Minimum Uncertainty and Entanglement

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We address the question, does a system A being entangled with another system B, put any constraints on the Heisenberg uncertainty relation (or the Schrödinger-Robertson inequality)? We find that for position and momentum, the equality of the uncertainty relation cannot be reached for entangled states. It cannot be reached for any two noncommuting observables, for finite dimensional Hilbert spaces if the Schmidt rank of the entangled state is maximal, barring the trivial cases of both the observables being zero. As a consequence, the lower bound of the uncertainty relation can never be attained for any two observables for qubits, if the state is entangled.

We start with the simplest bipartite entangled state we can think of, consisting of only two parts

$$|\Psi\rangle = c_1|\psi_1\rangle_A|\alpha_1\rangle_B + c_2|\psi_2\rangle_A|\alpha_2\rangle_B$$

(3)

where $|\psi_i\rangle_A$ are two states of system A, and $|\alpha_j\rangle_B$ are two orthonormal states of system B. The constants $c_1, c_2$ satisfy $|c_1|^2 + |c_2|^2 = 1$. The uncertainties in two observables $X_A \otimes 1_B$ and $Y_A \otimes 1_B$, in the entangled state $|\Psi\rangle$, are defined as

$$\langle \Delta X \rangle^2_{\Psi} = \langle \Psi | X^2 | \Psi \rangle \quad \langle \Delta Y \rangle^2_{\Psi} = \langle \Psi | Y^2 | \Psi \rangle$$

(4)

In what follows, we suppress the direct products explicitly as all our observables will be operating on system A. We also use the shorthand $\langle O \rangle_{\Psi}$ for $\langle \Psi | O | \Psi \rangle$, and $\langle O_A \rangle_i$ for $\langle \psi_i | O | \psi_i \rangle_A$.

One can relate the uncertainties in $|\Psi\rangle$ to the uncertainties of the observables in the states $|\psi_i\rangle$. For a generic observable $O$

$$\langle \Delta O \rangle^2_{\Psi} = \sum_i |c_i|^2 \langle \Delta O_i \rangle_A^2 + |c_1|^2|c_2|^2 \langle O_1 - \langle O \rangle_2 \rangle^2$$

(5)

where $\langle \Delta O_i \rangle_A^2 = \langle \psi_i | \tilde{O}^2 | \psi_i \rangle$. The product of uncertainties can be worked out to be

$$\langle \Delta X \rangle^2_{\Psi} \langle \Delta Y \rangle^2_{\Psi} = |c_1|^4 \langle \Delta X_1 \rangle_A^2 \langle \Delta Y_1 \rangle_A^2 + |c_2|^4 \langle \Delta X_2 \rangle_A^2 \langle \Delta Y_2 \rangle_A^2 + 2 |c_1|^2 |c_2|^2 \langle \Delta X_1 \rangle_A \langle \Delta Y_1 \rangle_A \langle \Delta X_2 \rangle_A \langle \Delta Y_2 \rangle_A + |c_1|^4 |c_2|^4 \langle (Y_1) - \langle Y \rangle \rangle^2 \langle (X_1) - \langle X \rangle \rangle^2 + |c_1|^2 |c_2|^2 \langle \Delta X_1 \rangle_A \langle \Delta Y_2 \rangle_A - \langle \Delta X_2 \rangle_A \langle \Delta Y_1 \rangle_A \rangle^2 + |c_1|^2 |c_2|^2 \sum_i \langle (X_1) - \langle X \rangle \rangle^2 \cdot \langle (Y_1) - \langle Y \rangle \rangle^2 + \sum_i |c_i|^2 \langle \Delta Y \rangle_i^2 \cdot \langle (X_1) - \langle X \rangle \rangle^2$$

(6)
It follows from (4) that the necessary conditions for the l.h.s. to reach its minimum value are (i) \( \langle X \rangle_1 = \langle X \rangle_2 \), (ii) \( \langle Y \rangle_1 = \langle Y \rangle_2 \), (iii) the states \( |\psi_i\rangle \) be minimum uncertainty states themselves, and (iv) \( \langle \Delta X \rangle_1 \langle \Delta Y \rangle_2 = \langle \Delta Y \rangle_1 \langle \Delta X \rangle_2 \). In the examples that follow we show that entangled states do not saturate the equality in HUR, for a wide range of familiar systems.

