Thermodynamical Properties of Apparent Horizon in Warped DGP Braneworld

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Abstract
In this paper we first obtain Friedmann equations for the $(n-1)$-dimensional brane embedded in the $(n+1)$-dimensional bulk, with intrinsic curvature term of the brane included in the action (DGP model). Then, we show that one can always rewrite the Friedmann equations in the form of the first law of thermodynamics, $dE = TdS + WdV$, at apparent horizon on the brane, regardless of whether there is the intrinsic curvature term on the brane or a cosmological constant in the bulk. Using the first law, we extract the entropy expression of the apparent horizon on the brane. We also show that in the case without the intrinsic curvature term, the entropy expressions are the same by using the apparent horizon on the brane and by using the bulk geometry. When the intrinsic curvature appears, the entropy of apparent horizon on the brane has two parts, one part follows the $n$-dimensional area formula on the brane, and the other part is the same as the entropy in the case without the intrinsic curvature term. As an interesting result, in the warped DGP model, the entropy expression in the bulk and on the brane are not the same. This is reasonable, since in this model gravity on the brane has two parts, one induced from the $(n+1)$-dimensional bulk gravity and the other due to the intrinsic curvature term on the brane.

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I. INTRODUCTION

The profound connection between the black hole physics and thermodynamics revealed in the 1970s’ inspired deep thinking on the relation between gravity and thermodynamics in general. A pioneer work on this respect was done by Jacobson a decade ago who showed that the gravitational Einstein equation can be derived from the relation between the horizon area and entropy, together with the Clausius relation $\delta Q = T \delta S$ [1]. The study on the relation between gravity and thermodynamics has been extended beyond the Einstein gravity. For the so called $f(R)$ gravity, Eling et al. [2] recently argued that the corresponding field equation describing gravity can be derived from thermodynamics by using the procedure in [1], but a treatment with nonequilibrium thermodynamics of spacetime is needed. In the framework of Gauss-Bonnet gravity, the scalar-tensor gravity and more general Lovelock gravity, this problem has also been studied in [3][4] respectively.

It is of great interest to generalize the study to the cosmological context. Attempts to disclose the connection between Einstein gravity and thermodynamics in the cosmological situation have been carried out in [5, 6, 7, 8, 9, 10]. It has been shown that the differential form of the Friedmann equation of the FRW universe at the apparent horizon can be rewritten in the form of the first law of thermodynamics [5]. Besides the usual FRW universe, it is also of interest to explore the deep connection between gravity and thermodynamics in the brane world cosmology [11].

In recent years, there are a lot of interest in the brane world scenario, based on the assumption that all matters in standard model of particle physics are confined on a surface (brane) embedded in a higher dimensional spacetime (bulk), while the gravitational field, in contrast, is usually considered to live in the whole spacetime. There are two main pictures in the brane world scenario. In the first picture which we refer as the Randall-Sundrum II model (RS II), a positive tension 3-brane embedded in an 5-dimensional AdS bulk and the crossover between 4D and 5D gravity is set by the AdS radius [12]. In this case, the extra dimension has a finite size. In another picture which is based on the work of Dvali, Gabadadze, Porrati (DGP model) [13, 14], a 3-brane is embedded in a spacetime with an infinite-size extra dimension, with the hope that this picture could shed new light on the standing problem of the cosmological constant as well as on supersymmetry breaking [13, 13]. The recovery of the usual gravitational laws in this picture is obtained by adding to the action of the brane an Einstein-Hilbert term computed with the brane intrinsic curvature. The presence of such a term in the action is generically induced by quantum corrections coming from the bulk gravity and its coupling with matter living on the brane and should be included in a large class of theories for self-consistency [14, 16].
It is worth applying the method developed in [5, 6, 11] to investigate the connection between gravity and thermodynamical properties of the apparent horizon of the universe in the framework of brane world scenarios. Gravity on the brane does not obey Einstein theory, thus the usual area formula for the black hole entropy does not hold on the brane. In addition, exact analytic black hole solutions on the brane have not been found until now, so that the relation between the braneworld black hole horizon entropy and its geometry is not known. It is expected that the connection between gravity and thermodynamics in the braneworld can shed some lights on understanding these problems. This is our main motivation to explore the thermodynamical properties of the Friedmann equation in the warped braneworld. Exact Friedmann equations on the brane for the RS II model have been derived by many authors (see e.g [17]). In this paper we will show that the differential form of the Friedmann equations on the \((n - 1)\)-dimensional brane embedded in an \((n + 1)\)-dimensional Minkowski or AdS spacetime can be written directly in the form of the first law of thermodynamics, \(dE = TdS + WdV\), on the apparent horizon. Using the first law, we will extract the entropy expression of the apparent horizon on the brane, which coincides with the result in [11] obtained by using the method of unified first law and the Clausius relation. When the intrinsic curvature appears on the brane, we will show that the entropy of apparent horizon on the brane is a sum of two terms, one is the area formula on the brane and the other is the entropy expression in the case without the intrinsic curvature term.

