Mechanisms of chaos onset in an inhomogeneous medium under cluster synchronization destruction

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Abstract. We show that in an inhomogeneous self-sustained oscillatory medium the destruction of perfect clusters of partial synchronization, that is induced both by varying the control parameter and by noise, leads to the onset of chaotic behaviour. We study the mechanisms of chaos formation in both cases. It is demonstrated that as parameters change, the transition to chaos in the deterministic medium can result from a hard (subcritical) period-doubling bifurcation and can be accompanied by intermittency. The noise-induced initiation of chaotic dynamics can be related to the existence of non-attracting chaotic motions in the vicinity of a regular regime.

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1. Introduction

Chaotic and turbulent regimes of oscillations in continuous active media have long attracted the attention of many researchers. The Ginzburg–Landau equation is considered to be one of the basic models of a self-sustained oscillatory medium. It can demonstrate various temporal chaotic and spatial non-regular oscillations over a wide range of parameter variation [1]–[4]. However, up to now, numerous studies of the dynamics of active media have typically been based on homogeneous deterministic models. In other words, system parameters have been assumed to be independent of spatial coordinates and the influence of noise has not been taken into account. The exceptions are the few works which have dealt with models of an inhomogeneous self-sustained oscillatory medium with discrete spatial coordinates (chains and lattices of non-identical self-sustained oscillators) [5]–[8]. A continuous model of an inhomogeneous self-sustained oscillatory medium was considered in [9]. The studies described in these papers were mainly related to global and partial (cluster) synchronization in an extended system. The frequency cluster formation was observed experimentally in some real extended self-sustained oscillatory systems, for example, when studying gastro-electric activity [10] and in the superficial Belousov–Zhabotinsky reaction proceeding in the presence of a temperature gradient [11]. The numerical simulation of inhomogeneous self-sustained oscillatory media by chains of self-sustained oscillators shows that the destruction of perfect frequency clusters can lead to non-regular temporal oscillations [8]. The linear analysis of the perturbation of similar oscillations in a continuous inhomogeneous medium, that was first made in [12], has revealed the presence of exponential instability as a direct indicator of the onset of dynamical chaos. However, the bifurcational mechanism of the appearance of chaotic self-sustained oscillations in an inhomogeneous medium when perfect clusters are destroyed has not been established. Our paper is mainly devoted to the analysis of this mechanism. Besides, noise-induced effects in an inhomogeneous medium have not also been studied. At the same time, inhomogeneity and random forces are inevitably present in any real medium. For this reason, it is quite important to identify a cooperative effect that both sources can have on system dynamics. For example, can the influence of noise cause chaotic behaviour in an inhomogeneous medium if the system dynamics are regular and without noise? In the present paper, we intend to show that such a case is possible.

2. Model and numerical methods

We study a self-sustained oscillatory medium that is governed by the Ginzburg–Landau equation with real parameters and a frequency depending linearly on the spatial coordinate

\[ a_t = \text{i}v(x)a + \frac{1}{2}(1 - |a|^2)a + ga_{xx} + \sqrt{2D}\xi(x, t), \]

where \( i = \sqrt{-1} \), \( a(x, t) \) is the complex amplitude of oscillations, and independent variables \( t \) and \( x \) are the time and the normalized spatial coordinate, respectively. \( a_t \) is the first time derivative, \( a_{xx} \) is the second derivative with respect to the spatial coordinate, and \( \xi(x, t) = n_1(x, t) + \text{i}n_2(x, t) \) is the normalized random force applied to each point of the medium. The imaginary and real components of the random force, \( n_1 \) and \( n_2 \) respectively, are assumed to be statistically independent Gaussian noise sources being temporarily and spatially uncorrelated, \( \langle n_j(x_1, t_1)n_k(x_2, t_2) \rangle = \delta_{jk}\delta(x_2 - x_1)\delta(t_2 - t_1), \) \( j, k = 1, 2 \), where \( \delta_{jk} \) is the Kronecker symbol, \( \delta(\cdots) \) is the Dirac function, and the brackets \( \langle \cdots \rangle \) denote statistical averaging. Parameter \( D \)
characterizes the intensity of random forcing that is considered constant in time and in space. In numerical simulation, the medium length was fixed $l = 50$. The diffusion coefficient $g$ was taken to be the same for all points of the medium. For $g \to 0$, oscillations in various points possess different frequencies defined by the function $\nu(x) = x\Delta/l$, where $\Delta$ is the maximal mismatch (the mismatch between the boundary points of the medium). The model of the medium (1) with linear mismatch can be treated as a limiting case of an inhomogeneous chain of quasi-harmonic self-sustained oscillators [8, 13] in passing to a continuous spatial coordinate. The model (1) without noise was recently studied in our work [12]. An analogous model of the medium was also considered earlier in [9]. The boundary conditions were set in the form: $a_i(x, t) \big|_{x=0,1} = 0$. The initial condition of the medium was randomly chosen to be near some homogeneous distribution $a_0 = \text{const.}$

