Kinetic-Theoretic Description based on Closed-Time-Path Formalism

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(November 14, 2018)

Utilizing a non-equilibrium Green function like the generalized Kadanoff-Baym ansatz, a systematic perturbative method is presented to calculate the expectation value of an arbitrary physical quantity under the restriction that the Wigner distribution function is fixed. It is shown that, in the diagrammatic expression of the quantity, a certain part of contributions can be eliminated due to the restriction. Together with the quantum kinetic equation, this method provides a basis for the kinetic-theoretical description.

I. INTRODUCTION

The non-equilibrium state of a dilute gas system is considered to be described by the one-particle distribution function (1PDF), and such an approach to the non-equilibrium system is called the ‘kinetic theory’. In the kinetic theory, the 1PDF is treated as the independent dynamical variable of the system, and all the physical quantities are determined by the 1PDF. The kinetic equation is an equation of motion of the 1PDF, and the dynamics in the kinetic theory is described by this equation.

A lot of works has been done on the derivation of the quantum kinetic equation (QKE) in the framework of the Green function technique. Perhaps, the most popular approach is the generalized Kadanoff-Baym (GKB) formalism, which utilizes an ansatz for the non-equilibrium Green function called the GKB ansatz. The GKB ansatz can be expressed by the usual equilibrium Keldysh Green function in which the equilibrium 1PDF is replaced by a non-equilibrium one.

An alternative approach is the counter-term method based on the CTP formalism, or on the thermo-field dynamics. In this approach, a counter-term, in which the non-equilibrium properties are included, is first introduced into the CTP or thermo-field Lagrangian, and then the unperturbed propagator gets the similar structure as the GKB ansatz.

Although the QKE can be derived in these approaches, the kinetic theory is not completely constructed because they do not give a proper method to express the expectation values of physical quantities in terms of the 1PDF. Of course, the expectation value of a physical quantity of interest can be calculated perturbatively by the usage of the non-equilibrium propagators mentioned above, and it becomes a functional of the 1PDF. But if we want to obtain a functional of the 1PDF, the value of the 1PDF must be fixed from the exterior, and the integrations over the microscopic fields should be carried out under the restriction due to the fixing of the 1PDF. This restriction has not been considered in the above formalisms, and hence they do not give a complete basis for the kinetic theory.

In this paper, we present a systematic perturbative method to calculate the expectation value of any physical quantity as a functional of the Wigner distribution function (WDF), which plays the role of the 1PDF in quantum theory. Our approach, is somewhat different from the GKB or the counter-term method. It is based on the inversion method. An inversion-method approach to derive the QKE was presented in Refs. and the problem of calculating the physical quantities in terms of the WDF is partly solved there; We first introduce an external source $J$ to probe the WDF $z$, and calculate the physical quantity as a functional of the source $Q[J]$. Then we express the source as a functional of the WDF ($J = J[z]$ which is an inversion of $z = z[J]$), and by substituting it into the above calculated $Q[J]$, we can write the quantity by the WDF. In this approach, the propagator is a functional of the external source $J$, and the perturbative calculation can be done without the restriction of fixing the WDF. The calculations in our formulation indeed generates different results from those obtained by the perturbative calculation using, e.g., the GKB ansatz. By the substitution of the inverted relation, some contributions from the diagrams, which will be present in the calculations with other formalisms, are canceled.

We show that the contributions which are canceled can be expressed by a corresponding time-ordered diagrams: The contributions from a diagram in the non-equilibrium theory can be classified by the temporal order of the vertices in the diagram, and to each temporal ordering of the vertices, a time-ordered diagram (called a configuration in the article) corresponds. Then if an obtained configuration can be separated into two parts by cutting two propagators at the same instant, the contribution from that configuration is canceled. Because the propagator used in our method has the form of the GKB ansatz, this will also provide a basis for the kinetic theoretic description in the GKB formalism.
In the course of proving this property, we reformulate the inversion method approach in the framework of the Legendre transformation. The definition of non-equilibrium generating functional is slightly modified in a way characteristic to the non-equilibrium theory, and the effective action is defined as the Legendre transformation of it. Then the diagrammatic rule for the effective action discussed in Ref. 15 can be utilized with an extension to the non-equilibrium case. By virtue of this, the QKE can also be expressed in a compact form which is finally given as (17).

In the next section, we summarize the inversion method approach to the QKE, and reformulate it in the terminology of Legendre transformation. Then in II B, the diagrammatic rule for the kinetic theory, and the rule to derive the QKE is in II C.

II. INVERSION METHOD APPROACH TO THE KINETIC EQUATION

In this section, we describe the inversion method approach to the QKE.

A. Probing source and the Green function

The system to be considered is the same as in Ref. 11; a non-relativistic bosonic field described by the Hamiltonian

\[ \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \]

and a spatially inhomogeneous initial density matrix \( \hat{\rho} \). We consider the case that the interaction \( \hat{H}_{\text{int}} \) can be treated perturbatively, and for simplicity, the initial correlation is not taken into account. See Ref. 10 for the treatment of the initial correlation.

