10^{-3} \) is only a possibility, whether Eve can actually do it depends on what her distribution \( p(K) \) is, which is obtained via her specific attack that gives her \( p(K) \). A QKD guarantee in terms of accessible information shows no \( I_E/n \) can exceed a designed level \( \epsilon \) when averaged over specific \( k \). Such guarantee leaves open the above possibility that cannot be further averaged out — Eve knows her \( p(K) \) which is fixed by her attack. Indeed, the possible large leakage from accessible information guarantee in composition security when Eve has quantum memory as given in ref [12] is exactly the same in this regard — it shows a serious compromise of security is not ruled out under the security condition given in [29].

A security proof via \( \delta_E \leq \epsilon \) or \( d \leq \epsilon \) is exactly of the same nature, and \( d \leq \epsilon \) is adequate only if one uses the mistaken interpretation (A’) in subsection IIIC instead of the correct (A). Unless \( I_E/n \) or \( \delta_E \) is close to \( 2^{-|K|} \), the QKD generated \( K \) is very far from a perfect \( K^p \) while a security proof of \( d \leq \epsilon \) has uncertain quantitative significance in terms of empirically meaningful probabilities. We will now discuss the numerical situation further specifically.

IVB. Actual Quantitative Guarantee

We summarize the status of the quantitative security guarantee situation in Table 1. Note that \( d \leq \epsilon \) is not listed because it has no clear security significance. The security parameter is \( \epsilon \) and smaller \( \epsilon \) means the system is more secure. Recall that raw security measures the information Eve has just from her attack before the key \( K \) is used, and composition security in this case refers only to the fraction \( f \) of deterministic bits that Eve could get on \( K \) from knowing the rest of \( K \) as in a KPA. The \( K \) are subsequences of the \( n \)-bit \( K, N = 2^n \). For the possible messages in the table other than “\( f \sim ? \)” are worst case leaks except for the composition security under \( I_E/n \leq \epsilon \). In that case the leak of \( f \sim \epsilon \) would also occur in some probabilistic form other than deterministic bits of \( K \) in the case of no quantum memory. With quantum memory \( f \sim \log_2 N \) has not been shown to be the worst scenario. It is important to observe that the fraction \( f \) of bits leaked can be distributed in any fashion and not just uniformly in \( K \). Thus, there could be a serious security breach even when \( f \) is very small. This shows the importance of semantic security.

In current experimental scenarios the final generated key in a single cycle, or round of QKD after error correction and privacy amplification, could have \( |K| \) in thousands of bits or more. The necessary message authentication shared secret key \( K^a \) that is needed to create a “public channel” in BB84 type protocols has never been explicitly integrated into the protocol and accounted for. It is reasonable to assume \(|K^a| \sim 100\) for each round is to be used with one of the current message authentication code, since \(|K^a| \geq 40\) is typically used for many such codes. Compared to the alternative use of \( K^a \) as the seed key \( K^m \) in a conventional cipher, it is clear that but for KPA there is little point in using a QKD generated \( K \) from the viewpoint of security guarantee. When the input data can be assumed uniformly random to Eve, such conventional cipher \( K' \) would give better protection than the QKD keys that can readily be generated in the foreseeable future. This is evident from the fact that even without taking (7) into account, the best current \( I_E/n \sim 2^{-10}[16] \) and \( \delta_E \) or \( d \) has apparently not been used in an actual experimental system while theoretical estimates give \( d = 10^{-5} \sim 2^{-17} [18] [19] \). In all these cases \( n = 10^3 \) or much larger. In the literature such numerical evaluation was never compared to the benchmark of a perfect \( K^p \).

One main problem in this connection is that almost every relevant quantity is “exponentially small” here. If one takes that to mean \( 2^{-\lambda n}, 0 < \lambda \leq 1 \), and \(|K| = n \), all depends on how large \( \lambda \) is. Indeed \( n \) is not “asymptotic” either in a real protocol. Thus, the actual security depends on the precise numerical values of the system parameter and one cannot capture the situation by a vague qualitative remark. Other than \( p_3(K) \) for identifying the whole \( K \), under (5) for \( \epsilon = 2^{-l} \) any large subset \( K'' \) of \( K \) may still be determined with a much
larger probability $2^{-l}$ than the uniform $|\tilde{K}|$ one of $2^{-|K|}$. It is clear that the incorrect statement $(A')$ in section IIIC is sorely needed for good security guarantee, but $(A)$ shows in a strong way that $(A')$ is forever unachievable.

