On the Asymptotic Superlinear Convergence of the Augmented Lagrangian Method for Semidefinite Programming with Multiple Solutions

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Abstract

Solving large scale convex semidefinite programming (SDP) problems has long been a challenging task numerically. Fortunately, several powerful solvers including SDPNAL, SDPNAL+ and QSDPNAL have recently been developed to solve linear and convex quadratic SDP problems to high accuracy successfully. These solvers are based on the augmented Lagrangian method (ALM) applied to the dual problems with the subproblems being solved by semismooth Newton-CG methods. Noticeably, thanks to Rockafellar’s general theory on the proximal point algorithms, the primal iteration sequence generated by the ALM enjoys an asymptotic Q-superlinear convergence rate under a second order sufficient condition for the primal problem. This second order sufficient condition implies that the primal problem has a unique solution, which can be restrictive in many applications. For gaining more insightful interpretations on the high efficiency of these solvers, in this paper we conduct an asymptotic superlinear convergence analysis of the ALM for convex SDP when the primal problem has multiple solutions (can be unbounded). Under a fairly mild second order growth condition, we prove that the primal iteration sequence generated by the ALM converges asymptotically Q-superlinearly, while the dual feasibility and the dual objective function value converge asymptotically R-superlinearly. Moreover, by studying the metric subregularity of the Karush-Kuhn-Tucker solution mapping, we also provide sufficient conditions to guarantee the asymptotic R-superlinear convergence of the dual iterate.

Keywords. Semidefinite programming, augmented Lagrangian, second order growth condition, metric subregularity

AMS subject classifications: 90C25, 90C33, 65K05

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1 Introduction

Let $\mathbb{S}^n$ be the space of $n \times n$ real symmetric matrices equipped with the standard trace inner product $\langle \cdot, \cdot \rangle$ and its induced Frobenius norm $\| \cdot \|$. We use $\mathbb{S}^n_+$ to denote the cone of $n \times n$ symmetric positive semidefinite matrices in $\mathbb{S}^n$. We write $X \succeq 0$ if $X \in \mathbb{S}^n_+$ and $X > 0$ if $X$ is symmetric positive definite.

Semidefinite programming (SDP) is an extremely important and active research area in modern optimization. Among various SDP models, the most fundamental one is the following standard primal linear SDP:

$$\min \left\{ \langle C, X \rangle \mid AX = b, \ X \in \mathbb{S}^n_+ \right\}, \tag{1}$$

where $A : \mathbb{S}^n \to \mathbb{R}^m$ is a linear map, $C \in \mathbb{S}^n$ and $b \in \mathbb{R}^m$ are given data. The dual of (1) is given by

$$\max \left\{ \langle b, y \rangle \mid A^* y + S = C, \ S \in \mathbb{S}^n_+ \right\}, \tag{2}$$

where $A^* : \mathbb{R}^m \to \mathbb{S}^n$ is the adjoint map of $A$.

The problem (1) arises frequently from the SDP relaxations of numerous NP-hard combinatorial optimization problems, such as frequency assignment problems [12], maximum stable set problems [14], quadratic assignment and binary integer quadratic problems [22], etc. As a consequence, much effort has been put into designing algorithms for solving large scale semidefinite programming (SDP) efficiently. It is widely recognized that interior point methods (IPMs) such as those implemented in [35, 39, 41] are highly successful in solving small and medium sized SDPs; see [38] for a nice survey on this topic. However, IPMs are generally inefficient for solving large scale SDPs due to their inherent poor computational scalability and expensive memory requirement. To overcome these drawbacks, various attempts on using first order methods to solve special classes of large SDPs have been made in recent years. These include the boundary point method [25], a directly extended alternating direction method of multipliers (ADMM) [12], a two-easy-block-decomposition hybrid proximal extragradient method [28], and a convergent multi-block ADMM [37]. While the first order methods just mentioned are reasonably efficient in solving some large scale SDPs, they may become inefficient when higher accuracy solutions are required and more seriously, they can fail badly when solving more difficult problems as demonstrated in [43].

In contrast, the solver SDPNAL [44] developed by Zhao, Sun and Toh, which made use of second-order information, is much more efficient in solving large SDPs to high accuracy. This powerful solver, designed for large scale linear SDP problems of the form (1), is based on an augmented Lagrangian method (ALM) applied to the dual problem (2) wherein the subproblems are approximately solved by the semismooth Newton-CG method. Extensive numerical experiments have shown that it is highly efficient for solving large scale SDPs with non-degenerate primal optimal solutions.

A more complicated linear SDP problem is the following so-called doubly nonnegative SDP:

$$\min \left\{ \langle C, X \rangle \mid AX = b, \ X \in \mathbb{S}^n_+, \ X \succeq 0 \right\}. \tag{3}$$

Even though it can be reformulated as a standard SDP by introducing additional constraints $X' = X$ and $X' \succeq 0$, the reformulated problem is usually degenerate and thus SDPNAL may fail to solve it efficiently. To overcome this difficulty, an enhanced version of SDPNAL, called SDPNAL+, was developed by Yang, Sun and Toh [43] recently. With a majorized semismooth Newton-CG method
for solving the inner subproblems in the ALM, the new solver can successfully compute solutions of high accuracy for large scale doubly nonnegative SDPs.

The dual based ALM coupled with a semismooth Newton-CG algorithm has also been extended to other classes of SDP problems. Jiang, Sun and Toh [18] have employed this approach to solve the least squares SDP problem:

\[
\min \left\{ \frac{1}{2} \| F X - d \|^2 + \langle C, X \rangle \mid AX = b, \quad X \in \mathbb{S}^n_+ \right\},
\]

(4)

where \( F : \mathbb{S}^n \to \mathbb{R}^{m'} \) is a linear map and \( d \in \mathbb{R}^{m'} \) is a given vector. Most recently, this idea is adopted by Li, Sun and Toh [21] for developing the solver QSDP NAL to deal with the following convex quadratic SDP (QSDP) with a given self-adjoint positive semidefinite operator \( P : \mathbb{S}^n \to \mathbb{S}^n \):

\[
\min \left\{ \frac{1}{2} \langle X, P X \rangle + \langle C, X \rangle \mid AX = b, \quad X \in \mathbb{S}^n_+ \right\}.
\]

(5)

It is well known that the ALM applied to the dual problem is equivalent to a proximal point algorithm (PPA) applied to its primal form [33]. So for all of the ALM based solvers mentioned above for solving SDP problems, the primal iteration is proven to converge asymptotically superlinearly to an optimal solution under a second order sufficient condition via [33, Proposition 3 and Theorem 2]. However, this second order sufficient condition can be restrictive because it fails to hold when the primal SDP problem has multiple solutions. For better understanding of the ALM for solving SDPs, in this paper we aim to remove this restriction by conducting extensive analysis on both the second order variational properties of the positive semidefinite cone and the metric subregularity of the solution mappings of linearly constrained convex SDPs. In particular, assuming that the problem admits a Karush-Kuhn-Tucker (KKT) point with a partial strict complementarity property, we prove that the sequence \( \{X^k\} \) generated by the ALM converges asymptotically Q-superlinearly, while the dual feasibility and the dual objective function value converge asymptotically R-superlinearly. We also study sufficient conditions for ensuring the metric subregularity of the KKT solution mapping, which is shown to guarantee the asymptotic R-superlinear convergence of the dual iteration sequence.

The remaining parts of this paper are organized as follows. In the next section, we introduce some definitions and preliminary results on variational analysis and maximal monotone operators. In Section 3, we conduct extensive studies on sufficient conditions for the metric subregularity of the solution mappings of linearly constrained convex SDPs. Section 4 is devoted to the asymptotic superlinear convergence analysis of the ALM for solving SDP problems. Under the existence of a strictly feasible solution, we also design new easy-to-implement stopping criteria for the ALM in this section. We conclude our paper and make some comments in the final section.

Below we list other symbols and notation to be used in our paper.

- Let \( \mathbb{R}^{m \times n} \) be the linear space of \( m \times n \) real matrices equipped with the inner product \( \langle X, Y \rangle = \text{tr}(X^T Y) \) for any \( X, Y \in \mathbb{R}^{m \times n} \). Here \( \text{tr}(\cdot) \) denotes the trace, i.e., the sum of all the diagonal entries, of a square matrix. Let \( \mathcal{O}^n \) be the set of \( n \times n \) orthogonal matrices. We also use \( 0_n \) and \( I_n \) to denote the \( n \times n \) zero matrix and identity matrix, respectively. For any \( X \in \mathbb{S}^n \), \( \lambda_{\max}(X) \) and \( \lambda_{\min}(X) \) represent the largest and the smallest eigenvalues of \( X \), respectively.

- We use \( X, Y, Z \) and \( W \) to denote some finite dimensional real Euclidean spaces. For any convex function \( p : X \to (-\infty, +\infty] \), we denote its effective domain as \( \text{dom}(p) := \{ x \in X \mid p(x) < \infty \} \) and its conjugate as \( p^*(u) := \sup_{x \in X} \langle x, u \rangle - p(x) \), \( u \in X \). For any \( x \in X \) and \( \rho > 0 \), we define
\( B_X(x, \rho) := \{ y \in X \mid \| y - x \| \leq \rho \} \). For any linear map \( \mathcal{A} : X \to Y \), we use \( \text{Range}(\mathcal{A}) \) to denote the range space of \( \mathcal{A} \).

- Let \( \alpha \subseteq \{1, \ldots, m\} \) and \( \beta \subseteq \{1, \ldots, n\} \) be two index sets. For any \( Z \in \mathbb{R}^{m \times n} \), we write \( Z_{\alpha \beta} \) to be the \( |\alpha| \times |\beta| \) sub-matrix of \( Z \) obtained by removing all the rows of \( Z \) not in \( \alpha \) and all the columns of \( Z \) not in \( \beta \). Denote \( \text{diag}(x_i \mid i \in \alpha) \) as the \( |\alpha| \times |\alpha| \) diagonal matrix whose \( i \)-th diagonal entry is the \( i \)-th component of \( x_\alpha \), \( i = 1, \ldots, |\alpha| \).

- Let \( D \subseteq X \) be a set. For any \( x \in X \), define \( \text{dist}(x, D) := \inf_{d \in D} \| x - d \| \). We let \( \delta_D(\cdot) \) to be the indicator function over \( D \), i.e., \( \delta_D(x) = 0 \) if \( x \in D \), and \( \delta_D(x) = \infty \) if \( x \not\in D \).

- If \( D \subseteq X \) is a convex set, we use \( \text{ri}(D) \) to denote its relative interior. For a given closed convex set \( D \subseteq X \), the metric projection of \( x \in X \) onto \( D \) is defined by \( \Pi_D(x) := \text{arg min}\{\| x - d \| \mid d \in D \} \). For any \( x \in D \), we use \( T_D(x) \) and \( N_D(x) \) to denote the tangent and normal cone of \( D \) at \( x \), respectively as in standard convex analysis [31]. If \( D \) is a closed convex cone, we use \( D^\circ \) and \( D^\ast \) to denote the polar of \( D \) and the dual of \( D \), respectively, i.e., \( D^\circ := \{ x \in X \mid \langle x, d \rangle \leq 0, \forall d \in D \} \) and \( D^\ast := -D^\circ \).

## 2 Preliminaries

Let \( F : X \rightrightarrows Y \) be a multi-valued mapping. The graph of the mapping \( F \) is defined as \( \text{gph}(F) := \{(x,y) \in X \times Y \mid y \in F(x)\} \). The following definition of metric subregularity is taken from [10, Section 3.8(3H)].

**Definition 2.1.** A multi-valued mapping \( F : X \rightrightarrows Y \) is said to be metrically subregular at \( \bar{x} \in X \) for \( \bar{y} \in Y \) with modulus \( \kappa \geq 0 \) if \( (\bar{x}, \bar{y}) \in \text{gph}(F) \) and there exist neighborhoods \( U \) of \( \bar{x} \) and \( V \) of \( \bar{y} \) such that

\[
\text{dist}(x, F^{-1}(\bar{y})) \leq \kappa \text{dist}(\bar{y}, F(x) \cap V), \; \forall x \in U, \tag{6}
\]

or equivalently, \( F \) is said to be metrically subregular at \( \bar{x} \) for \( \bar{y} \) with modulus \( \kappa \geq 0 \) if there exists a neighborhood \( U' \) of \( \bar{x} \) such that

\[
\text{dist}(x, F^{-1}(\bar{y})) \leq \kappa \text{dist}(\bar{y}, F(x)), \; \forall x \in U'. \tag{7}
\]

The next result, which provides a convenient way to check the metric subregularity of the subdifferential of a proper closed convex function, is proven in [11, Theorem 3.3].

**Proposition 2.1.** Let \( \mathcal{H} \) be a real Hilbert space endowed with the inner product \( \langle \cdot, \cdot \rangle \) and \( \theta : \mathcal{H} \to (-\infty, +\infty] \) be a proper lower semicontinuous convex function. Let \( \bar{v}, \bar{x} \in \mathcal{H} \) satisfy \( (\bar{x}, \bar{v}) \in \text{gph}(\partial \theta) \). Then \( \partial \theta \) is metrically subregular at \( \bar{x} \) for \( \bar{v} \) if and only if there exist a neighborhood \( U \) of \( \bar{x} \) and a constant \( \kappa > 0 \) such that

\[
\theta(x) \geq \theta(\bar{x}) + \langle \bar{v}, x - \bar{x} \rangle + \kappa \text{dist}^2(x, (\partial \theta)^{-1}(\bar{v})), \; \forall x \in U. \tag{8}
\]

A multi-valued mapping \( F : X \rightrightarrows Y \) is said to be polyhedral if its graph is the union of finitely many polyhedral convex sets. Below is a fundamental result from Robinson [30] on multi-valued polyhedral mappings.
Proposition 2.2. Let $F : X \rightrightarrows Y$ be a multi-valued polyhedral mapping and $(\bar{x}, \bar{y}) \in \text{gph}(F)$. Then $F$ is locally upper Lipschitz continuous at $\bar{x}$, i.e., there exist a constant $\kappa > 0$ and a neighborhood $\mathcal{U}$ of $\bar{x}$ such that

$$
F(x) \subseteq F(\bar{x}) + \kappa \|x - \bar{x}\| \mathcal{B}_Y(0, 1), \quad \forall \, x \in \mathcal{U}.
$$

In our subsequent discussions, we also need the concept of bounded linear regularity of a collection of closed convex sets, which can be found from, e.g., [3, Definition 5.6].

Definition 2.2. Let $D_1, D_2, \ldots, D_m \subseteq X$ be closed convex sets for some positive integer $m$. Suppose that $D := D_1 \cap D_2 \cap \ldots \cap D_m$ is non-empty. The collection $\{D_1, D_2, \ldots, D_m\}$ is said to be boundedly linearly regular if for every bounded set $B \subseteq X$, there exists a constant $\kappa > 0$ such that

$$
\text{dist}(x, D) \leq \kappa \max \{\text{dist}(x, D_1), \ldots, \text{dist}(x, D_m)\}, \quad \forall \, x \in B.
$$

A sufficient condition to guarantee the property of bounded linear regularity is established in [4, Corollary 3].

Proposition 2.3. Let $D_1, D_2, \ldots, D_m \subseteq X$ be closed convex sets for some positive integer $m$. Suppose that $D_1, D_2, \ldots, D_r$ are polyhedral for some $r \in \{0, 1, \ldots, m\}$. Then a sufficient condition for $\{D_1, D_2, \ldots, D_m\}$ to be boundedly linearly regular is

$$
\bigcap_{i=1, 2, \ldots, r} D_i \cap \bigcap_{i=r+1, \ldots, m} \text{ri} \, (D_i) \neq \emptyset.
$$

In the following, we shall present an equivalent result on the metric subregularity of maximal monotone operators. Consider the inclusion problem:

$$
0 \in \Gamma(x) + \mathcal{T}(x), \quad x \in X,
$$

where $\Gamma : X \rightarrow X$ is a continuous monotone operator and $\mathcal{T} : X \rightrightarrows X$ is a maximal monotone operator. Define the mapping $\mathcal{R} : X \rightarrow X$ by

$$
\mathcal{R} := \mathcal{I} - (\mathcal{I} + \mathcal{T})^{-1}(\mathcal{I} - \Gamma),
$$

where $\mathcal{I} : X \rightarrow X$ is the identity operator. Then one can see from [27] that

$$
x \in (\Gamma + \mathcal{T})^{-1}(0) \iff x \in \mathcal{R}^{-1}(0).
$$

The following proposition is on the equivalence between the metric subregularity of $\mathcal{R}$ and $\Gamma + \mathcal{T}$ at any $\bar{x} \in \mathcal{R}^{-1}(0)$ for the origin.

