Exponential Attractor for the Viscoelastic Wave Model with Time-Dependent Memory Kernels

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Abstract

The paper is concerned with the exponential attractors for the viscoelastic wave model in \( \Omega \subset \mathbb{R}^3 \):

\[
utt - h_t(0) \Delta u - \int_0^\infty \partial_s h_t(s) \Delta u(t - s) ds + f(u) = g,
\]

with time-dependent memory kernel \( h_t(\cdot) \) which is used to model aging phenomena of the material. Conti et al. (Am J Math 140(2):349–389, 2018a; Am J Math 140(6):1687–1729, 2018b) recently provided the correct mathematical setting for the model and a well-posedness result within the novel theory of dynamical systems acting on time-dependent spaces, recently established by Conti et al. (J Differ Equ 255:1254–1277, 2013), and proved the existence and the regularity of the time-dependent global attractor. In this work, we further study the existence of the time-dependent exponential attractors as well as their regularity. We establish an abstract existence criterion via quasi-stability method introduced originally by Chueshov and Lasiecka (J Dyn Differ Equ 16:469–512, 2004), and on the basis of the theory and technique developed in Conti et al. (2018a, b) we further provide a new method to overcome the difficulty of the lack of further regularity to show the existence of the time-dependent exponential attractor. And these techniques can be used to tackle other hyperbolic models.

Keywords Viscoelastic wave model · Time-dependent memory kernel · Exponential attractors · Time-dependent phase spaces · Longtime behavior of solutions

Mathematics Subject Classification 37L30 · 37L45 · 35B40 · 35B41 · 35L10
1 Introduction

In this paper, we investigate the existence of the exponential attractors for the following viscoelastic wave model with time-dependent memory kernel

$$u_{tt} - h_t(0)\Delta u - \int_0^\infty \partial_sh_t(s)\Delta u(t-s)ds + f(u) = g \quad \text{in} \quad \Omega \times (\tau, +\infty),$$

(1)

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with the smooth boundary $\partial \Omega$, the time-dependent function

$$h_t(s) = k_t(s) + k_\infty, \quad k_\infty > 0, \quad u(\tau - s) = \phi_\tau(s), \quad s > 0,$$

(2)

and where $k_t(\cdot)$ is convex and summable for every fixed $t$, $\phi_\tau(s)$ is an assigned function, together with the boundary and initial conditions

$$u|_{\partial\Omega} = 0, \quad (u(\tau), u_t(\tau)) = (u_\tau, v_\tau).$$

(3)

Model (1) arising from the theory of viscoelasticity was proposed by Conti, Danese, Giorgi and Pata [7] to describe the dynamics of aging materials because the memory kernel $h_t(\cdot)$ depends on time, and this feature allows to describe viscoelastic materials whose structural properties evolve over time, say, materials that undergo an aging process which can be reasonably depicted as a loss of the elastic response (for more details, one can see [5, 17, 24] and references therein). This translates into the study of dynamical systems acting on time-dependent spaces, according to the newly established theory by Conti, Pata and Temam [6], whose inspiration is on the basis of [1, 23].

The presence of a time-dependent kernel introduces essential difficulties in the analysis. When the memory kernels are independent of $t$, the classical method introduced by Dafermos [10, 11] is adding a new variable $\eta$ which is generated by the right-translation semigroup acting on the history space and satisfies a specific differential equation. Under this circumstance, there exist extensive researches on the well-posedness and the longtime behavior of Eq. (1), with $h_t(s) \equiv h(s)$, see [9, 10, 16, 18–20, 22] and references therein. However, this approach become useless for the models with time-dependent memory kernel because the phase spaces for the past history are time-dependent, which causes some problems even in the definition of the time derivative $\partial_t \eta$. Hence it is necessary to give a new definition and construct some new estimates about variable $\eta$.

Conti, Danese, Giorgi and Pata [7] introduce the time-dependent memory kernel

$$\mu_t(s) := -\partial_s k_t(s) = -\partial_s h_t(s),$$

and for simplicity let $k_\infty = 1$, $\eta_\tau(s) = u_\tau - \phi_\tau(s), s \in \mathbb{R}^+$. Then a simple calculation shows that problem (1)–(3) reads

$$\partial_{tt} u + Au + \int_0^\infty \mu_t(s)A\eta_t'(s)ds + f(u) = g,$$

(4)

where $A$ is the Laplacian with the Dirichlet boundary condition, with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, and

$$\eta_t'(s) = \begin{cases} u(t) - u(t-s), s \leq t - \tau, \\
\eta_\tau(s-t+\tau) + u(t) - u_\tau, s > t - \tau,
\end{cases}$$

(5)

with the initial condition

$$(u(\tau), \partial_t u(\tau), \eta_\tau^\tau) = (u_\tau, v_\tau, \eta_\tau).$$

(6)
By proposing a different notion of weak solution where the supplementary differential equation ruling the evolution of $\eta$ is replaced by (5), and establishing a family of integral inequalities rather than differential ones as before, Conti, Danese, Giorgi and Pata [7] first provided a global well-posedness result for problem (4)–(6). Then Conti, Giorgi and Pata [8] further focused on the asymptotic behavior of the weak solutions and proved the existence and the regularity of the time-dependent global attractor. The authors [7,8] developed the theory, along with the techniques in their works, and open the way to the longterm analysis of the solutions for the related model with time-dependent memory kernel.

We mention that when the memory kernel $h_t(s) \equiv h(s)$ (independent of $t$), Danese, Geredeli and Pata [9] have proved the existence of exponential attractors for Eq. (1) by using the abstract criterion given in their paper. The concept of exponential attractors was firstly introduced by Eden et al [13] in the Hilbert space (and later in the Banach space (cf. [14])), which have the advantage of being more stable than global attractors because they have finite fractal dimension and attract trajectories at an exponential rate (cf. [15,21] for a detail discussion).

However, to the best of the authors’ knowledge, there are no results on the exponential attractors for viscoelastic wave model (1) (or (4)) because of the absence of the exponential attractor theory in the time-dependent phase spaces and the technical difficulties arising from this kind of hyperbolic problem.

The purpose of this paper is to probe this question, and the motivation of this research comes from literatures [7,8]. For convenience, we use the same terminology used in [7,8]. The main strategies can be summarized as follows:

(i) We first give a proper notion of the time-dependent exponential attractors for the dynamical process acting on the time-dependent phase spaces, and provide an abstract existence criterion via quasi-stability method introduced originally by Chueshov and Lasiecka [2–4]. This criterion can be seen as an extension of that in [26], which provided an abstract result for the existence of the pullback exponential attractors.

(ii) We provide a new method to construct a special attracting family (rather than usual absorbing family) with higher regularity and forward invariance, and based on them to apply the abstract criterion to problem (4)–(6) to prove the existence and the regularity of the desired time-dependent exponential attractors.

It is worth mentioning that the existence of the time-dependent exponential attractor implies that the fractal dimension of the sections of time-dependent global attractor given by [8] are uniformly bounded, and the application of the abstract criterion is challenging because of the hyperbolicity of Eq. (1) (or (4)), which leads to non-further regularity of the solutions.

The main contributions of the current paper are that we provide a new method based on the compact attracting family to overcome the difficulty of the lack of further regularity, and to apply the abstract criterion established in this paper (See Theorem 1 as well as Corollary 1 and Corollary 2) to establish the existence of the time-dependent exponential attractor of problem (4)–(6) (see Theorem 3). And this technique can be exported to tackle other hyperbolic models.

The paper is organized as follows. In Sect. 2, we give the definition of the time-dependent exponential attractors and discuss their existence criterion at an abstract level. In Sect. 3, we quote the assumptions which are same with those in [7,8], and state the main theorem of the paper. In Sect. 4, we first quote some known results coming from literature [7,8], then based on them we further establish some new estimates which will play key roles for our proving the main theorem. In Sect. 5, we give the proof of the main theorem.
2 Time-Dependent Exponential Attractors

In this section, we first quote some notions of the time-dependent global attractor and the related results (cf. [6,8]), then give the definition of the time-dependent exponential attractor as well as an abstract criterion for its existence.

Definition 1 A two-parameter family of operators \( \{ U(t, \tau) : X_\tau \to X_t \mid t \geq \tau, \tau \in \mathbb{R} \} \) is called a process acting on time-dependent Banach spaces \( \{ X_t \}_{t \in \mathbb{R}} \) if (i) \( U(\tau, \tau) \) is the identity map on \( X_\tau \); (ii) \( U(t, s) U(s, \tau) = U(t, \tau) \) for all \( t \geq s \geq \tau \).

Let \( U(t, \tau) \) be a process acting on time-dependent Banach spaces \( \{ X_t \}_{t \in \mathbb{R}} \).

Definition 2 A family \( B = \{ B(t) \}_{t \in \mathbb{R}} \) is called a uniformly time-dependent absorbing set of the process \( U(t, \tau) \) if it is uniformly bounded, i.e.,

\[
\sup_{t \in \mathbb{R}} \| B(t) \|_{X_t} := \sup_{t \in \mathbb{R}} \sup_{\xi \in B(t)} \| \xi \|_{X_t} < +\infty,
\]

and for every \( R > 0 \), there exists a \( \tau_e = \tau_e(R) \geq 0 \) such that

\[
U(t, \tau) B(t) \subset B(t) \quad \text{as} \quad t - \tau \geq \tau_e,
\]

where and in the following \( B(t) = \{ \xi \in X_t \mid \| \xi \|_{X_t} \leq R \} \).

Definition 3 [6] A family \( A = \{ A(t) \}_{t \in \mathbb{R}} \) is called the time-dependent global attractor of the process \( U(t, \tau) \) if

(i) \( A(t) \) is compact in \( X_t \) for each \( t \in \mathbb{R} \);
(ii) \( A \) is pullback attracting, i.e., \( A \) is uniformly bounded and for every uniformly bounded family \( \mathcal{D} = \{ D(t) \}_{t \in \mathbb{R}} \),

\[
\lim_{\tau \to -\infty} \text{dist}_{X_t} (U(t, \tau) D(\tau), A(t)) = 0, \quad \forall t \in \mathbb{R},
\]

where

\[
\text{dist}_{X_t} (A, B) = \sup_{x \in A} \inf_{y \in B} \| x - y \|_{X_t}
\]

is the Hausdorff semidistance of the nonempty sets \( A, B \subset X_t \).

(iii) \( A \) is the smallest family with above mentioned properties (i) and (ii), i.e., if a family \( A_1 = \{ A_1(t) \}_{t \in \mathbb{R}} \) is of properties (i) and (ii), then \( A(t) \subset A_1(t) \) for all \( t \in \mathbb{R} \).

Now, we define the time-dependent exponential attractor, which is a generalization of the concept of the pullback exponential attractor, and give its existence criterion.

Definition 4 A uniformly bounded family \( \mathcal{E} = \{ E(t) \}_{t \in \mathbb{R}} \) is called a time-dependent exponential attractor of the process \( U(t, \tau) \) if

(i) \( E(t) \) is compact in \( X_t \) for each \( t \in \mathbb{R} \), and its fractal dimension in \( X_t \) is uniformly bounded, i.e.,

\[
\sup_{t \in \mathbb{R}} \dim_f (E(t); X_t) < +\infty.
\]

(ii) \( \mathcal{E} \) is semi-invariant, i.e., \( U(t, \tau) E(\tau) \subset E(t) \) for all \( t \geq \tau \).

(iii) There exists a positive constant \( \beta \) such that for every uniformly bounded family \( \mathcal{D} = \{ D(t) \}_{t \in \mathbb{R}} \),

\[
\text{dist}_{X_t} (U(t, t - \tau) D(t - \tau), E(t)) \leq C(D) e^{-\beta \tau}, \quad \forall \tau \geq (\tau(D), t \in \mathbb{R},
\]

where \( C(D), \tau(D) \) are positive constants depending only on \( D \).
Theorem 1 Let \( U(t, \tau) \) be a process acting on time-dependent Banach spaces \( \{X_t\}_{t \in \mathbb{R}} \). Assume that there exist a uniformly bounded family \( B = \{B(t)\}_{t \in \mathbb{R}} \) and a positive constant \( T \) such that

\[
(H_1) \quad B(t) \text{ is closed in } X_t \text{ for each } t \in \mathbb{R}, \text{ and } \quad U(t, t - \tau)B(t - \tau) \subset B(t), \quad \forall \tau \geq T; \tag{7}
\]

\[
(H_2) \quad \text{there exists a uniform Lipschitz constant } L_1 > 0 \text{ such that } \quad \|U(t, t - \tau)x - U(t, t - \tau)y\|_{X_t} \leq L_1\|x - y\|_{X_{t - \tau}}, \tag{8}
\]

for all \( x, y \in B(t - \tau), \tau \in [0, T] \text{ and } t \in \mathbb{R}; \)

\[
(H_3) \quad \text{there exist a Banach space } Z \text{ with compact seminorm } n_Z(\cdot), \text{ and a mapping } K_t : B(t - T) \to Z \text{ for each } t \in \mathbb{R} \text{ such that for any } x, y \in B(t - T),
\]

\[
\|K_t x - K_t y\|_Z \leq L\|x - y\|_{X_{t - T}}, \tag{9}
\]

\[
\|U(t, t - T)x - U(t, t - T)y\|_{X_t} \leq \eta\|x - y\|_{X_{t - T}} + n_Z(K_t x - K_t y), \tag{10}
\]

where \( \eta \in (0, 1/2), L > 0 \) are constants independent of \( t \).

