QUASI-PERIODIC SOLUTIONS FOR A CLASS OF BEAM EQUATION SYSTEM

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Abstract. In this paper, we establish an abstract infinite dimensional KAM theorem. As an application, we use the theorem to study the higher dimensional beam equation system

\[
\begin{align*}
    u_{1tt} + \Delta^2 u_1 + \sigma u_1 + u_1 u_2^2 &= 0 \\
    u_{2tt} + \Delta^2 u_2 + \mu u_2 + u_1^2 u_2 &= 0
\end{align*}
\]

under periodic boundary conditions, where \(0 < \sigma \in [\sigma_1, \sigma_2], 0 < \mu \in [\mu_1, \mu_2]\) are real parameters. By establishing a block-diagonal normal form, we obtain the existence of a Whitney smooth family of small amplitude quasi-periodic solutions corresponding to finite dimensional invariant tori of an associated infinite dimensional dynamic system.

1. Introduction. Both the CWB method [5, 6, 7, 8, 9, 10] and the infinite dimensional KAM theory [18, 21, 27] are important tools to study the existence of finite dimensional tori for Hamiltonian PDEs. The CWB method is a generalization of the Lyapunov-Schmidt reduction and the Newtonian method. The CWB method allows one to avoid explicitly using the Hamiltonian structure of the systems and can yield global solutions. The infinite dimensional KAM theory is the extension of classical KAM theory, its advantage is the construction of a local normal form in a neighborhood of the obtained solutions in addition to the existence of quasi-periodic solutions, the normal form method is used to understand the dynamics of the corresponding quasi-periodic solutions. Both the CWB method and the infinite dimensional KAM theory have been well developed for one dimensional Hamiltonian PDEs, see [1, 13, 14, 19, 20, 22, 24, 25, 26, 28, 29, 30] and the references therein.
In the case of higher dimensional PDEs, the first breakthrough is due to Bourgain [7, 8], he proved that the two dimensional Schrödinger equations admit small amplitude quasi-periodic solutions. The method was recently improved, Berti-Bolle [3, 4] proved the existence of quasi-periodic solutions for the wave equation and Schrödinger equations with finite differentiable nonlinearities and multiplicative potential on $T^d$ ($d \geq 1$). Geng-You [12] proved that the higher dimensional nonlinear beam equations and nonlocal Schrödinger equations admit small amplitude linearly stable quasi-periodic solutions. By a modified KAM theorem, Eliasson-Kuksin [11] constructed quasi-periodic solutions for more interesting higher dimensional Schrödinger equation. In 2011, Geng-Xu-You [12] proved an infinite dimensional quasi-periodic solutions for more interesting higher dimensional Schrödinger equation. In 2011, Geng-Xu-You [12] proved an infinite dimensional KAM theory, by which they obtained the existence of arbitrary finite dimensional invariant tori for a two dimensional nonlinear cubic Schrödinger equation under periodic boundary conditions, see also the recent generalization by Procesi-Xu [23] from two dimensional case to higher dimensional case.

But there is few result for quasi-periodic solutions of coupled system. Zhou [31] considers a class of higher dimensional nonlinear Schrödinger equation systems with real Fourier multiplier under periodic boundary condition:

$$\begin{cases}
-i\partial_t u = -\Delta u + M_\xi u + \partial_\xi G(|u|^2, |v|^2) \\
-i\partial_t v = -\Delta v + M_\sigma v + \partial_\sigma G(|u|^2, |v|^2)
\end{cases}$$

(1)

where $u = u(t, x), v = v(t, x), t \in \mathbb{R}, x \in \mathbb{T}_d, d \geq 2$. $G = G(a, b)$ is a real analytic function defined in a neighborhood of the origin in $\mathbb{R}^2$ with $G(0, 0) = \partial_\xi G(0, 0) = \partial_\sigma G(0, 0) = 0, M_\xi, M_\sigma$ are two real Fourier multipliers.

Grebert-Rocha [17] consider the system of coupled nonlinear Schrödinger equations on the torus

$$\begin{cases}
i\partial_t u + \partial_{xx} u = |v|^2 u, \\
i\partial_t v + \partial_{xx} v = |u|^2 v,
\end{cases}$$

(2)

They prove the above system exhibits both stable and unstable small KAM tori.

In this paper, we consider a class of higher dimensional nonlinear beam equation systems under periodic boundary condition:

$$\begin{cases}
u_{1tt} + \Delta^2 u_1 + \sigma u_1 + u_1 u_2^2 &= 0 \\
u_{2tt} + \Delta^2 u_2 + \mu u_2 + u_1^2 u_2 &= 0
\end{cases}$$

(3)

where $0 < \sigma \in \mathcal{N}_1 = [\sigma_1, \sigma_2], 0 < \mu \in \mathcal{N}_2 = [\mu_1, \mu_2]$ are real parameters, $u_1 = u_1(t, x), u_2 = u_2(t, x), t \in \mathbb{R}, x \in \mathbb{R}^d, d \geq 2$. Let $\phi_n(x) = \sqrt{1/(2\pi)^d} e^{i(n,x)}$ be eigenvectors of the operators $\Delta^2 + \sigma, \Delta^2 + \mu$ with periodic boundary conditions corresponding to eigenvalues $|n|^2 + \sigma, |n|^2 + \mu, n \in \mathbb{Z}^d$, respectively.

Similar to [16], the lattice points $\nu_1, \nu_2, \cdots, \nu_{b^*} \in \mathbb{Z}^d$ are called tangential sites in the following way: $\{\nu_1, \nu_2, \cdots, \nu_{b^*}\} \in \mathcal{J}$, where $\mathcal{J}$ is defined as follows:

$$\nu_j = (\nu_{j_1}, \cdots, \nu_{j_d}), \quad 1 \leq j \leq b^*,$$

$$\mathcal{J} = \left\{ (\nu_1, \nu_2, \cdots, \nu_{b^*}) : |\nu_{jk+1} | > 4d(\nu_{jk})^2, \quad 1 \leq k \leq d - 1, \right\}.$$  

Let $\mathcal{J} = \{i_1, i_2, \cdots, i_b\} \subset \mathcal{J}, \mathcal{S} = \{t_1, t_2, \cdots, t_b\} \subset \mathcal{J}, \ b, b^* \geq 2, b^* \geq b + \bar{b}, \mathcal{S} \cap \mathcal{S} = \emptyset.$
Now we state the main theorem as follows.

**Theorem 1.1.** Consider d-D beam equation system

\[
\begin{align*}
    u_{1tt} + \Delta^2 u_1 + \sigma u_1 + u_1 u_2^2 &= 0, \\
    u_{2tt} + \Delta^2 u_2 + \mu u_2 + u_1^2 u_2 &= 0.
\end{align*}
\]

(4)

\[ u_1(t, x_1 + 2\pi, \ldots, x_d) = \cdots = u_1(t, x_1, \ldots, x_d), \]

\[ u_2(t, x_1 + 2\pi, \ldots, x_d) = \cdots = u_2(t, x_1, \ldots, x_d), \]

where \( 0 < \sigma \in \mathcal{N}_1 = [\sigma_1, \sigma_2], \)

\[ 0 < \mu \in \mathcal{N}_2 = [\mu_1, \mu_2] \]

are real parameters. Then for fixed \( \mathcal{S}, \mathcal{S} \) and \( 0 < \gamma \ll 1, \)

there exists a Cantor subset \( \mathcal{O}_\gamma \subset \mathcal{N}_1 \times \mathcal{N}_2 \) with

\( \text{meas}(\mathcal{N}_1 \times \mathcal{N}_2 \setminus \mathcal{O}_\gamma) = O(\gamma^d), \)

such that for each \( (\sigma, \mu) \in \mathcal{O}_\gamma, \) the above nonlinear beam equation system admits a class of small amplitude quasi-periodic solutions of the form:

\[ u_1(t, x) = \sum_{j=1}^b u_{1j} (\omega_1 t, \ldots, \omega_b t) \phi_j(x), \quad u_2(t, x) = \sum_{j=1}^d u_{2j} (\tilde{\omega}_1 t, \ldots, \tilde{\omega}_d t) \phi_j(x), \]

where \( u_{1j} : \mathbb{T}^b \rightarrow \mathbb{R}, u_{2j} : \mathbb{T}^d \rightarrow \mathbb{R} \) and diophantine frequencies \( \omega = (\omega_1, \ldots, \omega_b), \tilde{\omega} = (\tilde{\omega}_1, \ldots, \tilde{\omega}_d) \) are close to the unperturbed frequencies \( (\sqrt{|i_1|^4 + \sigma}, \ldots, \sqrt{|i_b|^4 + \sigma}) \)

and \( (\sqrt{|i_1|^4 + \mu}, \ldots, \sqrt{|i_d|^4 + \mu}) \), respectively.

This paper is organized as follows: In Section 2, we give an infinite dimensional KAM theorem; in Section 3, we give its application to higher dimensional beam equation system. The proof of the KAM theorem is given in Sections 4, 5, 6.

2. An infinite dimensional KAM theorem for Hamiltonian partial differential equations. In this section, we introduce some notations and state an abstract KAM theorem. The KAM theorem can be applied to (4) to prove Theorem 1.1.

Given two sets \( \mathcal{S}, \tilde{\mathcal{S}} \subset \mathbb{Z}^d, d \geq 2, \)

\( \mathcal{S} = \{i_1, \cdots, i_b\}, \tilde{\mathcal{S}} = \{t_1, \cdots, \tilde{t}_d\}, b, \tilde{d} \geq 2, \)

\( \mathcal{S} \cap \tilde{\mathcal{S}} = \emptyset. \) Let \( \mathbb{Z}_d^b \) be the complementary set of \( \mathcal{S} \) in \( \mathbb{Z}^d \) and \( \mathbb{Z}_d^\tilde{d} \) be the complementary set of \( \tilde{\mathcal{S}} \) in \( \mathbb{Z}^d. \)

Denote \( q = (q_n)_{n \in \mathbb{Z}_d^d} \) with its conjugate \( \bar{q} = (\bar{q}_n)_{n \in \mathbb{Z}_d^d} \) and similarly \( p = (p_n)_{n \in \mathbb{Z}_d^d} \) with its conjugate \( \bar{p} = (\bar{p}_n)_{n \in \mathbb{Z}_d^d}. \) We introduce the weighted norm as follows:

\[ \|q\|_{a, \rho} = \sum_{n \in \mathbb{Z}_d^d} |q_n| |n|^{-\rho}, \quad \|p\|_{a, \rho} = \sum_{n \in \mathbb{Z}_d^d} |p_n| |n|^{-\rho}, \]

where \( |n| = \sqrt{n_1^2 + \cdots + n_d^2}, \) \( n = (n_1, \cdots, n_d) \in \mathbb{Z}_d^d \) for \( q \) and in \( \mathbb{Z}_d^d \) for \( p. \)

