Bound States of a System of Two Fermions on Invariant Subspace

J. I. Abdullaev, A. M. Toshturdiev*

Samarkand State University, University Boulevard 15, Samarkand, Uzbekistan
Email: jabdullaev@mail.ru, *atoshturdiev@mail.ru

Abstract

We consider a Hamiltonian of a system of two fermions on a three-dimensional lattice \( \mathbb{Z}^3 \) with special potential \( \hat{v} \). The corresponding Shrödinger operator \( \hat{H}(k) \) of the system has an invariant subspace \( \mathcal{L}^{123} \), where we study the eigenvalues and eigenfunctions of its restriction \( \hat{H}_{123}^k \). Moreover, there are shown that \( \hat{H}_{123}^k(k_1, k_2, \pi) \) has also infinitely many invariant subspaces \( \mathfrak{N}_{123}^n(n), n \in \mathbb{N} \), where the eigenvalues and eigenfunctions of eigenvalue problem

\[
\hat{H}(k_1, k_2, \pi) f = \epsilon f, \quad f \in \mathfrak{N}_{123}^n(n)
\]

are explicitly found.

Keywords

Hamiltonian, Fermion, Bound State, Shrödinger Operator, Invariant Subspace, Total Quasi-Momentum, Eigenvalue, Birman-Schwinger Principle

1. Introduction

The nature of bound states of two-particle cluster operators for small parameter values was first studied in detail by Minlos and Mamatov [1] and then in a more general setting by Minlos and Mogilner [2]. In [3], Howland showed that the Rellich theorem on perturbations of eigenvalues does not extend to the resonance theory. Studying bound states of a two-particle system Hamiltonian \( \hat{H} \) on the \( d \)-dimensional lattice \( \mathbb{Z}^d \) reduces to studying [2] [4] [5] [6] [7] the eigenvalues of a family of Shrödinger operators \( \hat{H}(k), k \in \mathbb{T}^d \), where \( k \) is the total quasi-momentum of a system. Moreover, eigenfunctions of \( \hat{H}(k) \) are interpreted as bound states of the Hamiltonian \( H \), and eigenvalues, as the bound state
energies. The bound states of $H$ of a system of two fermions on a one-dimensional lattice were studied in [4], a system of two bosons on a two-dimensional lattice was studied in [6], and perturbations of the eigenvalues of a two-particle Schrödinger operator on a one-dimensional lattice were studied in [8]. The finiteness of the number of eigenvalues of Schrödinger operator on a lattice was studied in the works [7] [9].

The discrete spectrum of the two-particle continuous Schrödinger operator

$$h_\lambda = -\Delta + \lambda V$$

was studied by many authors, with the conditions for the potential $V$ formulated in its coordinate representation. The condition for the finiteness of the set of negative elements of the spectrum and the absence of positive eigenvalues of $h_\lambda$ can be found in [10]. If $V \leq 0$, then the number of negative eigenvalues $N(\lambda)$ is a nondecreasing function of $\lambda \in (0, \infty)$, and each eigenvalue $z_n(\lambda)$ decreases on the half-axis $(0, \infty)$. It is known that when the coupling constant $\lambda$ decreases, the bound state energies of $h_\lambda$ tend to the boundary of the continuous spectrum (see [10]) and for some finite $\lambda$ are on the boundary. Two questions then arise: Does a bound or virtual state correspond to such a threshold state (i.e., is the corresponding wave function square-integrable)? And where do the bound states “disappear to” as $\lambda$ decreases further? The study of the first question was the subject in [11] [12]. Regarding the second question, it turns out that the bound state disappears by being absorbed into the continuous spectrum and becomes a resonance [5].

Here, we consider bound states of the Hamiltonian $\hat{H}$ (see (1)) of a system of two fermions on the three-dimensional lattice $\mathbb{Z}^3$ with the special potential $\hat{v}$ (see (5)). In other words, we study the discrete spectrum of a family of the Schrödinger operators $H(k), k = (k_1, k_2, k_3) \in \mathbb{T}^3$, (see (3)) corresponding to $\hat{H}$ in the invariant subspace $L_{123}^{-\infty}(\mathbb{T}^3)$.

Restriction of the operator $H(k)$ in the invariant subspace $L_{123}^{-\infty}(\mathbb{T}^3)$ is denoted by $H_{123}^-(k)$.

In the case $k = \pi := (\pi, \pi, \pi)$, the operator $H(\pi)$ has an infinite number of eigenvalues of the form $6 - v(n), n \in \mathbb{Z}$ and the essential spectrum consists of the single point 6. Here, the potential $\hat{v}$ is defined by (5) and $v: \mathbb{N} \to \mathbb{R}$ is a decreasing function on $\mathbb{N}$ and $\mathbb{V} \in \ell_2(\mathbb{N})$. These eigenvalues $z_n(\pi) = 6 - v(n), n \in \mathbb{N}$ are arranged in ascending order, $z_1(\pi) < \cdots < z_n(\pi) < \cdots$, and the smallest eigenvalue $z_1(\pi) = 6 - v(1)$ is threefold, $z_2(\pi) = 6 - v(2)$ is sevenfold, and the other eigenvalues $z_n(\pi) = 6 - v(n), n \geq 3$ are ninefold. All ninefold eigenvalues $z_n(\pi) = 6 - v(n), n \geq 3$ of the operator $H(\pi)$ are simple eigenvalues for the operator $H_{123}^-(\pi)$.

Further, we investigate eigenvalues and eigenfunctions of the restriction operator $H_{123}^-(k)$.

In the case $k = (k_1, k_2, \pi)$ the corresponding operator $H_{123}^-(k_1, k_2, \pi)$ has infinitely many invariant subspaces $\mathfrak{K}_{123}^-(n) := L_2(\mathbb{T}) \otimes L_2(\mathbb{T}) \otimes L^-(n), n \in \mathbb{N}$. It
is proved that the restriction $H_{123}^{\mathcal{R}}(k_1, k_2, \pi)$ of the operator $H_{123}^{\mathcal{R}}(k_1, k_2, \pi)$ in the invariant subspace $\mathcal{R}_{123}(n)$ has no more than one eigenvalue. If exists, it can be calculated explicitly. For every $(k_1, k_2) \in (-\pi, \pi)^2$ the operator $H_{123}^{\mathcal{R}}(k_1, k_2, \pi)$ has only a finite number of eigenvalues.

