A CONNECTION BETWEEN COVERS OF THE INTEGERS AND UNIT FRACTIONS

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Abstract. For integers \( a \) and \( n > 0 \), let \( a(n) \) denote the residue class \( \{ x \in \mathbb{Z} : x \equiv a \pmod{n} \} \). Let \( A \) be a collection \( \{a_s(n_s)\}_{s=1}^k \) of finitely many residue classes such that \( A \) covers all the integers at least \( m \) times but \( \{a_s(n_s)\}_{s=1}^{k-1} \) does not. We show that if \( n_k \) is a period of the covering function \( w_A(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}| \) then for any \( r = 0, \ldots, n_k-1 \) there are at least \( m \) integers in the form \( \sum_{s \in I} 1/n_s - r/n_k \) with \( I \subseteq \{1, \ldots, k-1\} \).

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1. Introduction

For an integer \( a \) and a positive integer \( n \), we use \( a(n) \) to denote the residue class \( \{ x \in \mathbb{Z} : x \equiv a \pmod{n} \} \). For a finite system

\[ A = \{a_s(n_s)\}_{s=1}^k \]

of residue classes, the function \( w_A : \mathbb{Z} \to \{0, 1, \ldots\} \) given by

\[ w_A(x) = |\{1 \leq s \leq k : x \in a_s(n_s)\}| \]

is called the covering function of \( A \). Clearly \( w_A(x) \) is periodic modulo the least common multiple \( N_A \) of the moduli \( n_1, \ldots, n_k \), and it is easy to verify the following well-known equality:

\[ \frac{1}{N_A} \sum_{x=0}^{N_A-1} w_A(x) = \sum_{s=1}^k \frac{1}{n_s}. \]
As in [7] we call \( m(A) = \min_{x \in \mathbb{Z}} w_A(x) \) the covering multiplicity of \( A \). For example,

\[
B = \{0(2), 0(3), 1(4), 5(6), 7(12)\}
\]

has covering multiplicity \( m(B) = 1 \), because the covering function is periodic modulo \( N_B = 12 \), and

\[
w_B(x) = \begin{cases} 
1 & \text{if } x \in \{1, 2, 3, 4, 7, 8, 10, 11\}, \\
2 & \text{if } x \in \{0, 5, 6, 9\}.
\end{cases}
\]

Let \( m \) be a positive integer. If \( w_A(x) \geq m \) for all \( x \in \mathbb{Z} \), then we call \( A \) an \( m \)-cover of the integers, and in this case we have the well-known inequality

\[
\sum_{k=1}^{s=1} \frac{1}{n_s} \geq m.
\]

(The term “1-cover” is usually replaced by the word “cover”.) If \( A \) is an \( m \)-cover of the integers but \( A_t = \{a_s(n_s)\} \) is not (where \([a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\} \) for \( a, b \in \mathbb{Z} \)), then we say that \( A \) forms an \( m \)-cover of the integers with \( a_t(n_t) \) irredundant. (For example, \( \{0(2), 1(2), 2(3)\} \) is a cover of the integers with distinct moduli. The topic of covers of the integers has been an active one in combinatorial number theory (cf. [3, 4]), and many surprising applications have been found (see, e.g., [1, 2, 9, 11]). The so-called \( m \)-covers and exact \( m \)-covers of the integers were systematically studied by the author in the 1990s.

Concerning the cover \( B \) given above one can easily check that

\[
\left\{ \sum_{n \in S} \frac{1}{n} : S \subseteq \{2, 3, 4, 6, 12\} \right\} = \left\{ 0, \frac{1}{12}, \ldots, \frac{11}{12} \right\} \cup \left\{ 1 + \frac{r}{12} : r = 0, 1, 2, 3, 4 \right\}.
\]

This suggests that for a general \( m \)-cover \( (1) \) of the integers we should investigate the set \( \left\{ \sum_{s \in I} 1/n_s : I \subseteq [1, k] \right\} \).

In this paper we establish the following new connection between covers of the integers and unit fractions.

**Theorem 1.** Let \( A = \{a_s(n_s)\}_{s=1}^{k} \) be an \( m \)-cover of the integers with the residue class \( a_k(n_k) \) irredundant. If the covering function \( w_A(x) \) is
periodic modulo $n_k$, then for any $r = 0, \ldots, n_k - 1$ we have

$$\left| \left\{ \left\lfloor \frac{1}{n_s} \sum_{s \in I} \frac{1}{n_s} \right\rfloor : I \subseteq [1, k - 1] \text{ and } \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} = \frac{r}{n_k} \right\} \right| \geq m, \quad (2)$$

where $\lfloor \alpha \rfloor$ and $\{ \alpha \}$ denote the integral part and the fractional part of a real number $\alpha$ respectively.

