DELETION-RESTRICTION FOR LOGARITHMIC FORMS ON MULTIARRANGEMENTS

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Abstract. We consider the behaviour of logarithmic differential forms on arrangements and multiarrangements of hyperplanes under the operations of deletion and restriction, extending early work of Günter Ziegler [24]. The restriction map of logarithmic forms to a hyperplane is not necessarily surjective, and we measure the failure of surjectivity in terms of commutative algebra of logarithmic forms and derivations. We find that the dual notion of restriction of logarithmic vector fields behaves similarly but inequivalently. A main result is that, if an arrangement is free, then any arrangement obtained by adding a hyperplane has the “dual strongly plus-one generated” property. One application is another proof of a main result of [4] characterizing when adding a hyperplane to a free arrangement remains free. A further application is to resolve two conjectures due to Ziegler, which we defer to a sequel [8].

1. Introduction

Let $V = \mathbb{K}^\ell$ and $S = \mathbb{K}[x_1, \ldots, x_\ell]$ its coordinate ring. Let $\text{Der } S := \oplus_{i=1}^\ell S \partial_{x_i}$, $\Omega^1_V := \oplus_{i=1}^\ell S dx_i$, and $\Omega^p_V = \wedge^p \Omega^1_V$. Let $\mathcal{A}$ be a central hyperplane arrangement: that is, a finite set of linear hyperplanes in $V$. For each $H \in \mathcal{A}$ fix $\alpha_H \in V^*$ such that $\ker \alpha_H = H$, and let $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H$. Then (the module of) logarithmic vector fields of $\mathcal{A}$ is defined to be

$$D(\mathcal{A}) := \{ \theta \in \text{Der } S \mid \theta(\alpha_H) \in S \alpha_H \ (\forall H \in \mathcal{A}) \},$$

and (the module of) logarithmic differential (1-)forms is

$$\Omega^1(\mathcal{A}) := \{ \omega \in \frac{1}{Q(\mathcal{A})} \Omega^1_V \mid \omega \wedge d\alpha_H \in \frac{\alpha_H}{Q(\mathcal{A})} \Omega^2_V \ (\forall H \in \mathcal{A}) \}.$$ 

These graded $S$-modules are mutually dual and reflexive of rank $\ell$ (see [19]), but in general they are not free. We say that $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, d_2, \ldots, d_\ell)$ provided that

$$D(\mathcal{A}) \cong \oplus_{i=1}^\ell S[-d_i] \quad \text{or, equivalently,} \quad \Omega^1(\mathcal{A}) \cong \oplus_{i=1}^\ell S[d_i].$$

The study of free arrangements has an extensive history: see, for example, [3, 5, 6, 8] for recent results and references. One of the subtleties of the theory is that deletion-contraction arguments, which are based on both removing and restricting to a hyperplane, generally work well only with additional hypotheses on freeness and
degrees of generation. For example, Terao’s Addition-Deletion Theorem [21] states that, if two of \( \mathcal{A}, \mathcal{A}' \) and \( \mathcal{A}^H \) are free, so is the third, provided a condition on the exponents is satisfied. Here, for a choice of \( H \in \mathcal{A} \), we let \( \mathcal{A}' := \mathcal{A} - \{H\} \), and

\[
\mathcal{A}^H := \{ L \cap H \mid L \in \mathcal{A}' \},
\]
an arrangement in \( H \subseteq V \).

At the heart of Terao’s argument there is the Euler exact sequence

\[
0 \to D(\mathcal{A}') \xrightarrow{\alpha H} D(\mathcal{A}) \xrightarrow{\rho^H} D(\mathcal{A}^H),
\]
where \( \rho^H \) is induced by the restriction map \( S \to S/(\alpha_H) \). Under his hypotheses, the map \( \rho^H \) is surjective, and logarithmic derivations on \( \mathcal{A} \) can be understood in terms of those on the smaller arrangements \( \mathcal{A}' \) and \( \mathcal{A}^H \). In general, though, the sequence (1.1) fails to be right exact. We show in Theorem 5.8 how to extend it to a long exact sequence.

The analogous exact sequence of logarithmic forms was first considered by Ziegler in [24]:

\[
0 \to \Omega^1(\mathcal{A}) \xrightarrow{\alpha H} \Omega^1(\mathcal{A}') \xrightarrow{i_H^*} \Omega^1(\mathcal{A}^H),
\]
The first map is given by multiplication by \( \alpha_H \), and the second by restricting a one-form along the inclusion of the hyperplane, \( i_H : H \to V \). Once again, the sequence (1.2) is right exact only under hypotheses that we elucidate in this paper. Since \( D(\mathcal{A}) \) and \( \Omega^1(\mathcal{A}) \) are dual, the two sequences have a similar flavour but also obvious differences. For example, the first author shows [6] that, if \( \mathcal{A} \) is free, then for any \( H \in \mathcal{A} \), that \( \text{pd}_S D(\mathcal{A}') \leq 1 \): see Theorem 1.4. On the other hand, even if \( \mathcal{A} \) is free, there are examples for which \( \text{pd}_S \Omega^1(\mathcal{A}') = \ell - 2 \), which equals the upper bound imposed by reflexivity.

**Multiarrangements.** The arguments we make in this article work in the somewhat more general context of multiarrangements, where each hyperplane \( H \) is assigned a nonnegative integer multiplicity, \( m(H) \). This is natural from various points of view: we refer to [25], [12], and to \( \S 2 \) for definitions. For any arrangement \( \mathcal{A} \) with multiplicity function \( m \in \mathbb{Z}^A \), Terao, Wakefield and the first author [12, Prop. 2.2] showed that the sequence (1.1) generalizes to a sequence

\[
0 \to D(\mathcal{A}, m') \xrightarrow{\alpha H} D(\mathcal{A}, m) \xrightarrow{\rho^H} D(\mathcal{A}^H, m^*),
\]
where \( m' \) is obtained from \( m \) by lowering the multiplicity of \( H \) by one, and \( m^* \) is the Euler multiplicity they introduce (Definition 2.1).

In Section \( \S 2 \) we show that the corresponding sequence for logarithmic forms exists and generalizes (1.2) in the same way:

**Proposition 1.1.** For any \( H \in \mathcal{A} \), there is an exact sequence

\[
0 \to \Omega^1(\mathcal{A}, m) \xrightarrow{\alpha H} \Omega^1(\mathcal{A}, m') \xrightarrow{i_H^*} \Omega^1(\mathcal{A}^H, m^*).\]
For multiarrangements, we obtain the following conditions for right exactness. We generalize this result to multiarrangements as follows. Once again, let \((A, m)\) be a multiarrangement and let \(m'\) be the vector obtained from \(m\) by lowering the multiplicity of \(H\) by one.

**Theorem 1.2** (Free surjection theorem). Suppose that \((A, m')\) is free for some hyperplane \(H \in A\). Then

\[
\rho^H : D(A, m) \to D(A^H, m^*)
\]

is surjective.

