Local density approximation for a perturbative equation of state

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Recent progress in experimental techniques has made it possible to investigate the properties of gases of bosonic atoms in traps at temperatures lower than that required to achieve Bose-Einstein condensation (BEC), as well as the properties of fermionic atoms at temperatures lower than the Fermi temperature. When the density of a homogeneous gas is small enough perturbation theory can be applied and the equation of state of repulsive bosons and two-component fermions can be expressed in terms of the expansion parameter \( g/\hbar \omega \), where \( g \) is the density and \( a_s \) is the s-wave scattering length. The method is commonly and successfully used to relate properties of an inhomogeneous system to the ones of the corresponding homogeneous system. In this way the predictions of known homogeneous theory can be tested experimentally in confined systems.

Alternatively, the problem can be seen from the other side. It might happen that it is possible to observe and investigate properties of a trapped system in an experiment, while an analytical description of the same physical system is not known even for a homogeneous case. Then the experimental study of a trapped system might shed light on the properties of a homogeneous system and allow us to judge the validity of the analytical approaches. For example, this is the case with the two-component Fermi gas in the BEC-BCS crossover. The unitarity regime was already observed in experiments (see, for example, [1, 2]), while there is a lack of a well-posed and complete theory. Another intriguing question is presence or absence of the Lee-Yang term, corresponding to a beyond mean field term etc.) can be predicted quite easily, no exact solutions in the presence of an external potential are generally known. If the number of particles is sufficiently large and the energy per particle is much larger than the level spacing of the harmonic oscillator, one can apply the local density approximation (LDA). This method is commonly and successfully used to relate properties of an inhomogeneous system to the ones of the corresponding homogeneous system. In this way the predictions of known homogeneous theory can be tested experimentally in confined systems.

The aim of the current work is to investigate how the expansion of the equation of state of a homogeneous system shows up in properties of a trapped system. In order to do so we assume that the equation of state of a homogeneous gas can be written as as series expansion in some limit. The knowledge of a series expansion of the equation of state provides a deep insight into the physical nature of a quantum system. Starting from a generic “perturbative” equation of state of a homogeneous ultracold gas we make predictions for the properties of the gas in the presence of harmonic confinement. The local density approximation is used to obtain the chemical potential, total and release energies, Thomas-Fermi size and density profile of a trapped system in three-, two-, and one-dimensional geometries. The frequencies of the lowest breathing modes are calculated using scaling and sum-rule approaches and could be used in an experiment as a high precision tool for obtaining the expansion terms of the equation of state. The derived formalism is applied to dilute Bose and Fermi gases in different dimensions and to integrable one-dimensional models. Physical meaning of expansion terms in a number of systems is discussed.

I. INTRODUCTION

The paper is organized as follows. In Section II we explain the local density approximation in an arbitrary number of dimensions and introduce dimensionless units. In Sec. III we introduce the “perturbative” equation of state and solve the local density approximation problem. Explicit expressions for the chemical potential, energy, release energy, size and density profile of a trapped gas are obtained. In Sec. IV we use scaling and sum-rule approaches for the prediction of the frequencies of collective oscillations. Cases of spherical, elongated, “cigar”- and “pancake”- shaped traps are considered. Formulas are derived for three-, two- and one-dimensional traps. The LDA is applied to relate the “perturbative” equation of state to frequencies of the breathing mode. In Sec. V we apply the developed formalism to a number of physical systems where perturbation expansion of the equation of state is known. These
II. LOCAL DENSITY APPROXIMATION

We consider a gas of \( N \) particles (bosons or fermions) at a density \( n = N/V \). We suppose that the density dependence of the chemical potential \( \mu_{\text{hom}} \) of the homogeneous system is known from either perturbation theory or from a fit to experimental data. Within the local density approximation one assumes (see, for example, [6]) that the chemical potential \( \mu \) of a trapped system is given by the sum of the local chemical potential, which is taken to be equal to the chemical potential of the uniform system \( \mu_{\text{hom}} \) at the corresponding density, and the external field \( V_{\text{ext}} \):

\[
\mu = \mu_{\text{hom}}(n) + V_{\text{ext}}(r) \tag{1}
\]

In the following we will always consider a harmonic external confinement for two basic reasons. First, this is the most widely used trapping potential in cold gases. Second, a quadratic potential can be used to describe the external potential close to its minimum. In order to consider the most general case we will assume that the trapping frequencies are different, so that the confinement is anisotropic \( V_{\text{ext}}(r) = \frac{1}{2}m\omega_x x^2 + \frac{1}{2}m\omega_y y^2 + \frac{1}{2}m\omega_z z^2 \).

The value of the chemical potential \( \mu \) of a trapped gas is fixed by the normalization condition

\[
N = \int n(r)dr = \iiint n_{\text{hom}}^{-1} \left[ \mu - \frac{1}{2}m\omega_x x^2 - \frac{1}{2}m\omega_y y^2 - \frac{1}{2}m\omega_z z^2 \right] dxdydz \tag{2}
\]

Here the density profile is obtained by inverting the density dependence of the local chemical potential \( n = \mu_{\text{hom}}^{-1} \).

The spatial extensions of the cloud, \( R_x, R_y, R_z \), depend on the value of the chemical potential and frequencies of the harmonic confinement:

\[
\mu = \frac{1}{2}m\omega_x^2 R_x^2 = \frac{1}{2}m\omega_y^2 R_y^2 = \frac{1}{2}m\omega_z^2 R_z^2 \tag{3}
\]

By expressing the distances in the trap in units of the size of the cloud: \( \tilde{r} = (x/R_x, y/R_y, z/R_z) \) it follows immediately that the frequencies of the trap enter in Eq. (2) only through the combination \( \omega_{ho} = (\omega_x \omega_y \omega_z)^{1/3} \) and the normalization condition can be conveniently written in terms of the averaged oscillator length \( a_{ho} = \sqrt{\hbar/m\omega_{ho}} \). Now the integral is to be taken inside a sphere of radius 1 and is symmetric with respect to \( \tilde{r} \).

