Centers of Braided Tensor Categories

Zhimin Liu∗†  Shenglin Zhu ‡§
Fudan University, Shanghai 200433, China

Abstract

Let \( C \) be a finite braided multitensor category. Let \( B \) be Majid’s automorphism braided group of \( C \), then \( B \) is a cocommutative Hopf algebra in \( C \). We show that the center of \( C \) is isomorphic to the category of left \( B \)-comodules in \( C \), and the decomposition of \( B \) into a direct sum of indecomposable \( C \)-subcoalgebras leads to a decomposition of \( B\text{-Comod}_C \) into a direct sum of indecomposable \( C \)-module subcategories.

As an application, we present an explicit characterization of the structure of irreducible Yetter-Drinfeld modules over semisimple quasi-triangular weak Hopf algebras. Our results generalize those results on finite groups and on quasi-triangular Hopf algebras.

KEYWORDS: Drinfeld Center, Braided tensor category, Automorphism braided group, Module category over monoidal category

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1 Introduction

The theory of module categories over a tensor category was introduced respectively by Bernstein’s [1], by Crane and Frenkel [8], and well-developed by Ostrik [27], by Etingof and Ostrik [12].

Let \( (\mathcal{M}, \otimes, a, \ell) \) be a semisimple module category over a finite multitensor category \( C \), and \( M \in \mathcal{M} \) be a generator of \( M \). It is proved in [27] [12] [10] that \( \mathcal{A} = \text{Hom} (M, M) \) is a semisimple algebra in \( C \), and the internal Hom functor \( F = \text{Hom}(M, \bullet) : \mathcal{M} \to \text{Mod}_C \mathcal{A} \) induces a \( C \)-module category equivalence. The proof is based on the fact that \( F \) is faithful.
and full, and essentially surjective on objects. In [18], for a right \(A\)-module \((U, q)\) in \(\mathcal{C}\) and a left \(A\)-module \((N, p)\) in \(\mathcal{M}\) the authors defined the tensor product \(U \otimes_A N\) and proved that the functor \(G = \bullet \otimes_A \mathcal{M} : \text{Mod}_C \rightarrow \mathcal{M}\) is a quasi-inverse of \(F\).

Let \(\mathcal{C}\) be a monoidal category. There is a well-known braided category construction \(\mathcal{Z}_l(\mathcal{C})\), called the Drinfeld center of \(\mathcal{C}\) (see [15]). The objects of \(\mathcal{Z}_l(\mathcal{C})\) are those objects of \(\mathcal{C}\) together with natural transformations satisfying a hexagon axiom. The center is a categorical version of the Hopf algebraic construction of the Drinfeld double. If \(H\) is a finite dimensional Hopf algebra over a field and \(\mathcal{C} = H \mathcal{M}\), then \(\mathcal{Z}_l(\mathcal{C})\) is equivalent to the Yetter-Drinfeld module category \(H_H^H \mathcal{YD}\).

Assume further that \(\mathcal{C}\) is braided. The center \(\mathcal{Z}_l(\mathcal{C})\) can be viewed as a right module category over \(\mathcal{C}\). If \(\mathcal{C}\) is multitensor with certain additional assumption, then there is a cocommutative \(\mathcal{C}\)-Hopf algebra \(U(\mathcal{C})\), coming from the braided reconstruction theory, which is named the automorphism braided group of \(\mathcal{C}\) by Majid [21, 20].

Let \((H, R)\) be a quasi-triangular Hopf algebra over a field \(k\) and \(\mathcal{C}\) be the braided tensor category \(H \mathcal{M}\). Then \(U(\mathcal{C}) = H\) [20], with the same algebra structure of \(H\) and an \(R\)-twisted coalgebra structure \(\Delta_R\). (For brevity, we denote the \(\mathcal{C}\)-coalgebra \((H, \Delta_R)\) by \(H_R\).) The Yetter-Drinfeld module category \(H_H^H \mathcal{YD}\) is equivalent to the relative module category \(H_R^H \mathcal{M}\) [32]. In [18], the authors have proved that each Yetter-Drinfeld submodule of \(H \in H_H^H \mathcal{YD}\) is a subcoalgebra of \(H_R\), and \(H\) admits a unique decomposition into the direct sum of indecomposable Yetter-Drinfeld submodules, while this decomposition coincides with the direct sum \((H, \Delta_R) = D_1 \oplus \cdots \oplus D_r\) of the indecomposable \(\mathcal{C}\)-subcoalgebras of \(H_R\). Furthermore, the tensor category

\[
H_H^H \mathcal{YD} \cong H_R^H \mathcal{M} = \oplus_{i=1}^{r} D_i \mathcal{M}
\]

is a canonical direct sum of indecomposable module categories over \(\mathcal{C}\), and by [27, 12, 10] each category \(D_i \mathcal{M}\) is equivalent to the category \(A_i \text{-Mod}_C\), where \(A_i = \text{Hom}_\mathcal{C}(M_i, M_i)\) for a nonzero object \(M_i \in D_i \mathcal{M}\).

Moreover, \(H_H^H \mathcal{YD}\) can also be viewed as a left module category over \(\mathcal{C}' = Vec_k\). In this case, internal Homs in \(H_H^H \mathcal{YD}\) are constructed concretely, and the structure of irreducible objects of \(H_H^H \mathcal{YD}\) are given in [18]. This structure theorem deduces the classical results on finite groups.

This paper is devoted to the study of the center \(\mathcal{Z}_l(\mathcal{C})\) of a finite braided multitensor category \(\mathcal{C}\). We develop a purely categorical version of the structure theorem on Yetter-Drinfeld modules for quasi-triangular Hopf algebras, which appeared in [18], extend the results to the center of finite braided multitensor categories. Explicitly, we prove that as module categories over \(\mathcal{C}\), \(\mathcal{Z}_l(\mathcal{C})\) is equivalent to the category \(U(\mathcal{C})\)-Comod\(\mathcal{C}\) of left \(U(\mathcal{C})\)-comodules in \(\mathcal{C}\), and the decomposition of \(U(\mathcal{C})\) into a direct sum of indecomposable \(\mathcal{C}\)-subcoalgebras leads to a decomposition of \(U(\mathcal{C})\)-Comod\(\mathcal{C}\) into a direct sum of indecomposable
$\mathcal{C}$-module subcategories, and each such indecomposable $\mathcal{C}$-module subcategory is equivalent to the category of left modules over a $\mathcal{C}$-algebra. And we present a characterization of the internal Hom for $U(\mathcal{C})\text{-Comod}_{\mathcal{C}}$.

It is known that any finite multifusion category is equivalent to the category of finite dimensional representations of a regular semisimple weak Hopf algebra [14][28]. The main results of this paper are applied to the study of Yetter-Drinfeld module for quasi-triangular weak Hopf algebras. An explicit characterization of the structure of irreducible Yetter-Drinfeld modules over semisimple quasi-triangular weak Hopf algebras will be given, which generalize those results on finite groups [9, 13] and on quasi-triangular Hopf algebras [18].

The paper is organized as follows. Section 2 recalls module categories, Drinfeld centers of monoidal categories. Section 3 discusses the center $Z_l(\mathcal{C})$ of a braided rigid category $\mathcal{C}$. Using graphical calculus, we prove that when the automorphism braided group $U(\mathcal{C})$ exists, the category $Z_l(\mathcal{C})$ is equivalent to the category $U(\mathcal{C})\text{-Comod}_{\mathcal{C}}$ of left $U(\mathcal{C})$-comodules in $\mathcal{C}$. In Section 4 we show that a decomposition of the automorphism braided group induces a decomposition of $Z_l(\mathcal{C})$ as $\mathcal{C}$-module subcategories. Section 5 is devoted to an application of the theory developed to weak Hopf algebras.

## 2 Preliminaries

### 2.1 Notations and Conventions

Throughout this paper, $k$ denotes a field, and $\text{Vec}_k$ denotes the category of finite dimensional vector spaces over $k$. For the basic theory of monoidal categories, the reader is referred to [10]. It is well-known that any monoidal category is equivalent to a strict one by MacLane’s strictness theorem [19], we assume that the monoidal categories considered are all strict.

Let $(\mathcal{C}, \otimes, 1)$ be a monoidal category. We will use graphical calculus to calculate in $\mathcal{C}$, representing morphisms by diagrams to be read downwards. Our references are [29, 16, 23]. We denote respectively the evaluations, the coevaluations for left dual $X^*$ and right dual $^*X$ of an object $X \in \mathcal{C}$ by

$$ev_X = \begin{array}{c} X^* \rule{0em}{2em} \otimes \rule{0em}{2em} X \\
\end{array}, \quad coev_X = \begin{array}{c} X \otimes \rule{0em}{2em} \otimes \rule{0em}{2em} X^* \\
\end{array}, \quad ev'_X = \begin{array}{c} X^* \otimes \rule{0em}{2em} \otimes \rule{0em}{2em} X \\
\end{array}, \quad coev'_X = \begin{array}{c} ^*X \otimes \rule{0em}{2em} \otimes \rule{0em}{2em} X \\
\end{array}.$$

If $\mathcal{C}$ is also braided, the braiding $c$ and its inverse $c^{-1}$ are denoted respectively by

$$c_{X,Y} = \begin{array}{c} X \otimes \rule{0em}{2em} \otimes \rule{0em}{2em} Y \\
\end{array}, \quad c^{-1}_{X,Y} = \begin{array}{c} Y \otimes \rule{0em}{2em} \otimes \rule{0em}{2em} X \\
\end{array}.$$
If $B$ is a Hopf algebra in $\mathcal{C}$, we denote its multiplication $m_B$, unit $u_B$, comultiplication $\Delta_B$, counit $\varepsilon_B$, antipode $S_B$ and the inverse $S_B^{-1}$ (if it exists) as follows:

$$m_B = \begin{tikzpicture}[baseline=(current bounding box.center),thick] \node (b) at (0,0) {$B$}; \node (a) at (-1,0) {$B$}; \node (c) at (1,0) {$B$}; \draw (a) .. controls +(up:0.5cm) and +(down:0.5cm) .. (c); \end{tikzpicture}, \quad u_B = \begin{tikzpicture}[baseline=(current bounding box.center),thick] \node (a) at (0,0) {$B$}; \node (b) at (-1,0) {$B$}; \node (c) at (1,0) {$B$}; \draw (a) .. controls +(up:0.5cm) and +(down:0.5cm) .. (c); \end{tikzpicture}, \quad \Delta_B = \begin{tikzpicture}[baseline=(current bounding box.center),thick] \node (a) at (0,0) {$B$}; \node (b) at (1,0) {$B$}; \node (c) at (-1,0) {$B$}; \draw (a) .. controls +(up:0.5cm) and +(down:0.5cm) .. (c); \end{tikzpicture}, \quad \varepsilon_B = \begin{tikzpicture}[baseline=(current bounding box.center),thick] \node (a) at (0,0) {$B$}; \node (b) at (-1,0) {$B$}; \node (c) at (1,0) {$B$}; \draw (a) .. controls +(up:0.5cm) and +(down:0.5cm) .. (c); \end{tikzpicture}, \quad S_B = \begin{tikzpicture}[baseline=(current bounding box.center),thick] \node (a) at (0,0) {$B$}; \node (b) at (-1,0) {$B$}; \node (c) at (1,0) {$B$}; \draw (a) .. controls +(up:0.5cm) and +(down:0.5cm) .. (c); \end{tikzpicture}, \quad S_B^{-1} = \begin{tikzpicture}[baseline=(current bounding box.center),thick] \node (a) at (0,0) {$B$}; \node (b) at (-1,0) {$B$}; \node (c) at (1,0) {$B$}; \draw (a) .. controls +(up:0.5cm) and +(down:0.5cm) .. (c); \end{tikzpicture}.$$

### 2.2 Module Categories

The Morita theory of module categories over a monoidal category was well developed by Ostrik and Etingof. For references, one can see [27, 10].

A left module category over a monoidal category $\mathcal{C}$ is a category $\mathcal{M}$ endowed with an action bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, an associativity constraint $a_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$ and a functorial unit isomorphism $\ell_M : 1 \otimes M \to M$, for $X,Y \in \mathcal{C}$, $M \in \mathcal{M}$, satisfying a pentagon axiom and a triangle axiom.

Similarly, one can define the notion of right module category over $\mathcal{C}$. Denote the opposite monoidal category of $\mathcal{C}$ by $\mathcal{C}^{op}$, which is the category $\mathcal{C}$ with reversed order of tensor product and inverted associativity isomorphism. Then a right $\mathcal{C}$-module category is a left module category over $\mathcal{C}^{op}$.

In the case that $\mathcal{C}$ is a multitensor category, we are interested in module categories over $\mathcal{C}$ with additional properties in the sense of [10, Definition 7.3.1]. That is, if we say $\mathcal{M}$ is a left module category over $\mathcal{C}$, we mean that $\mathcal{M}$ is a locally finite abelian category equipped with a structure of a left $\mathcal{C}$-module category, such that the module product bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is bilinear on morphisms and exact in the first variable.

In 2003, Ostrik [27] characterized semisimple indecomposable module categories over a fusion category $\mathcal{C}$. Later, Etingof and Ostrik [12] generalized that result to nonsemisimple case.

In the study of the structure of a module category $\mathcal{M}$ over a multitensor category $\mathcal{C}$, a basic tool is the internal Hom. We first recall this notion here. For objects $M_1, M_2, M_3$ of $\mathcal{M}$, the internal Hom of $M_1$ and $M_2$ is an object $\text{Hom}_\mathcal{M}(M_1, M_2)$ of $\mathcal{C}$ representing the contravariant functor $X \mapsto \text{Hom}_\mathcal{M}(X \otimes M_1, M_2) : \mathcal{C} \to \text{Vec}_k$, i.e., there exists a natural isomorphism

$$\eta_{\bullet, M_1, M_2} : \text{Hom}_\mathcal{M}(\bullet \otimes M_1, M_2) \xrightarrow{\sim} \text{Hom}_\mathcal{C}(\bullet, \text{Hom}(M_1, M_2)) \quad (2.1)$$

The evaluation morphism $ev_{M_1, M_2} = \eta^{-1}(id_{\text{Hom}(M_1, M_2)}) : \text{Hom}(M_1, M_2) \otimes M_1 \to M_2$ is obtained from the isomorphism

$$\text{Hom}_\mathcal{C}(\text{Hom}(M_1, M_2), \text{Hom}(M_1, M_2)) \xrightarrow{\sim} \text{Hom}_\mathcal{M}(\text{Hom}(M_1, M_2) \otimes M_1, M_2).$$
The multiplication (composition)\[
\mu_{M_1,M_2,M_3} : \text{Hom}(M_2, M_3) \otimes \text{Hom}(M_1, M_2) \to \text{Hom}(M_1, M_3)
\]
is defined as the image of the morphism\[
ev_{M_2,M_3}(id \otimes ev_{M_1,M_2}) a_{\text{Hom}(M_2,M_3),\text{Hom}(M_1,M_2),M_1}
\]
under the isomorphism\[
\text{Hom}_\mathcal{M} \left( (\text{Hom}(M_2, M_3) \otimes \text{Hom}(M_1, M_2)) \otimes M_1, M_3 \right) \xrightarrow{\cong} \text{Hom}_\mathcal{C} \left( \text{Hom}(M_2, M_3) \otimes \text{Hom}(M_1, M_2), \text{Hom}(M_1, M_3) \right).
\]
Then \( A = (\text{Hom}(M_1, M_1), \mu_{M_1,M_1,M_1}) \) is an algebra in \( \mathcal{C} \) with unit morphism \( u_{M_1} : 1 \to \text{Hom}(M_1, M_1) \) obtained from the isomorphism \( \text{Hom}_\mathcal{M} (M_1, M_1) \xrightarrow{\cong} \text{Hom}_\mathcal{C} (1, \text{Hom}(M_1, M_1)) \), and \( (\text{Hom}(M_1, M_2), \mu_{M_1,M_1,M_2}) \) is a natural right \( A \)-module in \( \mathcal{C} \).

