Hypothetical learning trajectory and students’ understanding of the concepts of the arithmetic sequence

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Abstract. A teacher’s role is to facilitate students in understanding knowledge in each learning. An ideal learning process is inseparable from the process of learning planning. The teacher needs to develop a lesson plan. Hypothetical Learning Trajectory (HLT) is a benchmark in the implementation of learning to anticipate every response given by the students in every learning problem. This study aims to prepare the students’ Hypothetical Learning Trajectory on Arithmetic Sequence and make the anticipation of teacher’s responses to the alleged students’ answers occurring in learning Arithmetic Sequence. The preparation of the HLT is a part of design research. This research focuses on the first stage, namely the preliminary design. The results of this study are in a form of HLT about the number patterns of the Arithmetic Sequence consisting of learning objectives, learning activities, and hypotheses of students’ learning processes. The teacher’s responses to each student's answer is different. When students have not been able to answer the problem correctly, the teacher can ask several directing questions that lead students to the expected answers. HLT can be used to facilitate teacher to knowledge development of the designed learning activities by paying attention to the development of students’ way of thinking.

1. Introduction

Learning can go well and achieve the expected learning goals as influenced by many factors. One of the factors influencing the course of learning begins with the plan of learning implementation. As expressed by Moore [1], planning is an important part of learning to achieve excellence in instruction. In learning mathematics, planning is also needed to make the steps of learning so that students can do it more effectively. The teacher plans learning that helps the students understand the abstract mathematical concepts. Cai et al [2] said that the teacher must prepare well-structured learning so that mathematics learning can run and be student-centered. According to Hiebert & Grouws [3], one of the factors influencing the students’ understanding concept is the way teacher teach. The teacher can prepare learning activities in a learning implementation plan or lesson plan.

However, what occurs in the field is that the teacher has not given his/her best in designing the lesson plans. As supported by a research from Gafoor [4], the problems of the teacher in planning the learning are selecting an appropriate learning experience to the students, deciding and allocating appropriate time to each learning, identifying and developing the appropriate teaching media. Furthermore, Gulten’s research [5] reported that the teacher faces difficulties to formulate the learning objectives and give concrete examples toward the objectives. The teacher finds the difficulties to
select the appropriate learning process and order the selected activities. It indicates that the reason why the teacher creates the lesson plan is only to fulfil the school administration. The lesson plans are designed in a standard form which contain a brief overview about the opening activities, the main activities, and the closing activities. Another thing contained is the summary of the materials. It is rare to see that the teacher prepares any alternative hypothesis toward problem-solving strategy used by the students [6]. Whereas, the alternative hypothesis of the problem-solving strategy can be used by the teacher to define any appropriate response to overcome the students’ difficulties. The alternative hypothesis of the problem-solving strategy is then called Hypothetical Learning Trajectory.

The students’ hypothetical learning trajectory is important as a reference for the teacher to be an illustration for students’ way of thinking that need to be prepared in the lesson plan. According to Retnawati [7], a well-prepared lesson plan will generate a well-done learning. The students’ hypothetical learning trajectory can be outlined in the lesson plan. According to Simon and Tzur [8], hypothetical learning trajectory (HLT) consists of the students’ objectives, mathematical assignments that will be used to introduce the learning, and hypotheses about students’ learning processes. According to Wilson, Mojica, Confrey [9], learning trajectory or mathematics learning trajectory supports teacher in creating models of students’ way of thinking and restructuring the teacher’s understanding of mathematics and the students’ logic. HLT is very useful for researchers, curriculum developers, and teachers. For the researchers, HLT is served as an illustration of how students’ understanding of a material can be used as a basis for starting other researches. For the curriculum developers, HLT explains the potential pathways for curriculum design. For the teachers, HLT can be a reference for students’ understanding of mathematical concepts.

As the importance of HLT is used to describe the students’ ways of thinking in learning a mathematical concept, this article will discuss some examples of HLT containing any important ideas in the Arithmetic Sequence. HLT comes up as an answer to questions about the development of students’ way of thinking and the teacher’s role in planning activities to respond to various students’ ways of thinking in learning Arithmetic Sequence. Based on the previous description, this article aims to compile the students' hypothetical learning trajectory on the Arithmetic Sequence and make the anticipation of the alleged students' answers that might occur in the learning of the Arithmetic Sequence.

2. Method

This research is a part of design research. There are three phases in design research: preparing an experiment (initial design), experimenting in class and implementing a retrospective analysis. This research focused on the first stage, namely the preliminary design. This design was included in the validation research. This research began by setting learning objectives, such as learning objectives in arithmetic material referring to the curriculum. The next stage is to develop the learning activities toward the learning objectives and make possible hypotheses arising from the students’ learning activities based on field observations and document analysis. This study tries to develop a learning design which is the HLT of Arithmetic Sequence.

