Erratum: Asymptotic work statistics of periodically driven Ising chains

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Received 18 November 2015
Accepted for publication 12 January 2016
Published 2 March 2016

Online at stacks.iop.org/JSTAT/2016/039901
doi:10.1088/1742-5468/2016/03/039901

1 The demonstration in appendix C, starting from the fourth line after equation C.5 to the end, is incorrect. This does not change the main results, but modifies some details. Here is the corrected version:

We see, from equation (7), that the symmetry relation

\[ \mathbb{H}_-(t) = \sigma_z \mathbb{H}_+(t) \sigma_z \]

is valid at all times. Because \( \sigma_z \) is a time-independent unitary transformation, equation (C.4) implies that \( \mathbb{H}_k^F = \mathbb{H}_k \mathbb{H}_k^F \sigma_z \). Because of equation (C.5), and the relations \( \text{Tr}[\mathbb{H}_k(t)] = 0 \), \( \sigma_z^4 = \sigma_z \), \( \sigma_z \sigma_z = -\sigma_z \), \( \sigma_z \sigma_z = -\sigma_y \), we can write the following second-order-in-\( k \) expansion\textsuperscript{6} of \( \mathbb{H}_k^F \)

\[ \mathbb{H}_k^F = \left( \frac{\hbar}{2} + a_x k^2 \right) \left( a_x - ia_y \right) k \left( a_x + ia_y \right) k \left( -\hbar - a_x k^2 \right) + \mathcal{O}(k^3) \]  

\textsuperscript{6} The vanishing of the trace of \( \mathbb{H}_k^F \) at any \( k \) comes from the vanishing of the trace of \( \mathbb{H}_a(t) \) and the formula \( \text{Tr}[\mathbb{H}_k^F] = \frac{1}{\tau} \int_0^\tau \text{Tr}[\mathbb{H}_a(t)] \mathrm{d}\tau \), which is a corollary of the Liouville’s theorem [49].
In general the coefficients $a_x$, $a_y$ and $a_z$ are non-vanishing; they can vanish in some cases, giving rise to interesting phenomena which we will discuss later. Whenever the resonance condition equation (C.3) is not fulfilled (hence $\tilde{h} \neq 0$), the second-order-in-$k$ expansion of Floquet modes (expressed in the basis $B = \{ \{0\}, \hat{c}^\dagger k \hat{c}^\dagger - k | 0 \} \}$) is

$$\begin{bmatrix} \phi^+_{\text{small } k} \end{bmatrix}_B = \begin{pmatrix} 1 - \frac{1}{8} \frac{a_y^2 + a_x^2}{\tilde{h}^2} k^2 \\ - \frac{1}{2} \frac{a_x + i a_y}{\tilde{h}} k \end{pmatrix}$$

and

$$\begin{bmatrix} \phi^-_{\text{small } k} \end{bmatrix}_B = \begin{pmatrix} \frac{1}{2} \frac{a_x - i a_y}{\tilde{h}} k \\ 1 - \frac{1}{8} \frac{a_x^2 + a_y^2}{\tilde{h}^2} k^2 \end{pmatrix},$$

(C.7)

which applies to the case $\tilde{h} > 0$; these two states should be exchanged if $\tilde{h} < 0$. Moving now to the initial Hamiltonian ground state $| \psi_k^{\text{gs}} \rangle$, the diagonalization of equation (7) (with $h(t) = h$) immediately gives (for $h > h_c$)

$$\begin{bmatrix} | \psi_k^{\text{gs}} \rangle \end{bmatrix}_B = \begin{pmatrix} \frac{i k}{2(h - h_c)} \\ 1 - \frac{8(h - h_c)^2}{k^2} \end{pmatrix}.$$ (C.8)

Hence, for the overlap $| r_k^+ |^2$ we find

$$| r_k^+ |^2 = | \langle \phi^+_k | \psi_k^{\text{gs}} \rangle |^2 = \frac{1}{4} \alpha^2 k^2 + \mathcal{O}(k^3) \quad \text{with} \quad \alpha^2 \equiv \left( \frac{1}{h - h_c} + \frac{a_y}{h} \right)^2 + \left( \frac{a_x}{h} \right)^2$$

which is indeed equation (B.6); this formula is valid also in the case $\tilde{h} < 0$ and $h_1 < h_c$. If $\tilde{h} < 0$ and $h_1 > h_c$, or $\tilde{h} > 0$ and $h_1 < h_c$, we find $| r_k^- |^2 = \alpha^2 k^2 + \mathcal{O}(k^3)$, but the crucial ingredient determining equation (A.9) is identical, since $\xi_k \simeq \alpha^2 k^2$ in both cases (see equation (A.2)).

