Abstract. We classify the Montesinos links up to mutation and 5-move equivalence, and obtain from this a Jones and Kauffman polynomial test for a Montesinos link.

Keywords: rational link, Montesinos link, pretzel link, \( n \)-move, Jones polynomial, Kauffman polynomial

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We call two links \( k \)-move equivalent, or simpler \( k \)-equivalent, if they are related by a sequence of \( k \)-moves (or their inverses). The case \( k = 2 \) is the usual crossing change. The cases \( k = 3, 4 \) were connected to two longstanding problems. The 3-move conjecture stated that all links are 3-move equivalent to a trivial link. It was refuted only recently in [DP]. The conjecture is known to be true for many links, for example 3-algebraic links [PT], links of braid index 4 and 5 [Ch] (latter with the exception of one equivalence class, which later provided a counterexample), and knots of weak genus two [St3]. The 4-move conjecture states that all knots are 4-move equivalent to the unknot. This conjecture remains open, though partial confirmations exist, and (in the general case) counterexamples are suspected (see [As, P2], and also [St3]).

The difficulty with the cases \( k = 3, 4 \) is that, while equivalence classes are rather few and large, (possibly exactly because of this) it is impossible to obtain essential information on them using polynomial (or other easy to compute) invariants.

However, the case \( k = 5 \) is different. Now there are some invariants that come from the Jones \( V \) and Kauffman \( F \) polynomial. Unfortunately, the equivalence classes become many, and their intersection with a meaningful family of links is difficult to describe. Recently, Przytycki-Dąbkowski-Ishiwata (see [I]) succeeded in determining the classes for rational (or 2-bridge) links. The aim of this paper is to extend their result to Montesinos links (up to mutation); see theorems 3.2 and 4.2.

The motivation for our work is to gain a condition on the polynomial invariants of Montesinos links. A few such conditions were obtained for partial classes, but all conditions for a general Montesinos link we have so far are inefficient, or not easy to test.

In [LT] semiadequate links were introduced. It was observed that Montesinos links are semiadequate, and that for such links one of the leading or trailing coefficients of the Jones polynomial \( V \) must be \( \pm 1 \). (In [St4] we understood also coefficients 2 and 3, and in particular proved that the Jones polynomial is non-trivial.) However, this property is not helpful as a Montesinos link test, because it is satisfied for many (other) links. Similarly impractical is the semiadequacy condition on the Kauffman \( F \) polynomial of [Th]. (The simplest knots whose polynomial shows negative “critical line” coefficients on either side have 15 crossings; see [St4].) For the Alexander polynomial no conditions whatsoever are known, that is, it is possible that every admissible Alexander polynomial is realized by a Montesinos link. In [LT] it was shown how to determine the crossing number of a Montesinos link from the \( V \) and \( F \) polynomial, but a(n extensive, though systematic) diagram verification is not promising either as a Montesinos link test.

Now, in contrast, since we can evaluate the 5-move invariants of the Jones and Kauffman polynomial on the classes of Montesinos links we obtain (theorems 4.1 and 6.1), we gain a condition on these polynomials, which turns out easy to verify. (It makes no assumption on the diagram we perform it on, except, of course, that one can evaluate the polynomials.) We will give some examples that show how to apply our test.

# Preliminaries

## Polynomial invariants

Let \( V \) be the Jones polynomial [J], \( Q \) the BLMH polynomial and \( F \) the Kauffman polynomial [Kf]. We give a basic description of these invariants.

Recall first the construction of the Kauffman bracket in [Kf2]. The Kauffman bracket \( \langle D \rangle \) of a diagram \( D \) is a Laurent polynomial in a variable \( A \), obtained by summing over all states the terms

\[
A^{#A - #B} \left( -A^2 - A^{-2} \right)^{|S|-1}.
\]  

A state is a choice of splittings of type \( A \) or \( B \) for any single crossing (see figure 1), \(#A\) and \(#B\) denote the number of type \( A \) (resp. type \( B \)) splittings and \(|S|\) the number of (disjoint) circles obtained after all splittings in a state.

The Jones polynomial of a link \( L \) is related to the Kauffman bracket of some diagram \( D \) of \( L \) by

\[
V_L(t) = \left( -t^{-3/4} \right)^{-w(D)} \langle D \rangle \bigg|_{A=t^{-1/4}}.
\]
2.2 Families of links

The Kauffman polynomial \([Kf]\) is usually defined via a regular isotopy invariant \(\Lambda(a, z)\) of unoriented links. We use here a slightly different convention for the variables in \(F\), differing from \([Kf, Th]\) by the interchange of \(a\) and \(a^{-1}\). Thus in particular we have the relation \(F(D)(a, z) = a^{w(D)} \Lambda(D)(a, z)\), where \(w(D)\) is the writhe of a link diagram \(D\), and \(\Lambda(D)\) is the writhe-unnormalized version of \(F\). \(\Lambda\) is given in our convention by the properties

\[
\Lambda\left(\begin{array}{c}
\Box
\end{array}\right) + \Lambda\left(\begin{array}{c}
\Box
\end{array}\right) = z \left( \Lambda\left(\begin{array}{c}
\Box
\end{array}\right) + \Lambda\left(\begin{array}{c}
\Box
\end{array}\right) \right),
\Lambda\left(\begin{array}{c}
\Box
\end{array}\right) = a^{-1} \Lambda\left(\begin{array}{c}
\Box
\end{array}\right); \quad \Lambda\left(\begin{array}{c}
\Box
\end{array}\right) = a \Lambda\left(\begin{array}{c}
\Box
\end{array}\right),
\Lambda\left(\begin{array}{c}
\Box
\end{array}\right) = 1.
\]

(4)

The BLMH polynomial \(Q\) is most easily specified by \(Q(z) = F(1, z)\).

2.2 Families of links

In the following we define rational, pretzel and Montesinos links according to Conway [Co].

\[
\begin{array}{ccccccc}
\Box & \Box & \Box & \Box & \Box & \Box & \Box \\
\infty & 0 & 1 & -1 & 4
\end{array}
\]

\[
\begin{array}{cccc}
\Box & \Box & \Box & \Box \\
\sum P, Q & \Box & \Box & \Box \\
\text{product } PQ & \Box & \Box & \Box \\
\text{closure } P & \Box & \Box & \Box \\
-2 & -3 & 4 & 2
\end{array}
\]

Figure 2: Conway’s tangles and operations with them. (The designation ‘product’ is very unlucky, as this operation is neither commutative, nor associative, nor is it distributive with ‘sum’. Also, ‘sum’ is associative, but not commutative.)

