Stochastic analysis & discrete quantum systems

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Abstract
We explore the connections between the theories of stochastic analysis and discrete quantum mechanical systems. Naturally these connections include the Feynman-Kac formula, and the Cameron-Martin-Girsanov theorem. More precisely, the notion of the quantum canonical transformation is employed for computing the time propagator, in the case of generic dynamical diffusion coefficients. Explicit computation of the path integral leads to a universal expression for the associated measure regardless of the form of the diffusion coefficient and the drift. This computation also reveals that the drift plays the role of a super potential in the usual super-symmetric quantum mechanics sense. Some simple illustrative examples such as the Ornstein-Uhlenbeck process and the multidimensional Black-Scholes model are also discussed. Basic examples of quantum integrable systems such as the quantum discrete non-linear hierarchy (DNLS) and the XXZ spin chain are presented providing specific connections between quantum (integrable) systems and stochastic differential equations (SDEs). The continuum limits of the SDEs for the first two members of the NLS hierarchy turn out to be the stochastic transport and the stochastic heat equations respectively. The quantum Darboux matrix for the discrete NLS is also computed as a defect matrix and the relevant SDEs are derived.

1 Introduction

Let $\hat{L}_0$ be a generic second order differential operator, and suppose $f(x,t)$ satisfies the time evolution equation:

$$\partial_t f(x,t) = \hat{L} f(x,t) = \left(\hat{L}_0 + u(x)\right) f(x,t),$$

and

$$\hat{L}_0 = \frac{1}{2} \sum_{i,j=1}^{M} g_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{M} b_j(x) \frac{\partial}{\partial x_j}, \quad g(x) = \sigma(x) \sigma^T(x)$$

(1.1)

where $\sigma(x)$ is a row vector.
where the diffusion matrix \( g(x) \) and the matrix \( \sigma(x) \) are in general dynamical (depending on the fields \( x_j \)) \( M \times M \) matrices, while \( x \) and the drift \( b(x) \) are \( M \) vector fields with components \( x_j, b_j \) respectively, and \( T \) denotes usual transposition. Within the quantum mechanics framework the \( \hat{L} \) operator may be seen as the Hamiltonian of a system of \( M \)-interacting particles and the time evolution problem \((1.2)\) can be thought of as a generalized imaginary time Schrödinger’s equation. Feynman’s path integral is a solution of \((1.2)\) that can be explicitly calculated via a time discretization scheme, and can be physically interpreted as the probability of the system to progress from an initial state configuration \( x_0 \) at time \( t_0 \) to a final configuration \( x_f \) at time \( t_f \). In the statistical physics language the imaginary time propagator provides the partition function of a given statistical system at temperature \( T \sim \frac{1}{t} \) (see for instance [11]).

In the quantum mechanical set up the time evolution problem \((1.2)\) is the commencing point, while from the stochastic analysis perspective the key object is a given stochastic differential equation (SDE) of the form

\[
dx = b(x_t)dt + \sigma(x_t)dw_t,
\]

which yields the generator \( \hat{L}_0 \) of the stochastic process and the corresponding time evolution problem via Itô’s formula. The SDE \((1.3)\) a priori determines the probabilistic measure in the Feynman-Kac formula, which in turn provides the associated conditional probability. Recall, \( w \) is also an \( M \)-vector with components \( w_j \) i.e. \( M \)-independent Wiener processes are considered: \( dw_i dw_j = \delta_{ij}dt \).

It would be instructive to briefly recall some background theory necessary here. Let us first review Itô’s formula; consider a function \( f(x, t) \), where \( x \) satisfies \((1.3)\) then

\[
df(x, t) = \frac{\partial f}{\partial t} dt + \sum_j \frac{\partial f}{\partial x_j} dx_j + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j.
\]

Taking into consideration that \( dx_j \) satisfy the SDE \((1.3)\), and \( dw_i dw_j = \delta_{ij}dt \), the expression \((1.4)\) becomes

\[
df(x, t) = \left( \frac{\partial}{\partial t} + \hat{L}_0 \right) f(x, t) dt + \sum_{i,j} \sigma_{ij}(x) \frac{\partial f(x, t)}{\partial x_i} dw_j
\]

yielding the generator \( \hat{L}_0 \). For a more detailed discussion on stochastic differential equations and in particular on Itô’s lemma and Itô’s calculus, Feynman-Kac formula, and Wiener processes we refer the interested reader to [2]-[7]. It is also useful to introduce the adjoint operator defined for any suitable function \( f(x, t) \):

\[
\hat{L}^*_0 f(x, t) = \frac{1}{2} \sum_{i,j=1}^M \frac{\partial^2}{\partial x_i \partial x_j} \left( g_{ij}(x) f(x, t) \right) - \sum_{j=1}^M \frac{\partial}{\partial x_j} \left( b_j(x) f(x, t) \right).
\]

The time evolution equation

\[
\partial_{t_1} f(x, t_1) = \hat{L}^*_0 f(x, t_1)
\]

is known as the Fokker-Plank or Kolmogorov forward equation \( t_1 \geq t_2 \), with known initial condition \( f(x, t_2) = f_{t_2}(x) \). The Kolmogorov backward equations \( t_2 \leq t_1 \), is given by

\[
- \partial_{t_2} f(x, t_2) = \hat{L}_0 f(x, t_2)
\]
with known final condition \( f(x, t_1) = f_{t_1}(x) \).

Formulating the path integral solution in the case when \( \hat{L} \) is a generic operator with non-constant (dynamical) diffusion coefficients represents a particular difficulty. More precisely, one has to deal with the situation of a non-Gaussian measure when computing the path integral in the standard way. To circumvent this complication we employ the notion of quantum canonical transformation and reduce the operator \( \hat{L} \) to a much simpler form with constant diffusion coefficients, but with an induced/effective drift. Moreover, by considering both \( \hat{L} \) and a “complementary” operator \( \hat{L}^s \) – it will be defined later in the text – we show that the drift plays the role of a super-potential, whereas \( \hat{L}, \hat{L}^s \) may be though of as “super-partners” in the typical super-symmetric quantum mechanics sense (see also e.g. \[8, 9\] and references therein). Indeed, (1.2) may be seen as a generalized imaginary time Schrödinger equation,

\[
\hat{L} = \sum_{j=1}^{M} D_{y_j}^2 + V(y)
\]

where \( D_{y_j} \) is a covariant derivative and \( V(y) \) is some induced (effective) potential. At the level of the associated SDEs the existence of two equivalent SDEs corresponding to \( \hat{L} \) in (1.2) and its simplified version with constant diffusion coefficient respectively is shown. Then Feynman’s path integral is evaluated for the reduced \( \hat{L}, \hat{L}^s \) operators and the findings obtained via the quantum canonical transform are confirmed and direct connections with Girsanov’s theorem [10] are made. The use of a precise time discretization scheme together with scaling arguments are essential for these connections.

We also compute explicitly the generic path integral with dynamical diffusion coefficients by \textit{a priori} requiring that the fields involved satisfy the underlying SDEs. This key assumption leads to a straightforward computation of the corresponding probabilistic measure, which turns out to be an infinite product of Gaussians. Hence the canonical transform implemented at the level of PDEs is also confirmed at the microscopic level of the path integral. It is worth emphasizing that in this framework solutions of the underlying SDEs are required in order to compute time expectation values (see also [11]). This formulation can be seen as an alternative to the customary Lagrangian formulation and perturbative methods used in quantum and statistical theories, or vice versa perturbative techniques can be employed for computing expectation values or solution of SDEs depending on the particular problem at hand.

Links between SDEs, quantum integrable systems and the Darboux-Bäcklund transformation [12] [13] [14] are also explored. To illustrate these associations between discrete quantum systems and SDEs we discuss typical exactly solvable discrete quantum system, such as the Discrete non-linear Schrödinger hierarchy, in particular the first two non-trivial members i.e. the the Discrete-self-trapping (DST) model and the discrete NLS (DNLS) model [15] [16], as well as the Heiseberg (XXZ) quantum spin chain [17] [18] [19]. Interesting connections between the DST model and the Toda chain are shown, and the SDEs associated to the DST model are solved using integrator factor techniques. We also consider both DST and DNLS models at the continuum limit, and we explicitly show that the associated SDEs turn to well known solvable SPDEs i.e. the stochastic transport and the stochastic heat equation respectively. The stochastic heat equation in particular can be further mapped
to the stochastic viscous Burgers equation (see for instance [20] and references therein). Moreover, the DST model in the presence of local defects is studied, the quantum Darboux-Bäcklund relations are derived for the defect and the corresponding SDEs are also identified. The algebraic picture is also engaged leading to a modified set of SDEs.

2 The quantum canonical transformation

The main aim in this section is to turn the dynamical diffusion matrix in (1.2) into identity. Indeed, we will show in what follows that the general \( \hat{L} \) operator can be brought into the less involved form:

\[
\hat{L} = \frac{1}{2} \sum_{j=1}^{M} \frac{\partial^2}{\partial y_j^2} + \sum_{j=1}^{M} \tilde{b}_j(y) \frac{\partial}{\partial y_j} + u(y)
\]  

(2.1)

with an induced drift \( \tilde{b}(y) \). This can be achieved via a simple change of the parameters \( x_j \), which in the Riemannian geometry sense is nothing but a change of frame, i.e. from curvilinear with a metric \( g \) to Euclidean. Indeed, let us introduce a new set of parameters \( y_j \) such that:

\[
dy_i = \sum_j \sigma^{-1}_{ij}(x) \, dx_j, \quad \det \sigma \neq 0,
\]  

(2.2)

then \( \hat{L} \) can be expressed in the form (2.1), and the induced drift components are given as

\[
\tilde{b}_k(y) = \sum_j \sigma^{-1}_{kj}(y)b_j(y) + \frac{1}{2} \sum_{j,l} \sigma_{jl}(y)\partial_{y_i} \sigma^{-1}_{kj}(y).
\]  

(2.3)

Bearing also in mind that \( \sum_j \sigma_{ij} \sigma^{-1}_{kj} = \delta_{kl} \), we can write in the compact vector/matrix notation:

\[
\tilde{b}(y) = \sigma^{-1}(y) \left( b(y) - \frac{1}{2} \left( \nabla_y \sigma^T(y) \right)^T \right), \quad \nabla_y = \left( \partial_{y_1}, \ldots, \partial_{y_M} \right)
\]  

(2.4)

where one first solves for \( x = x(y) \) via (2.2).

The latter statement can be seen in a more general algebraic (Hamiltonian) frame, which is more relevant for our purposes here, especially when considering the \( M \)-dimensional case, \( M \to \infty \) i.e. in a 1+1 field theoretical setting. Let \( X_j, P_j \) be elements of the Heisenberg-Weyl algebra:

\[
[X_i, P_j] = i \delta_{ij}.
\]  

(2.5)

where \( i = \sqrt{-1} \). The contribution of the commutator above gives rise to the second term in the induced drift (2.3), and this is clearly a purely “quantum” effect. We shall be able in fact to reproduce this quantum effect via the computation of Feynman’s path integral, and also we will be able to extract it at a fundamental SDE level. In the generic algebraic framework \( \hat{L} \) is then expressed as:

\[
\hat{L}(X, P) = -\frac{1}{2} \sum_{i,j} g_{ij}(X) P_i P_j + i \sum_j b_j(X) P_j + u(X).
\]
The elements $X$, $P$ are $M$ column vectors with components $X_j$, $P_j$, which in our setting are represented as:

$$X_j \mapsto x_j, \quad P_j \mapsto -i\partial x_j.$$  

Now consider the following quantum canonical transformation $\hat{L}(X, P) \mapsto \hat{L}(Y, P)$:

$$Y_i = \sum_j \int X_j^{-1}(X')\, dX'_j, \quad P_i = \sum_j \sigma_{ij}(X)\, P_j \quad (2.6)$$

where in the integral above and the derivative in (2.5) $X_j$ are formally treated as parameters. After the canonical transformation (2.6) is implemented the operator $\hat{L}$ reads as

$$\hat{L}(Y, P) = -\frac{1}{2} \sum_j P_j^2 + i \sum_j \tilde{b}_j(Y)\, P_j + u(Y)$$

and naturally $Y_j$, $P_j$ are also elements of the Heisenberg-Weyl algebra (2.5) represented as:

$$Y_j \mapsto y_j, \quad P_j \mapsto -i\partial y_j \quad (2.7)$$

This procedure is analogous to the quantum Darboux-Bäcklund transformation studied in [14]; similar results are presented in [12, 13], where the so-called Baxter’s $Q$-operator [21] is used as the generating function of the quantum canonical transformation.

