Stochastic quantization of a self-interacting nonminimal scalar field in semiclassical gravity

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Abstract

We employ stochastic quantization for a self-interacting nonminimal massive scalar field in curved spacetime. The covariant background field method and local momentum space representation are used to obtain the Euclidean correlation function and evaluate multi-loop quantum corrections through simultaneous expansions in the curvature tensor and its covariant derivatives and in the noise fields. The stochastic correlation function reproduces the well-known result by Bunch and Parker and is used to construct the effective potential in curved spacetime in an arbitrary dimension \( D \) up to the first order in curvature. Furthermore, we present a sample of numerical simulations for \( D = 3 \) in the first order in curvature.

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1. Introduction

The theory of quantum fields interacting with external sources has a special importance in modern theoretical physics. The reasons for this are basically twofold. First of all, there are situations where the external field is an inherent part of the system of interest. This is the case, for example, of semiclassical gravity, where the dynamics of quantum matter fields occurs on a classical metric background, a situation relevant for primordial cosmology. The applications in this setting range from creation of particles in the primordial universe [1, 2, 3, 4] (see Ref. [5] for a review and further references), to the back-reaction of quantum matter on the background [6], leading...
to the historically first and phenomenologically successful Starobinsky model of inflation [7, 8] (see also recent works on the quantum field theory backgrounds of this model [9, 10]). Second, external sources can be used as a tool for calculating observables, correlation functions or the effective action. In this respect, one recalls the background field method in perturbative quantum field theory (QFT), which provides technical advantages for calculating one-loop or higher-order loop quantum corrections.

An important case where higher-loop corrections are relevant in the context of cosmology is in the Higgs inflation model [11, 12, 13, 14, 15, 16], where two-loop corrections to the effective potential were shown to modify bounds for the Higgs mass. Another potentially relevant high-energy application of the nonperturbative effects is related to the problem of high energy (UV) Higgs stability, as discussed recently in [17] (see further references therein).

An example of the low-energy application where the nonperturbative effects are potentially important is related to quantum infrared (IR) vacuum effects of massive fields. It is known that such effects plausibly can produce a low-energy (IR) running of the observable cosmological constant [18]. However, one can not prove or disprove this due to technical limitations of the available theoretical methods. The analyses based on the hypothesis of quadratic decoupling [19, 20], covariance [21, 22] and dimensional arguments [23] provide a universal form of the IR running for cosmological and Newton constants, such that there remains a single parameter to be defined from the cosmological [24, 25] or astrophysical [26, 27] data. At the same time, it would be very important to have a quantum field theory based support (or ruling out) of this hypothesis. Unfortunately, the existing analytical methods of calculations in curved spacetime do not enable one to achieve such a verification.

A situation somehow similar to the one described above takes place in quantum chromodynamics (QCD). The traditional perturbative approach, based on an expansion in the coupling constant, becomes inapplicable in the IR sector of the theory. There, the framework of lattice QCD has had remarkable success in e.g. reproducing the masses of the ground-state light hadrons [28] and has become the standard source of insight into the IR sector of QCD. The theory is reformulated on a discrete Euclidean spacetime lattice and solved as a classical statistical mechanics problem employing Monte-Carlo methods [29]. Furthermore, when Monte-Carlo methods become inefficient, a most-used method is stochastic quantization, which also provides a promising alternative to deal with complex-action problems [30]. In stochastic quantization, quantum fluctuations are obtained with the use of the Langevin equation. A first attempt to derive the Schrödinger equation within such an approach is due to Nelson [31] and its modern incarnation dates back to the 1980’s with the work of Parisi and Wu [32], who introduced a fictitious time variable $\tau$ for the evolution of the Langevin equation. Within this approach, an Euclidean quantum field theory in $D$ dimensions is obtained as the equilibrium limit for $\tau \to \infty$ of a classical system in $D + 1$ dimensions coupled to
a random noise source with strength $\hbar$. When applied to gauge theories, stochastic quantization is useful to resolve certain issues related to gauge fixing ambiguities. One can see Refs. [33, 34] for early developments and applications.

Given its nonperturbative, Lorentz invariant nature, stochastic quantization is an alternative to the functional integral and it looks natural to formulate the approach in an arbitrary curved spacetime, dealing with the semiclassical approach to gravity. There is a vast literature on this subject; Refs. [35] provide recent reviews with extensive list of references, including on the applications to quantum gravity. One can also mention the important work of Ref. [36], where the correlation functions for a massive nonminimal scalar field were considered on de Sitter background by means of the Fokker-Planck equations. More recent publications, employing the Langevin equation, can be found in Refs. [37, 38].

In the present work we begin the systematic consideration of stochastic quantization of semiclassical gravity in an arbitrary curved spacetime. Starting from the covariant form of the Langevin equation for a massive self-interacting nonminimal scalar field in curved spacetime, we develop a computational scheme based on a local momentum representation to obtain multi-loop quantum corrections to the classical solution. The multi-loop corrections are obtained through a simultaneous expansion in the curvature tensor and in the noise fields. To test the formalism analytically, we derive the correlation function of the scalar field and the effective potential to first order in $\hbar$ and in the curvature tensor. The results are exactly the same as derived by other methods, confirming the correctness of the covariant formulation of the Langevin equation. In addition, we show that the multi-loop expansion converges to the full nonperturbative solution by comparing the noise-expanded and nonperturbative numerical solutions of the Langevin equation in the first order in curvature in the case of $D = 3$.

The paper is organized as follows. Sec. 2 gives a survey of deriving the curved space effective potential by means of the local momentum representation. In Sec. 3 the lowest-order derivation of the correlation function of the scalar field theory nonminimally coupled to gravity is presented. The calculation is performed analytically by perturbatively solving the Langevin equation using the Riemann normal coordinates. Furthermore, this result is used to obtain the effective potential to first order in curvature in the framework of the cutoff regularization scheme. In order to test the method and also to generalize a previous result obtained by means of the local momentum representation [42], here the calculation is performed in an arbitrary Euclidean dimension $D$. Section 4 presents results of the numerical simulations in $D = 3$. Finally, our Conclusions and Perspectives are presented in Sec. 5.