Proposition 2.4. Suppose that $\mathcal{R}^{-1}(0) \neq \emptyset$. Let $\bar{x} \in \mathcal{R}^{-1}(0)$. Consider the following two statements:

(a) $\mathcal{R}$ is metrically subregular at $\bar{x}$ for the origin with modulus $\kappa_1 \geq 0$ along with a neighborhood $\mathcal{B}_X(\bar{x}, \rho_1)$, i.e.,

$$
\text{dist}(x, \mathcal{R}^{-1}(0)) \leq \kappa_1 \|\mathcal{R}(x)\|, \quad \forall \, x \in \mathcal{B}_X(\bar{x}, \rho_1);
$$

(b) $\Gamma + \mathcal{T}$ is metrically subregular at $\bar{x}$ for the origin with modulus $\kappa_2 \geq 0$ along with a neighborhood $\mathcal{B}_X(\bar{x}, \rho_2)$, i.e.,

$$
\text{dist}(x, (\Gamma + \mathcal{T})^{-1}(0)) \leq \kappa_2 \text{dist}(0, (\Gamma + \mathcal{T})(x)), \quad \forall \, x \in \mathcal{B}_X(\bar{x}, \rho_2).
$$

Then, the inequality (4) implies the inequality (10) with $\rho_2 = \rho_1$ and $\kappa_2 = \kappa_1$. Conversely, if the inequality (10) holds and there exists $\tau \geq 0$ such that $\Gamma$ is Lipschitz continuous on $\mathcal{B}_X(\bar{x}, (1 + \tau)^{-1}\rho_2)$ with modulus $\tau$, then the inequality (4) holds with $\rho_1 = (1 + \tau)^{-1}\rho_2$ and $\kappa_1 = 1 + (1 + \tau)\kappa_2$. 

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Proof. In [8, Theorem 3.1], Dong proved the equivalence of parts (a) and (b), with $B_X(\tilde{x}, \rho_1)$ in [8] being replaced by $\{x \in X \mid \text{dist}(x, R^{-1}(0)) \leq \epsilon_1\}$ for some $\epsilon_1 > 0$ and $B_X(\tilde{x}, \rho_2)$ in [10] being replaced by $\{x \in X \mid \text{dist}(x, (\Gamma + T)^{-1}(0)) \leq \epsilon_2\}$ for some $\epsilon_2 > 0$, respectively. The proof of Proposition 2.1 can be conducted in a similar way as in [8, Theorem 3.1]. For brevity, we omit the details here. □

3 The metric subregularity of solution mappings

In this section, we shall discuss the metric subregularity of the solution mappings for solving linearly constrained convex semidefinite programming with multiple solutions. These problems can be cast into the following form:

$$
\begin{align*}
\min & \quad \theta(x) := h(Fx) + \langle c, x \rangle + p(x) \\
\text{s.t.} & \quad b - Ax \in Q',
\end{align*}
$$

where $F : X \to W$ and $A : X \to Y$ are given linear maps, $Q \subseteq Y$ is a given convex polyhedral cone, $c \in X$ and $b \in Y$ are given data, $p : X \to (-\infty, +\infty]$ is a closed proper convex function, $h : W \to (\infty, +\infty]$ is continuously differentiable on dom$(h)$, which is assumed to be a non-empty open convex set, and is also strictly convex on any convex subset of dom$(h)$. The dual of problem (11) can be written, in its equivalent minimization form, as

$$
\begin{align*}
\min & \quad \delta_Q(y) - \langle b, y \rangle + h^*(-w) + p^*(-s) \\
\text{s.t.} & \quad A^*y + F^*w + s = c.
\end{align*}
$$

For notational convenience, define $Z := Y \times W \times X$ and for any $(y, w, s) \in Y \times W \times X$, write $z := (y, w, s)$.

The Lagrangian function $l$ associated with problem (12) is given by

$$l(z, x) := \delta_Q(y) - \langle b, y \rangle + h^*(-w) + p^*(-s) + \langle x, A^*y + F^*w + s - c \rangle, \quad \forall \ (z, x) \in Z \times X. \quad (13)$$

Define the functions $\psi$ and $\phi$ by

$$
\begin{align*}
\psi(z) := \sup_{x \in X} l(z, x), & \quad \forall z \in Z, \\
\phi(x) := \inf_{z \in Z} l(z, x), & \quad \forall x \in X.
\end{align*}
$$

Moreover, we define the mapping $T_l : Z \times X \rightrightarrows Z \times X$ as

$$T_l(z, x) := \{(u, v) \in Z \times X \mid (u, -v) \in \partial l(z, x)\}, \quad \forall \ (z, x) \in Z \times X \quad (15)$$

and the mappings $T_\psi : Z \rightrightarrows Z$ and $T_\phi : X \rightrightarrows X$ as

$$
\begin{align*}
T_\psi(z) := -\partial \psi(z), & \quad \forall z \in Z, \\
T_\phi(x) := -\partial \phi(x), & \quad \forall x \in X.
\end{align*}
$$

Assume that problem (12) admits at least one optimal solution $(\bar{y}, \bar{w}, \bar{s}) \in Y \times W \times X$. Let $M_\psi(\tilde{z}) \subseteq X$ denote the set of Lagrangian multipliers corresponding to $\tilde{z}$, i.e., $\tilde{x} \in M_\psi(\tilde{z})$ if and only if $(\bar{y}, \bar{w}, \bar{s}, \bar{x})$ solves the following KKT system:

$$
\begin{align*}
\begin{cases}
0 \in -b + Ax + N_Q(y), & 0 \in Fx - \partial h^*(-w), & 0 \in x - \partial p^*(-s), \\
0 = c - (A^*y + F^*w + s), & (y, w, s, x) \in Z \times X.
\end{cases}
\end{align*}
$$

(17)
It can be easily checked that if \((\bar{z}, \bar{x}) = (\bar{y}, \bar{w}, \bar{s}, \bar{x})\) solves the following KKT system:
\[
\begin{cases}
0 \in c - A^*y + F^*\nabla h(Fx) + \tilde{c}p(x), \\
0 \in Ax - b + \mathcal{N}_Q(y),
\end{cases}
\]
conversely, if \((\bar{x}, \bar{y}) \in X \times Y\) solves \((\ref{eq:KKT})\), then for \(\bar{w} = -\nabla h(F\bar{x})\) and \(\bar{s} = c - A^*\bar{y} - F^*\bar{w}\), \((\bar{z}, \bar{x}) = (\bar{y}, \bar{w}, \bar{s}, \bar{x})\) solves the KKT system \((\ref{eq:KKT})\). Let \(\mathcal{M}_\phi(\bar{x}) \subseteq Z\) be the set of Lagrangian multipliers corresponding to \(\bar{x} \in \mathcal{M}_\phi(\bar{z})\).

### 3.1 The metric subregularity of \(\mathcal{T}_\phi\)

Assume that the KKT system \((\ref{eq:KKT})\) or \((\ref{eq:KKT2})\) admits at least one solution. It is known from \cite{31} Theorem 30.4 and Corollary 30.5.1 that \((\bar{z}, \bar{x}) \in Z \times X\) solves the KKT system \((\ref{eq:KKT})\) if and only if \(\bar{z} \in Z\) solves problem \((\ref{eq:problem})\) and \(\bar{x} \in X\) solves problem \((\ref{eq:subproblem})\). To further characterize \(\mathcal{T}_\phi^{-1}(0)\), we need the following invariant property of \(F\bar{x}\) over \(x \in \mathcal{T}_\phi^{-1}(0)\), whose proof readily follows from the well-known existing techniques in the literature \cite{26, 23, 40}.

**Lemma 3.1.** The value \(F\bar{x}\) is invariant over \(x \in \mathcal{T}_\phi^{-1}(0)\), i.e., for any \(x', x'' \in \mathcal{T}_\phi^{-1}(0)\), we have \(Fx' = Fx''\).  

Take an arbitrary point \(\bar{x} \in \mathcal{T}_\phi^{-1}(0)\) and denote
\[
\bar{\zeta} := F\bar{x}, \quad \bar{\eta} := F^*\nabla h(\bar{\zeta}) + c, \quad \bar{V} := \{x \in X \mid Fx = \bar{\zeta}\}.
\]

We define two multi-valued mappings \(G_1 : Y \ni X\) and \(G_2 : Y \ni X\) by
\[
G_1(y) := (\tilde{c}p)^{-1}(A^*y - \bar{\eta}), \quad G_2(y) := \{x \mid 0 \in Ax - b + \mathcal{N}_Q(y)\}, \forall y \in Y.
\]

Then, from \((\ref{eq:subproblem})\), \((\ref{eq:KKT})\), Lemma 3.1 and the arguments above Lemma 3.1, we immediately obtain the following useful observation for the optimal solution set \(\mathcal{T}_\phi^{-1}(0)\).

**Proposition 3.1.** Assume that \((\bar{y}, \bar{w}, \bar{s}, \bar{x}) \in Z \times X\) solves the KKT system \((\ref{eq:KKT})\). Then the optimal solution set \(\mathcal{T}_\phi^{-1}(0)\) to problem \((\ref{eq:problem})\) can be characterized as
\[
\mathcal{T}_\phi^{-1}(0) = \{x \in X \mid Fx = \bar{\zeta}, 0 \in \bar{\eta} + \tilde{c}p(x) - A^*\bar{y}, 0 \in Ax - b + \mathcal{N}_Q(\bar{y})\} = \bar{V} \cap G_1(\bar{y}) \cap G_2(\bar{y}).
\]

To analyse the metric subregularity of \(\mathcal{T}_\phi\), we will need the following assumption later.

**Assumption 3.1.** The following local growth conditions hold:
(i) For any \(w \in \text{dom}(h)\), there exist \(\kappa_1 > 0\) and a neighborhood \(W \subseteq \mathbb{W}\) of \(w\) such that
\[
h(w') \geq h(w) + \langle \nabla h(w), w' - w \rangle + \kappa_1\|w' - w\|^2, \forall w' \in W.
\]
(ii) For any \((x, v) \in \text{gph}(\tilde{c}p)\), there exist \(\kappa_2 > 0\) and a neighborhood \(U \subseteq X\) of \(x\) such that
\[
p(x') \geq p(x) + \langle v, x' - x \rangle + \kappa_2 \text{dist}^2(x', (\tilde{c}p)^{-1}(v)), \forall x' \in U.
\]
We say that for problem (11), the second order growth condition holds at an optimal solution \( \bar{x} \in T_{\phi}^{-1}(0) \) with respect to the set \( T_{\phi}^{-1}(0) \) if there exist \( \kappa > 0 \) and a neighborhood \( U \) of \( \bar{x} \) such that

\[
\theta(x) \geq \theta(\bar{x}) + \kappa \text{dist}^2(x, T_{\phi}^{-1}(0)), \quad \forall x \in U \cap \{x \in X \mid b-Ax \in Q^c\}. \tag{21}
\]

Consider an arbitrarily fixed point \( \bar{x} \in T_{\phi}^{-1}(0) \). It can be seen from Proposition 2.1 that the operator \( T_{\phi} \) is metrically subregular at \( \bar{x} \) for the origin if and only if the second order growth condition \( (21) \) holds at \( \bar{x} \) with respect to \( T_{\phi}^{-1}(0) \). Thus, we can study the metric subregularity of \( T_{\phi} \) at \( \bar{x} \) for the origin via the second order growth condition \( (21) \). The following lemma is convenient for our later discussions.

**Lemma 3.2.** Let \( \bar{x} \in T_{\phi}^{-1}(0) \) and \( \bar{y} \in M_{\phi}(\bar{x}) \). Then there exist a constant \( \kappa > 0 \) and a neighborhood \( U \) of \( \bar{x} \) such that

\[
\text{dist}(x, G_2(\bar{y})) \leq \kappa \text{dist}(b-Ax, N_Q(\bar{y})), \quad \forall x \in U. \tag{22}
\]

**Proof.** Define the subspace \( \Xi_1 \subseteq X \times Y \) and the polyhedral set \( \Xi_2 \subseteq X \times Y \) by

\[
\Xi_1 = \{(x,q) \in X \times Y \mid b-Ax = q\}, \quad \Xi_2 = \{(x,q) \in X \times Y \mid q \in N_Q(\bar{y})\}.
\]

Denote \( \tilde{G}_2 := \Xi_1 \cap \Xi_2 \), which is non-empty as \( (\bar{x}, b-A\bar{x}) \in \tilde{G}_2 \). Since \( \Xi_1 \) and \( \Xi_2 \) are polyhedral sets, we know from Proposition 2.3 that the collection \( \{\Xi_1, \Xi_2\} \) is boundedly linearly regular. Therefore, there exist a constant \( \kappa > 0 \) and a neighborhood \( U \) of \( \bar{x} \) such that

\[
\text{dist}((x, b-Ax), \tilde{G}_2) \leq \kappa (\text{dist}((x, b-Ax), \Xi_1) + \text{dist}((x, b-Ax), \Xi_2)) = \kappa \text{dist}(b-Ax, N_Q(\bar{y})).
\]

Thus, by noting that there exists \( (x', w') \in \tilde{G}_2 \) such that

\[
\text{dist}((x, b-Ax), \tilde{G}_2) = \sqrt{\|x-x'\|^2 + \|b-Ax-w'\|^2} \geq \|x-x'\| \geq \text{dist}(x, G_2(\bar{y})),
\]

we prove the conclusion of Lemma 3.2. \( \square \)

The following result, which is partially motivated by the recent paper \[45]\ and its further development in [11] for convex composite optimization problems regularized by the nuclear norm function of rectangular matrices, provides a general approach for proving the metric subregularity of \( T_{\phi} \) associated with problem (11) where the constraint \( b-Ax \in Q^c \) is present.

**Theorem 3.1.** Assume that \( T_{\phi}^{-1}(0) \) is non-empty. Suppose that Assumption 3.1 holds and that there exists \( (\bar{y}, \bar{w}, \bar{s}) \in T_{\psi}^{-1}(0) \) such that the collection of three sets \( \{\bar{V}, \bar{G}_1(\bar{y}), \bar{G}_2(\bar{y})\} \) is boundedly linearly regular. Then the second order growth condition \( (22) \) holds at any \( \bar{x} \in T_{\phi}^{-1}(0) \) with respect to the optimal solution set \( T_{\phi}^{-1}(0) \) for problem (11).

