On exact solutions of nonlinear acoustic equations

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I. INTRODUCTION

Propagation of sound pulses and sound beams in weakly nonlinear media with account of small curvature of wave fronts is of considerable interest for many applications in science and industry from geology to medicine (see, e.g. [1]). Mathematical theory of propagation of nonlinear sound pulses is often based on the study of the so-called Khokhlov-Zabolotskaya (KZ) equation. If we denote, say, the flow velocity of the medium at a point \(r\) at a moment \(t\) as \(u(r, t)\), then the KZ equation can be written in the form

\[
(u_t + (c_0 + \alpha u)u_x)_x + \frac{1}{2}c_0^2 \Delta u = 0,
\]

where \(x\) is the axis of propagation of a sound pulse, \(c_0\) is the sound velocity of linear waves in the medium under consideration, \(\Delta\) is the Laplace operator in directions normal to the propagation direction, i.e., \(\Delta = \partial_x^2 + \partial_y^2\) in 2D-geometry and \(\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2\) in 3D-geometry, and subscripts denote derivatives with respect to corresponding variables. In fact, this equation appeared first in the paper [2] in the context of the theory of compressible gas flow past a thin wing but at that time it was not related with the problem of propagation of strong sound pulses. Later it was derived by A.P. Sukhorukov in framework of nonlinear physics studies that were performing in R.V. Khokhlov group and was published in [3]. The name \textit{Khokhlov-Zabolotskaya equation} was coined by O.V. Rudenko and early history of this equation is described in much detail in his paper [4]. After the soliton theory arose in 70’s, equation (1) for a particular case of two spatial coordinated \((x\) and \(y\)) has drawn much attention as a dispersionless limit of the famous Kadomtsev-Petviashvili (KP) equation and the name \textit{KZ equation} has become commonly accepted.

Both for applications and mathematical theory of nonlinear waves it is important to have methods of finding physically reasonable solutions of the KZ equation and of other equations of the same type. First approximate solutions of this equation were studied already in the papers [3] (see also [5]). Methods related with the complete integrability of the KZ equation (dispersionless limits of the KP equation) were developed, for example, in [6–9] and in these papers some interesting exact solutions were presented. Another methods based on the contact geometry approach to nonlinear differential equations were developed in [10, 11] and they were applied, in particular, to the KZ equation and its particular solutions were found. However, all these methods are mathematically involved and hardly can be used in more general situations important for applications.

The aim of this paper is to present a simple and direct method of finding exact solutions of equations of KZ type. At first in section 2 we present derivation of particular solutions of Eq. (1) in two- and three-dimensional cases based on physical reasoning. These solutions coincide with those found in [9, 11] by much more complicated methods and their form suggests \textit{ansatz} for a generalization, developed in section 3, for finding exact solutions of other equations of this type. Our approach allows us to construct more general solutions than those found in the aforementioned papers and in the recent preprint [12]. In section 4 we illustrate the general theory by particular examples and in the last section 5 we present conclusions.

II. PHYSICAL DERIVATION OF THE EXACT SOLUTION OF THE KZ EQUATION

Although the KZ equation has a quite general nature, for definiteness we shall imply here a problem of propagation of a strong sound pulse through a gaseous medium.

In the main linear approximation the pulse propagates as a plane unidirectional wave, then the corresponding wave equation reduces to

\[
u_x + c_0 u_x = 0 \tag{2}
\]

with obvious solution

\[
u(x, t) = F(x - c_0 t), \tag{3}
\]

where \(F(X)\) is an arbitrary function that can be determined from the initial condition.
If we take into account weak nonlinear effects, then the sound velocity becomes dependent on the amplitude of the wave and the first approximation for the nonlinear correction to velocity of liquid particles in the wave profile is proportional to the amplitude \( u(x,t) \),

\[
\frac{dx}{dt} = c_0 + \alpha u(x,t),
\]

(4)

where the coefficient \( \alpha \) can be expressed in terms of the derivative of the pressure \( p \) on the specific volume \( V = 1/\rho \) (\( \rho \) being the gas density) as follows (see, e.g., [13]):

\[
\alpha = \frac{c_0^4}{2V^3} \left( \frac{\partial^2 V}{\partial p^2} \right)_s,
\]

where the derivative is calculated at the condition that the entropy \( s \) is constant. For example, in the case of a polytropic gas we have \( \alpha = (\gamma + 1)/2 \) where \( \gamma = c_p/c_v \) is the heat capacity ratio.

