Lower Bounds for Approximating Graph Parameters via Communication Complexity

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Abstract

We present a new framework for proving query complexity lower bounds for graph problems via reductions from communication complexity. We illustrate our technique by giving new proofs of tight lower bounds for several graph problems: estimating the number of edges in a graph, sampling edges from an almost-uniform distribution, estimating the number of $k$-cliques in a graph, and estimating the moments of the degree distribution of a graph.

1 Introduction

Large networks are everywhere, arising naturally in computer science, social science, biology, economics, and many other fields. Often, the large scale of these networks makes it infeasible to compute parameters of interest quickly. Further, many of these networks are dynamic. By the time an execution of an algorithm completes, the topology of the network may have changed significantly. Thus, exact graph parameters may not even be well-defined. The immense scale and dynamism of such networks motivates the study of sublinear-time and approximation algorithms. Such algorithms access a network via simple, local queries, and are guaranteed to return an approximate value of the desired property with reasonably good probability.

In this work, our goal is to prove lower bounds on the number of queries necessary to (approximately) solve various graph problems. In particular, we introduce a framework for deriving query complexity lower bounds for computations on graphs via reductions from communication complexity. Previous proofs of lower bounds typically relied upon careful analysis of statistical distances between families of graphs. In contrast, our technique is purely combinatorial. Once a “hard” communication problem is chosen, and a suitable (combinatorial) embedding of the hard problem is defined, query lower bounds follow from simple simulation arguments. Thus, we believe our techniques unify and simplify proofs of many existing lower bounds.

We apply our lower bound framework to the following graph problems: (1) estimating the number of edges, (2) sampling edges from an almost-uniform distribution, (3) estimating the number of $k$-cliques (in particular, triangles), and (4) estimating the moments of the degree distribution. Our technique yields tight lower bounds for each of these problems. Informally, each of these graph parameters is hard (i.e., requires many queries) because it is possible to have a relatively

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small subgraph with a large (constant fraction of the total) contribution to the graph parameter in question. Thus, in order to approximately compute the parameter, a large portion of the graph must be sampled.

For each of the problems listed above, our argument uses a reduction from the well-studied disjointness function in the two party communication model. Suppose two parties, Alice and Bob, hold subsets $A$ and $B$ (respectively) of $[N] = \{1, 2, \ldots, N\}$. A fundamental result in communication complexity states that in order for Alice and Bob to determine whether or not their sets are disjoint with constant probability bounded away from $1/2$, they must exchange $\Omega(N)$ bits. Informally, one should expect that the disjointness function is hard (i.e., requires a lot of communication) because an element in $A \cap B$ could be hidden anywhere in Alice’s and Bob’s inputs—essentially all possibilities must be checked. This idea is conceptually similar to the reason that the graph parameters above are hard to estimate: a small set of vertices with large contribution may be hidden anywhere in the graph. Our reductions make the connection between set disjointness and approximating graph parameters explicit, and reveal a unifying principle underlying the inherent difficulty in solving these problems.

1.1 Related Work

A model for query-based sublinear graph algorithms was first presented in the seminal work of Goldreich, Goldwasser, and Ron [10] in the context of property testing. Their model is appropriate for dense graphs, as only “pair queries” (i.e., queries of the form Do vertices $u$ and $v$ share an edge?) are allowed. An analogous model for sparse graphs was introduced by Goldreich and Ron in [11]. The sparse graph model allows degree queries (What is the degree of $v$?) and neighbor queries (Who is $v$’s $i^{th}$ neighbor?). Parnas and Ron [19] introduced the general graph model which allows all of the above queries—pair, degree and neighbor queries. The lower bounds we prove all apply to the general graph model.

The problem of estimating the average degree of a graph (or equivalently, the number of edges in a graph) was first studied by Feige [9]. In [9], Feige proves tight bounds on the number of degree queries necessary to estimate the average degree. In [12], Goldreich and Ron study the same problem, but in a model that additionally allows neighbor queries. In this model, they prove matching upper and lower bounds for the number of queries needed to estimate the average degree. In Theorem 4.1 we achieve the same lower bound as [12] for estimating the number of edges in a graph in the more general graph model that additionally allows pair queries. The related problem of sampling edges from an almost-uniform distribution was recently studied by Eden and Rosenbaum in [8]. They prove tight bounds on the number of queries necessary to sample an edge in a graph from an almost-uniform distribution. In Theorem 4.4 we present a new derivation of the lower bound presented in [8].

Eden et al. [5] prove tight bounds on the number of queries needed to estimate the number of triangles. Their results were generalized by Eden et al. in [6] to approximating the number of $k$-cliques for any $k \geq 3$. In Theorem 4.5 and 4.6 we give a new derivation of the lower of [6] (and a fortiori the lower bound of [5]).

In [14], Gonen et al. study the problem of approximating the number of $s$-star subgraphs, and give tight bounds on the number of (neighbor, degree, edge) queries needed to solve this problem. As noted by Eden et al. [7], counting $s$-stars is closely related to computing the $s^{th}$ moment of the degree sequence. In [7], they provide a simpler optimal algorithm for computing the $s^{th}$ moment of the degree sequence that has better or matching query complexity when the algorithm is also given an upper bound on the arboricity of the graph. In Theorems 4.9 and 4.10 we prove the lower bounds of [7] that build on and generalize the lower bounds of [14]. The recent paper of
Aliakbarpour et al. [1] proposes an algorithm for estimating the number of s-star subgraphs that is allowed uniformly random edge samples as a basic query as well as degree queries. Interestingly, the additional computational power afforded by random edge queries allows their algorithm to break the lower bound of [14].

Communication complexity was introduced in the seminal work of Yao [21]. We refer the reader to [16] for a thorough introduction to communication complexity. An optimal $\Omega(n)$ lower bound on the randomized communication complexity of the disjointness function (Theorem 2.5) was first proved by Kalyanasundaram and Schwentter [15], and independently by Razborov [20]. The communication lower bound for the disjointness function was first identified as a valuable tool to achieve (sublinear space) lower bounds in the streaming model by Alon et al. [2]. In [2], Alon et al. prove space lower bounds for estimating the frequency moments of a data stream. The communication complexity of the disjointness function was also used by Gonczarowski et al. [3] to prove query complexity lower bounds for the stable marriage problem.

Most closely related to our work are the papers of Blais et al. [3] and Blais et al. [4]. In [3], the authors present a framework for obtaining query lower bounds in the property testing model [10] from communication complexity. They focus on testing properties of boolean functions, and prove lower bounds for various problems, including testing $k$-linearity, monotonicity, concise representations, and juntas. The work of Blais et al. [4] extends the framework of [3] and proves lower bounds for distribution testing using reductions from the simultaneous message passing communication model. The framework we present for general graph problems is analogous to the frameworks of [3] and [4] for testing properties of functions and distributions, but our results are not directly comparable. One advantage of our framework over previous work is that we exploit shared public randomness in order to improve lower bounds by a logarithmic factor.

