ON SLIM DOUBLE LIE GROUPOIDS

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ABSTRACT. We prove that every slim double Lie groupoid with proper core action is completely determined by a factorization of a certain canonically defined "diagonal" Lie groupoid.

INTRODUCTION

A double groupoid is a set $B$ provided with two different but compatible groupoid structures. It is useful to represent the elements of $B$ as boxes that merge horizontally or vertically according to the groupoid multiplication into consideration. The vertical (respectively horizontal) sides of a box belong to another groupoid $V$ (resp. $H$). A double groupoid is slim if any box is determined by its four sides. The notion of double groupoids was introduced by Ehresmann [E63], and later studied in [B04, BJ04, BM92, BS76] and references therein.

The notion of double Lie groupoid was defined and investigated by K. Mackenzie [M92, M00]; see also [P77, M99, LW89] for applications to differential and Poisson geometry. In particular the question of the classification of double Lie groupoids was raised in [M92], see also [BM92]. In the latter article, a complete answer was given in the restricted case of locally trivial double Lie groupoids. More recently, a description in two stages of discrete double groupoids was given in [AN06b]. To state them, let us recall that a diagram over a pair of groupoids $V$ and $H$ is a triple $(D, j, i)$ where $D$ is a groupoid and $i : H \to D$, $j : V \to D$ are morphisms of groupoids (over a fixed set of points). The stages in [AN06b] are:

(a) Any double groupoid is an extension of slime double groupoid (its frame) by an abelian group bundle.

(b) The category of slim double groupoids, with fixed vertical and horizontal groupoids $V$ and $H$, satisfying the filling condition, is equivalent to the category of diagrams over $V'$ and $H$.

In this paper, we extend stage (b) to the setting of double Lie groupoids. In this context, instead of the filling condition, one requires that the double source map is a surjective submersion [M92]. As one may naturally expect, there are some topological and geometrical ingredients in our main Theorem 3.12 which says:

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The category of slim double Lie groupoids, with fixed vertical and horizontal Lie groupoids $V$ and $H$, and proper core action, is equivalent to the category of diagrams of Lie groupoids $(D, j, i)$ such that the maps $j$ and $i$ are transversal at the identities.

Our proof of this theorem relies on [AN06b, Theorem 2.8] and some topological and differentiable considerations such as properness of the core action on one side and a transversality condition on the morphisms involved in a diagram of Lie groupoids on the other. It is also possible to adjust stage (a) to the context of double Lie groupoids but we postpone the investigations to a future paper.

1. Preliminaries on Lie groupoids and double Lie groupoids

We denote a groupoid in the form $G \xrightarrow{s} P$, where $s$ and $e$ stand for ‘source’ and ‘end’ respectively; and the identity map is $id : P \rightarrow G$. Recall that a groupoid $G \xrightarrow{s} P$ is a Lie groupoid [M05], if $P$ and $G$ are smooth manifolds, $s$ and $e$ are surjective submersions and the other structural maps are smooth. The anchor of $G$ is the map $\chi : G \rightarrow P \times P$ given by $\chi(g) = (s(g), e(g))$.

We recall the following well known definition.

**Definition 1.1.** A left action of a groupoid $G \rightrightarrows P$ along a map $\varepsilon : N \rightarrow P$ is given by a map $G_{\varepsilon} \times_{\varepsilon} N \rightarrow N$, denoted by $(g, n) \mapsto gn$, which satisfies the following identities:

$\varepsilon(hy) = s(h), \quad \text{id}(\varepsilon(y)) y = y, \quad (gh)y = g(hy),$

for all $g, h \in G$ and $y \in N$ such that $e(g) = s(h)$ and $e(h) = \varepsilon(y)$.

The transformation or action groupoid $G \times_{\varepsilon} N \Rightarrow N$, associated with such an action, is the groupoid with set of arrows $G_{\varepsilon} \times_{\varepsilon} N$ and base $N$. The source and target maps are

$s' : G \times_{\varepsilon} N \rightarrow N, \quad \text{given by} \quad s'(g, n) = gn \quad \text{and} \quad e'(g, n) = n,$

respectively, and composition $(g, n)(h, m) = (gh, m)$.

**Remark 1.2.** If in the above definition $G \rightrightarrows P$ is a Lie groupoid, $N$ a smooth manifold and $\varepsilon : N \rightarrow P$ a smooth map, we define a left action by the same properties and the only extra requirement is the smoothness of the map that gives the action. The resulting action groupoid is again a Lie groupoid.

We remind now the definition of local bisections on a Lie groupoid.

**Definition 1.3.** Let $G \xrightarrow{s} P$ be a Lie groupoid and let $U \subseteq P$ be an open subset. A local bisection of $G$ on $U$ is a smooth map $\sigma : U \rightarrow G$ which is a section.
of $e$ such that $V := (s \circ \sigma)(U)$ is an open subset of $\mathcal{P}$ and $s \circ \sigma : U \to V$ is a diffeomorphism. Define $g^U = s^{-1}(U)$ and $g_U = e^{-1}(U)$. The local left and right translations induced by $\sigma$ are (respectively) the maps

$$
L_\sigma : g^U \to g^V, \quad g \mapsto \sigma(s(g))g; \quad \text{and}
$$

$$
R_\sigma : g_V \to g_U, \quad g \mapsto g\sigma((s \circ \sigma)^{-1}(e(g))).
$$

(1.1)

For more on bisections see [M05 Section 1.4].

1.1. Double Lie groupoids.

**Definition 1.4** (Ehresmann). A double groupoid is a groupoid object internal to the category of groupoids. That is, a double groupoid consist of a set $\mathcal{B}$ with two groupoid structures with bases $\mathcal{H}$ and $\mathcal{V}$, which are themselves groupoids over a common base $\mathcal{P}$, all subject to the compatibility condition that the structure maps of each structure are morphisms with respect to the other.

It is usual to represent a double groupoid $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ as a diagram of four related groupoids

$$
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{l} & \mathcal{V} \\
\downarrow & & \downarrow \\
\mathcal{H} & \xrightarrow{l} & \mathcal{P}
\end{array}
$$

where $t, b, l, r$ mean ‘top’, ‘bottom’, ‘left’ and ‘right’, respectively. We sketch the main axioms that these groupoids should satisfy and refer e. g. to [AN05 Section 2] and [AN06 Section 1] for a detailed exposition and other conventions.

