CONSERVATION LAWS AND GEOMETRY OF
PERTURBED COSET MODELS

Ioannis Bakas

Theory Division
CERN
CH–1211 Geneva 23
SWITZERLAND

ABSTRACT

We present a Lagrangian description of the $SU(2)/U(1)$ coset model perturbed by its first thermal operator. This is the simplest perturbation that changes sign under Krammers–Wannier duality. The resulting theory, which is a 2–component generalization of the sine–Gordon model, is then taken in Minkowski space. For negative values of the coupling constant $g$, it is classically equivalent to the $O(4)$ non–linear $\sigma$–model reduced in a certain frame. For $g > 0$, it describes the relativistic motion of vortices in a constant external field. Viewing the classical equations of motion as a zero curvature condition, we obtain recursive relations for the infinitely many conservation laws by the abelianization method of gauge connections. The higher spin currents are constructed entirely using an off–critical generalization of the $W_\infty$ generators. We give a geometric interpretation to the corresponding charges in terms of embeddings. Applications to the chirally invariant $U(2)$ Gross–Neveu model are also discussed.
1 Introduction

Integrable perturbations of 2–dim conformal field theories (CFT) have been studied extensively over the last few years. Zamolodchikov proved that there are perturbations of the form

\[ S = S_{\text{CFT}} + g \int \Phi \]  

(1.1)

which take us away from the critical point, but the resulting theory still possesses an infinite number of local conservation laws [1, 2]. \( \Phi \) is typically a local spinless field with conformal dimension \( \Delta_\Phi \) which is found in the operator algebra of the unperturbed theory \( S_{\text{CFT}} \). The coupling constant \( g \) is a dimensionful parameter of weight \( (1 - \Delta_\Phi, 1 - \Delta_\Phi) \).

In these cases, away from criticality, the chiral conservation laws of the CFT are replaced by

\[ \bar{\partial}A_s + \partial B_s = 0 \]  

(1.2)

for appropriately chosen non–chiral currents \( \{A_s\} \) and \( \{B_s\} \). Their existence is deeply related to the null vector conditions on the primary field \( \Phi \) driving the perturbation. Then, the integrals of motion off–criticality are

\[ Q_s = \int dzA_s - d\bar{z}B_s, \]  

(1.3)

with \( s \) ranging over an infinite set of values, depending on the model.

In this paper we study the geometry and the classical conservation laws of the simplest \( Z_N \)–invariant CFT, ie the \( SU(2)/U(1) \) parafermion coset model, perturbed by its first thermal operator \( \Phi = \epsilon_1 \). In the large \( N \) limit, where all our work will be concentrated, this theory is described by the action

\[ S = \int \frac{\partial u \bar{\partial} \bar{u} + \bar{\partial} u \partial \bar{u}}{1 - |u|^2} + g \left(1 - |u|^2\right), \]  

(1.4)

with \( |u|^2 \leq 1 \). The first term is the usual classical action for the \( SU(2)/U(1) \) coset model and can be obtained from the \( SU(2) \) gauged WZW model after performing the necessary gauge field integrations [3, 4, 5]. The potential term, which takes us away from the critical point while preserving the \( U(1) \) invariance of the theory, has conformal dimension \( 0 (= \lim_{N \to \infty} (2/N + 2)) \) and it corresponds to the first thermal operator of the unperturbed model. Many aspects of this model have been studied before, in connection with the relativistic theory of vortices in superfluids and other problems in field theory.

The classical equations of motion that follow from the action (1.4) are

\[ \partial \bar{\partial}u + \frac{\bar{\partial} u \partial u}{1 - |u|^2} + g \left(1 - |u|^2\right) = 0, \]  

(1.5)

\[ \partial \bar{\partial} \bar{u} + \frac{\bar{\partial} \bar{u} \partial u}{1 - |u|^2} + g \left(1 - |u|^2\right) = 0. \]  

(1.6)
This theory provides a 2–component generalization of the sine–Gordon model. Indeed, for \( u = \bar{u} = \cos \theta \), the classical equations of motion reduce to

\[
2 \partial \partial \theta = g \sin 2\theta \tag{1.7}
\]

and so the integrability of the full theory should generalize the results we already know for the sine–Gordon model.

Integrable deformations of \( Z_N \)–symmetric models of CFT have been studied extensively [6, 7]. However, the Lagrangian description and the geometric interpretation of these perturbations have not been addressed in all generality. We focus on the simplest perturbation that changes sign under Krammers–Wannier duality and examine its geometry, in the context of perturbed parafermion models. The recent developments in the geometric interpretation of various CFT coset models as exact theories of black holes [5], provide the main motivation for considering this question with a perturbation switched on. We also write the classical equations of motion as a zero curvature condition with spectral parameter. Then, the abelianization method of gauge connections [8, 9] is employed to construct classically the infinitely many local conservation laws of the theory away from criticality, in a systematic way. Generalizations to other perturbed coset models are possible, but go beyond the scope of the present work.

The Lagrangian approach to perturbed CFT coset models has been proven useful in other occasions for studying the integrability aspects of some special operators. For example, when a relevant perturbation by the \((1, 3)\) operator is applied to the minimal models of the Virasoro algebra with \( c \leq 1 \), the perturbed system is effectively described by the sine–Gordon equation (1.7), for appropriately chosen values of the coupling constant \( g \) [10]. In this case, the local conservation laws can be constructed systematically from the zero curvature formulation of the problem and they are related, as it is well known, with the Hamiltonian densities of the KdV hierarchy (ie, their flows are mutually commuting). More generally, affine Toda theories have provided a Lagrangian framework for looking at the problem of various integrable perturbations (see for instance [11]).

The perturbed theory (1.4) has an interesting physical and geometric interpretation, when it is defined on Minkowski space. Based on some old work by Lund and Regge [12, 13], we find that for \( g < 0 \) its classical equations of motion describe a reduced form of the \( O(4) \) non–linear \( \sigma \)–model in two dimensions (see also [14]). For \( g > 0 \), it describes in a certain gauge the relativistic motion of vortices in constant external field. To put it differently, the physical picture for \( g > 0 \) is that of a 4–dim bosonic string propagating in an axionic background of vortex type. At the conformal point \( g = 0 \), it describes the (transverse modes of the) free Nambu–Goto string in 4–dim Minkowski space in the orthonormal gauge (see also [15]).

We should mention for completeness (and as independent motivation) that the perturbed parafermion theory (1.4) is also interesting for the classical problem of two massless Fermi fields with contact 4–fermion interaction \((\bar{\psi}^\alpha \psi^\alpha)^2 - (\bar{\psi}^\alpha \gamma_5 \psi^\alpha)^2\) [16]. The Nambu–Jona-Lasinio model in 2–dim Minkowski space (or Gross–Neveu model) provides
a chirally invariant generalization of the (multi–component) Thirring model. Exploiting the symmetries of this model, we may reduce it in a frame where

\[ \psi_1^* \alpha_1 \psi_1^\alpha = g_1, \quad \psi_2^* \alpha_2 \psi_2^\alpha = g_2, \]

(1.8)

with \( g_1 \) and \( g_2 \) being constant. Here (1) and (2) denote the upper and lower components of the Lorentz spinors \( \psi^\alpha \), while summation over the fermion species \( \alpha = 1, 2 \) is implicitly assumed. Then, it can be verified [17] that the composite complex fields

\[ u = \frac{1}{2 \sqrt{g_1 g_2}} \sum_{\alpha=1}^2 \bar{\psi}^\alpha (1 + \gamma_5) \psi^\alpha, \quad \bar{u} = \frac{1}{2 \sqrt{g_1 g_2}} \sum_{\alpha=1}^2 \bar{\psi}^\alpha (1 - \gamma_5) \psi^\alpha \]  

(1.10)

satisfy eqs.(1.4), (1.5) with

\[ g = g_1 g_2. \]

(1.11)

If there is an arbitrary coupling constant in the 4–fermion interaction, it will enter multiplicatively in (1.11). Consequently, in the absence of the 4–fermion interaction, the unperturbed \( SU(2)/U(1) \) coset model is recovered.

This result is analogous to the well known relation between the Thirring and the sine–Gordon models [18]. Note in the present case that the chiral invariance of the Gross–Neveu model manifests as rotational \( (U(1) \) invariance) of the perturbed parafermion model. Indeed, in the variables (1.10), the chiral 4–fermion interaction is simply \( |u|^2 \). Breaking chiral invariance of the theory by adding a mass term \( \bar{\psi}^\alpha \psi^\alpha \), corresponds to introducing the \( U(1) \) violating term \( u + \bar{u} \) in the perturbed parafermion model. A useful consequence of our work is that the infinite many conservation laws of the theory (1.4) can be applied directly to the classical \( U(2) \) fermion system, using the transformation (1.10) in the special frame (1.8), (1.9). To the best of our knowledge, the systematic construction of the infinite many conserved charges of the Gross–Neveu model has not been carried out in detail. Although its integrable properties have been studied in general, the (off–critical) \( W_\infty \) structure of its currents has not been recognized so far.

Having presented an outline of the main issues and motivations of the present work, we describe briefly the organization of the remaining sections. In section 2, we examine the perturbation of the parafermion coset by its first thermal operator, in the framework of gauged WZW models. We also discuss the 1–soliton (and anti–soliton) solution that exists in the large \( N \) limit of the theory. In section 3, the physical interpretation and geometry of the classical theory (1.4) are subsequently described in Minkowski space for \( g > 0, g < 0 \) and \( g = 0 \). In section 4, the infinitely many local conservation laws are obtained by the abelianization method of gauge connections. It turns out that for generic values of the coupling constant \( g \), the higher spin currents of the theory are written in terms of an off–critical generalization of the \( W_\infty \) generators. In section 5, connections with KdV type equations are presented. In particular, the conserved densities of the perturbed model are identified with the Hamiltonian densities of the \((2–boson)\) KP hierarchy, thus generalizing the connection between the sine–Gordon equation and
SL(2) KdV hierarchy. Finally, in section 6, we present the conclusions and directions for future work. A geometrical interpretation of the conserved charges is also given in terms of embeddings.

