Singlet Generation in Mixed State Quantum Networks

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We study the generation of singlets in quantum networks with nodes initially sharing a finite number of partially entangled bipartite mixed states. We prove that singlets between arbitrary nodes in such networks can be created if and only if the initial states connecting the nodes have a particular form. We then generalize the method of entanglement percolation, previously developed for pure states, to mixed states of this form. As part of this, we find and compare different distillation protocols necessary to convert groups of mixed states shared between neighboring nodes of the network into singlets. In addition, we discuss protocols that only rely on local rules for the efficient connection of two remote nodes in the network via entanglement swapping. Further improvements of the success probability of singlet generation are developed by using particular forms of ‘quantum preprocessing’ on the network. This includes generalized forms of entanglement swapping and we show how such strategies can be embedded in regular and hierarchical quantum networks.

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I. INTRODUCTION

Quantum entanglement is one of the most notable features of quantum systems and has been accepted as a key resource for quantum information processing [1]. The distribution of entanglement through quantum networks is therefore essential for the future of a variety of applications, ranging from quantum cryptography to quantum teleportation and distributed quantum computing [2]. However, the generation of these entangled states faces a severe obstacle. Quantum channels such as free-space transmission or optical fibers are prone to loss and decoherence. This causes the desired maximally entangled states to degrade into mixtures and limits the distance over which the quantum information can be sent directly. To overcome these problems ‘quantum repeater’ schemes have been proposed [3, 4, 5, 6, 7, 8] which make use of the ability to ‘purify’ [9, 10] and ‘swap’ [11, 12] entanglement to maintain a high fidelity throughout. Quantum repeaters are a promising tool for entanglement distribution, particularly since the amount of required physical resources increases only polynomially with the distance [5], but operate in a 1D setup of network nodes. Real networks are typically two- (or higher-) dimensional networks was recently proposed by Acín et al. [13] in which ideas from classical bond percolation schemes applied to the network can improve the SCP [13, 15, 16, 17, 18, 19], and thus it is possible to apply bond percolation to lattices in which this would otherwise not be possible.

Clearly, the assumption of having a pure-state network is an idealization and in any practical situation the states connecting the nodes of the network will be mixed. In [20] the idea of entanglement percolation was applied to mixed states for the first time. In this paper we elaborate and extend the ideas presented in [20]. The networks we consider are composed of nodes, each of which can consist of several qubits, and may be connected by an arbitrary number of bipartite mixed states (see Fig. 1). We aim to create a perfect singlet between two arbitrary nodes in the network using a finite amount of resources, i.e. a finite number of initial states which are converted into a singlet, which distinguishes our and other entanglement percolation schemes from, e.g., the quantum repeater protocol where one aims to generate a state with high but non-unit entanglement fidelity. Particularly we structure the paper as follows.

In Sec. [11] we prove a necessary and sufficient condition that a perfect singlet can be generated in a network of arbitrary geometry the nodes of which are initially connected by bipartite mixed qubit states. We show that...
singlet generation between two nodes is possible if and only if they are connected by at least two ‘paths’ consisting of a particular class of states. These states arise naturally in systems undergoing amplitude damping. Thus our result is not only of theoretical but also of practical relevance. Unfortunately, the proof does not deliver an efficient scheme for singlet generation. We therefore specialize in the remaining sections on networks with regular geometry, i.e., lattices in 2D and 3D and devise generalizations of entanglement percolation to the mixed states described in Sec. II.

In Sec. III we briefly summarize the idea of classical entanglement percolation with pure states.

In Sec. IV we extend the concept of classical entanglement percolation to mixed states. To this end we consider regular networks where each node is connected to its neighboring nodes by a finite number of the mixed states introduced in Sec. II. We present two different distillation protocols which are used to convert these states into a singlet with a probability above the percolation threshold of a variety of lattice geometries. After the distillation, clusters of singlet-connected nodes emerge and we aim to create a singlet between two nodes in such a cluster by successive application of entanglement swapping. By communicating classically each node can determine if singlets exist between it and its neighboring nodes. This information can be communicated and stored classically in a central data processor. Typically one would then use this information to apply a path-finding algorithm which locates a suitable ‘path’ of singlets before swapping operations are performed. As an alternative to this we discuss a classical and a quantum protocol which merely require classical communication between neighboring nodes and basic computing within each node. The quantum protocol relies on the formation of many-qubit GHZ states via local operations and classical communication with neighboring nodes and subsequent measurements at all nodes except the ones to be left in the final singlet.

In Sec. V we show that the idea of ‘quantum preprocessing’ as it was successfully applied in pure state networks can be generalized to mixed states. In particular we devise a number of strategies on small networks which improve the SCP, and we show that these smaller networks can be embedded into larger networks to enable CEP which would otherwise not be possible. Furthermore, we discuss ‘hierarchical schemes’, i.e. networks which are defined iteratively and were first discussed in [13, 10]. Also in these cases it turns out that quantum methods outperform classical percolation. Finally, in Sec. VI we summarize and conclude.

II. SINGLET GENERATION WITHIN AN ARBITRARY MIXED STATE NETWORK

In this section we consider quantum networks of arbitrary geometry as shown in Fig. 1 where the qubits in the nodes are ‘connected’ by bipartite mixed states to qubits in other nodes. We will call a single bipartite mixed state an edge and the set of edges directly connecting two nodes a bond. Note that an edge connects exactly two qubits in different nodes. In the following we will prove that the generation of a perfect singlet between two arbitrary nodes A and B with finite probability in such a network is possible if and only if there are at least two paths of states linking A and B which have, up to local unitaries, the form

\[ \rho(\alpha, \gamma, \lambda) = \lambda |\alpha, \gamma \rangle \langle \alpha, \gamma | + (1 - \lambda) |01 \rangle \langle 01 | \]

where \( |\alpha, \gamma \rangle = \sqrt{\alpha} |00 \rangle + \sqrt{1 - \alpha - \gamma} |11 \rangle + \sqrt{\gamma} |01 \rangle \) and \( 0 \leq \lambda \leq 1 \). We show this by separately proving a necessary and sufficient condition which, together, prove the above statement.

**Necessary condition.** We split the network into two groups of nodes, \( \mathcal{A} \), containing A and a finite number of other nodes, and \( \mathcal{B} \), which consists of the rest of the network and particularly contains B. These groups are linked by a finite number of edges. A singlet can be established with finite probability, via local operations in the groups and classical communication between them, if and only if at least two of the states have the form (1). Appendix A contains a concise proof of this fact based on [21] which agrees with the result of Ref. [22], that,
in general, a singlet can not be generated with a finite probability from a finite number of mixed states.

