DEFORMATIONS OF $\mathbb{A}^1$-FIBRATIONS

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Abstract. Let $B$ be an integral domain which is finitely generated over a subdomain $R$ and let $D$ be an $R$-derivation on $B$ such that the induced derivation $D_m$ on $B \otimes_R R/m$ is locally nilpotent for every maximal ideal $m$. We ask if $D$ is locally nilpotent. Theorem 1.1 asserts that this is the case if $B$ and $R$ are affine domains. We next generalize the case of $G_a$-action treated in Theorem 1.1 to the case of $\mathbb{A}^1$-fibrations and consider the log deformations of affine surfaces with $\mathbb{A}^1$-fibrations. The case of $\mathbb{A}^1$-fibrations of affine type behaves nicely under log deformations, while the case of $\mathbb{A}^1$-fibrations of complete type is more involved (see Dubouloz-Kishimoto [3]). As a corollary, we prove the generic triviality of $\mathbb{A}^2$-fibration over a curve and generalize this result to the case of affine pseudo-planes of ML$_0$-type under a suitable monodromy condition.

Introduction

An $\mathbb{A}^1$-fibration $\rho : X \to B$ on a smooth affine surface $X$ to a smooth curve $B$ is given as the quotient morphism of a $G_a$-action if the parameter curve $B$ is an affine curve (see [8]). Meanwhile, it is not so if $B$ is a complete curve. When we deform the surface $X$ under a suitable setting (log deformation), our question is if the neighboring surfaces still have $\mathbb{A}^1$-fibrations of affine type or of complete type according to the type of the $\mathbb{A}^1$-fibration on $X$ being affine or complete. Assuming that the neighboring surfaces have $\mathbb{A}^1$-fibrations, the propagation of the type of $\mathbb{A}^1$-fibration is proved in Lemma 2.2 whose proof reflects the structure of the boundary divisor at infinity of an affine surface with $\mathbb{A}^1$-fibration. The stability of the boundary divisor under small deformations, e.g., the stability of the weighted dual graphs has been

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discussed in topological methods (e.g., [26]). Furthermore, if such property is inherited by the neighboring surfaces, we still ask if the ambient threefold has an \(\mathbb{A}^1\)-fibration or equivalently if the generic fiber has an \(\mathbb{A}^1\)-fibration.

The answer to this question is subtle. We consider first in the section one the case where each of the fiber surfaces of the deformation has an \(\mathbb{A}^1\)-fibration of affine type induced by a global vector field on the ambient threefold. This global vector field is in fact given by a locally nilpotent derivation (Theorem 1.1). If the \(\mathbb{A}^1\)-fibrations on the fiber surfaces are of affine type, we can show (Theorem 2.8), with the absence of monodromies of boundary components, that there exists an \(\mathbb{A}^1\)-fibration on the ambient threefold such that the \(\mathbb{A}^1\)-fibration on each general fiber surface is induced by the global one up to an automorphism of the fiber surface. The proof of Theorem 2.8 depends on Lemma 2.2 which we prove by observing the behavior of the boundary rational curves. This is done by the use of Hilbert scheme (see [21]) and by killing monodromies by étale finite changes of the base curve.

As a consequence, we can prove the generic triviality of an \(\mathbb{A}^2\)-fibration over a curve. Namely, if \(f : Y \to T\) is a smooth morphism from a smooth affine threefold to a smooth affine curve such that the fiber over every closed point of \(T\) is isomorphic to the affine plane \(\mathbb{A}^2\), then the generic fiber of \(f\) is isomorphic to \(\mathbb{A}^2\) over the function field \(k(T)\) of \(T\) and \(f\) is an \(\mathbb{A}^2\)-bundle over an open set of \(T\) (see Theorem 2.10). This fact, together with a theorem of Sathaye [29], shows that \(f\) is an \(\mathbb{A}^2\)-bundle over \(T\) in the Zariski topology.

The question on the generic triviality is also related to a question on the triviality of a \(k\)-form of a surface with an \(\mathbb{A}^1\)-fibration (see Problem 2.13). In the case of an \(\mathbb{A}^1\)-fibration of complete type, the answer is negative by Dubouloz-Kishimoto [3] (see Theorem 5.1).

Theorem 2.10 was proved by our predecessors Kaliman-Zaidenberg [16] in a more comprehensive way and without assuming that the base is a curve. The idea in our first proof of Theorem 2.10 is of more algebraic nature and consists of using the existence of a locally nilpotent derivation on the coordinate ring of \(Y\) and the second proof of using the Ramamujam-Morrow graph of the normal minimal completion of \(\mathbb{A}^2\) was already used in [16]. The related results are also discussed in the article [28]. We cannot still avoid the use of a theorem of Kambayashi [13] on the absence of separable forms of the affine plane.

Some of the algebro-geometric arguments using Hilbert scheme in the section two can be replaced by topological arguments using Ehresmann’s theorem which might be more appreciated than the use of the
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Hilbert scheme. But they are restricted to the case of small deformations. This is done in the section three.

In the section four, we extend the above result on the generic triviality of an $\mathbb{A}^2$-fibration over a curve by replacing $\mathbb{A}^2$ by an affine pseudo-plane of $\text{ML}_0$-type which has properties similar to $\mathbb{A}^2$, e.g., the boundary divisor for a minimal normal completion is a linear chain of rational curves. But we still need a condition on the monodromy. An affine pseudo-plane, not necessarily of $\text{ML}_0$-type, is a $\mathbb{Q}$-homology plane, and we note that Flenner-Zaidenberg [5] made a fairly exhaustive consideration for the log deformations of $\mathbb{Q}$-homology planes.

In the final section five, we observe the case of $\mathbb{A}^1$-fibration of complete type and show by an example of Dubouloz-Kishimoto [3] that the ambient threefold does not have an $\mathbb{A}^1$-fibration. But it is still plausible that the ambient threefold is affine-uniruled in the stronger sense that the fiber product of the ambient deformation space by a suitable lifting of the base curve has a global $\mathbb{A}^1$-fibration. But this still remains open.

We use two notations for the intersection of (not necessarily irreducible) subvarieties $A, B$ of codimension one in an ambient threefold. Namely, $A \cap B$ is the intersection of two subvarieties, and $A \cdot B$ is the intersection of effective divisors. In most cases, both are synonymous.

As a final remark, we note that a preprint of Flenner-Kaliman-Zaidenberg [6] recently uploaded on the web treats also deformations of surfaces with $\mathbb{A}^1$-fibrations.

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1. Triviality of deformations of locally nilpotent derivations

Let $k$ be an algebraically closed field of characteristic zero which we fix as the ground field. Let $Y = \text{Spec} B$ be an irreducible affine algebraic variety. We define the tangent sheaf $\mathcal{T}_{Y/k}$ as $\mathcal{H}om_{\mathcal{O}_Y}(\Omega^1_{Y/k}, \mathcal{O}_Y)$. A regular vector field on $Y$ is an element of $\Gamma(Y, \mathcal{T}_{Y/k})$. A regular vector field $\Theta$ on $Y$ is identified with a derivation $D$ on $B$ via isomorphisms

$$\Gamma(Y, \mathcal{T}_{Y/k}) \cong \text{Hom}_B(\Omega^1_{B/k}, B) \cong \text{Der}_k(B, B).$$

We say that $\Theta$ is locally nilpotent if so is $D$. In the first place, we are interested in finding a necessary and sufficient condition for $D$ to be locally nilpotent. Suppose that $Y$ has a fibration $f: Y \rightarrow T$. A natural question is to ask whether $D$ is locally nilpotent if the restriction of
$D$ on each closed fiber of $f$ is locally nilpotent. The following result shows that this is the case.\footnotemark

**Theorem 1.1.** Let $Y = \text{Spec } B$ and $T = \text{Spec } R$ be irreducible affine varieties defined over $k$ and let $f : Y \to T$ be a dominant morphism such that general fibers are irreducible and reduced. We consider $R$ to be a subalgebra of $B$. Let $D$ be an $R$-trivial derivation of $B$ such that, for each closed point $t \in T$, the restriction $D_t = D \otimes_R R/\mathfrak{m}$ is a locally nilpotent derivation of $B \otimes_R R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of $R$ corresponding to $t$. Then $D$ is locally nilpotent.

We need some preliminary results. We retain the notations and assumptions in the above theorem.

**Lemma 1.2.** There exist a finitely generated field extension $k_0$ of the prime field $\mathbb{Q}$ which is a subfield of the ground field $k$, geometrically integral affine varieties $Y_0 = \text{Spec } B_0$ and $T_0 = \text{Spec } R_0$, a dominant morphism $f_0 : Y_0 \to T_0$ and an $R_0$-trivial derivation $D_0$ of $B_0$ such that the following conditions are satisfied:

1. $Y_0, T_0, f_0$ and $D_0$ are defined over $k_0$.
2. $Y = Y_0 \otimes_{k_0} k, T = T_0 \otimes_{k_0} k, f = f_0 \otimes_{k_0} k$ and $D = D_0 \otimes_{k_0} k$.
3. $D_0$ is locally nilpotent if and only if so is $D$.

**Proof.** Since $B$ and $R$ are integral domains finitely generated over $k$, write $B$ and $R$ as the residue rings of certain polynomial rings over $k$ modulo the finitely generated ideals. Write $B = k[x_1, \ldots, x_r]/I$ and $R = k[t_1, \ldots, t_s]/J$. Furthermore, the morphism $f$ is determined by the images $f^*(\eta_j) = \varphi_j(\xi_1, \ldots, \xi_r)$ in $B$, where $\xi_i = x_i \pmod{I}$ and $\eta_j = t_j \pmod{J}$. Adjoin to $\mathbb{Q}$ all coefficients of the finite generators of $I$ and $J$ as well as the coefficients of the $\varphi_j$ to obtain a subfield $k_0$ of $k$. Let $B_0 = k_0[x_1, \ldots, x_r]/I_0$ and $R_0 = k_0[t_1, \ldots, t_s]/J_0$, where $I_0$ and $J_0$ are respectively the ideals in $k_0[x_1, \ldots, x_r]$ and $k_0[t_1, \ldots, t_s]$ generated by the same generators of $I$ and $J$. Furthermore, define the homomorphism $f_0^*$ by the assignment $f_0^*(\eta_j) = \varphi_j(\xi_1, \ldots, \xi_r)$. Let $Y_0 = \text{Spec } B_0, T_0 = \text{Spec } R_0$ and let $f_0 : Y_0 \to T_0$ be the morphism defined by $f_0^*$. The derivation $D$ corresponds to a $B$-module homomorphism $\delta : \Omega^1_{B/R} \to B$. Since $\Omega^1_{B/R} = \Omega^1_{B_0/R_0} \otimes_{k_0} k$, we can enlarge $k_0$ so that there exists a $B_0$-homomorphism $\delta_0 : \Omega^1_{B_0/R_0} \to B_0$ satisfying $\delta = \delta_0 \otimes_{k_0} k$. Let $D_0 = \delta_0 \cdot d_0$, where $d_0 : B_0 \to \Omega^1_{B_0/R_0}$ is the standard differentiation. Then we have $D = D_0 \otimes_{k_0} k$.

\footnotetext{The result is also remarked in [3, Remark 13].}
Let \( \Phi_0 : B_0 \to B_0[[u]] \) be the \( R_0 \)-homomorphism into the formal power series ring in \( t \) over \( B_0 \) defined by

\[
\Phi_0(b_0) = \sum_{i \geq 0} \frac{1}{i!} D_0^i(b_0) u^i.
\]

Let \( \Phi : B \to B[[u]] \) be the \( R \)-homomorphism defined in a similar fashion. Then \( \Phi_0 \) and \( \Phi \) are determined by the images of the generators of \( B_0 \) and \( B \). Since the generators of \( B_0 \) and \( B \) are the same, we have \( \Phi = \Phi_0 \otimes_{k_0} k \). The derivation \( D_0 \) is locally nilpotent if and only if \( \Phi_0 \) splits via the polynomial subring \( B_0[u] \) of \( B_0[[u]] \). This is the case for \( D \) as well. Since \( \Phi_0 \) splits via \( B_0[u] \) if and only if \( \Phi \) splits via \( B[[u]] \), \( D_0 \) is locally nilpotent if and only if so is \( D \).

\[\text{Lemma 1.3.} \] Let \( k_1 \) be the algebraic closure of \( k_0 \) in \( k \). Let \( Y_1 = \text{Spec} \ B_1 \) with \( B_1 = B_0 \otimes_{k_0} k_1 \), \( T_1 = \text{Spec} \ R_1 \) with \( R_1 = R_0 \otimes_{k_0} k_1 \) and \( f_1 = f_0 \otimes_{k_0} k_1 \). Let \( D_1 = D_0 \otimes_{k_0} k_1 \). Then the following assertions hold.

1. Let \( t_1 \) be a closed point of \( T_1 \). Then the restriction of \( D_1 \) on the fiber \( f_1^{-1}(t_1) \) is locally nilpotent.
2. \( D_1 \) is locally nilpotent if and only if so is \( D \).

\[\text{Proof.} \] (1) Let \( t \) be the unique closed point of \( T \) lying over \( t_1 \) by the projection morphism \( T \to T_1 \), where \( R = R_1 \otimes_{k_1} k \). (If \( m_1 \) is the maximal ideal of \( R_1 \) corresponding to \( t_1 \), \( m_1 \otimes_{k_1} k \) is the maximal ideal of \( R \) corresponding to \( t \).) Then \( F_t = f^{-1}(t) = f_1^{-1}(t_1) \otimes_{k_1} k \) and the restriction \( D_t \) of \( D \) onto \( F_t \) is given as \( D_{1,t_1} \otimes_{k_1} k \), where \( D_{1,t_1} \) is the restriction of \( D_1 \) onto \( f_1^{-1}(t_1) \). We consider also the \( R \)-homomorphism \( \Phi : B \to B[[u]] \) and the \( R_1 \)-homomorphism \( \Phi_1 : B_1 \to B_1[[u]] \). As above, let \( m \) and \( m_1 \) be the maximal ideals of \( R \) and \( R_1 \) corresponding to \( t \) and \( t_1 \). Then \( D_t \) gives rise to the \( R/m \)-homomorphism \( \Phi \otimes_R R/m : B \otimes_R R/m \to (B \otimes_R R/m)[[u]] \). Similarly, \( D_{1,t_1} \) gives rise to the \( R_1/m_1 \)-homomorphism \( \Phi_1 \otimes_{R_1} R_1/m_1 : B_1 \otimes_{R_1} R_1/m_1 \to (B_1 \otimes_{R_1} R_1/m_1)[[u]] \), where \( R/m = k \) and \( R_1/m_1 = k_1 \). Then \( \Phi \otimes_R R/m = (\Phi_1 \otimes_{R_1} R_1/m_1) \otimes_{k_1} k \). Hence \( \Phi \otimes_R R/m \) splits via \( (B \otimes_R R/m)[[u]] \) if and only if \( \Phi_1 \otimes_{R_1} R_1/m_1 \) splits via \( (B_1 \otimes_{R_1} R_1/m_1)[[u]] \). Hence \( D_{1,t_1} \) is locally nilpotent as so is \( D_t \).

2. The same argument as above using the homomorphism \( \Phi \) can be applied. \[\square\]

The field \( k_0 \) can be embedded into the complex field \( \mathbb{C} \) because it is a finitely generated field extension of \( \mathbb{Q} \). Hence we can extend the embedding \( k_0 \hookrightarrow \mathbb{C} \) to the algebraic closure \( k_1 \). Thus \( k_1 \) is viewed as a subfield of \( \mathbb{C} \). Then Lemma 1.3 holds if one replaces the extension
Lemma 1.4. Theorem 1.1 holds if $k$ is the complex field $\mathbb{C}$.

Proof. Let $Y(\mathbb{C})$ be the set of closed points which we view as a complex analytic space embedded into a complex affine space $\mathbb{C}^N$ as a closed set. Consider the Euclidean metric on $\mathbb{C}^N$ and the induced metric topology on $Y(\mathbb{C})$. Then $Y(\mathbb{C})$ is a complete metric space.

Let $b$ be a nonzero element of $B$. For a positive integer $m$, define a Zariski closed subset $Y_m(b)$ of $Y(\mathbb{C})$ by

$$Y_m(b) = \{ Q \in Y(\mathbb{C}) \mid D^m(b)(Q) = 0 \} .$$

Since $Q$ lies over a closed point $t$ of $T(\mathbb{C})$ and $D_t$ is locally nilpotent on $f^{-1}(t)$ by the hypothesis, we have

$$f^{-1}(t) \subset \bigcup_{m>0} Y_m(b) .$$

This implies that $Y(\mathbb{C}) = \bigcup_{m>0} Y_m(b)$. We claim that $Y(\mathbb{C}) = Y_m(b)$ for some $m > 0$. In fact, this follows by Baire category theorem, which states that if the $Y_m(b)$ are all proper closed subsets, its countable union cannot cover the uncountable set $Y(\mathbb{C})$. If $Y(\mathbb{C}) = Y_m(b)$ for some $m > 0$ then $D^m(b) = 0$. This implies that $D$ is locally nilpotent on $B$.