**Theorem 2.1** ([27, 12, 10]) Let \( \mathcal{M} \) be a semisimple module category over a finite multitensor category \( \mathcal{C} \). If \( M \in \mathcal{M} \) is a generator, then \( A = \text{Hom}(M, M) \) is a semisimple algebra in \( \mathcal{C} \). The functor \( F = \text{Hom}(M, \bullet) : \mathcal{M} \to \text{Mod}_\mathcal{C}-A \) given by \( V \mapsto \text{Hom}(M, V) \) is an equivalence of \( \mathcal{C} \)-module categories.

If assume further that \( \mathcal{M} \) is indecomposable, then every nonzero object \( M \) generates \( \mathcal{M} \), and the functor \( F = \text{Hom}(M, \bullet) : \mathcal{M} \to \text{Mod}_\mathcal{C}-A \) is an equivalence of \( \mathcal{C} \)-module categories.

Let \((A, m, u)\) be a \( \mathcal{C} \)-algebra. In [18], the authors defined left \( A \)-modules in \( \mathcal{M} \), by using the module category tensor, and give the \( A \)-tensor product of a right \( A \)-module \((U, q)\) in \( \mathcal{C} \) and a left \( A \)-module \((M, p)\) in \( \mathcal{M} \). Explicitly, a left \( A \)-module in \( \mathcal{M} \) is a pair \((M, p)\), where \( M \) is an object of \( \mathcal{M} \) and \( p : A \otimes M \to M \) is a morphism (in \( \mathcal{M} \)) satisfying two natural axioms,

\[
p(m \otimes id_M) = p(id_A \otimes p) a_{A,A,M}, \quad p(u \otimes id_M) = id_M,
\]

where \( a \) is the associativity constraint for \( \mathcal{M} \). For right \( A \)-module \((U, q)\) in \( \mathcal{C} \) and left \( A \)-module \((M, p)\) in \( \mathcal{M} \), the tensor product \( U \otimes_A M \) is the co-equalizer of the morphisms

\[
(U \otimes A) \otimes M \xrightarrow{q \otimes id_M} U \otimes M \xrightarrow{(id_U \otimes p)a_{U,A,M}} U \otimes_A M,
\]
i.e., the cokernel of the morphism \( q \otimes id_M - (id_U \otimes p) a_{U,A,M} \).

With all these terms, the authors presented a quasi-inverse for the equivalence \( F = \text{Hom}(M, \bullet) : \mathcal{M} \to \text{Mod}_\mathcal{C}-A \) given in Theorem 2.1

**Theorem 2.2** ([18, Theorem 4.3]) Let \( \mathcal{M} \) be a semisimple module category over a finite multitensor category \( \mathcal{C} \). Let \( M \) be a generator of \( \mathcal{M} \), then the functor \( G = \bullet \otimes_A M : \text{Mod}_\mathcal{C}-A \to \mathcal{M} \) is a quasi-inverse to the equivalence \( F = \text{Hom}(M, \bullet) : \mathcal{M} \to \text{Mod}_\mathcal{C}-A \).
2.3 The Drinfeld Center

Recall that the left Drinfeld center (left center) of a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$ is a category $\mathcal{Z}_l(\mathcal{C})$. An object of $\mathcal{Z}_l(\mathcal{C})$ is a pair $(Z, \gamma_{Z,\bullet})$ consisting of an object $Z \in \mathcal{C}$ and a natural isomorphism $\gamma_{Z,X} : Z \otimes X \to X \otimes Z$, $X \in \mathcal{C}$, such that

$$\gamma_{Z,X \otimes Y} = (id_X \otimes \gamma_{Z,Y}) (\gamma_{Z,X} \otimes id_Y), \quad \text{for all } X, Y \in \mathcal{C}. \quad (2.2)$$

A morphism from $(Z, \gamma_{Z,\bullet})$ to $(Z', \gamma_{Z',\bullet})$ is a morphism $f \in \text{Hom}_\mathcal{C} (Z, Z')$ such that

$$(id_X \otimes f) \gamma_{Z,X} = \gamma_{Z',X} (f \otimes id_X),$$

for all $X \in \mathcal{C}$. The right center $\mathcal{Z}_r(\mathcal{C})$ of $\mathcal{C}$ is a similar category with reversed order of tensor product in its definition.

For any $Z \in \mathcal{Z}_r(\mathcal{C})$, the objects $Z \otimes 1$ and $1 \otimes Z$ are identified with $Z$. Then (2.2) implies that $\gamma_{Z,1} = \gamma_{Z,1 \otimes 1} = (\gamma_{Z,1})^2$. Hence, one has

$$\gamma_{Z,1} = id_Z.$$

The center $\mathcal{Z}_l(\mathcal{C})$ is a braided monoidal category with braiding $c$ given by $c_{Z,Z'} = \gamma_{Z,Z'}$. Also, $\mathcal{Z}_r(\mathcal{C})$ is a braided monoidal category, which is isomorphic to $\mathcal{Z}_l(\mathcal{C})$ with the inverse braiding.

For the basic theory of the Drinfeld center, we refer to [15, 16].

By definition, an object of $\mathcal{Z}_l(\mathcal{C})$ is an object in $\mathcal{C}$ with a natural isomorphism satisfying (2.2). In fact if every object of $\mathcal{C}$ has a right dual, then identity (2.2) is sufficient for a natural transformation $\gamma_{Z,\bullet}$ to be a natural isomorphism.

**Lemma 2.3** Let $\gamma_{Z,\bullet} : Z \otimes \bullet \to \bullet \otimes Z$ be a natural transformation satisfying

$$\gamma_{Z,X \otimes Y} = (id_X \otimes \gamma_{Z,Y}) (\gamma_{Z,X} \otimes id_Y), \quad \text{for all } X, Y \in \mathcal{C}.\quad (2.2)$$

If $X \in \mathcal{C}$ has a right dual $^*X$, then

$$
(ev'_X \otimes id_Z) (id_X \otimes \gamma_{Z,^*X}) (\gamma_{Z,X} \otimes id_{^*X}) = id_Z \otimes ev'_X, \quad (2.3)
$$

$$(id_{^*X} \otimes \gamma_{Z,X}) (\gamma_{Z,^*X} \otimes id_X) (id_Z \otimes coev'_X) = coev'_X \otimes id_Z. \quad (2.4)$$

Moreover, if every object of $\mathcal{C}$ has a right dual, then $\gamma_{Z,\bullet}$ is a natural isomorphism.

**Proof.** By the naturality of $\gamma_{Z,\bullet}$, the equalities (2.3) and (2.4) are hold. If we denote

$\gamma_{Z,X} = \begin{array}{c}
\begin{array}{c}
Z \quad \begin{array}{c}
X
\end{array}
\
\begin{array}{c}
X
\end{array}
\begin{array}{c}
Z
\end{array}
\end{array}
\end{array}$,

then the pictorial transcriptions of (2.3) and (2.4) are respectively
In addition, the inverse of $\gamma_{Z,X}$ is given by

$$\gamma_{Z,X}^{-1} = (ev'_X \otimes id_Z \otimes id_X)(id_X \otimes \gamma_{Z,X} \otimes id_X)(id_X \otimes id_Z \otimes coev'_X).$$

The compositions of $\gamma_{Z,X}$ and $\gamma_{Z,X}^{-1}$ are computed as follows:

\[
\begin{align*}
\begin{tikzpicture}[baseline=-.5ex]
  \node (z) at (0,0) {$z$};
  \node (x) at (1,0) {$x$};
  \node (z') at (1,-1) {$z$};
  \node (x') at (0,-1) {$x$};
  \node (xx) at (2,0) {$x$};
  \node (z'x) at (2,-1) {$z$};
  \draw[->] (z) to [out=90, in=180] (x');
  \draw[->] (x) to [out=90, in=0] (z');
  \draw[->] (x) to [out=90, in=0] (xx);
  \draw[->] (xx) to [out=90, in=0] (z'x);
\end{tikzpicture}
&= id_{Z \otimes X},
\begin{tikzpicture}[baseline=-.5ex]
  \node (z) at (0,0) {$z$};
  \node (x) at (1,0) {$x$};
  \node (z') at (1,-1) {$z$};
  \node (x') at (0,-1) {$x$};
  \node (xx) at (2,0) {$x$};
  \node (z'x) at (2,-1) {$z$};
  \draw[->] (z) to [out=90, in=180] (x');
  \draw[->] (x) to [out=90, in=0] (z');
  \draw[->] (x) to [out=90, in=0] (xx);
  \draw[->] (xx) to [out=90, in=0] (z'x);
\end{tikzpicture}
&= id_{X \otimes Z},
\end{align*}
\]

where $\gamma_{Z,X}^{-1}$ is represented by the morphism in the dashed box. ■

Let $H$ be a finite dimensional Hopf algebra over $k$, and $C = H\mathcal{M}$ be the category of finite dimensional left $H$-modules. Then the center $Z_l(H\mathcal{M})$ of the monoidal category $H\mathcal{M}$ is isomorphic to the Yetter-Drinfeld category $^H_H\mathcal{YD}$ over $H$.

3 The Center of Braided Rigid Categories

In this section, $C$ will be a braided rigid category with a braiding $c$. We will show that under some representability assumption the Drinfeld center $Z_l(C)$ of $C$ is equivalent to the left comodule category of the automorphism braided group $U(C)$ of $C$, as right $C$-module categories.

Firstly, let’s recall Majid’s reconstruction [21, 20, 22] of the automorphism braided group $U(C)$. We require the representability assumption for modules [22 § 9.4]. That is, there exist an object $U(C) \in C$, and a natural isomorphism

$$\theta_V : \text{Hom}_C(V, U(C)) \to \text{Nat}(V \otimes id_C, id_C)$$

for any $V \in C$, and the maps

$$\begin{align*}
\theta^2_V & : \text{Hom}_C(V, U(C) \otimes U(C)) \to \text{Nat}(V \otimes id_C \otimes id_C), \\
\theta^3_V & : \text{Hom}_C(V, U(C) \otimes U(C) \otimes U(C)) \to \text{Nat}(V \otimes id_C \otimes id_C \otimes id_C),
\end{align*}$$

induced by $\alpha = \theta_{U(C)}(id_{U(C)})$ and the braiding $c$, are bijective. With graphical convention, we denote

$$\alpha_X = \begin{tikzpicture}[baseline=-.5ex]
  \node (x) at (0,0) {$x$};
  \fill[dashed] (0,-.5) rectangle (1,.5);
  \draw[->] (x) to [out=90, in=180] (x);
\end{tikzpicture}, \text{ or simply } \alpha_X = \begin{tikzpicture}[baseline=-.5ex]
  \node (x) at (0,0) {$x$};
  \fill[dashed] (0,-.5) rectangle (1,.5);
  \draw[->] (x) to [out=90, in=180] (x);
\end{tikzpicture}, \text{ for } X \in C.
Then $\theta^2_V$ and $\theta^3_V$ can be expressed graphically as follows. For $X, Y, Z \in \mathcal{C}$, $t \in \text{Hom}_\mathcal{C}(V, U(\mathcal{C}) \otimes U(\mathcal{C}))$ and $s \in \text{Hom}_\mathcal{C}(V, U(\mathcal{C}) \otimes U(\mathcal{C}) \otimes U(\mathcal{C}))$,

\[
\theta^2_V(t)_{X,Y} = \begin{array}{c}
\begin{array}{c}
V \\
X
\end{array}
\end{array} \quad , \quad \theta^3_V(s)_{X,Y,Z} = \begin{array}{c}
\begin{array}{c}
V \\
X
\end{array}
\end{array} \quad .
\] (3.1)

Then $U(\mathcal{C})$ is a Hopf algebra in the braided monoidal category $\mathcal{C}$, named the automorphism braided group of $\mathcal{C}$, which acts canonically on every object $X \in \mathcal{C}$ via $\alpha_X$. Write $B = U(\mathcal{C})$. Then with the graphical notations (see Page 4) the Hopf algebra structure on $B$ is determined by the diagrams (see [22, § 9.4])

\[
\theta_{B \otimes B}(m_B)_X = \begin{array}{c}
\begin{array}{c}
B \\
X
\end{array}
\end{array} \quad , \quad \theta_1(u_B)_X = \begin{array}{c}
\begin{array}{c}
X \\
X
\end{array}
\end{array} \quad , \quad \theta_2(\Delta_B)_{X,Y} = \begin{array}{c}
\begin{array}{c}
B \\
X
\end{array}
\end{array} = \alpha_{X \otimes Y}, \quad \varepsilon_B = \alpha_1,
\] (3.2)

\[
\theta_B(S_B)_X = \begin{array}{c}
\begin{array}{c}
X
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
X
\end{array}
\end{array} \quad = \begin{array}{c}
\begin{array}{c}
X
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
X
\end{array}
\end{array} \quad .
\] (3.3)

It is known that if an object $X$ of a braided monoidal category has a left dual $X^*$, then $X^*$ is naturally a right dual of $X$ with $ev'_X = ev_X \circ c_{X,X^*}$ and $coev'_X = c_{X^*,X}^{-1} \circ coev_X$. We will use the right rigidity of $\mathcal{C}$ to construct the inverse of $S_B$.

**Proposition 3.1** The antipode of $B$ is an isomorphism with its inverse $T_B = \begin{array}{c}
\begin{array}{c}
B
\end{array}
\end{array}$ given by

\[
\theta_B(S_B \circ T_B) = \theta_B(id_B) = \theta_B(T_B \circ S_B) .
\]

**Proof.** To show that $T_B$ be the inverse of $S_B$ is to show

\[
\theta_B(S_B \circ T_B) = \theta_B(id_B) = \theta_B(T_B \circ S_B) .
\]
The following graphical calculus yields the result:

\[
\begin{align*}
B X & \quad = \quad B X \\
& \quad = \quad B X \\
& \quad = \quad B X \\
& \quad = \quad B X
\end{align*}
\]

\[
\begin{align*}
B X & \quad = \quad B X \\
& \quad = \quad B X \\
& \quad = \quad B X \\
& \quad = \quad B X
\end{align*}
\]

\[
\begin{align*}
B X & \quad = \quad B X \\
& \quad = \quad B X \\
& \quad = \quad B X \\
& \quad = \quad B X
\end{align*}
\]

For any \( M, N \in \mathcal{C} \), we define

\[
\varphi_{M,N} : \text{Hom}_\mathcal{C} (M, B \otimes N) \to \text{Nat} (M \otimes id_\mathcal{C}, id_\mathcal{C} \otimes N),
\]

\[
\varphi^2_{M,N} : \text{Hom}_\mathcal{C} (M, B \otimes B \otimes N) \to \text{Nat} (M \otimes id_\mathcal{C} \otimes 2, id_\mathcal{C} \otimes 2 \otimes N)
\]

via

\[
\varphi_{M,N} (t)_X = \begin{pmatrix} M & X \\ t & X \\ N \end{pmatrix}, \quad \varphi^2_{M,N} (s)_{X,Y} = \begin{pmatrix} M & X & Y \\ s & X & Y \\ X & Y & N \end{pmatrix}, \tag{3.5}
\]

where \( X, Y \in \mathcal{C} \), \( t \in \text{Hom}_\mathcal{C} (M, B \otimes N) \) and \( s \in \text{Hom}_\mathcal{C} (M, B \otimes B \otimes N) \).

