Non-renormalization for the Liouville wave function

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Abstract. Using an exact functional method, within the framework of the
gradient expansion for the Liouville effective action, we show that the kinetic
term for the Liouville field is not renormalized.

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1. Introduction

It is known that the effective potential for the two-dimensional Liouville theory remains an exponential, with renormalized coupling constant and mass parameter \[1\]. This property respects the symmetry of the classical action, under which a translation in the Liouville field is equivalent to a change in the mass parameter.

We study here the wave function renormalization \(Z\) of the Liouville field, using an exact functional method, which leads to a self-consistent equation for the effective action (the proper graphs generator functional), in the spirit of a Schwinger–Dyson equation, and which is therefore not based on a loop expansion. The idea is to look at the evolution of the quantum theory with the amplitude of the central charge deficit \(Q^2\) of the Liouville theory \([2]\), since it was shown in \([3]\) that it is possible to obtain exact flows for the quantum theory with \(Q^2\). As we emphasize below, these flows are regularized by a fixed world sheet cut-off, unlike the Wilsonian approach. Using this method, it was already found in \([3]\), in the approximation where \(Z\) does not depend on the Liouville field, that \(Z\) does not get quantum corrections and keeps its classical value. We extend here this study to the more general situation where \(Z\) could be a polynomial of the Liouville field. This is the next step in order to have a complete picture, consistent with the gradient expansion. As we shall demonstrate below the result is similar to that of \([3]\): the wave function renormalization remains trivial, and the kinetic term for the Liouville field does not get dressed by quantum fluctuations.

We note that the functional approach used here, which serves as an alternative to Wilsonian renormalization, has proved to give new insights into the quantum structure of a theory, and led to non-trivial results in a variety of contexts so far, including scalar field theory \([4]\), quantum electrodynamics–dynamics \([5]\), Wess–Zumino \([6]\) and Kaluza–Klein \([7]\) models, time-dependent bosonic strings \([8]\). Finally, the two-dimensional sine-Gordon model was also discussed using this approach \([9]\), which is a Liouville-like model, with a complex coupling constant.

The structure of our paper is as follows: in section 2, we explain in some detail the functional method, already used in \([3]\), and derive the exact equations for the evolution of the potential and the wave function renormalization with the central charge deficit, \(Q^2\). The details of the derivations are given in appendix A. We emphasize the specific role played by the two-dimensional field theory in ensuring the wave function non-renormalization, and we give the solution for the corresponding effective potential. In section 3, we demonstrate the consistency of our results with the Wilsonian approach, where we explain that this trivial solution for \(Z\) is consistent with Wilsonian exact renormalization equations \([10, 11]\). We also check the consistency of our result with the known self-duality of the quantum theory in the change \(b \rightarrow b^{-1}\), where \(b\) is the coupling constant \([12]\) (see \([13]\) for reviews). Finally, in appendix B we show the equivalence between the Wilsonian and the one-particle-irreducible effective potentials.

2. Evolution equations

The bare action for the Liouville field, on a flat world sheet we assume in this work, reads:

\[
S = \int d^2 \xi \left\{ \frac{Q^2}{2} \partial_\alpha \phi \partial^\alpha \phi + \mu^2 e^\phi \right\}.
\]
where the amplitude of the kinetic term is controlled by the central charge deficit $Q^2$. Upon quantization of this theory, as we explain below [3], $Q^2$ controls the amplitude of quantum fluctuations:

1. for $Q^2 \gg 1$, the quadratic kinetic term dominates the bare Lagrangian and therefore the quantum theory is almost classical;
2. when $Q^2$ decreases, quantum fluctuations gradually appear in the system and the full quantum theory is obtained when $Q^2 \to$ finite constant.

Our motivation is to find the evolution of the proper graphs generator functional with $Q^2$, and therefore obtain information on the quantum theory.

2.1. Path integral quantization

In order to define the corresponding quantum theory, one first defines the partition function (assuming a Euclidean world sheet, as required for convergence of the respective path integral)

$$Z[j] = \int D[\phi] \exp \left\{-S - \int j \phi \right\} = \exp \{-W[j]\},$$

(2)

where $j$ is the source and $W$ is the connected graphs generator functional. Note that $Z$ is not normalized to 1 for a free theory, and possible field-independent terms in the effective theory will be disregarded later on. The classical field is defined as

$$\phi_c = \frac{\delta W}{\delta j},$$

(3)

and the proper graphs generator functional $\Gamma$, describing the quantum theory, is obtained as the Legendre transform of $W$:

$$\Gamma[\phi_c] = W[j] - \int d^2 \xi j \phi_c,$$

(4)

where the source $j$ is to be understood as a functional of $\phi_c$, found by inverting the definition (3).

