Fourier-Stieltjes algebras, decomposable Fourier multipliers and amenability

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Abstract

We prove that the Fourier-Stieltjes algebra $B(G)$ of a discrete group $G$ is isometrically isomorphic to the algebra $\mathfrak{M}^{\infty,\text{dec}}(G)$ of decomposable Fourier multipliers on the group von Neumann algebra $\text{VN}(G)$. In contrast, we show that $\mathfrak{M}^{\infty,\text{dec}}(G) \neq B(G)$ for some classes of non-discrete locally compact groups, whereas we prove that $\mathfrak{M}^{\infty,\text{dec}}(G) = B(G)$ holds for any (second-countable unimodular) inner amenable locally compact group. To establish these results, we leverage groupoid theory and examine whether a contractive projection preserving complete positivity exists from the space of normal completely bounded operators on $\text{VN}(G)$ onto the space $\mathfrak{M}^{\infty,\text{cb}}(G)$ of completely bounded Fourier multipliers. We provide an affirmative solution in the inner amenable case and demonstrate that such projections do not exist for (second-countable) non-amenable connected locally compact groups. Furthermore, we investigate whether the space $\mathfrak{M}^{p,\text{cb}}(G)$ of completely bounded Fourier multipliers on the noncommutative $L^p$-space $L^p(\text{VN}(G))$ is complemented in the space of completely bounded operators, where $1 \leq p \leq \infty$. Using doubling metrics on Lie groups and structural results from the solution to Hilbert’s fifth problem, we establish that any (second-countable unimodular) finite-dimensional amenable locally compact group admits compatible bounded projections at the levels $p = 1$ and $p = \infty$, which have applications to decomposable Fourier multipliers. Finally, we present a new characterization of amenability for (second-countable unimodular) locally compact groups.

Contents

1 Introduction .................................................. 2
  1.1 Introduction ........................................... 2
  1.2 Structure of the paper .................................. 7
2 Fourier-Stieltjes algebras and decomposable multipliers on $\text{VN}(G)$ .... 7
  2.1 Preliminaries ............................................ 7
  2.2 Fourier-Stieltjes algebras and decomposable multipliers on $\text{VN}(G)$ ... 16
3 Inner amenability .............................................. 22
  3.1 Preliminaries on inner amenability and amenability .............. 22
  3.2 Some characterizations of inner amenability .................... 24
4 Projections on the space of completely bounded Fourier multipliers ...... 27
  4.1 Preliminaries ............................................. 27
  4.2 Description of the approach ................................ 31
  4.3 Step 1: the mappings $P_j(T)$ ................................ 34
  4.4 Step 2: the symbol of $P_j(T)$ is Herz-Schur if $G$ is inner amenable ... 40

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1 Introduction

1.1 Introduction

The Fourier-Stieltjes algebra \( B(G) \) of a locally compact group \( G \) is a generalization of the algebra of bounded regular complex Borel measures of an abelian locally compact group to non-abelian groups. Since its introduction by Eymard in [Eym64], this commutative unital Banach algebra has become a central object in noncommutative harmonic analysis and is closely related to the unitary representation theory of \( G \). More precisely, the elements of \( B(G) \) are exactly the matrix coefficients of continuous unitary representations of \( G \) on complex Hilbert spaces, i.e.

\[
B(G) \overset{\text{def}}{=} \{ \langle \pi(\cdot) \xi, \eta \rangle_H : \pi \text{ is a unitary representation of } G \text{ on } H \text{ and } \xi, \eta \in H \}.
\]

The norm is defined by

\[
\|\varphi\|_{B(G)} \overset{\text{def}}{=} \inf \{ \|\xi\| \|\eta\| : \varphi = \langle \pi(\cdot) \xi, \eta \rangle_H \},
\]

where the infimum is taken over all \( \pi, \xi, \eta \) such that \( \varphi = \langle \pi(\cdot) \xi, \eta \rangle_H \). The operations of this algebra are pointwise multiplication and addition. Also note that \( B(G) \) is a complete invariant of \( G \), i.e. \( B(G_1) \) and \( B(G_2) \) are isometrically isomorphic as Banach algebras if and only if \( G_1 \) and \( G_2 \) are topologically isomorphic as locally compact groups as proved by Walter in [Wal74] (see also [Wal70] and [Kal18, Theorem 3.2.5 p. 99]).

Decomposable maps is a class of operators between \( \mathcal{C}^\ast \)-algebras generalizing completely positive maps. The class of decomposable maps is perhaps the most general class of tractable operators. If \( A \) and \( B \) are \( \mathcal{C}^\ast \)-algebras, recall that a linear map \( T : A \rightarrow B \) is called decomposable \([Haa85]\) if there exist linear maps \( v_1, v_2 : A \rightarrow B \) such that the linear map

\[
\Phi = \begin{bmatrix} v_1 & T \\ T^\circ & v_2 \end{bmatrix} : M_2(A) \rightarrow M_2(B), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} v_1(a) & T(b) \\ T^\circ(c) & v_2(d) \end{bmatrix}
\]

is completely positive, where \( T^\circ(c) \overset{\text{def}}{=} T(c^\ast)^\ast \). In this case, the maps \( v_1 \) and \( v_2 \) are completely positive and the decomposable norm of \( T \) is defined by

\[
\|T\|_{\text{dec, } A \rightarrow B} \overset{\text{def}}{=} \inf \{ \max\{\|v_1\|, \|v_2\|\} \},
\]

where the infimum is taken over all maps \( v_1 \) and \( v_2 \). See the books [BIM04], [Efr00] and [Pis03] for more information on this classical notion. We also refer to [ArK23] and [JuR04] for the analogue notion for operators acting on a noncommutative \( L^p \)-space \( L^p(M) \) associated to a von Neumann algebra \( M \) endowed with a normal semifinite faithful trace, for any \( 1 \leq p \leq \infty \). If

\[
\|T\|_{\text{dec, } A \rightarrow B} \overset{\text{def}}{=} \inf \{ \max\{\|v_1\|, \|v_2\|\} \},
\]

where the infimum is taken over all maps \( v_1 \) and \( v_2 \). See the books [BIM04], [Efr00] and [Pis03] for more information on this classical notion. We also refer to [ArK23] and [JuR04] for the analogue notion for operators acting on a noncommutative \( L^p \)-space \( L^p(M) \) associated to a von Neumann algebra \( M \) endowed with a normal semifinite faithful trace, for any \( 1 \leq p \leq \infty \). If
\( \mathcal{M} \) is approximately finite-dimensional (which is equivalent to injective), it is known that we have the isometric complex interpolation formula

\[
\text{Dec}(L^p(\mathcal{M})) = (\text{CB}(\mathcal{M}), \text{CB}(L^1(\mathcal{M})))^\frac{1}{p},
\]

for the Banach space \( \text{Dec}(L^p(\mathcal{M})) \) of decomposable operators acting on the noncommutative \( L^p \)-space \( L^p(\mathcal{M}) \), which is a combination of \cite{Pis95, Theorem 3.7} and the isometric identification \cite[Theorem 3.24 p. 41]{ArK23} between regular and decomposable operators. Here \( \text{CB}(L^1(\mathcal{M})) \) is the space of completely bounded operators acting on the Banach space \( L^1(\mathcal{M}) \) and the space \( \text{CB}(\mathcal{M}) \) is defined similarly.

Recall that the group von Neumann algebra \( \text{VN}(G) \) of a locally compact group \( G \) is the von Neumann algebra generated by the range \( \lambda(G) \) of the left regular representation \( \lambda \) of \( G \) on the complex Hilbert space \( L^2(G) \) and that the subspace \( \text{span}\{\lambda_s : s \in G\} \) is weak* dense in \( \text{VN}(G) \). If \( G \) is abelian, then the von Neumann algebra \( \text{VN}(G) \) is \( * \)-isomorphic to the algebra \( L^\infty(\hat{G}) \) of essentially bounded functions on the Pontryagin dual \( \hat{G} \) of \( G \). As fundamental models of quantum groups, these algebras play a crucial role in operator algebras. A Fourier multiplier acting on \( \text{VN}(G) \) is a weak* continuous linear operator \( T : \text{VN}(G) \to \text{VN}(G) \) that satisfies \( T(\lambda_s) = \varphi(s)\lambda_s \) for all \( s \in G \), for some measurable function \( \varphi : G \to \mathbb{C} \). In this case, we let \( M_\varphi := T \). Our first result, proved in Corollary 2.18 is the following statement.

**Theorem 1.1** The Fourier-Stieltjes algebra \( B(G) \) of a discrete group \( G \) is canonically isometrically isomorphic to the algebra \( \mathfrak{M}^{\infty, \text{dec}}(G) \) of decomposable Fourier multipliers on the group von Neumann algebra \( \text{VN}(G) \) via the map \( \varphi \mapsto M_\varphi \).

This identification further highlights the ubiquity of the Fourier-Stieltjes algebra \( B(G) \) for a discrete group \( G \). Indeed, we will show in Proposition 2.12 that for any locally compact group \( G \) there exists a well-defined injective contractive map from the Fourier-Stieltjes algebra \( B(G) \) into the space \( \mathfrak{M}^{\infty, \text{dec}}(G) \) of decomposable Fourier multipliers and we will also examine the surjectivity of this map. We will show that the following property

\[
\kappa(G) := \inf \left\{ \|P\|_{\text{CB}_{w^*}(\text{VN}(G))\to\text{CB}_{w^*}(\text{VN}(G))} : P \text{ is a bounded projection from } X \text{ onto } Y \right\}
\]

1. The subscript \( w^* \) means \( \text{weak* continuous} \). With the projection provided by \cite[Proposition 3.1 p. 24]{ArK23}, we could replace the space \( \text{CB}_{w^*}(\text{VN}(G)) \) by the space \( \text{CB}(\text{VN}(G)) \) in this definition.
2. This space is denoted sometimes \( M_0\mathcal{A}(G) \) or \( M_{\text{cb}}\mathcal{A}(G) \).
Indeed, in Proposition 2.15, we will prove that this property suffices to ensure that the previous inclusion $B(G) \hookrightarrow \mathfrak{M}^{\infty, \text{dec}}(G)$ is a bijection. In order to prove that this map is an isometry, we need a matricial generalization of property $(\kappa_\infty)$ (satisfied for any discrete group $G$) and surprisingly (at first sight) the use of results on groupoids. Note also that the existence of non-discrete and non-abelian locally compact groups with $(\kappa_\infty)$ (and even with the stronger property $(\kappa)$ of Definition 1.8) was a rather surprising result of the paper [ArK23] since the proof of property $(\kappa_\infty)$ of a discrete group $G$ is an average argument relying on the compactness of the compact quantum group $(\text{VN}(G), \Delta)$ defined by the group von Neumann algebra $\text{VN}(G)$ and its canonical coproduct $\Delta$. According to [ArK23, Theorem 6.38 p. 121], a second-countable pro-discrete locally compact group $G$ satisfies $\kappa_\infty(G) = 1$. With sharp contrast, we will observe in this paper (see Example 5.3), as announced in [ArK23], that the unimodular locally compact group $G = \text{SL}_2(\mathbb{R})$ does not have $(\kappa_\infty)$.

If $G$ is a locally compact group, with our result we can insert the space $\mathfrak{M}^{\infty, \text{dec}}(G)$ of decomposable Fourier multipliers acting on the von Neumann algebra $\text{VN}(G)$ in the classical contractive inclusion $B(G) \subset \mathfrak{M}^{\infty, \text{dec}}(G) \subset \mathfrak{M}^{\infty, \text{cb}}(G)$:

$$(1.7) \quad B(G) \subset \mathfrak{M}^{\infty, \text{dec}}(G) \subset \mathfrak{M}^{\infty, \text{cb}}(G).$$

It is known [Pis01, p. 54] that the equality $B(G) = \mathfrak{M}^{\infty, \text{cb}}(G)$ characterizes amenability for locally compact groups. This observation allows us to revisit another nice characterization of amenability of Lau and Paterson [LaP91, Corollary 3.2 p. 161] [Pat88a, p. 85], which is described by the next theorem.

**Theorem 1.3 (Lau-Paterson)** Let $G$ be a locally compact group. The following properties are equivalent.

1. The group von Neumann algebra $\text{VN}(G)$ is injective and $G$ is inner amenable.
2. $G$ is amenable.

Recall that a locally compact group $G$ equipped with a left Haar measure is inner amenable if there exists a conjugation-invariant state on the algebra $L^\infty(G)$. We introduce the following conjecture.

**Conjecture 1.4** Let $G$ be a locally compact group.

1. $G$ is inner amenable if and only if we have the equality $B(G) = \mathfrak{M}^{\infty, \text{dec}}(G)$.
2. The von Neumann algebra $\text{VN}(G)$ is injective if and only if we have $\mathfrak{M}^{\infty, \text{dec}}(G) = \mathfrak{M}^{\infty, \text{cb}}(G)$.

We will prove the «only if» part of the first assertion for second-countable unimodular locally compact groups by showing in Theorem 4.29 that inner amenability implies $\kappa_\infty(G) = 1$, and hence $B(G) = \mathfrak{M}^{\infty, \text{dec}}(G)$. We refer to Section 4.2 for a detailed presentation of our approach. This is our first main result.

**Theorem 1.5** Let $G$ be a second-countable unimodular locally compact group. If $G$ is inner amenable then we have $B(G) = \mathfrak{M}^{\infty, \text{dec}}(G)$.

The «only if» part of the second assertion of Conjecture 1.4 is true by a classical result of Haagerup [Haa85, Corollary 2.8 p. 201]. A consequence of our results is that the second point of Conjecture 1.4 is true for discrete groups, see Theorem 2.19, and also for second-countable unimodular inner amenable locally compact groups (see Corollary 5.4), i.e. we can state the following result.
Theorem 1.6 Let $G$ be a discrete group or a second-countable unimodular inner amenable locally compact group. Then the von Neumann algebra $\text{VN}(G)$ is injective if and only if we have $\mathfrak{M}^{\infty,\text{dec}}(G) = \mathfrak{M}^{\infty,\text{cb}}(G)$.

As a byproduct, we also obtain in Theorem 5.1 the following new characterization of amenability, which is in the same spirit as the characterization of Lau and Paterson, previously discussed in Theorem 1.3.

Theorem 1.7 Let $G$ be a second-countable unimodular locally compact group. Then the following are equivalent.

1. $\text{VN}(G)$ is injective and $G$ has $(\kappa_\infty)$.
2. $G$ is amenable.

Finally, we will observe in Example 2.17 that the first inclusion in (1.7) can also be strict, e.g.

for $G = \text{SL}_2(\mathbb{R})$. The converses of these results will need further investigations. We also give in Section 3.2 other characterizations of inner amenability for unimodular locally compact groups and we will use one of these in the proofs of our results.

Note that if $G$ is a unimodular locally compact group, there exists a canonical normal semifinite faithful trace on the group von Neumann algebra $\text{VN}(G)$, allowing to introduce the associated noncommutative $L^p$-space $L^p(\text{VN}(G))$ for any $1 \leq p < \infty$. In this context, we can introduce the space $\mathfrak{M}^{\infty,\text{cb}}(G)$ of completely bounded Fourier multipliers acting on the noncommutative $L^p$-space $L^p(\text{VN}(G))$.

We describe a new class of locally compact groups with the following property introduced in [ArK23, Definition 1.1 p. 3] which requires a bounded projection at the level $p = \infty$ and a compatible bounded projection at the level $p = 1$. The compatibility is taken in the sense of interpolation theory described in the books [BeL76] and [Tri95]. This compatibility property is crucial in a companion paper in order to describe the decomposable norm of Fourier multipliers acting on noncommutative $L^p$-spaces with the interpolation formula (1.5) and a classical argument.

Definition 1.8 We say that a locally compact group $G$ has property $(\kappa)$ if there exist compatible bounded projections $P_G^{\infty} : \text{CB}_{\text{w}^*}(\text{VN}(G)) \rightarrow \text{CB}_{\text{w}^*}(\text{VN}(G))$ and $P_G^1 : \text{CB}(L^1(\text{VN}(G))) \rightarrow \text{CB}(L^1(\text{VN}(G)))$ onto the subspaces $\mathfrak{M}^{\infty,\text{cb}}(G)$ and $\mathfrak{M}^{1,\text{cb}}(G)$, preserving the complete positivity. In this case, we introduce the constant

$$
(1.8) \quad \kappa(G) \overset{\text{def}}{=} \inf \max \left\{ \|P_G^{\infty}\|_{\text{CB}_{\text{w}^*}(\text{VN}(G)) \rightarrow \text{CB}_{\text{w}^*}(\text{VN}(G))}, \|P_G^1\|_{\text{CB}(L^1(\text{VN}(G))) \rightarrow \text{CB}(L^1(\text{VN}(G)))} \right\},
$$

where the infimum is taken on all admissible couples $(P_G^{\infty}, P_G^1)$ of projections. Finally, we let $\kappa(G) \overset{\text{def}}{=} \infty$ if the locally compact group $G$ does not have $(\kappa)$.

3. The following remark is important to note. If $P_G^{\infty} : \text{CB}_{\text{w}^*}(\text{VN}(G)) \rightarrow \text{CB}_{\text{w}^*}(\text{VN}(G))$ is a bounded projection onto the subspace $\mathfrak{M}^{\infty,\text{cb}}(G)$ then we can define a map $P_G^1 : \text{CB}(L^1(\text{VN}(G))) \rightarrow \text{CB}(L^1(\text{VN}(G)))$ by

$$
P_G^1(T) \overset{\text{def}}{=} (P_G^{\infty}(T^*)^*), \quad T \in \text{CB}(L^1(\text{VN}(G))).
$$

It is then easy to check that $P_G^1$ is a bounded projection preserving complete positivity onto the subspace $\mathfrak{M}^{1,\text{cb}}(G)$ and its norm is equal to the one of $P_G^{\infty}$. It is important to note that there is no evidence that the maps $P_G^{\infty}$ and $P_G^1$ are compatible in the sense of interpolation. Consequently, the properties $(\kappa_\infty)$ and $(\kappa)$ seem to be different.
The well-known average trick [Haa16, proof of Lemma 2.5] of Haagerup essentially implies that \( \kappa(G) = 1 \) for any discrete group \( G \), see [ArK23, Section 4.2]. In [ArK23, Proposition 6.43 p. 125] and [ArK23, Theorem 6.38 p. 121], it is proved that an abelian locally compact group satisfies \( \kappa(G) = 1 \) and that a second-countable pro-discrete locally compact group \( G \) satisfies \( \kappa(G) = 1 \). It is equally proved in [ArK23, Theorem 6.16 p. 96] that some class of second-countable unimodular locally compact groups approximable by lattice subgroups have \( (\kappa) \).

Another very significant result that we obtain in this paper is described in the following statement, see Corollary 4.33. Let us first recall that the concept of dimension of a suitable topological space can be defined using the small inductive dimension, the large inductive dimension, or the covering dimension. In the case of a locally compact group \( G \), these three notions of dimension coincide. We refer to Section 4.2 for more background.

**Theorem 1.9** A second-countable unimodular finite-dimensional amenable locally compact group \( G \) has property \( (\kappa) \).

An upper estimate of \( \kappa(G) \) is possible for some groups. For example, in the case of a second-countable unimodular totally disconnected amenable locally compact group, our method gives \( \kappa(G) = 1 \), which is a sharp result. Note that these results complement the result of our previous paper [ArK23]. From this point of view, totally disconnected locally compact groups behave better than Lie groups, phenomenon that we already noticed in [ArK23]. We refer to Section 4.2 for a detailed presentation of our approach.

The proof relies on the structure of finite-dimensional locally compact groups extracted from the solution to Hilbert’s fifth problem. More precisely, we use a version of Iwasawa’s local splitting theorem, which says that an \( n \)-dimensional second-countable locally compact group is locally isomorphic to the product of a totally disconnected compact group \( K \) and a Lie group \( L \) of dimension \( n \), to reduce the problem to totally disconnected groups and to connected Lie groups. It allows us to use doubling constants of the Carnot-Carathéodory metric of connected Lie groups for small balls to construct special suitable «noncommutative functions», which are crucial for our proof.

We will prove in Corollary 4.31 a different result for the case of a second-countable unimodular amenable locally compact group \( G \), using some other special «noncommutative functions». We obtain that the space \( \mathfrak{M}^{p,cb}(G) \) of completely bounded Fourier multipliers on the noncommutative \( L^p \)-space \( L^p(\text{VN}(G)) \) is contractively complemented in the space \( \text{CB}(L^p(\text{VN}(G))) \) of completely bounded operators acting on the Banach space \( L^p(\text{VN}(G)) \), by a contractive projection preserving the complete positivity. Note that this map is contractive which is better than the boundedness of the maps \( P^1_G \) and \( P^\infty_G \) provided by Theorem 1.9 (when it applies), but only for one value of \( p \). Note also that this property does not characterize amenability if \( 1 < p < \infty \) since a discrete group \( G \) such that the von Neumann algebra \( \text{VN}(G) \) is QWEP\(^4\) also satisfies this property, see [JR03, p. 334] and [ArK23, Theorem 4.2 p. 62].

**Theorem 1.10** Let \( G \) be a second-countable unimodular amenable locally compact group. Let \( 1 < p < \infty \) such that \( \frac{1}{p} \) is rational. Then there exists a contractive projection \( P^p_G : \text{CB}(L^p(\text{VN}(G))) \to \text{CB}(L^p(\text{VN}(G))) \) onto the subspace \( \mathfrak{M}^{p,cb}(G) \), preserving the complete positivity.

In contrast to Theorem 1.9, this result cannot be used to characterize the norms of decomposable multipliers on noncommutative \( L^p \)-spaces.

\(^4\) It is not clear if this assumption is removable or not. It is required by the use of vector-valued noncommutative \( L^p \)-spaces and the most general known theory needs the QWEP assumption.
1.2 Structure of the paper

To facilitate access to individual topics, each section is made as self-contained as possible. The paper is structured as follows.

Section 2.1 provides background on Fourier-Stieltjes algebras, groupoids, and operator algebras. In Proposition 2.2, we demonstrate that the space Dec(A, B) of decomposable operators between C*-algebras admits a canonical operator space structure. In Section 2.2, we prove in Proposition 2.12 that for any locally compact group G, there exists a well-defined injective completely contractive map from the Fourier-Stieltjes algebra B(G) into the space $M_\infty, \text{dec}(G)$ of decomposable Fourier multipliers on the group von Neumann algebra VN(G). Furthermore, we explore in Proposition 2.15 the relationship between the equality $B(G) = M_\infty, \text{dec}(G)$ and property ($\kappa_\infty$). We also show in Theorem 2.18 that the Fourier-Stieltjes algebra B(G) of a discrete group G is isometrically isomorphic to the algebra $M_\infty, \text{dec}(G)$ of decomposable Fourier multipliers. In Example 2.17, we demonstrate that for $G = \text{SL}_2(\mathbb{R})$, the inclusion $B(G) \subset M_\infty, \text{dec}(G)$ is strict.

In Section 3.1, we provide background on inner amenability and amenability. In Section 3.2, we establish various characterizations of inner amenability for unimodular locally compact groups, using asymptotically central nets of functions or inner Følner nets. These characterizations are employed in Section 4.4.

Section 4.1 offers background on measurable Schur multipliers and Plancherel weights on group von Neumann algebras. In Section 4.2, we outline the technical approach of this chapter. Section 4.3 presents the construction of some Schur multipliers derived from a (weak* continuous if $p = \infty$) completely bounded map $T: L^p(VN(G)) \rightarrow L^p(VN(G))$ acting on the noncommutative $L^p$-space $L^p(VN(G))$ of a second-countable unimodular locally compact group G. In Section 4.4, we show that the symbol can be chosen as a Herz-Schur symbol if the group G is inner amenable. Section 4.5 examines the symbols of these Schur multipliers for $p = 1$ and $p = \infty$. Section 4.6 explores the convergence of the symbols of Schur multipliers, while Section 4.7 focuses on Lie groups and totally disconnected locally compact groups, culminating in the proof in Section 4.9 that unimodular finite-dimensional amenable locally compact groups have property ($\kappa$). In Section 4.8, we construct a contractive projection from the space of completely bounded Schur multipliers $M^\text{cb}_G$ onto the subspace $M^\text{cb, HS}_G$ of Herz-Schur multipliers, in the case where G is amenable. This result will be used in Section 4.9, which contains our main complementation results.

In Section 5.1, Theorem 5.1 presents a new characterization of amenability for second-countable unimodular locally compact groups. In Example 5.3, we observe that the unimodular locally compact group $G = \text{SL}_2(\mathbb{R})$ does not have property ($\kappa_\infty$). Finally, in Section 5.2, we show that if there exists a bounded projection $Q: M^\infty_G \rightarrow M^\infty_G$ onto the space of completely bounded Herz-Schur multipliers $M^\infty_G$ over the space $B(L^2(G))$ of bounded operators on the Hilbert space $L^2(G)$, preserving the complete positivity for some second-countable unimodular locally compact group G such that the von Neumann algebra VN(G) is injective, then G must be amenable.

2 Fourier-Stieltjes algebras and decomposable multipliers on $VN(G)$

2.1 Preliminaries

Decomposable maps Recall that the notion of decomposable map is defined in (1.3). Consider some C*-algebras $A$, $B$ and $C$. Let $T_1: A \rightarrow B$ and $T_2: B \rightarrow C$ be some decomposable
the following elementary lemma.

By [Haa85, Proposition 1.3 (4) p. 177], any completely positive map \( T: A \to B \) between C*-algebras is decomposable and we have

\[
||T||_{\text{dec}, A \to B} = ||T||_{\text{cb}, A \to B} = ||T||_{A \to B}.
\]

It is known that the space \( \text{Dec}(A, B) \) of decomposable maps can be endowed with an operator space structure. To demonstrate this, we will use in the first proof of Proposition 2.12 the following elementary lemma.

**Lemma 2.1.** Let \( \mathcal{M}_1, \mathcal{M}_2 \) and \( \mathcal{N} \) be von Neumann algebras and let \( T: \mathcal{M}_1 \to \mathcal{M}_2 \) be a weak* continuous decomposable map. Then we have a well-defined weak* continuous decomposable map \( \text{Id}_\mathcal{N} \otimes T: \mathcal{N} \overline{\otimes} \mathcal{M}_1 \to \mathcal{N} \overline{\otimes} \mathcal{M}_2 \) and

\[
||\text{Id}_\mathcal{N} \otimes T||_{\text{dec}, \mathcal{N} \overline{\otimes} \mathcal{M}_1 \to \mathcal{N} \overline{\otimes} \mathcal{M}_2} \leq ||T||_{\text{dec}, \mathcal{M}_1 \to \mathcal{M}_2}.
\]

**Proof:** Note that the decomposable map \( T \) is completely bounded by [Haa85, Proposition 1.3 (3) p. 177]. By [BIM04, p. 40], we infer that we have a well-defined weak* continuous completely bounded map \( \text{Id}_\mathcal{N} \otimes T: \mathcal{N} \overline{\otimes} \mathcal{M}_1 \to \mathcal{N} \overline{\otimes} \mathcal{M}_2 \). By [Haa85, Remark 1.5 p. 183], the infimum in the definition of the decomposable norm given in (1.4) is actually a minimum. Consequently, there exist some linear maps \( v_1, v_2: \mathcal{M}_1 \to \mathcal{M}_2 \) such that the map

\[
\begin{bmatrix}
v_1 & T \\
v_2 & T^0
\end{bmatrix} : \mathcal{M}_2(\mathcal{M}_1) \to \mathcal{M}_2(\mathcal{M}_2)
\]

is completely positive with \( \max\{||v_1||, ||v_2||\} = ||T||_{\text{dec}, \mathcal{M}_1 \to \mathcal{M}_2} \). It is not difficult to see that we can suppose that \( v_1 \) and \( v_2 \) are weak* continuous by using [ArK23, Proposition 3.1 p. 24] as in the proof of [ArK23, Proposition 3.4 p. 26]. Then by [Li92, Proposition 4.3.7 p. 225] the tensor product

\[
\text{Id}_\mathcal{N} \otimes v_1 \otimes \text{Id}_\mathcal{N} \otimes T \otimes \text{Id}_\mathcal{N} \otimes v_2 : \mathcal{M}_2(\mathcal{N} \overline{\otimes} \mathcal{M}_1) \to \mathcal{M}_2(\mathcal{N} \overline{\otimes} \mathcal{M}_2)
\]

is a well-defined completely positive map. We deduce that the map \( \text{Id}_\mathcal{N} \otimes T: \mathcal{N} \overline{\otimes} \mathcal{M}_1 \to \mathcal{N} \overline{\otimes} \mathcal{M}_2 \) is decomposable with

\[
||\text{Id}_\mathcal{N} \otimes T||_{\text{dec}} \overset{(1.4)}{\leq} \max\{||\text{Id}_\mathcal{N} \otimes v_1||, ||\text{Id}_\mathcal{N} \otimes v_2||\} \leq \max\{||v_1||_{\text{cb}}, ||v_2||_{\text{cb}}\} \overset{(2.2)}{=} \max\{||v_1||, ||v_2||\} = ||T||_{\text{dec}, \mathcal{M}_1 \to \mathcal{M}_2},
\]

where we use in the first equality the complete positivity of the linear maps \( v_1 \) and \( v_2 \). ■

Finally, if \( A \) and \( B \) are C*-algebras, with \( B \) unital, we will show that the space \( \text{Dec}(A, B) \) of decomposable maps can be endowed with an operator space structure. To demonstrate this, suppose that \( [T_{ij}] \) belongs to the matrix space \( M_n(\text{Dec}(A, B)) \), where \( n \geq 1 \) is an integer. We
identify the matrix \([T_{ij}]\) with the map \(A \mapsto M_n(B), x \mapsto [T_{ij}(x)]\). We define a norm on the space \(M_n(\text{Dec}(A, B))\) by setting

\[
\|T_{ij}\|_{M_n(\text{Dec}(A, B))} \overset{\text{def}}{=} \|x \mapsto [T_{ij}(x)]\|_{\text{Dec}(A, M_n(B))}.
\]

In short, we make the identification \(M_n(\text{Dec}(A, B)) = \text{Dec}(A, M_n(B))\).

**Proposition 2.2** Let \(A\) and \(B\) be \(C^*\)-algebras, with \(B\) unital. When endowed with the matricial norms from (2.5), the Banach space \(\text{Dec}(A, B)\) acquires the structure of an operator space.

**Proof** : Let \(X, Y \in M_n\) and \([T_{ij}] \in M_n(\text{Dec}(A, B))\) for some integer \(n \geq 1\). Note that by [Pis03, Exercise 12.1 p. 251] the two-sided multiplication map \(u : M_n(B) \to M_n(B), y \mapsto (X \otimes 1_B) y (Y \otimes 1_B)\) is decomposable with \(\|u\|_{\text{dec}, M_n(B) \otimes M_n(B)} \leq \|X \otimes 1_B\|_{M_n(B)} \|Y \otimes 1_B\|_{M_n(B)}\). Using this observation in the equality, we obtain

\[
\|X[T_{ij}]Y\|_{M_n(\text{Dec}(A, B))} = \|x \mapsto X[T_{ij}(x)]Y\|_{\text{Dec}(A, M_n(B))} = \|x \mapsto (X \otimes 1_B)[T_{ij}(x)](Y \otimes 1_B)\|_{\text{Dec}(A, M_n(B))} \leq \|X \otimes 1_B\|_{M_n(B)} \|Y \otimes 1_B\|_{M_n(B)} \|x \mapsto [T_{ij}(x)]\|_{\text{Dec}(A, M_n(B))}.
\]

Let \([T_{ij}] \in M_n(\text{Dec}(A, B))\) and \([S_{kl}] \in M_m(\text{Dec}(A, B))\) for some integers \(n, m \geq 1\). Using [Pis20, Lemma 6.8 p. 118] in the second equality, we have

\[
\|T_{ij} \oplus S_{kl}\|_{M_{n+m}(\text{Dec}(A, B))} = \max \left\{ \|x \mapsto [T_{ij}(x)]\|_{\text{Dec}(A, M_n(B))}, \|x \mapsto [S_{kl}(x)]\|_{\text{Dec}(A, M_m(B))} \right\}.
\]

Now, it suffices to use Ruan’s theorem [Pis03, p. 35] or [EffR00, Proposition 2.3.6 p. 34].