One can avoid the use of Baire category theorem in the following way. Suppose that $Y_m(b)$ is a proper closed subset for every $m > 0$. Let $H$ be a general hyperplane in $\mathbb{C}^N$ such that the section $Y(\mathbb{C}) \cap H$ is irreducible, $\dim Y(\mathbb{C}) \cap H = \dim Y(\mathbb{C}) - 1$, and $Y(\mathbb{C}) \cap H = \bigcup_{m>0} (Y_m(b) \cap H)$ with $Y_m(b) \cap H$ a proper closed subset of $Y(\mathbb{C}) \cap H$ for every $m > 0$. We can further take hyperplane sections and find a general linear subspace $L$ in $\mathbb{C}^N$ such that $Y(\mathbb{C}) \cap L$ is an irreducible curve and $Y(\mathbb{C}) \cap L = \bigcup_{m>0} (Y_m(b) \cap L)$, where $Y_m(b) \cap L$ is a proper Zariski closed subset. Hence $Y_m(b) \cap L$ is a finite set, and $\bigcup_{m>0} (Y_m(b) \cap L)$ is a countable set, while $Y(\mathbb{C}) \cap L$ is not a countable set. This is a contradiction. Thus $Y(\mathbb{C}) = Y_m(b)$ for some $m > 0$. $\square$

Let $D$ be a $k$-derivation on a $k$-algebra $B$. It is called surjective if $D$ is so as a $k$-linear mapping. The following result is a consequence of Theorem 1.1.
Corollary 1.5. Let \( Y = \text{Spec } B, \ T = \text{Spec } R \) and \( f : Y \to T \) be the same as in Theorem 1.1. Let \( D \) be an \( R \)-derivation of \( B \) such that \( D_t \) is a surjective \( k \)-derivation for every closed point \( t \in T \). Assume further that the relative dimension of \( f \) is one. Then \( D \) is a locally nilpotent derivation and \( f \) is an \( \mathbb{A}^1 \)-fibration.

Proof. Let \( t \) be a closed point of \( T \) such that the fiber \( f^{-1}(t) \) is irreducible and reduced. By [9, Theorem 1.2 and Proposition 1.7], the coordinate ring \( B \otimes R/\mathfrak{m} \) of \( f^{-1}(t) \) is a polynomial ring \( k[x] \) in one variable and \( D_t = \partial/\partial x \), where \( \mathfrak{m} \) is the maximal ideal of \( R \) corresponding to \( t \). Then \( D_t \) is locally nilpotent. Taking the base change of \( f : Y \to T \) by \( U \hookrightarrow T \) if necessary, where \( U \) is a small open set of \( T \), we may assume that \( D_t \) is locally nilpotent for every closed point \( t \) of \( T \). By Theorem 1.1, the derivation \( D \) is locally nilpotent and hence \( f \) is an \( \mathbb{A}^1 \)-fibration. \( \square \)

2. Deformations of \( \mathbb{A}^1 \)-fibrations of affine type

In the present section, we assume that the ground field \( k \) is the complex field \( \mathbb{C} \). Let \( X \) be an affine algebraic surface which is normal. Let \( p : X \to C \) be an \( \mathbb{A}^1 \)-fibration, where \( C \) is an algebraic curve which is either affine or projective and \( p \) is surjective. We say that the \( \mathbb{A}^1 \)-fibration \( p \) is of affine type (resp. complete type) if \( C \) is affine (resp. complete). The \( \mathbb{A}^1 \)-fibration on \( X \) is the quotient morphism of a \( G_a \)-action on \( X \) if and only if it is of affine type (see [8]). We consider the following result on deformations. For the complex analytic case, one can refer to [19] and also to [12, p. 269].

Lemma 2.1. Let \( \overline{f} : \overline{Y} \to T \) be a smooth projective morphism from a smooth algebraic threefold \( \overline{Y} \) to a smooth algebraic curve \( T \). Let \( C \) be a smooth rational complete curve contained in \( \overline{Y}_0 = \overline{f}^{-1}(t_0) \) for a closed point \( t_0 \) of \( T \). Then the following assertions hold.

1. The Hilbert scheme \( \text{Hilb}(\overline{Y}) \) has dimension less than or equal to \( h^0(C, N_{C/\overline{Y}}) \) in the point \([C]\). If \( h^1(C, N_{C/\overline{Y}}) = 0 \) then the equality holds and \( \text{Hilb}(\overline{Y}) \) is smooth at \([C]\). Here \( N_{C/\overline{Y}} \) denotes the normal bundle of \( C \) in \( \overline{Y} \).
2. Let \( n = (C^2) \) on \( \overline{Y}_0 \). Then \( N_{C/\overline{Y}} \cong \mathcal{O}_C \oplus \mathcal{O}_C(n) \) provided \( n \geq -1 \).
3. Suppose \( n = 0 \). Then there exists an étale finite covering \( \sigma_2 : T' \to T \) such that the morphism \( \overline{f}_{T'} \) splits as
\[
\overline{f}_{T'} : \overline{Y} \times_T T' \xrightarrow{\varphi} V \xrightarrow{\sigma_1} T',
\]

\footnote{When we write \( t \in T \), we tacitly assume that \( t \) is a closed point of \( T \)}
where \( \varphi \) is a \( \mathbb{P}^1 \)-fibration with \( C \) contained as a fiber and \( \sigma_1 \) makes \( V \) a smooth \( T' \)-scheme of relative dimension one with irreducible fibers. Assume further that every smooth rational complete curve \( C' \) in \( Y_0 \) satisfies \((C' \cdot C) = 0\) provided \( C' \) is algebraically equivalent to \( C \) in \( Y \). Then the covering \( \sigma_2 : T' \to T \) is trivial, i.e., \( \sigma_2 \) is the identity morphism.

4. Suppose \( n = -1 \). Then \( C \) does not deform in the fiber \( Y_0 \) but deforms along the morphism \( f \) after an étale finite base change. Namely, there are an étale finite morphism \( \sigma : T' \to T \) and an irreducible subvariety \( Z \) of codimension one in \( Y' := Y \times_T T' \) such that \( Z \) can be contracted along the fibers of \( f : Y' \to T' \), where \( Y' \) is an irreducible smooth affine curve and \( f \) is the second projection of \( Y \times_T T' \) to \( T' \).

5. Assume that there are no \((-1)\)-curves \( E \) and \( E' \) in \( Y_0 \) such that \( E \cap E' \neq \emptyset \) and \( E \) is algebraically equivalent to \( E' \) as 1-cycles on \( Y \). Then, after shrinking \( T \) to a smaller open set if necessary, we can take \( Z \) in the assertion (4) above as a subvariety of \( Y \). The contraction of \( Z \) gives a factorization 
\[
\begin{array}{ccc}
Z & \xrightarrow{g} & T' \\
\sigma & \xrightarrow{\cdot} & T
\end{array}
\]
where \( g \) is a \( \mathbb{P}^1 \)-fibration, \( C \) is a fiber of \( g \) and \( \sigma \) is as above.

**Proof.** (1) The assertion follows from Grothendieck [7, Cor. 5.4].
(2) We have an exact sequence 
\[
0 \to N_{C/Y_0} \to N_{C/Y} \to N_{Y_0/Y}\big|_C \to 0,
\]
where \( N_{C/Y_0} \cong \mathcal{O}_C(n) \) and \( N_{Y_0/Y}\big|_C \cong \mathcal{O}_C \). The obstruction for this exact sequence to split lies in \( \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(n)) \cong H^1(C, \mathcal{O}_C(n)) \), which is zero if \( n \geq -1 \).

(3) Suppose \( n = 0 \). Then \( \dim_{[C]} \text{Hilb}(Y) = 2 \) and \([C]\) is a smooth point of \( \text{Hilb}(Y) \). Let \( H \) be a relatively ample divisor on \( Y/T \) and set \( P(n) := P_C(n) = h^0(C, \mathcal{O}_C(nH)) \) the Hilbert polynomial in \( n \) of \( C \) with respect to \( H \). Then \( \text{Hilb}^P(Y) \) is a scheme which is projective over \( T \). Let \( V \) be the irreducible component of \( \text{Hilb}^P(Y) \) containing the point \([C]\). Then \( V \) is a \( T \)-scheme with a morphism \( \sigma : V \to T \), \( \dim V = 2 \) and \( V \) has relative dimension one over \( T \). Furthermore, there exists a subvariety \( Z \) of \( Y \times_T V \) such that the fibers of the composite morphism 
\[
g : Z \hookrightarrow Y \times_T V \xrightarrow{p_2} V
\]
are curves on \( Y \) parametrized by \( V \). For a general point \( v \in V \), the corresponding curve \( C' := C_v \) is a smooth rational complete curve because \( P_{C'}(n) = P(n) \) and \((C')^2 = 0\) on \( Y_t = f^{-1}(t) \) with \( t = \sigma(v) \).
because \( \dim_{C'} \text{Hilb}(Y) = 2 \). In fact, if \((C')^2 \leq -1\) then the exact sequence of normal bundles in (2) implies \( h^0(C', N_{C'/Y}) \leq 1 \), which contradicts \( \dim_{C'} \text{Hilb}(Y) = 2 \). If \((C')^2 > 0\) then \( \dim_{C'} \text{Hilb}(Y) > 2 \), which is again a contradiction. So, \((C')^2 = 0\). Hence \( \text{Hilb}(Y) \) is smooth at \( t \). Since \( V \) is smooth over \( T \), \( \alpha_t \) induces an isomorphism from \( B_t \) to a connected component of \( V_t := \sigma^{-1}(t) \). This is the case if we take \( v \in V \) from a different connected component of \( V_t \). Let \( \sigma : V \xrightarrow{\sigma_1} T' \xrightarrow{\sigma_2} T \) be the Stein factorization of \( \sigma \). Then \( \sigma_2 \) is an étale finite morphism and \( \sigma_1 : V \to T' \) is a smooth morphism of relative dimension one with irreducible fibers. Furthermore, the morphism \( g \) above factors as a composite of \( T' \)-morphisms

\[
g : Z \hookrightarrow (\overline{Y} \times_T T') \times_T V \xrightarrow{\bar{p}_2} V,
\]

where \( Z \) is identified with \( \overline{Y} \times_T T' \) by the above construction. Hence \( g \) induces a \( T' \)-morphism \( \varphi : \overline{Y} \times_T T' \to V \) such that the composite \( \sigma_1 \cdot \varphi : \overline{Y} \times_T T' \to V \to T' \) is the pull-back \( \overline{f}_{T'} : \overline{Y} \times_T T' \to T' \) of the morphism \( \overline{f} \).

In the above argument, we take two curves \( C, C' \) corresponding two points \( v, v' \) in \( V_{t_0} \). Then \( C \) is algebraically equivalent to \( C' \) in \( \overline{Y} \), and hence \((C \cdot C') = 0\) by the hypothesis. So, \( C = C' \) or \( C \cap C' = \emptyset \). This implies that \( C' \) and \( C \) are the fibers of the same \( \mathbb{P}^1 \)-fibration \( \varphi_{t_0} : \overline{Y}_{t_0} \to \overline{B}_{t_0} \) and hence that \( V_{t_0} \) is irreducible. Namely, \( \sigma_{t_0}^{-1}(t_0) \) consists of a single point. Hence \( \deg \sigma_2 = 1 \), i.e., \( \sigma_2 \) is the identity morphism.

(4) Suppose \( n = -1 \). Then \( h^0(C, N_{C/Y}) = 1 \) and \( h^1(C, N_{C/Y}) = 0 \). Hence \( \text{Hilb}^P(Y) \) has dimension one and is smooth at \([C]\), where \( P(n) = P_C(n) \) is the Hilbert polynomial of \( C \) with respect to \( H \). Let \( T' \) be the irreducible component of \( \text{Hilb}^P(Y) \) containing \([C]\). Note that \( \dim T' = 1 \). Then we find a subvariety \( Z \) in \( \overline{Y} \times_T T' \) such that \( C \) is a fiber of \( g \) and every fiber of the \( T \)-morphism \( g = p_2|_{T'} : Z \to T' \) is a \((-1)\) curve in the fiber \( \overline{Y}_t \). In fact, the nearby fibers of \( C \) are \((-1)\) curves as a small deformation of \( C \) by \([12]\). Hence, by covering \( T' \) by small disks, we know that every fiber of \( g \) is a \((-1)\) curve. Further, the projection \( \sigma : T' \to T \) is a finite morphism as it is projective and \( T' \) is smooth because each fiber is a \((-1)\) curve in \( \overline{Y} \) (see the above argument for \([C]\)). Furthermore, \( \sigma \) is étale since \( \overline{f} \) is locally a product of the fiber and the base in the Euclidean topology. Hence \( \sigma \) induces a local isomorphism between \( T' \) and \( T \). This implies that \( \overline{Y} \times_T T' \) is a
smooth affine threefold and the second projection \( f : Y \times_T T' \to T' \) is a smooth projective morphism. Now, after an étale finite base change \( \sigma : T' \to T \), we may assume that \( Z \) is identified with a subvariety of \( \overline{Y} \). Since \( C \) is a \((-1)\) curve in \( \overline{Y}_0 \), it is an extremal ray in the cone \( \overline{NE}(\overline{Y}_0) \). Since \( C \) is algebraically equivalent to the fibers of \( g : Z \to T' \), it follows that \( C \) is an extremal ray in the relative cone \( \overline{NE}(Y/T) \).

Then it follows from [22, Theorem 3.25] that \( Z \) is contracted along the fibers of \( g \) in \( \overline{Y} \) and the threefold obtained by the contraction is smooth and projective over \( T \).

(5) Let \( \sigma^{-1}(t_0) = \{u_1, \ldots, u_d\} \) and let \( Z_{u_i} = Z \cdot (\overline{Y} \times \{u_i\}) \) for \( 1 \leq i \leq d \). Then the \( Z_{u_i} \) are the \((-1)\) curves on \( \overline{Y}_0 \) which are algebraically equivalent to each other as \(-1\)-cycles on \( \overline{Y} \). By the assumption, \( Z_{u_i} \cap Z_{u_j} = \emptyset \) whenever \( i \neq j \). This property holds for all \( t \in T \) if one shrinks to a smaller open set of \( t_0 \). Then we can identify \( Z \) with a closed subvariety of \( \overline{Y} \). In fact, the projection \( p : Z \hookrightarrow Y \times_Y T' \to Y \) is a \( T \)-morphism. For the point \( t_0 \in T \), the morphism \( p \otimes_{\mathcal{O}_T,t_0} \mathcal{O}_{T,t_0} \) with the completion \( \hat{\mathcal{O}}_{T,t_0} \) of \( \mathcal{O}_{T,t_0} \) is a direct sum of the closed immersions from \( Z \otimes_{\mathcal{O}_T,t_0} \hat{\mathcal{O}}_{T,t_0} \) into \( Y \otimes_{\mathcal{O}_T,t_0} \hat{\mathcal{O}}_{T,t_0} \) for \( 1 \leq i \leq r \). So, \( p \otimes_{\mathcal{O}_T,t_0} \mathcal{O}_{T,t_0} \) is a closed immersion. Hence \( p \) is a closed immersion locally over \( t_0 \) because \( \hat{\mathcal{O}}_{T,t_0} \) is faithfully flat over \( \mathcal{O}_{T,t_0} \). The rest is the same as in the proof of the assertion (4).

Let \( Y_0 \) be a smooth affine surface and let \( \overline{Y}_0 \) be a smooth projective surface containing \( Y_0 \) as an open set in such a way that the complement \( \overline{Y}_0 \setminus Y_0 \) supports a reduced effective divisor \( D_0 \) with simple normal crossings. We call \( \overline{Y}_0 \) a normal completion of \( Y_0 \) and \( D_0 \) the boundary divisor of \( Y_0 \). An irreducible component of \( D_0 \) is called a \((-1)\) component if it is a smooth rational curve with self-intersection number \(-1\). We say that \( \overline{Y}_0 \) is a minimal normal completion if the contraction of a \((-1)\) component of \( D_0 \) (if any) results the image of \( D_0 \) losing the condition of simple normal crossings.

Let \( f : \overline{Y} \to T \) be a smooth projective morphism from a smooth algebraic threefold \( \overline{Y} \) to a smooth algebraic curve \( T \) and let \( S = \sum_{i=1}^r \hat{S}_i \) be a reduced effective divisor on \( \overline{Y} \) with simple normal crossings. Let \( Y = \overline{Y} \setminus S \) and let \( f = f|_Y \). We assume that for every point \( t \in T \), the intersection cycle \( D_t = f^{-1}(t) \cdot S \) is a reduced effective divisor of \( \overline{Y}_t = f^{-1}(t) \) with simple normal crossings\(^3\) and \( Y_t = Y \cap \overline{Y}_t \) is an affine

\(^3\)In order to avoid the misreading, it is better to specify our definition of simple normal crossings in the case of dimension three. We assume that every irreducible component \( S_i \) of \( S \) and every fiber \( \overline{Y}_i \) are smooth and that analytic-locally at every intersection point \( P \) of \( S_i \cap S_j \) (resp. \( S_i \cap S_j \cap S_k \) or \( S_i \cap \overline{Y}_i \)), \( S_i \) and \( S_j \) (resp. \( S_i \cap S_k \) or \( S_i \cap \overline{Y}_i \)).
open set of $\overline{Y}_t$. For a point $t_0 \in T$, we assume that $\overline{Y}_{t_0} = \overline{Y}_0, D_{t_0} = D_0$ and $Y_{t_0} = Y_0$. A collection $(Y, \overline{Y}, S, \overline{f}, t_0)$ is called a family of logarithmic deformations of a triple $(Y_0, \overline{Y}_0, D_0)$. We call it simply a log deformation of the triple $(Y_0, \overline{Y}_0, D_0)$. Since $f$ is smooth and $S$ is a divisor with simple normal crossings, $(Y, \overline{Y}, S, \overline{f}, t_0)$ is a family of logarithmic deformations in the sense of Kawamata [17, 18].

