Dynamics of spherical gas bubbles in a cluster under an increase in the surrounding liquid pressure

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Abstract. The influence of the hydrodynamic interaction of spherical gas bubbles in linear, circular and ball-shaped clusters under a sudden increase in the surrounding liquid pressure is studied. The centers of the bubbles are located at the nodes of a uniform one-dimensional mesh on a segment of a straight line in the linear clusters, at the nodes of a uniform two-dimensional mesh inside a circular region of a plane in the circular clusters and at nodes of a uniform three-dimensional mesh inside a spherical region in the ball-shaped clusters. Initially, the liquid and the bubbles are at rest, the liquid pressure is 1 bar, the bubble radius is 0.25 mm, and the size of the mesh cells is 5 mm. The pressure rises by 0.8 bar. A mathematical model is applied, in which the dynamics of bubbles is described by the second-order ODEs in the radii of the bubbles and the position-vectors of their centers. It is shown that the radial oscillations of the bubbles in the clusters are significantly different from those of a single bubble. In particular, the period of their oscillations is longer, their damping is non-monotonic (with beats). The amplitudes of the oscillations and their beats are much greater in the central bubbles of the clusters than in the peripheral ones. The process of decay of the radial oscillations of the central bubbles in the circular and ball-shaped clusters begins with the phase of their amplification, during which the bubble pressure maximum becomes higher than the maximum achieved during the oscillations of a single bubble. Moreover, with growing number of bubbles in these clusters, the rate of damping of their oscillations decreases, and the oscillation beat frequency of both the central and peripheral bubbles increases.

1. Introduction
It is well known that the hydrodynamic interaction between bubbles can have a significant effect on their dynamics. For example, as a result of interaction, the bubbles can be attracted and repulsed [1], compressed more strongly [2–4], can move [5–8], deform [9–12], form stable configurations [5, 7, 13–15], increase the damage to surfaces of closely spaced solids [16, 17], etc. All this is of great practical importance. The most studied to date is the interaction between two bubbles, which are the simplest case of a bubble cluster [5, 6, 12, 18].

In this work, we study the influence of hydrodynamic interaction between spherical bubbles on their dynamics, depending on the type of the cluster (linear, circular, and ball-shaped ones), the number of bubbles forming it, and their position in the cluster (in the center, at the periphery) with an instantaneous increase in the liquid pressure. Similar studies for the case of a sudden growth in the liquid pressure were performed in [4, 19, 20]. Clusters of two structures were considered in [4]. The first structure was formed by three equally spaced spherical layers of bubbles located at the vertices of three concentric equally-oriented dodecahedrons (i.e., regular polyhedrons with 20 vertices), in the
center of which there was one more bubble. In the second structure, the bubbles were distributed stochastically inside a spherical volume, with a limited minimum initial distance between the bubbles. In this structure, one of the bubbles was in the center of the spherical volume. The clusters in [19] consisted of bubbles, one of which was located in the center, and the others at the vertices of one or a few regular polyhedrons nested inside each other. The clusters in [20] were formed by the bubbles located at the nodes of a uniform cubic mesh inside a spherical volume.

It should be noted that in the works [4, 19, 20], the bubbles were assumed spherical. The calculations by the authors of the present work (using the model of the dynamics of interacting bubbles [21], which takes into account the distortion of their spherical shape) showed that the hypothesis adopted in [4, 19, 20] that the interacting bubbles remain close to spherical is incorrect. Therefore, the results presented in those works can only be considered as model ones. In contrast to this, under the conditions considered in the present work, all bubbles during the entire interaction process become only weakly nonspherical, which was checked using the model of [21].

2. Problem statement

The dynamics of spherical gas (air) bubbles in linear (1D), circular (2D) and ball-shaped (3D) clusters (figure 1) under sudden increase in the pressure of the surrounding liquid (water-glycerin mixture) is considered. Initially (up to time $t = 0$) the bubbles and the liquid are at rest, the radii of the bubbles $R_k$ ($k$ is the bubble number, $1 \leq k \leq K$, $K$ is the number of bubbles in the cluster) are equal to $R_0 = 0.25$ mm, the liquid pressure $p_L$ is equal to 1 bar. In the linear cluster, the centers of the bubbles are located at the nodes of a uniform one-dimensional mesh on a segment of a straight line. In the circular cluster, the centers of the bubbles are at the nodes of a uniform two-dimensional mesh inside the circular region of a plane. In the ball-shaped cluster, the centers of the bubbles are located at the nodes of a uniform three-dimensional mesh inside a spherical region. In all the clusters, the distance between the centers of the nearest bubbles is $d_0 = 5$ mm. At $t = 0$, the liquid pressure $p_L$ rises instantly to 1.8 bar.