Denote a neighborhood of \( \mathbb{T}^b + \tilde{b} \times \{I = 0 \} \times \{J = 0 \} \times \{q = 0 \} \times \{\bar{q} = 0 \} \times \{p = 0 \} \times \{\bar{p} = 0 \} \)

by

\[ D(r, s) = \{ (\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}) : |\text{Im} \theta|, |\text{Im} \varphi| < r; |I|, |J| < s^2; \}

where \( |\cdot| \) means the sup-norm of complex vectors. Moreover, let \( \mathcal{O} \) be a compact subset of \( \mathbb{R}^2 \) of positive Lebesgue measure, and let \( C^{N,1}(\mathcal{O}) \) be \( N \)-order Lipschitz continuously differentiable function space; here the derivatives of function on \( \mathcal{O} \) are understood in the sense of Whitney, so the space \( C^{N,1}(\mathcal{O}) \) is also understood in the sense of Whitney.
Let $\alpha = \{\alpha_n\}_{n \in \mathbb{Z}_+^2}, \beta = \{\beta_n\}_{n \in \mathbb{Z}_+^2}, \tilde{\alpha} = \{\tilde{\alpha}_n\}_{n \in \mathbb{Z}_+^2}, \tilde{\beta} = \{\tilde{\beta}_n\}_{n \in \mathbb{Z}_+^2}, \alpha_n, \beta_n, \tilde{\alpha}_n, \tilde{\beta}_n \in \mathbb{N}$ with only finitely many non-zero components of positive integers. Denote $q^{\alpha} \bar{q}^{\tilde{\alpha}} = \prod_{n \in \mathbb{Z}_+^2} q^{\alpha_n} \bar{q}^{\tilde{\alpha}_n}, \bar{p}^{\beta} = \prod_{n \in \mathbb{Z}_+^2} \tilde{p}^{\beta_n}$ and let

$$F(\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}) = \sum_{k, l, \alpha, \beta, k, l, \tilde{\alpha}, \tilde{\beta}} F_{k\alpha\beta, l\tilde{\alpha}\tilde{\beta}}(\sigma, \mu)e^{i(k, \theta) + (\tilde{k}, \varphi)} I^l \bar{J}^i \bar{q}^{\beta} \bar{p}^{\tilde{\beta}},$$

where $(\sigma, \mu) \in \mathcal{O} \subset \mathcal{N}_1 \times \mathcal{N}_2$ is the parameter set, $F_{k\alpha\beta, l\tilde{\alpha}\tilde{\beta}}$ belongs to $C^{N,1}(\mathcal{O})$ in the parameters $\sigma$ and $\mu$. $k = (k_1, \ldots, k_b) \in \mathbb{Z}_+^b, \tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_b) \in \mathbb{Z}_+^b$, $l = (l_1, \ldots, l_b) \in \mathbb{N}_+^b, l = (l_1, \ldots, \tilde{l}_b) \in \mathbb{N}_+^b, I^l = I_1^{l_1} \cdots I_b^{l_b}, J^i = J_1^{i_1} \cdots J_b^{i_b}$. Denote the weighted norm of $F$ by

$$\|F\|_{D(r,s), \mathcal{O}} = \sup_{(\sigma, \mu) \in \mathcal{O}, \|q\|_{a, \rho}, \|\bar{q}\|_{a, \rho}, \|p\|_{a, \rho}, \|\bar{p}\|_{a, \rho}, s, 0 \geq s} \sum_{k, l, \alpha, \beta} |F_{k\alpha\beta, l\tilde{\alpha}\tilde{\beta}}(\sigma, \mu)\|_{\mathcal{O}} e^{(l_1+k_1)\tau} \times 2^{||l||+||k||} |q^{\alpha}\|_{\mathcal{O}} \|p^{\beta}\|_{\mathcal{O}}.$$

To function $F$, we associated a Hamiltonian vector field defined by

$$X_F = (F_I, F_J, -F_q, -F_{\bar{q}}, \{F_{q_n}\}_{n \in \mathbb{Z}_+^2}, \{-iF_{\bar{q}_n}\}_{n \in \mathbb{Z}_+^2}, \{iF_{p_n}\}_{n \in \mathbb{Z}_+^2}, \{-iF_{\bar{p}_n}\}_{n \in \mathbb{Z}_+^2}).$$

Its weighted norm is defined by

$$\|X_F\|_{D(r,s), \mathcal{O}} = \|F_I\|_{D(r,s), \mathcal{O}} + \|F_J\|_{D(r,s), \mathcal{O}} + \frac{1}{2} (\|F_q\|_{D(r,s), \mathcal{O}} + \|F_{\bar{q}}\|_{D(r,s), \mathcal{O}} + \|F_p\|_{D(r,s), \mathcal{O}} + \|F_{\bar{p}}\|_{D(r,s), \mathcal{O}})$$

$$+ \frac{1}{2} \left( \sum_{n \in \mathbb{Z}_+^2} \|F_{q_n}\|_{D(r,s), \mathcal{O}} |n|^{a} e^{|n|\rho} + \sum_{n \in \mathbb{Z}_+^2} \|F_{\bar{q}_n}\|_{D(r,s), \mathcal{O}} |n|^{\tilde{a}} e^{|n|\rho} \right)$$

$$+ \frac{1}{2} \left( \sum_{n \in \mathbb{Z}_+^2} \|F_{p_n}\|_{D(r,s), \mathcal{O}} |n|^{a} e^{|n|\rho} + \sum_{n \in \mathbb{Z}_+^2} \|F_{\bar{p}_n}\|_{D(r,s), \mathcal{O}} |n|^{\tilde{a}} e^{|n|\rho} \right).$$

**Remark.** We require $a > \tilde{a},$ i.e., the weight of vector fields is a litter heavier than that of $q, \bar{q}, p, \bar{p}$. The boundedness of $\|X_F\|_{D(r,s), \mathcal{O}}$ means $X_F$ sends a decaying $(q, p)$-sequence to a faster decaying sequence.

The normal form $H_0 = N + B$ with

$$N = \langle \omega(\sigma), I \rangle + \langle \tilde{\omega}(\mu), J \rangle + \sum_{n \in \mathbb{Z}_+^2} \Omega_n(\sigma) q_n \bar{q}_n + \sum_{n \in \mathbb{Z}_+^2} \tilde{\Omega}_n(\mu) p_n \bar{p}_n$$

$$B = \sum_{n \in \mathbb{Z}_+^2 \cap \mathbb{Z}_+^2} (a_n(\sigma, \mu) q_n \bar{p}_n + b_n(\sigma, \mu) \bar{q}_n p_n)$$

where $(\sigma, \mu) \in \mathcal{O}$ is the parameter. Notice that apart from integrable terms, $q_n$ and $\bar{p}_n, p_n$ and $q_n$ may also be coupled and as a result our normal form is in the form of block diagonal with each block of degree 2.

For this unperturbed system, it’s easy to see that it admits a special solution

$$(\theta, \varphi, 0, 0, 0, 0, 0, 0) \rightarrow (\theta + \omega t, \varphi + \tilde{\omega} t, 0, 0, 0, 0, 0, 0)$$

corresponding to an invariant torus in the phase space. Our goal is to prove that, after removing some parameters, the perturbed system $H = H_0 + P$ still admits invariant torus provided that $\|X_F\|_{D(r,s), \mathcal{O}}$ is sufficiently small. To archive this goal,
we require that the Hamiltonian $H$ satisfies some conditions:

(A1) **Nondegeneracy**: Suppose for $\forall (\sigma, \mu) \in \mathcal{O} \subset \mathbb{R}^2$

$$
\begin{bmatrix}
\frac{d\omega_1}{d\sigma} & \frac{d\omega_2}{d\sigma} & \cdots & \frac{d\omega_n}{d\sigma} \\
\frac{d^2\omega_1}{d\sigma^2} & \frac{d^2\omega_2}{d\sigma^2} & \cdots & \frac{d^2\omega_n}{d\sigma^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d^m\omega_1}{d\sigma^m} & \frac{d^m\omega_2}{d\sigma^m} & \cdots & \frac{d^m\omega_n}{d\sigma^m}
\end{bmatrix} \neq 0,
$$

and

$$
\begin{bmatrix}
\frac{d\omega_1}{d\mu} & \frac{d\omega_2}{d\mu} & \cdots & \frac{d\omega_n}{d\mu} \\
\frac{d^2\omega_1}{d\mu^2} & \frac{d^2\omega_2}{d\mu^2} & \cdots & \frac{d^2\omega_n}{d\mu^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d^m\omega_1}{d\mu^m} & \frac{d^m\omega_2}{d\mu^m} & \cdots & \frac{d^m\omega_n}{d\mu^m}
\end{bmatrix} \neq 0.
$$

Moreover, for some $N > \max\{n + 3, b + 3\}$, $\omega, \tilde{\omega}$ belongs to $C^{N,1}(\mathcal{O})$.

(A2) **Asymptotics of normal frequencies**: There exists $\ell > 0$ such that

$$
\Omega_n = |n|^2 + \Omega_n^*, \quad n \in \mathbb{Z}_1^n,
$$

$$
\tilde{\Omega}_n = |n|^2 + \tilde{\Omega}_n^*, \quad n \in \mathbb{Z}_2^n,
$$

where $|\Omega_n^*|_{C^{N,1}(\mathcal{O})} = o(|n|^{-\ell})$, $|\tilde{\Omega}_n^*|_{C^{N,1}(\mathcal{O})} = o(|n|^{-\ell})$.

(A3) **Non-resonance conditions and admissible tangential sites**:
Let

$$
A_n = \begin{pmatrix} \Omega_n \\ \Omega_n^* \end{pmatrix}, \quad n \in \mathbb{Z}_1^n \cap \mathbb{Z}_2^n
$$

and

$$
A_n = \begin{pmatrix} \Omega_n \\ \tilde{\Omega}_n \end{pmatrix}, \quad n \in \mathbb{Z}_1^n \setminus \mathbb{Z}_2^n,
$$

$$
A_n = \begin{pmatrix} \tilde{\Omega}_n \\ \Omega_n \end{pmatrix}, \quad n \in \mathbb{Z}_2^n \setminus \mathbb{Z}_1^n.
$$

For a fixed $\gamma > 0$ small enough and $\tau$ sufficiently large, we assume that either

$$
|\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle | \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad |k| + |\tilde{k}| \neq 0,
$$

$$
|\det((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) \cdot I + A_n) | \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau},
$$

$$
|\det((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) \cdot I + A_n \otimes I_2 + I_2 \otimes A_m^T) | \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau},
$$

$$
|\det((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) \cdot I + A_n \otimes I_2 - I_2 \otimes A_m^T) | \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau},
$$

here $A^T$ denotes the transpose of matrix $A$ and $I$ denotes the identity matrix, or

$$
\langle k, i \rangle + n + m \neq 0, \text{ for } (k, |n|, |m|) = (-e_j - e_l, |i_j|, |i_l|), \quad 1 \leq j < l \leq b,
$$

$$
\langle \tilde{k}, t \rangle + n + m \neq 0, \text{ for } (\tilde{k}, |n|, |m|) = (-e_j - e_l, |t_j|, |t_l|), \quad 1 \leq j < l \leq \tilde{b},
$$

$$
\langle k, i \rangle + n - m \neq 0, \text{ for } (k, |n|, |m|) = (-e_j + e_l, |i_j|, |i_l|), \quad 1 \leq j < l \leq b,
$$

$$
\langle \tilde{k}, t \rangle + n - m \neq 0, \text{ for } (\tilde{k}, |n|, |m|) = (-e_j + e_l, |t_j|, |t_l|), \quad 1 \leq j < l \leq \tilde{b},
$$

$$
\langle k, i \rangle + \langle \tilde{k}, t \rangle + n + m \neq 0, \text{ for } (k, \tilde{k}, |n|, |m|) = (-e_j - e_l, |i_j|, |t_l|), \quad 1 \leq j \leq b, \quad 1 \leq l \leq \tilde{b},
$$

$$
\langle k, i \rangle + \langle \tilde{k}, t \rangle + n - m \neq 0, \text{ for } (k, \tilde{k}, |n|, |m|) = (-e_j, e_l, |i_j|, |t_l|), \quad 1 \leq j \leq b, \quad 1 \leq l \leq \tilde{b}.
$$
where \( i = (i_1, \ldots, i_b) \), \( t = (t_1, \ldots, t_n) \) and \( e_j \) denotes \( b\) or \( \bar{b} \) vectors with its jth component being 1 and the other components being zero.

(A4) **Regularity**: \( B + P \) is real analytic with respect to \( \theta, \varphi, I, q, \bar{q}, p, \bar{p} \) and Whitney smooth with respect to \( (\sigma, \mu) \). And we have

\[
\|X_B\|_{D(r,s), \mathcal{O}} < 1, \quad \|X_P\|_{D(r,s), \mathcal{O}} < \varepsilon
\]

with \( \bar{a} > a \).

