Computational Relativistic Astrophysics With Adaptive Mesh Refinement: Testbeds

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We have carried out numerical simulations of strongly gravitating systems based on the Einstein equations coupled to the relativistic hydrodynamic equations using adaptive mesh refinement (AMR) techniques. We show AMR simulations of NS binary inspiral and coalescence carried out on a workstation having an accuracy equivalent to that of a 1025 regular unigrid simulation, which is, to the best of our knowledge, larger than all previous simulations of similar NS systems on supercomputers. We believe the capability opens new possibilities in general relativistic simulations.

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a. Introduction Numerical study of compact systems has received much attention due to observations in high-energy astronomy and the promise of gravitational wave astronomy. Most effort focuses on solving the Einstein equations with finite differencing methods. The main difficulty of this approach is that many general relativistic astrophysical processes of interest, e.g., processes involving black holes and neutron stars, require computational resources that are beyond what present day computers allow. The reasons that they are computationally demanding are: 1. the lack of symmetry in realistic astrophysical situations, requiring the solving of the full set of Einstein equations coupled to the general relativistic hydrodynamic (GRHydro) equations in 3+1 dimensional spacetime; and 2. the involvement of many length scales.

The difficulty of multiple length scales can be illustrated with the neutron star (NS) coalescence problem, one of our main systems of study. The length scales involved are: (i) A short length scale coming from the internal dynamics of a neutron star as a self gravitating object. One needs to resolve the density/pressure variation accurately enough to maintain a stable configuration in the Einstein theory. (ii) A longer length scale coming from the dynamics of two NSs moving under the influence of one another, i.e., the length scale of the orbital radius. (iii) The dynamical time scale of the system (the orbital period \(T\)) turns into a long length scale due to the dynamical nature of Einstein gravity (no such difficulty exists in Newtonian gravity, where one can evolve orbiting system more easily). The space surrounding the NSs within the corresponding length scale (the wavelength of the gravitational wave due to the orbital motion) needs to be covered in the computational domain, both for the extraction of the waveform and for an accurate dynamical evolution (the problem manifests itself as that the evolution can be affected by the outer boundary if put too close). (iv) The secular evolution time scale of the orbital motion turns into a resolution requirement for the numerical simulation as computational error accumulates. Our study in a previous paper [1] indicates that

1. To simulate a single isolated NS in a stable fashion with the Einstein equations requires a resolution on the order of 0.1\(M_0\), where \(M_0\) is the baryonic mass of the NS, for a typical equation of state (EOS).
2. To set initial data in a fashion (e.g., using the conformally flat quasi-equilibrium (CFQE) approach) that we can have some confidence of its being astrophysically relevant, the initial separation of the two NSs would have to be on the order of 50\(M_0\) (depending on the initial spin states of the two NSs).
3. To get inspiral dynamics without much artificial influence from the boundary of the computational domain, it has to be at least 0.5\(\lambda\) away, where \(\lambda\) is the gravitational wavelength of the system (assuming the present state of the art in setting outer boundary conditions for the constrained system of the Einstein equations).
4. To be able to accurately extract a gravitational waveform from the simulation, the computational domain should include up to 1\(\lambda\).
5. To be able to evolve the spiraling NSs within the convergence regime to the point of coalescence: This depends on the choice of initial configuration and the numerical method used. With all existing methods we know of, the longer in time one needs to stay within the convergence regime (i.e., the constraint violations and physical quantities converging with respect to increasing resolution throughout that time period), the finer the resolution has to be. In our simulation reported in [1], a 643\(^3\) simulation with \(\Delta x = 0.2M_0\), covering up to 0.28\(\lambda\) for orbiting NSs at an initial separation of 28\(M_0\) (with an angular frequency of \(\Omega = 0.012M_0^{-1}\)) the system remains in the convergence regime for only about half an orbit. Being so far off from our target of evolving to the coalescence point, an estimate of what might be needed would be meaningless.

The wavelength of a gravitational wave with orbital separation of 50\(M_0\) (cf., (2) above) is about 1,000\(M_0\). For a unigrid at 0.2\(M_0\) (cf., (1) above), the requirements (3) and (4) imply a grid of 5,000\(^3\). A computer with a
memory size capable of doing such a simulation will not be available in the near future.

Hence the biggest obstacle we encounter in NS coalescence simulation based on finite differencing of the Einstein equations is the need for a large number of grid points, which translates into large computer memory and long execution time. We need the adaptive mesh refinement (AMR) treatment: Use fine grid patches co-moving with the compact objects to satisfy the resolution required by (1), and a coarse grid extending to the local wave zone for (2), (3) and (4). Similar considerations have motivated much effort in this direction, see e.g., [2] for recent progress.

