Topology-controlled spectra of imaginary cubic oscillators in the large–ℓ approach

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Abstract
For quantum (quasi)particles living on complex toboggan-shaped curves which spread over \( N \) Riemann sheets the approximate evaluation of topology-controlled bound-state energies is shown feasible. In a cubic-oscillator model the low-lying spectrum is shown decreasing with winding number \( N \).
1 Introduction

In one of the most up-to-date reviews of the so called $\mathcal{PT}$–symmetric quantum mechanics [1] the existence has been emphasized of multiple connections between certain less usual differential Schrödinger equations for bound states and a number of important integrable models in statistical physics and/or in conformal quantum field theory [2]. This correspondence offers one of explanations of the current growth of interest in many apparently exotic bound-state problems

$$\left[ -\frac{d^2}{dq^2} + V_{\text{eff}}(q) \right] \varphi_n(q) = E_n \varphi_n(q) \quad (1)$$

where one considers, say [3], the three-parametric family of complex interactions

$$V_{\text{eff}}(q) = q^2 (iq)^{2M-2} - \alpha (iq)^{M-1} + \frac{\ell(\ell+1)}{q^2}, \quad M > 1. \quad (2)$$

Apparently, this manifestly non-Hermitian model can hardly find any direct applicability in quantum phenomenology. Fortunately, this first impression proved wrong. The relevance of similar non-selfadjoint models has recently been revealed not only in quantum mechanics (recall, e.g., the so called interacting boson models in nuclear physics [4]) but also in certain quantum field theories [5] and even in non-quantum nonlinear optics [6] (cf. also the other recent applications of non-Hermitian Hamiltonians collected in review papers [7, 8, 9] and in proceedings [10]).

Even though the study of similar models is quite common in mathematics [11], virtually all of the above-mentioned dedicated reviews of their role in physics only appeared very recently. Indeed, the manifestly non-selfadjoint problems of the form of Eq. (1) + (2) were usually considered “unphysical”. Their studies only appeared in the purely methodical context of perturbation theory where, typically (cf., e.g., [12]), differential Eq. (1) + (2) with $M < 2$ has been assumed integrated along the straight real line or along its shifted, complexified version

$$q = s - i\varepsilon \equiv q^{(0)}(s), \quad s \in (-\infty, \infty). \quad (3)$$

A decisive progress has been achieved when, in Ref. [3], the reality of the spectra generated by potentials (2) has been given a rather nontrivial proof requiring just the following very elementary sufficient condition

$$M + 1 + |2\ell + 1| > \alpha. \quad (4)$$

One appreciates, in particular, that for any coupling $\alpha$ and dominant exponent $2M$ this reality of the energies (i.e., in principle, their observability) will always be guaranteed once we select a sufficiently large strength $\ell \gg 1$ of the singularity.

The latter observation attracted our attention for two reasons. Firstly, we kept in mind that the presence of the singularity in the potential may imply that the individual wave functions $\varphi_n(q)$ become multsheeted when treated as analytically continued functions of complex “coordinate” $q$ [13]. In parallel, we imagined that the condition of physical acceptability (4) coincides with the condition of mathematical consistency of the so called $1/\ell$ perturbation expansions (for details and further references see Ref. [14] or Appendix A below).
In what follows we intend to make full use of the latter unexpected coincidence of the physical and mathematical appeal of the “strongly spiked”, $\ell \gg 1$ versions of potentials (2). An immediate motivation of our present return to the related Schrödinger Eq. (1) arose from Refs. [13, 15]. There, the topologically trivial integration path (3) has been replaced by the complex curves which were allowed to encircle the singularities of the potentials (cf., e.g., Refs. [16] and [17] or Sec. 2 below for a compact introduction to this new possibility of quantum model-building).

Unfortunately, even in the simplest models (1) the tobogganic bound-state problems were found extraordinarily hard to solve numerically [18, 19]. For this reason, no sufficiently reliable illustration of a tobogganic spectrum is at our disposal at present. At the same time, the above-mentioned tests of applicability of $1/\ell$ perturbation expansions to non-tobogganic non-Hermitian models [14] may be perceived as a source of new optimism and as a decisive encouragement of our present project of an approximative evaluation of the tobogganic low-lying spectra.