In quantum statistical physics, the natural alternative of the 1PDF will be the Wigner distribution function (WDF) defined as

\[ f_K(X, t) = \int \frac{d\Delta x}{V} e^{-iK \cdot \Delta x} \left\langle \hat{\psi}^\dagger(X - \frac{\Delta x}{2}, t) \hat{\psi}(X + \frac{\Delta x}{2}, t) \right\rangle, \]

where \( \hat{\psi}(x) = \frac{1}{\sqrt{V}} \sum_k e^{ik \cdot x} \hat{\psi}_k \) and the angular bracket implies the average over initial density matrix \( \hat{\rho} \); \( \langle \cdots \rangle = \text{Tr} \hat{\rho} \cdots \). As in Ref. 11, for the sake of perturbative calculation, it is more convenient to work with the Fourier transform of the WDF defined as

\[ x_{k,q}(t) \equiv \left\langle \hat{\psi}^\dagger_q(t) \hat{\psi}_k(t) \right\rangle = \int dx e^{-i(k-q)t} f_{k+q,x}(x, t), \]

to which we refer simply as the WDF in the following. Note that \( x_{k,q}^* = x_{q,k} \) holds due to the hermitian property of \( \hat{\rho} \).

Within the CTP formalism, Eq. (3) can be represented as

\[ x_{k,q}(t) \propto \int [d\psi_1 d\psi_2] \psi_q^\dagger(t) \psi_k(t) e^{\frac{i}{\hbar} \int_0^t dx (L(\psi_1) - L(\psi_2)) \left\langle \psi_1 | \hat{\rho} | \psi_2 \right\rangle}. \]

In the inversion method approach to the kinetic theory, we introduce a probing source \( J \) for \( z \), and calculate \( z[J] \) as a functional of the source. By inverting the relation as \( J = J[z] \), the QKE is obtained as an equation of motion for \( z \) by setting \( J = 0 \). According to Ref. 3, the proper way to introduce the source \( J_{k,q} \) is that the source is built into the quadratic form of the free part of the Lagrangian in (L) \( L_0(\psi_1) - L_0(\psi_2) = \sum_{k,q,ij} \psi_k^\dagger \psi_j \partial_{k,q} \psi_j^\dagger \psi_j \) by

\[ D_{k,q} = \begin{pmatrix} (i\hbar \partial_t - \epsilon_k) \hat{\delta}_{k,q} + iJ_{k,q}(t) & -iJ_{k,q}(t) \\ -iJ_{k,q}(t) & (i\hbar \partial_t - \epsilon_k) \hat{\delta}_{k,q} + iJ_{k,q}(t) \end{pmatrix}. \]

An inverse of this matrix leads to the 2 × 2-Green function, which is a functional of the source \( J \). From the relation \( DG^{(0)} = -i\hbar \), we get
Problem in this paper is to find what kind of the diagram should be retained in the evaluation of the QKE following from (9) after the removal of the source. This QKE is reduced to the usual Boltzmann equation after the Markovian and local approximations.

B. kinetic theoretic description in the inversion method

In the kinetic theory, the 1PDF is considered to be an independent dynamical variable and the kinetic equation describes its dynamics. This means a coarse graining from the microscopic field variables to the 1PDF. All other quantities should be expressed in terms of the 1PDF, and their dynamics should follow from the kinetic equation. Since such physical quantities may be defined microscopically by a temporary-local functions of the field variables as \( Q = Q(\psi) \), in order to obtain a complete framework of kinetic theory, we must express their expectation values as functionals of the 1PDF.

In the inversion method approach, this can be realized by first calculating perturbatively the expectation value \( Q(t) = \langle Q(t) \rangle \) with the use of the propagator \( -G^{(0)}[J] \) in previous subsection, and then by substituting the source written by the WDF as in (8) into the obtained functional. The former procedure will provide us with the expectation value as a functional of the source \( J \), and the latter reduces it into a functional of the WDF \( z \).

From some explicit calculations, we can see the following fact: After the substitution of \( J = J^{(0)}[z] + \Delta J[z] \) in the second step, if we expand the obtained expression around \( J^{(0)}[z] \), the contributions due to the perturbative correction \( \Delta J[z] \) cancels some part of the unperturbed contributions. Such an expansion can be expressed diagrammatically by the usage of a propagator \( -G^{(0)}[J = J^{(0)}[z]] \), and the above mentioned cancellation implies that some part of the diagram should be omitted if we want to evaluate the expectation value as a functional of the WDF.

From this observation, it is expected that simplified procedure to calculate the expectation value \( Q[z] \) exists. Our problem in this paper is to find what kind of the diagram should be retained in the evaluation of the \( Q[z] \) if the diagrams are written with the propagator \( -G^{(0)}[J = J^{(0)}] \). In fact, the propagator \( -G^{(0)}[J = J^{(0)}[z]] \) has the form of the GKB ansatz with the free particle approximation of the spectral function; it can be obtained by replacing \( z^{(0)}[J] \) in (8) by \( z \). So our consideration here also provides the way to calculate \( Q[z] \) in the GKB formalism.
C. Formulation with Legendre transformation

In order to discuss the problem settled in the previous subsection, it is convenient to rewrite the inversion method in the framework of the Legendre transformation formalism. The physical representation of the CTP formalism is introduced by a simple transformation of the variables from \( \psi_1 \) and \( \psi_2 \) to \( \psi_C \) and \( \psi_\Delta \), which is defined as

\[
\psi_C = \frac{\psi_1 + \psi_2}{2}, \quad \psi_\Delta = \psi_1 - \psi_2.
\]