2 WZW Description of the Model

We review first some standard results from the theory of $Z_N$ parafermions [19], in order to explain the form and the properties of the classical action (1.4). It is known that the field space of the $SU(2)_N$ WZW model (see also [20, 21]) contains $N+1$ invariant fields $\Phi^{(j)}$ with $j = 0, 1/2, 1, 3/2, \cdots, N/2; \Phi^{(0)}$ coincides with the identity operator. Each $\Phi^{(j)}$ is an $SU(2) \times SU(2)$ tensor with $(2j+1)^2$ components $\Phi^{(j)}_{m,\bar{m}}$ with $m, \bar{m} = -j, -j+1, \cdots, j-1, j$. These fields have conformal dimension $j(j+1)/(N+2)$ and $m, \bar{m}$ are the $U(1)$ charges of $\Phi^{(j)}_{m,\bar{m}}$ in the two chiral sectors of the theory. The principal fields $\phi_{2m,2\bar{m}}^{(2j)}$ of the $Z_N$ parafermion theory are related with the components of the WZW invariant fields by

$$\Phi^{(j)}_{m,\bar{m}}(z, \bar{z}) = \phi_{2m,2\bar{m}}^{(2j)}(z, \bar{z}) \exp \left\{ \frac{im}{\sqrt{N}} \chi(z) + \frac{i\bar{m}}{\sqrt{N}} \bar{\chi}(\bar{z}) \right\},$$

where $\chi$ and $\bar{\chi}$ are the $U(1)$ bosons of the two chiral sectors that are moded out in the construction of the $SU(2)_N/U(1)$ coset model. In this notation, $\sigma_k = \phi_{k,k}^{(k)}$ are the spin variables and $\mu_k = \phi_{k,-k}^{(k)}$ are the dual spin variables of the parafermion theory.

We are interested in the primary field with $j = 1$, in which case the 9 components of $\Phi^{(1)}$ can be naturally identified [20] with the matrix elements

$$\Phi^{(1)}_{ab} = Tr(g^{-1}T_a g T_b),$$

where $g, T$ are $SU(2)$ group elements and Lie algebra generators respectively. The thermal operators (sometimes also called energy operators) are defined to be the $U(1)$ neutral fields $\epsilon_j = \phi_{0,0}^{(2j)}$ and so

$$\epsilon_1 = Tr(g^{-1}\sigma_3 g \sigma_3).$$

In the large $N$ limit, the conformal dimension of all $\epsilon_j$ goes to zero. An important property of the thermal operators under the Krammers–Wannie duality $\sigma \leftrightarrow \mu$ is

$$\epsilon_j \rightarrow (-1)^j \epsilon_j.$$  

The Krammers–Wannier duality generalizes to arbitrary $N$ the relation between the low and high temperature phases of the Ising model. As we will see later, the change of sign in $\epsilon_1$ under $\sigma \leftrightarrow \mu$ implies, upon analytic continuation in Minkowski space, a duality relation between the $O(4)$ non–linear $\sigma$–model and the relativistic theory of vortices in a certain frame.
Consider now the Lagrangian description of the $SU(2)/U(1)$ coset model in terms of the $SU(2)$ gauged WZW model [3, 4, 5]. Gauging the diagonal $U(1)$ subgroup of $SU(2)$ we obtain the action

$$S = S_{WZW} + \frac{N}{2\pi} \int Tr \left( iA\partial gg^{-1} - i\bar{A}g^{-1}\partial g + Ag\bar{A}g^{-1} - A\bar{A} \right),$$

where the gauge fields $A$, $\bar{A}$ take values in $U(1)$. The classical equations of motion for $A$ and $\bar{A}$ are in component form

$$A = \frac{1}{2(1 - M_{33})} Tr(g^{-1}\partial g \sigma_3),$$

$$\bar{A} = -\frac{1}{2(1 - M_{33})} Tr(\bar{\partial}gg^{-1}\sigma_3),$$

where $M_{33}$ is given by

$$M_{33} = \frac{1}{2} Tr(g^{-1}\sigma_3g\sigma_3).$$

We fix the gauge by choosing $SU(2)$ group elements of the form

$$g = \begin{pmatrix} g_0 + ig_3 & ig_1 \\ ig_1 & g_0 - ig_3 \end{pmatrix},$$

with $g_0^2 + g_1^2 + g_3^2 = 1$. Then, in this unitary gauge, substituting the classical equations of motion for $A$ and $\bar{A}$ in (2.5), we obtain the action of the $SU(2)/U(1)$ coset model

$$S = \frac{N}{4\pi} \int \frac{\partial u\partial \bar{u} + \partial \bar{u}\partial u}{1 - |u|^2}$$

in terms of the complex variables

$$u = g_0 + ig_3, \quad \bar{u} = g_0 - ig_3.$$ 

Clearly, we have the condition $|u|^2 \leq 1$.

The classical action (2.10) differs from the ordinary $O(3)$ non–linear $\sigma$–model in that the target space metric is $(1 - |u|^2)^{-1}$ instead of $(1 - |u|^2)^{-2}$. As a result, the topology of the target space is not that of a round sphere but of two bell touching together at the rim $|u|^2 = 1$. The action (2.10) is valid only classically and therefore it provides a Lagrangian description of the parafermion theory in the large $N$ limit. Quantum mechanically there are $1/N$ corrections to the target space metric [22] and there is also a dilaton field from the path integral measure that insures conformal invariance at the critical point. In this paper the quantum mechanics of the problem will not be considered at all.

The perturbation of $S_{WZW}$ by the first thermal operator $\epsilon_1$ can be gauged similarly, with no extra effort, because $\epsilon_1$ is the neutral component of the primary WZW field $\Phi^{(1)}$. In the unitary gauge we have

$$\epsilon_1 = 2 M_{33} = 2 (2|u|^2 - 1)$$

(2.12)
and so the action we obtain in the large $N$ limit of the perturbed theory is essentially given by eq.(1.4), where $g$ is the coupling constant of the perturbation. In deriving (1.4) we have shifted the action by a constant. This adjusts the zero of the energy density and has no effect on the classical equations of motion. We also note that unlike the sine–Gordon model (1.7), the coupling constant $g$ of the 2–component generalization (1.5), (1.6) can not be made positive always. There are two phases in the perturbed theory, one for $g > 0$ and one for $g < 0$, related to each other by the Krammers–Wannier duality (2.4). The conformal (critical) point $g = 0$ is self–dual. Generalization of this formalism to other pertubations is also possible. It would be interesting to study the Lagrangian description and the geometric interpretation of perturbations driven by the higher thermal operators of parafermions, in a systematic way. The Lund–Regge formalism that will be adopted in the next section might be useful for handling the more general case as well. We hope to address these problems in the future.

At the classical level, the coupling constant $g$ can be normalized to 1 or $-1$, depending on its sign, with no loss of generality. This is possible because $g$ is dimensionful of weight $(1, 1)$. In Euclidean space, the two cases are related to each other by Krammers–Wannier duality, as it has already been pointed out. In Minkowski space, the sign of the coupling constant can change by interchanging the role of space and time coordinates. From now on we consider the theory (1.4) defined in Minkowski space and study its physical interpretation for $g > 0$, $g < 0$ and $g = 0$ in the light–cone coordinates

$$\partial = \partial_\sigma + \partial_\tau, \quad \bar{\partial} = \partial_\sigma - \partial_\tau.$$  

(2.13)

The space and time variables will be $\sigma$ and $\tau$ in all cases.

An important property of the perturbed parafermion theory is that it admits soliton solutions, in analogy with the sine–Gordon model. To make the comparison easier, we introduce the variables $\lambda$ and $\theta$

$$u = \cos \theta \ e^{i\lambda}, \quad \bar{u} = \cos \theta \ e^{-i\lambda}.$$  

(2.14)

In the static limit $\partial = \bar{\partial}$, the 1–soliton solution of the classical equations of motion (1.5), (1.6) was found long time ago [12]. For $g > 0$ (and normalized to 1) the 1–soliton is

$$\theta(z) = \sin^{-1}\left\{\frac{\sqrt{1 - A^2}}{\cosh(z\sqrt{1 - A^2})}\right\} = \sin^{-1}\left\{\sqrt{1 - A^2}\sin\left(2 \tan^{-1}\exp(z\sqrt{1 - A^2})\right)\right\},$$  

(2.15)

$$\lambda(z) = A \int^z dz' \tan^2 \theta(z'),$$  

(2.16)

where $A$ is constant. In the limit $A \to 0$ we have $\lambda = 0$ and the usual sine–Gordon 1–soliton solution $\theta = 2 \tan^{-1} \exp z$ is obtained (see for instance [23]). The corresponding anti–soliton solution is obtained by shifting $\theta \to \pi - \theta$. As we will see later, the parameter $A$ is the $U(1)$ charge of the solution and so the sine–Gordon model describes the zero charge sector of the theory. A localized lump travelling with constant velocity is obtained, as usual, by Lorentz transformation.
Multisolitons solutions can also be constructed, using standard techniques from soliton theory. The difference between $g > 0$ and $g < 0$ in the soliton solutions has been considered in ref. [24], together with some related issues. An analogous solution, for $g = 0$, was found by Bardacki et al. [4]. In the static limit of the theory,

$$\theta(z) = \sin^{-1} \left\{ \sqrt{1 - A^2 \sin z} \right\}$$

(2.17)

and $\lambda(z)$ also given by eq. (2.16), solve the classical equations of motion without the thermal perturbation.

The existence of soliton solutions in the perturbed parafermion model might have important consequences for the associated scattering problem. This sector has not been included so far in determining the $S$-matrix of the theory. Naive consideration of the large $N$ limit shows that $S = 1$ for this model [6, 25]. It is not clear at this point whether the soliton solutions persist for finite $N$ or whether they are characteristic of the large $N$ limit. It might be possible to address this question by considering the $1/N$ corrections to the classical action (1.4) from the gauged WZW point of view. In any event, the scattering of these solitons should be investigated separately in the future.

### 3 Geometry of the Thermal Perturbation

In this section we describe geometrically the classical equations of motion of the perturbed parafermion model. The main subject of the section is the embedding of a 2–dim surface $S$ with local coordinates $\sigma, \tau$ and metric

$$ds^2 = \cos^2 \theta d\sigma^2 + \sin^2 \theta d\tau^2$$

(3.1)

in a 3–dim space of constant curvature, which in turn is embedded in 4–dim flat space. The differential equations that the metric $g$ and the extrinsic curvature tensor $K$ of $S$ have to satisfy, in order to have a solution to the embedding problem, are given by the Gauss–Codazzi integrability conditions (see for instance [26]). These conditions admit a field theoretic interpretation which helps us to understand the physics and geometry of the first thermal perturbation of the parafermion coset. The technical details of the embedding will be considered later and they are intimately related with the physical interpretation of the perturbed coset model. The cases $g < 0$ and $g > 0$ are treated separately, since the results we obtain depend crucially on the sign of the coupling constant. In this framework we also revisit the critical point $g = 0$ and find that the $SU(2)/U(1)$ coset model provides an effective description of the transverse modes of the 4–dim Nambu–Goto string in the orthonormal gauge.