Our notations mainly follow those of Bunch and Parker [39]. The Riemann curvature tensor is
defined by
\[ R^\alpha_{\beta\mu
u} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\tau} \Gamma^\tau_{\beta\nu} - \Gamma^\alpha_{\nu\tau} \Gamma^\tau_{\beta\mu}, \]

and the Ricci tensor is \( R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} \). For performing quantum calculations, we assume analytic continuation to Euclidean spacetime, but use the notation \( \eta_{\mu\nu} \) for the flat spacetime metric. The determinant of the metric is \( g = \det (g_{\mu\nu}) \).

### 2. Effective potential in curved space: a brief review

The effective potential \( V_{\text{eff}}(\phi) \) is defined as the zeroth-order term in the derivative expansion of the effective action of a background scalar field \( \phi(x) \),
\[
\Gamma [\phi, g_{\mu\nu}] = \int d^Dx \sqrt{g} \left[ V_{\text{eff}}(\phi) + \frac{1}{2} Z(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \cdots \right],
\]
where \( D \) is the spacetime dimension. We are interested in the theory of a nonminimally coupled with gravity self-interacting massive scalar field \( \varphi \), with the action
\[
S = \int d^Dx \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{2} \left( m^2 + \xi R \right) \varphi^2 + V(\varphi) \right],
\]
where \( R \) is the scalar curvature and \( V(\varphi) \) is a potential, and \( m^2 > 0 \) or \( m^2 < 0 \), the latter being the case of spontaneous symmetry breaking. \( \xi R \varphi^2 \) is the nonminimal term and \( \xi \) is the nonminimal parameter, while \( m^2 \varphi^2 + V(\varphi) \) is the minimal part of the classical potential. The addition of the nonminimal term is necessary for the renormalizability of the quantum theory in \( D = 4 \) (see e.g. Ref. [40] for an introduction to the subject). Although for the analytic part of our work the form of the potential is not really essential, for the numerical calculations we use the function
\[ V(\varphi) = \lambda \varphi^4, \]
with \( \lambda > 0 \).

Within the background field method the scalar field is split into a classical field, \( \phi \), and a quantum fluctuation field \( \varphi \): \( \varphi \to \phi + \varphi \). The loop expansion of the effective action is
\[
\Gamma = S + \bar{\Gamma}^{(1)} + \cdots,
\]
where \( S \) is the classical action and \( \bar{\Gamma}^{(1)} \) is the one-loop contribution, given by
\[
\bar{\Gamma}^{(1)} = \frac{1}{2} \text{Tr} \ln \hat{H},
\]
where
\[
\hat{H} = \frac{1}{\sqrt{g(x)} \sqrt{g(x')}} \left. \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(x')} \right|_{\varphi=0} = -\Box + m^2 + \xi R + V''(\phi)
\]
is a bilinear form of the classical action and \( \Box = g^{\mu \nu} \nabla_\mu \nabla_\nu \) is the d’Alembert operator. Correspondingly, the effective potential is

\[
V_{\text{eff}}(\phi) = \frac{1}{2} m^2 \phi^2 + V(\phi) + \bar{V}^{(1)}(\phi) + \cdots, 
\]

where

\[
\int d^D x \sqrt{g} \bar{V}^{(1)}(\phi) = \frac{1}{2} \text{Tr} \ln \hat{H} \bigg|_{\phi = \text{const}}
\]

is the one-loop correction to the effective potential.

The flat-space part of effective potential was derived many times and in different ways starting from the work of Coleman and Weinberg [41]. In curved spacetime the potential can also be derived in different ways — see e.g. Ref. [40] for detailed discussions and further references. Here we rely on the local momentum representation method based on the Riemann normal coordinates, which is a useful formalism for mass-dependent calculations of local quantities, such as the effective potential [42]. The advantage of these special coordinates is that one can use flat-space methods of calculations, such as momentum representation, and apply an expansion in powers of the curvature tensor and its covariant derivatives at the point \( P \) to all relevant quantities (see e.g. Eqs. (41) and (42) below for expansion of the d’Alembert operator). The method is very efficient especially for deriving local quantities.

In the normal coordinates formalism, the spacetime metric \( g_{\alpha\beta}(x) \) and all related quantities are expanded near a point \( P(x') \), where the metric is supposed to be flat — this is not strictly necessary, albeit it is a useful restriction. At the first order in the curvature, we have the dimension-independent result [43],

\[
g_{\alpha\beta}(x) = \eta_{\alpha\beta} - \frac{1}{3} R_{\alpha\beta\gamma}(x') y^\gamma y^\nu + \cdots, 
\]

where \( x' \) are the coordinates of the point \( P \) and \( y^\mu = x^\mu - x'^\mu \) are deviations from this point. The object of our interest is the effective potential, which is certainly local, and therefore the local momentum representation method should work in any approach, including stochastic quantization.

As we have already mentioned, one of the simplest applications of this technique is the derivation of effective potential [42]. The main idea is to evaluate the curvature dependence of the functional determinant in Eq. (8) by means of the Green’s function \( \bar{G} \), defined as

\[
\hat{H} \bar{G} = -\delta^D(x - x'), 
\]

where \( \delta^D(x - x') \) is the Dirac delta function in the flat Minkowski spacetime. The Green’s function \( \bar{G} \) can be expressed in the local momentum representation through the use of normal Riemann
coordinates [43]. The crucial point behind Eq. (10) is that its r.h.s. does not depend on the metric tensor, hence one can use the relation

$$\text{Tr} \ln \hat{H} = - \text{Tr} \ln \tilde{G},$$

(11)
to obtain the dependence on the curvature tensor in Eq. (8). Since Eq. (11) is one of the central parts of the following consideration, let us review certain details of how to construct the equation for the Green’s function in curved spacetime.

