Classifying Approximation Algorithms: Understanding the APX Complexity Class

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Abstract

We are interested in the intersection of approximation algorithms and complexity theory, in particular focusing on the complexity class APX. Informally, APX ⊆ NPO is the complexity class comprising optimization problems where the ratio OPT(I)/ALG(I) ≤ c for all instances I. We will do a deep dive into studying APX as a complexity class, in particular, investigating how researchers have defined PTAS and L reductions, as well as the notion of APX-completeness, thereby clarifying where APX lies on the polynomial hierarchy. We will discuss the relationship of this class with FPTAS, PTAS, APX, log-APX and poly-APX. We will sketch the proof that Max 3-SAT is APX-hard, and compare this complexity class in relation to BPP, ZPP to elucidate whether randomization is powerful enough to achieve certain approximation guarantees and introduce techniques that complement the design of approximation algorithms such as through primal-dual analysis, local search and semi-definite programming. Through the PCP theorem, we will explore the fundamental relationship between hardness of approximation and randomness, and will recast the way we look at the complexity class NP. We will finish by looking at the “real world” applications of this material in Economics. Finally, we will touch upon recent breakthroughs in the Metric Travelling Salesman and asymmetric travelling salesman problem, as well original directions for future research, such as quantifying the amount of additional compute power that access to an APX oracle provides, elucidating fundamental combinatorial properties of log-APX problems and unique ways to attack the problem of whether the minimum set-cover problem is self-improvable.

Keywords: Approximation Algorithms, Polynomial Time Reduction, Provable Guarantees, Linear Programming, Randomisation, Hardness, Travelling Salesman, Oracle Reduction, Polynomial Hierarchy, Auction Design

Note to Reviewer. If pressed for time, recommended reading is comprises the sections "Introduction", "Definitions and Setup", "APX-Completeness Results", and "Randomization, Primal-Dual Methods and LP Relaxation". Since we provide a sketch of proofs for problems from many disparate domains of Computer Science, the reviewer is invited to examine problems that are of personal interest to them once they have read the main definitions.

Introduction

Approximation algorithms are ubiquitous in providing useful solutions to NP-hard optimization problems: while we cannot solve these completely in polynomial time, we can often solve arbitrary instances of these problems up to some constant c of the optimal output. The design and analysis of approximation algorithm is rich and rapidly-growing field, and has far reaching ramifications in many branches of Computer Science. In this paper we will attempt to outline and study the theoretical scope of approximation algorithms within the backdrop of Complexity Theory.

The APX complexity class is the class of approximation algorithms that allow for polynomial time approximation schemes, with approximation ratios bounded above by some constant c i.e. the set of all algorithms that can be “almost solved” to a particular degree. The intellectual history of this idea dates back to 1991, where a closely related class called
MaxSNP was introduced by Papadimitriou and Yannakakis in 1991. According to Scott Aaronson, APX was first formally defined by Ausiello et al in 1999. Some concrete examples can be found below:

**Load Balancing:** The goal of this problem is to distribute some set of tasks over a fixed set of resources as efficiently as possible, with close ties to the sub-field of Computer Science known as parallel computing. More formally, we are provided with a sequence of jobs of multiple sizes and are required to assign these jobs to some $n$ processors, under particular constraints, such that the load on the most burdened processor is minimised. This problem is not only within $APX$, but also lies within a subset of this complexity class called $PTAS$, which we will develop further.

Researchers in the Stanford Theory Group - Aggarwal, Motwani and Zhu actually demonstrated that the load re-balancing problem must also lie within $PTAS$. This is an extension of the traditional load balancing problem, where we are first given a potentially sub-optimal assignment of tasks to processors and are required to rearrange these jobs to reduce the objective.

**Vertex Cover:** A vertex cover is a set of nodes in a graph such that every edge has an endpoint in at least one set of nodes. This admits an easy 2-approximation and we can verify by proof that the 2-approximation is indeed tight.

**Travelling Salesman:** The travelling salesman problem seeks to find the shortest cost path that traverses through all of the nodes in a graph and plays a central role in the field of combinatorial optimisation. While it is an NP-hard problem and can be approximated to as close as within 1 percent, there exists a 2-approximation algorithm for this problem by considering the minimum spanning tree. Christofide’s algorithm is an improvement that allows for a 3/2-approximation algorithm where we perfect-match the odd degree vertices.

**Max-3SAT:** This problem plays a central role in the theory of the hardness of approximation. Given some cnf-formula $\phi$, with at most 3-variables per clause, we would like to find an assignment that satisfies the largest number of clauses. There are many theorems we can prove about the Max-3SAT problem in relation to the APX complexity class, in particular that it is $APX Complete$.

The following four complexity classes are related to $APX$, and defined analogously as follows:

**PTAS:** These are the algorithms that produces a solution with a factor $1 + \epsilon$, where $\epsilon > 0$. Lots of problems which do not have a polynomial time approximation scheme (PTAS) permit a polynomial time randomized approximation scheme (PRAS). We are able to get arbitrarily close to the optimum, provided that we are willing to accept the trade-off between the quality of the solution and time.

**APX-Intermediate:** These are problems that are within the polynomial-time approximation scheme, however are not $APX$-complete. Analogous problems in $NP$ would be the problems that lie within $NP$ but are not $NP$-complete.

One natural example of such a problem is the minimum bin-packing problem, which permits a $(1 + \epsilon) - OPT + 1$ approximation but is only asymptotically PTAS due to the extra +1 additive factor.

**f(n)-APX:** We can consider the complexity class that consists of all of the problems that can be approximated within some approximation ratio $O(f(n))$, which is dependent on $n$ and not necessarily constant.

There are multiple heuristics we can use to design approximation algorithms, which the reader may already be familiar with (either from 154/254 or classes on algorithms). These include the Greedy Algorithm, Local Search, Dynamic Programming, Convex Programming (LP relaxation and Semidefinite Programming relaxations), Primal-Dual Methods, Dual Fitting, Metric Embedding and Random Sampling. We will see examples of these heuristics and methods throughout the paper and hope to provide intuition as to when and where such heuristics should be used.