**Proof.** Let \( \bar{x} \in T_{\phi}^{-1}(0) \) be an arbitrary but fixed point. From Assumption 3.1 (b), we know that there exist \( \kappa_1 > 0 \) and a neighborhood \( U \) of \( \bar{x} \) such that

\[
p(x) \geq p(\bar{x}) + \langle A^*\bar{y} - \bar{\eta}, x - \bar{x} \rangle + \kappa_1 \text{dist}^2(x, (\partial p)^{-1}(A^*\bar{y} - \bar{\eta})), \quad \forall x \in U. \tag{22}
\]

Note that \( (b-A\bar{x}, \bar{y}) \in \text{gph}(N_Q^{-1}) \) and \( N_Q(\cdot) \) is a multi-valued polyhedral function. Thus, we can obtain from Proposition 2.2 that \( N_Q(\cdot) \) is locally upper Lipschitz continuous, which further implies...
the metric subregularity of $N^{-1}_Q$ at $b - Ax$ for $\bar{y}$ by definition. Now by shrinking the neighborhood $U$ if necessary, we know that there exists a constant $k'_1 > 0$ such that

$$\delta_Q(b - Ax) \geq \delta_Q(b - A\bar{x}) + \langle \bar{y}, b - Ax - (b - A\bar{x}) \rangle + k'_1 \text{dist}^2(b - Ax, N_Q(\bar{y})), \quad \forall x \in U.$$  \hspace{1cm} (23)

Moreover, the assumed bounded linear regularity of $\{\bar{G}_1(\bar{y}), \bar{G}_2(\bar{y})\}$ and the result in Proposition 3.1 imply that there exist $\kappa_2 > 0$ and $\kappa_3 > 0$, such that for any $x \in U$,

$$\begin{align*}
\text{dist}^2(x, T^{-1}_\phi(0)) &= \text{dist}^2\left(x, \bar{G}_1(\bar{y}) \cap T^{-1}_\phi(0)\right) \\
&\leq \kappa_2 \left(\text{dist}^2(x, \bar{G}_1(\bar{y})) + \text{dist}^2(x, \bar{G}_2(\bar{y}))\right) \\
&\leq \kappa_3 \left(\|F x - \bar{\zeta}\|^2 + \text{dist}^2(x, (\hat{\phi}p)^{-1}(A^*\bar{y} - \bar{\eta})) + \text{dist}^2(b - Ax, N_Q(\bar{y}))\right),
\end{align*}$$

where in the last inequality, the first term comes from Hoffman’s error bound [16] and the third term comes from Lemma 3.2. Then by Assumption 3.1 (b), shrinking $U$ if necessary, we know that there exists $\kappa_4 > 0$ such that for any $x \in U$,

$$h(F x) \geq h(\bar{\zeta}) + \langle \nabla h(\bar{\zeta}), F x - \bar{\zeta} \rangle + \kappa_4 \|F x - \bar{\zeta}\|^2.$$  \hspace{1cm} (24)

Summing up the inequalities (22), (23) and (25) and recalling that $\bar{\eta} = F^*\nabla h(F \bar{x}) + c$ in (16), we know that for any $x \in U \cap \{x \in X \mid b - Ax \in Q^*\}$,

$$\begin{align*}
\theta(x) &= \theta(x) + \delta_Q(b - Ax) \\
&\geq \theta(\bar{x}) + \langle F^*\nabla h(\bar{\zeta}) + c - \bar{\eta}, x - \bar{x} \rangle + \kappa_4 \|F x - \bar{\zeta}\|^2 + \kappa_1 \text{dist}^2(x, (\hat{\phi}p)^{-1}(A^*\bar{y} - \bar{\eta})) \\
&\quad + \kappa'_1 \text{dist}^2(b - Ax, N_Q(\bar{y})) \\
&\geq \theta(\bar{x}) + \min\{\kappa_1, \kappa'_1, \kappa_4\} \left(\|F x - \bar{\zeta}\|^2 + \text{dist}^2(x, (\hat{\phi}p)^{-1}(A^*\bar{y} - \bar{\eta})) + \text{dist}^2(b - Ax, N_Q(\bar{y}))\right) \\
&= \theta(\bar{x}) + \kappa^{-1}_3 \min\{\kappa_1, \kappa'_1, \kappa_4\} \text{dist}^2(x, T^{-1}_\phi(0)),
\end{align*}$$

which shows that the second order growth condition (21) holds at $\bar{x}$ with respect to $T^{-1}_\phi(0)$.  \hfill \Box

### 3.2 The metric subregularity of $T_\phi$ for SDP problems

In this subsection, we analyze the metric subregularity of $T_\phi$ associated with SDP problems where in (11), $X = S^n$ and $p(\cdot) = \delta_{S^n_+}(\cdot)$. The corresponding primal and dual forms now can be written, respectively, as

$$\begin{align*}
\min \quad h(F X) + \langle C, X \rangle + \delta_{S^n_+}(X) \\
\text{s.t.} \quad b - AX \in Q^* 
\end{align*}$$

and

$$\begin{align*}
\min \quad \delta_Q(y) - \langle b, y \rangle + h^*(-w) + \delta_{S^n_+}(S) \\
\text{s.t.} \quad A^*y + F^*w + S = C.
\end{align*}$$

In the following, we shall provide sufficient conditions to ensure that assumptions made in Theorem 3.1 hold for the SDP problems.

Let $\bar{X} \in S^n_+$ and $\bar{S} \in S^n_+$ satisfy $0 \in \bar{X} + \partial \delta_{S^n_+}(\bar{S})$, or equivalently, $\langle X, \bar{S} \rangle = 0$. Suppose that $\bar{Z} := \bar{X} - \bar{S}$ has its eigenvalues $\bar{\lambda}_1 \geq \bar{\lambda}_2 \geq \ldots \geq \bar{\lambda}_n$ being arranged in a non-increasing order. Denote

$$\alpha := \{i \mid \bar{\lambda}_i > 0, 1 \leq i \leq n\}, \quad \beta := \{i \mid \bar{\lambda}_i = 0, 1 \leq i \leq n\}, \quad \gamma := \{i \mid \bar{\lambda}_i < 0, 1 \leq i \leq n\}. \hspace{1cm} (28)$$
Then there exists an orthogonal matrix $\mathcal{P} \in \mathcal{O}^n$ such that

$$Z = \mathcal{P} \begin{pmatrix} \bar{X}_\alpha & 0 \\ -\bar{X}_\gamma & \end{pmatrix} \mathcal{P}^T, \quad \bar{X} = \mathcal{P} \begin{pmatrix} \bar{X}_\alpha & 0 \\ 0 \end{pmatrix} \mathcal{P}^T, \quad \bar{S} = \mathcal{P} \begin{pmatrix} 0_{|\alpha|} & \bar{X}_\gamma \\ 0 & \end{pmatrix} \mathcal{P}^T,$$

(29)

where $\bar{X}_\alpha = \text{diag}(\bar{\lambda}_i \mid i \in \alpha) > 0$ and $\bar{X}_\gamma = \text{diag}(|\bar{\lambda}_j| \mid j \in \gamma) > 0$. Denote $\mathcal{P} = [\mathcal{P}_\alpha \mathcal{P}_\beta \mathcal{P}_\gamma]$ with $\mathcal{P}_\alpha \in \mathbb{R}^{n \times |\alpha|}, \mathcal{P}_\beta \in \mathbb{R}^{n \times |\beta|}$ and $\mathcal{P}_\gamma \in \mathbb{R}^{n \times |\gamma|}$. Then we have

$$\mathcal{T}_{\mathcal{S}^n_+}(\bar{X}) = \{ H \in \mathbb{S}^n \mid [\mathcal{P}_\beta \mathcal{P}_\gamma]^T H [\mathcal{P}_\beta \mathcal{P}_\gamma] \geq 0 \},$$

$$\mathcal{T}_{\mathcal{S}^n_+}(\bar{S}) = \{ H \in \mathbb{S}^n \mid [\mathcal{P}_\alpha \mathcal{P}_\beta]^T H [\mathcal{P}_\alpha \mathcal{P}_\beta] \geq 0 \},$$

$$\mathcal{N}_{\mathcal{S}^n_+}(\bar{X}) = \{ H \in \mathbb{S}^n \mid [\mathcal{P}_\beta \mathcal{P}_\gamma]^T H [\mathcal{P}_\beta \mathcal{P}_\gamma] \leq 0, \mathcal{P}_\beta^T H \mathcal{P}_\gamma = 0 \},$$

$$\mathcal{N}_{\mathcal{S}^n_+}(\bar{S}) = \{ H \in \mathbb{S}^n \mid [\mathcal{P}_\alpha \mathcal{P}_\beta]^T H [\mathcal{P}_\alpha \mathcal{P}_\beta] \leq 0, \mathcal{P}_\beta^T H \mathcal{P}_\gamma = 0 \}. $$

For the convenience of later discussions, we also denote the critical cone of $\mathcal{S}^n_+$ at $\bar{S}$ associated with $\bar{X}$ as

$$\mathcal{C}_{\mathcal{S}^n_+}(\bar{S}, \bar{X}) := \mathcal{T}_{\mathcal{S}^n_+}(\bar{S}) \cap \mathcal{N}_{\mathcal{S}^n_+}(\bar{X}) = \{ H \in \mathbb{S}^n \mid [\mathcal{P}_\beta \mathcal{P}_\gamma]^T H [\mathcal{P}_\beta \mathcal{P}_\gamma] = 0, \mathcal{P}_\beta^T H \mathcal{P}_\gamma \geq 0 \}$$

and the critical cone of $\mathcal{S}^n_+$ at $\bar{X}$ associated with $\bar{S}$ as

$$\mathcal{C}_{\mathcal{S}^n_+}(\bar{X}, \bar{S}) := \mathcal{T}_{\mathcal{S}^n_+}(\bar{X}) \cap \mathcal{N}_{\mathcal{S}^n_+}(\bar{S}) = \{ H \in \mathbb{S}^n \mid [\mathcal{P}_\beta \mathcal{P}_\gamma]^T H [\mathcal{P}_\beta \mathcal{P}_\gamma] = 0, \mathcal{P}_\beta^T H \mathcal{P}_\gamma \geq 0 \}. $$

By noting that $\partial \mathcal{N}_{\mathcal{S}^n_+}(\bar{S}) = \mathcal{N}_{\mathcal{S}^n_+}(\bar{X})$, we immediate obtain the following results.

**Proposition 3.2.** Let $\bar{S} \in \mathcal{S}^n_+$ and $0 \in \bar{X} + \partial \mathcal{N}_{\mathcal{S}^n_+}(\bar{S})$. Suppose that $\bar{S}$ and $\bar{X}$ have the eigenvalue decompositions as in (22). Then it holds that:

(a) $\mathcal{N}_{\mathcal{S}^n_+}(\bar{S})$ is a polyhedral set if and only if $|\gamma| \geq n - 1$;

(b) $0 \in \bar{X} + \text{ri}(\mathcal{N}_{\mathcal{S}^n_+}(\bar{S}))$ if and only if $|\beta| = 0$, i.e., rank($\bar{X}$) + rank($\bar{S}$) $= n$.

Next, we shall prove the metric subregularity of $\partial \mathcal{N}_{\mathcal{S}^n_+}(\cdot)$ and $\partial \mathcal{N}_{\mathcal{S}^n_+}(\cdot)$, which is one of the key components in our subsequent analysis.

**Proposition 3.3.** Let $\bar{S} \in \mathcal{S}^n_+$ and $0 \in \bar{X} + \partial \mathcal{N}_{\mathcal{S}^n_+}(\bar{S})$. Then $\partial \mathcal{N}_{\mathcal{S}^n_+}(\cdot)$ is metrically subregular at $\bar{X}$ for $-\bar{S}$ and $\partial \mathcal{N}_{\mathcal{S}^n_+}(\cdot)$ is metrically subregular at $-\bar{S}$ for $\bar{X}$.

**Proof.** In the following, we shall prove the metric subregularity of $\partial \mathcal{N}_{\mathcal{S}^n_+}(\cdot)$ at $-\bar{S}$ for $\bar{X}$ and its counterpart regarding $\partial \mathcal{N}_{\mathcal{S}^n_+}$ can be obtained similarly. Without loss of generality, let $\bar{X}$ and $\bar{S}$ have the eigenvalue decompositions as in (24). According to Proposition 2.1 in order to prove the metric subregularity of $\partial \mathcal{N}_{\mathcal{S}^n_+}(\cdot)$ at $-\bar{S}$ for $\bar{X}$, it suffices to show that there exist a constant $\kappa > 0$ and a neighborhood $\mathcal{U}$ of $\bar{S}$ such that for any $S \in \mathcal{S}^n_+ \cap \mathcal{U}$,

$$0 \geq \langle \bar{X}, -S + \bar{S} \rangle + \kappa \text{dist}^2(-S, (\partial \mathcal{N}_{\mathcal{S}^n_+})^{-1}(\bar{X})) = \langle \bar{X}, -S + \bar{S} \rangle + \kappa \text{dist}^2(-S, \mathcal{N}_{\mathcal{S}^n_+}(\bar{X})).$$

(30)

If $|\alpha| = 0$, then $\bar{X} = 0$ and the inequality (30) holds automatically for any $\kappa > 0$ and any neighborhood $\mathcal{U}$ of $\bar{S}$. Thus, we only need to consider the case that $|\alpha| \neq 0$. Since the case that

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This result is part of the first author's PhD thesis [3] Section 2.5.2].
$|\gamma| = 0$ can be proved similarly as in the case for $|\gamma| \neq 0$, we only consider the latter case. Set $\rho := \frac{1}{2} \min\{1, |\lambda_j| \mid j \in \gamma\} > 0$. Let $S \in \mathbb{S}^n_+ \cap B_{\mathbb{S}^n}(\mathbf{S}, \rho)$ be arbitrarily chosen. We write $\widetilde{S} = \mathbf{T}^T \mathbf{T}$ and decompose $\widetilde{S}$ into the following form:

$$
\widetilde{S} = \begin{pmatrix}
\tilde{s}_{\alpha\alpha} & \tilde{s}_{\alpha\beta} & \tilde{s}_{\alpha\gamma} \\
\tilde{s}_{\alpha\beta}^T & \tilde{s}_{\beta\beta} & \tilde{s}_{\beta\gamma} \\
\tilde{s}_{\alpha\gamma}^T & \tilde{s}_{\beta\gamma}^T & \tilde{s}_{\gamma\gamma}
\end{pmatrix}.
$$

By the fact that $S \in \mathbb{S}^n_+$, we can easily check that

$$
\Pi_{\mathbb{S}^n_+}(\mathbf{T})(-S) = \mathbf{T}^T \begin{pmatrix}
0 & 0 & 0 \\
0 & \tilde{s}_{\beta\beta} & \tilde{s}_{\beta\gamma} \\
0 & \tilde{s}_{\beta\gamma}^T & \tilde{s}_{\gamma\gamma}
\end{pmatrix} \mathbf{T}.
$$

Thus, we have

$$
\text{dist}^2(-S, \mathbb{S}^n_+(-S)) = \|S - \Pi_{\mathbb{S}^n_+}(\mathbf{T})(-S)\|^2 = \|\tilde{S}_{\alpha\alpha}\|^2 + 2\|\tilde{S}_{\alpha\beta}\|^2 + 2\|\tilde{S}_{\alpha\gamma}\|^2. \quad (31)
$$

Next we proceed to estimate $\|\tilde{S}_{\alpha\alpha}\|$, $\|\tilde{S}_{\alpha\beta}\|$ and $\|\tilde{S}_{\alpha\gamma}\|$. By using the Bauer-Fike Theorem \[2\], one obtains that for any $i = 1, \ldots, |\gamma|$, $\text{dist}(\lambda_i(\tilde{S}_{\gamma\gamma}), \{|\lambda_j| \mid j \in \gamma\}) \leq \|\tilde{S}_{\gamma\gamma} - \lambda_i\| = \|\mathbf{T}_{\gamma\gamma}^T S \mathbf{T}_{\gamma\gamma} - \mathbf{T}_{\gamma\gamma}^T \mathbf{T} \| \leq \|S - \mathbf{T}\| \leq \rho.$

The above inequality further implies that $0 < \lambda_i(\tilde{S}_{\gamma\gamma}) \leq |\lambda_n| + \rho \leq |\lambda_n| + \frac{1}{2}$ for all $i = 1, \ldots, |\gamma|$. Thus, $\tilde{S}_{\gamma\gamma}$ is positive definite and $\lambda_{\max}(\tilde{S}_{\gamma\gamma}) \leq |\lambda_n| + \frac{1}{2}$. Note that $\|\tilde{S}_{\alpha\alpha}\| \leq \rho$ and $\|\tilde{S}_{\beta\beta}\| \leq \rho$ as $S \in B_{\mathbb{S}^n}(\mathbf{S}, \rho)$.

From the fact that $\tilde{S}_{\alpha\alpha} - \tilde{S}_{\alpha\gamma} \tilde{S}_{\gamma\gamma}^{-1} \tilde{S}_{\gamma\alpha} \geq 0$ (because $\tilde{S} \in \mathbb{S}^n_+$), we have

$$
\lambda_{\max}^{-1}(\tilde{S}_{\gamma\gamma}) \tilde{S}_{\alpha\gamma} \tilde{S}_{\gamma\gamma}^{-1} \tilde{S}_{\alpha\gamma} \tilde{S}_{\gamma\gamma}^{-1} \tilde{S}_{\alpha\gamma} \leq \tilde{S}_{\alpha\alpha}.
$$

Hence

$$
\|\tilde{S}_{\alpha\gamma}\|^2 = \text{tr}(\tilde{S}_{\alpha\gamma} \tilde{S}_{\alpha\gamma}^T) \leq \text{tr}(\tilde{S}_{\alpha\alpha}) \lambda_{\max}(\tilde{S}_{\gamma\gamma}) \leq \frac{|\lambda_n| + \frac{1}{2}}{\lambda_{|\alpha|}} \langle \tilde{S}_{\alpha\alpha}, \Lambda_{\alpha} \rangle. \quad (32)
$$

Moreover, we obtain from

$$
\begin{pmatrix}
\tilde{s}_{\alpha\alpha} & \tilde{s}_{\alpha\beta} \\
\tilde{s}_{\alpha\beta}^T & \tilde{s}_{\beta\beta}
\end{pmatrix} \geq 0
$$

that

$$
\tilde{s}_{ij}^2 \leq \tilde{s}_{ii} \tilde{s}_{jj} \leq \rho \tilde{s}_{ii} \leq \frac{1}{2\lambda_{|\alpha|}} \lambda_{|\alpha|} \tilde{s}_{ii}, \quad \forall i \in \alpha, j \in \beta,
$$

which implies that

$$
\|\tilde{S}_{\alpha\beta}\|^2 = \sum_{i \in \alpha, j \in \beta} \tilde{s}_{ij}^2 \leq \frac{\|\beta\|}{2\lambda_{|\alpha|}} \langle \tilde{S}_{\alpha\alpha}, \Lambda_{\alpha} \rangle. \quad (33)
$$

Let $\kappa := \frac{\lambda_{|\alpha|}}{2|\lambda_n| + 3/2 + \|\beta\|} > 0$. Then, in view of (31), (32), (33) and

$$
\|\tilde{S}_{\alpha\alpha}\|^2 \leq \rho \|\tilde{S}_{\alpha\alpha}\| \leq \rho \text{tr}(\tilde{S}_{\alpha\alpha}) \leq \frac{1}{2\lambda_{|\alpha|}} \langle \tilde{S}_{\alpha\alpha}, \Lambda_{\alpha} \rangle,
$$

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In addition, for any $p \in S^+_2 \cap \mathcal{B}_{S^2}(S, \rho)$,
\[
\langle X, -S + \bar{S} \rangle + \kappa \text{dist}^2(-S, N_{S^2+}(X)) = \langle \Lambda_\alpha, -\tilde{S}_{\alpha \alpha} \rangle + \kappa \left( \|\tilde{S}_{\alpha \alpha}\|^2 + 2\|\tilde{S}_{\alpha \beta}\|^2 + 2\|\tilde{S}_{\alpha \gamma}\|^2 \right)
\]
\[
\leq \langle \Lambda_\alpha, -\tilde{S}_{\alpha \alpha} \rangle + \kappa \left( \frac{1}{2\lambda|\alpha|} + \frac{|\beta|}{\lambda|\alpha|} + \frac{2|\lambda| + 1}{\lambda|\alpha|} \right) \langle \tilde{S}_{\alpha \alpha}, \Lambda_\alpha \rangle = \langle \Lambda_\alpha, -\tilde{S}_{\alpha \alpha} \rangle + \langle \tilde{S}_{\alpha \alpha}, \Lambda_\alpha \rangle = 0.
\]
Therefore, the inequality (30) holds for any $p \in S^+_2 \cap \mathcal{B}_{S^2}(S, \rho)$ and the proof is completed.

