Diagonalization of Certain Integral Operators II*

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Abstract

We establish an integral representations of a right inverse $s$ of the Askey-Wilson finite difference operator in an $L^2$ space weighted by the weight function of the continuous $q$-Jacobi polynomials. We characterize the eigenvalues of this integral operator and prove a $q$-analog of the expansion of $e^{ixy}$ in Jacobi polynomials of argument $x$. We also outline a general procedure of finding integral representations for inverses of linear operators.

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1 Introduction.

The Askey-Wilson divided difference operator $D_q$, \( \mathbb{H} \) is defined by

\[
(D_q f)(x) = \frac{\delta_q f(x)}{i(q^{1/2} - q^{-1/2}) \sin \theta}, \quad x = \cos \theta,
\]

where

\[
(\delta_q g)(e^{i\theta}) = g(q^{1/2}e^{i\theta}) - g(q^{-1/2}e^{i\theta}).
\]

Observe that \( i(q^{1/2} - q^{-1/2}) \sin \theta \) which appears in the denominator of (1.1) is \( \delta_q x \), \( x \) being the identity map: \( x \mapsto x \) evaluated at \( x \). Magnus [17] showed how the Askey-Wilson operator arises naturally from divided difference operators.

It is easy to that

\[
D_q T_n(x) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} U_{n-1}(x),
\]

where \( T_n(x) \) and \( U_n(x) \) are Chebyshev polynomials of the first and second kinds, respectively. Therefore \( D_q \) maps a polynomial of degree \( n \) to a polynomial of degree \( n - 1 \). As such \( D_q \) resembles the differential operator. It was observed in [5] and [15] that one can construct integral operators which are a right inverse to \( D_q \) on certain weighted spaces. In [15] Ismail and Zhang diagonalized the right inverse to \( D_q \) on weighted \( L_2 \) space on \([-1, 1]\) with Jacobi weights \((1 - x)\alpha (1 + x)\beta\) or \( q \)-ultraspherical weights

\[
w_{\beta; q}(x) = \frac{1}{\sqrt{1 - x^2}} \prod_{n=0}^{\infty} \frac{1 - 2(2x^2 - 1)q^n + q^{2n}}{1 - 2(2x^2 - 1)q^{n+\nu} + q^{2n+2\nu}}, \quad \beta = q^\nu, \; \nu > 0.
\]

Recall the notations

\[
(a; q)_0 := 1, \quad (a; q)_n := \prod_{j=1}^{n} (1 - a q^j), \quad n = 1, 2, \ldots, \text{ or } \infty,
\]

\[
(a_1, \ldots, a_m; q)_n = \prod_{k=1}^{m} (a_k; q)_n,
\]

for \( q \)-shifted factorials. We shall normally drop “; \( q \)” from the shifted factorials in (1.3) and (1.4) when this does not lead to any confusion. Thus for the purpose of this paper we will use

\[
(a)_n := (a; q)_n
\]
A basic hypergeometric series is

\[
\tau \phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right| q, z \right) = \tau \phi_s (a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)
\]

\[
:= \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n \left[ (-1)^n q^{-n(n-1)/2} \right]^{1+s-r}.
\]

A $q$-analog of Jacobi polynomials is given by

\[
P_n^{(\alpha, \beta)}(x; q) = \frac{(q^{\alpha+1}, -q^{\beta+1}; q)_n}{(q, -q; q)_n} \tau \phi_3 \left( \begin{array}{c} q^{-n}, q^{n+\alpha+\beta+1}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta} \\ q^{\alpha+1}, -q^{\beta+1}, -q \end{array} \right| q, q),
\]

where $x = \cos \theta$. The $P_n$'s are called the continuous $q$-Jacobi polynomials. In what follows we assume

\[
e^{i\theta} = x + \sqrt{x^2 - 1}, \quad e^{-i\theta} = x - \sqrt{x^2 - 1},
\]

where the sign of the square root in (1.8) is taken so that $\sqrt{x^2 - 1} \approx x$ as $x \to \infty$ in the complex plane. The normalization in (1.7) was introduced by Rahman in [18]. The original normalization used by Askey and Wilson in [13] is $P_n^{(\alpha, \beta)}(x|q)$, where

\[
P_n^{(\alpha, \beta)}(x; q) = \frac{(-q^{\alpha+\beta+1}; q)_n}{(-q; q)_n} q^{-an} P_n^{(\alpha, \beta)}(x|q^2).
\]

Both $\{P_n^{(\alpha, \beta)}(x; q)\}$ and $\{P_n^{(\alpha, \beta)}(x|q)\}$ are called continuous $q$-Jacobi polynomials because they are orthogonal with respect to an absolutely continuous measure. The orthogonality relation is

\[
\int_{-1}^{1} P_m^{(\alpha, \beta)}(x|q) P_n^{(\alpha, \beta)}(x|q) w_{\alpha,\beta}(x|q) \, dx = h_n^{(\alpha, \beta)}(q) \delta_{mn}.
\]

The weight function $w_{\alpha,\beta}(x|q)$ and the normalization constants $h_n^{(\alpha, \beta)}(q)$ are given by [13], (7.5.28), (7.5.30), (7.5.31).

\[
w_{\alpha,\beta}(\cos \theta|q^2) = \frac{(e^{2i\theta}, e^{-2i\theta}; q^2)_\infty (1 - x^2)^{-1/2}}{(q^{\alpha+1/2}e^{i\theta}, q^{\alpha+1/2}e^{-i\theta}, -q^{\beta+1/2}e^{i\theta}, -q^{\beta+1/2}e^{-i\theta}; q)_\infty},
\]

\[
h_n^{(\alpha, \beta)}(q) = \frac{2\pi (1 - q^{\alpha+\beta+1})(q^{(\alpha+\beta+2)/2}, q^{(\alpha+\beta+3)/2}; q)_\infty}{(q, q^{\alpha+1}, q^{\beta+1}, -q^{(\alpha+\beta+1)/2}, -q^{(\alpha+\beta+2)/2}; q)_\infty}
\]

\[
\cdot \frac{(q^{\alpha+1}, q^{\beta+1}, -q^{(\alpha+\beta+3)/2}; q)_n q^{n(2\alpha+1)/2}}{(1 - q^{2n+\alpha+\beta+1})(q, q^{\alpha+\beta+1}, -q^{(\alpha+\beta+1)/2}; q)_n}.
\]
Note that (7.5.31) in \[10\] contains a misprint where \(q^{n(2\alpha+1)/4}\) on the right-hand side should read \(q^n(2\alpha+1)/2^n\).

The Askey-Wilson operator acts on continuous \(q\)-Jacobi polynomials in a very natural way. It’s action is given by, \[10, (7.7.7)\],

\[
D_q f_n(x) = \frac{2q^{-n+2\alpha+5}/4(1-q^{\alpha+\beta+n+1})}{(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2})(1-q)} P^{(\alpha+1,\beta+1)}_{n-1}(x/q).
\]

Following \[13\] we define \(D_q\) densely on \(L_2[w_{\alpha,\beta}(x)|q]\) by

\[
D_q f \sim \sum_{n=1}^{\infty} \frac{2q^{-n+2\alpha+5}/4(1-q^{\alpha+\beta+n+1})}{(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2})(1-q)} f_n P^{(\alpha+1,\beta+1)}_{n-1}(x/q),
\]

if

\[
f \sim \sum_{n=0}^{\infty} f_n P^{(\alpha,\beta)}_n(x/q).
\]

Clearly \(D_q\) as defined by (1.14) and (1.15) maps a dense subset of \(L_2[w_{\alpha,\beta}(x)|q]\) into \(L_2[w_{\alpha+1,\beta+1}(x)|q]\). We are interested in finding a formal inverse to \(D_q\), that is we seek a linear operator \(T_{\alpha,\beta; q}\) which maps \(L_2[w_{\alpha+1,\beta+1}(x)|q]\) into \(L_2[w_{\alpha,\beta}(x)|q]\) such that \(D_q T_{\alpha,\beta; q}\) is the identity map on the range of \(D_q\). It is clear from (1.14) and (1.15) that we may require \(T_{\alpha,\beta; q}\) to satisfy

\[
(T_{\alpha,\beta; q} g)(x) \sim \sum_{n=0}^{\infty} \frac{(1-q)(-q^{(\alpha+\beta+1)/2}q^{1/2})}{2(1-q^{\alpha+\beta+n+2})} g_n P^{(\alpha,\beta)}_n(x/q)
\]

for

\[
g(x) \sim \sum_{n=0}^{\infty} g_n P^{(\alpha+1,\beta+1)}_n(x|q).
\]

It is easy to find a representation of \(T_{\alpha,\beta; q}\) as an integral operator. We use the orthogonality relation (1.10) to write \(g_n\) as

\[
\int_{-1}^{1} g(x) w_{\alpha+1,\beta+1}(x|q) P^{(\alpha+1,\beta+1)}_n(x|q) \, dx / h^{(\alpha+1,\beta+1)}_n(q)
\]

and formally interchange summation and integration in (1.16). The result is the formal definition

\[
(T_{\alpha,\beta; q} g)(x) = \int_{-1}^{1} K_{\alpha,\beta; q}(x,y) \, g(y) \, w_{\alpha+1,\beta+1}(y|q) \, dy,
\]
and the kernel \( K_{\alpha,\beta;q}(x, y) \) is given by

\[
K_{\alpha,\beta;q}(x, y) = \sum_{n=0}^{\infty} \frac{(1 - q)(-q^{(\alpha+\beta+1)/2}; q^{1/2})_n}{2(1 - q^{\alpha+\beta+n+2})h_n^{(\alpha+1,\beta+1)}(q)} q^{-n(2\alpha+1)/4} P_n^{(\alpha,\beta)}(x|q) P_n^{(\alpha+1,\beta+1)}(y|q) \tag{1.19}
\]

The purpose of this work is to study the spectral properties of the integral operator (1.18). It is worth noting that \( T_{\alpha,\beta;q} \) is linear but is not normal and a spectral theory of such operators is not readily available. Our main result is Theorem 3.1 which characterizes the eigenvalues of \( T_{\alpha,\beta;q} \) as zeros of a certain transcendental function. In order to prove Theorem 3.1 we proved several auxiliary results which may be of interest by themselves. First in §2 we solve the connection coefficient problem of expressing \( P_n^{(\alpha,\beta)}(x) \) in terms of \( \{P_j^{(\alpha+1,\beta+1)}(x)\} \). The solution of this connection coefficient problem is then used to expand \( w_{\alpha+1,\beta+1}(x|q) P_n^{(\alpha+1,\beta+1)}(x|q) \) in terms of \( \{w_{\alpha,\beta}(x|q) P_k^{(\alpha,\beta)}(x)\} \).

In Section 3 we used the latter expansion to find a tridiagonal matrix representation of \( T_{\alpha,\beta;q} \). In Section 3 we also find the eigenvalues and eigenfunctions \( g(x|\lambda) \) so that

\[ T_{\alpha,\beta;q} g = \lambda g. \]

The eigenvalues are multiples of the reciprocals of the zeros of a transcendental function \( X_{-1}^{(\alpha,\beta)}(1/x) \) defined in (5.13). Such a function is a \( q \)-analog of the confluent hypergeometric function \( _1F_1 \). The eigenfunctions are shown to have the orthogonal expansion

\[
\sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x; q) \tag{1.20}
\]

such that

\[
\sum_{n=1}^{\infty} h_n^{(\alpha,\beta)}(q) |a_n(\lambda|q)|^2 < \infty. \tag{1.21}
\]

We may normalize the eigenfunctions by choosing \( a_1(\lambda|q) = 1 \). With this normalization we prove, in Section 3, that \( a_n(\lambda|q) \) is a polynomial of degree \( n \). The \( a_n \)'s are \( q \)-analogs of polynomials studied by Walter Gautschi [11] and Jet Wimp [23]. An explicit formula for \( a_n(\lambda|q) \) is given in Section 4, see (3.11) and (4.1). The large \( n \) asymptotics of \( a_n(\lambda|q) \) in different parts of the complex plane are found in Section 5 and they are used to characterize the \( \lambda \)'s for which (1.21) holds.

In Section 5 we note that when \( \alpha \) and \( \beta \) are complex conjugates and are not real then the polynomials \( i^{-n} a_n(ix|q) \) are real orthogonal polynomials. They are orthogonal with respect to a discrete measure supported at the zeros of a transcendental function which we denoted by \( F_L(\eta, \rho; q) \). The function \( F_L(\eta, \rho; q) \) is a \( q \)-analog of the regular Coulomb wave function \( F_L(\eta, \rho), [1] \) Chapter 14]. Our analysis implies that the functions \( F_L(\eta, \rho; q) \) have only real and simple zeros. Neither the functions \( F_L(\eta, \rho; q) \) nor their zeros seem to have been studied before this work.
In Section 6 we prove that the eigenfunctions are constant multiples of the $q$-exponential function $E_q(x; -i, b)$, where

\begin{equation}
E_q(x; a, b) := \sum_{n=0}^{\infty} \frac{q^n}{(q)_n} \left( aq^{(1-n)/2} e^{i\theta}, aq^{(1-n)/2} e^{-i\theta} \right) b^n.
\end{equation}

The function $E_q$ was introduced in [13]. Since the eigenfunctions also have the orthogonal expansion (1.20) we obtain an identity valid on the spectrum of $T_{a,\beta; q}$. This identity is (6.13) and is a $q$-analog of

\begin{equation}
e^{ixy} = e^{-iy} \sum_{n=0}^{\infty} \frac{(\alpha + \beta + 1)^n}{(\alpha + \beta + 1)_{2n}} (2i)^n \left( \frac{n + \beta + 1}{2n + \alpha + \beta + 2} \right) \left( aq^{(1-n)/2} e^{i\theta}, aq^{(1-n)/2} e^{-i\theta} \right) P_n^{(a,\beta)}(x), \quad -1 < x < 1,
\end{equation}

see (10.20.4) in [3]. In Section 6 we use properties of basic hypergeometric functions to show that the above mentioned $q$-identity holds also off the spectrum of $T_{a,\beta; q}$. In Section 7 we give a second proof of (6.13) using a technique similar to what was used in [13] to prove the same result for the continuous $q$-ultraspherical polynomials. We also include in §7 a formal approach to finding the spectrum of certain integral operators of the type considered in this paper. In §8 we include some remarks on asymptotic results of Schwartz [20], Dickinson, Pollack and Wannier [7] and general remarks on this work.

