A one-dimensional soliton system of gauged Q-ball and anti-Q-ball

A. Yu. Loginov and V. V. Gauzhtein
Tomsk Polytechnic University, 634050 Tomsk, Russia
(Dated: January 3, 2019)

The (1 + 1)-dimensional gauge model of two complex self-interacting scalar fields that interact with each other through an Abelian gauge field and a quartic scalar interaction is considered. It is shown that the model has nontopological soliton solutions describing soliton systems consisting of two Q-ball components possessing opposite electric charges. The two Q-ball components interact with each other through the Abelian gauge field and the quartic scalar interaction. The interplay between the attractive electromagnetic interaction and the repulsive quartic interaction leads to the existence of symmetric and nonsymmetric soliton systems. Properties of these systems are investigated by analytical and numerical methods. The symmetric soliton system exists in the whole allowable interval of the phase frequency, whereas the nonsymmetric soliton system exists only in some interior subinterval. Despite the fact that these soliton systems are electrically neutral, they nevertheless possess nonzero electric fields in their interiors. It is found that the nonsymmetric soliton system is more preferable from the viewpoint of energy than the symmetric one. Both symmetric and nonsymmetric soliton systems are stable to the decay into massive scalar bosons.

PACS numbers: 11.27.+d, 11.10.Lm, 11.15.-q

I. INTRODUCTION

There are many field models possessing global symmetries and corresponding conserved Noether charges that admit the existence of nontopological solitons [1, 2]. The determining property of a nontopological soliton is that it is an extremum of the energy functional at a fixed value of the Noether charge. This feature of nontopological solitons leads to the characteristic time dependence \( \propto \exp(-i\omega t) \) of their fields. This nontrivial time dependence of the soliton’s field allows to avoid severe restrictions of Derrick’s theorem [3], so scalar nontopological solitons can exist in any number of spatial dimensions.

The simplest nontopological soliton, proposed in [4] and known as a Q-ball [5], has been found in a model of a complex scalar field possessing a global \( U(1) \) symmetry. Q-balls can also exist in scalar field models possessing a global non-Abelian symmetry [6, 7]. They are present in the minimal supersymmetric extension of the Standard Model having flat directions in the interaction potential of scalar fields. Q-balls are of great interest to cosmological models describing the evolution of the early Universe [10, 11].

There are also other types of nontopological solitons in global-symmetric field models. The most known of them is the nontopological soliton of the Friedberg-Lee-Sirlin model [12]. The model consists of two interacting scalar fields, one of which is real and the other is complex. It possesses a global \( U(1) \) symmetry and a renormalizable interaction potential. Another example is the nontopological soliton in the model of a massive self-interacting complex vector field [13].

In all of the examples given above, the existence of nontopological solitons is due to a global invariance of the corresponding Lagrangians, so the Noether charge of such solitons cannot be a source of a gauge field. At the same time, nontopological solitons also exist in field models possessing a local gauge invariance, both Abelian [14–19] and non-Abelian [20, 21]. The nontopological solitons [14–16] possess a long-range gauge electric field, and Noether charges of these solitons are proportional to their electric charges. However, all these electrically charged nontopological solitons are three-dimensional ones. This is because any one-dimensional or two-dimensional field configuration with a nonzero electric charge possesses infinite energy, as it follows from Gauss’s law and the expression for the electric field energy density. Nevertheless, there are electrically neutral low-dimensional soliton systems that have a nonzero electric field in their interiors. In particular, the two-dimensional soliton systems consisting of vortex and Q-ball components interacting through an Abelian gauge field have been described in [22, 23].

In the present paper, we research the (1 + 1)-dimensional gauge model of two complex self-interacting scalar fields interacting with each other through an Abelian gauge field and a quartic scalar interaction. In particular, it is found that symmetric and nonsymmetric soliton systems exist in the model. The soliton systems consist of two Q-ball components having opposite electric charges. The soliton systems are electrically neutral but nevertheless possess nonzero electric fields in their interiors. The paper is structured as follows. In Sec. II we describe briefly the Lagrangian and the field equations of the model under consideration. By means of the Hamiltonian formalism and the Lagrange multipliers method, the time dependence is established for the soliton system’s fields. Then, we give the ansatz used for solving the model’s field equations and establish the basic relation for the nontopological soliton system. In Sec. III we derive the system of nonlinear differential
equations for the ansatz functions and the expressions for the electromagnetic current density and the energy density in terms of these functions. Then, some general properties of the soliton system are established, its asymptotic properties are researched, and the virial relation for the soliton system is derived. In Sec. IV, we study properties of the soliton system in the thick-wall and thin-wall regimes and establish its stability to decay into free massive scalar bosons. In Sec. V, we briefly describe the procedure for numerical solving of a boundary value problem and discuss possible types of soliton solutions of the problem. The dependences of the energy and the Noether charge on the phase frequency are presented for both (symmetric and nonsymmetric) types of the soliton solutions. Then, we show the dependences of the symmetric soliton system’s energy and the energy difference between the symmetric and nonsymmetric soliton systems on the Noether charge. After that, we present the numerical results for the ansatz functions, the energy density, the electric charge density, and the electric field strength for the symmetric and nonsymmetric soliton systems.

Throughout the paper the natural units $c = 1, \hbar = 1$ are used.

II. THE LAGRANGIAN AND THE FIELD EQUATIONS

The (1 + 1)-dimensional gauge model we are interested in is described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_{\mu}\phi)^* D^{\mu}\phi + (D_{\mu}\chi)^* D^{\mu}\chi - V(|\phi|) - U(|\chi|) - W(|\phi|, |\chi|).$$

(1)

It describes the two complex scalar fields $\phi$ and $\chi$ that minimally interact with the Abelian gauge field $A_{\mu}$ through the covariant derivatives:

$$D_{\mu}\phi = \partial_{\mu}\phi + ieA_{\mu}\phi, \quad D_{\mu}\chi = \partial_{\mu}\chi + iqA_{\mu}\chi. \quad (2)$$

The scalar fields interact with each other and self-interact. The self-interaction potentials of the scalar fields have the form

$$V(|\phi|) = m_{\phi}^2 |\phi|^2 - \frac{g_{\phi}}{2} |\phi|^4 + \frac{h_{\phi}}{3} |\phi|^6, \quad (3)$$

$$U(|\chi|) = m_{\chi}^2 |\chi|^2 - \frac{g_{\chi}}{2} |\chi|^4 + \frac{h_{\chi}}{3} |\chi|^6, \quad (4)$$

whereas the interaction potential is

$$W(|\phi|, |\chi|) = \lambda |\phi|^2 |\chi|^2. \quad (5)$$

We suppose that the self-interaction potentials $V$ and $U$ admit the existence of usual non-gauged nontopological solitons (Q-balls) formed from the scalar fields $\phi$ and $\chi$, respectively. We also suppose that the potentials $V$ and $U$ possess global minima at $\phi = 0$ and $\chi = 0$, respectively. Then the parameters of the potentials satisfy the condition

$$\frac{m_{\phi}^2 h_{\phi}}{g_{\phi}} > \frac{3}{16}, \quad (6)$$

where the index $i = (\phi, \chi)$.