(A5) **Zero – momentum condition**: \( B + P \) belongs to a class of functions \( \mathcal{A} \) defined by

\[
f = \sum_{k \in \mathbb{Z}^b, \tilde{k} \in \mathbb{Z}^b, j \in \mathbb{N}^b, \alpha, \beta, \bar{\alpha}, \bar{\beta}} f_{k\tilde{k}j, \alpha\beta, \bar{\alpha}\bar{\beta}} e^{i(k,\varphi) + (\tilde{k},\bar{\varphi})} f(t) q^\alpha \bar{q}^\beta \bar{p}^\bar{\alpha} p^\bar{\beta}, \quad f \in \mathcal{A}
\]

implies

\[
f_{k\tilde{k}j, \alpha\beta, \bar{\alpha}\bar{\beta}} = 0, \quad \text{if} \quad \sum_{j=1}^{\tilde{b}} k_j j + \sum_{j=1}^{\tilde{b}} \tilde{k}_j j + \sum_{n \in \mathbb{Z}_1^d} (\alpha_n - \beta_n) n + \sum_{n \in \mathbb{Z}_1^d} (\bar{\alpha}_n - \bar{\beta}_n) n \neq 0.
\]

**Theorem 2.1.** Assume that the Hamiltonian \( H = N + B + P \) satisfies conditions (A1)-(A5), then there exists \( \varepsilon > 0 \), such that if \( \|X_B\|_{D(r,s), \mathcal{O}} < \varepsilon \), then the following holds: there exists a Cantor subset \( \mathcal{O}_\gamma \subset \mathcal{O} \) with \( \text{meas}(\mathcal{O} \setminus \mathcal{O}_\gamma) = O(\gamma^\delta) \) (\( \vartheta \) is a positive constant) and two maps which are analytic in \( \theta, \varphi \) and belonging to \( \mathcal{O}^{N,1}(\mathcal{O}_\gamma) \) in \( (\sigma, \mu) \)

\[
\Phi : \mathbb{T}^{b+\tilde{b}} \times \mathcal{O}_\gamma \rightarrow D(r,s), \quad \omega^* : \mathcal{O}_\gamma \rightarrow \mathbb{R}^{b+\tilde{b}}
\]

where \( \Phi \) is \( (\varepsilon/\gamma^{N+1}) \)-close to the trivial embedding \( \Phi_0 : \mathbb{T}^{b+\tilde{b}} \times \mathcal{O} \rightarrow \mathbb{T}^{b+\tilde{b}} \times \{0,0\} \times \{0,0\} \times \{0,0\} \) and \( \omega^* \) is \( \varepsilon \)-close to the unperturbed frequency \( (\omega, \bar{\omega}) \). Such that for all \( (\sigma, \mu) \in \mathcal{O}_\gamma \) and \( (\theta, \varphi) \in \mathbb{T}^{b+\tilde{b}} \), the curve \( t \rightarrow \Phi((\theta, \varphi) + \omega^* t, (\sigma, \mu)) \) is a quasiperiodic solution of the Hamiltonian equation governed by \( H = N + B + P \).

3. **Application to higher dimensional beam equation system.** Consider d-D beam equation systems

\[
u_{1tt} + B_1 u_1 + u_1 u_2^2 = 0, \quad B u_1 = (\Delta^2 + \sigma)^{1/2} u_1, \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \quad (9)
\]

\[
u_{2tt} + A_2 u_2 + u_1^2 u_2 = 0, \quad A u_2 = (\Delta^2 + \mu)^{1/2} u_2, \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \quad (10)
\]

\[
u_1(t, x_1 + 2\pi, \ldots, x_d) = \cdots = u_1(t, x_1, \ldots, x_d + 2\pi) = \cdots = u_1(t, x_1, \ldots, x_d),
\]

\[
u_2(t, x_1 + 2\pi, \ldots, x_d) = \cdots = u_2(t, x_1, \ldots, x_d + 2\pi) = \cdots = u_2(t, x_1, \ldots, x_d).
\]

Introducing \( v_1 = u_1t, v_2 = u_2t \), then (9) reads

\[
u_{1tt} = v_1
\]

\[
u_{1t} = -B_1 u_1 - u_1 u_2^2,
\]

(10) reads

\[
u_{2tt} = v_2
\]

\[
u_{2t} = -A_2 u_2 - u_1^2 u_2.
\]

Letting

\[
w_1 = \frac{1}{\sqrt{2}} B^{1/2} u_1 - i \frac{1}{\sqrt{2}} B^{-1/2} v_1, \quad w_2 = \frac{1}{\sqrt{2}} A^{1/2} u_2 - i \frac{1}{\sqrt{2}} A^{-1/2} v_2,
\]

\[
\in \mathbb{C}^d.
\]
Consider the equations set (11) in view of Hamiltonian and it could be rewritten as

\[
\begin{align*}
\frac{1}{t}w_{1t} &= Bw_1 + \frac{1}{\sqrt{2}} B^{-1/2} \{ B^{-1/2} (\frac{w_1 + \bar{w}_1}{\sqrt{2}}) \cdot [A^{-1/2} (\frac{w_2 + \bar{w}_2}{\sqrt{2}})]^2 \} \\
\frac{1}{t}w_{2t} &= Aw_2 + \frac{1}{\sqrt{2}} A^{-1/2} \{ [B^{-1/2} (\frac{w_1 + \bar{w}_1}{\sqrt{2}})]^2 \cdot A^{-1/2} (\frac{w_2 + \bar{w}_2}{\sqrt{2}}) \}
\end{align*}
\]  

(11)

where the function \( H \) is a Hamiltonian

\[
H = \int_{T^d} (Bw_1) \bar{w}_1 \, dx + \int_{T^d} (Aw_2) \bar{w}_2 \, dx \\
+ \frac{1}{2} \int_{T^d} \{ B^{-1/2} (\frac{w_1 + \bar{w}_1}{\sqrt{2}}) \}^2 \cdot [A^{-1/2} (\frac{w_2 + \bar{w}_2}{\sqrt{2}})]^2 \, dx.
\]

The operators \( B \) and \( A \) with periodic boundary conditions have an exponential basis \( \phi_n(x) = \sqrt{1/(2\pi)^d} e^{i \langle n, x \rangle} \) and corresponding eigenvalues \( \lambda_n = \sqrt{|n|^2 + \sigma} \) and \( \hat{\lambda}_n = \sqrt{|n|^2 + \mu} \), respectively. Let

\[
w_1 = \sum_{n \in \mathbb{Z}^d} q_n \phi_n(x), \quad w_2 = \sum_{n \in \mathbb{Z}^d} p_n \phi_n(x).
\]

System (12) is then equivalent to the lattice Hamiltonian equations

\[
\begin{align*}
\dot{q}_n &= i (\lambda_n q_n + \frac{\partial G}{\partial p_n}), \\
\dot{p}_n &= i (\lambda_n p_n + \frac{\partial G}{\partial q_n}),
\end{align*}
\]

with corresponding Hamiltonian function

\[
H = \Lambda + G
\]

\[
= \sum_{n \in \mathbb{Z}^d} \lambda_n q_n \bar{q}_n + \sum_{n \in \mathbb{Z}^d} \hat{\lambda}_n p_n \bar{p}_n + \frac{1}{2} \int_{T^d} \left[ \sum_{n \in \mathbb{Z}^d} \frac{q_n \phi_n + \bar{q}_n \phi_n}{\sqrt{2\lambda_n}} \right]^2 \left[ \sum_{n \in \mathbb{Z}^d} \frac{p_n \phi_n + \bar{p}_n \phi_n}{\sqrt{2\lambda_n}} \right]^2 \, dx.
\]  

(13)

Let \( q = (\cdots, q_n, \cdots)_{n \in \mathbb{Z}^d}, \quad \bar{q} = (\cdots, \bar{q}_n, \cdots)_{n \in \mathbb{Z}^d}, \quad p = (\cdots, p_n, \cdots)_{n \in \mathbb{Z}^d}, \quad \bar{p} = (\cdots, \bar{p}_n, \cdots)_{n \in \mathbb{Z}^d} \), then we may rewrite \( G \) as follows

\[
G = \frac{1}{2} \int_{T^d} \left[ \sum_{n \in \mathbb{Z}^d} \frac{q_n \phi_n + \bar{q}_n \phi_n}{\sqrt{2\lambda_n}} \right]^2 \left[ \sum_{n \in \mathbb{Z}^d} \frac{p_n \phi_n + \bar{p}_n \phi_n}{\sqrt{2\lambda_n}} \right]^2 \, dx = \sum_{\alpha,\beta,\tilde{\alpha},\tilde{\beta}} G_{\alpha\beta\tilde{\alpha}\tilde{\beta}} q^\alpha \bar{q}^\beta p^{\tilde{\alpha}} \bar{p}^{\tilde{\beta}}
\]

\[
G_{\alpha\beta\tilde{\alpha}\tilde{\beta}} = 0, \quad \text{if} \sum_{n \in \mathbb{Z}^d} (\alpha_n - \beta_n) n + \sum_{n \in \mathbb{Z}^d} (\tilde{\alpha}_n - \tilde{\beta}_n) n \neq 0.
\]  

(14)

Next, we consider the regularity of the gradient of \( G \). To this end, let \( l^{a,\rho} \) be the Banach space of all bi-infinite, complex valued sequences \( q = (\cdots, q_n, \cdots)_{n \in \mathbb{Z}^d}, \quad p = (\cdots, p_n, \cdots)_{n \in \mathbb{Z}^d} \) with finite weighted norm

\[
||q||_{a,\rho} = \sum_{n \in \mathbb{Z}^d} |q_n| |n|^a |e|^{n\rho}, \quad ||p||_{a,\rho} = \sum_{n \in \mathbb{Z}^d} |p_n| |n|^a |e|^{n\rho}.
\]

The convolution \( q * q' \) of two such sequences is defined by \( (q * q')_n = \sum_{m} q_{n-m} q'_m \).
Lemma 3.1. ([16] Lemma 3.1) For $a \ge 0, \rho > 0$, the space $l^{a,\rho}$ is a Banach algebra with respect to convolution of sequences and

$$\|q * q'\|_{a,\rho} \le c\|q\|_{a,\rho} \cdot \|q'\|_{a,\rho}$$

with a constant $c$ depending only on $a$.

Lemma 3.2. For $a \ge 0, \rho > 0$, the gradient $G_{\tilde{q}}, G_{\tilde{p}}$ are real analytic as maps from some neighborhood of the origin in $l^{a,\rho}$ into $l^{a+1}, \rho$, with

$$\|G_{\tilde{q}}\|_{a+1,\rho} = O(\|q\|_{a,\rho} \|p\|_{a,\rho}^2), \quad \|G_{\tilde{p}}\|_{a+1,\rho} = O(\|q\|_{a,\rho}^2 \|p\|_{a,\rho}).$$

The proof of lemma 3.2 is similar to lemma 3.2 of [16].

Introducing action-angle variable

$$q_n = \sqrt{T_n} e^{i(k,\theta)}, \quad \tilde{q}_n = \sqrt{T_n} e^{-i(k,\theta)}, \quad n \in S,$$

$$p_n = \sqrt{T_n} e^{i(k,\varphi)}, \quad \tilde{p}_n = \sqrt{T_n} e^{-i(k,\varphi)}, \quad n \in \tilde{S}.$$

The Hamiltonian (13) is now turned into

$$H = \langle \omega(\sigma), I \rangle + \langle \tilde{\omega}(\mu), J \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(\sigma) q_n \tilde{q}_n$$

$$+ \sum_{n \in \mathbb{Z}_2^d} \tilde{\Omega}_n(\mu) p_n \tilde{p}_n + P(\theta, \varphi, I, J, q, \tilde{q}, p, \tilde{p}; \sigma, \mu), \quad (15)$$

where $\omega(\sigma) = (\omega_1(\sigma), \ldots, \omega_b(\sigma))$, $\Omega_n(\sigma) = \lambda_n(\sigma), n \in \mathbb{Z}_1^d$,

$\tilde{\omega}(\mu) = (\tilde{\omega}_1(\mu), \ldots, \tilde{\omega}_b(\mu))$, $\tilde{\Omega}_n(\mu) = \tilde{\lambda}_n(\mu), n \in \mathbb{Z}_2^d$. Now

let’s verify conditions (A1)-(A5) for (15).

