Fermion production by a dependent of time electric field in de Sitter universe

Cosmin Crucean

West University of Timișoara,
V. Parvan Ave. 4 RO-300223 Timișoara, Romania

Abstract

Fermion production by the electric field of a charge on de Sitter expanding universe is analyzed. The amplitude and probability of pair production are computed. We obtain from our calculations that the modulus of the momentum is no longer conserved and that there are probabilities for production processes where the helicity is no longer conserved. The rate of pair production in an electric field is found to be important in the early universe when the expansion factor was large comparatively with the particle mass.

PACS numbers: 04.62.+v
I. INTRODUCTION

Pair production by electric field is known as the Schwinger effect [? , and even if this phenomenon was not yet observed, it might be possible to be experimentally proven in the next years. The method employed in obtaining this result is based on the non-perturbative structure of the QED vacuum physics, when strong fields are considered. However, pair production by a weak electric field is in principle possible if this field is coupled with strong gravity. This kind of conditions were achieved only in the early universe. In the present paper we want to prove that pair production by the electric field of a charge could have arisen in the early universe due to the large values of the expansion factor.

The problem of fermion production by a constant electric field on an expanding universe was the subject of a few investigations [19, 20, 27, 31]. The main results of the mentioned papers show that there are indeed nonvanishing rates for pair production in an electric field. However, pair production in the early universe was not addressed in [19, 20, 27] due to some constrains which are imposed to the particle mass, which must be larger than the expansion factor of the space.

It is important to specify that in our calculations we will use the electric field produced by a point charge (or a distribution of charges) in the de Sitter universe, which is a time dependent quantity. To the best of our knowledge, this kind of electric field was not considered in the investigations related to the pair production in an expanding de Sitter universe. The work of L. Parker [8–10] shows that we must expect a considerable amount of pair production from vacuum in the early universe. Then we must expect the phenomenon of pair production to be amplified in the presence of an external electric field coupled with gravity.

In the present paper, we want to prove that the electric field produced by a charge, or a distribution of charges can produce large amounts of fermions in the early expansion conditions. Our method is based on perturbations and was used in a number of investigations for calculating QED processes in de Sitter universe [16, 18]. To the best of our knowledge, the method based on perturbations was not used to calculate the probability of pair production in electric fields on de Sitter space. We will compute the amplitude corresponding to the process vacuum → e− + e+ in the presence of an electric field. The time reversed process can be also calculated, and represents the process in which the pair is absorbed by the electric
field. As we know, a point charge can produce an electric field with intensity decreasing with the square of distance to the source (charge). In flat space, this kind of electric field doesn’t produce pairs of fermions. However, in the early universe, this phenomenon can not be excluded.

The second section begins with a short review of the free fields theory. In the third section we compute the amplitude and probability of pair production in an electric field and we analyze the physical consequences of our calculations. The fourth section is dedicated to the calculation of the total number of produced fermions and in section five we present our conclusions.

II. PRELIMINARIES

The theory of the free Dirac field on de Sitter space is well known and in the Appendix is given only the form of the fundamental solutions, which have a definite momentum and helicity. These solutions will help us calculate the form of our amplitude.

We will focus in this section in establishing the form of the electric field in the de Sitter metric:

\[
ds^2 = dt^2 - e^{2\omega t} d\vec{x}^2 = \frac{1}{(\omega t_c)^2} (dt_c^2 - d\vec{x}^2),
\]

where \(\omega\) is the expansion factor (\(\omega > 0\)) and the conformal time is defined as \(t_c = -\frac{e^{-\omega t}}{\omega}\). The form of the line element allows us to choose the simple Cartesian gauge with the nonvanishing tetrad components

\[
e^0_0 = e^{-\omega t}; e^i_j = \delta^i_j e^{-\omega t}; e^0_i = e^{\omega t}; e^i_j = \delta^i_j e^{\omega t}
\]

so that \(e_\mu = e^\nu_\mu e_\nu\) and have the orthonormalization properties \(e_\mu e^\nu_\mu = \eta_{\mu\nu}\), \(e^\mu_\mu = \delta^\mu_\mu\), with respect to the Minkowski metric \(\eta = diag(1, -1, -1, -1)\).

The theory of the free electromagnetic field on de Sitter space is simple, if we recall that the de Sitter metric is conformal with the Minkowski metric. At a conformal transformation the metric tensor \(g_{\alpha\beta}\) can always be expressed in the form \(g_{\alpha\beta} = \Omega \eta_{\alpha\beta}, g'^{\alpha\beta} = \Omega^{-1} \eta^{\alpha\beta}\), where \(\eta_{\alpha\beta}\) is the Minkowski metric and \(\Omega = \frac{1}{(\omega t_c)^2}\). Using this transformation, it was shown that the four vector potential in de Sitter space can be expressed in terms of the corresponding four vector potential on Minkowski space as follows:

\[
A_\mu S = A_\mu M, A_\nu^S = \Omega^{-1} A_\nu^M.
\]
The electric field of a charge will be under the expansion conditions a time-dependent quantity. This can be seen from the expression of the electromagnetic field tensor in the de Sitter metric. The electromagnetic field tensor in de Sitter space can be expressed in terms of the electromagnetic field tensor from Minkowski space, in order to preserve the conformal invariance:

\[ F_{\mu\nu}^S = F_{\mu\nu}^M, \quad F_{\mu\nu}^S = \Omega^{-2} F_{\mu\nu}^M, \quad (4) \]

