Luttinger liquid universality in the time evolution after an interaction quench

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We provide strong evidence that the relaxation dynamics of one-dimensional, metallic Fermi systems resulting out of an abrupt amplitude change of the two-particle interaction has aspects which are universal in the Luttinger liquid sense: The leading long-time behavior of certain observables is described by universal functions of the equilibrium Luttinger liquid parameter and the renormalized velocity. We analytically derive those functions for the Tomonaga-Luttinger model and verify our hypothesis of universality by considering spinless lattice fermions within the framework of the density matrix renormalization group.

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The equilibrium low-energy physics of a large class of one-dimensional (1d), correlated, metallic Fermi systems is described by the Luttinger liquid (LL) phenomenon. The Tomonaga-Luttinger (TL) model is the effective low-energy fixed point model of the LL universality class and thus plays the same role as the free Fermi gas in Fermi liquid theory. This universality relies on the renormalization group (RG) irrelevance of contributions such as the momentum dependence of the two-particle interaction or the curvature of the single-particle dispersion which are present in microscopic models but ignored in the TL model. For a model falling into the LL universality class it is not necessary to explicitly compute thermodynamic observables and correlation functions if one is interested in the low-energy limit. One only needs to determine two numbers – the LL parameter $K$ and the renormalized velocity $v$ of the excitations – which fully characterize the low-energy physics of a spinless LL (on which we focus). Those depend on the band structure and filling as well as the amplitude and range of the two-particle interaction of the microscopic model at hand and can be extracted from the ground-state energy or ‘simple’ response functions. Thereafter, correlation functions at long length scales or thermodynamic quantities at low energies can be obtained by plugging in $K$ and $v$ into analytic expressions derived within the exactly solvable TL model.

The recent progress in experimentally controlling isolated many-body states, in particular in cold atomic gases, led to numerous theoretical studies on the dynamics of closed quantum systems resulting out of an abrupt change of the amplitude $U$ of the two-particle interaction. One assumes that the system is prepared in a canonical thermal state or, at temperature $T = 0$ on which we focus, the ground state of an initial Hamiltonian $H_i$ with a given $U_i$; often $U_i = 0$ is considered. At time $t = 0$ the interaction is quenched to $U_f$ and the time evolution is performed with the final Hamiltonian $H_f$. Fundamental questions discussed are

(a) Do some observables become stationary at large times? (b) How can they be classified (locality)? (c) Is it possible to compute their steady-state expectation values using an appropriate density matrix $\rho_{st}$? It was conjectured that in models with many integrals of motion, e.g. those which are solvable by Bethe ansatz (‘integrable’), $\rho_{st}$ is not of thermal but rather of generalized (‘Gibbs’) canonical form. This has been confirmed for models which can be mapped to effective noninteracting ones, in particular the TL model, and a variant of the latter, but a general prove is lacking. For generic models one generally expects $\rho_{st}$ to be thermal.

Here we address questions about the relaxation dynamics towards the steady state. Considering several models which in equilibrium fall into the LL class we ask: (A) Can the time evolution be characterized as universal in the LL sense? As model-dependent high energy processes matter at short to intermediate times one can only expect to find LL universality on large time scales – but even then it is not obvious whether the notion of RG irrelevance can be transferred from equilibrium to the nonequilibrium dynamics. (B) If LL universality is found, does it hold independently of the number of integrals of motion and thus independently of the expected nature of the steady state (generalized canonical versus thermal)? To answer those questions we proceed in steps. We compute the time evolution of the $Z$-factor jump of the momentum distribution function $n(k, t)$ at the Fermi momentum $k_F$ and the kinetic energy per length $\epsilon_{\text{kin}}(t)$ for the spinless TL model with arbitrary momentum dependent interactions $g_{2/4}(k)$ using bosonization. $\epsilon_{\text{kin}}(t)$ is defined as the expectation value of the initial Hamiltonian $H_i$ and thus describes how excitations die out. The quench is performed out of the noninteracting ground state. We analytically show that for any continuous $g_{2/4}(k)$ the long-time dynamics are given by