From time to time, we have to make a base change by an étale finite covering of a triple $(\overline{Y}, S, D, t_0)$. We call it simply a log deformation of affine type. Then $\rho$ is a divisor with simple normal crossings. Hence $(\overline{Y}, S, D, t_0)$ is a family of logarithmic deformations of the triple $(Y_0, \overline{Y}_0, D_0)$, where $t'_0 \in T'$ with $\sigma(t'_0) = t_0$.

We have the following result on logarithmic deformations of affine surfaces with $\mathbb{A}^1$-fibrations.

**Lemma 2.2.** Let $(Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of the triple $(Y_0, \overline{Y}_0, D_0)$. Then the following assertions hold.

1. Assume that $Y_0$ has an $\mathbb{A}^1$-fibration. Then $Y_t$ has an $\mathbb{A}^1$-fibration for every $t \in T$.

2. If $Y_0$ has an $\mathbb{A}^1$-fibration of affine type (resp. of complete type), then $Y_t$ has also an $\mathbb{A}^1$-fibration of affine type (resp. of complete type) for every $t \in T$.

**Proof.** (1) Note that $K_{\overline{Y}_t} = (K_{\overline{Y}} + \overline{f}_t) \cdot \overline{Y}_t = K_{\overline{Y}} \cdot \overline{Y}_t$ because $\overline{Y}_t$ is algebraically equivalent to $\overline{Y}_{t'}$ for $t' \neq t$. Then $K_{\overline{Y}_t} + D_t = (K_{\overline{Y}} + S) \cdot \overline{Y}_t$. By the hypothesis, $h^0(\overline{Y}_0, \mathcal{O}(n(K_{\overline{Y}_0} + D_0))) = 0$ for every $n > 0$. Then the semicontinuity theorem [11, Theorem 12.8] implies that $h^0(\overline{Y}_t, \mathcal{O}(n(K_{\overline{Y}_t} + D_t))) = 0$ for every $n > 0$. Hence $\pi(Y_t) = -\infty$. Since $Y_t$ is affine, this implies that $Y_t$ has an $\mathbb{A}^1$-fibration.

(2) Suppose that $Y_0$ has an $\mathbb{A}^1$-fibration $\rho_0 : Y_0 \to B_0$ which is of affine type. Then $\rho_0$ defines a pencil $\Lambda_0$ on $\overline{Y}_0$.

Suppose first that $\Lambda_0$ has no base points and hence defines a $\mathbb{P}^1$-fibration $\overline{\rho}_0 : \overline{Y}_0 \to \overline{B}_0$ such that $\overline{\rho}_0|_{Y_0} = \rho_0$ and $\overline{B}_0$ is a smooth completion of $B_0$. If $\overline{\rho}_0$ is not minimal, let $E$ be a $(-1)$ curve contained in a fiber of $\overline{\rho}_0$, which is necessarily not contained in $Y_0$. By Lemma 2.1 E extends along the morphism $\overline{f}$ if one replaces the base $T$ by a suitable étale finite covering $T'$ and can be contracted simultaneously with other $S_t, S_j$ and $S_k$ or $S_i$ and $\overline{Y}_t$ behave like coordinate hypersurfaces. Hence $S_t \cap S_j$ or $S_i \cap \overline{Y}_t$ are smooth curves at the point $P$. 
(−1) curves contained in the fibers $Y_t$ ($t \in T$). Note that this étale finite change of the base curve does not affect the properties of the fiber surfaces. Hence we may assume that all simultaneous blowing-ups and contractions as applied below are achieved over the base $T$.

The contraction is performed either within the boundary divisor $S$ or the simultaneous half-point detachments in the respective fibers $Y_t$ for $t \in T$. (For the definition of half-point detachment (resp. attachment), see for example [4]). Hence the contraction does not change the hypothesis on the simple normal crossing of $S$ and the intersection divisor $S \cdot Y_0$. Thus we may assume that $\rho_0$ is minimal. Since $B_0 \subsetneq B_0$, a fiber of $\rho_0$ is contained in a boundary component, say $S_1$. Then the intersection $S_1 \cdot Y_0$ as a cycle is a disjoint sum of the fibers of $\rho_0$ with multiplicity one. Hence $(S_1^2 \cdot Y_0)_{Y_0} = 0$. Since $Y_t$ and $Y_0$ are algebraically equivalent, we have $(S_1^2 \cdot Y_t) = 0$ for every $t \in T$. Note that $Y_t$ is also a ruled surface by Iitaka [12] and minimal by the same reason as for $Y_0$. Considering the deformations of a fiber of $\rho_0$ appearing in $S_1 \cdot Y_0$, we know by Lemma 2.1 that $S_1 \cdot Y_t$ is a disjoint sum of smooth rational curves with self-intersection number zero. Namely, $S_1 \cdot Y_t$ is a sum of the fibers of a $\mathbb{P}^1$-fibration. Here we may have to replace the $\mathbb{P}^1$-fibration $\rho_t$ by the second one if $Y_t \cong \mathbb{P}^1 \times \mathbb{P}^1$. In fact, if a smooth complete surface has two different $\mathbb{P}^1$-fibrations and is minimal with respect to one fibration, then the surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and two $\mathbb{P}^1$-fibrations are the vertical and horizontal fibrations. This implies that $Y_t$ has an $\mathbb{A}^1$-fibration of affine type.

Suppose next that $\Lambda_0$ has a base point, say $P_0$, and that the $\mathbb{A}^1$-fibration $\rho_0$ is of affine type. Then all irreducible components of $D_0 := S \cdot Y_0$ are contained in the members of $\Lambda_0$. Since the boundary divisor $D_0$ of $Y_0$ is assumed to be a connected divisor with simple normal crossings, there are at most two components of $S \cdot Y_0$ passing through $P_0$, and if there are two of them, they lie on different components of $S$ and $P_0$ lies on their intersection curve. In particular, if $S_1$ is a component of $S$ containing $P_0$, then $S_1 \cdot Y_0$ is a disjoint sum of smooth rational curves. Let $C_1$ be the component of $S_1 \cdot Y_0$ passing through $P_0$ and let $F_0$ be the member of $\Lambda_0$ which contains $C_1$. We may assume that $F_0$ is supported by the boundary divisor $D_0$. If $F_0$ contains a $(-1)$ curve $E$ such that $P_0 \notin E$, then $E$ extends along the morphism $\overline{f}$ and can be contracted simultaneously along $\overline{f}$ after the base change by an étale finite covering $T' \to T$. So, we may assume that every irreducible component of $F_0$ not passing $P_0$ has self-intersection number $\leq -2$ on $Y_0$. Then we may assume that $(C_1^2)_{Y_0} \geq 0$. In fact, if there are two irreducible components of $S \cdot Y_0$ passing through $P_0$ and belonging to
the same member $F_0$ of $\Lambda_0$, one of them must have self-intersection number $\geq 0$, for otherwise all the components of the member of $\Lambda_0$, after the elimination of base points, would have self-intersection number $\leq -2$, which is a contradiction. So, we may assume that the one on $S_1$, i.e., $C_1$, has self-intersection number $\geq 0$. Then the proper transform of $C_1$ is the unique $(-1)$ curve with multiplicity $> 1$ in the fiber corresponding to $F_0$ after the elimination of base points of $\Lambda_0$.

On the other hand, $S_1 \cdot Y_0$ (as well as $S_i \cdot Y_0$ if it is non-empty) is a disjoint sum of smooth rational curves, one of which is $C_1$. Let $n := (C_1^2)_{Y_0} \geq 0$. Then $\text{Hilb}^n(Y)$ has dimension $n + 2$ and is smooth at the point $[C_1]$. Since $C_1 \cong \mathbb{P}^1$ and $N_{C_1/Y} \cong \mathcal{O}(n) \oplus \mathcal{O}$, $C_1$ extends along the morphism $\overline{f}$. Namely, $\overline{f}|_{S_1} : S_1 \to T$ is a composite of a $\mathbb{P}^1$-fibration $\sigma_1 : S_1 \to T'$ and an étale finite morphism $\sigma_2 : T' \to T$, where $C_1$ is a fiber of $\sigma_1$. By the base change by $\sigma_2$, we may assume that $S_1 \cdot Y_0 = C_1$. In particular, $(C_1^2)_{S_1} = 0$.

Suppose that $C_2$ is a component of $F_0$ meeting $C_1$. Then $C_2$ is contained in a different boundary component, say $S_2$, which intersects $S_1$. Since $(H \cdot S_2 \cdot Y_0) > 0$, we have $(H \cdot S_2 \cdot Y_t) > 0$ for every $t \in T$, where $H$ is a relatively ample divisor on $Y$ over $T$. Furthermore, $S_2 \cdot Y_0$ is algebraically equivalent to $S_2 \cdot Y_t$. Note that $S_2 \cdot Y_0$ is a disjoint sum of smooth rational curves, one of which is the curve $C_2$ connected to $C_1$. By considering the factorization of $\overline{f}|_{S_2} : S_2 \to T$ into a product of a $\mathbb{P}^1$-fibration and an étale finite morphism as in the case for $S_1 \cdot Y_0$ and taking the base change by an étale finite morphism, we may assume that $S_2 \cdot Y_0 = C_2$. Hence we have

$$(S_1 \cdot S_2 \cdot Y_t) = (S_1 \cdot S_2 \cdot Y_0) = (C_1 \cdot (S_2 \cdot Y_0))_{Y_0} = (C_1 \cdot C_2)_{Y_0} = 1.$$  

This implies that $S_2 \cdot Y_t$ is irreducible for a general point $t \in T$. For otherwise, by the Stein factorization of the morphism $\overline{f}|_{S_2} : S_2 \to T$, the fiber $S_2 \cdot Y_t$ is a disjoint sum $A_1 + \cdots + A_s$ of distinct irreducible curves which are algebraically equivalent to each other on $S_2$. Since

$$1 = (S_1 \cdot S_2 \cdot Y_t) = ((S_1 \cdot S_2) \cdot (S_2 \cdot Y_t))_{S_2} = ((S_1 \cdot S_2) \cdot (A_1 + \cdots + A_s))_{S_2} = s((S_1 \cdot S_2) \cdot A_1),$$

we have $s = 1$ and $(S_1 \cdot S_2 \cdot A_1) = 1$. So, $\overline{f}|_{S_2} : S_2 \to T$ is now a $\mathbb{P}^1$-bundle and $(C_2^2)_{S_2} = 0$. This implies that $N_{C_2/Y} \cong \mathcal{O}(m) \oplus \mathcal{O}$ with $m = (C_2^2)_{Y_0} \leq -2$ and that $C_2$ extends along the morphism $\overline{f}$. We can argue in the same way as above with irreducible components of $F_0$ other than $C_1$.

Assume that no members of $\Lambda_0$ except $F_0$ have irreducible components outside of $Y_0$. If $C_1$ is shown to move on the component $S_i$ along
the morphism $\overline{f}$, we consider a component $C_{i+1}$ anew which meets $C_i$. Each of them is contained in a distinct irreducible boundary component of $S$ and extends along the morphism $\overline{f}$. Let $S_1, S_2, \ldots, S_r$ be all the boundary components which meet $\overline{Y}_0$ along the irreducible components of $F_0$. Then $\overline{Y}_i$ intersects $S_1 + S_2 + \cdots + S_r$ in an effective divisor which has the same form as $F_0$. Furthermore, we have

$$(S_i \cdot \overline{Y}_i)^2 = (S_i^2 \cdot \overline{Y}_0) = (S_i^2 \cdot \overline{Y}_0) = ((S_i \cdot \overline{Y}_0)^2)_{\overline{Y}_0}$$

for $1 \leq i \leq r$. Namely, the components $S_i \cdot \overline{Y}_i$ ($1 \leq i \leq r$) with the same multiplicities as $S_i \cdot \overline{Y}_0$ in $F_0$ is a member $F_i$ of the pencil $\Lambda_i$ lying outside of $Y_i$. This implies that $\Lambda_i$ has a base point $P_i$ and at least one member of $\Lambda_i$ lies outside of $Y_i$. So, the $\mathbb{A}^1$-fibration $\rho_i$ on $Y_i$ is of affine type.

If the pencil $\Lambda_0$ contains two members $F_0, F'_0$ such that the components $C_1, C'_1$ of $F_0, F'_0$ lie outside of $Y_0$ and pass through the point $P_0$, we may assume that $F_0$ is supported by the boundary components, while $F'_0$ may not. Then no other members of $\Lambda_0$ have irreducible components outside of $Y_0$ because $\overline{Y}_0 \setminus Y_0$ is connected. We can argue as above to show, after a suitable étale finite base change, that the member $F_0$ moves along the morphism $\overline{f}$, and further that every boundary component of $F'_0$ moves on a boundary component, say $S'_j$, as a fiber of $f|_{S'_j}: S'_j \to T$. Hence the pencil $\Lambda_1$ has the member $F_i$ corresponding to $F_0$ whose all components lie outside of $Y_i$ and the member $F'_i$ corresponding to $F'_0$. In fact, the part of $F'_i$ lying outside of $Y_i$ is determined as above, but since $Y_1 \cap F'_i$ is a disjoint union of the $\mathbb{A}^1$ which correspond to the $(-1)$ components of $F'_i$ (the half-point attachments), the member $F'_i$ is determined up to its weighted graph. This proof also implies that if $\rho_0$ is of complete type then $\rho_i$ is of complete type for every $t \in T$.

**Remark 2.3.** (1) In the above proof of Lemma 2.2, the case where the pencil $\Lambda_0$ has a base point $P_0$ on one of the connected components $S_1 \cap \overline{Y}_0$, say $C_1$, there might exist a monodromy on $Y$ which transform $\Lambda_0$ to a pencil $\Lambda'_0$ on $\overline{Y}_0$ having a base point $P'_0$ on a different connected component $C'_1$ of $S_1 \cap \overline{Y}_0$. However, we have $(C^2)_{\overline{Y}_0} \geq 0$ as shown in the proof, and $(C^2) = (C^2)_{\overline{Y}_0}$. Since $C'_1$ is contained in a member of $\Lambda_0$, whence $(C^2) < 0$. This is a contradiction. So, $S_1 \cap \overline{Y}_0$ is irreducible.

(2) In the step of the above proof of Lemma 2.2 where we assume that no members of $\Lambda_0$ except $F_0$ have irreducible components outside of $Y_0$, let $P'_0$ be a point on $C_1$, $S_1 \cdot \overline{Y}_1$ other than $P_0$ which is the base point of the given pencil $\Lambda_i$. Then there is a pencil $\Lambda'_1$ on $\overline{Y}_1$ which is
similar to $\Lambda_t$. In fact, note first that $\overline{Y}_t$ is a rational surface. Perform
the same blowing-ups with centers at $P'_t$ and its infinitely near points
as those with centers at $P_t$ and its infinitely near points which eliminate
the base points of $\Lambda_t$. Then we find an effective divisor $\tilde{F}'_t$ supported by
the proper transforms of $S'_i \cdot \overline{Y}_t (1 \leq i \leq r)$ and the exceptional curves
of the blowing-ups such that $\tilde{F}'_t$ has the same form and multiplicities
as the corresponding member $\tilde{F}_t$ in the proper transform $\tilde{\Lambda}_t$ of $\Lambda_t$ after
the elimination of base points. Then $(\tilde{F}'_t)^2 = 0$ and hence $\tilde{F}'_t$ is a fiber
of an $\mathbb{P}^1$-fibration on the blown-up surface of $\overline{Y}_t$. Then the fibers of the
$\mathbb{P}^1$-fibration form the pencil $\Lambda'_t$ on $\overline{Y}_t$ after the reversed contractions.
In fact, the surface $Y_t = \overline{Y}_t \setminus D_t$ is the affine plane with two systems
of coordinate lines given as the fibers of $\Lambda_t$ and $\Lambda'_t$. Hence the $\mathbb{A}^1$-
fibrations induced by $\Lambda_t$ and $\Lambda'_t$ are transformed by an automorphism
of $\overline{Y}_t$.

The following is one of the simplest examples of our situation.