Our results will solely involve tobogganic versions of the concrete toy model (2) possessing the single singularity in the origin $q = 0$ and considered just at a fixed sample exponent $M = 3/2$ and at the simplest coupling $\alpha = 0$. Under these assumptions our main attention will be paid to our Schrödinger Eq. (1) integrated along the whole family of nontrivial, “tobogganic” curves denoted by the symbol $q^{(N)}(s)$ and classified, in an exhaustive manner, by their winding number $N = 0, 1, 2, \ldots$ (cf. their sample in Figure 1). By definition (cf. Sections 2 and 3 below) these smooth complex curves will interconnect an $N-$plet of sheets of the Riemann surface $R$ supporting the analytic, multivalued wave functions $\phi_n(q)$.

![Figure 1: The complex straight contour $q^{(0)}(s)$ of eq. (3) (thin line) and its first two winding descendants $q^{(1)}(s)$ (medium curve) and $q^{(2)}(s)$ (thick curve).](image)

We shall reflect the dominance of the centrifugal repulsion near the origin as an encouragement of the use of the approximative large–$\ell$ method of the solution of Eq. (1). The essence of this method will be explained in Section 4. For our $M = 3/2$ (i.e., imaginary cubic) quantum toboggan the application of this method will finally be described in Section 5 complemented by a few comments and a brief summary in Section 6.
2 Topologically nontrivial integration paths

Several important theoretical reasons of the contemporary growing interest in non-tobogganic as well as tobogganic models sampled by Eqs. (1) + (2) + (3) deserve to be re-emphasized. One of the most typical ones lies in the emergence of several unexpected conceptual links between the ODE language and its correspondence to the physical theories of integrable statistical and field-theoretical models. In this sense, in a way mentioned in [1, 3], the integrable SUSY point of the sine-Gordon model has been found related to the value of the exponent $M = 1/2$ in (2). Similarly, the partnership with the YangLee model emerged at $M = 3/2$, etc (cf. section 5.6 in [7] for a “full dictionary”).

One should also mention the importance of the flow of inspiration in the opposite direction. Indeed, certain very specific questions asked in the context of conformal field theory [2] brought, as their direct consequence, the immediate turn of attention from the most elementary and regular $\ell = 0$ versions of Schrödinger Eq. (2) to its specific singular generalizations. A concrete example of the emerging family of relevant generic potentials with multiple centrifugal-type singularities may be found, e.g., as Eq. (1) in Ref. [2].

Naturally, after similar generalizations and incorporation of singularities even the ODE mathematics becomes perceivably less transparent (cf., e.g., the classification problem solved for the first nontrivial double-singularity tobogganic model in [16]). Indeed, under the presence of singularities in the potential $V_{\text{eff}}(q)$ the wave functions $\varphi_n(q)$ generated by the corresponding Schrödinger Eq. (1) get analytically continued to a topologically nontrivial, punctured and multisheeted Riemann surface $\mathcal{R}$. Even the description of this surface itself (characterized by its so called monodromy group [20]) strongly depends on the details of the form of the potential.

In the majority of the above-mentioned applications, the technical obstacle emerging with nontrivial monodromies is most often circumvented via an ad hoc constraint which guarantees a “trivialization” of the topological structure of $\mathcal{R}$ and of its monodromy [21]. An explicit form of the “trivialization” constraint is sampled, e.g., by Eq. (3) in Ref. [2] or by Eq. (6.31) in Ref. [7]. In this context, new an interesting questions emerge, of course, when one admits a nontrivial monodromy [13, 15].

The latter step of generalization is now in the center of our interest. In general, it would induce a rich menu of curves supporting the wavefunctions and connecting, in general, different sheets of the surface $\mathcal{R}$ [16]. In the spirit of an older letter [22], different spectrum may be attributed to each attachment of a new tobogganic curve to the same differential Schrödinger equation. In the two-singularity case, for illustration, the menu of these curves has been made available via a specific interactive demonstration using MATHEMATICA by Novotný [23].

In this context we feel curious what happens to the spectrum after the changes of topology in the most elementary though still nontrivial single-singularity case.

