Explicit Construction of Yang-Mills Instantons on ALE Spaces

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ABSTRACT

We describe the explicit construction of Yang-Mills instantons on Asymptotically Locally Euclidean (ALE) spaces, following the work of Kronheimer and Nakajima. For multicenter ALE metrics, we determine the abelian instanton connections which are needed for the construction in the non-abelian case. We compute the partition function of Maxwell theories on ALE manifolds and comment on the issue of electromagnetic duality. We discuss the topological characterization of the instanton bundles as well as the identification of their moduli spaces. We generalize the ’t Hooft ansatz to $SU(2)$ instantons on ALE spaces and on other hyper-Kähler manifolds. Specializing to the Eguchi-Hanson gravitational background, we explicitly solve the ADHM equations for $SU(2)$ gauge bundles with second Chern class $1/2$, $1$ and $3/2$.

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1 Introduction

Asymptotically Locally Euclidean (ALE) gravitational instantons have played a fundamental role in Euclidean quantum supergravity (for a review see \cite{1}). They induce non-perturbative effects which turn out to be very relevant for understanding the vacuum structure of the theory and are possibly responsible for the emergence of dynamical breaking mechanisms for supersymmetry (SUSY). From a mathematical point of view they are interesting non-compact hyper-Kähler manifolds which admit a complete classification, induced by the A-D-E classification of simply-connected Lie algebras.

In a recent work \cite{2} we have shown that ALE metrics can be incorporated in a fully supersymmetric solution of the four-dimensional heterotic string equations of motion with constant dilaton and zero torsion (for a review of SUSY solutions of low energy supergravities see \cite{3}). Thanks to the so-called standard embedding condition, this solution can be shown to be an exact one (corresponding to an exact $N = (4,4)$ superconformal field theory on the world-sheet, see also \cite{4, 5}), to all orders in (the σ-model coupling) $\alpha'$. To study whether these backgrounds might eventually lead to the formation of chiral condensates, instanton dominated correlation functions must be computed. For globally supersymmetric Yang-Mills (YM) theories with $N = 1, 2, 4$ we have performed this computation around the minimal instanton on the Eguchi-Hanson (EH) manifold \cite{6}, the simplest of all possible ALE spaces. The denomination minimal instanton comes from the value of its second Chern class, $\kappa = 1/2$, which is the minimal one allowed for $SU(2)$ bundles on the EH manifold. The minimal instanton plays on the EH background the same role as the t’Hooft-Polyakov instanton on flat space. It was also found, by quite different methods, in \cite{7}. The results of the instantonic computations presented in \cite{6} are analogous to those obtained in flat spacetime \cite{8}: the chiral condensates are constant and turn out to be

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proportional to the appropriate power of the renormalization group invariant scale of the theory, as dictated by naive dimensional counting.

The extension of instanton computations to fullfledged solutions of the heterotic string equations of motion \[4\], is not straightforward. It requires, as we said, the embedding of the spin connection into the gauge group, and it is based on the fact that the self-dual spin connection of an ALE gravitational background lies in the $SU(2)_L$ factor of the $SO(4)$ Lorentz group. The relevant $SU(2)$ gauge bundle is then the holomorphic tangent bundle to the ALE manifold. In the case of the EH background the dimension of the moduli space of the gauge connections on the tangent bundle was computed, via index formulae, to be 12 and its second Chern class turned out to be $\kappa = 3/2$ \[2\]. Computation of instanton effects requires the knowledge of the form of the gauge connection. This is the task we tackle in the present paper where we build all $SU(2)$ gauge bundles up to $\kappa = 3/2$ and clarify the quite intricate mathematics needed to do it. To carry out this program we will use recent results of Kronheimer and Nakajima who, in a series of beautiful papers \[9, 10, 11\], completely solved the problem of constructing self-dual YM connections on ALE manifolds along the lines of the analogous Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction \[12, 13\] of gauge instantons on flat space.

The plan of the paper is as follows. In section 2 we review the construction of ALE gravitational instantons. Using twistor techniques, Hitchin \[14\] has shown that ALE manifolds are smooth resolutions of algebraic varieties in $\mathbb{C}^3$. Simple singularities admit an A-D-E classification according to which, the class of multicenter metrics of Gibbons-Hawking (GH) \[15\] (to which the EH instanton \[16\] belongs) may be identified with the resolution of singularities of A-type. A general construction of all ALE manifolds was then worked out by Kronheimer \[17\]. These manifolds emerge as minimal resolutions
of $\mathbb{C}^2/\Gamma$, where the discrete subgroups $\Gamma$ of $SU(2)$ are in one-to-one correspondence with the extended Dynkin diagrams of simply-laced (i.e. A-D-E) simple Lie algebras. ALE spaces are explicitly obtained as hyper-Kähler quotients of flat Euclidean spaces. This allows, in principle, the determination of the hyper-Kähler metric on them. In practice only for ALE spaces of A-type, corresponding to $\Gamma = \mathbb{Z}_N$, the metric is known and it turns out to be diffeomorphic to the GH multicenter metric.

The hyper-Kähler quotient construction allows to identify a principal bundle over the ALE space whose natural connection has anti-self-dual curvature. To this principal bundle, a tautological vector bundle, $\mathcal{T}$, can be associated by a change of fiber. This bundle admits a decomposition under the action of $\Gamma$ into $r$ ($=\text{rank}(\Gamma)$) elementary bundles, $\mathcal{T}_i$, whose first Chern classes form a basis for the second cohomology group of the ALE space. For multicenter ALE metrics the bundles $\mathcal{T}_i$ admit abelian connections that we explicitly construct in section 3. They will be needed in the following to compute the gauge connections in non-abelian cases. As a byproduct we obtain a formula for the partition function of abelian gauge theories on ALE spaces, in terms of the level one characters of the affine Lie algebra of the type A-D-E, associated to the ALE space under consideration, and we show that electromagnetic duality, at least in its simplest form, does not hold on ALE spaces.

In Section 4 we recall the essential steps of the ADHM construction of gauge instantons on $\mathbb{R}^4$ and argue that this construction can be viewed as an hyper-Kähler quotient. This observation establishes a bridge with the Kronheimer-Nakajima (KN) construction of YM instantons on ALE spaces which is discussed immediately afterwards. We then elaborate on some known solutions, generalizing the ansätze of ’t Hooft [18] and Jackiw, Nohl and Rebbi [19] to four-dimensional hyper-Kähler manifolds (including ALE
gravitational instantons) and discuss some $\Gamma$-invariant instantons on $\mathbb{R}^4$.

In section 5 by studying the topological properties of the gauge instanton bundles, we are able to identify some already known solutions. We emphasize the role of the discrete group $\Gamma$ and compute the topological invariants of these bundles, the dimensions of their moduli spaces and, in the $SU(2)$ case, the indices of the relevant Dirac operators.

In section 6 we explicitly solve the ADHM equations for $SU(2)$ gauge bundles with second Chern class up to $\kappa = 3/2$ on the EH background. Following the KN-ADHM construction the problem is reduced to a bunch of conceptually simple but somewhat tedious algebraic manipulations.

In section 7 we discuss some properties of the moduli spaces of YM instanton connections on ALE spaces and exhibit the hyper-Kähler metric for the minimal instanton on the EH background and in section 8 we draw our conclusions.

In the appendix we show how to recover the explicit form of the EH metric, following the hyper-Kähler quotient construction.

2 Kronheimer Construction of ALE Instantons

2.1 Hyper-Kähler Quotients

A Riemannian manifold $X$ is said to be hyper-Kähler if it is equipped with three covariantly constant complex structures $I, J, K$, i.e. automorphisms of the tangent bundle satisfying the quaternionic algebra $I^2 = J^2 = K^2 = -1$ and $IJ = -K, JK = -I, KI = -J$. In this circumstance the metric $g$ on $X$ is said to be of Kähler type (or simply, Kähler) with respect to $I, J, K$ and
one can define three closed Kähler 2-forms

\[
\begin{align*}
\omega_I(V, W) &= g(V, IW) \\
\omega_J(V, W) &= g(V, JW) \\
\omega_K(V, W) &= g(V, KW)
\end{align*}
\] (2.1)

with \(V, W\) vector fields on \(X\). We will often denote the hyper-Kähler forms in (2.1) as \(\omega_i\), with \(i = I, J, K\). In four dimensions a simply-connected Riemannian manifold is hyper-Kähler when its Riemann curvature tensor is either self-dual or anti-self-dual\(^2\). A four-dimensional hyper-Kähler manifold is thus a gravitational instanton.

In this section we want to briefly recall the Kronheimer \([17]\) construction of a particular family of hyper-Kähler manifolds, the so called ALE gravitational instantons.

We begin by discussing a method to produce a hyper-Kähler manifold \(X\) of real dimensions \(d = 4(n - k)\) starting from a 4\(n\)-dimensional hyper-Kähler manifold \(\Xi\) \([20]\). The manifold \(X\) is obtained as the quotient of a real subspace of \(\Xi\) by some subgroup of the isometry group of \(\Xi\). In this procedure a central role is played by the moment map. By assumption \(\Xi\) admits vector fields \(V\), known as Killing vectors, which generate isometries, \(i.e.\) for which \(\mathcal{L}_V g = 0\), with \(\mathcal{L}_V\) the Lie derivative along \(V\). Killing vector fields are said to be triholomorphic if furthermore \(\mathcal{L}_V \omega_i = i_V d\omega_i + d(i_V \omega_i) = 0\), where \(i_V\) denotes contraction with \(V\). To each vector of this kind there correspond three Killing potentials, \(\mu^V_i\), which can be obtained by integrating

\(^2\)In the mathematical literature, instantons on complex manifolds denote YM connections whose field-strength is anti-self-dual in the natural orientation of the manifold, inherited from its complex structure. In these conventions, (ALE) gravitational instantons, which are complex hyper-Kähler manifolds, turn out to have anti-self-dual curvature. We will follow, instead, the physicists’ conventions of inverting the natural orientation and call instantons, either YM or gravitational, connections with self-dual curvature.
the equations

\[ d\mu_i^V = i_V \omega_i \quad (2.2) \]

following from \( d\omega_i = 0 \) and \( \mathcal{L}_V \omega_i = 0 \). In the absence of abelian factors, the integration constants may be fixed requiring \( W\mu_i^V = \mu_i^{[V,W]} \). The Killing potentials \( \{\mu_i^V\} = \mu_i \) define a hyper-Kähler moment map

\[ \mu_i : \xi \in \Xi \mapsto \mu_i^a(\xi) \in \mathbb{R}^3 \times \mathcal{G}^* \quad i = 1, 2, 3 \quad a = 1, \ldots, \text{dim}(G) \quad (2.3) \]

where \( \mathcal{G}^* \) is the dual to the Lie algebra, \( \mathcal{G} \), of the isometry group \( G \) generated by the triholomorphic vectors, \( V \). The moment maps in (2.3) can be suggestively reorganized in the combinations \( \mu_\mathbb{R} = \mu_3 \) and \( \mu_\mathbb{C} = \mu_1 + i\mu_2 \).

Let us make an example to clarify the meaning of these concepts. In Hamiltonian mechanics, there is a natural two form:

\[ \omega = dp^i \wedge dq^i \quad (2.4) \]

where the \( q^i \)'s and the \( p^i \)'s are the generalized coordinates and momenta. If we take the isometries to be translations, \( V = a^i \frac{\partial}{\partial q^i} \), then \( i_V \omega = a^i dp^i = d\mu \) and \( \mu \) is the usual linear momentum, whence the name. If the \( V \)'s were the generators of the rotation group, \( SO(3) \), then the moment map would be given by \( \vec{\mu} = \vec{q} \wedge \vec{p} \), i.e. by the angular momentum. It should be clear now that the moment map construction is a way of generalizing the procedure of identifying conserved quantities in classical mechanics.

Any hyper-Kähler manifold \( \Xi \) admitting a compact group \( G \) of \( k \) freely acting triholomorphic isometries, contains a hyper-Kähler submanifold \( X \) of real dimension

\[ \text{dim}(X) = \text{dim}(\Xi) - 4\text{dim}(G) = 4n - 4k \quad (2.5) \]

which is the hyper-Kähler quotient of \( \Xi \) with respect to \( G \). The construction of \( X \) proceeds in two steps. First a submanifold \( P_\zeta \) of dimension \( \text{dim}(P_\zeta) = \)
\text{dim}(\Xi) - 3\text{dim}(G) = 4n - 3k$, is identified as the level set of $3 \times k$ hyper-Kähler moment maps, \textit{i.e.}

\[ P_{\zeta} = \{ \xi \in \Xi \mid \mu_{a}^{\xi}(\xi) = \zeta_{a}^{a}, \quad i = 1, 2, 3 \quad a = 1, \ldots, k \} \quad (2.6) \]

When the $\zeta$’s belong to $\mathbb{R}^{3} \times \mathbb{Z}^{*}$, with $\mathbb{Z}^{*}$ the dual to the center of $G$, the hypersurface $P_{\zeta}$ is left invariant by the action of $G$. In fact $P_{\zeta}$ turns out to be a $G$-principal bundle. If one divides out the group $G$ from $P_{\zeta}$, one obtains a new hyper-Kähler manifold, $X_{\zeta} = P_{\zeta}/G$. Notice that the algebro-geometric quotient $N_{\zeta}/G^{C}$, with

\[ N_{\zeta} = \{ \xi \in \Xi \mid \mu_{c}^{a}(\xi) = \zeta_{c}^{a}, \quad a = 1, \ldots, k \} \quad (2.7) \]

and $G^{C}$ the complexification of $G$, is diffeomorphic to $X_{\zeta}$. As a by-product of this construction one finds that the curvature of the natural connection on the principal bundle $P_{\zeta}$ over $X_{\zeta}$ is self-dual \cite{20, 11}.

\section{2.2 A-D-E Classification of ALE Gravitational Instantons}

Kronheimer constructed all ALE gravitational instantons as particular hyper-Kähler quotients \cite{17}. Given a discrete Kleinian subgroup of $\text{SU}(2)$, \textit{i.e.} $\Gamma = \mathbb{Z}_{N}, D_{N}^{*}, O^{*}, T^{*}, I^{*}$, consider the flat hyper-Kähler space

\[ \Xi = (Q \otimes \text{End}(R))_{\Gamma} \quad (2.8) \]

where $Q(\sim \mathbb{C}^{2})$ is the vector space of the fundamental two-dimensional representation, $\rho_{Q}$, of $\Gamma$ and $\text{End}(R)$ is the set of endomorphisms of the vector space, $R$, of the regular representation, $\rho_{R}$, of $\Gamma$. $\Xi$ is the space of $\Gamma$-invariant pairs of endomorphisms of $R$. Since for the regular representation
\( \dim(R) = \dim(\Gamma) = |\Gamma| \), the elements of \( \Xi \) can be represented as doublets of the form

\[
\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}
\]

(2.9)

where \( \alpha, \beta \) are \( |\Gamma| \times |\Gamma| \) complex matrices satisfying the \( \Gamma \)-invariance property

\[
\begin{pmatrix}
\rho_R(\gamma) \alpha \rho_R(\gamma^{-1}) \\
\rho_R(\gamma) \beta \rho_R(\gamma^{-1})
\end{pmatrix}
= \rho_Q(\gamma)
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
\quad \gamma \in \Gamma
\]

(2.10)

The points \( \xi \) of the manifold \( \Xi \) can also be represented by quaternions of matrices

\[
\xi = \begin{pmatrix} \alpha & -\beta^t \\ \beta & \alpha^t \end{pmatrix}
\]

(2.11)

so that \( \Xi \) can also be viewed as the space \( \Xi = (T^* \otimes_{\mathbb{R}} \text{Endskew}(R))_\Gamma \), with \( T \) the quaternionic space and \( T^* \) its dual.