Unfortunately, application of AMR techniques in general relativistic astrophysics is more difficult than one might naively think. Although the theory and algorithms of mesh refinement are well established in computational science, and the numerical treatments of the Einstein and GRHydro equations have been extensively investigated by relativists and astrophysicists, after many years of intense effort by many research groups it has not been possible to put the two together for a fully general relativistic 3D AMR simulation. The main difficulty is that it involves huge infrastructures on both the computer science side and the physics side: it is difficult for computer scientists to dive into the complexity of the physics, and vice versa. As a rough representation of the complexity, in our code construction process, we have to integrate a 100,000 line mesh refinement code (GrACE [3]), a 85,000 line general relativistic astrophysics code (GR-Astro [4]) and a 500,000 line parallel computational library (Cactus Toolkit [5]) that GR-Astro makes use of. One central message of this paper is: We confirm that there is no vice versa. As a rough representation of the complexity, in our code construction process, we have to integrate a 100,000 line mesh refinement code (GrACE [3]), a 85,000 line general relativistic astrophysics code (GR-Astro [4]) and a 500,000 line parallel computational library (Cactus Toolkit [5]) that GR-Astro makes use of. One central message of this paper is: We confirm that there is no problem in enabling general relativistic AMR, the devils are all in the details.

In this paper we demonstrate for the first time that a full 3+1 dimensional simulation based on the Einstein theory can be carried out with AMR. Three sample systems are studied:

1. A NS moving at a speed of 0.5c described by the Einstein plus GRHydro equations. The validity of our AMR treatment is examined with convergence tests. Convergence tests are more complicated with AMR; three different kinds of convergence tests are presented: (i) simulations with increasing resolutions on all grid levels, (ii) simulations with added levels of refinement, and (iii) comparison to ungrid results. The investigation of a boosted star, which invokes all terms in the evolution equations, played an important role in our code construction process.

2. Two NSs coalescing with angular momentum ($L = 5.9M_{\odot}c^2$). The study demonstrates that our AMR treatment can (i) handle collisions and merging of not only NSs, but also grid patches, (ii) handle gravitational collapses, and (iii) simulate NS processes with an accuracy comparable to that of a ungrid run with resolution same as the resolution of the finest grid of the AMR run.

3. An inspiraling NS binary. The two NSs are covered by co-moving fine grid patches, with the coarsest grid covering a fraction of a wavelength of the system. We show an AMR simulation which is equivalent to a regular 1025^3 ungrid simulation, larger than any simulation of NS binary systems performed so far.

In the following sections we discuss these 3 simulations. The last section summarizes and discusses the next steps.

b. Boosted Neutron Star. We begin with a study of a NS moving across an otherwise empty space at a constant speed. Although the physical system is not changing in time beyond a uniform boost, the metric has complicated spacetime dependences due to the frame dragging effect. Accordingly, all coordinate quantities including those of the spacetime and matter are changing in a non-trivial manner (not just a uniform translation). In the simulation, we start with a configuration satisfying the Hamiltonian constraint (HC) and momentum constraint (MC) representing a NS boosted to 0.5c, and evolve it with the full set of dynamical Einstein equations coupled to the GRHydro equations. The system of equations as well as the conventions we use in this paper are given in [1].

The simulation provides a good test for our code as it invokes all terms in the equations, and is numerically a fully dynamical test.

The NS is described by a polytropic EOS: $P = (\Gamma - 1)\rho\epsilon$ with $\Gamma = 2$ ($P = k\rho^\Gamma$ for initial data, with $k = 0.0445c^2/\rho_n$, where $\rho_n$ is the nuclear density, approximately $2.3 \times 10^{14}$ g/cm^3). (All simulations reported in this paper use the same EOS.) The NS has a proper radius of $R = 12M_{\odot}$, an ADM mass of $1.4M_{\odot}$ and a baryonic mass $M_0 = \frac{1}{2} \int d^3x \sqrt{-g}\rho W = 1.49M_{\odot}$. (For these values of the parameters, the maximum stable NS configuration has an ADM mass of 1.79$M_{\odot}$ and a baryonic mass of 1.97$M_{\odot}$). The initial data is obtained by imposing a boost on the TOV solution ([1]). The evolution is carried out with the $\Gamma$ freezing shift and the “1 + log” lapse (for details of the shift and lapse conditions and method of implementations, see [1]).

