Recent Developments in Distribution Theory: A Brief Survey and Some New Generalized Classes of distributions

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Recent Developments in Distribution Theory: A Brief Survey and Some New Generalized Classes of distributions

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Abstract
The generalization of the classical distributions is an old practice and has been considered as precious as many other practical problems in statistics. These generalizations started with the introduction of the additional location, scale or shape parameters. In the last couple of years, this branch of statistics has received a great deal of attention and quite a few new generalized classes of distributions have been introduced. We present a brief survey of this branch and introduce several new families as well.

Keyword: Generalized classes of distributions; Exponentiated family; Marshall Olkin family; Transmuted family; Kumaraswamy family; Alpha power transformation; Zubair-G family; Construction of new families.

1. Introduction
The recent development in distribution theory stresses on problem solving faced by the researchers and proposes a variety of models so that lifetime data sets can be better assessed and investigated in different applied areas. In other words, there is a need to introduce useful models for the better exploration of the real phenomenon of nature. Nowadays, the trends and practices in proposing new probability models totally differ in comparison to the models suggested before 1997. One main objective for proposing, extending or generalizing (models or their classes) is to explain how the lifetime phenomenon arises in fields like physics, computer science, insurance, public health, medical, engineering, biology, industry, communications, life-testing and many others. The well-known and fundamental distributions such as exponential, Rayleigh, Weibull and gamma are very limited in their characteristics and are unable to show wide flexibility. For example, the exponential distribution is capable of modeling with constant hazard function, whereas, the Rayleigh distribution has increasing hazard function only. However, the Weibull is much flexible and capable of modeling with increasing, decreasing or constant hazard function.
Unfortunately, the Weibull model is not capable of modeling with non-monotonic (such as unimodal, modified unimodal or bathtub shaped) failure rate function. The gamma distribution does not have a closed form of cumulative distribution function (cdf) which causes difficulties in describing its mathematical properties. For complex phenomenon in human mortality studies, reliability studies, lifetime testing, engineering modeling, electronic sciences and biological surveys, the failure rate behavior can be bathtub, upside-down bathtub and other shaped but not usually monotone increasing or decreasing. Thus, in order to cope with both monotonic and non-monotonic failure rate shapes, researchers have proposed several generalized classes of distributions which are very flexible to study needful properties of the model and its fitness. In the last two decades, several generalization approaches were adopted and practiced, which have received increased attention.

The objectives of the present study are three-fold: Firstly, we present an up-to-date account of the extended classes of distributions for the readers of modern distribution theory. Secondly, this survey will motivate the researchers to fill up the gap and to furnish their work in remaining applied areas. Thirdly, we propose some new classes of distributions which might be helpful as a tutorial to the beginners of the generalized modeling art.

The rest of the article is organized as follows. In Section 2, some extended classes of distributions are reviewed. In section 3, we present some new families. Section 4 presents certain characterizations of the distributions listed in Section 3. Finally, concluding remarks are provided in Section 5.

2. Review of the existing family of distributions

In this section, we present up-to-date review of the extended families of distributions.

2.1. The exponentiated family of distributions

Mudholkar and Srivastava (1993) proposed another method of introducing an extra parameter to a two-parameter Weibull distribution. The cumulative distribution function of the Mudholkar and Srivastava (1993)’s proposed exponentiated family has the following form

\[ G(x; \alpha, \xi) = F(x; \xi)^\alpha, \quad \alpha, \xi > 0, \quad x \in \mathbb{R}, \]  

(1)

where \( \alpha > 0 \) is an extra shape parameter. Due to the presence of an extra shape parameter, the proposed exponentiated distributions are more flexible than the traditional models. Using (1), a number of modifications of the existing distributions have been proposed in the literature. A brief list of these modifications is presented in Table 1:
Table 1: Contributed work on exponentiated distributions