(a) Angular Momentum Operators: The HUR between, say, \( J_x \) and \( J_y \) reads

\[
(\Delta J_x)^2(\Delta J_y)^2 \geq \frac{1}{4} |\langle i | \hbar J_z | i \rangle|^2
\]

Consider the state

\[
|\Psi\rangle = c_1|m_1\rangle|\alpha_1\rangle + c_2|m_2\rangle|\alpha_2\rangle,
\]

where \( |m_1\rangle, |m_2\rangle \) are two of the eigenstates of \( J_z \). The uncertainties for \( |m\rangle \) are given by

\[
(\Delta J_x)^2 = (\Delta J_y)^2 = \frac{\hbar^2}{2}(j + 1 - m^2),
\]

which will be minimum for \( m = \pm j \).

Let \( |\psi\rangle \) can be entangled only if \( m_1 = +j \) and \( m_2 = -j \), or vice-versa. As the expectation values of both \( J_x, J_y \) in eigenstates of \( J_z \) are zero, this example satisfies all the conditions i-iv. The necessary conditions do not further restrict \( |\psi\rangle \).

But, \( \langle \Delta J_x \rangle^2 \cdot \langle \Delta J_y \rangle^2 = \frac{\hbar^2}{4} \) whereas \( |\langle J_z \rangle_\psi|^2 = j^2\hbar^2(\langle c_1 \rangle^2 - \langle c_2 \rangle^2)^2 \). Therefore, there can be equality in HUR only when one of the \( c_i \) vanishes, but then \( |\Psi\rangle \) is not entangled. Thus we conclude that

\[
(\Delta J_x)^2(\Delta J_y)^2 > \frac{1}{4} |\langle i | \hbar J_z | i \rangle|^2
\]

for entangled states of a system with fixed angular momentum.

(b) Heisenberg Algebra: Next we look at the position and momentum operators, \( X \) and \( P \) in one dimension. The HUR has the form

\[
(\Delta X)^2(\Delta P)^2 \geq \frac{\hbar^2}{4}.
\]

We consider an entangled state made up of two Gaussian states entangled with two orthogonal states of another system. The Gaussian states are described by

\[
\langle x | \psi_i \rangle = \frac{1}{(2\pi\sigma_i^2)^{1/4}} e^{i\pi x_i/\hbar} \exp \left(-\frac{(x - x_i)^2}{4\sigma_i^2}\right)
\]

Hence in this case \( \langle X \rangle_i = x_i, \langle P \rangle_i = p_i, \langle \Delta X \rangle_i = \sigma_i, \langle \Delta P \rangle_i = \frac{\sigma_i}{\hbar} \).

The conditions i-iv yield: \( x_1 = x_2, p_1 = p_2 \) and \( \sigma_1 = \sigma_2 \). But in that situation, both the Gaussians are identical, and hence the state saturating the equality in HUR is disentangled.

As yet another example consider an entangled state built out of energy-eigenstates of Harmonic oscillator,

\[
|\Psi\rangle = c_1|n_1\rangle|\alpha_1\rangle + c_2|n_2\rangle|\alpha_2\rangle,
\]

where the states \( |n_i\rangle \) satisfy \( H|n_i\rangle = (n_i + \frac{1}{2})\hbar \omega |n_i\rangle \), where \( H \) is the Hamiltonian of a Harmonic oscillator with frequency \( \omega \). Now, \( \langle n | X | n \rangle = \langle n | P | n \rangle = 0 \) for any \( |n\rangle \). Also, the uncertainties can be easily calculated to yield,

\[
(\Delta X)^2 = (2n + 1)\frac{\hbar}{2m\omega} \quad \text{and} \quad (\Delta P)^2 = (2n + 1)\frac{\hbar m\omega}{2}. \]

In this case too, all the conditions i-iv are satisfied.