In the multiplicity-free case, this was established earlier by the first author [7]. The dual version is analogous:

**Theorem 1.3** (Free surjection theorem 2). Suppose that \((A, m)\) is free. Then

\[
i^*_H : \Omega^1(A, m') \to \Omega^1(A^H, m^*)
\]

is surjective for all \(H \in A\).

In §5 we refine these results using duality. For example, if \(C_D\) denotes the cokernel of the injective map

\[ D(A, m') \xrightarrow{\alpha_H} D(A), \]

we show in Proposition 5.7 that the dual of \(C_D\) always equals \(\Omega^1(A^H, m^*)\). This leads to a long exact sequence extending (1.3). Parallel results exchange the roles of logarithmic derivations and forms.

**SPOG properties.** In §3, we consider arrangements which are, from the point of view of logarithmic derivations, the next simplest after free arrangements. These are arrangements for which \(D(A)\) is generated by \(\ell + 1\) elements which are subject to a single relation: see Definition 3.1 for the precise definition of strongly plus-one generated (SPOG) arrangements. The first author [6] showed the following:

**Theorem 1.4** (Theorem 1.13, [6]). Suppose \(A\) is a free arrangement. Then each deletion \(A'\) is either free or SPOG.

Here, we generalize this result to multiarrangements (Theorem 3.8). The analogous statement for logarithmic forms is known to be false, by Edelman–Reiner’s famous counterexample to Orlik’s conjecture: we refer to [6] for details. In this paper, we introduce a condition which we call dual SPOG (Definition 3.2), which turns out to be the appropriate replacement for “strongly plus-one generated”:

**Theorem 1.5.** Let \(A'\) be free. Then \(A\) is free or dual SPOG.

The multiarrangement version of this result appears as the second part of Theorem 3.8.

In §4, we apply our “SPOG” results in order to give a short new proof of the combinatorial addition-deletion theorem of Abe [4]. In §6, we introduce a uniform
way to understand the two Euler exact sequences together with SPOG-type results, using the concept of negative multiplicities.

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2. Restrictions of forms and surjectivity

In this section, we prove Proposition 1.1. For that, first let us recall the notation used in Proposition 1.1. A multiarrangement is the pair \((\mathcal{A}, m)\), where \(\mathcal{A}\) is an arrangement and \(m\) is a map \(m : \mathcal{A} \to \mathbb{Z}_{>0}\). One may think of a multiarrangement as a linear realization of a matroid without loops in which parallel elements are regarded as unordered: then \(\mathcal{A}\) is a realization of the underlying simple matroid, and the multiplicities \(m(H)\) keep track of the size of each equivalence class of parallel elements.

Let \(Q(\mathcal{A}, m) := \prod_{H \in \mathcal{A}} \alpha^m_H\) and \(|m| := \deg Q(\mathcal{A}, m)\). Then the module of logarithmic derivations is defined to be

\[
D(\mathcal{A}, m) := \{\theta \in \text{Der} S | \theta(\alpha_H) \in S\alpha^m_H \ (\forall H \in \mathcal{A})\},
\]

and (the module of) logarithmic differential \((1-)\)forms is

\[
\Omega^1(\mathcal{A}, m) := \{\omega \in \frac{1}{Q(\mathcal{A}, m)} \Omega^1 \mid \omega \wedge d\alpha_H \in \frac{\alpha^m_H}{Q(\mathcal{A}, m)} \Omega^1 \ (\forall H \in \mathcal{A})\}.
\]

They are again \(S\)-dual modules, and freeness and exponents are defined in the same way as for \(\mathcal{A}\). Note that \(\mathcal{A}\) can be regarded as a multiarrangement \((\mathcal{A}, 1)\).

Let us define the Euler exact sequence for multiarrangements. For this, we recall that Euler restriction has a nontrivial generalization in the presence of multiplicities.

**Definition 2.1** (12). Let \((\mathcal{A}, m)\) be a multiarrangement, \(H \in \mathcal{A}\) and let \(\delta_H : \mathcal{A} \to \{0, 1\}\) be the characteristic multiplicity of \(H\), i.e., \(\delta_H(L) = 1\) if \(H = L\), and 0 otherwise. Define \(m' := m - \delta_H\). Let \(X \in \mathcal{A}^H\). Then \(\mathcal{A}_X = \mathcal{A}_X^c \times \emptyset_{l-2}\), where \(\mathcal{A}_X^c\) is the essentialization of \(\mathcal{A}_X\) and \(\emptyset_{l-2}\) is the empty arrangement in \(\mathbb{R}^{l-2}\). Thus \(D(\mathcal{A}_X^c, m_X)\) has a basis \(\theta_X, \varphi_X\) such that \(\alpha_H \nmid \theta_X\) and \(\alpha_H \mid \varphi_X\). In this terminology, define the Euler multiplicity \(m^* : \mathcal{A}^H \to \mathbb{Z}_{>0}\) by

\[
m^*(X) := \deg \theta_X.
\]

We say that \((\mathcal{A}^H, m^*)\) is the Euler restriction of \((\mathcal{A}, m)\) onto \(H\).

**Remark 2.2.** When \(m \equiv 1\), then it is easy to see that \(m^* \equiv 1\).

Now let us prove the exactness of the sequence (1.4):

**Proof of Proposition 1.1**. It suffices to show that \(i_H^* (\omega) \in \Omega^1(\mathcal{A}^H, m^*)\) for \(\omega \in \Omega^1(\mathcal{A}, m')\).

Let \(Q_1 := Q(\mathcal{A}_X, m'_X)\) and \(Q_2 := Q(\mathcal{A}, m')/Q_1\). Then \(Q_2 \omega \in \Omega^1(\mathcal{A}_X, m'_X) = (\theta_X, \alpha_H \varphi_X)_S\). Thus \(i_H^* (\omega)\) is in the image of \(i_H^* (\theta_X),\) which is of the form \((\alpha_X)^{-d\theta_X} \varphi_X\). Hence \(i_H^* (\omega)\) has a pole of order \(\deg \theta_X = m^*(X)\), which completes the proof. \(\square\)
Recall that
\[ D^p(A, m) : = \{ \theta \in \wedge^p \text{Der } S \mid \theta(\alpha_H, f_2, \ldots, f_p) \in S\alpha_H^{m_H} (\forall H \in A, \forall f_2, \ldots, f_p \in S) \}, \]
\[ \Omega^p(A, m) : = \{ \omega \in \frac{1}{Q(A, m)}\Omega^p \mid (Q(A, m)/\alpha_H^{m_H})\omega \wedge d\omega_H \in \Omega^1_V (\forall H \in A) \}. \]

The following isomorphism is well-known when \( m = 1 \): see, e.g., [23].