For the resonant case $\tilde{h} = 0$ we find

$$\begin{bmatrix} \phi^+_{\text{small } k} \end{bmatrix}_B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \frac{1}{2} \frac{a_x}{\sqrt{a_x^2 + a_y^2}} k \\ - a_x + i a_y \left( 1 - \frac{1}{2} \frac{a_x}{\sqrt{a_x^2 + a_y^2}} k \right) \end{pmatrix} \sqrt{a_x^2 + a_y^2}$$

which (if $h_1 > h_c$) gives rise to...
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\[ \text{tr}_k \hat{\mathcal{E}}^2 = \frac{1}{2} \left( 1 - \frac{a_z}{\sqrt{a_x^2 + a_y^2}} \right) + \mathcal{O}(k^3). \]

This is indeed equation (B.5) with \( \beta = a_z/\sqrt{a_x^2 + a_y^2} \). We notice that this formula is valid if \( a_x - ia_y \neq 0 \). This is generally true, up to special cases where there is coherent destruction of tunnelling (CDT) [48]: here \( a_x = a_y = 0 \), and we fall back to equation (B.6). In the Supplemental material of [28] we show that, in the case of a sinusoidal driving \( h(t) = h_0 + A \cos(\omega_0 t + \phi_0) \), CDT occurs if \( h_0 = 1 \) and \( J_0(2A/\omega_0) = 0 \). More in general, if the resonance condition equation (C.3) is valid for \( l \neq 0 \), we can show—exactly with the same arguments used in [28]—that there is CDT whenever \( J_l(2A/\omega_0) = 0 \).
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Received 12 May 2015
Accepted for publication 29 July 2015
Published 26 August 2015

Abstract. We study the work statistics of a periodically-driven integrable closed quantum system, addressing in particular the role played by the presence of a quantum critical point. Taking the example of a one-dimensional transverse Ising model in the presence of a spatially homogeneous but periodically time-varying transverse field of frequency $\omega_0$, we arrive at the characteristic cumulant generating function $G(u)$, which is then used to calculate the work distribution function $P(W)$. By applying the Floquet theory we show that, in the infinite time limit, $P(W)$ converges, starting from the initial ground state, towards an asymptotic steady state value whose small-$W$ behaviour depends only on the properties of the small-wave-vector modes and on a few important ingredients: the time-averaged value of the transverse field, $h_0$, the initial transverse field, $h_i$, and the equilibrium quantum critical point $h_c$, which we find to generate a sequence of non-equilibrium critical points $h_{*l} = h_c + l\omega_0/2$, with $l$ integer. When $h_i \neq h_c$, we find a ‘universal’ edge singularity in $P(W)$ at a threshold.
value of $W_{\text{th}} = 2|\hat{h}_l - \hat{h}_c|$ which is entirely determined by $\hat{h}_l$. The form of that singularity—Dirac delta derivative or square root—depends on $\hat{h}_0$ being or not at a non-equilibrium critical point $\hat{h}_c$. On the contrary, when $\hat{h}_l = \hat{h}_c$, $G(u)$ decays as a power-law for large $u$, leading to different types of edge singularity at $W_{\text{th}} = 0$. Generalizing our calculations to the case in which we initialize the system in a finite temperature density matrix, the irreversible entropy generated by the periodic driving is also shown to reach a steady state value in the infinite time limit.

**Keywords:** spin chains, ladders and planes (theory), stationary states, quantum quenches

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doi:10.1088/1742-5468/2015/08/P08030
1. Introduction