It proves useful to introduce the covariant two-point function $G_c(x, x')$, defined by the equation

$$\left[ -\Box + m^2 + \xi R + V''(\phi) \right] G_c(x, x') = -\delta_c(x, x').$$

(12)

Here, $\delta_c(x, x')$ is the covariant delta function, which satisfies the two requirements

$$\delta_c(x, y) = \delta_c(y, x),$$

(13)

and

$$\int d^D y \sqrt{g(y)} f(y) \delta_c(x, y) = f(x).$$

(14)

It is not difficult to verify that the solution for this delta function is

$$\delta_c(x, x') = g^{-1/4} \delta^D(x - x') g'^{-1/4},$$

(15)

where $g = g(x)$, $g' = g(x')$ and $\delta^D(x - x')$ is the ordinary Dirac delta function in flat space. It is important to note that $\delta_c(x, x')$ is not a function of $x - x'$ and moreover all quantities in Eq. (12) are explicitly covariant and as such it is a genuine covariant equation.

Since the r.h.s. of Eq. (10) does not depend on curvature, we can use Eq. (15) to write

$$g^{1/4} \left[ -\Box + m^2 + \xi R + V''(\phi) \right] G_c(x, x') g'^{1/4} = -\delta^D(x - x').$$

(16)

This is not yet the final point in our construction, as the operator in the l.h.s. of this equation is different from the bilinear operator $\hat{H}$, Eq. (6). To make them the same, we follow Bunch and Parker [39] and introduce a new Green’s function $\tilde{G}(x, x')$ through the formula

$$G_c(x, x') = g^{-1/4} \tilde{G}(x, x') g'^{-1/4}.$$
Our goal is to introduce the expansion in normal coordinates and solve Eq. (18) order by order in a curvature expansion around flat spacetime. Regardless the fact that the covariant derivative commutes with \( g \) in general, this is not true order by order in the expansion in normal coordinates; see e.g. Eqs. (41) and (42), where clearly the terms at first order in curvature tensors are different. Therefore, the presence of the extra factors of \( g^{1/4} \) in Eq. (18) is an important aspect of our approach.

The solution of Eq. (18) for the free field case, \( V(\phi) = 0 \), has been given in Ref. [39]. The solution up to the first order in the curvature has the form

\[
\bar{G}(x, x') = \int \frac{d^Dk}{(2\pi)^D} e^{iky} \left[ \frac{1}{k^2 + m^2} - \left( \xi - \frac{1}{6} \right) \frac{R}{(k^2 + m^2)^2} \right].
\]

(19)

In the next section we reobtain this result by means of the Parisi and Wu stochastic quantization method.

3. Correlation function and effective potential in stochastic quantization

To implement the stochastic quantization, an extra coordinate \( \tau \), (the fictitious time, also known as Markov parameter), is introduced and the scalar field is supplemented with this additional coordinate, \( \phi(x) \rightarrow \phi(x, \tau) \). The dynamics of the field \( \phi(x, \tau) \) is described by a Langevin equation driven by a random white noise field \( \eta(x, \tau) \). Let us start by writing the Langevin equation in curved spacetime in a covariant form,

\[
\frac{\partial \phi(x, \tau)}{\partial \tau} = -\frac{1}{\sqrt{g}} \frac{\delta S}{\delta \phi(x, \tau)} + \eta_c(x, \tau),
\]

(20)

where \( S \) is the action defined previously in Eq. (3) and \( \eta_c(x, \tau) \) is the “covariant” white noise field. By covariant we mean that the gaussian white noise obeys the following correlations (generalized Einstein’s relations)

\[
\langle \eta(x, \tau) \rangle = 0,
\]

(21)

\[
\langle \eta(x, \tau) \eta(x', \tau') \rangle = 2\hbar \delta_c(x, x') \delta(\tau - \tau'),
\]

(22)

where \( \langle \cdots \rangle_{\eta} \) means the stochastic average. Note that in Eq. (22) there is a covariant delta function related to the spacetime coordinates \( x' \). Although we have been using and will continue to use \( \hbar = 1 \) in the following, we have written it explicitly in Eq. (22) to recall the quantum nature of the noise fields. The main idea of Parisi and Wu stochastic quantization [32] is that quantum field theory vacuum expectation values of the field \( \phi(x) \) are obtained from the stochastic averages of correlation functions of the field \( \phi(x, \tau) \) in the \( \tau \to \infty \) limit (the equilibrium limit). We are interested in evaluating the correlation function \( G_c(x, x') \). This correlation function is obtained as

\[
G_c(x, x') = \lim_{\tau \to \infty} \Delta(x, x'|\tau),
\]

(23)
where

$$\Delta(x, x'|\tau) = \langle \varphi(x, \tau) \varphi(x', \tau) \rangle_\eta - \langle \varphi(x, \tau) \rangle_\eta \langle \varphi(x', \tau) \rangle_\eta. \tag{24}$$

For the action in Eq. (3), the Langevin equation Eq. (20) reads

$$\frac{\partial \varphi(x, \tau)}{\partial \tau} = -\left[ -\Box + m^2 + \xi R \right] \varphi(x, \tau) - V'(\varphi(x, \tau)) + \eta_c(x, \tau). \tag{25}$$

