Network structures sustained by internal links and distributed lifetime of old nodes in stationary state of number of nodes

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Abstract. In network models that take into account growth properties, deletion of old nodes has a serious impact on degree distributions, because old nodes tend to become hub nodes. In this study, we aim to provide a simple explanation for why hubs can exist even in conditions where the number of nodes is stationary due to the deletion of old nodes. We show that an exponential increase in the degree of nodes is a natural consequence of the balance between the deletion and addition of nodes as long as a preferential attachment mechanism holds. As a result, the largest degree is determined by the magnitude relationship between the time scale of the exponential growth of degrees and lifetime of old nodes. The degree distribution exhibits a power-law form \( k^{-\gamma} \) with exponent \( \gamma = 1 \) when the lifetime of nodes is constant. However, various values of \( \gamma \) can be realized by introducing distributed lifetime of nodes.

1. Introduction
The knowledge of network structures provides a very fundamental framework for studying complex systems constructed by many constituents interacting with each other [1]. One of the most interesting properties commonly seen in real networks must be the existence of hub nodes whose degree (number of links incident to the node) is typically a power function of the network size. One simple explanation for the existence of hub nodes is based on the growth property of networks [2]. However, there are some real networks including social networks, in which network growth is attenuated by the continuous deletion of nodes [3]. In these networks, the deletion of old nodes is inevitable for reviewing how their structures developed, for deletion of old nodes has a serious impact on degree distributions of growing networks. Network structures in the stationary state of number of nodes have not been completely clarified even in the explanation for the existence of hub nodes.

In the growth model of networks where a new node attaches to pre-existing nodes with the probability proportional to the node at each time step (BA model [2]), the random deletion of a node at each time step leads to the divergence of the power-law exponent of degree distribution as the growth rate vanishes, and hub nodes disappear in the stationary state of number of nodes [4]. To maintain the power-law behavior in the degree distribution in the stationary state of number of nodes, other mechanism besides BA rule is needed. For example, Miura et.al [5] showed that the coagulation of nodes can maintain the power-law form of degree distribution even in the stationary state of number of nodes.
In the present study, we consider the selective deletion of old nodes (hub nodes) instead of the random deletion of nodes. In this case, it is obvious that the hub nodes disappear immediately by the deletion of nodes if only the BA rule is employed, because old nodes tend to become hub nodes in this rule. Here, we aim to provide a simple mechanism based on internal links [6] by which hubs can exist even in conditions where old nodes are being deleted. First, we will show that an exponential increase in the degree of nodes is a natural consequence of the constancy of the internal factor which describes the contribution of internal links to the rate of increase in degrees. Secondly, we will show that the values of the largest degree and the power-law exponent describing the degree distribution are determined by the magnitude relationship between the time scale of the exponential growth of degrees and lifetime of old nodes.

2. Effect of constant lifetime of nodes

To carry out a concrete examination of the effect of the deletion of old nodes on the degree distribution of networks, let us consider the following network model in which all nodes have the same lifetime.

(i) A complete graph with \( m \) nodes is prepared initially \((t = 1)\).

(ii) At each discrete time step \( t \), a vertex with \( m \) edges newly attaches to the network with a preferential attachment rule (each edge attaches to the pre-existing node with a probability proportional to the degree of the node [2]).

(iii) At each time step \( t \), two vertices are selected with a probability proportional to the product of degrees of the selected vertices [6], and if the selected vertices have not been linked, these two vertices are linked. If the two vertices have already been linked, nothing happens. This process repeats \( rN_t \) times, where \( N_t \) is the number of nodes at time \( t \), and \( 0 \leq r \leq 1 \) is a constant parameter.

(iv) Let \( t_i \) be the time at which node \( i \) attached to the network \((i = 1, 2, \cdots, t + m)\). When \( t - t_i > L_0 \) is satisfied, node \( i \) and the incident links are deleted, where \( L_0 \) is the constant life time of all nodes.

(v) \( t \) is increased by 1, and goes to step (ii).

