The spectrum of an operator associated with $G_2$–instantons with 1–dimensional singularities and Hermitian Yang-Mills connections with isolated singularities

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Abstract

This is the first step in an attempt at a deformation theory for $G_2$–instantons with 1–dimensional conic singularities. Under a set of model data, the linearization yields a self-adjoint first order elliptic operator $P$ on a certain bundle over $S^5$. As a dimension reduction, the operator $P$ also arises from Hermitian Yang-Mills connections with isolated conic singularities on a Calabi-Yau 3-fold.

Using the Quaternion structure in the Sasakian geometry of $S^5$, we describe the set of all eigenvalues of $P$ (denoted by $\text{Spec}P$). We show that $\text{Spec}P$ consists of finitely many integers induced by certain sheaf cohomologies on $\mathbb{P}^2$, and infinitely many real numbers induced by the spectrum of the rough Laplacian on the pullback endomorphism bundle over $S^5$. The multiplicities and the form of an eigensection can be described fairly explicitly.

Using the representation theory of $SU(3)$ and the subgroup $SU(1) \times SU(2)$, we show an example in which $\text{Spec}P$ and the multiplicities can be completely determined.

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1 Introduction

1.1 Overview

$G_2$–instantons (and projective $G_2$–instantons) are the analogue of both flat connections in dimension 3, and anti self-dual connections in dimension 4. Understanding the singularities of (projective) $G_2$–instantons plays an important role in the programs proposed by Donaldson-Thomas [4] and Donaldson-Segal [2] on higher dimensional gauge theory.

In conjunction with Jacob-Walpuski [9], to construct (projective) $G_2$–instantons with 1–dimensional singularities on twisted connected-sum $G_2$–manifolds via gluing, an important step is a deformation theory built upon a Fredholm theory for the linearized operator. In [20], it is shown that a Fredholm theory, and consequently a deformation theory, always exist for instantons with isolated singularities. However, the situation of instantons with 1–dimensional singularities is expected to be drastically different.

On a $G_2$–instanton with conic singularities along a circle, in the model setting, the linearized operator yields a self-adjoint elliptic operator $P$ on a certain bundle over $\mathbb{S}^5$ (see Lemma 2.14 below). The set of all eigenvalues of $P$, denoted by $\text{Spec} P$, plays a crucial role in the construction of a deformation theory. It determines the indicial roots of the linearization.

The purpose of this paper is to describe $\text{Spec} P$ and address the multiplicities.

1.2 Background of the operator $P$

Before stating the main results, we briefly recall some background on $G_2$–instantons with 1–dimensional singularities, and explain where the operator $P$ comes from. More details can be found in Section 2.
1.2.1 General Background

Let $M^7$ be a 7-dimensional manifold with a $G_2$-structure $\phi$, we denote the co-associative 4-form by $\psi$. Given a smooth Hermitian vector bundle $E$ over $M^7$, let $adE$ denote the bundle of skew-adjoint endomorphisms. Let $A$ be a unitary connection, and $\sigma$ be a section of $adE$. The pair $(A, \sigma)$ is called a $G_2$-monopole if the following equation is satisfied.

$$*(F_A \wedge \psi) + d_A \sigma = 0.$$

When $d_A \sigma = 0$, we say that $A$ is a $G_2$-instanton.

Let $\Omega^{\cdot}_{adE}$ denote the bundle of $adE$-valued $k$-forms. With gauge fixing, the linearization of $\Omega^{\cdot}_{adE}$ in $A$ is an elliptic operator $L_{A,\phi}$ which maps $C^\infty[M^7, \Omega_0^{\phi} \oplus \Omega_{adE}^1]$ to itself. Namely,

$$L_{A,\phi}[\sigma] = [d_A \sigma + *(d_A a \wedge \psi)],$$

where $\sigma \in C^\infty[M^7, \Omega_0^{\phi}]$, and $a \in C^\infty[M^7, \Omega_{adE}^1]$.

1.2.2 A holomorphic Hermitian triple and the associated data setting

We now introduce the simple manifolds and maps required.

**Definition 1.1.** Using the dimensions of the domain and range manifold as subscripts, we consider the following standard projection maps.

| Map          | Domain and Range                        |
|--------------|-----------------------------------------|
| $\pi_{7,6}$  | $(\mathbb{C}^3 \setminus O) \times \mathbb{R} \to \mathbb{C}^3 \setminus O$ |
| $\pi_{6,5}$  | $\mathbb{C}^3 \setminus O \to \mathbb{S}^6$ |
| $\pi_{5,4}$  | $\mathbb{S}^5 \to \mathbb{P}^2$       |
| $\pi_{6,4}$  | $\pi_{6,5} \cdot \pi_{6,5}$            |
| $\pi_{7,4}$  | $\pi_{7,6} \cdot \pi_{7,6}$            |
| $\pi_{7,5}$  | $\pi_{7,6} \cdot \pi_{7,6}$            |

**Definition 1.2.** The standard Hermitian metric on the universal bundle $O(-1) \to \mathbb{P}^2$ is $|Z_0|^2 + |Z_1|^2 + |Z_2|^2$. We denote it by $h_{O(-1)}$ (see [1]). For any integer $l \neq 0$, this metric induces uniquely a Hermitian metric $h_{O(l)}$ on $O(l)$. We call the Chern connection of $h_{O(l)}$ the standard connection.

We introduce the following terms.

**Definition 1.3.**

1. **Holomorphic Hermitian triple:** A triple $(E, h, A_O)$ consists of a holomorphic vector bundle $E \to \mathbb{P}^2$, a Hermitian metric $h$ on $E$, and the Chern connection $A_O$ is called a holomorphic Hermitian triple on $\mathbb{P}^2$.

2. **Hermitian Yang-Mills triple:** A holomorphic Hermitian triple on $\mathbb{P}^2$ is called a Hermitian Yang-Mills if the connection $A_O$ is Hermitian Yang-Mills.

3. **Irreducible Hermitian Yang-Mills triple:** A Hermitian Yang-Mills triple on $\mathbb{P}^2$ is called an irreducible Hermitian Yang-Mills if $A_O$ is irreducible and $\text{rank} E \geq 2$.

A holomorphic Hermitian (Hermitian Yang-Mills) triple on $\mathbb{P}^n$ and the standard connection on $O(l) \to \mathbb{P}^n$ are defined in the same manner. Nevertheless, except for the Kähler identity in Lemma [1] below, we only consider such a triple on $\mathbb{P}^2$. 
The bundle $(End_0 E)(l)$ is called the twisted traceless endomorphism bundle, and $(End E)(l)$ is called the twisted endomorphism bundle. We call the tensor product of the standard connection on $O(l)$ and $A_O$ the twisted connection. We also call the tensor product of the standard metric on $O(l)$ and $h$ the twisted metric. A section of $(End E)(l)$ is called a twisted endomorphism. The same applies to the pullbacks.

4. The associated data setting: Given a holomorphic Hermitian triple $(E, h, A_O)$ on $\mathbb{P}^2$. Throughout, the operator $P$ and the other bundle rough Laplacians are defined by the following data.

- On $S^5 \setminus O$, and $(C^3 \setminus O) \times S^1$, we consider the pullback of $A_O$ and the pullback Hermitian metric on the (pullback) endomorphism bundles.

On a twisted endomorphism bundle over $\mathbb{P}^2$, we consider the twisted connection and the twisted metric.

- Let

\[ \eta \triangleq d^c \log r \triangleq \sqrt{-1}(\bar{\partial} - \partial) \log r \]  

be the contact form on $S^5$. Throughout, we consider the Fubini-Study metric $d\mu$ on $\mathbb{P}^2$, the standard round metric on $S^5$, and the Euclidean metric on $C^3 \times S^1$ (and $C^3$).

1.2.3 The linearized operator under the model data

Suppose $A_O$ is not flat on $\mathbb{P}^2$, the pullback connection on $(C^3 \setminus O) \times S^1$ has conic singularity along the circle $O \times S^1$. This is the prototype of what we are interested in. The model linear problem for $G_2$—instantons with conic singularities along a circle is as follows.

On $C^3$, let $\omega_{C^3} = \frac{1}{2}(dZ_0 d\bar{Z}_0 + dZ_1 d\bar{Z}_1 + dZ_2 d\bar{Z}_2)$ be the standard Kähler form, and let $\Omega_{C^3} = dZ_0 d\bar{Z}_1 d\bar{Z}_2$ be the standard holomorphic volume form. The standard $G_2$—structure on $C^3 \times S^1$ is defined by

\[ \phi_{C^3 \times S^1} \triangleq ds \wedge \omega_{C^3} + Re\Omega_{C^3}. \]

The standard co-associative 4—form is $\psi_{C^3 \times S^1} \triangleq \frac{\omega^2_{C^3}}{2} - ds \wedge Im\Omega_{C^3}$. Given a holomorphic Hermitian triple $(E, h, A_O)$ on $\mathbb{P}^2$, the model linearized operator is defined as follows.

\[ \mathcal{L}_{A_O, \phi_{C^3 \times S^1}}[a_{C^3 \times S^1}^\sigma] = [d_{A_O, C^3 \times S^1}, a_{C^3 \times S^1}^\sigma] + d_{A_O, C^3 \times S^1}^* (a_{C^3 \times S^1} \wedge \psi_{C^3 \times S^1}), \]  

where $\sigma \in C^\infty[(C^3 \setminus O) \times S^1, \Omega^0_{\pi_{C^3}^*(adE)}]$ is a section of the adjoint bundle, and $a_{C^3 \times S^1} \in C^\infty[(C^3 \setminus O) \times S^1, \Omega^1_{\pi_{C^3}^*(adE)}]$ is an adjoint bundle-valued 1—form. A section $a_{C^3 \times S^1}$ of $\Omega^1_{\pi_{C^3}^*(adE)} \to (C^3 \setminus O) \times S^1$ can be split into

\[ a_{C^3 \times S^1} = a_s ds + a_{C^3}, \]  

where $a_s$ is a section of $\pi_{C^3}^*(adE)$, and $a_{C^3}$ is a section of $\pi_{C^3}^*(\Omega^1_{C^3}) \otimes \pi_{C^3}^*(adE)$ i.e. an adjoint bundle-valued 1—form without $ds$—component.

1.2.4 The fine splitting

We employ the Sasakian geometry of $S^5$, and aim at a more meticulous splitting with respect to the transverse Kähler structure.

We denote the contact distribution $Ker\eta$ by $D$. Let $\xi$ denote the standard Reed vector field on $S^5$ (tangential to orbit of the $U(1)$—multiplications). The orthogonal complement of
the contact form $\eta$ is denoted by $D^\ast$, we call it the contact co-distribution. We call a form $\theta$ semi-basic if $\xi \cdot \theta = 0$. A section of $D^\ast$ is precisely a semi-basic 1-form.

We define the finer splitting of a section in the domain of the linearized operator. In view of formula (5) and (6), let $u \triangleq r\sigma$, $a_s \triangleq ra_s$, we find

$$a_{C^3} = a_r \frac{dr}{r} + (a_\eta) \eta + a_0,$$

where

- $a_r$ and $a_\eta$ are sections of $\pi^*_{7,4}(\text{ad}E)$,
- $a_0$ is an adjoint bundle-valued semi-basic 1-form i.e. a 1-form with no $ds$, $dr$, or $\eta$-component.

For further calculation, we let $a_{S^5} \triangleq a_\eta(\eta) + a_0$. Fixing $r$ and $s$, both $a_{S^5}$ and $a_0$ are forms on $S^5$. We then obtain the splitting of the domain bundle of the linearized operator.

$$\Omega^0_{\pi^*_{7,4}(\text{ad}E)} \oplus \Omega^1_{\pi^*_{7,4}(\text{ad}E)} = [\pi^*_{7,5}(\text{ad}E)]^{\oplus 4} \oplus [\pi^*_{7,5}(D^\ast) \otimes \pi^*_{7,4}(\text{ad}E)] :$$

$$\begin{bmatrix} \sigma \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{r} \\ \frac{ds}{r} \\ \frac{dr}{r} \\ \eta \\ 1 \end{bmatrix} \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix}.$$  

**Definition 1.4.** Henceforth, let $\text{Dom}_{S^5}$ denote $[\pi^*_{7,4}(\text{ad}E)]^{\oplus 4} \oplus [D^\ast \otimes (\pi^*_{7,4}\text{ad}E)]$ as well as the space of smooth sections of the same bundle on $S^5$. Similarly, on $(\mathbb{C}^3 \setminus \mathcal{O}) \times S^1$, let $\text{Dom}_7$ denote the pullback $\pi^*_{7,5}\text{Dom}_{S^5}$ as well as the space of smooth sections. They are the “domain” of the operator $P$, and also of the linearized operator in (5).

**1.2.5 Introducing the operator $P$**

Now we can abbreviate Lemma 2.14 below: given a holomorphic Hermitian triple on $\mathbb{P}^2$, there exists isometries $I, K$ on the bundle $\text{Dom}_7$, and a self-adjoint elliptic operator $P$ on $\text{Dom}_{S^5}$, such that under the basis in (8), the following splitting holds.

$$L_{\mathcal{A}_0 \cdot \phi_{C^3 \times S^1}} = \frac{\partial}{\partial s} \circ I + K \circ \left( \frac{\partial}{\partial r} - \frac{P}{r} \right).$$

As a dimension reduction, in complex dimension 3, the operator $K \circ \left( \frac{\partial}{\partial r} - \frac{P}{r} \right)$ is the model linearized operator with gauge fixing for a Hermitian Yang-Mills monopole with isolated conic singularity (see Appendix E below). Thus, on a Calabi-Yau 3-fold, our results allow us to calculate the index of the linearized operator of Hermitian Yang-Mills monopoles with isolated conic singularities.

**1.3 The main theorem**

Throughout, $\text{Spec} P$ does not count the multiplicity of an eigenvalue (cf. the “$\text{Spec}^{\text{mult}}$” in Definition 3.17 below that counts multiplicity). This means an eigenvalue appears in $\text{Spec} P$ exactly once, thus it is a subset of $\mathbb{R}$. The same applies to $\text{Spec}(\nabla^\ast \nabla|_{S^5})$ below. We address the multiplicities separately.
Definition 1.5. (Eigenspaces) Given a holomorphic Hermitian triple on $\mathbb{P}^2$ and in the associated data setting, for any real number $\mu$, let

$$E_\mu P \overset{\triangle}{=} \text{Ker}(P - \mu \text{Id}).$$

Resultantly, $\mu \in \text{Spec} P$ if and only if $E_\mu P \neq \{0\}$.

On the adjoint bundle $\pi_5^* (\text{ad} E) \to \mathbb{S}^5$, let $\nabla^* \nabla$ and $\nabla^* \nabla|_{\mathbb{S}^5}$ both abbreviate the rough Laplacian defined by the pullback connection $A_0$ and the standard round metric on $\mathbb{S}^5$. The set of all its eigenvalues is denoted by $\text{Spec}(\nabla^* \nabla|_{\mathbb{S}^5})$. Let

$$E_\lambda (\nabla^* \nabla|_{\mathbb{S}^5}) \overset{\triangle}{=} \text{Ker}[(\nabla^* \nabla|_{\mathbb{S}^5}) - \lambda \text{Id}].$$

We still call $E_\mu P$ ($E_\lambda (\nabla^* \nabla|_{\mathbb{S}^5})$) the eigenspace with respect to $\mu$ ($\lambda$), though $\mu$ is not necessarily an eigenvalue. $\mu$ ($\lambda$) is an eigenvalue if and only if $E_\mu P \neq \{0\}$ ($E_\lambda (\nabla^* \nabla|_{\mathbb{S}^5}) \neq \{0\}$). This convention turns out to work well.

Notation Convention 1.6. The equal sign “$=$” between two vector spaces (bundles) always means at least a real isomorphism. The notation “$\dim$” means the real dimension.

Before stating the main theorem, we need two more notions.

Definition 1.7. Given a holomorphic Hermitian triple on $\mathbb{P}^2$, we define the following two subsets of $\mathbb{R}$.

- $S_{\nabla^* \nabla} \overset{\triangle}{=} \{\mu | (\mu^2 + 2\mu - 3) \in \text{Spec}(\nabla^* \nabla|_{\mathbb{S}^5}) \} \cup \{\mu | \mu^2 + 4\mu \in \text{Spec}(\nabla^* \nabla|_{\mathbb{S}^5}) \}$ i.e.

$$S_{\nabla^* \nabla} \overset{\triangle}{=} \cup_{\lambda \in \text{Spec}(\nabla^* \nabla|_{\mathbb{S}^5})} \{-1 + \sqrt{4 + \lambda}\} \cup \{-1 - \sqrt{4 + \lambda}\} \cup \{-2 + \sqrt{4 + \lambda}\} \cup \{-2 - \sqrt{4 + \lambda}\}.$$

- $S_{\text{coh}} \overset{\triangle}{=} \{l | l \text{ is an integer and } H^1[\mathbb{P}^2, (\text{End} E)(l)] \neq \{0\}\}$.

Apparentely, the set $S_{\nabla^* \nabla}$ is induced by the spectrum of the rough Laplacian, the set $S_{\text{coh}}$ consists of integers and is given by the sheaf cohomologies. Intuitively speaking, under natural assumptions, our main theorem “decomposes” $\text{Spec} P$ into the union of these two sets. The multiplicities can be described.

Theorem 1.8. Let $(E, h, A_0)$ be an irreducible Hermitian Yang-Mills triple on $\mathbb{P}^2$ with the associated data setting. Let $P$ be the first-order self-adjoint elliptic operator defined in $[9]$ (and Lemma 2.14 below).

I (Spectrum). The following spectral decomposition holds for $P$.

$$\text{Spec} P = S_{\nabla^* \nabla} \cup S_{\text{coh}}.$$ 

Consequently, the set $(\text{Spec} P) \cap (-3, 0)$ contains and only contains the two numbers $-1, -2$.

II (Multiplicities).

1. In view of Definition 1.5 of the eigenspaces and Remark 2.16 below on the complex structure of each $E_\mu P$, the following complex isomorphisms hold.

$$E_{-1} P = H^1[\mathbb{P}^2, (\text{End} E)(-1)], \ E_{-2} P = H^1[\mathbb{P}^2, (\text{End} E)(-2)].$$

2. If $\mu \in \text{Spec} P$ and $\mu$ is not an integer, the following real isomorphism holds.

$$E_\mu P = [E_{\mu^2 + 2\mu - 3}(\nabla^* \nabla|_{\mathbb{S}^5})]^{\mathbb{R}^2} \oplus [E_{\mu^2 + 4\mu}(\nabla^* \nabla|_{\mathbb{S}^5})]^{\mathbb{R}^2}.$$
3. If $\mu \in \text{Spec} P$, $\mu$ is an integer, but $\mu \neq -1$ or $-2$, then

$$
\dim \mathbb{E}_\mu P = 2 \dim \mathbb{E}_{\mu+2}\nabla^* \nabla |_{\mathbb{S}^5} + 2 \dim \mathbb{E}_{\mu+4}\nabla^* \nabla |_{\mathbb{S}^5} + 2 h^1[\mathbb{P}^2, (\text{End} E)(\mu)]
- 2 h^0[\mathbb{P}^2, (\text{End}_0 E)(\mu)] - 2 h^0[\mathbb{P}^2, (\text{End}_0 E)(-\mu - 3)].
$$

In particular, $\dim \ker P = 2 h^1[\mathbb{P}^2, \text{End} E]$. By the Enrique-Severi-Zariski Lemma, $S_{coh}$ is a finite set. Theorem 1.8. says that up to this finite set, $\text{Spec} P$ is induced by $\text{Spec}(\nabla^* \nabla |_{\mathbb{S}^5})$. We can reduce $\text{Spec}(\nabla^* \nabla |_{\mathbb{S}^5})$ to the spectrum of the rough Laplacians on the twisted traceless endomorphism bundles on $\mathbb{P}^2$ (see Formula 3.23). The spectrum of such bundle rough Laplacians is “easier” to understand, in a sense like Theorem 1.10 below.

The splitting $\text{Spec} P = S_{\nabla^* \nabla} \cup S_{coh}$ corresponds to that the domain $L^2(\mathbb{S}^5, \text{Dom}_{\nabla^* \nabla})$ is the direct sum of a finite dimensional subspace given by the cohomologies and the infinite-dimensional orthogonal complement. Please see Definition 3.13 below for detail.

Theorem 1.8. II says that the multiplicity of $-1$ and $-2$ are both equal to

$$
2 h^1[\mathbb{P}^2, (\text{End} E)(-1)] = 2 h^1[\mathbb{P}^2, (\text{End} E)(-2)],
$$

which are the real dimension of the cohomologies.

The identity $\dim \ker P = 2 h^1[\mathbb{P}^2, \text{End} E]$ in the end of Theorem 1.8 says that, as vector spaces over $\mathbb{R}$, the kernel of $P$ is isomorphic to the deformations space of the Hermitian Yang-Mills connection on $\mathbb{P}^2$.

Counting multiplicities, the eigenvalues of $P$ is symmetric with respect to $-\frac{3}{2}$ (see the Kodaira-Serre duality for eigenspaces in Lemma 3.24). Therefore, the eta invariant of $P$ can be calculated (for example, see [21, Proposition 5.4]).

The binomials $\mu^2 + 2\mu - 3$ and $\mu^2 + 4\mu$ are given by the Bochner formulas (see Lemma 2.21 below). The Hermitian Yang-Mills condition is required for these formulas. The irreducible Hermitian Yang-Mills condition is required for the Chern number inequality (see (95) below) that implies $-1$ and $-2$ must be eigenvalues.

Remark 1.9. The eigensections of $P$ admit an explicit form (109) in terms of the eigensections of $\nabla^* \nabla |_{\mathbb{S}^5}$. Please see Remark 3.29 and Lemma 3.27 below.

1.4 An example

As an application, if $E = T^\prime \mathbb{P}^2(k)$ is a twisted holomorphic tangent bundle of $\mathbb{P}^2$, we can completely determine $\text{Spec} P$. We call the Levi-Civita connection of the Fubini-Study metric on $T^\prime \mathbb{P}^2$ the Fubini-Study connection, and denote it by $\nabla^{FS}$.

1.4.1 Spectrum of the rough Laplacian

A step for the above goal is to determine the spectrum of the rough Laplacian. On the twisted endomorphism bundles $(\text{End} T^\prime \mathbb{P}^2)(l)$, the tensor product of the Fubini-Study connection (metric) and the standard connection (metric) on $O(l)$ is called the twisted Fubini-Study connection (metric), respectively. The same notions also apply to $T^\prime \mathbb{P}^2(k)$.

Theorem 1.10. In the setting of Theorem 1.8, let the irreducible Hermitian Yang-Mills triple be a twisted holomorphic tangent bundle $T^\prime \mathbb{P}^2(k)$ with the twisted Fubini-Study metric and connection.
For any integer \( l \), let the rough Laplacian \( \nabla^* \nabla \big|_{(\text{End}_0 T^{\mathbb{P}^2}(l) \to \mathbb{P}^2)} \) be defined by the twisted Fubini-Study connection and the Fubini-Study metric \( d\eta \). Then

\[
\text{Spec} \nabla^* \nabla \big|_{(\text{End}_0 T^{\mathbb{P}^2}(l) \to \mathbb{P}^2)} = \left\{ \frac{4}{3} (a^2 + b^2 + ab + 3a + 3b) - \frac{4}{3} l^2 - 8 \mid a, b \in \mathbb{Z}; \ a, b \geq 0; \right. \\
\left. \max(3 - a - 2b, b - a - 3) \leq l \leq \min(2a + b - 3, 3 + b - a) \right\}. \tag{10}
\]

Consequently, in the associated data setting,

\[
\text{Spec} \nabla^* \nabla \big|_{S^5} = \left\{ \frac{4}{3} (a^2 + b^2 + ab + 3a + 3b) - \frac{l^2}{3} - 8 \mid a, b, l \in \mathbb{Z}; \ a, b \geq 0; \right. \\
\left. \max(3 - a - 2b, b - a - 3) \leq l \leq \min(2a + b - 3, 3 + b - a) \right\}. \tag{11}
\]

The multiplicity of each eigenvalue in (10) and (11) is determined by Proposition 4.1.

Henceforth, we abbreviate \( \nabla^* \nabla \big|_{(\text{End}_0 T^{\mathbb{P}^2}(l) \to \mathbb{P}^2)} \) to \( \nabla^* \nabla \big|_{(\text{End}_0 T^{\mathbb{P}^2}(l))} \); and more generally, we abbreviate \( \nabla^* \nabla \big|_{(\text{End}_0 E(l) \to \mathbb{P}^2)} \) to \( \nabla^* \nabla \big|_{(\text{End}_0 E(l))} \).

In view of the spectral reduction in Formula 3.23 below, it suffices to prove (10). Because \( T^{\mathbb{P}^2} \) and \( O(l) \) are both \( SU(3) \)-homogeneous, we prove it by Peter-Weyl formulation.

- The numbers \( \frac{4}{3} (a^2 + b^2 + ab + 3a + 3b) \) and \( -8 \) therein arise from the Casimir operators of \( su(3) \) and \( su(2) \) on certain irreducible representations respectively. The number \( -\frac{l^2}{3} \) arises from the action of a certain element in the Cartan sub-algebra of \( su(3) \). Please see (100) and Formula 4.20, 4.25 below.

- The desired equation (10) means that these 3 terms are all the contributions from the representation theoretic quantities.

- The condition on \( a, b \) in (10) is the equivalence condition of that a certain irreducible \( SU(3) \)-representation appears as a summand in a certain infinite dimensional representation (see Fact 4.24 below).

It is not obvious to the author how to directly calculate \( \text{Spec} \nabla^* \nabla \big|_{S^5} \triangleq \text{Spec} \nabla^* \nabla \big|_{\pi^5_4(adE) \to S^5} \) on \( S^5 \).

### 1.4.2 The example of Spec\( P \)

Theorem 1.8 [1.10] and Proposition 4.1 below imply the following.

**Corollary 1.11.** In the setting of Theorem 1.8 and 1.10, still let the irreducible Hermitian Yang-Mills triple be a twisted holomorphic tangent bundle \( T^{\mathbb{P}^2}(k) \) with the twisted Fubini-Study metric and connection. Then

\[
S_{\text{coh}} = \{-1\} \cup \{-2\}.
\]

Consequently, let \( S_{\nabla, \nabla} \) be defined by Theorem 1.8 II and (11), the following splitting holds.

\[
\text{Spec} \: P = S_{\nabla, \nabla} \cup \{-1\} \cup \{-2\}.
\]

The first row of the following table contains all the eigenvalues of \( P \) in the closed interval \([-4, 1]\). The second row addresses the multiplicity of each.

| eigenvalue of \( P \) | \(-4\) | \(-2\sqrt{2} - 1\) | \(-2\) | \(-1\) | \(2\sqrt{2} - 2\) | \(1\) |
|----------------------|--------|---------------------|--------|--------|---------------------|--------|
| multiplicity         | 12     | 16                  | 6      | 6      | 16                  | 12     |

The multiplicities of the other eigenvalues are also determined by Proposition 4.1 and Theorem 1.8 II.
1.5 Sketch of the proof of Theorem 1.8 and Corollary 1.11

The ideas in proving Theorem 1.8 can be partially described as follows.

Our Sasaki-Quaternion structure on $S^5$ is a special case of a Sasaki-Einstein $SU(2)$—structure mentioned in [5, 4.1]. We employ more explicit information to prove the fine formula for $P$ (Lemma 2.14).

Then we seek for Bochner formulas (see Lemma 2.21 below). The observation is that the first and second row of $P^2 + 2P$ are “autonomous” i.e. they are independent of the unknowns corresponding to the other rows. The same is true for the third and fourth row of $P^2 + 4P$. This implies that if an eigensection does not correspond to an eigenvalue given by $Spec\nabla^*\nabla|_{S^5}$, the first 4—component of the eigensection (regarding the decomposition in $S^5$) must all be 0, then it can be identified with a certain sheaf cohomology class. Please see Theorem 3.15 below for more detail.

Corollary 1.11 is the direct consequence of Theorem 1.8, 1.10, Proposition 4.1, and a little bit of algebraic geometry (Lemma 4.26).

The paper is organized as follows. In Section 2, we fully employ the Sasaki-Quaternion structure of $(\mathbb{C}^3 \setminus O) \to S^5$ to prove the fine formula (Lemma 2.14) for the operator $P$. We prove Theorem 1.8 in Section 5. In Section 4.4, we prove Theorem 1.10 and Proposition 4.1 by representation theoretic methods, then combine them with Theorem 1.8 to prove Corollary 1.11. The Appendix collects some results obtained by routine calculations.

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2 A little bit of Sasakian geometry and the fine formula for the linearized operator

2.1 The Sasakian geometry of $S^5$

2.1.1 General conventions

We recall some general Riemannian geometry setting under which the required identities are established.

Given a Riemannian manifold $(M, g)$, for any tangent vector $v_x \in T_x M$, let $v_x^\perp$ denote the metric dual form in $T_x^* M$. Conversely, for any 1—form $\theta_x \in T_x^* M$, let $\theta_x \Lambda_M$ denote the metric dual vector in $T_x M$. Given a 1—form $h \in T_x^* M$ and a $p$—form $\Omega \in \wedge^p T_x^* M$, $p \geq 1$, we define the contraction by

$$h \Lambda g \Omega \triangleq h \Lambda_M \Lambda \Omega.$$ (13)

The superscript $\mathbb{C}$ on a (real) vector-bundle (vector space) means the complexification. Associated with the Riemannian metric, in the below, the tensor operators $\Lambda$ (contraction), $\Lambda$ (pulling up), $\Lambda$ (pushing down), $\parallel X$ (projection onto a real vector field $X$), $P_X$ (projection onto the orthogonal complement of a real vector field $X$), and the star operators $*_0$, $*_{\perp}$ etc are all extended $\mathbb{C}$—linearly onto the complexified tangent and co-tangent bundle.
2.1.2 Sasakian coordinate system

The purpose of this section is to define the Sasakian coordinate.

We denote the contact distribution $\text{Ker}\eta$ by $D$. Let $\xi$ denote the standard Reed vector field on $S^5$ (tangential to orbit of the $U(1)$—multiplications). Then $D = \xi^\perp$, and $\eta$ is the metric dual of $\xi$.

The form $\frac{d\eta}{2} = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2)$ is (the pullback of) the (Kähler-form of) the Fubini-Study metric on $\mathbb{P}^2$. This fixes the scaling of the Fubini-Study metric in this article. The pullback of the Fubini-Study metric $g_{FS}$ to $S^5$ is a metric on $D$, though it is not a metric on $S^5$. It induces a metric on the contact co-distribution $D^*$. Henceforth, we collectively call the form $\frac{d\eta}{2}$, and the metrics on $\mathbb{P}^2$, $D$, $D^*$ mentioned in the underlying paragraph the Fubini-Study metric.

It is well known that $D^* = \pi_{5,4}^* T^* \mathbb{P}^2$. On the other hand, because $\pi_{5,4}$ is a Riemannian submersion, the tangent map $\pi_{5,4,*}$ is an isometry $D \to T\mathbb{P}^2$ i.e.

$$g_{S^5}(v, w) = g_{FS}(\pi_{5,4,*}v, \pi_{5,4,*}w).$$

Using the Reeb vector field, we split the tangent bundle of $\mathbb{C}^3 \setminus O$ as

$$T^C(\mathbb{C}^3 \setminus O) = \text{span}(\frac{\partial}{\partial r}, \xi) \oplus \pi_{6,4}^* D^C. \quad (14)$$

Similarly, the tangent bundle of the 7—dimensional manifold $(\mathbb{C}^3 \setminus O) \times S^1$ splits as

$$T^C[(\mathbb{C}^3 \setminus O) \times S^1] = \text{span}(\frac{\partial}{\partial s}, \frac{\partial}{\partial r}, \xi) \oplus \pi_{7,4}^* D^C. \quad (15)$$

Each of the above two splittings is orthogonal with respect to the underlying Euclidean metric.

**The coordinate neighborhoods**

For any $\beta = 0, 1, \text{ or } 2$, we define the coordinate neighborhoods by the following.

$$U_{\beta, \mathbb{P}^2} \triangleq \{|Z| \in \mathbb{P}^2 | Z_\beta \neq 0\} \subset \mathbb{P}^2, \quad U_{\beta, \mathbb{C}^3} \triangleq \{Z \in \mathbb{C}^3 \setminus O | Z_\beta \neq 0\} \subset \mathbb{C}^3,$$

and $U_{\beta, S^5} \triangleq \{Z \in \mathbb{C}^3 \setminus O | Z_\beta \neq 0, \; |Z| = 1\} \subset S^5$. (16)

On each of the open sets in (16), we recall the well known complex coordinate functions by the following table.

| Coordinate neighborhoods | Part of the coordinate functions |
|--------------------------|---------------------------------|
| $U_{0, \mathbb{P}^2}$, $U_{0, S^5}$, $U_{0, \mathbb{C}^3}$ | $u_1 = \frac{Z_1}{Z_0}$, $u_2 = \frac{Z_2}{Z_0}$ |
| $U_{1, \mathbb{P}^2}$, $U_{1, S^5}$, $U_{1, \mathbb{C}^3}$ | $v_0 = \frac{Z_0}{Z_1}$, $v_2 = \frac{Z_2}{Z_1}$ |
| $U_{2, \mathbb{P}^2}$, $U_{2, S^5}$, $U_{2, \mathbb{C}^3}$ | $w_0 = \frac{Z_0}{Z_2}$, $w_1 = \frac{Z_1}{Z_2}$ |

(17)

In $U_{0, S^3}$, the complexification $D^* \mathbb{C}$ is spanned by $du_1$, $du_2$, $d\bar{u}_1$, $d\bar{u}_2$ everywhere. Defining the real coordinates $(x_i, \; i = 1, 2, 3, 4)$ by

$$u_1 = x_1 + \sqrt{-1} x_2, \; u_2 = x_3 + \sqrt{-1} x_4,$$

both the real vector bundle $D^*$ and the complex bundle $D^* \mathbb{C}$ are spanned by $dx_1$, $dx_2$, $dx_3$, $dx_4$. Similar facts also hold in the other two neighborhoods $U_{1, S^5}$ and $U_{2, S^5}$.
The coordinate maps

Based on the above table, on \( \mathbb{C}^3 \setminus O \), we define the Sasakian coordinate.