Verifying (A1): It is easy to see that

$$\begin{vmatrix}
\frac{d\omega_1}{d\sigma} & \frac{d\omega_2}{d\sigma} & \cdots & \frac{d\omega_b}{d\sigma} \\
\frac{d^2\omega_1}{d\sigma^2} & \frac{d^2\omega_2}{d\sigma^2} & \cdots & \frac{d^2\omega_b}{d\sigma^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d^b\omega_1}{d\sigma^b} & \frac{d^b\omega_2}{d\sigma^b} & \cdots & \frac{d^b\omega_b}{d\sigma^b}
\end{vmatrix} \neq 0,$$

$$\begin{vmatrix}
\frac{d\tilde{\omega}_1}{d\mu} & \frac{d\tilde{\omega}_2}{d\mu} & \cdots & \frac{d\tilde{\omega}_b}{d\mu} \\
\frac{d^2\tilde{\omega}_1}{d\mu^2} & \frac{d^2\tilde{\omega}_2}{d\mu^2} & \cdots & \frac{d^2\tilde{\omega}_b}{d\mu^2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{d^b\tilde{\omega}_1}{d\mu^b} & \frac{d^b\tilde{\omega}_2}{d\mu^b} & \cdots & \frac{d^b\tilde{\omega}_b}{d\mu^b}
\end{vmatrix} \neq 0.$$
\[(k,i) + \langle \hat{k},t \rangle + n + m \neq 0, \text{ for } (k,\hat{k},|n|,|m|) = (-e_j,-e_l,|i_j|,|t_l|),
1 \leq j \leq b, 1 \leq l \leq b,\]

\[(k,i) + \langle \hat{k},t \rangle + n - m \neq 0, \text{ for } (k,\hat{k},|n|,|m|) = (-e_j,e_l,|i_j|,|t_l|),
1 \leq j \leq b, 1 \leq l \leq b.\]

To check the non-resonance condition (A3), for convenience, we only verify the most complicate (8). Recall the structure of the Hamiltonian (15), now \(a_n = 0, b_n = 0,\) so we need to concentrate on each elements of the diagonal of the matrix

\[\langle (k,\omega) + \langle \hat{k},\bar{\omega} \rangle \rangle \cdot I + A_n \otimes I_2 - I_2 \otimes A_n^T\]

which means that we only need to verify

\[|\langle (k,\omega) + \langle \hat{k},\bar{\omega} \rangle \rangle + \sqrt{|n|^4 + \sigma} - \sqrt{|m|^4 + \mu} \geq \frac{\gamma}{(|k| + |\hat{k}|)^{\tau}}. \tag{16} \]

Similar to the proof of lemma 3.7 in [16], after excluding a subset with measure \(O(\gamma^d)\) of the parameter set, (16) follows and the condition (A3) is verified. Verifying (A4): It’s similar with the verification of condition (A4) in [16]. Verifying (A5): It’s very similar to that in [16]. Denote by \(e_n\) the infinite dimensional vector with the \(n\)-th component being 1 and the other components being zero, and \(k = (k_1, \ldots, k_b), k_j = \alpha_j - \beta_j, 1 \leq j \leq b, \) \(\bar{k} = (\bar{k}_1, \ldots, \bar{k}_b), \) \(\bar{k}_j = \tilde{\alpha}_j - \tilde{\beta}_j, 1 \leq j \leq \tilde{b},\) then due to (14),

\[G = \sum_{n \in \mathbb{Z}^d} (\alpha_n - \beta_n)n + \sum_{n \in \mathbb{Z}^d} (\tilde{\alpha}_n - \tilde{\beta}_n)n = 0\]

\[= \sum_B G_{\alpha \beta \tilde{\alpha} \tilde{\beta}} q^\alpha q^\beta q^{\tilde{\alpha}} q^{\tilde{\beta}}\]

\[\sum_{\alpha_n}^{\tilde{\alpha}_n, \beta_n} \sum_{\beta_{\tilde{a}_n}}^{\beta_{\tilde{b}_n}} \sum_{\tilde{\alpha}_{\tilde{n}}}^{\tilde{\alpha}_{\tilde{b}}} = \sum_B G_{k_\alpha k_\beta k_{\tilde{a}} k_{\tilde{b}}} l_j l_j e^{(k,\theta)} e^{(\hat{k},\bar{\theta})} q^\alpha q^\beta q^{\tilde{\alpha}} q^{\tilde{\beta}},\]

where \(B\) denotes \(B = \sum_{\sum_{n \in \mathbb{Z}^d} (\alpha_n - \beta_n)n + \sum_{\sum_{n \in \mathbb{Z}^d} (\tilde{\alpha}_n - \tilde{\beta}_n)n = 0,}\)

i.e. \(P \in A.\)
4. KAM step. We prove Theorem 2.1 by a KAM iteration which involves an infinite sequence of change of variables. Each step of KAM iteration makes the perturbation smaller than the previous step at the cost of excluding a small set of parameters. We have to prove the convergence of the iteration and estimate the measure of the excluded set after infinite KAM steps.

At the v-th step of the KAM iteration, we consider a Hamiltonian vector field with

\[ H_v = N_v + B_v + P_v(\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}; \sigma, \mu) \]

where

\[ N_v = \langle \omega_v(\sigma), I \rangle + \langle \tilde{\omega}_v(\mu), J \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n^v(\sigma)q_n\bar{q}_n + \sum_{n \in \mathbb{Z}_2^d} \tilde{\Omega}_n^v(\mu)p_n\bar{p}_n \]

\[ B_v = \sum_{n \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} (a_n^v(\sigma, \mu)q_n\bar{p}_n + b_n^v(\sigma, \mu)p_n\bar{q}_n) \]

with \( B_v + P_v \) is defined in \( D(r_v, s_v) \times \mathcal{O}_{v-1} \).

We construct a map

\[ \Phi_v: D(r_{v+1}, s_{v+1}) \times \mathcal{O}_v \to D(r_v, s_v) \times \mathcal{O}_{v-1}, \]

so that the vector field \( X_{H_v\circ \Phi_v} \) defined on \( D(r_{v+1}, s_{v+1}) \) satisfies

\[ \|X_{P_{v+1}}\|_{D(r_{v+1}, s_{v+1}) \times \mathcal{O}_v} = \|X_{H_v\circ \Phi_v} - X_{N_{v+1} + B_{v+1}}\|_{D(r_{v+1}, s_{v+1}) \times \mathcal{O}_v} \leq \varepsilon^\kappa, \quad \kappa > 1 \]

and the new Hamiltonians still satisfy conditions (A1)-(A5).

To simplify notations, in what follows, the quantities without subscripts refer to quantities at the v-th step, while the quantities with subscript + denote the corresponding quantities at the (v+1)-th step. Let’s consider the Hamiltonian defined in \( D(r, s) \times \mathcal{O} \):

\[ H = N + B + P \]

\[ = \langle \omega(\sigma), I \rangle + \langle \tilde{\omega}(\mu), J \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n(\sigma)q_n\bar{q}_n + \sum_{n \in \mathbb{Z}_2^d} \tilde{\Omega}_n(\mu)p_n\bar{p}_n \]

\[ + \sum_{n \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} (a_n(\sigma, \mu)q_n\bar{p}_n + b_n(\sigma, \mu)p_n\bar{q}_n) + P(\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}; \sigma, \mu, \varepsilon). \]

We assume that \((\sigma, \mu) \in \mathcal{O}\) satisfies (a suitable \( \tau > 0 \) which will be specified later)

\[ |\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle| \geq \frac{\gamma}{(|k| + |\tilde{k}|)^\tau}, \quad |k| + |\tilde{k}| \neq 0, \]

\[ |\det(\langle \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle \cdot I + A_n \rangle)\| \geq \frac{\gamma}{(|k| + |k|)^\tau}, \]

\[ |\det(\langle \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle \cdot I + A_n \otimes I_2 + I_2 \otimes A_m^T \rangle)\| \geq \frac{\gamma}{(|k| + |k|)^\tau}, \]

\[ |\det(\langle \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle \cdot I + A_n \otimes I_2 - I_2 \otimes A_m^T \rangle)\| \geq \frac{\gamma}{(|k| + |k|)^\tau}, \quad |k| + |\tilde{k}| + |n| - |m| \neq 0. \]

Moreover

\[ \|X_P\|_{D(r, s), \mathcal{O}} \leq \varepsilon. \]

Expand \( P \) into Fourier-Taylor series

\[ P = \sum_{k, k, l, l, \alpha, \beta, \tilde{\alpha}, \tilde{\beta}} P_{k\alpha\beta, k\tilde{\alpha}\tilde{\beta}} e^{i((\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle)I) + \tilde{\beta}q^\alpha q^{\beta} \bar{p}^\beta}, \]
by condition (A5), we get that

\[ P_{kl\alpha\beta, kl\alpha\beta} = 0, \quad \text{if } \sum_{j=1}^{b} k_j i_j + \sum_{j=1}^{b} \tilde{k}_j i_j + \sum_{\alpha \in \mathbb{Z}_2^d} (\alpha_n - \beta_n)n + \sum_{\alpha \in \mathbb{Z}_2^d} (\tilde{\alpha}_n - \tilde{\beta}_n)n \neq 0 \]

which means that when \( k = 0, \tilde{k} = 0 \), the terms \( q_n\bar{q}_m, q_n\bar{p}_m, p_n\bar{q}_m, p_n\bar{p}_m \) are absent when \( |n| = |m|, n \neq m \).

Let \( 0 < r_+ < r, s_+ = \frac{1}{2} s \varepsilon \frac{4}{3}, \varepsilon_+ = e^{-N-2}(r-r_+)^{-\varepsilon \frac{4}{3}} \), now we describe how to construct a subset \( \mathcal{O}_+ \subset \mathcal{O} \) and a change of variables \( \Phi : D_+ \times \mathcal{O}_+ = D(r, s) \times \mathcal{O} \to D(r, s) \times \mathcal{O} \) such that the transformed Hamiltonian \( H_+^N = H \circ \Phi = N_+ + B_+ + P_+ \) satisfies conditions (A1)-(A5) with new parameters \( \varepsilon_+, r_+, s_+ \) and with \( (\sigma, \mu) \in \mathcal{O}_+ \).

4.1. Homological equation. Expand \( P \) into Fourier-Taylor series

\[ P = \sum_{k, \tilde{k}, i, j, \alpha, \beta, \tilde{\alpha}, \tilde{\beta}} P_{kl\alpha\beta, kl\alpha\beta} e^{i(k, \theta) + (\tilde{k}, \varphi)} I_j^l q^\alpha \bar{q}^\beta \bar{p}^\tilde{\alpha} \bar{p}^\tilde{\beta}, \]

where \( k \in \mathbb{Z}_2^b, \tilde{k} \in \mathbb{Z}_2^b, l \in \mathbb{N}_0^b, i \in \mathbb{N}_{\tilde{b}} \) and the multi-indices \( \alpha, \beta, \tilde{\alpha}, \tilde{\beta} \) run over the set of all infinite dimensional vectors \( \alpha = (\cdots, \alpha_n, \cdots)_{n \in \mathbb{Z}_2^d}, \beta = (\cdots, \beta_n, \cdots)_{n \in \mathbb{Z}_2^d}, \tilde{\alpha} = (\cdots, \tilde{\alpha}_n, \cdots)_{n \in \mathbb{Z}_2^d}, \tilde{\beta} = (\cdots, \tilde{\beta}_n, \cdots)_{n \in \mathbb{Z}_2^d} \) with finitely many nonzero components of positive integers.