One obtains then a family of quantum theories, parametrized by $Q^2$; it was shown in [3] that the effective action $\Gamma$ satisfies the following exact evolution equation with $Q^2$ (we omit the subscript ‘c’ for the classical field)

$$\dot{\Gamma} = \frac{1}{2} \int d^2 \xi \partial_a \phi \partial^a \phi + \frac{1}{2} \text{Tr} \left\{ \frac{\partial}{\partial \xi_a} \frac{\partial}{\partial \xi^a} \left( \frac{\delta^2 \Gamma}{\delta \phi_c \delta \phi_c} \right)^{-1} \right\},$$

(5)

where the dot represents a derivative with respect to $Q^2$. The evolution equation (5) is exact and does not rely on any loop expansion: it is a self-consistent equation, in the spirit of a differential Schwinger–Dyson equation. We stress here that the trace appearing in equation (5) is regularized with a fixed world sheet cut-off $\Lambda$, and the running parameter is $Q^2$, unlike the Wilsonian approach where, for a fixed $Q^2$, one would study the evolution of $\Gamma$ with a running world sheet cut-off. As a consequence, the use of $Q^2$ cannot be compared with a cut-off function, as this parameter is independent of our world sheet regularizator.

We note here that, although a cut-off does not respect conformal invariance on the world sheet, our results will be consistent with conformal properties of the Liouville theory, since we
do not look at the evolution of the theory with $\Lambda$, and the latter will only appear as an overall constant factor defining the dressed mass scale $\mu_R$.

In the framework of the gradient expansion, which we adopt in this work, we consider the projection on a specific subspace of functionals in the functional space where $\Gamma$ lives, for which we assume the following form of the effective action

$$\Gamma = \int d^2 \xi \left\{ \frac{Z_Q(\phi)}{2} \partial_a \phi \partial^a \phi + V_Q(\phi) \right\}.$$  \hfill (6)

Note that the field-dependence of $Z$ does not contradict conformal invariance: since $\phi$ is dimensionless, no dimensionfull parameter appears in $Z$. As we show in appendix $A$, the evolution equations with $Q^2$ for the potential $V_Q(\phi)$ and the wave function renormalization $Z_Q(\phi)$ are

$$\dot{Z} = 1 + \frac{1}{8\pi} \left( \frac{Z'}{Z} \right)^2 \left[ 5 \ln \left( 1 + \frac{Z \Lambda^2}{V''} \right) - \frac{47}{6} \right] + \frac{7}{24\pi Z} \left( \frac{Z'}{Z} \right) \left( \frac{V'''}{V''} \right),$$ \hfill (7)

where $Z = Z_Q(\phi)$ and $V = V_Q(\phi)$, and a prime denotes derivative with respect to $\phi$.

As can be seen from the evolution equations (7), a solution where $Z$ does not depend on the Liouville field (i.e. $Z' = 0$) is consistent, for which case we also obtain $\dot{Z} = 1$, and therefore no renormalization of the wave function.

One could seek other solutions, different from $Z = Q^2$, but we will give below several arguments in favour of the uniqueness of the $\phi$-independent solution:

1. As discussed in section 3, the solution $Z = Q^2$ is consistent with an exact renormalization equation for the potential, using a sharp cut-off.

2. Also in the Wilsonian context, the Liouville theory has been studied using the average action formalism [10], based on a smooth cut-off procedure, thereby allowing the study of the evolution of the wave function renormalization. In this work, the wave function renormalization $Z_k(\phi)$, where $k$ is the running cut-off, does depend on the Liouville field, as a consequence of the initial condition of the flows, which is chosen so as to satisfy the respective Weyl–Ward identities. The authors argue, though, that the IR limit $k \to 0$ of the average action, which corresponds to the effective action we consider here, is consistent with this non-renormalization property.

3. We can imagine integrating the equation for $Z_Q$ numerically, starting from the initial condition $Z_Q(\phi) \simeq Q^2$ for $Q^2 \gg 1$, since the theory is then almost classical. The step $Q \to Q - dQ$ corresponds to $Q^2 \to Q^2 + dx$, with $dx = -2QdQ + dQ^2$, and we have then

$$Z_{Q-dQ} = Z_Q + dx(\dot{Z}_Q) = Z_Q + dx$$

because $Z' = 0$ for the initial condition. Therefore

$$Z_{Q-dQ} = Z_Q - 2QdQ + dQ^2 = (Q - dQ)^2.$$  

In this way we arrive, step by step, to the result that $Z = Q^2$ for any value of $Q$. 