The formalism we adopt here originates in the work by Lund and Regge [12, 13] (see also [14]), but since it is rather unknown we review briefly the main ideas applicable to

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*I thank T. Hollowood for some discussions on this point.

†Although we use the same symbol $g$ for the metric and the coupling constant, the distinction between the two will be obvious from the context they appear.
our model. The considerations are entirely local, with no proper reference to boundary conditions. Throughout this section, the perturbed parafermion model is defined on 2–dim Minkowski space. Analytic continuation to Minkowski space is neccessary, in order to describe the physics and geometry of the theory via embeddings.

(i) $g < 0$: In this case, the perturbed model (1.4) is classically equivalent to the 2–dim $O(4)$ non–linear $\sigma$–model, reduced in a certain frame. Recall that the $O(4)$ model consists of four scalar fields $\eta_i(\sigma,\tau) (i = 1, 2, 3, 4)$ interacting through the quadratic constraint

$$ (\eta_i)^2 = 1. \tag{3.2} $$

The solutions of the classical equations of motion

$$ (\partial_{\sigma}^2 - \partial_{\tau}^2)\eta_i + \left( (\partial_{\sigma}\eta_k)^2 - (\partial_{\tau}\eta_k)^2 \right)\eta_i = 0, \tag{3.3} $$

describe 2–dim surfaces (parametrized by $\sigma$ and $\tau$) embedded in the 3–dim sphere (3.2), which in turn is embedded in flat 4–dim Euclidean space. The embedding variables are simply $\eta_i$. It is natural to reduce the $O(4)$ non–linear $\sigma$–model in the frame

$$ (\partial_{\sigma}\eta_i)^2 + (\partial_{\tau}\eta_i)^2 = 1, \quad (\partial_{\sigma}\eta_i)(\partial_{\tau}\eta_i) = 0 \tag{3.4} $$

by exploiting the invariances of the classical theory. When the conditions (3.4) are satisfied, the induced metric on the 2–dim embedded surface is given by eq.(3.1), where

$$ \cos^2\theta = (\partial_{\sigma}\eta_i)^2, \quad \sin^2\theta = (\partial_{\tau}\eta_i)^2. \tag{3.5} $$

Note that the $O(4)$ model is defined in Minkowski space, while the associated embedding problem is entirely Euclidean.

The formulation of an embedding problem requires apart from the metric $g$, knowledge of the extrinsic curvature tensor $K$. Its components are defined to be

$$ K_{\sigma\sigma} = (\partial_{\sigma}^2\eta_i)Z_i^{(3)} \quad K_{\tau\tau} = (\partial_{\tau}^2\eta_i)Z_i^{(3)} \quad K_{\sigma\tau} = K_{\tau\sigma} = (\partial_{\sigma}\partial_{\tau}\eta_i)Z_i^{(3)}, \tag{3.6} $$

where $Z_i^{(3)} = Z_i^{(1)} \times Z_i^{(2)}$ with

$$ Z_i^{(1)} = \frac{1}{\cos\theta}(\partial_{\sigma}\eta_i), \quad Z_i^{(2)} = \frac{1}{\sin\theta}(\partial_{\tau}\eta_i). \tag{3.7} $$

We also choose $Z_i^{(4)} = \eta_i$ to complete an orthonormal tetrad in 4–dim Euclidean space, where the embedding takes place. The classical equations of motion of the reduced $O(4)$ non–linear $\sigma$–model are equivalent in the special frame (3.4) to

$$ K_{\sigma\sigma} = K_{\tau\tau}. \tag{3.8} $$

This is obtained by multiplying the classical equations of motion (3.3) with $Z_i^{(3)}$. Multiplication with $Z_i^{(1)}$, $Z_i^{(2)}$ and $Z_i^{(4)}$ leads to identities in the special frame (3.4). The reduction of the $O(4)$ model in this frame is analogous to the reduction of the chirally invariant $U(2)$ Gross–Neveu model, described in the introduction.
It is clear that the classical equations of motion of the reduced theory can be formulated as an embedding problem in the 3–dim sphere (3.2), with the extrinsic curvature satisfying the physical requirement (3.8). This embedding is possible provided that the Gauss–Codazzi conditions on $g$ and $K$ are satisfied. In the present case they assume the form

$$\partial_\tau (\tan \theta K_{\sigma\sigma}) = \partial_\sigma (\tan \theta K_{\sigma\tau}), \quad \partial_\sigma (\cot \theta K_{\sigma\sigma}) = \partial_\tau (\cot \theta K_{\sigma\tau}), \quad (3.9)$$

$$R = (g^{\alpha\beta} K_{\alpha\beta})^2 - g^{\alpha\gamma} g^{\beta\delta} K_{\alpha\beta} K_{\gamma\delta} + 2, \quad (3.10)$$

where $R$ is the scalar curvature of the 2–dim metric (3.1)

$$R = \frac{2}{\sin \theta \cos \theta} (\partial_\tau^2 \theta - \partial_\sigma^2 \theta). \quad (3.11)$$

The constant term in eq.(3.10) is the curvature contribution of the 3–dim sphere (3.2) where the embedding takes place. The first condition in eq.(3.9) implies the existence of a scalar field $\lambda$ such that $K_{\sigma\sigma} = \cot \theta \partial_\sigma \lambda$ and $K_{\sigma\tau} = \cot \theta \partial_\tau \lambda$. Introducing complex variables $u, \bar{u}$ as in eq.(2.13) and light–cone coordinates as in eq.(2.16), we obtain

$$K_{\sigma\sigma} + K_{\sigma\tau} = i \frac{u \partial \bar{u} - \bar{u} \partial u}{2 | u | \sqrt{1 - | u |^2}}, \quad K_{\sigma\sigma} - K_{\sigma\tau} = i \frac{u \partial \bar{u} - \bar{u} \partial u}{2 | u | \sqrt{1 - | u |^2}}. \quad (3.12)$$

Then, it is straightforward to verify that the remaining Gauss–Codazzi conditions become the classical equations of motion of the perturbed parafermion model in Minkowski space, with $g = -1$.

The reconstruction of $S$ (and hence $\eta_k$) can be done by solving the Gauss–Weingarten equations for the orthonormal vectors $Z^{(1)}, Z^{(2)}, Z^{(3)}$ and $Z^{(4)}$. These are first order linear differential equations, forming a system, whose compatibility conditions are provided by the Gauss–Codazzi equations (for details see [26]). The classical equivalence between the two theories, both defined in 2–dim Minkowski space, has been established with the aid of a Euclidean embedding problem. In this case, the curvature of the 3–dim sphere (3.2) determines the coupling constant (it turns out to be $-2g$) and hence $g$ is negative.

The physical interpretation of the perturbed theory with $g > 0$ will be addressed differently, without interchanging $\sigma \leftrightarrow \tau$. Following Lund and Regge [12], we find that the phase of the theory with $g > 0$ describes (in a certain gauge) the relativistic motion of vortices in a constant external field. We also adopt this picture in order to show that the transverse modes of the 4–dim Nambu–Goto string in the orthonormal gauge are effectively described by the conformal limit of the $SU(2)/U(1)$ coset model.

(ii) $g > 0$: Consider first the string action in 4–dim Minkowski space

$$S = -N \int \sqrt{-detg} \, d\sigma d\tau + \int A_{\mu}(X) \partial_\sigma X^\mu \partial_\tau X^\nu d\sigma d\tau - \frac{1}{4} \int F_\mu F^\mu d^4 y, \quad (3.13)$$

where $\sigma, \tau$ are local coordinates on the string world–sheet $\Sigma$ and $detg$ is the determinant of the 2–dim metric on $\Sigma$. The first term is the usual Nambu–Goto action, while the
second term represents a self–interaction of Kalb–Ramond type [27]. $A_{\mu\nu}(X)$ is an anti-
symmetric tensor (axionic background) and $F_\mu$ is defined to be

$$F_\mu = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} \partial_\rho A_{\nu\lambda}. \quad (3.14)$$

$X^{\mu}(\sigma, \tau) \ (\mu = 0, 1, 2, 3)$ are the embedding variables of the string in 4–dim Minkowski
space with signature $- + ++$. When the spatial components of $A_{\mu\nu}$ are linear in $X_i$
$(i = 1, 2, 3)$, ie

$$A_{ij} = \epsilon_{ijk} X^k, \quad (3.15)$$

this action describes the relativistic motion of vortices (strings) in a superfluid [12] (see
also [28] for the non–relativistic limit). In this case, $F^i$ is identified with the velocity of
the fluid

$$F^i = \epsilon^{ijk} \partial_j A_{k0} = v^i \quad (3.16)$$

and the last term in eq.(3.13) represents the hydrodynamic action of the superfluid.

Following Lund and Regge we study the relativistic motion of vortices in a uniform
static external field. We choose a Lorentz frame in which

$$X^0 = \tau, \quad F^i = 0 \quad (3.17)$$

and $F^0$ is constant and introduce the effective coupling constant of the theory

$$c = \frac{fF^0}{2N}. \quad (3.18)$$

At this point we do not normalize $c$ to 1, keeping its dependence on $f$ and $N$ explicitly.
We need this in order to understand the physical interpretation of the limit $c \rightarrow 0$ that
will be considered later. We also choose the orthonormal gauge for the induced metric
on the vortex (string) world–sheet, which leads to the quadratic constraints

$$(\partial_\sigma X_i)^2 + (\partial_\tau X_i)^2 = 1, \quad (\partial_\sigma X_i)(\partial_\tau X_i) = 0 \quad (3.19)$$

in the Lorentz frame (3.17). This gauge choice is analogous to the reduction of the $O(4)$
non–linear $\sigma$–model (3.4) introduced earlier. Then, the vortex dynamics is determined
entirely by the classical equations of motion

$$(\partial_\sigma^2 - \partial_\tau^2)X_i + 2c \left((\partial_\sigma \vec{X}) \times (\partial_\tau \vec{X})\right)_i = 0 \quad (3.20)$$

plus the constraints (3.19).