**Example 2.4.** Let $C$ be a smooth conic and let $S$ be the subvariety
of codimension one in $\mathbb{P}^2 \times C$ defined by
$$S = \{(P, Q) \mid P \in L_Q, Q \in C\},$$
where $L_Q$ is the tangent line of $C$ at $Q$. Let $Y = (\mathbb{P}^2 \times C) \setminus S$ and let
$f : Y \to C$ be the projection onto $C$. We set $T = C$ to fit the previous
notations. Set $\overline{Y} = \mathbb{P}^2 \times C$. Then $\overline{f} : \overline{Y} \to T$ is the second projection
and the boundary divisor $S$ is irreducible. For every point $Q \in C$,
$Y_Q := \mathbb{P}^2 \setminus L_Q$ has a linear pencil $\Lambda_Q$ generated by $C$ and $2L_Q$, which
induces an $\mathbb{A}^1$-fibration of affine type. The restriction $\overline{f}|_S : S \to T$ is a
$\mathbb{P}^1$-bundle. Let $C$ be defined by $X_0X_2 = X_1^2$ with respect to a system
of homogeneous coordinates $(X_0, X_1, X_2)$ of $\mathbb{P}^2$ and let $\eta = (1, t, t^2)$ be
the generic point of $C$ with $t$ an inhomogeneous coordinate on $C \cong \mathbb{P}^1$.
Then $L_\eta$ is defined by $t^2X_0 - 2tX_1 + X_2 = 0$. The generic fiber $Y_\eta$
of $f$ has an $\mathbb{A}^1$-fibration induced by the linear pencil $\Lambda_\eta$ whose general
members are the conics defined by $(X_0X_2 - X_1^2) + u(t^2X_0 - 2tX_1 + X_2)^2 = 0$,
where $u \in \mathbb{A}^1$. Indeed, the conics are isomorphic to $\mathbb{P}^1_{k(t)}$ since they
have the $k(t)$-rational point $(1, t, t^2)$, and $Y_\eta$ is isomorphic to $\mathbb{A}^2_{k(t)}$. This
implies that the affine threefold $Y$ itself has an $\mathbb{A}^1$-fibration.

In the course of the proof of Lemma 2.2, we frequently used the base
change by a finite étale morphism $\sigma : T' \to T$, where $T'$ is taken in such
a way that for every $t \in T$, the points $\sigma^{-1}(t)$ correspond bijectively
to the connected components of $S_t \cap \overline{Y}_t$, where $S_t$ is an irreducible
component of $S$. Suppose that $\deg \sigma > 1$. Let $\overline{Y}' = \overline{Y} \times_T T'$ and
$\mathcal{f}' = \mathcal{f} \times_T T'$. Note that $\mathcal{Y}$ is smooth because $\sigma$ is étale. The morphism $\sigma : T' \to T$ gives the Stein factorization $|f|_{S_i} : S_i \xrightarrow{\varphi} T' \xrightarrow{\sigma} T$. Then the subvariety $S_i$ is considered to be a subvariety of $\mathcal{Y}'$ via a closed immersion $(\text{id}_{S_i}, \varphi) : S_i \to S_i \times_T T' \hookrightarrow \mathcal{Y}' \times_T T'$. We denote it by $S'_i$. Let $t_1, t_2$ be points of $T'$ such that they correspond to the connected components $A, B$ of $S_i \cap \mathcal{Y}_t$, whence $\sigma(t_1) = \sigma(t_2) = t$. Then $A, B$ are the fibers of $S'_i$ over the points $t_1, t_2$ of $T'$. Hence $A$ and $B$ are algebraically equivalent in $\mathcal{Y}'$. Since $T'$ is étale over $T$, we say more precisely that they are \textit{étale-algebraically equivalent}. We have $(A^2)_{\mathcal{Y}_{t_1}} = (B^2)_{\mathcal{Y}_{t_1}}$. In fact, noting that $\mathcal{Y}'_{t_1}$ and $\mathcal{Y}'_{t_2}$ are algebraically equivalent in $\mathcal{Y}$ and that $\mathcal{Y}_{t_1}$ and $\mathcal{Y}_{t_2}$ are isomorphic to $\mathcal{Y}_{t}$, we have

\begin{align*}
(A^2)_{\mathcal{Y}_{t_1}} &= (A^2)_{\mathcal{Y}_{t_2}} = (S'_1 \cdot S'_2 \cdot \mathcal{Y}'_{t_1}) \\
&= (S'_1 \cdot S'_2 \cdot \mathcal{Y}'_{t_2}) = (B^2)_{\mathcal{Y}_{t_2}} = (B^2)_{\mathcal{Y}_{t_1}}
\end{align*}

Let $C$ be an irreducible curve in $\mathcal{Y}_0 \cap S$, say a connected component of $\mathcal{Y}_0 \cap S_1$ with an irreducible component $S_1$ of $S$. We say that $C$ has \textit{no monodromy} in $\mathcal{Y}$ if $|f|_{S_1} : S_1 \to T$ has no splitting $|f|_{S_1} : S_1 \xrightarrow{\sigma_1} T' \xrightarrow{\sigma_2} T$, where $\sigma_2$ is an étale finite morphism with $\deg \sigma_2 > 1$. Note that, after a suitable étale finite base change $\mathcal{Y} \times_T T'$, this condition is fulfilled. Namely, \textit{the monodromy is killed}. Concerning the extra hypothesis in Lemma \ref{Lemma 1} (3) and the possibility of achieving the contractions over the base curve $T$ in Lemma \ref{Lemma 1} (5), we have the following result.

\textbf{Lemma 2.5.} Let $(Y, \mathcal{Y}, S, \mathcal{f}, t_0)$ be a family of logarithmic deformation of the triple $(Y_0, \mathcal{Y}_0, D_0)$. Assume that $\mathcal{Y}_0$ has an $\mathbb{A}^1$-fibration of affine type. Let $\Lambda_0$ be the pencil on $\mathcal{Y}_0$ whose general members are the closures of fibers of the $\mathbb{A}^1$-fibration. Suppose that $\Lambda_0$ defines a $\mathbb{P}^1$-fibration $\varphi_0 : \mathcal{Y}_0 \to \mathcal{B}_0$. Suppose further that the section of $\varphi_0$ in $S \cap \mathcal{Y}_0$ has no monodromy in $\mathcal{Y}$. Then the following assertions hold.

\begin{enumerate}
  \item If $C$ is a fiber of $\varphi_0$ with $C \cap Y_0 \neq \emptyset$ and $C'$ is a smooth rational complete curve which is algebraically equivalent to $C$ in $\mathcal{Y}$, then $(C \cdot C') = 0$.
  \item There are no two $(-1)$ curves $E_1$ and $E_2$ such that they belong to the same connected component of the Hilbert scheme $\text{Hilb}(\mathcal{Y})$, $E_1$ is an irreducible component of a fiber of $\varphi_0$ and $E_1 \cap E_2 \neq \emptyset$.
\end{enumerate}

\textbf{Proof.} (1) Let $S_0$ be an irreducible component of $S$ such that $(S_0 \cdot F) = 1$ for a general fiber $F$ of $\varphi_0$. Then $S_0 \cap \mathcal{Y}_0$ contains a cross-section of $\varphi_0$. The assumption on the absence of the monodromy implies that
$S_0 \cap \overline{Y}_0$ is irreducible and is the section of $\varphi_0$. Note that $\varphi_0$ contains a fiber $F_\infty$ at infinity which is supported by the intersection of $\overline{Y}_0$ with the boundary divisor $S$ in $\overline{Y}$. Such a fiber exists by the assumption that the $\mathbb{A}^1$-fibration on $Y_0$ is of affine type. Since $S_0 \cap \overline{Y}_0$ gives the cross-section, $F_\infty$ is supported by $S \setminus S_0$. Since $C \cap (S \setminus S_0) = \emptyset$ and $C'$ is algebraically equivalent to $C$ in $\overline{Y}$, $C'$ does not meet the components of $S \setminus S_0$. Hence $C' \cap F_\infty = \emptyset$, and $C'$ is a component of a fiber of $\varphi_0$. So, $(C \cdot C') = 0$.

(2) Suppose that such $E_1$ and $E_2$ exist. Since $E_1$ and $E_2$ are algebraically equivalent 1-cycles on $\overline{Y}$, $E_1$ and $E_2$ have the same intersections with subvarieties of codimension one in $\overline{Y}$. We consider possible cases separately.

(i) Suppose that both $E_1$ and $E_2$ are contained in the fiber at infinity $F_\infty$. Since $E_1 \cap E_2 \neq \emptyset$, it follows that $F_\infty = E_1 + E_2$ with $(E_1 \cdot E_2) = 1$. If $E_1$ meets the section $S_0 \cap \overline{Y}_0$, then $(E_1 \cdot S_0) = 1$, whence $(E_2 \cdot S_0) = 1$ because $E_1$ and $E_2$ are algebraically equivalent in $\overline{Y}$. This is a contradiction. Hence $E_1 \cap E_2 = \emptyset$.

(ii) Suppose that only $E_1$ is contained in the fiber at infinity $F_\infty$. Take a smooth fiber $F_0$ of $\varphi_0$ with $F_0 \cap Y_0 \neq \emptyset$ and consider a deformation of $F_0$ in $\overline{Y}$. Then there exist an étale finite morphism $\sigma_2 : T' \to T$ and a decomposition of $\overline{f}_{T'} : \overline{Y} \times_T T' \dashrightarrow V \xrightarrow{\sigma_1} T'$ such that $F_0$ is a fiber of $\varphi$ (see Lemma 2.1 (3)). Let $B$ be an irreducible curve on $V$ such that $\varphi(F_\infty) \notin B$ and let $W = \varphi^{-1}(B)$. Note that $E_1$ and $E_2$ are also algebraically equivalent in $\overline{Y} \times_T T'$. Since $(E_1 \cdot W) = 0$ by the above construction, it follows that $(E_2 \cdot W) = 0$. This implies that $E_2$ is contained in a fiber of $\varphi_0$. Hence $E_1 \cap E_2 = \emptyset$.

(iii) Suppose that $E_1$ and $E_2$ are not contained in the fiber $F_\infty$. Then $E_1$ and $E_2$ are the fiber components of $\varphi_0$ because $(E_i \cdot F_\infty) = 0$ for $i = 1, 2$. If they belong to the same fiber, we obtain a contradiction by the same argument as in the case (i). If they belong to different fibers, then $E_1 \cap E_2 = \emptyset$.

The following example, which is due to one of the referees of this article, shows that Lemma 2.5 (1) does not hold without the monodromy condition on the section of $\varphi_0$.

**Example 2.6.** Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and $T' = \mathbb{A}^1$ which is the affine line minus one point and hence is the underlying scheme of the multiplicative group $G_m$. We denote by $\ell$ (resp. $M$) a general fiber of the first projection $p_1 : Q \to \mathbb{P}^1$ (resp. the second projection $p_2 : Q \to \mathbb{P}^1$). Let

\[^4\text{We note here that without the condition on the absence of the monodromy of the cross-section, the assertion fails to hold. See Example 2.6.}\]
$x$ (resp. $y$) be an inhomogeneous coordinate on the first (resp. the second) factor of $Q$. Set $\ell_\infty = p_1^{-1}(\infty)$ and $M_\infty = p_2^{-1}(\infty)$. We consider an involution $\iota$ on $Q \times T'$ defined by $(x, y, z) \mapsto (y, x, -z)$, where $z$ is a coordinate of $\mathbb{A}^1_x$. Let $Q'$ be the blowing-up of $Q$ with center $P_\infty := \ell_\infty \cap M_\infty$ and let $E$ be the exceptional curve. Then the involution $\iota$ extends to the threefold $Q' \times T'$ in such a way that $E \times T'$ is stable under $\iota$. Let $\overline{Y}$ be the quotient threefold of $Q' \times T'$ by this $\mathbb{Z}_2$-action induced by the involution $\iota$. Since the projection $p_2 : Q' \times T' \rightarrow T'$ is $\mathbb{Z}_2$-equivariant, it induces a morphism $\overline{f} : \overline{Y} \rightarrow T$, where $T = T'/\mathbb{Z}_2 \cong \mathbb{A}^1_x$. Let $S_1 = ((\ell_\infty \cup M_\infty) \times T')/\mathbb{Z}_2$, $S_2 = (E \times T')/\mathbb{Z}_2$ and $S = S_1 + S_2$. Further, let $Y = \overline{Y} \setminus S$ and $f : Y \rightarrow T$ the restriction of $\overline{f}$ onto $Y$. Then the following assertions hold.

1. The surfaces $S_1$ and $S_2$ are smooth irreducible surfaces intersecting normally.
2. Fix a point $t_0 \in T$ and denote the fibers over $t_0$ with the subscript 0. Then the collection $(Y, \overline{Y}, S, \overline{f}, t_0)$ is a family of logarithmic deformations of the triple $(Y_0, \overline{Y}_0, D_0)$, where $D_0 = S \cdot \overline{Y}_0$. 
3. For every $t \in T$, $S_1 \cap \overline{Y}_t$ is a disjoint union of two smooth curves $C_{1t}, C_{2t}$ and $S_2 \cap \overline{Y}_t$ is a smooth rational curve $C_{2t}$, where $(C_{1t} \cdot C_{2t}) = (C_{1t}' \cdot C_{2t}) = 1$ and $(C_{1t}^2) = (C_{1t}'^2) = (C_{2t}^2) = -1$. In particular, $C_{1t}$ is étale-algebraically equivalent to $C_{1t}'$, and hence has a non-trivial monodromy.
4. Each fiber $\overline{Y}_t$ is isomorphic to $Q'$ with $C_{1t}, C_{1t}'$ and $C_{2t}$ identified with the proper transforms of $M_\infty, \ell_\infty$ on $Q'$ and $E$.
5. Let $\varphi_t : \overline{Y}_t \rightarrow \mathbb{P}^1$ be the $\mathbb{P}^1$-fibration induced by the first projection $p_1 : Q \rightarrow \mathbb{P}^1$. Then a general fiber $\ell = p_1^{-1}(x)$ is algebraically equivalent to $M = p_2^{-1}(x)$ for $x \in T$.
6. For every $t \in T$, the affine surface $Y_t$ has an $\mathbb{A}^1$-fibration of affine type.

**Proof.** (1) Since $(Q \setminus (\ell_\infty \cup M_\infty)) = \text{Spec} k[x, y]$, the quotient threefold $V = (Q \times T')/\mathbb{Z}_2$ contains an open set $(\mathbb{A}^2 \times T')/\mathbb{Z}_2$, which has the coordinate ring over $k$ generated by elements $X = x + y$, $U = xy$, $Z = z^2$ and $W = (x - y)z$. Hence the open set is a hypersurface $W^2 = Z(X^2 - 4U)$. The quotient threefold $V$ has a similar open neighborhood of the image of the curve $\{P_\infty\} \times T'$. This can be observed by taking inhomogeneous coordinates $x', y'$ on $Q$ such that $x' = 1/x$ and $y' = 1/y$, where $\ell_\infty \cup M_\infty$ is given by $x'y' = 0$. If we put $W' = x' + y'$, $U' = x'y'$ and $W' = (x' - y')z$, the open neighborhood is defined by a similar equation $W'^2 = Z(X'^2 - 4U')$. Then the image of $(\ell_\infty \cup M_\infty) \times T'$ is given by $U' = 0$. Hence it has an equation $W'^2 = ZX'^2$. So, this is a
smooth irreducible surface. The curve $E$ has inhomogeneous coordinate $x'/y'$ (or $y'/x'$). Hence $E$ is stable under the involution $\iota$. Note that the involution $\iota$ has no fixed point because there are no fixed points on the factor $T'$. The surface $S_1$ is simultaneously contracted along $T$, and by the contraction, $\overline{Y}$ becomes a $\mathbb{P}^2$-bundle and the surface $S_2$ becomes an immersed $\mathbb{P}^1$-bundle. Then the assertion (1) follows easily.

(2) The threefold $\overline{Y}$ is smooth and $\overline{f}$ is a smooth morphism. In fact, every closed fiber of $f = \overline{f}|_Y : Y \to T$ is isomorphic to the affine plane.

(3) If $t = z^2$, $C_{1t}$ (resp. $C'_{1t}$) is identified with $M'_{\infty}$ (resp. $\ell'_{\infty}$) in $Q' \times \{z\}$ and $\ell_{\infty}$ (resp. $M_{\infty}$) in $Q' \times \{-z\}$ under the identification $\overline{Y}_t \cong Q' \times \{z\} \cong Q' \times \{-z\}$, where $\ell_{\infty}$ and $M_{\infty}$ are the proper transforms of $\ell_{\infty}$ and $M_{\infty}$ on $Q'$. Now the rest of the assertions are easily verified.

A sufficient condition on the absence of the monodromy in Lemma 2.5 is given by the following result.

**Lemma 2.7.** Let the notations and the assumptions be the same as in Lemma 2.5 and its proof. Let $S_0 \cap \overline{Y}_0 = C_{01} \cup \cdots \cup C_{0m}$. Suppose that $C_{01}$ is a section of the $\mathbb{P}^1$-fibration $\varphi_0$. If $(C_{0i}^2) \geq 0$, then $C_{01}$ has no monodromy in $\overline{Y}$. Namely, $m = 1$ and $S_0 \cap \overline{Y}_0$ is irreducible.

**Proof.** Suppose that $m > 1$. Note that $C_{02}, \ldots, C_{0m}$ are mutually disjoint and do not meet a general fiber of $\varphi_0$ because they lie outside $Y_0$ and a general fiber meets only $C_{01}$ in the boundary at infinity. This implies that $C_{02}, \ldots, C_{0m}$ are rational curves and the fiber components of $\varphi_0$. By the remark given before Lemma 2.5, we have $(C_{0i}^2) = (C_{01}^2) \geq 0$ for $2 \leq i \leq m$. Then $C_{02}, \ldots, C_{0m}$ are full fibers of $\varphi_0$ and hence they meet the section $C_{01}$. This is a contradiction. □

We prove one of our main theorems.