Nevertheless, equation (9) then assumes the form

\[
(\Delta X)^2(\Delta P)^2 = \frac{\hbar^2}{4} \sum_i |c_i|^2(2n_i + 1)
\]

where the states \( |n_i\rangle \) satisfy \( H|n_i\rangle = (n_i + \frac{1}{2})\hbar \omega |n_i\rangle \), where \( H \) is the Hamiltonian of a Harmonic oscillator with frequency \( \omega \). Now, \( \langle n | X | n \rangle = \langle n | P | n \rangle = 0 \) for any \( |n\rangle \). Also, the uncertainties can be easily calculated to yield,

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(\Delta X)^2 = (2n + 1)\frac{\hbar}{2m\omega} \quad \text{and} \quad (\Delta P)^2 = (2n + 1)\frac{\hbar m\omega}{2}. \]

In this case too, all the conditions i-iv are satisfied.

Finally, we consider the example of a continuous variable entangled state which is a superposition of an infinite number of parts.

\[
\Psi(x_A, x_B) = \frac{1}{\sqrt{\pi\Omega/\sigma}} e^{-(x_A - x_B)^2/\sigma^2} e^{-(x_A + x_B)^2/16\Omega^2}
\]

In the limit \( \sigma \to \infty \) and \( \Omega \to \infty \), the state reduces to the so-called EPR state considered by Einstein, Podolsky and Rosen [14]. The uncertainties in position and momentum of particle A (say) is given by

\[
\Delta X_A = \sqrt{\Omega^2 + 1/16\sigma^2}, \quad \Delta P_A = \hbar \sqrt{\sigma^2 + 1/16\Omega^2}.
\]

The minimum uncertainty equality is obtained only if \( \sigma = \frac{1}{4\Omega} \). But for these values, the state becomes disentangled, as one can see from (13).

Based on these examples, we had initially conjectured that the lower bound of the HUR and the SR cannot be obtained for any two observables if the state is entangled. However, Englert provides counter-examples to show situations where lower bound is obtained for entangled states [15].

In the following we will carry out a general analysis, and prove that there is wide class of scenarios in which this lower bound cannot be achieved by entangled states.

**GENERAL ANALYSIS**

**Finite Dimensional Hilbert Spaces**

Let \( |\psi\rangle \) be a pure entangled state of two quantum systems belonging to Hilbert spaces \( (H_A, H_B) \), with respective dimensionalities \( d_A, d_B \) and let \( d_A \leq d_B \). The entangled state \( |\psi\rangle \) admits a Schmidt decomposition

\[
|\psi\rangle = \sum_i c_i|a_i\rangle_A |b_i\rangle_B
\]

where \( |a_i\rangle_A, |b_i\rangle_B \) are orthonormal basis vectors in \( H_A, H_B \) respectively. The number \( s \leq d_A \) is called the
Now let us consider single-system observables acting on \( \mathcal{H}_A \). Treatment of when the observables act on \( \mathcal{H}_B \) is completely parallel. We are considering operators of the type \( O_A \otimes 1_B \). Consider a pair of such Hermitian operators \( X_A, Y_A \) that do not commute with each other, i.e \([X_A, Y_A] = C_A \neq 0\).

