Rapid Decay is Preserved by Graph Products

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Abstract

We prove that the rapid decay property (RD) of groups is preserved by graph products defined on finite simplicial graphs.

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1 Introduction

The main result proved in this paper is that the graph product of finitely many groups with the rapid decay property (RD) has the same property. This is stated formally in Section 4 as Theorem 4.1.

We call a function \( \ell : G \to \mathbb{R} \) a length function for a group \( G \) if it satisfies

\[
\ell(1_G) = 0, \quad \ell(g^{-1}) = \ell(g), \quad \ell(gh) \leq \ell(g) + \ell(h), \quad \forall g, h \in G.
\]

Following Jolissaint [17], a group \( G \) is said to have RD if the operator norm \( ||.||_\ast \) for the group algebra \( \mathbb{C}G \) is bounded by a constant multiple of the Sobolev norm \( ||.||_{2,r,\ell} \), a norm that is a variant of the \( l^2 \) norm weighted by a length function \( \ell \) for \( G \).

More precisely, RD holds for \( G \) if there are constants \( C, r \) and a length function \( \ell \) on \( G \) such that for any \( \phi, \psi \in \mathbb{C}G \),

\[
||\phi||_\ast := \sup_{\psi \in \mathbb{C}G} \frac{||\phi * \psi||_2}{||\psi||_2} \leq C||\phi||_{2,r,\ell}.
\]

Here, \( \phi * \psi \) denotes the convolution of \( \phi \) and \( \psi \), \( ||.||_2 \) the standard \( l^2 \) norm, and \( ||.||_{2,r,\ell} \) the Sobolev norm of order \( r \) with respect to \( \ell \). So we have:

\[
\phi * \psi(g) = \sum_{h \in G} \phi(h)\psi(h^{-1}g),
\]

\[
||\psi||_2 = \sqrt{\sum_{g \in G} |\psi(g)|^2},
\]

\[
||\phi||_{2,r,\ell} = \sqrt{\sum_{g \in G} |\phi(g)|^2(1 + \ell(g))^{2r}}.
\]
A brief introduction to RD is given in [2]; it is relevant to the Baum-Connes and Novikov conjectures (see an account in [22]). RD was originally studied by Jolissaint [17], after it emerged from work of Haagerup [11], who proved it for free groups. Jolissaint extended Haagerup’s methods to prove it for classical hyperbolic groups (i.e. discrete cocompact groups of isometries of hyperbolic space), and to prove that free and direct products of groups with RD inherit the property, as do subgroups; de la Harpe [14] extended Jolissaint’s arguments for classical hyperbolic groups to derive RD for word hyperbolic groups. More recently, Drutu and Sapir have shown that a group that is hyperbolic relative to a family of parabolic subgroups with RD must itself have RD [9]. In all this work the proofs focus on the factorisations of geodesic words. Other authors prove RD through examination of actions of the group [21, 2].

The graph product construction is a natural generalisation of both direct and free products. Given a finite simplicial graph $\Gamma$ with a group attached to each vertex, the associated graph product is the group generated by the vertex groups with the added relations that elements of groups attached to adjacent vertices commute; the representation of such a group as a graph product of directly indecomposable groups is proved to be unique [20].

Right-angled Artin groups [3] (also known as graph groups) and right-angled Coxeter groups arise in this way, as the graph products of infinite cyclic groups and cyclic groups of order 2 respectively, and have been widely studied; some groups with rather interesting properties arise via graph products, including a group (a subgroup of a right-angled Artin group) that has FP but is not finitely presented [1] and a group $(F_2 \times F_2)$ with insoluble subgroup membership problem [19]. Both right-angled Artin groups and right-angled Coxeter groups are already known to possess RD, through their actions on CAT(0) cube complexes [2] (indeed all finite rank Coxeter groups possess RD for this reason [6]).

Graph products were introduced by Green in her PhD thesis [10] where, in particular, a normal form was developed and the graph product construction was shown to preserve residual finiteness; this work was extended by Hsu and Wise in [13] where, in particular, right-angled Artin groups were shown to embed in right-angled Coxeter groups and hence to be linear. The preservation of semihyperbolicity, automaticity (as well as asynchronous automaticity and biautomaticity) and the possession of a complete rewrite system under graph products is proved in [12], necessary and sufficient conditions for the preservation of hyperbolicity in [18], the question of when the group is virtually free in [16], of orderability in [4]. Automorphisms and the structure of centralisers for graph products of groups have been the subject of recent study, and in particular graph products defined over random graphs have provoked some interest for their applications [8, 5, 4].

