Critical Behavior of Dynamically Triangulated Quantum Gravity in Four Dimensions

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Abstract

We performed detailed study of the phase transition region in Four Dimensional Simplicial Quantum Gravity, using the dynamical triangulation approach. The phase transition between the Gravity and Antigravity phases turned out to be asymmetrical, so that we observed the scaling laws only when the Newton constant approached the critical value from perturbative side. The curvature susceptibility diverges with the scaling index $-6$. The physical (i.e. measured with heavy particle propagation) Hausdorff dimension of the manifolds, which is $2.3$ in the Gravity phase and $4.6$ in the Antigravity phase, turned out to be $4$ at the critical point, within the measurement accuracy. These facts indicate the existence of the continuum limit in Four Dimensional Euclidean Quantum Gravity.
I Introduction

The Quantum Gravity remains the last unsolved mystery of twentieth century physics. Unlike, say, turbulence, where we at least know what is happening and what are the equations to solve, in Quantum Gravity we cannot be sure about the basic principles.

One would like to provide the meaning to the Euclidean functional integral corresponding to the Einstein Action:

$$Z(\lambda, G) = \int Dg \exp \left( - \int d^4x \sqrt{g} \left( \lambda - \frac{R}{G} \right) \right)$$

(1)

This integral formally diverges, since the curvature $R$ is not bounded (neither from above nor from below). Moreover, there is no unique way to define the continuous measure $Dg$ in infinite dimensional configuration space of metric fields. The perturbative approach does not lead to the renormalizable theory which means that the genuine nonperturbative definition should be given before any computations could be done.

These problems, however, don’t arise in Dynamical Triangulation (DT) approach to Quantum Gravity (QG). The DT technique was introduced several years ago in two dimensional case (QG2) as a generalization of Regge calculus \[1, 2, 3\]. Metric field was represented by simplicial complex built from equilateral simplexes, the integration over $g$ being defined as summation with equal weight over all simplicial complexes of given topology. In two dimensional case this procedure would correspond to the summation over all possible triangulations, which gave the name DT to the method.

In the original Regge-Ponzano approach, the lattice structure was fixed (usually regular), and link lengths were the fluctuating dynamical variables, subject to triangle inequalities. It does not seem possible to define general covariant measure in Regge calculus, in a sense, that there is no symmetry to take place of the diffeomorphism group of the classical continuum theory. Even though it is much easier to implement than DT, and one can obtain some exciting results with Regge Calculus \[4, 5\], their meaning is yet not clear. The big question is how do these results depend upon the chosen lattice structure. There is no apparent reason to believe in universality in case of broken general covariance.

The DT approach is, however, generally covariant by construction, with the simplex permutations as discrete analog of the diffeomorphism. When we sum over all connectivity matrices at given topology, all simplexes are equivalent, hence there is permutational symmetry. The labels, numbering simplexes, play the role of discrete coordinates, so the relabeling of the same simplicial manifold corresponds to diffeomorphism \[1\].

The local curvature of such system is defined as the deficit angle of the loop of simplexes surrounding two dimensional hinge, and it is bounded from above. The net curvature is also bounded from below at fixed volume, as we shall discuss later. Therefore, the functional integral is well defined, at least for fixed volume and fixed topology. If there is a continuum limit in such a theory, it will be a real physical limit.

The first confirmation of the method came from the Liouville Field Theory (LFT): the values of critical exponents computed analytically in the DT model were obtained from

\[1\] To be more precise, the diffeomorphism represent the subset of all permutations, which becomes smooth transformations in naive continuum limit.
purely continuous arguments. The early computer simulations were not accurate enough to verify these values, but at least proved the concept. The basic elementary moves found in [1] proved to be quite useful.

The later investigations of the issues of internal and external geometries in DT and its correspondence with LFT predictions/assumptions [8, 24] showed, that the DT model in continuum limit indeed corresponds to QG2. At the same time some inherent inconsistencies in the conventional view on the quantum geometry based on classical geodesic lines were found [9].