The computational grid is set up as follows: 1. The coarse grid has a resolution of $dx = 2.88M_{\odot}$ (4 points across the radius $R$ of the NS) covering a region of $(58M_{\odot})^3$. 2. Two levels of adaptive fine grid with $dx = 1.44M_{\odot}$ and $dx = 0.72M_{\odot}$ are set up. The adaptive grid is allowed to change in size and location as the refinement criteria dictate. 3. Two different refinement criteria have been studied: (i) value of matter density $\rho$, and (ii) amount of HC violation. Combinations of the two with “or” can be used. It turns out that for the neutron star studies it does not matter much which condition is used: the central region of the NS is at the same time the region of highest density, maximum HC violation and maximum evolution error. All simulations shown in this paper are obtained with (i).

Fig. [1][2] examine the validity of the simulation with 3 kinds of convergence tests. Fig. [1] shows the violation of the HC at $t = 28.8M_{\odot}$ along the $x$-axis for four different runs. The HC violation is calculated on the finest grid available for regions covered by more than one grid (as
for all HC plots in this paper). The resolutions of the runs are $41^3$, $49^3$, $65^3$, and $81^3$, corresponding to $dz = 2.88M_\odot$, $2.4M_\odot$, $1.8M_\odot$, and $1.44M_\odot$, respectively, on the base grid. (The notation $(41 \times 2 \times 2)^3$ indicates a 41$^3$ base grid and two levels of refinement with a refinement ratio of 2 each.) The results for the higher resolution runs have been scaled linearly. The plot demonstrates that the code is converging to first order, which is the expected rate of convergence as we used a high resolution shock capturing TVD scheme \[1\] in our hydrodynamic evolution which is first order at extremal points.

In fig. 2, we compare the HC violations of two runs at time $t = 28.8M_\odot$: (i) the $(81 \times 2 \times 2)^3$ run shown in fig. 1 and (ii) an $(81 \times 2)^3$ run with only one level of refinement covering the high density region. We see that the addition of a refinement level lowers the HC violation by a factor of 2 in the region of the extra grid level (where the HC violation is significant).

In fig. 3, we show the HC violations of three runs at $t = 28.8M_\odot$. The AMR run is again the $(81 \times 2 \times 2)^3$ one in fig. 1. The other two are unigrid runs, one $(81^3)$ at the resolution of the coarsest AMR grid ($dz = 1.44M_\odot$), and the other $(321^3)$ at the resolution of the finest AMR grid ($dz = 0.36M_\odot$). We see that the AMR run has exactly the same accuracy as the unigrid fine resolution run (the two lines coincide). This is an important point for our study: For NS simulations in this and the following sections, finite difference error is most significant in the high density region covered by the finest AMR grid; using coarser grids elsewhere does not affect the accuracy of the simulation. The fact that the error can be the same for an unigrid run and an AMR run with suitable fine grid coverage enables us to speak of the “unigrid equivalent” of an AMR run: a unigrid run with the resolution of the finest AMR grid.

The three kinds of convergence tests provide confidence in the validity of our AMR treatment.

c. Coalescing Neutron Stars. In this section we study the coalescence of two NS’s having baryonic masses and EOS as in the boosted star case above. The NSs have their equatorial plane on the $x$-$y$ plane and an initial center to center (points of maximum mass) separation of $5R$ in the $x$ and $0.83R$ in the $y$ directions ($R = 12M_\odot$). The NSs are boosted in the $+x$ direction with the total angular momentum of the system equal to $2.67M_\odot^2 = 5.9M_\odot^2$.

To determine the initial data we solve the Hamilton equations on a unigrid of $(385, 257, 257)$ at a resolution of $dz = 0.9M_\odot$. The initial metric and hydrodynamic data are then interpolated onto the AMR grids. The AMR simulation is then compared to the unigrid one. The angular momentum (and log lapse) are used in both cases.

In fig. 4 we show the lapse (represented as height fields) on the equatorial plane of the NSs at 3 different times $t = 0.612, 122.4M_\odot$ in the AMR simulation, with the grid structure superimposed (downsampled by a factor of 2, and only the inner part is shown). We see initially there are two separated fine grid patches. At $t = 61.2M_\odot$, the two NSs, as well as their respective fine patches, begin to merge. At $t = 122.4M_\odot$ a black hole has formed with the lapse dipping to 0.002 at the center, and the fine grid patches have completely merged and shrunk into a cube.

