EPTAS for stable allocations in matching games
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Abstract
Gale-Shapley introduced a matching problem between two sets of agents where each agent on one side has a preference over the agents of the other side and proved algorithmically the existence of a pairwise stable matching (i.e. no uncoupled pair can be better off by matching). Shapley-Shubik, Demange-Gale, and many others extended the model by allowing monetary transfers. In this paper, we study an extension [18] where matched couples obtain their payoffs as the outcome of a strategic game and more particularly a solution concept that combines Gale-Shapley pairwise stability with a constrained Nash equilibrium notion (no player can increase its payoff by playing a different strategy without violating the participation constraint of the partner). Whenever all couples play zero-sum matrix games, strictly competitive bi-matrix games, or infinitely repeated bi-matrix games, we can prove that a modification of some algorithms in [18] converge to an \( \varepsilon \)-stable allocation in at most \( O(\frac{1}{\varepsilon}) \) steps where each step is polynomial (linear with respect to the number of players and polynomial of degree at most 5 with respect to the number of pure actions per player).

Keywords. EPTAS · Stable Matching · Generalized Nash Equilibrium · Zero-sum Games · Infinitely repeated Games · Matching with Transfer

1 Introduction
The Gale and Shapley [16] two-sided market matching problem consists in finding a “stable” assignment between two different sets of agents \( M \) and \( W \) given that each agent on one side has an exogenous preference ordering over the agents of the other side. The marriage problem focuses on a coupling \( \mu \) that associates to each agent on one side at most one agent on the other side. The coupling \( \mu \) is stable if no uncoupled pair of agents \((m, w) \in M \times W\), both prefer to be matched rather than staying with their partners in \( \mu \). Gale and Shapley [16] used a “propose-dispose” algorithm to prove the existence of a stable matching. Roth and Vande Vate [37] studied a random process to find a stable matching from some arbitrary matching. Ma [29] proved that the Roth and Vande Vate algorithm does not find all the stable matchings. Recently, Dworczak [15] introduced a new class of algorithms called deferred acceptance with compensation chains algorithms (DACC)¹ in which both sides of the market can make offers and proved that a matching is stable if and only if it is the outcome of some DACC algorithm.

Knuth [26] proved a lattice structure over the set of stable matchings (briefly mentioned in Gale and Shapley [16]). Later, Gale and Sotomayor [17, 36] showed that the algorithm in which men are proposing and women are disposing outputs the best stable matching for men.

Balinski and Ratier [7] proposed an elegant directed graph approach to the stable marriage problem and characterized the stable matching polytope in the one-to-one setting through linear inequalities, proving that any feasible point of the polytope is a stable matching and vice versa. Baiou and Balinski continued the approach and characterized in [5] the polytope of stable matching in the one-to-many problem. Later, they extended the many-to-many model to allocation problems with divisible capacities [6].

¹A related class of algorithms was introduced by McVitie and Wilson [30] in 1971.
Shapley and Shubik \cite{ShapleyShubik1971} extended the model by allowing monetary transfers. Their model consists of a housing market with buyers and sellers, where each seller has a house to sell and each buyer is interested in buying a house. A solution to this problem is a pair \((\mu, p)\), with \(\mu\) a matching between sellers and buyers, and \(p\) a vector of positive monetary transfers from buyers to sellers for each house. Each seller \(i \in S\), has a cost of his/her house \(c_i\), and each buyer \(j \in B\), has a valuation \(h_{i,j}\) for \(j\)'s house. If seller \(i\) sells his/her house to \(j\) at price \(p_i \geq 0\), their payoffs are \(U_{i,j}(p_i) = p_i - c_i\) and \(V_{i,j}(p_i) = h_{i,j} - p_i\), respectively. Demange and Gale \cite{DemangeGale1981} generalized the problem as follows. If two agents \(i, j\) are paired, their payoffs are given by some strictly increasing and continuous payoff functions \(\phi_{i,j}(t)\) for \(i\), and \(\psi_{i,j}(-t)\) for \(j\), with \(t \in \mathbb{R}\) being the net transfer from \(i\) to \(j\) \((t \geq 0\) means that \(j\) pays \(t\) to \(i\) and \(t \leq 0\) means that \(i\) pays \(-t\) to \(j\)). Hatfield and Milgrom \cite{HatfieldMilgrom2005} extended the model by allowing couples to sign contracts.

In this article we will study the complexity of the stable matching game introduced in \cite{DemangeGale1981} which proposes to go further on the extension by supposing that members of a couple \((i, j)\) \(\in M \times W\) obtain their payoffs as the output of a strategic game \(G_{i,j} = (X_i, Y_j, U_{i,j}, V_{i,j})\), that they have to play, where \(X_i\) is \(i\)'s action/strategy set, \(Y_j\) is \(j\)'s action/strategy set, and \(U_{i,j}, V_{i,j} : X_i \times Y_j \to \mathbb{R}\) are the utility functions of \(i\) and \(j\), respectively. Hence, if man \(i\) and women \(j\) are married and if \(i\) chooses to play \(x_i\) and \(j\) chooses to play \(y_j\), \(i\)'s final utility is \(U_{i,j}(x_i, y_j)\) and \(j\)'s final utility is \(V_{i,j}(x_i, y_j)\). An outcome of the matching game, called a matching profile, is a triple \((\mu, x, y)\) with \(\mu\) a matching between \(M\) and \(W\), \(x = (x_i)_{i \in M} \in \prod_{i \in M} X_i\) a men’s strategy profile and \(y = (y_j)_{j \in W} \in \prod_{j \in W} Y_j\) a women’s strategy profile. For example, the Demange-Gale model above can be mapped into a matching game in which all couples play a strictly competitive game by letting \(G_{i,j} = (S, T, U_{i,j}, V_{i,j})\), with \(S = T = \mathbb{R}_+\), \(U_{i,j}(s, t) = \phi_{i,j}(t - s)\) and \(V_{i,j}(s, t) = \psi_{i,j}(s - t)\).

Stable matching is one of the most productive and applied models in game theory. Its considerable success earned Roth and Shapley the Nobel Prize in 2012. Stable matching games model allows expanding the range of applicability. For example, to attract a worker a firm can, in addition to the salary and the bonus, offer medical care insurance, childcare assistance, and a flexible schedule. When a university wants to recruit a professor, it can reduce the teaching duties, require a minimum number of top publications, or ask for some responsibilities in the department. The professor can try to publish in 4* journals, promise to be an excellent teacher, agree to apply to an EU grant, and accept to head the department. All those actions are individual decisions that can be put explicitly or implicitly in a contract but each agent is responsible for the actions he/she controls.

The first solution concept studied is applicable when a matched couple \((i, j)\) cannot commit on the agreed action profile before they play their game. In this case, for the players not to deviate from the intended actions, they must form a Nash equilibrium of \(G_{i,j}\). A matching profile \((\mu, x, y)\) (an allocation) is Nash stable if (a) all matched couples play a Nash equilibrium of their game \((b)\) no pair of uncoupled agents \((i', j')\) can jointly deviate to some Nash strategy profile \((x_{i'}, y_{j'})\) in their game \(G_{i',j'}\) that Pareto improves their payoffs and \((c)\) all players receive at least their individually rational payoff. In \cite{DemangeGale1981}, it has been proved algorithmically that Nash stable allocations exist as soon as all games \(\{G_{i,j}\}_{i \in M, j \in W}\) have non-empty and compact sets of Nash equilibria. As their algorithm necessitates the knowledge of the whole set of Nash equilibria of the games \(\{G_{i,j}\}_{i \in M, j \in W}\) and optimizes over that set, computing Nash stable allocations is NP-hard. Consequently, the actual paper focuses on the complexity of the second, and perhaps more interesting, solution concept.

In Demange-Gale’s model viewed as a Matching game (see above), Nash stability condition (a) above implies that all transfers are zero: \(x_i = y_j = 0\) for all \(i\) and \(j\), and pairwise stability conditions \((b)+(c)\) imply that the Nash stable outcomes are exactly the Gale-Shapley stable solutions of the game without transfers. For positive transfers to be sustainable, players should be able to commit:
an assumption explicitly or implicitly made in all matching with transfer or contract models in the literature.

Hence, the second solution concept we will consider corresponds to situations, like with monetary transfers, in which players can commit either by signing binding contracts or because the game is repeated and so any deviation from the agreed strategy profile at some stage is punished the next stage by the break of the relation. In that context, a matching profile \((\mu, x, y)\) is called externally stable if each player gets at least its individual rational payoff and, no pair of unmatched agents \((i', j')\) can jointly deviate to some strategy profile \((x'_i, y'_j)\) in their game \(G_{i', j'}\) that Pareto improves their payoffs. This is a natural extension of Gale-Shapley pairwise stability. A “propose-dispose” algorithm, inspired by the one in Demange, Gale and Sotomayor [14], allows in [18] to prove that, if all the strategic games \(G_{i,j}\), \((i, j) \in M \times W\) have compact strategy spaces and continuous payoff functions, the matching game admits an externally stable matching profile. In the actual paper, we show that, in general, the optimization problems involved in this algorithm are NP-hard. Happily, we can show that whenever all the games \(\{G_{i,j}\}_{i \in M, j \in W}\) are bi-matrix zero-sum games, bi-matrix strictly competitive games, or infinitely repeated bi-matrix games, every iteration of the propose-dispose algorithm is polynomial. Given that the algorithm converges in \(O(\frac{1}{\varepsilon})\) number of iterations before converging to an \(\varepsilon\)-externally stable allocation, we conclude that for these three classes, the propose-dispose algorithm is an efficient PTAS [10] (usually denoted EPTAS).

To refine external stability, a second condition is added. It is implied by the fact that the players must be optimally choosing their strategies subject to their partner’s acceptance to participate. Hence, a matching profile \((\mu, x, y)\) is internally stable if no player has a profitable deviation that preserves the stability of his/her couple (i.e. any profitable deviation decreases the partner’s payoff below his/her market outside option). Said differently, for every matched couple \((i, j)\), fixing \(y_j\), \(x_i\) maximizes \(i\)’s payoff under the constraint of participation by \(j\), and vice-versa.

We introduce an \(\varepsilon\)-version of the “strategy-profile modification” algorithm studied in [18] and prove that this new version converges to an \(\varepsilon\)-externally-internally stable matching in at most \(O(\frac{1}{\varepsilon})\) steps whenever all the games \(G_{i,j}\) are bi-matrix zero-sum, bi-matrix strictly competitive games or infinitely repeated bi-matrix games. Moreover, for the same class of games, every iteration of the algorithm is polynomial.

There are recent but few stable matching papers where, before the pairing takes place, players can alter their preferences. Hatfield and Milgrom [20] in 2005, studied the case of matching with contracts, a model that can formally be mapped to a matching game. A bi-level problem called the investment and competitive matching problem was introduced in 2015 by Nöldeke and Samuelson [32] where agents first make investments and then are matched into potentially productive partnerships. In their model, once an investment is made, it cannot be changed or be made dependent on the identity of the partner. For example, in the hope of being assigned later to a good university, a student can invest his free time to work hard to obtain good grades, and a university can invest in its reputation and the quality of its programs. Thus, Nöldeke and Samuelson’s model is complementary to the model of matching games as some investments can be fixed before matching takes place and some (as in the case of matching games) can be decided during the matching process, changed, and be dependent on the identity of the partner.

The main tool we use to show that the algorithms in [18] and the new variant introduced here are polynomial is the reduction to a polynomial number of linear programs [11] which are known to be solvable in polynomial time. The first polynomial algorithms were published by Khachiyan [25] and Karmarkar [23]. We will refer to the complexity result of Vaidya [39], who designed an algorithm for solving linear programming problems that takes \(O((n + m)^{1.5}nL)\) elementary operations in the worst case, where \(m\) is the number of constraints, \(n\) is the number of variables, and \(L\) is the number of bits used for encoding the data of the problem. The main tool we use to show the NP-hardness
of the algorithms is the reduction to quadratically constrained quadratic programming (QCQP) \cite{2} \cite{12} \cite{28} or to the Nash uniqueness problem \cite{19}.

The paper is structured as follows. Section 2 introduces the matching game model, gives several examples, describes the two main solution concepts, and states the two algorithms. Section 3 proves the convergence of the algorithms in at most $O(\frac{1}{\varepsilon})$ number of steps to an $\varepsilon$-approximation and show that the complexity of each iteration is polynomial when all the games are zero-sum bi-matrix games. Section 4 and Section 5 does the same for strictly competitive bi-matrix games and infinitely repeated bi-matrix games, respectively. Section 6 concludes. The appendix includes a numerical example of the algorithms exposed in the article.

## 2 Stable matching model

A matching game consists in two finite set of agents, $M$ and $W$, such that for any $(i,j) \in M \times W$, the pair can play a two-person strategic game $G_{i,j} := (X_i, Y_j, U_{i,j}, V_{i,j})$. For simplicity, agents in $M$ will be called men and agents in $W$ will be called women. In addition, men will be indexed by $i \in M$, while women by $j \in W$.

**Definition 2.1.** In this setting, a matching profile (or an allocation) consists in a triple $\pi = (\mu, x, y)$, where $\mu$ is a correspondence between the agents of $M$ and $W$ in which each agent in one side is associated to at most one agent in the other side, $x$ is a strategy profile for all the men in $M$, and $y$ is a strategy profile for all the women in $W$.

Given a matching profile $\pi = (\mu, x, y)$, matched players’ utilities are computed as the payoff that each of them obtains in their game. Formally, given a couple $(i,j) \in \mu$ that plays a strategy profile $(x_i, y_j)$, their utilities correspond to $u_i(\pi) := U_{i,j}(x_i, y_j)$ and $v_j(\pi) := V_{i,j}(x_i, y_j)$.

Players are endowed with individually rational payoffs (IRPs), as they may prefer to remain unmatched or no partner may want to match with them. Formally, an agent $i \in M$ (resp. $j \in W$) will accept to be matched if and only if his utility is at least $u_i(\pi) := U_{i,j}(x_i, y_j)$ and $v_j(\pi) := V_{i,j}(x_i, y_j)$.

**Example 1.** Consider a matching game with only one man $i$ and one woman $j$, both having strictly positive IRPs $u_i = v_j = \delta > 0$. Suppose that if they agree to match, they play a constant-sum game $G_{i,j} = (\mathbb{R}_+, \mathbb{R}_+, U_{i,j}, V_{i,j})$ with $U_{i,j}(x_i, y_j) = 10\delta - x_i + y_j$, $V_{i,j}(x_i, y_j) = x_i - y_j$, for $x_i, y_j \geq 0$. The utility function of the man represents his basic utility of $10\delta$ from being married with $j$ minus the money $x_i$ he pays plus the money $y_j$ he receives. The utility of the women is the money transfer she gets $x_i$ minus the money transfer she pays $y_j$ (her basic utility is normalized to zero). Notice that, if the players intend to play $x_i = y_j = 0$ (no transfer), $j$’s utility, if she agrees to marry, is $0$ which is strictly smaller than her IRP $\delta$. Therefore, her rational decision should be to refuse to marry unless $x_i - y_j \geq \delta$.