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4. We show in the next subsection that, for a field-independent $Z$, this non-renormalization property is possible in dimension $d = 2$ only, which gives a strong indication that the solution $Z = Q^2$ is the relevant one in the more general case studied here. This is also consistent with the world-sheet conformal-invariance restoring properties of the Liouville theory [2].

Finally, in the case of a curved world sheet, the bare action contains an additional term, linear in the Liouville field, and reads

$$S = \int d^2 \xi \sqrt{\gamma} \left\{ \frac{Q^2}{2} \gamma_{ab} \partial^a \phi \partial^b \phi + Q^2 R^{(2)} \phi + \mu^2 e^\phi \right\},$$

where $\gamma_{ab}$ is the world sheet metric, with determinant $\gamma$ and curvature scalar $R^{(2)}$. The gradient expansion for the effective action would then have to take into account this linear term in $\phi$, but because of the second functional derivative, which appears in the evolution equation (5), this additional linear term does not play a role in the generation of quantum fluctuations. It is in this sense that working from the beginning with flat world sheets suffices for our purposes.

2.2. Specificity of two dimensions ($d = 2$)

In this subsection, we go through the same steps as those described in appendix A, for a wave function renormalization independent of $\phi$, but in any dimension $d$. We show then that the renormalization of the wave function renormalization vanishes only for the case $d = 2$.

We assume that the effective action has the form

$$\Gamma = \int d^d \xi \left\{ \frac{Z Q}{2} \partial_a \phi \partial^a \phi + V_0(\phi) \right\},$$

such that its second functional derivative in configuration space reads:

$$\frac{\delta^2 \Gamma}{\delta \phi_\xi \delta \phi_\xi} = (-Z \partial_a \partial^a + V''(\phi)) \delta^{(2)}(\xi - \xi).$$

For the configuration $\phi = \phi_0 + 2 \rho \cos(k \xi)$, where $\phi_0$, $\rho$, $k$ are constants, the second functional derivative in Fourier space is:

$$\frac{\delta^2 \Gamma}{\delta \phi_p \delta \phi_q} = (Z p^2 + V'' + \rho^2 V^{(4)}) (2\pi)^2 \delta^{(2)}(p + q)$$

$$+ \rho V^{(3)} (2\pi)^2 \left( \delta^{(2)}(p + q + k) + \delta^{(2)}(p + q - k) \right)$$

$$+ \frac{\rho^2}{2} V^{(4)} \left( \delta^{(2)}(p + q + 2k) + \delta^{(2)}(p + q - 2k) \right)$$

$$+ \text{higher orders in } \rho.$$  

The inverse of this second functional derivative is calculated using

$$(A + B)^{-1} = A^{-1} - A^{-1} B A^{-1} + A^{-1} B A^{-1} B A^{-1} + \ldots,$$  

where $A$ stands for the diagonal contribution and $B$ for the off-diagonal one, proportional to $\rho$. The relevant term for the evolution of $Z$ is

$$\text{Tr} \left\{ p^2 A^{-1} B A^{-1} B A^{-1} \right\}$$

$$= \mathcal{A} \rho^2 \left( V^{(3)} \right)^2 \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{f^2(p)} \left( \frac{1}{f(p + k)} + \frac{1}{f(p - k)} \right)$$

$$= 2 \mathcal{A} \rho^2 \left( V^{(3)} \right)^2 I(k),$$

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where $A$ is the two-dimensional volume, $f(p) = Zp^2 + V''$ and

$$I(k) = \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{f^2(p)f(p + k)}. \quad (14)$$

The evolution of the wave function renormalization, $Z$, is proportional to the quadratic-order-in-$k$ part of $I(k)$, and we have

$$I(k) = I(0) + \int \frac{d^d p}{(2\pi)^d} \left\{ \frac{4Z^2p^2(k \cdot p)^2}{(Zp^2 + V'')^5} - \frac{Zk^2p^2}{(Zp^2 + V'')^4} \right\} + O(k^4)$$

$$= I(0) + k^2 Z^{-d/2} \int \frac{d^d p}{(2\pi)^d} \left\{ \frac{2p^4}{(p^2 + V'')^5} - \frac{p^2}{(p^2 + V'')^4} \right\} + O(k^4)$$

$$= I(0) + k^2 \frac{\pi^{d/2}}{(2\pi)^d} \frac{Z^{d/2}}{[V'']^{d/2}} \left\{ 2 \frac{\Gamma(3 - d/2)}{\Gamma(3)} - 2 \frac{\Gamma(5 - d/2)}{\Gamma(5)} - 2d \frac{\Gamma(4) - \Gamma(3 - d/2)}{\Gamma(4)} \right\} + O(k^4). \quad (15)$$

Using the property $\Gamma(n + 1) = n\Gamma(n)$, together with $\Gamma(1) = 1$, the expansion (15) can be written

$$I(k) = I(0) + k^2 \frac{\pi^{d/2}}{(2\pi)^d} \frac{Z^{d/2}}{[V'']^{d/2}} \Gamma(3 - d/2) \frac{d}{24} \left( \frac{d}{2} - 1 \right) + O(k^4), \quad (16)$$

which shows that the term of quadratic order in $k$ vanishes for $d = 2$ only. This is a strong indication that the solution $\dot{Z} = 1$ found previously is the relevant one.