We finish by providing another formula for the decomposable norm.

**Proposition 2.3** Consider a decomposable map \(T : A \to B\) between \(C^*\)-algebras. Then

\[
\|T\|_{\text{dec}, A \to B} = \inf \left\{ \|v_1\|_+^\top, \|v_2\|_+^\top \right\},
\]

where the infimum is taken over all maps \(v_1\) and \(v_2\) such that the operator \(\Phi\) introduced in (1.3) is completely positive.

**Proof** : The inequality \(\geq\) is obvious. Now, we show the reverse inequality, assume that the operator \(\Phi = \begin{pmatrix} v_1 & T \\ T^\circ & v_2 \end{pmatrix}\) of (1.3) is completely positive for some linear maps \(v_1, v_2\). Suppose that \(v_1 = 0\). Consider a positive element \(x \in A\). Since the element \(\begin{pmatrix} x & x \\ x & x \end{pmatrix}\) in \(M_2(A)\) is positive, we see that \(\begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}\) and \(\begin{pmatrix} x & x \\ x & x \end{pmatrix}\) are positive. By [Bha07, Proposition 1.3.2 p. 13], we infer that \(\|T(x)\| \leq 0\|v_2(x)\|_+^\top\). As every element of \(A\) is a linear combination of positive elements, we conclude that \(T = 0\).
So we can suppose $v_1 \neq 0$. For any $t > 0$ we define the positive matrix $A_t \overset{\text{def}}{=} \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}$ in $\mathbb{M}_2$. Then the linear map $\Phi_t: \mathbb{M}_2(A) \to \mathbb{M}_2(B)$, $x \mapsto A_t \Phi(x) A_t$ is also completely positive. For any $t > 0$, observe that $\Phi_t = \begin{pmatrix} v_1 & T \\ T^* & \frac{1}{4}v_2 \end{pmatrix}$. Hence $\|T\|_{\text{dec,}A \to B} \overset{\text{(1.4)}}{\leq} \inf_{t>0} \max \{t \|v_1\|, t^{-1} \|v_2\|\}$.

The choice $t = \frac{\|v_2\|^2}{2} \|v_1\|^{-1}$ gives the inequality $\|T\|_{\text{dec,}A \to B} \leq \|v_1\|^\frac{1}{2} \|v_2\|^\frac{1}{2}$.

Taking the infimum on $v_1, v_2$, we obtain $\|T\|_{\text{dec,}A \to B} \leq \inf \left\{ \|v_1\|^\frac{1}{2} \|v_2\|^\frac{1}{2} \right\}$.

\[
\begin{aligned}
\text{Example 2.4} & \quad \text{If the locally compact group } G \text{ is abelian then by [Fol16, Example p. 225] the } \\
& \quad \text{C}\text{-algebra } C^*(G) \text{ is } \ast\text{-isomorphic to the C}\text{-algebra } C_0(G), \text{ where } G \text{ is the Pontryagin dual of } G.
\end{aligned}
\]

\[
\begin{aligned}
\text{Fourier-Stieltjes algebras} & \quad \text{Recall that we defined the Fourier-Stieltjes algebra } B(G) \text{ of a locally compact group } G \text{ in (1.1). It is known that it is the complex linear span of the set of all continuous positive definite functions on } G, \text{ see [Kal18, Definition 2.1.5 p. 40]. Equipped with pointwise multiplication and addition } B(G) \text{ becomes a commutative unital Banach algebra by} \\
& \quad [Kal18, Theorem 2.1.11 p. 44]. \text{ Let } \mu_G \text{ be a left Haar measure of } G. \text{ If } \varphi \in B(G) \text{ then by [Eym64, p. 193] the linear form } \\
& \quad \omega_{\varphi}: L^1(G) \to \mathbb{C} \text{ defined by} \\
& \quad (2.7) \quad \omega_{\varphi}(f) \overset{\text{def}}{=} \int_G \varphi(s) f(s) d\mu_G(s) \\
& \quad \text{extends to a bounded linear form } \omega_{\varphi}: C^*(G) \to \mathbb{C} \text{ with } \|\varphi\|_{B(G)} = \|\omega_{\varphi}\|. \text{ It is well-known that each bounded linear form on } C^*(G) \text{ satisfies this description, i.e. we have } B(G) = C^*(G) \ast \text{isometrically, see [Eym64, p. 192] or [Kal18, p. 40]. Moreover, by [Fel60, Lemma 1.4 p. 370] and} \\
& \quad [Kal18, Theorem 1.6.1 p. 29] \text{ the linear form } L^1(G) \to \mathbb{C}, f \mapsto \int_G \varphi(s) f(s) d\mu_G(s) \text{ extends to a positive linear form on the C}\text{-algebra } C^*(G) \text{ if and only if } \varphi \text{ is a continuous positive definite function.}
\end{aligned}
\]

\[
\begin{aligned}
\text{Example 2.5} & \quad \text{If the locally compact group } G \text{ is abelian, recall that the Fourier transform } \\
& \quad \hat{\mu}: G \to \mathbb{C} \text{ of a bounded regular complex Borel measure } \mu \in M(\hat{G}) \text{ on the Pontryagin dual } \hat{G}
\end{aligned}
\]
is given by $$\hat{\mu}(s) \overset{\text{def}}{=} \int_G \chi(s) \, d\mu(\chi)$$ where $$s \in G$$. According to [Eym64, Exemple p. 92], $$B(G)$$ is the space of Fourier transforms $$\varphi = \hat{\mu}$$ of bounded regular complex Borel measures $$\mu \in M(G)$$ and $$\|\varphi\|_{B(G)} = \|\mu\|_{M(G)}$$.

We will use the next observation written without proof in [Eym64, p. 188]. For the sake of completeness, we give a proof.

**Proposition 2.6** Let $$G$$ be a locally compact group. Let $$\varphi \colon G \to \mathbb{C}$$ be a continuous positive definite function. We have

$$\|\varphi\|_{B(G)} = \varphi(e). \quad (2.8)$$

**Proof**: First proof if $$G$$ is discrete. Let $$G$$ be a discrete group. By [Dix77, 13.9.2 p. 303], the full C*-algebra $$C^*(G)$$ of $$G$$ is unital\(^5\). By (2.7), we have a positive linear form $$\omega_\varphi \colon C^*(G) \to \mathbb{C}$$, $$U(s) \mapsto \varphi(s)$$. So using [KaL97, Theorem 4.3.2 p. 256] in the second equality, we conclude that

$$\|\varphi\|_{B(G)} = \|\omega_\varphi\|_{C^*(G)^*} = \omega_\varphi(1) = \omega_\varphi(U(e)) = \varphi(e).$$

Second proof if $$G$$ is locally compact. Using [KaL87, Remark 2.1.10 p. 43], we know that $$\|\varphi\|_{B(G)} \geq \|\varphi\|_{C^*(G)^*} \geq \varphi(e)$$. Furthermore, with [Dix77, Theorem 13.4.5 p. 288], we can write $$\varphi = \langle \pi(\cdot)\xi, \xi \rangle_H$$, where $$\pi$$ is a continuous unitary representation of $$G$$ on some complex Hilbert space $$H$$ and $$\xi \in H$$. We deduce that $$\varphi(e) = \langle \xi, \xi \rangle_H = \|\xi\|_H^2 \geq \|\varphi\|_{B(G)}$$.  

**Fell’s absorption principle** Let $$G$$ be a locally compact group. Let $$\pi \colon G \to B(H)$$ be any continuous unitary representation of $$G$$. Recall Fell’s absorption principle, e.g. [KaL87, Lemma 5.5.3 p. 187] (see also [Pis03, Proposition 8.1 p. 149] for the discrete case). If $$1_H : G \to B(H)$$, $$s \mapsto \text{Id}_H$$ is the identity representation, we have a unitary equivalence

$$\lambda \otimes \pi \approx \lambda \otimes 1_H. \quad (2.9)$$

**Groupoids** We refer to [Hah78], [Muh90], [Pat03], [Pat04], [Pat99], [Ren97] [Ren80] and [RaW97] for background on groupoids. A groupoid is a set $$G$$ together with a distinguished subset $$G^{(2)} \subset G \times G$$, a multiplication map $$G^{(2)} \to G$$, $$(s,t) \mapsto st$$ and an inverse map $$G \to G$$, $$s \mapsto s^{-1}$$ such that

1. for any $$s \in G$$ we have $$(s^{-1})^{-1} = s$$,
2. if $$(s,t), (t,r) \in G^{(2)}$$ then $$(st, r)$$ and $$(s, tr)$$ belong to $$G^{(2)}$$ and $$(st)r = s(tr),$$
3. for any $$s \in G$$ we have $$(s, s^{-1}) \in G^{(2)}$$ and if $$s, r \in G$$ satisfies $$(s, r) \in G^{(2)}$$, we have $$s^{-1}(sr) = r$$ and $$(sr)r^{-1} = s$$.

We say that $$G^{(2)}$$ is the set of composable pairs. Second axiom shows that for products of three groupoid elements, there is no ambiguity in dropping the parentheses, and simply writing $$s tr$$ for $$(st)r$$. A groupoid $$G$$ is a group if and only if its unit space $$G^{(0)}$$ is a singleton.

---

5. Actually, by [Mil71] the full C*-algebra $$C^*(G)$$ of a locally compact group $$G$$ is unital if and only if $$G$$ is discrete.
Given a groupoid $G$ we shall write $G^{(0)} \overset{\text{def}}{=} \{s^{-1}s : s \in G\}$ and refer to elements of $G^{(0)}$ as units and to $G^{(0)}$ itself as the unit space. Since $(s^{-1})^{-1} = s$ for any $s \in G$, we also have $G^{(0)} = \{ss^{-1} : s \in G\}$. We define the range and domain maps $r, d: G \to G^{(0)}$ by

$$r(s) \overset{\text{def}}{=} ss^{-1} \quad \text{and} \quad d(s) \overset{\text{def}}{=} s^{-1}s, \quad s \in G.$$ 

For any $s, t \in G$ we have $(s, t) \in G^{(2)}$ if and only if $d(s) = r(t)$. For any unit $u \in G^{(0)}$, we let $G_u \overset{\text{def}}{=} r^{-1}(\{u\})$ and $G_u \overset{\text{def}}{=} d^{-1}(\{u\})$.

**Measured groupoids** A locally compact groupoid is a groupoid $G$ equipped with a locally compact topology, where the inversion map $s \mapsto s^{-1}$ is continuous, and the multiplication map $(s, t) \mapsto st$ is continuous with respect to the relative topology on $G^{(2)}$, considered as a subset of $G \times G$.

Following [Ren80, Definition 2.2 p. 16] and [Muh90, Definition 2.28 p. 24], a left Haar system for $G$ is defined as being a family $(\nu^u)_{u \in G^{(0)}}$ of positive Radon measures on $G$ such that

1. the support supp $\nu^u$ is $G_u$,
2. for any function $f \in C_c(G)$, the map $G^{(0)} \to \mathbb{C}$, $u \mapsto \int_G f \, d\nu^u$ is continuous,
3. for any function $f \in C_c(G)$ and any $s \in G$ we have $\int_G f(s(t)) \, d\nu^u(t) = \int_G f(t) \, d\nu^{r(s)}(t)$.

Roughly speaking, to each unit $u$ we associate a measure $\nu^u$ supported on $G_u$. With such system, the space $C_c(G)$ of continuous functions with compact support, endowed with the operations

$$(f \ast g)(s) \overset{\text{def}}{=} \int_G f(t)g(t^{-1}s) \, d\nu^{r(s)}(t), \quad f^*(s) \overset{\text{def}}{=} \frac{f(s^{-1})}{f(s)} \quad s \in G,$$

is a $*$-algebra, according to [Pat99, p. 38].

Let us additionally consider a positive Radon measure $\mu$ on the unit space $G^{(0)}$. Following [Ren80, Definition 3.1 p. 22] and [Pat99, p. 86], we can introduce the measure $\nu \overset{\text{def}}{=} \int_{G^{(0)}} \nu^u \, d\mu(u)$ induced on $G$ by $\mu$. The measure $\mu$ is said to be quasi-invariant if the measure $\nu$ is equivalent to its image by the inversion map $G \to G$, $s \mapsto s^{-1}$. A measured groupoid $(G, \nu, \mu)$ is a locally compact groupoid equipped with a left Haar system $\nu$ and a quasi-invariant measure $\mu$.

**Example 2.7** Every locally compact group $G$ can be viewed as a locally compact groupoid, with $G^{(0)} = \{e\}$, multiplication given by the group operation, and inversion by the usual group inverse. We obtain a measured groupoid with a left Haar measure and the Dirac measure as a quasi-invariant measure on $G^{(0)}$.

**Example 2.8** Let $X$ be a locally compact space. Set $G \overset{\text{def}}{=} X \times X$ and $G^{(2)} \overset{\text{def}}{=} \{(x, y), (y, z) : x, y, z \in X\}$. Moreover, for any $x, y, z \in X$ we define $(x, y)(y, z) \overset{\text{def}}{=} (x, z)$ and $(x, y)^{-1} \overset{\text{def}}{=} (y, x)$. We obtain the pair groupoid (or Brandt groupoid). We have $G^0 = \{(x, x) : x \in X\}$, which can be identified with $X$. Moreover, for any $x, y \in X$ we have $r(x, y) = x$ and $d(x, y) = y$. For any unit $u \in X$, we have $G_u = X \times \{u\}$ and $G_u = \{u\} \times X$.

If we equip $X$ with a positive Radon measure $\mu$, we can define for any unit $u \in X$ the measure $\nu^u \overset{\text{def}}{=} \delta_u \otimes \mu$ on $G$, where $\delta_u$ is the unit measure at $u$. In this case, the measure $\mu$ is quasi-invariant. If we consider the discrete space $X = \{1, \ldots, n\}$ for some integer $n \geq 1$, equipped with the counting measure $\mu_n$, we denote by $P_n$ the associated measured groupoid.
Example 2.9 If $G_1$ and $G_2$ are groupoids, it is clear that the product $G_1 \times G_2$ has a canonical structure of groupoid with $(G_1 \times G_2)^{(0)} = \{(x_1, y_1), (x_2, y_2) : (x_1, x_2) \in G_1^{(2)}, (y_1, y_2) \in G_2^{(2)}\}$,

\[(2.10)\]

\[(x, y)^{-1} = (x^{-1}, y^{-1}) \quad \text{and} \quad (x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2).\]

We have $(G_1 \times G_2)^{(0)} = G_1^{(0)} \times G_2^{(0)}$, $d(x, y) = (d(x), d(y))$ and $r(x, y) = (r(x), r(y))$

If $G_1 = P_n$ and $G_2 = G$ is a group, an element of the product $P_n \times G$ can be written under the form $(i, j, s)$ with $i, j \in \{1, \ldots, n\}$ and $s \in G$. We can see a complex function $F : P_n \times G \to \mathbb{C}$ as a $n \times n$ matrix-valued function $[F_{ij}]_{1 \leq i, j \leq n}$ on the group $G$, where $F_{ij} : G \to \mathbb{C}$, $s \mapsto F(i, j, s)$.

**von Neumann algebras and multipliers** The von Neumann algebra of a measured groupoid $(G, \lambda, \mu)$ is the von Neumann algebra generated by $\lambda(C_c(G))$, where $\lambda$ is the regular representation defined in [Ren80, p. 55] and [Pat99, pp. 93-94] of the measured groupoid $(G, \nu, \mu)$. If $G = P_n$ for some integer $n \geq 1$ then it is easy to check that the von Neumann algebra $\text{VN}(G)$ is *-isomorphic to the matrix algebra $M_n$.

Following [Ren97, Definition 3.1 p. 475], we say that a function $\varphi \in L^\infty(G)$ induces a bounded Fourier multiplier if it induces a weak* continuous$^7$ operator $\text{VN}(G) \to \text{VN}(G)$, $\lambda(f) \mapsto \lambda(\varphi f)$.

**Positive definite functions** Let $(G, \nu, \mu)$ be a measured groupoid. By [Ren97, Proposition 1.1 p. 457 and Definition 1.1 p. 458], a function $\varphi \in L^\infty(G)$ is said to be positive definite if for any integer $n \geq 1$ and any complex numbers $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, the inequality

\[(2.11)\]

\[\sum_{k,l=1}^n \alpha_k \overline{\alpha_l} \varphi(\gamma_k^{-1} \gamma_l) \geq 0\]

holds for $\mu$-almost all $u \in G^{(0)}$ and $\nu$-$\mu$-almost all $\gamma_1, \ldots, \gamma_n \in G^u$. We now naturally relate this condition to [ArtK23, Proposition 8.4 p. 166], where the proof holds for the case of a locally compact group endowed with a trivial cocycle.

**Lemma 2.10** Let $G$ be a locally compact group and let $n \geq 1$ be an integer. The $n \times n$ matrix-valued function $F = [F_{ij}]_{1 \leq i, j \leq n}$ in the space $L^\infty(P_n \times G)$ defines a positive definite function on the groupoid $P_n \times G$ if and only if for any integer $m \geq 1$, any elements $i_1, \ldots, i_m \in \{1, \ldots, n\}$, any $s_1, \ldots, s_m \in G$ and any complex numbers $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, we have the inequality

\[(2.12)\]

\[\sum_{k,l=1}^m \alpha_k \overline{\alpha_l} F_{i_k, j_l}(s_k^{-1} s_l) \geq 0.\]

**Proof**: Note that the unit space of the groupoid $P_n \times G$ identifies to $(P_n \times G)^{(0)} = P_n^{(0)} \times G^{(0)} = \{1, \ldots, n\}$. Fix some $q \in \{1, \ldots, n\}$. If $\gamma_k = (q, i_k, s_k)$ and $\gamma_l = (q, j_l, s_l)$ are elements of the groupoid $P_n \times G$ with $i_k, j_l \in \{1, \ldots, n\}$ and $s_k, s_l \in G$ then we have

\[\gamma_k^{-1} \gamma_l = (q, i_k, s_k)^{-1}(q, j_l, s_l) \stackrel{(2.10)}{=} (i_k, q, s_k^{-1})(q, j_l, s_l) \stackrel{(2.10)}{=} (i_k, j_l, s_k^{-1} s_l).\]

So the condition (2.11) translates to

\[(2.13)\]

\[\sum_{k,l=1}^m \alpha_k \overline{\alpha_l} F(i_k, j_l, s_k^{-1} s_l) \geq 0 \quad \text{i.e.} \quad \sum_{k,l=1}^m \alpha_k \overline{\alpha_l} F_{i_k, j_l}(s_k^{-1} s_l) \geq 0\]

$^7$. In [Ren97, Proposition 3.1 p. 474], «bounded» must be replaced with «weak* continuous» to ensure the correctness of the statement.
A characterization of functions of Fourier-Stieltjes algebras  In the case of groupoids, we caution the reader that there exist three notions of Fourier-Stieltjes algebra, introduced in the papers [Pat04], [Ren97] and [Rau97]. We refer to the excellent survey [Pat03] for more information. We require a specific case (for groups) of a result, essentially stated in [Ren97, Proposition 1.3 p. 459 and Lemma 1.1 p. 460] and [Pat04, Proposition 5 p. 1266] that is more generally stated for Fourier-Stieltjes algebras associated with measured groupoids. Unfortunately, the proof of [Ren97, Proposition 1.3 p. 459] is false (the result [Pat04, Proposition 5 p. 1266] is incomplete) and must be corrected. Consequently, we provide an argument sufficient for our purposes.

For the proof, we will use the notion of $G$-Hilbert bundle on a locally compact groupoid $G$, which is a Hilbert bundle $\mathcal{H}$ over its unit space $G^{(0)}$ such that there is a linear unitary operator $\pi_s : \mathcal{H}(d(s)) \to \mathcal{H}(r(s))$ for each $s \in G$ such that for all continuous bounded sections $\xi$ and $\eta$ of $\mathcal{H}$, the map $(\xi, \eta) : G \to B(\mathcal{H}(d(s)), \mathcal{H}(r(s)))$, $s \mapsto \langle \pi_s \xi(s), \eta(r(s)) \rangle_{\mathcal{H}(r(s))}$ is continuous, and the map $s \mapsto \pi_s$ is a groupoid homomorphism from $G$ into the isomorphism groupoid of the fibered set $\bigcup_{u \in G^{(0)}} \mathcal{H}_u$, see [Muh90, Chapter 1]. Finally, recall that if $\phi : G \to \mathbb{C}$ is a continuous function then $\phi$ is positive definite if and only if $\phi$ is of the form $(\xi, \xi)$ for some $G$-Hilbert bundle. This result is proved in [Pat04, Theorem 1 p. 1264].

**Proposition 2.11** Let $G$ be a locally compact group. A continuous function $\varphi : G \to \mathbb{C}$ belongs to the Fourier-Stieltjes algebra $B(G)$ if and only if there exists continuous positive definite functions $\psi_1, \psi_2 : G \to \mathbb{C}$ such that the matrix

$$
\begin{bmatrix}
\psi_1 & \varphi \\
\varphi^* & \psi_2
\end{bmatrix}
$$

defines a continuous positive definite function on the measured groupoid $P_2 \times G$. In this case, we have

$$
\|\varphi\|_{B(G)} = \inf \|\psi_1\|_{L^\infty(G)} \|\psi_2\|_{L^\infty(G)},
$$

where the infimum is taken over all $\psi_1$ and $\psi_2$ satisfying the previous condition.

**Proof** : $\Rightarrow$: Let $\varepsilon > 0$. Using (1.1) and (1.2), we can write $\varphi = \langle \pi(\cdot)\xi_1, \xi_2 \rangle_H$ for some vectors $\xi_1$ and $\xi_2$ in a complex Hilbert space $H$ and some continuous unitary representation $\pi$ of $G$ on $H$ with $\|\xi_1\|_H \|\xi_2\|_H \leq \|\varphi\|_{B(G)} + \varepsilon$. For any $s \in G$, we have

$$
\tilde{\varphi}(s) = \langle \pi(s^{-1})\xi_1, \xi_2 \rangle_H = \langle \xi_2, \pi(s)\xi_1^* \rangle_H = \langle \pi(s)\xi_2, \xi_1 \rangle_H.
$$

Hence $\tilde{\varphi} = \langle \pi(\cdot)\xi_2, \xi_1 \rangle_H$. Now, we introduce the continuous positive definite functions $\psi_1 \overset{\text{def}}{=} \langle \pi(\cdot)\xi_1, \xi_2 \rangle_H$ and $\psi_2 \overset{\text{def}}{=} \langle \pi(\cdot)\xi_2, \xi_1 \rangle_H$ on the group $G$. Now, we consider the Hilbert $(P_2 \times G)$-bundle $\mathcal{H}$ over the discrete space $(P_2 \times G)^{\text{discrete}} = \{1, 2\}$ defined by $\mathcal{H}_1 \overset{\text{def}}{=} H$ and $\mathcal{H}_2 \overset{\text{def}}{=} H$ and $\pi_{i,j,s} \overset{\text{def}}{=} \pi(s) : \mathcal{H}_j \to \mathcal{H}_i$ for any $i, j \in \{1, 2\}$ and any $s \in G$. For any $i \in \{1, 2\}$, we introduce the vector $\zeta(i) \overset{\text{def}}{=} \xi_i$. This defines a section $\zeta$ of the bundle $\mathcal{H}$. For any $i, j \in \{1, 2\}$ and any $s \in G$, we obtain

$$
(\zeta, \zeta)(i, j, s) = \langle \pi_{i,j,s}(\xi(d(i, j, s)), \zeta(r(i, j, s))) \rangle_{\mathcal{H}_i} = \langle \pi_{i,j,s}(\zeta(j), \zeta(i)) \rangle_{\mathcal{H}_i} = \langle \pi(s)\xi(j), \xi(i) \rangle_{\mathcal{H}_i}.
$$

We deduce that

$$
\begin{bmatrix}
\psi_1 & \varphi \\
\varphi^* & \psi_2
\end{bmatrix} = \begin{bmatrix}
\langle \pi(\cdot)\xi_1, \xi_1 \rangle_H & \langle \pi(\cdot)\xi_1, \xi_2 \rangle_H \\
\langle \pi(\cdot)\xi_2, \xi_1 \rangle_H & \langle \pi(\cdot)\xi_2, \xi_2 \rangle_H
\end{bmatrix} \overset{\text{by (2.15)}}{=} (\zeta, \zeta).
$$

8. The operator $L'(\gamma)$ of [Ren97, Proposition 1.3 p. 459] is not a unitary.
Consequently by [Pat04, Theorem 1 p. 1264], the continuous function $\frac{\psi_1}{\varphi} \frac{\varphi}{\psi_2}$ is positive definite on the measured groupoid $P_2 \times G$. Moreover, we have

$$\|\psi_1\|_{L^\infty(G)} \|\psi_2\|_{L^\infty(G)} |(2.8) = \psi_1(e)\frac{\psi_2(e)}{\varphi} = (\pi(e)\xi_1, \xi_1)_H(\pi(e)\xi_2, \xi_2)_H$$

$$= \|\xi_1\|_H \|\xi_2\|_H \leq \|\varphi\|_{B(G)} + \varepsilon.$$  

Since $\varepsilon > 0$ is arbitrary, we conclude that $\inf \|\psi_1\|_{L^\infty(G)} \|\psi_2\|_{L^\infty(G)} \leq \|\varphi\|_{B(G)}$.

$\Leftarrow$: Suppose that there exists some strong positive definite functions $\psi_1, \psi_2: G \to \mathbb{C}$ such that the matrix $F \overset{\text{def}}{=} \begin{bmatrix} \psi_1 & \varphi \\ \varphi & \psi_2 \end{bmatrix}$ defines a strong positive definite function on the measured groupoid $P_2 \times G$. By [Pat04, Theorem 1 p. 1264], there exists a Hilbert $(P_2 \times G)$-bundle $H$ over the discrete space $(P_2 \times G)^{(0)} = \{1, 2\}$, with groupoid homomorphism $(i, j, s) \mapsto \pi(i, j, s)$, and a section $\zeta: \{1, 2\} \to H$ such that $\begin{bmatrix} \psi_1 & \varphi \\ \varphi & \psi_2 \end{bmatrix} = (\zeta, \zeta)$. So we have two complex Hilbert spaces $H_1$ and $H_2$. Here $\pi(i, j, s): H_j \to H_i$ is a unitary operator. Note that $\pi(1, 1, e) = \text{Id}_{H_1}$ and $\pi(2, 2, e) = \text{Id}_{H_2}$. We consider the operator

$$P \overset{\text{def}}{=} \frac{1}{2} \begin{bmatrix} \text{Id}_{H_1} & \pi(1, 2, e) \\ \pi(2, 1, e) & \text{Id}_{H_2} \end{bmatrix}$$

acting on the Hilbert space $H_1 \oplus H_2$. It is easy to see that $P$ is a selfadjoint projection and it commutes with each operator $\frac{1}{2} \begin{bmatrix} \pi(i, 1, s) & \pi(i, 2, e) \\ \pi(2, 1, e) & \pi(2, 2, e) \end{bmatrix}$. We introduce the complex Hilbert space $H \overset{\text{def}}{=} P(H_1 \oplus H_2)$. Observe that an element $(x, y) \in H_1 \oplus H_2$ belongs to the subspace $H$ if and only if $\pi(1, 2, e)(y) = x$ and $\pi(2, 1, e)(x) = y$. Consequently, we can consider the operator

$$\tilde{\pi}_s \overset{\text{def}}{=} \frac{1}{2} \begin{bmatrix} \pi(1, 1, s) & \pi(1, 2, e) \\ \pi(2, 1, e) & \pi(2, 2, e) \end{bmatrix} |_{H}, \quad s \in G.$$  

It is easy to check that we have a continuous unitary representation of $G$ on the Hilbert space $H$. Indeed, for any $(x, y) \in H$, we have

$$\|\tilde{\pi}_s(x, y)\|_H^2 = \frac{1}{4} \left( \|\pi(1, 1, s)(x) + \pi(1, 2, e)(y)\|_{H_1}^2 + \|\pi(2, 1, e)(x) + \pi(2, 2, e)(y)\|_{H_2}^2 \right).$$

$$= \frac{1}{4} \left( \|\pi(1, 1, s)(x + \pi(1, 2, e)(y))\|_{H_1}^2 + \|\pi(2, 1, e)(\pi(2, 1, e)(x) + y)\|_{H_2}^2 \right).$$

$$= \frac{1}{4} \left( \|x + \pi(1, 2, e)(y)\|_{H_1}^2 + \|\pi(2, 1, e)(x) + y\|_{H_2}^2 \right) = \frac{1}{4} \left( \|2x\|_{H_1}^2 + \|2y\|_{H_2}^2 \right).$$

Thus we also have

$$\left( \frac{1}{2} \begin{bmatrix} \pi(1, 1, s) & \pi(1, 2, e) \\ \pi(2, 1, e) & \pi(2, 2, e) \end{bmatrix} \right) \left( \frac{1}{2} \begin{bmatrix} \pi(1, 1, s) & \pi(1, 2, e) \\ \pi(2, 1, e) & \pi(2, 2, e) \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} \pi(1, 1, s) & \pi(1, 2, e) \\ \pi(2, 1, e) & \pi(2, 2, e) \end{bmatrix} = P = \text{Id}_H.$$
We consider the vectors \( \xi \overset{\text{def}}{=} \sqrt{2}P(0, \zeta(2)) \) and \( \eta \overset{\text{def}}{=} \sqrt{2}P(\zeta(1), 0) \) in the space \( H \). Now, for any \( s \in G \) we observe that

\[
\varphi(s) = (\zeta, \zeta)(1, 2, s) = \langle \pi(1, 2, s) \zeta(d(1, 2, s)), \zeta(r(1, 2, s)) \rangle
= \langle \pi(1, 2, s) \zeta(2), \zeta(1) \rangle
= \frac{1}{2} \begin{pmatrix}
1 & \text{Id}_{H_1} & \pi(1, 2, e) \\
\pi(2, 1, e) & \text{Id}_{H_2} & \pi(2, 1, s)
\end{pmatrix}
\begin{pmatrix}
\pi(1, 1, s) \\
\pi(2, 1, s)
\end{pmatrix}
\begin{pmatrix}
0, \zeta(2) \\
(\zeta(1), 0)
\end{pmatrix}
= 2 \langle \tilde{\pi}_s P(0, \zeta(2)), P(\zeta(1), 0) \rangle_H = \langle \tilde{\pi}_s \sqrt{2}P(0, \zeta(2)), \sqrt{2}P(\zeta(1), 0) \rangle_H = \langle \tilde{\pi}_s \xi, \eta \rangle_H.
\]

Hence the function \( \varphi \) belongs to the Fourier-Stieltjes algebra \( B(G) \). Furthermore, if \( i \in \{1, 2\} \) we have

\[
\|\varphi\|_{B(G)} \overset{(1.2)}{=} \|\varphi\|_H = \|\xi\|_{\mathcal{H}_i} = \|\zeta(1)\|_{\mathcal{H}_i}.
\]

Next, observe that

\[
\|\xi\|_{H}^2 = \sqrt{2} \|P(0, \zeta(2))\|_{H}^2 = 2 \left( \frac{1}{4} \|\xi_{1,2,e}(\zeta(2))\|_{\mathcal{H}_1}^2 + \frac{1}{4} \|\zeta(2)\|_{\mathcal{H}_2}^2 \right)
= \frac{1}{2} \left( \|\zeta(2)\|_{\mathcal{H}_2}^2 + \|\zeta(2)\|_{\mathcal{H}_2}^2 \right) = \|\zeta(2)\|_{\mathcal{H}_2}^2.
\]

In the same way, we have \( \|\eta\|_{H}^2 = \|\zeta(1)\|_{\mathcal{H}_1}^2 \). Moreover, we have

\[
\|\varphi\|_{B(G)} \overset{(1.2)}{=} \|\xi\|_{H} \|\eta\|_{H} = \|\zeta(2)\|_{\mathcal{H}_2} \|\zeta(1)\|_{\mathcal{H}_1} \overset{(2.17)}{=} \|\psi_1\|_{L_{\infty}(G)} \|\psi_2\|_{L_{\infty}(G)}.
\]

\[\square\]

2.2 Fourier-Stieltjes algebras and decomposable multipliers on \( \text{VN}(G) \)

Let \( G \) be a locally compact group. Recall that by [DCH85, Corollary 1.8 (i) p. 465] or [Kal18, Corollary 5.4.1 p. 185] we have a contractive inclusion \( B(G) \subset \mathcal{M}^{\text{dec}}(G) \). This is even a complete contraction by [Sprt04, Corollary 4.3 p. 179], where we equip the Fourier-Stieltjes algebra \( B(G) \) with the dual operator space structure induced by the equality \( C^*(G)^* = B(G) \).