2 Preliminaries

2.1 Graph Query Models

Let $G = (V, E)$ be a graph where $n = |V|$ is the number of vertices and $m = |E|$ is the number of edges. We assume that the vertices $V$ are given distinct labels, say, from $[n] = \{1, 2, \ldots, n\}$. For $v \in V$, let $N(v)$ denote the set of neighbors of $v$, and $\deg(v) = |N(v)|$ is $v$’s degree. For each $v \in V$, we assume that $N(v)$ is ordered by specifying some arbitrary bijection $N(v) \rightarrow [\deg(v)]$ so that we may refer unambiguously to $v$’s $i$th neighbor. We let $\mathcal{G}_n$ denote the set of all graphs on $n$ vertices, together with all possible labelings of the vertices (from $[n]$) and all orderings of the neighbors of each vertex, and we define $\mathcal{G} = \bigcup_{n \in \mathbb{N}} \mathcal{G}_n$.

We consider algorithms that access $G$ via queries. In general, a query is an arbitrary function $q : \mathcal{G} \rightarrow \{0, 1\}^*$. We are interested in the following question: “Given a set $Q$ of allowable queries and a graph problem $P$ (e.g. a computing function, estimating a graph parameter, etc.), how many queries $q \in Q$ are necessary to compute $P$?”

In what follows, we allow the following sets of queries:

1. Neighbor query $\text{nbr}_i : V \rightarrow V \cup \{\emptyset\}$ for $i \in [n - 1]$, and the answer to $\text{nbr}_i(v)$ is $v$’s $i$th neighbor if $i \leq \deg(v)$ and $\emptyset$ otherwise.

2. Degree query $\text{d} : V \rightarrow [n - 1]$, and the answer to $\text{d}(v)$ is $v$’s degree.

3. Pair query $\text{pair} : V \times V \rightarrow \{0, 1\}$, and the answer to $\text{pair}(u, v)$ is either $(u, v) \in E$ or $(u, v) \notin E$. 


Taking \( Q \) to be the set of all neighbor, degree, and pair queries, we have \( Q = O(n^2) \). This query model is known as the \textit{general graph model} introduced in \cite{19}.

We wish to characterize the \textit{query complexity} of graph problems, that is, the minimum number of queries necessary to solve the problem. We consider randomized algorithms, and we assume the randomness is provided via a random string \( r \in \{0,1\}^N \). The number of queries made by an algorithm is generally a random variable, so we consider the \textit{expected} number of queries that an algorithm makes.

Most of our results are lower bounds on the number of queries necessary to estimate graph parameters.

**Definition 2.1.** A \textit{graph parameter} is a function \( f : \mathcal{G} \to \mathbb{R} \) that is invariant under any permutation of the vertices of each \( G \in \mathcal{G} \). Formally, \( f \) is a graph parameter if for every \( n \in \mathbb{N} \), \( G = (V,E) \in \mathcal{G}_n \) and every permutation \( \pi : [n] \to [n] \), the graph \( G_\pi = (V,E_\pi) \) defined by \( (v_\pi(i),v_\pi(j)) \in E_\pi \iff (v_i,v_j) \in E \) satisfies \( f(G_\pi) = f(G) \).

**Definition 2.2.** Let \( f : \mathcal{G}_n \to \mathbb{R} \) be a graph parameter, \( A \) an algorithm, and \( \varepsilon > 0 \). We say that \( A \) \textit{computes a (multiplicative) \((1 \pm \varepsilon)\)-approximation} of \( f \) if for all \( G \in \mathcal{G} \), the output of \( A \) satisfies \( \Pr_r(|A(G) - f(g)|) \leq \varepsilon f(g) \geq 2/3 \). Here, the probability is taken over the random choices of the algorithm \( A \) (i.e., over the random string \( r \)).

**Remark 2.3.** In the general graph model, every graph \( G \) can be explored using \( O(\max \{n,m\}) \) queries, for example, by using depth first search. Thus, we are interested in algorithms that perform \( o(\max \{n,m\}) \)—or even better, \( o(n) \)—queries.

### 2.2 Communication Complexity Background

In this section, we briefly review some background on two party communication complexity and state a fundamental lower bound for the disjointness function. We refer the reader to \cite{16} for detailed introduction.

We consider two party communication complexity in the following setting. Let \( f : \{0,1\}^N \times \{0,1\}^N \to \{0,1\} \) be a boolean function. Suppose two parties, traditionally referred to as Alice and Bob, hold \( x \) and \( y \), respectively, in \( \{0,1\}^N \). The \textit{communication complexity} of \( f \) is the minimum number of bits that Alice and Bob must exchange in order for both of them to learn the value \( f(x,y) \).

More formally, let \( \Pi \) be a communication protocol between Alice and Bob. We assume that \( \Pi \) is randomized, and that Alice and Bob have access to a shared random string, \( r \). We say that \( \Pi \) \textit{computes} \( f \) if for all \( x,y \in \{0,1\}^n \), \( \Pr_r[\Pi(x,y) = f(x,y)] \geq 2/3 \), where the probability is taken over all random strings \( r \). For fixed inputs \( x,y \in \{0,1\}^n \) and random string \( r \), we denote the number of bits exchanged by Alice and Bob using \( \Pi \) on input \((x,y)\) and randomness \( r \) by \( |\Pi(x,y;r)| \). The \textit{(expected) communication cost} of \( \Pi \) is defined by

\[
\text{cost}(\Pi) = \sup_{x,y} \mathbb{E}[|\Pi(x,y;r)|].
\]

Finally, the \textit{(randomized) communication complexity} of \( f \), denoted \( R(f) \), is the minimum cost of any protocol that computes \( f \):

\[
R(f) = \min \{ \text{cost}(\Pi) | \Pi \text{ computes } f \}.
\]

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1. Allowing only neighbor and degree queries is known as the "sparse graph" model, while only allowing pair queries is the "dense graph" model.

2. It is more common in the literature to define the cost in terms of the worst case random string \( r \) rather than expected. However, for our purposes it will be more convenient to use expected cost. Since we allow our protocols to err with (small) constant probability, this difference only affects the communication complexity by a constant factor.
The notion of communication complexity extends to partial functions in a natural way. That is, we may restrict attention to particular inputs for $f$ and allow $\Pi$ to have arbitrary output for all other values. Formally, we model this extension to partial functions via \textit{promises} on the input of $f$. Let $P \subseteq \{0, 1\}^N \times \{0, 1\}^N$. We say that a protocol $\Pi$ computes $f$ for the promise $P$ if for all $(x, y) \in P$, $\Pr_r(\Pi(x, y) = f(x, y)) \geq 2/3$. The communication complexity of a promise problem (or equivalently, a partial function) is defined analogously to the paragraph above.

One of the fundamental results in communication complexity is a linear lower bound for the communication complexity of the disjointness function. Suppose Alice and Bob hold subsets $A, B \subseteq [N]$, respectively. The disjointness function takes on the value 1 if $A \cap B = \emptyset$, and 0 otherwise. By associating $A$ and $B$ with their characteristic vectors in $\{0, 1\}^N$ (i.e. $x_i = 1 \iff i \in A$ and $y_j = 1 \iff j \in B$), we can define the disjointness function as follows.