In recent years, there have been many theoretical studies aimed at understanding the non-equilibrium dynamics of closed quantum systems [1, 2], inspired by a series of experiments on cold atomic gases, which are nearly isolated systems with long phase-coherence times, allowing for the study of a coherent quantum dynamics over long time scales [3–8]. These experiments have paved the way to addressing many fundamental questions, such as the role of integrability in thermalization following a quench [1] or the universal scaling of the defects generated when a system is driven across a quantum critical point [2, 9]. Usually, a quantum system is driven out of equilibrium by a slow ramping (annealing) or by a sudden quench of a parameter of the Hamiltonian (for example, the transverse magnetic field in the quantum Ising model discussed in this paper); the subsequent dynamical response of the system is encoded in several quantities, e.g. the Loschmidt echo [10–14], dynamical correlation functions [15], the growth of entanglement entropy [16], time evolution of observables [17–19] and dynamical response functions [20] following a sudden quench, or the change in diagonal entropy [21]. In parallel, in the context of the Jarzynski equality [22] and non-equilibrium fluctuation relations [23], the question of the emergence of thermodynamical laws from a finite quantum system driven out of equilibrium and the generation of irreversible entropy have been addressed in several recent works [24, 25].

One of the ways to characterize the dynamics of an out-of-equilibrium quantum system is to explore the statistics of the performed work, both at zero [26–28] and finite temperatures [29]. Given the non-equilibrium nature of the driving protocol, the work (W) is a stochastic variable and hence described by a probability distribution P(W). Recently, the W = 0 work distribution function P(W = 0) = |⟨Ψ₀|e⁻ⁱᴴₜ/ℏ|Ψ₀⟩|^², which can be viewed as the probability of doing no work in a time t during a double-quench process, has been shown to display non-analyticities as a function of t which can mark the existence of a sequence of dynamical phase transitions in real time [30]. Furthermore, the knowledge of P(W) enables us to obtain information about some universal features, by connecting it to the critical Casimir effect, for sudden quenches ending near the critical point [27]; in particular, there exists a power-law edge singularity in P(W), for small W, which is characterized by an exponent that is independent of the choice of protocol, but rather depends just on the initial and final values of the parameter being quenched [31]. It is usually convenient to define and work with the characteristic function G(u), obtained by Fourier transforming P(W). The characteristic function G(u) has been shown to be closely related to the Loschmidt echo of a quenched quantum system, both at zero [10, 26] and finite temperature [32].

In this paper we focus on a periodically driven integrable closed quantum system, namely the transverse Ising chain [33, 34], with a spatially homogeneous but time-periodic transverse field ℏ(τ) = ℏ(τ + 2π/ω₀), and calculate P(W) stroboscopically at the end of n complete periods τ = 2π/ω₀. It has been shown in the literature [35, 36] that the system reaches a periodic steady state in the limit n → ∞; the residual energy (which is in fact the first moment of P(W))—and indeed essentially any local observable—reaches a stationary value, when observed at times τₙ = nτ, in the thermodynamic
The system does not heat-up indefinitely, in spite of the driving, because the model is integrable, as discussed in [35]. We have observed a similar relaxation to a periodic steady condition also for a genuinely quantum non-local object, the so-called dynamical fidelity [37]. In the present work we will investigate what happens to $G(u)$, and hence to $P(W)$, under such periodic driving in the asymptotic limit $n \to \infty$, and address the question of the universal behaviour emerging in the small-$W$ region.

Working within the framework of the Floquet theory [38, 39], and assuming that the initial state is a Gibbs state at temperature $T$, we provide an analytical form of the characteristic function $G(u)$ in terms of Floquet quasi-energies and corresponding overlaps between the initial state and the Floquet eigenstates. $G(u)$ is then used to arrive at an exact expression of $P(W)$ at zero temperature: we demonstrate that indeed $P(W)$ also tends to ‘synchronize’ with the periodic driving in the limit $n \to \infty$, converging to a well-defined asymptotic work distribution function $P_\infty(W)$, whose small-$W$ behaviour depends only on the properties of the small-wavevector modes, ultimately controlled by the time-averaged value of the transverse field $h_0 = \tau^{-1} \int_0^\tau h(t) \, dt$, and its initial value $h_i = h(t = 0)$. All universal features of the work distribution are encoded into a singularity of $P(W)$ for small $W$. The position $W_{h_l} = 2l h_i - h_c l$ of the edge of that singularity depends only on the distance of $h_i$ from the equilibrium critical point $h_c$, while its detailed form—a Dirac delta derivative or a square root singularity—depends on $h_i$ being or not at a non-equilibrium critical point $h_{c_l}$—determined by $h_{*l} = h_c + l \omega_0 / 2$ with $l$ integer—where the Floquet spectrum turns out to be gapless. (The main Floquet resonance is for $h_0 = h_c$, corresponding to $l = 0$, but $l$ can also in principle be negative.) When $h_i = h_{c_l}$ the generating function $G(u)$ has a power law decay $1/(iu)^p$, which results into a step singularity in $P_\infty(W)$ at $W_{h_l} = 0$ for $p = 1$, or a mild $W^{p-1}$ increase for larger integer $p$. Figure 1 schematically sketches the various possibilities for $P_\infty(W)$ in a $h_i$-versus-$h_0$ phase diagram.