3. Result and discussion

According to Simon, the hypothetical learning trajectory (HLT) is the teacher’s assumption about the learning flow that might occur in the learnings in the classroom. Sarama et al. [10] the HLT has three parts namely a mathematical goal, a path of the development in which students develop to achieve that goal, and a series of teaching activities, or assignments adjusted to each level of thought in a path that help the students to develop higher levels of thinking. The learning trajectory illustrates what students develop over time through a series of jobs and roles, thus it brings more purposes and clarity to the concept of lifelong learning [11].

According to Simon & Tzur [8], the HLT formulation was based on the following assumptions:

- HLT formulation is based on the teacher’s understanding toward the students’ knowledge,
- HLT is a vehicle for the teacher to plan learning of a mathematical concept,
- mathematical assignments or activities are tools to present the learning toward a mathematical concept as well as a key part in the learning process,
as it is conjectural or hypothetical and not in accordance to the actual process, the teacher needs to modify every aspect of HLT into the sustainable ones.

HLT is a vehicle to plan learning certain mathematical concepts. Because of the hypothetical and inherently uncertain nature of this process, the teacher is regularly involved in modifying every aspect of HLT. Furthermore, the cycle occurs in Figure 1. HLT is based on the teacher’s understanding of the students’ knowledge when carrying out the previous learning. Moreover, the teacher determines the learning objectives. Based on the learning objectives, the teacher plans to learn activities. Next, the teacher makes an alternative hypothesis of the student learning trajectory, namely the hypothesis of various problem-solving strategies that might be done by the students. Based on it, the teacher can assess how students' understanding of the material to be taught and can conduct evaluations to improve the future learning. HLT formulation is important since it can facilitate the students to build their logic, and build the students’ way of thinking. HLT is used to guide the implementation of learning and help to overcome the students’ difficulties.

![Diagram of Hypothetical Learning Trajectory](image_url)

**Figure 1.** The cycle of developing a hypothetical learning trajectory

It is relatively not easy for the teacher to develop and implement the teaching materials containing the mathematical tasks based on the students’ way of thinking. Therefore, this article can be a reference for the teacher to develop or modify the students’ learning trajectories, which are used as a reference to develop the teaching materials and plan the learning process. This paper presents an example of the application of Hypothetical Learning Trajectory (HLT) for the study of the number patterns of the Arithmetic Sequence, namely:

### 3.1 Learning objectives
Based on the Curriculum 2013 of 2018 revision, the study of Arithmetic Sequence was taught in class XI semester 2. The learning objectives to be achieved are that the students can understand the number patterns of the Arithmetic Sequence and determine the contextual problem solving related to the Arithmetic Sequence.

### 3.2 Learning activities
Based on the learning objectives above, the researchers designed some appropriate learning activities. The learning is begun with the time when the teacher presents the opening greetings and checks the students’ attendance list as well as informs the learning objectives. Furthermore, the teacher gives the apperception of the pattern on the sequence of numbers that has been learned in class VIII of junior high school. Then, the teacher motivates the students by directing them to find the use of the Arithmetic Sequence in daily life. If the students cannot find the use of the Arithmetic Sequence, the
teacher gives the response by showing and explaining the pictures related to the Arithmetic Sequence in daily life. Next, the main activities contain two activities that will be carried out to achieve the learning objectives. In main activity 1, the students will be given problem 1 which aims to help the students to find the definitions of the Arithmetic Sequence. In main activity 2, the students will be given problem 2 related to daily life. This activity aims to make the students able to find the n-th term formula of the Arithmetic Sequence. When working on problems 1 and 2, there are several hypotheses of the students’ learning flow and the teacher’s alternative responses as discussed below.

3.3 The hypothesis of student learning flow and alternative teacher response
The examples of the hypothetical student learning flow and the teacher’s alternative responses in learning the number patterns in the Arithmetic Sequence are as follow:

3.3.1 Main activities 1
In main activity 1, the first problem is presented to help the students to find the concepts/definitions of the Arithmetic Sequence. The teacher gives several examples of the Arithmetic Sequence, then asks the students to identify the specific characteristics of the sequences given.

| Problem 1: Pay attention to the sequences below. Identify the characteristics of the sequences below! |
|---------------------------------------------------------------|
| 1) 2, 4, 6, 8, ...                                           |
| 2) 100, 90, 80, 70, ...                                     |
| 3) ![Image](image1.png)                                     |
| 4) ![Image](image2.png)                                     |

![Figure 2. Problem 1](image3.png)

The hypothesis of the students' responses when they are asked to identify the specific characteristics of arithmetic sequences are:

- The students answer that there are rows in which the order of numbers increases in value and there are sequences of the order whose numbers get smaller. This shows that the students can identify that each row has a certain pattern, however the students have not been able to identify that the rows have the same difference in each consecutive number. To overcome this answer, the teacher can ask the question: "How much is the difference from the first term to the second term? How much is the difference from the second term to the third term? Is it different from each number in the same sequence?" Then, the teacher can direct the students to write the difference of the first term and the second term, the second term and the third term, and so on, in each row of numbers. Furthermore, the teacher can direct the students to analyze and summarize the results obtained by students.