The role of the discrete subgroup \( \Gamma \subset SU(2) \) in (2.10), is clarified by the Mac Kay correspondence \cite{21} between Kleinian subgroups \( \Gamma \subset SU(2) \) and A-D-E extended Dynkin diagrams, \( \tilde{\Delta}_\Gamma \).

We recall that, given a representation \( \rho_w \) of \( \Gamma \), in order to find its decomposition, \( \rho_w = \bigoplus_{i=0}^{r-1} w_i \rho_i \), in irreducible representations of \( \Gamma \) (\( \rho_0 \equiv \rho \), where \( \rho_0 \) is the trivial representation), we must look at the extended Dynkin diagram, \( \tilde{\Delta}_\Gamma \), of the A-D-E Lie algebra associated to \( \Gamma \). In the above decomposition \( r \) is the rank of \( \Gamma \), \( i.e. \) the number of conjugacy classes of \( \Gamma \). The extended Dynkin diagram is constructed, starting from the standard Dynkin diagram, \( \Delta_\Gamma \), according to the following steps

\begin{enumerate}
\item an extra dot, corresponding to minus the highest root of the algebra
\end{enumerate}

(called the extended root, \( \alpha_0 \)), is added to \( \Delta_\Gamma \);
ii) a number, $n_i$, equal to half the sum of the numbers attributed to the neighbouring roots is associated to each dot of $\bar{\Delta}_\Gamma$, starting from the assignment $n_0 = 1$ made to the extended root.

In this way one finds $\alpha_0 = -\sum_{i=1}^{r-1} n_i \alpha_i$ and the numbers $n_i$ turn out to coincide with the dimensions of the irreducible representations $\rho_i$ of $\Gamma$. In the case $\Gamma = \mathbb{Z}_N$, the extended Dynkin diagram corresponds to that of the Lie algebra of $SU(N)$ (the algebra $A_{N-1}$ in the A-D-E classification), with all $n_i = 1$, as shown in Fig.1. As a consequence, $\rho_w$ decomposes in one-dimensional representations.

\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {$\oplus$};
  \node (2) at (-1,-1) {$\circ$};
  \node (3) at (1,-1) {$\circ$};
  \node (4) at (0,-2) {$\circ$};
  \draw (1) -- (2);
  \draw (1) -- (3);
  \draw (3) -- (4);
  \draw (2) -- (4);

defines
\end{tikzpicture}
\end{center}

**Figure 1**

Denoting by $R_i$ the space of the irreducible $n_i$-dimensional representations of $\Gamma$, the decomposition

$$Q \otimes R_i = \oplus_j A_{ij} R_j$$

(2.12)

makes the correspondence between Kleinian subgroups $\Gamma \subset SU(2)$ and A-D-E extended Dynkin diagrams quite explicit, as it turns out that $A_{ij} = 2\delta_{ij} - \bar{C}_{ij}$ with $\bar{C}_{ij}$ the extended Cartan matrix of $\bar{\Delta}_\Gamma$ \cite{22}. For later use we remark here that the vector $n = (n_0, n_1, \ldots, n_{r-1})$, whose components are the dimensions of the irreducible representations of $\Gamma$, belongs to the null
space of the extended Cartan matrix, i.e. \( \sum_{j=0}^{r-1} \tilde{C}_{ij} n_j = 0 \).  

The vector space carrying the regular representation admits the decomposition

\[
R = \bigoplus_i R_i \otimes \tilde{R}_i
\]

(2.13)

In the following we will be interested in the action of \( \Gamma \) by left multiplication only. Since left multiplication leaves \( \tilde{R}_i \) untouched, one can simply write \( R = \bigoplus_i R_i \otimes \mathbb{C}^{n_i} \). In this respect \( \Gamma \) can be taken to be a subgroup of the same \( SU(2)_L \) factor of the Lorentz group in which the self-dual spin connection of the resulting ALE instanton will lie.

Imposing \( \Gamma \)-invariance in the form of (2.10), one finds the decomposition

\[
\Xi = \bigoplus_{ij} A_{ij} \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})
\]

(2.14)

which shows that the (real) dimension of \( \Xi \) is

\[
\dim(\Xi) = 2 \sum_{i,j=0}^{r-1} A_{ij} n_i n_j = 4 \sum_{i=0}^{r-1} (n_i)^2 = 4|\Gamma|
\]

(2.15)

To prove (2.14) we write successively

\[
\Xi = (Q \otimes \text{End}(R))_\Gamma = (Q \otimes \text{Hom}(\bigoplus_i R_i \otimes \mathbb{C}^{n_i}, \bigoplus_j R_j \otimes \mathbb{C}^{n_j}))_\Gamma \\
= (Q \otimes (\bigoplus_{ij} \text{Hom}(R_i, R_j)))_\Gamma \otimes \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \\
= (\bigoplus_{ijk} A_{ik} \text{Hom}(R_k, R_j))_\Gamma \otimes \text{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})
\]

(2.16)

having used the decomposition (2.12). Since \( \text{Hom}(R_i, R_j)_\Gamma = \delta_{ij} \) thanks to Schur’s lemma, equation (2.14) immediately follows.

Using the representation (2.11), the three hyper-Kähler forms on \( \Xi \) can be written

\[
\omega_i = \text{Tr}(d\xi^i \wedge d\xi^i P) \quad i = 1, 2, 3
\]

(2.17)

\footnote{From now on, unless differently stated, all the sums and products over indices labelling the irreducible representations of \( \Gamma \) are understood to be extended from 0 to \( r - 1 \).}
where the $\sigma^p_i$'s are the standard Pauli matrices. They can be combined into a real $(1, 1)$-form and a complex $(2, 0)$-form as follows

$$\begin{align*}
\omega_R &= \text{Tr}(d\alpha \wedge d\alpha^\dagger) + \text{Tr}(d\beta \wedge d\beta^\dagger) \\
\omega_C &= \text{Tr}(d\alpha \wedge d\beta)
\end{align*}$$

(2.18)

The action of a generic element $g \in U(|\Gamma|)$ on $\xi \in \Xi$ is induced by the transformations

$$\alpha \to g\alpha g^\dagger, \quad \beta \to g\beta g^\dagger$$

(2.19)

and leaves invariant the hyper-Kähler forms (2.18) and the flat metric on $\Xi$. The representation space $R$ is naturally acted upon by $U(|\Gamma|)$. Requiring $\Gamma$-invariance on $R$ reduces $U(|\Gamma|)$ to $G' = \bigotimes_{i=0}^{r-1} U(n_i)$. The group of freely acting triholomorphic isometries of $\Xi$ that will be used to perform the hyper-Kähler quotient is $G = \bigotimes_{i=1}^{r-1} U(n_i)$, where the factor $U(n_0) = U(1)$, acting trivially on $\Xi$, has been eliminated. Using the invariance of (2.18) under the transformation (2.19), one can construct the hyper-Kähler moment maps which turn out to be $\mu_C = [\alpha, \beta]$ and $\mu_R = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]$ (2.20)

The level sets, $P_\zeta$ and $N_\zeta$, defined in (2.6) and (2.7), can be explicitly identified with

$$\begin{align*}
[\alpha, \beta] &= \zeta_C \\
[\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] &= \zeta_R
\end{align*}$$

(2.21)

where $\zeta \in \mathbb{R}^3 \otimes \mathbb{Z}^*$, with $\mathbb{Z}^*$ the dual to the center of the Lie algebra of $G$, i.e. $\zeta = \bigoplus_{i=0}^{r-1} \zeta_i \Pi_{n_i}$ with $\sum_{i=0}^{r-1} \zeta_i = 0$. The last step of the hyper-Kähler quotient can be eventually performed and the resulting space $X_\zeta = P_\zeta/G$ turns out to have dimension $\dim(X_\zeta) = \dim(\Xi) - 4\dim(G) = 4|\Gamma| - 4(|\Gamma| - 1) = 4$.
For generic values of the deformation parameters, $\zeta$, $X_\zeta$ is a smooth hyper-Kähler manifold. In the final form of the metric the parameters $\zeta_C$, which describe the deformations of the complex structure, can be reabsorbed through a non-analytic change of coordinates on $X$, and the choice $\zeta_C = 0$ can be made without loosing generality, much as in the case of gauge instanton connections where the parameters associated to global gauge transformations are not explicitly displayed.

On the contrary, setting the deformation parameters $\zeta_R = 0$, we get the orbifold $X_0 = \mathbb{C}^2/\Gamma$. Kronheimer has shown [17] that every ALE hyper-Kähler four-manifold $X_\zeta$ is diffeomorphic to the minimal resolution of $X_0$, thus proving the completeness of the construction we have just described.

An explicit example of the hyper-Kähler quotient construction is given in the appendix where the EH metric is derived.

3 Tautological Bundle and Abelian Instantons

As already mentioned, the principal bundle $P_\zeta$ admits a natural connection with self-dual curvature. By a change of fiber, a tautological vector bundle $\mathcal{T}$ with typical fiber $R$ can be associated to $P_\zeta$. This vector bundle is tautological in the sense that (see (2.8)) the points of the base manifold, $X$, are endomorphisms of $R$ itself. Under the action of $\Gamma$, the tautological bundle $\mathcal{T}$ admits the decomposition: $\mathcal{T} = \oplus_i \mathcal{T}_i \otimes \bar{R}_i$. Apart from the trivial bundle $\mathcal{T}_0$ associated to the trivial (one-dimensional) representation of $\Gamma$, the elementary bundles $\mathcal{T}_i$ admit self-dual connections which are asymptotic to flat connections with holonomy determined by the representation $\rho_i$. The first Chern classes, $c_1(\mathcal{T}_i), i \neq 0 (c_1(\mathcal{T}_0) = 0)$, form a basis of the cohomology.
group, $H^2(X, \mathbb{R})$, and satisfy

$$\int_X c_1(T_i) \wedge c_1(T_j) = (C^{-1})_{ij} \quad i, j = 1, \ldots, r - 1 \quad (3.1)$$

where $C^{-1}$ is the inverse of the Cartan matrix of the unextended Dynkin diagram, $\Delta_T$. The dual homology basis of $H_2(X, \mathbb{Z})$ consists of the two-spheres, $\Sigma_i$, which arise from the resolution of the exceptional set in $c^2/\Gamma$ \cite{14, 17}.

ALE manifolds associated to the discrete groups, $\Gamma = \mathbb{Z}_N$, i.e. to the Dynkin diagrams of Lie algebras of $A_{N-1}$ type, admit the multicenter (self-dual) metrics \cite{14, 15}

$$ds^2 = V^{-1}(\vec{x})(dt + \vec{\omega} \cdot d\vec{x})^2 + V(\vec{x})d\vec{x} \cdot d\vec{x} \quad (3.2)$$

with

$$V(\vec{x}) = \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i|} \quad (3.3)$$

The functions $V(\vec{x})$ represent the localized solutions of the equation $\nabla^2 V = 0$, which follows from the self-duality condition on the spin connection

$$\vec{\nabla} V = \vec{\nabla} \times \vec{\omega} \quad (3.4)$$

When, as in this case, the base manifold, $X$, is a multicenter ALE gravitational instanton, the explicit expression of the self-dual curvatures of $T_i$ may be found using the following argument. The $N - 1$ minimal surfaces, $\Sigma_i$, homologically equivalent to the uncontractible two-spheres, mentioned above, can be taken to be \cite{14}

$$\Sigma_i = \{(\vec{x}, t) \in X : \vec{x} = \vec{x}_i + \lambda(\vec{x}_{i+1} - \vec{x}_i); \lambda \in (0, 1), t \in (0, 4\pi)\} \quad (3.5)$$

The intersection form of these $N - 1$ two-spheres is given by (minus) the Cartan matrix $C$ of $A_{N-1}$. By identifying the periods of the three Kähler
forms along the $\Sigma_i$ with the moduli of the multicenter metrics (3.2), the
latter can be shown to coincide with the positions of the centers, $\bar{x}_i$. An explicit basis for $H^2(X, \mathbb{R})$, consisting of $N$ self-dual two-forms, may be
found starting from the ansatz [23]

$$F = E_l(\bar{x})(e^\alpha \wedge e^\beta + \frac{1}{2} \epsilon_{\alpha\beta\gamma} e^\gamma \wedge e^\delta) \quad \alpha, \beta = 1, 2, 3$$

(3.6)

with $E_l(\bar{x})$ functions of $\bar{x}$ to be determined by the closure condition $dF = 0$. If we choose the tetrad corresponding to (3.2) to be

$$e^0 = e^0_{\mu} dx^\mu = V^{-\frac{1}{2}} (dt + \vec{\omega} \cdot d\bar{x}), \quad e^l = e^l_{\mu} dx^\mu = V^{\frac{1}{2}} dx^l$$

(3.7)

the two-forms (3.6) become

$$F = E_l(\bar{x}) B_l$$

(3.8)

with

$$B_l = (dt + \vec{\omega} \cdot d\bar{x}) \wedge dx^l + \frac{1}{2} \epsilon_{\mu\nu} V dx^\mu \wedge dx^\nu$$

(3.9)

Imposing the closure condition, one finds the $N$ solutions $E_l^{(i)} = \nabla_l (V^{(i)}/V)$, where $V^{(i)} = 1/|x - x_i|$ and $\vec{\nabla} V^{(i)} = \vec{\nabla} \times \vec{\omega}^{(i)}$. From the definition of $V$ in (3.3), it is clear that $\sum_i E_l^{(i)} = 0$, so that only $N - 1$ out of the $N$ two-forms, $F^{(i)} = E_l^{(i)}(\bar{x}) B_l$ are linearly independent. A convenient choice of basis is $\mathcal{F}^i = \frac{1}{4\pi} (F^{(i)} - F^{(i+1)})$ for $i = 1, \cdots, N - 1$. In order to relate $\mathcal{F}^i$ to $c_1(T_i)$, which are dual to the chosen homology basis in (3.3), i.e. satisfy

$$\int_{\Sigma_j} c_1(T_i) = \delta^i_j$$

(3.11)

$$\int_{\Sigma_j} \mathcal{F}^i = C^{ij} \int_X \mathcal{F}^i \wedge \mathcal{F}^j = C^{ij}$$

(3.10)

Comparing (3.1) with (3.10), one concludes that the relation between the two bases $\{\mathcal{F}^i\}$ and $\{c_1(T_i)\}$ is

$$c_1(T_i) = \sum_j (C^{-1})_{ij} \mathcal{F}^j$$

(3.11)
In the KN construction of YM instantons on \(X\), which will be described in the next section, it will prove convenient to introduce the monopole (or better the abelian instanton) potentials

\[
A^i = A^{i}_\mu dx^\mu = \frac{1}{4\pi} V^{-\frac{1}{2}} \left( (V^{(i)} - V^{(i+1)}) e^\alpha + (\omega^{(i)} - \omega^{(i+1)}) \cdot e^\alpha \right)
\]  

(3.12)

such that locally \(F^i = dA^i\). Using (3.12), it will be similarly possible to write the self-dual two forms, \(c_1(T_i)\), in terms of the monopole potentials \(A^{T_i} = (C^{-1})_{ij} A^j\).