The nonlinear contribution \( \alpha u \) to the velocity leads to an additional displacement of the wave profile points and as a result the solution (3) should be modified as

\[
u = F[\alpha - (c_0 + \alpha u)t].
\]

(5)

This equation determines implicitly the dependence of \( u = u(x,t) \) on time and space coordinates for a given function \( F(X) \). In other words, the nonlinear correction transforms the linear equation (2) into the so-called Hopf equation

\[
u_t + (c_0 + \alpha u)u_x = 0
\]

(6)

whose characteristic equation coincides with Eq. (4).

For transition to the case of several space dimensions we have to take into account displacements caused by curvature of the wave pulse front. In a simplest way it can be done for cylindrical and spherical geometries, and such an additional displacement was calculated long ago by Landau [14] and Whitham [15] (see also section 102 in [13] and section 9.1 in [16]).

In cylindrically symmetric case a linear wave at large enough distance from the origin (i.e. for \( c_0 \tau \ll r \), where \( \tau \) is the pulse duration) can be represented by the asymptotic formula

\[
u \approx \frac{1}{\sqrt{r}} F(r - c_0t).
\]

(7)

After time \( t - t_0 = (r - r_0)/c_0 \) the pulse propagates due to nonlinear correction to velocity at the additional distance

\[
\delta r = \int_{t_0}^{t} \alpha u dt = \frac{\alpha F}{c_0} \int_{r_0}^{r} \frac{dr}{\sqrt{r}} = \frac{2\alpha F}{c_0} (\sqrt{r} - \sqrt{r_0}).
\]

Assuming \( r \gg r_0 \) and taking into account that in the additional term we can make a replacement \( F\sqrt{r} = ur \approx u_0 t \) and also replacing \( r \) by \( c_0 t \) in the coefficient \( 1/\sqrt{r} \), we modify (7) as follows:

\[
u = \frac{1}{\sqrt{t}} F[r - (c_0 + 2\alpha u)t].
\]

(8)

This formula describes evolution of a cylindrically symmetric weakly nonlinear wave.

In a similar way we start from the expression

\[
u = \frac{1}{r} F(r - c_0t)
\]

(9)

for the solution of a spherically symmetric linear wave equation and find the additional distance of propagation due to the nonlinear correction to the sound velocity,

\[
\delta r = \int_{t_0}^{t} \alpha u dt = \frac{\alpha F}{c_0} \int_{r_0}^{r} \frac{dr}{\sqrt{r}} = \frac{\alpha F}{c_0} \ln \frac{r}{r_0} \approx \alpha u t \ln \frac{t}{t_0}.
\]

After replacement with the same accuracy \( 1/r \rightarrow 1/(c_0t) \) we arrive at the expression

\[
u = \frac{1}{t} F \left[ r - (c_0 + \alpha u \ln \frac{t}{t_0}) \right],
\]

(10)

describing evolution of a spherically symmetric weakly nonlinear wave.

We can confirm validity of the expressions (8) and (10) by the perturbation theory calculations. As is known (see, e.g., [3]), in cylindrically or spherically symmetric geometries evolution of waves through weakly nonlinear medium is governed by the equation

\[
u_t + (c_0 + \alpha u)u_r + \kappa \frac{\nu}{t} = 0,
\]

(11)

where \( \kappa = 1/2 \) for cylindrical case and \( \kappa = 1 \) for spherical case. Making variables replacements

\[
u = \sqrt{\frac{t_0}{t}} U, \quad T = 2\sqrt{t_0}t, \quad X = r - c_0t
\]

(12)

in cylindrical case or

\[
u = \frac{U}{t}, \quad T = \ln \frac{t}{t_0}, \quad X = r - c_0t
\]

(13)

in spherical case, we transform Eq. (11) to the Hopf equation

\[
u_T + \alpha UU_X = 0
\]

(14)

with well-known solution

\[
u = \frac{1}{t} F(U)
\]

(15)

that is transformed to (8) and (10) after returning to the initial physical variables.