Equation (1) was integrated numerically by means of the finite difference method with regard to the influence of random forces [14] and according to an implicit scheme of forward and backward sweeps. The real amplitude $A(x, t)$ and the phase $\phi(x, t)$ of oscillations are calculated as follows:

$$A(x, t) = |a(x, t)| = \sqrt{\text{Re}(a)^2 + \text{Im}(a)^2},$$

$$\phi(x, t) = \arg a(x, t) = \arctg \frac{\text{Im}(a)}{\text{Re}(a)} \pm \pi k, \quad k = 0, 1, 2, \ldots.$$

The quantity $\pm \pi k$ is added as the phase changes continuously in time.

The numerically obtained data are used to calculate the following characteristics: the average frequency of oscillations in different points of the medium $\Omega(x)$, autocorrelation functions (ACF) of amplitude $A$ time oscillations in a chosen point $x$, and corresponding power spectra of fluctuations $\tilde{A}(t) = A(t) - \langle A(t) \rangle$, projections of phase trajectory on the plane $(\tilde{A}, H\tilde{A})$, where $H\tilde{A}(t)$ is the Hilbert conjugate process, and the maximal Lyapunov exponent $\lambda_1$ of the medium dynamics. These characteristics and methods of their calculation were considered in detail in [12].

3. Perfect cluster destruction and transition to chaos under variation of the control parameter

The regimes of perfect and imperfect frequency clusters in a deterministic non-homogeneous medium (1) were studied in detail in [12]. It was shown that in a certain region of parameter $\Delta$ and $g$ values one can observe regimes of partial synchronization, which are accompanied by frequency cluster formation. Perfect and imperfect clusters can be obtained by varying parameter values. In the regime of perfect clusters, the medium is divided into $M$ clusters (regions), each possessing strictly the equal average frequency of oscillations $\Omega_i$, where $i = 1, 2, \ldots, M$ is the cluster number. It is worth taking into account that it is the average values of the oscillation frequency that are used here. The spectrum of oscillations in each spatial point will contain the same set of frequency components. But the spectral powers of these components will be redistributed according to which cluster is considered. A continuous dependence of the average frequency on the spatial coordinate is typical for the regime of imperfect clusters. In this case, the notion of clusters appears to be quite conventional. Clusters can be associated with space regions for which average frequencies of oscillations are close to each other (gentle segments of the $\Omega(x)$ dependence). The frequency of the $j$th cluster can be taken as the value of $\Omega(x)$, that corresponds to the centre of a gentle segment. In the case of imperfect clusters, the medium
also exhibits intercluster areas relating to fast changes of the average frequency in space. Our study presented in [12] has shown that the character of system temporal behaviour depends on the form of cluster structure. The regime of perfect clusters is characterized by regular (periodic or quasi-periodic) oscillations. The destruction of perfect clusters leads to chaotic temporal dynamics.

The basic characteristics of oscillations in the fixed spatial point \(x = 25\) (the centre of the middle cluster) are illustrated in figure 1 for regimes of perfect and imperfect clusters. The linear analysis of stability has shown that the maximal Lyapunov exponent is positive in the regime of imperfect clusters. For example, for \(\Delta = 0.2\), \(g = 0.85\), it has the value \(\lambda_1 \approx 0.002\) [12].

**Figure 1.** Distribution of average frequencies, normalized power spectrum of oscillations \(\tilde{A}(t)\) and projection of phase trajectories on the plane \((\tilde{A}(t), H\tilde{A}(t))\) in the point \(x = 25\) in the regime of perfect clusters for \(\Delta = 0.02\), \(g = 1.0\) (a, b and c, respectively), and in the regime of imperfect clusters for \(\Delta = 0.2\), \(g = 0.85\) (d,e and f, respectively). Discretization steps are \(h_t = 0.01\) and \(h_x = 0.001\).
Now consider the details of transition from regular to chaotic dynamics when varying parameter \( g \) for fixed \( \Delta = 0.2 \). Figure 2 shows the maximal Lyapunov exponent as a function of the control parameter \( g \) in the range \( g \in [0.85, 1.0] \). For \( g = g_{\text{cr}} \approx 0.93 \), \( \lambda_1 \) becomes positive and further increases as the parameter \( g \) decreases. In figure 3, the basic characteristics of oscillations in the point \( x = 25 \) are presented for the regime of imperfect clusters near the threshold of chaos onset (\( g = 0.92 \)).