Definition 2.1 A tangle diagram is a diagram consisting of strands crossing each other, and having 4 ends. A rational tangle diagram is the one that can be obtained from the primitive Conway tangle diagrams by iterated left-associative product in the way displayed in figure 2. (A simple but typical example of is shown in the figure.)

Let the continued (or iterated) fraction \([s_1, \ldots, s_r]\) for integers \(s_i\) be defined inductively by \([s] = s\) and

\[
[s_1, \ldots, s_r] = s_r + \frac{1}{[s_1, \ldots, s_{r-1}]}.
\]
The rational tangle \( T(p/q) \) is the one with Conway notation \( c_1 c_2 \ldots c_n \), when the \( c_i \) are chosen so that
\[
\left[ [c_1, c_2, c_3, \ldots, c_n] \right] = \frac{p}{q}.
\]
Equation (5)

One can assume without loss of generality that \( (p,q) = 1 \), and \( 0 < q < |p| \). A rational (or 2-bridge) link \( S(p,q) \) is the closure of \( T(p/q) \).

Montesinos links (see e.g. [LT]) are generalizations of pretzel and rational links and special types of arborescent links. They are denoted in the form \( M(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}, e) \), where \( e, p_i, q_i \) are integers, \( (p_i, q_i) = 1 \) and \( 0 < |q_i| < p_i \). Sometimes \( e \) is called the integer part, and \( n \) the length of the Montesinos link. If \( e = 0 \), it is omitted in the notation.

To visualize the link, let \( p_i/q_i \) be continued fractions of rational tangles \( c_1,c_2 \ldots c_n \) with \( \left[ [c_1,c_2,c_3,\ldots,c_n] \right] = \frac{e}{q_i} \).

Then \( M(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}, e) \) is the link that corresponds to the Conway notation
\[
(c_1,1\ldots c_1,1), (c_1,2\ldots c_2), \ldots, (c_{1,n}\ldots c_n,n), e0.
\]

The defining convention is that all \( q_i > 0 \) and if \( p_i < 0 \), then the tangle is composed so as to give a non-alternating sum with a tangle with \( p_i+1 > 0 \). This defines the diagram up to mirroring.

An easy exercise shows that if \( q_i > 0 \) resp. \( q_i < 0 \), then
\[
M(\ldots, q_i/p_i, \ldots, e) = M(\ldots, (q_i + p_i)/p_i, \ldots, e \pm 1),
\]
i.e. both forms represent the same link (up to mirroring).

In the following the mirroring convention in the notation \( M(q_1/p_1, \ldots, q_n/p_n, e) \) will be so that a \( +1 \) integer twist in \( e \) is a crossing whose \( A \)-splicing gives an \( \infty \)-tangle. So for example, the positive (right-hand) trefoil is \( M(-3) \).

(If \( e = 0 \), use (6) to make it \( \pm 1 \), and specify the mirroring accordingly.)

A typical example is shown on figure 3.

![Figure 3: The Montesinos knot \( M(3/11, -1/4, 2/5, 4) \) with Conway notation \((213, -4, 22, 40)\).](image)

If the length \( n < 3 \), an easy observation shows that the Montesinos link is in fact a rational link. A pretzel link is a Montesinos link with all \( |q_i| = 1 \). Geometric properties of Montesinos links are discussed in detail in [BZ].

### 3 5-equivalence of Montesinos links

**Definition 3.1** We denote the equivalence of polynomials in \( \mathbb{Z}[t^{\pm 1/2}] \) modulo \( X = (t^5 + 1)/(t + 1) \) and multiplication with \( \pm t^{k/2} \) by \( \hat{=} \). By \( \tilde{V} \) we denote the equivalence class of a polynomial \( V \) under \( \hat{=} \). For \( \epsilon \in \mathbb{C} \), we denote by \( \hat{\epsilon} \) the set of complex numbers obtained from \( \epsilon \) by multiplication with a 20-th root of unity. If \( \hat{\epsilon}_1 = \hat{\epsilon}_2 \), we write also \( c_1 \simeq c_2 \).
Proposition 3.1 (Przytycki-Ishiwata [I]) \( \bar{V} \) is an invariant of 5-moves.

The roots of \( X = (t^5 + 1)/(t + 1) \) are the primitive 10-th roots of unity. These are two pairs of conjugate complex numbers. Since, when working with real coefficients, conjugate complex numbers are equivalent, we have two roots to work with, \( e^{\pi i/5} \) and \( e^{3\pi i/5} \). In particular we obtain from the previous proposition:

Corollary 3.1 (Przytycki-Ishiwata) For a link \( L \) and \( n = 1, 3 \), let \( x_n(L) = V(\bar{L}, e^{n\pi i/5}) \). Then the quantities \( \gamma_n = \bar{x}_n \), and in particular \( v_n = |x_n| \) are invariants of 5-moves.

Concerning \( Q \), we have the following easy to show fact (see for example lemma 6.1 below).

Proposition 3.2 \( Q(2\cos\frac{2\pi}{5}) \) and \( Q(2\cos\frac{4\pi}{5}) \) are invariants of 5-moves.

These values of \( Q \) are studied by Jones [J2] and Rong [R], who show that they are equal to \( \pm \sqrt{5} d(K) \), where \( d(L) := \dim_{\mathbb{Z}_5} H_1(D_L, \mathbb{Z}_5) \) is the number of torsion numbers divisible by 5 of the homology of the 2-branched cover \( D_L \). This special form makes the \( Q \) values of limited use as detectors of 5-move (in)equivalence, but nonetheless one should not write them off. In particular the sign can be useful in some cases.

Ishiwata [I] succeeded in determining the 5-move equivalence classes of rational tangles.