For any perturbation $\beta > 0$, the essential spectrum $\{\sigma\}$ of $H(\pi)$ becomes the essential spectrum $\sigma_{\text{ess}}(H(\pi-2\beta, \pi, \pi)) = [6 - 2\sin \beta, 6 + 2\sin \beta]$. If the potential $\hat{v}$ is of the form (5), the Shrödinger equation $H_{123}(\pi-2\beta, \pi, \pi) f = zf$, $f \in \mathcal{R}_{123}(n)$ can be exactly solved (see Theorem 1).

The Shrödinger equations $H(\pi-2\beta, \pi, \pi) f = zf$ and $H(\pi-2\beta, -\pi, \pi) f = zf$, $f \in \mathcal{R}_{123}(n)$ with small $\beta$ are solved by using methods invariant subspaces and operator theory.

2. Description of the Hamiltonian and Expansion in a Direct Integral

The free Hamiltonian $\hat{H}_0$ of a system of two fermions on a three-dimensional lattice $\mathbb{Z}^3$ usually corresponds to a bounded self-adjoint operator acting in the Hilbert space $\mathcal{H} \equiv \mathcal{H}^{\mathcal{R}}(\mathbb{Z}^3 \times \mathbb{Z}^3) := \{ f \in \ell_2(\mathbb{Z}^3 \times \mathbb{Z}^3) : f(x) = -f(y, x) \}$ by the formula

$$\hat{H}_0 = -\frac{1}{2m} \Delta_1 - \frac{1}{2m} \Delta_2.$$ 

Here, $m$ is the fermion mass, which we assume to be equal to unity in what follows, $\Delta_1 = \Delta \otimes I$ and $\Delta_2 = I \otimes \Delta$, where $I$ is the identity operator, and the lattice Laplacian $\Delta$ is a difference operator that describes a translation of a particle from a side to a neighboring side,

$$(\Delta \hat{\psi})(x) = \sum_{j=1}^{3} \left[ \hat{\psi}(x + e_j) + \hat{\psi}(x - e_j) - 2 \hat{\psi}(x) \right], \quad x \in \mathbb{Z}^3, \quad \hat{\psi} \in \ell_2(\mathbb{Z}^3),$$

where $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$ are unit vectors in $\mathbb{Z}^3$. The total Hamiltonian $\hat{H}$ acts in the Hilbert space $\mathcal{H}^{\mathcal{R}}(\mathbb{Z}^3 \times \mathbb{Z}^3)$ and is the difference of the free Hamiltonian $\hat{H}_0$ and the interaction potential $\hat{V}_2$ of the two fermions (see [8] [13]):

$$\hat{H} = \hat{H}_0 - \hat{V}_2,$$

where

$$(\hat{V}_2 \hat{\psi})(x, y) = \hat{v}(x - y) \hat{\psi}(x, y), \quad \hat{\psi} \in \ell_2^{\mathcal{R}}(\mathbb{Z}^3)^2 \equiv \ell_2^{\mathcal{R}}(\mathbb{Z}^3 \times \mathbb{Z}^3).$$

Hereafter, we assume that

$$\hat{v} \in \ell_2(\mathbb{Z}^3) \quad \text{and} \quad \hat{v}(x) = \hat{v}(-x) \geq 0 \quad \text{for all} \quad x \in \mathbb{Z}^3.$$  

Under this condition, the Hamiltonian $\hat{H}$ is a bounded self-adjoint operator in $\ell_2^{\mathcal{R}}(\mathbb{Z}^3)^2$.

We pass to momentum representation using the Fourier transform [2] [4] [7]

$$F : \ell_2^{\mathcal{R}}(\mathbb{Z}^3 \times \mathbb{Z}^3) \to L_2^{\mathcal{R}}(\mathbb{T}^3 \times \mathbb{T}^3).$$
The Hamiltonian \( H = H_0 - V = F \hat{H} F^{-1} \) in the momentum representation commutes with the unitary operators \( U_s \), given by
\[
(U_s f)(k_1, k_2) = \exp(-i(s, k_1 + k_2)) f(k_1, k_2), \quad f \in L_2^s(T^3 \times T^3).
\]

It follows that there exist decompositions of \( L_2^s(T^3 \times T^3) \) and the operators \( U_s \) and \( H \) into direct integrals (see [7] [9] and [10])
\[
L_2^s(T^3 \times T^3) = \int_{\mathbb{R}^3} \oplus L_2^s(F_k)dk, \quad U_s = \int_{\mathbb{R}^3} \oplus U_s(k)dk, \quad H = \int_{\mathbb{R}^3} \oplus \hat{H}(k)dk.
\]

Here,
\[
F_k = [(k_1, k_2) \in T^3 \times T^3 : k_1 + k_2 = k], \quad k \in T^3,
\]
and \( U_s(k) \) is an operator of multiplication by the function \( \exp(-i(s, k)) \) in \( L_2^s(F_k) \). The fiber operator \( \hat{H}(k) \) of \( H \) also acts in \( L_2^s(F_k) \) and is unitarily equivalent to \( H(k) := H_0(k) - V \), which is called the Schrödinger operator. This operator acts in the Hilbert space \( L_2^s(T^3) := \{ f \in L_2(T^3) : f(-q) = -f(q) \} \) by the formula
\[
(H(k)f)(q) = \varepsilon_k(q)f(q) - (2\pi)^\frac{3}{2} \int_{\mathbb{R}^3} \mathbf{v}(q-s)f(s)ds. \tag{3}
\]

The unperturbed operator \( H_0(k) \) is an operator of multiplication by the function
\[
\varepsilon_k(q) = \frac{k}{2} + q + \frac{k}{2} - q
\]
\[
= 6 - 2\cos\frac{k}{2}\cos q_1 - 2\cos\frac{k}{2}\cos q_2 - 2\cos\frac{k}{2}\cos q_3. \tag{4}
\]

From (3) and (4), it follows that
\[
H(k_1, k_2, k_3) = H(-k_1, k_2, k_3) = H(k_1, -k_2, k_3) = H(k_1, k_2, -k_3),
\]
so we can assume \( k_1, k_2, k_3 \in [0, \pi] \).

The perturbation operator \( V \) is an integral operator in \( L_2^s(T^3) \) with the kernel
\[
(2\pi)^\frac{3}{2} \mathbf{v}(q-s) = (2\pi)^\frac{3}{2}(F\mathbf{v})(q-s),
\]
and belongs to the class of Hilbert-Schmidt operators \( \Sigma_2 \).

