Stepping out of homogeneity in loop quantum cosmology

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Abstract
We explore the extension of quantum cosmology outside the homogeneous approximation using the formalism of loop quantum gravity. We introduce a model where some of the inhomogeneous degrees of freedom are present, providing a tool for describing general fluctuations of quantum geometry near the initial singularity. We show that the dynamical structure of the model reduces to that of loop quantum cosmology in the Born–Oppenheimer approximation. This result corroborates the assumptions that ground loop cosmology sheds some light on the physical and mathematical relation between loop cosmology and full loop quantum gravity, and on the nature of the cosmological approximation. Finally, we show that the non-graph-changing Hamiltonian constraint considered in the context of algebraic quantum gravity provides a viable effective dynamics within this approximation.

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1. Introduction

Loop quantum cosmology (LQC) provides the most successful physical application of loop gravity, and one of the most promising avenues toward the possibility of an empirical test of quantum gravity [1, 2]. The possibility of a fully consistent quantum description of ‘big-bang physics’, and the robustness of the bounce prediction represent a clear advance in our understanding of quantum gravitational physics within this theoretical framework. This impressive success opens a number of physical and mathematical questions: (i) can we include inhomogeneities? Inhomogeneities and their quantum fluctuations play a fundamental role in the currently fashionable cosmological scenario. Inhomogeneities can in principle be re-inserted at a later cosmological epoch, restricting the analysis of the Planck epoch to the
sole homogeneous degrees of freedom—but is this approximation sufficient? After all, what is very interesting is to precisely understand the configuration of the full fluctuating quantum geometry near the singularity itself. In other words: (ii) can we describe the actual quantum state of the geometry near the initial singularity beyond the homogeneous approximation? Perhaps this state could even teach us something directly about the emergence of the physical inhomogeneities of our universe. (iii) What is the true relation between full loop quantum gravity [3, 4] and LQC? The question has been addressed repeatedly [5] and concerns have been raised on whether the two theories are truly consistent—some simple minded ways of interpreting their relation have even been recently rigorously proven incorrect [6].

We address all these questions here. We do so by first analyzing the nature of the approximation on which cosmology itself—classical or quantum—is based. This is neither a low-energy nor a high-energy approximation, since cosmology appears to describe well very large distance features of our universe as well as its behavior in much higher energy-density regimes. The analysis leads us to the idea that the full theory may be consistently expanded by adding degrees of freedom one by one, starting from the cosmological ones. Accordingly, we define an approximated dynamics of the universe, inhomogeneous but truncated at a finite number of degrees of freedom, and we discuss its regime of validity. (For previous works on inhomogeneities in LQC, see [7–9]. See also [10].) This approximation includes and extends conventional cosmology, without however including the full infinite-dimensional field theory. For simplicity, we consider the Euclidean context, leaving the analysis of the Lorentzian case for the future. We also work in the compact case, which is conceptually simpler—that is, we assume that the topology of the spacial universe is that of a 3-sphere (a possibility which is still compatible with observations, and, according to some [11], it is even favored by them). The approximation we take can be intuitively interpreted as a truncation of all degrees of freedom to a finite order in a multipolar expansion of the fields on the topological 3-sphere. These degrees of freedom can be described using a fixed 3D compact triangulation $\Delta_n$, formed by $n$ tetrahedra.

The quantum kinematics of this system turns out to be described by the truncation of loop quantum gravity obtained by restricting the spin-network states to those based on a graph equal (or contained in) to the graph defined by the dual of $\Delta_n$. Within the approximation considered, the quantum dynamics can be described by the non-graph-changing version of the Hamiltonian constraint [12] that has been recently considered in the context of algebraic quantum gravity [15]. Thus, the non-graph-changing Hamiltonian constraint here plays the role of an effective dynamics, as originally suggested by Thiemann. In this way, one can try to define a quantum cosmological model for any given triangulation $\Delta_n$, with a number of degrees of freedom that increases with the complexity of $\Delta_n$.

We study here in detail the simplest nontrivial case, based on a triangulation $\Delta_2$ formed by the minimal triangulation of a 3-sphere: two tetrahedra glued along each face. The corresponding dual graph is formed by two nodes joined by four links. We show that the model is well defined and in particular the constraint algebra closes. We write the state space, the quantum operators and the Hamiltonian constraints of this model explicitly. The model represents a non-homogeneous quantum universe, where, say, we do not consider just the overall spatial average of a scalar field, but also its dipole moment. This provides a well-defined ‘first step out of homogeneity’ in loop quantum cosmology. We call this model a ‘dipole cosmology’, but the name should not be taken literally, as the gravitational degrees of freedom that are described are more ‘quadrupolar’ than dipolar: the spatial universe is split into two hemispheres (represented by the two tetrahedra), separated by a closed surface $\Sigma$.

4 ‘Maybe one could call the operator as formulated in this section an effective operator’. Reference [12], section 5.2.
but $\Sigma$ splits in turn into four large surfaces (the four triangles bounding the tetrahedra), whose areas provide degrees of freedom that can be roughly thought of as capturing the geometry of a 3D ellipsoid.

We then ask in which sense LQC is contained in the larger model. We argue that a proper way of addressing the problem is to interpret LQC as a first-order Born–Oppenheimer approximation. This is the approximation in which the effect of the inhomogeneities on the dynamics of the scale factor is small, as the effect of the electrons on the dynamics of the nuclei is small, in the Born–Oppenheimer approach to molecules [16]. (For other utilizations of the Born–Oppenheimer approximation in this context see [17].) We show concretely that taking the zero-order Born–Oppenheimer approximation of the $\Delta_2$ model yields the structure of the LQC dynamics. In particular, we recover the characteristic structure of the LQC Wheeler–DeWitt equation, defined as a 3-terms finite-difference equation in the scale factor. This derivation provides a prototype for understanding how LQG is contained in full loop quantum gravity. In particular, we derive here the quantization of the LQC $\mu$ parameter directly from LQG, without the need of an explicit recourse to the area-gap argument [1].

