Quantum folded string in $S^5$ and the Konishi multiplet at strong coupling

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Abstract: The Konishi superconformal multiplet is an important theoretical laboratory where one can test AdS/CFT methods to compute strong coupling corrections to the spectrum of superstrings in $AdS_5 \times S^5$. In particular, one can exploit integrability for finite charge states/operators. The multiplet ground state is a singlet operator with two simple descendants in the rank-1 sectors $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$ of $\mathcal{N} = 4$ super Yang-Mills theory. Recently, the next-to-leading quantum correction to the $\mathfrak{sl}(2)$ state has been computed. Here, we use the algebraic curve approach to determine the correction to the other state recovering universality of the correction inside the multiplet.
1. Introduction and result

AdS/CFT correspondence relates the spectrum of conformal dimensions of the $\mathcal{N} = 4$ SYM theory to the spectrum of $AdS_5 \times S^5$ superstring. In the planar limit integrability emerges and anomalous dimensions can be computed as eigenvalues of an integrable super spin chain by solving nested non-perturbative Bethe Ansatz equations. These equations are asymptotic, i.e., valid for states with large enough charges. Finite charge states are more difficult and their anomalous dimensions, including the so-called wrapping corrections, are captured by the Y-system successfully checked at strong coupling in the quasi-classical limit. At weak-coupling the leading order predictions from the Y-system agree with standard field theoretical calculations. At next-to-leading order they are also in agreement with the Lüscher corrections.

Beyond perturbation theory the Y-system can be treated numerically. The anomalous dimension of the states in the Konishi multiplet have been an important theoretical laboratory to test the method. In the Y-system was combined with the vacuum TBA equations to produce an infinite set of integral equations for the $sl(2)$ sector of the spectrum. They were then solved numerically for the simplest state in the Konishi multiplet.
The numerical approach starts in the weak-coupling regime and pushes the 't Hooft coupling \( \lambda \) to large values in order to extrapolate to the strong-coupling limit \([10,11]\). The prediction obtained in \([10]\) for the Konishi anomalous dimension \( \gamma \) is

\[
\gamma + 4 = 2.0004 \lambda^{1/4} + 1.99/\lambda^{1/4} + \ldots.
\] (1.1)

The leading coefficient agrees with the prediction of \([12]\) giving 2. This was also confirmed in a recent paper \([13]\) in the light-cone approach.

The problem with an analytical proof of a relation like (1.1) is only technical, but very hard. In particular, it is expected that the analytical structure of the Y-system at finite coupling \([5]\) becomes very complicated at strong coupling.

A very interesting approach, pioneered by A. Tseytlin and collaborators, is based on the semiclassical quantization of spinning string solutions with large charges and recently systematically applied to the problem of the Konishi multiplet in \([14,15]\). To explain the basic idea we can consider the simple case of the spinning folded string with two charges, the Lorentz spin \( S \) and R-charge \( J \). Let us introduce the ratios \( S = S/\sqrt{\lambda}, \, J = J/\sqrt{\lambda} \). If we expand at large \( \lambda \) and fixed \( S, J \), the expansion of the energy is of the form

\[
E \equiv \gamma + S + J = \sqrt{\lambda} E_0(S,J) + E_1(S,J) + \frac{1}{\sqrt{\lambda}} E_2(S,J) + \ldots.
\] (1.2)

If we now replace the ratios \( S, J \) by their definitions, fix \( S \) and \( J \), and re-expand at large \( \lambda \) we find that the above expansion turns into a power series of the type \([14]\)

\[
E = \lambda^{1/4} a_0 + \frac{1}{\lambda^{1/4}} a_2 + \ldots.
\] (1.3)

Here, the classical energy \( E_0 \) contributes to the first coefficient \( a_0 \) while both \( E_1 \), the one-loop \( \sigma \)-model correction, and \( E_0 \) contribute to the coefficient \( a_2 \). Eq. (1.3) is indeed the expected near-flat space large \( \lambda \) expansion for the energy of a finite charge state. Although the above result is obtained from a semiclassical calculation where \( S, J \) are always large, it is tempting to identify \( a_0, a_2 \) with the coefficients of the expansion of the finite charge state.

The advantage of this approach is that all calculations can be done by semiclassical methods in the string theory or, exploiting the integrability structures, by working with the simpler quasi-classical Y-system whose equivalence with the semiclassical computation has been established in \([5]\). The short string expansion of the energy for the \((S,J)\) folded string reads (see for instance \([14,15]\))

\[
E^{\text{str}}(S,J) = \sqrt{2 S} \lambda^{1/4} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( \frac{3 S}{8} + \frac{J^2}{4 S} + \frac{a_0^{(2)}}{\text{quantum}} \right) \right] + \ldots.
\] (1.4)