We now wish to make a brief detour on abelian gauge theory and discuss some issues concerning electromagnetic duality on ALE spaces. To this purpose let us consider a Maxwell theory with a \(\theta\)-term [24, 25]. The action for this theory is

\[
S = \frac{1}{e^2} \int_X F \wedge \ast F - \frac{i\theta}{8\pi^2} \int_X F \wedge F
\]

(3.13)

where \(F = dA\) and \(e\) is the electromagnetic coupling constant. The gauge field can be split into \(A = A_{cl} + A_{qu}\) where, in order to have a globally defined gauge connection on \(X\), \(A_{cl}\) is a linear combination of the instanton potentials (3.12) with integer coefficients. As a consequence, the classical part of the field strength \(F_{cl}\) will have in the basis (3.11) the expansion

\[
\int_{\Sigma_i} F_{cl} = 2\pi m^i
\]

(3.14)

The classical action takes the form

\[
S_{cl}[m^i] = \frac{4\pi^2}{e^2} m^i G_{ij} m^j - \frac{i\theta}{2} m^i Q_{ij} m^j = \left( \frac{4\pi^2}{e^2} - \frac{i\theta}{2} \right) m^i (C^{-1})_{ij} m^j
\]

(3.15)

because for ALE instantons \(G_{ij} = Q_{ij} = (C^{-1})_{ij}\), as it follows from the absence of anti-self-dual two-forms and (3.1). The partition function for this Maxwell theory is then given by

\[
Z_M = \frac{\det'(\Delta_F)}{\det'(\Delta_B)^\frac{1}{2}} \sum_{m^i \in \mathbb{Z}} e^{-S_{cl}[m^i]}
\]

(3.16)
where $\Delta_B$ and $\Delta_F$ are the kinetic operators for the quadratic fluctuations (the only ones present in an abelian theory) of boson and fermion fields respectively. As usual, the primes mean that the determinants are to be computed in the functional space orthogonal to the zero-modes. In the case under consideration, this restriction is immaterial since there are neither fermionic nor bosonic neutral zero-modes, as we shall see in section 5. Moreover, specializing to a supersymmetric theory, the functional determinants exactly cancel.

Putting $\tau = \frac{4\pi i}{e^2} + \frac{\theta}{2\pi}$ the partition function becomes

$$Z_M = \sum_{m^i} e^{i\pi \tau (m^i C_{ij}^{-1} m^j)} = \eta(\tau)^{N-1} \sum_{\Lambda} \chi_\Lambda(\tau) \tag{3.17}$$

where $\eta(\tau)$ is the Dedekind function and $\chi_\Lambda(\tau)$ are characters of the unitary representations of highest weight, $\Lambda$, of the affine Lie algebra associated to $\tilde{\Delta}_\Gamma$ at level one [26]. For $A$-type ALE spaces one has the affine Lie algebra of $SU(N)$ at level one. The generalized electromagnetic duality (S-duality) is induced by the transformation [24, 25]

$$S : \tau \to -\frac{1}{\tau} \tag{3.18}$$

The characters $\chi_\Lambda$ provide a unitary representation of $S$, in the sense that

$$\chi_\Lambda(-\frac{1}{\tau}) = \frac{1}{\sqrt{N}} \exp(2\pi i \Lambda \Lambda') \chi_{\Lambda'}(\tau) \tag{3.19}$$

From (3.19) and $\eta(-1/\tau) = \sqrt{\tau} \eta(\tau)$ it follows that, summing over $\Lambda$, the S-transformed partition function is proportional to $\chi_0(\tau)$, the character of the singlet representation of the affine Lie algebra $SU(N)$, hence (3.17) is not invariant under the transformation (3.18). The lack of S-invariance of a Maxwell theory on ALE spaces, unlike what happens on other hyper-Kähler manifolds [24] (e.g. the four-torus $T^4$ and the Kümmers third surface $K3$) is due to the properties of the lattice of the charges $m^i$. For $T4$ and
K3 the relevant lattices are $L_{(4,4)}$ and $L_{(3,19)}$ respectively, while for multi-center ALE instantons one has the weight lattice of $SU(N)$. The former are even, self-dual, Lorentzian lattices, while the latter is an $(N - 1)$-dimensional Euclidean lattice which is neither even nor self-dual. Additional terms in the action which arise upon coupling to gravity do not make the above result consistent with the expected modular transformation of $Z$ for compact manifolds \[25\], which reads

\begin{equation}
Z\left(-\frac{1}{\tau}\right) = \tau^u \bar{\tau}^v Z(\tau)
\end{equation}

where $u = (\chi_E - \tau_H)/4$ and $v = (\chi_E + \tau_H)/4$, $\chi_E$ being the Euler characteristic and $\tau_H$ the Hirzebruch signature of the manifold. For multicenter ALE metrics $\chi_E = N$ and $\tau_H = 1 - N$, so that $u = N/2 - 1/4$ and $v = 1/4$, and only the dominant powers of $\tau$ in (3.20) agree with the transformation properties of $Z_M$ under (3.18). In view of this result, it appears that the issue of S-duality on ALE spaces must be more carefully reconsidered \[27\].

\section{Kronheimer-Nakajima Construction of Yang-Mills Instantons on ALE Spaces}

\subsection{The ADHM Construction on $\mathbb{R}^4$}

Before embarking in the KN construction, we would like to recall few relevant facts about the ADHM construction on $\mathbb{R}^4$ which will be carried over to ALE spaces.

Self-dual $SU(2)$ connections on $S^4$, can be put into one to one correspondence with holomorphic vector bundles of rank 2 over $\mathbb{C}P^3$ admitting a reduction of the structure group to its compact real form. The ADHM construction \[12\] gives all these holomorphic bundles and consequently all $SU(2)$ connections on $S^4$. The construction is purely algebraic and we find
it more convenient to use, at the beginning, quaternionic notations. The points, \( x \), of the one-dimensional quaternionic space \( \mathbb{H} \equiv \mathbb{C}^2 \equiv \mathbb{R}^4 \) can be conveniently represented in the form \( x = x^\mu \sigma_\mu \), with \( \sigma_0 = \mathbb{1} \) and \( \sigma_r = i \sigma^P_r \). The conjugate of \( x \) is \( x^\dagger = x^\mu \sigma^\dagger_\mu \). A quaternion is said to be real if it is proportional to \( \mathbb{1} \) and imaginary if it has vanishing real part. The identity \( \sigma_\mu \sigma^\dagger_\nu = \delta^- \pm i \eta_{\mu \nu} \sigma^P_r \), where the \( \eta_{\mu \nu} \)'s are the 't Hooft symbols, can be used to write the one-instanton solution of Belavin, Polyakov, Schwarz and Tyupkin \[28\] in the form

\[
A = \Im \left( \frac{x^\dagger dx}{|x|^2 + \lambda^2} \right)
\]

where \( \Im(q) = (q - q^\dagger)/2 \). To find the number of moduli which parametrize this solution we act on it with the transformations of the symmetry group of the YM action, which turns out to be the conformal group \[5\]. We find in this way that only the five-parameter transformations of the type

\[
x \mapsto D = \mu(x_0 - x), \quad \mu \in \mathbb{R}, \quad x_0 \in \mathbb{H}
\]

deform the solution. The inclusion of the three parameters related to global \( SU(2) \) rotations in \( (4.2) \) can be accomplished by promoting \( \mu \) to be a quaternion.

The structure of \( (4.1) \) and \( (4.2) \) can be generalized to the case of instantons with arbitrary winding number, \( k \), by replacing in \( (4.1) \) \( x \) with a column of \( k + 1 \) quaternions, \( x \rightarrow U(x) \in \mathbb{H}^{k+1} \) and in \( (4.2) \) \( \mu \) with a \( k \)-dimensional quaternionic vector, \( \mu \rightarrow q \in \mathbb{H}^k \), and at the same time \( x_0 \) with a \( k \times k \) matrix of quaternions, \( x_0 \rightarrow a_0 \). By accomodating the vector \( q \) and the matrix \( a_0 \) in a single \( (k + 1) \times k \) matrix, \( a = \begin{pmatrix} a_0 \\ q \end{pmatrix} \), \( D \) in \( (4.2) \) can be generalized to \( D = a + bx \), where \( b \) is another \( (k + 1) \times k \) quaternionic matrix and \( bx \) means multiplication of each element of \( b \) by the quaternion \( x \).

\[5\]In quaternionic notations the conformal group is \( SL(2, \mathbb{H}) \).
The (anti-hermitean) gauge connection is written in the form

\[ A_\mu = U^\dagger \partial_\mu U \] (4.3)

where \( U \) is a \((k + 1) \times 1\) matrix of quaternions providing an orthonormal frame of \( \text{Ker} \mathcal{D}^\dagger \). In formulae

\[ \mathcal{D}^\dagger U = 0 \] (4.4)

\[ U^\dagger U = \mathbb{I}_2 \] (4.5)

where \( \mathbb{I}_2 \) is the two-dimensional identity matrix. The constraint (4.3) ensures that \( A_\mu \) is an element of the Lie algebra of the \( SU(2) \) gauge group. The condition of self-duality on the field strength of (4.3) is imposed by restricting the matrix \( \mathcal{D} \) to obey

\[ \mathcal{D}^\dagger \mathcal{D} = \Delta \otimes \mathbb{I}_2 \] (4.6)

with \( \Delta \) an invertible hermitean \( k \times k \) matrix (of complex numbers). In addition to the gauge freedom (right multiplication by a unitary quaternion) we have the freedom to left multiply the matrices \( a, b \) by a unitary \( k \times k \) quaternionic matrix. These symmetries that can be used to simplify the expressions of \( a, b \) and \( U \).

The extension of (4.1) to the case of gauge groups other than \( SU(2) \) is accomplished by further enlarging the dimensions of \( \mathcal{D} \), while equations (4.3), (4.4), (4.5) and (4.6) remain formally unchanged.

For instance, for \( Sp(2n) \) gauge groups \( a \) and \( b \) are \((k + n) \times k\) matrices of quaternions, where \( b \) can always be cast in the standard form

\[ b = \begin{pmatrix} \mathbb{I}_{k \times k} \\ O_{n \times k} \end{pmatrix} \] (4.7)

\(^6\)Using physicist’s nomenclature we call \( Sp(2n) \) the group of \( 2n \)-dimensional matrices, \( g \), with the property \( g^T J g = J \), where \( J \) is the symplectic (antisymmetric) metric.
by exploiting the symmetries of the construction, and $U$ becomes a $(k+n) \times n$ matrix of quaternions.

With minor changes a similar procedure can be used to construct $U(n)$ connections. In this case $U$ and $D$ can be taken as $(2k + n) \times n$ and $(2k + n) \times 2k$ complex matrices, respectively. To impose the constraint (4.6) it is convenient to split $D$ into two matrices, $D_r = a_r + b_r x$, $r = 1, 2$, of $(2k+n) \times k$ elements each. The condition $D^\dagger D \propto 1 1$ leads to

$$a_r^\dagger a_s = \nu \delta_{rs} \tag{4.8}$$

$$b_r^\dagger b_s = \nu' \delta_{rs}, \quad \epsilon_{rs} b_s^\dagger a_t = \epsilon_{st} a_s^\dagger b_r \tag{4.9}$$

where $\nu, \nu'$ are $k \times k$ hermitean matrices and $\epsilon_{rs}$ is the two-dimensional anti-symmetric tensor. Using (4.9) and the $U(k)$ symmetry of the construction, $a, b$ can be cast into form

$$a = \begin{pmatrix} A & -B^\dagger \\ B & A^\dagger \\ s & t^\dagger \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4.10}$$

where $A, B$ and $s, t^\dagger$ are $k \times k$ and $n \times k$ complex matrices respectively, subjected to the constraints coming from (4.8).

For $U(n)$ gauge groups, the set

$$M = \{ A, B; s, t \mid A, B \in \text{End}(V); s, t^\dagger \in \text{Hom}(V, W) \} \tag{4.11}$$

where $V, W$ are $k$-dimensional and $n$-dimensional complex vector spaces respectively, is often called a set of ADHM data. The real dimension of $M$ is $\dim(M) = 4k^2 + 4kn$.

It is very much inspiring to reformulate the ADHM construction in the language of the previous section. To this end we first notice that, in terms of the matrices $A, B, s, t^\dagger$, the condition for $D^\dagger D$ not to have off-diagonal blocks becomes

$$[A, B] + ts = 0 \tag{4.12}$$
while the conditions for the diagonal blocks of $D^\dagger D$ to be all proportional to the same $k \times k$ hermitean matrix, $\Delta$, is

$$([A, A^\dagger] + [B, B^\dagger]) - s^\dagger s + tt^\dagger = 0 \quad (4.13)$$

Observing that the three closed two-forms

$$\omega_C = \text{Tr}(dA \wedge dB) + \text{Tr}(dt \wedge ds)$$
$$\omega_R = \text{Tr}(dA \wedge dA^\dagger) + \text{Tr}(dB \wedge dB^\dagger) - \text{Tr}(ds^\dagger \wedge ds) + \text{Tr}(dt \wedge dt^\dagger) \quad (4.14)$$

are invariant under the transformations $A \mapsto gAg^\dagger$, $s \mapsto gsv^\dagger$ with $g \in U(k)$ and $v \in U(n)$ (and the same for $B, t^\dagger$), one concludes that the moment maps for the triholomorphic $U(k)$ isometries are exactly given by the l.h.s. of the ADHM equations $(4.12)$ and $(4.13)$. In the r.h.s. of these equations, there is no deformation parameter, such as the $\zeta$’s appearing in (2.6), as a consequence of the fact that the base manifold, $\mathbb{R}^4$, is undeformable. Equations $(4.12)$ and $(4.13)$ yield $3k^2$ real constraints on $M$. Taking into account the residual $U(k)$ invariance, the hyper-Kähler quotient of $M$ with respect to $U(k)$ has dimension $4k^2 + 4kn - 3k^2 - k^2 = 4kn$, and coincides with the framed moduli space of $SU(n)$ self-dual connections on $\mathbb{R}^4$.

### 4.2 The ADHM construction on ALE Spaces

The initial step in the KN construction (for $U(n)$ gauge groups) is to give a set of ADHM data $M = \{A, B; s; t; \xi\}$ where $\xi \in \Xi$, $A$ and $B$ are $\Gamma$-equivariant endomorphisms of a $k$-dimensional complex vector space, $V$, and $s, t^\dagger$ is a pair of homomorphisms between $V$ and an $n$-dimensional complex vector space $W$. Both $V$ and $W$ are $\Gamma$-modules, i.e. they admit the decomposition

\footnote{$\Gamma$-equivariance simply means that the matrices $\xi$, $A$ and $B$ are naturally decomposed into $n_i \times n_j$-dimensional blocks.}
\[ V = \oplus_i V_i \otimes R_i, \quad W = \oplus_i W_i \otimes R_i, \] 
where \( V_i \sim \mathbb{C}^{v_i}, \ W_i \sim \mathbb{C}^{w_i}. \) Therefore 
\[ k = \text{dim}(V) = \sum_i n_i v_i \quad n = \text{dim}(W) = \sum_i n_i w_i \quad (4.15) \]

where the lower case letters, \( v_i, w_i, \) stand for the dimensions of the corresponding vector spaces. Out of these data the matrix 
\[ \mathcal{D} = (A \otimes \mathbb{1} - \mathbb{1} \otimes \xi) \oplus \Psi \otimes \mathbb{1} \quad (4.16) \]
is constructed. In (4.16) we have used the definitions 
\[ A = \begin{pmatrix} A & -B^\dagger \\ B & A^\dagger \end{pmatrix} \in (T^* \otimes \text{End}(V))_{\mathfrak{g}} = \oplus_{i,j} A_{ij} \text{Hom}(V_i, V_j) \quad (4.17) \]
and 
\[ \Psi = (s \quad t^\dagger) \in \text{Hom}(S^+ \otimes V, W)_{\mathfrak{g}} = \oplus_i \text{Hom}(S^+ \otimes V_i, W_i) \quad (4.18) \]

We have arranged the matrices \( A \) and \( B \) into a quaternion of matrices \( A \), similarly to what has been done in (2.11) to represent the points \( \xi \) of the manifold \( \Xi \), or in (4.10). \( S^+ \) is isomorphic to the two-dimensional complex vector space, \( \mathbb{C}^2 \), and may be conveniently thought as the space of right-handed spinors.

