SU(1, 1) Coherent States For Position-Dependent Mass Singular Oscillators

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Abstract The Schrödinger equation for position-dependent mass singular oscillators is solved by means of the factorization method and point transformations. These systems share their spectrum with the conventional singular oscillator. Ladder operators are constructed to close the $su(1, 1)$ Lie algebra and the involved point transformations are shown to preserve the structure of the Barut-Girardello and Perelomov coherent states.

1 Introduction

There has been much interest in the construction of coherent states since the origin of quantum mechanics [1]. Following Glauber, the concept is useful in the quantum description of the correlation and coherence properties of light [2]. The Glauber states deal with the dynamical properties of the harmonic oscillator and possess three basic properties: they (1) are eigenfunctions of the annihilation operator belonging to complex eigenvalues, (2) are displaced versions of the ground state wave-function and (3) minimize the uncertainty relation between position and momentum. For systems other than the harmonic oscillator, such properties are used as different definitions of the coherent states and, in general, they are not equivalent to each other. Thus, the term coherent states (CS) has been used for a wide class of mathematical objects over the years. For instance, the generalized CS studied by Barut and Girardello [3] and by Perelomov [4] are based on the first and second properties of the Glauber states. In turn, the construction of CS as minimizing the uncertainty relation of a pair of observables is incidentally found as a secondary result for some special systems. Nevertheless, the most valuable property of the CS is that they can be studied for many systems in terms of the definition leading to the desirable result.

The purpose of this work is to draw the CS applications to the study of position-dependent mass (PDM) systems in quantum mechanics. Some previous results include the CS for the harmonic [5,6] and nonlinear [7,9] oscillators. We are interested in the construction of PDM Hamiltonians such that their spectrum is exactly the same as that of the conventional singular oscillator. We shall show that the description of these new systems obeys the $su(1, 1)$ algebra. The structure of the paper is as follows. In Section 2, expressions for the mappings and the operators leading to the relevant algebras are given.
The algebraic structure of the PDM linear oscillators is revisited in Section 3. Departing from the ladder operators of the linear case, in Section 4 we construct a new class of operators and show that they are connected in a natural way to the singular oscillator. It is shown that these operators are the generators of the \( su(1, 1) \) Lie algebra (the subject studied also in [10]). In Section 5 some applications are given and the corresponding CS are constructed a la Barut-Girardello and a la Perelomov. Some final remarks close the paper.

2 Factorization and point transformations

Consider a one-dimensional quantum system of mass \( m(x) \) acted by the potential \( V(x) \). A convenient expression for the Hamiltonian reads

\[
H_a = \frac{1}{2} m^a P m^b P m^a + V, \quad 2a + 2b = -1
\]

(1)

where \( P \) and the position operator \( X \) satisfy \( [X, P] = i\hbar \). This Hamiltonian admits the factorization

\[
H_a = AB + \epsilon, \quad A = B^\dagger = -\frac{i}{\sqrt{2}} m^a P m^b + \beta,
\]

(2)

with \( \epsilon \) a constant (in energy units) to be fixed. In the position representation (\( X = x \) and \( P = -i\hbar \frac{d}{dx} \)) the function \( \beta \) is a root of the Riccati equation:

\[
V - \epsilon = \frac{\hbar}{\sqrt{2}m} \left[ 2 \left( a + \frac{1}{4} \right) \left( \frac{m'}{m} \right) \beta - \beta' \right] + \beta^2, \quad \beta' \equiv \frac{d}{dx},
\]

(3)

and the factorization operators satisfy the commutation rule

\[
[A, B] = -\frac{\hbar^2}{m^3} \left( a + \frac{1}{4} \right) \left[ mm'' - \frac{3(m')^2}{2} \right] - \frac{2\hbar}{\sqrt{2}m} \beta'.
\]

(4)

The PDM problem can be simplified using the map \( \psi \to e^{g} \varphi \) and the change of the independent variable \( x \), ruled as follows:

\[
x \mapsto y = s(x), \quad y \mapsto x = s^{-1}(y).
\]

(5)

To define \( s \) as a bijection we shall assume that \( J(D_a) \neq 0 \), with \( D_a \equiv \text{Dom}(H_a) \) and \( J = s' \) (see e.g. [11]). The combination of transformations (5) produces

\[
y = \int J(x) dx + y_0, \quad J(x) = e^{2g(x)} = \left[ \frac{m(x)}{m_0} \right]^{1/2}
\]