The models defined here, and in particular the $\Delta_2$ dipole cosmology, are finite-dimensional quantum theories, that can be used for describing the inhomogeneous quantum geometry near the initial singularity and its quantum fluctuations, in any given arbitrary order. We leave the analysis of these models, in the classical or quantum domains, for further investigations.

This paper is organized as follows. In section 2, we discuss the nature of the approximation we take and its viability. In section 3, we define the quantum models for arbitrary $\Delta_n$. The $\Delta_2$ model is described in some detail in section 4. The Born–Oppenheimer approximation is introduced in section 5, where we show how the structure of LQC can be recovered. In section 6 we conclude pointing out some implications of the results presented, in particular their relation with Regge calculus cosmology [18], and with the approximation used in computing background-independent $n$-point functions [19].

2. Approximations in cosmology

Modern cosmology was born with Einstein’s 1917 seminal paper [20], where Einstein states the cosmological principle, according to which the dynamics of a homogeneous and isotropic space approximates that of our real universe. A certain vagueness lingers around the precise status of this principle presented under a variety of different lights in the literature. The question of whether the universe approaches homogeneity at large scale is an empirical question: the later evolution of observational cosmology appears to be corroborating it, as well as finding some precise quantitative limits to large scale homogeneity. Einstein’s principle is therefore a hypothesis, in the healthy tradition of hypothetical-deductive science. But what precisely is the hypothesis, and why is it relevant? The universe is obviously not homogeneous at every scale and not exactly homogeneous at any scale, except at its very largest scale, where it is homogeneous by definition. Why can we neglect the effect of the inhomogeneities on the dynamics of this largest scale, in a nonlinear theory such as general relativity, where all scales are coupled? In fact, this is precisely the hypothesis put forward by Einstein in 1917: that the universe happens to be in a state where the effect of the inhomogeneities on the dynamics of its largest scale, described by the scale factor, can be neglected in a first approximation. More generally, that the universe is in a configuration where the effect on the longest wavelength of the interaction with the shorter wavelengths is negligible. The significance of this hypothesis is becoming particularly clear today, since a number of contrary hypotheses are being explored, such as, in particular, the very intriguing possibility that the measured cosmological constant could be—in full or in part—the result of the effect of these shorter wavelengths on the largest
scale (see for instance [21]). In other words, the cosmological principle is the hypothesis that a certain approximation scheme is viable in general relativity, and that the universe happens to be in the regime where this approximation scheme is effective.

What precisely is this approximation? One is tempted to say that it is simply a long-distance one: it is defined by cutting-off the modes with wavelength shorter than a certain size \( L \). But this is imprecise, because \( L \) varies with the size itself of the scale factor, and can also be very small. The situation is easier to analyze in the context of a spatially closed universe; let us therefore assume in the following that we are in this context. Let \( a^3(t) \) be the volume of the universe at the cosmological time \( t \). Then the approximation on which cosmology is based is to neglect the dynamics of the wavelengths \( \lambda \) shorter than \( a(t) \). But \( a(t) \) itself can be large as well as small. If \( n \sim \frac{\pi}{a} \), the cosmological approximation is the first-order term, \( n = 1 \), in an expansion in small \( n \), which does not necessarily mean large \( \lambda \).

Consider the next terms in the expansion, say \( n = 2, 3, \ldots \) and so on. An immediate consequence of the cosmological principle (or a natural extension of the same) is the hypothesis that the dynamics of the long wavelengths modes of the universe is only weakly affected by its highest modes. We can implement this expansion by approximating the geometry of the universe not just by a maximally symmetric space, as in standard cosmology, but rather by a geometry described by a finite number of degrees of freedom. These can be labeled by the elements of a triangulation \( \Delta_n \) formed by a finite number \( n \) of the tetrahedra. If a maximally symmetric space represents the gravitational degrees of freedom of the universe averaged over the largest possible scale, the degrees of freedom of a fixed triangulation can be interpreted as a representation of the gravitational degrees of freedom of the universe averaged over the large scales, up to a certain degree in a mode expansion. In the following, we consider the classical and quantum description of the dynamics of a model of the universe defined in this manner. That is, we construct an effective theory where wavelengths \( \lambda < \lambda_0 \sim \frac{\pi}{n} \) are neglected.

3. The model

3.1. Classical theory

Fix an (oriented) triangulation \( \Delta_n \) of a (topological) 3-sphere, formed by \( n \) tetrahedra \( t \) glued by their triangles. We label the triangles with an index \( f \) (‘f’ for face) that runs from 1 to \( 2n \) (the number of faces is twice the number of tetrahedra). Associate a group element \( U_f \in SU(2) \) and a \( su(2) \) algebra element \( E_f \) with each oriented triangle \( f \). We take the convention that to the face \( f^{-1} \) obtained inverting the orientation of \( f \) is associated with the group element

\[
U_{f^{-1}} = U_f^{-1}
\]

and the algebra element

\[
E_{f^{-1}} = -U_f^{-1} E_f U_f.
\]

If \( \tau_i, i = 1, 2, 3 \) is a basis in \( su(2) \), we write \( E_f = E_f^i \tau_i \). We take \( U_f \) and \( E_f \) as phase space variables of a dynamical system, with the conventional Poisson brackets structure of a canonical lattice \( SU(2) \) Yang–Mills theory, that is

\[
\{U_f, U_{f'}\} = 0,
\]

\[
\{E_f^i, U_{f'}\} = \delta_{ff'}^{ij} \tau^j U_f,
\]

\[
\{E_f^i, E_{f'}^j\} = -\delta_{ff'} \epsilon^{ijk} E_f^k.
\]
In particular, writing the total volume of space. We have taken here $8\pi G$ of constraints: the gauge constraints with its natural symplectic structure. Let the dynamics of the system be defined by two sets of constraints: the gauge constraints

$$G_i \equiv \sum_{f \in \Sigma} E_f \sim 0,$$

where the sum is over the four faces of the tetrahedron, and the Hamiltonian constraints

$$C_i \equiv V^{-1}_i \sum_{\{f\}, f' \in \Sigma} \text{Tr}[U_{ff} E_{f'} E_{f}] \sim 0$$