Owing to step (iii), the network is accelerated in the time interval \( 0 < t \leq L_0 \), that is, the mean vertex degree increases with time [8]. Owing to (iv), however, \( N_t \) and the sum of all vertex degrees \( D_t \) take constant values \( N_{eq} = L_0 + m \approx L_0 \) and \( D_{eq} \) when \( t \gg L_0 \). (Isolated nodes will be counted in \( N_{eq} \) in later sections.) In this model, the increment of degree \( k_i \) of node \( i \) per unit time is evaluated by the following equation in terms of \( D_t \),

\[
\frac{dk_i}{dt} = \begin{cases} \frac{Bk_i}{t} + \frac{mk_i}{D_t}, & (t \leq L_0) \\ \tilde{B}k_i + \frac{mk_i}{D_{eq}} - f_k(t), & (t > L_0) \end{cases}
\]  

(1)

where \( B \) and \( \tilde{B} \) are internal factors corresponding to \( t \leq L_0 \) and \( t > L_0 \), which are defined by

\[
Bt^{-1} = 2rN_t(1 - \sum_{j \in n_i} k_j/D_t)/D_t, \quad \tilde{B} = 2rN_{eq}(1 - \sum_{j \in n_i} k_j/D_{eq})/D_{eq},
\]

(2)

(3)

where \( n_i \) is the set of nodes directly connected to node \( i \) [7]. In contrast to (2), the right hand side of (3) is independent of time, because \( D_{eq} \) is nearly constant. The second term in (1) is the contribution of external links, and \( f_k(t) \) represents the effect of the deletion of nodes on \( dk_i/dt \).

To ensure exact treatment of (1), there is a need to know to which nodes deleted nodes are linked. However, as a first approximation, we can evaluate the behavior of degrees \( k_i \) for
Figure 1. Numerical results for the time-dependence of degrees when \( r = 0.05, m = 2 \) and \( L_0 = 2000 \). Dashed lines indicate exponential increases \( \sim \exp \left\{ \left( \frac{\bar{B} + m/D_{eq}}{\bar{D}_{eq}} \right)(t - t_i) \right\} \) in which the numerical value of (3), \( \left( \frac{\bar{B} + m/D_{eq}}{\bar{D}_{eq}} \right) = 1/327.2 \), is substituted. The horizontal dotted line indicates a numerical value of \( k_{del} \) obtained by (6).

\[ t > t_i > L_0 \] using the assumption that the first term in (1) \( \bar{B} \) is constant and dominant enough to ignore \( f_k(t) \),

\[ \log k_i \simeq \left( \frac{\bar{B} + m/D_{eq}}{\bar{D}_{eq}} \right)(t - t_i). \quad \text{(for } t > t_i > L_0, \ t - t_i < L_0) \quad (4) \]

Specifically speaking, \( \bar{B} \) is a decreasing function with respect to \( k \) with a very small slope. However, approximated behaviors of \( k \) can be described by the value of \( \bar{B} \) at \( k \simeq 0 \) even if considering \( k \)-dependence of \( \bar{B} \) [7]. According to (4), the density of number of nodes with degree \( k \) is proportional to \( d \log k = dk/k \), because \( t_i \) is uniformly distributed in the time axis (see Fig. 1). As the result, the degree distribution is obtained as

\[ P(k)dk \sim \frac{dk}{k}. \quad (5) \]

Fig. 1 presents the numerical result for the time-dependence of degrees when condition \( e^{\bar{B}L_0} \gg 1 \) is satisfied. (Condition \( e^{\bar{B}L_0} \gg 1 \) must be satisfied for degree to survive to a certain size.) The deviation from the ideal behavior (4) in high degrees found in Fig. 1 can be understood by the following equation on the balance between the increase and decrease in the number of edges,

\[ \alpha rN_{eq} = \bar{k}_{del}, \quad (6) \]

where \( \alpha \) represents the probability that two nodes selected in step (iii) have not linked until time \( t \),

\[ \alpha = 1 - \sum_{i} \sum_{j \in n_i} k_i k_j / D_{eq}, \quad (7) \]

and \( \bar{k}_{del} \) is the mean degree of deleted nodes. The value of \( \bar{k}_{del} \) is smaller than \( e^{\bar{B}L_0} \), because \( e^{\bar{B}L_0} \) is an ideal maximum degree which can be realized by ignoring the effect of \( f_k(t) \). Power-law exponent 1 indicated in (5) is not adequate for describing the result in range \( \bar{k}_{del} < k < e^{\bar{B}L_0} \), because most of the nodes with degree \( \bar{k}_{del} < k < e^{\bar{B}L_0} \) are deleted. Fig. 2 shows that (5) describes the numerical result with considerable accuracy in range \( k < \bar{k}_{del} \), although the increase tendency of degrees described by (4) is perturbed by \( f_k(t) \) in Fig. 1. At least, condition \( 1 \ll \bar{k}_{del} \) must be satisfied for the power-law form to appear clearly in the degree distribution.