2.3. Solution for the potential

From now on, we consider $Z = Q^2$. The evolution equation (7) for the potential then becomes

$$\dot{V} = -\frac{V''}{8\pi Q^4} \ln \left( 1 + \frac{Q^2 \Lambda^2}{V''} \right), \quad (17)$$

where the quadratic divergence was disregarded, as it is field-independent.

Equation (17) has been studied in [3] for the specific regimes $Q^2 \to 0$ and $Q^2 \to \infty$. We give here some details of the derivation for finite values of $Q^2$. We therefore assume that

$$\frac{Q^2 \Lambda^2}{V''} \gg 1. \quad (18)$$

With this condition in mind, equation (17) is then satisfied by a potential of the form

$$V_Q(\phi) = \Lambda^2 v_Q \exp \left( \varepsilon_Q \phi \right), \quad (19)$$

where $v_Q$ and $\varepsilon_Q$ are dimensionless functions of $Q$ (for the condition (18) to be satisfied we need $v_Q \ll 1$). Indeed, plugging this ansatz into the evolution equation (17) gives, in the limit (18),

$$\dot{v} = -\frac{v \varepsilon^2}{8\pi Q^4} \ln \left( \frac{Q^2}{v \varepsilon^2} \right),$$

$$\dot{\varepsilon} = \frac{\varepsilon^3}{8\pi Q^4}. \quad (20)$$

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The latter evolution equation for $\varepsilon$ can be integrated exactly. The appropriate boundary condition is $\varepsilon \to 1$ when $Q^2 \to \infty$, since the system is then classical. The integration over $Q^2$ leads to

$$
\varepsilon Q = \sqrt{\frac{4\pi Q^2}{1 + 4\pi Q^2}}.
$$

We remind the reader that the solution (21) is exact in the framework of the gradient expansion (6), and is not based on a loop expansion. The evolution equation for $\nu_Q$ is not solvable exactly, and we thus leave the study of the potential amplitude for the next section, where this is achieved by means of a Wilsonian exact renormalization approach.

Before closing this section, we note that, for the specific conformal charge deficit $Q^2 = 8$, corresponding to $c = 1$ conformal field theories, there are two cosmological constant operators, dressing the identity $(\mu_1^2 + \mu_2^2 \phi) \exp(\sqrt{2} \phi)$ ([14] and references therein, where $\mu_1, \mu_2$ are constants. Our solution above cannot include the operator proportional to $\mu_2^2$, since we consider a continuous set of values for $Q^2$ and this operator exists only for a discrete isolated value. To incorporate this case, one should study the flow with respect to another parameter in the bare theory with fixed $Q^2 = 8$, such as $\alpha'$ or $\mu_i$.

3. Properties of the solution

3.1. Consistency with the Wilsonian picture

We now exhibit the Wilsonian properties of the solution (19), using the exact renormalization method of [15]. This approach uses a sharp cut-off, and it will be explained below that, as far as the potential term is concerned, the results are the same as those obtained with the usual smooth cut-off procedures. It can be therefore seen as a specific case of the studies made in [10].

We consider an initial two-dimensional bare theory, with running cut-off $\Lambda_0$. The effective theory defined at the scale $\Lambda - \delta \Lambda$ is derived by integrating the ultraviolet degrees of freedom from $\Lambda$ to $\Lambda - \delta \Lambda$. The idea of exact renormalization methods is to perform this integration infinitesimally, i.e. take the limit $\delta \Lambda / \Lambda \to 0$, which leads to an exact evolution equation for $S_{\Lambda}$. The procedure was detailed in [15], and here we reproduce only the main steps for clarity and completeness. Note that we consider here a sharp cut-off, which is possible only if we consider the evolution of the potential part of the Wilsonian action, as explained now.

