EQUIVARIANT COMMUTATIVE STRINGY COHOMOLOGY
RINGS ON ALMOST COMPLEX MANIFOLDS

BOHUI CHEN, CHENG-YONG DU, AND TIYAO LI

ABSTRACT. In this paper, motivated by Chen–Ruan’s stringy orbifold theory
on almost complex orbifolds, we construct a new cohomology ring $H_{G,cs}^\ast(X)$
for an equivariant almost complex pair $(X, G)$, where $X$ is a compact connected
almost complex manifold, $G$ is a connected compact Lie group which acts on
$X$ and preserves the almost complex structure.

1. INTRODUCTION

Stringy cohomology theory on orbifolds was motivated from physics and it is
discovered in mathematics by Chen–Ruan as the Chen–Ruan cohomology ring
$H_{CR}^\ast(M)$ ([12]) in 2004, where $M$ is an almost complex orbifold. For symplectic
orbifolds, this cohomology ring is the classical limit of the orbifold quantum co-
homology ring (cf. [11]). The main ingredient in Chen–Ruan’s ring structure is
the obstruction bundle whose fiber is interpreted as the cokernel of certain elliptic
operator. The computation of the obstruction bundle was later discovered by S.
Hu and the first author [10] for abelian case, and by Hu–Wang for general cases in
[23]. Their computations are crucial for our construction in this paper.

The ordinary equivariant cohomology theory could not detect the information
of subspace with finite stabilizer. Motivated by Chen–Ruan’s theory, it is expected
that if there exists a Chen–Ruan type equivariant theory when $X$ is an almost
(abbreviated) complex manifold which admits a connected compact Lie group $G$ action.
We may call such a theory as an equivariant stringy cohomology theory for the
pair $(X, G)$. When $G$ is a finite group or $G = T$ is a torus, the problem can be
perfectly solved. This is due to the fact that $[g, g] = 0$, where $g$ is the Lie algebra
of $G$. However, when $G$ is a connected non-commutative compact Lie group, the
problem is still open. In this paper, our goal is to construct such a theory.

We list the works related to equivariant stringy cohomology theory for the pair
$(X, G)$.

(1) When $G$ is finite, this is essentially the Chen–Ruan’s theory. Fantechi–
Göttscbe ([18]) and Jarvis–Kaufmann–Kimura ([25]) studied the pair $(X, G),
where $G$ is a finite group. They show that the theory constructed is same
as the Chen–Ruan’s.

2010 Mathematics Subject Classification. Primary 53D45, 55N91; Secondary 14N35, 14L30.
Key words and phrases. Almost complex manifold, Lie group action, equivariant commutative
stringy cohomology, symplectic reduction.

The first author was supported by National Natural Science Foundation of China (No.
11431001 and No. 11726607). The second author was supported by National Natural Science
Foundation of China (No. 11501390).
When $G$ is an abelian group, the equivariant stringy cohomology theory can be constructed by Goldin–Holm–Knutson ([22]) and it is applied to study the Kirwan morphisms for stringy cohomologies. The idea appeared also in [10]. Later, Becerra–Uribe [6] extended the constructions in [10, 25] to (twisted) K-theory for both cases that $G$ is either abelian or finite.

When $G$ is a non-abelian connected compact Lie group, there is an attempt made by Edidin–Jarvis–Kimura [17]. They considered the case that $G$ is a reducible algebraic group, acting on an algebraic space $X$, and $[X/G]$ is an orbifold (hence the action has finite stabilizer). They used the formal bundle $TX - g$ to constructed the obstruction bundle hence is essentially the theory of quotient orbifold $[X/G]$. Since such construction requires that $g \hookrightarrow TX$ on $X$, $X$ is very restrictive, e.g., $X$ can not be compact. There are some other works may relate to this problem and may shed light on it.

Let $(X, \omega, G, \mu)$ be a symplectic manifold with a hamiltonian $G$-action, and $\mu$ be the moment map. Let $M = X \sslash G$ be the symplectic reduction. It is known that the moduli spaces of symplectic vortices may give a Gromov–Witten type theory, which people call the Hamiltonian Gromov–Witten theory (cf. [8, 27, 9, 28, 20]). Recently, Chen–Wang–Wang’s work [13, 14] says that $L^2$-Hamiltonian Gromov–Witten theory implies a Gromov–Witten type theory on the orbifold $M$, and they also introduce a new equation, so called the augmented symplectic vortex equation, and use it to quantize the equivariant stringy cohomology for the pair $(X, G)$ when $G$ is abelian (cf. [15]). However, we do not know if there are some way to overcome the nonabelian case on this direction. Nevertheless, the work of symplectic vortices inspires us to consider certain moduli space of degree zero. The construction in this paper is based on this. Let $(X, G)$ be a compact almost complex manifold pair, i.e., $X$ is a compact almost complex manifold and $G$ is a connected compact Lie group acting on $X$ and preserving the almost complex structure. We construct a cohomology ring $\mathcal{H}^*_G,cs(X)$ for the pair which we call the equivariant commutative stringy cohomology ring. We make some remarks on the technical issues in our construction.

This paper is organized as follow. In §2 we define the equivariant commutative stringy cohomology group. In §3 we construct the obstruction bundle and the equivariant commutative stringy cohomology ring. Finally in §4 we study the relation between equivariant commutative stringy cohomology ring and symplectic reduction. As an application, we consider an orbifold $M$ which is a symplectic reduction of a hamiltonian system $(X, \omega, G, \mu)$ with $G$ being connected and compact; the equivariant commutative stringy cohomology ring of the pair $(X, G)$ induces a new stringy cohomology ring of the Chen–Ruan cohomology group of $M$. The
appendix A provides a technical result of the existence of equivariant volume form over certain bundles associated to a connected compact Lie group.

Acknowledgement. The authors thank Rui Wang and Yu Wang for useful discussions.

2. Equivariant Commutative Stringy Cohomology Group on Almost Complex Manifolds

2.1. Inertia manifolds. Let $G$ be a connected compact Lie group acting on an almost complex manifold $(X,J)$ and preserving the almost complex structure $J$. For an element $g \in G$, denote by $[g]$ the conjugate class of $g$ in $G$. Denote the unit element by $1 \in G$. Let $G_t$ denote the subspace of all finite order elements in $G$.

Definition 2.1. For $m \in \mathbb{Z}_{\geq 1}$, we set

$$G_t^m := \left\{ \bar{g} = (g_1, \ldots, g_m) \in G_t \times \cdots \times G_t \mid g_1, \ldots, g_m \in G_t/T \text{ for a maximal torus } T \text{ of } G \text{ and the subgroup } \langle \bar{g} \rangle = \langle g_1, \ldots, g_m \rangle \text{ generated by } \bar{g} \text{ is finite} \right\}$$

and

$$[G_t^m] := \{ [\bar{g}] \mid \bar{g} \in G_t^m \}$$

to be the set of conjugate classes of $m$-tuples in $G_t^m$ under the diagonal conjugation action of $G$, i.e.

$$[\bar{g}] = [g_1, \ldots, g_m] := \{ h\bar{g}h^{-1} := (h_1, \ldots, h_m) \mid h \in G \}.$$

When $m = 1$, $G_t^1 = G_t$, so we omit the superscript.

We endow each conjugate class $[\bar{g}]$ the subspace topology of $G_t^m$. For a $g \in G$ denote by $C_G(g)$ the centralizer of $g$ in $G$, or simply by $C(g)$. We also set

$$C(\bar{g}) := \bigcap_{i=1}^{m} C(g_i)$$

for a $\bar{g} = (g_1, \ldots, g_m)$. Then we have a diffeomorphism $[\bar{g}] \cong G/C(\bar{g})$.

Denote the $G$-action on $X$ by $gx$ for a $g \in G$ and an $x \in X$.

Definition 2.2. For each $[\bar{g}] \in [G_t^m]$, we set

$$X_{[\bar{g}]} := \{ (x, \bar{g}) \mid \bar{g} \in [\bar{g}], \bar{g} \cdot x = (g_1x, \ldots, g_mx) = (x, \ldots, x) \}$$

which is viewed as a submanifold of $X \times G_t^m$. We call $X_{[\bar{g}]}$ an $m$-sector of $(X,G)$. We call $X_{[\bar{g}]}$ a twisted sector of $(X,G)$ when $m = 1$ and $[\bar{g}] \neq [1]$, and $X_{[1]} = X \times \{ 1 \}$ the non-twisted sector.

For each $m \in \mathbb{Z}_{\geq 1}$ we set

$$I_G^m(X) := \bigsqcup_{[\bar{g}] \in [G_t^m]} X_{[\bar{g}]}$$

to be the disjoint union of $X_{[\bar{g}]}$ over $[\bar{g}] \in [G_t^m]$. We call $I_G^m(X)$ the $m$-inertia manifold of $(X,G)$. When $m = 1$ we call $I_G^1(X)$ the inertia manifold and denote it simply by $I_G(X)$.

$G$ acts on $I_G^m(X)$ by $h \cdot (x, \bar{g}) = (hx, h\bar{g}h^{-1})$. It preserves every $X_{[\bar{g}]}$. 

For a $[\tilde{g}] \in [G^m]$, fix a representative $\tilde{g} = (g_1, \ldots, g_m)$, let

$$X^{\tilde{g}} := \bigcap_{i=1}^m X^{g_i}$$

be the set of fixed loci of the subgroup $\langle \tilde{g} \rangle$ of $G$ that are generated by $g_1, \ldots, g_m$, where $X^{g_i}$ is the fixed loci of $g_i$-action on $X$. There is a $C(\tilde{g})$-action on the product space $X^{\tilde{g}} \times G$ given by

$$g \cdot (x, k) = (gx, k\tilde{g}^{-1}).$$

Let $X^{\tilde{g}} \times_{C(\tilde{g})} G$ be the quotient space of this action. Then $G$ acts on $X^{\tilde{g}} \times_{C(\tilde{g})} G$ by multiplying from the left to the second factor $G$.

**Lemma 2.3.** There is a $G$-equivariant diffeomorphism

$$\phi : X_{[\tilde{g}]} \to X^{\tilde{g}} \times_{C(\tilde{g})} G, \quad (x, k\tilde{g}^{-1}) \mapsto [k^{-1} \cdot x, k]$$

with inverse map given by $[x, h] \mapsto (h \cdot x, h\tilde{g}h^{-1})$.

There are natural maps between $m$-sectors $I_G^m(X)$, the inertia manifold $I_G X$ and the ambient manifold $X$. We list them as follow:

**Definition 2.4.**

1. For $m \geq 1$, $e : I_G^m(X) \to X$, $(x, \tilde{g}) \mapsto x$.
2. For $m \geq 2$ and $1 \leq i \leq m$,
   $$e_i : I_G^m(X) \to I_G(X), \quad (x, (g_1, \ldots, g_m)) \mapsto (x, g_i),$$
   and
   $$e_\infty : I_G^m(X) \to I_G(X), \quad (x, \tilde{g}) \mapsto (x, \tilde{g}_\infty),$$
   where $\tilde{g}_\infty := g_1 \cdots g_m$, and
   $$e_0 : I_G^m(X) \to I_G(X), \quad (x, \tilde{g}) \mapsto (x, \tilde{g}_\infty^{-1}),$$
3. For $m \geq 2$ and $1 \leq l \leq m$,
   $$e_{i_1, \ldots, i_l} : I_G^m(X) \to I_G(X), \quad (x, (g_1, \ldots, g_m)) \mapsto (x, (g_{i_1}, \ldots, g_{i_l})).$$
   When $l = 1$, we get those $e_i$ in (2).