The non-equilibrium phase transitions we find are reminiscent of those discussed in [40], but are different in many aspects, notably in residing at low frequency, as we will discuss.

An important aspect of these results is that the small-$W$ properties of $P_\infty(W)$ are determined (as we are going to show in detail in the paper) exclusively by the small-$k$ modes: the aspects of the dynamics independent of the details of the driving protocol rely only on the large wavelenght modes which encode the universal properties of the ground state in the static system [41, 42]. Remarkably, a similar fact can be observed in two coupled Luttinger liquids undergoing a quantum quench [43]: if the quenched operator is relevant in the renormalization group sense (and then affecting mainly the long wavelength modes) then the phase coherence evolves with a universal scaling function.

Finally we move to the finite temperature case: starting from an initial mixed state at finite temperature, we will show that the irreversible entropy generated during the periodic driving tends to ‘synchronize’ with the driving for $n \to \infty$. It saturates to a steady state value for large $\omega_0$, displaying a sequence of dips and peaks for small and intermediate values of $\omega_0$, respectively. We note in passing a recent study on the work statistics of a periodically driven system which has explored the universal properties of the rate function, that is found to satisfy a lower bound and has a zero when $W$ matches the residual energy [44].
The structure of the paper is as follows. In section 2 we summarize the basic definitions and properties of $P(W)$ and $G(u)$. Next, in section 3 we focus on the case of a quantum Ising chain undergoing a uniform generic periodic driving, showing in section 4 that the stroboscopic $G(u)$ and $P(W)$ converge to an asymptotic value (see appendix A for mathematical details). In section 5 we discuss the behaviour of the asymptotic $P(W)$ in the small-$W$ regime, showing that it is independent of the details of the driving protocol; we substantiate our analysis with numerical results obtained in the case of a sinusoidal driving. In section 6 we discuss briefly the case with finite temperature $T$ in the context of irreversible entropy generation, and in section 7 we draw our conclusions together with perspectives of future work. Technical details of our derivations are contained in three appendices.

2. The work distribution $P(W)$ and its characteristic function $G(u)$

We present here, for the readers’ convenience, some basic facts about the work distribution function, following [23, 45, 46]. Suppose that a closed quantum system undergoes a time-dependent driving such that its Hamiltonian is $\hat{H}(t)$, while the system was at the initial time $t_0 = 0$ in a given (possibly mixed) state $\hat{\rho}(0)$. If $p_n(0)$ is the probability that the system has energy $E_n(0)$ at the initial time, and $P(mt|n)$ the conditional probability that the system is observed to have energy $E_m(t_f)$ at some later time $t_f$, then the work distribution function is defined as:
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\[ P_t(W) = \sum_{n,m} \delta(W - E_n(t_t) + E_n(0)) P(mt_t|n0) p_n(0). \]  

(1)

To deal with the Dirac deltas appearing in the definition of \( P_t(W) \) it is convenient to study the Fourier transform of \( P_t(W) \), arriving at the characteristic function:

\[ G_t(u) = \int_{-\infty}^{+\infty} dW e^{iuW} P_t(W). \]  

(2)

With simple manipulations, and introducing the unitary evolution operator \( \hat{U}(t_t,0) \) for the closed quantum system, it results that:

\[ G_t(u) = \text{Tr}(\hat{U}^\dagger(t_t,0)e^{iu\hat{H}(t_t)}\hat{U}(t_t,0)e^{-iu\hat{H}(0)}\hat{\rho}(0)) = \text{Tr}(e^{iu\hat{H}(t_t)}e^{-iu\hat{H}(0)}\hat{\rho}(0)), \]  

(3)

where \( \hat{H}_{\text{f}}(t_t) = \hat{U}^\dagger(t_t,0)\hat{H}(t_t)\hat{U}(t_t,0) \) is the final Hamiltonian in Heisenberg representation, and we have assumed that the initial state is such that \([\hat{\rho}(0),\hat{H}(0)] = 0 \) (a Gibbs or micro-canonical state would do that). We note, in passing, that the quantum Jarzynski equality \([22,25]\) follows immediately from the previous expression by taking \( u = i\beta \) and assuming an initial Gibbs state \( \hat{\rho}(0) = e^{-\beta(\hat{H}(0)-F_0)} \):

\[ G_t(i\beta) = \int_{-\infty}^{+\infty} dW e^{-\beta W} P_t(W) = e^{-\beta(R-F)} , \]  

(4)

where \( F_0 \) and \( F_f \) correspond to the free energies of the initial and final equilibrium states, respectively.

