Almost complex structures on $S^2 \times S^2$

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Abstract

In this note we investigate the structure of the space $J$ of smooth almost complex structures on $S^2 \times S^2$ that are compatible with some symplectic form. This space has a natural stratification that changes as the cohomology class of the form changes and whose properties are very closely connected to the topology of the group of symplectomorphisms of $S^2 \times S^2$. By globalizing standard gluing constructions in the theory of stable maps, we show that the strata of $J$ are Fréchet manifolds of finite codimension, and that the normal link of each stratum is a finite dimensional stratified space. The topology of these links turns out to be surprisingly intricate, and we work out certain cases. Our arguments apply also to other ruled surfaces, though they give complete information only for bundles over $S^2$ and $T^2$.

1 Introduction

It is well known that every symplectic form on $X = S^2 \times S^2$ is, after multiplication by a suitable constant, symplectomorphic to a product form $\omega^\lambda = (1 + \lambda)\sigma_1 + \sigma_2$ for some $\lambda \geq 0$, where the 2-form $\sigma_i$ has total area 1 on the $i$th factor. We are interested in the structure of the space $J^\lambda$ of all $C^\infty \omega^\lambda$-compatible almost complex structures on $X$. Observe that $J^\lambda$ itself is always contractible. However it has a natural stratification that changes as $\lambda$ passes each integer. The reason for this is that as $\lambda$ grows the set of homology classes that can be represented by an $\omega^\lambda$-symplectically embedded 2-sphere changes. Since each such 2-sphere can be parametrized to be $J$-holomorphic for some $J \in J^\lambda$, there is a corresponding change in the structure of $J^\lambda$.

To explain this in more detail, let $A \in H_2(X, \mathbb{Z})$ be the homology class $[S^2 \times pt]$ and let $F = [pt \times S^2]$. (The reason for this notation is that we are

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thinking of $X$ as a fibered space over the first $S^2$-factor, so that the smaller sphere $F$ is the fiber.) When $\ell - 1 < \lambda \leq \ell$,

$$\omega^\lambda(A - kF) > 0,$$

for $0 \leq k \leq \ell$.

Moreover, it is not hard to see that for each such $k$ there is a map $\rho_k : S^2 \to S^2$ of degree $-k$ whose graph

$$z \mapsto (z, \rho_k(z))$$

is an $\omega^\lambda$-symplectically embedded sphere in $X$. It follows easily that the space

$$\mathcal{J}_k^\lambda = \{ J \in \mathcal{J}^\lambda : \text{there is a } J\text{-hol curve in class } A - kF \}$$

is nonempty whenever $k < \lambda + 1$. Let

$$\overline{\mathcal{J}}_k^\lambda = \bigcup_{m \geq k} \mathcal{J}_m^\lambda.$$ 

Because $(A - kF) \cdot (A - mF) < 0$ when $k \neq m > 0$, positivity of intersections implies that there is exactly one $J$-holomorphic curve in class $A - kF$ for each $J \in \mathcal{J}_k$. We denote this curve by $\Delta_J$.

\textbf{Lemma 1.1} The spaces $\mathcal{J}_k^\lambda, 0 \leq k \leq \ell$, are disjoint and $\overline{\mathcal{J}}_k^\lambda$ is the closure of $\mathcal{J}_k^\lambda$ in $\mathcal{J}^\lambda$. Further, $\mathcal{J}^\lambda = \overline{\mathcal{J}}_0^\lambda$.

\textbf{Proof:} It is well known that for every $J \in \mathcal{J}$ the set of $J$-holomorphic curves in class $F$ form the fibers of a fibration $\pi_J : X \to S^2$. Moreover, the class $A$ is represented by either a curve or a cusp-curve (i.e. a stable map). Since the class $F$ is always represented and $(mA + pF) \cdot F = m$, it follows from positivity of intersections that $m \geq 0$ whenever $mA + pF$ is represented by a curve. Hence any cusp-curve in class $A$ has one component in some class $A - kF$ for $k \geq 0$, and all others represent a multiple of $F$. In particular, each $J \in \mathcal{J}^\lambda$ belongs to some set $\mathcal{J}_k^\lambda$. Moreover, because $(A - kF) \cdot (A - mF) < 0$ when $k \neq m$ and $k, m \geq 0$, the different $\mathcal{J}_k^\lambda$ are disjoint.

The second statement holds because if $J_n$ is a sequence of elements in $\mathcal{J}_k^\lambda$, then the corresponding sequence of $J_n$-holomorphic curves in class $A - kF$ has a convergent subsequence whose limit is a cusp-curve in class $A - kF$. This limit has to have a component in some class $A - mF$, for $m \geq k$, and so $J \in \mathcal{J}_m^\lambda$ for some $m \geq k$. For further details see Lalonde–McDuff [LM], for example.

Here is our main result. Throughout we are working with $C^\infty$-maps and almost complex structures, and so by manifold we mean a Fréchet manifold.

A stratified space $\mathcal{X}$ we mean a topological space that is a union of a finite number of disjoint manifolds that are called strata. Each stratum $S$ has a
neighborhood $N_S$ that projects to $S$ by a map $N_S \to S$. When $N_S$ is given the induced stratification, this map is a locally trivial fiber bundle whose fiber has the form of a cone $C(L)$ over a finite dimensional stratified space $L$ that is called the link of $S$ in $X$. Moreover, $S$ sits inside $N_S$ as the set of vertices of all these cones.

**Theorem 1.2** (i) For each $1 \leq k \leq \ell$, $J^\lambda_k$ is a submanifold of $J^\lambda$ of codimension $4k - 2$.

(ii) For each $m > k \geq 1$ the normal link $L^\lambda_{m,k}$ of $J^\lambda_m$ in $J^\lambda_k$ is a stratified space of dimension $4(m - k) - 1$. Thus, there is a neighborhood of $J^\lambda_m$ in $J^\lambda_k$ that is fibered over $J^\lambda_m$ with fiber equal to the cone on $L^\lambda_{m,k}$.

(iii) The structure of the link $L^\lambda_{m,k}$ is independent of $\lambda$ (provided that $\lambda > m - 1$.)

The first part of this theorem was proved by Abreu in [A], at least in the $C^s$-case where $s < \infty$. (Details are given in §4.1 below.) The second and third parts follow by globalising recent work by Fukaya–Ono [FO], Li–Tian [LiT], Liu–Tian [LiuT1], Ruan [R] and others on the structure of the compactification of moduli spaces of $J$-holomorphic spheres via stable maps. We extend current gluing methods by showing that it is possible to deal with obstruction bundles whose elements do not vanish at the gluing point: see §4.2.3. Another essential point is that we use Fukaya and Ono’s method of dealing with the ambiguity in the parametrization of a stable map since this involves the least number of choices and allows us to globalize by constructing a gluing map that is equivariant with respect to suitable local torus actions: see §4.2.4 and §4.2.5.

The above theorem is the main tool used in [AM] to calculate the rational cohomology ring of the group $G^\lambda$ of symplectomorphisms of $(X, \omega^\lambda)$.

Observe that part (iii) states that the normal structure of the stratum $J^\lambda_k$ does not change with $\lambda$. On the other hand, it follows from the results of [AM] that the cohomology of $J^\lambda_k$ definitely does change as $\lambda$ passes each integer. Obviously, it would be interesting to know if the topology of $J^\lambda_k$ is otherwise fixed. For example, one could try to construct maps $J^\lambda_k \to J^\mu$ for $\lambda < \mu$ that preserve the stratification, and then try to prove that they induce homotopy equivalences $J^\lambda_k \to J^\mu$ whenever $\ell - 1 < \lambda \leq \mu \leq \ell$.

The most we have so far managed to do in this direction is to prove the following lemma that, in essence, constructs maps $J^\lambda_k \to J^\mu$ for $\lambda < \mu$. It is not clear whether these are homotopy equivalences for $\lambda, \mu \in (\ell - 1, \ell]$. It is convenient to fix a fiber $F_0 = pt \times S^2$ and define

$$J^\lambda_k(N(F_0)) = \{ J \in J^\lambda_k : J = J^{\text{split}} \text{ near } F_0 \},$$

where $J^{\text{split}}$ is the standard product almost complex structure.

**Lemma 1.3** (i) The inclusion $J^\lambda(N(F_0)) \to J^\lambda$ induces a homotopy equivalence $J^\lambda_k(N(F_0)) \xrightarrow{\simeq} J^\lambda_k$ for all $k < \lambda + 1$. 

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(ii) Given any compact subset $C \subset J^\lambda(N(F_0))$ and any $\mu > \lambda$ there is a map

$$\iota_{\lambda,\mu} : C \to J^\mu(N(F_0))$$

that takes $C \cap J^\lambda_k(N(F_0))$ into $J^\mu_k(N(F_0))$ for all $k$.

This lemma is proved in §2.

The next task is to calculate the links $L^\lambda_{m,k}$. So far this has been done for the easiest case:

**Proposition 1.4** For each $k \geq 1$ and $\lambda$ the link $L^\lambda_{k+1,k}$ is the 3-dimensional lens space $L(2k,1)$.

Finally, we illustrate our methods by using the stable map approach to confirm that the link of $J^\lambda_2$ in $J^\lambda$ is $S^5$, as predicted by part (i) of Theorem 1.2. Our method first calculates an auxiliary link $L_Z$ from which the desired link is obtained by collapsing certain strata. The $S^5$ appears in a surprisingly interesting way, that can be briefly described as follows.

Let $O(k)$ denote the complex line bundle over $S^2$ with Euler number $k$, where we write $C$ instead of $O(0)$. Given a vector bundle $E \to B$ we write $S(E) \to B$ for its unit sphere bundle. Note that the unit 3-sphere bundle

$$S(O(k) \oplus O(m)) \to S^2$$

decomposes as the composite

$$S(L_{P(k,m)}) \to P(O(k) \oplus O(m)) \to S^2$$

where $L_{P(k,m)} \to P(O(k) \oplus O(m))$ is the canonical line bundle over the projectivization of $O(k) \oplus O(m)$. In particular, the space $S(O(-1) \oplus C)$ can be identified with $S(L_{P(-1,0)})$. But $P(O(-1) \oplus C)$ is simply the blow up $CP^2 \# CP^2$, and its canonical bundle is the pullback of the canonical bundle over $CP^2$. We also consider the singular line bundle (or orbibundle) $L_Y \to L$ whose associated unit sphere bundle has total space $L(Y) = S^5$ and fibers equal to the orbits of the following $S^1$-action on $S^5$:

$$\theta \cdot (x, y, z) = (e^{i\theta}x, e^{i\theta}y, e^{2i\theta}z), \quad x, y, z \in C.$$

**Theorem 1.5** (i) The space $L_Z$ obtained by plumbing the unit sphere bundle of $O(-3) \oplus O(-1)$ with the singular circle bundle $S(L_Y) \to Y$ may be identified with the unit circle bundle of the canonical bundle over $P(O(-1) \oplus C) = CP^2 \# CP^2$.

(ii) The link $L^\lambda_{2,0}$ is obtained from $L_Z$ by collapsing the fibers over the exceptional divisor to a single fiber, and hence may be identified with $S^5$. Under this identification, the link $L^\lambda_{2,1} = RP^3$ corresponds to the inverse image of a conic in $CP^2$. 

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In his recent paper [K], Kronheimer shows that the universal deformation of the quotient singularity \( \mathbb{C}^2/(\mathbb{Z}/m) \) is transverse to all the submanifolds \( \mathcal{J}_k \) and so is an explicit model for the normal slice of \( \mathcal{J}_m \) in \( \mathcal{J} \). Hence one can investigate the structure of the intermediate links \( \mathcal{L}_{m,k}^\lambda \) using tools from algebraic geometry. It is very possible that it would be easier to calculate these links this way. However, it is still interesting to try to understand these links from the point of view of stable maps, since this is more closely connected to the symplectic geometry of the manifold \( X \).

Another point is that throughout we consider \( \omega^\lambda \)-compatible almost complex structures rather than \( \omega^\lambda \)-tame ones. However, it is easy to see that all our results hold in the latter case.

Other ruled surfaces

All the above results have analogs for other ruled surfaces \( Y \to \Sigma \). If \( Y \) is diffeomorphic to the product \( \Sigma \times S^2 \), we can define \( \omega^\lambda, \mathcal{J}_k^\lambda \) as above, though now we should allow \( \lambda \) to be any number \( \geq -1 \) since there is no symmetry between the class \( A = [\Sigma \times pt] \) and \( F = [pt \times S^2] \). In this case Theorem 1.2 still holds. The reason for this is that if \( u : \Sigma \to Y \) is an injective \( J \)-holomorphic map in class \( A - kF \) where \( k \geq 1 \), then the normal bundle \( E \) to the image \( u(\Sigma) \) has negative first Chern class so that the linearization \( Du \) of \( u \) has kernel and cokernel of constant dimension. (In fact, the normal part of \( Du \) with image in \( E \) is injective in this case. See Theorem 1' in Hofer–Lizan–Sikorav [HLS].) However, Lemma 1.1 fails unless \( \Sigma \) is a torus since there are tame almost complex structures on \( Y \) with no curve in class \([A]\). One might think to remedy this by adding other strata \( \mathcal{J}_{\lambda,k}^\lambda \) consisting of all \( J \) such that the class \( A + kF \) is represented by a \( J \)-holomorphic curve \( u : (\Sigma, j) \to Y \) for some complex structure \( j \) on \( \Sigma \). However, although the universal moduli space \( \mathcal{M}(A + kF, \mathcal{J}^\lambda) \) of all such pairs \((u, J)\) is a manifold, the map \((u, J) \to J \) is no longer injective: even if one cuts down the dimension by fixing a suitable number of points each \( J \) will in general admit several curves through these points. Moreover, as \( u \) varies over \( \mathcal{M}(A + kF, \mathcal{J}^\lambda) \) the dimension of the kernel and cokernel of \( Du \) can jump. Hence the argument given in §4.1 below that the strata \( \mathcal{J}_{\lambda,k}^\lambda \) are submanifolds of \( \mathcal{J}^\lambda \) fails on several counts.

In the case of the torus, \( \mathcal{J}_0^\lambda \) is open and so Lemma 1.1 does hold. However, it is not clear whether this is enough for the main application, which is to further our understanding of the groups \( G^\lambda \) of symplectomorphisms of \((Y, \omega^\lambda)\). One crucial ingredient of the argument in [AM] is that the action of this group on each stratum \( \mathcal{J}_{k}^\lambda \) is essentially transitive. More precisely, we show that the action of \( G^\lambda \) on \( \mathcal{J}_{k}^\lambda \) induces a homotopy equivalence \( G^\lambda/\text{Aut}(J_k) \to \mathcal{J}_{k}^\lambda \), where \( J_k \) is an integrable element of \( \mathcal{J}_{k}^\lambda \) and \( \text{Aut}(J_k) \) is its stabilizer. It is not clear whether this would hold for the stratum \( \mathcal{J}_{0}^\lambda \) when \( \Sigma = T^2 \). One might have to take into account the finer stratification considered by Lorek in [Lo]. He points
out that the space $\mathcal{J}_0^\lambda$ of all $J$ that admit a curve in class $A$ is not homogeneous. A generic element admits a finite number of such curves that are regular (that is $Du$ is surjective), but since this number can vary the set of regular elements in $\mathcal{J}_0^\lambda$ has an infinite number of components. Lorek also characterises the other strata that occur. For example, the codimension 1 stratum consists of $J$ such that all $J$-holomorphic $A$ curves are isolated but there is at least one where the kernel of $Du$ has dimension 3 instead of 2. (Note that these 2 dimensions correspond to the reparametrization group, since $Du$ is the full linearization, not just the normal component.)

Similar remarks can be made about the case when $Y \to \Sigma$ is a nontrivial bundle. In this case we can label the strata $\mathcal{J}_k^\lambda$ so that the $J \in \mathcal{J}_k^\lambda$ admit sections with self-intersection $-2k + 1$. Again Theorem 1.2 holds, but Lemma 1.1 may not. When $\Sigma = S^2$ the homology class of the exceptional divisor is always represented, so that $\mathcal{J}^\lambda = \mathcal{J}^1_1$. When $\Sigma = T^2$, the homology class of the section of self-intersection +1 is always represented. Thus $\mathcal{J}^\lambda = \mathcal{J}^1_{-1}$. Hence the analog of Lemma 1.1 holds in these two cases. Moreover all embedded tori of self-intersection +1 are regular (by the same result in [HLS]), which may help in the application to $\text{Symp}(Y)$.

We now state in detail the result for the nontrivial bundle $Y \to S^2$ since this is used in [AM]. Here $Y = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and so every symplectic form on $Y$ can be obtained from an annulus $A_{r,s} = \{z \in \mathbb{C}^2 : r \leq |z| \leq s\}$ by collapsing the boundary spheres to $S^2$ along the characteristic orbits. This gives rise to a form $\omega_{r,s}$ that takes the value $\pi s^2$ on the class $L$ of a line and $\pi r^2$ on the exceptional divisor $E$. Let us write $\omega^\lambda$ for the form $\omega_{r,s}$ where $\pi s^2 = 1 + \lambda, \pi r^2 = \lambda > 0$. Then the class $F = L - E$ of the fiber has size 1 as before, and $\mathcal{J}_k^\lambda, k \geq 1$, is the set of $\omega^\lambda$-compatible $J$ for which the class $E - (k - 1)F$ is represented.

**Theorem 1.6** When $Y = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ the spaces $\mathcal{J}_k^\lambda$ are Fréchet submanifolds of $\mathcal{J}^\lambda$ of codimension $4k$, and form the strata of a stratification of $\mathcal{J}^\lambda$ whose normal structure is independent of $\lambda$. Moreover, the normal link of $\mathcal{J}^\lambda_{k+1}$ in $\mathcal{J}_k^\lambda$ is the lens space $L(4k + 1, 1), k \geq 1$.

This paper is organised as follows. §2 describes the main ideas in the proof of Theorem 1.2. This relies heavily on the theory of stable maps, and for the convenience of the reader we outline its main points. References for the basic theory are for example [FO], [LiT] and [LiuT1]. §3 contains a detailed calculation of the link of $\mathcal{J}^\lambda_2$ in $\mathcal{J}^\lambda$. In particular we discuss the topological structure of the space of degree 2 holomorphic self-maps of $S^2$ with up to 2 marked points, and of the canonical line bundle that it carries. Plumbing with the orbibundle $L_Y \to Y$ turns out to be a kind of orbifold blowing up process: see §3.1. Finally, in §4 we work out the technical details of gluing that are needed to establish that the submanifolds $\mathcal{J}_k^\lambda$ do have a good normal structure. The basic method here is taken from McDuff–Salamon [MS] and Fukaya–Ono [FO].
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2 Main ideas

We begin by proving Lemma 1.3 since this is elementary, and then will describe the main points in the proof of Proposition 1.2.