**Lemma 2.3.** For all \( 0 \leq p \leq \ell \), we have
\[ \Omega^p(A, m) \cong D^{\ell-p}(A, m). \]

**Proof.** For \( I \subset [\ell] := \{1, \ldots, \ell\} \), let \( I^c := \{i \in [\ell] \mid i \notin I\} \). Define
\[ dx_I := \wedge_{i \in I} dx_i, \quad \partial_I := \wedge_{i \in I} \partial x_i, \]
and \([\ell]_p := \{I \in [\ell] \mid |I| = p\}\). Then the map
\[ \Omega^p(A, m) \ni \sum_{I \in [\ell]_p} (f_I/Q(A, m))dx_I \mapsto \sum_{I \in [\ell]_p} f_I \partial_I \in D^{\ell-p}(A, m) \]
is easily seen to be an isomorphism. \( \square \)

We introduce Euler sequences in higher degrees as follows:

**Proposition 2.4.** For any \( p \geq 0 \), there are exact sequences
\[ 0 \to D^p(A, m - \delta_H) \overset{\alpha_H^*}{\longrightarrow} D^p(A, m) \overset{\alpha_H^*}{\longrightarrow} D^p(A^H, m^*) \]
\[ 0 \to \Omega^p(A, m - \delta_H) \overset{i^*_H}{\longrightarrow} \Omega^p(A, m) \overset{i^*_H}{\longrightarrow} \Omega^p(A^H, m^*). \]

**Proof.** The first statement is proven in [18], Prop. 4.10, and the argument for the second is similar to that of Proposition [18]. We must show that \( i^*_H(\omega) \in \Omega^p(A^H, m^*) \) for any \( \omega \in \Omega^p(A, m - \delta_H) \). Let \( \alpha_H = x_1 \). Let \( X = \{x_1 = x_2 = 0\} \in A^H \), \( Q_1 := Q(A_X, (m - \delta_H)_X) \) and \( Q_2 := Q(A, m - \delta_H)/Q_1 \). Then \( Q_2 \omega \in \Omega^1(A_X, (m - \delta_H)_X) = \langle \omega^X_1, \alpha_H \omega^X_2, \partial x_3, \ldots, \partial x_\ell \rangle \). Thus \( i^*_H(\omega) \) is in the image of \( i^*_H(\omega^X_1) \), which is of the form \((\alpha_X)^{-\deg x_2} \partial x_2\). Hence \( i^*_H(\omega) \) has a pole of order \(-\deg \omega^X_1 = m^*(X)\), which completes the proof. \( \square \)

We will use the fact that logarithmic derivations and forms behave well under localization, following [19], §4.6. If \( p \) is a prime ideal of \( S \), let \( X(p) \) be the intersection of all hyperplanes of \( A \) containing \( V(p) \). Then
\[ \Omega^p(A, m)_p = \Omega^p(A_X, m_X)_p \text{ for all } p \geq 0, \]
where \( X = X(p) \), and analogously for derivations. In particular, the localization \( \Omega^1(A, m)_p \) is a free \( S_p \)-module if and only if the multiarrangement \((A_X, m_X)\) is free.

To prove Theorem [18] we need the following.
Proposition 2.5.

\[ H^0_0(\Omega^p(\overline{A}, m)) := \bigoplus_{j \in \mathbb{Z}} H^0_0(\Omega^p(\overline{A}, m)(j)) = \Omega^p(\overline{A}, m), \]

and

\[ H^0_0(D^p(\overline{A}, m)) := \bigoplus_{j \in \mathbb{Z}} H^0_0(D^p(\overline{A}, m)(j)) = D^p(\overline{A}, m). \]

Proof. The modules \( \Omega^p(\overline{A}, m) \) and \( D^p(\overline{A}, m) \) are reflexive, so their depth is at least two [16, Prop. 1.3], and the conclusion follows by comparing with local cohomology: see, e.g., [14, Cor. A1.13]. \( \square \)

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. The statement is clear if \( \ell \leq 2 \). We argue by induction on rank, using (2.3). For \( \ell \geq 3 \), this gives an exact sequence of sheaves on \( P^{\ell-1} \),

\[
0 \to \Omega^1(\overline{A}, m) \xrightarrow{\alpha} \Omega^1(\overline{A}, m - \delta_H) \xrightarrow{i^*_H} \Omega^1(\overline{A^H}, m^*) \to 0.
\]

Now we write the long exact sequence in cohomology, with the help of Proposition 2.5. By hypothesis, \((\overline{A}, m)\) is free, so by Proposition 2.5 the vector bundle \( \Omega^1(\overline{A}, m) \) is split. It follows that

\[
H^1_*(P^{\ell-1}, \Omega^1(\overline{A}, m)) = 0,
\]

since \( \ell \geq 2 \), which implies

\[
0 \to \Omega^1(\overline{A}, m) \xrightarrow{\alpha} \Omega^1(\overline{A}, m - \delta_H) \xrightarrow{i^*_H} \Omega^1(\overline{A^H}, m^*) \to 0
\]

is exact. \( \square \)

The same argument, using the fact that \( \Omega^p(\overline{A}, m) = \wedge^p \Omega^1(\overline{A}, m) \) and \( D^p(\overline{A}, m) \cong \wedge^p D(\overline{A}, m) \) when \( \overline{A} \) is free (see [19] for example), extends Theorem 1.3.

Corollary 2.6. For all \( p \geq 1 \),

1. The Euler restriction map \( \rho^p : D^p(\overline{A}, m) \to D^p(\overline{A^H}, m^*) \) in Proposition 2.4 is surjective if \((\overline{A}, m - \delta_H)\) is free.

2. The restriction map \( i^*_H : \Omega^p(\overline{A}, m - \delta_H) \to \Omega^p(\overline{A^H}, m^*) \) in Proposition 2.4 is surjective if \((\overline{A}, m)\) is free.

3. SPOG results for \( \Omega^1(\overline{A}) \)

It is known [6] that if \( \overline{A} \) is a free arrangement, then the structure of \( D(\overline{A}') \) is either free or “nearly” free in the following sense. We say that \( \overline{A}' \) is strongly plus-one generated (SPOG) if there is a minimal set of homogeneous generators \( \theta_E, \theta_2, \ldots, \theta_\ell, \varphi \) for \( D(\overline{A}') \) satisfying a relation, unique up to a nonzero scalar multiple,

\[
\sum_{i=2}^{\ell} f_i \theta_i + \alpha \varphi = 0
\]
where $0 \neq \alpha \in V^*$. This is to say that the generators of $D(A')$ have a single relation, and that at least one of the coefficients in that relation is linear. See [6] for details.

On the other hand, even if $\Omega^1(A)$ is free, $\Omega^1(A')$ need not even have projective dimension 1. We take a more systematic approach.

**Definition 3.1.** A graded module $S$-module $M$ of rank $\ell$ is *strongly plus-one generated (SPOG)* if there is a minimal free resolution of the following form:

$$0 \to S[d-1] \xrightarrow{(f_1, \ldots, f_\ell, \alpha)} \oplus_{i=1}^\ell S[d_i] \oplus S[d] \to M \to 0,$$

where $d, d_i \in \mathbb{Z}$ and $0 \neq \alpha \in V^*$. We call POexp($M$) := $(-d_1, \ldots, -d_\ell)$ the exponents of the SPOG module $M$, and $-d$ the level. We call the corresponding degree $d$-element the level element.