The next two lemmas will show connection between the category \( B\text{-Comod}_\mathcal{C} \) of left \( B \)-comodules in \( \mathcal{C} \) and the center \( \mathcal{Z}_l (\mathcal{C}) \) of \( \mathcal{C} \).
Lemma 3.2 For any $M, N \in C$, $\varphi_{M,N}$ and $\varphi_{M,N}^2$ are isomorphisms natural in both variables.

Proof. Since $C$ is rigid, there is a natural isomorphism

$$\text{Hom}_C (M, X \otimes N) \to \text{Hom}_C (M \otimes *N, X),$$

$$M \otimes t \mapsto M \otimes t \otimes N, \quad t \in \text{Hom}_C (M, X \otimes N) \quad (3.6)$$

with its inverse being

$$M \otimes s \mapsto M \otimes s \otimes N, \quad s \in \text{Hom}_C (M \otimes *N, X),$$

which induces an isomorphism

$$\text{Nat} (M \otimes *N \otimes id_C, id_C) \cong \text{Nat} (M \otimes id_C \otimes *N, id_C) \cong \text{Nat} (M \otimes id_C, id_C \otimes N),$$

$$(M \otimes *N \otimes id_C, id_C) \quad (3.7)$$

Then by a graphical calculation, $\varphi_{M,N}$ is the following composition

$$\text{Hom}_C (M, B \otimes N) \xrightarrow{(3.6)} \text{Hom}_C (M \otimes *N, B) \xrightarrow{\theta_{M \otimes *N}} \text{Nat} (M \otimes *N \otimes id_C, id_C) \xrightarrow{(3.7)} \text{Nat} (M \otimes id_C, id_C \otimes N),$$

which is clearly natural in both variables $M$ and $N$.

Similarly, $\varphi_{M,N}^2$ is the composition of the isomorphisms

$$\text{Hom}_C (M, B \otimes B \otimes N) \to \text{Hom}_C (M \otimes *N, B \otimes B) \xrightarrow{\theta_{M \otimes *N}^2} \text{Nat} (M \otimes *N \otimes id_C \otimes id_C, id_C \otimes id_C) \to \text{Nat} (M \otimes id_C \otimes id_C \otimes N),$$

which is natural in $M$ and $N$. ■
Lemma 3.3  Assume that $C$ is a braided rigid category, and the automorphism braided group $B = U(C)$ exists. For any $M \in C$ and a morphism $\rho_M : M \to B \otimes M$, let $\gamma_{M,\bullet} = \varphi_{M,M}(\rho_M)$ be the natural transformation from $M \otimes id_C$ to $id_C \otimes M$. Then

$$(\Delta_B \otimes id_M) \rho_M = (id_B \otimes \rho_M) \rho_M,$$

if and only if

$$\gamma_{M,X \otimes Y} = (id_X \otimes \gamma_{M,Y}) (\gamma_{M,X} \otimes id_Y), \text{ for any } X, Y \in C.$$  

**Proof.** It is clear that for any $X, Y \in C$, by the definition of $\varphi^2$ and (3.3) the morphism $\varphi^2_{M,M} ((\Delta_B \otimes id_M) \rho_M)_{X,Y}$ is expressed by the diagram

![Diagram 1](#)

while the morphism $\varphi^2_{M,M} ((id_B \otimes \rho_M) \rho_M)_{X,Y}$ is expressed by the diagram

![Diagram 2](#)

Since $\varphi^2_{M,M}$ is an isomorphism by Lemma 3.2, the equality

$$\gamma_{M,X \otimes Y} = (id_X \otimes \gamma_{M,Y}) (\gamma_{M,X} \otimes id_Y)$$

holds for all $X, Y \in C$ if and only if

$$(\Delta_B \otimes id_M) \rho_M = (id_B \otimes \rho_M) \rho_M.$$ 

For any coalgebra $D$ in $C$, let $D$-$Comod_C$ be the category of left $D$-comodules in $C$. Then $D$-$Comod_C$ is a natural right $C$-module category, where for any $(M, \rho_M) \in D$-$Comod_C$ and $X \in C$, the comodule morphism of $M \otimes X$ is

$$\rho_{M \otimes X} = \rho_M \otimes id_X : M \otimes X \to D \otimes M \otimes X. \tag{3.8}$$
Specially $B$-$\text{Comod}_C$ can be viewed as a right $C$-module category in this way.

The category $Z_l(C)$ is also a right $C$-module category, via the tensor functor $C \to Z_l(C), X \mapsto (X, c_{X,\cdot})$. Precisely, for an object $(Z, \gamma_{Z,\cdot})$ of $Z_l(C)$, $(Z \otimes X, \gamma_{Z \otimes X,\cdot})$ is an object of $Z_l(C)$ with $\gamma_{Z \otimes X,\cdot} = (\gamma_{Z,\cdot} \otimes \text{id}_X)(\text{id}_Z \otimes c_{X,\cdot})$.

Now we are ready to prove the main result of this section.

**Theorem 3.4** Let $C$ be a braided rigid category with representability assumption for modules. Let $B = U(C)$ be the automorphism braided group of $C$. For any $M \in C$, we have the following statements.

1) If $(M, \rho_M)$ is a left $B$-comodule in $C$, then $(M, \varphi_{M,M} (\rho_M))$ is an object of $Z_l(C)$.

2) If $(M, \gamma_M, \cdot)$ is an object of $Z_l(C)$, then $(M, \varphi^{-1}_{M,M} (\gamma_M, \cdot))$ is a left $B$-comodule in $C$.

Moreover, as right $C$-module categories $Z_l(C)$ and $B$-$\text{Comod}_C$ are equivalent via

$F : B$-$\text{Comod}_C \to Z_l(C), (M, \rho_M) \mapsto (M, \varphi_{M,M} (\rho_M))$,

$G : Z_l(C) \to B$-$\text{Comod}_C, (M, \gamma_M, \cdot) \mapsto (M, \varphi^{-1}_{M,M} (\gamma_M, \cdot))$.

**Proof.** If $(M, \rho_M) \in B$-$\text{Comod}_C$, then by Lemma 3.3

$$\varphi_{M,M} (\rho_M)_{X \otimes Y} = (\text{id}_X \otimes \varphi_{M,M} (\rho_M)_Y)(\varphi_{M,M} (\rho_M)_X \otimes \text{id}_Y),$$

so $\varphi_{M,M} (\rho_M)$ is an isomorphism by the rigidity of $C$ and Lemma 2.3. Thus $(M, \varphi_{M,M} (\rho_M))$ is an object of $Z_l(C)$.

Conversely, assume that $(M, \gamma_M, \cdot)$ is an object of $Z_l(C)$. Let $\rho_M = \varphi^{-1}_{M,M} (\gamma_M, \cdot)$, then $\gamma_M, \cdot = \varphi_{M,M} (\rho_M)$. Again by Lemma 3.3

$$(\text{id}_B \otimes \rho_M) \rho_M = (\Delta_B \otimes \text{id}_M) \rho_M.$$ 

In addition,

$$\text{id}_M = \gamma_{M,1} = \varphi_{M,M} (\rho_M)_1 = \begin{array}{c} M \\ \otimes \\ M \end{array}.$$ 

Thus $(M, \varphi^{-1}_{M,M} (\gamma_M, \cdot))$ is a $B$-comodule in $C$.

Moreover, let $(M, \rho_M), (N, \rho_N) \in B$-$\text{Comod}_C$ and $f : (M, \rho_M) \to (N, \rho_N)$ be a $B$-comodule map in $C$. Then we have

$$\varphi_{M,N} (\rho_N f) = \varphi_{M,N} ((\text{id}_B \otimes f) \rho_M),$$
that is,\\
\[
\begin{equation}
\begin{array}{c}
\begin{array}{c}
M \\
\downarrow \\
X \\
\downarrow \\
N
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\rho_N \\
\downarrow \\
X \\
\downarrow \\
N
\end{array}
\end{array}
\end{equation}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
M \\
\downarrow \\
X \\
\downarrow \\
N
\end{array}
\end{array}
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\rho_M \\
\downarrow \\
X \\
\downarrow \\
N
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{equation}
\end{array}
\]

which implies that\\
\[
F(f) = f : (M, \varphi_{M,M}(\rho_M)) \rightarrow (N, \varphi_{N,N}(\rho_N))
\]
is a map in \(Z_l(C)\).

If \((M, \gamma_{\bullet,M}), (N, \gamma_{\bullet,N}) \in Z_l(C)\), and \(f : (M, \gamma_{\bullet,M}) \rightarrow (N, \gamma_{\bullet,N})\) is a map in \(Z_l(C)\), then \(G(f) = f : (M, \varphi_{\bullet,M}^{-1}(\gamma_{\bullet,M})) \rightarrow (N, \varphi_{\bullet,N}^{-1}(\gamma_{\bullet,N}))\) is a map in \(B\)-Comod_\(C\). Clearly, \(FG = id\), and \(GF = id\). This establishes the equivalence of \(Z_l(C)\) and \(B\)-Comod_\(C\).

Finally we show that \(F\) is a \(C\)-module functor. Let \((M, \rho_M)\) be an object of \(B\)-Comod_\(C\). For all \(X \in C\), observe that
\[
\varphi_{M\otimes X, M\otimes X}(\rho_M \otimes id_X) = (\varphi_{M,M}(\rho_M) \otimes id_X)(id_M \otimes c_{X,\bullet}).
\]

Then \(F(M \otimes X) = F(M) \otimes X\), and thus \((F, s)\) is a \(C\)-module with \(s_{M,X} = id_{F(M \otimes X)}\).

**Remark 3.5** If \(C\) is not strict, the same argument of Theorem 3.4 is also true. The proof is similar but quite lengthy, we leave this for an interested reader.

## 4 The Center of Braided Multifusion Categories — A Decomposition Theorem

In this section, \(C\) will be a finite braided multitensor category. We assume that for \(C\) the module representability assumption holds. Let \(B\) be the automorphism braided group \(U(C)\). We will show that any direct sum decomposition of \(B\) in \(B\)-Comod_\(C\) induces a decomposition of the category \(B\)-Comod_\(C\) into a direct sum of \(C\)-module subcategories.

Let \(i : D \rightarrow C\) be a monomorphism in \(C\). Since the bifunctor \(\otimes : C \times C \rightarrow C\) is exact in both factors, \((D \otimes C, i \otimes id_C)\) and \((C \otimes D, id_C \otimes i)\) are subobjects of \(C \otimes C \in C\). We show that \((D \otimes D, i \otimes i)\) is the intersection of subobjects \(D \otimes C\) and \(C \otimes D\) in the following lemma, i.e., \((D \otimes D, i \otimes id_D, id_D \otimes i)\) is the pullback of the monomorphisms \(i \otimes id_C\) and \(id_C \otimes i\).
Lemma 4.1 Let $0 \rightarrow D_j \xrightarrow{i_j} C_j \xrightarrow{f_j} E_j$ ($j = 1, 2$) be exact sequences in $\mathcal{C}$. If $g : X \rightarrow C_1 \otimes C_2$ is a morphism in $\mathcal{C}$ with $(f_1 \otimes \text{id}_{C_2}) g = 0$ and $(\text{id}_{C_1} \otimes f_2) g = 0$, then there exists a unique morphism $h : X \rightarrow D_1 \otimes D_2$, such that $g = (i_1 \otimes i_2) h$.

Moreover, $(D_1 \otimes D_2, i_1 \otimes i_2)$ is the intersection of the subobjects $D_1 \otimes C_2$ and $C_1 \otimes D_2$.

Proof. Consider the diagram

It is trivial that the two bottom parallelograms commute. The exactness of the tensor product implies $(D_1 \otimes C_2, i_1 \otimes \text{id}_{C_2})$, $(C_1 \otimes D_2, \text{id}_{C_1} \otimes i_2)$ are respectively the kernel of $f_1 \otimes \text{id}_{C_2}$ and the kernel of $\text{id}_{C_1} \otimes f_2$. Since $(f_1 \otimes \text{id}_{C_2}) g = 0$ by assumption, there is a unique morphism $g_1 : X \rightarrow D_1 \otimes C_2$ such that $g = (i_1 \otimes \text{id}_{C_2}) g_1$. Then

$$(i_1 \otimes \text{id}_{E_2}) (i_1 \otimes \text{id}_{C_2}) g_1 = (i_1 \otimes \text{id}_{C_2}) g_1 = (\text{id}_{C_1} \otimes f_2) g_1 = 0,$$

and we have $(\text{id}_{D_1} \otimes f_2) g_1 = 0$, since $i_1 \otimes \text{id}_{E_2}$ is monic. So there exists a morphism $h : X \rightarrow D_1 \otimes D_2$ such that $g_1 = (\text{id}_{D_1} \otimes i_2) h$. It is clear that

$$g = (i_1 \otimes \text{id}_{C_2}) g_1 = (i_1 \otimes \text{id}_{C_2}) (\text{id}_{D_1} \otimes i_2) h = (i_1 \otimes i_2) h.$$

As the morphism $i \otimes i$ is monic, $h$ is unique. □

We have known from [20] that $B$ is $\mathcal{C}$-cocommutative in the sense that for every object $X \in \mathcal{C}$, the $B$-action $\alpha_X$ on $X$ satisfies the following identity

$$(\text{id}_B \otimes \alpha_X) (\Delta_B \otimes \text{id}_X) = (\text{id}_B \otimes \alpha_X) (c_{B,B} \otimes \text{id}_X) (\text{id}_B \otimes c_{X,B} \alpha_X) (\Delta_B \otimes \text{id}_X),$$

that is,

$$\alpha_X = \ldots \quad (4.1)$$
Note that \((B, \Delta_B) \in B\text{-Comod}_C\). The next proposition shows that a subobject (subcomodule) of \(B \in B\text{-Comod}_C\) is also a subcoalgebra of \(B\) in \(C\).

**Theorem 4.2** Let \(i : (D, \rho_D) \to (B, \Delta_B)\) be a subobject of \(B \in B\text{-Comod}_C\). Then

1) there exists a unique \(C\)-coalgebra structure on \(D\) such that \(i\) is a coalgebra morphism (i.e., \(D\) is a subcoalgebra of \(B\)),

2) the category \(D\text{-Comod}_C\) is a \(C\)-module subcategory of \(B\text{-Comod}_C\).

**Proof.**

1) Let \((E, f)\) be the cokernel of \(i\) in \(C\). Then \(f : B \to E\) and \(\ker f = i\). Since \(i\) is a \(B\)-comodule morphism, we have \((id_B \otimes f) \Delta_B i = (id_B \otimes f i) \rho_D = 0\).

We claim that \((f \otimes id_B) \Delta_B i = 0\). Since \(C\) is rigid, there exists a natural isomorphism

\[
\zeta : \text{Hom}_C (D, E \otimes B) \to \text{Nat} (D \otimes id_C, E \otimes id_C)
\]

via the composition of following isomorphisms

\[
\text{Hom}_C (D, E \otimes B) \xrightarrow{\theta_{E^* \otimes D}} \text{Hom}_C (E^* \otimes D, B) \xrightarrow{} \text{Nat} (E^* \otimes D \otimes id_C, id_C) \xrightarrow{} \text{Nat} (D \otimes id_C, E \otimes id_C).
\]

Denoted \(\rho_D\) by \(\bullet\), then \(\bullet = \bullet\). So we have

\[
\bullet = 0,
\]

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which shows that \( \zeta((f \otimes \text{id}_B) \Delta_Bi) = 0 \), where \( \zeta \) is the isomorphism \[4.2\]. Thus 
\( (f \otimes \text{id}_B) \Delta_Bi = 0 \). By Lemma \[4.1\] there exists a unique \( \Delta_D : D \to D \otimes D \), such that 
\( \Delta_Bi = (i \otimes i) \Delta_D \).