Our proof that the graph product construction preserves rapid decay will build on the methods used by Jolissaint for direct and free products [17]. We use a reformulation of RD due to Jolissaint, explained in Section 2 below, which compares norms on elements of $CG$ with restricted support. We examine in Section 3 the geodesic decompositions of elements in a graph product, and show that they satisfy particularly useful properties, which will be applied in our proof. Section 4 states our main result, Theorem 4.1, and reduces its proof to
the proof of a further technical condition, which is similar to that of Jolissaint’s reformulation, but easier to verify in the context of graph and free products; the same condition is used in [17] for free products. The remaining two sections are devoted to the proof of Proposition 4.3 thereby completing the proof of Theorem 4.1.

2 A reformulation of the rapid decay property

By [17] Lemma 2.1.3, any length function \( \ell \) on \( G \) is equivalent to one with \( \ell(G) \subseteq \mathbb{N} \) and \( \ell(g) > 0 \ \forall g \neq 1 \). Since RD is invariant under length equivalence, by [17] Remark 1.1.7, we shall assume from now on that all length functions have this property.

Given a length function \( \ell \) on \( G \), and \( k \in \mathbb{N} \), we define \( C_k(\ell) \) to be the set \( \{ g \in G | \ell(g) = k \} \). We write \( \chi_k \) for the characteristic function on \( C_k \), and for \( \phi \in \mathcal{C}G \), we write \( \phi \chi_k \) for the pointwise product \( \phi \chi_k \).

It is proved by Jolissaint [17, Proposition 1.2.6] that RD for \( G \) is equivalent to the following condition:

\[
(\ast) \quad \text{There exist } c,r > 0 \text{ such that } \forall \phi, \psi \in \mathcal{C}G, \ k,l,m \in \mathbb{N} : \\
\| (\phi_k \ast \psi_l)_m \|_2 \leq c \| \phi_k \|_2 \| \psi_l \|_2 \text{ if } |k - l| \leq m \leq k + l, \\
\| (\phi_k \ast \psi_l)_m \|_2 = 0 \text{ otherwise.}
\]

It follows from the properties of a length function that \( \| (\phi_k \ast \psi_l)_m \|_2 = 0 \) for \( m \) outside the range \( [k - l, k + l] \). Hence we shall establish RD by verifying the following condition:

\[
(\ast\ast) \quad \text{There exists a polynomial } P(x) \text{ such that } \forall \phi, \psi \in \mathcal{C}G, \ k,l,m \in \mathbb{N} : \\
|k - l| \leq m \leq k + l \ \Rightarrow \ \| (\phi_k \ast \psi_l)_m \|_2 \leq P(k) \| \phi_k \|_2 \| \psi_l \|_2.
\]

3 Graph products

Graph products of groups are studied in detail in [10], and we shall make use of the results from that thesis. Let \( \Gamma = (V,E) \) be a finite simplicial graph, together with vertex groups \( G_v \) for each \( v \in V \). The associated graph product \( G \) is defined to be the quotient of the free product of the groups \( G_v \) by the normal closure of all the commutators \( [g, g'] \) for which \( g \in G_v, \ g' \in G_w \) and \( \{v, w\} \) is an edge of the graph. We write \( G = G(\Gamma; G_v, v \in V) \).

Given such a group \( G \), we define \( K \) to be the set of cliques (including the empty set) of the associated graph \( \Gamma \), which we can identify with a set of subsets of the vertex set \( V \) of \( \Gamma \). We define \( K_m \) to be the subset of \( K \) of cliques of size \( m \).

Given any subset \( J \) of \( V \) we define \( G_J \) to be the subgroup of \( G \) generated by the elements of its subgroups \( G_v \) with \( v \in J \). It follows from [10] Proposition 3.31]
that $G_J$ is naturally isomorphic to the graph product defined by the induced subgraph of $\Gamma$ with vertex set $J$. If $J$ is a clique, then $G_J$ is the direct product of its vertex groups.