In fig. 5, we plot $g_{xx}$ along the $x$ axis at $t = 122.4M_\odot$ for the AMR simulation (dashed line) and the unigrid simulation (dagger) that has the same resolution as the finest AMR grid. We see the results coincide to high accuracy. All metric functions and hydro variables show the same degree of agreement even at this late time. In fig. 6 we compare the values of the (spatial) maximum of the AMR run. We see that the two simulations give basically the same results throughout the evolution.

d. Inspiring Neutron Stars. In this section we study the coalescence of two NS’s having baryonic masses and EOS as in the boosted star case above. The NSs have their equatorial plane on the $x$-$y$ plane and an initial center to center (points of maximum mass) separation of $5R$ in the $x$ and $0.83R$ in the $y$ directions ($R = 12M_\odot$). The NSs are boosted in the $+x$ direction with the total angular momentum of the system equal to $2.67M_\odot^2 = 5.9M_\odot^2$.

To determine the initial data we solve the Hamilton equations on a unigrid of $(385, 257, 257)$ at a resolution of $dz = 0.9M_\odot$. The initial metric and hydrodynamic data are then interpolated onto the AMR grids. The AMR simulation is then compared to the unigrid one. The angular momentum and log lapse are used in both cases.

In fig. 4 we show the lapse (represented as height fields) on the equatorial plane of the NSs at 3 different times $t = 0.612, 122.4M_\odot$ in the AMR simulation, with the grid structure superimposed (downsampled by a factor of 2, and only the inner part is shown). We see initially there are two separated fine grid patches. At $t = 61.2M_\odot$, the two NSs, as well as their respective fine patches, begin to merge. At $t = 122.4M_\odot$ a black hole has formed with the lapse dipping to 0.002 at the center, and the fine grid patches have completely merged and shrunk into a cube.

In fig. 5a, we plot $g_{xx}$ along the $x$ axis at $t = 122.4M_\odot$ for the AMR simulation (dashed line) and the unigrid simulation (dagger) that has the same resolution as the finest AMR grid. We see the results coincide to high accuracy. All metric functions and hydro variables show the same degree of agreement even at this late time. In fig. 6 we compare the values of the (spatial) maximum of the HC violations (which is one of the most sensitive measure of differences between runs) over time. We see that the two simulations give basically the same results throughout the evolution.

In this section we demonstrate that with AMR we can now carry out on a workstation (Dell Poweredge 1850) NS inspiral simulations that are beyond existing unigrid simulations on supercomputers.

The NSs are taken to be initially in a conformally flat quasi-equilibrium (CFQE) irrotational circular orbit with an orbital separation of $3.3R$. Each NS has a baryonic mass of $1.625M_\odot$ with the same EOS as before. The initial data is obtained by solving the CFQE equations using the pseudo-spectral code developed by the Meudon group \[15\], and imported onto the Cartesian grid structure in GR-Astro-AMR for dynamical evolutions, again with $\Gamma$ freezing shift and 1+ log lapse.

We show results from two AMR simulations. In the high resolution run with 4 levels of refinement (with refinement ratio 2), the finest level grid has 60 points across
FIG. 4: The lapse of coalescing NSs at 3 different times, with the grid structure superimposed, and with 1 to 2 downsampling (showing every other point). Only the inner part of the computational domain is shown.

We carried out (based on various kinds of convergence tests) for GR-Astro-AMR with a boosted NS. In sec. c we demonstrated that GR-Astro-AMR can be used to simulate NS coalescences and formation of black holes, with an accuracy comparable to that of an unigrid simulation using a resolution same as that of the finest grid in the AMR run. Sec. d showed an AMR simulation of an inspiraling NS binary carried out on a Dell workstation with 8 GB of memory, which is equivalent in accuracy to a 1025^3 unigrid run that requires over 1.2TB of memory. To the best of our knowledge this is larger than all previous simulations of similar systems on supercomputers.

In a future publication, we will extend the study in sec. c to analyze the amount of matter available for accretion after the NS coalescence/BH formation, as a function of the angular momentum of the system at the plunge point of the inspiral. We will extend the study in sec. d to determine astrophysically realistic initial data for inspiral, following the line initiated in [1]. These investigations require more computational resources than are available to us if they are to be carried out in unigrid.

There are many aspects in the GR-Astro-AMR code that need improvement as a computational infrastructure for general relativistic simulations. In the next steps, we will (i) develop the parallel capacity of GR-Astro-AMR, (ii) study the usage of different refinement criteria for other NS/BH processes, and (iii) enable the direct solving of elliptic equations on the grid hierarchy.

The code is developed with the intention of providing a
computational tool to the general relativistic astrophysics community. The unigrid version of GR-Astro has been released (available at [http://www.wugrav.wustl.edu](http://www.wugrav.wustl.edu)). GR-Astro-AMR will be released as soon as ready. We invite researchers to join us in making use of as well as further developing this code.

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