We have established via the quantum canonical transform that the general diffusion matrix can turn to the identity. Let us now consider both $L$ and $\hat{L}^*$, not to be confused with the adjoint operator $\hat{L}^\dagger$. The two operators coincide only when a purely imaginary drift $\tilde{b}$ is considered.

**Remark 1: the drift as super-potential**

We introduce the following compact notation: $\forall F$, define $F^D \in \{F, F^*\}$. We can bring the $\hat{L}^D$ operators to an even more familiar form by setting: $\hat{P} = P - i\tilde{b}(Y)$, where $Y_j$, $P_j$ are also elements of the Heisenberg-Weyl algebra (2.5). Then

$$\hat{L}^D = \frac{1}{2} \hat{P}^T \hat{P} + V^D(Y)$$

$$V^D(Y) = -\frac{1}{2} \tilde{b}_j^T(Y)\tilde{b}(Y) - \frac{1}{2} \nabla_Y \tilde{b}(Y) + u^D(Y) \quad (2.9)$$

$$u^s(Y) = u(Y) + \nabla_Y \tilde{b}(Y).$$

where $V^D$ are effective potentials, produced exclusively by the drift. In terms of differential operators then one obtains via:

$$\hat{L}^D = \frac{1}{2} \sum_j D^2_{y_j} + V^D(y) \quad (2.10)$$

where $D_{y_j} = \partial_{y_j} + \tilde{b}_j(y)$ is the covariant derivative. Interestingly $\tilde{b}$ turns out to be the super-potential that produces $V$, $V^*$ and satisfies the familiar Riccati equations (2.9).

The statements at the algebraic level will be explicitly confirmed via the evaluation of the path integrals associated to both operators $\hat{L}$, $\hat{L}^*$.
Remark 2: Riccati equation & Girsanov’s theorem
Note that the Riccati equation (2.9) emerging from the identification of the effective potential due to the presence of the drift, -and hence the covariant derivative (2.10)- is essentially the origin of Girsanov’s theorem on the change of probabilistic measure (we refer the interested reader to [2]-[7]) from the PDE point of view. This will be even more transparent in the next section from the explicit path integral computation.

2.1 Equivalence of SDEs
The generator $\hat{L}_0$ (1.2) as already discussed can be obtained via Itô’s lemma from the SDE (1.3), while the operator in the simpler equivalent form established previously:

$$\hat{L}_0 = \frac{1}{2} \sum_{j=1}^{M} \frac{\partial^2}{\partial y_j^2} + \sum_{j=1}^{M} \tilde{b}_j(y) \frac{\partial}{\partial y_j}$$

(2.11)
is obviously obtained via the following simpler SDE

$$dy_t = \tilde{b}(y_t)dt + dw_t,$$

(2.12)
where recall $\tilde{b}$ is defined in (2.3), (2.4).

The main aim now is to show that these two SDEs (1.3) and (2.12) are in fact equivalent. Indeed, let us start from (2.12), multiply both sides of the equation with $\sigma(y_t)$ and add Itô’s correction i.e.

$$\sigma(y_t)dy_t + \frac{1}{2}\sigma(y_t)dy_t = \sigma(y_t)\tilde{b}(y_t)dt + \frac{1}{2}\sigma(y_t)dy_t + \sigma(y_t)dw_t.$$  (2.13)

Recall the connection between Itô and Stratonovich calculus:

$$f(y_t) \circ dy_t = f(y_t)dy_t + \frac{1}{2}df(y_t)dy_t \quad \text{and} \quad dy_tdy_j = \delta_{ij}dt$$

where $f(y_t) \circ dy_t$ refers to Stratonovich calculus, and $f(y_t)dy_t$ to Itô. More comments on this relation will be given in the next subsection, where the discrete time scheme is discussed via the path integral computation. Recall also the definition of the modified drift $\tilde{b}$ (2.3):

$$\sigma(y_t)\tilde{b}(y_t)dt + \frac{1}{2}d\sigma(y_t)dy_t = b(y_t)dt,$$
then (2.13) becomes

$$\sigma(y_t) \circ dy_t = b(y_t)dt + \sigma(y_t)dw_t.$$  

We now set

$$dx_t = \sigma(y_t) \circ dy_t$$  (2.14)
and we immediately recover the original SDE

$$dx_t = b(x_t)dt + \sigma(x_t)dw_t.$$  

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after solving $y = y(x)$ through (2.14). Integration of (2.14) in the Stratonovich calculus frame – i.e. the usual calculus rules apply– yields
\[ x_i = \sum_j \int y_j \sigma_{ij}(y') \circ dy'_j, \]
and the latter as expected is nothing but the inverse of (2.9).

Naturally we could have started from (1.3) and end up to (2.12) via the reverse process. Indeed, we start from (1.3) we multiply the equation by $\sigma^{-1}$, add Itô’s correction $\frac{1}{2} \sigma^{-1}(x) dx$ at both sides of the equation, and use (2.14). Then setting $dy = \sigma^{-1}(x) \circ dx$ we end up to the desired equation (2.12). With this we conclude our proof on the equivalence of the two SDEs (1.3) and (2.12). Relevant results on the change of the diffusion matrix from the SDE point of view via the so called Lamperti transform are presented in [22], (regarding the Lamperti transform we refer the interested reader to [3, 4, 23]).

3 The path integral

3.1 Identity diffusion matrix

We have established in the previous section that if the generic dynamical diffusion matrix is invertible then $\hat{L}$ can be brought into a simple form with identity diffusion matrix via the quantum canonical transformation. We basically confirm the results of section 2 via the explicit computation of the relevant path integral using the semi-group property (see for instance [24]). We review in this subsection the computation of the path integral in the case when the diffusion matrix is constant, but in the presence of a general drift.

We explicitly compute in what follows the propagator associated to the operator $\hat{L}$ using the standard discrete time frame and make various interesting connections. More precisely: 1) We make a direct connection with the discrete time version of Cameron-Martin-Girsanov theorem on the change of the probabilistic measure. 2) We show that the path integral can be alternatively computed by assuming the existence of underlying discrete time SDEs. This fundamental assumption leads to a straightforward computation of the associated measure, which turns out to be a Gaussian. This result is generalized in the next subsection where dynamical diffusion coefficients are also considered. 3) We show that the drift plays the role of the super-potential with $\hat{L}$, $\hat{L}^\dagger$ being super-partners in the usual super-symmetric quantum mechanics sense, confirming the findings of the previous section.

Our starting point is the time evolution equation (1.3), (1.6), (2.1):
\[ \partial_t f(y, t) = \hat{L}^\dagger f(y, t), \]
we then explicitly compute the propagator $K(y_f, y_i|t)$:
\[ f(y, t) = \int \prod_{j=1}^M dy'_j K(y, y'|t, t') f(y', t') \]
\[ = \int \prod_{n=1}^N \prod_{j=1}^M dy_{jn} \prod_{n=1}^N K(y_{n+1}, y_n|t_{n+1}, t_n) f(y_1, t_1). \] (3.1)
In what follows we shall reproduce Itō’s correction as the result of quantum/statistical fluctuations—i.e. use of suitable scaling arguments—and also derive the corresponding SDEs via the exact computation of the propagator $K$, assuming no a priori knowledge of stochastic analysis. Within this frame the discrete time analogue of Girsanov’s theorem is a straightforward result.

Employing the standard time discretization scheme or the semi-group property, as shown above, (see also for instance [24, 25]) the path integral or time propagator can be expressed as

$$K(y_f, y_i | t) = \frac{1}{(2\pi)^{NM}} \int d\mathbf{y} d\mathbf{p} \exp \left[ -\frac{1}{2} \sum_{n=1}^{N} \sum_{j=1}^{M} p_{jn}^2 + i \sum_{n=1}^{N} \sum_{j=1}^{M} p_{jn} (\Delta y_{jn} - \delta \tilde{b}_{jn}(y)) + \delta \sum_{n=1}^{N} u_n(y) \right]$$

(3.2)

where we define

$$d\mathbf{y} = \prod_{n=2}^{N} \prod_{j=1}^{M} dy_{jn}, \quad d\mathbf{p} = \prod_{n=1}^{N} \prod_{j=1}^{M} dp_{jn},$$

where $\delta = t_{n+1} - t_n$ and with boundary conditions: $y_f = y_{N+1}$, $y_i = y_1$, $t_i = 0$, $t_f = t$; $p_{nj}$ are known as response variables. To obtain the latter formula we have inserted the unit $N$ times, $(\frac{1}{2\pi} \int d\mathbf{y} d\mathbf{p} e^{i\mathbf{p}(\mathbf{y} - \mathbf{a})} = 1)$, associated to each component $y_j$. After performing the Gaussian integrals with respect to the $p_{jn}$ parameters we conclude:

$$K(y_f, y_i | t) = \int d\mathbf{q} \exp \left[ -\sum_{j} \sum_{n} \frac{(\Delta y_{jn} - \delta \tilde{b}_{jn}(y))^2}{2\delta} + \delta \sum_{n} u_n(y) \right]$$

(3.3)

d$q = \frac{1}{(2\pi\delta)^{\frac{NM}{2}}} \prod_{n=2}^{N} \prod_{j=1}^{M} dy_{jn}$

where $f_n = f_n(y_n)$ and $\Delta y_{jn} = y_{jn+1} - y_{jn}$. At this stage we are dealing with an $M \times N$ space-time lattice; $n$: time indices, $j$: space indices.

We now expand the square in the exponential in (3.3), and consider the continuum time limit as follows:

$$\sum_{j} \sum_{n} b_{jn}(y) \Delta y_{jn} \to \int \tilde{b}^T(y_s) dy_s$$

$$\frac{\delta}{2} \sum_{j} \sum_{n} \tilde{b}_{jn}^2(y) \to \frac{1}{2} \int_0^t \tilde{b}^T(y_s) \tilde{b}(y_s) ds$$

$$\delta \sum_{n} u_n(y) \to \int_0^t u(y_s) ds,$$

where \( \int_0^t \tilde{b}^T(y_s) dy_s \) corresponds to an Itô type integral, due to the chosen discretization scheme above. We shall discuss the Itô-Stratonovich correspondence later in this subsection.
The propagator then takes the form

\[
K(y_f,0|t) = \int d\mathcal{M} \exp \left[ -\frac{1}{2} \int_0^t \tilde{b}^T(y_s)\tilde{b}(y_s)ds + \int_0^t \tilde{b}^T(y_s)du(y_s)ds \right]
\]

(3.4)

\[
= \int d\mathcal{P} \exp \left[ \int_0^t u(y_s)ds \right]
\]

(3.5)

where we define

\[
d\mathcal{P} = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{(2\pi \delta)^{MN}} \exp \left[ -\sum_{n=1}^N \sum_{j=1}^M \frac{(\Delta y_{jn})^2}{2\delta} \right] \prod_{n=1}^N \prod_{j=1}^M dy_{jn}.
\]

(3.6)

We can follow an alternative path to compute the propagator in a straightforward way by making explicit use of the corresponding SDEs. Recall expression (3.2) and assume that:

\[
\Delta y_n - \delta \tilde{b}_n(y) = \Delta w_n
\]

(3.7)

assuming also that \(w_n\) are independent of the vector fields \(y_n\), and also satisfy \(\Delta w_n, \Delta w_{jn} = \delta_{ij}\). The latter expression can be seen as the discrete time analogue of the underlying SDE. After performing the standard Gaussian integrals in (3.2) subject to (3.7) we conclude that

\[
K(y_f,y_i|t) = \int dm \exp \left[ \int_0^t u(y_s)ds \right],
\]

(3.8)

\[
dm = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{(2\pi \delta)^{MN}} \exp \left[ -\frac{1}{2\delta} \sum_{n=1}^N \Delta w_n^2 \right] \prod_{n=2}^N \prod_{j=1}^M dw_{jn}.
\]

The measure will be explicitly evaluated in the next subsection, where the more general of dynamical diffusion coefficients is discussed. Finally, expressions (3.4), (3.8) of conditional expected values, evaluated via two different approaches, provide essentially the Feynman-Kac formula, notice also that expressions (3.5) and (3.8) are the same. Explicit evaluation of the propagator via (3.4) naturally leads to the use of the Lagrangian formulation, which is usually implemented in quantum systems and quantum/statistical field theories. Evaluation of \(K(y,y'|t,t')\) on the other hand via (3.8) involves stochastic analysis methods and in particular the solution of the corresponding SDEs.