In [6], the SPOG property of $D(A)$ was introduced and studied, leading to Theorem 1.4 in the introduction. Let us study the analogue for logarithmic forms.

**Definition 3.2.** We say that $(A, m)$ is dual SPOG if $\Omega^1(A, m)$ is a SPOG module: that is, if there is a minimal free resolution

$$0 \to S[d-1] \to S[d] \oplus (\oplus_{i=1}^\ell S[d_i]) \to \Omega^1(A, m) \to 0,$$

so that POexp($\Omega^1((A, m))$) = $(-d_1, \ldots, -d_\ell)$ with level $-d$.

The following is immediate from the definition and a result of Denham and Schulze [13, Prop. 5.18] that the second Betti number $b_2(A, m)$ agrees with the second Chern class $c_2(\tilde{\Omega}^1(A))$. Note that the grading convention in [13] differs by a shift of one. Here, $b_2(A, m)$ is the coefficient of $t^{\ell-2}$ in the characteristic polynomial the multiarrangement introduced in [11, Def. 2.6].

**Proposition 3.3.** Suppose that $(A, m)$ is dual SPOG with exponents $(-d_1, -d_2, \ldots, -d_\ell)$ and level $-d$, for some $d_i, d \in \mathbb{Z}$. Then we have

$$|m| = 1 + d_1 + d_2 + \cdots + d_\ell,$$

$$b_2(A, m) = \sum_{1 \leq i < j \leq \ell} d_i d_j + \sum_{i=1}^\ell d_i - d + 1.$$

**Remark 3.4.** If $(A, m)$ is SPOG with exponents $(d_1, d_2, \ldots, d_\ell)$ and level $d$ (that is, if $D(A, m)$ is an SPOG module) then by the same arguments as in [6] and Proposition 3.3 we can show that

$$|m| = d_1 + \cdots + d_\ell - 1,$$

$$b_2(A, m) = \sum_{1 \leq i < j \leq \ell} d_i d_j - \sum_{i=1}^\ell d_i + d + 1.$$

These formulae are similar but not equivalent: we refer to Example 3.9 at the end of the section.
We saw that if \( \mathcal{A} \) is free, then \( \mathcal{A}' \) is SPOG; in the remainder of this section, we prove the analogous result for logarithmic forms, that if some deletion \( \mathcal{A}' \) is free, then \( \mathcal{A} \) is dual SPOG. We begin with a lemma.

For any multiarrangement \( (\mathcal{A}, m) \), let
\[
\omega = \sum_{i=1}^{\ell} \frac{f_i}{Q} dx_i
\]
in the usual notation, where \( Q := Q(\mathcal{A}, m) \) and each \( f_i \in S \).

**Lemma 3.5.** Let \( (\mathcal{A}, m) \) be a multiarrangement and \( H \in \mathcal{A} \). Assume \( \alpha_H = x_1 \).
For each hyperplane \( X \in \mathcal{A}^H \), let \( \eta_X, \omega_X \) be a basis for \( \Omega^1(\mathcal{A}_X, m_X) \) for which \( \eta_X \in \Omega^1(\mathcal{A}_X, m_X - \delta_H) \). Then
\[
f_1 \in (\alpha_H, \prod_{X \in \mathcal{A}^H} \alpha_X^{e_X}),
\]
where \( \tilde{X} \) denotes some hyperplane of \( \mathcal{A} \) for which \( X = \tilde{X} \cap H \).
Moreover, if we let \( e_X = -\deg \eta_X \) and \( d_X = -\deg \omega_X \), then the polynomial \( B := \prod_{X \in \mathcal{A}^H} \alpha_X^{e_X} \) satisfies \( \deg B/Q = -|m| + |m^s| \).

**Remark 3.6.** When \( m = 1 \), Lemma 3.5 is essentially the same as the Strong Preparation Lemma, [24, Thm. 5.1]. Independently, Lemma 3.5 when \( m = 1 \) is proved by Terao and the first author in [20].

**Proof of Lemma 3.5.** Let us fix \( X \in \mathcal{A}^H \) and assume that \( \alpha_H = x_1 \) and \( X = \{x_1 = x_2 = 0\} \). Clearly \( Q(\mathcal{A}, m) \subset Q_X \Omega^1(\mathcal{A}_X, m_X) \), where \( Q_X := Q(\mathcal{A}_X, m_X) \). Thus
\[
f_1 \in (g_1^X, h_1^X),
\]
where \( \eta_X = \sum_{i=1}^{\ell} (g_i^X/Q_X) dx_i \) and \( \omega_X = \sum_{i=1}^{\ell} (h_i^X/Q_X) dx_i \). By the choice of \( \eta_X \) and \( \omega_X \), the polynomial \( g_1^X \) is divisible by \( x_1 \), and \( h_1^X \) is not. We may assume that \( g_1^X \) and \( h_1^X \) are both polynomials in \( x_1, x_2 \). Then \( g_1^X = x_1 g \) and \( h_1^X = x_2^{|m_X| - d_X} = x_2^{e_X} \) for some \( g \), since \( e_x + d_X = |m_X| \), which proves the first assertion.

By definition of \( m^s \), it is clear that \( \deg B = |m^s| \), which completes the proof.

We note that Lemma 3.5 is the dual of the following important result, which we will also need for the main result of this section.

**Proposition 3.7 ([12], Lemma 3.4).** There is a homogeneous polynomial \( B \) of degree \( |m| - 1 - |m^s| \) such that
\[
\theta(\alpha_H) \in (\alpha_H^{m(H)}, B).
\]
for any \( \theta \in D(\mathcal{A}, m - \delta_H) \).

In the case where \( m = 1 \), the module \( D(\mathcal{A}, 1 - \delta_H) \) is free or SPOG whenever \( \mathcal{A} \) is free, by Theorem 1.4. Thus it is natural to ask, if \( \mathcal{A}' = (\mathcal{A}, 1 - \delta_H) \) is free, whether \( (\mathcal{A}, 1) = \mathcal{A} \) is either free or dual SPOG, and this is exactly our Theorem 1.6.
It is also natural to ask whether the same holds in the generality of multiarrangements. The multi-version of Theorem 1.4 could not be proven in [6], since the proof there used Ziegler restriction, which is intrinsic to simple arrangements. With the technique here, we are now able to determine completely the relationship between freeness, (dual)SPOG properties, and addition/deletion operations. Theorems 1.4, 1.5 are special cases of the following main result.

**Theorem 3.8.**

1. Suppose \((A, m)\) is free. Then \((A, m - \delta_H)\) is either free or SPOG.
2. Suppose \((A, m - \delta_H)\) is free. Then \((A, m)\) is either free or dual SPOG.