With two states of the form (1), \(\rho(\alpha, \gamma, \lambda)\) and \(\rho(\beta, \delta, \nu)\), we obtain a singlet with a finite probability by first performing two C-NOT gates locally, with the \(\rho(\beta, \delta, \nu)\) state’s qubits acting as the target qubits. These target qubits are then measured in the computational basis. If we find both qubits to be in the state \(|1\rangle\) we have generated a pure entangled state between the qubits that originally corresponded to the \(\rho(\alpha, \gamma, \lambda)\) state. We will refer to this measurement as the pure state conversion measurement (PCM). The state formed is

\[
|\alpha'\rangle \equiv |\alpha', \gamma = 0\rangle = \sqrt{\alpha'}|00\rangle + \sqrt{1 - \alpha'}|11\rangle, \tag{2}
\]

i.e. \(\alpha'\) is a Schmidt-coefficient that has the value

\[
\alpha' = \min(\alpha(1 - \beta - \delta), \beta(1 - \alpha - \gamma)). \tag{3}
\]

The probability that the PCM succeeds in generating this state is given by

\[
p_c = \lambda\nu(\alpha(1 - \beta - \delta) + \beta(1 - \alpha - \gamma)). \tag{4}
\]

For identical states, i.e. \(\alpha = \beta, \gamma = \delta\), the PCM already yields a singlet. Otherwise the state can be transformed into a singlet via the ‘Procrustean method’ \(23\) that converts any pure 2-qubit state \(|\alpha'\rangle\) into a singlet \(|1/2\rangle\) with a probability \(2\min(1 - \alpha', \alpha')\). The total success probability of generating a singlet is then given by the SCP

\[
p_{conv} = 2\lambda\nu \min(\alpha(1 - \beta - \delta), \beta(1 - \alpha - \gamma)) \tag{5}
\]

which coincides with the optimal probability for creating a singlet from two of these states \(21\).

We can perform this partition of the network in an arbitrary way, as long as one group contains \(A\) and the other contains \(B\). To be able to create a singlet between \(A\) and \(B\) via LOCC we must have at least two states of the form (1) in all possible partitions. This gives us a necessary condition that to create a singlet between two nodes with a non-zero probability there have to be at least two distinct ‘paths’ of edges of the form (1) connecting the corresponding nodes. In Fig. 1(a) this is indicated by two spatially distinct paths of bonds. The states of the qubits that are not contained in this path are irrelevant and can therefore be in arbitrary states.

**Sufficient condition.** In order to show this we make use of entanglement swapping. This operation can be performed in the setup shown in Fig. 2 and consists of performing a measurement in the standard Bell basis on the qubits located at \(C_2\) and LOCC which causes \(C_1\) and \(C_3\) to become entangled. If the edges are of the form (1), \(\rho(\alpha, \gamma, \lambda)\) and \(\rho(\beta, \delta, \nu)\), then there are four possible outcomes. The probabilities to obtain measurement outcomes corresponding to the Bell states \(|\Psi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}\) and \(|\Phi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}\) are

\[
p(|\Psi^\pm\rangle) = \frac{1}{2}(h_+ \lambda\nu + (1 - \beta - \delta)(1 - \lambda)\nu + \alpha\lambda(1 - \nu)) \tag{6}
\]

\[
p(|\Phi^\pm\rangle) = \frac{1}{2}(g_+ \sqrt{\alpha\delta + \gamma(1 - \beta - \delta)} + (1 - \gamma)\beta, \delta = 0) \tag{7}
\]

where

\[
h_\pm = \alpha\beta + (1 - \alpha - \gamma)(1 - \beta - \delta) + (\sqrt{\alpha\delta + \gamma(1 - \beta - \delta)} + (1 - \gamma)\beta, \delta = 0) \tag{8}
\]

\[
g_\pm = \gamma\beta + (1 - \alpha - \gamma)(1 - \beta - \delta) + (\sqrt{\gamma\delta + \alpha(1 - \beta - \delta)} + (1 - \gamma)\beta, \delta = 0) \tag{9}
\]

If we measure the qubits at \(B\) to be in the states \(|\Psi^\pm\rangle\) then we actually form another state,

\[
\rho\left(\frac{h_+\sqrt{\alpha\delta + \gamma(1 - \beta - \delta)} + (1 - \gamma)\beta, \delta = 0}{h_+, \nu h_+}, \lambda\nu h_+\right) \tag{10}
\]

of the form (1) between \(C_1\) and \(C_3\). Unfortunately for the other outcomes the states’ form is not generally maintained. Note that if \(\delta = \gamma = 0\) we can discard these cases by replacing the state with \(|01\rangle\) leading to an operation that transforms \(\rho(\alpha, 0, \lambda) \otimes \rho(\beta, 0, \nu)\) into

\[
\rho\left(\frac{\alpha\beta, \sqrt{\alpha\delta + \gamma(1 - \beta - \delta)} + (1 - \gamma)\beta, \delta = 0}{h_+, \nu h_+}, \lambda\nu h_+\right), \tag{11}
\]

which will be useful in Sec. IIIA We can therefore create a state of the form (1) with non-zero probability between two nodes of the network, e.g. \(A\) and \(B\) in Fig. 1 given that these nodes are connected by a path consisting of states of the same form. Two such states, originating from two paths, can then be converted into a singlet, using a PCM and the Procrustean procedure. Unfortunately, this scheme leads to an exponential decrease of entanglement fidelity \(5\), and thus success probability, with the number of swapping operations. Hence it is not an effective solution to the problem of long-distance entanglement distribution. In Sec. IV we will therefore introduce effective protocols which can be applied in regular network geometries and succeed in creating a singlet with a probability independent of distance.

Note that when entanglement swapping is done with pure states all of the outcomes can be used, and if these outcomes occur with probabilities \(p_m\) the pure state \(|\tilde{\alpha}\rangle\)
any two nodes in the cluster. In the theoretical case of an infinitely large lattice a cluster that is infinite in extent forms if and only if $p > p_c$, where $p_c$ is a lattice-dependent percolation threshold. This approximates the case for large but finite lattices where the threshold becomes more definitive as the size of the lattice increases. Values of $p_c$ for a number of lattice geometries are given in Table I. If each bond in a network consists of a single pure state $|\alpha\rangle$ we can calculate a threshold for $\alpha$ given by $2 \min(\alpha, 1 - \alpha) > p_c$. The probability that a node belongs to the infinite cluster is known as the percolation probability $\theta(p)$. Two randomly chosen nodes are both part of the infinite cluster with a probability $\theta(p)^2$ and thus can be connected over an arbitrary distance.

$$\theta(p)^2 = \left(\frac{\alpha + 1 - \alpha}{2}ight)^2$$

TABLE I: Threshold probabilities for various regular network geometries.

| Lattice                  | Threshold $p_c$ |
|--------------------------|-----------------|
| 2D Square                | 0.5             |
| 2D Triangular            | $2\sin(\pi/18) \approx 0.347$ |
| 2D Honeycomb             | $1 - 2\sin(\pi/18) \approx 0.653$ |
| 3D Simple Cubic          | 0.249           |
| 3D Face-Centered Cubic   | 0.120           |

It has been shown that CEP using pure states is not optimal and that by performing particular quantum preprocessing steps, particularly swapping operations on the lattice before converting to singlets, improvements can be achieved. These improvements include obtaining a geometry with a lower percolation threshold after the swapping operation and splitting the lattice into two, so that a higher percolation probability can be obtained \cite{13, 14, 15, 16, 17, 18}. Recently, another method, that transforms the initial bipartite network into a probabilistic multipartite network, has also been shown to yield an improvement \cite{19}.

IV. CLASSICAL ENTANGLEMENT PERCOLATION WITH MIXED STATES

In this section we extend CEP to mixed states. We consider regular lattices, e.g. triangular (see Fig. 4), square, or even lattices in higher dimensions. Bonds between network nodes are composed of multiple edges which satisfies the necessary condition proven in Sec. II. We assume that each bond is identical. When these bonds contain at least two states of the form (1) they can be converted into singlets by PCM followed by the Procrustean method. If the probability that a bond becomes a singlet exceeds the percolation threshold CEP is achieved. In the remainder of the paper we will assume that the states forming edges are of the form (1) with $\gamma = 0$. Setting $\gamma = 0$ is not a major restriction but allows us to keep the equations manageable. All protocols presented in this paper can also be performed if $\gamma \neq 0$. 