**Definition 2.1.** Let \( Z_\beta = |Z_\beta| e^{\sqrt{-1} \theta_\beta} \), \( \theta_\beta \in S^1 \triangleq \mathbb{R}/2\pi \mathbb{Z} \). On \( U_{\beta, \mathbb{C}^3} \), the Sasakian coordinate is defined to be the functions \( (r, \theta_\beta, u_j, u_k) \) which is a homeomorphism from \( U_{\beta, \mathbb{C}^3} \to \mathbb{R}^+ \times S^1 \times \mathbb{C}^2 \). Similarly, we also call the homeomorphism \( (\theta_\beta, u_j, u_k) \) from \( U_{\beta, \mathbb{S}^2} \) to the trivial circle bundle \( S^1 \times \mathbb{C}^2 \) the Sasakian coordinate.

Both the vector field \( \frac{\partial}{\partial \theta_\beta} \) and the form \( d\theta_\beta \) descend to the \( S^1 \)–component of \( U_{\beta, \mathbb{S}^5} \) \( (U_{\beta, \mathbb{C}^3}) \).

A prominent difference from the Sasakian coordinate to the holomorphic coordinate is that with respect to the standard complex structure on \( \mathbb{C}^3 \), \( \frac{\partial}{\partial \eta_i} \) \( (i = 1, 2) \) is not \( (1, 0) \) in Sasakian coordinate, but it obviously is \( (1, 0) \) in the holomorphic coordinate. In a Sasakian coordinate chart, the Reeb vector field can be described satisfactorily.

**Fact 2.2.** Let \( \beta = 0, 1, \) or \( 2 \), then \( \xi \equiv \frac{\partial}{\partial \theta_\beta} \) in \( U_{\beta, \mathbb{S}^5} \) under Sasakian coordinate.

**Proof of Fact 2.2.** It suffices to observe that for any point \( Z = (r, \theta_\beta, u_j, u_k) \in U_{\beta, \mathbb{C}^3} \), under the Sasakian coordinate, the scalar multiplication \( e^{\sqrt{-1}t}Z \) is given by the translation in the angular variable.

\[
e^{\sqrt{-1}t} \cdot (r, \theta_\beta, u_j, u_k) = (r, \theta_\beta + t, u_j, u_k) \quad \text{(where \( \theta_\beta + t \in \mathbb{R}/2\pi\mathbb{Z} \)).}
\]

\[\Box\]

### 2.1.3 The Sasaki-Quaternion structure

The purpose of this section is to introduce the Sasakian-Quaternion structure i.e. Lemma 2.7 below. Again, it is a special case of a Sasaki-Einstein \( SU(2) \)–structure mentioned in [5, 4.1]. For our purposes, we look for more explicit information.

**Preliminary**

We begin with a convenient convention.

**Convention 2.3. (pullback and descent)** Given a holomorphic Hermitian triple \((E, h, A_\beta)\) on \( \mathbb{P}^2 \), let \( \theta \) be a usual form or an endomorphism-valued form on \( \mathbb{P}^2, \mathbb{S}^2, \mathbb{C}^3 \setminus O \), or \((\mathbb{C}^3 \setminus O) \times S^1\). Abusing notation, we let \( \theta \) also denote any pullback or descent.

For example, the contact form \( \eta \) originally defined on \( \mathbb{C}^3 \setminus O \) also means the one on \( \mathbb{S}^3 \). The form \( \frac{dn}{2} \) a priori on \( \mathbb{S}^3 \) also means the Fubini-Study Kähler-form on \( \mathbb{P}^2 \).

Similar abusing of notations also applies to differential operators. For example, in (31) below, the exterior derivative on \( \mathbb{P}^2 \), denoted by \( dp_2 \), also means the local pullback operator on \( \mathbb{S}^3 \).

Next, we introduce a formula for the contact form under a Sasakian coordinate chart. We shall mainly work in \( U_{0, \mathbb{S}^5} \). This is because \( U_{0, \mathbb{S}^5} \) is dense and open in \( \mathbb{C}^3 \setminus O \), therefore it suffices to prove the desired identities therein.

For any \( \beta = 0, 1, \) or \( 2 \), let \( \phi_\beta \equiv \frac{r^2}{|Z_\beta|^2} = \frac{|Z_1|^2 + |Z_2|^2 + |Z_3|^2}{|Z_\beta|^2} \) be the Kähler potential of the Fubini-Study metric \( \frac{dn}{2} \) in \( U_{\beta, \mathbb{P}^2} \). Then

\[
|Z_\beta| = \frac{r}{\sqrt{\phi_\beta}}, \quad \text{and} \quad Z_\beta = \frac{r}{\sqrt{\phi_\beta}} e^{\sqrt{-1} \theta_\beta}.
\]  
**Formula 2.4.** For any \( \beta = 0, 1, \) or \( 2 \), \( \eta = d\theta_\beta + \frac{d\log \phi_\beta}{2} \) in \( U_{\beta, \mathbb{S}^5} \).

The above formula, whose proof is deferred to Appendix [A], is used in the proof of Lemma 2.5 below.
The semi-basic forms $G$ and $H$

Using the standard holomorphic volume form on $\mathbb{C}^3$, we now define the forms $G$ and $H$ leading to the Sasaki-Quaternion structure.

On the cone $\mathbb{C}^3 \setminus O$, the vector field $\frac{1}{2r}(\frac{\partial}{\partial r} - \sqrt{-1}\xi)$ is $(1,0)$. Contracting with the standard $(3,0)$–form on $\mathbb{C}^3$, we obtain a form in $\wedge^{(2,0)}D^\ast_{\mathbb{C}}$. Namely, the following Lemma holds.

**Lemma 2.5.** There exist (smooth) semi-basic 2–forms $H$ and $G$ which are sections of

$$\left(\wedge^{2,0} \oplus \wedge^{0,2}\right)D^\ast_{\mathbb{C}} \to S^5$$

with the following properties. Let $\Omega_{\mathbb{C}^3} \triangleq dZ_0dZ_1dZ_2$ be the standard holomorphic volume form on $\mathbb{C}^3$, we have

$$\Omega_{\mathbb{C}^3} = (r^2dr + \sqrt{-1}r^3\eta) \wedge H + (r^2\bar{\eta} - \sqrt{-1}r^2dr) \wedge G,$$

$$Re\Omega_{\mathbb{C}^3} = r^2dr \wedge H + r^3\eta \wedge G, \quad Im\Omega_{\mathbb{C}^3} = r^2\eta \wedge H - r^2dr \wedge G,$$

and

$$\frac{1}{2r^3}(r \frac{\partial}{\partial r} - \sqrt{-1}\xi)\Omega_{\mathbb{C}^3} \triangleq \Theta = H - \sqrt{-1}G.$$ (19)

Under the Sasakian coordinate in $U_{0,S^5}$, $U_{1,S^5}$, $U_{2,S^5}$ respectively, the following holds true for $G$, $H$, and $\Theta$.

| $U_{0,S^5}$ | $G = -\frac{1}{2\sqrt{-1}}(Z_0^3du_1dw_2 - \bar{Z}_0^3d\bar{u}_1d\bar{w}_2)$, | $H = \frac{1}{2}(Z_0^3du_1dw_2 + \bar{Z}_0^3d\bar{u}_1d\bar{w}_2)$, |
| $\Theta = Z_0^3d\bar{u}_1dw_2$. | |
| $U_{1,S^5}$ | $G = \frac{1}{2\sqrt{-1}}(Z_0^3du_0dv_2 - \bar{Z}_0^3d\bar{u}_0d\bar{v}_2)$, | $H = -\frac{1}{2}(Z_0^3du_0dv_2 + \bar{Z}_0^3d\bar{u}_0d\bar{v}_2)$, |
| $\Theta = -Z_0^3d\bar{u}_0dv_2$. | |
| $U_{2,S^5}$ | $G = -\frac{1}{2\sqrt{-1}}(Z_0^3du_0dw_1 - \bar{Z}_0^3d\bar{u}_0d\bar{w}_1)$, | $H = \frac{1}{2}(Z_0^3du_0dw_1 + \bar{Z}_0^3d\bar{u}_0d\bar{w}_1)$, |
| $\Theta = Z_0^3d\bar{u}_0dw_1$. | (22) |

The proof of the above Lemma is by routine calculation, and is also deferred to Appendix A.

In the Sasakian geometry of $\mathbb{C}^3 \setminus O \to S^5$, schematically speaking, the form $H - \sqrt{-1}G$ plays similar role as a no-where vanishing holomorphic volume form on a $K3$–surface. This is analogous to hyper-Kähler geometry.

We search for more properties of the forms $G$ and $H$. Let $\ast_0$ denote the Hodge star operator of the Fubini-Study metric on the exterior algebra of $D^\ast$ (see the paragraph about Fubini-Study metric in Section 2.1.2). Because $H$ and $G$ are both in $(\wedge^{(2,0)} \oplus \wedge^{(0,2)})D^\ast_{\mathbb{C}}$, they are $\ast_0$–self-dual i.e.

$$\ast_0G = G, \quad \ast_0H = H.$$ (23)

At the point $(1,0,0) \in S^5$,

$$G = -Im(du_1dw_2), \quad H = Re(du_1dw_2).$$ (24)

The Quaternion structure

Before proceeding, we stipulate the following.

**Convention 2.6.** We let $\lrcorner$ denote the contraction between two forms on $S^5$ under the standard round metric.

The same notation might also denote the usual contraction between a tangent vector and a form (without involving the Riemannian metric). Our rationale is that if a tensor operation or operator has no subscript for the domain, then it is on $S^5$. 

13
We now get into the crucial properties of $G$ and $H$. In view of the contraction \( (13) \), for any integer \( p \geq 1 \), let \( \lrcorner \) denote the contraction between $T^*S^5$ and $\wedge^p T^*S^5$ under the standard metric $g_{S^5}$ on $S^5$. At an arbitrary point on $S^5$, it is straightforward to verify the following (for example, under the Sasakian-Quaternion coordinate in Appendix B) so that $G$ and $H$ is of the canonical form \( (24) \).

\[
(a \lrcorner G) \lrcorner G = -a, \ (a \lrcorner H) \lrcorner H = -a.
\] (25)

Consequently, both $\lrcorner G$ and $\lrcorner H$ are almost complex structures on $D^\star$. From now on, let $J_G$, $J_H$ denote $\lrcorner G$, $\lrcorner H$. Let $J_0$ denote $\lrcorner \frac{\partial}{\partial t}$. They are all isometries.

We define the complex structures on $D$ naturally by the metric pulling up and down i.e.

\[
J_0(X) \triangleq [J_0(X^t)]_{\partial t}, \ J_H(X) \triangleq [J_H(X^t)]_{\partial t}, \ J_G(X) \triangleq [J_G(X^t)]_{\partial t}.
\] (26)

The Sasakian-Quaternion structure applies to the contact distribution $D$ as well.

Based on the above, we routinely verify our main Lemma in the underlying section.

**Lemma 2.7.** (The Sasakian-Quaternion structure) On $D^\star \to S^5$, $\pi_{7,5}^* D^\star \to (C^3 \setminus O) \times S^1$, and also on $D \to S^5$, $D \to (C^3 \setminus O) \times S^1$, the following holds.

\[
J_G J_H = J_0, \ J_H J_0 = J_G, \ J_0 J_G = J_H, \ J_0^2 = J_H^2 = J_G^2 = -Id.
\] (27)

Moreover, the Fubini-Study metric on $D$ and $D^\star$ is preserved by each of $J_0$, $J_H$, $J_G$.

Consequently, the above is true for the complexifications $D^{\star \mathbb{C}}$, $D^\mathbb{C}$, $\pi_{7,5}^* D^{\star \mathbb{C}}$, $\pi_{7,5}^* D^\mathbb{C}$.

Given a holomorphic Hermitian triple $(E, h, A_0)$ on $\mathbb{P}^2$, the identities in (27) hold for an endomorphism-valued semi-basic 1–form.

### 2.1.4 The Reeb Lie derivative and the transverse exterior derivative

To describe certain eigensections of the operator $P$, we need the two first order differential operators in Formula 2.8 and Definition 2.10.

**Formula 2.8.** Let $L_\xi$ denote the Lie derivative in the direction of the Reeb vector field. Then the followings is true.

\[
L_\xi H = 3G, \ L_\xi G = -3H, \ L_\xi (\frac{d\eta}{2}) = 0.
\] (28)

**Proof of Formula 2.8.** Differentiating (22) with respect to $\theta_0$, the equalities hold everywhere in $U_{0,S^5}$, therefore everywhere in $C^3 \setminus O$ by continuity.

Given a holomorphic Hermitian triple on $\mathbb{P}^2$, let $a_0$ be a $\pi_{7,5}^* End E$–valued semi-basic 1–form on $S^5$. The following holds by the formula for $G$ and $H$ (Lemma 22) and the local formula for the Reeb vector field (Fact 2.2).

\[
J_G L_\xi(a_0) = L_\xi J_G(a_0) + 3J_H(a_0), \ J_H L_\xi(a_0) = L_\xi J_H(a_0) - 3J_G(a_0)
\] (29)

**Notation Convention 2.9.** Most of the time, to avoid heavy notation, for an differential operator on the bundle, we shall suppress the subscript for the connection.

We now turn to the definition of a derivative operator with respect to the contact co-distribution $D^\star$. 

\[\text{14}\]
Definition 2.10. Given a holomorphic Hermitian triple \((E, h, A_O)\) on \(\mathbb{P}^2\), on the bundle \(\wedge^p D^* \otimes \pi_{\mathbb{A}}^* \text{End} E \to \mathbb{S}^5\) of endomorphism-valued semi-basic \(p\text{-forms}\), the transverse exterior derivative operator \(d_0\) is defined by
\[
d_0 \triangleq d - \eta \wedge L_\xi. \tag{30}\]

It turns out that if \(\theta\) is semi-basic, so is \(d_0\theta\). The Lie derivative \(L_\xi\) is well defined because the endomorphism bundle is pulled back from \(\mathbb{P}^2\).

The operator \(d_0\) admits a local splitting in the following sense.

For any \(\beta = 0, 1, 2\), let "\(b\)" be a section (form) of \(\wedge^p D^* \otimes \pi_{\mathbb{A}}^* \text{End} E \to U_{\beta, \mathbb{S}^5}\). For any \(\theta_\beta \in [0, 2\pi)\), the restriction \(b(\cdot, \theta_\beta)\) onto the \(\theta_\beta\)–slice is a form on \(U_{\beta, \mathbb{P}^2}\). We define \(d_{0, b}\) to be the (partial) exterior derivative in the direction of \(U_{\beta, \mathbb{P}^2}\). A priori, this partial exterior derivative is not defined globally on \(\mathbb{S}^5\) because it is not the product manifold \(\mathbb{P}^2 \times \mathbb{S}^1\). However, the open set \(U_{\beta, \mathbb{S}^5}\) is a trivial \(\mathbb{S}^1\)–bundle over \(U_{\beta, \mathbb{P}^2} \subset \mathbb{P}^2\). Employing Formula 2.21 for \(\eta\), this leads to the following splitting of \(d_0\) in \(U_{\beta, \mathbb{S}^5}\).

\[
d_0 b = db - \eta \wedge L_\xi b = d_{0, b} + \frac{1}{2}(\partial^* \log \phi_\beta) \wedge L_\xi b. \tag{31}\]

In view of the above decomposition, we have the further splitting
\[
d_0 = \partial_0 + \bar{\partial}_0, \quad \text{where} \quad \partial_0 = \partial_{0, b} + \frac{1}{2}(\partial_{0, b} \log \phi_\beta) \wedge L_\xi, \quad \text{and} \quad \bar{\partial}_0 = \bar{\partial}_{0, b} - \frac{1}{2}(\partial_{0, b} \log \phi_\beta) \wedge L_\xi. \tag{32}\]

Given an arbitrary semi-basic \((p, q)\)–form \(b\), \(\partial_0 b = (d_0 b)^{p+1, q}\), \(\bar{\partial}_0 b = (d_0 b)^{p, q+1}\). Hence, both of the two operators are globally defined on \(\mathbb{S}^5\).

For any \(\beta\), let \(x_i, i = 1, 2, 3, 4\) be the Euclidean coordinate functions on \(U_{\beta, \mathbb{S}^5} = \mathbb{S}^1 \times \mathbb{C}^2\). The identity
\[
\eta - d\theta_\beta = \eta(\frac{\partial}{\partial x_j}) dx^j
\]
is verified on the basis \((\xi = \frac{\partial}{\partial \theta_\beta}, \frac{\partial}{\partial x_j}, j = 1, 2, 3, 4)\). Given an endomorphism \(u\), we have
\[
d_0 u = (\partial u - \xi(u)\eta(\frac{\partial}{\partial x_j})) dx^j. \tag{33}\]

Similarly, given a semi-basic endomorphism-valued 1–form \(a_0 = \Sigma_{i=1}^4 a_i dx^i\), we have
\[
d_0 a_0 = (\partial a_i - \xi(a_i)\eta(\frac{\partial}{\partial x_j})) dx^j \wedge dx^i. \tag{34}\]

Remark 2.11. Our calculation for the bundle-valued forms remains true for usual forms, unless the irreducible condition is required. This is because we can simply let it be the trivial line bundle. For example, Definition 2.10 and its subsequent calculations hold for usual forms.

2.2 The deformation operator for \(G_2\)–instantons

2.2.1 Separation of variables and the “Quaternion” structure on \((\mathbb{C}^3 \setminus O) \times \mathbb{S}^1\)

The purpose of this section is to state the formula (Lemma 2.14) for the linearized operator of the \(G_2\)–instanton equation under the model data defined in (5).
**Formula 2.12.** ([36, Proposition 3.13]) In view of the splitting in (7), we have

\[
L_{A_O, \phi_{C^3 \times S^1}}[1, ds, 1] \cdot \begin{bmatrix}
\sigma \\
a_s \\
a_r \\
a_\eta \\
a_0 \\
\end{bmatrix} = [1, ds, 1] \cdot \left( (\frac{\partial}{\partial s} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & J_{C^3} \end{bmatrix} + \Box \begin{bmatrix} \sigma \\
a_s \\
a_r \\
a_\eta \\
a_0 \\
\end{bmatrix} \right)
\]  

(35)

where \( \Box \begin{bmatrix} \sigma \\
a_s \\
a_r \\
a_\eta \\
a_0 \\
\end{bmatrix} = \begin{bmatrix}
d^{*_{C^3}} a_{C^3} \\
(d_{C^3} a_{C^3}) \wedge \Omega_{C^3} \\
d_{C^3} \sigma - J_{C^3} (d_{C^3} a_{C^3}) + (d_{C^3} a_{C^3}) \wedge \Omega_{C^3} \\
\end{bmatrix} \).

To obtain a finer splitting, we need to generalize the Sasaki-Quaternon structure on \( S^5 \) in Lemma 2.7 to the domain bundle on the 7—dimensional manifold \( (C^3 \setminus O) \times S^1 \).

**Lemma 2.13.** (The “Quaternon” structure on \( (C^3 \setminus O) \times S^1 \)) Under the setting from (7) to (5) above, let the column vector \( \begin{bmatrix} u \\
a_s \\
a_r \\
a_\eta \\
a_0 \\
\end{bmatrix} \) represents the 5 components of \( \text{Dom}_{S^5} \), respectively.

Let

\[
I = \begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & J_0 \\
0 & 0 & 0 & 0 & J_G \\
\end{bmatrix}, \quad K = \begin{bmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & J_H \\
0 & 0 & 0 & 0 & -J_G \\
\end{bmatrix}
\]  

(36)

\[
T = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & J_G \\
\end{bmatrix}, \quad \text{and } \bar{T} = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -J_G \\
\end{bmatrix}
\]  

(37)

be the isometries of \( \text{Dom}_{S^5} \) (and \( \text{Dom}_r \)) acting on the column vector (by left multiplication). Then \( \bar{T}, K, T \) form an quaternion structure, i.e. all the pairwise multiplications anticommute, and the following is true.

\[
K T = I, \quad I K = T, \quad T I = K, \quad \text{and } I^2 = K^2 = T^2 = -I d_{\text{Dom}_{S^5}}.
\]  

(38)

Under Convention 2.6 on the tensor contractions, we state our main Lemma.

**Lemma 2.14.** Given a Hermitian Yang-Mills triple \((E, h, A_O)\) on \( \mathbb{P}^2 \), under the model setting in (5), the following formula for the model linearized operator holds true.

\[
L_{A_O, \phi_{C^3 \times S^1}} \left[ \begin{bmatrix}
\frac{1}{r} \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \right] \cdot \begin{bmatrix}
\frac{\partial}{\partial s} u \\
\frac{\partial}{\partial s} a_s \\
\frac{\partial}{\partial s} a_r \\
\frac{\partial}{\partial s} a_\eta \\
\frac{\partial}{\partial s} a_0 \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{r} \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \cdot \begin{bmatrix}
\frac{\partial}{\partial s} u \\
\frac{\partial}{\partial s} a_s \\
\frac{\partial}{\partial s} a_r \\
\frac{\partial}{\partial s} a_\eta \\
\frac{\partial}{\partial s} a_0 \\
\end{bmatrix} + \frac{\partial}{\partial r} \left( \begin{bmatrix}
1 & -L_{\xi} & 0 & 0 & -(d_{0})_x H \\
L_{\xi} & 1 & 0 & 0 & (d_{0})_x G \\
0 & 0 & -4 & -L_{\xi} & d_{0}^a \\
0 & 0 & L_{\xi} & 4 & -(d_{0})_x \frac{d_{0}^a}{2} \\
J_{H} d_{0} & -J_{G} d_{0} & d_{0} & J_{0} d_{0} & -L_{\xi} J_{0} \end{bmatrix} \right) \begin{bmatrix}
u \\
a_s \\
a_r \\
a_\eta \\
a_0 \\
\end{bmatrix}
\]  

(39)

where

\[
P = \begin{bmatrix}
u \\
a_s \\
a_r \\
a_\eta \\
a_0 \\
\end{bmatrix} = \begin{bmatrix}
1 & -L_{\xi} & 0 & 0 & -(d_{0})_x H \\
L_{\xi} & 1 & 0 & 0 & (d_{0})_x G \\
0 & 0 & -4 & -L_{\xi} & d_{0}^a \\
0 & 0 & L_{\xi} & 4 & -(d_{0})_x \frac{d_{0}^a}{2} \\
J_{H} d_{0} & -J_{G} d_{0} & d_{0} & J_{0} d_{0} & -L_{\xi} J_{0} \end{bmatrix}^{-1}
\]  

(40)

is a first-order self-adjoint elliptic operator on the bundle \( \text{Dom}_{S^5} \) (see Definition 1.4).
Under the tools established in the introduction, Section 2.1, and the generalized Quaternion structure in the above Lemma 2.13, the proof of the above Lemma is a routine and fairly tedious tensor calculation. We defer it to Appendix D.

**Convention 2.15.** Let the \([\bar{\pi}^*_{5,4}(adE)]^\mathbb{S}^4\) component be zero, a semi-basic \(\pi^*_7\) \(adE\)–valued 1–form \(a_0\) can be viewed as in \(\text{Dom}_{\mathbb{S}^5}\). The corresponding vector under the basis in (8) is

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
a_0
\end{bmatrix}
\]

i.e. the first 4 components are zero.

We note again that Lemma (2.14) does not require the connection \(A_O\) to be Hermitian Yang-Mills, but the Bochner formulas in the following section do i.e. it is not known whether they remain true if \(A_O\) is not Hermitian Yang-Mills.

### 2.2.2 Bochner formula for \(P\)

In the Quaternion structure, both \(I, K\) commute with \(\frac{\partial}{\partial r}\) and \(\frac{\partial}{\partial s}\). Using the Quaternion identities (38), we routinely verify the following formula for commutators between the operator \(P\) and \(I, K, T\).

\[
PI = IP, \quad KP + PK = -3K, \quad TP + PT = -3T.
\]

#### (41)

**Remark 2.16.** That \(P\) commutes with \(I\) makes \(I\) a complex structure of the eigenspaces of \(P\).

Consequently, we straight-forwardly verify the following formula for the square of \(L_{A_O, \phi_{\mathbb{C}^3 \times \mathbb{S}^1}}\).

\[
L^2_{A_O, \phi_{\mathbb{C}^3 \times \mathbb{S}^1}} = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2} - \frac{3}{r} \frac{\partial}{\partial r} + \frac{P^2 + 2P}{r^2}.
\]

#### (42)

We have another formula for \(L^2_{A_O, \phi_{\mathbb{C}^3 \times \mathbb{S}^1}}\) than the above.

**Remark 2.17.** In conjunction with the discussion below equation (2), because the linearized operator only depends on the projective connection induced, we denote the curvature form of the projective connection by \(F^0_{A_O}\) (or \(F^0_{A_0}\)).

In arbitrary dimension \(n\), on \(\mathbb{R}^n \setminus O\), let \(A_0\) be a pullback connection from \(\mathbb{S}^{n-1}\). Let \(\Box_{A_0}^{\text{dimn}} \triangleq \nabla^* \nabla + 2F^0_{A_0} \otimes_{\mathbb{R}^n}\) be the operator acting on \(\Omega^1(EndE) \to \mathbb{R}^n \setminus O\) (\(EndE\) is the pullback endomorphism bundle). Using the formula [20, Lemma 3.2] for the rough Laplacian \(\nabla^* \mathbb{R}^n \nabla^* \mathbb{R}^n\) on 1–forms, we find

\[
\Box_{A_0}^{\text{dimn}} = -\frac{\partial^2}{\partial r^2} - \frac{n - 3}{r} \frac{\partial}{\partial r} + \frac{\hat{B}_{0,\text{dimn}}}{r^2},
\]

#### (43)

where

\[
\hat{B}_{0,\text{dimn}} \begin{bmatrix}
a_r \\
a
\end{bmatrix} \triangleq \begin{bmatrix}
\nabla^{\mathbb{S}^{n-1}} a_r - 2d^* a + 2(n - 2)a \\
\nabla^{\mathbb{S}^{n-1}} a - 2d a_r + (n - 2)a + 2F^0_{A_0} \otimes_{\mathbb{S}^{n-1}} a
\end{bmatrix}.
\]

#### (44)

\(\Box_{A_0}^{\text{dimn}}\) is the “linearized” operator for the Yang-Mills equation with gauge fixing (cf. [21, 6.1]).

Then we go on to prove Theorem 3.15.
Formula 2.18. Given a Hermitian Yang-Mills triple on $\mathbb{P}^2$, in view of Lemma (2.14), and still let $\nabla^* \nabla$ denote the rough Laplacian on $\pi_*^* 4adE$, the following identity holds

$$P^2 + 2P = B_{0, \dim 6},$$

where

$$B_{0, \dim 6} = \begin{bmatrix} u \\ a_s \\ a_r \\ a \end{bmatrix} \triangleq \begin{bmatrix} \nabla^* \nabla u + 3u \\ \nabla^* \nabla a_s + 3a_s \\ \nabla^* \nabla a_r - 2d^* a + 8a_r \\ \nabla^* \nabla a - 2da_r + 4a + 2F^0_\varphi \otimes C_3 a \end{bmatrix}.$$  \hspace{1cm} (45)

In relation to the remark under Theorem 1.8, we need the Hermitian Yang-Mills condition in Lemma 2.18 but not in 2.14 because the formula [20, (33)] needs the pullback connection still let $\nabla$, and Lemma 3.2] (let $\nabla$).

Proof of Formula 2.18: The observation is that, using the usual Euclidean coordinates on $\mathbb{C}^3 \times S^1$ (induced from $\mathbb{R}^7$), we have another way to compute $L^2_{A_0, \phi_{C^3 \times S^1}}$ that yields

$$L^2_{A_0, \phi_{C^3 \times S^1}} = -\frac{\partial^2}{\partial s^2} - \frac{\partial^2}{\partial r^2} - \frac{3}{r} \frac{\partial}{\partial r} + \frac{B_{0, \dim 6}}{r^2}.$$ \hspace{1cm} (46)

The proof is complete comparing (46) with (12).

It remains to show (46). It directly follows from the identities in [20]. In view of the splitting in (6), the Bochner formula [20, (146)], which holds for projective $G_2$–instantons (because locally the connection form acts on endomorphisms-valued forms via Lie-bracket), yields

$$L^2_{A_0, \phi_{C^3 \times S^1}} \left( \frac{1}{r}, 1 \right) = \left[ u \quad a_s \quad a_r \quad a \right] = \begin{bmatrix} \nabla^* C^3 x S^1 \nabla C^3 x S^1 (u) \\ \nabla^* C^3 x S^1 \nabla C^3 x S^1 (a_s) \\ \nabla^* C^3 x S^1 \nabla C^3 x S^1 (a_r) \\ \nabla^* C^3 x S^1 \nabla C^3 x S^1 (a) \end{bmatrix}.$$ \hspace{1cm} (47)

We note that the proof of [20, (146)] is by Euclidean coordinates for the model $G_2$–structure, thus it also holds in our case as $\mathbb{C}^3 \times S^1$ possesses such coordinates for the standard $G_2$–structure $\phi_{C^3 \times S^1}$.

The point is that $\nabla^* C^3 x S^1 \nabla C^3 x S^1 = -\frac{\partial^2}{\partial s^2} + \nabla^* C^3 \nabla C^3$, and $ds$ is $\nabla C^3 x S^1$–parallel. We then compute

$$\nabla^* C^3 x S^1 \nabla C^3 x S^1 \left( \frac{a_s ds}{r} \right) = \left[ \nabla^* C^3 x S^1 \nabla C^3 x S^1 \left( \frac{a_s}{r} \right) \right] ds = \left[ -\frac{\partial^2}{\partial s^2} \left( \frac{a_s}{r} \right) + \nabla^* C^3 \nabla C^3 \left( \frac{a_s}{r} \right) \right] ds,$$

and the similar identity holds for $\nabla^* C^3 x S^1 \nabla C^3 x S^1 \left( \frac{u}{r} \right)$ i.e.

$$\nabla^* C^3 x S^1 \nabla C^3 x S^1 \left( \frac{u}{r} \right) = -\frac{\partial^2}{\partial s^2} \left( \frac{u}{r} \right) + \nabla^* C^3 \nabla C^3 \left( \frac{u}{r} \right).$$

Hence, in view of the splitting in (7), we find

$$L^2_{A_0, \phi_{C^3 \times S^1}} \left[ \frac{1}{r}, \frac{ds}{r}, 1 \right] = \begin{bmatrix} u \\ a_s \\ a_r \\ a \end{bmatrix} \triangleq \begin{bmatrix} \nabla^* C^3 x S^1 \nabla C^3 x S^1 (u) \\ \nabla^* C^3 x S^1 \nabla C^3 x S^1 (a_s) \\ \nabla^* C^3 x S^1 \nabla C^3 x S^1 (a_r) \\ \nabla^* C^3 x S^1 \nabla C^3 x S^1 (a) \end{bmatrix}.$$ \hspace{1cm} (48)

Now we view $\mathbb{C}^3 \setminus \{O\}$ as the real 6–dimensional flat cone over $S^5$. Using the formulas [20 (29) and Lemma 3.2] (let $n = 6$ therein) for the rough Laplacians $\nabla^* C^3 x S^1 a$, $r \nabla^* C^3 \nabla C^3 a$, and $r \nabla^* C^3 \nabla C^3 \left( \frac{u}{r} \right)$, the proof for (46) is complete.