3 Paths encircling single singularity

For the sake of definiteness we may define our curves of integration, say, by formula Nr. (10) of Ref. [17],

$$q(s) = q^{(N)}(s) = -i \left[i(s + i\varepsilon)^{2N+1}\right], \quad s \in (-\infty, \infty).$$

(5)
Without any change of the spectrum these curves may be also smoothly deformed of course (cf. Figures 5 or 6 in Ref. [1] for illustration). In this case we only have to preserve both the left and right asymptotic parts of the curve \( q^{(N)}(s) \) unchanged (a detailed discussion of this point may be also found in [1]).

From the point of view of physics (discussed in more detail, say, in [9, 17]) one of the most important features of the complexified quantum systems represented by Eqs. (1) + (2) on a line (3) (with \( \varepsilon \neq 0 \)) or on a curve (5) can be seen in the manifest loss of observability of their “coordinate” \( q \notin \mathbb{R} \). In principle, the only measurable quantity remains to be the bound-state energy spectrum. This clarifies the main purpose of considering our specific class of a non-Hermitian models at nonvanishing winding numbers \( N = 1, 2, \ldots \): We may expect that the new free parameter \( N \) opens a new, potential-independent way of the modification of the spectrum needed, say, during its better fit to some given experimental data (of course, just hypothetical data in our present methodical study).

In this broad framework the purpose of our present paper lies in finding a sample of quantitative analysis of the \( N \)–dependence of the energies. The two main sources of inspiration of such a study may be seen in the existence of a gap between the older, non-tobogganic results of Refs. [24] [reproduced also in Figure 8 of review [1] and using, incidentally, just \( \ell = \alpha = 0 \) in Eq.(2)] and the Bíla’s preliminary tobogganic numerical results which were presented in his very recent paper [19] and in his dissertation [25].

On the technical level the main ingredient of our present calculations will lie in the rectification transformation which will replace our imaginary cubic differential Schrödinger equation defined along one of the winding trajectories (5) by another, equivalent differential Schrödinger-type equation obtained via a suitable change of variables.

In a more explicit specification of such a transformation we shall

- distinguish between the non-winding curves where \( N = 0 \) [abbreviating \( q^{(0)}(s) \equiv y(s) \) in Eq. (3)] and the genuine winding-paths \( q^{(N)}(s) \) with \( N = 1, 2, \ldots \) [use Eq. (4) and abbreviate \( q^{(N)}(s) \equiv z(s) \) at any \( N \geq 1 \)],

- start from the initial cubic and tobogganic bound-state problem

\[
\left[-\frac{d^2}{dz^2} + \frac{i(\ell + 1)}{z^2} + i z^3\right] \varphi_n^{[N]}(z) = E_n^{[N]} \varphi_n^{[N]}(z),
\]

- employ the change of variables \( z = -i (iy)^{2N+1} \) or, in opposite direction, \( iy = (iz)^\alpha \) where \( \alpha = 1/(2N + 1) \) is rational, i.e., where several (logarithmic) Riemann sheets of the original variable \( z \) are mapped into the single (cut) complex plane of the new variable \( y \).

The detailed realization of such a project may be summarized as the direct use of the elementary substitution rules

\[
dz = (-1)^N (2N + 1) y^{2N} \, dy,
\]

\[
\frac{d}{dz} = \frac{(-1)^N}{(2N + 1) y^{2N}} \frac{d}{dy} \quad (7)
\]

\[
\varphi_n^{[N]}(z) = y^N \psi_n(y), \quad L = (2N + 1) \left( \ell + \frac{1}{2} \right) - \frac{1}{2} \quad (8)
\]
under which our original ordinary differential Eq. (6) becomes transformed into the following Sturm-Schrödinger [26] differential equation

$$\left[ -\frac{d^2}{dy^2} + \frac{L(L+1)}{y^2} + i (-1)^N (2N+1)^2 y^{10N+3} \right] \psi_n(y) = (2N+1)^2 y^{4N} E_n^{[N]} \psi_n(y).$$  

(9)

This change of variables realizes the one-to-one correspondence not only between the two respective (in general, logarithmic) Riemann surfaces of $z$ and $y$ but also between the original integration path (5) (living on the original, tobogganic $z$–surface) and the new integration path for $y$ which, by its present specification, coincides with the straight-line formula (3).