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1In fact, APX can be thought of as the closure of MaxSNP under PTAS Reduction
Ultimately, studying approximation algorithms provides greater nuance to the classification of NP-hard problems, which are otherwise equivalent to 3-SAT under the traditional polynomial-time reduction scheme. Whilst two NP-hard problems are on the surface, identical, we can dig deeper into the approximation algorithms that these problems merit in order to study how they relate to and differ from each other on a more fundamental level. One surprising example is the fact that whilst the problem of finding a minimum size vertex cover is the same as the problem of finding a maximum size independent set, when considering these two identical problems through the lens of approximation, we find that VERTEX COVER permits a 2-approximation algorithm whilst INDEPENDENT SET does not permit a constant-factor approximation at all unless \( P = NP \).

Definitions and Setup

**Definition.** Let \( \text{NPO} \) be the set of optimization problems in \( NP \). An optimization problem has the following components: an input, a notion of a valid solution that can be checked in polynomial time (e.g. an assignment satisfying some clauses in a boolean formula\(^2\)), and a value, denoted \( \text{VAL} \), for each valid solution (e.g. the number of satisfied clauses in a boolean formula). For an instance \( I \) of an optimization problem, let \( \text{OPT}(I) \) be the value of the optimal solution to \( I \), and let \( \text{ALG}(I) \) be the value produced by some approximation algorithm \( A \).

**Definition.** We say a Maximization problem \( L \) is in \( \text{APX} \) if \( L \in \text{NPO} \) and there exists polynomial-time approximation algorithm \( A \) such that for all instances \( I \) of \( L \), there exists \( c > 1 \) such that:

\[
\frac{\text{OPT}(I)}{\text{ALG}(I)} \leq c
\]

**Remark.** We can define this similarly for minimization problems, requiring that there exists \( c > 1 \) such that \( \frac{\text{ALG}(I)}{\text{OPT}(I)} \leq c \).

**Example.** Let Max3SAT be the the following: given a 3cnf formula \( \phi \), find an assignment satisfying the largest number of clauses. Then Max3SAT \( \in \text{APX} \).

**Proof.** Consider algorithm \( A \) that does the following:

- Let \( C \) be the number of clauses \( x_1 \) be an assignment with all variables to 1. Check the number of satisfied clauses.
- Let \( x_2 \) set all variables to 0. Check the number of satisfied assignments.
- Return \( \text{Max}\{x_1, x_2\} \)

Note that if a clause is satisfied when all variables are set to 1, then it must not be satisfied if they are all set to 0. The reverse is also true. Thus one of the assignments \( \{x_1, x_2\} \) must satisfy \( |C|/2 \) clauses. Thus \( \text{ALG}(I) \geq |C|/2 \). Since \( \text{OPT}(I) \leq |C| \), we have that \( \frac{\text{OPT}(I)}{\text{ALG}(I)} \leq 2 \).

**Definition.** Let \( L \) be a Maximization (resp. minimization) problem. An approximation scheme for \( L \) is a family of \( (1 - \epsilon) \) (resp. \( (1 + \epsilon) \)) approximation algorithms) for \( L \) for any \( 0 \leq \epsilon \leq 1 \). A Polynomial Time Approximation Scheme (PTAS) is an approximation scheme with time complexity of \( \text{poly}(n) \) in the size of input \( I \) (but not necessarily polynomial in \( 1/\epsilon \)).

We have claimed that \( \text{APX} \) is a complexity class. However, we have not proved its closure under any type of reductions. Intuitively, any reduction between problems in \( \text{APX} \) must preserve the 'approximability' of the problems. Additionally, we would like to define \( \text{APX} \)-Completeness as follows: If one \( \text{APX} \)-Complete problem has a PTAS, then all \( \text{APX} \) problems

\(^2\)We assume there always exists one valid solution. For example, in Max-CLIQUE, we can assume that if there are no cliques we should output a single vertex.
have a PTAS (the analog of $P$ in $NP$). This concept has been formalized in the following sense.

**Definition.** Let $P_1, P_2$ be optimization problems. We say $P_1$ is **L-reducible** to $P_2$ (i.e. $P_1 \leq_L P_2$) if:

- There exists a polynomial time computable function $f$ such that if $x$ is an instance of $P_1$, $f(x)$ is an instance of $P_2$.
- There exists a polynomial time computable function $g$ such that if $y$ is a solution to $f(x)$, $g(y)$ is a solution to $x$.
- There exists a constant $a$ such that for any instance $x$ of $P_1$:
  \[ \text{OPT}_{P_2}(f(x)) \leq a \cdot \text{OPT}_{P_1}(x) \]
- There exists a constant $b$ such that for any instance $x$ of $P_1$ with image $f(x)$ and solution $y$ to $f(x)$:
  \[ \left| \text{OPT}_{P_1}(x) - \text{VAL}_{P_1}(g(y)) \right| \leq b \cdot \left| \text{OPT}_{P_2}(f(x)) - \text{VAL}_{P_2}(y) \right| \]

In other words, if $P_1$ is L-reducible to $P_2$, then the value of the optimal solutions are within constants of each other, and there is a constant bound on the absolute error of some proposed solution to $P_2$ in terms of the absolute error of some proposed solution to $P_1$.

From the definitions, it is easy to check familiar properties of reductions (e.g. if $P_1 \leq_L P_2$, $P_2 \leq_L P_3$, then $P_1 \leq_L P_3$). We thus define **APX-completeness** in the same way as NP Completeness: to be APX-Complete, a problem $P$ must be in APX, and there must be a L-reduction from all other problems in APX to $P$. The following lemma establishes a meaningful way APX-completeness provides an analogue of NP-completeness:

**Lemma 1.** If $P_1$ and $P_2$ are optimization problems, $P_1 \leq_L P_2$, then a PTAS for $P_2$ implies that there is a PTAS for $P_1$.