The outline of the paper is as follows. In Sec. II we generalize Friedmann equations for the \((n - 1)\)-dimensional brane embedded in an \((n + 1)\)-dimensional bulk, with intrinsic curvature term of the brane included in the action. In Sec. III we study thermodynamical behavior of Friedmann equation in RS II braneworld scenario and extract the entropy of apparent horizon from the first law. In Sec. IV we extend our method for the warped DGP brane world scenario and find out the entropy associated with apparent horizon on the brane. The last section is devoted to summary and conclusions.

II. GENERAL FORMALISM

We consider an \((n - 1)\)-dimensional brane embedded in an \((n + 1)\)-dimensional spacetime with an intrinsic curvature term included in the brane action

\[
I_G = -\frac{1}{2\kappa^2_{n+1}} \int d^{n+1}x \sqrt{-\tilde{g}}R + \int d^{n+1}x \sqrt{-\tilde{g}}\mathcal{L}_m - \frac{1}{2\kappa^2_n} \int d^n x \sqrt{-g}R. \tag{1}
\]
The first term in (1) corresponds to the Einstein-Hilbert action in the \((n + 1)\)-dimensional bulk, where \(\tilde{g}_{AB}\) is the bulk metric and \(\tilde{R}\) is the \((n + 1)\)-dimensional scalar curvature. Similarly, the last term is the Einstein-Hilbert action for the induced metric \(g_{\mu\nu}\) on the brane with scalar curvature \(R\). The second term in (1) corresponds to the matter content. Aside from the bulk matter, we have included the contribution of the brane-localized matter, which can be rewritten as

\[
\int d^n x \sqrt{-g} (l_m - 2\lambda) \tag{2}
\]

where \(l_m\) is the lagrangian density of the brane matter fields, and \(\lambda\) is the brane tension (or cosmological constant). We emphasize here that this tension is not related to the presence of the intrinsic curvature term, and can in principle be tuned to be zero \[13, 14\]. Hereafter we assume that the brane cosmological constant is zero (if it does not vanish, one can absorb it in the stress-energy tensor of perfect fluid on the brane) and redefine

\[
\kappa_{n+1}^2 = 8\pi G_{n+1}, \quad \kappa_n^2 = 8\pi G_n, \quad \Lambda_{n+1} = -\frac{n(n-1)}{2\kappa_{n+1}^2 \ell^2}, \tag{3}
\]

where \(\Lambda_{n+1}\) is the \((n + 1)\)-dimensional bulk cosmological constant. We will consider \((n + 1)\)-dimensional spacetime metric of the form

\[
ds^2 = \tilde{g}_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{4}
\]

where \(y\) is the coordinate of the bulk. We assume the brane is located at \(y = 0\) and the bulk has \(\mathbb{Z}_2\) symmetry. Since we are interested in cosmological solutions, we take the metric in the form

\[
ds^2 = -N^2(t, y) dt^2 + A^2(t, y) \gamma_{ij} dx^i dx^j + dy^2 \tag{5}
\]

where \(\gamma_{ij}\) is a maximally symmetric \((n - 1)\)-dimensional metric for the surface \((t=\text{const.}, y=\text{const.})\), whose spatial curvature is parameterized by \(k = -1, 0, 1\). On every hypersurface \((y=\text{const})\), we have the metric of a FRW cosmological model. The metric coefficients \(A\) and \(N\) are chosen so that, \(N(t, 0) = 1\), \(A(t, 0) = a(t)\) and \(t\) is cosmic time on the brane. The \((n + 1)\)-dimensional Einstein equations take the form

\[
\tilde{G}_{AB} \equiv \tilde{R}_{AB} - \frac{1}{2} \tilde{R} \tilde{g}_{AB} = \kappa_{n+1}^2 \tilde{S}_{AB}. \tag{6}
\]

where \(\tilde{R}_{AB}\) is the \((n + 1)\)-dimensional Ricci tensor, and the tensor \(\tilde{S}_{AB}\) is the sum of the energy-momentum tensor \(\tilde{T}_{AB}\) of matter and the contribution coming from the scalar curvature of the brane. We denote this latter contribution by \(\tilde{U}_{AB}\). We have

\[
\tilde{S}_{AB} = \tilde{T}_{AB} + \tilde{U}_{AB} \tag{7}
\]
The energy-momentum tensor can be further decomposed into two parts