The \((2k + n)|\Gamma| \times 2k|\Gamma|\) matrix \( \mathcal{D} \) represents the linear map 
\[ \mathcal{D} : (S^+ \otimes V \otimes \mathcal{T}) \mapsto (Q \otimes V \otimes \mathcal{T}) \oplus (W \otimes \mathcal{T}) \quad (4.19) \]
where \( \mathcal{T} \) is the tautological bundle and \( Q \), defined after (2.8), is to be identified with \( S^- \), the space of left-handed spinors (the dual of \( S^+ \)). Remembering that the base manifold, \( X_\zeta \), is a smooth resolution of \( \mathbb{C}^2 / \Gamma \), one is led to consider only the \( \Gamma \)-invariant part of (4.19), \( \mathcal{D}_\Gamma \). When \( \mathcal{D} \) is restricted in this way, it becomes a \((2k + n) \times 2k\) matrix which plays the same role as the matrix \( \mathcal{D} \) of the previous section: more precisely in the matrix \( A \oplus \Psi \) we recognize the matrix \( a \) of (4.10).
In order to generalize (4.4) we introduce the adjoint of the $\Gamma$-restriction of (4.19) as the mapping

$$D^\dagger_\Gamma: V \oplus W \mapsto U$$

(4.20)

where

$$U \equiv S^+ \otimes (\bar{V} \otimes T)_\Gamma$$

$$V \equiv (\bar{Q} \otimes \bar{V} \otimes T)_\Gamma$$

$$W \equiv (\bar{W} \otimes T)_\Gamma$$

(4.21)

and $\bar{Q}, \bar{V}, \bar{W}$ denote the trivial (i.e. product) vector bundles over $X_\xi$ with fiber $Q, V, W$ respectively.

Once $\xi \in \Xi$ has been restricted to lie on the ALE space $X$ by imposing

$$[\alpha, \beta] = \zeta_C$$

$$[\alpha, \alpha^\dagger] + [\beta, \beta^\dagger] = \zeta_R$$

(4.22)

self-duality of the resulting YM connection follows from the condition $D^\dagger_\Gamma D_\Gamma = \Delta \otimes 1 \mathbb{I}$ with $\Delta$ a hermitean $k \times k$ matrix and $1 \mathbb{I}$ the $2 \times 2$ identity matrix in $S^+$. This is accomplished by imposing the deformed version of the ADHM equations (4.12) and (4.13)

$$[A, B] + ts = -\zeta_C$$

$$[A, A^\dagger] + [B, B^\dagger] - s^\dagger s + tt^\dagger = -\zeta_R$$

(4.23)

where $\zeta = \bigoplus_{i=0}^{r-1} \zeta_i \mathbb{I}_{v_i} \in \mathbb{R}^3 \otimes Z^*_V$, with $Z^*_V$ the dual to the center of the Lie algebra of $G_V = \otimes_i U(v_i)$ and the parameters $\zeta_i$ in the r.h.s of (4.23) are identified with those in the r.h.s. of (4.22), thanks to the homomorphism of $U(n_i)$ in $U(v_i)$.

In view of the following applications, we refer more rigorously in (4.21) to bundles, and not to vector spaces, because we will have to consider sections on them, i.e. locally defined functions of the variable $x \in X$. 

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The condition (4.4) allows to identify the instanton bundle $\mathcal{E}$ with

$$\mathcal{E} = \text{Ker} \mathcal{D}^\dagger_{\Gamma} \subset \mathcal{V} \oplus \mathcal{W} \quad (4.24)$$

$\mathcal{E}$ is then a complex vector bundle on $X_\zeta$, with typical fiber $W$ and rank $n = \text{dim}(W)$. The YM connection on $\mathcal{E}$ is given by

$$A_\mu = U^\dagger \nabla_\mu U \quad (4.25)$$

where $U$ represents an orthonormal frame of sections of $\text{Ker} \mathcal{D}^\dagger_{\Gamma}$, i.e. a $(2k + n) \times n$ complex matrix obeying $\mathcal{D}^\dagger_{\Gamma} U = 0$ and $U^\dagger U = 1_n$. Since $\tilde{Q}, \tilde{V}, \tilde{W}$ are flat bundles, the covariant derivative on $(\tilde{Q} \otimes \tilde{V} \otimes T)_{\Gamma}$ takes the form

$$\nabla_\mu = (\partial_\mu + A^T_\mu), \text{ with } A^T_\mu \text{ the (self-dual) connection on the tautological bundle } T.$$ For multicenter ALE metrics the abelian connections on $T_i$ are given in (3.12).

### 4.3 Particular Solutions on ALE Spaces and $\Gamma$-invariant Instantons

Not unlike what happens on $\mathbb{R}^4$, the generalized ADHM equations on ALE spaces can be solved explicitly only in the simplest cases.

On $\mathbb{R}^4$ for the group $SU(2)$ solutions with generic winding number $k$, but with a number of parameters (moduli) smaller than the dimension of the moduli space have been found. For example the well-known 't Hooft ansatz

$$A^r_\mu = -\bar{\eta}^r_{\mu \nu} \partial^\nu \log \phi, \text{ where } \phi_{x, \mu} = 1 + \sum_{i=1}^k \frac{\rho_i^2}{(x - x_i)^2} \quad (4.26)$$

contains only $5k$ parameters, the centers and sizes of the instantons, instead of the expected $8k - 3$. Similarly the conformally invariant solution of Jackiw,
Nohl and Rebbi (JNR) \[19\] with
\[
\phi_{JNR} = \sum_{i=1}^{k+1} \frac{\lambda_i^2}{(y - y_i)^2}
\] (4.27)
effectively depends only on \(5k + 4\) parameters, without a clear geometrical meaning. The expected number of moduli is matched by (4.26) for \(k = 1\) and by (4.27) for \(k = 1, 2\). Actually also for \(k = 3\) one can find self-dual connections with the right number of moduli by directly solving the ADHM equations \[30\].

It is not difficult to relate (4.26) and (4.27) to \(SU(2)\) instantons on orbifolds \(X_0 = \mathbb{R}^4/\Gamma\), whose smooth resolutions are precisely the ALE manifolds we are interested in. In the orbifold limit, one is lead to consider self-dual connections on \(\mathbb{R}^4\) satisfying \(A_\mu(\gamma x) = A^\gamma_\mu(x)\) for \(\gamma \in \Gamma\), i.e. invariant under \(\Gamma\) up to gauge transformations. The condition of strict \(\Gamma\)-invariance, \(\Omega_\gamma = 1, \forall \gamma \in \Gamma\), is particularly simple to achieve for \(\Gamma = \mathbb{Z}_2\), i.e. for the orbifold limit of EH instanton. One simply has to place the centers appearing in (4.26) or (4.27) in \(\mathbb{Z}_2\) symmetric configurations. In particular for \(k\) even one may take
\[
\phi = 1 + \sum_{i=1}^{k/2} \left( \frac{\rho_i^2}{(x - x_i)^2} + \frac{\rho_i^2}{(x + x_i)^2} \right)
\] (4.28)
and for \(k\) odd
\[
\phi = \sum_{i=1}^{(k+1)/2} \left( \frac{\lambda_i^2}{(y - y_i)^2} + \frac{\lambda_i^2}{(y + y_i)^2} \right)
\] (4.29)
The number of free parameters in (4.28) and (4.29) is clearly reduced with respect to (4.26) and (4.27) and it happens to coincide with the actual dimensions of the moduli space of the instanton connections only for bundles with second Chern class \(\kappa = 1/2, 1\) and \(3/2\), corresponding to \(k = 1, 2\) and \(3\) in the above formulae. The ansatze (4.28) and (4.29) can be generalized to the other multicenter ALE metrics, where they amount to place the centers in a \(\mathbb{Z}_N\) symmetric way.
When $X$ is a generic smooth ALE manifold, another possibility is to consider restricted classes of solutions of the self-duality equations. One starts from the very simple and inspiring ansatz valid for $SU(2)$ given in \cite{7} for the case of multicenter ALE metrics \((3.2)\) which reads

$$A_0 = \frac{1}{2} A_0^P \sigma^P = \frac{1}{2} \tilde{G} \cdot \bar{\sigma}^P \quad \tilde{A} = \frac{1}{2} (\tilde{\omega}(\tilde{G} \cdot \bar{\sigma}^P) - V(\tilde{G} \times \bar{\sigma}^P)) \quad (4.30)$$

where $V$ is defined in \((3.3)\) and $\tilde{G}$ is taken to be independent from the cyclic coordinate $t$. The general ($t$-independent) solution of the self-duality equations is given by \cite{7}

$$\tilde{G} = -V^{-1} \nabla \log(H) \quad (4.31)$$

with $H = \sum_{i=1}^{n} \lambda_i / |x - x_i|$. Regularity of the gauge action restricts the sum in $H$ to a subset of the $N$ centers, $\bar{x}_i$, appearing in $V$ ($n \leq N$). The resulting $SU(2)$ connections have second Chern class $\kappa = n - 1/N \leq N - 1/N$. For $n = N$, the bound is saturated, $\tilde{G}$ equals $-V^{-2} \nabla V$ and the $SU(2)$ gauge connection \((4.30)\) coincides with the self-dual spin connection on the tangent bundle to $X$ \cite{7}.

The ansatz \((4.30)\) may be cast into a form which more clearly resembles \((4.26)\) and \((4.27)\). In fact, introducing the inverse tetrad $E_a$ with

$$E_0 = E_0^\mu \partial_\mu = V^{1/2} \frac{\partial}{\partial t} \quad E_i = E_i^\mu \partial_\mu = V^{-1/2}(\frac{\partial}{\partial x^i} - \omega_i^a \frac{\partial}{\partial t}) \quad (4.32)$$

one finds that \((4.30)\) can be generalized to

$$A_\mu^a dx^\mu = -\bar{\eta}_{ab} e^a E^b(\log H) \quad (4.33)$$

with $H$ now not necessarily independent from the coordinate $t$. Substituting \((4.33)\) into the self-duality equation $F = *F$, one gets for $H$

$$\frac{\nabla_a \nabla^a H}{H} = 0 \quad (4.34)$$
where $\nabla_a$ is the covariant derivative on $X$. To derive (4.34) the three covariantly constant complex structures, $I, J, K$, introduced in subsection 2.1, have been identified with $\bar{\eta}^i_{ab}$, and we have used the equations $E_a H = E^\mu_a \partial_\mu H = \nabla_a H$ and $\nabla_a \nabla_b H = \nabla_b \nabla_a H$. The solutions of (4.34) can be written in the form $H(x) = H_0 + \sum_i^n G(x, x_i)$, where $H_0$ is a constant and $G(x, x')$ is the scalar propagator in the background of the multicenter metric. $G(x, x')$ has been explicitly computed by Page [31]. Depending on whether the constant $H_0$ is zero or not, (4.33) turns out to have second Chern class $\kappa = n - 1/N$ or $\kappa = n$, respectively. The ansatz (4.33) seems to be more flexible than the fullfledged solutions of the KN-ADHM equations as far as the computation of zero-modes is concerned. Moreover, the ansatz (4.33) can be used to generate self-dual $SU(2)$ connections on other four-dimensional hyper-Kähler manifolds, such as, for instance, the $K3$ surface.

5 Topological Properties of Yang-Mills Instanton Bundles on ALE Manifolds

5.1 General Setting

In this subsection we illustrate how to compute some topological invariants of the instanton bundle $E$, such as the first and second Chern class and the dimension of its moduli space. Recalling

$$V = \bigoplus_i V_i \otimes R_i$$
$$W = \bigoplus_i W_i \otimes R_i$$
$$\mathcal{T} = \bigoplus_i T_i \otimes R_i$$

we can write

$$U = S^+ \otimes (\bar{V} \otimes \mathcal{T})_\Gamma = S^+ \otimes (\bigoplus_i \bar{V}_i \otimes T_i)$$

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\[ \mathcal{W} = (W \otimes T)_R = \oplus_i W_i \otimes T_i \]
\[ \mathcal{V} = (\bar{Q} \otimes \bar{V} \otimes T)_R = \oplus_{i,k} A_{ik} \bar{V}_i \otimes T_k \]  
(5.2)

where we have used the decompositions (2.12), (2.13).

From these formulae the first and second Chern class of \( \mathcal{E} \) can be expressed as [9]

\[ c_1(\mathcal{E}) = c_1(\mathcal{V}) + c_1(\mathcal{W}) - c_1(\mathcal{U}) = \sum_i (w_i - \sum_j \tilde{C}_{ij} v_j) c_1(T_i) = \sum_i u_i c_1(T_i) \]
\[ c_2(\mathcal{E}) = \sum_i u_i c_2(T_i) + \frac{k}{|\Gamma|} \]  
(5.3)

with

\[ u_i \equiv w_i - \sum_j \tilde{C}_{ij} v_j \quad i = 1, \ldots, r - 1 \]  
(5.4)

In equations (5.3) and (5.2) the term with \( i = 0 \) does not actually appear in the sums over \( i \), because the triviality of \( T_0 \) implies \( c_1(T_0) = 0 \) and \( c_2(T_0) = 0 \).

Making use of the relation \( \sum_{j=0}^{r-1} \tilde{C}_{ij} n_j^2 = 0 \) and recalling that \( n_0 = 1 \), it is possible to consistently extend the definition (5.4) to \( i = 0 \), by putting

\[ u_0 = w_0 + \sum_{i \neq 0} n_i (w_i - u_i) \]  
(5.5)

If the structure group of the bundle \( \mathcal{E} \) is restricted to \( SU(n) \), then \( c_1(\mathcal{E}) = 0 \) and one immediately finds \( u_i = 0 \) for \( i = 1, \ldots, r - 1 \).

The rank of the instanton bundle, \( \mathcal{E} \), defined by (4.20) and (4.24) can be computed from the formula

\[ \text{rank} \mathcal{E} = \dim(\mathcal{E}) = \dim(\text{Ker}(D^\dagger)) = \dim(\mathcal{V}) + \dim(\mathcal{W}) - \dim(\mathcal{U}) \]  
(5.6)

Using \( A_{ij} = 2 \delta_{ij} - \tilde{C}_{ij} \), one finds as expected

\[ \dim(\mathcal{E}) = \sum_i w_i n_i = n \]  
(5.7)
with the integers \( w_i \) being the dimensions of the subspaces \( W_i \).

Recalling the general discussion after (4.14), it can be shown that the framed moduli space of connections on \( \mathcal{E}, \mathcal{M}_\mathcal{E} \), is itself a hyper-Kähler manifold. Indeed \( \mathcal{M}_\mathcal{E} \) is given by the equivalence class of data \( (A, \Psi) \in M \) satisfying the ADHM equations (4.23), up to transformations of \( G_V = \otimes_i U(v_i) \).