Schwarz inequality for the states \( X_A|\Psi\rangle, Y_A|\Psi\rangle \) gives
\[
\langle X_A^2 \rangle \psi Y_A^2 \psi \geq |\langle (X_A Y_A) \psi|^2 |
\]
(17)
The inner product occurring on the r.h.s. of (17) can be written as
\[
\langle X_A Y_A \rangle \psi = \frac{1}{2} \langle \{X_A, Y_A\} \rangle \psi + \frac{1}{2} \langle [X_A, Y_A] \rangle \psi
\]
(18)
The first term is purely real while the second term is purely imaginary. Hence (17) can be rewritten as
\[
\langle X_A^2 \rangle \psi Y_A^2 \psi \geq \frac{1}{4} |\langle C_A \rangle |^2 + \frac{1}{4} |\langle X_A, Y_A \rangle |^2
\]
(19)
We now consider the operators \( \tilde{X}_A = X_A - \langle X_A \rangle \psi, \tilde{Y}_A = Y_A - \langle Y_A \rangle \psi \) instead of the operators \( X_A, Y_A \) respectively. Then we can put together everything and write
\[
\langle \tilde{X}_A^2 \rangle \psi \cdot \langle \tilde{Y}_A^2 \rangle \psi \geq \frac{1}{4} |\langle \{X_A, Y_A\} \rangle |^2 + \frac{1}{4} |\langle [X_A, Y_A] \rangle |^2
\]
(20)
The l.h.s. of (20) is the same as \( (\Delta X_A)^2 (\Delta Y_A)^2 \). Thus, (20) is nothing but the SR inequality. The equality in (17) holds if, and only if, the vectors \( X_A|\Psi\rangle, Y_A|\Psi\rangle \) are parallel. That is, if there exists a complex number \( \gamma \) such that
\[
X_A|\Psi\rangle + \gamma Y_A|\Psi\rangle = 0
\]
(21)
This, in addition to leading to the equality in eqn. (17), further implies that
\[
\gamma_R \langle \{X_A, Y_A\} \rangle \psi + i \gamma_i \langle [X_A, Y_A] \rangle \psi = 0
\]
\[
(\Delta X_A)^2 = |\gamma|^2 (\Delta Y_A)^2
\]
(22)
Therefore, for the equality in HUR to be realised, the last term in (20), which is real, must also vanish in addition eqn. (21), but now for the new set of operators \( \tilde{X}_A, \tilde{Y}_A \): 
\[
\langle X_A - \langle X_A \rangle \psi \rangle |\Psi\rangle + \Gamma (Y_A - \langle Y_A \rangle \psi) |\Psi\rangle = 0
\]
(23)
This is possible only if \( \Gamma \) appearing in (23) is purely imaginary. For the SR case, however, \( \Gamma \) can be any complex number. Substituting (16) in (23):
\[
\sum_{i=1}^{s} c_i (\tilde{X}_A + \Gamma \tilde{Y}_A) |a_i\rangle |b_i\rangle = 0
\]
(24)
This can only be satisfied if
\[
\langle \{X_A - \langle X_A \rangle \psi \} + \Gamma (Y_A - \langle Y_A \rangle \psi) \rangle |a_i\rangle = 0
\]
(25)
for every \( i \). Taking the inner product of this equation with \( |a_i\rangle \), one gets:
\[
\langle (X_A)_i - \langle X_A \rangle \psi \rangle + \Gamma \langle (Y_A)_i - \langle Y_A \rangle \psi \rangle = 0
\]
(26)
for every \( i \). Here \( \langle O_A \rangle \) is the expectation value of \( O_A \) in \( |a_i\rangle \). But due to the real nature of all the expectation values, this is possible if and only if
\[
\langle X_A \rangle | = \langle X_A \rangle \psi \quad (Y_A)_i = \langle Y_A \rangle \psi
\]
(27)
But eqn. (25) is precisely the requirement that all the \( |a_i\rangle \) are also minimum uncertainty states for \( X_A, Y_A \). In addition, the second of the condition in eqn. (22) must be individually satisfied, which means \( (\Delta X_A)^2 (\Delta Y_A)^2 \) should be the same for all \( i = 1 \ldots s \). These constitute a generalization of conditions (i)-(iv) spelt out earlier.

Therefore, eqn. (25) is the key to whether entangled states can saturate the equality in the uncertainty relations (see also [15], [17]). What this equation means is that in the subspace spanned by \( |a_i\rangle \), the operators \( \tilde{X}_A, \tilde{Y}_A \) are zero. It is instructive to list a few possibilities at this stage:

(a) The operator \( R_A = \tilde{X}_A + \Gamma \tilde{Y}_A \) does not have any degenerate eigenfunctions. In this case entangled states can not saturate the equality;

(b) \( R_A \) has degenerate eigenstates but they also happen to be simultaneous eigenstates of both \( X_A, Y_A \). In this case the equality will be satisfied in a trivial way in the sense that all uncertainties vanish in \( |\Psi\rangle \).

Now, if the bipartite entangled state is such that \( s = d_A \), the subspace in which the operators \( \tilde{X}_A, \tilde{Y}_A \) span the entire Hilbert space \( \mathcal{H}_A \) and this will be a realisation of case (b) above. Now for qubits, the Hilbert space is 2-dimensional which is equal to the minimal Schmidt rank 2, required for a state to be entangled. Thus, our result implies that for qubits, the lower bound of HUR or SR cannot be attained, if the state is entangled.