All these maps are $G$-equivariant.

2.2. **Equivariant commutative stringy cohomology group.** We first describe a degree shifting. For each point $(x, g) \in I_G(X)$, $T_x X$ decomposes into eigen-spaces of $g$-action

$$T_x X = \bigoplus_{0 \leq j \leq \text{ord}(g) - 1} T_{x,g,j}, \tag{2.1}$$

where $\text{ord}(g)$ is the order of $g$, and $g$ acts on $T_{x,g,j}$ by multiplying

$$e^{2\pi i T_{x,g,j}}. \tag{2.2}$$

**Definition 2.5.** For each (twisted) sector $X_{[g]}$, the degree shifting is

$$\iota([g]) := \sum_{0 \leq j \leq \text{ord}(g) - 1} j \cdot \dim C T_{x,g,j}.$$ 

Over each connected component of $X_{[g]}$, this is a constant.

Now we define the equivariant commutative stringy cohomology group of $(X, G)$. 
The equivariant commutative stringy cohomology group of \((X, G)\) is
\[
\mathcal{H}^*_c(X, G) := \bigoplus_{[g] \in [G]} H^{*-2\chi([g])}_c(X, g).
\]

In this paper, we take \(\mathbb{R}\) as the coefficient field of (equivariant) cohomology group. One can also take \(\mathbb{Q}\) or \(\mathbb{C}\). In the following we abbreviate “equivariant commutative string cohomology” into “ECS-cohomology” for simplicity.

**Remark 2.7.** One should note that, up to now, all the definitions and constructions work without the assumption on the commutativity of the tuples \(g\), i.e. even if \(g_1, \ldots, g_m\) do not lie in a maximal torus of \(G\), all the definitions and constructions above work.

### 3. Equivariant Commutative Stringy Cohomology Rings on Almost Complex Manifolds

In this section we construct a ring structure over the ECS-cohomology group.

#### 3.1. Moduli space of degree zero maps

In this subsection we describe a moduli space of certain degree zero maps. Take a \([\bar{g}] \in [G^n]\) and a representative \(\bar{g} = (g_1, \ldots, g_m)\). In this subsection we always assume that \(m \geq 2\). Set \(g_0 := \bar{g}_{\infty}^{-1} = (g_1 \cdot \ldots \cdot g_m)^{-1}\). We have an \((m + 1)\)-tuple of positive integers \(r = (r_0, r_1, \ldots, r_m)\) with \(r_i = \text{ord}(g_i)\), which depends only on \([\bar{g}]\) not on the representatives.

There is an orbifold sphere \(S^2_{\text{orb}}\) with \((m + 1)\) orbifold points \(z = (z_0, z_1, \ldots, z_m)\) such that the isotropy group at \(z_i\) is \(\mathbb{Z}_{r_i}\), the cyclic group of order \(r_i\), for \(0 \leq i \leq m\). Denote this orbifold sphere by \(S_F := (S^2_{\text{orb}}, \bar{z}, \bar{r})\). \(S_F\) depends only on \([\bar{g}]\). Its orbifold fundamental group has a presentation given by (cf. \([5, 10, 12]\))
\[
\pi^{\text{orb}}_1(S_F) = \{\lambda_0, \lambda_1, \ldots, \lambda_m | \lambda_0 \cdot \ldots \cdot \lambda_m = 1, \text{ and } \lambda_i^{r_i} = 1\}.
\]

There is a group homomorphism
\[
\psi : \pi^{\text{orb}}_1(S_F) \to G
\]
given by \(\psi(\lambda_i) = g_i\), \(i = 0, 1, \ldots, m\). Let \(\bar{\lambda} = (\lambda_1, \ldots, \lambda_m)\). Then we could denote \(\psi\) by \(\psi(\bar{\lambda}) = \bar{g}\). Denote the kernel by \(N\), which also depends only on \([\bar{g}]\). The image of \(\psi\) is the subgroup \((\bar{g}) = (g_1, \ldots, g_m)\) of \(G\). By Definition 2.1, \((\bar{g})\) is a finite group. Therefore there is a smooth closed Riemann surface \(\Sigma\), and an orbifold covering (cf. \([5, 10, 12]\))
\[
\pi : \Sigma \to S_F
\]
with deck transformation group isomorphic to \((\bar{g})\) and \(\pi_1(\Sigma) = N\). Therefore \(\pi^{\text{orb}}_1(S_F)\) acts on \(\Sigma\) via the homomorphism \(\pi^{\text{orb}}_1(S_F) \to (\bar{g})\), and \(N\) acts on \(\Sigma\) trivially. Then we have
\[
\frac{\Sigma}{\pi^{\text{orb}}_1(S_F)/N} = \frac{\Sigma}{(\bar{g})} = S_F.
\]

One can change the representative \(\bar{g}\) of \([\bar{g}]\), hence \((\bar{g})\), without changing \(N\) and \(\Sigma\). Consider the following space
\[
\tilde{\mathcal{H}}_{\bar{g}} := \left\{(f, \psi) : (\Sigma, j; \pi^{\text{orb}}_1(S_F)) \to (X, J; G) \bigg| \begin{array}{c}
f(\Sigma) = X, \\
\psi(\bar{\lambda}) = \bar{g} \in [\bar{g}], \\
f \text{ is } J, j \text{ holomorphic and equivariant w.r.t. } \psi
\end{array} \right\}.
\]
The group $\text{Aut}(\Sigma, \pi_1^{\text{orb}}(S_\mathcal{R}))$ of holomorphic automorphisms of $\Sigma$ that commute with the $\pi_1^{\text{orb}}(S_\mathcal{R})$-action acts on $\mathcal{M}_{[\mathcal{g}]}$. Then we get the following moduli space

$$\mathcal{M}_{[\mathcal{g}]} := \mathcal{M}_{[\mathcal{g}]} / \text{Aut}(\Sigma, \pi_1^{\text{orb}}(S_\mathcal{R})).$$

One could view this as a combination of the moduli space of certain orbifold pseudo-holomorphic curves and the moduli space of certain symplectic vortices. On the other hand, $G$ acts on $\mathcal{M}_{[\mathcal{g}]}$ by transforming the images of $f$ and conjugating the images of $\psi$. This $G$-action commutes with the $\text{Aut}(\Sigma, \pi_1^{\text{orb}}(S_\mathcal{R}))$-action. Hence $G$ acts on $\mathcal{M}_{[\mathcal{g}]}$.

By allowing $[\mathcal{g}]$ varying in the whole $[G^m]$ we get a moduli space

$$\mathcal{M}_m = \bigsqcup_{[\mathcal{g}] \in [G^m]} \mathcal{M}_{[\mathcal{g}]}.$$

We have a natural map $\pi : \mathcal{M}_{[\mathcal{g}]} \to \mathcal{M}_{0,m+1}$ to the moduli space of $(m+1)$-marked smooth closed genus zero Riemann surfaces, by mapping an element $[(f, \psi) : (\Sigma, j, \pi_1^{\text{orb}}(S_\mathcal{R})) \to (X, J; G)]$ to the equivalent class of the coarse space of the orbifold sphere $S_\mathcal{R} = \Sigma / (\pi_1^{\text{orb}}(S_\mathcal{R})/N) = \Sigma / [\mathcal{g}]$. As the orbifold case (cf. [12]),

**Proposition 3.1.** We have $\mathcal{M}_{[\mathcal{g}]} \cong \mathcal{M}_{0,m+1 \times X_{[\mathcal{g}]}}$, hence $\mathcal{M}_m \cong \mathcal{M}_{0,m+1 \times I_G^n}$. Moreover, $\mathcal{M}_{[\mathcal{g}]}$ can be compactified into $\overline{\mathcal{M}}_{[\mathcal{g}]}$ with

$$\overline{\mathcal{M}}_{[\mathcal{g}]} \cong \overline{\mathcal{M}}_{0,m+1 \times X_{[\mathcal{g}]}}.$$

**Lemma 3.2.** We have obvious evaluation maps $ev_i : \overline{\mathcal{M}}_m \to I_G(X)$ for $i = 0, 1, \ldots, m$ and $\infty$ which factor through the following composition

$$\overline{\mathcal{M}}_m \cong \overline{\mathcal{M}}_{0,m+1 \times I_G^n} \xrightarrow{\text{proj}} I_G^n \xrightarrow{ev} I_G(X).$$

On $\mathcal{M}_m$, for $i = 0, 1, \ldots, m$, $ev_i$ is

$$[(f, \psi) : (\Sigma, j, \pi_1^{\text{orb}}(S_\mathcal{R})) \to (X, J; G)] \mapsto (f(\Sigma), g_i = \psi(\lambda_i))$$

and $ev_\infty$ is

$$[(f, \psi) : (\Sigma, j, \pi_1^{\text{orb}}(S_\mathcal{R})) \to (X, J; G)] \mapsto (f(\Sigma), g_\infty = \psi(\lambda_i^{-1})).$$

Restrict to each component we get $ev_i : \overline{\mathcal{M}}_{[\mathcal{g}]} \to X_{[\mathcal{g}]}$, for $i = 0, 1, \ldots, m$ and $\infty$.

**3.2. Obstruction bundle.** In this subsection we define an obstruction bundle $\mathcal{O}_m$ for each moduli space $\mathcal{M}_m$. Consider a component $\mathcal{M}_{[\mathcal{g}]} = \mathcal{M}_{0,m+1 \times X_{[\mathcal{g}]}}$, and an element

$$[(f, \psi) : (\Sigma, j, \pi_1^{\text{orb}}(S_\mathcal{R})) \to (X, G)] \in \mathcal{M}_{[\mathcal{g}]}$$

with image $f(\Sigma) = x \in X$ and $\psi(\lambda) = \mathcal{g} = (g_1, \ldots, g_m)$. Then $g_i \cdot x = x$ and there is a $([\mathcal{g}])$-equivariant elliptic complex

$$\tilde{\partial} : \Omega^{0,0}(\Sigma, f^*T_xX) \to \Omega^{0,1}(\Sigma, f^*T_xX)$$

over $\Sigma$. Consider its $([\mathcal{g}])$-invariant part, which is also an elliptic complex over $\Sigma$. Then we get a space

$$\mathcal{H}^{0,1}(\Sigma, f^*T_xX)^{([\mathcal{g}])}.$$ 

This forms a bundle $\mathcal{O}_{[\mathcal{g}]}$ over $\mathcal{M}_{[\mathcal{g}]}$, which is $G$-equivariant with respect to the induced $G$-action on $TX$. Denote the disjoint union of all $\mathcal{O}_{[\mathcal{g}]}$ by $\mathcal{O}_m$.

**Definition 3.3.** We call $\mathcal{O}_m$ the obstruction bundle over $\mathcal{M}_m$. 
We next give another description of $\mathcal{O}_m$, which is similar to the K-theory description of the Chen–Ruan obstruction bundle obtained by Chen–Hu [10] and Hu–Wang [23], see also [18, 22, 25, 6, 16]. Consider the element (3.1). As above, suppose that $\psi(\vec{\lambda}) = \vec{g} = (g_1, \ldots, g_m)$ and $f(\Sigma) = x$. As in previous subsection, we set $g_0 = g_\infty = (g_1 \cdots g_m)^{-1}$. Let $C(\langle \vec{g} \rangle)$ denote the center of the group $\langle \vec{g} \rangle$. By Definition 2.1, $\langle \vec{g} \rangle$ is finite and $C(\langle \vec{g} \rangle)$ is finite and $C(\langle \vec{g} \rangle) = \langle \vec{g} \rangle$.