In many of our calculations we found it advantageous to follow [10]. The only disadvantage is that we have to introduce some additional relations. Recall that the Askey-Wilson polynomials are defined by [10, (7.5.2)]

\begin{equation}
p_n(x; a, b, c, d|q) = (ab, ac, ad; q)_n a^{-n} \phi_3 \left( q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \left| q; q \right. \right).
\end{equation}

Their orthogonality relation is [10, (7.5.15), (7.5.16)], [4]

\begin{equation}
\int_{-1}^{1} \frac{h(x; 1, -1, q^{1/2}, -q^{1/2})}{h(x; a, b, c, d)} p_n(x; a, b, c, d) p_m(x; a, b, c, d) \frac{dx}{\sqrt{1-x^2}} = \kappa(a, b, c, d|q) \delta_{m,n},
\end{equation}

with

\begin{equation}
h(\cos \theta; a_1, a_2, a_3, a_4) := \prod_{j=1}^{4} (a_j e^{i\theta}, a_j e^{-i\theta})_{\infty},
\end{equation}

\begin{equation}
\delta_{m,n},
\end{equation}

with
and

\begin{equation}
\kappa(a, b, c, d|q) = 2\pi(abcd)_\infty/[(q, ab, ac, ad, bc, bd, cd)_\infty].
\end{equation}

We shall also use the notation

\begin{equation}
\hbar(\cos \theta; a) := (ae^{i\theta}, ae^{-i\theta})_\infty.
\end{equation}

Note that

\begin{equation}
\hbar(\cos \theta; 1, -1, q^{1/2}, q^{-1/2}) = (e^{2i\theta}, e^{-2i\theta})_\infty.
\end{equation}

Note also that the continuous $q$-Jacobi polynomials correspond to the identification of parameters

\begin{equation}
a = q^{(2\alpha+1)/4}, \quad b = q^{(2\alpha+3)/4}, \quad c = -q^{(2\beta+1)/4}, \quad d = -q^{(2\beta+3)/4}.
\end{equation}

In fact the polynomials $P_n^{(\alpha, \beta)}(x|q)$ of (1.7) have the alternate representation,

\begin{equation}
P_n^{(\alpha, \beta)}(x|q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} 4\phi_3 \begin{pmatrix}
q^{-n}, q^{n+\alpha+\beta+1}, q^{(2\alpha+1)/4} e^{i\theta}, q^{(2\alpha+1)/4} e^{-i\theta} \\
q^{\alpha+1}, q^{(\alpha+\beta+1)/2}, q^{(\alpha+\beta+2)/2}
\end{pmatrix}. \\
\left( q; q \right).
\end{equation}
2 Connection Coefficients

In this section we derive a $q$-analogue of the formula

\[
(2.1) \quad (1 - x^2)P_{n-1}^{(\alpha+1,\beta+1)}(x) = \frac{4(n + \alpha)(n + \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 1)}P_{n-1}^{(\alpha,\beta)}(x)
+ \frac{4n(\alpha - \beta)}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)}P_n^{(\alpha,\beta)}(x) - \frac{4(n + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}P_{n+1}^{(\alpha,\beta)}(x).
\]

Formula (2.1) is (1) in §37 of [19]. The Jacobi polynomials \( \{P_n^{(\alpha,\beta)}(x)\} \) are orthogonal with respect to \((1 - x)^\alpha(1 + x)^\beta\) and (2.1) is essentially the expansion of \((1 - x)^{\alpha+1}(1 + x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)\) in terms of \(\{(1 - x)^\alpha(1 + x)^\beta P_j^{(\alpha,\beta)}(x)\}_{j=0}^\infty\). The $q$-analogue of this question is to expand \(w_{\alpha+1,\beta+1}(x;q)P_{n-1}^{(\alpha+1,\beta+1)}(x;q)\) in terms of \(\{w_{\alpha,\beta}(x;q)P_j^{(\alpha,\beta)}(x;q)\}_{j=0}^\infty\). It is worth mentioning that the latter problem is equivalent to the expression of \(P_n^{(\alpha,\beta)}(x|q)\) in terms of \(\{P_j^{(\alpha+1,\beta+1)}(x|q)\}_{j=0}^\infty\).

This follows from the following known observation. If \(\{p_n(x; \lambda)\}\) are orthonormal with respect to a weight function \(w(x; \lambda)\) then the connection coefficient formula

\[
p_n(x; \lambda) = \sum_{j=0}^n c_{n,j}(\lambda, \mu)p_j(x; \mu)
\]

holds if and only if its dual, namely

\[
w(x; \mu)p_n(x; \mu) \sim \sum_{j=n}^\infty c_{j,n}(\lambda, \mu)w(x; \lambda)p_j(x; \lambda),
\]

holds. This latter fact follows from computing the Fourier coefficients of both sides.

The main result of this section is the following theorem

**Theorem 2.1** We have

\[
(2.2) \quad (1 - 2xq^{\alpha+1/2} + q^{2\alpha+1})(1 + 2xq^{\beta+1/2} + q^{2\beta+1})P_{n-1}^{(\alpha+1,\beta+1)}(x;q)
= \frac{(1 + q^{\alpha+\beta+n})(1 + q^{\alpha+\beta+n+1})(1 + q^{\alpha+n})(1 + q^{\beta+n})(1 - q^{\alpha+n})(1 - q^{\beta+n})}{(1 - q^{2n+\alpha+\beta})(1 - q^{2n+\alpha+\beta+1})}P_{n-1}^{(\alpha,\beta)}(x;q)
+ \frac{(1 + q^{\alpha+\beta+n+1})(1 + q^{\alpha+\beta+2n+1})(1 + q^n)(1 - q^{\alpha-\beta})(1 - q^n)^2(1 - q^{\alpha-\beta})}{(1 - q^{2n+\alpha+\beta})(1 - q^{2n+\alpha+\beta+2})}q^{\beta}P_n^{(\alpha,\beta)}(x;q) - \frac{(1 + q^n)^2(1 + q^{n+1})(1 - q^{n+1})(1 - q^{n+1})(1 - q^{n+1})}{(1 - q^{2n+\alpha+\beta+1})(1 - q^{2n+\alpha+\beta+2})}q^{\alpha+\beta}P_{n+1}^{(\alpha,\beta)}(x;q).
\]
For our purposes it is convenient to express (2.2) in the Askey-Wilson normalization $P_n^{(\alpha,\beta)}(x|q)$. The result is

\begin{equation}
(1 - 2xq^{\alpha+1/2} + q^{2\alpha+1})(1 + 2xq^{\beta+1/2} + q^{2\beta+1})P_n^{(\alpha+1,\beta+1)}(x|q^2)
= \frac{(1 - q^{2n+2})(1 - q^{2\beta+2n})(-q^{\alpha+\beta+1}; q)q^{n-1}P_n^{(\alpha,\beta)}(x|q^2)}{(q^{2n+\alpha+\beta}; q)q^{n-1}}
+ \frac{(-q^{\alpha+\beta+1}; q)_2(1 + q^{\alpha+\beta+2n})(1 - q^{2n})(1 - q^{\alpha-\beta})q^{\beta-\alpha+n-1}P_n^{(\alpha,\beta)}(x|q^2)}{(q^{2n+\alpha+\beta+1}; q)q^{n-1}}
- \frac{(-q^{\alpha+\beta+1}; q)_2(1 - q^{2n})(1 - q^{2n+2})q^{\beta-\alpha+n-1}P_n^{(\alpha,\beta)}(x|q^2)}{(q^{2n+\alpha+\beta+1}; q)q^{n-1}}.
\end{equation}

The rest of this section will be devoted to proving (2.2). Our proof is very technical and the reader who is more conceptually oriented is advised to turn to Section 3.

Our proof uses the Sears transformation [15, (III.15)]

\begin{equation}
4\phi_3\left( \begin{array}{c} q^{-n}, a, b, c \\ d, e, f \end{array} \mid q, q \right) = \frac{(e/a, f/a)_n}{(e, f)_n} q^n 4\phi_3\left( \begin{array}{c} q^{-n}, a, d/b, d/c \\ d, aq^{1-n}/c, aq^{1-n}/f \end{array} \mid q, q \right),
\end{equation}

where $def = abcq^{1-n}$. Note that the parameters $q^{-n}, a$ and $d$ remain invariant under the transformation (2.4).

We now proceed with the proof. We seek a connection coefficient formula of the type

\begin{equation}
(1 - 2b \cos \theta + b^2)(1 + 2c \cos \theta + c^2)4\phi_3\left( \begin{array}{c} q^{1-n}, bcq^{n+1}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta} \\ bq^{3/2}, -cq^{3/2}, -q \end{array} \mid q, q \right)
= \sum_{k=0}^{n+1} A_k 4\phi_3\left( \begin{array}{c} q^{-k}, bcq^k, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta} \\ bq^{1/2}, -cq^{1/2}, -q \end{array} \mid q, q \right),
\end{equation}

where $b := q^{\alpha+1/2}, c := q^{\beta+1/2}$.

The orthogonality relation (1.23) gives

\begin{equation}
A_k \kappa(q^{1/2}, b, -c, -q^{1/2}) \frac{(1 - bc)(q, -bc, -b\sqrt{q}, c\sqrt{q})_k}{(1 - bcq^{2k})(bc, -q, -c\sqrt{q}, b\sqrt{q})_k} q^k
= \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{h(\cos \theta; \sqrt{q}, bq, -cq, -\sqrt{q})}
\cdot 4\phi_3\left( \begin{array}{c} q^{-k}, bcq^k, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta} \\ bq^{1/2}, -cq^{1/2}, -q \end{array} \mid q, q \right).
\end{equation}
\begin{align*}
\cdot & \, \phi_3 \left( \begin{array}{c}
q^{1-n} \, bcq^{n+1},
q^{1/2}e^{i\theta},
q^{1/2}e^{-i\theta}
\end{array} \begin{array}{c}
bq^{3/2},
-cq^{3/2},
-q
\end{array} \begin{array}{c}
q,
q
\end{array} \right) d\theta.
\end{align*}

Apply the Sears transformation (2.4) with invariant parameters \(q^{-k}, bcq^k\) and \(-q\) to the first \(\phi_3\) in the above equation. The result is

\begin{align*}
\kappa(q^{1/2}, b, -c, -q^{1/2}) & \left( -q^k(1 - bc)(q, -bc) \right)_{Ak} \\
& = \sum_{r=0}^{k} \frac{(q^{-k}, bcq^k)_r}{(q, -q, cq^{1/2}, -bq^{1/2})_r} q^r \\
& \cdot \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta})_\infty}{h(\cos \theta; q^{1/2}, bq, -cq, -q^{1/2})} \, d\theta.
\end{align*}

The integral on the right-hand side of the formula (2.5) is

\begin{align*}
2\pi & (bcq^{r+s+3})_\infty \left( q, bq^{s+3/2}, -cq^{s+3/2}, -q^{r+s+1}, -bcq^2, -bq^{r+3/2}, cq^{r+3/2} \right)_\infty,
\end{align*}

which can be written as

\begin{align*}
\kappa(q^{1/2}, b, -c, -q^{1/2}) & \left( 1 + bc \right) \left( 1 + bcq \right) \left( 1 - qb^2 \right) \left( 1 - qc^2 \right) \\
& \cdot (bq^{3/2}, cq^{3/2})_r \left( bq^{3/2}, -cq^{3/2} \right)_s (-q)_{r+s}.
\end{align*}

Therefore (2.5) leads to

\begin{align*}
\frac{(1 - bcq)(1 - bcq^2)}{(-bc)_2(1 - qb^2)(1 - qc^2)} & \frac{(1 - bc)(q, -bc)_k}{(1 - bcq^{2k})(bc, -q)_k} (-q)^k A_k \\
& = \sum_{r=0}^{k} \frac{(q^{-k}, bcq^{k}, -bq^{3/2}, cq^{3/2})_r}{(q, bcq^2, cq^{1/2}, -bq^{1/2})_r} q^r \phi_2 \left( \begin{array}{c}
q^{1-n},
bcq^{n+1},
-q^{r+1}
\end{array} \begin{array}{c}
bcq^{r+3},
-q
\end{array} \begin{array}{c}
q,
q
\end{array} \right).
\end{align*}

The above \(\phi_2\) can be summed by the \(q\)-analog of the Pfaff-Saalschütz theorem, (II.12) in [10]. It’s sum is

\begin{align*}
\frac{(-bcq^2, q^{2+r-n})_{n-1}}{(-q^{1-n}, bcq^{r+3})_{n-1}}
\end{align*}
which clearly vanishes if \( r \leq n - 2 \). Thus \( A_k = 0 \) if \( k \leq n - 2 \). When \( k \geq n - 1 \), replace \( r \) by \( r + n - 1 \) in (2.6) and simplify the result to see that the right-hand side of (2.6) is

\[
\frac{(-bcq^2, q^{-k}, bcq^k, -bq^{3/2}, cq^{3/2})_{n-1}}{(-q^{1-n}, bcq^3, cq^{1/2}, -bq^{1/2})_{n-1}} q^{n-1} (bcq^{n+2})_{n-1}^{-1} \quad \phi_3
\]

When \( k = n - 1 \) the above \( \phi_3 \) is 1. If \( k = n, n + 1 \), the aforementioned \( \phi_3 \) has only 1, 2 terms; respectively. After some simplification we find

\[
A_n = \frac{(1 - bq^{1/2})(1 + cq^{1/2})(1 + bcq^{2n})(1 + bcq^n)(1 + q^n)(1 - b/c)}{(1 - bcq^{2n-1})(1 - bcq^{2n+1})} cq^{-1/2},
\]

\[
A_{n+1} = \frac{-(1 - bq^{1/2})(1 + cq^{1/2})(1 - bq^{n+1/2})(1 + cq^{n+1/2})(1 + q^n)(1 + q^{n+1})}{(1 - bcq^{2n})(1 - bcq^{2n+1})} (bcq^{-1}),
\]

\[
A_{n-1} = \frac{(1 - bq^{1/2})(1 + cq^{1/2})(1 + bq^{n-1/2})(1 - cq^{n-1/2})(1 + bcq^{n-1})(1 + bcq^n)}{(1 - bcq^{2n-1})(1 - bcq^{2n})}.
\]

Using the dual relationships mentioned at the beginning of this section we see that Theorem 2.1 is equivalent to the following theorem.

**Theorem 2.2** The connection coefficient formula

\[
P_{n}^{(\alpha, \beta)}(x|q) = \frac{q^{-n/2}(1 - q^{\alpha+n+1})(1 - q^{\alpha+n+2})}{(-q^{(\alpha+\beta+1)/2}; q^{1/2})_2(1 - q^{n+(\alpha+\beta+1)/2})(1 - q^{n+(\alpha+\beta+2)/2})} P_{n}^{(\alpha+1, \beta+1)}(x|q)
\]

\[
+ \frac{q^{(\alpha+\beta+2-n)/2}(1 - q^{\alpha+n+1})(1 + q^{n+(\alpha+\beta+1)/2})(1 - q^{(\alpha-\beta)/2})}{(-q^{(\alpha+\beta+1)/2}; q^{1/2})_2(1 - q^{n+(\alpha+\beta)/2})(1 - q^{n+(\alpha+\beta+2)/2})} P_{n-1}^{(\alpha+1, \beta+1)}(x|q)
\]

\[
- \frac{q^{(3\alpha+\beta+4-n)/2}(1 - q^{\alpha+n})(1 - q^{\beta+n})}{(-q^{(\alpha+\beta+1)/2}; q^{1/2})_2(1 - q^{n+(\alpha+\beta)/2})(1 - q^{n+(\alpha+\beta+1)/2})} P_{n-2}^{(\alpha+1, \beta+1)}(x|q)
\]

holds.
3 An Eigenvalue Problem

In this section we characterize the eigenvalue and the eigenfunction of the eigenvalue problem

\[(T_{\alpha,\beta;q}g)(x) = \lambda g(x).\]

This characterization is stated as Theorem 3.1 at the end of the present section.