In the next result, we strengthen this result by replacing the space \( \mathcal{M}^{\text{dec}}(G) \) of completely bounded Fourier multipliers on the von Neumann algebra \( \text{VN}(G) \) by the space \( \mathcal{M}^{\text{dec}}(\text{Dec}(\text{VN}(G))) \) of decomposable Fourier multipliers on \( \text{VN}(G) \). While the inclusion \( B(G) \subset \mathcal{M}^{\text{dec}}(G) \) is straightforward\(^{10}\), the contractivity of the inclusion \( B(G) \subset \mathcal{M}^{\text{dec}}(\text{Dec}(\text{VN}(G))) \) is new, even in the case where \( G \) is discrete. Here, we equip the space \( \mathcal{M}^{\text{dec}}(G) \) with the operator space structure induced by the one of the operator space \( \text{Dec}(\text{VN}(G)) \). For the proof, we will use the notion of a quasi-complete locally convex space. Recall that a locally convex space \( X \) is called quasi-complete if every bounded Cauchy net in \( X \) converges [Osb14, Definition 4.23 p. 107].

\(^{10}\) Indeed, if \( \varphi \in B(G) \) then we can write \( \varphi = \varphi_1 - \varphi_2 + i\varphi_3 - \varphi_4 \), where each \( \varphi_i \) is a continuous positive definite function. By [Kal18, Proposition 5.4.9 p. 184], each Fourier multiplier \( M_{\varphi_i} : \text{VN}(G) \rightarrow \text{VN}(G) \) is completely positive. Then it is immediate that the Fourier multiplier

\[
M_{\varphi} = M_{\varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)} = M_{\varphi_1} - M_{\varphi_2} + i(M_{\varphi_3} - M_{\varphi_4})
\]

is decomposable.
Proposition 2.12 Let $G$ be a locally compact group. The map $B(G) \to \mathfrak{M}_{\infty,dec}(G)$, $\varphi \mapsto M_{\varphi}$ is a well-defined injective complete contraction from the Fourier-Stieltjes algebra $B(G)$ into the space $\mathfrak{M}_{\infty,dec}(G)$ of decomposable Fourier multipliers.

Proof: We will present two distinct proofs. First proof: We begin with a purely group-theoretic argument. Let $\varphi \in B(G)$. By homogeneity, we can suppose that $\|\varphi\|_{B(G)} = 1$. We will use the associated linear form $\omega_{\varphi} : C^*(G) \to \mathbb{C}$, $\int_G f(s)U_s d\mu_G(s) \mapsto \int_G \varphi(s)f(s) d\mu_G(s)$ defined in (2.7). By [BlM04, Lemma A.2.2 p. 360], we can consider the unique weak* continuous extension $\tilde{\omega}_{\varphi} : W^*(G) \to \mathbb{C}$ on the von Neumann algebra $W^*(G)$, where we use here (and only in this step) the identification $W^*(G) = C^*(G)^{**}$ of [Dix77, p. 265]. We will prove that for any $s \in G$ the element $U_s$ belongs to the von Neumann algebra $W^*(G)$ and the equality

$$\tilde{\omega}_{\varphi}(U(s)) = \varphi(s), \quad s \in G. \quad (2.18)$$

Let $s \in G$ and let $\mathfrak{B}$ be a neighbourhood basis at $s$ constituted of compact neighbourhoods. For any $V \in \mathfrak{B}$, consider a positive continuous function $f_V : G \to \mathbb{R}^+$ on $G$ such that $\int_G f_V d\mu_G = 1$ with support contained in $V$. Then by [Bou04b, Corollary 3, VIII.17] the net $(\int_G f_V(t)U_t d\mu_G(t))$ converges to $U_s$ in the strong operator topology, and therefore also in the weak operator topology. Moreover, by [Bou04b, VIII.15] for any $V$ we have the estimate

$$\left\| \int_G f_V(t)U_t d\mu_G(t) \right\| \leq \int_G f_V d\mu_G = 1.$$

So the net $(\int_G f_V(t)U_t d\mu_G(t))_V$ is bounded. We deduce that the net $(\int_G f_V(t)U_t d\mu_G(t))_V$ converges to $U_s$ in the weak* topology by [Tak02, Lemma 2.5 p. 69]. In particular, we deduce that $U_s$ belongs to the von Neumann algebra $W^*(G)$. Moreover, on the one hand, we infer by weak* continuity of the linear form $\tilde{\omega}_{\varphi}$ that the net $(\tilde{\omega}_{\varphi}(\int_G f_V(t)U_t d\mu_G(t))_V)$ converges to $\tilde{\omega}_{\varphi}(U_s)$. On the other hand, using the continuity of the function $\varphi : G \to \mathbb{C}$ and [Bou04b, Corollary 2, VIII.17], in the limit process, we obtain

$$\tilde{\omega}_{\varphi}\left(\int_G f_V(t)U_t d\mu_G(t)\right) = \omega_{\varphi}\left(\int_G f_V(t)U_t d\mu_G(t)\right) \overset{(2.7)}{=} \int_G \varphi(t)f_V(t) d\mu_G(t) \rightarrow \varphi(s).$$

By uniqueness of the limit, we conclude that (2.18) is true. By Fell’s absorption principle (2.9) applied to the representation $U : G \to \mathcal{B}(H)$ instead of $\pi$, there exists a unitary $W : L^2(G, H) \to L^2(G, H)$ such that for any $s \in G$

$$W(\lambda_s \otimes \text{Id}_H)W^* = \lambda_s \otimes U_s.$$

By [Dix81, p. 9 and p. 25], we deduce that there exists a normal unital *-homomorphism $\Delta : \text{VN}(G) \to \text{VN}(G)\mathcal{B}W^*(G)$, $\lambda_s \mapsto \lambda_s \otimes U_s$. Since any Banach space is barreled [Osb14, Theorem 4.5 p. 97], we see by [Osb14, Corollary 4.25 (b) p. 107] that the weak* topology on the dual Banach space $\text{VN}(G)$ is quasi-complete. Consequently, by [Bou04a, Corollary 2, III p. 38], for any function $f \in C_c(G)$, the integral $\int_G f(s)\lambda_s d\mu_G(s)$ is a well-defined weak* integral. Using the weak* continuity of $\Delta$ together with [Bou04a, Proposition 1, VI.3] in the first equality, we deduce that

$$\Delta\left(\int_G f(s)\lambda_s d\mu_G(s)\right) = \int_G f(s)\Delta(\lambda_s) d\mu_G(s) = \int_G f(s)(\lambda_s \otimes U_s) d\mu_G(s). \quad (2.19)$$


Now, for any $s \in G$, we obtain again with [Bou04b, Proposition 1, VI.3] that

$$(\text{Id} \otimes \tilde{\omega}_\varphi) \circ \Delta \left( \int_G f(s) \lambda_s \, d\mu_G(s) \right) \overset{(2.19)}{=} (\text{Id} \otimes \tilde{\omega}_\varphi) \left( \int_G f(s) (\lambda_s \otimes U_s) \, d\mu_G(s) \right)
$$

and

$$(\int_G f(s)(\text{Id} \otimes \tilde{\omega}_\varphi)(\lambda_s \otimes U_s) \, d\mu_G(s)) \overset{(2.18)}{=} \int_G \varphi(f(s) \lambda_s) \, d\mu_G(s).$$

We conclude that the weak* continuous map $(\text{Id} \otimes \tilde{\omega}_\varphi) \circ \Delta$ is the Fourier multiplier $M_\varphi$ of symbol $\varphi$.

Note that the $*$-homomorphism $\Delta$ is decomposable since it is completely positive. According to [Efr00, Lemma 5.4.3 p. 96], the linear form $\tilde{\omega}_\varphi$ is equally decomposable with

$$||\tilde{\omega}_\varphi||_{\text{dec}} \overset{(2.3)}{=} ||\tilde{\omega}_\varphi||_{\text{cb}} = ||\tilde{\omega}_\varphi|| = 1,$$

where we use [Efr00, Corollary 2.2.3 p. 24] in the second equality. By Lemma 2.1, we deduce that we have a well-defined weak* continuous decomposable map $\text{Id} \otimes \tilde{\omega}_\varphi : \text{VN}(G) \overline{\otimes} \text{W}^*(G) \to \text{VN}(G)$. We conclude by composition that the linear map $M_\varphi = (\text{Id} \otimes \tilde{\omega}_\varphi) \circ \Delta : \text{VN}(G) \to \text{VN}(G)$ is decomposable and that

$$||M_\varphi||_{\text{dec}} = ||(\text{Id} \otimes \tilde{\omega}_\varphi) \circ \Delta||_{\text{dec}} \overset{(2.1)}{\leq} ||(\text{Id} \otimes \tilde{\omega}_\varphi)||_{\text{dec}} ||\Delta||_{\text{dec}} \overset{(2.4)}{\leq} ||\tilde{\omega}_\varphi||_{\text{dec}} ||\Delta||_{\text{dec}} \overset{(2.20)}{\leq} ||\Delta||_{\text{dec}} \overset{(2.2)}{\leq} 1.$$

Finally, it is easy to check that the map $B(G) \to \mathfrak{M}^{\infty, \text{dec}}(G)$, $\varphi \mapsto M_\varphi$ is also injective. For the complete contractivity, the argument is similar. Consider a matrix $[\varphi_{ij}] \in M_n(B(G))$.

We have a completely bounded map $[\varphi_{ij}] : C^*(G) \to M_n$. Using [BIM04, 1.4.8 p. 24], its unique weak* continuous extension $[\tilde{\omega}_{\varphi_{ij}}] : \text{W}^*(G) \to M_n$ is completely bounded with the same completely bounded norm. Note that this linear map is decomposable and its decomposable norm coincides with its completely bounded norm by (2.3). Finally, we can write $[M_{\varphi_{ij}}] = (\text{Id} \otimes [\tilde{\omega}_{\varphi_{ij}}]) \circ (\text{Id}_{M_n} \otimes \Delta)$.

**Second proof of the contractivity.** Now, we give a second proof using groupoids. Let $\varphi \in B(G)$ and $\varepsilon > 0$. By Proposition 2.11, there exists continuous positive definite functions $\psi_1$ and $\psi_2$ (hence bounded by [BHV08, Proposition C.4.2 p. 351]) such that the matrix $\begin{bmatrix} \psi_1 & \varphi \\ \varphi^\ast & \psi_2 \end{bmatrix}$ defines a continuous positive definite function $F$ on the groupoid $P_2 \times G$ with

$$\|\psi_1\|_{L^\infty(G)} \|\psi_2\|_{L^\infty(G)} \leq \|\varphi\|_{B(G)} + \varepsilon.$$  

Note the identification $\text{VN}(P_2 \times G) = \text{VN}(P_2) \overline{\otimes} \text{VN}(G) = M_2 \overline{\otimes} \text{VN}(G) = M_2(\text{VN}(G))$. By generalizing the very transparent argument of [BrO08, Proposition 5.6.16 p. 206] with [Arb24, Proposition 4.12 and Remark 4.12], we see that $F$ induces a completely positive multiplier on the von Neumann algebra $\text{VN}(P_2 \times G)$. This completely positive multiplier identifies to the map $\begin{bmatrix} M_{\psi_1} & M_\varphi \\ M_\varphi^\ast & M_{\psi_2} \end{bmatrix} = \begin{bmatrix} M_{\psi_1} & M_\varphi \\ M_\varphi^\ast & M_{\psi_2} \end{bmatrix} : M_2(\text{VN}(G)) \to M_2(\text{VN}(G))$. Note that the Fourier multipliers $M_{\psi_1}$ and $M_{\psi_2}$ are completely positive. We conclude that the Fourier multiplier $M_\varphi : \text{VN}(G) \to \text{VN}(G)$ is decomposable with

$$\|M_\varphi\|_{\text{dec}, \text{VN}(G) \to \text{VN}(G)} \overset{(2.6)}{\leq} \|M_{\psi_1}\|_{\text{VN}(G) \to \text{VN}(G)} \|M_{\psi_2}\|_{\text{VN}(G) \to \text{VN}(G)} \overset{(2.21)}{\leq} \|\psi_1\|_{L^\infty(G)} \|\psi_2\|_{L^\infty(G)} \|\varphi\|_{B(G)} + \varepsilon.$$
Now, we study the converse of Proposition 2.12 in Proposition 2.15 and in Theorem 2.18. We need the following result, which gives a description of the norm of the Fourier-Stieltjes algebra $B(G)$ for some suitable functions. Here, we denote by $P(G)$ the set of continuous positive definite functions on $G$, following [KaL18, Definition 1.4.18 p. 23].

**Proposition 2.13** Let $G$ be a locally compact group. Let $\varphi \in B(G)$ such that $\hat{\varphi} = \overline{\varphi}$. We have

\[ \|\varphi\|_{B(G)} = \inf \{ \varphi_1(e) + \varphi_2(e) : \varphi = \varphi_1 - \varphi_2, \varphi_1, \varphi_2 \in P(G) \}. \tag{2.22} \]

**Proof**: Suppose that $\varphi = \varphi_1 - \varphi_2$ for some continuous positive definite functions $\varphi_1, \varphi_2 : G \to \mathbb{C}$. We have

\[ \|\varphi\|_{B(G)} = \|\varphi_1 - \varphi_2\|_{B(G)} \leq\|\varphi_1\|_{B(G)} + \|\varphi_2\|_{B(G)} \overset{(2.8)}{=} \varphi_1(e) + \varphi_2(e). \]

Passing to the infimum, we obtain that $\|\varphi\|_{B(G)} \leq \inf \{ \varphi_1(e) + \varphi_2(e) : \varphi = \varphi_1 - \varphi_2, \varphi_1, \varphi_2 \in P(G) \}$. Indeed, by [Eym64, (2.7) p. 193] (or [KaL18, p. 41]) we have an equality in this last inequality and the infimum is a minimum.

**Remark 2.14** Suppose that the locally compact $G$ is abelian. For any real bounded regular Borel measure $\mu$ on the dual group $\hat{G}$, the previous result combined with Example 2.5 and (2.8) implies that

\[ \|\mu\|_{M(G)} = \inf \{ \|\mu_1\|_{M(G)} + \|\mu_2\|_{M(G)} : \mu = \mu_1 - \mu_2, \mu_1, \mu_2 \geq 0 \}. \tag{2.23} \]

We can replace the group $\hat{G}$ by a locally compact space $X$. Indeed, for any real bounded regular Borel measure $\mu$, we can decompose the measure $\mu$ with [Bou04a, III §1. 8 Corollary 2] as $\mu = \mu^+ - \mu^-$ for some bounded positive regular Borel measures $\mu^+$ and $\mu^-$ on $X$ with $\|\mu\|_{M(X)} = \|\mu^+\|_{M(X)} + \|\mu^-\|_{M(X)}$ and we can use a similar reasoning.

By adding property ($\kappa_\infty$) to the group $G$, we obtain a partial converse to Proposition 2.12. For the proof, we will use the folklore fact that says that the symbol of any bounded multiplier on the von Neumann algebra $VN(G)$ of a locally compact group $G$ is almost everywhere equal to a continuous function. This follows from the «regularity» of the Fourier algebra, established in [KaL18, Theorem 2.3.8 p. 53].

**Proposition 2.15** Let $G$ be a locally compact group. If $G$ has property ($\kappa_\infty$), then the linear map $B(G) \to \mathfrak{M}_{\infty, \text{dec}}(G)$, $\varphi \mapsto M_\varphi$ is a bijection from the Fourier-Stieltjes algebra $B(G)$ onto the space $\mathfrak{M}_{\infty, \text{dec}}(G)$ of decomposable multipliers. Moreover, if $\kappa_\infty(G) = 1$ and if the function $\varphi$ belongs to $B(G)$ and satisfies $\hat{\varphi} = \overline{\varphi}$, we have $\|\varphi\|_{B(G)} = \|M_\varphi\|_{\text{dec}, VN(G) \to VN(G)}$.

**Proof**: In Proposition 2.12, we established a (completely) contractive inclusion $B(G) \subset \mathfrak{M}_{\infty, \text{dec}}(G)$. We show the reverse inclusion. Suppose that $M_\varphi : VN(G) \to VN(G)$ is a decomposable Fourier multiplier (hence weak* continuous) with continuous symbol $\varphi : G \to \mathbb{C}$. We can write

\[ M_\varphi = T_1 + T_2 + i(T_3 - T_4) \tag{2.24} \]

for some completely positive maps $T_1, T_2, T_3, T_4 : VN(G) \to VN(G)$. By using the contractive projection $P_\varphi : B(VN(G)) \to B(VN(G))$ of [ArK23, Proposition 3.1 p. 24], which preserves the

11. Note that in this reference, the assumption “$\hat{\varphi} = \overline{\varphi}$” is missing.
complete positivity, as in the proof of [ArK23, Proposition 3.4 p. 26], we can suppose that these maps \( T_1, T_2, T_3, T_4 \) are weak* continuous since \( P_{w^*}(M_\varphi) = M_\varphi \). Using the bounded projection \( P_G^\infty : CB_{w^*}^{}(VN(G)) \to CB_{w^*}^{}(VN(G)) \) provided by property \((\kappa_\infty)\), we obtain

\[
M_\varphi = P_G^\infty (M_\varphi) = P_G^\infty (T_1 - T_2 + i(T_3 - T_4)) = P_G^\infty (T_1) - P_G^\infty (T_2) + i(P_G^\infty (T_3) - P_G^\infty (T_4)),
\]

where each \( P_G^\infty (T_k) : VN(G) \to VN(G) \) is a completely positive Fourier multiplier for some symbol \( \varphi_k : G \to \mathbb{C} \), i.e. \( P_G^\infty (T_k) = M_{\varphi_k} \). By [DCH85, Proposition 4.2 p. 487], the function \( \varphi_k \) is continuous and positive definite. We deduce that

\[
M_\varphi = M_{\varphi_1} - M_{\varphi_2} + i(M_{\varphi_3} - M_{\varphi_4}) = M_{\varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)}.
\]

We infer that \( \varphi = \varphi_1 - \varphi_2 + i\varphi_3 - \varphi_4 \). We conclude that the function \( \varphi \) belongs to the Fourier-Stieltjes algebra \( B(G) \). Hence we have an inclusion \( M_{\varphi} \subseteq B(G) \).

Now, we prove the second part of the statement assuming \( \kappa_\infty (G) = 1 \). Suppose that the function \( \varphi \) belongs to the Fourier-Stieltjes algebra \( B(G) \) and satisfies \( \widetilde{\varphi} = \overline{\varphi} \). This last condition means that the Fourier multiplier \( M_{\widetilde{\varphi}} : VN(G) \to VN(G) \) is adjoint preserving, i.e. \( M_{\widetilde{\varphi}} (x^*) = (M_{\varphi} (x))^* \) for any \( x \in VN(G) \). Let \( \varepsilon > 0 \). By [Haa85, Proposition 1.3 (1) p. 177] and [Haa85, p. 184], the equality \((2.25)\) gives \( M_{\varphi} \subseteq B(G) \). By [Haa85, Proposition 3.4 p. 26] and \((2.26)\) in the second equality, we see that

\[
\| \varphi \|_{B(G)} \leq \| \varphi_1 \|_{B(G)} + \| \varphi_2 \|_{B(G)} = (\| \varphi_1 \|_{B(G)} + \| \varphi_2 \|_{B(G)}) = \| M_{\varphi_1} + M_{\varphi_2} \|_{VN(G) \to VN(G)} = \| P_G^\infty (T_1) + P_G^\infty (T_2) \| = \| P_G^\infty (T_1 + T_2) \| \\
\leq (1 + \varepsilon) \| T_1 + T_2 \| \leq (1 + \varepsilon) \| M_\varphi \|_{VN(G) \to VN(G)}.
\]

Since \( \varepsilon > 0 \) is arbitrary, we deduce that \( \| \varphi \|_{B(G)} \leq \| M_\varphi \|_{VN(G) \to VN(G)} \). Combining with Proposition 2.12, we conclude that \( \| \varphi \|_{B(G)} = \| M_\varphi \|_{VN(G) \to VN(G)} \).

Now, we observe that the first inclusion in \((1.7)\) can be strict.

**Proposition 2.16** Let \( G \) be a non-amenable locally compact group such that the von Neumann algebra \( VN(G) \) is injective. Then there exists a decomposable Fourier multiplier \( T : VN(G) \to VN(G) \), which is not induced by an element \( \varphi \in B(G) \).

**Proof** : Since the von Neumann algebra \( VN(G) \) is injective, we have by [Haa85, Theorem 1.6 p. 184] the equality \( M_{\varphi} \subseteq B(G) \), so it is isometrically injective. Since the group \( G \) is not amenable, we know by an unpublished result of Ruan stated in [Pis01, p. 54] and [Spr04, p. 190] that \( B(G) \not\subseteq M_{\varphi} \). We conclude that \( B(G) \not\subseteq M_{\varphi} \).

\[ \square \]
Example 2.17 By [Con76, Corollary 7 p. 75], the von Neumann algebra $VN(G)$ of a second-countable connected locally compact group $G$ is injective. This result applies for example to the locally compact group $G = \text{SL}_2(\mathbb{R})$, which is non-amenable by [BHvO8, Example G.2.4 (i) p. 426]. We conclude that $B(G) \not\subset \mathfrak{M}^\infty,\text{dec}(G)$ in this case.

For discrete groups, a matricial improvement of property $(\kappa_\infty)$ is available in [ArK23, Theorem 4.2 p. 62]. Consequently, we can establish the following isometric result.

Theorem 2.18 Let $G$ be a discrete group. The map $B(G) \to \mathfrak{M}^\infty,\text{dec}(G)$, $\varphi \mapsto M_\varphi$ is an isometric isomorphism from the Fourier-Steiltjes $B(G)$ onto the algebra $\mathfrak{M}^\infty,\text{dec}(G)$ of decomposable multipliers on the von Neumann algebra $VN(G)$.

Proof: In Proposition 2.12, we have seen that we have a contractive inclusion $B(G) \subset \mathfrak{M}^\infty,\text{dec}(G)$. It suffices to show the reverse inclusion. Suppose that the Fourier multiplier $M_\varphi : VN(G) \to VN(G)$ is decomposable. By [Haa85, Remark 1.5 p. 183], there exist some linear maps $v_1, v_2 : VN(G) \to VN(G)$ such that the linear map $egin{bmatrix} v_1 & M_\varphi & v_2 \ M_\varphi & v_2 \ v_1 & M_\varphi & v_2 \end{bmatrix} : M_2(VN(G)) \to M_2(VN(G))$ is completely positive with $\max\{ \|v_1\|, \|v_2\| \} = \|M_\varphi\|_{\text{dec},VN(G)} = \|\varphi\|_{\text{VN}(G)}$. We can suppose that the completely positive maps $v_1$ and $v_2$ are in addition weak* continuous by using [ArK23, Proposition 3.1 p. 24].

Now, we consider the projection $P_{\text{dec}(G)} : CB_w(M_2(VN(G))) \to CB_w(M_2(VN(G)))$, preserving the complete positivity and contractive, provided by [ArK23, Theorem 4.2 p. 62]. The proof shows that in case it is applied to an element of special structure as $\begin{bmatrix} v_1 & M_\varphi & v_2 \\ M_\varphi & v_2 \ v_1 & M_\varphi & v_2 \end{bmatrix}$, the mapping is $P_{\text{dec}(G)} = \begin{bmatrix} P_{\text{dec}(G)}^\infty & P_{\text{dec}(G)}^\infty \\ P_{\text{dec}(G)}^\infty & P_{\text{dec}(G)}^\infty \end{bmatrix}$, where $P_{\text{dec}(G)}^\infty : CB_w(VN(G)) \to CB_w(VN(G))$ is the contractive projection onto the space of completely bounded Fourier multipliers, provided by [ArK23, Theorem 4.2 p. 62]. We obtain that the map

$$
(2.27) \quad \begin{bmatrix} P_{\text{dec}(G)}^\infty(v_1) & M_\varphi & P_{\text{dec}(G)}^\infty(v_2) \\ M_\varphi & v_2 \ P_{\text{dec}(G)}^\infty(v_2) \\ M_\varphi & v_2 \end{bmatrix} = \begin{bmatrix} P_{\text{dec}(G)}^\infty(v_1) & P_{\text{dec}(G)}^\infty(M_\varphi) & P_{\text{dec}(G)}^\infty(v_2) \\ P_{\text{dec}(G)}^\infty(M_\varphi) & P_{\text{dec}(G)}^\infty(v_2) \end{bmatrix} = P_{\text{dec}(G)}^{(1,2),G}(\begin{bmatrix} v_1 & M_\varphi & v_2 \\ M_\varphi & v_2 \ v_1 & M_\varphi & v_2 \end{bmatrix})
$$

is completely positive. Moreover, we have

$$
(2.28) \quad \max\{ \|P_{\text{dec}(G)}^\infty(v_1)\|, \|P_{\text{dec}(G)}^\infty(v_2)\| \} \leq \|P_{\text{dec}(G)}^\infty\| \max\{ \|v_1\|, \|v_2\| \} = \|M_\varphi\|_{\text{dec},VN(G)} = \|\varphi\|_{\text{VN}(G)}.
$$

We can write $P_{\text{dec}(G)}^\infty(v_1) = M_{\psi_1}$ and $P_{\text{dec}(G)}^\infty(v_2) = M_{\psi_2}$ for some continuous positive definite functions $\psi_1, \psi_2 : G \to C$. By [ArK23, Proposition 8.4 p. 166], the condition (2.12) is satisfied with

$$
\begin{bmatrix} \psi_1 & \varphi \\ \bar{\varphi} & \psi_2 \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \bar{\varphi} & \psi_2 \\ \psi_1 \end{bmatrix}
$$

By Lemma 2.10, we conclude that $F \overset{\text{def}}{=} \begin{bmatrix} \psi_1 & \varphi \\ \bar{\varphi} & \psi_2 \end{bmatrix}$ identifies to a continuous positive definite function on the groupoid $P_2 \times G$. According to Proposition 2.11, we obtain that the function $\varphi$ belongs to the Fourier-Steiltjes algebra $B(G)$. Moreover, using the well-known contractive inclusion $\mathfrak{M}^\infty(G) \subset L^\infty(G)$ of [KaL18, Proposition 5.1.2 p. 154] in the first inequality, we infer that

$$
\|\varphi\|_{B(G)} \overset{(2.14)}{\leq} \left\|\psi_1\right\|_{L^\infty(G)} \left\|\psi_2\right\|_{L^\infty(G)} \leq \max\{ \|M_{\psi_1}\|, \|M_{\psi_2}\| \}
$$

$$
= \max\{ \|P_{\text{dec}(G)}^\infty(v_1)\|, \|P_{\text{dec}(G)}^\infty(v_2)\| \} \overset{(2.28)}{\leq} \|M_\varphi\|_{\text{dec},VN(G)} = \|\varphi\|_{\text{VN}(G)}.
$$

\[\blacksquare\]
Finally, we prove the second part of Conjecture 1.4 in the discrete case. This result improves [ArK23, Proposition 3.32 (1) p. 51] which says that the second inclusion of (1.7) is strict for any non-amenable weakly amenable discrete group $G$. This result can be seen as a new characterization of amenability for discrete groups.

**Theorem 2.19** Let $G$ be a discrete group. The von Neumann algebra $VN(G)$ is injective if and only if we have $\mathfrak{M}^{\infty,\text{dec}}(G) = \mathfrak{M}^{\infty,\text{cb}}(G)$.

**Proof**: By Corollary 2.18, we have an isometric isomorphism $\mathfrak{M}^{\infty,\text{dec}}(G) = B(G)$. It suffices to use the result stated in [Pis01, p. 54], which says that $B(G) = \mathfrak{M}^{\infty,\text{cb}}(G)$ if and only if the group $G$ is amenable. For a discrete group $G$, the amenability is equivalent to the injectivity of the von Neumann algebra $VN(G)$ by [SiS08, Theorem 3.8.2 p. 51] (or Theorem 1.3). 

### 3 Inner amenability

#### 3.1 Preliminaries on inner amenability and amenability

We warn the reader that different notions of inner amenability coexist in the literature, see [Pat88a, p. 84] for more information. We say that a locally compact group $G$, equipped with a left Haar measure $\mu_G$, is inner amenable if there exists a state $m$ on the algebra $L^\infty(G)$ such that

\[
m(\text{int}_s f) = m(f)
\]

for any $s \in G$, where

\[
(\text{int}_s f)(t) \overset{\text{def}}{=} f(s^{-1}ts), \quad s, t \in G.
\]

It is worth noting that by [CrT17, Proposition 3.2 p. 2527], a locally compact group $G$ is inner amenable if and only if there exists a state $m$ on the group von Neumann algebra $VN(G)$ such that

\[
m(\lambda_s^* x \lambda_s) = m(x), \quad s \in G, x \in VN(G).
\]

Such a state is said to be $G$-invariant. According to [CrT17, Proposition 3.3 p. 2528], any closed subgroup $H$ of an inner amenable locally compact group $G$ is inner amenable. If in addition $H$ is normal then the group $G/H$ is also inner amenable by [LaP91, Proposition 6.2 p. 168].

**Example 3.1** Every amenable locally compact group $G$ is inner amenable. Indeed, by [Pie84, Theorem 4.19 p. 36] there exists a state $m$ on $L^\infty(G)$, which is two-sided invariant.

**Example 3.2** Following [Pal01, p. 1273], we say that a locally compact group $G$ is said to have an invariant neighborhood if there exists a compact neighbourhood $V$ of the identity $e$ in $G$ such that $V$ is stable under all inner automorphisms of $G$, i.e. $s^{-1}Vs = V$ for all $s \in G$. Such a group is said to be an IN-group. By [Pal01, Proposition 12.1.9 p. 1273], any IN-group $G$ is unimodular. Note that if $\mu_G$ is a Haar measure on an IN-group $G$, it is clear using [Bou04b, (31) and (33) VII.13] that the state $m: L^\infty(G) \to \mathbb{C}$, $f \mapsto \frac{1}{\mu_G(V)} \int_V f$ satisfies the equation (3.1). Hence an IN-group is inner amenable.

