Birational geometry of sextic double solids with a compound $A_n$ singularity

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Abstract

Sextic double solids, double covers of $\mathbb{P}^3$ branched along a sextic surface, are the lowest degree Gorenstein Fano 3-folds, hence are expected to behave very rigidly in terms of birational geometry. Smooth sextic double solids, and those which are $\mathbb{Q}$-factorial with ordinary double points, are known to be birationally rigid. In this article, we study sextic double solids with an isolated compound $A_n$ singularity. We prove a sharp bound $n \leq 8$, describe models for each $n$ explicitly and prove that sextic double solids with $n > 3$ are birationally non-rigid.

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1. Introduction

We work with projective varieties over $\mathbb{C}$. Classification of algebraic varieties is one of the fundamental goals in algebraic geometry. The Minimal Model Program says that every variety is birational to either a minimal model or a Mori fibre space. Two Mori fibre spaces are birational if they are connected by a sequence of Sarkisov links (see [Sar89], [Rei91], [Cor95], [HM13]). In the extreme case, the Mori fibre space is \textit{birationally rigid}, meaning that it is essentially the unique Mori fibre space in its birational class.

Examples of Mori fibre spaces include Fano varieties. The first birational rigidity result was in the seminal paper by Iskovskikh and Manin [IM71] for smooth quartic 3-folds in $\mathbb{P}^4$. A wealth of examples of birationally rigid varieties was given in [CPR00] and [CP17], by showing that every quasismooth member of the 95 families of Fano 3-folds that are hypersurfaces in weighted projective spaces is birationally rigid. One major consequence of birational rigidity is nonrationality. Birational rigidity remains an active area of research (see [Pro18], [Kry18], [AO18], [dF17], [CG17], [CS19], [EP18]).

Among smooth Fano 3-folds, the projective space has the highest degree (64), and sextic double solids, double covers of $\mathbb{P}^3$ branched along a sextic surface, have the least degree (2). In [Isk80], it is proved that smooth sextic double solids are birationally rigid. It is interesting to see how this changes as we impose singularities on the variety. The paper [Puk97] proved that sextic double solids stays birationally rigid if we impose an ordinary double point, meaning the 3-fold $A_1$ singularity $x_1^2 + x_2^2 + x_3^2 + x_4^2$. A sextic double solid can have up to 65 singular points (see [Bar96], [Bas06], [JR97], [Wah98]), and for each $n \leq 65$, there exists a sextic double solid with exactly $n$ ordinary double points and smooth otherwise (see [CC82]). A sextic double solid with only ordinary double points is birationally rigid if and only if it is factorial, which is true for example if it has at most 14 ordinary double points (see [CP10]).

The next natural question is to consider more complicated singularities in the Mori category. We study sextic double solids with an isolated \textit{compound} $A_n$ singularity, also called a $cA_n$ singularity, meaning that the general section through the point is the Du Val $A_n$ singularity $x_1 x_2 + x_3^{n+1}$. A $cA_n$ singularity is locally analytically given by $x_1 x_2 + h(x_3, x_4)$ where the least degree among monomials in $h$ is $n + 1$. The first main result of the paper is describing sextic double solids with an isolated $cA_n$ singularity.

\textbf{Theorem} (see Theorem A). If a sextic double solid has an isolated $cA_n$ point, then $n \leq 8$.
Moreover, in Theorem A we explicitly parametrize all sextic double solids with an
isolated $cA_n$ singularity for every $n \leq 8$. These form 11 families, as there are 4 families
for $cA_7$. A very general member of every family, except for the family 7.4, is a Mori fibre
space.

We say a few words on bounding the number of $cA_n$ singularities. It is clear that an
isolated $cA_n$ singularity has Milnor number at least $n^2$. Since the third Betti number
of a smooth sextic double solid is 104 (see [IP99, Table 12.2]), an argument similar to
[AK16, Section 3.2] shows that the total Milnor number of a sextic double solid which is
a Mori fibre space is at most 104. This gives the bounds that a Mori fibre space sextic
double solid can have up to 1 $cA_8$ singularity, or up to 2 $cA_7$ singularities, or up to 2
$cA_6$ singularities, ..., or up to 26 $cA_2$ singularities. We do not expect these bounds to
be sharp, as already for ordinary double points it gives an upper bound of 104, far from
the actual 65. Using Theorem A, it is possible to construct sextic double solids with a
$cA_8$ point, a $cA_3$ point and two ordinary double points with both total Milnor and total
Tjurina number at least 66.

The second main result is the following theorem:

**Theorem** (see Theorem B and Section 5.3). A general sextic double solid which is a Mori
fibre space with an isolated $cA_n$ singularity where $n \geq 4$ is not birationally rigid.

Birational non-rigidity for a sextic double solid $X$ is proved by describing a birational
model, meaning a Mori fibre space $T \rightarrow S$ such that $X$ and $T$ are birational. We find
the birational models by explicitly constructing a Sarkisov link for each family of sextic
double solids, under the generality conditions described in Condition 5.1. Table 1 gives an
overview of the Sarkisov links $X \leftarrow Y_0 \rightarrow Y_2 \rightarrow Z$ and the birational models, which are
either fibrations $Y_2 \rightarrow Z$ or Fano varieties $Z$. In the latter case, $Y_2 \rightarrow Z$ is a divisorial to
the given singular point. The morphism $Y_0 \rightarrow X$ is a divisorial contraction with centre
the $cA_n$ point. The birational maps $Y_0 \rightarrow Y_2$ are isomorphisms in codimension 1.

Note that when we say that a birational map $Y_0 \rightarrow Y_1$ is $k$ Atiyah flops, then we mean
that algebraically it is one flop, contracting $k$ curves to $k$ points and extracting $k$ curves,
and locally analytically around each of those points, it is an Atiyah flop. Similarly for
flips. Also note that the Sarkisov link to a sextic double solid with a $cA_4$ singularity
was already described in [Oka14, Section 9, No. 9], starting from a general quasismooth
complete intersection $X_{5,6} \subseteq \mathbb{P}(1,1,1,2,3,4)$.

We briefly describe the proof. The first step in the Sarkisov link starting from a Fano
variety $X$ is a divisorial contraction $Y \rightarrow X$. Kawakita described divisorial contractions
to $cA_n$ points locally analytically, showing that they are certain weighted blowups. To
construct Sarkisov links, we need a global description. In Proposition 4.5 and Lemma 4.8,
we show how to construct divisorial contractions to $cA_n$ points algebraically on affine
hypersurfaces, and use this in Section 5 to construct divisorial contractions $Y \rightarrow X$ for
(projective) sextic double solids $X$. Using unprojection techniques (see [PR04] for a general
theory of unprojection), we find an embedding of $Y$ inside a toric variety $T$, such that the
2-ray link of $T$ restricts to a Sarkisov link for $X$ (following [BZ10] and [AZ16]).

If we try the same methods as in the proof of Theorem B on sextic double solids with
a $cA_n$ singularity where $n \leq 3$, then we do not find any new birational models. More
precisely: a $(3,1,1,1)$-Kawakita blowup of a $cA_3$ singularity on a general Mori fibre space
sextic double solid initiates a Sarkisov link to itself $X \rightarrow X$. A $(2,2,1,1)$-Kawakita
blowup for a $cA_3$ singularity, a $(2,1,1,1)$-Kawakita blowup for a $x_1x_2 + x_3^3 + x_4^2$
singularity and the (usual) blowup for an ordinary double point on a general Mori fibre space sextic
Table 1. Birational models for general sextic double solids that are Mori fibre spaces with an isolated $cA_n$ singularity

| $cA_n$ | weighted blowup $\varphi$ | ----- | weighted blowup or fibration $\psi$ | $Z$ |
|--------|-----------------------------|-------|---------------------------------|-----|
| $cA_4$ | $(3, 2, 1, 1)$              | 10 Atiyah flops | $(\frac{1}{4}, \frac{1}{4}, \frac{3}{4})$ | $\frac{1}{2}(1, 1, 3) \in Z_{5,6}$ $\subseteq \mathbb{P}(1^4, 3)$ |
| $cA_5$ | $(3, 3, 1, 1)$              | 4 Atiyah flops | $(3, 3, 1, 1)$ | $cA_5 \in Z_6 \subseteq \mathbb{P}(1^4, 3)$, $X \not\cong Z$ if general |
| $cA_6$ | $(4, 3, 1, 1)$              | 2 Atiyah flops, then $(4, 1, 1, -2, -1; 2)$-flip | $(3, 1, 1, 1)$ | $cA_3 \in Z_5 \subseteq \mathbb{P}(1^4, 2)$ |
| $cA_7, 1$ | $(4, 4, 1, 1)$ | two $(4, 1, 1, -2, -1; 2)$-flips | $(1, 1, 1, 1)$ | ODP $\in Z_{3,4} \subseteq \mathbb{P}(1^4, 2^2)$ |
| $cA_7, 2$ | $(4, 4, 1, 1)$ | Atiyah flop, then two $(4, 1, -1, -3)$-flips | $(3, 3, 2, 1)$ | $cA_2 \in Z_{2,4} \subseteq \mathbb{P}(1^5, 2)$ |
| $cA_7, 3$ | $(4, 4, 1, 1)$ | 2 Atiyah flops | dP$_2$-fibration | $\mathbb{P}^1$ |
| $cA_8$ | $(5, 4, 1, 1)$ | $(4, 1, 1, -2, -1; 2)$-flip | $(3, 2, 2, 1, 5)$ | $cD_4 \in Z_{3,3} \subseteq \mathbb{P}(1^5, 2)$ |

The double solid initiate ‘bad links’, which end in either a non-terminal 3-fold or a K3-fibration. These are 2-ray links which are not Sarkisov links, where in the last step of the 2-ray game only $K$-trivial curves are contracted, leaving the Mori category. We expect that general Mori fibre space sextic double solids with a $cA_2$ singularity are birationally rigid, and with certain $cA_2$ or $cA_1$ singularities are birationally superrigid.

**Organization of the paper**

In Sections 2.1, 2.2 and 2.3, we give known results that we use respectively in Sections 3, 4 and 5. In Section 3, we construct a parameter space of sextic double solids in Theorem A with an isolated $cA_n$ singularity. In Section 4, we explain the relationship between algebraic and local analytic weighted blowups, and in Proposition 4.5 and the technical Lemma 4.8, show how to construct divisorial contractions to $cA_n$ points algebraically on affine hypersurfaces. In Section 5, we construct birational models for general sextic double solids which are Mori fibre spaces with an isolated $cA_n$ singularity where $n \geq 4$, thereby showing that they are not birationally rigid. We treat the 7 families separately. Appendix A contains the computer code, in particular the splitting lemma from singularity theory (Proposition 3.2) in Listing 1, which we use for constructing the parameter space and the divisorial contractions for sextic double solids.
2. Preliminaries

All varieties we consider are irreducible and over $\mathbb{C}$.

We study sextic double solids, which are double covers of the projective 3-space branched along a sextic surface. We use the following equivalent characterization:

**Definition 2.1.** A sextic double solid is an irreducible hypersurface given by the zero locus of $w^6 + g$ in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 3)$ with variables $x, y, z, t, w$, where $g \in \mathbb{C}[x, y, z, t, w]$ is a homogeneous polynomial of degree six.

2.1. Singularity theory

We recall some results from the singularity theory of complex analytic spaces and on terminal singularities.

All varieties we consider are irreducible and over $\mathbb{C}$.

We denote the variables on $\mathbb{C}^n$ by $x = (x_1, \ldots, x_n)$, where $n$ is a positive integer. Let $\mathbb{C}\{x\}$ denote the convergent power series ring. The zero set of an ideal $I$ is denoted by $\mathbb{V}(I)$, where $I$ is either an ideal of regular functions or holomorphic functions, depending on context. Given a regular or holomorphic function $f$ on a variety or space $X$, denote the non-zero locus of $f$ by $X_f$. Given positive integer weights $w = (w_1, \ldots, w_n)$ for $x$, we can write a non-zero polynomial or power series $f$ as a sum of its weighted homogeneous parts $f_i$. Then, the weight of $f$, denoted $wt(f)$, is the least non-negative integer $d$ such that $f_d \neq 0$. We define $wt(0) = \infty$. If $w = (1, \ldots, 1)$, then $d$ is called the multiplicity, denoted $mult(f)$. A hypersurface singularity is a complex analytic space germ (not necessarily irreducible or reduced) that is isomorphic to $(X, 0)$ where $X \subseteq \mathbb{C}^n$ is given by the zero set of some $f \in \mathbb{C}\{x\}$. A singularity is isolated if it has a smooth analytic punctured neighbourhood.

**Definition 2.2** ([GLS07, Definition 2.9]). Let $f, g \in \mathbb{C}\{x\}$.

(a) We say $f$ and $g$ are right equivalent if there exists a biholomorphic map germ $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g = f \circ \varphi$.

(b) We say $f$ and $g$ are contact equivalent if there exists a biholomorphic map germ $\varphi: (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and a unit $u \in \mathbb{C}\{x_1, \ldots, x_n\}$ such that $g = u(f \circ \varphi)$.

**Remark 2.3** ([GLS07, Remark 2.9.1(3)]). Two convergent power series $f, g \in \mathbb{C}\{x\}$ are contact equivalent if and only if the complex analytic space germs $(X, 0)$ and $(Y, 0)$ are isomorphic, where $X \subseteq \mathbb{C}^n$ is given by the zeros of $f$ and $Y \subseteq \mathbb{C}^n$ is given by the zeros of $g$.

We use the following proposition in Section 3 to parametrize sextic double solids with a $cA_1$ singularity:

**Proposition 2.4** ([GLS07, Remark 2.50.1]). Let $f, g \in \mathbb{C}\{x_1, \ldots, x_n\}$ be two contact equivalent power series with zero constant term. Then their multiplicity $m$ is the same and furthermore, $f_m$ and $g_m$ are the same up to an invertible linear change of coordinates.

We use the following proposition in Section 3 to construct sextic double solids with a $cA_n$ singularity where $n \geq 2$, as well as in Section 4 to describe weighted blowups of $cA_n$ points:

**Proposition 2.5.** Let $F = x_1^2 + \ldots + x_k^2 + f$ and $G = x_1^2 + \ldots + x_k^2 + g$, where $f$ and $g$ are convergent power series in $\mathbb{C}\{x_{k+1}, \ldots, x_n\}$ with zero constant term. Then $F$ and $G$ are contact (respectively, right) equivalent if and only if $f$ and $g$ are contact (respectively, right) equivalent.
Proof. By a result of Mather and Yau [MY82] (see also [GLS07, Theorem 2.26]), \( f \) and \( g \) are contact equivalent if and only the Tjurina algebras \( T_f \) and \( T_g \) are isomorphic. A simple computation shows that \( T_f \cong T_F \) and \( T_g \cong T_G \), which proves the proposition for contact equivalence.

The proof for right equivalence is similar. Namely, we use a statement analogous to [MY82]: two elements \( h, k \in \mathbb{C}\{x\} \) with zero constant term are right equivalent if and only if the Milnor algebras \( M_h \) and \( M_k \) are isomorphic as algebras over the ring \( \mathbb{C}\{t\} \), where \( t \) acts on \( M_h \), respectively \( M_k \), by multiplying by \( h \), respectively \( k \) (see [GLS07, Theorem 2.28]).

Reid defined in [Rei80, Definition 2.1] that a compound Du Val singularity is a 3-dimensional singularity where a hypersurface section is a Du Val singularity, also called a surface ADE singularity. The singularity is denoted \( cA_n \), \( cD_n \) or \( cE_n \), respectively, if the general hyperplane section is an \( A_n \), \( D_n \) or \( E_n \) singularity, respectively. Reid showed in [Rei83, Theorem 0.6] that a 3-dimensional hypersurface singularity is terminal if and only if it is an isolated compound Du Val singularity.

In this paper, we focus on the most general class of compound Du Val singularities, namely \( cA_n \) singularities. Since a surface \( A_n \) singularity is given by \( x^2 + y^2 + z^{n+1} \), we have the following almost immediate corollary:

**Corollary 2.6.** Let \( n \) be a positive integer. A singularity is of type \( cA_n \) if and only if it is isomorphic to the complex analytic space germ \( (X, 0) \) where \( X \subseteq \mathbb{C}^4 \) is given by the zero set of \( x^2 + y^2 + g_{\geq n+1}(z, t) \) with variables \( x, y, z, t \) where \( g \in \mathbb{C}\{z, t\} \) is a convergent power series of multiplicity \( n + 1 \).

The simplest example of a \( cA_1 \) singularity is the ordinary double point, given by \( x^2 + y^2 + z^2 + t^2 \). Note that terminal sextic double solids have only hypersurface singularities, therefore only \( cA_n \), \( cD_n \) and \( cE_n \) singularities.

### 2.2. Divisorial contractions

The first step in a Sarkisov link for a Fano variety is a divisorial contraction.

**Definition 2.7.** Let \( \varphi : Y \to X \) be a proper morphism with connected fibres between varieties with terminal singularities. We say \( \varphi \) is a divisorial contraction if the exceptional locus of \( \varphi \) is a prime divisor and \( -K_Y \) is \( \varphi \)-ample.

Theorem 2.10 says that divisorial contractions to \( cA_n \) points are weighted blowups. First, we recall the definition of a weighted blowup in both the algebraic and the analytic categories.

**Definition 2.8.** We say that two birational morphisms of varieties (or bimeromorphic holomorphisms of complex analytic spaces) \( \varphi : Y \to X \) and \( \varphi' : Y' \to X' \) are equivalent if there exist isomorphisms \( X \cong X' \) and \( Y \cong Y' \) such that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\varphi} & Y' \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
X & \xrightarrow{} & X'
\end{array}
\]

commutes. We say \( \varphi \) and \( \varphi' \) are locally equivalent if there exist isomorphic open subsets \( U \subseteq X \) and \( U' \subseteq X' \) containing the centres of the morphisms \( \varphi \) and \( \varphi' \) such that the restrictions \( \varphi|_{\varphi^{-1}U} : \varphi^{-1}U \to U \) and \( \varphi'|_{\varphi'^{-1}U'} : \varphi'^{-1}U' \to U' \) are equivalent.
If we consider the complex analytic space corresponding to a variety or when we wish to emphasize that we are working in the category of complex analytic spaces, we sometimes say **analytically equivalent** or **locally analytically equivalent**.

**Definition 2.9.** Let $w = (w_1, \ldots, w_n)$ be positive integers, called the weights of the blowup. Define a $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ by $\lambda \cdot (u, x_1, \ldots, x_n) = (\lambda^{-1}u, \lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n)$ and define $T$ by the geometric quotient $(\mathbb{C}^{n+1} \setminus \mathbb{V}(x_1, \ldots, x_n))/\mathbb{C}^*$ (or its analytification). Then, the morphism $\varphi : T \rightarrow \mathbb{C}^n$, $[u, x_1, \ldots, x_n] \mapsto (x_1u^{w_1}, \ldots, x_nu^{w_n})$ is called the $w$-blowup of $\mathbb{C}^n$ at the origin $0$. If $Z \subseteq \mathbb{C}^n$ is a subvariety (or a complex analytic subspace) containing $0$ and $\tilde{Z}$ is its strict transform, then the restriction $\varphi|_{\tilde{Z}} : \tilde{Z} \rightarrow Z$ is called the $w$-blowup of $Z$ at $0$. Let $\psi : Y \rightarrow X$ be a birational morphism of varieties (or bimeromorphic holomorphism of complex analytic spaces). Given an open subset $U \subseteq X$ containing the centre of $\psi$ and an isomorphism $U \cong X' \subseteq \mathbb{C}^n$ taking a point $P \in X$ to $0$, $\psi$ is called the $w$-blowup of $X$ at $P$ if the restriction $\psi|_{\psi^{-1}U} : \psi^{-1}U \rightarrow U$ is equivalent, through the given isomorphism $U \cong X'$, to the $w$-blowup of $X'$ at $0$.

Note that a weighted blowup crucially depends on the choice of coordinates, that is, on the isomorphism $U \cong X'$, even though it is not explicit in the notation.

Kawakita [Kaw03] described divisorial contractions to $cA_n$ points. Notational differences from [Kaw03, Theorem 1.13] are that below we have left out the description for $cA_1$ singularities and an exceptional case for $cA_2$. Also, we have written out the converse statement more explicitly (that being a Kawakita blowup implies that it is a divisorial contraction).

**Theorem 2.10 ([Kaw03, Theorem 1.13]).** Let $P$ be a $cA_n$ point where $n \geq 3$ of a variety $X$ with terminal singularities. Let $\varphi : Y \rightarrow X$ be a morphism of varieties such that the restriction $\varphi|_{Y \setminus E} : Y \setminus E \rightarrow X \setminus \{P\}$ is an isomorphism, where the reduced closed subvariety $E$ is given by $\varphi^{-1}\{P\}$. If $\varphi$ is a divisorial contraction, then $\varphi$ is locally analytically equivalent to the $(r_1, r_2, a, 1)$-blowup of $\mathbb{V}(x_1x_2 + g(x_3, x_4)) \subseteq \mathbb{C}^4$ at $0$ with variables $x_1, x_2, x_3, x_4$ where

1. $a$ divides $r_1 + r_2$ and is coprime to both $r_1$ and $r_2$,
2. $g$ has weight $r_1 + r_2$, and
3. the monomial $x_3^{(r_1 + r_2)/a}$ appears in $g$ with non-zero coefficient.

Moreover, any $\varphi$ which is locally analytically equivalent to a weighted blowup as above is a divisorial contraction, even for $n = 2$.

Any weighted blowup that is locally analytically equivalent to $\varphi$ in Theorem 2.10 for $n \geq 2$ is called a $(r_1, r_2, a, 1)$-Kawakita blowup, or simply a Kawakita blowup.

### 2.3. Sarkisov links

One of the possible outcomes of the minimal model program is a Mori fibre space:

**Definition 2.11.** A Mori fibre space is a morphism of normal projective varieties $\varphi : X \rightarrow S$ with connected fibres such that

1. $X$ is $\mathbb{Q}$-factorial and has terminal singularities,
2. the anticanonical class $-K_X$ is $\varphi$-ample,
3. $X/S$ has relative Picard number 1, and
4. $\dim S < \dim X$. 

7
If \( \dim S > 0 \), then we say \( \varphi \) is a \textit{strict} Mori fibre space.

The main examples of Mori fibre spaces we see in this paper are Fano 3-folds that are projective, \( \mathbb{Q} \)-factorial, with terminal singularities and Picard number 1, considered as a morphism over a point.

Any birational map between two Mori fibre spaces is a composition of Sarkisov links (see [Cor95] or [HM13]). Below, we describe the two possible types of Sarkisov links starting from a Fano variety.

**Definition 2.12.** A \textbf{Sarkisov link} of type I (respectively II) between a Fano variety \( X \) and a strict Mori fibre space \( Y_k \to Z \) (respectively Fano variety \( Z \)) is a diagram of the form

\[
Y_0 \longrightarrow \cdots \longrightarrow Y_k
\]

where \( X, Y_0, \ldots, Y_k, Z \) are normal, projective and \( \mathbb{Q} \)-factorial, the varieties \( X, Y_0, \ldots, Y_k \) have terminal singularities, \( Z \) has terminal singularities if it is 3-dimensional, \( X \) has Picard number 1, \( \varphi: Y_0 \to X \) is a divisorial contraction, \( Y_0 \to \cdots \to Y_k \) is a sequence of anti-flips, flops and flips, and \( \psi: Y_k \to Z \) is a strict Mori fibre space (respectively divisorial contraction). If we do not require the varieties \( X, Y_0, \ldots, Y_k \) (respectively \( X, Y_0, \ldots, Y_k, Z \)) to be terminal and we do not require \(-K_{Y_0}\) to be \( \varphi \)-ample and we do not require \(-K_{Y_k}\) to be \( \psi \)-ample but all the other properties hold, then the diagram above is called a \textbf{2-ray link} ([BZ10, Definition 2.1]).