In this expression, the terms labeled \textit{classical} come from the expansion of the classical energy. The one-loop corrections are fully encoded in the \textit{quantum} term \( a_0^{(2)} \). The algebraic
curve quantization procedure for an arbitrary $S$ and $J$ \cite{16, 17} leads to the result \cite{18}

$$a_{01}^{s(2)} = -\frac{1}{4}. \quad (1.5)$$

The Konishi state is associated with $S = J = 2$ and we obtain an analytical prediction for the coefficients in Eq. (1.3) in full agreement with the numerical results of \cite{10} (see also \cite{15, 19}),

$$a_0 = a_2 = 2. \quad (1.6)$$

It is very interesting to study the manifestation of superconformal invariance at the level of strong coupling corrections. The multiplet structure can be regarded as a consistency check of any method attempting to deal with such regime. This problem has been addressed in \cite{14, 15} from the perspective of semiclassical string quantization. Here, we would like to test the algebraic curve approach from this point of view. To this aim, we remind that quantum string states as well as dual gauge theory operators are highest weight states with Dynkin labels

$$[p_1, q, p_2]_{(s_L, s_R)}, \quad (1.7)$$

where, in terms of the classical charges $S_{1,2}$, $J_{1,2,3}$, the $so(4) = su(2) \oplus su(2)$ labels $(s_L, s_R)$ are given by $s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$ and the Dynkin labels $[p_1, q, p_2]$ of $su(4)$ are given by $p_{1,2} = J_2 \mp J_3$, $q = J_1 - J_2$.

With this notation, the singlet operator $\text{Tr}(\Phi^4)$ with bare dimension 2 is the top state $[0,0,0]_{(0,0)}$ of the Konishi multiplet. It has two superconformal descendants in the $sl(2)$ and $su(2)$ sectors given by the following states with bare dimension 4

$$\begin{array}{c|c|c}
\text{sector} & \text{state} & [p_1, q, p_2]_{(s_L, s_R)} \\
\hline
sl(2) & \text{Tr}(\Phi^4) & [0,2,0]_{(1,1)} \\
su(2) & \text{Tr}(\Phi^2) & [2,0,2]_{(0,0)} \\
\end{array} \quad (1.8)$$

The state in the $sl(2)$ sector has been worked out in details in \cite{18}. As is well known it is associated with a classical string solution represented by a string rotating in just one plane in $S^5$ with a spin in $AdS_6$ \cite{21}. We shall denote is as the $(S,J)$ folded string. The second state has been discussed in details in \cite{21, 22} and it is associated with a classical string rotating in two planes in $S^5$, the $(J_1, J_2)$ folded string \cite{23}. The two (classical) solutions are related by an analytic continuation connecting the respective string profiles and conserved charges. From the point of view of the Bethe Ansatz description, at least in the gauge theory, they are quite different. The folded $(S, J)$ string is described by a 2-cut solution with symmetric cuts on the real axis. Instead, the folded $(J_1, J_2)$ string is associated (at least at weak coupling) with a 2-cut solution with two cuts symmetric around the imaginary axis and with a non-trivial geometry. The special role of these particular very symmetric 2-cut solutions has been investigated in details in \cite{24}.

It is very interesting to pursue the duality in the context of the algebraic curve approach (or the equivalent quasi-classical Y-system). In particular, one would like to check
whether the multiplet structure is obeyed by the first non trivial strong coupling correction to the energy. A first analysis in this direction has been presented in [14, 15]. The one-loop corrected energy for the \( J_1, J_2 \) folded string takes a form similar to Eq. (1.4)

\[
E^{su(2)}(J_1, J_2) = \sqrt{2J_2} \lambda^{1/4} \left[ 1 + \frac{1}{\sqrt{\lambda}} \left( J_2 + \frac{J_1^2}{2} + a^{su(2)}_{01} \right) \right] + \ldots . \tag{1.9}
\]

The authors of [14, 15] conjectured that \( a^{su(2)}_{01} \) should be the same with an opposite sign, \( i.e. + \frac{1}{4} \), reflecting the opposite sign of the curvature of \( S^3 \) as compared to \( AdS_3 \). This proposal is consistent with similar behaviour of the correction for circular spinning strings [14,15]. For the Konishi representative with \( J_1 = J_2 = 2 \), the assignment \( a^{su(2)}_{01} = \frac{1}{4} \) leads to the same strong coupling correction as for the \( sl(2) \) Konishi descendant

\[
E^{S=2, J=2} = E^{J_1=2, J_2=2} = 2 \lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \ldots . \tag{1.10}
\]

Beyond the Konishi state, this choice is also consistent with the superconformal degeneracy of the states \( (S = 2, J) \) and \( (J_1 = J, J_2 = 2) \) \(^1\) because it predicts the same correction \(^2\)

\[
E^{S=2, J} = E^{J_1=J, J_2=2} = 2 \lambda^{1/4} + \frac{J^2 + 4}{4} \lambda^{1/4} + \ldots . \tag{1.11}
\]

Finally, as a further support of the conjecture \( a^{su(2)}_{01} = \frac{1}{4} \), we recall that an argument in [15] \(^3\) suggests that the independence of \( a_{01} \) on the charge ratio is not accidental and has instead a deep origin being related to the continuity of observables with respect to the addition of a small charge to the principal one \( (S \text{ or } J_2 \text{ for the two folded strings}) \).