From the hyper-Kähler quotient construction it follows that \( \mathcal{M}_\mathcal{E} = P_\mathcal{E}/G_V \), with \( P_\mathcal{E} \) a \( G_V \) principal bundle on \( \mathcal{M}_\mathcal{E} \), analogous to \( P_\zeta \) in (2.6). From (4.17) and (4.18), the real dimension of \( \mathcal{M}_\mathcal{E} \) is computed to be

\[
\dim(\mathcal{M}_\mathcal{E}) = \dim(M) - 4\dim(G_V) = \dim(P_\mathcal{E}) - 4\dim(G_V) = 2\sum_i \left( \sum_j A_{ij}v_iv_j + 4v_iw_i - 4v_i^2 \right) = 2\sum_i v_i(u_i + w_i) \tag{5.8}
\]

To clarify the ADHM construction in the general case let us first discuss few explicit examples which will be also useful for the applications presented in section 6. We start by noticing that, in order to get a finite gauge action, the instanton connection must be asymptotically equal to a pure gauge. However, the gravitational instanton background allows for the existence of non-trivial holonomies at infinity. This means that, if one parallel transports the typical fiber of the instanton bundle, \( \mathcal{E} \), around infinity, it will be transformed according to some representation of \( \Gamma \). This possibility is the main novelty in the KN construction on ALE spaces with respect to the standard ADHM construction on \( \mathbb{R}^4 \).

As a first example let us take \( W = R_i \) and \( V = \emptyset \). In this case the instanton bundle \( \mathcal{E} \) coincides with the elementary bundle \( \mathcal{T}_i \) \( (\mathcal{E} = \mathcal{T}_i) \), so that, recalling the decomposition formulae (5.1) and equations (5.4) and (5.5), one gets for the dimension vectors, \( v = 0 \) and \( u = w = (0, \ldots, 1, \ldots, 0) \), with the 1 in the \( i \)-th position. It follows from (5.8) that the moduli space of \( \mathcal{E} \) is 0-dimensional, \( i.e. \) the connection on the elementary bundle, \( \mathcal{T}_i \), has no free continuous parameter, as explicitly found in (3.12) for the case of multican-
ter ALE metrics. Strictly speaking the tautological bundles $\mathcal{T}_i$ correspond to singular limits of the KN construction in which $A, B, s, t$ are absent, while at the same time the ADHM equations (4.23) lose their force because the instanton connection is abelian.

The simplest non-abelian instanton bundle is the $SU(2)$ bundle associated to the natural (two-dimensional) homomorphism, $\rho_Q(\gamma)$, of $\Gamma$ in $SU(2)$. For $\Gamma = \mathbb{Z}_N$, the two-dimensional representation space decomposes according to $Q = R_1 \oplus \bar{R}_1$. The corresponding decomposition of $W$ is associated with the dimension vector $w = (0, 1, 0, \ldots, 0, 1)$. The homomorphism $\rho_Q(\gamma)$ of $\mathbb{Z}_N$ in $SU(2)$ implies that, if we parallel transport around infinity $m$ times a section $\phi$ of $E$, it transforms according to

$$
\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \text{Tr} \exp(i \oint A dx) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} e^{\frac{2\pi i}{N} m} & 0 \\ 0 & e^{-\frac{2\pi i}{N} m} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
$$

(5.9)

where $A$ is the connection on $E$ and $P$ denotes path-ordering. In order to identify $E$, one may recall that for $SU(2)$ bundles $c_1(E) = 0$. In view of (5.3) this condition is equivalent to $u_i = 0$ for $i \neq 0$. To determine $u_0$ and the vector $v$ we solve the system of $N - 1$ homogeneous equations (5.4). This leads to $v = (k_1 - 1, k_1, \ldots, k_1)$ and $u = (2, 0, \ldots, 0)$, with $k_1$ a positive integer. The different choices of $k_1$ correspond to different instanton numbers, because $c_2(E) = \kappa = k_1 - \frac{1}{N}$. Using (5.8), we find the dimension of the moduli space to be $\dim(\mathcal{M}_E) = 8\kappa + \frac{2}{N} - 4 = 8k_1 - 4$. For $k_1 = 1, \ldots, N$ these solutions are topologically equivalent to those in [7], generalized here by the ansatz (4.33), (4.34).

In the next section, we will devote particular attention to rank-two ($n = 2$) $SU(2)$ bundles on the EH instanton. In this case $\Gamma = \mathbb{Z}_2$, so that $R_1 = \bar{R}_1$ and (5.7) is satisfied by the three choices $w = (0, 2)$, $w = (2, 0)$ and $w = (1, 1)$. The last case will not be considered here because it corresponds to a non-vanishing first Chern class. In the other two cases one has $u = (2, 0)$. For
\( w = (0, 2) \), using (5.4) one finds \( v = (k_1 - 1, k_1) \) and (5.8) gives

\[
\dim(M_E) = 8k_1 - 4 \tag{5.10}
\]

The lowest \( k_1 \) values, \( k_1 = 1 \) and \( k_1 = 2 \), correspond to an instanton moduli space of dimensions 4 and 12 respectively. The non integral value of the corresponding second Chern classes, \( \kappa = 1/2 \) and \( \kappa = 3/2 \), is a consequence of the non-trivial holonomy of the connection at infinity, where the instanton bundle \( \mathcal{E} \) coincides with \( \mathcal{T}_1 \oplus \bar{\mathcal{T}}_1 \). \( \bar{\mathcal{T}}_1 \) is the bundle conjugate to \( \mathcal{T}_1 \), i.e. the one with monopole charge opposite to \( \mathcal{T}_1 \), as shown in (5.9).

For \( w = (2, 0) \), one finds \( v = (k_2, k_2) \) and \( \dim(M_E) = 8k_2 \). These \( SU(2) \) bundles descend from \( SU(2) \) bundles on \( \mathbb{R}^4 \) with even instanton number \( k = 2k_2 \). Indeed, the connection has trivial holonomy at infinity, i.e. it coincides with the trivial connection on the bundle \( \mathcal{T}_0 \oplus \mathcal{T}_0 \).

### 5.2 Index Theorems

Generalizing the inverse construction of Corrigan and Goddard [32], Kronheimer and Nakajima have also carried over to ALE spaces the difficult part of the ADHM construction, i.e. they have shown the uniqueness and completeness of the construction [3]. Using hard analysis on ALE spaces (mainly Sobolev spaces with properly defined norms) Kronheimer and Nakajima have shown that the ADHM data \( W_i, V_i \) are related to the space of bounded harmonic scalars, \( W_i = \mathcal{H}(\Delta, \mathcal{E} \otimes \mathcal{T}_i) \), and to the space of zero-modes of the Dirac operator, \( V_i = \text{Ker}(\mathcal{D}, \mathcal{E} \otimes \mathcal{T}_i) \)[9]. Apart from checking the above identifications in some explicit examples, we are not able to reproduce their results in any easy way so that we refer the reader interested in this aspect of the KN construction to the original paper [9]. We simply collect some relevant facts which may be helpful in the computation of instanton effects on ALE spaces.
No neutral spinor zero-modes are expected on ALE spaces since the index of the Dirac operator (for gauge singlets) is zero \[1\]. However the presence of charged spinor zero-modes is guaranteed by a non-vanishing index of the Dirac operator coupled to the gauge bundle \(\mathcal{E}\). No Dirac zero-mode of wrong chirality is expected as well as no (normalizable) zero-mode of the scalar Laplacian.

These results are obtained computing the classical topological invariants on ALE spaces. Standard formulae, which can be found in \[33\], lead for the Euler characteristic, \(\chi_\mathcal{E}\), and the Hirzebruch signature, \(\tau_H\), to the results

\[
\chi_\mathcal{E} = -\frac{1}{16\pi^2} \int \text{tr} R \wedge *R + \frac{1}{|\Gamma|} = r
\]

\[
\tau_H = -\frac{1}{24\pi^2} \int \text{tr} R \wedge R + \xi_s = r - 1
\]  \hspace{1cm} (5.11)

where \(r\) is the rank of \(\Gamma\). The number of gauge singlet spin \(1/2\) and \(3/2\) zero modes are given by

\[
I_{1/2} = \frac{1}{192\pi^2} \int \text{tr} R \wedge R + \xi_{1/2} = -\frac{\tau_H - \xi_s}{8} + \xi_{1/2}
\]

\[
I_{3/2} = -\frac{21}{192\pi^2} \int \text{tr} R \wedge R + \xi_{3/2} = \frac{21}{8}(\tau_H - \xi_s) + \xi_{3/2}
\]  \hspace{1cm} (5.12)

The so called \(G\)-index theorems allow the computation of the boundary corrections, \(\xi\)’s, for which one finds \[33\]

\[
\xi_{1/2} = \frac{1}{|\Gamma|} \sum_{\gamma \neq e} \frac{1}{2 - \chi_Q(\gamma)}
\]

\[
\xi_s = 4\xi_{1/2} - 1 + \frac{1}{|\Gamma|}
\]

\[
\xi_{3/2} = 3\xi_{1/2} - 2 + \frac{2}{|\Gamma|}
\]  \hspace{1cm} (5.13)

where \(\chi_Q\) is the character of the representation \(\rho_Q\) of \(\Gamma\) and the sum runs over all elements of \(\Gamma\) different from the identity.

For multicenter ALE spaces, where \(\Gamma = \mathbb{Z}_N\), the boundary corrections can be explicitly computed. Recalling that for \(\Gamma = \mathbb{Z}_N\) any element \(\gamma \in \mathbb{Z}_N\)
satisfies $\gamma^N = 1$, one gets $\chi_Q(\gamma) = \text{tr}[\text{diag}(\epsilon^{m/j}, \bar{\epsilon}^{m/j})] = 2\cos\left(\frac{2\pi m}{N}\right)$ with $\epsilon = \exp\left(\frac{4\pi i}{N}\right)$ and $m = 1, \ldots, N - 1$. From this result one finds

$$\xi_{\frac{1}{2}} = \frac{1}{4N} \sum_{m=1}^{N-1} \frac{1}{\sin^2\left(\frac{2\pi m}{N}\right)} = \frac{N^2 - 1}{12N}$$

$$\xi_s = \frac{(N - 1)(N - 2)}{3N}$$

(5.14)

We remark that $\xi_s$ is zero for flat space ($N = 1$) and for the EH gravitational instanton ($N = 2$).

The generalization of the index theorem for the spin $1/2$ complex to arbitrary representations, $T$, of the $SU(2)$ gauge group gives

$$I^{(2j+1)}_{\frac{1}{2}} = \dim T \left( -\frac{\tau + 1}{8} + \frac{1}{2}\xi_{\frac{1}{2}} + \frac{1}{|\Gamma|} \right) + \frac{\text{tr}_T(T^aT^b)}{\text{tr}_Q(T^aT^b)\epsilon_2(\mathcal{E})} + \xi_T^{(T)}$$

(5.15)

where $\dim T = (2j + 1)$ and the second term on the r.h.s. is the gauge bulk contribution. The boundary correction for $\Gamma = \mathbb{Z}_N$ is given by

$$\xi_T^{(T)} = \frac{1}{|\Gamma|} \sum_{\gamma \notin \epsilon} \frac{\chi_T(\gamma)}{2 - \chi_Q(\gamma)} = \frac{1}{N} \sum_{m=1}^{N-1} \frac{\sin(2j + 1)\frac{2\pi m}{N}}{2\sin\frac{2\pi m}{N}(1 - \cos\frac{2\pi m}{N})}$$

(5.16)

In analogy with the case of the fundamental two-dimensional representation, one can compute $\chi_T(\gamma)$ through the formula

$$\chi_T(\gamma) = \text{tr}[\text{diag}(\epsilon^{ mj}, \ldots, \epsilon^{- mj})] = \frac{\sin \left[ (2j + 1)\frac{2\pi m}{N} \right]}{\sin \left( \frac{2\pi m}{N} \right)}$$

(5.17)

Using the notation of the previous section, we finally find

$$I^{(2)}_{\frac{1}{2}} = k_1 - 1 = v_0 = \dim(V_0)$$

$$I^{(3)}_{\frac{1}{2}} = 4k_1 - 2 = \frac{1}{2}\dim(M_\epsilon)$$

(5.18)

These results generalize those obtained in [2]. Comparing the number of spin $1/2$ zero-modes in the adjoint representation of $SU(2)$ obtained there
with (5.18), we find that the solutions of the heterotic string equations of motion in the background of the EH gravitational instanton, fulfilling the standard embedding condition, must have instanton number $\kappa = 3/2$. For a generic ALE instanton, the $SU(2)$ connection corresponding to the standard embedding is obtained taking $\kappa = |\Gamma| - 1/|\Gamma|$, i.e. $k_1 = |\Gamma|$, from which $I_{3/2}^{(3)} = 4|\Gamma| - 2$ follows.

6 \textit{SU}(2) Gauge Instantons on the EH Manifold

We now specialize the discussion to $SU(2)$ instantons on the EH background expanding the results obtained in [34]. In this case $\Gamma = \mathbb{Z}_2$, the flat hyper-Kähler manifold $\Xi$ is $\mathbb{R}^8$ and the hyper-Kähler quotient is taken with respect to the group $G = U(1)$, since the two irreducible representations of $\mathbb{Z}_2$ are one-dimensional ($n_0 = n_1 = 1$). The metric on the EH instanton [16] is

$$ds_X^2 = \frac{(dr)^2}{1 - (\frac{a}{r})^4} + \frac{r^2}{4}(\sigma_x^2 + \sigma_y^2) + \frac{r^2}{4}(1 - (\frac{a}{r})^4)\sigma_z^2$$  \hspace{1cm} (6.1)

where the $\sigma_i$'s are the left-invariant one-forms of $SU(2)$. The formula (6.1) is explicitly derived in the appendix, using the hyper-Kähler quotient construction. For future use we record here also the expression of the $U(1)$ instanton connection on the EH manifold, computed in [16]

$$A^T = -\frac{a^2}{r^2}(d\psi + \cos \theta d\phi)$$  \hspace{1cm} (6.2)

(6.2) is obtained from (3.12), by changing to the variables used in (6.1).

We already observed that the condition $c_1(\mathcal{E}) = 0$ for rank two bundles can be satisfied in two different ways, with $u = (2, 0)$ in both cases. For bundles which descend from those with odd Chern class on $\mathbb{R}^4$ one has $w =$
(0, 2) and $v = (k_1 - 1, k_1)$, while for bundles with even Chern class on $\mathbb{R}^4$ one finds $w = (2, 0)$ and $v = (k_2, k_2)$, where $k_1, k_2$ are positive integers.

In the following we will see that for $k_1 = 1, 2$ and for $k_2 = 1$ the KN ADHM equations may be solved explicitly, showing in particular that in the limit $a \to 0$ (i.e. in the orbifold limit $X_\zeta \to X_0 = \mathbb{R}^4/\mathbb{Z}_2$) the solutions are invariant under $x \to -x$, i.e. are $\mathbb{Z}_2$-invariant instantons on $\mathbb{R}^4$.