The scope of this article is to study the complexity of the algorithms designed in \cite{18} to find a stable allocation of a matching game (formally defined below). To do so, our paper will focus on bi-matrix matching games, i.e. the setting in which every potential couple plays a finite game in mixed strategies or an infinitely repeated version of a bi-matrix game. Formally, for every couple
\((i,j) \in M \times W\), their strategy sets will be simplex \(X_i := \Delta(S_i), Y_j := \Delta(T_j)\), with \(S_i, T_j\) finite sets. In addition, for any strategy profile \((x,y) \in X_i \times Y_j\),
\[
U_{i,j}(x,y) := xA_{i,j}y = \sum_{s \in S_i} \sum_{t \in T_j} A_{i,j}(s,t)x_s y_t
\]
\[
V_{i,j}(x,y) := xB_{i,j}y = \sum_{s \in S_i} \sum_{t \in T_j} B_{i,j}(s,t)x_s y_t
\]
where \(A_{i,j}, B_{i,j} \in \mathbb{R}^{\left|S_i\right| \times \left|T_j\right|}\) are payoff matrices. Here are two examples.

**Example 2.** Consider a matching game with only one man \(i\) and one woman \(j\), both having strictly positive IRPs \(u_i = v_j = \delta > 0\). Suppose that if they agree to match, they play the following prisoners’ dilemma \(G_{i,j}\),

|     | C     | B     |
|-----|-------|-------|
| C   | 2\delta, 2\delta | 0, 3\delta |
| B   | 3\delta, 0       | -\delta, -\delta |

Table 1: Prisoners’ dilemma

Remark that, as the Nash equilibrium of \(G\) gives to the agents less than their IRPs, if they agree to match, they must cooperate with positive probability.

**Example 3.** As in the example just above, suppose that if \(i\) and \(j\) agree to match, they play the following coordination game,

|     | A     | B     |
|-----|-------|-------|
| A   | 4\delta, \delta/2 | 0, 0  |
| B   | 0, 0       | \delta/2, 4\delta |

Table 2: Coordination game

Remark that none of the pure Nash equilibria can be the outcome of an agreement as at least one of the players is worst off compared to being single.

The good notion of stability depends on the players’ ability to commit: in the first solution concept, players cannot be forced to respect the agreed actions. In the second one, players can commit (e.g. by signing binding contracts or because the game is infinitely repeated and so any deviation at some stage from the agreed stationary strategy profile can be immediately punished by a break of the relation).

### 2.1 Model without commitment: Nash stability

Suppose \(i\) and \(j\) agree to match with the promise to play actions \(x_i\) and \(y_j\) respectively. If no specific contract forces them to respect that agreement then, for \((x_i, y_j)\) to be stable, it must constitute a Nash equilibrium of \(G_{i,j}\).

**Definition 2.2.** A matching profile \(\pi = (\mu, x, y)\) is **internally Nash stable** if for any matched couple \((i,j) \in \mu\), \((x_i, y_j)\) is a Nash equilibrium of \(G_{i,j}\), i.e. \((x_i, y_j) \in N.E(G_{i,j})\).
Since each player can remain single or be coupled with a better partner, a pairwise stability condition à la Gale-Shapley must also be satisfied for a matching profile to be sustainable.

**Definition 2.3.** A matching profile \( \pi = (\mu, x, y) \) is **externally Nash stable** if,

(a) For any \( i \in M \) and \( j \in W \), \( u_i(\pi) \geq u_{i'} \) and \( v_j(\pi) \geq v_{j'} \),

(b) There is no \((i,j)\) \( \in M \times W \), not matched between them by \( \mu \), and no Nash equilibrium \((x'_i, y'_j) \in N.E(G_{i,j})\) such that \( U_{i,j}(x'_i, y'_j) > u_i(\pi) \) and \( V_{i,j}(x'_i, y'_j) > v_j(\pi) \).

A matching profile that is externally and internally Nash stable is called **Nash stable**. The existence of Nash stable allocations has been proved algorithmically in \cite{IS} for any setting in which the Nash equilibrium sets of all the games \( G_{i,j} \) are non-empty and compact (which is the case for bi-matrix matching games).

**Example 1.** Let \( \pi \) be a matching profile in Example 1. The only Nash equilibrium of the constant-sum game \( G_{i,j} \) is \( x_i = y_j = 0 \), as any transfer \( x_i > 0 \) (resp. \( y_j > 0 \)) is a strictly dominated strategy for \( i \) (resp. for \( j \)). Thus, if \( \pi \) is internally Nash stable and players \( i \) and \( j \) are matched, their payoffs are \( u_i(\pi) = 10\delta \) and \( v_j(\pi) = 0 \). However, \( j \) prefers to be single and getting \( \delta > 0 \). Therefore, \( \pi \) is Nash stable if and only if the players remain single.

Unfortunately, the algorithm for computing Nash stable allocations in \cite{IS} necessitates the knowledge of all Nash equilibria of all the finite games \( G_{i,j} \), which is a hard problem. Indeed, knowing all mixed Nash equilibria of a finite game allows to solve the uniqueness Nash equilibrium problem, proved to be NP-hard \cite{VAA, VAA2}. Consequently, the rest of the article focuses on the model with commitment, which happens to be more interesting mathematically and conceptually as it extends and refines the most known models with transfers and contracts.

A numerical example of computing Nash stable allocations in a matching game with transfers, as Example 1 using the algorithm designed in \cite{IS}, can be found in Appendix A.

### 2.2 Model with commitment: External-internal stability

Suppose that partners within a couple can commit to a specific action profile before playing their game. This allows them to enlarge their set of agreements (or feasible contracts) well beyond their set of Nash equilibria, leading naturally to the following stability notion.

**Definition 2.4.** A matching profile \( \pi = (\mu, x, y) \) is **externally stable** if,

(a) For any \( i \in M \) and \( j \in W \), \( u_i(\pi) \geq u_{i'} \) and \( v_j(\pi) \geq v_{j'} \),

(b) There is no \((i,j)\) \( \in M \times W \), not matched by \( \mu \), and no \((x'_i, y'_j) \in X_i \times Y_j\) such that \( U_{i,j}(x'_i, y'_j) > u_i(\pi) \) and \( V_{i,j}(x'_i, y'_j) > v_j(\pi) \).

Compared to Nash stability, players can now choose any feasible strategy profile.

**Definition 2.5.** An **externally stable** matching profile \( \pi = (\mu, x, y) \) is **internally stable** if for any couple \((i,j)\) matched by \( \mu \) and any \((s_i, t_j) \in X_i \times Y_j\), it holds,

(a) If \( U_{i,j}(s_i, y_j) > u_i(\pi) \) then, \((\mu, ((x'_{i'})_{i' \neq i}, s_i), y)\) is not externally stable,

(b) If \( V_{i,j}(x_i, t_j) > v_j(\pi) \) then, \((\mu, x, ((y'_{j'})_{j' \neq j}, t_j))\) is not externally stable.

When a matching profile is externally and internally stable, it will be called **stable**.
Example 1. Comeback again to the first leading example. External stability implies that players match and \( \delta \leq x_i - y_j \leq 9\delta \). If \( x_i - y_j > \delta \), decreasing slightly \( x_i \) increases \( i \)'s payoff and does not violate the participation constraint of \( j \). Thus, internal stability implies that \( x_i - y_j = \delta \). If \( y_j > 0 \), decreasing slightly \( y_j \) increases \( j \)'s payoff without violating the participation constraint of \( i \). Thus, a matching profile is externally and internally stable if and only if \( i \) and \( j \) agree to match, \( x_i = \delta \) and \( y_j = 0 \). This “intuitive” solution refines the continuum set of predictions of Shapley-Shubik and Demange-Gale (players match and \( x_i - y_j \in [\delta, 9\delta] \)).

Example 2. Consider again the prisoners’ dilemma matching game example. The only symmetric externally-internally stable allocation is the one in which agents are matched and cooperate with probability \( 1/2 \). Indeed, both receive exactly \( \delta \) as a payoff and any profitable deviation decreases the partner’s payoff below his/her IRP, breaking the stability of the couple.

Example 3. Consider again the cooperation matching game example. We find two symmetric externally and internally stable allocations, both of them with the players matched. In the first one each player plays the most preferred option with probability \( 2/3 \) while in the second one, each player plays the least preferred option with probability \( 2/3 \). In both cases, both agents receive exactly \( \delta \) as a payoff and any profitable deviation decreases the partner’s payoff below his/her IRP, breaking the couple.

Example 4. Consider a market with \( n \) buyers who can commit to a monetary transfer and one seller of an indivisible good. Buyer \( i = 1, ..., n \) has a value \( v_i \) for the good, and the seller the reservation price \( c \). If the seller accepts to contract with \( i \) at price \( p_i \), \( i \)'s utility is \( v_i - p_i \) and the seller’s utility is \( p_i - c \). Suppose \( v_1 > v_2 \geq ... \geq v_n \). If \( c > v_1 \) the unique stable allocation is when all the players remain single (given a price and a buyer, either the seller refuses to sell or the buyer refuses to buy). If \( v_1 \geq c \), external stability implies that the good is sold to the buyer 1 for a price \( p_1 \in [\max(c, v_2), v_1] \). In this continuum, the unique internally stable allocation is when \( p_1 = \max(c, v_2) \), which corresponds to the outcome of the second price auction.

As for Nash stable allocations, the existence of stable matching profiles is also guaranteed under certain assumptions [18]. Its proof uses two algorithms: a “propose-dispose” algorithm for computing an externally stable matching profile and then, a “strategy profiles modification” procedure for making it also internally stable. Notice, however, that the propose-dispose algorithm in [18] computes an \( \varepsilon \)-approximation of an externally stable allocation (Definition 2.6) while the second algorithm in [18] starts from an exact externally matching profile and assumes that there is an oracle that can compute an exact constrained Nash equilibrium. Such an oracle does not always exist and at best it can be obtained an \( \varepsilon \)-constrained Nash equilibrium. Hence, in this article, we will extend the analysis in [18] by defining an \( \varepsilon \)-strategy profiles modification version of the algorithm. This is necessary to guarantee a finite-time convergence and a polynomial-time approximation in several interesting classes of games.

The rest of this section introduces the two main algorithms and discusses their correctness and finiteness. The following sections prove their convergence and complexity when all couples play zero-sum games, strictly competitive games, or infinitely repeated matching games.

2.3 Algorithm 1 Propose-dispose

Consider a bi-matrix matching game \( \Gamma = (M_0, W_0, (G_{i,j} : i, j \in M \times W), u, v) \). First of all, we state the definition of an \( \varepsilon \)-externally stable matching profile.
Definition 2.6. A matching profile \( \pi = (\mu, x, y) \) is \( \varepsilon \)-externally stable if there is no \( (i, j) \in M_0 \times W_0 \), not matched by \( \mu \), and no \( (x'_i, y'_j) \in X_i \times Y_j \), such that \( U_{i,j}(x'_i, y'_j) > u_i(\pi) + \varepsilon \) and \( V_{i,j}(x'_i, y'_j) > v_j(\pi) + \varepsilon \).

One by one, agents at one side of the market propose to the agents on the other side an acceptable strategy profile to be played in their common game. If the proposed agent is unmatched, the proposal is automatically accepted and a new proposer is picked. If the proposed agent is matched, there is a competition between the proposer and the current partner.

Algorithm 1: \( \varepsilon \)-Externally stable allocation computation

\begin{algorithm}
\begin{itemize}
    \item \textbf{Proposal phase.} Let \( i \in M' \) be a proposer man. Given the current matching profile \( \pi \) (initially empty), the one generates a women’s payoff vector \( v(\pi) = (v_j(\pi))_{j \in W} \), \( i \) computes his optimal proposal as,
    \[
    (j, x, y) \in \text{argmax} \{U_{i,j}(x, y) : V_{i,j}(x, y) \geq v_j + \varepsilon, j \in W_0, (x, y) \in X_i \times Y_j\}
    \]  
    Problem (1) is always feasible as \( i \) can always propose to \( j_0 \). If \( j \) is single, \( i \) is automatically accepted and the algorithm picks a new proposer in \( M' \).
    \item \textbf{Competition phase.} If the proposed woman \( j \) is matched, namely with a man \( i' \), a competition between \( i \) and \( i' \) starts. In the stable marriage problem, the competition is the simple comparison between the places that \( i \) and \( i' \) occupy in \( j \)’s ranking. In the case of matching games, as agents have strategies, a competition is similar to a second-price auction. Let \( \beta_i, \beta' \) be the lowest payoff that \( i \) and \( i' \) are willing to accept for being with \( j \) (their reservation prices). \( i \)'s bid \( \lambda_i \) (and analogously the one of \( i' \)) is computed by,
    \[
    \lambda_i := \text{max} \{V_{i,j}(x, y) : U_{i,j}(x, y) \geq \beta_i, (x, y) \in X_i \times Y_j\}
    \]
    Then, the winner is the one with the highest bid. For ending the competition, the winner, namely \( i \), decreases his bid until matching the one of the loser. Formally, \( i \) solves,
    \[
    \text{max} \{U_{i,j}(x, y) : V_{i,j}(x, y) \geq \lambda_{i'}, (x, y) \in X_i \times Y_j\}
    \]
    The loser is included in \( M' \) and a new proposer is chosen.
\end{itemize}
\end{algorithm}

For a numerical example of Algorithm 1 check Appendix A

Theorem 2.1. Algorithm 1 outputs an \( \varepsilon \)-externally stable matching profile.
The theorem has been proved in [18]. The next proposition bounds the number of iterations of Algorithm 1. For every woman \( j \in W \), consider the difference between her best possible payoff and her individually rational payoff, plus the highest of all these values:

\[
V_j := \max\{B_{i,j}(s,t) - v_j : i \in M_0, s \in S_i, t \in T_j\}, \forall j \in W,
\]

\[
V^\text{max} := \max_{j \in W} V_j
\]

**Proposition 2.1.** The number of iterations of Algorithm 1 is at most \( \frac{1}{\varepsilon} V^\text{max} \).

*Proof.* By construction, the propose-dispose algorithm increases women’s payoff at each iteration by at least \( \varepsilon \). Then, the number of iterations is at most \( \frac{1}{\varepsilon} V^\text{max} \). \( \square \)

Remark that, as the strategy sets and the agent sets are finite, \( V^\text{max} \) can be computed in \( \sum_{j \in W} |T_j| \cdot \sum_{i \in M} |S_i| + |W| \) comparisons.

**Remark 2.1.** The set of externally stable matching profiles can be proved to be a semi-lattice (i.e. the max with respect to men of two externally stable allocations is an externally stable allocation). It becomes a full lattice when all couples play zero-sum games or strictly competitive games [18] (i.e. the max and the min of two externally stable allocations is an externally stable allocation). In addition, as for Gale-Shapley’s, it can be proved that Algorithm 1 outputs the highest element in the lattice with respect to the proposer side.