We will solve this equation by an expansion in powers of \( \hbar \). From Eq. (22), one can see that \( \eta = O(\hbar^{1/2}) \). We write for the field \( \varphi(x, \tau) \) the expansion

$$\varphi(x, \tau) = \phi(x) + \varphi^{(1)}(x, \tau) + \varphi^{(2)}(x, \tau) + \cdots, \tag{26}$$

where \( \phi(x) \) is the \( O(\hbar^0) \) classical background field and the \( \varphi^{(n)}(x, \tau) \) are quantum corrections of order \( O(\hbar^{n/2}) \). Using the expansion in the potential, one obtains

$$V'(\varphi(x, \tau)) = V'(\phi) + V''(\phi) \varphi^{(1)}(x, \tau) + \frac{1}{2} V'''(\phi) [\varphi^{(1)}(x, \tau)]^2 + \cdots, \tag{27}$$

where dots stand for next order contributions with \( n > 2 \). Replacing Eqs. (26) and (27) into the Langevin equation Eq. (25) and equating terms of the same order in \( \hbar \), one finds the following set of equations for the \( \varphi^{(n)} \):

$$\frac{\partial \phi(x)}{\partial \tau} = -\left[ -\Box + m^2 + \xi R \right] \phi(x) - V'(\phi), \tag{28}$$

$$\frac{\partial \varphi^{(1)}(x, \tau)}{\partial \tau} = -\left[ -\Box + m^2 + \xi R + V''(\phi) \right] \varphi^{(1)}(x, \tau) + \eta_c(x, \tau), \tag{29}$$

$$\frac{\partial \varphi^{(2)}(x, \tau)}{\partial \tau} = -\left[ -\Box + m^2 + \xi R + V''(\phi) \right] \varphi^{(2)}(x, \tau) - \frac{1}{2} V'''(\phi) [\varphi^{(1)}(x, \tau)]^2 \tag{30}$$

and similarly for \( n > 2 \). This set is formed by linear equations which can be, in principle, solved by iterations. We note that a noise expansion procedure was used previously in cosmology [44, 45, 46, 47] in a different context, not related to stochastic quantization. Ref. [48] is first publication related to noise perturbation in the context of stochastic quantization.

Although Eqs. (28)-(30) are linear equations, they can not be solved analytically in general when the classical solution is not a constant. Therefore, for the analytical calculation of the propagator and effective potential in the next section, we take \( \phi(x) \equiv \phi_{\text{cl}} = \text{const} \). The following initial conditions are assumed:

$$\varphi(x, 0) = \phi(x) + \varphi^{(1)}(x, 0) + \varphi^{(2)}(x, 0) + \cdots = \phi_{\text{cl}}, \tag{31}$$

which imply

$$\varphi^{(1)}(x, 0) = \varphi^{(2)}(x, 0) = \cdots = 0. \tag{32}$$
3.1. Correlation function

Using the noise expansion of the field Eq. (26) in the correlation function (24), one finds

\[ \Delta(x, x'|\tau) = \langle \phi^{(1)}(x, \tau) \phi^{(1)}(x', \tau) \rangle_\eta + \cdots, \]

(33)

thus, at one-loop level we can simply write

\[ \Delta^{(1 - \text{loop})}(x, x'|\tau) = \langle \phi^{(1)}(x, \tau) \phi^{(1)}(x', \tau) \rangle_\eta. \]

(34)

Therefore, to obtain the correlation function up to first order in \( R \), one just needs to solve the first order equation given in Eq. (29).

By the very same reasons discussed in Sec. 2, before starting the normal coordinates expansion one has to introduce a noncovariant correlation function \( \tilde{\Delta}(x, x'|\tau) \) through the relation

\[ \Delta(x, x'|\tau) = g^{-1/4} \tilde{\Delta}(x, x'|\tau) g^{-1/4}. \]

(35)

In the close analogy with Eq. (34), we also introduce a new field \( \bar{\phi} \) such that

\[ \tilde{\Delta}^{(1 - \text{loop})}(x, x'|\tau) = \langle \bar{\phi}^{(1)}(x, \tau) \bar{\phi}^{(1)}(x', \tau) \rangle, \]

(36)

where the relation between the old field variable \( \varphi \) and the new one \( \bar{\varphi} \) is given by \( \varphi = g^{-1/4} \bar{\varphi} \). By defining a new noise field \( \bar{\eta} \) through \( \eta = g^{-1/4} \bar{\eta} \), the noise correlation function reduces to

\[ \langle \bar{\eta}(x, \tau) \bar{\eta}(x', \tau') \rangle_\eta = 2 \delta^D(x - x') \delta(\tau - \tau'), \]

(37)

where \( \delta^D(x - x') \) is the flat spacetime delta function. In terms of the new variables, Eq. (29) becomes

\[ g^{-1/4} \frac{\partial \bar{\phi}^{(1)}(x, \tau)}{\partial \tau} = - \left[ -\Box + m^2 + \xi R + V''(\phi) \right] g^{-1/4} \bar{\phi}^{(1)}(x, \tau) + g^{-1/4} \bar{\eta}(x, \tau). \]

(38)

After multiplying both sides of this equation from the left by \( g^{1/4} \), one obtains

\[ \frac{\partial \bar{\phi}^{(1)}(x, \tau)}{\partial \tau} = - \left[ - g^{1/4} \Box g^{-1/4} + m^2 + \xi R + V''(\phi) \right] \bar{\phi}^{(1)}(x, \tau) + \bar{\eta}(x, \tau). \]

(39)

Equation (39) has an appropriate form for introducing the Riemann normal coordinates. In particular, we are interested in the terms that are of first order in the curvature. Using Eq. (9), the expansions of \( R \) and \( \Box \) up to first order in the curvature are given by

\[ R(x) = R(x') + \cdots, \]

(40)

\[ \Box = \partial^2 + \frac{1}{3} R^a_{\alpha \beta}(x') y^a y^\beta \partial_\alpha \partial_\beta - \frac{2}{3} R^a_{\beta}(x') y^\beta \partial_a + \cdots, \]

