LIFTING OF RECOLEMENTS AND GORENSTEIN PROJECTIVE MODULES

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Abstract. In the paper, we investigate the lifting of recollements with respect to Gorenstein-projective modules. Specifically, a homological ring epimorphism can induce a lifting of the recollement of the stable category of finitely generated Gorenstein-projective modules; the recollement of the bounded Gorenstein derived categories of some upper triangular matrix algebras can be lifted to the homotopy category of Gorenstein-projective modules. As a byproduct, we give a sufficient and necessary condition on the upper triangular matrix algebra $T_n(A)$ to be of finite CM-type for an algebra $A$ of finite CM-type.

1. Introduction

Derived categories, introduced by A.Grothendieck and J.L.Verdier [23], play an increasingly important role in various areas of mathematics, including representation theory, algebraic geometry, microlocal analysis and mathematical physics. Major topics of current interest include substructures of derived categories, such as bounded $t$-structures, which form the “skeleton” of Bridgeland’s stability manifold, as well as comparisons of derived categories.

Recollements of triangulated categories were introduced by Beilinson, Bernstein and Deligne [5] as a tool to get information about the derived category of sheaves over a topological space $X$ from the corresponding derived categories for an open subset $U \subseteq X$ and its complement $F = X \setminus U$. To any recollement one canonically associates the TTF triples. Moreover, it is natural to look for conditions under which recollements of triangulated categories at the “bounded” level lift to recollements at the “unbounded” level. Recently, M.Saorín and A.Zvonareva [22] gave a criterion for a recollement of triangulated subcategories to lift to a torsion torsion-free triple (TTF triple) of ambient triangulated categories with coproducts.

The study of Gorenstein homological algebra is due to Enochs and Jenda [10]. They introduced the concept of Gorenstein-projective modules, which are as a generalization of finitely generated modules of G-dimension zero over a two-sided Noetherian ring, in the sense of Auslander and Bridger [1]. The main idea of Gorenstein homological algebra is to replace projective modules by Gorenstein-projective modules, which is useful to study some Gorenstein properties. To intend to close a gap of the corresponding version of derived categories in Gorenstein homological algebra,
Gao and Zhang [16] introduce Gorenstein derived category. Recently, Zhang [25] characterized the recollement of the stable categories of Gorenstein-projective modules over the triangular matrix algebra. Gao-Xu [15] showed that the recollement of derived categories of algebras induced by some homological ring epimorphism produces a ladder of the stable categories of Gorenstein-projective modules over corresponding algebras.

In the paper, we investigate the lifting of recollements of bounded Gorenstein derived categories and the stable categories of Gorenstein-projective modules, respectively, where ladder is a crucial tool. We show that a homological ring epimorphism can induce a lifting of the recollement of the stable category of finitely generated Gorenstein-projective modules to non-finitely generated case (see Theorem 3.3). We prove that the recollement of the bounded Gorenstein derived categories of some upper triangular matrix algebra can be lifted to the homotopy category of Gorenstein-projective modules (see Theorem 3.10 and Example 3.7). As a byproduct, we give a sufficient and necessary condition on the upper triangular matrix algebra \( T_n(A) \) to be of finite CM-type for an algebra \( A \) of finite CM-type (see Proposition 3.9).

2. Preliminaries

In this section, we fix notations and recall some basic concepts.

Let \( A \) be an Artin algebra. Denote by \( A\text{-Mod} \) (resp. \( A\text{-mod} \)) the category of (resp. finitely generated) left \( A \)-modules, and \( A\text{-P} \) (resp. \( A\text{-proj} \)) the full category of (resp. finitely generated) projective \( A \)-modules. A module \( G \) of \( A\text{-Mod} \) (resp. \( A\text{-mod} \)) is Gorenstein-projective if there is an exact sequence

\[
\cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots
\]

in \( A\text{-P} \) (resp. \( A\text{-proj} \)), which stays exact after applying \( \text{Hom}_A(-, P) \) for each module \( P \) in \( A\text{-P} \) (resp. \( A\text{-proj} \)), such that \( G \cong \text{Ker} \ d^0 \) (see [10, 11]). Denote by \( A\text{-GP} \) (resp. \( A\text{-Gproj} \)) the full subcategories of Gorenstein-projective modules in \( A\text{-Mod} \) (resp. \( A\text{-mod} \)), and \( A\text{-GP}^\perp \) (resp. \( A\text{-Gproj}^\perp \)) the stable category of \( A\text{-GP} \) (resp. \( A\text{-Gproj} \)) that modulo \( A\text{-P} \) (resp. \( A\text{-proj} \)). Similarly, denote by \( A\text{-GI} \) the full subcategory of Gorenstein-injective modules in \( A\text{-Mod} \).