The tangent space $T_xX$ is a complex representation of $\langle \vec{g} \rangle$. We decompose $T_xX$ into direct sum of $\langle \vec{g} \rangle$-irreducible representations

$$T_xX = \bigoplus_{\lambda \in \hat{\langle \vec{g} \rangle}} T_{x,\lambda}.$$  

Note that for $0 \leq i \leq m$, $g_i$ acts on $T_{x,\lambda}$ for all $\lambda \in \hat{\langle \vec{g} \rangle}$. Since $\text{ord}(g_i) < \infty$, $T_{x,\lambda}$ decompose into eigen-spaces of $g_i$. Then we see that each eigen-space of $g_i$ is also a representation of $\langle \vec{g} \rangle$. Therefore by the irreducibility of $T_{x,\lambda}$, it is an eigen-space of $g_i$. So for $i = 0, 1, \ldots, m$, $g_i$ acts on each $T_{x,\lambda}$ by multiplying $e^{2\pi i w_{\lambda,i} \sqrt{-1}}$ for a number $w_{\lambda,i} \in (0, 1) \cap \mathbb{Q}$, called the weight\(^1\). Since $g_0g_1 \cdots g_m = 1$, one has

$$w_{\lambda,0} + w_{\lambda,1} + \ldots + w_{\lambda,m} = 0, 1, \ldots, m.$$  

Define $m + 1$ formal vector spaces

$$S_{g_i,x} := \bigoplus_{\lambda \in \hat{\langle \vec{g} \rangle}} w_{\lambda,i} T_{x,\lambda}, \quad \text{for } i = 0, \ldots, m.$$  

On the other hand, the normal spaces at $x$ of $X^{\vec{g}_i}$ and $X^{\vec{g}}$ in $X$ are

$$N_{g_i,x} = \bigoplus_{\lambda \in \hat{\langle \vec{g} \rangle}, w_{\lambda,i} > 0} T_{x,\lambda}, \quad \text{for } i = 0, 1, \ldots, m,$$

and

$$N_{\vec{g},x} = \bigoplus_{\lambda \in \hat{\langle \vec{g} \rangle}, \sum_{i=0}^m w_{\lambda,i} \geq 1} T_{x,\lambda}$$

respectively. It is direct to see that

$$S_{g_i,x} \oplus S_{g_i^{-1},x} = N_{g_i,x}.$$  

Moreover,

$$\iota([g_i]) = \text{rank}_C S_{g_i,x}.$$  

Lemma 3.4. The fiber of $\mathcal{O}_m$ over a point $[f, \psi]$ as in (3.1) is

$$\bigoplus_{i=0}^m S_{g_i,x} \otimes N_{\vec{g},x} = \bigoplus_{\lambda \in \hat{\langle \vec{g} \rangle}, \sum_{i=0}^m w_{\lambda,i} \geq 2} \left( \sum_{i=0}^m w_{\lambda,i} - 1 \right) T_{x,\lambda}.$$  

These vector spaces form a $G$-bundle over $\mathcal{M}_m$, and is isomorphic to $\mathcal{O}_m$ as a $G$-bundle.

Proof. After replacing the bundle $T\mathcal{G}^0$ by $TX$ in the proof of [23, Theorem 3.2] we get this lemma. □

\(^{1}\)Comparing with the eigen-space decomposition of $T_xX$ under the $g$ action in (2.1) and the action weight in (2.2).
3.3. **Equivariant commutative stringy product.** In this subsection we will use the obstruction bundle $\mathcal{O}_2$ over $\mathcal{M}_2 = I^2_2(X)$ to define the equivariant commutative stringy product (ECS-product) $\ast_{cs}$ over $\mathcal{M}_{G,cs}(X)$.

All maps in Definition 2.4 decompose naturally into compositions of $G$-equivariant embeddings and $G$-equivariant fiber bundle projections. For example, take a $[\vec{g} = (g_1, \ldots, g_m)] \in [G^n]$. For $1 \leq i_1 < \ldots < i_k \leq m$, set $\vec{g}_{i_1, \ldots, i_k} = (g_{i_1}, \ldots, g_{i_k})$, we have the following diagram

$$
\begin{array}{ccc}
X^{\vec{g}} \times_{C(\vec{g})} G & \xrightarrow{e_{i_1, \ldots, i_k}} & X^{\vec{g}_{i_1, \ldots, i_k}} \times_{C(\vec{g})} G \\
\downarrow & & \downarrow \\
G/C(\vec{g}) & \xrightarrow{p_{i_1, \ldots, i_k}} & G/C(\vec{g}_{i_1, \ldots, i_k})
\end{array}
$$

(3.6)

where

- $e_{i_1, \ldots, i_k}$ is obtained from the $C(\vec{g})$-equivariant inclusion $X^{\vec{g}} \hookrightarrow X^{\vec{g}_{i_1, \ldots, i_k}}$, hence is a $G$-equivariant embedding,
- $p_{i_1, \ldots, i_k} : X^{\vec{g}_{i_1, \ldots, i_k}} \times_{C(\vec{g})} G \to X^{\vec{g}_{i_1, \ldots, i_k}} \times_{C(\vec{g}_{i_1, \ldots, i_k})} G$ is the pull back bundle of $G/C(\vec{g}) \to G/C(\vec{g}_{i_1, \ldots, i_k})$, which is also a $G$-equivariant projection of fiber bundle.

Then by the natural $G$-equivariant diffeomorphism in Lemma 2.3, the above diagram decomposes the map $e_{i_1, \ldots, i_k} : X[\vec{g}] \to X[\vec{g}_{i_1, \ldots, i_k}]$ into the composition of $e_{i_1, \ldots, i_k}$ and $p_{i_1, \ldots, i_k}$.

Denote the normal bundle of the embedding $e_{i_1, \ldots, i_k}$ by $N_{e_{i_1, \ldots, i_k}}$. Denote the fiber-wise $G$-equivariant volume form of the fibration $G/C(\vec{g}) \to G/C(\vec{g}_{i_1, \ldots, i_k})$ by $\text{vol}(\vec{g}_{i_1, \ldots, i_k}, \vec{g})$.

(We will prove the existence of such $G$-equivariant volume form in the appendix, see Theorem A.3.) Then via the vertical map in Diagram (3.6) we pull this volume form to the $G$-equivariant fibration $p_{i_1, \ldots, i_k} : X^{\vec{g}_{i_1, \ldots, i_k}} \times_{C(\vec{g})} G \to X^{\vec{g}_{i_1, \ldots, i_k}} \times_{C(\vec{g}_{i_1, \ldots, i_k})} G$ to get the fiber-wise $G$-equivariant volume form for this fibration. We still denote the pull-back volume form by $\text{vol}(\vec{g}_{i_1, \ldots, i_k}, \vec{g})$.

All these notations above apply to other maps $e_i, e_{i, \infty}$ in Definition 2.4.

**Definition 3.5.** For $\alpha_i \in H^*_G(X[\vec{g}_i]), i = 1, 2$, we define the ECS-product of $\alpha_1$ and $\alpha_2$ to be

$$
\alpha_1 \ast_{cs} \alpha_2 := \sum_{[\vec{h}] = [\vec{h}_1, \vec{h}_2] \in [G^2]} p_{\infty, \ast} \left( e_{\infty, \ast}^* \alpha_1 \wedge e_{\infty, \ast}^* \alpha_2 \wedge e_G(\mathcal{O}_{\vec{g}_1}) \right) \wedge \text{vol}(\vec{h}_\infty, \vec{h})
$$

(3.7)

The RHS is a finite sum, since a maximal torus of $G$ intersects with every conjugate class $[g]$ in $G$ finitely. The push forward $e_{\infty, \ast}$ is obtained by wedging equivariant Thom form, and $p_{\infty, \ast}$ is obtained via integration along fiber (cf. [19, 21]). In particular, $p_{\infty, \ast}(\text{vol}(\vec{h}_{\infty}, \vec{h})) = 1$. Denote the summand on RHS by $(\alpha_1 \ast_{cs} \alpha_2)[\vec{h}]$.

**Remark 3.6.** The commutativity of $\vec{g}$ ensure the finiteness of the sum in the definition of “$\ast_{cs}$” and the existence of fiber-wise $G$-equivariant volume form $\text{vol}(\vec{h}_{\infty}, \vec{h})$.

Our main theorem in this section is
Theorem 3.7. The ECS-product $\ast_{cs}$ over $\mathcal{H}_G^{cs}(X)$ is associative.

Proof. Take $\alpha_i \in H^*_G(X_{[g_i]})$ for $i = 1, 2, 3$. We next show that

\[(\alpha_1 \ast_{cs} \alpha_2) \ast_{cs} \alpha_3 = \alpha_1 \ast_{cs} (\alpha_2 \ast_{cs} \alpha_3).\]

By definition, the LHS is

\[
(\alpha_1 \ast_{cs} \alpha_2) \ast_{cs} \alpha_3 = \left( \sum_{[\tilde{h}] = [h_1, h_2] \in [G_f^2], [h_1] = [g_i], i = 1, 2} \alpha_1 \ast_{cs} \alpha_2 \right) \ast_{cs} \alpha_3
\]

\[= \sum_{[\tilde{h}] = [h_1, h_2, h_3] \in [G_f^3], [h_1] = [g_i], i = 1, 2, 3} (\alpha_1 \ast_{cs} \alpha_2) \ast_{cs} \alpha_3]_{\tilde{h}_{1,2}} \ast_{cs} \alpha_3]_{\tilde{h}_{1,2,3}}
\]

where $\tilde{h}_{1,2} = (h_1, h_2)$ and $\tilde{h}_{1,2,3} = (h_1 h_2, h_3)$. The third equality follows from the fact that all maximal torus are conjugate and a maximal torus intersects with a conjugate class finitely. Similarly, the RHS is

\[
\alpha_1 \ast_{cs} (\alpha_2 \ast_{cs} \alpha_3) = \sum_{[\tilde{h}] = [h_1, h_2, h_3] \in [G_f^3], [h_1] = [g_i], i = 1, 2, 3} (\alpha_1 \ast_{cs} (\alpha_2 \ast_{cs} \alpha_3)]_{\tilde{h}_{1,2,3}}
\]

with $\tilde{h}_{2,3} = (h_2, h_3)$ and $\tilde{h}_{1,2,3} = (h_1 h_2, h_3)$.