Then, there exists a semi-invariant family \( \mathcal{E} = \{E(t)\}_{t \in \mathbb{R}} \) possessing the following properties:

(i) the section \( E(t) \subset B(t) \) is compact in \( X_t \) for each \( t \in \mathbb{R} \) and its fractal dimension in \( X_t \) is uniformly bounded, i.e.,

\[
\sup_{t \in \mathbb{R}} \dim_f(E(t); X_t) \leq \left[ \ln \left( \frac{1}{2\eta} \right) \right]^{-1} \ln m_Z \left( \frac{2L}{\eta} \right) < +\infty, \tag{11}
\]

where \( m_Z(R) \) is the maximal number of elements \( z_i \in \text{ the ball } \{z \in Z \|z\|_Z \leq R\} \text{ such that } n_Z(z_i - z_j) > 1, i \neq j; \)

(ii) there exist positive constants \( \beta, C \) and \( \tau_0 \) such that

\[
\text{dist}_{X_t}(U(t, t - \tau)B(t - \tau), E(t)) \leq Ce^{-\beta\tau} \quad \text{as } \quad \tau \geq \tau_0, \forall t \in \mathbb{R}. \tag{12}
\]

Corollary 1 Let the assumptions of Theorem 1 be valid. If the family \( B = \{B(t)\}_{t \in \mathbb{R}} \) is a uniformly time-dependent absorbing set of the process \( U(t, \tau) \), then the family \( \mathcal{E} = \{E(t)\}_{t \in \mathbb{R}} \) given in Theorem 1 is a time-dependent exponential attractor of the process \( U(t, \tau) \).

Proof For any uniformly bounded family \( D = \{D(t)\}_{t \in \mathbb{R}} \), there exist positive constants \( R \) and \( e(R) \) such that \( D(t) \subset B_t(R) \) for all \( t \in \mathbb{R} \) and

\[
U(t, t - \tau)D(t - \tau) \subset U(t, t - \tau)B_{t - \tau}(R) \subset B(t), \quad \forall \tau \geq e(R). \]

Thus, it follows from (12) that, for every \( t \in \mathbb{R}, \tau \geq e(R) + \tau_0, \)

\[
\text{dist}_{X_t}(U(t, t - \tau)D(t - \tau), E(t)) \leq \text{dist}_{X_t}(U(t, t - \tau + e(R))U(t - \tau + e(R), t - \tau)D(t - \tau), E(t)) \leq \text{dist}_{X_t}(U(t, t - \tau + e(R))B(t - \tau + e(R), E(t)) \leq Ce^{\beta e(R)}e^{-\beta\tau}.
\]

This completes the proof. \( \square \)

Corollary 2 Let the assumptions of Theorem 1 be valid. If the process \( U(t, \tau) \) possesses a uniformly time-dependent absorbing set \( \{B_t(R_1)\}_{t \in \mathbb{R}} \) possessing the following properties:
1. there is a positive constant $R_2 > R_1$ such that $B(t) \subset \mathbb{B}_t(R_2)$ for all $t \in \mathbb{R}$ and
\[
\|U(t, t - \tau)x - U(t, t - \tau)y\|_{X_t} \leq C_1 e^{\gamma\tau\|x - y\|_{X_{t-\tau}}}
\] (13)
holds for all $x, y \in \mathbb{B}_{t-\tau}(R_2)$, $\tau \geq 0$ and $t \in \mathbb{R}$, where $C_1$ and $\gamma$ are positive constants depending only on $R_2$.
2. there exist positive constants $\kappa, \tau_1$ and $C_0$ such that
\[
\text{dist}_{X_t}(U(t, t - \tau)\mathbb{B}_{t-\tau}(R_1), B(t)) \leq C_0 e^{-\kappa\tau}, \ \forall t \in \mathbb{R}, \ \tau \geq \tau_1.
\] (14)

Then the family $E = \{E(t)\}_{t \in \mathbb{R}}$ given in Theorem 1 is a time-dependent exponential attractor of the process $U(t, \tau)$.

**Proof** By the definition of uniformly time-dependent absorbing set, there exists a positive constant $e(R_1) > \tau_1$ such that for any given $\theta \in (0, 1),
\[
U(t, t - \tau)\mathbb{B}_{t-\tau}(R_1) \subset \mathbb{B}_{t-\tau\theta}(R_1) \subset \mathbb{B}_{t-\tau\theta}(R_2)
\]
for all $\tau \geq (1 - \theta)^{-1}e(R_1)$ and $t \in \mathbb{R}$. Thus, it follows from (12)–(13) and the fact: $B(t - \tau\theta) \subset \mathbb{B}_{t-\tau\theta}(R_2)$ that
\[
\text{dist}_{X_t}(U(t, t - \tau)\mathbb{B}_{t-\tau}(R_1), E(t))
\leq \text{dist}_{X_t}(U(t, t - \tau)U(t, t - \tau\theta)B(t - \tau\theta)\mathbb{B}_{t-\tau}(R_1), U(t, t - \tau)B(t - \tau\theta))
\leq C_1 e^{\gamma\theta\tau} \text{dist}_{X_t}(U(t, t - \tau\theta)B(t - \tau\theta), E(t))
\leq C_1 e^{\gamma\theta\tau} \text{dist}_{X_t}(U(t, t - \tau)\mathbb{B}_{t-\tau}(R_1), B(t - \tau\theta)) + C e^{-\beta\tau}
\leq C_0 C_1 e^{\gamma\theta\tau + \kappa\beta - \kappa\tau} + C e^{-\beta\tau} = (C + C_0 C_1) e^{-\beta\tau}
\]
for all $\tau \geq \tau_2 := \max \left\{ \theta^{-1} \tau_0, (1 - \theta)^{-1}e(R_1) \right\}$ and $t \in \mathbb{R}$, where
\[
\theta = \frac{\kappa}{2(\gamma + \kappa)} \in (0, 1) \text{ and } \beta' = \min \left\{ \frac{\kappa}{2}, \frac{\kappa\beta}{2(\gamma + \kappa)} \right\} > 0.
\]
Then, repeating the same argument as the proof of Corollary 1 and using estimate (15), we obtain that for any uniformly bounded family $D = \{D(t)\}_{t \in \mathbb{R}},
\[
\text{dist}_{X_t}(U(t, t - \tau)D(t - \tau), E(t)) \leq (C + C_0 C_1) e^{\beta' e(R)} e^{-\beta'\tau},
\]
for all $\tau \geq \tau_2 + e(R)$ and $t \in \mathbb{R}$. This completes the proof. \hfill \Box

**Proof of Theorem 1** For clarity and without loss of generality, we assume $T = 1$. Since $B = \{B(t)\}_{t \in \mathbb{R}}$ is a uniformly bounded family, we have
\[
B(t) \subset \mathbb{B}_t(R_0), \ \forall t \in \mathbb{R}
\]
for some positive constant $R_0$. Thus $N_t(B(t), R_0) = 1$ for all $t \in \mathbb{R}$, here $N_t(B, \epsilon)$ denotes the cardinality of minimal covering of the set $B \subset X_t$ by its closed subsets of diameter $\leq 2\epsilon$.

Conditions (H1)-(H3) show that
\[
U(m, n)B(n) \subset B(m), \ \forall m \geq n,
\] (16)
and for all $x, y \in B(n - 1)$ and $n \in \mathbb{Z},$
\[
\|U(n, n - 1)x - U(n, n - 1)y\|_{X_n} \leq L_1\|x - y\|_{X_{n-1}},
\] (17)
\[
\|K_n x - K_n y\|_{Z} \leq L\|x - y\|_{X_{n-1}},
\] (18)
\[
\|U(n, n - 1)x - U(n, n - 1)y\|_{X_n} \leq \eta\|x - y\|_{X_{n-1}} + n\xi(K_n x - K_n y).
\] (19)
Now we show the following formula for all \( n \in \mathbb{Z} \) by induction on \( k \in \mathbb{N} \)
\[
N_n(k) := N_n \left( U(n, n-k) B(n-k), (2\eta)^k R_0 \right) \leq \left[m_Z \left( \frac{2L}{\eta} \right) \right]^k. \tag{20}
\]

Let, for \( k = 1 \) and \( n \in \mathbb{Z} \),
\[
n(1) := \mathbb{N} \left\{ x_j \in B(n-1) \mid n_Z \left( K_n x_j - K_n x_i \right) > \eta R_0, \; j \neq i \right\},
\]
where \( \mathbb{N}\{\cdots\} \) denotes the maximal number of elements with the given properties. And let
\[
\mathcal{B}(n-1) = K_n B(n-1) = \{K_n x \mid x \in B(n-1)\}.
\]
Formula (18) implies that
\[
diam \left( \mathcal{B}(n-1); Z \right) \leq L \text{diam} \left( \mathcal{B}(n-1); X_{n-1} \right) \leq 2LR_0,
\]
\[
\mathcal{B}(n-1) \subset B_Z(y; 2LR_0) = \{z \in Z \mid \|z-y\|_Z \leq 2LR_0\} \text{ for some } y \in Z.
\]
By the linearity and compactness of the seminorm \( n_Z(\cdot) \),
\[
n(1) = \mathbb{N} \left\{ z_i \in \mathcal{B}(n-1) \mid n_Z \left( z_j - z_i \right) > \eta R_0, \; j \neq i \right\}
\]
\[
\leq \mathbb{N} \left\{ z_i \in B_Z(y; 2LR_0) \mid n_Z \left( z_j - z_i \right) > \eta R_0, \; j \neq i \right\}
\]
\[
= \mathbb{N} \left\{ z_i \in B_Z \left( 0, \frac{2L}{\eta} \right) \mid n_Z \left( z_j - z_i \right) > 1, \; j \neq i \right\}
\]
\[
= m_Z \left( \frac{2L}{\eta} \right) < +\infty. \tag{21}
\]
Consequently, there exists a maximal subset \( \{x_j\}_{j=1}^{n(1)} \) of \( \mathcal{B}(n-1) \) such that
\[
n_Z \left( K_n x_j - K_n x_i \right) > \eta R_0, \; j \neq i,
\]
and
\[
\mathcal{B}(n-1) = \bigcup_{j=1}^{n(1)} C_j \text{ with } C_j = \{x \in \mathcal{B}(n-1) \mid n_Z \left( K_n x - K_n x_j \right) \leq \eta R_0\},
\]
\[
U(n, n-1) \mathcal{B}(n-1) = \bigcup_{j=1}^{n(1)} U(n, n-1)C_j.
\]
Formula (19) implies that, for all \( x, y \in C_j \) and all \( j = 1, \ldots, n(1) \),
\[
\|U(n, n-1)x - U(n, n-1)y\|_{X_n}
\]
\[
\leq \eta\|x - y\|_{X_{n-1}} + n_Z \left( K_n x - K_n y \right)
\]
\[
\leq \eta \text{diam} \left( \mathcal{B}(n-1); X_{n-1} \right) + 2\eta R_0 \leq 4\eta R_0,
\]
which implies
\[
diam \left( U(n, n-1)C_j; X_n \right) \leq 4\eta R_0, \; j = 1, \ldots, n(1). \tag{22}
\]
Thus, the combination of (21) and (22) shows that
\[
N_n(1) = N_n \left( U(n, n-1) \mathcal{B}(n-1), 2\eta R_0 \right) \leq m_Z \left( \frac{2L}{\eta} \right),
\]
that is, formula (20) holds for \( k = 1 \) and \( n \in \mathbb{Z} \).
Assume that formula (20) holds for all $1 \leq k \leq k_0$ and $n \in \mathbb{Z}$. We prove that it also holds for $k = k_0 + 1$. Due to
\[
U(n, n - k_0 - 1)B(n - k_0 - 1) = U(n, n - 1)U(n - 1, n - k_0 - 1)B(n - k_0 - 1),
\]
and
\[
N_n(k_0) = N_n \left( U(n, n - k_0)B(n - k_0), (2\eta)^{k_0}R_0 \right) \leq \left[ m_Z \left( \frac{2L}{\eta} \right) \right]^{k_0}, \quad (23)
\]
for all $n \in \mathbb{Z}$, there exists a minimal covering \( \bigcup_{i=1}^{N_{n-1}(k_0)} F_i \) of $U(n - 1, n - k_0 - 1)B(n - k_0 - 1)$ by its closed subsets of diameter $\leq 2(2\eta)^{k_0}R_0$, that is,
\[
\bigcup_{i=1}^{N_{n-1}(k_0)} F_i = U(n - 1, n - k_0 - 1)B(n - k_0 - 1) \subset B(n - 1), \quad (24)
\]
\[
diam (F_i; X_{n-1}) \leq 2(2\eta)^{k_0}R_0, \quad i = 1, \ldots, N_{n-1}(k_0). \quad (25)
\]
Let
\[
\mathcal{F}_i = K_n F_i = \{ K_n z \mid z \in F_i \} \subset Z, \quad i = 1, \ldots, N_{n-1}(k_0).
\]
The combination of (18) and (25) shows that
\[
diam (\mathcal{F}_i; Z) \leq L \text{diam} (F_i; X_{n-1}) \leq 2L(2\eta)^{k_0}R_0,
\]
\[
\mathcal{F}_i \subset B Z \left( y_i; 2L(2\eta)^{k_0}R_0 \right) = \left\{ z \in Z \mid \| z - y_i \|_Z \leq 2L(2\eta)^{k_0}R_0 \right\},
\]
for some $y_i \in Z$. Hence,
\[
n_i(k_0 + 1) := \# \left\{ x_i \in F_i \mid n_Z \left( K_n x_j - K_n x_m \right) > \eta(2\eta)^{k_0}R_0, \ j \neq m \right\}
\]
\[
= \# \left\{ z_i \in \mathcal{F}_i \mid n_Z \left( z_j - z_m \right) > \eta(2\eta)^{k_0}R_0, \ j \neq m \right\}
\]
\[
\leq \# \left\{ z_i \in B Z \left( y_i; 2L(2\eta)^{k_0}R_0 \right) \mid n_Z \left( z_j - z_m \right) > \eta(2\eta)^{k_0}R_0, \ j \neq m \right\}
\]
\[
= \# \left\{ z_i \in B Z \left( 0; \frac{2L}{\eta} \right) \mid n_Z \left( z_j - z_m \right) > 1, \ j \neq m \right\}
\]
\[
= m_Z \left( \frac{2L}{\eta} \right) < +\infty. \quad (26)
\]
Consequently,
\[
F_i = \bigcup_{j=1}^{n_i(k_0+1)} C_j^i
\]
with
\[
C_j^i = \left\{ x \in F_i \mid n_Z \left( K_n x_i - K_n x_j^i \right) \leq \eta(2\eta)^{k_0}R_0 \right\},
\]
where \( \{ x_j^i \}_{j=1}^{n_i(k_0+1)} \) is the maximal subset of $F_i$ such that
\[
n_Z \left( K_n x_i^j - K_n x_j^i \right) > \eta(2\eta)^{k_0}R_0, \ l \neq j.
\]
It follows from (19) and (23)--(27) that
\[
diam \left( U(n, n - 1)C_j; X_n \right) \leq \eta \text{diam} (F_i; X_{n-1}) + (2\eta)^{k_0+1}R_0 \leq 2(2\eta)^{k_0+1}R_0,
\]
\[ U(n, n - k_0 - 1)B(n - k_0 - 1) = \bigcup_{i=1}^{N_{n-1}(k_0)} \bigcup_{j=1}^{n_i(k_0+1)} U(n, n - 1)C_j, \]

which imply that
\[
N_n(k_0 + 1) = N_n \left( U(n, n - k_0 - 1)B(n - k_0 - 1), (2\eta)^{k_0+1}R_0 \right)
\leq \sum_{i=1}^{N_{n-1}(k_0)} n_i(k_0 + 1) \leq \left[ \frac{2L}{\eta} \right]^{k_0+1}.
\]

Therefore, formula (20) is valid.