The following formula says the 3 forms yielding the Sasaki-Quaternion structure are all $d_0$–harmonic.
Formula 2.19. The following vanishing holds.

\[
d_0 \left( \frac{d\eta}{2} \right) = d_0 G = d_0 H = d_0 \Theta = d_0 \bar{\Theta} = 0. \tag{49}\]

Consequently, because they are all \( \ast_0 \) self-dual,

\[
d_0^\ast \left( \frac{d\eta}{2} \right) = d_0^\ast G = d_0^\ast H = d_0^\ast \Theta = d_0^\ast \bar{\Theta} = 0. \tag{50}\]

Proof of Formula 2.19: Routine calculation shows that the two individual terms in the formula \( (31) \) for \( d_0 \) are

\[
de_{\mathbb{P}^2}(Z_0^3 du_1 du_2) = -\frac{3}{2} \bar{\partial} \log \phi_0 \wedge (Z_0^3 du_1 du_2),
\]

and

\[
-\frac{1}{2}[(d_{\mathbb{P}^2}) \log \phi_0] \wedge \xi_\xi(Z_0^3 du_1 du_2) = \frac{3}{2} [\bar{\partial} \log \phi_0] \wedge (Z_0^3 du_1 du_2).
\]

Then \( (31) \) says that \( d_0(Z_0^3 du_1 du_2) = 0. \) Taking complex conjugate, we find

\[
d_0(\bar{Z}_0^3 d\bar{u}_1 d\bar{u}_2) = 0.
\]

Using the expression of \( G, H, \) and \( \Theta \) in \( (22), (19) \) holds in \( U_0, \mathbb{S}^5. \) Because they are both smooth forms, by continuity, \( (49) \) holds true everywhere on \( \mathbb{S}^5. \)

The \( \ast_0 \) self-duality of the forms \( \frac{d\eta}{2} \), \( G, H \) also yields the following identities.

**Formula 2.20.** \( d_0^\ast J_0(a_0) = d_0(a_0) \wedge \omega_0, \ d_0^\ast J_G(a_0) = d_0(a_0) \wedge G, \ d_0^\ast J_H(a_0) = d_0(a_0) \wedge H. \)

Proof of Formula 2.20: We only prove the second identity, the other two are similar. We calculate

\[
d_0^\ast (J_G a_0) \triangleq - \ast_0 d_0 \ast_0 (a_0 \wedge G) = - \ast_0 d_0 \ast_0 (a_0 \wedge G) = \ast_0 d_0 (a_0 \wedge G)
\]

(by the \( d_0 \) - closeness in Formula \( (219) \)).

Lemma 2.21. (Bochner formulas for \( P \)) Given a Hermitian Yang-Mills triple on \( \mathbb{P}^2, \) still in view of the 5 component separation in \( (7) \) andLemma 2.14 the following holds.

\[
(P^2 + 2P) \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} = \begin{bmatrix} \nabla^s \nabla u + 3u \\ \nabla^s \nabla a_s + 3a_s \\ \nabla^s \nabla a_r - 2d_0^\ast a_0 + 2L_\xi a_\eta + 8a_r \\ \nabla^s \nabla a_\eta - 2L_\xi a_r + 8a_\eta + 2d_0^\ast J_0(a_0) \\ \nabla^s \nabla a_0 - 2d_0 a_r - 2J_0(d_0 a_\eta) + 4a_0 + 2F_{A_0}^0 \otimes \mathbb{S}^5 a_0 \end{bmatrix}. \tag{50}\]

Consequently,

\[
(P^2 + 4P) \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} = \begin{bmatrix} \nabla^s \nabla u + 5u - 2L_\xi a_s - 2d_0^\ast J_H(a_0) \\ \nabla^s \nabla a_s + 5a_s + 2L_\xi u + 2d_0^\ast J_G(a_0) \\ \nabla^s \nabla a_r \\ \nabla^s \nabla a_\eta \\ \nabla^s \nabla a_0 + 4a_0 + 2F_{A_0}^0 \otimes \mathbb{S}^5 a_0 + 2J_H(d_0 u) - 2J_G(d_0 a_s) - 2L_\xi(J_0 a_0) \end{bmatrix}. \tag{51}\]
Here, we need the Hermitian Yang-Mills condition because we need the pullback of the connection to \((\mathbb{C}^3 \setminus O) \times S^1\) to be a projective \(G_2\)-instanton. Please see \([20]\) reduction from (32) to (33) using the projective instanton condition).

**Proof of Lemma 2.21:** We keep the identity \(P^2 + 2P = B_{0,dim6}\) in mind throughout. Formula (51) for \(P^2 + 4P\) is obtained simply by adding twice of \(P\) to (50) (see formula (40)). It suffices to prove (50).

First of all, formula (199) below for the operator \(d_{S^5}\) yields that row 3 of (43) is equal to row 3 of the desired formula (50).

On row 4 of the desired formula (50),

- formula A.2 below for the rough Laplacian says that the \(\eta\) component (co-efficient for \(\eta\)) of \(\nabla^* \nabla a_0\) is \([d_0^* \theta_0(\alpha_0)]\eta\);

- the \(\eta\)-component of \(-2d_s a_r\) is \(-2\eta \wedge L_\xi a_r\), and that of \(4\alpha\) is apparently \(4\alpha_\eta\eta\);

- Formula A.3 below says that the \(\eta\)-component of \(\nabla^* \nabla (\eta a_\eta)\) is \(4\alpha_\eta + \nabla^* \nabla a_\eta\).

The above facts amount to that the \(\eta\)-component of row 4 in (45) is

\[
\eta [\nabla^* \nabla a_\eta - 2L_\xi a_r + 8\alpha + 2d_0^* \theta_0(\alpha_0)],
\]

which exactly gives row 4 of the desired formula (50) as co-efficient of \(\eta\).

For row 5, by similar idea, we only have to observe that Formula A.3 below also says that the \(\pi_{\beta,4}(\mathbf{D}^* \otimes \mathbf{ad} E)\) (semi-basic) component of \(\nabla^* \nabla (\eta a_\eta)\) is \(-2J_0(d_0 a_\eta)\). This completes the proof of (50).  

\[\square\]

3  **Eigenvalues and eigenspaces of the operator \(P\) on \(S^5\), and the proof of Theorem 1.8**

3.1  **Fourier expansion with respect to the Reeb vector field**

Given a holomorphic Hermitian triple \((E, h, A_0)\) on \(\mathbb{P}^2\), the purpose of this section is to show that any (sufficiently regular) endomorphism on \(S^5\) admits a global “Fourier series” in terms of the trivializations \(s_{-k}\) of \(\pi_{\beta,4} O(-k) \to S^5\) (Lemma 3.3). The “Fourier”-co-efficient of \(s_{-k}\) is a section of \((\text{End} E)(k)\) over \(\mathbb{P}^2\).

The series is important in characterizing the vector spaces \(V_l\) (see Section 3.2 below), and in the spectral reduction from \(S^5\) to \(\mathbb{P}^2\) (see Formulas A.2 below).

This global Fourier-Series is equal to the usual Fourier-series in terms of \(e^{\sqrt{-1}k\theta}\) defined (locally) in \(U_{\beta,S^5}\), for any \(\beta\) among \(0, -1, -2\).

Let \(\nu \in C^1[S^5, \pi_{\beta,4} \text{End} E]\). For any \(\beta = 0, 1\) or \(2\), in \(U_{\beta,S^5}\), the usual Fourier expansion

\[
\nu = \sum_{k \in \mathbb{Z}} v_\beta(k)e^{\sqrt{-1}k\theta}\]

(52)

converges uniformly in \(U_{\beta,S^5}\) under the Hermitian metric (see Lemma A.2 below). For any \(k\), \(v_\beta(k)\) is a section of \(\pi_{\beta,4} \text{End} E \to U_{\beta,S^5}\).

Because on the overlap \(U_{\beta,S^5} \cap U_{\alpha,S^5}\), the function \(\frac{\sqrt{-1}k\theta}{\sqrt{-1}k\theta_0}\) is equal to \(\frac{Z_{\beta}}{Z_{\alpha}} \cdot \sqrt{\frac{\phi_\beta}{\phi_\alpha}}\), it is pulled back from the open set \(U_{\beta,\mathbb{P}^2} \cap U_{\alpha,\mathbb{P}^2}\) in \(\mathbb{P}^2\). Because \(\nu\) is globally defined, given a \(\beta\) and \(\alpha\) among \(0, 1, 2\), for any integer \(k\), on the overlap \(U_{\beta,S^5} \cap U_{\alpha,S^5}\), the \(k-\)th terms of the Fourier-Series satisfy the following.

\[
\nu_\beta(k)e^{\sqrt{-1}k\theta_\beta} = \nu_\alpha(k)e^{\sqrt{-1}k\theta_\alpha}.
\]

(53)
Pointwisely, for any \([Y] \in \mathbb{P}^2\), let \((X_0, X_1, X_2) \in \mathbb{C}^3 \setminus O\) belong to the line \([Y]\). Tautologically, we define the dual basis \((X_0, X_1, X_2)^\vee\) as the functional on \([Y]\) whose value at \((X_0, X_1, X_2)\) is equal to 1. Then

\[
(\lambda X_0, \lambda X_1, \lambda X_2)^\vee = \frac{1}{\lambda} (X_0, X_1, X_2)^\vee.
\]

**Definition 3.1.** Let

\[
s_{-1} \triangleq (X_0, X_1, X_2)
\]

be the standard unitary trivialization of \(\pi_5^4 |O(-1)| \to S^5\). Then for any integer \(l\),

\[
s_l \triangleq \begin{cases} 
    s_{-1}^{\otimes -1} & \text{when } l \leq 0 \\
    s_{-1}^{\otimes \geq l} & \text{when } l > 0
\end{cases}
\]

is the unitary trivialization of \(\pi_5^4 |O(l)| \to S^5\) with respect to the standard metric (see Definition 1.2).

We understand \(s_{-1}^{\otimes 0}\) and \([s_{-1}^{\otimes 0}]\) as the constant function 1 which trivializes the trivial rank 1 complex bundle on \(S^5\).

Let \(s_k \otimes s_{-k}\) denote the section of the trivial complex line bundle \(\pi_5^4 O(k) \otimes \pi_5^4 O(-k)\) over \(S^5\). Viewing \(\text{End} E\) as the tensor product between itself and \(\pi_5^4 O(k) \otimes \pi_5^4 O(-k)\), we find for any fixed \(k\) that

\[
\nu_\beta(k) e^{\sqrt{-1} k \theta_\beta} = [\nu_\beta(k) e^{\sqrt{-1} \theta_\beta} s_k] \otimes s_{-k}.
\]

By the transition condition (53), the section \(\nu_k\) of \(\text{End} E \to S^5\) defined piece-wisely by \(\nu_\beta(k) e^{\sqrt{-1} k \theta_\beta} s_k\) on \(U_{\beta, \mathbb{S}^5}\), is independent of the coordinate neighborhood \(U_{\beta, \mathbb{S}^5}\) chosen. Moreover, \(\nu_k\) is independent of \(\theta_\beta\) in \(U_\beta\), thus it descends to \(\mathbb{P}^2\). This means the section \(\nu_k\) is a globally defined section pulled back from \(\mathbb{P}^2\). We then have the global Fourier series.

\[
\nu = \sum_{k \in \mathbb{Z}} \nu_k \otimes s_{-k}.
\]

The negative sign in “\(-k\)” is to be consistent with the usual local Fourier-Series in (32) i.e. for any integer \(k\), and any \(\beta\) among 0, 1, 2, the following is true in \(U_{\beta, \mathbb{S}^5}\).

\[
\nu_\beta(k) e^{\sqrt{-1} k \theta_\beta} = \nu_k \otimes s_{-k}.
\]

**Definition 3.2.** Henceforth, the series (58) is called the Sasakian-Fourier series of \(\nu\). This is not the same object as the eigen Fourier expansion in Definition 3.21 below.

We are ready for the main lemma of this section.

**Lemma 3.3.** Given a holomorphic Hermitian triple \((E, h, A_0)\) on \(\mathbb{P}^2\), let \(\nu \in C^{10}(S^5, \pi_5^4 \text{End} E)\). For any \(k \in \mathbb{Z}\), there is an unique section \(\nu_k\) of \((\text{End} E)(k)\) such that the following holds.

Under the pullback Hermitian metric, the Sasaki-Fourier series \(\sum_k \nu_k \otimes s_{-k}\) converges uniformly to \(\nu\) on \(S^5\). Moreover, this series can be differentiated term by term by the Reeb Lie derivative \(L_\xi\) and the rough Laplacian \(\nabla^* \nabla\) i.e.

- \(\sum_k L_\xi(\nu_k \otimes s_{-k})\) is the Sasaki-Fourier series of \(L_\xi \nu\), and converges uniformly on \(S^5\) to \(L_\xi \nu\).

- \(\sum_k \nabla^* \nabla(\nu_k \otimes s_{-k})\) is the Sasaki-Fourier series of \(\nabla^* \nabla \nu\), and converges uniformly on \(S^5\) to \(\nabla^* \nabla \nu\).

The Sasaki-Fourier co-efficient \(\nu_k\) is \((\text{End} A_0 E)\)—valued if \(\nu\) is.

Please see the remark above Claim F.4 that not every operator can differentiate the Sasaki-Fourier Series term by term.

**Proof of Lemma 3.3.** It is a straightforward combination of the uniform convergence in Lemma F.2 and the term by term-wise differentiation in Claim F.4 below. □
3.2 Characterizing some $P$–invariant subspaces of sections on $S^5$ by sheaf cohomologies on $\mathbb{P}^2$

In this sub-section, we study the special class of eigensections of the operator $P$ consisted of $\pi_{5,4}^*(adE)$–valued semi-basic 1–forms i.e. an eigensection of which the first 4 endomorphism components are 0 (regarding the decomposition in (3)). These turn out to be a “building-block” of $\text{Spec}P$ (as in Theorem 1.8 above, also see Theorem 3.15 below).

**Definition 3.4.** Let $V_l \triangleq \{a_0 \in C^{10}[S^5, D^* \otimes \pi_{5,4}^*adE] | Pa_0 = la_0 \}$. This means

$$V_l = \{a_0 \in C^{10}[S^5, D^* \otimes \pi_{5,4}^*adE]|d_0a_0 \cdot H = d_0a_0 \cdot G = d_0^\omega a_0 = d_0a_0 \cdot \frac{\eta}{2} = 0, \text{ } L_\xi(J_0a_0) = -la_0 \},$$

$$L_\xi(J_0a_0) = -la_0 \}.$$ 

The above definition says that $V_l$ is a subspace of the eigenspace $\mathbb{E}_l P$. Elliptic regularity implies that any $a_0 \in V_l$ is smooth.

This sub-section is devoted to the proof of the following characterization of $V_l$.

**Proposition 3.5.** Given a holomorphic Hermitian triple on $\mathbb{P}^2$, for any integer $l$, the subspace $V_l$ (of the eigenspace) is isomorphic to

$$H^{0,1}[\mathbb{P}^2, (\text{End}_0E)((l))] \text{ (space of } \bar{\partial} \text{ – harmonic forms).}$$

Consequently, with respect to the complex structure $J_0$, $V_l$ is complex isomorphic to the sheaf cohomology $H^1[\mathbb{P}^2, (\text{End}_0E)((l))]$.

To prove the above proposition and for other purposes, it is useful to set the following convention.

**Notation Convention 3.6.** In conjunction with Notation Convention 1.6 when the two vector spaces are complex vector spaces of sections of a complex vector bundle (like the twisted endomorphism bundles), or when they are sheaf cohomologies etc, the “=” means a complex isomorphism. Otherwise, to say it is a complex isomorphism, the complex structure should be specified in a manner similar to Proposition 3.5.

3.2.1 The two term Sasaki-Fourier series for elements in $V_l$

We decompose any $\pi_{5,4}^*adE$–valued semi-basic 1–form $a_0$ into the $(1, 0)$ and $(0, 1)$–components

$$a_0 = a_0^{1,0} + a_0^{0,1}. \quad (61)$$

Then the condition

$$-L_\xi J_0(a_0) = la_0 \text{ (which is part of (60))} \quad (62)$$

yields that

$$la_0^{1,0} + la_0^{0,1} = la_0 = -L_\xi J_0(a_0) = -L_\xi (-\sqrt{-1}a_0^{1,0} + \sqrt{-1}a_0^{0,1}) = \sqrt{-1}L_\xi a_0^{1,0} - \sqrt{-1}L_\xi a_0^{0,1}. \quad (63)$$

Comparing $(1, 0)$ and $(0, 1)$–part of both sides, we find

$$L_\xi a_0^{0,1} = \sqrt{-1}la_0^{0,1}, \quad L_\xi a_0^{1,0} = -\sqrt{-1}la_0^{1,0}. \quad (63)$$
Consider the Fourier-expansions
\[ a_{0}^{0,1} = \sum_{k} a_{0}^{0,1}(k)s_{-k}, \quad a_{0}^{1,0} = \sum_{k} a_{0}^{1,0}(k)s_{-k}. \] (64)
where the summations are over all integers, and each coefficient is a \( \text{End}_{0}E \)-valued 1—form pulled back from \( \mathbb{P}^2 \).

Since
\[ L_{\xi}s_{-k} = \sqrt{-1}ks_{-k}, \] (65)
the following holds.
\[ L_{\xi}a_{0}^{0,1} = \sum_{k}\sqrt{-1}ka_{0}^{0,1}(k)s_{-k}, \quad L_{\xi}a_{0}^{1,0} = \sum_{k}\sqrt{-1}ka_{0}^{1,0}(k)s_{-k}. \] (66)

Compare the Sasaki-Fouriercoefficients of the first equation in (66) with the first identity in (63), we find that the eigenvalue \( l \) must be an integer, and \( a_{0}^{0,1}(k) = 0 \) if \( k \neq l \). Consequently,
\[ a_{0}^{0,1} = a_{0}^{0,1}(l)s_{-l}. \quad \text{Similarly,} \quad a_{0}^{1,0} = a_{0}^{1,0}(-l)s_{l}. \]
In summary, we have found

**Claim 3.7.** Suppose \( a_{0} \in C^{10}[\mathbb{S}^5, D^* \otimes \pi_{4}^{*}adE] \) satisfies equation (62) and \( a_{0} \neq 0 \), then the \( l \) therein is an integer. Moreover, in view of the \( (1, 0) \oplus (0, 1) \)—decomposition (61), we have
\[ a_{0}^{1,0} = c^{1,0}s_{l} + c^{0,1}s_{-l}, \] (67)
where \( c^{1,0} \) is an \( \text{End}_{0}E)(-l) \)—valued \( (1, 0) \)—form on \( \mathbb{P}^2 \), and \( c^{0,1} \) is an \( \text{End}_{0}E)(l) \)—valued \( (0, 1) \)—form on \( \mathbb{P}^2 \).

The above claim particularly means that there are only two non-zero terms in the Sasaki-Fourier series of \( a_{0} \) (if it satisfies (62)).

### 3.2.2 Equivalence between the conditions on \( d_{0}b \) and that \( b^{0,1} \) is \( \bar{\partial}_{0} \)—harmonic

The purpose of this section is to prove Lemma 3.9 on the equivalent characterization of the 4—different “\( d_{0}^{\theta} \)—closeness” in the defining conditions of (60).

We need the following simple algebraic fact. The 2—form \( H - \sqrt{-1}G = \Theta \in \Lambda^{2,0}D^{*,\mathbb{C}} \) is nowhere vanishing, and the complex dimension of \( D^{*}(1,0) \) is 2. Then the following holds elementarily.

**Fact 3.8.** Let \( p \in \mathbb{S}^5 \), \( \theta_{1} \in \Lambda^{(0,2)}D^{*,\mathbb{C}}|_{p} \), and \( \theta_{2} \in \Lambda^{(2,0)}D^{*,\mathbb{C}}|_{p} \).

- \( \theta_{1,\gamma}(H - \sqrt{-1}G) = 0 \) at \( p \) if and only if \( \theta = 0 \) at \( p \).
- \( \theta_{2,\gamma}(H + \sqrt{-1}G) = 0 \) at \( p \) if and only if \( \theta = 0 \) at \( p \).

Again, because \( \Theta \) is \( (2, 0) \), the contraction between any form in \( \Lambda^{(2,0)}D^{*,\mathbb{C}} \) with \( \Theta = H - \sqrt{-1}G \) is automatically 0. The same holds for the contraction between any form in \( \Lambda^{(0,2)}D^{*,\mathbb{C}} \) with \( \Theta = H + \sqrt{-1}G \).

In our convention, the Hodge star \( \ast_{0} \) is extended complex linearly. On semi-basic 1—forms, we define
\[ d_{0}^{\theta} = -\ast_{0}d_{0}\ast_{0}, \quad \partial_{0}^{\theta} = -\ast_{0}\partial_{0}\ast_{0}; \quad \bar{\partial}_{0}^{\theta} = -\ast_{0}\bar{\partial}_{0}\ast_{0}. \] (68)
Then the adjoint \( \bar{\partial}_{0} \) with respect to the Hermitian inner-product is \( \partial_{0}^{\theta} \), and that of \( \partial_{0} \) is \( \bar{\partial}_{0}^{\theta} \) (cf. the other notation convention in [6]).
Lemma 3.9. Given a holomorphic Hermitian triple on $\mathbb{P}^2$, let $b$ be a smooth section of $D^* \otimes \pi^*_5\mathcal{A}dE \to \mathbb{S}^5$, the following holds true.

\[ d_0b_\perp H = d_0b_\perp G = 0 \iff \bar{\partial}_0b^{0,1} = \partial_0b^{1,0} = 0; \]  
\[ d_0^{\ast}b = d_0b_\perp \frac{d\eta}{2} = 0 \iff \bar{\partial}_0^{\ast}b^{0,1} = \partial_0^{\ast}b^{1,0} = 0. \]  

Proof of Lemma 3.9: It is by simple and routine comparison of types of the forms. For the reader’s convenience, we still provide the detail. As follows, we can split $d_0b$ into $(0, 2), (2, 0)$, and $(1, 1)$ components.

\[ d_0b = \bar{\partial}_0b^{0,1} + \partial_0b^{1,0} + (\partial_0b^{0,1} + \bar{\partial}_0b^{1,0}). \]  

The form $H - \sqrt{-1}G$ is $(2, 0)$, thus among the 4 terms in (71), only the $(0, 2)$—component $\bar{\partial}_0b^{0,1}$ might have non-zero contraction with $H - \sqrt{-1}G$, the contraction between each of the other 3 terms and $H - \sqrt{-1}G$ vanishes. Combining the vanishing criteria in Fact 3.8 we find

\[ d_0b_\perp (H - \sqrt{-1}G) = 0 \iff \bar{\partial}_0b^{0,1} = 0. \]  

Similarly, because $H + \sqrt{-1}G$ is $(0, 2)$, the following holds true.

\[ d_0b_\perp (H + \sqrt{-1}G) = 0 \iff \partial_0b^{1,0} = 0. \]  

The proof of (69) is complete.

To prove (70), we first observe that

\[ \partial_0^{\ast}b^{1,0} = 0, \quad \bar{\partial}_0^{\ast}b^{0,1} = 0. \]  

To prove the first identity in (74), it suffices to notice that $\ast_0b^{1,0}$ is a $(2, 1)$—form and the complex dimension of $D^{\ast,(1,0)}$ (and of $D^{\ast,(0,1)}$) is 2, then $\partial_0\ast_0b^{1,0} = 0$. The proof for the other one is similar. Therefore

\[ d_0^{\ast}b = \partial_0^{\ast}b^{0,1} + \bar{\partial}_0^{\ast}b^{1,0}. \]  

Contracting $d_0b$ with $\frac{d\eta}{2}$ using the decomposition (71), still using the vanishing in (74), we find

\[ d_0b_\perp \frac{d\eta}{2} = d_0^{\ast}J_0b = d_0^{\ast}(\sqrt{-1}b^{0,1} - \sqrt{-1}b^{1,0}) = \sqrt{-1}\partial_0^{\ast}b^{0,1} - \sqrt{-1}\bar{\partial}_0^{\ast}b^{1,0}. \]  

Via the two different identities (75) and (76), the condition $d_0^{\ast}b = d_0b_\perp \frac{d\eta}{2} = 0$ is equivalent to that $\sqrt{-1}\partial_0^{\ast}b^{0,1} = 0 = \sqrt{-1}\bar{\partial}_0^{\ast}b^{1,0}$. The sign difference between (75) and (76) (caused by the complex structure $J_0$) is crucial. The proof for (70) is complete.

3.2.3 $d_0$—parallel of the trivialization of $\pi_5^*O(-1) \to \mathbb{S}^5$, and the proof of Proposition 3.5

The purpose of this section is to show that the map sending $a_0$ to $\omega^{0,1}$ (see Claim 3.7) is the desired isomorphism in Proposition 3.5 (identifying $V_l$ to the space of $\bar{\partial}$—harmonic $(0, 1)$ $End_0E$—valued forms on $\mathbb{P}^2$).

We need the following Lemma saying that the section $s_{-l}$ of $\pi_5^*O(l) \to \mathbb{S}^5$ is $d_0$—closed (parallel). It is not parallel under the (full) connection on $\pi_5^*O(l)$ unless $l = 0$.

Lemma 3.10. Under the pullback standard connection on $\pi_5^*O(l) \to \mathbb{S}^5$, the standard trivialization $s_l$ (see (56) and Definition 3.1) is $d_0$—closed i.e.

\[ \partial_0s_l = \bar{\partial}_0s_l = 0, \quad and \quad d_0s_l = 0. \]
Proof of Lemma 3.10: We only prove it when \( l = -1 \). When \( l = 1 \), it follows by dualizing. For arbitrary integer \( l \), it follows by Leibniz-rule with respect to tensor product.

It suffices to prove \( \partial_0 s_{-1} = \bar{\partial}_0 s_{-1} = 0 \) in \( U_{0,5}^2 \). The vanishing on the whole \( S^5 \) follows by continuity.

First, we routinely verify the following.

**Formula** 3.11. In \( U_{0,5}^2 \), \( \partial_0 Z_0 = -Z_0 \partial_{\bar{\mathcal{L}}} \log \phi_0 \), and \( \bar{\partial}_0 Z_0 = 0 \).

For the reader’s convenience, we still give the proof of Formula 3.11. It is routine to verify the following two identities by Formula (18) for \( Z_0 \) (which particularly says \( L_\xi Z_0 = \sqrt{-1} Z_0 \)), Fact 2.2 on the Reeb vector field, and Formula (32) for \( \partial_0, \bar{\partial}_0 \) etc.

\[
\begin{align*}
\partial_0 Z_0 &= \partial_{\bar{\mathcal{L}}} Z_0 + \frac{\sqrt{-1}}{2} (\partial_{\bar{\mathcal{L}}} \log \phi_0) \wedge (L_\xi Z_0) = -\frac{Z_0}{2} \partial_{\bar{\mathcal{L}}} \log \phi_0 - \frac{Z_0}{2} \partial_{\bar{\mathcal{L}}} \log \phi_0 \\
&= -Z_0 \partial_{\bar{\mathcal{L}}} \log \phi_0, \\
\bar{\partial}_0 Z_0 &= \bar{\partial}_{\mathcal{L}} Z_0 - \frac{\sqrt{-1}}{2} (\bar{\partial}_{\mathcal{L}} \log \phi_0) \wedge (L_\xi Z_0) = -\frac{Z_0}{2} \bar{\partial}_{\mathcal{L}} \log \phi_0 + \frac{Z_0}{2} \bar{\partial}_{\mathcal{L}} \log \phi_0 \\
&= 0.
\end{align*}
\]

We continue proving Lemma 3.10. In \( U_{0,5}^2 \), the trivialization \((1, u_1, u_2)\) of \( \pi_{*,4}^* O(-1) \) descends to \( U_{0,5^2} \). Then, under the Chern-connection of the standard metric, we find

\[
\bar{\partial}_0 (1, u_1, u_2) = \bar{\partial}_{\bar{\mathcal{L}}} (1, u_1, u_2) = 0,
\]

and

\[
\partial_0 (1, u_1, u_2) = \partial_{\bar{\mathcal{L}}} (1, u_1, u_2) = (\partial_{\bar{\mathcal{L}}} \log \phi_0) (1, u_1, u_2).
\]

Therefore,

\[
\bar{\partial}_0 (Z_0, Z_1, Z_2) = \bar{\partial}_0 [Z_0 (1, u_1, u_2)] = (\bar{\partial}_0 Z_0) (1, u_1, u_2) + Z_0[\bar{\partial}_{\bar{\mathcal{L}}} (1, u_1, u_2)] = 0.
\]

\[
\begin{align*}
\partial_0 (Z_0, Z_1, Z_2) &= \partial_0 [Z_0 (1, u_1, u_2)] = (\partial_0 Z_0) (1, u_1, u_2) + Z_0[\partial_{\bar{\mathcal{L}}} (1, u_1, u_2)] \\
&= -Z_0 \partial_{\bar{\mathcal{L}}} \log \phi_0 + Z_0 \partial_{\bar{\mathcal{L}}} \log \phi_0 \\
&= 0.
\end{align*}
\]

The proof is complete.

To extend the usual conjugate transpose of endomorphisms to twisted endomorphisms, the following convention helps, for example, in the proof of Proposition 3.3 and Proposition 3.30 below.

**Notation Convention** 3.12. For any integer \( k \), let \( \llcorner \cdot \rrcorner \) denote the conjugate linear map from \( \pi_{*,4}^* O(k) \) to \( \pi_{*,4}^* O(-k) \) defined by

\[
\llcorner s \rrcorner_k \triangleq s_{-k}.
\]

Let it applies distributively to a tensor product of the line bundles.

Let \((E, h, A_0)\) be a holomorphic Hermitian triple on \( \mathbb{P}^2 \), it is obvious that we can take conjugate transpose \( \llcorner \cdot \rrcorner \) of any endomorphism. Using the above conjugation, let the transpose only applies to the \( \End E \) part but not the line bundle part, we can also take \( \llcorner \cdot \rrcorner \) of any twisted endomorphism.

This is why we only work with endomorphisms and twisted endomorphisms.

Under the identification of the following 3 objects:
1 as a section of the trivial line bundle,
• \( s_k \otimes s_{-k} \) as a section of \( [\pi^*_{5,4}O(k)] \otimes [\pi^*_{5,4}O(-k)] \),
• and \( s_{-k} \otimes s_k \) as a section of \( [\pi^*_{5,4}O(-k)] \otimes [\pi^*_{5,4}O(k)] \),

the following diagram commutes (where the vertical maps are the conjugation).

\[
\begin{array}{ccc}
  s_k \otimes s_k & \rightarrow & s_k \otimes s_{-k} \\
  \downarrow & \downarrow & \downarrow \\
  s_k \otimes s_{-k} & \rightarrow & s_{-k} \otimes s_k \\
\end{array}
\]

Therefore, the conjugate transpose of a Fourier-Series of local form (left side of (59)) is equal to that of a series of global form (right side of (59)).

The conventions and equations established so far are at our disposal to characterize the subspaces \( V_l \).

**Proof of Proposition 3.5:** We show that the map

\[
\Gamma(b) \triangleq b^{0,1}s_l
\]

is the desired isomorphism \( V_l \rightarrow H^{0,1}[\mathbb{P}^2, (End_0E)(l)] \). Because \( J_0d^{0,1} = \sqrt{-1}d^{0,1} \) for any \((\pi^*_{5,4}End_0E)-valued\) \((0,1)\) semi-basic form \(d^{0,1}\), the \( \Gamma \) above is complex linear.

Let \( a_0 = b \) in Claim 3.7, \( \Gamma(b) \) is precisely the \( \alpha^{0,1} \) in the splitting (67).

Step 1: \( \Gamma(b) \) is a priori a \((EndE)(l)\)-valued \((0,1)\)-form. We show that it is \( \bar{\partial}_b \)-harmonic. By the \( \partial_b \) and \( \bar{\partial}_b \)-closeness of \( s_l \) in Lemma 3.10 we compute

\[
\bar{\partial}_b (b^{0,1}s_l) = (\bar{\partial}_b b^{0,1})s_l + b^{0,1}(\bar{\partial}_b s_l) = 0,
\]

and

\[
\begin{align*}
\partial^*_0 (b^{0,1}s_l) &= -\ast_0 \partial_0 [s_l \ast_0 (b^{0,1})] = -\ast_0 (\partial_0 s_l) \wedge \ast_0 (b^{0,1}) - s_l [\ast_0 \partial_0 \ast_0 (b^{0,1})] \\
&= -s_l [\ast_0 \partial_0 \ast_0 (b^{0,1})] = s_l \partial^*_0 b^{0,1} \\
&= 0.
\end{align*}
\]

Because \( b^{0,1}s_l \) descends to \( \mathbb{P}^2 \), the above vanishing implies that

\[
\bar{\partial}_b (b^{0,1}s_l) = \partial^*_0 (b^{0,1}s_l) = 0 \text{ i.e. } \Gamma(b) \in H^{0,1}[\mathbb{P}^2, O(l) \otimes End_0E].
\]

Step 2: We show that the following map \( \Gamma : H^{0,1}[\mathbb{P}^2, O(l) \otimes End_0E] \rightarrow V_l \) is the (two-sided) inverse of \( \Gamma \).

\[
\Gamma(d^{0,1}) \triangleq d^{0,1} \otimes s_{-l} - [d^{0,1} \otimes s_{-l}],
\]

where \( d^{0,1} \otimes s_{-l} \) is viewed as an \( \pi^*_{5,4}(End_0E) \)-valued semi-basic \( 1 \)-form on \( S^5 \).