**Remark 3:** Radon-Nikodym derivative & Girsanov’s theorem

Note that the ratio (3.5), (3.6)

\[
\frac{d\mathcal{M}}{d\mathcal{P}} = \exp \left[ -\frac{1}{2} \int_0^t \tilde{b}^T(y_s)\tilde{b}(y_s)ds + \int_0^t \tilde{b}^T(y_s)du(y_s)ds \right]
\]

turns out to be the Radon-Nikodym derivative in analogy to Girsanov’s theorem [10]. Indeed, as already pointed out there is an one to one correspondence with the results of section 2 especially in relation to the notions of the effective potential \(V\) (2.10) and the covariant derivative (2.10).

**Remark 4:** Itô vs Stratonovich calculus
We shall now reproduce the Itô-Stratonovich correspondence via distinct discretization schemes. The main reason we address this issue here is because Stratonovich type integrals follow the usual calculus rules, and are the ones that are usually employed in the path integral formulation in quantum/statistical physics. Let us focus again on the square of the exponential (3.3) and in particular on the following term:

$$\sum_n \tilde{b}_{jn} \Delta y_{jn} = \sum_n \tilde{b}^+_{jn} \Delta y_{jn} - \frac{1}{2} \sum_n \Delta \tilde{b}_{jn} \Delta y_{jn}$$  (3.9)

where $\tilde{b}^+_{jn} = \frac{1}{2}(\tilde{b}_{jn+1} + \tilde{b}_{jn})$ and $\Delta \tilde{b}_{jn} = \tilde{b}_{jn+1} - \tilde{b}_{jn}$. We now make the fundamental assumption in line with the ideas of suitable scaling and stochastic calculus. Indeed, assume that $\Delta y$ contains statistical or quantum fluctuations of leading order $\sqrt{\delta}$:

$$\Delta y_{jn} \sim \delta_{ij} \delta.$$  

After taking into consideration the scaling argument given above we can take the continuum time limit in (3.9) and conclude

$$\int \tilde{b}^T(y_s)dy_s = \int \tilde{b}^T(y_s) \circ dy_s - \frac{1}{2} \int_0^t \nabla_y \tilde{b}(y_s)ds$$

where as mentioned $\int_0^t \tilde{b}^T(y_s)dy_s$ corresponds to an Itô type integral and $\int_0^t \tilde{b}^T(y_s) \circ dy_s$ corresponds to a Stratonovich type integral, for which the customary calculus rules apply.

- Remark 5: super-symmetry & drift

Having introduced the Itô-Stratonovich correspondence at the microscopic level it is worth pointing out that the propagator $K^s$ for the “super-partner” $L^s$ (2.8) can be also evaluated and one concludes that the difference $D = K^s - K$ (3.5) provides indeed the suitable expression in the context of super-symmetric quantum mechanics, related also to the index theorem (see for instance [8, 9] and references therein),

$$D = 2 \int d\mathcal{\tilde{M}} \exp \left[ \int_0^t \int_0^t \nabla_y \tilde{b}(y_s)ds \right].$$

This result is also in line with the findings of section 2.

3.2 Dynamical diffusion matrix

We examine the general case with dynamical diffusion matrix (1.1), (1.2), although we showed its equivalence with a case of identity diffusion matrix and modified drift via the quantum canonical transform. The reason we view this case separately is because, as already mentioned, the underlying SDEs turn out to play a crucial role when computing the path integral. Indeed, instead of using the customary perturbative methods of computing the
path integral one can alternatively exploit the solutions of the relevant SDEs (see also [11]). Specifically, our main aim in this subsection is the explicit computation of the propagator in the more general case of dynamical diffusion coefficients and drift. We compute the path integral using two different approaches as in the previous subsection: 1) We employ the underlying SDEs as a fundamental assumption naturally emerging after performing the relevant Gaussian integrals in the propagator. 2) We generalize the description that led to the discrete time version of Girsanov’s theorem described in the preceding subsection, in the case of dynamical diffusion coefficients.

Following the prescription of the preceding subsection we can compute the corresponding path integral for the general case [1,2] via [1,5], [1,6] and express it in the compact vector/matrix notation:

\[ K(x_f, x_i|t) = \frac{1}{(2\pi)^{NM}} \int dx \, dp \, \exp \left[ -\frac{\delta}{2} \sum_{n=1}^{N} p_n^T g_n(x)p_n + \sum_{n=1}^{N} p_n^T (\Delta x_n - \delta b_n(x)) + \delta \sum_{n=1}^{N} u_n(x) \right] \]

where \( \Delta x_n = x_{n+1} - x_n \) and recall the diffusion matrix is \( g_n = \sigma_n \sigma_n^T \).

We come now to our main assumption, let

\[ \Delta x_n - \delta b_n(x) = \sigma_n(x) \Delta w_n \]

assuming also that \( w_n \) are independent of the vector fields \( x_n \), and also satisfy \( \Delta w_n \Delta w_{jn} = \delta_{ij} \). After performing the standard Gaussian integral:

\[ \int dp \, \exp \left[ -\frac{\delta}{2} p_n^T g_n(x)p_n + ip_n^T \sigma_n(x) \Delta w_n \right] = (\det \sigma_n(x))^{-1} \left( \frac{2\pi}{\delta} \right)^{\frac{N}{2}} e^{-\frac{1}{2} \Delta w_n^T \Delta w_n} \]

we conclude that

\[ K(x_f, x_i|t) = \int dm \, e^{f_{n_0}^T u(x_s) ds}, \]

\[ dm = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{(2\pi \delta)^{\frac{NM}{2}}} \exp \left[ -\frac{1}{2\delta} \sum_{n=1}^{N} \Delta w_n^T \Delta w_n \right] \prod_{n=1}^{N} (\det \sigma_n(x))^{-1} \prod_{j=1}^{M} \prod_{n=2}^{N} dx_{jn}, \]

however: \( (\det \sigma_n(x))^{-1} \prod_{j=1}^{M} dx_{jn} = \prod_{j=1}^{M} dw_{jn} \), which is a typical change of the volume element (see also relevant change of variables discussed in section 2 equ. (2.2)). We can now explicitly evaluate the measure from the explicit expression just above, which in the continuum limit becomes \(^1\)

\[ dm = dw \, e^{-\frac{1}{4} f_{n_0}^T \Delta w_n^T \Delta w_n}, \]

\(^1\)When keeping \( N \) finite we have from [8,10]

\[ dm = (\det \sigma(x_1))^{-1} \frac{1}{(2\pi \delta)^{\frac{M}{2}}} \exp \left[ -\frac{1}{2\delta} \sum_{n=1}^{N} \Delta w_n^T \Delta w_n \right] \]

The factor in front of \( dm^\cdot \) is nothing but the infinitesimal kernel \( K(x_2, x_1|\delta \to 0) = (\det \sigma(x_1))^{-1} \delta(\Delta w_1) \) compatible with the initial conditions of \( K \). The pre-factor \( (\det \sigma(x_1))^{-1} \) is essentially absorbed in the measure via the change of the volume element \( (\det \sigma(x_n))^{-1} \prod_{j=1}^{M} dx_{jn} = \prod_{j=1}^{M} dw_{jn} \). Equation (3.11) is then expressed in the form

\[ f(x_f, t) = \int dm \, e^{f_{n_0}^T u(x_s) ds} f(x_0, t_0), \]

which is precisely the Feynman-Kac formula.
and although we use the rather abused notation $\dot{w}$ this does not affect the validity of the result.

To compute the measure we consider the following boundary conditions: $w(s = 0) = 0$, $w(s = t) = w_t$ as well as the following Fourier representation on $[0, t]$ for $w_s$, i.e. Wiener’s representation of the Brownian path:

$$w_s = \frac{f_0}{\sqrt{t}} s + \sqrt{\frac{2}{t}} \sum_{k > 0} f_k \sin(\omega_k s), \quad \omega_k = \frac{2\pi k}{t}.$$

(3.13)

$f_0 = \frac{w_t}{\sqrt{t}}$ and $f_k$, $k \in \{0, 1, \ldots\}$ are $M$ vectors with components $f_{kj}$, $j \in \{1, 2, \ldots, M\}$ being standard random variables. This leads to

$$dm = \frac{e^{-\frac{1}{2}w_t^T w_t}}{(2\pi)^{M/2}} \ dm_0$$

$$dm_0 = \prod_{k \geq 1} \prod_{j=1}^M \frac{df_{kj}}{\sqrt{2\pi}} \exp[-\frac{1}{2} \sum_{k \geq 1} \sum_{j} f_{kj}^2].$$

(3.14)

The measure naturally turns out to be an infinite product of Gaussians regardless of the specific forms of the diffusion coefficients and the drift, which is indeed an elegant result. However, the price one pays is that solutions of the SDEs, available in general via iteration techniques, are now required. In any case this scheme can be seen as an alternative to the established perturbative approach employed in quantum mechanical systems and quantum field theories (see also [11]), therefore algebraic and numerical techniques developed in solving SDEs will be of great relevance (see e.g. [5, 26] and references therein). The significant observation is the existence of the heat kernel in front of $dm_0$. This boundary term plays a crucial role when computing explicitly the propagator or in general when computing time expectation values as will be transparent later in the text. In the more standard case where $\hat{L}$ is expressed as $\frac{1}{2} \sum_i \partial_x^2 x_i + u(x)$ the underlying SDE is

$$d\dot{x}_t = d\dot{w}_t \Rightarrow x_s = x_0 + w_s.$$ 

It is clear that in more complicated cases, i.e. in the presence of drift the relation between $w_t$ and $x_t$ may become significantly involved depending of the specific form of the associated SDEs. A few simple illustrative examples are discussed in section 4.

Alternatively, the problem described above can be thought of as a typical optimal control problem (feedback problem). More precisely, the minimization of the action $\int_0^t ds \left(\frac{1}{2} \eta^T \eta_t - u(x_s)\right)$ should be required subject to the equation $\dot{x}_t = b(x_t) + \sigma(x_t) \eta_t$. Then linearization methods, i.e. suitable perturbations around the optimal solution, can be applied leading to certain Riccati type equations. This point of view is naturally closer to the more established semi-classical Lagrangian approach. We shall report on these approaches in more detail in a separate publication [28].

Let us now rederive the path integral in the case of dynamical diffusion matrices repeating the computations of the previous subsection, which led to the discrete time version of Girsanov’s theorem. We start from the general expression (3.10) and perform the Gauss integrals (set $B_n(x) = \Delta x_n - \delta b_n(x)$)

$$\int dp \exp \left[ -\frac{\delta}{2} p_T \eta_T \eta_n - dp_n \right] = (\det \sigma_n(x)^{-1}) \left( \frac{2\pi}{\delta} \right)^{M/2} e^{-\frac{1}{2} \sigma^{-1}_n(x)p_Tp_n + \frac{1}{2} \sigma_n(x)p_TB_n(x)}.$$
Let us now focus on
\[ \sigma_n^{-1}(x)B_n(x) = \sigma_n^{-1}(x)\Delta x_n - \delta \sigma_n^{-1}(x)b_n(x) \]
\[ = (\sigma_n^+)^{-1}(x)\Delta x_n - \frac{1}{2}\Delta \sigma_n^{-1}(x)\Delta x_n - \delta \sigma_n^{-1}(x)b_n(x),\]
and in accordance to our usual notation: \((\sigma_n^+)^{-1} = \frac{1}{2}(\sigma_n^{-1} + \sigma_n)\), \(\Delta \sigma_n^{-1} = \sigma_n^{-1} - \sigma_n^{-1}\).

We set
\[ \Delta y_n = (\sigma_n^+)^{-1}(x)\Delta x_n, \quad (3.15)\]
which is the discrete time analogue of definition (2.14). Using (3.15) we conclude:
\[ \sigma_n^{-1}(x)B_n(x) = \Delta y_n - \frac{1}{2}\Delta \sigma_n^{-1}(y)\sigma_n(y)\Delta y_n - \delta \sigma_n^{-1}(y)b_n(y) \]
\[ = \Delta y_n - \delta \tilde{b}_n(y) \]
after solving \(x_n = x_n(y_n)\) via (3.15) as in the continuous time case discussed in section 2.