**Proof.** We will prove this for minimisation problems (for maximisation problems, the proof is similar). For $\epsilon > 0$, assume that there is a $1+\delta$ approximation for $P_2$, where $\delta = \frac{\epsilon}{2+\epsilon}$. We claim there is a $(1+\epsilon)$-approximation for $P_1$. Note that by the above definition, $\text{OPT}_{P_2}(f(x)) \leq a \cdot \text{OPT}_{P_1}(x)$ and also $\text{VAL}_{P_1}(g(y)) - \text{OPT}_{P_1}(x) \leq b \cdot (\text{VAL}_{P_2}(y) - \text{OPT}_{P_2}(f(x)))$. We then apply the following inequalities:

\[
\frac{\text{VAL}_{P_1}(g(y))}{\text{OPT}_{P_1}(x)} \leq \frac{\text{OPT}_{P_1}(x) + b \cdot (\text{VAL}_{P_2}(y) - \text{OPT}_{P_2}(f(x)))}{\text{OPT}_{P_1}(x)}
= 1 + b \cdot \frac{(\text{VAL}_{P_2}(y) - \text{OPT}_{P_2})}{\text{OPT}_{P_1}(x)}
\leq 1 + a \cdot b \cdot \frac{(\text{VAL}_{P_2}(y) - \text{OPT}_{P_2})}{\text{OPT}_{P_2}(x)}
\leq 1 + a \cdot b \cdot (1 + \delta - 1)
= 1 + \epsilon
\]

We can interpret the result as showing that APX forms an analogue of NP: in the case of NP, if one NP-Complete problem admits a polynomial time solution, they all do. In the case of APX, if one APX-complete problem admits a PTAS, they all do.
APX-Completeness Results

Here, we use Max-3SAT at as our gateway to prove some example problems are APX-Complete. In particular, we will use the PCP Theorem, which shows that for any $\epsilon > 0$, the $(7/8 + \epsilon)$ approximation of Max-3SAT is NP Hard. We first note that Max-3SAT is in APX since a $7/8$ approximation exists.

**Lemma 2. (PCP Theorem)** There exists $\rho < 1$ such that for all $L \in NP$ there is a polynomial time computable $f$ mapping strings to 3-cnf formulas such that:

\[
x \in L \Rightarrow OPT(f(x)) = 1
\]
\[
x \notin L \Rightarrow OPT(f(x)) < \rho
\]

Where for a 3-cnf formula $f(x)$, $OPT(f(x))$ returns the fraction of clauses that are satisfied by the Maximal satisfying assignment.

**Theorem 1.** Max-3SAT is APX-Hard

We will use the above two results without proof for now, but a proof sketch is provided in Theorem 1.

**Proof.** Let $P \neq NP$. Assume Max-3SAT admits a PTAS. Recall the definition of PTAS must require the existence of an $\epsilon$-approximation for any $\epsilon \in [0, 1]$. Let $\rho$ be as defined in Lemma 2 for an arbitrary problem $L$. Let $\epsilon = \frac{1-\rho}{2}$, noting that $1 - \epsilon > \rho$. Assume we have a $(1-\epsilon)$-approximation algorithm $A$ for Max-3SAT, that takes as input a 3-cnf formula and outputs a $(1-\epsilon)$ approximation of the Maximum number of clauses satisfiable. Then we can apply the reduction $f$ from Lemma 1 for the chosen language $L$ to an instance $x$, and note that $x \in L \Rightarrow OPT(f(x)) = 1 \Rightarrow A(f(x)) \geq 1 - \epsilon$. Analogously, $x \notin L \Rightarrow OPT(f(x)) < \rho \Rightarrow A(f(x)) \leq \rho < 1 - \epsilon$. So $x \in L$ if and only if $A(f(x)) \geq 1 - \epsilon$. So we have just used a polynomial-time approximation algorithm for Max-3SAT to exactly solve an arbitrary NP Problem, contradicting $P \neq NP$.

To complete the proof, note that by Lemma 1, any PTAS for some other APX-complete problems can be converted into a PTAS for Max-3SAT and hence can be used to solve arbitrary NP problems.

We now demonstrate that APX-complete problems comprise an interesting subset of optimization problems. We show that Max-2SAT is also APX-Complete. This should be not at all obvious, since 2SAT (regular boolean satisfiability for 2-cnf formulas) is in fact decidable in polynomial time!

**Theorem 2.** Max-2SAT is APX-Complete.

**Proof.** We construct an L-reduction Max-2SAT $\leq_L$ Max-3SAT. Let the transformation $f$ be defined as follows: for a 3-cnf clause $(x \lor y \lor z)$, $f$ returns:

\[
(x) \land (y) \land (z) \land (v) \land (\neg x \lor \neg y) \land (\neg x \lor \neg z) \land (\neg y \lor \neg z) \land (x \lor \neg v) \land (y \lor \neg v) \land (z \lor \neg v)
\]

It is straightforward (but tedious) to verify that if at least one of $(x, y, z)$ are true the above expression has a Maximum of 7 satisfied clauses (choosing $v$ appropriately). Conversely, if none of $(x, y, z)$ are true then the above expression has a Maximum of 6 satisfied clauses (setting $v = \text{FALSE}$). We now define $g$ as follows: for a given truth assignment $w$ to a 2-cnf formula created by $f(x)$, $g$ restricts to the assignments of the variables present in $x$ (i.e. it omits the assignment to the $v_i$’s in each clause, since the $v_i$’s are created by $f$). Now let $OPT(\varphi)$ count the Maximum number of satisfiable clauses of a k-CNF formula $\varphi$. Let $C$ be the number of clauses in a 3-cnf formula $x$. We note that:
\[
OPT(f(x)) = 6|C| + OPT(x) \leq 12OPT(x) + OPT(x) = 13OPT(x)
\]

Where the first equality comes from our characterization of \( f \) above, and the subsequent inequality comes from the fact that \( OPT(x) \geq \frac{|C|}{2} \), as argued earlier. So we have fulfilled the first criterion in the definition of L-reductions, with \( a = 13 \). To fulfill the second criteria, it suffices to note that \( f \) maintains the fact that for any assignment \( y \) to \( f(x) \),

\[
|OPT(x) - VAL(g(y))| = |OPT(f(x)) - VAL(y)|
\]

Thus taking \( b = 1 \) suffices to complete the proof.