Note that $n_k$ in Theorem 1 needn’t be the largest modulus among $n_1, \ldots, n_k$. In the case $m = 1$ and $n_k = N_A$, Theorem 1 is an easy consequence of [5, Theorem 1] as observed by the author’s twin brother Z. H. Sun. When $w_A(x) = m$ for all $x \in \mathbb{Z}$, the author [6] even proved the following stronger result:

$$\left| \left\{ I \subseteq [1, k - 1] : \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_k} \right\} \right| \geq \left( \frac{m - 1}{[a/n_k]} \right) \text{ for all } a = 0, 1, \ldots.$$

Given an $m$-cover $\{ a_s(n_s) \}_{s=1}^k$ of the integers with $a_k(n_k)$ irredundant, by refining a result in [7] the author can show that there exists a real number $0 \leq \alpha < 1$ such that (2) with $r/n_k$ replaced by $(\alpha + r)/n_k$ holds for every $r = 0, \ldots, n_k - 1$.

Here we mention two local-global results related to Theorem 1.

(a) (Z. W. Sun [5]) $\{ a_s(n_s) \}_{s=1}^k$ forms an $m$-cover of the integers if it covers $|\{ \sum_{s \in I} 1/n_s : I \subseteq [1, k] \}|$ consecutive integers at least $m$ times.

(b) (Z. W. Sun [10]) $\{ a_s(n_s) \}_{s=1}^k$ is an exact $m$-cover of the integers if it covers $|\bigcup_{s=1}^k \{ r/n_s : r \in [0, n_s - 1] \}|$ consecutive integers exactly $m$ times.

**Corollary 1.** Suppose that the covering function of $A = \{ a_s(n_s) \}_{s=1}^k$ has a positive integer period $n_0$. If there is a unique $a_0 \in [0, n_0 - 1]$ such that $w_A(a_0) = m(A)$, then for any $D \subseteq \mathbb{Z}$ with $|D| = m(A)$ we have

$$\left\{ \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} : I \subseteq [1, k] \text{ and } \sum_{s \in I} \frac{1}{n_s} \notin D \right\} \supseteq \left\{ \frac{r}{n_0} : r \in [0, n_0 - 1] \right\}.$$

**Proof.** Let $m = m(A) + 1$. Clearly $A' = \{ a_s(n_s) \}_{s=0}^k$ forms an $m$-cover of the integers with $a_0(n_0)$ irredundant. As $w_{A'}(x) - w_A(x)$ is the characteristic function of $a_0(n_0)$, $w_{A'}(x)$ is also periodic mod $n_0$. Applying Theorem 1 we immediately get the desired result. \qed

We will prove Theorem 1 in Section 3 with help from some lemmas given in the next section.
2. Several Lemmas

Lemma 1. Let $(1)$ be a finite system of residue classes with $m(A) = m$, and let $m_1, \ldots, m_k$ be any integers. If $f(x_1, \ldots, x_k)$ is a polynomial with coefficients in the complex field $\mathbb{C}$ and $\deg f \leq m$, then for any $z \in \mathbb{Z}$ we have

$$\sum_{I \subseteq [1,k]} (-1)^{|I|} f([1 \in I], \ldots, [k \in I]) e^{2\pi i \sum_{s \in I} (a_s-z)m_s/n_s}$$

$$= (-1)^k c(I_z) \prod_{s \in [1,k] \setminus I_z} \left( e^{2\pi i (a_s-z)m_s/n_s} - 1 \right),$$

where $[s \in I]$ takes $1$ or $0$ according as $s \in I$ or not, $I_z = \{1 \leq s \leq k : z \in a_s(n_s)\}$, and $c(I_z) = \prod_{s \in I_z} f(x_1, \ldots, x_k)$ is the coefficient of the monomial $\prod_{s \in I_z} x_s$ in $f(x_1, \ldots, x_k)$.

