Abstract

The relation between the charge of Lascoux-Schulenberger and the energy function in solvable lattice models is clarified. As an application, A.N.Kirillov’s conjecture on the expression of the branching coefficient of $\widehat{sl}_n/sl_n$ as a limit of Kostka polynomials is proved.
1. Introduction

The study of two dimensional solvable lattice models has provided many new links with other areas of mathematics and physics. In particular, one basic object of the models, referred to as the one dimensional configuration sum (1DCS) gives a generalization of Rogers-Ramanujan type identities [ABF] and is related with the characters of (quantum) affine Lie algebras [DJKMO]. The relation between 1DCS and affine Lie algebras was systematized using the theory of crystal bases by Kashiwara [K1,KMN1-2].

The weight function describing the 1DCS is called the energy function and it is determined from the \( q = 0 \) behavior of the R matrix. The theory of crystal bases gives a simple characterization of energy functions without need for explicit forms of R matrices [KMN1]. Obtaining this characterization played an essential role in proving the relations between 1DCS and string functions [KMN1] and the branching coefficients [DJO] of affine Lie algebras.

In this paper we add further application of the energy function to combinatorics and representation theory. In particular we obtain a new expression of the Kostka polynomials in terms of energy functions. (for instance see (1.1) below). This result was obtained while trying to generalize our previous work [NY1-2] on spinon bases to the case of \( \widehat{sl}_n \).

The Kostka polynomial \( K_{\lambda \mu}(q) \) is the connection coefficient between Schur function \( s_\lambda(x) \) and the Hall-Littlewood symmetric function \( P_\mu(x : q) \):

\[
s_\lambda(x) = \sum_{\mu} K_{\lambda \mu}(q) P_\mu(x : q).
\]

The Kostka polynomial can be understood in many ways, as the \( q \) analogue of weight multiplicities [L], the \( q \) analogue of Clebsch-Gordan coefficients [Kir2], the Kazhdan-Lusztig polynomials, [L] etc. Among the ways of expressing the Kostka polynomial, the combinatorial description of Lascoux-Schutzenberger [LS,B] takes the form

\[
K_{\lambda \mu}(q) = \sum_{T \in \mathcal{T}(\lambda, \mu)} q^{c(T)},
\]

where \( \mathcal{T}(\lambda, \mu) \) is the set of semi-standard tableaux of shape \( \lambda \) and weight \( \mu \), the non-negative integer \( c(T) \) is called the ”charge”, defined as a sum of the indices (for a precise definition, see §3 and [M] §3). Our main theorem (Theorem 4.1) gives an expression of each of the indices in terms of the energy function of \( U_q(\widehat{sl}_n) \). Hence the charge of Lascoux-Schutzenberger appears to have a representation theoretical meaning in quantum affine algebra.

To avoid notational complexity, in this introduction we illustrate our results with one of their applications. For the special case \( \mu = (k^l) \), combining our results and the theorem of
Lascoux-Schützenberger, we have

\[ K_{\lambda\mu}(q) = \sum_{(b_1, \ldots, b_1) \in T(\lambda, \mu)} q^{\sum_{j=1}^l jH_{k\Lambda_1k\Lambda_1}(b_{j+1}, b_j)}, \quad (1.1), \]

where the sum is taken over \((b_1, \cdots, b_1)\), which is a path (see §2 for precise definition) parametrizing the set of highest weight vectors with weight \(\lambda\) in the tensor product \(V_{k\Lambda_1} \otimes V_{k\Lambda_1}\), \(V_{k\Lambda_1}\) being the irreducible representation of \(\mathfrak{sl}_n\) with highest weight \(k\Lambda_1\). The function \(H_{k\Lambda_1k\Lambda_1}(\cdot, \cdot)\) is the corresponding energy function (see §3). The right hand side of (1.1) is nothing but the 1DCS, restricted to the highest weight elements. Hence by the theorem of [JMMO, KMN1] we have

**Theorem 1.1.** For a dominant integral weight \(\lambda = (\lambda_1, \cdots, \lambda_n) (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)\) of \(\mathfrak{sl}_n\), let us set

\[ V(k\Lambda_0 : \lambda) = \{ v \in V(k\Lambda_0) \mid e_jv = 0 (1 \leq j \leq n - 1), \text{wt}v = \lambda \}. \]

We assume \(|\lambda| \equiv 0 \mod n\). Then we have

\[ \text{tr}_{V(k\Lambda_0; \lambda)}(q^{-d}) = \lim_{N \to \infty} q^{-\frac{1}{2}kn(N-1)} K_{\lambda^{(N)}\mu^{(N)}}(q), \]

where \(\lambda^{(N)} = (\lambda_1 + kN - \frac{1}{n}|\lambda|, \cdots, \lambda_n + kN - \frac{1}{n}|\lambda|)\) and \(\mu^{(N)} = (k^{nN})\).

This formula was conjectured by A.N. Kirillov (Conjecture 4 in [Kir]).

The present paper is organized as follows. In §2 we briefly review the necessary results in crystal theory and its relation to combinatorics of Young tableaux, the charge, the index, and the Kostka polynomials. More precise definitions of crystals and related notation are given in the Appendix. We define the energy function and its explicit formula in §3. In §4 the statement of the main theorem of this paper and its applications are given. §5 is devoted to the proof of this main theorem.
2. Brief Review of Crystal Theory

Here we recapitulate some basic facts concerning the crystal base theory for the case of symmetric and anti-symmetric representations in $sl_n$ mainly through examples. For more details and precise definitions see the Appendix and [K3,KN,KMN1].

2.1. Crystal base

For a dominant integral weight $\lambda$ of $sl_n$, a crystal (base) $B_\lambda$ parametrizes a base of the irreducible representation $V_\lambda$. $B_\lambda$ can be thought as a set of semi-standard tableaux of shape $\lambda$ [KN]. For the symmetric $B_{k\Lambda_1}$ and anti-symmetric $B_{k\Lambda_k}$ cases ($1 \leq k \leq n$), we also use another representation, as seen in the following examples. (There are two reasons for this: (1) We will use semi-standard tableaux to represent a ”path”. (2) The energy function has a simple description in this representation.)

**Example 2.2.** Elements in $B_{\Lambda_2} = B_{\frac{1}{2}}$ for $sl_4$ are

$\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\end{array} \quad (= \begin{array}{c}
1 \\
2 \\
\end{array}) , \\
\begin{array}{c}
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\end{array} .
\end{array}$

In general there are $k$-dots in $n$-boxes for $B_{k\Lambda_k}$ in the $sl_n$ case. The boxes are labeled $1, 2, 3, ..., n$ starting from the top, and each has a weight $\epsilon_i = \Lambda_i - \Lambda_{i-1}$ $(\Lambda_0 = \Lambda_n = 0)$.

**Example 2.3.** There are four more elements in $B_{2\Lambda_1} = B_{\frac{1}{2}}$ for $sl_4$ :

$\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\end{array} \quad (= \begin{array}{c}
1 \\
2 \\
\end{array}) , \\
\begin{array}{c}
\bullet \\
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\bullet \\
\end{array} , \\
\begin{array}{c}
\bullet \\
\bullet \\
\end{array} .
\end{array}$

Note that for $B_{\Lambda_k}$, putting more than one dot in a box is forbidden, while for $B_{k\Lambda_1}$ it is permitted.

2.4. The action of $\tilde{f}_i$ and $\tilde{e}_i$
On $B_\lambda$ one can define the action of the operators $\tilde{e}_i$ and $\tilde{f}_i$ ($1 \leq i \leq n-1$) : $B_\lambda \to B_\lambda \sqcup \{0\}$. This is the $q = 0$ analog of the action of $sl_n$ generators. On $B_{\Lambda_k}$ and $B_{k\Lambda_1}$ the action can be described explicitly in the following manners [KN].

**Case 1:** $b \in B_{\Lambda_k}$.

(a) If the $i$-th box in $b$ is dotted and the $(i+1)$-th is not, then $\tilde{f}_i b$ is the element obtained by shifting the $i$-th dot to $(i+1)$-th box. Otherwise, $\tilde{f}_i b = 0$.

(b) If the $(i+1)$-th box in $b$ is dotted and the $i$-th is not, then $\tilde{e}_i b$ is the element obtained by shifting the $(i+1)$-th dot to $i$-th box. Otherwise, $\tilde{e}_i b = 0$.

**Case 2:** $b \in B_{k\Lambda_1}$.

(a) If there are dots in the $i$-th box in $b$, then $\tilde{f}_i b$ is the element obtained by shifting one of the $i$-th dots to the $(i+1)$-th box. Otherwise, $\tilde{f}_i b = 0$.

(b) If there are dots in the $(i+1)$-th box in $b$, then $\tilde{e}_i b$ is the element obtained by shifting one of the $(i+1)$-th dots to the $i$-th box. Otherwise, $\tilde{e}_i b = 0$.

One can also define the action of $\tilde{e}_0$ and $\tilde{f}_0$, which we identify with $\tilde{e}_n$ and $\tilde{f}_n$. Here the column is considered to be periodically continued with period $n$. Then the action of $\tilde{e}_n, \tilde{f}_n$ is defined by the above rules. In this way $B_\lambda$ is extended to a crystal for $\hat{sl}_n$ [KMN2]. This extension is crucial for the characterization of the energy function.

### 2.5. Tensor products

On the tensor product of crystals (direct product as a set), the actions of $\tilde{e}_i$ and $\tilde{f}_i$ are defined as follows. Since the action depends only on the configuration in the $i$-th and $(i+1)$-th box, only these two boxes will be considered.

In both symmetric and anti-symmetric cases, the rules are as follows:

(a) We neglect empty boxes $\square$.

(b) We remove singlet pairs of dots, that is

\[
\begin{pmatrix}
\circ \\
\circ
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
\circ \\
\circ
\end{pmatrix} \otimes
\begin{pmatrix}
\circ \\
\circ
\end{pmatrix}
\]

for the anti-symmetric case and

\[
\begin{pmatrix}
\bullet(\cdots)
\end{pmatrix}
\otimes
\begin{pmatrix}
\circ
\end{pmatrix}
\]

\begin{pmatrix}
\circ
\end{pmatrix}
\otimes
\begin{pmatrix}
\bullet(\cdots)
\end{pmatrix}
\]

(and others like this) for the symmetric case.
(c) We continue (a) and (b) as far as possible until we obtain something like

for the anti-symmetric case and

for the symmetric case.

(d) Then $\tilde{e}_i$ shifts a dot in the bottom box of the rightmost to the top box and $\tilde{f}_i$ shifts a dot in the top box of the leftmost to the bottom box. If there is no such dot, then $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$).

**Example 2.6.** Let $b \in B_{\Lambda_3} \otimes B_{\Lambda_2} \otimes B_{\Lambda_2} \otimes B_{\Lambda_1} \otimes B_{\Lambda_1} \otimes B_{\Lambda_1}$ as follows:

As far as the actions of $\tilde{e}_1$ and $\tilde{f}_1$ are concerned, $b$ is equivalent to

Then by the above rules (a) and (b), this can be reduced to the following:

and

by (a)
and finally
\[
\rightarrow \begin{array}{c}
\end{array}.
\]
Hence \(\tilde{e}_1\) acts on the last component of the tensor product, while \(\tilde{f}_1 b = 0\).