In this work, we consider the operator \( H(k) \) with the potential \( \mathbf{v} \) of the form
\[
\hat{\mathbf{v}}(n) = \hat{\mathbf{v}}(n_1, n_2, n_3) = \begin{cases} \mathbf{T}([n]), & |n_1| + |n_2| \leq 1 \\ 0, & |n_1| + |n_2| \geq 2 \end{cases} \tag{5}
\]
where \( [n] = |n_1| + |n_2| + |n_3| \). Supporter is in the cylinder:
\[
D = \{ n = (n_1, n_2, n_3) \in \mathbb{Z}^3 : n_2 \in \mathbb{Z}, |n_1| + |n_2| \leq 1 \}.
\]

Since for every function \( \hat{\psi} \in L_2^s((\mathbb{Z}^3)^2) \) the equality \( \hat{\psi}(x, x) = 0, x \in \mathbb{Z}^3 \) holds, then the value of the potential \( \hat{\mathbf{v}} \) at the origin can be set arbitrary, since it does not affect the result, for simplicity, we assume that \( \hat{\mathbf{v}}(0) = 0 \).
The function \( v : \mathbb{N} \to \mathbb{R} \) in (5) is decreasing in \( \mathbb{N} \), i.e.,
\[
\bar{v}(1) > \bar{v}(2) > \cdots
\]
and belongs to \( \ell_2(\mathbb{N}) \). The kernel \( v \), of the integral operator \( V \), i.e., the Fourier transform \( v(p) = (F\hat{v})(p) \), of the potential \( \hat{v} \), has the form
\[
v(p) := (F\hat{v})(p) = \frac{1}{(2\pi)^{\frac{3}{2}}} \sum_{n \in \mathbb{Z}^3} \hat{v}(n) e^{i(n \cdot p)}
\]
\[
= \frac{1}{(2\pi)^{\frac{3}{2}}} \left[ 2\bar{v}(1)(\cos p_1 + \cos p_2 + \cos p_3)
+ 2\bar{v}(2)(\cos 2p_3 + 2\cos p_1 \cos p_2 + 2\cos p_1 \cos p_3 + 2\cos p_2 \cos p_3)
+ 2\sum_{n=1}^{\infty} \bar{v}(n+2)(\cos(n+2)p_3 + 2\cos(n+1)p_1 \cos p_1 + \cos p_2)
+ 4\cos p_1 \cos p_2 \cos np_3 \right].
\]

**Eigenvalues of the operator** \( H(k) \). We note that the spectra of the operators \( H_0(k) \) and \( V \) are known. The operator \( H_0(k) \) does not have eigenvalues, its spectrum is continuous and coincides with the range of the function \( \epsilon_k \):
\[
\sigma(H_0(k)) = [m(k), M(k)], \text{ where } m(k) = \min_{q \in \mathbb{C}} \epsilon_q(k), M(k) = \max_{q \in \mathbb{C}} \epsilon_q(k).
\]

The spectrum of \( V \) consists of the set \( \{0, \bar{v}(n), n \in \mathbb{N}\} \). Under condition (2), the operator \( V \) is a Hilbert-Schmidt operator and is hence compact. By the Weyl theorem [10], the essential spectrum of \( H(k) \) coincides with the spectrum of \( H_0(k) \):
\[
\sigma_{\text{ess}}(H(k)) = [m(k), M(k)].
\]

If \( k = \pi \), then the spectrum of \( H(\pi) = 6I - V \) consists of eigenvalues of the form \( 6 - \bar{v}(n), n \in \mathbb{N} \) and the essential spectrum is \( \{6\} \). If \( k = \pi \) (for some \( j \in \{1, 2, 3\} \), then there exists a potential \( \hat{v} \) such that \( H(k) \) has an infinite number of eigenvalues outside the continuous spectrum (see [4] [14]).

We recall some notations and known facts. For any self-adjoint operator \( B \) acting in a Hilbert space \( \mathcal{H} \) without an essential spectrum to the right of \( \mu \in \mathbb{R} \), we let \( n(\mu, B) \) denote the number of its eigenvalues to the right of \( \mu \). We let \( N(k, z) \) denote the number of eigenvalues of \( H(k) \) to the left of \( z \leq m(k) \), i.e.,
\[
N(k, z) = n(-z, -H(k)),
\]
the number \( N(k, m(k)) \) in fact coincides with the number of eigenvalues outside the continuous spectrum of \( H(k) \). It follows from the self-adjointness of \( H(k) = H_0(k) - V \) and positivity of \( V \) that
\[
\sigma(H(k)) \cap (M(k), \infty) = \emptyset,
\]
and hence \( \sigma_{\text{disc}}(H(k)) \subset (-\infty, m(k)) \). Therefore we seek only eigenvalues \( z \) less than \( m(k) \).

For any \( k \in \mathbb{T}^3 \) and \( z < m(k) \), we define the integral operator
\[
G(k, z) = V^2 r_0(k, z) V^2,
\]
where \( r_0(k, z) \) is the resolvent of the unperturbed operator \( H_0(k) \). Under
condition (2), the operator $V$ is positive, and we let $\sqrt[\frac{1}{2}]{V}$ denote the positive square root of the positive operator $V$. A solution $f$ of the Schrödinger equation $H(k)f = zf$

and the fixed points $\varphi$ of $G(k, z)$ are connected by the relations

$$f = r_\varphi(k, z)\sqrt[\frac{1}{2}]{V}\varphi$$ \quad \text{and} \quad \varphi = \sqrt[\frac{1}{2}]{V}f.$$

The following proposition (the Birman-Schwinger principle) holds [9].

**Lemma 1.** The number of eigenvalues of $H(k)$ to the left of $z < m(k)$ coincides with the number of eigenvalues of $G(k, z)$ greater than unity, i.e., the equality

$$N(k, z) = n(1, G(k, z))$$

holds.

**Lemma 2.** If for some $k \in \mathbb{T}^3$ the limit operator

$$\lim_{z \to m(k)-} G(k, z) = G(k, m(k))$$

exists and is compact, then the equality

$$N(k, m(k)) = n(1, G(k, m(k)))$$

(8)

holds.

Equality (8) states that the number of eigenvalues of $H(k)$, to the left of $m(k)$ is equal to the number of eigenvalues of $G(k, m(k))$ greater than unity.

### 3. Invariant Subspaces of $H(k)$

In this section, we study the invariant subspaces with respect to the operator $H(k)$.