In this paper we perform an algebraic curve calculation of the correction and provide very convincing numerical evidence that the result \( a^{su(2)}_{01} = \frac{1}{4} \) proposed in [14, 15] is indeed correct.

2. Algebraic curve method for the \( AdS_5 \times S^5 \) superstring

The general construction of the algebraic curve for the \( AdS_5 \times S^5 \) superstring is discussed for instance in [25, 17]. Here, we summarize in a self-contained way the main results for the reader’s convenience.

2.1 Classical algebraic curve

The monodromy matrix of the Lax connection for the integrable dynamics of the \( AdS_5 \times S^5 \) superstring has eigenvalues

\[
\{ e^{i \tilde{p}_1}, e^{i \tilde{p}_2}, e^{i \tilde{p}_3}, e^{i \tilde{p}_4}, e^{i \tilde{p}_5}, e^{i \tilde{p}_6}, e^{i \tilde{p}_7}, e^{i \tilde{p}_8} \} \tag{2.1}
\]

\(^1\)It follows for instance by duality of the Bethe equations and adding roots at infinity to implement superconformal transformations.

\(^2\)The case \( S = 2, J = 3 \) has been confirmed by an independent TBA computation in [18].

\(^3\)We thank A. Tseytlin and R. Roiban for pointing out this issue.
The eigenvalues are roots of the characteristic polynomial and define an 8-sheeted Riemann surface. The classical algebraic curve has macroscopic cuts connecting various pairs of sheets. Around each cut, we have

\[ p_i^+ - p_j^- = 2\pi n_{ij}, \quad x \in C_{ij}^\pm, \tag{2.2} \]

where \( n \) is an integer associated with the cut. The possible combinations of sheets (a.k.a. polarizations) that are relevant for \( AdS_5 \times S^5 \) are

\[ i = \hat{1}, \hat{2}, \hat{1}, \hat{2}, \quad j = \hat{3}, \hat{4}, \hat{3}, \hat{4}. \tag{2.3} \]

The properties of the monodromy matrix implies (for folded configurations) the inversion properties

\[
\begin{align*}
\tilde{p}_{1,2}(x) &= -2\pi m - \tilde{p}_{2,1}(1/x), \quad m \in \mathbb{Z}, \\
\tilde{p}_{3,4}(x) &= +2\pi m - \tilde{p}_{4,3}(1/x), \\
\tilde{p}_{1,2,3,4}(x) &= -\tilde{p}_{2,1,4,3}(1/x).
\end{align*}
\tag{2.4}
\]

The poles of the connection plus Virasoro constraints implies the pole structure around the special points \( x = \pm 1 \)

\[
\{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4 | \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4\} \sim \frac{\{\alpha_\pm, \alpha_\pm, \beta_\pm, \beta_\pm | \alpha_\pm, \alpha_\pm, \beta_\pm, \beta_\pm\}}{x \pm 1}. \tag{2.5}
\]

Also, the asymptotic value at \( x \to \infty \) is related to the conserved charges as in \((Q = \frac{Q}{\sqrt{x}})\)

\[
\begin{pmatrix}
\tilde{p}_1 \\
\tilde{p}_2 \\
\tilde{p}_3 \\
\tilde{p}_4 \\
\end{pmatrix}
= \frac{2\pi}{x}
\begin{pmatrix}
+\epsilon - S_1 + S_2 \\
+\epsilon + S_1 - S_2 \\
-\epsilon - S_1 - S_2 \\
-\epsilon + S_1 + S_2 \\
\end{pmatrix}
+ \mathcal{O}(1/x^2), \tag{2.6}
\]

3. Fluctuations frequencies from the algebraic curve

The macroscopic cuts can be thought as the condensation of a large number of poles as it happens in semiclassical quantum mechanics for a large excitation number. We shall be interested in the effect of the addition of a single pole and in the shift \( p \to p + \delta p \) of the quasi-momenta. This insertion will compute the quantum fluctuations around the classical solution. From the definition of the action-angle variables for the integrable string, we deduce that residue of \( \delta p \) around such a pole has to be

\[ \delta p \sim \pm \frac{\alpha(x_p)}{x - x_p}, \quad \alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1}. \tag{3.1} \]

\(^4\)At weak coupling, the two points collapse and we end with the usual pole at \( x = 0 \) well known in the study of integrable spin chains.
The position of the poles can be found by solving (for generic \( n \)) the equation

\[
p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi \, n, \quad |x_n^{ij}| > 1,
\]

for all polarizations \((i, j)\) with \( i < j \) and the pairs

\[
S^e : (i, j) = (\bar{1}, \bar{3}), (\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (\bar{2}, \bar{4}),
\]

\[
Ad S^e : (i, j) = (\bar{1}, \bar{3}), (\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (\bar{2}, \bar{4}),
\]

Fermions : \((i, j) = (\bar{1}, \bar{3}), (\bar{1}, \bar{4}), (\bar{2}, \bar{3}), (\bar{2}, \bar{4})\),

(3.1)