Then the free part of the Lagrangian is rewritten as

\[
L_0(\psi_C, \psi_\Delta) = \sum_{k,q} \left( \psi^*_k C \psi_{k,\Delta} \right) \left( \begin{pmatrix} 0 & \ii \hbar \partial_t - \epsilon_k \delta_{k,q} \\ \ii J_{k,q}(t) \end{pmatrix} \right) \left( \begin{array}{c} \psi_{q,C} \\ \psi_{q,\Delta} \end{array} \right),
\]

where we have denoted the source \( J \) in \( \bar{W} \) as \( J_C \) for convenience. We can see the source \( J_C \) is simply coupled to \( \psi_\Delta \). Correspondingly, the Green function \( G^{(0)}[J_C] \) becomes

\[
G^{(0)}_{k,q}[t, s; J_C] = \left( \begin{array}{cc} g^{C}_{k,q}[t, s; J_C] & g^{R}_{k,q}(t, s) \\ g^{R}_{k,q}(t, s) & 0 \end{array} \right),
\]

where the respective components are defined as

\[
g^{C}_{k,q}[t, s; J_C] = \left( \frac{\theta(t-s)e^{-\ii \omega_k(t-s)}}{2} \right) - \theta(s-t)e^{\ii \omega_q(s-t)} \left( \frac{\theta(t-s)e^{-\ii \omega_k(t-s)} \delta_{k,q}}{2} \right),
\]

\[
g^{R}_{k,q}(t, s) = -\theta(t-s)e^{-\ii \omega_k(t-s)} \delta_{k,q},
\]

\[
g^{A}_{k,q}(t, s) = \theta(s-t)e^{\ii \omega_q(s-t)} \delta_{q,k}.
\]

To use the Legendre transformation formalism, we must introduce another source \( J_\Delta \) coupled to \( \psi^*_C \psi_C \), and define the generating functional \( W \) as

\[
e^{\ii \int \bar{W}[J_C, J_\Delta]} \equiv \int [d\psi_C d\psi_\Delta] \left\{ \langle \psi_{C,1} + \frac{\ii}{2} \psi_{\Delta,1} | \hat{\rho} | \psi_{C,1} - \frac{\ii}{2} \psi_{\Delta,1} \rangle 
\times e^{\ii \int \bar{L}_0(\psi_C, \psi_\Delta) - V(\psi_\Delta, \bar{\psi}_C) + \sum_{k,q} J_{k,q,\Delta} \psi^*_k C \psi_{q,C} + \bar{L}_0(\psi_C, \psi_\Delta) } \right\},
\]

where \( V \) is the interaction part of the CTP Lagrangian \( H_{\text{int}}(\bar{\psi}_1) - H_{\text{int}}(\bar{\psi}_2) \) written in terms of \( \psi_C \) and \( \psi_\Delta \). The source \( J_\Delta \) is unphysical in the sense that the expectation value of an hermitian operator is not guaranteed to be real under the existence of this source. It is just introduced so that we can write the WDF \( z \) by a derivative of the generating functional, and should be removed after all the calculation. For the same reason, we have introduced the source \( I_\Delta \) which is coupled to \( Q(\psi_C) \); replacement of \( \psi \) by \( \psi_C \) in the time-local composite operator \( Q(\psi) \) in which we are interested. Note that all the integrands in the exponent of \( \bar{W} \) are local in time.

Here we define two variables

\[
z_{k,q,C}(t) = \frac{\delta W[J_C, J_\Delta]}{\delta J_{k,q,\Delta}(t)}, \quad z_{k,q,\Delta}(t) = \frac{\delta W[J_C, J_\Delta]}{\delta J_{k,q,C}(t)}.
\]

When the sources are removed, \( z_C \) is reduced to \( \langle T \hat{\psi}^\dagger \hat{\psi} + \hat{\psi}^\dagger \hat{\psi} + \hat{T} \hat{\psi}^\dagger \hat{\psi} \rangle = z_{k,q} + \delta_{k,q}/2 \), and \( z_\Delta \) is to \( \langle (T \hat{\psi}^\dagger \hat{\psi} - \hat{\psi}^\dagger \hat{\psi} + \hat{T} \hat{\psi}^\dagger \hat{\psi}) \rangle = 0 \). Note that we regard \( \psi^* \psi \) as \( \psi^*(t + 0)\psi(t) \) in the course of the path integration. Particularly, \( z_{\Delta,2} = 0 \) is realized by removing only the unphysical source \( J_\Delta \), and in this case, \( z_C \) becomes a functional of \( J_C \) as \( z_{k,q,C}[J_C] + \delta_{k,q}/2 \). Of course, the non-equilibrium expectation value of symmetrized \( Q \) is obtained as a functional of the sources \( J_C \) by

\[
Q[t; J_C] = \left. \frac{\delta W[J_C, J_\Delta]}{\delta I_\Delta(t)} \right|_{J_\Delta = I_\Delta = 0}.
\]