The regime shown in figure 3 corresponds to weak chaos near the threshold of its onset. Indeed, the Lyapunov exponent value is not large, \( \lambda_1 \approx 0.0003 \). The presented characteristics testify that the transition from the regular limit set to the chaotic one occurs through a hard bifurcation. However, the effect of hysteresis has not been revealed. The form of \( A(t) \) oscillations indicates the presence of intermittency. During laminar phases of different durations, the amplitude of fluctuations \( \tilde{A}(t) \) first grows and then sharply decreases in the later part (figure 3(a)). The trajectory in the projection (\( \tilde{A}(t), \tilde{H}\tilde{A}(t) \)) rotates in the vicinity of the disappeared cycle, then leaves it occasionally and comes back again (figure 3(b)). The power spectrum now consists of a slightly increased pedestal and well pronounced spectral maxima at the subharmonic \( \Delta\Omega/2 \) and its multiple frequencies (figure 3(c)). Here, \( \Delta\Omega = \Omega_2 - \Omega_1 \) is the intercluster frequency. The correlation function decays slowly according to a weakly developed chaotic regime (figure 3(d)). The appearance of subharmonics of the basic frequency in the spectrum of fluctuations \( \tilde{A}(t) \) becomes clear if we consider a sequence of amplitudes \( A_1(n) \) of the process \( A(t) \) (figure 4(a)). It can be seen that the difference between values of successive amplitudes \( A_1(n) \) and \( A_1(n + 1) \) grows initially with time, then terminates abruptly and the oscillations return again to the vicinity of the periodic state. The two-fold map \( A_1(n + 2) = F[A_1(n)] \) (figure 4(b)) in the vicinity of an unstable fixed point has a part that is typical for the maps giving birth to III-type intermittency [15, 16], although as a whole not all points fall on the curve (this task cannot be rigorously reduced to a one-dimensional map). The III-type intermittency was studied for model one-dimensional maps and was related with a subcritical doubling bifurcation. However, such behaviour can be encountered in more complicated systems, including distributed ones. For example, this kind of dynamics was experimentally revealed in Rayleigh–Benar convection [15, 17]. The numerically estimated statistics of the duration of laminar phases are also close to the regularity typical for III-type intermittency (figure 4(c)). Let \( \varepsilon \) be the quantity characterizing the deviation of the control parameter from the

Figure 2. Maximal Lyapunov exponent of a dynamical regime of the medium (1) as a function of the control parameter \( g \) for \( \Delta = 0.02 \) and \( D = 0 \).
Figure 3. Characteristics of $A(t)$ oscillations of the deterministic medium (1) in the regime of imperfect clusters for $\Delta = 0.2$ and $g = 0.92$ in the fixed spatial point $x = 25$. (a) Time series $A(t)$, (b) projection of the trajectory on the plane $(\tilde{A}(t), H\tilde{A}(t))$ (the white closed curve corresponds to a limit cycle at $g = 0.93$), (c) normalized power spectrum of $\tilde{A}(t)$ oscillations, (d) normalized ACF.

bifurcational value divided by the basic period of oscillations. In the case being studied, we have $\varepsilon = (g - g_{cr})/T_1$, where $T_1 = 2\pi/\Delta\Omega$. It is known that the number of laminar phases $N$ whose duration exceeds $\tau_0$ for $\tau_0 > \varepsilon^{-1}$ can be described by the following relation:

$$N = C \exp (-2\varepsilon \tau_0), \quad C = \text{const.} \tag{2}$$

For the regime under consideration we have $\varepsilon = 0.000085$. The approximation of $N(\tau_0)$, obtained for this value of $\varepsilon$ according to (2), is depicted in figure 4(c) by the dashed line. It does not fit exactly the numerically built histogram but the decrement $\varepsilon$ is of the same order as in numerical experiments. The discrepancies appear to be connected with an insufficiently large number of laminar phases that have been taken into account when constructing the histogram.