The correction to the quasi-momenta will be \( \delta \hat{p}_i \) with the pole structure (3.1), regularity across the macroscopic cuts, and asymptotic behaviour (\( N_{ij} = \sum_n N_n^{ij} \) is the number of \((i, j)\) excitations)

\[
\delta \left( \\begin{array}{c}
\hat{p}_1 \\
\hat{p}_2 \\
\hat{p}_3 \\
\hat{p}_4 \\
\end{array} \right) = \frac{4\pi}{x \sqrt{\lambda}} \left( \begin{array}{c}
\frac{+\frac{1}{2} \delta \Delta + N_{14} + N_{13} + N_{12} + N_{14}}{+\frac{1}{2} \delta \Delta + N_{24} + N_{33} + N_{33} + N_{24}} \\
\frac{+\frac{1}{2} \delta \Delta - N_{23} + N_{13} - N_{13}}{+\frac{1}{2} \delta \Delta - N_{23} + N_{24} + N_{24} - N_{14}} \\
\frac{+\frac{1}{2} \delta \Delta - N_{14} + N_{23} + N_{24} - N_{14}}{+\frac{1}{2} \delta \Delta - N_{14} + N_{23} + N_{24} - N_{14}} \\
\end{array} \right) + \mathcal{O}(1/x^2),
\]

(3.6)

The precise values of the residues can be read off the definition of the action-angle variables and are

\[
\text{res}_{x = x_n^{ij}} \hat{p}_k = (\delta_{i, k} - \delta_{j, k}) \alpha(x_n^{ij}) N_n^{ij}, \quad \text{res}_{x = x_n^{ij}} \hat{p}_k = (\delta_{i, k} - \delta_{j, k}) \alpha(x_n^{ij}) N_n^{ij},
\]

(3.7)

where \( k = 1, 2, 3, 4, \) and \( i < j \) taking values \( \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{1}, \bar{2}, \bar{3}, \bar{4} \). The anomalous shift \( \delta \Delta \) can be written as a linear combination of the \( N_n^{ij} \) numbers

\[
\delta \Delta = \sum_{n, [ij]} N_n^{ij} \Omega_n^{ij}.
\]

(3.8)

This formula for \( \delta \Delta \) exhibits the classical frequencies \( \Omega_n^{ij} \) around the classical solution. These frequencies can be thought as normal mode frequencies. After quantization, and taking into account statistics, the one loop correction to the energy can be written as a sum over zero point energies

\[
\delta E = \frac{1}{2} \sum_{n, [ij]} (-1)^F \Omega_n^{ij}.
\]

(3.9)

### 3.1 Inversion symmetry and linear combinations of frequencies for rank-1 solutions

The inversion symmetry (3.4) implies the two important relations

\[
\Omega_{\bar{1} \bar{4}}(x) = -\Omega_{\bar{2} \bar{3}}(1/x) + \Omega_{\bar{2} \bar{3}}(0),
\]

(3.10)
\[ \Omega^{\text{14}}(x) = -\Omega^{\text{23}}(1/x) - 2. \]  

In addition, we have linear relations between the various \( \Omega^{ij} \) which can be easily read by representing a particular frequency connecting two sheets as the sum of the intermediate frequencies connecting an intermediate sheet. Assuming the top-down symmetry (valid for rank-1 solutions)

\[ p_{\bar{i}, \bar{j}, \bar{\ell}, \bar{\gamma}} = -p_{\bar{\delta}, \bar{\epsilon}, \bar{\theta}, \bar{\zeta}}, \]

one can prove that all the \( 8 + 8 \) physical frequencies can be written in terms of the two basic ones

\[ \Omega_S(x) = \Omega^{\text{23}}(x), \quad \Omega_A(x) = \Omega^{\text{33}}(x). \]

The final result is

\[
\begin{align*}
\Omega^{\text{14}} &= -\Omega_S(1/x) + \Omega_S(0), \\
\Omega^{\text{24}} &= \Omega^{\text{33}} = \frac{1}{2}[\Omega_S(x) - \Omega_S(1/x) + \Omega_S(0)], \\
\Omega^{\text{14}} &= -\Omega_A(1/x) - 2, \\
\Omega^{\text{24}} &= \Omega^{\text{33}} = \frac{1}{2}[\Omega_A(x) - \Omega_A(1/x) - 1], \\
\Omega^{\text{24}} &= \Omega^{\text{33}} = \frac{1}{2}[\Omega_A(x) - \Omega_A(1/x) - 1], \\
\Omega^{\text{14}} &= \Omega^{\text{14}} = \frac{1}{2}[-\Omega_S(1/x) - \Omega_A(1/x) + \Omega_S(0)] - 1, \\
\Omega^{\text{23}} &= \Omega^{\text{23}} = \frac{1}{2}[\Omega_S(x) + \Omega_A(x)].
\end{align*}
\]

4. Algebraic curve computation for the \((J_1, J_2)\) folded strings

4.1 Classical \((S, J)\) folded string in the short string limit

According to [22], the folded string rotating in \(AdS_5\) and \(S^5\) with angular momenta \(S\) and \(J\) can be analytically continued to the folded string rotating in \(S^5\) with two angular momenta \(J_1, J_2\) according to the replacement rule

\[ (E, J_1, J_2) \leftrightarrow (-E, -J, S). \]  