Then in a local Minkowski frame, equation (4) becomes, \( F_{\mu\nu}^\hat{S} = e^\hat{\mu}_\alpha e^\hat{\nu}_\beta F_{\alpha\beta}^S = e^\hat{\mu}_\alpha e^\hat{\nu}_\beta \Omega^{-2} F_{\alpha\beta}^M. \)

Now, if we consider the electric field produced by a charge \( Q \), or a distribution of charges \( \sum_i Q_i \), we will find in de Sitter space, that at distance \( |\vec{x}| \) from the charge, the intensity of the electric field is:

\[ \vec{E} = -\frac{\partial \vec{A}}{\partial t} = \frac{Q}{|\vec{x}|^2} e^{-2\omega t} \hat{n}, \quad (5) \]

where \( \hat{n} \) is the unit vector which gives the direction and orientation of the intensity of electric field. It is now simple to establish the form of the vector potential that will be associated to this electric field:

\[ \vec{A} = \frac{Q}{2\omega |\vec{x}|^2} e^{-2\omega t} \hat{n}. \quad (6) \]

This expression will be further used to calculate the amplitude of pair production by an electric field in de Sitter universe.

### III. TRANSITION AMPLITUDE AND PROBABILITY

In this section we will calculate the transition amplitude and probability, for pair production in an electric field and we will explore the physical consequences of our result. The form of the transition amplitude can be established using the same methods as in the flat space case, as was shown in [12, 16–18, 21, 22]. For pair production in an external electromagnetic field, supposing that the fields are coupled by the elementary electric charge \( e \), the expression of the transition amplitude is:

\[ A_{e^-e^+} = -ie \int d^4 x \left[ -g(x) \right]^{1/2} \hat{U}_{\vec{p}, \lambda}(x) \gamma^\gamma \cdot \vec{A}(x) V_{\vec{p}', \lambda'}(x) \quad (7) \]

where \( e \) is the unit charge of the field and the fields \( U_{\vec{p}, \lambda}(x), V_{\vec{p}', \lambda'}(x) \), are supposed to be exact solutions of the free Dirac equation in the momentum basis [4]. Using the fundamental solutions of the Dirac equation given in equation (25) from Appendix and the potential (6)
and setting \( Q = e \), we finally arrive at the amplitude:

\[
A_{e^- e^+} = -i \frac{e^2 \sqrt{pp'}}{32|\vec{p} + \vec{p}'|} \left[ -\text{sgn}(\lambda \lambda') e^{-\pi k} \int_0^\infty dzz^2 H_{\nu'}^{(2)}(pz) H_{\nu}^{(2)}(p'z) 
+ e^{\pi k} \int_0^\infty dzz^2 H_{\nu'}^{(2)}(pz) H_{\nu}^{(2)}(p'z) \right] \xi^*(\vec{p}')(\vec{\sigma} \cdot \vec{n})\eta\xi'(\vec{p}')
\]

\[(8)\]

The result of the spatial integral \[7\] was included in the above amplitude. In the temporal integral a new variable \( z = -tc \) was introduced and \( \text{sgn} \) is the signum function. To solve these integrals we use the relation between Hankel functions and Macdonald functions \[11, 12, 23\], arriving in this way at the integrals discussed in Appendix. The result of our calculations can be expressed in terms of unit step functions \( \theta \), Euler gamma functions \( \Gamma \) and Gauss hypergeometric functions \( _2F_1 \):

\[
A_{e^- e^+} = \frac{e^2}{64\pi|\vec{p} + \vec{p}'|} \left[ p^{-2} \theta(p - p') f_k \left( \frac{p'}{p} \right) + p'^{-2} \theta(p' - p) f_k^* \left( \frac{p}{p'} \right) 
- \text{sgn}(\lambda \lambda') \left( p^{-2} \theta(p - p') f_k \left( \frac{p'}{p} \right) + p'^{-2} \theta(p' - p) f_k^* \left( \frac{p}{p'} \right) \right) \xi^*(\vec{p}')(\vec{\sigma} \cdot \vec{n})\eta\xi'(\vec{p}'), \right.
\]

\[(9)\]

where the functions \( f_k \left( \frac{p}{p'} \right) \), which define the amplitude are given by:

\[
f_k \left( \frac{p}{p'} \right) = \left( \frac{p}{p'} \right)^{1-ik} \Gamma(2 - ik) \Gamma(1 + ik) _2F_1 \left( \frac{3}{2}, 2 - ik; 3, 1 - \left( \frac{p}{p'} \right)^2 \right)
= \frac{4}{\sqrt{\pi}} \left( \frac{p}{p'} \right)^{1-ik} \Gamma \left( ik - \frac{1}{2} \right) _2F_1 \left( \frac{3}{2}, 2 - ik; 3 - ik; \left( \frac{p}{p'} \right)^2 \right)
+ \frac{4}{\sqrt{\pi}} \left( \frac{p}{p'} \right)^{ik} \Gamma \left( \frac{1}{2} - ik \right) _2F_1 \left( \frac{3}{2}, 1 + ik; \frac{1}{2} + ik; \left( \frac{p}{p'} \right)^2 \right), \quad (10)
\]

where the second equality is obtained when we use \[28\] and the function \( f_k \left( \frac{p}{p'} \right) \) is obtained when one makes the substitution \( p \rightleftharpoons p' \). The parameter \( k = m/\omega \) is the particle mass/expansion factor ratio.

We will explore in the rest of the paper the physical consequences of our calculations. It is obvious from equation \[9\], that the only possible transitions are those in which the momenta of the electron and positron are not equal. From here we conclude that the law of conservation for the modulus of momentum is lost in this process. The background which enables the breaking of momentum conservation law is the external electric field, not the
FIG. 1: The real part of $f_k(\chi)$ as a function of $k$. The solid line is for $\chi = 0.1$ and the dashed line for $\chi = 0.9$.

dSitter geometry, because the geometry (1) has spatial translation invariance as an exact symmetry, so the momentum is conserved [26].

Further let us study the properties of our amplitude by drawing the graphs of the real and imaginary parts of $f_k \left( \frac{p}{p'} \right)$ (the analysis is similar for function $f_k \left( \frac{p'}{p} \right)$), as function of parameter $k = m/\omega$, for different values of the ratio $\frac{p}{p'} = \chi (= \frac{p'}{p})$. The parameter $k$ encodes the influence of space expansion upon the pair production process.
FIG. 2: The imaginary part of $f_k(\chi)$ as a function of $k$. The solid line is for $\chi = 0.1$ and the dashed line for $\chi = 0.9$.

FIG. 3: The real part of $f_k(\chi)$ as a function of $k$. The solid line is for $\chi = 0.001$ and the dashed line for $\chi = 0.00001$.

From our graphs we observe that both the real part and the imaginary part of the function $f_k\left(\frac{p}{p'}\right)$, are finite in origin and very convergent for large values of $k$ (see Figs.(1)-(2)). As the ratio of the momenta $\frac{p}{p'}$ takes small values, we observe that these functions become oscillatory (see Figs.(3)-(4)). This oscillatory behavior is the result of the behavior of Gauss hypergeometric functions as their algebraic argument $1 - \left(\frac{p}{p'}\right)^2$ approaches to one (or $\frac{p}{p'}$ approaches zero), combined with the oscillatory factors $\left(\frac{p}{p'}\right)^{1-ik}$. Squaring the amplitude and summing after final helicities $\lambda, \lambda'$, we obtain the probability of pair production.
FIG. 4: The imaginary part of $f_k(\chi)$ as a function of $k$. The solid line is for $\chi = 0.001$ and the dashed line for $\chi = 0.00001$.

We find that the probability of fermion pair production in an electric field is:

$$P_{e^-e^+} = \frac{1}{2} \sum_{\lambda' \lambda} |\mathcal{A}_{e^-e^+}|^2$$

$$= \frac{1}{2} \sum_{\lambda' \lambda} e^4 \frac{e^4}{(64)^2 \pi^2 |\bar{\vec{p}} + \vec{p}^\prime|^2} |\xi_{\lambda}^+(\bar{\vec{p}})(\vec{\sigma} \cdot \vec{n})\eta_{\lambda'}(\vec{p}^\prime)|^2 \{p^{'-4}\theta(p - p') \left[ 2 \left| f_k \left( \frac{p}{p'} \right) \right|^2 - \text{sgn}(\lambda \lambda') \left( f_k^2 \left( \frac{p}{p'} \right) + f_k^2 \left( \frac{p}{p'} \right) \right) + p^{'-4}\theta(p - p') \right] + p^4\theta(p - p') \left[ 2 \left| f_k \left( \frac{p}{p'} \right) \right|^2 - \text{sgn}(\lambda \lambda') \left( f_k^2 \left( \frac{p}{p'} \right) + f_k^2 \left( \frac{p}{p'} \right) \right) \right] \right\}. \quad (11)$$

The total probability is obtained integrating equation (11) after the final momenta $p, p'$:

$$P_{e^-e^+}^{\text{tot}} = \int P_{e^-e^+} \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3}.$$ From the general formula for probability (11), we observe that in the process of pair production by an electric field the helicity is conserved when $\lambda = -\lambda'$ and the law of helicity conservation is broken when $\lambda = \lambda'$. We must specify that the non-conservation of helicity is driven by the fermion mass and not by the de Sitter geometry \[26\]. Plotting our probability (11) as function of parameter $k$, we obtain the results from Figs. (5)-(8).
FIG. 5: $P_{e^-e^+}$ as a function of $k$ for $\chi = 0.9$. The dashed line represents the case of helicity conservation and the solid line represents the case when helicity is not conserved.

