Controllability for the Impulsive Semilinear Fuzzy Integrodifferential Equations with Nonlocal Conditions

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Abstract. In this paper, we study the controllability for the impulsive semilinear fuzzy integrodifferential control system with nonlocal conditions in $E_N$ by using the concept of fuzzy number whose values are normal, convex, upper semicontinuous and compactly supported interval in $E_N$.

1. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva [3] studied the existence and uniqueness of solution for the fuzzy differential equation on $E_N$ where $E_N$ is normal, convex, upper semicontinuous and compactly supported fuzzy sets in $R^n$. Seikkala [7] proved the existence and uniqueness of fuzzy solution for the equation $\dot{x}(t) = f(t, x(t))$, $x(0) = x_0$, where $f$ is a continuous mapping from $R^+ \times R$ and $x_0$ is a fuzzy number in $E^1$. Diamond and Kloeden [2] proved the fuzzy optimal control for the system $\dot{x}(t) = a(t)x(t) + u(t)$, $x(0) = x_0$, where $x()$, $u()$ are nonempty compact interval-valued functions on $E^1$. Kwun and Park [4] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in $E_N^1$ using by Kuhn-Tucker theorems. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently Park, Park and Kwun [6] find the sufficient conditions of nonlocal controllability for the semilinear fuzzy integrodifferential equations with nonlocal initial conditions.

In this paper we prove the existence and uniqueness of fuzzy solutions and find the sufficient conditions of controllability for the following impulsive semilinear fuzzy integrodifferential equations with nonlocal conditions:

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\[ \frac{dx(t)}{dt} = A[x(t) + \int_0^t G(t-s)x(s)ds] + f(t,x) + u(t), \quad t \in J = [0,1], \]  
(1)

\[ x(0) + g(x) = x_0 \in E_N, \]  
(2)

\[ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1,2, \ldots, m, \]  
(3)

where \( T > 0 \), \( A : J \rightarrow E_N \) is a fuzzy coefficient, \( E_N \) is the set of all upper semicontinuous convex normal fuzzy numbers with bounded \( \alpha \) -level intervals, \( f : J \times E_N \rightarrow E_N \) is a nonlinear continuous function, \( g : E_N \rightarrow E_N \) is a nonlinear continuous function, \( G(t) \) is an \( n \times n \) continuous matrix such that \( \frac{dG(t)x}{dt} \) is continuous for \( x \in E_N \) and \( t \in J \) with \( |G(t)| \leq K, \quad K > 0 \), with all nonnegative elements, \( u : J \rightarrow E_N \) is a control function and \( I_k \in C(E_N,E_N)(k=1,2,\ldots,m) \) are bounded functions, \( \Delta x(t_k) = x(t_k^+) - x(t_k^-) \), where \( x(t_k^+) \) and \( x(t_k^-) \) represent the left and right limits of \( x(t) \) at \( t = t_k \), respectively.

2. Existence and uniqueness of fuzzy solution

In this section we consider the existence and uniqueness of fuzzy solutions for the impulsive semilinear fuzzy integrodifferential equation with nonlocal conditions (1)-(3).

We denote the suprimum metric \( d_\infty \) on \( E^n \) and the suprimum metric \( H \) on \( C(J : E^n) \).

**Definition 1.** Let \( a,b \in E^n \), \( d_\infty(a,b) = \sup \{d_H([a]^\alpha,[b]^\alpha) : \alpha \in (0,1) \} \), where \( d_H \) is the Hausdorff distance.