We consider a Euclidean two-dimensional spacetime, and we assume that, for each value of the energy scale $\Lambda$, the Euclidean action $S_{\Lambda}$ has the form

$$
S_{\Lambda} = \int d^2 \xi \left\{ \frac{Z_{\Lambda}(\phi)}{2} \partial_\alpha \phi \partial^\alpha \phi + V_{\Lambda}(\phi) \right\}.
$$

The integration of the ultraviolet degrees of freedom is implemented in the following way. We write the dynamical fields $\phi = \phi_{IR} + \psi$, where the $\phi_{IR}$ is the infrared field with non-vanishing Fourier components for $|p| \leq \Lambda - \delta \Lambda$, and $\psi$ is the degree of freedom to be integrated out, with non-vanishing Fourier components for $\Lambda - \delta \Lambda < |p| \leq \Lambda$ only. An infinitesimal step of
the renormalization group transformation reads:

\[
\exp(-S_{\Lambda-\delta\Lambda}[\phi_{IR}]+S_{\Lambda}[\phi_{IR}]) = \exp(S_{\Lambda}[\phi_{IR}]) \int \mathcal{D}[\psi] \exp(-S_{\Lambda}[\phi_{IR}+\psi])
\]

\[
= \int \mathcal{D}[\psi] \exp \left( -\int_{\Lambda} \frac{\delta S_{\Lambda}[\phi_{IR}]}{\delta \psi(p)} \psi(p) - \frac{1}{2} \int_{\Lambda} \int_{\Lambda} \frac{\delta^2 S_{\Lambda}[\phi_{IR}]}{\delta \psi(p) \delta \psi(q)} \psi(p) \psi(q) \right) ,
\]

+ higher orders in \( \delta \Lambda \),

(23)

where \( \int_{\Lambda} \) represents the integration over Fourier modes for \( \Lambda - \delta \Lambda < |p| \leq \Lambda \). Higher order terms in the expansion of the action are indeed of higher order in \( \delta \Lambda \), since each integral involves a new factor of \( \delta \Lambda \). The only relevant terms are of first and second order in \( \delta \Lambda \) [15], which are at most quadratic in the dynamical variable \( \psi \), and therefore lead to a Gaussian integral. We then have

\[
\frac{S_{\Lambda}[\phi_{IR}]-S_{\Lambda-\delta\Lambda}[\phi_{IR}]}{\delta \Lambda} = \frac{1}{\delta \Lambda} \int \frac{d^2 p}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \delta S_{\Lambda}[\phi_{IR}][\delta S_{\Lambda}[\phi_{IR}]}{\delta \psi(p)} \left( \frac{\delta^2 S_{\Lambda}[\phi_{IR}]}{\delta \psi(p) \delta \psi(q)} \right)^{-1} \frac{\delta S_{\Lambda}[\phi_{IR}]}{\delta \psi(q)}
\]

\[-\frac{\text{Tr}_{\Lambda}}{2\delta \Lambda} \left\{ \ln \left( \frac{\delta S_{\Lambda}[\phi_{IR}]}{\delta \psi(p) \delta \psi(q)} \right) \right\} + O(\delta \Lambda),
\]

(24)

where the trace \( \text{Tr}_{\Lambda} \) is to be taken in the shell of thickness \( \delta \Lambda \), and is therefore proportional to \( \delta \Lambda \).

We are interested in the evolution equation for the potential only, for which it is sufficient to consider a constant infrared configuration \( \phi_{IR} = \phi_0 \), and this is the reason why a sharp cut-off can be used: the singular terms that could arise from the \( \theta \) function, characterizing the sharp cut-off, are not present, since the derivatives of the infrared field vanish. In this situation, the first term on the right-hand side of equation (24), which is a tree-level term, does not contribute: \( \delta S_{\Lambda}/\delta \psi(p) \) is proportional to \( \delta^2(p) \), and thus has no overlap with the domain of integration \( |p| = \Lambda \). We are therefore left with the second term, which arises from quantum fluctuations, and the limit \( \delta \Lambda \rightarrow 0 \) gives, with the ansatz (22),

\[
\partial_{\Lambda} V_{\Lambda}(\phi_0) - \partial_{\Lambda} V_{\Lambda}(0) = -\frac{\Lambda}{4\pi} \ln \left( \frac{Z_{\Lambda}(\phi_0) \Lambda^2 + V''_{\Lambda}(\phi_0)}{Z_{\Lambda}(0) \Lambda^2 + V''_{\Lambda}(0)} \right),
\]

(25)

equation (25) provides a resummation of all the loop orders, since it consists of a self-consistent equation. As a result, the evolution equation (25) is exact within the framework of the ansatz (22), and is independent of a loop expansion.