During the search for the energies $E_n^{[N]}$ the new, “rectified”, generalized eigenvalue problem (9) + (3) must still be addressed by purely numerical means in general. Still, the key merit of its choice is that one can employ some more or less standard methods. In particular, we intend to use here an appropriately adapted form of the large–$\ell$ approximation method of Ref. [14], to be first briefly summarized in Sec. 4 and subsequently applied to our problem in Sec. 5.

4 The large–$\ell$ approximation technique

The key ingredients of the approximations where the quantity $1/\ell$ remains small may be most easily illustrated via the non-tobogganic, single-sheeted version of our imaginary cubic oscillator, integrated along the straight-line path $q = q^{(0)}(s)$ of Eq. (3),

$$\left[ -\frac{d^2}{dq^2} + \frac{\ell(\ell+1)}{q^2} + iq^3 \right] \varphi_n^{[0]}(q) = E_n^{[0]} \varphi_n^{[0]}(q).$$  

(10)

The main idea of the approximative search for eigenvalues $E_n^{[0]}$ is based on the invariance of the spectrum during any parallel shift of the path $q^{(0)}(s)$ [1]. Thus, we are allowed to choose the free parameter $\varepsilon = \varepsilon_j$ in Eq. (3) in such a way that one and only one of the roots $q = Q = Q_j$, $j = 1, 2, \ldots, 5$ of the auxiliary equation $\partial_Q V_{\text{eff}}(Q) = 0$ will lie on the corresponding specific, $j$–dependent exceptional line of our “coordinates” in Eq. (11), $q^{(0)}(s) \to q_j^{(0)}(s)$.

In the next step we may rewrite our auxiliary equation in its fully explicit form $2\ell(\ell+1) = 3iQ^5$ and notice that in the vicinity of any of the quintuplet of its roots $Q = Q_j$, $j = 1, 2, \ldots, 5$ we may Taylor-expand our effective potential in terms of the new, shifted complex variable $\xi = q - Q$,

$$V_{\text{eff}}(q) = V_{\text{eff}}(Q) + \frac{1}{2} V''_{\text{eff}}(q) \xi^2 + \frac{1}{6} V'''_{\text{eff}}(q) \xi^3 + \ldots.$$  

(11)

Next, we have to take into account the available explicit formulae for our $V_{\text{eff}}$ and for all the roots

$$Q_j = -i \tau \exp \left( \frac{2i\pi(j-1)}{5} \right), \quad \tau = \left| \left( \frac{2}{3} \ell(\ell+1) \right)^{1/5} \right|, \quad j = 1, 2, \ldots, 5.$$  

(12)

Due to our assumption of the smallness of the quantities $1/\ell$ and/or $1/\tau$ we can choose $j = 1$ and simplify our Taylor series (11) to its much more explicit form

$$V_{\text{eff}}(q) = -\frac{5}{2} \tau^3 + \frac{15}{2} \frac{\tau}{\xi^2} - 5i \xi^3 + \mathcal{O}(\tau^{-1}).$$  

(13)
This formula shows that up to the negligible error term our potential coincides with the deep and real harmonic-oscillator well, complemented just by a purely imaginary cubic perturbation.

In the spirit of perturbation theory we shall accept our tentative choice of \( j = 1 \) in what follows. The more general discussion of difficulties emerging in connection with the use of the other roots with \( j > 1 \) may be found in Refs. [14] and [27]. As long as we now have the new independent variable \( \xi = q_1^{(0)}(s) - Q_1 \equiv s \) which is real, we may insert the Taylor series \([13]\) in our Schrödinger Eq. \([10]\) and rewrite it in its leading-order anharmonic-oscillator reincarnation

\[
\left[ -\frac{d^2}{d\xi^2} - \frac{5}{2} \tau^3 + \frac{15 \tau}{2} \xi^2 - 5 i \xi^3 + \mathcal{O}(\tau^{-1}) \right] \varphi^{[0]}(-i\tau + \xi) = E_n^{[0]} \varphi^{[0]}(-i\tau + \xi). \tag{14}
\]