### 6.1 $SU(2)$ Gauge Bundle with $c_2(E) = \frac{1}{2}$

This case was called the minimal $SU(2)$ instanton bundle in [34] because of its topological numbers $c_1(E) = 0$, $c_2(E) = 1/2$. Thanks to the simple form of the resulting gauge connection, instanton dominated correlators around this background could be explicitly computed in [3]. As we said in the previous section, the vector spaces $V, W$ admit a decomposition as in (5.1), with $w = (0, 2), v = (0, 1), u = (2, 0)$. In this case the matrices $A$ and $B$, appearing in (4.17) are simply absent. The map $\Psi$ is represented by a pair of $1 \times 2$ complex matrices, i.e. by two two-dimensional complex vectors. From (4.20) we see that $D_\Gamma^\dagger$ acts on the frame of sections, $U$, of the bundle $V \oplus W$ admitting the decomposition (5.2). Since $A_{ij}$ off diagonal, we notice from the last formula in (5.2) that, the sections of the space $\tilde{V}_1$ are coupled only to those of the trivial bundle, $T_0$. Moreover, the sections of $T$ are appropriately cast into the form of doublets, as in this way they are naturally acted upon by the elements of $\xi$, which are two by two matrices. Putting these observations together, we may write the complex sections $u_U$ appearing in $U$ in the form

$$u_U = \begin{pmatrix} \nu_1 \otimes \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix} \\ \nu_2 \otimes \begin{pmatrix} \phi_2 \\ 0 \end{pmatrix} \\ (\psi_1) \otimes \begin{pmatrix} 0 \\ \phi \end{pmatrix} \end{pmatrix} \quad \text{(6.3)}$$
where \( \nu_1, \nu_2 \in V_1; \phi \in \mathcal{T}_1; \phi_1, \phi_2 \in \mathcal{T}_0; \psi_1, \psi_2 \in \mathcal{W}_1 \). Following the arguments spelled out in the appendix, we see that \( \Gamma \)-invariance restricts the matrices \( \alpha \) and \( \beta \) in (2.11) to the form

\[
\alpha = \begin{pmatrix} 0 & x_1 \\ y_1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & x_2 \\ y_2 & 0 \end{pmatrix}
\]

(6.4)

where \( x_1, x_2, y_1, y_2 \) are four complex coordinates on \( \Xi = \mathbb{R}^8 \). The map \( \Psi \) in (4.18) can be represented as

\[
\Psi = \begin{pmatrix} s \ t^\dagger \end{pmatrix} = \begin{pmatrix} s_1 - \bar{t}_2 \\ s_2 \bar{t}_1 \end{pmatrix}
\]

(6.5)

where \( s_1, s_2, t_1, t_2 \) are four complex parameters.

Combining (6.4) with (6.5), one can construct the matrix \( D_\Gamma^\dagger \)

\[
D_\Gamma^\dagger = \begin{pmatrix} \alpha^\dagger & \beta^\dagger & s^\dagger \\ -\beta & \alpha & t \end{pmatrix}
\]

(6.6)

Acting with (6.6) on (3.3), it is easily seen that the non trivial equations coming from the condition \( D_\Gamma^\dagger u_U = 0 \), can be obtained from the reduced form of \( D_\Gamma^\dagger \)

\[
D_\Gamma^\dagger = \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{s}_1 & \bar{s}_2 \\ -y_2 & y_1 & -t_2 & t_1 \end{pmatrix}
\]

(6.7)

acting on a similarly reduced form of \( u_U \).

Putting \( \zeta_\infty = -a^2 \) and \( \zeta_C = 0 \) (see the discussion at the end of section 2.2) and introducing the definitions \( x^2 = |x_1|^2 + |x_2|^2 \) and \( s^2 = |s_1|^2 + |s_2|^2 \), the ADHM equations (4.23), are solved by \( y_1 = \lambda x_1, y_2 = \lambda x_2, t_1 = \mu s_1, t_2 = \mu s_2 \) with \( \lambda^2 = 1 + a^2/x^2 \), \( \mu^2 = 1 - a^2/s^2 \). By exploiting the \( U(1) \) isometries of the principal bundles \( P_\zeta \) over \( X_\zeta \) and \( \mathcal{P}_\varepsilon \) over \( \mathcal{M}_\varepsilon \), \( \lambda \) and \( \mu \) can be chosen to be real. The orthonormal frame \( U \) for the two-dimensional space \( \text{Ker}D_\Gamma^\dagger \) can
be arranged in the $2 \times 4$ matrix

$$U = \frac{1}{xs\sqrt{x^2 + s^2}} \begin{pmatrix} s^2x_1 & -\mu s^2\bar{x}_2 \\ s^2x_2 & \mu s^2\bar{x}_1 \\ -x^2s_1 & \lambda x^2\bar{s}_2 \\ -x^2s_2 & -\lambda x^2\bar{s}_1 \end{pmatrix}$$

where $x = \sqrt{x^2}$ and $s = \sqrt{s^2}$. In this setting $SU(2)$ gauge transformations correspond to $U \to U\Omega$ with $\Omega \in SU(2)$.

The last ingredient needed to compute the self-dual connection (4.25) is the abelian ($G = U(1)$) connection on the tautological bundle $\mathcal{T}$. Having reduced the frame $U$ to a pair of four-dimensional vectors, as explained before, we observe that the first two components of these vectors correspond to sections of $\mathcal{T}_0$ which is trivial by construction. The covariant derivative, $\nabla_\mu$, appearing in (4.25) can thus be written as a $4 \times 4$ diagonal matrix of the kind $\nabla_\mu = \text{diag} (\partial_\mu, \partial_\mu, \partial_\mu + iA_\mu^T, \partial_\mu + iA_\mu^T)$. Putting $x_i = |x_i|e^{i\alpha_i}$ (see the appendix), the connection one-form on $\mathcal{T}$ becomes

$$A_\mu^T dx^\mu = -\frac{a^2}{x^2} \frac{|x_1|^2d\alpha_1 + |x_2|^2d\alpha_2}{2x^2 + a^2}$$

and turns out to be a particular case of (A.12), when $P_\zeta$ is the $U(1)$ principal bundle over the EH manifold. In the coordinates employed in (6.1) and with a proper gauge choice, $A_\mu^T$ may be identified with the monopole potential (6.2). In the same coordinate system, inserting (6.8) in (4.25), one explicitly gets

$$A = A_\mu dx^\mu = i \begin{pmatrix} f(r)\sigma_z & g(r)\sigma_- \\ g(r)\sigma_+ & -f(r)\sigma_z \end{pmatrix}$$

where $\sigma_\pm = \sigma_x \pm i\sigma_y$ and

$$f(r) = \frac{t^2r^2 + a^4}{r^2(r^2 + t^2)} \quad g(r) = \frac{\sqrt{t^4 - a^4}}{r^2 + t^2}$$

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with \( t^2 = 2s^2 - a^2 \) and \( r^2 = 2x^2 + a^2 \). The resulting self-dual field strength is given by

\[
F = i \begin{pmatrix}
H(rdr \wedge \sigma_z + r^2\sigma_x \wedge \sigma_y) & G(\frac{r^2}{u} dr \wedge \sigma_+ + ur\sigma_- \wedge \sigma_z) \\
G(\frac{r^2}{u} dr \wedge \sigma_+ + ur\sigma_- \wedge \sigma_z) & -H(rdr \wedge \sigma_z + r^2\sigma_x \wedge \sigma_y)
\end{pmatrix}
\]  

(6.12)

with

\[
H = \frac{2 df}{r dr} = \frac{4(t^2r^4 + 2a^4r^2 + a^4t^2)}{r^4(r^2 + t^2)^2}
\]

\[
G = \frac{2u dg}{r^2 dr} = \frac{4u\sqrt{t^4 - a^4}}{r(r^2 + t^2)^2}
\]

(6.13)

Using (6.12), the second Chern class may be checked to be 1/2, as expected.

The connection (6.10) was previously found in [7] following a completely different procedure. In the limit \( a \to 0 \) (6.10) becomes a connection over \( \mathbb{R}^4/\mathbb{Z}_2 \) and coincides with the BPST instanton (in the singular gauge) with center located at \( x_0 = 0 \) and size \( t \) [13]. This is no surprise since, in this limit, this construction gives the usual ADHM construction for \( SO(3) \) bundles [30].

6.2 \( SU(2) \) Gauge Bundle with \( c_2(\mathcal{E}) = 1 \)

This case corresponds to the choice \( v = (1, 1), w = (2, 0), u = (2, 0) \). Following the same reasoning as in the previous section and the results in the appendix, the matrices in (4.17), (4.18) can be cast into the form

\[
A = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}
\]

(6.14)

We remark that, since \( w_1 = 0 \), the map \( t^\dagger \in \text{Hom}(V_1, W_1) \) is absent. Employing a notation similar to (5.3), we write the sections of the bundles of interest as

\[
\begin{pmatrix} \nu_0^1 \\ \nu_0^2 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \in \tilde{Q} \otimes \tilde{V}_0 \otimes \mathcal{T}_1
\]

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\[
\begin{pmatrix}
\nu^1_1 \\
\nu^2_1 \\
\psi^1_0 \\
\psi^2_0
\end{pmatrix}
\otimes
\begin{pmatrix}
\phi_0 \\
0
\end{pmatrix}
\in \bar{Q} \otimes \bar{V}_1 \otimes \mathcal{T}_0
\]
\[
\begin{pmatrix}
\nu^1_2 \\
\nu^2_2 \\
\psi^1_0 \\
\psi^2_0
\end{pmatrix}
\otimes
\begin{pmatrix}
\phi \\
0
\end{pmatrix}
\in \bar{W}_0 \otimes \mathcal{T}_0
\]  
(6.15)

By eliminating rows and columns of \(\mathcal{D}_\Gamma^\dagger\) that, when applied to (6.15) lead to no constraint, the linear map (4.19) can be restricted to the form

\[
\mathcal{D}_\Gamma = \begin{pmatrix}
y_1 & -\bar{x}_2 & a_1 & -\bar{b}_2 \\
y_2 & \bar{x}_1 & a_2 & \bar{b}_1 \\
b_1 & -\bar{a}_2 & x_1 & -\bar{y}_2 \\
b_2 & \bar{a}_1 & x_2 & \bar{y}_1 \\
s_1 & -\bar{t}_2 & 0 & 0 \\
s_2 & \bar{t}_1 & 0 & 0
\end{pmatrix}
\]  
(6.16)

We start by solving the ADHM equations (4.23) in the limit \(a \to 0\).

The condition \(\mathcal{D}^\dagger \mathcal{D} = \Delta \otimes \mathbb{1}\) with \(\Delta\) a hermitean matrix leads to \(b_i = a_i, t_i = s_i, y_i = x_i\). The adjoint of (6.16) can then be written as a matrix of quaternions

\[
\mathcal{D}_\Gamma^\dagger = \begin{pmatrix}
x^\dagger \\
p^\dagger \\
x^\dagger
\end{pmatrix}
\begin{pmatrix}
\mathcal{D} \\
q
\end{pmatrix}
\]  
(6.17)

where \(p, s\) and \(x\) are quaternions whose components are the complex numbers \(a_i, s_i\) and \(x_i\) \((i = 1, 2)\) respectively. To compute the kernel of the matrix (6.17) we rewrite it as \(\mathcal{D}_\Gamma^\dagger = (D, q)\) [30], where

\[
q = \begin{pmatrix}
s^\dagger \\
0
\end{pmatrix}, \quad D = \begin{pmatrix}
x^\dagger \\
p^\dagger \\
x^\dagger
\end{pmatrix}
\]  
(6.18)

Notice that \(q\) and \(D\) are respectively the restrictions of the maps \(\Psi^\dagger\) and \(\mathcal{A}^\dagger\), defined in (4.18) and (4.17). The reorganization of the ADHM data in quaternionic notation is always possible for \(a \to 0\). In fact, as we have already noticed, in this limit the construction we are describing becomes the usual ADHM construction on the orbifold \(\mathbb{R}^4/\mathbb{Z}_2\). In the same vein, we rewrite the
frame $U$ in (4.25) as $U = \begin{pmatrix} \nu \\ \vartheta \end{pmatrix}$, so that the matrix $q (D)$ will be acting on the quaternion $\vartheta (\nu)$.

The condition $D^\dagger U = 0$ yields

$$\nu = -D^{-1} q \vartheta$$

and $U = \begin{pmatrix} -D^{-1} q \vartheta \\ \vartheta \end{pmatrix}$. The unitarity constraint $U^\dagger U = \mathbb{I}$ then gives

$$\vartheta = \frac{1}{\sqrt{1 + q^\dagger (DD^\dagger)^{-1} q}} \vartheta$$

with $\vartheta^\dagger \vartheta = \mathbb{I}$. By an $SU(2)$ gauge transformation one can always set $\vartheta = \mathbb{I}$. Exploiting this freedom, (4.25) can be written in the form

$$A_\mu = \frac{1}{2} q^2 [(D^{-1} q)^\dagger \partial_\mu (D^{-1} q) - (\partial_\mu (D^{-1} q)^\dagger)(D^{-1} q)]$$

The only delicate point in these formulae is the computation of the matrix $D^{-1}$ with the property $D^{-1} D = 1$. As $D$ is a matrix of quaternions $D$ does not commute with $D^{-1}$. The left and the right inverse of $D$ are not necessarily equal. To compute the left inverse, $D^{-1}$, we observe that it is always possible to find matrices $E_k$ such that $E_k \ldots E_2 E_1 D = \mathbb{I}$ [38]. $E_k$ is a matrix representing one of the three possible elementary operations on rows

i) P: permutation of two rows

ii) A: addition of two rows

iii) M: multiplication of a row by a number

A matrix can be diagonalized by a finite number of P, A, M operations. Following the appropriate steps on the matrix $D$, one finds

$$D^{-1} = \frac{1}{\Phi} \begin{pmatrix} (x^2 + p^2) x^\dagger & -2(x \cdot p) p^\dagger \\ (x^2 + p^2) p^\dagger & (x^2 + p^2) x^\dagger - 2(x \cdot p) p^\dagger \end{pmatrix}$$

(6.22)
where $\Phi = (x - p)^2(x + p)^2$ and

$$U = \frac{1}{\sqrt{1 + \frac{p^2}{(x+p)^2} + \frac{p^2}{(x-p)^2}}} \left( \frac{1}{\sqrt{(x-p)^2}} \right) \left( \frac{1}{\sqrt{(x+p)^2}} \right) \tag{6.23}$$

In (6.22) $x \cdot p = \frac{1}{2}(x^\dagger p + p^\dagger x)$ is the scalar product of the two vectors $x^\mu$ and $p^\mu$, i.e. $\frac{1}{2}(\bar{x}_1 a_1 + \bar{x}_2 a_2 + \bar{\alpha}_1 x_1 + \bar{\alpha}_2 x_2 = x^\mu p_\mu$. As expected, the resulting gauge connection $A^a_\mu = -i\eta^a_{\mu\nu} \partial^\nu \ln \vartheta^2$ is invariant under the $\mathbb{Z}_2$ transformation $x \to -x$.