Definition 2.4. For any $x, y \in \{0, 1\}^N$, the \textit{disjointness function} is defined by the formula

$$\text{disj}(x, y) = \neg \bigvee_{i=1}^N x_i \land y_i.$$ 

The following lower bound for the communication complexity of disj was initially proved by Kalyansundaram and Schintger [15] and independently by Razborov [20]. All of the results presented in this paper rely on this fundamental lower bound.

Theorem 2.5 ([15, 20]). The randomized communication complexity of the disjointness function is $R(\text{disj}) = \Omega(N)$. This result holds even if $x$ and $y$ are promised to satisfy $\sum_{i=1}^N x_i y_i \in \{0, 1\}$ — that is, Alice’s and Bob’s inputs are either disjoint or intersect on a single point.

The promise in Theorem 2.5 is known as \textit{unique intersection}. In the remainder of the paper we will use a variant of the unique intersection problem that we refer to as the $k$-intersection problem.

Definition 2.6. Let $x, y \in \{0, 1\}^N$. We say that $x$ and $y$ are \textit{k-intersecting} if $\sum_{i=1}^N x_i y_i \geq k$. The $k$-intersection function is defined by the formula

$$\text{int}_k(x, y) = \begin{cases} 1 & \text{if } \sum_{i=1}^N x_i y_i \geq k \\ 0 & \text{otherwise.} \end{cases}$$

We now prove the following consequence of Theorem 2.5.

Corollary 2.7. $R(\text{int}_k) = \Omega(N/k)$. The result holds even if $x$ and $y$ are promised to satisfy $\sum_i x_i y_i \in \{0, k\}$.

Proof. The argument is by simulation. Specifically, we will show that any efficient protocol for $\text{int}_k$ yields an efficient protocol for disj. Suppose $\Pi$ is a protocol for the promise problem of the corollary with $\text{cost}(\Pi) = B$. For $x, y \in \{0, 1\}^{N/k}$, let $x^k, y^k \in \{0, 1\}^N$ denote the concatenation of $x$ and $y$ (respectively) repeated $k$ times. Observe that if $x$ and $y$ satisfy the unique intersection promise, then $x^k$ and $y^k$ satisfy the $k$-intersection promise. Further, $\text{int}_k(x^k, y^k) = 1$ if and only if $\text{disj}(x, y) = 1$. Since $\Pi$ computes $\text{int}_k$ for all $x', y' \in \{0, 1\}^N$ satisfying the $k$-intersection promise, $\Pi(x^k, y^k)$ computes disj on input $x, y$. Therefore, by Theorem 2.5, $\text{cost}(\Pi) = \Omega(N/k)$, which gives the desired result. \hfill $\square$
3 General Lower Bounds

In this section, we describe a framework for obtaining general query lower bounds from communication complexity. Let \( G_n \) denote the family of graphs on the vertex set \( V = [n] \), which we assume have labels 1 through \( n \). We will use \( g : G_n \to \{0, 1\} \) to denote a boolean function on \( G_n \).

**Definition 3.1.** Let \( P \subseteq \{0, 1\}^N \times \{0, 1\}^N \). Suppose \( f : P \to \{0, 1\} \) is an arbitrary (partial) function, and let \( g \) be a boolean function on \( G_n \). Let \( \mathcal{E} : \{0, 1\}^N \times \{0, 1\}^N \to \mathcal{G}_n \). We call the pair \( (\mathcal{E}, g) \) an embedding of \( f \) if for all \( (x, y) \in P \) we have \( f(x, y) = g(\mathcal{E}(x, y)) \).

For a general embedding \((\mathcal{E}, g)\) of a function \( f \), the edges of \( \mathcal{E}(x, y) \) can depend on \( x \) and \( y \) in an arbitrary way. In order for the embedding to yield meaningful lower bounds, however, each allowable query \( q \) should be computable from \( x \) and \( y \) with little communication.

**Definition 3.2.** Let \( q : \mathcal{G}_n \to \{0, 1\}^* \) be a query and \((\mathcal{E}, g)\) an embedding of \( f \). We say that \( q \) has communication cost at most \( B \) and write \( \text{cost}_\mathcal{E}(q) \leq B \) if there exists a (zero-error) communication protocol \( \Pi_q \) such that for all \((x, y) \in P \) we have \( \Pi_q(x, y) = q(\mathcal{E}(x, y)) \) and \( |\Pi_q(x, y)| \leq B \).

**Theorem 3.3.** Let \( Q \) be a set of allowable queries, \( f : P \to \{0, 1\} \), and \((\mathcal{E}, g)\) an embedding of \( f \). Suppose that each query \( q \in Q \) has communication cost relative to \( \mathcal{E} \) of \( \text{cost}_\mathcal{E}(q) \leq B \). Suppose \( A \) is an algorithm that computes \( g \) using \( T \) queries (in expectation) from \( Q \). Then the expected query complexity of \( A \) is \( T = \Omega(R(f)/B) \).

**Proof.** Suppose \( A \) computes \( g \) using \( T \) queries in expectation. From \( A \) we define a two party communication protocol \( \Pi_f \) for \( f \) as follows. Let \( x \) and \( y \) denote Alice and Bob’s inputs, respectively, and \( r \) their shared public randomness. Alice and Bob both invoke \( A \), letting \( r \) be the randomness of \( A \). Whenever \( A \) performs a query \( q \) that Alice and Bob cannot answer on their own, they communicate to the other party in order to determine the outcome of the query.\(^4\) That is, they invoke \( \Pi_q \) in order to compute the response \( a \) to query \( q \). The protocol terminates when \( A \) halts and returns an answer \( A \), at which point Alice and Bob determine their answer to \( f \) according to \( A \).

Since \( \Pr_r(A(G) = g(G)) \geq 2/3 \), and \( g(G) = f(x, y) \) it is clear that \( \Pi_f \) computes \( f \). Further, since each \( \Pi_q \) satisfies \( |\Pi_q| \leq B(q) \) (where if Alice and Bob can answer \( q \) without communication we let \( B(q) = 0 \)), we have \( \text{cost}(\Pi_f) = 2BT \). Since \( \text{cost}(\Pi_f) \geq R(f) \), we have \( T \geq R(f)/2T \), as desired.\(\square\)

Given the above Theorem, we suggest the following framework for proving graph query lower bounds.

1. Choose a “hard” communication problem \( f : P \to \{0, 1\} \).
2. Define functions \( \mathcal{E} : P \to \mathcal{G}_n \) and \( g : \mathcal{G}_n \to \{0, 1\} \) such that \((\mathcal{E}, g)\) is an embedding of \( f \) in the sense of Definition 3.1.
3. For each allowable query \( q \in Q \), bound \( B \), the number of bits that must be exchanged in order to simulate \( q \) given \( \mathcal{E} \).

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\(^4\)Since the randomness of \( A \) is the shared randomness of Alice and Bob they both witness the same execution of \( A \) and agree without communication on which query is being performed during each step. Further, since they both know the function \( \mathcal{E} \), they can individually determine if a query \( q \) cannot be answered by the other party. In this case, they invoke \( \Pi_q \).
4 Lower Bounds for Particular Problems

In this section, we derive lower bounds for particular problems. In all cases, we allow $Q$ to be the family of all degree, neighbor, and pair queries.