Theorem 3.1 (Ishiwata) Rational tangles are 5-move equivalent to one of the 12 basic tangles
\[
0, \infty, \pm 1, \pm \frac{1}{2}, \pm \frac{3}{2}, \frac{2}{5}, \frac{5}{2}.
\]

One obtains immediately from that the result for rational (2-bridge) links.

Corollary 3.2 (Ishiwata) Rational links are 5-move equivalent to one of the unknot, 2-component unlink, figure-8 knot or Hopf link.

This means in particular that \( v_1 \) and \( v_3 \) take only 4 values on rational links. (They are indeed different for the 4 classes, so the classes are distinct.) Nonetheless, some very strong conditions on polynomial invariants of 2-bride links are known. For the Alexander polynomial such a condition is formulated in [Mu]. The test of Kanenobu [K] using \( V \) and \( Q \) is particularly efficient; see also [St]. Here we use some of Ishiwata’s work to extend her result to Montesinos links. Our goal is to obtain conditions on the polynomial invariants of Montesinos links (which are much fewer in previous literature, and less efficient).

Theorem 3.2 A Montesinos link is equivalent up to 5-moves and mutations to one of

1. \( M(-1/2, \ldots, -1/2, 1/2, \ldots, 1/2) \) with \( k \geq 0 \) entries \( 1/2 \) and \( 0 \leq l \leq 4 \) entries \( -1/2 \), s.t. \( l + k \geq 3 \),
2. \( M(1/2, \ldots, 1/2, 2/5, \ldots, 2/5) \) with \( k \geq 0 \) entries \( 1/2 \), and \( 1 \leq l \) entries \( 2/5 \), s.t. \( l + k \geq 3 \),
3. the connected sum of some number (possibly no) of Hopf links, with possible trivial split components,
4. the connected sum of some number (at least one) of figure-8 knots with a link of Form 3, or
5. \( M(1/2, 1/2, 1/2, 1) \).

Proof. We apply first theorem 3.1 to reduce all tangles to the 12 basic ones. If one of the tangles is \( \infty \), we are down to the connected sum of rational links, and with corollary 3.2 have the third or fourth form.

So we assume henceforth that we have no \( \infty \) tangle. Then the 11 remaining basic tangles have non-integer parts \( \pm 1/2 \) and \( 2/5 \). We assume that at least three non-integer parts occur, otherwise we have again a rational link.
Now assume that at least one tangle $2/5$ occurs. Then we use the observation of Ishiwata (made in the proof of theorem 3.1) that the $2/5$ tangle acts (up to 5-moves) as an “annihilator” of integer twists. Since one can make all $-1/2$ to $1/2$ up to integer twists, we have the second form.

So assume below that we have only $\pm 1/2$ and possible integer twists. It is an easy observation that 5 copies of $1/2$ are 5-move equivalent to 5 copies of $-1/2$, so we can always make the $-1/2$ to be at most 4. Moreover, the normal form of Montesinos links teaches that if have $n \neq 0$ integer twists, then we can assume that all $\pm 1/2$ have the sign of $n$. (If $n = 0$ we have the first form.)

Assume first $n > 0$. Clearly we can achieve that $n < 5$. Now if the number $l$ of $1/2$ satisfies $l \geq 5 - n$, we can make the $n$ integer twists to $5 - n$ of opposite sign, and annihilate them making $5 - n$ of the $l$ copies of $1/2$ to $-1/2$. Since $0 < n < 5$ and $l \geq 3$, there is only one case where this procedure does not work, namely when $l = 3$ and $n = 1$. In this case we have the link of the fifth form. Otherwise we achieve the first form.

If $n < 0$ we can apply the same argument, since the family of links of the first form is invariant (by making the $-1/2$ to be $< 5$ under 5-equivalence) to the family of their mirror images. Also, the link in the fifth form (as well as all links of Forms 2, 3 and 4) is equivalent to its mirror image. □

**Remark 3.1** By omission of Form 2, we obtain an analogous result for pretzel links. The (not very natural) distinction of Forms 3 and 4 will become clear with theorem 4.1 below.

# 4 Jones polynomial of Montesinos links

## 4.1 Formulas for the Jones polynomial

We succeeded in evaluating $\bar{\nu}$ on the representatives in theorem 3.2, and so we have

**Theorem 4.1** For a Montesinos link $L$, the reduced Jones polynomial $\bar{\nu}_L$ equals one of

1. Form 1, with $k + l \geq 3, k, l \geq 0, l \leq 4$

$$V_1(k, l) = \frac{(-t - 1/t)^{k+l} + (1/t)^l(1-t)^{k+l}(t+1+1/t)}{-\sqrt{t} - \frac{1}{\sqrt{t}}}$$

2. Form 2, with $k + l \geq 3, k, l - 1 \geq 0$

$$V_2(k, l) = \frac{(1-t^2)^l(1-t)^k(t+1+1/t)}{-\sqrt{t} - \frac{1}{\sqrt{t}}}$$

3. Form 3, with $k, l \geq 0$

$$V_3(k, l) = \left(-\sqrt{t} - \frac{1}{\sqrt{t}}\right)^l \left(-t - \frac{1}{t}\right)^k$$

4. Form 4: 0

5. Form 5:

$$V_5 = \frac{(-t - 1/t)^3 - (1-t)^2(t+1+1/t)}{-\sqrt{t} - \frac{1}{\sqrt{t}}}$$

The proof of this theorem bases on routine Kauffman bracket skein module calculation, and we omit it. (The reader may consult [St2] for some explanation on this kind of calculation.) The given polynomials correspond (up to units) to the Jones polynomials of the knots, except for Form 2, where the numerator was simplified by reducing modulo $X = (t^5 + 1)/(t+1)$. (For the calculation, we also used $M(-1/2, \ldots, -1/2, 2/5, \ldots, 2/5)$, which is equivalent.) The fraction may therefore not be a genuine polynomial in $\mathbb{Z}[t^{\pm 1}]$. Instead it should be considered lying in the field $\mathbb{Z}[t^{\pm 1}]/ < X >$ (where the division is possible).
Remark 4.1 We should remark the following relation of these forms to \( d(L) \).