The Lagrangian (1) is invariant under the local gauge transformations

$$j_{\mu}^\phi = i \left[ \phi^* D^\mu D^\rho \phi - (D^\rho \phi)^* \phi \right], \quad (8a)$$

$$j_{\mu}^\chi = i \left[ \chi^* D^\mu \chi - (D^\mu \chi)^* \chi \right]. \quad (8b)$$

The presence of the two separately conserved Noether charges $Q_\phi = \int j_\phi^\mu dx$ and $Q_\chi = \int j_\chi^\mu dx$ is the result of the structure of the interaction potential $W$ and the neutrality of the Abelian gauge field $A_{\mu}$.

The field equation of the model are obtained by varying the action $S = \int \mathcal{L} dx$ in the corresponding fields:

$$D_{\mu} D^{\mu} \phi = \frac{\partial V}{\partial |\phi|} \frac{\phi}{|\phi|}, \quad D_{\mu} D^{\mu} \chi = \frac{\partial U}{\partial |\chi|} \frac{\chi}{|\chi|}, \quad \frac{\partial W}{\partial (|\phi|, |\chi|)} = 0, \quad (9)$$

$$D_{\mu} D^{\mu} \phi + \frac{\partial W}{\partial |\phi|} \frac{\phi}{|\phi|} + \frac{\partial W}{\partial |\chi|} \frac{\chi}{|\chi|} = 0, \quad (10)$$

$$\partial_{\mu} F^{\mu\nu} = j^\nu, \quad (11)$$

where the electromagnetic current $j^\nu$ is written in terms of two Noether currents $\mathcal{S}$

$$j^\nu = e j^\nu_\phi + q j^\nu_\chi. \quad (12)$$

The symmetric energy-momentum tensor of the model is written as

$$T_{\mu\nu} = -F_{\lambda\nu} F^\lambda_\mu + \frac{1}{4} g_{\mu\nu} F_{\lambda\rho} F^{\lambda\rho} + (D_{\mu}\phi)^* D_{\nu}\phi + (D_{\mu}\phi)^* D_{\nu}\phi$$

$$+ (D_{\mu}\chi)^* D_{\nu}\chi + (D_{\mu}\chi)^* D_{\nu}\chi - g_{\mu\nu} \left[ (D_{\mu}\phi)^* D_{\nu}\phi + (D_{\mu}\chi)^* D_{\nu}\chi \right]$$

$$- V(|\phi|) - U(|\chi|) - W(|\phi|, |\chi|), \quad (13)$$

so we have the following expression for the energy density

$$T_{00} = e = \frac{1}{2} E^2 + (D_t \phi)^* D_t \phi + (D_x \phi)^* D_x \phi$$

$$+ (D_t \chi)^* D_t \chi + (D_x \chi)^* D_x \chi$$

$$+ V(|\phi|) + U(|\chi|) + W(|\phi|, |\chi|). \quad (14)$$

By analogy with nontopological solitons, we find a solution of model (11) that is an extremum of the energy functional $E = \int E dx$ at a fixed value of the Noether
charge $Q_\chi = \int j_\chi^0 dx$. Such a solution is an unconditional extremum of the functional

$$F = \int \mathcal{E} dx - \omega \int j_\chi^0 dx = E - \omega Q_\chi,$$

where $\omega$ is the Lagrange multiplier. To determine the time dependence of the soliton solution, we will use the Hamiltonian formalism. We adopt the axial gauge in which the spatial component of the gauge potential vanishes: $A_x = A^1 = 0$. In this case, the gauge model is described in terms of the eight canonically conjugated fields: $\phi$, $\pi_\phi = (D_0 \phi)^*$, $\phi^*$, $\pi_{\phi^*} = D_0 \phi$, $\chi$, $\pi_\chi = (D_0 \chi)^*$, $\chi^*$, and $\pi_{\chi^*} = D_0 \chi$. Then, the Hamiltonian density has the form

$$\mathcal{H} = \pi_\phi \partial_t \phi + \pi_{\phi^*} \partial_t \phi^* + \pi_\chi \partial_t \chi + \pi_{\chi^*} \partial_t \chi^* - \mathcal{L}$$

$$= -\frac{1}{2} (\partial_x A_0)^2 + \pi_\phi \pi_{\phi^*} + \pi_\chi \pi_{\chi^*} + \partial_x \phi^* \partial_x \phi + \partial_x \chi^* \partial_x \chi$$

$$+ i e A_0 \{ \phi^* \pi_\phi - \phi \pi_{\phi^*} \} + i q A_0 \{ \chi^* \pi_\chi - \chi \pi_{\chi^*} \}$$

$$+ V (|\phi|^2) + U (|\chi|^2) + W (|\phi|, |\chi|), \quad (16)$$

where the time component $A_0$ is determined in terms of the canonically conjugated fields by Gauss’s law

$$\partial_t A_0 + ie \{ \phi^* \pi_\phi - \phi \pi_{\phi^*} \} + i q A_0 \{ \chi^* \pi_\chi - \chi \pi_{\chi^*} \} = 0. \quad (17)$$

Note that energy density (14) does not coincide with Hamiltonian density (16):

$$\mathcal{H} - \mathcal{E} = - (\partial_x A_0)^2 + ie A_0 \{ \phi^* \pi_\phi - \phi \pi_{\phi^*} \}$$

$$+ i q A_0 \{ \chi^* \pi_\chi - \chi \pi_{\chi^*} \}. \quad (18)$$

However, the integral of Eq. (18) over the space dimension vanishes for field configurations possessing finite energy and satisfying Gauss’s law (17). So, for such configurations

$$E = \int \mathcal{E} dx = H = \int \mathcal{H} dx. \quad (19)$$

It can be shown that the field equations (9) and (10) can be rewritten in the Hamiltonian form:

$$\partial_t \phi = \frac{\delta H}{\delta \pi_\phi}, \quad \partial_t \pi_\phi = -\frac{\delta H}{\delta \phi}, \quad \partial_t \pi_{\phi^*} = -\frac{\delta E}{\delta \phi}, \quad (20)$$

$$\partial_t \chi = \frac{\delta H}{\delta \pi_\chi}, \quad \partial_t \pi_\chi = -\frac{\delta H}{\delta \chi}, \quad \partial_t \pi_{\chi^*} = -\frac{\delta E}{\delta \chi}. \quad (21)$$

Further, the first variation of the functional $F$ vanishes for the soliton solution:

$$\delta F = \delta E - \omega \delta Q_\chi = 0,$$

where the first variation of the Noether charge $Q_\chi$ can be expressed in terms of the canonically conjugated fields

$$\delta Q_\chi = -i \int (\pi_\chi \delta \chi + \chi \delta \pi_\chi - c.c.) dx. \quad (23)$$