$$Z(t) \sim t^{-\gamma_{\text{LL}}(K)},$$

$$|d\epsilon_{\text{kin}}(t)/dt| \sim c(K, v)t^{-3}.$$
The decay of $Z(t)$ is governed by a universal exponent $\gamma_{st}(K)$ that depends on the equilibrium LL parameter $K$ only; $\epsilon_{\text{kin}}(t)$ features an asymptotic power law with an interaction-independent exponent but universal prefactor $\epsilon(K,v)$ determined by $K$ as well as by the renormalized velocity $v$. In equilibrium, this is the characteristic $K$- and $v$-dependence of correlation functions or of thermodynamic quantities, which \textit{a posteriori} motivates to consider $Z(t)$ and $\epsilon_{\text{kin}}(t)$ as representative examples. The notion of LL universality can now be defined in analogy to equilibrium: \textit{The quench dynamics is universal if Eqs. (3) describe the long-time relaxation for any model falling into the equilibrium LL universality class if the corresponding values for $K$ and $v$ are plugged in. To investigate this we compute $Z(t)$ for a 1d lattice of spinless fermions with nearest-neighbor hopping and interaction $\Delta$ \cite{24} as well as an extension of the latter including a next-to-nearest-neighbor interaction $\Delta_2$. We use the numerical time-dependent density-matrix RG (DMRG) \cite{25,27}. The model with $\Delta_2 = 0$ has many conserved quantities, is Bethe ansatz integrable, and thus $K$ as well as $v$ are known analytically \cite{25,26}. The $\Delta,\Delta_2$-model, however, is believed to be not exactly solvable. For $\Delta_2 > 0$ we extract $K$ and $v$ from equilibrium quantities (e.g. the small momentum density response function) using DMRG \cite{6,26,29}. Our data for the $Z$-factor agrees with Eq. (1a) for any interaction strength, filling factor, and irrespective of the integrability of the model. The results for $\epsilon_{\text{kin}}(t)$ are consistent with Eq. (11), but on the time scales accessible by DMRG the asymptotic behavior is still masked by oscillatory terms of higher order in $t^{-1}$. To unambiguously determine the prefactor of the $t^{-3}$-decay of the energy we resort to a numerical trick. Instead of performing the time evolution with $\exp(-iHt)$ we apply the imaginary time analogue $\exp(-H\tau)$. In this case the total energy per length $\epsilon(\tau)$ – which is no longer conserved – is the natural observable. For the TL model we show that the asymptotics is completely analogous to Eq. (11) with $t \rightarrow \tau$ and $\epsilon(K,v)$ replaced by a different function $\epsilon_{st}(K,v)$. For the lattice model, the $\tau^{-3}$-decay manifests over several orders of magnitude, and the prefactor agrees with the TL prediction.

This altogether provides strong evidence that questions (A) and (B) can be answered by ‘yes’. We conjecture that the universality of the quench dynamics also holds for other models falling into the equilibrium LL class.

\textit{The TL model} — After bosonizing \cite{1,2} the density of left and right moving fermions with a linear dispersion the Hamiltonian of the TL model is quadratic in operators $b_n^{(1)}$ which obey bosonic commutation relations:

$$H = \sum_{n>0} \left[ k_n \left( v_F + \frac{g_2(k_n)}{2\pi} \right) \left( b_n b_n^\dagger + b_n^\dagger b_n^\dagger \right) + k_n \frac{g_2(k_n)}{2\pi} \left( b_n b_n^\dagger + b_n^\dagger b_n^\dagger \right) \right]. \quad (2)$$

where $k_n = 2\pi n/L$, $n \in \mathbb{Z}$, $L$ denotes the chain length, and $v_F$ is the Fermi velocity. The two coupling functions (potentials) $g_{2/4}$ determine the strength of the scattering of fermions on different branches ($g_2$) and the same branch ($g_4$). Usually the $k$-dependence of $g_{2/4}$ is neglected and integrals are regularized in the ultraviolet through an ad hoc procedure \cite{1,2}. As the momentum dependence is RG irrelevant this is justified in equilibrium if all energy scales are sent to zero \cite{3}. For the quench dynamics – even at asymptotic times – it is, however, not clear if the same reasoning holds and we thus keep the full $k$-dependence and consider coupling functions. In fact, it was recently shown that the momentum dependence indeed affects the long-time dynamics of certain observables \cite{22}. For the system to be a LL in equilibrium we require that $0 < g_{2/4}(0) < \infty$ (repulsive interactions) and that $g_{2/4}(k)$ decay on a scale $k_c$. The Hamiltonian of Eq. (2) can be diagonalized to $H = \sum_{n\neq 0} \omega(k_n) \alpha_n^\dagger \alpha_n + E_{gs}$ by introducing new modes $\alpha_n = c(k_n) b_n + s(k_n) b_n^\dagger$ with