(41)
so that
\[
g^{1/4} \Box g^{-1/4} = \partial^2 + \frac{1}{6} R + \frac{1}{3} \left[ R_{\alpha\beta,\gamma} (x') y^\alpha y^\beta \partial_\mu \partial_\nu - R^\alpha_{\beta, \gamma} (x') y^\beta \partial_\alpha \right] + \cdots , \tag{42}
\]
where the derivatives are \( \partial_\alpha = \partial/\partial x^\alpha \), \( \partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu \) and \( \cdots \) stands for the terms of higher orders in the curvature and its covariant derivatives. Deriving the effective potentials, one can safely consider \( V'(\phi) = \text{const} \) and simply replace \( m^2 \) by \( \bar{m}^2 = m^2 + V''(\phi) \) in Eq. (39) and in the following. Using Eqs. (9) and (42) in Eq. (39), we obtain
\[
\frac{\partial \bar{\varphi}_0^{(1)} (x, \tau)}{\partial \tau} = - \left( -\partial^2 + \bar{m}^2 \right) \bar{\varphi}_0^{(1)} (x, \tau) - \left( \xi - \frac{1}{6} R \right) \bar{\varphi}_0^{(1)} (x, \tau)
+ \frac{1}{3} \left[ R_{\alpha\beta,\gamma} y^\alpha y^\beta \partial_\mu \partial_\nu - R^\alpha_{\beta, \gamma} y^\beta \partial_\alpha \right] \bar{\varphi}_0^{(1)} (x, \tau) + \cdots + \bar{\eta}(x, \tau) . \tag{43}
\]
Equation (43) is the generalized Langevin equation that we intend to solve. It contains an infinite expansion in the curvature \( R \) but it can be solved consistently order by order within the curvature expansion by expanding \( R \) and its covariant derivatives at the point \( P \). To do so, we write the field in the form
\[
\bar{\varphi}_0^{(1)} (x, \tau) = \varphi_0^{(1)} (x, \tau) + \bar{\varphi}_0^{(1)} (x, \tau) + \cdots , \tag{44}
\]
where \( \varphi_0^{(1)} \) is of \( O(\bar{m}^2) \). By direct substitution of this expansion into Eq. (43), we arrive at the equations for each \( \bar{\varphi}_n^{(1)} \),
\[
\frac{\partial \bar{\varphi}_0^{(1)} (x, \tau)}{\partial \tau} = - \left( -\partial^2 + \bar{m}^2 \right) \bar{\varphi}_0^{(1)} (x, \tau) + \bar{\eta}(x, \tau) , \tag{45}
\]
\[
\frac{\partial \bar{\varphi}_1^{(1)} (x, \tau)}{\partial \tau} = - \left( -\partial^2 + \bar{m}^2 \right) \bar{\varphi}_1^{(1)} (x, \tau) - \left( \xi - \frac{1}{6} R \right) \bar{\varphi}_0^{(1)} (x, \tau)
+ \frac{1}{3} \left[ R_{\alpha\beta,\gamma} y^\alpha y^\beta \partial_\mu \partial_\nu - R^\alpha_{\beta, \gamma} y^\beta \partial_\alpha \right] \bar{\varphi}_0^{(1)} (x, \tau) , \tag{46}
\]
and similarly for the \( \bar{\varphi}_n^{(1)} \) with \( n \geq 2 \). We note that the last term in Eq. (46) vanishes due to Lorentz invariance [39]. The corresponding curvature expansion of the one-loop correlation function Eq. (36) is then given by
\[
\tilde{\Delta}^{(1-\text{loop})} (x, x' | \tau) = \langle \bar{\varphi}_0^{(1)} (x, \tau) \bar{\varphi}_0^{(1)} (x', \tau) \rangle_\eta + 2 \langle \bar{\varphi}_0^{(1)} (x, \tau) \bar{\varphi}_1^{(1)} (x', \tau) \rangle_\eta + \cdots , \tag{47}
\]
where we used the fact that
\[
\langle \bar{\varphi}_0(x, \tau) \bar{\varphi}_1(x', \tau) \rangle_\eta = \langle \bar{\varphi}_0(x', \tau) \bar{\varphi}_1(x, \tau) \rangle_\eta , \tag{48}
\]
since in the first order in curvature one can simply make $x \leftrightarrow x'$ for the curvature expansions in the normal coordinates, see, e.g., Eq. (40). The first term in Eq. (47) corresponds to the correlation function in flat spacetime which is already known, while the second one is the first order in curvature correction which is the new object to evaluate.

We solve Eqs. (45) and (46) by using the Fourier transforms of the scalar and noise fields,

\[
\tilde{\varphi}_n^{(1)}(x, \tau) = \int \frac{d^Dk}{(2\pi)^D} e^{ikx} \tilde{\varphi}_n^{(1)}(k, \tau), \tag{49}
\]
\[
\tilde{\eta}(x, \tau) = \int \frac{d^Dk}{(2\pi)^D} e^{ikx} \tilde{\eta}(k, \tau), \tag{50}
\]

with $\langle \tilde{\eta}(k, \tau)\tilde{\eta}(k', \tau') \rangle_\eta = 2\hbar (2\pi)^D\delta^D(k + k')\delta(\tau - \tau')$, which follows from Eq. (22). The Fourier transformed Eqs. (45) and (46) are

\[
\frac{\partial \tilde{\varphi}_0^{(1)}(k, \tau)}{\partial \tau} = -\left(k^2 + \tilde{m}^2\right) \tilde{\varphi}_0^{(1)}(k, \tau) + \tilde{\eta}(k, \tau), \tag{51}
\]
\[
\frac{\partial \tilde{\varphi}_1^{(1)}(k, \tau)}{\partial \tau} = -\left(k^2 + \tilde{m}^2\right) \tilde{\varphi}_1^{(1)}(k, \tau) - \left(\xi - \frac{1}{6}\right) R \tilde{\varphi}_0^{(1)}(k, \tau), \tag{52}
\]

where we have used the fact that the last term in Eq. (46) vanishes. Let us introduce the retarded Green function $G_s(k, \tau)$ for the stochastic differential equation, i.e.,

\[
\left[\frac{\partial}{\partial \tau} + (k^2 + \tilde{m}^2)\right] G_s = \delta(\tau), \tag{53}
\]

with the boundary condition

\[
G_s(k, \tau) = 0, \quad \text{for} \quad \tau < 0. \tag{54}
\]