Let Not-all-equal-3SAT (NAE-3SAT) be the following problem: for a 3cnf formula \( \varphi \), is there a satisfying assignment such that for all clauses \( C_i \) in \( \varphi \) at least one variable is true and at least one variable is false? Max-NAE3SAT is defined similarly to MAX-3SAT for NAE-3SAT problems. The following 2 theorems are due to Papadimitriou and Yannakakis [10]:

**Theorem 3.** Max-NAE3SAT is APX-Complete.

**Proof.** We construct an L-reduction \( \text{Max-2SAT} \leq_L \text{Max-NAE3SAT} \). Let \( x \) be a 2-cnf formula. Define \( f(x) \) as follows: for each clause \( (a \lor b) \) in \( x \), create a NAE clause \( (a \lor b \lor c) \).

Without loss of generality (since flipping all the variables does not change the number of NAE clauses satisfied by truth assignment), we may set \( c \) to be identically false. Then a truth assignment with \( c \) false satisfies a NAE clause if and only if it satisfies \( (a \lor b) \). Thus \( OPT(f(x)) = OPT(x) \) and the L-reduction is trivial with \( a = b = 1 \).

Recall that for an undirected graph \( G = (V, E) \), MAX-CUT is the problem of finding a partition of the graph’s vertices into two complementary sets such that the number of edges between these sets is as large as possible.

**Theorem 4.** MAX-CUT is APX-Complete

**Proof.** We construct an L-reduction \( \text{Max-NAE3SAT} \leq_L \text{MAX-CUT} \). We first show this in the case of multiedge graphs (where two nodes can have several edges connecting them), the case of simple graphs is similar. Let \( x \) be an instance of MAX-3NAESAT with \( m \) clauses. Then we construct the following graph \( G \):

- There is one vertex corresponding to each variable \( a \) and another vertex corresponding to \( \neg a \) in \( x \).
- If \( (a \lor b \lor c) \) is a 3-cnf clause, we add edges \( (a, b), (b, c), (a, c) \) to \( E \). If a clause only has two literals \( (a \lor b) \), we add a parallel edge \( (a, b) \) twice to \( E \).
- For every variable \( a \), we add \( 2k \) parallel edges between \( a \) and \( \neg a \) where \( k \) is the number of times \( a \) or \( \neg a \) occurs in the clauses.

See Figure 1 for an example. Now note that given a truth assignment \( \tau \) that satisfies \( p \) clauses of \( x \). We can partition \( V \) into \( S, T \) disjoint by placing all nodes corresponding to true literals under \( \tau \) in \( S \) and all nodes corresponding to false literals in \( T \). Clearly, every NAE clause that is satisfied has two edges crossing the cut. Moreover, if a NAE clause is not satisfied (all true or all false), it has no edges crossing the cut. If there are \( l \) literal occurrences, then the cut contains \( 2l + 2p \) edges (since all the edges between \( a \) and \( \neg a \) must cross the cut).

Now assume we have a partition \( S, T \) of \( G \), where \( G \) is based on some MAX-3NAESAT formula. Note that to be a max cut, we would need variables that correspond to \( a \) and \( \neg a \) to be on opposite sides of the cut (otherwise we can move them across and weakly increase the cut). Likewise, for each of the \( p \) NAE-satisfiable clauses (which map to triangles of 3
vertices), a max cut would require having 2 edges of the triangle cross the cut. Now, note that setting an arbitrary one of $S, T$ to be true (all the literals inside are true) and the other to be false, yields a solution to MAX-3NAESAT. It is easy to see that for a max cut, the size of the cut would need to be $2l + 2p$, where $p$ is the number of satisfied clauses. Now note that $l \leq 3m$ (since each 3cnf clause has 3 literals) and $OPT(x) \geq m/2$ (by our earlier arguments). So $2p + 2l \leq 2m + 6l = 8l$ and hence $OPT(f(x)) \leq 8OPT(x)$. Moreover, for some arbitrary solution $y$ we have that $OPT(f(x)) - VAL(y) = (2l + 2m) - (2l + 2p) = 2(m - p)$ for some $p \leq m$. Casting back to NAE-3SAT, we find that $y$ satisfies $p$ many clauses. So $OPT(x) - VAL(g(y)) = m - p$. Thus choosing $a = 8, b = 2$, we have a L-reduction.

**Remark.** The above only proves the result for multigraphs, but there is a straightforward extension for simple graphs. However, it requires introducing the notion of expanders, which is outside the scope of this essay.

We have seen that many optimization problems that have NP-hard decision problem analogues are APX-hard. Indeed, using Max-3SAT, a number of older papers have already shown many familiar problems are APX-Complete. For example, let MIN-VERTEX-k be the problem of finding the Minimum vertex cover of a graph of bounded degree $\geq k$. Then it has been showing (Alimonti and Kann [1]) that MIN-VERTEX-k is APX-Complete for $k \geq 3$, and as an easy corollary MAX-IND-SET-k (the maximum independent set of a graph of degree $\geq k$) is also APX-Complete.

**Interactions of APX with the PH, The Hardness of Approximation**

Note that whilst complexity theory is well-suited to handle decision problems, many important combinatorial optimisation problems such as MINIMUM VERTEX-COVER, MAX-3SAT, MAX-CUT are all search problems. This is resolved by considering the following claim.

**Statement:** Note that if the $c, s$-gap problem of determining whether the optimum objective of some input $x$ to a combinatorial optimisation problem $P$ falls between the thresholds of $\leq s$ and $\geq c$ is NP-hard, then it is NP-hard to approximate $P$ to within a $c/s$ factor.

The PCP Theorem tells us about the fundamental relationship between Hardness and Approximation. In the 1990s, we found that the problem of proving the hardness of approximation is very closely related to the issue of whether we could reformulate proofs to be checked probabilistically through only a constant number of queries. We used this previously in our analysis of MAX-3-SAT, and in the next section we provide a proof outline and sketch.

