Modelling a torus cloud employing the van Wijngaarden Ansatz and the Gilmore equation

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Abstract. The present work analyses the dynamics of cloud cavitation in a compressible flow. Recent experiments highlight that a common cloud geometry for cavitation clouds is a horseshoe. This paper presents a calculation method describing the dynamics of a cavitation cloud applying analytical methods. With the Helmholtz vortex theorem in mind, the horseshoe is artificially completed to be a generic torus shaped cloud. Following van Wijngaarden, the mixture of cavitation bubbles and liquid inside the cloud is treated as a continuous medium, i.e. a homogenous model with the void fraction $\alpha$ and the single bubble radius $R$. By doing so, the bubble radii are a function of the radial position inside the cloud and time only. The individual bubble dynamics depend on the given dynamic and kinematic boundary conditions of the model. A pressure history at infinity and a strain rate or circulation represent these two types of excitations. The problem results in a system of non-linear differential equations consisting of the Gilmore equation, the continuity equation and the Euler equation. The Gilmore equation describes the dynamics of the individual bubbles inside the cloud whereas the continuity and Euler equations describe the movement and interaction of the bubbles. The resulting pressure and the bubble radii inside the cloud as well as the cloud radius are highlighted. They depend on the excitation of the cloud, the Mach number $M$, the Reynolds number $Re$, the Weber number $We$ and the polytropic index $k$.

1. Introduction
Depending on the Reynolds and the cavitation number, the cavitation regime transitions from sheet to cloud cavitation, as Pelz et al. [1] describe. This effect was observed in our experiments in a convergent-divergent nozzle [2]. With the Helmholtz vortex theorem in mind, a torus is considered to be a common cloud geometry inside turbomachines (cf. Kawanami et al., 1998 [3]; Keil et al., 2012 [4]; Pelz et al., 2014 [5]). The present work analyses the dynamics of a torus cloud employing the van Wijngaarden ansatz [6]. This method combines the continuity equation, the Euler equation and the Gilmore bubble equation to obtain a solution for a bubbly cloud. The flow inside the cloud is treated quasi one-dimensional leading to a single phase model introducing the share of vapour $\alpha$. A potential flow models the flow outside the cloud. The cloud is composed of individual bubbles whose radii are a continuous function of space inside the cloud and time. Furthermore, the pressure distribution inside the cloud is also a continuous function of space and time. Applying initial and boundary conditions to the flow, the complete cloud characteristics are analyzed. Hence this study generates a deeper understanding of the process of cloud cavitation and is applicable to numerical methods solving cavitating flow problems.
Figure 1. Idealized torus geometry for the interior of the cloud.

Other than previous investigations [7], this analysis takes the compressibility of the liquid into account.

2. Torus cloud model

The torus shown in Figure 1 is an idealized cloud geometry, which permits to investigate the dynamic behaviour of a torus cloud. Two geometrical quantities describe the torus. These are the major radius $\tilde{R}_T$, and the minor radius $\tilde{R}_{Cl}$. In the following analysis, quantities with a tilde are dimensional quantities and without tilde are dimensionless. As a generic model we consider the cloud in an infinite ambiance. Hence each cross section of the cloud behaves equally which leads to a quasi one dimensional problem.

It is important to mention: In contrast to many other approaches the entire flow (liquid and gas-phase) is treated in the framework of continuum mechanics. The flow field is divided into a field outside the cloud, in which cavitation is excluded and the compressible cloud interior.

In the interior of the cloud a mixture of vapor bubbles of radius $\tilde{R}(\tilde{r}, \tilde{t})$ with spatial and time coordinates $\tilde{r}$ and $\tilde{t}$, respectively, and inviscid fluid of constant density is assumed. The flow of this dispersion is modeled as a homogeneous medium neglecting the relative motion between the phases. This assumption is justified provided (i) the bubble size $\tilde{R}$ is small compared to the cloud size $\tilde{R}_{Cl}$, and (ii) the bubble density is small. The density of the mixture is then given by the void fraction $\alpha$, the densities of the vapor within the bubbles $\tilde{\rho}_V$ and the liquid density $\tilde{\rho}_L$:

$$\tilde{\rho} = \alpha \tilde{\rho}_V + (1 - \alpha) \tilde{\rho}_L \approx (1 - \alpha) \tilde{\rho}_L.$$  

(1)

The non-equilibrium bubble dynamics, which is essential to study the processes of cloud collapse, requires a description of the density on the geometry of the individual bubbles. This is achieved by a coupling the Gilmore bubble equation to mass and momentum balance equations for the mixture (cf. van Wijngaarden, 1964 [6]).