| S. No. | Year | Distribution                          | Author(s)                  |
|--------|------|---------------------------------------|----------------------------|
| 1      | 2001 | Exponentiated Exponential            | Gupta and Kundu (2001)     |
| 2      | 2005 | Exponentiated beta                     | Nadarajah (2005)           |
| 3      | 2005 | Exponentiated Pareto                   | Nadarajah (2005)           |
| 4      | 2006 | exponentiated lognormal                | Shirke and Kakde (2006)    |
| 5      | 2006 | exponentiated Fréchet                  | Nadarajah and Kotz (2006)  |
| 6      | 2006 | exponentiated Gumbel                   | Nadarajah (2006)           |
| 7      | 2007 | exponentiated Gamma                    | Nadarajah and Gupta (2007) |
| 8      | 2011 | exponentiated generalized gamma        | Cordeiro et al. (2011)     |
| 9      | 2013 | exponentiated Lomax Poisson            | Ramos et al. (2013)        |
| 10     | 2013 | exponentiated modified Weibull extension | Sarhan and Apaloo (2013)   |
| 11     | 2013 | exponentiated generalized class        | Cordeiro et al. (2013)     |
| 12     | 2016 | exponentiated Weibull-Pareto           | Afify et al. (2016)        |
| 13     | 2013 | exponentiated Kumaraswamy              | Lemonte et al. (2013)      |
| 14     | 2014 | Exponentiated Kumaraswamy-Dagum        | Huang and Oluyede (2014)   |
| 15     | 2014 | Exponentiated Half-Logistic family     | Cordeiro et al. (2014)     |
| 16     | 2015 | Exponentiated Power Lindley            | Ashour and Eltehiwy (2015) |
| 17     | 2015 | Exponentiated power Lindley            | Ashour and Eltehiwy (2015) |
| 18     | 2015 | exponentiated generalized modified Weibull | Aryal and Elbatal (2015)   |
| 19     | 2015 | Exponentiated Burr XII Poisson         | da Silva et al. (2015)     |
| 20     | 2015 | Exponentiated Generalized Gumbel       | Andrade et al. (2015)      |
| 21     | 2015 | exponentiated transmuted generalized Rayleigh | Nofal et al. (2015)       |
| 22     | 2015 | exponentiated flexible Weibull extension | El-Gohary et al. (2015)   |
| 23     | 2016 | Exponentiated Gumbel Type-2            | Okorie et al. (2016)       |
| 24     | 2016 | Exponentiated Gompertz Generated Family | Cordeiro et al. (2016)     |
| 25     | 2017 | Exponentiated Generalized Weibull Gompertz | El-Bassiouny et al. (2017)|
| 26     | 2017 | Exponentiated power Lindley Poisson    | Pararai et al. (2017)      |
| 27     | 2017 | Exponentiated inverse flexible Weibull | Morshedy and El-Bassiouny  |
| 28     | 2017 | Exponentiated Lomax Geometric          | Hassan and Abd-Allah (2017)|
| 29     | 2018 | Exponentiated Inverse Power Lindley    | Jan et al. (2018)          |
| 30     | 2018 | Exponentiated Weibull-Lomax            | Hassan and Abd-Allah (2018)|

2.2. The Marshall-Olkin family of distributions

Marshall and Olkin (1997) pioneered a simple method of adding a single parameter to a family of distributions and several authors used their method to extend well-known distributions in the last few years. If \( F(x; \xi) \) and \( F(x; \xi) \) denote the survival function (sf) and cumulative distribution function of a parent distribution depending on the vector parameter \( \xi \), then the sf of Marshall and Olkin (MO) family is defined by

\[
G(x; \sigma, \xi) = \frac{\sigma F(x; \xi)}{1 - \sigma F(x; \xi)}, \quad \xi, \sigma > 0, \ x \in \mathbb{R}.
\]
where, $\tilde{\sigma} = 1 - \sigma$. Clearly, for $\sigma = 1$, we obtain the baseline distribution, i.e., $\tilde{F}(x; \xi) = \tilde{G}(x; \xi)$.

Using (2), the extended versions of the existing distributions have been proposed. Based on the MO family, a detail review of the existing distributions is provided in Table 2:

### Table 2: Contributed work on Marshall-Olkin distributions

| S. No. | Year | Distribution                          | Author(s)               |
|--------|------|---------------------------------------|-------------------------|
| 1      | 2003 | Marshall Olkin Pareto                | Alice and Jose (2003)   |
| 2      | 2005 | Marshall Olkin extended Pareto       | Ghitany (2005)          |
| 3      | 2005 | Marshall Olkin semi Weibull          | Alice and Jose (2005)   |
| 4      | 2005 | Marshall Olkin Logistic              | Alice and Jose (2005)   |
| 5      | 2005 | Marshall Olkin extended Weibull      | Ghitany et al. (2005)   |
| 6      | 2007 | Marshall-Olkin gamma                 | Ristic et al. (2007)    |
| 7      | 2007 | Marshall-Olkin extended Lomax        | Ghitany (2007)          |
| 8      | 2009 | Marshall-Olkin beta                  | Jose et al. (2009)      |
| 9      | 2011 | Marshall-Olkin extended exponential  | Rao et al. (2011)       |
| 10     | 2010 | Marshall Olkin q-Weibull             | Jose et al. (2010)      |
| 11     | 2011 | Marshall-Olkin extended uniform      | Jose and Krishnu (2011) |
| 12     | 2013 | Marshall-Olkin Extended Log-Logistic | Gui (2013)              |
| 13     | 2013 | Marshall-Olkin Extended Zipf         | Casany and Casellas (2013) |
| 14     | 2013 | Marshall-Olkin power log-normal      | Gui (2013)              |
| 15     | 2013 | Marshall-Olkin extended Weibull      | Cordeiro and Lemonte (2013) |
| 16     | 2014 | Marshall-Olkin extended Weibull family | Santos-Neto et al. (2014) |
| 17     | 2014 | Marshall Olkin extended Burr type XII | Al-Saiari et al. (2014) |
| 18     | 2014 | Marshall-Olkin discrete uniform      | Sandhya and Prasanth (2014) |
| 19     | 2015 | Marshall-Olkin generalized exponential | Ristic and Kundu (2015) |
| 20     | 2015 | Marshall–Olkin exponential Weibull   | Pogány et al. (2015)    |
| 21     | 2016 | Marshall-Olkin Extended Burr Type III | Kumar (2016) |
| 22     | 2016 | Marshall-Olkin Flexible Weibull Extension | Mustafa et al. (2016) |
| 23     | 2016 | Marshall–Olkin gamma–Weibull         | Saboor and Pogány (2016) |
| 24     | 2016 | Marshall-Olkin Additive Weibull      | Afify et al. (2016)     |
| 25     | 2017 | Marshall-Olkin Extended Generalized Gompertz | Benkhelifa (2016) |
| 26     | 2017 | Marshall-Olkin Log-Logistic Extended Weibull | Lepetu et al. (2017) |
| 27     | 2017 | Marshall-Olkin Half Logistic         | Yeğen and Özel (2018)   |
| 28     | 2017 | Marshall-Olkin generalized Erlang-truncated | Yeğen and Özel (2018) |
| 29     | 2017 | Marshall-Olkin Burr X family         | Jamal et al. (2017)     |
| 30     | 2018 | Marshall-Olkin Extended Inverse Power Lindley | Hibatullah (2018) |
| 31     | 2018 | Marshall-Olkin Extended Inverse Weibull | Pakungwati et al. (2018) |
| 32     | 2018 | Marshall-Olkin Half Logistic         | Yeğen and Özel (2018)   |
| 33     | 2018 | Marshall-Olkin generalized-G family  | Yousof et al. (2018)    |