Moreover, we infer from (28) and (32) that
\[
V_k(n) \subseteq U(n, n - k)B(n - k) \subseteq B(n) \subseteq X_n,
\]
and every \( V_k(n) \) possessing the following properties:
\[
\text{Card} (V_k(n)) \leq \left[ \frac{2L}{\eta} \right]^k, \quad (28)
\]
\[
V_k(n) \subseteq U(n, n - k)B(n - k) \subseteq \bigcup_{h \in V_k(n)} \left\{ \mathbb{B}_n \left( 2(2\eta)^k R_0 \right) + \{h\} \right\}. \quad (30)
\]

For any given \( n \in \mathbb{Z} \), we define by induction that
\[
\begin{aligned}
E_1(n) &= V_1(n), \\
E_k(n) &= V_k(n) \cup \left[ U(n, n - 1)E_{k-1}(n - 1) \right], \quad k \geq 2, \quad (31)
\end{aligned}
\]

where \([\cdot]_X \) denotes the closure in \( X \). Thus it follows from (16) and (29)–(30) that
\[
E_k(n) = \bigcup_{l=0}^{k-1} U(n, n - l)V_{k-l}(n - l) \subseteq U(n, n - k)B(n - k), \quad (32)
\]
\[
U(n + 1, n)E_k(n) \subseteq E_{k+1}(n + 1), \quad (33)
\]
\[
E(n) \subseteq U(n, n - 1)B(n - 1) \subseteq B(n), \quad \forall n \in \mathbb{Z}, \quad k \geq 1. \quad (34)
\]

Moreover, we infer from (28) and (32) that
\[
\text{Card} (E_k(n)) \leq \sum_{l=0}^{k-1} \text{Card} (V_{k-l}(n - l)) \leq \sum_{l=0}^{k-1} \left[ \frac{2L}{\eta} \right]^{k-l} \leq \left[ \frac{2L}{\eta} \right]^{k+1}, \quad (35)
\]

for all \( n \in \mathbb{Z} \) and \( k \geq 1 \).

We show that the family \( \{E(n)\}_{n \in \mathbb{Z}} \) is of the following properties:

(i) Semi-invariance. By the Lipschitz continuity (17), formulas (31) and (33)–(34),
\[
U(n, l)E(l) \subseteq \left[ \bigcup_{k \geq 1} U(n, l)E_k(l) \right]_X.
\]
(iii) Boundedness of the fractal dimension. For any $\varepsilon > 0$, there exists a unique $k_\varepsilon \in \mathbb{N}^+$ such that
\[ 2(2\eta)^{k_\varepsilon} R_0 < \varepsilon \leq 2(2\eta)^{k_\varepsilon - 1} R_0. \] (38)
Obviously, $k_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. It follows from (16) and (32) that
\[ E_k(n) \subset U(n, n - k)B(n - k) \subset U(n, n - k_\varepsilon)B(n - k_\varepsilon), \forall k \geq k_\varepsilon, n \in \mathbb{Z}, \]
which implies that
\[ E(n) \subset \left( \bigcup_{k < k_\varepsilon} E_k(n) \right) \bigcup \left[ U(n, n - k_\varepsilon)B(n - k_\varepsilon) \right]_{X_n}, \forall n \in \mathbb{Z}. \]
Thus it follows from (20), (35) and (38) that
\[ N_n \left( E(n), \varepsilon \right) \leq N_n \left( E(n), 2(2\eta)^{k_\varepsilon} R_0 \right) \leq \sum_{k < k_\varepsilon} \text{Card} \left( E_k(n) \right) + N_n \left( U(n, n - k_\varepsilon)B(n - k_\varepsilon), 2(2\eta)^{k_\varepsilon} R_0 \right) \leq \sum_{k < k_\varepsilon} \left[ mZ \left( \frac{2L}{\eta} \right) \right]^{k_\varepsilon + 1} + N_n(k_\varepsilon) \leq 2 \left[ mZ \left( \frac{2L}{\eta} \right) \right]^{k_\varepsilon + 1} < +\infty. \] (39)
By the arbitrariness of $\varepsilon \in (0, 1)$, we see from (39) that $E(n)$ is a compact subset of $X_n$. Moreover, estimates (38)–(39) and a simple calculation shows that
\[ \frac{\ln N_n \left( E(n), \varepsilon \right)}{\ln \left( 1/\varepsilon \right)} \leq \frac{(k_\varepsilon + 1) \ln \left[ mZ \left( \frac{2L}{\eta} \right) \right] + \ln 2}{(k_\varepsilon - 1) \ln \left( 1/2\eta \right) - \ln(2R_0)}, \varepsilon \in (0, 1), \]
which implies that
\[ \dim_f \left( E(n); X_n \right) = \limsup_{\varepsilon \to 0^+} \frac{\ln N_n \left( E(n), \varepsilon \right)}{\ln \left( 1/\varepsilon \right)} \leq \left[ \ln \left( \frac{1}{2\eta} \right) \right]^{-1} \ln \left[ mZ \left( \frac{2L}{\eta} \right) \right], \forall n \in \mathbb{Z}. \] (40)
For any $t \in \mathbb{R}$, there exists a unique $n \in \mathbb{Z}$ such that $t \in [n, n + 1)$. Let
\[ E(t) = U(t, n)E(n). \] (41)
We claim that $\{ E(t) \}_{t \in \mathbb{R}}$ is the desired family.
(i) It follows from formulas (7) and (34) that
\[ E(t) = U(t, n)E(n) \subset U(t, n)U(n, n - 1)B(n - 1) \subset B(t), \ \forall t \in \mathbb{R}. \]
Moreover, by the Lipschitz continuity of \( U(t, n) : B(n) \subset X_n \to X_t \) (see (8)), and the compactness of \( E(n) \) in \( X_n \), we know that \( E(t) \) is a compact subset of \( X_t \) and by (40),
\[ \dim_f(E(t); X_t) \leq \dim_f(E(n); X_n) \leq \left[ \ln \left( \frac{1}{2\eta} \right) \right]^{-1} \ln \left[ m_Z \left( \frac{2L}{\eta} \right) \right]. \]

(ii) For any \( t \geq s \in \mathbb{R}, \) let \( t = n + t_1, \ s = m + s_1 \) for some \( n, m \in \mathbb{Z} \) and \( t_1, s_1 \in [0, 1) \). Then, by formula (41) and the semi-invariance of \( \{E(n)\}_{n \in \mathbb{Z}} \) (see (36)), we have
\[ U(t, s)E(s) = U(t, n)U(n, s)U(s, m)E(m) = U(t, n)U(n, m)E(m) \subset U(t, n)E(n) = E(t). \]

(iii) For any given \( t \in \mathbb{R} \) and \( \tau \geq 3 \), there exist \( n \in \mathbb{Z} \) and \( k_\tau \in \mathbb{N}^+ \) such that
\[ t \in [n, n + 1) \text{ and } \tau \in [k_\tau + 2, k_\tau + 3), \]
which imply that
\[ n - k_\tau - (t - \tau) \geq 1, \quad -k_\tau < -\tau + 3, \quad (42) \]
and by formula (7),
\[ U(t, t - \tau)B(t - \tau) = U(t, n)U(n, n - k_\tau)U(n - k_\tau, t - \tau)B(t - \tau) \subset U(t, n)U(n, n - k_\tau)B(n - k_\tau). \quad (43) \]
Thus we infer from the Lipschitz continuity (8), estimate (37) and formulas (41)–(42) that
\[ \text{dist}_{X_t}(U(t, t - \tau)B(t - \tau), E(t)) \]
\[ \leq \text{dist}_{X_t}(U(t, n)U(n, n - k_\tau)B(n - k_\tau), U(t, n)E(n)) \]
\[ \leq L_1 \text{dist}_{X_n}(U(n, n - k_\tau)B(n - k_\tau), E(n)) \]
\[ \leq 2L_1(2\eta)^{k_\tau}R_0 = 2L_1R_0e^{-\beta k_\tau} \leq Ce^{-\beta \tau}, \ \forall t \in \mathbb{R}, \ \tau \geq 3, \]
with \( \beta = \ln \frac{1}{2\eta} \) and \( C = 2L_1R_0e^3 \). This completes the proof.

\[ \square \]

**Remark 1** Theorem 1 and its Corollaries 1 and 2 are established, for simplicity, in a Banach space framework because of the definition of phase space \( \mathcal{H}_t \) in the time-dependent memory kernel problem. However, they are still valid if the family of Banach spaces \( \{X_t\}_{t \in \mathbb{R}} \) there is replaced by a family of normed linear spaces \( \{X_t\}_{t \in \mathbb{R}} \).

### 3 Preliminaries and Main Results on the Model (4)–(6)

For any \( \sigma \in \mathbb{R} \), we define the compactly nested Hilbert spaces
\[ H^\sigma = D(A^{\sigma/2}) \]
endowed with the inner products and the norms:
\[ \langle u, v \rangle_\sigma = \langle A^{\sigma/2}u, A^{\sigma/2}v \rangle_{L^2}, \quad \|u\|_\sigma = \|A^{\sigma/2}u\|_{L^2}, \]

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respectively, where and in the context the operator \( A \) is as shown in Eq. (4), and \( L^2 = L^2(\Omega) \).
The index \( \sigma \) will be omitted whenever zero. The symbol \( \langle \cdot, \cdot \rangle \) for the \( L^2 \)-inner product will also be used for the duality pairing between the dual spaces. We denote
\[
L^p = L^p(\Omega), \quad H = L^2, \quad H^1 = H_0^1(\Omega), \quad H^{-1} = H^{-1}(\Omega), \quad H^2 = H^2(\Omega) \cap H_0^1(\Omega),
\]
with \( p \geq 1 \). For every fixed time \( t \) and index \( \sigma \), we introduce the weighted \( L^2 \)-spaces, hereafter they are called memory spaces,
\[
\mathcal{M}^\sigma_t = L^2_{\mu_t}(\mathbb{R}^+; H^{\sigma+1}) = \left\{ \xi : \mathbb{R}^+ \to H^{\sigma+1} \mid \int_0^\infty \mu_t(s) \| \xi(s) \|^2_{\sigma+1} ds < \infty \right\}
\]
equipped with the weighted \( L^2 \)-inner products
\[
\langle \eta, \xi \rangle_{\mathcal{M}^\sigma_t} = \int_0^\infty \mu_t(s) \langle \eta(s), \xi(s) \rangle_{\sigma+1} ds.
\]
We define the extended memory spaces
\[
\mathcal{H}_t^\sigma = H^{\sigma+1} \times H^\sigma \times \mathcal{M}_t^\sigma
\]
equipped with the usual product norm
\[
\| (u, v, \eta) \|^2_{\mathcal{H}_t^\sigma} = \| u \|^2_{\sigma+1} + \| v \|^2_{\sigma} + \| \eta \|^2_{\mathcal{M}_t^\sigma}.
\]
For any \( r > 0 \), we denote by
\[
B_t^\sigma(r) = \{ z \in \mathcal{H}_t^\sigma \mid \| z \|_{\mathcal{H}_t^\sigma} \leq r \}
\]
the closed \( r \)-ball centered at zero of \( \mathcal{H}_t^\sigma \).

### 3.1 Assumptions and Well-Posedness

**Assumption 31** [8] (i) Let \( g \in H \) be independent of time, and let \( f \in C^2(\mathbb{R}) \) with \( f(0) = 0, \)
\[
| f''(u) | \leq c (1 + |u|) \quad \text{and} \quad \liminf_{|u| \to \infty} f'(u) > -\lambda_1, \tag{44}
\]
for some \( c \geq 0 \), where \( \lambda_1 > 0 \) is the first eigenvalue of \( A \).

(ii) The map \( (t, s) \mapsto \mu_t(s) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+ \) possesses the following properties:

\( (M_1) \) For every fixed \( t \in \mathbb{R} \), the map \( s \mapsto \mu_t(s) \) is nonincreasing, absolutely continuous and summable. We denote the total mass of \( \mu_t \) by \( \kappa(t) = \int_0^\infty \mu_t(s) ds \).