Let \( \varepsilon_D = \varepsilon_{BI} : D \to 1 \). We need to check that \( (D, \Delta_D, \varepsilon_D) \) is a \( C \)-coalgebra. First, it follows from the counit axiom of \( B \) that 
\( (\varepsilon_D \otimes i) \Delta_D = (\varepsilon_B \otimes \text{id}_B)(i \otimes i) \Delta_D = (\varepsilon_B \otimes \text{id}_B) \Delta_Bi = i \).

Since \( i \) is monic, 
\( (\varepsilon_D \otimes \text{id}_D) \Delta_D = \text{id}_D \). Similarly, 
\( (\text{id}_D \otimes \varepsilon_D) \Delta_D = \text{id}_D \). To show the coassociativity, it suffices to show that 
\( (i \otimes i \otimes i)(\Delta_D \otimes \text{id}_D) \Delta_D = (i \otimes i \otimes i)(\text{id}_D \otimes \Delta_D) \Delta_D \),

which follows directly from the coassociativity of \( B \). Consequently, \( (D, \Delta_D, \varepsilon_D) \) is a coalgebra in \( C \), and \( i : D \to B \) is a coalgebra map.

2) Let \( (V, \tilde{\rho}_V) \) be a left \( D \)-comodule in \( C \). Then \( V \) is a left \( B \)-comodule via \( \rho_V = (i \otimes \text{id}_V)\tilde{\rho}_V \). For \( V, W \in \text{D-Comod}_C \), \( \text{Hom}^D_C(V, W) = \text{Hom}^B_C(V, W) \). So the category \( D\text{-Comod}_C \) is a full subcategory of \( B\text{-Comod}_C \), and it’s clearly closed under the \( C \)-module product. Thus \( D\text{-Comod}_C \) is a \( C \)-module subcategory of \( B\text{-Comod}_C \).

In the following proposition, we give some equivalence conditions for the indecomposability of \( D\text{-Comod}_C \).

**Proposition 4.3** Let \( i : (D, \rho_D) \to (B, \Delta_B) \) be a subobject of \( B \in \text{B-Comod}_C \). Then the following statements are equivalent.

1) \( D \) is indecomposable in \( \text{B-Comod}_C \).

2) \( D \) is indecomposable in \( \text{D-Comod}_C \).

3) \( D \) is an indecomposable \( C \)-coalgebra.

4) The \( C \)-module category \( D\text{-Comod}_C \) is indecomposable.

**Proof.** Obviously, in \( C \), each subcoalgebra of \( D \) is a \( D \)-subcomodule of \( D \), and each \( D \)-subcomodule of \( D \) is a \( B \)-subcomodule of \( D \), so the implications \( (1) \Rightarrow (2) \Rightarrow (3) \) are clear.

Given a \( B \)-subcomodule \( (D_1, \rho_1) \) of \( D \) with monomorphism \( j : D_1 \to D \), \( D_1 \) is clearly a \( B \)-subcomodule of \( B \), and thus it’s a subcoalgebra of \( B \) by Theorem \[4.2\]. Thus there exists a coproduct \( \Delta_1 : D_1 \to D_1 \otimes D_1 \) in \( C \), such that 
\[(ij \otimes ij) \Delta_1 = \Delta_Bij, \rho_1 = (ij \otimes \text{id}_{D_1}) \Delta_1,\]
and a counit \( \varepsilon_1 = \varepsilon_Di_j = \varepsilon_Dj \). It’s easy to see that \( j : D_1 \to D \) is a coalgebra map.

If we assume further that \( j \) splits in \( B\text{-Comod}_C \) and \( p \) is a retraction of \( j \), then we have \( \Delta p = (p \otimes p) \Delta_D \). It follows that if \( D \) is decomposable in \( B\text{-Comod}_C \), then \( D \) is decomposable as \( C \)-coalgebras. So we get (3)\( \Rightarrow \)(1).

(2)\( \Rightarrow \)(4). Assume that \( D\text{-Comod}_C = \mathcal{M}_1 \oplus \mathcal{M}_2 \), where \( \mathcal{M}_1, \mathcal{M}_2 \) are nontrivial \( C \)-module subcategories of \( D\text{-Comod}_C \).

For any \( M \in D\text{-Comod}_C \), there exist \( M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2 \) such that \( M = M_1 \oplus M_2 \). If \( M, N \in D\text{-Comod}_C \), and \( f \in \text{Hom}_C^D(M, N) \), then \( f = (f_1, f_2) \), where \( f_1 \in \text{Hom}_{\mathcal{M}_1}(M_1, N_1) \); \( f_2 \in \text{Hom}_{\mathcal{M}_2}(M_2, N_2) \), \( M = M_1 \oplus M_2, N = N_1 \oplus N_2 \).

As an object of \( D\text{-Comod}_C \), \( D = D_1 \oplus D_2 \), where \( D_1 \in \mathcal{M}_1, D_2 \in \mathcal{M}_2 \). Take a nonzero object \( N \in \mathcal{M}_1 \), then the object \((D \otimes N, \rho_D \otimes id_N) \in D\text{-Comod}_C \), and \( \rho_N : N \to D \otimes N \) is a \( D \)-comodule map. For \( i = 1, 2 \), \((D \otimes N)_i = D_i \otimes N \in \mathcal{M}_i \), as \( \mathcal{M}_i \) is closed under right \( C \)-module product. So \( \rho_N = (\rho_N)_1 : N \to D_1 \otimes N \). Note that \( \rho_N \) is monic, so \( D_1 \neq 0 \).

Similarly, \( D_2 \neq 0 \). Hence, \( D \) is decomposable in \( D\text{-Comod}_C \).

(4)\( \Rightarrow \)(1). Assume that \( D = D_1 \oplus D_2 \) is a direct sum of \( B \)-subcomodules in \( C \). For \( j = 1, 2 \), let \( i_j : D_j \to D \) and \( p_j : D \to D_j \) be the canonical injections and projections. Then the direct sum \( D = D_1 \oplus D_2 \) can be viewed as in category \( D\text{-Comod}_C \), and also as \( C \)-coalgebras.

Given a left \( D \)-comodule \((M, \rho_M) \) in \( C \), define maps

\[ f_j = (\varepsilon_D_i j \otimes id_M) \rho_M, \quad j = 1, 2. \]

Since \( \varepsilon_D = \varepsilon_Di_j, f_1 + f_2 = (\varepsilon_D \otimes id_M) \rho_M = id_M \). We easily get that

\[ \rho_M f_j = (\varepsilon_D, i j \otimes id_D \otimes id_M) \rho_M = id_D \otimes \rho_M \rho_M = \rho_M. \]

Thus \( f_j f_l = \delta_{j l}f_j \), for \( j, l = 1, 2 \). It’s then easy to verify that \( f_1, f_2 \) are \( D \)-colinear. Therefore \( \{f_1, f_2\} \) is a complete set of orthogonal idempotents in \( \text{End}_C^D(M) \).

Now setting \( M_j = \text{Im} f_j \), we have \( M = M_1 \oplus M_2 \) as \( D \)-comodules. Let \( \rho_j \) be the \( D \)-coaction on \( M_j \). Then one may check that \( M_j \in D_j\text{-Comod}_C \) via

\[ \tilde{\rho}_j : M_j \xrightarrow{\rho_j} D \otimes M_j \xrightarrow{i_j \otimes id_M} D_j \otimes M_j, \]

and that \( \rho_j = (i_j \otimes id_M) \tilde{\rho}_j \).

Let \((N_1, \tilde{\rho}_{N_1}) \in D_1\text{-Comod}_C, (N_2, \tilde{\rho}_{N_2}) \in D_2\text{-Comod}_C \). Then \( N_j \) can be viewed as a natural left \( D \)-comodule via \( \rho_{N_j} = (i_j \otimes id_{N_j}) \tilde{\rho}_{N_j} \). For any \( f \in \text{Hom}_C^D(N_1, N_2) \), we have \( \rho_{N_2} f = (id_D \otimes f) \rho_{N_1} \). Applying \( i_2 p_2 \otimes id_{N_2} \) to both side, we get

\[ \rho_{N_2} f = (i_2 p_2 \otimes f) \rho_{N_1} = (i_2 p_2 i_1 \otimes f) \tilde{\rho}_{N_1} = 0, \]
and thus \( f = 0 \) and \( \text{Hom}_C^D(N_1, N_2) = 0 \). Similarly, \( \text{Hom}_C^D(N_2, N_1) = 0 \). So

\[
D\text{-Comod}_C = D_1\text{-Comod}_C \oplus D_2\text{-Comod}_C
\]

as \( C \)-module categories, and (4) \( \Rightarrow \) (1) is done. ■

Now assume that \( k \) is an algebraically closed field of characteristic zero, and \( C \) is a finite braided multifusion category. Note that \( C \) has a natural module category structure over \( C \otimes C^{\text{op}} \), and the dual category is the Drinfeld center \( Z_l(C) \) (see [12, Corollary 3.37]). It due to Etingof, Nikshych and Ostrik [11, Theorem 2.18] that for any module category \( \mathcal{M} \) over a multifusion category \( C \) the dual category \( C^\ast_{/\mathcal{M}} \) is semisimple. In particular, the Drinfeld center \( Z_l(C) \) of \( C \) is semisimple. By Theorem 3.4, the category \( B\text{-Comod}_C \cong Z_l(C) \) is semisimple.

As an object of \( B\text{-Comod}_C \), \( B \) is a direct sum of simple subobjects. By Proposition 4.3 each simple subobject of \( B \) is an indecomposable coalgebra in \( C \), and the following proposition is immediate.

**Proposition 4.4** Let \( C \) be a finite braided multifusion category over an algebraically closed field \( k \) of characteristic zero, and \( B \cong D_1 \oplus D_2 \oplus \cdots \oplus D_r \) be a direct sum of simple objects in \( B\text{-Comod}_C \), then \( Z_l(C) \cong B\text{-Comod}_C \cong \bigoplus_{j=1}^r D_j\text{-Comod}_C \) is a direct sum of indecomposable \( C \)-module subcategories.

If \( H \) is a semisimple quasi-triangular Hopf algebra and \( C = H\mathcal{M} \) is the category of finite dimensional representations, this decomposition has already appeared in the authors’ paper [18], as in the following example.

**Example 4.5** ([18]) Let \((H, R)\) be a semisimple quasi-triangular Hopf algebra. The automorphism braided group \( H_R \) of \( C = H\mathcal{M} \) is constructed as follows. As an \( H \)-module algebra, \( H_R = H \) with the left adjoint action \( \cdot_{ad} \). The comultiplication and antipode are defined by

\[
\Delta_R(h) = h_{(1)}S(R^2) \otimes R^1 \cdot_{ad} h_{(2)}, \quad S_R(h) = R^2 S(R^1 \cdot_{ad} h), \quad h \in H.
\]

The decomposition of the automorphism braided group \( H_R \) is the unique decomposition \( H_R = D_1 \oplus \cdots \oplus D_r \) of the minimal \( H \)-adjoint-stable subcoalgebras \( D_1, \ldots, D_r \) of \( H_R \), and the category \( H^r \mathcal{YD} = Z_l(C) \cong H^r\mathcal{M} \cong D_1\mathcal{M} \oplus \cdots \oplus D_r\mathcal{M} \) is a direct sum of indecomposable right \( C \)-module subcategories.

In literature [27, 12], the concept of internal Hom plays a crucial role in the study of module categories. Once the internal Hom is determined, Theorem 2.1, Theorem 2.2 can be applied to characterize indecomposable \( C \)-module subcategories.

Now let \( C \) be a multitensor category and \( D \) be a coalgebra in \( C \). Naturally, \( D\text{-Comod}_C \) is a right \( C \)-module category. We end this section by presenting a characterization of the internal Hom for \( D\text{-Comod}_C \).

First we need the notion of cotensor product over a coalgebra \( D \) in \( C \).
Definition 4.6 Let $M, N$ be respectively a right $D$-comodule and a left $D$-comodule in $\mathcal{C}$ with structure maps $\rho_M, \rho_N$. The cotensor product $M \square^C_D N$ of $M$ and $N$ over $D$ is the equalizer of the diagram

$$M \square^C_D N \subseteq M \otimes N \xrightarrow{\rho_M \otimes id_N} M \otimes D \otimes N.$$  \hfill (4.4)

That is, $M \square^C_D N$ is the kernel of the morphism $\rho_M \otimes id_N - id_M \otimes \rho_N$.

Let $(M, \rho_M) \in D\text{-Comod}_C$. Then $^*M$ has a natural right $D$-comodule structure $\rho_{^*M}$, which is the image of $\rho_M$ under the composition of the isomorphisms

$$\text{Hom}_C (M, D \otimes M) \xrightarrow{\cong} \text{Hom}_C (D^* \otimes M, M) \xrightarrow{\cong} \text{Hom}_C (^*M, ^*M \otimes D).$$

The graphical representation of $\rho_{^*M}$ is

$$\rho_{^*M} = \begin{array}{c}
\begin{array}{c}
M
\end{array}
\end{array}$$

\begin{array}{c}
\begin{array}{c}
^*M
\end{array}
\end{array}$$

\begin{array}{c}
\begin{array}{c}
M
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\rho_M
\end{array}
\end{array}$$


Proposition 4.7 Let $M, N \in D\text{-Comod}_C$. Then $\text{Hom}_C (M, N) = ^*M \square^C_D N$, i.e., the functor $^*M \square^C_D \bullet : D\text{-Comod}_C \to \mathcal{C}$ is a right adjoint of $M \otimes \bullet$.

Proof. It suffices to show that there is a natural isomorphism

$$\text{Hom}_C (X, ^*M \square^C_D N) \cong \text{Hom}_C (M \otimes X, N),$$

natural in $X \in \mathcal{C}$. We will show that the required isomorphism can be deduced from the composition

$$\text{Hom}_C (X, ^*M \square^C_D N) \xrightarrow{j_*} \text{Hom}_C (X, ^*M \otimes N) \xrightarrow{\cong} \text{Hom}_C (M \otimes X, N),$$ \hfill (4.5)

where $j : ^*M \square^C_D N \to ^*M \otimes N$ is the natural monomorphism in (4.4). We need to show that the image of this composition is equal to $\text{Hom}_C (M \otimes X, N)$. Let $f \in \text{Hom}_C (X, ^*M \otimes N)$, then $f \in \text{Im} j_*$ if and only if

$$(\rho_M \otimes id_N)f = (id_M \otimes \rho_N)f.$$ \hfill (4.6)
The graphical expression of (4.6) is

\[
\begin{align*}
\text{equivalently, under the isomorphism } & \text{Hom}_C (X, *M \boxtimes N) \xrightarrow{\cong} \text{Hom}_C (M \otimes X, N) \text{ the image of } f \text{ is in } \text{Hom}_C^D (M \otimes X, N) \subseteq \text{Hom}_C (M \otimes X, N). \text{ Hence, we get the isomorphism } \\
& \text{Hom}_C (X, *M \boxtimes N) \xrightarrow{\cong} \text{Hom}_C^D (M \otimes X, N), \text{ and the naturality in } X \text{ is obvious.} \quad \text{□}
\end{align*}
\]

**Remark 4.8** The internal Hom for the category Mod$_C$-A for a $C$-algebra $A$ was calculated by Etingof and Ostrik [12, Example 3.19]. The proposition is a dual version of their result.