By [ArK23, Proposition 6.36 p. 119], a locally compact group $G$ is pro-discrete if and only if it admits a basis $(X_i)$ of neighborhoods of the identity $e$ consisting of open compact normal subgroups. Consequently, pro-discrete locally compact groups are IN-groups. Moreover, according to [Pal01, Proposition 12.1.9 p. 1273], compact groups, locally compact abelian groups and discrete groups groups are IN-groups. These groups are therefore all inner amenable. In particular, inner amenability is significantly weaker than amenability.
Example 3.3 Recall that a topological group $G$ is of type I [BeH20, Definition 6.D.1 p. 196 and Proposition 7.C.I p. 219] if for any continuous unitary representation $\pi$ of $G$, the von Neumann algebra $\pi(G)''$ is of type I, hence injective by [Str81, Proposition 10.23 p. 144]. In particular, by Theorem 1.3 a second-countable locally compact group $G$ of type I is inner amenable if and only if it is amenable. We refer to [BeH20, Theorem 6.E.19 p. 208 and Theorem 6.E.20 p. 209] for an extensive list of locally compact groups of type I, including connected nilpotent locally compact groups and linear algebraic groups over a local field of characteristic 0.

Example 3.4 If a locally compact group $G$ is almost connected, i.e. $G/G_e$ is compact if $G_e$ is the connected component of the identity $e$, then its von Neumann algebra $\text{VN}(G)$ is injective by [Pat88b, p. 228]. Again, by Theorem 1.3 such a group is inner amenable if and only if it is amenable. This result in the connected case was first proved by Losert and Rindler in [LoR87, Theorem 1 p. 222] and proven again in [LaP91, Corollary 3.4 p. 161].

If $A$ and $B$ are two subsets of a set $E$, the notation $A \Delta B \overset{\text{def}}{=} (A - B) \cup (B - A)$ denotes here the symmetric difference of $A$ and $B$. Recall that

\begin{equation}
|1_A - 1_B| = 1_{A \Delta B}.
\end{equation}

We will use the following reformulation of [CPPR15, Lemma 8.6 p. 43], which is actually and essentially a variant of a classical trick in amenability theory used in [ElW17, pp. 364-365], [BHV08, pp. 441-442] and [Fre13, p. 410]. We give the two lines of calculus for the sake of completeness.

Lemma 3.5 Let $G$ be a locally compact group equipped with a left Haar measure $\mu_G$. Let $\varepsilon > 0$ and consider some positive functions $f, g_1, \ldots, g_n$ in the space $L^1(G)$ satisfying the inequality

\[ \sum_{k=1}^n \|f - g_k\|_{L^1(G)} < \varepsilon \text{ and } \|f\|_{L^1(G)} = 1. \]

Then there exists $t > 0$ such that

\begin{equation}
\sum_{k=1}^n \mu_G(\{f > t\} \Delta \{g_k > t\}) < \varepsilon \mu_G(\{f > t\}).
\end{equation}

Proof : For any $s \in G$ and integer $1 \leq k \leq n$, we have by [BHV08, Lemma G.5.2 p. 441] and (3.3) the equalities

\begin{equation}
\|f\|_{L^1(G)} = \int_0^\infty \mu_G(\{f > t\}) \, dt \text{ and } \|f - g_k\|_{L^1(G)} = \int_0^\infty \mu_G(\{f > t\} \Delta \{g_k > t\}) \, dt.
\end{equation}

We deduce that

\[ \int_0^\infty \sum_{k=1}^n \mu_G(\{f > t\} \Delta \{g_k > t\}) \, dt = \sum_{k=1}^n \int_0^\infty \mu_G(\{f > t\} \Delta \{g_k > t\}) \, dt \]

\[ \overset{(3.5)}{=} \sum_{k=1}^n \|f - g_k\|_{L^1(G)} < \varepsilon \|f\|_{L^1(G)} \overset{(3.5)}{=} \varepsilon \int_0^\infty \mu_G(\{f > t\}) \, dt. \]

The conclusion is obvious.

Convolution If $G$ is a unimodular locally compact group equipped with a Haar measure $\mu_G$, recall that the convolution product of two functions $f$ and $g$ is given, when it exists, by

\begin{equation}
(f \ast g)(s) \overset{\text{def}}{=} \int_G f(r)g(r^{-1}s) \, d\mu_G(r) = \int_G f(sr^{-1})g(r) \, d\mu_G(r).
\end{equation}
3.2 Some characterizations of inner amenability

Now, we introduce the following definition which is an «inner variant» of the well-known definition of the notion of «Følner net» in amenability theory.

**Definition 3.6** A locally compact group \( G \) is said to be inner Følner (in short \( G \in (IF) \)) if for every finite subset \( F \) of \( G \) there exists a net \( (V_j^F) \) of measurable subsets of \( G \) such that \( \mu(V_j^F) \in (0, \infty) \), with the property that for all \( s \in F \),

\[
\frac{\mu(V_j^F \Delta s^{-1}V_j^F \cdot s)}{\mu(V_j^F)} \xrightarrow{j \to \infty} 0.
\]

Now, we give different characterizations of inner amenability for unimodular locally compact groups. The equivalence between the first and the second point is sketched in [LoR87, Proposition 1 p. 222]. For the sake of completeness, we give a complete proof.

**Theorem 3.7** Let \( G \) be a unimodular locally compact group. The following are equivalent.

1. \( G \) is inner amenable.

2. There exists an asymptotically central net \( (f_j) \) of functions in the space \( L^1(G) \), i.e. for any \( s \in G \), we have

\[
\frac{\|f_j - \text{int}_s f\|_{L^1(G)}}{\|f_j\|_{L^1(G)}} \xrightarrow{j \to \infty} 0.
\]

3. \( G \) is inner Følner.

4. There exists a net \( (f_j) \) of positive functions in the space \( L^1(G) \) with \( f_G f_j \, d\mu = 1 \) such that for all \( s \in G \), we have \( \|f_j - \text{int}_s f\|_{L^1(G)} \xrightarrow{j \to \infty} 0 \).

5. The same property as before, but the \( f_j \)'s belong in addition to the space \( C_c(G) \) and are positive definite.

6. \( G \) is inner Følner and in addition the sets \( V_j^F \) can be chosen to be symmetric, open and containing \( e \).

Finally, the net \( (V_j^F) \) in the previous definition of inner Følner can be chosen to be a sequence.

**Proof** : 1. \( \implies \) 4.: Let \( m \in L^\infty(G)^* \) be an inner invariant mean. By [Pie84, Proposition 3.3 p. 25] (see also [EiW17, Lemma 10.16 p. 366]), we can approximate \( m \) in the weak* topology by a net \( (f_j) \) of functions in \( L^1(G) \) with \( f_j \geq 0 \) and \( \|f_j\|_{L^1(G)} = 1 \). For any \( s \in G \) and any \( g \in L^\infty(G) \), we have

\[
\langle \text{int}_s f_j, g \rangle_{L^\infty(G),L^\infty(G)} = \langle f_j, \text{int}_{-s^{-1}} g \rangle_{L^\infty(G),L^\infty(G)} \xrightarrow{j \to \infty} \langle m, \text{int}_{-s^{-1}} g \rangle = \langle m, g \rangle.
\]

and \( \langle f_j, g \rangle_{L^1(G),L^\infty(G)} \to \langle m, g \rangle_{L^\infty(G)^*,L^\infty(G)} \). With an \( \epsilon \)-argument, it follows that for any \( s \in G \) we have \( w - \lim_j (\text{int}_s f_j - f_j) = 0 \).

Since for convex sets the weak closure coincides with the norm closure [Meg98, Theorem 2.5.16 p. 216], we can replace \( f_j \) by some convex combinations to get \( \lim_j \|\text{int}_s f_j - f_j\|_{L^1(G)} = 0 \). This replacement can be seen in the following way. Let \( F \overset{\text{def}}{=} \{s_1, \ldots, s_n\} \) be a finite set
of $G$. According to the above, $(0, \ldots, 0)$ belongs to the weak-closure of the convex hull of \{$(\text{int}_{s_1}(f_j)) - f_j, \text{int}_{s_2}(f_j) - f_j, \ldots, \text{int}_{s_n}(f_j) - f_j$\}, hence to the $L^1(G)^n$ norm closure of this convex hull. Thus there exists a sequence $(g_k)_{k \in \mathbb{N}}$ in this convex hull converging to $(0, \ldots, 0)$ in norm. For any $k \in \mathbb{N}$, we can write

$$g_k = \sum_{\ell=1}^{L} \lambda_\ell (\text{int}_{s_1}(f_{j_\ell}) - f_{j_\ell}, \ldots, \text{int}_{s_n}(f_{j_\ell}) - f_{j_\ell})$$

$$= \left(\text{int}_{s_1}\left(\sum_{\ell=1}^{L} \lambda_\ell f_{j_\ell}\right), \ldots, \text{int}_{s_n}\left(\sum_{\ell=1}^{L} \lambda_\ell f_{j_\ell}\right)\right)$$

$$= (\text{int}_{s_1}(h_k) - h_k, \ldots, \text{int}_{s_n}(h_k) - h_k),$$

where $\lambda_\ell \geq 0$, $\sum_{\ell=1}^{L} \lambda_\ell = 1$ and where $h_k \overset{\text{def}}{=} \sum_{\ell=1}^{L} \lambda_\ell f_{j_\ell}$ still is a positive normalized element in $L^1(G)$. We can suppose that $\|\text{int}_s(h_k) - h_k\|_1 \leq \frac{1}{s}$ for any $s \in F$. Now write $h_k = h_{k,F}$, let $F$ vary in the set of finite subsets of $G$ directed by inclusion, so that $(h_{k,F})_{k,F}$ becomes a net in $L^1(G)$ such that $\|\text{int}_s(h_{k,F}) - h_{k,F}\|_1 \to 0$ as $(k,F) \to \infty$ for any $s \in G$.

4. $\implies$ 1.: Note that we have an isometric inclusion $L^1(G) \subset L^\infty(G)^*$. Consider a cluster point $m \in L^\infty(G)^*$ of this net for the weak* topology which is positive and satisfies clearly $m(1) = 1$. For any $f \in L^\infty(G)$ and any $s \in G$, we have

$$\left|\langle f_j, f \rangle_{L^1(G), L^\infty(G)} - \langle f_j, \text{int}_s f \rangle_{L^1(G), L^\infty(G)}\right| = \left|\langle f_j, f \rangle - \langle \text{int}_{s^{-1}} f_j, f \rangle\right| = \left|\langle f_j - \text{int}_{s^{-1}} f_j, f \rangle\right|$$

$$\leq \|f_j - \text{int}_{s^{-1}} f_j\|_{L^1(G)} \|f\|_{L^\infty(G)} \to 0.$$
Then for any $j$
\[
\|g_j\|_1 = \int_G g_j(s) \, d\mu(s) \overset{\text{3.9}}{=} \int_G \left( \int_G f_j(sr) f_j(r) \, d\mu(r) \right) \, d\mu(s) = \int_G \left( \int_G f_j(sr) \, d\mu(s) \right) f_j(r) \, d\mu(r) \\
= \int_G \left( \int_G f_j(u) \, d\mu(u) \right) f_j(r) \, d\mu(r) = \|f_j\|_1^2 = 1.
\]
Moreover, for any $t \in G$, we have
\[
(\text{int}_t g_j)(s) \overset{\text{3.2}}{=} g_j(t^{-1}st) \overset{\text{3.9}}{=} \int_G f_j(t^{-1}str) f_j(r) \, d\mu(r) = \int_G f_j(t^{-1}rt) \, d\mu(r).
\]
Thus we obtain
\[
\|g_j - \text{int}_t g_j\|_1 = \int_G |g_j(s) - (\text{int}_t g_j)(s)| \, d\mu(s) \\
\overset{\text{3.9} \text{ and } \text{3.10}}{=} \int_G \left| \int_G f_j(sr) f_j(r) - f_j(t^{-1}str) f_j(t^{-1}rt) \, d\mu(r) \right| \, d\mu(s) \\
\leq \int_G \left| \int_G f_j(sr) f_j(r) - f_j(t^{-1}rt) \, d\mu(r) \right| \, d\mu(s) + \int_G \left| f_j(t^{-1}str) f_j(t^{-1}rt) \, d\mu(r) \right| \, d\mu(s) \\
\leq \|f_j\|_1 \|f_j - \text{int}_t f_j\|_1 + \|f_j - \text{int}_t f_j\|_1 \|\text{int}_t f_j\|_1 = 2 \|f_j - \text{int}_t f_j\|_1.
\]
This shows that the $g_j$’s have the same normalisation and convergence property as the $f_j$’s. Moreover, by [HeR70, p. 281] the $g_j$’s are continuous positive definite functions with compact support.

5. $\implies$ 6.: Let $F = \{s_1, \ldots, s_n\}$ be a finite subset of $G$ and $\varepsilon > 0$. According to the fifth point and (3.8), choose some positive definite functions $f_j \in C_c(G)$ such that $\|f_j - \text{int}_{s_j} f_j\|_1 / \|f_j\|_1 < \varepsilon / \text{card } F$ for all $s \in F$. Using Lemma 3.5 with $n = \text{card } F$, $f = f_j$ and $g_k = \text{int}_{s_k} f_j$ and the subset $V \overset{\text{def}}{=} \{f_j > t\}$ of $G$, we deduce that for some suitable $t > 0$ and $s \in F$,
\[
\sum_{s \in F} \mu(V \Delta (s^{-1} Vs)) = \sum_{s \in F} \mu(\{f_j > t\} \Delta \{\text{int}_{s^{-1}} f_j > t\}) \overset{\text{3.4}}{<} \varepsilon \mu(\{f_j > t\}) = \varepsilon \mu(V).
\]
Therefore, the group $G$ is inner Følner. Moreover, since $f_j$ is continuous, $V$ is an open subset of $G$. Furthermore, since the function $f_j$ is positive definite, we have $\|f_j\|_\infty = f_j(e_G)$ by [KaL18, p. 23]. We deduce that $e_G$ belongs to $V$ since otherwise we would have $V = \emptyset$ and the previous strict inequality could not hold. Finally, [KaL18, Proposition 1.4.16 (ii) p. 22], we have $f_j = \hat{f}_j$ since $f_j \geq 0$. We conclude that $V$ is symmetric.

6. $\implies$ 3.: trivial.

We turn to the last sentence of the statement. So we assume that $G$ is an inner Følner group, such that for any finite subset $F$ of $G$ there exists a net $(V_\alpha^F)_\alpha$ of measurable subsets of $G$ such that $\mu(V_\alpha^F) \in (0, \infty)$, with the property that for all $s \in F$,
\[
(\text{3.11}) \quad \frac{\mu(V_\alpha^F \Delta (s^{-1} V_\alpha^F s))}{\mu(V_\alpha^F)} \xrightarrow{\alpha \to \infty} 0.
\]
We will construct a sequence \((W_j^F)\), indexed by \(j \in \mathbb{N}\), that satisfies the same convergence property \((3.11)\) as the \((V_\alpha^F)\). Start by putting \(\varepsilon = 1\). By \((3.11)\), for all \(s \in F\), there exists some \(\alpha(1, s)\) such that if \(\alpha \geq \alpha(1, s)\), then
\[
\frac{\mu(V_\alpha^F \Delta (s^{-1}V_\alpha^F s))}{\mu(V_\alpha^F)} \leq 1.
\]
Choose some \(\alpha(1) \geq \alpha(1, s)\) for all \(s \in F\) (directed set property) and put \(W_1^F \equiv V_\alpha^F\). Now, let \(\varepsilon = \frac{1}{2}\). Again by \((3.11)\), for all \(s \in F\), there exists some \(\alpha(\frac{1}{2}, s)\) such that if \(\alpha \geq \alpha(\frac{1}{2}, s)\), then
\[
\frac{\mu(V_\alpha^F \Delta (s^{-1}V_\alpha^F s))}{\mu(V_\alpha^F)} \leq \frac{1}{2}.
\]
Choose some \(\alpha(\frac{1}{2}) \geq \alpha(\frac{1}{2}, s)\) for all \(s \in F\) and put \(W_2^F \equiv V_\alpha^F\). Continue with \(\varepsilon = \frac{1}{4}, \frac{1}{8}, \ldots\) and obtain a sequence of subsets \((W_j^F)\) such that for all \(s \in F\) we have
\[
\frac{\mu(W_j^F \Delta (s^{-1}W_j^F s))}{\mu(W_j^F)} \leq \frac{1}{2^{j-1}}.
\]
For any \(s \in F\), we infer that
\[
\frac{\mu(W_j^F \Delta (s^{-1}W_j^F s))}{\mu(W_j^F)} \xrightarrow{j \to \infty} 0.
\]

4 Projections on the space of completely bounded Fourier multipliers

4.1 Preliminaries

Hilbert-Schmidt operators Let \(\Omega\) be a \(\sigma\)-finite measure space. We will use the space \(S_\Omega^\infty \equiv S_\Omega^\infty(L^2(\Omega))\) of compact operators, its dual \(S_\Omega^1\) and the space \(\mathcal{B}(L^2(\Omega))\) of bounded operators on the complex Hilbert space \(L^2(\Omega)\). If \(f \in L^2(\Omega \times \Omega)\), we denote the associated Hilbert-Schmidt operator by
\[
K_f: L^2(\Omega) \longrightarrow L^2(\Omega), \quad \xi \longmapsto \int_\Omega f(\cdot, y)\xi(y) \, dy.
\]
Using the notation \(\hat{f}(x, y) \equiv f(y, x)\), we have \((K_f)^* = K_{\hat{f}}\). Note that the linear map \(L^2(\Omega \times \Omega) \to S_\Omega^2, f \mapsto K_f\) is an isometry from the Hilbert space \(L^2(\Omega \times \Omega)\) onto the Hilbert space \(S_\Omega^2\) of Hilbert-Schmidt operators acting on the Hilbert space \(L^2(\Omega)\). This means that
\[
\text{Tr}(K_f K_g) = \int_{\Omega \times \Omega} f\hat{g}, \quad f, g \in L^2(\Omega \times \Omega).
\]
Schur multipliers acting on $S^p_{\Omega}$ Suppose that $1 \leq p \leq \infty$. We say that a measurable function $\varphi: \Omega \times \Omega \to \mathbb{C}$ induces a bounded Schur multiplier on the Schatten class $S^p_{\Omega}$ if for any $f \in L^2(\Omega \times \Omega)$ satisfying $K_f \in S^p_{\Omega}$ we have $K_{\varphi f} \in S^p_{\Omega}$ and if the map $S^p_{\Omega} \cap S^p_{\Omega} \to S^p_{\Omega}$, $K_f \mapsto K_{\varphi f}$ extends to a bounded map $M_{\varphi}$ from $S^p_{\Omega}$ into $S^p_{\Omega}$ called the Schur multiplier associated with $\varphi$. It is well-known [LaS11, Remark 1.4 p. 77] that in this case $\varphi \in L^\infty(\Omega \times \Omega)$ and that
\[
\|\varphi\|_{L^\infty(\Omega \times \Omega)} \leq \|M_{\varphi}\|_{S^p_{\Omega} \to S^p_{\Omega}}. \tag{4.3}
\]
We denote by $\mathfrak{M}^p_{\Omega}$ the space of bounded Schur multipliers on $S^p_{\Omega}$ and by $\mathfrak{M}^{p, cb}_{\Omega}$ the subspace of completely bounded ones.

Schur multipliers acting on $B(L^2(\Omega))$ We say that a function $\varphi \in L^\infty(\Omega \times \Omega)$ induces a Schur multiplier on $B(L^2(\Omega))$ if the map $S^p_{\Omega} \to B(L^2(\Omega))$, $K_f \mapsto K_{\varphi f}$ induces a bounded operator from $S^\infty_{\Omega}$ into $B(L^2(\Omega))$. In this case, the operator $S^\infty_{\Omega} \to B(L^2(\Omega))$, $K_f \mapsto K_{\varphi f}$ admits by [BIM04, Lemma A.2.2 p. 360] a unique weak* extension $M_{\varphi}: B(L^2(\Omega)) \to B(L^2(\Omega))$ called the Schur multiplier associated with $\varphi$. It is known that $M_{\varphi}$ induces a bounded map $M_{\varphi}: S^p_{\Omega} \to S^p_{\Omega}$ for any $1 \leq p \leq \infty$. We refer to the surveys [ToT10] and [Tod15] for more information. See also the papers [Arh24] and [Spr04].

Example 4.1 If the set $\Omega = \{1, \ldots, n\}$ is equipped with the counting measure, we can identify the space $B(L^2(\Omega))$ with the matrix algebra $M_n$. Then each operator $K_f$ identifies to the matrix $[f(i, j)]$. A Schur multiplier is given by a map $M_{\varphi}: M_n \to M_n$, $[f(i, j)] \mapsto [\varphi(i, j)f(i, j)]$.

By [Arh24, Proposition 4.3], the map $S^2_{\Omega} \to S^2_{\Omega}$, $K_f \mapsto K_f$ extends to an involutive normal $*$-antiautomorphism $R: B(L^2(\Omega)) \to B(L^2(\Omega))$. We introduce the following duality bracket
\[
\langle z, y \rangle_{B(L^2(\Omega)), S^1_{\Omega}} \defeq \text{Tr}(R(z)y), \quad z \in B(L^2(\Omega)), y \in S^1_{\Omega}, \tag{4.4}
\]
which is more suitable than the bracket $\langle z, y \rangle = \text{Tr}(zy)$ since we have
\[
\langle M_{\varphi}(z), y \rangle_{B(L^2(\Omega)), S^1_{\Omega}} = \langle z, M_{\varphi}(y) \rangle_{B(L^2(\Omega)), S^1_{\Omega}}, \quad z \in B(L^2(\Omega)), y \in S^1_{\Omega}, \tag{4.5}
\]
for any Schur multiplier $M_{\varphi}$ and since the operator space duality requires taking the opposite structure into account.

Herz-Schur multipliers Let $G$ be a (second-countable) unimodular locally compact group. Following [Spr04, p. 179], a bounded Schur multiplier $M_{\varphi}: B(L^2(G)) \to B(L^2(G))$ is a Herz-Schur multiplier if for any $r \in G$ we have $\varphi(sr, t) = \varphi(s, tr^{-1})$ for marginally almost all $(s, t)$ in $G \times G$. We define similarly the notion of Herz-Schur multiplier on $S^p_{G}$. We denote by $\mathfrak{M}^{p, cb}_{G}$ the subspace of $\mathfrak{M}^{p}_{G}$ of completely bounded Herz-Schur multipliers. We define similarly $\mathfrak{M}^{\infty, cb}_{G}$.

If $\varphi: G \to \mathbb{C}$, we introduce the function $\varphi^{HS}: G \times G \to \mathbb{C}$, $(s, t) \mapsto \varphi(st^{-1})$. By [BoF84] and [Spr04, Theorem 5.3 p. 181], the linear map $\mathfrak{M}^{\infty, cb}_{G} \to \mathfrak{M}^{\infty, cb}_{G}$, $M_{\varphi} \mapsto M_{\varphi^{HS}}$ is a surjective isometry. We let $M_{\varphi}^{HS} \defeq M_{\varphi^{HS}}$.

Plancherel weights Let $G$ be a locally compact group. A function $g \in L^2(G)$ is called left bounded [Haa78b, Definition 2.1] if the convolution operator $\Lambda(g): C_c(G) \to C_c(G)$, $f \mapsto g \ast f$
induces a bounded operator on the Hilbert space $L^2(G)$. The Plancherel weight $\tau_G : VN(G)^+ \to [0, \infty]$ is\footnote{This is the natural weight associated with the left Hilbert algebra $C_c(G)$.} defined by the formula

$$\tau_G(x) = \begin{cases} \|g\|_{L^2(G)}^2 & \text{if } x^\frac{1}{2} = \lambda(g) \text{ for some left bounded function } g \in L^2(G) \\ +\infty & \text{otherwise} \end{cases}$$

By [Haa78b, Proposition 2.9 p. 129] (see also [Ped79, Theorem 7.2.7 p. 236]), the canonical left ideal $n_{\tau_G} = \{ x \in VN(G) : \tau_G(x^*x) < \infty \}$ is given by

$$n_{\tau_G} = \{ \lambda(g) : g \in L^2(G) \text{ is left bounded} \}.$$ 

Recall that $m^*_{\tau_G}$ denotes the set $\{ x \in VN(G)^+ : \tau_G(x) < \infty \}$ and that $m_{\tau_G}$ is the complex linear span of $m^*_{\tau_G}$, which is a two-sided ideal of the group von Neumann algebra $VN(G)$. By [Haa78b, Proposition 2.9 p. 129] and [Str81, Proposition p. 280], we have

$$m_{\tau_G} = \{ \lambda(g) : g \in L^2(G) \text{ continuous and left bounded, } \lambda(g) \geq 0 \}.$$ 

By [Ped79, Proposition 7.2.8 p. 237], the Plancherel weight $\tau_G$ on the von Neumann algebra $VN(G)$ is tracial if and only if the locally compact group $G$ is unimodular, which means that the left Haar measure of $G$ and the right Haar measure of $G$ coincide. Now, in the sequel, we suppose that the locally compact group $G$ is unimodular.

We will use the involution $f^* (t) \overset{\text{def}}{=} \frac{f(t^{-1})}{f(t)}$. By [Kun58, Theorem 4 p. 530], if the functions $f, g \in L^2(G)$ are left bounded then $f * g$ and $f^*$ are left bounded and we have

$$\lambda(f)\lambda(g) = \lambda(f * g) \quad \text{and} \quad \lambda(f)^* = \lambda(f^*).$$

If $f, g \in L^2(G)$ it is well-known [Bon04b, VIII pp. 39-40] that the function $f * g$ is continuous and that we have $(f * g)(e) = (g * f)(e) = \int_G \hat{g} \hat{f} \, d\mu_G$, where $e$ denotes the identity element of $G$ and where $\hat{g}(s) \overset{\text{def}}{=} g(s^{-1})$. By [StZ75, (4) p. 282], if $f, g \in L^2(G)$ are left bounded, the operator $\lambda(g)^* \lambda(f)$ belongs to $m_{\tau_G}$ and we have the fundamental noncommutative Plancherel formula

$$\tau_G(\lambda(g)^* \lambda(f)) = \langle g, f \rangle_{L^2(G)},$$

which gives

$$\tau_G(\lambda(g) \lambda(f)) = \int_G \hat{g} \hat{f} \, d\mu_G = \langle g * f \rangle(e).$$

In particular, this formula can be used with any functions belonging to the space $L^1(G) \cap L^2(G)$.

If we introduce the subset $C_c(G) \overset{\text{def}}{=} \text{span} \{ g^* f : g, f \in L^2(G) \text{ left bounded} \}$ of the space $C(G)$ considered in [Ped79, p. 238], then we have

$$m_{\tau_G} = \lambda(C_c(G)).$$

In this context, $\tau_G$ can be interpreted as the functional that evaluates functions of $C_c(G)$ at the identity element $e_G$. While the formula $\tau_G(\lambda(h)) = h(e)$ appears meaningful for every function $h$ in $C_c(G)$, we caution the reader that, in general, it is not true that $\lambda(C_c(G)) \subseteq m_{\tau_G}$. Unfortunately, this misconception is frequently encountered in the literature.

**Noncommutative $L^p$-spaces**  In this paper, we focus on noncommutative $L^p$-spaces associated to semifinite von Neumann algebras. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace $\tau$. Let $S^+$ be the set of all $x \in \mathcal{M}_+$ such that...
$\tau(\text{supp}(x)) < \infty$, where $\text{supp}(x)$ denotes the support of $x$. Let $\mathcal{S}$ be the linear span of $\mathcal{S}^+$, then $\mathcal{S}$ is weak* dense $\star$-subalgebra of $\mathcal{M}$.

Suppose that $1 \leq p < \infty$. For any $x \in \mathcal{S}$, the operator $|x|^p$ belongs to $\mathcal{S}_+$ and we set

$$
(4.9) \quad \|x\|_{L^p(\mathcal{M})} \overset{\text{def}}{=} (\tau(|x|^p))^\frac{1}{p}.
$$

Here $|x| \overset{\text{def}}{=} (x^*x)^{\frac{1}{2}}$ denotes the modulus of $x$. It turns out that $||\|_{L^p(\mathcal{M})}$ is a norm on $\mathcal{S}$. By definition, the noncommutative $L^p$-space $L^p(\mathcal{M})$ associated with $(\mathcal{M}, \tau)$ is the completion of $(\mathcal{S}, ||\|_{L^p(\mathcal{M})})$. For convenience, we also set $L^{\infty}(\mathcal{M}) \overset{\text{def}}{=} \mathcal{M}$ equipped with its operator norm. Note that by definition, $L^p(\mathcal{M}) \cap \mathcal{M}$ is dense in $L^p(\mathcal{M})$ for any $1 \leq p < \infty$. See [PiX03] for more information on noncommutative $L^p$-spaces.

Furthermore, the trace $\tau$ uniquely extends to a bounded linear functional on the Banach space $L^1(\mathcal{M})$, still denoted by $\tau$. Actually, we have

$$
(4.10) \quad |\tau(x)| \leq \|x\|_{L^1(\mathcal{M})}, \quad x \in L^1(\mathcal{M}).
$$

Recall the noncommutative Hölder’s inequality. If $1 \leq p, q, r \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ then

$$
(4.11) \quad \|xy\|_{L^r(\mathcal{M})} \leq \|x\|_{L^p(\mathcal{M})}\|y\|_{L^q(\mathcal{M})}, \quad x \in L^p(\mathcal{M}), y \in L^q(\mathcal{M}).
$$

For any $1 \leq p < \infty$, let $p^* \overset{\text{def}}{=} \frac{p}{p-1}$ be the conjugate number of $p$. Applying (4.11) with $q = p^*$ and $r = 1$ together with (4.10), we obtain a linear map $L^{p^*}(\mathcal{M}) \to (L^p(\mathcal{M}))^*$, $y \mapsto \tau(xy)$, which induces an isometric isomorphism

$$
(4.12) \quad (L^p(\mathcal{M}))^* = L^{p^*}(\mathcal{M}), \quad 1 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{p^*} = 1.
$$

In particular, we may identify the Banach space $L^1(\mathcal{M})$ with the unique predual $\mathcal{M}_*$ of the von Neumann algebra $\mathcal{M}$.