![Figure 1](image-url)

**Figure 1.** (a) linear, (b) circular and (c) ball-shaped clusters.

The main attention is directed to the influence of the hydrodynamic interaction between the bubbles on their dynamics, depending on the number of the bubbles in the cluster, the type of the cluster (linear, circular, or ball-shaped ones) and the position of the bubbles in the cluster (in the center, at the periphery). For this purpose, the dynamics of the bubbles in the clusters is compared with that of a single bubble (under the similar conditions).

A mathematical model is used, in which the dynamics of bubbles in a cluster is governed by a system of the second-order ordinary differential equations in the radii of the bubbles $R_k$ and the position-vectors of their centers $p_k$. These equations are a special case (the case of spherical bubbles) of the equations of [21] and are written as follows
\[
R_k \dot{R}_k + \frac{3}{2} \ddot{R}_k - \frac{\dot{p}_k^2}{4} - \frac{p_{bb} - p_L}{\rho_L} + \frac{2\sigma}{\rho_L R_k} + \psi_{ok} + \Delta_k = 0
\]

\[
= \sum_{j=1, j\neq k}^K \left[ \frac{\dot{B}_{0j}}{d_{kj}} - \frac{R_j^2 \dot{p}_{kj} + 5 \dot{R}_j \dot{p}_{kj}}{2d_{kj}^3} \right] + \sum_{l=1, l\neq k}^K \frac{3B_{0j} B_{0l} \dot{p}_{kl}}{4d_{kl}^3 d_{kl}^3} \left( \frac{R_j^2 \ddot{R}_l - 2R_l \dot{R}_l \dot{R}_j}{R_k} \right) \left( \frac{R_j^2 \dot{p}_{kj}}{R_k} \right) \left( \frac{R_j^2 \dot{p}_{kj}}{R_k} \right) \left( \frac{R_j^2 \dot{p}_{kj}}{R_k} \right) \left( \frac{R_j^2 \dot{p}_{kj}}{R_k} \right) \left( \frac{R_j^2 \dot{p}_{kj}}{R_k} \right)
\]

Here the overdots and the primes mean differentiation with respect to time \( t \), \( B_{0k} = -R_k^2 \dot{R}_k \), \( \dot{p}_{kj} \) is the pressure in the \( k \)-th bubble, \( \dot{p}_L \) is the liquid density (\( \rho_L = 1156 \text{ kg/m}^3 \)), \( \sigma \) is the surface tension (\( \sigma = 0.07 \text{ N/m} \)), \( \psi_{ok} \), \( \psi_{ik} \), \( \Delta_k \) are the corrections taking into account the influence of the liquid compressibility, \( \Delta_k \) is the correction for the influence of the liquid compressibility. Equations (1), (2) are valid for not too small distances between the bubbles, otherwise their deformations must be taken into account.

It is taken that the pressure in the bubbles changes according to the adiabatic law

\[
p_{bl} = \left( \frac{p_L + 2\sigma}{R_0} \right)^\gamma,
\]

where \( \gamma \) is adiabatic exponent (\( \gamma = 1.4 \)).

The effects of the liquid viscosity and compressibility are assumed small and are described without taking into account the interaction between bubbles, so that the corrections \( \psi_{ok} \), \( \psi_{ik} \), and \( \Delta_k \) are determined by the following expressions

\[
\psi_{ok} = -\frac{4\nu_L \dot{R}_k}{R_k}, \quad \psi_{ik} = \frac{18\nu_L \dot{p}_k}{R_k}, \quad \Delta_k = -\frac{R_k}{c_L} \left( 4R_k \ddot{R}_k + \frac{R_k^2 - p_L}{\rho_L} \right) = \frac{R_k}{c_L} \left( \dot{p}_{bb} - \dot{p}_L \right) = \frac{4\nu_L \dot{R}_k}{R_k}.
\]

where \( \nu_L = \mu_L / \rho_L \) is the kinematic viscosity of the liquid (\( \mu_L = 0.011 \text{ Pa·s} \)), \( c_L \) is the speed of sound in the liquid (\( c_L = 1500 \text{ m/s} \)).

Equations (1), (2) are solved numerically by the Runge-Kutta method with a variable integration step.

3. Results

The results presented below were obtained using the model of the dynamics of interacting bubbles (1), (2), in which the bubbles are assumed purely spherical. To check the validity of this assumption, additional calculations were made using the model of [21], in which the bubbles are considered as weakly nonspherical. The analysis showed that under the conditions considered in the present work, the hypothesis about the sphericity of bubbles is acceptable.