Let $L_2^1(\mathbb{T}) = \{ f \in L_2(\mathbb{T}) : f(-p) = -f(p) \}$ be a subspace of the space $L_2(\mathbb{T})$, consisting of odd functions on $\mathbb{T} = [-\pi, \pi]$, and $L_2^2(\mathbb{T}) = \{ f \in L_2(\mathbb{T}) : f(-p) = f(p) \}$ be a subspace of $L_2(\mathbb{T})$, consisting of even functions on $\mathbb{T}$. In addition, we use the notation

$$L_{123}^1(\mathbb{T}^3) := L_2^1(\mathbb{T}) \otimes L_2^1(\mathbb{T}) \otimes L_2^1(\mathbb{T}), \quad L_{123}^2(\mathbb{T}^3) := L_2^2(\mathbb{T}) \otimes L_2^2(\mathbb{T}) \otimes L_2^2(\mathbb{T}).$$

Note that $L_{123}^1(\mathbb{T}^3)$ is a subspace of the space $L_{12}^0(\mathbb{T}^3)$. It is natural to expect the invariance of the subspace $L_{123}^1(\mathbb{T}^3)$ with respect to the operator $H(k)$. It turns out that this subspace is invariant under the operator $H(k)$, i.e. the following statement holds.

**Lemma 3.** Let the potential $\hat{v}$ have the form (5). Then the subspace $L_{123}^1(\mathbb{T}^3)$ is invariant under the action of $H(k)$.

**Proof.** We prove that this subspace is invariant first with respect to $H_0(k)$, and then with respect to $V$. It follows from representation (4) that the function $\varepsilon_k$ belongs to the subspace $L_{123}^1(\mathbb{T}^3)$, and it follows from the inclusion $f \in L_{123}^1(\mathbb{T}^3)$ that $\varepsilon_k f \in L_{123}^1(\mathbb{T}^3)$. This proves that $L_{123}^1(\mathbb{T}^3)$ is invariant with respect to $H_0(k)$.

Simple calculations show that the function (see (7))
\[ (If)(p_1, p_2, p_3) = \frac{1}{(2\pi)^3} \int v(p_1 - s_1, p_2 - s_2, p_3 - s_3) f(s_1, s_2, s_3) \, ds_1 \, ds_2 \, ds_3 \]

belongs to the subspace \( L_{123}(\mathbb{T}^3) \) for \( f \in L_{123}(\mathbb{T}^3) \). Hence, we prove the invariance of \( L_{123}(\mathbb{T}^3) \) with respect to \( V \) and it follows that \( L_{123}(\mathbb{T}^3) \) is invariant with respect to \( H(k) = H_0(k) - V \).

\( H_{123}(k) \) denotes the restriction of \( H(k) \) to the respective subspace \( L_{123}(\mathbb{T}^3) \). The action of \( H_0(k) \) is unchanged, the unperturbed operator \( H_0(k) \) is an operator of multiplication by the function \( \varepsilon_k \). We present the formula for \( V_{123} = V \varepsilon_{123} \) operator \( V \) acts on the element \( f \in L_{123}(\mathbb{T}^3) \) according to the formula

\[
(V_{123}f)(p) = \frac{1}{\pi} \sum_{n=1}^{\infty} \sqrt{n + 2} \int_{\mathbb{T}^3} \sin p_1 \sin p_2 \sin np_3 \sin q_1 \sin q_2 \sin nq_3 f(q) \, dq.
\]

Note that for \( k = \pi \), the spectrum of \( H(\pi) = 6I - V \) consists only of the eigenvalues \( 6, 6 - \varpi(n), n \in \mathbb{N} \) and the essential spectrum \( \{6\} \). Under condition (6) the number \( z_1(\pi) = 6 - \varpi(1) \) is a threefold eigenvalue of \( H(\pi) \), with the corresponding eigenfunctions

\[ \sin p_1, \sin p_2, \sin p_3, \]

the number \( z_2(\pi) = 6 - \varpi(2) \) is a sevenfold eigenvalue with the corresponding eigenfunctions

\[ \sin 2p_3, \cos p_1 \sin p_2, \cos p_1 \sin p_3, \sin p_1 \cos p_3, \cos p_2 \sin p_3, \sin p_2 \cos p_3, \]

for each \( n \geq 3 \), the number \( z_n(\pi) = 6 - \varpi(n) \) is a ninefold eigenvalue, and the corresponding eigenfunctions are

\[
\sin(n + 2)p_3, \sin p_1 \cos(n + 1)p_3, \sin p_2 \cos(n + 1)p_3, \\
\sin(n + 1)p_1 \cos p_3, \sin(n + 1)p_2 \cos p_3, \sin np_3 \cos p_1 \cos p_2, \\
\sin p_2 \cos p_1 \cos np_3, \sin p_2 \cos p_3 \cos np_1, \sin p_1 \sin p_2 \sin np_1.
\]

The number \( z_n(\pi) = 6 \) is an eigenvalue of an infinite multiplicity, and the corresponding eigenfunctions are

\[
\psi_{(n_1, n_2, n_3)}(p) = \sin n_1 p_1 \sin n_2 p_2 \sin n_3 p_3, \quad n_1, n_2 \geq 3.
\]

All ninefold eigenvalues \( z_n(\pi) = 6 - \varpi(n), n \geq 3 \) of the operator \( H(\pi) \) are simple eigenvalues for the operator \( H_{123}(\pi) \), and the number \( z_n(\pi) = 6 \) is an eigenvalue of an infinite multiplicity.

If the third coordinate \( k_3 \) of the total quasimomentum \( k \) is equal to \( \pi \), then the operator \( H(k_1, k_2, \pi) \) has infinitely many invariant subspaces \( \mathcal{R}_{123}(n), n \in \mathbb{N} \).

Next, we give a description of the invariant subspace \( \mathcal{R}_{123}(n), n \in \mathbb{N} \).

The system of functions

\[
\left\{ \psi_{n}(q) = \frac{1}{\sqrt{\pi}} \sin nq \right\}_{n \in \mathbb{N}}
\]
is an orthonormal basis in the space $L_2(\mathbb{T})$. Let us denote by $L(n), n \in \mathbb{N}$ the one-dimensional subspace spanned by the vector $\psi_n^a$. The space $L_2(\mathbb{T})$ can be decomposed into the direct sum

$$L_2(\mathbb{T}) = \bigoplus_{n=1}^{\infty} L(n).$$

This decomposition produces another decomposition

$$L_{123}(\mathbb{T}^2) = \bigoplus_{n=1}^{\infty} \left[ L_2(\mathbb{T}) \otimes L(n) \right] = \bigoplus_{n=1}^{\infty} \mathcal{R}_{123}(n),$$

where

$$\mathcal{R}_{123}(n) = L_2(\mathbb{T}) \otimes L(n), \quad L_2(\mathbb{T}) = L_2(\mathbb{T}) \otimes L_2(\mathbb{T}).$$

**Lemma 4.** Let the potential $\hat{v}$ have the form (5). Then the subspace $\mathcal{R}_{123}(n)$ is invariant under $H_{123}(k_1, k_2, \pi)$ for any $n \in \mathbb{N}$.