(6)

and

\[
\psi(x) = J^{1/2}(x) \varphi_s(y)
\]

(7)
where \( m_0 \) and \( y_0 \) are integration constants in proper units (hereafter we take \( y_0 = 0 \)). The function \( \varphi_* \) is the representation of \( \varphi \) in the \( y \)-space: \( \varphi(x) = \varphi(s^{-1}(y)) = (\varphi \circ s^{-1})(y) \equiv \varphi_*(y) \) and vice versa \( \varphi_* \circ s = \varphi \). A further simplification (see [5] for details) leads to the Schrödinger equation

\[
H^{(a)}_s \varphi_*(y) := \left[ -\left( \frac{\hbar^2}{2m_0} \right) \frac{d^2}{dy^2} + V_*(y) \right] \varphi_*(y) = E \varphi_*(y),
\]

(8)

with \( \mathcal{D}^{(a)}_s = \text{Dom}(H^{(a)}_s) \) and \( \text{Sp}(H_a) = \text{Sp}(H^{(a)}_s) \). Hence two general cases are distinguishable:

- **MDNT** (Mass-dependent null terms) The mass-function \( m(x) \) is a solution of the non-linear, second order differential equation

\[
m m'' - \left( \frac{7}{4} + a \right) (m')^2 = 0.
\]

(9)

A simple analysis shows that the roots of (9) leading to appropriate bijections \( s \) have the form

\[
m(x; a) = m_0 (x_0 + \lambda x)^{-4/(3+4a)}
\]

(10)

with \( a \) in the set

\[
\mathcal{A} = \left\{ a_0 = -1/4, a_n = \frac{1 - n}{4n} \right\}, \quad n \in \mathbb{N},
\]

(11)

and \( x_0 \) a dimensionless real constant while \( m_0 \) and \( \lambda \) are real constants expressed in mass and inverse of distance units respectively. Then the functions

\[
m_{(0)}(x) \equiv m(x; a_0) = \frac{m_0}{(x_0 + \lambda x)^2}, \quad s_{(0)}(x) = \frac{\ln(x_0 + \lambda x)}{\lambda}, \quad x \geq t_0 = -\frac{x_0}{\lambda},
\]

(12)

define an invertible mapping from \( \mathcal{D}_{a_0} \subseteq [t_0, +\infty) \) to \( \mathcal{D}^{(a_0)}_s \subseteq \mathbb{R} \) and the pair

\[
m_{(n)}(x) \equiv m(x; a_n) = m_0 (x_0 + \lambda x)^{-4n/(2n+1)}, \quad s_{(n)}(x) = \frac{(x_0 + \lambda x)^{1/(2n+1)}}{\lambda(2n+1)^{-1}}, \quad n \in \mathbb{N}
\]

(13)

corresponds to a bijection between \( \mathcal{D}_{a_n} \subseteq \mathbb{R} \) and \( \mathcal{D}^{(a_n)}_s \subseteq \mathbb{R} \).

- **MINT** (Mass-independent null terms) Given a properly defined mass-function \( m(x) \), the ordering of \( P \) and \( m(x) \) in (1) is *a priori* fixed as \( a = b = -1/4 \). For instance, the regular functions

\[
m_R(x) = \frac{m_0}{1 + (\lambda x)^2}, \quad s_R(x) = \frac{\text{arcsinh}(\lambda x)}{\lambda}, \quad \lambda \in \mathbb{R}
\]

(14)

define a bijection connecting \( \mathcal{D}_{-1/4} \subseteq \mathbb{R} \) to \( \mathcal{D}^{(-1/4)}_s \subseteq \mathbb{R} \). Another well behaved bijection (see Section [5]) is given by

\[
m_e(x) = m_0 e^{2\lambda x}, \quad s_e(x) = \frac{e^{\lambda x} - x_0}{\lambda},
\]

(15)

and connects \( \mathcal{D}_{-1/4} \subseteq \mathbb{R} \) to \( \mathcal{D}^{(-1/4)}_s \subseteq [t_0, +\infty) \).
3 Position-dependent mass linear oscillators