The solution of Eq. (53) is simply

\[
G_s(k, \tau) = \theta(\tau) e^{-(k^2 + \tilde{m}^2)\tau}, \tag{55}
\]

where $\theta(\tau)$ is the Heaviside step function. Using the initial conditions given in Eqs. (31) and (32), one obtains for the flat spacetime Langevin equation

\[
\tilde{\varphi}_0^{(1)}(k, \tau) = \int_0^\tau d\tau' e^{-(k^2 + \tilde{m}^2)(\tau - \tau')} \tilde{\eta}(k, \tau'), \tag{56}
\]

and for the order-$R$ equation

\[
\tilde{\varphi}_1^{(1)}(k, \tau) = -\left(\xi - \frac{1}{6}\right) R \int_0^\tau d\tau' e^{-(k^2 + \tilde{m}^2)(\tau - \tau')} \tilde{\varphi}_0^{(1)}(k, \tau'). \tag{57}
\]
With the explicit solutions, one can evaluate the one-loop correlation function: while the flat space time part is given by [33]

\[
\langle \bar{\varphi}_0^{(1)}(x, \tau) \varphi^{(1)}_0(x', \tau) \rangle_\eta = \int \frac{dB}{(2\pi)^D} \frac{e^{iky}}{k^2 + \bar{m}^2} \left[ 1 - e^{-2(k^2 + \bar{m}^2)\tau} \right],
\]

(58)

the order-R is

\[
\langle \bar{\varphi}_0^{(1)}(k, \tau) \varphi^{(1)}_1(k', \tau) \rangle_\eta = - \left( \frac{\xi}{6} \right) R \int \frac{dB}{(2\pi)^D} \frac{e^{iky}}{k^2 + \bar{m}^2} \left[ \frac{1 - e^{-2(k^2 + \bar{m}^2)\tau}}{2(k^2 + \bar{m}^2)^2} - \frac{\tau e^{-2(k^2 + \bar{m}^2)\tau}}{k^2 + \bar{m}^2} \right],
\]

(59)

where we used the expression given right below Eq. (50) for the stochastic average of the noise fields in momentum space. Therefore, the one-loop correlation function \( \bar{\Delta}^{(1-\text{loop})}(x, x'|\tau) \) at first order in the curvature tensor is

\[
\bar{\Delta}^{(1-\text{loop})}(x, x'|\tau) = \int \frac{dB}{(2\pi)^D} e^{iky} \left\{ \frac{1 - e^{-2(k^2 + \bar{m}^2)\tau}}{k^2 + \bar{m}^2} \right. \\
- \left. \left( \frac{\xi}{6} \right) R \left[ \frac{1 - e^{-2(k^2 + \bar{m}^2)\tau}}{(k^2 + \bar{m}^2)^2} - \frac{\tau e^{-2(k^2 + \bar{m}^2)\tau}}{k^2 + \bar{m}^2} \right] \right\},
\]

(60)

In the equilibrium limit, one has

\[
\lim_{\tau \to \infty} \bar{\Delta}^{(1-\text{loop})}(x, x'|\tau) = \int \frac{dB}{(2\pi)^D} e^{iky} \left[ \frac{1}{k^2 + \bar{m}^2} - \left( \frac{\xi}{6} \right) R \frac{1}{(k^2 + \bar{m}^2)^2} \right],
\]

(61)

which reproduces in the equilibrium limit the result of Bunch and Parker [39] for the momentum-space representation of the Feynman propagator \( \bar{G}(x, x') \), Eq. (19). It is remarkable that we could obtain this result in the new way, using stochastic quantization in curved space and normal coordinates.

3.2. Effective potential

After deriving Eq. (61), one can use the result for calculating the effective potential. Let us start considering the curvature expansion of bilinear operator in Eq. (6) and the Green’s function in Eq. (61):

\[
\hat{H} = \hat{H}_0 + \hat{H}_1 + \cdots, \quad \bar{G} = \bar{G}_0 + \bar{G}_1 + \cdots.
\]

(62)

Solving the relation \( \hat{H} \bar{G} = -\hat{1} \) in Eq. (10) order by order in the curvature tensor one arrives at

\[
\hat{H}_0 \bar{G}_0 = - \hat{1}, \quad \hat{H}_0 \bar{G}_1 + \bar{G}_0 \hat{H}_1 = 0.
\]

(63)

To extract the order-R part of Eq. (8), we can make the following transformation:

\[
\text{Tr } \ln \hat{H} = \text{Tr } \ln (\hat{H}_0 + \hat{H}_1 + \cdots) \\
= \text{Tr } \ln \hat{H}_0 + \text{Tr } \ln (\hat{1} + \hat{H}_0^{-1} \hat{H}_1 + \cdots) \\
= \text{Tr } \ln \hat{H}_0 + \text{Tr } \hat{H}_0^{-1} \hat{H}_1 + \cdots,
\]