Note that for uniform static external field, the last term in the action (3.13) can be
dropped out (it is just a number). The theory in this case describes a 4–dim bosonic
string in an axionic background of vortex type. For $f = 0$, ie $c = 0$, it describes the
propagation of a free bosonic string in 4–dim Minkowski space in the orthonormal gauge
with $X_0 = \tau$. It is important to realize that the Lund–Regge model for vortex dynamics in
a uniform static external field is classically equivalent to the perturbed parafermion coset
(1.4) with $g = c^2 > 0$, upon analytic continuation in Minkowski space. This formalism is
certainly unphysical for \( g < 0 \), since the coupling constant of the vortex self–interaction term will be imaginary in that case. We also point out that the results presented in the sequel are independent of the Lorentz frame \( X_0 = \tau \), up to Lorentz transformations in the \((\sigma, \tau)\) space.

To illustrate the vortex–like description of the perturbed parafermion model with \( g > 0 \) it is convenient to incorporate the classical equations of motion (3.20) in the integrability conditions of an embedding problem, in analogy with the \( g < 0 \) case. Let \( S \) be the projection of the string world–sheet \( \Sigma \) in the \( X_0 = \tau \) hyperplane\(^\dagger\). \( S \) is a Euclidean surface, which in the orthonormal frame has an induced metric given by eq.(3.1). We have

\[
(\partial_\sigma X^i)^2 = \cos^2 \theta, \quad (\partial_\tau X^i)^2 = \sin^2 \theta. \tag{3.21}
\]

In analogy with the \( g < 0 \) case, we also consider the extrinsic curvature tensor of \( S \),

\[
K_{\sigma\sigma} = (\partial_\sigma^2 X^i)Z_i^{(3)}, \quad K_{\tau\tau} = (\partial_\tau^2 X^i)Z_i^{(3)}, \quad K_{\sigma\tau} = (\partial_\sigma \partial_\tau X^i)Z_i^{(3)} \tag{3.22}
\]

with \( Z^{(3)} = Z^{(1)} \times Z^{(2)} \), where

\[
Z_i^{(1)} = \frac{1}{\cos \theta} \partial_\sigma X^i, \quad Z_i^{(2)} = \frac{1}{\sin \theta} \partial_\tau X^i. \tag{3.23}
\]

In terms of these variables, the classical evolution of vortices in the orthonormal frame is entirely determined by the embedding of \( S \) in the 3–dim Euclidean space \( X_0 = \tau \), which in turn is embedded in 4–dim Minkowski space. The components of the extrinsic curvature tensor have to satisfy the condition

\[
K_{\sigma\sigma} - K_{\tau\tau} + 2c \sin \theta \cos \theta = 0, \tag{3.24}
\]

which follows from the classical equations of motion (3.20) by multiplication with \( Z_i^{(3)} \). Multiplication with \( Z_i^{(1)} \) and \( Z_i^{(2)} \) leads to identities in the special frame (3.19). Hence, eq.(3.24) encodes all the information contained in eq.(3.20).

The embedding problem here is different in that the 3–dim space where the embedding takes place is flat and the extrinsic curvature satisfies the condition (3.24) rather than (3.8). The Gauss–Codazzi integrability conditions have to be satisfied, however, in order to have a solution to the problem. In this case they read

\[
\partial_\tau (\tan \theta (K_{\sigma\sigma} + K_{\tau\tau})) = 2 \partial_\sigma (\tan \theta K_{\sigma\tau}), \quad \partial_\sigma (\cot \theta (K_{\sigma\sigma} + K_{\tau\tau})) = 2 \partial_\tau (\cot \theta K_{\sigma\tau}), \tag{3.25}
\]

\[
R = (g^{\alpha\beta} K_{\alpha\beta})^2 - g^{\alpha\gamma} g^{\beta\delta} K_{\alpha\beta} K_{\gamma\delta}, \tag{3.26}
\]

where \( R \) is the curvature of the 2–dim metric (3.1), given again by eq.(3.11). Introducing light–cone variables \( z, \bar{z} \) and complex coordinates \( u, \bar{u} \) as before, we find that the first condition in eq.(3.25) implies

\[
K_{\sigma\sigma} + K_{\tau\tau} + 2 \cdot K_{\sigma\tau} = i \frac{u \partial \bar{u} - \bar{u} \partial u}{|u| \sqrt{1 - |u|^2}}, \quad K_{\sigma\sigma} + K_{\tau\tau} - 2 \cdot K_{\sigma\tau} = i \frac{u \bar{u} \partial u - \bar{u} \partial \bar{u}}{|u| \sqrt{1 - |u|^2}}. \tag{3.27}
\]

\(^\dagger\)To know \( \vec{X} \), it is sufficient to consider \( S \) for \( X^0 = \tau \).
It is straightforward to verify using eq. (3.24) that the remaining Gauss–Codazzi integrability conditions become the classical equations of motion of the perturbed parafermion model in Minkowski space, with $g = c^2$. The reconstruction of $S$ (and hence $X_i$) can be done by solving the corresponding Gauss–Weingarten equations for the orthonormal vectors $Z^{(1)}$, $Z^{(2)}$ and $Z^{(3)}$.

(iii) $g = 0$: When the coupling constant $f$ of the antisymmetric tensor $A_{\mu\nu}$ is zero, the theory reduces to the Nambu–Goto string propagating in 4–dim Minkowski space. As it has been pointed out already, in the orthonormal gauge with $X_0 = \tau$, the unperturbed $SU(2)/U(1)$ coset model, when it is defined in Minkowski space, describes the classical dynamics of the transverse modes of a free string. This result is very intriguing and probably it can be generalized to more arbitrary backgrounds. In a separate publication [15], the implications of this equivalence are analyzed in detail. The infinitely many conservation laws of the parafermion coset model can be easily applied to string dynamics to yield hidden symmetries (Backlund transformations) in their classical solution space. In the next section we analyze the structure of the conserved charges for arbitrary values of the coupling constant $g$ and establish their relation with $W_{\infty}$. Then, a geometric interpretation of the higher spin currents follows immediately from the embedding problem that describes the classical equations of motion of the theory.

The classical equivalence between 4–dim string theory and the perturbed parafermion model with $g \geq 0$ is not restricted to the special frame $X_0 = \tau$. For $X_0 = \tilde{\tau}$, where

$$\tilde{\tau} = \sinh \alpha \sigma + \cosh \alpha \tau, \quad \tilde{\sigma} = \cosh \alpha \sigma + \sinh \alpha \tau$$

(3.28)

are related by Lorentz transformation in $(\sigma, \tau)$ space, the embedding problem remains essentially the same. The projection of the string world–sheet is performed now in the $X_0 = \tilde{\tau}$ 3–space, but the classical equations of motion that follow from the Gauss–Codazzi conditions remain unchanged [12]. For this reason, the equivalence between the two theories is independent of the Lorentz frame. This issue does not arise for $g < 0$, because the physics of the problem does not require the choice of a Lorentz frame to formulate the corresponding embedding.

The soliton solution (2.14)–(2.15) of the classical equations of motion with $g > 0$ becomes relevant for string propagation in an axionic background of vortex type. It would be interesting to study further the localized properties of this soliton solution, directly in the string variables $X_i$. The issue of boundary conditions and the physical interpretation of coset model solutions has to be addressed properly in the future, in the context of string theory.

4 Local Conservation Laws

We turn now to the explicit construction of the infinitely many conservation laws of the perturbed parafermion theory for arbitrary values of the coupling constant $g$. For this, it
is convenient to rewrite the classical equations of motion (1.5), (1.6) as a zero curvature condition. We introduce the linear system of differential equations
\[
\partial \Phi = A \Phi, \quad \bar{\partial} \Phi = B \Phi,
\]
where \(A\) and \(B\) are 2 × 2 matrices depending on \(z, \bar{z}\) and a spectral parameter \(\lambda\)
\[
A = A_0 + \lambda E, \quad B = B_0 + \lambda^{-1} B_1.
\]
The spectral parameter should not be confused with the scalar field \(\lambda\) appearing in the definition (2.13) of \(u\) and \(\bar{u}\). \(A_0, B_0\) and \(B_1\) are taken to be
\[
A_0 = -\frac{1}{4 |u|^2 (1 - |u|^2)} \left( \begin{array}{cc}
(2|u|^2 - 1)(u \bar{\partial} u - \bar{u} \partial u) & 4i |u| \sqrt{1 - |u|^2} \bar{u} \partial u \\
4i |u| \sqrt{1 - |u|^2} u \partial \bar{u} & -(2|u|^2 - 1)(u \partial \bar{u} - \bar{u} \partial u)
\end{array} \right),
\]
\[
B_0 = \frac{u \bar{\partial} u - \bar{u} \partial u}{4 |u|^2 (1 - |u|^2)} E,
\]
\[
B_1 = \frac{g}{4} \left( \begin{array}{cc}
2 |u|^2 - 1 & -2i |u| \sqrt{1 - |u|^2} \\
2i |u| \sqrt{1 - |u|^2} & -(2|u|^2 - 1)
\end{array} \right)
\]
and \(E\) is the constant matrix
\[
E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Then, it may be easily verified that the compatibility (zero curvature) condition of the linear system (4.1),
\[
[\partial - A, \bar{\partial} - B] = 0,
\]
is equivalent to the classical equations of motion (1.5), (1.6), for all values of \(\lambda\).

The physical meaning of the spectral parameter \(\lambda\) can be understood in terms of the Lorentz transformation (3.28). Suppose that the linear system (4.1) is given initially with \(\lambda = 1\). The Lorentz transformation \((\sigma, \tau) \rightarrow (\bar{\sigma}, \bar{\tau})\) amounts to the rescaling \(\partial \rightarrow \exp(-\alpha) \partial, \bar{\partial} \rightarrow \exp(\alpha) \bar{\partial}\) and \(A_0 \rightarrow \exp(-\alpha) A_0, B_0 \rightarrow \exp(\alpha) B_0\). Absorbing the Lorentz factor, is equivalent to choosing \(\lambda = \exp(\alpha)\) in the gauge connections (4.2). Consequently, the independence of the classical equations of motion on \(\lambda\) is really a statement about Lorentz invariance. As we will see shortly, the presence of a spectral parameter in the linear system (4.1) is necessary to derive the infinitely many conservation laws of the theory in a systematic way. The results are valid equally well in Euclidean and Minkowski space.