**Remark 4.9** ([18]) If we take $C = _H \mathcal{M}$, the category of finite dimensional representations of a Hopf algebra $H$, and take $D$ an $H$-module coalgebra, then $\text{Hom} (M, N) = \text{Hom}_D^C (M, N)$, for any $M, N \in D$-Comod$_C$.

Recall that when the appropriated internal Hom objects exist, there are definitions of evaluation morphism, multiplication morphism of internal Homs. If in the case that we can identified the internal Homs as the cotensors the evaluation, the multiplication morphism has the following form.

For any $M_1, M_2, M_3 \in D$-Comod$_C$, the evaluation morphism is the composition

\[
ev'_{M_1, M_2} : M_1 \otimes (*M_1 \boxtimes_D M_2) \hookrightarrow M_1 \otimes *M_1 \otimes M_2 \xrightarrow{ev_{*M_1 \otimes id_{M_2}}} M_2,
\]

and the multiplication morphism of internal Hom is defined as the preimage of the map

\[
(*M_1 \boxtimes_D M_2) \otimes (*M_2 \boxtimes_D M_3) \hookrightarrow *M_1 \otimes M_2 \otimes *M_2 \otimes M_3 \xrightarrow{id_{*M_1} \otimes ev'_{M_2} \otimes id_{M_3}} *M_1 \otimes M_3,
\]

under the map

\[
j_* : \text{Hom}_C (**M_1 \boxtimes_D M_2) \otimes (**M_2 \boxtimes_D M_3), *M_1 \boxtimes_D M_3)
\to \text{Hom}_C (**M_1 \boxtimes_D M_2) \otimes (**M_2 \boxtimes_D M_3), *M_1 \otimes M_3),
\]

where $j : *M \boxtimes_D N \to *M \otimes N$ is the natural monomorphism. It makes $A = *M \boxtimes_D M$ into an algebra in $\mathcal{C}$, and $*M \boxtimes_D N$ a left $A$-module, for $M, N \in D$-Comod$_C$. Now apply theorem 2.1 and 2.2 with $\mathcal{M} = D$-Comod$_C$, we get:
Proposition 4.10 Let $C$ be a finite multitensor category, and let $D$ be a cosemisimple coalgebra in $C$. If $M$ is a generator of $D$-$\text{Comod}_C$, then $A = *M \square^C_D M$ is a semisimple algebra in $C$, and the functors

$$F = *M \square^C_D \bullet : D$-$\text{Comod}_C \to A$-$\text{Mod}_C$$

and $G = M \otimes_A \bullet : A$-$\text{Mod}_C \to D$-$\text{Comod}_C$

establish an equivalence between $C$-module categories $D$-$\text{Comod}_C$ and $A$-$\text{Mod}_C$.

To sum up, we have the following theorem.

Theorem 4.11 Let $C$ be a braided finite multitensor category, and $B$ be the automorphism braided group of $C$. As an object of $Z_l(C) = B$-$\text{Comod}_C$, write $B = B_1 \oplus \cdots \oplus B_r$ as a direct sum of indecomposable subobjects.

1) Then the decomposition $B = B_1 \oplus \cdots \oplus B_r$ is unique as a direct sum of indecomposable subobjects, and it is also unique as a direct sum of indecomposable $C$-subcoalgebras.

2) The category $Z_l(C)$ admits a unique decomposition

$$Z_l(C) = B_1$-$\text{Comod}_C \oplus \cdots \oplus B_r$-$\text{Comod}_C$$

into the direct sum of indecomposable $C$-module subcategories.

3) For each $1 \leq i \leq r$, let $M_i \in B_i$-$\text{Comod}_C$ be a nonzero object, and $A_i = *M_i \square^C_{B_i} M_i$. Then $F_i = *M_i \square^C_{B_i} \bullet : B_i$-$\text{Comod}_C \to A_i$-$\text{Mod}_C$ is an equivalence between $C$-module categories $B_i$-$\text{Comod}_C$ and $A_i$-$\text{Mod}_C$.

Proof. The statements 1) and 2) follows from Theorem 4.2, the proof of Proposition 4.3; 3) follows from Proposition 4.10. ■

5 An Application to Weak Hopf Algebras

In this section, we will visualize the results in the previous two sections by using the theory of weak Hopf algebras. Weak Hopf algebras was introduced by Böhm, Nill, and Szlachányi [5] and studied extensively by Nikshych and Vainerman [26]. The category of finite-dimensional representations of a semisimple weak Hopf algebra is a multifusion category. On the other hand, it is due to Hayashi [14] and Szlachányi [28] that any multifusion category is equivalent to the category of finite-dimensional representations of a regular semisimple weak Hopf algebra. For this reason, the theory of weak Hopf algebras is not merely good examples for categorical construction but also a helpful tool for discussing multifusion categories.
This section is arranged as follows. We begin by recalling some preliminaries of weak Hopf algebras in Section 5.1. We then discuss some properties of module (co)algebras for weak Hopf algebras in Section 5.2. Next, we consider the braided multitensor category $\mathcal{C} = H\mathcal{M}$, where $H$ is a quasi-triangular weak Hopf algebra. We present Majid’s braided reconstruction with $\mathcal{C} = H\mathcal{M}$, and obtain the automorphism braided group $U(\mathcal{C})$ in Section 5.3, and give the structure of irreducible Yetter-Drinfeld modules over $H$ in Section 5.4.

5.1 Preliminaries of Weak Hopf Algebras

Now we recall the definition of weak Hopf algebra, quasi-triangular weak Hopf algebra, and some basic properties. Our references are [5, 25]. We will use the sigma notation: $\Delta(h) = h(1) \otimes h(2)$ for coproduct and $\rho(m) = m(0) \otimes m(1)$ for right coaction (or $\rho(m) = m(-1) \otimes m(0)$ for left coaction).

A weak Hopf algebra $H$ over $k$ is a $k$-algebra and also a $k$-coalgebra with an antipode $S : H \to H$, such that

1) $\Delta(xy) = \Delta(x)\Delta(y),$

2) $\Delta^2(1) = 1(1) \otimes 1(2)1(1') \otimes 1(2') = 1(1) \otimes 1(1')1(2) \otimes 1(2'),$

3) $\varepsilon(xyz) = \varepsilon(xy(1))\varepsilon(y(2)z) = \varepsilon(xy(2))\varepsilon(y(1)z),$

4) $x(1)S(x(2)) = \varepsilon(1(1)x)1(2),$

5) $S(x(1))x(2) = 1(1)\varepsilon(x1(2)),$

6) $S(x) = S(x(1))x(2)S(x(3)),$

for all $x, y, z \in H$, where $\Delta$ is the coproduct and $\varepsilon$ is the counit.

The target and the source counital maps $\varepsilon_t, \varepsilon_s : H \to H$ are defined by

$$\varepsilon_t(x) = \varepsilon(1(1)x)1(2), \quad \varepsilon_s(x) = 1(1)\varepsilon(x1(2)),$$

for all $x \in H$. The images of these counital maps, denoted by $H_t = \varepsilon_t(H)$ and $H_s = \varepsilon_s(H)$.

A weak Hopf algebra $H$ is regular if the restriction of $S^2$ on $H_tH_s$ is identity map. We will always assume that the weak Hopf algebras we considered are regular.
If $H$ is a weak Hopf algebra, for all $g, h \in H$, the following conditions hold:

\begin{align*}
\varepsilon (hg) &= \varepsilon (\varepsilon_s (h) g) = \varepsilon (h \varepsilon_t (g)), \\
\varepsilon_t \circ S &= \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s, \\
\varepsilon_s \circ S &= \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t, \\
1_{(1)} h \otimes 1_{(2)} &= h_{(1)} \otimes \varepsilon_t (h_{(2)}), \\
1_{(1)} \otimes h_{1(2)} &= \varepsilon_s (h_{(1)}) \otimes h_{(2)}. 
\end{align*}

(5.1) (5.2) (5.3) (5.4) (5.5)

Let $H$ be a weak Hopf algebra, and $\mathcal{C} = _H\mathcal{M}$ be the category of finite dimensional left $H$-modules. For any $M, N \in _H\mathcal{M}$,

\[ M \otimes_t N = \Delta (1) (M \otimes_k N) \subseteq M \otimes_k N \]

with $H$-action given by $h \cdot (1_{(1)} m \otimes 1_{(2)} n) = h_{(1)} m \otimes h_{(2)} n$, for $h \in H$, $m \in M$, $n \in N$. The subalgebra $H_t$ is an $H$-module via $h \cdot z = \varepsilon_t (hz)$, where $h \in H$, $z \in H_t$. Furthermore, $H_t$ is the unit object of $\mathcal{C}$. The functorial unit isomorphism $l_X : H_t \otimes_t X \rightarrow X$ and $r_X : X \otimes_t H_t \rightarrow X$ are defined by

\[ l_X (1_{(1)} z \otimes 1_{(2)} x) = z x, \quad r_X (1_{(1)} x \otimes 1_{(2)} z) = S (z) x, \quad z \in H_t, x \in X. \]

Then $(_H\mathcal{M}, \otimes_t, H_t, l, r)$ is a monoidal category. Using the isomorphism $l_X$ and $r_X$ identifying $H_t \otimes_t X$, $X \otimes_t H_t$ and $X$, we see that the monoidal category $_H\mathcal{M}$ is strict. In addition, if $M \in _H\mathcal{M}$, then there is a canonical $H_t$-$H_t$-bimodule structure on $M$, such that $M \otimes_t N \cong M \otimes_{H_t} N$ (see. \[H\]).

The monoidal category $\mathcal{H}_M$ has left duality. For any $X \in \mathcal{H}_M$, the left dual of $X$ is $X^* = \text{Hom}_k (X, k)$, considered as an object of $\mathcal{H}_M$ via

\[ \langle hx^*, x \rangle = \langle x^*, S (h) x \rangle, \quad \forall x \in X, x^* \in X^*, h \in H. \]

The evaluation map $ev_X : X^* \otimes_t X \rightarrow H_t$ and the coevaluation map $coev_X : H_t \rightarrow X \otimes_t X^*$ are defined as follows:

\[ ev_X (1_{(1)} x^* \otimes 1_{(2)} x) = \langle x^*, 1_{(1)} x \rangle 1_{(2)}, \quad coev_X (z) = \sum_i z_{(1)} x_i \otimes z_{(2)} x_i^*, \]

where $\{ (x_i, x_i^*) \}_i$ is a dual basis of $X$.

Recall that a quasi-triangular weak Hopf algebra is a pair $(H, R)$, where $H$ is a weak Hopf algebra and $R \in \Delta^{op} (1) (H \otimes_k H) \Delta (1)$ satisfying the following conditions:

1) $R \Delta (h) = \Delta^{op} (h) R$, for all $h \in H$. 

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2) \( (id \otimes \Delta) R = R_{13} R_{12}, \ \ (\Delta \otimes id) R = R_{13} R_{23} \), where \( R_{12} = R \otimes 1, \ R_{23} = 1 \otimes R, \ R_{13} = R^1 \otimes 1 \otimes R^2 \).

3) There exists \( \bar{R} \in \Delta \) \((H \otimes_k H)\Delta^{\text{op}}(1)\) with \( R \bar{R} = \Delta^{\text{op}}(1), \ \bar{R} R = \Delta(1) \).

The element \( R \) is called an R-matrix of \( H \). We write \( R = R_i = R_i^1 \otimes R_i^2, \ \forall i \in \mathbb{N}^+ \).

Let \((H, R)\) be a quasi-triangular weak Hopf algebras, then the following properties hold:

\[
\begin{align*}
(\varepsilon_s \otimes id) (R) &= \Delta(1), \\
(id \otimes \varepsilon_s) (R) &= (S \otimes id) \Delta^{\text{op}}(1), \\
(S \otimes id) (R) &= \bar{R}, \\
\varepsilon_C(R) &= \varepsilon(t)(\varepsilon_s(h) \cdot c).
\end{align*}
\]

(5.6) \( (id \otimes \varepsilon_t) (R) = (S \otimes id) \Delta(1), \) \( (S \otimes S) (R) = R. \) \( \) (5.7) \( (S \otimes id) (R) = \bar{R}, \) \( (id \otimes S) (R) = R. \) \( \) (5.8)

It’s known from \([25, \text{Proposition 5.2}]\) that if \((H, R)\) is a quasi-triangular weak Hopf algebras, then the monoidal category \( \mathcal{C} = \mathcal{H} \mathcal{M} \) is braided with a braiding

\[
c_{M,N}(1_m \otimes 1_n) = R^2 l_2 n \otimes R^1 l_1 m, \ \text{for} \ m \in M, \ n \in N, \ \text{where} \ M, N \in \mathcal{H} \mathcal{M}.
\]

5.2 Module (Co)algebras and Weak Smash Products

If \( H \) is a Hopf algebra, a coalgebra \( C \) in the monoidal category \( \mathcal{C} = \mathcal{H} \mathcal{M} \) is simply a left \( H \)-module coalgebra, and left \( C \)-comodules in \( \mathcal{C} \) are left \((C, H)\)-Hopf modules. For weak Hopf algebras, some categorical notions and formulaic notions, including cotensors, need to be reconciled. From now on, \( H \) is a finite dimensional weak Hopf algebra over \( k \), and \( \mathcal{C} \) is the monoidal category \( \mathcal{H} \mathcal{M} \).

A \( k \)-coalgebra \((C, \Delta_C, \varepsilon_C)\) is a left \( H \)-module coalgebra \([3]\) if \( C \) is a left \( H \)-module via \( h \otimes c \mapsto h \cdot c \) and for all \( h \in H, \ c \in C \)

\[
\Delta_C(h \cdot c) = h(1) \cdot c(1) \otimes h(2) \cdot c(2),
\]

\( \varepsilon_C(h \cdot c) = \varepsilon_C(\varepsilon_s(h) \cdot c). \) \( \) (5.9) \( \) (5.10)

A left \( H \)-module \( M \) is a left \((C, H)\)-Hopf module \([3]\) if it is a left \( C \)-comodule such that the compatibility condition

\[
\rho(h \cdot m) = h(1) \cdot m_{(-1)} \otimes h(2) \cdot m_{(0)},
\]

(5.11) holds for any \( m \in M, \ h \in H \).

Observe that \( \boxed{\text{(5.9)}} \) implies \( \Delta_C(c) = 1_{(1)} \cdot c(1) \otimes 1_{(2)} \cdot c(2) \in C \otimes \epsilon C \), hence \( \boxed{\text{(5.9)}} \) holds \( \Leftrightarrow \Delta_C \) is a morphism in \( \mathcal{H} \mathcal{M} \). Analogously, \( \boxed{\text{(5.11)}} \) holds \( \Leftrightarrow \rho \) is a morphism in \( \mathcal{C} \).

Let \( M \) be a left \( H \)-module, the invariants of \( M \) is the subspace

\[
\text{Inv } M = \{ m \in M \mid h \cdot m = \varepsilon_t(h) \cdot m, \forall h \in H \}.
\]
Note that $M^* = \text{Hom}_k(M,k)$ is also a left $H$-module via $\langle h \cdot m^*, m \rangle = \langle m^*, S(h) \cdot m \rangle$, for $h \in H$, $m \in M$, $m^* \in M^*$. It is routine to check that

$$\text{Inv} M^* = \{ m^* \in M^* \mid \langle m^*, h \cdot m \rangle = \langle m^*, \varepsilon_s(h) \cdot m \rangle, \forall h \in H, m \in M \}.$$  

**Lemma 5.1** For any $M \in C$, the linear map $\beta_M : \text{Inv} M^* \to \text{Hom}_H(M,H_1)$ defined by

$$\beta_M(m^*)(m) = \langle m^*, 1_{(1)} \cdot m \rangle 1_{(2)}, \forall m \in M, m^* \in \text{Inv} M^*,$$

is an isomorphism.