Every element of the group $G$ can be written as a product $y_1 \cdots y_k$ for $k \geq 0$ and with each $y_i$ in a vertex group $G_{v_i}$; that is, each element of the group has a representation as a word over the set $S = \cup_{v \in V} (G_v \setminus \{1\})$. We call such a representation an expression, the elements $y_i$ the syllables of the expression, and say that the expression has syllable length $k$; we define the syllable length of $g$, $\lambda(g)$, to be the minimum of the syllable lengths of expressions for $g$, and say that an expression for $g$ is reduced if it has syllable length $\lambda(g)$. The function $\lambda$ is easily seen to be a length function, but it is not the one that we shall use to prove RD.

We define $\Lambda_k = \{g \in G : \lambda(g) = k\}$.

Note that when $G$ is a free product, every expression for which consecutive $y_i$s come from distinct vertex groups is reduced, and each reduced expression corresponds to a distinct element of $G$. That is not true in general. But it is proved in [10, Theorem 3.9] that any expression for $g$ can be transformed to any reduced expression by a sequence of replacements of the form $y'y \rightarrow yy'$ where $y, y'$ belong to commuting vertex groups, or $y'y'' \rightarrow y$, where $y'y'' = y$ is a relation holding between three elements of one of the vertex groups, or deletion of $yy''$ where the $y', y''$ are mutually inverse elements of a vertex group. Hence any reduced expression for an element $g$ involves the same syllables, but the order of the syllables in the expression is not determined.

We shall need to estimate the number of factorisations of elements in graph products. For $g \in \Lambda_{k+l}$, we use the notation

$$ F_k^l(g) := \{(g_1, g_2) \mid g = g_1 g_2, \lambda(g_1) = k, \lambda(g_2) = l, \}, $$

$$ F_k^l := \sup_{g \in \Lambda_{k+l}} |F_k^l(g)|. $$

Similarly, given a clique $J \in K$ and $g \in \Lambda_{k+l+|J|}$, we use the notation

$$ F^l_k(J, g) := \{(g_1, s, g_2) \mid g = g_1 sg_2, \lambda(g_1) = k, \lambda(g_2) = l, s \in G, \lambda(s) = |J|\}, $$

$$ F^l_k(J) := \sup_{g \in \Lambda_{k+l+|J|}} |F^l_k(J, g)|. $$

By considering factorisations of $g^{-1}$, we see that $F_k^l = F_k^l$ and $F^l_k(J) = F^l_k(J)$.

Let $g, h \in G$. We say that $h$ is a left divisor of $g$, if $\lambda(g) = \lambda(h) + \lambda(h^{-1}g)$, or, equivalently, if $h$ has a reduced expression $v$ that is a prefix of a reduced expression $w$ for $g$. We define right divisors similarly.

**Lemma 3.1.** For each $J \in K$, $F^l_k(J)$ is bounded by a polynomial in $\min(k, l)$.

**Proof.** Since $F^l_k(J) = F^l_k(J)$, it is sufficient to prove that $F^l_k(J)$ is bounded by a polynomial in $k$.

Let $g \in \Lambda_m$ with $m \geq k + l$, and consider factorisations $g = g_1 sg_2$ with $g_1 \in \Lambda_k$, $g_2 \in \Lambda_l$, and $s \in G_J$ with $\lambda(s) = |J|$.
If there is cancellation between \( s \) and there is a clique \( \Gamma = 0 \) then \( s \). Then pick a right divisor with respect to syllable length, such that \( \lambda \).

Similarly, we have an upper bound of \( |J| \) for the number of left divisors of length \( |J| \) for \( g \) for a given \( g_1 \). In fact, the bound for the number of choices for \( s \) is \( (|J|+1)^{|\frac{d}{2}} \), as \( s \in G \). The right divisor \( g_2 \) is then completely determined by \( g, g_1 \) and \( s \). This results in the inequality \( F_k^l(J) \leq (k+1)^{|J|}(|J|+1)^{|\frac{d}{2}} \), giving the required polynomial bound in \( k \).