Therefore, for \( s = d_A \), which is the maximum possible value for \( s \), the equality for entangled states can only be realised trivially. On the other hand, if \( s < d_A \), the above argument does not hold, and minimum uncertainty equality can be attained, as exemplified by Englert [15].

**States of fixed angular momentum**

Now we consider the finite dimensional Hilbert space of \( d_A = 2j + 1 \), spanned by angular momentum states with fixed value of \( J^2 = j(j + 1) \). We only consider the case where the operators are linear combinations of \( J_i \). The minimum uncertainty states in this case can be taken, without any loss of generality, to be the eigenstates \( |j, j \rangle, |j, -j \rangle \) of \( J_z \). For both these states, as already noted before, \( \langle J_x \rangle = \langle J_y \rangle = 0 \). Eqn. (25) reads, in this case
\[
\{ J_x + \Gamma J_y \} |j, \pm j \rangle = 0
\]
(28)
Decomposing $J_x + \Gamma J_y$ as

$$J_x + \Gamma J_y = \frac{1-i\Gamma}{2} J_+ + \frac{1+i\Gamma}{2} J_-$$  \hspace{1cm} (29)$$

where $J_{\pm}$ are the angular momentum ladder operators, and recalling

$$J_{\pm}|j, m\rangle = \sqrt{(j \pm m)(j \pm m + 1)}|j, m \pm 1\rangle$$  \hspace{1cm} (30)$$

it can easily be seen that both equations of eqn.(28) cannot be simultaneously satisfied. Specifically, $(j, j)$ satisfies it for $\Gamma = i$, and $(j, -j)$ satisfies it with $\Gamma = -i$. This proves that for the system under consideration no entangled state saturates either the HUR or SR equality, for $J_x$, $J_y$. However, one can have other observables for which the lower bound in the uncertainty relation can be achieved [15].

**Infinite Dimensional Hilbert Spaces**

When the Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$ are infinite dimensional, the whole analysis needs to be redone. A general treatment of the infinite dimensional case is beyond the scope of this paper. In fact, Englert [15] has given an ingenious counter-example to show that entangled states can indeed nontrivially saturate the equality for carefully chosen operators. Though we can show our results for other nontrivial choices of $X_A, Y_A$, here we shall only show what happens when $X, Y$ are the momentum $P$ and position $Q$ operators. There are two possibilities: (i) both $\mathcal{H}_A, \mathcal{H}_B$ are infinite dimensional, or (ii) only $\mathcal{H}_A$ is infinite dimensional. In both cases one gets the analog of (24) where now the index $i$ runs over both continuous and discrete labels, but $|\psi_n\rangle$ need not be mutually orthogonal. Since $[Q, P] = i\hbar$ (25) for HUR leads, for example in the position representation, to

$$-i\hbar \frac{d}{dq} + i\Gamma \Psi - (P)\Psi + i\Gamma (Q)\Psi \psi_n(q) = 0$$  \hspace{1cm} (31)$$

which requires all states $\psi_n(q)$ to be the same minimum uncertainty Gaussian state with position centered around $\langle Q \rangle\Psi$, momentum centered around $\langle P \rangle\Psi$ and with width $\Delta Q = \frac{\hbar}{\Gamma}$. Thus $|\Psi\rangle$ can not be entangled. For the SR case, since $\Gamma$ has both real and imaginary parts, the minimum uncertainty states acquire an additional phase $e^{i\Gamma \eta \hbar^2/2}$, but the corresponding $|\Psi\rangle$ is still disentangled.

**Multipartite Entanglement**

The general analysis for the bipartite case is enough to address the same issue for multipartite case also. The crucial issue is whether eqn.(25) admits degenerate solutions or not. If it does, the answer in both the bipartite and multipartite cases is the same, namely, entangled states can saturate the equality. This is so as one can build entangled states, bipartite or multipartite, with these distinct states. On the other hand, if eqn.(25) has only one solution, neither in the bipartite case nor in the multipartite case can entangled states saturate the equality.

In conclusion, we have shown that entanglement puts a bound on the product of uncertainties of non-commuting observables, for certain class of systems and states. Of particular significance is the result that for position and momentum, minimum uncertainty equality can never be attained, for entangled states.

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