FIG. 6: $P_{e^-e^+}$ as a function of $k$ for $\chi = 0.1$. The dashed line represents the case of helicity conservation and the solid line represents the case when helicity is not conserved.

From Figs. (5)-(8) we observe that the electric field of the type considered here can produce pair of fermions only in the early universe, when the expansion factor was sensibly larger than the particle mass ($\omega > m$). This result confirms the predictions from \cite{8–10}, were it was proven using a WKB approach, that the rate of pair production was important only in the early universe. As $\chi \to 0$, the probability becomes oscillatory in both conserving and non-conserving helicity cases. It is also important to observe that the probabilities for a helicity conserving/non-conserving processes becomes approximatively equal in the limit $\chi \to 0$ (see Fig. (8)). Also from Figs. (5)-(8), we observe that it is more likely to produce pairs of
fermions, which have the ratio of the momenta $\chi$ very small. To conclude, the presence of an electric field in this geometry will favour the processes in which for example, the electron momenta will be small comparatively to the positron momenta, with the specification that large momenta don’t refer here, to the relativistic case. As $\chi$ is close to unity, we observe that the probability of pair production become sensibly smaller (see Fig. (5)), than in the case of small $\chi$ (see Figs. (7),(8)). We also observe that our probability is proportional with a factor $|p + p'|^2$, from which it is clear that in the limit of large momenta our amplitude/probability will vanish. From here we conclude that only fermions with small momenta were produced.
in the presence of an electric field, due to the early expansion of the universe.

The results from Figs. \([5]-[5]\), also show that the probability of pair production in electric field of a charge id sensibly larger that the probability of pair production in Coulomb field studied in \([20]\). Also the present study show that there is a preferential direction of motion for the produced fermions which is the direction of the intensity of electric field, as we will see below.

Let us do a more detailed analysis in the helicity space. We use helicity bispinors \(\xi_\lambda(\vec{p})\), \(\eta_{\lambda'}(\vec{p}')\), for \(\lambda = \lambda'\) and \(\lambda = -\lambda'\), and restrict the analysis only to the square of our amplitude. In an orthogonal local frame \(\{\vec{e}_i\}\), we take the electron and positron momenta in the plane \((1, 2)\), denoting their spherical coordinates as \(\vec{p} = (p, \alpha, \beta)\), \(\vec{p}' = (p', \gamma, \delta)\) and we consider that the unit vector which gives the direction and orientation of the electric field, has the spherical coordinates \(\vec{n} = (1, \theta, \varphi)\), where \(\alpha, \gamma, \theta \in (0, \pi)\); \(\beta, \delta, \varphi \in (0, 2\pi)\).

Corresponding to the two cases of conserving/nonconserving helicity in the pair production process by an electric field, the bispinor product \(\xi_\lambda^+(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_{\lambda'}(\vec{p}')\) can be calculated. In the case of helicity conservation this gives:

\[
\begin{align*}
\xi_+^+(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_-^-(\vec{p}') &= \cos \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \cos(\theta) + e^{i(\varphi-\beta)} \sin \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \sin(\theta) \\
&\quad + e^{i(\varphi-\varphi)} \cos \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \sin(\theta) - e^{i(\delta-\delta)} \sin \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \cos(\theta), \\
\xi_-^+(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_+^-(\vec{p}') &= \cos \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \cos(\theta) + e^{-i(\varphi+\beta)} \sin \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \sin(\theta) \\
&\quad + e^{-i(\varphi-\varphi)} \cos \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \sin(\theta) - e^{-i(\delta+\delta)} \sin \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \cos(\theta).
\end{align*}
\]

(12)

For the helicity nonconserving case the result is:

\[
\begin{align*}
\xi_+^+(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_+^-(\vec{p}') &= e^{-i\delta} \cos \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \cos(\theta) + e^{i(\varphi-\beta-\delta)} \sin \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \sin(\theta) \\
&\quad + e^{-i\beta} \sin \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \cos(\theta) - e^{-i\varphi} \sin \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \sin(\theta), \\
\xi_-^+(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_-^-(\vec{p}') &= -e^{i\delta} \cos \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \cos(\theta) - e^{i(\varphi-\beta-\delta)} \sin \left(\frac{\alpha}{2}\right) \sin \left(\frac{\gamma}{2}\right) \sin(\theta) \\
&\quad - e^{-i\beta} \sin \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \cos(\theta) + e^{i\varphi} \sin \left(\frac{\alpha}{2}\right) \cos \left(\frac{\gamma}{2}\right) \sin(\theta).
\end{align*}
\]