In order to make the connection with the solution (19), we now consider the following ansatz

\[
Z_{\Lambda}(\phi_0) = Q^2, \quad V_{\Lambda}(\phi_0) = \Lambda^2 v_{\Lambda} \exp(\varepsilon \phi_0),
\]

(26)

where \( \varepsilon \) is the constant (21) and \( v_{\Lambda} \) depends on the running cut-off only. One should keep in mind here that \( Q^2 \) is now constant, whereas the cut-off \( \Lambda \) is running. When plugged in the
Wegner–Houghton equation (25), the ansatz (26) leads to

\[(2\Lambda v + \Lambda^2 \partial_\Lambda v) \left( \exp (\varepsilon \phi_0) - 1 \right) = -\frac{\Lambda}{4\pi} \ln \left( \frac{Q^2 + \varepsilon^2 v \exp (\varepsilon \phi_0)}{Q^2 + \varepsilon^2 v} \right). \quad (27)\]

One can see that this equation is consistent in the limit \(v \ll 1\) only, which we are interested in: keeping the first orders in \(v\), the \(\phi_0\)-dependence cancels out and the remaining equation is

\[2\Lambda v + \Lambda^2 \partial_\Lambda v = -\frac{\Lambda}{4\pi} Q^2 \varepsilon^2 v, \tag{28}\]

which is easily integrated to

\[v_\Lambda = \left( \frac{\mu}{\Lambda} \right)^{2+\varepsilon^2/(4\pi Q^2)}. \tag{29}\]

This solution indeed satisfies \(v \ll 1\), since we are interested in the regime of large cut-off, in the spirit of the condition (18). Taking into account the solution (21), the potential is finally

\[V_\Lambda(\phi) = \mu^2 \left( \frac{\mu}{\Lambda} \right)^{1/(1+4\pi Q^2)} \exp \left( \phi \sqrt{\frac{4\pi Q^2}{1+4\pi Q^2}} \right). \tag{30}\]

We stress again that this solution is not the result of a loop expansion. An important remark is in order here: it was possible to find the solution (30) of the Wilsonian exact renormalization group equation, because \(Z\) does not depend on \(\phi_0\). Indeed, it is the only possibility for the \(\phi_0\)-dependence to cancel in equation (27), at the first order in \(v\). This shows the consistency of the choice \(Z = Q^2\) made in the previous section.

Note that the solution (30) does not need to satisfy the evolution equation (17), since the Wilsonian potential defined in this section is not obtained by means of the Legendre transform as the potential defined in the previous section. The equivalence between these potentials is obtained in the limit where the running cut-off goes to 0 (this is well known [11], and we give in appendix B an alternative argument), but in this case, the expression (30) is not valid since it was derived in the limit of large cut-off.

Finally, the limit of the Wilsonian potential (30) when \(Q^2 \to \infty\), for fixed cut-off (in the spirit of section 2), gives the expected bare Liouville potential shown in equation (1).

3.2. Self duality

Due to conformal properties of the Liouville theory, the whole quantum theory can be recovered from the three point correlation function, which is given by the DOZZ formula [12]. The latter shows a self duality of the quantum theory under the change \(b \to b^{-1}\), where \(b\) plays the role of the coupling constant, and we show now how this duality appears in our result.

Starting from the classical action (1), we introduce the parameter \(q = 2\sqrt{\pi} Q\) and see that the change of dynamical variable \(\phi \to \tilde{\phi} = q \phi\) leads to the action

\[\tilde{S} = \int d^2 \xi \left\{ \frac{1}{8\pi} \partial_a \tilde{\phi} \partial^a \tilde{\phi} + \mu^2 e^{b \tilde{\phi}} \right\}, \tag{31}\]
where \( b = q^{-1} \) is the coupling constant of the theory. We therefore expect a duality of our solution when \( q \to q^{-1} \). This is indeed the case, since we have

1. from the result (21):
\[
\varepsilon_{q^{-1}} = \frac{\varepsilon_q}{q}, \tag{32}
\]

such that
\[
\exp (\varepsilon_{q^{-1}} \tilde{\phi}) = \exp (\varepsilon_q \phi); \tag{33}
\]

2. from the result (29), if we define \( w_q = v_\Lambda (\Lambda/\mu)^2 \):
\[
w_{q^{-1}} = (w_q)^{-q^2}. \tag{34}
\]

These properties of our solution are consistent with the known duality of quantum theory \([12]\) (see \([13]\) for reviews), more specifically the power-law transformation (34). The difference here is that, after the change \( q \to q^{-1} \), we need to redefine also the field, in addition to the amplitude \( w_q \). This is a consequence of our dealing with the proper graphs generator functional, from which the correlation functions are obtained after differentiation with respect to the field, and which should then lead to the duality properties of the DOZZ formula.