In the \((S, J)\) folded string, the two cuts of the elliptic curve are symmetrically placed along the real axis, \((a, b), (-a, -b)\), where \(1 < a < b\). The conserved quantities are given by the expressions [18]

\[
\begin{align*}
S &= 2 n g \frac{ab + 1}{ab} \left[ b \mathbb{E} \left( 1 - \frac{a^2}{b^2} \right) - a \mathbb{K} \left( 1 - \frac{a^2}{b^2} \right) \right], \\
J &= \frac{4 n g}{b} \sqrt{(a^2 - 1)(b^2 - 1)} \mathbb{K} \left( 1 - \frac{a^2}{b^2} \right). 
\end{align*}
\]
\[ E = 2n g \frac{ab - 1}{ab} \left[ b E \left( 1 - \frac{a^2}{b^2} \right) + a K \left( 1 - \frac{a^2}{b^2} \right) \right]. \]

The branch points can be expanded for small \( S \) and \( J \) according to

\[
\begin{align*}
a &= 1 + \frac{\rho^2 s^3}{8} + \frac{1}{128} \left( \rho^2 - \rho^4 \right) s^5 + \frac{\rho^2 (4\rho^4 - 22\rho^2 - 9)}{4096} s^7 + O \left( s^8 \right), \quad (4.3) \\
b &= 1 + 2s + 2s^2 + \frac{1}{8} \left( \rho^2 + 7 \right) s^3 + \frac{1}{4} \left( \rho^2 - 1 \right) s^4 + \frac{1}{256} \left( -2\rho^4 + 34\rho^2 - 85 \right) s^5 + O \left( s^6 \right).
\end{align*}
\]

Indeed, the associated charges are

\[
\begin{align*}
S &= 2\pi n g s^2 + O(s^3), \\
J &= 2\pi n g \rho s^3 + O(s^3), \quad (4.4) \\
E &= 4\pi n g s + \frac{1}{4} \pi g n \left( 2\rho^2 + 3 \right) s^3 - \frac{1}{128} s^5 \left( \pi g n \left( 4\rho^4 - 20\rho^2 + 21 \right) \right) + O \left( s^6 \right).
\end{align*}
\]

This \( s \sim \sqrt{S} \) and \( \rho = \frac{s}{S} \). More precisely, from \( \sqrt{\lambda} = 4\pi g \), we have

\[
s = \frac{S}{2\pi n g} = \frac{\sqrt{2S/n}}{\lambda^{1/4}}. \quad (4.5)
\]

The short string expansion of the energy is

\[
\frac{E}{n\sqrt{\lambda}} = s + \frac{1}{16} \left( 2\rho^2 + 3 \right) s^3 + \frac{1}{512} \left( -4\rho^4 + 20\rho^2 - 21 \right) s^5 + O \left( s^6 \right). \quad (4.6)
\]

### 4.2 Analytic continuation to the \( (J_1, J_2) \) folded string

In order to describe the \( (J_1, J_2) \) string, we apply the continuation (4.1) and are now led to study

\[
\begin{align*}
J_1 &= -2n g \frac{ab - 1}{ab} \left[ b E \left( 1 - \frac{a^2}{b^2} \right) + a K \left( 1 - \frac{a^2}{b^2} \right) \right], \\
J_2 &= 2n g \frac{ab + 1}{ab} \left[ b E \left( 1 - \frac{a^2}{b^2} \right) - a K \left( 1 - \frac{a^2}{b^2} \right) \right], \quad (4.7) \\
E &= -\frac{4n g}{b} \sqrt{(a^2 - 1)(b^2 - 1)} K \left( 1 - \frac{a^2}{b^2} \right).
\end{align*}
\]

We expand \( a, b \) around the point \(-1\). Introducing the small parameter \( s \), we find the expansion

\[
\begin{align*}
a &= -1 + is + \left( \frac{1}{2} - \frac{\rho}{2} \right) s^2 + \frac{1}{16} \left( 8\rho - 3 \right) s^3 + \frac{1}{16} \left( -2\rho^2 + 2\rho - 1 \right) s^4 + \\
&\quad + \frac{1}{512} \left( 32\rho^2 + 16\rho + 3 \right) s^5 + O \left( s^6 \right), \\
b &= \bar{a}. \quad (4.8)
\end{align*}
\]
Notice that this is precisely the short string limit of the double contour discussed in [21]. These branch points give

\[
\begin{align*}
J_2 &= 2n \pi g s^2 + O(s^6), \\
J_1 &= 2n \pi g \rho s^2 + O(s^7), \\
E &= 4n \pi g s + \frac{1}{4} g n \left( 2\pi \rho^2 + \pi \right) s^3 - \frac{1}{128} s^5 \left( \pi g n \left( 4\rho^4 - 28\rho^2 - 3 \right) \right) + O\left( s^6 \right).
\end{align*}
\]

Using again the relation (4.15) and identifying \( \rho = \frac{J_1}{J_2} \), we find the following expansion

\[
\frac{E}{n \sqrt{\lambda}} = s + \frac{1}{16} \left( 2\rho^2 + 1 \right) s^3 + \frac{1}{512} \left( -4\rho^4 + 28\rho^2 + 3 \right) s^5 + O\left( s^6 \right). \tag{4.10}
\]

It can be easily shown that this result is in full agreement with the general treatment in [24].