1. Form 1: \( d(L) \leq 1 \). Whether \( d(L) = 0 \) or \( 1 \) can then be decided by looking at the (5-divisibility of the) determinant. By a simple calculation we find

\[
d(L) = \begin{cases} 
0 & \text{if } 5 \nmid k - l \\
1 & \text{if } 5 \mid k - l 
\end{cases}
\]

2. Form 2: \( |d(L) - l| \leq 1 \). This follows because by a change from a 0 tangle to an infinity tangle, \( d(L) \) is altered by at most \( \pm 1 \), and we can obtain the connected sum \( L' \) of \( k \) Hopf links and \( l \) figure-8-knots, and \( d(L') = l \).

3. Form 3: \( l = d(L) \)

4. Form 4: \( d(L) > 0 \)

5. Form 5: By direct calculation the determinant is 20, so \( d(L) = 1 \).

For the following calculations, it is useful to compile a few norms of complex numbers that will repeatedly occur.

| norm of \( t \) | \( t = e^{\pi i/5} \) | \( t = e^{3\pi i/5} \) |
|-----------------|--------------------------|--------------------------|
| \( 1 - t \) | \( \frac{\sqrt{5} - 1}{2} \approx 0.618033988 \) | \( \sqrt{2 + \sqrt{3} - \sqrt{5}} \approx 1.6952970385 \) |
| \( 1 + t \) | \( \frac{\sqrt{2\sqrt{5} + 10}}{2} \approx 1.9021130325 \) | \( \sqrt{2 - \sqrt{3} - \sqrt{5}} \approx 1.0611637019 \) |
| \( 1 + t^2 \) | \( \frac{\sqrt{5} + 1}{2} \approx 1.618033988 \) | \( 3 - \sqrt{5} \approx 0.874032048897 \) |
| \( 1 - t^2 \) | \( \frac{\sqrt{5} - \sqrt{3}}{2} \approx 1.17557050458 \) | \( \frac{2\sqrt{5} + 10}{2} \approx 1.9021130325 \) |
| \( 1 + t + t^2 \) | \( \frac{\sqrt{5} + 3}{2} \approx 2.618033988 \) | \( 1 - \sqrt{3} - \sqrt{5} \approx 0.125967951102 \) |

Using this table, one easily sees

Corollary 4.1 The set

\[ \{ |V(e^{\pi i/5})|, |V(e^{3\pi i/5})| \} : L \text{ is a Montesinos link} \]

as a subset of \( \mathbb{R}^2 \) is discrete.

Discrete is to be understood here so that the intersection with any ball is finite. (It is in fact in size at most logarithmic in the radius of the ball.)

Proof. We can treat the families \( V_{1,2,3} \) separately. In case of \( V_{1,3} \), the invariant \( v_1 \) diverges (exponentially) when \( k + l \to \infty \). For \( V_2 \), take \( v_3 \).

\[ \square \]

The pretzel links are those for which Form 2 does not occur, and so we have

Corollary 4.2 The set

\[ \{ |V(e^{\pi i/5})| : L \text{ is a pretzel link} \} \subset \mathbb{R} \]

is discrete.

\[ \square \]

Note that this gives a very strong condition on the polynomials of pretzel or Montesinos links. For a general link, there is no particular feature of \( v_1 \) or \( v_3 \) to be expected. For arbitrary links they will be dense in \( \mathbb{R}_+ \). Jones claimed in [J] (and I wrote down an argument in [St3]) that \( v_1 \) is dense in an interval on closed 4-braids. On the opposite hand, while several other conditions on invariants of Montesinos links are known, these are all difficult to test and/or assume additional properties of the diagram.
For example a link $L$ is Montesinos if and only if $\pi_1(D_L)$ is finite (where $D_L$ is the double branched cover). While it is easy to gain a presentation of $\pi_1(D_L)$ from any diagram of $L$, the finiteness decision problem from this presentation is highly difficult.

Also, the work in [LT, Th] implies that any Montesinos link has a minimal (in crossing number) Montesinos diagram. However, it is not clear how to find this diagram starting from a given one. If the link is alternating, all minimal diagrams will be Montesinos diagrams. However, for non-alternating knots there may be even non-Montesinos minimal crossing diagrams; the knots $10_{145}$, $10_{146}$ and $10_{147}$ are examples.

In contrast, our result allows for a (not completely exact but) efficient test of the Montesinos property. It is in fact this condition, arising from Ishiwata’s result and the detection of 5-moves by the polynomials, that motivated the present work. We give some examples of the application of our condition in §5. First we show, however, that the invariants we have in hand suffice to distinguish all equivalence classes in theorem 3.2.

### 4.2 Completing the classification

**Theorem 4.2** No two forms in theorem 3.2 are 5-move equivalent. In other words, the links in theorem 3.2 determine exactly the equivalence classes of Montesinos links under mutation and 5-moves.

**Remark 4.2** The massive failure of polynomial invariants to detect mutation results in a serious lack of tools to examine 5-equivalence of mutants. While it is almost certain that such examples exist, probably no simpler invariants than the (almost incomputable) Burnside groups [DP] could be applied to show it. However, note that at least it is known that any mutant of a Montesinos link is again a Montesinos link, so that the study of 5-equivalence of mutants among Montesinos links would reduce to the mutants of the representative links in theorem 4.1. This is (basically) the question whether the 1/2,2/5 Montesinos tangle (the one with Conway notation 2,22) is 5-equivalent to its (mutant) 2/5,1/2 Montesinos tangle. In particular, our classification of 5-move equivalence of pretzel links (remark 3.1) applies without mutation.

**Proof.** We check case by case possible duplications of $\tilde{V}_L$ and $d(L)$. Often $V_1$ and $V_3$, or sometimes at least $\gamma_1$ and $\gamma_3$, already suffice to distinguish the forms. We will show that all these invariants coincide only if the links are identical (except the final case, where we are lead to apply the $Q$ polynomial to arrive at this conclusion). We remark also that $V_5 = V_1(4,-1)$. Although the value $l = -1$ in Form 1 does not make sense knot-theoretically, from the point of view of formal algebra this identity will allow us to handle Form 5 often in the same way as Form 1.

We also make the convention that is a comparison ‘Form x vs Form y’, the integers $k,l$ are parameters that correspond to ‘Form x’, while $k',l'$ correspond to ‘Form y’.

