GLOBAL WELL-POSEDNESS FOR ONE DIMENSIONAL
CHERN-SIMONS-DIRAC SYSTEM IN $L^p$

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Abstract. Time global wellposedness in $L^p$ for the Chern-Simons-Dirac equation in 1 + 1 dimension is discussed. The two types of quadratic terms are treated, null case and non-null case. The standard iteration arguments, different settings correspond to each cases respectively, are used for the proof. For the critical case in $L^1$, the mass concentration phenomena of the solutions is denied to show the time global solvability. The intrinsic estimate plays an important role in the proof. These arguments follow the work of Candy [3].

1. CHERN-SIMONS-DIRAC EQUATION IN ONE SPATIAL DIMENSION

We consider the Cauchy problem for the Chern-Simons-Dirac equation in one spatial dimension. Let $\psi = t(\psi_1(t, x), \psi_2(t, x)) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2$ and $A = t(A_0(t, x), A_1(t, x)) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be the unknown functions subject to the Cauchy problem for the Chern-Simons-Dirac system:

\[
\begin{align*}
\frac{i}{2} \gamma^\mu (\partial_\mu - i A_\mu) \psi &= m \psi, & t > 0, x \in \mathbb{R}, \\
\partial_t A_1 - \partial_x A_0 &= \psi^* \alpha \psi, & t > 0, x \in \mathbb{R}, \\
\partial_t A_0 - \partial_x A_1 &= 0, & t > 0, x \in \mathbb{R}, \\
\psi(0, x) &= \psi_0(x), & x \in \mathbb{R},
\end{align*}
\]

where $m > 0$ is a constant, $\mu = 0, 1$ and $\gamma^\mu$ are Dirac matrices

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The initial data $\psi_0 = t(\psi_{10}(x), \psi_{20}(x))$ and $A_0 = t(A_{00}(t, x), A_{10}(t, x))$. The coupling matrix $\alpha$ can be chosen either $\gamma^0, -i\gamma^1$ or identity matrix $I$. The conjugate $^*$ means $\psi^* = t\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2)$ for $\psi = t(\psi_1, \psi_2)$.

The Chern-Simons gauge theory coupled with the Dirac particles is described by the Chern-Simons-Dirac Lagrangean density:

\[
\mathcal{L} = \frac{k}{4} \varepsilon_{\mu\nu}\rho F_{\mu\nu} A_\rho + \psi^* \gamma^\mu (\partial_\mu - i A_\mu) \psi + m \psi^* \psi,
\]
where $\varepsilon_{\mu,\nu,\rho}$ denotes the cyclic change anti-symmetric tensor. From physical point of view, 1 + 2 dimensional model is natural and important in the gauge theory. On the other hand, it is known that the one spatial dimensional model for the Maxwell-Klein-Gordon system is integrable and the solution is expressed by the initial data explicitly if we consider the massless case. As a natural extension, the one spatial dimensional Chern-Simons-Dirac system may also have a favorable mathematical structure from the mathematical point of view. Unfortunately, it is not the integrable system, we may develop some mathematical theory on the global well-posedness which has a close nature to the massless Maxwell-Klein-Gordon system.

If we neglect the mass term $m\psi$ in the Dirac part, the system maintains a scaling invariant under the transform

$$
\psi_\lambda(t, x) = \lambda \psi(\lambda t, \lambda x), \\
A_\lambda(t, x) = \lambda A(\lambda t, \lambda x)
$$

for any $\lambda > 0$. Therefore it is natural to consider the well-posedness of the system in some Hilbert scale where the above scaling left invariant. Toward this idea, Bourneaveas-Candy-Machihara [3] considered the time local well-posedness in the space $C(I; H^{-1/2+}_0) \times C(I; H^{-1/2+})$, where $H^s = H^s(\mathbb{R})$ is the Sobolev space defined by $H^s(\mathbb{R}) = \{ f \in S^*; \langle \xi \rangle^s \hat{f}(\xi) \in L^2(\mathbb{R}) \}$, where $\hat{f}$ denotes the Fourier transform of $f$. We simply denote $H^{-1/2+} = H^{-1/2++}\varepsilon(\mathbb{R})$ for any small $\varepsilon > 0$.

On the other hand, for the global well-posedness, the charge conservation law for the Dirac part is used as an a priori estimate:

$$
\|\psi_1(t)\|_2^2 + \|\psi_2(t)\|_2^2 = \|\psi_1(0)\|_2^2 + \|\psi_2(0)\|_2^2.
$$

Hence the global well-posedness is restricted in the class of $L^2(\mathbb{R})$, while the scaling invariant critical space stays in $H^{-1/2}(\mathbb{R})$ in the case (1.1).

In this paper, we choose a different approach to consider the time global well-posedness of (1.1). One simple way is to consider the Banach space to construct the time local solution. Thanks to the good structure of the system, we may derive the time global a priori estimate in $L^p$, where $2 \leq p \leq \infty$ and we may show the global existence and well-posedness for the solution to (1.1) for any large data. Besides, if we restrict ourselves to particular nonlinear coupling cases $\alpha = \gamma^0$ or $\alpha = -i \gamma^1$, we may reach much further. Indeed, if we choose $\alpha = \gamma^0$ or $\alpha = -i \gamma^1$ then the nonlinear coupling term hold the null structure and the well-posedness result is improved.

If the coupling term is given by either of the matrices $\alpha = \gamma^0$ or $\alpha = -i \gamma^1$, then the coupling term is less interactive and it satisfies the null condition. In this case we may have the well-posedness result up to the critical Banach scale:

**Theorem 1.1** (the null case). Let $1 \leq p \leq \infty$ and $\alpha = \gamma^0$ or $\alpha = -i \gamma^1$ in (1.1). For any $\psi_0 \in L^p(\mathbb{R})$ and $A_0 \in L^p(\mathbb{R})$, there exists a global weak solution $(\psi, A) \in C([0, \infty); L^p) \times C([0, \infty); L^p)$ to (1.1) such that
(1) the solution is unique in
\[ \psi \in C([0, \infty); L^p), \quad \Lambda \in C([0, \infty); L^p), \]
\[ \psi^* \alpha \psi \in L^p_{\text{loc}}([0, \infty); L^p), \quad \Lambda \psi \in L^p_{\text{loc}}([0, \infty); L^p). \]

(2) The map from the data \((\psi_0, \Lambda_0)\) to the solution \((\psi, \Lambda)\) is Lipschitz continuous from \(L^p \times L^p\) to \(C([0, \infty); L^p) \times C([0, \infty); L^p)\).

We state next that the nonlinear coupling is given by \(\alpha = I\):

**Theorem 1.2** (the non-null case). Let \(1 \leq p \leq \infty\) and \(\alpha = I\) in (1.1). For any \(\psi_0 \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})\) and \(\Lambda_0 \in L^p(\mathbb{R})\), there exists a unique global weak solution \((\psi, \Lambda) \in C([0, \infty); L^p \cap L^\infty) \times C([0, \infty); L^p)\) to (1.1). The map from the data \((\psi_0, \Lambda_0)\) to the solution \((\psi, \Lambda)\) is Lipschitz continuous from \((L^p \cap L^\infty) \times L^p\) to \(C([0, \infty); L^p \cap L^\infty) \times C([0, \infty); L^p)\).

**Remark.** The interesting case in the above theorem is the scaling critical setting \(p = 1\), where the norm of the space maintains the scaling (1.3) invariant. This stands for the space is critical for the local and global well-posedness for the system (1.1). We note that neither the global well-posedness nor local well-posedness is yet established in the critical Sobolev space \(H^{-1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})\). It is also worth to remark that there is no any inclusion relation between the spaces \(L^1(\mathbb{R})\) and \(H^{-1/2}(\mathbb{R})\), while if we consider the case \(p > 1\) then it holds that \(L^p(\mathbb{R}) \subset H^{-1/2+\varepsilon}(\mathbb{R})\) where \(\varepsilon > 0\) is depending on \(p\).

The main part of our result is to consider the mass concentration phenomena. Since we treat the scaling critical case, it is required to show that the mass concentration phenomena; namely for any \(\delta > 0\)
\[ \liminf_{t \to T} \left( \int_{B_\delta(x_0)} |\psi(t, x)| \, dx + \int_{B_\delta(x_0)} |\Lambda(t, x)| \, dx \right) \geq \varepsilon_0 \]
for some \(\varepsilon_0 > 0\) for the solution \(\psi, \Lambda\) does not occur in \(L^1\). In order to show this, we derive the non-concentration lemma which is possible in aid of the intrinsic \(L^\infty\) estimate for the nonlinear part of the solution. This idea originally goes back to the \(L^\infty\) estimate due to Delgado [5]. Candy [4] extended this idea to the nonlinear component for the Dirac part and applied for the global well-posedness of the 1 dimensional Dirac equation with cubic nonlinearity in \(L^2(\mathbb{R})\). We extend the intrinsic \(L^\infty\) estimate to the Lebesgue spaces \(L^p\), where \(1 \leq p < \infty\). For this estimate, we essentially use the null structure of the nonlinear couplings for both Dirac and gauge equations.

We give the notation here. For any Banach space \(X\) with respect to \(x\) variable (space variable), we denote the Bochner space \(L^p(0, T; X)\) by the abbreviation \(L^p_T X\) for \(T > 0\) (see section 3). Then the local well-posedness of the system can be shown in \(L^p\) including the critical case \(p = 1\).
under the smallness condition on the data. The finite time propagation of the solution and the intrinsic $L^\infty$ estimate shows the existence of the large data solution in time globally.

Before closing this section, we state the each roles for the sections below. In section 3, we show the time local solution for small data by using standard iteration argument. In section 4, we discuss the finite speed of propagation of solutions to the transport equation, which is maybe well-known among the researchers of wave equation. Thanks to this property, we will remove the smallness assumption for the data to have time local solution. In section 5, we show that the solutions never concentrate anywhere, and we can extend those solutions to global time.

2. Bilinear estimate in $L^p$

For the proof, we reduce the system (1.1) into the diagonal form for the principal part. We let $\psi = \psi_1 \pm \psi_2$ and $A = A_0 \mp A_1$ to rewrite for the system (1.1) as

$$
\begin{align*}
\partial_t \psi_\pm \mp \partial_x \psi_\pm &= i A_\pm \psi_\pm - im\psi_\mp, \quad t \in \mathbb{R}, x \in \mathbb{R}, \\
\partial_t A_\pm \pm \partial_x A_\pm &= \mp P(\psi_+, \psi_-), \quad t \in \mathbb{R}, x \in \mathbb{R}, \\
\psi_\pm(0, x) &= \psi_{10}(x) \mp \psi_{20}(x), \quad x \in \mathbb{R}, \\
A_\pm(0, x) &= A_{00}(0) \mp A_{110}(0), \quad x \in \mathbb{R},
\end{align*}
$$

(2.1)

where

$$
P(\psi_+, \psi_-) = \begin{cases} 
\text{Re}(\psi_+ \bar{\psi}_-^*), & \alpha = \gamma^0, \\
\text{Im}(\psi_+ \bar{\psi}_-^*), & \alpha = -i \gamma^1, \\
(|\psi_+|^2 + |\psi_-|^2)/2, & \alpha = I.
\end{cases}
$$

(2.2)

We give an explicit formula for the solution for the inhomogeneous transport equation.