### 2.7. Parametrization of the highest weight vectors

In what follows we fix a partition \(\mu = (\mu_1, \cdots, \mu_l)\) and consider the tensor products

\[
\mathcal{H}_\mu = B_{\mu_1 \Lambda_1} \otimes \cdots \otimes B_{\mu_l \Lambda_1},
\]

and

\[
\mathcal{H}'_\mu = B_{\Lambda_{\mu_1}} \otimes \cdots \otimes B_{\Lambda_{\mu_l}}.
\]

Here we will give a canonical parametrization of the \(sl_n\) highest weight elements \(b\) (i.e. \(\tilde{e}_i b = 0\) for all \(1 \leq i \leq n - 1\)) in terms of semi-standard tableau.

### 2.8. Semi-standard tableau \(\mathcal{T}(\lambda, \mu)\)

For partitions \(\lambda\) and \(\mu\) (\(|\lambda| = |\mu|\)), let \(\mathcal{T}(\lambda, \mu)\) be the set of semi-standard tableaux of shape \(\lambda\) and weight \(\mu\).

**Example 2.9.** A semi-standard tableau of shape \(\lambda = (5, 4, 2)\) is a diagram of the form

\[
\begin{array}{cccc}
1_1 & 1_2 & 1_3 & 1_4 \\
1_5 & 2_1 & 1_2 & 1_3 \\
2_4 & 3_1 & 1_2
\end{array}
\]

where the numbers \(i_{n,m}\) satisfy the following conditions:

\[
i_{n,m+1} \geq i_{n,m}, \quad i_{n+1,m} > i_{n,m}.
\]

**Example 2.10.** The following tableau \(T\) has the shape \(\lambda = (5, 4, 2)\) and the weight \(\mu = (3^2, 2^2, 1)\).
Here, \( \#\{ (n, m) \mid i_{n,m} = j \} = \mu_j \).

We denote by \( \lambda' \) the transposition of the partition \( \lambda \) [M]. For each \( T \in \mathcal{T}(\lambda, \mu) \) one can construct a highest weight element \( b \) in \( \mathcal{H}_{\mu} \) [resp. \( \mathcal{H}'_{\mu} \)] of weight \( \lambda \) [resp. \( \lambda' \)] as in the following examples.

**Example 2.11.**
The highest weight element in \( \mathcal{H}_{\mu} \) corresponding to the tableau \( T \) in Example 2.10 is

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

The highest weight element in \( \mathcal{H}'_{\mu} \) corresponding to the tableau \( T \) in Example 2.10 is

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

Note that the correspondence makes sense iff \( n \geq l(\lambda) \) [resp. \( n \geq l(\lambda') \)].

**Theorem 2.12.** The element \( b \in \mathcal{H}_{\mu} \) [resp. \( \mathcal{H}'_{\mu} \)] is a highest weight element with weight \( \lambda \) [resp. \( \lambda' \)] if and only if it is an image of \( T \in \mathcal{T}(\lambda, \mu) \) under the above correspondence. This correspondence gives a bijection between the highest elements and the semi-standard tableaux. Proof. The statement is easily proved using the rules of the action of \( \tilde{e}_i \) described above. q.e.d.

**2.13. Review of the Kostka Polynomials**

Here we review the necessary basic facts concerning the symmetric functions, in particular Kostka polynomials. For more details, see Ref.[M].
2.14. Kostka number $K_{\lambda,\mu}$

For a partition $\lambda$, let $s_\lambda(x)$, $m_\lambda(x)$, $h_\lambda(x)$ and $e_\lambda(x)$ be the symmetric functions on $n$-variables $x = (x_1, \cdots, x_n)$. These are referred as the Schur, monomial, complete and elementary functions respectively. We are interested in their transition matrices.

The Kostka number $K_{\lambda,\mu}$ is defined by

$$s_\lambda(x) = \sum_\mu K_{\lambda,\mu} m_\mu(x),$$

that is, $K_{\lambda,\mu}$ is the multiplicity of weight $\mu$ in the irreducible highest weight representation $V_\lambda$.

By the duality relation

$$\sum_{\lambda} m_\lambda(x) h_\lambda(y) = \sum_{\lambda} s_\lambda(x) s_\lambda(y),$$

this definition is equivalent to

$$h_\mu(x) = \sum_{\lambda} K_{\lambda,\mu} s_\lambda(x).$$

Since $h_\mu = h_{\mu_1} \cdots h_{\mu_l}$ is the character of the tensor product of complete symmetric representations $V_{\mu_1 \Lambda_1} \otimes \cdots \otimes V_{\mu_l \Lambda_1}$, $K_{\lambda,\mu}$ is the multiplicity of $V_\lambda$ in this tensor product, i.e., the Clebsch-Gordan coefficient.

By applying the involution $\omega$, $\omega(s_\lambda) = s_{\lambda'}$, for which $\omega(h_\mu) = e_\mu$ to the above equation, we also have

$$e_\mu(x) = \sum_{\lambda} K_{\lambda,\mu} s_{\lambda'}(x),$$

that is, $K_{\lambda,\mu}$ is the multiplicity of $V_{\lambda'}$ in the tensor products $V_{\Lambda_{\mu_1}} \otimes \cdots \otimes V_{\Lambda_{\mu_l}}$.

Theorem 2.12 gives an explanation in terms of crystals for the well known fact that $K_{\lambda,\mu}$ is the number of semi-standard tableaux of shape $\lambda$ and weight $\mu$.

2.15. Kostka polynomial $K_{\lambda,\mu}(q)$

The Kostka polynomial $K_{\lambda,\mu}(q)$ is defined as the transition coefficient of the Schur function in terms of the Hall-Littlewood polynomial $P_\mu(x, q)$ :

$$s_\lambda(x) = \sum_\mu K_{\lambda,\mu}(q) P_\mu(x, q).$$

Since $P_\lambda(x, 1) = m_\lambda(x)$, $K_{\lambda,\mu}(q)$ is the $q$-analog of the Kostka number $K_{\lambda,\mu} = K_{\lambda,\mu}(1)$. 

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Though it is quite nontrivial from this definition, \( K_{\lambda,\mu}(q) \) is a polynomial in \( q \) with non-negative integer coefficients. This fact was proved by Lascoux-Schutzenberger by establishing the following combinatorial formula for Kostka polynomials.

**Theorem 2.16.** ([LS])

\[
K_{\lambda,\mu}(q) = \sum_{T \in \mathcal{T}(\lambda,\mu)} q^{c(T)},
\]

where the definition of the charge \( c : \mathcal{T} \to \mathbb{Z}_{\geq 0} \) is illustrated by the following example [LS,M].

**Example 2.17.** For the tableau \( T \) in Example 2.10, the charge is calculated by the following three steps.

In each step, the sequence \( 1, 2, 3, \ldots \) is selected from the top and right (periodically), and each suffix counts the "winding number" (= the number of times the top is revisited). Then index, \( \text{ind}(j) \), is defined as a sum of the winding numbers assigned to \( j \), i.e.,

\[
\text{ind}(1) = 0, \quad \text{ind}(2) = 1, \quad \text{ind}(3) = 1, \quad \text{ind}(4) = 3, \quad \text{ind}(5) = 1.
\]

Finally, the charge is the sum of all indices, \( c(T) = 6 \).

In the main Theorem, we will give a similar (but not the same) formula for the index (or \( K_{\lambda,\mu}(q) \)) from the point of view of crystal theory.

### 2.18. Skew Kostka polynomials

For the skew Young diagram \( \lambda/\nu \), the skew Kostka polynomial \( K_{\lambda/\nu,\mu}(q) \) was similarly defined by [KR] as the connection matrix between the skew Schur function \( s_{\lambda/\nu} \) and the Hall polynomials. It has the description [KR, B]

\[
K_{\lambda/\nu,\mu}(q) = \sum_{T \in \mathcal{T}(\lambda/\nu,\mu)} q^{c(T)},
\]

where \( \mathcal{T}(\lambda/\nu,\mu) \) is the set of semi-standard tableaux of shape \( \lambda/\nu \) and weight \( \mu \).
Similarly to the non-skew case ($\nu = \phi$), there exists a one to one correspondence between the semi-standard tableau $T$ in $\mathcal{T}(\lambda/\nu, \mu)$ and the highest element $b$ in $B_{\nu} \otimes \mathcal{H}_{\mu}$ of weight $\lambda$, [resp. $b'$ in $B_{\nu'} \otimes \mathcal{H}_{\mu}'$ of weight $\lambda'$]. In terms of paths, these correspond to the fusion path beginning at $\nu$ and ending at $\lambda$ [resp. $\nu'$ and $\lambda'$].
3. Energy Functions

3.1. General definition

Here we give definitions and descriptions of energy functions. The precise definitions of notation used here are explained briefly in the Appendix.

Let us consider two classical crystals, $B_1$ and $B_2$, for $\widehat{\mathfrak{sl}}_n$ satisfying the following conditions:

$$B_1 \otimes B_2 \text{ is connected.} \quad (3.1)$$

$$B_1 \otimes B_2 \text{ is isomorphic to } B_2 \otimes B_1. \quad (3.2)$$

**Definition 3.2.** Suppose that $b_1 \otimes b_2$ is mapped to $b'_2 \otimes b'_1$ by the isomorphism (3.2):

$$B_1 \otimes B_2 \simeq B_2 \otimes B_1,$$

$$b_1 \otimes b_2 \mapsto b'_2 \otimes b'_1.$$

The energy function $H$ of the crystal $B_1 \otimes B_2$ is a map

$$H : B_1 \otimes B_2 \longrightarrow \mathbb{Z}$$

which satisfies the following conditions. Suppose that $\tilde{e}_i(b_1 \otimes b_2) \neq 0$. Then

$$H(\tilde{e}_i(b_1 \otimes b_2)) = H(b_1 \otimes b_2) + 1 \text{ if } i = 0, \varphi_0(b_1) \geq \varepsilon_0(b_2) \text{ and } \varphi_0(b'_2) \geq \varepsilon_0(b'_1),$$

$$= H(b_1 \otimes b_2) - 1 \text{ if } i = 0, \varphi_0(b_1) < \varepsilon_0(b_2) \text{ and } \varphi_0(b'_2) < \varepsilon_0(b'_1),$$

$$= H(b_1 \otimes b_2) \text{ otherwise.}$$

We often denote $H(b_1, b_2)$ instead of $H(b_1 \otimes b_2)$ for the sake of simplicity.

By definition the energy function is constant on each connected component of $B_1 \otimes B_2$ as $\mathfrak{sl}_n$ crystal and is unique up to an additive constant by the condition (3.1). For $B_1 = B_2$ the energy function has been used extensively to evaluate the one point function of solvable lattice models [DJKMO,KMN1,DJO]. The study of spinon character formulas [NY1-2] leads us to consider the above slightly generalized energy functions.

**Remark 3.3.** (1). The definition of the energy function above follows from the property of the combinatorial $R$ matrix in a manner similar to the $B_1 = B_2$ case [KMN1].