**Definition 2.13.** A Fano 3-fold \( X \) that is a Mori fibre space is \textbf{birationally rigid} if for any Mori fibre space \( Y \to S \) such that \( X \) and \( Y \) are birational, we have that \( S \) is a point and \( X \) and \( Y \) are isomorphic.

In Section 5, we show that a general sextic double solid \( X \) with a \( cA_n \) singularity with \( n \geq 4 \) which is a Mori fibre space is not birationally rigid. We show this by explicitly constructing a Sarkisov link between \( X \) and another Mori fibre space. We find the Sarkisov link by restricting from a toric 2-ray link, as described in Construction 2.14.

See [Cox95] for the definition of Cox rings for toric varieties (where it is called the \textit{homogeneous coordinate ring}), and [HK00, Definition 2.6] for the definition of Cox rings for Mori dream spaces. Note that isomorphic varieties can have different Cox rings. By [Cox95, Theorem 3.7], closed subschemes of a toric variety \( T \) with only cyclic quotient singularities are given by homogeneous ideals in the Cox ring \( \text{Cox} T \), which is a polynomial ring.

**Construction 2.14.** Let \( X \) be a Fano variety embedded in a weighted projective space \( \mathbb{P} \), where \( X \) is a Mori fibre space, and let \( Y_0 \to X \) be a divisorial contraction from a projective \( \mathbb{Q} \)-factorial variety \( Y \). By [AK16, Lemma 2.9], the divisorial contraction \( Y_0 \to X \) is part of a Sarkisov link only if \( Y_0 \) is a Mori dream space.

By [HK00, Proposition 2.11], we can embed a Mori dream space \( Y_0 \) into a projective toric variety \( T_0 \) with cyclic quotient singularities such that the Mori chambers of \( Y_0 \) are unions of finitely many Mori chambers of \( T_0 \). Moreover, we can embed \( Y_0 \) in such a way that \( Y_0 \) is given by a homogeneous ideal \( I_Y \) in \( \text{Cox} T_0 \), and the toric 2-ray link

\[
T_0 \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_r
\]

\[
\mathbb{P} \quad T_{X_0} \quad \cdots \quad S_T
\]
restricts to a 2-ray link

\[
\begin{array}{cccccc}
Y_0 & \longrightarrow & Y_1 & \longrightarrow & \cdots & \longrightarrow & Y_r \\
X & \rightarrow & X_0 & \rightarrow & \cdots & \rightarrow & S_i,
\end{array}
\]

where each \( Y_i \subseteq T_i \) is given by the same ideal \( I_Y \subseteq \text{Cox} \, T_0 = \ldots = \text{Cox} \, T_r \), and \( X_i \subseteq T_{X_i} \) is given by the ideal \( I_Y \cap \mathbb{C}[v_0, \ldots, v_s] \), where \( T_{X_i} \) is given by \( \text{Proj} \mathbb{C}[v_0, \ldots, v_s] \) for some polynomials \( v_j \in \text{Cox} \, T_0 \) that depend on \( i \) (see [AZ16, Remark 4]). In this case, \( \text{Cox}(T_0)/I_Y \) is a Cox ring for \( Y_0 \) and it is said that \( I_Y \) 2-ray follows \( T_0 \) ([AZ16, Definition 3.5]).

Note that some of the small birational maps \( T_i \rightarrow T_{i+1} \) may restrict to isomorphisms \( Y_i \rightarrow Y_{i+1} \). If all the varieties \( Y_i \) are terminal and the anticanonical divisor \( -K_{Y_0} \) of \( Y_0 \) is inside the interior int(Mov \( Y_0 \)) of the movable cone, then the 2-ray link for \( Y_0 \) is a Sarkisov link (see [AK16, Lemma 2.9]), otherwise it is called a bad link.

In Section 5, where \( X \) is a sextic double solid and the centre of \( Y_0 \rightarrow X \) is a \( cA_n \) point, we use a projective version of Corollary 4.9 to construct the divisorial contraction \( Y_0 \rightarrow X \), which is the restriction of a toric weighted blowup \( \bar{T}_0 \rightarrow \mathbb{P} \). This gives us an embedding \( Y_0 \rightarrow \mathbb{V}(I_Y) \subseteq \bar{T}_0 \) where \( I_Y \) might not 2-ray follow \( T_0 \). We use unprojection to modify \( \bar{T}_0 \) to find an embedding \( Y_0 \rightarrow \mathbb{V}(I_Y) \subseteq T_0 \) such that \( I_Y \) 2-ray follows \( T_0 \). See [Rei00, Section 2.1] for a simple example of unprojection, and Sections 5.2, 5.5, 5.6 and 5.8 for applications of unprojection.

To explain the notation we use for 2-ray links, we do an example in detail, namely the 2-ray link for the ambient space of the sextic double solid with a \( cA_4 \) singularity in Section 5.2.

**Example 2.15** (2-ray link for \( \mathbb{P}(1, 1, 1, 1, 3, 5) \)). Denote the variables on \( \mathbb{P}(1, 1, 1, 1, 3, 5) \) by \( x, y, z, t, \alpha, \xi \). We perform the weighted blowup \( T_0 \rightarrow \mathbb{P}(1, 1, 1, 1, 3, 5) \) with weights \( (1, 1, 2, 3, 6) \) for variables \( y, z, t, \alpha, \xi \), where the centre is the point \( P_x = [1, 0, 0, 0, 0, 0] \).

We define \( T_0 \) as a geometric quotient. By a slight abuse of notation we denote the variables on \( \mathbb{C}^7 \) by \( u, x, y, z, \alpha, \xi, t \), repeating the symbols for \( u \) and \( t \). Define a \((\mathbb{C}^*)^2\)-action on \( \mathbb{C}^7 \) for all \((\lambda, \mu) \in (\mathbb{C}^*)^2\) by

\[
(\lambda, \mu) \cdot (u, x, y, z, \alpha, \xi, t) = (\mu^{-1} u, \lambda x, \lambda \mu y, \lambda \mu z, \lambda^3 \mu^3 \alpha, \lambda^5 \mu^5 \xi, \lambda \mu^2 t).
\]

Define the irrelevant ideal \( I_0 = (u, x) \cap (y, z, \alpha, \xi, t) \), and define \( T_0 \) by the geometric quotient \( \mathbb{C}^7 \setminus \mathbb{V}(I_0)/(\mathbb{C}^*)^2 \). We use the notation

\[
\begin{array}{cccccc}
u & u & x & y & z & \alpha & \xi & t \\
0 & 1 & 1 & 1 & 3 & 5 & 1 \\
-1 & 0 & 1 & 1 & 3 & 6 & 2
\end{array}
\]

for \( T_0 \). Note that we order the variables \( u, x, \ldots, t \) such that the corresponding rays \((\frac{1}{3}, 1), (\frac{1}{5}, 1), \ldots, (\frac{1}{5}, 1)\) are ordered anti-clockwise around the origin. The vertical bar indicates that the irrelevant ideal is \((u, x) \cap (y, z, \alpha, \xi, t)\). The Cox ring of \( T_0 \) is given by \( \text{Cox} \, T_0 = \mathbb{C}[u, x, y, z, \alpha, \xi, t] \). The weighted blowup \( T_0 \rightarrow \mathbb{P}(1, 1, 1, 1, 3, 5) \) is given by

\[
[u, x, y, z, \alpha, \xi, t] \mapsto [x, uy, uz, u^2 t, u^3 \alpha, u^6 \xi].
\]

We describe the cones of the toric variety \( T_0 \). By [HK00], \( T_0 \) is a Mori dream space. The Picard group of \( T_0 \) is generated by \( \mathbb{V}(u) \), the reduced exceptional divisor, and \( \mathbb{V}(x) \),
the strict transform of a plane not passing through $P_x$, which have bidegree $(0,1)$ and $(1,0)$, respectively. The variety $T_0$ is $\mathbb{Q}$-factorial, and any two divisors with the same bidegree are linearly equivalent. As in [BZ10, Section 4.1.3], the effective cone $\text{Eff}(T_0)$ is given by $\langle \mathcal{V}(u), \mathcal{V}(x) \rangle$, a cone in the group $N^1(T_0)$ of divisors of $T_0$ up to numerical equivalence with coefficients in $\mathbb{R}$. As in [AZ16, Section 3.2], the movable cone $\text{Mov}(T_0)$ is $\langle \mathcal{V}(x), \mathcal{V}(\xi) \rangle$, and it is divided into the nef cone $\text{Nef}(T_0) = \langle \mathcal{V}(x), \mathcal{V}(y) \rangle$ of $T_0$ and $\langle \mathcal{V}(y), \mathcal{V}(\xi) \rangle$, which is the pull-back of the nef cone of the small $\mathbb{Q}$-factorial modification $T_1$ of $T_0$. The cones $\langle \mathcal{V}(x), \mathcal{V}(y) \rangle$ and $\langle \mathcal{V}(y), \mathcal{V}(\xi) \rangle$ are called Mori chambers. The variety $T_1$ is defined by

$$T_1 = \left( \begin{array}{cccccc} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & 0 & 1 & 1 & 3 & 6 & 2 \end{array} \right).$$

Here, $T_1$ is the geometric quotient $(\mathbb{C}^7 \setminus I_1)/(\mathbb{C}^*)^2$, where the irrelevant ideal $I_1$ is given by $(u, x, y, z, \alpha) \cap (\xi, t)$, which is indicated by the position of the vertical bar in the action-matrix. The Cox ring of $T_1$ is equal to the Cox ring of $T_0$, namely $\text{Cox}T_1 = \mathbb{C}[u, x, y, z, \alpha, \xi, t]$.

The weighted blowup morphism $T_0 \to \mathbb{P}(1^3, 1, 1, 1, 3, 5)$ can be read off from the action-matrix of $T_0$. Consider the ray given by $\mathcal{V}(x)$ in $N^1(T_0)$. The union of the linear systems $|n \mathcal{V}(x)|$ where $n \geq 0$ has a $\mathbb{C}$-algebra basis $x, uy, uz, u^2t, u^3\alpha, u^6\xi$. So, the ample model (see [BCHM10, Definition 3.6.5]) of the divisor class $\mathcal{V}(x)$ is the morphism

$$T_0 \to \text{Proj} \bigoplus_{n \geq 0} H^0(T_0, \mathcal{O}_{T_0}(n \mathcal{V}(x))) = \text{Proj} \mathbb{C}[x, uy, uz, u^2t, u^3\alpha, u^6\xi] = \mathbb{P}(1, 1, 1, 1, 3, 5)$$

given by

$$[u, x, y, z, \alpha, \xi, t] \mapsto [x, uy, uz, u^2t, u^3\alpha, u^6\xi],$$

which is precisely the weighted blowup $T_0 \to \mathbb{P}(1, 1, 1, 1, 3, 5)$ given in Equation (2.1).

As in [BZ10, Section 2.1], there are two projective morphisms of relative Picard number 1 from $T_0$ up to isomorphisms, corresponding to the ample models of divisors in the two edges of the nef cone of $T_0$. The ample model of any divisor in the interior of the nef cone of $T_0$ gives an embedding of $T_0$ into a weighted projective space. The ample model of $\mathcal{V}(y) \in N^1(T_0)$ is given by

$$T_0 \to \text{Proj} \mathbb{C}[y, z, u, ut, x, xt] \subseteq \mathbb{P}(1, 1, 3, 5, 1, 6, 2)$$

$$[u, x, y, z, \alpha, \xi, t] \mapsto [y, z, \alpha, u, ut, x, xt].$$

\begin{figure}
\begin{center}
(a) Rays in $N^1(T_0)$ \hspace{1cm} (b) Ample models of $T_0$
\end{center}
\includegraphics{cone.png}
\caption{Cones of $T_0$}
\end{figure}
Denoting $T_{x_0} = \text{Proj} \mathbb{C}[y, z, \alpha, u\xi, ut, x\xi, xt]$, we see that the morphism $T_0 \to T_{x_0}$ contracts $V(\xi, t)$ to the surface $\mathbb{P}(1, 1, 3) \subseteq T_{x_0}$ and is an isomorphism elsewhere. The ample model of $V(y) \in N^1(T_1)$ is given similarly by 

$$T_1 \to \text{Proj} \mathbb{C}[y, z, \alpha, u\xi, ut, x\xi, xt] = T_{x_0},$$

contracting $V(u, x)$ to $\mathbb{P}(1, 1, 3)$. This induces a birational map $T_0 \dasharrow T_1$, a small $\mathbb{Q}$-factorial modification, given by 

$$[u, x, y, z, \alpha, \xi, t] \mapsto [u, x, y, z, \alpha, \xi, t].$$

The diagram $T_0 \to T_{x_0} \leftarrow T_1$ is a flop.

Note that multiplying the action-matrix of $T_0$ or $T_1$ with a matrix in $\text{GL}(2, \mathbb{Q})$ is equivalent to choosing a different basis for the group $(\mathbb{C}^*)^2$, so the geometric quotients $T_0$ and $T_1$ stay the same (see [Ahm17, Lemma 2.4]). If we multiply with a matrix with negative determinant, then we change the order of the rays in $N^1(T_0)$ from anti-clockwise to clockwise.

Similarly, there are only two projective morphisms of relative Picard number 1 from $T_1$: the contraction $T_1 \to T_{x_0}$ and the ample model of $V(\xi)$. We multiply the action-matrix of $T_1$ by the matrix \[
\begin{pmatrix}
6 & -5 \\
-2 & -1 \\
\end{pmatrix}
\]
with determinant 4 to find

$$T_1 : \begin{pmatrix}
u & x & y & z & \alpha & \xi & t \\
5 & 6 & 1 & 1 & 3 & 0 & -4 \\
1 & 2 & 1 & 1 & 3 & 4 & 0 \end{pmatrix}.$$

The ample model of $V(\xi)$ is given by

$$T_1 \to \mathbb{P}(1, 1, 1, 2, 3, 4) \quad [u, x, y, z, \alpha, \xi, t] \mapsto \left[t^{\frac{5}{2}}u, t^{\frac{1}{2}}y, t^{\frac{1}{2}}z, t^{\frac{3}{2}}x, t^{\frac{3}{2}}\alpha, \xi \right].$$

Note that this is a morphism of varieties despite having fractional powers (see [BB13]).

The 2-ray link that we have found for $\mathbb{P}(1, 1, 1, 1, 3, 5)$ is summarized by the diagram below.

For more examples on toric 2-ray links, see [BZ10, Section 4].

**3. Constructing sextic double solids with a $cA_n$ singularity**

In this section, we give a bound $n \leq 8$ for an isolated $cA_n$ singularity on a sextic double solid, and we explicitly describe all sextic double solids that contain an isolated $cA_n$ singularity where $n \leq 8$. The main tool we use for this is the splitting lemma from singularity theory, first introduced in [Tho72], which is used for separating the quadratic terms and the higher order terms of a power series.
3.1. Splitting lemma from singularity theory

Before we go into details, let us recall the statement of the splitting lemma. Here the statement is taken from [GLS07, Theorem 2.47], with a slight modification in notation. Specifically, we write \( v(x + p) \) instead of \( x + g \), where \( v \) is a unit in the power series ring and \( p \) does not depend on \( x \), as we use this form in Section 5 for constructing birational models.

**Theorem 3.1** (Splitting lemma). Let \( m \) be a positive integer and let \( y \) denote variables \((y_1, \ldots, y_m)\). Let \( f \in \mathbb{C}\{x, y\} \) be a convergent power series of multiplicity two, with degree two part of the form \( x^2 + \text{(terms in } y) \). Then, there exist unique \( v \in \mathbb{C}\{x, y\} \) and \( p, h \in \mathbb{C}\{[y]\} \), where \( v \) is a unit and the multiplicity of \( p \) is at least two, such that

\[
f = (v(x + p))^2 + h.
\]

Moreover, the power series \( h, p \) and \( v \) are absolutely convergent around the origin, and the multiplicity of \( h \) is at least two. It follows immediately that \( f \) is right equivalent to \( x^2 + h \).

**Proof.** It is proved in [GLS07, Theorem 2.47] that there exist unique \( g \in \mathbb{C}\{[x, y]\} \) and \( h \in \mathbb{C}\{[y]\} \), where the multiplicity of \( g \) is at least two, such that \( f = (x + g)^2 + h \). Moreover, it is proved that the power series \( g \) and \( h \) are absolutely convergent around the origin, and the multiplicity of \( h \) is at least two.

By the Weierstrass preparation theorem (see [GLS07, Theorem 1.6]), there exists a unique unit \( v \in \mathbb{C}\{x, y\} \) and a unique \( p \in \mathbb{C}\{y\} \) such that \( x + g = v(x + p) \). \(\square\)

Below we give explicit recurrent formulas for \( g, h, p, v \) of the splitting lemma in terms of the coefficients of \( f \), which is implemented in Listing 1 in Appendix A.

**Proposition 3.2** (Explicit splitting lemma). Below, we use the same notation as in the splitting lemma Theorem 3.1 and its proof. Denote

\[
f = \sum_{i,d \geq 0} x^i f_{i,d}, \quad g = \sum_{i,d \geq 0} x^i g_{i,d}, \quad h = \sum_{d \geq 0} h_d, \quad p = \sum_{d \geq 0} p_d, \quad v = \sum_{i,d \geq 0} x^i v_{i,d}
\]

where \( f_{i,d}, g_{i,d}, h_d, p_d, v_{i,d} \in \mathbb{C}\{y\} \) are homogeneous of degree \( d \). Then,

\[
g_{1,0} = 0,
\]

\[
g_{i,d} = \frac{1}{2} \left( f_{i+1,d} - \sum_{k=0}^{d} \min(i+1,i-d-k-1) \sum_{j=0}^{k} g_{j,k} g_{i+1-k,j-d-k} \right), \quad \text{if } (i, d) \neq (1, 0), \tag{3.1}
\]

\[
h_d = f_{0,d} - \sum_{j=0}^{d} g_{0,d-j}, \tag{3.2}
\]

\[
p_d = g_{0,d} - \sum_{j=2}^{d-1} v_{0,d-j} p_j, \tag{3.3}
\]

\[
v_{0,0} = 1,
\]

\[
v_{i,d} = g_{i+1,d} - \sum_{j=2}^{d} (v_{i+1,d-j} p_j), \quad \text{if } (i, d) \neq (0, 0). \tag{3.4}
\]

**Proof.** Taking the degree \( d \) part of the coefficient of \( x^{i+1} \) in \( f = (x + g)^2 + h \) where \( i \geq 0 \), we find Equation (3.1). Taking all degree \( d \) terms of \( f = (x + g)^2 + h \) that are not divisible by \( x \), we find Equation (3.2). Taking the degree \( d \) part of the coefficient of \( x^{i+1} \) in \( x + g = v(x + p) \) where \( i \geq 0 \), we find Equation (3.4), and taking all degree \( d \) terms not divisible by \( x \), we find Equation (3.3). \(\square\)
Example 3.3. Using the notation of Proposition 3.2, the first few homogeneous parts of $h$ are given in terms of coefficients of $f$ by

$$
\begin{align*}
    h_2 &= f_{0,2} \\
    h_3 &= f_{0,3} \\
    h_4 &= f_{0,4} - \frac{f_{1,2}^2}{4} \\
    h_5 &= f_{0,5} - \frac{f_{1,2}^2 f_{2,1}}{4} - \frac{f_{1,2} f_{1,3}}{2} \\
    h_6 &= f_{0,6} - \frac{f_{1,2}^2 f_{3,0}}{8} + \frac{f_{1,2}^2 f_{2,2}}{4} - \frac{f_{1,2}^2 f_{2,1}}{4} + \frac{f_{1,2} f_{1,3} f_{2,1}}{2} - \frac{f_{1,2} f_{1,4}}{2} - \frac{f_{1,3}^2}{4}.
\end{align*}
$$

3.2. Isolated $cA_n$ singularity

Now, we apply the explicit splitting lemma (Proposition 3.2) to the case we are most interested in, that is, sextic double solids. First, we describe the equation of a sextic double solid $X \subseteq \mathbb{P}(1, 1, 1, 3)$ that has a singular point $P$. The following argument shows that without loss of generality, we can move the singular point to $P_x = [1, 0, 0, 0, 0]$ using an automorphism of $\mathbb{P}(1, 1, 1, 1, 3)$. Denote $P = [P_0, P_1, P_2, P_3, P_4]$. Since $X$ does not contain the point $[0, 0, 0, 0, 1]$, there exists $0 \leq i \leq 3$ such that $P_i \neq 0$. After interchanging the variables if necessary, we find $P_0 \neq 0$. Now, the automorphism of $\mathbb{P}(1, 1, 1, 1, 3)$ given by

$$(x, y, z, t, w) \mapsto \left(\frac{P_1}{P_0}x, z - \frac{P_2}{P_0}x, t - \frac{P_3}{P_0}x, w - \frac{P_4}{P_0}x^3\right)$$

takes $P$ to $P_x$.

Below, the subindices denote degree and $\mathbb{V}(f)$ denotes the zero locus of a polynomial $f$.

Notation 3.4. Define the variety $X$ by

$$X: \mathbb{V}(f) \subseteq \mathbb{P}(1, 1, 1, 3),$$

with variables $x, y, z, t, w$ where

$$f = -w^2 + x^4 t^2 + x^4 z^2 + x^3 (4t^3 a_0 + 4t^2 a_1 + 2t a_2 + a_3) + x^2 (2t^4 b_0 + 2t^3 b_1 + 2t^2 b_2 + 2t b_3 + b_4) + x (2t^5 c_0 + 2t^4 c_1 + 2t^3 c_2 + 2t^2 c_3 + 2t c_4 + c_5) + t^6 d_0 + 2t^5 d_1 + t^4 d_2 + 2t^3 d_3 + t^2 d_4 + 2t d_5 + d_6,$$

where the polynomials $\xi_j, a_j, b_j, c_j, d_j \in \mathbb{C}[y, z]$ are homogeneous of degree $j$.

Now, define the following technical conditions:

1. $\xi_2 = 0$.
2. $a_3 = 0$.
3. $b_4 = a_2^2$.
4. $c_5 = 2a_2 b_3 - 4a_1 a_2^2$.
5. $d_6 = 2a_2 c_4 + b_3^2 - 8a_1 a_2 b_3 - 2a_2 b_2 + 4a_0 a_3^2 + 16a_1^2 a_2^2$. 

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7. There exist unique $q, r, s, e \in \mathbb{C}[y, z]$ where $r$ and $s$ do not have a common prime divisor, and $q$ and $e$ do not have a common prime divisor, such that

\[
\begin{align*}
  a_2 &= qr \\
  b_3 &= qs + 4a_1qr \\
  c_4 &= 2a_1qs - 6a_0q^2r^2 + 8a_1qr + er \\
  d_5 &= 2b_2qs - 8a_1^2qs - es - b_1q^2r^2 + c_3qr,
\end{align*}
\]

and $q, r, s, e$ are respectively homogeneous of degrees

(7.1) $0, 2, 3, 2$ and $q = 1$,
(7.2) $1, 1, 2, 3$ and the leading coefficient of $q$ under the ordering $y < z$ is one,
(7.3) $2, 0, 1, 4$ and $r = 1$, or
(7.4) $3, *, 0, 5$ and $r = 0$ and $s = 1$ (since $r = 0$, ‘$*$’ denotes that $r$ is homogeneous of any non-negative degree).