The normalization condition (2) can be conveniently rewritten in dimensionless units:

\[
\Lambda_D^{D/2} = \tilde{\mu}_D^{D/2} \int \mu_{\text{hom}}^{-1} \left[ \tilde{\mu} \Lambda_D (1 - \tilde{r}^2) \right] d\tilde{r}, \quad n(\tilde{r}) = 0 \text{ for } |\tilde{r}| > 1 \tag{4}
\]

Here the chemical potential \( \tilde{\mu} \) is expressed in units of \( \frac{1}{2}N^{1/D}\hbar\omega_{ho} \), which is a natural unit of energy in the trap, the density in a homogeneous system \( \mu_{\text{hom}}^{-1} \) is measured in units of \( a^{-D} \), where \( a \) is a length scale convenient for the homogeneous system (for example, it can be equal to the s-wave scattering length \( a_s \)), chemical potential (i.e. the argument of the inverse function \( \mu_{\text{hom}}^{-1} \)) is measured in units of \( \hbar^2/ma^2 \), \( D = 1, 2, 3 \) is the dimensionality of the system, and the LDA characteristic parameter \( \Lambda_D \) is defined as

\[
\Lambda_D = N^{1/D} \frac{a^2}{a_{ho}^D} \tag{5}
\]

From the Eq. (4) it becomes clear that within the LDA there is a scaling in terms of the characteristic parameter \( \Lambda_D \). In other words systems having a different number of particles and different frequencies of the harmonic confinement will have the same density profile and other LDA properties (once expressed in appropriate units as discussed above) if they have equal values of the parameter \( \Lambda_D \).

III. SOLUTION FOR “PERTURBATIVE” EQUATION OF STATE

In this Section we develop a general theory for a “perturbative” equation of state in a three-, two-, and one-dimensional systems. We start by writing down the equation of state \( \mu_{\text{hom}}(n) \). Assuming that a perturbation
approach can be applied to a homogeneous system, we suppose that $\mu_{\text{hom}}(n)$ is known as a series expansion, either at small or at high density, in terms of the expansion parameter ($\text{it can be gas parameter } na^D$ in the weakly interacting regime and $1/(na^D)$ in the regime of strong interactions):

$$\mu_{\text{hom}} = C_1(na^D)^{\gamma_1} + C_2(na^D)^{\gamma_2} + \ldots \frac{\hbar^2}{ma^2}. \quad (6)$$

In this formula the second term is considered to be small compared to the first term ($|C_2(na^D)^{\gamma_2}| \ll 1$) and will be treated perturbatively in the following. We will limit the series to the second term, as it is responsible for the main contribution to the dependence of the frequency of the breathing mode on the density, and we omit higher terms in the expansion \((6)\).

We obtain the value of the chemical potential of a trapped system by resolving the normalization condition \((4)\). The result can be conveniently expressed in terms of the characteristic parameter $A_D$ \((3)\).

$$\tilde{\mu} \equiv \frac{\mu}{2N^{1/D}\hbar\omega_{\text{ho}}} = C_\mu A_D^{\frac{D\gamma_1-2}{D\gamma_1+2}} \left( 1 + C_\mu^2 A_D^{\frac{D\gamma_2}{D\gamma_1+2}} + \ldots \right), \quad (7)$$

where constants $C_{\mu 1}$, $C_{\mu 2}$ are independent of the parameters of the trap and, instead, are directly related to the equation of state of a homogeneous system. In order to find them we evaluate the integral \((4)\) in three-, two-, and one-dimensional geometries. The explicit expressions for the coefficients $C_{\mu 1}$ and $C_{\mu 2}$ are given in Table I. The leading term in \((7)\) depends only on two parameters, $C_1$ and $\gamma_1$, of the homogeneous equation of state, while in order to obtain the next term one has to know four parameters: $C_1$, $\gamma_1$, $C_2$, $\gamma_2$. This type of behaviour is quite general and applies to majority of the quantities we estimate in a trapped gas, with the exception of the frequencies of collective oscillations, where the leading term depends only on $\gamma_1$ (see below).

| Dimensionality | $C_{\mu 1}$ | $C_{\mu 2}$ |
|---------------|-------------|-------------|
| 3D            | $2 \left( \frac{\mu}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \right)^{\frac{D\gamma_1-2}{D\gamma_1+2}}$ | $\left( \frac{\mu}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \right)^{\frac{D\gamma_1-2}{D\gamma_1+2}} \left( 1 + C_\mu^2 A_D^{\frac{D\gamma_2}{D\gamma_1+2}} + \ldots \right)$ |
| 2D            | $\left( \frac{2\pi C_1}{\gamma_1} \right)^{\frac{1}{\gamma_1}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \left( \frac{\mu}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \right)^{\frac{D\gamma_1-2}{D\gamma_1+2}}$ | $\left( \frac{2\pi C_1}{\gamma_1} \right)^{\frac{1}{\gamma_1}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \left( \frac{\mu}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \right)^{\frac{D\gamma_1-2}{D\gamma_1+2}} \left( 1 + C_\mu^2 A_D^{\frac{D\gamma_2}{D\gamma_1+2}} + \ldots \right)$ |
| 1D            | $\left( \frac{\mu}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \right)^{\frac{D\gamma_1-2}{D\gamma_1+2}}$ | $\left( \frac{\mu}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \right)^{\frac{D\gamma_1-2}{D\gamma_1+2}} \left( 1 + C_\mu^2 A_D^{\frac{D\gamma_2}{D\gamma_1+2}} + \ldots \right)$ |

TABLE I: Coefficients $C_{\mu 1}$ and $C_{\mu 2}$ of the expansion of the chemical potential $\mu$ of a trapped system \((4)\) obtained from evaluation of the normalization integral \((4)\). In a 3D system the integral is to be taken inside a sphere and can be evaluated explicitly if condition $(1+\gamma_2)/\gamma_1 + 1 > 0$ is satisfied (see, for example, \((3)\)). In a two-dimensional system, $D = 2$, the integral is to be calculated inside a circle of unit radius. There is a substitution $\int_0^\infty f(1-r^2)2\pi r \, dr = \int_0^1 \int_0^{2\pi} f(y)\pi dy$ which together with a particular form of the chemical potential \((\tilde{\mu})\) leads to an elementary integral. In a one-dimensional system, $D = 1$, the spatial integration over the coordinate $z$ goes from $-R$ to $R$. In contrast to the two-dimensional case no simplification is found and the final expression for the chemical potential $\mu$ \((4)\) will contain the gamma functions as in fully three-dimensional case. The integration was carried out under conditions $\gamma_1 + \gamma_2 + 1 > 0$.