**PCP Theorem:** There exists some $\epsilon_1 > 0$ such that there is no polynomial-time $(1 + \epsilon_1)$-approximation algorithm for MAX-3-SAT unless $P = NP$. [6]

**Proof. Sketch:** Given some instance $I$ of MAX$q$CSP, where $q$ is the constant in the PCP theorem, over variables $x_1, \ldots, x_n$ with $m$ constraints, we will show how to construct an instance $\phi_I$ of MAX-3-SAT with $m'$ clauses such that:
I is satisfiable ⇒ φ_I is satisfiable

\[ \text{opt}_{\text{Max}}q\text{CSP}(I) \leq \frac{m}{2} \Rightarrow \text{opt}_{\text{Max}}3\text{SAT}(\phi_I) \leq (1 - \epsilon_1)m' \]

This is because for every constraint \( f(x_{i_1}, \ldots, x_{i_q}) = 1 \) in \( I \), we will first construct an equivalent qCNF of size \( \leq 2^q \). Then we will "convert" clauses of length \( q \) to clauses of length \( 3 \), which can be done by introducing additional variables as in the standard reduction from \( k\text{SAT} \) to \( 3\text{SAT} \). Overall this transformation creates a formula \( \phi_I \) with at most \( q2^q \) 3CNF clauses for each constraint in \( I \), so the total number of clauses in \( \phi_I \) is at most \( q \cdot 2^q \cdot m \).

Now let us use the relation between the optimum of \( \phi_z \) as an instance of \( \text{MAX}3\text{SAT} \) and the optimum of \( I \) as an instance of \( \text{MAX}q\text{CSP} \).

If \( I \) is satisfiable, then the set of the \( x \) variables in \( \phi_I \) to the same values and set the auxiliary variables appropriately, then the assignment satisfies all clauses, and \( \phi_I \) is satisfiable.

If every assignment of \( I \) contradicts at least half of the constraints, then consider an arbitrary assignment to the variables of \( \phi_I \); the restriction of the assignment to the \( x \) variables contradicts at least \( m/2 \) constraints of \( I \) and so at least \( m/2 \) of the clauses of \( \phi_I \) are also contradicted. The number \( m' - m/2 \) is at most \( m'(1 - \epsilon_1) \) if we choose \( \epsilon_1 \leq \frac{1}{2\sqrt{q}} \).

The relationship between the statement of the PCP theorem and NP-hardness is at the surface surprising, but when we dig a little deeper we find that it is in fact intimately related to the Cook Levin Theorem, which tells us about the accepting computation paths of a non-deterministic Turing Machine with satisfying assignments to some Boolean Formula input. Once a certificate is given, the entire certificate is read and checked for membership.

The PCP theorem offers a sturdier representation of the certificate for NP languages that are, in some sense, impervious to changes to a single bit. The certificate can be rewritten so that it can be verified by probabilistically selecting some fixed number of locations with which are then examined. With this theorem, whole new class of reductions other than polynomial-time reductions are opened to us. We see that there is a fundamental relationship between the hardness of approximation and probabilistically checkable proofs.

Finally note that we have a spectrum of approximation algorithms of increasing size, where each set has some representative problem (Sauerwald, Cambridge Easter 2017):

**FPTAS:** Knapsack Subset-Sum

\[ \downarrow \]

**PTAS:** Scheduling Euclidean TSP

\[ \downarrow \]

**APX:** Vertex Cover, Maximum Cut Problem, Metric Travelling Salesman Problem

\[ \downarrow \]

**Log-APX:** Set-Cover Problem

\[ \downarrow \]

**Poly-APX:** Maximum Clique Problem

These inclusions are strict if and only if \( P \neq NP \). It is very fruitful to think about the relationship between these complexity classes and reductions that leave these complexity classes invariant. For example consider the following results [3] which we will not delve into the proofs of here:

**Claim 1:** No log-APX-complete and indeed no poly-APX problem can be self-improvable unless the polynomial-time hierarchy collapses.

**Self-Improvable:** A problem \( P \) is self-improvable if there exist two polynomial-time algorithms \( A_1 \) and \( A_2 \) such that for any instance \( a \in P \) and for any two rationals \( r_1, r_2 > 1 \)
that $a' = A_1(a, r_1, r_2)$ is also an instance of $P$ and for any $y' \in P_{r_2}(a')$ we have that $y' = A_2(a, y', r_1, r_2)$ must also be an instance of $P_{r_2}$. 

Claim 2: The MINIMUM BIN PACKING problem is not APX-complete unless the polynomial-time-hierarchy collapses.

Claim 3: Finding the vertices of the largest clique is more difficult than simply finding the vertices of a 2-approximation clique given that the polynomial-time-hierarchy does not collapse.

**APX-Hardness in Mechanism Design**

Economists and Computer Scientists are often interested in real-world allocation problems for which exact solutions are often NP-hard. For example, given $n$ prospective buyers for an indivisible good, each with some private valuation drawn from a distribution over $[0,1]^n$, how should we design an auction such that maximizes the seller's revenue, while encouraging bidder participation? In this section, we summarize, without too much detail, some interesting results in this area, in particular recent proofs by Papadimitriou et al. 

As a preliminary, we define an allocation problem with $n$ bidders for an indivisible good with valuations over $[0,1]^n$ drawn from a distribution $\phi(v)$. An auction consists of an allocation rule $x_i(v) = \{0,1\}$, which is an indicator of whether bidder $i$ is allocated the item, and a payment rule $p_i(v)$ (how much $i$ pays). We aim to maximize the auctioneer’s profit, i.e. $\sum_{i=1}^{n} E[\phi(v)p_i(v)]$ (the expected value of payments of each type of bidder, weighted by the distribution over the types). However, we also want the auction to have the following characteristics (here, $f(v_i, v_{-i})$ is used in the usual manner it is used in game theory, which is to say apply function $f$ to profile $v$ comprising of $v_i$ for individual $i$ and $v_{-i} = \{v_1, \cdots v_{i-1}, v_{i+1} \cdots v_n\}$ for all other non-$i$ individuals):

1. Incentive-Compatible: $\forall v_i, v'_i, v_{-i}: v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i x_i(v'_i, v_{-i}) - p_i(v'_i, v_{-i})$. In other words, player $i$ has no positive reason to imitate a player of valuation $v'_i$, regardless of the valuation profile of the other players.

2. Individually Rational: $\forall i, v_i, v_{-i}: v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq 0$. In other words, there is no situation in which player $i$ loses money by participating in the auction.

Let $3\text{OPTIMAL-AUCTION}$ be the following problem: given a 3-dimensional matrix that represents a discrete probability distribution $\phi(x, y, z)$ with finite support, design the optimal, incentive-compatible, individually rational auction and output its 3-dimensional allocation matrix. Intuitively, given a set of 3 players and some prior beliefs about their valuations, we are trying to design an auction that satisfies constraints (1) and (2).