The remarkable advantages of DT over continuum methods in two dimensions, caused the attempts to generalize it to higher dimensions. In three dimensions (QG3), the dynamical triangulations were simulated recently by three groups [10, 11, 12, 13]. All three simulations show the existence of statistical limit, which implies that the number of triangulations is exponentially bounded. In [13] the first order phase transition was observed, with two different values of average curvature, both finite in lattice units, which implies the absence of continuum geometry. This transition was later on investigated and confirmed on much larger systems in [13, 14]. In [10, 11] the average curvature was forced to continuum limit, by extra constraint, which corresponds to unstable phase of [13]. The absence of continuum limit in QG3 is not so surprising, as there are no gravitons in 3 dimensions.

In the four dimensional case the situation was different. The first simulations [16] revealed the existence of nontrivial phase transition, which looked like a second order one, with small hysteresis, the significance of which was not clear at that time. The existence of the second order phase transition would imply renormalizability and existence of continuum limit. These results were later qualitatively confirmed on much smaller statistics by two other groups [18, 19].

In this work we addressed ourselves to detailed investigation of the thermodynamics and internal geometry of QG4 in the critical region. We studied the four-spheres with up to $32K$ simplexes and observed, that the scaling laws, which characterize second order phase transition, exist only when the system approaches critical point from the perturbative phase (small gravitational constant and positive curvatures). When we approached the critical point from the nonperturbative phase of large negative curvatures, we did not find convincing evidence for the powerlike singularities. The physical geometry, measured on this system also showed very interesting behavior. The Hausdorff dimension $d_h$ of the QG4 manifolds in the perturbative phase is about $d_h \approx 2.3$, which correspond to “compactified” space. In the nonperturbative phase, $d_h \approx 4.6$, while at the phase transition point it is $d_h = 4$ within the accuracy of the measurements.

These results allow for the exciting physical interpretation: We simulated the asymptotically Euclidean QG4 from the first principles. Unlike 2 and 3 dimensional cases, we observed the existence of nontrivial continuum behavior. When the system approaches the critical point from the gravity side, the scaling properties of intrinsic quantum geometry of

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2 As for the topological gravity in 3 dimensions, most likely it corresponds to Regge-Ponzano-Turayev-Viro model, where the $6j$ symbols attached to tetrahedra eliminate necessity of summation over triangulations. The model then reduces to finite simplicial complex, say, 2 tetrahedra for the spherical topology, with no continuum space. Apparently, we are dealing with different model.
the manifolds presumably become the ones of flat four-dimensional space.

The paper is organized as follows. In the Section II we describe the kinematics of four-dimensional manifolds, basic steps of the Monte Carlo simulations, and the main observables. In the Section III we present the results, which are followed by the Discussion in the Section IV.

II Numerical method

The numerical technique of QG4 simulations is described in details in [16]. For the reader’s convenience we repeat the basic facts below.

We simulated the functional integration \([\mathbb{I}]\) by a Markov chain in the space of four dimensional simplicial manifolds. The manifolds are represented by simplicial complexes built from equilateral simplexes with unit volumes. Each 4D simplex has 5 tetrahedra faces, 5 vertices, 10 links and 10 triangles. Simplicial complex is a collection of these simplexes with pairwise identified faces. Simplicial manifold is a simplicial complex with the extra condition: for any given vertex, the set of simplexes sharing this vertex should form a spherical ball.

The dynamical triangulation approach is a modified version of Regge calculus [20]. The space in considered flat inside the \(D = 4\) simplexes, the curvature being concentrated in \(D - 2 = 2\) dimensional hinges, i.e. triangles. The angle between two tetrahedra-faces, sharing a triangle is

\[
\alpha = \arccos\left(\frac{1}{D}\right) = 1.3181161
\]  

The volumes of \(K\)-simplexes are given by

\[
V_K = a^K \Omega_K = \Omega_K \left(\frac{V_D}{\Omega_D}\right)^K, \quad \Omega_K = \sqrt{\frac{\Gamma(K+2)}{2^K \Gamma^3(K+1)}}
\]  

We are going to measure volumes in the units of \(V_4\), lengths \(-\) in \(a = 2 \ast 6^{\frac{1}{2}} \ast 5^{-\frac{1}{2}} V_4^{\frac{1}{2}}\), and curvature in \(\frac{V_4}{V_4} = 3 \ast 2^{\frac{1}{2}} \ast 5^{-\frac{1}{2}} V_4^{-\frac{1}{2}}\). The local curvature, i.e. deficit of angle on a triangle \(\triangle\) which is shared by \(N_4(\triangle)\) 4-simplexes equals to

\[
R(\triangle) = 2\pi - \alpha N_4(\triangle)
\]  