Combining Theorem 3.1 and Propositions 3.2 and 3.3, we obtain the following result.

**Corollary 3.1.** Suppose that Assumption 3.1 (a) holds for problem (26). Then $\mathcal{T}_\phi$ is metrically subregular at any $X \in \mathcal{T}_\phi^{-1}(0)$ for the origin under one of the following two conditions:

(i) there exists $(\bar{y}, \bar{w}, \bar{S}) \in \mathcal{M}_\phi(X)$ such that $\text{rank}(\bar{S}) \geq n - 1$;

(ii) there exist $X \in \mathcal{T}_\phi^{-1}(0)$ and $(\bar{y}, \bar{w}, \bar{S}) \in \mathcal{M}_\phi(X)$ such that $\text{rank}(X) + \text{rank}(\bar{S}) = n$.

### 3.3 The metric subregularity of $\mathcal{T}_t$ for SDP problems

In this subsection, we focus on the metric subregularity of $\mathcal{T}_t$ at a KKT point for the origin associated with the SDP problem (26) and its dual (27). Denote $E := Y \times W \times S^n \times S^n$. Consider a perturbed point $(u, V) \in E$ with $u := (u_1, u_2, U) \in Y \times W \times S^n$ and $V \in S^n$. Then $(y, w, S, X) \in \mathcal{T}_t^{-1}(u, V)$ if and only if $(y, w, S, X) \in E$ solves the following perturbed KKT system:

\[
\begin{cases}
  u_1 \in -b + AX + N_Q(y), & u_2 \in F_X - \partial \hat{h}^*(-w), & U \in X + N_{S^2+n}(S), \\
  V = C - (A^*y + F^*w + S), & (y, w, S, X) \in E.
\end{cases}
\]

We know from Corollary 3.1 that for problem (26), if there exists a KKT point satisfying the partial strict complementarity condition as in (ii) of Corollary 3.1 then $\mathcal{T}_\phi$ is metrically subregular at any $X \in \mathcal{T}_\phi^{-1}(0)$ for the origin. Naturally one may ask whether a corresponding result can be extended to $\mathcal{T}_t$ under the same assumptions. The answer is unfortunately negative, as can be seen from the following example.

**Example 1.** Consider the following SDP problem:

\[
\begin{align*}
\min \left\{ X_{22} + \delta_{S^2_+}(X) \mid 2X_{12} - X_{22} = 0 \right\} 
\end{align*}
\]  
(35)

and its dual (in its equivalent minimization format):

\[
\begin{align*}
\min \left\{ \delta_{S^2_+}(S) \mid S_{11} = 0, y + S_{12} = 0, -y + S_{22} = 1 \right\}.
\end{align*}
\]  
(36)

It is easy to check that the sets of optimal solutions to (35) and (36) are given, respectively, by

\[
\mathcal{T}_t^{-1}(0) = \left\{ X \in S^2_+ \mid X_{11} \geq 0, X_{22} = 0 \right\}, \quad \mathcal{T}_t^{-1}(0) = \left\{ (y, S) \in \mathbb{R} \times S^2_+ \mid y = 0, S_{11} = 0, S_{22} = 1 \right\}.
\]

In addition, for any $(\bar{y}, \bar{S}, \bar{X}) \in \mathcal{T}_t^{-1}(0) = \mathcal{T}_t^{-1}(0) \times \mathcal{T}_t^{-1}(0)$ with $\bar{X}_{11} > 0$, it holds that $\text{rank}(\bar{S}) = 1$ and $\text{rank}(\bar{X}) + \text{rank}(\bar{S}) = 2$. Therefore, we know from Corollary 3.1 that $\mathcal{T}_\phi$ is metrically subregular at any $X \in \mathcal{T}_\phi^{-1}(0)$ for the origin. However, $\mathcal{T}_t$ fails to be metrically subregular at $(\bar{y}, \bar{S}, \bar{X}) \in \mathcal{T}_t^{-1}(0)$.
with $\overline{X}_{11} = 0$. This can be seen as follows: for any $\epsilon > 0$, consider the perturbed points $u(\epsilon) := (0, (\epsilon \quad \epsilon \quad \epsilon)) \in \mathbb{R} \times S^2$ and $V(\epsilon) := \left(\begin{array}{c} -\epsilon \\ 0 \\ 0 \end{array}\right) \in S^2$. Then one can show from (34) that

$$(y(\epsilon), S(\epsilon), X(\epsilon)) := \left(\sqrt{\epsilon}, \left(\begin{array}{ccc} \epsilon & -\sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 + \sqrt{\epsilon} \end{array}\right), \left(\begin{array}{c} \epsilon \\ \epsilon \end{array}\right)\right) \in T^{-1}_I(u(\epsilon), V(\epsilon)).$$

Also one can readily verify that

$$\|(u(\epsilon), V(\epsilon))\| = 2\sqrt{2}\epsilon, \quad \text{dist}(X(\epsilon), T^{-1}_I(0)) = \left\|\left(\begin{array}{c} 0 \\ \epsilon \\ 2\epsilon \end{array}\right)\right\| = \sqrt{6}\epsilon,$$

$$\text{dist}((y(\epsilon), S(\epsilon)), T^{-1}_I(0)) = \left((\sqrt{\epsilon})^2 + \left\|\left(\begin{array}{c} \epsilon \\ -\sqrt{\epsilon} \\ \sqrt{\epsilon} \end{array}\right)\right\|^2\right)^{1/2} \geq 2\sqrt{\epsilon}.$$
subregularity of $\mathcal{T}_i$ for the convex QSDP problem, when the KKT solution point is assumed to be unique [15]. Define the KKT mapping $\mathcal{R} : \mathbb{E} \to \mathbb{E}$ associated with problem (27) as

$$
\mathcal{R}(y, w, S, X) := \begin{pmatrix}
    y - \Pi_Q(y + b - AX) \\
    \mathcal{F}X + \nabla h^*(-w) \\
    S - \Pi_{\mathcal{Q}_2}(S - X) \\
    C - (A^*y + \mathcal{F}^*w + S)
\end{pmatrix}, \quad \forall (y, w, S, X) \in \mathbb{E}.
$$

It has been shown in [15, Theorem 5.1] that for linear and least squares SDP problems, when $(\bar{y}, \bar{w}, \bar{S}, \bar{X}) \in \mathbb{E}$ is the unique solution to $\mathcal{R}(y, w, S, X) = 0$, then $\mathcal{R}$ is metrically subregular at $(\bar{y}, \bar{w}, \bar{S}, \bar{X})$ for the origin if and only if both the primal and dual second order sufficient conditions hold [2]. On the other hand, Proposition 2.4 says that the metric subregularity of $\mathcal{R}$ at $(\bar{y}, \bar{w}, \bar{S}, \bar{X})$ for the origin is equivalent to the metric subregularity of $\mathcal{T}_i$ at $(\bar{y}, \bar{w}, \bar{S}, \bar{X})$ for the origin. Thus, we consider the second order sufficient condition (37) for problem (27) when the solution set to problem (27) is not a singleton.

We need two perturbation properties, one on the SDP cone and the other on the polyhedral cone, before stating our main result on the metric subregularity of $\mathcal{T}_i$.

**Proposition 3.4.** Let $\bar{S} \in \mathbb{S}^n_+$ and 0 $\in \bar{X} + \partial \phi_{\mathbb{S}_+^n}(\bar{S})$. Suppose that $\bar{X}$ and $\bar{S}$ have the eigenvalue decompositions as in (24). Then for all $(X, S) \in \mathbb{S}^n \times \mathbb{S}^n$ satisfying 0 $\in X + \partial \phi_{\mathbb{S}_+^n}(S)$ and is sufficiently close to $(\bar{X}, \bar{S}) \in \mathbb{S}^n \times \mathbb{S}^n$, we have

$$
\tilde{X}_{\alpha\alpha} = \bar{X}_{\alpha\alpha} + O(\|\Delta X\|), \quad \tilde{X}_{\alpha\beta} = O(\|\Delta X\|), \quad \tilde{X}_{\alpha\gamma} = O(\min\{\|\Delta X\|, \|\Delta S\|\}),
$$

$$
\tilde{X}_{\beta\beta} = O(\|\Delta X\|), \quad \tilde{X}_{\beta\gamma} = O(\|\Delta X\|\|\Delta S\|), \quad \tilde{X}_{\gamma\gamma} = O(\|\Delta X\|\|\Delta S\|),
$$

$$
\tilde{S}_{\alpha\alpha} = O(\|\Delta X\|\|\Delta S\|), \quad \tilde{S}_{\alpha\beta} = O(\|\Delta X\|\|\Delta S\|), \quad \tilde{S}_{\alpha\gamma} = O(\min\{\|\Delta X\|, \|\Delta S\|\}),
$$

$$
\tilde{S}_{\beta\beta} = O(\|\Delta S\|), \quad \tilde{S}_{\beta\gamma} = O(\|\Delta S\|), \quad \tilde{S}_{\gamma\gamma} = \bar{X}_{\gamma\gamma} + O(\|\Delta S\|),
$$

$$
\langle \tilde{X}_{\beta\beta}, \tilde{S}_{\beta\beta} \rangle = \begin{cases} 
O(\|\Delta X\|\|\Delta S\|(\|\Delta X\| + \|\Delta S\|)) & \text{if } |\alpha| > 0, \\
O(\|\Delta X\|\|\Delta S\|^2) & \text{if } |\alpha| = 0,
\end{cases}
$$

where $\Delta X := X - \bar{X}$, $\Delta S := S - \bar{S}$, $\tilde{X} := P^TXP$ and $\tilde{S} := P^TS^TP$.

**Proof.** Let $\mu_1 > \ldots > \mu_r > 0$ and $0 < \nu_1 < \ldots < \nu_s$ be all the distinct eigenvalues of $\bar{X}$ and $\bar{S}$, respectively. Denote

$$
\alpha_i := \{ k \in \alpha \mid \lambda_k = \mu_i \}, \quad i = 1, \ldots, r, \quad \gamma_j := \{ k \in \gamma \mid \lambda_k = -\nu_j \}, \quad j = 1, \ldots, s.
$$

It is easy to see that for all $\Delta X$ and $\Delta S$ sufficiently small, there exists $\tilde{P} \in \mathcal{O}^n$ such that

$$
\tilde{X} = \tilde{P} \begin{pmatrix} \Lambda_\alpha & \Lambda_\beta & \Lambda_\gamma \\ 0_{|\alpha|} & 0_{|\beta|} & 0_{|\gamma|} \end{pmatrix} \tilde{P}^T, \quad \tilde{S} = \tilde{P} \begin{pmatrix} 0_{|\alpha|} & \Lambda_\beta & \Lambda_\gamma \\ 0_{|\beta|} & 0_{|\beta|} & 0_{|\gamma|} \end{pmatrix} \tilde{P}^T.
$$

[2] In fact, this characterization of metric subregularity for $\mathcal{T}_i$ is also true for the convex QSDP problem even if its least squares representation is not explicitly available computationally. For details, see [15].
where $\Lambda_0 > 0, \Lambda_1 \geq 0, \Lambda_1' \geq 0, \Lambda_2 > 0$ and $\langle \Lambda_2, \Lambda_1' \rangle = 0$. From [16] Lemma 4.12, for all $\Delta X$ and $\Delta S$ sufficiently small, there exist $\Theta_{\alpha_i} \in O[\alpha_i], i = 1, \ldots, r, \Theta' \in O[\beta + \gamma]$, $\Theta_\gamma \in O[\gamma], i = 1, \ldots, s$ and $\Theta'' \in O[\alpha + |\beta|]$ such that
\[
\tilde{P} = \begin{pmatrix} \Theta_\alpha & O(\|\Delta X\|) \\ O(\|\Delta X\|) & \Theta' \end{pmatrix} + O(\|\Delta S\|),
\]
where $\Theta_{\alpha_i} \in O[\alpha]$ and $\Theta_\gamma \in O[\gamma]$ are block-diagonal orthogonal matrices given by
\[
\Theta_{\alpha_i} := \begin{pmatrix} \Theta_{\alpha_1} & \cdots & \Theta_{\alpha_r} \end{pmatrix}, \quad \Theta_\gamma := \begin{pmatrix} \Theta_{\gamma_1} & \cdots & \Theta_{\gamma_s} \end{pmatrix}.
\]
Note that
\[
\Theta_{\alpha_i}^T \Theta_{\alpha_i} = \overline{\Theta}_{\alpha_i}, \quad \Theta_{\gamma}^T \Theta_{\gamma} = \overline{\Theta}_{\gamma}.
\]
By using (41) and the fact that for any $N \in \mathbb{R}^{[\beta] \times [\beta]}$,
\[
NN^T = I_{[\beta]} + O(\|\Delta X\| + \|\Delta S\|) \implies \exists \hat{N} \in O[\beta] \text{ such that } \hat{N} = N + O(\|\Delta X\| + \|\Delta S\|),
\]
we further obtain from (43) that there exists $\Theta_{\beta} \in O[\beta]$ such that
\[
\tilde{P} = \begin{pmatrix} \Theta_\alpha + O(\|\Delta X\|) & O(\|\Delta X\|) & \tilde{P}_{\alpha \gamma} \\ O(\|\Delta X\|) & \Theta_{\beta} + O(\|\Delta X\| + \|\Delta S\|) & O(\|\Delta S\|) \\ \tilde{P}_{\gamma \alpha} & O(\|\Delta S\|) & \Theta_\gamma + O(\|\Delta S\|) \end{pmatrix}
\]
with
\[
\tilde{P}_{\alpha \gamma} = O(\min(\|\Delta X\|, \|\Delta S\|)), \quad \tilde{P}_{\gamma \alpha} = O(\min(\|\Delta X\|, \|\Delta S\|)).
\]
It then follows from (43), (44) and the orthogonality of $\tilde{P}$ that for all $\Delta X$ and $\Delta S$ sufficiently small,
\[
0 = \tilde{P}_{\alpha \alpha} \tilde{P}_{\gamma \gamma} - \tilde{P}_{\alpha \beta} \tilde{P}_{\beta \gamma} = \Theta_\alpha + O(\|\Delta X\|) \tilde{P}_{\gamma \alpha} + O(\|\Delta X\| \|\Delta S\|) + \tilde{P}_{\alpha \gamma} (\Theta_{\beta} + O(\|\Delta S\|))
\]
with
\[
\tilde{P}_{\alpha \beta} = O(\min(\|\Delta X\|, \|\Delta S\|)) \quad \text{and} \quad \tilde{P}_{\beta \alpha} = O(\min(\|\Delta X\|, \|\Delta S\|)).
\]
By using (41)-(44), the definitions of $\tilde{X}$ and $\tilde{S}$ and the Bauer-Fike Theorem [2], we obtain that for all $\Delta X$ and $\Delta S$ sufficiently small,
\[
\tilde{X}_{\alpha \alpha} = \overline{\Theta}_{\alpha} + O(\|\Delta X\|), \quad \tilde{X}_{\alpha \beta} = O(\|\Delta X\|),
\]
\[
\tilde{X}_{\alpha \gamma} = \overline{\Theta}_{\alpha} \Theta_{\gamma} \tilde{P}_{\alpha \alpha} + O(\|\Delta X\| \|\Delta S\|) = O(\min(\|\Delta X\|, \|\Delta S\|)),
\]
\[
\tilde{X}_{\beta \beta} = O(\|\Delta X\|), \quad \tilde{X}_{\beta \gamma} = O(\|\Delta X\| \|\Delta S\|), \quad \tilde{X}_{\gamma \gamma} = O(\|\Delta X\| \|\Delta S\|).
\]
This completes the proof of the proposition.\\n
Next, we shall prove (39) and (41). In view of (45), (47) and (48), we know that for all \( \Delta X \) and \( \Delta S \) sufficiently small,
\[
\tilde{S}_{\alpha} = \tilde{P}_{\alpha} \Theta_{\gamma} T, \tilde{S} + O(\|\Delta X\|\|\Delta S\|) = O\left(\min\{\|\Delta X\|, \|\Delta S\|\}\right),
\]
which show that (38) holds.