From Eqs. (20), (21), (22), and (23), we obtain the following Hamilton field equations:

$$\partial_t \chi = \frac{\delta E}{\delta \pi_\chi} = \omega \frac{\delta Q_\chi}{\delta \pi_\chi} = -i \omega \chi, \quad (24)$$

$$\partial_t \chi^* = \frac{\delta E}{\delta \pi_{\chi^*}} = \omega \frac{\delta Q_\chi}{\delta \pi_{\chi^*}} = i \omega \chi, \quad (25)$$

while time derivatives of the other model’s fields are equal to zero. Thus, in the adopted gauge $A_x = 0$, only the scalar field $\chi$ has nontrivial time dependence, whereas the model’s fields $\phi$ and $A_0$ do not depend on time:

$$\phi (x, t) = f (x), \quad (26a)$$

$$\chi (x, t) = s (x) \exp (-i \omega t), \quad (26b)$$

$$A_\mu (x, t) = (a_0 (x), 0). \quad (26c)$$

From extremum condition (22), it follows that the soliton solution satisfies the important relation

$$\frac{dE}{dQ_\chi} = \omega, \quad (27)$$

where the Lagrange multiplier $\omega$ is some function of the Noether charge $Q_\chi$. Note that unlike Eqs. (25), relation (27) is gauge-invariant. Just as in the case of non-gauged nontopological solitons [1], relation (27) plays the primary role in the determining of properties of the gauged nontopological soliton system.

### III. SOME PROPERTIES OF THE SOLUTION

In Eqs. (20), $f (x)$ and $s (x)$ are some complex functions of the real argument $x$. Substituting Eqs. (20) into field equations (9) – (11), we obtain the system of ordinary nonlinear differential equations for the functions $a_0 (x)$, $f (x)$, and $s (x)$:

$$a''_0 (x) - 2 \left( e^2 |f (x)|^2 + q^2 |s (x)|^2 \right) a_0 (x) = 0 \quad (28)$$

$$+ 2 q \omega |s (x)|^2 = 0,$$

$$f'' (x) - \left( m^2_\phi - e^2 a_0 (x)^2 \right) f (x) = 0 \quad (29)$$

$$+ \left( g_\phi |f (x)|^2 - h_\phi |f (x)|^4 - \lambda |s (x)|^2 \right) f (x) = 0,$$

$$s'' (x) - \left( m^2_\chi - (\omega - qa_0 (x))^2 \right) s (x) = 0 \quad (30)$$

$$+ \left( g_\chi |s (x)|^2 - h_\chi |s (x)|^4 - \lambda |f (x)|^2 \right) s (x) = 0.$$
by global gauge transformations \[. \] Thus we can suppose without loss of generality that \( f(x) \) and \( s(x) \) are real functions of \( x \). Substituting Eqs. \((26)\) into Eq. \((12)\) and Eq. \((14)\), we obtain the electromagnetic current density and the energy density in terms of the real functions \( a_0(x), f(x), \) and \( s(x) \):

\[
j^\mu = (2q\omega s^2 - 2(q^2 s^2 + e^2 f^2) a_0, 0), \tag{31}
\]

\[
E = \frac{a_0^2}{2} + f^2 + s^2 + (\omega - qa_0)^2 s^2 + e^2 a_0^2 f^2 + V(f) + U(s) + W(f, s). \tag{32}
\]

The finiteness of the soliton system’s energy \( E = \int E \, dx \) leads to the following boundary conditions for the functions \( a_0(x), f(x), \) and \( s(x) \):

\[
a'_0(x) \xrightarrow{x \to -\infty} 0, \quad a'_0(x) \xrightarrow{x \to \infty} 0, \tag{33a}
\]

\[
f(x) \xrightarrow{x \to -\infty} 0, \quad f(x) \xrightarrow{x \to \infty} 0, \tag{33b}
\]

\[
s(x) \xrightarrow{x \to -\infty} 0, \quad s(x) \xrightarrow{x \to \infty} 0. \tag{33c}
\]

Let us discuss some general properties of the soliton system. The invariance of the Lagrangian \((11)\) under the charge conjugation leads to the invariance of system \((28) - (30)\) under the discrete transformation

\[
\omega, a_0, f, s \rightarrow -\omega, -a_0, f, s. \tag{34}
\]

From Eqs. \((31), (52),\) and \((54),\) it follows that the energy \( E = \int E \, dx \) is an even function of \( \omega \), whereas the Noether charges \( Q_\phi \) and \( Q_\chi \) are odd functions of \( \omega \):

\[
E(-\omega) = E(\omega),
\]

\[
Q_{\phi, \chi}(-\omega) = -Q_{\phi, \chi}(\omega). \tag{35}
\]

The Lagrangian \((11)\) is also invariant under the parity transformation. It follows that system \((28) - (30)\) is invariant under the space inversion: \( x \rightarrow -x \). Thus, if \( a_0(x), f(x), \) and \( s(x) \) is a solution of Eqs. \((28) - (30),\) then \( a_0(-x), f(-x), \) and \( s(-x) \) is also a solution. This fact, however, does not mean that \( a_0(x), f(x), \) and \( s(x) \) must be even functions of \( x \). Indeed, we shall see later that system \((28) - (30)\) together with boundary conditions \((33)\) has nonsymmetric soliton solutions.

Eq. \((28)\) can be written as \( a''_0 = -j_0, \) where \( j_0 \) is the electric charge density \((31)\). Integrating this equation over \( x \in (-\infty, \infty) \) and taking into account boundary conditions \((33)\), we conclude that the total electric charge of a field configuration with a finite energy vanishes:

\[
Q = eQ_\phi + qQ_\chi = 0. \tag{37}
\]

Substituting the power expansions for the functions \( a_0(x), f(x), \) and \( s(x) \) into Eqs. \((28) - (30),\) we obtain the asymptotic form of the solution as \( x \rightarrow 0 \):

\[
a_0(x) = a_0 + a_1x + \frac{a_2}{2!} x^2 + O(x^3), \tag{38a}
\]

\[
f_0(x) = f_0 + f_1x + \frac{f_2}{2!} x^2 + O(x^3), \tag{38b}
\]

\[
s_0(x) = s_0 + s_1x + \frac{s_2}{2!} x^2 + O(x^3), \tag{38c}
\]

where the next-to-leading coefficients

\[
a_2 = 2a_0 \left( e^2 f_0^2 + q^2 s_0^2 \right) - 2qa_0 s_0^2, \tag{39a}
\]

\[
f_2 = f_0 \left( m_0^2 - g_0 f_0^2 + h_0 f_0^4 - e^2 a_0^2 + \lambda s_0^2 \right), \tag{39b}
\]

\[
s_2 = s_0 \left( m_0^2 - (\omega - qa_0)^2 - q s_0 + h_0 s_0^3 \right) + \lambda f_0^2 \tag{39c}
\]