To use the variables \( z_C \) and \( z_\Delta \) as independent variables, we define the Legendre transformation of \( W \) by
\[ \Gamma[z_C, z_\Delta; I_\Delta] \equiv W[J_\Delta, J_C, I_\Delta] - \sum_{k, q} \int_{t_1}^{d_2} dt \left( J_{kq, \Delta} \delta_{kq, C} + J_{kq, C} \delta_{kq, \Delta} \right), \]  

where \( J_\Delta \) and \( J_C \) are functional of \( z_C \) and \( z_\Delta \), which are obtained by solving (18). Now we can obtain the expectation value of \( Q \) as a functional of the WDF \( z_C \) by

\[ Q[t; z_C] = \frac{\delta \Gamma[z_C, z_\Delta; I_\Delta]}{\delta z_\Delta(t)} \bigg|_{z_\Delta = I_\Delta = 0}. \]

Moreover, from an identity of the Legendre transformation, we have

\[ J_{kq, C}(t) = -\frac{\delta \Gamma[z_C, z_\Delta; I_\Delta]}{\delta z_{kq, C}(t)}, \quad J_{kq, \Delta}(t) = -\frac{\delta \Gamma[z_C, z_\Delta; I_\Delta]}{\delta z_{kq, \Delta}(t)}. \]

If we remove the unphysical source \( J_\Delta \), the first equation become an equation of motion of \( z_{kq, C} = z_{kq, C}[J_C] + \delta_{kq, C}/2 \) which corresponds to (18), and finally the removal of \( J_C \) reduces the equation of motion to the QKE. In this sense, \( \Gamma \) is referred to as the effective action.

### III. DIAGRAMMATIC RULE FOR KINETIC THEORY

Diagrammatic expression of the effective action \( \Gamma \) is well investigated. For an expectation value of non-local product \( \langle \hat{\psi}^\dagger(t) \hat{\psi}(s) \rangle \), the effective action is expressed simply by the two particle irreducible (2PI) diagrams. Here, 2PI diagram is a diagram which cannot be separated by cutting any pair of propagators. For the expectation value of a local product, such as the 1PDF, the situation is more complicated. In this subsection, we utilize the rules presented in Ref. [13] with a non-equilibrium extension, and clarify the meaning of the rule. By use of the rule, the QKE can also be rewritten in a compact form. For notational simplicity, the time arguments and wave-number indices will not explicitly be written if it is not misleading.

#### A. Diagrammatic expression of the effective action

First we consider a diagrammatic expansion of the generating functional \( W \). The building blocks of the diagram are a \( 2 \times 2 \)-propagator \( -\tilde{G}^{(0)} \) given below, an interaction vertex \( V(\psi_\Delta, \psi_C) \) given in (17) and an external leg \( Q(\psi_C) \) coupled to \( I_\Delta \). In the diagram, an arrow expresses the contraction operator

\[ -\sum_{k, q} \int dt ds \left( \frac{\delta}{\delta \psi_{kq, C}(t)} \right) \left( \frac{\delta}{\delta \psi_{kq, \Delta}(t)} \right) \tilde{G}^{(0)}_{kq}(t, s) \left( \frac{\delta}{\delta \psi_{q, C}(s)} \right) \left( \frac{\delta}{\delta \psi_{q, \Delta}(s)} \right). \]

The \( 2 \times 2 \)-Green function \( \tilde{G}^{(0)} \) is defined as an inverse of the matrix in bilinear form of the exponent in (17);

\[ \tilde{G}^{(0)}[J_C, J_\Delta] \equiv \frac{i}{\hbar} \left( \frac{J_\Delta}{i\hbar \partial_t - \epsilon} \right) \left( \frac{J_C}{\hbar \partial_t - \epsilon} \right)^{-1} = \begin{pmatrix} \tilde{g}^C & \tilde{g}^R \\ \tilde{g}^A & \tilde{g}^\Delta \end{pmatrix}, \]

where the tilde implies the unphysical case \( J_\Delta \neq 0 \). Using the physical case \( J_\Delta = 0 \) given in (14)-(16), the components of (24) can be written as

\[ \tilde{g}^C[J_\Delta, J_C] \equiv \begin{pmatrix} 1 - g^C[J_C, J_\Delta/\hbar] \end{pmatrix}^{-1} g^C[J_C], \]

\[ \tilde{g}^R[J_\Delta, J_C] \equiv \begin{pmatrix} 1 + g^C[J_C, J_\Delta/\hbar] \end{pmatrix} g^R, \]

\[ \tilde{g}^A[J_\Delta, J_C] \equiv \begin{pmatrix} 1 + \frac{J_\Delta}{\hbar g^C[J_C, J_\Delta]} \end{pmatrix} g^A, \]

\[ \tilde{g}^\Delta[J_\Delta, J_C] \equiv \begin{pmatrix} J_\Delta \frac{1}{\hbar} - \frac{J_\Delta}{\hbar g^C[J_C, J_\Delta]} \end{pmatrix} g^R, \]
with a short-hand notation. Of course \( \tilde{G}^{(0)} \) is reduced to \( G^{(0)} \) in (13) by setting \( J_{\Delta} = 0 \). Note that retarded or advanced nature of \( g^R \) or \( g^A \), respectively, is recovered only in the physical case \( J_{\Delta} = 0 \).