2.3. **Transmuted family of distributions**
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Shaw and Buckley (2009) pioneered another prominent method of adding a parameter into a family of distributions and several authors used their method to extend well-known distributions in the last couple of years. If \( F(x; \xi) \) denotes the cdf of a parent distribution depending on the vector parameter \( \xi \), then the cdf of the transmuted family is given by

\[
G(x; \lambda, \xi) = (1 + \lambda) F(x; \xi) - \lambda F(x; \xi)^2, \quad \xi > 0, \ [\lambda] \leq 1, \ x \in \mathbb{R}.
\]  

From (3), for \( \lambda = 0 \), we obtain the baseline distribution, i.e., \( F(x; \xi) = G(x; \xi) \). Using (3), the extended versions of the existing distributions have been proposed, for detail we refer to Tahir and Cordeiro (2016).

2.4. Cubic Transmuted family of distributions

Granzotto et al. (2017) proposed a new method of generating distributions called Cubic Transmutation method. Let \( X_1, X_2 \) and \( X_3 \) be independent and identically random variables with distribution \( F(x; \xi) \). Then, the ranking cubic transmutation map is given by

\[
G(x; \lambda_1, \lambda_2, \xi) = \lambda_1 F(x; \xi) + (\lambda_2 - \lambda_1) F(x; \xi)^2 + (1 - \lambda_2) F(x; \xi)^3, \quad \xi > 0, \ x \in \mathbb{R},
\]  

with \( \lambda_1 \in [0, 1] \) and \( \lambda_2 \in [-1, 1] \).

Recently, Aslam et al. (2018) proposed Cubic transmuted-G family by using the T-X idea of Alzaatreh (2013).

2.5. A General Transmuted family of distributions

Recently, Rahman et al. (2018) proposed a general transmuted family of distributions, is defined by

\[
G(x; \lambda_i, \xi) = F(x; \xi) + (1 - F(x; \xi)) \sum_{i=1}^{k} \lambda_i F(x; \xi)^i, \quad \xi > 0, \ x \in \mathbb{R},
\]  

with \( \lambda_i \in [-1, 1] \) for \( i = 1; 2; \cdots; k \) and \( -k \leq \sum_{i=1}^{k} \lambda_i \leq 1 \). The general transmuted family reduces to the base distribution for \( \lambda_i = 0 \) for \( i = 1; 2; \cdots; k \).

2.6. Kumaraswamy-G family of distributions

Kumaraswamy (1980) (for short Ku) proposed a two-parameter distribution on \((0,1)\), called Kumaraswamy distribution, is defined by

\[
G(x; \alpha, \beta, \xi) = 1 - (1 - x^\alpha)^\beta, \quad \xi > 0, \ x \in (0,1),
\]  

where \( \alpha > 0 \) and \( \beta > 0 \) are shape parameters. The density function corresponding to (6) is

\[
g(x; \alpha, \beta, \xi) = \alpha \beta x^{\alpha-1} (1-x^\alpha)^{\beta-1}, \quad x \in (0,1).
\]  

The Ku density has the same basic shape properties as to the beta distribution: \( \alpha > 1 \) and \( \beta > 1 \) (unimodal); \( \alpha < 1 \) and \( \beta < 1 \) (bathtub); \( \alpha > 1 \) and \( \beta \leq 1 \) (increasing); \( \alpha \leq 1 \) and \( \beta > 1 \) (decreasing) and \( \alpha = \beta = 1 \) (constant). Using (7), for an arbitrary baseline distribution function \( F(x; \xi) \), Cordeiro and Castro proposed the cdf of the Kumaraswamy-G (Ku-G) family.
Using (8), a number of modifications of the existing distributions have been proposed in the literature. A brief list of these modifications is presented in Table 3:

Table 3: Contributed work on Ku-G distributions.