Proof. Write $f(x_1, \ldots, x_k) = \sum_{j_1, \ldots, j_k \geq 0} c_{j_1, \ldots, j_k} x_1^{j_1} \cdots x_k^{j_k}$. Observe that

$$\sum_{I \subseteq [1,k]} (-1)^{|I|} f([1 \in I], \ldots, [k \in I]) e^{2\pi i \sum_{s \in I} (a_s-z)m_s/n_s}$$

$$= \sum_{j_1, \ldots, j_k \geq 0} c_{j_1, \ldots, j_k} \sum_{I \subseteq [1,k]} \left( \prod_{s=1}^{k} [s \in I]^{j_s} \times (-1)^{|I|} e^{2\pi i \sum_{s \in I} (a_s-z)m_s/n_s} \right)$$

$$= \sum_{j_1, \ldots, j_k \geq 0} c_{j_1, \ldots, j_k} \sum_{J(j_1, \ldots, j_k) \subseteq I \subseteq [1,k]} (-1)^{|I|} e^{2\pi i \sum_{s \in I} (a_s-z)m_s/n_s},$$

where $J(j_1, \ldots, j_k) = \{1 \leq s \leq k : j_s \neq 0\}$.

Let $z$ be any integer. If $I_z \not\subseteq J(j_1, \ldots, j_k)$, then

$$\sum_{J(j_1, \ldots, j_k) \subseteq I \subseteq [1,k]} (-1)^{|I|} e^{2\pi i \sum_{s \in I} (a_s-z)m_s/n_s} = 0$$

since

$$\sum_{I \subseteq [1,k] \setminus J(j_1, \ldots, j_k)} (-1)^{|I|} e^{2\pi i \sum_{s \in I} (a_s-z)m_s/n_s}$$

$$= \prod_{s \in [1,k] \setminus J(j_1, \ldots, j_k)} \left( 1 - e^{2\pi i (a_s-z)m_s/n_s} \right) = 0.$$
hence $I_z = J(j_1, \ldots, j_k)$ and $j_s = 1$ for all $s \in I_z$.

Combining the above we find that the left-hand side of (3) coincides with

$$
c(I_z) \sum_{I \subseteq [1,k]} (-1)^{|I|} e^{2\pi i \sum_{s \in I} (a_s - z) m_s/n_s} = c(I_z) (-1)^{|I_z|} e^{2\pi i \sum_{s \in I_z} (a_s - z) m_s/n_s} \prod_{s \in [1,k] \setminus I_z} \left( 1 - e^{2\pi i (a_s - z) m_s/n_s} \right)
$$

$$
= (-1)^k c(I_z) \prod_{s \in [1,k] \setminus I_z} \left( e^{2\pi i (a_s - z) m_s/n_s} - 1 \right).
$$

This proves the desired (3). □

**Lemma 2.** Let (1) be an $m$-cover of the integers with $a_k(n_k)$ irredundant, and let $m_1, \ldots, m_{k-1}$ be positive integers. Then, for any $0 \leq \alpha < 1$ we have $C_0(\alpha) = \cdots = C_{n_k-1}(\alpha)$, where $C_r(\alpha)$ (with $r \in [0, n_k - 1]$) denotes the sum

$$
\sum_{\{\sum_{s \in I} m_s/n_s\} = (\alpha + r)/n_k} (-1)^{|I|} \left( \frac{\sum_{s \in I} m_s/n_s}{m-1} \right) e^{2\pi i \sum_{s \in I} (a_s - a_k) m_s/n_s}.
$$

**Proof.** This follows from [7, Lemma 2]. □

**Lemma 3.** Let (1) be an $m$-cover of the integers with $a_k(n_k)$ irredundant. Suppose that $n_k$ is a period of the covering function $w_A(x)$. Then, for any $z \in a_k(n_k)$ we have

$$
\prod_{s \in [1,k] \setminus I_z} \left( 1 - e^{2\pi i (a_s - z)/n_s} \right) = \prod_{s \in I_z} \frac{n_k}{n_s} \times \prod_{t=1}^{n_k} \left( 1 - e^{2\pi i (t-a_k)/n_k} \right)^{w_A(t) - m}
$$

where $I_z = \{1 \leq s \leq k : z \in a_s(n_s)\}$.

**Proof.** Since $a_k(n_k)$ is irredundant, we have $w_A(z_0) = m$ for some $z_0 \in a_k(n_k)$. As the covering function of $A$ is periodic mod $n_k$, $|I_z| = w_A(z) = m$ for all $z \in a_k(n_k)$.