\( (M_2) \) For every \( \tau \in \mathbb{R} \), there exists a function \( K_\tau : [\tau, \infty) \to \mathbb{R}^+ \), summable on any interval \( [\tau, T] \), such that \( \mu_t(s) \leq K_\tau(t) \mu_t(s) \) for every \( t \geq \tau \) and every \( s > 0 \).

\( (M_3) \) For almost every fixed \( s > 0 \), the map \( t \mapsto \mu_t(s) \) is differentiable for all \( t \in \mathbb{R} \), and \( (t, s) \mapsto \mu_t(s) \in L^\infty(\mathcal{K}) \), \( (t, s) \mapsto \partial_t \mu_t(s) \in L^\infty(\mathcal{K}) \) for every compact set \( \mathcal{K} \subset \mathbb{R} \times \mathbb{R}^+ \).

\( (M_4) \) There exists a \( \delta > 0 \) such that \( \partial_t \mu_t(s) + \partial_s \mu_t(s) + \delta \kappa(t) \mu_t(s) \leq 0 \) for every \( t \in \mathbb{R} \) and almost every \( s > 0 \).

\( (M_5) \) The function \( t \mapsto \kappa(t) \) fulfills: \( \inf_{t \in \mathbb{R}} \kappa(t) > 0 \).

\( (M_6) \) The function \( t \mapsto \partial_t \mu_t(s) \) satisfies the uniform integral estimate:
\[
\sup_{t \in \mathbb{R}} \frac{1}{\kappa(t)^2} \int_0^\infty | \partial_t \mu_t(s) | ds < \infty.
\]
(M7) For every $t \in \mathbb{R}$, the function $s \mapsto \mu_t(s)$ is bounded about zero, with $\sup_{t \in \mathbb{R}} \frac{\mu_t(0)}{\varepsilon(t)^2} < \infty$.

(M8) For every $a < b \in \mathbb{R}$, there exists a $v > 0$ such that $\int_0^1 \frac{1}{v} \mu_t(s) ds \geq \frac{\epsilon(t)}{2}$ for every $t \in [a, b]$.

Remark 2  
(i) The conditions (M1)-(M8) are quoted from [8]. As is shown in [8], the following function
\[ \mu_t(s) = \frac{1}{\varepsilon(t)} e^{-\frac{s^2}{\varepsilon(t)}} \] with $\varepsilon(t) = \frac{1}{4} \left[ \frac{\pi}{2} - \arctan(t) \right]$ satisfies above mentioned conditions (M1)-(M8).

(ii) Condition (M2) implies the continuous embedding: $M^\sigma_{t, \tau} \hookrightarrow M^\sigma_{t, \tau}$ for all $\sigma \in \mathbb{R}$ and $t > \tau$, with
\[ \| \eta \|^2_{M^\sigma_{t, \tau}} \leq K_{\varepsilon(t)} \| \eta \|^2_{M^\sigma_{t, \tau}}, \quad \forall \eta \in M^\sigma_{t, \tau}. \]

Therefore, $H^\sigma_{t, \tau} \hookrightarrow H^\sigma_{t, \tau}$ for all $\sigma \in \mathbb{R}$ and $t > \tau$.

Now, we quote some known results in recent literatures [7,8], which are the bases of our arguments. Let us begin with the definition of weak solution.

Definition 5  [7] Let $T > \tau \in \mathbb{R}$, and $z_\tau = (u_\tau, v_\tau, \eta_\tau) \in \mathcal{H}_\tau$ be a fixed vector. A function
\[ z(t) = (u(t), \partial_t u(t), \eta^t) \in \mathcal{H}_t \] for a.e. $t \in [\tau, T]$ is called a weak solution of problem (4)–(6) on interval $[\tau, T]$ if $u(\tau) = u_\tau$, $\partial_t u(\tau) = v_\tau$ and

1. $u \in L^\infty(\tau, T; H^1)$, $\partial_t u \in L^\infty(\tau, T; H)$, $\partial_{tt} u \in L^1(\tau, T; H^{-1})$;
2. the function $\eta^t$ fulfills the representation formula (5);
3. the function $u(t)$ fulfills (4) in the weak sense, i.e.,
\[ \langle \partial_{tt} u(t), \phi \rangle + \langle u(t), \phi \rangle + \int_0^\infty \mu_t(s) \langle \eta^t(s), \phi \rangle ds + \langle f(u(t)), \phi \rangle = \langle g, \phi \rangle \]
for almost every $t \in [\tau, T]$ and every $\phi \in H^1$.

Theorem 2  [7] Let Assumption 31 be valid. Then for every $T > \tau \in \mathbb{R}$, and every $z_\tau = (u_\tau, v_\tau, \eta_\tau) \in \mathcal{H}_\tau$, problem (4)–(6) admits a unique weak solution $z(t) = (u(t), \partial_t u(t), \eta^t)$ on $[\tau, T]$ with
\[ (u, \partial_t u) \in C([\tau, T]; H^1 \times H), \quad \eta^t \in \mathcal{M}_t, \quad \forall t \in [\tau, T]. \]

Moreover, for any two weak solutions $z_1(t)$ and $z_2(t)$ on $[\tau, T]$ with $\| z_1(\tau) \|_{\mathcal{H}_\tau} + \| z_2(\tau) \|_{\mathcal{H}_\tau} \leq R$,
\[ \| z_1(t) - z_2(t) \|_{\mathcal{H}_t} \leq Q(R) e^{Q(R)(t-\tau)} \| z_1(\tau) - z_2(\tau) \|_{\mathcal{H}_\tau}, \quad \forall t \in [\tau, T], \] (45)
where $Q$ is an increasing positive function independent of $t$.

3.2 Main Results

Under Assumption 31, we define the mapping
\[ U(t, \tau) : \mathcal{H}_\tau \rightarrow \mathcal{H}_t, \quad U(t, \tau) z_\tau = z(t), \quad \forall z_\tau \in \mathcal{H}_\tau, \quad t \geq \tau, \]
where \( z(t) \) is the weak solution of problem (4–6) corresponding to the initial data \( z_\tau \in \mathcal{H}_\tau \). By Theorem 2, the two-parameter family \( \{ U(t, \tau) \mid t \geq \tau \} \) constitutes a process acting on time-dependent Banach spaces \( \{ \mathcal{H}_t \}_{t \in \mathbb{R}} \).

**Lemma 1** [8] Let Assumption 31 be valid. Then the process \( U(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t \) generated by problem (4–6) admits an invariant time-dependent global attractor \( A = \{ A(t) \}_{t \in \mathbb{R}} \), with \( A(t) \subset \mathcal{H}_t \) for each \( t \in \mathbb{R} \), and

\[
\sup_{t \in \mathbb{R}} \| A(t) \|_{\mathcal{H}_t} < \infty.
\]

Now, we state the main results of the paper, and its proof will be given in Sect. 5 after some delicate technique preparations in Sect. 4.

**Theorem 3** Let Assumption 31 be valid. Then the process \( U(t, \tau) : \mathcal{H}_\tau \to \mathcal{H}_t \) generated by problem (4–6) admits a time-dependent exponential attractor \( \mathcal{E} = \{ E(t) \}_{t \in \mathbb{R}} \), with \( E(t) \subset \mathcal{H}_t \) for each \( t \in \mathbb{R} \), and

\[
\sup_{t \in \mathbb{R}} \| E(t) \|_{\mathcal{H}_t} < \infty.
\]

**Corollary 3** Let Assumption 31 be valid. Then the fractal dimension of the invariant time-dependent global attractor \( A = \{ A(t) \}_{t \in \mathbb{R}} \) given by Lemma 1 is uniformly bounded, that is,

\[
\sup_{t \in \mathbb{R}} \dim_f (A(t); \mathcal{H}_t) \leq \sup_{t \in \mathbb{R}} \dim_f (E(t); \mathcal{H}_t) < +\infty.
\]

### 4 Some Key Estimates

We first quote some known results (Lemmas 2 to 6) coming from literature [7,8], which will be the stating point of our argument.

**Lemma 2** [8] (Gronwall-type lemma in integral form) Let \( \tau \in \mathbb{R} \) be fixed, and \( \Lambda : [\tau, \infty) \to \mathbb{R} \) be a continuous function. Assume that, for some \( \epsilon > 0 \) and every \( b > a \geq \tau \),

\[
\Lambda(b) + 2\epsilon \int_a^b \Lambda(y)dy \leq \Lambda(a) + \int_a^b q_1(y)\Lambda(y)dy + \int_a^b q_2(y)dy,
\]

where \( q_1, q_2 \) are locally summable nonnegative functions on \( [\tau, \infty) \) satisfying

\[
\int_a^b q_1(y)dy \leq \epsilon(b - a) + c_1 \text{ and } \sup_{t \geq \tau} \int_{t}^{t+1} q_2(y)dy \leq c_2,
\]

for some \( c_1, c_2 \geq 0 \). Then, we have

\[
\Lambda(t) \leq e^{c_1 \left[ |\Lambda(\tau)| + \epsilon(t - \tau) \right]} + \frac{c_2 e^\epsilon}{1 - e^{-\epsilon}}, \quad \forall t \geq \tau.
\]

We consider the following problem

\[
\partial_t p(t) + Ap(t) + \int_0^\infty \mu_t(s)A\psi^r(s)ds + \gamma(t) = 0, \quad t > \tau, \quad (46)
\]

\[
\psi^r(s) = \begin{cases} p(t) - p(t - s), & s \leq t - \tau, \\ \psi_t(s - t + \tau) + p(t) - p_{t, s} > t - \tau, \\ p(t) = p_{t, \tau}, \quad \partial_t p(t) = q_{t, \tau}, \quad \psi^\tau = \psi_t, \end{cases} \quad (47)
\]

\[
p(t) = p_{t, \tau}, \quad \partial_t p(t) = q_{t, \tau}, \quad \psi^\tau = \psi_t, \quad (48)
\]

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where $\gamma$ is a certain forcing term (possibly depending on $p$) and $(p_\tau, q_\tau, \psi_\tau) \in \mathcal{H}_\tau$. Assuming that problem (46)–(48) admits a global solution $(p(t), \partial_\tau p(t), \psi') \in \mathcal{H}_\tau$ for all $t \in [\tau, \infty)$.

**Lemma 3** [7] Let Assumption 31: (ii) be valid. For any fixed $\sigma \in \mathbb{R}$ and every $T > \tau \in \mathbb{R}$, if also

$$p \in W^{1,\infty}(\tau, T; H^{\sigma+1})$$

then for all $\tau \leq a \leq b \leq T$,

$$\|\psi^b\|^2_{M^\sigma_b} - \int_a^b \int_0^\infty [\partial_t \mu_\tau(s) + \partial_\tau \mu_\tau(s)] \|\psi'(s)\|_{\sigma+1}^2 ds dt
\leq \|\psi^a\|^2_{M^\sigma_a} + 2 \int_a^b \langle \partial_t p(t), \psi' \rangle_{M^\sigma_b} dt.$$

**Lemma 4** [8] Let Assumption 31: (ii) be valid, and the global solution $(p(t), \partial_\tau p(t), \psi')$ of problem (46)–(48) be sufficiently regular and let the functionals

$$\Phi(t) = 2(p(t), \partial_\tau p(t)),$$

$$\Psi(t) = -\frac{2}{\kappa(t)} \int_{-\infty}^t \mu_\tau(s) \psi'(s), \partial_\tau p(t) ds.$$  

Then, for every $\sigma \in (0, 1]$ and every $b > a \geq \tau$, we have

$$\Phi(b) + \left( 2 - \sigma \right) \int_a^b \|p(t)\|^2_{1t} dt \leq \Phi(a) + 2 \int_a^b \|\partial_\tau p(t)\|^2 dt + \frac{1}{\sigma} \int_a^b \kappa(t) \|\psi'\|^2_{M^\sigma_d} dt - 2 \int_a^b \langle \psi(t), p(t) \rangle dt,$$

and

$$\Psi(b) + \int_a^b \|\partial_\tau p(t)\|^2 dt \leq \Psi(a) - M \int_a^b \int_0^\infty [\partial_t \mu_\tau(s) + \partial_\tau \mu_\tau(s)] \|\psi'(s)\|_{\sigma+1}^2 ds dt$$

$$+ \frac{2}{\kappa(t)} \int_a^b \|p(t)\|^2_{1t} dt + \frac{C}{\kappa(t)} \int_a^b \|\psi'\|^2_{M^\sigma_d} dt$$

$$+ \int_a^b \frac{2}{\kappa(t)} \int_0^\infty \mu_\tau(s) \psi'(s), \gamma(t) ds dt,$$

where $M$ and $C$ are positive constants depending only on the structural assumptions on the memory kernel.

**Remark 3** (i) In the following in our arguments, Lemma 4 is always used to the Galerkin approximations which are of enough regularity.

(ii) By conditions (M1)–(M5) one easily sees that

$$|\Phi(t)| + |\Psi(t)| \leq C \| (p(t), \partial_\tau p(t), \psi') \|^2_{\mathcal{H}_\tau}, \quad \forall t \geq \tau.$$  

**Lemma 5** [8] Let Assumption 31 be valid, and $z_\tau \in \mathcal{H}_\tau$ with $\|z_\tau\|_{\mathcal{H}_\tau} \leq R$. Then there exist positive constants $\omega$ and $R_0$ such that

$$E(t, \tau) := \frac{1}{2} \|U(t, \tau) z_\tau\|^2_{\mathcal{H}_\tau} \leq Q(R)e^{-\omega(t-\tau)} + R_0, \quad \forall t \geq \tau.$$

That is, the family $\{\mathbb{B}_t(R_1)\}_{t \in \mathbb{R}}$, with $R_1 > \sqrt{2R_0}$ is a uniformly time-dependent absorbing set of the process $U(t, \tau)$. 