4.1 Counting and Sampling Edges

4.1.1 Counting Edges

We start by proving a result of [12] that gives a lower bound of $\Omega(n/\sqrt{\varepsilon m})$ for estimating the number of edges in a graph up to a $(1 \pm \varepsilon/2)$-multiplicative factor using degree, neighbor and pair queries. A similar result was first proven by Feige [9] for any $(2 + \varepsilon)$-multiplicative factor approximation algorithm for the number of edges using only degree and pair queries.

**Theorem 4.1** (Thm. 3.2 in [12]). Let $A$ be an algorithm that approximates the number of edges using $Q$ consisting of neighbor, degree, and pair queries. Specifically, for any $\varepsilon > 0$, on any input graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, $A$ outputs an estimate $\hat{m}$ satisfying $\Pr(|\hat{m} - m| < \varepsilon m) \geq 2/3$. Then the expected query complexity of $A$ is $\Omega(n/\sqrt{\varepsilon m})$.

**Proof.** To prove Theorem 4.1, we take $f = \text{disj}$, the disjointness function (Definition 2.4). For Step 2 in the framework above, the idea is to "hide" edges from the algorithm by possibly including a small subgraph that nonetheless contributes a constant fraction of the edges of $G$. Starting from an arbitrary graph $G'$ with $n'$ vertices and $m'$ edges, we can accomplish this by adding a clique with at least $\varepsilon m'$ edges—a clique on $2\sqrt{\varepsilon m'}$ vertices certainly suffices. In order to hide such a clique, we add to $G'$ an additional $n'$ vertices, partitioned into sets of size $k = n/2\sqrt{\varepsilon m'}$. Each set in the partition corresponds to a single bit of Alice and Bob's inputs in $\{0,1\}^k$. In the case where Alice and Bob's inputs are disjoint, the additional vertices will be isolated. On the other hand, if Alice and Bob's sets intersect, then at least one of the partitions of the new vertices will form a clique, thereby increasing the number edges in the graph beyond $(1 + \varepsilon)m'$. Thus, the output of any algorithm that approximates the number of edges in the new graph can be used to compute disj.

Formally, let $G' = (V', E')$ be an arbitrary fixed graph with $n' = |V'|$ and $m' = |E'|$. Let $V = V' \cup W$, where $W = [n']$. Take $n = |V| = 2n'$. We partition $W$ such that $W = W_1 \cup \cdots \cup W_k$ where $k = n'/2\sqrt{\varepsilon m'}$, and $W_i = \{(i-1)k + 1, \ldots, ki\}$. We define $E : \{0,1\}^k \times \{0,1\}^k \rightarrow G_n$ as follows. Given $x, y \in \{0,1\}^k$, all $u, v \in V$, we have $(u, v) \in E$ if and only if one of the following two conditions is satisfied:

\[
\begin{align*}
u, v & \in V' \quad \text{and} \quad (u, v) \in E' \\
u, v & \in W_i \quad \text{and} \quad i \in x \cap y.
\end{align*}
\]

That is, $W_i$ is a clique if $i \in x \cap y$ and a set of isolated vertices otherwise. See Figure 1 for an illustration.

Let the function $g : G \rightarrow \{0,1\}$ be defined by

\[
g(G) = \begin{cases} 1 & \text{if } |E| \leq m' \\ 0 & \text{if } |E| \geq (1 + \varepsilon)m'. \end{cases}
\]

First we argue that $(E, g)$ is an embedding of disj. To see this, note that if disj$(x, y) = 1$, then the second condition of Equation (1) is never satisfied. Therefore, $m = |E| = |E'| = m'$, so that $g(E(x, y)) = \text{disj}(x, y)$. On the other hand, if disj$(x, y) = 0$, say $x_i = y_i = 1$, then $W_i$ is a clique. Therefore, $m = |E| \geq (1 + \varepsilon)m'$.
Figure 1: The construction of the graph $G = \mathcal{E}(x, y)$. The top subgraph is an arbitrary graph with $n'$ and $m'$ edges, and there are $n'/\sqrt{\varepsilon m}$ additional subgraphs $W_i$, each of size $\sqrt{\varepsilon m}$. In case $x$ and $y$ intersect on the $i^{th}$ coordinate, the subgraph $W_i$ is a clique, and otherwise it is a set of isolated vertices.

We now bound the communication complexity required to answer queries from $Q$. For a query $q \in Q$, note that Alice and Bob can answer the query themselves (i.e., $q$ is independent of $x$ and $y$) unless $q$ is one of the following queries:

1. $\text{nbr}(v, j)$ with $v \in W_i$,
2. $d(v)$ with $v \in W_i$,
3. $\text{pair}(u, v)$ with $u, v \in W_i$.

If $q$ is any of the three queries above, then its answer can be determined using the following protocol $\Pi_q$: Alice sends $x_i$ to Bob, and Bob sends $y_i$ to Alice. If $\neg (x_i \land y_i)$ then Alice and Bob answer the query accordingly. Otherwise, Alice and Bob announce $x_i \land y_i$. Thus $|\Pi_q| = 2$ for all $q \in Q$. The theorem then follows from Theorem 3.3.

Remark 4.2. The lower bound presented above can easily be modified to allow additional restrictions on the input graph while still giving the same lower bound. For example, we may require that the input graph $G$ be connected or have small diameter. In this case, we can modify the construction by adding $O(n)$ edges, say, by connecting every vertex in the graph to some fixed vertex $v$. In this case, the argument above remains valid with minor modifications. Thus, the lower bound holds, for example, in the case where $G$ is promised to be a connected graph of diameter 2.

4.1.2 Sampling Edges

In this subsection, we prove a lower bound on the number of queries necessary to sample an edge in a graph $G = (V, E)$ from an “almost-uniform” distribution $D$ over $E$. Here, we use “almost uniform” in the sense of total variational distance:
Definition 4.3. Let $D$ and $D'$ be probability distributions over a finite set $X$. Then the total variational distance between $D$ and $D'$, denoted $\text{dist}_{TV}(D, D')$, is defined by

$$\text{dist}_{TV}(D, D') = \frac{1}{2} \sum_{x \in X} |D(x) - D'(x)|.$$ 

For $\varepsilon > 0$, we say that $D$ is $\varepsilon$-close to uniform if $\text{dist}_{TV}(D, U) \leq \varepsilon$ where $U$ is the uniform distribution on $X$ (i.e., $U(x) = 1/|X|$ for all $x \in X$).

Theorem 4.4. Let $0 < \varepsilon < 1/3$. Suppose $A$ is an algorithm that for any graph $G = (V, E)$ on $n$ vertices and $m$ edges returns an edge $e \in E$ sampled from a distribution $p$ that is $\varepsilon$-close to uniform using neighbor, degree, and pair queries. Then $A$ requires $\Omega(n/\sqrt{m})$ queries.

Proof. We use the same embedding $(\mathcal{E}, g)$ of disj described in the Section 4.1.1 with $k = n/2\sqrt{m}$ (i.e., setting $\varepsilon = 1$ in that section). Thus, if any $W_i$ is a clique, the induced subgraph on $W$ contains at least $m/2$ edges in $G$.