### 2.4 Algorithm 2: Strategy profiles modification

Let us start this section by introducing the notion of \( \varepsilon \)-internally stable matching profile. It recovers the definition in [18] when \( \varepsilon = 0 \).

**Definition 2.7.** An \( \varepsilon \)-externally stable matching profile \( \pi = (\mu, x, y) \) is \( \varepsilon \)-internally stable if for any couple \((i, j)\) matched by \( \mu \) and any \((s_i, t_j) \in X_i \times Y_j\), it holds,

(a) If \( U_{i,j}(s_i, y_j) > u_i(\pi) + \varepsilon \) then, \((\mu, ((x_{j'})_{j' \neq i}, s_i), y)\) is not \( \varepsilon \)-externally stable,

(b) If \( V_{i,j}(x_i, t_j) > v_j(\pi) + \varepsilon \) then, \((\mu, x, ((y_{j'})_{j' \neq j}, t_j))\) is not \( \varepsilon \)-externally stable.

Internal stability can be seen as a Nash equilibrium condition subject to a participation constraint, also referred as a constrained Nash equilibrium condition. Indeed, let \( \pi = (\mu, x, y) \) be a matching profile, and for each couple \((i, j) \in \mu\), define,

\[
u_i^\varepsilon := \max\{U_{i,j}(\bar{x}, \bar{y}) : j' \in W_0 \setminus \{j\}, \bar{x} \in X_i, \bar{y} \in Y_j\} \quad \text{and} \quad U_{i,j}(\bar{x}, \bar{y}) > v_j(\pi) + \varepsilon\},
\]

\[
u_j^\varepsilon := \max\{V_{i,j}(\bar{x}, \bar{y}) : i' \in M_0 \setminus \{i\}, \bar{x} \in X_i, \bar{y} \in Y_j\} \quad \text{and} \quad V_{i,j}(\bar{x}, \bar{y}) > u_i(\pi) + \varepsilon\}
\]

These values, called the players’ outside options, correspond to the \( \varepsilon \)-maximum payoffs that each agent can obtain outside of their couple, with someone that may accept them.

**Definition 2.8.** Given a couple \((i, j) \in M \times W\) and \((u_i^\varepsilon, v_j^\varepsilon)\) a pair of outside options, a strategy profile \((x', y') \in X_i \times Y_j\) is an \( \varepsilon \)\((u_i^\varepsilon, v_j^\varepsilon)\)-constrained Nash equilibrium (CNE) if it satisfies,

\[
U_{i,j}(x', y') + \varepsilon \geq \max\{U_{i,j}(x, y') : x \in X_i\} \quad \text{and} \quad U_{i,j}(x', y') + \varepsilon \geq v_j^\varepsilon\}
\]

\[
V_{i,j}(x', y') + \varepsilon \geq \max\{V_{i,j}(x', y) : y \in Y_j\} \quad \text{and} \quad V_{i,j}(x', y') + \varepsilon \geq u_i^\varepsilon\}
\]

The following theorem links internal stability with constrained Nash equilibria. It recovers the analogous result in [18] when \( \varepsilon = 0 \).
Theorem 2.2. An \( \varepsilon \)-externally stable matching profile \( \pi = (\mu, x, y) \) is \( \varepsilon \)-internally stable if and only if for any \((i,j) \in \mu, (x_i, y_j) \) is an \( \varepsilon \)-(\( u_i^\varepsilon, v_j^\varepsilon \))-constrained Nash equilibrium.

Proof. Suppose that all couples play constrained Nash equilibria. Let \((i,j) \in \mu \) and \((x_i, y_j) \) their \( \varepsilon \)-(\( u_i^\varepsilon, v_j^\varepsilon \))-CNE. Suppose there exists \( x'_i \in X_i \) such that \( U_{i,j}(x'_i, y_j) > U_{i,j}(x_i, y_j) + \varepsilon \). In particular, \( U_{i,j}(x'_i, y_j) > \max\{ U_{i,j}(s, y_j) : V_{i,j}(s, y_j) + \varepsilon \geq v_j^\varepsilon, s \in X_i \} \). Thus, \( V_{i,j}(x'_i, y_j) + \varepsilon < v_j^\varepsilon \). Let \( i' \) be the player that attains the maximum in \( v_j^\varepsilon \). Then, \((i', j) \) is a blocking pair of the external stability of \( \pi \). For player \( j \), the proof is analogous.

Conversely, suppose \( \pi \) is \( \varepsilon \)-internally stable. Let \((i,j) \in \mu \) and \((x_i, y_j) \) be their strategy profile. Then, for any \( x'_i \in X_i \) such that \( U_{i,j}(x'_i, y_j) > U_{i,j}(x_i, y_j) + \varepsilon \), it holds that \( V_{i,j}(x'_i, y_j) + \varepsilon < v_j^\varepsilon \). Thus, \( U_{i,j}(x_i, y_j) + \varepsilon \geq \max\{ U_{i,j}(s, y_j) : V_{i,j}(s, y_j) + \varepsilon \geq v_j^\varepsilon, s \in X_i \} \). For player \( j \), the proof is analogous. \( \Box \)

Constrained Nash equilibria are not guaranteed to exist in every bi-matrix game [18]. Due to this, we consider the following class of games.

Definition 2.9. A two-person game \( G = (X, Y, U, V) \) is called \( \varepsilon \)-feasible if for any pair of outside options \((u, v) \in \mathbb{R}^2 \), which admits at least one strategy profile \((x, y) \in X \times Y \) satisfying \( U(x, y) + \varepsilon \geq u, V(x, y) + \varepsilon \geq v \), there exists an \( \varepsilon \)-(\( u, v \))-constrained Nash equilibrium.

Theorem 2.3. The class of \( \varepsilon \)-feasible games includes zero-sum games with a value, strictly competitive games with an equilibrium, and infinitely repeated games.

The proof of Theorem 2.3 is given in [18] for \( \varepsilon = 0 \). Its proof for the case \( \varepsilon > 0 \) is given in the following sections with the complexity study of each of the class of games mentioned.

The strategy profiles modification algorithm (Algorithm 2) takes as an input an externally stable allocation and changes the strategy profiles of the couples by constrained Nash equilibria. The selection of the equilibrium is made by an oracle that depends on the games as seen later. For a numerical example of Algorithm 2, check Appendix A.

Algorithm 2: Strategy profiles modification, \( \varepsilon \)-version

| 1 Input: \( \pi \) an \( \varepsilon \)-externally stable allocation |
| 2 Output: \( \pi \) an \( \varepsilon \)-externally and \( \varepsilon \)-internally stable allocation |
| 3 repeat |
| 4 for \((i,j) \in \mu \) do |
| 5 Compute \( (u_i^\varepsilon, v_j^\varepsilon) \) and replace \((x_i, y_j) \) by \((x'_i, y'_j) \in \varepsilon\text{-CNE}(u_i^\varepsilon, v_j^\varepsilon) \) |
| 6 until Convergence; |

Theorem 2.4. If Algorithm 2 converges, its output is an \( \varepsilon \)-externally-externally stable allocation.

The convergence in the class of zero-sum matrix games with a value, strictly competitive bi-matrix games with an equilibrium, and infinitely repeated bi-matrix games as well as the polynomial complexity are studied in the following sections.

Proof. Theorem 2.2 guarantees that the output of Algorithm 2 is \( \varepsilon \)-externally stable, as all couples play constrained Nash equilibria. Regarding \( \varepsilon \)-external stability, we prove by induction over the number of iterations that \( \pi \) always remains \( \varepsilon \)-externally stable. Clearly, the base case holds as the input of Algorithm 2 is \( \varepsilon \)-externally stable. Suppose that at some iteration \( t \) the matching profile is \( \varepsilon \)-externally stable. Let \((i,j) \) be the couple that changed of strategy at iteration \( t \) and suppose that at iteration \( t+1 \) there exists an \( \varepsilon \)-blocking pair \((m, w) \) of \( \pi \). Necessarily, \( m = i \) or \( w = j \), otherwise...
\((m, w)\) would also be a blocking pair of \(\pi\) at iteration \(t\). Suppose, without loss of generality, that \(m = i\). Then, there exists \((x^*, y^*) \in X_i \times Y_w\) such that

\[
U_{i,w}(x^*, y^*) > u_i(\pi(t+1)) + \varepsilon \quad \text{and} \quad V_{i,w}(x^*, y^*) > v_w(\pi(t+1)) + \varepsilon
\]

where \(\pi(t+1)\) is the matching profile at iteration \(t+1\). Remark that \(u_i(\pi(t+1)) = U_{i,j}(x', y')\) with \((x', y') \in \varepsilon\text{-CNE}(u_i^j(t), v_j^i(t))\), where \((u_i^j(t), v_j^i(t))\) were the players' outside options at iteration \(t\). Since \(w \neq j\), her payoff did not change from iteration \(t\) to \(t+1\), so \(v_w(\pi(t+1)) = v_w(\pi(t))\). Adding this last fact to (6) it holds, \(u^i_\varepsilon(t) \geq U_{i,w}(x^*, y^*)\). Putting all together,

\[
U_{i,w}(x^*, y^*) > u_i(\pi(t+1)) + \varepsilon = U_{i,j}(x', y') + \varepsilon \geq u^i_\varepsilon(t) \geq U_{i,w}(x^*, y^*)
\]

from which we obtain a contradiction. \(\square\)

### 2.5 Complexity analysis in general bi-matrix matching games

The main issue on the complexity study of both algorithms is the presence of quadratically constrained quadratic programming (QCQP) problems. Remark that all problems solved during an iteration of Algorithm 1 as well as the computation of the outside options in Algorithm 2 have the following structure,

\[
\begin{align*}
\max & \quad xA y \\
\text{s.t.} & \quad xB y \geq c \\
& \quad x \in X, y \in Y
\end{align*}
\]

where \(A, B\) are real valued matrices, \(c \in \mathbb{R}\), and \(X, Y\) are simplex. Problem (7), in its general form, belongs to the class of quadratically constrained quadratic programming (QCQP) problems \([2, 12, 28]\). If the matrices \(A, B\) are negative semi-definite, Problem (7) is a convex problem and therefore, it can be solved in polynomial time. However, in the general case, Problem (7) is NP-hard. Due to this, the complexity analysis is studied in three classes of games in which \([18]\) proved that the algorithms converge to a stable allocation: zero-sum games, strictly competitive games, and infinitely repeated games.

### 3 Complexity analysis in zero-sum matching games

In this section, we suppose that all games \(G_{i,j}\) are finite zero-sum matrix games in mixed strategies and we call such a model a zero-sum matching game. This includes matching with linear but bounded transfers \([21]\).

The first subsection proves that each iteration of Algorithm 1 is polynomial (recall that it converges in at most \(\frac{1}{\varepsilon} V^{\max}\) steps to an \(\varepsilon\)-externally stable matching). The last subsection proves a similar result for Algorithm 2: it converges in \(O(\frac{1}{\varepsilon})\) number of steps and each iteration has polynomial complexity.

#### 3.1 Algorithm 1: Complexity in zero-sum games

Regarding the propose-dispose algorithm, we aim to prove the following result.

**Theorem 3.1 (Complexity).** Let \(i \in M\) be a proposer man. Let \(j\) be the proposed woman and \(i'\) her current partner. If \(i\) is the winner of the competition and \(L\) represents the number of bits required for encoding all the data, the entire iteration of Algorithm 1 has complexity,

\[
O \left( \left| W \right| + \left( \left| S_i \right| + \left| S_{i'} \right| \right) \cdot \sum_{j' \in W} \left| T_{j'} \right| \right) L
\]

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Remark 3.1. If there are at most $N$ players in each side and at most $k$ pure strategies per player, Theorem 3.1 proves that each iteration of Algorithm 1 takes $O(N + k^2)$ number of elementary operations in being solved, hence is polynomial. As Proposition 2.1 bounds the number of iterations of Algorithm 1 by $\frac{1}{\epsilon}V^{\text{max}}$, we conclude that computing an $\epsilon$-externally stable matching profile for a zero-sum matching game has complexity $O(\frac{1}{\epsilon}(N + k^2))$.

To prove Theorem 3.1 first of all we study Problem (7) and prove it to be polynomial. Then, we study each of the phases of an iteration of Algorithm 1.

The structure of zero-sum games allows to solve Problem (7) in a finite number of comparisons, as the matrix payoff in the constraint can be replaced by the one in the objective function. Formally, given a finite zero-sum game in mixed strategies $G$ as the matrix payoff in the constraint can be replaced by the one in the objective function. Formally, given a finite zero-sum game in mixed strategies $G = (X = \Delta(S), Y = \Delta(T), A)$, Problem (7) becomes,

$$\begin{align*}
\max xAy \\
\text{s.t. } xAy &\leq c \\
x \in X, y \in Y
\end{align*}$$

(8)

Notice that, without loss of generality, we can always take $c \leq \max A$, with $\max A$ being the highest coefficient of the matrix $A$, as we can always replace $c$ by $\min\{c, \max A\}$. Problem (8) is infeasible if $c < \min A$, with $\min A$ being the minimum coefficient of the matrix $A$. Indeed, as $x$ and $y$ are probability distributions, $xAy$ is a convex combination of its coefficients and it cannot be lower than $\min A$. On the other hand, if $\min A \leq c \leq \max A$, by continuity, there always exists $(x, y) \in X \times Y$ such that $xAy = c$. Even more, there always exists a pair $(x, y)$ in which at least one of the players plays a pure strategy. Let us prove this result first.

Proposition 3.1. Given a matrix payoff $A$ and $c \in \mathbb{R}$, with $\min A \leq c \leq \max A$, there always exists $(x, y) \in X \times Y$, such that $xAy = c$, with $x$ or $y$ being a pure strategy.

Proof. Let $s \in S$ be a pure strategy for player 1, such that there exist $t, t' \in T$, with $A(s, t) \leq c \leq A(s, t')$. Then, there exists $\lambda \in [0, 1]$ such that $\lambda A(s, t) + (1 - \lambda)A(s, t') = c$. Even more, $\lambda$ is explicitly given by

$$\lambda = \frac{c - A(s, t)}{A(s, t') - A(s, t)}$$

(9)

Suppose that such a $s$ does not exist, so for any $s \in S$, either $A(s, t) \leq c, \forall t \in T$, or $A(s, t) \geq c, \forall t \in T$. Let $t \in T$ be any pure strategy of player 2. Then, since $\min A \leq c \leq \max A$, there exists $s, s' \in S$ such that $A(s, t) \leq c \leq A(s', t)$. Thus, considering $\lambda$ given by,

$$\lambda = \frac{c - A(s, t)}{A(s', t) - A(s, t)}$$

(10)

it holds that $\lambda A(s, t) + (1 - \lambda)A(s', t) = c$. \qed

Notice that the solution of Problem (8) corresponds to any strategy profile $(x, y) \in X \times Y$, satisfying $xAy = c$. This yields the following proposition.

Proposition 3.2. Let $A$ be a matrix payoff and $c \in \mathbb{R}$, such that $\min A \leq c \leq \max A$. Then, solving Problem (8) has complexity $O(|S| \cdot |T|)$.