The energy of the trapped system can be obtained by integrating the chemical potential $E = \int_0^N \mu(N')dN'$. From Eqs. $(7)$ and $(4)$ we derive an explicit expression

$$\tilde{E} \equiv \frac{E}{2N^{1/D}\hbar\omega_{\text{ho}}} = \frac{N \left( D\gamma_1 + 2 \right) C_{\mu 1} A_D^{\frac{D\gamma_1-2}{D\gamma_1+2}} \left( 1 + (D\gamma_1 + 2\gamma_1 + 2)C_{\mu 2} A_D^{\frac{D\gamma_2}{D\gamma_1+2}} + \ldots \right) (D+2)}{D+2}$

$$ \quad \left( \frac{\mu}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma \left( \frac{1}{D} + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{D} \right)} \right)^{\frac{D\gamma_1-2}{D\gamma_1+2}} \left( 1 + C_\mu^2 A_D^{\frac{D\gamma_2}{D\gamma_1+2}} + \ldots \right) \quad (8)$$

The density profile of a trapped gas can be accessed experimentally by using an absorption imaging technique, when the shadow of the cloud of the cloud is imaged on a CCD camera. The size of the cloud can be extracted from such images. In our approach the size of the cloud along axis $\alpha$ ($\alpha = x, y, z$) is given by $R_\alpha = R\omega_{\text{ho}}/\omega_\alpha$, where the expression for the Thomas-Fermi radius $R$ is

$$R = N \pi \frac{1}{2} C_{\mu 1}^{1/2} A_D^{\frac{D\gamma_1-2}{D\gamma_1+2}} \left( 1 + C_{\mu 2}^2 A_D^{\frac{D\gamma_2}{D\gamma_1+2}} + \ldots \right) a_{\text{ho}} \quad (9)$$

The applicability of the local density approximation is that the energy of the gas \((8)\) is large compared to the interlevel spacing $E \gg \hbar\omega_{\text{ho}}$, so that the discretization of the levels in the trap can be ignored. The same condition it
terms of length means that the size of the gas \([9]\) is large compared to the oscillator length \(R \gg a_ho\). These conditions are usually satisfied for a large number of atoms. In a one-dimensional system the LDA works well even for a small number of atoms \([5]\), as effects of interactions are enhanced in low-dimensional systems compared to three-dimensional systems.

The density profile \(n(r)\) has a typical shape for the local density approximation:

\[
n(r) \approx \left[ \frac{\tilde{\mu} \Lambda_D}{2C_1} \left( 1 - \frac{r^2}{R^2} \right) \right]^{\frac{1}{\gamma_1}} - C_2 \left[ \frac{\tilde{\mu} \Lambda_D}{2C_1} \left( 1 - \frac{r^2}{R^2} \right) \right]^{\frac{1+\gamma_2}{\gamma_1}} + ...
\]

where the chemical potential is given by \([7]\) and the size of the cloud is as in \([3]\). For example, for an expansion in the mean-field regime \(\gamma_1 = 1\), the leading term has the shape of an inverted parabola. In the case of a 1D weakly interacting Fermi gas or Tonks-Girardeau gas, \(\gamma_1 = 2\), and it takes shape of a semicircle.

Another important spatial quantity is the mean square displacement \(\langle r^2 \rangle\). It is directly related to the potential energy of the harmonic confinement. The integration of \(r^2\) over the density profile \([10]\) can be performed and the result of such an integration is conveniently written in units of the size of the cloud:

\[
\frac{\langle r^2 \rangle}{R^2} = \frac{D \gamma_1}{2 + (2 + D) \gamma_1} \left( 1 + \frac{C_2 \gamma_2 \Gamma(1 + \frac{D}{2} + \frac{1}{\gamma_1}) \Gamma(1 + \frac{1+\gamma_2}{\gamma_1})}{\gamma_1 \Gamma(\frac{1}{\gamma_1}) \Gamma(2 + \frac{D}{2} + \frac{1+\gamma_2}{\gamma_1})} \left( \frac{\Lambda_D \tilde{\mu}}{2C_1} \right) \right)^{\frac{2}{D}}
\]

Another relevant quantity that can be accessed in experiments is the release energy defined as the difference of the total energy \(E\) of the trapped gas and the potential energy \(E_{ho} = \frac{1}{2} m \omega_{ho}^2 \langle r^2 \rangle\) of the confinement: \(E_{rel} = E - E_{ho}\). The potential energy of the confinement can be eliminated by switching off the trap and the energy of the expanding cloud is given by \(E_{rel}\). Combining together Eqs. \([8]\) and \([11]\) and after some algebra we obtain the following expression for the dimensionless release energy \(E_{rel} = E_{rel}/\frac{N}{2} \hbar \omega_{ho}\):

\[
\bar{E}_{rel} = \frac{2 C_\mu^1 \Lambda_D^{\frac{D+D_D}{\gamma_1}}}{2 + (2 + D) \gamma_1} \left[ 1 + \left( \frac{1 + \frac{1}{\gamma_1} - \frac{D (\gamma_2 - 1)}{2}}{1 + \frac{D}{2} + \frac{1+\gamma_2}{\gamma_1}} \right) C_{\mu 2} - \frac{D C_2 \gamma_2 \Gamma(1 + \frac{D}{2} + \frac{1}{\gamma_1}) \Gamma(1 + \frac{1+\gamma_2}{\gamma_1})}{2 \Gamma(\frac{1}{\gamma_1}) \Gamma(2 + \frac{D}{2} + \frac{1+\gamma_2}{\gamma_1})} \left( \frac{C_{\mu 1}}{2C_1} \right) \right]^{\frac{2}{D}} \frac{\Lambda_D^{\frac{2 D_D}{\gamma_1}}}{\Lambda_D^{\frac{2 D_D}{\gamma_1}}} \]

IV. FREQUENCIES OF COLLECTIVE OSCILLATIONS

A very important aspect of the LDA approach is that it can be used for the prediction of the frequencies of the collective oscillations. If oscillations are generated by displacing the center of the trap, the frequency of the resulting excitation is determined only by the harmonic confinement and is independent of the interactions between atoms. Thus, this type of excitation can not be used to probe the equation of state, but helps to measure the trap’s frequencies. Instead, if the collective mode is excited by changing slightly the frequency of the confinement, the resulting “breathing” oscillation depends on the speed of sound in the gas \(c = \sqrt{n/m} \partial \mu/\partial n\) and probes directly the equation of state.