$$\tilde{T}_{AB} = -\Lambda_{n+1}\tilde{g}_{AB} + T_{AB},$$

where $T_{AB}$, is the matter content on the brane ($y = 0$), which we assume in the form of a perfect fluid for homogenous and isotropic universe on the brane

$$T_{AB} = \delta^\mu_A \delta^\nu_B t_{\mu\nu} \delta(y), \quad t_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu},$$

where $u^\mu$, $\rho$ and $p$ being the perfect fluid velocity ($u^\mu u_\nu = -1$), energy density and pressure respectively. The possible contribution of a nonzero brane tension $\lambda$ will be assumed to be included in $\rho$ and $p$. The assumption that $\tilde{G}_{0y} = 0$, which physically means that there is no energy flow between the brane and the bulk, implies that $\tilde{G}_{0y} = 0$ vanishes. The nonvanishing components of $\tilde{U}_{AB}$ are

$$\tilde{U}_{00} = -\frac{\delta(y)}{2\kappa^2_n} (n-1)(n-2) \left( \frac{\dot{A}^2}{A^2} + \frac{k}{A^2} \right),$$

$$\tilde{U}_{ij} = -\frac{\delta(y)}{2\kappa^2_n} (n-2)(n-3)\gamma_{ij} \left[ \frac{A^2}{N^2} \left( \frac{\dot{A}^2}{A^2} + \frac{2 \dot{N} \dot{A}}{n-3NA} - \frac{2}{n-3} \frac{\dot{A}}{A} \right) - k \right],$$

where dot denotes derivative with respect to $t$. Our aim here is to obtain the Friedmann equation governing the cosmological evolution on the brane. Following [17] we find that a set of functions $A(t, y)$ and $N(t, y)$ satisfying in equation

$$\left( \frac{\dot{A}}{NA} \right)^2 + \frac{k}{A^2} - \frac{2\kappa^2_{n+1}}{n(n-1)} - \left( \frac{A'}{A} \right)^2 - C A^n = 0,$$

together with $\tilde{G}_{0y} = 0$ which leads $\dot{A}/N = A(t)$, will be solutions of the field equations in the bulk. In Eq. [12] prime denotes derivative with respect to $y$ and $C$ is an integration constant which is related to the $(n + 1)$-dimensional bulk Weyl tensor [18, 19] and will be zero in the cases of Minkowski and AdS bulk. To find the Friedmann equation on the brane we need to study the junction condition on the brane. The metric is required to be continuous across the brane. However its derivative with respect to $y$ can be discontinuous at $y = 0$. This will entail the existence of a Dirac delta function in the second derivative of the metric with respect to $y$ (see [20] for details).

Integrating the (00) and $(ij)$ components of the field equations [10] across the brane and imposing $\mathbb{Z}_2$ symmetry, the Junction conditions are shown as follows

$$\frac{2a'}{a} = -\frac{\kappa^2_{n+1}}{n-1} + \frac{\kappa^2_{n+1}(n-2)}{2\kappa^2_n} \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right),$$
\[ 2N'_+ = \kappa_{n+1}^2 \left( p + \frac{n-2}{n-1} \rho \right) - \frac{\kappa_{n+1}^2(n-2)}{2\kappa_n^2} \left( \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - 2\frac{\ddot{a}}{a} \right), \tag{14} \]

where \(2N'_+ = -2N'_-\) is the discontinuity of the first derivative. One may note that in the particular case \(n = 4\) these junction conditions reduce to those obtained in [21] for 3-brane embedded in a 5-dimensional bulk including intrinsic curvature term on the brane (DGP model). On the other hand, taking the limit \(\kappa_n \rightarrow \infty\) we obtain the junction conditions on the \((n-1)\)-brane embedded in an \((n+1)\)-dimensional bulk in RS II brane scenario. Using (13) in Eq. (12) on the brane \((y = 0)\) we have the generalized Friedmann equation

\[ \epsilon \sqrt{H^2 + \frac{k}{a^2}} - \frac{2\kappa_{n+1}^2 \Lambda_{n+1}}{n(n-1)} \frac{C}{a^n} = -\frac{\kappa_{n+1}^2(n-2)}{4\kappa_n^2} (H^2 + \frac{k}{a^2}) + \frac{\kappa_{n+1}^2}{2(n+1)} \rho, \tag{15} \]

where \(H = \dot{a}/a\) is the Hubble parameter on the brane and \(\epsilon = \pm 1\). For later convenience we choose \(\epsilon = 1\). Inserting boundary conditions (13) and (14) into the equation \(\tilde{G}_{0y} = 0\), we get the continuity equation for the perfect fluid confined on the brane

\[ \dot{\rho} + (n-1)H(\rho + p) = 0. \tag{16} \]

Eqs. (15) and (16), together with the equation of state \(p = p(\rho)\), describe completely the cosmological dynamics on the brane. We will use this equation and the Friedmann equation to obtain the first law of thermodynamics at the apparent horizon on the brane both in the RS II model and the DGP model.