**Proof.** Let $(f \psi_n^a)(p_1, p_2, p_3) = f(p_1, p_2) \psi_n^a(p_3)$, where $f \in L_2(\mathbb{T}^2)$, $\psi_n^a \in L(n)$ is an arbitrary element of $\mathcal{R}_{123}(n)$. We consider the action of $H_{123}(k_1, k_2, \pi) = H_0(k_1, k_2, \pi) - V_{123}$ on $f \psi_n^a$:

$$
(H_0(k_1, k_2, \pi) f \psi_n^a)(p) = \left[ \frac{6 - 2 \cos \frac{k_1}{2} \cos p_1 - 2 \cos \frac{k_2}{2} \cos p_2}{\sin q_1 \sin q_2} \right] f(p_1, p_2) \psi_n^a(p_3), \quad (9)
$$

$$
(V_{123} f \psi_n^a)(p) = \left[ \frac{\sqrt{(n+2)}}{\pi^2} \right] \int_{\mathbb{T}^2} \sin p_1 \sin q_1 \sin q_2 \sin q_1 \sin q_2 f(q_1, q_2) dq_1 dq_2 \psi_n^a(p_3). \quad (10)
$$

To obtain the last formula (10), we use the orthogonality of the system of functions $\{\psi_n^a\}_{n \in \mathbb{N}}$ in $L_2(\mathbb{T})$. Relations (9) and (10) imply the equality

$$
(H_{123}(k_1, k_2, \pi) f \psi_n^a)(p_1, p_2, p_3) = \left[ \frac{6 - 2 \cos \frac{k_1}{2} \cos p_1 - 2 \cos \frac{k_2}{2} \cos p_2}{\sin q_1 \sin q_2} \right] f(p_1, p_2) \psi_n^a(p_3) - \left[ \frac{\sqrt{(n+2)}}{\pi^2} \right] \int_{\mathbb{T}^2} \sin p_1 \sin q_1 \sin q_2 \sin q_1 \sin q_2 f(q_1, q_2) dq_1 dq_2 \psi_n^a(p_3) \quad (11)
$$

which completes the proof of the lemma.

We denote by $H_{123a}(k_1, k_2, \pi)$ restriction of the operator $H_{123}(k_1, k_2, \pi)$ in the invariant subspace $\mathcal{R}_{123}(n)$. Formula (11) shows that the restriction $H_{123a}(k_1, k_2, \pi)$ to the subspace $\mathcal{R}_{123}(n) = L_2(\mathbb{T}) \otimes L(n)$ has the form

$$
H_{123a}(k_1, k_2, \pi) = \left[ 2I + H_0(k_1, k_2) - \sqrt{(n+2)} V_{11} \right] \otimes I, \quad (12)
$$

where $I$ is the identity operator and $H^{(e)}(k) := 2I + H_0(k) - \sqrt{(n+2)} V_{11}$, $k = (k_1, k_2)$, is a two-dimensional two-particle operator acting in $L_2(\mathbb{T}^2)$ by
the formula
\[
\left( H_{123}^{(\alpha)} (k) f \right) (p) = (2 + \varepsilon_k (p)) f (p) - \frac{V(n + 2)}{\pi^2} \sin p_1 \sin p_2 \sin q_1 \sin q_2 f (q) d\mathbf{q},
\]
where \( \varepsilon_k (p) = 4 - 2 \cos \frac{k}{2} \cos p_1 - 2 \cos \frac{k}{2} \cos p_2 \), and \( V_{11} \) is a one-dimensional integral operator in \( L^2 \left( \mathbb{T}^2 \right) \) with the kernel

\[
v(p, q) = \frac{1}{\pi^2} \sin p_1 \sin p_2 \sin q_1 \sin q_2.
\]

Studying the eigenvalues of \( H_{123a}^{(\alpha)} (k_1, k_2, \pi) \) by representations (12) reduces to studying the eigenvalues of

\[
H_{123}^{(\alpha)} (k) = 2I + H_0 (k) - \overline{V} (n + 2) V_{11}, \quad k = (k_1, k_2)
\]

i.e. the three-dimensional problem reduces to the two-dimensional problem.

4. Eigenvalues of the Operator \( H_{123}^{(\alpha)} (k) \)

Our main goal in this section is to study the behavior of the nondegenerate eigenvalue \( \varepsilon_{n+2} (\pi) = 6 - \overline{V} (n + 2), n \in \mathbb{N} \) of \( H_{123}^{(\alpha)} (\pi) \) at small perturbations \( \beta \) \( (k_1 = \pi - 2\beta \) or \( k_2 = \pi - 2\beta \), i.e. the eigenvalues of \( H_{123}^{(\alpha)} (\pi - 2\beta, \pi, \pi) \) (or \( H_{123}^{(\alpha)} (\pi, \pi - 2\beta, \pi) \)) at small perturbations \( \beta \). The studying of the eigenvalues of \( H_{123}^{(\alpha)} (\pi - 2\beta, \pi, \pi) \) is reduced to study the eigenvalues of the operator \( H_{123a}^{(\alpha)} (\pi - 2\beta, \pi, \pi) \) for each fixed \( n \in \mathbb{N} \). In turn, the problem of studying the eigenvalues of the operator \( H_{123a}^{(\alpha)} (\pi - 2\beta, \pi, \pi) \) by virtue of (12) is reduced to study of the discrete spectrum of the operator

\[
H_{123}^{(\alpha)} (\pi - 2\beta, \pi, \pi) = 2I + H_0 (\pi - 2\beta, \pi, \pi) - \overline{V} (n + 2) V_{11}.
\]

Studying the eigenvalues of \( H_{123}^{(\alpha)} (\pi - 2\beta, \pi, \pi) \) and \( H_{123}^{(\alpha)} (\pi, \pi - 2\beta, \pi) \) reduces to studying the eigenvalues of \( H_\lambda (k) \) acting in \( L^2 \left( \mathbb{T}^2 \right) \) by the formula

\[
\left( H_\lambda (k) f \right) (p) = \varepsilon_k (p) f (p) - \frac{\lambda}{\pi^2} \sin p \sin q f (q) d\mathbf{q},
\]

\[
\varepsilon_k (p) = 2 - 2 \cos \frac{k}{2} \cos p.
\]

It is known that the essential spectrum of

\[
H_\lambda (\pi - 2\beta) = H_0 (\pi - 2\beta) - \lambda V, \quad \beta \in \left( 0, \frac{\pi}{2} \right]
\]

consists of a segment \([ m(\beta), M(\beta) ]\), where \( m(\beta) = 2 - 2 \sin \beta \), \( M(\beta) = 2 + 2 \sin \beta \).