8. Condition (7.1) holds and there exists a unique $A_0 \in \mathbb{C}$ and a unique polynomial $B_1 \in \mathbb{C}[y, z]$ homogeneous of degree 1 such that

\[
\begin{align*}
  e_2 &= 4A_0r_2 + b_2 - 6a_1^2 \\
  c_3 &= 6a_0s_3 - 4A_0s_3 + 4a_0a_1r_2 - 8A_0a_1r_2 + B_1r_2 + 2a_1b_2 - 4a_1^3 \\
  d_4 &= -2s_3B_1 + 16r_2^2A_0^2 - 8b_2r_2A_0 + 16a_2^2r_2A_0 + 4b_1s_3 \\
  &\quad - 8a_0a_1s_3 - 2b_0r_2^2 + 2c_2r_2 + b_2^2 - 4a_1^2b_2 + 4a_1^4.
\end{align*}
\]

Notation 3.4 describes 11 families of sextic double solids, namely when conditions 1 to $n$ are satisfied for some $n \leq 8$. For $n = 7$, there are 4 families. Below, general means ‘in a Zariski open dense subset of the family’, and very general means ‘outside a countable union of proper Zariski closed subsets of the family’. Using the above notation, we state the main theorem of this section, describing sextic double solids with an isolated $cA_n$ singularity.

**Theorem A.**

(a) If a sextic double solid has an isolated $cA_n$ singularity, then $n \leq 8$.

Furthermore, for every positive integer $n \leq 8$:

(b) Every sextic double solid with an isolated $cA_n$ singularity $P$ is isomorphic to some $X$ satisfying conditions 2 to $n$ in Notation 3.4, with the isomorphism sending $P \mapsto P_x = [1, 0, 0, 0, 0]$.

(c) Every $X$ that satisfies conditions 2 to $n$ in Notation 3.4 and has otherwise general coefficients is smooth outside a $cA_n$ singularity at $P_x$.

(d) A very general sextic double solid with an isolated $cA_n$ singularity is factorial and has Picard number 1, except for the family (7.4) in Notation 3.4. No member of the family (7.4) is $\mathbb{Q}$-factorial.

**Remark 3.5.** Note that if $n = 1$, then the set of conditions 2 to $n$ is empty, so every variety with a $cA_1$ singularity is isomorphic to some $X$ in Notation 3.4, and every $X$ in Notation 3.4 that has general coefficients has a $cA_1$ singularity at $P_x$ and is smooth elsewhere. Note that zero is homogeneous of every non-negative degree, so for example in condition (7.1) of Notation 3.4, the term $e$ can be zero. Also, note that in conditions (7.1) and (7.2), the terms $r$ and $s$ must both be non-zero, otherwise either $r$ or $s$ is a common divisor of both $r$ and $s$. 

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Before we prove Theorem A, we state a few lemmas needed for the proof.

**Lemma 3.6.** If \( X \) in Notation 3.4 satisfies conditions 2 to 6 and \( P_x \) is an isolated singularity of \( X \), then either \( a_2 \neq 0 \) or \( b_3 \neq 0 \).

*Proof.* If conditions 2 to 6 hold and \( a_2 = b_3 = 0 \), then \( a_3 = b_1 = c_5 = d_6 = 0 \). Let \( C \) be the curve defined by the ideal \((t, w, 2xc_4 + 2d_5)\). Taking partial derivatives, we see that every point of \( C \) is a singular point of \( X \). Since \( C \) passes through \( P_x \), \( X \) does not have an isolated singularity at \( P_x \), a contradiction. \( \frown \)

**Lemma 3.7.** Let \( r, s \in \mathbb{C}[y, z] \) have no common prime divisors, and let \( q \in \mathbb{C}[y, z] \) be non-zero. Let \( h_n \in \mathbb{C}[y, z] \) be of the form \( h_n = q^a(r^\beta C_r - s^\gamma C_s) \) where \( C_r, C_s \in \mathbb{C}[y, z] \) and \( \alpha, \beta, \gamma \) are non-negative integers. Then

\[ h_n = 0 \iff \text{there exists } C \in \mathbb{C}[y, z] \text{ such that } C_r = s^\gamma C \text{ and } C_s = r^\beta C. \]

**Lemma 3.8.** If \( X \) in Notation 3.4 satisfies conditions 2 to 7 and \( P_x \) is an isolated singularity of \( X \), then \( q \) and \( e \) do not have a common prime divisor in \( \mathbb{C}[y, z] \).

*Proof.* If \( q \) and \( e \) have a common prime divisor \( D \), then \( D \) divides \( a_2, b_3, c_4, d_5 \), and \( D^2 \) divides \( a_3, b_4, c_5, d_6 \). Let \( C \) be the curve defined by the ideal \((D, t, w)\). Taking partial derivatives, we see that \( X \) is singular at every point of \( C \), so \( P_x \) is not isolated, a contradiction. \( \frown \)

**Lemma 3.9.** Denote the parameter space of all possible \( f \) in Notation 3.4 satisfying conditions 2 to \( n \) by \( \mathcal{P}_n \). Denote the parameter space of all \( f \in \mathcal{P}_n \) where \( \mathcal{V}(f) \) has a singular point with \( x, t \) and \( w \)-coordinate zero by \( \mathcal{A}_n \). Then \( \dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2 \).

*Proof.* Let \( P = [0, \beta, \gamma, 0, 0] \) be a singular point of \( \mathcal{V}(f) \) where \( f \in \mathcal{A}_n \) and \( \beta, \gamma \in \mathbb{C} \). We find

\[
 f(P) = d_6(P), \quad \frac{\partial f}{\partial x}(P) = c_5(P), \quad \frac{\partial f}{\partial y}(P) = \frac{\partial d_6}{\partial y}(P), \quad \frac{\partial f}{\partial z}(P) = \frac{\partial d_6}{\partial z}(P), \quad \frac{\partial f}{\partial t}(P) = 2d_5(P). 
\]

Define the polynomial \( l = \gamma y - \beta z \). Since \( P \) is a singular point, we have

\[ l \text{ divides } c_5, d_5 \text{ and } d_6, \text{ and } l^2 \text{ divides } d_6. \tag{3.5} \]

We use the divisibility constraint (3.5) repeatedly below.

1. If \( n \leq 5 \), then there are no restrictions on \( d_6 \) or \( d_5 \) in \( \mathcal{P}_n \). For \( \mathcal{A}_n \), we have the restrictions that \( l^2 \mid d_6 \) and \( l \mid d_5 \). In particular, \( d_6 \) has a square factor which is also a factor of \( d_5 \). We find that \( \dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2 \).

2. If \( n = 6 \), then there are no restrictions on \( d_5, a_2, b_3 \) or \( c_4 \) in \( \mathcal{P}_n \). We have \( c_5 = a_2(2b_3 - a_1a_2) \) and \( d_6 = a_2 \cdot (\ldots) + b_3^2 \). Below we consider \( f \in \mathcal{A}_n \).

If \( l \) divides \( a_2 \), then using the divisibility constraint (3.5), we find that \( l \) divides \( b_3 \). So, there are at least two less degrees of freedom in choosing \( a_2, b_3 \) and \( d_5 \).

If \( l \) does not divide \( a_2 \), then \( l \) divides \( 2b_3 - a_1a_2 \), and from \( l \mid d_6 \) we find that \( l \) also divides \( 8c_4 - 4a_0b_3 + 8a_0b_3^2 + a_1^2a_2 \). So, after fixing \( a_0, a_1, a_2 \) and \( b_2 \), there are at least two less degrees of freedom in choosing \( b_3, c_4 \) and \( d_5 \).

In either case, we see that \( \dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2 \).
3. If $n = 7$, then
\[
\begin{align*}
c_5 &= 4q^2 r(2s + a_1 r) \\
d_5 &= -es + q(2b_2 s - a_1^2 s - 4b_1 q r^2 + c_3 r) \\
d_6 &= 4q(es^2 + q(s^2 + a_1 rs - 8a_0 q r^3 - b_2 r^2 + a_1^2 r^2)).
\end{align*}
\]

Let us consider $f \in \mathcal{A}_n$. If $l \mid q$, then since $q$ and $e$ are coprime, we have $l \mid r$ and $l \mid s$, a contradiction. If $l \mid r$, then since $l \mid d_6$, we find $l \mid s$, a contradiction. Therefore, $l$ divides neither $q$ nor $r$.

So, $l$ divides $2s + a_1 r$. Using $l^2 \mid d_6$, we see that $l^2$ divides $-32a_0 q^2 r - 4b_2 q + 3a_1^2 q + 4e$. After fixing $a_0, a_1, b_2, q$ and $r$, we see that there are at least two less degrees of freedom in choosing $s$ and $e$. So, we have $\dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2$.

4. If $n = 8$, then
\[
\begin{align*}
c_5 &= 2r_2(s_3 + 2a_1 r_2) \\
d_5 &= r_2(r_2 B_1 - 8s_3 A_0 - 8a_1 r_2 A_0 + 6a_0 s_3 - b_1 r_2 + 4a_0 a_1 r_2 + 2a_1 b_2 - 4a_1^2) \\
&\quad + s_3(b_2 - 2a_1^2) \\
d_6 &= r_2(8s_3^2 A_0 + 4a_1 s_3 - 8a_0 r_2^2 + 4a_1^2 r_2) + s_3^2
\end{align*}
\]

We consider $f \in \mathcal{A}_n$. If $l \mid r_2$, then $l \mid s_3$, a contradiction. So, $l$ divides $s_3 + 2a_1 r_2$. Since $l$ divides $d_6$, we have $l \mid r_3^2(a_0 - a_0)$. So, $A_0 = a_0$. Since $l$ divides $d_5$, we see that $l \mid r_3^2(B_1 - b_1)$. We find that the coefficients of $f$ have at least three less degrees of freedom, namely $A_0 = a_0$, and the polynomials $B_1, b_1$ and $s_3 + 2a_1 r_2$ have a common prime divisor. So, we have $\dim \mathcal{A}_n \leq \dim \mathcal{P}_n - 2$.

\[\square\]

**Lemma 3.10.** Denote the parameter space of all possible $f$ in Notation 3.4 satisfying conditions 2 to 11 by $\mathcal{P}_n$. Denote the parameter space of all $f \in \mathcal{P}_n$ such that $\mathbb{N}(f)$ has a singular point at $P_t = [0, 0, 0, 1, 0]$ by $\mathcal{B}_n$. Then, $\dim \mathcal{B}_n = \dim \mathcal{P}_n - 4$.

**Proof.** We find
\[
f(P_t) = d_0, \quad \frac{\partial f}{\partial x}(P_t) = 2c_0, \quad \frac{\partial f}{\partial y}(P_t) = 2\frac{\partial d_1}{\partial y}, \quad \frac{\partial f}{\partial z}(P_t) = 2\frac{\partial d_1}{\partial z}, \quad \frac{\partial f}{\partial t}(P_t) = 6d_0.
\]

For $\mathcal{B}_n$, we have $d_0 = c_0 = d_1 = 0$. So, there are 4 less degrees of freedom in choosing coefficients for $f \in \mathcal{B}_n$, therefore $\dim \mathcal{B}_n = \dim \mathcal{P}_n - 4$.

\[\square\]

To show $\mathbb{Q}$-factoriality and that the Picard number is 1, we use the following proposition by Namikawa:

**Proposition 3.11** ([Nam97, Proposition 2]). Let $X$ be a Fano 3-fold with Gorenstein terminal singularities and $D$ its general anti-canonical divisor. Then, the natural homomorphism $\text{Pic}(X) \to \text{Pic}(D)$ is an injection.

**Corollary 3.12.** In the notation of Proposition 3.11, let $X$ be smooth along $D$. Then $\text{Cl}(X) \to \text{Pic}(D)$ is an injection.

**Proof.** Let $U$ be the smooth locus of $X$. Since $X \setminus U$ is of codimension at least 2 in $X$, we have $\text{Cl}(X) \cong \text{Cl}(U)$. It follows from the proof of Proposition 3.11 that $\text{Pic}(U)$ injects into $\text{Pic}(D)$.

\[\square\]
**Proof of Theorem A.** Note that every $X$ above with a $cA_n$ singularity at $P_x$ is irreducible.

(b) We prove that every sextic double solid $Y \subseteq \mathbb{P}(1,1,1,1,3)$ with a $cA_n$ singularity is isomorphic to some $X$ above, with the isomorphism sending the $cA_n$ point to $P_x = [1,0,0,0,0]$. After applying a suitable automorphism of $\mathbb{P}(1,1,1,1,3)$, the $cA_n$ point is at $P_x$. From Corollary 2.6, we know that a $cA_n$ singularity is isomorphic to a complex analytic space germ $(\mathbb{V}(\alpha^2 + \beta^2 + H),0)$ with variables $\alpha, \beta, \gamma, \delta$ where $H \in \mathbb{C}\{\gamma, \delta\}$ has multiplicity $n+1$. Consider the affine patch $Y_x$, given by inverting $x$. Using Proposition 2.4, we find that after a suitable invertible linear coordinate change, $Y_x$ is given by $\mathbb{V}(-w^2 + t^2 + g_{2}(y,z,t))$ in $\mathbb{C}^4$ with variables $y,z,t,w$, where $g \in \mathbb{C}[y,z,t]$ has multiplicity at least two and the degree 2 part $g_2$ is contained in $\mathbb{C}[y,z]$. So, $Y$ has the required form $\mathbb{V}(f)$, proving the case $n = 1$.

Applying the splitting lemma on the affine patch where $x$ is non-zero, we find that $X$ is locally analytically of the form $\mathbb{V}(-w^2 + t^2 + h(y,z))$ in $\mathbb{C}^4$ with variables $y,z,t,w$ for some $h \in \mathbb{C}\{y,z\}$ of multiplicity at least two. Any $cA_n$ singularity is isomorphic to a complex analytic space germ $(\mathbb{V}(\alpha^2 + \beta^2 + H),0)$ with variables $\alpha, \beta, \gamma, \delta$ where $H \in \mathbb{C}\{\gamma, \delta\}$ has multiplicity $n+1$. By Proposition 2.5, $X$ has a $cA_n$ singularity at $P_x$ if and only if $h_2 = h_3 = \ldots = h_n = 0$ and $h_{n+1} \neq 0$ as polynomials in $\mathbb{C}[y,z]$. We have $h_2 = \xi_2$, so $h_2 = 0$ is equivalent to condition 2, namely $\xi_2 = 0$. We can show that $h_3 = a_3$, so $h_3 = 0$ is equivalent to condition 3, namely $a_3 = 0$. Similarly, using the explicit splitting lemma (Proposition 3.2), it is straightforward to compute that $h_2 = \ldots = h_n = 0$ is equivalent to satisfying conditions 2 to $n$ when $n \leq 6$, even if $P_x$ is not isolated. This proves part (b) for $n \leq 6$.

In the rest of the proof of part (b), using that fact that the singularity $P_x$ of $X$ is isolated, we show that $h_2 = \ldots = h_n = 0$ is equivalent to satisfying conditions 2 to $n$ for any $n \leq 8$.

By Lemma 3.6, either $a_2 \neq 0$ or $b_3 \neq 0$. We write $a_2 = qr$ and $b_3 = qs + 4a_1qr$, where $q \in \mathbb{C}[y,z]$ is a (homogeneous) greatest common divisor of $a_2$ and $b_3$, and $r$ and $s \in \mathbb{C}[y,z]$ have no common prime divisor. In the rest of the proof of parts (a) and (b), we repeatedly use Lemma 3.7.

If conditions 2 to 6 hold, then using the explicit splitting lemma (Proposition 3.2), we compute in Listing 3 of Appendix A that

\[ h_7 = q(r(-12a_0q^2rs + 4b_2qs - 2b_1q^2r^2 + 2c_3qr - 2d_5)) - s(2c_4 - 4a_1qs). \]

Using Lemma 3.7, we find that $h_7 = 0$ is equivalent to the existence of a homogeneous $e \in \mathbb{C}[y,z]$ such that

\[
\begin{align*}
c_4 &= 2a_1qs - 6a_0q^2r^2 + 8a_1^2qr + er, \\
d_5 &= 2b_2qs - 8a_1^2qs - es - b_1q^2r^2 + c_3qr.
\end{align*}
\]

We defined $q$ as a homogeneous greatest common divisor of $a_2$ and $b_3$. Every non-zero complex multiple of $q$ is another greatest common divisor. Therefore, there is redundancy in choosing $q,r,s,e$. We eliminate this redundancy by choosing $q = 1$ in condition (7.1), leading coefficient of $q$ equal to one in condition (7.2), $r = 1$ in condition (7.3), and $s = 1$ in condition (7.4). By Lemma 3.8, $q$ and $e$ have no common prime divisor in $\mathbb{C}[y,z]$.

This proves part (b) for $n = 7$, namely that $h_2 = \ldots = h_7 = 0$ is equivalent to conditions 2 to 7 if $P_x$ is an isolated singularity of $X$.

Now, we show that if $h_2 = \ldots = h_8 = 0$ and one of the conditions (7.2) to (7.4) holds, then the singularity $P_x$ is not isolated. In condition (7.2), we calculate that $h_8 + e^2t^2$ is
divide by $q$, giving $r = Cq$ for some $C \in \mathbb{C}$. Substituting into $h_8$, we compute that $h_8 - 2qes^2$ is divisible by $q^2$. Therefore $q$ and $s$ have a common prime divisor, giving that $r$ and $s$ have a common prime divisor, a contradiction. So, $P_x$ is not an isolated singularity of $X$. Conditions (7.3) and (7.4) are similar.

Hence, if $h_2 = \ldots = h_8 = 0$ and $P_x$ is an isolated singularity, then condition (7.1) holds. Using the explicit splitting lemma, we calculate $h_8$ in Listing 3 in Appendix A, and using Lemma 3.7, we can show that $h_2 = \ldots = h_8 = 0$ is equivalent to conditions 2 to 8.

(a) If conditions 2 to 8 are satisfied, then similarly to proof of part (b), $P_x$ being a $cA_n$ singularity where $n \geq 9$ implies that $h_9 = 0$. Using the explicit splitting lemma, we compute $h_9$ in Listing 3 in Appendix A, and using Lemma 3.7, we find that this implies that there exists $B_0 \in \mathbb{C}$ such that

$$A_0 = a_0$$

$$B_1 = b_1$$

$$d_3 = -s_3B_0 + 2b_0s_3 - 2a_2^3s_3 + c_1r_2 - 4a_0b_1r_2$$

$$+ 16a_2^2a_1r_2 + b_1b_2 - 4a_0a_1b_2 - 2a_2^2b_1 + 8a_2^3$$

$$c_2 = r_2B_0 - 6a_0^2r_2 + 2a_0b_2 + 2a_1b_1 - 12a_0a_1^2.$$ 

Substituting into $f$ gives

$$x^3a_3 + x^2b_1 + xc_5 + d_6 = (s_3 + 2a_1r_2 + xr_2)^2$$

$$x^3a_2 + x^2b_3 + xc_4 + d_5 = (s_3 + 2a_1r_2 + xr_2)(-2a_0r_2 + b_2 - 2a_1^2 + 2xa_1 + x^2).$$

Define the curve $C$ by the ideal $(w, t, s_3 + 2a_1r_2 + xr_2)$. Taking partial derivatives, we find that $X$ is singular at every point of $C$. Therefore, $P_x$ is not an isolated singularity of $X$.

(c) We consider varieties $X$ satisfying conditions 1 to $n$. We show that a general $X$ has no other singular points apart from $P_x = \{1, 0, 0, 0, 0\}$. Denote the parameter space of all possible $f$ in Notation 3.4 satisfying conditions 2 to $n$ by $P_n$.

If $P \neq P_x$ is a singular point of $V(f)$ with $t$-coordinate zero, then one of $y$ or $z$-coordinate is non-zero. A suitable change of coordinates of the form $x \mapsto x + \alpha y$ or $x \mapsto x + \alpha z$, where $\alpha \in \mathbb{C}$, takes the point $P$ to $P'$ with $x$, $t$, and $y$-coordinate zero. Note that this coordinate change fixes the point $P_x$, keeps the form of $f$ given in Notation 3.4, and $f$ will continue to satisfy conditions 2 to $n$ after this coordinate change. Using Lemma 3.9, we find that the parameter space of all $f$ such that $V(f)$ has a singular point $P \neq P_x$ with $t$-coordinate zero is at most $(\dim P_n - 1)$-dimensional.

If $P$ is a singular point of $V(f)$ with $t$-coordinate non-zero and $n \geq 2$, then a suitable change of coordinates given by $x \mapsto x + \alpha_x t$, $y \mapsto y + \alpha_y t$ and $z \mapsto z + \alpha_z t$, where $\alpha_x$, $\alpha_y$, $\alpha_z \in \mathbb{C}$, takes the point $P$ to $P_t = \{0, 0, 0, 1, 0\}$. Note that this coordinate change fixes the point $P_x$, keeps the form of $f$ given in Notation 3.4, and $f$ will continue to satisfy conditions 2 to $n$ after this coordinate change. Using Lemma 3.10, we find that the parameter space of all $f$ such that $V(f)$ has a singular point $P$ with $t$-coordinate non-zero is at most $(\dim P_n - 1)$-dimensional, under the condition $n \geq 2$. If $n = 1$, then instead we perform a suitable coordinate change given by $x \mapsto x + \alpha_x t$, $y \mapsto y + \alpha_y t$, $z \mapsto z + \alpha_z t$ and $t \mapsto t$ or $t \mapsto 2t$, composed with a coordinate change of the form $t \mapsto \beta_y y + \beta_z z + \beta_t t$ where $\beta_y$, $\beta_z$ and $\beta_t \in \mathbb{C}$ depend only on $\alpha_x$, $\alpha_y$, $\alpha_z$ and the coefficients of $x^2 t^2$, such that this composition takes the point $P$ to $P_t$, fixes the point $P_x$ and keeps the form of $f$ given in Notation 3.4. This extra coordinate change $t \mapsto \beta_y y + \beta_z z + \beta_t t$ is needed to keep the form of $f$ in Notation 3.4, namely to diagonalize the quadratic part with respect to $t$, that is, to remove the quadratic monomials $yt$ and $zt$, and set the coefficient of $t^2$ to one. Similarly,
using Lemma 3.10, we find that the parameter space of all \( f \) such that \( V(f) \) has a singular point \( P \) with \( t \)-coordinate non-zero is at most \( (\dim P_n - 1) \)-dimensional when \( n = 1 \).

This shows that a general \( X \) satisfying 2 to \( n \) is smooth outside a \( cA_n \) singularity at \( P_x \).

(d) Since \( X \) has local complete intersection singularities, it is Gorenstein ([Eis95, Corollary 21.19]). A terminal Gorenstein Fano 3-fold is \( \mathbb{Q} \)-factorial if and only if it is factorial [Kaw88, Lemma 6.3]. To see that in family (7.4) we do not have any \( \mathbb{Q} \)-factorial members, suffices to note that \( V(t, q - w) \) and \( V(t, q + w) \) are not Cartier on the sextic double solid.