V Conclusion and Outlook

In this paper we have seen that realistic QKD generated keys have inadequate raw security and no composition security guarantee. One may obtain much better raw security probability guarantee with conventional ciphers. The perception to the contrary is due mainly to a mistaken interpretation of the security criterion $d \leq \epsilon$, which has actually no clear operational security significance. One may view the logical/historical development on the information theoretic security of the generated key as follows. The Shannon limit on conventional key expansion leads to very poor composition security under known plaintext attack. This composition security predicament is rectified by QKD via the mutual information criterion when the attacker does not possess good quantum memory, but the problem remains when she does. At the same time the raw security guarantee worsens in concrete QKD protocols. Since $I_E/n \leq \epsilon \sim 2^{-\lambda|K|}$ needs to go down exponentially for linear bit improvement in the security of $K$, it does not appear promising to try a brute-force experimental approach for increased security of either the raw or composition kind. The criterion $\delta E \leq \epsilon$ may be used in a classical noise protocol, it provides good PKL security but has problems similar to $I_E/n$ for raw security. It is not clear how its quantum generalization may be developed and what its composition security would be in the presence of quantum memory. In the absence of adequate guarantee on both raw and composition security, QKD would lose its main claim of merit over conventional cryptography and would reduce in practice to a mere “art” similar in many ways to the latter. What can be done about it? The following four alternative routes may be suggested:

(i) At the expense of efficiency, it may be possible to improve security under (3)-(4) by appropriate privacy amplification. However, privacy amplification cannot improve $p_1$ [9,section IIID]. Due to the small $l$ that can be obtained, this does not look promising for a real protocol to get $l$ (near) uniform bits in $K$ from $p_1 \sim 2^{-l}$ even if possible. The effective key generation rate would be reduced from $r$ to $r' = rl/n$ for an $n$-bit $K$.

(ii) One may use more efficient key generation schemes from the KCQ approach [9] other than QKD with intrusion level estimation. The possibility of obtaining adequate security with such approach against all attacks can be explored.

(iii) One may limit the security to just the more realistic attacks that can be launched with foreseeable technology advance. This would rule out, in particular, joint attacks that involve actual quantum entanglement over several or more subsystems. The situation may then be reduced to that of a wiretap channel [1] and one may generate near-uniform $K$ with a nonvanishing final key rate [37].

(iv) One may limit the devices $E$ possesses to more realistic ones. In particular, this would exclude long and near-perfect quantum memory and help the composition security instantly. Devices that are totally free of the many limits that have been around for a long time may also be excluded. This restriction is in addition to and independent of that in (iii), as it concerns with device realization rather than unknown in-principle schematic realization although both can be brought under the general classification of limited technology.

Given the subtle modeling question in physical cryptography and especially in BB84 type protocols, it is not clear that (iii)-(iv) entail any loss of true security in a real world application as compared to the inadequate levels one may obtain in an ideal model that allows Eve all the physical possibilities. Note that all conventional cryptosystems are being currently deployed under equivalent assumptions to (iii)-(iv) on unavailable algorithms and computing power, which are of a mathematical nature instead of physical ones. It is not clear why mathematical presumptions are better than physical ones. One may argue the contrary in some situations. The clear advantage of physical cryptography is that it is difficult to launch an attack or to obtain just the “ciphertext”, in sharp contrast to conventional cryptosystems. It is possible that feature alone is enough to justify the deployment of physical cryptosystems in some applications.

| Criteria | $p_1(K) \leq \epsilon$ | $I_E/n \leq \epsilon$ | $\delta E \leq \epsilon$ |
|-----------|-------------------|-------------------|-------------------|
| Raw Security | $p_1(K) \sim \epsilon$, $p_1(K) = \epsilon + \frac{\epsilon}{|K|}$ | $p_1(K) = \epsilon + \frac{\epsilon}{|K|}$ | $p_1(K) = \epsilon + \frac{\epsilon}{|K|}$ |
| Composition Security | no quantum memory | $f \sim 1 - \epsilon$ | $f \sim \epsilon$ | $f \sim 0$ |
| fraction $f$ of $K$ revealed in PKL | with quantum memory | $f \geq 1 - \epsilon$ | $f \geq \log \frac{1}{\epsilon}$ | $f \sim ?$ |

TABLE I: Quantitative Security in QKD
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