| S. No. | Year | Distribution                      | Author(s)                          |
|--------|------|-----------------------------------|------------------------------------|
| 1      | 2010 | Kumaraswamy Weibull              | Cordeiro et al. (2010)             |
| 2      | 2011 | Kumaraswamy Generalized Gamma     | Pascoa et al. (2011)               |
| 3      | 2012 | Kumaraswamy-Log-Logistic          | Santana et al. (2012)              |
| 4      | 2012 | Kumaraswamy Pareto                | Pereira et al. (2012)              |
| 5      | 2012 | Kumaraswamy Gumbel                | Cordeiro et al. (2012)             |
| 6      | 2012 | Kumaraswamy Birnbaum-Saunders     | Saulo et al. (2012)                |
| 7      | 2013 | Kumaraswamy Generalized Logistic  | Shams (2013)                       |
| 8      | 2013 | Kumaraswamy Generalized Exponentiated Pareto | Shams (2013) |
| 9      | 2013 | Kumaraswamy generalized linear failure rate | Elbatal (2013) |
| 10     | 2013 | Kumaraswamy Pareto                | Elbatal (2013)                     |
| 11     | 2013 | Kumaraswamy Burr XII              | Paranaba et al. (2013)             |
| 12     | 2013 | Kumaraswamy Generalized Pareto    | Nadaraja and Eljabri (2013)        |
| 13     | 2014 | Kumaraswamy Inverse Rayleigh      | Roges et al. (2014)                |
| 14     | 2014 | Kumaraswamy-geometric distribution | Akinsete et al. (2014)             |
| 15     | 2014 | Kumaraswamy modified Weibull      | Cordeiro et al. (2014)             |
| 16     | 2014 | Kumaraswamy Lindley               | Merovci and Sharma (2014)          |
| 17     | 2014 | Kumaraswamy Inverse Weibull       | Shahbaz et al. (2014)              |
| 18     | 2014 | Kumaraswamy generalized Rayleigh  | Gomes et al. (2014)                |
| 19     | 2014 | Kumaraswamy exponentiated Lomax   | Elbatal and Kareem (2014)          |
| 20     | 2015 | Kumaraswamy Modified Inverse Weibull | Pararai et al. (2015)            |
| 21     | 2015 | Kumaraswamy Lindley-Poisson       | Alizadeh et al. (2015)             |
| 22     | 2015 | Kumaraswamy odd log-logistic      | Alizadeh et al. (2015)             |
| 23     | 2015 | Kumaraswamy Modified Inverse Weibull | Aryal and Elbatal (2015)         |
| 24     | 2016 | Kumaraswamy Gompertz Makeham      | Chukwu and Ogunde (2016)           |
| 25     | 2016 | Kumaraswamy Laplace               | Nassar (2016)                      |
| 26     | 2016 | Kumaraswamy Exponentiated Inverse Rayleigh | Haq (2016) |}

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\[
G(x; \alpha, \beta, \xi) = 1 - \left(1 - F(x; \xi)^\alpha\right)^\beta, \quad \alpha, \beta, \xi > 0, \quad x \in \mathbb{R}.
\]
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S. No. Year Distribution Author(s)
37 2017 Kumaraswamy transmuted exponentiated modified Al-Babtain et al. (2017)
38 2017 Kumaraswamy Half-Logistic Usman et al. (2017)
39 2018 Kumaraswamy Exponentiated U-Quadratic Muhammad et al. (2018)
40 2018 Kumaraswamy exponentiated Chen Khan et al. (2018)
41 2018 Kumaraswamy odd Burr-G family Nasir et al. (2018)
42 2018 Kumaraswamy Marshall-Okin Log-Logistic Cakmakyapan et al. (2018)

2.7. T-X Family approach
Eugene et al. (2002) introduced the beta generated method that uses the beta distribution with parameters $a$ and $b$ as the generator to develop the beta generated distributions. The distribution of a beta generated random variable $X$ is defined as

$$G(x; a, b, \xi) = \int_0^{F(x; \xi)} r(t) dt, \quad a, b, \xi > 0,$$  \hspace{1cm} (9)

where $r(t)$ is the pdf of a beta random variable and $F(x; \xi)$ is the cdf of any random variable $X$. Alzaatreh et al. (2013) proposed another method of generating families of continuous distributions called T-X family by replacing the beta pdf with a pdf, $b(t)$, of a continuous random variable and applying a function $W[F(x; \xi)]$ that satisfies some certain conditions.

Using the T-X idea, several new classes of distributions have been introduced in the literature. Table 4 provides some $W[F(x; \xi)]$ functions for some members of the T-X family.

