Amplitude Zeroes in Collinear Processes or What Is Left from a Factorizable 2d Model in Higher Dimensions.

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abstract

We show that for collinear processes, i.e. processes where the incoming and outgoing momenta are aligned along the same line, the S-matrix of the tree level 2+1 dimensional Thirring model factorizes: any S - matrix element is a product of $2 \rightarrow 2$ elements. In particular this means nullification of all collinear $2 \rightarrow n$ amplitudes for $n > 2$. 
The behavior of amplitudes for production of a large number of particles has recently been studied within the framework of perturbation theory. The tree and one loop level contribution to the $2 \to n$ amplitudes at threshold (i.e. all produced particles are at rest) has been calculated exactly for $\phi^4$ scalar theory (broken and unbroken phase) using various methods [1, 2]. The phenomenon of nullification was discovered: for all $n \geq n_0$ the amplitude $A(2 \to n)$ vanishes when all $n$ particles are produced at rest (threshold). In the unbroken phase $n_0 = 4$, while in the broken phase $n_0 = 2$. The phenomenon is rather exclusive and occurs only in special models [3]. For example a general scalar model with just two fields doesn’t exhibit nullification.

Argyres, Kleiss and Papadopoulous [2] proposed a general method based on constructing generating function for the amplitude $A_{1 \to n}$. The recursion relations for it reduces to a second order differential equation with a potential which is (in all known cases) one of the solvable reflectionless potentials of quantum mechanics (quantum field theory in $d = 1$). They pointed out that the study of the relation between nullification at kinematical threshold and integrability in $d = 2$ space time dimensions could provide us with a new insight into multiboson production processes.

In this note we consider a generalization of the above results in the following direction. Instead of considering all the particles at threshold i.e. with zero momentum we only restrict them to move along the same line. We consider the simplest 2d - integrable model, the Thirring model in higher dimensions. First we show that the 2d type factorization which immediately implies nullification of all $2 \to n$ collinear amplitudes for $n \geq 2$ is preserved on tree level 3d model for collinear processes despite nonelasticity. The phenomenon seems to be even more exclusive then the nullification. For example, (a nonsolvable in 2d) model like $\phi^4$ does not possesses such a property.\footnote{from the point of view of performing perturbation theory}
The massive Thirring model

\[ \mathcal{L} = \bar{\psi}(i\gamma \partial \psi - m)\psi + \frac{g}{2}(\bar{\psi}\gamma \mu \psi)^2 \]  

belongs to the special class of non-trivial 1+1 dimensional models with \( n \to m \) S-matrix elements factorizing into a product of \( 2 \to 2 \) elements, which in turn are purely elastic (other solvable models are nonlinear \( \sigma \) model, Gross - Neveu, ... \[5\]). The factorizability is a consequence of the infinite number of conservation laws that exist in these models. Together with the general properties of analyticity, crossing and unitarity of the S-matrix, it enables the exact determination of their S-matrix.

In higher dimensions, purely elastic \( 2 \to 2 \) nontrivial S-matrix necessarily contradicts relativistic invariance or basic analyticity properties of the S-matrix \[3\]. Also the factorization like that in 2d is not possible. However like in 2d one can get some hints about the structure of these theories and possible factorization studying the perturbation series. In two dimensions one is able to spot the factorization within perturbation series using so called ”cutting rules” \[7\]. It is interesting that there are cutting rules also in higher dimensions. A cutting rule states that any one loop diagram with \( n \) external legs in space time of dimension \( d \) can be reduced to products of tree level diagrams and a one loop diagram with \( d \) external lines. This feature gives rise to the hope that for some models even in \( d > 2 \) the S matrix can be built up (algebraically) from a finite number of independent functions \[8\]. In particular in three dimensions the only irreducible diagrams are the two (”fish”) and three (”triangle”) legged one loop diagrams along with the tree level diagrams. Thus, it is possible in principle that any S-matrix element of some model might be algebraically built up from \( 2 \to 2 \), \( 2 \to 4 \) and \( 3 \to 3 \) amplitudes.

\[2\]The \( d > 2 \) dimensional Thirring model is perturbatively nonrenormalizable (however it can be shown \[8\] that non-perturbatively in 3d four - Fermi interactions are in fact renormalizable). Note that the nullification occurs in nonrenormalizable models: \( \phi^4 \) is nonrenormalizable for \( d > 4 \). There is no direct relation between the nullification and the ultraviolet properties so one can gain information on the behavior of multiparticle amplitudes by considering the first few orders in perturbation theory even in apparently nonrenormalizable theories.
We start by showing the complete factorizability of the scattering amplitude for collinear processes at tree level. The first step in the induction is an explicit calculation of the $3 \to 3$ amplitude for such processes. The $3 \to 3$ amplitude at tree level is (see fig.1):

$$A_{33} = \frac{3g^2}{4} \sum_{\pi(p_{\pi_1},...,p_{\pi_6})} \bar{u}(p_{\pi_1})\gamma_{\mu}u(p_{\pi_4})\bar{u}(p_{\pi_2})\gamma^{\mu} \frac{i}{y_{\pi_1}y_{\pi_2}y_{\pi_4}m} \gamma_{\nu}u(p_{\pi_3})\bar{u}(p_{\pi_5})\gamma^{\nu}u(p_{\pi_6})$$

(2)

one thus has to show that for all values of collinear external momenta for which the propagator (see fig.1) is off shell the amplitude vanishes. Using the following decomposition of the propagator:

$$i\frac{p}{p - m} = P\frac{i}{p - m} + \delta(p^2 - m^2)$$

(3)

(where $P$ means principle value part) it is possible explicitly see that the principle value part of the amplitude vanishes for arbitrary collinear momenta. Note that this cancellation is non trivial because the restriction to collinear processes is by no means equivalent to the 1+1 dimensional interaction. The third component of the current $J_{\mu} = \bar{\psi}\gamma_{\mu}\psi$ gives a nonvanishing contribution to the amplitude. We have also checked the $3 \to 3$ and $4 \to 4$ amplitudes in the 3+1 dimensional Thirring model. The principle value part of these amplitudes does not vanish even for collinear processes in this case.