**Operator theory** Suppose that $1 \leq p < \infty$. Let $T: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ be any bounded operator. We will denote by $T^*$ the adjoint of $T$ defined by

$$
\tau(T(x)y) = \tau(xT^*(y)), \quad x \in L^p(\mathcal{M}), y \in L^{p^*}(\mathcal{M}).
$$

For any $1 \leq p \leq \infty$ and any $T: L^p(\mathcal{M}) \to L^p(\mathcal{M})$, we can consider the map $T^0: L^p(\mathcal{M}) \to L^p(\mathcal{M})$ defined by

$$
(4.13) \quad T^0(x) \overset{\text{def}}{=} T(x^*)^*, \quad x \in L^p(\mathcal{M}).
$$

If $p = 2$ and if we denote by $T^1: L^2(\mathcal{M}) \to L^2(\mathcal{M})$ the adjoint of $T: L^2(\mathcal{M}) \to L^2(\mathcal{M})$ in the usual sense of Hilbertian operator theory, that is

$$
\tau(T(x)y^*) = \tau(x(T^1(y))^*), \quad x, y \in L^2(\mathcal{M}),
$$

we see that

$$
(4.14) \quad T^1 = T^{*0}.
$$

30
4.2 Description of the approach

Suppose that $1 \leq p \leq \infty$ and let $G$ be a locally compact group. In this section, we present an approach for obtaining some bounded projections $P^p_G : CB(L^p(VN(G))) \rightarrow CB(L^p(VN(G)))$ onto the subspace $\mathcal{M}^{p,cb}(G)$ of completely bounded Fourier multipliers on $L^p(VN(G))$, beyond the case of discrete groups, for a suitable locally compact group. The methods are different from the ones of [ArK23] and complement the results of this paper. If $G$ is a locally compact group, we will use the fundamental unitary $W : L^2(G \times G) \rightarrow L^2(G \times G)$ in $\mathcal{B}(L^2(G) \overline{\otimes} VN(G))$ and its inverse $W^{-1}$ defined in [Vae01, Example 2.2.10 p. 26] (see also [Kus05, Remark 5.16 p. 150]) by

$$
(4.15) \quad (W\xi)(s, t) \overset{\text{def}}{=} \xi(s, s^{-1}t), \quad (W^{-1}\xi)(s, t) = \xi(s, st), \quad s, t \in G, \xi \in L^2(G \times G).
$$

Before going into the details, let us shortly present the roadmap of the proof of results of Section 4.9.

Suppose that the group $G$ is discrete and recall the well-known construction. Consider the coproduct $\Delta : VN(G) \rightarrow VN(G) \overline{\otimes} VN(G)$, $\lambda_s \mapsto \lambda_s \otimes \lambda_s$. This trace preserving normal unital injective $*$-homomorphism extends to a completely positive isometric map $\Delta_p : L^p(VN(G)) \rightarrow L^p(VN(G))$ for any $1 \leq p \leq \infty$. With the adjoint $(\Delta^*_p)^* : L^p(VN(G)) \rightarrow L^p(VN(G))$, the map $P^p_G : CB(L^p(VN(G))) \rightarrow CB(VN(G))$ defined by

$$
(4.16) \quad P^p_G(T) = (\Delta^*_p)^*(\text{Id}_{L^p(VN(G))} \otimes T)\Delta_p, \quad T \in CB(L^p(VN(G)))
$$

is a contractive projection from the Banach space $CB(L^p(VN(G)))$ onto the subspace $\mathcal{M}^{p,cb}(G)$ of completely bounded Fourier multipliers acting on $L^p(VN(G))$ which preserves the complete positivity (in the case $p = \infty$ replace $CB(L^p(VN(G)))$ by the space $CB_w(VN(G))$).

By [Vae01, p. 26] and [Str74, p. 267], we can factorize\textsuperscript{13} the coproduct as

$$
\Delta(x) = W(x \otimes 1)W^{-1}, \quad x \in VN(G).
$$

If $u, v \in VN(G)$, we can therefore rewrite the formula (4.16) as

$$
(4.17) \quad \left\langle P^p_G(T)u, v \right\rangle_{L^p(VN(G)), L^p(VN(G))} = \left\langle (\text{Id} \otimes T)\Delta_p(u), \Delta^*_p(v) \right\rangle_{L^p, L^p}.
$$

and finally

Now, if $G$ is a (second-countable unimodular) locally compact group and if $T : L^p(VN(G)) \rightarrow L^p(VN(G))$ is again a completely bounded map, we wish to replace one or both units $1$ of the formula (4.17) by suitable sequences $(x_j)$ and $(y_j)$ of elements which approximate $1$ in some sense. Actually, we start by replacing in (4.17) the elements $1$ by elements $x, y \in L^1(VN(G)) \cap VN(G)$ and $u, v$ by elements of $u \in S^p_G$ and $v \in S^p_G$. We will show that there exists a completely bounded Schur multiplier $M_{x, y, T} : S^p_G \rightarrow S^p_G$ (replace the Schatten class $S^p_G$ by the von Neumann algebra $\mathcal{B}(L^2(G))$ if $p = \infty$) such that

$$
(4.18) \quad \left\langle M_{x, y, T}(u), v \right\rangle_{S^p_G, S^p_G} = \left\langle (\text{Id} \otimes T)(W(u \otimes x)W^{-1}), W(v \otimes y)W^{-1} \right\rangle_{S^p_G(L^p(VN(G))), S^p_G(L^p(VN(G)))}
$$

for any suitable elements $u \in S^p_G$ and $v \in S^p_G$. Note that $x \in L^p(VN(G))$, $y \in L^p(VN(G))$ and that $W, W^{-1} \in \mathcal{B}(L^2(G)) \overline{\otimes} VN(G)$. Moreover, we will compute the symbol $\varphi_{x, y, T}$, belonging to $L^\infty(G \times G)$, of the Schur multiplier $M_{x, y, T}$ and we will get

$$
(4.19) \quad \varphi_{x, y, T}(s, t) = \tau_G(\lambda_t y \lambda_{s^{-1}} T(\lambda_s x \lambda_{t^{-1}})) \quad s, t \in G.
$$

\textsuperscript{13} Indeed, this factorization is the definition of the coproduct.
In the particular case of finite groups, these assertions are straightforward and we refer to the end of this section for a short proof of (4.18) and (4.19).

For the case of a locally compact group, this step unfortunately uses a painful approximation procedure described in Section 4.3 relying on a sequence \((M_{\phi_n})\) of completely bounded Fourier multipliers \(M_{\phi_n} : L^p(VN(G)) \to L^2(VN(G))\) which allows us to consider the completely bounded maps \(M_{\phi_n} T : L^p(VN(G)) \to L^2(VN(G))\) in order to reduce the problem to the level \(p = 2\).

We therefore obtain a map \(P_{x,y} : CB(L^p(VN(G))) \to CB(S_G^p),\ T \mapsto M_{x,y}T\) and it is easy to check that this map preserves the complete positivity. Introducing suitable sequences \((x_j)\) and \((y_j)\) of elements in \(L^1(VN(G)) \cap VN(G)\) which approximate the element 1 we obtain a sequence \((P_j)\) of linear maps \(P_j = P_{x_j,y_j} : CB(L^p(VN(G))) \to CB(S_G^p)\). One of the difficulties in this area is to construct suitable sequences with the chosen assumptions on the group \(G\).

Essentially, in the sequel we capture a cluster point of the bounded family \((P_j)\) and we obtain a bounded map \(P^{(1)} : CB(L^p(VN(G))) \to CB(S_G^p)\). Each map \(P^{(1)}(T)\) is a completely bounded Schur multiplier.

**Case where \(G\) is inner amenable and \(p = \infty\)** With a suitable choice of the sequences \((x_j)\) and \((y_j)\) provided by the inner amenability of \(G\), the map \(P^{(1)} : CB_{\infty}(VN(G)) \to CB(B(L^2(G)))\) is contractive and the Schur multiplier \(P^{(1)}(T) : B(L^2(G)) \to B(L^2(G))\) is a Herz-Schur multiplier for all weak* continuous completely bounded maps \(T : VN(G) \to VN(G)\). So, we can see the linear map \(P^{(1)}\) as a map \(P^{(1)} : CB_{\infty}(VN(G)) \to M_G^{\infty,\text{HS}} = M_G^{\infty,\text{cb,HS}}\). Now, it suffices to identify (completely) bounded Herz-Schur multipliers acting on the space \(B(L^2(G))\) isometrically with completely bounded Fourier multipliers acting on \(VN(G)\), while preserving the complete positivity. This step is well-known [BoF84] and true for any locally compact group \(G\) without amenability assumption. Denoting \(I : M_G^{\infty,\text{HS}} \to CB(VN(G))\) the associated isometry with range \(M_G^{\infty,\text{cb}}(G)\), the final contractive projection will be \(P_G^{\infty} \overset{\text{def}}{=} I \circ P^{(1)}\).

Indeed, in the case where \(T = M_{\phi} : VN(G) \to VN(G)\) is a Fourier multiplier we will prove that the symbol \(\phi_{j,T}\) of the Schur multiplier \(P_j(M_{\phi}) : B(L^2(G)) \to B(L^2(G))\) is equal to the symbol \(\phi_{\text{HS}} : (s,t) \mapsto \phi(st^{-1})\) for any \(j\). By passing the limit, \(P^{(1)}(M_{\phi}) = M_{\phi_{\text{HS}}}\) and finally

\[
P_G^{\infty}(M_{\phi}) = I \circ P^{(1)}(M_{\phi}) = I(M_{\phi_{\text{HS}}}) = M_{\phi}.
\]

So we obtain the property \((\kappa_{\infty})\) of Definition 1.2 for these groups with constant \(\kappa_{\infty}(G) = 1\).

**Case where \(G\) is finite-dimensional and amenable and simultaneous cases \(p = 1\) and \(p = \infty\)** In the case where the group \(G\) is in addition finite-dimensional and amenable, replacing the sequences \((x_j)\) and \((y_j)\) of the proof of the last case by new ones, we obtain linear maps \(P^{(1)}_p : CB(L^p(VN(G))) \to CB(S_G^p)\) for \(p = 1\) and \(p = \infty\) (replace \(CB(BL^\infty(VN(G)))\) by \(CB_{\infty}(VN(G))\) here an the sequel), that we see as maps \(P^{(1)}_p : CB(L^p(VN(G))) \to M_G^{p,\text{cb}}\).

The cost of this replacement of sequences is the non-contractivity of \(P^{(1)}_p\) but we obtain the compatibility of the maps \(P^{(1)}_\infty(T)\) and \(P^{(1)}_1(T)\). For the construction of the sequences \((x_j)\) and \((y_j)\), our approach relies on the structure of locally compact groups from the solution to Hilbert’s fifth problem which makes appear connected Lie groups in this context and the use of Carnot-Carathéodory metrics on connected Lie groups.

Now, we construct and use a contractive map \(Q : M_G^{p,\text{cb}} \to M_G^{p,\text{cb,HS}}\) from the space \(M_G^{p,\text{cb}}\) of Schur multipliers onto the subspace of Herz-Schur multipliers which preserves the complete positivity and the Herz-Schur multipliers\(^{14}\). In this essentially folklore step, we need

14. We can see \(Q\) as a contractive projection \(Q : M_G^{p,\text{cb}} \to M_G^{p,\text{cb}}\) onto the subspace \(M_G^{p,\text{cb,HS}}\) of completely bounded Herz-Schur multipliers.
the amenability of the group $G$ in sharp contrast with our previous work [ArK23]. Then put $P_p^{(2)} \overset{\text{def}}{=} Q \circ P_p^{(1)} : \CB(L^p(VN(G))) \to \mathfrak{M}_G^{cb,HS}$.

At present, it suffices with [CaS15] to identify completely bounded Herz-Schur multipliers isometrically with completely bounded Fourier multipliers, preserving the complete positivity. Denoting $I : \mathfrak{M}_G^{cb,HS} \to \CB(L^p(VN(G)))$ the associated isometry, the final contractive projection will be

$$P_p^G \overset{\text{def}}{=} I \circ P_p^{(2)} = I \circ Q \circ P_p^{(1)}.$$  

In the case where $T = M_φ$ is a Fourier multiplier, we will prove that the symbol $φ_{j,T}$, element in $L^\infty(\mathbb{G} \times G)$, of the completely bounded Schur multiplier $P_j(T)$ converges to the symbol $φ_{HS} : (s,t) \mapsto φ(st^{-1})$ for the weak* topology of of the dual Banach space $L^\infty(\mathbb{G} \times G)$.

We deduce that $P_p^G(M_φ) = I \circ Q \circ P_p^{(1)}(M_φ) = I \circ Q(M_{φ_{HS}}) = I(M_{φ_{HS}}) = M_φ$.

We conclude that we obtain the property $(κ)$ for totally disconnected groups, the method gives the sharp result $κ(G) = 1$.

**Case where $G$ is amenable and $1 < p < ∞$ with $\frac{p}{p'}$ being rational.** In the case where the group $G$ is amenable, using some sequences $(x_j)$ and $(y_j)$, we obtain a contractive linear map $P_p^{(1)} : \CB(L^p(VN(G))) \to \CB(S_p^G)$ which is better than the boundedness of the previous case, but only for one value of $p$. The method is similar to the previous case but we use [CaS15] (see also [NeR11]) instead of [BoF84] to identify completely bounded Herz-Schur multipliers isometrically with completely bounded Fourier multipliers (which require the amenability of $G$ once again).

**Particular case of finite groups: proof of (4.18) and (4.19)** If the group $G$ is finite and if $(e_i)$ is an orthonormal basis of the Hilbert space $ℓ^2_G$, then (4.15) translates to

$$W(e_t \otimes e_r) = e_t \otimes e_{tr}, \quad W^{-1}(e_t \otimes e_r) = e_t \otimes e_{t^{-1}r}, \quad t, r \in G.$$  

For any $i, j, s, t, u \in G$, we have

$$W(e_{st} \otimes λ_u)W^{-1}(e_i \otimes e_j) = W(e_{st} \otimes λ_u)(e_i \otimes e_{i^{-1}j}) = W(e_{st}e_i \otimes λ_u(e_{i^{-1}j})) = δ_{tui}W(e_s \otimes e_{ui^{-1}j}).$$  

Hence in $\mathcal{B}(ℓ^2(G))$\,⊗\,$VN(G)$, we have

$$W(e_{st} \otimes λ_u)W^{-1} = e_{st} \otimes λ_{sut^{-1}}.$$  

We deduce that

$$\text{(Id} \otimes T)(W(e_{st} \otimes λ_u)W^{-1}) = e_{st} \otimes T(λ_{sut^{-1}}).$$  

15. Actually, it is showed in [CaS15] that the map $I$ is a contraction (when $G$ is amenable), which is an isometry on a large subspace.
We infer that
\[
(\text{Tr} \otimes \tau_G)[(\text{Id} \otimes T)(W(e_{st} \otimes \lambda_x)W^{-1})(W(e_{ij} \otimes \lambda_{ij-1})W^{-1})]
\]
\[= (\text{Tr} \otimes \tau_G)[(e_{st} \otimes T(\lambda_{s_{ij-1}}))(e_{ij} \otimes \lambda_{ij-1})]
\]
\[= \text{Tr}(e_{st}e_{ij})\tau_G(T(\lambda_{s_{ij-1}})\lambda_{ij-1}) = \delta_t=1\delta_{s=x}\tau_G(\lambda_{t}\lambda_x\lambda_{j-1}T(\lambda_{s_{ij-1}})).
\]

By linearity, we deduce on the one hand for any \(x \in L^p(VN(G))\) and any \(y \in L^p(VN(G))\)
\[
(\text{Tr} \otimes \tau_G)[(\text{Id} \otimes T)(W(e_{st} \otimes x)W^{-1})(W(e_{ij} \otimes y)W^{-1})] = \delta_{t=1}\delta_{s=x}\tau_G(\lambda_{t}y\lambda_{j-1}T(\lambda_{x_{ij-1}}))
\]

On the other hand, if we consider the Schur multiplier \(M_{x,y,T} : S^p_G \rightarrow S^p_G\) with symbol (4.19), we have
\[
\langle M_{x,y,T}(e_{st}), e_{ij} \rangle_{S^p_G}^{(4.19)} = \delta_{t=1}\delta_{s=x}\tau_G(\lambda_{t}y\lambda_{j-1}T(\lambda_{x_{ij-1}})).
\]

**Remark 4.2** Note that with \(x = y = 1\), the Schur multiplier \(M_{x,y,T}\) is a Herz-Schur multiplier. See Section 4.4 for a generalization of this crucial observation.

### 4.3 Step 1: the mappings \(P_j(T)\)

In this section, we establish (4.18) and (4.19). We caution the reader that while this part is technically involved, the underlying idea is quite simple. Specifically, we reduce the computation to the case \(p = 2\), where Parseval’s identity can be applied.

Let \(G\) be a unimodular locally compact group. We denote by \(\text{Tr}_G\) and \(\tau_G\) the canonical traces \(\text{Tr}_G\) of the von Neumann algebras \(B(L^2(G))\) and \(VN(G)\). Suppose that \(G\) is second-countable and fix an orthonormal basis \(\{e_i\}\) of the Hilbert space \(L^2(G)\) such that each function \(e_i\) is continuous with compact support\(^{16}\). Note that by [BIM04, p. 40] we have a canonical identification \(B(L^2(G)) \overline{\otimes} VN(G) = M_{\infty}(VN(G))\). That means that an element \(X\) belonging to the von Neumann tensor product \(B(L^2(G)) \overline{\otimes} VN(G)\) identifies to a matrix \([x_{ij}]\) with entries in the von Neumann algebra \(VN(G)\). For any \(h \in L^2(G)\) and any integer \(k\), note that in \(L^2(G)\)
\[
(x_{kk}(h))(w) = \int_G \langle X(e_k \otimes h), (s, w) \rangle_{L^2(G)} d\mu_G(s), \quad h \in L^2(G), \text{ a.e. } w \in G.
\]
\[
(4.23)\quad \int_G \langle X(e_k \otimes h), (s, w) \rangle_{L^2(G)} d\mu_G(s),
\]
If \(1 \leq p < \infty\), we have by [Ps98] a similar isometry \(L^p(B(L^2(G)) \overline{\otimes} VN(G)) = S^p_G(L^p(VN(G)))\). Moreover, if \(X\) belongs to the intersection \(L^1(B(L^2(G)) \overline{\otimes} VN(G)) \cap \{B(L^2(G)) \overline{\otimes} VN(G)\}\) we have \(x_{kk} \in L^1(VN(G)) \cap VN(G)\) for any integer \(k\) and
\[
(4.24)\quad (\text{Tr}_G \otimes \tau_G)(X) = \sum_{k=1}^{\infty} \tau_G(x_{kk}).
\]
In the next result, we use the operator \(W\) in \(B(L^2(G)) \overline{\otimes} VN(G)\) and its inverse from (4.15).

**Lemma 4.3** Let \(G\) be a second-countable unimodular locally compact group.

1. Let \(\phi \in L^2(G \times G)\) such that \(K_\phi\) belongs to \(S^1_G\) and \(x \in L^1(VN(G)) \cap VN(G)\). Then \(W(K_\phi \otimes x)W^{-1}\) belongs to \(L^1(B(L^2(G)) \overline{\otimes} VN(G))\) and to \(B(L^2(G)) \overline{\otimes} VN(G)\).

---
\[^{16}\text{To demonstrate the existence of such an orthonormal basis, consider a sequence of continuous functions with compact support that is dense in } L^2(G),\text{ and apply the Gram-Schmidt procedure.}\]
2. If $f$ belongs to the space $C_c(G)$ and if $g$ belongs to the space $C_c(G) \ast C_c(G)$, we have for any integers $i, j$

\[ (4.25) \quad (\text{Tr}_G \otimes \tau_G)[W(K_\phi \otimes \lambda(g))W^{-1} \cdot (e_{ij}^* \otimes \lambda(f))] \]

\[ = \int_G \int_G \phi(s,t)\tau_G[\lambda_s \lambda(t)\lambda^{-1} \cdot \lambda(f)]e_i(s)e_j(t) \, d\mu_G(s) \, d\mu_G(t). \]

**Proof**: 1. The element $K_\phi \otimes x$ belongs to $S^1_0 \otimes [L^1(\mathcal{VN}(G)) \cap \mathcal{VN}(G)]$, hence to the space $L^1(\mathcal{B}(L^2(G)) \otimes \mathcal{VN}(G))$. Then the claim follows since $W$ and $W^{-1}$ belong to the space $\mathcal{B}(L^2(G)) \otimes \mathcal{VN}(G)$ and since $L^1(\mathcal{B}(L^2(G)) \otimes \mathcal{VN}(G)) \cap [\mathcal{B}(L^2(G)) \otimes \mathcal{VN}(G)]$ is an ideal of the von Neumann algebra $\mathcal{B}(L^2(G)) \otimes \mathcal{VN}(G)$.

2. By the first part, observe that the element $X \equiv W(K_\phi \otimes \lambda(g))W^{-1} \cdot (e_{ij}^* \otimes \lambda(f))$ belongs to the intersection $L^1(\mathcal{B}(L^2(G)) \otimes \mathcal{VN}(G)) \cap [\mathcal{B}(L^2(G)) \otimes \mathcal{VN}(G)]$. According to (4.24), we have

\[ (\text{Tr}_G \otimes \tau_G)[W(K_\phi \otimes \lambda(g))W^{-1}(e_{ij}^* \otimes \lambda(f))] = \sum_{k=1}^{\infty} \tau_G(x_{kk}) \]

Now, we want to compute $\tau_G(x_{kk})$ with (4.23). If $k \neq i$, we have

\[ X(e_k \otimes h) = W(K_\phi \otimes \lambda(g))W^{-1}(e_{ji} \otimes \lambda(f))(e_k \otimes h) = 0. \]

Hence $x_{kk} = 0$ in this case and therefore $\tau_G(x_{kk}) = 0$. Thus, we only need to consider $k = i$ in the sequel. Then replacing $r$ by $tv^{-1}s^{-1}r$ in the last equation for $h \in C_c(G)$

\[ (4.26) \quad (X(e_i \otimes h))(s, w) = (W(K_\phi \otimes \lambda(g))W^{-1}(e_{ji} \otimes \lambda(f))(e_i \otimes h))(s, w) \]

\[ = (W(K_\phi \otimes \lambda(g))W^{-1}(e_j \otimes \lambda(f))(h))(s, w) \]

\[ = (K_\phi \otimes \lambda(g))W^{-1}(e_j \otimes \lambda(f))(h))(s, s^{-1}w) \]

\[ = \int_G \int_G \phi(s, t)g(v)W^{-1}(e_j \otimes \lambda(f))(h)(t, v^{-1}s^{-1}w) \, d\mu_G(t) \, d\mu_G(v) \]

\[ = \int_G \int_G \phi(s, t)g(v)(e_j \otimes \lambda(f))(h)(t, tv^{-1}s^{-1}w) \, d\mu_G(t) \, d\mu_G(v) \]

\[ = \int_G \int_G \phi(s, t)e_j(t)g(v)f(r)(r^{-1}tv^{-1}s^{-1}w) \, d\mu_G(t) \, d\mu_G(v) \]

\[ = \int_G \int_G \phi(s, t)g(v)f(tv^{-1}s^{-1}r)h(r^{-1}w)e_j(t) \, d\mu_G(t) \, d\mu_G(v) \]

\[ = \int_G \int_G \phi(s, t)g(v)f(tv^{-1}s^{-1}r) \, d\mu_G(t) \, d\mu_G(v) \]

Hence for almost all $t \in G$

\[ (x_{ii}(h))(w) = \int_G (X(e_i \otimes h))(s, w)e_i(s) \, d\mu_G(s) \]

\[ = \int_G \int_G \int_G \phi(s, t)g(v)f(tv^{-1}s^{-1}r)h(r^{-1}w)e_i(s)e_j(t) \, d\mu_G(t) \, d\mu_G(v) \, d\mu_G(s). \]

So we obtain

\[ x_{ii} = \int_G \int_G \int_G \phi(s, t)g(v)f(tv^{-1}s^{-1}r)e_i(s)e_j(t) \, d\mu_G(t) \, d\mu_G(v) \, d\mu_G(s). \]
which identifies to the convolution operator $\lambda(k)$ where $k$ is the function defined by

\[(4.27) \quad k(r) \overset{\text{def}}{=} \int_G \int_G \int_G \phi(s, t)g(v)f(tu^{-1}s^{-1}r)e_i(s)e_j(t) \, d\mu_G(t) \, d\mu_G(v) \, d\mu_G(s).\]

We can easily evaluate the trace of the diagonal entry $x_{ii}$ which is an element in the space $L^1(VN(G)) \cap VN(G)$. Indeed, replacing $v$ by $s^{-1}v$ in the second equality, we have

\[
\tau_G(x_{ii}) = k(e) = \int_G \int_G \int_G \phi(s, t)g(s^{-1}v)f(tu^{-1}s^{-1})e_i(s)e_j(t) \, d\mu_G(t) \, d\mu_G(v) \, d\mu_G(s)
\]

\[
\quad = \int_G \int_G \int_G \phi(s, t)g(s^{-1}v)f(tu^{-1})e_i(s)e_j(t) \, d\mu_G(t) \, d\mu_G(v) \, d\mu_G(s)
\]

\[
\quad = \int_G \int_G \phi(s, t) \left( \int_G g(s^{-1}v)f(tu^{-1}) \, d\mu_G(v) \right) e_i(s)e_j(t) \, d\mu_G(t) \, d\mu_G(s)
\]

\[
\quad \overset{(4.27)}{=} \int_G \int_G \phi(s, t) \tau_G[\lambda_s \lambda(g)\lambda_{-1}(f)] e_i(s)e_j(t) \, d\mu_G(t) \, d\mu_G(s).
\]

In a similar way to Lemma 4.3, we have the following result.

**Lemma 4.4** Let $G$ be a second-countable unimodular locally compact group. Let $\phi \in L^2(G \times G)$ such that $K_\phi \in S^2_G$ and $x \in L^1(VN(G)) \cap VN(G)$. Suppose that $1 \leq p \leq 2$. Let $T : L^p(VN(G)) \to L^2(VN(G))$ be a completely bounded map. Then $(\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1})$ belongs to $L^2(B(L^2(G))\otimes VN(G))$. If $g \in C_c(G) \ast C_c(G)$ we have for any integers $i, j$ and any $f \in C_c(G)$

\[(4.28) \quad \int_G \int_G \phi(s, t) \tau_G[\lambda_s \lambda(g)\lambda_{-1}(f)] e_i(s)e_j(t) \, d\mu_G(s) \, d\mu_G(t).
\]

**Proof** : According to Lemma 4.3, the element $W(K_\phi \otimes \lambda(g))W^{-1}$ belongs to the space $L^1(B(L^2(G))\otimes VN(G)) \cap [B(L^2(G))\otimes VN(G)]$, hence to the Banach space $L^p(B(L^2(G))\otimes VN(G)) = S^p_G(L^p(VN(G)))$.

By the complete boundedness of $T : L^p(VN(G)) \to L^2(VN(G))$ and [Pis98, Lemma 1.7 p. 23], we infer that the element $(\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1})$ belongs to the space $S^p_G(L^2(VN(G)))$. Since $p \leq 2$, it belongs to the Banach space $S^p_G(L^2(VN(G))) = L^2(B(L^2(G))\otimes VN(G))$. We have immediately

\[
(\text{Tr}_G \otimes \tau_G)[(\text{Id} \otimes T)(W(K_\phi \otimes \lambda(g))W^{-1}) \cdot (e^*_ij \otimes \lambda(f))]
\]

\[
= (\text{Tr}_G \otimes \tau_G)[(W(K_\phi \otimes \lambda(g))W^{-1}) \cdot (e^*_ij \otimes T^*\lambda(f))].
\]

Now, it suffices to show that $(4.25)$ holds for generic elements in $L^p(VN(G))$ instead of $\lambda(f)$. This is indeed the case by density since both sides of $(4.25)$ are continuous as functions in $\lambda(f) \in L^p(VN(G))$.

**Remark 4.5** The two previous lemmas are true without the assumption «second-countable».

**Lemma 4.6** Let $G$ be a second countable unimodular locally compact group. Suppose that $1 \leq p \leq 2$. Let $T : L^p(VN(G)) \to L^p(VN(G))$ be a completely bounded map. There exists a
sequence $(M_{\phi_n})$ of bounded Fourier multipliers $M_{\phi_n} : \text{VN}(G) \to \text{VN}(G)$ such that $\phi_n \in C_c(G)$, $\|\phi_n\|_\infty \leq 1$, $M_{\phi_n} : L^p(\text{VN}(G)) \to L^2(\text{VN}(G))$ is completely bounded, satisfying for any $g \in C_c(G)$, any $\phi \in C_c(G \times G)$ such that $K_\phi \in S^1_{\text{VN}}$ and any sufficiently large $n$

$$M_{\phi_n}(\lambda(g)) = \lambda(g) \quad \text{and} \quad (\text{Id} \otimes M_{\phi_n})(W(K_\phi \otimes \lambda(g))W^{-1}) = W(K_\phi \otimes \lambda(g))W^{-1}.$$

**Proof:** Since the group $G$ is second-countable, we can consider a sequence $(K_n)$ of symmetric compacts in $G$ such that for any compact $K$ of $G$, one has $K \subset K_n$ for sufficiently large enough $n$. By [KaL18, Proposition 2.3.2 p. 50], for any $n$ there exists a function $\phi_n : G \to \mathbb{C}$ which is a finite linear combination of continuous positive definite functions with compact support with $0 \leq \phi_n \leq 1$ and $\phi_n(s) = 1$ for any $s \in K_n$. By essentially [KaL18, Proposition 5.4.9 p. 184] each function induces a completely bounded Fourier multiplier $M_{\phi_n} : \text{VN}(G) \to \text{VN}(G)$. Furthermore, since $\phi_n \in L^2(G)$ each map $M_{\phi_n} : L^1(\text{VN}(G)) \to L^2(\text{VN}(G))$ is completely bounded by [GJP17, Remark 2.4 p. 899] and duality. Now, it suffices to interpolate with $M_{\phi_n} : L^2(\text{VN}(G)) \to L^2(\text{VN}(G))$ to obtain a completely bounded Fourier multiplier $M_{\phi_n} : L^p(\text{VN}(G)) \to L^2(\text{VN}(G))$.

Now, let $g \in C_c(G)$ and $\phi \in C_c(G \times G)$ such that $K_\phi \in S^1_{\text{VN}}$. Consider the compact $K_\phi \overset{\text{def}}{=} \text{supp}\; g$ and some compacts $L_1, L_2$ of $G$ such that $\text{supp}\; \phi \subset L_1 \times L_2$ and let $L_1 \overset{\text{def}}{=} L_1 \cdot K \cdot L_2^{-1}$, which is also compact. Then for any sufficiently large enough $n$ such that $K \subset K_n$,

$$M_{\phi_n}(\lambda(g)) = \lambda(\phi_n g) = \lambda(g).$$

Moreover, consider some sufficiently large enough $n$ such that $L \subset K_n = \tilde{K}_n$. For any $s \in L_1$ and any $t \in L_2$, the element $\lambda_s \lambda(t) = 1$ has its Fourier support in $L_1 \cdot K \cdot L_2^{-1} = L$. Thus, $M_{\phi_n}(\lambda_s \lambda(t) = 1) = \lambda_s \lambda(t)$, $\lambda(t - 1)$. Then for any integers $i, j$ and any function $f \in C_c(G \ast C_c(G)$ we have $M_{\phi_n} = M_{\phi_n}$. Hence

$$\left(\text{Tr}_G \otimes \tau_G\right) \left[ (\text{Id} \otimes M_{\phi_n})(W(K_\phi \otimes \lambda(g))W^{-1}) \cdot (e^*_{ij} \otimes \lambda(f)) \right] = \left(\text{Id} \otimes M_{\phi_n}(W(K_\phi \otimes \lambda(g))W^{-1}) \cdot (e^*_{ij} \otimes \lambda(f)) \right).$$

Recall that $(e_n)$ is an orthonormal basis of the Hilbert space $L^2(G)$ such that each function $e_n$ is continuous with compact support. So the family $(\lambda(e_n))$ is an orthonormal basis of $L^2(\text{VN}(G))$ and $(e^*_{ij} \otimes \lambda(e_n))$ is an orthonormal basis of the Hilbert space $L^2(\mathcal{B}(L^2(G)) \otimes \text{VN}(G))$.