**Theorem 2.8.** Let $f : Y \to T$ be a morphism from a smooth affine threefold onto a smooth curve $T$ with irreducible general fibers. Assume that general fibers of $f$ have $\mathbb{A}^1$-fibrations of affine type. Then, after shrinking $T$ if necessary and taking an étale finite morphism $T' \to T$, the fiber product $Y' = Y \times_T T'$ has an $\mathbb{A}^1$-fibration which factors the morphism $f' = f \times_T T'$. Furthermore, suppose that there is a relative normal completion $\overline{f} : \overline{Y} \to T$ of $f : Y \to T$ satisfying the following conditions:

1. $(Y, \overline{Y}, S, \overline{f}, t_0)$ with $t_0 \in T$ and $S = \overline{Y} \setminus Y$ is a family of logarithmic deformations of $(Y_0, \overline{Y}_0, D_0)$ as above, where $Y_0 = f^{-1}(t_0), \overline{Y}_0 = \overline{f}^{-1}(t_0)$ and $D_0 = S \cdot \overline{Y}_0$. 


The given $\mathbb{A}^1$-fibration of affine type on each fiber $Y_t$ extends to a $\mathbb{P}^1$-fibration $\varphi_t : \overline{Y}_t \to \overline{B}_t$.

(3) A section of $\varphi_0$ in the fiber $Y_0$ lying in $D_0$ has no monodromy in $\overline{Y}$.

Then the given morphism $f : Y \to T$ is factored by an $\mathbb{A}^1$-fibration.

**Proof.** Embed $Y$ into a smooth threefold $\overline{Y}$ in such a way that $f$ extends to a projective morphism $\overline{f} : \overline{Y} \to T$. We may assume that the complement $S := \overline{Y} \setminus Y$ is a reduced divisor with simple normal crossings. Let $S = S_0 + S_1 + \cdots + S_r$ be the irreducible decomposition of $S$. For a general point $t \in T$, let $Y_t$ be the fiber $f^{-1}(t)$ and let $\rho_t : Y_t \to B_t$ be the given $\mathbb{A}^1$-fibration on $Y_t$. By the assumption, $B_t$ is an affine curve. We may assume that $Y_t$ is smooth and hence $B_t$ is smooth. Let $\overline{Y}_t$ be the closure of $Y_t$ in $\overline{Y}$ which we may assume to be a smooth projective surface with $t$ a general point of $T$. By replacing $T$ by a smaller Zariski open set, we may assume that $\overline{f}$ is a smooth morphism and that $S \cdot \overline{Y}_t$ is a divisor with simple normal crossings for every $t \in T$. Hence we may assume that the condition (1) above is realized.

For each $t \in T$, let $\Lambda_t$ be the pencil generated by the closures (in $\overline{Y}_t$) of the fibers of the $\mathbb{A}^1$-fibration $\rho_t$. If $\Lambda_t$ has a base point, we can eliminate the base points by simultaneous blowing ups on the boundary at infinity after an étale finite base change of $T$. In this step, we may have to replace, for some $t \in T$, the pencil $\Lambda_t$ by another pencil $\Lambda'_t$ which also induces an $\mathbb{A}^1$-fibration of affine type on $Y_t$ (see the proof of Lemma 2.2). So, we may assume that the condition (2) above is also satisfied.

If $S_0 \cap \overline{Y}_0$ contains a section of $\varphi_0$, we may assume by an étale finite base change that $S_0 \cap \overline{Y}_0$ is irreducible (see the remark before Lemma 2.5). So, we may assume that the condition (3) is satisfied as well.

Hence, we may assume from the beginning that three conditions are satisfied. The fibration $\rho_t$ extends to a $\mathbb{P}^1$-fibration $\varphi_t : \overline{Y}_t \to \overline{B}_t$ for every $t \in T$, where $\overline{B}_t$ is a smooth completion of $B_t$. For $t_0 \in T$, we consider the fibration $\varphi_0 : \overline{Y}_0 \to \overline{B}_0$. A general fiber of $\varphi_0$ meets one of the irreducible components, say $S_{0_0}$, of $S$ in one point. Then so does every fiber of $\varphi_0$ because $S_0 \cdot \overline{Y}_0$ is an irreducible divisor on $\overline{Y}_0$ and the fibers of $\varphi_0$ are algebraically equivalent to each other on $\overline{Y}_0$. Hence $S_0 \cdot \overline{Y}_0$ is a section. We claim that

(1) $\overline{Y}_t$ meets the component $S_0$ for every $t \in T$.

(2) After possibly switching the $\mathbb{A}^1$-fibrations if some $Y_t$ has two $\mathbb{A}^1$-fibrations, we may assume that for every $t \in T$, the fibers of the $\mathbb{P}^1$-fibration $\varphi_t$ on $\overline{Y}_t$ meet $S_0$ along a curve $A_t$ such that
is a cross-section of \( \varphi_t \) and hence \( \varphi_t \) induces an isomorphism between \( \overline{A}_t \) and \( \overline{B}_t \).

In fact, for a relatively ample divisor \( H \) of \( \overline{Y} \) over \( T \), we have \( (H \cdot S_0 \cdot \overline{Y}_t) > 0 \), whence \( (H \cdot S_0 \cdot \overline{Y}_t) > 0 \) for every \( t \in T \) because \( \overline{Y}_t \) is algebraically equivalent to \( \overline{Y}_0 \). This implies the assertion (1). To prove the assertion (2), we consider the deformation of a smooth fiber \( C \) of \( \varphi_0 \) in \( \overline{Y}_0 \). Since general fibers \( Y_t \) of \( f \) have \( \mathbb{A}^1 \)-fibrations of affine type, by Lemma 2.11 (3) and Lemma 2.5 (1), there is a \( \mathbb{P}^1 \)-fibration \( \varphi : \overline{Y} \to V \) such that \( C \) is a fiber of \( \varphi \). Then the restriction \( \varphi|_{\overline{Y}_0} \) is the \( \mathbb{P}^1 \)-fibration \( \varphi_0 \). For every \( t \in T \), the restriction \( \varphi|_{\overline{Y}_t} \) is a \( \mathbb{P}^1 \)-fibration on \( \overline{Y}_t \). If it is different from \( \varphi_t \), we replace \( \varphi_t \) by \( \varphi|_{\overline{Y}_t} \). Then \( (S_0 \cdot C') = (S_0 \cdot C) = 1 \) for a general fiber \( C' \) of \( \varphi_t \) because \( C' \) is algebraically equivalent to \( C \). The assertion follows immediately.

With the notations in the proof of Lemma 2.11, the isomorphisms \( \overline{A}_t \xrightarrow{\sim} V_t := \sigma^{-1}(t) \cong \overline{B}_t \) shows that the morphism

\[
S_0 \hookrightarrow \overline{Y} \xrightarrow{\varphi} V \xrightarrow{\sigma} T
\]

induces a birational \( T \)-morphism \( S_0 \to V \) and \( S_0 \) is a cross-section of \( \varphi \). It is clear that the boundary divisor \( S \) contains no other components which are horizontal to \( \varphi \). Hence \( Y \) has an \( \mathbb{A}^1 \)-fibration.

As a consequence of Theorem 2.8, we have the following result.

**Corollary 2.9.** Let \( f : Y \to T \) be a smooth morphism from a smooth affine threefold \( Y \) to a smooth affine curve \( T \). Assume that \( f \) has a relative projective completion \( \overline{f} : \overline{Y} \to T \) which satisfies the same conditions on the boundary divisor \( S \) and the intersection of each fiber \( \overline{Y}_t \) with \( S \) as set in Lemma 2.22. If a fiber \( Y_0 \) has a \( G_a \)-action, then there exists an étale finite morphism \( T' \to T \) such that the threefold \( Y' = Y \times_T T' \) has a \( G_a \)-action as a \( T' \)-scheme. Furthermore, if the relative completion \( \overline{f} : \overline{Y} \to T \) is taken so that the three conditions in Theorem 2.8 are satisfied, the threefold \( Y \) itself has a \( G_a \)-action as a \( T \)-scheme.

**Proof.** By Lemma 2.22 every fiber \( Y_t \) has an \( \mathbb{A}^1 \)-fibration of affine type \( \rho_t : Y_t \to B_t \), where \( B_t \) is an affine curve. As in the proof of Theorem 2.8 we may assume that the three conditions therein are satisfied. By the same theorem, \( Y \) has an \( \mathbb{A}^1 \)-fibration \( \rho : Y \to U \) such that \( f \) is factored as

\[
f : Y \xrightarrow{\rho} U \xrightarrow{\sigma} T \ ,
\]

where \( U_t := \sigma^{-1}(t) \cong B_t \) for every \( t \in T \). Then \( U \) is an affine scheme after restricting \( T \) to a Zariski open set. Then \( Y \) has a \( G_a \)-action by \( \mathbb{A}^1 \).
Given a smooth affine morphism $f : Y \to T$ from a smooth algebraic variety $Y$ to a smooth curve $T$ such that every closed fiber is isomorphic to the affine space $\mathbb{A}^n$ of fixed dimension, one can ask if the generic fiber of $f$ is isomorphic to $\mathbb{A}^n$ over the function field $k(T)$. If this is the case with $f$, we say that the generic triviality holds for $f$. In the case $n = 2$, this holds by the following theorem. If the generic triviality for $n = 2$ holds for $f : Y \to T$ in the setup of Theorem 2.10, a theorem of Sathaye [29] shows that $f$ is an $\mathbb{A}^2$-bundle in the sense of Zariski topology.

**Theorem 2.10.** Let $f : Y \to T$ be a smooth morphism from a smooth affine threefold $Y$ to a smooth affine curve $T$. Assume that the fiber $Y_t$ is isomorphic to $\mathbb{A}^2$ for every closed point of $T$. Then the generic fiber $Y_\eta$ of $f$ is isomorphic to the affine plane over the function field of $T$.

Hence $f : Y \to T$ is an $\mathbb{A}^2$-bundle over $T$ after replacing $T$ by an open set if necessary.

Before giving a proof, we prepare two lemmas where an integral $k$-scheme is a reduced and irreducible algebraic $k$-scheme and where a separable $K$-form of $\mathbb{A}^2$ over a field $K$ is an algebraic variety $X$ defined over $K$ such that $X \otimes_K K'$ is $K'$-isomorphic to $\mathbb{A}^2$ for a separable algebraic extension $K'$ of $K$.

**Lemma 2.11.** Let $p : X \to T$ be a dominant morphism from an integral $k$-scheme $X$ to an integral $k$-scheme $T$. Assume that the fiber $X_t$ is an integral $k$-scheme for every closed point $t$ of $T$. Then the generic fiber $X_\eta = X \times_T \text{Spec } k(T)$ is geometrically integral $k(T)$-scheme.

**Proof.** We have only to show that the extension of the function fields $k(X)/k(T)$ is a regular extension. Namely, $k(X)/k(T)$ is a separable extension, i.e., a separable algebraic extension of a transcendental extension of $k(T)$ and $k(T)$ is algebraically closed in $k(X)$. Since the characteristic of $k$ is zero, it suffices to show that $k(T)$ is algebraically closed in $k(X)$. Suppose the contrary. Let $K$ be the algebraic closure of $k(T)$ in $k(X)$, which is a finite algebraic extension of $k(T)$. Let $T'$ be the normalization of $T$ in $K$. Let $\nu : T' \to T$ be the normalization morphism which is a finite morphism. Then $p : X \to T$ splits as $p : X \xrightarrow{\nu^*} T' \xrightarrow{\nu} T$, which is the Stein factorization. Then the fiber $X_t$ is not irreducible for a general closed point $t \in T$, which is a contradiction to the hypothesis. $\square$

The following result is due to Kambayashi [13].

**Lemma 2.12.** Let $X$ be a separable $K$-form of $\mathbb{A}^2$ for a field $K$. Then $X$ is isomorphic to $\mathbb{A}^2$ over $K$. 

The following proof of Theorem 2.10 uses a locally nilpotent derivation and hence is of purely algebraic nature.

**Proof of Theorem 2.10.** Every closed fiber $Y_t$ has an $\mathbb{A}^1$-fibration of affine type and hence a $G_a$-action. By Corollary 2.9, there exists an étale finite morphism $T' \to T$ such that $Y' = Y \times_T T'$ has a $G_a$-action as a $T'$-scheme. Suppose that the generic fiber $Y_\eta'$ of $f_{T'}: Y' \to T'$ is isomorphic to $\mathbb{A}^2$ over the function field $k(T')$. Since $Y_\eta' = Y_\eta \otimes_{k(T)} k(T')$, it follows by Lemma 2.12 that $Y_\eta$ is isomorphic to $\mathbb{A}^2$ over $k(T)$.

Hence, we may assume from the beginning that $Y$ has a $G_a$-action which induces $\mathbb{A}^1$-fibrations on general closed fibers $Y_t$. The $G_a$-action on a $T$-scheme $Y$ is induced by a locally nilpotent derivation $\delta$ on the coordinate ring $B$ of $Y$, i.e., $Y = \text{Spec } B$. Let $T = \text{Spec } R$. Here $\delta$ is an $R$-trivial derivation on $B$, i.e., $Y = \text{Spec } B$. Let $A$ be the kernel of $\delta$. Since $B$ is a smooth $k$-algebra of dimension 3, $A$ is a finitely generated, normal $k$-algebra of dimension 2. The derivation $\delta$ induces a locally nilpotent derivation $\delta_t$ on $B_t = B \otimes_R R/m_t$, where $m_t$ is the maximal ideal of $R$ corresponding to a general point $t$ of $T$. We assume that $\delta_t \neq 0$.

Since $B_t$ is a polynomial $k$-algebra of dimension 2 by the hypothesis, $A_t := \text{Ker } \delta_t$ is a polynomial ring of dimension 1.

**Claim 1.** $A_t = A \otimes_R R/m_t$ if $\delta_t$ is nonzero.

**Proof.** Let $\varphi : B \to B[u]$ be the $k$-algebra homomorphism defined by

$$\varphi(b) = \sum_{i \geq 0} \frac{1}{i!} \delta^i(b) u^i.$$

Then $\text{Ker } \delta = \text{Ker } (\varphi - \text{id})$. Hence we have an exact sequence of $R$-modules

$$0 \to A \to B \xrightarrow{\varphi - \text{id}} B[u].$$

Let $\mathcal{O}_t$ be the local ring of $T$ at $t$, i.e., the localization of $R$ with respect to $m_t$, and let $\widehat{\mathcal{O}}_t$ be the $m_t$-adic completion of $\mathcal{O}_t$. Since $\widehat{\mathcal{O}}_t$ is a flat $R$-module, we have an exact sequence

$$0 \to A \otimes_R \widehat{\mathcal{O}}_t \to B \otimes_R \widehat{\mathcal{O}}_t \to (B \otimes_R \widehat{\mathcal{O}}_t)[u].$$

The completion $\widehat{\mathcal{O}}_t$ as a $k$-module decomposes as $\widehat{\mathcal{O}}_t = k \oplus \widehat{m}_t$, where $\widehat{m}_t = m_t \widehat{\mathcal{O}}_t$, the above exact sequence splits as a direct sum of exact sequences of $k$-modules

$$0 \to A \otimes_R k \to B \otimes_R k \to (B \otimes_R k)[u]$$

and

$$0 \to A \otimes_R \widehat{m}_t \to B \otimes_R \widehat{m}_t \to (B \otimes_R \widehat{m}_t)[u].$$
The first one is, in fact, equal to
\[ 0 \to A \otimes_R R/\mathfrak{m}_t \to B_t \xrightarrow{\varphi_t - \text{id}} B_t[u] , \]
where \( \varphi_t \) is defined by \( \delta_t \) in the same way as \( \varphi \) by \( \delta \). Hence \( \text{Ker} \delta_t = A \otimes_R R/\mathfrak{m}_t = A_t. \)

Let \( X = \text{Spec} A \) and let \( p : X \to T \) be the morphism induced by the inclusion \( R \hookrightarrow A \). Thus \( f : Y \to T \) splits as
\[ f : Y \xrightarrow{q} X \xrightarrow{p} T , \]
where \( q \) is the quotient morphism by the induced \( G_a \)-action on \( Y \).

**Claim 2.** Suppose that \( \delta_t \neq 0 \) for every \( t \in T \). Then \( X \) is a smooth surface with \( \mathbb{A}^1 \)-bundle structure over \( T \).

**Proof.** Note that \( R \) is a Dedekind domain and \( A \) is an integral domain. Hence \( p \) is a flat morphism. Since \( f \) is surjective, \( p \) is also surjective. Hence \( p \) is a faithfully flat morphism. Further, by Claim 1, \( X_t = \text{Spec} (A \otimes_R R/\mathfrak{m}_t) \) is equal to \( \text{Spec} A_t \) for every \( t \), which is isomorphic to \( \mathbb{A}^1 \). In fact, the kernel of a non-trivial locally nilpotent derivation on a polynomial ring of dimension 2 is a polynomial ring of dimension 1. The generic fiber of \( p \) is geometrically integral by Lemma 2.11. Hence, by [14, Theorem 2], \( X \) is an \( \mathbb{A}^1 \)-bundle over \( T \). In particular, \( X \) is smooth. \( \square \)

Let \( K = k(T) \) be the function field of \( T \). The generic fiber \( X_K = X \times_T \text{Spec} K \) is geometrically integral as shown in the above proof of Claim 2.