2.1 The effect of increasing $\lambda$

Proof of Lemma 1.3

Recall that $F_0$ is a fixed fiber $pt \times S^2$ and that

$$\mathcal{J}_k^\lambda(N(F_0)) = \{ J \in \mathcal{J}_k^\lambda : J = J_{\text{split}} \text{ near } F_0 \}.$$ 

We will also use the space

$$\mathcal{J}_k^\lambda(F_0) = \{ J \in \mathcal{J}_k^\lambda : J = J_{\text{split}} \text{ on } TF_0 \}.$$ 

Let $\mathcal{F}^\lambda$ be the space of $\omega^\lambda$-symplectically embedded curves in the class $F$ through a fixed point $x_0$. Because there is a unique $J$-holomorphic $F$-curve through $x_0$ for each $J \in \mathcal{J}^\lambda$ (see Lemma 1.1), there is a fibration

$$\mathcal{J}_k^\lambda(F_0) \rightarrow \mathcal{J}_k^\lambda \rightarrow \mathcal{F}^\lambda.$$ 

Since the elements of $\mathcal{J}_k^\lambda(F_0)$ are sections of a bundle with contractible fibers, $\mathcal{J}_k^\lambda(F_0)$ is contractible. Hence $\mathcal{F}^\lambda$ is also contractible. By using the methods of Abreu [A], it is not hard to show that the symplectomorphism group $G^\lambda = \text{Symp}_0(X, \omega^\lambda)$ of $(X, \omega^\lambda)$ acts transitively on $\mathcal{F}^\lambda$. Since the action of $G^\lambda$ on $\mathcal{J}_k^\lambda$ preserves the strata $\mathcal{J}_k^\lambda$, it follows that the projection $\mathcal{J}_k^\lambda \rightarrow \mathcal{F}^\lambda$ is surjective. Hence there are induced fibrations

$$\mathcal{J}_k^\lambda(F_0) \rightarrow \mathcal{J}_k^\lambda \rightarrow \mathcal{F}^\lambda.$$ 

This implies that the inclusion $\mathcal{J}_k^\lambda(F_0) \rightarrow \mathcal{J}_k^\lambda$ is a weak homotopy equivalence.

We now claim that the inclusion $\mathcal{J}_k^\lambda(N(F_0)) \rightarrow \mathcal{J}_k^\lambda(F_0)$ is also a weak homotopy equivalence. To prove this, we need to show that the elements of any compact set $K \subset \mathcal{J}_k^\lambda$ can be homotoped near $F_0$ to make them coincide with $J_{\text{split}}$. Since the set of tame almost complex structures at a point is contractible, this is always possible in $\mathcal{J}^\lambda$: the difficulty here is to ensure that $K$ remains in $\mathcal{J}_k^\lambda$ throughout the homotopy. Here is a sketch of one method. For each $J \in \mathcal{J}_k^\lambda$, let $\Delta_J$ denote the unique $J$-holomorphic curve in class $A - kF$. Then $\Delta_J$ meets
$F_0$ transversally at one point, call it $q_J$. For each $J \in \mathcal{K}$, isotop the curve $\Delta_J$ fixing $q_J$ to make it coincide in a small neighborhood of $q_J$ with the flat section $S^2 \times pt$ that contains $q_J$. (Details of an very similar construction can be found in [MP], Prop 4.1.C.) Now lift this isotopy to $J_k^\lambda$. Finally adjust the family of almost complex structures near $F_0$, keeping $\Delta_J$ holomorphic throughout.

This proves (i). Statement (ii) is now easy. For any compact subset $C$ of $\mathcal{J}^\lambda(\mathcal{N}(F_0))$ there is $\varepsilon > 0$ such that $J = J_{\text{split}}$ on the $\varepsilon$-neighborhood $\mathcal{N}_\varepsilon(F_0)$ of $F_0$. Let $\rho$ be a nonnegative 2-form supported inside the 2-disc of radius $\varepsilon$ that vanishes near 0, and let $\pi^*(\rho)$ denote its pullback to $\mathcal{N}_\varepsilon(F_0)$ by the obvious projection. Then every $J$ that equals $J_{\text{split}}$ on $\mathcal{N}_\varepsilon(F_0)$ is compatible with the form $\omega^\lambda + \kappa\pi^*(\rho)$ for all $\kappa > 0$. Since $\omega^\lambda + \kappa\pi^*(\rho)$ is isotopic to $\omega^\mu$ for some $\mu$, there is a diffeomorphism $\phi$ of $X$ that is isotopic to the identity and is such that $\phi^*(\omega^\lambda + \kappa\pi^*(\rho)) = \omega^\mu$. Moreover, because, by construction, $\pi^*(\rho) = 0$ near $F_0$, we can choose $\phi = \text{Id}$ near $F_0$. Hence the map $J \mapsto \phi^*(J)$ takes $\mathcal{J}^\lambda(\mathcal{N}(F_0))$ to $\mathcal{J}^\mu(\mathcal{N}(F_0))$. Clearly it preserves the strata $J_k$.

\[\square\]

### 2.2 Stable maps

From now on, we will drop $\lambda$ from the notation, assuming that $k < \lambda + 1$ as before. We study the spaces $\mathcal{J}_k$ and $\overline{\mathcal{J}}_k$ by exploiting their relation to the corresponding moduli spaces of $J$-holomorphic curves in $X$.

**Definition 2.1** When $k \geq 1$, $\mathcal{M}_k = \mathcal{M}(A - kF, \mathcal{J})$ is the universal moduli space of all unparametrized $J$-holomorphic curves in class $A - kF$. Thus its elements are equivalence classes $[h, J]$ of pairs $(h, J)$, where $J \in \mathcal{J} = \mathcal{J}^\lambda$, $h$ is a $J$-holomorphic map $S^2 \to X$ in class $A - kF$, and where $(h, J) \equiv (h \circ \gamma, J)$ when $\gamma : S^2 \to S^2$ is a holomorphic reparametrization of $S^2$. Similarly, we write $\mathcal{M}_0 = \mathcal{M}(A, x_0, \mathcal{J})$ for the universal moduli space of all unparametrized $J$-holomorphic curves in class $A$ that go through a fixed point $x_0 \in X$. Thus its elements are equivalence classes of triples $[h, z, J]$ with $z \in S^2$, $(h, J)$ as before, $h(z) = x_0$ and where $(h, z, J) \sim (h \circ \gamma, \gamma^{-1}(z), J)$ when $\gamma : S^2 \to S^2$ is a holomorphic reparametrization of $S^2$.

The next lemma restates part (i) of Theorem 1.2. The proof uses standard Fredholm theory for $J$-holomorphic curves and is given in §4.1. The only noteworthy point is that when $k > 0$ the almost complex structures in $\mathcal{J}_k$ are not regular. In fact, the index of the relevant Fredholm operator is $-(4k - 2)$. However, because we are in 4-dimensions the Fredholm operator has no kernel, which is the basic reason why the space of $J$ for which it has a solution is a submanifold of codimension $4k - 2$.

**Lemma 2.2** For all $k \geq 0$, the projection

$$\pi_k : \mathcal{M}_k \to \mathcal{J}_k : \ [h, J] \mapsto J$$
is a diffeomorphism of the Fréchet manifold $\mathcal{M}_k$ onto the submanifold $\mathcal{J}_k$ of $\mathcal{J}$. This submanifold is an open subset of $\mathcal{J}$ when $k = 0$ and has codimension $4k - 2$ otherwise.

Our tool for understanding the stratification of $\mathcal{J}$ by the $\mathcal{J}_k$ is the compactification $\overline{\mathcal{M}}(A - kF, \mathcal{J})$ of $\mathcal{M}(A - kF, \mathcal{J})$ that is formed by $J$-holomorphic stable maps. For the convenience of the reader we recall the definition of stable maps with $p$ marked points. We always assume the domain $\Sigma$ to have genus 0. Therefore it is a connected union $\bigcup_{i=0}^m \Sigma_i$ of Riemann surfaces each of which has a given identification with the standard sphere $(S^2, j_0)$. (Note that we consider $\Sigma$ to be a topological space: the labelling of its components is a convenience and not part of the data.) The intersection pattern of the components can be described by a tree graph with $m + 1$ vertices, one for each component of $\Sigma$, that are connected by an edge if and only if the corresponding components intersect. No more than two components meet at any point. Also, there are $p$ marked points $z_1, \ldots, z_p$ placed anywhere on $\Sigma$ except at an intersection point of two components. (Such pairs $(\Sigma, z_1, \ldots, z_p) = (\Sigma, z)$ are called semi-stable curves.)

Now consider a triple $(\Sigma, h, z)$ where $h : \Sigma \to X$ is such that $h^*([\Sigma]) = B$ and where the following stability condition is satisfied:

the restriction $h_i$ of the map $h$ to $\Sigma_i$ is nonconstant unless $\Sigma_i$ contains at least 3 special points.

(By definition, special points are either points of intersection with other components or marked points.) A stable map $\sigma = [\Sigma, h, z]$ in class $B \in H_2(X, \mathbb{Z})$ is an equivalence class of such triples, where $(\Sigma, h, z') \equiv (\Sigma, h \circ \gamma, z)$ if there is an element $\gamma$ of the group $\text{Aut}(\Sigma)$ of all holomorphic self-maps of $\Sigma$ such that $\gamma(z_i) = z_i'$ for all $i$. For example, if $\Sigma$ has only one component and there are no marked points, then $(\Sigma, h) \equiv (\Sigma, h \circ \gamma)$ for all $\gamma \in \text{Aut}(S^2) = \text{PSL}(2, \mathbb{C})$. Thus stable maps are unparametrized. We may think of the triple $(\Sigma, h, z)$ as a parametrized stable map. Almost always we will only consider stable maps that are $J$-holomorphic for some $J$. If necessary, we will include $J$ in the notation, writing elements as $\sigma = [\Sigma, h, z, J]$, but often $J$ will be understood.

Note that some stable maps $\sigma = [\Sigma, h, z, J]$ have a nontrivial reparametrization group $\Gamma_\sigma$. Given a representative $(\Sigma, h, z, J)$ of $\sigma$, this group may be defined as

$$\Gamma_\sigma = \{ \gamma \in \text{Aut}(\Sigma) : h \circ \gamma = h, \gamma(z_i) = z_i, 1 \leq i \leq p \}.$$ 

It is finite because of the stability condition. The points where this reparametrization group $\Gamma_\sigma$ is nontrivial are singular or orbifold points of the moduli space. Here is an example where it is nontrivial.

**Example 2.3** Let $\Sigma$ have three components, with $\Sigma_2$ and $\Sigma_3$ both intersecting $\Sigma_1$ and let $z_1$ be a marked point on $\Sigma_1$. Then we can allow $h_1$ to be constant without violating stability. If in addition $h_2, h_3$ have the same image curve, there is an automorphism that interchanges $\Sigma_2$ and $\Sigma_3$. Since nearby stable maps do
not have this extra symmetry, \([\Sigma, h, z_1]\) is a singular point in its moduli space. However, because marked points are labelled, there is no such automorphism if we put one marked point \(z_2\) on \(\Sigma_2\) and another \(z_3\) at the corresponding point on \(\Sigma_3\), i.e. so that \(h_2(z_2) = h_3(z_3)\). One can also destroy this automorphism by adding just one marked point \(z_0\) to \([\Sigma, h, z_1]\) anywhere on \(\Sigma_2\) or \(\Sigma_3\).

**Definition 2.4** For \(k \geq 0\) we define \(\overline{M}(A - kF, J)\) to be the space of all \(J\)-holomorphic stable maps \(\sigma = [\Sigma, h, J]\) in class \(A - kF\). Further, given any subset \(K\) of \(J\) we write
\[
\overline{M}(A - kF, K) = \cup_{J \in K} \overline{M}(A - kF, J).
\]

It follows from the proof of Lemma [1.1] that the domain \(\Sigma = \cup_{i=0}^p \Sigma_i\) of \(\sigma \in \overline{M}(A - kF, J)\) contains a unique component that is mapped to a curve in some class \(A - mF\), where \(m \geq k\). We call this component the stem of \(\Sigma\) and label it \(\Sigma_0\). Thus \(\overline{M}(A - kF, J_m)\) is the moduli space of all curves whose stem lies in class \(A - mF\). Note that \(\Sigma - \Sigma_0\) has a finite number of connected components called branches. If \(h_0\) is parametrized as a section, a branch \(B_w\) that is attached to \(\Sigma_0 = S^2\) at the point \(w\) is mapped into the fiber \(\pi^{-1}(w)\). In particular, distinct branches are mapped to distinct fibers.

The moduli spaces \(\overline{M}(A - kF, J)\) and \(\overline{M}(A - kF, J)\) have natural stratifications, in which each stratum is defined by fixing the topological type of the pair \((\Sigma, z)\) and the homology classes \([h_*(\Sigma_i)]\) of the components. Observe that the class \(A - mF\) of the stem is fixed on each stratum \(S\) in \(\overline{M}(A - kF, J)\). Hence there is a projection
\[
S \to J_m,
\]
whose fiber at \(J \in J_m\) is some stratum of \(\overline{M}(A - kF, J)\). Usually, in order to have a moduli space with a nice structure one needs to consider perturbed \(J\)-holomorphic curves. But, because we are working with genus 0 curves in dimension 4, the work of Hofer–Lizan–Sikorav [HLS] shows that all \(J\)-holomorphic curves are essentially regular. In particular, all curves representing some multiple \(mF\) of the fiber class are regular. Therefore each stratum of \(\overline{M}(A - kF, J)\) is a (finite-dimensional) manifold. The following result is an immediate consequence of Lemma [2.2].

**Lemma 2.5** Each stratum \(S\) of \(\overline{M}(A - kF, J)\) is a manifold and the projection \(S \to J_m\) is a locally trivial fibration.

**Definition 2.6** When \(k \geq 1\), we set \(\overline{M}_k = \overline{M}(A - kF, J)\). Further, \(\overline{M}_0 = \overline{M}(A, x_0, J)\) is the space of all stable maps \([\Sigma, h, z, J]\) where \([\Sigma, h, z]\) is a \(J\)-holomorphic stable map in class \(A\) with one marked point \(z\) such that \(h(z) = x_0\).

In the next section we show how to fit the strata of \(\overline{M}_k\) together by gluing to form an orbifold structure on \(\overline{M}_k\) itself.
2.3 Gluing

In this section we describe the structure of a neighborhood $N(\sigma) \subset \overline{\mathcal{M}}_k$ of a single point $\sigma \in \overline{\mathcal{M}}(A-kF, J_m)$. Suppose that $\sigma = [\Sigma, h, J]$, and order the components $\Sigma_i$ of $\Sigma$ so that $\Sigma_0$ is the stem and so that the union $\bigcup_{i \leq \ell} \Sigma_i$ is connected for all $\ell$. Then each $\Sigma_i, i > 0$ is attached to a unique component $\Sigma_j, j < i$ by identifying some point $w_i \in \Sigma_i$ with a point $z_i \in \Sigma_j$. At each such intersection point consider the “gluing parameter”

$$a_i \in T_{w_i} \Sigma_i \otimes C T_{z_i} \Sigma_j.$$

The basic process of gluing allows one to resolve the singularity of $\Sigma$ at the node $w_i = z_i$ by replacing the component $\Sigma_i$ by a disc attached to $\Sigma_j$ and suitably altering the map $h$. As we now explain, there is a 2-dimensional family of ways of doing this that is parametrized by (small) $a_i$.

**Proposition 2.7** Each $\sigma \in \overline{\mathcal{M}}(A-kF, J_m)$ has a neighborhood $N(\sigma)$ in $\overline{\mathcal{M}}_k$ that is a product $U_S(\sigma) \times (N(V_\sigma)/\Gamma_\sigma)$, where $U_S(\sigma) \subset \overline{\mathcal{M}}(A-kF, J_m)$ is a small neighborhood of $\sigma$ in its stratum $S$ and where $N(V_\sigma)$ is a small $\Gamma_\sigma$-invariant neighborhood of 0 in the space of gluing parameters

$$V_\sigma = \bigoplus_{i>0} T_{w_i} \Sigma_i \otimes C T_{z_i} \Sigma_j.$$

**Proof:** The proof is an adaptation of standard arguments in the theory of stable maps. The only new point is that the stem components are not regular so that when one does any gluing that involves this component one has to allow $J$ to vary in a normal slice $K_J$ to the submanifold $J_m$ at $J$. This analytic detail is explained in §4.2. What we will do here is describe the topological aspect of the proof.

First of all, let us describe the process of gluing. Given $a \in V_\sigma$, the idea is first to construct an approximately $J$-holomorphic stable map $(\Sigma_a, h_a, J)$ on a glued domain $\Sigma_a$ and then to perturb $h_a$ and $J$ using a Newton process to a $J_a$-holomorphic map $h_a : \Sigma_a \rightarrow X$ in $\overline{\mathcal{M}}(A-kF, K_J)$. We will describe the first step in some detail here since it will be used in §3. The analytic arguments needed for the second step are postponed to §4.

The glued domain $\Sigma_a$ is constructed as follows. For each $i$ such that $a_i \neq 0$, cut out a small open disc $\text{Int}D_{w_i}(r_i)$ in $\Sigma_i$ centered at $w_i$ and a similar disc $\text{Int}D_{z_i}(r_i)$ in $B_{z_i}$, where $r_i^2 = \|a_i\|$, and then glue the boundaries of these discs together with a twist prescribed by the argument of $a_i$. The Riemann surface $\Sigma_a$ is the result of performing this operation for each $i$ with $a_i \neq 0$. (When $a_i = 0$ one simply leaves the component $\Sigma_i$ alone.)

To be more precise, consider gluing $z \in \Sigma_0$ to $w \in \Sigma_1$. Take a Kähler metric on $\Sigma_0$ that is flat near $z$ and identify the disc $D_z(r)$ isometrically with the disc of radius $r$ in the tangent space $T_z = T_z(\Sigma_0)$ via the exponential map.
Take a similar metric on $(\Sigma_1, w)$. Then the gluing $\partial D_z(r) \to \partial D_w(r)$ may be considered as the restriction of the map

$$\Psi_a : T_z - \{0\} \to T_w - \{0\}$$

that is defined for $x \in T_z$ by the requirement:

$$x \otimes \Psi_a(x) = a, \ x \in T_z.$$

Thus, with respect to chosen identifications of $T_z$ and $T_w$ with $\mathbb{C}$, $\Psi_a$ is given by the formula: $x \mapsto a/x$ and so takes the circle of radius $r = \sqrt{\|a\|}$ into itself. This describes the glued domain $\Sigma_a$ as a point set. It remains to put a metric on $\Sigma_a$ in order to make it a Riemann surface. By hypothesis the original metrics on $\Sigma_0, \Sigma_1$ are flat near $z$ and $w$ and so may be identified with the flat metric $|dx|^2$ on $\mathbb{C}$. Since

$$\Psi_a(|dx|^2) = \left|\frac{a}{x}\right|^2 |dx|^2,$$

$\Psi_a(|dx|^2) = |dx|^2$ on the circle $|x| = r$. Hence, we may choose a function $\chi_r : (0, \infty) \to (0, \infty)$ so that the metric $\chi_r(|x|)|dx|^2$ is invariant by $\Psi_a$ and so that $\chi_r(s) = 1$ when $s > (1 + \varepsilon)r$, and then patch together the given metrics on $\Sigma_0 - D_z(2r)$ and $\Sigma_1 - D_z(2r)$ via $\chi_r(|x|)|dx|^2$.