Now fix an $[\tilde{h}] \in [G_f^3]$ with representative $\tilde{h} = (h_1, h_2, h_3)$. We compare the contribution $((\alpha_1 \ast_{cs} \alpha_2) \ast_{cs} \alpha_3)]_{\tilde{h}_{1,2,3}}$ with $(\alpha_1 \ast_{cs} (\alpha_2 \ast_{cs} \alpha_3)]_{\tilde{h}_{1,2,3}}$. We have the following commutative diagram of maps:

\[
\begin{array}{cccccc}
X^{h_1} \times C(h_1) & \xrightarrow{e_1} & X^{h_{1,2}} \times C(h_{1,2}) & \xrightarrow{e_{12}} & X^{h_{12}} \times C(h_{12}) & \xrightarrow{p_{12}} & X^{h_{12}} \times C(h_{12}) \\
X^{h_2} \times C(h_2) & \xrightarrow{e_2} & X^{h_{1,2}} \times C(h_{1,2}) & \xrightarrow{e_{12}} & X^{h_{12}} \times C(h_{12}) & \xrightarrow{p_{12}} & X^{h_{12}} \times C(h_{12}) \\
\end{array}
\]
Now we compute \(((\alpha_1 *_{cs} \alpha_2)_{[\tilde{h}_{1,2}]} *_{cs} \alpha_3)_{[\tilde{h}_{1,2}]}, \text{ and } (\alpha_1 *_{cs} (\alpha_2 *_{cs} \alpha_3)_{[\tilde{h}_{2,3}]})_{[\tilde{h}_{1,2}]}\).

First, since $\tilde{h}_\infty = (\tilde{h}_{1,2})_\infty$ we have

\[
((\alpha_1 *_{cs} \alpha_2)_{[\tilde{h}_{1,2}]} *_{cs} \alpha_3)_{[\tilde{h}_{1,2}]} = p_{\infty,*} \left[ \tilde{e}_{\infty,*} \left( e_1^* [(\alpha_1 *_{cs} \alpha_2)_{[\tilde{h}_{1,2}]}] \wedge e_3^* \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee) \right) \wedge \text{vol}(\tilde{h}_\infty, \tilde{h}_{1,2}) \right].
\]

For $e_1^* [(\alpha_1 *_{cs} \alpha_2)_{[\tilde{h}_{1,2}]}, by using the above commutative diagram we get

\[
e_1^* [(\alpha_1 *_{cs} \alpha_2)_{[\tilde{h}_{1,2}]}] = \tilde{e}_1^* \circ p_1^* [(\alpha_1 *_{cs} \alpha_2)_{[\tilde{h}_{1,2}]}].
\]

**Definition 3.5**

\[
e_1^* \circ p_1^* \circ p_{\infty,*} \left( \tilde{e}_{\infty,*} \left[ e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \wedge \text{vol}(h_{1,2}, \tilde{h}_{1,2}) \right] \right)
\]

\[
\overset{B}{=} \tilde{e}_1^* \circ p_{12,3,*} \left( p_{12,2,*} \circ \tilde{e}_{\infty,*} \left[ e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}) \right] \right)
\]

\[
\overset{A}{=} \tilde{e}_1^* \circ p_{12,3,*} \left( \tilde{e}_{\infty,*} \circ p_{12,2,*} \left[ e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}) \right] \right)
\]

\[
\overset{D}{=} p_{12,3,*} \left( \tilde{e}_1^* \circ \tilde{e}_{\infty,*} \circ p_{12,2,*} \left[ e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}) \right] \right)
\]

\[
\overset{C}{=} p_{12,3,*} \left( \tilde{e}_1^* \circ p_{12,2,*} \left[ e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}) \right] \right)
\]

\[
\wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}).
\]

where $E_{12,3}$ is the equivariant counterpart of the excess bundle (cf. [30]) for the intersection $X^{\tilde{h}_{1,2}} \cap X^{\tilde{h}_{1,2}} = X^{\tilde{h}}$ in $X^{\tilde{h}_{1,2}}$, that is

\[
E_{12,3} = \left( (NX^{\tilde{h}_{1,2}} | X^{\tilde{h}_{1,2}}) | X_{[\tilde{h}]} \right) \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}).
\]

Then we get

\[
((\alpha_1 *_{cs} \alpha_2)_{[\tilde{h}_{1,2}]} *_{cs} \alpha_3)_{[\tilde{h}_{1,2}]} = p_{\infty,*} \left[ \tilde{e}_{\infty,*} \left( \tilde{e}_{12,3,*} \left( e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee) \right) \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}) \right] \right).
\]

\[
\overset{E}{=} p_{\infty,*} \left[ \tilde{e}_{12,3,*} \left( e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_3^* \alpha_3 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}) \right] \right)
\]

\[
\overset{F}{=} p_{\infty,*} \left( \tilde{e}_{12,3,*} \left( e_1^* [\alpha_1 \wedge e_2^* \alpha_2 \wedge e_3^* \alpha_3 \wedge e_G(\mathcal{O}_{[\tilde{h}_{1,2}]}^\vee)] \right) \wedge \text{vol}(\tilde{h}_{1,2}, \tilde{h}) \right).
\]
Here we also have used commutative diagrams similar to
\[
\begin{array}{c}
\xymatrix{
X_{[h_{1,2}]} \cong X_{\tilde{h}_{1,2}} \times_{C(h_{1,2})} G \ar[r]^(0.6){e_1} & X_{h_1} \times_{C(h_1)} G \cong X_{[h_1]} \\
X_{[\tilde{h}]} \cong X_{\tilde{h}} \times_{C(\tilde{h})} G.
}
\end{array}
\]
By the same computation we get
\[
E \mapsto O \alpha \mapsto \alpha G
\]
Here we also have used commutative diagrams similar to
\[\text{Obviously, all these computations are G-equivariant.}\]
\[
\begin{align*}
\alpha_1 \ast_{cs} (\alpha_2 \ast_{cs} \alpha_3)_{[h_{1,2}]} & \mapsto (\alpha_1 \ast_{cs} (\alpha_2 \ast_{cs} \alpha_3)_{[h_{1,2}]} = (\alpha_1 \ast_{cs} (\alpha_2 \ast_{cs} \alpha_3)_{[h_{1,2}]})_{[h_{1,2}]}
\end{align*}
\]
for any $[\tilde{h}] \in [G]_\infty$ with $[h_i] = [g_i]$. Consequently, the ECS-product $*_c$ is associative.

\[ \circ \]

**Proposition 3.8.** The ECS-product $*_c$ preserves the shifted degree. Moreover, it is super-commutative, i.e.

\[ \alpha_1 *_c \alpha_2 = (-1)^{\deg \alpha_1 \deg \alpha_2} \alpha_2 *_c \alpha_1. \]

**Proof.** Using the notation in Definition 3.5. Then we need to show that

\[ \deg(\alpha_1 *_c \alpha_2)[\tilde{h}] + 2\iota([h_1 h_2]) = \deg \alpha_1 + 2\iota([g_1]) + \deg \alpha_2 + 2\iota([g_2]), \tag{3.9} \]

for $\tilde{h} = (h_1, h_2) \in G_2$ with $[h_i] = [g_i], i = 1, 2$. By the definition of $*_c$,

\[ \deg(\alpha_1 *_c \alpha_2)[\tilde{h}] = \deg \alpha_1 + \deg \alpha_2 + \text{rank} \Theta[\tilde{g}] + \text{rank} N X^\tilde{h}[X^{h_1 h_2}] = \deg \alpha_1 + \deg \alpha_2 + \text{rank} \Theta[\tilde{g}] + \text{rank} N_x h - \text{rank} N_x h_{1 h_2}. \]

By Lemma 3.4 we have

\[ \text{rank} \Theta[\tilde{g}] = \text{rank} S_x(h_1 h_2)^{-1} + \text{rank} S_x h_1 + \text{rank} S_x h_2 - \text{rank} N_x h. \]

Therefore by (3.4) we have

\[ \deg(\alpha_1 *_c \alpha_2)[\tilde{h}] = \deg \alpha_1 + \deg \alpha_2 + \text{rank} S_x(h_1 h_2)^{-1} + \text{rank} S_x h_1 + \text{rank} S_x h_2 - \text{rank} N_x h_{1 h_2} = \deg \alpha_1 + \deg \alpha_2 - \text{rank} S_x h_1 + \text{rank} S_x h_2. \]

Then (3.9) follows from (3.5), i.e. $\iota([g]) = \text{rank}_C S_g$. The second assertion follows from the fact that the rank of $\Theta$ and the degree of $vol(\tilde{h}, h)$ are both even.

**Example 3.9.** When $G = T$ is a torus, $\mathcal{H}^*_c(T, X) = \bigoplus_{t \in T_1} H_*^{s - 2\iota(t)}(X^t)$, where $T_1$ is the subgroup of finite order elements. The diagram (3.6) becomes

\[
\begin{array}{c}
X \xrightarrow{\tilde{e}_{i_1, \ldots, i_k}} X \xrightarrow{p_{i_1, \ldots, i_k} = \text{id}} X \xrightarrow{\text{pt}} \{\text{pt}\}
\end{array}
\]

So $e_{i_1, \ldots, i_k} = \tilde{e}_{i_1, \ldots, i_k}$ and the volume form is trivial. So

\[ \alpha_1 *_c \alpha_2 = e_{\infty, \ast} (e^h_1 \alpha_1 \wedge e^h_2 \alpha_2 \wedge e_G(\Theta \tilde{g})) \]

for $\alpha_i \in H_*^s(X^g), i = 1, 2$, where $\tilde{g} = (g_1, g_2)$. This product coincides with [22, Definition 3.3]. The ring $\mathcal{H}_c^*_c(X)$ is the $\Gamma$-subring of the inertia cohomology (cf. [22, Definition 6.11]).

4. **Equivariant commutative stringy cohomology ring and symplectic reduction**

In this section we consider the ECS-cohomology rings for hamiltonian symplectic manifolds and its relation with the Chen–Ruan cohomology of the symplectic reduction orbifolds.
Let \((X, \omega, G, \mu)\) be a hamiltonian system with \(G\) being connected and compact. By choosing a \(G\)-invariant, \(\omega\)-compatible almost complex structure \(J\) we get the ECS-cohomology ring for \((X, G)\)

\[
\mathcal{H}^*_G,cs(X) := \bigoplus_{[g] \in [G]} H^*_{G}(-2\iota([g]) (X_g), \ast_{cs}).
\]

This ring does not depend on the choices of \(J\), since the space of \(G\)-invariant, \(\omega\)-compatible almost complex structures on \(X\) is path connected.

Now suppose \(0 \in \mathfrak{g}\) is a regular value of \(\mu\). Denote the level set by \(Y := \mu^{-1}(0)\). The normal bundle of \(Y\) in \(X\) is a trivial bundle \(Y \times \mathfrak{g}^\ast\). By symplectic reduction, there is an induced symplectic form \(\omega_{\text{red}}\) on the reduction \(M := [Y/G] = X//G\). A \(G\)-invariant, \(\omega\)-compatible almost complex structure \(J\) on \(X\) induces an almost complex structure \(J\) on \(M\) that is compatible with \(\omega_{\text{red}}\). Then we get the Chen–Ruan cohomology ring for the symplectic orbifold \(M\).

In this section we first construct a ECS-cohomology ring \(\mathcal{H}^*_G,cs(Y)\) for the pair \((Y, G)\), and show that the natural inclusion \(i : Y \hookrightarrow X\) induces a surjective ring homomorphism \(i^* : \mathcal{H}^*_G,cs(X) \to \mathcal{H}^*_G,cs(Y)\) in in §4.1. Then in §4.2 we show that we have a natural group isomorphism \(\mathcal{H}^*_G,cs(Y) \cong H^*_CR(M)\), and in general it is not a ring isomorphism with respect to the ECS-product and the Chen–Ruan product. Moreover, we found that by modifying the hamiltonian system \((X, \omega, G, \mu)\) we could assign the Chen–Ruan cohomology group \(H^*_CR(M)\) infinite different ring structures that are different from the Chen–Ruan product, and the resulting ring structures are compatible with the ECS-cohomology ring. See Remark 4.9.