(13)
Taking now $\beta = \pi, \delta = \varphi = 0$ and squaring the bispinor product we obtain:

\[
|\xi_\lambda^\dagger(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_{\lambda'}(\vec{p}')|^2 = \begin{cases} 
|\cos(\theta) \cos\left(\frac{\alpha - \gamma}{2}\right) - \sin(\theta) \sin\left(\frac{\alpha - \gamma}{2}\right)|^2, & \text{for } \lambda = -\lambda' \\
|\cos(\theta) \sin\left(\frac{\alpha - \gamma}{2}\right) + \sin(\theta) \cos\left(\frac{\alpha - \gamma}{2}\right)|^2, & \text{for } \lambda = \lambda'.
\end{cases}
\]

(14)

From equation (14) if we set $\alpha = \pi, \gamma = \theta = 0$, corresponding to the situation when the electron-positron pair is generated on the direction of electric field, but they move in the opposite senses, we observe that the probability in the helicity conserving case is zero, while the probability of a process where helicity is not conserved becomes maxim. From here we can conclude that only the production processes that don’t conserve the helicity could produce pairs that move in opposite senses, increasing in this way the chances of separation between matter and anti-matter. Now if we set $\alpha = \gamma = \theta = 0$, corresponding to the situation when the electron-positron pair is emitted on the direction of the electric field, but they move in the same sense, we observe that the probability in the helicity conserving case becomes maxim and is zero in the helicity non-conserving case. In this case, it is more likely that the pair will annihilate than become separate. Also from (12), (13), (14), we observe that the electron-positron pair could be emitted in other directions, which make various angles with the direction of the electric field, but with smaller probabilities.

To conclude, in both helicity conserving/non-conserving cases, the most probable transitions are those where the pairs are emitted on the direction of the electric field.

IV. NUMBER OF FERMIONS

Let us translate our perturbational result in terms of density number of produced particles. In the vast majority of the papers related to the particle production, the outcome of the calculations is the density number of created particles. So our reader could ask how we will compute the density number, taking into account that we use a perturbative approach.

First let us comment on the previous results related to the density number of created particles. The most important result in this direction was obtained by L.Parker using a WKB approach [8–10]. This result shows for the first time that the density number of created particles was important only in the early universe, which corresponds to the inflation phase. However there are results concerning the density number of created particles
obtained using the method of Bogoliubov transformations. As was shown by Fulling [30], there are many cases of interest where this method gives divergent results. This happens because the density number of created particles is not a function of the final momenta and as a result of that, the integration after the final momenta will give a quantity which is linearly divergent. This important observation is valid for the results presented in [19, 20], where an integration after final momenta gives an infinite number of particles. So we cannot even speak about a regularization of the integral simply because we do not have any dependence on final momenta [19, 20]. In addition, when we compute the number of created particles in an expanding universe, this will be in general a time dependent quantity. This is the result of the fact that the volume will expand in time. For example, in de Sitter case discussed here, the physical spatial coordinates are $x_{ph}^i = x_i e^{\omega t}$, and from here we observe that volume will expand as $V_{ph} = V e^{3\omega t}$. The above mentioned problems seem to receive little attention despite their crucial importance. In [29] the density number of created particles was computed in a Robertson-Walker metric, and this is one of the few papers which takes into account the effect of the field interactions upon particle production processes. Moreover in [29], the correct density number of created particles was computed taking into account that the volume is a function of the expansion factor.

Here we give the main steps for computing the number of created fermions in our case. Squaring the amplitude and summing after the final helicities $\lambda, \lambda'$ we obtain the probability of pair production (11). Then following the result from [29], the ratio probability/volume, is the density number of fermions:

$$N = \frac{P}{V_{ph}} = \frac{1}{2} \sum_{\lambda \lambda'} \left| A_{e^+ e^-} \right|^2 \frac{V}{V e^{3\omega t}}.$$  \hspace{1cm} (15)

This quantity can be computed now explicitly using the equation for probability (11). The total number of produced fermions can be obtained integrating the density number above, after the final momenta $p, p'$:

$$N^{tot} = \frac{P^{tot}}{V_{ph}} = \int \frac{P}{V e^{3\omega t}} \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3}. \hspace{1cm} (16)$$

Here a few comments are in order. First our density number of produced fermions defined in (15), is dependent on the final momenta $p, p'$. However the integrals that result are not known in mathematics, these being a combination of two hypergeometric Gauss functions and a fraction of the type $\frac{p^\nu p'^\mu}{|p+p'|^2}$, where $\nu, \mu$ are complex numbers. Despite of these difficulties we will further give a way to calculate the total number of fermions.
First from our graphs for probability Figs. (5)-(8), we observe that there are larger probabilities to produce pairs of fermions which have the ratio of the momenta very small $\chi \ll 1$. This is the case when for example the positron momenta is large comparatively with the electron momenta ($p' \gg p$). In this case the ratio $\left( \frac{p}{p'} \right)^\alpha \simeq 0$, when $\alpha$ is a real number and the hypergeometric function become $\hypergeom{2}{1}{a, b; c; 0} = 1$. Then the final expression of the function $f_k$ for $p' \gg p$, is expressed only in purely imaginary powers:

$$f_k \left( \frac{p}{p'} \right) \simeq \frac{4}{\sqrt{\pi}} \left( \frac{p}{p'} \right)^{ik} \Gamma \left( \frac{1}{2} - ik \right) \Gamma(1 + ik).$$