Proof. The complexity of solving Problem (8) is basically the complexity of finding the pure strategies used in the proof of Proposition 3.1, as then it is enough with computing $\lambda$ by the respective expression. Let $S^+ := \{s \in S : \exists t \in T, A(s, t) \geq c\}$ and $S^- := \{s \in S : \exists t \in T, A(s, t) \leq c\}$. 

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Each of these sets is computed in $|S| \cdot |T|$ comparisons, as we have to check all coefficients in $A$. As $\min A \leq c \leq \max A$, both sets are non-empty. If $S^+ \cap S^- \neq \emptyset$, there exist $s \in S$ and $t, t' \in T$ such that $A(s, t) \leq c \leq A(s, t')$, so Equation (9) gives the solution desired. Otherwise, there exists $t \in T$ and $s, s' \in S$ such that $A(s, t) \leq c \leq A(s', t)$, and Equation (10) gives the solution desired. Computing the intersection of $S^+$ and $S^-$ has complexity $O(|S|)$. In either cases (the intersection is empty or not), finding the pure strategies needed for the convex combination takes at most $|T|$ comparisons. Finally, computing $\lambda$ requires a constant number of operations with respect to the sizes of the strategy sets. Summing all, we obtain the complexity result stated.

Remark 3.2. For being totally accurate, the complexity result of Proposition 3.2 should be accompanied by a constant $L_A$ regarding the number of bits required for encoding the problem.

From Proposition 3.2 we can compute the complexity of one iteration of Algorithm 1. Let $\pi$ be a matching profile at some iteration of Algorithm 1 and $i \in M$ be the single man that proposes next. His optimal proposal is computed by solving,

$$\max x A_{i,j} y$$
$$s.t. \ x A_{i,j} y \leq v_j(\pi) - \varepsilon$$
$$j \in W \cup \{j_0\}, x \in X_i, y \in Y_j$$

Problem (11) can be divided in $|W|$ optimization problems, each of them having the structure of (8). As a corollary of Proposition 3.2, we obtain the following result.

Corollary 3.1. The complexity of computing $i$’s optimal proposal is $O \left( |W| + |S_i| \cdot \sum_{j \in W} |T_j| \right)$.

Remark that once solved all $|W|$ optimization problems, the maximum between each of the solutions has to be computed, obtaining the extra complexity term $|W|$. Also, for each sub-optimization problem it has to be checked that $\min A_{i,j} \leq v_j(\pi) - \varepsilon \leq \max A_{i,j}$, which adds $|S_i| \cdot |T_j|$ more comparisons to the computation. However, as these terms are bounded by the ones already considered, they are omitted.

Consider the problem solved by a man $i$ during a competition, in the zero-sum game setting,

$$\min x A_{i,j} y$$
$$s.t. \ x A_{i,j} y \geq \beta_i$$
$$x \in X_i, y \in Y_j$$

where $\beta_i$ is $i$’s reservation price. Remark that $\beta_i$ can be computed by solving (11) leaving $j$ out of the feasible region. Then, solving (12) is reduced to solving a problem of type (8). Summing all, we obtain the following result.

Proposition 3.3. The computation of the reservation price of a man $i$ plus his bid $\lambda_i$ during a competition has complexity $O \left( |W| + |S_i| \cdot \sum_{j' \in W} |T_{j'}| \right)$.

For ending the competition phase, the winner of the competition, namely $i$, reduces his offer until matching the one of the loser, namely $i'$. If $\lambda_{i'}$ was the bid of $i'$, $i$ solves,

$$\max x A_{i,j} y$$
$$s.t. \ x A_{i,j} y \leq \lambda_{i'}$$
$$x \in X_i, y \in Y_j$$

Problem (13) has the same structure as problem (8) and therefore, its complexity is $O \left( |S_i| \cdot |T_j| \right)$. The proof of Theorem 3.1 is obtained by summing up all the previous results and taking into account the number of bits required for the problem.
3.2 Algorithm 2: Convergence and complexity in zero-sum games

We focus now in the computation of internally stable allocations. For that, we firstly prove that computing a CNE is a polynomial problem. Then, we bound the number of iterations of Algorithm 2 for finite zero-sum matching games. Suppose that for all couples \((i, j) \in M \times W\), their zero-sum game \(G_{i,j} = (X_i, Y_j, A_{i,j})\) has value \(w_{i,j}\). As for the \(\varepsilon = 0\) case, it can be proved that all games are always feasible and even more, if \((x_i, y_j)\) is a \((u_i, v_j)\)-CNE, for a pair of outside options \((u_i, v_j)\), it holds,

\[
x_iA_{i,j}y_j = \text{median}\{u_i - 2\varepsilon, w_{i,j}, v_j + 2\varepsilon\}
\]

This new formula for the value of a constrained Nash equilibrium will guarantee the convergence of Algorithm 2 in a finite number of iterations. First of all, we prove a useful lemma.

**Lemma 3.1.** Let \(s_1, s_2 \in S_i\) be two pure strategies for player \(i\). Consider a mixed strategy \(x \in X_i\) that only has \(s_1, s_2\) in its support. Let \(\tau \in (0,1)\) and \(y_r := (1-\tau)y + \tau y^*\). Suppose that \(x A_{i,j}y_r = u_i\) but \(s_1 A_{i,j}y_r, s_2 A_{i,j}y_r \neq u_i\). Then, we can always find \(\tau' \in (\tau,1)\), and a pure strategy \(s \in S_i\) such that \(s A_{i,j}y_{\tau'} = u_i\).

**Proof.** Notice that \(x A_{i,j}y_{\tau'} = x s_1 A_{i,j}y_r + x s_2 A_{i,j}y_r = u_i\) with \(x s_1 + x s_2 = 1\), \(x s_1, x s_2 \in [0,1]\). Then, as \(s_1 A_{i,j}y_r\) and \(s_2 A_{i,j}y_r\) are both different of \(u_i\), necessarily one of them must be higher than \(u_i\), and the other one lower than \(u_i\). Without loss of generality, suppose \(s_1 A_{i,j}y_r > u_i\) and \(s_2 A_{i,j}y_r < u_i\). Recalling that \(x A_{i,j}y^* = w_{i,j} < u_i\), and that \((x^*, y^*)\) is a saddle point, in particular \(s_1 A_{i,j}y_r \leq w_{i,j} < u_i\). As \(y_{\tau=1} = y^*\), by continuity, there exists \(\tau' \in (\tau,1)\) such that, \(s_1 A_{i,j}y_{\tau' = 1} < u_i = s_1 A_{i,j}y_r < s_1 A_{i,j}y_{\tau'}\).

**Theorem 3.2 (CNE Complexity).** Let \((i, j)\) be a couple and \(G_{i,j} = (X_i, Y_j, A_{i,j})\) be their bi-matrix zero-sum game with value \(w_{i,j}\). Let \((u_i, v_j)\) be a pair of outside options which admits at least one strategy profile \((x_0, y_0)\) satisfying \(u_i - \varepsilon \leq x_0 A_{i,j}y_0 \leq v_j + \varepsilon\). Then,

1. \(G_{i,j}\) is \(\varepsilon\)-feasible,
2. For any \((x', y') \in CNE(u_i, v_j)\), it holds \(x' A_{i,j}y' = \text{median}\{u_i - 2\varepsilon, w_{i,j}, v_j + 2\varepsilon\}\),
3. Computing a CNE \((x', y')\) has complexity

\[
O \left( \max\{|S_i|, |T_j|\}^{2.5} \cdot \min\{|S_i|, |T_j|\} \cdot L_{A_{i,j}} \right),
\]

where \(S_i, T_j\) are the pure strategy sets and \(L_{A_{i,j}}\) is the number of bits required for encoding the matrix \(A_{i,j}\).

**Proof.** Let \((x^*, y^*)\) be the optimal strategies of the players, i.e. the strategy profile that achieves the value of the game \(x^* A_{i,j}y^* = w_{i,j}\). We split the proof into three cases.

(1) Suppose that \(u_i - 2\varepsilon \leq w_{i,j} \leq v_j + 2\varepsilon\). In particular, the value of the game is feasible for both agents. Since it is also a saddle point so agents do not have profitable deviations, \((x^*, y^*)\) is also a \((u_i, v_j)\)-CNE. From Von Neumann’s theorem we know that \((x^*, y^*, w_{i,j})\) can be obtained from the solutions of the pair primal-dual problems,

\[
(P) \quad \min(c, x) \quad \text{subject to} \quad x A_{i,j} \geq b \quad x \geq 0
\]

\[
(D) \quad \max(b, y) \quad \text{subject to} \quad A_{i,j}y \leq c \quad y \geq 0
\]
where the variables satisfy $x \in X_i$, $y \in Y_j$, and the vectors $c, d$ are given and are equal to 1 in every coordinate. If $(x', y')$ is the primal-dual solution and $z$ is their optimal value, the optimal strategies of player $i$ and $j$ are given by $(x^*, y^*) = (x', y'/z)$, and they achieve the value of the game $w_{i,j}$.

The number of elementary operations needed for solving the primal-dual problem and computing $(x^*, y^*)$ is $O((|S_i| + |T_j|)^{1.5} \max\{|S_i|, |T_j|\}L_{i,j})$.

(2) Suppose that $w_{i,j} < u_i - 2\varepsilon \leq v_j + 2\varepsilon$. Consider the set $A(u_i) := \{x \in X_i : \exists y \in Y_j, xA_{i,j}y + 2\varepsilon \geq u_i\}$. The set $A(u_i)$ is non-empty as $(x_0, y_0)$ belongs to it. Consider the problem,

$$
\sup \{\inf \{xA_{i,j}y : xA_{i,j}y + 2\varepsilon \geq u_i, y \in Y\} : x \in A(u_i)\}
$$

Since the set $\{xA_{i,j}y + 2\varepsilon \geq u_i, y \in Y\}$, for a given $x$, is bounded, as well as the set $A(u_i)$, there exists a solution $(x, y)$ of (14). (14) is equivalent to solve at most $|T_j|$ linear programming problems, each of them with $|S_i|$ variables and 1 constraint, and then considering the highest value between them. Each problem takes $O(|S_i|^{2.5}L_t)$ elementary operations in being solved, with $L_t$ the number of bits required for encoding the problem for each $t \in T_j$. Therefore, (14) has a complexity of $O(|T_i| \cdot |S_i|^{2.5} \max_t\{L_t\})$.

By construction, it holds $xA_{i,j}y + 2\varepsilon \geq u_i$. Suppose $xA_{i,j}y + 2\varepsilon > u_i$. It follows,

$$xA_{i,j}y > u_i - 2\varepsilon > w_{i,j} = x^*A_{i,j}y^* \geq xA_{i,j}y^*$$

where the last inequality holds as $(x^*, y^*)$ is a saddle point. Then, there exists $y' \in (y, y^*)$ such that $xA_{i,j}y' = u_i - 2\varepsilon$. This contradicts that $(x, y)$ is solution of (14) as $y' \in A(u_i)$ and $xA_{i,j}y'$ is lower than $xA_{i,j}y$. Suppose finally, that $(x, y)$ is not a $(u_i, v_j)$-CNE and consider the problem,

$$
t := \sup \{\tau \in [0, 1] : y_r := (1 - \tau)y + \tau y^* \text{ and } \exists x_r \in X_i, x_rA_{i,j}y_r = u_i - 2\varepsilon\}
$$

Notice $t$ exists as for $\tau = 0$, $xA_{i,j}y = u_i - 2\varepsilon$. In addition, $y_t \neq y^*$ as $xA_{i,j}y^* < u_i - 2\varepsilon$ and $(x^*, y^*)$ is a saddle point. From Lemma 3.1 if $xA_{i,j}y_r = u_i$ for some value $\tau \in (0, 1)$, then there always exists a pure strategy $s \in S_i$ and $\tau \leq \tau' < 1$ such that $sA_{i,j}y_{\tau'} = u_i$. Thus, solving problem (15) is equivalent to solve each of the next linear problems,

$$
t_s := \sup \{\tau \in [0, 1] : y_r := (1 - \tau)y + \tau y^* \text{ and } sA_{i,j}y_r = u_i, \forall s \in S_i\}
$$

and then considering $t := \max_{s \in S_i} t_s$. Each $t_s$ can be computed in constant time over $|S_i|$ and $|T_j|$, as the linear programming problem associated has only one variable and one constraint. Finally, computing the maximum of all $t_s$ takes $|S_i|$ comparisons. We claim that $(x_t, y_t)$ is a $(u_i, v_j)$-CNE. Let $x' \in X_i$ such that $xA_{i,j}y_t \leq v_j + \varepsilon$. We aim to prove that $xA_{i,j}y_t \leq x_tA_{i,j}y_t + \varepsilon$. Suppose $xA_{i,j}y_t > x_tA_{i,j}y_t + \varepsilon$. It holds,

$$xA_{i,j}y^* \leq w_{i,j} = x^*A_{i,j}y^* < u_i - 2\varepsilon = x_tA_{i,j}y_t < x_tA_{i,j}y_t + \varepsilon \leq x'A_{i,j}y_t$$

Then, there exists $z \in X_i$ and $y' \in (y_t, y^*)$ such that $zA_{i,j}y' = u_i - 2\varepsilon$, contradicting that $t$ is solution of the supremum. Let $y' \in Y_j$ such that $xA_{i,j}y' + \varepsilon \geq u_i$. We aim to prove that $xA_{i,j}y' \geq x_tA_{i,j}y_t - \varepsilon$. Indeed,

$$x_tA_{i,j}y' \geq u_i - \varepsilon = u_i - 2\varepsilon + \varepsilon = x_tA_{i,j}y_t + \varepsilon > x_tA_{i,j}y_t - \varepsilon$$

(3) Suppose that $u_i - 2\varepsilon \leq v_j + 2\varepsilon < w_{i,j}$. Analogously to case 2, there exists a $(u_i, v_j)$-CNE $(x, y)$ satisfying $xA_{i,j}y = v_j + 2\varepsilon$. 

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Finally, the complexity given at the theorem’s state is obtained when taking the maximum complexity between the three cases.

As a corollary of Theorem 3.2 we obtain the following result.

**Corollary 3.2.** Given a matching profile \((\pi, x, y)\), computing all games’ values is a polynomial problem and its complexity is bounded by

\[
O\left( \sum_{(i,j) \in \mu} (|S_i| + |T_j|)^{1.5} \max\{|S_i|, |T_j|\} L_{A_{i,j}} \right)
\]

**Proof.** Let \((i, j) \in \mu\) be a matched couple and \(G_{i,j} = (X_i, Y_j, A_{i,j})\) be their zero-sum game. The proof of Theorem 3.2 in its first case proves that computing \(w_{i,j}\) takes at most \(O((|S_i| + |T_j|)^{1.5} \max\{|S_i|, |T_j|\} L_{A_{i,j}})\) elementary operations, where \(S_i, T_j\) are the players’ strategy sets and \(L_{A_{i,j}}\) is the number of bits required for encoding the matrix \(A_{i,j}\). Summing up over all the couples, we obtain the complexity stated.