We are now ready to solve the KN-ADHM equations for $a \neq 0$. Going back to (6.16), we write

$$D^\dagger \Gamma = \left( \begin{array}{cccc} y_1 & y_2 & \bar{b}_1 & \bar{b}_2 \\ -x_2 & x_1 & -a_2 & a_1 \\ \bar{a}_1 & \bar{a}_2 & \bar{x}_1 & \bar{x}_2 \\ -b_2 & b_1 & -y_2 & y_1 \end{array} \right) \tag{6.24}$$

The condition $D^\dagger D = \Delta \otimes \mathbb{I}$ with $\Delta$ a hermitean matrix implies $t_i = s_i$, $y_i = \lambda x_i$ and $b_i = \mu a_i$, where $\lambda^2 = 1 + a^2/x^2$ and $\mu^2 = 1 - a^2/p^2$ can be chosen to be real. As before $p$ is the quaternion made out of the two complex numbers $a_1, a_2$. It is again convenient to put $D^\dagger_\Gamma = (D, q)$, where $D$ is a $4 \times 4$ complex matrix and $q^\dagger = (s, 0)$ with $s$ the quaternion made out of $s_1, s_2$. The frame of sections $U$ can be put into the form

$$U = \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \vartheta_1 \\ \vartheta_2 \end{array} \right) \tag{6.25}$$

where $\nu_1, \nu_2$ and $\vartheta_1, \vartheta_2$ are four-dimensional and two-dimensional complex vectors, respectively. The constraint (4.4) becomes

$$D \nu_i = -\left( \begin{array}{c} s^\dagger \vartheta_i \\ 0 \end{array} \right) \tag{6.26}$$
and yields for \( \nu_i \)
\[
\nu_i = D^\dagger (DD^\dagger)^{-1} \left( \begin{array}{c} s^\dagger \vartheta_i \\ 0 \end{array} \right) 
\]  
(6.27)

Since \( DD^\dagger \) turns out to be made of real quaternions only, the inversion of this operator is easily performed and gives
\[
(DD^\dagger)^{-1} = \frac{1}{\Delta} \begin{pmatrix} (x^2 + p^2) \otimes \mathbb{I} & -c \otimes \mathbb{I} \\ \bar{c} \otimes \mathbb{I} & (x^2 + p^2) \otimes \mathbb{I} \end{pmatrix} 
\]  
(6.28)

where
\[
c = \lambda(a_1 \bar{x}_1 + a_2 \bar{x}_2) + \mu(x_1 \bar{a}_1 + x_2 \bar{a}_2) \\
\Delta = (x^2 + p^2)^2 - |c|^2 
\]  
(6.29)

Imposing the unitarity condition \( U^\dagger U = \mathbb{I} \) (i.e. \( \vartheta^\dagger \vartheta + \nu^\dagger \nu = \mathbb{I} \)) yields
\[
\vartheta = \sqrt{\Delta \over \Delta + s^2(x^2 + p^2)} \tilde{\vartheta} 
\]  
(6.30)

with \( \tilde{\vartheta}^\dagger \tilde{\vartheta} = \mathbb{I} \). Exploiting as before the \( SU(2) \) gauge symmetry one can set \( \tilde{\vartheta} = \mathbb{I} \), finally getting
\[
\nu_1 = \frac{1}{\Lambda} \begin{pmatrix} (x^2 + p^2)(\lambda x_1 \bar{s}_1 + s_2 x_2) - \bar{c}(a_1 \bar{s}_1 + s_2 a_2) \\ (x^2 + p^2)(\lambda x_2 \bar{s}_1 - s_2 x_1) - \bar{c}(a_2 \bar{s}_1 - s_2 a_1) \\ (x^2 + p^2)(\mu a_1 \bar{s}_1 + s_2 a_2) - \bar{c}(x_1 \bar{s}_1 + \lambda s_2 \bar{x}_2) \\ (x^2 + p^2)(\mu a_2 \bar{s}_1 - s_2 \bar{a}_1) - \bar{c}(x_2 \bar{s}_1 - \lambda s_2 \bar{x}_1) \end{pmatrix} 
\]  
(6.31)

\[
\nu_2 = \frac{1}{\Lambda} \begin{pmatrix} (x^2 + p^2)(\lambda x_1 \bar{s}_2 - s_1 \bar{x}_2) - \bar{c}(a_1 \bar{s}_2 - s_1 a_2) \\ (x^2 + p^2)(\lambda x_2 \bar{s}_2 + s_1 \bar{x}_1) - \bar{c}(a_2 \bar{s}_2 + s_1 a_1) \\ (x^2 + p^2)(\mu a_1 \bar{s}_2 + s_1 a_2) - \bar{c}(x_1 \bar{s}_2 + \lambda s_1 \bar{x}_2) \\ (x^2 + p^2)(\mu a_2 \bar{s}_2 - s_1 \bar{a}_1) - \bar{c}(x_2 \bar{s}_2 + \lambda s_1 \bar{x}_1) \end{pmatrix} 
\]  
(6.32)

where \( \Lambda = \sqrt{\Delta + s^2(x^2 + p^2)} \).

Together with (6.30), (6.31) and (6.32) allow to compute \( U \) and consequently the gauge connection. As expected, the solution depends on eight real parameters. These are the four complex numbers \( s_1, s_2, a_1 \) and \( a_2 \).
6.3 \textit{SU(2) Gauge Bundle with } c_2(\mathcal{E}) = \frac{3}{2} \textit{ }

This is the case corresponding to the deformation of the solution studied in \cite{2} which is obtained from the standard embedding of the spin into the gauge connection. This solution is identified by the choice \( w = (0, 2), u = (2, 0), v = (1, 2) \). \((1.17)\) is a \(2 \times 2\) matrix of the form \((6.14)\), but the entries are themselves matrices describing the maps \(\text{Hom}(V_i, V_j)\). Since \(v_1 = 1\) and \(v_2 = 2\), these are \(2 \times 1\) and \(1 \times 2\) matrices. Therefore the complex matrices \(A, B\) are of the form

\[
A = \begin{pmatrix} 0 & a_1 & a_2 \\ a_3 & 0 & 0 \\ a_4 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 & b_2 \\ b_3 & 0 & 0 \\ b_4 & 0 & 0 \end{pmatrix} \tag{6.33}
\]

The maps \(s, t^\dagger\) in \((4.18)\) belong to \((\oplus_i \text{Hom}(W_i, V_i))_\Gamma\) and can be represented by the \(2 \times 3\) matrices

\[
s = \begin{pmatrix} 0 & s_1 & s_2 \\ 0 & s_3 & s_4 \end{pmatrix}, \quad t^\dagger = - \begin{pmatrix} 0 & \bar{t}_1 & \bar{t}_2 \\ 0 & \bar{t}_3 & \bar{t}_4 \end{pmatrix} \tag{6.34}
\]

The zero entries correspond to maps involving the null space \(W_0\), which has dimension \(w_0 = 0\).

The section \(u_U\) can be written in this case

\[
u = \begin{pmatrix} \nu_0^1 \otimes \begin{pmatrix} 0 \\ \phi_1^1 \end{pmatrix} \\ \nu_1^1 \otimes \begin{pmatrix} 0 \\ \phi_0^1 \end{pmatrix} \\ \nu_2^1 \otimes \begin{pmatrix} 0 \\ \phi_2^1 \end{pmatrix} \\ \nu_0^2 \otimes \begin{pmatrix} 0 \\ \phi_1^2 \end{pmatrix} \\ \nu_1^2 \otimes \begin{pmatrix} 0 \\ \phi_0^2 \end{pmatrix} \\ \nu_2^2 \otimes \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \\ \psi_1 \otimes \begin{pmatrix} 0 \\ \phi \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}
\]

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The action of the term $\mathbb{1}_1 \otimes \xi^\dagger$, in (4.16), on $u_\nu$ is made more transparent if we assemble the first four rows of (6.35) into one doublet

$$
\begin{pmatrix}
\Sigma \\
\Lambda
\end{pmatrix} \equiv
\begin{pmatrix}
\nu_0^1 \otimes \phi_1^1 & \nu_1^1 \otimes \phi_0^1 \\
\nu_0^2 \otimes \phi_2^1 & \nu_1^2 \otimes \phi_0^1
\end{pmatrix}
$$

(6.36)

Consequently

$$
(\mathbb{1}_1 \otimes \xi^\dagger) 
\begin{pmatrix}
\Sigma \\
\Lambda
\end{pmatrix} = 
\begin{pmatrix}
\alpha^\dagger \Sigma + \beta^\dagger \Lambda \\
-\beta \Sigma + \alpha \Lambda
\end{pmatrix}
$$

(6.37)

Neglecting rows and columns of $D_1^\dagger$ that when acting on (6.35) lead to no constraint, the expression of $D_1^\dagger$ can be reduced to

$$
D_1^\dagger = 
\begin{pmatrix}
\bar{y}_1 & \bar{a}_3 & \bar{a}_4 & \bar{y}_2 & \bar{b}_3 & \bar{b}_4 & 0 & 0 \\
\bar{a}_1 & \bar{x}_1 & 0 & \bar{b}_1 & \bar{x}_2 & 0 & \bar{s}_1 & \bar{s}_3 \\
\bar{a}_2 & 0 & \bar{x}_1 & \bar{b}_2 & 0 & \bar{x}_2 & \bar{s}_2 & \bar{s}_4 \\
-x_2 & b_1 & -b_2 & x_1 & a_1 & a_2 & 0 & 0 \\
-b_3 & y_2 & 0 & a_3 & y_1 & 0 & t_1 & t_3 \\
-b_4 & 0 & y_2 & a_4 & 0 & y_1 & t_2 & t_4
\end{pmatrix}
$$

(6.38)

In the limit $a \to 0$ the ADHM equations (4.23) are satisfied by

$$
y_1 = x_1 \quad y_2 = x_2 \\
a_3 = a_1 \quad a_4 = a_2 \quad b_3 = b_1 \quad b_4 = b_2 \\
s_1 = \frac{a_2}{c} \quad s_2 = a_1 c \quad s_3 = \frac{b_2}{c} \quad s_4 = b_1 c
$$

(6.39)

with $c$ an arbitrary real number. By a reshuffling of rows and columns the map (6.38) can be rewritten as a matrix of quaternions

$$
D_1^\dagger = \begin{pmatrix} x & q_1 & q_2 & 0 \\ q_1 & x & 0 & q_2/c \\ q_2 & 0 & x & q_1 c \end{pmatrix} = (D, q)
$$

(6.40)
where \( x, q_1 \) and \( q_2 \) are quaternions whose components are the complex numbers \( x_i, a_i \) and \( b_i \) (\( i = 1, 2 \)), respectively. \( D \) is the \( 3 \times 3 \) left block of \( D_1^\dagger \).

Exploiting the quaternionic notation, we write the frame of sections \( U \) as

\[
U = \begin{pmatrix} \nu \\ \vartheta \end{pmatrix}
\]

(6.41)

where \( \nu \) and \( \vartheta \) are three-dimensional and one-dimensional quaternions respectively. The constraint (4.4) becomes

\[
D\nu = - \begin{pmatrix} s^1 \vartheta \\ 0 \end{pmatrix}
\]

(6.42)

and yields for \( \nu \)

\[
\nu = -D^{-1}q^\dagger \vartheta
\]

(6.43)

The solution can be expressed in terms of the left inverse of \( D, D^{-1} \), whose matrix elements are computed to be

\[
D_{11}^{-1} = \frac{1}{\Delta} x^2[(x^2 + q_1^2 + q_2^2)x - 2(x \cdot q_2)\bar{q}_2 - 2(x \cdot q_1)\bar{q}_1]
\]

\[
D_{12}^{-1} = \frac{1}{\Delta} [-2(x \cdot q_1)(x^2 + q_2^2)x + x^2(x^2 + q_1^2 + q_2^2)\bar{q}_1 + 2(x \cdot q_2)q_2q_1x]
\]

\[
D_{13}^{-1} = \frac{1}{\Delta} [-2(x \cdot q_2)(x^2 + q_1^2)x + x^2(x^2 + q_1^2 + q_2^2)\bar{q}_2 + 2(x \cdot q_1)q_1q_2x]
\]

\[
D_{21}^{-1} = \frac{1}{\Delta} [2x \cdot q_1(x^2 + q_2^2)]\bar{q}_1 - 2(x \cdot q_2)q_2\bar{q}_2]
\]

\[
D_{22}^{-1} = \frac{1}{\Delta} [4(x \cdot q_2)(x \cdot q_1)x - 2(x \cdot q_2)(x^2 + q_1^2)\bar{q}_1 + 2(x \cdot q_1)q_1^2q_2]
\]

\[
D_{23}^{-1} = \frac{1}{\Delta} [4(x \cdot q_2)(x \cdot q_1)x - 2(x \cdot q_2)(x^2 + q_1^2)\bar{q}_1 + 2(x \cdot q_1)q_1^2q_2]
\]

\[
D_{31}^{-1} = \frac{1}{\Delta} [-2(x \cdot q_2)(x^2 + q_1^2)x + x^2(x^2 + q_1^2 + q_2^2)\bar{q}_2 + 2(x \cdot q_1)xq_2\bar{q}_1]
\]

\[
D_{32}^{-1} = \frac{1}{\Delta} [4(x \cdot q_2)(x \cdot q_1)x - 2(x \cdot q_2)(x^2 + q_1^2)\bar{q}_2 + 2(x \cdot q_2)q_2^2\bar{q}_1]
\]

\[
D_{33}^{-1} = \frac{1}{\Delta} [(x^2 + q_1^2 + q_2^2)(x^2 + q_1^2)\bar{q}_2 + 2(x \cdot q_1)(q_1 \cdot q_2)
\]

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\[-2(x \cdot q_2)(x^2 + q_1^2)\bar{q}_2 - 2(x \cdot q_1)q_2^2\bar{q}_1]\]  \hspace{1cm} (6.44)

with

\[
\Delta = (x^2 + q_1^2 + q_2^2) - 4(x^2 + q_1^2)(x \cdot q_2)^2 - 4(x^2 + q_2^2)(x \cdot q_1)^2 + 8x \cdot q_1 x \cdot q_2 q_1 \cdot q_2 \]  \hspace{1cm} (6.45)

The last step of this long calculation is to fix the form of the quaternion \(\vartheta\) by the orthogonalization condition (4.5) and the symmetries of the construction. We do not report here the complicated, uninspiring expression of \(\vartheta\).

As expected the solution depends on twelve parameters. The eight real parameters of the two quaternions \(q_1, q_2\), the real constant \(c\) and three angles associated with global \(SU(2)\) rotations. The latter are left implicit in the construction. The \(a \neq 0\) case turns out to require quite a formidable number of algebraic manipulations. As our main interest lies in the computation of instanton dominated correlators for (locally) SUSY gauge theories, the \(a = 0\) solution appears to be sufficient for the \(N = 1, 2\) cases, as remarked in [6]. In this respect, it is interesting to note that the two-center solution derived from (4.33) and (4.34) has the right number of parameters to match the dimension of the moduli space for the case \(\kappa = 3/2\).

### 7 Moduli Spaces of Gauge Connections of Yang-Mills instantons on ALE manifolds

The KN construction establishes an isomorphism between the equivalence class of solutions of the ADHM equations on ALE spaces, \(X\), and the moduli space of gauge connections on instanton bundles, \(\mathcal{E}\). It can be proved that this isomorphism is an hyper-Kähler isometry and, as we already argued, the moduli space, \(\mathcal{M}_\mathcal{E}\), turns out to be a hyper-Kähler manifold [2] (for a pedagogical introduction to the structure of moduli spaces see [36]). The
metric $G$ on $\mathcal{M}_E$ is given by

$$G_{IJ} = \int_X d^4 x \sqrt{\det(g)} g^{\mu\nu} \delta_I^\mu \delta_J^\nu \delta_{ij}$$  \hspace{1cm} (7.1)$$

where $g$ is the metric on $X$ and $\delta A$ are the zero-mode fluctuations of the gauge fields around the instanton configuration. The three hyper-Kähler forms on $X$ can also be mapped into the moduli space, $\mathcal{M}_E$, by the formulae

$$\Omega^a_{IJ} = \int_X d^4 x \omega^a_{\mu\nu} \delta_I^\mu \delta_J^\nu \delta_{ij}$$  \hspace{1cm} (7.2)$$

The three resulting two-forms, $\Omega^a$, are closed and thus define a hyper-Kähler structure on the moduli space $\mathcal{M}_E$.