Then the generating functional \( W \) can be expressed as

\[
\frac{i}{\hbar} W[J, I_{\Delta}] = \text{Tr} \ln \tilde{G}^{(0)}[J] + \kappa[J, I_{\Delta}],
\]

where \( J \) expresses the set of \( J_{\Delta} \) and \( J_{C} \), and \( \kappa \) is the sum of all the connected diagrams constructed by the propagator \(-\tilde{G}^{(0)}[J] \), the vertex \( V \) and the external leg \( I_{\Delta} \). For simplicity, we suppress the argument \( I_{\Delta} \) in this subsection.

Next we evaluate (29) at \( J = J^{(0)}[z] + \Delta J[z] \) (cf. (10)), and substitute it into the definition (20) of the effective action \( \Gamma \). Expanding \( \Gamma \) around \( J = J^{(0)}[z] \) in terms of \( \Delta J[z] \), the terms linear in \( \Delta J \) are canceled, and we obtain

\[
\frac{i}{\hbar} \Gamma^{(0)}[z] = \text{Tr} \ln \tilde{G}^{(0)}[J^{(0)}] - \frac{i}{\hbar} \left\{ \left( J^{(0)} + \Delta J \right) z_{C} + \left( J^{(0)} + \Delta J_{C} \right) z_{\Delta} \right\}
\]

\[
= \frac{i}{\hbar} \Gamma^{(0)}[z] - \frac{1}{2\hbar^2} \left( \Delta J_{\Delta} \Delta J_{C} \right) \Delta_{2} \left( \frac{\Delta J_{\Delta}}{\Delta J_{C}} \right) + \kappa'[z],
\]

where \( \Gamma^{(0)}, \Delta_{2} \) and \( \kappa' \) are defined respectively by

\[
\frac{i}{\hbar} \Gamma^{(0)}[z] = \text{Tr} \ln \tilde{G}^{(0)}[J^{(0)}] - \frac{i}{\hbar} \left\{ \left( J^{(0)} + \Delta J \right) z_{C} + \left( J^{(0)} + \Delta J_{C} \right) z_{\Delta} \right\},
\]

\[
\Delta_{2}(t, s) = - \left( \hat{g}^{C}(t, s) \hat{g}^{C}(s, t) - i \hat{g}^{R}(t, s) \hat{g}^{A}(s, t) \right),
\]

\[
\kappa'[z] = \kappa[J[z]] + \text{Tr} \sum_{k \geq 2} \frac{1}{k} \left\{ \frac{1}{\hbar} \tilde{G}^{(0)}[J^{(0)}] \right\}^{k}.
\]

Eq. (31) corresponds to Eq. (3.21) in Ref. [13] and then, as it is proved in Ref. [15], the effective action can be expressed as

\[
\frac{i}{\hbar} \Gamma[z] = \mathcal{R}_{2} \left( \frac{i}{\hbar} W[J^{(0)}[z]] \right) - \frac{i}{\hbar} \sum_{k, q} \int_{\Delta}^{t_{1}} dt \left( \tilde{r}_{k, \Delta}^{(0)}[z_{k, C}] z_{k, C} + J_{k, C}^{(0)}[z] z_{k, \Delta} \right).
\]

Here, \( \mathcal{R}_{2} \) is a diagrammatic operation defined by the following process.

1. The first process of \( \mathcal{R}_{2} \) can be expressed schematically as

\[
\text{to which we refer as the 'cut-and-patch' operation: If there is a 2PR part in the diagram, separate the graph into two pieces by cutting the corresponding pair of the propagators. In each of the separated diagrams, make the resultant two external lines to contract } \psi_{C}(t) \psi_{C}(t) \text{ or } i \dot{\psi}_{\Delta}(t) \psi_{\Delta}(t), \text{ which we call the } z_{C} \text{- or } z_{\Delta} \text{-leg, respectively. Then reconnect the two diagrams by contracting their } z \text{-legs with } \Delta_{2}^{-1}.
\]

2. The second step is to carry out the procedure [11] in all possible ways, and sum up all the resulting diagrams including the original one.

For example,

\[
\mathcal{R}_{2} \left( \begin{array}{c}
\end{array} \right) = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} .
\]

where \( \Delta_{2}^{-1} \) is expressed by doubled lines. It should be emphasized that \( W \) is evaluated at \( J = J^{(0)}[z] \) in (29): In the diagrammatic expression of \( W \), the propagators are \(-\tilde{G}^{(0)}[J^{(0)}[z]] \) which acquire the form of the GKB ansatz when we set \( z_{\Delta} = 0 \).

As it was discussed in Ref. [13] the operation \( \mathcal{R}_{2} \) cancels some part of the 2PR diagrams, and in this sense, \( \Gamma \) has a modified 2PI property. It is reduced to the usual 2PI when we discuss an effective action of non-local operator \( \dot{\psi}_{\Delta}(t) \psi(s) \). In the next subsection, we will clarify what is the contents of this modified 2PI property.
B. Diagrammatic Rule for $Q[z_C]$

Recovering the argument $I_\Delta$ in (35), the expectation value $Q$ as a functional of the WDF is obtained from (21) as

$$Q[t; z_C] = R_2 \left( \frac{\delta W[J^{(0)}(z), I_\Delta]}{\delta I_\Delta(t)} \right) \bigg|_{z_\Delta = I_\Delta = 0}. \quad (38)$$

Note that $J^{(0)}[z]$ in (35) does not depend on $I_\Delta$. Diagrammatically, inside the operation $R_2$ is a sum of all the connected diagrams with one external point expressing $Q(\psi_C)$. For definiteness, we consider the case $\hat{Q} = \hat{\psi}_q^\dagger \hat{\psi}_q^\dagger \hat{\psi}_k \hat{\psi}_k$ as an example. In the following, since we have set $z_\Delta = I_\Delta = 0$ in (38), the form of the Green function $\tilde{G}^{(0)}[J^{(0)}[z]]$ is reduced to that of the GKB ansatz; Eq.(13) in which $z^{(0)}[J_C]$ is replaced by $z$.