Theorem 3.2 and Lemma 3.2 guarantees the convergence of Algorithm 2 for matching games in which players play zero-sum games with a value.

**Lemma 3.2.** Let \(\Gamma\) be a matching game where all strategic games are zero-sum games with a value. Let \(\pi = (\mu, x, y)\) be an externally stable matching and \((i, j)\) be a matched couple. Let \(w_{i,j}\) be the value of their game. Consider the sequence of outside options of \((i, j)\) denoted by \((u^\varepsilon_i(t), v^\varepsilon_j(t))_t\) with \(t\) being the iterations of Algorithm 2. If there exists \(t^*\) such that \(w_{i,j}^\varepsilon - u^\varepsilon_i(t) - 2\varepsilon \leq v^\varepsilon_j(t) + 2\varepsilon\), then the subsequence \((u^\varepsilon_i(t))_{t \geq t^*}\) (resp. \((v^\varepsilon_j(t))_{t \geq t^*}\)) decreases (resp. increases) at least \(\varepsilon\) at each step.

**Proof.** Suppose there exists an iteration \(t\) in which \(w_{i,j}^\varepsilon - u^\varepsilon_i(t) - 2\varepsilon \leq v^\varepsilon_j(t) + 2\varepsilon\), so couple \((i, j)\) switches its payoff to \(u^\varepsilon_i(t) - 2\varepsilon\). Let \((\hat{x}_i, \hat{y}_j)\) be the \(\varepsilon\)-constrained Nash equilibrium played by \((i,j)\) at iteration \(t\). Let \(t'\) be the next iteration in which \((i,j)\) moves. Since \((\hat{x}_i, \hat{y}_j)\) must be \(\varepsilon\)-(\(u^\varepsilon_i(t'), v^\varepsilon_j(t')\))-feasible, in particular it holds \(u^\varepsilon_i(t') \leq g_{i,j}(\hat{x}_i, \hat{y}_j) + \varepsilon = u^\varepsilon_i(t) - \varepsilon\). Therefore, the sequence of outside options starting from \(t\) decreases at least in \(\varepsilon\) at each step.

**Theorem 3.3 (Convergence).** Let \(\Gamma = (M_0, W_0, (G_{i,j} : i, j \in M \times W), u, v)\) be a matching game in which each game \(G_{i,j}\) is a zero-sum game with value \(w_{i,j}\). Let \(\pi = (\mu, x, y)\) be an \(\varepsilon\)-externally stable matching, input of Algorithm 2, the one defines a profile of outside options \((u_i, v_j)_{(i,j)\in \mu}\). Then, the number of iterations of Algorithm 2 is bounded by

\[
O\left( \frac{1}{\varepsilon} \max_{(i,j) \in \mu} \{u_i - w_{i,j}, w_{i,j} - v_j\} \right)
\]

**Proof.** At the beginning of Algorithm 2 all couples \((i,j)\) belong to one (not necessarily the same) of the following cases: \(u_i - 2\varepsilon \leq w_{i,j} \leq v_j + 2\varepsilon\), \(w_{i,j} \leq u_i - 2\varepsilon \leq v_j + 2\varepsilon\) or \(u_i - 2\varepsilon \leq v_j + 2\varepsilon \leq w_{i,j}\). In the first case, the couple plays a Nash equilibrium and never changes it afterwards. In the second case, as \(u_i\) is strictly decreasing for \(i\) (Lemma 3.2) and bounded from below by \(w_{i,j}\), his sequence of outside options converges in \(O\left(\frac{1}{\varepsilon}(u_i - w_{i,j})\right)\). Analogously, the sequences of outside options for \(j\) converges on the third case in finite time. Therefore, Algorithm 2 converges and its complexity comes from taking the maximum between all couples.
Remark that the result given in Corollary \[3.2\] corresponds to the complexity of computing the value of each game plus the optimal strategies that attain them. As we only need the values of the games for Theorem \[3.3\] the complexity of computing them may be much lower.

| Algorithm | Complexity per Iteration | \(N^0\) Iterations | Constants |
|-----------|--------------------------|--------------------|-----------|
| Algorithm \[1\] | \(O((N + k^2)L)\) | \(C_1/\varepsilon\) | \(C_1 \leq \max_{i,j}(\max A_{i,j} - \min A_{i,j})\) |
| Algorithm \[2\] | \(O(Nk^{3.5}L)\) | \(C_2/\varepsilon\) | \(C_2 \leq \max_{i,j}(\max A_{i,j} - \min A_{i,j})\) |

Table 3: Complexity zero-sum matching games, \(N\) number players per side, \(k\) strategies per player, \(L\) bits for encoding the data, and \(A_{i,j}\) payoff matrix of couple \((i, j)\)

4 Complexity analysis in strictly competitive matching games

The class \(\mathcal{S}\) of strictly competitive games, initially defined by Aumann \[3\], was fully characterized by Adler, Daskalakis, and Papadimitriou \[1\] in the bi-matrix case.

**Definition 4.1.** Given two \(m \times n\) matrices \(A, B\), we say that \(B\) is an **affine variant** of \(A\) if for some \(\lambda > 0\) and unrestricted \(\mu \in \mathbb{R}\), \(B = \lambda A + \mu U\), where \(U\) is \(m \times n\) all-ones matrix.

Adler et al. proved the following result.

**Theorem 4.1.** If for all \(x, x' \in X\) and \(y, y' \in Y\), \(xAy - x'Ay'\) and \(xBy - x'By'\) have the same sign, then \(B\) is an affine variant of \(A\). Even more, the affine transformation is given by,

\[
A = \frac{a_{\max} - a_{\min}}{b_{\max} - b_{\min}} [B - b_{\min}U] + a_{\min}U, \quad \text{with} \quad \begin{cases} a_{\max} := \max A, & a_{\min} := \min A \\ b_{\max} := \max B, & b_{\min} := \min B \end{cases}
\]

If \(a_{\max} = a_{\min}\), then it also holds that \(b_{\max} = b_{\min}\) (and vice-versa), in which case clearly \(A\) and \(B\) are affine variants.

4.1 Algorithm \[1\]: Complexity in strictly competitive games

Theorem 4.1 allows to extend the results on zero-sum games to all bi-matrix strictly competitive games in mixed strategies. First of all, notice that computing the affine transformations is a polynomial problem as it only needs to compute the coefficients \(a_{\min}, a_{\max}, b_{\min}, \text{and } b_{\max}\).

**Theorem 4.2.** Let \(\Gamma = (M_0, W_0, (A_{i,j}, -B_{i,j} : i, j \in M \times W), (u_i, v_j)\) be a matching game in which all \((A_{i,j}, -B_{i,j})\) are bi-matrix strictly competitive games. Let \(\Gamma'\) be the affine transformation of \(\Gamma\) in which all couples play zero-sum games. Then, computing \(\Gamma'\) has complexity

\[
O \left( \left| M \right| + \left| W \right| + \sum_{i \in M} \sum_{j \in W} \left| S_i \right| \cdot \left| T_j \right| \right)
\]

**Proof.** In order of obtaining \(\Gamma'\), besides computing all zero-sum games, we also need to compute all the new individually rational payoffs. Let \((i, j) \in M \times W\) be a potential couple that plays a strictly competitive game \((A_{i,j}, -B_{i,j})\). The complexity of computing their affine transformation to a zero-sum game \((B_{i,j}, -B_{i,j})\) is \(O(\left| S_i \right| \cdot \left| T_j \right|)\), as we need to compute \(a_{i,j}^{\max}, a_{i,j}^{\min}, b_{i,j}^{\max}, \text{and } b_{i,j}^{\min}\). Regarding the individually rational payoffs \((u_i, v_j)\), set \(\alpha_{i,j} := \frac{a_{i,j}^{\max} - a_{i,j}^{\min}}{b_{i,j}^{\max} - b_{i,j}^{\min}}\). We take \(\alpha_{i,j}\) so it is
always lower or equal than 1 (at least one of the two ways of taking the affine transformation guarantees this). Given \((x, y) \in X_i \times Y_j\) a strategy profile, notice that,

\[
x A_{i,j}y \geq u_i \iff x B_{i,j}y \geq \frac{u_i - (a_{i,j}^{\min} - b_{i,j}^{\min})}{a_{i,j}} \alpha_{i,j},
\]

\[
x (-B_{i,j})y \geq v_j \iff x B_{i,j}y \leq -v_j
\]

where we have used that \(xUy = 1\). Unlike a “standard” matching game in which each player has a unique IRP that works for all possible partner, in the transformed game \(\Gamma'\) men will have one IRP for each women, given by (16). Formally, let \(u'_{i,j} := \frac{u_i - (a_{i,j}^{\min} - b_{i,j}^{\min})}{a_{i,j}} \alpha_{i,j}, \forall i, j \in W\). Then, a man \(i\) accepts to be matched with a woman \(j\) if and only his payoff is greater or equal than \(u'_{i,j}\). Regarding women, it is enough considering \(v'_j := -v_j\). Computing each coefficient takes constant time on the size of the agent sets and strategy sets. Thus, the complexity of transforming the IRPs is \(O(|M| + |W|)\) plus some factor indicating the number of required bits.

The analysis of Algorithm 1’s complexity is not affected by the fact that men may have an IRPs for each woman. Thus, from Theorem 3.1 and Theorem 4.2, we conclude the following.

**Corollary 4.1.** Computing \(\varepsilon\)-externally stable allocations in bi-matrix strictly competitive matching games has complexity \(O(\frac{1}{\varepsilon}(N + k^2)L)\), where \(N\) bounds the number of players in the biggest side, \(k\) bounds the number of pure strategies per player and \(L\) is the number of bits required for encoding all the data.

### 4.2 Algorithm 2: Complexity and convergence in strictly competitive games

As in the zero-sum case, we start with the complexity of computing a constrained Nash equilibrium. Let \(G_{i,j} = (X_i, Y_j, A_{i,j}, -B_{i,j})\) be a bi-matrix strictly competitive game in mixed strategies and \((u_i, v_j)\) be a pair of outside options. Let \((x, y)\) be a \((u_i, v_j)\)-feasible strategy profile, that is,

\[
xA_{i,j}y + \varepsilon \geq u_i \quad \text{and} \quad x(-B)_{i,j}y + \varepsilon \geq v_j \iff x A_{i,j}y + \varepsilon \geq u_i \quad \text{and} \quad xB_{i,j}y \leq -v_j - \varepsilon
\]

Notice that,

\[
x A_{i,j}y + \varepsilon \geq u_i \iff x \left( \alpha_{i,j}[B_{i,j} - b_{i,j}^{\min}U] + a_{i,j}^{\min}U \right) y + \varepsilon \geq u_i
\]

\[
\iff \alpha_{i,j}x B_{i,j}y + (a_{i,j}^{\min} - b_{i,j}^{\min} \alpha_{i,j})x Uy + \varepsilon \geq u_i
\]

\[
\iff \alpha_{i,j}x B_{i,j}y + (a_{i,j}^{\min} - b_{i,j}^{\min} \alpha_{i,j}) + \varepsilon \geq u_i
\]

\[
\iff x B_{i,j}y + \varepsilon \geq \frac{u_i - (a_{i,j}^{\min} - b_{i,j}^{\min} \alpha_{i,j})}{\alpha_{i,j}} - \varepsilon \cdot \frac{1 - \alpha_{i,j}}{\alpha_{i,j}}
\]

Recall we have considered \(\alpha_{i,j} \in [0, 1]\). Thus, in the zero-sum game \(G'_{i,j} = (X_i, Y_j, B_{i,j})\), considering the pair \((u'_i, v'_j)\) of outside options given by,

\[
u'_i := \frac{u_i - (a_{i,j}^{\min} - b_{i,j}^{\min} \alpha_{i,j})}{\alpha_{i,j}} - \varepsilon \cdot \frac{1 - \alpha_{i,j}}{\alpha_{i,j}} \quad \text{and} \quad v'_j := -v_j,
\]

the sets of feasible strategy profiles, as well as the sets of CNE of \(G_{i,j}\) and \(G'_{i,j}\), coincide. Therefore, for computing a \((u_i, v_j)\)-constrained Nash equilibrium of the strictly competitive game, we can use the following scheme:
1. Compute the affine transformation from $A_{i,j}$ to $B_{i,j}$, and define the zero-sum game $G'_{i,j}$.
2. Consider the new outside options $(u'_i, v'_j)$ as in (18).
3. Compute an $\varepsilon$-$(u'_i, v'_j)$-CNE for the zero-sum game, namely $(x', y')$.

**Proposition 4.1.** The scheme above computes an $\varepsilon$-$(u'_i, v'_j)$-CNE of $G_{i,j}$.

**Proof.** Let $(x', y')$ be a $(u'_i, v'_j)$-CNE of the zero-sum game $G'_{i,j}$. Then, it holds that,

1. $v'_j + \varepsilon \geq x'B_{i,j}y' \geq u'_i - \varepsilon$
2. For any $x \in X_i$ such that $xB_{i,j}y' \leq v_j + \varepsilon$, it holds $(x - x')B_{i,j}y' \leq \varepsilon$
3. For any $y \in Y_j$ such that $x'B_{i,j}y + \varepsilon \geq u_i$, it holds $x'B_{i,j}(y' - y) \leq \varepsilon$

From (1) we obtain that $x'(-B_{i,j})y' \geq v'_j - \varepsilon = v_j - \varepsilon$, and $x'B_{i,j}y' \geq u'_i - \varepsilon$, which implies that $x'A_{i,j}y' \geq u_i - \varepsilon$, so $(x', y')$ is $(u_i, v_j)$-feasible in the game $G_{i,j}$. Let $x \in X_i$ such that $x(-B_{i,j})y' + \varepsilon \geq v_i$. Then, $xB_{i,j}y' \leq v'_i - \varepsilon$. From (2.1), $(x - x')B_{i,j}y' + \varepsilon$. Noticing that $\alpha_{i,j}(x - x')B_{i,j}y' = (x - x')A_{i,j}y'$, we obtain that $(x - x')A_{i,j}y' \leq \alpha_{i,j}\varepsilon \leq \varepsilon$, as $\alpha_{i,j}$ was taken lower of equal than 1. Analogously, suppose there is $y \in Y_j$ such that $x'A_{i,j}y + \varepsilon \geq u_i$. Then, $x'B_{i,j}y + \varepsilon \geq u'_i$. From (2.2), $x'(-B_{i,j})(y - y') \leq \varepsilon$. Therefore, $(x', y')$ is a CNE of $G_{i,j}$.

From this, and as a corollary of Theorem 3.2, we obtain the following result.

**Corollary 4.2.** Let $G_{i,j} = (X_i, Y_j, A_{i,j}, -B_{i,j})$ be a bi-matrix strictly competitive game and $(u_i, v_j)$ be a pair of outside options. The complexity of computing a $(u_i, v_j)$-constrained Nash equilibrium is

$$O \left( \max\{|S_i|, |T_j|\}^{2.5} \cdot \min\{|S_i|, |T_j|\} \cdot L_{i,j} \right)$$

with $L_{i,j}$ is the number of bits required for encoding the matrices $A_{i,j}$ and $B_{i,j}$.