\(\alpha\) not being rational part of \(\pi\) makes it impossible to construct a locally flat space from equilateral simplexes. Still, if the volume of the system, which is now equal to the number of 4D simplexes in the manifold, is large enough, the manifold can be flat in average.\(^3\) In such a complex, the average number of simplexes \(< N_4(\triangle) >\) sharing a triangle would be

\[
\frac{2\pi}{\alpha} = 4.7667921
\]  

\[5N_4 = 2N_3\]  

\(^3\)Anyway, the observable metric and curvature get renormalized due to strong quantum fluctuations.
\[
\sum_{k=0}^{4} (-1)^{D-k} N_k = 2
\]  
(7)

The manifold condition provides additional relations between the invariants \(N_k\). These relations were derived in [21] for simplicial manifolds of any dimension. We will need only the simplest one:

\[2N_1 - 3N_2 + 4N_3 - 5N_4 = 0\]  
(8)

This relation is independent from (6),(7). We, therefore, can choose the volume \(N_4\) and the number of vertices \(N_0\) as our parameters. Using (6),(7),(8) we can express the integral curvature of the manifold in terms of \(N_0, N_4\) as follows:

\[R = \sum_{\triangle} R(\triangle) = 2\pi N_2 - 10\alpha N_4 = 4\pi(N_0 + N_4 - 2) - 10\alpha N_4\]  
(9)

We simulated the functional integration (1) by Markov chain in the space of QG4 simplicial manifolds. As was mentioned in [11] the minimal set of elementary moves, which guarantees for this chain to be ergodic can be obtained from all possible decompositions of the boundary of \(D + 1\) simplex into two \(D\)-complexes. At \(D > 4\) this is a trivial corollary of Smale’s \(h\)-cobordism theorem. It means, that there are \(D + 1\) basic moves in \(D\)-dimensional gravity. For \(D = 4\) we brought them together in the table [1], where we, for the sake of simplicity, represented a simplex by the list of its vertices in the alphabetical order. These elementary moves are theoretically ergodic and allow to construct a Markov chain in the space of simplicial complexes. The moves described in the Table [1] can sometimes lead to degenerate triangulations, which are excluded from conventional definitions of simplicial manifold. For example, one could get several disconnected sets of 4-simplices, sharing the same triangle, or some other simplex of dimension \(k < 4\). So, like in QG2 and in QG3 simulations, we had to check the local geometry at each step and reject the improper moves. Let us stress, however, that the manifold restriction is not absolutely necessary from the continuum limit point of view, like in QG2 one could allow tadpoles and self-energy graphs. We eliminated degenerate configurations because it simplified the representation of triangulation in computer memory, and also because we expected the degenerate triangulations to delay the infinite volume limit.

We used the Grand Canonical algorithm proposed in [11], which is a modification of the algorithm introduced in [12]. The Action (1) was changed as follows:

\[
Z(\hat{\lambda}_4, \hat{\lambda}_0) = \sum_{\text{manifolds}} \exp \left( -\hat{\lambda}_4 N_4 - \hat{\lambda}_0 R - \Delta \lambda_4 (N_4 - \hat{N}_4)^2 - \Delta \lambda_0 (R - \hat{R})^2 \right) 
\]

\[
= \sum_{N_4, N_0} Z(N_4, N_0) \exp \left( -\hat{\lambda}_4 N_4 - \hat{\lambda}_0 R - \Delta \lambda_4 (N_4 - \hat{N}_4)^2 - \Delta \lambda_0 (R - \hat{R})^2 \right)
\]

where \(N_4\) is the volume of the system, \(R\) – the net curvature, and \(Z(N_4, N_0)\) – the microcanonical partition function, which is nothing but the number of the corresponding triangulations of the 4-sphere. The \(\hat{N}_4\) and \(\hat{R}\) are the desired volume and the net curvature.