There is one case in which $\mathcal{M}_E$ is completely known. To show this we have to use a very powerful theorem proven in [9, 10], that states that on any ALE manifold, $X$, there always exists a bundle, $\mathcal{E}$, with $c_1(\mathcal{E}) = 0, c_2(\mathcal{E}) = (|\Gamma|-1)/|\Gamma|$ such that the moduli space, $\mathcal{M}_E$ is four-dimensional and coincides with the base manifold $X$ itself. Furthermore there exists a point $m_0 \in \mathcal{M}_E$ at which $\mathcal{E}(m_0) = \mathcal{L} \oplus \bar{\mathcal{L}}$ with $\mathcal{L}$ a line bundle, implying that $\mathcal{E}$ is reducible at $m_0$. While on compact manifolds $m_0$ must be a singular point of the moduli space $\mathcal{M}_E$, this is not necessarily true on non-compact manifolds [10]. For EH manifold with metric (6.1) $r = a$ is such a point. Going back to our previous discussion and using this theorem, we see that the moduli space of connections on the minimal $SU(2)$ instanton (the one with $\kappa = 1/2$) has dimension four (from (5.10)) and precisely coincides with the EH manifold. Here we have a nice example of symmetry enhancement at a singular point of a moduli space. As depicted in Fig.2, the EH manifold is the smooth resolution of the singularity of the orbifold $\mathbb{C}^2/\mathbb{Z}_2$: the parameter $a$ appearing in (6.1) measures the distance from the singularity. In the limit $a \rightarrow 0$ the $SU(2) \times U(1)$ symmetry of the moduli space gets enhanced to $SU(2) \times SU(2)$.  

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In order to explicitly compute (7.1), one needs the four zero-modes fluctuations of the gauge fields around the minimal instanton. They are related to the global symmetries broken by the instanton background, \textit{i.e.} dilatations and \textit{SU}(2) rotations. Notice that translation zero-modes do not exist because the $Z_2$ identification $x \approx -x$ forbids global translations in the EH manifold $[16]$. We must look for zero modes that satisfy the background gauge condition and that, up to a local term of the form $D_\mu A$ $[39]$, have the form of derivatives of the gauge connection with respect to the four free parameters (collective coordinates) $[29]$ appearing in (7.11).

For the zero-mode associated to dilatations the situation is very simple, because the derivative of (7.10) with respect to $t$, \textit{i.e.}

$$\delta_0 A = \frac{\partial A}{\partial t} = \frac{it}{\sqrt{t^4 - a^4}} \left( \begin{array}{cc} f^3(r)u_{\sigma_z} & f^1(r)r_{\sigma_-} \\ f^1(r)r_{\sigma_+} & -f^3(r)u_{\sigma_z} \end{array} \right)$$

(7.3)

with

$$f^3(r) = \frac{2(t^2r^2 + a^4)}{r(r^2 + t^2)^2}$$

$$f^1(r) = \frac{2\sqrt{r^4 - a^4}\sqrt{t^4 - a^4}}{r(r^2 + t^2)^2}$$

(7.4)

automatically satisfies the background gauge condition, $D^\mu \delta_0 A_\mu = 0$. The zero-modes associated to global $SU(2)$ rotations

$$\left. \frac{\partial A^{\Omega(\theta_0)}}{\partial \theta_0^j} \right|_{\theta_0 = 0} = i[T_j, A]$$

(7.5)
are not transverse. In order to make them transverse one has to add a local term of the form $D_\mu \Lambda_j$ (3.3). Putting $\Lambda_j = (\theta_j - \theta_j^{(o)})$, the $\theta_j$ turn out to be the bounded harmonic scalars given by (34, 3)

$$\theta_3 = \frac{t^2 r^2 + a^4}{t^2 (r^2 + t^2)} \theta_3^{(o)}, \quad \theta_{1,2} = \frac{\sqrt{r^4 - a^4}}{r^2 + t^2} \theta_{1,2}^{(o)} \quad (7.6)$$

The resulting three transverse zero-modes have the final expression

$$\delta_j A^r_\mu = (D_\mu \theta_j)^r \quad (7.7)$$

The four transverse zero-modes $\delta_I A$ form an orthogonal basis with norms

$$||\delta_0 A||^2 = \frac{8\pi^2}{g^2} \frac{t^4}{t^4 - a^4}$$
$$||\delta_1 A||^2 = ||\delta_2 A||^2 = \frac{8\pi^2}{g^2} \frac{t^2}{t^4}$$
$$||\delta_3 A||^2 = \frac{8\pi^2}{g^2} \frac{t^4 - a^4}{t^2} \quad (7.8)$$

where we have introduced back the gauge coupling constant, $g$.

The framed moduli space of the minimal instanton is four-dimensional and locally looks like $\mathbb{R}^+ \times S^3/\mathbb{Z}_2$, with $t$ ($t \geq a$) playing the role of the radial variable. Near the identity element of $SU(2)$ the metric is given by

$$G_{IJ} = \int_X d^4 x \sqrt{\det(g_{EH})} g_{\mu\nu}^{EH} \delta_I A_\mu^r \delta_J A_\nu^s \delta_r^s \quad (7.9)$$

The metric $G_{IJ}$ may be transported to any element of the group $S^3/\mathbb{Z}_2$ by left translations and one may check that it is identical (up to an overall rescaling) to the metric in (6.4), after the substitutions

$$t \to r, \quad exp(i\theta^k T_k) \to exp(i\psi T_3) exp(i\theta T_2) exp(i\phi T_3) \quad (7.10)$$
8 Conclusions

In this paper we have tackled the problem of the explicit construction of self-dual gauge connections on ALE manifolds and we have carried out computations for the simplest of all these manifolds the EH gravitational instanton. The results obtained here should allow to generalize the computations of instanton dominated correlation functions performed in [3]. There we computed the gaugino condensate in a globally $N = 1$ supersymmetric YM theory on the background of the EH manifold by expanding the functional integral around the minimal $SU(2)$ instanton and we found a constant finite value as expected on supersymmetry grounds. The explicit calculation of instanton effects in a supersymmetric YM theory coupled to supergravity seems to be viable. The relevant Green functions contain a gravitational sector (represented by the field strength gravitino bilinear) and a gauge sector (represented by the gaugino bilinears) which are completely decoupled as argued in [3]. More explicitly, the result of the functional integration in the gauge sector is independent from the moduli of the gravitational background, $\zeta_C$ and $\zeta_R$. This observation suggests that it may suffice to solve the ADHM equations on ALE spaces in the orbifold limit, $\zeta_R = 0, \zeta_C = 0$. This, as we have seen, is a great simplification and it is the reason why we have presented separately many of the results for the EH background in the limit $a \to 0$. Unfortunately, even in this limit, the final form of the $SU(2)$ gauge connection which corresponds to the standard embedding solution, i.e. the one with $c_2(\mathcal{E}) = 3/2$ and $\dim(\mathcal{M}_\mathcal{E}) = 12$, looks quite involved, making the computation of the relevant zero-modes very complicated. A possible way out could be to perform these calculations by resorting to the two-center ansatz (4.33), (4.34).

It is interesting to note that the multi-center ansatz for self-dual $SU(2)$ gauge connections seems to work for any four-dimensional hyper-Kähler man-
ifold, since it essentially relies on the existence of three covariantly constant complex structures. It may then prove to be useful in the study of certain properties of the moduli spaces of YM instantons on the $K3$ surface, which seem to play a fundamental role in the issue of string-string duality in six dimensions and in the intriguing relation between $K3$ and ALE instantons.

As a side remark we have presented the computation of the partition sum for an abelian theory with a $\theta$-term, outlining the lack of S-invariance on ALE spaces, at least in the form which works for other compact hyper-Kähler manifolds [24, 25]. Still, the modular properties of the level-one characters of the affine A-D-E Lie algebras, appearing in the abelian partition function, as well as those of higher level, appearing in non-abelian cases [27], call for some deeper explanation.

\section*{A Appendix}

The hyper-Kähler quotient construction of the EH metric starts from $\Sigma = \mathbb{C}^4$ endowed with the flat metric

$$ds^2 = \sum_{i=1}^{2}(|dx_i|^2 + |dy_i|^2)$$ \hfill (A.1)

The metric (A.1) and the hyper-Kähler two-forms

$$\omega_C = \sum_{i=1}^{2} dx_i \wedge dy_i \quad \omega_R = \sum_{i=1}^{2} (dx_i \wedge d\bar{x}_i + dy_i \wedge d\bar{y}_i)$$ \hfill (A.2)

are invariant under

$$x_i \mapsto e^{i\omega} x_i \quad y_i \mapsto e^{-i\omega} y_i$$ \hfill (A.3)

The moment maps relative to this $U(1)$ triholomorphic isometry are

$$\mu_C = x_1 y_1 - x_2 y_2 \quad \mu_R = \sum_{i=1}^{2}(|x_i|^2 - |y_i|^2)$$ \hfill (A.4)

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Correspondingly the five-dimensional submanifold \( P_\zeta = \{ \xi \in \Xi | \mu_\mathbb{R}(\xi) = \zeta_\mathbb{R}, \mu_\mathbb{C}(\xi) = 0 \} \) is determined by

\[
0 = \zeta_\mathbb{C} = [\alpha, \beta] \\
a^2 = \zeta_\mathbb{R} = [\alpha, \alpha ^{\dagger}] + [\beta, \beta ^{\dagger}]
\]  

(A.5)

where \( \alpha, \beta \) are defined in (6.4). The EH instanton is the simplest of the ALE manifolds and corresponds to the choice \( \Gamma = \mathbb{Z}_2 \). The condition of \( \Gamma \)-invariance on the doublet of matrices \((\alpha, \beta)\) reads

\[
\begin{pmatrix}
\rho_\mathbb{R}(\gamma) \alpha \rho_\mathbb{R}(\gamma^{-1}) \\
\rho_\mathbb{R}(\gamma) \beta \rho_\mathbb{R}(\gamma^{-1})
\end{pmatrix} = \rho_\mathbb{Q}(\gamma)
\begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}
\]  

(A.6)

The matrices which satisfy the constraint (A.6) are 2 \( \times \) 2 matrices with only (complex) off-diagonal entries. Equations (A.5) are solved by \( y_1 = \lambda x_2, y_2 = \lambda x_1 \), where \( \lambda \) is a complex number with \( |\lambda|^2 = 1 + a^2/x^2 \) and \( x^2 = \sum_i |x_i|^2 \).

The metric on the submanifold \( P_\zeta \) then becomes

\[
ds^2_{P_\zeta} = \sum_{i=1}^{2}(|dx_i|^2 + |\lambda|^2|dx_i + x_i \frac{d\lambda}{\lambda}|^2)
\]  

(A.7)

from which, putting \( \lambda = |\lambda|e^{i\beta} \) and \( x_i = r_i e^{i\alpha_i} \), one obtains

\[
ds^2_{P_\zeta} = \sum_{i=1}^{2}[(dr_i^2 + (|\lambda|dr_i + r_i d|\lambda|)^2) + r_i^2(\alpha_i^2 + |\lambda|^2(\alpha_i + d\beta)^2)]
\]  

(A.8)

It is convenient to divide the line element into two parts \( ds^2_{P_\zeta} \equiv ds^2_r + ds^2_\Omega \), with \( ds^2_r \) containing the differentials of the variables \( r, |\lambda| \) and \( ds^2_\Omega \) the differentials of \( \alpha, \beta \). In fact \( ds^2_{\Omega} \) has still a residual \( G = U(1) \) invariance associated to a combination of the variables \( \alpha, \beta \). The Killing vector related to the corresponding cyclic variable is

\[
k = k^\mu \partial_\mu = \sum_i \left( \frac{\partial}{\partial \alpha_i} - \frac{\partial}{\partial \beta} \right)
\]  

(A.9)

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The next step is to construct $X_\zeta = P_\zeta / G$. As it is nicely explained in [20], from a physicist’s point of view, this construction can be regarded as a gauging procedure of a non-linear $\sigma$-model on $\mathbb{R}^d$ with Lagrangian density

$$L = \frac{1}{2} g^P_{\mu \nu}(\phi) \partial^a \phi^\mu \partial_a \phi^\nu$$

where $g^P_{\mu \nu}(\phi)$ is the metric on the target space $P$ with coordinates $\phi^\mu$ and $\partial_a = \partial / \partial x^a$, with $x^a$, $a = 1, \ldots, d$, coordinates on $\mathbb{R}^d$. Let $P$ admit a group of isometry $G = U(1)$ whose action on the fields $\phi^\mu$ is

$$\delta \phi^\mu = \epsilon k^\mu(\phi)$$

with $\epsilon$ constant. The global symmetry (A.11) can be promoted to a local one by introducing the connection

$$A_a = - \frac{\partial_a \phi^\mu g^P_{\mu \nu} k^\nu}{g^P_{\rho \sigma} k^\rho k^\sigma}$$

and substituting standard derivatives with covariant ones. The resulting gauged $\sigma$-model has target space $X = P / G$, the space of $G$ orbits on $P$ with $\dim(X) = \dim(P) - \dim(G)$ [20].

In the case under consideration $\mu$ runs from 1 to $\dim(P) = 5$ and the coordinates on $P$ may be taken to be $\phi^\mu = (r_1, r_2, \alpha_1, \alpha_2, \beta)$. The global isometry of $ds^2_{P_\zeta}$ is generated by the Killing vector (A.9), with $k^\mu = (0, 0, 1, 1, -2)$. From the expression of the gauged version of the lagrangian (A.10) one can read off the metric on $X = P / G$, namely

$$g^X_{\mu \nu} = g^P_{\mu \nu} - \frac{g^P_{\mu \rho} g^P_{\nu \sigma} k^\rho k^\sigma}{g^P_{\lambda \tau} k^\lambda k^\tau}$$

where $g^P_{\mu \nu}$ is the metric tensor in (A.8). By construction the metric (A.13) is transverse to the Killing vector and the local $U(1)$ isometry allows one to gauge $\beta$ to zero.

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Computing the connection \([A.12]\) and comparing it with \((6.2)\) suggest the further change of variables

\[
\alpha_1 = \frac{\psi + \varphi}{2}, \quad \alpha_2 = \frac{\psi - \varphi}{2}
\]  

(A.14)

As for the second part of the metric, \(ds_r^2\), after the change of variables

\[
\begin{align*}
\alpha_1 &= \sqrt{r^2 - a^2} \cos \frac{\theta}{2} e^{\frac{i \psi + \varphi}{2}} \\
\alpha_2 &= \sqrt{r^2 + a^2} \sin \frac{\theta}{2} e^{\frac{i \psi - \varphi}{2}} \\
\end{align*}
\]  

(A.15)

it can be written in the more familiar form

\[
ds_r^2 = \frac{(dr)^2}{1 - \left(\frac{a}{r}\right)^4} + \frac{r^2}{4} (d\theta)^2
\]  

(A.16)

Collecting all terms, we get the EH metric \((6.1)\).

To summarize we give the complete changes of variables from the original metric \((A.1)\) to the variables employed in \((6.1)\)

\[
\begin{align*}
x_1 &= \sqrt{\frac{r^2 - a^2}{2}} \cos \frac{\theta}{2} e^{\frac{i \psi + \varphi}{2}} \\
y_1 &= \sqrt{\frac{r^2 + a^2}{2}} \sin \frac{\theta}{2} e^{\frac{i \psi - \varphi}{2}} \\
x_2 &= \sqrt{\frac{r^2 - a^2}{2}} \sin \frac{\theta}{2} e^{\frac{i \psi - \varphi}{2}} \\
y_2 &= \sqrt{\frac{r^2 + a^2}{2}} \cos \frac{\theta}{2} e^{\frac{i \psi + \varphi}{2}}
\end{align*}
\]  

(A.17)
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