4. Discussion

In this work, we have analysed Liouville field theory on the world-sheet from the perspective of a novel functional method, suggested in [3]. In particular, we have demonstrated that the function \( Z(\phi) \) appearing in the Liouville kinetic term, is not renormalized, that is it preserves its classical form \( Z = Q^2 \) in the full quantum theory. As we have discussed, this is a specific feature of the two-dimensional field theory, and is not the case in general, e.g. in four dimensions \([4]\). In fact, this feature is also essential for maintaining the conformal properties of the Liouville field, in particular its role in restoring conformal symmetry \([2]\).

It should be stressed that in the present work we have assumed a functional dependence of the effective theory based on the gradient expansion, namely that \( Z(\phi) \) is only a polynomial function of the Liouville field \( \phi \) and not its world-sheet derivatives \( \partial \phi \). This assumption is dictated by the above-mentioned argument of conformal covariance of the Liouville action, which we wish to maintain in the full quantum theory \([2]\). It is remarked that in the case when one included such higher derivative terms, the allowed structures in the effective action should involve terms of the form
\[
Z_n(\phi) \partial_\alpha \phi \left( \frac{\partial_b \partial^b}{\mu^2} \right)^n \partial^\alpha \phi, \tag{35}
\]
where \( Z_n(\phi) \) are dimensionless polynomials of \( \phi \) and \( n \) is an integer. It remains to be seen whether in such cases the above-mentioned non-renormalization result is valid. However, we expect such terms not to be present, as their presence would appear in conflict with the standard conformal properties of the Liouville field. In this sense, we think that the analysis in the present paper is complete.

As a final remark, we note that for a curved world sheet, with a curvature scalar \( R^{(2)} \), replacing \( \mu^2 \) in (35), one could in principle have structures of the form
\[
Y_n(\phi) \partial_\alpha \phi \left( \frac{\partial_b \partial^b}{R^{(2)}} \right)^n \partial^\alpha \phi, \tag{36}
\]
where \(Y_n(\phi)\) are dimensionless polynomials of \(\phi\). However, such structures cannot appear for \(n \neq 0\), as they should vanish in the limit of flat world sheet \(R^{(2)} \to 0\), and since \(Y_n\) does not depend on the curvature scalar, it cannot vanish in this limit in order to leave the terms (36) finite.

Before closing we note that the above analysis can be extended to incorporate Liouville-dressed non-critical stringy \(\sigma\)-models, involving the coupling of \(X^\mu\) fields with the Liouville mode \(\phi\). In such a case there are more complicated potential terms, since for each non-conformal vertex operator \(V(X)\) of the non-critical string, there is a conformal-symmetry restoring factor \(e^{\alpha \phi}\), with \(\alpha\) the appropriate Liouville dimension [2], multiplying \(V(X)\), \(\int d^2 \sigma e^{\alpha \phi} V(X)\). Nevertheless, the application of the exact method for the Liouville sector and the associate \(Q^2\) flows applies to this case, with similar results, as far as the Liouville wave function non-renormalization is concerned.

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Appendix A. Derivation of the flow equations

With the following ansatz for the effective action,
\[
\Gamma = \int d^2 \xi \left\{ \frac{Z_\phi(\phi)}{2} \partial_a \phi \partial^a \phi + V_\phi(\phi) \right\}, \tag{A.1}
\]
the second functional derivative appearing in the evolution equation (5) is
\[
\frac{\delta^2 \Gamma}{\delta \phi(\xi) \delta \phi(\zeta)} = \left( -Z \partial_a \partial^a - \frac{Z'}{2} \partial_a \phi \partial^a \phi \right) (\xi - \zeta)
+ \left( -Z' \partial_a \partial^a \phi - Z' \partial_a \phi \partial^a + V'' \right) (\xi - \zeta). \tag{A.2}
\]
In order to derive the evolution equation for the potential \(V\), a constant configuration \(\phi = \phi_0\) is sufficient. However, since we are also interested in the evolution of \(Z\), we need a coordinate-dependent configuration, and thus we consider
\[
\phi(\xi) = \phi_0 + 2 \rho \cos(k \xi), \tag{A.3}
\]
where \(\rho\) and \(k\) are constants. The evolution equation for the potential \(V\) will then be obtained by identifying the terms independent of \(k\) in equation (5). On the other hand, the evolution equation for \(Z\) is obtained by identifying the terms proportional to \(\rho^2 k^2\).