Consider its quadratic truncation \( R \):

\[ R(\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}) = R^0 + R^1 + R^2 \]

where

\[ R^0 = \sum_{k, \tilde{k}, |k| + |\tilde{k}| \leq 1} (P_{kl\alpha\beta, kl\alpha\beta} e^{i(k, \theta) + (\tilde{k}, \varphi)}) I^l q^\alpha \bar{q}^\beta \bar{p}^\tilde{\alpha} \bar{p}^\tilde{\beta}, \] \hspace{1cm} (17)

\[ R^1 = \sum_{k, \tilde{k}, n} \left( P_{n, k} e^{i(k, \theta) + (\tilde{k}, \varphi)} q_n + P_{n, \tilde{k}} e^{i(k, \theta) + (\tilde{k}, \varphi)} \bar{q}_n \right) \]

\[ + \sum_{k, \tilde{k}, n} \left( P_{n, k} e^{i(k, \theta) + (\tilde{k}, \varphi)} p_n + P_{n, \tilde{k}} e^{i(k, \theta) + (\tilde{k}, \varphi)} \bar{p}_n \right), \] \hspace{1cm} (18)

and

\[ R^2 = R^2_{qq} + R^2_{qp} + R^2_{pp}, \] \hspace{1cm} (19)

where

\[ R^2_{qq} = \sum_{k, \tilde{k}, n, m} \left( P_{nm} e^{i(k, \theta) + (\tilde{k}, \varphi)} q_n q_m + P_{nm} e^{i(k, \theta) + (\tilde{k}, \varphi)} q_n \bar{q}_m \right) \] \hspace{1cm} (20)

\[ R^2_{qp} = \sum_{k, \tilde{k}, n, m} \left( P_{nm} e^{i(k, \theta) + (\tilde{k}, \varphi)} q_n p_m + P_{nm} e^{i(k, \theta) + (\tilde{k}, \varphi)} q_n \bar{p}_m \right) \] \hspace{1cm} (21)

\[ R^2_{pp} = \sum_{k, \tilde{k}, n, m} \left( P_{nm} e^{i(k, \theta) + (\tilde{k}, \varphi)} p_n p_m + P_{nm} e^{i(k, \theta) + (\tilde{k}, \varphi)} p_n \bar{p}_m \right). \] \hspace{1cm} (22)

We explain the coefficients in (17)-(22) as below (here \( e_n \) denotes the vector with the n-th component being 1 and the other components being zero): \( P_{kl\alpha\beta, kl\alpha\beta} = P_{k0\alpha\beta, k0\alpha\beta} P_{l0\alpha\beta, l0\alpha\beta} P_{kl, l0\alpha\beta} P_{k0\alpha\beta, k0\alpha\beta} \) with \( \alpha = e_n, \beta = 0, \tilde{\alpha} = 0, \tilde{\beta} = 0; P_{kl, l0\alpha\beta} = P_{k0\alpha\beta, k0\alpha\beta} \) with \( \alpha = 0, \beta = e_n, \tilde{\alpha} = 0, \tilde{\beta} = 0; \) the definitions of \( P_{kl\alpha\beta, kl\alpha\beta} \) are similar. \( P_{kl, l0\alpha\beta} = P_{k0\alpha\beta, k0\alpha\beta} \) with \( \alpha = e_n + e_m, \beta = 0, \tilde{\alpha} = 0, \tilde{\beta} = 0; \)

\( P_{l0\alpha\beta, l0\alpha\beta} = P_{k0\alpha\beta, k0\alpha\beta} \) with \( \alpha = 0, \beta = e_m, \tilde{\alpha} = 0, \tilde{\beta} = 0; \)
\[ P_{nm}^{\alpha_0,\beta} = P_{mn}^{\alpha_0,\beta} \] with \( \alpha = 0, \beta = e_n + e_m, \dot{\alpha} = 0, \dot{\beta} = 0 \); the definitions of \( P_{nm}^{\alpha_0,\beta} \) are similar. \( P_{nm}^{\alpha_1,\beta} \) are similar.

In the following, we will construct a Hamiltonian function \( F \) where

\[ F = \alpha + \beta = 0, \dot{\alpha} = e_n, \dot{\beta} = e_m; \] the definitions of \( F_{nm}^{\alpha_1,\beta} \) are similar.

Rewrite \( H \) as \( H = N + B + R + (P - R) \). Due to the choice of \( s_+ \ll s \) and the definition of the norm, it follows immediately that

\[ \|X_R\|_{D(r,s),C} \leq \|X_P\|_{D(r,s),C} \leq \varepsilon \]

and in \( D(r_+, s_+) \)

\[ \|X_{P-R}\|_{D(r,s_+),C} \leq c\varepsilon. \]

In the following, we will construct a Hamiltonian function \( F \) satisfying (A5) and with the same form of \( R \) defined in \( D_+ = D(r_+, s_+) \) such that the time one map \( X^1_B \) of the Hamiltonian vector field \( X_F \) defines a map from \( D_+ \) to \( D \) and puts \( H \) into \( H_+ \). Precisely, one has

\[ H \circ X^1_B = (N + B + R) \circ X^1_B + (P - R) \circ X^1_B \]

\[ = N + B + \{N + B, F\} + R + \int_0^1 (1 - t) \{\{N + B, F\}, F\} \circ X^1_B dt \]

\[ + \int_0^1 \{R, F\} \circ X^1_B dt + (P - R) \circ X^1_B. \]

So we get the linearized homological equation

\[ \{N + B, F\} + R = \hat{N} + \hat{B} \quad (23) \]

where

\[ \hat{N} = P_{0000,0000} + (\hat{\omega}, I) + (\hat{\omega}, J) + \sum_{n \in \mathbb{Z}^2} P_{nn}^{011,000} \hat{q}_n \hat{p}_n + \sum_{n \in \mathbb{Z}^2} P_{nn}^{000,011} \hat{p}_n \]

\[ \hat{\omega} = (P_{l}^{0100,0001})_{|l|=1}, \quad \hat{\omega} = (P_{l}^{0000,0101})_{|l|=1}, \]

\[ \hat{B} = \sum_{n \in \mathbb{Z}^2} (P_{nn}^{0101,0011} \hat{q}_n \hat{p}_n + P_{nn}^{0011,0101} \hat{q}_n \hat{p}_n). \]

We define \( N_+ = N + \hat{N}, B_+ = B + \hat{B} \) and

\[ P_+ = \int_0^1 (1 - t) \{\{N + B, F\}, F\} \circ X^1_B dt + \int_0^1 \{R, F\} \circ X^1_B dt + (P - R) \circ X^1_B. \quad (24) \]

We construct the Hamiltonian function \( F \) as below, with the same structure of \( R \):

\[ F(\theta, \varphi, I, J, q, \tilde{q}, p, \tilde{p}) = F^0 + F^1 + F^2 \]

with

\[ F^0 = \sum_{|\tilde{k}|+|\tilde{l}| \neq 0, |l|+|\tilde{l}| \leq 1} (F_{k0100,\tilde{k}000} l^I + F_{k000,\tilde{k}100} j^I) e^{i(l(k,\theta)+(\tilde{k},\varphi))} \quad (25) \]

\[ F^1 = \sum_{k,k',n} (F_{nk}^{k101,000} q_n + F_{nk}^{k011,000} q_{\tilde{n}} + F_{nk}^{k000,101} p_n + F_{nk}^{k000,011} p_{\tilde{n}}) e^{i(l(k,\theta)+(\tilde{k},\varphi))} \quad (26) \]

and

\[ F^2 = F^2_{qq} + F^2_{qp} + F^2_{pp} \quad (27) \]

where

\[ F^2_{qq} = \sum_{k,k',n,m} (F_{nm}^{k020,\tilde{k}00} q_n \tilde{q}_m + F_{nm}^{k011,000} q_n \tilde{q}_m + F_{nm}^{k000,202} q_n \tilde{q}_m) e^{i(l(k,\theta)+(\tilde{k},\varphi))}. \quad (28) \]
\[ F_{qp}^2 = \sum_{k,k',n,m} \left( F_{nm}^{k_2,10} q_n p_{m} + F_{nm}^{k_2,11} q_n \bar{p}_m + F_{nm}^{k_2,10} q_{k'n} \bar{p}_m + F_{nm}^{k_2,11} q_{k'n} \bar{p}_m \right) e^{i(k,\theta + (k',\phi))} \]  

(29)

\[ F_{pp}^2 = \sum_{k,k',n,m} \left( F_{nm}^{k_2,00} p_n p_{m} + F_{nm}^{k_2,10} p_n \bar{p}_m + F_{nm}^{k_2,10} q_{k'n} \bar{p}_m + F_{nm}^{k_2,11} q_{k'n} \bar{p}_m \right) e^{i(k,\theta + (k',\phi))}. \]  

(30)

Now (23) is turned into
\[ \{ N, F^0 \} + R^0 - P_{0000,0000} - (\hat{\omega}, I) - (\hat{\omega}, J) = 0, \]  

(31)

\[ \{ N + B, F^1 \} + R^1 = 0 \]  

(32)

and the most complicated
\[ \{ N + B, F^2 \} + R^2 = \sum_{n \in \mathbb{Z}_2^d} P_{011,000}^{011,000} q_n \bar{q}_n + \sum_{n \in \mathbb{Z}_2^d} P_{011,001}^{011,001} p_n \bar{p}_n + \sum_{n \in \mathbb{Z}_2^d} (F_{nn}^{k_2,001} q_{k'n} \bar{p}_m + F_{nn}^{k_2,010} q_{k'n} \bar{p}_m). \]  

(33)

Now we solve equations (31)-(33) one by one:

Solving (31): by the expansion (25), (31) is turned into
\[ i(\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) F_{100,000}^{k_10,000} = P_{100,000}^{k_10,000}, \quad |k| + |	ilde{k}| \neq 0, \]
\[ i(\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) F_{000,100}^{k_00,100} = P_{000,100}^{k_00,100}, \quad |k| + |	ilde{k}| \neq 0. \]

According to assumption (5) in condition (A3), one has
\[ |F_{100,000}^{k_10,000}|_\mathcal{O} \leq \gamma^{-N-2}(|k| + |	ilde{k}|)^{\gamma(N+2)+N+1} P_{100,000}^{k_10,000}|_\mathcal{O}, \quad |k| + |	ilde{k}| \neq 0, \]
\[ |F_{000,100}^{k_00,100}|_\mathcal{O} \leq \gamma^{-N-2}(|k| + |	ilde{k}|)^{\gamma(N+2)+N+1} P_{000,100}^{k_00,100}|_\mathcal{O}, \quad |k| + |	ilde{k}| \neq 0. \]

Solving (32): for convenience, we only describe the homological equation related to the elimination of term \( q_n, p_n, n \in \mathbb{Z}_2^d \cap \mathbb{Z}_2^d \), and the corresponding equation related to \( \bar{q}_n, \bar{p}_n \) with the estimate of its solution follow the same way. By the expansion (26), (32) is turned into
\[ i[ (\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_n) F_{n}^{k_10,000} + a_n F_{n}^{k_00,100} ] = P_{k_10,000}^{k_10,000}, \]
\[ i[ (\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_n) F_{n}^{k_00,100} + b_n F_{n}^{k_10,000} ] = P_{k_00,100}^{k_00,100}. \]

The coefficient matrix is just \( (\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) I + A_n \), so according to (6) in condition (A3), one has the estimate
\[ |F_{n}^{k_10,000}|_\mathcal{O}, |F_{n}^{k_00,100}|_\mathcal{O} < c_\gamma^{-N-2}(|k| + |	ilde{k}|)^{\gamma(N+2)+N+1} e^{-\gamma|k|+|	ilde{k}|} e^{-|n|}. \]  

(34)

For the case when \( n \in \mathbb{Z}_2^d \setminus \mathbb{Z}_2^d \), we eliminate the term \( q_n \), and the corresponding equation in (32) is
\[ i(\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_n) F_{n}^{k_10,000} = P_{k_10,000}^{k_10,000}, \]

the coefficient matrix is still \( (\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) I + A_n \) by the definition of \( A_n \). The case \( n \in \mathbb{Z}_2^d \setminus \mathbb{Z}_2^d \) is similar. So (34) still holds.