It is tacitly assumed in (3.1) that \(g\) belongs to the domain of \(T_{\alpha,\beta;q}\) and \(\lambda g\) belongs to its range.

Now assume

\[g(x) := g(x; \lambda|q) \sim \sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q).\]

Since \(g \in L^2[w_{\alpha,\beta}(x; q)]\) then (1.10) implies

\[\sum_{n=0}^{\infty} h_n^{(\alpha,\beta)}(q)|a_n(\lambda|q)|^2 < \infty.\]

The condition (3.3) and the eigenvalue equation (3.1) will characterize the eigenvalues \(\lambda\) and the eigenfunctions \(g\).

It is clear that

\[\lambda \sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q)\]

\[= \sum_{n=1}^{\infty} a_n(\lambda|q) \int_{-1}^{1} w_{\alpha+1,\beta+1}(x; q) K^{(\alpha,\beta)}(x, t) P_n^{(\alpha,\beta)}(t|q) \, dt\]

\[= \frac{(q, q^{\alpha+2}, q^{\beta+2}, -q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}})_\infty (1-q)}{4\pi (q^{\frac{\alpha+2}{2}}, q^{\frac{\beta+2}{2}})_\infty (1-q^{\alpha+\beta+3})} q^{-\frac{1}{2}(2\alpha+1)}\]

\[\cdot \sum_{n=1}^{\infty} a_n(\lambda|q) \sum_{k=0}^{\infty} \frac{1 - q^{\alpha+\beta+3+2k}}{1 - q^{\alpha+\beta+2+k}} \frac{(q^{\alpha+\beta+3}, q^{\frac{\alpha+\beta+3}{2}})_k}{(q^{\alpha+2}, q^{\beta+2}, -q^{\frac{\alpha+\beta+3}{2}})_k} q^{-\frac{1}{2}(2\alpha+1)} P_n^{(\alpha,\beta)}(x|q)\]

\[\cdot \int_{-1}^{1} w_{\alpha+1,\beta+1}(x; q) P_{n+1}^{(\alpha+1,\beta+1)}(t|q) P_n^{(\alpha,\beta)}(t|q) \, dt.\]

The next step is to evaluate the integral on the extreme right-hand side of (3.4). This will lead to a three term recurrence relation satisfied by the \(a_n\)'s. By (2.3), the integral on the right, denoted \(I_{k,n}\) with \(k\) replaced by \(k - 1\), is

\[I_{k,n} = \int_{-1}^{1} w(t; q^{\alpha/2+1/4}, q^{\alpha/2+3/4}, -q^{3/2+1/4}, -q^{3/2+3/4})\]
Substituting this in (3.5) we find that

\[ \lambda \sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q) \]

\[ = \frac{1 - q^{\alpha+k}}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} \left\{ \frac{1 - q^{\alpha+k}}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} q^{\frac{k-1}{2}} P_{k-1}^{(\alpha,\beta)}(t|q) \right\} \]

\[ + \frac{1 + q^{\frac{\alpha+2}{2}}}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} \left\{ \frac{(1 - q^{\alpha+k})(1 - q^{\beta+k})}{(1 - q^{\alpha+1})(1 - q^{\alpha+2})} q^{\frac{k-1}{2}} P_{k-1}^{(\alpha,\beta)}(t|q) \right\} \]

\[ + \frac{(1 - q^{k})(1 - q^{k+1})}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} \left\{ \frac{1 - q^{\alpha+k}}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} q^{\frac{k-1}{2}} P_{k+1}^{(\alpha,\beta)}(t|q) \right\} \]

\[ + \frac{(1 - q^{k})(1 - q^{k+1})}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} \left\{ \frac{1 - q^{\alpha+k}}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} q^{\frac{k-1}{2}} P_{k+1}^{(\alpha,\beta)}(t|q) \right\} \]

Now the orthogonality relation (1.10) is equivalent to

\[ \int_{-1}^{1} w(t; q^{\alpha/2+1/4}, q^{\alpha/2+3/4}, -q^{3/2+1/4}, -q^{3/2+3/4}) P_n^{(\alpha,\beta)}(t|q) P_j^{(\alpha,\beta)}(t|q) dt \]

\[ = \frac{2\pi(q^{\alpha+1}, q^{\alpha+3}; q)_{\infty}}{(q, q^{\alpha+1}, q^{\beta+1}, -q^{\alpha+1}, -q^{\alpha+2}; q)_{\infty}} \]

\[ \frac{(1 - q^{\alpha+\beta+1})(1 - q^{\alpha+1}, q^{\beta+1}, -q^{\alpha+\beta+3}; q)_{n}}{(1 - q^{\alpha+\beta+1+2n})(1 - q^{\alpha+\beta+1+2n}; q)_{n}} q^{\frac{1}{2}(2\alpha+1)} \delta_{n,j}. \]

Substituting this in (3.5) we find that

\[ I_{k,n} = \frac{2\pi(q^{\alpha+1}, q^{\alpha+3}; q)_{\infty}}{(q, q^{\alpha+1}, q^{\beta+1}, -q^{\alpha+1}, -q^{\alpha+2}; q)_{\infty}} \]

\[ \cdot \frac{(1 - q^{\alpha+\beta+1})(1 - q^{\alpha+1}, q^{\beta+1}, -q^{\alpha+\beta+3}; q)_{n}}{(1 - q^{\alpha+\beta+1+2n})(1 - q^{\alpha+\beta+1+2n}; q)_{n}} q^{\frac{1}{2}(2\alpha+1)} \delta_{n,k-1} \]

\[ + \frac{(1 + q^{\frac{\alpha+2}{2}})(1 - q^{\alpha+k})(1 - q^{\beta+k})}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} q^{\frac{k-1}{2}} \delta_{n,k} \]

\[ - \frac{(1 - q^{k})(1 - q^{k+1})}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} \delta_{n,k+1} \]

From (3.4) and (3.6) we have

\[ \lambda \sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q) \]

\[ = \frac{(1 - q)(1 - q^{\alpha+1})(1 - q^{\alpha+2})}{2(1 - q^{\alpha+1})(1 - q^{\alpha+2})} q^{\frac{1}{2}(2\alpha+1)} (1 + q^{\alpha+\beta+1})(1 + q^{\alpha+\beta+2}) \]
The recurrence relation for $P_n^{(\alpha,\beta)}(x|q)$ can be written as:

$$
\{ \begin{align*}
\sum_{k=1}^{\infty} \frac{1-q^{\alpha+\beta+1+2k}}{1-q^{\alpha+\beta+1+k}} & \left( (1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k}) \right) & \lambda \sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q) = \sum_{k=1}^{\infty} P_k^{(\alpha,\beta)}(x|q) \left[ \frac{(1-q)(1-q^{\alpha+\beta+k})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} a_{k+1}(\lambda|q) \right] \\
\lambda \sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q) = & \sum_{k=1}^{\infty} P_k^{(\alpha,\beta)}(x|q) \left[ \frac{(1-q)(1-q^{\alpha+\beta+k})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} a_{k+1}(\lambda|q) \right] \\
& + \frac{(1-q)(1-q^{\alpha+\beta+1+k})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} a_k(\lambda|q) \\
& - \frac{(1-q)(1-q^{\alpha+\beta+1+k})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} a_{k+1}(\lambda|q) \\
& + \frac{(1-q)(1-q^{\alpha+\beta+k+1})(1-q^{\beta+k+1})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} q^{(3\alpha+\beta+k+1)/2} a_{k+1}(\lambda|q) \\
& - \frac{(1-q)(1-q^{\alpha+\beta+k+1})(1-q^{\beta+k+1})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} q^{(3\alpha+\beta+k+1)/2} a_{k+1}(\lambda|q) \\
& - \frac{(1-q)(1-q^{\alpha+\beta+k+1})(1-q^{\beta+k+1})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} q^{(3\alpha+\beta+k+1)/2} a_{k+1}(\lambda|q) \\
& - \frac{(1-q)(1-q^{\alpha+\beta+k+1})(1-q^{\beta+k+1})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} q^{(3\alpha+\beta+k+1)/2} a_{k+1}(\lambda|q) \end{align*} \right\}
$$

After some simplification we find

$$
\lambda \sum_{n=1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q) = \sum_{k=1}^{\infty} P_k^{(\alpha,\beta)}(x|q) \left[ \frac{(1-q)(1-q^{\alpha+\beta+k})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} a_{k+1}(\lambda|q) \right]
$$

By equating the coefficients of $P_n^{(\alpha,\beta)}(x|q)$ on both sides of (3.8) we establish the following three-term recurrence relation for $a_n$'s

$$
\lambda \sum_{n=1}^{\infty} a_n(\lambda|q) q^{\frac{\alpha+\beta}{2}+\frac{k}{2}} = \frac{(1-q)(1-q^{\alpha+\beta+k+1})(1-q^{\beta+k+1})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} q^{(3\alpha+\beta+k+1)/2} a_{k+1}(\lambda|q)
$$

$$
- \frac{(1-q)(1-q^{\alpha+\beta+1+k})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} q^{\frac{\alpha+\beta+k}{2}} a_k(\lambda|q)
$$

$$
- \frac{(1-q)(1-q^{\alpha+\beta+k+1})(1-q^{\beta+k+1})}{2(1-q^{\alpha+\beta+1+k})(1-q^{\alpha+\beta+1+2k})} q^{\frac{3\alpha+\beta+k+1}{2}} a_{k+1}(\lambda|q), \quad k > 0.
$$
The limiting case \( q \to 1^- \) of the recursion relation (3.9) is

\[
(3.10) \quad -\lambda a_k(\lambda) = \frac{2(\alpha + 1 + k)(\beta + 1 + k)}{(\alpha + \beta + 2 + k)(\alpha + \beta + 2 + 2k)^2} a_{k+1}(\lambda) + \frac{2(\beta - \alpha)}{(\alpha + \beta + 2k)(\alpha + \beta + 2 + 2k)} a_k(\lambda) - \frac{2(\alpha + \beta + k)}{(\alpha + \beta + 2k - 1)(\alpha + \beta + 2k)} a_{k-1}(\lambda).
\]

for \( k > 0 \), which is (4.16) of [13].

It is clear from (3.3) that \( a_0(\lambda|q) = 0 \) and that \( a_1(\lambda|q) \) is arbitrary. It is also clear from (3.3) that \( a_k(\lambda|q)/a_1(\lambda|q) \) is a polynomial in \( \lambda \) of degree \( k - 1 \). It is more convenient to renormalize \( a_k(\lambda|q)/a_1(\lambda|q) \) in terms of monic polynomials. Thus we set

\[
(3.11) \quad a_{k+1}(\lambda|q) = \left(\frac{q^{\alpha+\beta+2}, q^{\frac{\alpha+\beta+4}{2}}, q^{\frac{\alpha+\beta+5}{2}}}{(q^{\alpha+2}, q^{\beta+2}; q)_k}\right)(-1)^k b_k(\lambda) 2\lambda q^{1/2} (1 - q)^{-(k^2/4 + (\alpha + \beta + 1 + k))}.
\]

There is no loss of generality in taking \( b_0(2\lambda/(1 - q)) = 1 \). In terms of the \( b_n \)'s, (3.8) becomes

\[
(3.12) \quad b_{k+1}(\mu) = b_k(\mu) \left[ \mu + \frac{(1 - q^{-\alpha})}{(1 - q^{-\alpha + 1 + k})(1 - q^{-\alpha + 1/2 + k})} \right] + \frac{(1 - q^{\alpha + 1 + k})(1 - q^{\beta + 1 + k})q^{\frac{\alpha + \beta + 1}{2}}}{(1 - q^{\frac{\alpha + \beta + 1}{2} + k})(1 - q^{\frac{\alpha + \beta + 2}{2} + k})} b_{k-1}(\mu),
\]

where

\[
(3.13) \quad \mu = \frac{2\lambda q^{1/2}}{1 - q}, \quad b_{-1}(\mu) = 0, \quad b_0(\mu) = 1.
\]

In Section 5 we shall determine the large \( n \) behavior of the polynomials \( b_n(x) \) and \( a_n(\lambda|q) \). These asymptotic results will be used to prove the following theorem.

**Theorem 3.1** The eigenvalue problem (3.1)-(3.3) has a countable infinite number of eigenvalues. The eigenvalues are \((1 - q)/2\) times the reciprocals of the roots of the transcendental equation

\[
(3.14) \quad (-p^{\alpha + 3/2}(1 - q)x/2; p)_{\infty} \Phi_1 \begin{pmatrix} p^{\alpha + 1}, (1 - q)x p^{1/2}/2 \mid p^{\beta + 1} \end{pmatrix} = 0, \quad p := q^{1/2}.
\]

Furthermore \( \lambda = 0 \) is not an eigenvalue and the eigenspaces are one dimensional.

**Theorem 3.2** Any eigenfunction \( g(x; \lambda|q) \) corresponding to an eigenvalue \( \lambda \) is a constant multiple of \( \mathcal{E}_q(x; -i, \lambda) \).
4 A q-Analog of Wimp’s Polynomials.

In this section we find an explicit solution of (3.12).

Theorem 4.1 The polynomial \( \{b_n(x)\} \) generated by (3.12) and (3.13) are given by

\[
(4.1) \quad b_n(\mu) = \sum_{j=0}^{n} \frac{(p^{-\beta-n-1}, -p^{-\alpha-n-1}; p)_j}{(p, p^{-2n-\alpha-\beta-2}; p)_j} (-1)^j p^{j/2} \mu^{n-j} \\
\cdot \Phi_3 \left( \begin{array}{c} p^{-j}, p^{2n+\alpha+\beta+3-j}, p^{\beta+1}, -p^{\alpha+1} \\ p^{\alpha+\beta+2}, p^{\alpha+\beta+2-j}, -p^{\alpha+n+2-j} \end{array} \right| p, p \right),
\]

where

\[
(4.2) \quad p := q^{1/2}.
\]

Proof. From (4.1) it is clear that \( b_0(\mu) = 1 \) and that \( b_1(\mu) \) satisfies (3.12) when \( k = 0 \). Since the solution of the initial value problem (3.12)-(3.13) must be unique, all we need to do is to verify that the right-hand side of (4.1) satisfies (3.12). The actual process of this verification is rather long and tedious.