In the final step of our considerations we introduce another very small auxiliary quantity \( \sigma = 1/\tau^{1/4} \) and rescale the coordinate \( \xi \to \sigma \xi \). This yields the final version of our approximate eigenvalue problem,

\[
\left[ -\frac{d^2}{d\xi^2} - \frac{5}{2} \sigma^{10} + \frac{15}{2} \xi^2 - 5 \sigma^5 i \xi^3 + \mathcal{O}(\sigma^6) \right] \varphi^{[0]}(-i\tau + \xi) = \sigma^2 E_n^{[0]} \varphi^{[0]}(-i\tau + \xi). \tag{15}
\]

As long as \( \sigma \ll 1 \) is very small, the leading-order version of this equation has the purely harmonic-oscillator form

\[
\left[ -\frac{d^2}{d\xi^2} - \frac{5}{2} \sigma^{10} + \frac{15}{2} \xi^2 + \mathcal{O}(\sigma^5) \right] \varphi^{[0]}(-i\tau + \xi) = \sigma^2 E_n^{[0]} \varphi^{[0]}(-i\tau + \xi) \quad \tag{16}
\]

which is exactly solvable.

As long as we are interested here just in the very global features of our sample of tobogganic spectrum (like, e.g., the leading-order winding-number-dependence of the removal of its degeneracy, etc), we shall skip here the construction of higher-order corrections completely. Interested readers may find an instructive sample of such calculations in Ref. [28]. In this case we may already conclude that the low-lying spectrum evaluates to the following formula

\[
E_n^{[0]} = -\frac{5 \tau^3}{2} + \sqrt{\frac{15 \tau}{2}} (2n + 1) + \mathcal{O}(\tau^{-3/4}), \quad n = 1, 2, \ldots
\]

with an asymptotically vanishing error term and predicting the perceivably negative and approximately equidistant leading-order spectrum.

## 5 The large–\( \ell \) approximants at tobogganic \( N > 0 \)

On the basis of our preceding non-tobogganic \( N = 0 \) results we are now prepared to move to the genuine tobogganic models with windings \( N = 1, 2, \ldots \) entering our rectified Sturm-Schrödinger differential Eq. \([9]\). Immediately, we may make use of the fact that one of the most convenient small parameters \( \rho = 1/(\ell + 1/2)^2 \) remains \( N \)--independent. This allows us to keep the \( N \)--dependence carried solely by the adapted shifts \( \varepsilon = \varepsilon^{(N)} \) in the straight lines \( y(s) \) of Eq. \([3]\).

In the first step we have to Taylor-expand our effective potential again,

\[
V_{\text{eff}}[y(x)] = \frac{L(L+1)}{y^2} + i (-1)^N (2N+1)^2 y^{10N+3}.
\]
In contrast to section 4 an amended notation must be introduced now since the original tobogganic (i.e., multisheeted) $z$–reference complex points $Q_j = q^{(N)}_j(0)$ differ from their transformed, Sturm-Schrödinger maps or partners $T_j = q^{(0)}_j(0)$. The latter points lie on the rectified, straight-line contours in the simpler complex $y$–plane with, possibly, just a single cut directed upwards.

The positions of the latter reference points will again be determined by the derivative-vanishing condition $\partial_y V_{\text{eff}}(y) = 0$. Thus, for our particular dynamical model each $L$–dependent root $y = T = T_j(L)$ of this equation will coincide with one of the $10N + 5$ distinct roots of the following elementary algebraic equation

$$2L(L + 1) = (2N + 1)^2(10N + 3) T^{10N+5}.$$ 

All of these roots form the vertices of a regular $(10N + 5)$–angle in complex plane. At $N = 0$ we would just return to the pentagon which has been obtained in the non-tobogganic imaginary cubic model above. At any other $N \geq 0$ we shall abbreviate

$$\tau = \tau^{(N)} = \left| \frac{2 L (L + 1)}{(2N + 1)^2(10N + 3)} \right|^{1/(10N+5)}$$

and set

$$T_1 = -i\tau, \quad T_j = T_{j-1} e^{2i\pi/(10N+5)}, \quad j = 2, 3, \ldots, 10N + 5.$$ 

In a way which parallels our above discussion of the model with $N = 0$ we use just the same exceptional root $T_1 = -i\tau$ giving the maximal size of the shift $\varepsilon = \tau$ in Eq. 3. This choice also offers the single isolated real minimum of our effective potential along this line so that we may work again with the adapted real independent variable $\xi = s$ everywhere in what follows.