(64)
where in the last line we have made the expansion of the logarithm and kept only terms of the first order in curvature. The first term in the last line of Eq. (64) is the well-known result for the flat spacetime effective potential. The second term corresponds to the first order in curvature correction $\bar{V}_1^{(1)}(\phi)$, which can be derived by using relations (63), transformed as follows

$$
\bar{V}_1^{(1)} = -\frac{1}{2} \int d^D x' \int d^D k \frac{1}{(2\pi)^D} \epsilon^{jk(x-x')} \int d^D p \frac{1}{(2\pi)^D} \epsilon^{ip(x'-x)} \bar{G}_0^{-1}(k) \bar{G}_1(p)
$$

so that

$$
\bar{V}_1^{(1)} = \left( \xi - \frac{1}{6} \right) \frac{\Omega^D}{2^D \pi^{D/2} D \Gamma(D/2) m^2} \frac{1}{(2\pi)^D k^2 + m^2}
$$

Now, we can use Eq. (61) and replace the explicit forms of $\bar{G}_0^{-1}(k)$ and $\bar{G}_1(k)$ in Eq. (66) to obtain

$$
\bar{V}_1^{(1)} = \left( \xi - \frac{1}{6} \right) \frac{\Omega^D}{2^D \pi^{D/2} D \Gamma(D/2) m^2} \frac{1}{(2\pi)^D k^2 + m^2}.
$$

For $D = 4$, recalling that $\bar{m}^2 = m^2 + V''(\phi)$, one obtains

$$
\bar{V}_1^{(1)}(\phi) = \bar{V}_1^{(1)}_{1-\text{div}}(\phi) + \bar{V}_1^{(1)}_{1-\text{fin}}(\phi),
$$

where

$$
\bar{V}_1^{(1)}_{1-\text{div}}(\phi) = \frac{1}{2(4\pi)^2} \left( \xi - \frac{1}{6} \right) \left[ \Omega^2 - (m^2 + V'') \ln \frac{\Omega^2}{m^2} \right],
$$

and

$$
\bar{V}_1^{(1)}_{1-\text{fin}}(\phi) = \frac{1}{2(4\pi)^2} \left( \xi - \frac{1}{6} \right) \left( m^2 + V'' \right) \ln \left( \frac{m^2 + V''}{m^2} \right).
$$

This result is precisely the one obtained in Ref. [42], where also the details of renormalization and related aspects were discussed in detail and hence will not be considered here. In a similar way, for $D = 3$, we obtain

$$
\bar{V}_1^{(1)}(\phi) = \bar{V}_1^{(1)}_{1-\text{div}}(\phi) + \bar{V}_1^{(1)}_{1-\text{fin}}(\phi),
$$
where

\[
\tilde{V}_{1-div}^{(1)}(\phi) = \frac{\Omega}{(2\pi)^2} \left( \xi - \frac{1}{6} \right),
\]

and

\[
\tilde{V}_{1-fin}^{(1)}(\phi) = -\frac{1}{8\pi} \left( \xi - \frac{1}{6} \right) \left( m^2 + V'' \right).
\]

4. Numerical simulations

The main advantage of the approach based on the Riemann normal coordinates is that the practical calculation is performed in flat space. In particular, this means we can use the well-known methods of lattice-regularized Langevin numerical simulations.

For this initial investigation, we perform numerical simulations for the simplest \( D = 3 \) case. Then the mass dimensions of the relevant quantities are as follows: \([\bar{\phi}] = 1/2\), \([\lambda] = 1\), \(\tau = -2\), \([\bar{\eta}] = 5/2\), and \([R] = 2\). We solve the Langevin equations on a \( N^3 \) lattice, with lattice spacing \( a \). The Langevin-time discretization is denoted \( \Delta \tau \). It is convenient to rescale all dimensionful quantities by \( a \), namely

\[
\bar{\phi} = \hat{\phi} a^{-1/2}, \quad x = \hat{x} a, \quad \bar{m} = \hat{m} a^{-1}, \quad \bar{\lambda} = \hat{\lambda} a^{-1}, \quad \bar{\eta} = \hat{\eta} a^{-5/2} e^{1/2}, \quad \tau = \hat{\tau} a^2, \quad \Delta \tau = \epsilon a^2.
\]

In terms of these rescaled quantities, the discretized Langevin equations Eqs. (45) and (46) are given (in \( \text{Itô calculus} \ [49] \) by

\[
\hat{\phi}_0^{(1)}(\hat{x}, \hat{\tau} + \epsilon) = \hat{\phi}_0^{(1)}(\hat{x}, \hat{\tau}) + \epsilon \left( \hat{\Box} - \hat{m}^2 \right) \hat{\phi}_0^{(1)}(\hat{x}, \hat{\tau}) + \sqrt{\epsilon} \hat{\eta}(\hat{x}, \hat{\tau}),
\]

\[
\hat{\phi}_1^{(1)}(\hat{x}, \hat{\tau} + \epsilon) = \hat{\phi}_1^{(1)}(\hat{x}, \hat{\tau}) + \epsilon \left( \hat{\Box} - \hat{m}^2 \right) \hat{\phi}_1^{(1)}(\hat{x}, \hat{\tau}) - \left( \xi - \frac{1}{6} \right) \hat{R} \hat{\phi}_0^{(1)}(\hat{x}, \hat{\tau}),
\]

with \( \hat{\Box} \hat{\phi} = \hat{\phi}(\hat{x} + \hat{\epsilon}) - 2\hat{\phi}(\hat{x}) + \hat{\phi}(\hat{x} - \hat{\epsilon}) \), where \( \hat{\epsilon} \) is a unit vector, and the noise correlation is

\[
\langle \hat{\eta}(\hat{x}, \hat{\tau}) \hat{\eta}(\hat{x}', \hat{\tau}') \rangle_{\hat{\eta}} = 2 \delta_{\hat{x},\hat{x}'} \delta_{\hat{\tau},\hat{\tau}'}.
\]

The other equations for \( \hat{\phi}_0^{(2)}, \hat{\phi}_1^{(2)} \ldots \) are discretized in the same manner.