Now we can turn to the KZ equation (1). We notice that the solutions (8) and (10) for small values of transverse coordinates \( y, z \) compared with the propagation distance \( x \) must be the solutions of the KZ equation in the first order of the series expansion with respect to a small parameter \( y^2 + z^2/x^2 \equiv (y^2 + z^2)/(c_0t)^2 \) (term with \( z^2 \) should be omitted in the cylindrical case). Thus, we arrive at the solution

\[
u = \frac{1}{\sqrt{t}} F \left[ x + \frac{y^2}{2c_0t} - (c_0 + 2\alpha u)t \right]
\]

(16)
of cylindrical KZ equation and
\[ u = \frac{1}{t} F \left[ x + \frac{y^2 + z^2}{2ct} - \left( c_0 + \alpha u \ln \frac{t}{t_0} \right) t \right] \] (17)
of spherical KZ equation. These solutions were found respectively in [9] and [10, 11] by different more complicated mathematical methods.

On one hand, the presented here derivation shows that solutions of multi-dimensional equations that take into account small curvature of wave fronts can be found from corresponding symmetrical solution by their series expansions with respect to small transverse coordinates. Therefore this method can be easily generalized on other forms of nonlinearity and number of transverse dimensions. On the other hand, substitutions (12) or (13) suggest that similar transformations can be done directly in the KZ equation leading to its exact solutions. We shall develop such a direct method for quite general class of nonlinear acoustic equations in the next section.

III. DIRECT METHOD

For simplicity of notation, we shall consider here the KZ equation in non-dimensional form,
\[ (u_t + u^nu_{x})_x + \frac{1}{2} \Delta_{\perp} u = 0, \] (18)
for generalized nonlinearity \( w^nu_x \) and arbitrary number \( N \) of transverse spatial coordinates \( z = (z_1, \ldots, z_N) \),
\[ \Delta_{\perp} = \frac{\partial^2}{\partial z_1^2} + \ldots + \frac{\partial^2}{\partial z_N^2}. \] (19)
Of course, in standard physical application we have \( N = 1 \) or \( N = 2 \). Besides that, we shall confine ourselves to the case of integer positive values of \( n \).

Solutions (16) and (17) as well as substitutions (12) and (13) suggest that it might be possible to obtain a wide class of solutions of the equation (19) with the use of the following ansatz
\[ u(t,x,z) = a(T)U(X,T), \]
\[ X = x + \varphi(t,z), \quad T = T(t). \] (20)
In these new variables the equation (18) takes the form
\[ [T_t U_T + (T_t a_T/a + \frac{1}{2} \Delta_{\perp} \varphi) U + (\varphi_t + \frac{1}{2} (\nabla \varphi)^2 + a^n U^n) U]_X = 0, \] (21)
where subscripts denote derivatives with respect to corresponding variables. If we assume that
\[ T_t = a^n, \quad \varphi_t + \frac{1}{2} (\nabla \varphi)^2 = ka^n, \quad \Delta_{\perp} \varphi = 2a^n (m-a_T/a), \]
where \( k \) and \( m \) are some constants, then Eq. (21) transforms to the equation
\[ (U_T + (U^n + k) U_X + mU)_X = 0. \] (22)
The term with \( k \) can be excluded by means of the additional replacement
\[ \varphi = \psi(t,z) + kT(t), \] (23)
where \( \psi \) satisfies the equations
\[ \psi_t + \frac{1}{2} (\nabla \psi)^2 = 0, \quad \Delta_{\perp} \psi = 2a^n (m-a_T/a), \] (24)
and evolution of \( U(X,T) \) defined in (20) and evolving according to Eq. (22) is governed by the generalized Hopf equation
\[ (U_T + U^n U_X + mU)_X = 0. \] (25)
The first equation in (24) coincides with the Hamilton-Jacobi equation for a free particle moving in \( N \)-dimensional space. Its particular solution can be easily found by separation of variables as a sum of "actions" corresponding to separate space coordinates. Most interesting for us solution (17) reads
\[ \psi(t,z) = \frac{1}{2} \sum_{p=1}^{N} \frac{z_p^2}{t - t_p}, \] (26)
where \( t_p, p = 1, \ldots, N \), are integration constants. Then substitution of this formula in the left-hand side of the second equation (24) yields
\[ a^{n-1} a_T - ma^n = - \frac{1}{2} \sum_{p=1}^{N} \frac{1}{t - t_p}. \]
Taking into account \( a^n = T_t, \) \( na^{n-1} a_T = (n/a) a_T T_t = n(a_t/a) = (a^n)_t/a^n \) and introducing \( y = a^n = T_t \), we arrive at the Bernoulli equation
\[ y' + \left( \frac{1}{2} \sum_{p=1}^{N} \frac{1}{t - t_p} \right) y = nmy^2 \]
that can be solved by a standard method to give
\[ y = T_t = -Ce^{-nmT(t)} \prod_{p=1}^{N} |t - t_p|^{-n/2} \] (27)
or
\[ (e^{-nmT(t)})^T = nmC \prod_{p=1}^{N} |t - t_p|^{-n/2}, \] (28)
where \( C \) is an integration constant. If \( m = 0 \) then Eq. (27) simplifies to
\[ T_t = -C \prod_{p=1}^{N} |t - t_p|^{-n/2}. \] (29)
Thus, we have reduced finding \( T(t) \) to integration of the function in the right-hand side of Eqs. (28) or (29). When \( T(t) \) is found, \( a(t) \) is determined as \( a(t) = (T(t))^{1/n} \).
As a result of the above calculations, the variables in (20) can be considered as known and it remains to find the solution of the generalized Hopf equation (25). Its integration with respect to $X$ gives at once