In §3 we need to understand what happens as $a$ rotates around the origin. It is not hard to check that if we write $a_\theta = e^{i\theta}a_z \otimes a_w$, where $a_z \in T_z, a_w \in T_w$ are fixed, and if $\Psi_{a_\theta}$ identifies the point $p_z$ on $\partial D_z(r)$ with $p_w$ on $\partial D_w(r)$ then

$$\Psi_{a_\theta}(e^{i\theta} p_z) = p_w.$$  

The next step is to define the approximately holomorphic map (or pre-gluing) $h_a : \Sigma_a \to X$ for sufficiently small $\|a\|$. The map $h_a$ equals $h$ away from the discs $D_z(r_i), D_w(r_i)$, and elsewhere is defined by using cut-off functions that depend only on $|a|$. To describe the deformation of $h_a$ to a holomorphic map one needs to use analytical arguments. Hence further details are postponed until §4.

We are now in a position to describe a neighborhood of $\sigma$. It is convenient to think of $V_\sigma$ as the direct sum $V'_\sigma \oplus V''_\sigma$ where $V'_\sigma$ consists of the summands $T_w, \Sigma_i \otimes C T_z, \Sigma_j$, with $j_i = 0$ and $V''_\sigma$ of the rest. Note that the obvious action of $\Gamma_\sigma$ on $V_\sigma$ preserves this splitting. (It is tempting to think that the induced action on $V'_\sigma$ is trivial since the elements of $\Gamma_\sigma$ act trivially on the stem. However, this need not be so since they may rotate branch components that are attached to the stem.) If we glue at points parametrized by $a'' \in V''_\sigma$ then the corresponding curves lie in some branch and are regular. Hence the result of gluing is a $J$-holomorphic curve (i.e. there is no need to perturb $J$). Further, because the gluing map $\tilde{G}$ is $\Gamma_\sigma$-equivariant, there is a neighborhood of $\sigma$ in $\overline{\mathcal{M}(A - kF, J_m)}$ of the form

$$\mathcal{U}''(\sigma) = \mathcal{U}S(\sigma) \times (\mathcal{N}(V''_\sigma)/\Gamma_\sigma),$$
where \( \mathcal{N}(V) \) denotes a neighborhood of 0 in the vector space \( V \).

When we glue with elements from \( V'_\sigma \), the homology class of the stem changes and so the result cannot be \( J \)-holomorphic since \( J \in \mathcal{J}_m \). We show in Proposition 4.4 that if \( K_J \) is a normal slice to the submanifold \( \mathcal{J}_m \) at \( J \) then for sufficiently small \( a \in V'_\sigma \) the approximately holomorphic map \( h_a : \Sigma_a \to X \) deforms to a unique \( J_a \) holomorphic map \( \tilde{G}(h_\sigma, a) \) with \( J_a \in K_J \). We show in Proposition 4.4 that if \( K_J \) is a normal slice to the submanifold \( \mathcal{J}_m \) at \( J \) then for sufficiently small \( a \in V'_\sigma \) the approximately holomorphic map \( h_a : \Sigma_a \to X \) deforms to a unique \( J_a \) holomorphic map \( \tilde{G}(h_\sigma, a) \) with \( J_a \in K_J \). Therefore, for each element \( \sigma'' = [\Sigma, h'' \sigma] \in U''(\sigma) \) there is a homeomorphism from some neighborhood \( N(V'_{a''}) \) onto a neighborhood of \( \sigma'' \) in \( \mathcal{N}(V''_{a''}) \). Moreover, if \( U''(\sigma) \) is sufficiently small, the spaces \( V''_{a''} \) can all be identified with \( V_a'' \) and it follows from the proof of Proposition 4.4 that the neighborhoods \( \mathcal{N}(V''_{a''}) \) can be taken to have uniform size and so may all be identified. Hence the neighborhood \( \mathcal{N}(\sigma) \) projects to \( U''(\sigma) \) with fiber at \( \sigma'' \) equal to \( N(V''_{a''})/\Gamma_{a''} \). In general, the groups \( \Gamma_{a''} \) are subgroups of \( \Gamma_{a''} \) that vary with \( a'' \): in fact they equal the stabilizer of the corresponding gluing parameter \( a'' \in V(\sigma'') \). However, since elements of \( U_\sigma(\sigma) \) lie in the same stratum they have isomorphic isotropy groups. It is now easy to check that the composite map

\[
\mathcal{N}(\sigma) \to U''(\sigma) \to U_\sigma(\sigma)
\]

has fiber \( \mathcal{N}(V_a)/\Gamma_a \) as claimed. \( \square \)

### 2.4 Moduli spaces and the stratification of \( \mathcal{J} \)

Since each stable \( J \)-curve in class \( A - kF \) has exactly one component in some class \( A - mF \) with \( m \geq k \), the projection \( \pi_k : \overline{\mathcal{M}}(A - kF, \mathcal{J}) \to \mathcal{J} \) has image \( \overline{\mathcal{J}}_k \). Consider the inverse image

\[
\overline{\mathcal{M}}(A - kF, \mathcal{J}_m) = \pi_k^{-1}(\mathcal{J}_m).
\]

The next result shows that we can get a handle on the structure of \( \overline{\mathcal{J}}_k \) by looking at the spaces \( \overline{\mathcal{M}}(A - kF, \mathcal{J}_m) \).

**Proposition 2.8** When \( k > 0 \) the projection

\[
\pi_k : \overline{\mathcal{M}}(A - kF, \mathcal{J}_m) \to \mathcal{J}_m
\]

is a locally trivial fibration whose fiber \( \mathcal{F}_J(m - k) \) at \( J \) is the space of all stable \( J \)-curves \( [\Sigma, h] \) in class \( A - kF \) that have as one component the unique \( J \)-holomorphic curve \( \Delta_J \) in class \( A - mF \). In particular, \( \mathcal{F}_J(m - k) \) is a stratified space with strata that are manifolds of (real) dimension \( \leq 4(m - k) \). Its diffeomorphism type depends only on \( k - m \).

**Proof:** Let us look at the structure of \( \mathcal{F}_J(m - k) = \pi_k^{-1}(J) \). The stem of each element \( [\Sigma, h, J] \in \mathcal{F}_J(m - k) \) is mapped to the unique \( J \)-curve \( \Delta_J \) in class \( A - mF \). Fix this component further by supposing that it is parametrized as
a section of the fibration $\pi_j : X \to S^2$ (where $\pi_j$ is as in Lemma 1.1). We may divide the fiber $F_j(m - k)$ into disjoint sets $Z_{D,j}$ each parametrized by a fixed decomposition $D$ of $m - k$ into a sum $d_1 + \ldots + d_p$ of unordered positive numbers. The elements of $Z_{D,j}$ are those with $p$ branches $B_{w_1}, \ldots, B_{w_p}$ where $h_{\ast}[B_{w_i}] = d_i[F]$. Thus $Z_{D,j}$ maps onto the configuration space of $p$ distinct (unordered) points in $S^2$ labelled by the positive integers $d_1, \ldots, d_p$ with sum $m - k$. Moreover this map is a fibration with fiber equal to the product

$$\prod_{i=1}^{d} \overline{M}_{0,1}(S^2, q, d_i)$$

where $\overline{M}_{0,1}(S^2, q, d)$ is the space of $J$-holomorphic stable maps into $S^2$ of degree $d$ and with one marked point $z$ such that $h(z) = q$. (This point $q$ is where the branch is attached to $\Delta_j$.) According to the general theory, $\overline{M}_{0,1}(S^2, q, d)$ is an orbifold of real dimension $4(d-1)$. It follows easily that $Z_{D,j}$ is an orbifold of real dimension $4(m - k) - 2p$. It remains to understand how the different sets $Z_{D,j}$ fit together, i.e. what happens when two or more of the points $w_i$ come together. This may be described by suitable gluing parameters as in Proposition 2.7. The result follows. (For more details see any reference on stable maps, eg [FO], [LiT], [LiuT].) An example is worked out in §3.2.4 below.

\[\text{Note}\] For an analogous statement when $k = 0$ see Proposition 3.3.

Our next aim is to describe the structure of a neighborhood of $\overline{M}(A - kF, J_m)$ in $\overline{M}_k = \overline{M}(A - kF, J)$. We will write $Z_j$ for the fiber $F_j(m - k)$ of $\pi_k$ that was considered above and set

$$Z = \bigcup_{J \in J_m} Z_J, \quad Z_D = \bigcup_{J \in J_m} Z_{D,J}.$$ 

(The letter $Z$ is used here because $Z$ is the “zero-section” of the space of gluing parameters $V$ constructed below.) Consider an element $\sigma = [\Sigma, h, J]$ that lies in a substratum $Z_S$ of $Z_D$ where $D = d_1 + \ldots + d_p$. Then $\Sigma$ has $p$ branches $B_1, \ldots, B_p$ that are attached at the distinct points $w_1, \ldots, w_p \in \Sigma_0$. Let $z_i$ be the point in $B_i$ that is identified with $w_i \in \Sigma_0$ and define

$$V_{\sigma} = \bigoplus_{i=1}^{p} T_{z_i}B_i \otimes_{C} T_{w_i} \Sigma_0.$$ 

As explained in Proposition 2.4 the gluing parameters $a \in V_{\sigma}$ (when quotiented out by $\Gamma_{\sigma}$) parametrize a normal slice to $Z_D$ at $\sigma$. (Note that previously $V_{\sigma}$ was called $V_{\sigma''}$.) We now want to show how to fit these vector spaces together to form the fibers of an orbibundle over $Z_D$. Here we must incorporate twisting

\[\text{2} A \text{ rank } k \text{ orbibundle } \pi : E \to Y \text{ over an orbifold } Y \text{ has the following local structure. Suppose that } \sigma \in Y \text{ has local chart } U \subset U/\Gamma_{\sigma} \text{ where the uniformizer } U \text{ is a subset of } \mathbb{R}^{n}. \text{ Then } \pi^{-1}(U) \text{ has the form } U \times \mathbb{R}^{k}/\Gamma_{\sigma} \text{ where the action of } \Gamma_{\sigma} \text{ on } \mathbb{R}^{n} \times \mathbb{R}^{k} \text{ lifts that on } \mathbb{R}^{n} \text{ and is linear on } \mathbb{R}^{k}. \text{ There is an obvious compatibility condition between charts: see [FO], §2.} \]
that arises from the fact that gluing takes place on the space of parametrized stable maps. Since this is an important point, we dwell on it at some length. For the sake of clarity, we will in the next few paragraphs denote parametrized stable maps by \( \tilde{\sigma} = (\Sigma, h) \) and the usual (unparametrized) maps by \( \sigma = [\Sigma, h] \). Further, \( \Gamma_{\tilde{\sigma}} \) denotes the corresponding realization of the group \( \Gamma_{\sigma} \) as a subgroup of \( \text{Aut}(\Sigma) \).

Recall that \( X \) is identified with \( S^2 \times S^2 \) in such a way that the fibration \( \pi_J : X \to S^2 \) whose fibers are the \( J \)-holomorphic \( F \)-curves is simply given by projection onto the first factor. Hence each such fiber has a given identification with \( S^2 \). Further, we assume that the stem \( h_{\sigma,0} : \Sigma_{\sigma,0} \to \Delta_J \) is parametrized as a section \( z \mapsto (z, \rho(z)) \). Hence we only have to choose parametrizations of each branch. Since each branch component has at least one special point, its automorphism group is either trivial or has the homotopy type of \( S^1 \). Let \( \text{Aut}'(\Sigma) \) be the subgroup of \( \text{Aut}(\Sigma) \) consisting of automorphisms that are the identity on the stem. Then the identity component of \( \text{Aut}'(\Sigma) \) is homotopy equivalent to a torus \( T^{k(S)} \). (Here \( S \) is the label for the stratum containing \( \sigma \).)

Let \( g \) be a \( \Gamma_{\tilde{\sigma}} \)-invariant metric on the domain \( \Sigma \) that is also invariant under some action of the torus \( T^{k(S)} \).

**Definition 2.9** The group \( \text{Aut}^K(\Sigma) \) is defined to be the subgroup of the isometry group of \((\Sigma, g)\) generated by \( \Gamma_{\tilde{\sigma}} \) and \( T^{k(S)} \). Note that \( \Gamma_{\tilde{\sigma}} \) is the semidirect product of a subgroup \( \Gamma'_{\tilde{\sigma}} \) of \( T^{k(S)} \) with a subgroup \( \Gamma''_{\tilde{\sigma}} \) that permutes the components of each branch. Further \( \text{Aut}^K(\Sigma) \) is a deformation retract of the subgroup \( p^{-1}(\Gamma''_{\tilde{\sigma}}) \) of \( \text{Aut}(\Sigma) \), where we consider \( \Gamma''_{\tilde{\sigma}} \) as a subgroup of \( \pi_0(\text{Aut}(\Sigma)) \) and

\[
p : \text{Aut}(\Sigma) \to \pi_0(\text{Aut}(\Sigma))
\]

is the projection. For a further discussion, see §4.2.4.

Let us first consider a fixed \( J \in J_k \). It follows from the above discussion that on each stratum \( Z_{S,J} \) there is a principal bundle

\[
Z^\text{para}_{S,J} \to Z_{S,J}
\]

with fiber \( \text{Aut}^K(\Sigma) \) such that the elements of \( Z^\text{para}_{S,J} \) are parametrized stable maps \( \tilde{\sigma} = (\Sigma, h) \). Since the space \( V_{\tilde{\sigma}} \) of gluing parameters at \( \tilde{\sigma} \) is made from tangent spaces to \( \Sigma \) there is a well defined bundle

\[
V^\text{para}_{S,J} \to Z^\text{para}_{S,J}
\]

with fiber \( V_{\tilde{\sigma}} \). Further the action of the reparametrization group \( \text{Aut}^K(\Sigma) \) lifts to \( V^\text{para}_{S,J} \) and we define \( V_{S,J} \) to be the quotient \( V^\text{para}_{S,J} / \text{Aut}^K(\Sigma) \). Thus there is a commutative diagram

\[
\begin{array}{ccc}
V^\text{para}_{S,J} & \to & V_{S,J} \\
\downarrow & & \downarrow \\
Z^\text{para}_{S,J} & \to & Z_{S,J}.
\end{array}
\]
where the right hand vertical map is an orbibundle with fiber $V_\sigma/\Gamma_\sigma$.

Now consider the space $Z_{D,J} = \cup_{S \in D} Z_{S,J}$. The local topological structure of $Z_{D,J}$ is given by gluing parameters as in Proposition 2.7. Observe that every $J$ is regular for the branch components so that the necessary gluing operations can be performed keeping $J$ fixed. The spaces $Z_{D,J}^{\text{para}}$, $V_{D,J}^{\text{para}}$ are defined similarly and clearly there is a vector bundle $V_{D,J}^{\text{para}} \to Z_{D,J}^{\text{para}}$.

We want to see that the union

$$V_{D,J} = \bigcup_{S \in D} V_{S,J},$$

has the structure of an orbibundle over $Z_{D,J}$. The point here is that the groups $\text{Aut}^K(\Sigma)$ change dimension as $\tilde{\sigma}$ moves from stratum to stratum. Hence we need to see that the local gluing construction that fits the different strata in $V_{D,J}$ together is compatible with the group actions. We show in §4.2.4 below that the gluing map $\tilde{G}$ can be defined at the point $\tilde{\sigma}$ to be $\text{Aut}^K(\Sigma)$-invariant, i.e. so that

$$\tilde{G}(h_\sigma,a) = \tilde{G}(h_\sigma \circ \theta^{-1}, \theta \cdot a),$$

where $\tilde{G}(h_\sigma,a)$ is the result of gluing the map $h_\sigma$ with parameters $a$. In the situation considered here, we are dividing the set of gluing parameters at $\tilde{\sigma}$ into two, and will write $a = (a_b, a_s)$ where $a_b$ are the gluing parameters at intersections of branch components and $a_s$ are those involving the stem component. As $h_\sigma$ moves within $Z_{D,J}^{\text{para}}$ we glue along $a_b$, considering $a_s$ to be part of the fiber $V_\sigma$. Moreover, if $\tilde{\sigma}' = (\Sigma_{a_b}, \tilde{G}(h_\sigma, a_b))$, Lemma 4.9 (ii) shows that $\tilde{G}$ can be constructed to be compatible with the actions of the groups $\text{Aut}^K(\Sigma)$ and $\text{Aut}^K(\Sigma')$ on the fibers $V_\sigma$ and $V_{\sigma'}$ of $V_{D,J}^{\text{para}}$ at $\tilde{\sigma}, \tilde{\sigma}'$. It follows without difficulty that the quotient

$$V_{D,J} \to Z_{D,J}$$

is an orbibundle.

Finally, one forms spaces

$$V_J = \bigcup_D V_{D,J}, \quad V = \bigcup_{J \in J_0} V_J$$

whose local structure is also described by appropriate gluing parameters as above. Forgetting the gluing parameters gives projections

$$V_J \to Z_J = F_J(m - k); \quad \mathcal{V} \to \mathcal{Z} = F(m - k),$$

and $Z_J, Z$ embed in $V_J$ and $\mathcal{V}$ as the “zero sections”. The map $V_J \to Z_J$ preserves the stratifications of both spaces. However it is no longer an orbibundle since the dimension of the fiber $V_\sigma$ depends on $D$. In fact, the way that the different sets $V_{D,J}$ are fitted together is best thought of as a kind of plumbing: see §3.2.4.
Example 2.10 Everything is greatly simplified when \( m - k = 1 \). Here there is only one decomposition \( \mathcal{D} \) and the space \( \mathcal{Z}_{\mathcal{D}, J} \) consists of just one stratum diffeomorphic to \( S^2 \). Moreover the bundle \( \mathcal{Z}_{\mathcal{D}, J}^{\text{para}} \to \mathcal{Z}_{\mathcal{D}, J} \) has a section with the following description. Choose \( J \in \mathcal{J}_{k+1} \) so that \( \pi_J : X \to S^2 \) is the standard projection onto the first factor and so that the graph \( h_0 \) of the map \( \rho_k : S^2 \to S^2 \) of degree \( -(k + 1) \) is \( J \)-holomorphic. Let \( \Sigma_0, \Sigma_1 \) be two copies of \( S^2 \) and for each \( w \in S^2 \) define \((\Sigma_w, h_w) \in \mathcal{Z}_{\mathcal{D}, J}^{\text{para}} \) by

\[
\Sigma_w = \Sigma_0 \cup_{w = \rho(w)} \Sigma_1, \quad h_w|_{\Sigma_0} = h_0, \quad h_w|_{\Sigma_1} : z \mapsto (w, z).
\]

Hence in this case \( V_J \) is a complex line bundle over \( \mathcal{Z}_J = S^2 \). To calculate its Chern class, observe that \( V_J \) can be identified with the space \( \bigcup_{w \in S^2} T_{\rho(w)} \Sigma_1 \otimes T_w(\Sigma_0) = (T S^2)^{-k-1} \otimes T S^2 \), and so has Chern class \( -2k \).

The following result is proved in §4.