1. Time-ordered configuration and causality

Before considering the operation $R_2$, we define terminologies ‘time-ordered configuration’ and ‘causality’.

In the non-equilibrium Green function technique, because the propagator depends explicitly on time, the evaluation of a diagram may be carried out as a function of time as follows. For all possible ways of time ordering of the vertices, the diagram is arranged in such a way that the vertices are put on the time axis from right to left. Then assigning the factors of propagators and vertices, each time ordering gives different contribution. In the following, we refer to the diagram with a fixed time ordering of the vertices as the ‘time-ordered configuration’ or simply the configuration.

For example, the diagram in Fig. 1 (a) can be arranged as the eight configurations shown in Fig. 1 (b) and (c). Other possible configurations can be eliminated by the following mechanism.

![FIG. 1. Examples of time-ordered configurations. The open circle expresses the external point Q. The diagram (a) can be arranged into eight configurations shown in (b) and (c). The difference between the groups (b) and (c) will be clarified later.](image)

The vertices in the diagram expresses $H_{\text{int}}(\psi_1) - H_{\text{int}}(\psi_2)$ rewritten by $\psi_\Delta$ and $\psi_C$, which is odd in $\psi_\Delta$ for generic $H_{\text{int}}$, and contains at least one $\psi_\Delta$ or $\psi_\Delta^\ast$. Then, we can conclude that the time-ordered configuration like Fig. 2, where a vertex is on the latest time, vanishes: Assuming the vertex of Fig. 2 is on time $t$, $\psi_\Delta(t)$ or $\psi_\Delta^\ast(t)$ therein must be contracted by $g^A(t, s)$ or $g^R(s, t)$ ($t > s$), respectively. (Recall that we are working in the physical case $J_\Delta = 0$.) and their advanced or retarded character leads to the vanishing. This implies that, when we calculate an expectation value of a physical quantity at time $t$, the interaction at time later than $t$ does not contribute since the configuration like Fig. 2 can not be avoided. In other word, the time-ordered configuration of the diagram for $Q$ must have the external point $Q$ on the latest time within the diagram. In this article, we call such a fact the ‘causality’.

![FIG. 2. Vanishing configuration due to the causality. (For definiteness, the four point interaction in (3) is considered. )](image)
Now we consider the meaning of the operation $\mathcal{R}_2$ in (38). For this sake, we first examine the cut-and-patch operation. Since we are considering the physical case $z_\Delta = 0$, the $\Delta\Delta$-component of $\tilde{G}^{(0)}$ as well as the $\Delta\Delta$-component of $\Delta_2$ vanish. Then, because $\Delta_2^{-1}$ can be written as

$$
\Delta_2^{-1} = -\begin{pmatrix} g_C g_C & ig_R g_A \\ ig_A g_R & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & i\{g_R^2\}^{-1} - \{g_R g_A\}^{-1} i\{g_C g_A\}^{-1} \\ i\{g_R^2\}^{-1} - \{g_R g_A\}^{-1} i\{g_C g_A\}^{-1} & 0 \end{pmatrix},
$$

(39)

the $\Delta\Delta$-component of $\Delta_2^{-1}$ disappears. Thus, in (38), the connection of two $z_C$-legs is absent.

Moreover, in (38), the connection of two $z_\Delta$-legs is forbidden by the following reason. Since there is only one external point $Q$ in each diagram for (38), the external point belongs to only one of the two sub-diagrams connected by $\Delta_2^{-1}$. Then, if both of the two sub-diagrams are connected by their $z_\Delta$-legs, the one which does not contain the external point $Q$ must vanish due to the causality: The $z_\Delta$-leg at time $t$ is produced by a pair of Green functions $g_R(s, t)$ and $g^A(t, s')$, which implies they must be connected to vertices at time $s$ and $s'$ later than $t$. So, if the sub-diagram does not have an external point, any time-ordered configuration of the sub-diagram must have a vertex which possesses at the latest time as shown in Fig. 3, and this gives a vanishing contribution due to the causality discussed in the previous subsection.

Fig. 3. The time-ordered configuration of sub-diagram with a $z_\Delta$-leg but without the external point. (for convenience, the time $s$ is chosen to be later than $s'$.)

Thus it is enough to consider the connection of $z_\Delta$-leg and $z_C$-leg, where the external point belongs to the sub-diagram with $z_\Delta$-leg. In contrast to $z_\Delta$-leg, the $z_C$-leg must be possessed on the latest time within the sub-diagram: Otherwise, some vertex must be possessed on the latest time because the sub-diagram with $z_C$-leg does not have the external point, and the configuration in Fig. 3 cannot be avoided. As the result, the time-ordered configurations for $Q$ has a generic form shown in Fig. 4. In the sub-diagram with $z_\Delta$-leg, the external point $Q$ is on the latest time, and the $z_\Delta$-leg is connected to the vertices on the latest time (need not be on the earliest time of the sub-diagram). On the other hand, the sub-diagram with $z_C$-leg has the leg on the latest time.