Finally, we conclude from \( \Lambda_{\beta} = O(\|\Delta X\|), \Lambda_{\beta}' = O(\|\Delta S\|), \langle \Lambda_{\beta}, \Lambda_{\beta}' \rangle = 0 \) and (46) that for all \( \Delta X \) and \( \Delta S \) sufficiently small and \( |\alpha| > 0 \),
\[
\langle \tilde{X}_{\beta\beta}, \tilde{S}_{\beta\beta} \rangle = \langle \tilde{P}_{\beta\beta} \Lambda_{\alpha} \tilde{P}_{\beta\alpha}, \tilde{P}_{\beta\beta} \Lambda_{\alpha} \tilde{P}_{\beta\alpha} \rangle + \langle \tilde{P}_{\beta\beta} \Lambda_{\alpha} \tilde{P}_{\beta\beta}, \Lambda_{\alpha} \tilde{P}_{\beta\beta} \rangle = O(\|\Delta X\|^2\|\Delta S\|) + O(\|\Delta X\|\|\Delta S\|^2)\]
\[
\qquad + O(\|\Delta X\|\|\Delta S\|) = O(\|\Delta X\|^2\|\Delta S\|^2) + O(\|\Delta X\|\|\Delta S\|^2) + O(\|\Delta X\|\|\Delta S\|^2)
\]
or for all \( \Delta X \) and \( \Delta S \) sufficiently small and \( |\alpha| = 0 \),
\[
\langle \tilde{X}_{\beta\beta}, \tilde{S}_{\beta\beta} \rangle = \langle \tilde{P}_{\beta\beta} \Lambda_{\alpha} \tilde{P}_{\beta\beta}, \tilde{P}_{\beta\beta} \Lambda_{\alpha} \tilde{P}_{\beta\beta} \rangle = O(\|\Delta X\|\|\Delta S\|^2).
\]
This completes the proof of the proposition. \( \square \)

Now we shall focus on the convex polyhedral cone \( \mathcal{Q} \). Without loss of generality, let \( Y := \mathbb{R}^m \). Let \( (\bar{y}, \bar{q}) \in \mathbb{R}^m \times \mathbb{R}^m \) satisfy \( \bar{q} \in \mathcal{N}_{\bar{Q}}(\bar{y}) \). We denote the critical cone of \( \mathcal{Q} \) at \( \bar{y} \) associated with \( \bar{q} \) and the critical cone of \( \mathcal{Q}^\circ \) at \( \bar{q} \) associated with \( \bar{y} \) as
\[
\mathcal{C}_{\bar{Q}}(\bar{y}, \bar{q}) := \mathcal{T}_{\bar{Q}}(\bar{y}) \cap \bar{q}^\perp, \quad \mathcal{C}_{\bar{Q}^\circ}(\bar{q}, \bar{y}) := \mathcal{T}_{\bar{Q}^\circ}(\bar{q}) \cap \bar{y}^\perp.
\]
It is easy to check the following relation:
\[
(\mathcal{C}_{\bar{Q}}(\bar{y}, \bar{q}))^\circ = \mathcal{C}_{\bar{Q}^\circ}(\bar{q}, \bar{y}). \tag{49}
\]
For subsequent discussions, we write the convex polyhedral cone \( \mathcal{C}_{\bar{Q}}(\bar{y}, \bar{q}) \) as
\[
\mathcal{C}_{\bar{Q}}(\bar{y}, \bar{q}) = \{ y \in \mathbb{R}^m \mid Qy \preceq 0 \}, \tag{50}
\]
where \( Q \) is some matrix in \( \mathbb{R}^{r \times m} \). Define a collection of index sets
\[
\mathcal{I}_Q := \{ i \in \{1, 2, \ldots, r\} \mid \exists y \in \mathbb{R}^m \text{ satisfying } Q_i y = 0, \forall i \in a \text{ and } Q_i y < 0, \forall i \notin a \}, \tag{51}
\]
where \( Q_i \) denotes the \( i \)-th row of \( Q \). Moreover, for each \( a \in \mathcal{I}_Q \), define a subspace \( L_a \in \mathbb{R}^m \) by
\[
L_a := \{ y \in \mathbb{R}^m \mid Q_i y = 0, \forall i \in a \}. \tag{52}
\]
Proposition 3.5. Let \((\bar{y}, \bar{q}) \in \mathbb{R}^m \times \mathbb{R}^m\) satisfy \(\bar{q} \in \mathcal{N}_Q(\bar{y})\) and suppose that \(\mathcal{C}_Q(\bar{y}, \bar{q})\) has the form \((50)\). Then for any \((y, q) \in \mathbb{R}^m \times \mathbb{R}^m\) satisfying \(q \in \mathcal{N}_Q(y)\) and is sufficiently close to \((\bar{y}, \bar{q})\), there exists \(a \in \mathcal{I}_Q\) such that
\[
\mathcal{C}_Q(\bar{y}, \bar{q}) \cap L_a \ni (y - \bar{y}) \perp (q - \bar{q}) \in \mathcal{C}_{Q^*}(\bar{q}, \bar{y}) \cap L_a^\perp.
\]

Proof. Note that for any \((q, y) \in \mathbb{R}^m \times \mathbb{R}^m\), the relation \(q \in \mathcal{N}_Q(y)\) can be equivalently written as \(y = \Pi_Q(y + q)\). Since \(Q\) is a convex polyhedral cone, by \([13\textbf{, Theorem 4.1.1}]\) we know that for all \((y, q)\) satisfying \(q \in \mathcal{N}_Q(y)\) and is sufficiently close to \((\bar{y}, \bar{q})\),
\[
y = \Pi_Q(y + q) = \Pi_Q(\bar{y} + \bar{q}) + \Pi_Q'(\bar{y} + \bar{q})y - \bar{y} + q - \bar{q} = \bar{y} + \Pi_{\mathcal{C}_Q(\bar{y}, \bar{q})}(y - \bar{y} + q - \bar{q}).
\]

It is also known from \([13\textbf{, Proposition 4.1.9}]\) that \(\Pi_{\mathcal{C}_Q(\bar{y}, \bar{q})} = \{\Pi_{L_a} \mid a \in \mathcal{I}_Q\}\). Then for all \((y, q)\) satisfying \(q \in \mathcal{N}_Q(y)\) sufficiently close to \((\bar{y}, \bar{q})\), there exists \(a \in \mathcal{I}_Q\) such that
\[
\Pi_{\mathcal{C}_Q(\bar{y}, \bar{q})}(y - \bar{y} + q - \bar{q}) = \Pi_{L_a}(y - \bar{y} + q - \bar{q}).
\]

Thus, the equation \((53)\) is equivalent to
\[
\mathcal{C}_Q(\bar{y}, \bar{q}) \cap L_a \ni (y - \bar{y}) \perp (q - \bar{q}) \in (\mathcal{C}_Q(\bar{y}, \bar{q}) \cap L_a)^\circ,
\]
which, together with \((50)\), completes the proof of this proposition.

Suppose that the KKT solution set to problem \((27)\) is non-empty. Consider an optimal solution \((\bar{y}, \bar{w}, \bar{S})\) to problem \((27)\) and \(X \in \mathcal{M}_\psi(\bar{y}, \bar{w}, \bar{S})\). Motivated by Propositions \(3.4\) and \(3.5\) in order to state our main result on the metric subregularity of \(\mathcal{T}_i\), we define the following joint ‘critical cone’ associated with problem \((27)\) and its constraints as
\[
\mathcal{C}(\bar{y}, \bar{w}, \bar{S}, X) := \{ (d_y, d_w, d_S, d_X) \in \mathbb{E} \mid \begin{array}{c}
d_y \in \mathcal{C}_Q(\bar{y}, \bar{w} - A\bar{X}), \ d_S \in \mathcal{C}_{S^*_+}(\bar{S}, \bar{X}), \ d_X \in \mathcal{C}_{S^*_+}(\bar{S}, \bar{X}), \\
-Ad_X \in \mathcal{C}_{Q^*}(\bar{b} - A\bar{X}, \bar{y}), \ \langle d_y, Ad_X \rangle = 0, \\
X^{1/2}d_S(S^{1/2})^\top + (X^{1/2})^\top d_X S^{1/2} = 0
\end{array} \}.
\]

Theorem 3.2. Let \((\bar{y}, \bar{w}, \bar{S})\) be an optimal solution to problem \((27)\) and \(X \in \mathcal{M}_\psi(\bar{y}, \bar{w}, \bar{S})\). Let \(\bar{q} = b - A\bar{X}\) and \(\mathcal{C}_Q(\bar{y}, \bar{q})\) have the form \((54)\). Define
\[
\{ \begin{array}{c}
\mathcal{K} := (\mathcal{C}_{S^*_+}(\bar{S}, \bar{X}))^*, \\
\Xi := \{ (d_y, d_w, d_S, d_X) \in \mathbb{E} \mid A^*d_y + F^*d_w + d_S = 0, \ (\nabla h^*)(-d_w; d_w) + F d_X = 0 \}.
\end{array} \}
\]

Assume that the following three conditions hold:
(i) the sets \(FK\) and \((A - I)(\mathcal{K}, \mathcal{C}_{Q^*}(\bar{q}, \bar{y}) \cap L_a^\perp)\) are closed for all \(a \in \mathcal{I}_Q\), where \(\mathcal{I}_Q\) and \(L_a\) are defined in \((51)\) and \((52)\);
(ii) \(\langle \Pi_{\mathcal{K}}(d_X), \Pi_{\mathcal{K}}(d_S) \rangle = 0\) for any \((d_y, d_w, d_S, d_X) \in \mathcal{C}(\bar{y}, \bar{w}, \bar{S}, X) \cap \Xi\), where the set \(\mathcal{C}(\bar{y}, \bar{w}, \bar{S}, X)\) is defined in \((54)\);
(iii) for problem \((27)\), the second order sufficient condition \((37)\) holds at \((\bar{y}, \bar{w}, \bar{S})\) with respect to the multiplier \(\bar{X}\).
Then there exist a constant $\kappa > 0$ and a neighborhood $\mathcal{U}$ of $(\bar{y}, \bar{w}, \bar{S}, \bar{X})$ such that for any $(u, V) \in \mathbb{E}$,

$$
\|(y, w, S) - (\bar{y}, \bar{w}, \bar{S})\| \leq \kappa \|(u, V)\|, \quad \forall (y, w, S, X) \in \mathcal{T}_1^{-1}(u, V) \cap \mathcal{U}.
$$

(55)

In addition, if $\nabla h^*(\cdot)$ is locally Lipschitz continuous at $-\bar{w}$ and there exists $\bar{X} \in \mathcal{M}_\kappa(\bar{y}, \bar{w}, \bar{S})$ such that $\text{rank} (\bar{X}) + \text{rank} (\bar{S}) = n$, then $\mathcal{T}_1$ is metrically subregular at $(\bar{y}, \bar{w}, \bar{X}, \bar{S})$ for the origin.

Proof. We shall first show that under the given conditions, there exist a constant $\kappa > 0$ and a neighborhood $\mathcal{U}$ of $(\bar{z}, \bar{X})$ with $\bar{z} := (\bar{y}, \bar{w}, \bar{S})$ such that (55) holds. Assume for the sake of contradiction that there exist sequences $\{(y^k, w^k, S^k, X^k)\} \in \mathbb{E}$ and $\{(u^k_1, u^k_2, U^k, V^k)\} \in \mathbb{E}$ with $k \geq 0$ such that $u^k := (u^k_1, u^k_2, U^k) \rightarrow 0$, $V^k \rightarrow 0$, $z^k := (y^k, w^k, S^k) \rightarrow \bar{z}$, $X^k \rightarrow \bar{X}$ with $(z^k, X^k) \in \mathcal{T}^{-1}_1(u^k, V^k)$ and

$$
t_k := \|z^k - \bar{z}\| \geq \rho_k \|(u^k, V^k)\|,
$$

for some $0 < \rho_k \rightarrow \infty$.

By restricting to an appropriate subsequence if necessary, we may assume that $(z^k - \bar{z})/t_k \rightarrow d_\bar{z}$ for some $0 \neq d_\bar{z} := (d_\bar{g}, d_\bar{w}, d_\bar{S}) \in \mathbb{Z}$. It is easy to see from the KKT optimality condition (54) that for all $k \geq 0$,

$$
0 = A^*(y^k - \bar{y}) + F^*(w^k - \bar{w}) + (S^k - \bar{S}) + V^k
$$

and for all $k$ sufficiently large,

$$
0 = \nabla h^*(-w^k) - F X^k + u^k_2
$$

$$
= \nabla h^*(-\bar{w}) - F \bar{X} + (\nabla h^*)'(-\bar{w}; -w^k + \bar{w}) + r^k - F(X^k - \bar{X}) + u^k_2
$$

$$
= (\nabla h^*)'(-\bar{w}; -w^k + \bar{w}) - F(X^k - \bar{X}) + r^k + u^k_2
$$

with some $r^k \in \mathcal{W}$ and $r^k = o(t_k)$ as $k \rightarrow \infty$. Dividing both sides of the equation (56) by $t_k$ and then taking limits, we obtain

$$
A^* d_\bar{g} + F^* d_\bar{w} + d_\bar{S} = 0.
$$

(58)

For the simplicity, we denote

$$
\Omega := \{X \in \mathbb{S}_n \mid \bar{P}_\alpha [\bar{P}_\beta]^T X [\bar{P}_\alpha [\bar{P}_\beta] = 0\}
$$

and for all $k \geq 0$,

$$
\begin{cases}
X_U^k := X^k - U^k, \\
\hat{X}_U^k := [\bar{P}^T X_U^k] [\bar{P}], \\
\tilde{S}^k := [\bar{P}^T S^k] [\bar{P}], \\
H^k := \Pi_{\Omega} ((X_U^k - \bar{X})/t_k), \\
G^k := (X_U^k - \bar{X})/t_k - H^k \in \mathcal{K}.
\end{cases}
$$

(59)

Using Proposition 2.3 and $0 \in \bar{X} + \delta_{\mathbb{S}_n^+}(\bar{S})$, $0 \in X_U^k + \delta_{\mathbb{S}_n^+}(S^k)$ for all $k \geq 0$, we deduce that for all $(X^k, S^k)$ sufficiently close to $(\bar{X}, \bar{S})$,

$$
\begin{cases}
\tilde{S}^k_{\alpha\alpha} = O(\|S^k - \bar{S}\|\|X^k - \bar{X}\|), \\
\tilde{S}^k_{\alpha\beta} = O(\|S^k - \bar{S}\|\|X_U^k - \bar{X}\|), \\
(\hat{X}_U^k)_{\beta\gamma} = O(\|S^k - \bar{S}\|\|X_U^k - \bar{X}\|), \\
(\hat{X}_U^k)_{\alpha\gamma} = O(\|S^k - \bar{S}\|\|X_U^k - \bar{X}\|),
\end{cases}
$$

which, together with the fact that $\tilde{S}^k_{\beta\beta} \in \mathbb{S}_n^+\bar{P}$, yields

$$
d_{\bar{S}} \in C_{\mathbb{S}_n^+}(\bar{S}, \bar{X}), \\
H_1 := \lim_{k \rightarrow \infty} H^k = \bar{P}^T \begin{pmatrix}
0 & 0 & -\bar{X}_\alpha (d_{\bar{S}})_{\alpha\gamma} \bar{X}_\gamma^{-1} \\
0 & 0 & 0 \\
(\bar{X}_\alpha (d_{\bar{S}})_{\alpha\gamma} \bar{X}_\gamma^{-1})^T & 0 & 0
\end{pmatrix} [\bar{P}],
$$

