Energy Spectrum of Vortex Tangle

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The energy spectrum of superfluid turbulence in the absence of the normal fluid is studied numerically. In order to discuss the statistical properties, we calculated the energy spectra of the 3D velocity field induced by dilute and dense vortex tangles respectively, whose dynamics is calculated by the Biot-Savart law. In the case of a dense tangle, the slope of the energy spectrum is changed at $k = 2\pi/l$, where $l$ is the intervortex spacing. For $k > 2\pi/l$, the energy spectrum has $k^{-1}$ behavior in the same manner as the dilute vortex tangle, while otherwise the slope of the energy spectrum deviates from $k^{-1}$ behavior. We compare the behavior for $k < 2\pi/l$ with the Kolmogorov law.

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1. INTRODUCTION

The particular attention is recently focused on the similarity between superfluid turbulence and conventional turbulence. In conventional turbulence, the energy spectrum $E(k)$ has the Kolmogorov form $k^{-5/3}$ for wavenumbers $k$ in an inertial range. Such spectrum was observed indirectly by Stalp et al. in superfluid $^4$He above 1.4K, in spite of the fact that the rotational flow in the superfluid component must take the form of discrete quantized vortex lines, and that there can be no conventional viscous dissipation in the superfluid component. This is understood by the idea that the superfluid and the normal fluid are likely to be coupled together by the mutual friction and to behave like a conventional fluid. Since the density of the normal fluid is negligible at mK temperatures, an important question now arises: even free from the normal fluid, is the superfluid turbulence still similar to the conventional turbulence or not?

Recently Nore et al. solved the 3D Gross-Pitaevskii equation and discussed similarity to the Kolmogorov law. However this simulation includes
such complicated compressible effects as the radiation of sound from the vortex lines, the interaction between vortex lines and sound, etc. In order to consider pure superfluid turbulence in a simpler situation, we study the energy spectrum of the 3D velocity field induced by the vortex tangle under the vortex filament model which describes the dynamics of incompressible fluid.

2. NUMERICAL CALCULATION

For superfluid $^4$He, the vortex filament model is very useful, because the vortex core radius $a \sim 10^{-8}$cm is microscopic and the circulation $\kappa = 9.97 \times 10^{-4}$cm$^2$/sec is fixed by quantum constraint. Thus the dynamics of vortices is calculated numerically by the Biot-Savart law which is described in our previous paper. In our calculation, a vortex filament is represented by a single string of points at a distance $\Delta \xi$ apart. When two vortices approach within $\Delta \xi$, it is assumed that they are reconnected. The computational sample is taken to be a cube of size $L = 1$cm. This calculation assumes the walls to be smooth and takes account of image vortices so that the boundary condition may be satisfied.

We will introduce the energy spectrum of the velocity field induced by the vortex filament. Using Parseval’s theorem $\int d\mathbf{k} |\hat{v}(\mathbf{k})|^2 = (2\pi)^{-3} \int dr |\mathbf{v}(r)|^2$, the kinetic energy can be written as

$$E = \frac{\rho_s}{2} \int dr |\mathbf{v}_s(r)|^2 = \frac{\rho_s}{2} (2\pi)^3 \int d\mathbf{k} |\hat{v}(\mathbf{k})|^2, \quad (1)$$

where $\rho_s$ is the superfluid density. By using the relation $\hat{v}(\mathbf{k}) = i\mathbf{k} \times \hat{\omega}(\mathbf{k})/|\mathbf{k}|^2$, we obtain the kinetic energy:

$$E = \frac{\rho_s}{2} (2\pi)^3 \int d\mathbf{k} \frac{|\hat{\omega}(\mathbf{k})|^2}{|\mathbf{k}|^2}, \quad (2)$$

where $\hat{\omega}(\mathbf{k})$ is the Fourier component of the vorticity $\omega(r) = \nabla \times \mathbf{v}(r)$. In the vortex filament model, the vorticity can be defined as $\omega(r) = \kappa \int d\xi s'(\xi) \delta(s(\xi) - r)$, so that $\hat{\omega}(\mathbf{k})$ is given by

$$\hat{\omega}(\mathbf{k}) = \frac{\kappa}{(2\pi)^3} \int d\xi e^{-i\mathbf{s}(\xi) \cdot \mathbf{k}} s'(\xi). \quad (3)$$

A vortex filament is represented by the parametric form $\mathbf{s} = s(\xi, t)$, where $\mathbf{s}$ refers to a point on the filament, the prime denotes differentiation with respect to the arc length $\xi$ and the integration is taken along the filament.
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The energy spectrum $E(k)$ is defined as $E = \int_0^\infty dk E(k)$. Finally we obtain the energy spectrum:

$$E(k) = \frac{\rho s k^2}{2(2\pi)^3} \int \frac{d\Omega_k}{|k|^2} \int d\xi_1 d\xi_2 s'(\xi_1) \cdot s'(\xi_2) e^{-i k \cdot (s(\xi_1) - s(\xi_2))},$$

where $d\Omega_k$ denotes the volume element $k^2 \sin \theta_k d\theta_k d\phi_k$ in the spherical coordinate. The energy spectrum $E(k)$ is calculated for the vortex configuration $s(\xi)$ obtained by the simulation of the dynamics.