We notice that \( \alpha \) is a \( \pi^*_{5,4}adE \)-valued semi-basic \( 1 \)-form if and only if \( \alpha^{1,0} = -\alpha^{0,1} \) (cf. [10] (2.15) VII). Hence \( \Gamma \) is injective. By (85), \( \Gamma \Gamma = Id \) automatically holds true. Then for any \( b \in V_l \), we find \((\Gamma \Gamma)b - b \in Ker \Gamma = \{0\} \). The injectivity says \((\Gamma \Gamma)b - b = 0 \). Then \( \Gamma \Gamma = Id, \) and \( \Gamma \) is the two-sided inverse of \( \Gamma \). The complex linearity of \( \Gamma \) then gives the complex linearity of \( \Gamma \).

Then proof is complete. \( \square \)
3.3 Describing $SpecP$: proof of Theorem I.8

3.3.1 The orthogonal complement of the eigen cohomology space

Again, we stress that $V_l$ is a subspace of the eigenspace $E_lP$.

Based on the above, we first restrict the operator $P$ onto a closed subspace of $L^2(S^5, Dom_{S^5})$ of finite co-dimension.

Definition 3.13. In view of the invariant subspace $V_l$ that is isomorphic to the cohomology (Proposition 3.5), we define the eigen cohomology space $V_{coh} \equiv \oplus_{l \in \mathbb{Z}} V_l$, which is isomorphic to the finite dimensional vector space $\oplus_l H^1[\mathbb{P}^2, (End_0E)(l)]$. We view $V_{coh}$ as a subspace of the Hilbert space $L^2(S^5, Dom_{S^5})$.

Remark 3.14. Apparently, $SpecP|_{V_{coh}} = S_{coh}$. Moreover, any element in $L^2(S^5, Dom_{S^5})$ with vanishing 5th row (bottom entry) lies in $V_{coh}^\perp$.

Because $P$ is self-adjoint, the orthogonal complement $V_{coh}^\perp$, a closed subspace of finite co-dimension, is still $P$–invariant. We let $P|_{V_{coh}^\perp}$ denote the restriction of $P$ to $V_{coh}^\perp$. It is convenient to work with $P|_{V_{coh}^\perp}$.

3.3.2 The crucial intermediate theorem

The theory for the $P$–invariant spaces $V_i$ is at our disposal to show the following result, which is a crucial building block for Theorem I.8.

Theorem 3.15. Given a Hermitian Yang-Mills triple on $\mathbb{P}^2$, $SpecP|_{V_{coh}^\perp} = S_{\nabla^*\nabla}$.

The above theorem means the spectrum of $P$ on $V_{coh}^\perp$ is exactly the part induced by the spectrum of the rough Laplacian. Before proving it, to intuitively understand the feature of the Bochner formulas of $P$ (Lemma 2.21), we introduce the notion of “autonomous”.

Definition 3.16. For any $i = 1, 2, 3, \text{ or } 4$,

- let $Row^i$ denote the injection from $\pi^*_5 \times_4 adE$ (or $\pi^*_7 \times_4 adE$) to the $i$-th row of $Dom_{S^5}$ ($Dom_7$).

- Let $v_i$ be the $i$–th variable (in the $i$–th row) of $Dom_{S^5}$. A linear differential operator $L$ on $Dom_{S^5}$ is said to be autonomous with respect to row $i$, if row $i$ of $L$ is

$$ (\nabla^*\nabla + kId)v_i $$

for some real constant $k$, and the other rows does not depend on $v_i$.

Another ingredient we need is the spectrum counted with multiplicities.

Definition 3.17. Let the operator be $P$, $P|_{V_{coh}^\perp}$, or $\nabla^*\nabla|_{S^5}$ (which means $\nabla^*\nabla|_{\pi^*_5 \times_4 adE \to S^5}$). Let $Spec^{mul}(\cdot)$ denote the set of eigenvalues counted with real multiplicity. This means if $\mu$ is an eigenvalue and the real dimension of the eigenspace is $m_\mu$, $\mu$ appears in $Spec^{mul}(\cdot)$ $m_\mu$ times.

Similarly, when the operator is $\nabla^*\nabla|_{\pi^*_5 \times_4 End_0E \to S^5}$ or $\nabla^*\nabla|_{(End_0E)(l)\to \mathbb{P}^2}$, let $Spec^{mul_c}(\cdot)$ denote the set of eigenvalues counted with complex multiplicity. This means if $\mu$ is an eigenvalue and the complex dimension of the eigenspace is $m_\mu$, $\mu$ appears in $Spec^{mul_c}(\cdot)$ $m_\mu$ times.
Remark 3.18. The complex bundle $End_\mathbb{C}E$ is the complexification of $adE$. Hence, for any $\lambda \in \text{Spec} \nabla^\ast \nabla|_{\mathbb{R}^5}$, $E_\lambda \nabla^\ast \nabla|_{\pi_{5,4}^\ast End_\mathbb{C}E \rightarrow \mathbb{S}^5}$ is the complexification of $E_\lambda \nabla^\ast \nabla|_{{\pi_{5,4}^\ast adE} \rightarrow \mathbb{S}^5}$.

Because the real dimension of a vector space is equal to the complex dimension of its complexification, the following holds.

$$\text{Spec}^{\text{mul}} \nabla^\ast \nabla|_{\pi_{5,4}^\ast adE \rightarrow \mathbb{S}^5} = \text{Spec}^{\text{mul}} \nabla^\ast \nabla|_{\pi_{5,4}^\ast End_\mathbb{C}E \rightarrow \mathbb{S}^5}.$$  

(86)

3.3.3 Proving the crucial intermediate theorem

To prove Theorem 3.15, we need two simple facts on irreducibility of the connection in a Hermitian Yang-Mills triple.

Fact 3.19. (\cite{10} VII, Proposition 4.14) Given a Hermitian Yang-Mills triple $(E, h, A_0)$ on $\mathbb{P}^2$, $E$ is stable if and only if $A_0$ is irreducible on $\mathbb{P}^2$ (i.e. there is no non-zero parallel section of $adE$).

Fact 3.20. In the setting of Fact 3.19, $A_0$ is irreducible on $\mathbb{P}^2$ if and only if $\pi_{5,4}^\ast A_0$ is irreducible on $\mathbb{S}^5$.

Fact 3.20 is a direct corollary of the spectral decomposition in Formula 3.23 below.

Proof of Theorem 3.15. The crucial observation is that by the Bochner formulas in Lemma 2.21, $P^2 + 2P$ is autonomous with respect to row 1, 2, and $P^2 + 4P$ is autonomous with respect to row 3, 4.

Step 1: $\text{Spec}P|_{V_{\text{coh}}^\perp} \subseteq S_{\nabla^\ast \nabla}$

Definition 3.21. Let $\{\phi_\mu, \mu \in \text{Spec}^{\text{mul}}(P|_{V_{\text{coh}}^\perp})\}$ be an eigenbasis with respect to $P|_{V_{\text{coh}}^\perp}$. The eigen expansion of any $L_2$-section of $\text{Dom}_{\nabla^\ast \nabla}$ is called the $P|_{V_{\text{coh}}^\perp}$-eigen Fourier expansion. Similar definition of eigen Fourier expansion applies to the operator $P$ itself (no restriction) and the other rough Laplacians. This is not the same object as the Sasaki-Fourier series in Definition 3.2 above.

Let $\lambda \in \text{Spec} \nabla^\ast \nabla|_{\mathbb{R}^5}$, for any non-zero $u_\lambda \in E_\lambda \nabla^\ast \nabla|_{\mathbb{R}^5}$, in view of Remark 3.14 we consider the $P|_{V_{\text{coh}}^\perp}$-eigen Fourier expansion:

$$\begin{bmatrix} u_\lambda \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sum_{\mu \in \text{Spec}^{\text{mul}}(P|_{V_{\text{coh}}^\perp})} u_{\lambda, \mu} \phi_\mu.$$

Using the Bochner formula \cite{50}, we calculate that

$$\sum_{\mu \in \text{Spec}^{\text{mul}}(P|_{V_{\text{coh}}^\perp})} (\mu^2 + 2\mu) u_{\lambda, \mu} \phi_\mu = [P^2 + 2P] \begin{bmatrix} u_\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (\nabla^\ast \nabla u_\lambda) + 3u_\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$  

(87)

$$= (\lambda + 3) \begin{bmatrix} u_\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} = \sum_{\mu \in \text{Spec}^{\text{mul}}(P|_{V_{\text{coh}}^\perp})} (\lambda + 3) u_{\lambda, \mu} \phi_\mu.$$
Comparing the non-zero coefficients $u_{\lambda,\mu}$ (which must exist because $u_{\lambda} \neq 0$), we find

$$\mu^2 + 2\mu = \lambda + 3.$$  

This means that for any $\lambda \in \text{Spec}(\nabla^*\nabla|_{S^5})$,

$$\text{Row}^1[\mathbb{E}_\lambda(\nabla^*\nabla|_{S^5})] \subseteq \bigoplus_{\mu, \mu - 2\lambda - 3 \in \text{Spec}(\nabla^*\nabla|_{S^5})} L^2 \mu P|_{V^\perp_{\text{coh}}}.$$  

Because $u_{\lambda}$ is an arbitrary non-zero eigenvector, and that the eigenbasis with respect to $\nabla^*\nabla$ is complete in $L^2[S^5, \pi_{S^5,4}^* \text{ad} E]$, we find

$$\{ \begin{bmatrix} u \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid |u| \in L^2[S^5, \pi_{S^5,4}^* \text{ad} E]\} \subseteq \bigoplus_{\mu, \mu + 2 \lambda - 3 \in \text{Spec}(\nabla^*\nabla|_{S^5})} L^2 \mu P|_{V^\perp_{\text{coh}}}.$$  

Similarly, using (50) again, we verify that

$$\{ \begin{bmatrix} 0 \\ a_s \\ 0 \\ 0 \\ 0 \end{bmatrix} \mid |a_s| \in L^2[S^5, \pi_{S^5,4}^* \text{ad} E]\} \subseteq \bigoplus_{\mu, \mu + 4 \lambda - 3 \in \text{Spec}(\nabla^*\nabla|_{S^5})} L^2 \mu P|_{V^\perp_{\text{coh}}}.$$  

Using (51) instead of (50), we verify

$$\{ \begin{bmatrix} 0 \\ 0 \\ a_r \\ 0 \\ 0 \end{bmatrix} \mid |a_r| \in L^2[S^5, \pi_{S^5,4}^* \text{ad} E]\} \subseteq \bigoplus_{\mu, \mu + 4 \lambda - 3 \in \text{Spec}(\nabla^*\nabla|_{S^5})} L^2 \mu P|_{V^\perp_{\text{coh}}}.$$  

and

$$\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_\eta \\ 0 \end{bmatrix} \mid |a_\eta| \in L^2[S^5, \pi_{S^5,4}^* \text{ad} E]\} \subseteq \bigoplus_{\mu, \mu + 4 \lambda - 3 \in \text{Spec}(\nabla^*\nabla|_{S^5})} L^2 \mu P|_{V^\perp_{\text{coh}}}.$$  

In summary, by Definition 1.7 of $S_{\nabla^*\nabla}$, we find

$$\{ \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ 0 \end{bmatrix} \mid |u, a_s, a_r, a_\eta| \in L^2[S^5, \pi_{S^5,4}^* \text{ad} E]\} \subseteq \bigoplus_{\mu \in \text{Spec}(S_{\nabla^*\nabla})} L^2 \mu P|_{V^\perp_{\text{coh}}}.$$
Then
\[ \bigoplus_{\mu \notin S_{\nabla^*}} L^2 E_{\mu} P|_{V_{coh}^\perp} \subseteq \{ u, a_s, a_r, a_\eta \in L^2[S^5, \pi_{5,4}^* a dE]\} \]

\[ \subseteq \{ |a_0 \in L^2[S^5, D^* \otimes \pi_{5,4}^* a dE]| \} \]  

(92)

The above implies that if \( \mu \) is an eigenvalue of \( P|_{V_{coh}^\perp} \) and \( \mu \notin S_{\nabla^*} \), any eigensection of \( \mu \) must be of the form
\[ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_0 \end{bmatrix} \] i.e. row 1 to row 4 must all vanish. Again, by Formula (40), if \( a_0 \) does not vanish, \( a_0 \in V_\mu \) for some integer \( \mu \). But \( a_0 \perp V_\mu \) by assumption. Then \( a_0 = 0 \).

The statement \( \text{Spec} P|_{V_{coh}^\perp} \subseteq S_{\nabla^*} \) is proved.

Step 2: \( \text{Spec} P|_{V_{coh}^\perp} \supseteq S_{\nabla^*} \).

Actually, Step 1 has already hinted on this direction. Equation (87) can be written as
\[ \begin{bmatrix} u_\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} = \Sigma_{\mu \in \text{Spec mul}(P|_{V_{coh}^\perp})} \mu = -1 + \sqrt{4 + \lambda} u_\lambda \mu \phi_\mu + \Sigma_{\mu \in \text{Spec mul}(P|_{V_{coh}^\perp})} \mu = -1 - \sqrt{4 + \lambda} u_\lambda \mu \phi_\mu. \]  

(93)

Because \( u_\lambda \) is non-zero, the above means at least one of \(-1 - \sqrt{4 + \lambda}\) and \(-1 + \sqrt{4 + \lambda}\) is an eigenvalue of \( P|_{V_{coh}^\perp} \). We show by contradiction that both of them must be eigenvalues. If not, \[ \begin{bmatrix} u_\lambda \\ 0 \\ 0 \\ 0 \end{bmatrix} \] is an eigensection of \( P \), thus \( d_0 u_\lambda = L_\xi u_\lambda = 0 \) by Formula (40). By Fact 3.20, irreducibility of the connection on \( \mathbb{P}^2 \) implies the irreducibility of the pullback connection on \( S^5 \), which yields \( u_\lambda = 0 \). This contradicts the hypothesis that \( u_\lambda \neq 0 \).

Likewise, regarding the other polynomial \( \mu^2 + 4\mu \), corresponding to the 3rd and 4th row of \( \text{Dom}_{S^5} \), yields the two eigenvalues \(-2 - \sqrt{4 + \lambda}\) and \(-2 + \sqrt{4 + \lambda}\).

We proved \( \text{Spec} P|_{V_{coh}^\perp} \supseteq S_{\nabla^*} \). The proof of Theorem 3.15 is complete.

3.3.4 Proving Theorem 1.8

To complete the proof, we need the formula for the sheaf cohomologies.
Lemma 3.22. Let $E$ be a holomorphic vector bundle on $\mathbb{P}^2$. For any integer $k$, we have
\begin{align}
h^1[\mathbb{P}^2, (\text{End}E)(k)] &= h^0[\mathbb{P}^2, (\text{End}E)(k)] + h^0[\mathbb{P}^2, (\text{End}E)(-k-3)] \\
&+ 2\rho c_2(E) - (r-1)c^l_2(E) - \frac{r^2(k+1)(k+2)}{2}.
\end{align}
Therefore, suppose additionally that $E$ is stable and rank $E \geq 2$, the following is true.
\begin{align}
h^1[\mathbb{P}^2, (\text{End}E)(-1)] &= h^1[\mathbb{P}^2, (\text{End}E)(-2)] = c_2(\text{End}E) > 0.
\end{align}

The Chern number inequality (95) is crucial in showing $-1$ and $-2$ are eigenvalues of $P$.
The proof of the above Lemma is simply by Riemann-Roch formula, Serre-duality, and a vanishing theorem proved by Kobayashi. We defer it to Appendix G.

**Proof of Theorem 1.8**: The splitting $\text{Spec}P = S_{\nabla^* \nabla} \cup S_{\text{coh}}$ follows simply from Remark 3.14 and Theorem 3.15.
Fact 3.19 says that the bundle $E$ on $\mathbb{P}^2$ must be stable under the irreducible Hermitian Yang-Mills conduction. The calculation of cohomology in Lemma 3.22 (95), the obvious fact that $V_\mu \subset E_\mu P$, and the identification in Proposition 3.5 says that $-1$ and $-2$ must be eigenvalues. Because of irreducibility of the connection on $\mathbb{P}^2$ and therefore on $\mathbb{S}^5$, the lowest eigenvalue in $\text{Spec}(\nabla^* \nabla|_{\mathbb{S}^5})$ is positive. Then the eigenvalues of $P$ other than $-1$ and $-2$ are either nonnegative or no larger than $-3$.
The proof is complete. □

3.4 The dimension reduction for the spectrum of bundle rough Laplacians

To study the multiplicity of each eigenvalue of $P$, we need the following reduction.

**Formula 3.23.** Given a holomorphic Hermitian triple $(E, h, A_O)$ on $\mathbb{P}^2$, under the associated data setting and counted with multiplicity, the following holds.
\[ \text{Spec}^{\text{mul}} \nabla^* \nabla|_{\pi^*_4 \text{ad} E \to \mathbb{S}^5} = \text{Spec}^{\text{mul}} \nabla^* \nabla|_{\pi^*_4 \text{End}_d E \to \mathbb{S}^5} \]
\[ = \{ \alpha_i + \ell^2 \mid \alpha_i \in \text{Spec}^{\text{mul}} \nabla^* \nabla|_{(\text{End}_d E)(\ell)} \}. \]

Consequently, without counting multiplicity,
\[ \text{Spec} \nabla^* \nabla|_{\mathbb{S}^5} \triangleq \text{Spec} \nabla^* \nabla|_{\pi^*_4 \text{ad} E \to \mathbb{S}^5} = \{ \alpha_i + \ell^2 \mid \alpha_i \in \text{Spec} \nabla^* \nabla|_{(\text{End}_d E)(\ell)} \}. \]

**Proof of Formula 3.23.** In view of identity (86), it suffices to show
\[ \text{Spec}^{\text{mul}} \nabla^* \nabla|_{\pi^*_4 \text{End}_d E \to \mathbb{S}^5} = \{ \alpha_i + \ell^2 \mid \alpha_i \in \text{Spec}^{\text{mul}} \nabla^* \nabla|_{(\text{End}_d E)(\ell)} \}. \]

Given an arbitrary $p \in \mathbb{S}^5$, let $v_i, i = 1 \ldots 4$ be the transverse geodesic coordinate vector fields near $p$ in Lemma 3.11 below. Because $\nabla v_i v_i = 0$ at $p$, the following formula for the rough Laplacian is true
\[ -(\nabla^* \nabla u)(p) = [\nabla v_i (\nabla v_i u)](p) + (L^2 u)(p). \]

Suppose $u \in C^{10}|_{\mathbb{S}^5, \pi^*_4 \text{End}_d E}$, the Fourier-expansion $u = \Sigma_{i \in \mathbb{Z}} u_i \otimes s_{-i}$ converges pointwisely on $\mathbb{S}^5$. Moreover, we can calculate the Laplacian term by term i.e. using that $d_0 s_{-1} = 0$ and that $u_t$ descends to $\mathbb{P}^2$, we find
\[ \nabla v_i (\nabla v_i u) = \Sigma_{i \ell} \nabla v_i \nabla v_i (u_t \otimes s_{-1}) = \Sigma_{i \ell} (\nabla v_i \nabla v_i u_t) \otimes s_{-1} \]
\[ = -\Sigma_{i \ell} i_{\nabla^* \mathbb{P}^2} (\nabla^2 u_t) \otimes s_{-1}. \]
Using $L_\xi u_l = 0$ again, and $L_\xi^2 s_{-l} = -l^2 s_{-l}$, we find
\[
\nabla^* \nabla u = \Sigma_l[(\nabla^* P^2 \nabla^2 + l^2)u_l] \otimes s_{-l}.
\] (99)

Then the desired result (96) follows by plugging an arbitrary eigensection $u$ into (99) and comparing the Sasaki-Fourier series of both hand sides. For the reader’s convenience, we still provide the full detail.

Let $\lambda \in \text{Spec} \nabla^* \nabla|_{\pi^*_5 A, \text{End}_0 E \to \mathbb{S}}$, and $u_\lambda$ is a (non-zero) eigenvector. Then (99) yields that
\[
\lambda u_\lambda = \Sigma_l u_{\lambda,l} s_l = \Sigma_l[(\nabla^* P^2 \nabla^2 + l^2)u_{\lambda,l}] \otimes s_{-l}.
\] (100)

Then either $u_{\lambda,l} = 0$ or $\nabla^* P^2 \nabla^2 u_{\lambda,l} = (\lambda - l^2)u_{\lambda,l}$. Therefore, we obtain a linear injection
\[
i : \mathbb{E}_\lambda(\nabla^* \nabla|_{\pi^*_5 A, \text{End}_0 E \to \mathbb{S}}) \to \bigoplus_{l, \lambda - l^2 \in \text{Spec} \nabla^* \nabla|_{(\text{End}_0 E)(l)}}(\mathbb{E}_{\lambda - l^2} \nabla^* \nabla|_{(\text{End}_0 E)(l)})
\]
sending $u_\lambda$ to its Sasaki-Fourier co-efficients. We note that only finitely many $l$ yield non-zero eigenspace $\mathbb{E}_{\lambda - l^2} \nabla^* \nabla|_{(\text{End}_0 E)(l)}$.

The map $i$ is obviously surjective because for any set of Sasaki-Fourier co-efficients
\[
\Sigma_l u_{\lambda,l} s_l \in \mathbb{E}_\lambda(\nabla^* \nabla|_{\pi^*_5 A, \text{End}_0 E \to \mathbb{S}}), \text{ and } i(\Sigma_l u_{\lambda,l} s_l) = \Sigma_l u_{\lambda,l}.
\]

The proof of (96) is complete. \hfill \Box

In conjunction with the splitting (98) by the frame, we call the operator
\[
\Delta_0 \triangleq \nabla^* \nabla + L_\xi^2
\]
the transverse rough Laplacian.

### 3.5 On the multiplicities of the eigenvalues of $P$: proof of Theorem 1.8

The first thing we can say on the multiplicities is the following.

**Lemma 3.24.** (Kodaira-Serre duality for eigenspaces) In conjunction with Remark 2.16, for any eigenvalue $\mu$ of $P$, both $K, T$ are real isomorphisms : $\text{Eigen}_\mu(P) \to \text{Eigen}_{-(\mu + 3)}(P)$ that anti-commute with the complex structure $I$ of the eigenspaces of $P$.

**Proof of Lemma 3.24:** This is a direct corollary of the commutators between $P$ and the isomorphisms $I, T, K$ in (11). \hfill \Box

That $K, T$ both anti-commute with $I$ is consistent with that the Serre-duality map is conjugate $\mathbb{C}$-linear.

We can go much further than Lemma 3.24: the multiplicity of each eigenvalue of $P$ can be determined.
3.5.1 The definition and formula for the projection map $\|\mathbf{e}_\mu P|_{V^\perp_{\text{coh}}}\|

The purpose of this section is the formula for a certain projection map (Lemma 3.27). This map is decisive in determining the multiplicities. Before defining it, we stress the following.

**Notation Convention 3.25.** In conjunction with that each number in $\text{Spec}^{\text{mul}}(\nabla^*\nabla|_{\mathbb{S}^5})$ generates 4 eigenvalues of $P$ (see Theorem L84), for any $\lambda \in \text{Spec}(\nabla^*\nabla|_{\mathbb{S}^5})$, let

$$
\mu_{\lambda^+} \triangleq -1 + \sqrt{4 + \lambda}, \quad \mu_{\lambda^-} \triangleq 1 - \sqrt{4 + \lambda}, \quad \mu_{\lambda^+} = -2 + \sqrt{4 + \lambda}, \quad \mu_{\lambda^-} = -2 - \sqrt{4 + \lambda}. \quad (101)
$$

For any $\mu \in S\nabla^*\nabla \subseteq \text{Spec} P$, at least one of the following holds.

- $\mu^2 + 2\mu - 3 \in \text{Spec}(\nabla^*\nabla|_{\mathbb{S}^5})$.
- $\mu^2 + 4\mu \in \text{Spec}(\nabla^*\nabla|_{\mathbb{S}^5})$.

We are ready to define the projection.

**Definition 3.26.** Let $(E, h, A_O)$ be an irreducible Hermitian Yang-Mills triple on $\mathbb{P}^2$. Suppose $\mu \in \text{Spec} P|_{V^\perp_{\text{coh}}}$. Let $\|\mathbf{e}_\mu P|_{V^\perp_{\text{coh}}}\|$ denote the orthogonal projection from

$$(\mathbb{E}_1 \nabla^*\nabla|_{\mathbb{S}^5})^\otimes 2 \oplus (\mathbb{E}_2 \nabla^*\nabla|_{\mathbb{S}^5})^\otimes 2$$

to $\mathbb{E}_\mu P|_{V^\perp_{\text{coh}}}$, that is factored as follows.

$$(\mathbb{E}_1 \nabla^*\nabla|_{\mathbb{S}^5})^\otimes 2 \oplus (\mathbb{E}_2 \nabla^*\nabla|_{\mathbb{S}^5})^\otimes 2 \rightarrow V^\perp_{\text{coh}} \rightarrow L^2(\mathbb{S}^5, \text{Dom}_{\mathbb{S}^5}) \rightarrow \mathbb{E}_\mu P|_{V^\perp_{\text{coh}}}.$$  

In conjunction with Remark 3.14, the range of the first arrow above consists of sections of the form

$$
\begin{bmatrix}
v \\
h \\
g \\
w \\
0
\end{bmatrix}
$$

which are automatically in $V^\perp_{\text{coh}}$, where

$$v, \ h \in \mathbb{E}_{\lambda_1} \nabla^*\nabla|_{\mathbb{S}^5}, \ \text{and} \ g, \ w \in \mathbb{E}_{\lambda_2} \nabla^*\nabla|_{\mathbb{S}^5}.$$  

We start from the first component of the domain bundle $\text{Dom}_{\mathbb{S}^5}$.

Suppose $v \in \mathbb{E}_{\lambda_1} \nabla^*\nabla|_{\mathbb{S}^5}$, still by the $P|_{V^\perp_{\text{coh}}}$-eigen Fourier expansion, there is an (unique) orthogonal splitting

$$
\begin{bmatrix}
u \\
0 \\
0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
u \\
a_s \\
ar \\
a_{a_0} \\
a_0
\end{bmatrix} + \begin{bmatrix}
\tilde{u} \\
a_s \\
-a_r \\
-a_{a_0} \\
-a_0
\end{bmatrix}, \quad (102)
$$

such that

$$
\begin{bmatrix}
u \\
a_s \\
ar \\
a_{a_0}
\end{bmatrix} \in \mathbb{E}_{\mu_{\lambda_1^+}} P|_{V^\perp_{\text{coh}}}, \ \text{and} \ \begin{bmatrix}
\tilde{u} \\
a_s \\
-a_r \\
-a_{a_0}
\end{bmatrix} \in \mathbb{E}_{\mu_{\lambda_1^-}} P|_{V^\perp_{\text{coh}}}.
$$

By the fine formula (10) for $P$, the eigensection conditions give the following system.

$$
\begin{align*}
u - L\xi a_s - (d_0 a_0)J H &= \mu_{\lambda_1^+} u, \\
L\xi u + a_s + (d_0 a_0)J G &= \mu_{\lambda_1^+} a_s, \\
-4a_r - L\xi a_{a_0} + d_0 a_0 &= \mu_{\lambda_1^+} a_r, \\
L\xi a_r - 4a_{a_0} - (d_0 a_0)J H &= \mu_{\lambda_1^-} a_r, \\
J_H d_0 u - J_G d_0 a_s + d_0 a_r + J_H d_0 a_{a_0} - L\xi J_0 (a_0) &= \mu_{\lambda_1^+} a_0, \\
J_H d_0 a_r + J_G d_0 a_{a_0} - J_H d_0 a_{a_0} + L\xi J_0 (a_0) &= \mu_{\lambda_1^-} a_0.
\end{align*}
$$  

(103)
The column on the left of the comma corresponds to that
\[
\begin{bmatrix}
u \\
an \\
av \\
a_0
\end{bmatrix} \in \mathbb{E}_{\mu_{\lambda_1}^+} P_{V_{coh}^\perp},
\]
the column on the right of the comma corresponds to that
\[
\begin{bmatrix}
\tilde{u} \\
-a_n \\
-a_r \\
-a_0
\end{bmatrix} \in \mathbb{E}_{\mu_{\lambda_1}^-} P_{V_{coh}^\perp}.
\]

In view of formula (101) for the 4 generated eigenvalues, we have
\[
\mu_{\lambda_1^+} - \mu_{\lambda_1^-} = 2\sqrt{4 + \lambda_1} \neq 0.
\]
Then, summing up the two equations in row 3 of (103), we find \(a_r = 0\). Similar operation for row 4 yields that \(a_n = 0\).

Then the last row of (103) simply becomes the following two equations.
\[
J_H d_0 u - J_G d_0 a_n = \mu_{\lambda_1^+} a_0, \quad J_H d_0 \tilde{u} + J_G d_0 a_n + L_G J_0 (a_0) = -\mu_{\lambda_1^-} a_0. \quad (104)
\]
Summing them up and using \(v = u + \tilde{u}\) (which is evident from (102)), we find
\[
J_H d_0 v = 2\sqrt{4 + \lambda_1} a_0 \quad \text{i.e.} \quad a_0 = \frac{J_H d_0 v}{2\sqrt{4 + \lambda_1}}.
\]
On the other hand, summing up the two equations in the first row of (103), we find
\[
v = \mu_{\lambda_1^+} u + \mu_{\lambda_1^-} \tilde{u}.
\]
Using \(v = u + \tilde{u}\) again, we conclude that
\[
u = \frac{1 - \mu_{\lambda_1^-}}{\mu_{\lambda_1^+} - \mu_{\lambda_1^-}} v = \left( \frac{1}{\sqrt{4 + \lambda_1}} + \frac{1}{2} \right) v, \quad \tilde{u} = \frac{\mu_{\lambda_1^+} - 1}{\mu_{\lambda_1^+} - \mu_{\lambda_1^-}} v = \left( -\frac{1}{\sqrt{4 + \lambda_1}} + \frac{1}{2} \right) v.
\]
Next, summing up the two equations in the second row of (103), we obtain
\[
a_s = \frac{L_G v}{2\sqrt{4 + \lambda_1}}.
\]
Then
\[
\begin{bmatrix}
u \\
0 \\
0 \\
0
\end{bmatrix} \|_{\mu_{\lambda_1^+} P_{V_{coh}^\perp}} \begin{bmatrix}
(\frac{1}{\sqrt{4 + \lambda_1}} + \frac{1}{2}) v \\
\frac{L_G v}{2\sqrt{4 + \lambda_1}} \\
0 \\
\frac{J_H d_0 v}{2\sqrt{4 + \lambda_1}}
\end{bmatrix}, \quad \begin{bmatrix}v \\
0 \\
0 \\
0\end{bmatrix} \|_{\mu_{\lambda_1^-} P_{V_{coh}^\perp}} \begin{bmatrix}
(\frac{-1}{\sqrt{4 + \lambda_1}} + \frac{1}{2}) v \\
\frac{-L_G v}{2\sqrt{4 + \lambda_1}} \\
0 \\
\frac{-J_H d_0 v}{2\sqrt{4 + \lambda_1}}
\end{bmatrix}.
\quad (105)
\]
The same method as above yields the following projection formulas for the other rows of \(\text{Dom} \mathbb{G}^5\).

- Suppose \(\lambda_1 \in \text{Spec}(\nabla^* \nabla |_{\mathbb{G}^5})\), for any \(h \in \mathbb{E}_{\lambda_1} \nabla^* \nabla |_{\mathbb{G}^5}\) such that \(h \neq 0\),
  \[
  \begin{bmatrix}
0 \\
h \\
0 \\
0
\end{bmatrix} \|_{\mu_{\lambda_1^+} P_{V_{coh}^\perp}} \begin{bmatrix}
-\frac{L_G h}{2\sqrt{4 + \lambda_1}} \\
(\frac{1}{\sqrt{4 + \lambda_1}} + \frac{1}{2}) h \\
0 \\
-\frac{L_G d_0 h}{2\sqrt{4 + \lambda_1}}
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
h \\
0 \\
0
\end{bmatrix} \|_{\mu_{\lambda_1^-} P_{V_{coh}^\perp}} \begin{bmatrix}
\frac{L_G h}{2\sqrt{4 + \lambda_1}} \\
(\frac{-1}{\sqrt{4 + \lambda_1}} + \frac{1}{2}) h \\
0 \\
\frac{-L_G d_0 h}{2\sqrt{4 + \lambda_1}}
\end{bmatrix}.
\quad (106)
\]

34
• Suppose $\lambda_2 \in \text{Spec}(\nabla^* \nabla|_{S^5})$, for any $g \in \mathbb{E}_{\lambda_2} \nabla^* \nabla|_{S^5}$ such that $g \neq 0$,

$$
\begin{bmatrix}
0 \\
0 \\
g \\
0
\end{bmatrix}
\parallel_{\lambda_2, +} \! \! \nabla V_{\text{coh}}
\begin{bmatrix}
0 \\
0 \\
\left(-\frac{1}{\sqrt{2}+\lambda_2} + \frac{1}{2}\right) g \\
2\sqrt{2}+\lambda_2
d_0
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
g \\
0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\parallel_{\lambda_2, -} \! \! \nabla V_{\text{coh}}
\begin{bmatrix}
0 \\
0 \\
\left(-\frac{1}{\sqrt{2}+\lambda_2} + \frac{1}{2}\right) g \\
2\sqrt{2}+\lambda_2
d_0
\end{bmatrix}.
$$

(107)

• Suppose $\lambda_2 \in \text{Spec}(\nabla^* \nabla|_{S^5})$, for any $w \in \mathbb{E}_{\lambda_2} \nabla^* \nabla|_{S^5}$ such that $w \neq 0$,

$$
\begin{bmatrix}
0 \\
0 \\
w \\
0
\end{bmatrix}
\parallel_{\lambda_2, +} \! \! \nabla V_{\text{coh}}
\begin{bmatrix}
0 \\
0 \\
\left(-\frac{1}{\sqrt{2}+\lambda_2} + \frac{1}{2}\right) w \\
2\sqrt{2}+\lambda_2
J_{d_0, w}
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
w \\
0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\parallel_{\lambda_2, -} \! \! \nabla V_{\text{coh}}
\begin{bmatrix}
0 \\
0 \\
\left(-\frac{1}{\sqrt{2}+\lambda_2} + \frac{1}{2}\right) w \\
2\sqrt{2}+\lambda_2
J_{d_0, w}
\end{bmatrix}.
$$

(108)

Summing up the formulas (105)–(108), we arrive on target.