**Lemma 2.1.** Let $f_0$ and $F = F(t, x)$ be a locally integrable functions. Then the solution to the transport equation

$$
\begin{align*}
\begin{cases} 
\partial_t u_\pm \pm \partial_x u_\pm &= F_\pm(t, x), \\
u_\pm(0) &= u_{\pm0}
\end{cases}
\end{align*}
$$

(2.3)

is given by

$$
u_\pm(t, x) = u_{\pm0}(x \mp t) + \int_0^t F_\pm(s, x \mp (t - s)) ds,
$$

(2.4)

where the signs $\pm$ are the same in upper case and lower case, respectively. Besides if $u_{\pm0} \in L^p$ and $F \in L^1(0, T; L^p)$, then $u_\pm \in C([0, T]; L^p)$.

We set the following the characteristic function on the triangle area. For $x_0 \in \mathbb{R}$ and $R > 0$,

$$
\chi_{\Omega_R(x_0)}(t, x) = \begin{cases} 
1, & (t, x) \in \Omega_R(x_0) \\
0, & (t, x) \notin \Omega_R(x_0)
\end{cases}
$$

where

$$
\Omega_R(x_0) = \{(t, x) \in \mathbb{R}^2 : |x - x_0| \leq R - t, 0 \leq t \leq R\}.
$$
We also set the characteristic function on the interval,

\[ \chi_{I_R(x_0)}(x) = \begin{cases} 1, & x \in I_R(x_0) \\ 0, & x \notin I_R(x_0) \end{cases} \]

where

\[ I_R(x_0) = \{ x \in \mathbb{R} : |x - x_0| \leq R \} \]

From (2.4) and the influence region, we can put the characteristic function into the integral. For either cases \( \pm \) in (2.4),

\[ \chi_{\Omega_R(x_0)}(t,x)u_\pm(t,x) = \chi_{\Omega_R(x_0)}(t,x)u_{\pm 0}(x \mp t) + \chi_{\Omega_R(x_0)}(t,x) \int_0^t F_\pm(s,x \mp (t-s))ds \]

(2.5) \[ = \chi_{\Omega_R(x_0)}(t,x)(\chi_{\Omega_R(x_0)}u_{\pm 0})(x \mp t) + \chi_{\Omega_R(x_0)}(t,x) \int_0^t (\chi_{\Omega_R(x_0)}F_\pm)(s,x \mp (t-s))ds \]

We show a bilinear estimate for the inhomogeneous transport equation, which plays an important role for the contraction mapping method. The estimate below with \( p = 2 \) was shown in [10]. The other cases \( 0 < p \leq \infty \) is shown in the same way.

**Lemma 2.2.** For \( 0 < p \leq \infty \), let \( F_\pm \in L^p(\mathbb{R}) \), with initial data \( u_{\pm 0} \in L^p \). Then the solutions \( u_+, u_- \) to the transport equation

\[ \begin{cases} \partial_t u_\pm \pm \partial_x u_\pm = F_\pm(t,x), \\ u_\pm(0) = u_{\pm 0} \end{cases} \]

are subject to the following estimate: For \( T > 0 \),

\[ \|u_+ - u_-\|_{L^p_T L^p} \leq \left( \frac{1}{2} \right)^{\frac{1}{p}} \left( \|u_{\pm 0}\|_{L^p} + \int_0^T \|F_+\|_{L^p} ds \right) \left( \|u_{\pm 0}\|_{L^p} + \int_0^T \|F_-\|_{L^p} ds \right). \]

From (2.5), we also have the bilinear estimates on the restricted domains,

\[ \|\chi_{\Omega_R(x_0)}u_+ - u_-\|_{L^p_T L^p} \leq \left( \frac{1}{2} \right)^{\frac{1}{p}} \left( \|\chi_{\Omega_R(x_0)}u_{\pm 0}\|_{L^p} + \int_0^T \|\chi_{\Omega_R(x_0)}F_+(s)\|_{L^p} ds \right) \cdot \left( \|\chi_{\Omega_R(x_0)}u_{\pm 0}\|_{L^p} + \int_0^T \|\chi_{\Omega_R(x_0)}F_-(s)\|_{L^p} ds \right). \]

**Proof of Lemma 2.2** From Lemma 2.1 we have for \( u_+ \) and \( u_- \) that

\[ u_\pm(t,x) = u_{\pm 0}(x \mp t) + \int_0^t F(s,x \mp t \pm s)ds. \]

Changing the variable \( x - t = y, x + t = z \), we see

\[ \|u_{\pm 0}(x-t)u_{\pm 0}(x+t)\|_{L^p_T L^p} \leq \left( \frac{1}{2} \right)^{\frac{1}{p}} \|u_{\pm 0}\|_{L^p} \|u_{\pm 0}\|_{L^p}. \]
We remark that the above estimate holds for $p = \infty$ with $(\frac{1}{2})^{\frac{1}{p}} = 1$. Analogously
\[
\left\| u_{+0}(x-t) \int_0^t F_-(s, x + t - s)ds \right\|_{L^p_x L^p_t} \leq \left( \int_0^T \left\| u_{+0}(x-t) F_-(s, x + t - s) \right\|_{L^p_x L^p_t} ds \right)^{\frac{1}{p}} \left( \int_0^T \| F_-(s, x - t + s) \|_{L^p_x L^p_t} ds \right)^{\frac{1}{p}}.
\]
The estimate for the term $u_{-0}(x+t) \int_0^t F_+(s, x - t + s)ds$ is similarly obtained. For the product of the external force term, we see in a similar way as
\[
\left\| \int_0^T \int_0^T \| F_+(s, x - t + s) F_-(s', x - t + s') \|_{L^p_x L^p_t} ds \right\|_{L^p_x L^p_t} \leq \left( \int_0^T \int_0^T \| F_+(s, s - z_+) F_-(s', s' - z_-) \|^p dz_+ dz_- \right)^{\frac{1}{p}} \int_0^T \int_0^T \| F_-(s', s - z_-) \|^p dz_+ dz_- \int_0^T \| F_+(s', s + z_+) \|^p d_+ dz_-
\]
Those estimates imply the desired bound (2.6).

We apply these estimates above to (2.5) to derive (2.7). □

3. Finite Speed of Propagation

We introduce the well known fact, that is, the finite speed of propagation of wave and transport equations.

Lemma 3.1. Suppose $(u_\pm, v_\pm) \in L^\infty_t L^\infty_x \times L^\infty_t L^\infty_x$ be a solution to (2.3), so satisfying
\begin{align}
(3.8) & \quad u_\pm(t, x) = u_{\pm0}(x \mp t) + \int_0^t F(u_\pm, v_\pm)(s, x \mp (t-s))ds, \\
(3.9) & \quad v_\pm(t, x) = v_{\pm0}(x \mp t) + \int_0^t G(u_\pm, v_\pm)(s, x \mp (t-s))ds,
\end{align}
respectively, where $F(u_\pm, v_\pm)$ and $G(u_\pm, v_\pm)$ are any quadratic form like $uu, uv, vv$, where $u$ and $v$ stand for any $u_+, u_-$ and $v_+, v_-$ respectively. If for some $x_0 \in \mathbb{R}$ and $R > 0$,
\[
u_{\pm0}(x) = v_{\pm0}(x) = 0, \quad |x - x_0| < R,
\]
then for any $0 < t < R$,
\[
u_\pm(t, x) = v_\pm(t, x) = 0, \quad |x - x_0| < R - t.
\]
In particular if $u_{\pm0} \equiv 0$ on $|x - x_0| < R$, then $u_{\pm}(t, x) \equiv 0$ in $|x - x_0| < R - t$. 

\[6\]
Proof of Lemma 3.1. Fix $x_0$ and $R$. For any $0 < t < R$, we define the following norm which is the supremum on the triangle $\Delta = \Delta(x_0, R)$,

$$\|f(t)\|_\Delta := \sup_{\{x:|x-x_0|<R-t\}} |f(t, x)|.$$ 

For any $0 < s < t$, we have

$$\{|x-t+s-x_0|<R-s\} \supset \{|x-x_0|<R-t\},$$

$$\{|x+t-s-x_0|<R-s\} \supset \{|x-x_0|<R-t\}.$$

We consider the integral equation (3.8) with $F(u_\pm, v_\pm) = u_\pm v_\pm$ but the initial data disappear from the assumption,

$$\|u_\pm(t)\|_\Delta \leq \sup_{\{x:|x-x_0|<R-t\}} \int_0^t |(u_\pm v_\pm)(s, x - t + s)| ds$$

$$\leq \int_0^t \sup_{\{x:|x-t+s-x_0|<R-s\}} |(u_\pm v_\pm)(s, x - t + s)| ds$$

$$\leq \|v_\pm\|_{L^\infty L^\infty} \int_0^t \|u_\pm(s)\|_\Delta ds.$$

Then Gronwall’s inequality implies $\|u_\pm(t)\|_\Delta = 0$. Other cases of the quadratic coupling follow similarly. 

From this argument of the proof of lemma, we may say if two solutions $(u_\pm^{(1)}, v_\pm^{(1)}), (u_\pm^{(2)}, v_\pm^{(2)})$ are coincide when initial time $u^{(1)}_{\pm0} = u^{(2)}_{\pm0}, v^{(1)}_{\pm0} = v^{(2)}_{\pm0}$ where $|x-x_0|<R$, then the solutions coincide $u^{(1)}_\pm = u^{(2)}_\pm$, $v^{(1)}_\pm = v^{(2)}_\pm$ where $|x-x_0|<R-t$. We check this for the case $F(u_\pm, v_\pm) = G(u_\pm, v_\pm) = u_\pm v_\pm$ only here.

$$\|u^{(1)}_\pm(t) - u^{(2)}_\pm(t)\|_\Delta \leq \sup_{\{x:|x-x_0|<R-t\}} \int_0^t |(u^{(1)}_\pm v^{(1)}_\pm - u^{(2)}_\pm v^{(2)}_\pm)(s, x - t + s)| ds$$

$$\leq \|u^{(1)}_\pm\|_{L^\infty L^\infty} \int_0^t \|v^{(1)}_\pm(s) - v^{(2)}_\pm(s)\|_\Delta ds + \|v^{(2)}_\pm\|_{L^\infty L^\infty} \int_0^t \|u^{(1)}_\pm(s) - u^{(2)}_\pm(s)\|_\Delta ds.$$

The same estimate gives $\|v^{(1)}_\pm(t) - v^{(2)}_\pm(t)\|_\Delta$ is bounded by the same right hand side of this. Add up of these two estimates and apply Gronwall’s inequality to get the conclusion.

4. INTRINSIC $L^\infty$ ESTIMATE FOR DIRAC PART

In this section, we show the $L^\infty$ estimate for the solution of the Dirac part of the system. We consider the diagonal form of the system (2.1), (2.2).