The definition of \(\tilde{b}\) in the expression above is apparently the discrete time analogue of (2.3).

Bearing also in mind the volume element change \(\det(\sigma_n(x))^{-1} \prod_{j=1}^M dx_{jn} = \prod_{j=1}^M dy_{jn}\), we conclude that the propagator (3.10) coincides with expression (3.3), thus all computations of subsection 3.1 apply and the findings of section 2 are once more confirmed.

### 3.2.1 The case \(\det \sigma = 0\)

We shall briefly discuss in what follows the special situation where \(\det \sigma = 0\), in this case the \(\sigma\)-matrix has at least one zero eigenvalue. Let us consider the quite general scenario, where \(\sigma\) is given as
\[ \sigma(x) = \begin{pmatrix} \sigma(x)_{m \times m} & 0_{(M-m) \times m} \\ 0_{m \times (M-m)} & 0_{(M-m) \times (M-m)} \end{pmatrix}. \]

Then the corresponding SDEs are expressed as follows
\[ dx_{ti} = b_i(x_t)dt + \sum_{j=1}^m \sigma_{ij}i(x_t)d\omega_{ij}, \quad i \leq m \]
\[ dx_{ti} = b_i(x_t)dt, \quad i > m. \quad (3.16)\]

The associated generator \(\hat{L}_0\) is then given as
\[ \hat{L}_0 = \sum_{i,j=1}^m g_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^M b_i(x) \frac{\partial}{\partial x_i}, \quad (3.17)\]
g = \(\sigma \sigma^T\).

Following the findings of section 4 and assuming that \(\det \sigma \neq 0\), we consider the following simple change of variables:
\[ dy_{ti} = \sum_{i,j=1}^m \sigma_{ij}^{-1}(x_t)dx_{ij}, \quad i \leq m \]
\[ dy_{ti} = dx_{ti}, \quad i > m, \]
then the generator takes the simpler form
\[ \hat{L}_0 = \sum_{i=1}^{m} \frac{\partial^2}{\partial y_i^2} + \sum_{i=1}^{m} \tilde{b}_i(y) \frac{\partial}{\partial y_i} + \sum_{i=m+1}^{M} b_i^{-}(y) \frac{\partial}{\partial y_i} \]
and the SDE (3.16) becomes
\[ dy_{ti} = \tilde{b}_i(y_t) dt + dw_{tj}, \quad i \leq m \]
\[ dy_{ti} = b_i^{-}(y_t) dt, \quad i > m, \]
\( \tilde{b} \) is an \( m \)-vector defined as
\[ \tilde{b}(y_s) = \sigma^{-1}(y_s) \left( b(y_s) - \frac{1}{2} (\partial_{y}^{-} \sigma^T(y_s))^T \right), \quad \partial_{y}^{-} = \left( \partial_{y_1}, \ldots, \partial_{y_m} \right). \]
and \( b^{-} \) is an \( M - m \) vector with components \( b_i(y), \quad i \in \{ m + 1, \ldots, M \} \). Typical systems with such a behavior are systems in the presence of discontinuities/defects or interfaces. The DNLS/DST model in the presence of local defects is such a system and will be examined in the subsequent section.

4 Lagrangian versus stochastic formulation

In this section we compute the propagators for certain prototype models via the Lagrangian and stochastic approaches. More precisely, in the next subsection we evaluate Feynman’s path integral (3.4), for a system of \( M \)-damped harmonic oscillators with linear drift i.e. the case of \( M \)-dimensional Ornstein-Uhlenbeck process using the standard Lagrangian point of view. In subsection 4.2 we review the computation of the propagator for the 1-dimensional harmonic oscillator via the stochastic approach (see also [6]).

We also comment on generic processes associated to more complicated SDEs as well as on possible future directions combining the two approaches i.e. Lagrangian approach and perturbation theory versus stochastic methods and use of solutions of SDEs. Here we basically consider simple prototype models such as the harmonic oscillator and the Ornstein-Uhlenbeck process, in order to review how the two approaches work. However the goal is the study of more involved examples, where both perturbative/semi-classical methods and the stochastic approach can be tested and compared.

4.1 Lagrangian approach: multidimensional Ornstein-Uhlenbeck process

In this subsection we compute the propagator (3.4) for a system of \( M \)-damped oscillators using the standard for physicists formulation i.e. the Lagrangian description. The quantum Hamiltonian in this case is given by (2.1) with:
\[ \tilde{b}(y) = -\Theta y, \quad u(y) = \frac{1}{2} \nu^T \Theta^T \Theta y, \]
where $\Theta$, $\hat{\Theta}$ are constant $M \times M$ matrices. Notice in this case we deal with a non-self-adjoint operator. From the stochastic point of view this is a multidimensional Ornstein-Uhlenbeck process (see also expression (3.4)) with relevant SDEs given by:

$$dy_t = -\Theta y dt + dw_t. \quad (4.1)$$

We shall express the path integral (3.4) in the familiar for physicists form (we have suppressed the time subscript $s$ in this subsection for simplicity)

$$K(y_f, y_i | t) = \int dq \exp \left[ - \int_0^t L(y, \dot{y}) ds \right]. \quad (4.2)$$

We choose to consider $y_i = 0$ and the Lagrangian is given (3.5), (2.9):

$$L(y, \dot{y}) = \frac{1}{2} \dot{y}^T \dot{y} + \frac{1}{2} \Omega^2 y_i y_j + \sum_{i,j} \Theta_{ij} \dot{y}_i y_j - \frac{1}{2} \nabla_y b(y) - u(y) \quad (4.3)$$

where $\Omega^2 = \Theta^T \Theta - \hat{\Theta}^T \hat{\Theta}$ and is by construction symmetric. Notice that in the special case where $\Theta$ is a symmetric matrix the third term of the expression above can be expressed as a total derivative giving rise to a purely boundary term after integration.

Let us now follow the typical Lagrangian approach in computing the path integral (4.2) and focus on the case where $\Theta$ is symmetric. Let $y = z + w$, where $z$ is the deterministic contribution i.e. the solution of the classical equations of motion and $w$ is the random contribution (quantum fluctuation) then we obtain

$$\exp \left[ - \int_0^t L(y, \dot{y}) ds \right] = \exp \left[ - \int_0^t \left( \tilde{L}(z, \dot{z}) ds + \tilde{L}(w, \dot{w}) + F(z, \dot{z}; w, \dot{w}) \right) ds \right] e^{-\frac{1}{2} z^T \Theta z + \frac{1}{2} z^T \Theta^T \dot{z}} \quad (4.4)$$

where we define

$$\tilde{L}(z, \dot{z}) = \frac{1}{2} \sum_i \dot{z}_i^2 + \frac{1}{2} \sum_{i,j} \Omega^2_{ij} z_i z_j. \quad (4.5)$$

The classical equations of motion are obtained via the Euler-Lagrange equations:

$$\frac{\partial \tilde{L}}{\partial \dot{z}_j} = \frac{\partial}{\partial s} \left( \frac{\partial \tilde{L}}{\partial \dot{z}_j} \right) \Rightarrow \ddot{z} - \Omega^2 z = 0. \quad (4.6)$$

The general solution of (4.6) is given by

$$z(s) = \sinh (s \Omega) \sinh (t \Omega)^{-1} z_f \quad (4.7)$$
where recall we have considered $z(0) = 0$, $z(t) = z_f$.

The last term inside the integral in (4.4) is linear in $w$, and disappears via the classical equations of motion, whereas the classical Lagrangian will produce only boundary terms due to (4.6). The path integral is then expressed as

$$ K(y_f, 0|t) = e^{-\frac{1}{2}z_f^Tz_f - \frac{1}{2}z_f^T\Theta z_f + \Theta t} \int dwe^{-\int_0^t \tilde{L}(w, \dot{w})ds} \tag{4.8} $$

$$ dw = \lim_{\delta \to 0} \lim_{N \to \infty} \frac{1}{(2\pi \delta)^{N/2}} \prod_{n=2}^{N} \prod_{j=1}^{M} dw_{jn}. $$

Let us first compute the classical contribution of the path integral (4.8) using the solution (4.7) of the classical equations of motion. Indeed, substituting $z_f$, $\dot{z}_f$ via (4.7) we immediately obtain

$$ e^{-\frac{1}{2}z_f^Tz_f - \frac{1}{2}z_f^T\Theta z_f + \Theta t} = \exp \left[ -\text{tr} \left( \frac{z_f^Tz_f}{2t} \left( \frac{1}{2} \cosh (t\Omega) - t\Omega \sinh (t\Omega)^{-1} + t\Theta \right) \right) \right] e^{\Theta t}. \tag{4.9} $$

To compute the quantum contribution of the path integral (4.8) we consider a Fourier transform expansion for the random part of the fields (see also subsection 3.2)

$$ w_s = \sqrt{\frac{2}{t}} \sum_{k \geq 1} \frac{\sin (\omega_k s)}{\omega_k} f_k, \quad \omega_k = \frac{k\pi}{t} \tag{4.10} $$

$f_k$ are $M$-vector fields with components $f_{kj}$. After some straightforward substitutions via (4.10) we obtain

$$ \int dwe^{-\int_0^t \tilde{L}(w, \dot{w})ds} = \frac{1}{(2\pi t)^{\frac{M}{2}}} \int df \exp \left[ -\frac{1}{2} \sum_{k \geq 1} \frac{t^2}{k^2\pi^2}\left( I_M + \frac{t^2}{k^2\pi^2}\Omega^2 \right)f_k \right], \tag{4.11} $$

$$ = \frac{1}{(2\pi t)^{\frac{M}{2}}} \det \left( I_M + \frac{t^2}{k^2\pi^2}\Omega^2 \right) \left( \frac{t^2}{k^2\pi^2}\right)^{-\frac{M}{2}} \tag{4.12} $$

As already mentioned earlier in the text the appearance of $\dot{y}$ in the expression above although perhaps perplexing is nothing but a convenient choice of notation. In any case, we are able to obtain the correct result, having suitably regularized the coefficients of the Fourier transform (4.10).

After performing the Gaussian integrals in (4.11) we express the quantum contribution as

$$ \int dwe^{-\int_0^t \tilde{L}(w, \dot{w})ds} = \frac{1}{(2\pi t)^{\frac{M}{2}}} \prod_{k \geq 1} \det \left( I_m + \frac{t^2}{k^2\pi^2}\Omega^2 \right)^{-\frac{1}{2}}. \tag{4.12} $$

Recalling the infinite product identity

$$ \prod_{k \geq 1} \left( 1 + \frac{a^2}{k^2} \right)^{-1} = \frac{\pi a}{\sinh (\pi a)} $$
we eventually obtain
\[\int d\omega e^{-\int_0^t \mathcal{L}(w,\dot{w})dt} = \frac{1}{(2\pi)^{\frac{d}{2}}} \det \left( (t\Omega)^{-1} \sinh (t\Omega) \right)^{-\frac{d}{2}}. \tag{4.13}\]

Putting together the classical and quantum contributions (4.9), (4.13) and recalling that \(y_{0,f} = z_{0,f}\) we conclude (4.8):
\[K(y_f,0|t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \det \left( (t\Omega)^{-1} \sinh (t\Omega) \right)^{-\frac{d}{2}} e^{\frac{e^{\Theta} t}{2}} \times \exp \left[ - \text{tr} \left( \frac{Y_f Y_T}{2t} [\cosh (t\Omega) t\Omega \sinh (t\Omega)^{-1} + t\Theta] \right) \right]. \tag{4.14}\]

In the case where the potential is zero \(\hat{\Theta} = 0\), we obtain the multidimensional Ornstein-Uhlenbeck propagator. In the one dimensional case in particular the generic expression above reduces to the familiar propagator, and can be easily generalized for the case where \(z_t = y, z_0 = x\). Indeed, then the corresponding solution of the classical equations of motion become
\[z(s) = y - x e^{-\Theta t} e^{\Theta s} - \frac{y - x e^{\Theta t}}{2\sinh (\Theta t)} e^{-\Theta s}\]
modifying the classical contribution to the path integral. This then leads to the expression for the one dimensional propagator:
\[K(y,x|t) = \sqrt{\frac{\Theta}{\pi(1-e^{-2\Theta t})}} \exp \left[ -\Theta (y - x e^{-\Theta t})^2 \right]. \tag{4.14}\]

It is clear that the Lagrangian description provides a straightforward way of computing the propagator of the Ornstein-Uhlenbeck process.