3.1. Features of the dynamics of bubbles in a cluster

Figure 2 illustrates the typical features of the influence of hydrodynamic interaction between bubbles on their dynamics in the clusters under the considered conditions. A circular cluster with 13 bubbles is
taken for illustration. In this and other figures, \( t^* = R_0 \sqrt{\rho_L / \rho_0} \approx 2.7 \times 10^{-5} \text{s} \) is the characteristic time, and \( p_b \) is the maximum pressure attained in a single bubble.

**Figure 2.** Damping of the pressure oscillations (a) in a single bubble and in the (b) central and (c) peripheral bubbles of a 2D cluster with \( K = 13 \). In (b), the inset shows the pressure change in the vicinity of the first collapse of the central (solid line) and peripheral (dashed line) bubbles of the cluster, as well as of the single bubble (dotted line).

It is seen in figure 2 that the pressure oscillations in a single bubble decay monotonically. Within model (1), (2), the damping of the oscillations of a single bubble is due to the influence of the liquid compressibility (acoustic radiation) and viscosity. The interaction between bubbles significantly changes the oscillation pattern. In particular, the bubbles in the cluster have an oscillation period longer than that of a single bubble. The damping of their oscillations is nonmonotonic, with beats with a frequency of about 1/6–1/7 of the frequency of the radial oscillations. The beats of the pressure oscillations in the central and peripheral bubbles are in antiphase. In the peripheral bubble, the damping begins with a phase of a more rapid decrease in the oscillation amplitude, while in the central bubble, it is preceded by a phase of more intensive oscillations. In the phase of intensifying oscillations, not only the maximum pressure increases, but also the minimum pressure decreases. The first collapse of the bubbles in the cluster is somewhat delayed (the collapse of the central bubble is delayed slightly more). In doing so, the central bubble collapses more slowly than the peripheral one, and the peripheral bubble more slowly than the single one.

### 3.2. Influence of the quantity of bubbles in a 1D cluster

Figure 3 demonstrates the influence of the hydrodynamic interaction between bubbles on their dynamics in the case of linear clusters. The clusters with 11, 23, and 57 bubbles are taken for demonstration. It is seen in figure 3 that in the case of a linear cluster, varying the number of bubbles has nearly no effect on the dynamics of the central bubble. In the central and peripheral bubbles of the cluster, as in a single bubble, the phase of the oscillation amplification does not occur. Furthermore, the bubble pressure oscillations in the clusters decay non-monotonically, with beats more pronounced in the central bubble. The oscillations of the peripheral bubble pressure decay with an increase in the number of bubbles a little faster than the pressure oscillations of the central bubble do. In the initial interval, where the damping of the bubble pressure oscillations is quite significant, the rate of the damping for the peripheral bubble is much greater than for the central one, and with an increase in the number of bubbles, this difference slightly increases. During the first collapse, the pressure maxima for cluster bubbles and for a single bubble are approximately the same. The first collapse of the cluster bubbles ends a little later than that of a single bubble. With growing number of bubbles in the cluster, the change in the maximum pressure during the first collapse is very small, while the difference in the duration of the first collapse of the central and peripheral bubbles slightly increases.
3.3. Influence of the quantity of bubbles in a 2D cluster

Figure 4 characterizes the influence of the number of bubbles on the results of their hydrodynamic interaction in the case of circular clusters. The clusters with 9, 21, and 57 bubbles are presented.

Figure 4. Change of the pressure in the central bubble of a 2D (a) $K = 9$, (b) $K = 21$ and (c) $K = 57$. The white lines show the envelopes of the extreme values of the pressure in the peripheral bubbles. The insets give the evolution of the bubble pressure in the vicinity of the first collapse of the central (solid lines) and peripheral (dashed lines) bubbles in the cluster, as well as of that of a single bubble (dotted lines).

The main features of the bubble pressure oscillation decay in the 2D clusters considered were outlined above when discussing the results presented in figure 2. Figure 4 illustrates the dependence of these features on the number of bubbles. It is seen, in particular, that with an increase in the number of bubbles, the range of the oscillations increases, mainly due to an increase in the degree of compression of the bubbles. With a larger number of bubbles, the oscillation damping rate decreases, and the beat frequency of the oscillations increases for both the central and peripheral bubbles. In doing so, regardless of the number of bubbles, the beats of the pressure oscillations of the central and peripheral bubbles remain in antiphase. With an increase in the number of bubbles, the difference between the duration of the first collapse of the central and peripheral bubbles and the duration of the first collapse of a single bubble, as well as the attained extreme pressures, increasingly grow. Along with that, the duration of the first collapse of the peripheral bubble increasingly exceeds the duration of the collapse of a single bubble, and the duration of the collapse of the central bubble more and more exceeds the
duration of the collapse of the peripheral bubble. With rising number of bubbles, the extreme pressures during the first collapse of the central and peripheral bubbles become increasingly higher and lower, respectively, than that during the first collapse of a single bubble.