**Definition 2.** Let \( x \in C(J : E^n) \), \( H_1(x,y) = \sup \{d_\infty(x(t),y(t)) : t \in J \} \).

**Definition 3.** The fuzzy process \( x : J \rightarrow E_N \) is a solution of equations (1)-(2) without the inhomogeneous term if and only if

\[ (x^\alpha_j)'(t) = \min \{A^\alpha_j(t)x^\alpha_j(t) + \int_0^t G(t-s)x^\alpha_j(s)ds, \quad i,j = 1,\ldots,r \}, \]

\[ (x^\alpha_j)'(t) = \max \{A^\alpha_j(t)x^\alpha_j(t) + \int_0^t G(t-s)x^\alpha_j(s)ds, \quad i,j = 1,\ldots,r \}, \]

\[ (x^\alpha_j)(0) = x^\alpha_{0j} - g^\alpha_j(x), \quad (x^\alpha_r)(0) = x^\alpha_{0r} - g^\alpha_r(x). \]

**H1** The inhomogeneous term \( f : J \times E_N \rightarrow E_N \) is a continuous function and satisfies a global Lipschitz condition \( d_H([f(x,y)]^\alpha,[f(x,y)]^\alpha) \leq c_1d_H([x]^\alpha,[y]^\alpha) \), for all \( x,y \in E_N \), and a finite positive constant \( c_1 > 0 \).

**H2** The nonlinear function \( g : E_N \rightarrow E_N \) is a continuous function and satisfies a global Lipschitz condition \( d_H([g(x)]^\alpha,[g(y)]^\alpha) \leq Ld_H([x]^\alpha,[y]^\alpha) \), for all \( x,y \in E_N \), and a finite positive constant \( L > 0 \).

**H3** \( S(t) \) is a fuzzy number satisfying for \( y \in E_N \), \( S^\prime(t)y \in C^1(J : E_N) \cap C(J : E_N) \) the equation

\[ \frac{d}{dt}S^\alpha(t)y = A[S^\alpha(t)y + \int_0^t G(t-s)S^\alpha(s)ys] = S(t)Ay + \int_0^t S(t-s)AG(s)ys, \quad t \in J \]  

such that \( [S^\alpha(t)] = [S^\alpha_y(t)], \)

\( S^\alpha(t), \quad S(0) = I \) and \( S^\alpha(t)(i=1,\ldots,r) \) is continuous. That is, there exists a constant \( c > 0 \) such that \( |S^\alpha(t)| \leq c \) for all \( t \in J \).
In order to define the solution of (1)-(3), we shall consider the space \( \Omega = \{ x : J \to E_N : x_k \in C(J_k, E_N) \} \), where \( J_k = (t_k, t_{k+1}] \), \( k = 0, 1, \ldots, m \), and there exist \( x(t_k^-) \) and \( x(t_k^+) \) (\( k = 1, \ldots, m \)), with \( x(t_k^-) = x(t_k^+) \).

Lemma 1. If \( x \) is an integral solution of (1)-(3) \( (u \equiv 0) \), then \( x \) is given by

\[
x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s)f(s, x(s))ds + \sum_{0<\xi_j} S(t-t_k)I_k(x(t_k^-)), \quad t \in J.
\]

Proof. Let \( x \) be a solution of (1)-(3). Define \( \omega(s) = S(t-s)x(s) \). Then we have that

\[
\frac{d\omega(s)}{ds} = -S(t-s)x(s) + S(t-s)\frac{dx(s)}{ds} = -A[S(t)x + \int_0^s G(t-s)S(s)x(s)ds] + S(t-s)\frac{dx(s)}{ds} = S(t-s)f(s, x(s)).
\]

Consider \( t_k < t, \quad k = 1, \ldots, m \). Then integrating the previous equation, we have

\[
\int_0^t \frac{d\omega(s)}{ds}ds = \int_0^t S(t-s)f(s, x(s))ds.
\]

For \( k = 1 \), \( \omega(t) - \omega(0) = \int_0^1 S(t-s)f(s, x(s))ds \) or \( x(t) = S(t)(x_0 - g(x)) + \int_0^1 S(t-s)f(s, x(s))ds \) for \( k = 2, \ldots, m \). Now for \( k = 2, \ldots, m \), we have that

\[
\int_0^1 \frac{d\omega(s)}{ds}ds + \int_0^2 \frac{d\omega(s)}{ds}ds + \ldots + \int_0^m \frac{d\omega(s)}{ds}ds = \int_0^m S(t-s)f(s, x(s))ds.
\]