Further we give some information about the eigenvalues and eigenfunctions of the operator \( H_\lambda (k) \). Combining Theorem 6.3 in [6], Theorem 5.10 in [15] and Lemmas 1 and 2 we obtain the following statement about eigenvalues of the operator \( H_\lambda (k) \).

**Lemma 5.** Let \( \beta \in \left( 0, \frac{\pi}{2} \right] \).
a) If $\lambda < \sin \beta$, then the operator $H_2(\pi - 2\beta)$ has no eigenvalues lying outside of the essential spectrum.

b) If $\lambda = \sin \beta$, then the left edge $m(\beta)$ of essential spectrum of the operator $H_2(\pi - 2\beta)$ is a resonance.

c) If $\lambda > \sin \beta$, then the operator $H_2(\pi - 2\beta)$ has a unique nondegenerate eigenvalue

$$z_\lambda(\beta) = 2 - \lambda - \frac{1}{\lambda} \sin^2 \beta$$

which lying in the left of the essential spectrum with corresponding normalized eigenfunction

$$f_\lambda(p) = \frac{C_\lambda \sin p}{2 - 2 \sin \beta \cos p - z_\lambda(\beta)} \in L_2^2(T).$$

Here $C_\lambda$ is the normalizing multiplicity.

d) The operator $H_2(\pi - 2\beta)$ has no embedded eigenvalues in the interval $(m(\beta), M(\beta))$.

Hilbert space $L_2^2(T^2) = L_2(T) \otimes L_2(T)$ can be written as a direct sum:

$$L_2(T) \otimes L_2(T) = L_2(T) \otimes L^1(1) \oplus (L_2(T) \otimes L^1(1))^\perp.$$

The following lemma establishes a connection between the operators $H_2(\pi - 2\beta)$ and $H_1(k)$.

**Lemma 6.** Let the potential $\hat{v}$ have the form (5). Then:

a) the subspace $(L_2(T) \otimes L^1(1))^\perp$ and its orthogonal complement $(L_2(T) \otimes L^1(1))^\perp$ are invariant under $H_2(\pi - 2\beta, \pi)$.

b) restriction of the operator $H_{12}^{(n)}(\pi - 2\beta, \pi)$ to the invariant subspace $(L_2(T) \otimes L^1(1))^\perp$ coincides with the unperturbed operator $H_{12}(\pi - 2\beta, \pi)$.

c) restriction of the operator $H_{12}^{(n)}(\pi - 2\beta, \pi)$ to the invariant subspace $L_2(T) \otimes L^1(1)$ can be represented as a tensor product:

$$H_{12}^{(n)}(\pi - 2\beta, \pi) = \left[4I + H_0(\pi - 2\beta) - \nabla(n + 2)V_1\right] \otimes I.$$

Here, $I$ is the identity operator, and $H_0(\pi - 2\beta) = H_0(\pi - 2\beta) - \lambda(n)V_1$, $\lambda(n) = \nabla(n + 2)$ is a one-dimensional two-particle operator acting in $L_2(T)$ by the formula (13).

This lemma is proved in the same way as the Lemma 4. In particular, part b) of the lemma implies that the operator $H_{12}^{(n)}(\pi - 2\beta, \pi)$ has no eigenfunctions in $(L_2(T) \otimes L^1(1))^\perp$. Thus, studying the eigenvalues of the operator $H_{12}^{(n)}(\pi - 2\beta, \pi)$ is reduced to studying eigenvalues of the operator $H_{12}^{(n)}(\pi - 2\beta, \pi) = H_0(\pi - 2\beta) - \lambda(n)V_1$.

From Lemmas 5 - 6 and tensor product (15) implies the following statement regarding operator $H_{12}^{(n)}(\pi - 2\beta, \pi)$.

**Theorem 1.** Let $\beta \in \left[0, \frac{\pi}{2}\right]$ and $n \in \mathbb{N}$.

a) If $\nabla(n + 2) < \sin \beta$, then the operator $H_{12}^{(n)}(\pi - 2\beta, \pi)$ has no eigenvalues lying outside of the essential spectrum.
b) If \( \mathcal{V}(n+2) = \sin \beta \), then the left edge \( m(\beta) \) of essential spectrum of the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) is a resonance.

c) If \( \mathcal{V}(n+2) > \sin \beta \), then the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) has a unique non-degenerate eigenvalue

\[
z_{123}^{(n)}(\pi - 2\beta, \pi) = 4 + z_{i(a)}(\beta) = 6 - \mathcal{V}(n+2) - \frac{1}{\mathcal{V}(n+2)} \sin^2 \beta, \tag{16}\]

which lies in the left of the essential spectrum and with the corresponding normalized eigenfunction

\[
f_{i(a)}^{(n)}(p_1, p_2) = \frac{\sin \frac{p_2}{\pi}}{\sqrt{\pi}} = f_{i(a)}^{(n)}(p_1) \psi_1(p_2) \in L_2(\mathbb{T}) \otimes L'(1),
\]

where \( f_{i(a)}^{(n)} \) is the normalized eigenfunction of the operator \( H_{i(a)}(\pi - 2\beta) \) corresponding to the eigenvalue \( z_{i(a)}(\beta) \), the operator \( H_{i(a)}(k) \) is defined by the formula (13).

d) The operator \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) has no embedded eigenvalues in the interval \( (m(\beta), M(\beta)) \).

Similar statement is true for the operator \( H_{123}^{(n)}(\pi, \pi - \beta) \). The eigenvalues of the operators \( H_{123}^{(n)}(\pi, \pi - \beta) \) and \( H_{123}^{(n)}(\pi - 2\beta, \pi) \) are same, but eigenfunctions differ with variable replacement \( p_1 \) and \( p_2 \). In other words, the operators \( H_{123}^{(n)}(k_1, k_2) \) and \( H_{123}^{(n)}(k_2, k_1) \) are unitary equivalent. Therefore, the operators \( H_{123a}(k_1, k_2, \pi) \) and \( H_{123a}(k_2, k_1, \pi) \) are unitary equivalent too.