In all other families apart from (7.4), a very general sextic double solid \( X \) satisfies that the hyperplane section \( V(x) \) is a very general sextic double plane. More precisely, fix a positive integer \( n \leq 8 \) and a connected component of the parameter space of sextic double solids with an isolated \( cA_n \) singularity described in Remark 3.13, other than \( cA_7 \) family 4. Consider the 28-dimensional parameter space of sextic double planes

\[
V(-w^2 + g) \subseteq \mathbb{P}(1, 1, 1, 3)
\]

with variables \( y, z, t, w \) where \( g \in \mathbb{C}[y, z, t] \) is homogeneous of degree 6, where the sextic double plane corresponds to a point in \( \mathbb{C}^{28} \) given by the coefficients of \( y^6, y^5z, \ldots, t^6 \). For \( cA_4, cA_5, cA_6 \) and \( cA_7 \) families 1–3, the image of the parameter space of sextic double solids, under taking the hyperplane section \( V(x) \), contains a Zariski open dense set of the parameter space of sextic double planes. We show this by computing the rank of the Jacobian matrix corresponding to this projection morphism, and showing it is 28 for some particular point. For \( cA_7 \) family 4, respectively \( cA_8 \), it gives set which is open dense in a subvariety of codimension 3, respectively 1. For \( cA_8 \), using additionally the coordinate transformation \( t \mapsto \alpha y + \beta z + t \) where \( \alpha, \beta \in \mathbb{C} \) on the image of the parameter space of sextic double solids, we get a Zariski open dense set of the parameter space of sextic double planes.

Since a very general sextic double plane has Picard number 1, by Corollary 3.12 a very general sextic double solid \( X \) has Class number 1, except for \( cA_7 \) family 4. Therefore, \( X \) is factorial and has Picard number 1.

**Remark 3.13.** Consider the tuples \( \eta \) of coefficients of \( \xi_2, a_i, b_i, c_i, d_i \) in Notation 3.4. These form the parameter space \( \mathbb{C}^{77} \). As in the proof of Theorem A, we can locally analytically write \( X \) with a \( cA_n \) singularity by \( V(wt + h) \subseteq \mathbb{C}^4 \) where \( h \in \mathbb{C}\{y, z\} \) has multiplicity \( n + 1 \). Requiring that all the coefficients of \( h_{n+1} \) are zero gives us \( n + 2 \) conditions. If these conditions are algebraically independent and if \( h_{n+2} \) is not zero after imposing these, then we would expect the parameter space for \( cA_n \) to have dimension \( n + 2 \) less. Counting parameters in Notation 3.4, we see that this is precisely the case. Table 2 shows the dimensions of the parameter spaces.

The parameter space for \( n + 1 \) lies in the Zariski closure of the parameter space for \( n \). Note that the parameter space for \( cA_7 \) has four connected components, each of dimension 44.

The automorphisms of \( \mathbb{P}(1, 1, 1, 1, 3) \) that keep the form of a general \( f \) when \( n \geq 2 \) are of the form

\[
\begin{pmatrix}
x \\
y \\
z \\
t \\
w
\end{pmatrix} \mapsto \begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
\alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
\alpha_9 & \alpha_{10} & \alpha_{11} & \alpha_{12}
\end{pmatrix} \begin{pmatrix}
x \\
y \\
z \\
t \\
w
\end{pmatrix} \pm 1.
\]

These automorphisms form a 10-dimensional algebraic group. When \( n = 1 \), instead of polynomials \( f \), we can consider polynomials \( F \) of the form \( -w^2 + x^4 E_2 + x^3 A_3 + x^2 B_4 + \)
\[ xC_5 + D_6 \] where \( E_i, A_i, B_i, C_i, D_i \in \mathbb{C}[y, z, t] \) are homogeneous of degree \( i \). The parameter space of polynomials \( F \) is 80-dimensional, and the automorphisms of \( \mathbb{P}(1, 1, 1, 1, 3) \) that keep the form of a general \( F \) form a 13-dimensional algebraic group. If the course moduli space of sextic double solids with an isolated \( cA_n \) singularity exists, then we expect it to have dimension 10 less than the parameter space in Notation 3.4.

**Table 2. Dimension of the space of sextic double solids with an isolated \( cA_n \)**

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|---|---|---|---|---|---|---|---|
| parameter space dim | 77 | 74 | 70 | 65 | 59 | 52 | 44 | 35 |
| expected moduli space dim | 67 | 64 | 60 | 55 | 49 | 42 | 34 | 25 |

**Remark 3.14.** In some cases it suffices if \( X \) is *general* in Theorem A part (d), as opposed to very general. For example, if \( n = 1 \), then a general \( X \) has only one singularity which is an ordinary double point, and every such \( X \) is factorial and has Picard number 1, that is, \( X \) is a Mori fibre space. If \( n = 2 \), then a general \( X \) has the singularity given by \( x_1x_2 + x_3^3 + x_4^3 \), and is a Mori fibre space by [CM04, Remark 1.2]. If \( n = 4 \), a general \( X \) is a Mori fibre space, since in Section 5.2 we construct a Sarkisov link to a complete intersection \( Z_{6,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4) \) which is a Mori fibre space if it is general.

### 3.3. Other \( cA_n \) singularities

Although the primary interest is in isolated \( cA_n \) singularities since these are terminal, it is also possible to study non-isolated singularities with the same methods.

**Remark 3.15.** It follows from the proof of Theorem A that every \( f \) that satisfies conditions 2 to 8 defines a sextic double solid with a singularity at \( P_r \) which is either \( cA_m \) (possibly non-isolated) where \( m \geq n \), or it is the germ \( (Z, 0) \) where \( Z = \mathbb{V}(x_1^2 + x_2^2) \subseteq \mathbb{C}^4 \) with variables \( x_1, x_2, x_3, x_4 \).

We describe a family of examples of sextic double solids with a non-isolated \( cA_n \) singularity for all \( 9 \leq n \leq 11 \).

**Proposition 3.16.** Let \( 9 \leq n \leq 11 \). If \( X \) in Notation 3.4 satisfies conditions 2 to 8, and in addition satisfies conditions 9 to \( n \) and does not satisfy condition \( n + 1 \) from the following:

9. there exists \( B_0 \in \mathbb{C} \) such that

\[
A_0 = a_0 \\
B_1 = b_1 \\
d_3 = -s_3B_0 + 2b_0s_3 - 2a_0^2s_3 + c_1r_2 - 4a_0b_1r_2 \\
+ 16a_0^2a_1r_2 + b_1b_2 - 4a_0a_1b_2 - 2a_1^2b_1 + 8a_0a_1^3 \\
c_2 = r_2B_0 - 6a_0^2r_2 + 2a_0b_2 + 2a_1b_1 - 12a_0a_1^2.
\]

10. \( B_0 = b_0 \)

\[
d_2 = 2c_0r_2 - 8a_0b_0r_2 + 16a_0^3r_2 + 2b_0b_2 - 4a_0^2b_2 + b_1^2 - 8a_0a_1b_1 - 4a_1^2b_0 + 24a_0^2a_1^2 \\
c_1 = 2a_0b_1 + 2a_1b_0 - 12a_0^2a_1.
\]
Remark 3.17. If $9 \leq n \leq 11$, then the variety $X$ in Proposition 3.16 is singular along the curve $C$: $\mathbb{V}(t, w, s_3+2a_1r_2+xr_2)$ passing through $P_x$ (see the proof of part (a) of Theorem A). We can compute that at a general point of $C$, the singularity is locally analytically $\mathbb{C}^1 \times \text{ODP}$, that is, it is isomorphic to the germ $(Z, 0)$ where $Z$ is $\mathbb{V}(x_1^2 + x_2^2 + x_3^2) \subseteq \mathbb{C}^4$ with variables $x_1, x_2, x_3, x_4$.

Example 4.2. Translating the point $P_t = [0, 0, 0, 1, 0]$ to $[1, 0, 0, 0, 0]$, we find similar conditions to Theorem A for having a $cA_n$ singularity at $P_t \in X$, which can be used to construct general sextic double solids with two $cA_n$ singularities. It is also easy to construct simple examples with only two $cA_3$ singularities, such as the variety below with $cA_3$ singularities at $P_x$ and at $P_t$,

$$\mathbb{V}(-w^2 + x^4t^2 + x^2t^4 + y^6 + z^6) \subseteq \mathbb{P}(1, 1, 1, 1, 3).$$

4. Weighted blowups

In this section, we discuss weighted blowups from both algebraic and local analytic points of view. In Proposition 4.5 we show that to check whether a weighted blowup is a Kawakita blowup (see Theorem 2.10), it suffices to compute the weight of the defining power series. Using this, in the technical Lemma 4.8 we show how to algebraically construct Kawakita blowups of $cA_n$ points on affine hypersurfaces.

4.1. Weight-respecting maps

Let $n$ and $m$ be positive integers. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ denote the coordinates on $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Choose positive integer weights for $x$ and $y$.

Definition 4.1. Let $X \subseteq \mathbb{C}^n$, $X' \subseteq \mathbb{C}^m$ be complex analytic spaces containing the origins. We say a biholomorphic map $\psi: X \to X'$ is **weight-respecting** if denoting its inverse by $\theta$, we can locally analytically around the origins write $\psi = (\psi_1, \ldots, \psi_m)$ and $\theta = (\theta_1, \ldots, \theta_n)$ where for all $i$ and $j$, the power series $\psi_j \in \mathbb{C}\{x\}$ and $\theta_i \in \mathbb{C}\{y\}$ satisfy $\text{wt}(\psi_j) \geq \text{wt}(y_j)$ and $\text{wt}(\theta_i) \geq \text{wt}(x_i)$.

It is known that a biholomorphic map taking the origin to the origin lifts to a unique biholomorphic map of the blown-up spaces under the usual weights $(1, \ldots, 1)$ (see for example [GLS07, Remark 3.17.1(4)])). It is easy to come up with examples where a biholomorphic map does not lift under weighted blowups. We give one example below.

Example 4.2. Let $X \subseteq \mathbb{C}^3$ be the complex analytic space given by $\mathbb{V}(f)$ where

$$f = x_2^2x_3 + x_1^3 + ax_1x_2^2 + bx_3^3$$

for some $a, b \in \mathbb{C}^*$. Define $X' \subseteq \mathbb{C}^3$ by $\mathbb{V}(f')$ where $f' = f(x_1, x_2, -x_2 + x_3)$. Choose weights $(1, 1, 2)$ for $(x_1, x_2, x_3)$. Then, $X$ and $X'$ are biholomorphic and $\text{wt } f = \text{wt } f'$, but the weighted blowups of $X$ and $X'$ are not locally analytically equivalent.
Proof. Let $\psi: X \rightarrow X'$ be any local biholomorphism taking the origin to the origin. Composing with a suitable weight-respecting biholomorphic map and using Lemma 4.3, it suffices to consider the case where $\psi$ is a linear biholomorphism. Since the elliptic curve defined by $f$ in $\mathbb{P}^2$ with variables $x_1, x_2, x_3$ has only two automorphisms, there are only four possibilities for a linear biholomorphism $X \rightarrow X'$, namely $(x_1, x_2, x_3) \mapsto (x_1, \pm x_2, \pm x_2 + x_3)$.

Let $Y \rightarrow X$ and $Y' \rightarrow X'$ be the $(1, 1, 2)$-blowups of $X$ and $X'$ respectively. Then $Y$ is given by $\mathbb{V}(g)$ where

$$g(u, x_1, x_2, x_3) = ux_2^2x_3 + x_1^3 + au^2x_1x_3^2 + bu^3x_3^3.$$ 

Denoting the points of $Y$ and $Y'$ by $[u, x_1, x_2, x_3]$, the lifted map $\psi_Y: Y \rightarrow Y'$ is given by $[u, x_1, x_2, x_3] \mapsto [u, x_1, \pm x_2, \pm x_2/u + x_3]$, which is not holomorphic on the exceptional locus $\mathbb{V}(u)$. \hfill \Box

On the other hand, a weight-respecting coordinate change does lift to weighted blowups:

**Lemma 4.3.** The weighted blowups of $X \subseteq \mathbb{C}^n$ and $X' \subseteq \mathbb{C}^m$ at the origin are analytically equivalent if there exists a weight-respecting biholomorphic map $X \rightarrow X'$ taking $0$ to $0$.

**Proof.** Let $\varphi: Y \rightarrow X$ and $\varphi': Y' \rightarrow X'$ be the weighted blowups at the origin and let $\psi: X \rightarrow X'$ be a weight-respecting biholomorphic map. We define the holomorphism $\psi_Y = (\psi_{Y,0}, \psi_{Y,1}, \ldots, \psi_{Y,m}): Y \rightarrow Y'$ by choosing $\psi_{Y,0} = u$ and for all $j \geq 1$, $\psi_{Y,j} = (\psi_j \circ \varphi)/u^{\text{wt}(\psi_j)}$. Similarly, we define $\theta: Y' \rightarrow Y$ by $\theta_0 = u$ and $\theta_i = (\psi_i^{-1} \circ \varphi')/u^{\text{wt}(\varphi')}$. Since $\psi$ and $\psi^{-1}$ are weight-respecting, the maps $\psi_Y$ and $\theta$ are indeed holomorphic.

The map $\theta \circ \psi_Y$ coincides with the identity map on a dense open subset of $Y$: namely, for all $[1, x] \in Y$, we have $(\theta \circ \psi_Y)[1, x] = \theta[1, \psi(x)] = [1, x]$. Since coincidence sets are closed, the map $\theta \circ \psi_Y$ is the identity. Similarly, $\psi_Y \circ \theta$ is the identity, giving $\theta = \psi_Y^{-1}$.

Also, we have $(\varphi' \circ \psi_Y)[1, x] = \psi(x) = (\psi \circ \varphi)[1, x]$, showing that the diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\psi_Y} & Y' \\
\downarrow\varphi & & \downarrow\varphi' \\
X & \xrightarrow{\psi} & X'
\end{array}$$

commutes. Therefore, $\varphi$ and $\varphi'$ are analytically equivalent. \hfill \Box

### 4.2. Kawakita blowup in analytic neighbourhoods

In the following, we focus on Kawakita blowups (see Theorem 2.10). Unlike Example 4.2, for $cA_n$ singularities, having the correct weight for the defining power series is enough for the local analytic equivalence of weighted blowups.

**Notation 4.4.** We choose positive integer weights $w = (r_1, r_2, a, 1)$ for variables $x = (x_1, x_2, x_3, x_4)$ on $\mathbb{C}^4$ and define $n = (r_1 + r_2)/a - 1$ such that

- $a$ divides $r_1 + r_2$ and is coprime to both $r_1$ and $r_2$,
- $r_1 \geq r_2$, and
- $n \geq 2$.

**Proposition 4.5.** Using Notation 4.4, let $f \in \mathbb{C}(x)$ be such that $\mathbb{V}(f)$ has an isolated $cA_n$ singularity at the origin and $f$ has weight $r_1 + r_2$. Then, the $w$-blowup of $\mathbb{V}(f) \subseteq \mathbb{C}^4$ is a $w$-Kawakita blowup.
Proof. First, we remind that the terms homogeneous, degree and multiplicity are with respect to the standard weights \((1, \ldots, 1)\). Let the quadratic part of \(f\) denote the homogeneous part of \(f\) of degree 2. After a suitable invertible linear weight-respecting coordinate change, the quadratic part of \(f\) is \(x_1x_2\).

We find that \(f = x_1x_2 + x_1G + H\), where \(G \in \mathbb{C}\{x_1, \ldots, x_4\}\) has weight at least \(r_2\) and multiplicity \(m \geq 2\), and \(H \in \mathbb{C}\{x_2, x_3, x_4\}\). The coordinate change \(x_2 \mapsto x_2 - G_m\), where \(G_m\) is the homogeneous degree \(m\) part of \(G\), takes \(f\) to \(x_1x_2 + x_1G' + H', \) where \(G'\) has multiplicity at least \(m + 1\). By induction, this defines the unique formal power series \(K \in \mathbb{C}\{[x_1, \ldots, x_4]\}\) of multiplicity at least \(2\) and weight at least \(r_2\) such that the transformation \(x_2 \mapsto x_2 + K\) takes \(f\) to the form \(x_1x_2 + H''\) where \(H'' \in \mathbb{C}\{[x_2, x_3, x_4]\}\). Similarly, we transform \(f\) into \(x_1x_2 + h\) where \(h \in \mathbb{C}\{[x_3, x_4]\}\), using \(x_1 \mapsto x_1 + L\) where \(L \in \mathbb{C}\{[x_2, x_3, x_4]\}\). At the end of the proof, we show how to find a convergent weight-respecting coordinate change.

Since the singularity is \(cA_n\) where \(n = (r_1 + r_2)/a - 1\), \(h\) must contain a monomial of degree \((r_1 + r_2)/a\). Since \(x_1x_2 + h\) has weight \(r_1 + r_2\), if \(a > 1\), then the coefficient of \(x_3^{(r_1+r_2)/a}\) in \(h\) is non-zero. If \(a = 1\), then after a suitable invertible linear coordinate change on \(\mathbb{C}\{x_3, x_4\}\), the coefficient of \(x_3^{(r_1+r_2)/a}\) in \(h\) is non-zero.

We found that we can transform \(f\) into the form \(x_1x_2 + h\) where the coefficient of \(x_3^{(r_1+r_2)/a}\) in \(h\) is non-zero, by using only weight-respecting coordinate changes. By Lemma 4.3, the weighted blowup of \(f\) is locally analytically equivalent to the weighted blowup of \(x_1x_2 + h\), which is precisely a Kawakita blowup.

Lastly, we discuss convergence. Instead of the coordinate changes \(x_2 \mapsto x_2 + K, x_1 \mapsto x_1 + L\), which might not be convergent, we do a coordinate change with truncated power series \(K_{<N}\) and \(L_{<N}\) of homogeneous parts of \(K\) and \(L\) of degree at most \(N\). The coordinate change \(\Psi: x_1 \mapsto x_1 + ix_2, x_2 \mapsto x_1 - ix_2\) takes \(x_1x_2\) into \(x_1^2 + x_2^2\). Now we use the splitting lemma, which gives a convergent coordinate change \(\Phi\) which respects the weighting when \(N\) is large enough, to give \(f\) the form \(x_1^2 + x_2^2 + h(x_3, x_4)\) where \(h\) converges. Applying \(\Psi^{-1}\), we get \(x_1x_2 + h\). Note that the coordinate changes \(\Psi\) and \(\Psi^{-1}\) might not respect the weighting \(w\), but the total coordinate change \(\Psi^{-1} \circ \Phi \circ \Psi\) is weight-respecting if \(N\) is large enough.

Given a variety \(X\) with an isolated \(cA_n\) point \(P\), we show that any two \(w\)-Kawakita blowups \(Y \to X\) and \(Y' \to X\) of the point \(P\) are locally analytically equivalent. Note that they need not be globally algebraically equivalent. For example, [CM04, Remark 2.4] describes two different \((2, 1, 1, 1)\)-Kawakita blowups of a \(cA_2\) singularity on a quartic 3-fold.

**Proposition 4.6.** Any two \(w\)-Kawakita blowups of locally biholomorphic singularities are locally analytically equivalent.

**Proof.** Let \(f = x_1x_2 + g(x_3, x_4)\) and \(f' = x_1x_2 + g'(x_3, x_4)\) be contact equivalent, where \(g, g' \in \mathbb{C}\{x_3, x_4\}\) have weight \(r_1 + r_2\) and \(x_3^{(r_1+r_2)/a}\) appears in both \(g\) and in \(g'\) with non-zero coefficient. Suffices to show that the \(w\)-blowups of \(\mathbb{V}(f) \subseteq \mathbb{C}^4\) and \(\mathbb{V}(f') \subseteq \mathbb{C}^4\) are locally analytically equivalent.

Since \(f\) and \(f'\) are contact equivalent, there exists a unit \(u \in \mathbb{C}\{x\}\) and a local biholomorphism \(\psi: (\mathbb{C}^4, 0) \to (\mathbb{C}^4, 0)\) such that \(f' = u(f \circ \psi)\). Note that \(f'\) and \(f \circ \psi\) have the same weight \(r_1 + r_2\), and \(x_3^{(r_1+r_2)/a}\) appears in \(f \circ \psi\) with non-zero coefficient. Since the germs \((\mathbb{V}(f'), 0)\) and \((\mathbb{V}(f \circ \psi), 0)\) are equal, it suffices to show that the \(w\)-blowups of \(\mathbb{V}(f)\) and \(\mathbb{V}(f \circ \psi)\) are locally analytically equivalent.
Using arguments similar to the proof of Proposition 4.5, we can find a weight-respecting biholomorphic map germ \( \psi: (\mathbb{C}^4, \mathbf{0}) \to (\mathbb{C}^4, \mathbf{0}) \) such that \( f \circ \psi \circ \theta \) is of the form \( x_1 x_2 + g'' \) where \( g'' \in \mathbb{C}\{x_3, x_4\} \) contains \( x^{(r_1 + r_2)/a} \) and has weight \( r_1 + r_2 \). It suffices to show that the \( w \)-blowups of \( \mathbb{V}(f \circ \psi \circ \theta) \) and \( \mathbb{V}(f) \) are locally analytically equivalent.

By Proposition 2.5, \( g \) and \( g'' \) are right equivalent, meaning there exists an automorphism \( \Phi \) of \( \mathbb{C}\{x_3, x_4\} \) such that \( \Phi(g) = g'' \). Since \( x_3^{(r_1 + r_2)/a} \) has non-zero coefficient in both \( g \) and \( g'' \), and both \( g \) and \( g'' \) have weight \( r_1 + r_2 \), the image of \( x_3 \) has weight \( a \) under both \( \Phi \) and \( \Phi^{-1} \). Define the biholomorphic map germ \( \varphi: (\mathbb{V}(f \circ \psi \circ \theta), \mathbf{0}) \to (\mathbb{V}(f), \mathbf{0}) \) by \( x \mapsto (x_1, x_2, \Phi(x_3), \Phi(x_4)) \). By Lemma 4.3, the \( w \)-blowups of \( \mathbb{V}(f \circ \psi \circ \theta) \subseteq \mathbb{C}^4 \) and \( \mathbb{V}(f) \subseteq \mathbb{C}^4 \) are locally analytically equivalent.

\[ \square \]

### 4.3. Kawakita blowups on affine hypersurfaces

In this section, we see how to construct weighted blowups for affine hypersurfaces with a \( cA_n \) singularity where \( n \geq 2 \) such that locally analytically they are Kawakita blowups.

Most \( cA_n \) singularities do not admit \((r_1, r_2, a, 1)\)-Kawakita blowups where \( a \geq 2 \). Below we define the type of an isolated \( cA_n \) singularity, which for \( n \geq 2 \) is equal to the highest integer \( a \) such that it admits some \((r_1, r_2, a, 1)\)-Kawakita blowup locally analytically. General sextic double solids with an isolated \( cA_n \) singularity have a type 1 \( cA_n \) singularity.

**Definition 4.7.** Let \((X, P)\) be the complex analytic space germ of an isolated \( cA_n \) singularity. Let \( a \) be the largest integer such that \((X, P)\) is isomorphic to some germ \((\mathbb{V}(x_1 x_2 + g), \mathbf{0})\) where \( g \in \mathbb{C}\{x_3, x_4\} \) has weight \( a(n + 1) \) under the weighting \((a, 1)\) for \((x_3, x_4)\). Then, we say that the \( cA_n \) singularity is of type \( a \).

It is not obvious how to globally algebraically construct a Kawakita blowup for variety with a \( cA_n \) singularity. We show this for affine hypersurfaces in the technical Lemma 4.8. We use a projectivization of Corollary 4.9 in Section 5 for constructing Kawakita blowups of sextic double solids.