**Claim 3.** The generic fiber \( Y_K = Y \times_T \text{Spec} K \) is isomorphic to \( \mathbb{A}^2_K \).

**Proof.** We consider \( q_K : Y_K \to X_K \), where \( X_K \cong \mathbb{A}^1_K \). We prove the following two assertions.

1. For every closed point \( x \) of \( X_K \), the fiber \( Y_K \times_{X_K} \text{Spec} K(x) \) is isomorphic to \( \mathbb{A}^1_{K(x)} \).
2. The generic fiber of \( q_K \) is geometrically integral.

Note that \( K(x) \) is a finite algebraic extension of \( K \). Let \( T' \) be the normalization of \( T \) in \( K' := K(x) \). We consider \( Y' := Y \times_T T' \) instead of \( Y \). Then the \( G_a \)-action on \( Y \) lifts to \( Y' \) and the quotient variety is \( X' = X \times_T T' \). Indeed, the normalization \( R' \) of \( R \) in \( K' \) is the coordinate ring of \( T' \) and is a flat \( R \)-module. Then the sequence of \( R' \)-modules
\[ 0 \to A \otimes_R R' \to B \otimes_R R' \xrightarrow{\varphi' - \text{id}} (B \otimes_R R')[u] \]
is exact, where \( \varphi' = \varphi \otimes_R R' \). Hence \( q_{K'} : Y_{K'} \to X_{K'} \), which is the base change of \( q_K \) with respect to the field extension \( K'/K \), is the quotient morphism by the \( G_a \)-action on \( Y_{K'} \), induced by \( \delta \). Since \( X_{K'} = X \times_T \text{Spec} K' \), there exists a \( K' \)-rational point \( x' \) on \( X_{K'} \) such that \( x \) is the image of \( x' \) by the projection \( X_{K'} \to X_K \). If the fiber of \( q_{K'} \) over \( x' \), i.e., \( Y_{K'} \times_{X_{K'}} (\text{Spec} K', x') \), is isomorphic to \( \mathbb{A}^1_{K'} \), then \( Y_K \times_{X_K} \text{Spec} K' \) is isomorphic to \( \mathbb{A}^1_K \) because \( Y_{K'} \times_{X_{K'}} \text{Spec} K' = Y_K \times_{X_K} \text{Spec} K' \). Thus we may assume that \( x \) is a \( K' \)-rational point. Let \( C \) be the closure of \( x \) in \( X \). Then \( C \) is a cross-section of \( \pi : X \to T \). Let \( Z := Y \times_X C \). Then \( q_C : Z \to C \) is a faithfully flat morphism such that the fiber \( q_C^{-1}(w) \) is isomorphic to \( \mathbb{A}^1 \) for every closed point \( w \in C \). In fact, \( q_C^{-1}(w) \) is the fiber of \( Y_t \to X_t \) over the point \( w \in C \), where \( t = p(w), Y_t \cong \mathbb{A}^1, X_t \cong \mathbb{A}^1 \) and \( X_t = Y_t/G_a \). By Lemma [2.11] (which is extended to a non-closed field \( K \)), the generic fiber of \( q_C \) is geometrically integral, and the generic fiber of \( q_C \), which is \( Y_K \times_{X_K} \text{Spec} K(x) \), is isomorphic to \( \mathbb{A}^1_K \), by [14, Theorem 2]. This proves the first assertion.

The generic point of \( X_K \) corresponds to the quotient field \( L := \text{Q}(A) \). Then it suffices to show that \( B \otimes_A \text{Q}(A) \) is geometrically integral over \( \text{Q}(A) \). Meanwhile, \( B \otimes_A \text{Q}(A) \) has a locally nilpotent derivation \( \delta \otimes_A \text{Q}(A) \) such that \( \text{Ker}(\delta \otimes_A \text{Q}(A)) = \text{Q}(A) \). Hence \( B \otimes_A \text{Q}(A) \) is a polynomial ring \( \text{Q}(A)[u] \) in one variable over \( \text{Q}(A) \) because \( \delta \otimes_A \text{Q}(A) \) has a slice. So, \( B \otimes_A \text{Q}(A) \) is geometrically integral over \( \text{Q}(A) \). Now, by [15, Theorem 1], \( Y_K \) is an \( \mathbb{A}^1 \)-bundle over \( X_K \cong \mathbb{A}^1_K \). Hence \( Y_K \) is isomorphic to \( A^2_K \). We have to replace \( T \) by an open set \( T \setminus F \), where \( F = \{ t \in T \mid \delta_t = 0 \} \). This completes the proof of Theorem [2.10] \( \square \)

We can prove Theorem [2.10] in a more geometric way by making use of a theorem of Ramanujam-Morrow on the boundary divisor of a minimal normal completion of the affine plane [27, 25]. The proof given below is explained in more precise and explicit terms in [16, Lemma 3.2]. In particular, the step to show that \( Y_K \cong \mathbb{P}^2_K \) and \( Y_K \cong \mathbb{A}^2_K \) is due to [loc.cit.].

**The second proof of Theorem 2.10.** Let \( f : Y \to T \) be as in Theorem 2.10. Let \( \mathcal{Y} \) be a relative completion such that \( \mathcal{Y} \) is smooth and \( f \) extends to a smooth projective morphism \( \mathcal{f} : \mathcal{Y} \to T \) with the conditions in Lemma 2.2 being satisfied together with \( S := \mathcal{Y} \setminus Y \). To obtain this setting, we may have to shrink \( T \) to a smaller open set of \( T \). As in the first proof and the proof of Lemma 2.2, we can apply an étale finite base change \( T' \to T \) by which the intersection \( S_t \cap \mathcal{Y}_t \) is irreducible for every irreducible component \( S_t \) of \( S \) and every \( t \in T \). In particular, we assume that \( \mathcal{Y}_t \) is a smooth normal completion of \( Y_t \) for
every \( t \in T \), where \( Y_t \) is isomorphic to \( \mathbb{A}^2 \). Fix one such completed fiber, say \( \overline{Y}_0 = f^{-1}(t_0) \), and consider the reduced effective divisor \( \overline{Y}_0 \setminus Y_0 \) with \( Y_0 = f^{-1}(t_0) \cong \mathbb{A}^2 \). Namely, \((Y, \overline{Y}, S, \overline{f}, t_0)\) is a log deformation of \((Y_0, \overline{Y}_0, D_0)\). If the dual graph of this divisor is not linear then it contains a \((-1)\)-curve meeting at most two other components of \( D_0 \) by a result of Ramanujam [27]. By (4) of Lemma 2.1, such a \((-1)\)-curve deforms along the fibers of \( \overline{f} \) and we get an irreducible component, say \( S_1 \), of \( S = \sum_{i=0}^r S_i \) which can be contracted. Repeating this argument, we can assume that all the dual graphs for \( \overline{Y}_t \setminus Y_t \), as \( t \) varies on the set of closed points of \( T \), are linear chains of smooth rational curves. By [25], at least one of these curves is a \((0)\)-curve. Fix such a \((0)\)-curve \( C_1 \) in \( \overline{Y}_0 \setminus Y_0 \). Then \( C_1 \) deforms along the fibers of \( f \) and forms an irreducible component, say \( S_1 \), of \( S \) by abuse of the notations. By the argument in the proof of Lemma 2.2, if \( C_2 \) is a component of \( \overline{Y}_0 \setminus Y_0 \) meeting \( C_1 \), it deforms along the fibers of \( f \) on an irreducible component, say \( S_2 \), of \( S \). Repeating this argument, we know that all irreducible components of \( \overline{Y}_0 \setminus Y_0 \) extend along the fibers of \( \overline{f} \) to form the irreducible components of \( S \) and that the dual graphs of \( \overline{Y}_t \setminus Y_t \) are the same for every \( t \in T \). Now let \( K \) be the function field of \( T \) over \( k \). We consider the generic fibers \( \overline{Y}_K \) and \( Y_K \) of \( \overline{f} \) and \( f \). Then the dual graph of \( \overline{Y}_K \setminus Y_K \) is the same linear chain of smooth rational curves as the closed fibers \( \overline{Y}_t \setminus Y_t \). Write \( \overline{Y}_0 \setminus Y_0 = \sum_{i=1}^r C_i \). If \( C_i \) and \( C_j \) meet for \( i \neq j \), then the intersection point \( C_i \cap C_j \) moves on the intersection curve \( S_i \cdot S_j \). Since any minimal normal completion of \( \mathbb{A}^2 \) can be brought to \( \mathbb{P}^2 \) by blowing ups and downs with centers on the boundary divisor, we can blow up simultaneously the intersection curves and blow down the proper transforms of the \( S_i \) according to the blowing ups and downs on \( \overline{Y}_0 \). Here we note that the begining center of blowing up is a point on a \((0)\)-curve \( C_1 \). In this case, we choose a suitable cross-section on the irreducible component \( S_1 \) which is a \( \mathbb{P}^1 \)-bundle in the Zariski topology because \( \dim T = 1 \). Note that if \( T \) is irrational, then the chosen cross-section may meet the intersection curves on \( S_1 \) with other components of \( S \). Then we shrink \( T \) so that the cross-section does not meet the intersection curves. If \( T \) is rational, \( S_1 \) is a trivial \( \mathbb{P}^1 \)-bundle, hence we do not need the procedure of shrinking \( T \). Thus we may assume that, for every \( t \in T \), \( \overline{Y}_t \) is isomorphic to \( \mathbb{P}^2 \) and \( \overline{Y}_t \setminus Y_t \) is a single curve \( C_t \) with \((C_t)^2 = 1 \). This implies that \( \overline{Y}_K \cong \mathbb{P}^2_K \) and \( Y_K \cong \mathbb{A}^2_K \). \( \square \)

In connection with Theorem 2.10, we can pose the following

**Problem 2.13.** Let \( K \) be a field of characteristic zero and let \( X \) be a smooth affine surface defined over \( K \). Suppose that \( X \otimes_K \overline{K} \) has an
A\(^1\)-fibration of affine type, where \(\overline{K}\) is an algebraic closure of \(K\). Does \(X\) then have an \(A\(^1\)\)-fibration of affine type?

If we consider an \(A\(^1\)\)-fibration of complete type, an example of Dubouloz-Kishimoto gives a counter-example to a similar problem for the complete type (see Theorem 5.1). In view of Example 2.6 and Theorem 2.8, we need perhaps some condition for a positive answer in the case of affine type which guarantees the absence of monodromy of a cross-section of a given \(A\(^1\)\)-fibration.

3. Topological arguments instead of Hilbert schemes

In this section we will briefly indicate topological proofs of some of the results in the section two. The use of topological arguments would make the cumbersome geometric arguments more transparent for the people who do not appreciate the heavy machinery like Hilbert scheme.

We will use the following basic theorem due to Ehresmann [30, Chapter V, Prop. 6.4].

**Theorem 3.1.** Let \(M\) be a connected differentiable manifold, \(S\) a closed submanifold, \(f : M \to N\) a proper differentiable map such that the tangent maps corresponding to \(f\) and \(f|_S : S \to N\) are surjective at any point in \(M\) and \(S\). Then \(f|_{M \setminus S} : M \setminus S \to N\) is a locally trivial fiber bundle with respect to the base \(N\).

Note that the normal bundle of any fiber of \(f\) is trivial. We can give a proof of Ehresmann’s theorem using this observation, and the well-known result from differential topology that given a compact submanifold \(S\) of a \(C^\infty\) manifold \(X\) there are arbitrarily small tubular neighborhoods of \(S\) in \(X\) which are diffeomorphic to neighborhoods of \(S\) in the total space of normal bundle of \(S\) in \(X\) [1, Chapter II, Theorem 11.14].

Now let \(\overline{f} : \overline{Y} \to T\) be a smooth projective morphism from a smooth algebraic threefold onto a smooth algebraic curve \(T\). Let \(\overline{Y}_t = \overline{f}^{-1}(t)\) be the fiber over \(t \in T\). Let \(S\) be a simple normal crossing divisor on \(\overline{Y}\) such that \(D_t := S \cap \overline{Y}_t\) is a simple normal crossing divisor for each \(t \in T\) and \(Y_t := \overline{Y}_t \setminus D_t\) is affine for each \(t\).

We can assume that \(\overline{f} : \overline{Y} \to T\) has the property that the tangent map is surjective at each point. It follows from Ehresmann’s theorem that all the surfaces \(\overline{Y}_t\) are mutually diffeomorphic. In particular, they have the same topological invariants like the fundamental group \(\pi_1\) and the Betti number \(b_i\). By shrinking \(T\) if necessary, we will assume that the restricted map \(\overline{f} : S_t \to T\) is smooth for each \(i\). For fixed \(i\) and
the intersection $S_t \cap \overline{Y}_t$ is a disjoint union of smooth, compact, irreducible curves. Let $C_{t_0, i}$ be one of these irreducible curves. Then for each $t$ which is close to $t_0$, there is an irreducible curve $C_{t, i}$ in $S_t \cap \overline{Y}_t$ and suitable tubular neighborhoods of $C_{t_0, i}$, $C_{t, i}$ in $\overline{Y}_{t_0}$, $\overline{Y}_t$ respectively are diffeomorphic by Ehresmann’s theorem. This implies that $C_{t_0, i}$ in $\overline{Y}_{t_0}$ and $C_{t, i}$ in $\overline{Y}_t$ are equal. This proves that the weighted dual graphs of the curves $D_t$ in $\overline{Y}_t$ are the same for each $t$.

Recall that if $X$ is a smooth projective surface with a smooth rational curve $C \subset X$ such that $C^2 = 0$ then $C$ is a fiber of a $\mathbb{P}^1$-fibration on $X$. If the irregularity $q(X) > 0$ then the Albanese morphism $X \to \text{Alb}(X)$ gives a $\mathbb{P}^1$-fibration on $X$ with $C$ as a fiber. By the above discussion the fiber surfaces $\overline{Y}_t$ have the same irregularity.

Suppose that $\overline{Y}_0$ has an $A^1$-fibration of affine type $f : Y_0 \to B$. If $\overline{f} : \overline{Y}_0 \to \overline{B}$ is an extension of $f$ to a smooth completion of $Y_0$ then, after simultaneous blowing ups and downs along the fibers of $\overline{f}$, we may assume that $D_0 := \overline{Y}_0 \setminus Y_0$ contains at least one $(0)$-curve which is a tip, i.e., the end component of a maximal twig of $D_0$. Since $D_t$ and $D_0$ have the same weighted dual graphs $D_t$ also contains a $(0)$-curve which is a tip of $D_t$. Hence, $Y_t$ also has an $A^1$-fibration of affine type. This proves the assertion (2) in Lemma 2.2.

We can also shorten the part of showing the invariance of the boundary weighted graphs in the second proof of Theorem 2.10. Suppose now that $f : Y \to T$ is a fibration on a smooth affine threefold $Y$ onto a smooth curve $T$ such that every scheme-theoretic fiber of $f$ is isomorphic to $\mathbb{A}^2$. We can embed $Y$ in a smooth projective threefold $\overline{Y}$ such that $f$ extends to a morphism $\overline{f} : \overline{Y} \to T$. By shrinking $T$ we can assume that $\overline{f}$ is smooth, each irreducible component $S_t$ of $\overline{Y}_t \setminus Y$ intersects each $\overline{Y}_t \setminus Y_t$ transversally, etc. By the above discussions, each $D_t := \overline{Y}_t \setminus Y_t$ has the same weighted dual graph. Since $Y_t$ is isomorphic to $\mathbb{A}^2$, we can argue as in the second proof of Theorem 2.10 using the result of Ramanujam-Morrow to conclude that $f$ is a trivial $\mathbb{A}^2$-bundle on a non-empty Zariski-open subset of $T$. This observation applies also to the proof of Theorem 4.6.

4. Deformations of $\text{ML}_0$ surfaces

For $i = 0, 1, 2$, an $\text{ML}_i$ surface is by definition a smooth affine surface $X$ such that the Makar-Limanov invariant $\text{ML}(X)$ has transcendence degree $i$ over $k$ [10]. In this section, we assume that the ground field $k$
is the complex field $\mathbb{C}$. Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a family satisfying the conditions of Lemma 2.2. Let $D_0 = S \cap \overline{Y}_0$.

**Lemma 4.1.** Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of $(Y_0, \overline{Y}_0, D_0)$. Assume that $D_0$ is a tree of smooth rational curves satisfying one of the following conditions.

(i) $D_0$ contains an irreducible component $C_1$ such that $(C_1^2) \geq 0$.
(ii) $D_0$ contains a $(-1)$ curve which meets more than two other components of $D_0$.

Then the following assertions hold after changing $T$ by an étale finite covering of an open set of $T$ if necessary.