The elements of $\mathcal{B}$ are called ‘boxes’ and will be denoted by

$$
A = \begin{array}{c}
A \\
\downarrow \\
\downarrow \\
\hline
r(A) \\
\hline
l(A) \\
b(A)
\end{array} \in \mathcal{B}.
$$

Here $r(A), b(A) \in \mathcal{H}$ and $l(A), r(A) \in \mathcal{V}$. The identity maps will be denoted $\text{id} : \mathcal{V} \to \mathcal{B}$ and $\text{id} : \mathcal{H} \to \mathcal{B}$. The product in the groupoid $\mathcal{B}$ with base $\mathcal{V}$ is called horizontal and denoted by $AB$ or $\{AB\}$, for $A, B \in \mathcal{B}$ with $r(A) = l(B)$. The product in the groupoid $\mathcal{B}$ with base $\mathcal{H}$ is called vertical and denoted by $\begin{bmatrix} A \\ B \end{bmatrix}$, for $A, B \in \mathcal{B}$ with $l(A) = t(B)$. This pictorial notation is useful to understand the products in the double structure. For instance, compatibility axioms between the horizontal and vertical products are described by
We omit the letter inside the box if no confusion arises. We also write $A^b$ and $A^v$ to denote the inverse of $A \in B$ with respect to the horizontal and vertical structures of groupoid over $B$ respectively. When one of the sides of a box is an identity, we draw this side as a double edge. For example, if $t(A) = \text{id}_p$, we draw $\begin{array}{c} t \\ \downarrow \\ A \\ \uparrow \\ \text{id}_p \end{array}$ and say that $t(A) \in P$.

**Definition 1.5** (Mackenzie, [M92]). A double groupoid is a *double Lie groupoid* if all four groupoids involved are Lie groupoids and the double source map

$$S : B \to \mathcal{H} \times \mathcal{V}, \quad A \mapsto S(A) = (t(A), l(A)),$$

is a surjective submersion.

For clarity, we shall say that a double groupoid is *discrete* if no Lie structure is present. A discrete double groupoid satisfies the *filling condition* when the double source map defined above is surjective. We refer the reader to [AN06b] for details.

**Definition 1.6** (Brown and Mackenzie, [BM92], [M92]). Let $(B; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a double Lie groupoid. The core groupoid $E(B)$ of $B$ is

$$E(B) = \{ E \in B : t(E), r(E) \in \mathcal{P} \}$$

with $s_E : E(B) \to \mathcal{P}$, $e_E(E) = bl(E)$, $c_E(E) = tr(E)$; identity map given by $\text{id}_p = \Theta_p := \text{id} \circ \text{id}(p)$; multiplication and inverse given by

$$E \circ F := \begin{pmatrix} \text{id}(F) & F \\ E & \text{id}(b(F)) \end{pmatrix}, \quad E^{-1} := (E \text{id}(E)^{-1} \text{id}) = \begin{pmatrix} \text{id}(E)^{-1} & 0 \\ 0 & \text{id}(E)^{-1} \text{id} \end{pmatrix},$$

$E, F \in E(B)$. That is, the elements of $E(B)$ are of the form $E = \begin{array}{c} t \\ \downarrow \\ A \\ \uparrow \\ \text{id}_p \end{array}$; the source gives the bottom-left vertex and the target gives the top-right vertex of the box. Clearly $s_E$ and $e_E$ are surjective submersions. Thus $E(B)$ becomes a Lie groupoid, differentiability conditions being easily verified because $E(B)$ is an embedded submanifold of $B$. 

1.2. **Coarse double groupoid.** Let \( P \) be a set and \( \mathcal{V}, \mathcal{H} \) be groupoids over \( P \). Let \( \Box(\mathcal{V}, \mathcal{H}) \) be the set \( \left( \mathcal{V}_{\times 1} \mathcal{H} \right)_{(t,r)} \times \left( \mathcal{H}_{r \times \mathcal{V}} \right) \); that is, \( \Box(\mathcal{V}, \mathcal{H}) \) is the set of quadruples \( \begin{pmatrix} x & f & g & y \end{pmatrix} \) with \( x, y \in \mathcal{H}, f, g \in \mathcal{V} \) such that

\[
\begin{align*}
l(x) &= t(f), \\
r(x) &= t(g), \\
l(y) &= b(f), \\
r(y) &= b(g).
\end{align*}
\]

If no confusion arises, we shall denote a quadruple as above by a box \( \begin{boxed} x \end{boxed} \). The collection \( \Box(\mathcal{V}, \mathcal{H}) \rightarrow \mathcal{H} \rightarrow \mathcal{V} \rightarrow \mathcal{P} \) forms a double groupoid in the obvious way, called the **coarse double groupoid** with sides in \( \mathcal{H} \) and \( \mathcal{V} \).

**Remark 1.7.**

(i) Let \( M, N \) and \( P \) be smooth manifolds, let \( f : M \rightarrow P \) and \( g : N \rightarrow P \) be smooth maps, we remind that \( f \) and \( g \) are called **transversal** at \( p = f(m) = g(n) \), for \( m \in M, n \in N \), if \( (T_m f)(T_m M) + (T_n g)(T_n N) = T_p P \). We said that \( f \) and \( g \) are **transversal** if they are transversal at any \( p \) as above.

(ii) Let \( \mathcal{V} \) and \( \mathcal{H} \) be Lie groupoids over the same manifold \( P \), then if the anchors maps \( \chi_\mathcal{V} : \mathcal{V} \rightarrow P \times P \) or \( \chi_\mathcal{H} : \mathcal{H} \rightarrow P \times P \) are transversal then \( \Box(\mathcal{V}, \mathcal{H}) \) is a double Lie groupoid \([BM92]\).

**Definition 1.8.** \([AN06b]\) A double groupoid \( (\mathcal{B} ; \mathcal{V}, \mathcal{H} ; \mathcal{P}) \) is **slim** if the morphism of (discrete) double groupoids \( \Pi : \mathcal{B} \rightarrow \Box(\mathcal{V}, \mathcal{H}) \) given by

\[
\Pi \begin{boxed} x \end{boxed} \begin{boxed} A \end{boxed} \begin{boxed} y \end{boxed} = \begin{boxed} x \end{boxed} \begin{boxed} f \end{boxed} \begin{boxed} g \end{boxed} \begin{boxed} y \end{boxed}, \quad \begin{boxed} f \end{boxed} \begin{boxed} A \end{boxed} \begin{boxed} y \end{boxed} \in \mathcal{B},
\]

is injective.