III. CLASSICAL ENTANGLEMENT PERCOLATION WITH PURE STATES

In this section we will briefly review the use of percolation for distributing singlets in pure state networks \cite{13, 14, 15, 16, 17}. A description of classical bond percolation can be found in Ref. \cite{24}. If the nodes are connected by pure states of the form $|\alpha\rangle$ they can be converted into singlets using the Procrustean method with a SCP $p = 2 \min(\alpha, 1 - \alpha)$. These singlets act as the bond in the bond percolation model \cite{38} and are distributed randomly with a probability $p$. The nodes that can be connected by a path of singlets form a cluster. By using entanglement swapping (see Sec. II) we can then generate a singlet between

![Entanglement percolation diagram](https://via.placeholder.com/150)

FIG. 3: Illustration of classical entanglement percolation in a square network. Pairs of qubits (black dots) in neighboring nodes (circles) are in identical, pure, partially entangled states (solid, black lines). The percolation scheme involves these entangled states being converted into singlets (dashed lines) with probability $p$. If $p$ exceeds the percolation threshold these form large clusters and we can obtain a singlet between any two qubits within a cluster by performing swapping operations.

with

$$\tilde{\alpha} = \frac{1}{2} \left(1 + \sqrt{1 - \alpha \beta(1 - \alpha)(1 - \beta)}\right)$$

is recovered by using classical communication and local unitaries.

![Singlet formation diagram](https://via.placeholder.com/150)

![Cluster formation diagram](https://via.placeholder.com/150)

![Entanglement swapping diagram](https://via.placeholder.com/150)
We will call states of the form (1) with \( \gamma = 0, \lambda \)
\begin{align}
\rho(\alpha, \lambda) &\equiv \rho(\alpha, \gamma = 0, \lambda), \tag{13}
\end{align}
purifiable mixed states (PMSs). Note that these states form the states of two entangled atomic ensembles in the DLCZ quantum repeater scheme [4].

A. Distillation Procedures

1. Distillable Subspace Scheme

We assume that each pair of neighboring nodes is connected by \( n \) PMSs and our aim is to distill these states into a singlet. The basic setup is shown in Fig. 5. To accomplish this we will use ideas proposed in Ref. [20]. Here the concept of a distillable subspace (DSS) is introduced as a subspace such that the local projection of the system state into this space is pure and entangled. Locating the DSS involves calculating the eigenvectors of the state with non-zero eigenvalues. To simplify notation we will represent the states at \( A \) and \( B \) using the decimal value of its binary form, i.e. for example \( [00110]_A[01001]_B = [6]_A[d]_B 
\)

As an example, in the case of \( n = 2 \) identical states \( \rho(\alpha, \lambda) \) the eigenvalues and corresponding eigenvectors are
\begin{align*}
\lambda^2 : \quad &\alpha|0\rangle_A^d|0\rangle_B^d + \sqrt{\alpha(1-\alpha)}|1\rangle_A^d|1\rangle_B^d + \\
&\sqrt{\alpha(1-\alpha)}|2\rangle_A^d|2\rangle_B^d + (1-\alpha)|3\rangle_A^d|3\rangle_B^d, \\
\lambda(1-\lambda) : \quad &\sqrt{\alpha(0)}|0\rangle_A^d|2\rangle_B^d + \sqrt{1-\alpha}|1\rangle_A^d|3\rangle_B^d, \\
\lambda(1-\lambda) : \quad &\sqrt{\alpha(0)}|1\rangle_A^d|1\rangle_B^d + \sqrt{1-\alpha}|2\rangle_A^d|3\rangle_B^d, \\
(1-\lambda)^2 : \quad &|0\rangle_A^d|3\rangle_B^d. \tag{14}
\end{align*}

If this is acted on by the projective measurement \( |1\rangle_A^d(1) + |2\rangle_A^d(2) \) at \( A \) and \( |1\rangle_B^d(1) + |2\rangle_B^d(2) \) at \( B \) the state remaining is \((|1\rangle_A^d|1\rangle_B^d + |2\rangle_A^d|2\rangle_B^d)/\sqrt{2}\). Both of these projective measurements only occur with probability
\begin{align}
p_{n=2} = 2\lambda^2 \alpha(1-\alpha). \tag{15}
\end{align}

Note that this is the same SCP as obtained for PCM [see Eq. (3)]. For this example there is no choice between entangled states to project out and if the original states are the same a maximally entangled state is automatically obtained. For states that are not identical this does not need to be the case.

An extension of this scheme to \( n \) identical copies of PMSs \( \rho(\alpha, \lambda) \) yields the SCP
\begin{align}
p_n = \sum_{m=0}^{n} \lambda^{n-m}(1-\lambda)^m \binom{n}{m} \\
\times \left( \sum_{k=1}^{n-m-1} \alpha^{n-m-k}(1-\alpha)^k \binom{n-m}{k} - 1 \right). \tag{16}
\end{align}

A derivation of this formula is given in Appendix [13]. As a particular example it is worthwhile to discuss the case of three states in more detail. In this case the measurement at \( A \) is given by a Positive Operator Valued Measure
The SCP is obtained by setting $B$ and creates a maximally entangled state with a certain probability. The measurement at $B$ then depends on this outcome and enters a maximally entangled state with a certain probability. The SCP is obtained by setting $n = 3$ in Eq. (15) and is given by

$$p_{n=3} = 3\lambda^2 \alpha (1 - \alpha).$$

Comparing this with the $n = 2$ case [cf. Eq. (13)] shows an increase in the success probability which can be seen in Fig. 8, where the dashed line represents the SCP for three identical states.

2. **Recycling scheme**

The SCP using the DSS scheme does generally increase with increasing $n$. However, the scheme does not make use of the available resources in the best way. Indeed, the SCP $p_n$ can be significantly improved by grouping $n$ identical PMSs into sets of $m$ and converting each of these sets into a singlet. For example for $n = 2$ we apply the PCM as described in Sec. II on pairs of states which converts them into singlets with a probability given by Eq. (16). If this fails for a given pair we may still find both measured qubits in the state $|0\rangle$ and have generated another PMS. This PMS can then be used again in another purification attempt. To be more precise, starting with $n$ copies of a state $\rho(\alpha, \lambda)$ (with $\alpha \geq 1/2$) we apply a 2-state purification protocol on groups of two. If no singlet is obtained the procedure is repeated on the remaining PMSs as illustrated in Fig. 7. The coefficients for the PMSs after $k$ repetitions, when no singlet is created, are given by

$$\alpha_k = \frac{\alpha_{k-1}^2}{1 - 2\alpha_{k-1} + 2\lambda_{k-1}^2},$$

$$\lambda_k = \frac{\lambda_{k-1}^2(1 - 2\alpha_{k-1} + 2\lambda_{k-1}^2)}{1 - 2\lambda_{k-1} + 2\lambda_{k-1}^2(1 - \alpha_{k-1} + \alpha_{k-1}^2)},$$

where $\alpha_0 = \alpha$ and $\lambda_0 = \lambda$. For states of the form $\rho(\alpha_k, \lambda_k) \otimes \rho(\alpha_k, \lambda_k)$ the probability of obtaining a PMS is $c_k = 1 - 2\lambda_k + 2(1 - \alpha_k + \alpha_k^2)\lambda_k^2$. If the PCM yields two qubits that are measured in different states the purification step between the two PMSs has completely failed. The probability of this is given by $f_k = 2\lambda_k(1 - \lambda_k)$. The probability of not generating a singlet using this recycling protocol on $n$ states of the form $\rho(\alpha_i, \lambda_i)$ is then found to be

$$F_n(i) = \sum_{k=0}^i \left( \binom{n}{k} \frac{1}{k} \right) f_k^{i-k} c_i^k F_k(i + 1),$$

where $F_0(i) = 1$. Consequently, the probability of successfully generating a singlet by applying the procedure to $n$ states of the form $\rho(\alpha, \lambda)$ is $1 - F_n(0)$ which is calculated iteratively. Examples are shown in Figs. 6 for $\alpha = 1/2$.