- **Form 1 vs Form 1:** We need to deal with
  \[ V_1(k,l) \cong V_1(k',l'). \]  \tag{8}  

Looking at $t = e^{\pi i/5}$, we find using the above values and $k+l \geq 3$,
  \[ \left| (-1/t) \left( (1-t)^{k+l} t + 1/t \right) \right| \leq \left( \frac{\sqrt{5}-1}{2} \right)^3 \frac{\sqrt{5}+3}{2} = \frac{\sqrt{5}-1}{2}. \]

Using this inequality and the analogous one with $k,l$ replaced by $k',l'$, and taking norms in (8) we find
  \[ \left| -t - \frac{1}{t} \right|^{k+l} \leq \sqrt{5} - 1. \]  \tag{9}  

But $\left| -t - \frac{1}{t} \right| = \frac{\sqrt{5}+1}{2}$, and $k+l, k'+l' \geq 3$, so we see that $k+l = k'+l'$. We will argue that then (8) gives $5 \mid l-l'$, so $(k,l) = (k',l')$, as desired.
Indeed, set $\tilde{k} = k + l = k' + l'$. Now it is more convenient to use $\gamma_n$ instead of $v_n$. So assume there are two different values $l_{1,2}$ of $l = 0, \ldots, 4$, such that $V_1(\tilde{k} - l, l)$ (are different but) differ by a 20-th root of unity. Now

$$|t | V_1(\tilde{k} - l, l) | \geq \left| t - \frac{1}{r} \right|^k - \left| 1 - t \right|^k \left| 1 + t + \frac{1}{r} \right|.$$ 

So $V_1(\tilde{k} - l_1, l_1) \cong V_1(\tilde{k} - l_2, l_2)$ implies that

$$|t + 1| \cdot |V_1(\tilde{k} - l_1, l_1) - V_1(\tilde{k} - l_2, l_2)| \geq 1 - e^{\pi/10} \cdot \left| t - \frac{1}{r} \right|^k - \left| 1 - t \right|^k \left| 1 + t + \frac{1}{r} \right|.$$ 

On the other hand, by the definition of $V_1$,

$$|t + 1| \cdot |V_1(\tilde{k} - l_1, l_1) - V_1(\tilde{k} - l_2, l_2)| \leq \frac{\sqrt{2 \sqrt{5} + 10}}{2} \cdot \left| 1 - t \right|^k \left| 1 + t + \frac{1}{r} \right|,$$

where the first factor on the right stands for the largest distance between two 5-th roots of unity (which are $-1/r$). Setting $t = e^{\pi/5}$, and combining the last two inequalities, we obtain

$$\frac{\sqrt{2 \sqrt{5} + 10}}{2} \cdot \left( \frac{\sqrt{5} - 1}{2} \right)^k \left( \frac{\sqrt{5} + 3}{2} \right) \geq 1 - e^{\pi/10} \cdot \left[ \left( \frac{\sqrt{5} + 1}{2} \right)^k - \left( \frac{\sqrt{5} - 1}{2} \right)^k \right].$$

So with $1 - e^{\pi/10} = \sqrt{2 - \sqrt{\frac{5+\sqrt{5}}{2}}}$, we have

$$\left( \frac{\sqrt{5} + 1}{2} \right)^k \leq \left( \frac{\sqrt{5} - 1}{2} \right)^{k-2} \left[ 1 + \frac{\sqrt{2 \sqrt{5} + 10}}{2 \sqrt{2 - \sqrt{\frac{5+\sqrt{5}}{2}}}} \right].$$

Evaluating the second factor on the right, we have

$$\left( \frac{\sqrt{5} + 1}{2} \right)^{2k-2} \leq 7.0798 \ldots,$$

so $k \leq 3$, that is, $k = 3$. These cases are easily checked directly. By direct calculation, we find that $e_1$ and $e_3$ distinguish the cases $l = 0, 3$ from those $l = 1, 2$. Then one verifies that

$$\left( \frac{V_1(l, 3 - l) (e^{\pi/5})}{V_1(3 - l, l) (e^{\pi/5})} \right)^{20} \neq 1$$

for $l = 0, 1$, and this case is finished.

- **Form 1 vs Form 2:** We assume $V_1(k, l) \cong V_2(k', l')$,

which means

$$\left( -t - \frac{1}{r} \right)^{k+l} + \left( -1 \right)^{l} \left( 1 - t \right)^{k+l} \left( t + 1 + \frac{1}{r} \right) \cong (1 - t)^k \left( t + 1 + \frac{1}{r} \right) \left( \frac{1 - t^2}{1-t} \right)^{l'}.$$ 

Using remark 4.1, we see that for a link $L$ fitting into both forms we have $d(L) \leq 1$, and so $l' \leq 2$. For $t = e^{\pi/5}$, the base of the rightmost exponent in (11) is of norm $> 1$. We take norms in (11) and bring the second summand on the l.h.s. to the right. We obtain analogously to (9)

$$\left| -t - \frac{1}{r} \right|^{k+l} \leq \frac{\sqrt{5} - 1}{2} \left( 1 + \frac{2 \sqrt{5} + 10}{4} \right) \approx 2.85 \ldots \left( \frac{\sqrt{5} + 1}{2} \right)^3,$$

which is impossible.
• Form 1 vs Form 3: Let $\tilde{k} = k' + l' \geq 0$. We must consider

\[
\left( -t - \frac{1}{t} \right)^{k+l} + \left( -t - \frac{1}{t} \right)^l \left( 1 + \frac{1}{t} \right) = \left( -t - \frac{1}{t} \right)^{\tilde{k}} \left( -\sqrt{t} - \frac{1}{\sqrt{t}} \right)^{l'}.
\]  

(12)

By remark 4.1 we have $l' = 0$ if $5 \mid k - l$ and $l' = 1$ otherwise. So

\[
|1 - t^{k+l}| + t + \frac{1}{t} \leq |t^2 + 1|^{k+l} + |t + 1| \cdot |t^2 + 1|^{\tilde{k}} \max \left( 1, \frac{|t + 1|}{|t^2 + 1|} \right).
\]