are determined in terms of the three leading coefficients \( a_0, f_0, s_0 \) and the model’s parameters. The next coefficients \( a_n, f_n, s_n, \) where \( n = 3, 4, 5, \ldots \) are determined by the six leading coefficients \( a_0, f_0, s_0, a_1, f_1, s_1 \) and the model’s parameters. It can be easily shown that if the coefficients \( a_1, f_1, \) and \( s_1 \) vanish, all the other coefficients with an odd \( n \) also vanish, and we have an even solution of Eqs. \((28) - (30)\) at large \( x \) together with corresponding boundary conditions \((33)\) lead us to the asymptotic form of the solution as \( x \rightarrow \pm \infty:\)

\[
f(x) \sim f_{\pm \infty} \exp \left( \mp \tilde{m}_\phi \pm x \right), \tag{40a}
\]

\[
s(x) \sim s_{\pm \infty} \exp \left( \mp \tilde{m}_\chi \pm x \right), \tag{40b}
\]

\[
a_0(x) \sim a_{\pm \infty} + a_{\pm \infty}^2 \frac{e^2 f_{\pm \infty}^2}{2 \tilde{m}_{\phi \pm}^2} \times \exp \left( \mp \tilde{m}_\phi \pm x \right) - (\omega - qa_{\pm \infty}) \times \frac{q s_{\pm \infty}^2}{2 \tilde{m}_{\chi \pm}} \exp \left( \mp \tilde{m}_\chi \pm x \right), \tag{40c}
\]

where the mass parameters \( \tilde{m}_\phi \pm \) and \( \tilde{m}_\chi \pm \) are defined by the relations:

\[
\tilde{m}_\phi^2 = m_0^2 - e^2 a_{\pm \infty}^2, \tag{41}
\]

\[
\tilde{m}_\chi^2 = m_0^2 - (\omega - qa_{\pm \infty})^2. \tag{42}
\]

From Eqs. \((41)\) and \((42),\) we obtain the upper boundaries on the absolute values of \( a_0(\pm \infty) = a_{\pm \infty} \) and \( \omega:\)

\[
|a_0(\pm \infty)| < \frac{m_0}{e}, \quad |\omega| < m_\chi + \frac{q}{e} m_\phi. \tag{43}
\]

From Eqs. \((38) - (40,\) it follows that there may be two types of solutions: the symmetric one for which \( f(-x) = f(x), s(-x) = s(x), a_0(-x) = a_0(x) \) and the nonsymmetric one that does not possess this property. For a symmetric solution, the series coefficients \( a_n, f_n, \) and \( s_n \) with an odd \( n \) vanish, and so in Eqs. \((40) - (42),\) the asymptotic parameters corresponding to \( x \rightarrow -\infty \) are equal to those corresponding to \( x \rightarrow \infty.\)

If the values of the model’s parameters are fixed, then the behavior of a nonsymmetric solution \( f(x), s(x), \) \( a_0(x) \) as \( x \rightarrow 0 \) is determined by the six parameters \( a_0, f_0, s_0, a_1, f_1, \) and \( s_1 \) in Eqs. \((35)\). The behavior of the nonsymmetric solution as \( x \rightarrow \pm \infty \) is also determined by the six parameters in Eqs. \((40) - (42),\) namely \( a_{\pm \infty}, f_{\pm \infty}, \) and \( s_{\pm \infty} \) as \( x \rightarrow -\infty \) and \( a_{\pm \infty}, f_{\pm \infty}, \) and \( s_{\pm \infty} \) as \( x \rightarrow \infty.\) Thus we have twelve free parameters in all. The continuity condition for \( f(x), s(x), a_0(x) \) and their derivatives \( f'(x), s'(x), a'_0(x) \) at arbitrary \( x < 0 \) give us six equations. A similar condition at arbitrary \( x > 0 \) provides
us with another six equations. Therefore, we shall have twelve equations for determining the twelve parameters. According to [24], this fact is an argument in favor of the existence of the nonsymmetric solution for the boundary value problem in some range of the model’s parameters. Of course, similar arguments can also be applied to a symmetric solution.

Any solution of field equations (39) – (41) is an extremum of the action \( S = \int \mathcal{L} \, dx dt \). At the same time, the Lagrangian density (1) does not depend on time in the case of field configurations (20). It follows that any solution of Eqs. (28) – (30), satisfying boundary conditions (33), is an extremum of the Lagrangian \( L = \int \mathcal{L} \, dx \). Let \( a_0 (x), \ f (x), \) and \( s (x) \) be a solution of system (28) – (30), satisfying boundary conditions (33). After the scale transformation of the solution’s argument \( x \to \lambda x \), the Lagrangian \( L \) becomes a function of the scale parameter \( \lambda \). The function \( L (\lambda) \) has an extremum at \( \lambda = 1 \), so its derivative with respect to \( \lambda \) vanishes at this point: \( dL/d\lambda |_{\lambda=1} = 0 \). From this equation, we obtain the virial relation for the soliton system:

\[
E^{(E)} + E^{(P)} - E^{(G)} - E^{(T)} = 0, \tag{44}
\]

where

\[
E^{(E)} = \int \frac{\alpha}{2} \, f^2(x) \, dx \tag{45}
\]

is the electric field’s energy,

\[
E^{(G)} = \int \left( f^2 + s^2 \right) \, dx \tag{46}
\]

is the gradient part of the soliton’s energy,

\[
E^{(T)} = \int \left( (\omega - qa_0)^2 s^2 + e^2a_0^2 f^2 \right) \, dx \tag{47}
\]

is the kinetic part of the soliton’s energy, and

\[
E^{(P)} = \int \left( V (f) + U (s) + W (f, s) \right) \, dx \tag{48}
\]

is the potential part of the soliton’s energy.

The obvious equality \( E = E^{(E)} + E^{(T)} + E^{(G)} + E^{(P)} \) and virial relation (14) lead to the following representations for the soliton system’s energy:

\[
E = 2 \left( E^{(T)} + E^{(G)} \right), \tag{49}
\]

\[
E = 2 \left( E^{(P)} + E^{(E)} \right). \tag{50}
\]

Integrating the term \( \alpha^2 / 2 \) in Eq. (32) by parts and using Eqs. (28), (31), and (33), we obtain one more representation for the energy

\[
E = \frac{1}{2} \omega Q_\chi + E^{(G)} + E^{(P)}, \tag{51}
\]

which, in turn, leads to the relation between the Noether charge \( Q_\chi \), the electric field’s energy \( E^{(E)} \), and the kinetic energy \( E^{(T)} \):

\[
\omega Q_\chi = 2 \left( E^{(E)} + E^{(T)} \right). \tag{52}
\]

IV. THE THICK-WALL AND THIN-WALL REGIMES OF THE SOLITON SYSTEM

In this section, we research properties of the symmetric soliton solution in two extreme regimes. In the thick-wall regime, the mass parameters \( \bar{m}_\phi \) and \( \bar{m}_\chi \) tend to zero, leading to a spatial spreading of the soliton system. This fact and Eqs. (11) and (12) lead to the following limiting values of the potential \( a_0 (\infty) \) and the phase frequency \( \omega \) in the thick-wall regime:

\[
|a_0 (\infty)| = \frac{m_\phi}{e}, \quad \omega_{tk} = \text{sgn} (a_0 (\infty)) \left( \frac{m_\chi + q e}{m_\phi} \right). \tag{53}
\]