To have further understanding about the nature of apparent horizon we rewrite more explicitly, the metric of homogenous and isotropic FRW universe on the brane in the form

\[ ds^2 = h_{\mu\nu} dx^\mu dx^\nu + \tilde{r}^2 d\Omega_{n-2}^2, \tag{17} \]

where \(\tilde{r} = a(t)r, \ x^0 = t, x^1 = r\), the two dimensional metric \(h_{\mu\nu} = \text{diag} (-1, a^2/(1 - kr^2))\) and \(d\Omega_{n-2}\) is the metric of \((n-2)\)-dimensional unit sphere. Then, the dynamical apparent horizon, a marginally trapped surface with vanishing expansion, is determined by the relation \(h^{\mu\nu} \partial_\mu \tilde{r} \partial_\nu \tilde{r} = 0\), which implies that the vector \(\nabla \tilde{r}\) is null on the apparent horizon surface. The apparent horizon has been argued to be a causal horizon for a dynamical spacetime and is associated with gravitational entropy and surface gravity [22, 23]. The explicit evaluation of the apparent horizon for the FRW universe gives the apparent horizon radius

\[ \tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}. \tag{18} \]
The associated temperature on the apparent horizon can be defined as \( T = \frac{\kappa}{2\pi} \), where \( \kappa \) is the surface gravity

\[
\kappa = \frac{1}{\sqrt{-h}} \partial_a \left( \sqrt{-h} h^{ab} \partial_b \tilde{r} \right),
\]

Then one can easily show that the surface gravity at the apparent horizon of FRW universe can be written as

\[
\kappa = \frac{1}{\tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right).
\]

In the remain parts of this paper we will apply the result of this section to investigate the thermodynamical properties of apparent horizon in the braneworld scenario. We will show that the first law of thermodynamics on the apparent horizon can be extracted directly from the Friedmann equations for both AdS and Minkowski bulk and therefore we can extract an entropy relation on the brane.

III. THERMODYNAMICS OF APPARENT HORIZON IN RS II MODEL

Let us begin by RS II model in which no intrinsic curvature term on the brane contributes in the action. Taking the limit \( \kappa_n \to \infty \), while keeping \( \kappa_{n+1} \) finite, the equation (15) reduces to the Friedmann equation in the RS II brane world model

\[
H^2 + \frac{k}{a^2} - \frac{2\kappa_{n+1}^2 \Lambda_{n+1}}{n(n-1)} - \frac{C}{a^n} = \frac{\kappa_{n+1}^4 4(n-1)^2 \rho^2}{4(n-1)^2 \rho^2}.
\]

If one invokes the standard assumption that the energy density on the brane can be separated into two contributions, the ordinary matter component, \( \rho_b \), and the brane tension, \( \lambda > 0 \), such that \( \rho = \rho_b + \lambda \), (after fine tuning between the brane tension and the bulk cosmological constant), then one can recover the Friedmann equation presented in [11] for \( n \)-dimensional RS braneworld Scenario. We recall that one can interpret the constant \( C \) coming from the \( n+1 \)-dimensional bulk Weyl tensor. Since we are interested here in flat (Minkowskian) and conformally flat (AdS) bulk spacetime, the bulk Weyl tensor will vanish, so we set \( C = 0 \) in the following discussions.

A. Brane embedded in Minkowski bulk

We begin by the simplest case, namely Minkowski bulk, in which \( \Lambda_{n+1} = 0 \), so we can rewrite the Friedmann equation (21) in the simple form

\[
H^2 + \frac{k}{a^2} = \frac{\kappa_{n+1}^4}{4(n-1)^2 \rho^2}.
\]
In terms of the apparent horizon radius, we can rewrite the Friedmann equation (22) on the brane as

$$\frac{1}{\tilde{r}_A} = \frac{4\pi G_{n+1}}{n-1} \rho,$$  \hspace{1cm} (23)

where we have used Eq. (3). Taking differential form of equation (23) and using the continuity equation (16), one can get the differential form of the Friedmann equation on the brane

$$\frac{1}{4\pi G_{n+1}} \frac{d\tilde{r}_A}{\tilde{r}_A^2} = H(\rho + p) dt.$$  \hspace{1cm} (24)

Multiplying both sides of the equation (24) by a factor $(n-1)\Omega_{n-1}^{n-2} \tilde{r}_A^{n-2}$, and using the expression (20) for the surface gravity, after some simplification one can rewrite this equation in the form

$$\frac{\kappa}{2\pi} \frac{(n-1)}{2G_{n+1}} \Omega_{n-1}^{n-2} d\tilde{r}_A = (n-1)\Omega_{n-1}^{n-2} \tilde{r}_A^{n-2} H(\rho + p) dt - \frac{(n-1)}{2} \Omega_{n-1}^{n-2} (\rho + p) d\tilde{r}_A.$$  \hspace{1cm} (25)