\footnote{As it was mentioned above, the conventional normalization of curvature contains the extra factor \(3 \times 2\pi \times 5^- \times \approx 2.84\), which we dropped.}
respectively. This modification allows to take some guess values of $\lambda_i$ and to measure the functions $\lambda_i(N_4, R)$, $i = 0, 4$ directly from the saddle-point equations.

$$\frac{\partial \ln(Z(N_4, N_0))}{\partial N_4} \equiv \lambda_4(N_4, R) = \hat{\lambda}_4 + 2\Delta\lambda_4(N_4 - \hat{N}_4) \tag{11}$$

$$\frac{\partial \ln(Z(N_4, N_0))}{\partial R} \equiv \lambda_0(N_4, R) = \hat{\lambda}_0 + 2\Delta\lambda_0(R - \hat{R}) \tag{12}$$

Varying $\Delta\lambda_i$ one can smoothly interpolate between Grand Canonical and Micro Canonical simulations. From one hand, we would like the system to perform large steps in the phase space, i.e. to allow $N_4$ and $N_0$ to vary in wide enough limits. From the other hand, we want to be far from the finite size effects and to collect sufficient statistics near $\hat{N}_i$, where the saddle point approximations (11, 12) are valid. For these technical reasons we used $\Delta\lambda_4 = \Delta\lambda_0 = 0.005$.

One could regard these equations as parametric relations between average values of $N_4, R$ and desired values $\hat{N}_4, \hat{R}$. Taking different values of $\hat{N}_4, \hat{R}$ we could measure average values and use above relations to find the effective values $\lambda_4(N_4, R), \lambda_0(N_4, R)$. The results should be independent of auxiliary parameters $\hat{\lambda}_0, \hat{\lambda}_4, \Delta\lambda_4, \Delta\lambda_0$ as long as we stay in the region of large volumes and curvatures and small fluctuations, so that the saddle point equations apply. The disadvantage of this method would be the necessity to interpolate functions of two variables $N_4, R$ given at scattered points. We preferred to fine tune $\hat{N}_4, \hat{R}$ so that they coincide with average values. Then we could plot fixed volume curves in the $\hat{\lambda}_0, \hat{\lambda}_4$ plane, and observe the convergence to the infinite volume limit.

We performed our simulations in the following way: $\hat{\lambda}_0$ was slowly varied from large negative values (perturbative phase) to small positive values (nonperturbative phase) and back, keeping $N_4$ fixed. This way, we simulated each value of $\lambda_0$ twice, coming from different directions. If there were a first order phase transition, or we did not thermalize the system

| move                  | smplxs before | smplxs after | $\Delta N_0$ | $\Delta N_1$ | $\Delta N_2$ | $\Delta N_3$ | $\Delta N_4$ |
|-----------------------|---------------|--------------|-------------|-------------|-------------|-------------|-------------|
| bar. subd.            | ABCDE         | ABCDF, BCDEF, ABDEF, ABCEF, ACDEF | 1           | 5           | 10          | 10          | 4           |
| bar. subd. removal    | ABCDF, BCDEF, ABDEF, ABCEF, ACDEF | ABCDE       | -1          | -5          | -10         | -10         | -4          |
| flip                  | ABCDE, ABCDF, ABCEF, ACDEF | BCDEF, ACDEF, ABDEF | 0           | 0           | 0           | 0           | 0           |
| two-four exchange     | ABCDE, BCDEF | ABCDF, ABCEF, ACDEF, ABDEF | 0           | 1           | 4           | 5           | 2           |
| four-two exchange     | ABCDF, ABCEF, ACDEF, ABDEF | ABCDE, BCDEF | 0           | -1          | -4          | -5          | -2          |

Table 1: Ergodic moves in QG4
at given $\lambda_0$ we would observe hysteresis. We observed some hysteresis in the beginning \[16\], but we were not sure, what was the reason for it. By performing longer runs, we succeeded in eliminating it, which became our thermalization criteria. Since we were interested in the critical point vicinity, we had to take a fine grid in $\lambda_0$ (usually 60 values) \[5\]. In order to make the error bars in the measurements of $< R(\lambda_0) >$ and $< \lambda(\lambda_0) >$ smaller than the variation of these observables between the neighboring values of $\lambda_0$ we had to perform 160000 sweeps at each value of $\lambda_0$).

III Results

There were two major aspects, we were concerned about in our simulations: thermodynamical properties of QG4 near the phase transition and physical geometry at the critical point.

Using the formulae (11), (12) we obtained the curves for $< R(\lambda_0) >$ and $\lambda(\lambda_0)$ for different values of volume $N_4$ (see Fig. 1, 2).