(60)
where \( \delta_S := T^Fd_SF \). From Proposition 3.5 and \( b - A\overline{X} \in \mathcal{N}_Q(\bar{y}), u^k + b - AX^k \in \mathcal{N}_Q(y^k) \) for each \( k \), we may assume, by passing to a subsequence if necessary, that there exists \( a \in \mathcal{I}_Q \) such that for all \( k \geq 0 \),

\[
C_Q(\bar{y}, b - A\overline{X}) \cap L_a \ni (y^k - \bar{y}) \perp (u^k_1 - AU^k)/t_k - A(H^k + G^k) \in C_{Q^*}(b - A\overline{X}, \bar{y}) \cap L_a^{\perp}.
\]

This further implies that

\[
C_Q(\bar{y}, b - A\overline{X}) \cap L_a \ni d_{\bar{y}} \perp ((u^k_1 - AU^k)/t_k - A(H^k + G^k)) \in C_{Q^*}(b - A\overline{X}, \bar{y}) \cap L_a^{\perp}.
\] (61)

In view of (57), (61) and the definitions of \( H^k \) and \( G^k \) in (55), it follows that for \( k \) sufficiently large,

\[
\left\{
\begin{array}{l}
(u^k_1 - AU^k)/t_k - AH^k = AG^k + ((u^k_1 - AU^k)/t_k - A(H^k + G^k)) \\
\in (A, \mathcal{I})(K, C_{Q^*}(b - A\overline{X}, \bar{y}) \cap L_a^{\perp}),
\end{array}
\right.
\]

\[
(\nabla h^*)(-\bar{w}; - (w^k - \bar{w}))/t_k - FH^k - (FU^k - \tau^k - u^k_2)/t_k = \mathcal{F}G^k \in \mathcal{F}K.
\]

Since \((A, \mathcal{I})(K, C_{Q^*}(b - A\overline{X}, \bar{y}) \cap L_a^{\perp}) \) and \( \mathcal{F}K \) are assumed to be closed and that (60) holds, there exist \( q \in C_{Q^*}(b - A\overline{X}, \bar{y}) \cap L_a^{\perp} \) and \( H_2 \in K \) such that

\[-AH_1 = AH_2 + q, \quad -(\nabla h^*)(-\bar{w}; d_{\bar{w}}) - FH_1 = FH_2.\] (62)

Let \( d_{\overline{X}} := H_1 + H_2 \). Then we can obtain from (68) and (60)-(62) that \((d_{\bar{y}}, d_{\bar{w}}, d_{\overline{X}}, d_{\overline{S}}) \in C(\bar{y}, \bar{w}, \overline{S}, \overline{X}) \cap \Xi \). Furthermore, by using condition (ii) in this theorem, we have that \( 0 \neq (d_{\bar{y}}, d_{\bar{w}}, d_{\overline{S}}) \in C_{Q^*}(\bar{y}, \bar{w}, \overline{S}) \)

and

\[
\langle d_{\bar{w}}, (\nabla h^*)(-\bar{w}; d_{\bar{w}}) \rangle + 2\mathcal{Y}_S(\overline{X}, d_{\overline{S}}) = \langle d_{\bar{w}}, -\mathcal{F}d_{\overline{X}} \rangle + 2\langle \overline{X}, d_{\overline{S}}d_{\overline{S}} \rangle + \langle \lambda_{\alpha}, (\overline{d}_{\overline{S}})_{\alpha\gamma} - (\overline{d}_{\overline{S}})_{\alpha\gamma} \rangle + \langle (\overline{\Pi}_{\mathcal{K}}(d_{\overline{S}}), (\overline{\Pi}_{\mathcal{K}}(d_{\overline{S}})) = 0,\rangle
\]

which contradicts the asserted second order sufficient condition (37) at \((d_{\bar{y}}, d_{\bar{w}}, d_{\overline{S}}) \) for \( \overline{X} \). This contradiction shows that there exist a constant \( \kappa > 0 \) and a neighborhood \( \mathcal{U} \) of \((\bar{y}, \bar{w}, \overline{S}, \overline{X}) \) such that (57) holds.

Next we shall show that if there exists \( \hat{X} \in \mathcal{T}_h^{-1}(0) \) such that \( \text{rank}(\hat{X}) + \text{rank}(\overline{S}) = n \), then \( \mathcal{T}_h \) is metrically subregular at \((\bar{z}, \overline{X}) \) for the origin, or in view of Definition 2.1, equivalently to show that there exist a constant \( \kappa' > 0 \) and a neighborhood \( \mathcal{U}' \) of \((\bar{y}, \bar{w}, \overline{S}, \overline{X}) \) such that for any \((u, V) := (u_1, u_2, U, V) \in \Xi,\)

\[
\text{dist}((y, w, S, X), \mathcal{T}_h^{-1}(0)) \leq \kappa'(\|u, V\|), \quad \forall (y, w, S, X) \in \mathcal{T}_h^{-1}(u, V) \cap \mathcal{U}'.
\] (63)

Denote

\[
\Delta_1 := \{ X \in \mathbb{S}^n \mid b - AX \in \mathcal{N}_Q(\bar{y}) \}, \quad \Delta_2 := \{ X \in \mathbb{S}^n \mid \mathcal{F}X - \nabla h^*(-\bar{w}) = 0 \},
\]

\[
\Delta_3 := \{ X \in \mathbb{S}^n \mid X \in \mathcal{N}_{\mathbb{S}^n}(\overline{-S}) \}.
\]

Then one has \( \mathcal{T}_h^{-1}(0) = \Delta_1 \cap \Delta_2 \cap \Delta_3 \) and \( \hat{X} \in \Delta_1 \cap \Delta_2 \cap \text{ri} (\Delta_3) \). Thus, we obtain from Proposition 2.3 that there exists a constant \( \kappa_1 > 0 \) such that for any \((y, w, S, X) \in \mathcal{U},\)

\[
\text{dist}(X, \mathcal{T}_h^{-1}(0)) \leq \kappa_1(\text{dist}(X, \Delta_1) + \text{dist}(X, \Delta_2) + \text{dist}(X, \Delta_3)).
\] (64)
Consider an arbitrary but fixed point \((y, w, S, X) \in \mathcal{T}_1^{-1}(u, V) \cap \mathcal{U}\). From Proposition 3.2 and the fact that \(u_1 + b - AX \in \mathcal{N}_Q(y)\), we see that there exist constants \(\kappa_2 > 0\) and \(\kappa_2' > 0\) such that

\[
\text{dist}(X, \Delta_1) \leq \kappa_2 \text{dist}(b - AX, \mathcal{N}_Q(y)) \\
\leq \kappa_2(\|b - AX - (u_1 + b - AX)\| + \text{dist}(u_1 + b - AX, \mathcal{N}_Q(y))) \\
\leq \kappa_2'(\|u_1\| + \text{dist}(y, \mathcal{N}_Q^{-1}(u_1 + b - AX))) \leq \kappa_2'(\|u_1\| + \|y - \bar{y}\|),
\]

where the second inequality comes from the global Lipschitz continuity of \(\text{dist}(\cdot, \mathcal{N}_Q(y))\) with modulus 1. Using Hoffman’s error bound [16] and the assumption on the local Lipschitz continuity of \(\nabla h^*\) at \(-\bar{w}\), shrinking \(\mathcal{U}\) if necessary, there exist constants \(\kappa_3 > 0\) and \(\kappa_3' > 0\) such that

\[
\text{dist}(X, \Delta_2) \leq \kappa_3 \|FX - \nabla h^*(-\bar{w})\| \\
\leq \kappa_3(\|FX - \nabla h^*(-w)\| + \|\nabla h^*(-w) - h^*(-\bar{w})\|) \\
\leq \kappa_3'(\|u_2\| + \|w - \bar{w}\|).
\]

Since \(\partial \mathcal{S}^n_\infty(\cdot) = \mathcal{N}_\mathcal{S}^n_\infty(\cdot)\) has been proven to be metrically subregular at \(\bar{X}\) for \(-\bar{S}\) in Proposition 3.3 and \(-S \in \mathcal{N}_\mathcal{S}^n_\infty(X - U)\), we obtain, by shrinking \(\mathcal{U}\) if necessary, that there exists a constant \(\kappa_4 > 0\) such that

\[
\text{dist}(X, \Delta_3) \leq \|X - (X - U)\| + \text{dist}(X - U, \mathcal{N}_\mathcal{S}^n_\infty(-\bar{S})) \\
\leq \|U\| + \kappa_4 \text{dist}(-\bar{S}, \mathcal{N}_\mathcal{S}^n_\infty(X - U)) \leq \max\{1, \kappa_4\}(\|U\| + \|S - \bar{S}\|).
\]

Therefore, combining the inequality (55) and the inequalities (64)-(67) and, we show that there exists a constant \(\kappa'\) along with a neighborhood \(\mathcal{U}'\) such that (63) holds. This completes the proof of this proposition.

Below, we make some comments on Theorem 3.2.

**Remark 3.1.** As one can see, the proof of Theorem 3.2 is complicated due to the non-polyhedral nature of the positive semidefinite cone. Here, we have adopted some ideas from the nonlinear programming literature [8, 19, 17] on the proof of the metric subregularity of \(\mathcal{T}_1\) to our context. It is easy to verify via Theorem 3.2 that for Example 7, the operator \(\mathcal{T}_1\) is metrically subregular at any \((\bar{y}, \bar{S}, \bar{X}) \in \mathcal{T}_1^{-1}(0)\) with \(\bar{X}_{11} > 0\) for the origin. The failure of the metric subregularity of \(\mathcal{T}_1\) at \((\bar{y}, \bar{S}, \bar{X})\) with \(\bar{X}_{11} = 0\) for the origin is due to the violation of the second order sufficient condition at \((\bar{y}, \bar{S})\) for \(\bar{X}\).

The assumed condition (i) in Theorem 3.2 holds automatically if \(|\beta| = 0\) or \(|\beta| = 1\), in which case the set \(\mathcal{K}\) is a polyhedral cone. The polyhedral cones and positive semidefinite cones are “nice cones” in the terminology of Pataki [27, Definition 1], where the author also characterized the closedness of these cones under linear transformations [28, Theorem 1.1]. It is also clear that if \((\bar{y}, \bar{w}, \bar{S}, \bar{X})\) satisfies the partial strict complementarity, i.e., \(|\beta| = 0\), then condition (ii) in Theorem 3.2 holds automatically. One weaker sufficient condition than this partial strict complementarity to ensure the validity of condition (ii) is that either \(\Pi_{\mathcal{K}}(d_S) = 0\) or \(\Pi_{\mathcal{K}}(d_X) = 0\) for any \((d_y, d_w, d_S, d_X) \in \mathcal{C}(\bar{y}, \bar{w}, \bar{S}, \bar{X}) \cap \Xi\).

To illustrate the metric subregularity results proved in Theorem 3.2, we provide the following example, which is first considered in [17] for different purposes.
Example 2. Consider the following convex quadratic SDP problem:

$$\min \left\{ \frac{1}{2} \langle I_2, X \rangle - \frac{1}{2} + \delta_{\mathbb{S}^2_+}(X) \mid \langle A, X \rangle \leq 1 \right\},$$  \hspace{1cm} (68)

whose dual (in its equivalent minimization form) can be written as

$$\min \left\{ \delta_{\mathbb{R}_+}(y) - y + \frac{1}{2} w^2 + w + \delta_{\mathbb{S}^2_+}(S) \mid y A + w I_2 + S = 0 \right\},$$  \hspace{1cm} (69)

where \( A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \). Problem (69) has a unique solution \( (\bar{y}, \bar{w}, \bar{S}) = (0,0,0) \). The critical cone of problem (69) at \( (\bar{y}, \bar{w}, \bar{S}) \) is given by

$$\mathcal{T}_{\phi}^{-1}(0) = \{ X \in \mathbb{S}^2_+ \mid \langle A, X \rangle \leq 1, \langle I_2, X \rangle = 1 \}.$$

Since \( |\beta| \leq 1 \) for all \( X \in \mathcal{T}_{\phi}^{-1}(0) \), we see that condition (i) also holds. Therefore, by Theorem 3.2 we know that \( \mathcal{T}_i \) is metrically subregular at any \( (\bar{y}, \bar{w}, \bar{S}, \bar{X}) \in \mathcal{T}_{\phi}^{-1}(0) \) for the origin, even though the partial strict complementarity condition fails at \( \bar{X} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( \bar{X} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).

4 Asymptotic superlinear convergence of the ALM with multiple solutions

In this section, we study the asymptotic superlinear convergence of the ALM for solving problem (27). First, we need to state the PPA considered by Rockafellar [34]. Let \( \mathcal{T} : X \rightarrow X \) be a maximal monotone operator. Consider the following inclusion problem:

$$0 \in \mathcal{T}(\xi), \ \forall \xi \in X.$$

Given a sequence of scalars \( c_k \uparrow c_{\infty} \leq \infty \) and a starting point \( \xi^0 \in X \), the \((k+1)\)-th iteration of the PPA takes the form of

$$\xi^{k+1} \approx (\mathcal{I} + c_k \mathcal{T})^{-1}(\xi^k), \ \forall k \geq 0.$$  \hspace{1cm} (70)

For each \( k \geq 0 \), denote

$$e^{k+1} := (\mathcal{I} + c_k \mathcal{T})^{-1}(\xi^k) - \xi^{k+1}.$$

In one of his seminal works [34], Rockafellar suggested the following criteria for computing \( e^{k+1} \) approximately:

(A) \( \|e^{k+1}\| \leq \varepsilon_k, \ \varepsilon_k \geq 0, \ \sum_{k=1}^{\infty} \varepsilon_k < \infty, \)

(B) \( \|e^{k+1}\| \leq \eta_k \|\xi^{k+1} - \xi^k\|, \ \eta_k \geq 0, \ \sum_{k=1}^{\infty} \eta_k < \infty. \)

The next theorem concerning the convergence of the PPA essentially comes from Rockafellar [34] with an important extension made by Luque [24] on the rate of convergence without assuming the uniqueness of the solutions. For our later developments, here we make a further extension by relaxing Luque’s condition [24] (2.1), which can be too restrictive in our context, in particular when the optimal solution set to problem (26) is unbounded. Note that for the case \( e^k \equiv 0 \) for all \( k \geq 0 \), this relaxation has also been discussed by Leventhal [20].

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Theorem 4.1. Assume that $\mathcal{T}^{-1}(0) \neq \emptyset$. Let $\{x^k\}$ be an infinite sequence generated by the PPA (70) with stopping criterion (A). Then the following statements hold.