$E = \rho V$ is the total energy of the matter inside the $(n-1)$-sphere of radius $\tilde{r}_A$ on the brane, where $V = \Omega_{n-1}^{n-2}$ is the volume enveloped by $(n-1)$-dimensional sphere with the area of apparent horizon $A = (n-1)\Omega_{n-1}^{n-2}$ and $\Omega_{n-1} = \frac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}$. Taking differential form of the relation $E = \rho \Omega_{n-1}^{n-1}$ for the total matter energy inside the apparent horizon on the brane, we get

$$dE = (n-1)\Omega_{n-1}^{n-2} \rho d\tilde{r}_A + \Omega_{n-1}^{n-2} \rho d\tilde{r}_A.$$  \hspace{1cm} (26)

Using the continuity relation (16) we obtain

$$dE = (n-1)\Omega_{n-1}^{n-2} \rho d\tilde{r}_A - (n-1)H(\rho + p)\Omega_{n-1}^{n-1} dt.$$  \hspace{1cm} (27)

Substituting this relation into (25), and using the relation between temperature and the surface gravity, we get the first law of thermodynamics on the apparent horizon

$$dE = T dS + W dV,$$  \hspace{1cm} (28)

where $W = (\rho - p)/2$ is the matter work density which is defined by $W = -\frac{1}{2} \epsilon_{\mu\nu} h_{\mu\nu}$ \hspace{1cm} (22), and the entropy of the apparent horizon on the brane is now given by

$$S = \int_0^{\tilde{r}_A} dS = \frac{(n-1)\Omega_{n-1}}{2G_{n+1}} \int_0^{\tilde{r}_A} \tilde{r}_A^{n-2} d\tilde{r}_A = \frac{2\Omega_{n-1}^{n-1}}{4G_{n+1}}.$$  \hspace{1cm} (29)

Let us note that the entropy obeys the area formula of horizon in the bulk (the factor 2 comes from the $Z_2$ symmetry in the bulk). This is due to the fact that because of the absence of the negative
cosmological constant in the bulk, no localization of gravity happens on the brane. As a result, the gravity on the brane is still \((n + 1)\)-dimensional.

On the other hand, from the global point of view, the apparent horizon on the brane can be extended into the bulk. Since the gravity in the bulk is described by pure \((n + 1)\)-dimensional Einstein theory, then one can obtain the entropy of the apparent horizon by the area formula in the bulk. We directly calculate the area of the apparent horizon which extends into the bulk, and give the entropy expression of apparent horizon from the bulk geometry. Setting \(\Lambda_{n+1} = C = 0\) in Eq. \(\text{(12)}\), this equation reduces to

\[A'^2 - \alpha^2(t) - k = 0,\]  

with the solution

\[A(t, y) = a(t)(1 - \frac{|y|}{\tilde{r}_A}) = a(t)f(\tilde{r}_A, y),\]  

where we have used \(\alpha(t) = \dot{A}/N = \dot{a}\), since \(N(t, 0) = 1\). The function \(f(\tilde{r}_A, y)\) has a positive root at \(y_0 = \tilde{r}_A\). Then, the area of the apparent horizon in the \((n + 1)\)-dimensional bulk (assuming \(\mathbb{Z}_2\) symmetry) can be written as (see [11] for details)

\[A = 2 \times (n - 1)\Omega_{n-1}\tilde{r}_A^{n-2} \int_0^{y_0} f^{n-2}(\tilde{r}_A, y)dy = 2\Omega_{n-1}\tilde{r}_A^{n-1}.\]  

(32)

According to the \((n + 1)\)-dimensional area formula, we obtain the entropy in the bulk

\[S = \frac{A}{4G_{n+1}} = \frac{2\Omega_{n-1}\tilde{r}_A^{n-1}}{4G_{n+1}}.\]  

(33)

This expression for the entropy is exactly the same as the entropy expression \(\text{(29)}\) which we have derived at the apparent horizon on the brane from the first law of thermodynamics.