(2). In [KMN2] the crystal $B_{k\Lambda_i}$ (for $\widehat{\mathfrak{sl}}_n$) is proved to be a perfect crystal of level $k$ (Definition 4.6.1 [KMN1]). Thus if $B_1$ and $B_2$ are crystals of the form $B_{k\Lambda_i}$, we can prove that the
condition (3.2) is a consequence of (3.1) by an argument similar to Proposition 4.3.1 in [KMN1]. In this paper we do not use this fact since we explicitly construct the isomorphism for the cases we consider.

Example 3.4. Let us consider the case of \( \hat{sl}_3, B_{\Lambda_1} \otimes B_{\Lambda_2} \). Here the energy function is given by the following. \( H = 0 \) on the elements

\[
\begin{array}{cccc}
\bullet & \bullet & \circ & \circ \\
\circ & \circ & \bullet & \bullet \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

and

\[
\begin{array}{cccc}
\bullet & \bullet & \circ & \circ \\
\circ & \circ & \bullet & \bullet \\
\circ & \circ & \circ & \circ \\
\end{array}
\]

\( H = 1 \) on

\[
\begin{array}{cccc}
\bullet & \circ & \circ & \circ \\
\circ & \bullet & \circ & \circ \\
\circ & \circ & \bullet & \circ \\
\circ & \circ & \circ & \bullet \\
\end{array}
\]

The crystal graph of the tensor product is written in Figure 3.1. (0 arrows are omitted.) The property of the energy function is obvious from this figure.

3.5. Explicit forms of the isomorphism \( \iota \) and energy function \( H \)
First we demonstrate the connectivity for relevant cases.

**Proposition 3.6.** \( B_{\Lambda_k} \otimes B_{\Lambda_l} \) is connected.
Proof. Since $B_{\Lambda_k}$ and $B_{\Lambda_l}$ are both perfect crystals of level one, $B_{\Lambda_k} \otimes B_{\Lambda_l}$ is connected by Lemma 4.6.2 in [KMN1]. q.e.d.

**Proposition 3.7.** $B_{k\Lambda_1} \otimes B_{l\Lambda_1}$ is connected.

Proof. Since $B_{k\Lambda_1} \otimes B_{l\Lambda_1}$ is a union of highest weight crystals as a \{1, \ldots, n - 1\} crystal, any element is connected to the element of the form

$$1 \otimes \begin{bmatrix} i_1 & \cdots & i_l \end{bmatrix}.$$  

By applying suitable $\tilde{f}_i$'s ($1 \leq i \leq n - 1$) to this element, we arrive at the element of the form

$$1 \otimes \begin{bmatrix} 1 & \cdots & 1 & n & \cdots & n \end{bmatrix}.$$

Let $m$ be the number of $n$ in the second component. Then if we apply $\tilde{f}_0^m$ to this element, we arrive at

$$1 \otimes \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix}.$$

q.e.d.

By these propositions, if there exist isomorphisms

$$\iota : B_{\Lambda_k} \otimes B_{\Lambda_l} \to B_{\Lambda_l} \otimes B_{\Lambda_k},$$

$$\iota : B_{k\Lambda_1} \otimes B_{l\Lambda_1} \to B_{l\Lambda_1} \otimes B_{k\Lambda_1},$$

then the $\iota$'s are unique, and the corresponding energy function is unique up to additive constants. Now we shall give the explicit rules for the isomorphism and the energy function. We assume $k \geq l$. The proofs for these rules are given in §3.15.

**Example 3.8.** The following are examples of the isomorphism

$$\iota : B_{\Lambda_3} \otimes B_{\Lambda_2} \to B_{\Lambda_2} \otimes B_{\Lambda_3}.$$
For the energy function, $H = 0$ for the first example and $H = -1$ for the second.

**Example 3.9.** The following are examples of the isomorphism

\[
\iota : B_{3\Lambda_1} \otimes B_{2\Lambda_1} \rightarrow B_{2\Lambda_1} \otimes B_{3\Lambda_1}.
\]

For the first example $H = 0$, and for the second $H = 2$.

**Rule 3.10.** The case of $B_{\Lambda_k} \otimes B_{\Lambda_l}$ ($k \geq l$).

The general procedure to obtain the isomorphism and energy function is as follows.

1. Pick the dot $\bullet_a$ which has the highest position in the 2nd column and connect it with the dot $\bullet'_a$ in the first column which has the highest position among all dots whose positions are not higher than $\bullet_a$. If there is no such dot in the 1st column, return to the top, assuming the periodic boundary condition. We call such a pair a "winding".

2. Repeat the same procedure for the remaining unconnected dots $l - 1$-times.

3. The isomorphism $\iota$ is obtained by sliding the remaining $(k - l)$ unpaired dots in the 1st column to the 2nd.

4. The energy function is $(-1)$ times the number of "winding" pairs.

**Rule 3.11.** The case of $B_{\Lambda_k} \otimes B_{\Lambda_1}$ ($k \geq l$).

In this case the rule to determine the isomorphism $\iota$ and energy function $H$ is almost the same as that in the anti-symmetric case. The only differences are in 1. and 4.

1. The partner $\bullet'_a$ is the dot which has the lowest position among all dots whose positions are higher than that of $\bullet_a$. If there is no such dot return to the bottom.

4. The energy function is simply the number of "winding" pairs.

**Definition 3.12.** We refer to the line appearing in Rules 3.10 and 3.11 (connecting two dots) as the $H$-line.

Any way of drawing the $H$-lines is allowed as long as it represents Rules 3.10 and 3.11 correctly, however, we sometimes use the following special way of drawing $H$-lines, especially in the proof of Lemma 3.16 and in Section 5 for the sake of convenience. This special way is as follows. Starting from a dot in the 2nd column, first go horizontally left and then go
vertically down, until you find its partner in the first column. Then, if necessary, round up outside of the diagram and again go vertically down from the top.

3.13. The relation of energy functions
We denote by $H_{\Lambda_k \Lambda_l}$ and $H_{k \Lambda_1 \Lambda_1}$ the energy functions of $B_{\Lambda_k} \otimes B_{\Lambda_l}$ and $B_{k \Lambda_1} \otimes B_{l \Lambda_1}$, respectively, which are explicitly described above. They are characterized by the normalization $H_{\Lambda_k \Lambda_l} = 0$ on

$$
\begin{array}{cc}
1 & 1 \\
\vdots & \vdots \\
k & l
\end{array}
$$

and $H_{k \Lambda_1 \Lambda_1} = l$ on

$$
\begin{array}{cccc}
1 & \cdots & 1 & \cdots & 1
\end{array}
$$

Consider the semi-standard tableau $T \in \mathcal{T}(\lambda, \mu)$, and let $b = b_1 \otimes \cdots \otimes b_r$ and $\tilde{b} = \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_r$ be the corresponding highest weight elements in $B_{\Lambda_{\mu_1}} \otimes \cdots \otimes B_{\Lambda_{\mu_r}}$ and $B_{\mu_1 \Lambda_1} \otimes \cdots \otimes B_{\mu_r \Lambda_r}$, respectively. We define $b_i^{(j)}$ and $\tilde{b}_i^{(j)}$ as in the beginning of Section 4. Then we have

**Proposition 3.14.** The following relation holds:

$$
H_{\Lambda_j \Lambda_i}(b_j \otimes b_i^{(j+1)}) + H_{j \Lambda_1 \Lambda_1}(\tilde{b}_j \otimes \tilde{b}_i^{(j+1)}) = 0. \quad (3.3)
$$

Proof. Comparing the evaluation rule of $H_{\Lambda_j \Lambda_i}(b_j \otimes b_i^{(j+1)})$ and $H_{j \Lambda_1 \Lambda_1}(\tilde{b}_j \otimes \tilde{b}_i^{(j+1)})$ and the corresponding isomorphism rule on the semi-standard tableau $T$, one finds that they are the same (see, for example, both rules by Examples 2.10 and 2.11). Hence beginning from $b$ and $\tilde{b}$, we have equation (3.3) inductively. q.e.d.

3.15. Proof of Rules for $\iota$ and the energy function
Here we prove that the rules described above give the correct isomorphism $\iota$ and energy function. In Rule 3.10 and 3.11, we made pairs of dots starting from the top. However, this choice is not essential. In fact we can again consider these rules by changing the order in which the $H$ lines are drawn. Then one can prove

**Lemma 3.16.** The isomorphism $\iota$ and the energy function determined by the above rules are independent of the order of drawing lines.
Proof. (Anti-symmetric case.) Suppose there exists a dot (say ●_{L_1}) in one of the left boxes that is unpaired in one ordering (say A) and paired in another (say B). Let ●_{R_1} be the partner of ●_{L_1} in B, and let ●_{L_2} be the partner of ●_{R_1} in A. Then ●_{L_2} must be paired with some dot (say ●_{R_2}) in B since the R_1L_1 line already passes through ●_{L_2} (Figure 3.2). Let us prove that this process of determining R_1L_2R_2 \cdots does not stop. Suppose that we already have the R_jL_j line in B. Let us consider the following statement for R_j.

(\text{Ass})_j For each box in the left column in B whose position is not higher than that of R_j and not lower than that of L_1, there is a line passing through it. (Not necessarily the end point.)

Here we assume periodic boundary conditions (pbc). This implies, for example, that if R_j is below L_1 in the non-pbc sense, then the position specified by (\text{Ass})_j is like that in Figure 3.3. Suppose that (\text{Ass})_j holds. Let us prove that there then exist L_{j+1} and R_{j+1} for which (\text{Ass})_{j+1} holds. By definition L_{j+1} exists. Note that the position of L_{j+1} is above that of L_1 and not higher than that of R_j, since L_1 is not paired in A. Hence, in B, some line should already pass through L_{j+1} by (\text{Ass})_j, and R_{j+1} is determined. If the position of L_{j+1} is above that of L_1 in the non-pbc sense, then (\text{Ass})_{j+1} obviously holds by (\text{Ass})_j and the drawing rule (Figure 3.4). If the position of L_{j+1} is lower than the position of L_1 in the non-pbc sense, again (\text{Ass})_{j+1} obviously holds (Figure 3.5-6). By definition R_1R_2 \cdots are all distinct. This is a contradiction, since the number of boxes in the right column is finite. Therefore the set of end points of H lines is independent of the order in which they are drawn. This proves that the rule determining i is independent of the order of drawing lines.

Next let us prove that the value of the energy function H determined by the above rule is independent of the order of drawing the H lines.

Let l_0, \cdots, l_n be the horizontal lines in the diagram numbered from the top down. To each l_j we assign the non-negative integer h_J(l_j) as the number of H lines crossing l_j (Figure 3.7). Here the subscript J specifies an order. The value of the energy function H w.r.t. J is given by h_J(l_0). By the rule of drawing H lines, h_J(l) is subject to the following relation:

\begin{equation}
(\text{Rel}) \quad h_J(l_{j+1}) = h_J(l_j) + e^r_j - e^l_j,
\end{equation}

where e^l_j is the number of dots which are end points of the H lines and are between l_j and l_{j+1}. Similarly e^r_j is the number of dots between l_j and l_{j+1}. In particular e^r_j, e^l_j = 0, 1.