(a) For any $x \in \mathcal{T}^{-1}(0)$, it holds that

\[ \|x^{k+1} + e^{k+1} - \xi\| \leq \|x^k - \xi\| - \|x^{k+1} - x^k\| \leq \|x^{k+1} - \xi\|, \quad \forall k \geq 0. \]  

(b) The whole sequence $\{x^k\}$ converges to some $x^\infty \in \mathcal{T}^{-1}(0)$. Assume that $\mathcal{T}$ is metrically subregular at $x^\infty$ for the origin with modulus $\kappa > 0$. If in the PPA, the criterion (B) is also employed, then there exists $k_0 > 0$ such that for all $k > k_0$, $\eta_k < 1$ and

\[ \text{dist}(x^{k+1}, \mathcal{T}^{-1}(0)) \leq \theta_k \text{dist}(x^k, \mathcal{T}^{-1}(0)), \]  

where

\[ 1 > \theta_k = (\kappa/\sqrt{\kappa^2 + c_k^2 + 2\eta_k})(1 - \eta_k)^{-1} \rightarrow \theta_\infty = \kappa/\sqrt{\kappa^2 + c_\infty^2} \quad (\theta_\infty = 0 \text{ if } c_\infty = \infty). \]

\[ \text{Proof.} \quad \text{The inequality (71) in part (a) follows directly from [34, (2.11)]. Note that by Definition 2.1, the metric subregularity of } \mathcal{T} \text{ at } x^\infty \text{ for the origin with modulus } \kappa > 0 \text{ is equivalent to the existence of a neighborhood } \mathcal{U} \text{ of } x^\infty \text{ such that for all } w \in \mathcal{X}, \]

\[ \text{dist}(x, \mathcal{T}^{-1}(0)) \leq \kappa\|w\|, \quad \forall x \in \mathcal{T}^{-1}(w) \cap \mathcal{U}. \]  

Thus, to prove (72) in part (b), one can use a similar proof as in [24, Theorem 2.1] except for replacing condition (2.1) in [24] by condition (73) with some neighborhood $\mathcal{U}$ of $x^\infty$. For brevity, we omit the details here. \hfill \square

Denote $\mathbb{D} := Q \times W \times S_n$. For convenience, we rewrite problems (26) and (27) in the following equivalent forms, respectively:

\[ \max \quad -h(FX) - \langle C, X \rangle \]
\[ \text{s.t.} \quad b - AX \in Q^*, \quad X \in S^*_n \]  

and

\[ \min \quad \vartheta(z) := -\langle b, y \rangle + h^*(-w) \]
\[ \text{s.t.} \quad A^*y + F^*w + S = C, \quad z \in \mathbb{D}. \]  

Let $\inf \vartheta$ be the optimal value of $\vartheta$ for problem (75). The Lagrangian function $l$ for problem (75) now takes the form of

\[ l(z, X) := -\langle b, y \rangle + h^*(-w) + \langle X, A^*y + F^*w + S - C \rangle, \quad \forall (z, X) \in \mathbb{D} \times S^n. \]

The functions $\psi$ and $\phi$ defined in (13) can be rewritten as

\[ \psi(z) := \sup_{X \in S^n} l(z, X), \quad \forall z \in \mathbb{D}, \quad \phi(X) := \inf_{z \in \mathbb{D}} l(z, X), \quad \forall X \in S^n \]

while the mappings $\mathcal{T}_\psi$, $\mathcal{T}_\phi$ and $\mathcal{T}_\psi$ in (15) and (16) can be reformulated as

\[ \mathcal{T}_\psi(z) := \partial \psi(z), \quad \forall z \in \mathbb{D}, \quad \mathcal{T}_\phi(X) := -\partial \phi(X), \quad \forall X \in S^n. \]
Let $c > 0$ be a positive parameter. For any $X \in S^n$, the augmented Lagrangian function associated with problem (75) is given by

$$L_c(z, X) := l(z, X) + \frac{c}{2}\|A^*y + F^*w + S - C\|^2, \forall z \in \mathbb{D}.$$  

Given a sequence of scalars $c_k \uparrow c_{\infty} \leq \infty$ and a starting point $X^0 \in S^n$, for $k \geq 0$, the $(k + 1)$-th iteration of the ALM is given by

$$\begin{cases}
z^{k+1} = \arg\min\{\zeta_k(z) := L_c(z, X^k) \mid z \in \mathbb{D}\}, \\
X^{k+1} = X^k + c_k(A^*y^{k+1} + F^*w^{k+1} + S^{k+1} - C).
\end{cases} \quad (76)$$

It is easy to check that if $\bar{z} \in \text{arg min}\{\zeta_k(z) \mid z \in \mathbb{D}\}$, then we must have $\bar{S} = \Pi_{S^n}(C - A^*\bar{y} - F^*\bar{w} - c_k^{-1}X^k)$. This motivates us to define for any $k \geq 0$,

$$S_k(y, w) := \Pi_{S^n}(C - A^*y - F^*w - c_k^{-1}X^k), \forall (y, w) \in \mathbb{Q} \times \mathbb{W}.$$  

Thus, for $k \geq 0$, the $(k + 1)$-th iteration of the ALM in (76) can be computed in the following manner

$$\begin{cases}
(y^{k+1}, w^{k+1}) \approx \arg\min\{\zeta_k(y, w, S_k(y, w)) \mid y \in \mathbb{Q}, w \in \mathbb{W}\}, \\
S^{k+1} = S_k(y^{k+1}, w^{k+1}), \\
X^{k+1} = X^k + c_k(A^*y^{k+1} + F^*w^{k+1} + S^{k+1} - C) \\
= \Pi_{S^n}(X^k + c_k(A^*y^{k+1} + F^*w^{k+1} - C)).
\end{cases} \quad (77)$$

In accordance with Rockafellar’s work in [33], we shall terminate the subproblem for solving $z^{k+1}$ in (76) by the following three criteria:

- $(A') \quad \zeta_k(z^{k+1}) - \inf \zeta_k \leq \varepsilon_k^2/2c_k, \quad \varepsilon_k \geq 0, \quad \sum_{k=0}^{\infty} \varepsilon_k < \infty,$
- $(B') \quad \zeta_k(z^{k+1}) - \inf \zeta_k \leq (\eta_k^2/2c_k)\|X^{k+1} - X^k\|^2, \quad \eta_k \geq 0, \quad \sum_{k=0}^{\infty} \eta_k < \infty,$
- $(B''') \quad \text{dist}(0, \partial\zeta_k(z^{k+1})) \leq (\eta_k' / c_k)\|X^{k+1} - X^k\|, \quad 0 \leq \eta_k' \rightarrow 0.$

A notable result of Rockafellar [33] shows that the ALM in (76) for solving the dual problem (27) with criteria $(A')$ and $(B')$ can be viewed as the PPA applied to $T_\phi = \partial\phi$ as in (70) with stopping criteria $(A)$ and $(B)$. This will help us to obtain the global convergence and the asymptotic superlinear convergence rates of the ALM for solving problem (27). But first, we need the following simple property.

**Proposition 4.1.** Let $\{(z^k, X^k)\}$ be a sequence generated by the ALM (76) under criterion $(B')$. Then for all $k \geq 0$ such that $\eta_k < 1$, it holds that

$$\|X^{k+1} - X^k\| \leq (1 - \eta_k)^{-1}\text{dist}(X^k, T_{\phi}^{-1}(0)).$$

**Proof.** By using Theorem 4.1 (a) and criterion $(B')$, we get for all $k \geq 0$ that

$$\|X^{k+1} - X^k\| \leq \|X^{k+1} + e^{k+1} - X^k\| + \|e^{k+1}\| \\
\leq \text{dist}(X^k, T_{\phi}^{-1}(0)) + (2c_k(\zeta_k(z^{k+1}) - \inf \zeta_k))^{1/2} \\
\leq \text{dist}(X^k, T_{\phi}^{-1}(0)) + \eta_k\|X^{k+1} - X^k\|,$$

where the second term in the second inequality comes from [33, Proposition 6]. Thus, the conclusion of this proposition holds. \qed
Theorem 4.2. Assume that the optimal solution set $T_{\phi}^{-1}(0)$ to problem (26) is non-empty. Let $\{(z^k, X^k)\}$ be an infinite sequence generated by the ALM in (78) with stopping criterion (A'). Then, the whole sequence $\{X^k\}$ is bounded and converges to some $X^\infty \in T_{\phi}^{-1}(0)$, and the sequence $\{z^k\}$ satisfies for all $k \geq 0$, $z^k \in \mathbb{D}$ and

$$\|A^* y^{k+1} + F^* w^{k+1} + S^{k+1} - C\| = c_k^{-1} \|X^{k+1} - X^k\| \to 0,$$

(79)

$$\vartheta(z^{k+1}) - \inf \vartheta \leq \zeta_k(z^{k+1}) - \inf \zeta_k + (1/2c_k)(\|X^k\|^2 - \|X^{k+1}\|^2).$$

(80)

Moreover, if problem (27) admits a non-empty and bounded solution set, then the sequence $\{z^k\}$ is also bounded, and all of its accumulation points are optimal solutions to problem (27).

Regarding the convergence rates of the ALM, we have the following results.

(a) If $T_\phi$ is metrically subregular at $X^\infty$ for the origin with modulus $\kappa_\phi > 0$, then under criterion (B'): there exists $\tilde{k} \geq 0$ such that for all $k \geq \tilde{k}$, $\eta_k < 1$ and

$$\text{dist}(X^{k+1}, T_{\phi}^{-1}(0)) \leq \theta_k \text{dist}(X^k, T_{\phi}^{-1}(0)),$$

(81)

$$\|A^* y^{k+1} + F^* w^{k+1} + S^{k+1} - C\| \leq \tau_k \text{dist}(X^k, T_{\phi}^{-1}(0)),$$

(82)

$$\vartheta(z^{k+1}) - \inf \vartheta \leq \tau_k' \text{dist}(X^k, T_{\phi}^{-1}(0)),$$

(83)

where

$$1 > \theta_k = (\kappa_\phi/\sqrt{\kappa_\phi^2 + c_k^2} + 2\eta_k)(1 - \eta_k)^{-1} \to \theta_\infty = \kappa_\phi/\sqrt{\kappa_\phi^2 + c_\infty^2} (\theta_\infty = 0 \text{ if } c_\infty = \infty),$$

$$\tau_k = c_k^{-1}(1 - \eta_k)^{-1} \to \tau_\infty = 1/c_\infty (\tau_\infty = 0 \text{ if } c_\infty = \infty),$$

$$\tau_k' = \tau_k(\eta_k^2 \|X^{k+1} - X^k\| + \|X^{k+1}\| + \|X^k\|)/2 \to \tau_\infty' = \|X^\infty\|/c_\infty (\tau_\infty' = 0 \text{ if } c_\infty = \infty).$$

(b) If in addition to (B') and the metric subregularity of $T_\phi$ at $X^\infty$ for the origin, one has (B''), $T_{\phi}^{-1}(0)$ is non-empty and bounded and the following condition on $T_\ell$: there exist two constants $\kappa_l \geq 0$ and $\epsilon > 0$ such that for any $(z, X) \in \mathbb{Z} \times S^n$ satisfying $\text{dist}((z, X), T_{\phi}^{-1}(0) \times \{X^\infty\}) \leq \epsilon$,

$$\text{dist}((z, X), T_{\phi}^{-1}(0)) \leq \kappa_l \text{dist}(0, T_\ell(z, X)).$$

(84)

Then there exists $\tilde{k} \geq 0$ such that for all $k \geq \tilde{k}$, $\eta_k < 1$ and

$$\text{dist}(z^{k+1}, T_{\phi}^{-1}(0)) \leq \theta_k' \text{dist}(X^k, T_{\phi}^{-1}(0)),$$

(85)

where

$$\theta_k' = \kappa_l c_k^{-1}(1 + \eta_k')(1 - \eta_k')^{-1} \to \theta_\infty' = \kappa_l/c_\infty (\theta_\infty' = 0 \text{ if } c_\infty = \infty).$$

Proof. The convergence on the sequences $\{X^k\}$ and $\{z^k\}$ follows from [33, Theorem 4]. The inequality (79) can be obtained by the definition of $X^{k+1}$ in (78) and the convergence of $\{X^k\}$. By noticing of $\vartheta(z^{k+1}) - \zeta_k(z^{k+1}) = (1/2c_k)(\|X^k\|^2 - \|X^{k+1}\|^2)$ and $\inf \zeta_k \leq \inf \vartheta$ [33, (4.16)-(4.17)], we get (80).

Next, we prove the results on the convergence rates.

(a) The inequality (81) follows from Theorem 4.1 (b) directly. By combining (80), (79), (81), Proposition 4.1 and criterion (B'), we can obtain the inequalities (82) and (83).

(b) Since $T_{\phi}^{-1}(0)$ is assumed to be non-empty and bounded, we know that the sequence $\{z^k\}$ is
bounded and dist(z^k, T_\psi^{-1}(0)) \to 0. Thus there exists \tilde{k} \geq 0 such that for all \tilde{k} \geq k, \eta_k < 1 and dist((z^{k+1}, X^{k+1}), T_\psi^{-1}(0) \times \{X^\infty\}) \leq \epsilon. By using condition [33], we have for all \tilde{k} > k,
\begin{align*}
\text{dist}(z^{k+1}, T_\psi^{-1}(0)) &\leq \delta_k \text{dist}(0, T_l(z^{k+1}, X^{k+1})) \\
&\leq \delta_k (\text{dist}^2(0, \partial_\psi(z^{k+1}) + c_k^{-2} \|X^{k+1} - X^k\|^2)^{1/2} \\
&\leq \delta_k (1 + \eta_k) c_k^{-1} \|X^{k+1} - X^k\| \\
&\leq \delta_k (1 + \eta_k) c_k^{-1} (1 - \eta_k)^{-1} \text{dist}(X^k, T_\psi^{-1}(0)),
\end{align*}
where the third inequality comes from [33] (4.21), the forth inequality is due to criterion (B') and the last inequality follows from Proposition 4.1. Thus for \tilde{k} > k, the inequality (55) holds.

**Remark 4.1.** In Theorem 4.2, under the metric subregularity of T_\phi at X^\infty for the origin, the sequence \{X^k\} is proved to converge Q-(super)linearly to the optimal solution set T_\phi^{-1}(0) to problem (26), while the feasibility and the objective function value of problem (75) converge at least R-(super)linearly. For the asymptotic R-superlinear convergence of the iteration sequence \{z^k\} itself, one has to impose a stronger condition on \bar{T}_l as in part (b). In numerical computations one does not need \epsilon_k to converge to +\infty, instead one can just progressively choose \epsilon_k to be large enough, such as \epsilon_k \approx \kappa_\phi, to achieve fast linear convergence. Of course one does not know \kappa_\phi in practice, and hence the adjustment of \epsilon_k to achieve fast linear convergence is always an important issue in the practical implementation of the ALM. The metric subregularity of T_\phi at X^\infty for the origin is satisfied in one of the two situations in Corollary 3.7. Another situation for ensuring T_\phi to be metrically subregular at X^\infty for the origin is when the function h is twice continuously differentiable and the “no-gap” second order sufficient condition holds at X^\infty [33, Theorem 3.137]. Thus, we can see that the metric subregularity of T_\phi at X^\infty for the origin is quite mild. However, the metric subregularity of T_\phi can be more restrictive (refer to Remark 3.7).

### 4.1 On the implementable stopping criteria for solving the ALM subproblems

In this subsection, we shall study the implementation issues for applying the ALM to solve problem (75). While it is relatively easy to implement criterion (B') [33] (4.6), it can be a challenging task to execute criteria (A') and (B') since the value inf \psi_\kappa is not available. In the following, we shall take the least squares SDP problem with equality constraints, i.e., h(w) = \frac{1}{2} \|w - d\|^2 for any w \in \mathbb{W} with given d \in \mathbb{W} and Q = Y in problem (74), as an example to illustrate how to implement criteria (A') and (B'). Denote \mathcal{X} := \{X \in \mathbb{S}_+^n \mid AX - b = 0\} as the feasible set to problem (74) in this case. Here, we always assume that there exists a strictly feasible point \hat{X} \in \mathcal{X} such that
\begin{equation}
AX - b = 0, \quad \hat{X} > 0.
\end{equation}
Denote \sigma_{\min}(A) as the smallest positive singular value of A and define
\begin{equation}
\hat{\mu} := \sigma_{\min}^{-1}(A) \max \left\{ \lambda_{\min}^{-1}(\hat{X}), 1 + \lambda_{\min}^{-1}(\hat{X}) \|\hat{X}\| \right\}.
\end{equation}
The following proposition provides an upper bound for the distance of an X \in \mathbb{S}_+^n to the set \mathcal{X}.

**Proposition 4.2.** Let X \in \mathbb{S}_+^n be given. Then
\begin{equation}
\|X - \Pi_{\mathcal{X}}(X)\| \leq \sigma_{\min}^{-1}(A)(1 + \lambda_{\min}^{-1}(\hat{X}) \|X - \hat{X}\|) \|b - AX\| \leq \hat{\mu}(1 + \|X\|)\|b - AX\|.
\end{equation}
Proof. Denote
\[ u := \mathcal{A}X - b, \quad \Delta X := -\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^\dagger u, \quad \bar{X} := X + \Delta X. \]
Since \( b, u \in \text{Range}(\mathcal{A}) \), it holds that
\[ \mathcal{A}\bar{X} = \mathcal{A}X + \mathcal{A}\Delta X = b + u - \mathcal{A}\mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^\dagger u = b, \quad \|\Delta X\| \leq \sigma_{\min}^{-1}(\mathcal{A})\|u\| . \]
Define
\[ \tau := \frac{\|u\|}{\|u\| + \sigma_{\min}(\mathcal{A})\lambda_{\min}(\bar{X})}, \quad X' := (1 - \tau)\bar{X} + \tau X. \]
Obviously \( \tau \in [0, 1] \) and \( X' \in \mathcal{X} \). Therefore, we obtain that
\[
\|X - \Pi_{\mathcal{X}}(X)\| \leq \|X - X'\| \leq \|\Delta X\| + \tau\|X - \bar{X}\| \leq \sigma_{\min}^{-1}(\mathcal{A})(1 + \lambda_{\min}^{-1}(\bar{X})\|X - \bar{X}\|)\|u\|
\leq \sigma_{\min}^{-1}(\mathcal{A})(1 + \lambda_{\min}^{-1}(\bar{X})\|\bar{X}\| + \lambda_{\min}^{-1}(\bar{X})\|X\|)\|u\|
\leq \sigma_{\min}^{-1}(\mathcal{A})\max\left\{\lambda_{\min}^{-1}(\bar{X}), 1 + \lambda_{\min}^{-1}(\bar{X})\|\bar{X}\|\right\}(1 + \|X\|)\|u\|,
\]
which completes the proof of this proposition.

For any \( k \geq 0 \), denote \( f_k(X) := -h(\mathcal{F}X) - \langle C, X \rangle - \|X - X^k\|^2/2c_k \) for \( X \in \mathbb{S}^n \).