Let us comment on the case where \(\Theta\) contains an anti-symmetric part as well, i.e. \(\Theta = \Sigma + A\) where \(\Sigma\) is the symmetric part and \(A\) is the anti-symmetric one. From a physical viewpoint this case corresponds to the presence of an angular momentum term in the Lagrangian, whereas from the computational point of view there are two non-trivial contributions when computing the propagator. First the classical equations of motion are modified due to the existence of the anti-symmetric part. More precisely, from the Euler-Lagrange equations one obtains, the following classical equations of motion:
\[\ddot{z} + 2A \dot{z} - \Omega^2 z = 0,\]
thus the classical contribution (4.14) is also modified accordingly. More importantly there is a highly non-trivial contribution when computing the quantum part (4.13), indeed when \(\Theta\) is symmetric as described above the term \(e^{-\int_0^t ds \sum_{i,j} \Theta_{ij} (w_i w_j - w_j w_i)}\) gives a purely boundary contribution. However, when there is an anti-symmetric part then an extra non-trivial term in the integral (4.13) appear, which should be taken into consideration. The computation of the quantum contribution when \(\Theta\) is anti-symmetric and for \(\Theta = \hat{\Theta}\) is essentially equivalent to the problem of computing the characteristic function for the Levy area:
\[A = \int d\mathbf{m}_0 \exp \left[ \sum_{i>j} \Theta_{ij} \int (w_i dw_j - w_j dw_i) \right]. \tag{4.15}\]
where $d m_0$ is given in (3.14), and $w_s$ is given in (3.13) (see also e.g. [27] and references therein). This is a significant issue, which however will be addressed in a separate publication [28].

4.2 Stochastic approach: one dimensional harmonic oscillator

Our goal in this section is to rederive the propagator for the one dimensional quantum harmonic oscillator using probabilistic techniques, although the analytical form of the Feynman’s path integral via the Lagrangian (semi-classical) formulation is widely known in the quantum physics community, given by the celebrated Mehler formula. Consider the single particle quantum harmonic oscillator, with potential $u(x) = \omega^2 x^2$ for $x \in \mathbb{R}$, where $\omega > 0$ is a fixed parameter. We derive the formula here from the probabilistic perspective. Using (3.8), (3.14), we evaluate:

$$K(y, x|t) = \int d m \exp \left( -\frac{1}{2} \omega^2 \int_0^t |x_s|^2 \, ds \right).$$

For $x = 0$ this is a standard quadratic functional of Brownian motion studied extensively by Pitman and Yor, see for example [6].

We now derive this result for $x \neq 0$. We assume that $w_s$ is a standard Wiener process starting from 0. Then the required Brownian bridge satisfying $x_0 = x$ and $x_t = y$ is given by (3.15) $w_t = y - x$ and as Wiener proved a standard Wiener process has the Fourier representation on $[0, t]$ given in (3.13), where $f_n$ are normal random variables, $(w_t = \sqrt{t} f_0)$. Substituting this Fourier series into the prescription for $x_s$ above

$$x_s = x + \frac{2}{t} (y - x) + \sqrt{\frac{2t}{\pi}} \sum_{n \geq 1} \frac{1}{n} f_n \sin \left( \frac{n \pi s}{t} \right).$$

Let us directly substitute the Fourier series for $x_s$ into the integral in the exponent. After performing standard trigonometric integrals we conclude

$$-\frac{1}{2} \omega^2 \int_0^t |x_s|^2 \, ds = -\frac{1}{2} \sum_{n \geq 1} \left( a_n f_n^2 + b_n f_n \right) - \frac{1}{2} c,$$

where we define

$$a_n := \frac{\omega^2 t^2}{\pi^2 n^2}, \quad b_n := \frac{\omega^2 (2t)^{3/2}}{\pi^2 n^2} (x - (-1)^n y) \quad \text{and} \quad c := \frac{\omega^2 t}{3} (x^2 + xy + y^2).$$

The expectation for a generic term in the sum, namely

$$E \left( \exp \left( -\frac{1}{2} a_n \xi^2 - \frac{1}{2} b_n \xi \right) \right),$$

with $\xi$ a standard normal random variable. Directly computing
we find
\[
E \left( \exp\left( -\frac{1}{2}a_n\xi^2 - \frac{1}{2}b_n\xi \right) \right) \\
= \frac{1}{\sqrt{2\pi}} \exp\left( \frac{1}{4} \omega^4 b_n^2 \right) \int_\mathbb{R} \exp\left( -\frac{1}{2}(1 + \omega^2 a_n)(\xi + C)^2 \right) \, d\xi \\
= \exp\left( \frac{1}{2} \omega^4 b_n^2 \right) \cdot \frac{1}{(1 + \omega^2 a_n)^{1/2}},
\]
where \( C = \omega^2 b_n / (2(1 + \omega^2 a_n)) \).

Substituting these last expressions into the expectation of interest, we find
\[
E \left( \exp\left( -\frac{1}{2}\omega^2 \int_0^t |x_s|^2 \, ds \right) \right) \\
= \exp\left( -\frac{\omega^4 t}{6} (x^2 + xy + y^2) \right) \left( \prod_{n=1}^{\infty} \frac{n^2}{n^2 + \omega^2 t^2 / \pi^2} \right)^{1/2} \exp\left( \frac{(2t)^3 \omega^4}{8 \pi^4} \sum_{n=1}^{\infty} \frac{n^{-4}(x - (-1)^n y)^2}{1 + \omega^2 t^2 / (n^2 \pi^2)} \right) \\
= \exp\left( -\frac{\omega^4 t}{6} (x^2 + xy + y^2) \right) \left( \frac{\omega t}{\sinh (\omega t)} \right)^{1/2} \exp\left( t^2 \omega^4 \sum_{n=1}^{\infty} \frac{x^2 + y^2 - 2(-1)^n xy}{n^2(n^2 + \omega^2 t^2 / \pi^2)} \right),
\]
where we have substituted for \( a_n \) and \( b_n \) from the expressions above and also used Euler’s formula for the infinite product shown.

We now focus on the terms in the exponent of the final factor. By direct computation we observe that
\[
t^2 \omega^4 \sum_{n=1}^{\infty} \frac{x^2 + y^2 - 2(-1)^n xy}{n^2(n^2 + \omega^2 t^2 / \pi^2)} \\
= \frac{t^2 \omega^4}{\pi^2} \left( x^2 + y^2 \right) \left( \frac{\pi^2}{6} + \frac{\pi^2}{\omega^2 t^2} \frac{1}{2} (1 - \omega t \coth (\omega t)) + 2xy \left( \frac{\pi^2}{6} + \frac{\pi^2}{\omega^2 t^2} \frac{1}{2} (1 - \omega t \coth (\omega t)) \right) \\
- xy \left( \frac{\pi^2}{6} + \frac{4\pi^2}{\omega^2 t^2} \frac{1}{2} (1 - \frac{1}{2} \omega t \coth (\frac{1}{2} \omega t)) \right) \right).
\]

Suitably combining the above contributions we conclude that the quadratic functional of the Brownian bridge associated with the scalar single particle quantum harmonic oscillator can be then expressed as follows
\[
E \left( \exp\left( -\frac{1}{2}\omega^2 \int_0^t |x_s|^2 \, ds \right) : x_t = y, x_0 = x \right) \\
= \left( \frac{\omega t}{\sinh (\omega t)} \right)^{1/2} \exp\left( \frac{\omega}{2} \left( x^2 + y^2 \coth (\omega t) + \frac{2xy}{\sinh (\omega t)} + \frac{(x - y)^2}{2t} \right) \right),
\]
which gives
\[
K(y, x|t) = \left( \frac{\omega}{2\pi \sinh (\omega t)} \right)^{1/2} \exp\left( -\frac{\omega}{2} \left( x^2 + y^2 \coth (\omega t) + \frac{2xy}{\sinh (\omega t)} \right) \right).
\]

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and this is precisely Mehler’s formula. We observe that the proof above reveals that the physicist’s proof of Mehler’s formula via the action $S[x]$ is much more succinct. Note that the extension of Mehler’s formula for the $M$-particle harmonic oscillator by means of probabilistic techniques is given in [28], but also the results of the previous section from the semi-classical point of view are relevant when considering the case of zero drift.

### 4.2.1 Examples of stochastic processes

In general, one of the main aims is the computation of expectation values using the universal expression (3.12), (3.14) and solutions of the underlying SDEs. In particular, in statistical and quantum physics the quantities of significance are (we use here the familiar in physics community notation for expectation values $\langle O \rangle$):

$$\langle O(x_s) \rangle = E_t \left( O(x_s) e^{\int_0^t u(x_s) ds} \right), \quad 0 \leq s \leq t$$

(4.16)

where we define via (3.12), (3.14)

$$E_t \left( O(x_s) \right) = \int dw_t \ dm \ O(x_s) \quad 0 \leq s \leq t.$$  

(4.17)

Formula (4.16) can be practically used provided that solutions of the associated SDEs (SPDEs in the continuum space limit, i.e. in quantum/statistical field theories) are available so that the fields $x_{sj}$ are expressed in terms of the normal variables $w_{sj}$. Alternatively, in the cases of generic drift and/or potential in order to obtain the propagator and expectation values one can in principle start form (3.4) and apply semi-classical methods, perturbation theory, and Green’s function techniques as is customary in quantum and statistical field theory.

Let us briefly discuss below a few illustrative examples of simple processes, in the absence of potential $u$, which will be also used later in the text when discussing two fundamental integrable quantum systems, i.e. the DST and XXZ models. In the examples below we use solutions of the underlying SDEs in order to compute expectation values.

1. **Multidimensional Brownian motion with constant diffusion matrix**

   The corresponding SDEs and solutions are given by

   $$dy_t = \sigma dw_t \Rightarrow y_{sj} = y_{0j} + \sum_j \sigma_{ij} w_{sk}$$

   (4.18)

   where $\sigma$ is a non-dynamical (i.e. does not depend on the fields $y_j$) $M \times M$ matrix. Let us compute the first couple of expectation values (first couple of moments): $E_t(y_{sj})$ and $E_t(y_{si}, y_{s'j})$. Indeed, one readily finds via (4.17)

   $$E_t(y_{sj}) = y_{0j},$$

   $$E_t(y_{si}, y_{s'j}) = E_t(y_{0i}, y_{0j}) + E_t(\sum_{k,l} \sigma_{ik} \sigma_{jl} w_{sk} w_{s'l}) = y_{0i} y_{0j} + g_{ij} s', \quad s' \leq s$$

   (4.19)
and we have used the fact that
\[ E_t(w_{sj}) = 0, \quad (4.19) \]
\[ E_t(w_{si}w_{sj}) = \delta_{ij}s', \quad s' \leq s. \quad (4.20) \]

Equation (4.19) is obvious via (4.17), an explicit proof of (4.20) is presented below.

We focus on the one dimensional case, but the multidimensional generalization is straightforward. This expectation value can be readily computed using the fact that \( w_s - w_{s'} \) and \( w_{s'} \) are independent increments, which leads to:
\[ E_t(w_s w_{s'}) = E_t(w_s^2) = s', \quad s' \leq s. \quad (4.21) \]

However, let us show explicitly (4.21), via the use of (4.17) and Wiener’s representation (3.13). Let \( 0 \leq s' \leq s \leq t \):
\[
E_t(w_s w_{s'}) = \int dw_t \prod_{k \geq 1} \frac{dt_k}{\sqrt{2\pi t}} \exp \left[ -\frac{w_t^2}{2t} - \sum_{k \geq 1} \left( \frac{t_k^2}{2t} \right) \left( \frac{w_t^2}{t} \right) s s' + \frac{2}{t} \sum_{k \geq 1} \frac{\sin(\omega_k s) \sin(\omega_k s')}{\omega_k^2} \right]
\]
\[
= \frac{ss'}{t} - t \sum_{k \geq 1} \frac{\cos \left( \frac{2k\pi (s+s')}{2t} \right) - \cos \left( \frac{2k\pi (s-s')}{2t} \right)}{k^2\pi^2}.
\]

Taking into consideration the identity
\[ \sum_{k \geq 1} \frac{\cos \left( \frac{2k\pi x}{2t} \right)}{k^2\pi^2} = B_2(x), \quad 0 \leq x \leq 1, \]
where \( B_2 \) is the second Bernoulli polynomial \( B_2(x) = x^2 - x + \frac{1}{6} \), we conclude
\[ E_t(w_s w_{s'}) = \frac{ss'}{t} - t \left( B_2(\frac{s+s'}{2t}) - B_2(\frac{s-s'}{2t}) \right) = s', \]
which immediately leads to \( E_t((w_s - w_{s'})^2) = s - s' \).