The set (h_J(l_0), \cdots, h_J(l_{n-1})) is the same for any J up to an overall additive constant by (\text{Rel}), since, as previously proven, the set of end points of H lines is independent of J. We prove that h_J(l_j) does not depend on J. To this end it is sufficient to prove the existence of l_j for which h_J(l_j) = 0. In fact for two orders J_1 and J_2, if there is an i such that
\[ h_{J_1}(l_i) = h_{J_2}(l_i) + m, \ m > 0, \] then by the above comment,
\[ h_{J_1}(l_k) = h_{J_2}(l_k) + m \quad \text{for any} \ k. \]

Since \( h_{J_2}(l_k) \geq 0 \) for any \( k \), this contradicts the existence of \( l_j \) with \( h_{J_1}(l_j) = 0 \). So let us prove the existence of such \( l_j \).

Let \( \bullet \) (say \( L \)) be the end point (in the left column) of a winding line and is lower than all other such points. Let \( \bullet \) (say \( R \)) be the point in the right column from which a winding line starts and is higher than all other such points. We claim that there exists a horizontal line \( l \) between the underline of the box \( L \) and the overline of the box \( R \) for which \( h_{J}(l) = 0 \).

Suppose that such an \( l \) does not exist. Then the situation is like that in Figure 3.8. We proceed with our proof according to this figure. Note first that the order of \( \alpha_2 \) is smaller than or equal to that of \( \alpha_1 \) by the definition of \( R \) and \( L \). Next note that

- the order of \( \alpha_3 \) is larger than that of \( \alpha_1 \).
- the order of \( \alpha_4 \) is larger than that of \( \alpha_3 \).
- the order of \( \alpha_5 \) is larger than that of \( \alpha_4 \).
- the order of \( \alpha_2 \) is larger than that of \( \alpha_5 \).

Hence the order of \( \alpha_2 \) is larger than \( \alpha_1 \), which is a contradiction. Thus in the anti-symmetric case, the value of the energy function given by our rule does not depend on the order of drawing lines.

(Symmetric case.) The proof of the symmetric case reduces to the anti-symmetric case in the following way. Let us make the following change in the diagram (see Figure 3.9):

1. Within each box, align the \( \bullet \) vertically. Then, shift the left column down by one box. Create a rectangular diagram from the present one by adding two boxes.
2. Draw horizontal lines such that each \( \bullet \) is in one box.
3. Reverse up and down in the diagram.

Now for the final diagram, the drawing rule for \( H \) lines is the same as that in the anti-symmetric case. Hence the independence of the value of the \( H \) function and the isomorphism \( \iota \) follows from the anti-symmetric case. q.e.d.

\textbf{Proposition 3.17.} The map \( \iota \) described in the above rules gives the isomorphism of crystals, i.e., it commutes with the actions of \( \tilde{f}_i \) and \( \tilde{e}_i \).

Proof. (Anti-symmetric case.) In the following proof (also for the symmetric case) we prove the commutativity with \( \tilde{f}_i \). The case for \( \tilde{e}_i \) is similarly proved. There are four cases of the
configurations of the $i$-th boxes,
\[
\begin{align*}
&\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
& \bigotimes \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
\bigotimes & \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
\bigotimes & \bigotimes \\
\hline
\end{array}.
\end{align*}
\]

The first case is obvious ($\tilde{f}_i=0$). For the 2nd case, it is easy to see that the $i$-th and $(i+1)$-th configuration of $b \otimes b'$ is invariant under $\iota$ regardless of the $(i+1)$-th configuration in $b \otimes b'$. Hence $\iota$ commutes with $\tilde{f}_i$. For the 3rd case, possible nontrivial patterns (i.e., those in which the $i$-th and $(i+1)$-th configurations change under $\iota$) are
\[
\begin{align*}
&\begin{array}{|c|c|}
\hline
\bigotimes & \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
& \bigotimes \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
\bigotimes & \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
\bigotimes & \bigotimes \\
\hline
\end{array},
\end{align*}
\]
and
\[
\begin{align*}
&\begin{array}{|c|c|}
\hline
\bigotimes & \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
& \bigotimes \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
\bigotimes & \bigotimes \\
\hline
\end{array},
\end{align*}
\]
One can easily check the commutativity for all cases. For the last case, there is only one nontrivial pattern :
\[
\begin{align*}
&\begin{array}{|c|c|}
\hline
\bigotimes & \\
\hline
\end{array}, & &
\begin{array}{|c|c|}
\hline
& \bigotimes \\
\hline
\end{array},
\end{align*}
\]
which also commutes with $\tilde{f}_i$.

(Symmetric case.) For $b \otimes b'$, The action of $\tilde{f}_i$ is determined by the number of dots in $i$-th boxes that are not connected to $(i+1)$-th dots. If we note that the rule for $\iota$ preserves these numbers, the commutativity with $\tilde{f}_i$ is easily proved. q.e.d.

**Proposition 3.18.** The value of the energy function $H$ given by Rules 3.10 and 3.11 has the correct properties of an energy function.

Proof. (Anti-symmetric case.) The invariance of the value of $H$ under the non-trivial (i.e. non-zero) action of $\tilde{e}_i$ and $\tilde{f}_i(i \neq 0)$ can easily be checked. Let us demonstrate the property of $H$ under the action of $\tilde{f}_0$. There are three patterns (say, $RR$, $LR$ and $LL$) for the nontrivial action of $\tilde{f}_0$ on $b \otimes b'$ and $\iota(b \otimes b')$, where $LR$ means $\tilde{f}_0$ acts on the left component of $b \otimes b'$ and the right component of $\iota(b \otimes b')$.

The property to be proved is
\[
H(\tilde{f}_0(b \otimes b')) - H(b \otimes b') = \begin{cases} 
1 & \text{ (for } RR), \\
0 & \text{ (for } LR), \\
-1 & \text{ (for } LL). 
\end{cases}
\]
This can be checked case by case as in Figure 3.10. In the figure, the boxes are the $n(\equiv 0)$-th and 1st ones.

(Symmetric case.) In this case the invariance under the action of $\tilde{e}_i$ and $\tilde{f}_i(i \neq 0)$ can also be easily checked. Here, there are also three patterns of the $\tilde{f}_0$-action. The property to be proved is the same as that above. This can also be checked case by case, as in Figure 3.11. q.e.d.

Sometimes it is useful to represent the isomorphism $\iota : b_1 \otimes b_2 \rightarrow b'_2 \otimes b'_1$ and the value of the energy function $H(b_1, b_2) = h$ by the following diagram

```
\begin{array}{ccc}
b'_1 & | & b'_2 \\
| & -h- & b_2 \\
\end{array}
```

For a sequence of crystals $B_1, \cdots, B_i$ the isomorphism

$$
\iota : (B_1 \otimes B_2 \otimes \cdots \otimes B_{i-1}) \otimes B_i \rightarrow B_i \otimes (B_1 \otimes B_2 \otimes \cdots \otimes B_{i-1})
$$

is defined by the composition of adjacent isomorphisms as

```
\begin{array}{ccc}
b'_1 & | & b'_2 & | & b'_i-1 \\
| & -h_1- & b^{(1)}_i & -h_2- & \cdots & -h_{i-1}- & b_i = b^{(i)}_i \\
\end{array}
```

This $L$-operator like construction plays an essential role in our representation of the index in terms of the energy function. That is, if the sequence $b_1 \otimes b_2 \otimes \cdots \in B_1 \otimes B_2 \otimes \cdots$ is the highest weight element corresponding to a given semi-standard tableau $T$, the index of $i$-th component $\text{ind}(i)$ is given by

$$
\text{ind}(i) = \pm (h_1 + h_2 + \cdots + h_{i-1}),
$$

where $+$ [resp. $-$] is for the symmetric [resp. anti-symmetric] case.

**Example 3.19.**

The following diagram indicates the way of calculating the $\text{ind}(4)$ for the highest weight element in Example 2.11.

```
(134) | (123) | (45)
(12) -1- (14) -0- (13) -2- (45)
(123) | (124) | (13)
```

19
where (13) represents
4. Index in Terms of the Energy Function

Let \( T \) be a semistandard tableau in \( T(\lambda, \mu) \) of shape \( \lambda \) and weight \( \mu = (\mu_1, \ldots, \mu_l) \). Let \( b_1 \otimes \cdots \otimes b_l \) in \( B_{\Lambda_{\mu_1}} \otimes \cdots \otimes B_{\Lambda_{\mu_l}} \) and \( \tilde{b}_1 \otimes \cdots \otimes \tilde{b}_l \) in \( B_{\mu_1 \Lambda_1} \otimes \cdots \otimes B_{\mu_l \Lambda_l} \) be the corresponding highest weight elements. We define \( b^{(i)}_j \) and \( \tilde{b}^{(i)}_j \) by

\[
B_{\Lambda_{\mu_1}} \otimes \cdots \otimes B_{\Lambda_{\mu_i}} \cong B_{\Lambda_{\mu_1}} \otimes \cdots \otimes B_{\Lambda_{\mu_i+1}} \cong \cdots \cong B_{\Lambda_{\mu_1}} \otimes \cdots \otimes B_{\Lambda_{\mu_{i-1}+1}}
\]

\[
b_1 \otimes \cdots \otimes b_i \mapsto b_1 \otimes \cdots \otimes b_i^{(i-1)} \otimes b_{i+1}' \mapsto \cdots \mapsto b_1^{(i)} \otimes b_1' \otimes \cdots \otimes b_i'.
\]

We set \( b^{(i)}_i = b_i \). Then our main theorem is

**Theorem 4.1.** For \( 1 \leq i \leq l \) the following relation holds:

\[
\text{ind}(i) = - \sum_{j=1}^{i-1} H_{\Lambda_{\mu_j} \Lambda_{\mu_i}} (b_j, \tilde{b}^{(j+1)}_j),
\]

\[
= \sum_{j=1}^{i-1} H_{\mu_j \Lambda_1 \mu_i \Lambda_1} (\tilde{b}_j, \tilde{b}^{(j+1)}_j).
\]

The proof of this theorem is given in the next section. As a consequence of this theorem and Theorem 2.16 of Lascoux-Schutzenberger (and [KR,B] for the skew case) we have

**Corollary 4.2.** There exist the following two expressions of \( K_{\lambda, \mu}(q) \):

\[
K_{\lambda, \mu}(q) = \sum_{b_1 \otimes \cdots \otimes b_l \in T(\lambda, \mu)} q^{-\sum_{i=1}^{l} \sum_{j=1}^{i-1} H_{\Lambda_{\mu_j} \Lambda_{\mu_i}} (b_j, b_i^{(j+1)})},
\]

\[
= \sum_{\tilde{b}_1 \otimes \cdots \otimes \tilde{b}_l \in T(\lambda, \mu)} q^{\sum_{i=1}^{l} \sum_{j=1}^{i-1} H_{\mu_j \Lambda_1 \mu_i \Lambda_1} (\tilde{b}_j, \tilde{b}^{(j+1)}_j)}.
\]