### 2. Diagrams of Groupoids

**Definition 2.1.** \([AN06b]\) Let \( \mathcal{V} \) and \( \mathcal{H} \) be groupoids over the same base \( \mathcal{P} \). A **diagram** over \( \mathcal{H} \) and \( \mathcal{V} \) is a triple \( (\mathcal{D}, j, i) \) where \( \mathcal{D} \) is a groupoid over \( \mathcal{P} \) and \( i : \mathcal{H} \rightarrow \mathcal{D}, j : \mathcal{V} \rightarrow \mathcal{D} \) are morphisms of groupoids over \( \mathcal{P} \).

If \( \mathcal{V} \) and \( \mathcal{H} \) are Lie groupoids, then a **diagram of Lie groupoids** over \( \mathcal{H} \) and \( \mathcal{V} \) is a diagram of groupoids, such that \( \mathcal{D} \) is a Lie groupoid and \( i, j \) are smooth.

To each diagram of groupoids we can associate a discrete double groupoid, denoted by \( \Box(\mathcal{D}, j, i) \) and defined as follows: the boxes in \( \Box(\mathcal{D}, j, i) \) are of the form

\[
A = h \begin{boxed} x \end{boxed} \begin{boxed} g \end{boxed} \in \Box(\mathcal{V}, \mathcal{H}),
\]
with \(x,y \in \mathcal{H}, g,h \in \mathcal{V}\), such that
\[
i(x)j(g) = j(h)i(y) \quad \text{in } \mathcal{D}.
\]

**Definition 2.2.** [AN06b] A diagram of groupoids \((\mathcal{D}, j, i)\), over \(\mathcal{V}\) and \(\mathcal{H}\) is called a \((\mathcal{V}, \mathcal{H})\)-factorization of \(\mathcal{D}\), if \(\mathcal{D} = j(\mathcal{V})i(\mathcal{H})\).

Our aim is to determine when \(\square(\mathcal{D}, j, i)\) is a double Lie groupoid. We define two maps. The first one is the composition \(\mathcal{H} \times_\mathcal{V} \mathcal{V} \xleftarrow{\iota \times j} \mathcal{D} \times_\mathcal{S} \mathcal{D} \xrightarrow{m} \mathcal{D}\), i.e.
\[
\Phi : \mathcal{H} \times_\mathcal{V} \mathcal{V} \to \mathcal{D}, \quad (x,g) \mapsto i(x)j(g),
\]
where \(e, s, \text{ and } m\) are the end, source and multiplication maps of \(\mathcal{D}\). The second one is
\[
\Psi : \mathcal{V} \times_\mathcal{H} \mathcal{H} \to \mathcal{D}, \quad (f,y) \mapsto j(f)i(y).
\]
Since \(t, b, l \text{ and } r\) are surjective submersions, we have that the fiber products involved in the above maps, \(\mathcal{V} \times_\mathcal{H} \mathcal{V}\) and \(\mathcal{H} \times_\mathcal{V} \mathcal{V}\) are embedded submanifolds of \(\mathcal{V} \times \mathcal{H}\) and \(\mathcal{H} \times \mathcal{V}\) respectively, and since \(i\) and \(j\) are smooth then \(\Phi\) and \(\Psi\) are also smooth. With the above maps
\[
\square(\mathcal{D}, j, i) = (\mathcal{V} \times_\mathcal{H} \mathcal{H}) \psi \times \Phi (\mathcal{H} \times_\mathcal{V} \mathcal{V}),
\]
and from general theory of transversality [L02] Prop. 2.5 if \(\Phi\) and \(\Psi\) are transversal, then \(\square(\mathcal{D}, j, i)\) is an embedded submanifold of \((\mathcal{V} \times_\mathcal{H} \mathcal{H}) \times (\mathcal{H} \times_\mathcal{V} \mathcal{V})\).

**Lemma 2.3.** Let \((\mathcal{D}, j, i)\) be a diagram of Lie groupoids. If \(i\) and \(j\) are transversal at the identities, then \(\Psi\) and \(\Phi\) defined above are submersions.

**Proof:** We take \(((f,y),(x,g)) \in (\mathcal{V} \times_\mathcal{H} \mathcal{H}) \times (\mathcal{H} \times_\mathcal{V} \mathcal{V})\) such that \(\Psi(f,y) = \Phi(x,g)\) i.e. \(j(f)i(y) = i(x)j(g)\). Now, by [L02] Prop. 2.5], we have
\[
T_{(x,g)}(\mathcal{H} \times_\mathcal{V} \mathcal{V}) = \{(Y,X) \in (T_x \mathcal{H}) \times (T_g \mathcal{V})/ (T_x \mathcal{V}) = (T_g \mathcal{V}) \}
\]
Let \(W \in T_{(x,g)}(\mathcal{H} \times_\mathcal{V} \mathcal{V})\). We need to prove that there is \((X_1,Y_1)\) belonging to \(T_{(f,y)}(\mathcal{V} \times_\mathcal{H} \mathcal{H})\) such that
\[
T_{(f,y)}\Psi(X_1,Y_1) = (T_{(j(f),i(y))}m)(T_{(f,y)}j \times i)(X_1,Y_1)
\]
\[
= (T_{(j(f),i(y))}m)((T_f j)(X_1),(T_i Y_1)).
\]
We know that in \(\mathcal{H} \rightrightarrows \mathcal{P}\) there is a local bisection \(\tau : U \to \mathcal{H}\) with \(r(y) \in U \subseteq \mathcal{P}\) open and \(\tau(r(y)) = y\) [M05] Prop. 1.4.9]. Since \(\tau\) is a bisection, it induces local left and right translations defined as follows. Set \(V = (l \circ \tau)(U)\), open in \(\mathcal{P}\), \(\mathcal{H}^U = l^{-1}(U)\) and \(\mathcal{H}^l U = r^{-1}(U)\) (the same for \(V\)), and
\[
L_{\tau} : \mathcal{H}^U \to \mathcal{H}^V, \quad z \mapsto \tau(l(z))z \quad \text{and} \quad R_{\tau} : \mathcal{H}^V \to \mathcal{H}^U, \quad z \mapsto z \tau((l \circ \tau)^{-1}r(z))
\]
Define the map \(\tau_{\mathcal{D}} : \mathcal{U} \to \mathcal{D}\) by \(i \circ \tau\) and using that \(i\) is a groupoid morphism note that it is a local bisection of \(\mathcal{D}\). Also note that \(\tau_{\mathcal{D}}(e(i(y))) = (i \circ \tau)(r(y)) = i(y)\).
In the same way there is a local bisection \( \sigma : U' \to \mathcal{V} \) such that \( \sigma_{\mathcal{F}'}(b(f)) = f \) with \( U' \subseteq \mathcal{D} \) open and \( b(f) \in U' \). Again this induces a bisection in \( \mathcal{D} \), \( \sigma_{\mathcal{D}} : U' \to \mathcal{D} \) such that \( \sigma_{\mathcal{D}}(e(j(f))) = j(f) \). Let \( (X_1, Y_1) \in T_{(f, y)}(\mathcal{V}_b \times \mathcal{V}_y) \). Then by Xu’s formula for product in a tangent groupoid \([M05\text{, Theorem 1.4.14}]\) we obtain:

\[
T_{(f, y)} \Psi(X_1, Y_1) = (T_{(j(f), i(y))}m)((T_f j)(X_1), (T_f i)(Y_1)) = (T_{i(y)} L_{\sigma_{\mathcal{D}}})(T_f i)(Y_1) + (T_{j(f)} R_{\mathcal{D}})(T_f j)(X_1)
\]

\[
- (T_{i(y)} L_{\sigma_{\mathcal{D}}})(T_{id_p Y}(L_{\mathcal{D}} T_{\sigma_{\mathcal{D}}}) (T_{i(y)} Y_{\mathcal{D}}))(z),
\]

where we write \( z = (T_{j(f)} e)(T_f j)(X_1) = (T_{i(y)} s)(T_f i)(Y_1) \).

Now \( (T_{j(f)} i(y) L_{\sigma_{\mathcal{D}}}^{-1})(W) \in T_{i(y)} \mathcal{D} \) because we have

\[
L_{\sigma_{\mathcal{D}}}^{-1}(j(f)i(y)) = \sigma_{\mathcal{D}}^{-1}(s(j(f)i(y)))(j(f)i(y)) = \sigma_{\mathcal{D}}(s \circ \sigma_{\mathcal{D}})^{-1}(s(j(f)i(y)))^{-1} j(f)i(y) = \sigma_{\mathcal{D}}(e(j(f)))^{-1} j(f)i(y) = j(f)^{-1} j(f)i(y) = i(y).
\]

In analogous way, we have \( (T_{i(y)} R_{\mathcal{D}}^{-1})(T_{j(f)} i(y) L_{\sigma_{\mathcal{D}}}^{-1})(W) \in T_{id_p Y}(\mathcal{D}) \) since

\[
R_{\mathcal{D}}^{-1}(i(y)) = i(y) \tau_{\mathcal{D}}^{-1}((s \circ \tau_{\mathcal{D}})^{-1} e(i(y))) = i(y) \tau_{\mathcal{D}}^{-1}((s \circ \tau_{\mathcal{D}})^{-1} e(i(y)))^{-1} = i(y) \tau_{\mathcal{D}}^{-1} e(i(y))^{-1} = i(y) - i(y) = Id_{\mathcal{D}} s(i(y)) = Id_{\mathcal{D}} (l(y)).
\]

Denote \( p = l(y) \) since \( i \) and \( j \) are transversal at \( Id_{\mathcal{D}} (p) \) then

\[
T_{id_p \mathcal{D}} = (T_{id_p i})(T_{id_p \mathcal{D}}(Y)) + (T_{id_p j})(T_{id_p \mathcal{D}}(Y))
\]

and in consequence we can find \( X \in T_{id_p \mathcal{D}}X, Y \in T_{id_p \mathcal{D}}Y \) such that

\[
(T_{i(y)} R_{\mathcal{D}}^{-1})(T_{j(f)} i(y) L_{\sigma_{\mathcal{D}}}^{-1})(W) = (T_{id_p i})(X) + (T_{id_p j})(Y).
\]

Thus, if we consider the vectors

\[
X' = X + (T_p Id_{id_p})(T_{id_p t})(Y), \quad Y' = Y + (T_p Id_{id_p})(T_{id_p l})(X),
\]

a direct calculation shows that

\[
(T_{i(y)} R_{\mathcal{D}}^{-1})(T_{j(f)} i(y) L_{\sigma_{\mathcal{D}}}^{-1})(W) = (T_{id_p i})(X') + (T_{id_p j})(Y') - (T_{id_p b})(Z)
\]

where \( Z = (T_{id_p b})(Y) + (T_{id_p l})(X) \). Since

\[
(L_{\mathcal{D}} \circ R_{\mathcal{D}} \circ i) = (L_{\mathcal{D}} \circ i \circ R_{\mathcal{D}}), \quad (L_{\mathcal{D}} \circ R_{\mathcal{D}} \circ j) = (R_{\mathcal{D}} \circ j \circ L_{\mathcal{D}}),
\]
we may apply $\left( T_{(y)} L_{\sigma_d} \right) (T_{Id_p} R_{\tau_o})$ to both sides of (2.1), and arrive to
\[ W = \left( T_{(y)} L_{\sigma_d} \right) (T_{y} i) (Y_1) + \left( T_{(y)} R_{\tau_o} \right) (T_{f j} f) (X_1) - \left( T_{(y)} L_{\sigma_d} \right) (T_{Id_p} R_{\tau_o}) (T_{p Id_d}) (Z), \]
where $X_1 = (T_{Id_p} L_{\sigma_d}) (Y'_1)$, $Y_1 = (T_{Id_p} R_{\tau_o}) (X')$. It is clear that $(T_{f j} b) (X_1) = Z = (T_{l} l) (Y_1)$, thus $T_{(f,j)} \Psi (X_1, Y_1) = W$.

We prove that $\Phi$ is a submersion in the same way.

From the above result we obtain the following immediate consequence.

**Theorem 2.4.** Let $(D, j, i)$ be a $(\mathcal{P}, \mathcal{H})$-factorization of the Lie groupoid $D$. If $i$ and $j$ are transversal at the identities, then $\square(D, j, i)$ is an slim double Lie groupoid.