If the trap is spherical, then the mean square displacement \(\langle r^2 \rangle\) can be used to estimate the frequencies of the collective modes. An important point is that the breathing mode is naturally excited by a closely related operator \(\sum_i r_i^2\). In this case the sum rules approach can be applied leading to a compact expression for the frequency of the breathing mode

\[
\Omega_{sph}^2 = -2 \langle r^2 \rangle \frac{\partial \langle r^2 \rangle}{\partial \omega^2}\]

This formula was obtained in \([4]\) while analyzing collective frequencies of a one-dimensional Lieb-Liniger system. The applicability of formula \([13]\) is not restricted to a one-dimensional system and remains valid in two- and three-dimensional systems.

Calculation of the derivative of the mean square displacement \([11]\) with respect to the square of the trap frequency leads to the following expression for the frequency \(\Omega\) of the breathing mode in a spherical trap:

\[
\frac{\Omega_{sph}^2}{\omega_{ho}^2} = (2 + D \gamma_1) \left[ 1 + \frac{D \gamma_2}{2} \left( C_{\mu 2} + \frac{D C_2 \gamma_2 \Gamma(1 + \frac{D}{2} + \frac{1}{\gamma_1}) \Gamma(1 + \frac{1+\gamma_2}{\gamma_1})}{\gamma_1 \Gamma(\frac{1}{\gamma_1}) \Gamma(2 + \frac{D}{2} + \frac{1+\gamma_2}{\gamma_1})} \left( \frac{C_{\mu 1}}{2C_1} \right) \right] \right]^{\frac{2}{D}} \frac{\Lambda_D^{\frac{2 D_D}{\gamma_1}}}{\Lambda_D^{\frac{2 D_D}{\gamma_1}}} + ...
\]

The constant term in the expression for the frequencies \([14]\) depends on the power of the leading contribution to the homogeneous chemical potential \([6]\) and on the dimensionality of the system. The dependence on parameter \(\Lambda_D\) enters only in the subleading term which we are able to obtain with this perturbative calculation.
An alternative method is needed in order to study collective modes in anisotropic traps where the formula \[13\] is not applicable. As proved in \[10\], the scaling approach allows us to calculate the frequencies of the collective oscillations with extremely high precision in anisotropic harmonic confinement. The method is based on using a scaling ansatz for the time-evolution of the density profile \(n(x, y, z, t) \propto n(a_0(t)x, a_0(t)y, a_0(t)z)\) and solving the hydrodynamic equations under the assumption that the amplitude of oscillations is small. The oscillation frequencies \(\omega\) in a 3D elongated anisotropic trap with trap frequencies \((\omega_x, \omega_y, \omega_z)\) is not analytically known. It is commonly proposed to be linear in the parameter \(K_Fa_s\) under the assumption that the amplitude of oscillations is small. The oscillation frequencies \(\omega\) in a 3D elongated anisotropic trap with trap frequencies \((\omega_x, \omega_y, \omega_z)\) are given by

\[
\Omega^2_{3D} = \frac{1}{4} \left[ 2\Xi \omega_1^2 + (2 + \Xi) \omega_2^2 \pm \sqrt{(2\Xi \omega_1^2 + (2 + \Xi) \omega_2^2)^2 + 16(2 - 3\Xi)\omega_1^2 \omega_2^2} \right]
\]

(15)

The natural parameter \(\Xi\) is related to the average of the compressibility \(mc^2 = n\partial \mu_{\text{hom}} / \partial n\) and the potential energy of the harmonic oscillator: \(\Xi = D \langle mc^2 \rangle / \langle E_{ho} \rangle\) \[11, 12\]. We evaluate the parameter \(\Xi\) explicitly:

\[
\Xi = 2(1 + \gamma_1) + \frac{2C_2 \gamma_2(\gamma_1 + \gamma_2) \Gamma(1 + \frac{\gamma_1}{2\gamma_2}) \Gamma(2 + \frac{D}{2} + \frac{1}{\gamma_1}) (C_{\mu 1} \frac{2\Xi}{D\gamma_1})}{\Gamma(\frac{1}{\gamma_1}) \Gamma(2 + \frac{D}{2} + \frac{1}{\gamma_1})} \sim D \rightarrow \infty
\]

(16)

From Eq. \[15\] it follows that the solution in a a spherical trap \(\omega_1 = \omega_2 \equiv \omega_{ho}\) can be simplified. There are two frequencies: \(\Omega^2_{3D} / \omega_{ho}^2 = 2\) (center of the mass mode) and \(\Omega^2_{3D} / \omega_{ho}^2 = 3/2\) (breathing mode). Direct calculation shows that the sum rules prediction for the spherical trap, Eq. \[15\] with \(D = 3\), coincides with the predictions of Eq. \[15\]. In an elongated trap, \(\omega_z \neq \omega_1\), it is possible to excite axial \(\Omega_3\) and radial \(\Omega_2\) modes separately. In a “cigar”-shaped trap \(\omega_z \ll \omega_1\) the frequencies of these modes are \(\Omega_2^2 / \omega_2^2 = 3 - 2/\Xi\) and \(\Omega_3^2 / \omega_1^2 = \Xi\) correspondingly. Another possible configuration is a “pancake” trap, where \(\omega_z \gg \omega_1\). In this situation the frequencies are \(\Omega_2^2 / \omega_2^2 = 1 + \Xi/2\) and \(\Omega_3^2 / \omega_1^2 = 6 - 16/(2 + \Xi)\).

We can generalize this scaling method to systems with low dimensionality. We find that in a two-dimensional system the frequencies of small oscillations are given by

\[
\Omega^2_{2D} = \frac{1}{4} \left[ (2 + \Xi)(\omega_1^2 + \omega_2^2) \pm \sqrt{(2 + \Xi)^2(\omega_1^2 + \omega_2^2)^2 - 32\Xi \omega_1^2 \omega_2^2} \right]
\]

(17)

In a spherical trap the frequencies are equal to: \(\Omega^2_{2D} / \omega_{ho}^2 = 2\) (center of mass mode) and \(\Omega^2_{3D} / \omega_{ho}^2 = \Xi\) (breathing mode). Again, the frequency of one type of excitation is defined only by the the trap frequency, while the frequency of the other mode can be used to study the equation of state of a homogeneous system. The result obtained for a spherical trap is the same as from Eq. \[15\] and \(D = 2\). In an elongated trap \((\omega_z \ll \omega_1)\) one obtains the following expressions for the frequencies of collective excitations in terms of the parameter \(\Xi:\n\Omega_{1D}^2 / \omega_1^2 = 1 + \Xi/2\), \(\Omega_{2D}^2 / \omega_2^2 = 4\Xi/(2 + \Xi)\).