First, we shall rewrite (4.1) in the form

\[
(4.3) \quad b_n(\mu) = \sum_{j=0}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} (-1)^{j+k} p^{j/2} \mu^{n-j},
\]

where

\[
(4.4) \quad A_k = \frac{(p^{\beta+1}, -p^{\alpha+1}; p)_k}{(p, p^{\alpha+\beta+2}; p)_k}, \quad B_k^{(n)} = \frac{(p^{-\beta-n-1}, -p^{-\alpha-n-1}; p)_k}{(p, p^{-2n-\alpha-\beta-2}; p)_k}.
\]

Since \( A_0 = B_0^{(n)} = 1 \), we then have

\[
(4.5) \quad b_{n+1}(\mu) = \mu^{n+1} + \sum_{j=1}^{n+1} \sum_{k=0}^{j} A_k B_{j-k}^{(n+1)} (-1)^{j+k} p^{j/2} \mu^{n+1-j}
\]

\[
= \mu \left[ b_n(\mu) - \sum_{j=1}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} (-1)^{j+k} p^{j/2} \mu^{n-j} + \sum_{j=0}^{n+1} \sum_{k=0}^{j} A_k B_{j-k}^{(n+1)} (-1)^{j+k} p^{j/2} \mu^{n+1-j} \right],
\]

\[16\]
where the last line is obtained by separating $\mu^n$ from the rest of the series on the right-hand side of (4.1). Our first aim is to bring the same factor of $b_n(\mu)$ as is shown in (3.12), so we make a further separation of the series on (4.5) and find that

$$b_{n+1}(\mu) = b_n(\mu) \left[ \mu + \frac{(1 - p^{\beta-\alpha})(1 + p^{\alpha+\beta+3+2n})}{(1 - p^{\alpha+\beta+2+2n})(1 - p^{\alpha+\beta+2+4n})} p^{\alpha+n+3/2} \right] + c_n(\mu),$$

where

$$c_n(\mu) = \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n)} (-1)^{j+k} p^{\frac{j+1}{2}} \mu^{n-j} - \sum_{j=0}^{n} \sum_{k=0}^{j+1} A_k B_{j+1-k}^{(n+1)} (-1)^{j+k} p^{\frac{j+1}{2}} \mu^{n-j} - \frac{(1 - p^{\beta-\alpha})(1 + p^{\alpha+\beta+3+2n})p^{\alpha+n+1}}{(1 - p^{\alpha+\beta+2+2n})(1 - p^{\alpha+\beta+2+4n})} \sum_{j=0}^{n} \sum_{k=0}^{j} A_k B_{j-k}^{(n)} (-1)^{j+k} p^{\frac{j+1}{2}} \mu^{n-j}.$$

The rest of the exercise is to show that $c_n(\mu)$ is actually a multiple of $b_{n-1}(\mu)$, the same multiple as in (3.12). The coefficient of $\mu^n$ in $c_n(\mu)$ is

$$p^{j/2} \sum_{k=0}^{j+1} [B_{1-k}^{(n)} - B_{1-k}^{(n+1)}] (-1)^k A_k - \frac{(1 - p^{\beta-\alpha})(1 + p^{\alpha+\beta+3+2n})p^{\alpha+n+3/2}}{(1 - p^{\alpha+\beta+2+2n})(1 - p^{\alpha+\beta+2+4n})} \sum_{j=0}^{n} \sum_{k=0}^{j+1} (-1)^{j+k} A_k B_{j+1-k}^{(n)} p^{(j+2)/2} \mu^{n-j-1} + d_n(\mu),$$

which vanishes by the use of (4.4), verifying that $c_n(\mu)$ is a polynomial of degree $n - 1$ in $\mu$. We thus have

$$c_n(\mu) = \frac{(1 + p^{\alpha+\beta+3+2n})(1 - p^{\beta-\alpha})p^{\alpha+n+1}}{(1 - p^{\alpha+\beta+2+2n})(1 + p^{\alpha+\beta+4+2n})} \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} (-1)^{j+k} A_k B_{j+1-k}^{(n)} p^{(j+2)/2} \mu^{n-j-1} + d_n(\mu),$$

where

$$d_n(\mu) := \sum_{j=0}^{n-2} \sum_{k=0}^{j+2} (-1)^{j+k+1} A_k B_{j+2-k}^{(n)} p^{(j+2)/2} \mu^{n-j-1} - \sum_{j=0}^{n-1} \sum_{k=0}^{j+2} (-1)^{j+k+1} A_k B_{j+2-k}^{(n+1)} p^{(j+2)/2} \mu^{n-j-1}.$$

Since

$$B_{j+2-k}^{(n)} - B_{j+2-k}^{(n+1)} = \frac{(p^{-\beta-n-1}; p^{-\alpha-n-1}; p)_{j+1-k}}{(p; p)_{j+1-k}(p^{-2n-\alpha-\beta-4}; p)_{j+4-k}} \cdot \left[ p^{-2n-\alpha-\beta-3}(1 - p^{j-k+1})(1 - p^{-2n-\alpha-\beta-4}) - p^{-n-\beta-2}(1 - p^{\beta-\alpha})(1 - p^{-4n-2\alpha-2\beta-5+j-k}) \right],$$

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we find that the coefficient of \( \mu^{n-1-j} \) in \( d_n(\mu) \) is, for

\[
0 \leq j \leq n - 2,
\]

\[
(4.11) \quad \frac{(1 - p^{-\beta-n-1})(1 + p^{-\alpha-n-1})p^{-2n-\alpha-\beta-2}}{(p^{-2n-\alpha-\beta-3}; p)_3} \sum_{k=0}^{j} A_k B_{j-k}^{(n-1)} (-1)^{j+k} p^{j/2} - (1 - p^{\beta-\alpha}) p^{-\beta-n-2}
\]

\[
\cdot \sum_{k=0}^{j+1} A_k (1 - p^{-4n-2\beta-5+j-k}) (-1)^{j+k} p^{(j+2)/2} \frac{(p^{-\beta-n-1}, -p^{-\alpha-n-1}; p)_{j+1-k}}{(p; p)_{j+1-k}(p^{-2n-\alpha-\beta-4}; p)_{j+4-k}}.
\]

Now we combine the second series in (4.11) with the coefficients of \( \mu^{n-1-j} \) in the first series on the right-hand side of (4.8) which, by virtue of the identity

\[
\frac{(1 + p^{\alpha+\beta+3+2n})p^{\alpha+n+1}}{(1 - p^{\alpha+\beta+2+2n})(1 - p^{\alpha+\beta+4+2n})(p^{-2n-\alpha-\beta-2}; p)_{j+1-k}} - \frac{p^{-\beta-n-2}(1 - p^{-4n-2\beta-5+j-k})}{(p^{-2n-\alpha-\beta-4}; p)_{j+4-k}}
\]

results in the series

\[
- \frac{(1 - p^{\beta-n})(1 - p^{-\beta-n-1})(1 + p^{-\alpha-n-1})p^{-\beta-n-1}}{(1 - p^{2n+\alpha+\beta+2})(p^{-2n-\alpha-\beta-3}; p)_3} \sum_{k=0}^{j} A_k B_{j-k}^{(n-1)} (-1)^{k+j} p^{j/2}.
\]

Adding this to the first series in (4.11) we find, after a straightforward calculation, that the coefficient of \( \mu^{n-j-1} \), \( 0 \leq j \leq n - 2 \) in \( c_n(\mu) \) of (4.8) is

\[
(4.12) \quad \frac{(1 - p^{2\alpha+2n+2})(1 - p^{2\beta+2n+2})p^{\alpha+\beta+2n+2}}{(1 - p^{2n+\alpha+\beta+2})(1 - p^{2n+\alpha+\beta+1})(1 - p^{2n+\alpha+\beta+2})^2} \sum_{k=0}^{j} A_k B_{j-k}^{(n-1)} (-1)^{k+j} p^{j/2}.
\]

Finally, collecting the \( j = n - 1 \) term from the series in (4.8) and (4.9) we find that the constant term in (4.8) is given by

\[
(4.13) \quad \frac{(1 - p^{\beta-\alpha})(1 + p^{\alpha+\beta+2n+3})p^{\alpha+n+1}}{(1 - p^{2n+\alpha+\beta+2})(1 - p^{2n+\alpha+\beta+4})} \sum_{k=0}^{n} A_k B_{n-k}^{(n)} (-1)^{n+k-1} p^{(n+1)/2}
\]

\[
- \sum_{k=0}^{n+1} A_k B_{n+1-k}^{(n+1)} (-1)^{n+k} p^{(n+1)/2} =: f_n,
\]

say. By (4.4), we get

\[
(4.14) \quad f_n = \frac{(p^{-\beta-n-1}, -p^{-\alpha-n-1}; p)^{n}}{(p; p)_{n+1}(p^{-2n-\alpha-\beta-4}; p)_{n+2}}(-1)^{n-1} p^{(n+1)/2}
\]
Combining (4.12) and (4.17) we find that
\[ c \]
where
\[ (4.15) \]
To simplify the expression on the right side of (4.14) we first denote
\[ \phi_n := 4\phi_3 \left( \begin{array}{c|c} p^{-n}, p^{n+\alpha+\beta+3}, p^{\beta+1}, -p^{\alpha+1} \\ \hline p^{\alpha+\beta+2}, p^{\beta+2}, -p^{\alpha+2} \end{array} \right| p, p \]
and then use the contiguous relation of Askey and Wilson, see [10, Ex 7.5]:
\[ (4.15) \quad \phi_{n+1} = -\frac{B}{A} \phi_n - \frac{C}{A} \phi_{n-1}, \]
where
\[ (4.16) \quad A = p^{\alpha+\beta-3n+4}(1 - p^{n+\alpha+\beta+3})(1 - p^{2n+\alpha+\beta+2})(1 - p^{n+\alpha+\beta+2}) \cdot (1 - p^{n+\beta+2})(1 + p^{n+\alpha+2}), \]
\[ C = -p^{2\alpha+2\beta+6-3n}(1 - p^n)(1 - p^{2n+\alpha+\beta+4})(1 - p^{n+1})(1 - p^{n+\alpha+1})(1 + p^{n+\beta+1}), \]
\[ B = -C - A + p^{\alpha+\beta-3n+4}(p^{2n+\alpha+\beta+2})^3(1 - p^{\beta+1})(1 + p^{\alpha+1}). \]
Substituting (4.15) in (4.14) we find after some simplification that the coefficients of \( \phi_n \) cancel out, so that
\[ (4.17) \quad f_n = -\frac{C}{A} (-1)^{n-1} p^{(n+1)/2} \frac{(p^{-\beta-n-2}, -p^{-\alpha-n-2}; p)_{n+1}}{(p, p^{2n-\alpha-\beta-4}; p)_{n+1}} \phi_{n-1} \]
\[ = \frac{(1 - p^{2\alpha+2n+2})(1 - p^{2\beta+2n+2})p^{\alpha+\beta+2n+2}}{(1 - p^{2n+\alpha+\beta+3})(1 - p^{2n+\alpha+\beta+1})(1 - p^{2n+\alpha+\beta+2})^2} \sum_{k=0}^{n-1} A_k B_{n-1-k}^{(n-1)} (-1)^{k+n-1} p^{(n-1)/2}. \]
Combining (4.12) and (4.17) we find that \( c_0(\mu) \) is the same as the second term on the right side of (3.12) (with \( k \) replaced by \( n \)). This completes the proof of Theorem 4.1.
5 Properties of \( \{b_n(x)\} \).

In this section we derive asymptotic formulas for the polynomials \( \{b_n(x)\} \) in different parts of the complex \( x \)-plane. We also investigate a closely related set of orthogonal polynomials and record their associated continued \( J \)-fraction.

Recall that we normalized \( \{b_n(x)\} \) of (4.1) by

\[
(5.1) \quad b_0(x) := 1.
\]

To exhibit the dependence of \( b_n(x) \) on the parameters \( \alpha \) and \( \beta \) we shall use the notation \( b_n^{(\alpha,\beta)}(x) \) instead of \( b_n(x) \). Our first result concerns the limiting behavior of \( \{b_n^{(\alpha,\beta)}(x)\} \).

**Theorem 5.1** The limiting relation

\[
(5.2) \quad \lim_{n \to \infty} x^n b_n^{(\alpha,\beta)}(1/x) = \frac{(p^{\alpha+1}, -p^{\alpha+3/2}; p)_{\infty}}{(p^{\alpha+\beta+2}; p)_{\infty}} 2\phi_1 \left( \begin{array}{c} p^{\alpha+1}, p^{1/2}x \\ p^{\alpha+3/2}x \end{array} \bigg| p, p^{\beta+1} \right),
\]

holds uniformly on compact subsets of the complex \( x \)-plane.

**Proof.** We may use Tannery’s theorem (the discrete version of the Lebesgue bounded convergence theorem) to let \( n \to \infty \) in (4.1) after multiplying it by \( x^{-n} \). Thus the sequence \( \{x^{-n} b_n^{(\alpha,\beta)}(x)\} \) will have a finite limit if the series

\[
\sum_{j=0}^{\infty} (-x)^{-j} p^{j/2} \sum_{k=0}^{j} \left( \begin{array}{c} p^{-j}, p^{\beta+1}, -p^{\alpha+1} \\ 0, p^{\alpha+\beta+2} \end{array} \bigg| p, p \right)
\]

converges. Therefore

\[
(5.3) \quad \lim_{n \to \infty} x^{-n} b_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{\infty} (-x)^{-j} p^{j/2} \sum_{k=0}^{j} \left( \begin{array}{c} p^{-j}, p^{\beta+1}, -p^{\alpha+1} \\ p^{\alpha+\beta+2}, p \end{array} \bigg| p, p \right) p^k,
\]

if the right-hand side exists. In the above sum interchange the \( j \) and \( k \) sums and replace \( j \) by \( j+k \) to see that the right-hand side of (5.3) is

\[
\sum_{k=0}^{\infty} \left( \begin{array}{c} p^{\beta+1}, -p^{\alpha+1}; p \end{array} \bigg| p^{\alpha+\beta+2}, p \right) x^{-k} p^{k/2} \sum_{j=0}^{\infty} (-x)^{-j} p^{j/2}.
\]

The \( j \) sum is \( (p^{1/2}/x; p)_{\infty} \) by Euler’s sum [10, (II.2)]. This shows that

\[
(5.4) \quad \lim_{n \to \infty} x^{-n} b_n^{(\alpha,\beta)}(x) = (p^{1/2}/x; p)_{\infty} 2\phi_1 \left( \begin{array}{c} -p^{\alpha+1}, p^{\beta+1} \\ p^{\alpha+\beta+2} \end{array} \bigg| p, p^{1/2}/x \right),
\]

\[20\]
uniformly on compact subsets of the open disc \( \{ x : |x| < p^{1/2} \} \). On the other hand from Theorem 8.1 we know that

\[
\lim_{n \to \infty} x^n b_n^{(\alpha, \beta)}(1/x)
\]

exists uniformly on compact subsets of the complex plane and is an entire function of \( x \). The Heine transformation, [10, (III.1)]

\[
(5.5) \quad 2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} 2\phi_1(c/b, az; q, b),
\]

implies

\[
(p^{1/2}/x; p)_\infty 2\phi_1\left( \frac{-p^{\alpha+1}, p^{\beta+1}}{p^{\alpha+\beta+2}} \bigg| p, p^{1/2}/x \right) = \frac{(p^{\beta+1}, -p^{\alpha+3/2}/x; p)_\infty}{(p^{\alpha+\beta+2}; p)_\infty} 2\phi_1\left( \frac{p^{\alpha+1}, p^{1/2}/x}{-p^{\alpha+3/2}/x} \bigg| p, p^{\beta+1} \right).
\]

Therefore (5.2) holds in the interior of \( \{ x : |x| = p^{1/2} \} \) and analytic continuation establishes the validity of (5.2) on compact subsets of the complex plane. This completes the proof.