3. ENERGY SPECTRUM OF VORTEX TANGLE

In order to make clear the statistical properties of superfluid turbulence, we calculated the energy spectra of the 3D velocity field for dilute and dense vortex tangles respectively.

First we discuss the energy spectrum for a dilute vortex tangle. This calculation of the dynamics is made by the space resolution $\Delta \xi = 4.58 \times 10^{-3}$ cm and the time resolution $\Delta t = 6.25 \times 10^{-5}$ sec. Figure 1(a) shows the initial configuration of four vortices. Four rings move toward the center of the cube by their self-induced velocity to make the reconnection. After the reconnection the four rings move outside oppositely (Fig. 1(b), (c), (d)). The time evolution of the energy spectrum is shown in Fig. 2. The slope of the spectra is not strongly affected by the vortex configuration. The energy spectrum $E(k)$ is proportional to $k^{-1}$, which comes from the velocity field of an isolated vortex.

Next the energy spectrum for a dense vortex tangle is discussed. This calculation of the dynamics is made by the resolution $\Delta \xi = 1.83 \times 10^{-2}$ and $\Delta t = 1.0 \times 10^{-3}$ sec. Figure 3(a) shows the initial configuration of the vortex tangle where the direction of the circulation of upper vortices is opposite to that of lower vortices (Taylor-Green vortex). The vortices become tangled through lots of reconnections (Fig. 3(b), (c) and (d)). The energy spectra are shown in Fig. 4. In Fig. 4(a) the spectrum may reflect an artifact of the initial configuration. However through the chaotic dynamics, these spectra reach the equilibration (Fig. 4(d)). Figure 5 shows that the time averaged spectrum of five configurations around $t = 73.0$ sec. The slope is changed at $k \approx 20$ cm$^{-1}$ and at $k = 2\pi/l$, where $l$ is the intervortex spacing. For $k > 2\pi/l$, the energy spectrum has $k^{-1}$ behavior in the same manner as the dilute vortex tangle, by the contribution of the isolated vortex lines. For $2\pi/L < k < 20$ cm$^{-1}$, the spectrum is proportional to $k^{-5/2}$ by the effect of boundary. The spectrum for $20$ cm$^{-1} < k < 2\pi/l$ is associated with velocity field at scales larger than the intervortex spacing, and this slope is very similar to the Kolmogorov law.
Fig. 1. Collision of four rings at $t=0$ sec(a), $t=6.25$ sec(b), $t=12.50$ sec(c) and $t=18.75$ sec(d). At $t=6.12$ sec, two vortices reconnect.

Fig. 2. Evolution of the energy spectrum of four vortices: $t=0$ sec(solid), $t=6.25$ sec(long-dashed), $t=12.50$ sec(dot-dashed) and $t=18.75$ sec(dashed).
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Fig. 3. Time evolution of the vortex tangle at $t=0$ sec(a), $t=30.0$ sec(b), $t=50.0$ sec(c) and $t=73.0$ sec(d).

Fig. 4. The energy spectra of the vortex tangle at $t=0$ sec(a), $t=30.0$ sec(b), $t=50.0$ sec(c) and $t=73.0$ sec(d).
4. CONCLUSIONS

This work studies numerically the energy spectrum of the superfluid turbulence without the mutual friction. For \( k > 2\pi/l \), the spectrum can be attributed to the contribution of the individual vortex lines. For \( 2\pi/L < k < 20 \text{ cm}^{-1} \), the spectrum depends on the boundary. In the intermediate range \( 20 \text{ cm}^{-1} < k < 2\pi/l \), the slope of the spectrum is very similar to the Kolmogorov law. The calculation of the spectrum of a denser tangle is in progress so that the intermediate range may be extended.

REFERENCES

1. W.F. Vinen, Phys. Rev. B 61, 1410 (2000).
2. S.R. Stalp, L. Skrbek and R.J. Donnelly, Phys. Rev. Lett. 82, 4831 (1999).
3. C. Nore, M. Abid and M.E. Brachet, Phys. Rev. Lett. 78, 3896 (1997); Phys. Fluids 9, 2644 (1997).
4. C.F. Barenghi and D.C. Samuels, Phys. Rev. B 60, 1252 (1999).
5. M. Tsubota, T. Araki and S.K. Nemirovskii, Phys. Rev. B 62, 11751 (2000); T. Araki and M. Tsubota, J. Low Temp. Phys. 121, 405 (2000).
6. D. Kivotides, J.C. Vassilicos, D.C. Samuels and C.F. Barenghi, Phys. Rev. Lett. 86, 3080 (2001).