Obviously, the states do not necessarily need to be split into pairs. For example we can separate all of the states into sets of three and apply the three-state DSS distillation. In case of failure this can yield a PMS state as well, which can then be used in later distillation steps. There are a variety of ways to combine the three-state distillation with the two-state recycling scheme. Here we concentrate on the straightforward approach which only uses the three-state distillation on every level of the recycling scheme. The results are shown in Fig. 8. As can be seen in most cases the two-state recycling scheme has a
higher chance of success and because of this we will focus on the pairing arrangement in this paper.

B. Percolation Thresholds

Using the purification procedures described above we can apply CEP, as described in Sec. II, for lattice networks with multi-edged bonds. In most cases it is advantageous to use the two-state recycling scheme, except for $n = 3$ where the DSS scheme should be used. From Fig. 6 it can be seen that the SCP increases with the number of edges per bond and this allows for a larger range of values for $\lambda$ and $\alpha$ such that CEP is successful. For double edged bonds the optimal probability of generating a singlet is given by

$$0 \leq 2\lambda^2\alpha(1 - \alpha) \leq 1/2.$$  \hspace{1cm} (22)

When the bonds are composed of three edges, i.e. three PMSs between nodes, we have

$$0 \leq 3\lambda^2\alpha(1 - \alpha) \leq 3/4.$$  \hspace{1cm} (23)

By comparing these ranges to the percolation thresholds we see that a basic successful setup is a double bonded triangular lattice (see Fig. 4). The double bonds can be converted to singlets and if the chance of this is larger than the percolation threshold an infinite cluster will form. A singlet can then be created between any two nodes within the cluster. Thus percolation occurs if

$$2\lambda^2\alpha(1 - \alpha) > 2\sin(\pi/18) \approx 0.347.$$  \hspace{1cm} (24)

However, the singlet conversion probability never exceeds $1/2$ for two states. Therefore we would require more states in other geometries. For example, if we have three-edged bonds between each neighboring node we can apply CEP to a square lattice. This is because there are parameters such that $3\lambda^2\alpha(1 - \alpha) > 1/2$. Analogously, CEP is also possible in honeycomb lattices with three edges per bond.

C. Local Processing Strategies

The process of creating singlets, randomly replacing the initial network bonds, can be run if each node can only communicate classically with their neighbors. Each node then knows if a qubit that it contains is part of a singlet after this procedure has finished. This information can be stored classically within a node but after the bonds are distilled we are faced by the problem of finding a set of singlets that connect our requested nodes, $A$ and $B$.

If all of the singlet generation data is collected by a ‘controller’ then an efficient path finding algorithm can be applied to determine a suitable ‘path’ of singlets linking the nodes. An example of a suitable algorithm would be a Dijkstra scheme such as the A* path finding algorithm. The path information can then be used to instruct the correct nodes to perform swapping. The swapping operations are performed in order from node $A$ to $B$, so that the measurement outcomes only need to be communicated along the chain, between neighboring nodes. However this procedure requires one classical computer to have complete knowledge of the network. Instead, it is interesting to note that this does not need to be the case as there are algorithms which do not require any more classical communication than this ‘controller’ method, indeed they do not require a central ‘controller’ at all. This can be done not only classically but also via a quantum algorithm using multipartite entanglement which we will introduce below.

A classical path-finding method would use a type of breadth-first search algorithm called a burning algorithm. Node $A$ sends a ‘burning’ signal to its neighboring nodes connected by singlets. These nodes keep a record of where they received the signal from and send out an identical signal to the other nodes that they are connected to. We say that the node has ‘burned’. If it has already received a signal from a different node then the additional signal is ignored. This continues outwards from $A$, ‘burning’ the nodes. Once node $B$ receives the signal it replies to the node it came from with a ‘swapping’ message. This node can then perform a swapping operation and send another ‘swapping’ signal, together
with the Bell-measurement outcome, back to the node it received a ‘burning’ signal from. The path can then be traced back along the nodes with swapping performed at each step until node $A$ is reached. Both $A$ and $B$ can determine if the protocol has been successful. However, $A$ and $B$ may not be in the same cluster and they do not know if the protocol has failed when the network is of infinite size. This is not a problem for finite networks, containing $N$ nodes, as $A$ and $B$ can time the steps taken and if these exceed $2(N-1)$ they both know they are not in the same cluster.

Note that no extra information actually needs to be transmitted. We can combine the burning algorithm with the process of transmitting the distillation protocol information. For example, in a double edged network of identical edges, $A$ can perform her PCM and if $|1\rangle$ is the outcome she assumes she has a singlet and sends a burning signal to the node that would contain the singlet’s other qubit. If a node receives this signal it can perform its PCM and determine if there is a singlet there. When there is and if it is the first instance for the node it should record that entry qubit and repeat the process, performing a PCM on the remaining qubits and sending signals to those with the $|1\rangle$ outcome. Once $B$ receives a signal it can check that a singlet has been created with a PCM and then send a swapping signal back as before. During the swapping, a node can use the Bell-measurement information received to indicate that a swapping is required so no explicit ‘swapping’ signal is required either. All of this information transfer would have been necessary as well if a controller algorithm would have been used. Hence the generation of the singlet can be accomplished by defining rules for each node and allowing them to run with nearest neighbor classical communication. This is fundamentally different to the controller process and has made use of parallel computation to find a path that no single node has full knowledge of.

We will now consider an alternative, quantum algorithm that is based on the burning algorithm and makes use of multipartite entanglement in the network. The protocol starts after we attempted to convert all bonds into singlets and every node has knowledge about its singlet connections to nearest neighbors. We build up a progressively larger multi-qubit GHZ state, defined by $|GHZ_n\rangle = (|0\rangle_1 ... |0\rangle_n + |1\rangle_1 ... |1\rangle_n)/\sqrt{2}$, spread between the ‘burned’ nodes by adding qubits in each burning step. Building up such a state requires joining two GHZ states, $|GHZ_n\rangle$ and $|GHZ_m\rangle$, to create $|GHZ_{n+m-1}\rangle$ (note that a singlet equals $|GHZ_2\rangle$). This is done by performing a CNOT gate between a qubit in $|GHZ_n\rangle$ and a target qubit in $|GHZ_m\rangle$, measuring the target qubit in the Z-basis, communicating the measurement result to the other qubits in $|GHZ_m\rangle$ and performing a unitary operation on them depending on the outcome. Now we perform the same process as for the ‘burning algorithm’, however, as each node is ‘burned’ it is connected to the GHZ state spread over the previously burned nodes. The process to do this is illustrated in Fig. 9 and consists of joining the singlets partially contained in that node to the GHZ state. Within each node one qubit is left entangled with the GHZ state. After this operation has been run for a maximum of $N-1$ times all of the nodes in the cluster containing $A$ have a qubit from a single GHZ state.