We next determine the asymptotic behavior of \( b_n^{(\alpha, \beta)}(x) \) at \( x = 0 \) and in \( \{ x : 0 < |x| \leq p^{1/2} \} \).

**Theorem 5.2** We have for \( \alpha \neq \beta \),

\[
(5.6) \quad b_n^{(\alpha, \beta)}(0) \approx C p^{n^2/2} u^n \quad \text{as } n \to \infty,
\]

where \( C \) is a nonzero constant and \( |u| < 1 \).

**Proof.** Clearly

\[
(5.7) \quad b_n^{(\alpha, \beta)}(0) = \frac{(p^{\beta-n-1}, -p^{\alpha-n-1}; p)_n}{(p, p^{-2n-\alpha-\beta-2}; p)_n} (-1)^n p^{n/2}
\]

\[
\cdot 4\phi_3\left( \frac{p^{-n}, p^{n+\alpha+\beta+3}, p^{\beta+1}, -p^{\alpha+1}}{p^{\alpha+\beta+2}, p^{\beta+2}, -p^{\alpha+2}} \bigg| p, p \right).
\]

Ismail and Wilson [14] proved that if \( |z| < 1 \) then

\[
(5.8) \quad 4\phi_3\left( \frac{q^{-n}, abcdq^{-n-1}, az, a/z}{ab, ac, ad} \bigg| q, q \right) \approx \left( \frac{a}{z} \right)^n \frac{(az, bz, cz, dz; q)_\infty}{(z^2, ab, ac, ad; q)_\infty},
\]

as \( n \to \infty \). Now apply (5.8) with

\[
a = ip^{1+(\alpha+\beta)/2}, \quad b = -ip^{1+(\alpha+\beta)/2}, \quad c = -ip^{1-(\beta-\alpha)/2}, \quad d = ip^{1-(\alpha-\beta)/2},
\]

\[
z = \begin{cases} 
ip^{(\alpha-\beta)/2} & \text{if } \alpha > \beta \\
-ip^{(\beta-\alpha)/2} & \text{if } \beta > \alpha.
\end{cases}
\]
Theorem 5.4

This is not straightforward and requires some preliminary results.

Corollary 5.3

The values of $C$ and $u$ in (5.6) are given by

\[
(5.9) \quad u = \begin{cases} 
  p^{\beta+1} & \text{if } \alpha > \beta \\
  -p^{\alpha+1} & \text{if } \beta > \alpha
\end{cases}, \\
C = \begin{cases} 
  \frac{(-p^{\alpha+1}; p)^{\infty}}{(1+p^{\alpha-\beta}; p)^{\infty}} & \text{if } \alpha > \beta \\
  \frac{(-p^{\beta+1}; p)^{\infty}}{(1+p^{\beta-\alpha}; p)^{\infty}} & \text{if } \beta > \alpha.
\end{cases}
\]

The only case left now is to determine the large $n$ behavior of $b^{(\alpha, \beta)}(0)$ on the zeros of

\[
(5.10) \quad F(x) := \frac{(p^{\beta+1}, -p^{\alpha+3/2}/x; p)^{\infty}}{(p^{\alpha+3/2}, p)^{\infty}} \cdot \phi_1 \left( \begin{array}{c} p^{\alpha+1}/x \\ -p^{\alpha+3/2}/x \\ p, p^{\beta+1} \end{array} \right).
\]

This is not straightforward and requires some preliminary results.

Theorem 5.4

The function

\[
(5.11) \quad Y^{(\alpha, \beta)}_k(x) = (-x)^{-k} \frac{(p^{2\alpha+4}, p^{2\alpha+4}; p^2)^k p^{k(\alpha+\beta+3+k)}}{(p^{\alpha+3}, p^{\alpha+\beta+3}, p^{\alpha+\beta+4}, p^{\alpha+\beta+5}; p^2)^k} \cdot_2 \phi_1 \left( \begin{array}{c} -p^{\alpha+2+k}, p^{\beta+2+k} \\ p^{\alpha+\beta+2k+4} \end{array} \bigg| p, \frac{p^{1/2}}{x} \right),
\]

satisfies the three term recurrence relation (3.12), with $p = q^{1/2}$.

To prove Theorem 5.4 we used MACSYMA to first find the multiple of the $_2\phi_1$ then verified that the $Y^{(\alpha, \beta)}_k$ of (5.11) indeed satisfies (3.12) by equating coefficients of powers of $1/x$. Theorem 5.4 is also a limiting case of a result of Gupta, Ismail and Masson [12], as will be explained in §8.

It readily follows from Theorem 5.4 that

\[
X^{(\alpha, \beta)}_\nu(x) = (-x)^{-\nu} (p^{1/2}/x; p)^{\infty} \cdot_2 \phi_1 \left( \begin{array}{c} -p^{\alpha+2+\nu}, p^{\beta+2+\nu} \\ p^{\alpha+\beta+2\nu+4} \end{array} \bigg| p, \frac{p^{1/2}}{x} \right)
\]

satisfies the three term recurrence relation

\[
(5.12) \quad \frac{(1-q^{\nu+\alpha+\beta+4}/2)(1-q^{\nu+\alpha+\beta+4})}{(1-q^{\nu+\alpha+\beta+3}/2)(1-q^{\nu+\alpha+\beta+4}/2)} X^{(\alpha, \beta)}_{\nu+1}(x) = X^{(\alpha, \beta)}_\nu(x) + \frac{(1-q^{\nu+\alpha+\beta+2}/2)(1-q^{\nu+\alpha+\beta+3}/2)}{(1-q^{\nu+\alpha+\beta+2}/2)(1-q^{\nu+\alpha+\beta+4}/2)} q^{(\nu+\alpha+\beta+3)/2} X^{(\alpha, \beta)}_{\nu-1}(x).
\]
The Heine transformation (5.5) yields the alternate representation

\[
X_{\nu}^{(\alpha,\beta)}(x) = (-x)^{-\nu} \frac{(p^{\alpha+\nu+2}, -p^{\alpha+\nu+5/2}/x; p)_{\infty}}{(p^{\alpha+\beta+2\nu+1}; p)_{\infty}} \cdot \phi_1 \left( \begin{array}{c} p^{\nu+\alpha+2} \frac{p^{1/2}/x \quad | \quad p, p^{\beta+\nu+2}}{-p^{\alpha+\nu+5/2}/x} \end{array} \right).
\]

According to Theorem 4.5 of [13],

\[
C_{\nu}C_{\nu+1} \cdots C_{\nu+n-1}X_{\nu+n}^{(\alpha,\beta)}(x) = b_n^{(\alpha+\nu,\beta+\nu)}(x)X_{\nu}^{(\alpha,\beta)}(x) + b_{n-1}^{(\alpha+\nu+1,\beta+\nu+1)}(x)X_{\nu-1}^{(\alpha,\beta)}(x),
\]

where \( C_{\nu} \) is the coefficient of \( X_{\nu+n}(x) \) in (5.12), see (4.28) in [13].

**Theorem 5.5** Let \( \xi \) be a zero of \( F(x) \) of (5.10). The large \( n \) behavior of \( b_n^{(\alpha,\beta)}(\xi) \) is

\[
b_n^{(\alpha,\beta)}(\xi) \approx \frac{\xi^{-(n+\alpha+\beta+3)/2} \cdot X_0^{(\alpha,\beta)}(\xi)}{(q^{(\alpha+\beta+3)/2}; q)_{\infty} (q^{(\alpha+\beta+4)/2}; q)_{\infty}^2}.
\]

**Proof.** It is clear from (5.10) and (5.13) that \( F(x) = 0 \) if and only if \( X_{\nu-1}^{(\alpha,\beta)}(x) = 0 \). The recurrence relation (5.14) shows that if \( X_{\nu}^{(\alpha,\beta)}(x) \) and \( X_{\nu-1}^{(\alpha,\beta)}(x) \) vanish at \( x = \zeta \) then \( X_{\nu+n}^{(\alpha,\beta)}(\zeta) = 0 \) for all \( n, \ n = 2, 3, \ldots \). But it is obvious that

\[
X_{\nu}^{(\alpha,\beta)}(x) \approx (-x)^{-n}, \quad \text{as } n \to \infty.
\]

Thus \( X_{\nu}^{(\alpha,\beta)}(x) \) and \( X_{\nu-1}^{(\alpha,\beta)}(x) \) have no common zeros. Now (5.14) implies

\[
b_n^{(\alpha,\beta)}(\xi) \approx \frac{(q^{\alpha+2}; q)_{\infty} q^{n(n+\alpha+\beta+3)/2} X_{\nu}^{(\alpha,\beta)}(\xi)}{(q^{(\alpha+\beta+3)/2}; q)_{\infty} (q^{(\alpha+\beta+4)/2}; q)_{\infty}^2 X_0^{(\alpha,\beta)}(\xi)}.
\]

and we have established (5.15).

We are now in a position to prove Theorem 3.1.

**Proof of Theorem 3.1.** From (1.12) and (3.11) it follows that

\[
h_{n+1}^{(\alpha,\beta)}(q) |a_{n+1}(\lambda|q)|^2 = O(q^{-n(n+3\alpha+2\beta-1/2)/2} |b_n^{(\alpha,\beta)}(2\lambda q^{1/2}/(1-q)|^2).
\]

Now (5.2), (5.6), (5.9) and (5.15) show that \( \sum_1^{\infty} h_n^{(\alpha,\beta)}(q) |a_n(\lambda|q)|^2 \) converges if and only if \( 2q^{1/2}\lambda/(1-q) \) is a zero of \( X_{\nu-1}^{(\alpha,\beta)}(x) \). The eigenspaces are one dimensional since the eigenfunction with an eigenvalue \( \lambda \) must be given by

\[
g(x|\lambda) = \sum_{1}^{\infty} a_n(\lambda|q) P_n^{(\alpha,\beta)}(x|q).
\]
Next we prove that the polynomials \( \{i^{-n}b_n^{(\alpha,\beta)}(ix)\} \) are orthogonal on a bounded countable set when \( \alpha \) and \( \beta \) are not real and are complex conjugates.

Set

\[(5.19) \quad s_n^{(\alpha,\beta)}(x) := i^{-n}b_n^{(\alpha,\beta)}(ix).\]

The three term recurrence relation (3.12) leads to the following three term recurrence relation for the \( s_n \)'s.

\[(5.20) \quad s_{n+1}^{(\alpha,\beta)}(x) = \left[ x + \frac{(q^{(\alpha-\beta)/4} - q^{(\beta-\alpha)/4})q^{(2n+\alpha+\beta)/4}}{i(1 - q^{n+1+(\alpha+\beta)/2})(1 - q^{n+2+(\alpha+\beta)/2})} \right] s_n^{(\alpha,\beta)}(x)

- \frac{(1 - q^{n+\alpha+1})(1 - q^{n+\beta+1})q^{n+1+(\alpha+\beta)/2}}{(1 - q^{n+1+(\alpha+\beta)/2})(q^{n+2+(\alpha+\beta)/2}; q^{1/2})_3}s_{n-1}^{(\alpha,\beta)}(x).\]

We also have the initial conditions

\[(5.21) \quad s_0^{(\alpha,\beta)}(x) := 1, \quad s_1^{(\alpha,\beta)}(x) := 0.\]

When

\[(5.22) \quad \alpha = \overline{\beta}, \quad \text{Im} \alpha \neq 0, \quad \text{Re} \alpha > -1,\]

then the coefficient of \( s_n^{(\alpha,\beta)}(x) \) in (5.19) is real for \( n \geq 0 \) and the coefficient of \( s_{n-1}^{(\alpha,\beta)}(x) \) is negative for \( n > 0 \). Thus the \( s_n \)'s are orthogonal with respect to a positive measure, say \( d\psi \). The coefficients in the recurrence relation (5.19) are bounded. Thus we can apply Markov's theorem \[21\], namely

\[(5.23) \quad \lim_{n \to \infty} (s_n^{(\alpha,\beta)}(x))^*/s_n^{(\alpha,\beta)}(x) = \int_{-\infty}^{\infty} \frac{d\psi(t)}{x - t}, \quad \text{Im} x \neq 0,\]

where \( (s_n^{(\alpha,\beta)}(x))^* \) is a solution to (5.20) satisfying the initial conditions

\[(5.24) \quad (s_0^{(\alpha,\beta)}(x))^* := 0, \quad (s_1^{(\alpha,\beta)}(x))^* := 1.\]

It is easy to see that

\[(5.25) \quad (s_n^{(\alpha,\beta)}(x))^* = s_{n-1}^{(\alpha+1,\beta+1)}(x).\]

Therefore (5.2), (5.18) and (5.23) give

\[(5.26) \quad \int_{-\infty}^{\infty} \frac{d\psi(t)}{x - t} = \frac{(p^{\alpha+\beta+2}; p)_2}{(1 - p^{\beta+1})(1 - ip^{\alpha+3/2}/x)}

\[\cdot \frac{2\phi_1(p^{\alpha+2}, -ip^{1/2}/x; ip^{\alpha+5/2}/x; p, p^{\beta+2})}{2\phi_1(p^{\alpha+1}, -ip^{1/2}/x; ip^{\alpha+3/2}/x; p, p^{\beta+1})}.\]
Recall that the Coulomb wave function $F_L(\eta, \rho)$ are defined in terms of a confluent hypergeometric function as [1]

\begin{equation}
F_L(\eta, \rho) := 2^L e^{-\pi\eta/2} \frac{\Gamma(L + 1 + i\eta)}{\Gamma(2L + 2)} \rho^{L+1} e^{-i\rho} \mathbf{F}_1(L + 1 - i\eta; 2L + 2; 2i\rho)
\end{equation}

As $q \to 1^-$ the polynomials $(1 - q)^n s_n^{(a, b)}(x/(1 - q))$ tend to the Wimp polynomials, [23]. The right hand side of (5.2) in the case of Wimp’s polynomials is $F_L(\eta, \rho)$. This suggests defining a $q$-analog of $F_L(\eta, \rho)$ by

\begin{equation}
F_L(\eta, \rho; q) := (iq^{1/2} \rho; q)_\infty 2\phi_1\left(\begin{array}{c}
-q^{L+1+i\eta+1}, q^{L+1-i\eta+1}
\\
q^{2L+2}
\end{array} \middle| q, iq^{1/2} \rho\right).
\end{equation}

where $L$ and $\eta$ are real parameters. Observe that the iterate of the Heine transformation [10, (III.3)]

\begin{equation}
2\phi_1(a, b; c; q, z) = \frac{(abz/c; q)_\infty}{(z; q)_\infty} 2\phi_1(c/a, c/b; c, q, abz/c).
\end{equation}

shows that $F_L(\eta, \rho; q)$ is real when $\rho$ is real, as in the case of $F_L(\eta, \rho)$. 