$$s^2(k) = \frac{1}{2} \left[ 1 + \frac{\dot{g}_4(k)}{W(k)} \right] = c^2(k) - 1, \quad \omega(k) = v_F |k| W(k) = v_F |k| \sqrt{(1 + \dot{g}_4(k))^2 - \dot{g}_2^2(k)},$$

where $\dot{g}_{2/4} = g_{2/4}/(2\pi v_F)$, and $E_{gs}$ denoting the ground

FIG. 1: (Color online) Time evolution of the Z-factor out of the noninteracting ground state of a 1d metallic Fermi system after switching on two-particle terms at time $t = 0$. Dashed lines show the universal asymptotic power law $t^{-\gamma_{st}(K)}$ with an exponent determined by the equilibrium LL parameter $K$. (a) TL model. The plots displays box-like two-particle interactions $g(k)$ of strength $g = g(0)$; the asymptotics are universal for any $g(k)$. Time is given in units of $(\nu v_F)^{-1}$. (b, c, Inset) Spinless lattice fermions of Eq. (6) at filling $\nu$ featuring nearest ($\Delta$) and next-nearest ($\Delta_2$) neighbor interactions.
state energy. The LL parameter and the renormalized velocity read
\[ K = \sqrt{\frac{1 + \hat{\gamma}_4(0) - \hat{\gamma}_2(0)}{1 + \hat{\gamma}_4(0) + \hat{\gamma}_2(0)}}, \quad v = v_F W(0). \tag{4} \]

As our initial state we take the noninteracting ground state \( |\text{vac}(b)\rangle \) which is given by the vacuum \( |\text{vac}\rangle \) with respect to the \( b_n \). Expectation values of the time-evolved state \( |\Psi(t)\rangle = \exp(-iHt) |\text{vac}\rangle \) can be computed straightforwardly using the simple time dependence of the eigenmode operators \( a_n^{(1)} \) and their linear dependence on the \( b_{i,n}^{(1)} \)[22].

After bosonizing the fermionic field operator [1,2] the Z-factor \( Z(t) = \lim_{t \to \infty} \langle k \rangle \) can be obtained (taking \( L \to \infty \)) \[ Z(t) = \exp \left\{ -\int_0^\infty \frac{4s^2(k) c^2(k)}{k} (1 - \cos[2\omega(k)t]) \right\}. \]

Independent of the form of \( g_{2/4}(k) \) (even for potentials with a discontinuous jump to zero at \( k_c \)) the large-time behavior is given by Eq. (4) with \( \gamma_{st} = (K^2 + K^{-2} - 2)/4 \); it manifests on the (nonuniversal) scale \( (v_F k_c)^{-1} \). Figure 2(a) shows \( Z(t) \) obtained by numerically performing the integral for a simple box shaped potential \( \hat{g}_2(k) = \hat{\gamma}_4(k) = g\Theta(k_c - |k|)/2 \) of varying amplitude \( g \). The asymptotic power-law is modulated by oscillations which decay faster than \( t^{-\gamma_{st}} \).