With the bubble population $\tilde{\eta}$ that is the number of bubbles per volume of liquid, the void fraction can be expressed by the bubble volume $\tilde{V}$ which is a continuous spatial function:

$$1 - \alpha = \frac{1}{1 + \tilde{\eta} \tilde{V}} = \frac{\tilde{\rho}}{\tilde{\rho}_L} = \varrho.$$  

(2)
One assumes that no bubbles will be created or destroyed, which is equivalent to a vanishing change of bubble population along the particle path, hence \( \alpha = \alpha(\hat{R}) \). In the following, the quantities are non-dimensionalized using the liquid density \( \hat{\rho}_L \), the equilibrium bubble radius \( \hat{R}_0 \) as length, the mean channel velocity \( \hat{U} \) and therefore \( \hat{R}_0/\hat{U} \) as time. This leads to a dimensionless radial coordinate \( r = \hat{r}/\hat{R}_0 \) and time \( t = \hat{t}\hat{U}/\hat{R}_0 \). The flow within the cloud interior is not divergence free and not irrotational. The mixture is assumed to be inviscid but not the surrounding of the single bubble.

Unknown quantities of the problem are the individual bubble radii \( R = R(r,t) \) and the pressure inside the cloud \( C_p := 2(\hat{p} - \hat{p}_\infty)/\hat{\rho}_L\hat{U}^2 = C_p(r,t) \). The spacial distribution \( r = r(t) \) itself is a function of time, as the cloud changes in size due to the dynamic and kinematic excitation.

The system of equations is closed by van Wijngaarden’s continuums-mechanical form of the Gilmore equation. This form follows from an application of the Gilmore equation along the particle path and couples the pressure resulting from the bubble motion to the momentum equation and the bubble size to the mixture density:

\[
R\hat{R} \left( 1 - M\hat{R} \right) + \frac{3}{2} \hat{R}^2 \left( 1 - \frac{M}{3}\hat{R} \right) = H \left( 1 + M\hat{R} \right) + MR \left( 1 - M\hat{R} \right) \hat{H},
\]

with the Mach number \( M \) and the pressure coefficient \( H := \int_0^\infty d\hat{p}/\hat{p} \). Here, \( \hat{X} \) represents the derivative of the function \( X \) in time \( t \). The coefficient \( \hat{H} \) is a function of the bubble radii \( R \), the dimensionless pressure inside the cloud \( C_p \) and the dimensionless pressure at the bubble surface

\[
C_p = \frac{2(\hat{p} - \hat{p}_\infty)}{\hat{\rho}_L\hat{U}^2} = \sigma(1 - R^{-3k}) - \frac{8}{Re} \frac{\hat{R}}{R} - \frac{4}{We}(R^{-1} - R^{-3k})
\]

and is solved applying Tait’s law (Gilmore 1952, [8]). Equation 4 introduces the dimensionless parameters cavitation number \( \sigma \), Reynolds number \( Re \), Weber number \( We \) and polytropic index \( k \) which are constant.

To describe the cloud interior, i.e. its pressure \( C_p \), bubble radii \( R \) and coordinate \( r \), two additional equations are necessary to close the system of equations. The missing equations are the Euler equation

\[
\hat{u} = \hat{r} = -\frac{1}{2(1 - \alpha)} \frac{\partial C_p}{\partial \hat{r}}
\]

and the continuity equation

\[
\hat{\alpha} = (1 - \alpha) \frac{\partial u}{\partial \hat{r}}.
\]

It is important to note that the non-linear coupled system of equations (3), (5), (6) represents an ordinary system of equations for the variables \( R, C_p, r \) in time \( t \). A first order finite difference method describes the derivatives in space \( r \). A variable-step, variable-order solver based on the numerical differentiation formulas of order 1-5 solves the system of equations (3), (5), (6). The solver discretizes the inner part of the cloud into \( N \) supporting points. By doing so, the components of the initial vector \( r(t = 0) \) increase linearly from \( r(t = 0) \) at \( i = 1 \) to \( r(t = 0) \) at \( i = N = R_{CI} \).