Recall from [4] that an Artin algebra \( A \) is of finite Cohen-Macaulay type (resp. finite CM-type for simply), if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein-projective modules. Recall from [17] that an Artin algebra \( A \) is Gorenstein if \( \text{inj.dim}_A A < \infty \) and \( \text{inj.dim}_A A < \infty \). Recall from [3, 7] that an Artin algebra is called virtually Gorenstein if \( (A\text{-GP})^\perp = (A\text{-GI})^\perp \), where

\[
(A\text{-GP})^\perp := \{ X \in A\text{-Mod} \mid \text{Ext}_A^i(G, X) = 0, \forall i > 0 \text{ and } \forall G \in A\text{-GP} \}
\]

and

\[
(A\text{-GI})^\perp := \{ Y \in A\text{-Mod} \mid \text{Ext}_A^i(Y, I) = 0, \forall i > 0 \text{ and } \forall I \in A\text{-GI} \}.
\]

Note that a Gorenstein algebra is virtually Gorenstein, but in general, the converse is not true.

Now we write \( C^b(A\text{-Mod}) \), \( K^b(A\text{-Mod}) \) and \( D^b(A\text{-Mod}) \) (resp. \( C^b(A) \), \( K^b(A) \) and \( D^b(A) \)) for the bounded complex category, bounded homotopy category and bounded derived category of \( A\text{-Mod} \) (resp. \( A\text{-mod} \)), respectively. Denote by
Let \( \mathcal{D} \) be a triangulated category. From [18], a torsion pair in \( \mathcal{D} \) is a pair of full subcategories \((\mathcal{X}, \mathcal{Y})\), which are closed under direct summands, and satisfy \( \text{Hom}_\mathcal{D}(\mathcal{X}, \mathcal{Y}) = 0 \) and \( \mathcal{D} = \mathcal{X} \ast \mathcal{Y} \), where
\[
\mathcal{X} \ast \mathcal{Y} := \{ M \in \mathcal{D} \mid \exists \text{ a triangle } X \to Y \to M \to X[1] \text{ with } X \in \mathcal{X}, Y \in \mathcal{Y} \}.
\]
A torsion pair is called a t-structure when \( \mathcal{X}[1] \subseteq \mathcal{X} \) (\( \equiv \mathcal{Y}[-1] \subseteq \mathcal{Y} \)) (see [5]). A TTF triple on \( \mathcal{D} \) is a triple \((\mathcal{U}, \mathcal{V}, \mathcal{W})\) of full subcategories of \( \mathcal{D} \) such that \((\mathcal{U}, \mathcal{V})\) and \((\mathcal{V}, \mathcal{W})\) are t-structures on \( \mathcal{D} \).

Let \( \mathcal{D}, \mathcal{X} \) and \( \mathcal{Y} \) be triangulated categories. Recall from [5] that \( \mathcal{D} \) is said to be a recollement of \( \mathcal{X} \) and \( \mathcal{Y} \) if there are six triangle functors as in the following diagram
\[
\begin{array}{ccc}
\mathcal{Y} & \xrightarrow{j^*} & \mathcal{D} \\
i^* & \downarrow{i_*} & \downarrow{j_*} \\
\mathcal{X} & \xleftarrow{i^!} & \mathcal{D} & \xrightarrow{j^!} & \mathcal{X}
\end{array}
\]
such that
1. \((i^*, i_*), (i_*, i^!), (j_!, j^*), (j^*, j_*))\) are adjoint pairs;
2. \(i_*, j_! \text{ and } j_*\) are fully faithful;
3. \(j^*i_* = 0\);
4. For each \( Z \in \mathcal{D} \), the counits and units give rise to distinguished triangles:
\[
j_!j^*Z \to Z \to i_*i^*Z \to \quad \text{and} \quad i_*i^!Z \to Z \to j_*j^*Z \to.
\]
We simply denote the recollement by \((\mathcal{Y} \equiv \mathcal{D} \equiv \mathcal{X})\). It is well known that TTF triples are in bijection with (equivalence classes of) recollement data [5, 19, 20]. If there is a recollement as above, then \((\text{Im } j, \text{Im } i_!, \text{Im } j_*)\) is a TTF triple in \( \mathcal{D} \). Conversely, if \((\mathcal{X}, \mathcal{Y}, \mathcal{Z})\) is a TTF triple in \( \mathcal{D} \), then one obtains a recollement as above, where \( j : \mathcal{X} \hookrightarrow \mathcal{D} \) and \( i_* : \mathcal{Y} \hookrightarrow \mathcal{D} \) are the inclusion functors.
Two recollements

\[
\begin{array}{cccccc}
Y & \xrightarrow{i^*} & \mathcal{D} & \xrightarrow{j_1} & \mathcal{X} \\
\downarrow{i_*} & & \downarrow{j_*} & & \downarrow{i} \\
Y' & \xrightarrow{i'^*} & \mathcal{D}' & \xrightarrow{j'_1} & \mathcal{X}' \\
\downarrow{i'_*} & & \downarrow{j'_*} & & \downarrow{i'} \\
\end{array}
\]

are said to be equivalent, if \((\text{Im } j_1, \text{Im } i_*, \text{Im } j_*) = (\text{Im } j'_1, \text{Im } i'_*, \text{Im } j'_*)\).