In the thick-wall regime, where \( \bar{m}_\phi^2 \approx \bar{m}_\chi^2 \to 0 \), we undertake the following scale transformation of the fields and the \( x \)-coordinate:

\[
f (x) = \Delta \bar{f} (\bar{x}), \quad s (x) = \Delta \bar{s} (\bar{x}), \quad a_0 (x) = \frac{m_\phi}{e} + \frac{\Delta^2}{m_\phi^2} \bar{a}_0 (\bar{x}), \quad x = \Delta^{-1} \bar{x}, \tag{54}
\]

where the scale factor \( \Delta \) is defined as

\[
\Delta^2 = m_\phi^2 - e^2 a_0^2 (\infty) \approx m_\chi^2 - (\omega - qa_0 (\infty))^2 \approx \kappa^2 (\omega_{tk}^2 - \omega^2). \tag{55}
\]

In Eq. (55), the factor \( \kappa \) is expressed in terms of the scalar particles’ masses and the gauge coupling constants:

\[
\kappa = e \left( \frac{m_\phi m_\chi}{(e m_\phi + q m_\chi) (e m_\chi + q m_\phi)} \right)^{1/2}. \tag{56}
\]

Let us consider the functional \( F \), which has been defined in Eq. (15). This functional is related to the energy functional by means of Legendre transformation: \( F (\omega) = E (Q_\chi) - \omega Q_\chi \). On field configuration (33), the functional \( F \) can be written as

\[
F (\omega) = \Delta^3 \bar{F} + O (\Delta^5), \tag{57}
\]

where the functional \( \bar{F} \) does not depend on \( \omega \):

\[
\bar{F} = \int \left[ f' (\bar{x})^2 + s' (\bar{x})^2 + \bar{f} (\bar{x})^2 + \bar{s} (\bar{x})^2 \right] d\bar{x}. \tag{58}
\]

In the thick-wall regime, the phase frequency \( \omega \) tends to the limiting value \( \omega_{tk} \), so the parameter \( \Delta \) vanishes, and it is possible to ignore the higher-order terms in \( \Delta \) in Eq. (57). Using known properties of Legendre transformation, we obtain sequentially

\[
Q_\chi (\omega) = - \frac{dF (\omega)}{d\omega} = 3\bar{F} \kappa^3 \omega \left( \omega_{tk}^2 - \omega^2 \right)^{1/2}, \tag{59}
\]

\[
E (\omega) = F (\omega) - \omega \frac{dF (\omega)}{d\omega} = \bar{F} \kappa^3 \left( 2 \omega^2 + \omega_{tk}^2 \right) \left( \omega_{tk}^2 - \omega^2 \right)^{1/2}. \tag{60}
\]
From Eqs. (59) and (60), we obtain the dependence of the energy $E$ on the Noether charge $Q_X$ in the thick-wall regime:

$$E(Q_X) = \omega_{tk} Q_X - \frac{1}{54} F_{\phi}^2 \frac{1}{\omega_{tk}^3} Q_X^3 + O(Q_X^5). \quad (61)$$

We see from Eqs. (37), (59), (60), and (61) that the energy $E$ and the Noether charges $Q_\phi$ and $Q_X$ of the soliton system tend to zero in the thick-wall regime. Further, Eq. (61), basic relation (27), and the inequality $\omega^2 < \omega_{tk}^2$ lead to the conclusion that $E(Q_X) < \omega_{tk} Q_X$ for all values of $Q_X$. From Eqs. (36), (37), and (53) it follows that $\omega_{tk} Q_X$ is equal to $m_\phi |Q_\phi| + m_X |Q_X|$, which, in turn, is the rest energy of the neutral plan-wave configuration formed from the charged scalar $\phi$ and $\chi$-particles. Hence, the symmetric soliton system is stable to decay into the scalar $\phi$ and $\chi$-particles.

The second extremal regime of the symmetric soliton system is the thin-wall regime in which the absolute value of the phase frequency tends to some minimum value $\omega_{tn}$. In the thin-wall regime, the spatial size of the soliton system increases indefinitely, with the result that its energy $E$ and Noether charges $Q_\phi$ and $Q_X$ also tend to infinity. In the thin-wall regime, when the spatial size of the soliton system $L \to \infty$, the gradient operator gives a factor proportional to $L^{-1}$. Therefore, we can ignore the electric field’s energy (35) and the gradient energy (46) in comparison with the kinetic energy (47) and the potential energy (38). Then, from Eq. (44) it follows that the following limiting relation holds in the thin-wall regime:

$$\lim_{\omega \to \omega_{tn}} \frac{E(T)}{E(P)} = 1,$$

and, as a consequence,

$$\lim_{\omega \to \omega_{tn}} \frac{2E(T)}{E} = \lim_{\omega \to \omega_{tn}} \frac{2E(P)}{E} = 1. \quad (63)$$

Further, electric charge density (31) tends to zero in the thin-wall regime, since the soliton system’s electric charge is strictly equal to zero, whereas its spatial size tends to infinity. Then, using Eqs. (31), (37), and (38), we obtain the limiting relation

$$\lim_{\omega \to \omega_{tn}} \frac{2E(T)}{Q_X} = \lim_{\omega \to \omega_{tn}} \frac{E}{Q_X} = \omega_{tn}, \quad (64)$$

which is consistent with basic relation (27) and Eq. (52).

V. NUMERICAL RESULTS

The system of differential equations (28) – (30) with boundary conditions (33) is the mixed boundary value problem on the infinite interval $x \in (-\infty, \infty)$. This boundary value problem can be solved only by numerical methods. In this paper, the boundary value problem was solved using the MAPLE package [25] by the method of finite differences and subsequent Newtonian iterations. Equations (27), (37), and (52) were used to check the correctness of numerical solutions.

Let us discuss possible types of solutions of the boundary value problem. If the quartic coupling constant $\lambda$ and the electromagnetic coupling constants $e$ and $g$ are set equal to zero, then the Lagrangian (11) will describe the system of two self-interacting complex scalar fields that, however, do not interact with each other. In this case, the boundary value problem has the solution describing a system of two non-interacting non-gaunted one-dimensional Q-balls. Generally, these two Q-balls have different shapes and can be at an arbitrary distance from each other, so the solution will not be symmetric. However, the situation changes when the electromagnetic interaction is turned on. In this case, from Eq. (37) it follows that the electric charges of two Q-ball components are equal in magnitude, but opposite in sign. It is important to note that the electric charges of two gauged Q-balls are conserved separately owing to the neutrality of the Abelian gauge field. Since the opposite electric charges attract each other, the initially nonsymmetric soliton system transits to a symmetric one. Now we turn on the quartic interaction between the two complex scalar fields $\phi$ and $\chi$ by letting the coupling constant $\lambda$ be some positive value. From Eq. (3), it follows that the energy of the quartic interaction increases with the increase of overlap between the Q-ball components of the soliton system and is negligible at large separations between the Q-ball components. Such a behavior of the quartic interaction corresponds to a short-range repulsive force between the Q-ball components, while the electromagnetic long-range attractive force results in the confinement of the the Q-ball components. One would expect that for a sufficiently large positive coupling constant $\lambda$, the action of these opposite forces leads to an equilibrium nonsymmetric soliton configuration, which is the solution of boundary value problem (28) – (30) and (33). Indeed, we shall see later that such a nonsymmetric soliton solution really exists.