Solving (33): for convenience, we only describe the most complicated equation related to the elimination of the terms: \( q_n q_{m}, q_n \bar{p}_m, p_n q_{m}, p_n \bar{p}_m, n, m \in \mathbb{Z}_2^d \cap \mathbb{Z}_2^d \) with \( |n| \neq |m| \). The corresponding homological equation is
\[ i( (\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) |I + M| X = Y \]  

(35)
where
\[ X = (F_{nm}^{k11, k00}, F_{nm}^{k10, k01}, F_{nm}^{k01, k10}, F_{nm}^{k00, k11})^T, \]
\[ Y = (P_{nm}^{k11, k00}, P_{nm}^{k10, k01}, P_{nm}^{k01, k10}, P_{nm}^{k00, k11})^T, \]
and the $4 \times 4$ matrix $M$ is
\[
\begin{pmatrix}
-\Omega_n + \Omega_m & b_m & -a_n & 0 \\
-a_m & -\Omega_n + \hat{\Omega}_m & 0 & -a_n \\
-b_n & 0 & -\hat{\Omega}_n + \Omega_m & b_m \\
0 & -b_n & a_m & -\hat{\Omega}_n + \hat{\Omega}_m
\end{pmatrix}
\]
so the coefficient matrix of (35) is just
\[
\langle (k, \omega) + (\hat{k}, \hat{\omega}) \rangle I + A_n \otimes I_2 - I_2 \otimes A_m^T,
\]
according to assumption (8) in condition (A3), one has
\[
|F_{nm}^{k11, k00}|_{\mathcal{O}}, |F_{nm}^{k10, k01}|_{\mathcal{O}}, |F_{nm}^{k01, k10}|_{\mathcal{O}}, |F_{nm}^{k00, k11}|_{\mathcal{O}} 
\leq c\gamma^{-N-2}(|k| + |\hat{k}|)^{r(N+2)+N+1}\varepsilon e^{-\langle (|k|+|\hat{k}|)r e^{-(|n|+|m|)}\rangle}.
\]
For equation concerning elimination of terms $q_n q_m, p_n p_m, p_n q_m, p_n p_m, n, m \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d$, and the equation concerning elimination of terms $q_n q_m, q_n q_m, p_n q_m, p_n p_m, n, m \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d$, similar estimates follow by making use of (8). When $n \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, m \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d$, we have the linear equation concerning the elimination of $q_n q_m, p_n q_m, p_n p_m, n, m \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d$,
\[
i \langle (k, \omega) + (\hat{k}, \hat{\omega}) - \Omega_n + \hat{\Omega}_m \rangle F_{nm}^{k10, k01} + a_m F_{nm}^{k11, k00} = P_{nm}^{k10, k01},
\]
\[
i \langle (k, \omega) + (\hat{k}, \hat{\omega}) - \Omega_n + \Omega_m \rangle F_{nm}^{k11, k00} + b_m F_{nm}^{k10, k01} = P_{nm}^{k11, k00},
\]
the coefficient matrix is still in the form
\[
\langle (k, \omega) + (\hat{k}, \hat{\omega}) \rangle I + A_n \otimes I_2 - I_2 \otimes A_m^T,
\]
so the same estimates still hold. Similarly we could work on the case $n \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d, m \in \mathbb{Z}_1^d \setminus \mathbb{Z}_2^d$.

4.2. Estimate on the coordinate transformation.

Lemma 4.1. Let $D_i = D(r_+ + \frac{1}{4}(r - r_+), 0 < i \leq 4$, then one has
\[
\|X_{D_i}\|_{D_5, \mathcal{O}} \leq c\gamma^{-N-2}(r - r_+)^{-c\varepsilon}.
\]

Lemma 4.2. Let $\eta = \varepsilon^{1/3}, D_{\eta} = D(r_+ + \frac{1}{4}(r - r_+), \frac{1}{4}\varepsilon \eta), 0 < i \leq 4$. If $\varepsilon \ll c[\gamma^{N+2}(r - r_+)^{\frac{1}{2}}$, one has
\[
\phi_{D_{\eta}}^t : D_2 \eta \rightarrow D_3 \eta, -1 \leq t \leq 1
\]
and
\[
\|\phi_{D_{\eta}}^t - Id\|_{D_3} \leq c\gamma^{-N-2}(r - r_+)^{-c\varepsilon}.
\]
The proof of Lemma 4.1 and 4.2 is similar to that in [16], so we omit it.
4.3. Estimate of the new perturbations. Recall the definition of new perturbation

\[ P_+ = \int_0^1 \{ R(t), F \} \circ \phi_F^t dt + (P - R) \circ \phi_F^t \]

with \( R(t) = (1-t)\{(N + B, F) + R\} + tR \). So

\[ X_{P_+} = \int_0^1 (\phi_F^t)'^* X_{(R(t), F)} dt + (\phi_F^t)' X_{P - R} \]

By lemma 4.1, one has

\[ \| D\phi_F^t - Id \|_{D_{1n}} \leq c_\gamma N^{-2}(r - r_+)^{-\varepsilon}, \quad -1 \leq t \leq 1 \]

so we get \( \| D\phi_F^t \|_{D_{1n}} \leq 2, \quad -1 \leq t \leq 1 \), and

\[ \| X_{(R(t), F)} \|_{D_{2n}} \leq c_\gamma N^{-2}(r - r_+)^{-\varepsilon} \varepsilon^2 \]

Combining with \( \| X_{P - R} \|_{D_{2n}} \leq c_\eta \varepsilon \gamma + c_\gamma N^{-2}(r - r_+)^{-\varepsilon} \varepsilon^2 < \varepsilon_+ \).

4.4. Estimate for the new normal form. Due to the special form of \( P \) defined in (A5), the terms \( q_n \tilde{q}_m, q_n \tilde{p}_m, p_n \tilde{q}_m, p_n \tilde{p}_m \) with \( |n| = |m|, n \neq m \) are absent, which means that our normal form has a simpler form (we omit the constant term in the normal form part):

\[ N_+ = N + \langle \hat{\omega}, I \rangle + \langle \hat{\omega}_+, J \rangle + \sum_{n \in \mathbb{Z}^d_+} P_{011,000}^{001} q_n \tilde{q}_n + \sum_{n \in \mathbb{Z}^d_+} P_{010}^{001,011} p_n \tilde{p}_n \]

\[ = \langle \omega_+, I \rangle + \langle \tilde{\omega}_+, J \rangle + \sum_{n \in \mathbb{Z}^d_+} \Omega_+^n q_n \tilde{q}_n + \sum_{n \in \mathbb{Z}^d_+} \bar{\Omega}_+^n p_n \tilde{p}_n \]

\[ + \sum_{n \in \mathbb{Z}^d_+} (a_+^n q_n \tilde{p}_n + b_+^n q_n p_n) \]

where

\[ \omega_+ = \omega + P_{0100,0000}, \quad \tilde{\omega}_+ = \tilde{\omega} + P_{0000,0100}, \]

\[ \Omega_+^n = \Omega_n + P_{011,0000}^{001}, \quad n \in \mathbb{Z}^d_+, \]

\[ \bar{\Omega}_+^n = \bar{\Omega}_n + P_{010,0011}^{001}, \quad n \in \mathbb{Z}^d_+, \]

\[ a_+^n = a_n + P_{010,0011}^{001}, \quad b_+^n = b_n + P_{010,0010}^{001}, \quad n \in \mathbb{Z}^d_+ \cap \mathbb{Z}^d_+. \]

So with the help of the regularity of \( X_F \) and Cauchy estimate, set \( \delta = \min\{\tilde{\alpha} - a, l\} \), then we have

\[ |\omega_+ - \omega|, \quad |\tilde{\omega}_+ - \tilde{\omega}| < \varepsilon, \]

\[ |\Omega_+^n - \Omega_n|, \quad |\bar{\Omega}_+^n - \bar{\Omega}_n|, \quad |a_+^n - a_n|, \quad |b_+^n - b_n| < \varepsilon |n|^{-\delta}. \]

It follows that

\[ |\langle k, \omega_+ \rangle + \langle \tilde{k}, \tilde{\omega}_+ \rangle| \geq |\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle| - \varepsilon(|k| + |\tilde{k}|) \geq \frac{\gamma_+}{(|k| + |\tilde{k}|)^2}, \]

\[ |\text{det}( (\langle k, \omega_+ \rangle + \langle \tilde{k}, \tilde{\omega}_+ \rangle) \cdot I + A_+^n) | \]

\[ \geq |\text{det}( (\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle) \cdot I + A_n) | - c\varepsilon(|k| + |\tilde{k}|) \geq \frac{\gamma_+}{(|k| + |\tilde{k}|)^2}, \]
\[ | \det( (k, \omega) + (\tilde{k}, \tilde{\omega})) \cdot I + A_n \otimes I_2 + I_2 \otimes (A_n^+)^T | \geq | \det((k, \omega) + (\tilde{k}, \tilde{\omega})) \cdot I + A_n \otimes I_2 + I_2 \otimes A_{n+1}^T | | c \varepsilon (|k| + |\tilde{k}|) \]
\[
\geq \frac{\gamma_+}{(|k| + |\tilde{k}|)^2}
\]

and
\[
| \det((k, \omega) + (\tilde{k}, \tilde{\omega})) \cdot I + A_n \otimes I_2 - I_2 \otimes (A_n^+)^T | | c \varepsilon (|k| + |\tilde{k}|)
\]

This means that in the next KAM step, small denominator conditions are automatically satisfied for $|k| + |\tilde{k}|$ where $K \tau+1 \varepsilon < c(\gamma - \gamma_+)$.  

4.5. **Verification of condition (A5) after transformation.** The assumption $P \in A$ defined in (A5) is used to guarantee that the normal form at each KAM step has the same form as in the first step. To complete one KAM step, we need to prove the new perturbation $P_+$ still has the special form defined in (A5), i.e., $P_+ \in A$. Recall its definition (24) and it could be rewritten as
\[
P_+ = P - R + \{P, F\} + \frac{1}{2!} \{\{N + B, F\}, F\} + \frac{1}{3!} \{\{P, F\}, F\} + \cdots + \frac{1}{m!} \{\cdots \{P, F\}, F\} + \cdots .
\]  

Sine $P \in A$, we have $R, P - R \in A$. By the definition of $F$ one has $F \in A$. Now we only need to prove that the second line of (37) is also in $A$. We know $N, B, F, P \in A$, so we only need to prove the following lemma:

**Lemma 4.3.** If $G(\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}), H(\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}) \in A$, then $B(\theta, \varphi, I, J, q, \bar{q}, p, \bar{p}) = \{G, H\} \in A$.