**Proposition 4.7** Let $G$ be a second-countable unimodular locally compact group. Suppose that $1 \leq p \leq \infty$. Let $T : L^p(\text{VN}(G)) \to L^p(\text{VN}(G))$ be a completely bounded operator (normal if $p = \infty$). Let $\phi, \psi \in L^2(G \times G)$ such that $K_\phi, K_\psi \in S^1_{\text{VN}}$, and $x, y \in L^1(\text{VN}(G)) \cap \text{VN}(G)$. With the symbol

$$\varphi_{x,y,T}(s,t) \overset{\text{def}}{=} \tau_G \left( \lambda_s y \lambda_{t^{-1}} T(\lambda_s x \lambda_{t^{-1}}) \right).$$
we have

\[ (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle_{S_G^2(L^p(VN(\mathcal{G})))} = \langle M_{\psi,x,T}(K_\phi), K_\psi \rangle_{S_G^2}. \]

Finally, if \( p = \infty \), the same holds for any \( x \in VN(\mathcal{G}) \) and if \( p = 1 \), the same holds for any \( y \in VN(\mathcal{G}) \).

Proof : Note first that by a simple duality argument, we can suppose that \( 1 \leq p \leq 2 \) and that the functions \( \phi \) and \( \psi \) belong to the space \( C_c(G \times \mathcal{G}) \).

We start with the case where the operator \( T \) also induces a completely bounded map \( T : L^p(VN(\mathcal{G})) \to L^2(VN(\mathcal{G})) \). Then by Lemma 4.3 and Lemma 4.4, the elements \( W(K_\psi \otimes y)W^{-1} \) and \( (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}) \) belong to the Hilbert space \( L^2(B(L^2(\mathcal{G})) \otimes VN(\mathcal{G})) \). So the left-hand side of (4.31) is well-defined, and can be calculated with Parseval's formula and the orthonormal basis \((e_{ij} \otimes \lambda(e_k))_{i,j,k}.\) With Lemma 4.4, we get

\[ \langle (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle_{S_G^2(L^2(VN(\mathcal{G})))} \]

\[ = \sum_{i,j,k} \langle \text{Tr}_G \otimes \tau_G \rangle [(\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1})(e_{ij} \otimes \lambda(e_k))] \]

\[ \times \langle \text{Tr}_G \otimes \tau_G \rangle [W(K_\psi \otimes y)W^{-1}(e_{ij} \otimes \lambda(e_k))] \]

\[ = \sum_{i,j,k} \int_G \phi(s,t) \tau_G [\lambda_s x \lambda_{t-1} \lambda^*(\lambda(e_k))] \overline{e_i(s)e_j(t) \mu_G(s) d\mu_G(t)} \]

\[ \times \int_G \psi(s,t) \tau_G [\lambda_s y \lambda_{t-1} \lambda^*(\lambda(e_k))] d\mu_G(s) d\mu_G(t). \]

Since the functions \((s,t) \mapsto \tau_G [\lambda_s x \lambda_{t-1} \lambda^*(\lambda(e_k))] \phi(s,t)\) and \((s,t) \mapsto \tau_G [\lambda_s y \lambda_{t-1} \lambda^*(\lambda(e_k))] \psi(s,t)\) belong to the Hilbert space \( L^2(G \times \mathcal{G}) \), we can use the orthonormal basis \((\mathcal{E}_i \otimes e_j)_{i,j}\) of the space \( L^2(G \times \mathcal{G}) \) and reduce in the previous expression the sum over \( i,j \) and the integral over \( s,t \), then which becomes

\[ \sum_k \int_G \int_G \phi(s,t) \tau_G [\lambda_s x \lambda_{t-1} \lambda^*(\lambda(e_k))] \overline{e_i(s)e_j(t) \mu_G(s) d\mu_G(t)} \]

\[ \times \int_G \psi(s,t) \tau_G [\lambda_s y \lambda_{t-1} \lambda^*(\lambda(e_k))] d\mu_G(s) d\mu_G(t). \]

Recall that \( T^\dagger = (4.14) \) \( T^\dagger \) where \( T^\dagger \) is the hilbertian adjoint. We fix \( s, t \in \mathcal{G} \) and calculate using Parseval's identity in the fourth equality

\[ \sum_k \tau_G [\lambda_s x \lambda_{t-1} \lambda^*(\lambda(e_k))] \overline{\tau_G [\lambda_s y \lambda_{t-1} \lambda^*(\lambda(e_k))]} = \sum_k \langle \lambda_s x \lambda_{t-1} \lambda^*(\lambda(e_k)) \rangle L_2 \overline{\langle \lambda_s y \lambda_{t-1} \lambda^*(\lambda(e_k)) \rangle L_2} \]

\[ = \sum_k \langle \lambda_s x \lambda_{t-1} \lambda^*(\lambda(e_k)) \rangle L_2 \overline{\langle \lambda_s y \lambda_{t-1} \lambda^*(\lambda(e_k)) \rangle L_2} = \sum_k \langle T(\lambda_s x \lambda_{t-1}, \lambda(e_k)) \rangle \overline{\langle \lambda_s y \lambda_{t-1} \lambda^*(\lambda(e_k)) \rangle} \]

\[ = \langle T(\lambda_s x \lambda_{t-1}, \lambda_s y \lambda_{t-1}) \rangle L_2(VN(\mathcal{G})) = \tau_G [\lambda_s y^* \lambda_{s-1} T(\lambda_s x \lambda_{t-1})]. \]

Note that \( \|T(\lambda_s x \lambda_{t-1})\|_2 \leq C \) and \( \|\lambda_s y \lambda_{t-1}\|_2 \leq C \) justifies that we can integrate over \( s \) and \( t \).

Consequently, we obtain

\[ \langle (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle_{S_G^2(L^2(VN(\mathcal{G})))} \]

\[ = \int_{G \times \mathcal{G}} \tau_G [\lambda_s y^* \lambda_{s-1} T(\lambda_s x \lambda_{t-1})] \phi(s,t) \overline{\psi(s,t) d\mu_G(s) d\mu_G(t)} \]

\[ \overset{(4.2)}{=} \text{Tr}(K_{\phi,x,T}K_\psi) = \langle M_{\phi,x,T}(K_\phi), K_\psi \rangle_{S_G^2}. \]
Thus we have shown the formula

\[
(4.31) \quad \langle (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle_{S^p_G(L^2(VN(G)))} = \langle M_{\varphi_{x,y,T}}(K_\phi), K_\psi \rangle_{S^p_G},
\]

under the additional assumption that the linear map \( T \) defines a completely bounded operator \( L^p(VN(G)) \to L^2(VN(G)) \). Replacing both brackets, this formula translates to (4.30).

For the general case, we use the first part of Lemma 4.6 with the approximation sequence \((M_{\varphi_n})\) of Fourier multipliers \( M_{\varphi_n} : VN(G) \to VN(G) \). By a density argument, we can assume that \( y \in \lambda(C_c(G)) \). According to Lemma 4.6, the composition \( M_{\varphi_n}T : L^p(VN(G)) \to L^2(VN(G)) \) is completely bounded. So the first part of the proof applies to this operator. We obtain

\[
\langle (\text{Id} \otimes M_{\varphi_n}T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle = \langle M_{\varphi_n}(K_\phi), K_\psi \rangle_{S^p_G},
\]

with \( \varphi_n(s,t) \overset{\text{def}}{=} \tau_G(\lambda_s y^* \lambda_{x^{-1}}M_{\varphi_n}(\lambda_y \lambda_{t^{-1}})) \). By the approximation from Lemma 4.6, we have

\[
\langle (\text{Id} \otimes M_{\varphi_n}T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle
\]

\[= \langle (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}), (\text{Id} \otimes M_{\varphi_n}^*)(W(K_\psi \otimes y)W^{-1}) \rangle_{n \to \infty} \to \langle (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle.
\]

Again with Lemma 4.6, we have

\[
\varphi_n(s, t) = \langle M_{\varphi_n}T(\lambda_s x \lambda_{t^{-1}}), \lambda_y \lambda_{t^{-1}} \rangle
\]

\[= \langle T(\lambda_s x \lambda_{t^{-1}}), M_{\varphi_n}^*(\lambda_y \lambda_{t^{-1}}) \rangle_{n \to \infty} \to \langle T(\lambda_s x \lambda_{t^{-1}}), \lambda_y \lambda_{t^{-1}} \rangle^{(4.29)} = \varphi_{x,y,T}(s, t),
\]

pointwise in \( s, t \in G \). Even stronger, if \( s, t \) vary in given compacts, \( \varphi_n(s, t) = \varphi(s, t) \) for \( n \) sufficiently large. Since the functions \( \varphi \) and \( \psi \) are assumed to belong to the space \( C_c(G \times G) \), we obtain then \( \langle M_{\varphi_n}(K_\phi), K_\psi \rangle = \langle M_{\varphi}(K_\phi), K_\psi \rangle \) if \( n \) sufficiently large. In summary, we have established the formula

\[
\langle (\text{Id} \otimes T)(W(K_\phi \otimes x)W^{-1}), W(K_\psi \otimes y)W^{-1} \rangle = \langle M_{\varphi_{x,y,T}}(K_\phi), K_\psi \rangle.
\]

\[\]

**Lemma 4.8** Suppose that \( 1 \leq p \leq \infty \). Let \( T : L^p(VN(G)) \to L^p(VN(G)) \) be a completely bounded operator (weak* continuous if \( p = \infty \)). For any elements \( x \) and \( y \) in the space \( L^1(VN(G)) \cap VN(G) \), we have the estimate

\[
(4.32) \quad \| M_{\varphi_{x,y,T}} \|_{cb,S^p_G \to S^p_G} \leq \| T \|_{cb,L^p(VN(G)) \to L^p(VN(G))} \| x \|_{L^p(VN(G))} \| y \|_{L^p(VN(G))},
\]

with the usual convention if \( p = 1 \) or \( p = \infty \). If \( p = \infty \) (resp. \( p = 1 \)), we can also take \( x \in VN(G) \) (resp. \( y \in VN(G) \)). Moreover, if the linear map \( T \) is completely positive then the Schur multiplier \( M_{\varphi_{x,y,T}} \) is also completely positive.

**Proof**: By [Pis95, Definition 2.1], the duality [Pis98, Theorem 4.7 p. 49] and Plancherel formula

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39
(4.7), we have according to Proposition 4.7, 
\[
\|M_{\varphi_{x,y,T}}\|_{cb,S_G^p \to S_G^p} \\
\leq \sup \left\{ \sum_{ij} \langle (Id \otimes T)(W(K_{\phi_{ij}} \otimes x)W^{-1}), W(K_{\phi_{ij}} \otimes y)W^{-1} \rangle : \|K_{\phi_{ij}}\|_p, \right\} \\
\|\|K_{\phi_{ij}}\|_p, \leq 1 \right\} \\
\leq \|Id \otimes Id \otimes T\|_{B(S_p(S_G^p(L^p(VN(G)))))} \|W\|_\infty \|x\|_p \|W^{-1}\|_{\infty} \|W\|_{\infty} \|y\|_p, \|W^{-1}\|_{\infty}} \\
\leq \|T\|_{cb,L^p(VN(G))} \|x\|_{L^p(VN(G))} \|y\|_{L^p(VN(G))}.
\]

If the linear map \(T : L^p(VN(G)) \to L^p(VN(G))\) is completely positive (and weak* continuous if \(p = \infty\), then for any positive elements \([K_{\phi_{ij}}]\) and \([K_{\psi_{ij}}]\) we have
\[
\sum_{ij} \langle (Id \otimes T)(W(K_{\phi_{ij}} \otimes x)W^{-1}), W(K_{\phi_{ij}} \otimes y)W^{-1} \rangle \geq 0.
\]
Indeed, the map \(Id \otimes Id \otimes T\) preserves the positivity and \([W(K_{\phi_{ij}} \otimes x)W^{-1}]\) and \([W(K_{\psi_{ij}} \otimes y)W^{-1}]\) are positive. We infer by [ArK23, Lemma 2.6 p. 13] that \([M_{\varphi_{x,y,T}}K_{\phi_{ij}}]\) is positive, hence the map \(M_{\varphi_{x,y,T}}\) is completely positive. 

4.4 Step 2: the symbol of \(P_f(T)\) is Herz-Schur if \(G\) is inner amenable

In the following result, we show that if the group \(G\) is inner amenable, we are able to appear Herz-Schur multipliers. Recall that \(\varphi_{x,y,T}\) is defined in (4.29).

**Lemma 4.9** Let \(G\) be a second-countable unimodular inner amenable locally compact group. Let \(F\) be a finite subset of \(G\) and let \((V_f^F)_j\) be a sequence of subsets of \(G\) satisfying the last point of Theorem 3.7. Consider a weak* continuous completely bounded map \(T : VN(G) \to VN(G)\). With the notation (4.29), we let
\[
y_j^F \overset{\text{def}}{=} c_j^F |\lambda(1_{V_f^F})|^2 \quad \text{and} \quad \phi_{f,T}^F \overset{\text{def}}{=} \varphi_{1,y_j^F,T},
\]
where \(c_j^F > 0\) is the normalisation to have \(\|y_j^F\|_{L^1(VN(G))} = 1\). Then any weak* cluster point \(\phi_{f,T}^F\) of the sequence \((\phi_{f,T}^F)_j\) satisfies
\[
\phi_{f,T}^F(sr,tr) = \phi_{f,s,t}^F, \quad s,t \in G, \ r \in F.
\]
Moreover, any weak* cluster point of such \((\phi_{f,T}^F)_F\), where the finite subsets \(F\) of \(G\) are directed by inclusion, is also a Herz-Schur symbol.

**Proof** : For any \(s,t \in G\) and any \(r \in F\), we have
\[
\phi_{f,T}^F(sr,tr) - \phi_{f,s,t}^F = \phi_{1,y_j^F,T}(sr,tr) - \phi_{1,y_j^F,T}(s,t)
\]

\[= \tau_G(\lambda_{t r} y_j^F \lambda_{(t r)^{-1}} T(\lambda_{t r} \lambda_{(t r)^{-1}}) - \tau_G(\lambda_{t r} y_j^F \lambda_{(t r)^{-1}} T(\lambda_{t r} \lambda_{(t r)^{-1}}))
\]

\[= \tau_G(\lambda_t \lambda_{r} y_j^F \lambda_{r^{-1}} \lambda_{(r^{-1})} T(\lambda_{r^{-1}}) - \tau_G(\lambda_t \lambda_{r} y_j^F \lambda_{r^{-1}} \lambda_{(r^{-1})} T(\lambda_{r^{-1}}))
\]

\[= \tau_G(\lambda_t \lambda_{r} y_j^F \lambda_{r^{-1}} - y_j^F \lambda_{r^{-1}} T(\lambda_{r^{-1}}))
\]

40
If we can show that
\[ (4.34) \quad \| \lambda r y_j^{r_l} \lambda r - y_j \|_{L^1(VN(G))} \xrightarrow{j \to \infty} 0, \]
then we will obtain the pointwise convergence \( \phi_{j,T}^F(sr, tr) - \phi_{j,T}^F(s, t) \to 0 \) as \( j \to \infty \), for fixed \( s, t \in G \) and \( r \in F \). Since \( \phi_{j,T}^F(sr, tr) - \phi_{j,T}^F(s, t) \) is uniformly bounded in the Banach space \( L^\infty(G \times G) \), by dominated convergence, it follows that this sequence converges for the weak* topology to 0 in the space \( L^\infty(G \times G) \). Thus, if \( \phi_T^F \) is a cluster point of \( (\phi_{j,T}^F)_j \), it is easy to check by a \( \epsilon/2 \)-argument, using the weak* continuity of translations on \( L^\infty \), that \( \phi_T^F(sr, tr) = \phi_T^F(s, t) \) for any \( s, t \in G \) and \( r \in F \). It remains to show (4.34).

First, for any \( j \) we have
\[ (4.35) \quad (c_j^F)^{-1} = \| \lambda(1_{V_r})^2 \|_1 = \| \lambda(1_{V_r}) \|_{L^2(VN(G))}^2 = \| 1_{V_r} \|_{L^2(G)}^2 = \mu(V_r^F). \]

Now, observe by unimodularity in the second equality
\[ \| \lambda |(1_{V_r})^2| \lambda r - \lambda(1_{V_r}) | \|_1 \]
\[ = \| \lambda(1_{V_r} * 1_{V_r}) \lambda r - \lambda(1_{V_r} * 1_{V_r}) | \|_1 \]
\[ \leq \| \lambda(1_{V_r} * 1_{V_r}(r^{-1}(c)r) - 1_{V_r} * 1_{V_r}) \|_1 \]
\[ \leq \| \lambda(1_{V_r} * 1_{V_r} r^{-1}(c)r) - 1_{V_r} * 1_{V_r} r^{-1} - 1_{V_r} * 1_{V_r} \|_1 \]
\[ \leq \| \lambda(1_{V_r} * 1_{V_r} r^{-1} - 1_{V_r} * 1_{V_r} r^{-1}(c)r) \|_1 + \| \lambda(1_{V_r} * (1_{V_r} r^{-1} - 1_{V_r})) \|_1. \]

We estimate the second summand with unimodularity by
\[ \| 1_{V_r} \|_{L^2(G)} \| 1_{V_r} r^{-1} - 1_{V_r} \|_{L^2(G)} \]
\[ \leq \mu(V_r^F)^{1/2} \mu(V_r^F r^{-1} \Delta V_r^F)^{1/2}. \]

Now, we manipulate the first summand. By remplacing \( t \) by \( t r^{-1} \) and \( t \) by \( r^{-1} t \) in the fourth equality, we obtain
\[ (4.36) \quad 1_{V_r} * 1_{V_r} r^{-1}(s) - 1_{V_r} * 1_{V_r} r^{-1}(sr) \]
\[ \leq \int_G 1_{V_r}(t) 1_{V_r} r^{-1}(t-s) - 1_{V_r}(t) 1_{V_r} r^{-1}(t-s) \mu_G(t) \]
\[ = \int_G 1_{V_r}(t) 1_{V_r} r^{-1}(t-s) \mu_G(t) - \int_G 1_{V_r}(t) 1_{V_r} r^{-1}(t-s) \mu_G(t) \]
\[ = \int_G 1_{V_r}(t r^{-1}) 1_{V_r} r^{-1}(t-s) \mu_G(t) - \int_G 1_{V_r}(t r^{-1}) 1_{V_r} r^{-1}(t-s) \mu_G(t) \]
\[ = \int_G (1_{V_r} - 1_{V_r}) (t) 1_{V_r} r^{-1}(t-s) \mu_G(t) \]
\[ \leq \mu(V_r^F r^{-1} \Delta V_r^F)^{1/2} \mu(V_r^F r^{-1} \Delta V_r^F)^{1/2}. \]

Using the invariance of \( \mu_G \), we therefore obtain the following estimate for the first summand:
\[ \| \lambda(1_{V_r} * 1_{V_r} r^{-1} - 1_{V_r} * 1_{V_r} r^{-1}(c)r) \|_1 \]
\[ \leq \| 1_{V_r} \|_{L^2(G)} \| 1_{V_r} r^{-1} - 1_{V_r} \|_{L^2(G)} \]
\[ \leq \mu(V_r^F r^{-1} \Delta V_r^F)^{1/2} \mu(V_r^F r^{-1} \Delta V_r^F)^{1/2}. \]

Combining the two estimates, we obtain
\[ \| \lambda y_j \lambda r^{-1} - y_j \| \leq 2 r_f F \mu(V_r^F r^{-1}) \mu(V_r^F)^{1/2} \]
\[ \leq 2 \left[ \frac{\mu(V_r^F r^{-1})}{\mu(V_r^F)} \right]^{1/2} \mu(V_r^F)^{1/2} \]
\[ \xrightarrow{j \to \infty} 0. \]
according to the inner amenability assumption.

Now, for any finite subset $F$ of the group $G$, we fix a weak* cluster point $\phi_T^F$ of the net $(\phi_T^F)$. Let $\phi_T$ be a weak* cluster point of $(\phi_T^F)_F$. Then for any function $f \in L^1(G \times G)$ and any $r \in G$, the function $f(\cdot r^{-1}, \cdot r^{-1})$ belongs to the space $L^1(G \times G)$. Moreover, using unimodularity in the first and third steps and the first part of the proof valid for $F$ containing $\{r\}$ in the forth step, we obtain

\[
\langle \phi_T(\cdot r, \cdot r), f \rangle_{L^\infty(G \times G), L^1(G \times G)} = \langle \phi_T, f(\cdot r^{-1}, \cdot r^{-1}) \rangle_{L^\infty, L^1} = \lim_{F \to \infty} \langle \phi_T^F, f(\cdot r^{-1}, \cdot r^{-1}) \rangle_{L^\infty, L^1}.
\]

We deduce that the function $\phi_T$ is a Herz-Schur symbol.

\section{Step 2: the symbol of $P_j(T)$ for a Fourier multiplier $T$ if $p = \infty$ or $p = 1$}

We start with the case $p = \infty$. Let $T = M_\phi : \mathrm{VN}(G) \to \mathrm{VN}(G)$ be a completely bounded Fourier multiplier. If $x \in \mathrm{VN}(G)$ and $g \in L^1(\mathrm{VN}(G)) \cap \mathrm{VN}(G)$, recall that the symbol $\varphi_{x,y,T}$ is defined in (4.29).

**Lemma 4.10** Let $G$ be a second-countable unimodular locally compact group. Consider a completely bounded Fourier multiplier $T = M_\phi : \mathrm{VN}(G) \to \mathrm{VN}(G)$. Let $y$ be a positive element in the space $L^1(\mathrm{VN}(G)) \cap \mathrm{VN}(G)$ such that $\tau_G(y) = 1$. We have

\[(4.37) \varphi_{1,y,T}(s, t) = \phi(st^{-1}), \quad s, t \in G.\]

**Proof**: For any $s, t \in G$, we have

\[
\varphi_{1,y,T}(s, t) = \tau_G(y_\lambda^{-1} M_\phi(\lambda_s x \lambda_t^{-1})) = \phi(st^{-1}) \tau_G(y_\lambda^{-1} \lambda_s x \lambda_t^{-1})
\]

\[= \phi(st^{-1}) \tau_G(y) = \phi(st^{-1}).\]

\section{Example 4.11}

Let $g$ be a continuous function with compact support on $G$ with $\|g\|_{L^2(G)} = 1$.

With $y \overset{\text{def}}{=} (g^* \ast g)$, the assumptions of Lemma 4.10 are satisfied by (4.6) and since $\tau_G(y) = \tau_G(\lambda(g^*) \ast g(y)) = \|g\|^2_{L^2(G)} = 1$.

We continue with the case $p = 1$. We can prove the following similar result.

**Lemma 4.12** Let $G$ be a second-countable unimodular locally compact group. Consider a completely bounded Fourier multiplier $T = M_\phi : L^1(\mathrm{VN}(G)) \to L^1(\mathrm{VN}(G))$. Let $x$ be a positive element in the space $L^1(\mathrm{VN}(G)) \cap \mathrm{VN}(G)$ such that $\tau_G(x) = 1$. We have

\[(4.38) \varphi_{x,1,T}(s, t) = \phi(st^{-1}), \quad s, t \in G.\]

**Proof**: For any $s, t \in G$, we have

\[
\varphi_{x,1,T}(s, t) = \tau_G(\lambda_s x \lambda_t^{-1} M_\phi(\lambda_s x \lambda_t^{-1})) = \tau_G(\lambda_s x \lambda_t^{-1} M_\phi(\lambda_s x \lambda_t^{-1})))
\]

\[= \tau_G(M_\phi(\lambda_s x \lambda_t^{-1})) = \phi(st^{-1}) \tau_G(\lambda_s x \lambda_t^{-1})) = \phi(st^{-1}) \tau_G(x) = \phi(st^{-1}).\]
4.6 Step 2: convergence of the symbols for a multiplier $T$ with arbitrary symbol

We show that for a suitable choice of sequences of functions, we obtain the convergence of the desired Herz-Schur symbol.

**Proposition 4.13** Let $G$ be a second-countable unimodular locally compact group. Suppose $1 \leq p \leq \infty$. Consider some completely bounded Fourier multiplier $T = M_{\phi} : L^p(VN(G)) \to L^p(VN(G))$. Let $(f_j)$ and $(g_j)$ be nets of positive functions with compact support belonging to the space $C_c(G)$ such that if $x_j \overset{\text{def}}{=} \lambda(f_j)$, $y_j \overset{\text{def}}{=} \lambda(g_j)$ we have

- $\|x_j\|_{L^p(VN(G))} \|y_j\|_{L^p(VN(G))} \leq C$ for all $j$ for some positive constant $C$,
- $\tau_G(x_j y_j) = 1$ for all $j$,
- $\text{supp} f_j \to \{e\}$ or $\text{supp} g_j \to \{e\}$.

Moreover, let

$$
\phi_{j,T}(s,t) \overset{\text{def}}{=} \varphi_{x_j,y_j,T}(s,t) = \tau_G(\lambda(x_j) \lambda^{-1}(\lambda s \lambda^{-1}))(s,t) \in G.
$$

Then the sequence $(\varphi_{j,T})$ of elements in the space $L^\infty(G \times G)$ converges for the weak* topology to the function $\phi^{HS} : (s,t) \mapsto \phi(st^{-1})$.

**Proof**: For any $j$ and almost all $s,t \in G$, we have using a change of variables in the last equality

$$
\begin{align*}
\phi_{j,T}(s,t) &\overset{(4.39)}{=} \tau_G(y_j \lambda^{-1} \lambda s \lambda^{-1} \lambda t) = \tau_G(\lambda(x_j) \lambda^{-1} \lambda s \lambda^{-1} \lambda t) \\
&\overset{(4.29)}{=} \tau_G(\lambda(g_j) \lambda^{-1} \lambda s \lambda^{-1} \lambda t) = \tau_G(\lambda(t^{-1} \cdot s) M_{\phi}(\lambda(f_j) \lambda^{-1} \lambda t)) \\
&\overset{(4.7)}{=} \int_G g_j(t^{-1} u^{-1} s) \phi(u)f_j(s^{-1} u t) d\mu_G(u) = \int_G \phi(st^{-1}) g_j(u^{-1}) f_j(u) d\mu_G(u).
\end{align*}
$$

By Lemma 4.8 and (4.19), we deduce that $\|M_{\phi_{j,T}}\|_{\text{cb}, S^p_G \to S^p_G} \leq C$ for any $j$. Using the inequality (4.3), we see that the net $(\phi_{j,T})_j$ of functions is uniformly bounded in the Banach space $L^\infty(G \times G)$. Thus to check the claimed weak* convergence, it suffices by [Don98, Proposition 1.21 p. 8] to test against a function $h \in C_c(G \times G)$. We suppose that $\text{supp} f_j \to \{e\}$. Since

$$
\int_G g_j f_j d\mu_G \overset{(4.7)}{=} \tau(\lambda(g_j) \lambda(f_j)) = \tau(x_j y_j) = 1,
$$

we have, using unimodularity in a change of variables, and with the notation $K_j = \text{supp} h \cup \{(s,t) : \exists u \in \text{supp} f_j : (su^{-1},t) \in \text{supp} h\}$

$$
\begin{align*}
\int_{G \times G} (\phi_{j,T}(s,t) - \phi^{HS}(s,t)) h(s,t) d\mu_G(s) d\mu_G(t) &\overset{(4.40)}{=} \left| \int_G \int_G \left( \phi(su^{-1}) - \phi(st^{-1}) \right) g_j(u)f_j(u) h(s,t) d\mu_G(u) d\mu_G(s) d\mu_G(t) \right| \\
&= \left| \int_G \int_G \phi(st^{-1}) g_j(u)f_j(u) \left( h(su^{-1},t) - h(s,t) \right) d\mu_G(u) d\mu_G(s) d\mu_G(t) \right| \\
&\leq \sup \left\{ |h(su^{-1},t) - h(s,t)| : u \in \text{supp} f_j, (s,t) \in \text{supp} h, (su^{-1},t) \in \text{supp} h \right\} \\
&\cdot \int_{K_j} \int_G \left| \phi(st^{-1}) \right| f_j(u) g_j(u) d\mu_G(u) d\mu_G(s) d\mu_G(t) \\
&= \sup \left\{ |h(su^{-1},t) - h(s,t)| : u \in \text{supp} f_j, (s,t) \in \text{supp} h, (su^{-1},t) \in \text{supp} h \right\} \\
&\cdot \int_{K_j} |\phi(st^{-1})| d\mu_G(s) d\mu_G(t) \to 0,
\end{align*}
$$

43
since $K_j$ is contained in a fixed compact, so that the last integral is uniformly bounded in $j$, and $h$ was supposed to be continuous. We can use a similar reasoning if $\text{supp } g_j \to \{e\}$. ■

**Example 4.14** Let $G$ be a second-countable unimodular locally compact group. Consider some value $p \in (1, \infty)$ and assume that $\frac{1}{p} + 1$ is rational. That is, $p = \frac{1}{r} + 1$ is rational, which implies that both $\frac{1}{p}$ and $\frac{1}{p'}$ are also rational. Therefore, there exist integers $l, m, n \geq 1$ such that $\frac{1}{p} = \frac{m}{n}$ and $\frac{1}{p'} = \frac{l}{n}$. Consequently, $\frac{1}{p} = m$ and $\frac{1}{p'} = l$ are integers. Consider a sequence $(k_j)$ of positive functions belonging to the space $C_c(G)$ with $\text{supp } k_j \to \{e\}$. For each integer $j$, we define the function $h_j \overset{\text{def}}{=} k_j^* \ast k_j$. We can suppose that $\|\lambda(h_j)\|_{L^n(V_{\text{N}}(G))} = 1$. We let

\[
(4.41) \quad x_j \overset{\text{def}}{=} (\lambda(h_j))^m \quad \text{and} \quad y_j \overset{\text{def}}{=} (\lambda(h_j))^l.
\]

Note that by (4.6) these elements belong to $m_{r_G}$, as defined in (4.8). Then the sequences $(x_j)$ and $(y_j)$ satisfy the assumptions of Proposition 4.13. Indeed, the $x_j$ and $y_j$ are positive and we have

\[
\|x_j\|_p \overset{(4.41)}{=} \| (\lambda(h_j))^m \|_p = \| \lambda(h_j) \|^\frac{m}{l} \|_p = \| \lambda(h_j) \|_{\mathbb{P}}^\frac{m}{l} = 1
\]

and similarly

\[
\|y_j\|_{p'} \overset{(4.41)}{=} \| (\lambda(h_j))^l \|_{p'} = \| \lambda(h_j) \|^\frac{l}{m} \|_{p'} = \| \lambda(h_j) \|_{\mathbb{P}}^\frac{l}{m} = 1.
\]

Finally, we observe that

\[
\tau_G(x_jy_j) = \tau_G(\lambda(h_j)^m\lambda(h_j)^l) = \tau_G(\lambda(h_j)^n) = \|\lambda(h_j)\|^n_{\mathbb{P}} = 1.
\]

Note that these sequences depend on $p$.

**4.7 Step 2: totally disconnected and finite-dimensional groups**

In order to achieve a complementation that ensures the compatibility of the resulting projection $P^p_G$ for different values of $p$, one needs to select different sequences than those defined in Example 4.14. This will be achieved in Corollary 4.23 in the case where the locally compact group $G$ is finite-dimensional.