The kinetic energy per length \( e_{\text{kin}}(t) \) reads (\( L \to \infty \))
\[ e_{\text{kin}}(t) = \frac{v_F}{2\pi} \int_0^\infty \frac{d k k^4 s^2(k) c^2(k) \{ 1 - \cos[2\omega(k)t]\}}{k}. \tag{5} \]

The steady-state value is obtained by dropping the oscillatory term which averages out for \( t \to \infty \). For continuous coupling functions \( g_{2/4}(k) \) of range \( k_c \) asymptotic analysis yields Eq. (4) as the leading term in the long-time limit [22]; the coefficient is given by \( \epsilon(K, v) = \gamma_{st}(K) v_F/(4\pi v^2) \). Figure 2(a) shows the derivative of \( e_{\text{kin}} \) for \( \hat{g}_2(k) = \hat{\gamma}_4(k) = g(k) \), a Gaussian potential \( g(k) = g \exp(-|k/k_c|^2)/2 \) as well as a quartic potential \( g(k) = g/(1 + |k/k_c|^2)^2 \) and varying interaction strengths. As either \( g(0) \) or the lowest nonvanishing Taylor expansion order of \( g(k) - g(0) \) increases, the amplitude of an oscillatory term which decays faster than the leading one becomes stronger. The (nonuniversal) scale on which the asymptotic \( t^{-\gamma_{st}} \)-behavior dominates thus heavily depends on the strength and type of potential at hand [compare the inset and the main part of Figure 2(a)].

**Microscopic lattice model**—As a next step we provide strong evidence that Eqs. (1) describe the long-time relaxation dynamics of any model which in equilibrium falls into the LL universality class. To this end, we consider spinless lattice fermions,
\[ H = \sum_{j} \left[ \frac{1}{2} c_j^\dagger c_{j+1} + \text{H.c.} + \Delta n_j n_{j+1} + \Delta_2 n_j n_{j+2} \right], \tag{6} \]

with \( n_j = c_j^\dagger c_j - 1/2 \). We study the quench dynamics using an infinite-system DMRG algorithm [23,32]. We determine \( |E_{gs}\rangle \) by applying an imaginary time evolution \( \exp(-\tau H)|E_{gs}\rangle \) to a random initial matrix product state with a fixed matrix dimension \( \chi \) until the energy has converged to typically \( 8 - 10 \) relative digits. Operators \( \exp(-H) \) are factorized by a second or fourth order Trotter decomposition. Thereafter, we compute the real time evolution \( |\Psi(t)\rangle = \exp(-i\tau H)|E_{gs}\rangle \) in presence of the two-particle terms \( \Delta \) and \( \Delta_2 \). \( \chi \) is dynamically increased in order to maintain a fixed discarded weight. We carefully ensure that the latter is chosen small enough (and that the initial \( \chi \) is large enough) to obtain numerically exact results.

The time evolution of the momentum distribution
universal in the LL sense. The time derivative of the kinetic energy (the expectation value of $H$) $\langle E_{gs}^0 \rangle$ towards the interacting ground state $|E_{gs}\rangle$. The main part shows DMRG data for the $\tau$-derivative of the total energy in the lattice model. The $\tau^{-3}$-decay predicted by bosonization manifests over several orders of magnitude. The prefactor agrees with the TL model, one can easily access large imaginary times using DMRG, and the $\tau^{-3}$-decay manifests over several orders of magnitude. This is illustrated in Fig. 3. The prefactor (shown in the insets) agrees with $\epsilon_\text{kin}(K,v)$ (the latter depends on $K$ and $v$ only; thus, one does not need to consider ratios) for all interactions and fillings. The dynamics of the total energy at large $\tau$ is universal.

**Conclusion** — We have obtained exact expressions for the time evolution of the $Z$-factor and of the kinetic energy after an interaction quench within the Tomonaga-Luttinger model. For any continuous two-particle potential, their long-time asymptotics $Z \sim t^{-\gamma_0}$, $\text{de}_\text{kin}/dt \sim \epsilon(K,v)/t^3$ are universal functions of the LL parameter $K$ and the renormalized velocity $v$. We studied a similar scenario for spinless lattice fermions using DMRG; for large times, $Z(t)$ and $\epsilon_\text{kin}(t)$ are described by the above expressions. This provides strong evidence that the relaxation dynamics after an interaction quench within any model that falls into the equilibrium LL universality class has aspects which are universal in the LL sense.