**Proposition 4.3.** Assume that \( \mathcal{A} : \mathbb{S}^n \to \mathbb{Y} \) is surjective and that condition (S0) is satisfied. Let \( \bar{\mu} \) be given by (87) and \( \bar{\nu} \) be any positive constant. Suppose that for some \( k \geq 0 \), \( \varepsilon_k > 0 \), \( \eta_k > 0 \) and \( X^k \in \mathbb{S}^n_+ \) is not an optimal solution to problem (74). Let \( \{z^{k,j}\}_{j \geq 0} \) be any sequence such that \( \zeta_k(z^{k,j}) \to \inf \zeta_k \) with \( (y^{k,j}, w^{k,j}) \in \mathbb{Y} \times \mathbb{W} \) and \( S^{k,j} = S_k(y^{k,j}, w^{k,j}) \), where \( S_k(\cdot) \) is defined as in (77).

For any \( j \geq 0 \), let
\[
X^{k,j} := \Pi_{\mathbb{S}^n_+}(X^k + c_k(\mathcal{A}^*y^{k,j} + \mathcal{F}^*w^{k,j} - C)),
\]
\[
w^{j} := \mathcal{A}X^{k,j} - b, \quad t_j := \bar{\nu}^{-1}\min\{\varepsilon_k^2/2c_k, (\eta_k^2/2c_k)\|X^{k,j} - X^k\|^2\}.
\]
Then there exists \( j \) such that for any \( j \geq j \)
\[
\zeta_k(z^{k,j}) - f_k(X^{k,j}) \leq t_j, \quad (1 + \|X^{k,j}\|)\|u^{k,j}\| \leq \min\left\{1, \sqrt{c_k}, \sqrt{t_j}/\|\nabla f_k(X^{k,j})\|\right\}\sqrt{t_j}.
\]
Consequently, for all \( j \geq j \)
\[
\zeta_k(z^{k,j}) - \inf \zeta_k \leq \left(1 + \bar{\mu} + \frac{1}{2}\lambda_{\max}(\mathcal{F}^*\mathcal{F})\mu^2 + \frac{1}{2}\mu^2\right)t_j.
\]

**Proof.** Since \( f_k(\cdot) \) is strongly concave, \( \zeta_k(z^{k,j}) \to \inf \zeta_k \) and condition (S0) is satisfied with \( \mathcal{A} \) being surjective, we know from [32, Theorems 17 & 18] that the two sequences \( \{z^{k,j}\} \) and \( \{X^{k,j}\} \) are bounded, and \( \{X^{k,j}\} \) converges to some point \( X^{k,\infty} \in \mathcal{X} \) such that \( f_k(X^{k,\infty}) = \inf \zeta_k \). One can easily prove that \( X^{k,\infty} \neq X^k \), because otherwise \( X^k \) and any accumulation point of \( \{z^{k,j}\} \) forms a KKT solution point to problems (74) and (75), which would contradict our assumption that \( X^k \) is not an optimal solution to problem (74). Thus, for all \( j \) sufficiently large, \( t_j \) is bounded away from 0. Then, there exists \( j \geq 0 \) such that for all \( j \geq j \), the two inequalities in (88) hold.
By using Proposition 4.2 and (88), we get for all \( j \geq \bar{j} \) that
\[
\zeta_k(z^{k,j}) - \inf \zeta_k 
\leq \zeta_k(z^{k,j}) - f_k(\Pi_X(X^{k,j})) = (\zeta_k(z^{k,j}) - f_k(X^{k,j})) + (f_k(X^{k,j}) - f_k(\Pi_X(X^{k,j})))
\leq t_j - \langle \nabla f_k(X^{k,j}), \Pi_X(X^{k,j}) - X^{k,j} \rangle + \frac{1}{2}\langle \Pi_X(X^{k,j}) - X^{k,j}, (F^*F + c_k^{-1}I)(\Pi_X(X^{k,j}) - X^{k,j}) \rangle
\leq t_j + \bar{\mu}\|\nabla f_k(X^{k,j})\|(1 + \|X^{k,j}\|)\|u^{k,j}\| + \frac{1}{2}(\lambda_{\max}(F^*F) + c_k^{-1})\bar{\mu}^2(1 + \|X^{k,j}\|)^2\|u^{k,j}\|^2
\leq t_j + \bar{\mu}t_j + \frac{1}{2}\lambda_{\max}(F^*F)\bar{\mu}^2t_j + \frac{1}{2}\bar{\mu}^2t_j.
\]
This completes the proof of the proposition.

Proposition 4.3 says that if \( A \) is surjective and condition (86) is satisfied, we can use the following criterion to replace \((A') \) and \((B') \) with some \( \bar{\nu} > 0 \):
\[
\begin{align*}
(A') \quad & \left\{ \begin{array}{l}
\zeta_k(z^{k+1}) - f_k(X^{k+1}) \leq \bar{\nu}^{-1}\varepsilon_k^2/2c_k, \\
(1 + \|X^{k+1}\|)\|u^{k+1}\| \leq \min \left\{ 1, \sqrt{c_k}, \sqrt{t_{k,1}}/\|\nabla f_k(X^{k+1})\| \right\} \sqrt{t_{k,1}},
\end{array} \right. \quad \varepsilon_k \geq 0, \quad \sum_{k=0}^{\infty} \varepsilon_k < \infty,
\end{align*}
\]
\[
\begin{align*}
(B') \quad & \left\{ \begin{array}{l}
\zeta_k(z^{k+1}) - f_k(X^{k+1}) \leq \bar{\nu}^{-1}(\eta_k^2/2c_k)\|X^{k+1} - X^k\|^2, \\
(1 + \|X^{k+1}\|)\|u^{k+1}\| \leq \min \left\{ 1, \sqrt{c_k}, \sqrt{t_{k,2}}/\|\nabla f_k(X^{k+1})\| \right\} \sqrt{t_{k,2}},
\end{array} \right. \quad \eta_k \geq 0, \quad \sum_{k=0}^{\infty} \eta_k < \infty,
\end{align*}
\]
where \( t_{k,1} := \bar{\nu}^{-1}\varepsilon_k^2/2c_k, \ t_{k,2} := \bar{\nu}^{-1}(\eta_k^2/2c_k)\|X^{k+1} - X^k\|^2 \) and \( u^{k+1} := AX^{k+1} - b \). By taking \( \bar{\nu} = 1 + \bar{\mu} + \frac{1}{2}\lambda_{\max}(F^*F)\bar{\mu}^2 + \frac{1}{2}\bar{\mu}^2 \), we see that criteria \((A') \) and \((B') \) are satisfied as long as both \((A') \) and \((B') \) are true. Actually, by taking \( \bar{\nu} \) to be any positive constant satisfying \((A') \) and \((B') \), we obtain a sequence \( \{z^k, X^k\} \) that achieves \((A') \) and \((B') \) with \( \{\varepsilon_k\} \) and \( \{\eta_k\} \) being replaced by \( \{\sqrt{\bar{\nu}\varepsilon_k}\} \) and \( \{\sqrt{\bar{\nu}\eta_k}\} \), respectively. All the results in Theorem 4.2 are valid with these two new sequences \( \{\sqrt{\bar{\nu}\varepsilon_k}\} \) and \( \{\sqrt{\bar{\nu}\eta_k}\} \). As far as we know, these implementable criteria of the ALM are new.

Note that while the assumption on the existence of \( \hat{X} \) in (80) is crucial to our analysis in Proposition 4.3, the assumption on the surjectivity of the linear operator \( A : S^n \rightarrow \mathbb{Y} \) is not essential as we can always redefine \( \mathbb{Y} = \text{Range}(A) \), to make \( A \) to be surjective from \( S^n \) to \( \mathbb{Y} \). Additionally, it is not difficult to extend the results in Proposition 4.3 to the case that the constraint set \( X \) in (74) is replaced by \( \{X \in S^n_+ \mid A_1X = b_1, \ b_2 - A_2X \in Q' \} \) for some convex polyhedral cone \( Q' \), if one assumes the existence of \( \hat{X} > 0 \) such that \( A_1\hat{X} = b_1 \) and \( b_2 - A_2\hat{X} \in \text{int}(Q') \).

5 Concluding discussions

In this paper, we have established asymptotic superlinear convergence results of the ALM for solving SDP problems with multiple solutions. These results can be used to explain the success of the solvers SDPNAL, SDPNAL+ and QSDDPINAL for solving linear and convex quadratic SDP problems. There are several important issues that are worth further investigations. For example, it is interesting to study whether our results can be extended to other optimization problems with nonpolyhedral constraints besides SDP problems. Another important line of research is to further characterize the metric subregularity of \( T \) other than the ones stated in Corollary 3.1 or the “no-gap” second order sufficient condition. We also believe that it is worth the effort to investigate the metric subregularity of \( T \) under weaker conditions than the ones given in Theorem 3.2. Last but not least, one may ask how much the obtained results can help in finding ways to improve the efficiency
of the existing solvers for solving SDP problems, or even better, to obtain new and more efficient solvers for solving general large scale convex optimization problems.

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**References**

[1] F.J.A. Artacho and M.H. Geoffroy. Characterization of metric regularity of subdifferentials. *Journal of Convex Analysis*, 15(2):365–380, 2008.

[2] F.L. Bauer and C.T. Fike. Norms and exclusion theorems. *Numerische Mathematik*, 2(1):137–141, 1960.

[3] H.H. Bauschke and J.M. Borwein. On projection algorithms for solving convex feasibility problems. *SIAM Review*, 38(3):367–426, 1996.

[4] H.H. Bauschke, J.M. Borwein and W. Li. Strong conical hull intersection property, bounded linear regularity, Jameson’s property (G), and error bounds in convex optimization. *Mathematical Programming*, 86(1):135–160, 1999.

[5] J.F. Bonnans and A. Shapiro. Perturbation Analysis of Optimization Problems. *Springer, New York*, 2000.

[6] Y. Cui. Large Scale Composite Optimization Problems with Coupled Objective Functions: Theory, Algorithms and Applications. *PhD Thesis, Department of Mathematics, National University of Singapore*, January 2016.

[7] C. Ding, D.F. Sun and L.W. Zhang. Characterization of the robust isolated calmness for a class of conic programming problems. *arXiv preprint arXiv:1601.07418*, 2016.

[8] Y.D. Dong. An extension of Luque’s growth condition. *Applied Mathematics Letters*, 22(9):1390–1393, 2009.

[9] A.L. Dontchev and R.T. Rockafellar. Characterizations of Lipschitz stability in nonlinear programming. In *Mathematical Programming With Data Perturbations*, Marcel Dekker, New York, 1997, 65-82.

[10] A.L. Dontchev and R.T. Rockafellar. Implicit Functions and Solution Mappings. *Springer, New York*, 2009.

[11] D. Drusvyatskiy and A.S. Lewis. Error bounds, quadratic growth, and linear convergence of proximal methods. *arXiv preprint arXiv:1602.06661*, 2016.

[12] A. Eisenblätter, M. Grötschel and A.M. Koster. Frequency planning and ramifications of coloring. *Discussiones Mathematicae Graph Theory*, 22(1):51–88, 2002.

[13] F. Facchinei and J.S. Pang. Finite-dimensional Variational Inequalities and Complementarity Problems: Volume I. *Springer, New York*, 2003.

[14] M. Grötschel, L. Lovász and A. Schrijver. Relaxations of vertex packing. *Journal of Combinatorial Theory, Series B*, 40(3):330–343, 1986.

[15] D.R. Han, D.F. Sun and L.W. Zhang. Linear rate convergence of the alternating direction method of multipliers for convex composite quadratic and semi-definite programming. *arXiv preprint arXiv:1508.02131*, 2015.

[16] A.J. Hoffman. On approximate solutions of systems of linear inequalities. *Journal of Research of the National Bureau of Standards*, 49(4):263–265, 1952.
[17] A.F. Izmailov, A.S. Kurennoy and M.V. Solodov. A note on upper Lipschitz stability, error bounds, and critical multipliers for Lipschitz-continuous KKT systems. *Mathematical Programming*, 142(1-2):591–604, 2013.

[18] K.F. Jiang, D.F. Sun and K.-C. Toh. Solving nuclear norm regularized and semidefinite matrix least squares problems with linear equality constraints. In *Discrete Geometry and Optimization*, Springer, 2013, 133–162.

[19] D. Klatte. Upper Lipschitz behavior of solutions to perturbed $C^{1,1}$ programs. *Mathematical Programming*, 88(2):285–311, 2000.

[20] D. Leventhal. Metric subregularity and the proximal point method. *Journal of Mathematical Analysis and Applications*, 360(2):681–688, 2009.

[21] X.D. Li, D.F. Sun and K.-C. Toh. QSDPNAL: A two-phase proximal augmented Lagrangian method for convex quadratic semidefinite programming. *arXiv preprint arXiv:1512.08872*, 2015.

[22] L. Lovász and A. Schrijver. Cones of matrices and set-functions and 0-1 optimization. *SIAM Journal on Optimization*, 1(2):166–190, 1991.

[23] Z.-Q. Luo and P. Tseng. On the linear convergence of descent methods for convex essentially smooth minimization. *SIAM Journal on Control and Optimization*, 30(2):408–425, 1992.

[24] F.J. Luque. Asymptotic convergence analysis of the proximal point algorithm. *SIAM Journal on Control and Optimization*, 22(2):277–293, 1984.

[25] J. Malick, J. Povh, F. Rendl and A. Wiegele. Regularization methods for semidefinite programming. *SIAM Journal on Optimization*, 20(1):336–356, 2009.

[26] O.L. Mangasarian. A simple characterization of solution sets of convex programs. *Operations Research Letters*, 7(1):21–26, 1988.

[27] G.J. Minty. Monotone (nonlinear) operators in Hilbert space. *Duke Mathematical Journal*, 29(3):341-346, 1962.

[28] R.D. Monteiro, C. Ortiz and B.F. Svaiter. A first-order block-decomposition method for solving two-easy-block structured semidefinite programs. *Mathematical Programming Computation*, 6(2):103–150, 2014.

[29] G. Pataki. On the closedness of the linear image of a closed convex cone. *Mathematics of Operations Research*, 32(2):395–412, 2007.

[30] S.M. Robinson. Some continuity properties of polyhedral multifunctions. *Mathematical Programming Study*, 14:206–214, 1981.

[31] R.T. Rockafellar. Convex Analysis. *Princeton University Press*, 1970.

[32] R.T. Rockafellar. Conjugate Duality and Optimization, Volume 14. *SIAM, Philadelphia*, 1974.

[33] R.T. Rockafellar. Augmented Lagrangians and applications of the proximal point algorithm in convex programming. *Mathematics of Operations Research*, 1(2):97–116, 1976.

[34] R.T. Rockafellar. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization*, 14(5):877–898, 1976.

[35] J.F. Sturm. Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4):625–653, 1999.

[36] D.F. Sun and J. Sun. Semismooth matrix-valued functions. *Mathematics of Operations Research*, 27(1):150–169, 2002.
[37] D.F. Sun, K.-C. Toh and L.Q. Yang. A convergent 3-block semi-proximal alternating direction method of multipliers for conic programming with 4-type constraints. *SIAM Journal on Optimization*, 25(2):882–915, 2015.

[38] M.J. Todd. Semidefinite optimization. *Acta Numerica 2001*, 10:515–560, 2001.

[39] K.-C. Toh, M.J. Todd and R.H. Tüttüncü. SDPT3 - a Matlab software package for semidefinite programming, version 1.3. *Optimization Methods and Software*, 11(1-4):545–581, 1999.

[40] P. Tseng. Approximation accuracy, gradient methods, and error bound for structured convex optimization. *Mathematical Programming*, 125(2):263–295, 2010.

[41] R.H. Tüttüncü, K.-C. Toh and M.J. Todd. Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical Programming*, 95(2):189–217, 2003.

[42] Z.W. Wen, D. Goldfarb and W.T. Yin. Alternating direction augmented Lagrangian methods for semidefinite programming. *Mathematical Programming Computation*, 2(3-4):203–230, 2010.

[43] L.Q. Yang, D.F. Sun and K.-C. Toh. SDPNAL+: A majorized semismooth Newton-CG augmented Lagrangian method for semidefinite programming with nonnegative constraints. *Mathematical Programming Computation*, 7(3):1–36, 2015.

[44] X.Y. Zhao, D.F. Sun and K.-C. Toh. A Newton-CG augmented Lagrangian method for semidefinite programming. *SIAM Journal on Optimization*, 20(4):1737–1765, 2010.

[45] Z.Z. Zhou and A.M.C. So. A unified approach to error bounds for structured convex optimization problems. *arXiv preprint arXiv:1512.03518*, 2015.