4.1. ECS-cohomology ring for \((Y, G)\). Following the Definition 2.2 we define the \(m\)-sectors of \((Y, G)\).

**Definition 4.1.** For \(m \in \mathbb{Z}_{\geq 1}\), we set the \(m\)-sector of \((Y, G)\) to be \(I^m_G(Y) := \bigsqcup_{[g] \in [G]^m} Y_{[g]}\) with

\[
Y_{[g]} := \{(y, \bar{g}) \in Y \times G \mid \bar{g} \cdot y = (y, \ldots, y)\}.
\]

When \(m = 1\) we also omit the superscript. \(G\) also acts on \(Y_{[g]}\).

Let \(Y^{\bar{g}} = \cap_{i=1}^m Y^{g_i}\) be the fixed loci of \(\bar{g} = (g_1, \ldots, g_m)\). We also have a \(G\)-equivariant diffeomorphism \(Y_{[\bar{g}]} \cong Y^{\bar{g}} \times_C Y^{\bar{g}} G\).

Let \(i : Y \hookrightarrow X\) being the inclusion map as submanifold. Then one can see that for \(m \in \mathbb{Z}_{\geq 1}\), \(I^m_G(Y)\) is a \(G\)-invariant submanifold of \(I^m_G(X)\). We denote the inclusion also by \(i\). We could restrict those natural maps between \(I^m_G(X)\) in Definition 2.4 to \(I^m_G(Y)\), and get natural maps between \(I^m_G(Y)\). We use the same notations. For example we have the commutative diagram

\[
\begin{array}{ccc}
I^2_G(X) & \xrightarrow{e_1, e_2, e_\infty} & I_G(X) \\
\text{Id} & & \text{Id} \\
I^2_G(Y) & \xrightarrow{e_1, e_2, e_\infty} & I_G(Y),
\end{array}
\]

with the lower \(e_1, e_2\) and \(e_\infty\) being the restriction of the upper \(e_1, e_2\) and \(e_\infty\) respectively.

We restrict the degree shifting \(\iota([g])\) to \(I_G(Y)\) and still denote it by \(\iota([g])\).
Definition 4.2. We define the ECS-cohomology group of \((Y, G)\) as

\[
\mathcal{H}_{G,cs}^*(Y) := \bigoplus_{[g] \in [G]} H^{*-2\ell([g])}(Y_{[g]}).
\]

We restrict the obstruction bundle \(\mathcal{O}_2^Y\) over \(I_2^G(X)\) to \(I_2^G(Y)\), and denote it by \(\mathcal{O}_2^Y\). On the other hand, there is a \(G\)-equivariant bundle \(\mathcal{C}_2^Y\) over \(I_2^G(Y)\) whose fiber at a point \((y, \tilde{g})\) is \(c_{\tilde{g}, \infty} \otimes c_{\tilde{g}}\), where \(\tilde{g}_{\infty} = g_1 g_2\) for \(\tilde{g} = (g_1, g_2)\); \(c_{\tilde{g}, \infty}\) and \(c_{\tilde{g}}\) are the dual of the Lie algebras of the centralizer \(C(\tilde{g}_{\infty})\) and \(C(\tilde{g})\) respectively. We set

\[
\mathcal{O}_2^X := \mathcal{O}_2^Y \oplus \mathcal{C}_2^Y,
\]

and denote the components of \(\mathcal{O}_2^Y\), \(\mathcal{O}_2^X\) and \(\mathcal{C}_2^Y\) over \(Y_{[\tilde{g}]}\) by \(\mathcal{O}_2^Y_{[\tilde{g}]}\), \(\mathcal{O}_2^X_{[\tilde{g}]}\) and \(\mathcal{C}_2^Y_{[\tilde{g}]}\) respectively.

We next define the product over \(\mathcal{H}_{G,cs}^*(Y)\). We also decompose the natural map \(e_{\infty} : Y_{[\tilde{g}]} \to Y_{[\tilde{g}_{\infty}]}\) for \(\tilde{g} = (g_1, g_2)\), into

\[
Y_{\tilde{g}} \times_{C(\tilde{g})} G \xrightarrow{\tilde{e}_{\infty}} Y_{\tilde{g}_{\infty}} \times_{C(\tilde{g}_{\infty})} G \xrightarrow{p_{\infty}} Y_{\tilde{g}_{\infty}} \times_{C(\tilde{g}_{\infty})} G.
\]

The projection \(p_{\infty} : Y_{\tilde{g}_{\infty}} \times_{C(\tilde{g}_{\infty})} G \to Y_{\tilde{g}_{\infty}} \times_{C(\tilde{g}_{\infty})} G\) is also the pull back of the bundle \(p_{\infty} : G/C(\tilde{g}_{\infty}) \to G/C(\tilde{g}_{\infty})\). Therefore there is a fiberwise \(G\)-equivariant volume form for \(p_{\infty}\), which is also denoted by \(\text{vol}(\tilde{g}_{\infty}, \tilde{g})\).

Definition 4.3. For \(\alpha_i \in H^*(Y_{[\tilde{g}]}), i = 1, 2,\) the ECS-product of them is

\[
\alpha_1 \ast_{cs} \alpha_2 := \sum_{\tilde{h} = [h_1, h_2] \in [G], [\tilde{g}] = [h_1], i = 1, 2} p_{\infty,*} \left[ \tilde{e}_{\infty,*} \left( e_{\infty} \left( e_{1}^{*} \alpha_1 \wedge e_{2}^{*} \alpha_2 \wedge e_{G}(\mathcal{O}_2^Y_{[\tilde{g}]})) \wedge \text{vol}(\tilde{h}_{\infty}, \tilde{h}) \right) \right) \right]
\]

Theorem 4.4. The ECS-product \(\ast_{cs}\) over \(\mathcal{H}_{G,cs}^*(Y)\) is associative and preserves the shifted degree. Moreover it is supper commutative.

Proof. The proof is similar to the proof of the associativity of \(\ast_{cs}\) over \(\mathcal{H}_{G,cs}^*(X)\) in Theorem 3.7 and Proposition 3.8. The main part is to prove that over a 3-sector \(Y_{[\tilde{g}]}, \) with \(\tilde{h} = (h_1, h_2, h_3), \tilde{h}_{1,2} = (h_1, h_2), \tilde{h}_{1,2,3} = (h_1 h_2, h_3), \tilde{h}_{2,3} = (h_2, h_3)\) and \(\tilde{h}_{1,2,3} = (h_1, h_2 h_3),\) there is a \(G\)-equivariant isomorphism

\[
E_{12,3}^Y \oplus \mathcal{O}_2^Y_{[h_{1,2}]}|_{Y_{[\tilde{g}]}} \oplus \mathcal{O}_2^Y_{[h_{1,2,3}]}|_{Y_{[\tilde{g}]}} \cong E_{23}^Y \oplus \mathcal{O}_2^Y_{[h_{2,3}]}|_{Y_{[\tilde{g}]}} \oplus \mathcal{O}_2^Y_{[h_{1,2,3}]}|_{Y_{[\tilde{g}]}}\] (4.2)

where \(E_{12,3}^Y\) and \(E_{23}^Y\) denote equivariant counterpart of the excess bundles for \(Y\), which is similar to the \(E_{12,3}\) and \(E_{23}\) for \(X\). For example

\[
E_{12,3}^Y = \left[(NY^h_{h_1,2})|_{Y_{[\tilde{g}]} \otimes NY^h_{\tilde{h}}|_{Y_{h_{1,2,3}}} \times_{C(\tilde{h})} G. \right.
\]

Since the normal bundle of \(Y\) is \(X\) is the trivial bundle \(Y \times g^*\), one have

\[
E_{12,3}^Y = E_{12,3}|_{Y_{[\tilde{g}]} \otimes c_{h_1,2}^* \otimes c_{h_{1,2},3}^* \otimes c_{h_{1,12}}^* \otimes c_{h_{1,2}}^*},
\]

\[
E_{12,3}^Y = E_{12,3}|_{Y_{[\tilde{g}]} \otimes c_{h_{2,3}}^* \otimes c_{h_{1,2}}^* \otimes c_{h_{1,23}}^* \otimes c_{h_{2,3}}^*},
\]

where for example, \(c_{h_{1,2}}^*\) is the dual of the Lie algebra of the centralizer \(C(h_{1,2})\) and \(h_{12} = h_1 h_2 = (h_{1,2})_{\infty}\).
From the computation in the proof of Theorem 3.7 and (3.8), we see that over a point \((y, \vec{h}) \in Y_{\{\vec{h}\}}\), we have an equality of fibers
\[
E_{1,2,3}^i \oplus \delta h^\ast \rho_{[1,2]} \big|_{Y_{\{\vec{h}\}}} \oplus \delta h^\ast \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}}
= E_{1,2,3} \big|_{Y_{\{\vec{h}\}}} \oplus \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}}
\]
\[
\oplus C_{[1,2]} \big|_{Y_{\{\vec{h}\}}} \oplus \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}}
\]
\[
= E_{1,2,3} \big|_{Y_{\{\vec{h}\}}} \oplus \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}}
\]
\[
= E_{1,2,3} \big|_{Y_{\{\vec{h}\}}} \oplus \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}}
\]
\[
(3.8)
\]
\[
= E_{1,2,3} \big|_{Y_{\{\vec{h}\}}} \oplus \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}}
\]
\[
\oplus C_{[1,2]} \big|_{Y_{\{\vec{h}\}}} \oplus \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}}
\]

Obviously, this equality is \(G\)-equivariant. Then we get the \(G\)-equivariant isomorphism of bundles (4.2) over \(Y_{\{\vec{h}\}}\). Therefore \(\ast cs\) is associative on \(\mathcal{H}_{cs}^\ast(Y, G)\).

The second assertion follows from the same computation in Proposition 3.8 and the fact that for an \(\vec{h} \in G_2^\ast\),
\[
[NX^j X^{h_{12}}] \big|_{Y_{\{\vec{h}\}}} \oplus NY^\vec{h} X^{h_{12}} \times C(\vec{h}) G
\]
\[
= ([NY^\vec{h}] X^{h_{12}}] \big|_{Y_{\{\vec{h}\}}} \oplus NY^\vec{h} X^{h_{12}} \times C(\vec{h}) G
\]
\[
= \left[ h_{12} \ominus \rho_{[1,2]} \big|_{Y_{\{\vec{h}\}}} \times C(\vec{h}) G
\]
\[
= \left[ \rho_{[1,2,3]} \big|_{Y_{\{\vec{h}\}}} \times C(\vec{h}) G
\]
\[
(4.3)
\]

In fact, by this equation we get that for \(\alpha_i \in H^\ast_G(Y_{\{y_i\}})\) we have
\[
\text{deg}(\alpha_1 \ast cs \alpha_2) + \iota(\langle g_1, g_2 \rangle)
\]
\[
\text{Definition 4.3}
\]
\[
\text{deg} \alpha_1 + \text{deg} \alpha_2 + \text{rank} \delta h^\ast \rho_{[1,2]} + \iota(\langle g_1, g_2 \rangle) + \text{rank} \rho_{[1,2]} + \text{rank} NY^\vec{h} X^{h_{12}}
\]
\[
(4.3)
\]
\[
\text{deg} \alpha_1 + \text{deg} \alpha_2 + \text{rank} \delta h^\ast \rho_{[1,2]} + \iota(\langle g_1, g_2 \rangle) + \text{rank} NY^\vec{h} X^{h_{12}}
\]
\[
\text{Proposition 3.8}
\]
\[
\text{deg} \alpha_1 + \iota(\langle y_1 \rangle) + \text{deg} \alpha_2 + \iota(\langle g_2 \rangle)
\]

Since the degree of \(e_G(\delta h^\ast \rho_{[1,2]})\) and \(\text{vol}(\vec{h}, \tilde{\vec{h}})\) are both even, the third assertion follows. The proof is accomplished.