Proposition 2.11 There is a neighborhood \( \mathcal{N}_V(\mathcal{Z}) \) of \( \mathcal{Z} \) in \( V \) and a gluing map

\[
\mathcal{G} : \mathcal{N}_V(\mathcal{Z}) \to \overline{\mathcal{M}}(A - kF, \mathcal{J})
\]

that maps \( \mathcal{N}_V(\mathcal{Z}) \) homeomorphically onto a neighborhood of \( \overline{\mathcal{M}}(A - kF, \mathcal{J}_m) \) in \( \overline{\mathcal{M}}(A - kF, \mathcal{J}) \).

It follows from the construction of \( \mathcal{G} : \mathcal{N}_V(\mathcal{Z}) \to \overline{\mathcal{M}}(A - kF, \mathcal{J}_m) \) outlined in Proposition 2.7 that the stem of the glued map \( \mathcal{G}(\sigma, a) \) lies in the class \( A - (m - \sum_i n_i)F \) where the indices \( i \) label the branches \( B_i \) of \( \sigma \) and \( n_i \) is defined as follows. If \( a_i = 0 \) then \( n_i = 0 \). Otherwise, if \( \Sigma_{j_i} \) is the component of \( B_i \) that meets \( \Sigma_0 \) then \( n_i \) is the multiplicity of \( h_{j_i} \), that is \( [h(\Sigma_{j_i})] = n_iF \). Let \( \mathcal{N}_p \) denote the set of all elements \((\sigma, a) \in \mathcal{N}_V(\mathcal{Z}) \) such that the stem of the glued map lies in class \( A - pF \). In other words,

\[
\mathcal{N}_p = (\pi_k \circ \mathcal{G})^{-1} \mathcal{J}_p.
\]

Clearly, \( \mathcal{N}_p \) is a union of strata in the stratified space \( \mathcal{N}_V(\mathcal{Z}) \). Further, when \( k > 0 \) the map \( \pi_k : \mathcal{G}(\mathcal{N}_p) \to \mathcal{J}_p \) is a fibration with fiber \( \mathcal{F}(p - k) \). The next proposition follows immediately from Proposition 2.8.

Proposition 2.12 The link \( \mathcal{L}_{m,k} \) is the finite-dimensional stratified space obtained from the link of \( \mathcal{Z}_J \) in \( \mathcal{V}_J \) by collapsing the fibers of the projections \( \mathcal{V}_J \cap \mathcal{N}_p \to \mathcal{J}_p \) to single points.
Proof of Proposition 1.4

We have to show that the link $L_{k+1,k}$ is the lens space $L(2k,1)$. We saw in Example 2.10 that $V_J$ is a line bundle with Chern class $-2k$. In this case there is only one nontrivial stratum in $N_V(Z)$, namely $N_k$, which is the complement of the zero section. Moreover, the map $\pi_k \circ G$ is clearly injective. Hence by the above lemma $L_{k+1,k}$ is simply the unit sphere bundle of $V_J$ and so is a lens space as claimed.

3 The link $L_{2,0}$ of $J_2$ in $J_0$

In this section we illustrate Proposition 2.12 by calculating the link $L_{2,0}$. We know from Lemma 2.2 that $L_{2,0} = S^5$. The general theory of §2 implies that $L_{2,0}$ can be obtained from the link $L_Z$ of the zero section $Z_J$ in the stratified space $V_J$ of gluing data by collapsing certain strata. When looked at from this point of view, the $S^5$ appears in quite a complicated way that was described in Theorem 1.3. We begin here by explaining the plumbing construction, and then discuss how this relates to $L_Z$.

3.1 Some topology

Recall that $S(L_P) \to \mathcal{P}(O(k) \oplus O(m))$ is the unit circle bundle of the canonical line bundle $L_P$ over the projectivization $\mathcal{P}(O(k) \oplus O(m))$.

Lemma 3.1 The bundle $S(L_P) \to \mathcal{P}(O(-1) \oplus \mathbb{C})$ can be identified with the pullback of the canonical circle bundle $S(L_{can}) \to \mathbb{C}P^2$ over the blowdown map $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^2$.

Proof: It is well known that $\mathcal{P}(O(-1) \oplus \mathbb{C})$ can be identified with $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to \mathbb{C}P^2$. Indeed the section $S_- = \mathcal{P} \{0\} \oplus \mathbb{C}$ has self-intersection $-1$, while $S_+ = \mathcal{P} \{0\} \oplus \{0\}$ has self-intersection $1$. Further, the circle bundle $L_P$ is trivial over $S_-$ and has Euler class $-1$ over $S_+$ and over the fiber class. The result follows.

The space we are interested in is formed by plumbing a rank 2 bundle $E \to S^2$ to a line bundle $L \to Y$, where $\dim(Y) = 4$. This plumbing $E \# L$ is the space obtained from the unit disc bundles $D(E) \to S^2$ and $D(L) \to Y$ by identifying the inverse images of discs $D^2, D^4$ on the two bases in the obvious way: the disc fibers of $D(E) \to S^2$ are identified with with flat sections of $D(L)$, and flat sections of $D(E)$ over $D^4$ are identified with fibers of $D(L)$. There is a corresponding plumbing $S(E) \# S(L)$ of the two sphere bundles $S^3 \to S(E) \to S^2$ and $S^1 \to S(L) \to Y$, obtained by cutting out the inverse images of open discs in the two bases and appropriately gluing the boundaries. The resulting space $S(E) \# S(L)$ is the link of the core $S^2 \cup Y$ in the plumbed bundle $E \# L$.
Lemma 3.2 Let $L_{\text{can}} \to \mathbb{CP}^2$ be the canonical line bundle and $E = \mathcal{O}(k) \oplus \mathcal{O}(m)$. Then $E \ncong L_{\text{can}}$ may be identified with the blow-up of $\mathcal{O}(k+1) \oplus \mathcal{O}(m+1)$ at a point on its zero section. Hence

$$S(E) \ncong S(L_{\text{can}}) = S((\mathcal{O}(k+1) \oplus \mathcal{O}(m+1)).$$

Proof: First consider the structure of the blow up $\tilde{\mathbb{C}}^3$ of $\mathbb{C}^3 = \mathbb{C} \times \mathbb{C}^2$ at the origin. The fibration $\pi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}$ induces a fibration $\tilde{\pi} : \tilde{\mathbb{C}}^3 \to \mathbb{C}$.

Clearly, the inverse image $\tilde{\pi}^{-1}(z)$ of each point $z \neq 0$ is a copy of $\mathbb{C}^2$ while $\tilde{\pi}^{-1}(0)$ is the union of the exceptional divisor together with the set of lines in the original fiber $\pi^{-1}(0)$. Let $\lambda$ be the line in $\mathbb{C} \times \mathbb{C}^2$ through the origin and the point $(1, a, b)$. Lift $\lambda$ to the blow-up and consider its intersection with $\tilde{\pi}^{-1}(S^1) = \pi^{-1}(S^1) \subset \mathbb{C} \times \mathbb{C}^2$.

where $S^1$ is the unit circle in $\mathbb{C}$. This intersection consists of the points $(e^{it}, e^{it}a, e^{it}b)$, hence it is these circles (rather than the circles $(e^{it}, a, b)$) that bound discs in the blowup. Therefore, if we think of the blowup $\tilde{\mathbb{C}}^3$ as the plumbing of the bundle $\pi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}$ with $L_{\text{can}}$, the original trivialization of $\pi$ differs from the trivialization (or product structure) near $\pi^{-1}(0)$ that is used to construct the plumbing.

Now recall that

$$\mathcal{O}(k) = D^+ \times \mathbb{C} \cup_\alpha D^- \times \mathbb{C},$$

where $D^+, D^-$ are 2-discs, with $D^+$ oriented positively and $D^-$ negatively, and where the gluing map $\alpha$ is given by

$$\alpha : \partial D^+ \times \mathbb{C} \to \partial D^- \times \mathbb{C} : (e^{it}, w) \mapsto (e^{it}, e^{-ikt}w).$$

It follows easily that the blowup of $D(\mathcal{O}(k+1) \oplus \mathcal{O}(m+1))$ at a point on its zero-section is obtained by plumbing the disc bundle $D(\mathcal{O}(k) \oplus \mathcal{O}(m))$ with $D(L_{\text{can}})$. This proves the first statement. The second statement is then immediate. \(\square\)

We are interested in plumbing not with $L_{\text{can}} \to \mathbb{CP}^2$ but with a particular singular line bundle (or orbibundle) $L_Y \to Y$. This means that the unit circle bundle $S(L_Y) \to Y$ is a Seifert fibration with a finite number of singular (or multiple) fibers. In our case, there is an $S^1$ action on $S(L_Y)$ such that the fibers of the map $S(L_Y) \to Y$ are the $S^1$-orbits. In fact, we can identify $S(L_Y)$ with $S^5$ in such a way that the $S^1$ action is

$$\theta : (x, y, z) = (e^{i\theta}x, e^{i\theta}y, e^{2i\theta}z), \quad x, y, z \in \mathbb{C}.$$

Thus there is one singular fiber that goes through the point $(0, 0, 1)$. All other fibers $F$ are regular. For each such $F$ there is a diffeomorphism of $S^5$ that takes...
$F$ to the circle $\gamma_0 = (e^{i\theta}, e^{i\theta}, 0)$. Identify a neighborhood of $\gamma_0$ with $S^1 \times D^4$ in such a way that

$$S^5 = S^1 \times D^4 \cup D^2 \times S^3,$$

with the identity map of $S^1 \times S^3$ as gluing map. Then, in these coordinates near $\gamma_0$ the fibers of $S(L_Y)$ are (diffeomorphic to) the circles

$$\gamma_x = \{(\theta, A_\theta(x)) \in S^1 \times D^4 : A_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{2i\theta} \end{pmatrix} \}.$$

By way of contrast, the fibers of $S^5$ with the Hopf fibration have neighborhoods fibered by the circles

$$\gamma'_x = \{(\theta, A'_\theta(x)) \in S^1 \times D^4 : A'_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \}.$$

The next result shows that plumbing with $S(L)$ is a kind of twisted blowup.

**Proposition 3.3** Let $L_Y \to Y$ be the orbibundle described in the previous paragraph. Then the manifold obtained by plumbing $S(\mathcal{O}(k) \oplus \mathcal{O}(m))$ with a regular fiber of $S(L_Y)$ is diffeomorphic to $S(\mathcal{O}(k + 2) \oplus \mathcal{O}(m + 1))$.

**Proof:** We may think of plumbing as the result of a surgery that matches the flat circles $S^1 \times pt$ in the copy of $S^1 \times S^3$ in $S(\mathcal{O}(k) \oplus \mathcal{O}(m))$ with the circles $\gamma_x$ in the neighborhood of a regular fiber $\gamma_0$ of $S(L_Y)$. We would get the same result if we matched the circles

$$\delta_x = \{(\theta, A''_\theta(x)) \in S^1 \times S^3 : A''_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix} \}$$

in $S^1 \times S^3 \subset S(\mathcal{O}(k) \oplus \mathcal{O}(m))$ with the circles $\gamma'_x$ in the standard (Hopf) $S^5$. But if we trivialize the boundary of $S(\mathcal{O}(k) \oplus \mathcal{O}(m)) - D^2 \times S^3$ by the circles $\delta_x$ we get the same as if we trivialized the boundary of $S(\mathcal{O}(k + 1) \oplus \mathcal{O}(m)) - D^2 \times S^3$ in the usual way by flat circles. Thus

$$S(\mathcal{O}(k) \oplus \mathcal{O}(m)) \equiv S(L_Y) = S(\mathcal{O}(k + 1) \oplus \mathcal{O}(m)) \equiv S(L_{can}) = S(\mathcal{O}(k + 2) \oplus \mathcal{O}(m + 1)).$$

There is a question of orientations here: do we have to add or subtract 1 from $k$ to compensate for the extra twisting in $S(L_Y)$? One can check that it is correct to add 1 by using the present approach to give an alternate proof of the previous lemma. For, if we completely untwisted the circles in the neighborhood of $\gamma_0$ (thereby increasing the twisting of the other side by an additional 2), we would be doing the trivial surgery in which the attaching map is the identity. Note also that because the sum $\mathcal{O}(k) \oplus \mathcal{O}(m)$ depends only on $k + m$ we could equally well have put the extra twist on the other factor. $\square$
3.2 Structure of the pair \((\mathcal{V}_J, \mathcal{Z}_J)\)

Our aim is to prove the following proposition, where \(\mathcal{V}_J\) is the space of gluing parameters for a fixed \(J \in \mathcal{J}_2\) that describes the link of the space of \((A - 2F)\)-curves in the space of (pointed) \(A\)-curves.

**Proposition 3.4** The link \(L_Z\) of the zero section \(Z_J\) in the stratified space \(\mathcal{V}_J\) is constructed by plumbing \(S(\mathcal{O}(-3) \oplus \mathcal{O}(-1))\) to \(S(L_Y)\). Hence

\[
L_Z = S(\mathcal{O}(-1) \oplus \mathcal{C}).
\]

We are now not quite in the situation described in Proposition 2.8 because we are including the open stratum \(\mathcal{J}_0\) of \(\mathcal{J}\). This means that we have to replace the space \(\overline{\mathcal{M}}_k = \overline{\mathcal{M}}(A - kF, \mathcal{J})\) by a space \(\overline{\mathcal{M}}_0\) of curves of class \(A\) that go through the fixed point \(x_0\). Since we are interested in working out the structure of the fiber of the projection \(\overline{\mathcal{M}}_0 \to \mathcal{J}\) at a point \(J \in \mathcal{J}_2\), we will choose \(x_0\) so that it does not lie on the unique \(J\)-holomorphic \((A - 2F)\)-curve \(\Delta_J\) and then define \(\mathcal{M}_0\) to be the space \(\mathcal{M}(A, x_0, \mathcal{J})\) in Definition 2.6. Let \(\pi_0\) denote the projection

\[
\pi_0 : \overline{\mathcal{M}}_0 \to \mathcal{J}
\]

and set \(\overline{\mathcal{M}}_0(\mathcal{J}_m) = \pi_0^{-1}(\mathcal{J}_m)\) as before. It is not hard to see that the following analog of Proposition 2.8 holds.

**Proposition 3.5** (i) Let \(J \in \mathcal{J}_m\) be any almost complex structure such that the unique \(J\)-holomorphic \((A - mF)\)-curve \(\Delta_J\) does not go through \(x_0\). Then the projection

\[
\pi_k : \overline{\mathcal{M}}_0(\mathcal{J}_m) \to \mathcal{J}_m
\]

is a locally trivial fibration near \(J\), whose fiber \(F_J(0, m)\) is the space of all stable \(J\)-curves \([\Sigma, u]\) in class \(A\) that have \(\Delta_J\) as one component and go through \(x_0\). In particular, \(F_J(0, m)\) is a stratified space whose strata are orbifolds of (real) dimension \(\leq 4m - 2\).

(ii) The singular fibers of \(\pi_k : \overline{\mathcal{M}}_0(\mathcal{J}_m) \to \mathcal{J}_m\) occur at points \(J\) for which \(x_0 \in \Delta_J\). For such \(J\), \(\pi_k^{-1}(J)\) can be identified with the space \(F_J(m)\) described in Proposition 2.8.

As before, we now construct a pair \((\mathcal{V}_J, \mathcal{Z}_J)\) that describes a neighborhood of \(\overline{\mathcal{M}}_0(\mathcal{J}_m)\) in \(\overline{\mathcal{M}}_0\). We will concentrate on the case \(m = 2\) and will suppose that \(x_0 \notin \Delta_J\). We further normalize \(J\) by requiring that the projection \(\pi_J\) along the \(J\)-holomorphic \(F\)-curves is simply the projection onto the first factor \(S^2\). We write \(q_0 = \pi_J(x_0)\).

§3.2.1 The bundle \(\mathcal{V}_{2,J} \to \mathcal{Z}_{2,J}\).

Observe first that \(\mathcal{Z}_J\) has two subsets: \(\mathcal{Z}_{1,J}\) consisting of all stable \(A\)-maps \([\Sigma, z_0, h]\) that are the union of the \((A - 2F)\)-curve \(\Delta_J\) with a double covering
of the fiber $F_0$ through $x_0$, and $Z_{2,J}$ consisting of all stable $A$-maps $[\Sigma, z_0, h]$ that are the union of $\Delta_J$ with two distinct fibers. We will call these sets $Z_{i,J}$ strata. This is accurate as far as $Z_{2,J}$ is concerned, but strictly speaking $Z_{1,J}$ is a union of strata. (Recall that the strata are determined by the topological type of the marked domain $[\Sigma, z_0]$ and the homology class of the images of its components under $h$.)

Let us first consider $Z_{2,J}$. Since $h(z_0) = x_0$ always, one of the two fibers has to be $F_0$ and the other moves. Therefore, the stratum $Z_{2,J}$ maps onto $S^2 - \{q_0\}$. It is convenient to compactify $Z_{2,J}$ by adding a point $\sigma_*$ that projects to $q_0$. The domain $\Sigma$ of $\sigma_*$ has 4 components with $\Sigma_0, \Sigma_2, \Sigma_3$ all meeting $\Sigma_1$ and a marked point $z_0 \in \Sigma_3$. The map $h_0 : \Sigma_0 \to \Delta_J$ parametrizes $\Delta_J$ as a section, $h_1$ takes $\Sigma_1$ onto the point $F_0 \cap \Delta_J$, and $h_2, h_3$ have image $F_0$ with $h_3(z_0) = x_0$. The argument of Example 2.10 gives the following result.

**Lemma 3.6** The space $V_{2,J}$ of gluing parameters over $Z_{2,J} \cup \{\sigma_*\} = S^2$ is the bundle $O(-2) \oplus \mathbb{C}$.

### §3.2.2 The stratum $Z_{1,J}$

This gives half of $L_2$. The other half comes from the link of the orbifold $Z_{1,J}$ in $V_{1,J}$. Thus the next step is to look at $Z_{1,J}$.

Let $p_0, p_1$ be two distinct points on $F_0 = S^2$, with $p_0 = x_0$ and $p_1 = \Delta_J \cap F_0$. Then $Z_{1,J}$ is the orbifold

$$Z_{1,J} = Y = \overline{\mathcal{M}}_{0,2}(S^2, p_0, p_1, 2)$$

of all stable maps to $S^2$ with two marked points $z_0, z_1$ that are in the class $2[S^2]$ and are such that $h(z_0) = p_0, h(z_1) = p_1$. We will also need to consider the space $\overline{\mathcal{M}}_{0,0}(S^2, 2)$ of genus 0 stable maps of degree 2 into $S^2$ that have no marked points and the space

$$\tilde{Y} = \overline{\mathcal{M}}_{0,3}(S^2, p_0, p_1, p_2, 2)$$

of all degree 2 stable maps to $S^2$ with three marked points $z_0, z_1, z_2$ such that $h(z_0) = p_0, h(z_1) = p_1, h(z_2) = p_2$.

**Lemma 3.7** (i) $\overline{\mathcal{M}}_{0,0}(S^2, 2)$ is a smooth manifold diffeomorphic to $\mathbb{C}P^2$.

(ii) $\tilde{Y} = \overline{\mathcal{M}}_{0,3}(S^2, p_0, p_1, p_2, 2)$ is a smooth manifold diffeomorphic to $\mathbb{C}P^2$.