- The students' answers for the first to fourth row are continuously different, namely 2, 10, 3, and 2. The students have been able to identify that each consecutive number has the same difference, but the students have not been able to distinguish that the difference in each successive number can be negative. To overcome this answer, the teacher can ask the question: "How do you get the difference in these rows?" Then, the teacher can ask the students to add the difference obtained to the last number so that they get the next number.

- The students answer that each consecutive number has the same difference, that is 2, -10.3, -2. For example, in rows 2, 4, 6, 8, ... it always increases 2 numbers from a number to the next number. In rows of 100,90,80,70, ... it always decreases 10 numbers from a number to the number afterwards. It shows that the students have been able to identify the special characteristics of the rows given. Then, the teacher informs us that the characteristics of the ranks mentioned earlier are those of the Arithmetic Sequence. After that, the teacher directs the students to deduce the definition of the
Arithmetic Sequence, namely the arrangement of the numbers that have certain pattern or rule between one number and the next number, where the difference between two consecutive terms is always fixed. After the students understand the Arithmetic Sequence concept, the teacher can direct the students to identify and understand the elements that are known from the Arithmetic Sequence, such as the first syllable, the difference and the number of terms in the sequence. And then, the teacher can give problem 2 to determine the $n^{th}$ term formula in the Arithmetic Sequence.

3.3.2 Main activities 2
In main activity 2, the teacher presents problem 2 which aims to make the students able to find the n-th term formula of the Arithmetic Sequence. Problem 2 is presented in the form of the students’ worksheets done in groups.

Pay attention to the problem below!

**Problem 2**
If the first stair (the most bottom one) is 15 cm high and each of the next stair is 20 cm high, how tall is the stairs if there are 8 stairs?
Determine the pattern of the sequence!

**Figure 3. Problem 2**

a. What information is known about the problem?
What does the question ask?

**Figure 4. Question (a) in student worksheet**

The hypothesis of the students’ responses to identify the information that is known about the problem 2 are:
- The students do not or cannot answer the question correctly. To overcome this answer, the teacher asks the students to discuss with friends in group.
- The students answer that the first stair is 15 cm high, and the next stair is 20 cm high. The question asked is the height of 8 steps and the pattern of the sequence. Then, the teacher appreciates and continues the learning process.

b. To determine the height of 8 steps, the above problems can be sorted into:
- Height of one step = ...
- Height of two steps =...
- Height of three steps =...
- Height of four steps =...
- Height of five steps =...
- Height of six steps =...
- Height of seven steps =...
- Height of eight steps =...

**Figure 5. Question (b) in student worksheet**

The hypothesis of the students' responses to determine the height of 8 steps are:
- The students have not been able to answer the question correctly. The students answer it by adding 20 cm to each subsequent stair. However, the students ignore that the first stair is 15 cm high. The height of one step to the height of eight steps is 20, 40, 60, 80, 100, 120, 140, 160 cm high. In conclusion, the height of the stairs if there are 8 steps is 160 cm high. To overcome this answer,
the teacher can ask the question, "What is the height of the first stair?" Then, the teacher asks the students to look back at the information.

- The students misunderstand the purpose of the problem. The students think that what is asked is the height of the eighth stairs, not the height of the first stair. The students answered that the height of one stair to the height of eight stairs is 15, 20, 20, 20, 20, 20, 20, 20 cm. In conclusion, the height of the stairs if there are 8 steps is 20 cm high. To overcome this answer, the teacher asks the question, "What is the difference between the height of two stairs and the height of the second stair?" Then the teacher can explain it by using pictures.

- The students can answer the questions correctly. The students already understand that the height of the first stair is 15 cm high as the first syllable. And the next stair is obtained by adding 20 cm. The height of one stair to the height of eight stairs is 15, 35, 55, 75, 95, 115, 135, 155 cm. In conclusion, the height of the stairs if there are 8 stairs is 155 cm high. Then, the teacher appreciates and continues the learning process.

**Figure 6.** Question (c) in student worksheet

The hypothesis of the students' responses to arrange the numbers into rows of numbers are:

- The students' answers are incorrect. To overcome this answer, the teacher asks the students to recheck the previous process.