### 4.3 Construction of the \( p_2 \) quasi-momentum

In [18], the reader can find the explicit non-trivial quasimomentum \( p_2 \) for the \((S, J)\) folded string. Following our approach based on the analytic continuation, we can look for a suitable continuation of it. As we can verify \textit{a posteriori} (see the Appendix), this procedure gives the sphere quasimomentum \( p_2 \). The result is (written here with the standard branch line assignment for the square root)

\[
p_2 = \pi n - i \Delta \left( \frac{2abJ_2}{(b-a)(ab+1)} \right) F_1(x) - \frac{\Delta (a-b)}{2g \sqrt{(a^2-1)(b^2-1)}} F_2(x),
\]

\[
F_1(x) = i \mathcal{F} \left( i \sinh^{-1} \left( -\frac{a-b}{a+b} \frac{a-x}{a+x} \frac{(a-b)^2}{(a+b)^2} \right) \right),
\]

\[
F_2(x) = i \mathcal{E} \left( i \sinh^{-1} \left( -\frac{a-b}{a+b} \frac{a-x}{a+x} \frac{(a-b)^2}{(a+b)^2} \right) \right),
\]

where

\[
\begin{align*}
J_1 &= +2n g \frac{ab-1}{ab} \left[ b \mathcal{E} \left( 1 - \frac{a^2}{b^2} \right) + a \mathcal{K} \left( 1 - \frac{a^2}{b^2} \right) \right], \\
J_2 &= -2n g \frac{ab+1}{ab} \left[ b \mathcal{E} \left( 1 - \frac{a^2}{b^2} \right) - a \mathcal{K} \left( 1 - \frac{a^2}{b^2} \right) \right], \\
\Delta &= -\frac{4n g}{b} \sqrt{(a^2-1)(b^2-1)} \mathcal{K} \left( 1 - \frac{a^2}{b^2} \right).
\end{align*}
\]

This expression is valid provided

\[
\text{Re}(a), \text{Im}(a) > 0, \quad b = -\bar{x}. \tag{4.15}
\]
It can be checked that
\[ p(a) = p(\bar{a}) = n\pi, \quad p(-a) = p(-\bar{a}) = -n\pi, \quad p(\infty) = 0. \quad (4.16) \]

Setting
\[ a = 1 + i s + \frac{1}{2} (\rho - 1) s^2 + \frac{1}{16} i (8\rho - 3) s^3 + \frac{1}{16} \left( 2\rho^2 - 2\rho + 1 \right) s^4 + \frac{1}{512} i \left( 32\rho^2 + 16\rho + 3 \right) s^5 + O(s^6), \quad (4.17) \]
we recover the expansion (4.15) and (4.16). Notice again that in this section $b = -\bar{a}$ and not $b = \bar{a}$ as in the previous section. This is necessary to have the correct cut structure.

The full set of quasi-momentum is obtained by completing the sphere quasi-momenta with the relations
\[ p_2(x) = -p_3(x) = -p_4(1/x) = p_4(1/\bar{x}), \quad (4.18) \]
and by assigning the following AdS quasi-momenta (following from the absence of cuts in the AdS sheets)
\[ p_{1\hat{\beta}} = -p_{3\hat{\alpha}} = \frac{\Delta}{2g} \frac{x}{x^2 - 1} = \frac{2\pi E}{x^2 - 1}. \quad (4.19) \]

The sphere quasi-momentum $p_2$ defined in (4.14) has branch cuts along small arcs of circumference with radius $|a|$. A typical plot of it has the form shown in Fig. (1) where we can see the cuts and the singularity around $x = \pm 1$. Actually, these are not the physical branch cuts [26].

![Figure 1: Typical form of $p_2$. In the plot, we can recognize the cuts along arcs of circumference $|x| = |a|$ as well as the poles at $x = \pm 1$.](image-url)
4.4 Fluctuation energies for the \((J_1, J_2)\) folded string