25
6 An Expansion Formula.

The purpose of this section is to give a direct proof of the eigenfunction expansion (6.13).

Set

\[ E_q(x; a, r) = \sum_{m=0}^{\infty} a_m p_m(x; b, b \sqrt{q}, -c, -c \sqrt{q}). \]

(6.1)

In order to compute \( a_m \) we need to evaluate the integrals

\[ J_m(a; r) := \int_{-1}^{1} w(x; b, b \sqrt{q}, -c, -c \sqrt{q}) p_m(x; b, b \sqrt{q}, -c, -c \sqrt{q}) E_q(x; a, r) dx, \]

(6.2)

when \( a = -i \). We shall keep the parameter \( a \) in (6.2) free till the end then we specialize the result by choosing \( a = -i \). It is clear that

\[ J_m(a; r) = \sum_{n=0}^{\infty} q^{n^2/4} r^n I_{m,n}(a, b, c), \]

(6.3)

where

\[ I_{m,n}(a, b, c) := \int_{-1}^{1} w(x; b, b \sqrt{q}, -c, -c \sqrt{q}) p_m(x; b, b \sqrt{q}, -c, -c \sqrt{q}) h(x; a q^{1/2} t^2) dx. \]

(6.4)

Formulas (6.3.2) and (6.3.9) in [10] imply

\[ \int_{-1}^{1} w(x; \alpha, \beta, \gamma, \delta) h(x; g) h(x; f) dx = 2 \pi (\alpha g, \beta g, \gamma g, \delta g, \alpha f, \beta f, \gamma f, \delta f, g^2, q)_\infty \]

\[ \cdot {}_8 \phi_7(g^2/q; g/\alpha, g/\beta, g/\gamma, g/\delta, g/f; q, \alpha \beta \gamma \delta f/g). \]

Using the \( 4 \phi_3 \) representation of \( p_n \) in (6.4) we obtain

\[ I_{m,n}(a, b, c) = \frac{2 \pi (a b q^{(1-n)/2}, a b q^{1-n/2}, -a c q^{1-n/2}, -a c q^{1-n/2}; q)_\infty}{(q, b^2 \sqrt{q}, -b c, -b c \sqrt{q}, -b c \sqrt{q}, -b c, c^2 \sqrt{q}, a b q^{(1+n)/2}; q)_\infty} \]

\[ \cdot \frac{(a q^2, b^2 c^2 q^{n+1}; q)_\infty}{(a b q^{1+n/2}, -a c q^{1+n/2}, -a c q^{1+n/2}, a^2 q^{1-n}; q)_\infty} \]

\[ \cdot \sum_{j=0}^{m} \frac{(q^{-m}, b^2 c^2 q^n, a b q^{(n+1)/2})_j q^j}{(q, b^2 c^2 q^{n+1}, a b q^{1-n/2})_j} \]

\[ \cdot {}_8 \phi_7(a^2 q^{-n}; a q^{-j-(n-1)/2}/b, a q^{-n/2}/b, -a q^{(1-n)/2}/c, -a q^{-n/2}/c, q^{-n}; q, b^2 c^2 q^{j+n+1}). \]
We now apply Watson’s transformation formula which expresses a terminating very well-poised $\phi_7$ as a multiple of a terminating balanced $\phi_3$, \cite[(III.17)]{[2]}. Thus

$$8W_7(a^2q^{-n}; aq^{-j-(n-1)/2}/b, aq^{-n/2}/b, -aq^{(1-n)/2}/c, -aq^{-n/2}/c, q^{-n}; q, b^2c^2q^{j+n+1})$$

$$= \frac{(a^2q^{1-n}, c^2q^{1/2})_n}{(-acq^{(1-n)/2}, -acq^{(2-n)/2})_n} 4\phi_3 \left( \begin{array}{c} q^{-n}, -aq^{-n/2}/c, -aq^{(1-n)/2}/c, b^2q^{j+1/2} \\ q^{-n+1/2}/c^2, abq^{(1-n)/2}, abq^{1-n/2} \end{array} \right| q, q \right).$$

We then apply the Sears transformation (2.4) with invariant parameters $q^{-n}, -aq^{-n/2}/c, q^{-n+1/2}/c^2$. After some simplification we obtain

$$8W_7(a^2q^{-n}; aq^{-j-(n-1)/2}/b, aq^{-n/2}/b, -aq^{(1-n)/2}/c, -aq^{-n/2}/c, q^{-n}; q, b^2c^2q^{j+n+1})$$

$$= \frac{(a^2q^{1-n}, c^2q^{1/2}, -q^{-n}/bc, -q^{-n+1/2}/bc)_n}{(-acq^{(1-n)/2}, -acq^{1-n/2}, abq^{(1-n)/2}, -abq^{1-n/2})_n}$$

$$\frac{(-bcq^{n+1/2}, abq^{(1-n)/2})_j}{(abq^{(n+1)/2}, -bcq^{1/2})_j} (-acb^2q^{(n+1)/2}j)$$

$$4\phi_3 \left( \begin{array}{c} q^{-n}, q^{-n-j}/b^2c^2, -aq^{-n/2}/c, -aq^{-n+1/2}bcq/c^2 \\ -q^{-n}/bc, -q^{-n-j+1/2}/bc, q^{-n+1/2}/c^2 \end{array} \right| q, q \right).$$

The substitution of the right-hand side of (6.6) for the $8W_7$ in (6.5) and some simplification lead to

$$I_{m,n}(a, b, c) = \kappa(b, c) \frac{(c^2q^{1/2}, -bcq^{1/2}, -bcq)_n}{(qb^2c^2)_n} (-a/c)^n q^{-n^2/2}$$

$$\cdot \sum_{j=0}^{m} \frac{(q^{-m}, b^2c^2q^n, -bcq^{n+1/2})_j}{(q, b^2c^2q^{n+1}, -bcq^{1/2})_j} q^j$$

$$\cdot 4\phi_3 \left( \begin{array}{c} q^{-n}, q^{-n-j}/b^2c^2, -aq^{-n/2}/c, -aq^{-n+1/2}bcq/c^2 \\ -q^{-n}/bc, -q^{-n-j+1/2}/bc, q^{-n+1/2}/c^2 \end{array} \right| q, q \right),$$

where

$$\kappa(b, c) = \frac{2\pi(qb^2c^2)_\infty}{(q, b^2q^{1/2}, -bc, -bcq^{1/2}, -bcq^{1/2}, -bcq, c^2q^{1/2})_\infty}$$

$$= \frac{2\pi(bcq^{1/2}, bcq)_\infty}{(q, b^2q^{1/2}, -bc, -bcq^{1/2}, c^2q^{1/2})_\infty}. $$
Replace the $\phi_3$ by its series definition with summation index $k$. Then interchange the $j$ and $k$ sums to obtain

$$I_{m,n}(a, b, c) = \kappa(b, c)\frac{(c^{2}q^{1/2}, -bcq^{1/2}, -bcq)_{n}}{(qb^{2}c^{2})_{n}}(-a/c)^{n}q^{-n^{2}/2}$$

$$\cdot \sum_{k=0}^{n} \frac{(q^{-n}, -aq^{-n/2}/c, -q^{-n/2}/ac, q^{-n/b^{2}c^{2}})_{k}}{(q, -q^{-n/bc}, q^{-n+1/2}/c^{2}, -q^{-n+1/2}/bc)_{k}}q^{k}$$

$$\cdot \phi_2\left(\begin{array}{c}
q^{-m}, b^{2}c^{2}q^{m}, -bcq^{n-k+1/2} \\
-\frac{aq^{-m}}{bc}, -q^{-n+1/2}m, -bcq^{1/2}
\end{array} \right) \cdot \phi_2\left(\begin{array}{c}
q^{-n}, q^{-m-n}/b^{2}c^{2}, -aq^{-n/2}/c, -q^{-n/2}/ac \\
-q^{-n}/bc, -q^{-n+1/2}/bc, q^{-n+1/2}/c^{2}
\end{array} \right).$$

The $\phi_2$ can now be summed by the q-analog of the Pfaff-Saalschütz theorem \([10, (II.12)]\). Its sum is

$$(q^{k-n}, q^{-m+1/2}/bc)_{m}/(-bcq^{1/2}, q^{k-m-n}/b^{2}c^{2})_{m},$$

which vanishes for all $k, 0 \leq k \leq n$ if $m > n$. Thus we get

$$I_{m,n}(a, b, c) = \frac{(q^{-n}, -q^{-m+1/2}/bc)_{m}}{(-bcq^{1/2}, q^{-m-n}/b^{2}c^{2})_{m}}\kappa(b, c)\frac{(c^{2}q^{1/2}, -bcq^{1/2}, -bcq)_{n}}{(qb^{2}c^{2})_{n}}(-a/c)^{n}q^{-n^{2}/2}$$

$$\cdot \phi_3\left(\begin{array}{c}
q^{-m}, q^{-m-n}/b^{2}c^{2}, -aq^{-n/2}/c, -q^{-n/2}/ac \\
-q^{-n}/bc, -q^{-n+1/2}/bc, q^{-n+1/2}/c^{2}
\end{array} \right).$$

The relationship (6.3) and the observation $I_{m,n}(a, b, c) = 0$ if $n < m$ show that $J_{m}(a; r)$ is given by

$$\frac{q^{m^{2}/4}}{(q)_{m}} \sum_{n=0}^{m} q^{n^{2}/4}(rq^{m/2})^{n} \frac{(r^{m+1/2})_{n}}{(q^{m+1})_{n}} I_{m,n+m}(a, b, c).$$

Applying (6.8) we find after some simplification

$$J_{m}(a, r) = \kappa(b, c)q^{m^{2}/4}(-abr)^{m}\frac{(c^{2}q^{1/2}, -bcq^{1/2}, -bcq)_{m}}{(qb^{2}c^{2})_{2m}}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(c^{2}q^{m+1/2}, -bcq^{m+1/2}, -bcq^{m+1})_{n}}{(q, b^{2}c^{2}q^{2m+1})_{n}}\left(-\frac{ar}{c}q^{-m/2}\right)^{n}$$

$$\cdot q^{-n^{2}/4}\phi_3\left(\begin{array}{c}
q^{-n}, q^{-2m-n}/b^{2}c^{2}, -aq^{-(n+m)/2}/c, -q^{-(n+m)/2}/ac \\
-q^{-n-m}/bc, -q^{-(n-m+1)/2}bc, q^{-n-m+1}/c^{2}
\end{array} \right).$$
Finally we apply Sears transformation (2.4) to the 
\[ 4\phi_3 \left( q^{-n/2}, q^{-m-n/2}/bc, -aq^{-(n+m)/2}/c, -q^{-(n+m)/2}/ac \right| q^{1/2}, q^{1/2} \) 
by reversing the sum in the first \( 4\phi_3 \). After some straightforward manipulations we establish 
\[ J_m(a, r) = \kappa(b, c) \left( \frac{c^2 q^{1/2}; q}{bc q^{1/2}, bc q; q} \right) m \frac{q^{-m/2}}{q^{1/2}, q^{1/2}} \sum_{n=0}^{\infty} \left( -cq^{(m+1)/2}/a, -acq^{(m+1)/2}/q^{1/2} \right) \left( -\frac{ar}{c} q^{-m/2-1/4} \right) n \left( q^{-n/2}, -bcq^{(m+1)/2}, cq^{(m+1)/2}, -cq^{(m+1)/2}/ac \right| q^{1/2}, q^{1/2} \) \]

Finally we apply Sears transformation (2.4) to the \( 4\phi_3 \) in (6.10) with invariant parameters \( q^{-n}, cq^{(m+1)/2}/a, bcq^{m+1/2} \). This enables us to cast (6.10) in the form 
\[ J_m(a, r) = \kappa(b, c) \left( \frac{c^2 q^{1/2}; q}{bc q^{1/2}, bc q; q} \right) m \frac{q^{-m/2}}{q^{1/2}, q^{1/2}} \sum_{n=0}^{\infty} \left( -aq^{1/4}, -q^{1/4}/a; q^{1/2} \right) n \left( ar \right) n \left( q^{-n/2}, -q^{-n/2}, cq^{(m+1)/2}, -dq^{(m+1)/2}/bc \right| q^{1/2}, q^{1/2} \) \]

It is evident from (6.11) that \( J_m(a; r) \) is a double series. When \( a^2 = -1 \) we have been able to reduce the right-hand side of (6.11) to a single series. To see this, replace the \( 4\phi_3 \) in (6.11) by 
its defining series then interchange the sums. The result is 
\[ J_m(-i; r) = \kappa(b, c) \left( \frac{c^2 q^{1/2}; q}{bc q^{1/2}, bc q; q} \right) m \left( irq^{1/2}; q \right)_{\infty} \left( iqr \right)_{\infty} q^{m^2/4} \]
\[ \cdot \phi_1 \left( \begin{array}{c} cq^{m/2+1/4}, -bq^{m/2+1/4} \\ bcq^{m+1/2} \end{array} \middle| q^{1/2}, i r \right). \]

We have tried to express \( J_m(a; r) \) of (6.11) as a single sum for general \( a \) but this does not seem to be possible except when \( a = \pm i \).

Now the orthogonality relation (1.10) gives for the case \( a = -i \)

\[ (6.12) \quad \kappa(b, c)a_m = J_m(-i; r) \frac{(1 - b^2c^2q^{2m})(b^2c^2, b^2q^{1/2}, -bc; q)_m}{(1 - b^2c^2)(q, c^2q^{1/2}, -bcq; q)_m} b^{-2m}. \]

Thus we established the expansion formula

\[ (6.13) \quad \mathcal{E}_q(x; -i, r) = \sum_{m=0}^{\infty} a_m p_m(x; b, bq^{1/2}, -c, -cq^{1/2}), \]

with the \( a_m \)'s given by

\[ (6.14) \quad a_m = \frac{(b^2c^2, b^2q^{1/2}; q)_m (irq^{1/2}; q)_\infty (ir/b)^m q^{m^2/4}}{(q, bcq^{1/2}, bc; q)_m (ir; q)_\infty} (i r/b)^m q^{m^2/4} \phi_1 \left( \begin{array}{c} cq^{m/2+1/4}, -bq^{m/2+1/4} \\ bcq^{m+1/2} \end{array} \middle| q^{1/2}, i r \right). \]
7 A Formal Approach

We now formalize the procedure followed in Section 3 and used earlier in [13]. Let $S \subset C^k$ and assume that for every $A \in S$, the sequence of polynomials $\{p_n(x; A)\}_{0}^{\infty}$ are orthogonal with respect to a measure with a nontrivial absolutely continuous component. By $A + 1$ we mean $(1 + a_1, \ldots, 1 + a_k)$ if $A = (a_1, \ldots, a_k)$. We will assume that $A + 1 \in S$ whenever $A \in S$. Let the orthogonality relation of the $p_n$’s be

$$\int_{-\infty}^{\infty} p_n(x; A) p_m(x; A) d\mu(x; A) = h_n(A) \delta_{m,n}. \tag{7.1}$$

Assume that $D$ is an operator defined on polynomials by linearity and by its action on the basis $\{p_n(x; A)\}$ via

$$D p_n(x; A) = \xi_n(A) p_{n-1}(x; A + 1). \tag{7.2}$$

Furthermore assume that the support of $\mu'(x; A) = \frac{d\mu(x; A)}{dx}$ is the same for all $A \in S$ and that we know the connection coefficients in

$$p_n(x; A) = \sum_{j=0}^{n} c_{n,j} p_j(x; A + 1). \tag{7.3}$$

Therefore

$$c_{n,j} = \frac{1}{h_j(A + 1)} \int_{-\infty}^{\infty} p_n(x; A) p_j(x; A + 1) d\mu(x; A + 1). \tag{7.4}$$

The formula dual to (7.3) is

$$p_n(x; A + 1) \mu'(x; A + 1) = \sum_{m=n}^{\infty} \frac{h_n(A + 1)}{h_m(A)} c_{m,n} p_m(x; A) \mu'(x; A), \tag{7.5}$$

holding on the interior of the support of $\mu'(x; A)$.