We describe the notation for Lemma 4.8. Choose \( n \geq 2 \) and weights

\[ w = wt(\alpha, \beta, x_3, x_4) = (r_1, r_2, a, 1) \]

as in Notation 4.4. Let \( F \in \mathbb{C}[x_1, x_2, x_3, x_4] \) have multiplicity at least 3, and let

\[ f = -x_1^2 + x_2^2 + F \]

be such that \( \mathbb{V}(f) \subseteq \mathbb{C}^4 \) has terminal singularities and has a \( cA_n \) singularity of type at least \( a \) at the origin. Let \( q, w \) be the power series when splitting with respect to \( x_1 \) (Theorem 3.1), and \( p, v \) be the power series when splitting with respect to \( x_2 \), that is,

\[ f = -((x_1 + q)w)^2 + ((x_2 + p)v)^2 + h \]

(4.1)

where \( q \in \mathbb{C}\{x_2, x_3, x_4\} \) and \( p \in \mathbb{C}\{x_3, x_4\} \) both have multiplicity at least 2, and \( w \in \mathbb{C}\{x_1, x_2, x_3, x_4\} \) and \( v \in \mathbb{C}\{x_2, x_3, x_4\} \) are units, and \( h \in \mathbb{C}\{x_3, x_4\} \) has multiplicity at least 3. Choose weights

\[ w' = wt(\alpha, x_1, x_2, x_3, x_4) = (r_1, r_2, m, \min(r_2, \text{mult } p), a, 1) \]

for the variables on \( \mathbb{C}^6 \), where \( m = \min(r_2, \text{mult } q) \). If \( a > 1 \), then perform a coordinate change on \( x_3, x_4 \) for \( f \) such that \( h \) has weight \( r_1 + r_2 \). Writing a power series \( s \in \mathbb{C} \)
\(\mathbb{C}\{x_1,x_2,x_3,x_4\}\) as a sum of its \(\mathbf{w}'\)-weighted homogeneous parts \(s = \sum_{i=0}^{\infty} s_i\), let \(s_{<k}\) denote \(\sum_{i<k} s_i\) and \(s_{\geq k}\) denote \(\sum_{i\geq k} s_i\). Define the ideal
\[
I = (f, -\alpha + (x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2}, -\beta + x_2 + p_{<r_2}),
\]
where \(v_{<r_1-r_2}\) is defined to be \(1\) when \(r_1 = r_2\), and \(w_{<r_1-m}\) is defined to be \(1\) when \(r_1 = m\). Note that the affine varieties \(\mathbb{V}(f) \subseteq \mathbb{C}^4\) and \(\mathbb{V}(I) \subseteq \mathbb{C}^6\) are isomorphic.

**Lemma 4.8.** Using the notation above, the \(\mathbf{w}'\)-blowup of \(\mathbb{V}(I)\) is a \(\mathbf{w}\)-Kawakita blowup.

**Proof.** The morphism
\[
\varphi: \mathbb{C}^4 \rightarrow \mathbb{C}^4
\]
\[(x_1, x_2, x_3, x_4) \mapsto ((x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2}, x_2 + p_{<r_2}, x_3, x_4)\]
has a local analytic inverse \(\varphi^{-1}\), given by
\[
\varphi^{-1}: (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}^4, 0)
\]
\[(\alpha, \beta, x_3, x_4) \mapsto ((\alpha - (\beta - p_{<r_2} + p_{<r_1})v')u - q', \beta - p_{<r_2}, x_3, x_4)\]
where \(u \in \mathbb{C}\{\alpha, \beta, x_3, x_4\}\) is a unit, \(v' = v_{<r_1-r_2}(\beta - p_{<r_2}, x_3, x_4)\) and \(q' = q_{<r_1}(\beta - p_{<r_2}, x_3, x_4)\). Define the map germ
\[
\psi: (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}^6, 0)
\]
\[(\alpha, \beta, x_3, x_4) \mapsto (\alpha, \beta, \varphi^{-1}(\alpha, \beta, x_3, x_4)).\]
The restriction of \(\psi\) to \(\mathbb{V}(I) \rightarrow \mathbb{V}(f \circ \psi)\) is a weight-respecting local biholomorphism, whose inverse is a projection. Therefore, the \(\mathbf{w}\)-blowup of \(\mathbb{V}(f \circ \psi)\) is equivalent to the \(\mathbf{w}'\)-blowup of \(\mathbb{V}(I)\). If the \(\mathbf{w}\)-weight of \(f \circ \psi\) is \(r_1 + r_2\), then by Proposition 4.5, the \(\mathbf{w}'\)-blowup of \(\mathbb{V}(f \circ \psi)\) is the \(\mathbf{w}\)-Kawakita blowup map germ. Using Equation (4.1), it suffices to show that
\[
\text{wt}[(x_1 + q)w + (x_2 + p)v] = r_1 \quad (4.2)
\]
\[
\text{wt}[-(x_1 + q)w + (x_2 + p)v] = r_2. \quad (4.3)
\]
Since \(\psi\) is weight-respecting, we have
\[
\text{wt}[(x_1 + q)w_{\geq r_1-m} \circ \psi] \geq r_1
\]
\[
\text{wt}[q_{>r_1}w_{<r_1-m} \circ \psi] \geq r_1
\]
\[
\text{wt}[(x_2 + p)v_{\geq r_1-r_2} \circ \psi] \geq r_1
\]
\[
\text{wt}[p_{>r_1}v_{<r_1-r_2} \circ \psi] \geq r_1.
\]
Since \((x_1 + q_{<r_1})w_{<r_1-m} + (x_2 + p_{<r_1})v_{<r_1-r_2} \circ \psi = \alpha\), this proves Equation (4.2). Using in addition that \(\text{wt}[(x_2 + p_{<r_1})v_{<r_1-r_2} \circ \psi] = r_2\), Equation (4.3) follows. \(\square\)

**Corollary 4.9.** Using the notation above, if \(F \in \mathbb{C}[x_2, x_3, x_4]\), or equivalently, if \(q = 0\) and \(w = 1\), then define the ideal \(J \subseteq \mathbb{C}[\alpha, \beta, x_2, x_3, x_4]\) by
\[
J = (-\alpha - (x_2 + p_{<r_1})v_{<r_1-r_2})^2 + x_2^2 + F, -\beta + x_2 + p_{<r_2},
\]
where \(v_{<r_1-r_2}\) is defined to be \(1\) if \(r_1 = r_2\). Then, \(\mathbb{V}(J)\) and \(\mathbb{V}(f)\) are isomorphic affine varieties, and the \((r_1, r_2, \min(r_2, \text{mult} F), a, 1)\)-blowup of \(\mathbb{V}(J)\) is a \(\mathbf{w}\)-Kawakita blowup. If in addition \(r_1 = r_2\), then define the ideal \(J' \subseteq \mathbb{C}[x_1, \beta, x_2, x_3, x_4]\) by
\[
J' = (f, -\beta + x_2 + p_{<r_2}).
\]
Then, \(\mathbb{V}(J')\) and \(\mathbb{V}(f)\) are isomorphic affine varieties, and the \((r_1, r_2, \min(r_2, \text{mult} F), a, 1)\)-blowup of \(\mathbb{V}(J')\) is a \(\mathbf{w}\)-Kawakita blowup.
Proof. The isomorphism between $V(I)$ and $V(J)$ is a projection, with inverse given by $x_1 \mapsto \alpha - (\beta - p_{< r_2} + p_{< r_1})v_{r_1 - r_2}$, which is weight-respecting. If $r_1 = r_2$, the isomorphism between $V(J)$ and $V(J')$ is given by $x_1 \mapsto \alpha - \beta$, which is weight-respecting.

The power series $p, v, q, w$ can be expressed in terms of the coefficients of $F$ using the explicit splitting lemma, Proposition 3.2.

5. Birational models of sextic double solids

In this section, we prove Theorem B on birational non-rigidity of certain sextic double solids. First, we give generality conditions we use.

Condition 5.1. Let the sextic double solid $X$ be given as in Notation 3.4, and let $\mathbb{P}(1, 1, 3)$ have variables $y, z, w$ and $\mathbb{P}^1$ have variables $y, z$. Then we have the following conditions, depending on the family that $X$ lies in:

(a) $V(2wa_2 + c_5, w^2 - d_6) \subseteq \mathbb{P}(1, 1, 3)$ is 10 distinct points,
(b) $V(a_2, -w^2 + d_6) \subseteq \mathbb{P}(1, 1, 3)$ is 4 distinct points,
(c) $c_4 - 2a_1b_3 - a_2b_2 + 2a_0a_2^2 + 6a_2^2$ is non-zero, and $V(a_2) \subseteq \mathbb{P}^1$ is two distinct points, and for both of these points $P$, either $b_3(P)$, $c_4(P)$ or $d_5(P)$ is non-zero,
(d) $V(-c_2 + 4a_0r_2 + b_2 - 6a_2^2) \subseteq \mathbb{P}^1$ is two distinct points,
(e) $r_1$ and $q_1$ are coprime in $\mathbb{C}[y, z]$,
(f) $q_2 \in \mathbb{C}[y, z]$ is not a square,
(g) $a_0 \neq A_0$.

Theorem B. A sextic double solid, which is a Mori fibre space containing an isolated $cA_n$ singularity with $n \geq 4$ and satisfying Condition 5.1, has a Sarkisov link starting with a weighted blowup of the $cA_n$ point.

We treat each of the 7 families separately. We use the notation in Construction 2.14 and Example 2.15 for the 2-ray links. We write the $cA_4$ case in more detail. Below, when we say that a birational map is $k$ Atiyah flops, then we mean that the base of the flop is $k$ points, above each we are contracting a curve and extracting a curve, and locally analytically above each of the points it is an Atiyah flop (see [Rei92, Section 1.3] for Atiyah flop). Similarly for flips. Below, for a morphism $\Phi : T_0 \to \mathbb{P}$, $\Phi^* : \text{Cox}\mathbb{P} \to \text{Cox}T_0$ denotes a corresponding $\mathbb{C}$-algebra homomorphism of Cox rings (described explicitly in the proof of Proposition 5.4).

5.1. Singularities after divisorial contraction

Before proving Theorem B, we show that for any Kawakita blowup $Y_0 \to X$ (Theorem 2.10) of a sextic double solid $X$ with an isolated $cA_n$ singularity, the variety $Y_0$ has only up to two singular points if $X$ is general, which are quotient singularities. We do not give the generality conditions of Proposition 5.3 explicitly. We do not use this proposition in the proof of Theorem B. First, we give an elementary lemma:

Lemma 5.2. Let $a, b \in \mathbb{C}[y, z]$ be non-zero homogeneous polynomials with $\deg a \geq \deg b$ such that for every homogeneous polynomial $c \in \mathbb{C}[y, z]$ of degree $\deg a - \deg b$, the polynomial $a + bc$ is divisible by the square of a linear form. Then $a$ and $b$ are both divisible by the square of the same linear form.
Proof. Suffices to prove that for non-zero polynomials \( f, g \in \mathbb{C}[x], \) if \( f + \lambda g \) has a repeated root for all \( \lambda \in \mathbb{C}, \) then \( f \) and \( g \) have a common repeated root. This holds if there exists \( x_0 \in \mathbb{C} \) which is as a repeated root of \( f + \lambda g \) for infinitely many \( \lambda. \) Since \( g \) and \( f/g + \lambda \) have only finitely many repeated roots, the claim follows. \( \square \)

Proposition 5.3. Let \( X \) be a general sextic double solid with an isolated \( cA_n \) singularity \( P \) and \( Y_0 \to X \) a divisorial contraction with centre \( P, \) which is a \((r_1, r_2, 1, 1)\)-Kawakita blowup. Then, \( Y_0 \) has a quotient singularity \( 1/r_1(1, 1, r_1 - 1) \) if \( r_1 > 1 \) and a quotient singularity \( 1/r_2(1, 1, r_2 - 1) \) if \( r_2 > 1, \) and is smooth elsewhere.

Proof. By Theorem A, a general \( X \) is smooth outside \( P. \) So, it suffices to show that \( Y_0 \) has only up to two quotient singularities on the exceptional divisor and is smooth elsewhere. Since \( Y_0 \to X \) is a \((r_1, r_2, 1, 1)\)-Kawakita blowup, we can consider the local analytic coordinate system around \( P \) where \( X \) is given by \( wt + h(y, z) \) where \( h \in \mathbb{C}\{y, z\} \) has multiplicity \( n + 1. \) The variety \( Y_0 \) is locally analytically around the exceptional divisor given by \( wt + \frac{1}{r_1-1} h(uy, uz) \) inside the geometric quotient \( (\mathbb{C}^5 \setminus \mathcal{V}(w, t, y, z))/\mathbb{C}^* \) where the \( \mathbb{C}^*\)-action is given by \( \lambda \cdot (u, w, t, y, z) = (\lambda^{-1}u, \lambda r_1w, \lambda^2t, \lambda y, \lambda z). \) Taking partial derivatives, the singular locus of \( Y_0 \) is given by

\[
\text{Sing} \ Y_0 = \mathcal{V} \left( u, w, t, h_{n+1}, \frac{\partial h_{n+1}}{\partial y}, \frac{\partial h_{n+1}}{\partial z}, h_{n+2} \right) \cup \{ P_w \}_{if \ r_1 > 1} \cup \{ P_t \}_{if \ r_2 > 1},
\]

where \( h_i \) denotes the homogeneous degree \( i \) part of \( h, \) and \( P_w \) and \( P_t \) are the points \([0,1,0,0,0]\) and \([0,0,1,0,0]\), respectively. To prove the claim, it suffices to show that if \( X \) is general, then no square of a linear form divides \( h_{n+1}, \) that is, \( h_{n+1} \) is squarefree.

Considering the 11 families of Theorem A separately, it is easy to compute using the explicit splitting lemma (Proposition 3.2) and Lemma 5.2 that \( h_{n+1} \) is squarefree when \( X \) is general. For example, for a \( cA_8 \) singularity, we compute that

\[
h_9 = Q - 2d_3r_2^3 = 8(a_0 - A_0)s_3^3 + r_2R,
\]

where \( Q, R \in \mathbb{C}[y, z] \) are homogeneous of degrees 9 and 7 respectively, and \( Q \) does not contain the polynomial \( d_3. \) If the affine cone of \( Y_0 \) is not smooth for a general \( X, \) then the affine cone of \( Y_0 \) is singular for all \( X. \) In that case, Lemma 5.2 shows that a prime factor of \( r_2 \) divides \( h_9, \) which implies that it divides \( s_3, \) a contradiction. Similarly for the other 10 families. \( \square \)

5.2. \( cA_4 \) model

Note that Okada described a Sarkisov link starting from a general complete intersection \( Z_{5,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4) \) to a sextic double solid (see entry No. 9 of the table in [Oka14, Section 9]). We show the converse:

Proposition 5.4. A sextic double solid with a \( cA_4 \) singularity satisfying Condition 5.1 has a Sarkisov link to a complete intersection \( Z_{5,6} \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4), \) starting with a \((3, 2, 1, 1)\)-blowup of the \( cA_4 \) point, then 10 Atiyah flops, and finally a Kawamata divisorial contraction (see [Kaw96]) to a terminal quotient \( 1/4(1, 1, 3) \) point. Under further generality conditions (Proposition 5.3), \( Z \) is quasismooth.
Proof. We exhibit the diagram below.

\[
\begin{array}{c}
Y_0 \xrightarrow{10 \times 1(1,1,-1)} Y_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
cA_4 \subset X \subset \mathbb{P}(1^4,3) \quad X_0 \quad 1/4(1,1,3) \subset \mathbb{P}(3,2,3,4)
\end{array}
\]

The corresponding diagram for the ambient toric spaces is given in detail in Example 2.15. First, we describe the sextic double solid \(X\). By Theorem A, any sextic double solid \(\tilde{X}\) with an isolated \(cA_4\) singularity can be given by

\[
\tilde{X} : \mathbb{V}(f) \subset \mathbb{P}(1,1,1,1,3)
\]

with variables \(x, y, z, t, w\) where

\[
f = -w^2 + x^4t^2 + 2x^3ta_2 + x^2a_2^2 + x^2tB_3 + xC_5 + D_6,
\]

where \(a_2 \in \mathbb{C}[y, z]\) is homogeneous of degree 2, and \(A_i, B_i, C_i, D_i \in \mathbb{C}[y, z, t]\) are homogeneous of degree \(i\). Define the bidegree \((5, 6)\) complete intersection \(X\), isomorphic to \(\tilde{X}\), by

\[
X : \mathbb{V}(f, -x\xi + \alpha^2 - D_6) \subset \mathbb{P}(1,1,1,1,3,5)
\]

with variables \(x, y, z, t, \alpha, \xi\), where

\[
f = -\xi + 2\alpha a_2 + 2\alpha xt + x^2t^2A_1 + xtB_3 + C_5.
\]

The isomorphism is given by

\[
\tilde{X} \rightarrow X
\]

\[
[x, y, z, t, w] \mapsto [x, y, z, t, \alpha', 2\alpha' a_2 + 2\alpha' xt + x^2t^2A_1 + xtB_3 + C_5]
\]

where \(\alpha' = w + x^2t + xa_2\), with inverse

\[
[x, y, z, t, \alpha, \xi] \mapsto [x, y, z, t, \alpha - x^2t - xa_2].
\]

We describe the divisorial contraction \(\varphi : Y_0 \rightarrow X\). Define the toric variety

\[
T_0 : \begin{pmatrix} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & 0 & 1 & 1 & 3 & 6 & 2 \end{pmatrix},
\]

as in Example 2.15. Let \(\Phi\) be the ample model of \(\mathbb{V}(x)\), that is,

\[
\Phi : T_0 \rightarrow \mathbb{P}(1,1,1,1,3,5)
\]

\[
[u, x, y, z, \alpha, \xi, t] \mapsto [x, uy, uz, u^2t, u^3\alpha, u^6\xi].
\]

Let \(Y_0\) be the strict transform of \(X\). Let \(\Phi^*\) denote the corresponding \(\mathbb{C}\)-algebra homomorphism, namely

\[
\Phi^* : \mathbb{C}[x, y, z, t, \alpha, \xi] \rightarrow \mathbb{C}[u, x, y, z, \alpha, \xi, t]
\]

\[
\Phi^* : x \mapsto x, \quad y \mapsto uy, \quad z \mapsto uz, \quad t \mapsto u^2t, \quad \alpha \mapsto u^3\alpha, \quad \xi \mapsto u^6\xi.
\]
Define
\[ A_Y = A_1(y, z, ut), \quad B_Y = B_3(y, z, ut), \quad C_Y = C_5(y, z, ut), \quad D_Y = D_6(y, z, ut) \]
and define the polynomial \( g = \Phi^* f / u^5 \), that is,
\[ g = -u\xi + 2\alpha a_2 + 2\alpha xt + x^2t^2 A_Y + xtB_Y + C_Y. \]
Then, \( Y_0 \) is given by
\[ Y_0 : \mathcal{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (g, -x\xi + \alpha^2 - D_Y). \]
We will see later that \( I_Y \) 2-ray follows \( T_0 \). Note that there exist other ideals that define the same variety \( Y_0 \subseteq T_0 \) (see [Cox95, Corollary 3.9]), but where the ideal might not 2-ray follow \( T_0 \). Also note that we have not (and do not need to) prove that the ideal \( I_Y \) is saturated with respect to \( u \), although in general, saturating might help in finding the ideal that 2-ray follows \( T_0 \). The morphism \( Y_0 \to X \) is the restriction of \( T_0 \to \mathbb{P}(1, 1, 1, 3, 5) \). Locally, \((Y_0)_x \to X_x \) is the \((3, 2, 1, 1)\)-blowup of \( \mathcal{V}(f') \subseteq \mathbb{C}^4 \) with variables \( \alpha, t, y, z \), where
\[ f' = -\alpha^2 + 2\alpha a_2 + 2\alpha t + t^2 A_1 + tB_3 + C_5 + D_6. \]
Since \( \text{wt } f' = 5 \), by Proposition 4.5, \((Y_0)_x \to X_x \) is a \((3, 2, 1, 1)\)-Kawakita blowup.

The first diagram in the 2-ray game for \( Y_0 \) is 10 Atiyah flops, under Condition 5.1. We describe the diagram \( Y_0 \to X_0 \leftrightarrow Y_1 \) globally. Multiplying the action matrix of \( T_0 \) by the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \), define
\[ T_1 : \begin{pmatrix} u & x & y & z & \alpha & \xi & t \\ 0 & 1 & 1 & 1 & 3 & 5 & 1 \\ -1 & -1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \]
Define \( Y_1 \) by \( \mathcal{V}(I_Y) \subseteq T_1 \). Define the morphisms \( Y_0 \to X_0 \) and \( Y_1 \to X_0 \) as the ample models of \( \mathcal{V}(y) \). The exceptional locus of \( Y_0 \to X_0 \) is \( E_0^- = \mathcal{V}(\xi, t) \subseteq Y_0 \), the exceptional locus of \( Y_1 \to X_0 \) is \( E_1^+ = \mathcal{V}(u, x) \subseteq Y_1 \), and the base of the flop is
\[ \{ P_i \} = \mathcal{V}(2\alpha a_2 + C_5(y, z, 0), \alpha^2 - D_6(y, z, 0)) \subseteq \mathbb{P}(1, 1, 3) \subseteq X_0, \]
where \( \mathbb{P}(1, 1, 3) \) has variables \( y, z, \alpha \). If \( a_2, C_5(y, z, 0) \) and \( D_5(y, z, 0) \) are general enough, that is, if Condition 5.1 is satisfied, then the base of the flop is 10 points \( \{ P_i \}_{1 \leq i \leq 10} \), and both \( E_0^- \) and \( E_1^+ \) are 10 disjoint curves mapping to \( \{ P_i \}_{1 \leq i \leq 10} \).

We show that locally analytically, the diagram \( Y_0 \to X_0 \leftrightarrow Y_1 \) is 10 Atiyah flops. Let \( P \in X_0 \) be any point in the base of the flop. Then, \( P \) has either \( y \) or \( z \) coordinate non-zero. We consider the case where the \( y \)-coordinate is non-zero, the other case is similar. Since the base of the flop is 10 points, the point \( P \) is smooth in \( \mathbb{P}(1, 1, 3) \). By the implicit function theorem, we can locally analytically equivariantly express \( \alpha \) and \( z \) in terms of the variables \( u, x, \xi, t \) on the patches \((Y_0)_y, (X_0)_y \) and \((Y_1)_y \). So, the flop \( Y_0 \to X_0 \leftrightarrow Y_1 \) is locally analytically a \((1, 1, -1, -1)\)-flop, the so-called Atiyah flop, around \( P \).