Now fix $z \in a_k(n_k)$. Since $w_A(x)$ is periodic modulo $n_k$, by [8, Lemma 2.1] we have the identity

$$
\prod_{s=1}^{k} \left( 1 - y^{N/n_s} e^{2\pi i a_s/n_s} \right) = \prod_{t=1}^{n_k} \left( 1 - y^{N/n_k} e^{2\pi it/n_k} \right)^{w_A(t)},
$$
where \( N = N_A \) is the least common multiple of \( n_1, \ldots, n_k \). Putting \( y = r^{1/N} e^{-2\pi i z/N} \) where \( r \geq 0 \), we then get that

\[
\prod_{s=1}^{k} \left( 1 - r^{1/n_s} e^{2\pi i (a_s - z)/n_s} \right) = \prod_{t=1}^{n_k} \left( 1 - r^{1/n_k} e^{2\pi i (t - z)/n_k} \right)^{w_A(t)}.
\]

Therefore

\[
\prod_{s \in [1,k] \setminus I_z} \left( 1 - e^{2\pi i (a_s - z)/n_s} \right)
\]

\[
= \lim_{r \to 1} \prod_{s \in [1,k] \setminus I_z} \left( 1 - r^{1/n_s} e^{2\pi i (a_s - z)/n_s} \right)
\]

\[
= \lim_{r \to 1} \prod_{t=1}^{n_k} \frac{(1 - r^{1/n_k} e^{2\pi i (t - a_k)/n_k})^{w_A(t)}}{\prod_{s \in I_z} (1 - r^{1/n_s} e^{2\pi i (a_s - z)/n_s})}
\]

\[
= \prod_{s \in I_z} n_s \times \lim_{r \to 1} \prod_{t=1}^{n_k} (1 - r^{1/n_k} e^{2\pi i (t - a_k)/n_k})^{w_A(t) - 1},
\]

and hence the desired result follows. \( \square \)

3. Proof of Theorem 1

In the case \( k = 1 \), we must have \( m = 1 \) and \( n_k = 1 \); hence the required result is trivial. Below we assume that \( k > 1 \).

Let \( r_0 \in [0, n_k - 1] \) and \( D = \{d_n + r_0/n_k : n \in [1, m - 1]\} \), where \( d_1, \ldots, d_{m-1} \) are \( m - 1 \) distinct nonnegative integers. (If \( m = 1 \) then we set \( D = \emptyset \).) We want to show that there exists an \( I \subseteq [1, k - 1] \) such that \( \{\sum_{s \in I} 1/n_s\} = r_0/n_k \) and \( \sum_{s \in I} 1/n_s \notin D \).

Define

\[
f(x_1, \ldots, x_{k-1}) = \prod_{d \in D} \left( \frac{x_1}{n_1} + \cdots + \frac{x_{k-1}}{n_{k-1}} - d \right).
\]

(An empty product is regarded as 1.) Then \( \deg f = |D| = m - 1 \). For any \( z \in a_k(n_k) \), the set \( I_z = \{1 \leq s \leq k : z \in a_s(n_s)\} \) has cardinality \( m \) since \( a_k(n_k) \) is irredundant and \( w_A(x) \) is periodic mod \( n_k \). Observe that the coefficient

\[
c_z = \left[ \prod_{s \in I_z \setminus \{k\}} x_s \right] f(x_1, \ldots, x_{k-1}) = \left[ \prod_{s \in I_z \setminus \{k\}} x_s \right] \left( \sum_{s=1}^{k-1} \frac{x_s}{n_s} \right)^{m-1}\]

\[
= \left[ \prod_{s \in I_z \setminus \{k\}} x_s \right] \left( \sum_{s=1}^{k-1} \frac{x_s}{n_s} \right)^{m-1}.
\]
coincides with \((m - 1)! / \prod_{s \in I \setminus \{k\}} n_s\) by the multinomial theorem. For \(I \subseteq [1, k - 1]\) we set
\[ v(I) = f([1 \in I], \ldots, [k - 1 \in I]). \]

As \(|\{1 \leq s \leq k - 1 : x \in a_s(n_s)\}| \geq \deg f\) for all \(x \in \mathbb{Z}\), in view of Lemmas 1 and 3 we have
\[
\sum_{I \subseteq [1, k - 1]} (-1)^{|I|} v(I) e^{2\pi i \sum_{s \in I} (a_s - z)/n_s}
= (-1)^{k-1} c_z \prod_{s \in [1, k-1] \setminus I} \left( e^{2\pi i (a_s - z)/n_s} - 1 \right)
= (-1)^{m-1}(m-1)! \prod_{s \in I} n_s \times \prod_{t=1}^{n_k} \left( 1 - e^{2\pi i (t-a_k)/n_k} \right)^w A(t) - m = C,
\]
where \(C\) is a nonzero constant not depending on \(z \in a_k(n_k)\).