These plots clearly demonstrate the difference between Gravity and Antigravity phases. In the Gravity phase ($\lambda_0 < \lambda_0^{critical}$) the curves converge to some infinite volume limit. At large negative $\lambda_0$ this phase apparently corresponds to perturbative continuum gravity. In the Antigravity phase, however, $−< R(\lambda_0) >$ grows with volume, being bounded only by the lattice finite size effects. The physical behavior, therefore, exists only on the Gravity side of the phase transition, the Antigravity side corresponding to the branched polymer phase and formation of baby-universes.

Susceptibility measurements (see Figure 3) showed very clearly, that the phase transition is not symmetric. Since the phase transition did not look like the traditional second order transition, we did not want to make our conclusions based on these noisy data or on extrapolation from 5-6 points. We collected large statistics and fitted $\lambda_0$ with the fifth degree polynomial $\lambda_0 \approx P_5(< R >)$, like in Mean Field theory. Susceptibility $\chi$ is calculated from the derivative of $P_5$ as follows:

$$\chi = -\frac{d < R >}{d\lambda_0} = -\frac{1}{P_5'(< R > (\lambda_0))} \quad (13)$$

The critical value of $\lambda_0^c$ is obtained from the (real part of the nearest to the real axis) zero of polynomial $P_5'$ and corresponds to the maximum of $\chi$. We illustrated this procedure on Figure 4.

We performed the finite size scaling analysis of $\chi^c$, fitting it with the law $a + bN_4^\mu$. The fit showed that $\chi^c$ grows as small power of $N_4$, $\mu \approx 0.14$, which is consistent with the logarithmic growth. The fit of $\lambda_0^c$ with $a + bN_4^\nu$ was consistent with the hypothesis $\nu \approx -0.5$ (see Figure 4).

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5 We could, of course, simulate very few points and perform Ferrenberg-Swendsen interpolation like in \[19\]. This method, however, uses the assumption of regular second order phase transition and its accuracy is questionable. We decided, therefore, to employ a straightforward approach, which allowed us to observe the asymmetry of the phase transition.
Table 2: Measurements of $\gamma$ for different system sizes and extrapolation to the infinite volume.

The behavior of $\chi$ near the singularity is described by the critical index $\gamma$:

$$(\chi^c - \chi) \sim |\lambda^c_0 - \lambda_0|^{-\gamma}$$

This observable had rather big errors in our simulations, this is why we decided to calculate $\gamma$ directly from the $<R>$ measurements:

$$|R^c - R| \sim |\lambda^c_0 - \lambda_0|^{1-\gamma}$$

This observable was the most interesting one. The phase transition turned out to be asymmetric, with unequal $\gamma<$ and $\gamma>$ on the different sides of $\lambda^c_0$ (see Figure 5). The measurements of $\gamma$ for different system sizes are brought together in the table 2.

The weak singularity in the Antigravity phase $\gamma_\sim \approx .1$, is consistent with the visual observation, that the lines of average curvature are almost straight lines to the right of the inflection point.

After separating the phases and investigating the thermodynamics of the phase transition, we can ask a natural question: what are these phases all about? In the spin models, hot phase means completely random orientation of neighboring spins, while the cold phase is dominated by large domains. What can be the order parameter in the pure gravity? There is nothing, but geometry fluctuations in this theory, so can geometry itself be the order parameter?

As we already mentioned in the first paper [16], the internal Hausdorff dimension $d_h$, measured in physical definition of distances, has completely different value in both phases and can be considered as an order parameter for QG4, in a sense that in the Gravity phase the space tends to compactify, so that $d_h < 4$ whereas in the Antigravity phase the space tends to branch like the tree, so that $d_h > 4$. If the DT model describes continuum gravity, then we must have $d_h = 4$ at the transition point.

Therefore, the local parameter

$$\eta = \lim_{l \to 0} \frac{V(l)}{l^4}$$

(16)
vanishes in the Antigravity phase, goes to $\infty$ in the Gravity phase, and remains finite at the critical point. This is the closest we could get to the order parameter.