Table 4. Some members of the T-X family

| $W[F(x; \xi)]$ | Range of $T$ | Members of T-X family |
|----------------|-------------|-----------------------|
| $F(x; \xi)$    | [0, 1]      | Beta-G (Eugene et al., 2002), Mc-G (Alexander et al., 2012) |
| $-\log[F(x; \xi)]$ | $(0, \infty)$ | Gamma-G Type-2 (Ristić and Balakrishnan, 2012) |
| $-\log[1-F(x; \xi)]$ | $(0, \infty)$ | Gamma-G Type-1 (Zografos and Balakrishnan, 2009) |
| $F(x; \xi) / 1-F(x; \xi)$ | $(0, \infty)$ | Gamma-G Type-3 (Torabi and Montazeri, 2012) |
| $-\log[1-F^a(x; \xi)]$ | $(0, \infty)$ | Exponentiated T-X (Alzaghal et al., 2013) |
| $\log\left[\frac{F(x; \xi)}{1-F(x; \xi)}\right]$ | $(-\infty, \infty)$ | Logistic-G (Torabi and Montazeri, 2014) |
| $\log[-\log[1-F(x; \xi)]]$ | $(-\infty, \infty)$ | The Logistic-X Family (Tahir et al., 2015) |
| $-\log[1-F(x; \xi)] / 1-F(x; \xi)$ | $(0, \infty)$ | New Weibull-X Family (Ahmad et al., 2018) |
2.8. **Alpha Power Transformation**

Mahdavi and Kundu (2017) proposed a new method for introducing statistical distributions via the cdf given by

\[
G(x;\alpha,\xi) = \frac{\alpha^{F(x;\xi)} - 1}{\alpha - 1}, \quad \alpha,\xi > 0, \; \alpha \neq 1, \; x \in \mathbb{R}.
\] (10)

Using (10), some new extensions of the parent distributions have been introduced. A list of distributions based on alpha power transformation is provided in Table 5.

Table 5: Contributed work on alpha power transformation.

| S. No. | Year | Distribution | Author(s) |
|-------|------|--------------|-----------|
| 1     | 2017 | Alpha power exponential Weibull | Rahman and El-Bassiouny (2017) |
| 2     | 2018 | Alpha power inverted exponential | Unal et al. (2018) |
| 3     | 2018 | Alpha power transformed Lindley | Dey et al. (2018) |
| 4     | 2018 | Alpha power inverse Weibull | Ramadan and Walaa (2018) |
| 5     | 2019 | Alpha power transformed inverse Lindley | Dey et al. (2019) |
| 6     | 2019 | Alpha power transformed Frechet | Nasiru et al. (2019) |
| 7     | 2019 | Alpha power transformed power Lindley | Hassan et al. (2019) |

2.9. **The Zubair-G family**

Recently, Ahmad (2018) proposed another method for generating new distributions via the cdf given by

\[
G(x;\alpha,\xi) = \frac{e^{\alpha F(x;\xi)} - 1}{e^\alpha - 1}, \quad \alpha,\xi > 0, \; x \in \mathbb{R}.
\] (11)

Using (11), some new modified versions of the parent distributions have been proposed. A list of distributions based on the Zubair-G method is provided in Table 6.

Table 6: Contributed work on the Zubair-G family.

| S. No. | Year | Distribution | Author(s) |
|-------|------|--------------|-----------|
| 1     | 2019 | \(\alpha\)-Zubair-G family | Kyurkchiev et al. (2019) |
| 2     | 2019 | Zubair-G distribution with baseline Lomax | Pavlov et al. (2019) |
| 3     | 2019 | Zubair-G distribution with baseline Ghosh–Bourguignon’s extended Burr XII | Rahneva et al. (2019) |
3. New Proposed Families

As we discussed in Section 2, the distribution theory has received serious consideration in the literature. We carry further this branch of statistics and propose some new methods for generating new distributions. We can define a general form of cdf via the expression

\[ G(x; \xi) = \frac{e^{\alpha R(x)} - 1}{e^{\alpha} - 1}, \quad \xi > 0, \ x \in \mathbb{R}, \]  

(12)

where, \( R(x; \xi) \) is a baseline cdf. We can also take \( R(x; \xi) \) as any function of cdf, which obey the properties of cdf, or we may combine two or more distribution functions to propose a new class of distributions. For the sake of simplicity we omit the dependency on the vector parameter and we simply write

\[ G(x) = \frac{e^{\alpha R(x)} - 1}{e^{\alpha} - 1}, \quad \alpha > 0, \ x \in \mathbb{R}. \]  

(13)

Taking \( R(x) = F(x)^\alpha \) in (13), we arrive at the Zubair-G distribution.

3.1. The extended Zubair-G family

In this sub-section, we define a new family of distributions, called the extended Zubair-G (EZ-G) family via taking \( R(x) = \alpha F(x)^\gamma + \beta F(x) \) in (12). The cdf of the EZ-G family is given by

\[ G(x) = \frac{e^{\alpha R(x)^\gamma + \beta F(x)} - 1}{e^{\alpha + \beta} - 1}, \quad \alpha, \beta > 0, \ x \in \mathbb{R}, \]  

(14)

where \( \alpha > 0 \) and \( \beta > 0 \) are the additional parameters. The density corresponding to (14) is

\[ g(x) = \frac{f(x) (2 \alpha F(x) + \beta) e^{\alpha R(x)^\gamma + \beta F(x)}}{e^{\alpha + \beta} - 1}, \quad x \in \mathbb{R}. \]  

(15)

Using (15), we can generate the extended version of the existing distributions. We discuss some special sub-models of the EZ-G class by considering \( F(x; \xi) \) as the cdf of the baseline model. In Table 7, we define \( R(x) \) for the sub-models of the EZ-G class of distributions.