A ladder of recollements \(\mathcal{L}\) is a finite or infinite diagram of triangulated categories and triangle functors

\[
\begin{array}{cccccc}
\vdots & & \ldots & & \vdots \\
\downarrow{i_2} & & \downarrow{j_2} & & \downarrow{i_2} \\
\mathcal{D}' & \xrightarrow{j_1} & \mathcal{D} & \xrightarrow{i_0} & \mathcal{D}' \\
\downarrow{\ldots} & & \downarrow{\ldots} & & \downarrow{\ldots} \\
\mathcal{D}' & \xrightarrow{j_0} & \mathcal{D} & \xrightarrow{i_1} & \mathcal{D}' \\
\downarrow{\ldots} & & \downarrow{\ldots} & & \downarrow{\ldots} \\
\vdots & & \ldots & & \vdots \\
\end{array}
\]

such that any consecutive rows form a recollement (see [2, Section 3]; [6, Section 1.5]). The height of a ladder is the number of recollements contained in it (counted with multiplicities).

Let \(\mathcal{D}\) be a triangulated category with coproducts. A triangulated subcategory closed under arbitrary coproducts is called a localizing subcategory. Given a class \(S\) of objects of \(\mathcal{D}\), denote by \(\text{Loc}_{\mathcal{D}}(S)\) the smallest localizing subcategory containing \(S\). A compact object is an object \(M \in \mathcal{D}\) such that \(\prod_{i \in I} \text{Hom}_{\mathcal{D}}(M, N_i) \cong \text{Hom}_{\mathcal{D}}(M, \prod_{i \in I} N_i)\), for each family of objects \((N_i)_{i \in I}\). We say that \(\mathcal{D}\) is compactly generated when it has a generating set of compact objects. Denote by \(\mathcal{D}^c\) the subcategory of compact objects of \(\mathcal{D}\).

For a class of objects \(S\) in \(\mathcal{D}\), we consider the following subcategories of \(\mathcal{D}\):

\[
S^\perp = \{ M \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}(S, M) = 0 \text{ for any } S \in S \}
\]

and

\[
^\perp S = \{ M \in \mathcal{D} \mid \text{Hom}_{\mathcal{D}}(M, S) = 0 \text{ for any } S \in S \}.
\]

We say that \(S\) generates a torsion pair in \(\mathcal{D}\) if \((^\perp(S^\perp), S^\perp)\) is a torsion pair. If moreover \(S\) is a set of compact objects, \((^\perp(S^\perp), S^\perp)\) is called compactly generated.

### 3. Lifting of TTF triples

In this section, we investigate the lifting of recollements of bounded Gorenstein derived categories and the stable categories of Gorenstein-projective modules, respectively, where ladder is a crucial tool. Moreover, we give a sufficient and necessary condition on the upper triangular matrix algebra \(T_n(A)\) to be of finite CM-type for an algebra \(A\) of finite CM-type.

**Definition 3.1.** ( [21, Definition 2]) Let \(\mathcal{D}\) be a triangulated category and \(\mathcal{D}'\) the full triangulated subcategory of \(\mathcal{D}\). We say that a TTF triple \((U, V, W)\) in \(\mathcal{D}\)
restrict to or is a lifting of a TTF triple \((U', V', W')\) in \(\mathcal{D}'\), if we have
\[(U \cap \mathcal{D}', V' \cap \mathcal{D}', W' \cap \mathcal{D}') = (U', V', W').\]

Recall that, given thick subcategories \(Y' \subseteq Y\), \(\mathcal{D}' \subseteq \mathcal{D}\) and \(\mathcal{X}' \subseteq \mathcal{X}\) and recollements
\[
\begin{align*}
\xymatrix{ Y \ar[r]^{i_*} \ar[d]_{i^!} & \mathcal{D} \ar[r]^{j_*} \ar[d]_{j^!} & \mathcal{X} \ar[d]_{j^*} \\
Y' \ar[r]_{i'^*} & \mathcal{D}' & \mathcal{X}' }
\end{align*}
\]

We say that the recollement \((Y \equiv \mathcal{D} \equiv \mathcal{X})\) restricts, up to equivalence, to the recollement \((Y' \equiv \mathcal{D}' \equiv \mathcal{X}')\), or that the recollement \((Y' \equiv \mathcal{D}' \equiv \mathcal{X}')\) lifts, up to equivalence, to the recollement \((Y \equiv \mathcal{D} \equiv \mathcal{X})\), when \((\text{Im } j_! \cap \mathcal{D}', \text{Im } i_* \cap \mathcal{D}', \text{Im } j_* \cap \mathcal{D}') = (\text{Im } j'_* \cap \text{Im } i'_*, \text{Im } j'_*).

Next we are interest in compactly generated triangulated categories. The lemma we give below is implicit in [22, Theorem 3.3]. In the following, when we say TTF quadruple \((U, V, W, Z)\), we mean that both \((U, V, W)\) and \((V, W, Z)\) are TTF triples. TTF tuple can be similarly defined.

**Lemma 3.2.** Let
\[
\begin{align*}
\xymatrix{ Y \ar[r]^{i_*} \ar[d]_{i^!} & \mathcal{D} \ar[r]^{j_*} \ar[d]_{j^!} & \mathcal{X} \ar[d]_{j^*} \\
Y' \ar[r]_{i'^*} & \mathcal{D}' & \mathcal{X}' }
\end{align*}
\]
be a ladder of recollements of height 2, and \(Y, D, X\) the thick subcategories of compactly generated triangulated category \(Y, D, X\) which contain the respective subcategories of compact objects.