**Remark 4.5.** From the construction of \((\mathcal{H}_{cs}^\ast(Y, \ast cs), \ast cs)\) we see that, we do not need the ambient symplectic manifold \(X\) to be compact, but only the level set \(Y\) being compact. That is, even if \(X\) is noncompact, the construction above works.

**Theorem 4.6.** The inclusion map \(i : I_G(Y) \hookrightarrow I_G(X)\) induces a degree preserved, surjective ring homomorphism
\[
i^\ast : (\mathcal{H}_{cs}^\ast(X), \ast cs) \rightarrow (\mathcal{H}_{gs}^\ast(Y), \ast cs).
\]

**Proof.** By Definition 2.6 and Definition 4.2, we see that \(i^\ast\) preserves the shifted degree. The rest proof consists of two parts.

**i^\ast is surjective.** First note that \(i : I_G(Y) \hookrightarrow I_G(X)\) decomposes into a disjoint union of
\[
i_{[g]} : Y_{[g]} \cong Y^g \times C(g) G \hookrightarrow X_{[g]} \cong X^g \times C(g) G, \quad i_{[g]}([y, \vec{h}]) = [i_{[g]}(y, \vec{h})]
\]
over all \([g] \in [G]\), where \(g \in [g]\) is a representative of the conjugate class \([g]\), and \(i_g\) is the inclusion map \(i_g : Y^g \to X^g\) that embeds \(Y^g\) as a \(C(g)\)-invariant submanifold of \(X^g\). Second, note that we have a commutative diagram

\[
\begin{array}{ccc}
H_G^*(Y^g) & \xrightarrow{i_g^*} & H_G^*(X^g) \\
\cong & & \cong \\
H^*_C(Y^g) & \xrightarrow{i_g^*} & H^*_C(X^g).
\end{array}
\]

Finally note that \((X^g, \omega)\) is a symplectic submanifold of \((X, \omega)\), and \(C(g)\) acts on it in a hamiltonian fashion with moment map being

\[\mu_{|X^g} : X^g \to \mathfrak{c}^*_g \subseteq \mathfrak{g}^*,\]

where \(\mathfrak{c}^*_g\), the dual of the Lie algebra of \(C(g)\). Then 0 is also a regular value and \((\mu_{|X^g})^{-1}(0) = Y^g\). Therefore that the classical Kirwan map is surjective implies that \(i_g^*\) is surjective. So is \(i_g^*[g]\).

\textbf{i* is a ring homomorphism.}\] Take \(\alpha_1 \in H^*(X_{[g_1]}^i)\) and \(\alpha_2 \in H^*(X_{[g_2]}^i)\), and \(\tilde{h} = (h_1, h_2) \in G^2_i\) such that \([h_2] = [g_2]\). Recall that \(\tilde{h}_\infty = h_1 h_2\). We have the following commutative diagram

\[
\begin{array}{ccc}
X^\tilde{h} & \xrightarrow{i} & X^\tilde{h}_\infty \\
\downarrow & & \downarrow \\
Y^\tilde{h} & \xrightarrow{i} & Y^\tilde{h}_\infty
\end{array}
\]

with all vertical arrows being inclusions. This gives us the commutative diagram

\[
\begin{array}{ccc}
X^\tilde{h} \times_{C(\tilde{h})} G & \xrightarrow{e_\infty} & X^\tilde{h}_\infty \times_{C(\tilde{h})} G \\
\downarrow & & \downarrow \\
Y^\tilde{h} \times_{C(\tilde{h})} G & \xrightarrow{\tilde{e}_\infty} & Y^\tilde{h}_\infty \times_{C(\tilde{h})} G \\
\downarrow & & \downarrow \\
X^\tilde{h}_\infty \times_{C(h_{12})} G & \xrightarrow{\tilde{p}_\infty} & X^\tilde{h}_\infty \times_{C(h_{12})} G
\end{array}
\]

Note that we have \(i^* \circ p_{\infty,*} = p_{\infty,*} \circ i^*\) for the second square and

\[i^* \circ \tilde{e}_{\infty,*} (\cdot) = \tilde{e}_{12,*} \left\{ i^* (\cdot) \right\} \cap \tilde{e}_{12,*} \left\{ \left(\langle (NX^\tilde{h} | X^\tilde{h}_\infty) | Y^\tilde{h} \right) \cap NY^\tilde{h} | Y^\tilde{h}_\infty \right\} \times_{C(\tilde{h})} G \right\}
\]

for the first square. In fact \((NX^\tilde{h} | X^\tilde{h}_\infty) | Y^\tilde{h} \cap NY^\tilde{h} | Y^\tilde{h}_\infty\) is the excess bundle for the intersection of \(X^\tilde{h} \cap Y^\tilde{h}_\infty = Y^\tilde{h}\) in \(X^\tilde{h}_\infty\). By (4.3),

\[\left(\langle (NX^\tilde{h} | X^\tilde{h}_\infty) | Y^\tilde{h} \right) \cap NY^\tilde{h} | Y^\tilde{h}_\infty \right\} \times_{C(\tilde{h})} G = \mathfrak{c}^*_\tilde{h}
\]

Therefore for we have

\[
\begin{array}{ll}
\text{Definition 3.5} & \begin{cases} 
\alpha_1 \star_{cs} \alpha_2 | [\tilde{h}] \\
p_{\infty,*} \circ \tilde{e}_{\infty,*} (\cdot) \left\{ \left(\langle (NX^\tilde{h} | X^\tilde{h}_\infty) | Y^\tilde{h} \right) \cap NY^\tilde{h} | Y^\tilde{h}_\infty \right\} \times_{C(\tilde{h})} G \right\}
\end{cases} \\
\text{B} & \begin{cases} 
p_{\infty,*} \circ \tilde{e}_{\infty,*} (\cdot) \left\{ \left(\langle (NX^\tilde{h} | X^\tilde{h}_\infty) | Y^\tilde{h} \right) \cap NY^\tilde{h} | Y^\tilde{h}_\infty \right\} \times_{C(\tilde{h})} G \right\}
\end{cases}
\end{array}
\]
by projecting $Y$ the singular cohomology group of its inertia orbifold. This implies $C/\pi$ induces a projection over the Borel construction $I$-action on $Y$, hence has finite stabilizers. Then for $g \in G_\ell$ we have $H^*_C(M) = \bigoplus_{[g] \in [G_\ell]} H^*(Y^g/C(g)).$

By the definition of ECS-cohomology group, we have

$$H^*_C(Y) = \bigoplus_{[g] \in [G_\ell]} H^{*-2\chi([g])}_C(Y).$$

Note that there is a natural projection map

$$\pi : I_C(Y) \to \text{IM}$$

by projecting $Y_{[g]} \cong Y^g \times_{C(g)} G$ to $Y^g/C(g)$ for each $[g] \in [G_\ell]$. For each $[g] \in [G_\ell]$, $\pi$ induces a projection over the Borel construction

$$\pi : (Y^g \times_{C(g)} G) \times_G EG \to Y^g/C(g),$$

which we still denote by $\pi$. Since $0 \in g^*$ is regular, the $G$-action on $Y$ is locally free, so is the $C(g)$-action on $Y^g$, hence has finite stabilizers. Then for $g \in G_\ell$ we have a group isomorphism

$$\pi^* : H^*(Y^g/C(g)) \xrightarrow{\cong} H^*_C(Y^g),$$

since the fiber of $\pi$ in (4.5) is rationally acyclic. By summing over $[G_\ell]$ we get a group isomorphism

$$\pi^* : H^*_C(M) \to \mathcal{H}^*_C(Y).$$

In general, for every $m \in \mathbb{Z}_{>1}$ there is a projection map from the $m$-inertia manifold $I^m_C(Y)$ to the $m$-inertia orbifold $I^m M$ of $M$

$$\pi^m : I^m_C(Y) \to I^m M = \bigsqcup_{[g] \in [G^m]} M_{[g]} = \bigsqcup_{[g] \in [G^m]} Y^g/C(g),$$

which maps $Y_{[g]} = Y^g \times_{C(g)} G$ to $Y^g/C(g)$, where $[G^m]$ is the set of conjugate classes of $m$-tuples of elements$^2$ of $G$. When $m = 1$, $\pi^1 = \pi$ in (4.4). Note that

$^2$Here we do not need the assumption on the finiteness of the orders since now the $G$-action on $Y$ is local freely. For an infinite order element $g$, the fixed loci $Y^g$ is empty.
generally the map $\pi^m$ is not surjective for $m \geq 2$. The image of $\pi^m$ is
\[
\bigsqcup_{[\bar{g}] \in [G^m]} M_{[\bar{g}]}.
\]
There would be $m$-sector $M_{[\bar{g}]}$ of $M$ with $\bar{g} \notin G^m$. We denote the image of $\pi^m$ by $I^m M$. By only using $I^m M$ we could modify the Chen–Ruan product over $H^*_C R (M)$ by setting
\[
\alpha_1 \bar{\cup} CR \alpha_2 = \sum_{[\bar{g}] \in [G^m]} \bar{\epsilon}_i (\bar{\epsilon}_i^1 \alpha_1 \wedge \bar{\epsilon}_i^2 \alpha_2 \wedge e(\mathcal{O}^CR_{[\bar{g}]})
\]
for $\alpha_1, \alpha_2 \in H^*_C R (M)$, where $\bar{\epsilon}_i : Y^\bar{g}/C(\bar{g}) \to Y^{g_i}/C(g_i)$ with $i = 1, 2, \infty$, are the evaluation maps, and $\mathcal{O}^CR_{[\bar{g}]}$ is the Chen–Ruan obstruction bundle over the 2-sector $M_{[\bar{g}]}$. This is a truncation of the original Chen–Ruan products in $[12]$. Since $[G^m]$ is closed under multiplication (see the proof of Theorem 3.7) we see that $\bar{\cup} CR$ also gives rise to a ring structure over $H^*_C R (M)$. We call the resulting ring the commutative Chen–Ruan cohomology ring, and denote it by $H^*_{C R, cs}(M)$. When $G = T$ is a torus, this is just the Chen–Ruan cohomology ring.

**Proposition 4.7.** When $G$ is a nonabelian connected compact Lie group, generally the group isomorphism
\[
\pi^* : H^*_{C R, cs}(M) \to \mathcal{H}^*_{G, cs}(Y)
\]
does not preserves the shifted degree and the products, hence is not a ring isomorphism.