(17)

The total probability will be calculated for the helicity nonconserving case and we will consider the most probable transitions which produce pairs that move on the direction of electric field. In the previous section we see that only the helicity nonconserving processes could produce pairs that are separable. This is equivalently to take $\vec{p}\vec{p}' = |\vec{p}||\vec{p}'| \cos(\alpha + \gamma) = -|\vec{p}||\vec{p}'|$, corresponding to an angle between momenta vectors equal with $\pi$ (helicity nonconservation) and from here $|\vec{p} + \vec{p}'|^2 \simeq p'^2 - 2pp'$. For $\lambda = \lambda'$ the summation after helicity bispinors become:

$$\sum_\lambda |\xi^+_{\frac{1}{2}}(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_\lambda(\vec{p}')(\vec{\sigma} \cdot \vec{n})|^2 = |\xi^+_{\frac{1}{2}}(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_\frac{1}{2}(\vec{p}')(\vec{\sigma} \cdot \vec{n})|^2 + |\xi^+_{-\frac{1}{2}}(\vec{p})(\vec{\sigma} \cdot \vec{n})\eta_{-\frac{1}{2}}(\vec{p}')(\vec{\sigma} \cdot \vec{n})|^2$$

(18)

and if we consider that the most probable transitions in the helicity nonconserving case are obtained when $\alpha = \pi$, $\gamma = \theta = 0$, corresponding to the situation when electron and positron move in opposite senses, the sum after helicity bispinors (18), simplify to a numeric factor. For the momenta modulus integrals we consider that $p' \in (0, \infty)$ and we restrict the limits of integration for $p$ to $p \in (\mu, \Lambda)$. The cutoff for the upper limit of integration in the electron momenta $p$ is in accord with our suppositions that $p \ll p'$. We also take the lower limit of integration for $p$ to be $\mu$.

Using (17) we compute the quantity that defines our probability. For $p' \gg p$, we neglect the terms $\left( \frac{p}{p'} \right)^\alpha \simeq 0$, (with $\alpha$ a real number) and we take only terms with purely imaginary powers, the result being:

$$2 \left| f_k \left( \frac{p}{p'} \right) \right|^2 - f_k^2 \left( \frac{p}{p'} \right) - f_k^* \left( \frac{p}{p'} \right) \simeq \frac{16}{\pi} \left[ 2 \left| \Gamma \left( \frac{1}{2} - ik \right) \right|^2 |\Gamma(1 + ik)|^2 \right.
\left. - \Gamma^2 \left( \frac{1}{2} - ik \right) \Gamma(1 + ik) \left( \frac{p}{p'} \right)^{2ik} - \Gamma^2 \left( \frac{1}{2} + ik \right) \Gamma^2(1 - ik) \left( \frac{p}{p'} \right)^{-2ik} \right].$$

(19)
We will note $A = \Gamma^2 \left( \frac{1}{2} - i k \right) \Gamma^2 (1 + ik)$, $B = \Gamma^2 \left( \frac{1}{2} + i k \right) \Gamma^2 (1 - ik)$. Then the total number of produced fermions in volume unit for the case when electron-positron pair move in opposite senses (helicity nonconserving case), but on the direction of the electric field vector, for $p' >> p$ is:

$$N_{\text{tot}} = \frac{e^4}{256 \pi^3 e^{3 \omega t}} \int \frac{d\Omega_p}{(2\pi)^3} \int \frac{d\Omega_{p'}}{(2\pi)^3} \times \Re\left\{ \int_{\mu}^{\Lambda} \left[ \int_{0}^{\infty} \frac{dp'}{p'^2(p'^2 - 2pp')} \left( -A \left( \frac{p}{p'} \right)^{2ik} - B \left( \frac{p}{p'} \right)^{-2ik} \right) \right] p'^2 dp' \right\},$$

where $\Re\{\ldots\}$ is the real part of our result. This is because we integrate the ratio of the momenta at imaginary powers and the final result will be an imaginary number. The second observation here is that we neglected the first term in (19) due to the fact that will be proportional when replaced in our probability with $\left( \frac{p}{p'} \right)^2$, which is negligible. The integrals that help us to establish the expression for total number of fermions are given in Appendix.