Let the commutator \([A, B] = -\hbar \omega_0\), with \(\omega_0\) in frequency units. The \(\beta\)-function is easily found to be

\[
\beta = \frac{\omega_0}{\sqrt{2}} \int x m^{1/2} dr - \frac{\hbar}{\sqrt{2}} \left( a + \frac{1}{4} \right) \left( \frac{m'}{m^{3/2}} \right) + \beta_0
\]

with \(\beta_0\) an integration constant which will be omitted in the sequel. The identification \(\epsilon = \hbar \omega_0 / 2\), after introducing (16) in the Riccati equation (3), leads to a very simple form of the potential in the \(y\)-representation

\[
V(x) = \frac{\omega_0^2}{2} \left( \int x m^{1/2} dr \right)^2 = \frac{m_0 \omega_0^2}{2} \left( \int J dr \right)^2 = \left( \frac{m_0 \omega_0^2}{2} \right) y^2 \equiv V_\ast(y).
\]

Thereby the transformation (6)-(7) tunes to the potentials \(V(x)\) exhibiting the equidistant energies \(\hbar \omega_0 (n + 1/2)\) of a constant mass quantum oscillator. Since \(s\) is a bijection, (17) admits another lecture: given \(V_\ast(y) = \frac{m_0 \omega_0^2}{2} y^2\), the mapping (6)-(7) leads to a potential \(V(x) = \frac{m_0 \omega_0^2}{2} (s(x))^2\) which is isospectral to the harmonic oscillator for the masses \(m(x)\) allowed by the rule \(s\).

It is convenient to introduce a dimensionless notation by taking \([E] = \hbar \omega_0\) and \([L] = \sqrt{\hbar/(m_0 \omega_0)}\) as the units of energy and distance respectively. Hence, the potential (17) reads \(V_\ast(y) = \frac{1}{2} y^2 [E]\), with \(y\) a real number such that \(y = y[L]\). From now on we drop the dimensions and use the same symbol for the physical and the dimensionless variables. We shall return to the expressions with units only if necessary. In each case, the notation will be self-consistent. The same holds for the subscript \("\ast\) labelling the representation of functions and operators in the \(y\)-space.

The introduction of (16) in (2) cancels the explicit dependence of the factorization operators \(A\) and \(B\) on the ordering label \(a\). We have

\[
A = a_+ + \frac{1}{2} \left( \frac{d}{dy} \ln J \right), \quad B = a_- - \frac{1}{2} \left( \frac{d}{dy} \ln J \right), \quad [A] = [B] = \sqrt{|E|/2}
\]

with \(a_- (a_+)\) the conventional annihilation (creation) operator of the quantum oscillator in the \(y\)-space

\[
a_+^\dagger = a_- := \frac{d}{dy} + y, \quad [a_-, a_+] = 2, \quad a_+ a_- = 2N, \quad [2N, a_\pm] = \pm 2a_\pm.
\]

Here \(N\) is the Fock’s number operator. A most convenient relationship between \(A\), \(B\) and \(a_\pm\) is easily calculated to read

\[
AJ^{1/2} = J^{1/2} a_+, \quad BJ^{1/2} = J^{1/2} a_-.
\]

Hence, the action of \(A\) and \(B\) in the \(\varphi\)-space can be established as

\[
A\psi = (J^{1/2} a_+) \varphi, \quad B\psi = (J^{1/2} a_-) \varphi.
\]
The Hamiltonian \( H_a \) is clearly isospectral to the one-dimensional quantum oscillator

\[
H_a \psi = (AB + 1) \psi = A \left( J^{1/2} a_- \right) \varphi + \psi = J^{1/2} (2N + 1) \varphi
\]

\[
= J^{1/2} \left[-\frac{d^2}{dy^2} + y^2 \right] \varphi = J^{1/2} H^{(a)} \varphi = (2n + 1) \psi,
\]

where \([H_a] = [H^{(a)}] = [E]/2\).

4 Position-dependent mass singular oscillators

Some consequences of the commutation relations (19) are that

\[
[a_-, f] = f_y, \quad [a_+, f] = -f_y, \quad [2N, f] = -f_{yy} - 2f_y \frac{d}{dy}, \quad f_y \equiv \frac{df}{dy}
\]

with \( f \) a differentiable function of the position. Moreover, since \([a_2, a_-] = 8 H^{(a)}\) we can introduce the operators

\[
c_+ := a_+^2 + f, \quad c_- := a_-^2 + f
\]

(24) to get

\[
[c_-, c_+] = 8 \left[ H^{(a)} + \left( \frac{y}{2} \right) f_y \right] := 8 h^{(a)}.
\]

(25)

The straightforward calculation shows that given \( f(y) = -\frac{g_0}{2y^2} \), with \( g_0 \) a real constant, the operators \( k_0 = h^{(a)}/4, k_\pm = c_\pm/4 \) close the \( su(1,1) \) algebra

\[
[k_-, k_+] = 2k_0, \quad [k_0, k_\pm] = \pm k_\pm.
\]