After a somewhat tedious calculation one obtains the Taylor-series asymptotic approximation of $V_{\text{eff}}(y)$ at any $N$,

$$V_{\text{eff}}(y) = -\frac{1}{2} (2N + 1)^2(10N + 5) \tau^{10N+3} + \omega^{2} \tau^{10N+1} \xi^2 + \mu(N) \tau^{10N} \xi^3 + O(\tau^{10N-1})$$

with

$$\omega^{2} = (2N + 1) \cdot \sqrt{\frac{(10N + 3)(10N + 5)}{2}}.$$ 

Similar closed formulae can also be derived for $\mu(N)$, etc. This means that our rectified Schrödinger equation will acquire the following approximate $N$–dependent asymptotic form

$$\left[ - \frac{d^2}{d\xi^2} + \omega^{2} \tau^{10N+1} \xi^2 + \mu(N) \tau^{10N} \xi^3 + \ldots \right] \psi_n(-i\tau + \xi) =$$

$$= \left\{ E_n^{(N)} (2N + 1)^2 \left[ \tau^{4N} + \ldots \right] + \frac{1}{2} (2N + 1)^2(10N + 5) \tau^{10N+3} \right\} \psi_n(-i\tau + \xi).$$

In a preliminary step of analysis we notice that the anharmonic-oscillator corrections proportional to the second coupling $\mu(N)$ will remain asymptotically negligible and that they may be ignored. Thus, in the asymptotic regime where $\tau \gg 1$ we shall apply the same considerations as tested in Appendix A.
In an explicit error analysis we rescale $\xi \rightarrow \sigma \xi$ where $\sigma = \tau - (10N + 1)/4$. This leads to the equation

$$\left[ -\frac{d^2}{d\xi^2} - \frac{5}{2\sigma^{10}} + \frac{15}{2} \xi^2 - 5 \sigma^5 i \xi^3 + \mathcal{O}(\sigma^6) \right] \varphi^{[N]}(-i\tau + \xi) = \sigma^2 E^{[N]}_n \varphi^{[N]}(-i\tau + \xi)$$

which shows that although the asymptotic order of magnitude of the energies remains $n-$ and $N-$independent, $E^{[N]}_n \propto \mathcal{O}(\ell^{6/5})$, there is in fact no degeneracy in $N$ and that also the degeneracy in $n$ becomes removed by the next-order correction (cf. illustrative Figure 2).

In a more explicit asymptotic study of our model we have to keep in mind that our large auxiliary parameter $\tau$ also changes with the winding number $N$, $\tau = \tau(N)$. The source of this dependence does not lie only in the manifest presence of $N$ in formula (17) but also in the $N-$dependence of the upper-case parameter $L$. This is a mere abbreviation of the expression $L = L(N, \ell) = (2N + 1)(\ell + 1/2) - 1/2$ in which one recognizes the presence of the small variable $\rho = 1/(\ell + 1/2)^2$ used in Figures 2 or 3. Thus, the physics hidden behind the asymptotics in $L$ can be re-read as a combined centrifugal-repulsion effect where both the coupling strength $1/\rho$ and the number of rotations $N$ appear to act in a coherent manner.

At a fixed topological characteristic $N$ of the tobogganic quantum system in question our asymptotically simplified equations indicate that the level-splitting will predominantly be mediated by the harmonic-oscillator term $\xi^2$ in Eq. (19). Thus, the leading-order equidistance of the levels (cf. Figure 3) will only be broken, in the next-order approximation, by the cubic term $\mathcal{O}(\xi^3)$. An explicit estimate of the size of the latter correction is still feasible. From Eq. (19) one immediately deduces that the anharmonic-oscillator part of our potential will contribute by the corrections of the order $\mathcal{O}(\tau^{-(10N+5)/4})$. They may be perceived, say, as a higher-order modification of the coefficient $\omega^{[N]}$.