For the configuration (A.3), the left-hand side of equation (5) reads
\[
\dot{\Gamma} = A \left\{ \dot{V} + \rho^2 \dot{V}'' + \rho^2 k^2 \dot{Z} + \cdots \right\}, \tag{A.4}
\]
where $\mathcal{A}$ is the world sheet surface area, and higher orders in $\rho$ were not written explicitly. The second derivative of $\Gamma$, in Fourier components and to order $\rho^2$, is

$$\frac{\delta^2 \Gamma}{\delta \phi_p \delta \phi_q} = A_{pq} + B_{pq}, \text{ with }$$

$$A_{pq} = \left\{ p^2 Z + V'' + \rho^2 \left[ (p^2 + k^2) Z'' + V^{(4)} \right] \right\} (2\pi)^2 \delta^{(2)}(p+q)$$

$$B_{pq} = \rho \left[ (p^2 - k p + k^2) Z' + V^{(3)} \right](2\pi)^2 \delta^{(2)}(p+q+k)$$

$$+ \frac{\rho^2}{2} \left[ (p^2 + 2k p + 3k^2) Z'' + V^{(4)} \right](2\pi)^2 \delta^{(2)}(p+q+2k)$$

$$+ \frac{\rho^2}{2} \left[ (p^2 - 2k p + 3k^2) Z'' + V^{(4)} \right](2\pi)^2 \delta^{(2)}(p+q-2k). \quad (A.5)$$

We see that $A$ is diagonal in Fourier space, whereas $B$ is not. The inverse is then expanded in powers of $\rho$, using

$$\left( \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \right)^{-1}_{pq} = (A^{-1})_{pq} - (A^{-1} B A^{-1})_{pq} + (A^{-1} B A^{-1} B A^{-1})_{pq} + \cdots \quad (A.6)$$

which finally gives, to order $\rho^2$,

$$\text{Tr} \left\{ \partial_p \partial_q \left( \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \right)^{-1} \right\} = \mathcal{A} \left\{ \frac{\Lambda^2}{8\pi Z} - \frac{V''}{8\pi Z^2} \ln \left( 1 + \frac{Z \Lambda^2}{V''} \right) \right\} + \rho^2 \mathcal{A} \left\{ \cdots \right\}$$

$$+ \rho^2 k^2 \mathcal{A} \left\{ \frac{5}{4\pi Z} \left( \frac{Z'}{Z} \right)^2 \ln \left( 1 + \frac{Z \Lambda^2}{V''} \right) - \frac{47}{24\pi Z} \left( \frac{Z'}{Z} \right)^2 + \frac{7}{12\pi Z} \frac{Z'}{Z} V^{(3)} \right\}. \quad (A.7)$$

In the above expression, the term proportional to $\rho^2$ and independent of $k$ is not relevant, as it leads to the evolution equation for $V''$. We checked, though, that this evolution is consistent with the one for $V$. The evolution equations for $V$ and $Z$ are then those given in equations (7), after identification with the left-hand side of (A.4).

**Appendix B. Equivalence between Wilsonian and one-particle-irreducible effective potentials**

For a constant IR configuration $\phi_0$, the Wilsonian effective potential $U_{\text{Wils}}$ is defined by

$$\exp \left( i V U_{\text{Wils}}(\phi_0) \right) = \int \mathcal{D}[\phi] \exp \left( i \mathcal{S}[\phi_0 + \phi] \right). \quad (B.1)$$
where $V$ is the volume of space time, $S$ is the bare action defined at some cut-off $\Lambda$, and the dynamical variable $\phi$ which is integrated out has non-vanishing Fourier components for $|p| \leq \Lambda$. One can also write the previous definition as

$$\exp(iV U_{\text{Wils}}(\phi_0)) = \int D[\phi] \exp(iS[\phi]) \delta \left( \int_x \phi - V \phi_0 \right)$$

$$= \int D[\phi] \exp(iS[\phi]) \int_j \exp \left( ij \int_x (\phi - \phi_0) \right), \quad (B.2)$$

where $\int_x$ denotes the integration over space time, $j$ is a real variable, and $\int_j$ denotes the integration over $j$. Using the notations of section 2, this expression can be written as

$$\exp(iV U_{\text{Wils}}(\phi_0)) = \int_j Z[j] \exp \left( -i \int_x j \phi_0 \right)$$

$$= \int_j \exp \left( iW[j] - i \int_x j \phi_0 \right)$$

$$= \int_j \exp (i\Gamma[\phi_0])$$

$$= \int_j \exp (iV U_{\text{1PI}}(\phi_0)), \quad (B.3)$$

where $U_{\text{1PI}}$ is the one-particle-irreducible effective potential. Note that $j$ plays the role of a constant source for the field $\phi$, leading to the constant classical field $\phi_0$. In the last expression, the integration over $j$ leads to a multiplicative constant, as $\phi_0$ is fixed. Disregarding the $\phi_0$-independent terms, we then obtain

$$U_{\text{Wils}}(\phi_0) = U_{\text{1PI}}(\phi_0). \quad (B.4)$$

Note that for the above argument to be valid it is essential that we work in Minkowski space time, since the delta function in equation (B.2) is expressed in terms of its Fourier transform.

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