**Proof.** First $\beta_M$ is an $H$-module map, since for any $m^* \in \text{Inv} M^*$, $h \in H$, $m \in M$,

$$\beta_M(m^*)(h \cdot m) = \langle m^*, (1_{(1)}h) \cdot m \rangle 1_{(2)} = \langle m^*, h_{(1)} \cdot m \rangle \varepsilon (h_{(2)})
= \langle m^*, \varepsilon_s(h_{(1)}) \cdot m \rangle \varepsilon (h_{(2)})
= \langle m^*, 1_{(1)} \cdot m \rangle \varepsilon (h_{(2)})
= h \cdot (\beta_M(m^*)(m)).$$

On the other hand, for all $f \in \text{Hom}_H(M,H_1)$,

$$\varepsilon(f(h \cdot m)) = \varepsilon(h \cdot f(m)) = \varepsilon(hf(m))
= \varepsilon(\varepsilon_s(h)f(m)) = \varepsilon(f(\varepsilon_s(h) \cdot m)).$$

So $\beta_M$ has a well-defined inverse $\beta_M^{-1} : \text{Hom}_H(M,H_1) \to \text{Inv} M^*$ by the formula

$$\langle \beta_M^{-1}(f), m \rangle = \varepsilon(f(m)).$$

\[ \blacksquare \]

**Remark 5.2** For $M, N \in C$, and $m^* \in \text{Inv} M^*$, $m \in M$, $n \in N$, we have

$$\langle (\beta_M(m^*))(m), n \rangle = \langle m^*, 1_{(1)} \cdot m \rangle 1_{(2)} \cdot n, \quad (5.12)$$

$$n \cdot (\beta_M(m^*)(m)) = 1_{(1)} \cdot n \langle m^*, 1_{(2)} \cdot m \rangle. \quad (5.13)$$

Now we are able to reconcile the notion of $C$-coalgebras with the notion of left $H$-module coalgebra.

**Lemma 5.3** A triple $(C, \Delta_C, \varepsilon_C)$, where $C \in C$, $\Delta_C \in \text{Hom}_C(C, C \otimes_C C)$, $\varepsilon_C \in \text{Hom}_C(C, H_1)$, is a coalgebra in $C$ if and only if $(C, \Delta_C, \beta^{-1}_C \varepsilon_C)$ is a left $H$-module coalgebra. In addition, if $C$ is a $C$-coalgebra, then the category $C$-$\text{Comod}_C$ of left $C$-comodules in $C$ is equal to the category $C_H M$ of left relative $(C, H)$-Hopf modules.
**Proof.** The equivalence of the counit axioms follows from (5.12) and (5.13). $lacksquare$

Let $D$ be a $C$-coalgebra. For $(M, \rho_M) \in \text{Comod}_C - D$, $(N, \rho_N) \in D - \text{Comod}_C$, we will show that the cotensor product $M \boxtimes_N D$ in $C$ (see Definition 4.6) is exactly the classical cotensor product $M \square D$ with the diagonal $H$-module action.

**Proposition 5.4** Let $D$ be a coalgebra in $C$. Let $(M, \rho_M)$ be a right $D$-comodule in $C$, and $(N, \rho_N)$ be a left $D$-comodule in $C$. Then $M \square D$ is an $H$-submodule of $M \otimes_t N$, and $M \square D = M \square D$.

**Proof.** For any $x = \sum_i m_i \otimes n_i \in M \square D N$, we have

$$x = \sum_i m_i \cdot \varepsilon_D (m_i(1)) \otimes n_i$$

$$= \sum_i m_i \cdot \varepsilon_D (1 \cdot n_i(1)) \otimes 1 \cdot n_i(0)$$

$$= \sum_i m_i \cdot \varepsilon (1 \cdot n_i(1)) \otimes 1 \cdot n_i(0)$$

$$= \sum_i (m_i \cdot \varepsilon_D(n_i)) \cdot S(1 \cdot n_i(1)) \otimes 1 \cdot n_i(0)$$

$$= \sum_i 1 \cdot (m_i \cdot \varepsilon_D(n_i)) \otimes 1 \cdot n_i(0)$$

$$= \Delta(1) \cdot x.$$

Thus $M \square D$ is a subspace of $M \otimes_t N$. Now we get the following commutative diagram

$$
\begin{array}{ccc}
M \square D & \xleftarrow{\rho_M \otimes \text{id}_N} & M \otimes D \otimes_t N \\
\| & & \| \\
M \square D & \xleftarrow{\rho_M \otimes \text{id}_N - \text{id}_M \otimes \rho_N} & M \otimes D \otimes_k N
\end{array}
$$

and the result is clear. $lacksquare$

Let $(C, \Delta_C, \varepsilon_C)$ be a left $H$-module coalgebra and $F : C, M \to \tilde{C} M$ be the forgetful functor. Then by references [3] Proposition 3.3 and [6] Proposition 2.1, the functor $F$ has a left adjoint functor $\text{Ind} : \tilde{C} M \to C, M$. For completeness, we include the structure here.

Naturally, $C^*$ can be considered as a left $H^*$-comodule $(C^*, \rho_{C^*})$. Define $\text{Ind} : C, M \to \tilde{C} M$ as follows: $\text{Ind}(W) = (H \otimes W) \rho_{C^*} (\varepsilon_C) = H \leftarrow \varepsilon_C(1) \otimes W \cdot \varepsilon_C(0)$ as $k$-space, which is the subspace of $H \otimes k W$ generated by elements of the form $\langle \varepsilon_C, h(1) \cdot w(0) \rangle h(2) \otimes w(0)$. The left $H$-action and left $C$-coaction $\rho$ on $\text{Ind}(W)$ are given by the formulas

$$x \left( \langle \varepsilon_C, h(1) \cdot w(1) \rangle h(2) \otimes w(0) \right) = \langle \varepsilon_C, h(1) \cdot w(1) \rangle x h(2) \otimes w(0)$$

$$\rho \left( \langle \varepsilon_C, h(1) \cdot w(1) \rangle h(2) \otimes w(0) \right) = h(1) \cdot w(1) \otimes h(2) \otimes w(0),$$

26
where \( x, h \in H, \ w \in W \).

**Lemma 5.5** (cf. [3, Proposition 3.3], [6, Proposition 2.1]) Let \( C \) be a left \( H \)-module coalgebra. Then the functor \( \text{Ind} : C\mathcal{M} \rightarrow C_H\mathcal{M} \) is a left adjoint of the forgetful functor \( F : C_H\mathcal{M} \rightarrow C\mathcal{M} \).

Let \( D \) be a coalgebra in \( C \). We have already known that \( D \) can be viewed as a \( k \)-coalgebra. If \( D \) is a cosemisimple \( C \)-coalgebra, one may ask whether \( D \) is cosemisimple as a \( k \)-coalgebra. We will show that it’s true under the assumption that \( H \) is cosemisimple, and this result will be used to present the structure of the irreducible Yetter-Drinfeld modules.

Let \( A \) be a left \( H \)-module algebra. Then the smash product \( A \# H \) is defined on the \( k \)-space \( A \otimes_{H_1} H \), where \( H \) is a left \( H_1 \)-module via multiplication and \( A \) is a right \( H_1 \) module via

\[
a \cdot z = S^{-1}(z) \cdot a, \quad a \in A, \ z \in H_1.
\]

Let \( a \# h \) denote the class of \( a \otimes h \) in \( A \# H \). The multiplication of \( A \# H \) is given by

\[
(a \# h)(b \# g) = a(h^{(1)} \cdot b) \# h^{(2)}g, \quad a, b \in A, h, g \in H,
\]

and the unit is \( 1_A \# 1_H \).

Observed that \( A \# H \) is a left \( H^* \)-module algebra via

\[
f \cdot (a \# h) = a \# (f \hookrightarrow h), \quad f \in H^*, a \in A, h \in H.
\]

The following duality theorem was shown by Nikshych [24].

**Lemma 5.6** ([24, Theorem 3.3]) There is an algebra isomorphism between the algebras \( (A \# H) \# H^* \) and \( \text{End} (A \# H)_A \), where \( A \# H \) is a right \( A \)-module via multiplication.

In the case when \( H \) is a Hopf algebra, it has been proved by Blattner and Montgomery [2] that \( (A \# H) \# H^* \cong M_n (A) \), where \( n = \dim H \). While if a weak Hopf algebra \( H \) is not free over \( H_1 \), \( (A \# H) \# H^* \) might not be isomorphic to a matrix algebra over \( A \). However, we have that \( (A \# H) \# H^* \) is Morita-equivalent to \( A \).

Consider \( A \) as a regular right \( A \)-module and a left \( H_1 \)-module via the left \( H \)-action. For \( z \in H_1, a, b \in A, \)

\[
z \cdot (ab) = (z^{(1)} \cdot a) (z^{(2)} \cdot b) = ((1^{(1)}z) \cdot a) (1^{(2)} \cdot b) = (z \cdot a) b,
\]
hence \( A \) is an \( H_1 \)-\( A \) bimodule.
Proposition 5.7 Let $H$ be a finite dimensional weak Hopf algebra, and $A$ be a left $H$-module algebra. Then as right $A$-modules

$$A \# H \cong H \otimes_{H^t} A,$$

and the algebra $(A \# H) \# H^*$ is Morita-equivalent to $A$.

**Proof.** Define a map

$$\Phi : H \otimes_{H^t} A \to A \# H \text{ via } h \otimes a \mapsto (1_A \# h)(a \# 1_H).$$

$\Phi$ is well-defined, since for $z \in H^t$, $a \in A$ and $h \in H$,

$$\begin{align*}
(1_A \# h z)(a \# 1_H) &= \left((h_{(1)} z_{(1)}) \cdot a\right) \# h_{(2)} z_{(2)} \\
&= \left((h_{(1)} 1_{(1)} z) \cdot a\right) \# h_{(2)} 1_{(2)} \\
&= (h_{(1)} \cdot (z \cdot a)) \# h_{(2)} \\
&= (1_A \# h)((z \cdot a) \# 1_H).
\end{align*}$$

Clearly $\Phi$ is an $A$-module map. Observe that for all $a \in A$, $h \in H$,

$$\begin{align*}
((1_A \# h_{(2)})(S^{-1}(h_{(1)}) \cdot a) \# 1_H)) &= (h_{(2)} S^{-1}(h_{(1)}) \cdot a) \# h_{(3)} \\
&= (S^{-1}(\varepsilon_t(h_{(1)})) \cdot a) \# h_{(2)} \\
&= a \# \varepsilon_t(h_{(1)}) h_{(2)} \\
&= a \# h.
\end{align*}$$

Then $\Phi$ has a well-defined inverse, namely, $a \# h \mapsto h_{(2)} \otimes S^{-1}(h_{(1)}) \cdot a$.

Since $H^t$ is semisimple and $H$ is a faithful $H^t$-module, $H$ is a progenerator of $M_{H^t}$. Hence, $H \otimes_{H^t} A$ is a progenerator of $M_A$. Now by Lemma 5.6, we have $(A \# H) \# H^* \cong \text{End}(A \# H)_A \cong \text{End}(H \otimes_{H^t} A)_A$, which implies that $(A \# H) \# H^*$ is Morita-equivalent to $A$. ■

Corollary 5.8 Let $H$ be a finite dimensional cosemisimple weak Hopf algebra, and $A$ be a left $H$-module algebra. If the algebra $A \# H$ is semisimple, then $A$ is also semisimple.

**Proof.** Since $A \# H$ and $H^*$ are semisimple, $(A \# H) \# H^*$ is semisimple by a Maschke-type theorem for weak Hopf algebras [30, Theorem 1]. Thus $A$ is semisimple by Proposition 5.7. ■

5.3 Automorphism Braided Group for Quasi-triangular Weak Hopf Algebras

Let $(H, R)$ be a quasi-triangular weak Hopf algebra. The goal of this subsection is to present Majid’s braided reconstruction with $C = H M$, and characterize the automorphism braided group $U(C)$ of the braided rigid monoidal category $C$. 28
We need some preliminary steps. First, take \( B = C_H(H_s) \), the centralizer of \( H_s \). It’s known from [5, Proposition 2.11] that \( 1_1 \otimes S(1_2) \in H_s \otimes_k H_s \) is a separable idempotent of \( H_s \), then

\[
B = C_H(H_s) = \{ 1_1 h S(1_2) \mid h \in H \}.
\]

Now we can consider \( B \) as an object of \( \mathcal{C} \) via the left \( H \)-adjoint action \( \cdot_{ad} \), namely, \( h \cdot_{ad} b = h(1_1) b S(h(2)) \), \( \forall h \in H, b \in B \). We will show that \( B = U(\mathcal{C}) \), the automorphism braided group of \( \mathcal{C} \).

The next step is to find an action \( \alpha \in \text{Nat} (B \otimes_t id_C, id_C) \). For any \( X \in \mathcal{C} \), define a map

\[
\alpha_X : B \otimes_t X \to X, \quad 1_1 \cdot_{ad} b \otimes 1_2 x \mapsto bx.
\]

Since for any \( b \in B, x \in X \), \( bx = b1_1 S(1_2) 1_3 x = (1_1 \cdot_{ad} b) 1_2 x \), \( \alpha_X \) is well-defined. Next, we check that each \( \alpha_X \) is a morphism in \( \mathcal{C} \). In fact, we have

\[
\alpha_X \left( h \left( 1_1 \cdot_{ad} b \otimes 1_2 x \right) \right) = \alpha_X \left( h \left( 1_1 \cdot_{ad} b \otimes h(2) x \right) \right) = h(1_1) b S(h(2)) h(3) x
\]

\[
= h(1_1) S(h(2)) h(3) bx = h(bx) = h \alpha_X \left( 1_1 \cdot_{ad} b \otimes 1_2 x \right),
\]

for all \( h \in H, b \in B, x \in X \). The naturality of \( \alpha \) is obvious. Now given \( f \in \text{Hom}_\mathcal{C}(V, B) \), we define \( \theta_V(f) \in \text{Nat} (V \otimes_t id_C, id_C) \) via \( \theta_V(f)_X = \alpha_X(f \otimes_t id_X) \), for all \( X \in \mathcal{C} \).

**Lemma 5.9** The natural transformation

\[
\theta : \text{Hom}_\mathcal{C}(\bullet, B) \to \text{Nat} (\bullet \otimes_t id_C, id_C)
\]

\[
V \leadsto \theta_V : \text{Hom}_\mathcal{C}(V, B) \to \text{Nat} (V \otimes_t id_C, id_C), \quad \theta_V(f)_X = \alpha_X(f \otimes_t id_X),
\]

is an isomorphism with inverse given by

\[
\theta_V^{-1} : \text{Nat} (V \otimes_t id_C, id_C) \to \text{Hom}_\mathcal{C}(V, B),
\]

\[
\delta \mapsto \delta_H \left( 1_1 (\bullet) \otimes 1_2 \right),
\]

where \( H \in \mathcal{C} \) is considered as the left regular representation.