(1) The given recollement can lift to a ladder of recollements of height 3:
\[
\begin{align*}
\xymatrix{ \hat{Y} \ar[r]^{i'_*} \ar[d]_{i'_!} & \hat{\mathcal{D}} \ar[r]^{j'_*} \ar[d]_{j'_!} & \hat{X} \ar[d]_{j'_*} \\
Y' \ar[r]_{i'^*_*} & \mathcal{D}' & \mathcal{X}' }
\end{align*}
\]

(2) The functors \(i^!\) and \(j_*\) preserve compact objects, i.e. \(i^!(\hat{\mathcal{D}}^c) \subseteq \hat{Y}^c\) and \(j_*(\hat{X}^c) \subseteq \hat{\mathcal{D}}^c\). Moreover, \(\text{Im } i_*\) cogenerates \(\text{Loc}_{\hat{\mathcal{D}}^c}(i_*(\hat{Y}^c))\) and \(\text{Im } j_*\) cogenerates \(\text{Loc}_{\hat{D}^c}(j_*(\hat{X}^c))\), or \(\mathcal{D}\) cogenerates \(\hat{\mathcal{D}}\).

If one of the above two conditions is satisfied, then the TTF quadruple \((\text{Im } j_!, \text{Im } i_*, \text{Im } j_*\), \text{Im } i)\) lifts to a TTF quadruple \((U, V, W, Z)\) in \(\hat{\mathcal{D}}\) such that:

(a): \((U, V), (V, W)\) and \((W, Z)\) are compactly generated;

(b): \(j_!(\hat{X}^c) = U \cap \hat{\mathcal{D}}^c\), \(i_!(\hat{Y}^c) = V \cap \hat{\mathcal{D}}^c\) and \(j_!(\hat{X}^c) = W \cap \hat{\mathcal{D}}^c\).

**Proof.** It can be seen from [22, Theorem 3.3] that when condition (1) or (2) is satisfied, there are two lifted TTF triples \((U, V, W)\) and \((V, W, Z)\) in \(\hat{\mathcal{D}}\), such that (a) and (b) hold. \(\Box\)
Theorem 3.3. Let $A$ be a Gorenstein algebra, and $\lambda: A \rightarrow B$ a homological ring epimorphism which induces a recollement of derived categories

\[
\begin{array}{cccc}
D(B\text{-Mod}) & \xrightarrow{i^*} & D(A\text{-Mod}) & \xrightarrow{j^*} \\
i_\sigma & \quad & j_\sigma & \\
D(C\text{-Mod}) & \xrightarrow{i^!} & & \xrightarrow{j^!} \\
\end{array}
\]

of algebras $B$, $A$ and $C$ such that $j_\sigma$ restricts to $D^b(C\text{-mod})$. Assume that $\text{pd}_A B < \infty$, then the TTF tuple in $A\text{-Gproj}$ lifts to a TTF tuple in $A\text{-GP}$.

Proof. Following [15, Theorem 3.1], there is an unbounded ladder

\[
\begin{array}{cccc}
\vdots & \xrightarrow{i^*} & \vdots & \xrightarrow{j^*} \\
i_\sigma & \quad & j_\sigma & \\
\vdots & \xrightarrow{i^!} & \vdots & \xrightarrow{j^!} \\
B\text{-Gproj} & \xrightarrow{i_\sigma} & A\text{-Gproj} & \xrightarrow{j_\sigma} \\
\end{array}
\]

Since the functors $i_\sigma$ and $j_\sigma$ in this ladder have right adjoints, they preserve coproducts, and so the functors $i^*$ and $j^*$ preserve compact objects. Furthermore, it follows from [8, Theorem 4.1] that $A\text{-GP}$ is a compactly generated triangulated category, and $(A\text{-GP})^c = A\text{-Gproj}$. For

\[
(A\text{-GP})^b := \{ M \mid \text{Hom}_{A\text{-GP}}(X, M[k]) = 0 \text{ for each } X \in A\text{-Gproj} \text{ and } |k| \gg 0 \},
\]

we have that $A\text{-Gproj} \subseteq (A\text{-GP})^b$. This means from [22, Corollary 3.12] that $A\text{-Gproj}$ cogenerates $A\text{-GP}$. By Lemma 3.2, we have that the TTF quadruple $(\text{Im} j_\sigma, \text{Im} i_\sigma, \text{Im} j_\sigma, \text{Im} i_\sigma)$ lifts to a TTF quadruple $(U, V, W, Z)$ in $A\text{-GP}$ such that (a) and (b) hold. Consequently, we get the lifting of TTF tuple in the unbounded ladder. \qed

The lemma given below will be very useful.

Lemma 3.4. ([22, Corollary 3.5]) Let $\mathcal{Y}$, $\mathcal{D}$ and $\mathcal{X}$ be compactly generated triangulated categories and suppose that we have a recollement

\[
\begin{array}{cccc}
\mathcal{Y} & \xrightarrow{i^*} & \mathcal{D} & \xrightarrow{j^*} \\
i_\sigma & \quad & j_\sigma & \\
\end{array}
\]

Then the TTF triple $(\text{Im} j_\sigma, \text{Im} i_\sigma, \text{Im} j_\sigma)$ in $\mathcal{D}^c$ lifts to a TTF triple $(U, V, W)$ in $\mathcal{D}$, where the torsion pairs $(U, V)$ and $(V, W)$ are compactly generated.