Figure 2. Numerical result for degree distribution at \( t = 10000 \) using the same parameters with Fig. 1. The bold line indicates a fitting curve with slope \(-1.06\).
3. Effect of distribution of lifetime of nodes
By introducing the distribution of lifetime of nodes, the impact of deletion of nodes on the degree distribution will decrease, because, for the same value of internal factor, the maximum degree can become larger as the lifetime of nodes tends to be longer. To carry out concrete calculation, we employed the following lifetime distribution that expresses that node \( i \) is not deleted until time \( t(t > t_i) \),

\[
P(t - t_i) = \begin{cases} 
1 & \text{(for } t - t_i \leq L_0) \\
\exp\left\{-\frac{(t - t_i - L_0)}{\tau}\right\} & \text{(for } t - t_i > L_0) 
\end{cases}
\]  

(8)

\[
N_{t}, k_{\text{max}}
\]

Figure 3. Numerical results for the time dependences of the number of nodes and the maximum degree when \( r = 0.05, m = 2 \) and \( L_0 = 2000 \). The bold lines and dashed lines indicate cases when \( \tau = 0 \) and when \( \tau = 500 \), respectively.

In Fig. 3, the time dependences of the number of nodes and maximum degrees for case \( \tau = 0 \) are compared with that for case \( \tau = 500 \). As shown in the figure, the distributed lifetime of nodes raises the maximum degree. As a result, the range of \( k \) where the power law is found widens (Fig. 4). For node \( i \) in condition \( t - t_i > L_0 \), the degree distribution is generalized by introducing probability (8) to,

\[
P(k)dk = \frac{P(t - t_i)dk}{k} \\
\sim k^{-1/(\tilde{B} + m/D_{eq})\tau - 1},
\]  

(9)

(10)

where we have used (4), \( t - t_i = \log k_i/(\tilde{B} + m/D_{eq}) \). As the result, we obtain,

\[
\gamma = 1/(\tilde{B} + m/D_{eq})\tau + 1.
\]  

(11)

The relation (11) shows that the value of \( \gamma \) is determined by the balance between the magnitude of the internal factor and range of distribution of lifetime of nodes. The power-law behavior disappears as \( \tau \to 0 \) (in \( k > \tilde{k}_{\text{del}} \)), and \( \gamma \) approaches to 1 as \( \tilde{B} + m/D_{eq} \) becomes large. Fig. 5 and Fig 6 illustrate a typical result for the time-dependence of degrees and the degree distribution when \( \tau \neq 0 \) and \( \tilde{k}_{\text{del}} \) is not that large. The validity of relation (10) is confirmed for parameters used in the numerical calculation. Note that the relation (10) is valid for \( k > \tilde{k}_{\text{del}} \), but not for \( k < \tilde{k}_{\text{del}} \). The value of \( \gamma \) remains the same as the case \( \tau = 0 \) for small \( k \).
Figure 5. The numerical results for the time-dependence of degrees when $r = 0.01$, $\tau = 1000$, $m = 2$ and $L_0 = 2000$. The dashed lines indicate exponential increases $\sim \exp\left(\left(\tilde{B} + m/D_{eq}\right)(t - t_i)\right)$ in which $(\tilde{B} + m/D_{eq}) = 1/722.8$ numerically obtained by (3) is substituted. The horizontal dotted line indicates the numerical value of $\tilde{k}_{\text{del}}$ numerically obtained by (6).

Figure 6. Numerical result for degree distribution at $t = 20000$ using the same parameters with Fig. 5. The guide line indicates a power-law form with (11) in which the numerical result, $\gamma = 722.8/1000 + 1 \simeq 1.72$, is substituted.

4. Summary
We have examined network structures in the stationary state of number of nodes where the rate of addition of new nodes and deletion of old nodes is balanced. We explained the exponential increase in vertex degrees by the constancy of the internal factor which describes the contribution of internal links to the rate of increase in degrees. If the internal factor $\tilde{B}$ is large enough to develop the value of degrees before the node is deleted, that is, if condition $\alpha \nu \tilde{N}_{eq} = \tilde{k}_{\text{del}} \gg 1$ is satisfied, hub nodes with order $\tilde{k}_{\text{del}}$ can exist, and the degree distribution can have a power-law form with exponent $\gamma = 1$, even though the deletion of old nodes causes deletion of hub nodes. When the lifetime of nodes is governed by an exponential decayed distribution, the rate of deletion of hub nodes decreases, and the power-law form of degree distribution with exponent $\gamma > 1$ is maintained by the balance between the magnitude of the internal factor and range of distribution of lifetime of nodes.

In this paper, we employed global links to describe internal links that connect pre-existing nodes. However, there are other mechanisms that emerge as an internal factor that maintains the power-law behavior in accelerating networks, such as intermediary process [7]. Local links may provide different features from networks examined here, which we plan to investigate in future studies.

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