**Lemma 3.2.** Suppose that \( g, h_1, h_2 \in G \) with \( g = g_1 h_1 h_2, h_1 \in \Lambda_\kappa, h_2 \in \Lambda_1, \) and \( g \in \Lambda_{\kappa+1-q} \) with \( q \geq 0 \). Then \( h_1 = g \) \( g_1 \) \( s_1 w \) and \( h_2 = g \) \( w^{-1} \) \( s_2 \). where:

1. \( s_1, s_2 \in J \) for some \( J \in \mathcal{K}, \) and \( \lambda(s_1) = \lambda(s_2) = \lambda(s_1 s_2) = |J| \).
2. \( q = |J| + 2 \lambda(w) \).

**Proof.** Suppose first that there is no cancellation between \( h_1 \) and \( h_2 \), that is, when we reduce the expression \( h_1 h_2 \), no syllable becomes equal to the identity. Then pick a right divisor \( s_1 \) of \( h_1 \), and a left divisor \( s_2 \) of \( h_2 \), both minimal with respect to syllable length, such that \( \lambda(s_1 s_2) = \lambda(s_1) + \lambda(s_2) - q \). (If \( q = 0 \) then \( s_1 \) and \( s_2 \) are empty.) Minimality ensures that every syllable of \( s_1 \) (or \( s_2 \)) must merge without cancelling with a symbol of \( s_2 \) (or \( s_1 \)). So \( \lambda(s_1) = \lambda(s_2) = \lambda(s_1 s_2) \), the syllables within each of \( s_1 \) and \( s_2 \) must commute, and there is a clique \( J \in \Gamma \) of size \( q \) such that \( s_1, s_2, s_1 s_2 \in G \). If there is cancellation between \( h_1 \) and \( h_2 \), then we can find a right divisor \( w \) of
such that \( w^{-1} \) is a left divisor of \( h_2 \). We choose \( w \) of maximal syllable length, so that \( h_1 = h'_1 w \) and \( h_2 = w^{-1} h'_2 \) for some \( h'_1, h'_2 \in G \), and no cancellation occurs between \( h'_1 \) and \( h'_2 \). Applying the above argument to \( h'_1 \) and \( h'_2 \), we find a clique \( J \) of size \( q - 2\lambda(w) \), a right divisor \( s_1 \) of \( h'_1 \), and a left divisor \( s_2 \) of \( h'_2 \), with \( \lambda(s_1) = \lambda(s_2) = \lambda(s_1 s_2) = |J| \).

### 4 Main result

We prove

**Theorem 4.1.** Suppose that \( G = G(\Gamma; G_v, v \in V) \) is a graph product of groups defined with respect to a finite simplicial graph \( \Gamma = (V, E) \), and suppose that each vertex group \( G_v \) satisfies RD. Then \( G \) satisfies RD.

The proof generalises Jolissaint’s proof in [17] of RD for free products, which is itself a generalisation of Haagerup’s proof for free groups. We use the result of [17, Lemma 2.1.2], which implies (although it is not stated explicitly) that RD is preserved by taking direct products, and rely on the existence of polynomial bounds relating to factorisation in graph products proved in Lemma 3.1 and Lemma 3.2.

The following easy consequence of the Cauchy-Schwarz inequality will be used frequently.

**Lemma 4.2.** For any positive integer \( M \) and real numbers \( a_1, \ldots, a_M \),

\[
\left( \sum_{i=1}^{M} a_i \right)^2 \leq M \left( \sum_{i=1}^{M} a_i^2 \right).
\]

The proof of Theorem 4.1 that follows depends on Proposition 4.3, which is stated within the proof; we defer the proof of that to Section 6.

**Proof of Theorem 4.1.** For each \( v \in V \) we choose a length function \( \ell_v \) on \( G_v \) with respect to which \( G_v \) has rapid decay. Then, given \( g \in G \) and a reduced expression \( g = y_1 \cdots y_k \) for \( G \), with \( y_j \in G_{v_j} \), we define

\[
\ell(g) = \sum_{j=1}^{k} \ell_{v_j}(y_j).
\]

That \( \ell \) is both well defined and a length function follows easily from [10, Theorem 3.9]. On subgroups \( G_J \) with \( J \in \mathcal{K} \), we see that \( \ell \) restricts to \( \ell_J \), defined in the same way as a sum of functions \( \ell_{v_j} \) with \( v_j \in J \); it follows from [17, Lemma 2.1.2] that \( G_J \) satisfies RD with respect to \( \ell_J \).