Fig. 4. The generic structure of the diagram for $Q[z]$.

Let us see the joint of $z^k\{z_\Delta\}$-leg at $t$ and $z^q\{z_C\}$-leg at $s$ by $i\{g_R g_A\}^{-1}_{kk'qq'}(t, s)$. The $z^k\{z_\Delta\}$-leg at time $t$ is produced by a pair of propagators which can be written as

$$
i g_R^{R} g_A^{A} (t', t) g_R^{R} (t', t'') = -i\theta(t' - t'') e^{-i\omega_k (t' - t'')} g_R^{R} (t'', t) g_A^{A} (t', t'').$$

(40)

On the other hand, the $z^q\{z_C\}$-leg at time $s$ is produced by a pair of propagators, one of which is $-g_{R,q}^{R} (s, s')$ or $-g_{q,q'}^{C} (s, s')$ and the other is $-g_{R,q}^{A} (s'', s)$ or $-g_{q,q'}^{C} (s'', s)$. As discussed above, $z_C$-leg is non-zero only when $s > s', s''$, and the following relations hold in this case:

$$
\theta(t' - s) e^{-i\omega_q (t' - s)} g_{R,q}^{R} (s, s') = \frac{g_{R,q}^{R} (t', s')}{g_{q,q'}^{C} (s'', s)},
$$

$$
\theta(t'' - s) e^{i\omega_q (t'' - s)} g_{q,q'}^{C} (s'', s) = \frac{g_{q,q'}^{C} (t'', s'')}{g_{R,q}^{R} (t', s')}.
$$

(41)
With the aid of Eqs. (40)-(41), we have
\[
\sum_{k,k',q,q'} \int dt ds i g_{k,k'}^{R}(t', t) A_{k,k'}^{A}(t, t') i \{ -g_{q,q'}^{R}(s',s') \} \frac{1}{k_{k,k'}^{C}(t, s)} A_{q,q'}^{A}(s', s) = -g_{k,q}^{R/C}(t', s') A_{q,k'}^{A/C}(s', t''),
\]
(42)
where \( t', t'' > s', s'' \) holds. This implies that the joint of \( z_{\Delta} \) and \( z_{C} \)-legs can simply be expressed as
\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{fig.png}
\end{array}
\]
(43)
Note that, on the rhs of (43), the time order of the vertices is restricted unlike usual diagrams: The vertices originally connected to the \( z_{\Delta} \)-leg are on later time than those originally connected to \( z_{C} \)-leg. This implies that the diagram on the rhs of (43) contains only the configurations which can be separated into two parts by cutting the pair of propagators at the same instant. In this sense, we call such a time-ordered configuration ‘instantaneous-2PR configuration’.

For instance, the configurations shown in Fig. 1 (a), we only need to calculate the contributions of the instantaneous-2PR configurations shown in Fig. 1 (b), and can eliminate the instantaneous-2PR ones in (b). More simple example can be seen in Ref. 11 (explicit calculations was shown above, the cut-and-patch process just restricts the time ordering of the vertices, and except the signature, the contribution produced by cut-and-patch is included in the original diagram. Then we should count how many pairs of propagators, such a configuration appears in a diagram where \( k \) of the corresponding \( N \) pairs of the propagators are cut-and-patched. There are \( N C_{k} \) ways of choosing \( k \) pairs, and the signature \( (-1)^{k} \) is assigned. Thus through the total process of \( R_{2} \), the instantaneous-2PR configuration appears \( \sum k (-1)^{k} N C_{k} = 0 \) times.

As the result, the operation \( R_{2} \) on a diagram implies that we can eliminate the instantaneous-2PR configurations which will be produced from the original diagram.

For example, considering (38) in the fourth order of the perturbation, when we evaluate the diagram shown in Fig. 1 (a), we only need to calculate the contributions of the instantaneous-2PI configurations shown in Fig. 1 (b), and can eliminate the instantaneous-2PR ones in (b). More simple example can be seen in Ref. 11 (explicit calculations are shown in Ref. 12), where the four-point function is calculated up to the first order of the perturbation. There appear tadpole diagrams, but their contributions are canceled when the four-point function is expressed in terms of the WDF. From the viewpoint of our rule, the contribution from the tadpole diagrams in Ref. 11 can be eliminated by the \( R_{2} \) operation in (38) because all of the time-ordered configurations produced from those diagrams are instantaneous-2PR.

C. Quantum kinetic equation

Finally, we summarize the rule for deriving the QKE. The physical source \( J_{C} \) as a functional of \( z_{C} \) is obtained by setting \( z_{\Delta} = 0 \) in the first equation of (22) since this condition is equivalent to \( J_{\Delta} = 0 \). With the use of (35), it can be expressed as
\[
J_{C}[t; z_{C}] = J_{C}^{(0)}[t; z_{C}] - \int ds \frac{\delta J_{\Delta}^{(0)}(s)}{\delta z_{\Delta}(t)} \bigg|_{z_{\Delta}=0} R_{2} \left( \frac{\delta W}{\delta J_{\Delta}(s)} \right)_{J=J^{(0)}[z_{\Delta}=0]} - z_{C}(s)
\]
(44)
and the QKE for the WDF \( z_{C} \) is obtained by setting \( J_{C}[t; z_{C}] = 0 \).