There is no way of having an anisotropic trap in a one-dimensional system, so the result of the scaling approach

\[
\Omega_{1D}^2 / \omega_1^2 = 1 + \Xi/2
\]

(18)

coincides with the prediction of a spherical trap Eq. \[15\] and \(D = 1\).

It worth noticing that for a given equation of state \[19\] the dependence on the number of particles \(N, a_{ho, a}\) (i.e. the interaction strength) enters only in the second term. This means that by using only the leading term in \[19\] one obtains no dependence on \(\Lambda_{3D}\) in \[10\] and it is necessary to consider the next term to describe changes in the frequency. This justifies the necessity of our perturbative approach \[16\].

V. DISCUSSION

As the frequencies of the collective oscillations can be measured in experiments with high precision, formulae \[14, 10\] provide a very useful tool for the investigation of the equation of state. As an example, the description of the two-component Fermi gas close to a Feshbach resonance (unitary regime) is a very non-trivial problem as there is no obvious small parameter that can be used to construct an analytically solvable expansion theory (see Fig. \[1\]). This regime has been studied in recent experiments \[12, 13, 14\]. The s-wave scattering length in the unitarity regime becomes larger than any other length-scale in the system \(|a_s| \rightarrow \infty\) \((\Lambda_{3D} \rightarrow \infty)\) and, essentially, it drops out of the problem. The only relevant scale is the density and as a result the gas has the same scaling behavior as an ideal Fermi gas. This allows one to fix immediately the leading power \(\gamma_1 = 2/3\), although the next term in the expansion is not analytically known. It is commonly proposed to be linear in the parameter \(x = -1 / k_Fa_s\) \[12, 16, 17, 18\]. Thus the expansion of the chemical potential can be written as \(\mu(x) = (\xi + Sx)h^2k_F^2/2m\), where the dependence on the density is accounted for by the Fermi momentum \(k_F = (3\pi^2 n)^{1/3}\). We find that the predictions of the sum-rule
approach (see Table I) coincide in the unitary regime with the results of 
[16], where the authors start directly from the hydrodynamic equations and solve them for this particular choice of the equation of state. The second row of the Table I shows the predictions in the Bardeen-Cooper-Schrieffer (BCS) limit $a_s \to 0$ ($\Lambda_{3D} \to 0$), where fermions experience a weak attraction and the equation of state can be found perturbatively starting from a weakly interacting Fermi gas. The leading term corresponds to an ideal Fermi gas and scales as $n^{2/3}$ with the density. This leads to the same leading term in the frequencies of the collective oscillations as at unitarity, although the next term [11, 20] is different and arises from the interparticle interaction of a normal Fermi gas.

The situation in the Bose-Einstein condensation (BEC) regime $a_s \to +0$ ($\Lambda_{3D} \to 0$) is extremely interesting. In this limit a two-body bound state (molecules) of atoms of different spin is formed. The main contribution to the energy comes from the negative binding energy of molecules. As the binding energy depends only on the interaction energy and is independent of the density limit, it does not contribute to the collective excitations and this term is omitted in Table I. The first leading term depending on the density corresponds to a dilute non-ideal Bose gas of molecules, which play the role of composite bosons with mass $2m$ and density $n/2$. The s-wave scattering length $a_m$ of such composite bosons was found directly by solving the four-body scattering problem [21] and was estimated to be $a_m = 0.6 a_s$. It should be noted that this result differs from the commonly used mean-field BCS theory which gives $a_m = 2 a_s$ (see, for example, [22, 23, 24]). The nature of the of the next term in the equation of state is still an open question and only a precise experimental measurement can give a definitive answer. This term might have a bosonic nature and correspond to the beyond mean-field correction [25] to the equation of state of composite bosons as proposed in [26] (see, also, [27]). Alternatively, the next correction might be of a fermionic nature as argued in [28].

An ab initio quantum Monte Carlo study of the many-body problem made it possible to obtain the equation of state numerically [29]. Firstly, it was shown that the leading term has a mean-field nature and agrees with a molecule-molecule scattering length equal to $a_m = 0.6 a_s$ (very recently this result was also recovered from a diagrammatic approach [30]). Secondly, the Monte Carlo study suggests the presence of the bosonic beyond mean-field correction. As inferred from Table I and shown in Fig. I this would mean that the frequencies of the collective oscillation should increase above the mean-field value by making the interaction between molecules stronger. To date there has been no experimental corroboration of this result. In the available experimental work [13] the frequency decreases, but the deep BEC regime has not yet been achieved. A more precise experimental study should be performed, in which case Eq. (10) can be used to fit the frequencies of the collective oscillations and thus provide insight into the nature of the equation of state.

We provide a relationship between the characteristic LDA parameter $\Lambda_{3D}$ and the value of the Fermi momentum in the center of the trap $k_{F,max}$ which is a natural quantity for a fermionic system. The leading term (which is sufficient for the applicability of Eq. (10)) is given by

$$\Lambda_{3D} = (2\pi C_1)^{1/2} \left( \frac{\Gamma \left(1 + \frac{1}{\gamma_1}\right)}{\Gamma \left(\frac{5}{2} + \frac{1}{\gamma_1}\right)} \right)^{1/3} \left( \frac{k_{F,max}^2}{3\pi^2} \right)^{\frac{1}{3} + \frac{2\gamma_1}{3}}$$