Then \( \omega(t_k^-) - \omega(0) + \omega(t_2^-) - \omega(t_1^+) + \ldots - \omega(t_k^-) + \omega(t) = \int_0^m S(t-s)f(s, x(s))ds \) if and only if

\[
\omega(t) = \omega(0) + \int_0^m S(t-s)f(s, x(s))ds + \sum_{0<\xi_j} \omega(t_k^-) - \omega(t_k^+).
\]

Hence \( x(t) = S(t)(x_0 - g(x)) + \int_0^m S(t-s)f(s, x(s))ds + \sum_{0<\xi_j} S(t-t_k)I_k(x(t_k^-)) \), which proves the lemma.

Assume the following:

(H4) There exists \( d_k, \quad k = 1, \ldots, m \), such that \( d_H([I_k(x(t_k^-))]^a, [I_k(y(t_k^-))]^a) \leq d_k d_H([x(t)]^a, [y(t)]^a) \), where \( \sum_{k=1}^m d_k = \bar{d}, \quad t \in J \).

(H5) \( c(L\bar{d} + c_1 T) < 1 \).

Theorem 1. Let \( T > 0 \), and hypotheses (H1)-(H5) hold. Then, for every \( x_0 \in E_N \), problem (1)-(3) \( (u \equiv 0) \) has a unique solution \( x \in \Omega \).

Proof. For each \( \xi(t) \in \Omega, \quad t \in J \) define

\[
(\Phi \xi)(t) = S(t)(x_0 - g(\xi)) + \int_0^t S(t-s)f(s, \xi(s))ds + \sum_{0<\xi_j} S(t-t_k)I_k(\xi(t_k^-)).
\]

Thus, \( (\Phi \xi)(t) : J \to \Omega \) is continuous, and \( \Phi : \Omega \to \Omega \).

It is obvious that fixed points of \( \Phi \) are solution for the problem (1)-(3) \( (u \equiv 0) \). For \( \xi(t), \eta(t) \in \Omega \), we have

\[
d_H([(\Phi \xi)(t)]^a, [(\Phi \eta)(t)]^a)
\]
\[ \leq cLd_H \left( ([\xi(t)]^a, [\eta(t)]^a) + cc_1 \int_0^1 d_H \left( ([\xi(s)]^a, [\eta(s)]^a) \right) ds + c\bar{d}d_H \left( ([\xi(t)]^a, [\eta(t)]^a) \right) \right). \]

Therefore,
\[
d_{\infty,1}((\Phi \xi(t), (\Phi \eta)(t))) = \sup_{t \in (0,1)} d_H \left( ([\Phi \xi(t)]^a, ([\Phi \eta](t)]^a) \right)
\]

\[
\leq cLd_{\infty,1}((\xi(t), \eta(t))) + cc_1 \int_0^1 d_{\infty,1}(\xi(s), \eta(s)) ds + c\bar{d}d_{\infty,1}([\xi(t)]^a, [\eta(t)]^a).
\]

Hence
\[
H_1(\Phi \xi, \Phi \eta) = \sup_{t \in J} d_{\infty,1}((\Phi \xi(t), (\Phi \eta)(t))) \leq c(L\bar{d} + c_1 T)H_1(\xi, \eta).
\]

By hypotheses (H5), \( \Phi \) is a contraction mapping. By the Banach fixed point theorem, (4) has a unique fixed point \( x \in \Omega \).

### 3. Nonlocal controllability

In this section, we show the controllability for the control system (1)-(3).

The control system (1)-(3) is related to the following fuzzy integral system:
\[
x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s)(f(s, x(s)) + u(s)) ds + \sum_{0 \leq t_k < t} S(t-t_k)I_{x_k}(x(t_k))
\]

for \( t \in J, \ t \neq t_k (k = 1, 2, \ldots, m) \), where \( S(t) \) satisfies (H3).