**Proof.**

1: Assume that \((A, m - \delta_H)\) is not free. Recall that, by Lemma 2.3, there is an identification \(\Omega^{\ell-1}(A, m - \delta_H) \cong D(A, m - \delta_H)\) by the correspondence

\[
\sum_{i=1}^{\ell} f_i Q(A, m - \delta_H) dx_i \mapsto \sum_{i=1}^{\ell} f_i \partial x_i.
\]

Here again, \(dx^i := \wedge_{j \neq i} dx_j\). Since \(\text{rank} A^H = \ell - 1\), it is the case that

\[
\Omega^{\ell-1}(A^H, m^*) = (1/Q(A^H, m^*)) dx,
\]

where

\[
dx := \wedge_{i=2}^{\ell} dx_j
\]

and \(H := \ker x_1\). By Proposition 2.4 and Corollary 2.6, we have the exact sequence

\[
0 \to \Omega^{\ell-1}(A, m) \xrightarrow{\alpha_H^*} \Omega^{\ell-1}(A, m - \delta_H) \xrightarrow{i_H^*} \Omega^{\ell-1}(A^H, m^*) \to 0.
\]

Hence there is some \(\omega \in \Omega^{\ell-1}(A, m - \delta_H)\) such that \(i_H^*(\omega) = (1/Q(A^H, m^*)) dx\), thus \(\deg \omega = -|m^*|\). Moreover, in the expression

\[
\omega = \sum_{i=1}^{\ell} (f_i/Q(A, m - \delta_H)) dx^i,
\]

the fact that \(i_H^*(\omega) \neq 0\) shows \(x_1 \nmid f_1\). Under the identification with derivations, \(\omega\) corresponds to

\[
\theta = \sum_{i=1}^{\ell} f_i \partial x_i \in D(A, m - \delta_H)
\]

with \(x_1 \nmid f_1\). Since \(\deg \theta = |m| - 1 - |m^*|\) and \(\theta \notin D(A, m)\), Proposition 4.7 shows that, for all \(\varphi \in D(A, m - \delta_H)\), there is \(f \in S\) such that \(\varphi - f \theta \in D(A, m) = \oplus_{i=1}^{\ell} S \theta_i\), where \(\theta_1, \ldots, \theta_{\ell}\) form a basis for \(D(A, m)\). Hence

\[
D(A, m - \delta_H) = \langle \theta_1, \ldots, \theta_{\ell}, \theta \rangle.
\]

Since \((A, m - \delta_H)\) is not free, this is a minimal set of generators. Thus it suffices to determine the relation among these this set of generators. Since \(\alpha_H \theta \in D(A, m)\), there is the unique relation in degree \(d + 1 := |m| - |m^*|\). This is to say that \(D(A, m - \delta_H)\) is SPOG with \(\text{POexp}(D(A, m - \delta_H)) = \exp(A, m)\) and level \(|m| - 1 - |m^*|\).
(2): Assume that \((A, m)\) is not free. Recall that \(\Omega^1(A, m) \cong D^{\ell-1}(A, m)\) by Lemma 2.3. Let \(\partial^i := \wedge_{j \neq i} \partial_{x_j}\). Now consider the exact sequence

\[
0 \rightarrow D^{\ell-1}(A, m - \delta_H) \overset{\alpha_H}{\rightarrow} D^{\ell-1}(A, m) \overset{\rho^H}{\rightarrow} D^{\ell-1}(A^H, m^*) \rightarrow 0
\]

in Proposition 2.4. Since \(\dim_k H = \ell - 1\), the module \(D^{\ell-1}(A^H, m^*)\) has rank 1 and is generated by \(Q(A^H, m^*)\partial\), where

\[
\partial := \wedge_{i=2}^{\ell} \partial_{x_i}
\]

assuming again that \(H = \ker x_1\). By Proposition 2.4, Corollary 2.6 and the freeness of \(A^i\), we have the exact sequence

\[
0 \rightarrow D^{\ell-1}(A, m - \delta_H) \overset{\alpha_H}{\rightarrow} D^{\ell-1}(A, m) \overset{\rho^H}{\rightarrow} D^{\ell-1}(A^H, m^*) \rightarrow 0.
\]

In particular, \(\rho^H\) is surjective, so there is some \(\theta \in D^{\ell-1}(A, m)\) such that \(\rho^H(\theta) = Q(A^H, m^*)\partial\), and it has \(\deg \theta = |m^*|\). Since \(\theta(x_1, x_{i_2}, \ldots, x_{i_{\ell-1}}) \in S\alpha_{H}^m\) for all \(\{i_2, \ldots, i_{\ell-1}\} \subset \{2, \ldots, \ell\}\), if we express

\[
\theta = \sum_{i=1}^{\ell} f_i \partial^i,
\]

for some \(f_i\)'s, we have \(x_i^m \parallel f_i\) unless \(i = 1\). Since \(\rho^H(\theta) \neq 0\), though, \(x_1 \parallel f_1\), under the correspondence with forms (Lemma 2.3), \(\theta\) corresponds to

\[
\omega = \sum_{i=1}^{\ell} \frac{f_i}{Q(A, m)} dx_i \in \Omega^1(A, m)
\]

with \(x_1 \parallel f_1\). Since \(\deg \omega = -|m| + |m^*| = \deg B/Q(A, m)\) and \(\omega\) has a pole along \(H\), Lemma 3.5 shows that, for all \(\eta \in \Omega^1(A, m)\), there is \(f \in S\) such that \(\omega - f\eta \in \Omega^1(A, m - \delta_H) = \wedge_{i=1}^{\ell} S\omega_i\). Hence

\[
\Omega^1(A, m) = \langle w_1, \ldots, w_\ell, \omega \rangle.
\]

Since \((A, m)\) is not free, they form a minimal set of generators. Thus it suffices to determine the relation among these generators. Since \(\alpha_H \omega \in \Omega^1(A, m - \delta_H)\), there is the unique relation in degree \(-|m| + 1 + |m^*| = -d\). Once again, \(\Omega^1(A, m)\) is SPOG with \(P\text{Oexp}(D(A, m)) = -\exp(A, m - \delta_H)\) and level \(-|m| + |m^*|\).

\(\Box\)

Example 3.9. Consider the graphic arrangement \(A = A(W_4)\) of 8 hyperplanes given by the “wheel” graph

\[
W_4 = \begin{array}{c}
\bullet \\
/ \\
/ \\
\bullet
\end{array}
\]

By adding an edge, one obtains a deletion of the graphic arrangement \(K_5\), which is free. By deleting any of the last four outer edges, one obtains a chordal graphic arrangement,
which is supersolvable, hence also free. However, $\mathcal{A}$ is not free, so by Theorem 3.3, it is both SPOG and dual SPOG. A Macaulay2 [15] computation shows $D(A)$ has exponents $(1, 2, 3, 3)$ with level $d = 3$ and $\Omega^1(\mathcal{A})$ has exponents $(-2, -2, -2, -1)$ and level $d = -2$. Using Proposition 3.3, we compute $b_1(A) = 8 = (2 + 2 + 2 + 1) + 1$ and
\[ b_2(A) = (4 + 4 + 4 + 2 + 2 + 2) + (2 + 2 + 2 + 1) - 2 + 1 \]
\[ = 24 \]
while Remark 3.4,
\[ b_1(A) = (1 + 2 + 3 + 3) - 1 \quad \text{and} \quad b_2(A) = (2 + 3 + 3 + 6 + 6 + 9) - (1 + 2 + 3 + 3) + 3 + 1 \]
\[ = 24 \]
again.