The system of differential equations (28) – (30) depends on the ten dimensional parameters: $\omega$, $e$, $g$, $m_\phi$, $m_X$, $g_\phi$, $g_X$, $h_\phi$, $h_X$, and $\lambda$. It is readily seen, however, that the dimensionless functions $a_0(x)$, $f(x)$, and $s(x)$ can depend only on nine independent dimensionless combinations of these parameters. Therefore without loss of generality, we can choose the mass $m_\phi$ of the scalar $\phi$-particle as the energy unit. We consider a general case in which the corresponding dimensionless parameters are values of the same order: $\tilde{e} = e/m_\phi = 0.2$, $\tilde{g} = g/m_\phi = 0.2$, $\tilde{m}_\phi = m_\phi/m_\phi = 1.25$, $\tilde{g}_\phi = g_\phi/m_\phi = 1$, $\tilde{g}_X = g_X/m_\phi = 1.5$, $\tilde{h}_\phi = h_\phi/m_\phi = 0.22$, $\tilde{h}_X = h_X/m_\phi = 0.31$, and $\tilde{\lambda} = \lambda/m_\phi^2 = 0.2$.

Figures 1 and 2 present the dependences of the soliton’s dimensionless energy $\tilde{E} = m_\phi^{-1} E$ and Noether charge $Q_X$ on the dimensionless phase frequency $\tilde{\omega} = m_\phi^{-1} \omega$. The most striking feature of
FIG. 1. The dependence of the dimensionless soliton energy $\tilde{E} = m_\omega^{-1} E$ on the dimensionless phase frequency $\tilde{\omega} = m_\omega^{-1} \omega$. The solid curve corresponds to the symmetric soliton system, and the dashed curve corresponds to the nonsymmetric one.

FIG. 2. The dependence of the soliton Noether charge $Q_\chi$ on the dimensionless phase frequency $\tilde{\omega} = m_\omega^{-1} \omega$. The solid curve corresponds to the symmetric soliton system, and the dashed curve corresponds to the nonsymmetric one.

the symmetric and nonsymmetric soliton solutions. Indeed, it has been found numerically that the symmetric soliton solution exists in the range from the minimum value $\tilde{\omega}_{\text{min}} = 1.4079916$, which we managed to reach by numerical methods, to the maximum value $\tilde{\omega}_{\text{tk}} = 2.25$. On the contrary, the nonsymmetric soliton solution exists only in the interval from the left bifurcation point $\tilde{\omega}_{\text{lb}} = 1.409$ to the right one $\tilde{\omega}_{\text{rb}} = 1.601$. We also see that the two types of curves have intersection points at $\tilde{\omega}_{11}$ and $\tilde{\omega}_{12}$ in Figs. 1 and 2, respectively. These intersection points are slightly different: $\tilde{\omega}_{11} = 1.4504025$, whereas $\tilde{\omega}_{12} = 1.4504280$. In each of the figures, the solid and dashed curves bound the two regions, which connect at the intersection points. Using Eq. (27), it can easily be shown that the areas of these regions are equal to each other, so we have the relations:

$$\int_{\tilde{\omega}_{\text{lb}}}^{\tilde{\omega}_{\text{rb}}} [Q_\chi (\tilde{\omega}) - Q_\chi (\tilde{\omega})] d\tilde{\omega} = 0,$$  \hspace{1cm} (65)

and

$$\int_{\tilde{\omega}_{\text{lb}}}^{\tilde{\omega}_{\text{rb}}} [-\tilde{E}_a (\tilde{\omega}) + \tilde{E}_a (\tilde{\omega})] d\tilde{\omega} = 0,$$  \hspace{1cm} (66)

which were checked numerically.

When $\tilde{\omega}$ tends to its minimal value $\tilde{\omega}_{\text{min}}$, the symmetric soliton system goes into the thin-wall regime. In this regime, the energy $\tilde{E}$, the Noether charges $Q_\chi$ and $Q_\phi$, and the effective spatial size $L$ of the symmetric soliton system increase indefinitely. In particular, we found numerically that $\tilde{E}$, $Q_\chi$, $Q_\phi$, and $L$ increase logarithmically as $\tilde{\omega} \to \tilde{\omega}_{\text{tn}}$,

$$\tilde{E} \sim -\tilde{\omega}_{\text{tn}} B \ln (\tilde{\omega} - \tilde{\omega}_{\text{tn}}),$$  \hspace{1cm} (67)

$$Q_\chi \sim -B \ln (\tilde{\omega} - \tilde{\omega}_{\text{tn}}),$$  \hspace{1cm} (68)

$$Q_\phi \sim B \frac{q}{e} \ln (\tilde{\omega} - \tilde{\omega}_{\text{tn}}),$$  \hspace{1cm} (69)

$$L \sim -C \ln (\tilde{\omega} - \tilde{\omega}_{\text{tn}}),$$  \hspace{1cm} (70)

where $B$ and $C$ are some positive constants, and the limiting thin-wall phase frequency $\tilde{\omega}_{\text{tn}} = 1.4079869$. Note that this numerical estimation of $\tilde{\omega}_{\text{tn}}$ is slightly less than the minimal value $\tilde{\omega}_{\text{min}} = 1.4079916$, which was reached by numerical methods. Note also that in the thin-wall regime, the behavior of $E$, $Q_\chi$, $Q_\phi$, and $L$ is similar to that of the corresponding values of the one-dimensional non-gauged Q-ball, as it follows from Eqs. (67) – (70) and (A9) – (A11).

When $\tilde{\omega}$ tends to its maximal value $\tilde{\omega}_{\text{tk}}$, the symmetric soliton system goes into the thick-wall regime. In this regime, the soliton system is spread out over one-dimensional space, while the amplitudes of the scalar fields $\phi$ and $\chi$ tend to zero as $(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{1/2}$ in accordance with Sec. [IV] It was found numerically that in the thick-wall regime, $\tilde{E}$, $Q_\chi$, and $Q_\phi$ also tend to zero as $(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{1/2}$, whereas the effective spatial size $L$ diverges as $(\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{-1/2}$:

$$\tilde{E} \sim -\frac{b}{e} (\tilde{\omega}_{\text{tk}} - \tilde{\omega})^2,$$  \hspace{1cm} (71)

$$Q_\chi \sim -b (\tilde{\omega}_{\text{tk}} - \tilde{\omega})^2,$$  \hspace{1cm} (72)

$$Q_\phi \sim -b \frac{q}{e} (\tilde{\omega}_{\text{tk}} - \tilde{\omega})^2,$$  \hspace{1cm} (73)

$$L \sim c (\tilde{\omega}_{\text{tk}} - \tilde{\omega})^{-1/2}.$$  \hspace{1cm} (74)

From Eqs. (71) – (74) and (A5) – (A7), it follows that the behavior of $E$, $Q_\chi$, $Q_\phi$, and $L$ is similar to that of the
FIG. 3. The dependence of the dimensionless energy \( \tilde{E} = m^{-1} E \) of the symmetric soliton system on the Noether charge \( Q_\chi \) (solid curve). The dash-dotted line is the straight line \( \tilde{E} = \tilde{\omega}_{\text{th}} Q_\chi = (1 + m_\chi/m_\phi) Q_\chi \).