where the sum is over all the couples of (non-necessarily distinct) faces at each tetrahedron, $U_{ff'} = U_{f1} U_{f2} U_{f3} U_{f4}$ where $t_{ff'} = \{f, f_1, f_2, \ldots, f_{n-1}\}$ is the link of the oriented faces around the edge where $f$ and $f'$ join, and

$$V^2_i = \frac{1}{4} \sum_{f, f' \in \Sigma} \text{Tr}[E_{f} E_{f'} E_{f''}]$$

where the sum is over the four unordered triplets of distinct faces at the tetrahedron and $\{f, f', f''\}$ has positive orientation. Note that $V^2_i = \text{Tr}[E_{f} E_{f'} E_{f''}]$ because of (6). This concludes the definition of the dynamical systems we want to consider. In section 4 we study one of these systems in more detail, and we make sure the constraint algebra closes. We leave the analysis of the constraint algebra in the general case for future developments.

This dynamical system can be interpreted as a cosmological approximation to the dynamics of the geometry of a closed universe. To see this, consider real Ashtekar fields $A^a_i(x)$ and $E^{ia}(x)$, with their standard Poisson algebra (see for instance [3]), on a 3D surface $\Sigma$ with the $S_3$ topology. (The index $a$ is a 3D (abstract) tangent index.) Let $\Delta_n$ be a triangulation of $\Sigma$ and $\Delta^a_n$ a dual of the triangulation. We interpret $U_f$ as the parallel transport of the Ashtekar connection $A^a_i(x)$ along the link $e_f$ of $\Delta^a_n$ dual to the triangle $f$, and $E_f$ as the flux $\Phi_f$ of the Ashtekar’s electric field $E^{ia}(x)$ across the triangle $f$, parallel transported to the center of the tetrahedron. That is, $E_f = U_{i1} \Phi_f U_{i1}^{-1}$ and $E_{f'} = -U_{i2} \Phi_f U_{i2}$, where $U_f = U_{f1} U_{f2} U_{f3}$ and $U_{f1}$ and $U_{f2}$ are the holonomies of the two segments $e_{f1}, e_{f2}$ into which the face $f$ cuts the link $e_f$. Then the Poisson brackets of $U_f$ and $E_f$ defined in this manner turn out to be precisely (3,4,5). The algebra of these variables has been computed by Thiemann in [14]. In particular, note that the origin of (5) is the fact that $\Phi_f$ is parallel transported to the center of each tetrahedron. (Note also that (5) follows from (3), (4) and the Jacobi identity.)

The gauge constraint (6) generates the correct internal gauge transformations on these variables. If the triangulation is sufficiently fine, (7) approximates the Ashtekar's (Euclidean part of the) Hamiltonian constraint $\text{Tr}[F_{ab} E^a E^b] / \sqrt{\det E} \sim 0$, where $F_{ab}$ is the curvature of $A_a$. Note the absence of the second term of the usual discretization $\text{Tr} \left( (U_{ff'} - U_{ff''}) E_{f'} E_{f''} \right)$ of the Hamiltonian constraint. The $U_{ff'}^{-1}$ is usually subtracted in order to subtract the first term in the small-curvature expansion $U \sim \exp \int_a A \sim 1 + \|\alpha^2 F_{ab} + O(|\alpha|^4 A^2)\|$, but the subtraction is not needed because this term does not contribute to $C_t$ thanks to (6). We do not know if this observation has already been made in the literature. An explicit calculation shows that $C_t$ is real. We use this nonstandard form of the Hamiltonian constraint because, as we shall see, it simplifies the constraint algebra later on.

Finally, $V_t$ is (proportional to) the volume of the tetrahedron $t$ and we call $V = \sum_t V_t$ the total volume of space. We have taken here $8\pi G_{\text{Newton}}$ and the speed of light to be unit,
and we have chosen the Immirzi parameter $\gamma = 1$ (we are in the Euclidean context); what follows needs to be extended to the more interesting Lorentzian case with real $\gamma$ and the full Hamiltonian constraint.

Thus, the dynamical model described above can be interpreted as a discretization of Euclidean general relativity on a fixed triangulation of space. The quantization of this discretization is known to be related to a truncation of LQG on a finite (abstract) graph \[13\].

The constraint (7) corresponds to the non-graph-changing version of the Hamiltonian constraint, as that utilized in algebraic quantum gravity \[15\]. This is the only viable alternative in the present context, where we have reduced the degrees of freedom of the gravitational field to a fixed number. This version of the constraint approximates the classical Hamiltonian constraint if $U_{ff'}$, namely the parallel transport of $A_x$ along the loop $\alpha$ dual to the link $l_{ff'}$, approximates $I + |\alpha|^2 F_{ab}$. It is important to note that this happens not only if the length of the loop is small, but also for large loops if $A_x$ is small. Hence near flat spacetime the approximation can be good even for coarse triangulations. Misunderstanding of this fact has generated the erroneous idea that low-curvature spacetime needs to be approximated by fine triangulations.

Alternatively, one can interpret the data $(\Delta_n, U_f, E_f)$ as a description of a piecewise flat Regge geometry, where the curvature of the connection is concentrated on the edges of $\Delta$. The quantity $U_{ff'}$ gives then the curvature at the corresponding edge. However, observe this interpretation is slightly misleading here, since the variables of the model are better understood as macroscopic quantities averaging over local degrees of freedom. Therefore the flatness of the individual tetrahedra does not need to be taken literally.

It is simple to couple a family of multifingered ‘clock’ variables, one per node. The simplest choice \[1\] is an (ultra-local) scalar field $\phi_t$ with a value $\phi_t$ and conjugate momentum $p_{\phi_t}$ at each node, with the overall Hamiltonian constraint given by

$$\frac{1}{V_t} \sum_{ff' \in t} \text{Tr} [U_{ff'} E_f E_f] + \frac{\kappa}{2V_t} p_{\phi_t}^2 \sim 0,$$

where $\kappa$, proportional to the Newton constant $G$, determines the matter–gravity coupling. The role of this field is double. First, it keeps track of evolution in a background-independent manner, namely it models a physical clock. Second, it represents in a simplified manner the matter content of the universe. Replacing this field with a more realistic description is viable here: ultralocality can be eliminated by adding a difference term; while Yang–Mills and fermion fields have a particularly straightforward description in this language \[3\].