Thus, it can be assumed that perfect frequency clusters are destroyed due to a hard bifurcation of the limit set. This case is similar to a subcritical period-doubling bifurcation of a limit cycle in finite-dimensional systems. With this, the non-attracting chaotic set already existing in the vicinity of regular attractor becomes attracting. Such a bifurcational mechanism must result in hard desynchronization of average frequencies of medium elements belonging to the same cluster. Dependence of the difference between average frequencies in points $x_1 = 25$ and $x_2 = 20$ on the control parameter $g$ is shown in figure 5 and confirms this assumption.
4. Noise-induced chaos

Our numerical studies have shown that the effect of noise on an inhomogeneous medium causes perfect clusters to be destroyed. The medium (1) in this situation, as well as in many other respects, behaves in a similar way to an inhomogeneous chain of self-sustained oscillators [13]. The linear analysis of stability of self-sustained oscillations in medium (1) has demonstrated that noise-induced breakdown of perfect clusters is accompanied by the appearance of chaotic temporal dynamics. In figure 6(a) the maximal Lyapunov exponent $\lambda_1$ is presented as a function of the noise intensity. This dependence was obtained for the parameters $\Delta = 0.2$, $g = 1.0$ corresponding to the existence of three perfect clusters at $D = 0$ (see figure 1(a)). The effect of noise on perfect cluster structure is exemplified in figure 6(b).

From the graph presented in figure 6(a) it is seen that when weak noise affects the medium (1) with regular deterministic dynamics, the maximal Lyapunov exponent $\lambda_1$ becomes positive.
Figure 5. Difference of average frequencies in two points of the same cluster $\delta = \Omega_{25} - \Omega_{20}$ as a function of the parameter $g$ when the regime of perfect clusters is destroyed. $\Delta = 0.2, D = 0$.

Figure 6. Effect of noise on the dynamics of inhomogeneous medium (1) in the regime of perfect clusters for $\Delta = 0.2, g = 1.0$. (a) Maximal Lyapunov exponent $\lambda_1$ as a function of the noise intensity $D$, and (b) imperfect frequency clusters for $D = 0.5 \times 10^{-4}$.

and originally grows as the noise intensity $D$ increases. Then $\lambda_1$ stops growing. The maximal value $\lambda_1 = 0.00072 \pm 10^{-5}$ is obtained for $D = 0.0005$. This is certainly a small magnitude but we can conclude that exponential instability is present in the system being studied. Indeed, our numerical computations have demonstrated that the perturbation norm logarithm slowly but definitely increases. This growth is similar to the corresponding dependence obtained for the chaotic regime without noise, presented in [12]. Besides, it should be taken into account that oscillations in the medium (1) are ‘slowly’ modulated, which explains the slow growth of the perturbation. It is worth noting that $\lambda_1$ values are also not large in chaotic regimes without noise.

Analyse in more detail the noise-induced transition of the medium to a chaotic behaviour. The basic characteristics of oscillations in the point $x = 25$ are presented in figure 7 for $\Delta = 0.2, g = 1.0$ and $D = 0.5 \times 10^{-4}$, which correspond to a weak noise-induced chaos with Lyapunov exponent $\lambda_1 = 0.00018 \pm 10^{-5}$. The trajectory on the plane $(\tilde{A}(t), H\tilde{A}(t))$ is mainly rotated in the vicinity of a limit cycle being realized at $D = 0$, but can go sufficiently far away from this region (figure 7(b)). Such behaviour indicates that in the system without noise some hyperbolic non-attracting subset exists in the vicinity of the regular solution.
Figure 7. Characteristics of noise-induced chaotic oscillations in medium (1) for $\Delta = 0.2$, $g = 1.0$ and $D = 0.5 \times 10^{-4}$ in the fixed spatial point $x = 25$. (a) Time series of $A(t)$ oscillations, (b) projection of the trajectory on the plane $(\tilde{A}(t), H\tilde{A}(t))$ (the white curve corresponds to a limit cycle at $D = 0$), (c) normalized power spectrum of $\tilde{A}(t)$ oscillations, and (d) normalized ACF.

In the presence of weak noise, trajectories can be induced to move along this subset. A similar mechanism of noise-induced chaos is well known for many finite-dimensional systems (see, e.g. [18]).