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Following the standard method given in [12], we split the nonlinearity \( f \) into the sum
\[
f(u) = f_0(u) + f_1(u),
\]
where \( f_1 \in C^2(\mathbb{R}) \) is globally Lipschitz with \( f_1(0) = 0 \), while \( f_0 \in C^2(\mathbb{R}) \) vanishes inside \([-1, 1]\) and
\[
|f''_0(u)| \leq C|u|, \quad f'_0(u) \geq 0
\]
for some positive constant \( C \). Then we decompose the solution \( U(t, \tau)z_\tau \) as the sum
\[
U(t, \tau)z_\tau = U_0(t, \tau)z_\tau + U_1(t, \tau)z_\tau,
\]
where \( U_0(t, \tau)z_\tau = (v(t), \partial_t v(t), \xi^t) \) solves the problem
\[
\begin{cases}
\partial_{tt}v(t) + Av(t) + \int_0^\infty \mu_t(s)A\xi^t(s)ds + f_0(v(t)) = 0, \\
U_0(\tau, \tau)z_\tau = z_\tau,
\end{cases}
\]
where
\[
\xi^t(s) = \begin{cases}
v(t) - v(t-s), & s \leq t - \tau, \\
\xi_\tau(s - t + \tau) + v(t) - v_\tau, & s > t - \tau,
\end{cases}
\]
and \( U_1(t, \tau)z_\tau = (w(t), \partial_t w(t), \zeta^t) \) solves the problem
\[
\begin{cases}
\partial_{tt}w(t) + Aw(t) + \int_0^\infty \mu_t(s)A\zeta^t(s)ds + f_0(u(t)) - f_0(v(t)) + f_1(u(t)) = 0, \\
U_1(\tau, \tau)z_\tau = 0,
\end{cases}
\]
where
\[
\zeta^t(s) = \begin{cases}
w(t) - w(t-s), & s \leq t - \tau, \\
\zeta_\tau(s - t + \tau) + w(t) - w_\tau, & s > t - \tau.
\end{cases}
\]

**Lemma 6** [8] Let Assumption 31 be valid, and \( z_\tau \in \mathcal{H}_\tau \) with \( \|z_\tau\|_{\mathcal{H}_\tau} \leq R \). Then for every \( t \geq \tau \),
\[
\begin{align*}
\|U_0(t, \tau)z_\tau\|^2_{\mathcal{H}_\tau} & \leq Q(R) e^{-\omega(t-\tau)}, \\
\|U_1(t, \tau)z_\tau\|^2_{\mathcal{H}_\tau^{1/3}} & \leq Q(R),
\end{align*}
\]
hereafter \( \omega > 0 \) is as shown in Lemma 5.

Now, based on Lemmas 2 to 6, we further establish some new estimates which will play key roles for our applying Theorem 1 and its corollary to problem (4)–(6) to establish the existence of the time-dependent exponential attractors.

**Lemma 7** Let Assumption 31 be valid, and \( z_\tau \in \mathcal{H}_\tau^{1/3} \) with \( \|z_\tau\|_{\mathcal{H}_\tau} \leq R \). Then for every \( t \geq \tau \), we have
\[
\|U(t, \tau)z_\tau\|^2_{\mathcal{H}_\tau^{1/3}} \leq Q(R + \|z_\tau\|_{\mathcal{H}_\tau^{1/3}}) e^{-\omega(t-\tau)} + Q(R).
\]

**Proof** We first make a priori estimates to the solutions of problem (4)–(6). For any \( t \geq \tau \), we define the functionals of the solutions \( (u(t), \partial_t u(t), \eta^t) = U(t, \tau)z_\tau \) as follows:
\[
\begin{align*}
\mathcal{E}_{1/3}(t, \tau) &= \frac{1}{2}\|U(t, \tau)z_\tau\|^2_{\mathcal{H}_\tau^{1/3}}, \\
\mathcal{L}_{1/3}(t) &= L_{1/3}(t) + \|\eta^t\|^2_{\mathcal{M}_\tau^{1/3}}, \\
\Lambda_{1/3}(t) &= \mathcal{L}_{1/3}(t) + 2\epsilon \left[ \Phi(t) + 4\Psi(t) \right],
\end{align*}
\]
where \( \epsilon \in (0, 1] \), \( \Phi \) and \( \Psi \) are as shown in (49)–(50), with \( (p(t), \partial_t p(t), \psi^t) = (A^{1/6}u(t), \partial_t A^{1/6}u(t), A^{1/6}v^t) \), and the functional
\[
L_{1/3}(t) = \|u(t)\|_{L^3}^2 + \|\partial_t u(t)\|_{L^3}^2 + 2\langle \gamma(t), A^{1/6}u(t) \rangle,
\]
with \( \gamma(t) = A^{1/6} (f(u(t)) - g) \). By condition (44) and Lemma 5, we have
\[
\|f(u)\| \leq C \|1 + |u|^3\| \leq C (1 + \|u\|_{L^6}^3) \leq C (1 + \|u\|^3) \leq Q(R),
\]
which implies that
\[
2\|\gamma(t), A^{1/6}u(t)\| \leq 2 [\|f(u)\| + \|g\|] \|u\|_{H^1/2} \leq \frac{1}{4} \|u(t)\|_{L^3}^2 + Q(R),
\]
and hence,
\[
\frac{3}{2} \mathcal{E}_{1/3}(t, \tau) - Q(R) \leq \mathcal{L}_{1/3}(t) \leq \frac{5}{2} \mathcal{E}_{1/3}(t, \tau) + Q(R).
\]
By formula (53),
\[
|\Phi| + |\Psi| \leq C \mathcal{E}_{1/3}(t, \tau), \quad \forall t \geq \tau.
\]
The combination of (61) and (63)–(64) yields
\[
\mathcal{E}_{1/3}(t, \tau) - Q(R) \leq A_{1/3}(t) \leq 3 \mathcal{E}_{1/3}(t, \tau) + Q(R)
\]
for \( \epsilon > 0 \) suitably small. Taking the multiplier \( 2A^{1/3} \partial_t u \) in Eq. (4) gives
\[
\frac{d}{dt} L_{1/3}(t) + 2\langle \eta', \partial_t u(t) \rangle \mathcal{M}^1_{1/3} = 2 \langle f'(u(t))\partial_t u(t), A^{1/3}u(t) \rangle := I_1(t) + I_2(t) + I_3(t),
\]
where
\[
I_1 = 2 [\left\langle f_0'(u) - f_0'(v) \right\rangle \partial_t u, A^{1/3}u],
I_2 = 2 [\left\langle f_0'(v)\partial_t u, A^{1/3}u \right\rangle],
I_3 = 2 [\left\langle f_1'(u)\partial_t u, A^{1/3}u \right\rangle].
\]
By Lemma 5, estimates (55), (58)–(59) and the Sobolev embedding
\[
H^1 \hookrightarrow L^6, \quad H^{4/3} \hookrightarrow L^{18}, \quad H^{2/3} \hookrightarrow L^{18/5}, \quad H^{1/3} \hookrightarrow L^{18/7},
\]
we have
\[
|I_1| \leq C \int_\Omega (|u| + |v|) |w||\partial_t u|A^{1/3}u|dx
\]
\[
\leq C \left( \|u\|_{L^6} + \|v\|_{L^6} \right) \|w\|_{L^{18}} \|\partial_t u\| \|A^{1/3}u\|_{L^{18/5}}
\]
\[
\leq \alpha \|u\|^2_{4/3} + \frac{Q(R)}{\alpha}, \quad \forall \alpha \in (0, 1],
\]
\[
|I_2| \leq C \int_\Omega |v|^2 |\partial_t u|A^{1/3}u|dx
\]
\[
\leq C \|v\|^2_{L^6} \|\partial_t u\|_{L^{18/7}} \|A^{1/3}u\|_{L^{18/5}}
\]
\[
\leq C \|v\|^2 \left[ \|\partial_t u\|^2_{1/3} + \|u\|^2_{4/3} \right],
\]
and
\[
|I_3| \leq C \|\partial_t u\| \|A^{1/3}u\| \leq Q(R) \|u\|_{4/3} \leq \alpha \|u\|^2_{4/3} + \frac{Q(R)}{\alpha}, \quad \forall \alpha \in (0, 1].
\]
Inserting above estimates into (66) and making use of (63) receive
\[
\frac{d}{dt} L_{1/3}(t) + 2(\eta', \partial_t u)_{\mathcal{A}^{1/3}_t} \leq q_1(t)L_{1/3}(t) + q_1(t) + \frac{Q(R)}{\alpha}, \quad \forall \alpha \in (0, 1],
\]
where \( q_1(t) = C(\alpha + \|v(t)\|_1^2) \). Integrating inequality (68) over \([a, b]\), with \( b \geq a \geq \tau \), yields
\[
L_{1/3}(b) + 2 \int_a^b (\eta', \partial_t u(t))_{\mathcal{A}^{1/3}_t} dt \\
\leq L_{1/3}(a) + \int_a^b q_1(t) L_{1/3}(t) dt + \int_a^b \left( q_1(t) + \frac{Q(R)}{\alpha} \right) dt, \quad \forall \alpha \in (0, 1].
\]
It follows form Lemma 3 (taking \( \sigma = 1/3 \)) there that
\[
\|\eta^b\|_{\mathcal{A}^{1/3}_b}^2 - \int_a^b \int_0^\infty [\partial_t \mu_t(s) + \partial_s \mu_t(s)] \|\eta'(s)\|_{4/3}^2 ds dt \\
\leq \|\eta^a\|_{\mathcal{A}^{1/3}_a}^2 + 2 \int_a^b \langle \partial_t u(t), \eta' \rangle_{\mathcal{A}^{1/3}_t} dt.
\]
Adding above two inequalities together we obtain
\[
L_{1/3}(b) - \int_a^b \int_0^\infty [\partial_t \mu_t(s) + \partial_s \mu_t(s)] \|\eta'(s)\|_{4/3}^2 ds dt \\
\leq L_{1/3}(a) + \int_a^b q_1(t) L_{1/3}(t) dt + \int_a^b \left( q_1(t) + \frac{Q(R)}{\alpha} \right) dt, \quad \forall \alpha \in (0, 1].
\]
Exploiting estimates (51)–(52) (taking \( \sigma = \frac{1}{20} \) there), we get
\[
\Phi(b) + 4\Psi(b) + \frac{7}{4} \int_a^b \|u(t)\|_{4/3}^2 dt + 2 \int_a^b \|\partial_t u(t)\|_{1/3}^2 dt \\
\leq \Phi(a) + 4\Psi(a) - 4M \int_a^b \int_0^\infty [\partial_t \mu_t(s) + \partial_s \mu_t(s)] \|\eta'(s)\|_{4/3}^2 ds dt \\
+ C \int_a^b \kappa(t) \|\eta'\|_{\mathcal{A}^{1/3}_t}^2 dt - 2 \int_a^b \langle \gamma(t), A^{1/6} u(t) \rangle dt \\
+ 8 \int_a^b \frac{1}{\kappa(t)} \int_0^\infty \mu_t(s) \langle A^{1/6} \eta'(s), \gamma(t) \rangle ds dt.
\]
Due to
\[
\|\gamma(t)\|_{-1} = \|A^{-1/3}(f(u) - g)\| \leq C(\|f(u(t))\| + \|g\|) \leq Q(R),
\]
we have
\[
-2 \int_a^b \langle \gamma(t), A^{1/6} u(t) \rangle dt \leq 2 \int_a^b \|\gamma(t)\|_{-1} \|u(t)\|_{4/3} dt \\
\leq \frac{1}{4} \int_a^b \|u(t)\|_{4/3}^2 dt + Q(R)(b - a),
\]
and by conditions \((M_1)\) and \((M_3)\), we have
\[
8 \int_a^b \frac{1}{\kappa(t)} \int_0^\infty \mu_t(s) \langle A^{1/6} \eta'(s), \gamma(t) \rangle ds dt
\]
The combination of (69) and (73) yields

$$
\leq 8 \int_a^b \frac{1}{\kappa(t)} \|\gamma(t)\|_{L^1} \left( \int_0^\infty \mu_{t}(s) \|\eta_{t}(s)\|_{L^4} ds \right) dt
$$

$$
\leq Q(R) \int_a^b \frac{1}{\kappa(t)} \sqrt{\kappa(t)} \|\eta_{t}\|_{\mathcal{M}_{t}^{1/3}} dt
$$

$$
\leq Q(R)(b - a) + Q(R) \int_a^b \kappa(t) \|\eta_{t}\|^2_{\mathcal{M}_{t}^{1/3}} dt.
$$

Inserting estimates (71)–(72) into (70) and making use of condition (M5) and (65) turn out

$$
\Phi(b) + 4\Psi(b) + \int_a^b \Lambda_{1/3}(t) dt
$$

$$
\leq \Phi(a) + 4\Psi(a) - 4M \int_a^b \int_0^\infty [\partial_t \mu_{t}(s) + \partial_s \mu_{t}(s)] \|\eta_{t}(s)\|_{L^4}^2 ds dt
$$

$$
+ Q(R) \int_a^b \kappa(t) \|\eta_{t}\|^2_{\mathcal{M}_{t}^{1/3}} dt + Q(R)(b - a).
$$

The combination of (69) and (73) yields

$$
\Lambda_{1/3}(b) + 2\epsilon \int_a^b \Lambda_{1/3}(t) dt + J
$$

$$
\leq \Lambda_{1/3}(a) + \int_a^b q_1(t) \mathcal{L}_{1/3}(t) dt + Q(R) \int_a^b \left( q_1(t) + \frac{1}{\alpha} \right) dt
$$

$$
\leq \Lambda_{1/3}(a) + \frac{5}{2} \int_a^b q_1(t) \Lambda_{1/3}(t) dt + Q(R) \int_a^b \left( q_1(t) + \frac{1}{\alpha} \right) dt,
$$

where

$$
J = - (1 - 8\epsilon M) \int_a^b \int_0^\infty [\partial_t \mu_{t}(s) + \partial_s \mu_{t}(s)] \|\eta_{t}(s)\|_{L^4}^2 ds dt
$$

$$
- 2\epsilon Q(R) \int_a^b \kappa(t) \|\eta_{t}\|^2_{\mathcal{M}_{t}^{1/3}} dt
$$

$$
\geq \delta (1 - 8\epsilon M) - 2\epsilon Q(R) \int_a^b \kappa(t) \|\eta_{t}\|^2_{\mathcal{M}_{t}^{1/3}} dt \geq 0
$$

for \( \epsilon \in (0, 1] \) suitably small. Taking \( \alpha : C\alpha = \omega \leq \epsilon \), a simple calculation shows that

$$
\frac{5}{2} \int_a^b q_1(t) dt \leq C\alpha(b - a) + Q(R) \int_a^b e^{-\omega(t - t)} dt \leq \omega(b - a) + Q(R),
$$

$$
\sup_{t \geq t} \int_{t_{i+1}}^{t_{i+1}} Q(R) \left( q_1(s) + \frac{1}{\alpha} \right) ds \leq Q(R) \left( \omega + \frac{Q(R)}{\omega} \right) = Q(R).
$$

Therefore, applying Lemma 2 to (74) and making use of (65) give the conclusion of Lemma 7.