3.2. Quantum theory

The quantization of the model is immediate. Following what is done in lattice QCD, a quantum representation of the observable algebra (3–5) is provided by the Hilbert space $H_{aux} = L_2[SU(2)^{2n}, dU_f]$ where $dU_f$ is the Haar measure. The states have the form $\psi(U_f)$. The operators $U_f$ are diagonal and the operators $E_f$ are the left invariant vector fields on each $SU(2)$. The operators $E_{ff'}$ then turn out to be the right invariant vector fields. The operator associated with the volume $V_t$ turns out to be the standard loop quantum gravity volume operator that is constructed in terms of $E_f$. The states that solve the gauge constraint (6) are labeled by SU(2) spin networks on the graph $\Delta'_n$, which has a node for each tetrahedron and a

6 ‘Ultralocal’ means that we have dropped the spatial derivative terms in the Hamiltonian of the matter field.
link for each face of $\Delta_n$. A basis of these is given by states $|j_f, t, \phi_t\rangle$, where $f = 1, \ldots, 2n$ and $t = 1, \ldots, n$ range over the links and the nodes of the graph. These are defined by

$$\psi_{j_f, t, \phi_t} \equiv |j_f, t, \phi_t\rangle \equiv \otimes_f \Pi^{(1)}(U_f) \otimes \otimes_t \iota_t \quad (10)$$

where $\Pi^{(1)}(U)$ are the matrix elements of the spin-$j$ representation of SU(2) and '·' indicates the contraction of the indices of these matrices with the indices of the intertwiners $\iota_t$ dictated by the graph $\Delta_1^\ast_n$. For details, see [3].

With a scalar field, the Hilbert space becomes $H_{aux} = L^2[\text{SU}(2)^{2n}, dU_f] \otimes L^2[R^n]$, with a (generalized) basis $|j_f, t, \phi_t, \phi_t\rangle$ and the states can be written in the form

$$\psi(j_f, t, \phi_t, \phi_t) \equiv \langle j_f, t, \phi_t, \phi_t | \psi \rangle. \quad (11)$$

In this basis the operator $\phi_t$ is diagonal while $p_{\phi_t} = -i \partial/\partial \phi_t$.

If all constraints are first class, they can be quantized à la Dirac. The quantum Hamiltonian constraint can be defined in two alternative forms. The first, à la Thiemann is obtained rewriting (7) in the Thiemann-like form

$$C_t = \sum_{f \neq f'} \text{Tr}[U_{ff'} U_{f'f}^{-1} \{U_{ff'}, \text{Tr}[E_{ff'} E_{f'f'}]\}] \sim 0, \quad (12)$$

and then defining the corresponding quantum operator by replacing the Poisson bracket with the commutator. Here the sum is over all ordered triples of $f$'s (the terms where, say, $f = f'$ vanish because of the second trace). The second possibility is to write directly the quantum operator which corresponds to the regularization of the Hamiltonian constraint used earlier in loop quantum gravity [22]

$$\tilde{C}_t = V_t C_t = \sum_{f \neq f'} \text{Tr}[U_{ff'} E_{f'} E_{f}] \sim 0. \quad (13)$$

This form is more handy in the present context. Multiplying (9) by $V_t$, we can rewrite the full quantum constraint equations in the form

$$S_t \psi = \left(\frac{\hbar}{2} p^2_{\phi_t} + \tilde{C}_t\right) \psi = 0. \quad (14)$$

This set of $n$ Hamiltonian constraints can be combined into a single one, introducing a 'lapse' $N = \{N_t\}$ and writing

$$S(N) \psi = \sum_i N_i S_i \psi = 0, \quad \forall N. \quad (15)$$

This concludes our definition of a family of finite-dimensional inhomogeneous quantum cosmologies. We have one of these for each triangulation $\Delta_n$ of a 3-sphere. The hypothesis that we put forward is that they give an approximate description of the quantum behavior of our inhomogeneous universe, increasingly accurate with $n$. This hypothesis can be seen as following naturally from the cosmological principle.

4. ‘Dipole’ cosmology

Consider the simple case obtained by taking $n = 2$ and the natural triangulation of a 3-sphere obtained by gluing two tetrahedra by all their faces. $\Delta_2^* n$ is then the graph formed by two nodes joined by four links The gravitational variables are $(U_f, E_f)$, $f = 1, 2, 3, 4$. We have two Hamiltonian constraints, one per each node, which we call $C_1$ and $C_2$. Define

$$\tilde{C} = V_1 C_1 + V_2 C_2 \quad (16)$$

7 An interesting possibility, which we leave to the reader, is also to consider the $n = 1$ case defined by the graph $\Delta_1^* n = \square$, namely a single tetrahedron with two couples of faces identified.
and rewrite the constraints in the equivalent form

\[
S = \tilde{C} + \frac{\kappa}{2} (p_{\phi_1}^2 + p_{\phi_2}^2) \sim 0.
\]

\[
D = (V_1 C_1 - V_2 C_2) + \frac{\kappa}{2} (p_{\phi_1}^2 - p_{\phi_2}^2) \sim 0.
\]

But using (1) and (2), it is easy to see that

\[
V_1 C_1 = V_2 C_2.
\]

Therefore

\[
D \text{ reads simply } D = p_{\phi_2}^2 - p_{\phi_1}^2 \sim 0,
\]

which shows immediately that the Poisson bracket algebra between the two Hamiltonian constraints closes. This is essentially due to the fact that the two gravitational parts of the constraints coincide in this simple case. We do not know what happens with generic triangulations, or with spatial derivative terms in the scalar field.