1. Every irreducible component of $D_0$ deforms along the fibers of $\overline{f}$. Namely, if $D_0 = \sum_{i=1}^{r} S_i$ is the irreducible decomposition, then, for every $1 \leq i \leq r$, there exists an irreducible component $S_i$ of $S$ such that $\overline{f}|_{S_i} : S_i \to T$ has the fiber $(\overline{f}|_{S_i})^{-1}(t_0) = C_i$. Furthermore, $S = \sum_{i=1}^{r} S_i$.

2. For $t \in T$, let $C_{i,t} = (\overline{f}|_{S_i})^{-1}(t)$. Then $D_t = \sum_{i=1}^{r} C_{i,t}$ and $D_t$ has the same weighted graph on $\overline{Y}_t$ as $D_0$ does on $\overline{Y}_0$.

3. For every $i$, $\overline{f}|_{S_i} : S_i \to T$ is a trivial $\mathbb{P}^1$-bundle over $T$.

**Proof.** By a suitable étale finite base change of $T$, we may assume that $S_i \cap \overline{Y}_0$ is irreducible for every irreducible component $S_i$ of $S$. Then the argument is analytic locally almost the same as in the proof for the assertion (2) of Lemma 2.2. Consider the deformation of $C_1$ along the fibers of $\overline{f}$, which moves along the fibers because $(C_1^2) \geq -1$. Then the components of $D_0$ which are adjacent to $C_1$ also move along the fibers of $\overline{f}$. Once these components of $D_0$ move, then the components adjacent to these components move along the fibers of $\overline{f}$. Since $D_0$ is connected because $Y_0$ is affine, all the components of $D_0$ move along the fibers of $\overline{f}$. If $S$ contains an irreducible component which does not intersect $\overline{Y}_0$, it is a fiber component of $\overline{f}$. Then we remove the fiber by shrinking $T$. This proves the assertion (1).

Let $S = \sum_{i=1}^{r} S_i$ be the irreducible decomposition of $S$. As shown in (1), $S_i \cap \overline{Y}_0 \neq \emptyset$ for every $i$. Then $S_i \cap \overline{Y}_t \neq \emptyset$ as well by the argument in the proof of Lemma 2.2.

Note that $((S_i \cdot \overline{Y}_t)^2)_{\overline{Y}_t} = (S_i^2 \cdot \overline{Y}_t) = (S_i^2 \cdot \overline{Y}_0) = (((S_i \cdot \overline{Y}_0)^2)_{\overline{Y}_0}$ because $\overline{Y}_t$ is algebraically equivalent to $\overline{Y}_0$. Hence $D_0$ and $D_t$ have the same dual graphs.

In order to prove the following result, we use Ehresmann’s theorem, which is Theorem 3.1.
Lemma 4.2. Let \( \mathcal{F} = (Y, Υ, S, f, t_0) \) be a log deformation of \((Y_0, Υ_0, D_0)\) which satisfies the same conditions as in Lemma 4.1. Assume further that \( p_g(Υ_0) = q(Υ_0) = 0 \). Then the following assertions hold.

1. \( \text{Pic}(Y_t) \cong \text{Pic}(Y_0) \) for every \( t \in T \).
2. \( \Gamma(Y_t, \mathcal{O}_{Y_t}) \cong \Gamma(Y_0, \mathcal{O}_{Y_0}) \) for every \( t \in T \).

**Proof.** Since \( p_g \) and \( q \) are deformation invariants, we have \( p_g(Υ_t) = q(Υ_t) = 0 \) for every \( t \in T \). The exact sequence

\[
0 \to \mathbb{Z} \to \mathcal{O}_{Υ_t} \to \mathcal{O}_{Υ_t}^\times \to 0
\]

induces an exact sequence

\[
H^1(Y_t, \mathcal{O}_{Υ_t}) \to H^1(Υ_t, \mathcal{O}_{Υ_t}^\times) \to H^2(Υ_t; \mathbb{Z}) \to H^2(Y_t, \mathcal{O}_{Υ_t})
\]

Since \( p_g(Υ_t) = q(Υ_t) = 0 \), we have an isomorphism

\[
H^1(Y_t, \mathcal{O}_{Υ_t}^\times) \cong H^2(Υ_t; \mathbb{Z}).
\]

Now consider the canonical homomorphism \( θ_t : H_2(D_t; \mathbb{Z}) \to H_2(Υ_t; \mathbb{Z}) \), where \( H_2(Υ_t; \mathbb{Z}) \cong H_2(Υ_t; \mathbb{Z}) = \text{Pic}(Υ_t) \) by the Poincaré duality. Then \( \text{Coim } θ_t = \text{Pic}(Y_t) \) and \( \text{Ker } θ_t = \Gamma(Y_t, \mathcal{O}_{Y_t}^\times)/k^* \).

Let \( N \) be a nice tubular neighborhood of \( S \) with boundary in \( Υ \). The smooth morphism \( f : Υ \to T \) together with its restriction on the \((N, ∂N)\) gives a proper differential mapping which is surjective and submersive. By Theorem 3.1, it is differentiably a locally trivial fibration. Namely, there exists a small disc \( U \) of \( t_0 \) in \( T \) and a diffeomorphism \( ϕ_0 : Υ_0 \times U \overset{≅}{\to} (f)^{-1}(U) \) such that its restriction induces a diffeomorphism

\[
ϕ_0 : (N \cap Υ_0) \times U \overset{≅}{\to} (f|_{N})^{-1}(U).
\]

For \( t \in U \), noting that \( U \) is contractible and hence \( H_2(Υ_0 \times U; \mathbb{Z}) = H_2(Υ_0; \mathbb{Z}) \) and \( H_2((N \cap Υ_0) \times U; \mathbb{Z}) = H_2(N \cap Υ_0; \mathbb{Z}) \), the inclusions \( Υ_t \hookrightarrow (f)^{-1}(U) \) and \( N \cap Υ_0 \hookrightarrow (f|_{N})^{-1}(U) \) induces compatible isomorphisms

\[
p_t : H_2(Υ_t; \mathbb{Z}) \to H_2((f)^{-1}(U); \mathbb{Z}) \overset{(ϕ^{-1})^*}{\to} H_2(Υ_0 \times U; \mathbb{Z}) = H_2(Υ_0; \mathbb{Z})
\]

and its restriction \( q_t : H_2(N \cap Υ_t; \mathbb{Z}) \overset{≅}{\to} H_2(N \cap Υ_0; \mathbb{Z}) \). Since \( S \) and hence \( D_t \) are strong deformation retracts of \( N \) and \( N \cap Υ_t \) respectively, the isomorphism \( q_t \) induces an isomorphism \( r_t : H_2(D_t; \mathbb{Z}) \overset{≅}{\to} H_2(N \cap Υ_t; \mathbb{Z}) \).
$H_2(D_0; \mathbb{Z})$ such that the following diagram

$$
\begin{array}{ccc}
H_2(D_t; \mathbb{Z}) & \xrightarrow{\theta_t} & H_2(Y_t; \mathbb{Z}) \\
\downarrow \rho_t & & \downarrow \rho_t \\
H_2(D_0; \mathbb{Z}) & \xrightarrow{\theta_0} & H_2(Y_0; \mathbb{Z})
\end{array}
$$

This implies that $\text{Pic}(Y_t) \cong \text{Pic}(Y_0)$ and $\Gamma(Y_t, O_{Y_t}^*) \cong \Gamma(Y_0, O_{Y_0}^*)$. If $t$ is an arbitrary point of $T$, we choose a finite sequence of points $\{t_0, t_1, \ldots, t_n = t\}$ such that $t_i$ is in a small disc $U_{i-1}$ around $t_{i-1}$ ($1 \leq i \leq n$) for which we can apply the above argument. \[\blacksquare\]

**Remark 4.3.** By a result of W. Neumann [26, Theorem 5.1], if $X$ is a normal affine surface, $D$ an SNC divisor at infinity of $X$ which does not contain any $(-1)$-curve meeting at least three other components of $D$ and all whose maximal twigs are smooth rational curves with self-intersections $\leq -2$, then the boundary 3-manifold of a nice tubular neighborhood $N$ of $D$ determines the dual graph of $D$. If we use the local differentiable triviality of a tubular neighborhood $N$, this result of Neumann shows that the weighted dual graph of $D_t$ is deformation invariant.

According to [10, Lemmas 1.2, 1.4], we have the following property and characterization of $\text{ML}_0$-surface.

**Lemma 4.4.** Let $X$ be a smooth affine surface and let $V$ be a minimal normal completion of $X$. Then the following assertions hold.

1. $X$ is an $\text{ML}_0$-surface if and only if $\Gamma(X, O_X^*) = k^*$ and the dual graph of the boundary divisor $D := V - X$ is a linear chain of smooth rational curves.
2. If $X$ is an $\text{ML}_0$-surface, $X$ has an $\mathbb{A}^1$-fibration, and any $\mathbb{A}^1$-fibration $\rho : X \to B$ has base curve either $B \cong \mathbb{P}^1$ or $B \cong \mathbb{A}^1$. If $B \cong \mathbb{P}^1$, $\rho$ has at most two multiple fibers, and if $B \cong \mathbb{A}^1$, it has at most one multiple fiber.

The following result is a direct consequence of the above lemmas.

**Theorem 4.5.** Let $\mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0)$ be a log deformation of $(Y_0, \overline{Y}_0, D_0)$, where $Y_0$ is an $\text{ML}_0$-surface. Then $Y_t$ is an $\text{ML}_0$-surface for every $t \in T$.

**Proof.** If $S \cap \overline{Y}_t$ contains a $(-1)$ curve, then it deforms along the fibers of $\overline{f}$ after an étale finite base change of $T$, and these $(-1)$ curves are contracted simultaneously by Lemma 2.1. Hence we may assume that $\overline{Y}_t$ is a minimal normal completion of $Y_t$ for every $t \in T$. By Lemma 4.4, $D_0 := S \cap \overline{Y}_0$ is a linear chain of smooth rational curves.
Hence \( D_t := S \cap \overline{Y}_t \) is also a linear chain of smooth rational curves. By Lemma 4.2, \( \Gamma(Y_t, \mathcal{O}^*_t) = k^* \) for every \( t \in T \) because \( \Gamma(Y_0, \mathcal{O}^*_0) = k^* \). So, \( Y_t \) is an \( ML_0 \)-surface by Lemma 4.4. 

A smooth affine surface \( X \) is, by definition, an affine pseudo-plane if it has an \( \mathbb{A}^1 \)-fibration of affine type \( p : X \rightarrow \mathbb{A}^1 \) admitting at most one multiple fiber of the form \( m\mathbb{A}^1 \) as a singular fiber (see [24] for the definition and relevant results). An affine pseudo-plane is a \( \mathbb{Q} \)-homology plane, its Picard group is a cyclic group \( \mathbb{Z}/m\mathbb{Z} \) and there are no non-constant invertible elements. An \( ML_0 \)-surface is an affine pseudo-plane if the Picard number is zero.

If \( \overline{X} \) is a minimal normal completion of an affine pseudo-plane \( X \), the boundary divisor \( D = \overline{X} - X \) is a tree of smooth rational curves, which is not necessarily a linear chain. By blowing-ups and blowing-downs with centers on the boundary divisor \( D \), we can make the completion \( \overline{X} \) satisfy the following conditions [24, Lemma 1.7].

(i) There is a \( \mathbb{P}^1 \)-fibration \( \overline{p} : \overline{X} \rightarrow \mathbb{P}^1 \) which extends the \( \mathbb{A}^1 \)-fibration \( p : X \rightarrow \mathbb{A}^1 \).

(ii) The weighted dual graph of \( D \) is

\[
\begin{array}{c}
(0) & \quad \ell & \quad M \\
& & \quad (0) & \quad A
\end{array}
\]

(iii) There is a \( (-1) \) curve \( F_0 \) (called feather) such that \( F_0 \cap X \cong \mathbb{A}^1 \) and the union \( F_0 \cup A \) is contractible to a smooth rational curve meeting the image of the component \( M \).

Note that \( X \) is an \( ML_0 \)-surface if and only if \( A \) is a linear chain. We then call \( X \) an affine pseudo-plane of \( ML_0 \)-type.

If we are given a log deformation \( (Y, \overline{Y}, S, \overline{f}, t_0) \) of the triple \( (Y_0, D_0, Y_0) \), it follows by Ehresmann’s fibration theorem that \( p_q \) and the irregularity \( q \) of the fiber \( \overline{Y}_t \) is independent of \( t \). Furthermore, by Lemma 2.2 \( Y_t \) has an \( \mathbb{A}^1 \)-fibration if \( Y_0 \) has an \( \mathbb{A}^1 \)-fibration. So, we can expect that \( Y_t \) is an affine pseudo-plane if so is \( Y_0 \). Indeed, we have the following result.

**Theorem 4.6.** Let \( \mathcal{F} = (Y, \overline{Y}, S, \overline{f}, t_0) \) be a log deformation of \( (\overline{Y}_0, D_0, Y_0) \). Assume that \( Y_0 \) is an affine pseudo-plane. Then the following assertions hold.

1. \( Y_t \) is an affine pseudo-plane for every point \( t \in T \).
2. Assume that \( Y_0 \) is an affine pseudo-plane of \( ML_0 \)-type. Assume further that the boundary divisor \( D_0 \) in \( \overline{Y}_0 \) has the same
weighted dual graph as above. Then \( f : Y \to T \) is a trivial bundle with fiber \( Y_0 \) after shrinking \( T \) if necessary.

**Proof.** (1) We have only to show that \( Y_t \) is an affine pseudo-plane for a small deformation of \( Y_0 \). After replacing \( T \) by an étale finite covering, we may assume that \( \bar{Y}_t \) is a minimal normal completion of \( Y_t \) for every \( t \in T \). Then, by Lemma 4.1, the boundary divisor \( D_t = S \cap \bar{Y}_t \) has the same weighted dual graph as shown above for \( D_0 \). Hence \( Y_t \) has an \( \mathbb{A}^1 \)-fibration of affine type. By Lemma 4.2, \( \text{Pic} (Y_t) \cong \text{Pic} (Y_0) \) which is a finite cyclic group. This implies that \( Y_t \) is an affine pseudo-plane.

(2) Consider the completion \( \bar{Y}_0 \) of \( Y_0 \). We may assume that \( \bar{Y}_0 \) is a minimal normal completion of \( Y_0 \). In fact, a \((-1)\)-curve contained in the boundary divisor \( D_0 \) which meets at most two other components of \( D_0 \) deforms to the nearby fibers and contracted simultaneously over the same \( T \) by Lemma 2.5, (2). Note that every fiber \( Y_t \) has an \( \mathbb{A}^1 \)-fibration of affine type by Lemma 2.2. As in the proof of Lemma 2.2, by performing simultaneous (i.e., along the fibers of \( f \)) blowings-ups and blowings-downs on the boundary \( S \), we may assume that \( \bar{Y}_0 \) has the weighted dual graph \( \ell \to M \to A \) as specified in the condition (ii) above, where \( A \) is a linear chain by the hypothesis. To perform a simultaneous blowing-up, we may have to choose as the center a cross-section on an irreducible component \( S_i \) which is a \( \mathbb{P}^1 \)-bundle over \( T \). If such a cross-section happens to intersect the curve \( S_i \cap S_j \) with another component \( S_j \), we shrink \( T \) to avoid this intersection (see the remark in the second proof of Theorem 2.10). Note that the interior \( Y \) (more precisely, the inverse image of \( f \) of the shrunken \( T \)) is not affected under these operations. Then the \((0)\) curve \( \ell \) defines a \( \mathbb{P}^1 \)-fibration \( \varphi : \bar{Y} \to V \) (see Lemma 2.5, (1)). In particular, \( \ell \) moves in an irreducible component, say \( S_{-1} \), of \( S \). The \((0)\) curve \( M \) moves along the fibers of \( \bar{f} \) in an irreducible component, say \( S_0 \), of \( S \). By Lemma 4.1, the curves in \( A \) move along the fibers of \( \bar{f} \) and fill out the irreducible components \( S_1, \ldots, S_r \) of \( S \). Hence \( S = S_{-1} \cup S_0 \cup S_1 \cup \cdots \cup S_r \) and \( D_t = S \cdot \bar{Y}_t \) has the same weighted dual graph as \( D_0 \).