**Lemma 3.27.** In the setting of Definition 3.26, for any $\mu \in \mathbb{S}_{\nabla^* \nabla} \subset \text{SpecP}$, let

$$
\lambda_1 \doteq \mu^2 + 2\mu - 3 \quad \text{and} \quad \lambda_2 \doteq \mu^2 + 4\mu.
$$

Suppose $v$, $h \in \mathbb{E}_{\lambda_1} \nabla^* \nabla|_{S^5}$ and $g$, $w \in \mathbb{E}_{\lambda_2} \nabla^* \nabla|_{S^5}$, the following projection formula is true.

$$
\begin{bmatrix}
v \\
h \\
g \\
w
\end{bmatrix}
\parallel_{\mu} \! \! V_{\text{coh}}
\begin{bmatrix}
-L_c h \\
L_c v \\
-L_c g \\
L_c w
\end{bmatrix}
\begin{bmatrix}
\left(-\frac{1}{\mu+1} + \frac{1}{2}\right) v \\
\left(-\frac{1}{\mu+1} + \frac{1}{2}\right) h \\
\left(-\frac{1}{\mu+2} + \frac{1}{2}\right) g \\
\left(-\frac{1}{\mu+2} + \frac{1}{2}\right) w
\end{bmatrix}.
$$

(109)

If $\lambda_1 \notin \text{Spec} \nabla^* \nabla|_{S^5}$, then $v$, $h$ must be 0. This is precisely an advantage in defining eigenspaces for all real numbers. The rationale is that if it is not an eigenvalue, then all “eigensections” are 0. The same applies to $\lambda_2$.

### 3.5.2 Surjectivity of the projection map $\parallel_{\mu} \! \! V_{\text{coh}}$

The purpose of this section is to prove the following.

**Proposition 3.28.** In the setting of Definition 3.26, the orthogonal projection $\parallel_{\mu} \! \! V_{\text{coh}}$ is surjective.

**Remark 3.29.** The surjectivity particularly means that any eigensection of $P$ must be of the form (109).

**Proof of Proposition 3.28.** The condition

$$
\begin{bmatrix}
u \\
a_s \\
a_r \\
a_\eta \\
a_0
\end{bmatrix}
\parallel_{\mu} \! \! V_{\text{coh}}
\begin{bmatrix}
v \\
h \\
g \\
w \\
0
\end{bmatrix}.
$$

(110)
Therefore, the condition \( \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} \in Coker \ ||E_\mu P||_{V_{coh}^\perp} \) is equivalent to that \((111)\) holds for all

\[
\begin{bmatrix}
  v \\
  h \\
  g \\
  w \\
  0
\end{bmatrix} \in (E_{A_1} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_2} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_3} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_4} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_5} \nabla \cdot \nabla |_{G^2})
\]

and

\[
\begin{bmatrix}
  u \\
  a_s \\
  a_r \\
  a_\eta \\
  a_0
\end{bmatrix} \in E_\mu P|_{V_{coh}^\perp}.
\]

The eigensection condition (the left of comma in system \((103)\)) again says that

\[
-L_\xi a_s - (d_0 a_0) \cdot H = (\mu - 1)u,
\quad L_\xi u + (d_0 a_0) \cdot G = (\mu - 1) a_s,
\]

\[
-L_\xi a_\eta + d_0^0 a_0 = (\mu + 4) a_r,
\quad L_\xi a_r - (d_0 a_0) \frac{d_\eta}{2} = (\mu + 4) a_\eta.
\]

Plugging \((112)\) into \((111)\), we find

\[
\begin{align*}
&< h, a_s > + < v, u > + < w, a_\eta > + < g, a_r > = 0.
\end{align*}
\]

Because \( \begin{bmatrix} v \\ h \\ g \\ w \\ 0 \end{bmatrix} \in (E_{A_1} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_2} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_3} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_4} \nabla \cdot \nabla |_{G^2}) \oplus (E_{A_5} \nabla \cdot \nabla |_{G^2}) \) is arbitrary, we find

\[
u = a_s = a_r = a_\eta = 0.
\]

That \( \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} \in V_{coh}^\perp \) implies \( a_0 = 0 \) as well, therefore \( Coker \ ||E_\mu P||_{V_{coh}^\perp} = \{0\} \).
3.5.3 Kernel of the projection $\|_E\mu P|_{V^\perp_{\text{coh}}}$

The purpose of this section is to describe the kernel of the projection.

**Proposition 3.30.** In the setting of Definition 3.26, the following holds.

1. When $\mu$ is not an integer, the orthogonal projection $\|_E\mu P|_{V^\perp_{\text{coh}}}$ is injective.

2. When $\mu$ is an integer,

$$Ker \|_E\mu P|_{V^\perp_{\text{coh}}} = H^0[\mathbb{P}^2, (\text{End}_0 E)(\mu)] \oplus H^0[\mathbb{P}^2, (\text{End}_0 E)(-\mu - 3)].$$

(114)

**Proof of Proposition 3.30:** Suppose

$$\begin{pmatrix} v \\ h \\ g \\ w \\ 0 \end{pmatrix} \in (\mathbb{E}_{\lambda_1} \nabla^* \nabla|_{S^2})^{\oplus 2} \oplus (\mathbb{E}_{\lambda_2} \nabla^* \nabla|_{S^2})^{\oplus 2}, \text{ and } \begin{pmatrix} v \\ h \\ g \\ w \\ 0 \end{pmatrix} \|_E\mu P|_{V^\perp_{\text{coh}}} = 0.$$ (115)

By projection formula (109), the vanishing (115) is equivalent to

$$\begin{align*}
-L_\xi h + \left( \frac{1}{\mu + 1} + \frac{1}{2} \right) v &= 0, \\
-L_\xi w + \left( \frac{1}{\mu + 2} + \frac{1}{2} \right) g &= 0, \\
L_\xi v + \left( \frac{1}{\mu + 1} + \frac{1}{2} \right) h &= 0, \\
L_\xi g + \left( \frac{1}{\mu + 2} + \frac{1}{2} \right) w &= 0, \\
\text{and} \quad -L_\xi d_0 h + \frac{J_G d_0 h}{2(\mu + 1)} + \frac{J_H d_0 v}{2(\mu + 1)} + \frac{d_0 g}{2(\mu + 2)} + \frac{J_0 d_0 w}{2(\mu + 2)} &= 0.
\end{align*}$$ (116)

We note again that Theorem 1.8 says none of $\mu, \mu + 1, \mu + 2, \mu + 3$ is 0. Hence row 1 of (116) is equivalent to

$$L_\xi^2 h = -(\mu + 3)^2 h, \quad v = \frac{L_\xi h}{\mu + 3}. \quad (117)$$

Similarly, row 2 of (116) is equivalent to

$$L_\xi^2 w = -\mu^2 w, \quad g = \frac{L_\xi w}{\mu}. \quad (118)$$

**Part I:** Suppose $\mu$ is not an integer.

Because the eigenvalues of $-L_\xi^2$ are squares of integers, but $\mu$ is not an integer, we find by (117) and (118) that

$$\begin{pmatrix} v \\ h \\ g \\ w \\ 0 \end{pmatrix} = 0.$$
Part II: Suppose $\mu$ is an integer.

In this case, (117) and (118) do not force the eigensection to vanish. We show that this is how the space of holomorphic sections come into play.

Similarly to the proof of the two term expansion in Claim 3.7, equation (117) implies that the Sasaki-Fourier series of $h$ only has 2-terms i.e.

$$h = h_{\mu+3}s_{-(\mu+3)} + h_{-(\mu+3)}s_{\mu+3}.$$ \hspace{1cm} (119)

Moreover, because $L_k \xi = -\sqrt{-1}k s_k$, we find

$$v = \frac{L_k h}{\mu+3} = \sqrt{-1}[h_{\mu+3}s_{-(\mu+3)} - h_{-(\mu+3)}s_{\mu+3}].$$ \hspace{1cm} (120)

Consequently, the transverse differential of $h$ is

$$d_0 h = [d_{\bar{P}_2} h_{\mu+3}]s_{-(\mu+3)} + [d_{\bar{P}_2} h_{-(\mu+3)}]s_{\mu+3},$$ \hspace{1cm} (121)

and that of $v$ is

$$d_0 v = \sqrt{-1}\{[d_{\bar{P}_2} h_{\mu+3}]s_{-(\mu+3)} - [d_{\bar{P}_2} h_{-(\mu+3)}]s_{\mu+3}\}. $$ \hspace{1cm} (122)

Hence,

$$J_0 d_0 v = \sqrt{-1}\{[d_{\bar{P}_2} d_{\bar{P}_2} h_{\mu+3}]s_{-(\mu+3)} - [d_{\bar{P}_2} d_{\bar{P}_2} h_{-(\mu+3)}]s_{\mu+3}\}.$$ \hspace{1cm} (123)

Equation (121) and (123) amount to

$$d_0 h + J_0 d_0 v = 2((\bar{d}_{\bar{P}_2} h_{\mu+3})s_{-(\mu+3)} + (\bar{d}_{\bar{P}_2} h_{-(\mu+3)})s_{\mu+3}).$$ \hspace{1cm} (124)

Because $G$ is minus the imaginary part of the form $\Theta$ i.e.

$$G = \frac{\Theta - \bar{\Theta}}{2\sqrt{-1}},$$ \hspace{1cm} (125)

we calculate that

$$-J_G d_0 h + J_H d_0 v = -J_G (d_0 h + J_0 d_0 v)$$

$$= -2\{(\bar{d}_{\bar{P}_2} h_{\mu+3})_{\partial} G s_{-(\mu+3)} + [\bar{d}_{\bar{P}_2} h_{-(\mu+3)}]_{\partial} G s_{\mu+3}\}$$

$$= \sqrt{-1}\{[\bar{d}_{\bar{P}_2} h_{\mu+3}]_{\partial} G s_{-(\mu+3)} - [\bar{d}_{\bar{P}_2} h_{-(\mu+3)}]_{\partial} G s_{\mu+3}\}. $$ \hspace{1cm} (126)

In the above, we used again that the contraction between a (1,0)-form and a (2,0)-form vanishes, and that the contraction between a (0,1)-form and a (0,2)-form vanishes.

Using (118), the following two term expansions hold as well.

$$w = w_\mu s_{-\mu} + w_{-\mu} s_\mu, \quad g = \sqrt{-1}(w_\mu s_{-\mu} - w_{-\mu} s_\mu).$$ \hspace{1cm} (127)

By similar derivation as of (124), we find

$$d_0 g + J_0 d_0 w = 2\sqrt{-1}[(\bar{d}_{\bar{P}_2} w_\mu) s_{-\mu} - (\bar{d}_{\bar{P}_2} w_{-\mu}) s_\mu].$$ \hspace{1cm} (128)

In the light of (126) and (128), the last equation in (116) reads

$$\frac{\sqrt{-1}}{2(\mu+1)}\{[\bar{d}_{\bar{P}_2} h_{\mu+3}]_{\partial} G s_{-(\mu+3)} - [\bar{d}_{\bar{P}_2} h_{-(\mu+3)}]_{\partial} G s_{\mu+3}\}$$

$$+ \frac{\sqrt{-1}}{2(\mu+2)}[(\bar{d}_{\bar{P}_2} w_\mu) s_{-\mu} - (\bar{d}_{\bar{P}_2} w_{-\mu}) s_\mu]$$

$$= 0.$$
The $(1,0)$ and $(0,1)$ part of (129) should both vanish. This yields the following.

\[-[(\partial_{\bar{z}} h_{-(\mu+3)}) \lhd \Theta] s_{\mu+3} - \frac{2(\mu+1)}{(\mu+2)} (\partial_{\bar{z}} w_{-\mu}) s_{\mu} = 0; \tag{130}\]

\[[\partial_{\bar{z}} h_{\mu+3}) \lhd \Theta] s_{-(\mu+3)} + \frac{2(\mu+1)}{(\mu+2)} (\partial_{\bar{z}} w_{\mu}) s_{-\mu} = 0. \tag{131}\]

Using that
- $d_0 \Theta = d_0 \bar{\Theta} = 0$ (see Formula 2.19),
- any power of $s_{-1}$ or its conjugation is $d_0$-closed (Lemma 3.10),
- and that the curvature form $F_{A^0}$ is $(1,1)$,

we find that

\[[\partial_{\bar{z}} h_{-(\mu+3)}) \lhd \Theta] s_{\mu+3} = \ast_0 [(\partial_{\bar{z}} h_{-(\mu+3)}) \wedge \Theta] s_{\mu+3} \text{ is } \bar{\partial}_0^* - \text{closed}, \]

and

\[[\partial_{\bar{z}} h_{\mu+3}) \lhd \Theta] s_{-(\mu+3)} = \ast_0 [(\partial_{\bar{z}} h_{\mu+3}) \wedge \Theta] s_{-(\mu+3)} \text{ is } \partial_0^* - \text{closed}. \]

Plugging the above into (130) and (131), we see that

\[(\partial_{\bar{z}} w_{-\mu}) s_{\mu} \text{ is } \bar{\partial}_0^* - \text{closed, and } (\partial_{\bar{z}} w_{\mu}) s_{-\mu} \text{ is } \partial_0^* - \text{closed.} \]

Because both $\partial_{\bar{z}} w_{-\mu}$ and $\partial_{\bar{z}} w_{\mu}$ are forms on $\mathbb{P}^2$, and $\bar{\partial}_0^*$ ($\partial_0^*$) are equal to the usual $\partial_{\bar{z}}^{\mathbb{C}^2}$ ($\partial_{\bar{z}}^{\mathbb{C}^2}$) on such forms, we find

\[\bar{\partial}_{\bar{z}}^{\mathbb{C}^2} \partial_{\bar{z}} w_{-\mu} = 0 \text{ and } \partial_{\bar{z}}^{\mathbb{C}^2} \partial_{\bar{z}} w_{\mu} = 0. \tag{132}\]

Integrating by parts on $\mathbb{P}^2$ shows

\[\partial_{\bar{z}} w_{-\mu} = 0 \text{ and } \partial_{\bar{z}} w_{\mu} = 0. \tag{133}\]

Plugging (133) back into (130) and (131), because the $(2,0)$-form $\Theta$ is no-where vanishing (its real and imaginary parts are both complex structures), we find

\[\partial_{\bar{z}} h_{-(\mu+3)} = 0, \text{ and } \partial_{\bar{z}} h_{\mu+3} = 0. \tag{134}\]

Then

\[w_{\mu} \in H^0[\mathbb{P}^2, (\text{End}_0 E)(\mu)] \text{ and } h_{-(\mu+3)} \in H^0[\mathbb{P}^2, (\text{End}_0 E)(-\mu-3)]. \]

The derivation so far has given a map

\[Q : \text{Ker} \|_{\mathbb{P}^2_{\mu} \nu_{\mathbb{C}^2}} \to H^0[\mathbb{P}^2, (\text{End}_0 E)(\mu)] \oplus H^0[\mathbb{P}^2, (\text{End}_0 E)(-\mu-3)] \]

defined by

\[Q \begin{bmatrix} v \\ h \\ g \\ w \\ 0 \end{bmatrix} \triangleq (w_{\mu}, h_{-(\mu+3)}). \]
In a similar manner to (85), reversing the above arguments yields the obvious inverse of $Q$. For the reader's convenience, we still give the detail.

For any $w_{\mu} \in H^0[\mathbb{P}^2, (\text{End}_0 E)(\mu)]$ and $h_{-(\mu+3)} \in H^0[\mathbb{P}^2, (\text{End}_0 E)(-\mu - 3)]$, let

$$w_{-\mu} \triangleq w_{\mu}, \quad h_{\mu+3} \triangleq h_{-(\mu+3)}.$$

The inverse $Q^{-1}(w_{\mu}, h_{-(\mu+3)})$ is simply the matrix defined by conditions (119), (120), and (127):

$$\begin{bmatrix}
v & h \\
g & w \\
0 & 0
\end{bmatrix}$$

The spectral reduction in Formula 3.23 and the identification in Lemma H.2 below says that

$$v \text{ and } h \in \mathbb{E}_{\mu^2+2\mu-3}(\nabla^* \nabla |_{\mathbb{S}^5}) = \mathbb{E}_{\lambda_1}(\nabla^* \nabla |_{\mathbb{S}^5}), \quad g \text{ and } w \in \mathbb{E}_{\mu^2+4\mu}(\nabla^* \nabla |_{\mathbb{S}^5}) = \mathbb{E}_{\lambda_2}(\nabla^* \nabla |_{\mathbb{S}^5});$$

which means

$$\begin{bmatrix}
v \\
h \\
g \\
w
\end{bmatrix} \in \text{Ker } \|\mathbb{E}_\mu P|_{V_{\text{coh}}} \|.$$

The desired identification (114) is proved.

3.5.4 Proof of Theorem 1.8

Our understanding of the projection map determines the multiplicities.

Proof of Theorem 1.8. In view of Definition 3.13 and the characterization of $V_i$ in Proposition 3.5, $\text{Spec}P|_{V_{\text{coh}}}$ are those integers $l$ such that $V_i = H^1[\mathbb{P}^2, (\text{End}_0 E)(l)]$ is nonzero, and the eigenspace is $V_i$. It suffices to determine the multiplicities of $P$ restricted to $V_{\text{coh}}^\perp$.

In conjunction with Theorem 3.15 and the last paragraph of the proof of Theorem 1.8, neither $-1$ nor $-2$ is in $\text{Spec}P|_{V_{\text{coh}}^\perp} = S_{\mathcal{V}^* \mathcal{V}}$. The eigenspaces of $-1$ and $-2$ are $V_{-1}$ and $V_{-2}$ respectively. Then in view of Remark 2.16 and the complex isomorphism in Proposition 3.5, the eigenspaces of $-1$ and $-2$ are complex isomorphic to $H^1[\mathbb{P}^2, (\text{End} E)(-1)]$ and $H^1[\mathbb{P}^2, (\text{End} E)(-2)]$ respectively. Theorem 1.8.1 is proved.

In view of the spectral advantage in Theorem 3.15 restricted to $V_{\text{coh}}^\perp$, the multiplicity of an eigenvalue

$$\mu \in S_{\mathcal{V}^* \mathcal{V}} \subset \text{Spec}P|_{V_{\text{coh}}^\perp}$$

is completely determined by the surjective map

$$\|\mathbb{E}_\mu P|_{V_{\text{coh}}^\perp} : (\mathbb{E}_{\lambda_1} \nabla^* \nabla |_{\mathbb{S}^5})^{\mathbb{S}^2} \oplus (\mathbb{E}_{\lambda_2} \nabla^* \nabla |_{\mathbb{S}^5})^{\mathbb{S}^2} \to \mathbb{E}_\mu P|_{V_{\text{coh}}^\perp} \text{ in Definition 3.26}$$ (135)
When $\mu$ is not an integer, the vanishing of $\text{Ker} \parallel E^\mu P|_{V_{coh}^\perp}$ in Proposition 3.30.1 says that (135) is an isomorphism. This finishes the proof of Theorem 1.8. When $\mu$ is an integer, the characterization (114) of $\text{Ker} \parallel E^\mu P|_{V_{coh}^\perp}$ in Proposition 3.30.2 finishes the proof of Theorem 1.8.

II 2.

When $\mu$ is an integer, the characterization (114) of $\text{Ker} \parallel E^\mu P|_{V_{coh}^\perp}$ in Proposition 3.30.2 finishes the proof of Theorem 1.8.

II 3.

4 Spectral theory of the rough Laplacian on the bundle over $S^5$, and the proof of Theorem 1.10, Proposition 4.1, and Corollary 1.11.

Theorem 1.8 reduces the spectrum of $P$ to spectrum of the rough Laplacian on $S^5$. The purpose of this section is to prove Theorem 1.10 and Proposition 4.1, i.e. to completely characterize the eigenvalues of the rough Laplacian on $\pi^*_B(End_0 T^* P^2) \to S^5$. This directly leads to Corollary 1.11.

**Proposition 4.1.** (Multiplicities of the eigenvalues in Theorem 1.10)

Under the special data in Theorem 1.10, for any number $\lambda \in \text{Spec} \nabla^* \nabla|_{(End_0 T^* P^2)(l)}$, let the set $S^{l}_{\lambda}$ be defined by

$$S^{l}_{\lambda} \triangleq \{(a,b) \in \mathbb{Z}^0 \times \mathbb{Z}^0 \mid \frac{4}{3}(a^2 + b^2 + ab + 3a + 3b) - \frac{4}{3}l^2 - 8 = \lambda, \text{ and } \max(3 - a - 2b, b - a - 3) \leq l \leq \min(2a + b - 3, 3 + b - a)\}.$$  

The (complex) multiplicity of any $\lambda \in \text{Spec} \nabla^* \nabla|_{(End_0 T^* P^2)(l)}$ is

$$\Sigma S^{l}_{\lambda} \frac{(a+1)(b+1)(a+b+2)}{2}.$$  

(136)

The (real) multiplicity of any $\lambda \in \text{Spec} \nabla^* \nabla|_{S^5}$ is

$$\Sigma_{l,\lambda-P \in \text{Spec} \nabla^* \nabla|_{(End_0 T^* P^2)(l)}} S^{l}_{\lambda-P} \frac{(a+1)(b+1)(a+b+2)}{2}.$$  

(137)

Section 4.1–4.3 below are devoted to the proof of Theorem 1.10 and Proposition 4.1. The proof of Corollary 1.11 is completed in Section 4.4.

4.1 Background on reductive homogeneous spaces and homogeneous vector bundles

The representation theoretic method for Theorem 1.10 has been well recorded in literature. For example, see [13] and [3]. To be self-contained, we recall the background tailored for our purpose.

4.1.1 Killing reductive homogeneous spaces

Our references for this section are [12] Section 2: geometry of homogeneous spaces] and [17] Section 5, page 13.

All the group actions below will be left actions unless otherwise specified. All the $G$—actions below are smooth. The “·” between a group element and a vector (in a representation space) means the underlying action (which should be clear from the context).
Definition 4.2. (Reductive homogeneous space) Let $G$ be a compact semi-simple matrix Lie group, and $K$ be a closed matrix Lie subgroup of $G$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of $G$ and $K$ respectively. Let $m$ be a subspace of $g$ such that $\mathfrak{g} = m \oplus \mathfrak{k}$. The manifold $M = G/K$ is called a reductive homogeneous space with respect to $m$ if $\text{Ad}_K m \subseteq m$ (which means that for any $k \in K$ and $X \in m$, $\text{Ad}_k X \in m$).

In practice, we hide the “$m$” and abbreviate it to reductive homogeneous space.

At an arbitrary point $gK \in M$, any $X \in \mathfrak{g}$ generates a tangent vector $X^*$ in the following way.

$$X^*(gK) = \frac{d}{dt}|_{t=0}(\exp tX)gK.$$  \hfill (138)

Let $E$ be a homogeneous bundle over a reductive homogeneous space $M = G/K$. Let $e$ denote the identity element in $G$ (and $K$). Let the base point $o \in M$ be $eK$. The natural map $\rho : G \times_K E_o \to E$ defined by $\rho(g, v) = g \cdot v$ is a $G$–equivariant isomorphism (covering identity diffeomorphism of $M$).

On the tangent bundle, let $\tau$ denote the natural isomorphism $G \times_{K, \text{Ad}} m \to TM$ defined by

$$\tau(g, X) \triangleq g_*[X^{|eK}] = (\text{Ad}_g X)^*|_{gK}.$$  \hfill (139)

The tautological isomorphism $\tau_{\text{taut}} : m \to T_o M$ is defined by $\tau_{\text{taut}}(X) = X^{|o}$.

The set of $G$–invariant Riemannian metrics on $M$ is bijective to the set of $\text{Ad}_K$–invariant inner products on $m$. For example, under the semi-simple condition on $G$, the restriction to $m$ of a negative scalar multiple of the Killing form of $G$ yields a $G$–invariant metric on $M$. This example leads to the notion of a Killing reductive homogeneous space, which is a Riemannian manifold.

Definition 4.3. A reductive homogeneous space $M = G/K$ with a $G$–invariant Riemannian metric $(\ ,\ )$ is called a Killing reductive homogeneous space with respect to $(\ ,\ )$ and $m$ if the following holds.

- Let $B$ be the Killing form on $\mathfrak{g}$. With respect to the inner product $-B$, $m$ is perpendicular to the Lie algebra $\mathfrak{k}$ of $K$.
- The restriction of $-B$ on $m$ is a (constant) real scalar multiple of the inner product $(\ ,\ )_m$ corresponding to $(\ ,\ )$.

We usually abbreviate it to Killing reductive homogeneous space, and denote it by $[M, (\ ,\ )_m]$. The following existence of a certain kind of frames enables us to calculate the bundle rough Laplacian by Casimir operators.

Lemma 4.4. Let $[M, (\ ,\ )_m]$ be a Killing reductive homogeneous space equipped with the Levi-Civita connection, and let $(e_i, \ i = 1, \ldots, \dim M)$ be an orthonormal basis of $m$. Then for any $i$,

$$\nabla_{e_i} e_i^* = 0 \text{ at } o = eK.$$  \hfill (140)

Consequently, for any $g \in G$, $[\text{Ad}_g(e_i)]^* = g_* (e_i^*)$ is an orthonormal frame at $gK$ such that

$$\nabla_{[\text{Ad}_g(e_i)]^*}[\text{Ad}_g(e_i)]^* = 0 \text{ at } gK.$$  \hfill (141)

We do not know whether Lemma 4.4 holds in general if the reductive homogeneous space is not Killing.

The proof of Lemma 4.4 is routine Riemannian geometry calculation. It is deferred to Appendix II below.
4.1.2 Homogeneous vector bundles

Based on the notion of a reductive homogeneous space, we briefly recall the homogeneous bundles.

**Definition 4.5.** Let $M = G/K$ be a reductive homogeneous space. A (smooth) vector-bundle $E \to M$ is said to be $G$–homogeneous if the left action of $G$ on $G/K$ can be lifted to a compatible action of $G$ on $E$.

We usually suppress the “$G$” and call it a homogeneous vector bundle.

The space of smooth sections of a homogeneous vector bundle can be identified to an $\infty$–dimensional $G$–representation.

**Definition 4.6.** (Sections of a homogeneous bundle) Let $\rho_\mathcal{E} : K \to GL(\mathcal{E})$ be a (real or complex) $K$–representation. We consider the associated vector bundle $E = G \times_{K, \rho_\mathcal{E}} \mathcal{E}$.

Let $C^\infty(G, \mathcal{E})$ denote the space of all smooth $\mathcal{E}$–valued functions on $G$, and let $C^\infty_{K, \rho_\mathcal{E}}(G, \mathcal{E})$ be the subspace of $K$–invariant functions i.e. the functions $f$ such that

$$f(gk) = \rho_\mathcal{E}(k^{-1})f(g) \triangleq k^{-1} \cdot f(g). \tag{142}$$

We sometimes suppress the representation $\rho_\mathcal{E}$ in the notation $C^\infty_{K, \rho_\mathcal{E}}(G, \mathcal{E})$.

A section $u$ of $E$ defines uniquely a $K$–invariant function in $C^\infty_K(G, \mathcal{E})$, denoted by $\tilde{u}$. The converse is also true. The correspondence between $u$ and $\tilde{u}$ is given by

$$u(gK) = (g, \tilde{u}) \text{ for any } g \in G. \tag{143}$$

The same correspondence also holds pointwisely: for any $u \in E|_{gK}$, there is an unique $K$–invariant function $\tilde{u}$ defined on the $K$–orbit passing through $g$ such that $u = (g, \tilde{u})$.

The left regular representation of $G$ on $C^\infty_K(G, \mathcal{E})$ is defined by

$$[L(a) \cdot f](g) = f(a^{-1}g) \text{ for any } a, g \in G.$$ 

We also have the right regular representation of $G$ on $C^\infty(G, \mathcal{E})$:

$$[R(a) \cdot f](g) = f(ga) \text{ for any } a, g \in G.$$ 

Though we do not have a proof in this paper, we expect that in general, $C^\infty_K(G, \mathcal{E}) \subset C^\infty(G, \mathcal{E})$ is not necessarily an invariant subspace under the right regular representation, though it apparently is invariant under the left regular representation.

For further references, see [13, Section 5.1] and [1, III.6].

Regarding the reductive homogeneous spaces and homogeneous bundles defined so far, it does not harm to keep the following routine convention in mind.

**Convention on the $G$–equivariant isomorphism:** from here to the end of Section 4.4.

- the equal signs “$=$” between reductive homogeneous spaces or sections of homogeneous bundles, and
- any correspondence/identification between homogeneous connections, or between sections of homogeneous bundles, or between reductive homogeneous spaces

are via the underlying $G$–equivariant isomorphism or diffeomorphism.
4.1.3 A useful identity

For any $X \in m$, the following lemma calculates the invariant function corresponding to the vector field $X^*$.

Let $[\cdot]_m$ be the projection to $m$ with respect to the directly sum $\mathfrak{g} = m \oplus \mathfrak{k}$.

Lemma 4.7. Under the conditions and setting in Definition 4.2, let $\tilde{m}$ denote the linear map $m \to C^\infty(G, m)$ defined by

$$\tilde{m}(X)(g) \triangleq [g^{-1}Xg]_m.$$  

Then for any $X \in m$, $\tilde{m}(X)$ is $AdK$–invariant. This means that $\tilde{m}$ is actually a linear map $m \to C^\infty_{K, Ad}(G, m)$.

As a vector field on $M = G/K$, via the isomorphism $\tilde{m}(X)$ corresponds to $X^*$ i.e.

$$X^*(gK) = [Ad_g\tilde{m}(X)]^*(gK) = \tau(g, [\tilde{m}(X)](g)) \text{ at any point } gK \in M.$$  

(144)

Proof of Lemma 4.7. Because $(Ad_K)m \subseteq m$, $Ad_K$ preserves the splitting $Y = Y_m + Y_\mathfrak{k}$ i.e.

$$[(Ad_k)Y]_m = (Ad_k)[Y]_m \text{ for any } k \in K.$$  

(145)

Thus $[\tilde{m}(X)](gk) = [k^{-1}g^{-1}Xgk]_m = Ad_{k^{-1}}[g^{-1}Xg]_m$.

To prove the second part, for any $Y \in \mathfrak{g}$, let $Y = [Y]_m + [Y]_\mathfrak{k}$ (where $[Y]_\mathfrak{k}$ is the $\mathfrak{k}$–component of $Y$), we calculate

$$\{Ad_g[\tilde{m}(X)]\}(g) = g[g^{-1}Xg]_m g^{-1} = gg^{-1}Xgg^{-1} - g[g^{-1}Xg]_\mathfrak{k}g^{-1} = X - g[g^{-1}Xg]_\mathfrak{k}g^{-1}.$$  

Because $[g^{-1}Xg]_\mathfrak{k} \in \mathfrak{k}$, the tangent vector $\{g[g^{-1}Xg]_\mathfrak{k}g^{-1}\}^*$ at $gK$ is equal to 0. Then (144) is proved. 

\[\square\]

4.1.4 Rough Laplacian on a homogeneous vector bundle and the Casimir operator

The purpose of this section is to show Formula 4.11 of the rough Laplacian in terms of the Casimir operator.

Definition 4.8. (Casimir operator associated with a basis) Let $\mathfrak{g}$ be a Lie algebra, and $\mathcal{B} = (e_i, \ i = 1 \ldots \dim \mathfrak{g})$ be a basis of $\mathfrak{g}$. Let $\rho : \mathfrak{g} \to gl(\mathcal{E})$ be a representation of $\mathfrak{g}$. Then we define the Casimir operator with respect to the basis $\mathcal{B}$ by

$$Cas^\mathcal{B}_{\mathfrak{g}, \rho} \triangleq \Sigma^\mathfrak{g}_{i=1} \rho(e_i)\rho(e_i).$$

The notion of a Casimir operator associated with a basis is more general than the notion of an usual Casimir operator. Our definition is tailored for our purpose: on the former, $\Sigma^\mathfrak{g}_{i=1} e_i \cdot e_i$ is an element in the universal enveloping algebra, but is not in general required to be in the center; on the contrary, the later must be in the center.

Moreover, in the underlying definition, we do not require $G$ to be semi-simple, though it indeed is in the case of interest. We do not need the Killing form either. All we need is a basis of the Lie algebra.