The gravitational Hilbert space is \( L_2[SU(2)^4] \) and a basis of spin network states that solve the gauge constraint is given by the states \(|j_1, j_2, j_3, j_4, \iota_1, \iota_2\rangle \). The action of one gravitational Hamiltonian constraint on a state gives

\[
\tilde{C} |j_f, \iota_t\rangle = \sum_{j_{f'} \iota_{t'}} C_{j_{f'} j_f \iota_{t'} \iota_t} |j_{f'} \iota_{t'}\rangle.
\]

where, each term of the sum comes from one of the terms in the sum in \( f \) and \( f' \) in (13). More explicitly, we have

\[
C_{12}(j_1, j_2, j_3, j_4, \iota_1, \iota_2) = \sum_{\epsilon, \delta = \pm 1} C^{\epsilon \delta \iota_1 \iota_2}_{j_f j_{f'} \iota_t} \left( j_1 + \frac{\epsilon}{2}, j_2 + \frac{\delta}{2}, j_3, j_4, \iota_1', \iota_2' \right),
\]

because the operator \( U_{12} = U_1 U_2^{-1} \) in (13) multiplies the terms \( \Pi^b(U_1) \) and \( \Pi^b(U_2) \) and

\[
U \Pi^b(U) = \Pi^{1/2}(U) \Pi^b(U) = c_{+} \Pi^{1/2}(U) + c_{-} \Pi^{-1/2}(U).
\]

The matrix elements \( C_{j_{f'} j_f \iota_{t'} \iota_t}^{\epsilon \delta} \) can be computed with a straightforward exercise in recoupling theory from (13), and with some more algebra, from (12). In a different notation, in terms of the wavefunction components, we can write

\[
\tilde{C} \psi(j_f, \iota_t) = \sum_{\epsilon_j = 0, \pm 1} C_{j_{f'} j_f \iota_{t'} \iota_t}^{\epsilon \delta} \psi\left( j_f + \frac{\epsilon_j}{2}, \iota_t \right),
\]

where \( C_{j_{f'} j_f \iota_{t'} \iota_t}^{\epsilon \delta} \) vanishes unless \( \epsilon_j = 0 \) for two and only two of the four \( j \)'s. The scalar field variables are \( \phi_1, \phi_2 \). Taking these into account leads to the wavefunctions \( \psi(j_f, \iota_t, \phi_t) \), and (14) gives the dynamical equations

\[
\left( \frac{\partial^2}{\partial \phi_1^2} + \frac{\partial^2}{\partial \phi_2^2} \right) \psi(j_f, \iota_t, \phi_t) = \frac{2}{\kappa} \sum_{\epsilon_j = 0, \pm 1} C_{j_{f'} j_f \iota_{t'} \iota_t}^{\epsilon \delta} \psi\left( j_f + \frac{\epsilon_j}{2}, \iota_t \right).
\]

\[
\frac{\partial^2}{\partial \phi_1^2} \psi(j_f, \iota_t, \phi_t) = \frac{\partial^2}{\partial \phi_2^2} \psi(j_f, \iota_t, \phi_t).
\]

The coefficients \( C \) can be computed explicitly from recoupling theory. They vanish unless two \( \epsilon_j \)'s are zero. Equations (25, 26), defined on the Hilbert space \( H_2 = L_2[SU(2)^4/SU(2)^2] \otimes L_2[R^2] \) define a quantum cosmological model which is just one step out of homogeneity.
5. Born–Oppenheimer approximation and LQC

We now ask if and how LQC is contained in the model defined above. The state space $H_2$ contains a subspace that could be identified as a homogeneous universe. This is the subspace $H_{\text{hom}} \subset H_2$ spanned by the states $|j, j, j, j, \iota, \iota, \phi, \phi\rangle$ where $\iota$ is the eigenstate of the volume that better approximates the volume of a classical tetrahedron whose triangles have area $j$. However, the dynamical equations (25, 26) do not preserve this subspace. This is physically correct, because the inhomogeneous degrees of freedom cannot remain sharply vanishing in quantum mechanics, due to Heisenberg uncertainty. Therefore it would be incorrect to search for states that reproduce LQG exactly, within this model. In which sense then can a quantum homogeneous cosmology make sense?

On the basis of the above discussion on the cosmological principle, the answer should be clear. The cosmological principle is the hypothesis that in the theory there is a regime where the inhomogeneous degrees of freedom do not affect too much the dynamics of the homogeneous degrees of freedom, and that the state of the universe happens to be within such a regime. In other words, the homogeneous degrees of freedom can be treated as ‘heavy’ degrees of freedom, in the sense of the Born–Oppenheimer approximation, and the inhomogeneous one can be treated as ‘light’ ones. Let us therefore separate explicitly the two sets of degrees of freedom. This can be done as follows.

First, change variables from the group variables $U_f \in SU(2)$ to the algebra variables $A_f \in su(2)$, defined by $\exp A_f = U_f$. Following what is done in loop quantum cosmology [23], let us fix a fiducial $su(2)$ element $\omega_f \in su(2)$ for each face $f$. (This can be interpreted as the logarithm of the holonomy of the fiducial connection along the link dual to $f$.) We choose for simplicity a fiducial connection normalized as $|\omega_f| = 1$, and such that the four vectors $\omega_f$ are normal to the faces of a regular tetrahedron centered at the origin of $su(2) \sim R^3$. Using this, we can decompose our variables as

$$A_f = c \omega_f + a_f,$$
$$E_f = p \omega_f + h_f. \quad (27, 28)$$

We need two conditions in order to fix this decomposition uniquely. First, we require that $p$ is determined by the total volume

$$V = p^\frac{3}{2}. \quad (29)$$

Second, we require that $c$ is its conjugate variable, that is

$$\{c, p\} = \frac{8\pi G}{3}. \quad (30)$$

The variable $c$ can then be identified with the corresponding variable used in quantum cosmology. We also define $\Delta V = V_2 - V_1$, so that $V_{1,2} = \frac{1}{2} (V \pm \Delta V)$.