The next step is to use induction to prove that all $n \to n$ amplitudes factorize. We thus assume that $P\mathcal{A}_{nn}^{tr} = 0$ for all $n \leq N - 1$ and prove $P\mathcal{A}_{NN} = 0$ . We shall only outline the proof here since essentially it is the same as Berg’s [9] for the 1+1 dimensional model. One fixes $N - 2$ of the momenta and considers the amplitude $A_{NN}$ as a function of the remaining independent variables, say $p_1$. We let $p_1$ take complex values and denote $A_{NN} = w(z)$. If the amplitude is not constant it satisfies an irreducible algebraic equation of the form

$$\sum_{j,k=0}^{r,s} a_{ij}w^jz^k = 0$$

(4)

where $r \geq 1$ and $a_{kk} \neq 0$ for at least one $k$. This equation defines a compact Riemann surface $F$ [[10]]. For each point on this surface one can choose local coordinates such
that $w$ is a single valued function on it. On the whole surface the number of poles and zeroes of $w$ is equal. Using the induction hypothesis it is now possible to show that the maximum degree of divergence of $\mathcal{P}A_{NN}$ is $\sim z^2$ for $z \to \infty$. By Fermi statistics $w$ has at least $N - 1$ zeroes and therefore $\mathcal{P}A_{NN} = 0$ for $N \geq 3$. As a check we also calculated the $4 \to 4$ amplitude for various collinear momenta and verified that the principle part of the amplitude vanishes.

The factorization of the S-matrix for collinear processes automatically forbids particle production. This is because any such processes is related by crossing to an $n \to n$ amplitude where one of the intermediate propagators is off-shell. In particular threshold production i.e. $2 \to n$ where $n > 2$ and the produced particles have spatial momentum zero is a special case of a collinear process. It is clear from the previous discussion that the amplitude for such processes in the Thirring model vanishes. We have also checked what happens when one of the particles momenta is off the line. In this case the principle value of the amplitude deviates from zero. This means that simple factorization to $3 \to 3$ and $2 \to 2$ amplitudes does not occur in this model. A similar calculation for the 3+1 dimensional Thirring model shows no sign of this property (i.e even for collinear processes $\mathcal{P}A_{NN} \neq 0$).

In the two dimensions one can extend this proof to all orders in loop expansion by using a cutting rule due to Källen and Toll [7]. This cutting rule states that any $n$-legged one loop boson diagram (see fig. 2) can be written as a product of tree level graphs where each term in the sum is multiplied by a one loop integral with two external legs and momentum assignment determined by the cut. It is clearly seen then that if tree level factorization has been proven no particle production is possible through any one loop diagram. The Källen-Toll cutting rule has obvious generalizations to higher dimensions.\footnote{since any fermionic one loop diagram can be written as a sum over bosonic diagrams and their derivatives with respect to external momenta this is true also for such diagrams}
For an $m$ legged loop diagram ($m > d$) in $d = 3$ it reads (see fig. 2):

$$
\int \frac{d^3 k}{(2\pi)^3} \prod_{j=1}^{m} \frac{1}{(k - p_j)^2 - m^2 + i\epsilon} = \frac{1}{2} \sum_p \sum_{s=\pm} \left[ T_s^p \int \frac{1}{(2\pi)^3} \prod_{i \in P} \left( \frac{1}{(k - p_i)^2 - m^2 + i\epsilon} \right) \right] \quad (5)
$$

where the sum is over all partitions $P = \{\{\pi_1, \pi_2\}\}$ with $\{\pi_1, \pi_2\} \in \{1, \ldots m\}$ and

$$
T_s^p = \prod_{r \in P} \left( \frac{1}{(k^s_r - p_r)^2 - m^2} \right)
$$

where $k^s_{ij}$ are the two solutions of $(k - p_i)^2 = m^2$ ($i \in P$). Any one loop diagram can thus be written as a sum over tree level diagrams multiplied by a one loop diagram with three external lines with momentum assignment determined by the cuts. We therefore see that since triangle diagram necessarily allows particle production, non particle number preserving processes are inevitable, in agreement with the general theorems [6].

To conclude, we considered generalizations of some integrable models to higher dimensions. It was shown that for the 3d massive Thirring model collinear factorization survives at the tree level. The nullification of the threshold $2 \rightarrow n$ amplitudes is an immediate consequence of this property and follows from the collinear factorizability property. The assumption of simple inelastic factorization as implied by the generalization of the cutting rule (e.g. in $d = 3$ writing $A_{4\rightarrow4}$ as a product of $A_{2\rightarrow2}$ and $A_{3\rightarrow3}$), proved to fail for the Thirring model. However, it would be interesting if such a model could be found.

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Figure 1: The $3 \rightarrow 3$ amplitude. The wavy line represents a fictitious photon.

Figure 2: The Kallen-Toll cutting rule. A reduction of the triangle diagram in 1+1 d. B reduction of the box diagram in 2+1 d.