**Proof.** By Lemma 2.3 we have that $\Phi$ and $\Psi$ are transversal, thus $\square(D, j, i)$ is an embedded submanifold of $(\mathcal{P}_{b \times 1} \mathcal{H}) \times (\mathcal{H}_{r \times 1} \mathcal{V})$. Since $\Phi$ and $\Psi$ are surjective submersions, both projections of $\square(D, j, i) = (\mathcal{P}_{b \times 1} \mathcal{H}) \mathcal{V} \times \Phi (\mathcal{H}_{r \times 1} \mathcal{V})$, to the first and second components are surjective submersions and the same is true for the projections from the fiber products $\mathcal{P}_{b \times 1} \mathcal{H}$ and $\mathcal{H}_{r \times 1} \mathcal{V}$, then the top, bottom, left and right maps from $\square(D, j, i)$ are surjective submersions and the same for the double source map. It is clear that the compositions, the identities maps and the inversions maps are smooth.

3. **Diagonal Groupoid Associated to a Slim Double Lie Groupoid**

From now on and until Lemma 3.4 all groupoids are discrete.

3.1. **Diagonal groupoid.** In this section we recall from [AN06b] the construction of the diagonal groupoid. Let $\mathcal{B}$ be a double groupoid that satisfies the filling condition and let $\mathcal{P} \oplus \mathcal{H}$ be the free product (over $\mathcal{P}$) of the vertical and horizontal groupoids, see [AN06b] and [H71]. If $h \in \mathcal{B}$ we denote $[A] := xgy^{-1}h^{-1} \in \mathcal{P} \oplus \mathcal{H}$. Then $J_\oplus(\mathcal{B})$ is the subgroupoid of $\mathcal{P} \oplus \mathcal{H}$ generated by $\{ [A] | A \in \mathcal{B}\}$. As $s_\oplus([A]) = e_\oplus([A]) = t(A)$ we have that the groupoid $J_\oplus(\mathcal{B}) \supseteq \mathcal{P}$ is in fact a group bundle. We know that the group bundle $J_\oplus(\mathcal{B})$ is a normal subgroupoid of $\mathcal{P} \oplus \mathcal{H}$ [AN06b] lemma 3.5.

Assume that $(\mathcal{B}; \mathcal{P}, \mathcal{H}; \mathcal{P})$ is slim; then the associated diagonal groupoid is $\mathcal{D}(\mathcal{B}) = \mathcal{P} \oplus \mathcal{H} / J_\oplus(\mathcal{B})$. If we compose the natural inclusions of $\mathcal{P}$ and $\mathcal{H}$ in $\mathcal{P} \oplus \mathcal{H}$ with the projections on $\mathcal{D}(\mathcal{B})$ we get two groupoid morphisms:

\[ i: \mathcal{H} \rightarrow \mathcal{D}(\mathcal{B}) \quad \text{and} \quad j: \mathcal{P} \rightarrow \mathcal{D}(\mathcal{B}). \]

Thus we have a diagram $(\mathcal{D}(\mathcal{B}), i, j)$. Our aim is to give another presentation of the diagonal groupoid as a quotient of $\mathcal{P}_{b \times 1} \mathcal{H}$. 

Proposition 3.1. Let $(\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})$ be a slim double groupoid that satisfies the filling condition. We define on $\mathcal{V} \times_1 \mathcal{H}$ the following relation $\sim_\mathcal{B}$:

$$(v_1, h_1) \sim_\mathcal{B} (v_2, h_2) \text{ if and only if } r(h_1) = r(h_2), t(v_1) = t(v_2) \text{ and } v_1 h_1^{-1} v_2^{-1} \in J_\mathcal{B}(\mathcal{B}).$$

then $\sim_\mathcal{B}$ is an equivalence relation and the map

$$\phi : \mathcal{V} \times_1 \mathcal{H} / \sim_\mathcal{B} \to \mathcal{D}(\mathcal{B}), \quad [v, h] \mapsto j(v) i(h)$$

is well defined and is a bijection (of quivers over $\mathcal{P}$).

Proof: Clearly, $\sim_\mathcal{B}$ is an equivalence relation; we denote $\mathcal{G} := (\mathcal{V} \times_1 \mathcal{H}) / \sim_\mathcal{B}$.

Let $[f_1, x_1] = [f_2, x_2]$ both in $\mathcal{G}$, then $f_1 x_1, x_2^{-1} f_2^{-1} \in J_\mathcal{B}(\mathcal{B})$, so $\overline{f_1 x_1} = \overline{f_2 x_2}$ in $\mathcal{D}(\mathcal{B})$; where $\overline{w}$ denotes the image of $w$ under $i$ if $w$ belongs to $\mathcal{H}$, or under $j$ if $w$ belongs to $\mathcal{V}$. This proves that $\phi$ is well defined. Suppose that $g, g' \in \mathcal{V}, x, x' \in \mathcal{H}$ is any collection satisfying $\overline{g x} = \overline{g' x'}$. Then $g' x' x^{-1} g^{-1} \in J_\mathcal{B}(\mathcal{B})$, hence $[g', x'] = [g, x]$. Therefore $\phi$ is injective.

To prove that $\phi$ is surjective, let $d \in \mathcal{D}(\mathcal{B})$. Then $d = \overline{d_1 d_2 \cdots d_n}$ with $d_i$ an element of $\mathcal{V}$ or $\mathcal{H}$. Let $d_i \in \mathcal{H}, d_{i+1} \in \mathcal{V}$ with $r(d_i) = t(d_{i+1})$. Since $\mathcal{B}$ satisfies the filling condition, the corner

$$(3.1)$$

$\begin{tikzpicture}
  \node (d) at (0,0) {$\overline{d_i}$};
  \node (y) at (0,-1) {$\overline{y}$};
  \node (d) at (0,-2) {$\overline{d_{i+1}}$};
  \draw (d) -- (y) -- (d);
\end{tikzpicture}$$

can be completed to a box in $\mathcal{B}$, i.e. there exists $B \in \mathcal{B}$ such that

$$B = \overline{f \bord{d_i \cdots d_{i+1}}}.$$ 

Thus, $d_id_{i+1}y^{-1}f^{-1} \in J_\mathcal{D}(\mathcal{B})$ and $\overline{d_i \cdots d_{i+1}} = \overline{f \bord{y}}$. So, we can commute the $d_i$’s in $d = \overline{d_1 d_2 \cdots d_n}$ in such a way that we can obtain $d = \overline{g x}$ with $g \in \mathcal{V}$ and $x \in \mathcal{H}$ and

$$b(g) = l(x).$$

This proves that $\phi$ is surjective. 