For any fixed \( \tau \in \mathbb{R} \) and \( z_{\tau} \in \mathcal{H}_{\tau}^{1/3} \) with \( \|z_{\tau}\|_{\mathcal{H}_{\tau}^{1/3}} \leq R \), we still write

$$
U(t, \tau)z_{\tau} = U_0(t, \tau)z_{\tau} + U_1(t, \tau)z_{\tau}.
$$

(75)

where \( U_0(t, \tau) \) and \( U_1(t, \tau) \) are the solution operators of problems (56) and (57), respectively, with

$$
f_0(u) = 0 \quad \text{and} \quad f_1(u) = f(u).
$$
It follows from Lemma 7 that, for every \( z_\tau \in \mathcal{H}_\tau^{1/3} \) with \( \|z_\tau\|_{\mathcal{H}_\tau^{1/3}} \leq R \),
\[
\|U(t, \tau)z_\tau\|_{\mathcal{H}_\tau^{1/3}}^2 \leq Q(R), \quad \forall t \geq \tau.
\]  
(76)

Thus, by using the same argument as Lemma 8.1 in [8], we have (the proof is omitted here)

**Lemma 8** Let Assumption 31 be valid, and \( z_\tau \in \mathcal{H}_\tau^{1/3} \) with \( \|z_\tau\|_{\mathcal{H}_\tau^{1/3}} \leq R \). Then for every \( t \geq \tau \),
\[
\|U_0(t, \tau)z_\tau\|_{\mathcal{H}_\tau^{1/3}}^2 \leq Q(R)e^{-\omega(t-\tau)},
\]  
(77)
\[
\|U_1(t, \tau)z_\tau\|_{\mathcal{H}_\tau^{1/3}}^2 \leq Q(R).
\]  
(78)

Based on Lemma 8, we give a further delicate estimate.

**Lemma 9** Let Assumption 31 be valid, and \( z_\tau \in \mathcal{H}_\tau^{1/3} \) with \( \|z_\tau\|_{\mathcal{H}_\tau^{1/3}} \leq R \). Then
\[
\|U(t, \tau)z_\tau\|_{\mathcal{H}_\tau^{1/3}}^2 \leq Q \left( R + \|z_\tau\|_{\mathcal{H}_\tau^{1/3}} \right) e^{-\omega(t-\tau)} + Q(R), \quad \forall t \geq \tau.
\]
(Proof) For any \( t \geq \tau \), we define the functionals of the solutions \((u(t), \partial_t u(t), \eta') = U(t, \tau)z_\tau\) as follows:
\[
E_1(t, \tau) = \frac{1}{2}\|U(t, \tau)z_\tau\|_{\mathcal{H}_\tau^{1/3}}^2,
\]
(79)
\[
L_1(t) = L_1(t) + \|\eta'\|_{\mathcal{M}_1}\|
\]
(80)
\[
A_1(t) = L_1(t) + 2\epsilon \left[ \Phi(t) + 4\Psi(t) \right],
\]
where \( \epsilon \in (0, 1] \), \( \Phi \) and \( \Psi \) are defined as in (49)–(50), with \( (p(t), \partial_t p(t), \psi') = (A^{1/2}u(t), \partial_t A^{1/2}u(t), A^{1/2}\eta') \), and the functional
\[
L_1(t) = \|u(t)\|_2^2 + \|\partial_t u(t)\|_1^2 + 2\|\gamma(t), A^{1/2}u\|, \quad \text{with} \quad \gamma(t) = A^{1/2}(f(u) - g).
\]
By (62),
\[
2\|\gamma(t), A^{1/2}u\| \leq 2\|\gamma(t)\|_{-1} A^{1/2}u_1 \leq C \left[ \|f(u)\| + \|8\| \right] \|u\|_2 \leq \frac{1}{4} \|u(t)\|_2^2 + Q(R).
\]
Consequently,
\[
\frac{3}{2}E_1(t, \tau) - Q(R) \leq L_1(t) \leq \frac{5}{2}E_1(t, \tau) + Q(R).
\]  
(81)

It follows from (53) that
\[
|\Phi(t)| + |\Psi(t)| \leq C E_1(t, \tau), \quad \forall t \geq \tau,
\]  
(82)
which combining with (80)–(81) gives
\[
E_1(t, \tau) - Q(R) \leq A_1(t) \leq 3E_1(t, \tau) + Q(R)
\]  
(83)
for \( \epsilon \in (0, 1] \) suitably small. Taking the multiplier \( 2A\partial_t u \) in Eq. (4) gives
\[
\frac{d}{dt}L_1(t) + 2\langle \eta', \partial_t u(t) \rangle_{\mathcal{M}_1} = 2\left( f'(u(t))\partial_t u(t), Au(t) \right) \leq C \left( 1 + \|u(t)\|_{L_{1/2}}^2 \right) \|\partial_t u(t)\|_{L^{18/7}} \|Au(t)\| \leq Q(R)\|u(t)\|_2 \leq \alpha \|u(t)\|_2^2 + \frac{Q(R)}{\alpha}, \quad \forall \alpha \in (0, 1], \ t \geq \tau.
\]
where we have used condition (44), Sobolev embedding (67) and formula (76). Integrating above inequality over \([a, b]\), with \(b \geq a \geq \tau\), gives

\[
L_1(b) + 2 \int_a^b \langle \eta' , \partial_t u(t) \rangle_{\mathcal{M}_1} dt \leq L_1(a) + \alpha \int_a^b \|u(t)\|_2^2 dt + \frac{Q(R)}{\alpha} (b-a), \quad \forall \alpha \in (0, 1].
\]

Applying Lemma 3 (taking \(L_1 = 2\)) there) yields

\[
\|\eta\|_{\mathcal{M}_1}^2 - \int_a^b \int_0^\infty \left[ \partial_t \mu_t(s) + \partial_s \mu_t(s) \right] \|\eta'(s)\|_2^2 ds \, dt \\
\leq \|\eta\|_{\mathcal{M}_1}^2 + 2 \int_a^b \langle \partial_t u(t), \eta' \rangle_{\mathcal{M}_1} dt.
\]

Adding above two inequalities together turns out

\[
\mathcal{L}_1(b) - \int_a^b \int_0^\infty \left[ \partial_t \mu_t(s) + \partial_s \mu_t(s) \right] \|\eta'(s)\|_2^2 ds \, dt \\
\leq \mathcal{L}_1(a) + \alpha \int_a^b \|u(t)\|_2^2 dt + \frac{Q(R)}{\alpha} (b-a), \quad \forall \alpha \in (0, 1].
\]

Now, we give the estimates of the last term in the right hand side of the corresponding formulas (51)–(52), respectively. By (62),

\[
-2 \int_a^b \langle \gamma(t), A^{1/2} u(t) \rangle dt \leq \frac{1}{20} \int_a^b \|u(t)\|_2^2 dt + Q(R)(b-a),
\]

and by conditions \((M_1), (M_5)\),

\[
4 \int_a^b \frac{2}{\kappa(t)} \int_0^\infty \mu_t(s) \langle A^{1/2} \eta'(s), \gamma(t) \rangle ds \, dt \\
\leq 8 \int_a^b \frac{1}{\kappa(t)} \|\gamma(t)\|_2 \left( \int_0^\infty \mu_t(s) \|\eta'(s)\|_2^2 ds \right) \, dt \\
\leq Q(R)(b-a) + Q(R) \int_a^b \kappa(t) \|\eta'\|_{\mathcal{M}_1}^2 dt.
\]

Exploiting (51)–(52) (taking \(\sigma = \frac{1}{20}\) there) and making use of above two estimates, we obtain

\[
\Phi(b) + 4 \Psi(b) + \int_a^b \Lambda_1(t) dt + \frac{1}{5} \int_a^b \|u(t)\|_2^2 dt \\
\leq \Phi(a) + 4 \Psi(a) - 4M \int_a^b \int_0^\infty \left[ \partial_t \mu_t(s) + \partial_s \mu_t(s) \right] \|\eta'(s)\|_2^2 ds \, dt \\
+ Q(R) \int_a^b \kappa(t) \|\eta'\|_{\mathcal{M}_1}^2 dt + Q(R)(b-a).
\]

Taking \(\alpha = \frac{2\epsilon}{5}\), the combination of (84) and (85) gives

\[
\Lambda_1(b) + 2\epsilon \int_a^b \Lambda_1(t) dt + J_1 \leq \Lambda_1(a) + \frac{Q(R)}{\epsilon}(b-a),
\]

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where we have used condition (M₄) and the fact that
\[
J₁ = -(1 - 8εM) \int_{a}^{b} \int_{0}^{∞} [\partial_t \mu_1(s) + \partial_s \mu_1(s)] \|\eta'(s)\|_2^2 \, ds \, dt \\
- 2εQ(R) \int_{a}^{b} \kappa(t) \|\eta'(s)\|_{\mathcal{M}_τ^1}^2 \, dt \\
\geq (δ(1 - 8εM) - 2εQ(R)) \int_{a}^{b} \kappa(t) \|\eta'(s)\|_{\mathcal{M}_τ^1}^2 \, dt \geq 0
\]
for ε > 0 sufficiently small. Applying Lemma 2 (with q₁ = 0, q₂ = \frac{Q(R)}{ε} there) to (86) and making use of (83) give the conclusion of Lemma 9. □

**Lemma 10** Let Assumption 31 be valid. Then for any \(z_{1τ}, z_{2τ} \in \mathcal{H}_τ^1\) with \(\|z_{iτ}\|_{\mathcal{H}_τ^1} \leq R, i = 1, 2\),
\[
\|U(t, τ)z_{1τ} - U(t, τ)z_{2τ}\|_{\mathcal{H}_τ^1}^2 \leq C e^{-κ(t-τ)} \|z_{1τ} - z_{2τ}\|_{\mathcal{H}_τ^1}^2 + Q(R)e^{τ - τ} \int_{τ}^{t} \|\bar{u}(s)\|^2 \, ds,
\]
where C and κ are positive constants independent of R, and
\[
\bar{z}(t) = (\bar{u}(t), \partial_t \bar{u}(t), \bar{η}^f) = z_{1}(t) - z_{2}(t),
\]
with \(z_{i}(t) = (u_{i}(t), \partial_t u_{i}(t), η_{i}^f) = U(t, τ)z_{iτ}, i = 1, 2\).

**Remark 4** Lemma 10 implies that the process \(U(t, τ): \mathbb{B}^1_τ(R) \subset \mathcal{H}_τ \rightarrow \mathcal{H}_τ\) is quasi-stable for all \(R > 0\).

**Proof of Lemma 10** We still split the solution \(U(t, τ)z_{iτ}\) into the sum
\[
U(t, τ)z_{iτ} = U_0(t, τ)z_{iτ} + U_1(t, τ)z_{iτ} \\
= (v_{i}(t), \partial_t v_{i}(t), ξ_{i}^f) + (w_{i}(t), \partial_t w_{i}(t), ξ_{i}^f), i = 1, 2,
\]
where \(U_0(t, τ)z_{iτ}\) and \(U_1(t, τ)z_{iτ}\) solves problem (56) and (57) (with \(f_0 = 0\) and \(f_1 = f\) there), respectively. Then \(\bar{v} = v_1 - v_2\) solves
\[
\begin{cases}
\partial_t \bar{v} + A\bar{v} + \int_{0}^{∞} \mu_1(s) A\xi^f(t)(s) \, ds = 0, \quad t > τ, \\
(\bar{v}(t), \partial_t \bar{v}(t), η^f) = z_{1τ} - z_{2τ},
\end{cases}
\]
with
\[
\xi^f(s) = \begin{cases}
\bar{v}(t) - \bar{v}(t - s), & s \leq t - τ, \\
\bar{ξ}_τ(s - t + τ) + \bar{v}(t) - \bar{v}_τ, & s > t - τ.
\end{cases}
\]
And \(\bar{w} = w_1 - w_2\) solves
\[
\begin{cases}
\partial_t \bar{w} + A\bar{w} + \int_{0}^{∞} \mu_2(s) A\xi^f(s) \, ds + f(u_1) - f(u_2) = 0, \quad t > τ, \\
(\bar{w}(t), \partial_t \bar{w}(t), ξ^f) = 0,
\end{cases}
\]
with
\[
\xi^f(s) = \begin{cases}
\bar{w}(t) - \bar{w}(t - s), & s \leq t - τ, \\
\bar{ξ}_τ(s - t + τ) + \bar{w}(t) - \bar{w}_τ, & s > t - τ.
\end{cases}
\]
It follows from Lemmas 8 and 9 that
\[
\|z_{i}(t)\|_{\mathcal{H}_τ^1} + \|U_0(t, τ)z_{iτ}\|_{\mathcal{H}_τ^1} + \|U_1(t, τ)z_{iτ}\|_{\mathcal{H}_τ^1} \leq Q(R), \quad t \geq τ.
\]
Hence we can take the multiplier $2\partial_t \tilde{v}$ in Eq. (87) and obtain
\[
\|\tilde{v}(b)\|_{1}^{2} + \|\partial_t \tilde{v}(b)\|_{1}^{2} + 2 \int_{a}^{b} \langle \tilde{\xi}^{t}, \partial_t \tilde{v}(t) \rangle_{\mathcal{M}_{1}} dt = \|\tilde{v}(a)\|_{1}^{2} + \|\partial_t \tilde{v}(a)\|_{1}^{2},
\]  
(90)
for all $b \geq a \geq \tau$. By Lemma 3 (taking $\sigma = 0$ there),
\[
\|\tilde{\xi}^{b}\|_{\mathcal{M}_{b}}^{2} - \int_{a}^{b} \int_{0}^{\infty} [\partial_t \mu_{t}(s) + \partial_s \mu_{t}(s)] \|\tilde{\xi}^{t}(s)\|_{1}^{2} ds dt \leq \|\tilde{\xi}^{a}\|_{\mathcal{M}_{a}}^{2} + 2 \int_{a}^{b} \langle \partial_t \tilde{v}(t), \tilde{\xi}^{t} \rangle_{\mathcal{M}_{b}} dt.
\]  
(91)