Before proceed in our calculations for the number of fermions we need to make a few clarifications. We observe that the integrand will contain momenta at imaginary powers and when integrated the final result will be an imaginary number. This is the result of the fact that we equal with unity the hypergeometric functions which contain imaginary arguments. For these reasons we must interpret the total number of fermions as the real part of our result. Then the final result for total number of fermions in volume unit will be:

$$N_{\text{tot}} = \frac{\alpha^2}{512 \pi^4 \sinh(2\pi k)} (\ln \Lambda - \ln \mu) e^{-3\omega t} \times \Re\left\{ i \Gamma^2 \left( \frac{1}{2} - ik \right) \Gamma^2 (1 + ik) e^{-2\pi k} 2^{-2ik} - i \Gamma^2 \left( \frac{1}{2} + ik \right) \Gamma^2 (1 - ik) e^{2\pi k} 2^{2ik} \right\} = \frac{\alpha^2 e^{-3\omega t}}{512 \pi^4} g_k(\mu, \Lambda),$$

where we note

$$g_k(\mu, \Lambda) = \Re\left\{ i \Gamma^2 \left( \frac{1}{2} - ik \right) \Gamma^2 (1 + ik) e^{-2\pi k} 2^{-2ik} - i \Gamma^2 \left( \frac{1}{2} + ik \right) \Gamma^2 (1 - ik) e^{2\pi k} 2^{2ik} \right\} \times \frac{(\ln \Lambda - \ln \mu)}{\sinh(2\pi k)}.$$

We see that the total number of fermions is a time dependent quantity and decreases exponentially in time. This is natural since the expansion factor decreases also with time. From here we can conclude that the effect presented in this paper could be observable only
FIG. 9: $g_k(\mu, \Lambda)$ as a function of $k$. The dashed line represents the case when $\mu = 0.001$, $\Lambda = 1000$ and the solid line represents the case when $\mu = 0.000001$, $\Lambda = 1000000000$.

in strong gravitational fields. Another interesting observation is that the total probability and total number of fermions contains both ultraviolet and infrared divergences that are logarithmical.

Plotting the function $g_k(\mu, \Lambda)$ in terms of parameter $k$ for different values of $\mu$, $\Lambda$ we obtain the results presented in Figs. (9)-(10). From these graphs we observe that the number of fermions produced by the electric field considered here was important only in the early universe. When the upper limit in momentum integration is large comparatively with the lower limit $\Lambda \gg \mu$, one can observe that the number of produced fermions is sensibly bigger than in the case when $\Lambda \simeq \mu$. 
FIG. 10: $g_k(\mu, \Lambda)$ as a function of $k$. The dashed line represents the case when $\mu = 3$, $\Lambda = 4$ and the solid line represents the case when $\mu = 2$, $\Lambda = 10$.

V. CONCLUSIONS

We compute the amplitude/probability for pair production by the electric field of a charge in the de Sitter expanding universe. The probability of pair production is nonvanishing only in the early universe when the expansion factor was large. From our calculations we recover the Minkowski limit, where the amplitude and probability for this process vanish. We found that that the most probable transitions are those in which the pairs are emitted on the direction of the electric field.

We also translate our perturbational calculations in terms of density number of fermions. A method to compute the total number of fermions in the case when the ratio $\frac{p}{p'}$ is close to zero, was proposed. The total number of produced fermions was obtained and we prove that is nonvanishing only in the early universe.

VI. APPENDIX

Let us introduce unit normalized helicity spinors [6] for an arbitrary momentum vector $\vec{p}$: $\xi_\lambda(\vec{p})$ and $\eta_\lambda(\vec{p}) = i \sigma_2 [\xi_\lambda(\vec{p})]^*$

$$\tilde{\sigma} \vec{p} \xi_\lambda(\vec{p}) = 2p\lambda \xi_\lambda(\vec{p}) \quad (23)$$
with \(\lambda = \pm 1/2\) and where \(\vec{\sigma}\) are the Pauli matrices and \(p = |\vec{p}|\). The particle spinors have the form

\[
\xi_{\pm}^{\mu}(\vec{p}) = \sqrt{p^3 + p} \left( \frac{1}{p_1 + ip_2} \right) \xi_{\mp}^{\mu}(\vec{p}), \quad \xi_{\pm}^{\mu}(\vec{p}) = \sqrt{-p^3 + p} \left( \frac{-p_1 + ip_2}{p_3 + p} \right). \tag{24}
\]

Then, the positive/negative frequency modes of momentum \(\vec{p}\) and helicity \(\lambda\) derived in \([4]\), assuming gamma matrices in Dirac representation, are \([1, 4, 5]\):

\[
U_{\vec{p},\lambda}(t, \vec{x}) = \frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}} \left( \frac{1}{2} e^{\pi k/2} H_{\nu}^{(1)}(\frac{2}{\omega} e^{-\omega t}) \xi_{\lambda}(\vec{p}) \right) e^{i\vec{p} \cdot \vec{x} - 2\omega t}, \tag{25}
\]

where \(H_{\nu}^{(1)}(z)\) are Hankel functions of the first kind and \(k = \frac{m}{\omega}, \nu_{\pm} = \frac{1}{2} \pm i k\). The negative frequency modes will be obtained via charge conjugation operation \(V_{\vec{p},\lambda}(t, \vec{x}) = i\gamma^0 (\vec{U}_{\vec{p},\lambda}(t, \vec{x}))^T\).