**Theorem 5.** (Papadimitriou and Pierrakos) $3\text{OPTIMAL-AUCTION}$ is APX-Hard.

**Proof Sketch.** The paper [9] proves the following:

1. Let $\phi(x, y, z)$ denote the distribution that governs the valuations of player 1, 2, and 3 (possibly correlated). The argument uses a discrete distribution, but the paper also shows that the proof carries through for continuous distributions under certain modest assumptions (e.g. Lipschitz continuity of $\phi$).

2. Now, we define functions $f, g, h$ for players 1, 2 and 3 as follows: $f(x, y, z) = \max_{x' \geq x, y \geq y, z \geq z} \{x \cdot \sum_{x' > x} \phi(x', y, z) - \sum_{x' > x} f(x', y, z), 0\}$, normalizing $f(1, y, z) = \phi(1, y, z)$ (analogously for player 2 and 3). See the paper for an interpretation of this function - essentially, we can imagine the allocation function $x_i$ as partitioning the 3D grid of valuations $(x, y, z)$ into 4 disjoint subsets (corresponding to areas where we allocate the good to player 1, 2, 3, and no one). $f$ is the added expected profit of adding a small area to the set allocated to player 1.

3. Given a discrete 3D grid with coordinates $(x, y, z)$, with $f(x, y, z) > 0$, define a segment as an interval (sequence of points) starting at $(x, y, z)$ and including all nodes $(x', y, z)$
with \( x' \geq x \). We define the weight of a segment as \( \sum_{x'>x} f(x', y, z) \). We define segments for the \( y \) and \( z \) dimensions analogously. We now define the following problem:

**3-SEGMENT**: Given a discrete probability distribution \( \phi(x, y, z) \) on a 3D grid, with weights defined as above, find a subset of non-intersecting segments with a maximum sum of weights.

4. Now, it is possible to demonstrate that \( 3\text{-OPTIMAL-AUCTION} \leq L \text{-3-SEGMENT} \). That is the main part of the proof, and we will not reproduce it here. It may be instructive to consider how the 2D case of \( 3\text{-OPTIMAL-AUCTION} \) (2 players instead of 3) is \( L \)-reducible to finding the Maximum Weight Independent Set on a bipartite graph, and try to generalize this to 3 dimensions.

5. To show \( 3\text{-SEGMENT} \) is \( \text{APX-Hard} \), we now consider another new problem:

**3CAT-SAT**. Let 3CAT-SAT be the 3SAT problem input restricted such that there are three categories of variables, \( \{x_i\}, \{y_i\} \) and \( \{z_i\} \) and a total of \( n_x + n_y + n_z \) variables. The 3SAT formula has \( m \) clauses of the following form: every clause has at most one literal from every type of \( x, y, z \).

It is easy to show that Max-3CAT-SAT is NP-hard to approximate better than some constant (use the PCP Theorem). In fact, we have an explicit bound: 3CAT-SAT is inapproximable at \( 103/104 + \epsilon \) for any \( \epsilon \) (i.e. there is no PTAS unless \( P = \text{NP} \)).

6. Now, demonstrate that \( \text{Max-3CAT-SAT} \leq L \text{-3SEGMENT} \). The main idea is to create 2 types of segments, corresponding to literals and clauses (and also "scaffold-segments" to ensure we only have segments in the desired directions). Through the use of intersecting segments, we can enforce rules such as "literal \( x \) cannot be both true and false".

Thus, researchers have shown that interesting "real-world" problems that economists may care about are \( \text{APX-Hard} \), and indeed this could be interpreted as a failure of the theory of mechanism design - if we introduce concepts like Incentive Compatibility, and later discover that implementing an auction that preserves these concepts cannot be done efficiently, what is the point? But the flip-side of the above is that it is \textit{plausible} that in fact \( 103/104 \) is the tightest upper bound on the quality of a polynomial time approximation of the 3-person auction (known approximations are around \( 1/2 \) of optimal revenue), and this paves the way for either better deterministic algorithms (or more pessimistically, a tighter upper bound). It may also be interesting to consider how the quality of the approximation bound for \( k\text{-OPTIMAL-AUCTION} \) decreases as \( k \geq 3 \) becomes larger (doing so using the method of Papadimitriou and Pierrakos would require considering segments in 4-dimensions, where their intuitive geometric arguments will be harder to apply).

**Randomization, Primal-Dual Methods and LP Relaxation**

Recall that often we may have an exact algorithm to a particular problem, but still may want to discover some strong approximation algorithm for the same problem with lower time and spatial requirements; this often comes up when the input size is extremely large, such as with the enormous data-sets that are produced in many modern-day industries. In this section, through examples, we will consider some important heuristics that are frequently used to construct approximation algorithms with certain guarantees. Whilst the Greedy Heuristic is useful, we can often push this technique to one step better through more nuanced methods such as LP-relaxation, finding randomised approximation algorithms and then subsequently de-randomising them.

Randomisation is an extremely powerful technique when studying the design of approximation algorithms. When randomization was first introduced via the renowned AKS primality testing paper, it provided a whole new class of algorithms we could use to attack problems. Through the introduction of a coin flip, randomization offers an improvement in speed and spatial requirements. The \textit{BPP} complexity class is the set of randomized algorithms that
can be run in polynomial time, and whilst it is clear that $P$ lies within $BPP$, it is still unknown (but conjectured to be true) whether $P = BPP$.

There are parallels that we can draw between the notion of randomization and the notion of approximation algorithms. Recall that $P = P_{\text{random}}$ (i.e. randomization does not lend us the ability to save time for problems in $P$), and that the analogous problem for space $L = L_{\text{random}}$ is currently an unresolved problem but conjectured to be true. This section we will explore the question of whether:

$$APX = APX_{\text{random}}$$

gives us greater power for finite approximation algorithms. The notion of error in randomized algorithms is intimately tied to the notion of error in approximation algorithm, with a correct/probability 1 implies determinism for randomised algorithms and solving the complete problem in polynomial time for approximation algorithms.