We prove rapid decay with respect to the length function \( \ell \) by verifying the condition (***) of Section 2. But rather than prove that directly we shall deduce it from a condition on the length function \( \lambda \), that is stated as Proposition 4.3.
Recall that $\Lambda_k$ is defined as the set $\{g \in G : \lambda(g) = k\}$. We now define $\chi(k)$ to be the characteristic function on $\Lambda_k$ and $\phi(k)$ to be the pointwise product $\phi \cdot \chi(k)$. In general a function labelled with the subscript $(k)$ is understood to have support on $\Lambda_k$.

We shall prove the following in Section 6.

**Proposition 4.3.** $\exists c, r > 0$ such that for all $\phi, \psi \in CG, k, l, m \in \mathbb{N}$, and $|k - l| \leq m \leq k + l$,

$$||((\phi_k) \ast \psi_{(l)})_{(m)}||_2 \leq c||\phi_k||_2 r \ell ||\psi_{(l)}||_2.$$

To derive our main result from this proposition, we shall use it to deduce the condition (***) of Section 2, that is, we shall deduce the similar condition in which restrictions to $\Lambda_k, \Lambda_l, \Lambda_m$ are replaced by restrictions to $C_k, C_l, C_m$. (But note that both of the length functions $\Lambda$ and $\ell$ are involved in the proposition statement.) Jolissaint’s proof of RD for free products follows exactly the same strategy, and the argument below is basically his, with some slight modification of notation to match our own. We suppose that $k, l, m$ are fixed in the appropriate range. We write $\phi'$ rather than $\phi_k$ and $\psi'$ rather than $\psi_l$, to make our notation less cumbersome.

Since $\ell(g) \geq 1$ for all $g \in G \setminus \{1\}$, we have $C_k \subseteq \bigcup_{j=0}^k A_j$ and $C_l \subseteq \bigcup_{i=0}^l A_i$, and hence $\phi' = \sum_{j=0}^k \phi'_{(j)}$, $\psi' = \sum_{i=0}^l \psi'_{(i)}$. Similarly, for fixed $j$, we have $||(\phi'_{(j)} \ast \psi')(m)(g)|| \leq ||\sum_{p=0}^m (\phi'_{(j)} \ast \psi')(p)(g)||$, for all $g \in G$. (Note that we dropped the restriction to $C_m$ on the right hand side of this inequality.) Hence

$$||(\phi'_{(j)} \ast \psi')(m)||_2^2 \leq \sum_{p=0}^m ||(\phi'_{(j)} \ast \psi')(p)||_2^2.$$

Now, in any product of group elements $g_1, g_2 = g$ with $g_1 \in A_i, g_2 \in A_j, g \in A_p$, we must have $p \leq i + j, i \leq p + j, j \leq p + i$. Hence in each of the terms $(\phi'_{(j)} \ast \psi')(p)$ in the sum on the right hand side of the above inequality, for fixed $j$ and $p$, the support of $\psi'$ lies in the union of the $A_i$ with $|j - p| \leq i \leq j + p$, and so

$$\sum_{p=0}^m ||(\phi'_{(j)} \ast \psi')(p)||_2^2 \leq \sum_{p=0}^m \left| \sum_{i=|j-p|}^{j+p} (\phi'_{(j)} \ast \psi'_{(i)})(p) \right|^2$$

Since there are at most $2j + 1$ values of $i$ in each of the ranges $[|j - p|, j + p]$, we can use Lemma 4.2 to bound the right hand side above by

$$(2j + 1) \sum_{p=0}^m \left| \sum_{i=|j-p|}^{j+p} (\phi'_{(j)} \ast \psi'_{(i)})(p) \right|^2.$$