Look first at $t = e^{\pi i/5}$. Since $|t + 1| = \sqrt{2 - \sqrt{3 - \sqrt{5}}}$ and $|t^2 + 1| = \sqrt{3 - \sqrt{5}} < 1$, we have the second alternative in the maximum. Then

\[
\left( 2 + \sqrt{3 - \sqrt{5}} \right) - \sqrt{2 - \sqrt{3 - \sqrt{5}}} + \sqrt{3 - \sqrt{5}} \cdot \left( 2 - \sqrt{3 - \sqrt{5}} \right).
\]

So

\[
\sqrt{2 + \sqrt{3 - \sqrt{5}}} \leq \frac{\sqrt{3 - \sqrt{5}}^4 + 2 - \sqrt{3 - \sqrt{5}}}{\sqrt{3 - \sqrt{5}}}.
\]

and by calculation

\[
1.6952970385^{k+l} \leq 15.5273358421,
\]

so $k + l \leq 5$. Now look at $t = e^{\pi i/5}$. Taking norms in (12), and this time minimizing w.r.t. $l' = 0, 1$, we find

\[
\left( \frac{\sqrt{3} + 1}{2} \right)^{k+l} + \left( \frac{\sqrt{3} - 1}{2} \right)^{k+l} = \left( \frac{\sqrt{3} + 1}{2} \right)^{k+l} + \left( \frac{\sqrt{3} + 1}{2} \right)^{2-k-l} \geq \left( \frac{\sqrt{3} + 1}{2} \right)^k \frac{2 \sqrt{5} + 10}{2}.
\]

The biggest l.h.s. evaluates at $k + l = 5$ to $\approx 11.326237921249$, and then

\[
\left( \frac{\sqrt{3} + 1}{2} \right) \leq \frac{2 \cdot 11.326237921249}{2 \sqrt{5} + 10} \approx 5.9545556584,
\]

so $\tilde{k} \leq 3$. So it remains to check the cases $k' + l' \leq 3$, $3 \leq k + l \leq 5$ (with $k, l, k' \geq 0$, $l \leq 4$; and $l' = 1$ if $5 \mid k - l$ and $l' = 0$ otherwise). It is easy to perform (still better by computer) these handful of comparisons; $\nu_3$ distinguishes all such $V_1(k, l)$ and $V_3(k', l')$.

• Form 1 vs Form 4: The same estimate as with 'Form 1 vs Form 1' works.

• Form 1 vs Form 5: We can apply the same argument as with 'Form 1 vs Form 1', since we did not use that $l > 0$ there. Then again we need to check only $\tilde{k} = 3$. Direct calculation shows that $e_1$ and $e_3$ distinguish the case $l = -1$ (in Form 5) from $l = 0, 3$ and $l = 1, 2$.

• Form 2 vs Form 2:

\[
(1 - t^2)^{l'} (1 - t)^{k'} = (1 - t^2)^{l'} (1 - t)^{k'}.
\]

Since $|l - d(L)|, |l' - d(L)| \leq 1$, we have $|l - l'| \leq 2$. Since for $l = l'$ we are easily done, assume w.l.o.g. $l' - l \in \{1, 2\}$. Then

\[
|1 - t|^{k-k'} \in \left\{ \frac{1}{|1 - t^2|}, \frac{1}{|1 - t^2|^2} \right\}.
\]

For $t = e^{\pi i/5}$ the numbers on the right are 0.850.., 0.723.., while the sequence on the left is 1, 0.618.., 0.381.., 0.236.. etc.
For Form 1, one should take polynomial. The preceding discussion explains how to proceed to test the Montesinos property of some link $5$ Applications

- Form 2 vs Form 3: We have to check

$$\left(1 - t^2\right)^l \left(1 - t\right)^k \left(t + 1 + \frac{1}{t}\right) \leq \left(-\sqrt{7} - \frac{1}{\sqrt{7}}\right)^{l+1} \left(-t - \frac{1}{t}\right)^k.$$  

Now by remark 4.1, $|l - l'| \leq 1$, so $\bar{l} := l' + 1 - l = 0, 1, 2$. Taking norms in (13) and using $\bar{l} = 0, 1, 2$, we have

$$|1 - t|^{l+k} = \frac{|t + 1| |t^2 + 1|^{k'}}{|1 + t + \frac{1}{t}|}.$$  

Now for $t = e^{\pi i/5}$, we get

$$\left(\frac{\sqrt{5} - 1}{2}\right)^{k+l} = \frac{\left(\sqrt{5} + 1\right)^{k'} \left(\sqrt{2\sqrt{5} + 10}\right)^{\bar{l}}}{\sqrt{5} + 3} = \frac{\left(\sqrt{5} + 1\right)^{k'-2} \left(\sqrt{2\sqrt{5} + 10}\right)^{\bar{l}}}{\sqrt{5} + 3}.$$  

Now the r.h.s. is at least $\left(\frac{\sqrt{5} - 1}{2}\right)^2$, while the l.h.s. for $k + l \geq 3$ is at most $\left(\frac{\sqrt{5} - 1}{2}\right)^3$, a contradiction.

- Form 2 vs Form 4: trivial

- Form 5 vs Form 2: We apply the same argument as with ‘Form 1 vs Form 2’.

- Form 3 vs Form 3: If $V_3(k, l) \equiv V_3(k', l')$, we can assume $|l - l'| \leq 2$ as in remark 4.1, so, as in ‘Form 2 vs Form 2’, we have

$$\left\lfloor \frac{1}{t} \right\rfloor \bar{l}^{-k} \bar{l}^{k'} \in \left\{ \frac{1}{1 + t} \right\}.$$  

For $t = e^{\pi i/5}$, we have $|\frac{1}{t} - t| \approx 1.61803398$ and $|1 + t| \approx 1.9021130325$, so the terms on the right are approx. 0.525 and 0.276, while the left side ranges in 0.618, 0.381, 0.236 etc.