| S. No. | Baseline model | \( R(x) \) | Proposed model       | Status  |
|-------|----------------|----------|----------------------|---------|
| 1     | Weibull        | \( \alpha (1 - e^{-q^{\gamma}})^\gamma + \beta (1 - e^{-q^{\gamma}}) \) | EZ-Weibull | New     |
| 2     | Lomax          | \( \alpha (1 - (1 + bx)^{-\alpha^{-1}})^{-\alpha^{-1}} + \beta (1 - (1 + bx)^{-\alpha^{-1}}) \) | EZ-Weibull | New     |
| 3     | uniform        | \( \alpha \left( \frac{x}{\eta} \right)^{\gamma} + \beta \left( \frac{x}{\eta} \right) \) | EZ-uniform | New     |
| 4     | Exponential    | \( \alpha (1 - e^{-q^{\gamma}})^\gamma + \beta (1 - e^{-q^{\gamma}}) \) | EZ- exponential | New     |
| 5     | Rayleigh       | \( \alpha (1 - e^{-q^{\gamma}})^\gamma + \beta (1 - e^{-q^{\gamma}}) \) | EZ-Rayleigh | New     |
3.2. The Cosine-$X$ family of distributions

Taking $R(x) = 1 - \cos\left(\frac{\pi}{2} F(x)\right)$ in (9), we define the cosine-$X$ family as

$$G(x) = \frac{e^{1 - \cos\left(\frac{\pi}{2} F(x)\right)} - 1}{e - 1}, \quad x \in \mathbb{R},$$

(16)

The pdf corresponding to (16), is given by

$$g(x) = \frac{\pi}{(e-1)^2} f(x) \sin\left(\frac{\pi}{2} F(x)\right) e^{1 - \cos\left(\frac{\pi}{2} F(x)\right)}, \quad x \in \mathbb{R}.$$  

(17)

3.3. The Cosine exponentiated-$X$ family of distributions

A random variable $X$ is said to follow the Cosine exponentiated-$X$ distribution if its cdf is given by

$$G(x) = \frac{e^{1 - \cos\left(\frac{\pi}{2} F(x)\right) - \alpha}}{e - 1}, \quad \alpha > 0, \quad x \in \mathbb{R},$$

(18)

with pdf

$$g(x) = \frac{\alpha \pi}{(e-1)^2} f(x) F(x)^{a-1} \sin\left(\frac{\pi}{2} F(x)\right) e^{1 - \cos\left(\frac{\pi}{2} F(x)\right)}, \quad x \in \mathbb{R}.$$  

(19)

3.4. The extended Cosine-$X$ family of distributions

A random variable $X$ is said to follow the extended Cosine-$X$ (for short ‘EC-$X$’) if its cdf is given by

$$G(x) = \frac{e^{\alpha\left[1 - \cos\left(\frac{\pi}{2} F(x)\right)\right]^2} - 1}{e^\alpha - 1}, \quad \alpha > 0, \quad x \in \mathbb{R},$$

(20)

with density function

$$g(x) = \frac{\pi}{e^\alpha - 1} f(x) \left[1 - \cos\left(\frac{\pi}{2} F(x)\right)\right] e^{\alpha\left[1 - \cos\left(\frac{\pi}{2} F(x)\right)\right]^2}, \quad \alpha > 0, \quad x \in \mathbb{R}.$$  

(21)

3.5. The extended Cosine exponentiated-$X$ family of distributions

A random variable $X$ is said to follow the extended cosine exponentiated-$X$ (for short ‘ECE-$X$’) distribution, if its cdf is given by
Let \( R(x) = \alpha \left(1 - \cos \left( \frac{x}{2} F(x) \right) \right)^2 + \beta \left(1 - \cos \left( \frac{F(x)}{2} \right) \right) \) in (14), we define another extended cosine-X family (for short ‘AEC-X’) family as

\[
G(x) = \frac{e^{\alpha \left(1 - \cos \left( \frac{x}{2} F(x) \right) \right)^2 + \beta \left(1 - \cos \left( \frac{F(x)}{2} \right) \right)}}{e^{\alpha + \beta} - 1}, \quad \alpha, \beta > 0, \quad x \in \mathbb{R}. \tag{24}
\]

The pdf of the AEC-X family can easily be obtained by simply differentiating (24).

3.7. **Another extended Cosine exponentiated-X family**

Taking \( R(x) = \alpha \left(1 - \cos \left( \frac{x}{2} F(x) \right) \right)^2 + \beta \left(1 - \cos \left( \frac{F(x)}{2} \right) \right) \) in (14), we introduce another extended cosine exponentiated-X (for short ‘AECE-X’) via the cdf

\[
G(x) = \frac{e^{\alpha \left(1 - \cos \left( \frac{x}{2} F(x) \right) \right)^2 + \beta \left(1 - \cos \left( \frac{F(x)}{2} \right) \right)}}{e^{\alpha + \beta} - 1}, \quad \alpha, \beta, a > 0, \quad x \in \mathbb{R}. \tag{25}
\]

By differentiating (25), we get the density function of the AECE-X family.