2. Multi-dimensional geometric Brownian motion (Black-Scholes model)

In this case the SDEs read as
\[ dx_{ij} = b_{ij} x_{ij} dt + x_{ij} \sum_k S_{jnk} dw_{tk} \quad (4.22) \]
both \( b_{ij} \) and \( S_{ij} \) are non-dynamical quantities. The obvious change of variable \( x_i = e^{y_i} \), also in the spirit of the quantum canonical transform, leads to the simplified version of the equations above
\[ dy_{ij} = \tilde{b}_{ij} dt + \sum_k S_{jnk} dw_{tk} \Rightarrow y_{sj} = y_{0j} + \tilde{b}_j s + \sum_k S_{ik} w_{sk} \quad (4.23) \]
where \( \tilde{b}_j = b_j - \frac{\partial g_0}{\partial y_j} \). It is then straightforward to compute expectation values via the solutions (4.23) and (4.19), (4.20)
\[ E_t(x_{sj}) = E_t(e^{y_{sj}}) = x_{0j} e^{\tilde{b}_j s}, \]
and similarly, using the definition (4.17) and the change of variables we obtain:
\[ E_t(x_t; x_{t_j}) = x_0 x_0^i e^{(b_i + b_j) t} e^{b_j t}. \]

As in the previous example we find the propagator via the universal expressions (3.12), (3.14) and (4.23). From the solution of (4.23)
\[ w_t = S^{-1}(\Delta y - \bar{b} t) \]
here \( \Delta y = y_t - y_0 \). Then substituting the above in (3.14) we conclude that the propagator is (3.12)
\[ K(y_t, y_0 | t) = e^{-\frac{1}{2} t (\Delta y - \bar{b} t)^T S^{-1} (\Delta y - \bar{b} t)} \frac{\det S}{(2\pi t)^{\frac{M}{2}}}. \]
The propagator can also be expressed in terms of the vector field \( x_t \), recall \( y_j = \log x_j \).

5 Discrete integrable quantum systems & SDEs

In this section we explore some interesting links between discrete quantum systems and SDEs. The correspondence between the quantum equations of motion and the SDEs is discussed and typical examples of exactly solvable quantum systems and the associated SDEs are presented.

To illustrate the main ideas of the quantum canonical transformation and the quantum equations of motion in connection with SDEs we present below typical examples of integrable quantum systems: the discrete NLS model and the Heisenberg (XXZ) quantum spin chain. As in the classical case integrability at the quantum level can be also described via the Lax pair \((L_m, A_m)\) which depends on quantum fields and a spectral parameter in general. The Lax pair satisfies the quantum auxiliary problem and hence the zero curvature condition
\[ \frac{d L_m(\lambda)}{dt} = A_{m+1}(\lambda) L_m(\lambda) - L_m(\lambda) A_m(\lambda) \]
providing the quantum equations of motion in analogy to Heisenberg’s picture [12] [13]. The Lax pair formulation ensures the existence of many conserved charges. Indeed, define
\[ t(\lambda) = \text{tr}_0 T_0(\lambda) \]
where \( T \) is the monodromy -a discrete path integral– is given as
\[ T_0(\lambda) = \mathbb{L}_{0M}(\lambda) \mathbb{L}_{0M-1}(\lambda) \ldots \mathbb{L}_{01}(\lambda). \] (5.1)

Using the quantum zero curvature condition and assuming periodic boundary conditions it is shown in straightforward manner that: \( \frac{d t(\lambda)}{d \lambda} = 0 \), i.e. \( t \) is the generator of the conserved quantities: \( t(\lambda) = \sum_k \frac{1}{k} \).

Let us clarify the index notation in the expressions above, the indices 0 (auxiliary) and \( m \in \{1, \ldots, M\} \) (quantum) correspond to the position in the \( M + 1 \) tensor product
\[ \mathbb{L}_{0m}(\lambda) = \sum_{a,b} c_a^{(0)} \otimes I^{(1)} \ldots \otimes I^{(m-1)} \otimes j_a^{(m)}(\lambda) \otimes \ldots \otimes I^{(M)} \]
where $e_{ab}$ are in general $N \times N$ matrices with elements: $(e_{ab})_{cd} = \delta_{ac}\delta_{bd}$. Notice that the quantum indices are suppressed in the expression for the monodromy (5.1) for brevity.

The algebraic formulation of quantum integrability on the other hand is based on the existence of a quantum $R$-matrix that satisfies the Yang-Baxter equation [19, 21]. This provides a rather stronger frame in the sense that guarantees the existence of many charges in involution, i.e. ensures the existence of a family of mutually commuting quantum charges: $[t(\lambda), t(\lambda')] = 0$. In this setting the $L$ operator satisfies the fundamental algebraic relation [18, 19],

$$R(\lambda_1 - \lambda_2) (\mathbb{I} \otimes L(\lambda_1)) (L(\lambda_2) \otimes \mathbb{I}) = (L(\lambda_2) \otimes \mathbb{I}) (\mathbb{I} \otimes L(\lambda_1)) R(\lambda_1 - \lambda_2) \quad (5.2)$$

$R(\lambda) \in \text{End}(V \otimes V)$ and $L(\lambda) \in \text{End}(V \otimes \mathcal{A})$; $\mathcal{A}$ is the underlying deformed algebra defined by (5.2). More specifically, equation (expressed in the index free notation) (5.2) acts on $V \otimes V \otimes \mathcal{A}$

$$R(\lambda) = \sum_{a,b,c,d} R_{ab,cd}(\lambda) e_{ab} \otimes e_{cd} \otimes \mathbb{I}$$

$$\mathbb{I} \otimes L(\lambda) = \sum_{i,j} e_{ab} \otimes \mathbb{I} \otimes l_{ab}(\lambda)$$

$$L(\lambda) \otimes \mathbb{I} = \sum_{a,b} \mathbb{I} \otimes e_{ab} \otimes l_{ab}(\lambda),$$

$l_{ij}(\lambda) \in \mathcal{A}$ and $T(\lambda) \in \text{End}(V \otimes \mathcal{A}^\otimes M)$, i.e. the monodromy is a tensor representation of (5.2). In both examples we are considering here the associated $R$-matrix is the Yangian or the trigonometric XXZ $R$ matrix (see [18, 19] and references therein).

### 5.1 The quantum discrete NLS hierarchy

We start our analysis with the DNLS model, with the corresponding Lax operator given by [12, 16]

$$L_j(\lambda) = \begin{pmatrix} \lambda + N_j & z_j \\ Z_j & 1 \end{pmatrix}.\quad (5.3)$$

$N_j = \Theta_j + z_jZ_j$. Using the fact that $L$ satisfies (5.2) together with the quantum zero curvature condition one can also construct the $A$ operators for the whole hierarchy [14]. In particular, the $A$-operators associated to the second and third conserved charges of the hierarchy is

$$A^{(1)}_j(\lambda) = \begin{pmatrix} \lambda & z_j \\ Z_{j-1} & 0 \end{pmatrix}, \quad A^{(2)}_j(\lambda) = \begin{pmatrix} \lambda^2 - z_jZ_{j-1} & \lambda z_j - z_jN_j + z_{j+1} \\ \lambda Z_{j-1} - Z_{j-1}N_{j-1} + Z_{j-2} & z_jZ_{j-1} \end{pmatrix}.\quad (5.3)$$

Notice that the first charge gives the numbers of particles of the DNLS model (corresponds to a system of $M$ harmonic oscillators), the second charge corresponds to the momentum of the model also known as the DST mode [15], whereas the third charge is the DNLS Hamiltonian [16]. We shall examine below the second and third charges, identify the corresponding SDEs, and study their continuum limits, which produce the stochastic transport and stochastic heat equation correspondingly.
The DST model

Let first focus on the second charge (momentum) is given by \[ H^{(1)} = \frac{1}{2} \sum_{j=1}^{M} z_j^2 Z_j^2 + \sum_{j=1}^{M} (c_j z_j - z_{j+1}) Z_j \]

and via (5.2) one obtains: \[ [z_i, Z_j] = -\delta_{ij}. \] The equations of motion can be now derived via the zero curvature condition or Heisenberg’s equation and they read as

\[ \frac{dz_j}{dt} = (c_j z_j - z_{j+1}) + z_j^2 Z_j. \] (5.4)

Consider now the following map

\[ z_j \mapsto x_j, \quad Z_j \mapsto \partial x_j \] (5.5)

then the Hamiltonian is expressed as:

\[ H^{(1)} = \frac{1}{2} \sum_{j=1}^{M} x_j^2 \partial^2 x_j + \sum_{j=1}^{M} (c_j x_j - x_{j+1}) \partial x_j \]

and we have considered periodic boundary conditions \( x_j = x_{j+M} \). The Hamiltonian is apparently of the form (1.2), and the corresponding set of SDEs are given by

\[ dx_{tj} = (c_j x_{tj} - x_{tj+1}) dt + x_{tj} dw_{tj}. \] (5.6)

If we compare the SDEs (5.6) with the quantum equations of motion (5.4) we observe that the last term in (5.4) is replaced by the multiplicative noise in (5.6). It is worth noting that in the special case that \( c_j \gg 1 \) the second term in the drift is neglected and the SDEs reduce to the ones of the multidimensional Black-Scholes model (4.2).

Let us apply the quantum canonical transformation as described in section 2. Indeed, we define the new variables \( y_j \) via (2.2)

\[ dy_i = x_i^{-1} dx_i \Rightarrow x_i = A_i e^{y_i} \]

where \( A_i \) are integration constants. Then the Hamiltonian of the DST model can be re-expressed in terms of the new variables as a Hamiltonian with identity diffusion matrix

\[ H^{(1)} = \frac{1}{2} \sum_{j=1}^{M} \partial^2 y_j + \sum_{j=1}^{M} (C_j + B_j e^{y_{j+1} - y_j}) \partial y_j, \] (5.7)

\( C_j = c_j - \frac{1}{2}, \quad B_j = -A_j A_j^{-1} \), and the corresponding set of SDEs are

\[ dy_{tj} = (C_j + B_j e^{y_{tj+1} - y_{tj}}) dt + dw_{tj}. \]

Connection with the quantum Darboux transforms introduced in [14] would be also a very interesting direction to pursue. Note that in [13, 14] an alternative version of the quantum discrete NLS model is studied i.e. the so called q-Boson or quantum Ablowitz-Ladik model.
Recall that in the classical case the Darboux-Bäcklund transformation \[29, 30\] provides an efficient way to find solutions of integrable non-linear PDEs. In particular, the transformation connects solutions of the same or different integrable PDEs. At the quantum level the transformation connects different realizations of the underlying quantum algebra. The pertinent question is how this transformation affects the associated SDEs. For instance does the quantum Darboux transformation connects solutions of different SDEs? This is a significant open question, which needs to be systematically addressed. In fact, a novel generalization of the “dressing” method has been introduced for classical PDEs \[31\], whereas a similar formulation is suitably extended in the context of SPDEs \[32\]. For a relevant discussion on stochastic Bäcklund transformation see also \[33\].

An interesting observation can be now made on the connection of the DST model and the Toda chain. Let us also consider the adjoint operator of (5.7), which reads as

\[
H(1)^\dagger = \frac{1}{2} \sum_{j=1}^{M} \partial^2_{y_j} - \sum_{j=1}^{M} \hat{B}_j(y) \partial_{y_j} + \sum_{j=1}^{M} \hat{B}_j e^{y_{j+1} - y_j}
\]

\[
\hat{B}_j = B_j - B_{j-1}. 
\]

Then the self-adjoint operator \( H = \frac{1}{2}(H^{(1)} + H^{(1)^\dagger}) \) is expressed as

\[
H = \frac{1}{2} \sum_{j=1}^{M} \partial^2_{y_j} + \sum_{j=1}^{M} \hat{B}_j e^{y_{j+1} - y_j},
\]

which is nothing but the Hamiltonian of the Toda chain.