- The students' answers are correct, which is 15, 35, 55, 75, 95, 115, 135, 155. Then, the teacher appreciates and continues the learning process.

| Look for the difference between two consecutive number terms | Fill in the blank! |
|-------------------------------------------------------------|--------------------|
| \( u_2 - u_1 = \ldots - \ldots = \ldots \)                | \( u_1 = \ldots \) |
| \( u_3 - u_2 = \ldots - \ldots = \ldots \)                | \( u_2 = \ldots = \ldots + \ldots = \ldots + ( \ldots )( \ldots ) \) |
| \( u_4 - u_3 = \ldots - \ldots = \ldots \)                | \( u_3 = \ldots = \ldots + \ldots = \ldots + ( \ldots )( \ldots ) \) |
| \( u_5 - u_4 = \ldots - \ldots = \ldots \)                | \( u_4 = \ldots = \ldots + \ldots = \ldots + ( \ldots )( \ldots ) \) |
| \( u_6 - u_5 = \ldots - \ldots = \ldots \)                | \( u_5 = \ldots = \ldots + \ldots = \ldots + ( \ldots )( \ldots ) \) |
| \( u_7 - u_6 = \ldots - \ldots = \ldots \)                | \( u_6 = \ldots = \ldots + \ldots = \ldots + ( \ldots )( \ldots ) \) |
| \( u_8 - u_7 = \ldots - \ldots = \ldots \)                | \( u_7 = \ldots = \ldots + \ldots = \ldots + ( \ldots )( \ldots ) \) |

So, the pattern of lines of the problem above is ....

**Figure 7.** Question (d) in students’ worksheet
The hypothesis of the students' responses to find the pattern of lines of the problem are:

- The students' answer are incorrect. To overcome this answer, the teacher directs the students to associate the difference between two successive terms to answer the right column.
- The students’ answer are correct, as shown in figure 8. The students have been able to link the difference between successive tribes to determine the pattern of ranks. Then, the teacher appreciates and continues the learning process.

| Look for the difference between two consecutive number terms | Fill in the blank! |
|-------------------------------------------------------------|--------------------|
| \( u_2 - u_1 = 35 - 15 = 20 \)                          | \( u_1 = 15 \)     |
| \( u_2 - u_2 = 55 - 35 = 20 \)                          | \( u_2 = 35 = 15 + 20 \) = 15 + (1)(20) |
| \( u_3 - u_3 = 75 - 55 = 20 \)                          | \( u_3 = 55 = 15 + 40 \) = 15 + (2)(20) |
| \( u_4 - u_4 = 95 - 75 = 20 \)                          | \( u_4 = 75 = 15 + 60 \) = 15 + (3)(20) |
| \( u_5 - u_5 = 115 - 95 = 20 \)                         | \( u_5 = 95 = 15 + 80 \) = 15 + (4)(20) |
| \( u_6 - u_6 = 135 - 115 = 20 \)                        | \( u_6 = 115 = 15 + 100 \) = 15 + (5)(20) |
| \( u_7 - u_7 = 155 - 135 = 20 \)                        | \( u_7 = 135 = 15 + 120 \) = 15 + (6)(20) |
|                                                               | \( u_8 = 155 = 15 + 140 \) = 15 + (7)(20) |

So, the pattern of lines of the problem above is \( u_n = 20n - 5 \)

**Figure 8.** Students’ right answers for point d

After that, the teacher directs the students to generalize the \( n \)th term formula of the Arithmetic Sequence.

e. If the difference between two terms of the same sequence of numbers is assumed to be \( b \) and the first term is assumed to be \( a \), then the general formula of the \( n \)th term of the arithmetic sequence is

\[
 u_n = \ldots + (\ldots - \ldots)(\ldots)
\]

with \( b = \ldots - \ldots \)

**Figure 9.** Question (e) in student worksheet

The hypothesis of the students' responses to find the general formula of the \( n \)th term of the arithmetic sequence are:

- The students’ answers are incorrect. To overcome this answer, the teacher asks the students to re-read the information for the first syllable, the difference, and the number of the syllables. Then, the teacher directs the students to relate it.
- The students’ answers are correct, as below:

\[
 u_n = a + (n - 1)(b) \\
\text{with } b = u_n - u_{n-1}
\]
4. Conclusions
Hypothetical learning trajectory consists of three constituent components, namely learning objectives, learning activities, and students' alleged thoughts. These trajectories are empirically-supported hypotheses about the level or way of thinking, knowledge, and skills in using the knowledge, which the students will most likely experience when they learn mathematics and the students expect to achieve or exceed the goals of learning. The teacher's response to each student's way of thinking is different. When the students have not been able to answer the problem correctly, the teacher can ask some questions that lead the students to the expected answers. HLT of mathematics learning supports the teacher to create the students' thinking model and restructure the teacher's understanding of mathematics and the students' mathematical logic. Because HLT is still speculative, the teacher can discover new things during the learning process, so the teacher must continue to improve the HLT that has been prepared after its implementation in the classroom.

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