Remark 3.2. $\phi$ induces a structure of groupoid on $\mathcal{G} = \mathcal{V} \times_1 \mathcal{H} / \sim_\mathcal{B}$ by:

- the source and the target projections are

  $$s : \mathcal{G} \to \mathcal{P}, \quad [v, h] \mapsto t(v) \quad e : \mathcal{G} \to \mathcal{P}, \quad [v, h] \mapsto r(h).$$

- The inclusion map is $\text{id} : \mathcal{P} \to \mathcal{G}, \quad p \mapsto \text{id}_p = [\text{id}_p, \text{id}_p]$.

- The partial multiplication is $[v_1, h_1][v_2, h_2] = [v_1 f, z h_2]$ where $\begin{tikzpicture}
  \node (z) at (0,0) {$z$};
  \draw (0,0) -- (0,-2);
\end{tikzpicture}$.
The inverse is $[v, h]^{-1} = [f^{-1}, z^{-1}]$ where $v f h \in B$.

If $\mathcal{V}$ and $\mathcal{H}$ are Lie groupoids, then $b$ and $l$ are surjective submersions, thus $\mathcal{V} \times_1 \mathcal{H}$ is an embedded submanifold of $\mathcal{V} \times \mathcal{H}$. We will prove in Theorem 3.9 that $\mathcal{V} \times_1 \mathcal{H} / \sim_B$ is a Lie groupoid under certain conditions.

Now we recall a lemma very useful for our purposes.

**Lemma 3.3.** [AN06b, Lemma 3.8] Let $(B, \mathcal{V}, \mathcal{H}; P)$ be a slim double groupoid that satisfies the filling condition. Let $f \in \mathcal{V}$ and $x \in \mathcal{H}$ such that:

- $l(x) = b(f)$ and $t(f) = r(x)$,
- $fx \in J_{\otimes}(B) \subset \mathcal{V} \otimes \mathcal{H}$.

Then there exists $E \in E(B)$ such that $E = f x$.

By this lemma, we have an alternative description of $\sim_B$. Indeed, $(f_1, x_1) \sim_B (f_2, x_2)$ if and only if $f_1 x_1 f_2^{-1} \in J_{\otimes}(B)$. Because $J_{\otimes}(B)$ is a normal subgroupoid, we have that $(f_1, x_1) \sim_B (f_2, x_2)$ if and only if $f_2^{-1} f_1 x_1 x_2^{-1} \in J_{\otimes}(B)$.

Hence

$$(3.2) \quad (f_1, x_1) \sim_B (f_2, x_2) \iff \text{there exist } E \in E(B), \ E = f_2^{-1} f_1 x_1 x_2^{-1} \in B.$$ 

Thus, the graph of the relation $\sim_B$ is

$$R = \{(f_1, x_1, f_2, x_2) \in (\mathcal{V} \times_1 \mathcal{H}) \times (\mathcal{V} \times_1 \mathcal{H}) | \exists E \in E(B), \ E = f_2^{-1} f_1 x_1 x_2^{-1}\}$$

where

$\eta : \mathcal{V} \times_1 \mathcal{H} \to \mathcal{P} \times \mathcal{P}, \ (f, x) \mapsto (t(f), r(x))$.

We conclude that the relation $\sim_B$ is determined by the core groupoid of $B$.

**Lemma 3.4.** Let $(B, \mathcal{V}, \mathcal{H}; P)$ be a slim double groupoid satisfying the filling condition. If $(f_1, x_1), (f_2, x_2) \in \mathcal{V} \times_1 \mathcal{H}$, then $(f_1, x_1) \sim_B (f_2, x_2)$ if and only if there exist $A, B \in B$ such that

$$A = f_1 x_1 g, \quad B = f_2 x_2 g.$$

Proof. In fact, if

$$A = f_1 x_1 g, \quad B = f_2 x_2 g$$

are in $B$,.
then \(xg_{x_1}^{-1}f_{x_1}^{-1} \in J_\otimes(B)\) and \(xg_{x_2}^{-1}f_{x_2}^{-1} \in J_\otimes(B)\), taking inverse of the first and composing, it follows that \(f_1x_1x_2^{-1}f_2^{-1} \in J_\otimes(B)\), i.e. \((f_1, x_1) \sim_B (f_2, x_2)\).

Reciprocally, if \((f_1, x_1) \sim_B (f_2, x_2)\) then by (3.2), there is \(E \in \mathbb{E}(B)\) such that \(E = f_2^{-1}f_1 \begin{array}{|c|} \hline x_2 \hline \end{array} \). The filling condition guarantees that given \(x_2 \in \mathcal{H}\) and \(f_2 \in \mathcal{V}\) with \(l(x_2) = b(f_2)\), there is a box \(B' \in \mathcal{B}\) with \(t(B') = x_2\) and \(l(B') = f_2^{-1}\). Let \(B' = f_2^{-1} \begin{array}{|c|} \hline x_2 \hline \end{array} g^{-1}\), and let \(A' := \left\{ \begin{array}{|c|c|} \hline B' & \mathbf{id}(x_2) \hline \hline \mathbf{id}(f_2) & A' \hline \end{array} \right\} = f_1^{-1} \begin{array}{|c|} \hline x_1 \hline \end{array} g^{-1}\). Let \(A, B\) be the vertical inverses of \(A'\) and \(B'\) respectively. Thus \(A = f_1 \begin{array}{|c|} \hline x \hline \end{array} g\) and \(B = f_2 \begin{array}{|c|} \hline x \hline \end{array} g\) are both in \(\mathcal{B}\) and we get the result. \(\Box\)

3.2. **The core action.** We recall that a continuous map \(f : X \rightarrow Y\), between two topological spaces \(X\) and \(Y\), is said to be *proper* if the inverse image of a compact subset of \(Y\) is compact.

**Definition 3.5.** A Lie groupoid \(\mathbb{G} \rightrightarrows \mathcal{P}\) is *proper* if the anchor map is proper. An action of a Lie groupoid \(\mathbb{G}\) on a smooth manifold \(Z\) is *proper* if the action groupoid \(\mathbb{G} \ltimes Z\) is proper.

The following proposition is useful to decide when an action is proper. For details and more on proper actions, see [T04].

**Proposition 3.6.** Let \(\mathbb{G} \rightrightarrows \mathcal{P}\) be a Lie groupoid. Let \(Z\) be a smooth manifold endowed with a left action of \(\mathbb{G}\), then \(\mathbb{G}\) acts properly on \(Z\) if the anchor map \((s, e) : \mathbb{G} \ltimes Z \rightarrow Z \times Z\) is closed and \(\forall z \in Z\), the stabilizer of \(z\) is compact. \(\Box\)

We shall need the following proposition from [AN06b, Prop. 1.1].