We introduce the decomposition for the solution of the Dirac equation $\psi_\pm$ such as

$$\psi_\pm = \psi_{L\pm} + \psi_{N\pm},$$
where each component of solution satisfies the following equations, respectively:

\[
\begin{align*}
\partial_t \psi_{L\pm} \pm \partial_x \psi_{L\pm} &= iA_\pm \psi_{L\pm}, & t \in \mathbb{R}, x \in \mathbb{R}, \\
\partial_t \psi_{N\pm} \pm \partial_x \psi_{N\pm} &= iA_\mp \psi_{N\pm} - im\psi_\mp, & t \in \mathbb{R}, x \in \mathbb{R}, \\
\partial_t A_\pm \pm \partial_x A_\pm &= -P(\psi_\pm, \psi_\mp), & t \in \mathbb{R}, x \in \mathbb{R}, \\
\psi_{L\pm}(0) &= \psi_{\pm 0}, \quad \psi_{N\pm}(0) = 0,
\end{align*}
\]

(4.1)

where the nonlinear coupling is given by any of \((2.2)\). We then show that the nonlinear coupling part \(\psi_N^\pm\) satisfies the intrinsic \(L^\infty\) bound as follows:

**Proposition 4.1** (Intrinsic \(L^\infty\) estimate). Let \(\psi_{L\pm}(t, x), \psi_{N\pm}(t, x)\) be smooth functions solving the above system \((4.1)\). Then it satisfies that

\[
|\psi_{L\pm}(t, x)| = |\psi_{\pm 0}(x \mp t)|
\]

and

\[
\|\psi_{N+}(t)\|_{L^p} + \|\psi_{N-}(t)\|_{L^p} \leq m\left(\|\psi_{+0}\|_{L^p} + \|\psi_{-0}\|_{L^p}\right)(e^{mt} + t - 1).
\]

**Corollary 4.2.** Let \(\psi_{\pm}(t, x)\) be smooth functions solving the system \((2.1), (2.2)\). Then it satisfies that

\[
\|\psi_{\pm}(t)\|_{L^p} \leq C\left(\|\psi_{+0}\|_{L^p} + \|\psi_{-0}\|_{L^p}\right)(e^{mt} + t).
\]

**Proof of Proposition 4.1.** Suppose that \((\psi_{L\pm}, \psi_{N\pm})\) is smooth function and satisfies the above system: Then noticing

\[
\partial_t |\psi_{L\pm}|^2 + \partial_x |\psi_{L\pm}|^2 = 2\Re(\overline{\psi}_{L\pm}(\partial_t \psi_{L\pm} + \partial_x \psi_{L\pm})) = 2\Re(iA_-|\psi_{L\pm}|^2) = 0,
\]

we deduce

\[
|\psi_{L\pm}(t, x)| = |\psi_{\pm 0}(x - t)|.
\]

Similarly

\[
\partial_t |\psi_{N\pm}|^2 + \partial_x |\psi_{N\pm}|^2 = 2\Re(iA_-|\psi_{N\pm}|^2 - im\psi_{N\mp}) = 2\text{Im}(\psi_{N\mp}).
\]

While

\[
\partial_t |\psi_{N\pm}|^2 + \partial_x |\psi_{N\pm}|^2 = 2|\psi_{N\mp}|(\partial_t |\psi_{N\mp}| + \partial_x |\psi_{N\mp}|).
\]

Hence if \(|\psi_{N\mp}| \neq 0\), then

\[
\partial_t |\psi_{N\mp}| + \partial_x |\psi_{N\mp}| = m\text{Im}\left(\frac{\psi_{-\psi_{N\pm}}}{|\psi_{N\mp}|}\right).
\]

Form Lemma \((2.1)\) it follows that

\[
|\psi_{N\mp}(t, x)| = m\int_0^t \text{Im}\left(\frac{\psi_{-\psi_{N\mp}}}{|\psi_{N\mp}|}\right)(s, x - t + s)ds.
\]

(4.6)

Analogously for \(\psi_-\),

\[
|\psi_{L\mp}(t, x)| = |\psi_{-0}(x + t)|,
\]

\[
|\psi_{N\mp}(t, x)| = m\int_0^t \text{Im}\left(\frac{\psi_{+\psi_{N\mp}}}{|\psi_{N\mp}|}\right)(s, x + t - s)ds.
\]
By the Hölder inequality,
\[
|\psi_{N+}(t, x)| \leq m \int_0^t |\psi_-(s, x - t + s)| ds
\]
\[
\leq m \int_0^t |\psi_0(2s + x - t)| + |\psi_{N-}(s, x - t + s)| ds
\]
\[
\leq mt^{1/p'}\|\psi_0\|_{L^p} + m \int_0^t \|\psi_{N-}(s)\|_{L_x^\infty} ds,
\]
where $\frac{1}{p} + \frac{1}{p'} = 1$. Applying similar estimate for $\psi_-$, we derive that
\[
\|\psi_{N+}(t)\|_{L_x^\infty} + \|\psi_{N-}(t)\|_{L_x^\infty} \leq mt^{1/p'}(\|\psi_0\|_{L^p} + \|\psi_0\|_{L^p})
\]
\[
+ m \int_0^t \|\psi_{N+}(s)\|_{L_x^\infty} + \|\psi_{N-}(s)\|_{L_x^\infty} ds.
\]
Gronwall’s inequality now implies that
\[
\|\psi_{N+}(t)\|_{L_x^\infty} + \|\psi_{N-}(t)\|_{L_x^\infty} \leq m(\|\psi_0\|_{L^p} + \|\psi_0\|_{L^p})(e^{mt} + t - 1).
\]
From (4.6), we also have for $1 \leq p < \infty$ that
\[
\|\psi_{N+}(t)\|_{L^p} \leq mt\|\psi_0\|_{L^p} + \int_0^t \|\psi_{N-}(s)\|_{L^p} ds
\]
and hence we obtain
\[
\|\psi_{N+}(t)\|_{L_x^p} + \|\psi_{N-}(t)\|_{L_x^p}
\]
\[
\leq mt(\|\psi_0\|_{L^p} + \|\psi_0\|_{L^p}) + \int_0^t \|\psi_{N+}(s)\|_{L_x^p} + \|\psi_{N-}(s)\|_{L_x^p} ds.
\]
We again conclude that
\[
\|\psi_{N+}(t)\|_{L_x^p} + \|\psi_{N-}(t)\|_{L_x^p} \leq m(\|\psi_0\|_{L^p} + \|\psi_0\|_{L^p})(e^t - 1).
\]
Combining (4.8) and (4.10), we obtain the desired estimate. \qed

We should note that the above estimates in Proposition 4.1 are available with any nonlinear coupling with the Chern-Simons gauge part, $\alpha = \gamma^0, -i\gamma^1$ and $I$.

5. **Time global well-posedness for the null case**

In this section, we give the proof of Theorem 1.1. Here we consider the null interaction case $\alpha = \gamma^0$ i.e.,
\[
\begin{aligned}
\partial_t \psi_{\pm} \pm \partial_x \psi_{\pm} &= iA \psi_{\pm} - im\psi_{\mp}, & t \in \mathbb{R}, x \in \mathbb{R},

\partial_t A_{\pm} \pm \partial_x A_{\pm} &= \mp\text{Re}(\psi_{\pm} \bar{\psi}_{\mp}), & t \in \mathbb{R}, x \in \mathbb{R},

(\psi_{\pm}, A_{\pm})|_{t=0} &= (\psi_{\pm,0}, A_{\pm,0}).
\end{aligned}
\]
The other case $\alpha = -i\gamma^1$ is considered as an equivalent form and can be estimated in a similar manner. We give the proof for the subcritical case $L^p, p > 1$ and the critical case $L^1$ separately.
5.1. Null and subcritical case. Our aim in this subsection is the following,

**Theorem 5.1.** Let \( \alpha = \gamma^0 \) or \(-i\gamma^1\). Let \( 1 < p \leq \infty \). For any \((\psi_{\pm 0}, A_{\pm 0}) \in L^p \times L^p\), there exists a global weak solution \((\psi_{\pm}, A_{\pm}) \in C([0, \infty); L^p) \times C([0, \infty); L^p)\) to (5.1) such that the solution is unique in

\[
\psi_{\pm} \in C([0, \infty); L^p), \quad A_{\pm} \in C([0, \infty); L^p),
\]

\[
A_{\pm} \psi_{\pm} \in L^p_{loc}(0, \infty; L^p), \quad \psi_{\pm} \tilde{\psi}_{\mp} \in L^p_{loc}(0, \infty; L^p).
\]

The map from the data \((\psi_{\pm 0}, A_{\pm 0})\) to the solution \((\psi_{\pm}, A_{\pm})\) is Lipschitz continuous from \(L^p \times L^p\) to \(C([0, \infty); L^p) \times C([0, \infty); L^p)\).

**Proof of Theorem 5.1.** We set \(1 < p \leq \infty\). We consider the following recurrence of successive approximation of the solution. Let \(\{\psi^{(n)}_{\pm}, A^{(n)}_{\pm}\}_{n=1,2,...}\) solve

\[
\begin{aligned}
\partial_t \psi^{(n+1)}_{\pm} + \partial_x \psi^{(n+1)}_{\pm} &= iA^{(n)}_{\pm} \psi^{(n)}_{\pm} - i\frac{m}{\alpha} \psi^{(n)}_{\pm}, \\
\partial_t A^{(n+1)}_{\pm} + \partial_x A^{(n+1)}_{\pm} &= \mp \text{Re}(\psi^{(n)}_{\pm} \tilde{\psi}^{(n)}_{\pm}), \\
(\psi^{(n)}_{\pm}, A^{(n)}_{\pm})|_{t=0} &= (\psi_{\pm 0}, A_{\pm 0})
\end{aligned}
\]

with the first step \((\psi^{(0)}_{\pm}, A^{(0)}_{\pm}) = (0, 0)\). Now we consider the integral equation derived from Lemma 2.1

\[
\psi^{(n+1)}_{\pm}(t, x) = \psi_{\pm 0}(x \mp t) + i \int_0^t (A^{(n)}_{\pm} \psi^{(n)}_{\pm} - m\psi^{(n)}_{\pm})(s, x \mp t + s) ds,
\]

\[
A^{(n+1)}_{\pm}(t, x) = A_{\pm 0}(x \mp t) - \int_0^t (\text{Re}(\psi^{(n)}_{\pm} \tilde{\psi}^{(n)}_{\pm}))(s, x \mp t + s) ds.
\]

We set

\[
M = 2(\|\psi_{+0}\|_{L^p} + \|\psi_{-0}\|_{L^p} + \|A_{+0}\|_{L^p} + \|A_{-0}\|_{L^p})
\]

and assume that \(M\) is sufficiently small. We then recall the function space defined by

\[
X^p_T \equiv \left\{ (\phi, \mathbb{B}) \in C([0, T); L^p) \times C([0, T); L^p); \quad \|\phi\|_{L^p_T L^p} + \|\mathbb{B}\|_{L^p_T L^p} \leq M \right\}.
\]