(iii) The forgetful map $f : \tilde{Y} \to Y$ may be identified with the 2-fold cover that quotients $\mathbb{C}P^2$ by the involution $\tau : [x : y : z] \mapsto [x : y : -z]$. In particular, $Y$ is smooth except at the point $\sigma_{01} = f([0 : 0 : 1])$ that has the local chart $\mathbb{C}^2/(x, y) = (-x, -y)$. This point $\sigma_{01}$ is the stable map $[S^2, h, z_0, z_1]$ where the critical values of $h$ are at $p_0$ and $p_1$. 

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Proof: (i) The space $\overline{M}_{0,0}(S^2,2)$ has two strata. The first, $S_1$, consists of self-maps of $S^2$ of degree 2, and the second, $S_2$, consists of maps whose domain has two components, each taken into $S^2$ by a map of degree 1. The equivalence relation on each stratum is given by precomposition with a holomorphic self-map of the domain. It is not hard to check that each equivalence class of maps in $S_1$ is uniquely determined by its two critical values (or branch points). Since these can be any pair of distinct points, $S_1$ is diffeomorphic to the set of unordered pairs of distinct points in $S^2$. On the other hand there is one element $\sigma_w$ of $S_2$ for each point $w \in S^2$, the correspondence being given by taking $w$ to be the image under $h$ of the point of intersection of the two components.

If $\sigma_{\{x,y\}}$ denotes the element of $S_1$ with critical values $\{x,y\}$, we claim that $\sigma_{\{x,y\}} \to \sigma_w$ when $x,y$ both converge to $w$. To see this, let $h_{\{x,y\}} : S^2 \to S^2$ be a representative of $\sigma_{\{x,y\}}$ and let $\alpha_{\{x,y\}}$ be the shortest geodesic from $x$ to $y$. (We assume that $x,y$ are close to $w$.) Then $h_{\{x,y\}}^{-1}(\alpha_{\{x,y\}})$ is a circle $\gamma_{\{x,y\}}$ through the critical points of $h_{\{x,y\}}$. This is obvious if $h_{\{x,y\}}$ is chosen to have critical points at $0,\infty$ and if $x = 0, y = \infty$ since $h_{\{x,y\}}$ is then a map of the form $z \mapsto az^2$. It follows in the general case because Mobius transformations take circles to circles. Hence $h_{\{x,y\}}$ takes each component of $S^2 - \gamma_{\{x,y\}}$ onto $S^2 - \alpha_{\{x,y\}}$. If we now let $x,y$ converge to $w$, we see that $\sigma_{\{x,y\}}$ converges to $\sigma_w$.

The above argument shows that $\overline{M}_{0,0}(S^2,2)$ is the quotient of $S^2 \times S^2$ by the involution $(x,y) \mapsto (y,x)$. This is well known to be $\mathbb{C}P^2$. In fact, it is easy to check that the map

$$H : ([x_0 : x_1], [y_0 : y_1]) \mapsto [x_0 y_0 : x_1 y_1 : x_0 y_1 + x_1 y_0 - x_0 y_0 - x_1 y_1]$$

induces a diffeomorphism from the quotient to $\mathbb{C}P^2$. Under this identification the stratum $S_2 = H(diag)$ is the quadric $(u + v + w)^2 = 4uv$ (where we use coordinates $[u : v : w]$ on $\mathbb{C}P^2$). Further, if we put

$$p_0 = [0 : 1], \quad p_1 = [1 : 0], \quad p_2 = [1, 1],$$

the set of points in $\overline{M}_{0,0}(S^2,2) = \mathbb{C}P^2$ consisting of maps that branch over $p_i$ is a line $\ell_i$, the image by $H$ of $(S^2 \times p_i) \cup p_i \times S^2$. Thus

$$\ell_0 = \{u = 0\}, \quad \ell_1 = \{v = 0\}, \quad \ell_2 = \{w = 0\}.$$ 

Note finally that all stable maps in $\overline{M}_{0,0}(S^2,2)$ are invariant by an involution: for example the map $z \mapsto z^2$ is invariant under the reparametrization $z \mapsto -z$. Since all elements have the same reparametrization group, $\overline{M}_{0,0}(S^2,2)$ is smooth. However, this will no longer be the case when we add two marked points.

(ii) Now consider the forgetful map

$$\phi_3 : \overline{M}_{0,3}(S^2, p_0, p_1, p_2, 2) \to \overline{M}_{0,0}(S^2, 2).$$

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For a general point of $\overline{M}_{0,0}(S^2, 2)$, that is a point where neither branching point is at $p_0$, $p_1$ or $p_2$, $\phi_{30}$ is 4 to 1. To see this, note that for $i = 0, 1, 2, z_i$ can be either of the points that get mapped to $p_i$ which seems to give an 8-fold cover. However, because $h$ has degree 2, $h$ is invariant under an involution $\gamma$ of $S^2$ that interchanges the two inverse images of a generic point. Hence the cover is 4 to 1, and the covering group is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. When just one branching point is at some $p_i$, $\phi_{30}$ is 2 to 1, and when both branching points are at some $p_i$, it is 1 to 1. This determines $\phi_{30}$. In fact, with the above identification for $\overline{M}_{0,0}(S^2, 2) = CP^2$, $\phi_{30}$ is the map

$$\phi_{30} : CP^2 \to CP^2 : [x : y : z] \mapsto [x^2 : y^2 : z^2].$$

Note that the inverse image of $S_2 = \{4uv = (u + v + w)^2\}$ consists of the 4 lines $x = y = iz = 0$.

These components correspond to the 4 different ways of arranging 3 points on the two components of the stable maps in $S_2$. Note further that none of the points in $\overline{M}_{0,0,3}(S^2, p_0, p_1, p_2, 2)$ are invariant by any reparametrization of their domains. Hence all points of this moduli space are smooth.

(iii) Similar reasoning shows that the forgetful map

$$\phi_{30} : Y = \overline{M}_{0,2}(S^2, p_0, p_1, 2) \to \overline{M}_{0,0}(S^2, 2)$$

is a 2-fold cover branched over $\ell_0 \cup \ell_1$. Hence we may identify $Y$ as

$$Y = \{[u : v : w : t] \in CP^3 : t^2 = uv\}$$

where the cover $\phi_{30} : Y \to CP^2$ forgets $t$. There is one point in $Y$ that is invariant under a reparametrization of its domain, namely the point $\sigma_{01}$ corresponding to the map $h : S^2 \to S^2$ that branches at $p_0$ and $p_1$. In the above coordinates on $Y$,

$$\sigma_{01} = \phi_{30}^{-1}(\ell_0 \cap \ell_1) = [0 : 0 : 1 : 0].$$

It is also easy to check that

$$\phi_{32} : \overline{M}_{0,3}(S^2, p_0, p_1, p_2, 2) = CP^2 \to Y$$

has the formula

$$\phi_{32}([x : y : z]) = [x^2 : y^2 : z^2 : xy].$$

Since $\phi_{32} \circ \tau([x : y : z]) = \phi_{32}([x : y : -z]) = \phi_{32}([x : y : z])$, $\phi_{32}$ is equivalent to quotienting out by $\tau$ as claimed.

§3.2.3 The bundle $V_{1,j} \to Z_{1,j}$.

Now we consider the structure of the orbibundles of gluing parameters over $\bar{Y} = \overline{M}_{0,3}(S^2, p_0, p_1, p_3, 2)$ and $Z_{1,j} = Y = \overline{M}_{0,2}(S^2, p_0, p_1, 2)$. We will call the first $\bar{L} \to \bar{Y}$ and the second $L_Y \to Y$. In both cases the fiber at the stable map $[\Sigma, h, z_i]$ is the tangent space $T_{z_i} \Sigma$. 24
Lemma 3.8  (i) The orbibundle $\tilde{L} \to \tilde{Y}$ is smooth and may be identified with the canonical line bundle $L_{\text{can}}$ over $\tilde{Y} = CP^2$.

(ii) The orbibundle $L_Y \to Y$ is smooth except at the point $\sigma_{01}$. It can be identified with the quotient of $L_{\text{can}}$ by the obvious lift $\bar{\tau}$ of $\tau$.

(iii) The set $S(L_Y)$ of unit vectors in $L_Y$ is smooth and diffeomorphic to $S^5$. The orbibundle $S(L_Y) \to Y$ can be identified with the quotient of $S^5$ by the circle action

$$\theta \cdot (x, y, z) = (e^{i\theta}x, e^{i\theta}y, e^{2i\theta}z).$$

Proof: Since $\tilde{Y}$ is smooth, the general theory implies that $\tilde{L}$ is smooth. Therefore, it is a line bundle over $CP^2$ and to understand its structure we just have to figure out its restriction to one line. It is easiest to consider one of the lines $x \pm y \pm \bar{z} = 0$ that lie over $S_2$. Recall that $\sigma_w \in S_2$ is the stable map $[\Sigma_w, h_w]$ with domain $\Sigma_w = S^2 \cup_{\gamma \equiv 0} S^2$ and where $h$ is the identity map on each component. Suppose we look at the line in $\tilde{Y}$ whose generic point has $z_1$ on one component of $\Sigma_w$ and $z_0, z_2$ on the other. Then the bundle $\tilde{L}$ has a natural trivialization over the set $\{w \in S^2 : w \neq z_0, z_1, z_2\}$. It is not hard to check that this trivialization extends over the points $z_0, z_2$ but that one negative twist is introduced when $z_1$ is added. The argument is very similar to the proof of Lemma 3.8 below, and is left to the reader.

(ii) It follows from the general theory that $L_Y \to Y$ is smooth over the smooth points of $Y$. Moreover, at $\sigma_{01} = [S^2, h]$ the automorphism $\gamma : S^2 \to S^2$ such that $h \circ \gamma = h$, $\gamma(z_i) = z_i$ acts on $T_{z_1}S^2$ by the map $v \mapsto -v$. (To see this, note that we can identify $S^2$ with $C \cup \{\infty\}$ in such a way that $z_0 = p_0 = 0$, $z_1 = p_1 = \infty$. Then $h(z) = z^2$, and $\gamma(z) = -z$.) Hence the local structure of $L$ at $\sigma_{01}$ is given by quotienting the trivial bundle $D^4 \times C$ by the map $(x, y) \times v \mapsto (-x, -y) \times -v$.

This is precisely the structure of the quotient of $\tilde{L}$ by $\bar{\tau}$ at the singular point. Moreover, we can identify $S(L_Y)$ with $S^5/\tau$ globally since $L_Y \to Y$ pulls back to $\tilde{L} \to \tilde{Y}$ under the map $\tilde{Y} \to Y$.

The quotient $S^5/\tau$ is smooth except possibly at the fixed points $(x, y, 0)$ of $\tau$. Since $S(L_Y)$ is smooth at these points, $S(L_Y)$ is smooth everywhere. It may be identified with $S^5$ by the map

$$S(L_Y) \equiv S^5/\tau \to S^5 : (x, y, z) \mapsto (x \sqrt{1 + |z|^2}, y \sqrt{1 + |z|^2}, z^2).$$

The last statement may be proved by noting that the formula

$$(x, y, z) \mapsto [x^2 : y^2 : z : xy] \in CP^3$$

defines a diffeomorphism from the orbit space of the given circle action to $CP^2/\tau = Y$.

§3.2.4 Attaching the strata.
The next step is to understand how the two strata \( \mathcal{V}_{1,J} \) and \( \mathcal{V}_{2,J} \) fit together. The two zero sections \( \mathcal{Z}_{1,J} \) and \( \mathcal{Z}_{2,J} \) intersect at the point \( \sigma_0 \). Recall that the domain \( \Sigma \) of \( \sigma_0 \) has 4 components with \( \Sigma_0, \Sigma_2, \Sigma_3 \) all meeting \( \Sigma_1 \) and a marked point \( z_0 \in \Sigma_3 \). The map \( h_0 : \Sigma_0 \rightarrow \Delta_J \) parametrizes \( \Delta_J \) as a section, \( h_1 \) takes \( \Sigma_1 \) onto the point \( x_1 = F_0 \cap \Delta_J \), and \( h_2, h_3 \) have image \( F_1 \) with \( h_3(z_0) = x_0 \). The stratum of \( \mathcal{Z}_{1,J} \) containing \( \sigma_0 \) consists just of this one point. Hence the local coordinates of \( \sigma_0 \) in \( \mathcal{Z}_{1,J} \) are given by two gluing parameters \((a_0, a_1)\). If we write \( z_{ij} \) for the point \( \Sigma_i \cap \Sigma_j \), these are

\[
(a_0, a_1) \text{ where } a_0 \in T_{z_{12}} \Sigma_1 \otimes T_{z_{12}} \Sigma_2, \quad a_1 \in T_{z_{13}} \Sigma_1 \otimes T_{z_{13}} \Sigma_3, \ldots
\]

Similarly, the local coordinates for a neighborhood of \( \sigma_0 \) in \( \mathcal{V}_{1,J} \) are

\[
(b, a_0, a_1)
\]

where \((a_0, a_1)\) are as before and \( b \in T_{z_{01}} \Sigma_0 \otimes T_{z_{01}} \Sigma_1 \) is a gluing parameter at the point \( z_{01} \) where the component \( \Sigma_0 \) mapping to \( \Delta_J \) is attached. On the other hand the natural coordinates for a neighborhood of \( \sigma_0 \) in \( \mathcal{V}_{2,J} \) are triples \((w, b, a)\) where \( b \) is a gluing parameter at the point \( z_{03} \) where the component \( \Sigma_0 \) that maps to \( \Delta_J \) is attached to the fixed fiber \( \Sigma_3 \), \( w \) is the point where the moving fiber \( \Sigma_2 \) (the one not containing \( z_0 \)) is attached to \( \Sigma_0 \) and \( a \) is a gluing parameter at \( w \).

**Lemma 3.9** The attaching map \( \alpha \) at \( \sigma_0 \) has the form \((b, a_0, a_1) \mapsto (w_0, b_0 a_0, b a_1)\), where \( b \neq 0 \) and \( \|b\| \) is small. Here the map \( b \mapsto w_0 \) identifies a small neighborhood of 0 in \( T_{z_{01}} \Sigma \) with a neighborhood of \( x_1 \) in \( \Delta_J \) in the obvious way.

**Proof:** The attaching of \( \mathcal{Z}_{1,J} \) to \( \mathcal{Z}_{2,J} \) comes from gluing at the point \( z_{01} \) via the parameter \( b \). Thus we are gluing the “ghost component” \( \Sigma_1 \) to the component \( \Sigma_0 \) that maps to \( \Delta_J \) in the space of stable \( A \)-curves that are holomorphic for a fixed \( J \). (It is only when one glues at \( a_0 \) or \( a_1 \) that one changes the homology class of the curve \( \Delta_J \) and hence has to change \( J \).) In particular, we can forget the components \( \Sigma_2, \Sigma_3 \) of the domain \( \Sigma \) of \( \sigma_0 \), retaining only the points \( z_{12}, z_{13} \) on \( \Sigma_1 \) where they are attached. Therefore we can consider the domain of the attaching map \( \alpha \) to be the 2-dimensional space

\[
\{[\Sigma_0 \cup_{z_{01}} \Sigma_1, z_{12}, z_{13}, h_\Delta; b] : b \in C = T_{z_{01}} \Sigma_0 \otimes T_{z_{01}} \Sigma_1\},
\]

and its range to be the space of all elements \([\Sigma_0, q_0, w, h_\Delta, J] \) where \( \Sigma_0 = S^2 \) and \( w \) moves in a small disc about \( x_1 \). Here, the map \( h_\Delta : \Sigma_0 \rightarrow \Delta_J \) is fixed and parametrizes \( \Delta_J \) as a section. We can encode this by picking two points \( q_1, q_2 \) in \( \Sigma_0 \) that are different from \( q_0 = \pi_J(x_0) \) and then considering \( h_\Delta \) to be the map that takes these two marked points to two other fixed points on \( \Delta_J \). Thus the attaching map \( \alpha \) is equivalent to the following map \( \alpha' \) that attaches different strata in the moduli space \( \overline{M}_{0,4}(S^2) \) of 4 marked points on \( S^2 \):

\[
\alpha' : \{(\Sigma_0 \cup \Sigma_1, q_1, q_2, z_{12}, z_{13}; b) : b \in C\} \rightarrow \{([\Sigma_0, q_1, q_2, z_{12}, z_{13}] \in \overline{M}_{0,4}(S^2)\}.
\]

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Here, each $\Sigma_i$ is a copy of $S^2$ as before. On the left $q_1, q_2$ are two marked points on $\Sigma_0$ and $z_{12}, z_{13}$ are two marked points on $\Sigma_1$. On the right, we should consider the three points $q_1, q_2, z_{13}$ to be fixed, while $z_{12} = w$ moves, since this corresponds to our previous trivialization of the neighborhood of $\sigma_*$ in $V_{2,J}$. Thus $\alpha'$ may be considered as a map taking $b$ to $w_b = z_{12} \in \Delta_J$.

It remains to check that as $b$ moves once (positively) around 0, $w_b$ moves once positively around $z_{13}$. This follows by examining the identification of the glued domain

$$\Sigma_b = (\Sigma_0 - D(z_{01})) \cup_{pl_b} (\Sigma_1 - D(z_{01}))$$

with $\Sigma_0 = S^2$. Observe that the two points $q_1, q_2$ in $\Sigma_0 - D(z_{01})$ and the single point $z_{12}$ in $\Sigma_1 - D(z_{01})$ must be taken to the corresponding three fixed points on $S^2 = \Sigma_1$. Hence the identification on $\Sigma_0 - D(z_{01})$ is fixed, while that on $\Sigma_1 - D(z_{01})$ can rotate about $z_{13}$ as $b$ moves. Hence, when $b$ moves round a complete circle, so does $w_b$. It remains to check the direction of the rotation.

Now, as we saw in Proposition 2.7, as $b$ moves once round this circle positively as seen from $z_{01}$, the point $p_b$ on $\partial D(z_{01}) \subset \Sigma_1$ that is matched with a fixed point $p$ on $\Sigma_0 - D(z_{01})$ moves once positively round $\partial D(z_{01})$. In order to line up $p_b$ with $p$, $\Sigma_1$ must be rotated in the opposite direction, i.e. positively as seen from the fixed point $z_{13}$. Hence $w_b$ rotates positively round $z_{13}$.