**Congestion Problem:** Consider the following optimisation problem, the input being some directed graph $G = (V, E)$ with positive integer edge capacities $c_e$ and a set of source-sink pairs $(s_i, t_i)$ where each $(s_i, t_i)$ is a pair of vertices such that $G$ contains at least one path from $s_i$ to $t_i$. Our goal is to output a list of paths $P_1, \ldots, P_k$ such that $P_i$ is a path from $s_i$ to $t_i$. The load on edge $e$, denoted by $l_e$, is defined to be the number of paths $P_i$ that traverse edge $e$. The congestion of edge $e$ is the ratio $l_e/c_e$ and the algorithm’s objective is to minimize congestion. With this problem, we will exhibit the power of using LP-relaxation to guide our process of designing approximation algorithms. Below is the study of crafting the approximation algorithm for this problem from [4].

**Claim:** (Without Proof) The congestion minimization problem is NP-hard.

Now we will construct an approximation algorithm with LP-relaxation. First consider some decision variable $x_{i,e}$ for each $i = 1, \ldots, k$ and each $e \in E$ denoting whether or not $e$ belongs to $P_i$. We will allow this variable to take fractional values. The resulting linear program can be written using $\delta^+(v)$ to denote the set of edges leaving $v$ and $\delta^-(v)$ denoting the set of edges entering $v$.

We can therefore formulate the congestion problem as the following LP:

$$\begin{align*}
\text{minimize} & \quad \sum_{j=1}^{m} w_j x_j \\
\text{subject to} & \quad \sum_{j: e \in S_j} x_j \geq 1, \quad i = 1, \ldots, n \\
& \quad x_j \in \{0, 1\}, \quad j = 1, \ldots, m
\end{align*}$$

Where $(x_{i,e})$ is a $\{0,1\}$-valued vector obtained from a collection of paths $P_1, \ldots, P_k$ by setting $x_{i,e} = 1$ for all $e \in P_i$. The first constraint ensures that $P_i$ is a path from $s_i$ to $t_i$, while the second one ensures that the congestion of each edge is bounded above by $r$.

The following approximation algorithm not only solves the linear program, but also performs post-processing of the solution to obtain a probability distribution over the paths for each terminal pair $(s_i, t_i)$ and outputs an independent random sample from each of these distributions.

To describe the post-processing step, we observe that the first LP constraint says that for every $i \in \{1, \ldots, k\}$ the values $x_{i,e}$ define a network flow of value 1 from $s_i$ to $t_i$. Now we will define a flow to be acyclic if there is no directed cycle with positive amount of flow on each edge of $C$. The first step of post-processing is to make the flow $(x_{i,e})$ acyclic for each $i$. If there is a directed cycle $C$ such that $x_{i,e} > 0$ for every edge $e \in C$, then we can let $\delta = \min \{x_{i,e} | e \in C\}$ and modify $x_{i,e}$ to $x_{i,e} - \delta$ for every $e \in C$. This modified solution still satisfies all of the LP constraints and has strictly fewer variables $x_{i,e}$ taking nonzero values.

After finitely many such modifications, we arrive at a solution in which each of the flow $(x_{i,e})$ where $1 \leq i \leq k$ is acyclic. Since this modified solution is also an optimal solution of
the linear program, we may assume without loss of generality that in our original solution \( (x_{i,e}) \) the flow was a-cyclic for each \( i \).

Next for each \( i \in \{1, \cdots, k\} \) we take the a-cyclic flow \( (x_{i,e}) \) and represent it as a probability distribution over paths from \( s_i \) to \( t_i \), i.e. a set of ordered pairs \((P, \pi_P)\) such that \( P \) is a path from \( s_i \) to \( t_i \) and \( \pi_P \) is a positive number interpreted as the probability of sampling \( P \) and the sum of probabilities \( \pi_P \) over all paths \( P \) is equal to 1. The distribution can be constructed using the following:

**Algorithm 1: Path Distribution Algorithm**

We are given source \( s_i \) and sink \( t_i \), acyclic flow \( x_{i,e} \) of value 1 from \( s_i \) to \( t_i \);

Initialize \( D_i = 0 \);

**while** \( \exists \) path \( P \) from \( s_i \) to \( t_i \) such that \( x_{i,e} > 0 \) for all \( e \in P \) **do**

\[
\begin{align*}
\pi_P &= \min\{x_{i,e} | e \in P\}; \\
D_i &= D_i \cup \{(P, \pi_P)\};
\end{align*}
\]

**end**

**for** \( e \in P \) **do**

\[
\begin{align*}
x_{i,e} &= x_{i,e} - \pi_P
\end{align*}
\]

**end**

Return \( D_i \);

Where each iteration of the **while** loop strictly reduces the number of edges with \( x_{i,e} > 0 \) and hence the algorithm must terminate after selecting at most \( m \) paths. When it terminates, the flow \( (x_{i,e}) \) has value 0, and it is a-cyclic because \( (x_{i,e}) \) was initially acyclic and we never inserted a nonzero amount of flow on an edge whose flow was initially zero. Therefore the only acyclic flow of value zero is the zero flow, and when the algorithm terminates we must have that \( x_{i,e} = 0 \) for all \( e \).

Each time we selected a path \( P \), we decreased the value of the flow by exactly \( \pi_P \). The value was initially 1 and finally 0, so the sum of \( \pi_P \) over all paths \( P \) is exactly 1. For any given edge \( e \), the value \( x_{i,e} \) decreased by exactly \( \pi_P \) each time we selected a path \( P \) containing \( e \), and hence the combined probability of all paths containing \( e \) is exactly \( x_{i,e} \).

Now performing the post-processing algorithm for each \( i \), we obtain that the probability distributions \( D_1, \cdots, D_k \) over paths from \( s_i \) to \( t_i \) with the property that the probability of a random sample from \( D_i \) traversing edge \( e \) is equal to \( x_{i,e} \). Now we draw one independent random sample from each of these \( k \) distributions and output the resulting \( k \)-tuple of paths \( P_1, \cdots, P_k \). We claim that with probability at least \( 1/2 \), the parameter \( \max_{e \in E} \{c_e/c_e\} \) is at most \( \alpha \), where \( \alpha = 2 \log(2m) \). This follows by direct application of Corollary 2 of the Chernoff bound. For any given edge \( e \), we can define independent random variables \( X_1, \cdots, X_k \) by specifying that:

\[
X_i = (c_e \cdot r)^{-1}
\]

These are independent and the expectation of their sum is \( \sum_{i=1}^{k} \frac{1}{c_e} \) which is at most 1 because of the second LP constraint above. Applying Corollary 2 with \( N-2m \), we find that the probability of \( X_1 + \cdots + X_k \) exceeding \( \alpha \) is at most \( 1/(2m) \). Summing the probabilities of these failure events for each of the \( M \) edges of the graph, we find that the probability at least \( 1/2 \) none of the failure events occur and \( \max_{e \in E} \{c_e/c_e\} \) is bounded above by \( \alpha r \).