**Proof.** For the first assertion take a $g \in G_1$. Without loss of generality, we assume that $Y^g$ is connected. The degree shifting of $\mathcal{H}^*_{G, cs}(Y)$ associated to $[g]$ is obtained from the $g$-action on $T_x X$ for some $x \in Y^g \subseteq X^g$. The degree shifting of $H^*_C R (M)$ associated to $[g]$ is obtained from the $g$-action on $T_{[x]} M$, where $[x]$ means the orbit of $x$ in $Y$. It is well-known that
\[
T_x X = T_{[x]} M \oplus \mathfrak{g}_C,
\]
with $\mathfrak{g}_C = \mathfrak{g} \oplus \mathfrak{g}^*$. Suppose that $\mathfrak{g}_C$ decompose into eigen-spaces of $g$-action
\[
\mathfrak{g}_C = \bigoplus_{0 \leq i \leq \text{ord}(g) - 1} \mathfrak{g}_{C,i}
\]
Then the difference between the degree shifting for $\mathcal{H}^*_{G, cs}(Y)$ and the degree shifting for $H^*_C R (M)$ associated to $[g]$ is
\[
\sum_{0 \leq i \leq \text{ord}(g) - 1} \frac{i}{\text{ord}(g)} \text{dim}_C \mathfrak{g}_{C,i},
\]
which is in general nonzero\(^4\).

We next consider the second assertion. Take two classes $\alpha_i \in H^*(M_{[g_i]}) = H^*(Y^{g_i}/C(g_i))$ for $i = 1, 2$. Denote by $\beta_i := \pi^* \alpha_i$, the images in $H^*_C (g_i)(Y^{g_i})$. Set $\bar{g} = (g_1, g_2)$. Then we have
\[
\alpha_1 \bar{\cup} CR \alpha_2 = \sum_{[\bar{h}] = ([h_1, h_2]) \in [G^2],\, [h_i] = [g_i], i = 1, 2} \bar{\epsilon}_{\infty,*} \left( \bar{\epsilon}_1^1 \alpha_1 \wedge \bar{\epsilon}_1^2 \alpha_2 \cup e(\mathcal{O}^CR_{[\bar{h}]}) \right).
\]

\(^3\)See for example [12, 5, 10, 23] the expression of Chen–Ruan product.

\(^4\)When $G$ is abelian, the adjoint $g$-action on $\mathfrak{g}_C$ is trivial. Hence the difference is zero.
On the other hand
\[ \beta_1 \ast_{cs} \beta_2 = \sum_{[\tilde{h}]= ([h_1, h_2]) \in [G^2]} p_{\infty, \ast} \left( \tilde{e}_{\infty, \ast} (e_{1, \ast}^* \beta_1 \wedge e_{2, \ast}^* \beta_2 \wedge e_G (\tilde{\mathcal{O}}_Y^{|\tilde{h}|})) \wedge \text{vol}(\tilde{h}_\infty, \tilde{h}) \right). \]

We next show that \( \beta_1 \ast_{cs} \beta_2 \neq \pi^*(\alpha_1 \cup CR \alpha_2) \) generally. We compare their components for every \([\tilde{h}] = ([h_1, h_2]) \in [G^2] \) satisfying \([h_i] = [g_i], i = 1, 2.\)

For an \( \tilde{h} \in G^2 \) we have a commutative diagram
\[ \begin{array}{c}
Y_{[\tilde{h}]} = (Y^R \times_{C(\tilde{h})} G) \\
\downarrow \pi \\
M_{[\tilde{h}]} = Y^R/C(\tilde{h})
\end{array} \]
\( \begin{array}{c}
\tilde{e}_\infty \\
\downarrow p \quad \quad \downarrow \pi \\
\tilde{e}_\infty \\
\end{array} \)
\[ \begin{array}{c}
M_{[\tilde{h}_\infty]} = Y^R/C(\tilde{h}_\infty). 
\end{array} \] (4.7)

Denote the normal bundle of \( \tilde{e}_\infty \) by \( N_{\tilde{e}_\infty} \), and the normal bundle of \( \tilde{e}_\infty \) by \( N_{\tilde{e}_\infty} \) for simplicity. We have seen that over \( Y_{[\tilde{h}]} \) there is a bundle \( \mathcal{E}_{[\tilde{h}]} \). Denote its dual bundle by \( \mathcal{E}^*_ {[\tilde{h}]} \). Then we have
\[ \pi^* N_{\tilde{e}_\infty} \oplus \mathcal{E}^*_ {[\tilde{h}]} = N_{\tilde{e}_\infty}. \]

We also pull back the Chen–Ruan obstruction bundle to \( Y_{[\tilde{h}]} \). Take a point \( x \in Y^R \).

Then we get a point \([x] \in Y^R/C(\tilde{h}) \subseteq \Gamma M \) via the natural projection \( \pi : T^2_G(Y) \to \Gamma M \). The fiber of \( \tilde{\mathcal{O}}_Y^{|\tilde{h}|} \) at \((x, \tilde{h})\) is
\[ \mathcal{E}^*_ {[\tilde{h}]}(x, \tilde{h}) \oplus \bigoplus_{\lambda \in (\tilde{h})} \sum_{i=0}^{m} w_{\lambda, i} \cdot T_{x, \lambda} \]
where \( \bigoplus_{\lambda \in (\tilde{h})} T_{x, \lambda} \) is the irreducible decomposition of \( T_x X \) under the \( \langle \tilde{h} \rangle \)-action.

Recall that \( T_x X = T_{[x]} M \oplus \mathfrak{g}_C \), and by the computation in [23, Theorem 3.2], the obstruction bundle \( \tilde{\mathcal{O}}_G \) has fiber over \([x]\) being
\[ \bigoplus_{\lambda \in (\tilde{h}), \sum_{i=0}^{m} w_{\lambda, i} \geq 2} \sum_{i=0}^{m} w_{\lambda, i} - 1 \cdot T_{x, \lambda} \]
where \( \bigoplus_{\lambda \in (\tilde{h})} T_{x, \lambda} \) is the irreducible decomposition of \( T_{[x]} M \) under the \( \langle \tilde{h} \rangle \)-action.

One see that
\[ \mathcal{E}^*_ {[\tilde{h}]}(x, \tilde{h}) + \bigoplus_{\lambda \in (\tilde{h}), \sum_{i=0}^{m} w_{\lambda, i} \geq 2} \sum_{i=0}^{m} w_{\lambda, i} - 1 \cdot \mathfrak{g}_C, \lambda \]
forms a \( G \)-bundle over \( Y_{[\tilde{h}]} \), where \( \bigoplus_{\lambda \in (\tilde{h})} \mathfrak{g}_C, \lambda \) is the irreducible decomposition of \( \mathfrak{g}_C \) under the \( \langle \tilde{h} \rangle \)-action. We denote this bundle by \( V_{[\tilde{h}]} \).

Then over \( Y_{[\tilde{h}]} \) we have
\[ \tilde{\mathcal{O}}_Y^{|\tilde{h}|} = \pi^* \tilde{\mathcal{O}}_G \oplus V_{[\tilde{h}]} \]

Therefore
\[ (\beta_1 \ast_{cs} \beta_2)_{[h]} = p_{\infty, \ast} \left( e_{1, \ast}^* \pi^* \alpha_1 \wedge e_{2, \ast}^* \pi^* \alpha_2 \wedge \Theta_G (\pi^* N_{\tilde{e}_\infty}) \wedge \Theta_G (\mathcal{E}_{[\tilde{h}]}^*) \right) \]
\[ \wedge e_G(V_{[\tilde{h}]}) \wedge e_G(\pi^* \Theta_{[\tilde{h}]}) \wedge \text{vol}(\tilde{h}_\infty, \tilde{h}). \]

On the other hand, by the triangle in the commutative diagram (4.7) we have
\[ \pi^*(\alpha_1 \bigcup_{CR} \alpha_2)_{[\tilde{h}]^*} = p_{\infty, \ast} \left( p^* \left[ \tilde{e}_{\infty, \ast} \left( \tilde{e}_1^* \alpha_1 \wedge \tilde{e}_2^* \alpha_2 \cup e(\Theta_{CR}) \right) \right] \right) \wedge \text{vol}(\tilde{h}_\infty, \tilde{h}). \]

It is direct to see that \( e_1^* \pi^* \alpha_1 \wedge e_2^* \pi^* \alpha_2 \wedge \Theta_G(\pi^* N_{\tilde{e}_\infty}) \wedge e_G(\pi^* \Theta_{CR}) \) corresponds to the \( p^* \left[ \tilde{e}_{\infty, \ast} \left( \tilde{e}_1^* \alpha_1 \wedge \tilde{e}_2^* \alpha_2 \cup e(\Theta_{CR}) \right) \right] \). Therefore, the difference between \( \pi^*(\alpha_1 \bigcup_{CR} \alpha_2)_{[\tilde{h}]^*} \) and \( (\beta_1 \ast_{cs} \beta_2)_{[\tilde{h}]^*} \) is determined by \( \Theta_G(\mathfrak{c}_{[\tilde{h}]}) \vee e_G(V_{[\tilde{h}]}) \). When \( G \) is non-abelian, these two bundles have non-zero equivariant characteristics generally. Therefore \( \ast \) is not a ring homomorphism in general.

\textbf{Example 4.8.} When \( G = T \) is a torus,
- \( \pi^* \) preserves the shifted degree, since the \( T \)-action on its Lie algebra is trivial;
- \( p_{\infty} = id \) and \( e_{\infty} = \tilde{e}_{\infty} \) for the decomposition \( e_{\infty} = p_{\infty} \circ \tilde{e}_{\infty} \) (cf. (3.6));
- \( \Gamma^m \mathfrak{m} = \Gamma^m \mathfrak{m} \) and \( \bigcup_{CR} \) coincides with the original Chen–Ruan product \( \bigcup_{CR} \);
- the commutative diagram (4.7) reduces to

\[
\begin{array}{ccc}
Y^\tilde{h} \times_T T & \overset{e_{\infty}}{\longrightarrow} & Y^\tilde{h} = \bigcup_{CR} T \\
\downarrow{\pi} & & \downarrow{\pi} \\
Y^\tilde{h}/T & \overset{e_{\infty}}{\longrightarrow} & Y^\tilde{h}/T.
\end{array}
\]

Then we see that both \( \mathfrak{c}_{[\tilde{h}]} \) and \( V_{[\tilde{h}]} \) are zero bundle. Therefore, for this case \( \pi^* \) is a ring isomorphism between \( \mathcal{H}^*_{G, cs}(Y) \) and the Chen–Ruan cohomology ring \( H^*_{CR}(\mathfrak{m}) \). In fact, for this case, \( Y \) is a \( T \)-equivariant stable almost complex manifold, and \( \mathcal{H}^*_{G, cs}(Y) \) is the \( T \)-subring of the inertia cohomology. Then by combining with Theorem 4.6 we recover [22, Corollary 6.12].

\textbf{Remark 4.9.} Via the group isomorphism (4.6), we could transfer the product \( \ast_{cs} \) to \( H^*_{CR}(\mathfrak{m}) \). With this new ring structure, we get a surjective ring homomorphism
\[ \mathcal{H}^*_{G, cs}(X) \xrightarrow{\ast} \mathcal{H}^*_{G, cs}(Y) \xrightarrow{\cong} \left( H^*_{CR}(\mathfrak{m}), \ast_{cs} \right). \]

We view this as a Kirwan morphism for ECS-cohomology ring.