The general structure of quantum fluctuations around symmetric 2-cut \(\mathfrak{su}(2)\) solutions has been investigated in detail in [17]. The fluctuations of quasi momenta with excitation of type \((\tilde{2}, \tilde{3})\) (with \(N_{\tilde{2}\tilde{3}} = 1\)) at \(z\) and excitation of type \((\bar{2}, \bar{3})\) (with \(N_{\bar{2}\bar{3}} = 1\)) at \(y\) have the general form

\[
\delta p_{\pm}^z = \frac{\alpha(z)}{z-x} + \frac{\delta \alpha_-}{x-1} + \frac{\delta \alpha_+}{x+1},
\]

\[
\delta p_{\pm}^z = \frac{1}{f(x)} \left[ -\frac{f(y) \alpha(y)}{x-y} + \frac{\delta \alpha_- f(1)}{x-1} + \frac{\delta \alpha_+ f(-1)}{x+1} - \frac{4\pi x}{\sqrt{\lambda}} + A \right],
\]

where \(\delta \alpha_\pm\) and \(A\) are constants and \(f(x)^2 = (x-a)(x-b)(x-c)(x-d)\). Using the inversion relations and replacing in the asymptotic condition we easily find

\[
\delta \Delta = \Omega_S(y) + \Omega_A(z),
\]

\[
\Omega_A(x) = \frac{2}{x^2 - 1} \left( 1 + x f(1) - f(-1) \right),
\]

\[
\Omega_S(x) = \frac{4}{f(1) + f(-1)} \left( \frac{f(x)}{x^2 - 1} - 1 \right).
\]

The considered solutions have the additional symmetry

\[
p_2 = -p_3, \quad p_4 = -p_5, \quad p_6 = -p_7 = -p_4.
\]

This we can identify all frequencies with the above pairing of indices. Consistency requires the following relation which indeed is true for the above expressions

\[
\Omega_A(x) + \Omega_A(1/x) + 2 = 0.
\]

We end with the following simple expressions for the six independent frequencies

**Bosonic fluctuations**

\[
\Omega_S = \Omega_{\tilde{2}\tilde{3}}^S, \\
\Omega_S^\bar{2} = \Omega_{\bar{2}\bar{3}}^{\bar{2}} = -\Omega_S(1/x) + \Omega_S(0), \\
2 \times \Omega_{\tilde{S} \tilde{S}} = \Omega_{\tilde{2}\tilde{4}} = \Omega_{\bar{2}\bar{4}} = \Omega_{\bar{2}\bar{3}} = \frac{1}{2} [\Omega_S(x) - \Omega_S(1/x) + \Omega_S(0)], \\
4 \times \Omega_A = \Omega_{\tilde{2}\tilde{4}} = \Omega_{\bar{2}\bar{4}} = \Omega_{\tilde{2}\tilde{3}} = \Omega_{\bar{2}\bar{3}} = \Omega_{\tilde{3}\tilde{3}} = \Omega_{\bar{3}\bar{3}},
\]

**Fermionic fluctuations**

\[
4 \times \Omega_F = \Omega_{\tilde{2}\tilde{4}} = \Omega_{\bar{2}\bar{4}} = \Omega_{\tilde{2}\tilde{3}} = \Omega_{\bar{2}\bar{3}} = \frac{1}{2} [\Omega_A(x) - \Omega_S(1/x) + \Omega_S(0)],
\]

\[
4 \times \Omega_F = \Omega_{\tilde{2}\tilde{4}} = \Omega_{\bar{2}\bar{4}} = \Omega_{\tilde{2}\tilde{3}} = \Omega_{\bar{2}\bar{3}} = \frac{1}{2} [\Omega_S(x) + \Omega_A(x)].
\]
5. Evaluation of the one-loop correction

The standard way to compute the one-loop energy \((5.5)\) is to write the sum over the mode number \(n\) as a contour integral

\[
\delta E = \frac{1}{2} \sum_{ij} (-1)^{F_{ij}} \int \frac{dx}{2\pi i} \left( \Omega^{ij}(x) \partial_x \log \sin \frac{p_i - p_j}{2} \right)
\]

The integral is conveniently computed by deforming the contour in two pieces:

a) The unit circumference \(|x| = 1\),

b) a contour surrounding the cut in the \((\bar{z}, \bar{s})\) plane.

The (a) contribution is rather easy. All singularities cancel as a consequence of the ultraviolet finiteness of the correction. The (b) contribution is less trivial since it requires some insight about how to deform the contour integration around the cut. In the simplest case with mode number \(n = 1\), the one relevant for Konishi, we find the structure of excitations for the \((\bar{z}, \bar{s})\) polarization shown in Fig. (2). Apart from the cut endpoints, we only find poles on the real axis. All but one of them can be grouped in an infinite sequence \(\{x_k\}\) that accumulates at \(x = 1\) with \(|x_k - 1| \sim 1/k\) for large \(k\). Then, there is a somewhat different pole at \(x = \xi\) whose position depends on \(\rho\) and moves to infinity as \(\rho \to 1\). Similar poles can be found for \(\text{Re} \, x < 0\) and are not drawn.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fluctuations.png}
\caption{Sketch of the fluctuations poles appearing for the \(n = 1\) folded string. Crosses denote an infinite sequence accumulating at the point \(x = 1\). The point \(\xi\) is a somewhat separate pole whose position depends on \(\rho\) and moves to infinity as \(\rho \to 1\). Similar poles can be found for \(\text{Re} \, x < 0\) and are not drawn.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fluctuations2.png}
\caption{Plot of \(\text{Re} \partial_x \log \sin p_{\bar{s}}\) at \(s = 1/10\) and \(\rho = 2\). The left and right panels differ by the range of \(p_{\bar{s}}\).
\end{figure}

\footnote{Notice that for \(n > 1\), the structure of fluctuations becomes more complicated. In particular, there are \(n - 1\) additional fluctuations near each branch cut endpoint. We shall not discuss configurations with \(n > 1\) here. For these states one has to identify the precise contour integration around the cut.}
Figure 3: $\text{Re}\Omega_s$ for $s = \frac{1}{2}$ and $\rho = 1$. The pinches are an infinite sequence of poles condensating around $x = 1$ plus two poles at $x = a, \bar{a}$. The various white lines are artifacts of the plot.