To define the distance we consider the propagation of massive test particle as first suggested by [9]. When the propagator is exponentially small, it decays as

$$G_{12} \to A(l_{12}) \exp(-ml_{12})$$

(17)

where $l_{12}$ is the geodesic distance between two points. The factor $A$ is not universal, but the mass $m$ is. This is the physical mass of the particle, with the gravitational renormalization. This renormalization is **multiplicative** rather than additive, as it follows from the fact that discrete Action

$$\sum_{<ij>} (\phi_i - \phi_j)^2 + \sum_i m_0^2 \phi_i^2 = \sum_{ij} \left( (m_0^2 + D + 1) \delta_{ij} - S_{ij} \right) \phi_i \phi_j$$

(18)

where $S_{ij}$ is the simplexes connectivity matrix, gets additional translational symmetry

$$\phi_i \to \phi_i + \text{const}$$

(19)

at vanishing bare mass $m_0$. In other words, at vanishing bare mass there is always the zero mode $\phi_i = \text{const}$, so that the physical mass would also vanish. In this respect the gravity is simpler then the usual gauge theory where there was additive renormalization of the scalar particle mass.

In terms of the random walk the propagator

$$G_{12} = \left( (m_0^2 + D + 1) \hat{I} - \hat{S} \right)^{-1} = \frac{1}{D + 1 + m_0^2} \sum_{k=1}^{\infty} \left( \frac{\hat{S}}{D + 1 + m_0^2} \right)^k$$

(20)

corresponds to the process of hopping from simplex to one of its $D + 1$ neighbors with probability $(D + 1 + m_0^2)^{-1}$. The net hopping probability

$$\frac{D + 1}{D + 1 + m_0^2} \equiv \exp(-m)$$

(21)

defines the particle mass in the sum over paths

$$G_{12} = \sum_{\Gamma_{12}} \exp(-ml[\Gamma_{12}])$$

(22)

One could use two heavy propagators with masses $m, m'$ to define the geodesic distance

$$l_{12} = \frac{\ln G_{12} - \ln G'_{12}}{m' - m}$$

(23)

If one takes the local limit after the large mass limit, he will obtain the ordinary lattice geodesics. This metric, which we called “mathematical”, does not have a continuum limit: it develops parametrically large amount of singularities, $d_h$ grows with the test-sphere size etc. [6, 8, 10].
However, if the local limit is taken before the large mass limit, then physical geometry differs from mathematical one. The entropy of the paths leads to effective transverse thickness of the dominant set of paths. This thickness is not related to the cutoff, and therefore the thick path is not sensitive to the lattice artifacts, such as the embryonic Universes, observed in the mathematical geometry.

We used the “Black-Box” Multigrid method to solve the Poisson equation on random manifolds (see [16] for details) and obtained the volume-radius histogram for the manifolds near the critical point (Figure 6). We fitted them with the scaling law $<V(r)> \approx a + br^{d_h}$. The result was quite remarkable: as we already mentioned above, $d_h = 2.3$ far in the Gravity phase and $d_h = 4.6$ in the Antigravity phase, but at the phase transition we found $d_h = 4$, just as expected!

IV Discussion

For somebody with the Classical Relativity background our model must look like a mathematical toy, nothing to do with real 4D gravity. All we did, we generated random collection of simplexes with spherical topology and fixed number of vertices. How could that be the same, as ”sum over all graviton loops” of continuum theory?

Well, this miracle already happened in QG2 where random triangulation was proven to reproduce the ”sum over all Liouville loops”. The general covariant theory looks much simpler without gauge fixing. As for effective infrared Lagrangean, such as phenomenological Einstein Gravity, it cannot be built in the Quantum Field Theory, but rather should come out dynamically.

Let us now speculate about the physical meaning of our results. At this point there are several possibilities. First of all, it is still possible that there is no critical behavior. One have to simulate much larger systems and check the trends we observed at the relatively small systems of several thousand simplexes. In particular, one have to check the observed asymmetry of the critical point. Would the average curvature continue to go down to $-\infty$ in the Antigravity phase? Would the scaling law for the curvature in the Gravity phase remain the same for large systems?

Also, at larger systems we should be able to observe the running Newton constant: would it grow or decrease with scale? In other words, is our fixed point infrared or ultraviolet stable? If the $2 + \epsilon$ approach [17] could be taken as a guide to the QG4, then there must be the ultraviolet stable fixed point, corresponding to the conformal field theory, and the infrared stable fixed point corresponding to zero gravity. At our modest scales all we could see, is the ultraviolet one, but with one more decade of scales one should see the decrease of effective Newton constant.