- Form 3 vs Form 4: trivial

- Form 5 vs Form 3: By the same norm estimate as in ‘Form 1 vs Form 3’, we are left with $k' + l' \leq 3$, and since $d(L) = 1$ in Form 5, also $l' = 1$, so $k' \leq 2$. So it remains to check $0 \leq k' \leq 2$, $l' = 1$. A check of $V_3$ and the three possible $V_3(k', l')$ rules out $k' = 1, 2$. The case $k' = 0$, however, is not ruled out; neither it is by $V_1$, and in fact not even by $\bar{V}$. It is the question of 5-equivalence of the link $7^3_4$ in [Ro, appendix] (which is the last link in theorem 4.1) and the 2-component unlink. This problem was encountered also in Ishiwata’s tabulation [I] of 5-equivalence of links up to 9 crossings. The problem is now resolved using the $Q$ polynomial. Clearly for both links $d(L) = 1$. However, exactly the sign, which $Q(2 \cos 2\pi/5)$ contains additionally, manages to distinguish the two links.

- In Form 4 vs Form 4, Form 4 vs Form 5 and Form 5 vs Form 5 there is nothing to do, and so our proof is complete.

5 Applications

The preceding discussion explains how to proceed to test the Montesinos property of some link $L$ using the Jones polynomial.

For Form 1, one should take $t = e^{\pi i/5}$. Then $|t^2 + 1| > 1 > |1 - t|$. Increase $m = k + l$ from 3 on, as long as

$$|t^2 + 1|^m \leq |t + 1| \cdot v_1(L) + |1 - t|^3 \cdot |t + 1 + 1/t|.$$
If
\[ |t^2 + 1|^m - |t + 1| \cdot v_1(L) | \leq |1 - t|^m \cdot |t + 1 + 1/t|, \]
then compare \( \bar{V} \) with \( \bar{V}_l(k, l) = \bar{V}_l(m - l, l) \) for \( 0 \leq l \leq 4 \). Note that we have a restriction on \( l \) from \( d(L) \). While we cannot determine \( d(L) \) from the Jones polynomial, it still leaves a “trace” in form of the condition \( 5^{d(L)} | \det(L) = |V_L(-1)| \). So among the 5 possible \( l \) we can exclude the cases where 5 divides exactly one of \( k - l = m - 2l \) and \( \bar{V}_l(-1) \), but not the other.

For Form 2, one should take \( t = e^{3\pi i/5} \). Then \( |t^2 - 1| > |1 - t| > 1 \). Now we increase first \( l \) from 1 onward as long as \( 5^{d(L)} \mid V_L(-1) \) (because of the relation to \( d(L) \)), and
\[ |t^2 - 1| \cdot |t + 1 + 1/t| \leq |t + 1| \cdot v_3(L). \]
For such \( l \), iterate \( k \) from \( \max(0, 3 - l) \) onward, as long as
\[ |t^2 - 1| \cdot |1 - t| \cdot |t + 1 + 1/t| \leq |t + 1| \cdot v_3(L). \]
If equality holds, compare \( \bar{V} \) with \( \bar{V}_2(k, l) \).

With a similar procedure one tests Form 3, now using \( t = e^{2\pi i/5} \). (Then \( |t^2 + 1|, |1 + t| > 1 \), and we must have \( 5^{d(L)} \mid \det(L) \).) Forms 4 and 5 (latter actually being redundant, since equivalent, as far as \( V \) can tell, to Form 3 for \( k = 0, l = 1 \) are tested by direct comparison.

If one fails to find \( \bar{V} \) in these forms, one can conclude that \( L \) is not (even 5-equivalent to) a Montesinos link. I wrote a computer program that performs this test, and show its output on (say) non-alternating 10 crossing knots:

10 124 match form 3 for k=2, l=0 10 145 match form 2 for k=2, l=1
10 125 match form 3 for k=2, l=1 10 146 match form 2 for k=2, l=1
10 126 match form 3 for k=2, l=0 10 147 match form 2 for k=2, l=1
10 127 match form 3 for k=2, l=0 10 148 match form 1 for k=1, l=2
10 128 match form 1 for k=2, l=1 10 149 match form 1 for k=1, l=2
10 129 form 5 10 150
10 130 match form 1 for k=0, l=3 10 151
10 131 match form 1 for k=1, l=2 10 152
10 132 form 5 10 153
10 133 match form 1 for k=2, l=1 10 154
10 134 match form 1 for k=3, l=0 10 155 match form 2 for k=1, l=2
10 135 match form 1 for k=3, l=0 10 156 match form 2 for k=1, l=2
10 136 match form 2 for k=1, l=2 10 157
10 137 match form 2 for k=1, l=2 10 158 match form 2 for k=1, l=2
10 138 match form 2 for k=1, l=2 10 159
10 139 match form 3 for k=1, l=0 10 160
10 140 match form 3 for k=0, l=0 10 161 match form 2 for k=1, l=2
10 141 match form 3 for k=0, l=0 10 162 form 5
10 142 form 5 10 163 match form 2 for k=3, l=1
10 143 match form 3 for k=1, l=0 10 164 match form 2 for k=1, l=2
10 144 match form 3 for k=0, l=0 10 165

Whenever no form is found, the Montesinos property is ruled out. This happens for 9 of the last 18 knots; the first 24 knots are Montesinos. Our test thus seems relatively efficient. It can surely not be perfect, since it is invariant under 5-moves, and also sporadic duplications of Jones polynomials occur. However, it seems easier to perform than all previously known Montesinos link tests (at least for general diagrams, and as long as the Jones polynomial can be calculated).

1In this table, we use the numbering of [Ro, appendix], but the mirroring convention in [HT]; a conversion to Rolfsen’s mirroring can be found on my website.
6 Invariants of the Kauffman polynomial

Now we study the 5-move invariants of the Kauffman polynomial to enhance the test. The following is easy to see:

**Lemma 6.1** Let \( n, m \in \mathbb{Z}_k \), so that \( n \neq \pm m \) and \( w = e^{2\pi i/n} \neq \pm 1, \pm i \). Then \( F(a, z) \) for \( a = e^{2\pi i/k} \) and \( z = w + w^{-1} = 2\cos 2\pi n/k \), up to multiplication with (powers of) \( a \), is a \( k \)-move invariant.