It follows from Proposition 4.3 that, for $|j - i| \leq p \leq j + i$,

$$||(\phi'_{(j)} \ast \psi'_{(i)})(p)||_2 \leq c||\phi'_{(j)}||_{2,r,\ell} ||\psi'_{(i)}||_2.$$
For other values of \( p \), the left hand side is zero. Hence
\[
||(φ_j * ψ')_m||_2^2 \leq c^2(2j + 1)||φ_j||^2_{2,r,ℓ} \sum \sum_{p=0}^{m} j + p ||ψ_i||_2^2.
\]
Since \( |i − j| \leq p \leq i + j \), for a given value of \( i \), there are at most \( 2j + 1 \) values of \( p \) in the above summation, and so we have
\[
||(φ_j * ψ')_m||_2^2 \leq c^2(2j + 1)^2||φ_j||^2_{2,r,ℓ} \sum_{i=0}^{j+m} ||ψ_i||_2^2 \leq c^2(2j + 1)^2||φ_j||^2_{2,r,ℓ}||ψ'||_2^2.
\]
Now, using the triangle inequality together with Lemma 4.2 again, we have
\[
||(φ_0 * ψ')_m||_2^2 = ||(φ' * ψ')_m||_2^2 \leq \left( \sum_{j=0}^{k} ||(φ'_j * ψ')_m|| \right)^2
\leq (k + 1) \sum_{j=0}^{k} ||(φ'_j * ψ')_m||^2
\leq c^2(k + 1) \sum_{j=0}^{k} (2j + 1)^2||φ'_j||^2_{2,r,ℓ}||ψ'||_2^2
\leq c^2(k + 1)(2k + 1)^2||φ_k||^2_{2,r,ℓ}||ψ||_2^2 = P(k)||φ_k||^2||ψ||_2^2,
\]
where the polynomial \( P \) has degree \( 3 + 2r \). So we have deduced (**)...

The proof of the theorem will be complete once Proposition 4.3 is proved. □

5 Technicalities of the proof of Proposition 4.3

In an attempt to make the proof of Proposition 4.3 more readable we start with some technical results and definitions.

For a function \( φ(k) \) and \( p \geq 0 \), we can define functions \( φ_{(k-p)}^{(p)} \) and \( (p)φ_{(k-p)} \) by

\[
φ_{(k-p)}^{(p)}(u) = \begin{cases} \sqrt{\sum_{w \in Λ_p} |φ(k)(uw)|^2} & \text{if } u \in Λ_{k-p} \\ 0 & \text{otherwise} \end{cases}
\]
\[
(p)φ_{(k-p)}(u) = \begin{cases} \sqrt{\sum_{w \in Λ_p} |φ(k)(w^{-1}u)|^2} & \text{if } u \in Λ_{k-p} \\ 0 & \text{otherwise} \end{cases}
\]

Note that these functions are non-negative real-valued; we shall sometimes make use of that fact as we bound sums.

Lemma 5.1.
\[
||φ_{(k-p)}^{(p)}||_2^2 \leq F_{k-p}^p||φ(k)||_2^2 \quad \text{and} \quad ||(p)φ_{(k-p)}||_2^2 \leq F_{k-p}^p||φ(k)||_2^2.
\]
Proof.

\[ ||\phi_{(k-p)}^p||^2 = \sum_{u \in \Lambda_{k-p}} \sum_{w \in \Lambda_p} |\phi_{(k)}(uw)|^2 \]
\[ \leq F_{k-p}^p \sum_{h \in \Lambda_k} |\phi_{(k)}(h)|^2 \]
\[ = F_{k-p}^p ||\phi_{(k)}||^2 \]

The second inequality follows similarly, since \( F_{k-p}^p = F_{p}^{k-p} \).

For \( 0 \leq i \leq k \) and \( g \in \Lambda_{k-1} \), we define functions \( \phi_{(i)}^q \) and \( g_{(i)} \) by

\[ \phi_{(i)}(v) = \begin{cases} \phi_{(k)}(vg) & \text{if } vg \in \Lambda_k \\ 0 & \text{otherwise} \end{cases} \]
\[ g_{(i)}(v) = \begin{cases} \phi_{(k)}(gv) & \text{if } gv \in \Lambda_k \\ 0 & \text{otherwise} \end{cases} \]

### 6 Proof of Proposition 4.3

Proof. Suppose that \( m = k + l - q \), with \( q \geq 0 \), and that \( g \in \Lambda_{k+l-q} \). By Lemma 3.2, for each factorisation of \( g \) as a product \( h_1 h_2 \) with \( h_1 \in \Lambda_k, h_2 \in \Lambda_l \), there is a 5-tuple \((g_1, g_2, s_1, s_2, w)\) of elements with syllable lengths \( k - q + p, l - q + p, q - 2p, q - 2p, p \), for which \( h_1 = g_1 s_1 w, h_2 = w^{-1} s_2 g_2, s := s_1 s_2 \in G_J \) has syllable length \( q - 2p \), and \( s_1, s_2, s \) are all elements of \( G_J \) for some \( J \in \mathcal{K}_{q-2p} \).