(26)

The operator \( h^{(a)} \) in (25) is the Hamiltonian of the singular oscillator:

\[
h^{(a)} = H^{(a)} + \frac{g_0}{2y^2} = -\frac{d^2}{dy^2} + y^2 + \frac{g_0}{2y^2}, \quad [g_0] = [L]^4.
\]

(27)

Remark that the potential \( V(y) = y^2 + \frac{g_0}{2y^2} \) admits an infinite point spectrum if \( g_0 > -1/2 \) (see e.g. Chs. III.18 and V.35 of [12]). Moreover, the presence of the centrifugal-like term \( \frac{g_0}{2y^2} \) constrains the domain of definition of \( h^{(a)} \) to be \( D_s^{(a)} = [0, +\infty) \), and the wave-functions of \( h^{(a)} \) are necessarily equal to zero at the origin. In this way, if \( g_0 = 0 \), only the odd linear-oscillator functions are recovered (see Section 4.1). On the other hand, from (20) and (22) one obtains the expressions for the generators in the PD M case:

\[
C_+ = A^2 - \frac{g_0}{2s^2(x)}, \quad C_- = B^2 - \frac{g_0}{2s^2(x)}, \quad h_a = H_a + \frac{g_0}{2s^2(x)}.
\]

(28)

Notice that the relationships

\[
C_\pm J^{1/2} = J^{1/2} C_\pm, \quad h_a J^{1/2} = J^{1/2} h^{(a)}.
\]

(29)

imply that the operators (28) also close the \( su(1,1) \) algebra provided that \( K_0 = h_a/4 \) and \( K_\pm = C_\pm/4 \).
4.1 Physical Solutions

Let us introduce the mappings $\varphi \rightarrow y^2 e^{-y^2/2}u(y)$ and $y^2 \rightarrow z$ so that the eigenvalue problem $h^{(a)}\varphi = E\varphi$ is mapped to the Kummer equation [13]:

$$zu_{zz} + \left(\ell + \frac{1}{2} - z\right)u_z - \frac{1}{4}(2\ell + 1 - E)u = 0, \quad \ell^2 - \ell - \frac{g_0}{2} = 0. \quad (30)$$

Then, for each $\ell_\pm = \alpha_\pm = \frac{1}{2}(1 \pm \sqrt{1+2g_0})$, we have a general expression of the form

$$\varphi_\pm = \lambda^{(1)}_\pm y^{\alpha_\pm}e^{-y^2/2}F_1(a,c,y^2) + \lambda^{(2)}_\pm y^{1-\alpha_\pm}e^{-y^2/2}F_1(\tilde{a},\tilde{c},y^2), \quad (31)$$

with $4a = 2\alpha_\pm + 1 - E$, $2c = 2\alpha_\pm + 1$, $4\tilde{a} = 3 - 2\alpha_\pm - E$ and $2\tilde{c} = 3 - 2\alpha_\pm$. The straightforward calculation shows that both of the general expressions (31) lead to the same physical solution if $-\frac{1}{2} < g_0$. In such a case we omit $\varphi_-$ and take $\lambda^{(2)}_+ = 0$, $a = -n$ and $\alpha_+ = \alpha$ to get

$$\varphi_n(y) = c_n y^\alpha e^{-y^2/2}F_1(-n, \alpha + \frac{1}{2}, y^2)$$
$$= \left(\frac{2n!}{(\alpha+n+1/2)}\right)^{1/2} y^\alpha e^{-y^2/2}L_n^{(\alpha-1/2)}(y^2), \quad n = 0, 1, 2, \ldots \quad (32)$$

where $L_n^{(\gamma)}(x)$ are the Generalized Laguerre Polynomials [14]. The set of energies is then defined by

$$E_n = 4n + 2 + \sqrt{1+2g_0} = 4n + 2\alpha + 1 = 4(k + n), \quad n = 0, 1, 2, \ldots \quad (33)$$

The change $y \mapsto y/\sqrt{2}$ in $h^{(a)}$ makes clear that the set (33) corresponds to the quantum oscillator spectrum $E_n = 2n + 1$, shifted by $\alpha - 1/2$. This last case has been discussed in [5] for $g_0 = 1/2$. On the other hand, if $g_0 = 0$ then $\alpha = 1$ and we have

$$\varphi_n(y) = \frac{1}{\pi^4\sqrt{(2n+1)!}}e^{-y^2/2}He_{2n+1}(\sqrt{2}y), \quad n = 0, 1, 2, \ldots \quad (34)$$

with $He_{2n+1}(x)$ the odd Hermite Polynomials [14] and $E_n = 4n + 3$. That is, the wavefunctions (32) are reduced to the odd-oscillator ones as $g_0 \rightarrow 0$.