Consider the following procedure, which we call algorithm $A'$: repeat $A$ 7 times to get edge samples $e_1, \ldots, e_7$. If at least one $e_i$ satisfies $e_i \in W \times W$, return 0, otherwise return 1. We claim that this procedure computes $g$ (on the range of $\mathcal{E}$). To see this, suppose $g(G) = 0$, i.e., at least one of the $W_i$ is a clique so that $W \times W$ contains at least $m'$ edges. Thus the fraction of edges in $W \times W$ is at least $1/2$, so any invocation of $A$ must return an edge $e \in W \times W$ with probability at least $1/2 - 1/3 = 1/6$. Therefore, if $g(G) = 0$, the probability that algorithm $A'$ returns 1 (i.e., that no edge $e \in W \times W$ is sampled) is at most $(1 - 1/6)^7 < 1/3$. On the other hand, if $g(G) = 1$, then the procedure will always return 1, as $G$ contains no edges in $W \times W$.

As in the proof of Theorem 4.1, algorithm $B$ requires $\Omega(n/\sqrt{m})$ queries. Since $A'$ invokes $A$, $O(1)$ times with no additional queries, the same is true of algorithm $A$. \qed

4.2 Counting $r$-cliques

In this section, we prove lower bounds for the problem of estimating the number of $r$-cliques in a graph. Recall that vertices $v_1, v_2, \ldots, v_r \in V$ are a $k$-clique if $(v_i, v_j) \in E$ for all $1 \leq i < j \leq r$. We denote the number of $r$-cliques in $G$ by $C_r$.

Theorem 4.5. Any multiplicative approximation algorithm for the number of $k$-cliques in a graph must perform $\Omega\left(n/C_r^{1/r}\right)$ queries.

The proof of Theorem 4.5 is almost identical to that of Theorem 4.1 with a slight change in parameters and hence we omit it.

Theorem 4.6. Any multiplicative approximation algorithm for the number of $r$-cliques in a graph must perform $\Omega\left(\min\left\{m, \frac{m^{r/2}}{C_r^{(c\gamma)^r}}\right\}\right)$ queries for some constant $c$.

Proof. We describe how to reduce the problem of $k$-intersection to the problem of approximating the number of $r$-cliques in a graph, where the value of $k$ depends on the value of $C_r$. We consider two cases depending on the relation between $C_r$ and $m$. Observe that since for every graph $G$ with $m$ edges $C_r(G) \leq m^{r/2}$, these two cases indeed cover the entire range of possible values of $C_r(G)$. 

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The case $C_r > m^{(k-2)/2}$. Given two strings $x$ and $y$ in $\{0, 1\}^M$ we define the graph $G = \mathcal{E}(x, y)$ as follows. $G$ consists of $r + 2$ sets $\{A, A', B, B', S^1, \ldots, S^{r-2}\}$ of size $\sqrt{M}$ and $n - (r + 2)\sqrt{M}$ additional isolated vertices. We say that the sets $A$ and $B$ are corresponding sets to $A'$ and $B'$, respectively. There is a complete bipartite graph between any two sets $S_i, S_j$ for $i \neq j \in [r - 2]$, and a complete bipartite graph between $A$ and $S_i$ and $B$ and $S_i$ for every $i \in [r - 2]$. Finally, there are $2M$ additional edges in $A \cup A' \cup B \cup B'$ that depend on the strings $x$ and $y$ as detailed below.

Let $A = \{a_1, \ldots, a_{\sqrt{M}}\}$, $A' = \{a'_1, \ldots, a'_{\sqrt{M}}\}$ and similarly for $B$ and $B'$. We think of every coordinate $\ell \in M$ as a pair $(i, j) \in [\sqrt{M}] \times [\sqrt{M}]$ if $x_\ell \land y_\ell = 0$, then the two edges $(a_i, a'_j)$ and $(b_j, b'_i)$ are in the graph. Otherwise, if $x_\ell \land y_\ell = 1$, then the edges $(a_i, b_j)$ and $(a'_j, b'_i)$ are in the graph. The ordering on the neighbors of each $a_i \in A$ is determined as follows: the $j^{th}$ neighbor of $a_i$ is either $a'_j$ or $b_j$. The ordering on the neighbors of vertices in $A'$, $B$, and $B'$ are analogous. Observe that the number of edges in the graph is $m = \Theta((r^2 - 2)M)$, independently of $x$ and $y$. See Figure 2 for an illustration.

The case $C_r < m^{(r-2)/2}$. As before, $G$ consists of $r + 2$ sets $\{A, A', B, B', S^1, \ldots, S^{r-2}\}$ of size $\sqrt{M}$ and $n - (r + 2)\sqrt{M}$ additional isolated vertices. In this case as well the subgraphs $S^1, \ldots, S^{r-2}$ are all connected to $A$ and $B$, but here they are no longer each connected to one another, rather they are constructed as to contain $\Theta(C_r)$ $(r - 2)$-cliques. The construction of edges inside the subgraph $A \cup A' \cup B \cup B'$ is exactly as before. Finally, in this case $k = 1$, so that $f = \text{int}_1 = \text{disj}$. Hence, if $x$ and $y$ intersect, then the single edge in $A \cup B$ contributes $\Theta(C_r)$ $r$-cliques to the graph, so that $C_r(G) = \Theta(C_r)$. Otherwise, if $x$ and $y$ are disjoint, $C_r(G) = 0$.

Claim 1. Let $g : \mathcal{G}_n \to \{0, 1\}$ be the partial function defined by

$$g(G) = \begin{cases} 1 & \text{if } C_r(G) = 0 \\ 0 & \text{if } C_r(G) \geq k \cdot M^{(r-2)/2} \end{cases}$$

Then $(\mathcal{E}, g)$ is an embedding of $\text{int}_k$.

Proof. We only prove the claim for the case $C_r = k \cdot m^{(r-2)/2}$, as the proof for the second case is analogous. If $x$ and $y$ are disjoint, any $r$ vertices either (1) contains at least two vertices from $A \cup B$; (2) contain at least one vertex from $A' \cup B'$; (3) contain two vertices belonging to the same set $S_i$ for some $i \in \sqrt{M}$; (4) contain an isolated vertex. In all cases there are at least two vertices that are not connected by an edge, implying that there are no $r$-cliques in the graph.

In the case that the $x$ and $y$ have a non-empty intersection, for every $\ell \in \sqrt{M} \times \sqrt{M}$ such that $x_\ell \land y_\ell = 1$, if $\ell = (i, j)$ then the edge $(a_i, b_j)$ participates in $M^{(r-2)/2}$ $r$-cliques. Hence $C_r(G) = k \cdot M^{(r-2)/2}$.

We now argue that each neighbor, degree, and pair query and be efficiently answered by Alice and Bob.

Claim 2. For the function $\mathcal{E}$ above and each $q \in Q$, $\text{cost}_{\mathcal{E}}(q) = O(1)$.

We prove the claim of each type of query separately, and again only consider the case that $C_r \geq m^{(r-2)/2}$, as the other case is analogous.