**Proof.** $G, H$ are sums of a series of monomials with the form:
\[
G_{kla} \tilde{a}_{\tilde{k}la} e^{i(k, \theta) + (\tilde{k}, \varphi)} I^1 J^1 q^1 q^2 \bar{q} \bar{p} \tilde{b}^1 \tilde{b}^2 \tilde{p} \tilde{p}, H_{uvn} \tilde{a}_{\tilde{a}n} \tilde{a}_{\tilde{v}n} e^{i(u, \theta) + (\tilde{v}, \varphi)} I^1 J^1 q^1 q^2 \tilde{q} \tilde{p} \tilde{p} \tilde{p} \tilde{p}
\]  

satisfying
\[
\sum_{j=1}^b k_j j + \sum_{j=1}^b \tilde{k}_j t_j + \sum_{n \in \mathbb{Z}^2} (\alpha_n - \beta_n) n + \sum_{n \in \mathbb{Z}^2} (\tilde{\alpha}_n - \tilde{\beta}_n) n = 0 \tag{39}
\]
\[
\sum_{j=1}^b u_j j + \sum_{j=1}^b \tilde{u}_j t_j + \sum_{n \in \mathbb{Z}^2} (\gamma_n - \delta_n) n + \sum_{n \in \mathbb{Z}^2} (\tilde{\gamma}_n - \tilde{\delta}_n) n = 0 \tag{40}
\]

Recall the definition of Poisson bracket:
\[
\{ G, H \} = \left( \frac{\partial G}{\partial \theta_1}, \frac{\partial H}{\partial \varphi_1} \right) - \left( \frac{\partial G}{\partial \varphi_1}, \frac{\partial H}{\partial \theta_1} \right) + \left( \frac{\partial G}{\partial \theta_2}, \frac{\partial H}{\partial \varphi_2} \right) - \left( \frac{\partial G}{\partial \varphi_2}, \frac{\partial H}{\partial \theta_2} \right)
\]
\[
+ i \sum_{n \in \mathbb{Z}^2} \left( \frac{\partial G}{\partial p_n}, \frac{\partial H}{\partial q_n} \right) - \left( \frac{\partial G}{\partial q_n}, \frac{\partial H}{\partial p_n} \right) + i \sum_{n \in \mathbb{Z}^2} \left( \frac{\partial G}{\partial \bar{p}_n}, \frac{\partial H}{\partial \bar{q}_n} \right) - \left( \frac{\partial G}{\partial \bar{q}_n}, \frac{\partial H}{\partial \bar{p}_n} \right). \tag{41}
\]

Let’s consider the Poisson bracket of the two terms in (38) and for convenience, we just omit the coefficients (assume the coefficients to be 1). Denote the first
one of (38) by $A$ and the second one $B$. Obviously $AB \in A$, we have $\frac{\partial A}{\partial v_j} \frac{\partial A}{\partial v_j} = ik_i q_j I_i^{-1} AB$ if $q_j > 0$, and be 0 if $q_j = 0$. So we conclude that $\langle \frac{\partial G}{\partial q}, \frac{\partial H}{\partial q} K \rangle \in A$ and similarly we have $\langle \frac{\partial G}{\partial q}, \frac{\partial H}{\partial q}, I \rangle \in A$. For the remaining terms, $\frac{\partial G}{\partial q}, \frac{\partial H}{\partial q} = a_n \delta_n(q_n q_n)^{-1} AB$ if $a_n, \delta_n > 0$ and be 0 otherwise. So we conclude that $\frac{\partial G}{\partial q}, \frac{\partial H}{\partial q} \in A$, and similarly, $\frac{\partial G}{\partial q}, \frac{\partial H}{\partial q} \in A$. To sum up, we have $\{G, H\} \in A$.

By lemma 4.3, then conclusion $P_+ \in A$ follows.

5. **Iteration lemma and convergence.** For any given $s, \varepsilon, r, \gamma$ and for all $v \geq 1$, we define the following sequences

\[
r_v = r(1 - \frac{v+1}{2^i}), \\
\varepsilon_v = c\gamma^{-N-2}(r_v - r_v)^{-1} c_v^{1/3}, \\
\gamma_v = \gamma(1 - \frac{v+1}{2^i}), \quad \eta_v = \varepsilon_v^{1/3}, \\
s_v = \frac{1}{4} \eta_v - s_v - 1 = 2^{-2v}(\Pi_{i=1}^{v-1} \varepsilon_i)^{1/2} s_0, \\
K_v = c(\varepsilon_v^{-1}(\gamma_v - \gamma_{v+1}))^{1/4}, \quad D_v = D(r_v, s_v),
\]

where $c$ is a constant, and the parameters $r_0, \varepsilon_0, \gamma_0, s_0$ and $K_0$ are defined to be $r, \varepsilon, \gamma, s$ and 1 respectively.

5.1. **Iteration lemma.** The preceding analysis can be summarized as follows:

**Lemma 5.1.** Let $\varepsilon$ be sufficiently small, $v \geq 0$. Suppose that:

(1) \[ N_v + B_v = \langle \omega_v, I \rangle + \langle \tilde{\omega}_v, J \rangle + \sum_{n \in \mathbb{Z}^d} \Omega_n q_n \tilde{q}_n + \sum_{n \in \mathbb{Z}^d} \tilde{\Omega}_n p_n \tilde{p}_n \]

is a normal form with parameters $(\sigma, \mu)$ satisfying

\[
| \langle k, \omega_v \rangle + \langle \tilde{k}, \tilde{\omega}_v \rangle | \geq \frac{\gamma_v}{(|k| + |\tilde{k}|)^{\tau}}, \quad |k| + |\tilde{k}| \neq 0
\]

and

\[
| \det( \langle k, \omega_v \rangle + \langle \tilde{k}, \tilde{\omega}_v \rangle \cdot I + A_n^v ) | \geq \frac{\gamma_v}{(|k| + |\tilde{k}|)^{\tau}},
\]

\[
| \det( \langle k, \omega_v \rangle + \langle \tilde{k}, \tilde{\omega}_v \rangle \cdot I + A_n^v \otimes I_2 + I_2 \otimes (A_n^v)^T ) | \geq \frac{\gamma_v}{(|k| + |\tilde{k}|)^{\tau}},
\]

\[
| \det( \langle k, \omega_v \rangle + \langle \tilde{k}, \tilde{\omega}_v \rangle \cdot I + A_n^v \otimes I_2 - I_2 \otimes (A_n^v)^T ) | \geq \frac{\gamma_v}{(|k| + |\tilde{k}|)^{\tau}},
\]

\[
|k| + |\tilde{k}| + |n| - |m| \neq 0
\]

on a closed set $\mathcal{O}_v$ of $\mathbb{R}^2$, where

\[
A_n = \begin{pmatrix}
\Omega_n & a_n \\
b_n & \tilde{\Omega}_n
\end{pmatrix}, \quad n \in \mathbb{Z}^d \cap \mathbb{Z}^d
\]

and

\[
A_n = \Omega_n, \quad n \in \mathbb{Z}^d \setminus \mathbb{Z}^d, \\
A_n = \tilde{\Omega}_n, \quad n \in \mathbb{Z}^d \setminus \mathbb{Z}^d.
\]
where $\omega_2(\Omega^v, \tilde{\Omega}^v)$ are $C^{N,1}$ smooth in $(\sigma, \mu)$ satisfying

$$|\omega_2 - \omega_{2-1}|_{C^0}, |\tilde{\omega}_2 - \tilde{\omega}_{2-1}|_{C^0} < \varepsilon_{v-1},$$

$$|\Omega^v_2 - \Omega_{2-1}^v|_{C^0}, |\tilde{\Omega}_2^v - \tilde{\Omega}_{2-1}^v|_{C^0}, |a_n^v - a_{n-1}^v|_{C^0}, |b_n^v - b_{n-1}^v|_{C^0} < \varepsilon_{v-1} |n|^{-\delta}.$$ 

(3) $P_v$ has the special form defined in (A5) and

$$\|X_{P_v}\|_{D(r_v, s_v), C_0} \leq \varepsilon_v.$$ 

Then there is a subset $\mathcal{O}_{v+1} \subset \mathcal{O}_v$,

$$\mathcal{O}_{v+1} = \mathcal{O}_v \setminus \bigcup_{|k|+|\tilde{k}|>K_v} \mathcal{R}_k^{v+1}(\gamma_{v+1}),$$

where

$$\mathcal{R}_k^{v+1}(\gamma_{v+1}) = \{ (\sigma, \mu) \in \mathcal{O}_v : |\langle k, \tilde{\omega}_{v+1} \rangle + \langle k, \tilde{\omega}_{v+1} \rangle | < \frac{\gamma_{v+1}}{|k| + |\tilde{k}|}; \text{ or}$$

$$|\text{det}(\langle k, \tilde{\omega}_{v+1} \rangle + \langle k, \tilde{\omega}_{v+1} \rangle) \cdot I + A_k^{v+1} | < \frac{\gamma_{v+1}}{|k| + |\tilde{k}|}; \text{ or}$$

$$|\text{det}(\langle k, \tilde{\omega}_{v+1} \rangle + \langle k, \tilde{\omega}_{v+1} \rangle) \cdot I + A_k^{v+1} \otimes I_2 + I_2 \otimes (A_k^{v+1})^T | < \frac{\gamma_{v+1}}{|k| + |\tilde{k}|}; \text{ or}$$

$$|\text{det}(\langle k, \tilde{\omega}_{v+1} \rangle + \langle k, \tilde{\omega}_{v+1} \rangle) \cdot I + A_k^{v+1} \otimes I_2 - I_2 \otimes (A_k^{v+1})^T | < \frac{\gamma_{v+1}}{|k| + |\tilde{k}|},$$

$$|k| + |\tilde{k}| + |m| - |n| \neq 0 \}$$

with $\omega_{v+1} = \omega_v + P_{0100,0000}^v, \tilde{\omega}_{v+1} = \tilde{\omega}_v + P_{0000,0100}^v$ and a symplectic transformation of variables:

$$\Phi_v : D(r_{v+1}, s_{v+1}) \times \mathcal{O}_v \to D(r_v, s_v),$$

such that on $D(r_{v+1}, s_{v+1}) \times \mathcal{O}_v$, $H_{v+1} = H_v \circ \Phi_v$ has the form:

$$H_{v+1} = \langle \omega_{v+1}, I \rangle + \langle \tilde{\omega}_{v+1}, I \rangle + \sum_{n \in \mathbb{Z}_1^d} \Omega_n^{v+1} q_n \bar{q}_n + \sum_{n \in \mathbb{Z}_2^d} \tilde{\Omega}_n^{v+1} p_n \bar{p}_n$$

$$+ \sum_{n \in \mathbb{Z}_1^d \cap \mathbb{Z}_2^d} (a_n^{v+1} q_n \bar{p}_n + b_n^{v+1} \bar{q}_n p_n)$$

with

$$|\omega_{v+1} - \omega_v|_{C^0}, |\tilde{\omega}_{v+1} - \tilde{\omega}_v|_{C^0} < \varepsilon_v,$$

$$|\Omega_n^{v+1} - \Omega_n^v|_{C^0}, |\tilde{\Omega}_n^{v+1} - \tilde{\Omega}_n^v|_{C^0}, |a_n^{v+1} - a_n^v|_{C^0}, |b_n^{v+1} - b_n^v|_{C^0} < \varepsilon_v |n|^{-\delta}.$$ 

Also $P_{v+1}$ has the special form defined in (A5) and

$$\|X_{P_{v+1}}\|_{D(r_{v+1}, s_{v+1}), C_0} \leq \varepsilon_{v+1}.$$
5.2. Convergence. Suppose that the assumption of Theorem 2.1 are satisfied. Recall that \( \varepsilon_0 = \varepsilon, r_0 = r, s_0 = s, \gamma_0 = \gamma, N_0 = N, B_0 = B, P_0 = P \),

\[
O_0 = \{ (\sigma, \mu) \in \mathcal{O} : |(k, \omega) + (\hat{k}, \hat{\omega})| \geq \frac{\gamma_0}{|k| + |\hat{k}|}, |k| + |\hat{k}| \neq 0; \]

\[
|\det( (k, \omega) + (\hat{k}, \hat{\omega}) ) \cdot I + A_n^u | \geq \frac{\gamma_0}{|k| + |\hat{k}|};
\]

\[
|\det( (k, \omega) + (\hat{k}, \hat{\omega}) ) \cdot I + A_n^u \otimes I_2 + I_2 \otimes (A_m^u)^T | \geq \frac{\gamma_0}{|k| + |\hat{k}|};
\]

\[
|\det( (k, \omega) + (\hat{k}, \hat{\omega}) ) \cdot I + A_n^u \otimes I_2 - I_2 \otimes (A_m^u)^T | \geq \frac{\gamma_0}{|k| + |\hat{k}|},
\]

\(|k| + |\hat{k}| + |n| - |m| \neq 0 \}
\]

is a bounded positive-measure set in \( \mathbb{R}^2 \).