With alternative scenario of asymptotically free Gravity with $R^2$ terms one would observe the growth of effective Newton constant. This scenario seems unlikely though, as we did not introduce large $R^2$ terms in the bare Action, as we should have done in the asymptotically free theory. Clearly, we are studying the strong gravity at small distances, with zero or at least small $R^2$ terms in effective Action. The apparent coexistence of continuum space with $d_h = 4$ at large scales with nontrivial thermodynamical scaling indices fits the first scenario
Some authors [18, 19] claim to obtain direct correspondence between the results of DT and Regge calculus [4, 5]. We would like to emphasize, that we don’t see any similarity. And not only the numbers are different ($\gamma > \approx 0.4$ in [5], $\gamma < \approx 0.6$, $\gamma > \approx 0$ in our simulations). There is no weak coupling phase in the Regge calculus, but this was the only physical phase we observed in our simulations! The phase, where the simulations of [5] were performed does not have a continuum limit in our model, and there is no significant singularity when one approaches the critical point from this side.

The observed scaling laws in our model imply, that there are some effective massless fields at the critical point. The natural candidates are gravitons, but we have not proved this yet. All we know, are the scaling laws, but not the structure of the operator algebra. The graviton has spin 2, but we have first to find the global $O(4)$ rotational symmetry to be able to talk about spin.

The problem is that initially we do not have any fields in our theory, but rather pointers from each simplex to its neighbors. These pointers are our only degrees of freedom, which makes theory easy to implement on the computer, but hard to interpret analytically. We need the reper fields and the spin connections to define the $O(4)$ group.

There is the natural local reper in any simplicial complex, namely, each simplex has $D + 1 = 5$ unit vectors $E^{(A)}_{\mu}$, $A = 1, \ldots, 5$ normal to its faces. They are linearly dependent, namely, they sum up to zero in our case of equilateral simplexes. Each of above-mentioned pointers is associated with corresponding normal vector. One could redefine the pointers without changing the simplicial complex, namely, one could rotate each simplex around its center, and then reconnect with neighbors. These local rotations form the discrete group, which is our counterpart of the local $O(4)$ symmetry. As usual in the lattice theory, we hope that in continuum limit only quadratic Casimir invariant of this group would remain relevant, which would raise the symmetry up to the full $O(4)$ group, at least globally.

We could now add matter to our system, by placing the matter fields in the middles of simplexes and coupling them to the neighbors in an obvious way, using this reper. The spin connections $\Omega_{<ij>}$ belonging to the corresponding representation of the simplex rotation group, should be associated with each pair $<ij>$ of neighbors, so that local rotations of the reper would result in the gauge transformations of the spin connections.

There are various alternative treatments of the reper and spin connections. One could classically relate them to the metric, say, by taking proper initial values at the beginning of the creation of the simplicial manifold, when it reduced to the pair of simplexes (for the spherical topology), and then rotating them correspondingly at each of above local canonical moves. Alternatively, one could treat them as independent dynamical variables, and simulate them along with the triangulation. In presence of dynamical spinning matter, these variables would interact with triangulation.

There are many interesting things to do with this model, like studying Wilson loops corresponding to the spin connections, or the physical Newton constant defined as susceptibility with respect to mass of the test particle. Or one could study physical geodesic lines formed by paths of heavy particles, and measure sum of angles of geodesic triangles. These studies would hopefully bring us closer to real physical questions, which we do not know how to
formulate so far: what is unitarity, and how to come back to Minkowskian space.

And the last point. In QG2 the basic moves were proven to be ergodic. There is also a proof [1], that any two configurations of the same genus can be connected by the number of moves linearly bounded by the volume. It means, that we have finite probability to reach any configuration of given volume in simulations. In the higher dimensions the situation is more complicated. There is little doubt, that the basic moves are ergodic, even though there is no $h$-cobordism theorem in four dimensions. The problem of complexity is, however, very much unclear. It has been proven [28], as a corollary of the Novikov theorem of unrecognizability of a sphere in five and higher dimensions, that the number of moves, required to transform one configuration into another is not bounded by any computable function of the configuration volume. It means, that the problem is ergodic in infinite time but it is infinitely complex. There is no rigorous proof of this conjecture in four dimensions, but it seems to be the same. This theorem, however, does not provide any idea about the measure of the fraction of configurations, which can not be reached in a functional time, so it can as well be zero. Besides, we are interested only in the configurations, which are in a sense close to the perturbative limit, so we might as well say, that we are not interested in these weird branches. The problem, however, is still open.