3.4. Influence of the quantity of bubbles in a 3D cluster

Figure 5 shows the influence of the hydrodynamic interaction between bubbles on their dynamics in the case of ball-shaped clusters. The clusters with 7, 19, and 57 bubbles are taken for illustration. One can see in figure 5 that the main features of damping of the radial oscillations of bubbles in the 3D clusters considered are in many respects similar to those characteristic of the case of the 2D clusters (and significantly different from those typical for the case of the 1D clusters). The same applies to the dependence of these features on the number of bubbles. At the same time, the oscillation beat amplitude in the peripheral bubbles in the case of the 3D clusters is noticeably smaller than in the case of the 2D clusters. In addition, the tendencies (mentioned above when considering the 2D clusters) in the difference between the duration of the first collapse of a single bubble, the central and peripheral bubbles of the 2D clusters, as well as the trends in the difference between the corresponding extreme pressure values attained in the bubbles of the 2D clusters, in the case of the 3D clusters not only persist but are even slightly increasing. This, in particular, is evidenced by a comparison of the corresponding curves in figure 4c and figure 5c.

Figure 5. Damping of the pressure oscillations in the central bubble of a 3D cluster with (a) $K = 7$, (b) $K = 19$ and (c) $K = 57$. The white lines show the envelopes of the extreme values of the pressure in the peripheral bubbles. The insets give the evolution of the bubble pressure in the vicinity of the first collapse of the central (solid lines) and peripheral (dashed lines) bubbles in the cluster, as well as of that of a single bubble (dotted lines).

3.5. Extreme compression of bubbles in the clusters

Figure 6 illustrates the dependence of the extreme compression of bubbles in the linear, circular and ball-shaped clusters on the number of bubbles forming these clusters (in the range up to $K = 81$). It is seen in figure 6 that for the linear cluster, an increase in the number of bubbles has practically no effect on the pressure maxima in both the central and peripheral bubbles, while for the circular and ball-shaped clusters its effect is very significant. In particular, the extreme pressures attained in the central bubbles of the 2D and 3D clusters at the end of their first collapse increase monotonically and relatively uniformly. Along with that, the increase rate in the case of the 3D cluster is noticeably higher than in the case of the 2D cluster. On the other hand, the maximum pressures attained in the central bubble during the entire process of decay rises with growing number of bubbles in the case of the 2D and 3D clusters not so uniformly. Moreover, as the number of bubbles in the 3D clusters rises, the pressure maximum first quite rapidly increases, then is sharply reduced and after that rather strongly decreases (at $K = 81$) to a value lower than that in the case of the 2D clusters. The latter is due to the fact that at $K = 81$ the beats are rather disorderly.
Figure 6. Influence of the number of bubbles in clusters on the maximum pressure (referred to the maximum pressure in a single bubble) attained in the (a, b) central and (c) peripheral bubbles of the 1D, 2D and 3D clusters (a, c) at the end of their first collapse and (b, c) during the whole process of oscillations.

For the peripheral bubbles of the 2D and 3D clusters, the pressure maxima are reached at the end of the first collapse. An increase in the number of bubbles leads to their much weaker change than the variation of the pressure maximum in the central bubbles. With an increase in the number of bubbles, the maximum pressure in the peripheral bubbles of the 2D and 3D clusters tends to decrease.

4. Conclusions
The dynamics of the central and peripheral gas bubbles in linear, circular and ball-shaped clusters and the dynamics of a similar single bubble under a sharp increase in liquid pressure have been compared. The centers of the bubbles are located at the nodes of a uniform one-dimensional mesh on a segment of a straight line in the linear cluster, at the nodes of a uniform two-dimensional mesh inside a circular region of a plane in the circular cluster, and at nodes of a uniform three-dimensional mesh inside a spherical region in a ball-shaped cluster. The number of bubbles in the clusters is varied from 3 to 81.

It is shown that for the bubbles in the clusters under consideration, compared to a single bubble, the bubble pressure oscillation period is longer, the damping is non-monotonous (with beats), and the first collapse is slightly delayed.

In the central bubbles of the clusters, the amplitude of the oscillations and the amplitude of their beats are much greater than in the peripheral bubbles.