Let us now derive the solution of the set of SDEs (5.6) introducing suitable integrator factors (we refer the interested reader to \[4\] on integrator factors in SDEs). Let us consider the general set of SDES

\[
dx_j = b_j(x) dt + x_j dW_j,
\]

for any drift \( b \). We introduce the following set of integrator factors:

\[
F_j(t) = \exp \left( - \int_0^t dw_{sj} + \frac{1}{2} \int_0^t ds \right) \quad (5.8)
\]

and define the new fields: \( y_{ij} = F_j(t)x_{ij} \), then one obtains a differential equation for the vector field \( y \). Indeed, from (5.6), (5.8)

\[
d(F_j(t)^{-1} y_{ij}) = dx_{ij} \Rightarrow \frac{dy_{ij}}{dt} = F_j(t) b_j (F_k^{-1}(t)y_{ik}) \quad (5.9)
\]

and we bear in mind that the usual calculus rules apply for the LHS of the first equation above.

Let us focus now on the SDEs of the DST model \[5.6\]; in this case \[5.9\] leads to the ODE:

\[
\frac{dy_t}{dt} = A(t)y_t,
\]

where the \( M \times M \) matrix \( A \) is given as (we have set \( c_j = 1 \) in \[5.6\])

\[
A(t) = I - B = \sum_{j=1}^{M} \left( e_{jj} - B_j(t) e_{jj+1} \right),
\]

\[25\]
and we define
\[ B_j(t) = F_j(t)F_{j+1}^{-1}(t) = \exp(w_{t+1} - w_t). \] (5.10)

The formal solution of the latter linear problem is the path ordered exponential (monodromy):
\[ y_t = \mathcal{P} \exp \left( \int_0^t A(s) ds \right) y_0, \]
which can be expressed as a formal series expansion:
\[ \mathcal{P} \exp \left( \int_0^t A(s) ds \right) = \sum_{n=0}^{\infty} \int_0^t \int_0^{t_n} \ldots \int_0^{t_2} dt_n dt_{n-1} \ldots dt_1 A(t_n) A(t_{n-1}) \ldots A(t_1), \]
t \geq t_n \geq t_{n-1} \ldots \geq t_2.

Note that \( A \) is an upper triangular local matrix, thus products of the matrices preserve the triangular structure, but not the locality. In fact, this formal series expansion provides time-like non-local charges of the theory in analogy to the non-local charges associated to representations of deformed algebras in quantum and classical integrable models.

It will be instructive to consider the DST model (5.4) and the respective SDEs (5.6) in the continuum limit. Let us set \( c_j = 1 \) in (5.6), then after suitably re-scaling the fields and considering the thermodynamic limit \( M \to \infty, \delta \to 0 (\delta \sim \frac{1}{M}) \) we obtain
\[ x_{tj} \to \varphi(x, t) \]
\[ \frac{x_{tj+1} - x_{tj}}{\delta} \to \partial_x \varphi(x, t) \]
\[ \delta \sum_j f_j \to \int dx \ f(x) \]
\[ w_{tj} \to W(x, t). \] (5.11)

a brief discussion on the representation of the Brownian sheet \( W(x, t) \) (space-time white noise) is presented in the last section. In the continuum limit the Hamiltonian of the DST model (5.6) becomes the Hamiltonian of an 1 + 1 dimensional quantum field theory
\[ H^{(1)}_c = \int dx \left( \frac{1}{2} \varphi^2(x) \frac{\partial^2}{\partial_x^2} \varphi(x) - \partial_x \varphi(x) \dot{\varphi}(x) \right), \]
where the issue of the ultra locality of underlying algebra has been taken into consideration: \( [\varphi(x), \dot{\varphi}(y)] = \delta(x-y), \frac{\partial}{\partial \varphi(x)} \) and the SDEs (5.6) become the stochastic transport equation with multiplicative noise:
\[ \partial_t \varphi(x, t) = -\partial_x \varphi(x, t) + \varphi(x, t) \dot{W}(x, t) \]
where \( \dot{W}(x, t) = \frac{dW(x)}{dt} \). It is worth noting that in general, the continuum limit of SDEs will lead to non-local SPDEs, given that the diffusion matrix is generically a full matrix, and the drift describes in principle non local interactions. However, in the case of diagonal (or slightly off diagonal) diffusion matrices and local drift one obtains local SPDEs as the example described above.
The DNLS model

We come now to the next member of the DNLS Hierarchy i.e. the quantum DNLS model (see e.g. [16] and [40] and references therein). We directly express the Hamiltonian in terms of differential operators after taking into consideration the map (5.5):

\[ H^{(2)} = \frac{1}{2} \sum_{j=1}^{M} \left( x_j (x_{j+1} - x_j) \partial^2_{x_j} + x^2_{j+1} \partial_{x_j} \partial_{x_{j+1}} - (x_j - 2x_{j+1} + x_{j+2}) \partial_{x_j} \right) \quad (5.12) \]

and we read the diffusion matrix with entries

\[ g_{jj} = x_j (x_{j+1} - x_j), \quad g_{jj+1} = g_{j+1j} = x^2_{j+1}, \quad g_{ij} = 0, \quad |i - j| > 1, \quad (5.13) \]

as well as drift components

\[ b_j(x) = -\frac{1}{2}(x_j - 2x_{j+1} + x_{j+2}), \quad \text{which in the continuum space limit discussed below provides a second derivative of the field.} \]

Using the dictionary described above (5.11) we can readily write down the continuum limit of the DNLS Hamiltonian

\[ H^c_{(2)} = \frac{1}{2} \int dx \left( \varphi^2(x) \dot{\varphi}(x) - \partial^2_x \varphi(x) \dot{\varphi}(x) \right) \quad (5.14) \]

as well as the continuum limit of the DNLS SDEs, which is nothing but the stochastic heat equation with multiplicative noise:

\[ \partial_t \varphi(x, t) = -\frac{1}{2} \partial^2_x \varphi(x, t) + \varphi(x, t) \dot{W}(x, t). \quad (5.15) \]

The latter equation is solvable, and can be also mapped to the stochastic viscous Burgers equation (see for instance [20]). Indeed, we set: \( \varphi = e^h, \ u = \partial_x h \) then (5.15) becomes,

\[ \begin{align*}
\partial_t h(x, t) &= -\frac{1}{2} \partial^2_x h(x, t) - \frac{1}{2} (\partial_x h(x, t))^2 + \dot{W}(x, t) \\
\partial_t u(x, t) &= -\frac{1}{2} \partial^2_x u(x, t) - u(x, t) \partial_x u(x, t) + \partial_x \dot{W}(x, t).
\end{align*} \quad (5.16) \]

The latter is precisely the viscous Burgers equation with additive noise.

5.2 The XXZ quantum spin chain

We briefly discuss another prototype integrable model, the XXZ spin \( \frac{1}{2} \) quantum spin chain and the associated SDEs. The XXZ Hamiltonian has the familiar form

\[ H = -\frac{1}{2} \sum_{j=1}^{M} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right) \quad (5.17) \]

\( \sigma^x, \sigma^y, \sigma^z \) are the familiar 2×2 Pauli matrices, which form a two dimensional representation of the \( \mathfrak{sl}_2 \) algebra, \( \Delta = \cosh(\mu) \), \( \mu \) the anisotropy parameter. The \( \mathfrak{sl}_2 \) algebra has generators \( S^j, j \in \{1, 2, 3\} \) that satisfy:

\[ [S^i, S^j] = 2i\epsilon_{ijk}S^k. \quad (5.18) \]
The spin $S \in \mathbb{R}$ representation of the algebra in terms of differential operators is given as

$$
S^1 \mapsto -(x^2 - 1)\partial_x + S(x + x^{-1})
$$

$$
S^2 \mapsto -i\left((x^2 + 1)\partial_x + S(x - 1 - x)\right)
$$

$$
S^3 \mapsto 2x\partial_x.
$$

(5.19)

We can replace the Pauli matrices in (5.17) with the spin $\frac{1}{2}$ two dimensional representation of $sl_2$ expressed in terms of differential operators (5.19). Then (5.17) becomes a typical diffusion reaction Hamiltonian

$$
H = \sum_{j=1}^{M-1} \left( (x_{j+1} x_j + x_j + 1) \partial_{y_j} \partial_{y_{j+1}} + \frac{1}{2} (x_j (x_{j+1} + x_j) - (x_{j+1} + x_j)) \right)
$$

$$
- \frac{1}{4} (x_{j+1} x_j - x_j x_{j+1})
$$

(5.20)

from which one immediately reads the diffusion matrix $g$ as well as the drift $b$, and thus the associated set of SDEs (1.3). Recall also $g = \sigma \sigma^T$ one then can identify $\sigma$ and hence $\sigma^{-1}$ in order to apply the results of section 2.

- The Ising model

Two special cases of interest arise as suitable limits of the XXZ spin chain: the XX model when $\Delta = 0$ which describes the free fermion point of the sine-Gordon model, and the Ising model when $\Delta \to \infty$. Let us focus here on the simple case of the Ising model in the presence of an external longitudinal magnetic field, then the Hamiltonian reads as (we have considered the open spin chain for convenience):

$$
H = \frac{1}{2} \sum_{j=1}^{M-1} x_j x_{j+1} \partial_{y_j} \partial_{y_{j+1}} + \frac{1}{2} (x_j (x_{j+1} + x_j)) \partial_{y_j} - \frac{i}{4} e^{x_j} \partial_{y_{j+1}}^2.
$$

It is convenient in this simple example to change variables: $x_j = e^{y_j}$, then the Hamiltonian turns into one with constant diffusion coefficient as well as constant drift

$$
H = \frac{1}{2} \sum_{j=1}^{M-1} \partial_{y_j} \partial_{y_{j+1}} + \xi \sum_{j=1}^{M} \partial_{y_j} - \frac{i}{4} \partial_{y_{j+1}}^2.
$$

The diffusion matrix now is constant and it reads as:

$$
g = \frac{1}{2} \sum_{j=1}^{M-1} (e_{jj+1} + e_{j+1 j}) = \frac{i}{2} e_{MM} \Rightarrow \sigma = \frac{1}{\sqrt{2i}} \left( \sum_{j=1}^{M} e_{jj} + i \sum_{j=1}^{M-1} e_{jj+1} \right)
$$

hence the SDEs are given as

$$
dy_{tj} = \xi dt + \frac{1}{\sqrt{2i}} (dw_{tj} + i dw_{t, j+1}), \quad 1 \leq j \leq M - 1,
$$

$$
dy_{tM} = \xi dt + \frac{1}{\sqrt{2i}} dw_{tM}.
$$
The solutions of the associated SDEs are then easily determined:

\[ y_{sj} - y_{0j} = \xi_s + \frac{1}{\sqrt{2i}}(w_{sj} + iw_{sj+1}) \quad 1 \leq j \leq M - 1 \]

\[ y_{sM} - y_{0M} = \xi_s + \frac{1}{\sqrt{2i}}w_{sM}. \]

The variables \( x_j = e^{y_j} \) are also immediately identified via the solution above. Expectation values can be now readily computed using the solutions above as described in section 3 (see also the example in section 4.1).

The addition of a transverse magnetic field, e.g. \( h = \xi \sum_j (S^1 - iS^2) \), to the free Hamiltonian leads to a set of SDEs with the same constant diffusion matrix as above, but with drift components proportional to \( e^{y_j} \). The solution of the associated SDEs in this case is a more involved problem, but it is the first task in order to be able to identify expectation values.

The \( \mathfrak{sl}_2 \) algebra can be also expressed in terms of the variables \( y_j, \partial_y \) and thus the XXZ Hamiltonian can be re-expressed as

\[
H = \sum_j \left( \cosh (y_j - y_{j+1}) - \Delta \right) \partial_{y_j} \partial_{y_{j+1}} + \sum_j \left( \sinh (y_j - y_{j+1}) + \sinh (y_j - y_{j-1}) \right) \partial_{y_j} - \frac{1}{2} \sum_j \cosh (y_j - y_{j+1}).
\]

Both expressions (5.20), (5.21) may be useful especially when considering the continuum limit of the model and making connections with certain quantum field theories, this is an important aspect which however will be studied separately. Relevant recent results on stochastic XXZ spin chain are presented in [34].