**Proposition 3.7.** Let \((\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P})\) be a slim double groupoid. Define \(\gamma : \mathcal{B} \rightarrow \mathcal{P}\), \(\gamma(A) = lb(A)\), the bottom-left vertex of \(A\).

(a). There is an action of the core groupoid \(\mathbb{E}(\mathcal{B})\) on \(\gamma : \mathcal{B} \rightarrow \mathcal{P}\) given by

\begin{equation}
E \mapsto A := \left\{ \begin{array}{|c|c|} \hline \mathbf{id}(A) & A \hline \hline E & \mathbf{id}(b(A)) \hline \end{array} \right\}, \quad A \in \mathcal{B}, E \in \mathbb{E}.
\end{equation}

(b). Let \(B \in \mathcal{B}\). Then the stabilizer \(\mathbb{E}(\mathcal{B})^B\) is trivial and the orbit of \(B\) is \(O_B = \{A \in \mathcal{B} : t(A) = t(B), r(A) = r(B)\}\). \(\Box\)

The above results enable us to state and proof the following lemma.
Lemma 3.8. Let \( (\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P}) \) be a slim (discrete) double groupoid. Define the map \( \eta : \mathcal{V}_{b \times 1} \mathcal{H} \rightarrow \mathcal{P} \) by \( \eta(f,x) = b(f) = l(x) \). Then \( E(\mathcal{B}) \) acts on \( \eta \) by

\[
(3.4) \quad E \triangleright (f,x) = (f l(E), b(E)x), \quad \text{when } \eta(f,x) = e_{E(\mathcal{B})}(E).
\]

The quotient space \( \mathcal{V}_{b \times 1} \mathcal{H} / E(\mathcal{B}) \) coincides with \( \mathcal{D}(\mathcal{B}) \).

Proof. Since \( b(f l(E)) = bl(E) = s_{E(\mathcal{B})}(E) \) and \( l(b(E)x) = lb(E) = s_{E(\mathcal{B})}(E) \) the map \( \triangleright : E(\mathcal{B}) \times_{E(\mathcal{B})} \mathcal{V}_{b \times 1} \mathcal{H} \rightarrow \mathcal{V}_{b \times 1} \mathcal{H} \) is well defined. That \( \triangleright \) is an action is straightforward, in fact,

\[
(E \circ F) \triangleright (f,x) = (f l(E \circ F), b(E \circ F)x) = (f l(F)l(E), b(E)b(F)x) = E \triangleright (f l(F), t(F)x) = E \triangleright (F \triangleright (f,x)).
\]

Also, \( \eta(E \triangleright (f,x)) = \eta(f l(E), b(E)x) = l(b(E)x) = lb(E) = e_{E(\mathcal{B})}(E) \).

For the second part, if \( (f,x) \sim_{\mathcal{B}} (g,y) \), then there are \( A, B \in \mathcal{B} \) such that \( f = \begin{array}{c} z \\ \hline x \\ \hline y \end{array} \quad \text{and} \quad \begin{array}{c} A \\ \hline B \end{array} \), see Lemma 3.4. Then, by Proposition 3.7, there exists a box \( E \in E(\mathcal{B}) \) such that \( A = E \rightarrow B \), in consequence, \( x = b(E)y \) and \( f = g l(E) \).

Conversely, if \( A, B \in \mathcal{B} \) and there exists \( E \in E(\mathcal{B}) \) with \( b(A) = b(E)b(B) \) and \( l(A) = l(B)l(E) \), then the boxes \( E \rightarrow B \) and \( B \) have the same top and right sides. By Lemma 3.4 we have \( (l(E \rightarrow B), b(E \rightarrow B)) \sim_{\mathcal{B}} (l(B), b(B)) \), that is \( (l(A), b(A)) \sim_{\mathcal{B}} (l(B), b(B)) \). From this we conclude that given \( (f,x), (g,y) \in \mathcal{V}_{b \times 1} \mathcal{H} \), \( (f,x) \sim_{\mathcal{B}} (g,y) \) iff \( \exists E \in E(\mathcal{B}) \) with \( f = g l(E) \) and \( x = b(E)y \). Thus the quotient coincides with the diagonal groupoid. \qed

The action \( (3.4) \) will be called the core action of \( E(\mathcal{B}) \) on \( \mathcal{V}_{b \times 1} \mathcal{H} \). Let \( \pi : \mathcal{V}_{b \times 1} \mathcal{H} \rightarrow \mathcal{V}_{b \times 1} \mathcal{H} / E(\mathcal{B}) \) be the projection determined by \( (3.4) \).

Theorem 3.9. Let \( (\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{P}) \) be a slim double Lie groupoid. If the core action is proper, then \( \mathcal{D}(\mathcal{B}) \) is a Lie groupoid over \( \mathcal{P} \).

Proof. Since \( \mathcal{B} \) is slim, the action \( (3.4) \) is free. Hence, if the action is proper, then the quotient \( \mathcal{V}_{b \times 1} \mathcal{H} / E(\mathcal{B}) \) has a unique manifold structure such that the projection \( \pi : \mathcal{V}_{b \times 1} \mathcal{H} \rightarrow \mathcal{V}_{b \times 1} \mathcal{H} / E(\mathcal{B}) \) is a surjective submersion [D07, Theorem 3.3.1]. Thus, \( \mathcal{D}(\mathcal{B}) \) is a Lie groupoid over \( \mathcal{P} \). In fact, the structure maps are described in Remark 3.2 using local sections of \( \pi \), it is clear that the source and target maps are surjective submersions and that the other structural maps are smooth. \qed
Lemma 3.10. The maps $i$ and $j$ defined above are transversal at the identities.