Immediately, we can draw the following conclusions.

Corollary 3.5. Let $\Lambda = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ be a Gorenstein Artin algebra such that $M$ is a projective $A$-module. Then the TTF triple in $\Lambda\text{-Gproj}$ can lift to a TTF triple in $\Lambda\text{-GP}$.
Proof. On one hand, since \( \Lambda \) is Gorenstein and \( M \) is \( A \)-projective, we have from [25, Theorem 2.2] that \( A \) and \( B \) are Gorenstein. Furthermore, it follows from [8, Theorem 4.1] that \( \Lambda\text{-GP}, \ A\text{-GP} \) and \( B\text{-GP} \) are compactly generated triangulated categories, and moreover, we have the following equalities:

\[
(\Lambda\text{-GP})^c = \Lambda\text{-Gproj}, \ (A\text{-GP})^c = A\text{-Gproj}, \ (B\text{-GP})^c = B\text{-Gproj}.
\]

On the other hand, there exists a recollement by [25, Theorem 3.5] as follows:

\[
\begin{array}{ccc}
A\text{-Gproj} & \overset{i^*}{\longrightarrow} & \Lambda\text{-Gproj} \\
\downarrow i^! & & \downarrow j^! \\
\Lambda\text{-Gproj} & \underset{j^*}{\longrightarrow} & B\text{-Gproj}.
\end{array}
\]

Thus we get from Lemma 3.4 that the TTF triple \((\text{Im}\ j^!, \text{Im}\ i^*, \text{Im}\ j^*)\) in \(\Lambda\text{-Gproj}\) lifts to a TTF triple \((U, V, W)\) in \(\Lambda\text{-GP}\), and that the torsion pairs \((U, V)\) and \((V, W)\) are compactly generated. \(\square\)

Lemma 3.6. Let \( A, B \) and \( C \) be virtually Gorenstein algebras of finite CM-type. Assume that \(D^b_{gp}(A)\) admits the following recollement:

\[
\begin{array}{ccc}
D^b_{gp}(B) & \overset{i^*}{\longrightarrow} & D^b_{gp}(A) \\
\downarrow i^! & & \downarrow j^! \\
D^b_{gp}(A) & \underset{j^*}{\longrightarrow} & D^b_{gp}(C).
\end{array}
\]

Then the TTF triple \((\text{Im}\ j^!, \text{Im}\ i^*, \text{Im}\ j^*)\) in \(D^b_{gp}(A)\) lifts to a TTF triple \((U, V, W)\) in \(K(A\text{-GP})\).

Proof. Since \( A, B \) and \( C \) are virtually Gorenstein algebras of finite CM-type, we get from [12, Theorem 3.2] that three homotopy categories \(K(A\text{-GP}), K(B\text{-GP})\) and \(K(C\text{-GP})\) are compactly generated, and moreover,

\[
(K(A\text{-GP}))^c \cong D^b_{gp}(A), \quad (K(B\text{-GP}))^c \cong D^b_{gp}(B)
\]

and

\[
(K(C\text{-GP}))^c \cong D^b_{gp}(C).
\]

Thus the result follows from Lemma 3.4. \(\square\)

Example 3.7. Let \( A \) be a Gorenstein algebra of finite CM-type, and \( T_n(A) \) the upper triangular matrix algebra of \( A \). Then the following hold.

1. \(D^b_{gp}(T_n(A))\) admits a recollement:

\[
\begin{array}{ccc}
D^b_{gp}(T_{n-1}(A)) & \overset{i^*}{\longrightarrow} & D^b_{gp}(T_n(A)) \\
\downarrow i^! & & \downarrow j^! \\
D^b_{gp}(T_n(A)) & \underset{j^*}{\longrightarrow} & D^b_{gp}(A).
\end{array}
\]

2. The TTF triple \((\text{Im}\ j^!, \text{Im}\ i^*, \text{Im}\ j^*)\) in \(D^b_{gp}(T_n(A))\) lifts to a TTF triple \((U, V, W)\) in \(K(T_n(A)-\text{GP})\).
Proof. (1) Since $T_n(A)$ admits a ladder of recollements of l-height 2 and r-height 4 as follows:

$$
\begin{array}{ccc}
T_{n-1}(R)-\text{mod} & \xrightarrow{i} & T_n(R)-\text{mod} \\
\downarrow & & \downarrow \\
\text{R-mod} & \xrightarrow{r} & \end{array}
$$

such that $q$ and $i$ are exact. It follows from [14, Theorem 7.4] that $l$, $e$ and $r$ preserve Gorenstein-projective modules. Moreover, by definitions of $q$, $i$ and $p$, we have that they preserve Gorenstein projective modules. It induces the following recollement of Gorenstein derived categories:

$$
\begin{array}{ccc}
D^b_{\text{gp}}(T_{n-1}(A)) & \xrightarrow{D^b(q)} & D^b_{\text{gp}}(T_n(A)) \\
\downarrow & & \downarrow \\
D^b_{\text{gp}}(A) & \xrightarrow{D^b(r)} & D^b_{\text{gp}}(A).
\end{array}
$$