For solving our integrals with Hankel functions we use their relations with Macdonald functions \([1, 12, 23]\):

\[
H_{\nu}^{(1,2)}(z) = \mp \left( \frac{2i}{\pi} \right) e^{\pi i \nu/2} K_{\nu}(\mp iz). \tag{26}
\]

In this way we arrive at the integrals of the type \([23]\):

\[
\int_0^\infty dz z^{-\lambda} K_{\mu}(az)K_{\nu}(bz) = 2^{-2-\lambda} a^{-\mu+\lambda+1} b^\nu \frac{\Gamma\left(1-\lambda+\mu+\nu\right)}{\Gamma(1-\lambda)} \frac{\Gamma\left(1-\lambda-\mu+\nu\right)}{\Gamma(1-\lambda)} \frac{1-\lambda+\mu+\nu}{2} \frac{1-\lambda-\mu+\nu}{2} 2F_1\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}; 1-\lambda; 1-\frac{b^2}{a^2}\right),
\]

\[
Re(a + b) > 0, \quad Re(\lambda) < 1 - |Re(\mu)| - |Re(\nu)|. \tag{27}
\]

In our case \(\lambda = -1\) and the second condition for convergence is satisfied. We also observe that in our case \(a, b\) are complex and for solving our integrals we add to \(a\) a small real part \(a \rightarrow a + \epsilon\), with \(\epsilon > 0\) and in the end we take the limit \(\epsilon \rightarrow 0\). This assure the convergence of our integral and will correctly define the unit step functions and \(f_k\) functions.

For the computation of the total number of particles we use the next relation between hypergeometric functions:

\[
2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} 2F_1(a, b; a + b - c + 1; 1 - z) + (1 - z)^{c-a-b} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} 2F_1(c - a, c - b; c - a - b + 1; 1 - z). \tag{28}
\]
The integrals that helps us to calculate the total number of fermions are:

$$
\int_0^\infty \frac{dp'}{p'^2 - 2pp'} = \mp \frac{i\pi 2^{\pm 2ik}}{8p^3 \sinh(2\pi k)} \left( \frac{-1}{p} \right)^{\pm 2ik} ; \int_\mu^\Lambda \frac{dp}{p} = \ln(\Lambda) - \ln(\mu). \quad (29)
$$

[1] P. Candelas and D.J. Raine, Phys.Rev. D 12, 965 (1975).
[2] E. Schrödinger, Physica 6, 899 (1939).
[3] C.W.Misner, K.S.Thorne and J.A.Wheeler, Gravitation (W.H.Freeman and Company New York, 1973).
[4] I.I.Cotăescu, Phys.Rev. D 65, 084008 (2002).
[5] A.O.Barut and I.H.Duru, Phys.Rev. D 36, 3705 (1987).
[6] S.Drell and J.D.Bjorken, Relativistic Quantum Fields (Mc Graw-Hill Book Co., New York 1965).
[7] L.Landau and E.M.Lifsit, Theorie Quantique Relativiste (Mir Moscou 1972).
[8] L.Parker, Phys.Rev.Lett. 21, 562 (1968).
[9] L.Parker, Phys.Rev. 183, 1057 (1969).
[10] L.Parker, Phys.Rev.D 3, 346 (1971).
[11] M.Abramowitz and I.A.Stegun, Handbook of Mathematical Functions (Dover, New York, 1964).
[12] G.N.Watson, Theory of Bessel Functions (Cambridge University Press, 1922).
[13] N. D. Birrel and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge 1982).
[14] S. Weinberg, The Quantum Theory of Fields (Cambridge University Press, Cambridge, 1995).
[15] C.Itzykson, J.B.Zuber, Quantum Field Theory (Mc Graw-Hill Inc. 1980).
[16] K.H.Lotze, Nucl.Phys.B 312, 673 (1989).
[17] K.H.Lotze, Class.Quant.Grav. 2, 351 (1988).
[18] K.H.Lotze, Class.Quant.Grav. 5, 595 (1985).
[19] V.M.Villalba, Phys.Rev.D 52, 3742 (1995).
[20] J.Garriga, Phys.Rev.D 49, 6343 (1994).
[21] I.L. Buchbinder, E.S. Fradkin and D.M. Gitman, Fortschr. Phys. 29, 187 (1981).
[22] I. L. Buchbinder and L. I. Tsaregorodtsev, Int. J. Mod. Phys. A 7, 2055 (1992).
[23] I.S.Gradshteyn and I.M.Ryzhik *Table of integrals, series and products* (Academic Press, 2007).

[24] I. I. Cotăescu and C. Crucean, Prog.T.Phys **124**, 1051 (2010).

[25] J. Schwinger, Phys. Rev. **82**, 664 (1950).

[26] C. Crucean, Phys.Rev.D **85**, 084036 (2012).

[27] J. Martin, Lect.Notes Phys. **738**, 193 (2008).

[28] J.Haro and E.Elizalde, J.Phys. A **41**, 372003 (2008).

[29] N. D. Birrel, P. C. W. Davies and L. H. Ford, J.Phys. A **13**, 961 (1980).

[30] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, 1989).

[31] S. Haouat and R. Chekireb, arXiv:hep-th/1207.4342, (2012).