More generally, for the skew Kostka polynomial \( K_{\lambda/\nu, \mu}(q) (\nu \subset \lambda) \), we have

\[
K_{\lambda/\nu, \mu}(q) = \sum_{b_1 \otimes \cdots \otimes b_l \in T(\lambda/\nu, \mu)} q^{-\sum_{i=1}^{l} \sum_{j=1}^{i-1} H_{\Lambda_{\mu_j} \Lambda_{\mu_i}} (b_j, b_i^{(j+1)})},
\]

\[
= \sum_{\tilde{b}_1 \otimes \cdots \otimes \tilde{b}_l \in T(\lambda/\nu, \mu)} q^{\sum_{i=1}^{l} \sum_{j=1}^{i-1} H_{\mu_j \Lambda_1 \mu_i \Lambda_1} (\tilde{b}_j, \tilde{b}^{(j+1)}_j)}.
\]
As an immediate corollary of Corollary 4.2 we have

**Corollary 4.3.** For a dominant integral weight $\lambda$ of $\mathfrak{sl}_n$ let us set

$$V(k\Lambda_i : \lambda) = \{v \in V(k\Lambda_i) \mid e_j v = 0 (1 \leq j \leq n - 1), \text{ wt} v = \lambda\}.$$ 

We assume $|\lambda| \equiv ki \mod n$. Let $\lambda^{(N)}(\lambda_1 + kN - \frac{1}{n}(|\lambda| - ki), \cdots, \lambda_n + kN - \frac{1}{n}(|\lambda| - ki))$ and $\mu^{(N)}(k^{nN+i})$, where $\lambda = (\lambda_1, \cdots, \lambda_n)$ ($\lambda_1 \geq \cdots \geq \lambda_n \geq 0$). Then we have

$$\text{tr}_{V(k\Lambda_i : \lambda)}(q^{-d}) = \lim_{N \to \infty} q^{-A_N} K_{\lambda^{(N)} \mu^{(N)}}(q),$$

where

$$A_N = \frac{1}{2}knN(N-1) + kNi.$$

The case $i = 0$ (Theorem 1.1) was conjectured by A.N. Kirillov (Conjecture 4 in [Kir]). For the level one modules we have another expression of the branching function as a limit of Kostka polynomials.

**Corollary 4.4.** Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be a partition such that $\lambda_1 \leq n$ and $|\lambda| \equiv 0 \mod n$. Set $\lambda^{(N)}(n, \cdots, n, \lambda_1, \cdots, \lambda_l), \mu^{(N)}(k^{nN})$, where the $n$ in front of $\lambda_1$ appears $kN - |\lambda|$ times. Then for any $1 \leq k \leq n - 1$ we have

$$\text{tr}_{V(\Lambda_0; \lambda)}(q^{-d}) = \lim_{N \to \infty} q^{-A_N^k} K_{\lambda^{(N)} \mu^{(N)}}(q^{-1}),$$

where $\lambda'$ is the transpose of $\lambda$, and

$$A_N^k = \sum_{j=1}^{nN-1} j H_{\lambda_1 \lambda_k} (p_{j+1}^{gr(k)}, p_j^{gr(k)}).$$

The path $p^{gr(k)} = (p_j^{gr(k)})$ is the ground state path which we explicitly describe below.

In the path realization of $B(\Lambda_0)$ by $B_{\Lambda_k}$, the ground state path $p^{gr(k)} = (p_j^{gr(k)}), p_j^{gr(k)} \in B_{\Lambda_k}$ is given explicitly by

$$p_j^{gr(k)} = \begin{cases} (n-j)k + 1 \\ \vdots \\ (n-j+1)k \end{cases}.$$
To understand this correctly we set the following rules.

(1) The numbers in boxes are considered modulo $n$.
(2) We impose periodic boundary conditions.

By these rules one can consider $p_{j}^{gr(k)}$ as a uniquely determined semi-standard tableau, an element of $B_{\Lambda_{k}}$.

\section{Proof of Corollary 4.3}

Let us consider the case $\mu_{j} = k$ in the second equation for $K_{\lambda,\mu}(q)$ in Corollary 4.2. In this case $\tilde{b}_{i}^{(j)} = \tilde{b}_{j}$ for any $i$ and $j$. Hence we have

$$K_{\lambda,\mu}(q) = \sum_{\tilde{b}_{1}^{(j)} \cdots \tilde{b}_{1} \in T(\lambda,\mu)} q^{\sum_{j=1}^{l} j H_{k\Lambda_{k}k\Lambda_{1}}(\tilde{b}_{j+1}, \tilde{b}_{j})}.$$  \hfill (4.1)

Recall that $\tilde{b}_{1} \otimes \cdots \otimes \tilde{b}_{1} \in T(\lambda,\mu)$ if and only if $\tilde{b}_{1} \otimes \cdots \otimes \tilde{b}_{1}$ is a highest weight element with weight $\lambda$ as an $sl_{n}$ crystal. Therefore

$$K^{(n)}_{\lambda} \mu(n)(q) = \sum_{b_{nN+i} \cdots \tilde{b}_{1} \in H(N)} q^{\sum_{j=1}^{nN+i-1} j H_{k\Lambda_{k}k\Lambda_{1}}(b_{j+1}, b_{j})},$$  \hfill (4.2)

$$H(N) = \{ b \in B \otimes (nN+i) | \tilde{e}_{j}(b) = 0 (1 \leq j \leq n - 1), wtb = \lambda \}.$$

We note that $B(k\Lambda_{i}) \simeq B(k\Lambda_{0}) \otimes B_{k\Lambda_{1}}^{\otimes(nN+i)}$ by the path realization. Using this isomorphism and the definition of a path, if we set

$$H(\infty) = \{ b \in B(k\Lambda_{i}) | \tilde{e}_{j}(b) = 0 (1 \leq j \leq n - 1), wtb = \lambda \},$$

we have

$$H(\infty) = \bigcup_{N=1}^{\infty} H(N),$$

$$H(N) \subset H(N+1).$$

The embedding of $H(N)$ into $H(\infty)$ is given by sending $b \in H(N)$ to $b_{k\Lambda_{0}} \otimes b \in H(\infty)$. Now by the results of [JMMO,KMN1],

$$\text{tr}_{V(k\Lambda_{i};\lambda)}(q^{-d}) = \sum_{b \in H(\infty)} q^{\omega(b)} = \lim_{N \to \infty} \sum_{b \in H(N)} q^{\omega_{N}(b)},$$

where

$$\omega(b) = \sum_{j=1}^{N} j \left( H_{k\Lambda_{1}k\Lambda_{1}}(b_{j+1}, b_{j}) - H_{k\Lambda_{1}k\Lambda_{1}}(p_{j}^{gr}, p_{j}^{gr}) \right),$$

$$\omega_{N}(b) = \sum_{j=1}^{nN+i} j \left( H_{k\Lambda_{1}k\Lambda_{1}}(b_{j+1}, b_{j}) - H_{k\Lambda_{1}k\Lambda_{1}}(p_{j}^{gr}, p_{j}^{gr}) \right),$$  \hfill (4.3)
\( b = (b_j)_{j=1}^{\infty} \) is the path realization and \( p^{gr} = (p^{gr}_j)_{j=1}^{\infty} \) is the ground state path given by
\[
p^{gr}_j = (i + 1 - j)^k,
\]
\[
j^k = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
\hline & & & & & & & & & & & \\
\hline
& & & & & & & & & & & \\
\hline
\end{array}
\]

We consider \( i + 1 - j \) modulo \( n \). Note that if \( b_{nN+i} \otimes \cdots \otimes b_1 \in \mathcal{H}(N) \), then \( b_{nN+i} = 1^k = p^{gr}_{nN+i} \) by the highest weight condition. Note also that if \( b \in \mathcal{H}(N) \) corresponds to \( (b_j)_{j=1}^{\infty} \in \mathcal{H}(\infty) \), then \( b_j = p^{gr}_j \) for all \( j \geq nN + i + 1 \) by definition. Thus if we set
\[
A_N = \sum_{j=1}^{nN+i-1} j \mathcal{H}_{k\Lambda_1k\Lambda_1}(p^{gr}_{j+1}, p^{gr}_j),
\]
we have from (4.2) and (4.3)
\[
q^{-A_N} K_{\lambda(N),\mu(N)}(q) = \sum_{b \in \mathcal{H}(N)} q^{\omega_N(b)}.
\]

We can evaluate \( A_N \) explicitly using
\[
\mathcal{H}_{k\Lambda_1k\Lambda_1}((j-1)^k, j^k) = k \delta_{j1}
\]
as
\[
A_N = \frac{1}{2} kn(N-1) + kNi.
\]
q.e.d.

4.6. Proof of Corollary 4.4

The proof of Corollary 4.4 is similar to that of Corollary 4.3. In this case we use the first expression of \( K_{\lambda,\mu}(q) \) in Corollary 4.2. Then we have
\[
K_{\lambda(N),\mu(N)}(q) = \sum_{b_{nN} \otimes \cdots \otimes b_1 \in \mathcal{H}(N)} q^{\sum_{j=1}^{nN-1} j \mathcal{H}_{k\Lambda k\Lambda}(b_{j+1}, b_j)},
\]
where
\[
\mathcal{H}(N) = \{ b \in B_{k\Lambda_k}^{\otimes nN} | \tilde{e}_j(b) = 0 (1 \leq j \leq n-1), \text{wt}b = \lambda' \}.
\]

This time we use the path realization of \( B(\Lambda_0) \) by \( B_{\Lambda_k} \). Then
\[
B(\Lambda_0) = \{ b = (b_j)_{j=1}^{\infty} | b_j \in B_{\Lambda_k}, b_j = p^{gr(k)}_j (j >> 0) \}.
\]

Define
\[
\mathcal{H}(\infty) = \{ b \in B(k\Lambda_k) | \tilde{e}_j(b) = 0 (1 \leq j \leq n-1), \text{wt}b = \lambda' \}.
\]

Again we have the injection \( \mathcal{H}(N) \rightarrow \mathcal{H}(\infty) \) sending \( b \) to \( b_{\Lambda_0} \otimes b \). Using the result [KMN1]
\[
\text{tr}_{V(\Lambda_0,\Lambda)}(q^{-d}) = \sum_{b \in \mathcal{H}(\infty)} q^{\omega(b)},
\]
we have Corollary 4.4. Here \( \omega(b) \) is similarly defined by replacing \( H_{k\Lambda_1k\Lambda_1}(\cdot, \cdot) \) by \( H_{\Lambda_k\Lambda_k}(\cdot, \cdot) \) in (4.3). q.e.d.
5. Proof of Theorem 4.1

In this section we give a proof of Theorem 4.1. We first reduce the statement of the theorem to a more tractable one.

5.1. Local index

Let us introduce the local version of the index.