FIG. 4. The dependence of the energy difference \( \Delta \tilde{E} = \tilde{E}_s - \tilde{E}_a \) between the symmetric and nonsymmetric soliton solutions on the Noether charge \( Q_\chi \).

The nonsymmetric character of the soliton system is obvious from Figs. 5 and 6. The most interesting feature of the nonsymmetric soliton system is the presence of the unidirectional electric field in its interior, as for a plane capacitor. From Fig. 5, it follows that the charged scalar \( \phi \) and \( \chi \)-particles can acquire the energy equal to \(-e\Delta a_0 \approx 0.32m_\phi \) in the electric field of the nonsymmetric soliton system. Note that this energy is comparable with the scalar particles’ masses. Lighter particles (e.g. light charged fermions) passing through the interior of the nonsymmetric soliton system can be accelerated to relativistic velocities and energies.

In Fig. 6 we can see the dependence of the symmetric soliton solution for the dimensionless phase frequency \( \tilde{\omega} = 1.5 \). whereas Fig. 5 presents the energy and electric charge densities and the electric field strength corresponding to

FIG. 5. The nonsymmetric numerical solution for \( f(\tilde{x}) \) (solid curve), \( s(\tilde{x}) \) (dashed curve), and \( \tilde{e}_0(\tilde{x}) \) (dotted curve). The dimensionless phase frequency \( \tilde{\omega} = 1.5 \).
it follows that the energy and electric charge densities are symmetric with respect to the center of the soliton system, while the electric field strength is antisymmetric. For positive \( \omega \), it is directed from the soliton system’s center, so it attracts negatively charged particles and repels positively charged ones. For negative \( \omega \), it is directed to the soliton system’s center, so the roles of negatively and positively charged particles are interchanged. For positive (negative) \( \omega \), the form of the electromagnetic potential \( a_0 \) corresponds to a potential well for negatively (positively) charged particles. It follows that bound fermionic and bosonic states can exist in the electric field of the symmetric soliton system.

VI. CONCLUSION

In the present paper, the one-dimensional nontopological soliton system consisting of two self-interacting complex scalar fields has been investigated. The scalar fields interact with each other through the Abelian gauge field and the quartic scalar interaction. The finiteness of the energy of the one-dimensional soliton system leads to its electric neutrality, so its two scalar components have opposite electric charges. The neutrality of the Abelian gauge field leads to the separate conservation of the electric charges of these scalar components. The interplay between the attractive electromagnetic interaction and the repulsive quartic interaction leads to the existence of symmetric and nonsymmetric soliton systems.

The symmetric soliton system exists in the whole allowable interval of the phase frequency \( \omega \). When \( \omega \) tends to its minimal (maximal) value, the symmetric soliton system goes into the thin-wall (thick-wall) regime. In the thin-wall regime, the energy, the Noether charges, and the spatial size of the symmetric soliton system tend to infinity. In the thick-wall regime, the spatial size of the symmetric soliton system also tends to infinity, but the energy and the Noether charges tend to zero. In contrast to this, the nonsymmetric soliton system exists only in some interior subinterval between the minimal and maximal allowable phase frequencies \( \omega_{tn} \) and \( \omega_{tk} \). It follows that there exists an interval of the Noether charge \( Q_x \) (and, consequently, an interval of the Noether charge \( Q_\phi = -q e^{-1} Q_x \)), where the symmetric and nonsymmetric soliton systems coexist. In all this interval, the energy of the nonsymmetric soliton system turns out to be less than that of the symmetric soliton system, so the symmetric soliton system can turn into the nonsymmetric
one through quantum tunneling. Both symmetric and nonsymmetric soliton systems are stable to decay into massive scalar $\phi$ and $\chi$-bosons.

Despite the fact that the soliton system is electrically neutral, it nevertheless possesses a nonzero electric field in its interior. Note that the electric fields of the symmetric and nonsymmetric soliton systems are essentially different. The electric field of the nonsymmetric soliton system is unidirectional in its interior, like the electric field of a plane capacitor. It can accelerate light particles up to relativistic velocities and energies. In contrast, the electric field of the symmetric soliton system corresponds to the electromagnetic potential of a potential well. In such an electric field, the existence of bound bosonic and fermionic states is possible.

It is known [1, 12] that the field configuration of a nontopological soliton composed only of scalar fields can be described in terms of a mechanical analogy. For the one-dimensional case, it corresponds to the motion of a particle with the unit mass in the time mass $m$ in the conservative force field of a certain potential. The dimension of space in which the particle moves is equal to the number of scalar fields constituting the nontopological soliton. Using this analogy, one can easily explain the behavior of the pure scalar nontopological soliton both in the thin-wall and in the thick-wall regimes. Moreover, one can easily determine whether a soliton solution can exist for any values of the model’s parameters. At the same time, system of differential equations (28)–(30) describing the soliton system of the present paper has no interpretation in terms of any mechanical analogy. For this reason, the existence of the soliton system should be established for any given set of the model’s parameters by means of numerical methods.

Finally, let us stress the specific character of the (1+1)-dimensional electromagnetic field. Its characteristic feature is the absence of nondiagonal terms of the electromagnetic stress-energy tensor. This is because the magnetic field does not exist in (1+1)-dimensions, so the Poynting vector vanishes there. Therefore, the (1+1)-dimensional electromagnetic field can not transfer any energy or momentum. Instead, the scalar fields’ kinetic energy can transform to the one-dimensional electric field’s energy, which, in turn, can transform back to the scalar fields’ energy. Note also that in (1+1)-dimensions, the potential energy of two oppositely charged particles is proportional to the distance between them, so the electromagnetic interaction is confining there. Thus, we can conclude that the (1+1)-dimensional electromagnetic interaction is similar to an elastic string. The only difference is that there is no energy and momentum transfer in the one-dimensional electric field, whereas in the elastic string waves can transfer energy and momentum. The behaviour of the (1+1)-dimensional electromagnetic field is completely determined by Gauss’s law, which is not a dynamic field equation but is the condition imposed on an initial field configuration. Indeed, in the adopted gauge $A_x = 0$, Gauss’s law does not contain time derivatives of the electromagnetic potential $A_\nu$. In this connection, it can be said that the (1+1)-dimensional electromagnetic field is not a dynamic one.

ACKNOWLEDGMENTS

The research is carried out at Tomsk Polytechnic University within the framework of Tomsk Polytechnic University Competitiveness Enhancement Program grant.