5 Applications

Each of the pairs $(V(x), m(x))$, with $V(x) = s^2(x) + g_0/2s^2(x)$ describe a PDM quantum system sharing its spectrum with a particle of mass $m_0$ subject to the singular oscillator interaction. We give some examples below.

- **MDNT** For $a = a_0 \in \mathcal{A}$, we have

$$V_1(x) = \frac{\ln^2(x_0 + \lambda x)}{\lambda^2} + \frac{g_0\lambda^2}{2\ln^2(x_0 + \lambda x)}, \quad (35)$$
with the mapping from $D_{a_0}$ to $D_{a_0}^∗ = [0, +∞)$ and vice versa. On the other hand, if $a = a_n ∈ A, n ∈ N$, the potential is given by

$$V_2(x) = \left(\frac{2n + 1}{\lambda}\right)^2 (x_0 + λx)^{(2n+1)/2} + \frac{g_0}{2} \left(\frac{\lambda}{2n + 1}\right)^2 (x_0 + λx)^{-2/(2n+1)}$$

(36)

with $D_{a_n} = [t_0, +∞)$ and $D_{a_n}^∗ = [0, +∞)$ connected by (13).

- **MINT** If we fix $a = -1/4$, the transformation (14) connects $D_{-1/4} = [0, +∞)$ with $D_{a_n}^{(-1/4)} = [0, +∞)$ and leads to the potential

$$V_3(x) = \frac{\text{arcsinh}^2(λx)}{λ^2} + \frac{g_0λ^2}{2\text{arcsinh}^2(λx)}.$$

(37)

In turn, for $x_0 = 1$, transformation (15) defines the mapping from $D_{-1/4} = [0, +∞)$ to $D_{a_n}^{(-1/4)} = [0, +∞)$ and the potential

$$V_4(x) = \frac{4e^{λx}}{λ^2} \sinh^2\left(\frac{λx}{2}\right) + \left(\frac{g_0λ^2}{8}\right) e^{-λx} \cosh^2\left(\frac{λx}{2}\right), \quad x > 0.$$

(38)

In Fig. 1 the global behavior of the above defined potentials is shown for specific values of the parameters. Fig. 2 shows the eigenfunctions of potentials $V_1(x)$ and $V_4(x)$. In the former case the functions are expanded towards infinity by preserving the shape of the constant-mass ones at short distances. In the second case they are squeezed into the vicinity of the origin of coordinates. As expected, no change in the normalization is found. Details can be appreciated in Fig. 3 where one of the squeezed probability densities of $V_4(x)$ is contrasted with its constant-mass equivalent. As a final remark, the above results make clear that $κ = \frac{1}{2} (\frac{1}{2} + α)$ determines the representation of $SU(1, 1)$ we are dealing with (see e.g. [15] and [16]).

![Figure 1](image1.png)

**Figure 1:** The PDM potentials $V_1$, $V_2$, $V_3$, together with the singular oscillator of constant mass (Sing). Potential $V_2(x)$ has been depicted with $n = 1$. The first four energy levels (En) are included as reference. In all cases $g_0 = 2$, $λ = 1$ and $x_0$ is fixed to give $D_a = [0, +∞)$. Vertical and horizontal axis are respectively in $ℏω_0/2$ and dimensionless units.

### 5.1 New $SU(1, 1)$ coherent states

Similar to the case of $c_+, c_-$ and $h^{(a)}$, the operators $C_±$ intertwine the Hamiltonian $h_a$ with itself, shifted by 4 units of the energy. In other words, $C_±$ work as ladder operators
when acting on the wave-functions of $h_a$. We get $C_-$ $\psi_0 = 0$ and
\[
C_\pm \psi_n = \gamma_\pm(n) \psi_{n\pm 1}, \quad \gamma_\pm(n) = \sqrt{E_{n+1}E_n + 3 - 2g_0}.
\]
(39)

It is convenient to rewrite the coefficients $\gamma_\pm$ as follows
\[
\gamma_+(n) = 4 \sqrt{(n+1)(n+2\kappa)}, \quad \gamma_-(n) = 4 \sqrt{n(n-1+2\kappa)}.
\]
(40)