The last morphism \( Y_1 \to Z \) in the link for \( X \) is a divisorial contraction. Multiplying the action matrix of \( T_6 \) by the matrix \( \begin{pmatrix} 6 & -5 \\ 2 & -1 \end{pmatrix} \) with determinant 4, we see that
\[ T_1 \simeq \begin{pmatrix} u & x & y & z & \alpha & \xi & t \\ 5 & 6 & 1 & 1 & 3 & 0 & -4 \\ 1 & 2 & 1 & 1 & 3 & 4 & 0 \end{pmatrix}. \]
Let $Y_1 \to Z$ be the ample model of $\frac{1}{4}V(\xi)$, that is,

$$Y_1 \to Z \quad [u, x, y, z, \alpha, \xi, t] \mapsto \left[ t^\frac{5}{4}u, t^\frac{1}{4}y, t^\frac{1}{4}z, t^\frac{3}{2}x, t^\frac{3}{4}\alpha, \xi \right].$$

Then $Z$ is the bidegree $(5, 6)$ complete intersection

$$Z : \mathcal{V}(h, -x\xi + \alpha^2 - D_6(y, z, u)) \subseteq \mathbb{P}(1, 1, 1, 2, 3, 4)$$

with variables $u, y, z, x, \alpha, \xi$, where the $h$ is given by applying the $\mathbb{C}$-algebra homomorphism $t \mapsto 1$ to $g$. The morphism $Y_1 \to Z$ contracts the exceptional divisor $\mathcal{V}(t) \subseteq Y_1$ to the point $P_\xi = [0, 0, 0, 0, 0, 1]$. On the quasiprojective patch $(Y_1)_{\xi}$, we can express $u$ and $x$ locally analytically equivariantly in terms of $y, z, \alpha, t$. So, the morphism $Y_1 \to Z$ is locally analytically the Kawamata weighted blowdown (see [Kaw96]) to the terminal quotient singular point $P_\xi$ of type $1/4(1, 1, 3)$.

**Remark 5.5.** We explain below how we found the variety $X$. We start with the variety $\hat{X}$, given by Theorem A. Next, we perform the coordinate change $\hat{X} \to \bar{X}$ given in Equation (4.4) of Corollary 4.9, with $(r_1, r_2, a, 1) = (3, 2, 1, 1), p_2 = a_2$ and $v_0 = 1$. We see that $\hat{X}$ is isomorphic to $\bar{X}$:

$$\mathcal{V}(\bar{f}) \subseteq \mathbb{P}(1, 1, 1, 1, 3)$$

with variables $x, y, z, t, \alpha$, where

$$\bar{f} = \alpha(-\alpha + 2x^2 + 2x \alpha) + x^3 \alpha^2 A_1 + x^2 t B_3 + x C_5 + D_6.$$

We construct a $(3, 2, 1, 1)$-Kawakita blowup $\bar{Y}_0 \to \bar{X}$. Define the toric variety $\bar{T}_0$ by

$$\bar{T}_0 : \begin{pmatrix} u & x & y & z & \alpha & t \\ 0 & 1 & 1 & 1 & 3 & 1 \\ -1 & 0 & 1 & 1 & 3 & 2 \end{pmatrix}.$$

In other words, $\bar{T}_0$ is given by the geometric quotient

$$\bar{T}_0 = \mathbb{C}^6 \setminus \mathcal{V}((u, x) \cap (y, z, \alpha, t)) / (\mathbb{C}^*)^2.$$

Let $\bar{\Phi}$ be the ample model of $\mathcal{V}(x)$, and let $\bar{Y}_0 \subseteq \bar{T}_0$ be the strict transform of $\bar{X}$. By Corollary 4.9, $\bar{Y}_0 \to \bar{X}$ is a $(3, 2, 1, 1)$-Kawakita blowup. Alternatively, we use Proposition 4.5 on the patch $(\bar{Y}_0)_x \to \bar{X}_x$ to show it is a $(3, 2, 1, 1)$-Kawakita blowup, like in Proposition 5.4.

We show that $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$. We describe the next (and the final) map in the 2-ray game for $\bar{T}_0$. Acting by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, we can write $\bar{T}_0$ by

$$\bar{T}_0 \simeq \begin{pmatrix} u & x & y & z & \alpha & t \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 2 & 1 & 1 & 3 & 0 \end{pmatrix}.$$

The ample model of the divisor $\mathcal{V}(y)$ is the weighted blowup

$$\bar{T}_0 \to \mathbb{P}(1, 1, 1, 2, 3)$$

$$[u, x, y, z, \alpha, t] \mapsto [y, z, ut, x, t, \alpha].$$
where the centre is the surface $\mathbb{P}(1,1,3)$ given by $\mathbb{V}(u, x) \subseteq \mathbb{P}(1, 1, 1, 2, 3)$ with variables $y, z, u, x, \alpha$. Above every point in $\mathbb{P}(1,1,3)$, the fibre is $\mathbb{P}^1$. Define

$$\bar{Z} : \mathbb{V}(\bar{h}) \subseteq \mathbb{P}(1, 1, 1, 2, 3)$$

where

$$\bar{h} = \alpha(-u\alpha + 2x^2 + 2x\alpha_2) + x^3 A_Z + x^2 B_Z + x C_Z + u D_Z,$$

where

$$A_Z = A_1(y, z, u), \quad B_Z = B_3(y, z, u), \quad C_Z = C_5(y, z, u), \quad D_Z = D_6(y, z, u).$$

We show that when restricting the weighted blowup to $\bar{Y}_0 \to \bar{Z}$, the exceptional locus is 1-dimensional. After restricting to $\bar{Y}_0$, the exceptional divisor $\mathbb{V}(t)$ becomes $\mathbb{V}(t, x(2\alpha_2 + C_5(y, z, 0)) + u(-\alpha^2 + D_6(y, z, 0)))$. By Condition 5.1, there are exactly 10 points $P_1, \ldots, P_{10} \in \mathbb{P}(1,1,3) \subseteq \bar{Z}$ such that $2\alpha_2 + C_5(y, z, 0)$ and $-\alpha^2 + D_6(y, z, 0)$ have a common solution. Above each of those points, the fibre is $\mathbb{P}^1$. Above any other point, the fibre is just one point. Therefore, the morphism $\bar{Y}_0 \to \bar{Z}$ contracts 10 curves onto 10 points, and is an isomorphism elsewhere. This shows that $\bar{Y}_0$ does not 2-ray follow $\bar{T}_0$, since a 2-ray link ends with either a fibration or a divisorial contraction.

The problem with the previous embedding was that $\bar{g}$ belonged to the irrelevant ideal $(u, x)$. We “unproject” the divisor $\mathbb{V}(u, x)$, to embed $\bar{Y}_0$ into a toric variety $\bar{T}_0$ such that $\bar{Y}_0$ 2-ray follows $\bar{T}_0$. The varieties $\bar{Y}_0 \subseteq \bar{T}_0$ are defined as in the proof of Proposition 5.4. We see that $\bar{Y}_0$ is isomorphic to $\bar{Y}_0$ through the map

$$[u, x, y, z, \alpha, t] \mapsto [u, x, y, z, \alpha, \alpha^2 - DY_x, t].$$

The map is a morphism, since we have the equality

$$\alpha^2 - D_Y x = \frac{2\alpha_2 + 2\alpha xt + x^2 t^2 A_Y + xt B_Y + C_Y}{u}$$

in the field of fractions of $\mathbb{C}[u, x, y, z, \alpha, t]$, and either $x$ or $u$ is non-zero at every point of $\bar{T}_0$. For more details on this kind of “unprojection”, see [Rei00, Section 2] or [PR04, Section 2.3].

Now, the coordinate change $\bar{Y}_0 \to Y_0$ induces a coordinate change $\bar{X} \to X$, where $X$ is defined as in the proof of Proposition 5.4.

5.3. $cA_5$ model

**Proposition 5.6.** A sextic double solid $X$ which is a Mori fibre space with a $cA_5$ singularity satisfying Condition 5.1 has a Sarkisov link to a sextic double solid $Z$ with a $cA_5$ singularity, starting with a $(3, 3, 1, 1)$-blowup of the $cA_5$ point in $X$, then four Atiyah flops, and finally a $(3, 3, 1, 1)$-blowdown to a $cA_5$ point. If in addition $c_4$ is general after fixing $a_1$, $b_1$ and $d_6$ in Notation 3.4, then $X$ and $Z$ are not isomorphic. Under further generality conditions, both $X$ and $Z$ are smooth outside the $cA_5$ point.

**Proof.** We exhibit the diagram below.

$$\xymatrix{ Y_0 \ar[rd]_{(3,3,1,1)} \ar[rr]^{4x(1,1,-1,-1)} & & Y_1 \ar[ld]^{(3,3,1,1)} \\
 cA_5 \in X \subseteq \mathbb{P}(1^4,3) & & X_0 & & cA_5 \in Z \subseteq \mathbb{P}(1^4,3) }$$

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We construct $X$ and a $(3, 3, 1, 1)$-Kawakita blowup $Y_0 \to X$. Using Theorem A, and performing the coordinate change in Equation (4.5) of Corollary 4.9 (with $p_2 = a_2$), we can write a sextic double solid $X$ with a $cA_5$ singularity by

$$X : \mathbb{V}(f, -\beta + xt + a_2) \subseteq \mathbb{P}(1, 1, 1, 2, 3),$$

with variables $x, y, z, t, \beta, w$ where

$$f = -w^2 + x\beta(2b_3 - 4\beta a_1 + 8xta_1 + x\beta) + 4x^3t^3a_0 + x^2t^2B_2 + xtC_4 + D_6,$$

where $B_i, C_i, D_i \in \mathbb{C}[y, z, t]$ are homogeneous of degrees $i$. Define $T_0$ by

$$T_0 : \begin{pmatrix} u & x & y & z & w & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 3 & 3 & 2 \end{pmatrix}.$$ 

Let $\Phi : T_0 \to \mathbb{P}(1, 1, 1, 1, 2, 3)$ be the ample model of $\mathbb{V}(x)$, $Y_0 \subseteq T_0$ the strict transform of $X$, and $Y_0 \to X$ the restriction of $\Phi$. Then, $Y_0$ is given by

$$Y_0 : \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^6, -u\beta + xt + a_2),$$

and $Y_0 \to X$ is a $(3, 3, 1, 1)$-Kawakita blowup.

We show that the first map in the 2-ray game for $Y_0$ is a flop, locally analytically 4 Atiyah flops, under Condition 5.1. Acting by the matrix $(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix})$, we find

$$T_0 \cong \begin{pmatrix} u & x & y & z & w & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 2 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$ 

The base of the flop in $\mathbb{P}(1, 1, 3) \subseteq X_0$ is given by $\mathbb{V}(a_2, -w^2 + D_6(y, z, 0)) \subseteq \mathbb{P}(1, 1, 3)$. If $a_2$ and $D_6(y, z, 0)$ are general, that is, Condition 5.1 is satisfied, then this is exactly 4 points. In this case, any such point $P$ is a smooth point in $\mathbb{P}(1, 1, 3)$. Consider the case where the $y$-coordinate of $P$ is non-zero, the case where $z$ is non-zero is similar. Locally analytically equivariantly, we can express $z$ and $w$ in terms of $u, x, \beta, t$ in $Y_0$, $X_0$ and $Y_1$. So, the diagram $Y_0 \to X_0 \to Y_1$ is locally analytically four Atiyah flops.

The last map in the 2-ray game of $Y_0$ is a weighted blowdown $Y_1 \to Z$. After acting by $(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix})$ on the initial matrix of $T_0$, we find that $T_1$ is given by

$$T_1 : \begin{pmatrix} u & x & y & z & w & \beta & t \\ 2 & 3 & 1 & 1 & 3 & 0 & -1 \\ 1 & 2 & 1 & 1 & 3 & 1 & 0 \end{pmatrix}.$$ 

We see that $Z \subseteq \mathbb{P}(1, 1, 1, 1, 2, 3)$ with variables $\beta, u, y, z, x, w$ is given by the ideal

$$I_Z = (h, -u\beta + x + a_2),$$

where $h$ is given by sending $t$ to 1 in $\Phi^* f/u^6$, namely

$$h = -w^2 + x\beta(2b_3 - 4u\beta a_1 + 8xa_1 + x\beta) + 4x^3a_6 + x^2B_2 + xC_4 + D_Z$$

and

$$B_Z = B_3(y, z, u), \quad C_Z = C_4(y, z, u), \quad D_Z = D_6(y, z, u).$$
Substituting $x = u\beta - a_2$ into $h$, we find that $Z$ is a sextic double solid. Applying the explicit splitting lemma (Proposition 3.2), we find that the complex analytic space germ $(Z, P_\beta)$ is isomorphic to $(\mathcal{V}(h_{\text{ana}}), 0) \subseteq (\mathbb{C}^4, 0)$ with variables $w, u, y, z$, where

$$h_{\text{ana}} = -w^2 + u^2 + d_6 - (b_3 - 2a_1a_2)^2 + (\text{h.o.t in } y, z),$$

where (h.o.t in $y, z$) stands for higher order terms in the variables $y, z$. So, $P_\beta \in Z$ is a $cA_3$ singularity. On the patch where $\beta$ is non-zero, we can substitute $u = xt + a_2$, so the morphism $(Y_1)_\beta \to Z_\beta$ is a weighted blowup of a hypersurface given by a weight 6 polynomial. By Proposition 4.5, $Y_1 \to Z$ is a $(3, 3, 1, 1)$-Kawakita blowup.

We show that $X$ and $Z$ are not isomorphic when $a_2 \neq 0$ and $c_4$ is general, using a dimension counting argument similar to [GLS07, Theorem 2.55]. Using the explicit splitting lemma, we find that the complex analytic space germ $(X, P_x)$ is isomorphic to $(\mathcal{V}(f_{\text{ana}}), 0) \subseteq (\mathbb{C}^4, 0)$ with variables $w, t, y, z$ where

$$f_{\text{ana}} = -w^2 + t^2 + d_6 - 2a_2c_4 + 2a_2^2b_2 - 4a_0a_2^3 - (b_3 - 4a_1a_2)^2 + (\text{h.o.t in } y, z).$$

If $X$ and $Z$ are isomorphic, then this implies that the complex analytic space germs $(X, P_x)$ and $(Z, P_\beta)$ are isomorphic, implying by Propositions 2.5 and 2.4 that the degree 6 parts of $f_{\text{ana}}(0, 0, y, z)$ and $h_{\text{ana}}(0, 0, y, z)$ are the same up to an invertible linear coordinate change on $y, z$. Fixing $a_0, a_1, a_2, b_2, b_3$ and $d_6$, we see that $h_{\text{ana}}(0, 0, y, z)$ is fixed, but $f_{\text{ana}}(0, 0, y, z)$ has 5 degrees of freedom. Since there are only 4 degrees of freedom in picking an element of $\text{GL}(\mathbb{C}, 2)$, the polynomials $f_{\text{ana}}(0, 0, y, z)$ and $h_{\text{ana}}(0, 0, y, z)$ are not related by an invertible linear coordinate change when $c_4$ is general. This shows that if $X$ is general, then the varieties $X$ and $Z$ are not isomorphic.  

\[ \square \]

5.4. $cA_6$ model

Proposition 5.7. A sextic double solid that is a Mori fibre space with a $cA_6$ singularity satisfying Condition 5.1 has a Sarkisov link to a hypersurface $Z_5 \subseteq \mathbb{P}(1, 1, 1, 1, 2)$ with a $cA_3$ singularity, starting with a $(4, 3, 1, 1)$-blowup of the $cA_6$ point, then two $(1, 1, -1, -1)$-flops, then a $(4, 1, 1, -2, -1, 2)$-flip, and finally a $(2, 2, 1, 1)$-blowdown to a $cA_3$ point. Under further generality conditions, the singular locus of $Z$ consists of 3 points, namely the $cA_3$ point, the $1/2(1, 1, 1)$ quotient singularity and an ordinary double point.

Proof. We exhibit the diagram below.

$\xymatrix{ & Y_0 \ar[r]^{2 \times (1,1,-1,-1)} & Y_1 \ar[r]^{(4,1,1,-2,-1,2)} & Y_2 \ar[dl]_{(3,1,1,1)}, & \text{cA}_6 \in X \subseteq \mathbb{P}(1^4,3) \ar[dl]_{X_0} \ar[l]^{\times (4,3,1,1)} \ar[r]_{\times (3,1,1,1)} & \text{cA}_3 \in Z_5 \subseteq \mathbb{P}(1^4, 2) \ar[dl]_{X_1} \ar[l]^{\times (4,3,1,1)} }$

We construct $X$ and a $(4, 3, 1, 1)$-Kawakita blowup $Y_0 \to X$. Using Theorem A and Corollary 4.9 with $p_2 = a_2$ and $p_3 = b_3 - 4a_1a_2$, we can write a sextic double solid $X$ with a $cA_6$ singularity by

$$X: \mathcal{V}(f, -\beta + xt + a_2) \subseteq \mathbb{P}(1, 1, 1, 1, 2, 3),$$

with variables $x, y, z, t, \beta, w$ where

$$f = \alpha(-\alpha + 2(b_3 - 4\beta a_1 + 4xta_1 + x\beta)) + 2\beta(c_4 - \beta b_2 + 2xtb_2 + 2x\beta a_1 + 2\beta^2 a_0 - 6xt\beta a_0 + 6x^2t^2 a_0) + x^2t^3B_1 + xt^2C_3 + tD_3.$$
where \( B_i, C_i, D_i \in \mathbb{C}[y, z, t] \) are homogeneous of degree \( i \). Define \( T_0 \) by

\[
T_0: \begin{pmatrix}
  u & x & y & z & \alpha & \beta & t \\
  0 & 1 & 1 & 1 & 3 & 2 & 1 \\
 -1 & 0 & 1 & 1 & 4 & 3 & 2
\end{pmatrix}.
\]

Let \( \Phi: T_0 \to \mathbb{P}(1, 1, 1, 2, 3) \) be the ample model of \( \mathbb{V}(x) \), \( Y_0 \subseteq T_0 \) the strict transform of \( X \), and \( Y_0 \to X \) the restriction of \( \Phi \). Then, \( Y_0 \) is given by

\[
Y_0: \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/\ell^7, -u\beta + xt + a_2),
\]

and \( Y_0 \to X \) is a \((4, 3, 1, 1)\)-Kawakita blowup.

We show that the first diagram \( Y_0 \to X_0 \leftarrow Y_1 \) in the 2-ray game for \( Y_0 \) is locally analytically two Atiyah flops under Condition 5.1, namely that \( \mathbb{V}(a_2) \subseteq \mathbb{P}^1 \) with variables \( y, z \) consists of exactly two points, and for both of the points \( P \), either \( b_3(P) \), \( c_4(P) \) or \( d_5(P) \) is non-zero, where \( D_5 = t^5d_0 + 2t^4d_1 + t^3d_2 + 2t^2d_3 + td_4 + 2d_5 \). Acting by the matrix \( \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \), we find

\[
T_0: \begin{pmatrix}
  u & x & y & z & \alpha & \beta & t \\
  3 & 4 & 1 & 1 & 0 & -1 & -2 \\
 -1 & -1 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

Under the above condition, after a suitable linear change of coordinates on \( y, z \), we find that \( a_2 = yz \). Let \( P = \mathbb{V}(z) \in \mathbb{P}^1 \subseteq X_0 \), the case where \( P = \mathbb{V}(y) \) is similar. On the patch where \( y \) is non-zero, we can substitute \( z = u\beta - xt \). The contracted locus is \( \mathbb{P}^1 \cong \mathbb{V}(\alpha, \beta, t) \subseteq (Y_0)_y \), and the extracted locus is \( \mathbb{V}(u, x) = \mathbb{V}(u, x, yb_3(1, 0) + \beta c_4(1, 0) + td_5(1, 0)) \subseteq (Y_1)_y \). By Condition 5.1, we can express one of \( \alpha, \beta, t \) equivariantly locally analytically in the other variables. So, the flop diagram \( Y_0 \to X_0 \leftarrow Y_1 \) is locally analytically a \((1, 1, -1, -1)\)-flop above both of the points.

We show that the next diagram in the 2-ray game of \( Y_0 \) is a \((4, 1, 1, -2, -1; 2)\)-flop (this is case (1) in [Bro99, Theorem 8]). The toric variety \( T_1 \) is given by

\[
T_1: \begin{pmatrix}
  u & x & y & z & \alpha & \beta & t \\
  3 & 4 & 1 & 1 & 0 & -1 & -2 \\
 -1 & -1 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

The base of the flip is \( P_\alpha = [0, 0, 0, 0, 1, 0, 0] \). On the patch where \( \alpha \) is non-zero, we can express \( u \) locally analytically and equivariantly in terms of \( x, y, z, \beta, t \). After substitution, the ideal is principal, with generator \( f' = -\beta \cdot (2x + \ldots) + xt + a_2 \). Under Condition 5.1, \( a_2 \) has a non-zero coefficient in \( f' \), so the flip diagram corresponds to case (1) in [Bro99, Theorem 8]. The flips contracts a curve containing a \( 1/4(1, 1, 3) \) singularity and extracts a curve containing a \( 1/2(1, 1, 1) \) singularity and an ordinary double point. The ordinary double point on \( Y_2 \) is at \( [u_0, 0, 0, 0, 2, 1, 1] \) for some \( u_0 \in \mathbb{C} \).

We show that the last map in the 2-ray game of \( Y_0 \) is a weighted blowup \( Y_2 \to Z \), where \( Z \) is isomorphic to a hypersurface \( Z_5 \subseteq \mathbb{P}(1, 1, 1, 1, 2) \) with variables \( u, y, z, \beta, \alpha \). Acting by the matrix \( \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \) on the initial action-matrix of \( T_0 \), we find that \( T_2 \) is given by

\[
T_2: \begin{pmatrix}
  u & x & y & z & \alpha & \beta & t \\
  2 & 3 & 1 & 1 & 1 & 0 & -1 \\
 1 & 2 & 1 & 1 & 2 & 1 & 0
\end{pmatrix}.
\]
Define the bidegree $(5,2)$ complete intersection $Z: \mathbb{V}(h,a_2 - u\beta + x) \subseteq \mathbb{P}(1,1,1,2,2)$ with variables $u,y,z,\beta,x,\alpha$, where
\[
\begin{align*}
h &= \alpha(-u\alpha + 2(b_3 - 4u\beta a_1 + 4xa_1 + x\beta)) \\
&\quad + 2\beta(c_1 - u\beta b_2 + 2xb_2 + 2x\beta a_1 + 2u^2\beta^2 a_0 - 6ux\beta a_0 + 6x^2a_0) \\
&\quad + x^2B_Z + xC_Z + D_Z,
\end{align*}
\]
where
\[
B_Z = B_1(y,z,u), \quad C_Z = C_3(y,z,u), \quad D_Z = D_5(y,z,u).
\]
The morphism $Y_2 \to Z$ given by the ample model of $\mathbb{V}(\beta)$ is a weighted blowdown with centre $P_\beta$ and exceptional locus $\mathbb{V}(t)$. Substituting
\[
x = u\beta - a_2 \tag{5.1}
\]
into $h$, we find that $Z$ is isomorphic to a hypersurface $Z_5 \subseteq \mathbb{P}(1,1,1,2)$ with variables $u,y,z,\beta,\alpha$. The substitution $(5.1)$ does not lift onto $Y_2$. Instead, on the patch $Z_\beta$, we can substitute $u = (a_2 + x)/\beta$. This substitution lifts to $(Y_2)_\beta$. By Condition 5.1, $P_\beta \in Z$ is a $cA_3$ singularity and the hypersurface $Z_3$ is given by a weight 4 polynomial. By Proposition 4.5, $(Y_2)_\beta \to Z_\beta$ is a $(3,1,1,1)$-Kawakita blowup.

Note that $Z$ has an ordinary double point at $[u_0,0,0,1,2]$ for some $u_0 \in \mathbb{C}$.