Degree queries. The degree of each vertex $v \in V$ is independent of $x$ and $y$ in $\mathcal{E}(x, y)$. Thus, these queries can be answered by Alice and Bob without communication.
**Pair query.** Observe that only pairs of vertices in $A \cup A' \cup B \cup B'$ depend on $x$ and $y$. Thus all other pair queries can be answered without communication. If $v_i \in A$ and $w_j \in B$, then Alice and Bob can answer pair$(v_i, w_j)$ by exchanging $x_{i,j}$ and $y_{i,j}$ using 2 bits of communication. Specifically, $(v_i, w_j) \in E$ if and only if $x_{i,j} = y_{i,j} = 1$. Similarly, pair queries of the form $(v, w) \in A' \times B'$, $A \times B'$, $A' \times B$ can be answered with 2 bits of communication each.

**Neighbor query.** Again, we need only consider edges within the set $A \cup A' \cup B \cup B'$—all other edges are independent of $x$ and $y$. Such queries can be answered analogously to pair queries. For example, the $j$th neighbor of $a_i$ is either $a'_j$ or $b_j$, depending on the value of $x_{\ell} \wedge y_{\ell}$ for $\ell = (i, j)$. This predicate can be computed by Alice and Bob exchanging $x_{\ell}$ and $y_{\ell}$, at a cost of 2 bits. Other neighbor queries can be answered similarly.

Observe that any algorithm $A$ that computes a constant factor approximation to $C_r$ can be used to compute $g$: if $A$ returns a positive number, then $g(G) = 0$; otherwise, $g(G) = 1$. Thus, Theorem 3.3 together with Claims 1 and 2 above imply that the expected number of queries used by $A$ satisfies

$$q = \Omega(R(int_k)/O(1)) = \Omega(M/k) = \Omega\left(\frac{M^{r/2}}{C_r}\right) = \Omega\left(\frac{m^{r/2}}{(c \cdot r)^r \cdot C_r}\right).$$

The third equality holds by (the proof of) Claim 1, while the final inequality holds by the relation $m = \Theta(\binom{\ell}{2} M)$. Finally, the theorem follows for every possible value of $C_r$ by taking $k$ to be integer that minimizes $|C_r - k \cdot m^{(r-2)/2}|$ or $k = 1$ in case $C_r < m^{(r-2)/2}$.

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![Figure 2: An illustration of the two graph constructions of $E(x, y)$ for $r = 4$ and $C_r = \Theta(k\cdot M^{(r-2)/2})$. If $x_{(i,j)} \wedge y_{(i,j)}$ then the dashed blue edges $(a_i, b_j), (a'_i, b'_j)$ are in $E$, and otherwise, if $\neg(x_{(i,j)} \wedge y_{(i,j)})$, then the red curly edges $(a_i, a'_i), (b_j, b'_j)$ are in $E.$](image-url)
4.3 Estimating Moments of the Degree Distribution

Let $M_s = \sum_{v \in V} d^s(v)$ denote the $s^{th}$ moment of the degree distribution of a graph. The problem of estimating $M_s$ in sublinear time in the general graph model was first studied in [14] for general graphs and later in [1] in a slightly different model which allows degree queries as well as access to uniform edge samples. In [7], Eden et al. generalize the results of [14] to graphs with bounded arboricity in the general graph model. The arboricity of a graph $G$ is a measure of its sparseness, that is essentially equal to the maximum of the average degree over all subgraphs $S$ of $G$ as proved in [17, 18]. In [7], Eden et al. observe that the hardness of the moments estimation problem arises from the existence of small (hidden) dense subgraphs, and exploit the fact that in graphs with bounded arboricity no such dense subgraphs can exist. This allows them to devise an algorithm for estimating the moments that has improved query complexity when the algorithm is given an upper bound on the arboricity of the graph.

We prove the result of [7] which gives a lower bound of

$$\Omega \left( \frac{n\alpha^{1/s}}{M_s^{1/s}} + \min \left\{ \frac{n\cdot\alpha}{M_s^{1/s}}, \frac{n^s\cdot\alpha}{M_s^{1/s}}, \frac{n^{s-1}/s}{M_s^{1-1/s}} \right\} \right)$$

for the problem of estimating $M_s$ when the graph has $n$ vertices and arboricity at most $\alpha$. We note that for every graph it holds that $\alpha \leq \sqrt{m}$, and hence if an upper bound on the arboricity is not known, then one can substitute $\alpha = \sqrt{m}$ and get the lower bound of estimating $M_s$ for general graphs. For more details see [7].

We start with the following useful claim and definition.

**Claim 4.7** (Claim 12 and Footnote 4 in [7]). For any graph $G$ with arboricity $\alpha(G)$,

$\frac{M_s(G)}{n^s} \geq \alpha(G) \leq M_s(G)^{1/\alpha}$.

**Definition 4.8.** For a value $\tilde{M}_s$, we let $g_M : G_n \rightarrow \{0, 1\}$ be defined by

$$g_M(G) = \begin{cases} 0 & \text{if } M_s(G) \leq \tilde{M}_s \\ 1 & \text{if } M_s(G) \geq c \cdot \tilde{M}_s, \end{cases}$$

where $c$ is a fixed constant to be determined later.

**Theorem 4.9** (Thm. 7 in [7]). Let $G$ be a graph over $n$ vertices and with arboricity $\alpha$, and let $A$ be a constant-factor approximation algorithm for $M_s(G)$ with allowed queries $Q$, consisting of neighbor, degree, and pair queries. The expected query complexity of $A$ is $\Omega \left( \frac{n\alpha^{1/s}}{M_s^{1/s}(G)} \right)$.

**Proof.** The proof uses an embedding of disj, and is similar to the proof of Theorem 4.1. We modify the construction of the hidden subgraph so as to contribute a constant factor of $M_s$ to the resulting graph without increasing the arboricity of the graph. For a fixed $\tilde{M}_s$, let we take $H$ to be a complete bipartite graph between sets of vertices of size $(c \cdot \tilde{M}_s/\alpha)^{1/\alpha}$ and $\alpha$ ($H$ will play the role of the cliques in the proof of Theorem 4.1). Then $M_s(H) = \alpha \cdot (c \cdot \tilde{M}_s/\alpha) + (c \cdot \tilde{M}_s/\alpha)^{1/\alpha} \cdot \alpha^s \geq c\tilde{M}_s$. Further, the arboricity $H$ is $\alpha$.

The graph $G = (V, E) = E(x, y)$ is constructed from an arbitrary graph $G' = (V', E')$ on $n'$ vertices where $M_s(G') = \tilde{M}_s$. We then form $V = V' \cup W$, where $|W| = n'$. For fixed $k$, we partition

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*To see this, note that the edges of $H$ can be partitioned into $\alpha$ trees: each tree consists of the edges incident with a single vertex the $\alpha$-sized side of the bipartition.*
$W$ into subsets each of size $k$, $W_1 \cup \cdots \cup W_{n/k}$, where $k = (c \cdot \widetilde{M}_s / \alpha)^{1/s} + \alpha$. As in the proof of Theorem 4.1, each $W_i$ will be either a set of isolated vertices (if $i \notin x \cap y$) and isomorphic to $H$ (if $i \in x \cap y$). The calculation of $M_s(H)$ above implies that in the intersecting case $M_s(G) \geq (1+c)\widetilde{M}_s$, while $M_s(G) = \widetilde{M}_s$ in the disjoint case. The remainder of the proof of Theorem 4.9 is analogous to Theorem 4.1.