The $su(1,1)$ CS for the PDM singular oscillator are now constructed as solutions of the equation $C_- \Phi_z = z \Phi_z$, $z \in \mathbb{C}$. We get
\[
\Phi_z(x) = \left(\frac{|z|}{4}\right)^{\kappa-\frac{1}{2}} \left[\frac{J(x)}{I_{2\kappa-1}(|z|/2)}\right]^{1/2} \sum_{\ell=0}^{\infty} \frac{(z/4)^\ell}{\sqrt{\ell!\Gamma(\ell+2\kappa)}} \varphi_\ell(s(x))
\]
(41)

where $I_\nu(z)$ is the $\nu$-order modified Bessel function of the first kind [14]. The constant mass case is recovered by taking $m(x) = m_0$, then $J = 1$ and $s(x) = x$. A final change $z \mapsto 4z$ leads from (41) to the well known generalized coherent states of Barut and Girardello [3].

In a similar form we get the involved Perelomov $SU(1,1)$ coherent states:
\[
\Phi_z(x) = J^{1/2}(x)(1 - |4z|^2)^\kappa \sum_{\ell=0}^{\infty} (4z)^\ell \left[\frac{\Gamma(\ell+2\kappa)}{\ell!\Gamma(2\kappa)}\right]^{1/2} \varphi_\ell(s(x))
\]
(42)

with the same recipe to recover the classical results [4]. In the description of Perelomov, each $SU(1,1)$ CS is connected with a point in the coset space $SU(1,1)/U(1)$. 
Thereby, the construction \((42)\) corresponds to the applying of the unitary operators \(\Omega(\xi) \in SU(1, 1)/U(1)\) to the lowest state \(\psi_0\). Here \(4z = (\xi/|\xi|) \tanh |\xi|\) so that \(|4z| < 1\). The details can be consulted in \([4]\) (see also \([16]\)).

Both of the above derived sets of CS preserve the form of the constant-mass case. Important properties like the resolution of the identity are found to be similar to the conventional case. These states also evolve in time without dispersion because their energy eigenvalues are equally spaced. Moreover, the quadratures \(X(X, P) = (K_+ + K_-)/2\) and \(P(X, P) = i(K_+ - K_-)/2\) appear to satisfy the commutator \([X, P] = iK_0\). In this sense, the Barut-Girardello CS can be considered as the quadrature states minimizing the inequality relation

\[
\Delta X \Delta P \geq \frac{1}{8} |\langle h_a \rangle|.
\]

(43)

Some other PDM coherent states minimizing similar inequality relations are constructed in \([3,9]\). The relevant aspect is that the point transformations analyzed in Section \(2\) simplify the construction of the PDM coherent states to be practically the same method as in the constant-mass case.

## 6 Concluding remarks

The Darboux transformation of the singular oscillator with a varying frequency and constant mass \(V(y, t) = \omega^2(t)y^2 + \frac{g_0}{2y^2}\) has been studied in \([17]\). As a result, it was shown that the CS belonging to \(V(y, t)\) are essentially unchanged, just as the results reported here for the stationary case and masses varying with the position \(m(x)\). A combination of these approaches would be applicable in the case of a time-dependent frequency \(\omega(t)\) and a position-dependent mass \(m(x)\). It is then expected a similar result: the Barut-Girardello and Perelomov CS will preserve their global properties after the transformations. In this context, it is important to remark that the \(\beta\)-function defined in \([16]\) obeys the fact that the operators \(A\) and \(B\) are taken to fulfill the oscillator algebra \([A, B] = -\hbar \omega_0\). That is, the function in \([16]\) corresponds to a particular solution of the Riccati equation \((3)\). As it is well known, general solutions give rise to different algebras (see e.g. \([18]\) and \([19]\)). These algebras have been applied in the construction of a new kind of CS connected with the linear oscillator and its Susy-partners \([20,21]\). Quite recently, it has been shown that non-linear Susy algebras can be linearized to exhibit the Heisenberg-Weyl structure. In particular, the \(SU(1,1)\) algebra as connected with the infinite well was analyzed \([21]\). Then, the higher order Susy transformations can be also studied for the position-dependent mass systems we have presented in this work. The classical models of the harmonic oscillator and the Pöschl-Teller potentials have been also useful in the solving of mass-dependent systems \([22]\) and in the construction of CS \([23]\). Of particular interest, the study of the Wigner function gives rise to a better understanding of the PDM coherent states \([6,24]\). Further insights are in progress.

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