The combination of (90) and (91) gives
\[
\|(\tilde{v}(b), \partial_t \tilde{v}(b), \tilde{\xi}^{b})\|_{\mathcal{H}_{b}}^{2} - \int_{a}^{b} \int_{0}^{\infty} [\partial_t \mu_{t}(s) + \partial_s \mu_{t}(s)] \|\tilde{\xi}^{t}(s)\|_{1}^{2} ds dt \leq \|(\tilde{v}(a), \partial_t \tilde{v}(a), \tilde{\xi}^{a})\|_{\mathcal{H}_{a}}^{2}, \ \forall b \geq a \geq \tau.
\]  
(92)

In order to obtain the sufficient regularity of the solutions needed for applying Lemma 4, we use the following approximating technique. We denote by
\[
\left(v_{in}(t), \partial_t v_{in}(t), \xi_{in}^{t}\right), \ i = 1, 2
\]
the Galerkin approximations of $(v_{i}(t), \partial_t v_{i}(t), \xi_{i}^{t}), i = 1, 2$, with initial data
\[
(v_{in}(	au), \partial_t v_{in}(	au), \xi_{in}^{\tau}) \rightarrow z_{i\tau} \text{ in } \mathcal{H}_{\tau}, \ i = 1, 2.
\]  
(93)

It follows from (92)–(93) and condition $(M_{4})$ that
\[
\lim_{n \to \infty} \|(v_{in}(t), \partial_t v_{in}(t), \xi_{in}^{t}) - (v_{i}(t), \partial_t v_{i}(t), \xi_{i}^{t})\|_{\mathcal{H}_{t}} \leq \lim_{n \to \infty} \|(v_{in}(t), \partial_t v_{in}(t), \xi_{in}^{t}) - z_{i\tau}\|_{\mathcal{H}_{\tau}} = 0, \ \forall t \geq \tau, \ i = 1, 2.
\]  
(94)

For every $n \in \mathbb{N}$, let $\bar{v}_{n} = v_{in} - v_{2n}, \bar{\xi}_{n} = \xi_{in}^{\tau} - \xi_{in}^{\tau}$ and
\[
\tilde{\xi}_{n}^{t}(s) = \begin{cases} \bar{v}_{n}(t) - \bar{v}_{n}(t-s), & s \leq t - \tau, \\ \bar{\xi}_{n}(s-t+\tau) + \bar{v}_{n}(t) - \bar{v}_{n}, & s > t - \tau. \end{cases}
\]

Obviously, formula (92) holds for all $(\bar{v}_{n}, \partial_t \bar{v}_{n}, \tilde{\xi}_{n}^{t}), n \in \mathbb{N}$.

For every $\epsilon \in (0, 1]$, we introduce the functional
\[
\Lambda_{n}^{\epsilon}(t) = \|(\bar{v}_{n}(t), \partial_t \bar{v}_{n}(t), \tilde{\xi}_{n}^{t})\|_{\mathcal{H}_{t}}^{2} + 2\epsilon \left[\Phi_{n}(t) + 4\Psi_{n}(t)\right], \ n \in \mathbb{N},
\]
where the functionals $\Phi_{n}$ and $\Psi_{n}$ are defined by formulas (49)–(50), with
\[
(p(t), \partial_t p(t), \psi^{t}) = (\bar{v}_{n}(t), \partial_t \bar{v}_{n}(t), \tilde{\xi}_{n}^{t}) \text{ and } \gamma(t) = 0 \text{ in (46)}.
\]

Thus, it follows from formula (53) that
\[
\frac{1}{2} \|(\bar{v}_{n}(t), \partial_t \bar{v}_{n}(t), \tilde{\xi}_{n}^{t})\|_{\mathcal{H}_{t}}^{2} \leq \Lambda_{n}^{\epsilon}(t) \leq \frac{3}{2} \|(\bar{v}_{n}(t), \partial_t \bar{v}_{n}(t), \tilde{\xi}_{n}^{t})\|_{\mathcal{H}_{t}}^{2}, \ n \in \mathbb{N}
\]  
(95)
for $\epsilon > 0$ sufficiently small. And by Lemma 4 (taking $\sigma = 1/20$ and $\gamma(t) = 0$ there) and a simple calculation we obtain
\[
\Phi_{n}(b) + 4\Psi_{n}(b) + \frac{7}{4} \int_{a}^{b} \|\bar{v}_{n}(t)\|_{1}^{2} dt + 2 \int_{a}^{b} \|\partial_t \bar{v}_{n}(t)\|^{2} dt
\]
\[
\leq \Phi_n(a) + 4\Psi_n(a) + C \int_a^b \kappa(t) \|\xi'_n\|_{\mathcal{M}_t}^2 dt
\]
\[
-4M \int_a^b \int_0^\infty \left[ \partial_t \mu_t(s) + \partial_s \mu_t(s) \right] \|\xi'_n(s)\|_1^2 ds dt.
\]

(96)

The combination of (92) and (96) gives

\[
A_v^n(b) + 2\epsilon \int_a^b A_v^n(t) dt \leq A_v^n(a)
\]

for \(\epsilon > 0\) suitably small, where we have used condition \((M_4)\). Hence applying Lemma 2, with \(q_1 = q_2 = 0\) there, and making use of (95), we obtain

\[
\|\tilde{v}_n(t), \partial_t \tilde{v}_n(t), \tilde{\xi}'_n\|_{7t}^2 \leq 3e^{-\epsilon(t-t_\tau)} \|\tilde{v}_n(\tau), \partial_t \tilde{v}_n(\tau), \tilde{\xi}'_n\|_{7t}^2,
\]

(97)

for all \(t \geq \tau\) and \(n \in \mathbb{N}\). Thus, by (93)–(94) and formula (97), we have

\[
\|\tilde{v}(t), \partial_t \tilde{v}(t), \tilde{\xi}'\|_{7t}^2 = \lim_{n \to \infty} \|\tilde{v}_n(t), \partial_t \tilde{v}_n(t), \tilde{\xi}'_n\|_{7t}^2
\]
\[
\leq 3e^{-\epsilon(t-t_\tau)} \lim_{n \to \infty} \|\tilde{v}_n(\tau), \partial_t \tilde{v}_n(\tau), \tilde{\xi}'_n\|_{7t}^2
\]
\[
= 3e^{-\epsilon(t-t_\tau)} \|z_{1\tau} - z_{2\tau}\|_{7t}^2.
\]

(98)

Taking into account estimate (89), we can use multiplier \(2\partial_t \tilde{w}\) in Eq. (88) and arrive at

\[
\frac{d}{dt} \left[ \|\tilde{w}(t)\|_1^2 + \|\partial_t \tilde{w}(t)\|_1^2 \right] + 2 \langle \tilde{\xi}', \partial_t \tilde{w}(t) \rangle_{\mathcal{M}_t}
\]
\[
= 2(f(u_2) - f(u_1), \partial_t \tilde{w}(t))
\]
\[
\leq C \left( 1 + \|u_1(t)\|_{L_\infty}^2 + \|u_2(t)\|_{L_\infty}^2 \right) \|\tilde{u}(t)\| \|\partial_t \tilde{w}(t)\|
\]
\[
\leq Q(R) \|\tilde{u}(t)\|_1^2 + \|\partial_t \tilde{w}(t)\|_1^2,
\]

which implies that

\[
\|\tilde{w}(t)\|_1^2 + \|\partial_t \tilde{w}(t)\|_1^2 + 2 \int_\tau^t \langle \tilde{\xi}', \partial_t \tilde{w}(s) \rangle_{\mathcal{M}_s} ds
\]
\[
\leq Q(R) \int_\tau^t (\|\tilde{u}(s)\|_1^2 + \|\partial_t \tilde{w}(s)\|_1^2) ds, \quad \forall t \geq \tau,
\]

where we have used condition (44), estimate (89) and the Sobolev embedding \(H^2 \hookrightarrow L^\infty\). Thus, making use of condition \((M_4)\) and Lemma 3 (with \(\sigma = 0\) there), we obtain

\[
\|\tilde{w}(t), \partial_t \tilde{w}(t), \tilde{\xi}'\|_{7t\tau}^2 \leq Q(R) \int_\tau^t \|\tilde{u}(s)\|_1^2 ds + \int_\tau^t \|\tilde{u}(s), \partial_t \tilde{w}(s), \tilde{\xi}'\|_{7t\tau}^2 ds.
\]

Applying the Gronwall inequality to above estimate gives

\[
\|\tilde{w}(t), \partial_t \tilde{w}(t), \tilde{\xi}'\|_{7t\tau}^2 \leq Q(R)e^{t-\tau} \int_\tau^t \|\tilde{u}(s)\|_1^2 ds, \quad \forall t \geq \tau.
\]

(99)
The combination of (98) and (99) yields
\[
\|U(t, \tau)z_{1\tau} - U(t, \tau)z_{2\tau}\|^2_{\mathcal{H}_t} \\
\leq C \left[ \|\tilde{u}(t), \tilde{v}(t), \tilde{z}'(t)\|^2_{\mathcal{H}_t} + \|\tilde{w}(t), \tilde{v}(t), \tilde{z}'(t)\|^2_{\mathcal{H}_t} \right] \\
\leq Ce^{-\varepsilon(t-\tau)}\|z_{1\tau} - z_{2\tau}\|^2_{\mathcal{H}_t} + Q(R)e^{\varepsilon(t-\tau)} \int_\tau^t \|\tilde{u}(s)\|^2 ds, \quad t \geq \tau,
\]
which completes the proof. \qed

5 Proof of the Main Result

The purpose of this section is to prove Theorem 3 by applying the abstract criteria obtained in Sect. 2. This argument is challenging because of the hyperbolicity of the problem, which translates into a loss of additional regularity of its solutions, so we put forward a new technique to overcome this difficulty. To this end, we first establish a specially pullback attracting family.

Lemma 11 Let Assumption 31 be valid. Then there exists a family \( \{B(t)\}_{t \in \mathbb{R}} \), with \( B(t) \subset \mathcal{H}_t \) for each \( t \in \mathbb{R} \), possessing the following properties:

(i) for every \( t \in \mathbb{R} \), the section \( B(t) \) is closed in \( \mathcal{H}_t \) and

\[
B(t) \subset \mathbb{B}_t(R_0) \cap \mathbb{B}_t^{\frac{1}{3}}(R)
\] (100)

for some constants \( R > 0 \) and \( R_0 > R_1 \), where \( R_1 \) is given by Lemma 5;

(ii) there exist positive constants \( \kappa \) and \( \tau_1 \) such that

\[
\text{dist}_{\mathcal{H}_t}(U(t, \tau)B(R_1), B(t)) \leq Q(R_1)e^{-\kappa(t-\tau)}, \quad \forall \tau \leq t - \tau_1, \quad t \in \mathbb{R};
\] (101)

(iii) there exists a positive constant \( T_1 \) such that

\[
U(t, \tau)B(t) \subset B(t), \quad \forall \tau \leq t - T_1, \quad t \in \mathbb{R}.
\] (102)

Proof For any \( \tau \in \mathbb{R} \) and \( z_\tau \in \mathbb{B}_\tau(R_1) \), it follows from Lemma 6 that

\[
\|U_0(t, \tau)z_\tau\|^2_{\mathcal{H}_t} \leq Q(R_1)e^{-\omega(t-\tau)} \quad \text{and} \quad \|U_1(t, \tau)z_\tau\|^2_{\mathcal{H}_t^{1/3}} \leq Q(R_1), \quad \forall t \geq \tau,
\]

which implies that there exists a positive constant \( R_1 \) depending only on \( R_1 \) such that

\[
\text{dist}_{\mathcal{H}_t}(U(t, \tau)\mathbb{B}_\tau(R_1), \mathbb{B}_t^{1/3}(R_1)) \leq Q(R_1)e^{-\omega(t-\tau)}, \quad \forall t \geq \tau.
\] (103)

Similarly, we infer from Lemma 8 that, for any \( z_\tau \in \mathbb{B}_t^{1/3}(R_1) \),

\[
\|U_0(t, \tau)z_\tau\|^2_{\mathcal{H}_t} \leq Q(R_1)e^{-\omega(t-\tau)} \quad \text{and} \quad \|U_1(t, \tau)z_\tau\|^2_{\mathcal{H}_t^{1/3}} \leq Q(R_1), \quad t \geq \tau.
\]