**B. Brane embedded in AdS bulk**

In subsection III A, we have assumed that the bulk cosmological constant is absent. Here we leave that assumption, and further we suppose that \(\Lambda_{n+1} < 0\). Using Eq. \(\text{(3)}\) the Friedmann equation \(\text{(21)}\) can be written as

\[\sqrt{H^2 + \frac{k}{a^2} + \frac{1}{\ell^2}} = \frac{4\pi G_{n+1}}{n - 1}\rho.\]  

(34)

In terms of the apparent horizon radius we have

\[\rho = \frac{n - 1}{4\pi G_{n+1}} \sqrt{\frac{1}{\tilde{r}_A^2} + \frac{1}{\ell^2}},\]  

(35)
Taking differential form of the equation (35) and using the continuity equation (16), one gets the differential form of the Friedmann equation on the brane

$$H(\rho + p)dt = \frac{\ell}{4\pi G_{n+1}r_A^2} \frac{d\tilde{r}_A}{\sqrt{\tilde{r}_A^2 + \ell^2}}.$$  (36)

Again, multiplying both sides of equation (36) by a factor \((n - 1)\Omega_{n-1}\tilde{r}_A^{n-1} (1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})\), and using Eqs. (20) and (27), after some simplifications one can rewrite this equation in the form

$$dE - WdV = \kappa \frac{(n - 1)\ell \Omega_{n-1}\tilde{r}_A^{n-2}}{2G_{n+1}} \frac{d\tilde{r}_A}{\sqrt{\tilde{r}_A^2 + \ell^2}}.$$  (37)

This expression is nothing, but the first law of thermodynamics at the apparent horizon on the brane, namely \(dE = TdS + WdV\). We can define the entropy associated with the apparent horizon on the brane as

$$S = \frac{(n - 1)\Omega_{n-1}}{2G_{n+1}} \int_0^{\tilde{r}_A} \tilde{r}_A^{n-2} \frac{d\tilde{r}_A}{\sqrt{\tilde{r}_A^2 + \ell^2}}.$$  (38)

Finally, the explicit form of the entropy at the apparent horizon can be obtained by integrating (38). The result is

$$S = \frac{2\Omega_{n-1}\tilde{r}_A^{n-1}}{4G_{n+1}} \times 2F_1 \left( \frac{n - 1}{2}, \frac{1}{2}, \frac{n + 1}{2}, -\frac{\tilde{r}_A^2}{\ell^2} \right),$$  (39)

where \(2F_1(a, b, c, z)\) is a hypergeometric function. This expression for the entropy is in complete agreement with the entropy expression obtained in [11] by using the method of unified first law and Clausius relation. We stress here that, in order to obtain an entropy expression on the brane one does not need to use the unified first law and Clausius relation. One can directly write down the Friedmann equation in the form of first law and extract the entropy expression. Also, it is easy to check that if one writes the Friedmann equation in RS II model in the form of [11] \((\rho = \rho_b + \lambda)\), then following the above method one can rewrite the Friedmann equation directly in the form of the first law, with entropy expression given by (39). It is worth noticing when \(\tilde{r}_A \ll \ell\), which physically means that the size of the extra dimension is very large if compared with the apparent horizon radius, one recovers the \((n + 1)\)-dimensional area formula for the entropy on the brane \(S = 2\Omega_{n-1}\tilde{r}_A^{n-1}/4G_{n+1}\). The factor 2 comes from the \(\mathbb{Z}_2\) symmetry in the bulk. This is an expected result since in this regime we have a quasi- Minkowski bulk and we have shown in the previous subsection that for a RS II brane embedded in the Minkowski bulk, the entropy on the brane follows the \((n + 1)\)-dimensional area formula in the bulk. One can also extend the apparent horizon on the brane into the bulk and determine the apparent horizon entropy by using
the \((n + 1)\)-dimensional area formula in the bulk. To do this, we put \(C = 0\) in Eq. (12) and use Eq. (3), so

\[ A'^2 - \frac{A^2}{\ell^2} - \alpha^2(t) - k = 0. \]  

(40)

This equation has a solution of the form (see [17] for details)

\[ A(t, y) = a(t) \left( -\frac{1}{2} \frac{\ell^2}{r_A^2} + \left( 1 + \frac{1}{2} \frac{\ell^2}{r_A^2} \right) \cosh \left( \frac{2y}{\ell} \right) \right) - \sqrt{1 + \frac{\ell^2}{r_A^2} \sinh \left( \frac{2|y|}{\ell} \right)} \right)^{\frac{1}{2}}. \]

(41)

The function \(f(r_A, y)\) has a positive root at \(y_0 = \ell \sinh^{-1} \left( \frac{r_A}{\ell} \right) \). Then, the area of the apparent horizon in the \((n + 1)\)-dimensional bulk (assuming \(Z_2\) symmetry) can be written as

\[ A = 2 \times (n - 1) \Omega_{n-1} r_A^{n-2} \int_{0}^{y_0} f^{n-2}(r_A, y) dy \]

\[ = 2 \times \Omega_{n-1} r_A^{n-1} \times 2F_1 \left( \frac{n - 1}{2}, 1, \frac{n + 1}{2}, -\frac{r_A^2}{\ell^2} \right). \]

(42)