We show the following bound,

\[
\|\psi^{(n)}_{\pm}\|_{L^p_T L^p}, \quad \|A^{(n)}_{\pm}\|_{L^p_T L^p} \leq M
\]

for all \(n = 1, 2, ...\). Besides

\[
\|\psi^{(n)}_{\pm} A^{(n)}_{\pm}\|_{L^p_T L^p}, \quad \|\psi^{(n)}_{\pm} \tilde{\psi}^{(n)}_{\pm}\|_{L^p_T L^p} \leq M^2
\]

for all \(n = 1, 2, ...\). We show (5.6) and (5.7) by induction with respect to \(n\). It is obvious with \(n = 0\). It follows from the first equation of the integral form (5.4) and Hölder inequality that

\[
\|\psi^{(n+1)}_{\pm}\|_{L^p_T L^p} \leq \|\psi_{+0}\|_{L^p} + \|A^{(n)}_{-\pm} \psi^{(n)}_{\pm}\|_{L^p_T L^p} + mT\|\psi^{(n)}_{-}\|_{L^p_T L^p}
\]

\[
\leq \frac{M}{2} + T^{1-\frac{1}{p}} M^2 + mTM \leq M.
\]
The last inequality holds for sufficiently small $T$. Similarly one may obtain $\|\psi^{(n+1)}\|_{L^\infty_t L^p_x} \leq M$. We also have for small $T$ that
\[
\|A_+^{(n+1)}\|_{L^\infty_t L^p_x} \leq \|A_{0+}\|_{L^p_x} + \|\psi_+^{(n)}\|_{L^1_t L^p_x} \leq \frac{M}{2} + T^{1-\frac{1}{p}} M^2 \leq M
\]
and similarly $\|A_-^{(n+1)}\|_{L^\infty_t L^p_x} \leq M$. According to Lemma 2.2 we then obtain that
\[
\|\psi_+^{(n+1)} A_-^{(n+1)}\|_{L^p_x} \leq \left(\frac{1}{2}\right)^\frac{1}{p} \left( \|\psi_0\|_{L^p_x} + \|A_-^{(n)} \psi_+^{(n)}\|_{L^1_t L^p_x} + mT \|\psi_-^{(n)}\|_{L^\infty_t L^p_x} \right)
\times \left( \|A_{-0}\|_{L^p_x} + \|\psi_+^{(n)} \psi_-^{(n)}\|_{L^1_t L^p_x} \right)
\leq \left(\frac{1}{2}\right)^\frac{1}{p} \left( \frac{M}{2} + T^{1-\frac{1}{p}} M^2 + mT \right) \left( \frac{M}{2} + T^{1-\frac{1}{p}} M^2 \right) \leq M^2
\]
for small $T > 0$. Analogously we obtain $\|\psi_+^{(n+1)} A_+^{(n+1)}\|_{L^p_x} \leq M^2$. Furthermore we see that
\[
\|\psi_+^{(n+1)} \psi_-^{(n+1)}\|_{L^p_x} \leq \left(\frac{1}{2}\right)^\frac{1}{p} \left( \|\psi_0\|_{L^p_x} + \|A_+^{(n)} \psi_+^{(n)}\|_{L^1_t L^p_x} + mT \|\psi_+^{(n)}\|_{L^\infty_t L^p_x} \right)
\times \left( \|\psi_0\|_{L^p_x} + \|A_-^{(n)} \psi_-^{(n)}\|_{L^1_t L^p_x} + mT \|\psi_-^{(n)}\|_{L^\infty_t L^p_x} \right)
\leq \left(\frac{1}{2}\right)^\frac{1}{p} \left( \frac{M}{2} + T^{1-\frac{1}{p}} M^2 + mT \right)^2 \leq M^2
\]
for small $T > 0$. We have obtained (5.6) and (5.7) for any $n$.

We next estimate the difference $\psi_+^{(n+1)} - \psi_+^{(n)}$ and $A_+^{(n+1)} - A_+^{(n)}$. We set
\[
d((\psi_+^{(n)}, A_+^{(n)}), (\phi_+^{(n)}, B_+^{(n)})) = \|\psi_+ - \phi_+\|_{L^\infty_t L^p_x} + \|\psi_+ A_+ - \phi_+ B_+\|_{L^p_x} + \|A_+ - B_+\|_{L^\infty_t L^p_x} + \|\psi_+ \psi_+ - \phi_+ \phi_+\|_{L^p_x}
\]
We will show $d((\psi_+^{(n+1)}, A_+^{(n+1)}), (\psi_+^{(n)}, A_+^{(n)})) \leq \frac{1}{2^n}$, that is
\[
\|\psi_+^{(n+1)} - \psi_+^{(n)}\|_{L^p_x} + \|\psi_+^{(n+1)} A_+^{(n+1)} - \psi_+^{(n)} A_+^{(n)}\|_{L^p_x} + \|A_+^{(n+1)} - A_+^{(n)}\|_{L^\infty_t L^p_x} + \|\psi_+^{(n+1)} \psi_+^{(n+1)} - \psi_+^{(n)} \psi_+^{(n-1)}\|_{L^p_x} \leq \frac{1}{2^n},
\] (5.8)
for small $T > 0$ by induction. Indeed
\[
\|\psi_+^{(n+1)} - \psi_+^{(n)}\|_{L^p_x} \leq T^{1-\frac{1}{p}} \|A_-^{(n)} \psi_+^{(n)} - A_-^{(n-1)} \psi_+^{(n-1)}\|_{L^p_x} + mT \|\psi_+^{(n)} - \psi_+^{(n-1)}\|_{L^\infty_t L^p_x}
\]
and
\[
\|A_+^{(n+1)} - A_+^{(n)}\|_{L^\infty_t L^p_x} \leq T^{1-\frac{1}{p}} \|\psi_+^{(n)} \psi_+^{(n)} - \psi_+^{(n)} \psi_+^{(n-1)}\|_{L^p_x}
\]
We estimate by the triangle inequality,
\[
\|\psi_+^{(n+1)} A_-^{(n+1)} - \psi_+^{(n)} A_-^{(n)}\|_{L^p_x} \leq \|\psi_+^{(n+1)} (A_-^{(n+1)} - A_-^{(n)})\|_{L^p_x} + \|A_-^{(n)} (\psi_+^{(n+1)} - \psi_+^{(n)})\|_{L^p_x}
\]
and estimate each term by Lemma 2.2
\[
\|\psi_+^{(n+1)} (A_-^{(n+1)} - A_-^{(n)})\|_{L^p_x}
\leq \left(\frac{1}{2}\right)^\frac{1}{p} \left( \|\psi_0\|_{L^1} + \|A_-^{(n)} \psi_+^{(n)}\|_{L^1_t L^p_x} + mT \|\psi_-^{(n)}\|_{L^\infty_t L^p_x} \right) T^{1-\frac{1}{p}} \|\psi_+^{(n)} \psi_+^{(n)} - \psi_+^{(n)} \psi_+^{(n-1)}\|_{L^p_x}
\leq \left(\frac{1}{2}\right)^\frac{1}{p} \left( \frac{M}{2} + mTM \right) T^{1-\frac{1}{p}} \|\psi_+^{(n)} \psi_+^{(n)} - \psi_+^{(n)} \psi_+^{(n-1)}\|_{L^p_x},
\]
and
\[ \|A_-(n)(\psi^+(n+1) - \psi^+(n))\|_{L^p_t L^p} \]
\[ \leq \frac{1}{2} \left( \frac{M}{2} + M^2 \right) (T^{1-\frac{2}{p}} \|A_-(n)\|_{L^p_t L^p} + mT \|\psi^+ - \psi^-(n-1)\|_{L^p_t L^p}). \]

We estimate by the triangle inequality,
\[ \|\psi^+(n+1) - \psi^+(n)\|_{L^p_t L^p} \leq \|\psi^+(n+1) - \psi^-(n)\|_{L^p_t L^p} + \|\psi^+(n) - \psi^+(n)\|_{L^p_t L^p} \]
and estimate each term by Lemma 2.2
\[ \|\psi^+(n+1) - \psi^+(n)\|_{L^p_t L^p} \leq \frac{1}{2} \left( \frac{M}{2} + M^2 + mTM \right) (T^{1-\frac{2}{p}} \|A^+(n) - \psi^-(n-1)\|_{L^p_t L^p} + mT \|\psi^+ - \psi^+(n-1)\|_{L^p_t L^p}). \]

We have obtained (5.8) for any \( n \) for small \( T \). Therefore we have obtained the time local existence of solution. Here we remark the existence time for small data. Next we discuss the uniqueness of solutions. We suppose two solutions \((\psi_\pm, A_\pm), (\phi_\pm, B_\pm)\) with same initial data to (5.1) satisfy the condition (5.2). Then there exists \( M_1, M_2 \) such that
\[ \|\psi_\pm\|_{L^\infty_t L^p}, \|A_\pm\|_{L^\infty_t L^p} \leq M_1, \|A_\pm\|_{L^p_t L^p}, \|\psi_\pm\|_{L^p_t L^p} \leq M_1, \]
\[ \|\phi_\pm\|_{L^\infty_t L^p}, \|B_\pm\|_{L^\infty_t L^p} \leq M_2, \|\phi_\pm\|_{L^p_t L^p}, \|\phi_\pm\|_{L^p_t L^p} \leq M_2. \]
We then derive the following from the same estimates which have concluded (5.8),
\[ d((\psi_\pm, A_\pm), (\phi_\pm, B_\pm)) \leq \frac{1}{2} d((\psi_\pm, A_\pm), (\phi_\pm, B_\pm)) \]
for small \( T > 0 \), which implies the two solutions coincide to each other.

If we set the different initial data \((\psi_{\pm 0}, A_{\pm 0})\) and \((\phi_{\pm 0}, B_{\pm 0})\) for \((\psi_\pm, A_\pm)\) and \((\phi_\pm, B_\pm)\) respectively, from the same estimates again, we have
\[ d((\psi_\pm, A_\pm), (\phi_\pm, B_\pm)) \leq \frac{1}{2} d((\psi_\pm, A_\pm), (\phi_\pm, B_\pm)) + C(\|\psi_{\pm 0} - \phi_{\pm 0}\|_{L^p} + \|A_{\pm 0} - B_{\pm 0}\|_{L^p}) \]
for small \( T > 0 \), which implies the map from the initial data to the solution is Lipschitz continuous.

5.2. Null and critical case. Our aim in this subsection is the following,

**Theorem 5.2.** Let \( \alpha = \gamma^0 \) or \(-i\gamma^1\). For any \((\psi_{\pm 0}, A_{\pm 0}) \in L^1 \times L^1\), there exists a global weak solution \((\psi_\pm, A_\pm) \in C([0, \infty); L^1) \times C([0, \infty); L^1)\) to (5.1) such that the solution is unique in
\[ \psi_\pm \in C([0, \infty); L^1), \quad A_\pm \in C([0, \infty); L^1), \]
\[ A_\pm \psi_\pm \in L^1_{\text{loc}}(0, \infty; L^1), \quad \psi_\pm \psi_\pm \in L^1_{\text{loc}}(0, \infty; L^1). \]
The map from the data \((\psi_{\pm 0}, A_{\pm 0})\) to the solution \((\psi_{\pm}, A_{\pm})\) is Lipschitz continuous from \(L^1 \times L^1\) to \(C([0, \infty); L^1) \times C([0, \infty); L^1)\).