Similar statement can relatively be formulated for the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta) \). For this purpose, we introduce the following notation. Through

\[
\Delta_n(\beta, z) = 1 - \frac{\mathcal{V}(n+2)}{\pi^2} \int_{\mathbb{T}} \frac{\sin^2 p_1 \sin^2 p_2 \, dp_1 \, dp_2}{2 + 2 \sin \beta \cos p_1 - \sin \beta \cos p_2 - z}
\]

we denote the Fredholm determinant of the operator \( I - \mathcal{V}(n+2) V_{11}(\beta, z) \), where \( r_{0}(\beta, z) \) is the resolvent of the operator \( 2I + H_0(\pi - 2\beta, \pi - 2\beta) \), and \( V_{11} \) is an integral operator with the kernel

\[
v(p, q) = \frac{1}{\pi} \sin p_1 \sin p_2 \sin q_1 \sin q_2.
\]

Through \( C_{11}^{(n)} \) denote the value of the following integral:

\[
C_{11}^{(n)} = \left. \frac{1}{\pi^2} \right| \frac{\sin^2 p_1 \sin^2 p_2 \, dp_1 \, dp_2}{2 - \cos p_1 - \cos p_2} = \frac{\left| \psi_1(p_1) \right|^2 \left| \psi_1(p_2) \right|^2 \, dp_1 \, dp_2}{2\varepsilon(p)}
\]

Simple calculations reveal the following approximate value \( C_{11}^{(n)} \approx 0.302347 \).

**Theorem 2.** Let \( \beta \in \left(0, \frac{\pi}{2}\right], \ n \in \mathbb{N} \).

a) If \( \mathcal{V}(n+2) < \frac{\sin \beta}{C_{11}^{(n)}} \), then the operator \( H_{123}^{(n)}(\pi - 2\beta, \pi - 2\beta) \) has no eigenvalues lying outside of the essential spectrum.

b) If \( \mathcal{V}(n+2) = \frac{\sin \beta}{C_{11}^{(n)}} \), then the left edge \( m(\beta) = 6 - 4\sin \beta \) of the spectrum
of the operator \( H^{(q)}_{n} (\pi - 2\beta, \pi - 2\beta) \) is an eigenvalue.

c) If \( \nabla (n+2) > \frac{\sin \beta}{C_{11}} \), then the operator \( H^{(q)}_{n} (\pi - 2\beta, \pi - 2\beta) \) has a unique nondegenerate eigenvalue \( z^{(n)}_{123} (\pi - 2\beta, \pi - 2\beta) \) below the essential spectrum.

d) The operator \( H^{(q)}_{123} (\pi - 2\beta, \pi - 2\beta) \) has no embedded eigenvalues in the interval \( (m(\beta), M(\beta)) \).

This theorem is proved in similar way as Lemma 5. There are some differences:

1) In the Theorem 2, the eigenvalue \( z^{(n)}_{123} (\pi - 2\beta, \pi - 2\beta) \) was calculated with the accuracy of \( \beta^{2} \):

\[
z^{(n)}_{123} (\pi - 2\beta, \pi - 2\beta) = 6 - \nabla (n+2) - \frac{2}{\nabla (n+2)} \sin^{2} \beta + O(\beta^{4})
\]

and corresponding normalized eigenfunction has the form

\[
f^{(n)}_{123} (p_{1}, p_{2}) = \frac{C_{n}(\beta) \sin p_{1} \sin p_{2}}{6 - 2 \sin \beta \cos p_{1} - 2 \sin \beta \cos p_{2} - z^{(n)}_{123} (\pi - 2\beta, \pi - 2\beta)} \in L_{12}^{2} (T^{2}),
\]

where \( C_{n}(\beta) \) is the normalizing multiplicity.

2) Left edge \( m(\beta) = 6 - 2 \sin \beta \) of the essential spectrum is a resonance for the operator \( H^{(q)}_{123} (\pi - 2\beta, \pi) \), but for the operator \( H^{(q)}_{123} (\pi - 2\beta, \pi - 2\beta) \) the left edge \( m(\beta) = 6 - 4 \sin \beta \) of the essential spectrum is the eigenvalue, i.e. the equation \( H^{(q)}_{123} (\pi - 2\beta, \pi - 2\beta) f = m(\beta) f \) has a non-trivial solution

\[
f (p_{1}, p_{2}) = \frac{C \sin p_{1} \sin p_{2}}{2 - \cos p_{1} - \cos p_{2}}
\]

and it belongs to \( L_{12}^{2} (T^{2}) \).

5. Conclusions

1) We have shown that the operator \( H^{(q)}_{123} (k_{1}, k_{2}, \pi) \) has infinitely many invariant subspaces \( \mathcal{U}^{(q)}_{123} (n), n \in \mathbb{N} \). It has been proved that if condition \( \nabla (n+2) > \sin \beta \) holds then the operator \( H^{(q)}_{123} (\pi - 2\beta, \pi, \pi) \) has a unique simple eigenvalue \( z^{(n)}_{123} (\pi - 2\beta, \pi) \) of the form (16), otherwise, the operator has no eigenvalues outside of the essential spectrum. A similar statement holds for the operator \( H^{(q)}_{123} (\pi - 2\beta, \pi - 2\beta, \pi) \).

2) Without loss of generality it can be assumed that \( \nabla (3) \leq 1 \). Since, if \( \nabla (3) > 1 \) then it follows from \( \lim_{n \to \infty} \nabla (n) = 0 \) that there exists a number \( m \in \mathbb{N} \) such that \( \nabla (m+2) \leq 1 \) and monotonicity of \( \nabla \) implies that \( \nabla (n) > 1 \) for \( n = 3, 4, \cdots, m+1 \), and in this case, the eigenvalues \( z^{(n)}_{123} (\pi - 2\beta, \pi, \pi) \), \( n = 1, 2, \cdots, m-1 \) of \( H^{(q)}_{123} (\pi - 2\beta, \pi, \pi) \) exist for all \( \beta \in [0, \pi/2] \).