By the above,
\[
N_A C = \sum_{x=0}^{N_A-1} \sum_{I \subseteq [1, k-1]} (-1)^{|I|} v(I) e^{2\pi i \sum_{s \in I} (a_s - a_k - n_k x)/n_s}
= \sum_{I \subseteq [1, k-1]} (-1)^{|I|} v(I) e^{2\pi i \sum_{s \in I} (a_s - a_k)/n_s} \sum_{x=0}^{N_A-1} e^{-2\pi i x \sum_{s \in I} n_k/n_s}
\]
and hence
\[
C = \sum_{I \subseteq [1, k-1]} (-1)^{|I|} v(I) e^{2\pi i \sum_{s \in I} (a_s - a_k)/n_s} = \sum_{r=0}^{n_k-1} C_r,
\]
where
\[
C_r = \sum_{I \subseteq [1, k-1]} (-1)^{|I|} \prod_{d \in D} \left( \sum_{s \in I} \frac{1}{n_s} - d \right) e^{2\pi i \sum_{s \in I} (a_s - a_k)/n_s}.
\]

Let \(r \in [0, n_k - 1]\). Write
\[
P_r(x) = \prod_{d \in D} \left( x + \frac{r}{n_k} - d \right) = \sum_{n=0}^{m-1} c_{n,r} \binom{x}{n}
\]
where \(c_{n,r} \in \mathbb{C}\). By comparing the leading coefficients, we find that
\[c_{m-1,r} = (m-1)!\]. Observe that
\[C_r = \sum_{I \subseteq [1, k-1]} \sum_{\{\sum_{s \in I} 1/n_s\} = r/n_k} (-1)^{|I|} \left| \left( \frac{\sum_{s \in I} 1/n_s}{n} \right) \right| e^{2\pi i \sum_{s \in I} (a_s - a_k)/n_s}
\]
\[= \sum_{n=0}^{m-1} c_{n,r} \sum_{I \subseteq [1, k-1]} \sum_{\{\sum_{s \in I} 1/n_s\} = r/n_k} (-1)^{|I|} \left( \frac{\sum_{s \in I} 1/n_s}{n} \right) e^{2\pi i \sum_{s \in I} (a_s - a_k)/n_s}
\]
\[= c_{m-1,r} \sum_{I \subseteq [1, k-1]} \sum_{\{\sum_{s \in I} 1/n_s\} = r/n_k} (-1)^{|I|} \left( \frac{\sum_{s \in I} 1/n_s}{m-1} \right) e^{2\pi i \sum_{s \in I} (a_s - a_k)/n_s};
\]
in taking the last step we note that if \(0 \leq n < m-1\) then
\[\sum_{I \subseteq [1, k-1]} \sum_{\{\sum_{s \in I} 1/n_s\} = r/n_k} (-1)^{|I|} \left( \frac{\sum_{s \in I} 1/n_s}{n} \right) e^{2\pi i \sum_{s \in I} a_s/n_s} = 0\]
by [5, Theorem 1] (since \(\{a_s(n_s)\}_{s=1}^{k-1}\) is an \((m-1)\)-cover of the integers). By Lemma 2 and the above,
\[C_r = (m-1)! \sum_{I \subseteq [1, k-1]} \sum_{\{\sum_{s \in I} 1/n_s\} = 0} (-1)^{|I|} \left( \frac{\sum_{s \in I} 1/n_s}{m-1} \right) e^{2\pi i \sum_{s \in I} (a_s - a_k)/n_s}
\]
does not depend on \(r \in [0, n_k-1]\). Combining the above we obtain that
\[n_k C_{r_0} = \sum_{r=0}^{n_k-1} C_r = C \neq 0\]
So there is an \(I \subseteq [1, k-1]\) for which \(\{\sum_{s \in I} 1/n_s\} = r_0/n_k\); \(\sum_{s \in I} 1/n_s \not\in D\) and hence \(\sum_{s \in I} 1/n_s \not\in \{d_n : n \in [1, m-1]\}\). This concludes our proof.

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