One can suppose that the effect of noise will be especially strong near the boundaries of the region where the perfect cluster structure exists. Consider, for example, $A(t)$ oscillations in point $x = 25$ for $g = 0.927$, which correspond to a three-step perfect cluster structure in the boundary of its destruction. The influence of weak noise of intensity $D = 10^{-5}$ leads to instability with Lyapunov exponent $\lambda_1 = 0.00010 \pm 10^{-5}$. In the presence of noise, the phase trajectory on the plane $(\tilde{A}(t), H\tilde{A}(t))$ wanders throughout the chaotic set in the vicinity of the periodic trajectory of the deterministic system (figure 8(a)). The power spectrum of $\tilde{A}(t)$ oscillations exhibits a subharmonic of the basic frequency $\Delta\Omega/2$ (figure 8(b)), that has also been observed when perfect clusters are destroyed in the deterministic system (figure 3(c)). Thus, the noise ‘activates’ the same chaotic set that becomes attracting as a result of the hard bifurcation occurring in the deterministic case.

5. Conclusions

In the present paper, we have established the following new facts concerning the dynamics of an inhomogeneous self-sustained oscillatory medium. In the first place, we have revealed
Figure 8. Characteristics of noise-induced chaotic oscillations in medium (1) for $\Delta = 0.02$, $g = 0.927$ and $D = 10^{-5}$ in the fixed spatial point $x = 25$. (a) Projection of the trajectory on the plane $(\hat{A}(t), H\hat{A}(t))$ (the white curve reflects a limit cycle at $D = 0$), and (b) normalized power spectrum of $\hat{A}(t)$ oscillations.

one of the possible mechanisms of chaos onset in a continuous inhomogeneous medium, that is related with the breakdown of perfect clusters. This mechanism involves a hard bifurcation being similar to the subcritical period-doubling bifurcation in finite-dimensional systems. The bifurcation is accompanied by the III-type intermittency. Secondly, it has been shown that noise in the continuous inhomogeneous medium can destroy perfect frequency cluster structures. This is accompanied by the appearance of chaotic temporal dynamics.

In our current research, we have been restricted by the study of cluster structure only when we fix the frequency mismatch $\Delta$ and vary the diffusion parameter $g$ in some interval of its values. Two questions remain open. Is it possible that for other values of the parameters the perfect cluster destruction and the transition to chaos are realized according to another bifurcational mechanism? Can the noise-induced destruction of perfect clusters in an inhomogeneous medium always lead to a chaotic dynamics? How can spatial correlations or dependence of the noise intensity on the spatial coordinate change the observed phenomena? Our further studies will enable us to answer these questions.

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References

[1] Shraiman B I, Pumir A, Van Saarlos W, Hohenberg P C, Chaté H and Holen M 1992 Physica D 57 241–248
[2] Cross M C and Hohenberg P C 1993 Rev. Mod. Phys. 65 851–1112
[3] Chaté H 1994 Nonlinearity 7 185–204

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[4] Aranson I S and Kramer L 2002 Rev. Mod. Phys. 74 99–143
[5] Sakaguchi H, Shinomoto S and Kuramoto Y 1987 Prog. Theor. Phys. 77 1005–10
[6] Strogatz S H and Mirollo R E 1988 Physica D 31 143–68
[7] Aronson D G, Ermentrout G B and Kopell N 1990 Physica D 41 403–49
[8] Osipov G V and Sushchik M M 1998 Phys. Rev. E 58 7198–207
[9] Ermentrout G B and Troy W C 1986 SIAM J. Appl. Math. 46 359–67
[10] Diamant N E and Bortoff A 1969 Am. J. Physiol. 216 301–7
[11] Winfree A T 1980 The Geometry of Biological Time (New York: Springer)
[12] Anishchenko V S, Vadivasova T E, Okrokvertskhov G A, Akopov A A and Strelkova G I 2005 Int. J. Bifurcation and Chaos 15 3661–73
[13] Vadivasova T E, Strelkova G I and Anishchenko V S 2001 Phys. Rev. E 63 036225
[14] Garcia-Ojalvo J and Sancho J M 1999 Noise in Spatially Extended Systems (New York: Springer)
[15] Berge P, Pomeau Y and Vidal Ch 1984 Order within Chaos (New York: Wiley)
[16] Schuster H G 1984 Deterministic Chaos (Weinheim: Physik)
[17] Dubois M, Rubio M A and Berge P 1983 Phys. Rev. Lett. 51 1446–9
[18] Anishchenko V S and Herzel H 1988 Z. Angew. Math. Mech. 68 317–8