According to the \((n + 1)\)-dimensional area formula, we obtain the entropy in the bulk

\[ S = \frac{A}{4G_{n+1}} = \frac{2\Omega_{n-1} r_A^{n-1}}{4G_{n+1}} \times 2F_1 \left( \frac{n - 1}{2}, 1, \frac{n + 1}{2}, -\frac{r_A^2}{\ell^2} \right). \]

(43)

This expression for the entropy is exactly the same as the entropy expression (39) which we have derived at the apparent horizon on the brane from the first law of thermodynamics. Indeed, we have shown that for RS II brane world embedded in \((n + 1)\)-dimensional (Minkowski) AdS bulk, the entropy in the bulk is exactly the same as the entropy expression associated with apparent horizon on the brane. This is in agreement with the arguments in [24].

IV. THERMODYNAMICS OF APPARENT HORIZON IN DGP MODEL

In the previous section, we have studied thermodynamical behavior of Friedmann equation at apparent horizon for RS II brane world embedded in \((n + 1)\)-dimensional Minkowski and AdS bulks and have showed that the Friedmann equation can be written directly in the form of first law. We have found an explicit expression for the entropy of the apparent horizon on the brane. In this section, we are going to extend the discussion to the case in which the intrinsic curvature term of the brane is included in the action, namely DGP brane world. The generalized Friedmann equation for DGP model is given in Eq. (15). Again, we are interested in studying DGP brane world embedded in the Minkowski and AdS bulks, so \(C = 0\).
A. Brane embedded in Minkowski bulk

In the Minkowski bulk, \( \Lambda_{n+1} = 0 \), and the Friedmann equation (15) reduces to the form

\[
\sqrt{H^2 + \frac{k}{a^2}} = -\frac{\kappa_{n+1}^2}{4\kappa_n^2} (n-2)(H^2 + \frac{k}{a^2}) + \frac{\kappa_{n+1}^2}{2(n-1)} \rho. \tag{44}
\]

In terms of the apparent horizon radius, we can rewrite this equation in the form

\[
\rho = \frac{(n-1)(n-2)}{2\kappa_n^2} \frac{1}{\tilde{r}_A^2} + \frac{2(n-1)}{\kappa_{n+1}^2} \frac{1}{\tilde{r}_A}. \tag{45}
\]

Taking the differential form of the equation (45) and using the continuity equation (16), one gets the differential form of the Friedmann equation on the brane

\[
H(\rho + p) dt = \frac{n-2}{8\pi G_n} \frac{d\tilde{r}_A}{\tilde{r}_A^3} + \frac{1}{4\pi G_{n+1}} \frac{d\tilde{r}_A}{\tilde{r}_A^2}, \tag{46}
\]

where we have used Eq. (3). Now, we multiply both sides of the equation (46) by a factor \((n-1)\Omega_{n-1} \tilde{r}_A^{n-1} \left(1 - \frac{\tilde{r}_A}{2M_A}\right)\), and use Eqs. (20) and (27), then we can rewrite this equation in the form of the first law

\[
dE - W dV = \frac{\kappa}{2\pi} (n-1)\Omega_{n-1} \left(\frac{(n-2)\tilde{r}_A^{n-3}}{4G_n} + \frac{\tilde{r}_A^{n-2}}{2G_{n+1}}\right) d\tilde{r}_A = T dS, \tag{47}
\]

where the entropy can be given by

\[
S = (n-1)\Omega_{n-1} \int_0^{\tilde{r}_A} \left(\frac{(n-2)\tilde{r}_A^{n-3}}{4G_n} + \frac{\tilde{r}_A^{n-2}}{2G_{n+1}}\right) d\tilde{r}_A = (n-1)\Omega_{n-1} \tilde{r}_A^{n-2} \frac{2\Omega_{n-1} \tilde{r}_A^{n-1}}{4G_{n+1}} = S_n + S_{n+1}. \tag{48}
\]

It is interesting to note that in this case the entropy can be regarded as a sum of two area formulas; one (the first term) corresponds to the gravity on the brane and the other (the second term) to the gravity in the bulk. This indeed reflects the fact that there are two gravity terms in the action of DGP model.