For a fixed \( \beta \in (0, \pi/2] \) there exists \( m \in \mathbb{N} \) such that \( \sin \beta \in (\nabla (m+3), \nabla (m+2)) \) and the operator \( H^{(q)}_{123} (\pi - 2\beta, \pi, \pi) \) has \( m \) non-degenerate eigenvalues outside of the essential spectrum (see Theorem 1):
\[
\begin{align*}
z_{123}^{(1)} (\pi - 2\beta, \pi, \pi) &= z_{123}^{(1)} (\pi - 2\beta, \pi) = 6 - \nu(3) - \frac{1}{\nu(3)} \sin^2 \beta, \\
z_{123}^{(2)} (\pi - 2\beta, \pi, \pi) &= z_{123}^{(2)} (\pi - 2\beta, \pi) = 6 - \nu(4) - \frac{1}{\nu(4)} \sin^2 \beta, \\
& \vdots \\
z_{123}^{(m)} (\pi - 2\beta, \pi, \pi) &= z_{123}^{(m)} (\pi - 2\beta, \pi) = 6 - \nu(m + 2) - \frac{1}{\nu(m + 2)} \sin^2 \beta.
\end{align*}
\]

The corresponding normalized eigenfunctions are of the forms:

\[
\begin{align*}
f_{123}^{(1)}(p_1, p_2, p_3) &= f_{123}^{(1)}(p_1) \psi_1^+(p_2) \psi_1^-(p_3) \in L_2 (\mathbb{T}) \otimes L^1 (1) \otimes L^1 (1), \\
f_{123}^{(2)}(p_1, p_2, p_3) &= f_{123}^{(2)}(p_1) \psi_1^+(p_2) \psi_2^+(p_3) \psi_3^- \in L_3 (\mathbb{T}) \otimes L^1 (1) \otimes L^1 (2), \\
& \vdots \\
f_{123}^{(m)}(p_1, p_2, p_3) &= f_{123}^{(m)}(p_1) \psi_1^+(p_2) \psi_m^+(p_3) \psi_m^- \in L_m (\mathbb{T}) \otimes L^1 (1) \otimes L^1 (m),
\end{align*}
\]

where, \( f_{123}^{(m)} \) is the normalized eigenfunction of the operator \( H_{123}^{(m)}(\pi - 2\beta) \) corresponding to the eigenvalue \( z_{123}^{(m)}(\beta) \) and the operator \( H_{123}^{(m)}(k) \) is defined by the formula (13), \( \lambda(m) = \nu(m + 2) \).

The eigenvalues of the operators \( H_{123}(\pi - 2\beta, \pi, \pi) \) and \( H_{123}(\pi, \pi - 2\beta, \pi) \) are same but eigenfunctions differ with variable replacement \( p_1 \) and \( p_2 \). In other words, the operators \( H_{123}(\pi - 2\beta, \pi, \pi) \) and \( H_{123}(\pi, \pi - 2\beta, \pi) \) are unitary equivalent.

In the case \( \sin \beta = \nu(m + 2) \), the left edge \( m(\beta) = 6 - 2\sin \beta \) of the essential spectrum is a resonance of the operator \( H_{123}(\pi - 2\beta, \pi, \pi) \) (see Theorem 1).

3) Let for some \( m \in \mathbb{N} \) the relation \( \sin \beta \in (\nu(m + 3)C_{11}^{-1}, \nu(m + 2)C_{11}^{-1}) \) hold then the operator \( H_{123}(\pi - 2\beta, \pi - 2\beta, \pi) \) has \( m \) nondegenerate eigenvalues outside the essential spectrum (see Theorem 2) and for small \( \beta \):

\[
\begin{align*}
z_{123}^{(1)} (\pi - 2\beta, \pi - 2\beta, \pi) &= z_{123}^{(1)} (\pi - 2\beta, \pi - 2\beta) = 6 - \nu(3) - \frac{2}{\nu(3)} \sin^2 \beta + O(\beta^4), \\
z_{123}^{(2)} (\pi - 2\beta, \pi - 2\beta, \pi) &= z_{123}^{(2)} (\pi - 2\beta, \pi - 2\beta) = 6 - \nu(4) - \frac{2}{\nu(4)} \sin^2 \beta + O(\beta^4), \\
& \vdots \\
z_{123}^{(m)} (\pi - 2\beta, \pi - 2\beta, \pi) &= z_{123}^{(m)} (\pi - 2\beta, \pi - 2\beta) = 6 - \nu(m + 2) - \frac{2}{\nu(m + 2)} \sin^2 \beta + O(\beta^4).
\end{align*}
\]

The corresponding normalized eigenfunctions are of the forms:

\[
\begin{align*}
f_{123}^{(1)}(p_1, p_2, p_3) &= f_{123}^{(1)}(p_1) \psi_1^+(p_2) \psi_1^-(p_3) \in L_2 (\mathbb{T}) \otimes L^1 (1), \\
f_{123}^{(2)}(p_1, p_2, p_3) &= f_{123}^{(2)}(p_1) \psi_1^+(p_2) \psi_2^+(p_3) \psi_3^- \in L_3 (\mathbb{T}) \otimes L^1 (2),
\end{align*}
\]
\[ f^{(m)}_{123}(p_1, p_2, p_3) = f^{(m)}_{123}(p_1, p_2)\psi^m(p_3) \in L^2(T^3) \otimes L^m(m), \]

where \( f^{(m)}_{123} \) is the normalized eigenfunction of the operator \( H^{123}_{m} (\pi - 2\beta, \pi - 2\beta) \) corresponding to the eigenvalue \( z^{123}_{m} (\pi - 2\beta, \pi - 2\beta) \) defined by the formula (17).

In the case \( \sin \beta = \sqrt{(m + 2)}C_{11}^{m} \), the left edge \( m(\beta) = 6 - 4\sin \beta \) of the essential spectrum is the eigenvalue of \( H^{123}_{m} (\pi - 2\beta, \pi - 2\beta, \pi) \) (see Theorem 2) with the corresponding eigenfunction

\[ f(p) = \frac{C \sin p_1 \sin p_2}{2 - \cos p_1 \cos p_2} \cdot \sin mp_3 \in L^2(T^3) \otimes L^m(m). \]

**Remark 1.** If the potential \( \hat{v} \) is even in all arguments \( p_1, p_2, p_3 \) and the condition \( \hat{v} \in \ell_2(\mathbb{Z}^3) \) holds, then the statements of Lemmas 3 - 4 remain valid.

**Remark 2.** If \( k_1 \neq \pi \), then the subspaces \( \mathcal{H}_{123}(n), n \in \mathbb{N} \) are not invariant under the operator \( H^{123}_{n_1, n_2, n_3} \).

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**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

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