**Definition 4.** The equation (5) is nonlocal controllable if, there exists \( u(t) \) such that the fuzzy solution \( x(t) \) of (5) satisfies \( x(T) = x^1 - g(x) \), i.e., \( [x(T)]^a = [x^1 - g(x)]^a \), where \( x^1 \) is target set.

We assume that the linear control system with respect to semilinear control system (5) is nonlocal controllable. Then
\[
x(T) = S(T)(x_0 - g(x)) + \int_0^T S(T-s)(f(s, x(s)) + u(s)) ds + \sum_{0 \leq t_k < T} S(T-t_k)I_{x_k}(x(t_k)) = x^1 - g(x)
\]

\( [x(T)]^a = [(x^1)]^a - g^a(x), (x^1)^a - g^a(x) \).

Define the \( \alpha \)-level set of fuzzy mapping \( G : P(R) \rightarrow E_N \) by
\[
G^a(v) = \begin{cases} 
\int_0^T S^a(T-s)v(s) ds, & v \in \bar{\Gamma}_a \\
0, & \text{otherwise}
\end{cases}
\]

where \( \bar{\Gamma}_a \) is closure of support of \( u \). Then there exists \( G_i^a (i = l, r) \) such that
\[
G_i^a(v_i) = \int_0^T S^a(T-s)v_i(s) ds, \quad v_i(s) \in [u_i^a(s), u_i^1(s)],
\]
\[
G_r^a(v_r) = \int_0^T S^a(T-s)v_r(s) ds, \quad v_r(s) \in [u_r^1(s), u_r^a(s)].
\]

We assume that \( G_l^a, G_r^a \) are bijective mapping. Hence \( \alpha \)-level set of \( u(s) \) is
\[
[u(s)]^a = [u_l^a(s), u_r^a(s)] = \left( [G_l^a]^{-1}(x^1) - g^a(x) - S^a_0(T)(x^0_0 - g^a(x)) - \sum_{0 \leq t_k < T} S^a_0(T-t_k)I_{x_k}^a(x(t_k)) \right)
\]
\[
-G_r^a(x^1) - g^a(x) - S^a_0(T)(x^0_0 - g^a(x)) - \sum_{0 \leq t_k < T} S^a_0(T-t_k)I_{x_k}^a(x(t_k))).
\]

Thus we can introduce \( u(s) \) of semilinear system.
\[ [u(s)]^\alpha = [u_I^\alpha(s), u_R^\alpha(s)] \]

\[ \begin{aligned}
&= \left[ (G_1^\alpha)^{-1}((x^1)^\varepsilon - g_I^\alpha(x)) - \int_0^s S_I^\alpha(T-s)f_I^\alpha(s,x(s))ds - \sum_{0<\tau_k<s} S_I^\alpha(T-\tau_k)I^\alpha_\tau(x(\tau_k)), \\
&\quad - \int_0^s S_I^\alpha(T-s)f_I^\alpha(s,x(s))ds - \sum_{0<\tau_k<s} S_I^\alpha(T-\tau_k)I^\alpha_\tau(x(\tau_k)), \\
&\quad (G_r^\alpha)^{-1}((x^1)^\varepsilon - g_r^\alpha(x)) - \int_0^s S_r^\alpha(T-s)f_r^\alpha(s,x(s))ds - \sum_{0<\tau_k<s} S_r^\alpha(T-\tau_k)I^\alpha_\tau(x(\tau_k))) \right] \end{aligned} \]

Then substituting this expression into the equation (5) yields \( \alpha \)-level of \( x(T) \).

We now set

\[ \Phi(x) = S(t)(x_0 - g(x)) + \int_0^s S(t-s)f(s,x(s))ds + \sum_{0<\tau_k<s} S(t-\tau_k)I_k(x(\tau_k)) + \int_0^s S(t-s)\tilde{G}^{-1} \]

\[ \times \left((x^1)^\varepsilon - g_I^\alpha(x) - \int_0^s S(t-s)f(s,x(s))ds - \sum_{0<\tau_k<s} S(t-\tau_k)I_k(x(\tau_k))ds \right) \]

where the fuzzy mappings \( \tilde{G}^{-1} \) satisfied above statements.