![FIG. 10: After the singlets are formed we can repeatedly extend a GHZ state from node A. This procedure uses the operation shown in Fig. 9 to add qubits to the GHZ state. The black squares depict qubits that are part of the GHZ state. Arrows represent a message to add nodes to the GHZ state along singlet paths. Each node keeps a record of the node from which it received this message from, symbolized here by a white dot. When a node cannot extend the GHZ state any further (highlighted by a dashed outline) it measures its qubits in the X basis (open squares) and sends this information back towards $A$ (thin arrows) along the route recorded. Certain nodes are selected beforehand not to perform the measurement (here $A$ and $B$) and these will form the resulting GHZ state. At each node the incoming data can be combined and sent back along one path if the routes branch. Once this data returns to $A$ a phase operation can be performed on the qubit there to correct for any errors and the final GHZ state (here a singlet between $A$ and $B$) will remain.](image-url)
At each node a record is kept of the bond via which it has been included into the GHZ state. If there is a singlet between two nodes that are being burned then the singlet is ignored. Furthermore we add the rule that whenever a node can not extend the GHZ state anymore X basis measurements are performed along the recorded path back to A. This removes a qubit from the GHZ state but introduces a phase error in the remaining GHZ state depending on the outcomes of the measurement. The information about these measurement outcomes has to be sent back along the path to A. Whenever the route back branches, the measurement outcome is sent in one way and a message corresponding to ‘no phase error occurrence’ is sent to the others. At each node the returning process is paused until all of the bonds it sent a burning signal to provide it with the phase information. At nodes A and B we do not perform the X measurement. Finally after A receives all of the phase information a phase correction can be performed and we obtain a singlet between A and B. In Fig. 11 an example is given to illustrate the protocol.

V. QUANTUM PREPROCESSING

Despite being a very effective method, it is known that CEP in a network of pure states can be improved by certain quantum ‘pre-processing’ strategies, and therefore CEP is not optimal [12, 13, 14, 15, 16, 17, 18, 19]. In the following we show that this is also the case in mixed-state networks.

A. Swapping procedure

To start with we generalize the swapping arrangement shown in Fig. 2 previously studied for pure states [12, 13]. In this arrangement we have two 2-qubit states that both have a qubit in a common node. If the two states are pure states $|\alpha\rangle$ and $|\beta\rangle$, with $\alpha \geq 1/2$ and $\beta \geq 1/2$, we can obtain a singlet by swapping and then converting the resulting pure state into a singlet with a total probability of $2 \min((1-\alpha), (1-\beta))$ which turns out to be the optimal probability. Particularly CEP, which consists here of the Procrustean method followed by entanglement swapping, always has a smaller SCP of $4(1-\alpha)(1-\beta)$. Note that the optimal probability is equal to that of converting the least entangled of the two bonds into a singlet using the procrustean scheme [12].

To generalize this to mixed states we must consider double-edged bonds, each consisting of two PMSs, as illustrated in Fig. 11 since otherwise singlet generation would not be possible. Introducing more than one edge between the nodes allows us to concentrate the entanglement at different stages which gives rise to three different possibilities:

I CEP - As previously described, the bonds are converted to singlets and then swapping is performed over the resulting states.

II Direct swapping - This applies entanglement swapping twice and then the resulting states are converted into a singlet.

III Hybrid swapping - Here we distill a state of higher entanglement in each bond (but not necessarily a singlet) leading, if successful, to a single (partially) entangled pure state in each bond. This is followed by entanglement swapping and the Procrustean scheme to create a singlet.

Each of these possibilities uses the swapping operation at different stages as illustrated in Fig. 11. The exact implementations for the procedures depend on the types of states used. We will first apply each of them on a network of pure states and compare the SCPs. We then generalize to PMSs and show that direct and hybrid swapping can outperform CEP.

1. Pure states

If we start with bonds made of pure states $|\alpha\rangle$ and $|\beta\rangle$ we must have a way to convert each bond into a singlet in order to apply CEP(I). The method and highest possible probability to accomplish this are given by Majorization [20] with a probability $p = \min(1, 2(1-\alpha\beta))$ [13, 16].
CEP applies this operation on each bond and if both bonds are converted into singlets swapping can be performed and the operation is a success. Therefore CEP succeeds with a probability \( \min(1, 2(1 - \alpha \beta))^2 \).

Our second method, direct swapping(II), is simply the application of the procedure for bonds containing one edge twice. If either generates a singlet the procedure succeeds. This gives a SCP of \( 1 - (1 - 2(1 - \alpha))(1 - 2(1 - \beta)) \). There are adjustments we could make, for example use the results of Majorization to convert both of the states into a singlet with the highest possible probability, however all of these have a smaller SCP than CEP for a range of parameters.

Finally, the hybrid swapping(III) method concentrates each bond to one pure state, \( \max(1/2, \alpha \beta) \), with certainty. This concentration procedure is also found using results from Majorization theory \[30\]. Afterwards there is one pure state in each bond, as discussed previously, and we can then perform the strategy with optimal success probability \( \min(1, 2(1 - \alpha \beta)) \), i.e. swapping over the pure states followed by the Procrustean method. We can actually consider the setup as a bipartite system between \( A \) and \( BC \). The Majorization results then give the best possible probability of generating a maximally entangled 2-qubit state between these systems as \( \min(1, 2(1 - \alpha \beta)) \) which means that it must be the highest possible probability for any method to succeed.

Figure 12 shows the probabilities in all three cases and we can see that CEP is outperformed for a vast range of parameters by both other strategies. In hybrid swapping (III), we have used multi-edged bonds to create pure states with the highest probability before applying entanglement swapping. We will refer to all strategies that have this property as ‘hybrid’. This probability is unity for initial pure states but for mixed states the initial conversion of bonds to pure states is probabilistic, so when the conversion fails the bond is destroyed.

2. Purifiable Mixed States

We will now investigate if similar improvements can be obtained with PMSs, i.e. if the bonds between the nodes are composed of \( \rho(\alpha, \lambda) \) and \( \rho(\beta, \nu) \). Again we will see that hybrid swapping provides the highest SCP.

I CEP

The classical percolation scheme involves performing a PCM described in Sec. II twice and each succeeds with a probability given by Eq. (5) which simplifies to

\[
p_{\text{conv}} = 2\lambda \nu \min(\alpha(1 - \beta), \beta(1 - \alpha)).
\]

To perform a swapping operation yielding a singlet, between nodes \( A \) and \( C \) we must succeed for both bonds which gives the total chance of success

\[
p_{\text{CEP}} = (2\lambda \nu \min(\alpha(1 - \beta), \beta(1 - \alpha)))^2,
\]

by simply squaring Eq. (26). In this case the swapping operation is the final step of the protocol.

II Direct swapping

In our 2-edged setup we perform the swapping operation introduced in Sec. II twice and there are two choices to do this if the states are not identical. Either we perform the swapping over the identical states \( \rho(\alpha, \lambda) \otimes \rho(\alpha, \lambda) \) or we perform the operation on the states \( \rho(\alpha, \lambda) \otimes \rho(\beta, \nu) \). When we swap over identical states we obtain the state

\[
\rho \left( \frac{\alpha^2}{1 - 2\alpha + 2\alpha^2}, \lambda^2 (1 - 2\alpha + 2\alpha^2) \right),
\]

together with a further state where \( \beta \) is replacing \( \alpha \) and \( \nu \) is replacing \( \lambda \). Note that Eq. (27) is obtained by setting \( \gamma = \delta = 0 \) and \( \alpha = \beta, \lambda = \nu \) in Eq. (11). This pair of states can then be transformed into a singlet with a probability

\[
p_{\text{II}} = 2\lambda^2 \nu^2 \min(\alpha^2(1 - \beta)^2, \beta^2(1 - \alpha)^2)
\]

which is calculated using Eq. (25). In the case where we swap over non-identical states \( \rho(\alpha, \lambda) \otimes \rho(\beta, \nu) \) we obtain two states of the form (11) with \( \gamma = \delta = 0 \). These can be converted into a singlet with probability

\[
p_{\text{II}} = 2\lambda^2 \nu^2 \alpha \beta (1 - \alpha)(1 - \beta).
\]

This is always larger than \( p_{\text{II}} \) and thus swapping with non-identical states should be preferred.