Now consider a \((-1)\) curve \( F_0 \) on \( \bar{Y}_0 \). By Lemma 2.1, \( F_0 \) moves along the fibers of \( \bar{f} \) and fills out a smooth irreducible divisor \( F \) which meets transversally an irreducible component \( S_i \) \((1 \leq i \leq r) \). In fact, the feather \( F_0 \) is unique on \( Y_0 \) and \( (S_i \cdot F \cdot \bar{Y}_t) = (S_i \cdot F \cdot \bar{Y}_0) = 1 \). Let \( S_1 \) be the component of \( S \) meeting \( S_0 \). Let \( F_t = F \cap \bar{Y}_t \) and \( S_{j,t} = S_j \cap \bar{Y}_t \) for every \( t \in T \). Then \( F_t + \sum_{j=2}^r S_{j,t} \) is contractible to a smooth point \( P_t \) lying on \( S_{1,t} \). After performing simultaneous
elementary transformations on the fiber $\ell$ which is the fiber at infinity of the $\mathbb{A}^1$-fibration of the affine pseudo-plane $Y_t$, we may assume that $P_t$ is the intersection point $S_0 \cap S_1$. By applying Lemma 2.5 (2) repeatedly, we can contract $F$ and the components $S_2, \ldots, S_r$ simultaneously. Let $Z$ be the threefold obtained from $Y$ by these contractions. Then $Z$ has a $\mathbb{P}^1$-fibration $\psi: Z \to V$ and the image of $S_0$ is a cross-section. Let $g = \sigma \cdot \psi: Z \to V$ (see Lemma 2.1 (3) for the notations). For every $t \in T$, $Z_t := g^{-1}(t)$ is a minimal $\mathbb{P}^1$-bundle with a cross-section $S_0$. Since $(S_{0,t})^2 = 0$, $Z_t$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then $Z$ is a trivial $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $T$ after shrinking $T$ if necessary. In fact, $Z$ with the images of $S_0$ and $S_1$ removed is a deformation of $\mathbb{A}^2$, which is locally trivial in the Zariski topology by Theorem 2.10. We may assume that $\psi: Z \to V$ is the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times T$ onto the second and the third factors. Choose a section $S'_0$ of $\psi$ which is disjoint from the image $\overline{S}_0$ of $S_0$. Then there is a non-trivial $\mathbb{G}_m$-action on $Z$ along the fibers of $\psi$ which has $S_0$ and $S'_0$ as the fixed point locus.

Now reverse the contractions $Y \to Z$. The center of the first simultaneous blowing-up with center $S_0 \cap S_1$ and the centers of the consecutive simultaneous blowing-ups except for the blowing-up which produces the component $F$ are $\mathbb{G}_m$-fixed because the blowing-ups are fiberwise sub-divisional. Only the center $Q_t$ of the last blowing-up on $Y_t$ is non-subdivisional. Let

$$\varphi: Y \xrightarrow{\sigma} Y_1 \xrightarrow{\sigma_1} Z$$

be the factorization of $\varphi$ where $\sigma$ is the last non-subdivisional blowing-up. By the construction, the natural $T$-morphism $\overline{f}_1 : Y_1 \to T$ is a trivial fibration with fiber $(Y_1)_0 = \overline{f}_1^{-1}(t_0)$. Then there exists an element $\{\rho_t\}_{t \in T}$ of $G_m(T)$ such that $\rho_t(Q_{t_0}) = Q_t$ for every $t \in T$ after shrinking $T$ if necessary. Here note that the $G_m$-action is non-trivial on the component with the point $Q_t$ thereon, for otherwise the $G_m$-action is trivial from the beginning. Then these $\{\rho_t\}_{t \in T}$ extend to a $T$-isomorphism $\tilde{\rho}: \overline{Y}_0 \times T \to Y$, which induces a $T$-isomorphism $Y_0 \times T \to Y$. Hence $Y$ is trivial.

\section{5. Deformations of $\mathbb{A}^1$-fibrations of complete type}

In the setting of Theorem 2.8, if the $\mathbb{A}^1$-fibration of a general fiber $Y_t$ is of complete type, we do not have the same conclusion. This case is treated in a recent work of Dubouloz and Kishimoto [3]. We consider this case by taking the same example of cubic surfaces in $\mathbb{P}^3$ and explain how it is affine-uniruled.
Taking a cubic hypersurface as an example, we first observe the behavior of the log Kodaira dimension for a flat family of smooth affine surfaces. Let $\mathbb{P}^3$ be the dual projective 3-space whose points correspond to the hyperplanes of $\mathbb{P}^3$. We denote it by $T$. Let $S$ be a smooth cubic hypersurface in $\mathbb{P}^3$ and let $\mathcal{W} = S \times T$ which is a codimension one subvariety of $\mathbb{P}^3 \times T$. Let $\mathcal{H}$ be the universal hyperplane in $\mathbb{P}^3 \times T$, which is defined by $\xi_0 X_0 + \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3 = 0$, where $(X_0, X_1, X_2, X_3)$ and $(\xi_0, \xi_1, \xi_2, \xi_3)$ are respectively the homogeneous coordinates of $\mathbb{P}^3$ and $T$. Let $\mathcal{D}$ be the intersection of $\mathcal{W}$ and $\mathcal{H}$ in $\mathbb{P}^3 \times T$. Let $\pi : \mathcal{W} \to T$ be the projection and let $\pi_D : \mathcal{D} \to T$ be the restriction of $\pi$ onto $\mathcal{D}$. Then $\pi$ and $\pi_D$ are the flat morphism. For a closed point $t \in T$, $\mathcal{W}_t = \pi^{-1}(t)$ is identified with $S$ and $\mathcal{D}_t = \pi_D^{-1}(t)$ is the hyperplane section $S \cap \mathcal{H}_t$ in $\mathbb{P}^3$, where $\mathcal{H}_t$ is the hyperplane $\tau_0 X_0 + \tau_1 X_1 + \tau_2 X_2 + \tau_3 X_3 = 0$ with $t = (\tau_0, \tau_1, \tau_2, \tau_3)$. Let $\mathcal{X} = \mathcal{W} \setminus \mathcal{D}$ and $p : \mathcal{X} \to T$ be the restriction of $\pi$ onto $\mathcal{X}$. Then $\mathcal{X}_t = p^{-1}(t)$ is an affine surface $S \setminus (S \cap \mathcal{H}_t)$.

Since $S$ is smooth, the following types of $S \cap \mathcal{H}_t$ are possible. In the following, $F = 0$ denotes the defining equation of $S$ and $H = 0$ does the equation for $\mathcal{H}_t$.

1. A smooth irreducible plane curve of degree 3.
2. An irreducible nodal curve, e.g., $F = X_0(X_1^2 - X_2^2) - X_2^3 + X_0^2 X_3 + X_3^3$ and $H = X_3$.
3. An irreducible cuspidal curve, e.g., $F = X_0 X_1^2 - X_2^3 + X_3(X_0^2 + X_1^2 + X_2^2 + X_3^2)$ and $H = X_3$.
4. An irreducible conic and a line which meets in two points transversally or in one point with multiplicity two. In fact, let $\ell$ and $D$ be respectively a line and an irreducible conic in $\mathbb{P}^2$ meeting in two points $Q_1, Q_2$, where $Q_1$ is possibly equal to $Q_2$. Let $C$ be a smooth cubic meeting $\ell$ in three points $P_i$ ($1 \leq i \leq 3$) and $D$ in six points $P_i$ ($4 \leq i \leq 9$), where the points $P_i$ are all distinct and different from $Q_1, Q_2$. Choose two points $P_1, P_2$ on $\ell$ and four points $P_i$ ($4 \leq i \leq 7$) on $D$. Let $\sigma : S \to \mathbb{P}^2$ be the blowing-up of these six points. Let $\ell', D'$ and $C'$ be the proper transforms of $\ell, D$ and $C$. Then $S$ is a cubic hypersurface in $\mathbb{P}^3$ and $K_S \sim -C'$. Since $\ell' + D' \sim C'$, it is a hyperplane section of $S$ with respect to the embedding $\Phi_{|C'|} : S \hookrightarrow \mathbb{P}^3$.

5. Three lines which are either meeting in one point or not. Let $\ell_i$ ($1 \leq i \leq 3$) be the lines. Let $Q_1 = \ell_1 \cap \ell_3$ and $Q_2 = \ell_2 \cap \ell_3$. In the setting of (4) above, we consider $\ell = \ell_3$ and $D = \ell_1 + \ell_2$. So, if $Q_1 = Q_2$, three lines meet in one point. Choose a smooth cubic $C$ meeting three lines in nine distinct points $P_i$ ($1 \leq i \leq 9$) other than $Q_1, Q_2$. Choose six points from the $P_i$, two points
lying on each line. Then consider the blowing-up in these six points. The rest of the construction is the same as above.

Note that if $S$ is smooth $S \cap \mathcal{H}_t$ cannot have a non-reduced component. In fact, the non-reduced component is a line in $\mathcal{H}_t$. Hence we may write the defining equation of $S$ as

$$F = X_0^2(aX_1 + X_0) + X_3G(X_0, X_1, X_2, X_3) = 0,$$

where $G = G(X_0, X_1, X_2, X_3)$ is a quadratic homogeneous polynomial and $a \in k$. We understand that $a = 0$ if the non-reduced component has multiplicity three. By the Jacobian criterion, it follows that $S$ has singularities at the points $G = X_0 = X_3 = 0$.

The affine surface $\mathcal{X}_t$ has log Kodaira dimension 0 in the cases (1), (2), (4) with the conic and the line meeting in two distinct points and (5) with non-confluent three lines, and $-\infty$ in the rest of the cases.

Although $p : \mathcal{H} \to T$ is a flat family of affine surfaces, the log Kodaira dimension drops to $-\infty$ exactly at the points $t \in T$ where the boundary divisor $S \cap \mathcal{H}_t$ is not a divisor with normal crossings. This accords with a result of Kawamata concerning the invariance of log Kodaira dimension under deformations (cf. [18]).

If $\kappa(\mathcal{X}_t) = -\infty$, then $\mathcal{X}_t$ has an $A_1$-fibration. We note that if $\kappa(\mathcal{X}_t) = 0$ then $\mathcal{X}_t$ has an $A_1^*$-fibration. In fact, we consider the case where the boundary divisor $\mathcal{D}_t$ is a smooth cubic curve. Then $S$ is obtained from $\mathbb{P}^2$ by blowing up six points $P_i$ ($1 \leq i \leq 6$) on a smooth cubic curve $C$. Choose four points $P_1, P_2, P_3, P_4$ and let $\Lambda$ be a linear pencil of conics passing through these four points. Let $\sigma : S \to \mathbb{P}^2$ be the blowing-up of six points $P_i$ ($1 \leq i \leq 6$). The proper transform $\sigma^*\Lambda$ defines a $\mathbb{P}^1$-fibration $f : S \to \mathbb{P}^1$ for which the proper transform $C' = \sigma'(C)$ is a 2-section. Since $\mathcal{X}_t$ is isomorphic to $S \setminus C'$, $\mathcal{X}_t$ has an $A_1^*$-fibration.

Looking for an $A_1$-fibration in the case $\kappa(\mathcal{X}_t) = -\infty$ is not an easy task. Consider, for example, the case where $X = \mathcal{X}_t$ is obtained as $S \setminus (Q \cup \ell)$, where $Q$ is a smooth conic and $\ell$ is a line in $\mathbb{P}^2$ which meet in one point with multiplicity two. As explained in the above, such an $X$ is obtained from $\mathbb{P}^2$ by blowing up six points $P_1, \ldots, P_6$ such that $P_1, P_2$ lie on a line $\bar{\ell}$ and $P_3, P_4, P_5, P_6$ are points on a conic $\bar{Q}$. Then the proper transforms on $S$ of $\bar{\ell}, \bar{Q}$ are $\ell, Q$. Consider the linear pencil $\tilde{\Lambda}$ on $\mathbb{P}^2$ spanned by $2\bar{\ell}$ and $\bar{Q}$. Then a general member of $\Lambda$ is a smooth conic meeting $\bar{Q}$ in one point $\bar{Q} \cap \bar{\ell}$ with multiplicity four. The proper transform $\Lambda$ of $\tilde{\Lambda}$ on $S$ defines an $A_1$-fibration on $X$.

The following result of Dubouloz-Kishimoto except for the assertion (4) was orally communicated to one of the authors (see [3]).
Theorem 5.1. Let $S$ be a cubic hypersurface in $\mathbb{P}^3$ with a hyperplane section $S \cap H$ which consists of a line and a conic meeting in one point with multiplicity two. Let $Y = \mathbb{P}^3 \setminus S$ which is a smooth affine threefold. Then the following assertions hold.

1. $\overline{\tau}(Y) = -\infty$.
2. Let $f : Y \to \mathbb{A}^1$ be a fibration induced by the linear pencil on $\mathbb{P}^3$ spanned by $S$ and $3H$. Then a general fiber $Y_t$ of $f$ is a cubic hypersurface $S_t$ minus $Q \cup \ell$, where $Q$ is a conic and $\ell$ is a line which meet in one point with multiplicity two. Hence $\overline{\tau}(Y_t) = -\infty$ and $Y_t$ has an $\mathbb{A}^1$-fibration.
3. $Y$ has no $\mathbb{A}^1$-fibration.
4. There is a finite covering $T'$ of $\mathbb{A}^1$ such that the normalization of $Y \times_{\mathbb{A}^1} T'$ has an $\mathbb{A}^1$-fibration.

Proof. (1) Since $K_{\mathbb{P}^3} + S \sim -4H + 3H = -H$, it follows that $\overline{\tau}(Y) = -\infty$.

(2) The pencil spanned by $S$ and $3H$ has base locus $Q \cup \ell$ and its general member, say $S_t$, is a cubic hypersurface containing $Q \cup \ell$ as a hyperplane section. It is clear that $S_t \setminus (Q \cup \ell) = Y_t$. Hence, as explained above, $Y_t$ has an $\mathbb{A}^1$-fibration.

(3) Let $\tau : \tilde{S} \to \mathbb{P}^3$ be the cyclic triple covering of $\mathbb{P}^3$ ramified totally over the cubic hypersurface $S$. Then $\tilde{S}$ is a cubic hypersurface in $\mathbb{P}^4$ and $\tau^*(S) = 3\tilde{H}$, where $\tilde{H}$ is a hyperplane in $\mathbb{P}^4$. The restriction of $\tau$ onto $Z := \tilde{S} \setminus \tilde{S} \cap \tilde{H}$ induces a finite étale covering $\tau_Z : Z \to Y$. Suppose that $Y$ has an $\mathbb{A}^1$-fibration $\varphi : Y \to T$. Then $T$ is a rational surface. Since $\tau_Z$ is finite étale, this $\mathbb{A}^1$-fibration $\varphi$ lifts up to an $\mathbb{A}^1$-fibration $\tilde{\varphi} : Z \to \tilde{T}$. By [2], $\tilde{S}$ is unirational and irrational. Hence $\tilde{T}$ is a rational surface. This implies that $Z$ is a rational threefold. This is a contradiction because $\tilde{S}$ is irrational.

(4) There is an open set $T$ of $\mathbb{A}^1$ such that the restriction of $f$ onto $f^{-1}(T)$ is a smooth morphism onto $T$. By abuse of the notations, we denote $f^{-1}(T)$ by $Y$ anew and the restriction of $f$ onto $f^{-1}(T)$ by $f$. Hence $f : Y \to T$ is a smooth morphism. Let $K = k(t)$ be the function field of $T$ and let $Y_K$ be the generic fiber. Let $\overline{K}$ be an algebraic closure of $K$. Then $Y_{\overline{K}} := Y_K \otimes_K \overline{K}$ is identified with $S_{\overline{K}} \setminus (Q \cup \ell)$, where $S_{\overline{K}}$ is a cubic hypersurface in $\mathbb{P}^3_{\overline{K}}$ defined by $F_K = F_0 + tX_3^3 = 0$. Here $t$ is a coordinate of $\mathbb{A}^1$ and $(X_0, X_1, X_2, X_3)$ is a system of homogeneous coordinates of $\mathbb{P}^3$ such that $F_0(X_0, X_1, X_2, X_3) = 0$ is the defining equation of the cubic hypersurface $S$ and the hyperplane $H$ is defined by $X_3 = 0$. Then $Y_{\overline{K}}$ is obtained from $\mathbb{P}^3_{\overline{K}}$ by blowing up six $\overline{K}$-rational points in general position (two points on the image of $\ell$ and
four points on the image of $Q$). As explained earlier, there is an $\mathbb{A}^1$-fibration on $Y_\mathbb{K}$ which is obtained from conics on $\mathbb{P}^2_\mathbb{K}$ belonging to the pencil spanned by $Q$ and $2\ell$. This construction involves six points on $\mathbb{P}^2_\mathbb{K}$ to be blown up to obtain the cubic hypersurface $S_\mathbb{K}$ and four points (the point $Q \cap \ell$ and its three infinitely near points). Hence there exists a finite algebraic extension $K'/K$ such that all these points are rational over $K'$. Let $T'$ be the normalization of $T$ in $K'$. Let $Y' = Y \otimes_K K'$. Then $Y'$ has an $\mathbb{A}^1$-fibration.

Based on the assertion (4) above, we propose the following conjecture.

**Conjecture 5.2.** Let $f : Y \to T$ be a smooth morphism from a smooth affine threefold $Y$ onto a smooth affine curve $T$ such that every closed fiber $Y_t$ has an $\mathbb{A}^1$-fibration of complete type. Then there exists a finite covering $T'$ of $T$ such that the normalization of $Y \times_T T'$ has an $\mathbb{A}^1$-fibration.

**Remark 5.3.** The conjecture 5.2 is true if Theorem 2.8 holds after an étale finite base change of $T$ in the case where the general fibers of $f$ have $\mathbb{A}^1$-fibrations of complete type. A main obstacle in trying to extend the proof in the case of $\mathbb{A}^1$-fibrations of affine type is to show that, with the notations in the proof of Theorem 2.8, the locus of base points $P_t$ of the linear pencil $\Lambda_t$ on $Y_t$ with $t$ varying in an open neighborhood of $t_0 \in T$ (resp. the loci of infinitely near base points) is a cross-section of the $\mathbb{P}^1$-bundle $f|_{S_1} : S_1 \to T$ (resp. the exceptional $\mathbb{P}^1$-bundle).

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