Since \( R_1 \) depends only on \( R_1 \), we can find a constant \( R_2 \) depending only on \( R_1 \) such that

\[
\text{dist}_{\mathcal{H}_t}(U(t, \tau)\mathbb{B}_t^{1/3}(R_1), \mathbb{B}_t^1(R_2)) \leq Q(R_1)e^{-\omega(t-\tau)}, \quad \forall t \geq \tau.
\] (104)

It follows from Definition 2 and Lemma 5 that there exists a positive constant \( e(R_1) \) depending only on \( R_1 \) such that

\[
U(t, \tau)\mathbb{B}_\tau(R_1) \subset \mathbb{B}_\tau(R_1), \quad \forall \tau \leq t - e(R_1).
\] (105)
Let $\theta = \frac{\omega}{Q(R_1) + 2\omega}$. Obviously, 
\[ \theta \in (0, 1) \] and 
\[ -\omega \theta = -\omega + (Q(R_1) + \omega) \theta. \]

We infer from formula (105) that
\[ U ((1 - \theta)t + \theta \tau, \tau) \mathbb{B}_\tau (R_1) \subset \mathbb{B}_\tau ((1 - \theta)t + \theta \tau) (R_1), \ \forall \tau \leq t - e_1, \quad (106) \]
where $e_1 = \frac{e(R_1)}{1 - \theta} > 0$. Thus, it follows from Theorem 2 and formula (103)-(106) that
\begin{align*}
\text{dist}_\mathcal{H}_t \left( U(t, \tau) \mathbb{B}_\tau (R_1), \mathbb{B}_\tau (R_2) \right) \\
\leq \text{dist}_\mathcal{H}_t \left( U(t, (1 - \theta)t + \theta \tau) U(((1 - \theta)t + \theta \tau, \tau) \mathbb{B}_\tau (R_1), U(t, (1 - \theta)t + \theta \tau) \mathbb{B}_{(1 - \theta)t + \theta \tau} (R_1) \right) \\
+ \text{dist}_\mathcal{H}_t \left( U(t, (1 - \theta)t + \theta \tau) \mathbb{B}_{(1 - \theta)t + \theta \tau} (R_1), \mathbb{B}_\tau (R_2) \right) \\
\leq Q(R_1) \exp(Q(R_1)(t - \tau)) \text{dist}_\mathcal{H}_t ((1 - \theta)t + \theta \tau) \left( U((1 - \theta)t + \theta \tau, \tau) \mathbb{B}_\tau (R_1), \mathbb{B}_{(1 - \theta)t + \theta \tau} (R_1) \right) \\
+ Q(R_1)e^{-\omega t(t - \tau)} \\
\leq Q(R_1) e^{-\omega t(t - \tau)}, \ \forall \tau \leq t - e_1. \quad (107)
\end{align*}

For every $z \in \mathbb{B}_\tau (R_2)$,
\[ \| z \|_{\mathcal{H}_t} \leq \lambda_1^{-1/2} \| z \|_{\mathcal{H}_t}, \quad \forall t \in \mathbb{R}, \]
which implies
\[ \mathbb{B}_\tau (R_2) \subset \mathbb{B}_t \left( \lambda_1^{-1/2} R_2 \right) \subset \mathbb{B}_t (R_3) \text{ and } \mathbb{B}_\tau (R_1) \subset \mathbb{B}_t (R_3), \ \forall t \in \mathbb{R}, \quad (108) \]
where $R_3 = R_1 + \lambda_1^{-1/2} R_2$ depends only on $R_1$. By Lemma 5 and formula (108), there exists a constant $e_2 > 0$ such that
\[ U(t, \tau) \mathbb{B}_\tau (R_3) \subset \mathbb{B}_t (R_1) \subset \mathbb{B}_t (R_3), \ \forall \tau \leq t - e_2. \quad (109) \]

It follows from Lemma 7 that for any $z_\tau \in \mathbb{B}_\tau (R_3) \cap \mathcal{H}_t^{1/3}$,
\[ \| U(t, \tau)z_\tau \|_{\mathcal{H}_t^{1/3}}^2 \leq Q \left( R_3 + \| z_\tau \|_{\mathcal{H}_t^{1/3}} \right) e^{-\omega t(t - \tau)} + R_4, \ \forall t \geq \tau, \quad (110) \]
where the positive constant $R_4$ depends only on $R_1$.

Similarly, for every $z \in \mathbb{B}_\tau (R_2)$, we have
\[ \| z \|_{\mathcal{H}_t^{1/3}} \leq \lambda_1^{-1/3} \| z \|_{\mathcal{H}_t^{1/3}} \leq \lambda_1^{-1/3} R_2, \ \forall t \in \mathbb{R}, \]
which implies
\[ \mathbb{B}_\tau (R_2) \subset \mathbb{B}_t^{1/3} \left( \lambda_1^{-1/3} R_2 \right) \subset \mathbb{B}_t^{1/3} (R_5), \ \forall t \in \mathbb{R}, \quad (111) \]
where $R_5 = R_4 + \lambda_1^{-1/3} R_2$ depends only on $R_1$. It follows from formula (110) that there exists a positive constant $e_3$ such that
\[ U(t, \tau) \left[ \mathbb{B}_\tau (R_3) \cap \mathbb{B}_t^{1/3} (R_5) \right] \subset \mathbb{B}_t^{1/3} (R_5), \ \forall \tau \leq t - e_3. \quad (112) \]

Lemma 9 shows that for any $z_\tau \in \mathbb{B}_\tau^{1/3} (R_5) \cap \mathcal{H}_t^{1/3}$,
\[ \| U(t, \tau)z_\tau \|_{\mathcal{H}_t^{1/3}}^2 \leq Q \left( R_5 + \| z_\tau \|_{\mathcal{H}_t^{1/3}} \right) e^{-\omega t(t - \tau)} + R_6, \ \forall t \geq \tau, \quad (113) \]
where the positive constant $\mathcal{R}_6$ depends only on $R_1$. Obviously,
\begin{equation}
\mathbb{B}_t^1(\mathcal{R}_2) \subset \mathbb{B}_t^1(\mathcal{R}_7) \quad \text{with} \quad \mathcal{R}_7 = \mathcal{R}_2 + \mathcal{R}_6, \quad \forall t \in \mathbb{R}.
\end{equation}
(114)
Thus formula (113) implies that there is a positive constant $e_4$ such that
\begin{equation}
U(t, \tau) \left[ \mathbb{B}_t^{1/3}(\mathcal{R}_5) \cap \mathbb{B}_t^1(\mathcal{R}_7) \right] \subset \mathbb{B}_t^1(\mathcal{R}_7), \quad \forall \tau \leq t - e_4.
\end{equation}
(115)
Let
\begin{equation*}
B(t) = \mathbb{B}_t(\mathcal{R}_3) \cap \mathbb{B}_t^{1/3}(\mathcal{R}_5) \cap \mathbb{B}_t^1(\mathcal{R}_7), \quad \forall t \in \mathbb{R}.
\end{equation*}
We show that \( \{B(t)\}_{t \in \mathbb{R}} \) is the desired family.

(i) Obviously, for every $t \in \mathbb{R}$, $B(t)$ is closed in $\mathcal{H}_t$ and
\begin{equation*}
B(t) \subset \mathbb{B}_t^1(\mathcal{R}_3) \cap \mathbb{B}_t^1(\mathcal{R}_7),
\end{equation*}
that is, conclusion (100) is valid, with $\mathcal{R}_0 = \mathcal{R}_3 > R_1$ and $\mathcal{R} = \mathcal{R}_7$.

(ii) It follows from formulas (108), (111) and (114) that $\mathbb{B}_t^1(\mathcal{R}_2) \subset B(t)$ holds for all $t \in \mathbb{R}$. Then we infer from estimates (107) that
\begin{equation*}
\text{dist}_{\mathcal{H}_t}(U(t, \tau)B(t), B(t)) \leq \text{dist}_{\mathcal{H}_t}(U(t, \tau)\mathbb{B}_t^1(\mathcal{R}_1), \mathbb{B}_t^1(\mathcal{R}_2)) \leq Q(R_1)e^{-\omega \theta(t-\tau)}, \quad \forall \tau \leq t - e_1,
\end{equation*}
that is, formula (101) holds, with $\kappa = \omega \theta$ and $\tau_1 = e_1$.

(iii) Taking $T_1 = \max\{e_2, e_3, e_4\}$ and making use of formulas (109), (112) and (115) yield
\begin{align*}
U(t, \tau)B(\tau) &\subset U(t, \tau)\mathbb{B}_t^1(\mathcal{R}_3) \subset \mathbb{B}_t^1(\mathcal{R}_3), \\
U(t, \tau)B(\tau) &\subset U(t, \tau)\left[ \mathbb{B}_t^1(\mathcal{R}_3) \cap \mathbb{B}_t^{1/3}(\mathcal{R}_5) \right] \subset \mathbb{B}_t^{1/3}(\mathcal{R}_5), \\
U(t, \tau)B(\tau) &\subset U(t, \tau)\left[ \mathbb{B}_t^{1/3}(\mathcal{R}_5) \cap \mathbb{B}_t^1(\mathcal{R}_7) \right] \subset \mathbb{B}_t^1(\mathcal{R}_7), \quad \forall \tau \leq t - T_1, \; t \in \mathbb{R}.
\end{align*}
Therefore,
\begin{equation*}
U(t, \tau)B(\tau) \subset \mathbb{B}_t^1(\mathcal{R}_3) \cap \mathbb{B}_t^{1/3}(\mathcal{R}_5) \cap \mathbb{B}_t^1(\mathcal{R}_7) = B(t), \quad \forall \tau \leq t - T_1, \; t \in \mathbb{R}.
\end{equation*}
This completes the proof.

\[\square\]

**Proof of Theorem 3** It follows from Lemma 11 that the family $\{B(t)\}_{t \in \mathbb{R}}$ is uniformly bounded in $\{\mathcal{H}_t\}_{t \in \mathbb{R}}$, $B(t)$ is closed in $\mathcal{H}_t$ for each $t \in \mathbb{R}$, and there exists a positive constant $T > T_1$ such that $\eta^2 = Ce^{-\kappa T} < \frac{1}{4}$ and
\begin{equation*}
U(t, t - \tau)B(t - \tau) \subset B(t), \quad \forall t \in \mathbb{R}, \; \tau \geq T,
\end{equation*}
where $T_1$ is as shown in Lemma 11. It follows from formula (45) that for any $t \in \mathbb{R}$,
\begin{equation}
\|U(t, t - \tau)z_1 - U(t, t - \tau)z_2\|_{\mathcal{H}_t} \leq Q(R_0)e^{Q(R_0)\tau}\|z_1 - z_2\|_{\mathcal{H}_{t-\tau}}
\end{equation}
for all $z_1, z_2 \in \mathbb{B}_t(\mathcal{R}_0)$ and $\tau \geq 0$. Taking $L_1 = Q(R_0)e^{Q(R_0)T}$, we infer from (116) that
\begin{equation*}
\|U(t, t - \tau)z_1 - U(t, t - \tau)z_2\|_{\mathcal{H}_t} \leq L_1\|z_1 - z_2\|_{\mathcal{H}_{t-\tau}},
\end{equation*}
for all $z_1, z_2 \in \mathbb{B}_t(\mathcal{R}_0)$ and $\tau \in [0, T]$.
Define the space
\begin{equation*}
Z = \left\{ u \in L^2(0, T; H^1) \mid \partial_t u \in L^2(0, T; H) \right\}
\end{equation*}
equipped with the norm
\[ \|u\|_Z = \| (u, \partial_t u) \|_{L^2(0,T; H)} \]
Obviously, \( Z \) is a Banach space. And the functional
\[ n_Z(u) = \mathcal{Q}(\mathcal{R}_0 + T) \|u\|_{L^2(0,T; H)} \]
is a compact semi-norm on \( Z \) (cf. [25]). For any given \( t \in \mathbb{R} \), we define the mapping
\[ \mathcal{K}_t : B(t - T) \to Z, \quad \mathcal{K}_t z = u(\cdot + t - T), \quad \forall z \in B(t - T), \]
where \( u(\cdot + t - T) \) means \( u(s + t - T), s \in [0, T] \), and
\[ \left( u(s + t - T), \partial_t u(s + t - T), \eta^{s+t-T} \right) = U(s + t - T, t - T)z. \]
Lemma 10 shows that
\[ \|U(t, t - T)z_1 - U(t, t - T)z_2\|_{\mathcal{H}_t} \leq \eta \|z_1 - z_2\|_{\mathcal{H}_{t-T}} + n_Z (K_t z_1 - K_t z_2), \]
and we infer from formulas (45) and (100) that
\[
\|K_t z_1 - K_t z_2\|^2_Z \leq \int_0^T \|U(s + t - T, t - T)z_1 - U(s + t - T, t - T)z_2\|^2_{\mathcal{H}_{t-T}} ds
\leq e^{\mathcal{Q}(\mathcal{R}_0)T}\|z_1 - z_2\|^2_{\mathcal{H}_{t-T}}, \quad \forall z_1, z_2 \in B(t - T), \quad t \in \mathbb{R}.
\]
Thus the family \( \mathcal{B} = \{ B(t) \}_{t \in \mathbb{R}} \) satisfies conditions \((H_1)-(H_3)\) of Theorem 1.
Moreover, by Lemma 5, \( \{\mathcal{B}_t(R_1)\}_{t \in \mathbb{R}} \) is a uniformly time-dependent absorbing set of the process \( U(t, \tau) \). And formulas (100)–(101) and (116) show that \( \{\mathcal{B}_t(R_1)\}_{t \in \mathbb{R}} \) satisfies the conditions of Corollary 2. Therefore, the process \( U(t, \tau) \) has a time-dependent exponential attractor \( \mathcal{E} = \{ E(t) \}_{t \in \mathbb{R}} \), with \( E(t) \subset B(t) \subset \mathbb{B}_1^1(\mathcal{R}) \) for each \( t \in \mathbb{R} \).

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