We should add that another specific feature of the tobogganic problem emerges...
in connection with the nontriviality of the right-hand-side “weight” term

\[ W(y) = (2N + 1)^2 y^{4N} \]

entering Eq. (9) at \( N \geq 1 \). Fortunately, the scaling argument can be re-applied here to give the estimate of the first non-constant \( \xi \)-dependent correction to \( W(T) \) in the form \( \mathcal{O}(\tau^{-(14N+7)/4}) \). This correction is even smaller than the contribution of the anharmonic force so that it can either be neglected (on the present level of precision) or, whenever needed, calculated and incorporated, say, by the appropriately adapted systematic perturbation techniques as described in Ref. [14].

We may now return to Eq. (18) and summarize our observations in the closed formula for the energies,

\[
E_{n}^{[N]} = -\frac{10N + 5}{2} \tau^{6N+3} + \frac{2n + 1}{2N + 1} \cdot \sqrt{\frac{(10N + 3)(10N + 5)}{2}} \tau^{N+1/2} + \mathcal{O}(\tau^{-(6N+3)/4}) \tag{20}
\]
giving the low-lying spectrum at \( n = 1, 2, \ldots \). This is our final result. Once we recollect that \( \tau \propto \ell^{2/(10N+5)} \) this formula reconfirms our above estimate that \( E_{n}^{[N]} \propto \ell^{6/5} \), i.e., that in the dominant order the power-law exponent of \( \ell \) does not vary with \( N \). In this setting the formula specifies the numerical factor and clarifies the mechanism of the removal of the degeneracy between topologically nonequivalent toboggans.

In the subdominant, next-order level of precision the next numerical coefficient in the spectral formula will still retain an explicit form. The dominant part of the distance \( G^{(N)} = E_{n+1}^{[N]} - E_{n}^{[N]} \) between levels appears to have the power-law form with another constant exponent, \( G^{(N)} \propto \ell^{1/5} \). This is well illustrated by Figure 3 where we see how the differences between constants \( \lim_{\ell \to \infty} G^{(N)} / \ell^{1/5} \) remove the topology-independence of the gaps, i.e., their leading-order degeneracy in \( N \).

This effect slightly weakens with the growth of winding number \( N \geq 1 \). It is worth adding that the function \( G^{(N)} / \ell^{1/5} \) of \( N \) happens to have its maximum at \( N = 1/2 \).
so that the observed monotonicity in $N$ does not apply to the “anomalous”, non-winding curve $G^{(0)}/\ell^{1/5}$ which, incidentally, lies somewhere in between $G^{(2)}/\ell^{1/5}$ and $G^{(3)}/\ell^{1/5}$.

6 Summary

Our present paper offers the first quantitative confirmation of the hypothesis that the spectrum of quantum toboggans can be well controlled not only by the potential and, independently, by the suitably complexified asymptotic boundary conditions but, in addition, also by some suitable topological, non-asymptotic characteristics of the curves of the complexified “coordinates”. These characteristics were shown to lead to the potentially useful changes in the spectra.

For the sake of brevity we analyzed only the simplest nontrivial quantum toboggans which are characterized by the presence of a single, isolated branch point singularity of $\psi(q)$ in the origin. In this case the non-equivalent toboggans can be classified by the mere integer winding number $N$. Of course, this does not mean that the analysis of the spectrum is easy. For illustration we may mention the recent diploma work [18] where, for several potentials, the stability and applicability of a few most common numerical methods has been shown restricted just to the non-winding dynamical regime with $N = 0$.

More stable though still preliminary numerical results are available in Ref. [19] where the first encouraging samples of the real spectra of toboggans were presented using $N \leq 2$. Although this analysis remained restricted just to the smallest exponents $M \in (1,3)$, an empirical confirmation of the reality of the energies at $M = 2$ has been achieved. This seems to be in correspondence with Ref. [20] where the exceptional role of integer exponents $M$ in certain non-tobogganic models has been revealed.

Intuitively one could expect that in the tobogganic cases the domain of a robust reality of the energies will move to the higher exponents $M \geq M^{(N)}$. Indeed, at $N = 0$, the explicit numerical confirmation of the rightward shift of the boundary of the interval of the acceptable exponents can be found in Refs. [21]. Unfortunately, our present approach is unsuitable for an analysis of similar phenomena since, e.g., Eq. [11] controls the reality of the energies just via the insensitive sum of a given $M$ with our very large $|2\ell + 1|$.