(2) Since $A$ is Gorenstein, it follows that $T_n(A)$ and $T_{n-1}(A)$ are Gorenstein. Since $A$ is CM-finite, it follows from [9, Example 3.2] that $T_n(A)$ and $T_{n-1}(A)$ are CM-finite. Thus we can get from Lemma 3.9 that the TTF triple in $D^b_{\text{gp}}(T_n(A))$ can lift to a TTF triple in $K(T_n(A)-\text{GP})$. \qed

In fact, we will give a criterion for the upper triangulated matrix algebra $T_n(A)$ to be of finite CM-type for the Gorenstein algebra $A$. Before this, we need to make some preparations.

Construction. By an $(A-\text{Gproj})$-module, we mean an additive contravariant functor $F : A-\text{Gproj} \rightarrow \text{Ab}$, where $\text{Ab}$ denotes the category of abelian groups.

We say that $X$ lies in the homotopy equivalence classes of $Y$, which means that the projective resolutions of $X$ and $Y$ are homotopy equivalent. We denote by $\text{ht}(A-\text{mod})$ the category of the homotopy equivalence classes of $A$-modules. Put

$$p\text{Gp}^{\leq n}(A) := \{ X \in \text{ht}(A-\text{mod}) \mid \exists \text{ a proper Gorenstein-projective resolution } 0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_0 \rightarrow X \rightarrow 0 \}$$

and

$$p\text{d}^{\leq n}((A-\text{Gproj})-\text{mod}) := \{ F \in \text{ht}((A-\text{Gproj})-\text{mod}) \mid \exists \text{ a projective resolution } 0 \rightarrow (-, G_n) \rightarrow (-, G_{n-1}) \rightarrow \cdots \rightarrow (-, G_0) \rightarrow F \rightarrow 0 \}.$$

Let $X \in p\text{Gp}^{\leq n}(A)$. Consider the proper Gorenstein-projective resolution of $X$:

$$0 \rightarrow G_n \xrightarrow{d_n} G_{n-1} \rightarrow \cdots \rightarrow G_1 \xrightarrow{d_1} G_0 \rightarrow X \rightarrow 0.$$

Then we get the following exact sequence:

$$0 \rightarrow (-, G_n) \xrightarrow{(-, d_n)} (-, G_{n-1}) \rightarrow \cdots \rightarrow (-, G_1) \xrightarrow{(-, d_1)} (-, G_0) \rightarrow F_X \rightarrow 0.$$
given by $F(X) = F_X$. For a morphism $\alpha : X \to X'$ in $\text{pGp}^{\leq n}(A)$, there is the following commutative diagram

$$
\begin{array}{cccccccccc}
0 & \rightarrow & G_n & \xrightarrow{d_n} & G_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \rightarrow & G_1 & \xrightarrow{d_1} & G_0 & \xrightarrow{\alpha} & X & \rightarrow & 0 \\
& & \downarrow{\alpha_n} & & \downarrow{\alpha_{n-1}} & & \cdots & & \downarrow{\alpha_1} & & \downarrow{\alpha_0} & & \downarrow{\alpha} & & \\
0 & \rightarrow & G'_n & \xrightarrow{d'_n} & G'_{n-1} & \xrightarrow{d'_{n-1}} & \cdots & \rightarrow & G'_{1} & \xrightarrow{d'_1} & G'_0 & \xrightarrow{\alpha} & X' & \rightarrow & 0.
\end{array}
$$

We define $F(\alpha) : F_X \to F_{X'}$ to be the unique morphism $(-, \alpha_0)$, which is obtained by making the following diagram commutative

$$
\begin{array}{cccccccccccc}
0 & \rightarrow & (-, G_n) & \xrightarrow{(-, d_n)} & (-, G_{n-1}) & \xrightarrow{(-, d_{n-1})} & \cdots & \rightarrow & (-, G_1) & \xrightarrow{(-, d_1)} & (-, G_0) & \xrightarrow{(-, \alpha)} & F_X & \rightarrow & 0 \\
& & \downarrow{(-, \alpha_n)} & & \downarrow{(-, \alpha_{n-1})} & & \cdots & & \downarrow{(-, \alpha_1)} & & \downarrow{(-, \alpha_0)} & & \downarrow{-, \beta} & & \\
0 & \rightarrow & (-, G'_n) & \xrightarrow{(-, d'_n)} & (-, G'_{n-1}) & \xrightarrow{(-, d'_{n-1})} & \cdots & \rightarrow & (-, G'_{1}) & \xrightarrow{(-, d'_1)} & (-, G'_0) & \xrightarrow{(-, \alpha)} & F_{X'} & \rightarrow & 0.
\end{array}
$$

The composition of morphisms will be defined naturally. Clearly, $F$ is an additive functor.

Denote by $\text{Mor}_n(A)$ the morphism category of $A$. It is well-known that there is an equivalence between $T_n(A)$-mod and $\text{Mor}_n(A)$. Denote by $S_n(A)$ the full subcategory of $\text{Mor}_n(A)$ consisting of all monomorphisms.