**Proof of Theorem 5.2.** A big difference between the following bound and the previous one is that the function space is \(L^1\) where the solution is scaling invariant and there is no room to produce the \(T^*, \varepsilon > 0\) in this case. So the proof below is not so straight like the sub-critical case above. We make some steps for the proof. We show the time local existence of solution for small data. We can remove this smallness condition of the initial data by finite speed of propagation of wave and transport equation. We observe the non-concentration property of solution which implies the time global existence of solution. We show the uniqueness of solution by cut-off solution by which we can treat the solution is small.

5.2.1. **Local existence of solution with small size initial data.** In this subsubsection, we will obtain a solution with small data. We use the same scheme \((5.3)\), that is \((5.4)\), and \(M\) of \((5.5)\). We consider \(M\) is sufficiently small for a while. We shall show

\[(5.10) \quad \|
\psi^{(n)}_f \|_{L^\infty_T L^1} , \| A^{(n)}_\pm \|_{L^\infty_T L^1} \leq M \]

for any \(n\) by the induction argument again. We also obtain the following bounds

\[(5.11) \quad \| \psi^{(n)}_f A^{(n)}_\pm \|_{L^1_T L^1} , \| A^{(n)}_\pm \psi^{(n)}_\mp \|_{L^1_T L^1} \leq M^2 \]

for any \(n\). We start to estimate

\[\|
\psi^{(n+1)}_+ \|_{L^\infty_T L^1} \leq \|
\psi^{(n)}_+ + A^{(n)}_- \psi^{(n)}_+ \|_{L^1_T L^1} + m T \|
\psi^{(n)}_- \|_{L^\infty_T L^1} \leq M + M^2 + m T M \leq M. \]

The last inequality holds for sufficiently small \(M\) and \(T\). Similarly we can obtain \(\|
\psi^{(n+1)}_- \|_{L^\infty_T L^1} \leq M\). We estimate

\[\|
A^{(n+1)}_+ \|_{L^\infty_T L^1} \leq \|
A^{(n)}_+ \|_{L^1_T L^1} + \|
\psi^{(n)}_- \psi^{(n)}_+ \|_{L^1_T L^1} \leq \frac{M}{2} + M^2 \leq M. \]

Similarly \(\|
A^{(n+1)}_- \|_{L^\infty_T L^1} \leq M\). From Lemma 2.2, we have

\[\|
\psi^{(n+1)}_+ A^{(n+1)}_- \|_{L^1_T L^1} \leq \frac{1}{2} (\|
\psi^{(n)}_+ \|_{L^1_T L^1} + \|
A^{(n)}_- \psi^{(n)}_+ \|_{L^1_T L^1} + m T \|
\psi^{(n)}_- \|_{L^\infty_T L^1} (\|
A^{(n)}_\mp \psi^{(n)}_\mp \|_{L^1_T L^1} )) \leq \frac{1}{2} (\|
\psi^{(n+1)}_+ \|_{L^1_T L^1} + M^2 + m T M ) \left( \frac{M}{2} + M^2 \right) \leq M^2. \]

Similarly, \(\|
\psi^{(n+1)}_+ A^{(n+1)}_- \|_{L^1_T L^1} \leq M^2\). We estimate

\[\|
\psi^{(n+1)}_+ \psi^{(n+1)}_- \|_{L^1_T L^1} \leq \frac{1}{2} (\|
\psi^{(n)}_+ \|_{L^1_T L^1} + \|
A^{(n)}_- \psi^{(n)}_+ \|_{L^1_T L^1} + m T \|
\psi^{(n)}_- \|_{L^\infty_T L^1} ) \times (\|
\psi^{(n)}_- \|_{L^1_T L^1} + \|
A^{(n)}_\pm \psi^{(n)}_- \|_{L^1_T L^1} + m T \|
\psi^{(n)}_+ \|_{L^\infty_T L^1} ) \leq \frac{1}{2} (\|
\psi^{(n)}_+ \|_{L^1_T L^1} + M^2 + m T M )^2 \leq M^2. \]

We have obtained \((5.10)\) and \((5.11)\) for any \(n\).
We next estimate the difference $\psi_\pm^{(n+1)} - \psi_\pm^{(n)}$ and $A_\pm^{(n+1)} - A_\mp^{(n)}$. We set
\begin{equation}
(5.12) \quad d((\psi_\pm, A_\pm), (\phi_\pm, B_\pm)) = \|\psi_\pm - \phi_\pm\|_{L_\infty^p L^p} + 3\|\psi_\pm A_\pm - \phi_\pm B_\pm\|_{L_\infty^p L^p} + \|A_\pm - B_\pm\|_{L_\infty^p L^p} + 3\|\psi_\pm A_\mp - \phi_\pm B_\mp\|_{L_\infty^p L^p}.
\end{equation}
We will show $d((\psi_\pm^{(n+1)}, A_\pm^{(n+1)}), (\psi_\pm^{(n)}, A_\pm^{(n)})) \leq \frac{1}{2^n}$, that is
\begin{equation}
(5.13) \quad \|\psi_\pm^{(n+1)} - \psi_\pm^{(n)}\|_{L_\infty^p L^1} + 3\|\psi_\pm^{(n+1)} A_\pm^{(n+1)} - \psi_\pm^{(n)} A_\mp^{(n)}\|_{L_\infty^p L^1} + \|A_\pm^{(n+1)} - A_\pm^{(n)}\|_{L_\infty^p L^1} + 3\|\psi_\pm^{(n+1)} A_\pm^{(n+1)} - \psi_\pm^{(n)} A_\mp^{(n)}\|_{L_\infty^p L^1} \leq \frac{1}{2^n}
\end{equation}
for any $n$ by the induction argument. Here it comes from a technical reason to put 3 before the norms of square terms. We show the estimates of upper sign only; $+$ of $\pm$ and $-$ of $\mp$.
\begin{equation}
(5.14) \quad \|\psi_\pm^{(n+1)} - \psi_\pm^{(n)}\|_{L_\infty^p L^1} \leq \|A_\pm^{(n)} \psi_\pm^{(n)} - A_\mp^{(n)} \psi_\pm^{(n)}\|_{L_\infty^p L^1} + mT\|\psi_\mp^{(n)} - \psi_\pm^{(n-1)}\|_{L_\infty^p L^1},
\end{equation}
and
\begin{equation}
(5.15) \quad \|A_\pm^{(n+1)} - A_\pm^{(n)}\|_{L_\infty^p L^1} \leq \|\psi_\pm^{(n)} \psi_\mp^{(n)} - \psi_\pm^{(n-1)} \psi_\mp^{(n-1)}\|_{L_\infty^p L^1}.
\end{equation}
We use triangle inequality
\begin{equation}
(5.16) \quad \|\psi_\pm^{(n+1)} A_\mp^{(n+1)} - \psi_\pm^{(n)} A_\mp^{(n)}\|_{L_\infty^p L^1} \leq \|\psi_\pm^{(n+1)} (A_\pm^{(n+1)} - A_\mp^{(n)}\|_{L_\infty^p L^1} + \|A_\mp^{(n)} (\psi_\pm^{(n+1)} - \psi_\mp^{(n)}\|_{L_\infty^p L^1},
\end{equation}
\begin{equation}
(5.17) \quad \|\psi_\pm^{(n+1)} (A_\mp^{(n+1)} - \psi_\mp^{(n)}\|_{L_\infty^p L^1} \leq \|\psi_\pm^{(n+1)} (\psi_\pm^{(n+1)} - \psi_\mp^{(n)}\|_{L_\infty^p L^1} + \|\psi_\mp^{(n)} (\psi_\pm^{(n+1)} - \psi_\pm^{(n)}\|_{L_\infty^p L^1}.
\end{equation}
We estimate each term by Lemma 2.2
\begin{equation}
(5.18) \quad \|\psi_\pm^{(n+1)} (A_\mp^{(n+1)} - A_\mp^{(n)}\|_{L_\infty^p L^1} \leq \frac{1}{2} \left(\|\psi_\pm^{(n)}\|_{L_\infty^p L^1} + \|A_\pm^{(n)} \psi_\pm^{(n)}\|_{L_\infty^p L^1} + mT\|\psi_\mp^{(n)}\|_{L_\infty^p L^1}\|\psi_\pm^{(n)} - \psi_\mp^{(n-1)}\|_{L_\infty^p L^1} \right) \leq \frac{1}{2} \left(\frac{M}{2} + M^2 + mTM\right)\|\psi_\pm^{(n)}\|_{L_\infty^p L^1} + \|\psi_\mp^{(n)}\|_{L_\infty^p L^1}.
\end{equation}
We make $\frac{1}{2} \left(\frac{M}{2} + M^2 + mTM\right)$ small sufficiently. Similarly
\begin{equation}
(5.19) \quad \|A_\mp^{(n)} (\psi_\pm^{(n+1)} - \psi_\mp^{(n)}\|_{L_\infty^p L^1} \leq \frac{1}{2} \left(\frac{M}{2} + M^2 + mTM\right)\|A_\pm^{(n)} \psi_\pm^{(n)} - A_\mp^{(n-1)} \psi_\pm^{(n-1)}\|_{L_\infty^p L^1} + mT\|\psi_\mp^{(n)} - \psi_\mp^{(n-1)}\|_{L_\infty^p L^1},
\end{equation}
\begin{equation}
(5.20) \quad \|\psi_\pm^{(n+1)} (\psi_\pm^{(n+1)} - \psi_\mp^{(n)}\|_{L_\infty^p L^1} \leq \frac{1}{2} \left(\frac{M}{2} + M^2 + mTM\right)\|\psi_\pm^{(n)} A_\pm^{(n)} - A_\pm^{(n-1)} \psi_\pm^{(n-1)}\|_{L_\infty^p L^1} + mT\|\psi_\pm^{(n)} - \psi_\pm^{(n-1)}\|_{L_\infty^p L^1},
\end{equation}
\begin{equation}
(5.21) \quad \|\psi_\mp^{(n)} (\psi_\pm^{(n+1)} - \psi_\mp^{(n)}\|_{L_\infty^p L^1} \leq \frac{1}{2} \left(\frac{M}{2} + M^2 + mTM\right)\|A_\pm^{(n)} \psi_\pm^{(n)} - A_\mp^{(n-1)} \psi_\pm^{(n-1)}\|_{L_\infty^p L^1} + mT\|\psi_\mp^{(n)} - \psi_\mp^{(n-1)}\|_{L_\infty^p L^1}.
\end{equation}
We have obtained (5.13) for any $n$. So we have obtained the time local solution with the small data.
We remark if the initial data is small, we have already obtained that the solution map from initial data is Lipshitz continuous as well,
\[ d((\psi_\pm, A_\pm), (\phi_\pm, B_\pm)) \leq C(\|\psi_\pm - \phi_\pm\|_{L^1} + \|A_\pm - B_\pm\|_{L^1}). \]

5.2.2. Local existence of solution with general size initial data. We remove the smallness condition for initial data by using the finite speed of propagation of solution.