Finally, from Theorem 3.3 and Theorem 4.1 we deduce the following.

**Corollary 4.3.** Algorithm 2 ends in a finite number of iterations of order $O(\frac{1}{\varepsilon})$ in bi-matrix strictly competitive games.

| Algorithm | Complexity per Iteration | $N^0$ Iterations | Constants |
|-----------|--------------------------|------------------|-----------|
| Algorithm 1 | $O((N + k^2)L)$ | $C_1/\varepsilon$ | $C_1 \leq \max_{i,j}(\max A_{i,j} - \min A_{i,j})$ |
| Algorithm 2 | $O(Nk^{3.5}L)$ | $C_2/\varepsilon$ | $C_2 \leq \max_{i,j}(\max A_{i,j} - \min A_{i,j})$ |
| Affine Transformation | $O(N^2k^2)$ | - | - |

Table 4: Complexity strictly competitive matching games, $N$ number players per side, $k$ strategies per player, $L$ bits for encoding the data, $A_{i,j}$ payoff matrix of the zero-sum game of $(i,j)$.

## 5 Complexity analysis in infinitely repeated matching games

### 5.1 Model and preliminary results

For each couple, we consider a bi-matrix stage game that is played in discrete time $k = \{1, \ldots, K, \ldots\}$ a large number of periods where, after observing the past history of plays $h_k = ((s_1, t_1), \ldots, (s_{k-1}, t_{k-1}))$, players can chose, eventually at random, an action in their set of actions. Formally, for each couple
Let \((i, j) \in M \times W\), let \(G_{i,j} = (X_i, Y_j, A_{i,j}, B_{i,j})\) be a finite bi-matrix game in mixed strategies, with \(X_i = \Delta(S_i), Y_j = \Delta(T_j)\), where all matrices have only rational entries. Given \(K \in \mathbb{N}\), consider the \(K\)-stages game \(G_{i,j}^K\) defined by the payoff functions,

\[
U_{i,j}(K, \sigma_i, \sigma_j) = \frac{1}{K} \mathbb{E}_\sigma \left[ \sum_{k=1}^{K} A_{i,j}(s_k, t_k) \right], \quad V_{i,j}(K, \sigma_i, \sigma_j) = \frac{1}{K} \mathbb{E}_\sigma \left[ \sum_{k=1}^{K} B_{i,j}(s_k, t_k) \right],
\]

where \(\sigma_i : \bigcup(S_i \times T_j)_{i,j}^\infty_{k=1} \rightarrow X_i\) is a behavioral strategy for man \(i\) and \(\sigma_j : \bigcup(S_i \times T_j)_{i,j}^\infty_{k=1} \rightarrow Y_j\) is a behavioral strategy for woman \(j\). The strategy profiles are pure if \(\sigma_i : \bigcup(S_i \times T_j)_{i,j}^\infty_{k=1} \rightarrow S_i\) and \(\sigma_j : \bigcup(S_i \times T_j)_{i,j}^\infty_{k=1} \rightarrow T_j\). We will be interested by infinitely repeated games \(G_{i,j}^\infty\) in which the players are \(\varepsilon\)-uniformly optimizing i.e. there is \(\sigma^\varepsilon\) and \(K^\varepsilon\) such that for every \(K \geq K^\varepsilon\), \(\sigma^\varepsilon\) is an \(\varepsilon\)-constrained Nash equilibrium in \(G_{i,j}^K\).

Before studying Algorithm 1 and Algorithm 2 (next subsections), we aim to prove the following useful proposition.

**Proposition 5.1.** Let \((i, j) \in M \times W\) be a couple and \(c \in \mathbb{R}\) such that \(c \leq \max B_{i,j}\). Then, solving Problem 1 in \(G_{i,j}^\infty\) has complexity \(O\left(\left|S_i \cdot |T_j|\right|^{2.5} L_{i,j}\right)\), where \(S_i, T_j\) are the sets of pure strategies of the players and \(L_{i,j}\) is the number of bits required for encoding \(G_{i,j}\).

For proving Proposition 5.1 we need the following lemma.

**Lemma 5.1.** Let \((i, j) \in M \times W\) and \((\bar{u}, \bar{v}) \in \mathbb{R}^2\) be a payoff vector. If \((\bar{u}, \bar{v}) \in \text{co}\{ (A_{i,j}(s,t), B_{i,j}(s,t)) : s \in S_i, t \in T_j \}\), then there is a pure strategy profile \(\sigma\) of the infinitely repeated game \(G_{i,j}^\infty\) that achieves \((\bar{u}, \bar{v})\). In addition, the number of elementary operations used for the computation of \(\sigma\) is bounded by \(O\left(\left|S_i \cdot |T_j|\right|^{2.5} L_{i,j}\right)\), where \(L_{i,j}\) is the number of bits required for encoding the matrices \(A_{i,j}\) and \(B_{i,j}\).

**Proof.** Consider the following system with \(|S_i \cdot |T_j|\) variables and three linear equations,

\[
\sum_{s,t} A_{i,j}(s,t) \lambda_{s,t} = \bar{u}, \\
\sum_{s,t} B_{i,j}(s,t) \lambda_{s,t} = \bar{v}, \quad \lambda \in \Delta(S_i \times T_j) \tag{19}
\]

System (19) can be solved in \(O\left(\left|S_i \cdot |T_j|\right|^{2.5} L_{i,j}\right)\) elementary operations. Since matrices \(A_{i,j}\) and \(B_{i,j}\) have rational entries, the solution has the form \((\lambda_{s,t})_{s,t} = \left(\frac{p_{s,t}}{q_{s,t}}\right)_{s,t}\) with each \(p_{s,t}, q_{s,t} \in \mathbb{N}\). Let \(N_\lambda = \text{lcm}(q_{s,t} : (s,t) \in S_i \times T_j)\) be the least common multiple of all denominators. The number of elementary operations for computing \(N_\lambda\) is bounded by \(O\left(\left|S_i \cdot |T_j|\right|^2\right)\). Enlarge each fraction of the solution so all denominators are equal to \(N_\lambda\), i.e. \(\lambda = \left(\frac{p_{s,t}}{N_\lambda}\right)_{s,t}\). Suppose that \(S_i = \{s_1, s_2, ..., s_I\}\) and \(T_j = \{t_1, t_2, ..., t_J\}\). Let \(\sigma\) be the strategy profile in which players play \((s_1, t_1)\) the first \(p'_{s_1,t_1}\)-stages, then \((s_1, t_2)\) the next \(p'_{s_1,t_2}\)-stages, then \((s_1, t_3)\) the next \(p'_{s_1,t_3}\)-stages and so on until playing \((s_1, t_J)\) during \(p'_{s_1,t_J}\)-stages, and then they repeat all infinitely. By construction, \((U_{i,j}(\sigma), V_{i,j}(\sigma)) = (\bar{u}, \bar{v})\). \(\square\)

Let us illustrate the previous lemma with an example.

\(^2\text{co}(A)\) refers to the convex envelope of the set \(A\).
Example 2. Recall the prisoners’ dilemma example for \( \delta = 1 \). Suppose that a couple repeats this game infinitely many times. Figure 1 shows the convex envelope of the pure payoff profiles.

Consider \((\bar{u}, \bar{v}) = (1, 1) \in \text{conv}(A_{i,j}, B_{i,j})\), represented in the figure by the star. Notice that \((1, 1)\) can be obtained as the convex combination of \(-1, -1\) and \(3, 0\). Therefore, \((i, j)\) can guarantee to obtain \((1, 1)\) in their infinitely repeated game by playing \((B, B)\) the first round, \((C, B)\) the second round, \((B, C)\) the third round and \((C, C)\) the fourth round, and cycling like that infinitely many times. As every 4 rounds the couple obtains \((1, 1)\), in the limit, their average payoff converges to \((\bar{u}, \bar{v})\).

Figure 1: Prisoners’ dilemma payoff profiles

Proof of Proposition 5.1. Consider the following optimization problem,

\[
\max_{\lambda \in \Delta(S_i \times T_j)} \sum_{s \in S_i, t \in T_j} A_{i,j}(s, t) \lambda_{s,t}
\]

\[
\text{s.t. } \sum_{s \in S_i, t \in T_j} B_{i,j}(s, t) \lambda_{s,t} \geq c
\]

Problem (20) is a linear programming problem with \(|S_i| \cdot |T_j|\) variables and two constraints. Its optimal value \((\bar{u}, \bar{v})\) coincides with the optimal value of Problem (7). Therefore, any strategy profile \(\sigma\) that achieves \((\bar{u}, \bar{v})\), is a solution of (7). The complexity stated in the proposition is obtained from solving (20) and applying Lemma 5.1 for computing \(\sigma\).

5.2 Algorithm 1: Complexity in infinitely repeated games

Proposition 5.1 allows us to prove the main result of this section.

Theorem 5.1 (Complexity). Let \(i \in M\) be a proposer man. Let \(j\) be the proposed woman and \(i'\) her current partner. If \(i\) is the winner of the competition, the entire iteration of Algorithm 7 has complexity,

\[
O \left( |W| + |S_i|^{2.5} \sum_{j' \in W} |T_{j'}|^{2.5} L_{i,j'} + |S_{j'}|^{2.5} \sum_{j' \in W} |T_{j'}|^{2.5} L_{i,j'} \right),
\]

where \(L_{m,w}\) is the number of bits required for encoding the payoff matrices of the couple \((m, w)\).

Proof. The optimal proposal problem is split in \(|W|\) problems, each of them having the complexity stated in Proposition 5.1. Thus, the optimal proposal computation has complexity,

\[
O \left( |W| + \sum_{j' \in W} (|S_i| \cdot |T_{j'}|)^{2.5} L_{i,j'} \right)
\]

Computing the reservation price and the bid of each competitor has exactly the same complexity than the optimal proposal computation, considering the respective set of strategies. Finally, the problem solved by the winner has complexity \(O \left( (|S_i| \cdot |T_j|)^{2.5} L_{i,j} \right)\). Summing up all, we obtain the complexity stated in the theorem.

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Remark 5.1. If there are at most \(N\) players in each side and at most \(k\) pure strategies per player, Theorem 5.1 proves that each iteration of Algorithm 2 takes \(O(Nk^5L)\) number of elementary operations with \(L\) the number of bits required for encoding all the data, hence it is polynomial. As Proposition 2.4 bounds the number of iterations of Algorithm 2 by \(\frac{1}{\epsilon}V^{\max}\), we conclude that computing an \(\epsilon\)-externally stable matching profile for \(G_{i,j}^\infty\) has complexity \(O(\frac{1}{\epsilon}(Nk^5L))\).

5.3 Algorithm 2: Convergence and complexity in infinitely repeated games

Before studying the complexity of Algorithm 2, we need further definitions, specific to infinitely repeated games. Let \(G_{i,j} = (X_i, Y_j, A_{i,j}, B_{i,j})\) be a bi-matrix stage game in mixed strategies and \(G_{i,j}^\infty\) its infinitely repeated version.

Definition 5.1. Consider the set of feasible payoffs \(co(A_{i,j}, B_{i,j}) := \{(A_{i,j}(s,t), B_{i,j}(s,t)) \in \mathbb{R}^2: (s,t) \in S_i \times T_j\}\). Define the punishment levels of \(i\) and \(j\) respectively by \(\alpha_i := \min_{y \in Y_j} \max_{x \in X_i} xA_{i,j}y, \beta_j := \min_{x \in X_i} \max_{y \in Y_j} xB_{i,j}y\). Define the set of uniform equilibrium payoffs as \(E_{i,j} := \{(\bar{u}, \bar{v}) \in co(A_{i,j}, B_{i,j}): \bar{u} \geq \alpha_i, \bar{v} \geq \beta_j\}\)

Actually, we know from the Folk theorem of Aumann-Shapley [4] that \(E_{i,j}\) is exactly the set of uniform equilibrium payoff of the \(G_{i,j}^\infty\). The proof is simple, it is enough to play the pure strategy profile that achieves some \((\bar{u}, \bar{v}) \in E_{i,j}\) as in the proof of Lemma 5.1 and, whenever a player deviates, the other player punishes him/her.

Definition 5.2. For every pair of outside options \((u_i, v_j)\) and \(\epsilon > 0\), the acceptable payoffs set is denoted \(E_{u_i,v_j}^\epsilon := co(A_{i,j}, B_{i,j}) \cap \{(u, v) \in \mathbb{R}^2: u + \epsilon \geq u_i, v + \epsilon \geq v_j\}\).

Definition 5.3. A strategy profile \(\sigma = (\sigma_i, \sigma_j)\) is an \(\epsilon\)-(\(u_i, v_j\))-CNE of \(G_{i,j}^\infty\) if,

1. \(\forall \epsilon > 0, \exists K_0, \forall K \geq K_0, \forall (\tau_i, \tau_j)\),
   
   \((a)\) if \(U_{i,j}(K, \tau_i, \sigma_j) > U_{i,j}(K, \sigma) + \epsilon\) then, \(V_{i,j}(K, \tau_i, \sigma_j) + \epsilon < v_j\),
   
   \((b)\) if \(V_{i,j}(K, \tau_i, \sigma_j) > V_{i,j}(K, \sigma) + \epsilon\) then, \(U_{i,j}(K, \tau_i, \sigma_j) + \epsilon < u_i\)

2. \((U_{i,j}(K, \sigma), V_{i,j}(K, \sigma)) \xrightarrow{K \to \infty} (U_{i,j}(\sigma), V_{i,j}(\sigma)) \in \mathbb{R}^2\) with \(U_{i,j}(\sigma) + \epsilon \geq u_i, V_{i,j}(\sigma) + \epsilon \geq v_j\)

The set of \(\epsilon\)-constrained Nash equilibrium payoffs is denoted \(E_{u_i,v_j}^{ce}\).

After all these definitions, we are going to prove the following theorem about the complexity of computing \(\epsilon\)-CNE.

Theorem 5.2 (CNE Complexity). Let \(G_{i,j}^\infty\) be an infinitely repeated game as defined above. Given any players’ outside options \((u_i, v_j)\) in \(\mathbb{R}^2\) such that \(E_{u_i,v_j}^\epsilon\) is non-empty, the complexity of computing a \((u_i, v_j)\)-constrained Nash equilibrium is at most, \(O(|S_i| \cdot |T_j|)^2.5L_{i,j}\), where \(L_{i,j}\) is the number of bits required for encoding the data of the stage game \(G_{i,j}\).

This theorem is the consequence of the following lemmas. First, from the Folk theorem of Aumann-Shapley [4], we have the following.

Lemma 5.2. Any vector payoff in \(E_{i,j} \cap E_{u_i,v_j}^\epsilon\) belongs to \(E_{u_i,v_j}^{ce}\).