Appendix A: The one-dimensional non-gauged Q-ball

Here we collect formulae concerning the one-dimensional non-gauged Q-ball in the model of a self-interacting complex scalar field with the six-order self-interaction potential $V(\phi) = m^2 |\phi|^2 - g |\phi|^4 / 2 + h |\phi|^6 / 3$. Note that an analytical Q-ball solution exists only in the (1+1)-dimensional case [1], where it can be written as

$$\phi(t, x) = \frac{2}{\sqrt{g}} \sqrt{m^2 - \omega^2} 
\times \left( 1 + \left( 1 - \frac{m^2 - \omega^2}{m^2 - \omega_{tn}^2} \right)^{\frac{1}{2}} \right)^{-\frac{1}{2}}
\times \cosh \left( 2 \sqrt{m^2 - \omega^2} (x - x_0) \right)
\times \exp (-i \omega (t - t_0)). \quad (A1)$$

In Eq. (A1), the squared phase frequency $\omega^2 \in (\omega_{tn}^2, m^2)$, where

$$\omega_{tn}^2 = m^2 \left( 1 - \frac{3}{16} \frac{g^2}{m^2 h} \right). \quad (A2)$$

The Noether charge and the energy of the one-dimensional Q-ball can be expressed in a rather compact form:

$$Q = 4\omega \sqrt{\frac{3}{h}} \arctanh \left( \frac{m^2 - \omega_{tn}^2}{m^2 - \omega^2} \right)^{\frac{1}{2}}
- \left( \frac{m^2 - \omega_{tn}^2}{m^2 - \omega^2} - 1 \right)^{\frac{1}{2}}, \quad (A3)$$

and

$$E = \omega Q - 2 \sqrt{\frac{3}{h}} (\omega^2 - \omega_{tn}^2)
\times \arctanh \left( \frac{\sqrt{m^2 - \omega_{tn}^2} - \sqrt{\omega^2 - \omega_{tn}^2}}{\sqrt{m^2 - \omega_{tn}^2} + \sqrt{\omega^2 - \omega_{tn}^2}} \right)^{\frac{1}{2}}
\times \sqrt{\frac{3}{h}} (m^2 - \omega^2) (m^2 - \omega_{tn}^2). \quad (A4)$$
Let us present the expressions of the Noether charge $Q$ and the energy $E$ in two extreme regimes. In the thick-wall regime, the squared phase frequency tends to its maximum value: $\omega^2 \rightarrow m^2$. Using Eqs. (A3) and (A4), we obtain the expressions of the soliton’s Noether charge and energy in the thick-wall regime:

$$Q = \text{sgn} (\omega) \frac{2\sqrt{6}m^{3/2}}{h(m^2 - \omega_{\text{tn}}^2)} \sqrt{\delta} \times \left(1 - \frac{7m^2 - 15\omega_{\text{tn}}^2}{12m(m^2 - \omega_{\text{tn}}^2)}\delta + O(\delta^2)\right), \quad (A5)$$

and

$$E = \frac{2\sqrt{6}m^{5/2}}{h(m^2 - \omega_{\text{tn}}^2)} \sqrt{\delta} \times \left(1 - \frac{11m^2 - 19\omega_{\text{tn}}^2}{12m(m^2 - \omega_{\text{tn}}^2)}\delta + O(\delta^2)\right), \quad (A6)$$

where the variable $\delta = m - |\omega|$. Furthermore, Eq. (A1) leads to the soliton’s width at half-height in the thick-wall regime:

$$L = \frac{\cosh^{-1}(7)}{\sqrt{2}} \frac{1}{\sqrt{m^0}} + \frac{1}{4\sqrt{2}} \times \left(\frac{\sqrt{3m}}{m^2 - \omega_{\text{tn}}^2} + \cosh^{-1}(7) \frac{m^{5/2}}{m^2 - \omega_{\text{tn}}^2}\right) \sqrt{\delta} + O(\delta^2). \quad (A7)$$

Using Eqs. (A5) and (A6), we obtain the dependence of $E$ on $Q$ in the thick-wall regime:

$$E = m |Q| - \frac{h}{3!} \frac{m^2 - \omega_{\text{tn}}^2}{12m^3} |Q|^3 + O(|Q|^5). \quad (A8)$$

From Eqs. (A5) – (A7), it follows that in the thick-wall regime, the soliton’s Noether charge and energy vanish as $\sqrt{\delta}$, whereas the soliton’s effective size diverges as $1/\sqrt{\delta}$.

In the thin-wall regime, the squared phase frequency tends to its minimum value: $\omega^2 \rightarrow m^2$. In this regime, the Noether charge, the energy, and the width at half-height of the one-dimensional Q-ball behave as follows:

$$Q = \text{sgn} (\omega) \sqrt{\frac{3}{h} \omega_{\text{tn}}} \left[\ln \left(\frac{2(m^2 - \omega_{\text{tn}}^2)}{\omega_{\text{tn}}^2}\right) - \sqrt{\frac{2\omega_{\text{tn}}\delta}{m^2 - \omega_{\text{tn}}^2}} + O(\delta)\right], \quad (A9)$$

$$E = \sqrt{\frac{3}{h} \omega_{\text{tn}}^2} \left\{\ln \left(\frac{2(m^2 - \omega_{\text{tn}}^2)}{\omega_{\text{tn}}^2}\right) + \frac{m^2}{\omega_{\text{tn}}^2} - 1 - \sqrt{\frac{2\omega_{\text{tn}}\delta}{m^2 - \omega_{\text{tn}}^2}} + O(\delta)\right\}, \quad (A10)$$

and

$$L = 2^{-1}(m^2 - \omega_{\text{tn}}^2)^{1/2} \ln \left(\frac{18(m^2 - \omega_{\text{tn}}^2)}{\omega_{\text{tn}}^5}\right) + \sqrt{\frac{2}{3}} \frac{\sqrt{\omega_{\text{tn}}^2}}{m^2 - \omega_{\text{tn}}^2} + O(\delta), \quad (A11)$$

where the variable $\delta = |\omega| - \omega_{\text{tn}}$. From Eqs. (A9) and (A10), we obtain the dependence of $E$ on $Q$ in the thin-wall regime:

$$E = \omega_{\text{tn}} |Q| + \sqrt{\frac{3}{h}} (m^2 - \omega_{\text{tn}}^2) + O\left(\exp\left(-\sqrt{\frac{h}{3}} |Q|\right)\right). \quad (A12)$$

From Eqs. (A9), (A10), and (A11), it follows that the Noether charge, the energy, and the effective size of the one-dimensional Q-ball logarithmically diverge in the thin-wall regime.
115, 32 (1976).
[22] A. Yu. Loginov, Phys. Lett. B 777, 340 (2018).
[23] A. Yu. Loginov and V. V. Gauzshtein, Phys. Lett. B 784, 112 (2018).

[24] V. Rubakov, Classical Theory of Gauge Fields (Princeton University Press, Princeton, 2002).

[25] Maple User Manual Maplesoft, Waterloo, Ontario (2014).