We next give an alternate proof for Theorem 8 in [7]. The function $\mathcal{E}$ we construct yields graphs that are almost identical to the constructions in the original proof. We note that some details in the original proof are omitted in [7] as the argument in [7] is based on the proofs of the second and third items of Theorem 5 in [14].

**Theorem 4.10 (Theorem 8 in [7]).** Let $G$ be a graph over $n$ vertices and with arboricity $\alpha$, and let $\mathcal{A}$ be a constant-factor approximation algorithm for $M_s(G)$ with allowed queries $Q$, consisting of neighbor, degree, and pair queries. The expected query complexity of $\mathcal{A}$ is

$$\Omega \left( \min \left\{ \frac{\alpha(G) \cdot n}{M_s(G)^{1/s}}, \frac{n^{1-1/s}}{\alpha(G) \cdot n^s}, \frac{\alpha(G) \cdot n^s}{M_s(G)}, \frac{n^{s-1/s}}{M_s(G)^{1-1/s}} \right\} \right).$$

**Proof.** Once again, we reduce from the problem of set disjointness. Let $f = \text{disj}$ for $N \in \left\{ \frac{\alpha(G) \cdot n}{M_s(G)^{1/s}}, \frac{n^{1-1/s}}{\alpha(G) \cdot n^s}, \frac{\alpha(G) \cdot n^s}{M_s(G)}, \frac{n^{s-1/s}}{M_s(G)^{1-1/s}} \right\}$. In order to prove the different lower bounds we describe a function $\mathcal{E} : \{0, 1\}^N \times \{0, 1\}^N \rightarrow \mathcal{G}_n$ that depends on the relations between $n, \alpha$ and $M_s$, and so that $n' = \Theta(n), \alpha(G) = \alpha$ and the value of $M_s(G)$ is some constant factor of $\widetilde{M}_s$ depending on whether or not $x$ and $y$ are disjoint.

As in [7], we divide the proof into two cases depending on the relation between $\widetilde{M}_s$ and $n$. In each case consider two sub-cases that depend on the relation between $\alpha$ and $(\widetilde{M}_s / n)^{1/s}$.

**The case $M_s^{1/s} \leq n/c$ for some constant $c > 4$** We start with some intuition behind the construction of the graphs. Let $G$ be a graph $G = A \cup B \cup R \cup W$ as follows. $A \cup B$ is a $d$-regular bipartite subgraph over $2n$ vertices (where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$), $R$ is a clique over $\alpha$ vertices and $W$ consists of $d/\ell$ subgraphs $W_1, \ldots, W_{d/\ell}$ of $c$ isolated vertices each. In the case where $i \in x \cap y$, we modify the graph above such that $W_i$ consists of extremely high degree vertices so that its contribution to the $s$th moment amounts to a constant factor of $M_s$.

Specifically, consider the following embedding $\mathcal{E} : \{0, 1\}^{d/\ell} \times \{0, 1\}^{d/\ell} \rightarrow \mathcal{G}_n$. Given $x, y \in \{0, 1\}^{d/\ell}$, $G$ is a graph over $n' = 2n + c \cdot (d/\ell) + \alpha$ vertices, where the exact values of $d$ and $\ell$ depend on the sub-case at hand, and will be set later. We show that in all cases $n' = \Theta(n)$ and $\alpha(G) = \alpha$. Depending on whether or not $x$ and $y$ intersect, $M_s(G) \leq \widetilde{M}_s$ or $M_s(G) \approx c \cdot \widetilde{M}_s$.

For every vertex in $A \cup B$, we think of its set of neighbors as divided into $d/\ell$ blocks of size $\ell$ each. If the sets are disjoint, then all the neighbors in every block are connected to vertices on the corresponding side of the bipartite subgraph. Thus, for $a_i \in A$ a vertex’s neighbors are $b_{i, b_{i+1}(\text{mod} \ n)}, \ldots, b_{i+d(\text{mod} \ n)}$ (and analogously for $b_i \in B$). Otherwise, if $x$ and $y$ intersect on the $j$th index, then the $j$th block of neighbors of $a_i$ is in $W_j$ (rather than in $B$). In this case, the degree of every vertex in $W_j$ is $2n \cdot \ell/c$. The identity of the neighbors of $a_i \in A \cup B$ in $W_j$ depend on the value of $i$, where the first (by order of indices) $2\ell \cdot n/c$ vertices in $A \cup B$ are connected to the first $\ell$ vertices in $W_j$, and so on. See Figures 3 and 4 for an illustration.

In the first sub-case $\alpha < (\widetilde{M}_s / n)^{1/s}$ we set $\ell = [c \cdot \widetilde{M}_s^{1/s} / 2n]$ and $d = \alpha$. If $x$ and $y$ are disjoint then $M_s(G) = 2n \cdot \alpha^s + \alpha^{s+1} \leq 3\widetilde{M}_s$, where the last inequality is due to Claim 4.7, and $\alpha(G) = \alpha$ due to the set $R$. Otherwise, if $x$ and $y$ intersect on index $j$, then the vertices of $W_j$ all have degree...
Figure 3: The construction of the graph \( G = \mathcal{E}(x, y) \). The subgraph \( A \cup B \) is either a bipartite graph. The subgraph \( R \) is a clique of size \( \alpha \) and additionally there are \( d/\ell \) subgraphs \( W_1, \ldots, W_{d/\ell} \) of size \( c \) each, for some small constant \( c \). In the case that \( x \) and \( y \) are disjoint \( A \cup B \) is \( d \)-regular bipartite subgraph and all the sets \( W_i \) are independent sets. In case \( x \) and \( y \) intersect on the \( j \)th coordinate, \( A \cup B \) is a \((d-\ell)\)-regular bipartite subgraph, all vertices in \( A \cup B \) have \( \ell = \lceil c \cdot \tilde{M}_s^{1/\ell} / 2n \rceil \) neighbors in \( W_j \), and all vertices in \( W_j \) have \( \tilde{M}_s^{1/\ell} \) in \( A \cup B \). All other subgraphs \( W_i \) for \( i \neq j \) are independent sets. Hence, in the former case \( M_s(G) \leq 3 \tilde{M}_s \) and in the latter, \( M_s(G) \approx c \tilde{M}_s \).

\[
2n \cdot \ell/c = \tilde{M}_s^{1/\ell} \quad \text{and therefore} \quad M_s(G) = c \cdot M_s^{1/\ell} + \alpha^{s+1} \approx c \cdot \tilde{M}_s \quad \text{(by Claim 4.7)} \quad \text{and} \quad \alpha(G) = \alpha \quad \text{(here too it can be verified that} \ R \text{the subgraph that maximizes the average degree)}.
\]

In the second sub-case, \( \alpha \geq (\tilde{M}_s/n)^{1/\ell} \), we set \( \ell = \lceil c \cdot \tilde{M}_s^{1/\ell} / 2n \rceil \) as before, but now we get \( d = \lceil (\tilde{M}_s/n)^{1/\ell} \rceil \). Hence, in case \( x \) and \( y \) intersect we get \( M_s(G) = c \cdot M_s + 2n \cdot (c \cdot \tilde{M}_s^{1/\ell} / 2n)^s + \alpha^{s+1} \approx c \cdot \tilde{M}_s \), and otherwise \( M_s(G) = 2n \cdot (c \cdot \tilde{M}_s^{1/\ell} / 2n)^s + \alpha^{s+1} \leq 3 \cdot \tilde{M}_s \). Still in both cases \( \alpha(G) = \alpha \).