For ease of notation we now define, for \( s \in G_J \),

\[ \mathcal{F}(J, s) := \{(s_1, s_2) \in G_J \times G_J : s = s_1 s_2, \lambda(s_1) = \lambda(s_2) = |J|\}. \]

Now for any \( g \in \Lambda_{k+l-q} \), \( |\phi_{(k)}(\psi_{(l)}(g))| \) is bounded above by

\[ \sum_{p=1}^{[q/2]} \sum_{J \in \mathcal{K}_{q-2p}} \sum_{(g_1, g_2, s_1, s_2) \in \mathcal{F}(J, s) \in \mathcal{F}_{k-q+p}(J, g)} |\phi_{(k)}(g_1 s_1 w)\psi_{(l)}(w^{-1} s_2 g_2)|, \]

by the triangle inequality.

By Cauchy-Schwarz

\[ \sum_{w \in \Lambda_p} |\phi_{(k)}(g_1 s_1 w)\psi_{(l)}(w^{-1} s_2 g_2)| \leq \sqrt{\sum_{w \in \Lambda_p} |\phi_{(k)}(g_1 s_1 w)|^2} \sqrt{\sum_{w \in \Lambda_p} |\psi_{(l)}(w^{-1} s_2 g_2)|^2} \]
\[ = \phi_{(k-p)}^p(g_1 s_1) \times (p) \psi_{(l-p)}^p(s_2 g_2) \]
\[ = g_{(2p-2)}^p(s_1) \times (p) \psi_{(q-2p)}^{2p}(s_2). \]
Then we apply the Lemma 4.2 to see that

\[ G \]

Further,

\[ \sum_{(s_1, s_2) \in F(J, s)} g_1 \phi^{(p)}_{(q-2p)}(s_1) \times (p) \psi^{g_2}_{(q-2p)}(s_2) \leq g_1 \phi^{(p)}_{(q-2p)} * (p) \psi^{g_2}_{(q-2p)}(s), \]

where the convolution here is over \( G_J \), not over \( G \).

Then we apply the Lemma \[12] to see that

\[ \left| \phi(k) * \psi(l)(g) \right|^2 \leq \left( \sum_{p=1}^{\lfloor q/2 \rfloor} \sum_{J \in K_{q-2p}} g_1 \phi^{(p)}_{(q-2p)} * (p) \psi^{g_2}_{(q-2p)}(s) \right)^2 \]

\[ \leq \text{MF}(k, q, l) \sum_{p=1}^{\lfloor q/2 \rfloor} \sum_{J \in K_{q-2p}} \left( g_1 \phi^{(p)}_{(q-2p)} * (p) \psi^{g_2}_{(q-2p)}(s) \right)^2. \]

where \( \text{MF}(k, q, l) := \sum_{p=1}^{\lfloor q/2 \rfloor} \sum_{J \in K_{q-2p}} \mathcal{F}^{(q-2p)}(J, g) \). Since there are only finitely many cliques \( J \), it follows from Lemma \[3.1] that \( \text{MF}(k, q, l) \) is bounded by \( Q(k) \) for some polynomial \( Q \). Hence

\[ \left\| \left( \phi \psi \right)(\Lambda_m) \right\|_2^2 = \sum_{g \in \Lambda_m} \left| \phi(k) * \psi(l)(g) \right|^2 \]

\[ \leq Q(k) \sum_{p=1}^{\lfloor q/2 \rfloor} \sum_{J \in K_{q-2p}} \left( g_1 \phi^{(p)}_{(q-2p)} * (p) \psi^{g_2}_{(q-2p)}(s) \right)^2 \]

\[ = Q(k) \sum_{p=1}^{\lfloor q/2 \rfloor} \sum_{J \in K_{q-2p}} \left\| \phi^{(p)}_{(q-2p)} * (p) \psi^{g_2}_{(q-2p)}(s) \right\|_{2, G_J}^2. \]