3. Boundary conditions

To solve the system of equations highlighted in section 2 additional boundary conditions are required. These boundary conditions primary influence the behaviour of the cavitation cloud. In the following these boundary conditions are introduced as dynamic and kinematic excitations. Dynamic excitations feature pressure excitations and kinematic excitations represent imposed strain rates or circulation.
Boundary conditions affect the outer streamline of the cloud, i.e. the cloud radius $R_{C_l} = r|_{i=N}$ and its radial velocity $\hat{R}_{C_l} = u|_{i=N}$. The dynamic excitation is imposed on the cloud by a dimensionless pressure

$$C_{p,\infty} = \frac{C_{p,\min}}{2} \left[ 1 - \cos \left( \frac{2\pi t}{t_E} \right) \right] \quad \text{for } \frac{t}{t_E} \leq 1,$$

$$C_{p,\infty} = 0 \quad \text{for } \frac{t}{t_E} > 1$$

(7)

at infinity, with the excitation time $t_E$. A minimum excitation pressure of $C_{p,\min} = -1.5$ is used. The excitation function is harmonic and falls below the dimensionless vapor pressure $-\sigma$, reaching its minimum at $t = t_E/2$.

The kinematic excitation is applied at $t = 0$ by an initial circulation $\Gamma$ or at $t \geq 0$ by a continuously imposed strain rate $\dot{\varepsilon}$. Applying a circulation, the cloud follows a rigid body circulation

$$u_{\varphi 0} = \frac{\Gamma}{2\pi R_{C_l 0}^2}.$$  

(8)

Hence, the rotation inside the cloud is constant at $t = 0$ and the rotation outside the cloud equals zero.

Another kinematic boundary condition is the strain rate. The applied strain rate must be in accordance with the Euler equation and hence reads

$$\varepsilon(t) = \ln \left( (\dot{\varepsilon} - 1) \frac{t}{t_E} + 1 \right) \quad \text{for } \frac{t}{t_E} \leq 1,$$

$$\varepsilon(t) = \dot{\varepsilon} \quad \text{for } \frac{t}{t_E} > 1$$

(9)

with the constant maximum strain $\dot{\varepsilon} = \dot{\varepsilon}(\dot{\varepsilon}_0, t_E)$ being a function of the initial strain rate and the excitation time.

These boundary conditions affect the outer shell of the cavitation cloud i.e. the pressure at the $N$th element of the unknown pressure vector $C_{p}$. For a spherical cloud, the Bernoulli equation solves the boundary condition, leading to

$$C_{p,i=N} - C_{p,\infty} = 2 \left( R_{C_l} \hat{R}_{C_l} + \frac{3}{2} \frac{\hat{R}_{C_l}^2}{R_{C_l}} \right).$$

(10)

As there is no potential for a closed vortex ring, the Bernoulli equation does not yield a surface pressure. To derive the pressure, integrating the Euler equation reads

$$C_{p,i=N} - C_{p,\infty} = 2 \int_{N}^{\infty} \frac{\partial \hat{u}}{\partial t} \cdot d\bar{x} - u_{\varphi N}^2 = 2 \int_{N}^{\infty} \frac{\partial \hat{u}}{\partial t} \cdot d\bar{x} - R_{C_l}^2 - \frac{1}{4\pi^2 R_{C_l}^2}.$$  

(11)

Equation 11 describes the surface pressure of the cloud for the irrotational outer flow. It is important to note that for this part of the analysis the flow is three-dimensional. Here, the velocity $\hat{u}$ and its derivative $\partial \hat{u}/\partial t$ are still unknown. To obtain a solution, this analysis separates $\hat{u}$ in two parts. First, a velocity $\hat{u}_s$ due to a singular source distribution inside the toroidal cloud. Second, a velocity $\hat{u}_\Gamma$ due to vorticity. The superposition of both velocities yields $\hat{u} = \hat{u}_s + \hat{u}_\Gamma$.