III Hybrid swapping

The hybrid method requires a concentration procedure to be performed (yielding a single pure state...
FIG. 13: Success probability to generate a singlet between the end nodes of the swapping setup shown in Fig. 11 for the classical scheme (solid line), direct swapping (dotted line), and the hybrid scheme (dashed line). Each bond initially contains the states $\rho(\alpha, \lambda)$ and $\rho(1/2, \lambda)$. We have indicated the percolation threshold of a face-centered cubic network.

In each bond (which is given here by PCM). However, in contrast to the pure state case discussed above, if $\alpha = \beta$ we obtain singlets (in which case the method is identical to CEP) and, generally, the operation succeeds with a finite probability given by Eq. (4). For non-identical PMSs PCM yields two non-maximally entangled pure states which are then used for entanglement swapping followed by the Procrustean method. The probability of succeeding in converting both of the bonds to pure states is

$$p_c^2 = \lambda^2 \nu^2 (\alpha(1 - \beta) + \beta(1 - \alpha))^2.$$  (30)

These pure states have largest Schmidt coefficient

$$\hat{\alpha} = \frac{\max(\alpha(1 - \beta), \beta(1 - \alpha))}{(\alpha(1 - \beta) + \beta(1 - \alpha))}. $$  (31)

So, by using the SCP in single edged swapping with pure states we find that we can convert this pair of states into a singlet between the end nodes with probability

$$2(1 - \hat{\alpha}) = 2\min(\alpha(1 - \beta), \beta(1 - \alpha)) (\alpha(1 - \beta) + \beta(1 - \alpha)).$$  (32)

Hence, the overall probability of succeeding with this scheme is

$$p_h = 2\lambda^2 \nu^2 [\alpha(1 - \beta) + \beta(1 - \alpha)] \times \min[\alpha(1 - \beta), \beta(1 - \alpha)].$$  (33)

If we compare the success probability of direct swapping, $p_d$, to the probability of success in the classical percolation scheme, $p_{CEP}$, it can be seen that classical percolation is more likely to succeed in producing a singlet if

$$2\min(\alpha(1 - \beta), \beta(1 - \alpha)) > \max(\alpha(1 - \beta), \beta(1 - \alpha)).$$  (34)

But the ratio of the success probability for the classical scheme against the hybrid protocol, $p_h$, is

$$\frac{p_{CEP}}{p_h} = \frac{2\min(\alpha(1 - \beta), \beta(1 - \alpha))}{(\alpha(1 - \beta) + \beta(1 - \alpha))}. $$  (35)

Whenever $\alpha \neq \beta$ this is less than one and there is an improvement over the classical percolation scheme. Furthermore, the hybrid scheme is more likely to succeed than direct swapping. In Fig. 13 we compare the probabilities of success for all schemes. As can be seen, hybrid swapping leads to the highest success probability.

Hybrid swapping can be used in sections of larger networks to allow percolation to take place. A simple example is a face-centered cubic (FCC) network, where every bond is split into two 2-edged bonds (see Fig. 13). When the above schemes are applied at the nodes linking two 2-edged bonds the FCC network is recovered. Percolation is possible in these 3D networks with a threshold of approximately $\approx 0.12$. Since the classical scheme always gives a smaller success probability than the hybrid scheme there are cases where the hybrid scheme allows

FIG. 14: Illustration of entanglement percolation in a 3D network. The circles represent nodes containing qubits and the lines represent bonds containing pairs of two-qubit entangled states (the edges are not shown). The 3D network can be transformed into a face-centered cubic network by performing the swapping operations (see Fig. 9) over the smaller nodes. For some bond parameters the hybrid scheme allows percolation to occur where classical percolation fails.
FIG. 15: Application of the hybrid scheme in a square network. This involves transforming the PMSs into pure states probabilistically and then applying a suitable pure state procedure (see text). In the case shown all of the conversions are successful. When this happens a swapping operation can be performed and the resulting states distilled into a singlet.

the percolation threshold to be exceeded but the classical scheme does not (see Fig. 13).

B. Square Protocol

CEP can also be improved on by using the hybrid strategy in a 2D square network, as shown in Fig. 15. Each bond is converted into a pure state, $|\hat{\alpha}\rangle = \sqrt{\alpha}|00\rangle + \sqrt{1-\alpha}|11\rangle$, by using PCM which is successful with a probability $p_c$ on each bond. If this yields only two states $|\hat{\alpha}\rangle$ having a common node ($B$ or $C$), entanglement swapping can be performed followed by the procrustean scheme. If all four PCMs succeed the resulting states can be connected (e.g. at nodes $B$ and $C$) via a slightly modified version of entanglement swapping, the so-called XZ-swapping [15]. For this swapping operation the Bell measurement that usually has both qubits measured in the $Z$ basis now measures one in the $X$ basis. After this measurement unitaries are again applied to return the state into Schmidt form. The results of the Bell measurement have an equal probability, $p_m = 1/4$, for all outcomes $m$. Performing this operation twice on the square leads to two pure states (between $A$ and $D$) of the form $|\tilde{\alpha}\rangle$, with $\tilde{\alpha} = (1 + \sqrt{1 - 16\alpha^2(1-\alpha^2)})/2$. These can be distilled into a singlet with probability $\min[1, 2(1 - \tilde{\alpha}^2)]$ by using the protocol based on Majorization [30]. The overall chance of succeeding in generating a singlet is then given by

$$p_{sq} = 4p_c^2(1-p_c^2)(1 - \tilde{\alpha}) + p_c^4\min(1, 2(1 - \tilde{\alpha}^2)).$$  \hspace{1cm} (36)

When attempting to accomplish the same scheme using CEP we succeed with a probability of $\tilde{p}_{CEP} = 1 - (1 - p_{CEP})^2$ which can be significantly smaller than Eq. (36), as shown in Fig. 16.

Again, this improved strategy may enable an infinite cluster to form when applied to larger networks. An example is shown in Fig. 17. Here the square protocol recovers a triangular lattice. If the conversion of the squares into singlets succeeds with a probability exceeding the percolation threshold an infinite cluster forms. In Fig. 16 it can be seen that the hybrid scheme exceeds the threshold for a triangular network in cases where CEP does not.

C. Hierarchical Networks

Small networks like the square configuration discussed above can be extended to larger networks in an iterative fashion. Networks formed in this way from pure states were considered in [15]. Again the probability of successfully creating a singlet was shown to be larger when quantum strategies were used instead of CEP. However, the scheme with the highest probability is still unknown for these ‘hierarchical’ networks. Here we will consider two
different hierarchical networks with two edges per bond. Each of these contains the square network at some iteration level. We determine the SCP when using CEP in both cases which we then compare to the hybrid strategy. As it turns out, the hybrid scheme outperforms CEP.