$x$. The left plot focuses on the region near $x = 1$ and shows the regular infinite sequence of poles $\{x_k\}$. The right plot shows that there is a pole at $x = \xi = 1.74078$ well separated from the other poles. Its contribution is non zero and must be included.

In order to evaluate the cut integral we continue the quasi-momentum the right in the complex plane. We deform suitably the integration contour and compute the discontinuity taking into account the jump of sign of $f(x)$ across the physical cut. From the computational point of view it is convenient to compute the integral along the dashed polygonal $\Gamma$ in Fig. (3) and evaluate separately the contribution of the special pole $\xi$. This is particularly important for values of $\rho$ near 1, the Konishi case, when $\xi$ is large.

We collect in Fig. (4) a few sample numerical values of the one-loop correction evaluated at the special values of the ratio of the two spins $\rho = 1, \frac{3}{2}, 2$ as a function of the spin parameter $s$. The independence of the $s \to 0$ limit with respect to $\rho$ is clear. A simple polynomial fit to a similar larger set of points provides the following estimate of the $s \to 0$
limit
\[ \lim_{\rho \to \infty} \frac{\delta E}{\rho} = 0.24999999(1). \]  
(5.2)

where the error represents the dependence on \( \rho \). Our computations clearly provides strong evidence for the correctness of the exact result \( \delta_{G_2}^{su(2)} = \frac{1}{4} \).

6. Conclusions

In this paper we have computed the strong coupling next-to-leading correction to the energy of the quantum folded string with two angular momenta \( J_{1,2} \) in \( S^5 \) in the limit where \( J_2 \) is large with fixed ratio \( J_1/J_2 \) and small \( J_{1,2}/\sqrt{\lambda} \) (semiclassical short string limit). This state is expected to capture at semiclassical level the properties of the \( su(2) \) descendent of the Konishi state. It belongs to the same multiplet as the analogous state dual to the folded string with spin in \( AdS \) and angular momentum in \( S^5 \). The correction should be the same as a consequence of superconformal symmetry.

We performed the computation by exploiting the algebraic curve method proposed in [17]. We computed the one-loop correction numerically with high precision and confirmed the conjecture proposed in [14, 15].

A natural continuation of this work is of course to perform a similar analysis for the small circular strings solutions considered in [14, 15] in order to prove universality of the next-to-leading strong coupling correction for more states with bare dimension 4 in the Konishi multiplet. This analysis is in progress.

Finally, the structure of multiplets beyond Konishi seems unclear at the moment so it is important to collect as much data as possible to see if there are degeneracies in energy for various other semiclassical states with different quantum numbers. The algebraic curve approach is clearly a powerful tool in this respect.

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A. Asymptotics of \( p_2 \)

Evaluating \( p_2 \) at large \( x \) we find

\[ p_2(x) \xrightarrow{x \to \infty} \frac{J_2 - J_1}{2g} \frac{2\pi}{x} (J_2 - J_1), \]
(A.1)

\[ p_2(1/x) \xrightarrow{x \to 0} -\frac{J_2 + J_1}{2g} x. \]
(A.2)
Comparing with the general asymptotic behaviour
\[
\begin{pmatrix}
\tilde{p}_1 \\
\tilde{p}_2 \\
\tilde{p}_3 \\
\tilde{p}_4
\end{pmatrix}
= \frac{2\pi}{x}
\begin{pmatrix}
+\epsilon - S_1 + S_2 \\
+\epsilon + S_1 - S_2 \\
-\epsilon - S_1 - S_2 \\
-\epsilon + S_1 + S_2
\end{pmatrix}
\begin{pmatrix}
+\epsilon + S_1 - S_2 \\
+\epsilon + S_1 - S_2 \\
-\epsilon - S_1 - S_2 \\
+\epsilon + S_1 + S_2
\end{pmatrix}
+ O(1/x^2),
\]
(A.3)

and with the inversion properties
\[
\tilde{p}_{1,2}(x) = -\tilde{p}_{2,1}(1/x) + 2\pi m,
\]
(A.4)
\[
\tilde{p}_{3,4}(x) = -\tilde{p}_{4,3}(1/x) + 2\pi m,
\]
(A.5)
\[
\tilde{p}_{1,2,3,4}(x) = -\tilde{p}_{2,1,3,4}(1/x),
\]
(A.6)

we identify the correct asymptotic behaviour of \(\tilde{p}_2 = -\tilde{p}_3\) after exchanging \(J_1 \leftrightarrow J_2\).

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