To complete the proof of the lemma, we must understand how the gluing parameters $a_0, a_1$ fit into this picture. Since nothing is happening in the vertical (i.e. fiberwise) direction, we may consider the $a_i$ to be elements of the following tangent spaces:

$$a_0 \in T_{z_{12}}\Sigma_1, \quad a_1 \in T_{z_{13}}\Sigma_1.$$ 

As $b$ rotates positively, the image of $a_0$ in the glued curve rotates once positively in the tangent space of $z_{12}$, and $a_1 \in T_{w_b}\Sigma_1$ also rotates once with respect to the standard trivialization of the tangent spaces $T_{w_b}\Sigma_1 \subset T(\Sigma_1)|_{D(z_{12})}$. Hence result.

\[ \text{Proof of Proposition 3.4} \]

We have identified the orbibundle $V_{1,J} \to Z_{1,J}$ with $L \to Y$ and the bundle $V_{2,J} \to Z_{2,J}$ with $\mathcal{O}(-2) \oplus \mathbb{C} \to S^2$. The previous lemma shows that these are attached by first twisting $V_{1,J}$ to $\mathcal{O}(-3) \oplus \mathcal{O}(-1)$ and then plumbing it to $L$. Hence

$$L_Z = S(\mathcal{O}(-3) \oplus \mathcal{O}(-1)) \# S(L_Y)$$

as claimed. The identification of the latter space with $S(\mathcal{O}(-1) \oplus \mathbb{C})$ follows from Proposition 3.3. \[ \Box \]

### 3.3 The projection $\mathcal{V}_J \to \mathcal{J}$

In order to complete the calculation of the link $L_{2,0}$ of $\mathcal{J}_2$ in $\mathcal{J}$ it remains to understand the projection $\mathcal{V}_J \to \mathcal{J}$. This is 1-to-1 except over the points of
In $\mathcal{V}_{2,j}$ it is clearly the points with zero gluing parameter at the moving fiber that get collapsed. Thus the subbundle $R_-$ of the circle bundle $S(L_P) \rightarrow \mathcal{P}(\mathcal{O}(−2) \oplus \mathbb{C})$ that lies over the (rigid) section $S_-=\mathcal{P}(\{0\} \oplus \mathbb{C})$ must be collapsed to a single circle. The subbundle $R_+$ lying over the other section $S_+=\mathcal{P}(\mathcal{O}(−2) \oplus \{0\})$ maps to a family of distinct elements in $\mathcal{J}_1$.

The story on $\mathcal{V}_{1,j}$ is, of course, more complicated. Here the points that concern us are the maps in $\mathcal{S}_2$ where the branch points coincide. Thus, if we identify $\overline{\mathcal{M}}_{0,0}(S^2,2)$ with $\mathbb{CP}^2$ as in Lemma 3.7, these are the points of the quadric $Q = \{(u + v + w)^2 = 4uv\}$. Note that the attaching point $\sigma_* \in \overline{\mathcal{M}}_{0,2}(S^2,p_0,p_1,2)$ sits over

$$[1:0:-1] = \ell_1 \cap Q \in \mathbb{CP}^2 = \overline{\mathcal{M}}_{0,0}(S^2,2).$$

The lift of $Q$ to $\overline{\mathcal{M}}_{0,2}(S^2,2)$ has two components $Q_\pm$, given by the intersection $Y \cap H_\pm$ where $H_\pm$ is the hyperplane $2t = \pm(u + v + w)$. Since we can assign these at will, we will say that $Q_-$ corresponds to elements with the two marked points $z_0,z_1$ on the same component of $\Sigma_w = \Sigma_0 \cup w = w \Sigma_1$ and that $Q_+$ corresponds to elements with $z_0,z_1$ on different components. Then, when one glues at $z_1$ the resulting $A$-curve is the union of an $(A - F)$-curve with an $F$ curve. It is not hard to check that the points on $Q_-$ give rise to a $(A - F)$-curve through $x_0$, which is independent of $w$, while those on $Q_+$ give rise to a varying $(A - F)$-curve that meets the $J_w$-holomorphic fiber through $x_0$ at the point $w$. Note that the intersection $Q_+ \cap Q_-$ consists of two points, $p_* = [1:0:-1:0]$ (corresponding to $\sigma_*)$ and $q_* = [0:1:-1:0]$. Moreover, in the coordinates $(a_0,a_1)$ of a neighborhood of $\sigma_*$ used in Lemma 3.9 above,

$$\{(a_0,a_1): a_0 = 0\} \subset Q_-, \quad \{(a_0,a_1): a_1 = 0\} \subset Q_+.$$ 

This confirms that when $\mathcal{V}_{2,j} \otimes \mathcal{O}(-1) = \mathcal{O}(-3) \oplus \mathcal{O}(-1)$ is plumbed to $\mathcal{V}_{1,j}$, $Q_-$ is plumbed to the subbundle $\{0\} \oplus \mathcal{O}(-1)$ corresponding to $R_-$ and $Q_+$ is plumbed to the subbundle $\mathcal{O}(-3) \oplus \{0\}$ corresponding to $R_+$.

Let $S(Q_\pm) \rightarrow Q_\pm$ denote the restriction of $S(L_Y) \rightarrow Y$ to $Q_\pm$. Then the plumbing $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \# S(L_Y)$ contains the plumbings $R_- \# S(Q_-)$, and $R_+ \# S(Q_+)$. 

**Lemma 3.10** (i) $R_- \# S(Q_-) = S(\mathcal{C}) = S^2 \times S^1$ and $R_+ \# S(Q_+) = S(\mathcal{O}(-2))$.

(ii) The subsets $R_- \# S(Q_-)$ and $R_+ \# S(Q_+)$ of $\mathcal{O}(-3) \oplus \mathcal{O}(-1) \# S(L_Y)$ intersect in a circle.

**Proof:** Since $Q_-$ and $Q_+$ do not meet the singular point of $Y$, both bundles $S(Q_\pm) \rightarrow Q_\pm$ have Euler number $-1$. Hence

$$R_- \# S(Q_-) = S(\mathcal{O}(-1)) \# S(\mathcal{O}(-1)) = S(\mathcal{C}) = S^2 \times S^1,$$
and
\[ R_+ \equiv S(Q_+) = S(O(-3)) \equiv S(3) = S(O(-2)). \]

This proves (i). To prove (ii) note that the inverse image (in \( S(\mathbb{Q}) \)) of the intersection point \( p_* = [1 : 0 : -1] \) of \( Q_- \) with \( Q_+ \) disappears under the plumbing. But the other one remains. \( \square \)

**Proof of Theorem 1.5**

It follows from part (i) of the preceding lemma that it is possible to collapse the subset \( R_- \equiv S(Q_-) \) of \( L \) to a single circle. Moreover, it is not hard to see that under the identification of \( L \equiv S(O(-3) \oplus O(-1)) \) with \( S(\mathbb{Q}) \), this collapsing corresponds to collapsing the circle bundle over the exceptional divisor. Since the intersection of \( R_- \equiv S(Q_-) \) with \( R_+ \equiv S(Q_+) \) is a single circle, this collapsing does not affect \( R_+ \equiv S(Q_+) \). Note that \( R_+ \equiv S(Q_+) \) is the inverse image of some 2-dimensional submanifold of \( \mathbb{C}P^2 \). Because \( R_+ \equiv S(Q_+) = S(O(-2)) \) this submanifold must be a quadric. \( \square \)

### 4 Analytic arguments

In §4.1 we prove the (easy) Lemmas 2.2. §4.2 contains a detailed analysis of gluing. The exposition here is fairly self-contained, though some results are quoted from [MS] and [FO].

#### 4.1 Regularity in dimension 4

The theory of \( J \)-holomorphic spheres in dimension 4 is much simplified by the fact that any \( J \)-holomorphic map \( h : S^2 \to X \) that represents a class \( A \) with \( A \cdot A \geq -1 \) is regular, i.e. the linearized delbar operator
\[
Dh : W^{1,p}(h^*(TX)) \to L^p \left( \Lambda_{j}^{0,1}(S^2) \otimes h^*(TX) \right)
\]
is surjective. This remains true even if \( h \) is a multiple covering. (For a proof see Hofer–Lizan–Sikorav [HLS]. The notation is explained in §3.1 below.) Therefore, regularity is automatic: one does not have to perturb the equation in order to achieve it. The analogous statement when \( A \cdot A < -1 \) is that \( \text{Coker} Dh \) always has rank equal to \( 2 + 2A \cdot A \). As is shown below, this almost immediately implies that the \( \mathcal{J}_k \) are submanifolds of \( \mathcal{J} \).

**Proof of Lemma 2.2**

We begin by proving that \( \mathcal{J}_k \) is a Fréchet manifold. This is obvious when \( k = 0 \), since \( \mathcal{J}_0 \) is an open subset of \( \mathcal{J} \). For \( k > 0 \), let \( \mathcal{C}_k \) denote the space of all symplectically embedded spheres in the class \( A - kF \), and let \( \mathcal{C}_k(\mathcal{J}) \) be the bundle over \( \mathcal{C}_k \) whose fiber at \( C \) is the space of all smooth almost complex structures on \( C \) that are compatible with \( \omega|_C \). Then \( \mathcal{C}_k(\mathcal{J}) \) fibers over \( \mathcal{C}_k \) and
it is easy to check that both spaces are Fréchet manifolds. (Note that $C_k$ is an open submanifold in the space of all embedded spheres in the class $A - kF$. Because these spheres are not parametrized the tangent space to $C_k$ at $C$ is the space of all sections of the normal bundle to $C$.) Further $J_k$ fibers over $C_k(J)$ with fiber at $(C, J|_C)$ equal to all $\omega$-compatible almost complex structures that restrict to $J$ on $T C$. This proves the claim.

To see that $\pi_k$ is bijective when $k > 0$ note that each $J \in J_k$ admits a holomorphic curve in class $A - kF$ by definition, and that this curve is unique by positivity of intersections. A similar argument works when $k = 0$ since the curves in $M(A, J)$ are constrained to go through $x_0$. Hence $M_k$ inherits a Fréchet manifold structure from $J_k$.

To show that $J_k$ is a submanifold of $J$ when $k > 0$ we must use the theory of $J$-holomorphic curves, as explained in Chapter 3 of [MS] for example. Let $M^s_k, J^s_k, J^s$ denote the similar spaces in the $C^s$-category for some large $s$. These are all Banach manifolds. It is easy to check that the tangent space $T_J J^s$ is the space $\text{End}(TX, \omega, J)$ of all $C^s$-sections $Y$ of the endomorphism bundle of $TX$ such that $$JY + YJ = 0, \quad \omega(Yx, y) = \omega(x, Yy).$$

These conditions imply that $\omega(Yx, x) = \omega(Yx, Jx) = 0$ for all $x$. It follows easily that $Y$ is determined by its value on a single nonzero vector $x$ that it has to take to the $\omega$-orthogonal complement to the $J$-complex line through $x$.

Observe further that there is an exponential map $\exp : T_J J^s \rightarrow J^s$ that preserves smoothness and is a local diffeomorphism near the zero section.

Next, note that the tangent space $T_{[h, J]} M^s_k$ is the quotient of the space of all pairs $(\xi, Y)$ such that
$$Dh(\xi) + \frac{1}{2} Y \circ dh \circ j = 0 \quad \text{(*)}$$
by the 6-dimensional tangent space to the reparametrization group $\text{PSL}(2, \mathbb{C})$. Here $j$ is the standard almost complex structure on $S^2$ and $Dh$ is the linearization of the delbar operator that maps the Sobolev space of $W^{1, p}$-smooth sections of $h^*(TX)$ to anti-$J$-holomorphic 1-forms, viz:
$$Dh : W^{1, p}(S^2, h^*(TX)) \rightarrow L^p(\Lambda^{0,1}_J(S^2, h^*(TX))), \quad (1)$$
where the norms are defined using the standard metric on $S^2$ and a metric on $TX$.

In [HLS], Hofer–Lizan–Sikorav show how to interpret elements of $\text{Ker } Dh$ and of $\text{Ker } Dh^*$ (where $Dh^*$ is the formal adjoint) as $J$-holomorphic curves in their own right. Using the fact that the domain is a sphere and that $X$ has dimension 4, they then use positivity of intersections to show that $\text{Ker } Dh$ is trivial when
\(k > 0\), i.e. it consists only of vectors that generate the action of \(\text{PSL}(2, \mathbb{C})\). Hence \(\text{Ker } Dh^*\) is a bundle over \(\mathcal{M}_k^s \cong \mathcal{J}_k^s\) of rank \(4k - 2 = -\text{index } Dh\), and it is not hard to see that it is isomorphic to the normal bundle of \(\mathcal{J}_k^s\) in \(\mathcal{J}^s\). In other words

\[
T_J \mathcal{J}^s = T_J \mathcal{J}_k^s \oplus \text{Ker } Dh^*.
\]

To see this, observe that the map

\[
i : Y \mapsto \frac{1}{2} Y \circ dh \circ j
\]

maps \(T_J \mathcal{J}^s\) onto the space of \(C^\infty\)-sections of \(\Lambda_{0,1}^s(S^2, h^*(TX))\), and that the kernel of this projection consists of elements \(Y\) that vanish on the tangent bundle to the image of \(h\) and so lie in \(T_J \mathcal{J}_k^s\) whenever \([h, J] \in \mathcal{M}_k^s\). It follows from equation (1) above that the image of \(T_J \mathcal{J}_k^s\) under this projection is precisely equal to the image of \(Dh\), and so has complement isomorphic to \(\text{Ker } Dh^*\). (For more details on all this, see the Appendix to [A].)

It now remains to show that \(\mathcal{J}_k\) is a submanifold of \(\mathcal{J}\) whose normal bundle has fibers \(\text{Ker } Dh^*\). This means in particular that the codimension of \(\mathcal{J}_k\) is \(-\text{ind } Dh = 4k - 2\). We therefore have to check that each point in \(\mathcal{J}_k\) has a neighborhood \(U\) in \(\mathcal{J}\) that is diffeomorphic to the product \((U \cap \mathcal{J}_k) \times \mathbb{R}^{4k - 2}\). It is here that we use the exponential map \(\exp\). Clearly, one can use \(\exp\) to define such local charts for \(\mathcal{J}_k^s\) in \(\mathcal{J}^s\). The point here is that the derivative of the putative chart will be the identity along \((U \cap \mathcal{J}_k^s) \times \{0\}\) and so by the implicit function theorem for Banach manifolds will be a diffeomorphism on a neighborhood. Then, because \(\text{Ker } Dh^*\) consists of \(C^\infty\) sections when \(J\) is \(C^\infty\) and because \(\exp\) respects smoothness, this local diffeomorphism will take \((U \cap \mathcal{J}_k) \times \mathbb{R}^{4k - 2}\) onto a neighborhood of \(J\) in \(\mathcal{J}\).

\[
\blacksquare\]

4.2 Gluing

The next task is to complete the proof of Propositions 2.7 and 2.11. The standard gluing methods are local and work in the neighborhood of one stable map, and so our main problem is to globalize the construction. The first step in doing this is to show that one can still glue even when the elements of the obstruction bundle are nonzero at the gluing point. We will use the gluing method of McDuff–Salamon [MS] and Fukaya–Ono [FO]. Much of the needed analysis appears in [MS] but the conceptual framework of that work has to be enlarged to include the idea of stable maps as in Hofer–Salamon [HS]. No doubt the other gluing methods can be adapted to give the same results.

Our aim is to construct a gluing map

\[
\mathcal{G} : \mathcal{N}_\mathcal{V}(\mathcal{Z}) \to \overline{\mathcal{M}}(A - kF, \mathcal{J})
\]

where \(\mathcal{Z} = \overline{\mathcal{M}}(A - kF, \mathcal{J}_m)\) is the space of stable maps in class \(A - kF\) with one component in class \(A - mF\), and \(\mathcal{N}_\mathcal{V}(\mathcal{Z})\) is a neighborhood of \(\mathcal{Z}\) in the space \(\mathcal{V}\) of
gluing parameters. Choose once and for all a $(4m-2)$-dimensional subbundle $K$ of $T_J|J_m$ that is transverse to $J_m$. As explained in §3.1 above the exponential map $\exp$ maps a neighborhood of the zero section in $K$ diffeomorphically onto a neighborhood of $J_m$ in $J$. For each $J \in J_m$ let

$$K_J < J$$

be the slice through $J$ (i.e. the image under $\exp$ of a small neighborhood $N_J(K)$ of 0 in the fiber of $K$ at $J$). We shall prove the following sharper version of Proposition 2.11.

**Proposition 4.1** Fix $J \in J_m$ and let $N_J(Z_J)$ be the fiber of the map $N_J(Z) \to J_m$ at $J$. Then, if the neighborhood $N_J(Z)$ is sufficiently small, there is a homeomorphism

$$G_J : N_J(Z_J) \to \mathcal{M}(A-kF,K_J)$$

onto a neighborhood $\mathcal{M}(A-kF,J)$ in $\mathcal{M}(A-kF,K_J)$. Moreover, the union of all the sets $\text{Im } G_J, J \in J_m$, is a neighborhood of $\mathcal{M}(A-kF,J_m)$ in $\mathcal{M}(A-kF,J)$.

Let $\pi_J : N_J(Z_J) \to Z_J$ denote the projection. We will first construct the map $G_J$ in the fiber at one point $\sigma = [\Sigma_\sigma, h_\sigma, J]$ of $Z_J$ and then how to fit these maps together to get a global map over $N_J(Z_J)$ with the stated properties. For the next paragraphs (until §4.2.4) we will fix a particular representative $h_\sigma : \Sigma_\sigma \to X$ of $\sigma$, and we will define $G$ as a map into the space of parametrized stable maps. In order to understand a full neighborhood of $\sigma$ we will have to glue not only at points where the branches meet the stem $\Sigma_0$ but also at points internal to the branches. Therefore, for the moment we will forget the stem-branch structure of our stable maps and consider the general problem of gluing, at the points $z_i \in \Sigma_{10} \cap \Sigma_{11}$ with parameter

$$a = \oplus_i a_i \in \bigoplus_i T_{z_i} \Sigma_{11} \otimes T_{z_i} \Sigma_{10}.$$ 

**§4.2.1: Construction of the pregluing $h_a$**

We showed in Proposition 2.11 above how to construct the glued domain $\Sigma_a$. Since this construction depends on a choice of metric on $\Sigma$, we must assume that the domain $\Sigma$ of each stable map is equipped with a Kähler metric that is flat near all double points and is invariant under the action of the isotropy group $\Gamma_\sigma$. Fukaya–Ono point out in [FO] §9 that it is possible to choose such a metric continuously over the whole moduli space: one just has to start at the strata containing elements $\sigma$ with the largest number of components, extend the choice of metric near these strata by using the gluing construction (which is invariant by $\Gamma_\sigma$) and then continue inductively, strata by strata. In what follows we will assume this has been done. We will also suppose that the cutoff functions $\chi_r$ used to define $\Sigma_a$ have been chosen once and for all.
The approximately holomorphic map \( h_a : \Sigma_a \to X \) is defined from \( h_\sigma \) by using cutoff functions. As before, we write \( r_i \) or simply \( r \) instead of \( \sqrt{|a_i|} \). Hence if \( R \) is as in [FO] or [MS], \( r = 1/R \). We choose a small \( \delta > 0 \) once and for all so that \( r/\delta \) is still small. Set \( x_i = h_\sigma(z_i) \). Then, for \( \alpha = 0, 1 \) define

\[
\begin{align*}
  h_a(z) &= h_\sigma(z) \text{ for } z \in \Sigma_{\sigma a} - D_{z_i}^\alpha(2r/\delta) \\
  &= x_i \text{ for } z \in D_{z_i}^\alpha(r/\delta) - D_{z_i}^\alpha(r)
\end{align*}
\]

and interpolate on the annulus \( D_{z_i}^\alpha(2r/\delta) - D_{z_i}^\alpha(r/\delta) \) in \( \Sigma_{\sigma a} \) by setting

\[
h_a(z) = \exp_{x_i}(\rho(\delta|z|/r)\xi_{\sigma a}(z)),
\]

where \( \rho \) is a smooth cut-off function that equals \( 1 \) on \([2, \infty)\) and \( 0 \) on \([0, 1]\), and the vectors \( \xi_{\sigma a}(z) \in T_{x_i}X \) exponentiate to give \( h_\sigma(z) \) on \( \Sigma_{\sigma a} \):

\[
h_\sigma(z) = \exp_{x_i}(\xi_{\sigma a}(z)), \text{ for } z \in D_{z_i}^\alpha(2r/\delta).
\]

The whole expression is defined provided that \( 2r/\delta \) is small enough for the exponential maps to be injective.