Inserting the decomposition described above into the quantum Hamiltonian constraint (12) gives

$$C_t = \frac{1}{2} \sum_{ff'\ell\ell'} \text{Tr}[e^{a_f - a_{f'}} e^{-a_f \ell + a_{f'} \ell'} e^{c \omega_f \ell + a_{f'} \ell'} e^{c \omega_f \ell'} e^{c \omega_{f'} \ell'} e^{c \omega_{f'} \ell} [e^{c \omega_f \ell'} + e^{c \omega_{f'} \ell'} \ell', V \pm \Delta V]]. \quad (31)$$

8 This is only a convenient rewriting of the holonomies, not really a return of the connection as main variable.

9 The definition of the variable $c$ is given here only in this implicit form. A more explicit expression for this variable would be more clarifying.
Let us now decompose this constraint into two parts, the first of which depends only on the homogeneous variable \( c \). This can be done keeping only the first term of the expansion of the exponentials in \( a_f \) and \( a_f' \), and only the \( V \) term in the volume term. That is, we write
\[
C_t = \frac{1}{2} C^{\text{hom}} + C^{\text{in}} \tag{32}
\]
where
\[
C^{\text{hom}} = \sum_{f f' f'' \in \mathcal{G}} \text{Tr}[e^{c \omega_f} e^{-c \omega_f'} e^{-c \omega_{f''}} [e^{c \omega_{f''}}, V]] = \frac{1}{V} \tilde{C}^{\text{hom}}. \tag{33}
\]
The interpretation of this split is transparent: \( C^{\text{hom}} \) gives the gravitational energy in the homogeneous degree of freedom, while \( C^{\text{in}} \) gives the sum of the energy in the inhomogeneous degrees of freedom and the interaction energy between the two sets of degrees of freedom.

Following Born and Oppenheimer, let us now make the hypothesis that the state can be rewritten in the form
\[
\psi(U_f, \phi_t) = \psi^{\text{hom}}(c, \phi) \psi^{\text{inh}}(c, \phi; a_f, \phi_-), \tag{34}
\]
where the variation of \( \psi^{\text{inh}} \) with respect to \( c \) and \( \phi \) can be neglected at the first order. Here \( \psi^{\text{hom}} \) represents the quantum state of the homogeneous cosmological variables, while \( \psi^{\text{inh}} \) represents the quantum state of the inhomogeneous fluctuations over the homogeneous background \((c, \phi)\). Inserting the Born–Oppenheimer ansatz (34) into (15), and taking \( N_1 = N_2 \), we have the equation
\[
\kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{inh}} - \tilde{C}^{\text{inh}} \psi^{\text{inh}} - \tilde{C}^{\text{inh}} \psi^{\text{hom}} \psi^{\text{inh}} = 0. \tag{35}
\]
Dividing by \( \psi^{\text{inh}} \psi^{\text{inh}} \) this gives
\[
\frac{\kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{inh}} - \kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{inh}}}{\psi^{\text{inh}}} = - \frac{\kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{inh}}}{\psi^{\text{inh}}} \psi^{\text{inh}} + \frac{\tilde{C}^{\text{inh}} \psi^{\text{inh}}}{\psi^{\text{inh}}} \psi^{\text{inh}}. \tag{36}
\]
Since the left-hand side of this equation does not depend on the inhomogeneous variables, there must be a function \( \rho(c, \phi) \) such that
\[
\kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{inh}} = - \frac{\kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{inh}}}{\psi^{\text{inh}}} \psi^{\text{inh}} + \rho \psi^{\text{inh}} = 0, \tag{37}
\]
\[
\kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{inh}} + \rho \psi^{\text{inh}}. \tag{38}
\]
The second equation is the Schrödinger equation for the inhomogeneous modes in the background homogeneous cosmology \((c, \phi)\), where \( \rho(c, \phi) \) plays the role of energy eigenvalue. The first equation is the quantum Friedmann equation for the homogeneous degrees of freedom \((c, \phi)\), corrected by the energy density \( \rho(c, \phi) \) of the inhomogeneous modes [21]. At the order zero of the approximation, where we disregard entirely the effect of the inhomogeneous modes on the homogeneous modes, we obtain
\[
\kappa^2 \frac{\partial^2}{\partial \phi^2} \psi^{\text{hom}} = - \tilde{C}^{\text{inh}} \psi^{\text{hom}}. \tag{39}
\]
Let us now analyze the action of the operator \( C^{\text{hom}} \), defined in (33). Note that \( c \) multiplies the generator of a \( U(1) \) subgroup of \( SU(2)^4 \). Therefore it is a periodic variable \( c \in [0, 4\pi] \). We can therefore expand the states \( \psi^{\text{hom}}(c, \phi) \) in Fourier sum
\[
\psi^{\text{hom}}(c, \phi) = \sum_{\mu} \psi(\mu, \phi) e^{i \mu c / 2}. \tag{40}
\]
where $\mu$ is an integer. The basis of states $\langle \psi | \mu \rangle = e^{i\mu c/2}$ in the gravitational sector of the $\psi_{\text{hom}}$’s state space satisfies
\[
p^1 |\mu\rangle = k \mu^2 |\mu\rangle \tag{41}\]
\[-4 \sin^2(c/2)|\mu\rangle = |\mu - 2|\rangle + |\mu - 2|\rangle, \tag{42}\]
which we shall use below. Here $k = \left(\frac{4\pi G\gamma}{c}\right)^2$. The homogeneous Hamiltonian constraint (33) can be rewritten as
\[
C_{\text{hom}} = \sum_{ff'f''} \text{Tr}\left[\left\langle \cos \frac{c}{2} \frac{\partial}{\partial e} + 2 \sin \frac{c}{2} \omega_f \right| \left(\cos \frac{c}{2} \frac{\partial}{\partial e} - 2 \sin \frac{c}{2} \omega_f \right) e^{-c\omega_{ff'}} |\psi_{\text{hom}}(V)\rangle\right]
= \sum_{ff'f''} \text{Tr}\left[\left(\cos \frac{c}{2} \frac{\partial}{\partial e} + 2 \sin \frac{c}{2} \cos \frac{c}{2} (\omega_f - \omega_{ff'f''}) + 4 \sin^2 \frac{c}{2} \omega_f \omega_f \right) e^{-c\omega_{ff'}} \right.
\times |\psi_{\text{hom}}(V)\rangle]. \tag{43}\]
Consider the action of the last factor on the state $|\mu\rangle$
\[
e^{-c\omega_{ff'}} |\psi_{\text{hom}}(V)\rangle e^{i\mu c/2} = p^1 e^{i\mu c/2} - e^{-c\omega_{ff'}} p^1 e^{c\omega_{ff'}} e^{i\mu c/2}
= \left(-i\frac{8\pi G\gamma}{3} \frac{\partial}{\partial e}\right)^2 e^{i\mu c/2} - e^{-c\omega_{ff'}} \left(-i\frac{8\pi G\gamma}{3} \frac{\partial}{\partial e}\right)^2 e^{i\mu(2-\omega_{ff'})}
= k \mu^2 e^{i\mu c/2} - e^{c\omega_{ff'}} k (\mu \frac{\partial}{\partial e} - i2\omega_{ff'})^2 e^{i\mu(2-\omega_{ff'})}
= k (\mu^2 \frac{\partial}{\partial e} - (\mu \frac{\partial}{\partial e} - i2\omega_{ff'})^2) e^{i\mu c/2}. \tag{44}\]
Now observe that we can write
\[
(\mu \frac{\partial}{\partial e} - i2\omega_{ff'})^2 = \alpha(\mu) \frac{\partial}{\partial e} + \beta(\mu) \omega_f, \tag{45}\]
where the coefficients $\alpha(\mu)$ and $\beta(\mu)$ can be easily computed squaring this equation. We write $\tilde{\alpha}(\mu) = \mu^2 - \alpha(\mu)$. Bringing everything together the only term that survives is
\[
C_{\text{hom}} e^{i\mu c/2} = k \left[\tilde{\alpha}(\mu) + \left(\frac{7}{4} \tilde{\alpha}(\mu) + 3\frac{1}{2} \beta(\mu) \right) \sin^2 \frac{c}{2} \right] e^{i\mu c/2}
= k \left[C^+(\mu) e^{i\mu c/2} + C^-(\mu) e^{-i\mu c/2} + C^0(\mu)\right] e^{i\mu c/2} \tag{46}\]
where $C^+(\mu) = C^-(\mu) = -\frac{7}{8} \tilde{\alpha}(\mu) - 2\sqrt{3} \beta(\mu)$ and $C^0(\mu) = \frac{11}{8} \tilde{\alpha}(\mu) + 2\sqrt{3} \beta(\mu)$. Using (42), this gives
\[
C_{\text{hom}} |\mu\rangle = k C^+(\mu) |\mu + 2\rangle + k C^0(\mu) |\mu\rangle + k C^-(\mu) |\mu - 2\rangle. \tag{47}\]
The full equation (39) can be written as
\[
C^+(\mu) \Psi(\mu + 2, \phi) + C^0(\mu) \Psi(\mu, \phi) + C^-(\mu) \Psi(\mu - 2, \phi) + \frac{k}{2k^2 \mu^{3/2}} \frac{\partial^2}{\partial \phi^2} \Psi(\mu, \phi) = 0. \tag{48}\]
where we have written $\Psi$ for $\psi_{\text{hom}}$. Equation (48) has the structure of the LQC dynamical equation. Thus, LQC appears in the zero-order Born–Oppenheimer approximation of a loop quantum gravity quantization of a finite number of degrees of freedom of the gravitational field truncated according to the approximation dictated by the cosmological principle.