**Dimensions of topological spaces** Recall that three notions of dimension of a suitable topological space $X$ exist: the small inductive dimension, the large inductive dimension and the covering dimension. These dimensions are defined, for example, in [Eng89, Chapter 7]. Recall the definition of small inductive dimension. Let $X$ be a regular topological space. We say that $\text{ind } X = -1$ if $X$ is empty. If $n$ is a natural number, then we say that $\text{ind } X \leq n$ if for every point $x \in X$ and each neighborhood $V$ of $x$ in $X$ there exists a open set $U$ included in $V$ such that $x \in U$ and such that the boundary $\partial U$ satisfies $\text{ind } \partial U \leq n - 1$. We say that $\text{ind } X = n$ if $\text{ind } X \leq n$ and $\text{ind } X \leq n - 1$ does not hold. Finally, we say that $\text{ind } X = \infty$ if the inequality $\text{ind } X \leq n$ does not hold for any integer $n$. If two regular topological spaces $X$ and $Y$ are homeomorphic then $\text{ind } X = \text{ind } Y$. We refer to the book [Eng89] for more information.

By [Eng89, Theorem 7.3.3, p. 404], these notions coincide when $X$ is metrizable and separable. Note that a second-countable locally compact group $G$ satisfies this property\(^\text{17}\). Indeed, Arhangle’skii and Pasynkov showed in [Arh60] and [Pas60] that these notions coincide for an arbitrary locally compact group $G$. We refer to the survey [ArM18, p. 205] for more information.

\(^\text{17}\). Such a group is metrizable by [CoH16, Theorem 2.1.2 p. 20] and second-countable topological spaces are separable by [Eng80, Corollary 1.3.8 p. 25]. See also [CoH16, Theorem 2.1.10 p. 15], which presents a characterization of locally compact spaces which are second-countable.
Example 4.15 According to [Eng89, p. 360], a topological space $X$ is called zero-dimensional if it is a non-empty $T_1$-space with a basis of open-and-closed subsets. If $X$ is locally compact and paracompact, it is equivalent, by [Eng89, Theorem 6.2.10, p. 362], to say that $X$ is totally disconnected\(^{18}\), meaning it contains no connected subspace with more than one point. Furthermore, [Eng89, Theorem 7.1.12, p. 388] shows that this is also equivalent to $\text{ind } X = 0$. It is worth noting that every metrizable space is paracompact, as stated in [Eng89, Theorem 5.1.3, p. 300].

Example 4.16 By [Str06, Remark 39.5 (d) p. 283], a finite-dimensional locally compact group $G$ is a Lie group if and only if it is locally connected\(^{19}\). See also [Pon66, Theorem 70, p. 337] for the compact case.

We need background on local isomorphisms since we will use Iwasa’s local splitting theorem, which provides some local isomorphism.

Local isomorphisms Recall that two topological groups $G$ and $H$ are said to be locally isomorphic [Bou98, p. 224] if there exist open neighborhoods $V$ and $W$ of the identity elements $e_G$ and $e_H$ and a homeomorphism $f: V \rightarrow W$ satisfying $f(xy) = f(x)f(y)$ for all $x, y \in V$ such that $xy \in V$ and if $g$ is the mapping inverse to $f$, then for each pair of points $x', y' \in W$ such that $x'y' \in W$, we have $g(x'y') = g(x')g(y')$. We say that $f$ is a local isomorphism of $G$ with $H$.

The following result from [Bou04b, pp. 18-19] describes the relationship between Haar measures and local isomorphisms.

Lemma 4.17 Let $G$ and $G'$ be locally isomorphic locally compact groups via a local homeomorphism $f: V \rightarrow W$. Consider a left Haar measure $\mu_G$ of $G$ and its restriction $\mu_V^G$ on $V$. Then $f(\mu_V^G)$ is the restriction of a unique left Haar measure on $G'$. We caution the reader that the property of being unimodular is not preserved under local isomorphisms. For an example of a non-unimodular locally compact group $G$ that is locally isomorphic to the unimodular locally compact group $\mathbb{R}$, see [Bou04b, Exercise 5, VII.78].

Splitting theorem We will use the following form [Glu60, Theorem B p. 92] of Iwasa’s local splitting theorem. See also [Tao14, Exercise 1.6.8 p. 122] and [Pon66, Theorem 70 p. 337] for a version for the particular case of compact groups.

Theorem 4.18 Every second-countable finite-dimensional locally compact group is locally isomorphic to the product of a totally disconnected compact group and a connected Lie group.

Doubling metric measure spaces A Borel regular measure $\mu$ on a metric space $(X, \text{dist})$ is called a doubling measure [HKST15, p. 76] if every ball in $X$ has positive and finite measure and if there exists a constant $c \geq 1$ such that

\[(4.42) \quad \mu(B(x, 2r)) \leq c \mu(B(x, r)), \quad x \in X, \ r > 0.\]

Here $B(x, r) \overset{\text{def}}{=} \{y \in X : \text{dist}(x, y) < r\}$ is the open ball with radius $r$ centred at $x$. We call the triple $(X, \text{dist}, \mu)$ a doubling metric measure space if $\mu$ is a doubling measure on $X$. Such a space

\(^{18}\) In [Eng89, p. 360], the term “hereditarily disconnected” is used for this notion.

\(^{19}\) This result is stronger than [Tao14, Exercise 1.6.9 p. 122], which says without proof that a locally compact group $G$ is a Lie group if and only if it is first-countable, locally connected and finite-dimensional. Moreover, the notion of dimension of [Tao14, Exercise 1.6.9 p. 122] is different.
X is separable as a topological space by [HKST15, p. 76]. We refer to the paper [SoT19] for more information on the least doubling constant \( \inf\{c \in (4.42) : \mu \text{ doubling measure on } (X, \text{dist})\} \) of a metric space \((X, \text{dist})\).

We introduce and will use the weaker notion of «doubling measure for small balls» replacing the inequality (4.42) by

\[
(4.43) \quad \mu(B(x, 2r)) \leq c \mu(B(x, r)), \quad x \in X, \ r \in (0, \frac{1}{2}].
\]

**Carnot-Carathéodory distances** Consider a connected Lie group \( G \) equipped with a left Haar measure \( \mu_G \) and identity element \( e \). We consider a finite sequence \( X \overset{\text{def}}{=} (X_1, \ldots, X_m) \) of left invariant vector fields, the generated Lie algebra of which is the Lie algebra \( \mathfrak{g} \) of the Lie group \( G \) such that the vectors \( X_1(e), \ldots, X_m(e) \) are linearly independent. We say that it is a family of left invariant Hörmander vector fields. Let \( \gamma : [0, 1] \rightarrow G \) be an absolutely continuous path such that \( \dot{\gamma}(t) \) belongs to the subspace \( \text{span}\{X_1|_{\gamma(t)}, \ldots, X_m|_{\gamma(t)}\} \) for almost every \( t \in [0, 1] \). If \( \dot{\gamma}(t) = \sum_{k=1}^m \gamma_k(t) X_k|_{\gamma(t)} \) for almost every \( t \in [0, 1] \), where each \( \gamma_k \) is measurable, we can define the length of \( \gamma \) by

\[
\ell(\gamma) \overset{\text{def}}{=} \int_0^1 \left( \sum_{k=1}^m |\gamma_k(t)|^2 \right)^{1/2} dt,
\]

which belongs to \([0, \infty]\). For any \( s, s' \in G \), there exists such a path \( \gamma : [0, 1] \rightarrow G \) with finite length such that \( \gamma(0) = s \) and \( \gamma(1) = s' \). If \( s, s' \in G \) then we define the Carnot-Carathéodory distance

\[
(4.44) \quad \text{dist}_{CC}(s, s') \overset{\text{def}}{=} \inf_{\gamma(0)=s,\gamma(1)=s'} \ell(\gamma)
\]

between \( s \) and \( s' \) to be the infimum of the length of all such paths with \( \gamma(0) = s \) and \( \gamma(1) = s' \). Then it is known that \( \text{dist}_{CC} \) is a left invariant distance on \( G \), inducing the same topology as the one of \( G \), see [VSCC92, Proposition III.4.1 p. 39] and [DrER03, pp. 22-23].

By [VSCC92, p. 124] there exist \( c_1, c_2 > 0 \) and \( d \in \mathbb{N} \) such that for all \( r \in (0, 1] \) we have

\[
(4.45) \quad c_1 r^d \leq \mu_G(B(e, r)) \leq c_2 r^d.
\]

The integer \( d \) is called the local dimension of \((G, X)\). We infer that there exists \( c > 0 \) such that \((4.43) \) is satisfied, i.e. \( \mu_G \) is a doubling measure for small balls. By [BEM13, Proposition 2.4 p. 199], the metric measure space \((G, \text{dist}_{CC}, \mu_G)\) is a doubling metric measure space if and only if the Lie group \( G \) has polynomial growth. Recall finally that the connected component of a Lie group is second-countable by [HiN12, Proposition 9.1.15 p. 293].

**Construction of some neighborhoods** We start with a technical result.

**Lemma 4.19** Let \( G \) be a second-countable locally compact group equipped with a left invariant distance \( \text{dist} \) and a doubling left Haar measure \( \mu_G \) for small balls. There exists a sequence \( \{B_j\} \) of open balls \( B_j \overset{\text{def}}{=} B(e, r_j) \) with decreasing radius \( r_j \rightarrow 0 \) satisfying

\[
(4.46) \quad \mu_G(B_j)^3 \leq c^3 \int_{B_j} \mu_G(B_j \cap sB_j)^2 d\mu_G(s), \quad j \geq 1,
\]

where \( c \) is a constant satisfying (4.43).
Proof: Assume that $0 < \varepsilon < 1$. For any integer $j \geq 1$, we introduce the ball $B_j \overset{\text{def}}{=} B(e, \frac{\varepsilon}{j})$. We have

$$\mu_G(B_j) = \mu_G(B(e, \frac{\varepsilon}{j})) \overset{(4.44)}{\leq} c \mu_G(B(e, \frac{\varepsilon}{2j})) = c \mu_G(B_{2j}).$$

For any element $s$ in the open ball $B_{2j} = B(e, \frac{\varepsilon}{2j})$, we will show that

$$B(e, \frac{\varepsilon}{2j}) \subset B(e, \frac{\varepsilon}{j}) \cap sB(e, \frac{\varepsilon}{j}), \quad \text{i.e.} \quad B_{2j} \subset B_j \cap sB_j.$$

Indeed, if $r \in G$ satisfies $\text{dist}(e, r) < \frac{\varepsilon}{2j}$, we have obviously $r \in B(e, \frac{\varepsilon}{j})$ and using left invariance of the distance, we obtain

$$\text{dist}(e, s^{-1}r) = \text{dist}(s, r) \leq \text{dist}(s, e) + \text{dist}(e, r) \leq \frac{\varepsilon}{2j} + \frac{\varepsilon}{2j} = \frac{\varepsilon}{j}.$$ So $s^{-1}r \in B(e, \frac{\varepsilon}{j})$ and consequently $r \in sB(e, \frac{\varepsilon}{j})$. So the claim (4.48) is proved. For any integer $j \geq 1$, we deduce that

$$c^3 \int_{B_{2j}} \mu_G(B_j \cap sB_j)^2 \, d\mu_G(s) \geq c^3 \int_{B_{2j}} \mu_G(B_j \cap sB_j)^2 \, d\mu_G(s) \overset{(4.48)}{=} c^3 \int_{B_{2j}} \mu_G(B_{2j})^2 \, d\mu_G(s) = c^3 \mu_G(B_{2j})^3 \overset{(4.47)}{\geq} \mu_G(B_j)^3.$$

We continue by proving another technical result for totally disconnected groups.

Lemma 4.20 Let $G$ be a second-countable totally disconnected locally compact group equipped with a left Haar measure $\mu_G$. Then there exists a basis $(K_j)$ of symmetric open compact neighborhoods $K_j$ of $e$ such that

$$\mu_G(K_j)^3 = \int_{K_j} \mu_G(K_j \cap sK_j)^2 \, d\mu_G(s), \quad j \geq 1.$$

Proof: According to Van Dantzig’s theorem [HeR79, (7.7) Theorem p. 62] or [CoH16, Theorem 2.6.4 p. 44], $G$ admits a basis $(K_j)$ of open compact subgroups. Clearly, each $K_j$ is symmetric, being a group, is a neighborhood of $e$, being an open subset, and of finite measure due to its compactness. Since we have assumed $G$ to be second-countable, it follows from the proof of [CoH16, Theorem 2.6.4 p. 44] that the basis can be chosen as a sequence. Since $K_j$ is a subgroup, we have $K_j \cap sK_j = \emptyset$ for $s \notin K_j$ and $K_j \cap sK_j = K_j$ for $s \in K_j$. Thus,

$$\int_{K_j} \mu_G(K_j \cap sK_j)^2 \, d\mu_G(s) = \int_{K_j} \mu_G(K_j)^2 \, d\mu_G(s) = \mu_G(K_j)^3.$$ Using a version of Iwasawa’s local splitting theorem, we are now able to obtain a result for finite-dimensional locally compact groups.

Lemma 4.21 Let $G$ be a second-countable finite-dimensional locally compact group equipped with a left Haar measure $\mu_G$. Then there exists a basis $(V_j)$ of symmetric open neighborhoods $V_j$ of $e$ and a constant $c > 0$ such that

$$\mu_G(V_j)^3 \leq c^3 \int_{V_j} \mu_G(V_j \cap sV_j)^2 \, d\mu_G(s), \quad j \geq 1.$$
**Proof:** We denote by $n$ the dimension of $G$. According to Theorem 4.18, $G$ is locally isomorphic to the product of a totally disconnected compact group $K$ and a Lie group $L$ of dimension $n$. So there exists a neighborhood $V$ of $e_G$ which is homeomorphic to the direct product $W \times U$ of a neighborhood $W$ of the neutral element $e_K$ and an open neighborhood $U$ of the neutral element $e_L$. We identify $V$ with $W \times U$. By Lemma 4.17, we can choose left Haar measures $\mu_K$ and $\mu_L$ on the groups $K$ and $L$ such that

$$\mu_G(A \times B) = \mu_K(A)\mu_L(B), \quad A \subset W, \; B \subset U. \tag{4.51}$$

Next, we consider the left invariant metric on the connected Lie group $L$ given by the Carnot-Carathéodory distance (4.44) with respect to some fixed sequence of left invariant vector fields. We consider a neighborhood basis sequence $(K_j)$ of open compact subgroups of $K$ whose existence is guaranteed by Van Dantzig’s theorem, see the proof of Lemma 4.20. Furthermore, we let $(B_j)$ be a sequence as in Lemma 4.19 for the Lie group $L$. For any integer $j \geq 1$, we put

$$V_j \overset{\text{def}}{=} K_j \times B_j \tag{4.52}$$

and we can suppose that $K_j \subset W$ and that $B_j \subset U$ for any integer $j \geq 1$. Recall that each $K_j$ resp. each ball $B_j$ is symmetric, being a subgroup resp. a ball with respect to a left invariant metric. We conclude that $V_j$ is symmetric as well.

In view of the previous local product structure of the Haar measure described in (4.51), we deduce that for any integer $j$

$$c^3 \int_{V_j} \mu_G(V_j \cap sV_j)^2 \, d\mu_G(s) \overset{(4.52)}{=} c^3 \int_{K_j \times B_j} \mu_G(V_j \cap sV_j)^2 \, d\mu_K \times \mu_L(s) \tag{4.53}$$

$$\overset{(4.54)}{=} c^3 \int_{K_j \times B_j} \mu_G \left( (K_j \times B_j) \cap (r, t) (K_j \times B_j) \right) ^2 \, d\mu_K(r) \, d\mu_L(t) \tag{4.55}$$

$$\overset{(4.56)}{=} c^3 \int_{K_j \times B_j} \mu_K(K_j \cap rK_j)^2 \mu_L(B_j \cap tB_j)^2 \, d\mu_K(r) \, d\mu_L(t) \tag{4.57}$$

$$\overset{(4.58)}{=} c^3 \int_{K_j} \mu_K(K_j \cap rK_j)^2 \, d\mu_K(r) \int_{B_j} \mu_L(B_j \cap tB_j)^2 \, d\mu_L(t) \tag{4.59}$$

$$\overset{(4.60)}{=} \mu_K(K_j)^3 \mu_G(B_j)^3 \overset{(4.51)}{=} \mu_G(K_j \times B_j)^3 \overset{(4.52)}{=} \mu_G(V_j)^3. \tag{4.61}$$

Now, we show the interest of the previous lemmas.

**Proposition 4.22** Let $G$ be a second-countable unimodular locally compact group. Suppose that $1 \leq p \leq \infty$. Let $(V_j)$ be a basis of symmetric neighborhoods of $e$ and a constant $c > 0$ such that

$$\mu_G(V_j)^3 \leq c^3 \int_{V_j} \mu_G(V_j \cap sV_j)^2 \, d\mu_G(s), \quad j \geq 1. \tag{4.53}$$

Moreover, we put

$$f_j \overset{\text{def}}{=} 1_{V_j} \ast 1_{V_j}, \quad x_j \overset{\text{def}}{=} a_j \lambda(f_j) \quad \text{and} \quad y_j \overset{\text{def}}{=} b_j \lambda(f_j) \tag{4.54}$$

Note that $\text{dist}_{CC}(e, s) = \text{dist}_{CC}(s^{-1}, e)$. 

48
with
\[(4.55) \quad a_j \overset{\text{def}}{=} \| \lambda(f_j) \|_{p^*} \| \lambda(f_j) \|_2^2 \quad \text{and} \quad b_j \overset{\text{def}}{=} \| \lambda(f_j) \|_{p^*}^{-1}.
\]

Then the sequences \((x_j)\) and \((y_j)\) satisfy the assumptions from Proposition 4.13. More precisely, we have \(\| x_j \|_p \leq c^3\) and \(\| y_j \|_{p^*} = 1\).

**Proof:** We denote by \(\mu_G\) a Haar measure on the group \(G\). Consider any measurable subset \(V\) of \(G\) of measure \(\mu_G(V) \in (0, \infty)\). Recall the isometric complex interpolation formula \(L^p(V\mathcal{N}(G)) = (L^\infty(V\mathcal{N}(G)), L^2(V\mathcal{N}(G)))^{1 \over 2}_p\) of [PiX03, (2.1) p. 1466]. In particular, we have by [Lun18, Corollary 2.8 p. 53], the inequality \(\| \cdot \|_{L^p(V\mathcal{N}(G))} \leq \| \cdot \|_{L^1(V\mathcal{N}(G))}^{1 \over 2} \| \cdot \|_{L^2(V\mathcal{N}(G))}^{1 \over 2}\). Using this inequality and Young’s inequality [HeR79, Corollary 20.14 p. 293] in the second inequality, we obtain
\[
\| \lambda(1_{V^{-1}} * 1_V) \|_p \overset{(4.6)}{=} \| \lambda(1_{V^{-1}}) \lambda(1_V) \|_p = \| \lambda(1_{V^{-1}}) \|_p^2 \leq \| \lambda(1_V) \|_p^2 \leq \| \lambda(1_V) \|_\infty^{2 \over 2} \| \lambda(1_V) \|_2^2 = \| \lambda(1_V) \|_2 \| \lambda(1_{V^{-1}}) \|_2 \leq \mu_G(V)^{1 \over 2} \cdot \mu_G(V)^{1 \over 2} = \mu_G(V)^{1 \over 2} \cdot \mu_G(V)^{1 \over 2} = \mu_G(V)^{1 \over 2} \cdot \mu_G(V)^{1 \over 2}.
\]

Taking \(V = V_j\) and using the previous calculation also for \(p^*\) in place of \(p\), we deduce with Lemma 4.21 that
\[(4.56) \quad \| \lambda(1_{V_j} * 1_{V_j}) \|_p \cdot \| \lambda(1_{V_j} * 1_{V_j}) \|_{p^*} \leq \mu_G(V_j)^{1 \over 2} \cdot \mu_G(V_j)^{1 \over 2} = \mu_G(V_j)^{1 \over 2} \cdot \mu_G(V_j)^{1 \over 2} = \mu_G(V_j)^{1 \over 2} \cdot \mu_G(V_j)^{1 \over 2} = \mu_G(V_j)^{1 \over 2} \cdot \mu_G(V_j)^{1 \over 2}.
\]

On the other hand, using \(V_j = V_j^{-1}\), we obtain
\[
\| \lambda(1_{V_j} * 1_{V_j}) \|_2 \overset{(3.6)}{=} \int_G \left| \int_G 1_{V_j}(t)1_{V_j}(t^{-1}s) d\mu_G(t) \right|^2 d\mu_G(s)
\]
\[
= \int_G \left| \int_G 1_{V_j}(t)1_{V_j}(t^{-1}s) d\mu_G(t) \right|^2 d\mu_G(s) = \int_G \left| \int_G 1_{V_j}(t)1_{sV_j}(t) d\mu_G(t) \right|^2 d\mu_G(s)
\]
\[(4.57) \quad = \int_G \mu_G(V_j \cap sV_j)^2 d\mu_G(s).
\]

Combining (4.56) and (4.57), we see that
\[(4.58) \quad \| \lambda(1_{V_j} * 1_{V_j}) \|_p \cdot \| \lambda(1_{V_j} * 1_{V_j}) \|_{p^*} \leq c^3 \| \lambda(1_{V_j} * 1_{V_j}) \|_2^2.
\]

Note that with the choice of \(a_j\) and \(b_j\), we finally obtain
\[
\| x_j \|_p \overset{(4.54)}{=} \| a_j \lambda(f_j) \|_p \overset{(4.55)}{=} \| \lambda(f_j) \|_p \cdot \| \lambda(f_j) \|_2^{-2} \| \lambda(f_j) \|_p \overset{(4.54)(4.58)}{=} c^3
\]
and \(\| y_j \|_{p^*} \overset{(4.54)}{=} \| b_j \lambda(f_j) \|_{p^*} = 1\), as well as \(\tau_G(x_j y_j) \overset{(4.55)}{=} \| \lambda(f_j) \|_2^{-2} \tau_G(\lambda(f_j)^2) = 1\).

A combination of Proposition 4.22 and the previous lemmas gives the next result, which is the main result of this section.

**Corollary 4.23** 1. Let \(G\) be a second-countable finite-dimensional unimodular locally compact group equipped with a Haar measure \(\mu_G\). Then there exist sequences \((x_j)\) and \((y_j)\) satisfying the assumptions from Proposition 4.13. More precisely, we have \(\| x_j \|_p \leq c^3\) and \(\| y_j \|_{p^*} = 1\) for any integer \(j\) for some constant \(c > 0\).
2. Let \( G \) be a second-countable totally disconnected unimodular locally compact group equipped with a Haar measure \( \mu_G \). Then there exist sequences \( \{x_j\} \) and \( \{y_j\} \) satisfying the assumptions from Proposition 4.13 with \( \|x_j\|_p \leq 1 \) and \( \|y_j\|_{p^*} = 1 \) for any integer \( j \).

4.8 Step 3: the projection \( Q : M_G^{p,\text{cb}} \to M_G^{p,\text{cb},\text{HS}} \) on the space of Herz–
Schur multipliers

The first part of the following result says that the unit ball of the space \( M_G^p \) of measurable Schur multipliers is closed for the weak* topology of the dual Banach space \( L_\infty(\Omega \times \Omega) \).

**Lemma 4.24** Let \( \Omega \) be a \( \sigma \)-finite measure space. Suppose that \( 1 \leq p \leq \infty \). Let \( \{M_{\phi_j}\} \) be a bounded net of bounded Schur multipliers on the Schatten class \( S^p_\Omega \) and suppose that \( \phi \) is an element in \( L_\infty(\Omega \times \Omega) \) such that the net \( \phi_j \) converges to \( \phi \) for the weak* topology of \( L_\infty(\Omega \times \Omega) \). Then the function \( \phi \) induces a bounded Schur multiplier on \( S^p_\Omega \). Moreover, the net \( \{M_{\phi_j}\} \) converges to the operator \( M_{\phi} \) for the weak operator topology of the space \( \mathcal{B}(S^p_\Omega) \) (point weak* topology if \( p = 1 \)) and

\[
\|M_{\phi}\|_{S^p_{\Omega} \to S^p_{\Omega}} \leq \liminf_{j \to \infty} \|M_{\phi_j}\|_{S^p_{\Omega} \to S^p_{\Omega}}.
\]

A similar statement is true upon replacing «bounded» by «completely bounded» and the norm \( \|\cdot\|_{S^p_{\Omega} \to S^p_{\Omega}} \) by the norm \( \|\cdot\|_{\text{cb},S^p_{\Omega} \to S^p_{\Omega}} \).

**Proof**: Consider some functions \( f, g \in L^2(\Omega \times \Omega) \) such that \( K_f \in S^p_\Omega \) and \( K_g \in S^p_{\Omega^*} \). Note that we have \( fg \in L^1(\Omega \times \Omega) \). For any \( j \), we have

\[
\left| \int_{\Omega \times \Omega} \phi_j f \bar{g} \right| \overset{(4.2)}{=} \left| \langle M_{\phi_j}(K_f), K_g \rangle_{S^p_{\Omega} \to S^p_{\Omega^*}} \right| \leq \|M_{\phi_j}(K_f)\|_{S^p_{\Omega}} \|K_g\|_{S^p_{\Omega^*}}
\]

\[
\leq \|M_{\phi_j}\|_{S^p_{\Omega} \to S^p_{\Omega}} \|K_f\|_{S^p_{\Omega}} \|K_g\|_{S^p_{\Omega^*}}.
\]

Passing to the limit, we obtain

\[
\left| \langle K_f, K_g \rangle_{S^p_{\Omega} \to S^p_{\Omega^*}} \right| \overset{(4.2)}{=} \left| \int_{\Omega \times \Omega} \phi f \bar{g} \right| \leq \liminf_{j \to \infty} \|M_{\phi_j}\|_{S^p_{\Omega} \to S^p_{\Omega}} \|K_f\|_{S^p_{\Omega}} \|K_g\|_{S^p_{\Omega^*}}.
\]

By density, we conclude that the function \( \phi \) induces a bounded Schur multiplier on \( S^p_\Omega \) with the estimate (4.59) on the norm of this operator. Using again the weak* convergence of the net \( \{\phi_j\} \), we see that for any functions \( f, g \in L^2(\Omega \times \Omega) \) such that \( K_f \in S^p_\Omega \) and \( K_g \in S^p_{\Omega^*} \)

\[
\text{Tr} \left( (M_{\phi} - M_{\phi_j})(K_f)K_g \right) = \text{Tr} \left( K_{(\phi - \phi_j)f}K_g \right) \overset{(4.2)}{=} \int_{\Omega \times \Omega} (\phi - \phi_j)f \bar{g} \to 0.
\]

By density, using an \( \frac{2}{p} \)-argument and the boundedness of the net, we conclude\(^{21}\) that the net \( \{M_{\phi_j}\} \) converges to the operator \( M_{\phi} \) for the weak operator topology of \( \mathcal{B}(S^p_\Omega) \) (point weak* topology if \( p = 1 \)).

\(^{21}\) More precisely, if \( X \) is a Banach space, if \( E_1 \) is dense subset of \( X \), if \( E_2 \) is a dense subset of \( X^* \) and if \( \{T_j\} \) is a bounded net of \( \mathcal{B}(X) \) with an element \( T \in \mathcal{B}(X) \) such that \( \langle T_j(x), x^* \rangle \to (T(x), x^*) \) for any \( x \in E_1 \) and any \( x^* \in E_2 \), then the net \( \{T_j\} \) converges to \( T \) for the weak operator topology of the space \( \mathcal{B}(X) \).
By Banach-Alaoglu’s theorem, there exists a subnet of \((S_\phi^p)_n\) such that \(\lim_{n \to \infty} \phi_n \to \phi\) in the weak* topology. The second point is also similar.

Passing to the limit, we obtain
\[
\left| \sum_{k,l=1}^N \int_{\Omega \times \Omega} \phi_j f_{k,l} \hat{g}_{k,l} \right| \leq \lim \inf_{j \to \infty} \| M_{\phi_j} \|_{cb, S_\phi^p \to S_\phi^p} \| [K_{f_{k,l}}] \|_{M_{\phi_j}(S_\phi^p)} \| [K_{g_{k,l}}] \|_{S_\phi^p(S_\phi^p)}.
\]

We deduce that the function \(\phi\) induces a completely bounded Schur multiplier on the Schatten space \(S_\phi^p\) with the suitable estimate on the completely bounded norm.

If \(1 \leq p < \infty\), note that the Schatten space \(S_\phi^p\) is a dual Banach space. So the Banach space \(CB(S_\phi^p)\) is also a dual space with predual \(S_\phi^p \otimes S_\phi^p\), where \(\otimes\) denotes the operator space projective tensor product and the duality bracket is given by
\[
\langle T(x), y \rangle_{S_\phi^p \otimes S_\phi^p} = \langle T(x \otimes y) \rangle_{S_\phi^p \otimes S_\phi^p}.
\]

**Lemma 4.25** Let \(\Omega\) be a \(\sigma\)-finite measure space.

1. Let \(1 \leq p < \infty\). Then the space \(M_{\phi_j}^{p, cb}(\Omega)\) of completely bounded Schur multipliers is weak* closed in \(CB(S_\phi^p)\) and the space \(M_\phi^{p, cb}(\Omega)\) of bounded Schur multipliers is weak* closed in \(B(S_\phi^p(\Omega))\).

2. The space \(M^{p, cb}_{\phi_j}(\Omega) = M_{\phi_j}(\Omega)\) of (completely) bounded Schur multipliers is weak* closed in the space \(CB(S_\phi^p(\Omega), B(L^2(\Omega)))\).

**Proof**: We start by proving the first assertion. By the Banach-Dieudonné theorem [Hol75, p. 154], it suffices to show that the closed unit ball of the space \(M_{\phi_j}^{p, cb}(\Omega)\) is weak* closed in the space \(CB(S_\phi^p(\Omega))\). Let \((M_{\phi_j})_j\) be a net in that unit ball converging for the weak* topology to some completely bounded map \(T: S_\phi^p \to S_\phi^p\). We have for any \(j\) the inequality
\[
\| M_{\phi_j} \|_{cb, S_\phi^p \to S_\phi^p} \leq 1.
\]

By Banach-Alaoglu’s theorem, there exists a subnet of \((\phi_j)\) converging for the weak* topology to some function \(\phi \in L^\infty(\Omega \times \Omega)\). It remains to show that \(T = M_{\phi}\). By (4.60), we have \(\langle M_{\phi_j}, x \rangle \to \langle T(x), y \rangle\) for any \(x \in S_\phi^p\) and any \(y \in S_\phi^p\). That means that the net \((M_{\phi_j})_j\) converges to \(T\) for the weak operator topology (point weak* topology if \(p = 1\)). By Lemma 4.24, the net \((M_{\phi_j})_j\) converges to \(M_{\phi}\). We conclude by uniqueness of the limit that \(T = M_{\phi}\).

The statement on the space \(M^{p, cb}_{\phi_j}(\Omega)\) can be proved in a similar manner, using the predual \(S_\phi^p \otimes S_\phi^p\) of the dual Banach space \(B(S_\phi^p)\), where \(\otimes\) denotes the Banach space projective tensor product. The second point is also similar.

The following is essentially folklore. The case \(p = \infty\) is explicitly proved in [SpT02, Proposition 5.2 p. 375] and [Spr04, Corollary 5.4 p. 183] with a slightly different method relying on the use of an invariant mean. We sketch a proof since it is important for us.
Proposition 4.26 Let \( G \) be an amenable unimodular locally compact group. Suppose that \( 1 \leq p \leq \infty \). Then there exists a contractive projection \( Q : \mathfrak{M}^{p,cb}_G \to \mathfrak{M}^{p,cb}_G \) onto the space \( \mathfrak{M}^{p,cb}_G \) of completely bounded Herz-Schur multipliers acting on the Schatten space \( S^p_G \) (\( B(L^2(G)) \)) if \( p = \infty \), preserving the complete positivity. Moreover, the obtained projections are compatible for all different values of \( 1 \leq p \leq \infty \).