The first hierarchical network we consider is based on the ‘Diamond’ lattice, which at each stage replaces its bonds by the square network. The geometry for the first three iterations is shown in Fig. 18. The aim is to create a singlet between $A$ and $B$ and if we apply CEP the probability of succeeding at each level is given by the iterative formula

$$p_{i}^{\text{Diamond}} = 1 - (1 - p_{i-1}^2)^2,$$

starting with $p_1^{\text{Diamond}} = p_{\text{conv}}$.

The second hierarchical network we consider is the ‘Tree’ network which is again built on the square configuration. For these networks an iteration is formed by creating two copies of the previous iteration and linking the bottom-left and top-right corner of the square to separate nodes $A$ and $B$ as shown in Fig. 19. Again, we wish to generate a singlet between the opposite corner nodes ($A$ and $B$) and CEP generates a singlet with a probability

$$p_{i}^{\text{Tree}} = 1 - (1 - p_{i-1}^2 p_{\text{conv}})^2,$$

where $p_0 = 1$.

Now we wish to see whether the hybrid scheme gives a larger SCP in these networks. Once again, the hybrid scheme we consider starts by converting all of the bonds into identical non-maximally entangled pure states probabilistically. If the conversion fails on a bond then the bond is destroyed. This results in a network containing random pure state bonds. Each of these bonds contains one edge. Ideally we would then apply a pure state protocol yielding the highest SCP between the intended nodes, however, this protocol is not known in the general case [15]. Instead we apply a procedure which performs $XZ$-swapping in cases when two bonds each have a qubit in the same node (except if these nodes are $A$ or $B$). However, we also distill pure states into states with more entanglement whenever two edges form between two nodes and before performing further swapping. Finally, once one state is obtained between $A$ and $B$, the procrustean procedure is used to create a singlet.

We applied this protocol to the hierarchical diamond and tree networks. For the second and third iterations of the diamond lattice the probabilities of creating a singlet are given in Fig. 20 together with the probabilities using CEP. This comparison was also made for the first, second and third iteration of the tree network and the results are shown in Fig. 21. These examples all illustrate an improvement in the probability of forming a singlet when using the hybrid method rather than classical percolation.

VI. CONCLUSION

We have demonstrated that within lattice networks, where the nodes are connected by multiple bipartite mixed states, percolation strategies can be applied for
is given in Ref. 31 for a bit-flip noise model. Progress in this direction has also been accomplished by generating 3D thermal cluster states using Werner states 32, 33.

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APPENDIX A: PROOF OF SINGLET DISTILLATION REQUIREMENT

In this appendix we give a concise proof for a necessary and sufficient condition to be able to create a singlet out of entangled mixed states using LOCC. We allow the states to be arbitrary bipartite states which are shared between two nodes and all operations are LOCC. A similar proof, but partly restricted to identical states, was given in 21.

Lemma A.1. If a quantum state $\rho_{ab}$ can be distilled to a pure state, $|\Psi\rangle$, then any state with the same range $R(\rho_{ab})$ is also distillable to this state with non-zero probability.

Proof. The general form of the state is

$$\rho_{ab} = \sum_{i=1}^{N} p_i |\psi_i\rangle\langle \psi_i|,$$

with $p_i > 0$, $\sum_{i} p_i = 1$ and $|\psi_i\rangle \in H_A \otimes H_B$. If the state is distillable to a pure state, $|\Psi\rangle$, there exist linear operators $M_A$ and $N_B$, with $M_A M_A^\dagger \leq I$, $N_B N_B^\dagger \leq I$, such that

$$M_A \otimes N_B \rho_{ab} M_A^\dagger \otimes N_B^\dagger = p |\Psi\rangle\langle \Psi|$$

(A2)

or

$$M_A \otimes N_B |\psi_i\rangle\langle \psi_i| M_A^\dagger \otimes N_B^\dagger \propto |\Psi\rangle\langle \Psi|$$

(A3)

This can be summarized as

$$\Rightarrow M_A \otimes N_B |\psi_i\rangle = q_i |\Psi\rangle,$$

(A5)

where at least one $q_i$ is non-zero as otherwise the operator fails to distill $\rho_{ab}$. If this condition is satisfied the operation distills the mixed state into $|\Psi\rangle$. Now given another state $\rho_{ab}^\prime$ with the same range as $\rho_{ab}$. We have that

$$\tilde{\rho}_{ab} = \sum_{i=1}^{M} \tilde{p}_i |\tilde{\psi}_i\rangle\langle \tilde{\psi}_i|$$

(A6)
with $\tilde{\psi}_i = \sum_{j=0}^{N} a_{i,j} |\psi_j\rangle$ and $|\psi_i\rangle = \sum_{j=0}^{M} b_{i,j} \tilde{\psi}_i$. This then gives

$$M_A \otimes N_B |\tilde{\psi}_i\rangle = M_A \otimes N_B \sum_{j=0}^{N} a_{i,j} |\psi_j\rangle$$

(A7)

$$= \sum_{j=0}^{N} a_{i,j} q_j |\Psi\rangle = \tilde{q}_i |\Psi\rangle.$$  \hspace{1cm} (A8)

The value of one $\tilde{q}_i$ must be non-zero as otherwise all $\tilde{q}_i$ are zero and this contradicts the fact that the operator distills $\rho_{ab}$. Hence the protocol also distills $\rho_{ab}$. □

Lemma A.2. For $n$ 2-qubit states to be distillable into a pure singlet at least two 2-qubit states cannot have a range spanned by product states.

Proof. If a 2-qubit state has a range that can be spanned by product states then a separable state with this range exists. If there are $n$ states each with a range spanned by product states the system state would have a range equivalent to a separable state formed by all of these 2-qubit separable states. Since it is impossible to distill a pure entangled state from any separable state it is impossible to distill a pure entangled state from $n$ two qubit states each with a range spanned by product states. Similarly, if one of the 2-qubit states does not have a range spanned by product states, but all of the other states do, the range is equivalent to the range formed from a separable state and one mixed 2-qubit state. This can not be distilled into a pure singlet as it would contradict the result in [22]. Hence at least two states can not have a range spanned by product states to be able to distill a pure entangled state. □

We now need to look at the two qubit states that satisfy this property. The states with rank one are already pure and if they have rank four the range can be spanned by product states. Similarly, if the state has rank three it can also be spanned by product states. This can be seen by considering the subspace orthogonal to a general state $\sqrt{\alpha} |00\rangle + \sqrt{1-\alpha} |11\rangle$. This space is spanned by $\{|01\rangle, |10\rangle, \sqrt{(1-\alpha)(1+\alpha)} (|0\rangle + |1\rangle) / \sqrt{2}\}$ and these are all product states. The last states to consider are those of rank 2, which fall into two categories [34]. The range is either spanned by product states $\{|00\rangle, \sqrt{(1-\alpha)}(|\bar{0}\rangle - \sqrt{1-\alpha}|\bar{1}\rangle)(\sqrt{p(\bar{0})} + \sqrt{1-p(\bar{1})})\}$ or $\{|00\rangle, \sqrt{\alpha}(|\bar{0}\rangle + \sqrt{\alpha}|\bar{1}\rangle)\}$, where only states that have a range containing one product state are the mixed states satisfying the condition. All mixed rank two states of two qubits can be considered to be the mixed state formed by tracing out a third qubit from a pure three qubit system. The classifications of these 3 qubit systems is given in [35, 36, 37] and for the range of the mixed system to contain one product state the three qubit state belongs to the W class. This class can always be written as $\sqrt{\lambda} |\Phi\rangle |1\rangle + \sqrt{1-\lambda} |00\rangle |0\rangle$ with

$$|\Phi\rangle = \sqrt{\alpha} |01\rangle + \sqrt{\beta} |10\rangle + \sqrt{\gamma} |00\rangle, \alpha + \beta + \gamma = 1.$$  \hspace{1cm} (A9)

By tracing out one qubit and using local operations the 2-qubit state that can not be spanned by product states has the form

$$\rho = \lambda |\psi\rangle \langle \psi| + (1-\lambda) |01\rangle \langle 01|,$$  \hspace{1cm} (A10)

where

$$|\psi\rangle = \sqrt{\alpha} |00\rangle + \sqrt{\beta} |11\rangle + \sqrt{\gamma} |01\rangle, \alpha + \beta + \gamma = 1.$$  \hspace{1cm} (A11)

So the only states that can be purified into a perfect singlet, given finite copies, are of this form.