But now, since RD holds with respect to \( \ell_J \) in each of the groups \( G_J \), we have

\[ \left\| \phi^{(p)}_{(q-2p)} * (p) \psi^{g_2}_{(q-2p)} \right\|_{2; G_J} \leq c^2_J \left\| \phi^{(p)}_{(q-2p)} \right\|_{2, r, \ell_J; G_J}^2 \left\| (p) \psi^{g_2}_{(q-2p)} \right\|_{2; G_J}^2. \]

We deduce easily from this that

\[ \sum_{g_1 \in \Lambda_{q-2p}, \ g_2 \in \Lambda_{q-2p}} \left\| \phi^{(p)}_{(q-2p)} * (p) \psi^{g_2}_{(q-2p)} \right\|_{2; G_J}^2 \leq c^2_J \sum_{g_1 \in \Lambda_{q-2p}} \left\| \phi^{(p)}_{(q-2p)} \right\|_{2, r, \ell_J; G_J}^2 \times \sum_{g_2 \in \Lambda_{q-2p}} \left\| (p) \psi^{g_2}_{(q-2p)} \right\|_{2; G_J}^2. \]
Then, for \( c = \max c_J \) and \( r = \max r_J \), we have

\[
\sum_{p=|q/2|}^{p=q/2} \sum_{J \in \mathcal{K}_{q-2p}} \left( \sum_{p=1}^{p=|q/2|} \sum_{g_1 \in \Lambda_{k-q+p}} \left( \sum_{g_2 \in \Lambda_{l-q+p}} \| \phi_1^{(p)}(q_{2p}) \|_{2,r,l,G_J} \sum_{g_2 \in \Lambda_{l-q+p}} \| \psi_2^{(p)}(q_{2p}) \|_{2,G_J} \right) \right) \leq c^2 \sum_{p=1}^{p=|q/2|} \sum_{J \in \mathcal{K}_{q-2p}} \left( \sum_{g_1 \in \Lambda_{k-q+p}} \left( \sum_{g_2 \in \Lambda_{l-q+p}} \| \phi_1^{(p)}(q_{2p}) \|_{2,r,l,G_J} \sum_{g_2 \in \Lambda_{l-q+p}} \| \psi_2^{(p)}(q_{2p}) \|_{2,G_J} \right) \right) \times \sum_{p=1}^{p=|q/2|} \sum_{J \in \mathcal{K}_{q-2p}} \left( \sum_{g_2 \in \Lambda_{l-q+p}} \| \psi_2^{(p)}(q_{2p}) \|_{2,G_J} \right).
\]

But

\[
= \sum_{p=|q/2|}^{p=q/2} \sum_{J \in \mathcal{K}_{q-2p}} \sum_{g_1 \in \Lambda_{k-q+p}} \sum_{g_2 \in \Lambda_{l-q+p}, s \in G_J, w \in \Lambda_p} |\phi(k)(g_1 sw)|^2 (1 + \ell_J(s))^{2r}.
\]

Since \( |q/2| \leq k \) and the number of sets \( J \) is bounded, Lemma 3.1 implies that the number of factorisations of a fixed \( g' \in \Lambda_k \) as \( g_1 sw \) in the above sum is at most \( P(k) \) for some polynomial \( P \). So the sum is bounded above by

\[
P(k) \sum_{g' \in \Lambda_k} |\phi(k)(g')|^2 (1 + \ell(g'))^{2r} = P(k) |\phi(k)|^2_{2,r,l},
\]

and similarly

\[
\sum_{p=|q/2|}^{p=q/2} \sum_{J \in \mathcal{K}_{q-2p}} \sum_{g_1 \in \Lambda_{k-q+p}} \sum_{g_2 \in \Lambda_{l-q+p}} \| \psi_2^{(p)}(q_{2p}) \|_{2,G_J} \leq P(k) |\psi(l)|^2_{2}.\]

Hence

\[
\| (\phi(k) * \psi(l)) \|_{2}^2 \leq c^2 Q(k) P(k) |\phi(k)|^2_{2,r,l} |\psi(l)|^2_{2} \leq c^2 |\phi(k)|^2_{2,r+\deg(Q)+2\deg(p),l} |\psi(l)|^2_{2},
\]

where the final inequality uses the fact that \( k \leq \ell(g) \) for all \( g \in \Lambda_k \). This completes the proof of Proposition 4.3. \( \square \)
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