Later it will be useful to consider the corresponding map \( h_{\sigma,r} \) with domain \( \Sigma_a \). This map equals \( h_a \) on \( \Sigma - \cup z_i D_{z_i}^\alpha(r_i) \) and is set equal to \( x_i \) on each disc \( D_{z_i}^\alpha(r_i) \). Note that \( h_{\sigma,r} : \Sigma \to X \) converges in the \( W^{1,p} \)-norm to \( h_\sigma \) as \( r \to 0 \).

### §4.2.2 Construction of the gluing \( \tilde{G}(h_\sigma, a) \).

Let

\[\mathcal{N}_0(W_\sigma) = \mathcal{N}_0((W^{1,p}(\Sigma_a, h_\sigma^*(TX)))\]

be a small neighborhood of \( 0 \) in \( W^{1,p}(\Sigma_a, h_\sigma^*(TX)) \). Note that, if \( \Sigma_a \) has several components \( \Sigma_{a,j} \), the elements \( \sigma \) of \( W_\sigma \) can be considered as collections \( \xi_j \) of sections in \( W^{1,p}(\Sigma_{a,j}, (h_{\sigma,j})^*(TX)) \) that agree pairwise at the points \( z_i \). (This makes sense since the \( \xi_j \) are continuous.) Further, we may identify \( \mathcal{N}_0(W_\sigma) \) via the exponential map with a neighborhood of \( h_a \) in the space of \( W^{1,p} \)-maps \( \Sigma_a \to X \). We will write \( h_{a,\xi} \) for the map \( \Sigma_a \to X \) given by:

\[
h_{a,\xi}(z) = \exp_{h_a(z)}(\xi(z)), \quad z \in \Sigma_a.
\]

Recall that \( \mathcal{N}_f(K) \) is a neighborhood of \( 0 \) in the fiber of \( K \). Given \( Y \in \mathcal{N}_f(K) \) we will write \( J_Y \) for the almost complex structure \( \exp(Y) \) in the slice \( K_f \). Now consider the locally trivial bundle \( \mathcal{E} = \mathcal{E}_a \to \mathcal{N}_0(W_\sigma) \times \mathcal{N}_f(K) \) whose fiber at \((\xi, Y)\) is

\[
\mathcal{E}_{(\xi, Y)} = L^p(\Lambda^{0,1}\Sigma_a) \otimes J_Y h_{a,\xi}^*(TX)).
\]

We wish to convert the pregluing \( h_a \) to a map that is \( J_Y \)-holomorphic for some \( Y \) by using the implicit function theorem for the section \( F_\sigma \) of \( \mathcal{E}_a \) defined by

\[
\mathcal{F}_a(\xi, Y) = \overrightarrow{\partial}_{J_Y}(h_{a,\xi}).
\]

\footnote{The logic is that one chooses \( \delta > 0 \) small enough for certain inequalities to hold, and then chooses \( r \leq r(\delta) \). See Lemma \[\text{[insert lemma number]}\] below.}

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Note that $F_a(\xi, Y) = 0$ exactly when the map $h_{a, \xi}$ is $J_Y$-holomorphic.

The linearization $\mathcal{L}(F_a)$ of $F_a$ at $(0, 0)$ equals

$$\mathcal{L}(F_a) = D(h_a) \oplus \iota_a : W^{1, p}(h^*_a(TX)) \oplus K \to L^p(\Lambda^{0, 1}(\Sigma_a) \otimes J h^*_a(TX)),$$

where $\iota_a$ is defined by $\iota_a(Y) = \frac{1}{2} Y \circ dh_a \circ j$ as in equation (2) in §4.1.

**Lemma 4.2** Suppose that there is a continuous family of right inverses $Q_a$ to $\mathcal{L}(F_a)$ that are uniformly bounded for $\|a\| \leq r_0$. Then, there is $r_1 > 0$ such that for all $a$ satisfying $\|a\| \leq r_1$ there is a unique element $(\xi_a, Y_a) \in \text{Im} Q_a$ such that

$$F_a(\xi_a, Y_a) = 0.$$

Moreover, $(\xi_a, Y_a)$ depends continuously on the initial data.

**Proof:** This follows from the implicit function theorem as stated in 3.3.4 of [MS]. It also uses Lemma A.4.3 of [MS]. See also [FO] §11.

We will construct the required family $Q_a$ in §4.2.3. By the above lemma, this allows us to define the gluing map.

**Definition 4.3** We set $\tilde{G}(h_\sigma, a) = (\Sigma_a, h_a, \xi_a, J Y_a)$ where $(\xi_a, Y_a)$ is the unique element in the above lemma. Further $G(h_\sigma, a) = [\Sigma_a, h_a, \xi_a, J Y_a]$.

The next proposition states the main local properties of the gluing map $G$.

**Proposition 4.4** Each $\sigma \in Z_J$ has a neighborhood $N_V(\sigma)$ in $\mathcal{V}_J$ such that the map

$$N_V(\sigma) \to \overline{\mathcal{M}}(A - kF, K_J) : (\sigma', a') \mapsto G(h_{\sigma'}, a')$$

takes $N_V(\sigma)$ bijectively onto an open subset in $\overline{\mathcal{M}}(A - kF, K_J)$. Moreover this map depends continuously on $J \in J_m$.

**Proof:** This is a restatement of Theorem 12.9 in [FO]. Note that the stable map $G(h_{\sigma'}, a')$ depends on the choice of representative $(\Sigma', h_{\sigma'})$ of the equivalence class $\sigma' = [\Sigma', h_{\sigma'}]$. However, it is always possible to choose a smooth family of such representatives in a small enough neighborhood of $\sigma$ in $Z_J$. (This point is discussed further in §4.2.4.) Moreover, if $\sigma$ is an orbifold point (i.e. if $\Gamma_\sigma$ is nontrivial), then $h_\sigma$ is $\Gamma_\sigma$-invariant and one can define $\tilde{G}$ so that it is equivariant with respect to the natural action of $\Gamma_\sigma$ on the space of gluing parameters $a$ and its action on a neighborhood of $\sigma$ in the space of parametrized maps. The composite $G$ of $\tilde{G}$ with the forgetful map is therefore $\Gamma_\sigma$-invariant. (Cf. the discussion before Lemma 4.9.) This shows that $G$ is well defined.

One proves that it is a local homeomorphism as in [FO] §13, 14, and we will say no more about this except to observe that our adding of $K$ to the domain of $Dh_\sigma$ is equivalent to their replacement of the range of $Dh_\sigma$ by the quotient $L^p/\iota_a(K)$.

\[\square\]
§4.2.3 Construction of the right inverses $Q_\alpha$

This is done essentially as in A.4 of [MS] and §12 of [FO]. However, there are one or two extra points to take care of, firstly because the stem of the map $h_\sigma$ is not regular, so that the restriction of $Dh_\sigma$ to $\Sigma_0$ is not surjective, and secondly because the elements of the normal bundle $K \to J_m$ do not necessarily vanish near the points $x_i$ in $X$ where gluing takes place.

For simplicity, let us first consider the case when $\Sigma$ has just two components $\Sigma_0, \Sigma_1$ intersecting at the point $w$, and that $h_\sigma$ maps $\Sigma_0$ onto the $(A - kJ)$-curve $\Delta J$ and $\Sigma_1$ onto a fiber. (For the general case see Remark 4.8.) Then the linearization of $\partial J$ at $h_\sigma$ has the form

$$Dh_\sigma : W^{1,p}(\Sigma, h_\sigma^*(TX)) \to L^p(\Lambda^1_0(\Sigma) \otimes h_\sigma^*(TX)).$$

Here the domain consists of pairs $(\xi_0, \xi_1)$, where $\xi_j$ is a $W^{1,p}$-smooth section of the bundle $h_\sigma^*(TX) \to \Sigma_j$, subject to the condition $\xi_0(w) = \xi_1(w)$, and the range consists of pairs of $L^p$-smooth $(0, 1)$-forms over $\Sigma_j$ with values in $h_\sigma^*(TX)$ and with no condition at $w$. For short we denote this map by

$$Dh_\sigma : W^{1,p} \to L_{\sigma_0} \oplus L_{\sigma_1}.$$

Recall from the discussion before Proposition 4.1, that we chose $K$ so that $Dh_{\sigma_0} \oplus \iota_0 : W_{\sigma_0} \oplus K \to L_{\sigma_0}$ is surjective and $\iota_0 : K \to L_{\sigma_0}$ is injective. (All maps $\iota$ are defined as in equation (2); it should be clear from the context what the subscripts mean.)

**Lemma 4.5** There are constants $c, r_0 > 0$ so that the following conditions hold for all $r < r_0$:

(i) $\iota_0$ is injective for all $\|a\| \leq r$;

(ii) the projection $pr_K : L_{\sigma_0} \to K$ that has kernel $\text{Im} D_{\sigma_0}$ and satisfies $pr_K \circ \iota_0 = \text{id}_K$ has norm $\leq c$;

(iii) for all $Y \in K$ and $j = 0, 1$

$$\left( \int_{D^j(r)} |t_j(Y)|^p \right)^{1/p} \leq \frac{1}{12c} \left( \int_{\Sigma_j} |t_j(Y)|^p \right)^{1/p},$$

where $D^j(r)$ is the disc in $\Sigma_j$ on which gluing takes place and integration is with respect to the area form defined by the chosen Kähler metric on $\Sigma_\sigma$.

**Proof:** There is $c$ so that (ii) holds because $\text{Im} D_{\sigma_0}$ is closed and $\text{Im} \iota_0$ is finite dimensional. Then there is $r_0 = r_0(c)$ satisfying (i) and (iii) since the elements of $K$ are $C^\infty$-smooth (as are the elements of $\mathcal{J}$).  

\end{proof}
Lemma 4.6 The operator

\[ Dh_\sigma \oplus (\iota_0, \iota_1) : W_\sigma \oplus K \to L_{\sigma_0} \oplus L_{\sigma_1}, \]

is surjective and has kernel \( \ker Dh_\sigma \).

Proof: We know from the proof of Lemma 2.2 that

\[ Dh_{\sigma_0} \oplus \iota_0 : W^{1,p}(\Sigma_0, h_{\sigma_0}^*(TX)) \oplus K \to L^p(\Lambda^{0,1}(\Sigma_0) \otimes J h_{\sigma_0}^*(TX)) = L_{\sigma_0}, \]

is surjective. Similarly, \( Dh_{\sigma_1} \) is surjective. Therefore, to prove surjectivity we just need to check that the compatibility condition \( \xi_0(w) = \xi_1(w) \) for the elements of \( W_\sigma \) causes no problem. However, the pullback bundle \( h_{\sigma_1}^*TX \) splits naturally into the sum of a line bundle with Chern class \( 2d \) (where \( d \geq 0 \) is the multiplicity of \( h_{\sigma,1} \)) and a trivial line bundle, the pullback of the normal bundle to the fiber \( \text{Im} h_{\sigma,1} \). Hence there is a element \( \xi_1 \) of \( \ker Dh_{\sigma_1} \) with any given value \( \xi_1(w) \) at \( w \). The result follows. Note that an appropriate version of this argument applies for all \( \sigma \), not just those with two components, since there is just one condition to satisfy at each double point \( z \) of \( \Sigma \) and the maps \( \ker Dh_{\sigma_j} \to C^2 : \xi \mapsto \xi(z) \) are surjective for \( j > 0 \). The second statement holds because \( \iota_0 \) is injective. \( \blacksquare \)

Note that the right inverse \( Q_\sigma \) to \( Dh_\sigma \oplus (\iota_0, \iota_1) \) is completely determined by choosing a complement to the finite dimensional subspace \( \ker Dh_\sigma \) in \( W_\sigma \).

Consider the composite

\[ pr_{\sigma_0} : L_{\sigma_0} \oplus L_{\sigma_1} \to L_{\sigma_0} \to K \]

where the second projection is as in Lemma 4.3 (ii). The fiber \( (pr_{\sigma_0})^{-1}(Y) \) at \( Y \) has the form \( (\text{Im} Dh_{\sigma_0} + \iota_0(Y)) \oplus L_{\sigma_1} \), and we write

\[ Q^Y_\sigma : \text{Im} Dh_{\sigma_0} \oplus L_{\sigma_1} + (\iota_0(Y), \iota_1(Y)) \to W_\sigma \]

for the restriction of \( Q_\sigma \) to this fiber.

We now use the method of [MS] A.4 to construct an approximate right inverse \( Q_{a,\text{app}} \) to

\[ L(F_a) = Dh_a \oplus \iota_a : W_a \oplus K \to L_a, \]

where

\[ W_a = W^{1,p}(h_a^*(TX)), \quad L_a = L^p(\Lambda^{0,1}(\Sigma_a) \otimes J h_a^*(TX)). \]

It will be convenient to use the approximations \( h_{\sigma,r} : \Sigma_\sigma \to X \) to \( h_{\sigma} \) that were defined at the end of §4.2.1 where \( r^2 = \|a\| \). We write \( h_{\sigma,j,r} \) for the restriction of \( h_{\sigma,r} \) to the component \( \Sigma_j \). Since \( h_{\sigma,r} \) converges \( W^{1,p} \) to \( h_{\sigma} \) as \( r \to 0 \), \( Dh_{\sigma,r} \) has a uniformly bounded inverse

\[ Q_{\sigma,r} : L_{\sigma_0,r} \oplus L_{\sigma_1,r} \to W_{\sigma,r} \oplus K. \]
(In the notation of [MS], $Q_{σ,r} \equiv Q_{a,σ,r,v}$. As above there is a projection $p_{r,σ,σ} : L_{σ_0} ⊕ L_{σ_1} → K$ and we write $Q_{σ,σ}^Y$ for the restriction of $Q_{σ,r}$ to the fiber over $Y$.

As a guide to defining $Q_{a,app}$ consider the following diagram of spaces

$$
\begin{array}{ccc}
W_{σ,r} ⊕ K & \leftarrow & L_{σ_r} ⊕ L_{σ_1,σ} \\
\downarrow & & \uparrow \\
W_a ⊕ K & \leftarrow & Q_{a,app} \uparrow L_a,
\end{array}
$$

where the maps are given by:

$$
(ξ_1, ξ_1, Y) \triangleleft (ξ_0, η_1) \quad (ξ, Y) ∈ W_a ⊕ K \quad Q_{a,app} η ∈ L_a.
$$

We define the horizontal arrow $Q_{a,app}$ by following the other three arrows. Here $η_a$, for $α = 0, 1$, is the restriction of $η$ to $Σ_{σ,α} = D_{σ,α}^α(r)$ extended by $0$ as in [MS]. Note that the $η_a$ are in $L^p$ even though they are not continuous. Next, decompose

$$
η_0 = η_0' + i_{σ,σ}(Y) ∈ (Im D_{σ,α}) + i_{σ,σ}(K) = L_{σ,α},
$$

$$
η_1 = η_1' + i_{σ,σ}(Y) ∈ L_{σ_1,σ}.
$$

Then $(ξ_0, ξ_1) = Q_{σ,σ}(η_0', η_1')$. Note that $ξ_0(w) = ξ_1(w) = v$ say. We then define the section $ξ$ by putting it equal to $ξ_α$ on $Σ_{σ,α} = D_{σ,α}^α(r)$ for $α = 0, 1$ and then extending it over the neck using cutoff functions so that it equals $ξ_0 + ξ_1 - v$ on the circle $∂D_{σ,α}^α(r) = ∂D_{σ_1,σ}^α(r) ⊂ Σ_a$. In the formula below we think of the gluing map $Ψ_a$ of Proposition 6.11 as inducing identifications

$$
Ψ_a : A_0 = D_{σ,α}^α(r/δ) - D_{σ,α}^α(r) → D_{σ,α}^α(r) - D_{σ,α}^α(rδ),
$$

$$
Ψ_a : A_1 = D_{σ,α}^α(r) - D_{σ,α}^α(rδ) → D_{σ,α}^α(r/δ) - D_{σ,α}^α(r).
$$

Then $ξ$ is given by

$$
ξ(z) = \begin{cases} 
ξ_0(z) + (1 - β(z/δ))(ξ_1(Ψ_a(z)) - v) & \text{if } z ∈ A_0, \\
ξ_1(Ψ_a(z)) + (1 - β(z/δ))(ξ_0(z) - v) & \text{if } z ∈ A_1
\end{cases}
$$

where $β : C → [0, 1]$ is a cutoff function that $= 1$ if $|z| ≤ δ$ and $= 0$ for $|z| ≥ 1$.

The next lemma is the analog of Lemma A.4.2 in [MS]. It shows that $Q_{a,app}$ is the approximate inverse that we are seeking. The norms used are the usual $L^p$-norms with respect to the chosen metric on $Σ_a$ and the glued metrics on $Σ_a$. Note that we suppose that the metrics on $Σ_a$ agree with the standard model $χ_α(|z|)|dz|^2$ on the annuli $D_{σ,α}^α(r/δ) - D_{σ,α}^α(rδ)$ (where $r = ||a||^2$) so that $Ψ_a$ is an isometry.
Lemma 4.7 For all sufficiently small δ there is r(δ) > 0 and a cutoff function β such that for all η ∈ L_a, ∥a∥ ≤ r(δ)^2, we have

\[ \|(Dh_a \oplus \iota_a)Q_{a,app}\eta - \eta\| \leq \frac{1}{2}\|\eta\|. \]

Proof: It follows from the definitions that

\[ (Dh_a \oplus \iota_a)Q_{a,app}\eta = \eta \]

on each set \( \Sigma_j - D_w^j(r/\delta) \). (Observe that \( h_{\sigma_j,a} = h_a \) on this domain so that \( \iota_{\sigma_j,a} = \iota_a \) here.) Therefore, we just have to consider what happens on the subannuli

\[ A_0 = D_w^0(r/\delta) - D_w^0(r), \quad \Psi_a(A_1) = \Psi_a(D_w^0(r) - D_w^0(r/\delta)) \]

of \( \Sigma_a \). In this region the maps \( h_{\sigma_j,a} \) as well as the glued map \( h_a \) are constant so that the maps \( \iota_{\sigma_j,a}, \iota_a \) are constant. Further, the linearizations \( Dh_{\sigma_j,a} \) and \( Dh_a \) are all equal and on functions coincide with the usual \( \overline{\partial} \)-operator. We will consider what happens in \( \Psi_a(A_1) \), leaving the similar case of \( A_0 \) to the reader.