$$U_T + U^n U_X + mU = g(T),$$

(30)

where $g(T)$ is an arbitrary function to be determined from the initial conditions. Equation (30) can be solved by a standard method of characteristics. Along characteristic curve starting at the point $U_0(X)$ at $T = 0$ we have

$$U_T + mU = g(T)$$

and this linear differential equation can be easily solved to give

$$U = (U_0(X) + G(T))e^{-mT}$$

(31)

where we denoted

$$G(T) = \int_0^T e^{m\tau} g(\tau) d\tau.$$  

(32)

The characteristic curve is determined by the equation

$$\frac{dX}{dT} = U^n = (U_0 + G(T))^n e^{-nmT} = \sum_{q=0}^{n} \binom{n}{q} U_0^{n-q} G^q(T) e^{-nmT}$$

and its integration yields

$$X = X_0 + \sum_{q=0}^{n} \binom{n}{q} U_0^{n-q} \int_0^T G^q(\tau) e^{-nm\tau} d\tau.$$  

(33)

Let the initial distribution $U(X)$ at $T = 0$ be given by the function $U_0 = F^{-1}(X_0)$, or in implicit form by the function $X_0 = F(U_0)$; then exclusion of $X_0$ and $U_0$ from (33) gives the final result:

$$X = F(U e^{mT} - G(T)) + \sum_{q=0}^{n} \binom{n}{q} (U e^{mT} - G(T))^{n-q} \int_0^T G^q(\tau) e^{-nm\tau} d\tau.$$  

(34)

This equation determines implicitly $u$ as a function of $x, z, t$ through variables

$$X = x + kT(t) + \frac{1}{2} \sum_{p=1}^{N} \frac{z_p^2}{t - t_p},$$

$$T = T(t), \quad u = a(T)U(X, T)$$

in terms of two arbitrary functions $F(U)$ and $G(T)$ which have to be found from the initial conditions.

IV. EXAMPLES

First of all, let us check that our general formulas reproduce the known solutions (16) and (17).

In case of a cylindrical KZ equation

$$(u_t + uu_x)_x + \frac{1}{2} u_{yy} = 0$$

(36)

written here in standard non-dimensional notation we choose $n = 1, N = 1, k = m = 0, t_1 = 0$. Then we have $a(t) = T_t = t^{-1/2}$ and $X = x + y^2/(2t)$. Consequently $T(t) = 2t^{1/2}$ and $U = t^{1/2}u$. Hence Eq. (36) has a solution

$$x + \frac{y^2}{2t} - 2tu = F^{-1}(t^{1/2}u)$$

or

$$u = \frac{1}{\sqrt{t}} F \left( x + \frac{y^2}{2t} - 2tu \right)$$

(37)

which up to notation coincides with Eq. (16).