Definition 5.2. Let $k_1, k_2, k_3$ satisfy $k_1 \geq k_2 \geq k_3$. Take any element $a_1 \otimes a_2 \otimes b$ in $B_{\Lambda k_1} \otimes B_{\Lambda k_2} \otimes B_{\Lambda k_3}$. Let

$$a_1 = \begin{array}{c}
         i_1 \\
         \vdots \\
         i_{k_1}
\end{array}, \quad a_2 = \begin{array}{c}
         i'_1 \\
         \vdots \\
         i'_{k_2}
\end{array}$$

and specify the order on $i_1, \cdots, i_{k_1}$, that is, fix a bijection $J$ $J : \{1, \cdots, k_1 \} \rightarrow \{i_1, \cdots, i_{k_1} \}$.

We simply call $J$ the order of $a_1$. Then we define the local index $\text{ind}_{(a_1,J)}(a_2)$ of $a_2$ with respect to $J$ as follows:

1. Let us write $a_1 \otimes a_2$ as a two column diagram with dotted boxes as in the previous sections. Namely, the first column is $a_1$ and the second column is $a_2$ (see example below). Begin from the $J(1)$-th dot in $a_1$. Look for the dot in $a_2$ which is no lower than this dot. If there is no such dot in $a_2$, look for the first dot in $a_2$ from the bottom. Connect these two dots by a line.

2. Ignore the dots already connected by a line and continue the process 1 for $J(2), \cdots, J(k_2)$-th dots in $a_1$.

3. Define $\text{ind}_{(a_1,J)}(a_2)$ as the number of lines which connect dots in $a_1$ with lower dots in $a_2$.

Next the index $\text{ind}_{(a_1,J)}(b)$ is defined as follows. By assigning the number $j$ to the dot in $a_2$ which is joined by a line with the $J(j)$-th dot in $a_1$, we can define the order of $a_2$. Let us represent this order by $\bar{J}$, which gives the bijection

$$\bar{J} : \{1, \cdots, k_2 \} \rightarrow \{i'_{1}, \cdots, i'_{k_2} \}.$$
Then we define

\[ \text{ind}_{(a_1,J)}(b) = \text{ind}_{(a_1,J)}(a_2) + \text{ind}_{(a_2,J)}(b). \]

Let us give an example.

**Example 5.3.** For \( sl_4 \), consider the element \( a_1 \otimes a_2 \otimes b \) in \( B_{\Lambda_3} \otimes B_{\Lambda_2} \otimes B_{\Lambda_1} \) given by

\[
\begin{align*}
  a_1 &= \begin{array}{c}
  1 \\
  2 \\
  3
  \end{array},
  a_2 &= \begin{array}{c}
  1 \\
  3
  \end{array},
  a_2 &= \begin{array}{c}
  2
  \end{array}.
\end{align*}
\]

Let us consider the order \( J \) given by

\[ J(1) = 2, \quad J(2) = 3, \quad J(3) = 1. \]

Then the diagram in this case is given by **Figure 5.1**. From this we have

\[ \text{ind}_{(a_1,J)}(a_2) = 0, \quad \text{ind}_{(a_2,J)}(b) = \text{ind}_{(a_1,J)}(b) = 1. \]

Let us introduce terminology associated with the above definition of the local index.

**Definition 5.4.** We call the line which is drawn in the definition of the local index the LS-line. A LS-line is called a ”down line” if it connects two dots such that the position of the dot in the left column is higher than that of the dot in the right column. Other LS-lines are called ”up lines”. We draw a down line in such a way that it first goes right and then goes up to arrive at the top horizontal line. Then, rounding outside the diagram to arrive at the bottom horizontal line, it finally goes up again to arrive at the ending point (see **Figure 5.1**). Thus a down line is ”winding”. The up line should be similarly drawn.

5.5. Reduction of the statement

Let \( b_1 \otimes \cdots \otimes b_l \in B_{\Lambda_{\mu_1}} \otimes \cdots \otimes B_{\Lambda_{\mu_l}} \) be an element from \( T(\lambda, \mu) \). Note that

\[
\begin{array}{c}
  1 \\
  \vdots \\
  \mu_1
\end{array}
\]

\[ b_1 = \begin{array}{c}
  1 \\
  \vdots \\
  \mu_1
\end{array} \]
since $b_1 \otimes \cdots \otimes b_l$ is a highest weight element (for $\mathfrak{sl}_n$).

We define the order $J_1$ of $b_1$ from the bottom up, that is

$$J_1(i) = \mu_1 + 1 - i.$$ 

The order $J_1$ of $b_1$ induces the order $J_2, \cdots, J_l$ of $b_2, \cdots, b_l$, respectively, as in the definition of the local index. By definition of the local index and the index of Lascoux-Schutzenberger, we have

$$\text{ind}(b_1) := \text{ind}(1) = 0$$

$$\text{ind}(b_i) := \text{ind}(i) = \text{ind}(b_{i-1}, J_{i-1})(b_2) + \cdots + \text{ind}(b_{i-1}, J_{i-1})(b_i).$$

In order to prove Theorem 4.1, it is sufficient to prove the equation

$$\text{ind}(b_i^{(j+1)}) - \text{ind}(b_i^{(j)}) = -H_{\Lambda_{\mu_j} \Lambda_{\mu_i}}(b_j, b_i^{(j+1)}). \quad (5.1)$$

In fact, summing up (5.1) in $j$ we have

$$\text{ind}(b_i) - \text{ind}(b_i^{(2)}) = -\sum_{j=2}^{i-1} H_{\Lambda_{\mu_j} \Lambda_{\mu_i}}(b_j, b_i^{(j+1)}). \quad (5.2)$$

Using this and the following lemma we have Theorem 4.1.

**Lemma 5.6.** Let us assume $k \geq k'$. If $a$ is the highest weight element of $B_{\Lambda_k}$, we have, for any $b \in B_{\Lambda_{k'}}$,

$$\text{ind}(a, J_a)(b) = -H_{\Lambda_k \Lambda_{k'}}(a, b),$$

where $J_a$ is the order of $a$ such that $J_a(i) = k + 1 - i$.

Proof. Since

$$a = \begin{array}{c}
1 \\
\vdots \\
k
\end{array}$$

the lemma follows immediately from the rule of the energy function. q.e.d.

Now to prove (5.1) it is sufficient to prove

**Proposition 5.7.** For any order $J$ of $b_{j-1}$ we have

$$\text{ind}(b_{j-1}, J)(b_i^{(j+1)}) - \text{ind}(b_{j-1}, J)(b_i^{(j)}) = -H_{\Lambda_{\mu_j} \Lambda_{\mu_i}}(b_j, b_i^{(j+1)}).$$
Finally, in order to prove this proposition, it is sufficient to prove

**Proposition 5.8.** Let us assume \( k'' \geq k' \geq k \). Taking any element \( a_1 \otimes a_2 \otimes b \) in \( B_{\Lambda_{k''}} \otimes B_{\Lambda_{k'}} \otimes B_{\Lambda_{k}} \), we define \( b' \) and \( a_2' \) by the isomorphism

\[
B_{\Lambda_{k''}} \otimes B_{\Lambda_{k'}} \otimes B_{\Lambda_{k}} \xrightarrow{\sim} a_1 \otimes a_2 \otimes b \rightarrow a_1 \otimes b' \otimes a_2'.
\]

Then for any order \( J \) of \( a_1 \) we have

\[
\text{ind}_{(a_1,J)}(b) - \text{ind}_{(a_1,J)}(b') = -H_{\Lambda_{k'},\Lambda_{k}}(a_2,b).
\]

### 5.9. Proof of Proposition 5.8

**Lemma 5.10.** For any order \( J \) of \( a_1 \) we have

\[
\text{ind}_{(a_1,J)}(b) - \text{ind}_{(a_1,J)}(a_2) \geq -H_{\Lambda_{k'},\Lambda_{k}}(a_2,b).
\]

Proof. Note that the left hand side of the above inequality is nothing but \( \text{ind}_{(a_2,J)}(b) \), by Definition 5.2. Thus we consider \( a_2 \otimes b \) below. Let us set

\[
k_2 = -H_{\Lambda_{k'},\Lambda_{k}}(a_2,b).
\]

Then

\[
k_2 = \sharp(\text{winding H-lines}).
\]

By the rule of drawing the LS-line and H-line, we easily have (see **Figure 5.2**)

\[
\sharp(\text{up line}) \leq \sharp(\text{non-winding H-lines}) = k - k_2,
\]

which is equivalent to

\[
\sharp(\text{down line}) = k - \sharp(\text{up line}) \geq k_2.
\]

q.e.d.

By Lemma 5.10 we can write

\[
\text{ind}_{(a_1,J)}(b) - \text{ind}_{(a_1,J)}(a_2) = k_2 + l
\]

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for some non-negative integer \( l \). Then the proof of Proposition 5.8 reduces to proving the equation

\[
\text{ind}_{(a_1, J)}(b') - \text{ind}_{(a_1, J)}(a_2) = l. \tag{5.3}
\]

The proof of (5.3) is divided into several steps.

Step 1.
Let us consider the following situation (Figure 5.3).

1. In Figure 5.3, the line starting from \( A \) is a down line and below it there are no \( \bullet \) from which a down line starts.
2. The line entering \( \bullet B \) is a down line and above it there are no \( \bullet \) into which a down line enters.

**Lemma 5.11.** \( \bullet B \) is below \( A \).

**Proof.**

i. If the LS-line starting from \( A \) and that ending in \( \bullet B \) are the same, the lemma is obvious by the drawing rule of LS-lines.

ii. Suppose that the LS-line starting from \( A \) and the LS-line ending in \( \bullet B \) are distinct. By the definition of \( A \), the starting point of the LS-line ending in \( \bullet B \) is above \( A \). Now suppose that \( \bullet B \) is above \( A \). Then the order of the line connected to \( \bullet B \) is smaller than that of \( A \). In fact, if this were not the case, the LS-line starting from \( A \) would stop at \( \bullet B \) or below it (see Figure 5.4). This contradicts the assumption that the line from \( A \) is down. Then there is no place where the line starting from \( A \) ends, since this line must end between the bottom and \( \bullet B \) by the definition of \( \bullet B \). q.e.d.

**Lemma 5.12.** Between \( A \) and \( \bullet B \) there exists \( C \) in \( a_2 \) whose bottom line does not intersect any LS-line (see Figure 5.5). \( C \) may coincide with \( A \), but must be distinct from \( \bullet B \).

**Proof.** Suppose that such a \( C \) does not exist. In this case the situation is like that in Figure 5.6. We proceed with the proof following this picture. The order of the LS-line \( \alpha_1 - \alpha_5 \) is as follows:

- The order of \( \alpha_3 \) is smaller than that of \( \alpha_1 \).
- The order of \( \alpha_4 \) is smaller than that of \( \alpha_3 \).
- The order of \( \alpha_5 \) is smaller than that of \( \alpha_4 \).
- The order of $\alpha_2$ is smaller than that of $\alpha_5$.

Hence the order of $\alpha_2$ is smaller than that of $\alpha_1$. This contradicts the fact that the end point of the line $\alpha_1$ is lower than the position of $B$. Note that the general case is to consider the $m$ LS-lines $\alpha_1 - \alpha_m$ and is proved analogously to the case above. q.e.d.