For any initial data \( \psi_{\pm 0}, A_{\pm 0} \in L^1 \) and any \( M > 0 \), there exists \( r > 0 \) such that
\[ \sup_{x \in \mathbb{R}} \int_{|x-y| < r} (|\psi_{\pm 0}(y)| + |A_{\pm 0}(y)|) dy < M. \]

With this \( r \), we split the real line for the variable \( x \) in the following two ways,
\[ \mathbb{R} = \bigcup_{j \in \mathbb{Z}} I_1^j = \bigcup_{j \in \mathbb{Z}} I_2^j, \quad I_1^j = [jr, (j + 1)r], \quad I_2^j = [(j + \frac{1}{2})r, (j + \frac{3}{2})r] \]

For each \( k = 1, 2, j \in \mathbb{Z} \), if we replace the initial data \( \psi_{\pm 0} \) and \( A_{\pm 0} \) by \( \chi_{I_1^j} \psi_{\pm 0} \) and \( \chi_{I_2^j} A_{\pm 0} \) respectively, the data is small and we could find the corresponding solution \( \psi_{\pm k,j}^{k,j}, A_{\pm k,j}^{k,j} \) with the uniform existence time \( T \) with respect to \( k \) and \( j \) from the argument in the previous subsubsection. We consider \( (t, x) \) plane in \( \mathbb{R}^2 \) and in which the two squares \( S^{1,j} \) and \( S^{2,j} \),
\[ S^{1,j} = \{(t, x) \in \mathbb{R}^2 : 0 \leq t < r, j + \frac{1}{4} \leq x < j + \frac{3}{4}\}, \]
\[ S^{2,j} = \{(t, x) \in \mathbb{R}^2 : 0 \leq t < r, j + \frac{3}{4} \leq x < j + \frac{5}{4}\}. \]

These are all disjointed and satisfy
\[ \bigcup_{k=1,2} \bigcup_{j \in \mathbb{Z}} S^{k,j} = \{(t, x) \in \mathbb{R}^2 : 0 \leq t < r\}. \]

We also remark these squares are included in the triangles
\[ S^{1,j} \subset \{(t, x) \in \mathbb{R}^2 : |x - (j + \frac{1}{2})r| < \frac{r}{2} - t\} =: \Omega^{1,j}, \]
\[ S^{2,j} \subset \{(t, x) \in \mathbb{R}^2 : |x - (j + 1)r| < \frac{r}{2} - t\} =: \Omega^{2,j}. \]

We define the functions, for \( 0 < t < \min\{T, \frac{r}{2}\} \),
\[ \psi_\pm(t, x) = \sum_{k=1,2} \sum_{j \in \mathbb{Z}} \chi_{S^{k,j}}(t, x) \psi_{\pm}^{k,j}(t, x), \quad A_\pm(t, x) = \sum_{k=1,2} \sum_{j \in \mathbb{Z}} \chi_{S^{k,j}}(t, x) A_{\pm}^{k,j}(t, x). \]

In the following, we shall show that \( \psi_{\pm}^{k,j} \) is a solution of \( (5.1) \). In order to do this, it is sufficient to prove that the solution which is restricted in \( S^{k,j} \), that is,
\[ \chi_{S^{k,j}}(t, x) \psi_{\pm}(t, x) = \chi_{S^{k,j}}(t, x) \psi_{\pm}^{k,j}(t, x), \quad \chi_{S^{k,j}}(t, x) A_{\pm}(t, x) = \chi_{S^{k,j}}(t, x) A_{\pm}^{k,j}(t, x) \]
does not influenced by changing the initial data on the out of the interval \( I^j_k \). We apply the equality (2.3) to (2.4),

\[
(5.24) \quad \chi_{\Omega^{k,j}} \psi(t, x) = \chi_{\Omega^{k,j}}(x_{I^j_k} \psi_{\pm}(x + t) + i \chi_{\Omega^{k,j}} \int_0^t \left( (\chi_{\Omega^{k,j}} A_{\pm}) (\chi_{\Omega^{k,j}} \psi_{\pm} - m(\chi_{\Omega^{k,j}} \psi_{\pm})) \right)(x, s + t \pm s) ds, \\
\chi_{\Omega^{k,j}} A_{\pm}(t, x) = \chi_{\Omega^{k,j}}(x_{I^j_k} A_{\pm}(x + t) + \chi_{\Omega^{k,j}} \int_0^t \left( \Re((\chi_{\Omega^{k,j}} \psi_{\pm})(\chi_{\Omega^{k,j}} \psi^*_{\pm})) \right)(x, s + t \pm s) ds.
\]

If \((t, x) \in S^{k,j} \subset \Omega^{k,j}\), then all \( \chi_{\Omega^{k,j}}(t, x) = 1 \) above. We can observe that the solution \((\chi_{\Omega^{k,j}} \psi_{\pm}, \chi_{\Omega^{k,j}} A_{\pm}) = (\psi_{\pm}, A_{\pm})\) of (5.24) won’t change when the out side of \( I^j_k \) for the initial data \((\psi_{\pm, 0}, A_{\pm, 0})\) changes.

To this end, we simply check that the solution (5.23) is in \( L^\infty_T L^1 \). We estimate

\[
\|\psi_{\pm}\|_{L^\infty_T L^1} + \|A_{\pm}\|_{L^\infty_T L^1} \leq C \sum_{k=1,2} \sum_{j \in \mathbb{Z}} \|x_{I^j_k} \psi_{\pm, 0}\|_{L^\infty_T L^1} + \|x_{I^j_k} A_{\pm, 0}\|_{L^\infty_T L^1},
\]

\[
\leq C \left( \|\psi_{\pm, 0}\|_{L^1} + \|A_{\pm, 0}\|_{L^1} \right).
\]

5.2.3. **Uniqueness of solution.** We now show the uniqueness of the solutions. We consider two solutions \((\psi_{\pm}, A_{\pm})\) and \((\phi_{\pm}, B_{\pm})\) to (5.11) satisfying

\[
(5.25) \quad \psi_{\pm}, A_{\pm}, \phi_{\pm}, B_{\pm} \in L^\infty_T L^1, \quad \psi_{\pm} A_{\mp}, \phi_{\pm} B_{\mp}, \psi_{\pm} \phi_{\mp}, \phi_{\pm} \phi_{\mp} \in L^1_T L^1.
\]

Fix \( x_0 \in \mathbb{R} \). For any \( M > 0 \), there is \( R \) such that

\[
(5.26) \quad \|\chi_{\Omega_R(x_0)} \psi_{\pm} A_{\mp}\|_{L^1_T L^1}, \quad \|\chi_{\Omega_R(x_0)} \psi_{\pm} \psi_{\mp}\|_{L^1_T L^1}, \quad \|\chi_{\Omega_R(x_0)} \phi_{\pm} B_{\mp}\|_{L^1_T L^1}, \quad \|\chi_{\Omega_R(x_0)} \phi_{\pm} \phi_{\mp}\|_{L^1_T L^1} \leq M^2.
\]

We have from (2.4)

\[
\|\chi_{\Omega_R(x_0)} \psi_{\pm}\|_{L^\infty_T L^1} \leq \|\chi_{I_R(x_0)} \psi_{\pm, 0}\|_{L^1} + \|\chi_{\Omega_R(x_0)} \psi_{\pm} A_{\mp}\|_{L^1_T L^1} + mT \|\chi_{\Omega_R(x_0)} \psi_{\mp}\|_{L^\infty_T L^1}.
\]

We remark the each terms here is finite since (5.25). For the sufficiently small \( T \), we have from (5.26)

\[
(5.27) \quad \|\chi_{\Omega_R(x_0)} \psi_{\pm}\|_{L^\infty_T L^1} + \|\chi_{\Omega_R(x_0)} \psi_{\mp}\|_{L^\infty_T L^1} \leq \frac{M}{2} + 4M^2 \leq M
\]

by resizing \( M \) sufficiently small. The same estimates for \( \chi_{\Omega_R(x_0)} \phi_{\pm} \) hold. We also estimate

\[
\|\chi_{\Omega_R(x_0)} A_{\pm}\|_{L^\infty_T L^1} \leq \|\chi_{I_R(x_0)} A_{\pm, 0}\|_{L^1} + \|\chi_{\Omega_R(x_0)} \psi_{\pm} \psi_{\mp}\|_{L^1_T L^1} \leq \frac{M}{2} + M^2 \leq M.
\]

The same estimates for \( \chi_{\Omega_R(x_0)} B_{\pm} \) hold. In the long run, we may think they are small data and small solutions to the problem. From the similar estimates (5.14), (5.15), (5.16), (5.17), (5.18), (5.19), (5.20) and (5.21) by using (2.7), we have

\[
d((\chi_{\Omega_R(x_0)} \psi_{\pm}, \chi_{\Omega_R(x_0)} A_{\pm}), (\chi_{\Omega_R(x_0)} \phi_{\pm}, \chi_{\Omega_R(x_0)} B_{\pm})) \\
\leq \frac{1}{2} d((\chi_{\Omega_R(x_0)} \psi_{\pm}, \chi_{\Omega_R(x_0)} A_{\pm}), (\chi_{\Omega_R(x_0)} \phi_{\pm}, \chi_{\Omega_R(x_0)} B_{\pm}))
\]

where \( d((\psi_{\pm, 0}, A_{\pm}), (\phi_{\pm, 0}, B_{\pm})) \) is defined in (5.12). Therefore we obtained the uniqueness in the region \( \Omega_R(x_0) \) at least, but this is sufficient.
5.2.4. Global existence of solution. By the previous subsubsection, we have obtained the time local solution to \([5.1]\) on \([0, T]\). We repeat this argument to have a solution started with initial data \((\psi_\pm(T), A_\pm(T))\). If we have the condition of \([5.22]\) for each step, we can derive the global solution. It is sufficient for this to show the following a priori estimate. For any \(T > 0\) and any \(\varepsilon > 0\), there exists \(r = r(T; \|\psi_\pm\|_{L^1}, \|A_\pm\|_{L^1}) > 0\) such that

\[
sup_{0 < t < T} \sup_{x \in \mathbb{R}} \int_{|x-y|<r} |\psi_\pm(t, y)| + |A_\pm(t, y)| dy < \varepsilon.
\]

(5.28)

where \((\psi_\pm, A_\pm)\) is the solution for \([5.1]\) in \(C([0, T]; L^1) \times C([0, T]; L^1)\).