Note that both the XXZ and DNLS models are exactly solvable system and their spectra and eigenstates are known via the Bethe ansatz formulation [18, 19] (see also [35]-[37]). A complete basis of Bethe states is available and the time evolution of the wave function may be given by

\[
f(x, t) = \sum_k e^{-E_k t} \phi_k(x)
\]

where \( E_k \) are the energy eigenvalues. Expectation values are then computed in the space of square integrable functions (for normalized wave functions):

\[
\langle O(x) \rangle = \sum_{k,j} \int dx \ O(x) e^{(E_k - E_j)t} \phi_k^*(x) \phi_l(x).
\]

### 5.3 Quantum Darboux transformation & defects

It will be useful for our purposes here to consider the DST model in the presence of integrable local defects [38]-[40]. We will derive the associated quantum Darboux-Bäcklund transform for the DNLS model as a defect matrix and shall also identify the corresponding SDEs. The monodromy in this case is modified as (the auxiliary index is suppressed below for brevity)

\[
T(\lambda) = L_M \ldots D_m(\lambda) \ldots L_1(\lambda)
\]

29
where we consider the defect matrix $D$ to be of the generic form

$$D_m(\lambda) = \begin{pmatrix} \lambda + \alpha_m & \beta_m \\ \gamma_m & \lambda - \alpha_m \end{pmatrix}.$$ 

The matrix $D$ will be explicitly derived via the zero curvature condition on the defect point i.e. the $t$-part of the quantum Darboux-Bäcklund transformation \[14\].

In the algebraic scheme preserving integrability requires that $D$ also satisfies (5.2), hence it turns out that $\alpha$, $\beta$, $\gamma$ are the generators of $sl_2$. Integrability in the weaker sense via the Lax pair description on the other hand leads to the zero curvature condition for $D$ (the $t$-part of the Bäcklund transformation (5.22))

$$\frac{dD_m(\lambda)}{dt} = \tilde{\mathcal{A}}_m(\lambda)D_m(\lambda) - D_m(\lambda)\mathcal{A}_m(\lambda) \quad (5.22)$$

where $\tilde{\mathcal{A}}_m$ in general is

$$\tilde{\mathcal{A}}_m(\lambda) = \begin{pmatrix} \lambda & \tilde{z}_m \\ \tilde{Z}_{m-1} & 0 \end{pmatrix},$$

but in our case $\tilde{\mathcal{A}}_m = \mathcal{A}_{m+1}$. We first focus on the Lax pair description and we assume that $\mathcal{A}_j$ are given in (5.3) for all the sites of the chain, that is we assume the existence of the fields $z_m$, $Z_m$. Then solving (5.22) we obtain the quantum Darboux-Bäcklund relations:

$$\beta_m = z_m - \tilde{z}_m$$

$$\gamma_m = \tilde{Z}_{m-1} - Z_{m-1}$$

$$\alpha_m^2 = \zeta - (\tilde{z}_m - \tilde{z}_m)(\tilde{Z}_{m-1} - Z_{m-1}),$$

where the expression for $\alpha$ follows from requiring the quantum determinant of the matrix $D$ is a constant. In our case here where we treat $D$ as a defect matrix: $\tilde{z}_m = z_{m+1}$, $\tilde{Z}_{m-1} = Z_m$.

The corresponding conserved charge in the presence of the local defects then reads as [35]

$$H = \frac{1}{2} \sum_{j \neq m} x_j^2 \partial^2 x_j + \left( \sum_{j \neq m} c_j x_j - \sum_{j \neq m, m-1} x_{j+1} \right) Z_j - z_{m+1} Z_{m-1}$$

$$- \beta_m Z_{m-1} - \gamma_m z_{m+1} + \frac{\alpha_m^2}{2},$$

then via relations (5.23) and recalling the map (5.3) the Hamiltonian is rewritten

$$H = \frac{1}{2} \sum_{j \neq m} x_j^2 \partial^2 x_j + \left( \sum_{j \neq m} c_j x_j - \sum_{j \neq m, m-1} x_{j+1} \right) \partial x_j + \frac{1}{2} \left( x_{m+1} \partial x_{m-1} - x_m \partial x_m - x_{m+1} \partial x_m - x_m \partial x_{m-1} + \zeta \right).$$

One can immediately read the corresponding SDEs, indeed for $j \neq m, m-1$ they are given by (5.6) and

$$dx_{tm-1} = \left( c_{m-1} x_{tm-1} + \frac{1}{2} (x_{tm+1} - x_{tm}) \right) dt + x_{tm-1} dw_{tm-1}$$

$$dx_{tm} = -\frac{1}{2} (x_{tm+1} + x_{tm}) dt.$$
We conclude from the equations above that the \( \sigma \) matrix in this case is not invertible as it has one zero eigenvalue, in particular \( \sigma = \text{diag}(x_1, x_2, \ldots, x_{m-1}, 0, x_{m+1}, \ldots, x_M) \), therefore the setting described in subsection 3.2.1 can be implemented.

Let us also employ the algebraic setting, in this case \( D \) is a representation of the algebra \((5.2)\), therefore as mentioned \( \alpha, \beta, \gamma \) are elements of \( \mathfrak{sl}_2 \) expressed in the Chevalley-Serre basis:

\[
\beta = \frac{1}{2} (S^1 - iS^2), \quad \gamma = \frac{1}{2} (S^1 + iS^2), \quad \alpha = \frac{1}{2} S^3.
\]

In the spin \( S \) representation \((5.19)\) they are given by the map:

\[
\beta_m \mapsto -x_m^2 \partial x_m + S x_m, \quad \gamma_m \mapsto \partial x_m + S x_m^{-1}, \quad \alpha_m \mapsto x_m \partial x_m,
\]

and the Hamiltonian \((5.24)\) becomes

\[
H = \frac{1}{2} \sum_{j \neq m} x_m^2 \partial x_j + \left( \sum_{j \neq m} c_j x_j - \sum_{j \neq m, m-1} x_{j+1} \right) \partial x_j
\quad - x_{m+1} \partial x_{m-1} + \left( x_m^2 \partial x_m - S x_m \right) \partial x_{m-1} - \left( \partial x_m + S x_m^{-1} \right) x_{m+1} + \frac{1}{2} x_m^2 \partial x_m + \frac{1}{2} x_m \partial x_m.
\]

The corresponding SDEs are given as before via \((5.6)\) for \( j \neq m, m-1 \), and

\[
dx_{m-1} = \left( c_{m-1} x_{m-1} - x_{m+1} - S x_{m} \right) dt + x_{m-1} dw_{m-1} + \sqrt{2} x_{m} dw_m
\]
\[
dx_m = \left( \frac{1}{2} x_m - x_{m+1} \right) dt + x_m dw_m + \sqrt{2} x_{m} dw_{m-1}.
\]

Notice that the diffusion matrix is not diagonal anymore and interestingly the \( \sigma \) matrix is invertible, therefore the quantum canonical transformation can be implemented subject to certain modifications associated to the presence of the defect. This model is integrable in the strong sense, and the Bethe ansatz techniques can be applied so the spectrum and eigenstates are available. It is also worth noting that in the algebraic setting one obtains deformed \( A \) operators around the defect point:

\[
A_m(\lambda) = \begin{pmatrix}
\lambda & \beta_m + z_{m+1} \\
Z_{m-1}^{-1} & 0
\end{pmatrix}, \quad A_{m-1} = \begin{pmatrix}
\lambda & z_{m+1} \\
\gamma_m + Z_{m-1}^{-1} & 0
\end{pmatrix}.
\]

The operators above become the usual bulk ones \((5.3)\) only in the continuum limit, via analyticity conditions implemented around the defect point. These analyticity conditions provide exactly Bäcklund type relations for the fields \([38]\). The classical equations of motion, which structurally coincide with the quantum ones are available for the model in the presence of local defects.

6 Generalizations & comments

We may now discuss certain generalizations associated to discrete quantum systems as well as quantum field theories. Let us first consider the obvious extension from SDEs associated
to vector fields $x$ to SDEs for generic matrix or tensor fields. Indeed, let us consider the
tensor field $Y$ with components $Y_{i_1...i_d}$, $d \in \mathbb{N}$, then the generator $\hat{L}_0$ is expressed

$$\hat{L}_0 = g_{i_1...i_d;j_1...j_d}(Y) \frac{\partial^2}{\partial Y_{i_1...i_d} \partial Y_{j_1...j_d}} + b_{i_1...i_d}(Y) \frac{\partial}{\partial Y_{i_1...i_d}}, \quad i_k \in \{1, \ldots, M\}.$$  

$g_{i_1...i_d;j_1...j_d}$ and $b_{i_1...i_d}$ are the generalized tensor diffusion coefficients and drift components
respectively, and we use the standard convention where repeated indices are summed. In
the $d = 2$ case for instance the generator above is associated to the partition function of a
matrix model (see e.g. review articles [41, 42] and references therein, and [43] in relation to
integrable models).

The SDEs associated to the generator -via the generalized Itô formula- are then given as

$$dY_{t_1...i_d} = b_{i_1...i_d}(Y_t) dt + \sigma_{i_1...i_d;j_1...j_d}(Y_t) dw_{t_1...j_d}$$

provided that the tensor Wiener processes satisfy:

$$dw_{t_1...i_d} dw_{t_1...j_d} = \delta_{i_1 j_1} \ldots \delta_{i_d j_d} dt,$$

and thus $g_{i_1...i_d;j_1...j_d} = \sigma_{i_1...i_d;k_1...k_d} \sigma_{j_1...j_d;k_1...k_d}$. In the continuum limit ($M \to \infty$), which
is of particular interest when studying quantum/statistical field theories at finite temperature
and SPDEs, the tensor fields become continuous space random fields depending on $t$ and
the continuum space parameters $x \in \mathbb{R}^d$, i.e

$$Y_{t_1...i_d} \to \varphi(x, t), \quad w_{t_1...i_d} \to W(x, t)$$

and the SDEs become SPDEs; $W(x, t)$ are the multi-dimensional Wiener fields or Brownian
sheets (white space-time noise). This description is also in line with the notion of stochastic
quantization in quantum field theory (see for example [44, 45]).

We come now to the issue of representing the Brownian sheet (we refer the interested
reader to [46]). Let $e_k$, $k \in \mathbb{N}$ be an orthonormal basis of a separable Hilbert space $\mathcal{H}$
consisting of eigen-vectors of an operator $Q$ with corresponding eigenvalues $\lambda_k$. Then the $\mathcal{H}$
valued stochastic process $W_t$, $t \in [0, t]$ (a $Q$-Wiener process) is represented as

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k(t) e_k$$

where $\beta_k(t)$ are independent real valued Brownian motions. Let us present as an example
the one spatial dimensional case, then the Wiener field, which is is periodic and square
integrable in $[-L, L]$ is

$$W(x, t) = \frac{\sqrt{L}}{\pi} \sum_{n \geq 1} \frac{1}{n} \left( X^{(n)}_t \cos \left( \frac{n \pi x}{L} \right) + Y^{(n)}_t \sin \left( \frac{n \pi x}{L} \right) \right),$$

(6.1)

where $X^{(n)}_t$, $Y^{(n)}_t$ are independent Brownian motions. In this case we have used as the operator
$Q$ the inverse Laplacian: $Q = (\partial^2)^{-1}$. The representation of higher dimensional Brownian
sheets becomes a more complicated issue involving multidimensional Fourier transforms. We
have already considered two fundamental examples i.e. the DST and DNLS models that
in the continuum limit led to the stochastic transport and the stochastic heat equations respectively.

The computation of expectation values via the solution of the associated SDEs for the DST (DNLS) and XXZ models is one of our main future goals. We have already at our disposal a formal series solution for the DST model, whereas solutions for the XXZ SDEs (or special cases such as the XX model) need to be derived using for instance the change of variables introduced in section 2. Moreover, we have discussed the issue of space like defects as a way to tackle the generic case where \( \det \sigma = 0 \), and we have provided a particular example i.e. the DST model in the presence of local defects. A relevant significant issue is the effect of non-trivial space and time like boundary conditions on the form of the generator of the Itô process as well as the form of the SDEs. This case is very interesting especially when one requires that the boundary conditions preserve the integrability of the model.

Extending the ideas on the construction and solution of non-linear non-local PDEs developed in [31] (i.e. generalizing the notion of the Darboux-dressing transform) in the case of SPDEs will lead to a modified scheme for producing and solving certain types of SDEs/SPDEs [32]. Also, the study of the possible connections between the algebraic structures arising when solving SDEs [26], and the deformed algebras associated to integrable systems is a particularly interesting direction to pursue.

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