3.8. **The extended transmuted-G family**

Let \( T(x) \) be the cdf of the transmuted distribution family. Then we define the extended transmuted-G family (for short ‘ET-G’) by taking \( R(x) = \alpha T(x)^2 + \beta T(x) \) in (14), as follows

\[
G(x) = \frac{e^{\alpha T(x) + \beta T(x)}}{e^{\alpha + \beta} - 1}, \quad \alpha, \beta, \xi > 0, \quad x \in \mathbb{R}. \tag{26}
\]

3.9. **The extended Kumaraswamy-G family**

Let \( K(x) \) be the cdf of the Kumaraswamy distributions. Then, we define the extended Kumaraswamy family (for short ‘EKu-G’) by taking \( R(x) = \alpha K(x)^2 + \beta K(x) \) in (14), as follows

\[
G(x) = \frac{e^{\alpha K(x) + \beta K(x)}}{e^{\alpha + \beta} - 1}, \quad \alpha, \beta, \xi > 0, \quad x \in \mathbb{R}. \tag{27}
\]

3.10. **The alpha power transformed Cosine-X family**

We define an extended form of the alpha power transformed family by

\[
G(x) = \frac{\alpha^{R(x)} - 1}{\alpha - 1}, \quad \alpha > 0, \quad \alpha \neq 1, \quad x \in \mathbb{R}, \tag{28}
\]
where, $R(x)$ may be any function of cdf satisfying the conditions stated in section 2. Here, we define a new family, called the alpha power transformed cosine-$X$ (for short ‘APTC-$X$’) family by taking $R(x) = 1 - \cos\left(\frac{\pi}{2} F(x)\right)$ in (28).

$$G(x) = \frac{\alpha^{-1}}{\alpha - 1} - 1, \quad \alpha > 0, \alpha \neq 1, x \in \mathbb{R}. \quad (29)$$

The pdf of the APTC-$X$ can easily be obtained by simply differentiating (29).

### 3.11. The alpha power transformed Cosine exponentiated-$X$ family

A random variable $X$ is said to have the alpha power transformed cosine exponentiated-$X$ (for short ‘APTCE-$X$’) family, if its cdf is given by

$$G(x) = \frac{\alpha^{-1}}{\alpha - 1} - 1, \quad \alpha, \alpha > 0, \alpha \neq 1, x \in \mathbb{R}, \quad (30)$$

with density function

$$f(x) = \frac{\alpha \pi}{2(\alpha - 1)} f(x) F(x)^{\alpha - 1} \sin\left(\frac{\pi}{2} F(x)\right) \alpha^{-1} \cos\left(\frac{\pi}{2} F(x)\right), \quad x \in \mathbb{R}. \quad (31)$$

### 3.12. The extended alpha power transformed-$X$ family

Taking $R(x) = \alpha F(x)^{\alpha} + \beta F(x)$ in (28), we introduce the extended alpha power transformed-$X$ (for short ‘EAPT-$X$’) via the cdf

$$G(x) = \frac{\alpha^{-1}}{\alpha - 1} - 1, \quad \alpha, \alpha, \beta > 0, \alpha \neq 1, x \in \mathbb{R}. \quad (33)$$

By differentiating (33), we get the density function of the EAPT-$X$ family.

### 4. Characterization Results

In designing a stochastic model for a particular modeling problem, an investigator will be vitally interested to know if their model fits the requirements of a specific underlying probability distribution. To this end, the investigator will rely on the characterizations of the selected distribution. Thus, the problem of characterizing a distribution is an important problem in various fields and has recently attracted the attention of many researchers. Consequently, various characterization results have been reported in the literature. These characterizations have been established in different directions. This section deals with various characterizations of 12 proposed distributions listed in Section 3. These characterizations are based on a simple relationship between two truncated moments. It should be mentioned that one important advantage of our characterization is that the cdf need not have a closed, and moreover, it depends on the solution of a first order differential equation, which provides a bridge between probability and differential equation. In the subsection 4.1 we provide the characterizations of the Extended Zubair-G (EZ-G) family of distributions. Similar characterizations can be stated for the other 11 distributions.

#### 4.1. Characterizations based on two truncated moments
This subsection deals with the characterizations of the EZ-G distribution based on the ratio of two truncated moments. Our first characterization employs a theorem of Glänzel (1987); see Theorem 1 of Appendix A.

**Proposition 4.1.** Let \( X : \Omega \to \mathbb{R} \) be a continuous random variable and let \( q_1(x) = 1 \) and \( q_2(x) = e^{\alpha F(x)^2 + \beta F(x)} \) for \( x \in \mathbb{R} \). Then, the random variable \( X \) has pdf (12) if and only if the function \( \eta \) defined in Theorem 1 is of the form
\[
\eta(x) = \frac{1}{2}(e^{x+\beta} + e^{\alpha F(x)^2 + \beta F(x)}), \quad x \in \mathbb{R}.
\]

**Proof.** Suppose the random variable \( X \) has pdf (12), then
\[
(1 - F(x)) E(q_1(X) | X \geq x) = \frac{1}{e^{x+\beta} - 1}(e^{x+\beta} - e^{\alpha F(x)^2 + \beta F(x)}), \quad x \in \mathbb{R},
\]
and
\[
(1 - F(x)) E(q_2(X) | X \geq x) = \frac{1}{2(e^{x+\beta} - 1)}(e^{2(x+\beta)} - e^{2\alpha F(x)^2 + \beta F(x)}), \quad x \in \mathbb{R}.
\]