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[1] T. Giamarchi, Quantum Physics in One Dimension (New York: Oxford University Press, 2003).
[2] K. Schönhammer in Interacting Electrons in Low Dimensions ed. by D. Baeriswyl (Dordrecht: Kluwer Academic Publishers, 2005).
[3] V. Meden, Phys. Rev. B 60, 4571 (1999).
[4] A. Imamov, T.L. Schmidt, and L.I. Glazman, arXiv:1110.1374v1.
[5] F.D.M. Haldane, Phys. Rev. Lett. 45, 1358 (1980).
[6] E. Ejima, F. Gebhard, and S. Nishimoto, Europhys. Lett. 70, 492 (2005).
[7] I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
[8] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. 83, 863 (2011).
[9] A. Faribault, P. Calabrese, and J.-S. Caux, J. Stat. Mech., P03018 (2009).
[10] J. Mossel and J.-S. Caux, New J. Phys. 12, 055028 (2010).
[11] V. Gritsev, T. Rostunov, and E. Demler, J. Stat. Mech., P05012 (2010).
[12] M. Rigol, V. Dunjko, V. Yurovsky, and M. Olshanii, Phys. Rev. Lett. 98, 050405 (2007).
[13] P. Calabrese and J. Cardy, J. Stat. Mech., P06008 (2007).
[14] M. Eckstein and M. Kollar, Phys. Rev. Lett. 100, 120404 (2008).

[15] M. Kollar and M. Eckstein, Phys. Rev. A 78, 013626 (2008).

[16] D. Rossini, A. Silva, G. Mussardo, and G.E. Santoro, Phys. Rev. Lett. 102, 127204 (2009).

[17] P. Calabrese, F.H.L. Essler, and M. Fagotti, Phys. Rev. Lett. 106, 227203 (2011).

[18] A discussion of the steady-state resulting out of single-particle parameter quench in noninteracting bosonic or fermionic systems is given in T. Barthel and U. Schollwöck, Phys. Rev. Lett. 100, 100601 (2008).

[19] M.A. Cazalilla, Phys. Rev. Lett. 97, 156403 (2006).

[20] D.M. Kennes and V. Meden, Phys. Rev. B 82, 085109 (2010).

[21] A. Mitra and T. Giamarchi, Phys. Rev. Lett. 107, 150602 (2011).

[22] J. Rentrop, D. Schuricht, and V. Meden, accepted for publication in New J. Phys. (2012), arXiv:1203.0932.

[23] To eliminate the steady-state value of \( e_{\text{kin}}(t) \) we take the derivative.

[24] S.R. Manmana, S. Wessel, R.M. Noack, and A. Muramatsu, Phys. Rev. Lett. 98, 210405 (2007).

[25] U. Schollwöck, Ann. Phys. 326, 96 (2011).

[26] S. R. White, Phys. Rev. Lett. 69, 2863 (1992).

[27] G. Vidal, Phys. Rev. Lett. 93, 040502 (2004); S. R. White and A. E. Feiguin, Phys. Rev. Lett. 93, 076401 (2004); A. Daley, C. Kollath, U. Schollwöck, and G. Vidal, J. Stat. Mech., P04005 (2004).

[28] S. Qin, M. Fabrizio, L. Yu, M. Oshikawa, and I. Affleck, Phys. Rev. B 56, 9766 (1997).

[29] C. Karrasch and J. E. Moore, in preparation.

[30] We focus on parameter regimes for which the model is a LL in equilibrium [1, 2].

[31] B. Dóra, M. Haque, and G. Zarán, Phys. Rev. Lett. 106, 156406 (2011).

[32] G. Vidal, Phys. Rev. Lett. 98, 070201 (2007).

[33] This holds as long as the interaction is smaller than a critical scale beyond which \( \omega(k) \) of Eq. 8 develops stationary points [22]. This scale depends on the form of the \( g_{2/4}(k) \).

[34] This result can straightforwardly be obtained by writing

\[
|\text{vac}(b)\rangle = \prod_{n>0} \sum_{l>0} a^{(n)}_l |l\rangle_m |l\rangle_n - |l\rangle_n,
\]

where \( |l\rangle_m \) is the \( l \)-th oscillator eigenstate with respect to the eigenmode given by \( \alpha_m^{(l)} \) and applying \( \exp(-\tau H) \). The expansion coefficients are \( a^{(n)}_l = (-1)^l [s(k_n)/c(k_n)]^l/c(k_n) \).