**Proof (sketch).** Consider the generating function

\[
f(a, z, x) = \sum_{j=0}^{\infty} \Lambda(A_j)(a, z)x^j,
\]

where \( A_j \) are link diagrams with a twist tangle of \( j \) crossings (that is, a tangle with Conway notation \( j \), as on the right of (1) for \( j = 3 \)). Use the relations (4) to rewrite \( f \) as a rational function in \( x \), determined by \( A_{0,1,\infty} \). Finally analyze for what values of \( a \) and \( z \) (for which \( F(a, z) \) makes sense), the zeros \( x \) of the denominator polynomial are distinct \( k \)-th roots of unity. \( \square \)

In the case of \( k = 5 \) this gives up to conjugacy four values. For \( m = 1 \) we have the two (equivalent under the interchange of \( \pm \sqrt{5} \)) Jones-Rong values of \( Q \) in proposition 3.2. The other two values are \( F(e^{\pm 2\pi i/5}, 2\cos 4\pi/5) \) and \( F(e^{\pm 2\pi i/5}, 2\cos 2\pi/5) \). They are equivalent in the same sense, but as with \( v_1 \) and \( v_3 \) the behaviour of the complex norm under this equivalence could be drastic, so one should not \textit{a priori} discard one. (One could do so, though, for one Jones-Rong value, because its special form implies that the \( \pm \sqrt{5} \) equivalence results at most in a change of sign.)

To evaluate \( F \) on the links in theorem 3.2, we make again a skein module calculation, this time using the Kauffman polynomial (\textit{not Kauffman bracket!}) skein relation (4). In the following we assume that \( a \) and \( z \) are as specified in lemma 6.1 for \( k = 5 \), and consider \( F(a, z) \) up to powers of \( a \).

We start with determining the coefficients

\[
< 1/2 > = \begin{cases} \Lambda(a) & \text{if } a > 0 \\ \Lambda(b) & \text{if } a < 1 \\ \Lambda(c) & \text{if } a < \infty \end{cases}
\]

\[
< 2/5 > = \begin{cases} \Theta(a) & \text{if } a > 0 \\ \Theta(b) & \text{if } a < 1 \\ \Theta(c) & \text{if } a < \infty \end{cases}
\]

of the tangles \( T = < 1/2 > \) and \( T' = < 2/5 > \) in the Kauffman skein module (not to be confused with the Kauffman bracket skein module).

By taking the closure of the sum of the tangle \( T = < 1/2 > \) resp. \( T' = < 2/5 > \) with the 0, \( \infty \) and \( -1 \) tangles, we obtain a linear equation system that determines the Kauffman skein module coefficients of \( T \) and \( T' \).

Let first \( a_1 = \frac{1}{a} + a \), and

\[
T_2 = -1 + \frac{a_1}{a}, \quad H = az_1 - T_2, \quad G_4 = (1 - a_1^2) - za_1 + z^2a_1^2 + z^3a_1, \quad G_3 = (-1/az - 2a) + za_1/a + z^3a_1
\]

be the writhe-unnormalized polynomials of the 2-component unlink, Hopf link, figure-8 knot, and negative trefoil resp.

Let

\[
M = \begin{pmatrix} T_2 & a & 1 \\ 1 & \frac{1}{a} & T_2 \\ \frac{1}{a} & T_2 & a \end{pmatrix}
\]

be the matrix that represents the closures of the three skein module generating tangles \( < 0 >, < 1 > \) and \( < \infty > \), and

\[
v = \begin{pmatrix} a^2 & a^2H \\ H & G_4 \\ \frac{1}{a} & G_3 \end{pmatrix}
\]
be the to-result polynomials for the closures of the the tangles $T$ and $T'$. Then
\[
\begin{pmatrix} A & B & C \\ D & E & F \end{pmatrix} = (M^{-1} \cdot v)^T,
\]
and by calculation we find
\[
(A, B, C) = (za, z, -1)
\]
\[
(D, E, F) = \left( -a^2 + z^2 + a^2 z^2 + a z^3, \ -z + \frac{z^2}{a} + z^3, \ -\frac{z}{a} - z^2 \right).
\]
A routine (and so omitted) further calculation leads to the formulas we wish (the link in Form 5 can be evaluated directly).

**Theorem 6.1** For $a$ and $z$ as in lemma 6.1 for $k = 5$, the values $F(a, z)$ of the links in theorem 3.2 are given, up to powers of $a$, as follows.

For Form 1 we have with $k = n, l = n'$:
\[
F_1(n, n') = \frac{1}{T_2} \left[ H^{n' + n} + a^{n - n'} z^{n + n'} \left( \sum_{j=0}^{n} \sum_{j'=0}^{n'} \binom{n}{j} \binom{n'}{j'} W_1((j - j') \mod 5) \right) \right],
\]
where
\[
W_1(0) = -1 + T_2^2,
\]
\[
W_1(\pm 1) = -a \mp 2 + T_2,
\]
\[
W_1(\pm 2) = -a \mp 1 + a \mp 2 HT_2.
\]

For Form 2 we have
\[
F_2(n, n') = \frac{1}{T_2} \left[ G_4^{n'} H^n + \sum_{j=0}^{n} \sum_{j'=0}^{n'} \binom{n}{j} \binom{n'}{j'} A^{n-j} B^{j} D^{n-j'} E^{j'} W_2((j + j') \mod 5) \right],
\]
where
\[
W_2(0) = -1 + T_2^2,
\]
\[
W_2(\pm 1) = -a \mp 1 + a \pm 1 T_2,
\]
\[
W_2(\pm 2) = -a \pm 2 + HT_2.
\]
For Form 3 and Form 4,
\[
F_3(n, n', n'') := G_4^n H^{n'} T_2^{n''}.
\]
If $n > 0$, we have Form 4; if $n = 0$, Form 3 with $k = n'$, $l = n''$.

These expressions look slightly more complicated than those for $V$, but are still straightforward to evaluate. Using the formulas, we can for example rule out 10155 and 10161 from being Montesinos. The calculation in §5 shows that if 10155 were Montesinos, it must be in the 5-equivalence class of a Montesinos link corresponding to the polynomial $F_2(1, 2)$ (for example 10136). But the $F$ polynomial invariants of $F(10155)$ are different from those of $F_2(1, 2)$. The same argument rules out 10161. The other 7 undecided knots remain, and it is suspectable that they are 5-equivalent to Montesinos links.

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