We start to prove \((5.28)\). We use the decomposition \(\psi_\pm = \psi_{L\pm}^0 + \psi_{N\pm}^0\). From the intrinsic \(L^\infty\) estimate Proposition \([4.1]\) we see that

\[
\|\psi_{N\pm}(t)\|_{L^\infty} + \|\psi_{N \pm}(t)\|_{L^\infty} \leq m(\|\psi_+0\|_{L^1} + \|\psi_-0\|_{L^1})(e^{mt} + t - 1) \\
\leq e^{mT} + T - 1,
\]

(5.29)

Hence by choosing \(r > 0\) properly small depending on the right hand side of \((5.29)\), we have

\[
\sup_{x \in \mathbb{R}} \int_{|x-y|<r} |\psi_\pm(t, y)| dy \leq \sup_{x \in \mathbb{R}} \int_{|x-y|<r} |\psi_{\pm0}(t \mp y)| dy + r(e^{mT} + T - 1) < \varepsilon.
\]

This provides the non-concentration estimate for Dirac part \(\psi_\pm\). We decompose also \(A_\pm\) such like \(A_\pm = A_{L\pm}^N + A_{N\pm}^N\)

\[
A_{L\pm}^N(t, x) = A_{\pm0}(x \mp t), \quad A_{N\pm}^N(t, x) = \mp \int_0^t \text{Re}(\psi_+\psi_-)(s, x \mp t \pm s) ds.
\]

We need to estimate the integrand of \(A_{N\pm}^N\), that is \(\psi_+\psi_- = \psi_{L+}\psi_{L-} + \psi_{L+}\psi_{N-} + \psi_{N+}\psi_{L-} + \psi_{N+}\psi_{N-}\).

\[
\int_0^t |\psi_{L+}\psi_{L-}(s, x - t + s)| ds = \int_0^t |\psi_+0(x - t)| \int_0^t |\psi_-0(x - t + 2s)| ds \\
\leq |\psi_+0(x - t)| \|\psi_-0\|_{L^1},
\]

\[
\int_0^t |\psi_{L+}\psi_{N-}(s, x - t + s)| ds \leq |\psi_+0(x - t)| \|\psi_-N\|_{L^\infty t} \\
\leq |\psi_+0(x - t)| (e^{mT} + T - 1) T,
\]

\[
\int_0^t |\psi_{N+}\psi_{L-}(s, x - t + s)| ds \leq \|\psi_+N\|_{L^\infty} \int_0^t |\psi_-0(x - t + 2s)| ds \\
\leq (e^{mT} + T - 1) \|\psi_-0\|_{L^1},
\]

\[
\int_0^t |\psi_{N+}\psi_{N-}(s, x - t + s)| ds \leq \|\psi_+N\|_{L^\infty} \|\psi_-N\|_{L^\infty T} \leq (e^{mT} + T - 1)^2 T.
\]

Gathering all the estimates, we obtain

\[
\int_0^t |\psi_+\psi_-|(s, x - t + s) ds \leq C|\psi_+0(x - t)| + C(T).
\]
Therefore we conclude that
\[
\sup_{x \in \mathbb{R}} \int_{|x-y| < r} |A_{\pm}(t, y)| dy \\
\leq \sup_{x \in \mathbb{R}} \int_{|x-y| < r} |A_{\pm 0}(t \mp y)| + C|\psi_{\pm 0}(t - y)| + C|\psi_{-0}(t + y)| + C(T) dy < \varepsilon
\]
and this shows the estimate for the term of \(A_{\pm}\).

\[\square\]

6. Time global well-posedness for the non-null case

In this section we consider the non-null case and establish the time local well-posedness for (1.1) when \(\alpha = I\). We consider here that

\[
\frac{\partial \psi_{\pm} \pm \partial_x \psi_{\pm}}{\partial t} = iA_{\pm} \psi_{\pm} - im\psi_{\mp} \quad \text{and } t \in \mathbb{R}, x \in \mathbb{R},
\]

(6.1)

6.1. Non-null and subcritical case. Our aim in this subsection is the following,

**Theorem 6.1.** Let \(\alpha = I\). Let \(1 < p \leq \infty\). For any \((\psi_{\pm 0}, A_{\pm 0}) \in (L^p \cap L^{\infty}) \times L^p\), there exists a unique global weak solution \((\psi_{\pm}, A_{\pm}) \in C([0, \infty); L^p \cap L^{\infty}) \times C([0, \infty); L^p)\) to (6.1). The map from the initial data to the solution is the Lipschitz continuous from \((L^p \cap L^{\infty}) \times L^p \to C([0, T]; L^p \cap L^{\infty}) \times C([0, T]; L^p)\).

**Proof of Theorem 6.1.** We consider the following recurrence of successive approximation:

Let \(\{\psi^{(n)}_{\pm}, A^{(n)}_{\pm}\}_{n=1, 2, \ldots}\) solve

\[
\begin{aligned}
\partial_t \psi^{(n+1)}_{\pm} \pm \partial_x \psi^{(n+1)}_{\pm} &= iA^{(n)}_{\pm} \psi^{(n+1)}_{\pm} - im\psi^{(n+1)}_{\mp}, \\
\partial_t A^{(n+1)}_{\pm} \pm \partial_x A^{(n+1)}_{\pm} &= \frac{1}{2}(|\psi^{(n)}_{\pm}|^2 + |\psi^{(n)}_{\mp}|^2), \\
(\psi^{(n)}_{\pm}, A^{(n)}_{\pm})|_{t=0} &= (\psi_{\pm 0}, A_{\pm 0}).
\end{aligned}
\]

(6.2)

with the first step \((\psi^{(1)}_{\pm}, A^{(1)}_{\pm}) = (\psi_{\pm 0}, A_{\pm 0})\). We should note that this scheme has different form from the former one (5.3) not only by the nonlinear coupling but also the recurrence suffix on the equation of \(\psi\). The aim for this is to use the calculation for the intrinsic estimates. Now then we introduce the integral equation derived from Lemma [2.1]

\[
\begin{aligned}
\psi^{(n+1)}_{\pm}(t, x) &= \psi_{\pm 0}(x \mp t) + i \int_0^t (A^{(n)}_{\pm} \psi^{(n+1)}_{\pm} - m\psi^{(n+1)}_{\mp}) (s, x \mp t \pm s) ds, \\
A^{(n+1)}_{\pm}(t, x) &= A_{\pm 0}(x \mp t) \pm \frac{1}{2} \int_0^t (|\psi^{(n)}_{\pm}|^2 + |\psi^{(n)}_{\mp}|^2) (s, x \mp t \pm s) ds.
\end{aligned}
\]

(6.3)

Letting

\[M = 2(\|\psi_{\pm 0}\|_{L^p} + \|\psi_{\pm 0}\|_{L^{\infty}} + \|A_{\pm 0}\|_{L^p}),\]

we define

\[X^p_T \equiv \{ (\phi, B) \in C([0, T]; L^p) \times C([0, T]; L^p); \quad \|\phi\|_{L^\infty L^p} + \|B\|_{L^\infty L^p} \leq M \}. \]
It is straightforward to show that $X^p_T$ is a complete metric space by the metric induced by the norm:

$$\| (\phi, B) \|_{X^p_T} \equiv \sup_{t \in [0,T]} \| \phi(t) \|_{L^p} + \sup_{t \in [0,T]} \| B(t) \|_{L^p}.$$ 

For that purpose we show the following estimate for the nonlinear coupling:

**Lemma 6.2.** For the sequence $\{ \psi_\pm^{(k)} + A_\pm^{(k)} \}_{k=1}^{n-1} \subset X^p_T$ with $n = 2, 3, \ldots$, it holds that, for small $T > 0$

\begin{align*}
(6.4) & \quad \| \psi_\pm^{(n)} \|_{L^\infty_T (L^p \cap L^\infty)} \leq M, \\
(6.5) & \quad \| \psi_\pm^{(n)} \|^2 \|_{L^p T L^p} \leq M^2, \\
(6.6) & \quad \| A_\pm^{(n)} \|_{L^\infty_T L^p} \leq M, \\
(6.7) & \quad \| A_\pm^{(n)} \psi_\pm^{(n+1)} \|_{L^\infty_T L^p} \leq M^2.
\end{align*}

**Proof of Lemma 6.2.** Assuming $\{ \psi_\pm^{(k)} + A_\pm^{(k)} \}_{k=1}^{n-1} \subset X^p_T$ and satisfying the estimate (6.5) up to $k \leq n - 1$, we decompose $\psi_\pm^{(n)}$ into two components $\psi_\pm^{(n)} + \psi_\mp^{(n)}$ and it follows from the intrinsic $L^\infty$ estimate (Proposition 4.1) that

\begin{align*}
(6.8) & \quad \| \psi_\pm^{(n)}(t) \|_{L^p \cap L^\infty} \leq C e^T \| \psi_0^\pm \|_{L^p}.
\end{align*}

for $0 < t < T$. We should note here that the right hand side of the recurrence scheme (6.2) is linear in $\psi_\pm^{(n)}$ which is crucial for obtaining the estimate (6.8). While we have from (6.2) that

\begin{align*}
(6.9) & \quad \| \psi_\pm^{(n)} \|_{L^\infty_T (L^p \cap L^\infty)} = \| \psi_0^\pm \|_{L^p \cap L^\infty} \leq \frac{1}{4} M.
\end{align*}

Combining (6.8) and (6.9) we obtain (6.4), and which implies (6.5) by Hölder inequality. Since $A_\pm^{(n)}$ is given by the solution formula, we compute from (6.4) that

\begin{align*}
\| A_\pm^{(n+1)} \|_{L^\infty_T L^p} \leq & \| A_0^\pm \|_{L^p} + \frac{1}{2} \int_0^T \left( \| \psi_\pm^+ \|_{L^\infty L^p} + \| \psi_\pm^- \|_{L^\infty L^p} \right) ds \\
\leq & \frac{1}{2} M + C T M^2 \leq M
\end{align*}

for small $T > 0$. (6.4) and (6.6) imply (6.7). □

We now show that the recurrence scheme converges to the solution of the Chern-Simons-Dirac system:

**Proposition 6.3.** Let $\{ \psi_\pm^{(n)} \}_{n=1}^\infty$ and $\{ A_\pm^{(n)} \}_{n=1}^\infty$ defined by recurrence scheme (6.2). Then there exists $\psi_\pm, A_\pm \in C([0, \infty); L^p)$ such that

$$\psi_\pm^{(n)} \to \psi_\pm, \quad n \to \infty,$$

$$A_\pm^{(n)} \to A_\pm, \quad n \to \infty$$

in $C([0, T); L^p)$ and $(\psi_\pm, A_\pm)$ solves the system (6.1).
Proof of Proposition 6.3. We derive the estimates for the difference $\psi^{(n+1)}_\pm - \psi^{(n)}_\pm$ and $A^{(n+1)}_\pm - A^{(n)}_\pm$ as

\[
\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L_p^\infty} + \|A^{(n+1)}_\pm - A^{(n)}_\pm\|_{L_p^\infty} \leq \frac{1}{2^T},
\]

and the sequence is concluded as the Cauchy sequence. Taking $L_p$ norm to

\[
\psi^{(n+1)}_\pm - \psi^{(n)}_\pm = \int_0^t (iA^{(n)}_+ \psi^{(n+1)}_\pm - iA^{(n-1)}_+ \psi^{(n)}_\pm)ds
+ \int_0^t (im\psi^{(n+1)}_\pm - im\psi^{(n)}_\pm)ds,
\]

we see by Hölder inequality that

\[
\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L_p^\infty} \leq T^{1-\frac{1}{p}}\|A^{(n)}_+ \psi^{(n+1)}_\pm - A^{(n-1)}_+ \psi^{(n)}_\pm\|_{L_p^L_p^p} + mT\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L_p^\infty}
\]