**Lemma 3.8.** ([24, Corollary 4.1(ii)]) Let $A$ be a Gorenstein algebra. Then

$T_n(A)$-Gproj $= S_n(A)$-Gproj.

We are ready to give the criterion for $T_n(A)$ to be of finite CM-type.

**Proposition 3.9.** Let $A$ be an Artin algebra. Then there are equivalence of additive categories for any positive integer $n$

$$
F : \text{pGp}^{\leq n}(A) \rightarrow \text{pd}^{\leq n}((A\text{-Gproj})\text{-mod}).
$$

Consequently, if $A$ is Gorenstein, then $T_n(A)$ is of finite CM-type if and only if $\text{pd}^{\leq n}((A\text{-Gproj})\text{-mod})$ is of finite type.

**Proof.** We first show that $F$ is full. Let $X$ and $X'$ be in $\text{pGp}^{\leq n}(A)$ and $\bar{\alpha} : F(X) \to F(X')$ the morphism in $\text{pd}^{\leq n}((A\text{-Gproj})\text{-mod})$. Obviously, this morphism can be lifted to the morphism between the projective resolutions of $F(X)$ and $F(X')$, which are induced by the proper Gorenstein-projective resolutions of $X$ and $X'$ respectively. Then by Yoneda’s lemma there exists the morphism $\alpha : X \to X'$ such that $F(\alpha) = \bar{\alpha}$.

Next we show that $F$ is faithful. Let $\alpha : X \to X'$ be a morphism in $\text{pGp}^{\leq n}(A)$ such that $F(\alpha) = 0$. This implies that the morphism $\bar{\alpha} : F_X \to F_{X'}$ is null-homotopic. From the following commutative diagram

$$
\begin{array}{cccccccccccc}
0 & \rightarrow & (-, G_n) & \xrightarrow{(-, d_n)} & (-, G_{n-1}) & \xrightarrow{(-, d_{n-1})} & \cdots & \rightarrow & (-, G_1) & \xrightarrow{(-, d_1)} & (-, G_0) & \xrightarrow{\alpha} & F_X & \rightarrow & 0 \\
& & \downarrow{(-, \alpha_n)} & & \downarrow{(-, \alpha_{n-1})} & & \cdots & & \downarrow{(-, \alpha_1)} & & \downarrow{(-, \alpha_0)} & & \downarrow{-, \beta} & & \\
0 & \rightarrow & (-, G'_n) & \xrightarrow{(-, d'_n)} & (-, G'_{n-1}) & \xrightarrow{(-, d'_{n-1})} & \cdots & \rightarrow & (-, G'_{1}) & \xrightarrow{(-, d'_1)} & (-, G'_0) & \xrightarrow{\alpha} & F_{X'} & \rightarrow & 0.
\end{array}
$$

and Yoneda’s lemma, there exists a series of morphisms $h_i : G_i \to G'_{i+1}$ for $i = 0, 1, \ldots, n-1$ such that $\alpha_0 = d'_1h_0, \alpha_i = h_{i-1}d_i + d'_{i+1}h_i$ and $\alpha_n = h_{n-1}d_n$ for
\( i = 1, 2, \cdots, n - 1 \), as shown in the diagram below:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & G_n & \rightarrow & G_{n-1} & \rightarrow & \cdots & \rightarrow & G_1 & \rightarrow & G_0 \\
\downarrow & \alpha_n & & \downarrow & h_{n-1} & \alpha_{n-1} & & \downarrow & \alpha_1 & & \downarrow & h_0 & \alpha_n \\
0 & \rightarrow & G'_n & \rightarrow & G'_{n-1} & \rightarrow & \cdots & \rightarrow & G'_1 & \rightarrow & G'_0.
\end{array}
\]

This implies that \( \alpha \) is null-homotopic, and hence \( \mathcal{F} \) is faithful.

Finally, we show that \( \mathcal{F} \) is dense. Let \( F \in \text{pd}^{\leq n}((A-\text{proj})\text{-mod}) \). Taking a projective resolution of \( F \) as follows

\[
0 \rightarrow (-, G_m) \rightarrow (-, G_{m-1}) \rightarrow \cdots \rightarrow (-, G_1) \rightarrow (-, G_0) \rightarrow F \rightarrow 0
\]

with \( m \leq n \), we get the exact sequence

\[
0 \rightarrow G_m \rightarrow G_{m-1} \rightarrow \cdots \rightarrow G_1 \xrightarrow{d_0} G_0
\]

which is the proper Gorenstein-projective resolution of \( \text{Coker} d_0 \), and \( \mathcal{F}(\text{Coker} d_0) = F \). \( \square \)

**Theorem 3.10.** Let \( A, B \) and \( C \) be virtually Gorenstein algebras of finite CM-type, and

\[
\begin{array}{ccccccccc}
K(B-GP) & \xrightarrow{i_*} & K(A-GP) & \xrightarrow{j_*} & K(C-GP) \\
\downarrow & i^{-} & & \downarrow & j^{-} & & \downarrow & i^{-} & & \downarrow & j^{-} \\
D_{gp}^b(B) & \xrightarrow{i_*} & D_{gp}^b(A) & \xrightarrow{j_*} & D_{gp}^b(C);
\end{array}
\]

a ladder of recollements of height two. The following are equivalent:

1. The recollement can restrict to a recollement

\[
D_{gp}^b(B) \xrightarrow{i_*} D_{gp}^b(A) \xrightarrow{j_*} D_{gp}^b(C);
\]

2. The associated TTF triple \( (\text{Im} j_, \text{Im} i, \text{Im} j_*) \) can restrict to \( D_{gp}^b(A) \).

If either of the above conditions holds, then \( A \) is Gorenstein if and only if \( B \) and \( C \) are also.