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6. Conclusions and perspectives

We leave a number of questions open. In particular: the extension to real Immirzi parameter $\gamma$, the inclusion of more realistic matter fields and the detailed comparison with the LQC quantization of the homogeneous universe, in particular in relation to the $\bar{\mu}$ quantization scheme [1]. The detailed relation between the difference equation we obtain and that obtained in LQC needs to be investigated in detail. The relation with more conventional cosmological perturbation expansion (see [8]) also needs to be investigated. In particular, a hypothesis considered in this paper is that an inhomogeneous cosmology can be approximated by the model presented, with a finite number of degrees of freedom. This hypothesis can in principle be tested already in the classical domain; we have not done so in this paper and we plan to do so in the future. Another issue that we leave completely open is that of the diffeomorphism constraint. In the dipole cosmology on which this paper focuses, one may expect that this constraint does not appear, for the same reason for which it does not appear in homogeneous cosmology, that is, the model describes the gravitational field is a gauge fixed symmetric coordinate system determined by the symmetries of the configuration, and only physical degrees of freedom are left free. However, the issue becomes more relevant as the complexity of the triangulation increases, and it is strictly related to the issue of the closure of the constraints. Finally, a better understanding is required of the explicit relation between the homogeneous states $\psi(\mu, \phi)$ and the spin network basis states $\psi(j_f, \iota, \phi)$ in the inhomogeneous model. This would shed light on the construction and the corresponding approximation. The explicit relation between these states and the states in the inhomogeneous model state space is not straightforward, and will be investigated in detail elsewhere.

We observe that the discussion on the cosmological principle given in section 2 and the observation on the use of coarse triangulations in section 3 also have a bearing on the discussion on the viability of the recent computations of background-independent $n$-point functions [19]. The conclusion of the discussion was the possibility of representing a large universe in terms of a coarse triangulation, and therefore, in the quantum theory, in terms of states based on a graph with a small number of nodes. In cosmology, as in the calculation of the $n$-point functions, we observe that general relativity admits an expansion in which a region of the length scale $L$ can be approximated by neglecting wavelengths much smaller than $L$, and described by spin networks with a small number of nodes in the quantum theory. In other words, the intuition that dynamics of nearly flat space can only be described using states with a high number of nodes is misleading. If this was the case, cosmology itself would be ill conceived.