It is not hard to check that for \( z \in A_1 \)

\[ Dh_a(\xi_0(z)) = \eta'_0(z) = -\iota_a(Y), \]
\[ Dh_a(\xi_1(\Psi_az)) = \eta'_1(\Psi_az) . \]

Let us write \( \beta_r \) for the function \( \beta_r(z) = \beta(z/r) \). Then, if \( r^2 = \|a\| \) and \( (\xi, Y) = Q_{a,app}\eta \), we have for \( z \in A_1 \)

\[ (Dh_a\xi + \iota_aY - \eta)(\Psi_az) = \eta'_1(\Psi_az) + (1 - \beta_r)(-\iota_aY - Dh_a(v))(z) - \overline{\partial}(\beta_r) \otimes (\xi_0 - v)(z) + (\iota_aY - \eta)(\Psi_az) \]
\[ = (\beta_r - 1)(\iota_aY + Dh_a(v))(z) - \overline{\partial}(\beta_r) \otimes (\xi_0 - v)(z) . \]

Therefore, taking the \( L^p \)-norm

\[ \| (Dh_a\xi + \iota_aY - \eta) \circ \Psi_a \|_{L^p,A_1} \leq \|\iota_a(Y)\|_{L^p,A_1} + \|Dh_a(v)\|_{L^p,A_1} + \|\overline{\partial}(\beta_r) \otimes (\xi_0 - v)\|_{L^p,A_1} . \]

If \( r \) is sufficiently small we can, by Lemma 14.2 suppose that \( \|\iota_a(Y)\|_{L^p,A_1} \leq \|\eta\|/12 \). Moreover, because \( v \) is a constant section \( Dh_a \) acts on \( v \) just by its zeroth order part and so there are constants \( c_1, c_2 \) such that

\[ \|Dh_a(v)\|_{L^p,A_1} \leq c_1\|v\|_{\text{area}A_1}^{1/p} \leq c_2\|v\|_{W^{2/p}} . \]

Furthermore, by [MS] Lemma A.1.2, given any \( \varepsilon > 0 \) we can choose \( \delta_\varepsilon > 0 \) and \( \beta \) so that

\[ \|\overline{\partial}(\beta_r) \otimes (\xi_0 - v)\|_{L^p,A_1} \leq \varepsilon \|\xi_0 - v\|_{W^{1,p}} . \]
for all $\delta \leq \delta_\varepsilon$. Hence
\[
\|D h_a \xi + t_a Y - \eta\|_{L^p, A_1} \leq (c_2 r^{2/p} + \varepsilon)(\|v\| + \|\xi_0 - v\|_{W^{1,p}}) + \|\eta\|/12 \\
\leq c_3 (r^{2/p} + \varepsilon)(\|\eta_0, \eta_1\|_{L^p} + \|\eta\|)/12 \\
\leq c_4 (r^{2/p} + \varepsilon)(\|\eta_0, \eta_1\|_{L^p} + \|\eta\|)/12 \\
= \left(c_4 (r^{2/p} + \varepsilon) + 1/12\right)\|\eta\|
\]
where the second inequality holds because of the uniform estimate for the right inverse $Q^Y_{\sigma,r}$ and the third inequality holds because the projection of $L_{\sigma_0,r} \oplus L_{\sigma,1}$ onto the subspace $\text{Im } Dh_{\sigma,r} \oplus L_{\sigma,1}$ is continuous. Then if we choose $\delta_\varepsilon$ so small that $c_4 \varepsilon < 1/12$ and $r \ll \delta_\varepsilon$ so small that $c_4 r < 1/12$ we find
\[
\| (D h_a \xi + t_a Y - \eta) \circ \Psi_a \|_{L^p, A_1} \leq 1/4\|\eta\|.
\]
Repeating this for $A_0$ gives the desired result. \hfill \Box

Finally, we define the right inverse $Q_a$ by setting
\[
Q_a = Q_{a,\text{app}} ((D h_a \oplus t_a)Q_{a,\text{app}})^{-1}.
\]
It follows easily from the fact that the inverses $Q_{\sigma,r}$ are uniformly bounded for $0 < r \leq r_0$ that the $Q_a$ are too.

It remains to remark that the above construction can be carried out in such a way as to be $\Gamma_\sigma$-equivariant. The only choice left unspecified above was that of the right inverse $Q_{\sigma,r}$. This in turn is determined by the choice of a subspace $R_{\sigma,r}$ of
\[
W^{1,p}((\Sigma, h^*_{\sigma,r}(TX))
\]
complementary to the kernel of $Dh_{\sigma,r}$. But since $\Gamma_\sigma$ is finite, we can arrange that $R_{\sigma,r}$ is $\Gamma_\sigma$-equivariant. For example, since $Dh_{\sigma,r}$ is a finite-dimensional space consisting of $C^\infty$ sections, we can take $R_{\sigma,r}$ to be the $L^2$-orthogonal complement of $D h_{\sigma,r}$ defined with respect to a $\Gamma_\sigma$-invariant norm on $h^*_{\sigma,r}(TX)$ [1]. Note that because $h_{\sigma,r} = h_{\sigma,r} \circ \gamma$ for $\gamma \in \Gamma_\sigma$, we can obtain a $\Gamma_\sigma$-invariant norm on $h^*_{\sigma,r}(TX)$ by integrating the pull-back by $h_{\sigma,r}$ of any norm on the tangent bundle $TX$ with respect to a $\Gamma_\sigma$-invariant area form on the domain $\Sigma_\sigma$. We can achieve this uniformly over $Z_f$ by choosing a suitable metric on each domain $\Sigma_\sigma$ as described at the beginning of §4.2.1.

---

4 As pointed out in [FO], the map \[
\xi \mapsto \xi - \sum_j \langle \xi, e_j \rangle e_j, \quad \xi \in W_{\sigma,a}
\]
is well defined whenever $e_1, \ldots, e_p$ is a finite set of $C^\infty$-smooth sections.
Remark 4.8 If one is gluing two branch components $\Sigma_{\ell_j}, j = 0, 1$ of $\Sigma$ then both linearizations $Dh_{\ell_j}$ are surjective and one can construct the inverse $Q_\alpha$ to have image in $W_\alpha$, thus forgetting about the summand $K$. The general gluing argument combines both these cases. If one is gluing at $N$ different points then one needs to choose $r$ so small that one has an inequality of the form

$$\|Dh_\alpha \xi + \iota_\alpha Y - \eta\|_{L^p, A} \leq 1/4N\|\eta\|,$$

on each of the $2N$-annuli $A$. Note that the number of components of $\Sigma$ is bounded above by some number that depends on $m$ (where $J \in J_m$). Hence there is always $r_0 > 0$ such that gluing at $\sigma$ is possible for all $r < r_0$, provided that one is looking a family of parametrized stable maps $\sigma = (\Sigma, h_\sigma, J)$ that is compact for each $J$ and where $J \in J_m$ is bounded in $C^\infty$-norm.

This completes the proof of Lemma 4.3 and hence of Proposition 4.4.

§4.2.4 Aut$^K(\Sigma)$-equivariance of $\tilde{G}$

Note that there is an action of $S^1$ on the pair $(h_\sigma, \alpha)$ that rotates one of the components (say $\Sigma_1$) of $\Sigma = \Sigma_0 \cup \Sigma_1$ fixing the intersection point $w = \Sigma_0 \cap \Sigma_1$. We claim that by choosing an invariant metric on $\Sigma$ we can make the whole construction invariant with respect to the action of this compact group, i.e. so that as unparametrized stable maps

$$[\Sigma_\alpha, \tilde{G}(h_\sigma, \alpha)] = [\Sigma_{\theta\alpha}, \tilde{G}(h_{\sigma \cdot \theta^{-1}}, \theta \cdot \alpha)].$$

For then there is an isometry $\psi$ from the glued domain $\Sigma_\alpha$ to $\Sigma_{\theta\alpha}$ such that

$$h_\alpha = h^{\theta}_{\theta\alpha} \circ \psi,$$

where $h^\theta_{\theta\alpha}$ denotes the pregluing of $h^\theta_{\theta\alpha} = h_{\sigma \cdot \theta^{-1}}$ with parameter $b$. There is a similar formula for the maps $h_{\sigma, r}$. It is not hard to check that the rest of the construction can be made compatible with this $S^1$ action. It is important here to use the Fukaya–Ono choice of $R_{\sigma, r}$ as described above, instead of cutting down the domain of $Dh$ fixing the images of certain points as in [LiT], [LiuT1], [Sieb].

More generally, consider a parametrization $\tilde{\sigma} = (\Sigma, h)$ and an arbitrary element $\sigma \in Z_J$. Recall that a component of $\Sigma$ is said to be unstable if it contains less than three special points, i.e. points where two components of $\Sigma$ meet. Each unstable branch component has at least one special point where it attaches to the rest of $\Sigma$ and so the identity component of its automorphism group has the homotopy type of a circle. Therefore, if there are $k$ such unstable components, the torus group $T^k$ is a subgroup of $\text{Aut}(\Sigma)$. It is not hard to see that if the automorphism group $\Gamma_{\tilde{\sigma}}$ of $\tilde{\sigma}$ is nonzero, we can choose the action of $T^k$ to be $\Gamma_{\tilde{\sigma}}$-equivariant so that the groups fit together to form the compact group Aut$^K(\Sigma)$ of Definition 2.4.
Note further that if \((\Sigma', h')\) is obtained from \(\hat{\sigma} = (\Sigma, h)\) by gluing, then \(\text{Aut}^K(\Sigma')\) can be considered as a subgroup of \(\text{Aut}^K(\Sigma)\). To see this, suppose for example that \(\Sigma'\) is obtained by gluing \(\Sigma_i\) to \(\Sigma_j\) with parameter \(a\), and that both these components have at most one other special point. Then we can choose metrics on \(\Sigma_i \cup \Sigma_j\) that are invariant under an \(S^1\) action in each component and so that the glued metric on \(\Sigma_a\) is invariant under the action of an \(S^1\) in \(\text{Aut}^K(\Sigma')\). Note that the diagonal subgroup \(S^1 \times S^1\) of \(\text{Aut}^K(\Sigma)\) acts trivially on the gluing parameters at the double point \(\Sigma_i \cap \Sigma_j\) since it rotates in opposite directions in the two tangent spaces. It is now easy to check that if we write \(\hat{\theta}\) for the image of \(\theta \in S^1\) in the diagonal subgroup of \(S^1 \times S^1\) then

\[
(\Sigma_a, h' \circ \theta) = (\Sigma_a, \tilde{\mathcal{G}}(h, a) \circ \theta) = (\Sigma_a, \tilde{\mathcal{G}}(h \circ \hat{\theta}, a)).
\]

Observe also that if \(b\) is a gluing parameter at the intersection of \(\Sigma_i\) with some other component \(\Sigma_k\) of \(\hat{\sigma}\), then it can also be considered as a gluing parameter for \(\hat{\sigma}'\). Moreover under this correspondence \(\hat{\theta} \cdot b\) corresponds to \(\theta \cdot b\).

These arguments prove the following result.

**Lemma 4.9** Let \(\hat{\sigma} = (\Sigma, h)\) and suppose that the metric on \(\Sigma\) is \(\text{Aut}^K(\Sigma)\)-invariant. Then the following statements hold.

(i) The composite \(\tilde{\mathcal{G}}\) of \(\tilde{\mathcal{G}}\) with the forgetful map into the space of unparametrized stable maps is \(\text{Aut}^K(\Sigma)\)-invariant.

(ii) Divide the set \(P\) of double points of \(\Sigma\) into two sets \(P_b, P_s\) and correspondingly write the gluing parameter \(a\) as \(a_b + a_s\). Suppose that \((\Sigma', h') = (\Sigma_a, \tilde{\mathcal{G}}(h, a_b))\), and consider \(a_s\) as a gluing parameter at \(\hat{\sigma}'\). Then one can choose metrics and choose the groups \(\text{Aut}^K(\Sigma), \text{Aut}^K(\Sigma')\) so that there is an inclusion

\[
\text{Aut}^K(\Sigma') \to \text{Aut}^K(\Sigma) : \theta \mapsto \hat{\theta}
\]

such that

\[
(\Sigma_a, h' \circ \theta^{-1} ; \theta \cdot a_s) = (\Sigma_{a_b}, \tilde{\mathcal{G}}(h \circ \hat{\theta}^{-1}, a_b) ; \hat{\theta} \cdot a_s).
\]

Further, this can be done continuously as \(a_b\) (and hence \(h'\)) varies, and smoothly if \(\Sigma_a\) varies in a fixed stratum.

**Remark 4.10** In their new paper [LiuT2] §5, Liu–Tian also develop a version of gluing that is invariant with respect to a partially defined torus action.

### §4.2.5 Globalization

The preceding paragraphs construct the gluing map \(\tilde{\mathcal{G}}(h_\sigma, a)\) over a neighborhood \(\mathcal{N}(\sigma)\) of one point \(\sigma \in \mathcal{Z}_J\). We now show how to define a gluing map \(\mathcal{G}_J : \mathcal{N}_J(\mathcal{Z}_J) \to \overline{\mathcal{M}}(A - kF, \mathcal{K}_J)\) on a whole neighborhood \(\mathcal{N}_J(\mathcal{Z}_J)\) of \(\mathcal{Z}_J\) in the space of gluing parameters \(\mathcal{V}_J\). The only difficulty in doing this lies in choosing a suitable parametrized representative \(s(\sigma) = (\Sigma, h_\sigma)\) of the equivalence class...
\[ \sigma = [\Sigma, h] \] as \( \sigma \) varies over \( \mathcal{Z}_J \). In other words, in order to define \( \tilde{G}(h_\sigma, a) \) we need to choose a parametrization \( h_\sigma : \Sigma \to X \) of the stable map \( \sigma \), and now we have to choose this consistently as \( \sigma \) varies. We now show that although we may not be able to make a singlevalued choice \( s(\sigma) = h_\sigma \) continuously over \( \mathcal{Z}_J \), we can find a section that at each point is well defined modulo the action of a suitable subgroup of \( \text{Aut}^K(\Sigma) \). More precisely, we claim the following.

**Lemma 4.11** We may choose a continuous family of metrics \( g_\sigma \) on \( \Sigma_\sigma \) for \( \sigma \in \mathcal{Z}_J \) and a family of parametrizations \( s(\sigma) \) for each \( \sigma \in \mathcal{Z}_J \) such that

1. \( s(\sigma) \) consists of a \( G_\sigma \)-orbit of maps \( h_\sigma : \Sigma_\sigma \to X \) and \( g_\sigma \) is \( G_\sigma \)-invariant, where \( G_\sigma \subset \text{Aut}^K(\Sigma) \);
2. the assignment \( \sigma \to s(\sigma) \) is continuous in the sense that near each \( \sigma \) there is a (singlevalued) continuous map \( \sigma \to h_\sigma \in s(\sigma) \) whose restriction to each stratum is smooth. Moreover, \( g(\sigma) \) varies smoothly on each stratum.

**Proof:** The strata in \( \mathcal{Z}_J \) can be partially ordered with \( \mathcal{S}' \leq \mathcal{S} \) if there is a gluing that takes an element in the stratum \( \mathcal{S} \) to that in \( \mathcal{S}' \), i.e. if the stratum \( \mathcal{S} \) is contained in the closure of \( \mathcal{S}' \). If \( \mathcal{S} \) is maximal under this ordering and \( \sigma \in \mathcal{S} \), then each branch component in \( \Sigma \) is mapped to a fiber by a map of degree \( \leq 1 \).

It is easy to check that in this case there is a unique identification of the domain \( \Sigma_\sigma \) with a union of spheres such that the map \( h_\sigma \) is either constant or is the identity map on each branch component and a section on the stem: cf. Example 2.10. We assume this done, and then extend the choice of parametrization to a neighborhood of each of these maximal strata by gluing.

We now start extending our choice \( s(\sigma) = h_\sigma \) of parametrization to the whole of \( \mathcal{Z}_J \) by downwards induction over the partially ordered strata. Clearly we can always choose a parametrization modulo the action of \( \text{Aut}^K(\Sigma) \). In order for the image of the fiber \( \pi_\gamma^{-1}(\sigma) = \{(\sigma, a) : |a| < \varepsilon \} \) under the gluing map to be independent of this choice, we need the metric \( g_\sigma \) on \( \Sigma_\sigma \) to be \( \text{Aut}^K(\Sigma_\sigma) \)-invariant. This choice of metric can be assumed to be smooth as \( \sigma \) varies in a stratum. However, it cannot always be chosen continuously as \( \sigma \) goes from one stratum to another. For example, if \( \sigma \) has one component \( \Sigma_i \) with 3 special points at 0, 1, \( \infty \) and that is glued to some component \( \Sigma_{j_i} \) at 1 with gluing parameter \( a \), then the resulting component \( \Sigma_a \) is unstable if \( \Sigma_{j_i} \) has no other special points. But for small \( |a| \) the metric on \( \Sigma_a \) is determined by the metrics on \( \Sigma = \Sigma_i \cup \Sigma_{j_i} \) by the gluing construction and cannot be chosen to be \( S^1 \)-invariant. On the other hand, if both \( \Sigma_i \) and \( \Sigma_{j_i} \) have at most one other special point then the glued metric on \( \Sigma_a \) will be \( S^1 \)-invariant provided that the original metrics on \( \Sigma_i, \Sigma_{j_i} \) were also \( S^1 \)-invariant.

The above remarks show that suitable \( g^S, s(\sigma)^S \) and \( G^S_\sigma \) can be defined over each stratum \( \mathcal{S} \), and in particular over maximal strata. If \( g_\sigma, s(\sigma) \) and \( G_\sigma \) are already suitably defined over some union \( Y \) of strata, then the above remarks about gluing show that they can be extended to a neighborhood \( \mathcal{U}(Y) \) of \( Y \). Let us write \( g^S_\sigma, s(\sigma)^g \) and \( G^g_\sigma \) for the objects obtained by gluing when \( \sigma \in \mathcal{U}(Y) \).
Then, if $\beta : U(Y) \cup S \to [0,1]$ is a smooth cutoff function that equals 0 near $Y$ and 1 near the boundary of $U(Y)$, set

$$g_\sigma = (1 - \beta(\sigma))g_{\sigma}^d + \beta(\sigma)g_{\sigma}^S, \sigma \in S$$

$$s(\sigma) = \begin{cases} s(\sigma)^S, & \text{if } \beta(\sigma) = 1, \\ s(\sigma)^d, & \text{otherwise,} \end{cases}$$

$$G_\sigma = \begin{cases} G_{\sigma}^S, & \beta(\sigma) = 1, \\ G_{\sigma}^d, & \text{otherwise.} \end{cases}$$

It is easy to check that the required conditions are satisfied.

**Proof of Proposition 4.1**

By Lemmas 4.9 and 4.11 there is a well defined continuous gluing map

$$\mathcal{G}_J : \mathcal{N}_V(Z_J) \to \overline{\mathcal{M}}(A-kF,K_J).$$

that restricts on $Z_J$ to the inclusion. Therefore, because $Z_J$ is compact, the injectivity of $\mathcal{G}_J$ on a small neighborhood $\mathcal{N}_V(Z_J)$ follows from the local injectivity statement in Proposition 4.4. Similarly, the local surjectivity of Proposition 4.4 implies that the image of $\mathcal{G}_J$ is open in $\overline{\mathcal{M}}(A-kF,K_J)$. Note that all the restrictions made on the size of $\mathcal{N}_V(Z_J)$ vary smoothly with $J$ (and involve no more than the $C^2$ norm of $J$). Hence $\bigcup_J \text{Im} \mathcal{G}_J$ is an open subset of $\overline{\mathcal{M}}(A-kF,J)$.

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