Now using Lemma 5.12 we modify our two column diagram in the following manner. Take any $C$ as in Lemma 5.12. We cut the diagram by the bottom horizontal line of $C$ and put the lower half of this diagram over the upper half (see Figure 5.7). We can naturally transplant the LS-lines to this new diagram (see Figure 5.8). We also call those new lines LS-lines. Then the LS-lines for this new diagram have the following properties:
- The new LS-lines go up only.
- Down lines in the old diagram correspond bijectively to new lines which intersect the top horizontal line of the old diagram.

In the following we consider only the new diagram, unless otherwise stated, and call the top horizontal line of the old diagram the "bold line" in the new diagram.

Step 2.
To each $\bullet$ in the right column and part of $\bullet$ in the left column we attach the symbol $u$ or $d$ by the following rule:

1. Begin with the lowest $\bullet$ in the right column. In the left column look for the nearest $\bullet$ which is no higher than this $\bullet$, and connect these two $\bullet$ with a line. If this line crosses the bold line then we attach the symbol $d$ to the original $\bullet$ in the right column as : $\bullet d$. Otherwise we attach the symbol $u$ as : $\bullet u$.
2. Ignore the pair already joined by a line and continue the process 1 for the 2nd,3rd,... $\bullet$ from the bottom successively.
3. For $\bullet$ in the left column which are connected by a line to some $\bullet$ in the right column, we attach $u$ or $d$ to match the corresponding symbol of $\bullet$ in the right column.
4. Finally we attach $v$ to any $\bullet$ which has neither a $u$ nor a $d$ as : $v \bullet$.

In processes 1 and 2 above, the lines never "wind". In fact let us consider process 1. There exists an up LS-line whose end point is the lowest $\bullet$ in the right column. Let us name its starting point $A$. Then the chosen $\bullet$ in the left column in process 1 is not below $A$. Hence the line drawn in process 1 cannot wind. The same reasoning is applied to each case of process 2.

**Definition 5.13.** We refer to the line drawn above the $H$-line for the new diagram.
After finishing this process, the symbols $u$ and $d$ have the following properties (see Figure 5.9):

1. All $d$ in the right column are over the bold line.
2. All $d$ in the left column are under the bold line.
3. There is no $v$ which is under the bold line and is over the lowest $d$.
4. Let $u = A$ be the $j$-th $u$ counting from the bottom (we count only $u$’s). If $A$ sits lower than the lowest $d$, then the number of LS-lines starting from $u$ in the left column whose positions are no higher than $A$ is greater than or equal to $j$.

Properties 1-3 are obvious. Let us prove property 4. Let us name $u$ from bottom to up as $A_1, A_2, \cdots$. Let $A_N$ be the highest $A_i$ which is lower than the lowest $d$. Let us also name $u$ from the bottom up as $B_1, B_2, \cdots$. Let us first prove that property 4 holds for $j = N$. Note that the position of $B_N$ is not lower than that of $A_N$ and is lower than the lowest $d$. In fact if $B_N$ is lower than $A_N$, then there are already $N$ $u$’s below $A_N$. This is impossible. By the drawing order of $H$ lines $B_N$, should be below the lowest $d$. Moreover, there are no $v$ which are above $A_N$ and not above $B_N$ by the drawing rule of $H$ lines. Therefore no starting point of $H$ lines which end at $B_1, B_2, \cdots$ can be above $A_N$. Thus property 4 holds for $j = N$.

Now suppose that property 4 holds for all $j' (j \leq j' \leq N)$ and does not hold for $j - 1$. Then there should be a $v$ between $A_{j-1}$ and $A_j$ from which an $H$ line starts. Let $C$ be the lowest $v$ which sits between $A_{j-1}$ and $A_j$. Then $B_{j-1}$ is no lower than $A_{j-1}$ but is lower than $C$. In fact the position of $B_{j-1}$ cannot be below that of $A_{j-1}$ by the same reasoning as in the case of $A_N$ and $B_N$. By the drawing rule of $H$ lines (or $u$), $B_{j-1}$ must be below $C$. As a consequence all starting points of $H$ lines ending at $B_1, \cdots, B_{j-1}$ are under $A_{j-1}$. Thus property 4 holds for $j - 1$. This is a contradiction. q.e.d.

Note that an old $H$ line drawn by the above order is naturally transplanted as a new $H$ line. By this transplant the winding $H$ line is interpreted as the new $H$ line which crosses the ”bold line”.

Step 3.

Let us set

$$\text{The number of } u \text{ over the bold line in the right column } = N^\text{up}_R(u).$$

$$\text{The number of } u \text{ under the bold line in the right column } = N^\text{down}_R(u).$$
Then we have the following statements.

The number of $u \boxed{\quad}$ over the bold line in the left column = $N_{R}^{up}(u)$.
The number of $u \boxed{\quad}$ under the bold line in the left column = $N_{R}^{down}(u)$.
The number of $d \boxed{\quad}$ under the bold line in the left column = $k_2$.

Here $k_2$ is the number of winding $H$ lines in $a_2 \otimes b$ in our assumption. Hence we have

$$\#(u \boxed{\quad} \text{under the bold line}) + \#(d \boxed{\quad} \text{under the bold line}) = N_{R}^{down}(u) + k_2. \ (5.4)$$

On the other hand,

- The number of LS-lines whose starting points are under the bold line and end points are over the bold line is $k_2 + l$ by assumption.
- The number of LS-lines whose starting and end points are both under the bold line is $N_{R}^{down}(u)$.

Consequently,

$$(\text{the number of LS-lines whose starting points are under the bold line}) = k_2 + l + N_{R}^{down}(u). \ (5.5)$$

To this point we have considered LS-lines starting from the left column $a_2$. Now we consider the LS-lines coming into $a_2$ from $a_1$.

**Lemma 5.14.** If we remove all $v \boxed{\quad}$ from the left column, among the LS-lines going into the left column, $l$ LS-lines ”newly cross the bold line”. Here, for example, in Figure 5.10, the line $\alpha_3$ ”newly crosses the bold line” after removing $v \boxed{\quad}$.

Proof. We first note the following property on the movement of the end position of LS-lines while removing $v \boxed{\quad}$’s:

If we remove $v \boxed{\quad}$, then the end positions of LS-lines in the left column shift upward only under the periodic boundary condition.

(see Figure 5.11). More strongly, we can prove

**Lemma 5.15.** There does not occur any winding of lines (in the new diagram) after removing $v \boxed{\quad}$ compared with the situation before removing $v \boxed{\quad}$.

Proof. In this proof we consider $a_2 \otimes b$. So the starting point of an LS-line is again a point in $a_2$. Let us consider the highest $d \boxed{\quad} (\text{say } D)$ over the bold line and $L$ the (horizontal) overline.
of the box $D$ (Figure 5.12). Then any $\square$ in the left column which is above the bold line and under $L$ is assigned the symbol $u$ by definition of the symbol $\square d$. Moreover any $\square$ above $L$ in the right column is assigned the symbol $u$ by definition of $L$. Let $A_1, \cdots, A_N$ be the starting points of LS-lines over $L$ numbered from the bottom up and $l_1, \cdots, l_n$ the corresponding LS-lines. Let $B_1, \cdots, B_N$ be the end points of $\{l_i\}$ numbered from the bottom up (Figure 5.13).

In the following we use the notation $A \leq B$ for two boxes $A$ and $B$ meaning the position of $A$ is equal to or lower than that of $B$. The symbol $A < B$ is similarly defined.

Note that $A_j \leq B_j$. In fact if $A_j > B_j$, then there are $j$ LS-lines going from $\{A_i | A_i < B_j < A_j\}$ to $B_1, \cdots, B_j$. This is impossible. Note also the following. The $\square$ in the left column which is joined by an $H$ line to some $\square u$ over $L$ is always over $L$ by definition of the rule of drawing $H$ lines in the present order. So we can consider the part over $L$ separately. Temporarily, we consider only this part unless otherwise stated. Then, by the first part of the proof of Lemma 3.16, the set of $u \square$ (over $L$) is independent of the order of drawing $H$ lines (whose starting points are over $L$). Now let us prove the following claim

There exists a subset $\{C_1, \cdots, C_N\}$ of $\{u \square\}$ such that $C_1 < \cdots < C_N$ and $A_1 \leq C_1, \cdots, A_N \leq C_N$. (5.7)

In fact let us define the order $1, 2, \cdots, N$ to $B_N, \cdots, B_1$. The order $N + 1, N + 2, \cdots$ is irrelevant to the following argument and arbitrarily assigned. Let $\bar{B}_N, \cdots, \bar{B}_1$ be the partners of $B_N, \cdots, B_1$ by $H$ lines in the presently specified order. Then by definition

$$\bar{B}_1 < \cdots < \bar{B}_N.$$ 

Since $A_j \leq B_j$, as we already proved, $A_j \leq \bar{B}_j$. Hence we can choose $C_j = \bar{B}_j$. Thus (5.7) is proved.

Let us return to the proof of Lemma 5.15. From this point, we again consider the entire diagram. Under the bold line in the left column we have

$$\sharp(u \square \text{ and } d \square ) - \sharp(\text{starting points of LS-lines})$$

$$= (k_2 + N_R^{\text{down}}(u)) - (k_2 + l + N_R^{\text{down}}(u))$$

$$= -l.$$ 

Hence, over the bold line in the left column,

$$\sharp(u \square \text{ and } d \square ) - \sharp(\text{starting points of LS-lines}) = l.$$
Since, in the left column as a whole, the following equation must hold
\[ \sharp(u \circlearrowleft \text{ and } d \circlearrowright) = \sharp(\text{starting points of LS-lines}). \]

Let us prove the impossibility of winding by considering two cases.

Case 1. The case that the line (say \( L \)) which winds for the first time (in the order of drawing LS-lines) ”newly crosses the bold line”.

Suppose that this occurs. Then every \( u \circlearrowleft \) and \( d \circlearrowright \) over the bold line is occupied by some LS-lines before drawing \( L \). Hence \( l \) lines already have ”newly crossed the bold line” before drawing \( L \), and \( L \) is the \( l+1 \)-th line which ”newly crosses the bold line”. But this is impossible unless a line already has wound before drawing \( L \) by property 4 in Step 2.

Case 2. The case that the line (say \( L \)) which winds for the first time does not ”newly cross the bold line”.

In this case \( l + 1 \) lines should ”newly cross the bold line” before drawing \( L \). In fact let \( A \) be the starting point of \( L \) before removing \( v \circlearrowleft \). Then \( A \) should be \( v \circlearrowright \). If this is not the case \( L \) cannot wind. If there exists \( u \circlearrowleft \) or \( d \circlearrowright \) below \( A \) and above \( L \) which is not occupied before drawing \( L \), then \( L \) cannot wind by (5.7). Hence before drawing \( L \) every \( u \circlearrowleft \) and \( d \circlearrowright \) over the bold line must become the starting points of some LS-lines. Hence \( l + 1 \) lines should already ”newly cross the bold line” before drawing \( L \). But this is impossible by the same reasoning as that in Case 1. q.e.d.