In a similar way in case of spherical KZ equation

$$(u_t + uu_x)_x + \frac{1}{2} (u_{yy} + u_{zz}) = 0$$

(38)

we choose $n = 1, N = 2, k = m = 0, t_1 = t_2 = 0$ and obtain $a(t) = T_t = 1/t$, $X = x + (y^2 + z^2)/(2t)$; consequently $T(t) = \ln(t/t_0), U = tu$ and Eq. (38) has a solution

$$x + \frac{y^2 + z^2}{2t} - t \ln \left( \frac{t}{t_0} \right) u = F^{-1}(tu)$$

or

$$u = \frac{1}{t} F \left( x + \frac{y^2 + z^2}{2t} - t \ln \left( \frac{t}{t_0} \right) u \right)$$

(39)

which again upon notation coincides with Eq. (17).

Now we notice that the constants $t_0$ in (26) and other formulas can have either sign what means that we can consider sound pulses focused in some transverse directions and defocused in the other directions. Exact solutions of the KZ equation for such situations, to the best of our knowledge, have not been considered earlier and we shall apply here our approach to a problem of this kind.

Thus, we wish to find the solution of the KZ equation (35) modeling propagation of a nonlinear sound pulse that is defocused in $y$ direction and focused in $z$ direction. To this end, we take in the above formulas $n = 1, N = 2, m = 0, k = 0, t_1 < 0, t_2 > 0, g(T) = 0$, so that

$$a(t) = \frac{dT}{dt} = \frac{1}{\sqrt{(t + |t_1|)(t_2 - t)}}$$

(40)

and, consequently,

$$T(t) = 2 \left( \arctan \sqrt{\frac{t + |t_1|}{t_2 - t}} - \arctan \sqrt{\frac{|t_1|}{t_2}} \right),$$

(41)
where the integration constant is chosen in such a way that \( T(0) = 0 \). We assume here that \( 0 \leq t < t_2 \). The self-similar variable has now the form

\[
X = x + \frac{1}{2} \left( \frac{y^2}{t + |t_1|} - \frac{z^2}{t_2 - t} \right).
\]

The variable \( u(x, y, z, t) \) is expressed in terms of \( U(X, T) \) as

\[
u(x, y, z, t) = a(T)U(X, T) = \frac{U(X, T)}{\sqrt{(t + |t_1|)(t_2 - t)}}.
\]

where \( U \) obeys the Hopf equation

\[
U_T + UU_X = 0.
\]

We assume that at \( t = 0 \) the distribution of \( u_0(x, y, z) \) depends on a single self-similar variable

\[
u_0(x, y, z) = F^{-1} \left( x + \frac{1}{2} \left( \frac{y^2}{t_1} - \frac{z^2}{t_2} \right) \right).
\]

Then the solution of Eq. (44) can be written as \( X - UT = F(U) \) or, returning to the original variables,

\[
x + \frac{1}{2} \left( \frac{y^2}{t + |t_1|} - \frac{z^2}{t_2 - t} \right) = 2\sqrt{(t + |t_1|)(t_2 - t)}
\]

\[
\times \left( \arctan \sqrt{\frac{t + |t_1|}{t_2 - t}} - \arctan \sqrt{\frac{|t_1|}{t_2}} \right) u
\]

\[
= F(\sqrt{(t + |t_1|)(t_2 - t)}/(|t_1|t_2)) u.
\]

This formula determines implicitly \( u \) as a function of space coordinates at any moment of time \( t \) in the interval \( 0 \leq t < t_2 \). It is worth noticing that this restriction makes it impossible to take the limit \( t_1 = t_2 = 0 \) and to reproduce the solution (49).

It is convenient to represent this solution in a parametric form. To this end we notice that the initial \( u_0 \) given in the form (43) is a function of a single variable

\[
X_0 = x + \frac{1}{2} \left( \frac{y^2}{|t_1|} - \frac{z^2}{t_2} \right).
\]

The corresponding value \( U = u_0/\sqrt{|t_1|t_2} \) is constant along characteristic \( X = X_0 + UT \), that is

\[
x + \frac{1}{2} \left( \frac{y^2}{t + |t_1|} - \frac{z^2}{t_2 - t} \right) = X_0 + 2\sqrt{|t_1|t_2}
\]

\[
\times \left( \arctan \sqrt{\frac{t + |t_1|}{t_2 - t}} - \arctan \sqrt{\frac{|t_1|}{t_2}} \right) u_0(X_0).
\]