We now wish to describe the spectrum of a formal inverse to $D$. Note that we can define $D$ densely on $L^2(d\mu(x; A))$ by (7.2) provided that the polynomials $p_n(x; A)$ are dense in $L^2(d\mu(x; A))$ for all $A \in S$. This suggests that we define $D^{-1}$ via

$$D^{-1} \sum_{n=0}^{\infty} a_n p_n(x; A + 1) := \sum_{n=0}^{\infty} \frac{a_n p_{n+1}(x; A)}{\xi_{n+1}(A)} \tag{7.6}$$

This motivates the definition

$$(T_A f)(x) := \int_{-\infty}^{\infty} f(t) \left[ \sum_{n=0}^{\infty} \frac{p_{n+1}(x; A) p_n(t; A + 1)}{\xi_{n+1}(A) h_n(A + 1)} \right] d\mu(t; A + 1), \tag{7.7}$$
if $f \in L^2(d\mu(x; A + 1))$.

The next step is to consider the eigenvalue problem

$$T_A g = \lambda g, \quad g(x) \approx \sum_{n=0}^{\infty} a_n(\lambda; A) p_n(x; A).$$

(7.8)

In order for (7.8) to hold it is necessary that $g$ lies in the range of $T_A$, hence (7.6) shows that $a_0(\lambda; A) = 0$. Therefore (7.7) and (7.8) yield

$$\sum_{n=1}^{\infty} a_n(\lambda; A) p_n(x; A) = \sum_{n=1}^{\infty} \frac{p_{n+1}(x; A)}{\xi_{n+1}(A)} \sum_{k=n}^{\infty} \frac{a_k(\lambda; A)}{h_n(A+1)} \int_{-\infty}^{\infty} p_k(t; A)p_n(t; A + 1)d\mu(t; A + 1).$$

Thus we have established

$$\lambda \sum_{n=1}^{\infty} a_n(\lambda; A) p_n(x; A) = \sum_{n=1}^{\infty} \frac{p_{n+1}(x; A)}{\xi_{n+1}(A)} \sum_{k=n}^{\infty} c_{k,n} a_k(\lambda, A).$$

(7.9)

Now (7.9) implies the recurrence relation

$$\lambda \xi_n(A) a_n(\lambda; A) = \sum_{k=n-1}^{\infty} c_{k,n-1} a_k(\lambda, A).$$

(7.10)

Observe that (7.10) transformed the eigenvalue problem (7.8) to the discrete eigenvalue problem (7.10). When $c_{n,k} = 0$ for $k < n - r$ for a fixed $r$ then (7.10) is the eigenvalue equation of a matrix with at most $r + 1$ nonzero entries in each row. The cases analyzed in [15] and in this paper are the cases when

$$c_{n,k} = 0 \quad \text{for } k < n - 2, \ n = 2, 3, \ldots.$$  

(7.11)

Note that $c_{n,n} \neq 0$. When (7.11) holds then (7.10) reduces to

$$\lambda \xi_n(A) a_n(\lambda; A) = c_{n-1,n-1} a_{n-1}(\lambda; A) + c_{n-1,n} a_n(\lambda; A) + c_{n,n+1} a_{n+1}(\lambda; A).$$

(7.12)

For example in the case when the $p_n$’s are the ultraspherical polynomials $C_n^\nu(x)$ we have [19, §144],

$$\frac{d}{dx} C_n^\nu(x) = 2\nu C_{n-1}^{\nu+1}(x), \quad 2(n + \nu)C_n^\nu(x) = 2\nu[C_n^{\nu+1}(x) - C_{n-2}^{\nu+1}(x)].$$

(7.13)

Therefore

$$\xi_n(\nu) = 2\nu, \ c_{n,n} = -c_{n,n-2} = \nu/(\nu + n), \ c_{n,n-1} = 0,$$  

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and (7.12) becomes

\[(7.14) \quad 2 \lambda a_n(\lambda; \nu) = \frac{a_{n-1}(\lambda; \nu)}{(\nu + n - 1)} - \frac{a_{n+1}(\lambda; \nu)}{(\nu + n + 1)},\]

which is (2.11) in [13].

The procedure just outlined is very formal but can be justified if both \(p_n(x; A)\) and \(p_n(x; A+1)\)
are dense in \(L^2(d\mu(x; A)) \cap L^2(d\mu(x; A+1))\).

Another approach to the same problem is to think of \(T_A\) as a right inverse to \(D\). In other words
\(DT_A\) is the restriction of the identity operator to the range of \(D\). Thus (7.8) is equivalent to

\[(7.15) \quad g(x; \lambda) = \lambda D g(x; \lambda).\]

Now the use of the orthogonal expansion of \(g\) and formulas (7.2) and (7.15) implies that
\(\lambda \xi_{n+1}(A) a_{n+1}(\lambda, A)\) is the projection of \(\sum_{k=1}^{\infty} a_k(\lambda; A) p_k(x; A)\) on the space spanned by \(p_n(x; A+1)\). Therefore (7.3) implies (7.12).

In [13] it was observed that the eigenfunction expansion \(\sum_{k=1}^{\infty} a_k(\lambda; A) p_k(x; A)\) can be extended to values of \(\lambda\) off the discrete spectrum of the operator under consideration. This is also the case with the expansion formula (6.13). We now attempt to find such an expansion in general.

We seek functions \(\{F_n(\lambda; A)\}\) such that the function \(E_A(x; \lambda)\),

\[(7.16) \quad E_A(x; \lambda) := \sum_{k=0}^{\infty} F_k(\lambda; A) p_k(x; A),\]

satisfies

\[(7.17) \quad \lambda D E_A(x; \lambda) = E_A(x; \lambda).\]

Observe that (7.16) reminds us of (7.15), hence under the assumption (7.11), (7.3) and (7.17) imply that the \(F_n\)'s must satisfy a recursion relation similar to (7.12), that is

\[(7.18) \quad \lambda \xi_n(A) F_n(\lambda; A) = c_{n-1, n-1} F_{n-1}(\lambda; A) + c_{n, n-1} F_n(\lambda; A) + c_{n+1, n-1} F_{n+1}(\lambda; A).\]

In order to maintain a parallel course with the results in [13] and with the notation of Section 5 we will renormalize the \(a_n\)'s and \(F_n\)'s in order to put (7.12) in monic form and change (7.18) to a recursion with the coefficient of \(y_{n-1}\) equal to unity. Keeping in mind that \(a_0(x; \lambda) = 0\) and that \(a_1(x; \lambda)\) is a multiplicative constant which we may take to be unity, we set

\[(7.19) \quad a_n(\lambda; A) = \prod_{j=0}^{n-2} \frac{\xi_{j+1}(A)}{c_{j+2, j}} u^{n-1} b_{n-1}(\lambda u; A) \quad n > 1, \quad a_1(x; \lambda) = b_0(x; \lambda) = 1.\]
and

\begin{equation}
F_n(\lambda; A) = \prod_{j=0}^{n-1} \frac{c_{ij}}{\xi_{j+1}(A)} u^{n-1} G_{n-1}(\lambda u; A), \quad n > 0 \quad F_0(\lambda; A) = G_{-1}(\lambda u; A).
\end{equation}

Here \( u \) is a free normalization factor at our disposal and may depend on \( A \). Thus

\begin{equation}
\lambda_n(\lambda; A) = b_{n+1}(\lambda; A) + B_n(A)b_n(\lambda; A) + C_n(A)b_{n-1}(\lambda; A),
\end{equation}

and

\begin{equation}
\lambda G_n(\lambda; A) = C_{n+1}(A)G_{n+1}(\lambda; A) + B_n(A)G_n(\lambda; A) + G_{n-1}(\lambda; A),
\end{equation}

hold with

\begin{equation}
B_n(A) := \frac{uc_{n+1,n}}{\xi_{n+1}(A)}, \quad C_n(A) = \frac{u^2c_{n,n+1,n-1}}{\xi_n(A)\xi_{n+1}(A)}.
\end{equation}

We are seeking a solution to (7.18) that makes (7.16) converge on sets of \( \lambda \)'s containing the spectrum. Since the \( p_n \)'s are given we need to choose the \( F_n \)'s to be as small as possible, that is choose \( F_n \) to be the minimal solution, if it exists. Recall that a solution \( w_n \) of (7.18) is minimal if \( w_n = o(v_n) \) where \( v_n \) is any other linearly independent solution of the same recurrence relation, [16]. It is clear that a minimal solution of (7.18) exists if and only if (7.22) has a minimal solution. The minimal solution may change form in different regions of the parameter or variable space, [12]. When \( B_n(A) \to 0 \) and \( C_n(A) \to 0 \) as \( n \to \infty \) then (7.22) has a minimal solution, see Theorem 4.55 in [16].

In many cases we encounter a fortuous situation where we can choose \( u \) in (7.23) such that

\begin{equation}
B_n(A) = B_0(A + n), \quad C_n(A) = C_0(n + A).
\end{equation}

Therefore

\begin{equation}
G_n(\lambda; A) = G_0(\lambda; A + n) = G_{-1}(\lambda; A + n + 1).
\end{equation}

When (7.24) holds Theorem 4.5 of [15] comes in handy. This latter theorem is

**Theorem 7.1** Let \( f(x; \nu + A) \) be a multi-parameter family of functions satisfying

\begin{equation}
C_{\nu+A}f(x; \nu + A + 1) = (A_{\nu+A}x + B_{\nu+A})f(x; \nu + A) \pm f(x; \nu + A - 1),
\end{equation}

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and let \( \{f_{n,\nu+A}(x)\} \) be a sequence of polynomials defined by

\[
\begin{align*}
(7.27) & \quad f_{0,\nu+A}(x) = 1, \quad f_{1,\nu+A}(x) = A_{\nu+A}x + B_{\nu+A}, \\
(7.28) & \quad f_{n+1,\nu+A}(x) = (A_{n+\nu+A}x + B_{n+\nu+A})f_{n,\nu+A}(x) \pm C_{n+\nu+A-1}f_{n-1,\nu+A}(x).
\end{align*}
\]

Then

\[
(7.29) \quad C_{\nu+A}C_{\nu+A+1}\cdots C_{\nu+A+n-1}f(x;\nu+A+n) = f_{n,\nu+A}(x)f(x,\nu+A) \pm f_{n-1,\nu+A+1}(x)f(x;\nu+A-1).
\]

Theorem 7.1 establishes

\[
(7.30) \quad C_{-1}(A) C_{-1}(A+1) \cdots C_{-1}(A+n-1) G_0(\lambda; A + n) = b_n(\lambda; A) G_0(\lambda; A) + b_{n-1}(\lambda; A + 1) G_{-1}(\lambda; A),
\]

where we used \( G_{-1}(\lambda; A) = G_0(\lambda; A - 1) \), see (7.25).

Now assume in addition to (7.26) that \( B_n(A) \to 0 \) and \( C_n(A) \to 0 \), hence the minimal solution to (7.22) exists. Let \( \{G_n(\lambda; A)\} \) be the minimal solution to (7.22). According to Pincherle’s theorem, [16], the continued \( J \)-fraction associated with (7.21) converges to a constant multiple of \( G_0(\lambda; A)/G_{-1}(\lambda; A) \). Therefore the eigenvalues of the infinite tridiagonal matrix associated with (7.21) are the zeros of \( G_{-1}(\lambda; A) \) which are not zeros of \( G_0(\lambda; A) \). If \( G_{-1}(\lambda; A) = 0 \) then (7.30) indicates that the series in (7.16) is a multiple of the series (7.8) and the multiplier does not depend on \( x \) but may depend on \( \lambda \). This explains the relationship between the expansion representing the eigenfunction \( g \) in (7.8) and the expansion in (7.16) which is expected to be valid for a range of \( \lambda \) wider than the spectrum of \( \mathcal{D} \).

Finally we apply the preceding outline to the case of continuous \( q \)-Jacobi polynomials and give another proof of (6.13)-(6.14). In the case under consideration

\[
(7.31) \quad \xi_n(\alpha, \beta) := \frac{2q^{-n+(\alpha+5/2)/2}(1 - q^{n+\alpha+\beta+1})}{(1 - q)(-q^{(\alpha+\beta+1)/2}; q^{1/2})_2},
\]

\[
(7.32) \quad c_{n,n} := \frac{q^{-n/2}(1 - q^{\alpha+\beta+n+1})(1 - q^{\alpha+\beta+n+2})}{(-q^{(\alpha+\beta+1)/2}; q^{1/2})_2(1 - q^{n+(\alpha+\beta+1)/2})(1 - q^{n+(\alpha+\beta+2)/2})},
\]

\[
(7.33) \quad c_{n,n-1} := \frac{q^{(\alpha+\beta+2-n)/2}(1 - q^{\alpha+\beta+n+1})(1 + q^{n+(\alpha+\beta+1)/2})(1 - q^{(\alpha-\beta)/2})}{(-q^{(\alpha+\beta+1)/2}; q^{1/2})_2(1 - q^{n+(\alpha+\beta)/2})(1 - q^{n+(\alpha+\beta+2)/2})},
\]

\[
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\]
and
\begin{equation}
\tag{7.34}
c_{n,n-2} = - \frac{q^{(3\alpha+\beta+4-n)/2}(1-q^{\alpha+n})(1-q^{\beta+n})}{(-q^{(\alpha+\beta+1)/2};q^{1/2})_{2}(1-q^{\alpha+\beta+1}/2)(1-q^{n+\beta+1}/2)}.
\end{equation}

With the choice
\begin{equation}
\tag{7.35}
u = \frac{2q^{1/2}}{1-q}
\end{equation}
we find
\begin{equation}
\tag{7.36}
B_n(\alpha,\beta) = B_0(\alpha+n,\beta+n) = - \frac{(1-q^{\beta+n})(1+q^{\alpha+\beta+1+n})q^{(n+\alpha+3)/2}}{(1-q^{\alpha+\beta+1+n})(1-q^{\alpha+\beta+1+n})}q^{(n+\alpha+3)/2}
\end{equation}
and
\begin{equation}
\tag{7.37}
C_n(\alpha,\beta) = C_0(\alpha+n,\beta+n) = - \frac{(1-q^{\alpha+1+n})(1-q^{\beta+1+n})q^{n+\alpha+\beta+1}}{(1-q^{\alpha+\beta+1+n})(1-q^{\alpha+\beta+1+n})^2(1-q^{\alpha+\beta+1+n})}
\end{equation}

Therefore (5.13) yields
\begin{equation}
\tag{7.38}
G_n(\lambda;\alpha,\beta) = G_0(\lambda;\alpha+n,\beta+n) = (-\lambda)^{-(\alpha+\beta)/2}X_n(\alpha,\beta)(\lambda).
\end{equation}

When \(A = (\alpha,\beta)\) and \(p_n(x;A)\) are the continuous \(q\)-Jacobi polynomials \(P_n^{(\alpha,\beta)}(x|q)\) we will denote \(E_A(x;\lambda)\) by \(E_{\alpha,\beta}(x;\lambda)\). Now with \(D = D_q\) formula (7.16) becomes
\begin{equation}
\tag{7.39}
E_{\alpha,\beta}(x;\lambda) = \sum_{n=0}^{\infty} q^{n(n-2\alpha)/4} \frac{(q^{\alpha+\beta+1};q)_n}{(q^{\alpha+\beta+1}/2,q^{1/2})_{2n}} \left(\frac{2q^{1/2}}{1-q}\right)^n G_{n-1} \left(\frac{2\lambda q^{1/2}}{1-q};\alpha,\beta\right) P_n^{(\alpha,\beta)}(x|q).
\end{equation}

We now find another solution to (7.17) and prove a uniqueness theorem for solutions of (7.17). We then equate \(E_{\alpha,\beta}(x;\lambda)\) and the second solutions to (7.17) and establish (6.13).