Notice that \( \Phi(x(T)) = x^1 - g(x) \), which means that the control \( u(t) \) steers the equation (5) from the origin to \( x^1 - g(x) \) in time \( T \) provided we can obtain a fixed point of nonlinear operator \( \Phi \).

Assume that the following hypotheses:

(H6) Linear system of equation (5) \( (f = 0) \) is nonlocal controllable.

(H7) \( c(L + \bar{T} + T(c_1 + cc_1 + \bar{T}T)) < 1 \).

**Theorem 2.** Suppose that (H1)-(H7) are satisfied. Then the equation (5) is nonlocal controllable.

**Proof.** We can easily check that \( \Phi \) is continuous function from \( \Omega \) to itself. For \( x, y \in \Omega \),

\[ \begin{aligned}
&d_H\left( [\Phi(x)]^\alpha, [\Phi(y)]^\alpha \right) \leq cLd_H(\{x(t)\}^\alpha, \{y(t)\}^\alpha) + c\bar{T}d_H(\{x(t)\}^\alpha, \{y(t)\}^\alpha) \\
&\quad + cc_1\int_0^T d_H(\{x(s)\}^\alpha, \{y(s)\}^\alpha)ds + c(cc_1 + \bar{T})\int_0^T d_H(\{x(r)\}^\alpha, \{y(r)\}^\alpha)drds \\
\end{aligned} \]

Therefore,

\[ \begin{aligned}
d_x(\Phi(x), \Phi(y)) &= \sup_{x \in (0,1]} d_H(\{[\Phi(x)]^\alpha, [\Phi(y)]^\alpha \} ) \\
&\leq cLd_x(x, y) + c\bar{T}d_x(x, y) + cc_1\int_0^T d_x(x(s), y(s))ds + c(cc_1 + \bar{T})\int_0^T d_x(x(r), y(r))drds.
\end{aligned} \]

Hence

\[ H_\varepsilon(\Phi, \Phi) = \sup_{x \in [0,T]} d_x(\Phi(x), \Phi(y)) \leq c((L + \bar{T}) + T(c_1 + cc_1 + \bar{T}))H_\varepsilon(x, y). \]

By hypotheses (H7), \( \Phi \) is a contraction mapping. Hence, by the Banach fixed point theorem, (5) has a unique fixed point \( x \in \Omega \).

**4. Example**

Consider the semilinear one dimensional heat equation on a connected domain \( (0,1) \) for a material with memory, boundary condition \( x(t,0) = x(t,1) = 0 \) and with initial condition \( x(0,z) \)
Let $A = \frac{\partial^{2}}{\partial z^{2}}, \sum_{k=1}^{P} c_{k} x(t_{k}, z) = g(x), \Delta t(k_{t}, z) = \Delta x(k_{t})$, \( x(t_{k}^{+}, z) - x(t_{k}^{-}, z) = I_{k}(x(t_{k})) \)

and $G(t-s) = e^{-(t-s)}$, then the balance equation becomes

$$\frac{dx(t)}{dt} = 2[x(t) - \int_{0}^{t} e^{-(t-s)} x(s)ds] + 2\alpha x(t)^{2} + u(t), \quad t \in J, \quad t \neq t_{k},$$

(7) \hspace{1cm} x(0) + g(x) = x_{0} \in E_{N}, \hspace{1cm} (8)

$$\Delta x(k_{t}) = I_{k}(x(t_{k})), \quad k = 1, 2, \cdots, m, \hspace{1cm} (9)$$