(19)

| Limit          | $C_1$ | $\gamma_1$ | $C_2$ | $\gamma_2$ | $\Omega^2 / \omega^2$ | $\Omega^2 / \omega^2$ | $\Omega^2 / \omega^2$ | $\Omega^2_{sph} / \omega^2$ |
|----------------|------|------------|------|------------|------------------|------------------|------------------|------------------|
| BCS            | 2    | 4          | 2    | 10         | $8192\sqrt{2}\Lambda_{3D}^{1/2}$ | $4096\sqrt{2}\Lambda_{3D}^{1/2}$ | $4096\sqrt{2}\Lambda_{3D}^{1/2}$ | $315\sqrt{2}$         |
| unitary        | 2    | 3          | 2    | 10         | $250\sqrt{2}$ | $128\sqrt{2}$ | $128\sqrt{2}$ | $128\sqrt{2}$         |
| BEC            | $2\pi a_m$ | 1          | 16$\sqrt{2a_{3D}}$ | $3\sqrt{7}$ | 1 $+ 15/5105a_m^{1/3}A_{3D}^{1/5}$ | $15/5105a_m^{1/5}A_{3D}^{1/3}$ | $21/5105$ | $15/5105a_m^{1/5}A_{3D}^{1/3}$ |
| Bose gas      | $4\pi$ | 32         | $3\sqrt{2}$ | 1 $+ 15/5105a_m^{1/3}A_{3D}^{1/5}$ | $15/5105a_m^{1/5}A_{3D}^{1/3}$ | $21/5105$ | $15/5105a_m^{1/5}A_{3D}^{1/3}$ |

TABLE II: Summary for three-dimensional systems. The first column labels the physical system under considerations: limits for the two-component Fermi gas in the BEC-BCS crossover and weakly-interacting system of bosons. Columns 2-5: coefficients of the expansion of the equation of state defined as in Eq. (9). Columns 6-7: frequencies of radial and axial oscillations in a “cigar”-shaped trap (note that in a 3D system the parameter $\Omega^2$ is $\Xi = \Omega^2 / \omega^2$). Column 8: frequency of the breathing mode in a spherical trap. The notation $a_m$ in the BEC regime denotes the molecule-molecule scattering length in units of the atomic scattering length. In the same regime we omit the contribution to the chemical potential from the binding energy, as it is independent of the density and, thus, is not important for the LDA.

Low-dimensional geometries are advantageous for the investigation of the energetic properties of a gas as the reduced phase space enhances the role of interactions. As a result, the frequencies of collective oscillations vary over
a larger range, thus facilitating possible experimental measurements. We consider a quasi-two-dimensional system of bosons in the regime of small densities. We assume that the gas is weakly bound in two-directions \((x, y)\) and is subjected to a tight confinement in \(z\) direction. The density profile in a uniform system can be approximated as \(n(x, y, z) = n_{2D} \exp(-z^2)/\sqrt{\pi} a_{osc}\) where \(n_{2D}\) is the two-dimensional density. Locally the equation of a rarefied three-dimensional bose gas can be applied (see Table II) and the energy of the gas is estimated by integrating out the variable corresponding to the tight confinement

\[
E_{2D}(n_{2D}) = \int n_{3D}(x, y, z) \, n(x, y, z) \, dz / \int n(x, y, z) \, dz.
\]

The chemical potential, obtained by differentiating the energy, is reported in Table III. We calculate frequencies of collective oscillations by exploiting Eqs. (16-17). The results for a spherical and very elongated traps are presented in Table III. We note that the quasi two-dimensional equation of state is valid when the scattering length \(a\) is small compared to the oscillator length of the tight confinement, so that the scattering process is still three-dimensional. In a “pure” two-dimensional system the equation of state contains logarithmic terms \(^{31}\) and our approach does not apply.

| system          | \(C_1\) | \(C_2\) | \(\gamma_1\) | \(\gamma_2\) | \(\Omega_{1D}^2 / \omega_z^2\) | \(\Omega_{2D}^2 / \omega_z^2\) | \(\Omega_{ss}^2 / \omega_z^2\) |
|-----------------|---------|---------|--------------|--------------|-------------------------------|-------------------------------|-------------------------------|
| Q2D Bose gas    | 2\sqrt{2}\pi | 1 | 64 | 1/2 | 3 + \(2^{9/8} 192 A_{1D}^{1/2}\) / \(35 \sqrt{5\pi}/8\) | 8 + \(2^{9/8} 256 A_{1D}^{1/2}\) / \(35 \sqrt{5\pi}/8\) | 4 + \(2^{9/8} 384 A_{1D}^{1/2}\) / \(35 \sqrt{5\pi}/8\) |

TABLE III: Equation of state of a quasi-two-dimensional bose gas in the weakly interacting regime. The expansion parameter is \(n_{2D}a^3 / a_{osc}\). It is assumed that \(a \ll a_{osc}\). The meaning of the columns is the same as in Table II.

In Table IV we summarize the examples of bosonic as well as fermionic systems in (quasi-) one-dimensional geometry. The Lieb-Liniger model \(^{32}\) describes a gas of repulsive bosons interacting with a contact \(\delta\)-potential. The equation of state of such a gas can be expanded in the weakly interacting limit \(\Lambda_{1D} \to \infty\) (one notes that in a waveguide small values of the three-dimensional \(s\)-wave scattering length \(a_{1D}\) correspond to small values of the one-dimensional \(s\)-wave scattering length \(a_{1D}\) as in this limit \(a_{1D} = -a_{1D}^2 / a_{osc}\)). The leading term in the chemical potential \(\mu\) is linear in the density and corresponds to the mean-field Gross-Pitaevskii theory. Zero point oscillations of quasi-particles contribute to the next term and in a three-dimensional system corresponds to the Lee-Yang correction \(^{20}\).

We applied the Bogoliubov theory to a one-dimensional system and found that the corresponding term coincides with the second term of the expansion of Lie-Liniger equation of state. In the opposite limit of strong correlations \(\Lambda_{1D} \to 0\) (Tonks-Girardeau regime) the leading term in \(\mu\) equals the Fermi energy of ideal spinless fermions due to a very peculiar property of a Tonks-Girardeau gas experiencing “fermionization”, as the wavefunction of impenetrable point-like bosons can be mapped directly onto a wavefunction of a fermionic system \(^{33}\). Another one-dimensional system, the Calogero-Sutherland gas \(^{34, 35}\) has a functionally similar equation of state (see Table IV parameter \(\lambda\) is related to the strength of the \(1/|z|^3\) interaction). In order to understand the physical meaning of the subleading correction we will consider for a moment a system of hard-rods, \(i.e.\) impenetrable bosons of a size \(|a_{1D}|\). Its equation of state can be obtained from the energy of a Tonks-Girardeau gas by reducing the density by the excluded volume \(n \to n - N|a_{1D}|\). The leading term is again given by the energy of an ideal Fermi gas of spinless particles and the next