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References

[1] D.V. Boulatov, V.A.Kazakov, I.K. Kostov, A.A. Migdal, Nuclear Physics B275 [FS17] (1986), 641.

[2] A.Billoire, F.David, Nuclear Physics B275 [FS17] (1986), 617.

[3] J.Ambjorn, B.Durhuus, J.Fr’echlich,Nuclear Physics B270 [FS16] (1986), 457.

[4] H.W.Hamber, Nuclear Physics B99A (proc. suppl.) (1991).

[5] H.W.Hamber, Preprint UCI-90-60, November 1990.

[6] M.Agishtein, A.A.Migdal, International Journal of Modern Physics C 1, No.1, April (1990), 165.

[7] M.Agishtein, L. Jacobs, A.A.Migdal, J. Richardson, Mod. Phys. Lett. A, 5, 12, (1990).

[8] M.E. Agishtein, A.A. Migdal, Nuclear Physics B350 (1991), 690-728.
[9] F.David, Preprint RU-91-25.

[10] M.E.Agishtein, A.A. Migdal, Mod. Phys. Lett. A Vol. 6, No 20 (1991) pp. 1863-1884, see also Errata.

[11] M.Agishtein, A.A.Migdal, Numerical simulations of three dimensional quantum gravity, Nuclear. Phys. B, Proc. Suppl., 25A (1992), pp. 1-7.

[12] J. Ambjørn, S.Varsted, Niels Bohr Inst. Preprint NBI-HE-91-17.

[13] D.V. Boulatov, A. Krzywicki, preprint LPTHE Orsay 91/35.

[14] J. Ambjørn, S.Varsted, Niels Bohr Inst. Preprint NBI-HE-91-45.

[15] J. Ambjørn, D.V. Boulatov, A. Krzywicki, S.Varsted, Niels Bohr Inst. Preprint NBI-HE-91-46.

[16] M.E.Agishtein, A.A.Migdal, Simulations of four dimensional quantum gravity. PUPT-1287. October 1991.

[17] H. Kawai, M. Ninomiya, Nuclear Physics B336, pp. 115-145, (1990).

[18] J. Ambjorn, J. Jurkiewicz, Preprint NBI-HE-91-47.

[19] S.Varsted, Preprint UCSD/PTH 92/03, January-92.

[20] Misner C.W., Thorne K.S., Wheeler J.A., Gravitation, Freeman and Company, New York,1973.

[21] N.H.Christ, R.Feinberg, T.D.Lee, Nuclear Physics B202, pp. 89-125, (1982).

[22] D.V.Boulatov, V.A. Kazakov, Phys. Lett., 184b,(1987), pp. 247-252.

[23] D.V.Boulatov, V.A. Kazakov, Preprint NBI-HE-88-42, (1988).

[24] M.Agishtein, R.BenAv, A.A.Migdal, S.Solomon, Mod. Phys. Lett. A Vol. 6, No. 12, (1991), pp 1115-1132.

[25] M.Agishtein, R.BenAv, A.A.Migdal, S.Solomon, Mod. Phys. Lett. A Vol. 6, No. 12, (1991), pp 1115-1132.

[26] M.Gross, S.Varsted, preprint NBI-HE-91-33.

[27] M.E. Agishtein, A.A. Migdal, in preparation.

[28] R. BenAv, A. Nabutovsky, private communication.
Figure 1: \( < R(\lambda_0) > \) for \( N_4 = 1024 - 12000 \).
Figure 2: \( \langle \lambda(\lambda_0) \rangle \) for \( N_4 = 1024 - 12000 \).
Figure 3: Direct measurements of susceptibility $\chi$. Fit of $\lambda_0$ with the polynomial of $< R >$ and susceptibility curve obtained from this fit.
Figure 4: Finite size scaling analysis for $\chi^c$ and $\lambda^c_0$. 
Figure 5: Fit of the critical exponent $\gamma$: $\log |R^c - R| \approx |\lambda^c_0 - \lambda_0| * (1 - \gamma)$. The indexes on the different sides of the phase transition are clearly different.
Figure 6: Geometry at the critical point: Hausdorff histogram: log – log plot of the average volume inside a geodesic sphere versus its radius. Fit of the Hausdorff histogram by the power law: $< V(r) > \sim r^4$. 