**Proof.** For \( V \in \mathcal{C} \), \( \delta \in \text{Nat} (V \otimes_t id_C, id_C) \), we get a linear map \( \delta_H \left( 1_1 (\bullet) \otimes 1_2 \right) : V \to H, v \mapsto \delta_H \left( 1_1 v \otimes 1_2 \right) \). We first show that the image of \( \delta_H \left( 1_1 (\bullet) \otimes 1_2 \right) \) lies in \( B \). For any \( X \in \mathcal{C} \) and \( x \in X \), the map \( x_r : H \to X, h \mapsto hx \) is a morphism in \( \mathcal{C} \). Since \( \delta \) is natural under the morphism \( H \to X \), we have

\[
\delta_X \left( 1_1 v \otimes 1_2 x \right) = \delta_H \left( 1_1 v \otimes 1_2 \right) x.
\]

(5.14)
Specially take $X = H$, then for any $y \in H_s$ we have

$$
\delta_H \left( (1_1 v \otimes 1_2) y = \delta_H \left( (1_1 v \otimes 1_2) y \right) = \delta_H \left( 1_1 v \otimes 1_2 y \right) \right) = \delta_H \left( y (1_1 v \otimes 1_2) \right)
$$

and thus $\delta_H \left( (1_1 v \otimes 1_2) \right) \in B$, for all $v \in V$. Also by (5.14), for $h \in H$, $v \in V$,

$$
\delta_H \left( (1_1 h v \otimes 1_2) \right) = \delta_H \left( h_1 (v \otimes \varepsilon_t (h_2)) \right) = \delta_H \left( h_1 (v \otimes h_2) S(h_3) \right)
$$

hence $\delta_H \left( (1_1 (\bullet) \otimes 1_2) \right) \in \text{Hom}_C (V, B)$. Now define

$$
\xi_V : \text{Nat} \left( V \otimes id_C, id_C \right) \rightarrow \text{Hom}_C (V, B),
$$

$$
\delta \mapsto \delta_H \left( (1_1 (\bullet) \otimes 1_2) \right).
$$

Finally, we show that $\xi_V$ is the inverse for $\theta_V$. If $f \in \text{Hom}_C (V, B)$, then for $v \in V$,

$$
\xi_V (\theta_V (f)) (v) = \theta_V (f) \left( (1_1 v \otimes 1_2) \right) = f \left( (1_1 v) 1_2 \right) = (1_1 \cdot \text{ad} f (v)) 1_2
$$

$$
= 1_1 f (v) S (1_2) 1_3 = f (v) 1_1 S (1_2) 1_3 = f (v).
$$

Conversely, if $\delta \in \text{Nat} \left( V \otimes id_C, id_C \right)$, then

$$
\theta_V (\xi_V (\delta)) (1_1 v \otimes 1_2) x = \alpha_X \left( \xi_V (\delta) (1_1 v) \otimes 1_2 x \right) = \alpha_X \left( (1_1 \cdot \text{ad} \xi_V (\delta) (v) \otimes 1_2) x \right)
$$

$$
= \xi_V (\delta) (v) x = \delta_H \left( (1_1 v \otimes 1_2) \right) x = \delta_X \left( (1_1 v \otimes 1_2) \right) x.
$$

Thus $\xi_V = \theta_V^{-1}$. ■

We will show that $B = U (C)$. In fact, the representable conditions for modules, as stated in Section 3, are satisfied. To present the reconstruction, we need the inverse of the natural transformation $\theta^2$ determined by the diagram (3.1).

**Lemma 5.10** For any $V \in C$, the morphism

$$
\theta^2_V : \text{Hom}_C (V, B \otimes B) \rightarrow \text{Nat} \left( V \otimes id_C \otimes id_C, id_C \otimes id_C \right)
$$

$$
\theta^2_V (f)_{X,Y} = (\alpha_X \otimes \alpha_Y) (id_B \otimes id_B, c_{B,X} \otimes id_Y) (f \otimes id_B \otimes id_Y),\; X,Y \in C
$$

is an isomorphism with inverse

$$
\xi^2_V : \text{Nat} \left( V \otimes id_C \otimes id_C, id_C \otimes id_C \right) \rightarrow \text{Hom}_C (V, B \otimes B)
$$

$$
\xi^2_V (\delta) (v) = v_1 [S (R^2) \otimes R^1] \cdot \text{ad} v_2,
$$

where $v_1 [v_2] = \delta_{H,H} \left( (1_1 v \otimes 1_2) \otimes 1_3 \right) \in H \otimes H$. 30
Proof. If \( f \in \text{Hom}_C (V, B \otimes_t B) \), for \( X, Y \in \mathcal{C}, x \in X, y \in Y, v \in V \),

\[
\theta^2_V (f)_{X,Y} \left( (1_{(1)} v \otimes 1_{(2)} x \otimes 1_{(3)} y) \right) = 1_{(1)} f (v)^{[1]} R^2 x \otimes 1_{(2)} \left( R^1 \cdot_{ad} f (v)^{[2]} \right) y = f (v)^{[1]} 1_{(1)} R^2 x \otimes 1_{(2)} S (1_{(3)}) \left( R^1 \cdot_{ad} f (v)^{[2]} \right) y = f (v)^{[1]} 1_{(1)} R^2 x \otimes \left( (1_{(2)} R^1) \cdot_{ad} f (v)^{[2]} \right) y = f (v)^{[1]} R^2 x \otimes \left( R^1 \cdot_{ad} f (v)^{[2]} \right) y,
\]

where \( f (v) = f (v)^{[1]} \otimes f (v)^{[2]} \in B \otimes_t B \). Given \( \delta \in \text{Nat} (V \otimes_t id_C \otimes id_C) \), from the naturality of \( \delta \), we have

\[
\delta_{X,Y} \left( (1_{(1)} v \otimes 1_{(2)} x \otimes 1_{(3)} y) \right) = v_{[1]} x \otimes v_{[2]} y. \tag{5.15}
\]

Applying \( \tag{5.15} \) with \( X = Y = H \), we have

\[
(hv)^{[1]} \otimes (hv)^{[2]} = \delta_{H,H} \left( (1_{(1)} hv \otimes 1_{(2)} \otimes 1_{(3)}) = \delta_{H,H} (h_{(1)} v \otimes h_{(2)} S (h_{(3)}) \otimes h_{(4)}) \right) = h_{(1)} \delta_{H,H} \left( (1_{(1)} v \otimes 1_{(2)} S (h_{(3)}) \otimes 1_{(3)} S (h_{(2)}) \right) = h_{(1)} \left( v_{[1]} S (h_{(3)}) \otimes v_{[2]} S (h_{(2)}) \right) = h_{(1)} \left( v_{[1]} S (h_{(3)}) \otimes v_{[2]} \right) \cdot_{ad} v_{[2]}, \tag{5.16}
\]

for \( h \in H \). Take \( h = 1 \), we get

\[
v_{[1]} \otimes v_{[2]} = 1_{(1)} v_{[1]} S (1_{(3)}) \otimes 1_{(2)} \cdot_{ad} v_{[2]} \tag{5.17}.
\]

Then

\[
1_{(1)} \cdot_{ad} \left( v_{[1]} S (R^2) \right) \otimes 1_{(2)} \cdot_{ad} \left( R^1 \cdot_{ad} v_{[2]} \right) = 1_{(1)} v_{[1]} S (1_{(2)} R^2) \otimes (1_{(3)} R^1) \cdot_{ad} v_{[2]} \]

\[
= 1_{(1)} v_{[1]} S (1_{(3)}) S (R^2) \otimes R^1 \cdot_{ad} (1_{(2)} \cdot_{ad} v_{[2]}) \]

\[
\overset{\text{5.17}}{=} v_{[1]} S (R^2) \otimes R^1 \cdot_{ad} v_{[2]},
\]

and thus \( \xi^2_V (\delta) (v) \in B \otimes_t B \).

Next, we show that \( \xi^2_V (\delta) \in \text{Hom}_C (V, B \otimes_t B) \). For \( h \in H, v \in V \),

\[
\xi^2_V (\delta) (hv) = (hv)^{[1]} S (R^2) \otimes R^1 \cdot_{ad} (hv)^{[2]} \]

\[
\overset{\text{5.16}}{=} h_{(1)} v_{[1]} S (h_{(3)}) S (R^2) \otimes (R^1 h_{(2)}) \cdot_{ad} v_{[2]} = h_{(1)} v_{[1]} S (h_{(2)} R^2) \otimes (h_{(3)} R^1) \cdot_{ad} v_{[2]} = h_{(1)} \cdot_{ad} \left( v_{[1]} S (R^2) \right) \otimes h_{(2)} \cdot_{ad} (R^1 \cdot_{ad} v_{[2]}) = h \xi^2_V (\delta) (v),
\]

as expected, and thus \( \xi^2_V : \text{Nat} (V \otimes_t id_C \otimes id_C) \rightarrow \text{Hom}_C (V, B \otimes_t B) \) is well-defined.
Now, we only need to check that \( \xi^2_v \) and \( \theta^2_v \) are mutual inverses. First, let \( f \in \text{Hom}_C(V, B \otimes \Delta B) \), then \( \theta^2_v (f)_{H,H} (1(1)v \otimes 1(2) \otimes 1(3)) = f(v)[1] R^2 \otimes (R^1 \cdot \text{ad } f(v)[2]) \). So we have that

\[
\xi^2_v \left( \theta^2_v (f) \right) (v) = f(v)[1] R^2 \otimes (R^1 \cdot \text{ad } f(v)[2]) \\
= f(v)[1] S(R^2) R^1 \cdot \text{ad } f(v)[2] \\
= f(v)[1] S(1(1)) \otimes 1(2) \cdot \text{ad } f(v)[2] \\
= 1(1) \cdot \text{ad } f(v)[1] \otimes 1(2) \cdot \text{ad } f(v)[2] = f(v),
\]

for \( v \in V \). On the other hand, for \( \delta \in \text{Nat}(V \otimes \Delta \theta \otimes \Delta \theta) \),

\[
\theta^2_v \left( \xi^2_v (\delta) \right)_{X,Y} (1(1)v \otimes 1(2)x \otimes 1(3)y) = (\xi^2_v (\delta) (v))^{[1]} R^2 x \otimes (R^1 \cdot \text{ad } (\xi^2_v (\delta) (v))^{[2]}) y \\
= v[1] S(R^2) R^1 \cdot \text{ad } v[2] \otimes (R^1 \cdot \text{ad } v[2]) y \\
= v[1] S(R^2) R^1 \cdot \text{ad } v[2] \otimes (S(R^1) R^1) \cdot \text{ad } v[2] y \\
= v[1] S(1(2)) x \otimes (1(1) \cdot \text{ad } v[2]) y \\
\stackrel{[5.17]}{=} 1(1)v[1] S(1(3)) S(1(2)) x \otimes (1(1) \cdot \text{ad } v[2]) y \\
= 1(1)v[1] S(1(3)) x \otimes (1(2) \cdot \text{ad } v[2]) y \\
= v[1] x \otimes v[2] y \delta_{X,Y} (1(1)v \otimes 1(2)x \otimes 1(3)y).
\]

Thus we have \( \xi^2_v = (\theta^2_v)^{-1} \). ■

We summarize the above discussion in the next theorem, and provide the concrete multiplication, comultiplication, etc.

**Theorem 5.11** Let \((H, R)\) be a quasi-triangular weak Hopf algebra. Then the automorphism braided group of \( C = H \cdot \mathcal{M} \) is the object \( B = (C_H(H), \cdot \text{ad}) \in C \) with Hopf algebra structure in \( C \) defined as follows.

1) The multiplication \( m_B : B \otimes \Delta B \rightarrow B \) and the unit \( u_B : H \rightarrow B \) are defined by

\[
m_B \left( 1(1) \cdot \text{ad } a \otimes 1(2) \cdot \text{ad } b \right) = ab, \quad u_B (z) = z, \quad \forall a, b \in B, z \in H_1.
\]

2) The comultiplication \( \Delta_B : B \rightarrow B \otimes \Delta B \) and the counit \( \varepsilon_B : B \rightarrow H_1 \) are defined by

\[
\Delta_B (b) = b_{(1)} S(R^2) \otimes R^1 \cdot \text{ad } b_{(2)}, \quad \varepsilon_B (b) = \varepsilon_1 (b).
\]

3) The antipode \( S_B : B \rightarrow B \) is defined by

\[
S_B (b) = R^2 S(R^1 \cdot \text{ad } b).
\]
Proof. To show the theorem, we use $\alpha, \theta, \theta^2$ to compute the Hopf algebra structure on $B$, determined by the diagrams (3.2, 3.4). As before, we use the unit isomorphisms $l_X$ and $r_X$ identifying $H_t \otimes X$ and $X \otimes H_t$ with $X$, for any $X \in \mathcal{C}$. According to (3.2), the multiplication and the unit are characterized by

$$m_B(1_{(1)} \cdot_{ad} a \otimes 1_{(2)} \cdot_{ad} b) x = a(bx), \quad u_B(z) x = zx, \quad \forall a, b \in B, \quad z \in H_t, \quad x \in X.$$ 

Take $X = H$ and $x = 1$ be the unit of $H$, then

$$m_B(1_{(1)} \cdot_{ad} a \otimes 1_{(2)} \cdot_{ad} b) = ab, \quad u_B(z) = z.$$ 

And the counit is characterized according to (3.3) by

$$\varepsilon_B(b) = b \cdot 1 \cdot \varepsilon_t(b), \quad \forall b \in B.$$ 

Again, the comultiplication is characterized according to (3.3) by

$$\theta^2_B(\Delta_B)_{X,Y}(1_{(1)} \cdot_{ad} b \otimes 1_{(2)} x \otimes 1_{(3)} y) = b_{(1)} x \otimes b_{(2)} y, \quad \forall x \in X, y \in Y,$$

so we apply $\xi^2_B$, and get

$$\Delta_B(b) = \xi^2_B(\theta^2_B(\Delta_B))(b) = b_{(1)} S(R^2) \otimes R^1 \cdot_{ad} b_{(2)}, \quad \forall b \in B.$$ 

Finally, the antipode $S_B$ is characterized according to (3.4) by

$$\theta_B(S_B)_X(1_{(1)} \cdot_{ad} b \otimes 1_{(2)} x)$$

$$= r_X(i_X \otimes t ev_X)(i_X \otimes t \alpha \cdot \otimes i_X)(c_{B,X} \otimes t i_X \otimes t i_X)(1_{(1)} \cdot_{ad} b \otimes t \text{coev}_X(1_{(2)} \cdot 1) \otimes t 1_{(3)} x)$$

$$= r_X(i_X \otimes t ev_X)(i_X \otimes t \alpha \cdot \otimes i_X) \left( \sum_{i} c_{B,X} (1_{(1)} \cdot_{ad} b \otimes 1_{(2)} x_i) \otimes 1_{(3)} x^{*}_i \otimes 1_{(4)} x \right)$$

$$= r_X(i_X \otimes t ev_X) \left( \sum_{i} 1_{(1)} R^2 x_i \otimes \alpha \cdot (1_{(2)} R^1) \cdot_{ad} b \otimes 1_{(3)} x^{*}_i \otimes 1_{(4)} x \right)$$

$$= r_X \left( \sum_{i} 1_{(1)} R^2 x_i \otimes ev_X(1_{(2)} (R^1 \cdot_{ad} b)(x^{*}_i \otimes 1_{(3)} x)) \right)$$

$$= r_X \left( \sum_{i} 1_{(1)} R^2 x_i \otimes 1_{(2)} \cdot _{1_{(1)}} (R^1 \cdot_{ad} b)(x^{*}_i, 1_{(1)} x) 1_{(2)} \right)$$

$$= r_X \left( 1_{(1)} R^2 R^2 S(R^1 b S(R^1)) \right) 1_{(1)} x \otimes 1_{(2)} \cdot _{1_{(2)}}$$

$$= S(1_{(2)} R^2 R^2 S(R^1 b S(R^1)) 1_{(1)} x$$

$$= S^2(1_{(1)} R^2 R^2 S(b S(R^1)) S(R^1) S(1_{(2)} x$$

$$= 1_{(1)} R^2 R^2 S(b S(R^1)) S(1_{(2)} R^1) x$$

$$= R^2 S(R^1 \cdot_{ad} b) x,$$
where \( \{(x_i, x_i^*)\}_i \) is a dual basis of \( X \). Then one can apply \( \xi_B \) to \( \theta_B (S_B) \), getting the formula as stated.