Consider the static limit \(\partial = \bar{\partial}\) first. It follows immediately from \([\partial - A, \bar{\partial} - B] = 0\) that the quantities
\[
I_n = \frac{1}{2} Tr(A - B)^n, \quad n = 1, 2, 3, \ldots,
\]
13
are conserved, i.e. $\partial I_n = 0$. This is a well known result for 1–dim integrable systems. If the number of degrees of freedom is finite, the quantities $I_n$ will not be all independent. For the present model we find that only $I_2$ is independent, because

$$I_{2k-1} = 0, \quad I_{2k} = (I_2)^k, \quad k = 1, 2, 3, \cdots.$$  \hspace{1cm} (4.9)

Explicit calculation shows that

$$I_2 = \frac{1}{4} Q^2 - H - \lambda Q + \frac{g}{4 \lambda} Q + \frac{1}{2} g + \lambda^2 + \frac{g^2}{16 \lambda^2};$$  \hspace{1cm} (4.10)

where

$$H = \frac{\partial u \partial \bar{u}}{1 - |u|^2} + g |u|^2, \quad Q = \frac{u \partial \bar{u} - \bar{u} \partial u}{1 - |u|^2}. \hspace{1cm} (4.11)$$

Since $\partial I_2 = 0$ for all values of $\lambda$, we arrive at the conservation laws

$$\partial H = 0 = \partial Q \hspace{1cm} (4.12)$$

in the static limit of the theory. $H$ is the energy and $Q$ is the $U(1)$ charge of a given configuration $u, \bar{u}$. For the 1–soliton (2.14), (2.15), with $g$ normalized to 1, we find that $H = 1$ and $Q = 2A$, provided that the charge of the theory is made real by multiplication with $i$. This gives a physical interpretation to the arbitrary parameter $A$ that determines the soliton solution. Solitons with zero $U(1)$ charge are simply sine–Gordon solitons.

In the general case, the conserved quantities of the theory are not given by eq.(4.8) and a field theoretical prescription is required to obtained the currents associated with the 2–dim zero curvature condition (4.7). The abelianization method of gauge connections [8, 9] provides the algorithm for their systematic construction. The main idea is to introduce a family of $\lambda$–dependent gauge transformations

$$T = 1 + \sum_{i=1}^{\infty} \lambda^{-i} t_i, \hspace{1cm} (4.13)$$

where $t_i$ depend on $z, \bar{z}$ and take values in $GL(2)$ with $det \ T \neq 0$. Then, it is always possible to choose $\{t_i\}$ appropriately so that the gauge transformed connections

$$T^{-1}(\partial - A_0 - \lambda E) T \equiv \partial + \sum_{j=0}^{\infty} \lambda^{-j} \tilde{A}_j - \lambda E, \hspace{1cm} (4.14)$$

$$T^{-1}(\bar{\partial} - B_0 - \lambda^{-1} B_1) T \equiv \bar{\partial} - \sum_{j=0}^{\infty} \lambda^{-j} \tilde{B}_j \hspace{1cm} (4.15)$$

commute with the constant matrix $E$ (4.6), i.e

$$[\tilde{A}_j, \ E] = 0 = [\tilde{B}_j, \ E] \hspace{1cm} (4.16)$$

for all values of $j$. Since $E$ is not the identity matrix, $\{\tilde{A}_j\}$ and $\{\tilde{B}_j\}$ commute among themselves and the zero curvature condition (4.7) yields upon gauge transformation an infinite number of functionally independent non–trivial conservation laws

$$\bar{\partial} \tilde{A}_j + \partial \tilde{B}_j = 0, \quad j = 0, 1, 2, \cdots. \hspace{1cm} (4.17)$$
one to each order in the $1/\lambda$ expansion. The presence of a spectral parameter in the zero curvature formulation of the problem is indeed essential. The validity of the equations for all values of $\lambda$ implies order by order the infinite hierarchy of conservation laws (4.17). For $E$ diagonal, as it is the case here, the abelianization method amounts to the diagonalization of the gauge connections $A$ and $B$. It is important to emphasize that the diagonalization of $B$ is not automatic and it can be achieved only on–shell.

Having presented the essential ingredients of this method, we may proceed with explicit calculations. Note first that the diagonalization of $A$ implies an infinite set of conditions. We have

$$\tilde{A}_0 + [E, t_1] + A_0 = 0 \quad (4.18)$$

to order $\lambda^0$, while for $j > 0$ we obtain the recursive relations

$$\tilde{A}_j + [E, t_{j+1}] + A_0 t_j - \partial t_j + \sum_{k=0}^{j-1} t_{j-k} \tilde{A}_k = 0, \quad (4.19)$$

one to each order in $\lambda^{-j}$. Since all $\tilde{A}_j$ have to commute with $E$, their most general form is

$$\tilde{A}_j = \begin{pmatrix} h_j^{(1)} & 0 \\ 0 & h_j^{(2)} \end{pmatrix}, \quad j \geq 0. \quad (4.20)$$

The problem now is to solve these recursive relations, in order to determine the form of $\tilde{A}_j$ and the gauge transformation $T$.

Note that the solution to the diagonalization problem of $A$ is not uniquely determined. There is the freedom to set the diagonal elements of all $t_i$ equal to zero, with no loss of generality. Indeed, any $2 \times 2$ matrix can be written in the form $X + [E, Y]$, where $X$ is a diagonal matrix and $Y$ off–diagonal. Taking this decomposition into account in the $\tilde{A}_j + [E, t_{j+1}]$ part of the recursive relations (4.18), (4.19), it can be easily seen that for diagonal $\tilde{A}_j$, as they should be, $t_{j+1}$ can be chosen so that their diagonal elements are zero. This gauge choice fixes the form of $t_i$ and $\tilde{A}_j$ uniquely and the solution that results from the recursive relations has no free parameters; everything is functionally dependent on the matrix elements of $A_0$. This gauge choice will be assumed from now on.

Gauge transformations with non–zero diagonal elements, modify the solution for $\tilde{A}_j$ by total $\partial$–derivative terms, via the recursive relations (4.19). This modification is supplemented in $B_j$ by substracting the $\bar{\partial}$–derivative of these additional terms. This has no effect on the local conservation laws (4.17) and it reflects the freedom we have in writing down the corresponding currents. We also note that the gauge transformation $T$ is not neccessarily restricted to $SL(2)$, but it can take values in $GL(2)$. In the gauge where the diagonal elements of all $t_i$ are set equal to zero, the determinant of $T$ is

$$detT = 1 - t_1^{12} t_2^{21} \lambda^{-2} - (t_1^{12} t_2^{21} + t_1^{21} t_2^{12}) \lambda^{-3} + \cdots. \quad (4.21)$$

Therefore, it is natural to expect that for $j \geq 2$ the matrices $\tilde{A}_j$ will not be traceless. We could have chosen a gauge (order by order in $1/\lambda$) so that $detT = 1$, to preserve the
traceless condition on all gauge connections. However, the two gauges are related to each other by the $U(1)$ element of $GL(2)$, which is proportional to the $2 \times 2$ unit matrix,

$$1 + \frac{1}{2} t_1^{12} t_1^{21} \lambda^{-2} + \frac{1}{2} (t_1^{12} t_1^{21} + t_1^{21} t_1^{12}) \lambda^{-3} + \cdots$$  \hspace{1cm} (4.22)$$

and the difference on the diagonalized gauge connections $\tilde{A}, \tilde{B}$ is therefore a total derivative trace term. We choose to work with a gauge in $GL(2)$, rather than $SL(2)$, in order to simplify the form of the recursive relations.

From eq.(4.18) we have immediately

$$h_0^{(1)} = -A_{01} = -h_0^{(2)}$$  \hspace{1cm} (4.23)$$

and

$$t_1^{12} = -\frac{1}{2} A_0^{12}, \quad t_1^{21} = \frac{1}{2} A_0^{21},$$  \hspace{1cm} (4.24)$$
in terms of the matrix elements of $A_0$. This is the initial data for the recursive relations with $j > 0$. To bring the gauge connection $A$ into the desired form, the following system of equations has to be iterated:

$$h_j^{(1)} = -A_0^{12} t_j^{21}, \quad h_j^{(2)} = -A_0^{21} t_j^{12},$$  \hspace{1cm} (4.25)$$

$$2 \ t_{j+1}^{12} = -A_0^{11} t_j^{12} + \partial t_j^{12} - \sum_{k=0}^{j-1} h_k^{(2)} t_{j-k}^{12},$$  \hspace{1cm} (4.26)$$

$$2 \ t_{j+1}^{21} = -A_0^{11} t_j^{21} - \partial t_j^{21} + \sum_{k=0}^{j-1} h_k^{(1)} t_{j-k}^{21},$$  \hspace{1cm} (4.27)$$

These relations follow from eq.(4.19), using the diagonal form (4.20) for $\tilde{A}_j$ and the off–diagonal gauge for $t_j$.

The solution (4.23) for the matrix elements of $\tilde{A}_0$ can be written in terms of $u$ and $\bar{u}$ as

$$h_0^{(1)} + h_0^{(2)} = 0, \quad h_0^{(1)} - h_0^{(2)} = \frac{2 |u|^2 - 1}{2 |u|^2 (1 - |u|^2)} (u \partial \bar{u} - \bar{u} \partial u).$$  \hspace{1cm} (4.28)$$

On the other hand, since $\tilde{B}_0 = B_0$, we derive to this order the non–chiral conservation law

$$\bar{\partial} J + \partial \bar{J} = 0,$$  \hspace{1cm} (4.29)$$

where

$$J = \frac{u \partial \bar{u} - \bar{u} \partial u}{1 - |u|^2} = 2(h_0^{(1)} - h_0^{(2)}) - \partial \left( \log \frac{u}{\bar{u}} \right),$$  \hspace{1cm} (4.30)$$

$$\bar{J} = \frac{\bar{u} \partial u - u \partial \bar{u}}{1 - |u|^2} = 2(B_0^{11} - B_0^{22}) + \bar{\partial} \left( \log \frac{u}{\bar{u}} \right).$$  \hspace{1cm} (4.31)$$

The $\lambda^0$ conserved currents are written conveniently in this form, in order to identify $J$ and $\bar{J}$ with the two components of the $U(1)$ current associated with the symmetry of the action (1.4) under $u \rightarrow u e^{i \epsilon}$ and $\bar{u} \rightarrow \bar{u} e^{-i \epsilon}$. Note that the conservation law (4.29) is
independent of the coupling constant $g$, simply because the term $|u|^2$ in the potential of the action is invariant under $U(1)$ rotations. In the static limit $\partial = \bar{\partial}$, we have $J = \bar{J} = Q$ and the conservation law $\partial Q = 0$ is recovered.