On the side of the advantages of our present large–$\ell$ recipe we may mention the transparency of the determination of the unique and optimal line (3) of complex coordinates. In the leading order the use of this path reduces the dominant part of the effective interactions $V_{eff}(q)$ to the solvable harmonic oscillator. In addition, whenever needed, the series of subdominant corrections can be evaluated in recurrent manner. In this sense the message of our present paper is non-numerical, showing that the low-lying levels of our toboggans can be given by closed formula and that they appear to vary sufficiently strongly with the variation of the topological winding number $N$.

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Appendix A. The large-$\ell$ approximation in an exactly solvable model

Let us consider, in a test of the method, the exactly solvable example of the linear harmonic oscillator

\[ V_{\text{eff}}^{(HO)}(q(s)) = \frac{\ell(\ell+1)}{(s-i\epsilon)^2} + \omega^2(s-i\epsilon)^2 \]  

(21)

in the strongly spiked dynamical regime, $\ell \gg 1$. In a preparatory step of the analysis at large $\ell$ let us Taylor-expand this effective potential near its (complex) stationary point $q = Q^{(HO)}$ such that

\[ \partial_q V_{\text{eff}}^{(HO)}(q) = 0. \]

For our model every such a point is determined by the elementary algebraic equation

\[ \ell(\ell+1) = \omega^2 T^4 \]

with an easy solution,

\[ Q_j = Q_j = (\tau) j, \quad j = 1, 2, 3, 4, \quad \tau = \sqrt{\frac{1}{\omega} \sqrt{\ell(\ell+1)}}. \]  

(22)

All of these four candidates for the (in general, complex) point where $V_{\text{eff}}$ attains its (local) minimum also satisfy the sufficient condition since the real part of the second derivative of the effective potential remains $Q-$independent and positive at all $q = Q$ since $V_{\text{eff}}''(Q) = 8\omega^2$.

In the next step let us define the distances $\xi = q - Q$ from the minima and generate the Taylor series

\[ V_{\text{eff}}(q) = V_{\text{eff}}(Q) + \frac{1}{2} V_{\text{eff}}''(Q) \xi^2 + \frac{1}{6} V_{\text{eff}}'''(Q) \xi^3 + \ldots. \]

For our toy model this gives

\[ V_{\text{eff}}(q) = 2\omega^2 Q^2 + 4\omega^2 \xi^2 + \frac{4}{Q} \omega^2 \xi^3 + O(Q^{-2}). \]

Under the assumption that $\ell$ is large, i.e., $|Q| \gg 1$, the insertion of this truncated series would transform our bound-state problem (1) into its approximatively isospectral partner.

The unrestricted freedom of the choice of the parameter $\epsilon > 0$ in Eq. (3) enables us to set $\epsilon = \tau$ and to let the line $q^{(0)}(s)$ cross the point $Q_1 = -i\tau$. This is equivalent to the replacement of our initial Schrödinger equation (1), at all the sufficiently large $|\ell| \gg 1$ and/or $\tau \gg 1$, by the new equation

\[ \left[ -\frac{d^2}{d\xi^2} - 2\omega^2 \tau^2 + 4\omega^2 \xi^2 + O(\tau^{-1}) \right] \varphi_n(-i\tau + \xi) = E_n(\omega) \varphi_n(-i\tau + \xi). \]  

(23)

In the leading-order approximation this replacement leads to the new closed formula

\[ E_n(\omega) = -\omega \sqrt{2(\ell+1)^2 - 1} + 2\omega (2n + 1) + O(\ell^{-2/3}) , \quad n = 1, 2, \ldots. \]

It is rather amazing to see how well this $\ell \gg 1$ estimate reproduces the exact low-lying spectrum

\[ E_n(\omega) = \omega (4n + 1 - 2\ell) , \quad n = 1, 2, \ldots, n_{\text{max}}, \quad n_{\text{max}} < \ell + 1/2 \]

as derived, for our toy model (21), in Ref. [30].