4. Combinatorial dependency of the addition theorem

By using the exact sequence and SPOG-theorem, we can give another proof of the combinatorial dependency of the addition theorem originally shown in [14]:

**Theorem 4.1** ([4], Theorem 1.4). Let $H \in \mathcal{A}, \mathcal{A}' := \mathcal{A} \setminus \{H\}$. Assume that $\mathcal{A}'$ is free. Then $\mathcal{A}$ is free if and only if $|\mathcal{A}'_X| - |\mathcal{A}'_X^H| = 0$ for all $X \in L(\mathcal{A}'^H)$.

**Proof.** The “only if” part is clear. Let us prove the “if” part. The statement is clear if $\ell \leq 3$, by, e.g., [2]. Thus we can use the induction on $\ell$. By the induction hypothesis and the assumption, we may assume that $\mathcal{A}$ is locally free: that is, $\mathcal{A}_X$ is free for all $X \in L(\mathcal{A}_X^H) \setminus \{0\}$. Assume that $\mathcal{A}$ is not free. Then by Theorem 1.5, $\Omega^1(\mathcal{A})$ is SPOG.

Let $0 \to S[d - 1] \xrightarrow{f} \bigoplus_{i=1}^{\ell} S[d_i] \oplus S[d] \to \Omega^1(\mathcal{A}) \to 0$ be a minimal free resolution. Here we assume that $-1 = d_1 \geq d_2 \geq \cdots \geq d_\ell$. We may assume that $\mathcal{A}'$ is essential, thus $-1 > d_2 \geq \cdots \geq -d_\ell$. Since the generating set is minimal, we may assume that the coefficient of the degree $(-1)$-element in the relation is zero by the direct sum decomposition. So
\[ \sum_{i=2}^{\ell} g_i \omega_i + \alpha H \omega = 0, \]
where $d\alpha_L/\alpha_L, \omega_2, \ldots, \omega_\ell$ form a basis for $\Omega^1(\mathcal{A}')$ and $\omega \in \Omega^1(\mathcal{A})$ a level element. By the assumption on $|\mathcal{A}'| - |\mathcal{A}_X^H|$, $\deg \omega = \deg \omega_i$ for some $i \geq 2$. Thus $g_j = 0$ for all $j$ with $-d_j < -d_i$. If there is some $k$ with $-d_k = -d$ and $g_k \neq 0$, then we may replace $\omega_k$ by $\alpha H \omega$ and $\omega_1, \omega_2, \ldots, \omega_{k-1}, \omega, \omega_{k+1}, \ldots, \omega_\ell$ form a basis for $\Omega^1(\mathcal{A})$. Thus such $g_k = 0$. Namely, if we replace $d_i$ in such a way that $d_i < d_{i+1}$, then $g_j = 0$ for all $j \leq i$. Then let $p$ be a prime ideal containing $g_{i+1}, \ldots, g_\ell, \alpha H$. Then $p$ determines a non-empty point in $\text{Proj}(S)$, and at this point $f_p = 0$, so taking the residue field of the minimal free resolution above,
\[ k_p^{\ell+1} \cong \Omega^1(\mathcal{A}) \otimes k_p \cong k_p^\ell \]
since $\mathcal{A}$ is locally free, a contradiction. \qed
In this section, we compare the behavior of the Euler restriction maps of logarithmic derivations and logarithmic forms, respectively.

First let us prove the following.

**Proposition 5.1.** For all \( S \)-modules \( M \) and all \( p \geq 0 \),
\[
\text{Ext}_S^p(M, S[1]) \cong \text{Ext}_S^{p+1}(M, S).
\]
In particular, we have isomorphisms
\[
\text{Ext}_S^1(D(A^H), S[1]) \cong \Omega^1(A^H) \quad \text{and} \quad \text{Ext}_S^1(\Omega^1(A^H), S[1]) \cong D(A^H).
\]

**Proof.** We apply the base change spectral sequence, using the fact that \( 0 \rightarrow S[-1] \xrightarrow{\alpha_H} S \rightarrow S \rightarrow 0 \) is a free resolution of \( S \) over \( S \).

Once again, let \((A, m)\) be a multiarrangement for which \( m_H > 0 \), for some \( H \in A \).

Let \( m' = m - \delta_H \), and \( m^* \) the Euler multiplicity on \( A^H \). Since the Euler restriction map \( \rho^H \) need not be surjective, we will denote its image by \( C_D(m) \) or just \( C_D \) if the choice of \( H \) and \( m \) are understood. Then we have a short exact sequence of graded \( S \)-modules
\[
0 \rightarrow D(A, m')[1] \xrightarrow{\alpha_H} D(A, m) \rightarrow C_D \rightarrow 0
\]
and an inclusion of \( S \)-modules \( C_D \rightarrow D(A^H, m^*) \) which is an isomorphism precisely when the restriction map \( \rho^H \) is surjective. These are the logarithmic derivations on \( (A^H, m^*) \) which can be lifted to \( (A, m') \).

Dually, let \( C_{\Omega, H}(m) \) or just \( C_{\Omega} \) denote the image of the restriction map \( i_H^*: \Omega^1(A, m') \rightarrow \Omega^1(A^H, m^*) \). This gives a short exact sequence of graded \( S \)-modules
\[
0 \rightarrow \Omega^1(A, m)[1] \xrightarrow{\alpha_H} \Omega^1(A, m') \rightarrow C_{\Omega} \rightarrow 0
\]
and an inclusion of \( S \)-modules \( C_{\Omega} \rightarrow \Omega^1(A^H, m^*) \) which is an isomorphism precisely when \( i_H^* \) is surjective. These are the logarithmic 1-forms on \( A^H \) which are restrictions of logarithmic 1-forms on \( (A, m) \).

Keeping track of local information leads to the following definitions.

**Definition 5.2.** Let \( A \) be a central arrangement of rank \( \ell \) and \( m \in \mathbb{Z}^A_{\geq 0} \).

1. Let
   \[
   \text{NF}(A, m) := \{ p \in (\text{spec } S): D(A, m)_p \text{ is not free} \},
   \]
   the nonfree locus of \( (A, m) \).

2. For each \( H \in A \), let
   \[
   \text{ND}(A, m, H) := \{ p \in (\text{spec } S): (C_D)_p \neq D(A^H, m^*)_p \}.
   \]
   We will call this the \( D \)-exceptional set of \( (A, m, H) \).