\section{5.5. $cA_7$ family 1 model}

\begin{proposition}
A Mori fibre space sextic double solid with a $cA_7$ singularity in family 1 satisfying Condition 5.1 has a Sarkisov link to $Z_{3,4} \subseteq \mathbb{P}(1,1,1,2,2)$ with an ordinary double point, starting with a $(4,4,1,1)$-blowup of the $cA_7$ point, then two $(4,1,1,-2,-1,2)$-flips, and finally a blowdown (with standard weights $(1,1,1,1)$) to an ordinary double point. Under further generality conditions, $Z$ has exactly five singular points, namely two $1/2(1,1,1)$ singularities and three ordinary double points.
\end{proposition}

\begin{proof}
We exhibit the diagram below.

\begin{align*}
(4,4,1,1) &\quad \xymatrix{ Y_0 \ar[r] \ar[dr] & Y_1 \ar[r]_{2 \times (4,1,1,-2,-1,2)} \ar[dl] & Y_2 \ar[dr] \ar[dl] \ar[r] & (1,1,1,1) } \\
cA_7 \in X \subseteq \mathbb{P}(1^4,3) &\quad \xymatrix{ \ar[r] & X_0 \ar[r] & \text{ODP} \in Z_{3,4} \subseteq \mathbb{P}(1^4,2^2) } \\
& \quad \text{We construct $X$ and a $(4,4,1,1)$-Kawakita blowup $Y_0 \to X$. We can write a sextic double solid $X$ with an isolated $cA_7$ singularity in family 1 by}
\end{align*}

\[
X: \mathbb{V}(f,\beta - xt - r_2, \gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,2,3,3)
\]

with variables $x,y,z,t,\beta,\gamma,w$, where
\[
f = -w^2 + \gamma^2 - 2t\gamma e_2 + 2\beta^2 e_2 + 2t\beta c_3 + 4t\gamma b_2 - 2\beta^2 b_2 - 2t\beta^2 b_1 + 4xt^2\beta b_1 \\
+ 2x^2t^4b_0 - 16t\gamma a_1^2 + 16\beta^2 a_1^2 + 4\beta^2 a_1 - 8\beta^2 a_0 + 12xt\beta^2 a_0 + xt^3C_2 + t^2D_4,
\]

where $C_i, D_i \in \mathbb{C}[y,z,t]$ are homogeneous of degree $i$. Define $T_0$ by
\[
T_0: \begin{pmatrix}
u & x & y & z & w & \gamma & \beta & t \\
0 & 1 & 1 & 3 & 3 & 2 & 1 & -1 \end{pmatrix}
\]

\[
0 & 1 & 1 & 4 & 4 & 3 & 2
\]

\[35\]
Define $Y_0$ by

$$Y_0: \mathbb{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^* f/u^8, u\beta - r_2 - xt, u\gamma - s_3 - x\beta).$$

The ample model of $\mathbb{V}(x) \subseteq Y_0$ is a $(4, 4, 1, 1)$-Kawakita blowup $Y_0 \to X$.

We show that the diagram $Y_0 \to X_0 \leftarrow Y_1$ induces an isomorphism $Y_0 \to Y_1$. Acting by the matrix \( \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \), we find

$$T_0 \cong \begin{pmatrix} u & x & y & z & w & \gamma & \beta & t \\ 3 & 4 & 1 & 1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$ 

Define $T_1$, respectively $T_2$, with the same action as $T_0$ but with irrelevant ideal $(u, x, y, z) \cap (w, \gamma, \beta, t)$, respectively $(u, x, y, z, w, \gamma) \cap (\beta, t)$. Define $Y_1 \subseteq T_1$ and $Y_2 \subseteq T_2$ by the same ideal $I_Y$. The base of the flop $T_0 \to T_{X_0} \leftarrow T_1$ restricts to $\mathbb{V}(r_2, s_3) \subset \mathbb{P}^1 \subseteq X_0$, which is empty. Therefore, $Y_0 \to X_0$ and $X_0 \leftarrow Y_1$ are isomorphisms.

We show that the next diagram $Y_1 \to X_1 \leftarrow Y_2$ in the $2$-ray game of $Y_0$ is locally analytically two $(4, 1, 1, -2, -1; 2)$-flips. The only monomials in $\Phi^* f/u^8$ that are not in $(u, x, y, z, \beta, t)$ are $-w^2$ and $\gamma^2$. Therefore, the base of the flip is two points, $[1, 1]$ and $[-1, 1] \in \mathbb{P}^1$ with variables $w$ and $\gamma$ inside $X_1$. We make a change of coordinates $w' = w - \gamma$, respectively $w'' = w + \gamma$, for the flip above $[1, 1]$, respectively $[-1, 1]$. On the patch where $\gamma$ is non-zero, we can substitute $u = s_3 + x\beta$ in $\Phi^* f/u^8$, and express $w'$ locally analytically and equivariantly above $[1, 1]$, respectively $[-1, 1]$, in terms of $x, y, z, \beta, t$. After projecting away the variables $u$ and $w'$, we are left with the principal ideal $(\beta s_3 - r_2 + x\beta^2 - xt)$. Since it contains both $r_2$ and $xt$, by case (1) in [Bro99, Theorem 8], it is a terminal $(4, 1, 1, -2, -1; 2)$-flip above both $[1, 1]$ and $[-1, 1]$. The flip contracts two curves, both containing a $1/4(1, 1, 3)$ singularity, and extracts two curves, both containing a $1/2(1, 1, 1)$ singularity and a $cA_1$ singularity. The $cA_1$ points are both ordinary double points if $r_2$ is not a square of a linear form, and are both $3$-fold $A_2$ singularities (given by $x_1^2 + x_2^2 + x_3^2 + x_4^2$) otherwise. On $Y_2$, the $cA_1$ singularities are at $[0, 0, 0, 0, 1, 1, 1, 1]$ and $[0, 0, 0, 0, -1, 1, 1, 1]$.

We show that the last map in the link for $X$ is a divisorial contraction $Y_2 \to Z'$. Acting by the matrix $\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$ on the initial action-matrix of $T_0$, we see that

$$T_2 \cong \begin{pmatrix} u & x & y & z & w & \gamma & \beta & t \\ 2 & 3 & 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 0 \end{pmatrix}.$$ 

Define $Z' \subseteq \mathbb{P}(1, 1, 1, 2, 2, 2)$ with variables $u, y, z, \beta, w, \gamma, x$ by the ideal $I_{Z'}$, where $I_{Z'}$ is the image of the ideal $I_Y$ under the homomorphism $t \mapsto t$. Let $Y_2 \to Z'$ be the ample model of $\mathbb{V}(\beta)$. On the affine patch $Z'_\beta$, we can express $u$ and $x$ locally analytically and equivariantly in terms of $y, z, w, \gamma, \beta, t$. This coordinate change lifts to $Y_2$. By Condition 5.1, we can compute that $P_\beta \subseteq Z'$ is an ordinary double point, and $Y_2 \to Z'$ is locally analytically the (usual) blowup with centre $P_\beta$.

The variety $Z'$ is isomorphic to a complete intersection $Z_{3, 4} \subseteq \mathbb{P}(1^4, 2^2)$, by projecting away from $x$. The variety $Z$ is given by

$$Z_{3, 4}: \mathbb{V}(-s_3 + \beta r_2 + u\gamma - u\beta^2, h) \subseteq \mathbb{P}(1, 1, 1, 1, 2, 2)$$

with variables $u, y, z, \beta, w, \gamma$, where

$$h = -w^2 + \gamma^2 + 2 \beta b_0 r_2^2 - 4 \beta b_1 r_2 - 4 u \beta b_0 r_2 - 12 \beta^2 a_0 r_2 - 2 \gamma e_2 + 2 \beta^2 e_2 + 2 \gamma b_0 - 2 \beta^2 b_2 + 2 u \beta^2 b_1 + 2 u^2 \beta^2 b_0 - 16 \gamma a_1^2 + 16 \beta^2 a_1^2 + 4 \beta \gamma a_1 + 4 u \beta^3 a_0 + (u \beta - r_2) C_Z + D_Z,$$
where $C_Z = C_2(y,z,u)$ and $D_Z = D_4(y,z,u)$. The variety $Z$ has two $cA_1$ singularities at $[0,0,0,1,1]$ and $[0,0,0,1,-1,1]$. □

**Remark 5.9.** We explain how we found the variety $X$. Using $p_2 = r_2$ and $p_3 = s_3$, we can write a sextic double solid with an isolated $cA_7$ in family 1 by $X$: $\mathbb{V}(f, x^2t + xr_2 + s_3 - \gamma)$ inside $\mathbb{P}(1,1,1,1,3,3)$ with variables $x, y, z, t, w, \gamma$, where $f$ is given as in Theorem A. The $(1,1,4,4,2)$-blowup $\bar{Y}_0 \to \bar{X}$ for variables $y, z, w, \gamma, t$ is a $(4,4,1,1)$-Kawakita blowup, but the 2-ray game of $\bar{Y}_0$ does not follow the ambient toric variety $\bar{T}_0$. Namely, the toric anti-flip $\bar{T}_0 \to \bar{T}^0 \leftarrow \bar{T}_1$ restricts to $Y_0 \to X_0 \leftarrow Y_1$, where $Y_0 \to X_0$ is an isomorphism and $X_0 \leftarrow Y_1$ extracts $\mathbb{P}^2$, a divisor on $Y_1$. The reason why $Y_0$ was not the correct variety is that one of the generators of the ideal of $Y_0$ is $\bar{g}_1 = x^2t + xr_2 + us_3 - u\gamma$, which is inside the irrelevant ideal $(u, x)$. We find the correct variety $\bar{Y}_0$ by “unprojecting” $\bar{g}_1 = 0$ with respect to $x, u$. By “unprojection”, we mean the coordinate change $\bar{Y}_0 \to Y_0$, an isomorphism. See [Rei00, Section 2] or [PR04, Section 2.3] for more details on this type of unprojection. This coordinate change induces the coordinate change $\bar{X} \to X$, where $X$ is given as in the proof of Proposition 5.8.

### 5.6. $cA_7$ family 2 model

**Proposition 5.10.** A Mori fibre space sextic double solid with a $cA_7$ singularity in family 2 satisfying Condition 5.1 has a Sarkisov link to a complete intersection $Z_{2,4} \subseteq \mathbb{P}(1,1,1,1,1,2)$ with a $cA_2$ singularity, starting with a $(4,4,1,1)$-blowup of the $cA_7$ point, followed by an Atiyah flop, then two $(4,1,-1,-3)$-flips, and finally a $(3,3,2,1)$-blowdown to a $cA_2$ point. Under further generality conditions, the variety $Z$ is smooth outside the $cA_2$ point.

**Proof.** We exhibit the diagram below.

$$
\begin{array}{ccc}
(4,4,1,1) & \xrightarrow{2 \times (-4,-1,1,3)} & (3,3,2,1) \\
cA_7 \in X \subseteq \mathbb{P}(1,4,3) & \xrightarrow{(1,1,-1,1)} & cA_2 \in Z_{2,4} \subseteq \mathbb{P}(15,2) \\
\xrightarrow{2 \times (-4,-1,1,3)} & \xrightarrow{\Phi} & \\
Y_0 & \to & Y_2 \\
Y_1 & \to & Y_3 \\
X_0 & \to & X_1 \\
\end{array}
$$

We describe the sextic double $X$. Define $X \subseteq \mathbb{P}(1,1,1,1,2,3,3,3)$ with variables $x, y, z, t, \beta, w, \gamma, \xi$ by the ideal

$$
I_X = (f - 2e_3\xi, \beta - q_1r_1 - xt, \gamma - q_1s_2 - x\beta, -\xi + ts_2 - \beta r_1),
$$

where

$$
f = -w^2 + \gamma^2 + 2t\beta c_3 + 4t\gamma b_2 - 2\beta^2 b_2 - 2t\beta^2 b_1 + 4tx^2 b_1 + 2x^2 t^4 b_0 - 16t\gamma a_1 + 16\beta^2 a_1 + 4\beta^2 a_1 - 8\beta^3 a_0 + 12xt\beta^2 a_0 + xt^2 C_2 + t^2 D_4
$$

where $C_i, D_i \in \mathbb{C}[y, z, t]$ are homogeneous of degree $i$.

We describe the weighted blowup $Y_0 \to X$, restriction of $\Phi: T_0 \to \mathbb{P}(1,1,1,1,2,3,3,3)$. Define $T_0$ by

$$
\begin{array}{cccccccc}
u & x & y & z & w & \gamma & \beta & \xi & t \\
0 & 1 & 1 & 3 & 3 & 2 & 3 & 1 \\
-1 & 0 & 1 & 1 & 4 & 4 & 3 & 5 & 2
\end{array}
$$

Define $Y_0 \subseteq T_0$ by the ideal $I_Y$ with the 6 generators

$$
g - 2e_3\xi, \quad u\beta - q_1r_1 - xt, \quad w\gamma - q_1s_2 - x\beta,
$$

$$
-u\xi + ts_2 - \beta r_1, \quad -x\xi + \beta s_2 - \gamma r_1, \quad -q_1\xi + t\gamma - \beta^2,
$$

37
where \( g = \Phi^* f / u^8 \). On the affine patch \( X_x \), we can express \( \beta, t \) and \( \xi \) in terms of \( w, \gamma, y, z \), to get a hypersurface in \( \mathbb{C}^4 \) given by \( f_{\text{hyp}} \in \mathbb{C}[w, \gamma, y, z] \). Note that these coordinate changes lift to \( (Y_0)_x \). Since \( f_{\text{hyp}} \) has weight 8, by Proposition 4.5, \( Y_0 \rightarrow X \) is a \((4, 4, 1, 1)\)-Kawakita blowup.

We show that the first diagram \( Y_0 \rightarrow X_0 \leftarrow Y_1 \) in the 2-ray game of \( Y_0 \) is an Atiyah flop, provided that \( r_1 \) and \( q_1 \) are coprime in \( \mathbb{C}[y, z] \). Acting by the matrix \( \left( \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{array} \right) \) on the action-matrix of \( T_0 \), define \( T_1 \) by

\[
T_1: \begin{pmatrix}
  u & x & y & z & w & \gamma & \beta & \xi & t \\
  3 & 4 & 1 & 1 & 0 & 0 & -1 & -3 & -2
\end{pmatrix}.
\]

Define \( Y_1 \subseteq T_1 \) by the ideal \( I_Y \). The base of the flop is \( \mathbb{V}(q_1) \subseteq \mathbb{P}^1 \) with variables \( y, z \), which is one point. Perform a suitable invertible linear coordinate change on \( y, z \) such that \( q_1 = z \) and \( r_1 = y \). Since \( u^\beta - q_1 r_1 - x t \) is in \( I_Y \), we can substitute \( z = u^\beta - x t \) on the patch where \( y \) is non-zero. The coefficients of \( \beta \) in \( -u \xi + ts_2 - 3y \in I_Y \) and \( \gamma \) in \( -x \xi + \beta s_2 - \gamma y \in I_Y \) are non-zero on the patch where \( y \) is non-zero. Therefore, we can locally analytically equivariantly express \( \beta \) and \( \gamma \) in terms of \( u, x, y, t \). After substituting \( z, \beta, \gamma \), we find that the diagram \( Y_0 \rightarrow X_0 \leftarrow Y_1 \) is locally analytically the Atiyah flop.

The next diagram in the 2-ray game of \( Y_0 \) is the flip \( Y_1 \rightarrow X_1 \). The base of the flip is \( \mathbb{V}(\gamma^2 - u^2) \subseteq \mathbb{P}^1 \) with variables \( w, \gamma \), which are two points \([1, 1] \) and \([-1, 1] \). We consider the point \( P = [1, 1] \), the flip for the other point is similar. Perform a coordinate change \( w' = w - \gamma \). On the patch where \( \gamma \) is non-zero, we find \( u = q_1 s_2 + x \beta \) and \( t = q_1 \beta^2 \). Writing \( q_1 = z \) and \( r_1 = y \) as before, we find \( y = -x \xi + \beta s_2 \). We are left with the principal ideal in \( \mathbb{C}[x, z, w', \beta, \xi] \) generated by \( -w'(2 + w') + \) terms not involving \( w' \). So, we can locally analytically equivariantly express \( w' \) in terms of \( x, z, \beta, \xi \). So, the diagram \( Y_1 \rightarrow X_1 \leftarrow Y_2 \) is locally analytically two \((-4, -1, 1, 3)\)-flips.

The next diagram in the toric 2-ray game \( T_2 \rightarrow T_2 \leftarrow T_3 \) restricts to isomorphisms \( Y_2 \rightarrow X_2 \leftarrow Y_3 \). The reason is that the base of the toric flip \( P_3 \) restricts to an empty set in \( X_2 \), since \( I_Y \) contains the polynomial \( t \gamma - \beta^2 - q_3 \xi \).

We show that the last diagram in the 2-ray game of \( Y_0 \) is a divisorial contraction \( Y_3 \rightarrow Z \). Multiplying the action-matrix of \( T_1 \) by \( \left( \begin{array}{ccc} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{array} \right) \), we see that \( T_3 \) is given by

\[
T_3: \begin{pmatrix}
  u & x & y & z & w & \gamma & \beta & \xi & t \\
  3 & 5 & 2 & 2 & 3 & 3 & 1 & 0 & -1
\end{pmatrix}.
\]

Consider the variety \( Z \subseteq \mathbb{P}(1, 1, 1, 1, 2, 2, 2) \) with variables \( \xi, u, y, z, \beta, x, w, \gamma \) where \( Y_3 \rightarrow Z \) is the ample model of \( \mathbb{V}(\xi) \). On the patch \( Z_\xi \), we can substitute \( u = s_2 - \beta r_1 \), \( x = \beta s_2 - \gamma r_1 \) and \( z = \gamma - \beta^2 \), and compute that \( Z_\xi \) is a hypersurface given by a weight 6 polynomial, with a \( cA_2 \) singularity at \( P_\xi \in Z_\xi \), of type at least 2 (see Definition 4.7). These substitutions lift to \( (Y_3)_\xi \), showing that \( Y_3 \rightarrow Z \) is a \((3, 3, 2, 1)\)-Kawakita blowup with centre \( P_\xi \). If the coefficients are general, namely when

\[
-2e_\beta + 8\beta^4 a_0 r_\beta - 2\beta^2 b_\beta + 12\beta^2 a_\beta^2 \in \mathbb{C}[y, \beta]
\]

is not a full square, where \( r_\beta = r_1(y, -\beta^2) \), \( e_\beta = e_3(y, -\beta^2) \), \( a_\beta = a_1(y, -\beta^2) \) and \( b_\beta = b_2(y, -\beta^2) \), then the point \( P_\xi \) is exactly of type 2.

The variety \( Z \) is isomorphic to a complete intersection \( Z_{2,4} \subseteq \mathbb{P}(1, 1, 1, 1, 2) \) with variables \( u, y, z, \beta, \xi, w \). We see this by substituting \( x = u^\beta - q_1 r_1 \) and \( \gamma = q_1^2 + \beta^2 \). We
find that $Z$ is isomorphic to $Z_{2,4}: \mathcal{V}(-u\xi + s_2 - \beta r_1, h)$, where
\[
\begin{align*}
    h &= -w^2 + \xi^2 q_1^2 - 2e_3\xi + \beta^4 + 2b_0 q_1^2 - 4\beta b_1 q_1 r_1 - 4u\beta b_0 q_1 r_1 - 12\beta^2 a_0 q_1 r_1 + 4\xi b_2 q_1 \\
    &\quad - 16\xi a_1^2 q_1 + 4\beta a_1 q_1 + 2\beta^2 q_1 + 2\beta^3 a_1 + 4u\beta^3 a_0 + (u\beta - q_1 r_1) C_Z + D_Z,
\end{align*}
\]
where $C_Z = C_2(y, z, u)$ and $D_Z = D_4(y, z, u)$. \hfill \Box \\

**Remark 5.11.** We explain below how we found the embedding of $X$. Using Theorem A and the coordinate change in $cA_7$ family 1, we can write a sextic double solid $\bar{X}$ with an isolated $cA_7$ in family 2 by
\[
\bar{X}: \mathcal{V}(f - 2e_3(ts_2 - \beta r_1), \beta - xt - q_1 r_1, \gamma - x\beta - q_1 s_2) \subseteq \mathbb{P}(1, 1, 1, 2, 3, 3)
\]
with variables $x, y, t, \beta, \gamma, w$.

We construct a $(4, 4, 1, 1)$-Kawakita blowup $\bar{Y}_0 \to \bar{X}$. Define $\bar{T}_0$ by

\[
\bar{T}_0: \begin{pmatrix} u & x & y & z & w & \gamma & \beta & t \\ 0 & 1 & 1 & 1 & 3 & 3 & 2 & 1 \\ -1 & 0 & 1 & 1 & 4 & 4 & 3 & 2 \end{pmatrix}.
\]

Let $T_0 \to \mathbb{P}(1, 1, 1, 2, 3, 3)$ be the ample model of $\mathcal{V}(x)$ and $Y_0 \subseteq T_0$ the strict transform of $X$. Then $\bar{Y}_0$ is given by the ideal $I_{\bar{Y}} = (\bar{g}_1, \ldots, \bar{g}_5)$, where
\[
\begin{align*}
    \bar{g}_1 &= u^3 + 2e_3(\beta r_1 - ts_2), \quad \bar{g}_2 = u^2 - q_1 r_1 - xt, \quad \bar{g}_3 = u\gamma - q_1 s_2 - x\beta, \\
    \bar{g}_4 &= x^3 + 2e_3(\gamma r_1 - ts_2), \quad \bar{g}_5 = q_1 g + 2e_3(\beta^2 - t\gamma)).
\end{align*}
\]

The morphism $\bar{Y}_0 \to \bar{X}$ is a $(4, 4, 1, 1)$-Kawakita blowup, as can be checked on the patch $(\bar{Y}_0)_x \to \bar{X}_x$.

Note that we do not prove that $I_{\bar{Y}}$ is saturated with respect to $u$. In fact, the saturation will not be $I_{\bar{Y}}$ if we do not use assume some generality conditions, similarly to $cA_6$ and $cA_7$ family 1. As a heuristic argument to see why $I_{\bar{Y}}$ might be saturated in the general case ("general" meaning a Zariski open dense set of the parameter space), we can use computer algebra software, pretend that $a_i, b_i, c_i, d_i, q_i, r_i, s_2, e_3$ are algebraically independent variables of a polynomial ring over $\mathbb{Q}$ or $\mathbb{Z}_p$ for a large prime $p$, and calculate that the saturation in that case indeed equals the ideal $I_{\bar{Y}}$.

Similarly to the diagram $Y_0 \to X_0 \leftarrow Y_1$ in the proof of Proposition 5.10, the diagram $\bar{Y}_0 \to \bar{X}_0 \leftarrow \bar{Y}_1$ is an Atiyah flop, provided $r_1$ and $q_1$ are coprime.