If there are two states of this form we know that the system is distillable since the procedure given in Sec. [11] succeeds in the distillation.

APPENDIX B: THE DISTILLABLE SUBSPACE SCHEME

To extend the DSS scheme to $n$ PMSs, $\rho(\alpha, \lambda)$, we first need to describe the $2^n$ non-zero eigenvalues and their eigenvectors. These correspond to different combinations of $n-1$ $|\alpha\rangle$ terms and $l$ $|01\rangle$ terms. Then taking the decimal representation of the local states we can label each of these eigenvectors by the decimal difference between the values at each location. This difference $y$ in binary gives the location of the $|01\rangle$ terms. For example, in the case of two identical PMSs these are

$$y = 0 = 00: \{|\alpha\rangle\},$$

$$y = 2 = 10: \{|01\rangle\},$$

$$y = 1 = 01: \{|\alpha\rangle|01\rangle\},$$

$$y = 3 = 11: \{|01\rangle|01\rangle\}.$$  \hspace{1cm} (B1)

and $y$ takes all of the values from 0 to $2^n - 1$. Now we define $m(x)$ to be the number of 1s in the binary representation of $x$ and $T_y = \{x : x \wedge y = 0, 0 \leq x < 2^n, x \in \mathbb{N}\}$ ($\wedge$ is the bitwise AND operation).

Then $m = m(y)$ and all of the terms in a non-zero eigenvector are of the form

$$\sum_{x \in T_y} \sqrt{\lambda^{n-m(x)}(1-\lambda)^{m(x)}} |x\rangle_A^d |x+y\rangle_B^d,$$  \hspace{1cm} (B2)

with eigenvalue $\lambda^{n-1}(1-\lambda)^l$.

From this structure we can project out an entangled state if we measure the operator $|c\rangle_A^d |d\rangle_A^d + |d\rangle_A^d |c\rangle_A^d$ at A and then $|c+y\rangle_B^d |d+y\rangle_B^d + |d+y\rangle_B^d |c+y\rangle_B^d$ at B, when $c \in T_y$, $d \in T_y$, $d > c$ and as long as there are no other terms of the form $|c\rangle_A^d |d+y\rangle_B^d + |d\rangle_A^d |c+y\rangle_B^d$, $|c\rangle_A^d |d+y\rangle_B^d$ or $|d\rangle_A^d |c+y\rangle_B^d$ in any non-zero eigenstate.

The term $|c\rangle_A^d |d+y\rangle_B^d + |d\rangle_A^d |c+y\rangle_B^d$ can not appear in one eigenstate since all of the terms must have the same $y$ value and this would require $c$ to be equal to $d$. The state $|c\rangle_A^d |d+y\rangle_B^d$ lies in one if and only if if $\exists w \in \mathbb{N}$, $0 \leq w \leq 2^n - 1$ such that $c + w = d + y$ and $c \in T_w$. Similarly for $|d\rangle_A^d |c+y\rangle_B^d$ but this case can not occur
since $c \in T_y$ and $d \in T_y$ means that $w \geq y$ and $d + w > c + y$. If we assume that $\exists w \in \mathbb{N}$, $0 \leq w \leq 2^n - 1$ such that $c + w = d + y$ and $c \in T_y$, this would mean that $d = c + k$ and that $c^w k = 0$ for some $k > 0$. Both of these results then give that $k^w y = 0$ and $w = k + y$. So, if $w = y + k = d + y - c$ such that $c^w k = 0$ cannot be satisfied we create a maximally entangled state.

Now we have a choice of ways of creating these measurements. One particular way involves the definition of sets $\mathcal{S}_k = \{x : m(x) = k, 0 \leq x < 2^n, x \in \mathbb{N}\}$ and $\mathcal{J}_{a,b} = \{x : x^a (a \text{ OR } b) = 0, 0 \leq x < 2^n, x \in \mathbb{N}\}$. Then the protocol consists of performing a POVM $P_{k,a,b} = C_k(\langle a\rangle^A_A + \langle b\rangle^B_B)$ at location $A$ with $a, b \in \mathcal{S}_k$, $a \neq b$ and $0 < k < n$. For $k = 0$ and $n$ we define $P_{k,a,b} = |2^k - 1\rangle\langle 2^k - 1|$ and when these outcomes occur the procedure has failed. Here $C_k$ is a factor to ensure that

$$\sum_{k=0}^{n} P_{k,a,b} = I.$$  

(B3)

With this outcome at location $A$ another POVM is done at location $B$ given the operators $Q_d = C_d(\langle a + d\rangle^A_A + \langle b + d\rangle^B_B)$ ($d \in \mathcal{J}_{a,b}$) and $F = I - \sum_{d \in \mathcal{J}_{a,b}} Q_d$. If the outcome here is $F$ the protocol has failed, otherwise we have obtained a maximally entangled state. This protocol works since $a, b \in T_y$ for all $y \in \mathcal{J}_{a,b}$ but there is no $w = k + y$ such that $k^w a = 0$ and $b + y = a + y + k$, since if there we would have $b = a + k$ but $m(a + k) \neq m(b)$.

The probability of succeeding is given by Eq. (16) which comes from considering a particular eigenstate with parameter $y$. In this eigenstate there are different pairing terms, $|a\rangle^A_A|a + y\rangle^B_B + |b\rangle^A_A|b + y\rangle^B_B$, with $a, b \in \mathcal{S}_k$. The number of possible measured operators from this eigenstate is given by

$$N_2 = \binom{n - m(y)}{k} \binom{n - m(y)}{k} - 1. \quad \text{(B5)}$$

Note that the pairings in the eigenstate are twice as likely to occur than the ones with just an overlap and these have been counted twice in this sum. The probability that starting with an eigenstate (parameter $y$) we succeed is then

$$\frac{2N_1}{N_2} = \frac{(n - m(y))}{(n - m(y)) - 1}, \quad \text{(B6)}$$

given that we have measured the operator $S_k$ and the probability of this was

$$P_k = \binom{n - m(y)}{k} \alpha^{n - m(y) - k} (1 - \alpha)^k. \quad \text{(B7)}$$

By summing over these we have, given we start with an eigenstate with $m(y) = m$, the probability of succeeding to be

$$\sum_{k=1}^{n-m-1} \alpha^{n-m-k} (1 - \alpha)^{k} \binom{n-m}{k} \binom{n-m}{k} - 1 \quad \text{(B8)}$$

and these eigenstates occur with probability

$$\lambda^{n-m}(1 - \lambda)^m \binom{n}{m}. \quad \text{(B9)}$$

We have not counted $k = 0, n - m$ since they never contribute to the success probability. Then by summing over all of these we get the result in Eq. (16).

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[38] Note that within the notation of this paper a ‘bond’ corresponds to a set of states connecting two nodes and not to a perfect singlet.