Further,
\[
\eta(x) q_1(x) - q_2(x) = \frac{1}{2}(e^{x+\beta} - e^{\alpha F(x)^2 + \beta F(x)}) > 0, \quad \text{for} \ x \in \mathbb{R}.
\]

Conversely, if \( \eta \) is of the above form, then
\[
s'(x) = \frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{f(x)(2\alpha F(x) + \beta)e^{\alpha F(x)^2 + \beta F(x)}}{e^{x+\beta} - e^{\alpha F(x)^2 + \beta F(x)}}, \quad x \in \mathbb{R},
\]
and hence
\[
s(x) = -\log\left(e^{x+\beta} - e^{\alpha F(x)^2 + \beta F(x)}\right), \quad x \in \mathbb{R}.
\]

Now, in view of Theorem 1, \( X \) has density (12).

**Corollary 4.1.** Let \( X : \Omega \to \mathbb{R} \) be a continuous random variable and let \( q_1(x) \) be as in Proposition 4.1. The random variable \( X \) has pdf (12) if and only if there exist functions \( q_2 \) and \( \eta \) defined in Theorem 1 satisfying the following differential equation
\[
\frac{\eta'(x) q_1(x)}{\eta(x) q_1(x) - q_2(x)} = \frac{f(x)(2\alpha F(x) + \beta)e^{\alpha F(x)^2 + \beta F(x)}}{e^{x+\beta} - e^{\alpha F(x)^2 + \beta F(x)}}, \quad x \in \mathbb{R}.
\]

**Corollary 4.2.** The general solution of the differential equation in Corollary 4.1 is
\[
\eta(x) = \left(e^{x+\beta} - e^{\alpha F(x)^2 + \beta F(x)}\right) \left[-\int f(x)(2\alpha F(x) + \beta)e^{\alpha F(x)^2 + \beta F(x)}(1 - F(x; \xi))^{\alpha+1}(q_1(x))^{-1} q_2(x) dx + D\right],
\]
where \( D \) is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1 with \( D=1/2 \). However, it should also be noted that there are other triplets \( (q_1(x), q_2(x), \eta(x)) \) satisfying the conditions of Theorem 1.

5. **Concluding Remarks**

The need of compounding and generalizing distributions were first felt in the financial and actuarial science and later in many other fields which researchers adopted this approach.
for lifetime and reliability modeling. In this way, the possible available compound and generalized G-classes are surveyed and using these basic principles nearly 12 new classes are proposed. The goal of providing a variety of new class classes is to test the flexibility of the proposed models to cope with the data available in complex situations. The parameters inducted in this way might be helpful in describing the phenomenon generated from real-lifetime data sets. We expect that these distributions will be an addition to the art of constructing useful probability models. One can imagine its motivation and usefulness in the fields which are not touched so far. Lastly, we offer more choices to the learners and practitioners of modeling to compare different models and to illustrate usefulness of old and new classes of distributions.

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Appendix A.

Theorem 1. Let $(\Omega, F, P)$ be a given probability space and let $H = [d; e]$ be an interval for some $d < e$ ($d = -\infty$; $e = \infty$ might as well be allowed). Let $X: \Omega \rightarrow H$ be a continuous random variable with the distribution function $F$ and let $q_1(x)$ and $q_2(x)$ be two real functions defined on $H$ such that

$$E(q_2(X) | X \geq x) = E(q_1(X) | X \geq x)\eta(x), \quad x \in H,$$

is defined with some real function $\eta$. Assume that $q_1, q_2 \in C^1(H), \eta \in C^2(H)$ and $F$ is twice continuously differentiable and strictly monotone function on the set $H$. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of $H$. Then $F$ is uniquely determined by the functions $q_1$, $q_2$ and $\eta$ particularly

$$F(x) = \int_a^x \frac{\eta'(u)}{\eta(u)q_1(u)-q_2(u)} \exp(-s(u))du,$$
where the function $s(u)$ is a solution of the differential equation $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$ and $C$ is the normalization constant, such that $\int_H dF = 1$.

Note that the result, however, holds also when the interval $H$ is not closed, since the condition is on the interior of $H$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence ((see, Glänzel (1990)), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions $q_{1n}$, $q_{2n}$ and $\eta_n$ ($n \in \mathbb{N}$) satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1$, $q_{2n} \to q_2$ for some continuously differentiable real functions $q_1$ and $q_2$. Let, finally, $X$ be a random variable with distribution $F(x)$. Under the condition that $q_{1n}$ and $q_{2n}$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence $X_n$ converges to $X$ in distribution if and only if $\eta_n$ converges to $\eta$, where

$$\eta(x) = \frac{E(q_2(X) | X \geq x)}{E(q_1(X) | X \geq x)}.$$  

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions $q_1$, $q_2$ and $\eta$ respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Levy-Smirnov distribution if $\alpha \to \infty$ as was pointed out in Glänzel and Hamedani (2001).

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions $q_1$, $q_2$, and, specially, $\eta$ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose $\eta$ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.