Therefore, if the intersection in Lemma 5.2 is non-empty, there exists a uniform equilibrium payoff profile \((\bar{u}, \bar{v})\) that belongs to \(E_{u_i,v_j}^{ce}\). Combining with Lemma 5.1 we obtain a uniform equilibrium that achieves \((\bar{u}, \bar{v})\) with the complexity stated in Theorem 5.2. The following lemma provides sufficient conditions for that intersection to be non-empty.
Lemma 5.3. Let \((u_i, v_j)\) be a pair of outside options such that \(E_{u_i, v_j}^\varepsilon\) is non-empty. Then, \(E_{u_i, v_j}^\varepsilon\) is non-empty if either \(u_i - \varepsilon \geq \alpha_{i,j}\) and \(v_j - \varepsilon \geq \beta_{i,j}\), or \(u_i - \varepsilon < \alpha_{i,j}\) and \(v_j - \varepsilon < \beta_{i,j}\).

Proof. In the first case, \(E_{u_i, v_j}^\varepsilon \subseteq E_{i,j}\), thus the intersection between them is equal to \(E_{u_i, v_j}^\varepsilon\), which is non-empty. In the second case, \(E_{i,j} \subseteq E_{u_i, v_j}\) and therefore, the intersection is non-empty. \(\square\)

This yields to the two following missing cases.

Lemma 5.4. Let \((u_i, v_j)\) be a pair of outside options such that \(E_{u_i, v_j}^\varepsilon\) is non-empty. Then, computing an \(\varepsilon\)-CNE has complexity \(O((|S_i| \cdot |T_j|)^{2.5}L_{i,j})\) either if \(u_i - \varepsilon \geq \alpha_{i,j}\) and \(v_j - \varepsilon < \beta_{i,j}\), or \(u_i - \varepsilon < \alpha_{i,j}\) and \(v_j - \varepsilon \geq \beta_{i,j}\).

Proof. Suppose the first case, \(u_i - \varepsilon \geq \alpha_{i,j}\) and \(v_j - \varepsilon < \beta_{i,j}\). Let \(\varepsilon\) be a strategy in \(G^\infty_{i,j}\) that achieves \((\bar{u}, \bar{v})\)-CNE. Indeed, it is feasible as their limit payoff profile is \((\bar{u}, \bar{v} + \varepsilon)\). In addition, remark that \(i\) does not have profitable deviations as \(j\) punishes him and \(\bar{u} \geq u_i - \varepsilon \geq \alpha_{i,j}\). Finally, let \(K \in \mathbb{N}\) and \(\varepsilon > \varepsilon\) such that \(j\) can deviate at time \(K\) and get \(v' \geq (\bar{v} + \varepsilon)\). Let \(u'\) be the payoff of \(i\) until the stage \(K\). Notice that \((u', v') \in co(A_{i,j}, B_{i,j})\) since \((u', v')\) is an average payoff profile of the \(K\)-stage game. Suppose that \(u' \geq u_i - \varepsilon\), so \((u', v') \in E_{u_i, v_j}^\varepsilon\). Then, \(\bar{v} \geq v' \geq \bar{v} + \varepsilon + \varepsilon \geq \bar{v}\), which is a contradiction. Therefore, \(u' < u_i - \varepsilon\). Thus, \(\sigma'\) is an \(\varepsilon\)-(\(u_i, v_j\))-CNE. For the second case in which \(u_i - \varepsilon < \alpha_{i,j}\) and \(v_j - \varepsilon \geq \beta_{i,j}\), the argument is analogous. \(\square\)

As all the possible cases are covered by Lemma 5.3 and Lemma 5.4 we conclude the proof of Theorem 5.2 regarding constrained Nash equilibria. We focus next in the convergence of Algorithm 2 for infinitely repeated games.

Theorem 5.3 (Convergence). Let \(\pi = (\mu, \sigma_M, \sigma_W)\) be an \(\varepsilon\)-externally stable matching profile. Let \((u_i, v_j)_{(i,j) \in \mu}\) be the outside options obtained from \(\pi\). Then, there exists an oracle for computing CNE such that, starting from \(\pi\), Algorithm 2 converges in at most

\[
O\left(\frac{1}{\varepsilon} \left(\max_{(i,j) \in \mu} \{\max\{\alpha_{i,j} - u_i, \beta_{i,j} - v_j\}\}\right)\right)
\]

iterations, where \(\alpha_{i,j}, \beta_{i,j}\) are the punishment levels of the couple.

Proof. Let \((i,j) \in \mu\) be a couple and \((u_i, v_j)\) be their outside options at the beginning of Algorithm 2. Notice that one of the following four cases must hold:

1. \(u_i - \varepsilon \leq \alpha_{i,j}\) and \(v_j - \varepsilon \leq \beta_{i,j}\),
2. \(u_i - \varepsilon \geq \alpha_{i,j}\) and \(v_j - \varepsilon \geq \beta_{i,j}\),
3. \(u_i - \varepsilon \geq \alpha_{i,j}\) and \(v_j - \varepsilon < \beta_{i,j}\),
4. \(u_i - \varepsilon < \alpha_{i,j}\) and \(v_j - \varepsilon \geq \beta_{i,j}\)

Let \(F_{i,j} := E_{i,j} \cap E_{u_i, v_j}^\varepsilon\) and suppose it is non-empty. Then, there exists a feasible uniform equilibrium for \((i,j)\), so the couple changes only once of strategy profile and never again. Suppose \(F_{i,j}\) is empty. Necessarily it must hold case (3) or (4). Suppose \(u_i - \varepsilon \geq \alpha_{i,j}\) and \(v_j - \varepsilon < \beta_{i,j}\) and
consider the oracle given in the proof of Lemma 5.4. Then, the couple passes to gain \((\bar{u}, \bar{v} + \varepsilon)\), where \(\bar{v} = \max\{v : \exists u, (u, v) \in E_{u,v}^\varepsilon\}\) and \(\bar{u} \in \{u : (u, \bar{v}) \in E_{u,v}^\varepsilon\}\). Suppose that at some later iteration \((i, j)\) changes again the strategy profile they play. Let \((u'_i, v'_j)\) be their new outside options and consider again \(F_{i,j} := E_{i,j} \cap E_{u'_i,v'_j}\). If \(F_{i,j}\) is non-empty, the couples passes to play a feasible uniform equilibrium. Otherwise, the oracle computes a new payoff profile \((\bar{u}', \bar{v}')\) such that \(\bar{v}' = \max\{v : \exists u, (u, v) \in E_{u,v}^\varepsilon\}\) and \(\bar{u}' \in \{u : (u, \bar{v}') \in E_{u,v}^\varepsilon\}\). Since \(\pi\) remains all the time \(\varepsilon\)-externally stable, it holds \(u' \leq \bar{u} + \varepsilon, v' \leq (\bar{v} + \varepsilon) + \varepsilon\). Therefore, \((\bar{u}, \bar{v} + \varepsilon) \in E_{u,v}^\varepsilon\) and then, \(\bar{v}' \geq \bar{v} + \varepsilon\). We conclude that at each iteration, either the couple changes to play a feasible uniform equilibrium, or player \(j\) increases her payoff in at least \(\varepsilon\). If case (4) holds, the conclusion is the same: at each iteration, either the couple plays a feasible uniform equilibrium or player \(i\) increases by at least \(\varepsilon\) his payoff. Thus, we obtain the number of iterations given in the statement of the theorem by considering the worst possible case.

**Remark 5.2.** If there are at most \(N\) players in each side and at most \(k\) pure strategies per player, Theorem 5.2 proves that each iteration of Algorithm 2 takes \(O(Nk^5L)\) number of elementary operations in being solved, with \(L\) the number of bits required for encoding all the data, hence is polynomial. As Theorem 5.3 bounds the number of iterations of Algorithm 3 by \(\frac{1}{\varepsilon}C\), with \(C\) a constant, we conclude that computing an \(\varepsilon\)-internally stable matching profile for an infinitely repeated matching game has complexity \(O\left(\frac{1}{\varepsilon}(Nk^5L)\right)\).

| Algorithm | Complexity per Iteration | \(N^0\) Iterations | Constants |
|-----------|--------------------------|---------------------|-----------|
| Algorithm 1 | \(O(Nk^5L)\) | \(C_1/\varepsilon\) | \(C_1 \leq \max_{i,j}\{\max B_{i,j} - \min B_{i,j}\}\) |
| Algorithm 2 | \(O(Nk^5L)\) | \(C_2/\varepsilon\) | \(C_2 \leq \max_{i,j}\{\max\{D_i, D_j\}\} D_i := \max A_{i,j} - \min A_{i,j} D_j := \max B_{i,j} - \min B_{i,j}\) |

Table 5: Complexity infinitely repeated games, \(N\) number players per side, \(k\) strategies per player, \(L\) bits for encoding the data, \(A_{i,j}, B_{i,j}\) payoff matrices of the stage game of \((i, j)\)

6 Conclusion

This paper provides an efficient PTAS \(^4\) to compute \(\varepsilon\)-stable allocations in several classes of matching games. The model of matching games covers and refines the various existing matching models with and without transfers, but can be applied much beyond the classical range of applications.

Nowadays it is well documented \(^4\) that an energy consumer cares about who produces the energy (he/she may prefer a local producer), the type of energy (sunlight, wind, water), the guarantee that his/her demand will be covered and, of course, the price. On the other hand, a producer can promise to a consumer that, for example, at least 40% of the energy received is green or is produced by some local producer; can decide to invest in different technologies and/or in the capacity of production for each production site; and a pricing policy. The model studied in this article, solution concepts, and algorithms can be used in green energy matching platforms \(^4\) to match consumers and producers.

\(^4\)Several studies show the consumers’ willingness to pay for green energy. See for example \(10, 31, 35\) for the US, \(27\) for Greece, \(33\) for Japan, and \(40\) for Korea.

\(^4\)Several electronic platforms to trade green energy around the world: Germany: https://sonnenbatterie.de/en/sonnenCommunity, USA: https://medium.com/le-lab/transactive-grid-le-r%C3%A9seau-d%C3%A9nergie-intelligent-%C3%A0-brooklyn-ead550918cc2, Netherlands: https://vandebron.nl, UK:
such that (1) consumers total demands for a given producer are less than its production, (2) each consumer is allocated the best possible price, contact, or technology for her from that producer, (3) no producer is better off by offering another contract, price or investing in a cheaper technology, and (4) no unmatched producer-consumer can increase their utility by contracting. Conditions (2) and (3) are the internal stability notions, (4) is the external stability condition.

At a more theoretical level, the matching game model and stability notions are closely related to the strategic network formation model and the associated Nash pairwise stability notion studied in Bich and Morhaim [8], itself motivated by a large body of literature (e.g. Jackson and Wolinsky [22]). This model considers strategic interactions in which each player decides with whom he wants to create a link and, simultaneously, which action to play. A link is formed between two agents only if both agree (and have an interest) to form it. A player’s payoff function depends on the adjacency matrix of the network of links and the profile of all players’ actions. Hence, the stable matching game model can be seen as a particular network game model where only bi-party graphs are possible and a link is formed if a man and a woman agree to match.

More precisely, Bich and Morhaim [8] use Browder fixed point theorem to prove the existence of a mixed Nash pairwise equilibrium. Matching game model –where only bipartite graphs can be formed– has a constructive proof of existence (algorithmic), and the stability notion is much stronger than Bich-Morhaim’s (in their case, when a new couple form, they cannot change their strategies. Also, when a player deviates to a new strategy, he/she assumes that his/her deviation will not break the stability of the couple). This raises several interesting questions: how to compute their solutions using the tools exposed in our article and how to prove the existence of stable allocations in their model?

Another natural extension of the model of stable matching games is a dynamic matching setting where each player can, at each time, marry, divorce, or change the partner, and if they match at some period they play a stage game. The solution developed in Section 5 where couples play an infinitely repeated game can be used to construct a usual uniform Nash equilibrium in the dynamic matching model, but there may be other uniform equilibria in which players can divorce and change the partner from one period to the other. This interesting extension is delegated to a future study. Remark it may also have some practical interest, e.g. it can be applied in the green energy platforms problem described above as electricity production and consumption are usually uncertain, and so the platforms must dynamically change and update their allocations.

Finally, observe that all known algorithms of matching with transfers (e.g. Kelso and Crawford [24] and Demange et al. [14]) converges to an ε-pairwise stable matching but not to an exact stable matching. Consequently, finding an algorithm that converges to an exact externally stable matching is an interesting open problem even in the simple matching game model with linear transfers.

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A numerical example

Consider a matching game with three men $M = \{i_1, i_2, i_3\}$ and three women $W = \{j_1, j_2, j_3\}$, who have the following preferences,

$$
A = \begin{pmatrix}
83 & 85 & 99 \\
74 & 13 & 15 \\
58 & 49 & 54
\end{pmatrix}
\quad B = \begin{pmatrix}
69 & 6 & 28 \\
88 & 2 & 70 \\
72 & 18 & 9
\end{pmatrix}
$$

$A(i, j)$ (resp. $B(i, j)$) represents the fixed utility that $i \in M$ (resp. $j \in W$) receives if he (resp. she) matches with $j$ (resp. $i$). Suppose that all agents have null IRPs, i.e. $(u_i, v_j) = (0, 0), \forall (i, j) \in M \times W$. Suppose in addition that if a couple $(i, j)$ is created, as in Example 1, each agent can decide for a linear monetary transfer to the partner. Formally, if $x_i, y_j \geq 0$ are $i$ and $j$ respective monetary transfers, their utilities are $U_{i,j}(x_i, y_j) = A(i, j) - x_i + y_j$ and $V_{i,j}(x_i, y_j) = B(i, j) + x_i - y_j$.

A.1 Computing a Nash stable allocation

As already discussed, internal Nash stable allocations for matching with transfer problems correspond to matching profiles in which no agent makes a positive transfer. Due to this, computing a Nash stable allocation is equivalent to Gale-Shapley’s stable allocations where the ordinal preferences are deduced from the matrices $A$ and $B$. Algorithm 3 is equivalent to Gale-Shapley’s algorithm in which men are proposing, as competitions are reduced to preference comparisons. We can summarize Algorithm 3 in that case in the following pseudo-code.

Algorithm 3: Propose-dispose algorithm

1. Set $M' \leftarrow \{i_1, i_2, i_3\}$ as the set of single men and $v_j(\pi) = 0, \forall j \in W$
2. while $M' \neq \emptyset$ do
   3. Let $i \in M'$ and $j$ such that $A(i, j) = \arg\max\{A(i, \cdot) : B(i, j) > v_j(\pi)\}$
   4. if $j$ is single then
      5. $i$ is automatically accepted
   6. else
      7. $j$ chooses $\arg\max\{A(i, j); A(\mu_i, j)\}$ and the loser is included in $M'$

Let us run Algorithm 3 iteration by iteration.

Iter 1. Suppose $i_1$ is the first proposer man. Looking at the first row of $A$, we see that $i_1$ prefers to propose to $j_2$, who accepts him.

Iter 2. Suppose $i_3$ is the second proposer. From $A(i_3, \cdot)$, $i_3$ proposes to $j_1$, who accepts him as she is single.