\[
\begin{align*}
\Omega_{1D}^2 = & 2^{9/8} 192 A_{1D}^{1/2} / 35 \sqrt{5\pi}/8 \\
\Omega_{2D}^2 = & 2^{9/8} 256 A_{1D}^{1/2} / 35 \sqrt{5\pi}/8 \\
\Omega_{ss}^2 = & 2^{9/8} 384 A_{1D}^{1/2} / 35 \sqrt{5\pi}/8
\end{align*}
\]
correction is given in Table IV. It was shown in 36 that a lowest gas-like state of attractive bosons \( a_{1D} > 0 \) in the limit of small \( a_{1D} \) (super-Tonks gas) has the same leading term in the expansion of the equation of state as a system of hard-rod. The reason is that the node of a two-body low-energy scattering solution lies at \( a_{1D} \) and for larger distances is essentially the same as in a system of hard-rod. Instead, for a repulsive gas with a contact interaction in the vicinity of the Tonks-Girardeau limit, the analytic continuation of the scattering solution has a node at \(-|a_{1D}|\). One finds that the subleading term in the equation of state is equal to the “excluded volume” correction, but differs in sign (see Table IV). Of course, the discussed terms in the chemical potential of the Lieb-Liniger gas coincides with the ones found by solving iteratively the Bethe ansatz integral equations. Still we find it important to understand the physical origin of the terms of the expansion.

| Limit                    | \( C_1 \) | \( \gamma_1 \) | \( C_2 \) | \( \gamma_2 \) |
|-------------------------|----------|----------------|----------|----------------|
| Lieb-Liniger: weak interaction | 2        | \( -\sqrt{2}/\pi \) | -1/2     | \( 3 + \frac{32^{1/3}}{\pi^{2/3}} \sqrt{2}/\Lambda_{1D} \) |
| Lieb-Liniger: strong interaction | \( \pi^2/2 \) | 2 | -8/3 | 14 - \( \frac{128\sqrt{2}}{15\pi} \Lambda_{1D}^{1/2} \) |
| Attractive Fermi gas: strong interaction | \( \pi^2/2 \) | 2 | 2/3 | 14 + \( \frac{64\sqrt{2}}{15\pi^2} \Lambda_{1D}^{1/2} \) |
| Attractive Fermi gas: weak interaction | \( \pi^2/8 \) | 2 | -8/π² | -14 + \( \frac{32}{3\pi^2} \Lambda_{1D}^{1/2} \) |
| Repulsive Fermi gas: strong interaction | \( \pi^2/2 \) | 2 | -8 ln(2)/3 | 14 - \( \frac{128\sqrt{2}}{15\pi^2} \Lambda_{1D}^{1/2} \) |
| Repulsive Fermi gas: weak interaction | \( \pi^2/8 \) | 2 | 8/π² | -14 - \( \frac{32}{3\pi^2} \Lambda_{1D}^{1/2} \) |
| Gas of Hard-Rods | \( \pi^2/2 \) | 2 | 8/3 | 14 + \( \frac{128\sqrt{2}}{15\pi^2} \Lambda_{1D}^{1/2} \) |
| Calogero-Sutherland | \( \lambda^2 \pi^2/2 \) | 2 | 8/3 | 14 + \( \frac{128\sqrt{2}}{15\pi^2} \Lambda_{1D}^{1/2} \) |

TABLE IV: Summary of some of the one-dimensional models where the expansion of the equation of state is known. The first column names the considered systems. The coefficients of the expansion are given in columns 2-5. The last column gives the predictions (18) for the oscillation frequencies. The parameter \( \Lambda_{1D} \) is defined by 39. Note that the presence of a term in the chemical potential independent of the density (e.g. binding energy of a molecule) does not modify the frequencies of oscillations and is ignored.

Considering a system of one-dimensional fermions and limiting ourselves to the s-wave scattering we immediately find that atoms of the same spin do not interact due to the Pauli exclusion principle. Instead, fermions of different spin are allowed to interact and the interaction can be either attractive or repulsive. The equation of state can be found by solving the integral Bethe ansatz equations as discussed for attractive \( 37, 38 \) and repulsive \( 39 \) interactions. The numerical solution for the LDA was carried out in 40 and is compared here to the series expansions obtained from Eq. (18). Again, there is interesting physics behind the expansion (19). In the weakly interacting limit (both attraction and repulsion) the leading term equals the energy of an ideal two-component Fermi gas. The next correction is linear in density and describe the mean-field interactions between particles of different spin. In the limit of strong repulsion the main contribution to \( \mu \) coincides with the chemical potential of a one-component ideal 1D Fermi gas with \( N \) atoms, as the strong repulsion between atoms of different spin plays the role of an effective Pauli principle 41. An attractive interaction in one-dimension immediately leads to the formation of a bound state (molecule). As the binding energy is independent of the density and does not contribute to the collective oscillations, this term is omitted in Table IV. In the low-density limit the internal structure of a molecule of two fermions can be neglected and be considered as a composite boson. Since in this limit the interaction between such composite bosons is repulsive and strong (Tonks-Girardeau limit of \( N/2 \) bosons of mass \( 2m \)), the system can be mapped to \( N/2 \) spinless fermions.

So the leading behavior is again of a fermionic nature. The next term was obtained correctly for the first time in 41, 40 and is reported in Table IV. In Figs. 2, 3 we visualize how the terms of the EOS expansion show up in the frequencies of collective modes. The Fig. 2 shows the frequencies for bosonic systems, although the Fig. 3 presents results for systems of two-component fermions. We compare the predictions of the expansion, Eq. (18), (dashed lines) with the numerical solution for the exact equation of state (solid lines). As it can be seen from the Figures, the region where the perturbative results hold is quite large, making possible the extraction of the expansion terms of the equation of state from the frequencies of collective oscillations. The accuracy is better for the expansions at low density, since if the expansion holds for the center of the trap it will certainly be valid close to the borders of the trap where the density is smaller (see, e.g. Fig. 3).
FIG. 2: Frequency of the breathing mode in a number of one-dimensional bosonic systems. The frequencies for the Lieb-Liniger model were obtained numerically in [9]. The collective excitations of a gas of hard-rods were discussed in [36] in their relation to the excitations of a gas in the super-Tonks regime. The dashed lines correspond to the expressions given in Table IV.