Using eqs. (4.24) and (4.25), we obtain the following expression for $\tilde{A}_1$ in terms of $u$ and $\bar{u}$:

$$h^{(1)}_1 + h^{(2)}_1 = 0, \quad h^{(1)}_1 - h^{(2)}_1 = \frac{\partial u \partial \bar{u}}{1 - |u|^2}.$$ (4.32)

Explicit calculation also shows that

$$\tilde{B}_1 = B_1 + [B_0, t_1] - \bar{\partial} t_1 = \frac{1}{4} g (2|u|^2 - 1) E,$$ (4.33)

where the second part of the equation is valid only on–shell. Therefore, to order $\lambda^{-1}$, the conservation law for the $zz$ and $\bar{z}\bar{z}$ components of the stress–energy–tensor of the theory is obtained,

$$\bar{\partial} \left( \frac{\partial u \partial \bar{u}}{1 - |u|^2} \right) + g \partial |u|^2 = 0.$$ (4.34)

At the conformal point $g = 0$, this reduces to the chiral conservation law for the $zz$ component of the stress–energy tensor of the $SU(2)/U(1)$ coset model. In the static limit, the result $\partial H = 0$ is also recovered.

Iteration of the recursive relations for $j \geq 2$ yields higher order non–chiral conservation laws, all depending on the coupling constant $g$. In the static limit they all reduce to the conservation of $H$ and $Q$, but in the field theory case we are considering now they turn out to be functionally independent and hence new. To understand their nature, in general, it is convenient to compare them with the infinitely many symmetries of the $SU(2)/U(1)$ coset model with $g = 0$. In the latter case, it is known that there is a chiral $W_\infty$ symmetry, whose generators are bilinear in the parafermions of the model [29–31]. In particular, introducing the parafermion currents

$$\psi_+ = \frac{\partial u}{\sqrt{1 - |u|^2}} V_+, \quad \psi_- = \frac{\partial \bar{u}}{\sqrt{1 - |u|^2}} V_-,$$ (4.35)

where

$$V_\pm = \exp \left( \pm \frac{1}{2} \int dz J - d\bar{z} \bar{J} \right)$$ (4.36)

are defined in terms of the non–chiral $U(1)$ current of the theory, the $W_\infty$ generators (in a quasi–primary basis and up to an overall normalization) are

$$W_s = \sum_{k=0}^{s-2} \frac{(-1)^k}{s-1} \binom{s-1}{k+1} \binom{s-1}{s-k-1} \partial^k \psi_+ \partial^{s-k-2} \psi_-,$$ (4.37)

with $s = 2, 3, 4, \cdots$. For $g = 0$, $\psi_+$ and $W_s$ are all chirally conserved, provided that the classical equations of motion of the $SU(2)/U(1)$ coset model are satisfied. For $g \neq 0$ this is not true, but it still makes sense to define parafermion currents $\psi_\pm$ as in (4.35) and (4.36), because the potential $|u|^2$ is $U(1)$–invariant.
The conservation laws that result from the diagonalization of the gauge connections of the perturbed $SU(2)/U(1)$ coset model can be written systematically for all $g$, using the off-critical generalization of the $W_\infty$ generators (4.37) in terms of the parafermions $\psi_\pm$. Note that for $g \neq 0$, $\partial W_s$ cannot be always brought in the form $\partial X$ for appropriately chosen $X$. Although this is true for $s = 2$, for higher spin currents it is not so. To describe the conserved currents of the theory in terms of $\{W_s\}$ off-criticality, we have to introduce appropriate polynomial combinations of the generators, depending on $s$. It is the abelianization method that provides the algorithm for writing down these higher (non-chiral) conservation laws of the theory in the form (4.17).

We find that for $j \geq 2$, the trace of $\tilde{A}_j$ is a total derivative of currents, which are composed from the subleading components $h_{j-1}^{(1)} - h_{j-1}^{(2)}$, $h_{j-2}^{(1)} - h_{j-2}^{(2)}$, etc. Hence, the only functionally independent conservation laws are obtained by considering $h_{j}^{(1)} - h_{j}^{(2)}$. For example, for the first few values of $j$ (apart from the ones already discussed) we find

$$h_2^{(1)} + h_2^{(2)} = -\frac{1}{4} \partial (h_1^{(1)} - h_1^{(2)}), \quad (4.38)$$

$$h_3^{(1)} + h_3^{(2)} = -\frac{1}{2} \partial (h_2^{(1)} - h_2^{(2)}), \quad (4.39)$$

$$h_4^{(1)} + h_4^{(2)} = -\frac{1}{32} \partial \left( 24(h_3^{(1)} - h_3^{(2)}) - (h_1^{(1)} - h_1^{(2)})^2 - \partial^2 (h_1^{(1)} - h_1^{(2)}) \right) \quad (4.40)$$

and so on. As for the $h_{j}^{(1)} - h_{j}^{(2)}$ components of $\tilde{A}_j$, the results of the calculation are summarized in the appendix for the first few values of $j$. The complexity of the expressions increases considerably for higher values of $j$, but they are all calculable order by order. Unfortunately, no closed expression for arbitrary $j$ is available at the moment. The form of $h_{j}^{(1)} - h_{j}^{(2)}$ indeed simplifies considerably, when they are written in terms of the parafermions off-criticality. Extending the definition of $W_s$ to all values of the coupling constant $g$ and introducing the explicit dependence of $A_0$ on $u$ and $\bar{u}$, we obtain after some lengthy calculation

$$h_1^{(1)} - h_1^{(2)} = W_2, \quad (4.41)$$

$$h_2^{(1)} - h_2^{(2)} = -\frac{1}{4} W_3, \quad (4.42)$$

$$h_3^{(1)} - h_3^{(2)} = \frac{1}{20} \left( W_4 + 5(W_2)^2 + \frac{3}{2} \partial^2 W_2 \right), \quad (4.43)$$

$$h_4^{(1)} - h_4^{(2)} = -\frac{1}{112} (W_5 + 21W_2W_3 + 6 \partial^2 W_3) \quad (4.44)$$

and so on. This summary shows that $h_{j}^{(1)} - h_{j}^{(2)}$ are functionally related to the $W_s$ currents, as advertized. Note that these polynomial combinations are independent of $g$.

The form of the infinitely many conservation laws of the theory (with arbitrary $g$) is determined completely, by performing the gauge transformation $T$ on the gauge connection $B$ as well. The iteration of the recursive relations also determines $\{t_i\}$ order by order. Explicit results for the first few values of $j$ are given in the appendix, together with $\tilde{B}_j$. It can be verified that the matrices $\tilde{B}_j$ are all diagonal on-shell, as required.
by the abelianization procedure. The final result for the local conservation laws of the theory (discarding total derivative terms and overall normalization factors) is

\[ \bar{\partial}W_3 + g \partial(u\partial \bar{u} - \bar{u}\partial u) = 0, \quad (4.45) \]

\[ \bar{\partial}(W_4 + 5W_2^2) + g \left( \partial^2|u|^2 + 5 \frac{2|u|^2 - 1}{1 - |u|^2} \partial u \partial \bar{u} \right) = 0, \quad (4.46) \]

\[ \bar{\partial}(W_5 + 21W_2W_3) + g \partial \left( u\partial^3 \bar{u} - \bar{u}\partial^3 u - \frac{6 - 13|u|^2}{1 - |u|^2} (\partial u \partial^2 \bar{u} - \partial \bar{u} \partial^2 u) \right) + 7 \frac{2 - 3|u|^2}{(1 - |u|^2)^2} (u\partial \bar{u} - \bar{u}\partial u) \partial u \partial \bar{u} \right) = 0. \quad (4.47) \]

The higher order conservation laws can be constructed in a similar fashion, but it is computationally difficult to find closed expressions for them, in general. Of course, we also have a complementary set of non–chirally conserved currents which is obtained by interchanging \( u \leftrightarrow \bar{u} \) and \( \partial \leftrightarrow \bar{\partial} \).

In the conformal limit \( g \to 0 \), the chiral \( W_\infty \) algebra is recovered, but in a polynomial basis. Its commutation relations are not linear in this basis, but we know that there exists the quasi–primary basis (4.37) where the linear structure of the algebra is manifest. For \( g \neq 0 \) it does not make sense to talk about the algebra of the non–chiral currents, because the symmetries they generate are global; it is only at the conformal point that global symmetries are promoted to local. What makes sense, however, is to consider the corresponding charges for all \( s = 1, 2, 3, \ldots \), which are conserved and in involution. Since they are infinitely many of them, we have a rigorous method (and an algorithm) for establishing the complete integrability of the parafermion model perturbed by its first thermal operator, in the large \( N \) limit.

The path ordered exponential

\[ P \exp \left( \int_{P_1}^{P_2} d\bar{z}A + d\bar{z}B \right) \quad (4.48) \]

is independent of the path joining two space–time points \( P_1 \) and \( P_2 \), provided that \( A, B \) satisfy the zero curvature condition (4.7). Since the expression (4.48) depends only on the end points \( P_1 \) and \( P_2 \), we may use it to obtain an alternative description of the conserved charges, by considering paths joining two spatially far apart points at different times. The transformation to the diagonal gauge connections \( \tilde{A}, \tilde{B} \) makes the path ordering unnecessary. Then, the \( 1/\lambda \) expansion of the new variables yields the infinite set of conservation laws we have already discussed. The generating function for the conservation laws can be obtained from the transition matrix of the theory, using the inverse scattering method (see [32] and references therein). This method has been applied to the model (1.4) by Lund [13] and Kulish [33], in order to construct the corresponding action–angle variables. In this regard, our results are complementary to theirs, but more explicit. The main point of this section was the realization that \( W_\infty \) and the off–critical
generalization of its generators in the parafermionic representation (4.37) determine the structure of the higher spin conservation laws completely.