**Proof.** Since \( A, B \) and \( C \) are virtually Gorenstein algebras of finite CM-type, we get from [12, Theorem 3.2] that three homotopy categories \( K(A-GP), K(B-GP) \) and \( K(C-GP) \) are compactly generated, and moreover,

\[
(K(A-GP))^c \cong D_{gp}^b(A), \quad (K(B-GP))^c \cong D_{gp}^b(B)
\]

and

\[
(K(C-GP))^c \cong D_{gp}^b(C).
\]

This means the equivalence of the two conditions by [22, Proposition 3.13].

Under the condition, Theorem 3.5 in [13] tells us that \( A \) is Gorenstein if and only if so are \( B \) and \( C \). \( \square \)
REFERENCES

[1] M.Auslander, M.Bridger, Stable Module Theory, Mem. Amer. Math. Soc., vol. 94, Amer. Math. Soc., Providence, RI, 1969.
[2] L.Angeleri Hügel, S.Koenig, Q.H.Liu, D.Yang, Ladders and simplicity of derived module categories, J. Algebra 472(2017), 15-66.
[3] A.Beligiannis A. Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, J. Algebra 288(2005), 137-211.
[4] A.Beligiannis, On algebras of finite Cohen-Macaulay type, Adv. Math. 226(2011), 1973-2019.
[5] A.Beilinson, J.Bernstein, P.Deligne, Faisceaux Pervers, Analysis and topology on singular spaces, I (Luminy, 1981), 5-171, Asterisque 100 Soc. Math. France, Paris 1982.
[6] A.Beilinson, V.Ginsburg, V.Schechtman, Koszul duality, J. Geom. Phys. 5(1988), 317-350.
[7] A.Beligiannis, I.Reiten, Homological and Homotopical Aspects of Torsion Theories, Providence: Amer. Math. Soc. 2007.
[8] X.W.Chen, Relative singularity categories and Gorenstein-projective modules, Math. Nachr. 284(2011), 199-212.
[9] H.Eshraghi, R.Hafezi, Sh.Salarian, Z.W.Li, Gorenstein projective modules over triangular matrix rings, Algebra Colloq. 23(2016), 97-104.
[10] E.E.Enochs, O.M.G.Jenda, Gorenstein injective and projective modules, Math. Z. 220(1995), 611-633.
[11] E.E.Enochs, O.M.G.Jenda, Relative homological algebra, Volume 1 (Second revised and extended edition), De Gruyter Expositions in Mathematics, 30, Walter de Gruyter GmbH and Co. KG, Berlin, 2011.
[12] N.Gao, On homotopy categories of Gorenstein modules: compact generation and dimensions, Homology Homotopy Appl. 17(2015), 13-24.
[13] N.Gao, Gorensteinness, homological invariants and Gorenstein derived categories, Sci. China Math. 60(2017), 431-438.
[14] N.Gao, S.Koenig, C. Psaroudakis, Ladders of Recollements of Abelian Categories, J. Algebra 579(2021), 256-302.
[15] N.Gao, X.J.Xu, Homological epimorphisms, compactly generated t-structures and Gorenstein-projective modules, Chinese Ann. Math. Ser. B 39(2018), 47-58.
[16] N.Gao, P.Zhang, Gorenstein derived categories, J. Algebra 323(2010), 2041-2057.
[17] D.Happel, On Gorenstein algebras. In: Representation Theory of Finite Groups and Finite-Dimensional Algebras, Progress in Mathematics, vol. 95. Basel: Birkhäuser, 1991, 389-404.
[18] O.Iyama, Y.Yoshino, Mutations in triangulated categories and rigid Cohen-Macaulay modules, Invent. Math. 172(2008), 117-168.
[19] A.Neeman, Triangulated Categories, Ann. of Math. Stud., vol. 148, Princeton University Press, 2001.
[20] P.Nicolás, On torsion torsionfree triples, PhD thesis, Universidad de Murcia, 2007.
[21] P.Nicolás, M.Saorín, Lifting and restricting recollement data, Appl. Categ. Structures 19(2011), 557-596.
[22] M.Saorín, A.Zvonareva, Lifting of recollements and gluing of partial silting sets, Proc. Roy. Soc. Edinburgh 152(2022), 209-257.
[23] J.L.Verdier, Des categories derivees abeliennes, Asterisque 239(1996).
[24] P.Zhang, Monomorphism categories, cotilting theory, and Gorenstein-projective modules, J. Algebra 339(2011), 181-202.
[25] P.Zhang, Gorenstein-projective modules and symmetric recollements, J. Algebra 388(2013), 65-80.