The $\alpha$- level set of fuzzy number $\tilde{2}$ is $[\tilde{2}]^{\alpha} = [\alpha + 1 - \alpha]$ for all $\alpha \in [0,1]$. Then $\alpha$- level sets of $f(t, x(t))$ is $[\{f(t, x(t))\}]^{\alpha} = \{\alpha + 1\}(x_{\alpha}(t))^{2}, (3 - \alpha)(x_{0}(t))^{2}\}$. Further, we have

$$d_{H}([\{f(t, x(t))\}]^{\alpha}, [\{f(t, y(t))\}]^{\alpha})$$

$$= d_{H}([\{\alpha + 1\}(x_{\alpha}(t))^{2}, (3 - \alpha)(x_{0}(t))^{2}, (\{\alpha + 1\}(y_{\alpha}(t))^{2}, (3 - \alpha)(y_{0}(t))^{2}])$$

$$= \max\{3\alpha(x_{\alpha}(t)^{2} - y_{\alpha}(t)^{2})^{2} + (3 - \alpha)(x_{0}(t)^{2} - y_{0}(t)^{2})^{2}\}$$

$$= \alpha \cdot d_{H}([x(t)]^{\alpha}, [y(t)]^{\alpha}),$$

where $c_{1}$ satisfies the inequality in hypothesis (H1), and also

$$d_{H}([g(x)]^{\alpha}, [g(y)]^{\alpha}) = d_{H}([\sum_{k=1}^{P} c_{k}(x(t_{k}))^{\alpha}, \sum_{k=1}^{P} c_{k}(y(t_{k}))^{\alpha}])$$

$$\leq \sum_{k=1}^{P} c_{k} \max d_{H}([x(t_{k})]^{\alpha}, [y(t_{k})]^{\alpha}) = Ld_{H}([x(t)]^{\alpha}, [y(t)]^{\alpha}),$$

where $L$ is satisfies the inequality in hypothesis (H2). Therefore $f$ and $g$ satisfy the global Lipschitz condition. Then all the conditions stated in Theorem 1 are satisfied, so the problem (7)-(9) has a unique fuzzy solution.

Let initial value $x_{0}$ is $\tilde{0}$. Target set is $X^{1} = \tilde{2}$. The $\alpha$- level set of fuzzy number $\tilde{0}$ is $[\tilde{0}]^{\alpha} = [\alpha - 1 - \alpha], \alpha \in (0,1]$. We introduce the $\alpha$-level set of $u(s)$ of equation (7)-(9).

$$[u(s)]^{\alpha} = [u_{\alpha}(s), u_{\alpha}(s)]$$

$$= [\tilde{G}_{T}^{-1}((\alpha - 1) - \sum_{k=1}^{P} c_{k} x_{\alpha}(t_{k}) - S_{\alpha}(T)((\alpha - 1) - \sum_{k=1}^{P} c_{k} x_{\alpha}(t_{k}))) - \int_{0}^{T} S_{\alpha}(T-s)u(\alpha + 1)$$

$$\times(x_{\alpha}(s))^{2} ds - \sum_{0 \leq t_{k} < T} S_{\alpha}(T-t_{k})I_{\alpha}(x(t_{k})),$$

$$\tilde{G}_{T}^{-1}((3 - \alpha) - \sum_{k=1}^{P} c_{k} x_{\alpha}(t_{k}) - S_{\alpha}(T)((3 - \alpha) - \sum_{k=1}^{P} c_{k} x_{\alpha}(t_{k}))) - \int_{0}^{T} S_{\alpha}(T-s)(3-\alpha)$$

$$\times(x_{\alpha}(s))^{2} ds - \sum_{0 \leq t_{k} < T} S_{\alpha}(T-t_{k})I_{\alpha}(x(t_{k}))).$$

Then substituting this expression into the integral system with respect to (7)-(9) yields $\alpha$-level set of $x(T)$.

$$[x(T)]^{\alpha} = [((\alpha + 1) - \sum_{k=1}^{P} c_{k} x_{\alpha}(t_{k}),(3 - \alpha) - \sum_{k=1}^{P} c_{k} x_{\alpha}(t_{k})) = [\tilde{2} - \sum_{k=1}^{P} c_{k} x(t_{k})]^{\alpha}.$$
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