3. For each \( H \in A \), let
   \[
   \text{N}\Omega(A, m, H) := \{ p \in (\text{spec } S): (C_{\Omega})_p \neq \Omega^1(A^H, m^*)_p \}.
   \]
   We will call this the \( \Omega \)-exceptional set of \( (A, m, H) \).
Remark 5.3. The loci above are all Zariski closed sets. For example, for any pair \((A, H)\), we have \((p, m) \in \text{ND}(A, H)\) if and only if \(p\) is in the support of the quotient of \(C_D \twoheadrightarrow D(A^H, m^*)\).

Remark 5.4. Since \(X \mapsto D(A_X)\) is a local functor, in the sense of Orlik–Terao \([19, \text{Def. 4.121}]\), the non-free locus is a union of subspaces from \(L(A)\). The map \(\text{NF}(A) \to L(A)\) given by

\[ p \mapsto \bigcap_{H \in A: H \supseteq V(p)} H \]

has an upward-closed image in \(L(A)\). To do: check if a similar statement holds for \(\text{ND}(A, H)\) and \(\text{NΩ}(A, H)\).

Remark 5.5. Clearly \(p \in \text{NF}(A)\) if and only if the arrangement \(A_X(p)\) is not free. Since the multiarrangement \((A_X, m_X)\) is free for all flats \(X\) of rank \(\leq 2\) and multiplicities \([12, \text{Prop. 1.1}]\), the set \(\text{NF}(A, m)\) has codimension at least 3 in \(\text{spec } S\).

Proposition 5.6. For any triple \((A, m, H)\) we have

\[ \text{ND}(A, m, H) \subseteq \text{NF}(A, m') \quad \text{and} \quad \text{NΩ}(A, m, H) \subseteq \text{NF}(A, m). \]

In particular, \(\text{ND}(A, m, H)\) and \(\text{NΩ}(A, m, H)\) both have codimension at least 2 in \(\text{spec } S\).

Proof. We apply Theorems 1.2 and 1.3, respectively, to the arrangements \((A_X(p), m_X(p))\), for each prime ideal \(p\).

Although the inclusions \(C_D \hookrightarrow D(A^H, m^*)\) and \(C_\Omega \hookrightarrow \Omega^1(A^H, m^*)\) may be strict, their duals are isomorphisms. Let \(-^\vee = \text{Hom}_S(-, \overline{S})\). For each \(m \in \mathbb{Z}_A^\geq 0\), \(c_D(m) := c_D(A, m, H)\) and \(c_\Omega(m) := c_\Omega(A, m, H)\) denote the codimensions of \(\text{ND}(A, m, H)\) and \(\text{NΩ}(A, m, H)\) in \(\text{spec } S\), respectively, with the convention that \(\text{codim } \emptyset = \infty\).

Proposition 5.7. For any triple \((A, m, H)\), we have

\[ \Omega^1(A^H, m^*) \cong C_D^\vee, \quad \text{and} \quad D(A^H, m^*) \cong C_\Omega^\vee. \]

More precisely, the natural maps

\[ \text{Ext}^p_S(D(A^H, m^*), \overline{S}) \to \text{Ext}^p_S(C_D, \overline{S}) \quad \text{and} \]

\[ \text{Ext}^p_S(\Omega^1(A^H, m^*), \overline{S}) \to \text{Ext}^p_S(C_\Omega, \overline{S}) \]

are isomorphisms for \(p \leq c_D(m) - 2\) and \(p \leq c_\Omega(m) - 2\), respectively. The maps are monomorphisms for \(p = c_D(m) - 1\) and \(p = c_\Omega(m) - 1\), respectively.

Proof. Let \(G\) denote the quotient,

\[ (5.3) \quad 0 \to C_D \to D(A^H, m^*) \to G \to 0. \]

By definition, localizations of the module \(G\) are zero outside of \(\text{ND}(A, m, H)\). Since \(\overline{S}\) is Cohen–Macaulay, this implies that \(\text{Ext}^p_S(G, \overline{S}) = 0\) for \(p < \text{codim } \text{ND}(A, m, H)\): see, for example, [17, 17.1].

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We use the isomorphisms $\text{Ext}^i(\cdot, S)$ applied to (5.3), noting that for $p = 0$, $D(A^H, m^*)^{\gamma} \cong \Omega^1(A^H, m^*)$.

The corresponding properties of $C_D$ are proven in the same way.

Using the sequences (5.1) and (5.2), this gives a measure of when the Euler restriction maps are surjective.

**Theorem 5.8.** For any arrangement triple $(A, m, H)$, the Euler sequences extend to the following exact sequences, for $p = c_{\Omega}(m) − 1$ and $p = c_{D}(m) − 1$, respectively:

\[
0 \to D(A, m')[-1] \xrightarrow{\alpha_S} D(A, m) \to D(A^H, m^*) \to
\]
\[
\to \text{Ext}^1_S(\Omega^1(A, m'), S)[-1] \to \text{Ext}^1_S(\Omega^1(A, m), S) \to \text{Ext}^1_S(\Omega^1(A^H, m^*), S) \to
\]
\[
\to \cdots \to
\]
\[
\to \text{Ext}^p_S(\Omega^1(A, m'), S)[-1] \to \text{Ext}^p_S(\Omega^1(A, m), S) \to \text{Ext}^p_S(C_D, S)
\]

and

\[
0 \to \Omega^1(A, m)[-1] \xrightarrow{\alpha_S} \Omega^1(A, m') \to \Omega^1(A^H, m^*) \to
\]
\[
\to \text{Ext}^1_S(D(A, m), S)[-1] \to \text{Ext}^1_S(D(A, m'), S) \to \text{Ext}^1_S(D(A^H, m^*), S) \to
\]
\[
\to \cdots \to
\]
\[
\to \text{Ext}^p_S(D(A, m), S) \to \text{Ext}^p_S(D(A, m'), S) \to \text{Ext}^p_S(C_D, S).
\]

**Proof.** To prove the first statement, we apply $\text{Hom}_S(\cdot, S)$ to the short exact sequence (5.2) and consider the long exact sequence

\[
0 \cong \text{Hom}_S(C_D, S) \to \text{Hom}_S(\Omega^1(A, m'), S) \to \text{Hom}_S(\Omega^1(A, m), S)[1] \to
\]
\[
\to \text{Ext}^1_S(C_D, S) \to \text{Ext}^1_S(\Omega^1(A, m'), S) \to \text{Ext}^1_S(\Omega^1(A, m), S)[1] \to
\]
\[
\to \text{Ext}^1_S(C_D, S) \to \cdots
\]

We use the isomorphisms $\text{Ext}^{i+1}_S(C_D, S) \cong \text{Ext}^i_S(C_D, S)[1]$ for each $i$, by Proposition 5.1 and $\text{Ext}^i_S(C_D, S) \cong \text{Ext}^i_S(\Omega^1(A^H, m^*), S)$ for $i \leq c_{\Omega} − 2$, by Proposition 5.7, to obtain the first exact sequence shown above. For the second statement, we start with (5.1) instead.