It is obvious that, different hamiltonian systems would have the same symplectic reduction orbifold. For example, Let \((X_i, \omega_i, G_i, \mu_i)\) be two hamiltonian systems for \( i = 1, 2 \), and \( 0 \in \mathfrak{g}_i^* \) be regular values of \( \mu_i \) for \( i = 1, 2 \). Suppose that the reduction symplectic orbifolds satisfy
\[ \left[ \mu_i^{-1}(0)/G_i \right] \cong \left[ \mu_2^{-1}(0)/G_2 \right] \cong \mathfrak{m}. \]

Then we get two ring structures over the Chen–Ruan cohomology group \( H^*_CR(\mathfrak{m}) \) via the group isomorphisms
\[ \mathcal{H}^*_{G_i, cs}(\mu_i^{-1}(0)) \cong H^*_CR(\mathfrak{m}), \quad \text{for} \quad i = 1, 2. \]

Then by the same computations in the proof of Proposition 4.7 we see that these two induced ring structure are not the same in general.
So, if a symplectic orbifold $M$ is a symplectic reduction of a hamiltonian system $(X, \omega, G, \mu)$, we could get infinite ring structures over its Chen–Ruan cohomology group by simply enlarging the hamiltonian system $(X, \omega, G, \mu)$ into

$$(X \times T^*H, \omega \oplus d\lambda, G \times H, \mu \oplus \mu_H)$$

for every connected compact Lie group $H$, where $(T^*H, d\lambda, H, \mu_H)$ is the canonical hamiltonian system associated to the cotangent bundle of $H$, and $\lambda$ is the Liouville form (cf. [7]).

Appendix A. Existence of equivariant volume form

In this appendix we show the existence of the equivariant volume form that we used in the definition of commutative stringy product. Let $G$ be a connected compact Lie group and $T$ be one of its maximal torus. Let $g$ and $t$ be their Lie algebra.

Proposition A.1. For any $h \in T$, there exists $\alpha \in t$ such that $C(h) = C(\alpha)$.

Corollary A.2. $G/C(h)$ is a Kähler manifold. Moreover, it has a $G$-equivariant volume form $\Omega^G_h$.

Proof. Since $G/C(\alpha)$ is a (co)-adjoint orbit, it is Kähler and the $G$-action on it is Hamiltonian. Denote its Kähler form by $\omega_\alpha$ and $\omega_\alpha + \mu_\alpha$ be its equivariant extension, where $\mu_\alpha$ is the moment map for the Hamiltonian $G$-action on $G/C(\alpha)$. Then $(\omega_\alpha + \mu_\alpha)^d$ is an equivariant volume form, where $d$ is the complex dimension of $G/C(\alpha)$.

Proof of Proposition A.1. Since $T$ is abelian, its adjoint action on $g$ induces a splitting

$g = t \oplus C_1 \oplus \cdots \oplus C_k$

and the action is given by $1 \oplus \phi_1 \oplus \cdots \oplus \phi_k$. Let $T_I = \{t \in T | \phi_i(t) = 1\}$, and for any $I \subseteq \{1, \ldots, k\}$, set

$$T_I = \bigcap_{i \in I} T_i$$

and $T_\emptyset = T$. Then $T$ is stratified by

$$T'_I = T_I \setminus \bigcup_{J \supseteq I} T_J.$$

Similarly, let $t_I$ be the Lie algebra of $T_I$ and $t'_I$ forms a stratification of $t$.

Suppose that $h \in T'_I$, then choose an $\alpha \in t'_I$. It is sufficient to show that $c(h) = c(\alpha)$ which implies that $C(h) = C(\alpha)$, since both $C(h)$ and $C(\alpha)$ are connected subgroup of $G$ (cf. [24, Corollary 2, §3.1]). In fact,

$$c(h) = \{\xi \mid h\xi h^{-1} = \xi\}$$

Since $h \in T'_I$, it fixes $t$ and all $C_i$ for $i \in I$. Hence $c(h) = t \oplus \bigoplus_{i \in I} C_i$. Similarly, $c(\alpha)$ is the same space.

Theorem A.3. Suppose that $h_1, h_2 \in T$ and $\vec{h} = (h_1, h_2)$. The fibration $p_1 : G/C(h_1, h_2) \to G/C(h_1)$ admits a fiber-wise $G$-equivariant volume form $\text{vol}(\vec{h}, h_1)$, so $p_1^*(\text{vol}(\vec{h}, h_1)) = 1$. 


Proof. The fiber of $p_1 : G/C(h_1, h_2) \to G/C(h_1)$ is $C(h_1)/C(h_1, h_2)$. Set $K = C(h_1)$. Since $h_1$ and $h_2$ commutes, $h_2 \in K$. Hence $C(h_1, h_2) = C_K(h_2)$ and $C(h_1)/C(h_1, h_2) = K/C_K(h_2)$.

By Corollary A.2, there exists a $K$-equivariant volume form $\Omega^K_{h_2}$ on $K/C_K(h_2)$. Note that

$$G/C(h_1, h_2) = K/C_K(h_2) \times_K G.$$ 

For each point $[h] \in G/C(h_1)$, write the fiber over $[h]$ by $F_{[h]}$. Then by setting $\text{vol}(\tilde{h}, h_1)|_{F_{[h]}} = h^*\Omega^K_{h_2}$ for each point $[h] \in G/C(h_1)$ we get this volume form $\text{vol}([h], h_1)$. We can also get this volume form $\text{vol}(\tilde{h}, h_1)$ via the canonical isomorphism

$$H^*_G(G/C(h_1, h_2)) = H^*_G(K/C_K(h_2) \times_K G) \cong H^*_K(K/C_K(h_2)).$$

By induction on the length of $\tilde{h} = (h_1, \ldots, h_n)$, this theorem could be generalized to $G/C(\tilde{h}) \to G/C(\tilde{h}_{i_1}, \ldots, i_k)$ and $G/C(\tilde{h}) \to G/C(\tilde{h}_\infty)$. Therefore all kinds of fibrations used in the definition of equivariant commutative stringy products have fiber-wise $G$-equivariant volume forms.

References

[1] A. Adem, F. R. Cohen, J. M. Gómez, Commuting elements in central products of special unitary groups, Proc. Edinb. Math. Soc., 56 (2013), No. 1, 1–12.
[2] A. Adem, F. R. Cohen, E. Torres-Giese, Commuting elements, simplicial spaces, and filtrations of classifying spaces, Math. Proc. Cambridge Philos. Soc., 152 (2012), No. 1, 91–114.
[3] A. Adem, J. Gómez, On the structure of spaces of commuting elements in compact Lie groups, in Configuration Spaces, 1–26, A. Bjorner, F. Cohen, C. De Concini, C. Procesi and M. Salvetti, eds., Springer, 2012.
[4] A. Adem, J. Gómez, A classifying space for commutativity in Lie groups, Algebr. Geom. Topol., 15 (2015), 493–535.
[5] A. Adem, J. Leida, Y. Ruan, Orbifolds and stringy topology. Cambridge Tracts in Mathematics 171, Cambridge University Press, 2007.
[6] E. Becerra, B. Uribe, Stringy product on twisted orbifold $K$-theory for abelian quotients, Trans. A. M. S., 361 (2009), No. 11, 5781–5803.
[7] A. Cannas da Silva, Lectures on symplectic geometry, Lecture Notes in Mathematics 1764, Springer-Verlag Berlin Heidelberg, 2008.
[8] K. Cieliebak, A. Gaio, D. Salamon, J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions, IMRN, 10 (2000), 831–882.
[9] K. Cieliebak, A. Gaio, I. Mundet i Riera, D.A. Salamon, The symplectic vortex equations and invariants of Hamiltonian group actions, J. Symplectic Geom., 1 (2002), No. 3, 543–645.
[10] B. Chen, S. Hu, A de Rham model of Chen–Ruan cohomology ring of abelian orbifolds, Math. Ann. 336 (2006), No. 1, 51–71.
[11] W. Chen, Y. Ruan, Orbifold Gromov–Witten theory, Cont. Math., 310 (2002), 25–85.
[12] W. Chen, Y. Ruan, A new cohomology theory for orbifold, Commun. Math. Phys. 248 (2004), 1–31.
[13] B. Chen, B.-L. Wang, R. Wang, $L^2$-moduli spaces of symplectic vortices on Riemann surfaces with cylindrical ends, arXiv:1405.6387, 2014.
[14] B. Chen, B.-L. Wang, R. Wang, $L^2$-symplectic vortices and Hamiltonian Gromov–Witten invariants, in preparation.
[15] B. Chen, B.-L. Wang, R. Wang, Augmented symplectic vortices, Hamiltonian equivariant Gromov–Witten invariants and quantization of Kirwan morphisms, in preparation.
[16] C.-Y. Du, T.-Y. Li, Chen–Ruan cohomology and stringy orbifold $K$-theory for stable almost complex orbifolds, preprint 2017.
[17] D. Edidin, T. J. Jarvis, T. Kimura, Logarithmic trace and orbifold products, Duke Math. J., 153 (2010), No. 3, 427–473.
[18] B. Fantechi, L. Gottsche, Orbifold cohomology for global quotients, Duke Math. J., 117 (2003), No. 2, 197-227.
[19] V. Guillemin, V. Ginzburg, Y. Karshon, Moment maps, cobordisms, and Hamiltonian group actions, AMS, Providence, RI, 2002.
[20] A. Gaio, D. Salamon, Gromov–Witten invariants of symplectic quotients and adiabatic limits. J. Symplectic Geom., 3 (2005), No. 1, 55–159.
[21] V. W. Guillemin, S. Sternberg, Supersymmetry and Equivariant de Rham Theory, Springer Science & Business Media, 1999.
[22] R. Goldin, T. S. Holm, A. Kautzoon, Orbifold cohomology of torus quotients, Duke Math. J., 139 (2007), No. 1, 89-139.
[23] J. Hu, B.-L. Wang, Delocalized Chern character for stringy orbifold K-theory. Trans. Amer. Math. Soc. 365 (2013), 6309–6341.
[24] W.-Y. Hsiang, Lectures on Lie groups, World Scientific Publishing Co. Pte. Ltd., 1998.
[25] T. J. Jarvis, R. Kaufmann, T. Kimura, Stringy K-theory and the Chern character, Invent. Math., 168 (2007), 23–81.
[26] T.-Y. Li, B. Chen, Hamiltonian Gromov–Witten Invariants on C^{n+1} with S^1-action, Acta Math. Sin. (Engl. Ser.), 32 (2016), No. 3, 309–330.
[27] I. Mundet i Riera, A Hitchin–Kobayashi correspondence for Kahler fibrations, J. Reine Angew. Math., 528 (2000), 41–80.
[28] I. Mundet i Riera, Hamiltonian Gromov–Witten invariants, Topology, 42 (2003), 525–553.
[29] D. McDuff, D. Salamon, J-holomorphic curves and symplectic topology, Colloquium Publications, vol. 52, Amer. Math. Soc., Providence, RI, 2004.
[30] D. Quillen, Elementary proof of some results of cobordism theory using Steenrod operators, Adv. Math., 7 (1971), 29–56.

School of Mathematics and Yangtze Center of Mathematics, Sichuan University, Chengdu 610065, China
E-mail address: bohui@cs.wisc.edu

School of Mathematics, Sichuan Normal University, Chengdu 610068, China
E-mail address: cyd9966@hotmail.com

School of Mathematics, Chongqing Normal University, Chongqing 401331, China
E-mail address: litiyao@ sina.com