We show that $I_{\bar{Y}}$ does not 2-ray follow $\bar{T}_0$, namely that the diagram $\bar{Y}_1 \to \bar{X}_1 \leftarrow \bar{Y}_2$ contracts a curve and extracts a divisor. Acting by the matrix \( \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \) on the action-matrix of $\bar{T}_0$, define $\bar{T}_1$ by

\[
\bar{T}_1: \begin{pmatrix} u & x & y & z & w & \gamma & \beta & t \\ 3 & 4 & 1 & 1 & 0 & 0 & -1 & -2 \\ -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},
\]

and define $\bar{Y}_1 \subseteq \bar{T}_1$ by the zeros of $I_{\bar{Y}}$. We consider the toric flip $\bar{T}_1 \to \bar{T}_{\bar{X}_1} \leftarrow \bar{T}_2$ and restrict it to $\bar{Y}_1 \to \bar{X}_1 \leftarrow \bar{Y}_2$. Since $I_{\bar{Y}}$ is the zero ideal when restricting to $\mathcal{V}(u, x, y, z, \beta, t)$, the base $\mathbb{P}_1 \subseteq \bar{T}_{\bar{X}_1}$ of the toric flip restricts to $\mathbb{P}_1 \subseteq \bar{X}_1$ with variables $w, \gamma$. The morphism $\bar{Y}_1 \to \bar{X}_1$ contracts a curve $\mathbb{P}_1$ to both of the points $[1, 1]$ and $[1, -1]$ in the base $\mathbb{P}_1 \subseteq \bar{X}_1$ and is an isomorphism elsewhere. The morphism $\bar{X}_1 \leftarrow \bar{Y}_2$ extracts a curve $\mathbb{P}^1$ for every
point in the base $\mathbb{P}^1 \subseteq \widetilde{X}_1$, so extracts a divisor on $\widetilde{Y}_2$. The diagram $\widetilde{Y}_1 \to \widetilde{X}_1 \leftarrow \widetilde{Y}_2$ is not a step in the 2-ray game of $\widetilde{Y}_0$, so $I_{\mathcal{V}}$ does not 2-ray follow $\widetilde{T}_0$. The reason for this was that the ideal $I_{\mathcal{V}}$ is contained in $(u, x, y, z)$, so the surface $\mathcal{V}(u, x, y, z) \subseteq \mathcal{T}_2$ exists on $\widetilde{Y}_2$, but does not exist on $\widetilde{T}_1$.

We “unproject” $\tilde{g}_1 = \tilde{g}_4 = \tilde{g}_5 = 0$ with respect to $u, x, y, z$ in $\widetilde{Y}_1 \subseteq \widetilde{T}_1$, to find a variety $Y_1 \subseteq T_1$. We explain below what we mean by this. We can write the system of equations

$$
\begin{pmatrix}
\beta r_1 - ts_2 \\
g \\
0 \\
\end{pmatrix} = 0
$$

If the above equations hold, then we have

$$
\begin{pmatrix}
0 \\
g \\
0 \\
\end{pmatrix} = 0
$$

Calculating the determinants and dividing by $-g^2$, we find the equalities

$$
\frac{ts_2 - \beta r_1}{u} = \frac{\beta s_2 - \gamma r_1}{x} = \frac{t\gamma - \beta^2}{q_1} = \frac{g}{2e_3}, \quad (5.3)
$$

between elements of the field of fractions of $\mathbb{C}[u, x, y, z, w, \gamma, \beta, t]/I_{\mathcal{V}}$. Using the Equations 5.3, we see that the morphism $\tilde{Y}_1 \to Y_1$ given by

$$
[u, x, y, z, w, \gamma, \beta, t] \mapsto [u, x, y, z, w, \gamma, \beta, \frac{ts_2 - \beta r_1}{u}, t]
$$

is an isomorphism, where $Y_1$ is described in the proof of Proposition 5.10.

The coordinate change $\tilde{Y}_1 \to Y_1$ induces an isomorphism $\tilde{X} \to X$, giving the variety $X$.

5.7. cA$_7$ family 3 model

**Proposition 5.12.** A Mori fibre space sextic double solid with a cA$_7$ singularity in family 3 satisfying Condition 5.1 has a Sarkisov link to a degree 2 del Pezzo fibration, starting with a $(4, 4, 1, 1)$-blowup of the cA$_7$ point and followed by two Atiyah flops.

**Proof.** We exhibit the diagram below.

First, we define $X$ and a $(4, 4, 1, 1)$-Kawakita blowup $Y_0 \to X$. Any sextic double solid with an isolated cA$_7$ family 3 can be given by a bidegree $(6, 2)$ complete intersection

$$
X : \mathcal{V}(f, -\xi + ts_1 - q_2 - xt) \subseteq \mathbb{P}(1, 1, 1, 1, 2, 3)
$$
with variables $x, y, z, t, \xi, w$, where
\[
f = -w^2 + x^2\xi^2 - 2\xi e_4 + \xi^2(s_1^2 + 4a_1s_1 + 2xs_1 - 2b_2 + 16a_1^2 + 4xa_1 + 8a_0) \\
+ t(ts_1^2 + 4ta_1s_1^2 - 8t^2a_0s_1^2 - 2xs_1^2 + 2tb_2s_1^2 - 2t^2b_1s_1^2 - 8\xi a_1s_1^2 + 24t\xi a_0s_1^2 \\
+ 12xt^2a_0s_1^2 - 2x\xi s_1^2 + 24ts_t s_1 + 4txt^2b_1s_1 - 16\xi a_1s_1 - 4x\xi a_1s_1 \\
- 24\xi^2a_0s_1 - 24xt\xi a_0s_1 - 2\xi c_3 - 4x\xi b_2 - 2\xi^2b_1 - 4xt\xi b_1 + 2x^2t^3b_0 + 16\xi a_1^2 \\
+ 12x^2\xi^2a_0 + xt^2C_2 + tD_4),
\]
where $C_i, D_i \in \mathbb{C}[y, z, t]$ are homogeneous of degree $i$. Define
\[
T_0 : \begin{pmatrix}
u & x & y & z & w & \xi & t \\
0 & 1 & 1 & 1 & 3 & 2 & 1 \\
-1 & 0 & 1 & 1 & 4 & 4 & 2
\end{pmatrix}.
\]
Define $\Phi : T_0 \to \mathbb{P}(1, 1, 1, 1, 2, 3)$ by the ample model of $\mathcal{V}(x)$, and define $Y_0$ as the strict transform of $X$. Then, $Y_0$ is given by
\[
Y_0 : \mathcal{V}(I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = (\Phi^*f/u^8, -u^2\xi + uts_1 - q_2 - xt),
\]
Using Proposition 4.5, we see that $Y_0 \to X$ is a $(4, 4, 1, 1)$-Kawakita blowup.

We describe the flop $Y_0 \to X_0 \to Y_1$. Multiplying the action-matrix of $T_0$ by $(\frac{1}{t} \quad 1)\), we find
\[
T_0 \approx \begin{pmatrix}
u & x & y & z & w & \xi & t \\
1 & 1 & 0 & 0 & -1 & -2 & -1 \\
-1 & 0 & 1 & 1 & 4 & 4 & 2
\end{pmatrix}.
\]
The base of the flop is given by $\mathcal{V}(q_2) \subseteq \mathbb{P}^1 \subseteq X_0$. After a suitable coordinate change on $y, z$, we find $q_2 = yz$. Consider the flop over $\mathcal{V}(y)$, the flop over the other point is similar. Since $q_2$ and $e_4$ have no common divisor, on the patch where $z$ is non-zero, we can express $y$ and $\xi$ locally analytically equivariantly in terms of $u, x, t, w$. So, $Y_0 \to X_0 \to Y_1$ is locally analytically two Atiyah flops.

The morphisms $Y_1 \to X_1 \to Y_2$ are isomorphisms, since $w^2$ has a non-zero coefficient in $\Phi^*f/u^8$.

We show that $Y_2$ is a degree 2 del Pezzo fibration. Multiplying the original action-matrix of $T_0$ by the matrix $(\frac{1}{t} \quad 0 \quad 1)$ with determinant $-1$, we find
\[
T_2 : \begin{pmatrix}
u & x & y & z & w & \xi & t \\
0 & 1 & 1 & 1 & 3 & 2 & 1 \\
1 & 2 & 1 & 1 & 2 & 0 & 0
\end{pmatrix}.
\]
The ample model of $\mathcal{V}(t)$ is
\[
Y_2 \to \mathbb{P}(2, 1) \\
[u, x, y, z, w, \xi, t] \mapsto [\xi, t].
\]
Since $\mathbb{P}(2, 1)$ is isomorphic to $\mathbb{P}^1$, we see that $Y_2$ is a fibration onto $\mathbb{P}^1$. On the patch $(Y_2)_t$, we can substitute $x = u s_1 - q_2 - u^2\xi$, to find that the general fibre is a weighted degree 4 hypersurface in $\mathbb{P}(1, 1, 1, 2)$, so a degree 2 del Pezzo surface. \qed
5.8. cA₈ model

**Proposition 5.13.** A Mori fibre space sextic double solid with a cA₈ singularity satisfying Condition 5.1 has a Sarkisov link to a complete intersection \( Z_{3,3} \subseteq \mathbb{P}(1,1,1,1,2) \) with a cD₄ singularity, starting with a (5, 4, 1, 1)-blowup of the cA₈ point, followed by a (4, 1, 1, −1, −2; 2)-flip, and finally a (3, 2, 2, 1, 5)-blowdown to the cD₄ singularity. Under further generality conditions, the singular locus of \( Z \) consists of 3 points, namely the cD₄ point, the 1/2(1, 1, 1) singularity and an ordinary double point.

**Proof.** We exhibit the diagram below.

\[
\begin{array}{ccc}
Y_0 \xrightarrow{\sim} Y_1 \xrightarrow{(4,1,1,-1,-2:2)} Y_2 \xrightarrow{\sim} Y_3 & \xrightarrow{(3,2,2,1,5)} cD_4 \subseteq Z_{3,3} \subseteq \mathbb{P}(5,2) \\
\xrightarrow{(1,1,4,5)} cA_8 \subseteq X_6 \subseteq \mathbb{P}(1^4,3) & \xrightarrow{X_1} \\
\end{array}
\]

First, we describe \( X \) and the weighted blowup \( Y_0 \to X \). A sextic double solid with a cA₈ singularity can be given by a multidegree \((6, 2, 3)\) complete intersection

\[
X : \mathbb{V} (f, \beta - xt - r_2, \gamma - x\beta - s_3) \subseteq \mathbb{P}(1,1,1,1,2,3,3),
\]

with variables \(x, y, z, t, \beta, \gamma, \xi\) where

\[
f = 8\beta^3(A_0 - a_0) + \xi(-\xi + 2\gamma - 8tA_0r_2 + 2tb_2 - 4t^2 + 4\beta a_1) + t(-16t\beta A_0^2r_2 + 2t\beta c_2 + 4t\gamma b_1 - 2\beta^2b_1 - 2t\beta^2b_0 + 4xt^2b_0 - 8t\gamma a_0a_1 + 8\beta^2a_0a_1 + 12\beta\gamma a_0 - 2t\gamma B_1 + 2\beta_2B_1 + 16t\beta A_0^2 - 16xt^2\beta A_0^2 - 8\beta\gamma A_0 + xt^2C_1 + t^2D_3)
\]

where \( C_i, D_i \in \mathbb{C}[y, z, t] \) are homogeneous of degree \(i\). Note that \( B_1 \in \mathbb{C}[y, z] \). Define

\[
T_0 : \begin{pmatrix} u & x & y & z & \beta & \xi & t \\ 0 & 1 & 1 & 3 & 2 & 3 & 1 \\ -1 & 0 & 1 & 4 & 3 & 5 & 2 \end{pmatrix}.
\]

Let \( \Phi : T_0 \to \mathbb{P}(1,1,1,1,2,3,3) \) be the ample model of \( \mathbb{V}(x) \) and let \( Y_0 \subseteq T_0 \) be the strict transform of \( X \). Then \( Y_0 \) is given by

\[
Y_0 : \mathbb{V} (I_Y) \subseteq T_0 \quad \text{where} \quad I_Y = \left( \Phi^* f / u^9, u\beta - xt - r_2, u\gamma - x\beta - s_3 \right),
\]

and \( Y_0 \to X \) is a (5, 4, 1, 1)-Kawakita blowup.

The first diagram in the 2-ray game of \( T_0 \) restricts to a isomorphisms \( Y_0 \to X_0 \leftarrow Y_1 \), since \( r_2 \) and \( s_3 \) are coprime.

The second diagram in the 2-ray game of \( T_0 \) restricts to a \((4, 1, 1, -1, -2; 2)\)-flip \( Y_1 \to X_1 \leftarrow Y_2 \). Define the toric variety \( T_1 \) by multiplying the action matrix of \( T_0 \) by the matrix \( \left( \begin{array}{ccc} 4 & -3 \\ 3 & -2 \end{array} \right) \),

\[
T_1 : \begin{pmatrix} u & x & y & z & \gamma & \beta & \xi & t \\ 3 & 4 & 1 & 1 & 0 & -1 & -3 & -2 \\ 2 & 3 & 1 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}.
\]

On the patch where \( \gamma \) is non-zero, we have \( u = x\beta + s_3 \) and we can write \( \xi \) locally analytically equivariantly in terms of \(x, y, z, \beta, t\). We are left with the hypersurface given by \( x\beta^2 + \beta s_3 - xt - r_2 \) in \( \mathbb{C}^5 \) with variables \(x, y, z, \beta, t\) with weights \((4, 1, 1, -1, -2)\). The
polynomial contains $xt$ and $r_2$, so this corresponds to case (1) in [Bro99, Theorem 8], a $(4, 1, 1, -1, -2; 2)$-flip. Similarly to Proposition 5.8, the flip contracts a curve containing a $1/4(1, 1, 3)$ singularity, and extracts a curve containing a $1/2(1, 1, 1)$ singularity and a $cA_1$ singularity, which is an ordinary double point if $r_2$ is not a square and is a 3-fold $A_2$ singularity otherwise. The $cA_1$ singularity on $Y_2$ is at $[0, 0, 0, 1, 1, -2a_0, 1]$. The third diagram in the 2-ray game of $T_0$ restricts to isomorphisms $Y_2 \rightarrow X_2 \leftarrow Y_3$, under Condition 5.1, namely that $a_0 \neq A_0$. On the patch where $\beta$ is non-zero, the base of the toric flip restricts to $\mathcal{V}(A_0 - a_0, u, x, y, z, \gamma, \xi, t) \subseteq X_2$.

We describe the weighted blowdown $Y_3 \rightarrow Z$. Multiplying the action matrix of $T_0$ by the matrix $\left( \begin{smallmatrix} 3 & 5 & 2 & 2 & 3 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{smallmatrix} \right)$, the toric variety $T_3$ is given by

$$T_3: \left( \begin{array}{cccccc} u & x & y & z & \gamma & \beta \\ 3 & 5 & 2 & 2 & 3 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 \end{array} \right).$$

The ample model of $\mathcal{V}(\xi)$ is $Y_3 \rightarrow Z$ where $Z$ is the tridegree $(3, 2, 3)$ complete intersection

$$Z: \mathcal{V}(h, u\beta - x - r_2, u\gamma - x\beta - s_3) \subseteq \mathbb{P}(1, 1, 1, 1, 1, 2, 2)$$

with variables $u, y, z, \beta, \xi, x, \gamma$, where

$$h = 8\beta^3(A_0 - a_0) + \xi(-u\xi + 2\gamma - 8A_0r_2 + 2b_2 - 4a_1^2 + 4\beta a_1) - 16\beta A_0^2r_2 + 2\beta c_2 + 4\gamma b_1 - 2\beta b_2 - 2u\beta_2 b_0 + 4x\beta b_0 - 8\gamma a_0 a_1 + 8\beta^2 a_0 a_1 + 12\beta a_0 - 2\beta B_1 + 2\beta^2 B_1 + 16u\beta^2 A_0^2 - 16x\beta A_0^2 - 8\gamma A_0 + xC_Z + D_Z$$

where $C_Z = C_1(y, z, u)$ and $D_Z = D_3(y, z, u)$. Substituting $x = u\beta - r_2$, we see that $Z$ is isomorphic to a complete intersection of bidegree $(3, 3)$ in $\mathbb{P}(1^5, 2)$ with variables $u, y, z, \beta, \xi, \gamma$. The variety $Z$ has a $cA_1$ singularity at $[0, 0, 0, 1, -2a_0, 1]$. We can compute that the point $P_\xi \in Z$ is a $cD_4$ point, by showing the complex analytic space germ $(Z, P_\xi)$ is isomorphic to $(\mathcal{V}(u^2 + 2\beta r_2 - s_3 + h.o.t), 0) \subseteq (\mathbb{C}^2, 0)$ with variables $u, \beta, y, z$, where h.o.t are higher order terms in $y, z, \beta$. We can compute that $Y_3 \rightarrow Z$ is the divisorial contraction to a $cD_4$ point described in [Yam18, Theorem 2.3].

A. Computer code

The code below is for the computer algebra system Maxima [Max18]. To use the splitting lemma library, copy the code below to a file named “Splitting lemma.mac”, start Maxima in the same folder as that file, and load the library using \texttt{load("Splitting lemma.mac")}. Alternatively, just copy-paste the code below to Maxima.

Listing 1. Splitting lemma library

```maxima
/* Language: Maxima 5.42.1 */
splitting(str, poly, inDeg, splitVar, dummyVar, outDeg) := block(
    /* Assume f[0, 0] = f[1, 0] = f[0, 1] = f[1, 1] = 0 and f[2, 0] = 1 */
    [simpPoly, outFun, outPoly],
    local(f, h, g, p, v),
    simpPoly : ratexpand(poly),
    /* Memoizing functions f[i, d] instead of f(i, d) for performance */
    f[i, d] := coeff(coeff(simpPoly, splitVar, i), dummyVar, inDeg-i-d),
```
/* Use apply + makelist instead of sum to avoid dynamic scoping issues */

h[d] := ratexpand(
    f[0, d] - apply("+", makelist(g[0, j]*g[0, d-j], j, 2, d-2))
),
g[i, d] := if i = 1 and d = 0 then 0
    else ratexpand(1/2*(f[i+1, d] - apply("+", makelist(apply("+", makelist(
g[j, k]*g[i+1-j, d-k], j, max(0, 2-k), min(i+1, i+d-k-1)), k, 0, d))))),
p[d] := ratexpand(g[0, d] - apply("+", makelist(v[0, d-j]*p[j], j, 2, d-1))),
v[i, d] := if i = 0 and d = 0 then 1
    else ratexpand(g[i+1, d] - apply("+", makelist(v[i+1, d-j]*p[j], j, 2, d))),

for i : 1 thru 4 do
    if str = ["h", "g", "p", "v"][i] then outFun : [h, g, p, v][i],
    outPoly : if member(str, ["h", "p"]) then
        apply("+", makelist(dummyVar^(outDeg-d)*outFun[d], d, 0, outDeg))
    else if member(str, ["g", "v"]) then
        apply("+", makelist(apply("+", makelist(
dummyVar^(outDeg-k) * splitVar^i * outFun[i, k-i], i, 0, k)), k, 0, outDeg))
    else
        "Splitting error: first argument must be 'h', 'g', 'p' or 'v'.",
    return(outPoly)
);

We give an example below how to use the splitting lemma library.

Listing 2. Splitting lemma example — quartic surface

/*
 * Language: Maxima 5.42.1
 * Example of a quartic surface in projective space with an
 * A_{19} singularity. We use the splitting lemma twice to verify that
 * the singularity type is A_{19}, so it is locally analytically given
 * by x^2 + y^2 + z^20.
 * The quartic polynomial is taken from M.-Kato, I.-Naruki, "Depth
 * of rational double points on quartic surfaces", Proc.-Japan
 * Acad.-Ser.-A Math.-Sci.-
 * textbf{58} (1982), no 2, p 72--75.
 * doi:10.3792/pjaa.58.72,
 * \url{https://projecteuclid.org/euclid.pjaa/1195516147}.
 * Here t is the dummy homogenizing variable, x and y are the splitting
 * variables. We check the singularity type of the point [0, 0, 0, 1].
 */

load("Splitting lemma.mac");
f : 1/16*(16*(x^2 + y^2)*t^2 + 32*x*z^2*t - 16*y^3*t + 16*z^4 - 32*y*z^3
    + 8*(2*x^2 - 2*x*y + 5*y^2)*z^2 + 8*(2*x^3 - 5*x^2*y - 6*x*y^2 - 7*y^3)*z
    + 8*(2*x^2 - 2*x*y + 5*y^2)*z^2 + 8*(2*x^3 - 5*x^2*y - 6*x*y^2 - 7*y^3)*z

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We use the code below in Section 3 to find the equations of sextic double solids with a $cA_n$ singularity.

Listing 3. Construct sextic double solids with a $cA_n$ singularity

```maxima
/* Language: Maxima 5.42.1 */
load("Splitting lemma.mac");
splitSDS(poly, n) := subst(1, x, splitting("h", poly + w^2, 6, t, x, n));
fGen : -w^2 + x^4*t^2 + x^3*(4*t^3*a_0 + 4*t^2*a_1 + 2*t*a_2 + a_3) + x^2*(2*t^4*b_0 + 2*t^3*b_1 + 2*t^2*b_2 + 2*t*b_3 + b_4) + x*(2*t^5*c_0 + 2*t^4*c_1 + 2*t^3*c_2 + 2*t^2*c_3 + 2*t*c_4 + c_5) + t^6*d_0 + 2*t^5*d_1 + t^4*d_2 + 2*t^3*d_3 + t^2*d_4 + 2*t*d_5 + d_6;
```

```maxima
h_3 = splitSDS(fGen, 3);
sub3(poly) := ratexpand(subst(0, a_3, poly));
h_4 = splitSDS(sub3(fGen), 4);
sub4(poly) := ratexpand(subst(a_2^2, b_4, sub3(poly)));
h_5 = splitSDS(sub4(fGen), 5);
sub5(poly) := ratexpand(subst(2*a_2*b_3 - 4*a_1*a_2^2, c_5, sub4(poly)));
h_6 = splitSDS(sub5(fGen), 6);
sub6(poly) := ratexpand(subst(2*a_2*c_4 + b_3^2 - 8*a_1*a_2*b_3 - 2*a_2^2*b_2 + 4*a_0*a_2^3 + 16*a_1^2*a_2^2, d_6, sub5(poly)));
h_7 = splitSDS(sub6(fGen), 7);
sub7(poly) := ratexpand(subst(q*r, a_2, subst(q*s + 4*a_1*q*r, b_3, subst(2*a_1*q*s - 6*a_0*q^2*r^2 + 8*a_1^2*q*r + e*r, c_4, subst(2*b_2*q*s - 8*a_1^2*q*s - e*s - b_1*q^2*r^2 + c_3*q*r, d_5, sub6(poly))))));
```

```maxima
h_8Family1 = splitSDS(sub71(fGen), 8);
sub71(poly) := ratexpand(subst(1, q, subst(r_2, r, subst(s_3, s, subst(e_2, e, subst(poly))))));
h_8Family2 = splitSDS(sub72(fGen), 8);
sub72(poly) := ratexpand(subst(q_1, q, subst(r_1, r, subst(s_2, s, subst(e_3, e, subst(poly))))));
h_8Family3 = splitSDS(sub73(fGen), 8);
sub73(poly) := ratexpand(subst(1, r, subst(q_2, q, subst(s_1, s, subst(e_4, e, subst(poly))))));
h_8Family4 = splitSDS(sub74(fGen), 8);
```

+ 20*x^4 + 44*x^3*y + 65*x^2*y^2 + 40*x*y^3 + 41*y^4)

sub8(poly) := ratexpand(
    subst(4*A_0*r_2 + b_2 - 6*a_1^2, e_2,
    subst(r_2*B_1 - 4*s_3*A_0 + 6*a_0*s_3 + 4*a_0*a_1*r_2 - 2*a_1*e_2 + 4*a_1*b_2
              - 16*a_1^3, c_3,
    subst(-2*s_3*B_1 + 16*r_2^2*A_0^2 - 8*b_2*r_2*A_0 + 16*a_1^2*r_2*A_0
              + 4*b_1*s_3 - 8*a_0*a_1*s_3 - 2*b_0*r_2^2 + 2*c_2*r_2 + b_2^2 - 4*a_1^2*b_2
              + 4*a_1^4, d_4,
    sub71(poly))))
); h_9 = splitSDS(sub8(fGen), 9);

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