On this surface in 3D-space for fixed \( t \) the dependent variable \( u \) takes the value

\[
u = \sqrt{\frac{|t_1|t_2}{(t + |t_1|)(t_2 - t)}} u_0(X_0).
\]

Thus the above solution is parameterized by a parameter \( X_0 \) on which depend both \( u \) and the self-similar variable \( X \). Exclusion of \( X_0 = F(\sqrt{(t + |t_1|)(t_2 - t)}/(|t_1|t_2)) \) and \( u_0(X_0) \) from (47) and (49) reproduces Eq. (46). As we see, in the limit \( t \to t_2 \) the amplitude goes to infinity as \( u \propto 1/\sqrt{t_2 - t} \) due to focusing.

We illustrate this evolution for the case of the initial distribution

\[
u_0(x, y, z) = \frac{1}{1 + (x + \frac{1}{2} (y^2 - z^2))^2}.
\]

The profile of the pulse along the \( x \)-axis for \( y = 0, z = 0 \) is shown in Fig. 1 for several moments of time. On the contrary to evolution of defocused pulse, now its amplitude increases at the axis and the profile steepens due by virtue of nonlinear effects.

V. CONCLUSION

We have demonstrated in this paper that exact nonlinear solutions of the KZ type equations can be obtained in framework of a simple enough ansatz leading to partial separation of variables: evolution in transverse direction reduces effectively to free propagation of rays that is governed by simple Hamilton-Jacobi equation whereas nonlinear effects are described by ordinary differential equation that can be integrated in a closed form. It is important that our method is effective not only in case of completely integrable situations, when the KZ equation represents a dispersionless limit of the KP equation, but also in multi-dimensional geometries and generalized nonlinearities. This allows one to obtain exact solutions in many realistic situations.

From physical point of view, this method combines nonlinear effects with linear diffraction for wave fronts
described by multi-dimensional paraboloid surfaces with arbitrary signs and values of curvature radii. In practically most important 3D case, this yields the exact solutions for focused/defocused pulses with engineered phase fronts. One may hope that such solutions can find many applications in science and technology.

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[1] Nonlinear Acoustic, eds. M. F. Hamilton, D. T. Blackstock, (Academic Press, San Diego, California, 1998).
[2] C. Lin, E. Reissner, and U. S. Tsien, J. Math. Phys. 27, 220 (1948).
[3] E. A. Zabolotskaya and R. V. Khokhlov, Akustich. Zh., 15, 40 (1969) [Sov. Phys. Acoust. 15, 35 (1969)]; 16, 49 (1970) [Sov. Phys. Acoust. 16, 39 (1970)].
[4] O. V. Rudenko, Akustich. Zh., 56, 452 (2010) [Acoust. Phys., 56, 457 (2010)].
[5] O. V. Rudenko and S. I. Soluyan, Theoretical Foundations of Nonlinear Acoustics, (Plenum, New York, 1977).
[6] Y. Kodama, Phys. Lett. A 129, 223 (1988).
[7] Y. Kodama and J. Gibbons, Phys. Lett. A 135, 167 (1989).
[8] E. V. Ferapontov and K. R. Khusnutdinova, Comm. Math. Phys., 248, 187 (2004).
[9] S. V. Manakov and P. M. Santini, J. Phys. A: Math. Theor., 44, 405203 (2011).
[10] V.V. Lychagin, Uspekhi Mat. Nauk, 1979, 34, 137, (1979) [Russian Math. Surveys, 34, 149 (1979)].
[11] A. Kushner, V. Lychagin, and V. Rubtsov, Contact Geometry and Nonlinear Differential Equations, (Cambridge University Press, 2007).
[12] F. Santucci and P. M. Santini, preprint arXiv:1512.05187 (2015).
[13] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, (Pergamon, Oxford, 1987).
[14] L. D. Landau, Prikl. Matem. Mekh., 9, 286 (1945) [L. D. Landau, Collected Papers, p. 437, (Gordon and Breach, New York, 1965)].
[15] G. B. Whitham, Proc. Roy. Soc. Lond. A 203, 571 (1950).
[16] G. B. Whitham, Linear and Nonlinear Waves, (Wiley, New York, 1974).
[17] I. Yehorchenko, preprint arXiv:1412.1889 (2014).