Definition 4.9. Let $M = G/K$ be a reductive homogeneous space with respect to $m$. We view $G$ as a $K$–principal bundle over $M$. We define the left invariant principal connection of $m$ to be the connection of which the horizontal distribution at $g \in G$ is $gm$ (viewed as subspace of left invariant vector fields). On an associated bundle, the connection given by this horizontal distribution is called the connection induced by $m$, or simply the induced connection.
Definition 4.10. Let \((M, \langle \cdot, \cdot \rangle_m)\) be a Killing reductive homogeneous space. A basis \(B_g = (e_i, \ i = 1 \ldots \dim g)\) for the Lie algebra \(g\) is called triply orthonormal if

- \(B_g\) is orthonormal with respect to a negative real scalar multiple of the Killing-form;
- the set of vectors \(B_k = (e_i, \ i = 1 + \dim m, \ldots, \dim g)\) form a basis for the Lie algebra \(\mathfrak{k}\) of \(K\);
- the set of vectors \(B_m = (e_i, \ i = 1, \ldots, \dim m)\) form an orthonormal basis of \(m\) with respect to \(\langle \cdot, \cdot \rangle_m\).

In view of the above 3 definitions and the notation \(\tilde{\cdot}\) in Definition 4.6 for the invariant function in terms of a section, we prove the formula for the rough Laplacian.

**Formula 4.11.** Let \((M, \langle \cdot, \cdot \rangle_m)\) be a Killing reductive homogeneous space with a triply orthonormal basis \(B_g\) for the Lie algebra \(g\). Let \(\rho : K \to GL(\mathcal{E})\) be a (real or complex) representation of \(K\). On the homogeneous bundle \(G \times_{K, \rho} \mathcal{E}\), the following holds with respect to induced connection.

\[-(\nabla^{\ast} \nabla u) = (\text{Cas}_{g, L}^B - \text{Cas}_{k, \rho}^E)\tilde{u}.\]

**Remark 4.12.** The second operator \(\text{Cas}_{k, \rho}^E\) acts on the value of \(\tilde{u}\).

**Proof of Formula 4.11.** Let \(\tilde{Y}\) denote the horizontal lift of a tangent vector \(Y\), the Kobayashi-Nomizu formula [11] Vol I, Chap III, page 115] says that

\[\tilde{Y}(\tilde{u})(g) = [\nabla Y u]_{gK}(g).\]  

(146)

For any \(e_i \in m\), the vector \(ge_i\) (the value of the left invariant vector field) at \(g\) is the horizontal lift of \([Ad_g(e_i)]^*\) at \(gK\), using the vanishing (111), we compute

\[-(\nabla^{\ast} \nabla u)(g) = \sum_{i=1}^{\dim M} \{[Ad_g(e_i)]^* \cdot [Ad_g(e_i)]^*\} \tilde{u}(g) = \sum_{i=1}^{\dim M} [R_\ast(e_i) R_\ast(e_i) \tilde{u}](g)\]

\[= \sum_{i=1}^{\dim G} [R_\ast(e_i) R_\ast(e_i) \tilde{u}](g) - \sum_{i=1}^{\dim G} [R_\ast(e_i) R_\ast(e_i) \tilde{u}](g)\]

\[= [\text{Cas}_{g, L}^B \tilde{u}](g) - [\text{Cas}_{k, \rho}^E \tilde{u}](g).\]  

(147)

For any \(g\) and \(\tilde{u} \in C^\infty(G, \mathcal{E})\), we compute

\[(\text{Cas}_{g, L}^B \cdot \tilde{u})(g) = [\sum_{i=1}^{\dim G} R_\ast(e_i) R_\ast(e_i) \tilde{u}](g) = \frac{d^2}{dsdt}|_{s=0} \tilde{u}(g \exp^{s e_i} \exp^{t e_i})\]

\[= \frac{d^2}{dsdt}|_{s=0} \tilde{u}(g \exp^{s e_i} g^{-1} \cdot g \exp^{t e_i} g^{-1} \cdot g) = \sum_{i=1}^{\dim G} \{[L_\ast(Ad_g e_i) L_\ast(Ad_g e_i)] \tilde{u}](g)\]

\[= \sum_{i=1}^{\dim G} ([L_\ast(e_i) L_\ast(e_i)] \tilde{u})(g)\]  

(148)

This means that on \(C^\infty(G, \mathcal{E})\) (not requiring \(K\)–invariance), the Casimir operator of the right regular representation coincides with the Casimir of the left regular representation.

Because of the \(K\)–invariance of \(\tilde{u}\), we have \(R(e_i) \tilde{u} = -\rho(e_i) \tilde{u}\) (acting on the value of \(\tilde{u}\)). Hence

\[\text{Cas}_{k, \rho}^E \cdot \tilde{u} = \sum_{i=1}^{\dim K} R_\ast(e_i) R_\ast(e_i) \tilde{u} = \sum_{i=1}^{\dim K} \rho_\ast(e_i) \rho_\ast(e_i) \tilde{u} = \text{Cas}_{k, \rho}^E \tilde{u}.\]  

(149)

Applying (148) and (149) to the two individual terms in (147), the desired formula is proved. 

\(\square\)
4.2 The standard connection on the homogeneous bundle 
\[ [EndT^{1,0}(\mathbb{P}^2)](l) \]

The purpose of this section is to interpret \( EndT^1\mathbb{P}^2(l) \) as a homogeneous bundle over the homogeneous space \( \mathbb{P}^2 \), and show that the twisted Fubini-Study connection corresponds to the standard horizontal distribution \( m_{\mathbb{P}^2} \) defined in Section 4.2. Please see Proposition 4.15 for the main statement of this section.

4.2.1 The horizontal distribution \( m_{\mathbb{P}^2} \)

Recall that \( \mathbb{P}^2 = SU(3)/S[U(1) \times U(2)] \). Let the subspace \( m_{\mathbb{P}^2} \subset su(3) \) be spanned by the following 4 matrices.

\[
e_1 \triangleq X_1 \triangleq \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_2 \triangleq Y_1 \triangleq \begin{bmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\[
e_3 \triangleq X_3 \triangleq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad e_4 \triangleq Y_3 \triangleq \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}.
\]

It admits a natural complex structure

\[
JX_1 = -Y_1, \quad JY_1 = X_1, \quad JX_3 = -Y_3, \quad JY_3 = X_3.
\]

(151)

Then \( m_{\mathbb{P}^2} \) is naturally isomorphic to \( m_{\mathbb{P}^2}^{(1,0)} \) (the \((1,0)\)–part of the complexification of \( m_{\mathbb{P}^2} \)). The isomorphism is given by the natural injection \( m_{\mathbb{P}^2} \to m_{\mathbb{P}^2} \otimes \mathbb{C} \) composed by the projection to the \((1,0)\)–part. \( m_{\mathbb{P}^2}^{(1,0)} \) is spanned by the vectors

\[
s_1 = \frac{1}{2}(X_1 + iY_1), \quad s_2 = \frac{1}{2}(X_3 + iY_3).
\]

(152)

It is routine to verify that \( m_{\mathbb{P}^2} \) is preserved by \( Ad_{S[U(1) \times U(2)]} \). Thus, with respect to the Fubini-Study metric and \( m_{\mathbb{P}^2} \), \( \mathbb{P}^2 \) is a Killing reductive homogeneous space.

As complex vector bundles, the homogeneous bundle \( SU(3) \times S[U(1) \times U(2)]_{ad} m_{\mathbb{P}^2}^{(1,0)} \) is \( SU(3) \)–equivariantly isomorphic to the holomorphic tangent bundle \( T^{1,0}(\mathbb{P}^2) \).

4.2.2 Interpreting the line bundle \( O(l) \to \mathbb{P}^n \) as an associated bundle

\( SU(n+1) \) acts on \( \mathbb{C}^{n+1} \setminus \{O\} \) which is the total space of \( O(-1) \). Let \( S[U(1) \times U(n)] \) denote the subgroup of block-diagonal matrices in \( SU(n+1) \) i.e. the matrices in \( SU(n+1) \) such that the only non-zero entry in the first row and the first column is the \((1,1)\)–entry. This means

\[
\begin{bmatrix}
 e^{\sqrt{-1} \theta} & 0 & \ldots & 0 \\
 \vdots & \ddots & \ddots & \ddots \\
 0 & \ldots & \ldots & \ldots \\
 \vdots & \ddots & \ddots & \ddots \\
 0 & \ldots & \ldots & \ldots \\
\end{bmatrix}
\]

that the matrices have the form

\( \tau_S : S[U(1) \times U(n)] \to U(1) \) maps a matrix in \( S[U(1) \times U(n)] \) to its \((1,1)\)–entry.

For any integer \( l \), let \( \rho_l \) denote the 1–dimensional complex representation of \( U(1) \) i.e. \( \rho_l(e^{\sqrt{-1} \theta}) = e^{\sqrt{-1} l \theta} \in GL(1, \mathbb{C}) \). Abusing notation, we still let \( \rho_l \) denote the \( S[U(1) \times U(2)] \) representation \( \rho_l \cdot \tau_S \).
As a homogeneous space, $\mathbb{P}^n$ is $SU(n+1)/S[U(1) \times U(n)]$. The projection map is defined by

$$\pi : SU(n+1) \to \mathbb{P}^n. \quad \pi(A) \triangleq A \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \text{ for any } A \in SU(n).$$

(153)

Obviously, $S[U(1) \times U(n)]$ is the isotropy group of the point $A \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{P}^n$.

The action of the isotropy group $S[U(1) \times U(n)]$ on $Span[1,0,\ldots,0]$, the fiber of $O(-1)$ at $[1,0,\ldots,0]$, factors through the standard representation $\rho_1$ of $U(1)$. Hence, the universal bundle $O(-1) \to \mathbb{P}^n$ is $SU(n+1)-$equivariantly isomorphic to the homogeneous bundle $SU(n+1) \times S[U(1) \times U(n)]_{\rho_1} \mathbb{C}$.

More generally, for any integer $l$, $O(l) \to \mathbb{P}^n$ is $SU(n+1)-$equivariantly isomorphic to $SU(n+1) \times S[U(1) \times U(n)]_{\rho_{-l}} \mathbb{C}$.

4.2.3 Characterizing the connection of interest

The main proposition of Section 4.2 is a direct corollary of the following two lemmas addressing the horizontal distribution corresponding to the standard $SU(3)-$invariant connections.

**Lemma 4.13.** Via the $SU(3)-$equivariant isomorphism

$$SU(3) \times S[U(1) \times U(2)]_{\rho_1} \mathbb{C} = O(-1) \to \mathbb{P}^2,$$

the connection induced by $m_{\mathbb{P}^2}$ corresponds to the standard connection (see Definition 1.2).

**Lemma 4.14.** Via the $SU(3)-$equivariant isomorphism

$$SU(3) \times S[U(1) \times U(2)]_{\rho_1} \mathbb{C} = O(-1) \to \mathbb{P}^2,$$

the connection induced by $m_{\mathbb{P}^2}$ corresponds to the Fubini-Study connection.

The proof of the above two Lemmas is deferred to Appendix J and K. We are ready for our main proposition about the standard connection on $[EndT^{1,0}(\mathbb{P}^2)](l)$.

**Proposition 4.15.** On the homogeneous bundle $[EndT^{1,0}(\mathbb{P}^2)](l)$, the tensor product of the Fubini-Study connections (on $T^{1,0}(\mathbb{P}^2)$ and its dual) and the standard connection on $O(l)$ is induced by the horizontal distribution $m_{\mathbb{P}^2}$.

**Proof of Proposition 4.15.** Using Lemma 4.13, the standard connection on $O(l) \to \mathbb{P}^2$, obtained by the dual and/or tensor product of the standard connection on $O(-1)$, is induced by $m_{\mathbb{P}^2}$. Using Lemma 4.14, the associated connection on the holomorphic co-tangent bundle $\Omega_{\mathbb{P}^2}^1$, therefore the (tensor product) Fubini-Study connection on $[EndT^{1,0}(\mathbb{P}^2)]$, are induced by $m_{\mathbb{P}^2}$. Then the tensor product connection on $[EndT^{1,0}(\mathbb{P}^2)](l)$ is induced by $m_{\mathbb{P}^2}$.
4.3 Representation theory of $SU(3)$ and $S[U(1) \times U(2)]$, and the proof of Theorem \[1.10] and Proposition \[4.1]

From here to the end of Section \[4.3\] given a vector space $V$, the symbol $|_V$ means “as an endomorphism of $V$” or “as a representation on $V$”.

4.3.1 The representation of $S[U(1) \times U(2)]$ on $End_{\mathbb{P}^2}^{(1,0)} \otimes \mathbb{C}$ and its “Casimir” operator

The purpose of this section is to prove Formula \[1.20\] on the Casimir operator of the subgroup $S[U(1) \times U(2)]$ of $SU(3)$ (see Section \[4.2.2\] for the definition of the subgroup).

As a subgroup of $SU(3)$, the Lie algebra of $S[U(1) \times U(2)]$ is spanned by

$$
\begin{align*}
e_5 & \triangleq \hat{H}_1 \triangleq \frac{1}{\sqrt{3}} \begin{bmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{bmatrix}, \quad e_6 \triangleq H_2 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}, \\
e_7 & \triangleq X_2 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad e_8 \triangleq Y_2 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}.
\end{align*}
$$

Definition 4.16. Let $B_{s[u(1) \times u(2)]} \triangleq \{e_5, e_6, e_7, e_8\}$ be the basis for the Lie algebra $s[u(1) \times u(2)]$ of $S[U(1) \times U(2)]$.

Remark 4.17. $SU(2)$ is isomorphic to the subgroup in $S[U(1) \times U(2)]$ of block diagonal matrices with $(1,1)$—entry equal to 1. Henceforth, let $SU(2)$ denote this subgroup. Then $su(2)$ is spanned by $H_2$, $X_2$, $Y_2$ ($e_6$, $e_7$, $e_8$). We denote this basis of $su(2)$ by $B_{su(2)}$.

The first columns of $(e_1, e_2, e_3, e_4)$ form an orthonormal set of vectors in $\mathbb{R}^6$ (see \[150\]). Thus in view of the formula for the Euclidean metric (Kähler form) $\omega_{23}$ in Table \[194\], it is straightforward to verify that the quadruple $(e_1, e_2, e_3, e_4)$ is an orthonormal basis of the inner product $\langle \cdot, \cdot \rangle_{m_{22}}$ induced by the Fubini-Study metric $\frac{dm}{2}$.

Let $B_{su(3)}$ denote the basis $(e_i, i = 1...8)$ of $su(3)$. According to the previous paragraph, it is triply orthonormal on $\mathbb{P}^2$. That $\hat{H}_1$ is the form in \[154\] is important for this triple orthogonality.

Let $V_d$ be the space of all degree 2 homogeneous polynomials of 2—complex variables. Let $\rho_{V_d}: su(2) \rightarrow gl(V_d)$ be the irreducible representation of $su(2)$ on $V_d$. With respect to the notation convention in Definition \[4.18\] the Casimir operator obeys the following formula

$$
Cas_{s[u(1) \times u(2)]_{V_d}} = -(d^2 + 2d)I_d|_{V_d}. \quad \text{When } d = 2, \quad Cas_{s[u(1) \times u(2)]_{V_2}} = -8I_2|_{V_2}.
$$

We routinely verify the following identities on $ad_{su(2)|_{m_{22}^{(1,0)}}}$.

$$
[H_2, s_1] = i s_1, \quad [H_2, s_2] = -i s_2, \quad [X_2, s_1] = -s_2, \quad [X_2, s_2] = s_1, \quad [Y_2, s_1] = i s_2, \quad [Y_2, s_2] = i s_1.
$$

Therefore, under the basis $(s_1, s_2)$ of $m_{22}^{(1,0)}$, the representation $ad_{su(2)}$ is given by

$$
\begin{align*}
ad_{H_2} \cdot (s_1, s_2) &= (s_1, s_2) \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad ad_{X_2} \cdot (s_1, s_2) = (s_1, s_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
ad_{Y_2} \cdot (s_1, s_2) &= (s_1, s_2) \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\end{align*}
$$

Let an element in $su(2)$ be represented by its lower block $2 \times 2$ (which is exactly the standard form of $su(2)$), the above identities mean that under the basis $(s_1, s_2)$, $ad_{su(2)}|_{m_{22}^{(1,0)}}$ is the standard representation of $su(2)$.

Based on the above discussion, we can characterize $End_{\mathbb{P}^2}^{(1,0)}$ as an $su(2)$—representation.
Lemma 4.18. In view of Remark 4.17, the $su(2)$ representation on $\text{End}_0 m_{1,0}^{(1,0)}$ inherited from $s[u(1) \times u(2)]$ is 3-dimensional and irreducible. Consequently, it is equivalent to $\rho_v$.

Proof of Lemma 4.18: Because $\text{ad}_{su(2)} | m^{(1,0)}_{\mathbb{R}^1}$ is (equivalent to) the standard representation of $su(2)$ (see [137]), it suffices to show that the $su(2)$-representation on $\text{End}_0 \mathbb{C}^2$ induced by the standard representation is irreducible.

By definition, we routinely verify that the $su(2)$-representation on $\text{End}_0 \mathbb{C}^2$ extends complex linearly to the adjoint representation of $s\ell(2, \mathbb{C})$. Because $s\ell(2, \mathbb{C})$ is simple, the adjoint action must be irreducible. Therefore it is also irreducible as a $su(2)$-representation. The proof is complete.

To calculate the Casimir operator of $S[U(1) \times U(2)]$, it remains to understand the adjoint action of $e_5 = \hat{H}_1$ on $m^{(1,0)}_{\mathbb{R}^1}$.

Lemma 4.19. $\text{ad}_{\hat{H}_1} | m^{(1,0)}_{\mathbb{R}^1} = -\sqrt{3} i \text{Id} | m^{(1,0)}_{\mathbb{R}^1}$.

Proof of Lemma 4.19: We straight-forwardly verify the following.

$$[\hat{H}_1, X_1] = \sqrt{3} Y_1, \ [\hat{H}_1, Y_1] = -\sqrt{3} X_1, \ [\hat{H}_1, X_3] = \sqrt{3} Y_3, \ [\hat{H}_1, Y_3] = -\sqrt{3} X_3.$$  

Then $[\hat{H}_1, s_1] = -\sqrt{3} i s_1$, $[\hat{H}_1, s_2] = -\sqrt{3} i s_2$.

Using that the representation of $su(2)$ on $\mathbb{C}$ is trivial (the image is the 0-endomorphism), we compute the $su(2)$-Casimir on the representation of interest.

$$\sum_{i=6}^{8} [(\text{ad} \otimes \rho_{-1} \otimes \rho_{-1})_*(e_i)(\text{ad} \otimes \rho_{-1} \otimes \rho_{-1})_*(e_i)] |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \mathbb{C}}$$

$$= \sum_{i=6}^{8} [(\text{ad}_* (e_i)(\text{ad}_* (e_i))] |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \text{Id} | \mathbb{C}}$$

$$= -8 \text{Id} |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \mathbb{C}}.$$  

(158)

Elementary calculation yields the action of $e_5$ via $\rho$:  

$$\rho_{-l}(e_5) | \mathbb{C} = -\frac{2li}{\sqrt{3}} \text{Id} | \mathbb{C}, \ \text{consequently} \ [\rho_{-l}(e_5)\rho_{-l}(e_5)] | \mathbb{C} = -\frac{4l^2}{3} \text{Id} | \mathbb{C}. \ (159)$$

On the other hand, Lemma 4.19 says that $\text{ad}_{e_5} | m^{(1,0)}_{\mathbb{R}^1}$ is a (complex) scalar multiple of the identity. Thus $\text{ad}_{e_5} |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1}} = 0$. We obtain

$$[(\text{ad} \otimes \rho_{-1} \otimes \rho_{-1})_*(e_5)(\text{ad} \otimes \rho_{-1} \otimes \rho_{-1})_*(e_5)] |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \mathbb{C}} = \text{Id} |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \text{Id} | \mathbb{C}}$$

$$= -\frac{4l^2}{3} \text{Id} |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \mathbb{C}}.$$  

(160)

Combining (158) and (160), we arrive at the following.

Formula 4.20.

$$\text{Cas}_{s[u(1) \times u(2)]} \otimes \rho_{-l} |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \mathbb{C}} \triangleq \sum_{i=5}^{8} [(\text{ad} \otimes \rho_{-1} \otimes \rho_{-1})_*(e_i)(\text{ad} \otimes \rho_{-1} \otimes \rho_{-1})_*(e_i)] |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \mathbb{C}}$$

$$= (-8 - \frac{4l^2}{3}) \text{Id} |_{\text{End}_0 m^{(1,0)}_{\mathbb{R}^1} \otimes \mathbb{C}}.$$  

(161)
4.3.2 The translation between two conventions of $SU(3)$–representations

Let $W_{1,0}$ be the standard representation of $su(3)$ on $\mathbb{C}^3$, and $W_{0,1}$ be the dual representation of $W_{1,0}$. Let $W_{a,b}$ be the irreducible representation generated by the highest weight vector in the tensor product representation $W_{1,0}^{\otimes a} \otimes W_{0,1}^{\otimes b}$ (see [8, II.5]). Any irreducible representation of $su(3)$ is equivalent to $W_{a,b}$ for some integer $a, b \geq 0$ ($W_{0,0}$ is the 1–dimensional trivial representation).

Notation Convention 4.21. In [8], a $SU(3)$ irreducible representation is labelled by an integer linear combination of the two weights $x_1^*$, $x_2^*$. We denoted it by $V_{m_1x_1^*+m_2x_2^*}^{SU(3)}$, and this is said to be the *Ikeda-Taniguchi convention*. In [3], such integer linear combinations also label the irreducible $S[U(1) \times U(2)]$–representations. We denote it by $V_{k_1x_1^*+k_2x_2^*}^{S[U(1) \times U(2)]}$.

We need the following translation from the Ikeda-Taniguchi convention to the (usual) $W_{a,b}$–convention.

**Lemma 4.22.** The irreducible representation $W_{a,b}$ of $SU(3)$ is isomorphic to Ikeda-Taniguchi’s $V_{(a+b)x_1^*+bx_2^*}^{SU(3)}$.

**Proof of Lemma 4.22:** It is an algebra exercise to verify that in Ikeda-Taniguchi convention, the standard representation $W_{1,0}$ of $su(3)$ has highest weight $x_1^*$, the dual representation $W_{0,1}$ has highest weight $-x_3^*$, which is equal to $x_1^* + x_2^*$.

The highest weight of a (possibly multiple) tensor product of irreducible $su(3)$–representations is the sum of the highest weight of each one. Thus, the highest weight of $W_{1,0}^{\otimes a} \otimes W_{0,1}^{\otimes b}$ (in Ikeda-Taniguchi convention) is $ax_1^* - bx_3^*$, which is equal to $(a + b)x_1^* + bx_2^*$. Because $W_{a,b}$ is the irreducible representation generated by the highest weight vector in $W_{1,0}^{\otimes a} \otimes W_{0,1}^{\otimes b}$, the highest weight of $W_{a,b}$ is the same i.e. $(a + b)x_1^* + bx_2^*$.

4.3.3 The irreducible $S[U(1) \times U(2)]$–representation $End_0m_{p_2}^{(1,0)} \otimes \mathbb{C}$ in Ikeda-Taniguchi convention

The Cartan sub-algebra $\mathcal{Y}_{su(3)}$ of $su(3)$ consists of diagonal traceless matrices with purely imaginary diagonal entries. Let $x_i^*$ maps any matrix in $\mathcal{Y}_{su(3)}$ to its $i$–th diagonal entry. Then $x_1^*$, $x_2^*$, $x_3^*$ are roots. They are subject to the relation $x_1^* + x_2^* + x_3^* = 0$. According to [8, Section 5, Page 529], the partial ordering is determined by

$$x_1^* > x_2^* > 0 > x_3^*.$$  

Moreover, $m_{p_2}^{(1,0)}$ has highest weight $x_2^* - x_1^*$, and the dual $m_{p_2}^{(1,0),*}$ has highest weight $x_1^* - x_3^*$ (see [8, page 532, (iii)]). Then the highest weight of $End_0m_{p_2}^{(1,0)} = m_{p_2}^{(1,0)} \otimes m_{p_2}^{(1,0),*}$ is $x_1^* + 2x_2^*$, the sum of the highest weights of $m_{p_2}^{(1,0)}$ and $m_{p_2}^{(1,0)}$.

**Lemma 4.23.** The tensor product representation $Ad \otimes \rho_{-1} : S[U(1) \times U(2)] \rightarrow GL(End_0m_{p_2}^{(1,0)} \otimes \mathbb{C})$ is equivalent to $V_{(1+1)x_1^*+2x_2^*}^{S[U(1) \times U(2)]}$.

**Proof of Lemma 4.23:** Because $S[U(1) \times U(2)]$ is a subgroup of $SU(3)$ having the same Cartan sub-algebra $\mathcal{Y}_{su(3)}$, the highest weight on $End_0m_{p_2}^{(1,0)}$ is the same as the highest weight as a $SU(3)$–representation, which is equal to $x_1^* + 2x_2^*$. Because $\mathbb{C}$ is 1–dimensional, the only weight for $\rho_{-1}$ is $-lx_1^*$. Then in Ikeda-Taniguchi convention, the representation $Ad \otimes \rho_{-1}$ is denoted by $V_{(1+1)x_1^*+2x_2^*}^{S[U(1) \times U(2)]}$.

50
4.3.4 The infinite dimensional $SU(3)$--representation of invariant functions

Using the translation in (1.22) between two different conventions, we re-state the result of Ikeda-Taniguchi in the following.

**Fact** 4.24. (Ikeda-Taniguchi [8, Proposition 5.1, Proposition 1.1])

Let $l$ be an integer, and let $a, b$ be non-negative integers. $W_{a,b}$ appears as an irreducible summand in $C_{\text{st}[U(1) \times U(2)]}^∞(SU(3), \text{End}_OM_{p_2}^{(1,0)} \otimes \mathbb{C})$ if and only if

$$\max(3 - a - 2b, b - a - 3) \leq l \leq \min(2a + b - 3, 3 + b - a).$$

(162)

**Proof of Fact 4.24.** The representation $SU(3)$--representation $W_{a,b}$ is also a representation of the subgroup $SU(1) \times SU(2)$ by restriction. The Frobenius reciprocal theorem (for example, see [8, Proposition 1.1]) implies that the following two conditions are equivalent.

- As $SU(3)$--representations, $W_{a,b}$ appears as an irreducible summand in $C_{\text{st}[U(1) \times U(2)]}^∞(SU(3), \text{End}_OM_{p_2}^{(1,0)} \otimes \mathbb{C})$.
- As $SU(1) \times SU(2)$--representations, $(\text{End}_OM_{p_2}^{(1,0)} \otimes \mathbb{C}, \text{Ad} \otimes \rho_{-1})$ appears as an irreducible summand in $W_{a,b}$.

It suffices to determine for which $a, b$ the latter happens.

[8, Proposition 5.1] states that $V_{k_1x_1^2+k_2x_2^2}$ appear as an irreducible summand in $V_{m_1x_1^2+m_2x_2^2}$ if and only if the following holds.

$$m_1 \geq k_2 + k \geq m_2 \geq 0, \text{ and } k_1 = m_1 + m_2 - k_2 - 3k.$$  

(163)

Because of Lemma 4.22 and 4.23, to verify the second bullet point above, it suffices to let $m_1 = a + b, m_2 = b, k_1 = -l + 1, k_2 = 2$. Then the second bullet point holds if and only if

$$a + b \geq 2 + k \geq b \geq k \geq 0, \text{ and } 3k = a + 2b - 3 + l.$$  

(164)

Elementary calculation shows (164) is equivalent to (162). □

4.3.5 Proof of Theorem 1.10 and Proposition 4.1

In conjunction with the notation convention in Definition 4.8 above, the known formula for the quadratic Casimir operator of $su(3)$ states:

**Formula** 4.25. $Cas_{\text{su}(3)}^{B_{\text{su}(3)}}(W_{a,b}) = \sum_{i=1}^{8} (e_i|W_{a,b}) \cdot (e_i|W_{a,b}) = (-\frac{4}{3}a^2 - \frac{4}{3}b^2 - 4a - 4b - \frac{4}{3}ab)Id.$

The tools at our disposal now can be assembled to achieve our goal.

**Proof of Theorem 1.10 and Proposition 4.1** We first prove Theorem 1.10 (10). It is a direct corollary of Fact 4.24 on the irreducible summand of the infinite dimensional representation, Formula 4.25 for the Casimir operator of $su(3)$, Formula 4.20 for $Cas_{\text{su}(3)}^{B_{\text{su}(3)}}(W_{a,b})$, and the general Formula 4.11 for rough Laplacian on a homogeneous bundle over a Killing reductive homogeneous space.

Because of the $SU(3)$--equivariant isomorphism:

$$(\text{End}_OM_{p_2}^{(1,0)} \otimes \mathbb{C}) \rightarrow SU(3) \times SU(1) \times SU(2) \rightarrow (\text{End}_OM_{p_2}^{(1,0)} \otimes \mathbb{C}),$$

the general formula 4.11 for $G = SU(3)$, $K = SU(1) \times SU(2)$, and $\rho = Ad \otimes \rho_{-1}$ says that the spectrum of the rough Laplacian is equal to the spectrum of

$$-Cas_{\text{su}(3), L}^{B_{\text{su}(3)}} + Cas_{\text{su}(3)}^{B_{\text{su}(3)}}(W_{a,b}) \rightarrow SU(3) \times SU(1) \times SU(2) \rightarrow (\text{End}_OM_{p_2}^{(1,0)} \otimes \mathbb{C}),$$

(165)
on the space $C^{\infty}_{S(U(1) \times U(2))} \otimes \text{End}_{0}m^{(1,0)}_{\P^2}$ of invariant functions.

On the whole $C^{\infty}_{S(U(1) \times U(2))} \otimes \text{End}_{0}m^{(1,0)}_{\P^2}$, by Formula 4.20

$Cas_{s[u(1) \times u(2)], Ad \otimes \rho \cdot l}(SU(3), End_{0}m^{(1,0)}_{\P^2} \otimes \C)$, acts by $-(\frac{4}{3}l^2 + 8)Id$. In the Peter-Weyl formulation (see the presentation in [13, Section 5.1]), as $SU(3)$—representations, on each irreducible summand $W_{a,b}$ of $C^{\infty}_{S(U(1) \times U(2))} \otimes \text{End}_{0}m^{(1,0)}_{\P^2}$, Formula 4.25 says that the action of $-Cas_{s[u(3), L]}$ is the scalar multiplication by $\frac{4}{3}(a^2 + b^2 + ab + 3a + 3b)Id$. Then on the irreducible summand $W_{a,b}$, the action (165) is the scalar multiplication by

$$\frac{4}{3}(a^2 + b^2 + ab + 3a + 3b) - \frac{4}{3}l^2 - 8.$$ 

Fact 4.24 says that $W_{a,b}$ appears as an irreducible summand if and only if the condition on the right side of (10) holds. The proof of Theorem 1.10 (10) is complete.

Hence, Theorem 1.10 (11) directly follows from the spectral splitting in Formula 3.23 and Theorem 1.10 (10): we only need to add $l^2$ to the $\frac{4}{3}(a^2 + b^2 + ab + 3a + 3b) - \frac{4}{3}l^2 - 8$ in (10).

Next, we address the multiplicities. It is evident from the first 4 paragraphs in the underlying proof that the eigenspace of any $\lambda_i \in \text{Spec} \nabla^* \nabla|_{(End_{0}T^{\P^2})}\P$ is isomorphic to the direct sum of all those $W_{a,b}$ such that

- $W_{a,b}$ is a summand in $C^{\infty}_{S(U(1) \times U(2))} \otimes \text{End}_{0}m^{(1,0)}_{\P^2}$ i.e. the conditions for $a, b$, on the right side of (10) holds;
- the value of $\frac{4}{3}(a^2 + b^2 + ab + 3a + 3b) - \frac{4}{3}l^2 - 8$ is equal to $\lambda_i$.

In the terminology of Proposition 4.1, the above precisely means that $(a, b) \in S^{l}_{\lambda_i}$. The proof of Proposition 4.1 (136) is complete.

Hence, Proposition 4.1 (137) follows by (136), and the spectral splitting (96) counting multiplicity.

4.4 The proof of Corollary 1.11

Corollary 1.11 can be proved using Theorem 1.10 1.8 and the following Lemma on cohomology.

**Lemma 4.26.** $h^{1}[\P^2, (\text{End}T^{\P^2})(l)] = \begin{cases} 3 & \text{if } l = -1 \text{ or } -2, \\ 0 & \text{otherwise.} \end{cases}$

Consequently, $h^{0}[\P^2, (\text{End}T^{\P^2})(l)] = \begin{cases} \frac{3(l+3)}{2} & \text{if } l > 0, \\ 0 & \text{if } l \leq 0. \end{cases}$

The proof of Lemma 4.25 is completely routine via Euler sequence and Bott formula for sheaf cohomology on $\P^n$ (see [14]). We defer it to Appendix G.