B. Brane embedded in AdS bulk

For the AdS bulk, \( \Lambda_{n+1} < 0 \), we can write the Friedmann equation (15) in the form

\[
\sqrt{H^2 + \frac{k}{a^2} + \frac{1}{\ell^2}} = -\frac{\kappa_{n+1}^2}{4\kappa_n^2} (n-2)(H^2 + \frac{k}{a^2}) + \frac{\kappa_{n+1}^2}{2(n-1)} \rho, \tag{49}
\]
where we have used Eq. \ref{eq:apparent_horizon_radius}. In terms of the apparent horizon radius, this equation can be rewritten as

$$\rho = \frac{(n-1)(n-2)}{2 \kappa_{n}^2} \frac{1}{r_{A}^2} + \frac{2(n-1)}{\kappa_{n+1}^2} \frac{1}{r_{A}^2 + \ell^2}. \quad (50)$$

If one takes the differential form of the equation (50), after using Eqs. \ref{eq:apparent_horizon_radius} and \ref{eq:apparent_horizon_radius_bulk}, one gets the differential form of the Friedmann equation on the brane

$$H(\rho + p)dt = \frac{n - 2}{8 \pi G_{n}} \frac{d\tilde{r}_{A}}{\tilde{r}_{A}^3} + \frac{\ell}{4 \pi G_{n+1}} \frac{d\tilde{r}_{A}}{\tilde{r}_{A}^2} \frac{1}{\sqrt{\tilde{r}_{A}^2 + \ell^2}}. \quad (51)$$

Again, multiplying both sides of the equation (51) by a factor $(n-1)\Omega_{n-1}\tilde{r}_{A}^{n-1} \left(1 - \frac{\tilde{r}_{A}^2}{\kappa_{n+1}}\right)$, and using Eqs. \ref{eq:apparent_horizon_area} and \ref{eq:apparent_horizon_volume}, one gets

$$dE - W dV = \frac{\kappa}{2\pi} (n-1)\Omega_{n-1} \left(\frac{(n-2)\tilde{r}_{A}^{n-3}}{4 G_{n}} + \frac{\ell}{2 G_{n+1}} \frac{\tilde{r}_{A}^{n-2}}{\sqrt{\tilde{r}_{A}^2 + \ell^2}}\right) d\tilde{r}_{A}. \quad (52)$$

One can immediately see that this equation has the form of the first law $dE = T dS + W dV$, if one writes entropy associated with the apparent horizon on the brane as

$$S = (n-1)\Omega_{n-1} \int_{0}^{\tilde{r}_{A}} \left(\frac{(n-2)\tilde{r}_{A}^{n-3}}{4 G_{n}} + \frac{\ell}{2 G_{n+1}} \frac{\tilde{r}_{A}^{n-2}}{\sqrt{\tilde{r}_{A}^2 + \ell^2}}\right) d\tilde{r}_{A}. \quad (53)$$

Integrating this equation one gets the explicit form for the entropy at the apparent horizon

$$S = \frac{(n-1)\Omega_{n-1}\tilde{r}_{A}^{n-2}}{4 G_{n}} + \frac{2\Omega_{n-1}\tilde{r}_{A}^{n-1}}{4 G_{n+1}} \times {}_{2}\text{F}_{1}\left(\frac{n-1}{2}, \frac{1}{2}, \frac{n+1}{2}, -\frac{\tilde{r}_{A}^2}{\ell^2}\right). \quad (54)$$

Again, we see that in the warped DGP brane model embedded in the AdS bulk, the entropy associated with the apparent horizon on the brane has two parts, $S = S_{n} + S_{n+1}$. The first part $S_{n}$, which follows the $n$-dimensional area law on the brane and the second part $S_{n+1}$ which is the same as the entropy expression obtained by extension the apparent horizon into the bulk (see Eq. \ref{eq:apparent_horizon_area_bulk}) and therefore obeys the $(n+1)$-dimensional area law in the bulk.

\section{V. CONCLUSION}

To summarize, we have showed that the Friedmann equations on the $(n-1)$-dimensional brane embedded in an $(n+1)$-dimensional spacetime can be written directly in the form of the first law of thermodynamics, $dE = T dS + W dV$. Note that here $E$ is not the Misner-Sharp energy, but the matter energy $\rho V$ inside the apparent horizon and they are equal only in Einstein gravity \ref{eq:einstein_gravity}.\ref{eq:einstein_gravity}.\ref{eq:einstein_gravity}.
This procedure leads to extract an expression for the entropy at the apparent horizon on the brane, which is useful in studying the thermodynamical properties of the black hole horizon on the brane. We have discussed several cases including whether there is or not a cosmological constant in the bulk and whether there is or not an intrinsic curvature term on the brane. Interestingly enough, we have noted that when the cosmological constant vanishes in the bulk and the intrinsic curvature term is absent, the entropy of apparent horizon on the brane obeys the area formula (29) in the bulk. This is actually expected since the brane looks like a domain wall moving in a Minkowski spacetime; no localization of gravity happens in this case, unlike the case of RS II model, where the localization of gravity occurs due to the negative cosmological constant in the bulk. Another interesting point we found in this paper is that when the intrinsic curvature of the brane is added to the action, the entropy expression of the apparent horizon will include a term which satisfies the area formula on the brane. This can also be understood because the Einstein-Hilbert term on the brane contributes the area term.

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