We can now finish the proof of Lemma 5.14. In fact it follows from property 4 in Step 2, (5.6) and Lemma 5.15 that the number of LS-lines ”newly crossing the bold line” is
\[ k_2 + l + N_{\text{down}}(u) - (k_2 + N_{\text{down}}(u)) = l \]
after removing \( v \circlearrowleft \). q.e.d.

Step 4.
Let us prove (5.3). We consider the old diagram here (i.e., not that one with the lower and upper halves reversed). When we remove all \( v \circlearrowleft \) from \( a_2 \), the following four changes can occur for LS-lines connecting \( a_1 \) and \( a_2 \) (the following are \( \square \) in \( a_1 \) and the LS-lines coming from it):

(1) \( \square \uparrow \) up line \implies \( \square \uparrow \) up line.
(2) \( \square \uparrow \) up line \implies \( \square \downarrow \) down line.
(3) \( \square \searrow \) down line \( \Rightarrow \square \nearrow \) up line.
(4) \( \square \searrow \) down line \( \Rightarrow \square \swarrow \) down line.

In truth, case (3) cannot occur by the rule for \( H \)-lines.

**Lemma 5.16.** If and only if the situation is that of case (2), the LS-line starting from \( \square \) in \( a_1 \) will "newly cross the bold line" after removing all \( v \square \) from \( a_2 \).

Proof. This is obvious from (5.6) and the definition of the word "newly cross the bold line". q.e.d.

Lemma 5.16 states

\[
\text{ind}_{(a_1, J)}(b') = \text{ind}_{(a_1, J)}(a_2) + \#\{\text{LS-line of type (2)}\}.
\]

Combining this with Lemma 5.14, we have (5.3). q.e.d.
6. Summary and Discussion

In this paper we have given an expression for the charge, or more precisely the index of Lascoux-Schutzenberger, in terms of the energy function. Since the energy function is determined from the crystals over quantum affine algebra, our theorem provides a representation theoretical meaning of the charge. As a corollary of our description of charge, we proved Kirillov’s conjecture on the relation between Kostka polynomials and the branching functions of the vacuum representation of $\widehat{sl}_n$ w.r.t. $sl_n$.

While preparing the manuscript we came to know about the paper [LLT] by Lascoux et al. In this paper the charge is also described in terms of crystal graphs. The relation between their description and ours is still under investigation. Here we only remark that their $d_i(t)$ defining the statistics $d(t)$ looks similar to the energy function, in particular in the case $\mu = (k^{n+1})$ for $sl_{n+1}$. However, they are not identical since $d_i(t)$ is determined from the $i$ string through $t$, while the energy function can actually depend on other data. In any case it is an interesting problem to clarify the relation between the two descriptions of charge.

After this manuscript is completed, there appeared a paper by Dasmahapatra [D] that contains some results similar to ours.

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We wish to thank A.N. Kirillov for stimulating discussions and informing us of the paper [LLT].
Appendix - Fundamentals of Crystal Theory

We briefly give here precise definitions of notions and notation used in the main text. For more details we refer to [KMN1,K3]. The crystals defined below are called seminormal in [K3] and are the same as the weighted crystal in [KMN1]. We restrict ourselves to the case of $A_n^{(1)}$.

We denote by $\{\alpha_0, \ldots, \alpha_n\}$ the set of simple roots, by $\{h_0, \ldots, h_n\}$ the set of simple coroots so that $(h_i, \alpha_j)$ is the generalized Cartan matrix of the $A_n^{(1)}$ type, by $\{\Lambda_0, \ldots, \Lambda_n\}$ the set of fundamental weights, by $\delta = \alpha_0 + \cdots + \alpha_n$ the null root, by $P = \mathbb{Z}\Lambda_0 + \cdots + \mathbb{Z}\Lambda_n + \mathbb{Z}\delta$ the weight lattice, and by $P^* = \mathbb{Z}h_0 + \cdots + \mathbb{Z}h_n + \mathbb{Z}d$ the dual weight lattice. The pairing of $P$ and $P^*$ is given by

$$\langle h_i, \Lambda_j \rangle = \delta_{ij}, \quad \langle d, \delta \rangle = 1, \quad \text{others} = 0.$$  

**Definition A.1.** A crystal $B$ (for $\widehat{\mathfrak{sl}_{n+1}}$) is a set with a map

$$(wt : B \rightarrow P, \tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\}, \varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z}_{\geq 0}, i = 0, \ldots, n)$$

satisfying the following axioms.

1. If $b \in B$ and $\tilde{e}_ib \in B$, then $\text{wt}(\tilde{e}_ib) = \text{wt}b + \alpha_i$.
2. If $b \in B$ and $\tilde{f}_ib \in B$, then $\text{wt}(\tilde{f}_ib) = \text{wt}b - \alpha_i$.
3. For $b \in B$, $\varphi_i(b) = \max\{n \geq 0 | \tilde{f}_ib \neq 0\}$, $\varepsilon_i(b) = \max\{n \geq 0 | \tilde{e}_ib \neq 0\}$.
4. $\varphi_i(b) - \varepsilon_i(b) = \langle h_i, \text{wt}b \rangle$.
5. $\tilde{e}_i0 = \tilde{f}_i0 = 0$.
6. For $b_1, b_2 \in B$, $\tilde{f}_ib_1 = b_2$ is equivalent to $b_1 = \tilde{e}_ib_2$.

Let us set $P_{cl} = P/\mathbb{Z}\delta$. A crystal similarly defined for $P_{cl}$ rather than $P$ is called a classical crystal. In the main text we use the term crystal to refer to a classical crystal as long as there is no chance of confusion. A crystal similarly defined for $\tilde{P} = \mathbb{Z}\Lambda_1 + \cdots + \mathbb{Z}\Lambda_n$ and $\tilde{\varepsilon}_i, \tilde{\varphi}_i, \varepsilon_i (1 \leq i \leq n)$ is a crystal for $\widehat{\mathfrak{sl}_{n+1}}$. For a subset $J$ of $\{0, 1, \cdots, n\}$, a crystal $B$ (for $\widehat{\mathfrak{sl}_{n+1}}$) is called a $J$ crystal if the index of $\tilde{\varepsilon}_i, \tilde{\varphi}_i, \varepsilon_i$ is restricted to $J$. In particular, for $J = \{1, \cdots, n\}$, the $J$-crystal is sometimes called a $\mathfrak{sl}_{n+1}$ crystal.

If $(L, B)$ is a crystal base of a representation $V$ of $U_q(\mathfrak{sl}_{n+1})$ or $U_q(\mathfrak{sl}_n)$, $B$ is a crystal by the obvious definitions of $wt, \tilde{e}_i, \tilde{f}_i, \varphi_i, \varepsilon_i$.

**Definition A.2.** A morphism $\psi : B_1 \rightarrow B_2$ of two crystals is a map $\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ satisfying the following conditions:

1. $\psi(0) = 0$. 

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2. $\psi$ commutes with all $\tilde{e}_i$ and $\tilde{f}_i$.
3. For any $b \in B_1$ $\text{wt}\psi(b) = \text{wt}b$.

The morphism defined here is called ”strict” in [K3]. A morphism is called an isomorphism if the associated map $\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ is bijective.

For two crystals $B_1$ and $B_2$ we can define the tensor product $B_1 \otimes B_2$. Here we only give the description of the action of $\tilde{e}_i$ and $\tilde{f}_i$.

\[
\begin{align*}
\tilde{e}_i(b_1 \otimes b_2) &= \tilde{e}_i b_1 \otimes b_2 \quad \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
&= b_1 \otimes \tilde{e}_i b_2 \quad \varphi_i(b_1) < \varepsilon_i(b_2), \\
\tilde{f}_i(b_1 \otimes b_2) &= \tilde{f}_i b_1 \otimes b_2 \quad \varphi_i(b_1) > \varepsilon_i(b_2), \\
&= b_1 \otimes \tilde{f}_i b_2 \quad \varphi_i(b_1) \leq \varepsilon_i(b_2).
\end{align*}
\]

Remark A.3.

1. A finite dimensional representation of $\widehat{sl}_{n+1}$ does not necessarily have a crystal base. For example, the representation of $U_q(\widehat{sl}_3)$ obtained from the adjoint representation of $U_q(sl_3)$ by the evaluation homomorphism $U_q(\widehat{sl}_3) \rightarrow U_q(sl_3)$ does not have a crystal base. In particular, the crystal of the adjoint representation of $U_q(sl_3)$ cannot be extended to the normal crystal (of [K3]) for $\widehat{sl}_3$.

2. Let $B_{l\Lambda_k}$ be the crystal of the irreducible highest weight representation of $U_q(sl_n)$ with highest weight $l\Lambda_k$. It is proved in [KMN2] that $B_{l\Lambda_k}$ can be extended to a classical crystal $B$ for $\widehat{sl}_n$ by defining a suitable action of $\tilde{e}_0$ and $\tilde{f}_0$. This crystal $B$ has the symmetry of a Dynkin diagram automorphism. Namely, even if we permute, in a cyclic way, the colors of the crystal graph $B$, the resulting crystal is isomorphic to $B$. 

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Figure 3.6

Figure 3.7
Figure 3.8
Figure 3.9
|       | $b \otimes b'$ | $\tilde{f}_0(b \otimes b')$ | $\zeta(b \otimes b')$ |
|-------|----------------|-----------------------------|----------------------|
| RR    | \[
\begin{array}{c}
\tilde{f}_0 \\
\rightarrow -1
\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] |
| LL    | \[
\begin{array}{c}
\tilde{f}_0 \\
\rightarrow +1
\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] |
| LR    | \[
\begin{array}{c}
\tilde{f}_0 \\
\rightarrow 0
\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] |
| RR    | \[
\begin{array}{c}
\tilde{f}_0 \\
\rightarrow +1
\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] |
| RR    | \[
\begin{array}{c}
\tilde{f}_0 \\
\rightarrow -1
\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] | \[
\begin{array}{c}

\end{array}
\] |

Figure 3.10
Figure 3.11
Figure 5.5

Figure 5.6
Figure 5.7

Figure 5.8
Figure 5.9

Figure 5.10
Figure 5.11
**Figure 5.12**

**H-lines**

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| u | v |   |   |   |   |   |
| u |   |   |   |   |   |   |
| v |   |   |   |   |   |   |
|   |   |   |   |   |   |   |
| u |   |   |   |   |   |   |
| u |   |   |   |   |   |   |
| d |   |   |   |   |   |   |
| d |   |   |   |   |   |   |
| u |   |   |   |   |   |   |
| u |   |   |   |   |   |   |
|   |   |   |   |   |   |   |
| v |   |   |   |   |   |   |

**Figure 5.13**

**LS-lines**

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
| A₂ u | v |   |   |   |   |   |
| u |   |   |   |   |   |   |
| A₁ v |   |   |   |   |   |   |
|   |   |   |   |   |   |   |
| A₂ u | v |   |   |   |   |   |
| u | A₁ v |   |   |   |   |   |
|   |   |   |   |   |   |   |
| d |   |   |   |   |   |   |
| d |   |   |   |   |   |   |
| u |   |   |   |   |   |   |
| u |   |   |   |   |   |   |
|   |   |   |   |   |   |   |
| v |   |   |   |   |   |   |

Figure 5.13