VI. CONCLUSIONS

It quite often happens that a perturbation theory for a homogeneous system can be developed analytically, although no solution for a system in an external confinement is known. If the number of particles in a trapped system is large enough (and this requirement often can be easily fulfilled in experiments) the local density approximation can be applied. We assume that the series expansion of the dependence of the chemical potential on density is known for a homogeneous system and we find the properties of a gas confined in an anisotropic trap. We consider three-, two-, and one-dimensional systems. We obtain the energetic properties such as the chemical potential, total energy of the gas, potential energy of the confinement. The calculated release energy can be accessed in expansion experiments. Spatial properties such as the density profile and size of the cloud are predicted and can be measured by imaging techniques in current experiments.

The frequencies of the collective oscillations in a harmonic trap are predicted from a hydrodynamic theory. We apply a scaling approach to obtain frequencies in three-, two-, and one-dimensional systems in traps with arbitrary aspect ratio $\omega_1/\omega_2$. In spherical, “cigar”- and “pancake”-shaped geometries the expressions are simplified. In a spherical trap an equivalent method, the sum-rule approach, can be conveniently applied and leads to analogous expressions. Exploiting the obtained expressions and using our solution to the LDA problem we relate the frequencies of the breathing modes to the expansion of the equation of state of a homogeneous system in an explicit way.

We argue that the obtained results are highly relevant for the experimental study of the equations of state of
trapped gases. The expansion of the equation of state can be found by a best fit of our formulas to experimentally measured quantities (for example, collective oscillation frequencies). This would provide an important insight into the physics of the systems under the study. As an example, there is an open and intriguing question of what will be the nature of the beyond-mean field term in the equation of state of the recently observed condensation of molecules. Will it have a bosonic or fermionic nature? Both proposals can be found in the literature and only a high precision experiment can provide a definitive result.

We apply the derived formalism to a number of three-dimensional systems (two-component Fermi gas in the BEC-BCS crossover, dilute Bose gas), quasi two-dimensional bose gas, quasi one-dimensional systems (Lieb-Liniger gas, gas of hard-rods, two-component fermionic gas, Calogero-Sutherland gas). A comparison with numerical solutions is provided. Finally we provide an physical interpretation of the expansion terms in a number of one-dimensional systems.

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[1] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. 92, 120401 (2004).
[2] J. Kinast, A. Turlapov, J. E. Thomas, Q. Chen, J. Stajic, and K. Levin (2005), 10.1126/science.1109220.
[3] M. Greiner, C. A. Regal, and D. S. Jin, Nature 426, 537 (2003).
[4] S. Jochim, M. Bartenstein, A. Altmeyer, G. Hendl, S. Riedl, C. Chin, J. H. Denschlag, and R. Grimm, Science 302, 2101 (2003).
[5] M. W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, S. Gupta, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. 91, 250401 (2003).
[6] L. P. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Oxford University Press, Oxford, 2003).
[7] I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Series and Products (Academic, New York, 1980).
[8] G. E. Astrakharchik and S. Giorgini, Phys. Rev. A 66, 053614 (2002).
[9] C. Menotti and S. Stringari, Phys. Rev. A 66, 043610 (2002).
[10] G. E. Astrakharchik, R. Combescot, X. Leyronas, and S. Stringari, Phys. Rev. Lett. 95, 030404 (2005).
[11] G. E. Astrakharchik, Ph.D. thesis, University of Trento, Italy (2004).
[12] J. Kinast, A. Turlapov, and J. E. Thomas, Phys. Rev. A 70, 051401(R) (2004).
[13] M. Bartenstein, A. Altmeyer, S. Riedl, S. Jochim, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. 92, 203201 (2004).
[14] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Phys. Rev. Lett. 92, 150402 (2004).
[15] A. Bulgac and G. Bertsch, Phys. Rev. Lett. 94, 070401 (2005).
[16] R. Combescot and X. Leyronas, Phys. Rev. Lett. 93, 138901 (2004).
[17] H. Heiselberg, Phys. Rev. Lett. 93, 040402 (2004).
[18] E. Y. Kim and A. L. Zubarev (2005), cond-mat/0502651.
[19] K. Huang and C. Yang, Phys. Rev. 105, 767 (1957).
[20] T. D. Lee and C. N. Yang, Phys. Rev. 105, 1119 (1957).
[21] D. S. Petrov, C. Salomon, and G. Shlyapnikov, Phys. Rev. Lett. 93, 090404 (2004).
[22] J. R. Engelbrecht, M. Randeria, and C. A. R. S. de Melo, Phys. Rev. B 55, 15153 (1997).
[23] A. J. Leggett, Modern Trends in the Theory of Condensed Matter (Springer-Verlag, Berlin, 1980).
[24] P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985).
[25] T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. 106, 1135 (1957).
[26] L. P. Pitaevskii and S. Stringari, Phys. Rev. Lett. 81, 4541 (1998).
[27] N. Manini and L. Salasnich, Phys. Rev. A 71, 033625 (2005).
[28] R. Combescot and X. Leyronas, Europhys. Lett. 68, 762 (2004).
[29] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini, Phys. Rev. Lett. 93, 200404 (2004).
[30] I. V. Brodsky, A. V. Klaptsov, M. Y. Kagan, R. Combescot, and X. Leyronas (2005), cond-mat/0507240.
[31] M. Schick, Phys. Rev. A 3, 1067 (1971).
[32] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[33] M. Girardeau, J. Math. Phys. (N.Y.) 1, 516 (1960).
[34] F. Calogero, J. Math. Phys. 10, 2191 (1969).
[35] B. Sutherland, J. Math. Phys. 12, 246 (1971).
[36] G. E. Astrakharchik, J. Boronat, J. Casulleras, and S. Giorgini (2004), cond-mat/0405225.
[37] M. Gaudin, Phys. Lett. 24A, 55 (1967).
[38] V. Y. Krivnov and A. A. Ovchinnikov, Sov. Phys. JETP 40, 781 (1975).
[39] C. N. Yang, Phys. Rev. Lett. 19, 1312 (1967).
[40] G. E. Astrakharchik, D. Blume, S. Giorgini, and L. P. Pitaevskii, Phys. Rev. Lett. 93, 050402 (2004).
[41] A. Recati, P. O. Fedichev, W. Zwerger, and P. Zoller, Phys. Rev. Lett. 90, 020401 (2003).