Iter 3. Finally, $i_2$ proposes as he is the only remaining man. From the payoff matrix, $i_2$ proposes to $j_1$ and has to compete against $i_3$. From the first column of $A$, the winner is $i_2$. $i_3$ becomes single and proposes again.

Iter 4. $i_3$ computes his optimal proposal. As $B(i_3, j_1) \leq v_{j_1}(\pi)$, $i_3$ proposes to his second best option, $j_3$. However, $j_3$ is already matched with $i_1$, who is better ranked than $i_3$ according to $j_3$’s preferences, so $i_3$ is rejected.

Iter 5. Finally, $i_3$ proposes to $j_2$ as she is his only feasible proposal, who accepts him, and the algorithm ends.
The output is $\mu = ((i_1, j_3), (i_2, j_1), (i_3, j_2))$. We see that $i_1$ and $i_2$ are matched with their best possible options, so they will never prefer to change to another partner. Regarding $i_3$, although he would prefer to change to another partner, neither $j_1$ nor $j_3$ would accept him, as each of them prefers their actual partners. We conclude that $\pi = (\mu, 0, 0)$ is a Nash stable matching profile where $(0, 0)$ means that there the transfer vector $(x, y)$ is equal to zero.

### A.2 Computing an externally stable allocation

Unlike Nash stability, as it will be seen now, the external stability concept will output a matching profile in which (some) agents will make a positive transfer. First, let’s recall the payoff matrices and the propose-dispose algorithm.

\[
A = \begin{bmatrix} 83 & 85 & 99 \\ 74 & 13 & 15 \\ 58 & 49 & 54 \end{bmatrix} \quad B = \begin{bmatrix} 69 & 6 & 28 \\ 88 & 2 & 70 \\ 72 & 18 & 9 \end{bmatrix}
\]

**Algorithm 4:** Propose-dispose algorithm

1. Set $M' \leftarrow \{i_1, i_2, i_3\}$ as the set of single men and $v_j(\pi) = 0, \forall j \in W$
2. \textbf{while} $M' \neq \emptyset$ \textbf{do}
3. \hspace{1em} Let $i \in M'$ and $(j, x_i, y_j) \in \arg\max\{A(i, j') - x_i + y_j : B(i, j') + x_i - y_j > v_j(\pi) + \varepsilon : j' \in \{j_1, j_2, j_3\}\}$
4. \hspace{2em} \textbf{if} $j$ is single \textbf{then}
5. \hspace{3em} $i$ is automatically accepted
6. \hspace{2em} \textbf{else}
7. \hspace{3em} $i$ and $\mu_i$ compete for $j$: Each of them computes
\[
\max\{B(i, j) + x_i - y_j : A(i, j) - x_i + y_j \geq \beta_i\}
\]
8. \hspace{3em} with $\beta_i$ being $i$’s reservation price. The highest bid wins and the winner decreases
\[9. \hspace{1em} \text{the bid until matching the one of the loser}\]

Let $\varepsilon = 1$ and let us consider the same order that for the previous section:

**Iter 1.** Suppose $i_1$ is the first proposer man. He solves
\[
\max\{A(i_1, j) - x_{i_1} + y_j : B(i_1, j) + x_{i_1} - y_j \geq \varepsilon, j \in W\} = 151
\]
\[\iff \max\{A(i_1, j) + B(i_1, j) - \varepsilon : j \in W\} = \max\{83 + 69 - 1, 85 + 6 - 1, 99 + 28 - 1\} = 151 \]

Therefore, the optimal proposal is $(j_1, 0, 68)$ i.e. $i_1$ proposes to $j_1$ and takes the highest possible profit from their transfer game by offering to $j_1$ exactly her IRP. Since $j_1$ is single, he accepts him automatically.

**Iter 2.** Suppose $i_3$ proposes next. Similarly, he solves the problem (notice that still all women have 0 payoff),
\[
\max\{A(i_3, j) - x_{i_3} + y_j : B(i_3, j) + x_{i_3} - y_j \geq \varepsilon, j \in W\} = 129
\]
\[\iff \max\{A(i_3, j) + B(i_3, j) - \varepsilon : j \in W\} = \max\{58 + 72 - 1, 49 + 18 - 1, 54 + 9 - 1\} = 129 \]

\[5\text{Any transfer profile } (x_i, y_j) \text{satisfying } -x_i + y_j = 68 \text{ is also an optimal proposal. Taking, in particular, an optimal proposal in which one of the agents makes a null transfer will be useful for the next section.}\]
Therefore, the optimal proposal for $i_3$ is $(j_1, 0, 71)$. However, since $j_1$ is already matched, $i_1$ and $i_3$ compete. First of all, notice that $i_3$’s reservation price is equal to 66, as it corresponds to the second highest value in the calculation of his optimal proposal. Similarly, $i_1$’s reservation price is equal to 126. Let us compute the bid of $i_1$,

$$\lambda_{i_1} = \max\{B(i_1, j_1) + x_{i_1} - y_{j_1} : A(i_1, j_1) - x_{i_1} + y_{j_1} \geq 126\}$$

$$\iff \lambda_{i_1} = \max\{69 + x_{i_1} - y_{j_1} : 83 - x_{i_1} + y_{j_1} \geq 126\}$$

$$\iff \lambda_{i_1} = \max\{69 + x_{i_1} - y_{j_1} : x_{i_1} - y_{j_1} \leq -43\} = 69 - 43 = 26$$

Similarly for $i_3$,

$$\lambda_{i_3} = \max\{72 + x_{i_3} - y_{j_1} : 58 - x_{i_3} + y_{j_1} \geq 66\}$$

$$\iff \lambda_{i_3} = \max\{72 + x_{i_3} - y_{j_1} : x_{i_3} - y_{j_1} \leq -8\} = 72 - 8 = 64$$

Since $i_3$’s bid is the highest, he wins the competition. Finally, he decreases his bid until matching the one of $i_1$,

$$\max\{A(i_3, j_1) - x_{i_3} + y_{j_1} : B(i_3, j_1) + x_{i_3} - y_{j_1} \geq \lambda_{i_1}\}$$

$$\iff \max\{58 - x_{i_3} + y_{j_1} : 72 + x_{i_3} - y_{j_1} \geq 26\}$$

$$\iff \max\{58 - x_{i_3} + y_{j_1} : x_{i_3} - y_{j_1} \leq 46\} = 58 + 46 = 104$$

Therefore, the final transfer profile between $i_3$ and $j_1$ is $(x_{i_3}, y_{j_1}) = (0, 46)$. Notice that $j_1$ reaches exactly the previous bid of $i_1$. As this is the highest payoff that $i_1$ is willing to offer her, he will not propose to her if he is the next proposer man.

Iter 3. Suppose $i_2$ proposes next. His optimal proposal is,

$$\max\{A(i_2, j) - x_{i_2} + y_j : B(i_2, j) + x_{i_2} - y_j \geq \epsilon, j \in \{j_2, j_3\}, B(i_2, j_2) + x_{i_2} - y_{j_2} \geq 26 + \epsilon\}$$

$$\iff \max\{A(i_2, j_1) + B(i_2, j_1) - 26 - \epsilon, A(i_2, j_2) + B(i_2, j_2) - \epsilon, A(i_2, j_3) + B(i_2, j_3) - \epsilon\}$$

$$\iff \max\{74 + 88 - 26 - 1, 13 + 2 - 1, 15 + 70 - 1\} = 135$$

Thus, $i_2$ proposes to $j_1$ the transfer profile $(0, 61)$, and has to compete against $i_3$. The reservation prices of the men are $\beta_{i_2} = 94$ and $\beta_{i_3} = 66$. Then, the bids are equal to,

$$\lambda_{i_2} = \max\{B(i_2, j_1) + x_{i_2} - y_{j_1} : A(i_2, j_1) - x_{i_2} + y_{j_1} \geq \beta_{i_2}\} = 68$$

$$\lambda_{i_3} = \max\{B(i_3, j_1) + x_{i_3} - y_{j_1} : A(i_3, j_1) - x_{i_3} + y_{j_1} \geq \beta_{i_3}\} = 64$$

Since $\lambda_{i_2}$ is the highest value, $i_2$ is the winner of the competition. Finally, he decreases his offer for matching the one of $i_3$,

$$\max\{A(i_2, j_1) - x_{i_2} + y_{j_1} : B(i_2, j_1) + x_{i_2} - y_{j_1} \geq \lambda_{i_3}\} = 98$$

and so, the final transfer profile between $i_2$ and $j_1$ is $(x_{i_2}, y_{j_1}) = (0, 24)$.

Iter 4. Suppose $i_1$ proposes next. Since $j_1$’s payoff is too high for offering her a profitable (for $i_1$) contract, $i_1$ proposes to his next best option, $j_3$ the transfer profile $(x_{i_1}, y_{j_3}) = (0, 27)$. Since $j_3$ is single, she accepts automatically.

Iter 5. Finally, $i_3$ proposes again as he is the only single man. Since he lost the competition for $j_1$, he is not able to propose her again and increase her payoff by $\epsilon$ without violating his own reservation price. Thus, he offers to his second-best option $j_2$ the transfer profile $(0, 17)$. Since she is single, $i_3$ is automatically accepted and the algorithm stops.
The output of Algorithm 1 is \( \pi = (\mu, x, y) \) with \( \mu = ((i_1, j_3), (i_2, j_1), (i_3, j_2)) \) and \( (x, y) = ((0, 0, 0), (24, 17, 27)) \). The final payoffs of the players are, \( (u(\pi), v(\pi)) = ((126, 98, 66), (64, 1, 1)) \). For checking the \( \varepsilon \)-external stability of \( \pi \), we compute the outside options of one side and check that none of them is \( \varepsilon \) higher than the players’ payoff. Formally, for \( i \in M \) we check that,

\[
u_i^\varepsilon = \max\{A(i, j) - x_i + y_j : B(i, j) + x_i - y_j \geq v_j(\pi) + \varepsilon, j \in W \setminus \{\mu_i\}\} \leq u_i(\pi) + \varepsilon
\]

\[
\iff \nu_i^\varepsilon = \max\{A(i, j) + B(i, j) - v_j(\pi) - \varepsilon, j \in W \setminus \{\mu_i\}\} \leq u_i(\pi) + \varepsilon
\]

We obtain that \( u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) = (89, 83, 65) \). Since men’s payoff in \( \pi \) is strictly higher than their outside options, none of them has the incentive to change of partner. Therefore, we conclude that \( \pi \) is \( \varepsilon \)-externally stable, for \( \varepsilon = 1^6 \)

### A.3 Computing an externally-internally stable allocation

The externally stable matching profile \( \pi \) found by Algorithm 1 at the previous section is known to be the most preferred stable allocation by the proposer side. However, there is a continuum of strategy profiles that achieve the same payoffs that \( \pi \). Indeed, for any \( \delta \in [0, \min\{y_{j_1}, y_{j_2}, y_{j_3}\}] \), considering the shifted matching profile \( \pi_\delta = (\mu, x + \delta, y - \delta) \) gives the same payoffs to the agents and therefore, it remains \( \varepsilon \)-externally stable. However, from this continuum of solutions, there is only one internally stable.

Let us start this section by studying the constrained Nash equilibria. Recall that \( (x_i, y_j) \in \mathbb{R}_+^2 \) is an \( \varepsilon \)-\((u_i^\varepsilon, v_j^\varepsilon)\)-CNE if and only if,

\[
U_{i,j}(x_i, y_j) + \varepsilon \geq \max\{U_{i,j}(s, y_j) : V_{i,j}(s, y_j) + \varepsilon \geq v_j^\varepsilon, s \geq 0\} \iff V_{i,j}(x_i, y_j) \leq v_j^\varepsilon
\]

\[
V_{i,j}(x_i, y_j) + \varepsilon \geq \max\{V_{i,j}(x_i, t) : U_{i,j}(x_i, t) + \varepsilon \geq u_i^\varepsilon, t \geq 0\} \iff U_{i,j}(x_i, y_j) \leq u_i^\varepsilon
\]

In words, as players’ profitable deviations are equivalent to decreasing their transfers, a couple plays a constrained Nash equilibrium if each of them is receiving no more than their outside option, so transfers are shifted to the minimum possible without breaking the external stability.

We state the pseudo-code of Algorithm 2 for this problem.

**Algorithm 5: Strategy profiles modification, \( \varepsilon \)-version**

1. repeat
2. \hspace{1em} for \((i, j) \in \mu\) do
   3. \hspace{2em} Compute \((u_i^\varepsilon, v_j^\varepsilon)\) and reduce \((x_i, y_j)\) s.t. \(U_{i,j}(x_i, y_j) = u_i^\varepsilon\) and \(V_{i,j}(x_i, y_j) = v_j^\varepsilon\).
4. until Convergence;

Let us run the strategy profiles modification algorithm iteration by iteration.

**Iter 1.** Let \((i_1, j_3)\) be the first couple. Since \(i_1\) is not making any transfer, only \(j_3\) decreases hers. Recall \(u_{i_1}^\varepsilon = 65\), therefore \(j_3\) passes to transfer \(y_{j_3}: A(i_1, j_3) + y_{j_3} = u_{i_1}^\varepsilon \iff y_{j_3} = 65 - 99 = -34\). Since transfers are non-negative, \(j_3\) decreases hers until 0. The new payoffs of the couple become \((u_{i_1}(\pi), v_{j_3}(\pi)) = (99, 28)\). Notice that the couple has passed to play the Nash equilibrium of their game, as this one is a feasible strategy profile.

**Iter 2.** Consider the second couple \((i_2, j_1)\). Again, only \(j_1\) can deviate as \(i_2\) already plays a best reply. We obtain that \(u_{i_2}^\varepsilon = 15\) thus, \(j_1\) decreases \(y_{j_1}\) such that \(A(i_2, j_1) + y_{j_1} = 15\), so \(y_{j_1}\) becomes

---

6. As the payoff matrices have integer values, the solution found is indeed a 0-externally stable allocation

7. In reality, as we are working with \( \varepsilon \)-external stability, players’ payoffs can be decreased until \( \varepsilon \) less than their outside options
0. The new payoffs of the couple is \((u_{i_2}(\pi), v_{j_1}(\pi)) = (74, 88)\), that corresponds to their Nash equilibrium payoff.

**Iter 3.** Consider the last couple \((i_3, j_2)\). It follows \(u_{i_3}^e = 41\) so \(j_2\) decreases \(y_{j_2}\) such that \(A(i_3, j_2) + y_{j_2} = 41\). Thus, \(y_{j_2} = 0\) and the third couple also passes to play a Nash equilibrium, getting as payoffs \((u_{i_3}(\pi), v_{j_2}(\pi)) = (49, 18)\).

As all couples switch to play the Nash equilibrium of their game, the algorithm stops. Hence starting from the best externally stable matching for men, when we apply algorithm 2, we find the best solution for men in the model without transfers (e.g. the Gale-Shapley original model). This property is not always true: in Example 1, in the unique externally-internally stable matching, the man offers a positive transfer \(\delta\) to the woman.