We recall that, if the deletion $(A, m')$ is free, the Euler sequence of derivations is right exact (Theorem 1.2). The Euler sequence of 1-forms (1.4), on the other hand, extends to a four-term exact sequence

\[
0 \to \Omega^1(A, m') \to \Omega^1(A, m) \to \Omega^1(A^H, m^*) \to \text{Ext}^1_S(D(A, m), S) \to 0.
\]

As usual, the dual observation is obtained by exchanging $m'$ and $m$.

We arrive at a slight refinement of Proposition 5.0.
Corollary 5.9. For any arrangement triple \((A,m,H)\), we have
\[
\text{ND}(A,m,H) \subseteq \text{supp} \Ext^1_S(\Omega^1(A,m'),S), \quad \text{and} \\
\text{NΩ}(A,m,H) \subseteq \text{supp} \Ext^1_S(D(A,m),S).
\]
Equality occurs on the top if \((A,m)\) is free, and on the bottom if \((A,m')\) is free.

In particular, if an arrangement \((A,m)\) contains hyperplanes \(H_1\) and \(H_2\) for which both deletions are free, then \(\text{NΩ}(A,m,H_1) = \text{NΩ}(A,m,H_2)\): that is, all free deletions produce the same \(\Omega\)-exceptional set. Since \(\text{NΩ}(A,m,H) \subseteq H = \Spec(\mathcal{S})\) for each \(H\), if \((A,m - \delta_H)\) is free, then
\[
\text{NΩ}(A,m,H) \subseteq \bigcap_{K \in A:\ (A,m - \delta_K) \text{ is free}} K.
\]

Example 5.10. Consider the simple arrangement of 10 hyperplanes in \(\mathbb{C}^4\) defined by the columns of the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

\(\mathcal{A}\) is free, so by Theorem 5.8 we have \(C_\Omega \cong \Omega^1(A^H)\) for each \(H\): that is, the \(\Omega\)-exceptional set is empty for each \(H\).

Numbering the hyperplanes in order, computation shows
\[
\begin{array}{|c|c|c|}
\hline
i & \text{NF}(A - \{H_i\}, \mathcal{S}) & \text{ND}(A - \{H_i\}, \mathcal{S}) \\
\hline
1, 2, 3, 7, 8, 9 & 1\text{-dimensional} & \{0\} \\
4, 5, 6 & \emptyset & \emptyset \\
10 & \{0\} & \emptyset \\
\hline
\end{array}
\]

E.g., for \(H = H_{10}\), we see that, although the deletion \(\mathcal{A}'\) is not free, the restriction map in the Euler sequence \((1.1)\) is surjective. The long exact sequence above shows that \(\Ext^1_S(\Omega^1(A'), S) = 0\), and
\[
\Ext^1_S(\Omega^1(A^H), S)[1] \cong \Ext^2_S(\Omega^1(A'), S).
\]

Since \(\mathcal{A}^H\) is not free, these must both be nonzero.

6. Viewpoint from negative multiplicities

Now we have two exact sequences:
\[
0 \rightarrow D(A, m - \delta_H) \xrightarrow{\alpha_H} D(A, m) \xrightarrow{\delta_H} D(A^H, m^*), \\
0 \rightarrow \Omega^1(A, m) \xrightarrow{\alpha_H} \Omega^1(A, m - \delta_H) \xrightarrow{\delta_H} \Omega^1(A^H, m^*).
\]

They are similar, but not the same, and their differences work well to prove Theorems 5.8 and 5.8. So it is natural to ask whether there are some way to treat these two
different sequences in a uniform manner. One way to do this is to introduce notion of logarithmic differentials and forms for arrangements with negative multiplicities.

**Definition 6.1.** A (generalized) multiplicity \( m \) on \( \mathcal{A} \) is a map

\[
m: \mathcal{A} \to \mathbb{Z}_{\geq 0}
\]
or

\[
m: \mathcal{A} \to \mathbb{Z}_{\leq 0}.
\]

A pair \( (\mathcal{A}, m) \) is called the (generalized) multiarrangement.

Negative multiplicities are first considered in [10] for Coxeter arrangements, and studied in [22]. They also appear in the context of well-generated complex reflection groups in [9]. The most general setup can be found in [1] (unpublished): in the first three papers, attention is restricted to multiplicities of the form from Definition 6.1, while in [1], arbitrary multiplicities \( m: \mathcal{A} \to \mathbb{Z} \) are considered. In any case, one assumes that \( V \) has a nondegenerate inner product, \( I: V^* \to V \), with which one identifies bases \( \{ \partial x_i \} \) and \( \{ dx_i \} \). For multiplicities as in Definition 6.1, we can naturally define its logarithmic modules by \( S \)-linearly extending \( I \) to an isomorphism \( I: \Omega^1 V \to \text{Der}(S) \).

**Definition 6.2.** For a generalized multiarrangement \( (\mathcal{A}, m) \), its logarithmic derivation module \( \mathcal{D}(\mathcal{A}, m) \) is defined by

\[
\mathcal{D}(\mathcal{A}, m) := \mathcal{D}(\mathcal{A}, m) \text{ if } m(\mathcal{A}) \subset \mathbb{Z}_{\geq 0} \text{ and } \mathcal{D}(\mathcal{A}, m) := I(\Omega^1(\mathcal{A}, -m)) \text{ if } m(\mathcal{A}) \subset \mathbb{Z}_{\leq 0}.
\]

In this context, it holds that

\[
\Omega^1(\mathcal{A}, m) \cong \mathcal{D}(\mathcal{A}, -m),
\]

\[
\Omega^1(\mathcal{A}, m - \delta_H) \cong \mathcal{D}(\mathcal{A}, -m + \delta_H).
\]

Since by definition, for all \( m_1, m_2 \) with \( m_1 \leq m_2 \) (which is to say that \( m_1(H) \leq m_2(H) \) for all \( H \in \mathcal{A} \)), it holds that

\[
\mathcal{D}(\mathcal{A}, m_1) \supset \mathcal{D}(\mathcal{A}, m_2).
\]

In this notation, our previous results can be combined in a uniform statement:

**Theorem 6.3.** Let \( m \) be a generalized multiplicity in the sense of Definition 6.2. There is an exact sequence

\[
0 \to \mathcal{D}(\mathcal{A}, m - \delta_H) \to \mathcal{D}(\mathcal{A}, m)^{\rho_H = \text{res}_H} \to \mathcal{D}(\mathcal{A}^H, m^*).
\]

The sequence is right exact if \( (\mathcal{A}, m - \delta_H) \) is free.

Similarly, Theorem 3.8 can be expressed as follows.

**Theorem 6.4.** Let \( m \) be a generalized multiplicity in the sense of Definition 6.2. If \( (\mathcal{A}, m) \) is free, then \( (\mathcal{A}, m - \delta_H) \) is either free or SPOG.
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