**Proof of Corollary 1.11** Under the setting of Theorem 1.8 when

$$\pi_{5,4}^{*} \text{End}E = \pi_{5,4}^{*} \text{End}(T^{\P^2})$$

is equipped with the pullback Fubini-Study connection, Lemma 4.26 means that except when $l \neq -1$ or $-2$, the sheaf cohomology has no contribution to $\text{Spec}P$. On the other hand, Theorem 1.10 addresses the source $\text{Spec} \nabla^* \nabla|_{S^{5}}$ of the other part of $\text{Spec}P$.

We need the fact that any $W_{a,b}$ appears in the infinite-dimensional representation at most once. This is because the representation of the associated bundle is irreducible (see Lemma
Please see the Frobenius reciprocal theorem (stated in [8, Proposition 1.1]), and also [8, Proposition 5.1].

We seek for those eigenvalues of $\nabla^*\nabla|_{\mathbb{S}^5}$ that is strictly less than 8. When $l \geq 3$, because of the “$+l^2$” in Formula 3.23 (96), the eigenvalues of $\nabla^*\nabla|_{\mathbb{S}^5}$ generated are $\geq 9$. Thus, it suffices to assume $-2 \leq l \leq 2$ and seek for those eigenvalues of $\nabla^*\nabla|_{(\text{End}_0T^*\mathbb{P}^2)(l)}$ that is strictly less than 8.

Under the conditions on $a$, $b$ in Theorem 1.10 (10), elementary calculation shows that this can only happen for the following values of $l$, $a$, $b$.

- $l = 0$, $(a, b) = (1, 1)$. In this case, the corresponding eigenvalue of $\nabla^*\nabla|_{\text{End}_0T^*\mathbb{P}^2}$ is 4, the eigenspace is isomorphic to $W_{1, 1}$.
- $l = 1$, $(a, b) = (2, 0)$. In this case, the corresponding eigenvalue of $\nabla^*\nabla|_{(\text{End}_0T^*\mathbb{P}^2)(1)}$ is 4, the eigenspace is isomorphic to $W_{2, 0}$.
- $l = -1$, $(a, b) = (0, 2)$. In this case, the corresponding eigenvalue of $\nabla^*\nabla|_{(\text{End}_0T^*\mathbb{P}^2)(-1)}$ is 4, the eigenspace is isomorphic to $W_{0, 2}$.

Then, still according to Formula 3.23 (96), the above three cases generate the numbers 4 and 5 in $\text{Spec}\nabla^*\nabla|_{\mathbb{S}^5}$. The multiplicity of 4 is equal to $\text{dim}W_{1, 1} = 8$, the multiplicity of 5 is equal to $\text{dim}W_{2, 0} + \text{dim}W_{0, 2} = 12$.

The number 4 in $\text{Spec}\nabla^*\nabla|_{\mathbb{S}^5}$ generates the following values in $\text{Spec}P$.

$$2\sqrt{2} - 1, \ 2\sqrt{2} - 2, \ -1 - 2\sqrt{2}, \ -2 - 2\sqrt{2}.$$ 

The number 5 in $\text{Spec}\nabla^*\nabla|_{\mathbb{S}^5}$ generates the following values in $\text{Spec}P$.

$$1, \ 2, \ -4, \ -5.$$ 

Apparently, among the above 8 numbers, $2\sqrt{2} - 2$ and 1 are the only ones in the interval $(0, 1]$.

Because $2\sqrt{2} - 2$ is not an integer, its multiplicity is 16 i.e. twice of the multiplicity of 4 $\in \text{Spec}\nabla^*\nabla|_{\mathbb{S}^5}$.

The other number 1 is an integer, in view of Lemma 4.26, the multiplicity is

$$2\text{dim}E_0(\nabla^*\nabla|_{\mathbb{S}^5}) - 2h^0[\mathbb{P}^2, (\text{End}_0T^*\mathbb{P}^2)(1)] = 24 - 12 = 12.$$ 

The proof of Table (12) is complete. This is a sample of how the multiplicity of each eigenvalue of $P$ is determined.

\[\square\]

**Appendices**

**A Elementary Sasakian geometry**

Still let $D$ denote the contact distribution, and let $D^*$ denote the contact co-distribution. In the main body and in the subsequent Appendix, we frequently appeal to the following identities on Sasakian geometry of $\mathbb{S}^5 \rightarrow \mathbb{P}^2$ (please also see [18]).

For any section $X$ of the contact distribution $D$, $\nabla_X \xi = J_0(X)$, $\nabla_X \eta = [J_0(X)]^2$. \ (166)
We also have the following formulas for Hessians of the Reeb vector field and contact form.
For any point \( p \in S^5 \) and \( X, Y \in D|_p \), the following is true.
\[
(\nabla^2 \xi)(X,Y) = -\langle X, Y > \xi, (\nabla^2 \eta)(X,Y) = -\langle X, Y > \eta. 
\]
Consequently, \( \nabla^* \nabla \eta = 4\eta. \)

(167)

For the point-wise calculations in the proof of Lemma 2.14 and others, it is helpful to have a transverse geodesic frame in the following sense.

**Lemma A.1.** (Properties of a transverse geodesic frame) Let \( (x_i, i = 1, \ldots, 4) \) be a Kähler geodesic coordinate with respect to the (Fubini-Study) metric \( \frac{d\eta}{2} \) at (near) an arbitrary point \( [p] \in \mathbb{P}^2 \). Then, for any \( \beta \) among 0, 1, 2 such that \( [p] \in U_{\beta,\mathbb{P}^2} \), the following vector fields
\[
[\xi; v_i] \triangleq \frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})\xi, i = 1, 2, 3, 4
\]
is a frame near the Reeb orbit \( \pi^{-1}_{5,4}[p] \), and is orthonormal on \( \pi^{-1}_{5,4}[p] \). Moreover, the following holds on the Reeb orbit.
\[
(\nabla_{v_i}v_i)|_{\pi^{-1}_{5,4}[p]} = 0, \ [\nabla_{v_i}(j_0v_i)]|_{\pi^{-1}_{5,4}[p]} = -\xi.
\]

Near the Reeb orbit, we call the \((v_i, i = 1, 2, 3, 4)\) above a transverse geodesic frame. It is generated by the geodesic coordinate on \( \mathbb{P}^2 \).

**Proof of Lemma A.1:** We first show
\[
[v_i, v_j] = [d\eta(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i})]\xi. \tag{168}
\]
Because the Reeb vector-field \( \xi \) is a coordinate vector field in \( U_{\beta,\mathbb{P}^2} \) for any \( \beta \) = 0, 1, or 2, we have the vanishing
\[
[\xi, \frac{\partial}{\partial x_i}] = 0 \text{ for any } i. \tag{169}
\]
Then we calculate
\[
[v_i, v_j] = [\frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})\xi, \frac{\partial}{\partial x_j} - \eta(\frac{\partial}{\partial x_j})\xi] = [-\frac{\partial}{\partial x_i} \eta(\frac{\partial}{\partial x_j}) + \frac{\partial}{\partial x_j} \eta(\frac{\partial}{\partial x_i})]\xi
\]
\[
= [d\eta(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i})]\xi. \tag{170}
\]
The proof of (168) is complete.
The vanishing (169) above implies the following vanishing.
\[
[\xi, v_i] = 0 \text{ for any } i. \tag{171}
\]
The identity (168) implies that the Lie bracket of \( v_i \) and \( v_j \) is perpendicular to both \( v_i \) and \( v_j \). Then, using the Koszul formula [15, page 25], we find
\[
2\langle \nabla_{v_i}v_j, v_k \rangle = v_i\langle v_j, v_k \rangle - v_k\langle v_i, v_j \rangle + v_j\langle v_k, v_i \rangle. \tag{172}
\]
We recall the following formula for the standard metric on \( S^5 \).
\[
g_{S^5} = \pi^*_{5,4}g_{FS} + \eta \otimes \eta. \tag{173}
\]
Then (172) implies
\[
2 \langle \nabla_v v, v_k \rangle = \frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right)_{\mathbb{P}^2} - \frac{\partial}{\partial x_k} \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right)_{\mathbb{P}^2} + \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right)_{\mathbb{P}^2}
\]
\[
\begin{align*}
&= 2 \langle \nabla^{FS}_{\pi_5,4} \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \rangle_{\mathbb{P}^2} - 2 \langle \nabla^{FS}_{\pi_5,4,\ast} v, \pi_5,4,\ast v, \pi_5,4,\ast v_k \rangle_{\mathbb{P}^2} \\
&= 2 \langle (\pi_5,4,\ast \nabla^{FS})_v, v, v_k \rangle
\end{align*}
\]
Moreover, the Lie bracket identity (171) and the Koszul formula yield that
\[
2 \langle \nabla_v v, v, \xi \rangle = \langle [v, v], \xi \rangle = d\eta \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right).
\]
Thus the identities (173) and (175) yield
\[
\nabla_v v = (\pi^* \nabla^{FS})_v, v + \xi \left( d\eta \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right) \right).
\]
Via the tangent map \(\pi_5,4,\ast : D \to T\mathbb{P}^2\) which is an isometry, we verify that \(J_0 = \pi_5,4,\ast J_{\mathbb{P}^2}\). This implies that \(J_0 v_1 = v_2\), \(J_0 v_3 = v_4\), because \(J_{\mathbb{P}^2} \frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_2}\), \(J_{\mathbb{P}^2} \frac{\partial}{\partial x_3} = \frac{\partial}{\partial x_4}\). Using that \((\nabla^{FS}_{\pi_5,4} \frac{\partial}{\partial x_i})|_{[p]} = 0\), the following holds true.
\[
\nabla_v (J_0 v) = (\pi_5,4,\ast \nabla^{FS})_v J_0 v + \xi \left( d\eta \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right) \right) = \nabla^{FS}_{\pi_5,4} (J_{\mathbb{P}^2} \frac{\partial}{\partial x_i}) + \xi \left( d\eta \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right) \right)
\]
\[
\begin{align*}
&= -\xi \\
&\text{on the Reeb orbit } \pi_5,4,\ast [p].
\end{align*}
\]
Similarly, we compute
\[
(\nabla_v v)|_q = 0 + \xi \left( d\eta \left( \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right) \right)|_q = 0.
\]
The proof is complete. \(\square\)

The transverse geodesic frame helps in proving the following two formulas which are applied in the proof of the Bochner formulas (see Lemma 2.21 above).

**Formula A.2.** In the setting of Lemma 2.21, \((\nabla^* \nabla a_0)(\xi) = 2d^a_0 J_0(a_0)\).

**Proof of Formula A.2:** Using that \(a_0\) is semi-basic i.e. \(a_0(\xi) = 0\), Leibniz-rule yields
\[
0 = (\nabla^* \nabla)[a_0(\xi)] = (\nabla^* \nabla a_0)(\xi) - 2tr(\nabla a_0 \otimes \nabla \xi) + a_0(\nabla^* \nabla \xi).
\]
Because \(\nabla^* \nabla \xi = 4\xi\) (see (167)), we find \(a_0(\nabla^* \nabla \xi) = 0\), hence
\[
(\nabla^* \nabla a_0)(\xi) = 2tr(\nabla a_0 \otimes \nabla \xi).
\]
Therefore, at an arbitrary \(p \in \mathbb{S}^5\), let \(v_i\) be a transverse geodesic frame given by Lemma A.1 using \(\nabla \xi = 0\), we compute
\[
2tr(\nabla a_0 \otimes \nabla \xi)|_p = 2\Sigma_{i=1}^4 (\nabla\nabla a_0)(\nabla\nabla \xi)|_p = 2\Sigma_{i=1}^4 (\nabla_i a_0)(J_0 v_i)|_p
\]
\[
\begin{align*}
&= (2\Sigma_{i=1}^4 \nabla_i a_0(J_0 v_i)) - 2a_0(\Sigma_{i=1}^4 \nabla_i(J_0 v_i))|_p = 2\Sigma_{i=1}^4 \nabla_i[a_0(J_0 v_i)]|_p \\
&= -2\Sigma_{i=1}^4 \nabla_i[(J_0 a_0)(v_i)]|_p \\
&= 2d^a_0 J_0(a_0)|_p.
\end{align*}
\]
The proof is complete by the above two identities. \(\square\)
Formula A.3. In the setting of Lemma 2.21, \([\nabla^* \nabla(\eta a)\eta] = \eta(4a_\eta + \nabla^* \nabla a_\eta) - 2J_0(\partial_0 a_\eta)\).

Proof of formula A.3: We still work with a transverse geodesic frame \(v_i\ (i = 1, 2, 3, 4)\) at an arbitrary \(p \in \mathbb{S}^5\). Using the fundamental identities \([167]\), we calculate

\[
\nabla^* \nabla(\eta a_\eta) = (\nabla^* \nabla a_\eta)\eta + a_\eta \nabla^* \nabla \eta - 2\text{tr}(\nabla a_\eta \otimes \nabla \eta) = (\nabla^* \nabla a_\eta)\eta + 4a_\eta \eta - 2\text{tr}(\nabla a_\eta \otimes \nabla \eta).
\]

In view of the local formula \([33]\) for \(\partial_0 a_\eta\), the transverse geodesic frame yields

\[
(\nabla v_i a_\eta) dx_i = (\partial_0 a_\eta)|_p.
\]

Using the vanishing \(\nabla \xi \eta = 0\) and the formula \(\nabla X \eta = [J_0(X)|^0]|^2\), the trace term in the above identity can be additionally analyzed as follows.

\[
\begin{align*}
\text{tr}(\nabla a_\eta \otimes \nabla \eta)|_p &= [\Sigma_{i=1}^4 \nabla v_i a_\eta \otimes \nabla v_i \eta + L_\xi a_\eta \otimes \nabla \xi \eta]|_p = [\Sigma_{i=1}^4 \nabla v_i a_\eta \otimes \nabla v_i \eta]|_p \\
&= J_0(dx^i)(\nabla v_i a_\eta)|_p \\
&= J_0(\partial_0 a_\eta)|_p \text{ by } [182].
\end{align*}
\]

The desired identity follows.

Elementary identity calculations establish the formula for the contact form \(\eta\), and the formula for \(G\) and \(H\).

Proof of Formula 2.4: It suffices to check it in \(U_{0,\mathbb{C}^3}\), the proof is similar in \(U_{1,\mathbb{C}^3}\) and \(U_{2,\mathbb{C}^3}\). We first have \(Z_0 \frac{\partial}{\partial z_1} = \frac{2z_1}{2r} \frac{\partial}{\partial r} + \frac{\partial}{\partial z_1}\) and \(Z_0 \frac{\partial}{\partial z_2} = \frac{2z_2}{2r} \frac{\partial}{\partial r} + \frac{\partial}{\partial z_2}\). Using \(\eta = \frac{1}{2} e^\phi \log(|Z_0|^2 + |Z_1|^2 + |Z_2|^2) \) (see definition \([4]\)),

we find

\[
\eta(\frac{\partial}{\partial u_1}) = \eta(Z_0 \frac{\partial}{\partial Z_1}) = - \frac{\sqrt{-1}\bar{u}_1}{2\phi_0} = \frac{\sqrt{-1}}{2} \frac{\partial \log \phi_0}{\partial u_1}.
\]

Similarly, we have \(\eta(\frac{\partial}{\partial u_2}) = - \frac{\sqrt{-1}}{2} \frac{\partial \log \phi_0}{\partial u_2}\). Taking conjugation, we then obtain

\[
\eta(\frac{\partial}{\partial u_1}) = \frac{\sqrt{-1}}{2} \frac{\partial \log \phi_0}{\partial u_1}, \quad \eta(\frac{\partial}{\partial u_2}) = \frac{\sqrt{-1}}{2} \frac{\partial \log \phi_0}{\partial u_2}.
\]

The proof is complete by observing that \(\eta\) coincides with \(d\theta_0 + \frac{e^\phi \log \phi_0}{2}\) on the basis \(\frac{\partial}{\partial \theta_0}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial \bar{u}_1}, \frac{\partial}{\partial \bar{u}_2}\) for \(T^C\mathbb{S}^5\).

Proof of Lemma 2.5: We routinely verify in \(U_{0,\mathbb{C}^3}\) that

\[
\frac{dZ_0}{Z_0} = \frac{1}{Z_0} d\left(\frac{r e^{-\theta_0}}{\sqrt{-1}\phi_0}\right) = \frac{dr}{r} - \frac{d \log \phi_0}{2} + \sqrt{-1} d\theta_0
\]

\[
= \frac{dr}{r} - \frac{d \log \phi_0}{2} + \frac{1}{\sqrt{-1}} \frac{1}{2} \frac{e^\phi \log \phi_0}{d\theta_0} \text{ by Formula 2.4}
\]

When \(i = 1, 2\), we calculate

\[
\frac{dZ_i}{Z_0} = \frac{d(Z_0 u_i)}{Z_0} = du_i + u_i \frac{dZ_0}{Z_0}.
\]
Then,
\[ \Omega_{C^3} = dZ_0dZ_1dZ_2 = Z_0^3 \cdot \frac{dZ_0}{Z_0} \frac{dZ_1}{Z_0} \frac{dZ_2}{Z_0} = Z_0^3 \cdot \left[ du_1 + u_1\left(\frac{dZ_0}{Z_0}\right)\right] \wedge \left[ du_2 + u_2\left(\frac{dZ_0}{Z_0}\right)\right]. \]

\[ = Z_0^3 \cdot \frac{dZ_0}{Z_0} \wedge du_1 \wedge du_2 \]

\[ = Z_0^3 \cdot \frac{dr}{r} \wedge du_1 \wedge du_2 + \sqrt{-1}Z_0^3 \cdot \eta \wedge du_1 \wedge du_2. \]

The last inequality above uses that \( d\log \phi_0, \ d^c \log \phi_0 \) are both pulled back from \( U_0, \mathbb{P}^2 \subset \mathbb{P}^2 \), therefore \( d\log \phi_0 \wedge du_1 \wedge du_2 = d^c \log \phi_0 \wedge du_1 \wedge du_2 = 0 \).

In \( U_{0,C^3} \), the proof of (19) and the first row in (22) is complete. The proof is similar in \( U_{1,C^3} \) and \( U_{2,C^3} \).

**B The Sasaki-Quaternion coordinate on \( \mathbb{S}^5 \)**

The purpose of this section is to introduce a simple coordinate system under which Lemma 2.14 can be proved.

Because both the \((1, 0)\) vector field \( \frac{1}{2\sqrt{-1}} \left( \frac{\partial}{\partial s} - \frac{\partial}{\partial t} \right) \) and the standard \((3, 0)\)–form \( dZ_0dZ_1dZ_2 \) are invariant under the \( SU(3)\)–action on \( \mathbb{C}^3 \), by definition (21), so is \( H \) and \( G \).

**Fact** B.1. For any \( \chi \in SU(3) \) acting on \( \mathbb{S}^5 \), \( \chi^* G = G \), \( \chi^* H = H \).

**Definition B.2.** For any point \([Z] \in \mathbb{P}^2\), let \( \chi \in SU(3) \) be an element mapping \([Z]\) to \([1, 0, 0]\), let \( z_1 = \chi^* u_1 \), \( z_2 = \chi^* u_2 \) be the pullback coordinate system near \([Z]\). Because the Fubini-Study metric is \( SU(3)\)–invariant, \((z_1, z_2)\) is a Kähler geodesic coordinate at \([Z]\) i.e. for any \( i, j \) among 1, 2, \( \nabla^{FS} \frac{\partial}{\partial z_i} = \nabla^{FS} \frac{\partial}{\partial z_j} = 0 \).

On \( \mathbb{S}^5 \), we call \((z_1, z_2)\) a \( Sasaki-Quaternion \) \( coordinate \) of the Reeb orbit \( \pi_{5,4}^{-1}([Z]) \).

Under such a coordinate, on the Reeb orbit, both \( G \) and \( H \) take the canonical form

\[ G = -Im(dz_1dz_2), \ H = Re(dz_1dz_2). \]

Moreover, it yields a transverse geodesic frame in the sense of Lemma A.1.

**C The usual separation of variable: proof of Formula 2.12**

**Proof of Formula 2.12.** It is completely routine. To be self-contained, we still show the detail. In view of the splitting (15), we find

\[ d_{C^3 \times S^1} a_{C^3 \times S^1} = (d_{C^3} a_3) \wedge ds + d_{C^3} a_{C^3} - \frac{\partial a_{C^3}}{\partial s} \wedge ds \]

\[ \text{and} \]

\[ d^*_{C^3 \times S^1} a_{C^3 \times S^1} = -\frac{\partial a_3}{\partial s} + d^*_{C^3} a_{C^3}. \]

Using the tensor identity

\[ \ast_{C^3 \times S^1} [d_{C^3 \times S^1} a_{C^3 \times S^1} \wedge \psi_{C^3 \times S^1}] = (d_{C^3 \times S^1} a_{C^3 \times S^1}) \ast_{C^3 \times S^1} \phi_{C^3 \times S^1}, \]

\[ \text{by (184)} \]
it suffices to calculate the right side of (183) term-wisely as follows.

\[ (d_{C^3 \times S^1} a_{C^3 \times S^1}) \cdot C^3 \times S^1 (\alpha_{C^3} \wedge ds) = -(d_{C^3} a_{a}) \cdot C^3 \omega_{C^3} + (d_{C^3} a_{a} \cdot C^3 \omega_{C^3}) ds + \frac{\partial a_{C^3}}{\partial s} \cdot C^3 \omega_{C^3}. \] (188)

\[ (d_{C^3 \times S^1} a_{C^3 \times S^1}) \cdot C^3 \times S^1 \Re \Omega = (d_{C^3} a_{a}) \cdot C^3 \Re \Omega. \] (189)

The contraction \( \cdot C^3 \omega_{C^3} \) is the complex-structure \( J_{C^3} \) on \( \Omega^1[ad(E)] \). Summing (188) and (189) up, we arrive at the following.

\[ (d_{C^3 \times S^1} a_{C^3 \times S^1}) \cdot C^3 \times S^1 \phi_{C^3 \times S^1} \]

\[ = -J_{C^3}(d_{C^3} a_{a}) + (d_{C^3} a_{a} \cdot C^3 \omega_{C^3}) ds + J_{C^3}(\frac{\partial a_{C^3}}{\partial s}) + (d_{C^3} a_{a}) \cdot C^3 \Re \Omega. \]

Using the above, the easy identity \( d_{C^3 \times S^1} \sigma = (\frac{\partial \sigma}{\partial s}) ds + d_{C^3} \sigma \), and definition (5) of \( L_{AO, \phi_{C^3 \times S^1}} \), we obtain the following.

\[ L_{AO, \phi_{C^3 \times S^1}} \begin{bmatrix} \sigma \\ a_{a} ds + a_{C^3} \end{bmatrix} = \begin{bmatrix} \frac{\partial \sigma}{\partial s} + d_{C^3} a_{C^3} \\ \frac{\partial \sigma}{\partial s} + (d_{C^3} a_{a}) \cdot C^3 \omega_{C^3} \end{bmatrix} ds + \\
J_{C^3}(\frac{\partial a_{C^3}}{\partial s}) + d_{C^3} \sigma - J_{C^3}(d_{C^3} a_{a}) + (d_{C^3} a_{a}) \cdot C^3 \Re \Omega. \]

The desired formula follows.

\[ \square \]

D The fine separation of variable: proof of Lemma 2.14

We prove Lemma 2.14 by computing each row in the operator \( \square \) (see Formula 2.12). We first recall the following splitting.

\[ a_{C^3} = a_{0} + (a_{\eta} \eta) + a_{r} \frac{dr}{r}. \] (190)

Given a section \( a \) of \( \wedge^q T^* S^5 \) and a section \( b \) of \( \wedge^p T^* S^5 \) such that \( 5 \geq q \geq p \), we need the following identity.

\[ a_{\cdot C^3} b = \frac{1}{r^{2p}} a_{\cdot S^5} b. \]

Employing the splitting

\[ d_{C^3} = d_{0} + \eta \wedge L_{\xi} + dr \wedge L_{\frac{\partial}{\partial r}}, \] (191)

we calculate

\[ d_{C^3} a_{C^3} = (d_{0} + \eta \wedge L_{\xi} + dr \wedge L_{\frac{\partial}{\partial r}})[a_{0} + (a_{\eta} \eta) + a_{r} \frac{dr}{r}] \]

\[ = d_{0} a_{0} + (2a_{\eta}) \frac{dr}{2} + \eta \wedge (L_{\xi} a_{0} - d_{0} a_{\eta}) + \frac{dr}{r} \wedge (r \frac{\partial a_{0}}{\partial r} - d_{0} a_{r}) \]

\[ + (\frac{dr}{r} \wedge \eta)(r \frac{\partial a_{0}}{\partial r} - L_{\xi} a_{r}). \] (192)

Via the splitting (191), using \( \sigma = \frac{u}{r} \), \( a_{a} = \frac{a_{s}}{r} \), we routinely verify the following two identities.

\[ d_{C^3} \sigma = \frac{d_{0} u}{r} + \eta \wedge L_{\xi} u + (\frac{\partial u}{\partial r} - \frac{u}{r} \frac{dr}{r}); \\
d_{C^3} a_{a} = \frac{d_{0} a_{s}}{r} + \eta \wedge L_{\xi} a_{s} + (\frac{\partial a_{s}}{\partial r} - \frac{a_{s}}{r} \frac{dr}{r}). \] (193)
Employing the table

| \( \omega_{C3} = r dr \wedge \eta + r^3 \frac{d\eta}{2} \) |
|-----------------|
| \( d\text{Vol}_{P2} = \frac{1}{2} (\frac{d\eta}{r})^2 \) |
| \( d\text{Vol}_{P5} = \eta \wedge d\text{Vol}_{P2} \) |
| \( d\text{Vol}_{C3} = r^3 dr \wedge \eta \wedge d\text{Vol}_{P2} \) |

(194)

via (193), we calculate the contraction

\[
(d_{C3} a_s) \omega_{C3} = \left[ \frac{d_0 a_s}{r} + \frac{\eta \wedge L_s a_s}{r} + \left( \frac{\partial a_s}{\partial r} - \frac{a_s}{r} \right) \frac{dr}{r} \right] \omega_{C3} + r^3 \frac{d\eta}{2}
\]

(195)

Employing the commutator identities (29) and formula (192) for \( d_{C3} a_{C3} \), we calculate

\[
d_{C3} a_{C3} \omega_{C3} = \frac{1}{r} \left( \frac{d_0 a_s}{r} \right) \frac{d\eta}{2} + \frac{\eta \wedge L_s a_s}{r} + \left( \frac{\partial a_s}{\partial r} - \frac{a_s}{r} \right) \frac{dr}{r} + \frac{r^3 d\eta}{2} \eta.
\]

Assembling identity (193), (195), and (196), we can characterize row 3 of the operator \( \Box \).

**Formula D.1.** In view of Formula 2.12, on the third row of the operator \( \Box \), we have

\[
\frac{d r}{r} \cdot \left[ \frac{d_0 a_s}{r} \right] + \frac{\eta \wedge L_s a_s}{r} + \left( \frac{\partial a_s}{\partial r} - \frac{a_s}{r} \right) \frac{dr}{r} + \frac{r^3 d\eta}{2} \eta.
\]

Next, we calculate the first and second row of \( \Box \).

**Formula D.2.** The following two identities hold.

\[
d_{C3} a_{C3} = \frac{1}{r} \frac{\partial a_s}{\partial r} - \frac{4 a_s}{r^2} - \frac{L_s a_s}{r^2} + \frac{d_0 a_s}{r^2}.
\]

(197)

\[
d_{C3} a_{C3} \omega_{C3} = \frac{1}{r^2} d_0 a_s \frac{d\eta}{2} - \frac{L_s a_s}{r^2} + \frac{1}{r} \frac{\partial a_s}{\partial r} + \frac{4 a_s}{r^2}.
\]

(198)

In particular, under the splitting \( a_{s_5} = a_{s_5} + a_0 \),

\[
d_{s_5} a_{s_5} = -L_s a_0 + a_0.
\]

(199)

**Proof of Formula D.2.** The volume forms in table (194) imply the following two identities.

\[
\ast_{C3} \eta = -r^3 dr \wedge d\text{Vol}_{P2}.
\]

(200)

\[
\ast_{C3} a_0 = r^3 dr \wedge \eta \wedge \ast_0 a_0.
\]

(201)
Since $d\eta$ is a section of $\Lambda^{(1,1)} \otimes D^*$, but $G$ is a section of $[\Lambda^{(2,0)} \oplus \Lambda^{(0,2)}] \otimes D^*$—valued, we find the following vanishing

$$(d\eta) \omega(G \wedge \eta) = 0.$$  \hfill (202)

Using the above 3 elementary identities, we calculate $d_r^* c_3 a_{c_3}$ according to the 3—terms in the fine splitting (190):

$$d_r^* c_3 (a_r \frac{dr}{r}) = - * c_3 d c_3 * c_3 (a_r \frac{dr}{r}) = - * c_3 d c_3 (r^4 a_r d V \omega_{G^3})$$

$$= - * c_3 \frac{1}{r^3} \frac{\partial (r^4 a_r)}{\partial r} r^5 dr \wedge dV \omega_{G^5} = - \frac{1}{r^3} \frac{\partial (r^4 a_r)}{\partial r}$$

$$= - \frac{1}{r^3} \frac{\partial a_r}{\partial r} - \frac{4a_r}{r^2}.$$ \hfill (203)

$$d_r^* c_3 (a_\eta \eta) = - * c_3 d c_3 * c_3 (a_\eta \eta) = * c_3 d c_3 (a_\eta r^3 dr \wedge dV \omega_{G^2})$$

$$= L_\xi a_\eta * c_3 (r^3 \eta \wedge dr \wedge dV \omega_{G^2})$$

$$= - \frac{L_\xi a_\eta}{r^2}.$$ \hfill (204)

$$d_r^* c_3 a_0 = - * c_3 d c_3 * c_3 a_0 = - * c_3 d c_3 (r^3 dr \wedge \eta \wedge *_0 a_0) = - * c_3 (r^3 dr \wedge \eta \wedge d_0 *_0 a_0)$$

$$= - \frac{1}{r^3} * c_3 (r^5 d r \wedge \eta \wedge d_0 *_0 a_0) = - \frac{1}{r^2} *_0 d_0 *_0 a_0$$

$$= \frac{d_0^* a_0}{r^2}.$$ \hfill (205)

Identity (197) follows simply by summing up (203), (204), and (205).

Because $d_{c_3} a_{c_3} \omega_{c_3} = d_{c_3}^* J_{c_3} a_{c_3}$, using

$$J_{c_3} (a_r \frac{dr}{r} + a_\eta \eta + a_0) = a_r \eta - a_\eta \frac{dr}{r} + J_0 a_0,$$

identity (198) follows from (197) replacing $a_\eta$ by $a_r$, $a_r$ by $-a_\eta$, and $a_0$ by $J_0 a_0$ therein. \hfill \Box

The formulas established so far can be assembled into the desired formula of $P$.

**Proof of Lemma 2.14** Still in view of Formula 2.12 it is natural to classify the terms in the fine splitting of $\Box$ into 3 kinds of terms: those only involving $\frac{\partial}{\partial r}$ (derivative in $r$), those only involving $L_\xi$ (derivative along the Reeb vector field), and those only involving $d_0$.

We carry out the above scheme. Using

- the formula for the isometries $K$ and $T$ in Lemma 2.13,

- Formula D.2 for the first and second row of $\Box$,

- Formula D.1 for the third row of $\Box$,

we find the following fine splitting for $\Box$:

$$\Box = \frac{\partial}{\partial r} K + \frac{L_\xi T}{r} + \frac{B_0}{r},$$ \hfill (206)

where

$$B_0 \begin{bmatrix} u \\ a_s \\ a_r \\ a_\eta \\ a_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -4 & 0 & d_0^* a_0 \\ 0 & 0 & 0 & 4 & (d_0^* , \frac{d_0}{\partial r}) \\ -1 & 0 & 0 & 0 & (d_0^* , \frac{d_0}{\partial H}) \\ 0 & 1 & 0 & 0 & (d_0^* , \frac{d_0}{\partial G}) \\ d_0 & -J_0 d_0 & -J_H d_0 & -J_G d_0 & 3 J_H \end{bmatrix}.$$ \hfill (207)
It is then routine to verify, with the help of the second commutator identity in (29), that $P \triangleq K(B_0 + L_\xi T)$ is equal to the one given by (40). Hence
\[ \Box = K(\frac{\partial}{\partial r} - \frac{P}{r}). \tag{208} \]
The proof is complete. \hfill \Box

**E Digression to Hermitian Yang-Mills connections**

On a smooth Hermitian vector bundle $E$ over a Calabi-Yau $3-$fold $(X, \omega, \Omega)$, a triple $(A, \sigma, u)$ consisted of a smooth connection $A$ and two smooth sections $\sigma$ and $u$ of $adE$ is called a Hermitian Yang-Mills monopole if it satisfies the following equations
\[ F_A \llcorner \text{Re}\Omega + d_A \sigma - J(d_A u) = 0, \quad F_A \llcorner \omega = 0. \tag{209} \]
Suppose the Calabi-Yau is compact, the closeness of the holomorphic $(3, 0)$–form $\Omega$ and the Kähler form $\omega$ actually implies $F_A \llcorner \text{Re}\Omega = 0$, $d_A \sigma = d_A u = 0$, i.e. $F_A$ is $(1, 1)$, and $A$ is a Hermitian Yang-Mills connection.