The systematic construction of the conserved currents was based on the Langrangian description of the model and the zero curvature formulation of its classical equations of motion, rather than the null–vector conditions on the parafermion first thermal operator. The quantum mechanical generalization of these results should be straightforward, but computationally difficult. The quantum inverse scattering method could be used to understand the relation with the null–vector conditions in a systematic way. We think that the cohomological framework of Feigin and Frenkel [34] might be also appropriate for investigating further the quantum mechanical formulation of the abelianization method.

5 KP–like Structure of the Currents

The conservation laws of the perturbed parafermion theory can be described systematically in terms of the KP hierarchy. We claim that the currents $h_j^{(1)} - h_j^{(2)}$, with $j \geq 1$, can be interpreted as Hamiltonian densities of the KP hierarchy, thus providing a more direct way for their explicit construction. This generalizes the well known relation between the conserved densities of the sine–Gordon model and the $SL(2)$ KdV hierarchy, to the full coset theory.

Recall that the KP hierarchy is formulated in terms of the pseudo–differential operator

$$L = \partial + q_1 \partial^{-1} + q_2 \partial^{-2} + q_3 \partial^{-3} + \cdots .$$

(5.1) The KP flows are defined to be

$$\partial_t L = [(L^r)_+, L],$$

(5.2) with $r = 1, 2, 3, \ldots$ (see for instance [35]). These equations admit a Hamiltonian description with

$$\mathcal{H}_r = \frac{1}{r} \text{res} L^r$$

(5.3) being the Hamiltonian density of the $r$–th flow. We have explicitly

$$\mathcal{H}_1 = q_1,$$

(5.4) $$\mathcal{H}_2 = q_2 + \frac{1}{2} \partial q_1,$$

(5.5) $$\mathcal{H}_3 = q_3 + q_1^2 + \partial(q_2 + \frac{1}{4} \partial q_1),$$

(5.6) $$\mathcal{H}_4 = q_4 + 3 q_1 q_2 + \frac{3}{4} \partial(2q_3 + q_1^2 + \partial q_2 + \frac{1}{6} \partial^2 q_1)$$

(5.7) and so on. Moreover, when the flows (5.2) are written in terms of the composite fields $\mathcal{H}$, they become

$$\partial_t \mathcal{H}_r = \partial \Theta_{r, j},$$

(5.8)
where $\Theta_{r,j}$ are composite fields of $\{q_i\}$ depending on $r$, $j$. It is for this reason that the charges $H_j = \int \mathcal{H}_j$ with $j \geq 1$ are all in involution and the KP system is integrable.

The formulation (5.8) of the KP flows should be compared with the local conservation laws of the perturbed parafermion theory. In the parafermion case, $h_j^{(1)} - h_j^{(2)}$ plays the role of $\mathcal{H}_j$ and $\bar{\partial}$ replaces $\partial_r$. Therefore, it is natural to identify $h_j^{(1)} - h_j^{(2)}$ with the Hamiltonian densities of the KP hierarchy. Of course, the equivalence between the two theories is not exact, because in the KP case the $\Theta$’s can be rewritten as local functionals of the $\mathcal{H}$’s, while in the parafermion model this is not so. To describe the KP–like structure of the parafermion currents, we identify the generators $W_s$ as

$$W_s = q_{s-1}, \quad s = 2, 3, 4, \ldots$$

and postulate

$$h_j^{(1)} - h_j^{(2)} = \frac{1}{2j-1} \mathcal{H}_j. \quad (5.10)$$

This correspondence is understood modulo total derivative terms, which are irrelevant anyway in both the KP flows and the parafermion conservation laws, since they lead to trivial field redefinitions.

To illustrate the validity of eq.(5.10), we have to use the right normalization for the generators $W_s$. Following earlier work on the $W_\infty$ symmetry [29–31], we scale $W_s$, as given by eq.(4.37), by multiplication with

$$B(s) = q^{s-2} \frac{2^{s-3}s!}{(2s-3)!!} \quad (5.11)$$

and choose $q = -1/4$. Then, in terms of the rescaled variables, eqs.(4.41)–(4.44) become

$$h_1^{(1)} - h_1^{(2)} = W_2, \quad (5.12)$$

$$h_2^{(1)} - h_2^{(2)} = \frac{1}{2} W_3, \quad (5.13)$$

$$h_3^{(1)} - h_3^{(2)} = \frac{1}{4} (W_4 + W_2^2 + \frac{3}{10} \partial^2 W_3), \quad (5.14)$$

$$h_4^{(1)} - h_4^{(2)} = \frac{1}{8} (W_5 + 3 W_2 W_3 + \frac{6}{7} \partial^2 W_3) \quad (5.15)$$

and so on. Comparison with the expressions (5.4)–(5.7) shows that up to total derivative terms (which are gauge dependent in the abelianization method, anyway), eq.(5.10) is correct provided that $W_s = q_{s-1}$ after the rescaling. We have verified eq.(5.10) for higher values of $j$ as well.

The relation we find between the currents of the perturbed parafermion theory and the Hamiltonian densities of the KP hierarchy, generalizes the results we already know for the conservation laws of the sine–Gordon model. The sine–Gordon model arises as a special case of the perturbed parafermion theory, obtained for $u = \bar{u}$. As we have already mentioned, this model describes only the neutral sector of our general theory, since both components $J$, $\bar{J}$ of the $U(1)$ current are identically zero for $u = \bar{u}$. All higher spin
currents \( W_s \) with odd values of \( s \) also vanish in this sector, as it can be readily seen from eq.(4.37). The only non–trivial conservation laws have even spin. Setting \( u = \bar{u} = \cos \theta \), we obtain \( \psi_+ = \psi_- = -\partial \theta \). The conservation law (4.34) for \( j = 2 \) reduces to

\[
\bar{\partial} \left( (\partial \theta)^2 \right) + g \partial (\cos^2 \theta) = 0.
\] (5.16)

For \( j = 4 \), eq.(4.46) becomes

\[
\bar{\partial} \left( 5(\partial \theta)^4 - 3(\partial^2 \theta)^2 + 2 \partial \theta \partial^3 \theta \right) + g \partial \left( 3 \cos 2 \theta (\partial \theta)^2 - \sin 2 \theta \partial^2 \theta \right) = 0.
\] (5.17)

Similar expressions result for (even) higher values of \( j \), which are all consistent with the sine–Gordon equation (1.7).

It is well known that the currents \( h_j^{(1)} - h_j^{(2)} \) of the sine–Gordon model, coincide with the Hamiltonian densities of the KdV hierarchy (see for instance [8]). In this case, the pseudo–differential operator \( L \) in eq.(5.1) satisfies the condition \( L^2 = \partial^2 + v \) and \( q_i \) are all functionally dependent on the KdV field \( v \). We have explicitly \( q_1 = v/2, q_2 = -\partial v/4, q_3 = (\partial^2 v - v^2)/8 \), etc. The relation between \( v \) and \( \theta \) is given by the Miura map. Flows with odd values of \( j \) are trivial, since \( H_j \) become total \( \partial \)–derivatives. For the even values of \( j \), the KdV reduction of eq.(5.10) describes in fact the relation between the conservation laws of the two integrable theories.

The validity of eq.(5.10), not only in the neutral sector \( u = \bar{u} \), but in the full perturbed parafermion theory with arbitrary \( U(1) \) charge, suggests that there should be a 2–component reduction of the KP hierarchy, which actually describes the structure of the currents \( h_j^{(1)} - h_j^{(2)} \) completely. The 2–component reduction in question is the non–linear Schrodinger hierarchy, as it has been pointed out recently [36], using a somewhat different method. The Lagrangian description of the parafermion coset perturbed by the first thermal operator is the essential ingredient that links our work with theirs. The non–linear Schrodinger hierarchy is practically the same as the 2–boson realization of the KP hierarchy [37]. The latter was defined following earlier work on the bosonic realization of \( W_\infty \)–type algebras [38].

We also point out for completeness that a close relation between the non–linear Schrodinger equation and vortex dynamics was discovered several years ago [39], along different lines; vortex filaments were interpreted as solitons. This result should not be surprising, given the fact that the perturbed parafermion coset (with \( g > 0 \)) in 2–dim Minkowski space describes the relativistic motion of vortices in constant external field, via the Lund–Regge formalism. We think, however, that the role of the non–linear Schrodinger equation in vortex dynamics should be investigated further and in connection with perturbed conformal field theories.

We note finally, as a side remark, that it would be interesting to examine the Hamiltonian structure of the local conservation laws of the perturbed coset model. In analogy with the KP hierarchy, there might exist a Poisson bracket so that \( \bar{\partial} \tilde{A}_j + \partial \tilde{B}_j = 0 \) could be written as

\[
\bar{\partial} \tilde{A}_j = \{ \Omega_j , \tilde{A}_j \},
\] (5.18)
when \( j \geq 1 \) and \( g \neq 0 \), for appropriately chosen Hamiltonian \( \Omega_j \). \( \Omega_j \) might have a natural interpretation in the \( SU(2)/U(1) \) coset model, which could clarify the structure and physical interpretation of the higher spin conservation laws even further.

6 Discussion and Conclusions

In this paper we have studied systematically the Lagrangian description, physical interpretation and geometry of the parafermion coset model perturbed by its first thermal operator. The present work is entirely confined at the classical level and in the large \( N \) limit of the theory. Quantum mechanical aspects and \( 1/N \) corrections are interesting to consider and we hope to return to them in a separate publication. The main tool in our study has been provided by the geometric framework of Lund and Regge, upon analytic continuation of the coset theory in 2–dim Minkowski (base) space. Reviving these geometric ideas in present day conformal field theories seems to be advantageous for understanding the Lagrangian description of their integrable perturbations. Generalizations of these results to other coset models are certainly of great interest. It seems that the main problem in the general case is to understand the zero curvature formulation of the gauged WZW model (with or without perturbations) and its geometric interpretation as Gauss–Codazzi integrability conditions for an embedding problem.

Many of the issues considered here have been studied before, but with different motivations in mind. Their relevance in 2–dim field theory and string theory might be quite general, not limited to the present models. We expect that the correspondence between different string backgrounds and integrable 2–dim field theories could be developed further. Then, a systematic approach to the problem of hidden symmetries in string theory might result, as advocated in ref.[15]. We conclude with some applications of the results we have obtained so far.