Therefore, in both sub-cases, \( g(\mathcal{E}(x, y)) = \text{disj}(x, y) \), and \( \mathcal{E}, g \) is an embedding of \( f \).

It remains to prove that Alice and Bob can answer queries of \( A \) according to \( \mathcal{E}(x, y) \) efficiently.

- For a degree query \( q = d(v) \), if \( v \in A \cup B \) then Alice and Bob answer \( d(v) = d \), and if \( v \) is in \( R \), then Alice and Bob answer \( d(v) = \alpha \). If \( v \in W_j \) for some \( j \in d/\ell \) then Alice and Bob communicate to each other whether or not \( j \) is in their set. If \( x \land y \) then Alice and Bob announce that the sets intersect. Otherwise, if \( j \not\in x \land y \), they answer \( d(v) = 0 \).

- For a neighbor query \( q = \text{nbr}_r(v) \), if \( v \in t_i \in R \) then Alice and Bob answer \( \text{nbr}_r(v) = t_{i+r(\text{mod} \ n)} \). If \( v = a_i \in A \) then Alice and Bob communicate to each other whether the \( j \)th index is in their set for \( j = [r/\ell] \) (since the \( r \)th neighbor of \( a_i \) belongs to the \( ([r/\ell]) \)th block of \( a_i \)’s neighbors). If \( x \land y \) then Alice and Bob determine that \( \text{nbr}_r(v) \) is the \( k \)th vertex in \( W_j \) where \( k = j \mod [r/\ell] \). Otherwise, if \( x \land y = 0 \), they answer \( b_{i+r(\text{mod} \ n)} \). The case that \( v = b_i \in B \) is analogous. Finally, if \( v \in W_j \) then Alice and Bob communicate to each other whether the \( j \)th index is in their set and as before, if \( j \in x \land y \) then they announce \( \text{nbr}_r(v) \) is the appropriate \( a_i \in A \), and otherwise they answer \( \text{nbr}_r(v) = \emptyset \).
that the loss of generality that expected query complexity of $A$

Figure 4: An illustration of the set of neighbors of the vertex $a_i \in A$ for the graph $\mathcal{E}(x, y)$. $a_i$

For a pair query $q = (v_i, v_r)$ if the two vertices are in $R$ then Alice and Bob answer $(v_i, v_r) \in E$

and if only one of the vertices is in $R$ then they answer $(v_i, v_r) \notin E$. If the two vertices are in
different sides of $A \cup B$ then assume without loss of generality that $v_i \in A$, $v_r \in B$ and $i < r$. If $r - i \notin [d]$ then Alice responds pair$(v_i, v_r) = 1$. Otherwise, Alice and Bob exchange $x_j$

and $y_j$ for $j = \lceil (i - r)/\ell \rceil$ (since $v_r$ in the $j^{th}$ block of $v_i$’s neighbors). If $x_j \land y_j = 1$ then they

find $(v_i, v_r) \notin E$, and otherwise they answer $(v_i, v_r) \in E$.

It holds by the above, that every query made by the algorithm can be answered by $O(1)$
communication, and since $A$ can be used to compute $q_M$, it follows from Theorem 3.3 that the
expected query complexity of $A$ is $\Omega(d/\ell) = \Omega\left(\min\left\{\frac{n^{1/s}}{M_s^{1/s}}, n^{1-1/s}\right\}\right)$.

The case $M_s^{1/s} > n$ The construction of the graphs in this case is very similar to the previous
case, except that now the sizes of the sets $C_1, \ldots, C_{d/\ell}$ is increased to $k = \lceil c \cdot M_s/(2n)^s \rceil$ for a
small constant $c$, and their potential contribution to the degree of the vertices of $A \cup B$, $\ell$ is also
increased to $k$. Hence, if $x$ and $y$ intersect on the $j^{th}$ index, then the degree of the vertices in $W_j$

is $2n$ and the subgraph $W_j \cup (A \cup B)$ is a complete bipartite graph.

As before, in the sub-case $\alpha < (\widetilde{M}/n)^{1/s}$, we set $d = \alpha$ and in the sub-case $\alpha \geq (\widetilde{M}/n)^{1/s}$
we set $d = \lceil (\widetilde{M}/n)^{1/s} \rceil$. As noted in the proof of Theorem 8 in [7], we may assume without
loss of generality that $\widetilde{M}_s \leq n^s \cdot \alpha/c'$ for a sufficiently large constant $c'$ since otherwise the lower
bound $\Omega\left(n^s \cdot \alpha/(\widetilde{M}_s)\right)$ becomes trivial. Therefore, indeed $d > \ell$ as required by our construction.

Similarly we can assume without loss of generality that $\widetilde{M}_s \leq n^s \cdot \alpha/c'$ or else the lower bound
$\Omega\left(n^s \cdot \alpha/(\widetilde{M}_s)\right)$ becomes trivial. By the above settings, in the first sub-case, if $x$ and $y$ are disjoint
then $\alpha(G) = G$ and $M_s(G) = 2n \cdot \alpha^s + \alpha^{s+1} \leq 3\tilde{M}$, where the last inequality is by Claim 4.7. Otherwise, if $x$ and $y$ intersect on the $j$th index, then the vertices of $W_j$ all have degree $2n \cdot \ell/k$ and therefore $M_s(G) = k \cdot (2n \cdot \ell/k)^s + 2n \cdot \alpha^s + \alpha^{s+1} = 2c \cdot \tilde{M} + 2n \cdot \alpha^s + \alpha^{s+1} \approx c\tilde{M}$. Also by the choice of parameters above, the average degree in the subgraph $W_j \cup A \cup B$ is now $k$, but since we assumed $\tilde{M} \leq n^s \cdot \alpha/c'$, it still holds that $R$ is the subgraph that maximizes the average and that $\alpha(G) = \alpha$.

In the second sub-case $\alpha > (M_s/n)^{1/s}$, we have that if $x$ and $y$ are disjoint, then $M_s(G) = 2n \cdot (M/n) + \alpha^{s+1} \leq 3\tilde{M}$, and if $x$ and $y$ do intersect then $M_s(G) = k \cdot (2n \cdot \ell/s)^s + 2n \cdot (M/s) + \alpha^{s+1} \approx \tilde{M}$. Here too, regardless of $x$ and $y$, $\alpha(G) = \alpha$.

As in the former case, it can be easily verified that Alice and Bob can answer any query made by $A$ from $Q$ using $O(1)$ communication, and it follows that the expected running time of $A$ is $\Omega(d/\ell) = \Omega(d/k) = \Omega\left(\min\left\{\frac{\alpha \cdot n^s}{M_s}, \frac{n^{s-1/s}}{M_{s-1}}\right\}\right)$ as desired.

\begin{flushright}
\Box
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