Given a holomorphic Hermitian triple on $\mathbb{P}^2$. With gauge fixing, the linearized operator with respect to the associated data setting on $\mathbb{C}^3 \setminus O$ is precisely the operator $\Box$ which is part of $L_{A_0, \phi_{\mathbb{C}^3 \times S^1}}$ (see Formula 2.12). It of course depends on $P$ (see (208)).

**F Fourier series re-visited**

We first recall an elementary fact on uniform convergence of the usual Fourier series.

**Lemma F.1.** There exists a positive function $[\epsilon(N), \ N \in \mathbb{Z}^+]$ such that $\lim_{N \to \infty} \epsilon(N) = 0$ and the following holds. Let $f \in W^{1,2}(S^1)$ and let its Fourier series be $\Sigma_k f_k e^{\sqrt{-1}k\theta}$, then
\[ \Sigma_{|k| \geq N} |f_k e^{\sqrt{-1}k\theta}| \leq \epsilon(N) + \frac{1}{\sqrt{N}} |f|_{W^{1,2}(S^1)^2}. \tag{210} \]

**Proof of Lemma F.1:** We estimate simply by Cauchy-Schwartz inequality that
\[ |f_k e^{\sqrt{-1}k\theta}| \leq \frac{1}{k^2} + k^\frac{3}{2} \frac{f_k^2}{2}. \tag{211} \]

Then, $\xi(N) \triangleq \Sigma_{N \geq 1} \frac{1}{k^2}$ satisfies the desired conditions. Moreover, on the other term in (211), we estimate
\[ \Sigma_{N \geq 1} k^\frac{3}{2} f_k^2 \leq \frac{1}{\sqrt{N}} \Sigma_{N \geq 1} k^2 f_k^2 \leq \frac{1}{\sqrt{N}} |f|_{W^{1,2}(S^1)}^2. \]

The desired estimate (210) follows. \hfill \Box

Under the assumption $f \in W^{1,2}(S^1)$, it is well known that its Fourier-series converges uniformly to $f$. Based on the above bound on the remainder, we provide an ingredient for Lemma 3.3

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Lemma F.2. In the setting of Lemma F.1, let $\nu \in C^1(S^5, \pi_{5,4}^*EndE)$. Under the pullback Hermitian metric on $\pi_{5,4}^*EndE$, for any $\beta = 0, 1, or 2$, the Sasaki-Fourier Series $\Sigma k \nu_\beta(k)e^{\nu_\beta} \rightarrow k\beta$ converges uniformly to $\nu$ on $U_{\beta,S^5}$. The equivalent global series $\Sigma k \nu_k \otimes s_{-k}$ converges uniformly to $\nu$ on $S^5$.

Proof of Lemma F.2: Under the pullback connection from $EndE \rightarrow \mathbb{P}^2$, $L_\xi = \nabla_\xi$ on the sections of $\pi_{5,4}^*EndE$. Thus the $C^1$-condition implies that $L_\xi \nu \in C^0(S^5, EndE)$. Because $\xi = \frac{\partial}{\partial \theta_\beta}$ in $U_{\beta,S^5}$, fixing $u_1, u_2$ in the Sasakian coordinate, under a unitary trivialization, $\nu \in W^{1,2}(\theta_\beta)$ (as a function of $\theta_\beta$) in the Sasakian coordinate, under a unitary trivialization, $\nu \in W^{1,2}(\theta_\beta)$ (as a function of $\theta_\beta$).

Remark F.3. Let $\nu_{-k}$ denote the $-k$-th term $\nu_k \otimes s_{-k}$ in the Fourier-series. The value of $(\nu)_{-k}$ on an arbitrary Reeb orbit only depends on the value of $\nu$ on the same Reeb orbit.

In the setting of Lemma F.1, let $c(\theta)$ be a smooth function, the operator $c(\theta) \frac{\partial f}{\partial \theta}$ in general can not differentiate the Sasaki-Fouriers series by term i.e. in general

$$[c(\theta) \frac{\partial f}{\partial \theta}]_k \neq [c(\theta) \frac{\partial f}{\partial \theta}]f_k,$$

where the subscript $\cdot_k$ means the $k$-th Fourier-coefficient. The next result shows that this is not the case for the two operators we are interested in.

Claim F.4. Still in the setting of Lemma F.1, for any $\nu \in C^1(S^5, \pi_{5,4}^*EndE)$, in view of the notation $(\cdot)_{-k}$ in Remark F.3, for the Sasaki-Fourier coefficients,

$$(\nabla^* \nabla \nu)_k = \nabla^* \nabla (\nu)_k, \text{ and } (L_\xi \nu)_k = L_\xi (\nu)_k.$$ 

Proof of Claim F.4: It suffices to prove the two identities under the local Fourier-series i.e. the left hand side of (58). We only need to work near each Reeb orbit.

The identity for $L_\xi$ holds because $\xi = \frac{\partial}{\partial \theta}$ in $U_{\beta,S^5}$, and the usual Fourier Series in $\theta_\beta$ can be differentiated term by term with respect to $\theta_\beta$.

To prove the identity for $\nabla^* \nabla$, for any $[Z] \in \mathbb{P}^2$, we need a transverse geodesic frame $[v_i = \frac{\partial}{\partial x_i} - \eta(\frac{\partial}{\partial x_i})]\xi, i = 1, 2, 3, 4$ near the Reeb orbit $\pi_{5,4}^{-1}[Z]$. Because $\xi[\eta(\frac{\partial}{\partial x_i})] = 0$, $\eta(\frac{\partial}{\partial x_i})$ is independent of $\theta_\beta$, and that the connection is also pullback from $\mathbb{P}^2$, for each $i$, we find

$$(\nabla v_i)_k = (\nabla_{\pi_{5,4}^{-1}}(\eta(\frac{\partial}{\partial x_i})\xi))_k = \nabla_{\pi_{5,4}^{-1}}(\eta(\frac{\partial}{\partial x_i})\xi)_k = \nabla v_i \nu_k \in \text{the domain of } v_i.$$

Because $\nabla^* \nabla \nu = \nabla v_i \nabla v_i \nu$ on the Reeb orbit $\pi_{5,4}^{-1}[Z]$, in view of Remark F.3

$$\nabla^* \nabla \nu)_k = \nabla^* \nabla \nu|_{\pi_{5,4}^{-1}[Z]} = \nabla^* \nabla (\nu)_{-k} = [\nabla v_i \nabla v_i (\nu)_{-k}]|_{\pi_{5,4}^{-1}[Z]}$$

The proof is complete.
G Some algebro-geometric calculations

Let $\omega_{O(1)}$ be $\sqrt{-1}/2\pi$ times the curvature form of the standard metric on $O(1) \to \mathbb{P}^2$. Then $\omega_{O(1)}$ represents $c_1[O(1)]$, and $\frac{dh}{2} = \pi \omega_{O(1)}$ (cf. [8, page 142 and 30], watch out the difference of our scaling from the one therein). Throughout this article, we call $\pi \omega_{O(1)} (\frac{dh}{2})$ the Fubini-Study metric, and denote it by $\omega_{FS}$. The same applies to $\mathbb{P}^n$ as well (still let $\eta \triangleq d\log r$, $r$ is the distance to the origin in $\mathbb{C}^{n+1}$).

**Proof of Lemma 7.22** It suffices to show that

$$\chi[\mathbb{P}^2, (\text{End}E)(k)] = \frac{r^2k^2}{2} + r^2 + \frac{3kr^2}{2} - 2rc_2(E) + (r - 1)c_1^2(E).$$

(212)

Because $K_{\mathbb{P}^2} = O(-3)$, when $k \geq 0$, Serre duality says that

$$h^2[\mathbb{P}^2, (\text{End}E)(k)] = h^0[\mathbb{P}^2, (\text{End}E)(-k - 3)].$$

(213)

Then the desired identity (71) follows from the formula for Euler Characteristic (212).

We go on to prove the corollary (95) using (94). Since $E$ is Hermitian Einstein, then $E^* \text{ is also Hermitian Einstein (see [10, V, Proposition 7.7]). Combining [10, IV, Proposition 1.4]}, we find that $(\text{End}E)(-k - 3)$ is Hermitian Einstein i.e. there exists a metric such that the curvature $F$ satisfies

$$\frac{\sqrt{-1}}{2\pi} r_{ij}^F \omega_{O(1)}^{ij} = \mu Id_E \text{ for some real constant } \mu.$$ 

(214)

When $k \geq -2$,

$$\mu = \frac{\text{deg}[(\text{End}E)(-k - 3)]}{r^2\text{Vol}(\mathbb{P}^2)} < 0.$$ 

This means the mean curvature form defined in [10, IV, below (1.3)] is negative definite, hence the vanishing theorem [10, III, Theorem 1.9] implies that

$$h^0[\mathbb{P}^2, (\text{End}E)(-k - 3)] = 0. \text{ Therefore } h^2[\mathbb{P}^2, (\text{End}E)(k)] = 0.$$ 

(215)

Then (212) and (215) imply the desired identity $h^1[\mathbb{P}^2, (\text{End}E)(k)] = c_2(\text{End}E)$. Furthermore, the usual Chern number inequality says $c_2(\text{End}E) \geq 0$. Because $\mathbb{P}^2$ is simply connected and $\text{rank} E \geq 2$, if $c_2(\text{End}E) = 0$, [19, Theorem 8.1] says that $E$ can not be simple, which contradicts stability of $E$. Thus $c_2(\text{End}E) > 0$.

Now we prove the Euler characteristic formula (212). Since $c_1(\text{End}E) = 0$, by [10, II, (1.10)], we compute

$$ch[\mathbb{P}^2, (\text{End}E)(k)] = \frac{ch[\mathbb{P}^2, O(k)] \cdot ch[\mathbb{P}^2, \text{End}E]}{2}.$$ 

(216)

$$= \left\{1 + c_1[O(k)] + \frac{c_2[O(k)]}{2}\right\} \{r^2 + c_1(\text{End}E) + \frac{1}{2}[c_1^2(\text{End}E) - 2c_2(\text{End}E)]\}.$$ 

$$= r^2 + kr^2[\omega_{O(1)}] - c_2(\text{End}E) + \frac{r^2k^2}{2}[\omega_{O(1)}]^2.$$ 

$$= k^2 \omega_{O(1)} + \frac{r^2k^2}{2}[\omega_{O(1)}]^2.$$ 

The well known formula for Todd class states (for example, see [10, page 288]):

$$Td(\mathbb{P}^2) = 1 + \frac{3[\omega_{O(1)}]}{2} + [\omega_{O(1)}]^2.$$ 

(217)

We compute

$$Td(\mathbb{P}^2) \cdot ch[(\text{End}E)(k)]$$ 

$$= r^2 + \left(\frac{3r^2}{2} + kr^2\right)\omega_{O(1)} - c_2(\text{End}E) + \left[\frac{r^2k^2}{2} + r^2 + \frac{3kr^2}{2}\right][\omega_{O(1)}]^2.$$ 

(218)
In conjunction with the remark on \( \omega_{O(1)} \) above the underlying proof, the following holds.

\[
[\omega_{O(1)}] = c_1[O(1)], \text{ hence } \int_{\mathbb{P}^2} [\omega_{O(1)}]^2 = \int_{\mathbb{P}^2} [c_1(O(1))]^2 = 1. \tag{219}
\]

Using Hirzebruch Riemann-Roch theorem, we integrate \cite{218} to obtain

\[
\chi[\mathbb{P}^2, (\text{End}E)(k)] = \int_{\mathbb{P}^2} Td(\mathbb{P}^2) \cdot ch[\mathbb{P}^2, (\text{End}E)(k)]
= \frac{r^2k^2}{2} + r^2 + \frac{3kr^2}{2} - c_2(\text{End}E). \tag{220}
\]

The proof of \cite{212} is complete. \( \square \)

**Proof of Lemma 4.26** We only prove the formula for \( h^1[\mathbb{P}^2, (\text{End}T^r\mathbb{P}^2)(l)] \), the formula for \( h^0[\mathbb{P}^2, \{\text{End}_0(T^r\mathbb{P}^2)\}(l)] \) thereupon follows by Riemann-Roch (see Lemma 3.22).

On \( \mathbb{P}^2 \), we tensor the Euler-Sequance

\[ 0 \to O \to O^{\oplus 3}(1) \to T^r\mathbb{P}^2 \to 0 \]

by the sheaf \( \Omega^1(l) \), the local freeness of \( \Omega^1(l) \) yields the exactness of the following.

\[ 0 \to \Omega^1(l) \to [\Omega^1(l+1)]^{\oplus 3} \to (\text{End}T^r\mathbb{P}^2)(l) \to 0. \tag{221} \]

Hence we have the following exact sequence of cohomologies

\[ \ldots \to H^1[\mathbb{P}^2, \Omega^1(l+1)]^{\oplus 3} \to H^1[\mathbb{P}^2, (\text{End}T^r\mathbb{P}^2)(l)] \to H^2[\mathbb{P}^2, \Omega^1(l)] \to \ldots \tag{222} \]

By Bott formula of sheaf cohomology on complex projective spaces (see \cite[Section 1.1]{14}), when \( l \geq 0 \), both \( H^1[\mathbb{P}^2, \Omega^1(l+1)]^{\oplus 3} \) and \( H^2[\mathbb{P}^2, \Omega^1(l)] \) vanish. Then \( H^1[\mathbb{P}^2, (\text{End}T^r\mathbb{P}^2)(l)] \) vanishes if \( l \geq 0 \).

When \( l = -1 \),

\[ H^1[\mathbb{P}^2, (\Omega^1)^{\oplus 3}] = \{H^1[\mathbb{P}^2, \Omega^1]\}^{\oplus 3} = \mathbb{C}^3, \quad H^2[\mathbb{P}^2, \Omega^1(-1)] = 0. \]

Thus \( H^1[\mathbb{P}^2, (\text{End}T^r\mathbb{P}^2)(-1)] = \mathbb{C}^3 \). By Serre-duality, we find

\[ H^1[\mathbb{P}^2, (\text{End}T^r\mathbb{P}^2)(-2)] = \mathbb{C}^3 \], and \( H^1[\mathbb{P}^2, (\text{End}T^r\mathbb{P}^2)(l)] = 0 \) if \( l \leq -3 \).

\( \square \)

**H Kähler identity for vector bundles**

The usual Kähler identity says that on a Kähler manifold, the Laplace-Beltrami operator (on functions) is twice of the \( \bar{\partial} \)–Laplacian. The Lemma below is a straight-forward generalization to bundle case. Though we do not know whether it is stated explicitly in literature, the proof is completely routine. Please see a related calculation in \cite[III.1]{10}.

**Lemma H.1.** Let \( \Xi \) be a holomorphic Hermitian vector bundle over a Kähler manifold \( (X, \omega) \). Let \( A \) denote the Chern connection. Then

\[
\nabla_A^* \nabla_A = 2\partial_A \bar{\partial}_A + 2\pi \cdot \frac{-1}{2\pi} F_A \omega. \tag{223}
\]

Consequently, let \( (E, h, A) \) be a Hermitian Yang-Mills triple on \( \mathbb{P}^n \). In view of the convention for the Kähler metric in the first paragraph of Appendix \[G\] (above the proof of
Lemma [3.23], we consider the Fubini-Study metric $\omega_{FS}$. On the twisted endomorphism bundle $(\text{End}E)(l)$, under the tensor product of $A$ and the standard connection on $O(l)$ (the twisted connection), suppressing the subscripts for the connection as usual, we have

$$\nabla^* \nabla = 2\partial^* \overline{\partial} + 2nl \cdot \text{Id.} \quad (224)$$

In particular, when $n = 2$,

$$\nabla^* \nabla = 2\partial^* \overline{\partial} + 4l \cdot \text{Id.} \quad (225)$$

Proof of Lemma [H.7]: At an arbitrary point $p \in X$, let $(z_j, j = 1, \ldots, n)$ be a Kähler geodesic coordinate for the metric $\omega$. By definition, we have for any section $\varphi$ of $\Xi$ that

$$\nabla^* \nabla \varphi = -2\Sigma_j (\varphi_{jj} + \overline{\varphi}_{jj}) = -4\Sigma_j \varphi^2 + 2\Sigma_j F_{A,jj} \cdot \varphi = 2\partial^*_A \overline{\partial}_A \varphi + 2\Sigma_j F_{A,jj} \cdot \varphi \text{ at } p. \quad (226)$$

Please compare it to the usual Kähler identity in [6, Chap 0.7, page 106]. To complete the proof of (223), it suffices to observe that $\Sigma_j F_{A,jj} = 2\nabla \omega_{FS}$. To prove (224), based on (223), we contract the following by $\omega_{FS}$.

$$F_A = [F_E, \cdot] \otimes \text{Id}_{O(l)} + \text{Id}_E \otimes F_{O(l)} \quad (227)$$

The Hermitian Yang-Mills condition says that $[F_E, \omega_{FS}, \cdot]$ acts by 0–endomorphism on $\text{End}E$, using $c_1(O(l)) = \omega_{O(1)}$, the following holds as endomorphisms on $(\text{End}E)(l)$.

$$\sqrt{-1} F_{A,\omega_{FS}} = \sqrt{-1} \frac{1}{2\pi} \text{Id}_E \otimes (F_{O(l),\omega_{FS}}) = \frac{(nl)\text{Id}}{\pi} \left( \frac{\int \omega_{O(1)}^n}{\int \omega_{FS}^n} \right)$$

We should notice that the $\pi$ factor in $\omega_{FS} = \pi \omega_{O(1)}$ produces the “$\pi$” in the denominator of the last line above. The proof of (224) is complete. \hfill \Box

The above Kähler identity relates the space of holomorphic sections to a certain eigenspace of the rough Laplacian.

**Lemma H.2.** (A holomorphic section is an eigensection of the rough Laplacian) Let $(E, h, A_0)$ be a Hermitian Yang-Mills triple on $\mathbb{P}^2$. For any nonnegative integer $l$,

$$\mathbb{E}_l \nabla^* \nabla |_{(\text{End}_0E)(l)} = H^0[\mathbb{P}^2, (\text{End}_0E)(l)]. \quad (228)$$

Moreover, the isomorphism “$\cong$” above is an actual equality: a holomorphic section of $(\text{End}_0E)(l)$ is an eigensection of $\nabla^* \nabla |_{(\text{End}_0E)(l)}$ with respect to the eigenvalue $4l$, and vice versa.

The proof is straightforward by formula (225).

**I  Calculations on a Killing reductive homogeneous space**

**Proof of Lemma [4.4]** For any $V, X, Y \in \mathfrak{g}$, the usual Koszul formula [15, page 25] says

$$2\langle \nabla_V X^*, Y^* \rangle = V^* \langle X^*, Y^* \rangle - Y^* \langle V^*, X^* \rangle + X^* \langle Y^*, V^* \rangle \quad (229)$$

$$+ \langle [V^*, X^*], Y^* \rangle - \langle [X^*, Y^*], V^* \rangle + \langle Y^*, V^* \rangle, X^* \rangle.$$
Because $V^*, X^*, Y^*$ are Killing vector fields, we find
\[
\begin{align*}
V^* \langle X^*, Y^* \rangle &= \langle [V^*, X^*], Y^* \rangle + \langle X^*, [V^*, Y^*] \rangle, \\
Y^* \langle V^*, X^* \rangle &= \langle [Y^*, V^*], X^* \rangle + \langle V^*, [Y^*, X^*] \rangle, \\
X^* \langle Y^*, V^* \rangle &= \langle [X^*, Y^*], V^* \rangle + \langle Y^*, [X^*, V^*] \rangle.
\end{align*}
\]

Plugging the above into (229), we find
\[
2 \langle \nabla_{V^*} X^*, Y^* \rangle = \langle [V^*, X^*], Y^* \rangle - \{\langle [X^*, [Y^*, V^*]] + \langle V^*, [Y^*, X^*] \rangle \}. (230)
\]

Next, for any $V, X, Y \in m$, we show that the condition of Killing homogeneous space implies
\[
\langle X^*, [Y^*, V^*] \rangle + \langle V^*, [Y^*, X^*] \rangle = 0 \text{ at } eK. (231)
\]

Proposition 2.1] says that $[X^*, Y^*] = -[X, Y]^* \text{ at } eK$ for any $X, Y \in g$. Then at $eK,$
\[
\begin{align*}
\langle X^*, [Y^*, V^*] \rangle + \langle V^*, [Y^*, X^*] \rangle &= -\langle X^*, [Y, V]^* \rangle - \langle V^*, [Y, X]^* \rangle \\
&= -\langle X, [[Y, V]]_m \rangle_m - \langle V, [[Y, X]]_m \rangle_m \\
&= -\langle X, [Y, V] \rangle_\mathfrak{g} - \langle V, [Y, X] \rangle_\mathfrak{g} \text{ (because } X, Y, V \in m, \text{ and } m \perp \mathfrak{f}) \\
&= 0 \text{ (because } \langle \cdot, \rangle_\mathfrak{g} \text{ is a scalar multiple of the Killing form). (232)}
\end{align*}
\]

In row 2 of (232), the inner bracket $[Y, V]$ means the Lie bracket, while the outer means the projection to $m$ according to the reductive splitting. The identity (231) is proved.

For any $V, X \in m$, plugging (231) back into (230), because $Y \in m$ is also arbitrary, we find
\[
\nabla_{V^*} X^* = \frac{1}{2} [V^*, X^*] \text{ at } eK. (233)
\]

Therefore, for any $V \in m$, $\nabla_{V^*} V^* = 0$ at $eK$.

Because $g$ acts as an isometry (thus it preserves the Levi-Civita connection), equation (141) holds at $gK$. \hfill \Box

### J The standard connection on $O(l) \to \mathbb{P}^2$: proof of Lemma 4.13

To prove Lemma 4.13, we need the $K$–invariant function corresponding to the local defining section of $O(-1)$.

We recall (153) for the natural map $\pi : SU(3) \to \mathbb{P}^2$.

**Lemma J.1.** In $U_{0,\mathbb{P}^2} = \{[Z_0, Z_1, Z_2] \in \mathbb{P}^2 | Z_0 \neq 0 \}$, the defining section $(1, u_1, u_2)$ of $O(-1)$ corresponds to the $\text{Span}[1, 0, 0]$–valued function $\alpha = (\frac{1}{g_1}, 0, 0)$ on $SU(3)$, where $g_{11}$ is the $(1, 1)$–entry of $g \in SU(3)$. This means
\[
\begin{bmatrix}
1 \\
u_1 \\
u_2
\end{bmatrix}
\begin{bmatrix}
[g] = (g, \alpha) \text{ for all } g \in \pi^{-1}U_{0,\mathbb{P}^2}.
\end{bmatrix}
\]

**Remark J.2.** $\alpha$ is obviously $S[U(1) \times U(2)]$–invariant i.e. $\alpha(gk) = k^{-1}\alpha(g)$ for any $k \in S[U(1) \times U(2)]$. Moreover, $U_{0,\mathbb{P}^2}$ is invariant under the action of $S[U(1) \times U(2)]$. 

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Proof of Lemma 4.11: For any \( g \in SU(3) \), it suffices to compute at \( g \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g_1^1 \\ g_2^1 \\ g_3^1 \end{bmatrix} \) that
\[
(g, \alpha) = g \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} g_1^1 \\ g_2^1 \\ g_3^1 \end{bmatrix} \begin{bmatrix} 1 \\ g_1^2 \\ g_2^2 \\ g_3^2 \end{bmatrix} \begin{bmatrix} 1 \\ g_1^3 \\ g_2^3 \\ g_3^3 \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix}.
\]

Proof of Lemma 4.13: On \( O(-1) = SU(3) \times S_{U(1) \times U(n)}, \rho_1 \mathbb{C} \), the induced connection \( d_{\text{induced}} \) is \( SU(3) \)-invariant, so is the standard connection \( d_{\text{Chern}} \). It suffices to verify that they coincide at the base point \( o \triangleq \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{P}^2 \) under the trivialization \( s_0 = \begin{bmatrix} 1 \\ u_1 \\ u_2 \end{bmatrix} \).

The standard connection on \( O(-1) \) yields that \( d_{\text{Chern}} s_0 = (\partial \log \phi_0) s_0 \). Consequently, by the definition of the Kähler potential \( \phi_0 \) above (18), we find
\[
d_{\text{Chern}} s_0 = 0 \text{ at } o.
\]

All the elements in \( m_{\mathbb{P}^2} \) have vanishing \((1,1)\)-entry (see (150)). Therefore, for any \( X \in m_{\mathbb{P}^2} \),
\[
\alpha(e^{iX}) = \begin{bmatrix} 1 + O(t^2) \\ 0 \\ 0 \end{bmatrix}.
\]

This implies \( X(\alpha) = 0 \) at \( e \in SU(3) \). Lemma 4.11 means that \( s_0 = (g, \alpha) \). Because \( m_{\mathbb{P}^2} \) is horizontal, we find
\[
d_{\text{induced},X} s_0 = 0 \text{ at } o \text{ for any } X \in m_{\mathbb{P}^2}.
\]

Thus, the induced connection coincides with the Chern connection at the base point. \( \square \)

K The horizontal distribution of the Fubini-Study connection on the holomorphic tangent bundle: proof of Lemma 4.14

Definition K.1. For any \( X \in m_{\mathbb{P}^2} \), let \( X^* \mathbb{C} \) be the projection of the real vector field \( X^* \) to \( T^* \mathbb{P}^2 \) (see the material from (151) to (152) for the projection, and see (138) for the definition of \( X^* \)).

To prove Lemma 4.14 we need another form for the vector fields \( X_1^* \mathbb{C} , Y_1^* \mathbb{C} , X_3^* \mathbb{C} , Y_3^* \mathbb{C} \).

Formula K.2. In view of the basis (150) of \( m_{\mathbb{P}^2} \),
\[
X_1^* \mathbb{C} = \pi_{5,4,*}(Z_1 \frac{\partial}{\partial Z_0} - Z_0 \frac{\partial}{\partial Z_1}). \quad \text{In } U_{0,\mathbb{P}^2}, \quad X_1^* \mathbb{C} = -(1 + u_1^2) \frac{\partial}{\partial u_1} - u_1 u_2 \frac{\partial}{\partial u_2}.
\]
\[
Y_1^* \mathbb{C} = \sqrt{-1}\pi_{5,4,*}(Z_1 \frac{\partial}{\partial Z_0} + Z_0 \frac{\partial}{\partial Z_1}). \quad \text{In } U_{0,\mathbb{P}^2}, \quad Y_1^* \mathbb{C} = \sqrt{-1} (1 - u_2^2) \frac{\partial}{\partial u_1} - \sqrt{-1} u_1 u_2 \frac{\partial}{\partial u_2}.
\]
\[
X_3^* \mathbb{C} = \pi_{5,4,*}(Z_2 \frac{\partial}{\partial Z_0} - Z_0 \frac{\partial}{\partial Z_2}). \quad \text{In } U_{0,\mathbb{P}^2}, \quad X_3^* \mathbb{C} = -u_1 u_2 \frac{\partial}{\partial u_1} - (1 + u_2^2) \frac{\partial}{\partial u_2}.
\]
\[
Y_3^* \mathbb{C} = \sqrt{-1}\pi_{5,4,*}(Z_2 \frac{\partial}{\partial Z_0} + Z_0 \frac{\partial}{\partial Z_2}). \quad \text{In } U_{0,\mathbb{P}^2}, \quad Y_3^* \mathbb{C} = -\sqrt{-1} u_1 u_2 \frac{\partial}{\partial u_1} + \sqrt{-1} (1 - u_2^2) \frac{\partial}{\partial u_2}.
\]
Consequently, \( s_1^* = -\pi_{5,4,*}(Z_0 \frac{\partial}{\partial Z_1}) \), \( s_2^* = -\pi_{5,4,*}(Z_0 \frac{\partial}{\partial Z_2}) \). In \( U_{0,\mathbb{P}^2} \),

\[
s_1^* = -\frac{\partial}{\partial u_1}, \quad s_2^* = -\frac{\partial}{\partial u_2}.
\]

Proof of Formula \([K.3] \): We verify that \( e^{tX_1} \left[ \begin{array}{l} 1 \\ u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{l} \cos t + (\sin t)u_1 \\ -\sin t + (\cos t)u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{l} 1 \\ -\sin t + (\cos t)u_1 \\ \cos t + (\sin t)u_2 \end{array} \right]. \) Thus, when \( t \) is sufficiently small with respect to \( u_1 \), the following holds on \( \mathbb{P}^2 \).

\[
e^{tX_1} \left[ \begin{array}{l} 1 \\ u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{l} \cos t + (\sin t)u_1 \\ -\sin t + (\cos t)u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{l} 1 \\ -\sin t + (\cos t)u_1 \\ \cos t + (\sin t)u_2 \end{array} \right]. \tag{236}
\]

Then the identity

\[
X_1^{\ast, C} = \frac{d}{dt} \bigg|_{t=0} e^{tX_1} \left[ \begin{array}{l} 1 \\ u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{l} 0 \\ -(1 + u_1^2) \\ -u_1 u_2 \end{array} \right] = -(1 + u_1^2) \frac{\partial}{\partial u_1} - u_1 u_2 \frac{\partial}{\partial u_2}
\]

holds in \( U_{0,\mathbb{P}^2} \). While the vector \( \left[ \begin{array}{l} 1 \\ u_1 \\ u_2 \end{array} \right] \) above means a point in \( U_{0,\mathbb{P}^2} \subset \mathbb{P}^2 \), the vector

\[
\left[ \begin{array}{l} 0 \\ -(1 + u_1^2) \\ -u_1 u_2 \end{array} \right] \]

above means a \((1,0)\) tangent vector (at the point).

By continuity of both \( X_1^{\ast, C} \) and \(-\pi_{5,4,*}(Z_0 \frac{\partial}{\partial Z_1}) + \pi_{5,4,*}(Z_1 \frac{\partial}{\partial Z_0}) \), they are identical everywhere on \( \mathbb{P}^2 \).

Employing the following identities of matrix exponentials,

\[
e^{tY_1} = \left[ \begin{array}{ccc} \cos t & \sqrt{-1} \sin t & 0 \\ -\sqrt{-1} \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{array} \right], \quad e^{tX_3} = \left[ \begin{array}{ccc} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{array} \right], \quad e^{tY_3} = \left[ \begin{array}{ccc} \cos t & 0 & \sqrt{-1} \sin t \\ 0 & 1 & 0 \\ \sqrt{-1} \sin t & 0 & \cos t \end{array} \right], \tag{238}
\]

similar computations as \(236\) and \(237\) show that in \( U_{0,\mathbb{P}^2} \),

\[
Y_1^{\ast, C} = \sqrt{-1}(1 - u_1^2) \frac{\partial}{\partial u_1} - \sqrt{-1} u_1 u_2 \frac{\partial}{\partial u_2}, \quad X_3^{\ast, C} = -u_1 u_2 \frac{\partial}{\partial u_1} - (1 + u_2^2) \frac{\partial}{\partial u_2}, \quad Y_3^{\ast, C} = -\sqrt{-1} u_1 u_2 \frac{\partial}{\partial u_1} + \sqrt{-1}(1 - u_2^2) \frac{\partial}{\partial u_2}. \tag{239}
\]

As below \(237\), the 3 formulas respectively for \( Y_1^{\ast, C}, X_3^{\ast, C}, Y_3^{\ast, C} \) follow by continuity. \( \square \)

Proof of Lemma \([4.13] \): Similarly to the proof of Lemma \([4.13] \) because both connections are left invariant, it suffices to show that they are identical at the base point \( o \).

The Fubini-Study co-variant derivatives of both \( \frac{\partial}{\partial u_1} \) and \( \frac{\partial}{\partial u_2} \) are 0 at \( o \). Using Formula \([K.2] \) we find

\[
\nabla^{FS} s_1^* = \nabla^{FS} s_2^* = 0 \text{ at } o. \tag{240}
\]
In view of the correspondence in Lemma 4.7 at \( e \in SU(3) \), for any \( X, Y \in m_{\mathbb{P}^2} \), we compute the ordinary derivative

\[
[Y \tilde{m}_{\mathbb{P}^2}(X)](e) = -[[Y, X]]_{m_{\mathbb{P}^2}}.
\]

On the right hand side of the above, the inner bracket is the Lie bracket, the outer one is the projection to \( m_{\mathbb{P}^2} \).

We straight-forwardly verify \([m_{\mathbb{P}^2}, m_{\mathbb{P}^2}] \subseteq s[u(1) \times u(2)]\). Then \([Y \tilde{m}_{\mathbb{P}^2}(X)](e) = 0\). The correspondence (141) and Kobayashi-Nomizu formula (146) again yields that

\[
(\nabla_Y^{\text{induced}} X^*)(|o) = 0.
\]

On the complexification, this means for any \( s \in m^{(1,0)}_{\mathbb{P}^2} \), \( \nabla^{\text{induced}} s^* = 0 \) at \( o \).

Then \( \nabla^{\text{induced}} \) coincides with \( \nabla^{FS} \) at the base point \( o \). By \( SU(3) \)-invariance, they coincide everywhere on \( \mathbb{P}^2 \).

\[\square\]

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