The assumptions of the iteration lemma are satisfied when \( \varepsilon_0 \) and \( \gamma \) are sufficiently small. Inductively, we obtain the following sequences:

\[
O_{v+1} \subset O_v, \Psi^v = \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_v : D(r_{v+1}, s_{v+1}) \times O_v \to D(r_0, s_0), \quad v \geq 0,
\]

\[
H \circ \Psi^v = H_{v+1} = N_{v+1} + B_{v+1} + P_{v+1}.
\]

Let \( \hat{\mathcal{O}} = \bigcap_{v=0}^{\infty} O_v \). By the help of lemma 4.2, we conclude that \( N_v, \Psi^v, D\Psi^v, \omega_v, \tilde{\omega}_v \) converge uniformly on \( D(\frac{1}{2} r, 0) \times \hat{\mathcal{O}} \) with

\[
N_\infty + B_\infty = \langle \omega_\infty, I \rangle + \langle \tilde{\omega}_\infty, J \rangle + \sum_{n \in \mathbb{Z}^d} \Omega_n^\infty q_n \bar{q}_n + \sum_{n \in \mathbb{Z}^d} \bar{\Omega}_n^\infty p_n \bar{p}_n
\]

\[
+ \sum_{n \in \mathbb{Z}^d \cap \mathbb{Z}^d} (a_n^\infty q_n \bar{p}_n + b_n^\infty \bar{q}_n p_n).
\]

Since \( \varepsilon_{v+1} = c \gamma^{-N-2}(r_v - r_{v+1})^{-c} \varepsilon_v^{1/3} \), it follows that \( \varepsilon_{v+1} \to 0 \) provided that \( \varepsilon \) is small enough.

Let \( \phi^t_H \) be the flow of \( X_H \). Since \( H \circ \Psi^v = H_{v+1}, \) we have

\[
\phi^t_H \circ \Psi^v = \Psi^v \circ \phi^t_{H_{v+1}}.
\]

The uniform convergence of \( \Psi^v, D\Psi^v, \omega_v, \tilde{\omega}_v \) and \( X_{H_m} \) implies that the limits can be taken on both sides of (42). Hence, on \( D(\frac{1}{2} r, 0) \times \hat{\mathcal{O}} \) we get

\[
\phi^t_H \circ \Psi^\infty = \Psi^\infty \circ \phi^t_{H_m}.
\]

and

\[
\Psi^\infty : D(\frac{1}{2} r, 0) \times \hat{\mathcal{O}} \to D(r, s) \times \mathcal{O}.
\]

It follows that from (43) that

\[
\psi^t_H(\Psi^\infty(T^{b+b} \times \{ \sigma, \mu \})) = \Psi^\infty(T^{b+b} \times \{ \sigma, \mu \})
\]

for \( (\sigma, \mu) \in \hat{\mathcal{O}} \). This means that \( \Psi^\infty(T^{b+b} \times \{ \sigma, \mu \}) \) is an embeded torus which is invariant for the original perturbed Hamiltonian system at \( (\sigma, \mu) \in \hat{\mathcal{O}} \). We remark that the frequencies \( (\omega_\infty(\sigma), \tilde{\omega}_\infty(\mu)) \) associated to \( \Psi^\infty(T^{b+b} \times \{ \sigma, \mu \}) \) are slightly different from the unperturbed frequencies \( (\omega(\sigma), \tilde{\omega}(\mu)) \).

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6. **Measure estimate.** For notational convenience, let $\mathcal{O}_{-1} = \mathcal{O}, K_{-1} = 0$. Then at the $v$-th step of KAM iteration, we have to exclude the following resonant set:

$$
\mathcal{R}^v = \bigcup_{|k|+|\tilde{k}| > K_{-1}, n, m} (\mathcal{R}^v_{kk} \cup \mathcal{R}^v_{kk,n} \cup \mathcal{R}^v_{kk,nm} \cup \mathcal{R}^v_{kk,nm})
$$

where

$$
\mathcal{R}^v_{kk} = \{ (\sigma, \mu) \in \mathcal{O} : \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle < \frac{\gamma_v}{(|k|+|\tilde{k}|)^r} \}
$$

$$
\mathcal{R}^v_{kk,n} = \{ (\sigma, \mu) \in \mathcal{O} : \det ( \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle ) \cdot I + A^v_{n} - I_2 \otimes (A^v_{n})^T ) | < \frac{\gamma_v}{(|k|+|\tilde{k}|)^r}, \}
$$

$$
\mathcal{R}^v_{kk,nm} = \{ (\sigma, \mu) \in \mathcal{O} : \det ( \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle ) \cdot I + A^v_{n} \otimes I_2 - I_2 \otimes (A^v_{n})^T ) | < \frac{\gamma_v}{(|k|+|\tilde{k}|)^r}, \}
$$

$$
\mathcal{R}^v_{kk,nm} = \{ (\sigma, \mu) \in \mathcal{O} : \det ( \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle ) \cdot I + A^v_{n} \otimes I_2 + I_2 \otimes (A^v_{n})^T ) | < \frac{\gamma_v}{(|k|+|\tilde{k}|)^r}, \}
$$

**Remark 1.** From Section 4.4, one has that at the $v$-th step, small divisor conditions are automatically satisfied for $|k| \leq K_{-1}$. Hence, we only need to excise the above resonant set $\mathcal{R}^v$. Note that due to the special form of the perturbation (A5), there are not the terms of the form $\sum_{n \neq m} P_{nm}^{k,10} q_n \tilde{q}_m$, $\sum_{n \neq m} P_{nm}^{k,10} q_n \tilde{p}_m$, $\sum_{n \neq m} P_{nm}^{k,10} q_n \tilde{p}_m$ and $\sum_{n \neq m} P_{nm}^{k,10} q_n \tilde{p}_m$ in the perturbation $P_v$, thus we need not to consider small divisors $\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega^v_n - \Omega^v_m$ with $k = 0, \tilde{k} = 0, n \neq m$, where $\Omega^v_n$ denotes $\Omega^v_n$, $\Omega^v_m$ denotes $\Omega^v_m$, $\Omega^v$ denotes $\Omega^v_m$, $\Omega^v_{\tilde{m}}$ denotes $\Omega^v_m$ or $\Omega^v_m$, which is crucial for this paper.

**Lemma 6.1.** *(lemma 8.4 of [2])* Let $g : \mathcal{I} \to \mathbb{R}$ be $b + 3$ times differentiable and assume that:

(1) $\forall \xi \in \mathcal{I}$ there exists $s \leq b + 2$ such that $g^{(s)}(\xi) > B$,

(2) there exists $A$ such that $|g^{(s)}(\xi)| < A$ for $\forall \xi \in \mathcal{I}$ and $\forall s$ with $1 \leq s \leq b + 3$.

Define

$$
\mathcal{I}_h = \{ \xi \in \mathcal{I} : |g(\xi)| \leq h \}
$$

then

$$
\frac{\text{meas}(\mathcal{I}_h)}{\text{meas}(\mathcal{I})} \leq \frac{A}{B} 2(2 + 3 + \cdots + (b + 3) + 2B^{-1})h^{1/(b+3)}.
$$

Here we only consider the most complicated case $\mathcal{R}^v_{kk,nm}$, the other cases can be handled in the same way.

As

$$
|a_n|, |b_n| < \varepsilon |n|^{-\delta}, \quad |a_n|, |b_m| < \varepsilon |m|^{-\delta},
$$

thus the main part of

$$
| \det ( \langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle ) \cdot I + A^v_{n} \otimes I_2 - I_2 \otimes (A^v_{n})^T ) |
$$

is

$$
\langle (k, \omega) + (\tilde{k}, \tilde{\omega}) + \Omega^v_n - \Omega^v_m \rangle^4,
$$

where $\Omega^v_n$ denotes $\Omega_n$ or $\tilde{\Omega}_n$, $\Omega^v_m$ denotes $\Omega_m$ or $\tilde{\Omega}_m$. 
Then $R_{kk,nm}^{v,-}$ is equivalent to

$$R_{kk,nm}^{v,-} = \{ (\sigma, \mu) \in \mathcal{O}_{v-1} : |\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + \Omega_n - \Omega_m| < c_1 \frac{\gamma_v^\frac{1}{12}}{(|k| + |\tilde{k}|)^{\frac{7}{12}}} \}. $$

Similar to Lemma 4.4 in [16], by lemma 6.1, we obtain

**Lemma 6.2.** For $\mathcal{S} = \{ i_1, \ldots, i_b \}, \mathcal{S} = \{ t_1, \ldots, t_b \}, \mathcal{S} \cap \mathcal{S} = \emptyset$, we have

$$\text{meas}(R_{kk,nm}^{v,-*}) < c_2 \left| n \right|^{8(b + \tilde{b}) + 12} \left| m \right|^{8(b + \tilde{b}) + 12} \frac{1}{\gamma_v^{(b + \tilde{b} + 3)}}.$$ 

**Lemma 6.3.**

$$\text{meas}(\bigcup_{k, \tilde{k}, n, m} R_{kk,nm}^{v,-*}) < c\gamma_v^{\vartheta'}, \quad \vartheta' > 0.$$ 

**Proof.** Suppose that $|n|^2 - |m|^2 = l \geq 0$. If $l > c(|k| + |\tilde{k}|)$, $R_{kk,nm}^{v,-*} = \emptyset$; if $l < c(|k| + |\tilde{k}|)$, one has

$$|\Omega_n - \Omega_m - l| = O(|m|^{-1}).$$

It follows that

$$R_{kk,nm}^{v,-*} \subset Q_{k,b,lm_0} = \{ (\sigma, \mu) : |\langle k, \omega \rangle + \langle \tilde{k}, \tilde{\omega} \rangle + l| < c_1 \frac{\gamma_v^\frac{1}{12}}{(|k| + |\tilde{k}|)^{\frac{7}{12}}} + O(|m|^{-1}) \}. $$

Moreover, $Q_{kk,nm} \subset Q_{k,b,lm_0}$ for $|m| \geq |m_0|$. From lemma 6.1 and 6.2, one has

$$\text{meas}(\bigcup_{l \leq c(|k| + |\tilde{k}|)} R_{kk,nm}^{v,-*}) \leq \sum_{l \leq c(|k| + |\tilde{k}|)} \text{meas}(R_{kk,nm}^{v,-*}) + \text{meas}(Q_{k,b,lm_0})$$

$$< c \left[ \frac{1}{2\gamma_v^{(b + \tilde{b} + 3)}} \left| m_0 \right|^{8(2b + \tilde{b}) + 3 + C(d)} \right] \frac{1}{(|k| + |\tilde{k}|)^{\frac{7}{12}} - 4(b + \tilde{b}) - 6} + O\left(\frac{1}{\left| m_0 \right|^{1/4}}\right)$$

$$\leq c \left[ \frac{1}{2\gamma_v^{(b + \tilde{b} + 3)}} \left| m_0 \right|^{8(2(b + \tilde{b}) + 3) + C(d)} \right] \frac{1}{(|k| + |\tilde{k}|)^{\frac{7}{12}} - 4(b + \tilde{b}) - 6} + O\left(\frac{1}{\left| m_0 \right|^{1/4}}\right)$$

By choosing

$$|m_0| = \left( \frac{(|k| + |\tilde{k}|)^{\frac{7}{12}} - 4(b + \tilde{b}) - 6}{\gamma_v^{(b + \tilde{b} + 3)}} \right)^{1/8(2b + \tilde{b}) + 3 + C(d) + \frac{1}{b + \tilde{b} + 3}}$$

and

$$\vartheta' = \frac{1}{(b + \tilde{b} + 3)^2 (8(2b + \tilde{b}) + 3) + C(d) + \left( b + \tilde{b} + 3 \right) (5b + \tilde{b} + 6)}$$

so that

$$\tau > (b + \tilde{b})(b + \tilde{b} + 3)^2 \left[ 8(2b + \tilde{b}) + 3 \right] + (b + \tilde{b} + 3) \left( 5b + \tilde{b} + 6 \right)$$
then
\[
\operatorname{meas}\left( \bigcup_{k, \tilde{k}, |n| \neq |m|} \mathcal{R}^{\nu, -, \tau}_{kk, nm} \right) \leq \sum_{k, \tilde{k}} \frac{\tau^{d'} n^{d'}}{((|k|+|\tilde{k}|)^{1/b+3})^{1/8(2b+3)} + C(d+1)} \leq c^{d'} \nu.
\]
As a consequence, we obtain
\[
\operatorname{meas}(\mathcal{R}^\nu) < c^{d'} \nu.
\]
\[\square\]

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