Proof. Let $p \in \mathcal{P}$. Take a tangent vector $Z \in T_{(\text{id}_p, \text{id}_p)} \mathcal{D}(\mathcal{B})$, where $[\text{id}_p, \text{id}_p] = \pi(\text{id}_p, \text{id}_p)$. Since $\pi$ is a surjective submersion, there is $(U, W) \in T_{(\text{id}_p, \text{id}_p)}(\mathcal{V}' \times_1 \mathcal{H})$ such that $T_{(\text{id}_p, \text{id}_p)} \pi(U, W) = Z$. Choose
\[ Y = U \in T_{(\text{id}_p)} \mathcal{V}, \quad X = W - (T_p \text{id}_\mathcal{H})(T_{(\text{id}_p, \text{id}_p)})(U) \in T_{(\text{id}_p)} \mathcal{H}. \]

It is clear that
\[ (T_{(\text{id}_p, \text{id}_p)} \tilde{j})(Y) = (U, (T_p \text{id}_\mathcal{H})(T_{(\text{id}_p, \text{id}_p)})(U)), \]
and
\[ (T_{(\text{id}_p, \text{id}_p)} \tilde{i})(X) = ((T_p \text{id}_\mathcal{V})(T_{(\text{id}_p, \text{id}_p)})(X), X). \]

We compute
\[ (T_p \text{id}_\mathcal{V})(T_{(\text{id}_p, \text{id}_p)})(X) = (T_p \text{id}_\mathcal{V})(T_{(\text{id}_p, \text{id}_p)})(W) - (T_p \text{id}_\mathcal{V})(T_{(\text{id}_p, \text{id}_p)})(T_p \text{id}_\mathcal{H})(U) \]
\[ = (T_p \text{id}_\mathcal{V})(T_{(\text{id}_p, \text{id}_p)})(W) - (T_p \text{id}_\mathcal{V})(T_{(\text{id}_p, \text{id}_p)})(U) = 0; \]
then
\[ (T_{(\text{id}_p, \text{id}_p)} \tilde{i})(X) = (0, X). \]

In consequence we have
\[ (T_{(\text{id}_p, \text{id}_p)} \tilde{j})(Y) + (T_{(\text{id}_p, \text{id}_p)} \tilde{i})(X) = (U, (T_p \text{id}_\mathcal{H})(T_{(\text{id}_p, \text{id}_p)})(U)) + (0, X) \]
\[ = (U, (T_p \text{id}_\mathcal{H})(T_{(\text{id}_p, \text{id}_p)})(U)) + W - (T_p \text{id}_\mathcal{V})(T_{(\text{id}_p, \text{id}_p)})(U) \]
\[ = (U, W). \]

Then if we apply $T_{(\text{id}_p, \text{id}_p)} \pi$ to both sides of the above equation we arrive to
\[ (T_{(\text{id}_p, \text{id}_p)} \tilde{j})(Y) + (T_{(\text{id}_p, \text{id}_p)} \tilde{i})(X) = Z, \]
that is, the maps $i$ and $j$ are transversal at the identities. \hfill \Box

Let $(\mathcal{D}, j, i)$ be a $(\mathcal{V}', \mathcal{H})$-factorization. The underlying manifold to the core groupoid of $\mathcal{B} = \Box(\mathcal{D}, j, i)$ is $\mathcal{V}' \times_1 \mathcal{H} = \{(h, y) \mid j(h^{-1}) = i(y)\}$. The core action on $\mathcal{V}' \times_1 \mathcal{H}$ is given by
\[ (h, y) \triangleright (f, x) = (fh, yx) \quad \text{when} \quad \eta(f, x) = t(h) = r(y); \]
the proof of (3.5) follows from the definition (3.4).

Lemma 3.11. The core action (3.5) is proper.
Proof. Since the action (3.5) is free, in order to prove that it is proper, we only need to check that the anchor map of the respective action groupoid

\[(s,t): (\mathcal{V}^op \times \mathcal{H}) \times (\mathcal{V} \times \mathcal{H}) \to (\mathcal{V} \times \mathcal{H}) \times (\mathcal{V}^op \times \mathcal{H})\]

is closed, see Proposition 3.6. Let \(A \subseteq (\mathcal{V} \times \mathcal{H})\) be a closed set and consider a sequence \(\{(f_n, x_n, g_n, y_n)\}_{n \in \mathbb{N}}\) in \(A\) such that the sequence

\[\{(s,t)(f_n, x_n, g_n, y_n)\}_{n \in \mathbb{N}} = \{(g_n f_n, x_n y_n, g_n, y_n)\}_{n \in \mathbb{N}}\]

converges to \((a, b, g, y) \in (\mathcal{V} \times \mathcal{H})\) by Theorem 3.12. Fix \(N\), and consider \(\{a, g, y\} \in (\mathcal{V} \times \mathcal{H})\). We need to see that \((a, b, g, y) \in (s,t)(A)\). Clearly, \(g_n \xrightarrow{n \to \infty} g, \ y_n \xrightarrow{n \to \infty} y, \ g_n f_n \xrightarrow{n \to \infty} a\) and \(x_n y_n \xrightarrow{n \to \infty} b\).

Hence \((f_n, x_n, g_n, y_n) \xrightarrow{n \to \infty} (g^{-1}a, b y^{-1}, g, y)\); since \(A\) is closed, we conclude that \((g^{-1}a, b y^{-1}, g, y) \in A\). Now \((a, b, g, y) = (s,t)(g^{-1}a, b y^{-1}, g, y) \in (s,t)(A)\) by a direct calculation, hence \((s,t)(A)\) is closed. \(\square\)

Finally, we arrive to our main result.

**Theorem 3.12.** Fix \(\mathcal{V}\) and \(\mathcal{H}\). The assignments \(\mathcal{B} \mapsto \mathcal{D}(\mathcal{B})\) and \(\mathcal{D}_j(i) \mapsto \square(\mathcal{D}_j(i))\) determine mutual category equivalences between

(a) The category of slim double Lie groupoids \((\mathcal{B}; \mathcal{V}, \mathcal{H}; \mathcal{D})\) with proper core action, and

(b) The category of \((\mathcal{V}, \mathcal{H})\) -factorizations of Lie groupoids \((\mathcal{D}, j, i)\) such that the maps \(i\) and \(j\) are transversal at the identities.

**Proof.** The equivalence of categories at the discrete level was obtained in [AN06b].

If \(\mathcal{B}\) is a double Lie groupoid as in (a), then the associated \((\mathcal{V}, \mathcal{H})\) -factorization \((\mathcal{D}(\mathcal{B}), j, i)\) is a Lie groupoid with \(i\) and \(j\) transversal to the identities, by Theorem 3.9 and Lemma 3.10 respectively.

Conversely, if we begin with a \((\mathcal{V}, \mathcal{H})\) -factorization of Lie groupoids as in (b), then the associated double groupoid is a slim double Lie groupoid as is required in (a), by Theorem 2.4 and Lemma 3.11. \(\square\)

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