The idea of describing cosmological evolution using Regge calculus has been explored in the past. See for instance [18]. Here we have adapted this idea to quantum cosmology, where it turns out to be particularly suitable for a loop quantization. The construction in [18] indicates that it is possible to have a 4D triangulation sliced by 3D triangulations equal to one another, and therefore suggests the possibility of writing a spinfoam version of the models introduced here. For instance, consider the kinematics of the $\Delta_2$ model and let $A(j_{ab}, t_\omega)$, $a = 1, \ldots, 5$ be the vertex amplitude of a spinfoam model, as for instance that introduced in [13, 24]. We can interpolate between two $\Delta_2$ with the triangulation $\Delta_5$ defined by the boundary of a four-simplex. In fact, collapsing four of the five tetrahedra of $\Delta_5$ into a single one gives precisely $\Delta_2$. This collapse is a 4–1 Pachner move, which can be realized interpolating a four-simplex. Therefore we can have a transition $\Delta_2 \rightarrow \Delta_2$ via an intermediate $\Delta_5$. See figure 1.

Accordingly, we can write a $\Delta_2 \rightarrow \Delta_2$ transition amplitude as the $\Delta_2 \rightarrow \Delta_5 \rightarrow \Delta_2$ amplitude defined by the transition amplitude $A(j_{ab}, t_\omega)$ for each move. For instance

$$A(j_f, t_i; j'_f, t'_i) = \delta_{j_f, j'_f} \sum_{\iota_f, \iota_i} A(j_f, j_i, t_c, \iota_f, \iota_i; j'_f, t'_i) A(j_i, j_f, j_{ab}, t_c, t_2, \iota_2)$$

(49)
where \( c = 1, 2, 3 \). Repeating this step four times, over different points of the triangulation generates spacetime with the \( S_3 \times [0, 1] \) topology [18]. Is this dynamics related to the canonical one defined here?

In summary, the models presented in section 3 open a systematic way for describing the inhomogeneous degrees of freedom in quantum cosmology. In particular, they open the possibility of checking whether the bounce scenario that is characteristic of the homogeneous theory survives in an inhomogeneous context. The simplest possibility is to analyze the cosmological evolution in the simple \( \Delta_3 \) model described in section 4, defined by the two equations (25) and (26). Do these equations govern semiclassical wave packets undergoing the cosmological bounce? If the answer to the above question is positive, the solution would also provide a concrete description of the fluctuating geometry at the bounce. The state \( \psi(j_f, i_t, \phi_t = \phi_{\text{bounce}}) \) would give such a description explicitly. It is tempting to begin speculating on the possible cosmological role of the fluctuations of the inhomogeneous degrees of freedom at (or near) the bounce. Could they play a role in structure formation? For inflation?

The derivation of the structure of the LQC dynamical equation presented in section 5 sheds light on the relation between LQC and full loop quantum gravity. In particular, we have argued that the first should be searched as a Born–Oppenheimer approximation, as suggested by the hypothesis that goes under the name of cosmological principle.

Finally, we point out the existence of the term \( \rho(c, \phi) \) in the quantum Friedmann equation (28). Its physical interpretation is clear: it represents the back reaction of the quantum fluctuating inhomogeneous degrees of freedom on the dynamics of the scale factor. Again, it is tempting to begin speculating on the possible cosmological role of this energy density. Does it play a role in structure formation? For inflation? In relation to the cosmological constant?

Appendix. A simpler Hamiltonian constraint

Instead of using the form (12) of the Hamiltonian constraint, we can also use the simpler densitized form (13) which corresponds to the regularization of the Hamiltonian constraint used earlier in loop quantum gravity in [22]. Here we show that this regularization gives a simpler dynamical equation, but with the same structure as in LQC.
In the quantum theory, the decomposition (28) becomes

\[ L_f = \omega_f \frac{\partial}{\partial c} + \tilde{L}_f \]  

(A.1)

where \( \tilde{L}_f c = 0 \). Inserting this decomposition into the Hamiltonian constraint (13) gives

\[ \tilde{C}_t = \sum_{ff' \in t} \text{Tr} \left[ e^{c \omega_f - a_f} e^{-c \omega_{f'} - a_{f'}} \left( \omega_f \frac{\partial}{\partial c} + \tilde{L}_f \right) \left( \omega_{f'} \frac{\partial}{\partial c} + \tilde{L}_{f'} \right) \right]. \]  

(A.2)

As before, we decompose this constraint into two parts, the first of which depends only on the homogeneous variable \( c \). This can be done keeping only the first term of the expansion of the exponentials in \( a_f \) and \( a_{f'} \), and only the term quadratic in \( \frac{\partial}{\partial c} \) in the derivative part. That is

\[ \tilde{C}_t^{\text{hom}} = \sum_{ff' \in t} \text{Tr} \left[ e^{c \omega_f} e^{-c \omega_{f'}} \omega_f \omega_{f'} \right] \frac{\partial^2}{\partial c^2} \equiv \frac{1}{2} \tilde{C}_t^{\text{hom}}. \]  

(A.3)

This can be rewritten as

\[ \tilde{C}_t^{\text{hom}}(v, \phi) = \frac{1}{2} \left[ -\mu^2 \tilde{C}_t^{\text{hom}}(\mu + 2, \phi) + 2 \mu^2 \tilde{C}_t^{\text{hom}}(\mu, \phi) - \mu^2 \tilde{C}_t^{\text{hom}}(\mu - 2, \phi) \right]. \]  

(A.5)

Bringing everything together, the full equation (39) reads

\[ C^+(\mu) \tilde{C}_t^{\text{hom}}(\mu + 2, \phi) + C^0(\mu) \tilde{C}_t^{\text{hom}}(\mu, \phi) + C^- (\mu) \tilde{C}_t^{\text{hom}}(\mu - 2, \phi) + \frac{\partial^2}{\partial \phi^2} \tilde{C}_t^{\text{hom}}(\mu, \phi) = 0 \]  

(A.6)

where the coefficient takes the simple form \( C^0(\mu) = -\frac{1}{2} \frac{\partial^2}{\partial \phi^2} \tilde{C}_t^{\text{hom}}(\mu, \phi) \). We do not know if this simple equation gives the same phenomenology as that used in LQC.

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