To obtain the explicit expression of \( \delta J_{\Delta}/\delta z_{\Delta} \) in (44), we differentiate the identity \( z = z^{(0)}[J^{(0)}[z]] \) with respect to \( z \), and obtain
\[
\begin{align*}
\begin{pmatrix}
\frac{\delta J_{\Delta}^{(0)}}{\delta z_{C}} & \frac{\delta J_{\Delta}^{(0)}}{\delta z_{\Delta}} \\
\frac{\delta J_{\Delta}^{(0)}}{\delta z_{C}} & \frac{\delta J_{\Delta}^{(0)}}{\delta z_{\Delta}}
\end{pmatrix}
\bigg|_{z_{\Delta}=0}
& = \begin{pmatrix}
\frac{\delta z_{C}^{(0)}}{\delta J_{\Delta}} & \frac{\delta z_{\Delta}^{(0)}}{\delta J_{\Delta}} \\
\frac{\delta z_{C}^{(0)}}{\delta J_{\Delta}} & \frac{\delta z_{\Delta}^{(0)}}{\delta J_{\Delta}}
\end{pmatrix}^{-1} \\
& = \begin{pmatrix}
\frac{\delta J_{\Delta}^{(0)}}{\delta z_{C}} & \frac{\delta J_{\Delta}^{(0)}}{\delta z_{\Delta}} \\
\frac{\delta J_{\Delta}^{(0)}}{\delta z_{C}} & \frac{\delta J_{\Delta}^{(0)}}{\delta z_{\Delta}}
\end{pmatrix}^{-1} \\
& = J^{(0)}[z_{\Delta}=0].
\end{align*}
\]
(45)
Because \( z_{C}^{(0)}[t; J_{\Delta}, J_{C}] \) and \( z_{\Delta}^{(0)}[t; J_{\Delta}, J_{C}] \) are given by \( -\hat{g}^{C}(t, t) \) and \( -i\hat{g}^{\Delta}(t, t) \), respectively, their derivatives can be calculated using the definitions (29) and (28). Then we can see that the rhs of (45) is nothing but \( \Delta_{z}^{-1} \) (multiplied by \( i\hbar \)) given in (39), and \( \delta J_{\Delta}^{(0)}/\delta z_{\Delta} \) is reduced to
Thus, in the rhs of (44), the last term in the braces compensates for the first term (cf. (3)), and the QKE can simply be written as

$$\{ \hbar \partial_t + i (\epsilon_k - \epsilon_q) \} \left\{ R_2 \left( z_{q,k}; \delta J(0) \right) \right\} = 0. \tag{47}$$

The QKE can be derived by calculating the instantaneous-2PI configurations of the diagrams for $z_{q,k}$, and by operating $\{ \hbar \partial_t + i (\epsilon_k - \epsilon_q) \}$. This rule can be confirmed by the example in Ref. [11] (details are in Ref. [12]). There, it is explicitly shown that the tadpole diagrams for $z_C$ are canceled through the process of inversion. According to our rule, the tadpole diagrams in Ref. [11] must vanish because they necessarily lead to instantaneous-2PR configurations.

IV. DISCUSSIONS

We have presented a systematic method to calculate an expectation value $Q(t)$ of some physical quantity $\hat{Q}$ as a functional of the WDF $z$. Using the propagator $-G^{(0)}(\epsilon_q)$, which has a form of the GKB ansatz, the precise expression of $Q(z)$ is obtained by eliminating the instantaneous-2PR configurations from the calculations. This is due to a restriction which must be taken into account in the course of the perturbative calculation: the integration over the microscopic field variable must be carried out in a way so that the value of the WDF is fixed. Together with the QKE, which can also be expressed in a compact form by the use of instantaneous-2PI property (cf. (47)), this method provides us with a complete framework for the quantum kinetic theory.

As pointed out in Ref. [11], the method presented here can straightforwardly be used in the GKB formalism. What we have used for the propagator is a GKB ansatz with the free-particle approximation of the spectral function. The GKB ansatz is defined for a more general form of the spectral function, which implies a corresponding renormalization of the free part of the Lagrangian. Even using more generic form of the spectral function, our method is applicable if the conditions (3)-(41) are held with the replacement of the free-particle spectral function $g^{A}(t,s)$ by renormalized one $a(t,s)$. (These conditions are nothing but the semi-group property discussed in Ref. [11].) Other parts of the proof are based on the retarded or advanced character of the propagators which is not affected by the use of generic spectral function. Thus, even in the generic GKB formalism, where the diagrammatic rule may be different due to the renormalization, the instantaneous-2PR configuration can be eliminated if the semi-group property is held for the GKB ansatz.

Note that our method is not valid for the time correlation function of $\hat{Q}$ such as $\langle \hat{Q}(t)\hat{Q}(s) \rangle$ because we have used the condition that the external point expressing $Q(\psi_c)$ appears only once in the diagram. For the calculation of the time correlation function of the composite operator, some of the instantaneous-2PR configuration may not be canceled.

ACKNOWLEDGEMENTS

The author is very grateful to Prof. R. Fukuda for helpful comments.

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