The process of decay of radial oscillations of the central bubbles in the circular and ball-shaped clusters begins with the phase of their amplification, during which the bubble pressure maximum becomes even higher than the maximum achieved during the oscillations of a single bubble. In the case of the linear cluster, the initial phase with such an increase in the compression of the central bubble does not arise.

With growing number of bubbles, the rate of decay of their oscillations in the circular and ball-shaped clusters decreases, and the oscillation beat frequency of both the central and peripheral bubbles increases. In addition, in all the clusters under consideration, the first collapse of the peripheral bubble becomes increasingly more prolonged than the collapse of a single bubble, and the collapse of the central bubble is increasingly more prolonged than the collapse of the peripheral bubble. With rising number of bubbles in the case of the circular and ball-shaped clusters, the difference between the bubble pressure maxima achieved during the first collapse increases. At the same time, in comparison with a single bubble, the maximum pressure in the peripheral bubble is increasingly reduced, and that in the central bubble, on the contrary, increasingly grows.

References
[1] Bjerknes V F K 1906 Field of Force (New York: Columbia Univ. Press)
[2] Van Wijngaarden L 1966 On the collective collapse of a large number of gas bubbles in water
[3] Wang Y-C and Brennen C E 1999 Numerical Computation of Shock Waves in a Spherical Cloud of Cavitation Bubbles Journal of Fluids Engineering 121 4 pp 872–80

[4] Gubaidullin A A and Gubkin A S 2015 Peculiarities of the dynamic behavior of bubbles in a cluster caused by their hydrodynamic interaction Thermophysics and Aeromechanics 22 4 pp 453–62

[5] Doinikov A A 2001 Translational motion of two interacting bubbles in a strong acoustic field Phys. Rev. E 64 2 026301

[6] Harkin A, Kaper T J and Nadim A 2001 Coupled pulsation and translation of two gas bubbles in a liquid Journal of Fluid Mechanics 445 pp 377–411

[7] Lauterborn W and Kurz T 2010 Physics of bubble oscillations Rep. Prog. Phys. 73 106501

[8] Aganin A A and Davletshin A I 2016 A refined model of spatial interaction of spherical gas bubbles Izvestia Ufimskogo Nauchnogo Tsentra RAN (Proceedings of the RAS Ufa Scientific Centre) 4 pp 9–13

[9] Chahine G L and Duraiswami R 1992 Dynamical Interactions in a Multi-Bubble Cloud Journal of Fluids Engineering 114 4 pp 680–6

[10] Blake J R, Robinson P B, Shima A and Tomita Y 1993 Interaction of two cavitation bubbles with a rigid boundary J Fluid Mech. 255 pp 707–21.

[11] Bremond N, Arora M, Dammer S M and Lohse D 2006 Interaction of cavitation bubbles on a wall Physics of Fluids 18 12 121505

[12] Aganin A A and Davletshin A I 2009 Simulation of interaction of gas bubbles in a liquid with allowing for their small asphericity Matematicheskoe modelirovanie 21 6 pp 89–102

[13] Pelekasis N A, Gaki A, Doinikov A and Tsamopoulos J A 2004 Secondary Bjerknes forces between two bubbles and the phenomenon of acoustic streamers Journal of Fluid Mechanics 500 pp 313–47

[14] Konovalova S and Akhatov I S 2005 Structure formation in acoustic cavitation Multiphase Sci. Technol. 17 pp 343–71.

[15] Aganin A A and Davletshin A I 2013 Interaction of spherical bubbles with centers located on the same line Matematicheskoe modelirovanie 25 12 pp 3–18

[16] Pearsall I S 1972 Cavitation (London: Mills and Boon Limited)

[17] Philipp A and Lauterborn W 1998 Cavitation erosion by single laser-produced bubbles J. Fluid Mech. 361 pp 75–116

[18] Pandey V 2019 Asymmetry and sign reversal of secondary Bjerknes force from strong nonlinear coupling in cavitation bubble pairs Phys. Rev. E 99 042209

[19] Aganin I A and Davletshin A I 2020 Dynamics of interacting bubbles located in the center and vertices of regular polyhedra Journal of Physics: Conference series (Scopus) 1588 012001

[20] Aganin I A and Davletshin A I 2020 Dynamics of gas bubbles inside a ball-like area at the nodes of a uniform cubic mesh under sudden liquid pressure rise Lobachevskii Journal of Mathematics 41 7 pp 1148–54

[21] Aganin A A and Davletshin A I 2018 Equations of interaction of weakly non-spherical gas bubbles in liquid Lobachevskii Journal of Mathematics 39 8 pp 1047–52