(6.11)

\[
\leq T^{1-\frac{1}{p}}(\|A^{(n)}_+ (\psi^{(n+1)}_\pm - \psi^{(n)}_\pm)\|_{L_p^L_p^p} + \|(A^{(n)}_+ - A^{(n-1)}_+) \psi^{(n)}_\pm\|_{L_p^L_p^p})
+ mT\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L_p^\infty}
\]

for small $T > 0$. The first term of the right hand side of (6.11) is estimated as follows. By Lemma 2.2 and (6.5),

(6.12)

\[
\|A^{(n)}_+ (\psi^{(n+1)}_\pm - \psi^{(n)}_\pm)\|_{L_p^L_p^p} \leq \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{2}\|A_0\|_p + M^2\right) T^{1-\frac{1}{p}}\|A^{(n)}_+ \psi^{(n+1)}_\pm - A^{(n-1)}_+ \psi^{(n)}_\pm\|_{L_p^L_p^p} + m\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L_p^L_p^p}
\]

and then separating

(6.13) $\|A^{(n)}_+ (\psi^{(n+1)}_\pm - \psi^{(n)}_\pm)\|_{L_p^L_p^p} \leq \|A^{(n)}_+ (\psi^{(n+1)}_\pm - \psi^{(n)}_\pm)\|_{L_p^L_p^p} + \|(A^{(n)}_+ - A^{(n-1)}_+) \psi^{(n)}_\pm\|_{L_p^L_p^p}$

we see that the second term of this is the left hand side of (6.12). Since $p > 1$, we choose $T > 0$ small enough such that

\[
L = \left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\frac{1}{2}\|A_0\|_p + M^2\right) T^{1-\frac{1}{p}} < 1,
\]

we have

(6.14)

\[
\|A^{(n)}_+ (\psi^{(n+1)}_\pm - \psi^{(n)}_\pm)\|_{L_p^L_p^p} \leq \frac{L}{1-L} \left(\|\psi^{(n)}_\pm (A^{(n)}_+ - A^{(n-1)}_+)\|_{L_p^L_p^p} + m\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L_p^L_p^p}\right)
\]

By plugging all (6.12), (6.13) and (6.14) into (6.11), we can replace the estimate (6.11) by the following,

(6.15)

\[
\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L_p^\infty} \leq CT^{1-\frac{1}{p}}\|A^{(n)}_+ (A^{(n)}_+ - A^{(n-1)}_+) \psi^{(n)}_\pm\|_{L_p^L_p^p}.
\]
We estimate by Lemma 2.2
\[ \| (A^{(n+1)}_+ - A^{(n)}_+) \|_{L^p_t L^p} \]
\[ \leq T^{1 - \frac{1}{p}} \| \psi_+^{(n)} \|^2 + \| \psi_-^{(n)} \|^2 - (|\psi_+^{(n-1)}|^2 + |\psi_-^{(n-1)}|^2) \|_{L^p_t L^p} \]
\[ \times \left( \frac{1}{2} \right)^n \left( \| \psi_0 \|_{L^1} + T^{1 - \frac{1}{p}} \| A^{(n-1)}_- \|_{L^p_t L^p} + mT \| \psi_-^{(n)} \|_{L^\infty_t L^p} \right) \]
\[ \leq \left( \frac{1}{2} \right)^n \left( \frac{M}{2} + M^2 + mTM \right) T^{1 - \frac{1}{p}} \| \psi_+^{(n)} \|^2 + \| \psi_-^{(n)} \|^2 - (|\psi_+^{(n-1)}|^2 + |\psi_-^{(n-1)}|^2) \|_{L^p_t L^p}. \]

We estimate
\[ \| |\psi_+^{(n)}|^2 + |\psi_-^{(n)}|^2 - (|\psi_+^{(n-1)}|^2 + |\psi_-^{(n-1)}|^2) \|_{L^p_t L^p} \]
\[ \leq \max \left( \| \psi_+^{(n)} \|_{L^\infty_t L^\infty}, \| \psi_-^{(n-1)} \|_{L^\infty_t L^\infty} \right) \| \psi_+^{(n)} \| - |\psi_+^{(n-1)}| \|_{L^p_t L^p} \]
\[ + \max \left( \| \psi_-^{(n)} \|_{L^\infty_t L^\infty}, \| \psi_-^{(n-1)} \|_{L^\infty_t L^\infty} \right) \| \psi_-^{(n)} \| - |\psi_-^{(n-1)}| \|_{L^p_t L^p} \]
\[ \leq C \| \psi_0 \|_{L^\infty} \left( \| \psi_+^{(n)} - \psi_-^{(n-1)} \|_{L^p_t L^p} + \| \psi_-^{(n)} - \psi_-^{(n-1)} \|_{L^p_t L^p} \right). \]

Finally we obtain that
\[ \| \psi_+^{(n+1)} - \psi_+^{(n)} \|_{L^\infty_t L^p} \leq C \left( \frac{M}{2} + M^2 + mTM \right) T^{1 - \frac{1}{p}} \| \psi_+^{(n)} - \psi_-^{(n-1)} \|_{L^p_t L^p} + \| \psi_-^{(n)} - \psi_-^{(n-1)} \|_{L^p_t L^p}, \]

for small \( T > 0 \). We also obtain the estimate similar to \( \| \psi_+^{(n+1)} - \psi_+^{(n)} \|_{L^\infty_t L^p} \).

On the other hand, taking \( L^\infty L^p \) norm for the difference of the integral equation
\[ A^{(n+1)}_+ - A^{(n)}_+ = \frac{1}{2} \int_0^t (|\psi_+|^2 + |\psi_-|^2) ds + \frac{1}{2} \int_0^t (|\psi_+^{(n-1)}|^2 + |\psi_-^{(n-1)}|^2) \]
we have from (6.17) that
\[ \| A^{(n+1)}_+ - A^{(n)}_+ \|_{L^\infty_t L^p} \leq C T^{1 - \frac{1}{p}} \left( \| \psi_+^{(n)} - \psi_-^{(n-1)} \|_{L^p_t L^p} + \| \psi_-^{(n)} - \psi_-^{(n-1)} \|_{L^p_t L^p} \right)^2. \]

We have desired estimate for \( A^{(n+1)}_+ - A^{(n)}_+ \) for sufficiently small \( T > 0 \). Combining two estimates (6.18) and (6.19), one may easily obtain that the recurrence formula produces the uniform bound estimate and it is direct to conclude that the sequence \( \{ \psi_+^{(n)}, A^{(n)}_+ \}_{n=1}^\infty \) is the Cauchy sequence in \( L^\infty(0, T; L^p) \times L^\infty(0, T; L^p) \). Hence there exists a pair of the limit function \( (\psi_+, A_+) \) that solves the integral equation (6.1). This proves Proposition 6.3.

As we saw that \( \psi_\pm \in L^\infty_t (L^p \cap L^\infty) \) and \( A_\pm \in L^\infty_t L^p \) imply \( |\psi_\pm|^2, A_\pm \psi_\mp \in L^p_t L^p \), we don’t need to have extra spaces which the solution belongs to obtain the uniqueness result, and Lipschitz continuous result for the solution map as well. We conclude Theorem 6.1.

6.2. Non-null and critical case. Our aim in this subsection is the following,

**Theorem 6.4.** Let \( \alpha = I \). For any \((\psi_\pm, A_\pm) \in (L^1 \cap L^\infty) \times L^1\), there exists a unique global weak solution \((\psi_\pm, A_\pm) \in C([0, \infty); L^1 \cap L^\infty) \times C([0, \infty); L^1) \) to (6.1). The map from the initial data to the solution is the Lipschitz continuous from \((L^1 \cap L^\infty) \times L^1 \to C([0, T); L^1 \cap L^\infty) \times C([0, T); L^1)\).
**Proof of Theorem 6.4.** We use the same argument with the case of null and critical case in subsection 5.2. We can make $L^1$ norm of initial data small as much as we want by splitting the initial data to small pieces. We remark here that we can’t make $\|\psi_0\|_{L^\infty}$ small even if we split the support of the functions. But it doesn’t cause a trouble. We know the solutions $\|\psi_\pm\|_{L^\infty L^1}$, $\|A_\pm\|_{L^\infty L^1}$ are small when the initial data $\|\psi_0\|_{L^1}$, $\|A_0\|_{L^1}$ are small. So $\|\psi_\pm^2\|_{L^1 L^1}$, $\|A_\pm^2\|_{L^1 L^1}$ are small by Hölder inequality as we saw. We follow the same ways with the proof of Proposition 6.3 but $T^{1-p} = 1$. We can make $(\frac{1}{2}\|A_0\|_p + M^2)$ in (6.12) and $(\frac{M}{T} + M^2 + mTM)$ in (6.18) as small as we like. So we can obtain for any small $C > 0$,

$$\|\psi^{(n+1)}_\pm - \psi^{(n)}_\pm\|_{L^\infty L^1} \leq \frac{C}{2^n} \tag{6.20}$$

provided that the induction assumption holds

$$\|\psi^{(n)}_\pm - \psi^{(n-1)}_\pm\|_{L^\infty L^1} + \|A^{(n)}_\pm - A^{(n-1)}_\pm\|_{L^\infty L^1} \leq \frac{1}{2^{n-1}}$$

and small $M$. From (6.19) and (6.20) with small $C$, we have

$$\|A^{(n+1)}_\pm - A^{(n)}_\pm\|_{L^\infty L^1} \leq \frac{1}{2^{n+1}}$$

which concludes (6.10) with $p = 1$. We obtain the time local existence.

For the time global existence, we show (5.28) again for this non-null setting. We know $\psi_\pm \in L^\infty_T L^\infty$, and

$$|A_\pm(t, x)| \leq |A_\pm(0, x \mp t)| + \int_0^t |\psi_\pm|^2(s, x \mp t \pm s)ds
\leq |A_\pm(0, x \mp t)| + T\|\psi_\pm^2\|_{L^\infty_T L^\infty}.$$ 

These imply (5.28). We conclude the proof of Theorem 6.4.

\[\square\]

**Remark.** Since the procedure for proving the existence and continuity for the global solution is based on the local arguments, the global well-posedness for the (1.1) can be generalized into the space of the locally uniformly class $L^p_{loc \text{ unif}}(\mathbb{R})$, where

$$L^p_{loc \text{ unif}}(\mathbb{R}) = \left\{ f \in L^1_{loc}(\mathbb{R}); \sup_{K \subset \subset \mathbb{R}} \|f\|_{L^p(K)} < \infty \right\}.$$ 

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