Now, \( r \) is a lower bound on the parameter \( \max_{e \in E} \{c_e/c_e\} \) for any \( k \)-tuple of paths with the specified source-sink pairs, since any such \( k \)-tuple defines a valid LP solution and \( r \) is the optimum value of the LP. Therefore, our randomized algorithm achieves approximation factor \( \alpha \) with probability at least \( 1/2 \).

This example perfectly exhibits the power of Linear Programming reformulation and randomisation to construct algorithms with certain approximation guarantees.

**Randomized 2-approximation for Max-Cut:** Recall that whilst the minimum-cut problem permits a polynomial-time solution, the maximum-cut problem is NP-hard. This is the problem of outputting some partition of a graph \( G \) into two subsets such that the total weight of all of the edges with one vertex in each of the partitioned sets is maximised. We have the following randomised algorithm:

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1. For each vertex \( v \), we randomly place \( v \) in the first set with probability 0.5 and the second set with probability 0.5.

With the above simple randomised algorithm, we obtain the expected weight of our cut as:

\[
E\left( \sum_{e \in E(A_1, A_2)} w_e \right) = \sum_{e \in E} w_e \cdot P(e \in E(A_1, A_2)) = \frac{1}{2} \sum_{e \in E} w_e
\]

Now we clearly have that this expectation is at least half of the weight of the maximum cut. Now we can run this expectation-guaranteed algorithm multiple times, subsequently outputting multiple cuts, and outputting the cut with the largest weight. With this method, we can actually boost the success probability by some guaranteed amount.

We can subsequently derandomize this algorithm with either pairwise independent hashing or conditional expectations to obtain an analogous deterministic algorithm with the same approximation guarantee with at most a polynomial-time factor increase in the running time. More details can be found in [4].

**Semidefinite Programming to Improve Max-Cut Approximation:** Semidefinite Programming is an optimisation problem that attempts to maximise a particular linear function over some set of symmetric positive semi-definite \( n \times n \) matrices subject to some linear inequality constraints. Goemans and Williamson used this technique to improve the 0.5 performance guarantee to a 0.87856 performance guarantee. This was a very famous result that requires more technology and we will refer the reader to [8] to learn more. The high level approach is as follows:

1. Reformulate the Max-Cut Problem as a Quadratic Optimisation Problem
2. Solve some SDP that is the relaxation of the original Max-Cut Problem
3. Once we obtain an approximation of the semi-definite program, attempt to construct some approximation of the original

**The Set Cover Problem - Greedy, Layering and LP-based Approach:** Given a universe \( U \) of \( n \) elements, a collection of subsets of \( U \), \( S = \{S_1, \ldots, S_k\} \) and some cost function \( c : S \to \mathbb{Q}^+ \), find a minimum cost sub-collection of \( S \) that covers all of the elements of \( U \). [12]

Studying the set cover problem illuminates many areas and techniques of approximation algorithms. There are multiple approximation algorithms that are classified into either achieving an \( O(\log n) \) approximation ratio or a ratio of \( m \), where \( m \) is the number of occurrences of the most frequent element within some set.

Again, we see that the follow Greedy Approach achieves the former logarithmic approximation bound. The precise analysis of this algorithm is studied in [12].

**Greedy:** Select the set that covers most of the points, throw out all of the points that are already covered. Repeat this process.

**LP-Based Approach:** We will first reformulate the minimum set-cover problem as a linear program:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} x_i \\
\text{subject to} & \quad \sum_{j \in T_i} x_i \geq 1, \quad j = 1, \ldots, n \\
& \quad x_i \in \{0, 1\}, \quad i = 1, \ldots, m
\end{align*}
\]

Note that if we relax the last condition to \( x_i \in [0, 1] \), then we are able to solve the above Linear Program in polynomial time. We can use this to formulate the randomized rounding...
approximation algorithm which gives us an $O(\log n)$ approximation guarantee:

**Algorithm 2: Randomized-Rounding Algorithm for the Set-Cover Problem**

$$S = \text{EMPTY};$$
$$p = \text{LP Solution};$$

**while** $i \leq d \cdot \log n$ **do**

$$S^{(i)} : \text{Select Every Set } i \in [m] \text{ with probability } p_i;$$

$$S : S \cup S^{i};$$

**end**

Return $S$;

We can analyse the guarantees of this algorithm with the union bound, Markov’s inequality to first bound the probability that $S$ is not a set cover, and subsequently show that this is indeed an $O(\log n)$ approximation, more detail is found in [11].

**Discussion and Future Research Directions**

This paper has demonstrated the central importance of the APX complexity class within the realm of Theoretical Computer Science. NP-Hard Optimization Problems are a rich field of study, and the approximations that they permit are even more so. One question that occurred to the authors was trying to formalize a notion of an "APX-Oracle": for example, a Turing Machine able to make oracle queries of the form "what is OPT($I$)?" that are correct for some approximation ratio $c$ that depends on the problem. This is closely related to the idea of "query complexity": the set of problems that can be solved with some $f(n)$ queries to a NP oracle, which is already well-defined in the literature.

We hope that through writing this report, we have clarified the fundamental interaction between approximation algorithms and randomisation, and whether the introduction of randomisation could boost our approximation capability. Techniques such as Primal-Dual analysis, greedy methods, randomisation and LP relaxation were used to perturb these problems within the APX-complexity class in order to study this class of problems in greater depth. The design of effective approximation algorithms is still a very vibrant area of research. We hope that we have offered some basic heuristics on how to construct such algorithms and have helped to communicate the significance of these approximation algorithms.

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