**Remark 5.12** A braided Hopf algebra structure on \( B = C_H (H_s) \) was constructed explicitly in [17]. By Theorem 5.11 we know that the braided Hopf algebra \( B \) they given is exactly the automorphism braided group of the braided multitensor category \( \mathcal{C} \).

### 5.4 Structure of Yetter-Drinfeld Modules over Quasi-triangular Weak Hopf Algebras

In this last subsection, we study the structure of Yetter-Drinfeld modules over a finite dimensional quasi-triangular weak Hopf algebra \((H, R)\). We will characterize the simple Yetter-Drinfeld modules in \( H H \mathcal{YD} \) in the case when the category \( H H \mathcal{YD} \) is semisimple, extending the results in [18] to a weak Hopf algebra version.

Böhm [3] generalized the notion of Yetter-Drinfeld modules to weak Hopf algebras. A left-left \( H \)-Yetter-Drinfeld modules \( M \) is a vector space with an \( H \)-action and an \( H \)-coaction satisfying the following conditions:

\[
\rho (m) = m_{(-1)} \otimes m_{(0)} \in H \otimes M,
\]

\[
h_{(1)} m_{(-1)} \otimes h_{(2)} m_{(0)} = (h_{(1)} m)_{(-1)} h_{(2)} \otimes (h_{(1)} m)_{(0)},
\]

for all \( h \in H, m \in M \). Denote by \( H H \mathcal{YD} \) the category of the left-left Yetter-Drinfeld module over \( H \).

If \( H \) is weak Hopf algebra with bijective antipode, then the category \( H H \mathcal{YD} \) is a braided monoidal category, and it’s isomorphic to the left center \( Z_l (H M) \) as braided monoidal categories (see [7]). Here is a brief description of the connecting functors. For an object \((M, \gamma_{M,*}) \in Z_l (H M)\), the map

\[
\rho : M \to H \otimes_M M, \quad \rho (m) = \gamma_{M,H} (1_{(1)} m \otimes 1_{(2)}) \tag{5.18}
\]

gives a left \( H \)-coaction on \( M \), which makes \( M \) into a Yetter-Drinfeld module in \( H H \mathcal{YD} \).

Conversely, for \( M \in H H \mathcal{YD} \), define a natural transformation \( \gamma_{M,*} \) by

\[
\gamma_{M,X} (1_{(1)} m \otimes 1_{(2)} x) = m_{(-1)} x \otimes m_{(0)}, \text{ for } X \in H M, x \in X, m \in M, \tag{5.19}
\]

then \((M, \gamma_{M,*})\) is an object of \( Z_l (H M) \).

Now let \((H, R)\) be a quasi-triangular weak Hopf algebra. In Section 5.3 we have proved that \( B = C_H (H_s) \) is the automorphism braided group of \( C = H M \). It was shown in [31] that the Yetter-Drinfeld module category \( H H \mathcal{YD} \) is isomorphic to the category of left \( B \)-comodules for the braided Hopf algebra \( B = C_H (H_s) \). We now give a categorical interpretation, as an application of Theorem 3.4 and Theorem 5.11.
Proposition 5.13 (cf. [31, Theorem 2.5]) There is an equivalence from the category $B$-$\text{Comod}_C$ of left $B$-comodules to the category $\mathcal{H}_H^H\mathcal{YD}$:

$$\mathcal{F}: B\text{-Comod}_C \to \mathcal{H}_H^H\mathcal{YD}, \ (M, \rho_R) \mapsto (M, \rho),$$

where the left $H$-coaction $\rho : M \to H \otimes_t M$ is defined by $\rho (m) = \sum m^{(-1)} R^2 \otimes R^1 m^{(0)}$, for $m \in M$. The quasi-inverse of $\mathcal{F}$ is

$$\mathcal{G}: \mathcal{H}_H^H\mathcal{YD} \to B\text{-Comod}_C, \ (M, \rho) \mapsto (M, \rho_R),$$

where the left $B$-coaction $\rho_R : M \to B \otimes M$ is defined by $\rho_R (m) = \sum m^{(-1)} S (R^2) \otimes R^1 m^{(0)}$, for $m \in M$. Here we use the notation $\rho_R (m) = m^{(-1)} \otimes m^{(0)}$ for left $B$-coaction to distinguish the $H$-coaction $\rho (m) = m^{(-1)} \otimes m^{(0)}$.

**Proof.** Note that $\mathcal{Z}_t (C) \cong \mathcal{H}_H^H\mathcal{YD}$. For any $(M, \rho_R) \in B\text{-Comod}_C$, one can easily check that

$$\varphi_{M,M} (\rho_R)_H \ (1_{(1)} m \otimes 1_{(2)}) = m^{(-1)} R^2 \otimes R^1 m^{(0)}$$

for $m \in M$, where $\varphi_{M,M}$ is defined as [3.5]. Then combining Theorem 3.4 with (5.18), $\mathcal{F}$ is an equivalence. Conversely, to show $\mathcal{G}$ is the quasi-inverse of $\mathcal{F}$, it is enough to verify that for any $(M, \rho) \in \mathcal{H}_H^H\mathcal{YD}$, $\rho_R = \varphi_{M,M}^{-1} (\gamma_{M\bullet})$, where $\gamma_{M\bullet}$ is defined as [5.19]. For $m \in M$, $x \in X$, we have

$$\varphi_{M,M} (\rho_R)_X \ (1_{(1)} m \otimes 1_{(2)}x) = m^{(-1)} S (R^2) R_2 x \otimes R_2^1 R_1^1 m^{(0)}$$

$$= m^{(-1)} S (R^2 R_1^2) x \otimes S (R_2^1) R_1^1 m^{(0)}$$

$$= m^{(-1)} S (1_{(2)}) x \otimes 1_{(1)} m^{(0)}$$

$$= 1_{(1)} m^{(-1)} S (1_{(2)}) x \otimes 1_{(2^\prime)} 1_{(1)} m^{(0)}$$

$$= 1_{(1)} m^{(-1)} S (1_{(3)}) x \otimes 1_{(2)} m^{(0)}$$

$$= m^{(-1)} x \otimes m^{(0)}$$

$$= \gamma_{M,X} \ (1_{(1)} m \otimes 1_{(2)}x).$$

Now the result follows from Theorem 3.4. 

The coproduct $\Delta_B$ of the braided group $B$ can be considered canonically as a coassociative coproduct in $\text{Vec}_k$, via

$$\Delta_B: B \to B \otimes_t B \leftrightarrow B \otimes_k B.$$

We have known in Section 5.2 that $(B, \Delta_B, \varepsilon_B)$ is a left $H$-module coalgebra, and $B$-$\text{Comod}_C = \mathcal{H}_H^H\mathcal{M}$. We will use the notation $\Delta_B (b) = b^{(1)} \otimes b^{(2)}$ to distinguish the original coproduct $\Delta (b) = b^{(1)} \otimes b^{(2)}$ of $H$. 

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From now on, we assume that \( k \) is algebraically closed of characteristic zero, and \((H, R)\) is a semisimple quasi-triangular weak Hopf algebra over \( k \). It was shown by Etingof, Nikshych and Ostrik that \( H \) is cosemisimple \([11]\) and the category \( \mathcal{H} \mathcal{YD} \) is also semisimple. Then by the dual of Corollary \([5,8]\), the \( k \)-coalgebra \((B, \Delta_B, \varepsilon_B)\) is cosemisimple.

A subcoalgebra \( D \) of \((B, \Delta_B, \varepsilon_B)\) is called \( H \)-adjoint-stable if \( H \cdot \text{ad} D \subseteq D \). Clearly, \((B, \cdot \text{ad}, \Delta_B) \in \mathcal{H} \mathcal{YD} \). For any Yetter-Drinfeld submodule \( D \) of \( B \), it follows from Theorem \([4,2]\) that \( D \) is an \( H \)-adjoint-stable subcoalgebra of \( B \).

**Proposition 5.14** Let \((H, R)\) be a quasi-triangular weak Hopf algebra. Then there is a unique decomposition

\[
B = D_1 \oplus \cdots \oplus D_r
\]

of minimal \( H \)-adjoint-stable subcoalgebras \( D_1, \ldots, D_r \) of \( B \). It coincide with the decomposition of simple Yetter-Drinfeld modules.

Moreover, the decomposition of \( B \) as direct sum of simple Yetter-Drinfeld modules is unique, and the category

\[
\mathcal{H} \mathcal{YD} \cong B \text{-Comod}_C = \bigoplus_{j=1}^r D_j \text{-Comod}_C
\]

is a direct sum of indecomposable \( C \)-module subcategories.

**Proof.** This follows from Proposition \([5,13]\) and Proposition \([4,4]\) \( \blacksquare \)

Let \( D \) be a minimal \( H \)-adjoint-stable subcoalgebra of \( B \). Next we give the structure of the indecomposable right \( C \)-module category \( D\text{-Comod}_C \), applying Proposition \([4,7]\) and \([4,10]\) on \( C = H \mathcal{M} \).

For finite dimensional vector space \( M, N \), one usually identifies \( M^* \otimes_k N \) with \( \text{Hom}_k (M, N) \) via

\[
(m^* \otimes n)(m) = \langle m^*, m \rangle n, \forall m \in M, n \in N, m^* \in M^*,
\]

where \( M^* = \text{Hom}_k (M, k) \) is the dual vector space. Then by Proposition \([5,4]\) for two objects \( M_1, M_2 \in D\text{-Comod}_C \), the internal Hom \( \text{Hom} (M_1, M_2) = *M_1 \Box_D M_2 \cong M_1 \Box_D M_2 \cong \text{Hom}^D (M_1, M_2) \), is the set of \( D \)-comodule map from \( M_1 \) to \( M_2 \). As an object of \( _H \mathcal{M} \), the left \( H \)-action on \( \text{Hom}^D (M_1, M_2) \) is given by

\[
(h \cdot f)(m_1) = \sum h_{(2)} f (S^{-1}(h_{(1)}) m_1),
\]

where \( h \in H, f \in \text{Hom}^D (M_1, M_2), m_1 \in M_1 \). It’s not difficult to verify that the evaluation map \( \text{ev}^{M_1, M_2} : M_1 \otimes \text{Hom}^D (M_1, M_2) \to M_2 \) is exactly the regular evaluation map. In particular, the internal endomorphism \( \text{Hom} (M_1, M_1) = \text{End}^D (M_1)^{op} \). As a consequence of Proposition \([4,10]\) we have:
Proposition 5.15 Let \(D\) be a minimal \(H\)-adjoint-stable subcoalgebra of \(B\). For any nonzero \(M \in D\text{-Comod}_C\), the algebra \(A = \text{End}^D (M)^{op}\) in \(C\) is semisimple, and the functors

\[
F = \text{Hom}^D (M, \bullet) : D\text{-Comod}_C \to A\text{-Mod}_C,
\]

\[
G = M \otimes_A \bullet : A\text{-Mod}_C \to D\text{-Comod}_C
\]

establish an equivalence of \(C\)-module categories between \(D\text{-Comod}_C\) and \(A\text{-Mod}_C\).

We will give another characterization of the category \(D\text{-Comod}_C\) by viewing it as a left module category over the tensor category \(\text{Vec}_k\) with \(X \otimes M = X \otimes_k M\), for any \(X \in \text{Vec}_k\), \(M \in D\text{-Comod}_C\). \(X \otimes_k M\) is an object of \(D\text{-Comod}_C\) via the \(H\)-action and \(D\)-coaction on the right tensor and \(M\). For objects \(M_1, M_2 \in D\text{-Comod}_C\), and \(X \in \text{Vec}_k\), the restriction of the canonical isomorphism

\[
\text{Hom}_k (X, \text{Hom}^D (M_1, M_2)) \cong \text{Hom}_k (X \otimes_k M_1, M_2)
\]
on \(\text{Hom}_k (X, \text{Hom}^D (M_1, M_2))\) induces a natural isomorphism

\[
\text{Hom}_k (X, \text{Hom}^D (M_1, M_2)) \cong \text{Hom}^D (X \otimes_k M_1, M_2),
\]

Thus the internal \(\text{Hom}_H (M_1, M_2) = \text{Hom}^D_H (M_1, M_2)\), and the evaluation map \(ev_{M_1, M_2}\) is indeed the regular evaluation map.

Applying Theorem 2.2 to the module category \(D\text{-Comod}_C\) over \(\text{Vec}_k\), we get:

**Theorem 5.16** Let \(D\) be a minimal \(H\)-adjoint-stable subcoalgebra of \(B\). If \(0 \neq W \in D\mathcal{M}\), then \(A_W = \text{End}^D_H (\text{Ind} (W))\) is a semisimple \(k\)-algebra, and the functors

\[
F = \text{Hom}^D_H (\text{Ind} (W), \bullet) : D\text{-Comod}_C \to \mathcal{M}_{A_W},
\]

\[
G = \bullet \otimes_{A_W} (\text{Ind} (W)) : \mathcal{M}_{A_W} \to D\text{-Comod}_C
\]
establish an equivalence of between \(D\text{-Comod}_C\) and \(\mathcal{M}_{A_W}\).

Furthermore, any irreducible object \(V \in D\text{-Comod}_C\) is isomorphic to \(U \otimes_{A_W} \text{Ind} (W)\), for some simple right \(A_W\)-module \(U\).

**Proof.** The proof we give here is similar to that of [18, Proposition 5.2]. To apply Theorem 2.2 to the category \(\mathcal{M} = D\text{-Comod}_C\) and the object \(M = \text{Ind} (W) \in \mathcal{M}\), we only need to verify that \(\text{Ind} (W)\) generates the module category \(\mathcal{M}\) over \(\text{Vec}_k\). For any simple object \(V \in \mathcal{M}\), we claim that the internal \(\text{Hom} (M, V) = \text{Hom}^D_H (M, V)\) is nonzero. Since \(\text{Ind}\) is the left adjoint of the forgetful functor \(\text{Vec}_k \rightarrow D\mathcal{M}\), it suffices to show \(\text{Hom}^D (W, V) \neq 0\). Let \(D' = \text{span} \{v^* \rightarrow v \mid v \in V, v^* \in V^*\}\) with \(v^* \rightarrow v = \sum v^{(-1)} \langle v^*, v^{(0)} \rangle\). It is easy to check
that $D'$ is a nonzero left coideal of $D$ and is also an $H$-submodule under the left $H$-action $\cdot_{ad}$. Since $D$ is irreducible in $\underline{H}YD$, then $D' = D$. So there exists a surjection $V^{(n)} \to D \to 0$ in $^H_D M$ for some $n \in \mathbb{N}^+$. Since $D$ is cosemisimple, there exists an injection $0 \to D \to V^{(n)}$ in $^H_D M$. Take a simple $D$-subcomodule $W'$ of $W$, then $W'$ is isomorphic to a simple left coideal of $D$. So there exists a left $D$-comodule injection $j : W' \to V$. Thus $\text{Hom}^D (W, V) \neq 0$, and the claim follows. Then by the isomorphism
\[
\text{Hom}^D_H (\text{Hom}^D_H (M, V) \otimes M, V) \cong \text{Hom} (\text{Hom}^D_H (M, V), \text{Hom}^D_H (M, V)) \neq 0,
\]
so the evaluation morphism $\text{Hom}^D_H (M, V) \otimes M \to V$ is a surjection in $\mathcal{M}$. Hence $M$ is a generator, and the result follows. ■

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