**Lemma 7.2** Assume that \(f(x)\) is an entire function of the complex variable \(x\). If
\begin{equation}
\tag{7.40}
D_q f(x) = \frac{iyq^{1/4}}{1-q} f(x),
\end{equation}
then \(f(x)\) is unique up to a multiplicative function of \(y\).
Lemma 7.2 is essentially Lemma 3.5 in [15].

A calculation gives

\[
\mathcal{D}_q \mathcal{E}_q(x; a, b) = \frac{-2abq^{1/4}}{1-q} \mathcal{E}_q(x; a, b).
\]

Therefore Lemma 7.2 implies

**Theorem 7.3** If \( f \) satisfies the assumptions in Lemma 7.2 then

\[
f(x) = w(y) \mathcal{E}_q(x; -i, y/2).
\]

**Theorem 7.4** The function \([-2\lambda q^{1/2}/(1-q)]^{(\alpha+\beta)/2} E_{\alpha,\beta}(x; \lambda)\) does not depend on \( \alpha \) or \( \beta \).

**Proof.** In general we have

\[
E_A(x; \lambda) = \sum_{n=0}^{\infty} F_n(\lambda; A) p_n(x; A)
\]

\[
= \sum_{n=0}^{\infty} F_n(\lambda; A) \left[ c_{n,n}p_n(x; A + 1) + c_{n,n-1}p_{n-1}(x; A + 1) + c_{n,n-2}p_{n-2}(x; A + 1) \right]
\]

\[
= \sum_{n=0}^{\infty} p_n(x; A + 1) \left[ c_{n,n} F_n(\lambda; A) + c_{n+1,n} F_{n+1}(\lambda; A) + c_{n+2,n} F_{n+2}(\lambda; A) \right]
\]

\[
= \sum_{n=0}^{\infty} p_n(x; A + 1) \lambda \xi_{n+1}(A) F_{n+1}(\lambda; A)
\]

\[
= \lambda u \sum_{n=0}^{\infty} p_n(x; A + 1) u^{n-1} \prod_{j=0}^{n} \frac{c_{j,j}}{\xi_{j+1}(A)} \xi_{n+1}(A) G_n(\lambda u; A)
\]

\[
= \lambda \sum_{n=0}^{\infty} c_{n,n} p_n(x; A + 1) u^n \prod_{j=0}^{n} \frac{c_{j,j}}{\xi_{j+1}(A)} G_{n-1}(\lambda u; A + 1).
\]

In the case of continuous \( q \)-Jacobi polynomials the last equation gives

\[
E_{\alpha,\beta}(x; \lambda) = \lambda \sum_{n=0}^{\infty} \frac{q^{-n/2}(1-q^{\alpha+\beta+n+1})(1-q^{\alpha+\beta+n+2})}{(-q^{(\alpha+\beta+1)/2}; q^{1/2})_2(1-q^{n+(\alpha+\beta+1)/2})(1-q^{n+(\alpha+\beta+2)/2})}
\]

\[
\cdot q^{n(2\alpha-1)/4} \frac{(q^\alpha+\beta+1; \lambda q)_n}{(q^{(\alpha+\beta+1)/2}; q^{1/2})_{2n}} \left( \frac{2q^{1/2}}{1-q} \right)^n G_{n-1} \left( \frac{2\lambda q^{1/2}}{1-q}; \alpha + 1, \beta + 1 \right) P_n^{(\alpha,\beta)}(x|q).
\]
Therefore

\[ E_{\alpha,\beta}(x; \lambda) = \left[ \frac{-2\lambda q^{1/2}(1-q)(q^{\alpha+\beta+1}; q)_2}{(q^{\alpha+\beta+1/2}, -q^{\alpha+\beta+1/2}; q^{1/2})_2} \right] E_{\alpha+1,\beta+1}(x; \lambda). \]

The above functional equation can be put in the form

\[ \left[ -2\lambda q^{1/2}(1-q) \right]^{(\alpha+\beta)/2} E_{\alpha,\beta}(x; \lambda) = \left[ -2\lambda q^{1/2}(1-q) \right]^{(\alpha+\beta+4)/2} E_{\alpha+2,\beta+2}(x; \lambda), \]

and we have

\[ \left[ -2\lambda q^{1/2}(1-q) \right]^{(\alpha+\beta)/2} E_{\alpha,\beta}(x; \lambda) = \lim_{m \to \infty} \left[ -2\lambda q^{1/2}(1-q) \right]^{2m+(\alpha+\beta)/2} E_{2m+\alpha,2m+\beta}(x; \lambda). \]  

Now substitute the right-hand sides of (7.38) and (7.39) for \( G_n \) and \( E_{\alpha,\beta} \) in the right-hand side of (7.45) to get

\[ \left[ -2\lambda q^{1/2}(1-q) \right]^{2m+(\alpha+\beta)/2} E_{2m+\alpha,2m+\beta}(x; \lambda) \approx \lambda \sum_{n=0}^{\infty} q^{n^2/4} (-\lambda)^{-n} q^{-n\alpha/2} P_{\alpha,\beta}^{(\alpha,\beta)}(x|q). \]

But (1.24) and (1.31) imply

\[ P_{\alpha,\beta}^{(\alpha,\beta)}(x|q) \approx \frac{q^{n\alpha/2}}{(q; q)_n} p_n(x; 0, 0, 0|q) = \frac{q^{n\alpha/2}}{(q; q)_n} H_n(x|q), \]

where \( \{H_n(x|q)\}_{0}^{\infty} \) are the continuous \( q \)-Hermite polynomials. Thus the limit on the right-hand side of (7.45) exists and we have established

\[ \left[ -2\lambda q^{1/2}(1-q) \right]^{(\alpha+\beta)/2} E_{\alpha,\beta}(x; \lambda) = \sum_{n=0}^{\infty} \frac{q^{n^2/4} (-\lambda)^{-n}}{(q; q)_n} H_n(x|q). \]

This proves Theorem 7.4.

**Corollary 7.5** We have

\[ \left[ -2\lambda q^{1/2}(1-q) \right]^{(\alpha+\beta)/2} E_{\alpha,\beta}(x; \lambda) = (\lambda^{-2}; q^2)_\infty \mathcal{E}_q(x; -i, i/\lambda). \]

Corollary 7.5 follows from [13] where Ismail and Zhang proved that the right-hand sides of (7.47) and (7.48) are equal.
Theorem 7.6 The expansion of $E_q(x; -i, r)$ in a continuous $q$-Jacobi series is given by (6.13) where $b = q^{(2\alpha+1)/4}$ and $c = q^{(2\beta+1)/4}$.

Proof. From Theorem 7.3 and Corollary 7.4 we see that the right-hand side of (6.13) is $w(r)E_q(x; -i, r)$. Furthermore $w(r)$ does not depend on $\alpha$ or $\beta$ since neither $E_q(x; -i, r)$ nor the right-hand side of (6.13) depend on $\alpha$ or $\beta$. Now $w$ can be found by letting $\alpha$ and $\beta$ tend to $\infty$ then use (7.48).
8 Remarks.

In 1940 Schwartz published an interesting paper [20] containing the following result.

**Theorem 8.1** Let \( \{p_{n,\nu}(x)\} \) be a family of monic polynomials generated by

\[
\begin{align*}
(8.1) & \quad p_{0,\nu}(x) = 1, \quad p_{1,\nu}(x) = x + B_{\nu}, \\
(8.2) & \quad p_{n+1,\nu}(x) = (x + B_{n+\nu}) p_{n,\nu}(x) + C_{n+\nu} p_{n-1,\nu}(x).
\end{align*}
\]

If both

\[
\sum_{n=0}^{\infty} |B_{n+\nu} - a| < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |C_{n+\nu}| < \infty
\]

hold, then \( x^n p_{n,\nu}(a+1/x) \) converges on compact subsets of the complex plane to an entire function.

It is clear from (8.1) and (8.2) that \((p_{n,\nu}(x))^* = p_{n-1,\nu+1}(x)\), hence the continued \( J \)-fraction associated with (8.1) and (8.2) converges to a meromorphic function of \( 1/x \) and the convergence is uniform on compact subsets of the complex plane which neither contain the origin nor contain poles of the limiting function. Schwartz illustrated his theory by applying it to the Lommel polynomials and he mentioned their orthogonality relation.

Somehow Schwartz’s interesting paper [20] was not noticed and neither his results were quoted nor his paper was cited in the standard modern references on orthogonal polynomials [6], [9], [21] and continued fractions, [22], [16]. Many of Schwartz’s results were later rediscovered by others. Dickinson, Pollack and Wannier [7] rediscovered the special case \( B_{n+\nu} = 0 \) of Schwartz’s theorem. It is worth noting that if \( B_{n+\nu} = 0 \) then a theorem of Van Vleck, Theorem 4.55 in [16], states that \( C_{n+\nu} \to 0 \) suffices to establish the uniform convergence of the continued \( J \)-fraction associated with (8.1) and (8.2) to a meromorphic function. The convergence being uniform on compact subsets of the complex plane which neither contain the origin nor contain poles of the limiting function.

In [12] it was proved that

\[
(8.4) \quad X_n^{(5)}(x) := \left( -\frac{D}{x ABC} \right)^n \frac{\left(Dq^{2n}, Dq^{2n-1}, -q^n/x\right)_\infty}{(Aq^n, Bq^n, Cq^n, Dq^n/A, Dq^n/B, Dq^n/C)_\infty}
\]

\[
 \cdot \phi_2 \left( \begin{array}{c}
 Aq^n, Bq^n, Cq^n \\
 Dq^{2n}, -q^n/x
\end{array} \left| q, -\frac{D}{x ABC} \right. \right).
\]
satisfies the three term recurrence relation

(8.5) \[ Z_{n+1}(x) = (x - a_n) Z_n(x) - b_n Z_{n-1}(x), \]

with

(8.6) \[ a_n := -\frac{D}{ABC} - q^{n-1} \frac{(1 - Dq^n/A)(1 - Dq^n/B)(1 - Dq^n/C)}{(1 - Dq^{2n-1})(1 - Dq^{2n-2})} \]

\[ + \frac{D}{ABC} \frac{(1 - Aq^{n-1})(1 - Bq^{n-1})(1 - Cq^{n-1})}{(1 - Dq^{2n-1})(1 - Dq^{2n})}, \]

and

(8.7) \[ b_n := -\frac{D}{ABC} q^{n-2}(1 - Aq^{n-1})(1 - Bq^{n-1})(1 - Cq^{n-1}) \]

\[ \cdot \frac{(1 - Dq^{n-1}/A)(1 - Dq^{n-1}/B)(1 - Dq^{n-1}/C)}{(1 - Dq^{2n-1})(1 - Dq^{2n-2})^2(1 - Dq^{2n-3})}. \]

We next identify (3.12) as a limiting case \( A \to \infty \) of (8.5) with \( B = -C \). When \( A \to \infty \), it is easy to see that

(8.8) \[ a_n \to -\frac{q^{n-1}(1 + Dq^{2n-1})(1 - D/B^2)}{(1 - Dq^{2n-2})(1 - Dq^{2n})}, \]

\[ b_n \to \frac{Dq^{2n-3}(1 - B^2q^{2n-2})(1 - D^2q^{2n-2}/B^2)}{B^2(1 - Dq^{2n-1})(1 - Dq^{2n-2})^2(1 - Dq^{2n-3})}. \]

We replace \( q \) by \( q^{1/2} \) then identify the parameters \( A, B, C, D \) as

(8.9) \[ B = q^{1+\alpha/2} = -C, \quad D = q^{2+\alpha/2}, \quad A \to \infty. \]

It is not difficult to see that if \( Z_n(x) \) satisfies

(8.9) \[ Z_{n+1}(x) = (x + a'_n) Z_n(x) - b'_n Z_{n-1}(x), \]

with

(8.10) \[ a'_n = -\frac{q^{(n-1)/2}(1 + q^{n+\alpha/2+3/2})(1 - q^{(\beta-\alpha)/2})}{(1 - q^{n+1+\alpha/2})(1 - q^{n+2+\alpha/2})}, \]

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\[ b'_n = \frac{q^{n+(\beta-\alpha-3)/2} (1 - q^{n+\alpha+1}) (1 - q^{n+\beta+1})}{(1 - q^{n+(\alpha+\beta+1)/2}) (1 - q^{n+1+(\alpha+\beta)/2})^2 (1 - q^{n+(\alpha+\beta+3)/2})}. \]

It then follows that

\[(8.11) \quad Y_n(x) := q^{(2\alpha+5)n/4} Z_n(xq^{-(2\alpha+5)/4})\]

satisfies (3.12) with \( \mu \) replaced by \( x \). This relationship between solutions of (8.8) and (3.12) enables us to take advantage of the detailed study of solutions of (8.5) contained in [12]. For example one can obtain the minimal solution to (3.12) by inserting the values of \( A, B, C, D \) of (8.8) into \( X_n^{(5)} \) of [12], which remains a minimal solution. This gives an alternate derivation of the form of \( Y_k^{(\alpha,\beta)}(x) \) of (5.11).
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