The infinitely many conservation laws of the perturbed parafermion model have a natural geometric interpretation in terms of embeddings. To treat all cases together, irrespectively of the sign of the coupling constant \( g \), we summarize the expressions (3.12) and (3.27) for the extrinsic curvature as follows

\[
K_+ \equiv 2 (K_{\sigma\sigma} + K_{\sigma\tau}) = i \frac{u \partial \bar{u} - \bar{u} \partial u}{|u| \sqrt{1 - |u|^2}} - 2 \kappa c |u| \sqrt{1 - |u|^2},
\]

\[
K_- \equiv 2 (K_{\sigma\sigma} - K_{\sigma\tau}) = i \frac{u \partial \bar{u} - \bar{u} \partial u}{|u| \sqrt{1 - |u|^2}} - 2 \kappa c |u| \sqrt{1 - |u|^2}.
\]

The parameter \( \kappa \) is a step function: for \( g < 0 \) it is \( \kappa = 0 \), while for \( g \geq 0 \) we have \( \kappa = 1 \) and \( \kappa^2 = g \). The metric (3.1) of the 2–dim Euclidean surface \( S \) that arises in the geometric description of the theory via embeddings, has components

\[
g_{\sigma\sigma} = |u|^2, \quad g_{\tau\tau} = 1 - |u|^2.
\]
It is also convenient to introduce the notation \( g_- = g_{\sigma\sigma} - g_{\tau\tau} \).

It is easy to verify that the conservation law of the \( U(1) \) current (4.29) reads as
\[
\bar{\partial} \left( \sqrt{\frac{g_{\sigma\sigma}}{g_{\tau\tau}}} K_+ + \kappa c g_- \right) + \partial \left( \sqrt{\frac{g_{\sigma\sigma}}{g_{\tau\tau}}} K_- + \kappa c g_- \right) = 0. \tag{6.4}
\]

The conservation law of the energy–momentum tensor becomes
\[
\bar{\partial} \left( \left( K_+ + 2\kappa c \sqrt{\text{det}g} \right)^2 + \frac{1}{4} \left( \frac{\partial g_-}{\text{det}g} \right)^2 \right) + 2g \partial g_- = 0, \tag{6.5}
\]
while similar (but considerably more complicated) expressions can be obtained for the higher spin currents by rewriting the generators \( W_s \) in terms of the extrinsic curvature and the metric. The calculation is straightforward and yields step by step the infinitely many integrals of the underlying embedding problem. Their dependence on the original field variables is obtained using the defining relations for the metric and the extrinsic curvature tensors in terms of \( \eta_i(\sigma,\tau) \) or \( X_i(\sigma,\tau) \), depending on the sign of the coupling constant \( g \). This geometric interpretation is advantageous for understanding the physical meaning of the infinitely many charges associated with the classical evolution of the reduced \( O(4) \) non–linear \( \sigma \)–model or the string propagation in a 4–dim axionic background of vortex type respectively.

For \( g = 0 \), the currents \( W_s \) become chiral and generate the \( W_\infty \) algebra. This acts as a hidden symmetry on the transverse modes of the 4–dim free bosonic string in the orthonormal gauge. More details on \( W_\infty \) Backlund transformations in string theory can be found in ref.[15]. The lesson for string theory (with or without axionic background) is that the classical description of its transverse modes as a perturbed \( SU(2)/U(1) \) coset model (with \( g > 0 \) or \( g = 0 \) respectively), implies an infinite number of conservation laws at the classical level. It would be interesting to study the quantum mechanical implications of this result and try to generalize the correspondence between integrable systems living in the string world–sheet and string dynamics, to more arbitrary backgrounds in four and higher dimensions. Also, the relevance of this formalism to the quantum theory of vortices in (relativistic) superfluids should be investigated further, using the perturbed gauged WZW model. Generalization to other integrable perturbations and higher rank coset models should be interesting as well.

Finally, it is straightforward to apply our results to the reduced \( U(2) \) Gross–Neveu model. As was explained in the introduction, in the special frame (1.8), (1.9), the local conservation laws of this model can be obtained from the perturbed parafermion coset by the simple substitution (1.10). Generalization to the \( U(N) \) case is an interesting problem which might be related to integrable perturbations of Grassmannian coset models, \( SU(2)/U(1) \) being the simplest one. The \( 1/N \) expansion of the Gross–Neveu model is particularly interesting in this framework, since \( \bar{\psi}^\alpha \psi^\alpha \) develops a non–vanishing vacuum expectation value which breaks chiral invariance. This result might have a useful interpretation in quantum coset model calculations.
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APPENDIX

In this appendix we summarize the results of the iteration of the recursive relations (4.25)–(4.27) for the first few $h_j^{(1)} - h_j^{(2)}$, in terms of the matrix elements of $A_0$.

\[ h_2^{(1)} - h_2^{(2)} = A_0^{11} A_0^{12} A_0^{21} + \frac{1}{4} (A_0^{12} \partial A_0^{21} - A_0^{21} \partial A_0^{12}), \quad (A.1) \]

\[ h_3^{(1)} - h_3^{(2)} = \frac{1}{4} A_0^{12} A_0^{21} \left( A_0^{12} A_0^{21} - 4(A_0^{11})^2 \right) - \frac{1}{8} (A_0^{12} \partial^2 A_0^{21} + A_0^{21} \partial^2 A_0^{12}) \]
\[ - \frac{1}{2} A_0^{11} (A_0^{12} \partial A_0^{21} - A_0^{21} \partial A_0^{12}), \quad (A.2) \]

\[ h_4^{(1)} - h_4^{(2)} = A_0^{11} A_0^{12} A_0^{21} \left( (A_0^{11})^2 - \frac{3}{4} A_0^{12} A_0^{21} \right) + \frac{3}{16} A_0^{12} A_0^{21} (A_0^{12} \partial A_0^{12} - A_0^{12} \partial A_0^{21}) \]
\[ + \frac{3}{4} A_0^{11} \left( A_0^{12} \partial (A_0^{11} A_0^{21}) - A_0^{21} \partial (A_0^{11} A_0^{12}) \right) + \frac{1}{16} (A_0^{12} \partial^3 A_0^{21} - A_0^{21} \partial^3 A_0^{12}) \]
\[ + \frac{1}{8} \left( 2A_0^{12} A_0^{21} \partial^2 A_0^{11} + 3(\partial A_0^{11})\partial (A_0^{12} A_0^{21}) + 3A_0^{11} (A_0^{12} \partial^2 A_0^{21} + A_0^{21} \partial^2 A_0^{12}) \right). \quad (A.3) \]

Similarly for $t_i$ we have

\[ t_2^{12} = \frac{1}{2} A_0^{11} A_0^{12} - \frac{1}{4} \partial A_0^{12}, \quad (A.4) \]

\[ t_2^{21} = -\frac{1}{2} A_0^{11} A_0^{21} - \frac{1}{4} \partial A_0^{21}, \quad (A.5) \]

\[ t_3^{12} = -\frac{1}{2} A_0^{12} \left( (A_0^{11})^2 - \frac{1}{4} A_0^{12} A_0^{21} \right) + \frac{1}{4} A_0^{12} \partial A_0^{11} + \frac{1}{2} A_0^{11} \partial A_0^{12} - \frac{1}{8} \partial^2 A_0^{12}, \quad (A.6) \]

\[ t_3^{21} = \frac{1}{2} A_0^{21} \left( (A_0^{11})^2 - \frac{1}{4} A_0^{12} A_0^{21} \right) + \frac{1}{4} A_0^{21} \partial A_0^{11} + \frac{1}{2} A_0^{11} \partial A_0^{21} + \frac{1}{8} \partial^2 A_0^{21}, \quad (A.7) \]

\[ t_4^{12} = \frac{1}{2} A_0^{11} A_0^{12} \left( (A_0^{11})^2 - \frac{3}{4} A_0^{12} A_0^{21} \right) - \frac{3}{4} A_0^{11} \partial (A_0^{11} A_0^{12}) \]
\[ + \frac{1}{16} A_0^{12} (A_0^{12} \partial A_0^{21} + 4A_0^{21} \partial A_0^{12}) - \frac{1}{16} \partial^3 A_0^{12} \]
\[ + \frac{1}{8} \left( 3A_0^{11} \partial^2 A_0^{12} + 3(\partial A_0^{11})\partial A_0^{12} + A_0^{12} \partial^2 A_0^{11} \right). \quad (A.8) \]

\[ t_4^{21} = -\frac{1}{2} A_0^{11} A_0^{21} \left( (A_0^{11})^2 - \frac{3}{4} A_0^{12} A_0^{21} \right) - \frac{3}{4} A_0^{11} \partial (A_0^{11} A_0^{21}) \]
\[ + \frac{1}{16} A_0^{21} (A_0^{21} \partial A_0^{12} + 4A_0^{12} \partial A_0^{21}) - \frac{1}{16} \partial^3 A_0^{21} \]
\[ - \frac{1}{8} \left( A_0^{21} \partial^2 A_0^{11} + 3(\partial A_0^{11})\partial A_0^{21} + 3A_0^{11} \partial^2 A_0^{21} \right). \quad (A.9) \]
For $\tilde{B}_j$ we have

$$\tilde{B}_2 = [B_1, t_1] + [B_0, t_2] + t_1 [t_1, B_0] - \partial t_2 + t_1 \partial t_1,$$  \hspace{1cm} (A.10)

$$\tilde{B}_3 = [B_0, t_3] + [B_1, t_2] + t_1 ([t_2, B_0] + [t_1, B_1])$$
$$- \partial t_3 + t_1 \partial t_2 + (t_2 - (t_1)^2) (\partial t_1 + [t_1, B_0]),$$  \hspace{1cm} (A.11)

$$\tilde{B}_4 = [B_0, t_4] + [B_1, t_3] + t_1 [t_3, B_0] + t_1 [t_2, B_1] + t_1 \partial t_3$$
$$+ (t_2 - (t_1)^2) ([t_2, B_0] + [t_1, B_1] + \partial t_2) - \partial t_4$$
$$+ (t_3 - t_1 t_2 - t_2 t_1 + (t_1)^3) ([t_1, B_0] + \partial t_1).$$  \hspace{1cm} (A.12)
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