The potential at the location $\bar{x}$ surrounding a ring source with the specific strength $q = q(R_{C_l}, \varepsilon)$ depending on the cloud radius and the strain yields

$$\Phi_s(\bar{x}, t) = -q(R_{C_l}, \varepsilon) \int_{0}^{2\pi} \frac{R_{\Gamma}}{\bar{x} - \bar{x}'} \, d\theta,$$  

(12)
with the reference point of the source \( \vec{x}' \) and the notation following figure 1. Following its definition \( \vec{u}_s = \nabla \Phi_s \) calculates the corresponding velocity field. Lacking a potential for a closed vortex ring an analogous procedure to calculate \( \vec{u}_\Gamma \) is not applicable. Hence, this investigation uses the Biot-Savart law to calculate
\[
\vec{u}_\Gamma = \frac{\Gamma}{2\pi} \int_0^{2\pi} \frac{R_T \vec{e}_\theta \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d\theta.
\] (13)

By doing so, equations (11), (12), (13) give an expression for the boundary condition \( C_p|_{i=N} \) of a toroidal cloud, taking into account both dynamic and kinematic excitations. Here, the pressure \( C_p \) still is a function of the angle \( \varphi \). As within the scope of this analysis, a one-dimensional model describes the torus geometry, averaging \( C_p \) yields
\[
\bar{C}_p|_{i=N} = \frac{1}{2\pi} \int_0^{2\pi} C_p|_{i=N} \, d\varphi
\] (14)
as boundary condition of a toroidal cloud and its one-dimensional model.

The main focus of this investigation is presenting the model itself. In the following figure 2 highlights the radius and pressure distribution inside the cloud for a given set of excitation parameters. Until the collapse at \( t/t_E \approx 1.6 \) the individual bubbles grow synchronously. At the collapse the pressure \( C_p \) reaches its maximum value. As the outer bubbles are the first to collapse, a shock wave propagates into the cloud. For \( t/t_E > 1.6 \) the model shows an asynchronous behaviour of the bubbles inside the cloud. Future studies will investigate the influence of different parameters on the dynamics of the cavitation cloud.

**Figure 2.** Bubble radius \( R \) and pressure \( C_p \) inside the cloud for \( Re = 20, k = 1.4, We = 200, \sigma = 1, M = 10^{-2}, \alpha_0 = 10^{-2}, \beta_0 = 2, \beta_E = 10, C_{p,\text{min}} = -1.5, \dot{\varepsilon} = 0, \Gamma = 0. \)

### 4. Time scales

The problem contains three time scales influencing the properties of the system. The first time scale is the excitation time \( t_E \) being introduced in section 3. The second and third time scales...
are the typical time of an individual bubble and the cloud,

\[ t_B = \frac{2\pi}{\omega_B} = 2\pi \left[ \frac{3k}{2} \frac{\sigma}{\rho} + (3k - 1) \frac{2}{W_e} \right]^{-1/2}, \]  
\[ t_{Cl} = \frac{R_{Cl}}{\omega_B} \sqrt{3\alpha_0(1 - \alpha_0)}, \]

given by the natural bubble frequency (Plesset & Prosperetti, 1977 [9]) and by the phase velocity of the dispersion, with the initial void fraction \( \alpha_0 \).

The relation between these time scales influence the dynamic behaviour of the cavitation cloud. With three time scales one obtains two linearly independent dimensionless relations. As a first relation the interaction parameter

\[ \beta_0 := R_{Cl0} \alpha_0 (1 - \alpha_0) \sim \left( \frac{t_{Cl}}{t_B} \right)^2 \]  

is used, cf. Brennen 2005 [10]. This parameter influences and specifies the initial cloud radius \( R_{Cl0} \). The second relation describes the excitation related to the typical time of the cloud

\[ \beta_E := \frac{t_E}{t_{Cl}}, \]  

By doing so an operating point depends on \( \beta_0 \) and \( \beta_E \) as well as the magnitude of the dynamic and kinematic excitations \( C_{p_{\text{min}}} \) and \( \hat{\epsilon}, \Gamma \) respectively.

5. Summary and Conclusion

This study presents a model analysing the dynamics of a toroidal cavitation cloud under the influence of an external dynamic excitation. The cloud is composed of smaller bubbles and treated as a continuous medium. Hence, the bubble radii \( R \) are a function of space and time inside the cloud. Van Wijngaarden’s continuum-mechanical form of the Gilmore equation including the Euler equation and the continuity equation solve this problem. By doing so, this model presents an analytical approach describing a cavitation cloud. Bubble radius \( R \), pressure \( C_p \) and spatial coordinate inside the cloud \( r \) depend on the applied dynamic and kinematic excitations.

Further investigations on this field such as analyzing dominating frequencies, influences of the share of vapor, the excitation characteristics or other parameters will provide additional data leading to a better understanding of cloud cavitation.

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