We consider the one-loop correlation function and the average value of the field. We concentrate on the \( R \)-dependent part of the correlation function, i.e. the second term in Eq. (47) (for simplicity, we consider the case \( x = x' \)):

\[
\hat{\Delta}_R(\hat{x}, \hat{x}|\tau) = 2 \langle \hat{\phi}_0^{(1)}(\hat{x}, \hat{\tau}) \hat{\phi}_1^{(1)}(\hat{x}, \hat{\tau}) \rangle_{\hat{\eta}},
\]
The right panel of the figure shows the results of the simulation for the average value of the field \( \langle \hat{\phi}(\hat{x}) \rangle \) at the minima of the classical potential, \( \hat{m} \), defined in Eq. (81), at the orders \( \hbar \) and \( \hbar^2 \).

The exact result in terms of lattice variables, corresponding to the second term in Eq. (61), is given by

\[
\hat{\Delta}_R^{\text{exact}}(\hat{x}, \hat{y}) = \frac{1}{16N^3} \sum_{n_1, n_2, n_3=0}^{N-1} \left\{ \frac{1 - e^{-2\hbar(\hat{k}^2 + \hat{m}^2)}}{[d(\hat{m}, n_1, n_2, n_3)]^2} - \frac{2\hbar e^{-2\hbar(\hat{k}^2 + \hat{m}^2)}}{d(\hat{m}, n_1, n_2, n_3)} \right\},
\]

(80)

where \( d(\hat{m}, n_1, n_2, n_3) = (\hat{m}/2)^2 + \sum_{i=1}^3 \sin^2(n_i \pi/N) \). We also consider the average of the scalar field as a function of \( \tau \),

\[
\langle \hat{\phi}(\hat{x}) \rangle = \frac{1}{N^3} \sum_{n_1, n_2, n_3=0}^{N-1} \langle \hat{\phi}(\hat{x}, \hat{y}) \rangle_\eta.
\]

(81)

We solve Eqs. (76) and (77) with \( \epsilon = 10^{-4} \) on a \( N^3 = 16^3 \) lattice. The values of the lattice mass and coupling constant are chosen \( \hat{m} = 1 = \hat{\lambda} \). The classical background field is the constant field at the minimum of the classical potential, \( \hat{\phi}_{\text{cl}} = \hat{m}/\hat{\lambda}^{1/2} = 1 \). The corresponding bare “Higgs mass” is \( \hat{m}_{\text{cl}}^2 = -\hat{m}^2 + 3\hat{\lambda}\hat{\phi}_{\text{cl}}^2 = 2\hat{m}^2 = 2 \).

Fig. 1 shows on the left panel the results of the numerical simulations (with stochastic averages taken over 100 noise realizations) for the \( R \)–dependent part of the correlation function at the one-loop order for different values of \( (\xi - 1/6)\hat{R} \). The corresponding analytical results are also shown in Fig. 1. It is easy to see that the lattice simulations reproduce very well the analytical results. The right panel of the figure shows the results of the simulation for the average value of the field \( \hat{\phi} \), defined in Eq. (81). Results are shown for the one- and two-loop contributions, i.e. \( O(\hbar) \) and \( O(\hbar^2) \). More specifically, the \( O(\hbar^2) \) contribution is given by \( \langle (\hat{\phi}_{\text{cl}} + \hat{\phi}_{(2)} + \hat{\phi}_{(4)} + \hat{\phi}_{(6)}) \rangle_\eta \), the
\( O(\hbar) \) contribution stops at the terms \( \phi^{(2)}_0 \) and \( \phi^{(2)}_1 \) in the sum. The results shown are for \( (\xi - 1/6)\hat{R} = 0.5 \); different values of this parameter lead to qualitatively the same results. For the parameters chosen, the figure reveals that the two-loop contribution leads to a decrease of the equilibrium value, although the effect is not large. Changing parameters, like the values of the mass and coupling constant, leads to qualitatively similar results, although some quantitative change takes place.

5. Conclusions and Perspectives

The formalism of stochastic quantization enables one to go beyond the scope of the usual perturbation theory, in particular by numerically solving Langevin equation. We presented a construction of this equation for the self-interacting scalar field in an arbitrary curved background. The solution of the Langevin equation can be carried on either analytically or numerically, by means of the local momentum representation.

In the analytical part of the work we used the Langevin equation to reproduce the known result for the effective potential of self-interacting scalar field in curved space in the dimension \( D = 4 \). This fact confirms that the equivalence between stochastic quantization and path integral also holds in curved space. Furthermore, we derive the effective potential in an arbitrary dimension \( D \) in curved space. To the best of our knowledge this result is original.

The main point of the paper is the complementarity of the analytic and numerical approaches, especially the possibility of using numerical simulations based on lattice methods in curved spacetime. This possibility is facilitated by the use of the local momentum representation, which enables one to obtain the curved space results by making calculations in flat space, associated with a given point \( P \).

From the exercise described in Sec. 4 it is clear that the numerical simulations can be carried out to an arbitrary order in \( \hbar \). A reason for that is that the Langevin equations for the different orders in \( \hbar \) have the same structure, they are linear and can be solved iteratively. Furthermore, a fully nonperturbative numerical simulation can be performed by performing an expansion in curvature only and not expanding in \( \hbar \), i.e. no expansion in the noise field.

Given the limited scope of this first publication regarding numerical simulations, we do not pursue such analyses further here. It is planned to make a more complete analysis in a future publication for the realistic case of \( D = 4 \) for which, in particular, the continuum limit and renormalization will be studied.

The progress in nonperturbative methods in curved spacetime would pave the way to various applications. Perhaps the most relevant step would be the development of nonperturbative methods of evaluating the effective action, which is a generalization of effective potential for the non-constant background field. Another possible development of the approach which we presented...
above is related to the gauge-independent quantization of the theories with unbroken, softly broken and even hardly broken gauge symmetries. We expect to deal with the mentioned issues in future works.

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