**Proof**: Let \( (F_j) \) be a Følner net in \( G \) provided by the amenability of the group \( G \). For any \( K_f \in S^2_G \cap S^p_G \), the map \( G \to S^p_G, r \mapsto \text{Ad}(\rho_r)(K_f) \) is continuous since the composition of operators is strongly continuous on bounded sets by [EFHN15, Proposition C.19 p. 517] (recall the notation \( \text{Ad}(\rho_r)(x) = \rho_r x \rho_r^{-1} \)). Similarly for any \( M_\phi \in \mathfrak{M}^{cb}_G \), the map \( G \to S^p_G, r \mapsto [\text{Ad}(\rho_r^*)M_\phi \text{Ad}(\rho_r)](K_f) \) is also continuous, hence Bochner integrable on the compact \( F_j \).

Now, for any \( K_f \in S^2_G \cap S^p_G \) and any \( M_\phi \in \mathfrak{M}^{cb}_G \), put

\[
Q_f(M_\phi)(K_f) = \frac{1}{\mu_G(F_j)} \int_{F_j} [\text{Ad}(\rho_r^*)M_\phi \text{Ad}(\rho_r)](K_f) \, d\mu_G(r).
\]

For any \( K_f \in S^2_G \cap S^p_G \) and any completely bounded Schur multiplier \( M_\phi : S^p_G \to S^p_G \), we have

\[
\|Q_f(M_\phi)(K_f)\|_{S^p_G} = \frac{1}{\mu_G(F_j)} \left\| \int_{F_j} [\text{Ad}(\rho_r^*)M_\phi \text{Ad}(\rho_r)](K_f) \, d\mu_G(r) \right\|_{S^p_G} \\
\leq \frac{1}{\mu_G(F_j)} \int_{F_j} \| [\text{Ad}(\rho_r^*)M_\phi \text{Ad}(\rho_r)](K_f) \|_{S^p_G} \, d\mu_G(r) \\
\leq \|M_\phi\|_{S^p \to S^p} \|K_f\|_{S^p_G}.
\]

A similar argument shows that \( \|Q_f(M_\phi)\|_{cb,S^p \to S^p_G} \leq \|M_\phi\|_{cb,S^p \to S^p_G} \). Consequently, we have a well-defined contractive map \( Q_f : \mathfrak{M}^{p,cb}_G \to \text{CB}(S^p_G), M_\phi \mapsto \frac{1}{\mu_G(F_j)} \int_{F_j} [\text{Ad}(\rho_r^*)M_\phi \text{Ad}(\rho_r)] \, d\mu_G(r) \).

If the linear map \( M_\phi \) is completely positive then observe that the map \( [\text{Ad}(\rho_r^*)M_\phi \text{Ad}(\rho_r)] \) is also completely positive. Thus the map \( Q_f \) preserves the complete positivity. It is easy to check that \( Q_f(M_\phi) \) is a Schur multiplier with symbol

\[
\phi_f(s,t) = \frac{1}{\mu_G(F_j)} \int_{F_j} \phi(s,r) \, d\mu_G(r)
\]

(Gelfand integral in \( L^\infty(G \times G) \)).

We continue with the case \( 1 \leq p < \infty \). Since the space \( \mathfrak{M}^{p,cb}_G \) is weak*-closed in \( \text{CB}(S^p_R) \), the space \( \mathfrak{M}^{p,cb}_G \) is a dual Banach space. Hence \( B(\mathfrak{M}^{p,cb}_G, \mathfrak{M}^{p,cb}_G) \) is a dual space. By Banach-Alaoglu’s theorem, the uniformly bounded net \( (Q_f) \) admits a weak* accumulation point that we denote by \( Q \) which is obviously a contraction. So we can suppose that \( Q_f \to Q \) for the weak* topology. So, for each completely bounded Schur multiplier \( M_\phi : S^p_G \to S^p_G \) this implies that \( Q(M_\phi) = \lim_f Q_f(M_\phi) \) in the weak operator topology. Recall that the weak* topology on \( \text{CB}(S^p_G) \) coincides on bounded subsets with the point weak* topology. Since \( Q_f(M_\phi) \) belongs to the space \( \mathfrak{M}^{p,cb}_G \) and since the latter space is weak*-closed in \( \text{CB}(S^p_G) \) according to Lemma 4.25, we obtain that \( Q(M_\phi) \) also belongs to the space \( \mathfrak{M}^{p,cb}_G \). Since each map \( Q_f \) preserves complete positivity, by [ArK23, Lemma 2.10 2, p. 15], the map \( Q \) also preserves complete positivity.

For any completely bounded Schur multiplier \( M_\phi : S^p_G \to S^p_G \), it remains to show that \( Q(M_\phi) \) is in addition a Herz-Schur multiplier. That is, for any \( r_0 \in G \) we have to show that \( \lim_f M_{\phi_j(sr_0, tr_0)} = \lim_f M_{\phi_j(s,t)} \). Fix some \( r_0 \in G \) and some \( j \). Using the Følner condition in
the last line, we have

\[
\left\| M_{\phi}(x,t) - M_{\phi(r(\nu,r_0)\cdot)} \right\|_{cb,S_G^p \rightarrow S_G^q} \\
\overset{(4.62)}{=} \left\| \frac{1}{\mu_G(F_j)} \int_{F_j} M_{\phi(\nu,r)} \, d\mu_G(r) - \frac{1}{\mu_G(F_j)} \int_{F_j} M_{\phi(\nu,r_0,t) \cdot r_0} \, d\mu_G(u) \right\|_{cb,S_G^p \rightarrow S_G^q} \\
= \frac{1}{\mu_G(F_j)} \left\| \int_{F_j} M_{\phi(\nu,r)} \, d\mu_G(r) - \int_{\nu F_j} M_{\phi(\nu,r)} \, d\mu_G(r) \right\|_{cb,S_G^p \rightarrow S_G^q} \\
\leq \frac{1}{\mu_G(F_j)} \int_{F_j \setminus \nu F_j} \left\| M_{\phi(\nu,r)} \right\|_{cb,S_G^p \rightarrow S_G^q} \, d\mu_G(r) \\
= \frac{\mu_G(F_j \setminus \nu F_j)}{\mu_G(F_j)} \left\| M_{\phi} \right\|_{cb,S_G^p \rightarrow S_G^q} \xrightarrow{j \to \infty} 0.
\]

Using the weak* lower semicontinuity of the norm [Meg98, Theorem 2.6.14 p. 227], we infer that

\[
\left\| Q(M_\phi) - Q(M_{\phi(\nu,-) \cdot}) \right\|_{cb,S_G^p \rightarrow S_G^q} \leq \liminf_j \left\| M_{\phi(j,x,t)} - M_{\phi(x,t)} \right\|_{cb,S_G^p \rightarrow S_G^q} = 0.
\]

Finally, it is easy to see that \(Q(M_{\phi(\nu,-) \cdot}) = Q(M_\phi)\). The case \(p = \infty\) is similar.

In order to obtain that the mappings \(Q^{(p)}: \mathcal{M}_G^{cb} \rightarrow \mathcal{M}_G^{cb}\) are compatible for different values of \(1 \leq p < \infty\), it suffices to observe that we can choose the indices \(j'\) in the converging subnet \(Q^{(p)}_{j'}\) independent of \(p\), in the same manner as done in the proof of Corollary 4.33 below by means of an argument relying on Tychonoff’s theorem. The proof is complete. \(\square\)

Remark 4.27 We have a similar result for spaces of bounded Schur multipliers.

Remark 4.28 If \(G\) is compact, the proof is simpler. We do not need to use an approximation procedure. See [SpT02, Proposition 2.3 p. 365] for the case \(p = \infty\).

4.9 Steps 1-3 put together: theorems of complementation

Let \(G\) be a locally compact group. Note that the space \(\mathcal{B}(CB(VN(G)), CB(S_G^\infty, B(L^2(G))))\) is a dual space and admits the predual

\[
(4.63) \quad CB(VN(G)) \hat{\otimes} (S_G^\infty \hat{\otimes} S_G^1),
\]

where \(\hat{\otimes}\) denotes the Banach space projective tensor product and where \(\hat{\otimes}\) denotes the operator space projective tensor product. The duality bracket is given by

\[
(4.64) \quad \langle P, T \otimes (x \otimes y) \rangle = \langle P(T)x, y \rangle_{B(L^2(G)), S_G^1}.
\]

Now, we prove one of our main results.

Theorem 4.29 Let \(G\) be a second-countable unimodular inner amenable locally compact group. Then \(G\) has property \((\kappa_\infty)\) with \(\kappa_\infty(G) = 1\). More precisely, there exists a contractive projection \(P_G^\infty: CB_w^\infty(VN(G)) \rightarrow CB_w(VN(G))\) preserving the complete positivity onto the space \(\mathcal{M}_G^{cb}(G)\) of completely bounded Fourier multipliers on the group von Neumann algebra \(VN(G)\).
Proof: Fix some finite subset $F$ of the group $G$. We can consider a sequence $(V^F_j)_j$ of subsets of $G$ satisfying the last point of Theorem 3.7. As in Lemma 4.9, if $T: VN(G) \rightarrow VN(G)$ is a weak* continuous completely bounded map then we consider the elements $y_j^F \overset{(4.33)}{=} \phi_j^F(4.33) \lambda(1_{V^F_j})^2$ in $L^1(VN(G)) \cap VN(G)$ and the symbol $\phi_j^F \overset{(4.33)}{=} \phi_{1,y_j^F}^T$. Recall that $\|y_j^F\|_{L^1(VN(G))} = 1$.

**Step 1** Consider the mapping $P^F : CB(VN(G)) \rightarrow CB(S_G^\infty, B(L^2(G)))$, $T \mapsto M_{\phi_j^F, \rho_{w^*}^T}$, where the projection $P_{w^*} : CB(VN(G)) \rightarrow CB(VN(G))$, preserving the complete positivity, is defined in [ArK23, Proposition 3.1 p. 24]. By Lemma 4.8, we have the estimate

$$\|M_{\phi_j^F, T}\|_{cb} \leq \|M_{\phi_{1,y_j^F}^F, T}\|_{cb} \overset{(4.33)}{=} \|T\|_{cb, VN(G) \rightarrow VN(G)}.$$ 

Hence the maps $P_j^F$ belong to the unit ball of the space $B(CB(VN(G)), CB(S_G^\infty, B(L^2(G))))$. By Banach-Alaoglu’s theorem, we can introduce a weak* accumulation point $P^F : CB(VN(G)) \rightarrow CB(S_G^\infty, B(L^2(G)))$. So, we have a net $(P_{j(k)}^F)$ which converges to $P^F$ in the weak* topology. Taking into account (4.64), this implies that the bounded net $(P_{j(k)}^F(T))$, that is $(M_{\phi_{j(k)}^F, \rho_{w^*}^T})$, converges in the point weak* topology of the space $CB(S_G^\infty, B(L^2(G)))$ to $P^F(T)$. Since the weak* topology on the space $CB(S_G^\infty, B(L^2(G)))$ coincides, essentially by the same proof as the one of [Pan02, Lemma 7.2 p. 85], on bounded subsets with the point weak* topology, we conclude by the second part of Lemma 4.25 that the map $P^F(T): S_G^\infty \rightarrow B(L^2(G))$ is itself a Schur multiplier. Note that by the weak* lower semicontinuity of the norm [Meg98, Theorem 2.6.14 p. 227], we have

$$\|P^F\|_{CB(VN(G)) \rightarrow CB(S_G^\infty, B(L^2(G)))} \leq \liminf_{k \rightarrow \infty} \|P_{j(k)}^F\|_{CB(VN(G)) \rightarrow CB(S_G^\infty, B(L^2(G)))} \leq 1.$$ 

We next show that the map $P^F$ preserves the complete positivity. Suppose that the map $T$ is completely positive. Using Lemma 4.8, we see that each map $M_{\phi_j^F, \rho_{w^*}^T}$ is completely positive. Since $P^F(T)$ is the limit in the point weak* topology of the $M_{\phi_j^F, \rho_{w^*}^T}$’s, the complete positivity of $M_{\phi_j^F, \rho_{w^*}^T}$ carries over to that of $P^F(T)$ by [ArK23, Lemma 2.10 p. 15].

Now, we consider a weak* accumulation point $P^{(1)} : CB(VN(G)) \rightarrow CB(S_G^\infty, B(L^2(G)))$ of the net $(P_j^F)_F$ and by the same reasoning as before, the map $P^{(1)}(T): S_G^\infty \rightarrow B(L^2(G))$ is again a completely bounded Schur multiplier and preserves the complete positivity. The map $P^{(1)}$ is contractive.

**Step 2** For any weak* continuous completely bounded map $T: VN(G) \rightarrow VN(G)$, we claim that the map $P^{(1)}(T): S_G^\infty \rightarrow B(L^2(G))$ is in fact a Herz-Schur multiplier. It is easy to check that the weak* convergence of a subnet of $(M_{\phi_j^F, \rho_{w^*}^T})_j$ to $M_{\phi^F}$ implies that $\phi^F$ is a cluster point of $(\phi_{j^F, \rho_{w^*}^T})_j$ for the weak* topology of the dual space $L^\infty(G \times G)$. In the same manner, the symbol $\phi$ of the Schur multiplier $P^{(1)}(T)$ is a cluster point of $(\phi^F)_F$ for the weak* topology of $L^\infty(G \times G)$. Thus according to Lemma 4.9, the function $\phi$ is a Herz-Schur symbol. This is the step where we use the assumption of inner amenability on the group $G$.

**Step 3** By [BoF84] and [Spr04, Theorem 5.3 p. 181], we have an isometric map $I : 2\mathcal{N}_G^{\text{ch, HS}} \rightarrow CB_{w^*}(VN(G)), M_G^{\text{HS}} \rightarrow M_\phi$ with range $2\mathcal{N}_\infty^{\text{ch, HS}}(G)$, preserving the complete positivity. Indeed, on the one hand, recall that by [ArK23, Proposition 6.11 p. 90], a Fourier multiplier
$M_{\varphi}: \text{VN}(G) \to \text{VN}(G)$ is completely positive if and only if $\varphi$ is equal almost everywhere to a continuous positive definite function, i.e. the kernel $(s, t) \mapsto \varphi(st^{-1})$ is of positive type [BHV08, p. 351] and that a bounded Fourier multiplier necessarily has a symbol that is equal almost everywhere to a continuous function (see the discussion [ArK23, p. 85]). On the other hand, the Herz-Schur multiplier $M_{\varphi}^{\text{HS}}$ is completely positive if and only if $\varphi^{\text{HS}}: (s, t) \mapsto \varphi(st^{-1})$ is equal almost everywhere to a bounded and measurable function of positive type (see [Arb24, Proposition 3.3 p. 781 and Remark 4.8 p. 785]).

We introduce the linear map $P \overset{\text{def}}{=} I \circ P^{(1)}: \text{CB}_{w^*}(\text{VN}(G)) \to \text{CB}_{w^*}(\text{VN}(G))$ with values in the space $\mathcal{M}_{\infty}^{\text{ch}}(G)$ of completely bounded Fourier multipliers. By composition, this map is contractive and preserves the complete positivity.

Finally, if $T = M_{\varphi}: \text{VN}(G) \to \text{VN}(G)$ is a completely bounded Fourier multiplier, then for any $j$ the symbol $\phi_{j,T}^F$ of the Schur multiplier $P_j^F(T): S_{\varphi}^\infty \to B(L^2(G))$ is given by

$$\phi_{j,T}^F(s, t) \overset{(4.33)}{=} \varphi_1, y_j^F(s, t) \overset{(4.29)}{=} \tau_G(y_j^F \lambda_{s^{-1}} T(\lambda_{s^{-1}}) \lambda_t) = \varphi(st^{-1}) \tau_G(y_j^F \lambda_{s^{-1}} \lambda_{s^{-1}} \lambda_t) = \varphi(st^{-1}).$$

Thus $P^{(1)}(T) = P^F(T) = M_{\varphi}^{\text{HS}}$ and $P(T) = I \circ P^{(1)}(T) = I(M_{\varphi}^{\text{HS}}) = M_{\varphi} = T$.

Now, we state the following general theorem of complementation of the space of completely bounded Fourier multipliers. The proof uses in a crucial way that completely bounded Herz-Schur multipliers acting on $S_{\varphi}^\infty$ are in one-to-one (linear, norm and order) correspondence with completely bounded Lp-Fourier multipliers, thanks to the amenability of the group $G$.

**Theorem 4.30** Let $G$ be a second-countable unimodular amenable locally compact group. Suppose $1 < p < \infty$. Let $(f_j)$ and $(g_j)$ be nets of positive functions with compact support belonging to the space $C_v(G)$ such that if $x_j \overset{\text{def}}{=} \lambda(f_j)$, $y_j \overset{\text{def}}{=} \lambda(g_j)$ we have

- for some positive constant $C$ we have $\|x_j\|_{L^p(\text{VN}(G))}, \|y_j\|_{L^p(\text{VN}(G))} \leq C$ for all $j$,
- $\tau_G(x_j,y_j) = 1$ for all $j$,
- supp $f_j \to \{e\}$ or supp $g_j \to \{e\}$.

Then there exists a bounded projection $P_G^p: \text{CB}(L^p(\text{VN}(G))) \to \text{CB}(L^p(\text{VN}(G)))$ onto the space $\mathcal{M}_{\infty}^{\text{ch}}(G)$ of completely bounded Fourier multipliers with the properties

1. $\|P_G^p\| \leq C^2$,
2. $P_G^p(T)$ is completely positive whenever $T$ is completely positive.

**Proof**: The proof consists of several steps.

**Step 1** Recall that the function $\phi_{j,T} \overset{\text{def}}{=} \varphi_{x_j,y_j,T}$ belonging to the space $L^\infty(G \times G)$ is defined in $(4.39)$. Consider the linear map $P_j: \text{CB}(L^p(\text{VN}(G))) \to \text{CB}(S_{\varphi}^\infty)$, $T \mapsto M_{\phi_{j,T}}$. Using Lemma 4.8, we obtain the estimate

$$\|P_j(T)\|_{\text{ch},S_{\varphi}^\infty \to S_{\varphi}^\infty} \overset{(4.32)}{\leq} \|T\|_{\text{ch},L^p(\text{VN}(G)) \to L^p(\text{VN}(G))} \|x_j\|_{L^p(\text{VN}(G))} \|y_j\|_{L^p(\text{VN}(G))}$$

$$\leq C^2 \|T\|_{\text{ch},L^p(\text{VN}(G)) \to L^p(\text{VN}(G))}.$$

55
4.26
4.24
4.14
4.25
ArK23
CaS15
4.29
4.13
4.8

\[ \text{By Lemma (2)}, \] the result is true for any completely bounded Herz-Schur multiplier on \( S_G^{cb} \).

Note that the proof of the previous result can be adapted to the cases \( p = 1 \) and \( p = \infty \) using Section 4.5. It is worth mentioning that the construction of the projection in the case \( p = \infty \) differs from that in Theorem 4.29.

Combining this result with Example 4.14, we obtain the following result.

22. The result [CaS15, Corollary 5.3 p. 7008] is stated for some class of symbols. However, the proof shows that the result is true for any completely bounded Herz-Schur multiplier on \( S_G^{cb} \).
Corollary 4.31 Let $G$ be a second-countable unimodular amenable locally compact group. Let $1 < p < \infty$ such that $\frac{1}{p}$ is rational. Then there exists a contractive projection

$$P^p_G : \text{CB}(L^p(VN(G))) \to \text{CB}(L^p(VN(G)))$$

on the space $\mathcal{M}^{cb}(G)$ of completely bounded Fourier multipliers such that the map $P^p_G(T)$ is completely positive whenever $T$ is completely positive.

**Remark 4.32** Going through the different steps of the proof, one sees that the projection in the Corollary 4.31 has the property that $P^p_G(T)^* = P^q_G(T^*)$ for any completely bounded map $T : L^p(VN(G)) \to L^q(VN(G))$ and any $1 \leq p < \infty$. Here, we emphasize that $P = P^p_G$ depends on $1 \leq p \leq \infty$. However, it is not clear that the maps $P^p_G(T)$ and $P^q_G(T)$ coincide for $1 \leq p, q \leq \infty$ and an operator $T$ acting on both $L^p(VN(G))$ and $L^q(VN(G))$.

Using Theorem 4.30, we obtain the following important property of compatibility of projections.

**Corollary 4.33** Any second-countable unimodular finite-dimensional amenable locally compact group $G$ has property (s). More precisely, for all $1 \leq p \leq \infty$ there exists a bounded injection

$$P^p_G : \text{CB}(L^p(VN(G))) \to \text{CB}(L^p(VN(G)))$$

onto the subspace $\mathcal{M}^{cb}(G)$ (resp $P^\infty_G : \text{CB}(w^* (VN(G))) \to \text{CB}(w^* (VN(G)))$ on $\mathcal{M}^{\infty,cb}(G)$) with the properties

1. $\|P^p_G\| \leq C$, where the constant $C$ depends on $G$ but not on $p$.
2. $P^p_G(T)$ is completely positive whenever the map $T$ is completely positive.
3. If $T$ belongs to $\text{CB}(L^p(VN(G)))$ and to $\text{CB}(L^q(VN(G)))$ for two values $1 \leq p, q \leq \infty$, then the Fourier multipliers $P^p_G(T)$ and $P^q_G(T)$ are compatible mappings coinciding on $L^p(VN(G)) \cap L^q(VN(G))$, i.e. have the same symbol.

**Proof** : Consider first the case $1 < p \leq \infty$. It suffices to pick the sequence $(f_j)$ in $C_u(G)$ and the associated $x_j = a_j \lambda(f_j)$, $y_j = b_j \lambda(f_j)$ from Corollary 4.22. Then apply this corollary together with Theorem 4.30 to deduce the two first points. Let us show the last point 3. Due to the specific choice of $x_j = a_j \lambda(f_j)$ and $y_j = b_j \lambda(f_j)$ with $a_j \cdot b_j$ being independent of $p$ (see Corollary 4.22), we deduce that $P^p_G(T)K_\phi = P^q_G(T)K_\phi$ for $K_\phi \in S^p \cap S^q$, where $P^p_G$ and $P^q_G$ denote the mappings from Step 1 of the proof of Theorem 4.30. Indeed, the symbol of the Schur multiplier $P^p_G(T)$ is

$$\phi_{j,T}(s,t) = a_j b_j T(\lambda_s \lambda(f_j), \lambda_t \lambda(f_j))$$

Consider the product

$$X \overset{\text{def}}{=} \prod_{1 \leq p < \infty} B(\text{CB}(L^p(VN(G))), \text{CB}(S^p_G)) \times B(\text{CB}(w^* (VN(G))), \text{CB}(S^\infty_G, B(L^2(G))))$$

which is a topological space when equipped with the product topology of the weak* topology. Then $P_j = (P^p_j)_{1 \leq p < \infty}$ lies in a compact subspace of $X$ for all $j$, since $\|P^p_j\| \leq C$ with a constant independent of $p$ and Tychonoff’s theorem for the product of compact spaces applies here. Thus, the net $(P^p_j)$ admits an accumulation point in $X$, that is to say that for the same
subnet \( j(k) \) for all \( p \), we have \( P^{(1), p} = \lim_{j(k)} P^{p}_{j(k)} \). We infer that \( P^{(1), p}(T)K_{\phi} = P^{(1), q}(T)K_{\phi} \) for any \( K_{\phi} \in S^p \cap S^q \), where \( P^{(1), p} \) and \( P^{(1), q} \) denote the mappings from Step 1 of the proof of Theorem 4.30. Since the mapping \( Q \) from Proposition 4.26 is compatible for different values \( p \) and \( q \), also \( P^{(2), p} \) and \( P^{(2), q} \) from Step 2 of the proof of Theorem 4.30 are compatible. Finally, since the mappings \( I = I_p \) from Step 3 of the proof of Theorem 4.30 are compatible for different values \( p \) and \( q \), the mappings \( P^{p}_{G}(T) \) and \( P^{q}_{G}(T) \) are compatible for any \( T \).

Example 4.34 Let \( G \) be a second-countable unimodular finite-dimensional amenable locally compact group. Consequently, Corollary 4.33 applies. Suppose that \( G \) is in addition totally disconnected. Then the projections \( P^{p}_{G} \) in that result are in fact contractions. Indeed, an inspection of the proof of Corollary 4.33 (see also Step 1 of the proof of Theorem 4.30) shows that \( \|P^{p}_{G}\| \leq \sup_j \|x_j\|_p \|y_j\|_p^{*} \). According to Corollary 4.23, the right-hand side is less than 1.

Therefore, Corollary 4.33 gives a variant of [ArK23, Theorem 6.38 p. 121]. Note that Corollary 4.33 needs that the group \( G \) is amenable in the cases \( p = \infty \) and \( p = 1 \) in contrast to [ArK23, Theorem 6.38 p. 121].

Remark 4.35 We have resisted to the temptation to write a matricial version of Theorem 4.29 in the spirit of [ArK23, Theorem 4.2 p. 62]. It is likely that the same method works.

Remark 4.36 We make no attempt with non-unimodular locally compact groups. It is likely that the same strategy works in the case \( p = \infty \). We leave this case as an exercise for the reader.

5 Final remarks

5.1 A characterization of the amenability of unimodular locally compact groups

We present a new characterization of amenability, in the same spirit as that of Theorem 1.3. Recall that property \((\kappa_\infty)\) is defined in Definition 1.2.

Theorem 5.1 Let \( G \) be a second-countable unimodular locally compact group. Then the following are equivalent.

1. The von Neumann algebra \( \text{VN}(G) \) is injective and \( G \) has property \((\kappa_\infty)\).

2. The group \( G \) is amenable.

Moreover, the implication 1. \( \Rightarrow \) 2. is true without the assumption «second-countable unimodular».

Proof : 1. \( \Rightarrow \) 2. Since the locally compact group \( G \) has property \((\kappa_\infty)\), Proposition 2.15 gives the equality

\[
\mathfrak{M}^{\infty, \text{dec}}(G) = B(G).
\]

Since \( \text{VN}(G) \) is injective, by [Haa85, Theorem 1.6 p. 184] each completely bounded operator \( T: \text{VN}(G) \to \text{VN}(G) \) is decomposable with \( \|T\|_{\text{cb}, \text{VN}(G) \to \text{VN}(G)} \overset{(2.3)}{=} \|T\|_{\text{dec}, \text{VN}(G) \to \text{VN}(G)} \). In particular, we have \( \mathfrak{M}^{\infty, \text{cb}}(G) = \mathfrak{M}^{\infty, \text{dec}}(G) \) isometrically. We deduce that

\[
\mathfrak{M}^{\infty, \text{cb}}(G) = B(G).
\]
Using a result stated in [Pis01, p. 54] (see also [Spr04, p. 190]), which says that this equality is equivalent to the amenability of $G$, we conclude that the group $G$ is amenable.

2. $\Rightarrow$ 1. If the locally compact group $G$ is amenable, then by Theorem 1.3 the von Neumann algebra $VN(G)$ is injective. Note that the group $G$ is inner amenable by Example 3.1. Consequently, for the second property, it suffices to use Theorem 4.29.

As a result, we can provide explicit examples of locally compact groups that do not satisfy property $(k_\infty)$.

**Corollary 5.2** Any non-amenable second-countable connected locally compact group $G$ does not have property $(k_\infty)$.

**Proof**: By [Con76, Corollary 7 p. 75], the von Neumann algebra $VN(G)$ of a second-countable connected locally compact group is injective. Suppose that the group $G$ has property $(k_\infty)$. By the implication 1. $\Rightarrow$ 2. of Theorem 5.1, we obtain that $G$ is amenable, i.e. a contradiction.

**Example 5.3** This result applies for example to the connected locally compact group $SL_2(\mathbb{R})$, which is non-amenable by [BHV08, Example G.2.4 (i) p. 426] and unimodular by [Lan75, p. 4], contradicting the observation [DFSW16, Remark 7.6 p. 24], stated for unimodular locally compact quantum groups.

The following result generalizes Theorem 2.19 when second countability is not assumed.

**Corollary 5.4** Let $G$ be a second-countable unimodular inner amenable locally compact group. Then the von Neumann algebra $VN(G)$ is injective if and only if we have $\mathcal{M}_\infty^{\text{dec}}(G) = \mathcal{M}_\infty^{\text{cb}}(G)$.

**Proof**: The «only if» part is true by a result of Haagerup [Haa85, Corollary 2.8 p. 201]. Now, assume that $\mathcal{M}_\infty^{\text{dec}}(G) = \mathcal{M}_\infty^{\text{cb}}(G)$. By Proposition 2.15 and Theorem 4.29, we have a bijection from the space $\mathcal{M}_\infty^{\text{cb}}(G)$ onto the Fourier-Stieltjes algebra $B(G)$. So the group $G$ is amenable by a result stated in [Pis01, p. 54]. We conclude that the von Neumann algebra $VN(G)$ is injective by Theorem 1.3.

### 5.2 Complementation of the space of bounded Herz-Schur multipliers and amenability

Consider a locally compact group $G$. To know whether the amenability of $G$ is characterized by the complementation of the space $\mathcal{M}_G^{\text{HS}}$ of (completely) bounded Herz-Schur multipliers in the space $\mathcal{M}_G^{\infty}$ of (completely) bounded Schur multipliers over $\mathcal{B}(L^2(G))$ is a well-known open question, explicitly stated in [Spr04, p. 184]. Now, we present the first progress on this classical question.

**Proposition 5.5** Let $G$ be a second-countable unimodular locally compact group such that the von Neumann algebra $VN(G)$ is injective. Suppose that there exists a bounded projection $Q: \mathcal{M}_G^{\infty} \to \mathcal{M}_G^{\infty}$ onto the space $\mathcal{M}_G^{\infty,\text{HS}}$ of bounded Herz-Schur multipliers over the space $\mathcal{B}(L^2(G))$, preserving the complete positivity. Then the group $G$ is amenable. Conversely, if $G$ is amenable, then such a $Q$ exists according to Proposition 4.26.

23. We would like to thank Adam Skalski for his confirmation of this problem in this small remark by email on his own initiative. The results of the nice paper [DFSW16] remain correct.
Proof: Since the von Neumann algebra $VN(G)$ is injective, it suffices by Theorem 5.1 to show that the group $G$ has property $(\kappa_\infty)$. Now, we follow the proof of Theorem 4.30. For the first step, we use Lemma 4.10 and we define the linear map $P^{(1)}_G: CB_{\infty}(VN(G)) \to CB(B(L^2(G)))$, $T \mapsto M_{\psi_1, y, T}$ for some positive element $y$ in the space $L^1(VN(G)) \cap VN(G)$ such that $\tau_G(y) = 1$.

The second step of the proof is addressed using the existence of the projection $Q$. If $p = \infty$, the third step is done without amenability. So we obtain property $(\kappa_\infty)$.

Example 5.6 By Example 3.4, a second-countable almost connected unimodular locally compact group satisfies the assumptions of the previous result: that is, $VN(G)$ is injective. In particular, this result applies to any connected unimodular Lie group: such a Lie group is amenable if and only if a projection $Q: M_\infty^G \to M_\infty^G$, $HS_G$ preserving the complete positivity exists.

Declaration of interest None.

Competing interests The authors declare that they have no competing interests.

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64