3. The uniform periodically driven quantum Ising chain

Let us now specialize our discussion to the quantum Ising chain in a uniform time-periodic transverse field (although some progress might be done in the general non-homogeneous case). The Hamiltonian we consider is \([41]\):

\[ \hat{H}(t) = -\frac{J}{2} \sum_{j=1}^{L} [\hat{\sigma}_j^+\hat{\sigma}_{j+1}^- + h(t)\hat{\sigma}_j^z], \]  

(5)

where \( h(t) = h(t+\tau) \) is a generic uniform-in-space transverse field which is periodically driven with frequency \( \omega_0 = 2\pi/\tau \) around the average value

\[ h_0 = \frac{1}{\tau} \int_0^\tau h(t) \, dt. \]  

(6)

The Hamiltonian can always be recast in the form of a quadratic fermionic model, thanks to the Jordan–Wigner transformation \([47]\). Upon Fourier transforming in space to the relevant Jordan–Wigner fermions, we get:

\[ \hat{H}^+(t) = \sum_k^{ABC} \hat{H}_k(t) = \sum_k^{ABC} \begin{bmatrix} \hat{c}_k^\dagger & \hat{c}_{-k} \end{bmatrix} \begin{bmatrix} \mathbb{H}_k(t) \end{bmatrix} \begin{bmatrix} \hat{c}_k \\ \hat{c}_{-k}^\dagger \end{bmatrix}, \]  

(7)

where the \( 2 \times 2 \) matrix \( \mathbb{H}_k(t) \) has the form:
with $\epsilon_k(t) = b(t) - \cos k$ and $\Delta_k = \sin k$, having set $J = 1$. Notice that the previous Hamiltonian is really only a part of the total $\hat{H} = \hat{H}^+ + \hat{H}^-$, i.e. the part living in the subspace with even fermion-parity, for which anti-periodic boundary conditions (ABC) apply, and $k = (2n - 1)\pi/L$ with $n = 1 \cdots L/2$. This is certainly enough for describing the ground state and the dynamics starting from the ground state. At finite temperature, the contributions due to the extra odd-fermion-parity term, $\hat{H}^-$, corresponding to periodic boundary conditions $k$-values, are automatically accounted for when we transform the sum over $k$ into an integral over the half of the Brillouin zone $k \in [0, \pi]$.

We assume that the system is, at time $t = 0$, in the Gibbs ensemble at temperature $T$ for the Hamiltonian $\hat{H}(0)$. Following a standard procedure [35–37], the initial problem is diagonalized by introducing new fermionic operators $\hat{c}^\dagger_k = u_k \hat{c}^\dagger_k + v_k \hat{c}^\dagger_{-k}$ in terms of which the BCS-ground state for each $k$ reads $|\psi^\text{gs}_k\rangle = [u_k + v_k \hat{c}^\dagger_{-k}]|0\rangle$, with energy $-E_k = -\sqrt{\epsilon_k^2(0) + \Delta_k^2}$, and the excited state is $|\psi^\text{ex}_k\rangle = \gamma_k^\dagger |\psi^\text{gs}_k\rangle = [v_k + u_k \hat{c}^\dagger_{-k}]|0\rangle$, with energy $+E_k$. If we are interested in calculating $G_{t=\tau}(u)$ at integer multiples of the period $\tau = 2\pi/\omega_0$, it is sufficient to construct the Floquet modes $|\phi^\pm_k(0)\rangle$ with corresponding Floquet quasi-energies $\pm \mu_k$; this must be done, in general, by a numerical integration of the Schrödinger equation, for each $k$, over a single period [35, 37]. The outcome of that calculation provides the relevant overlaps $r^\pm_k = \langle \phi^\pm_k(0) | \psi^\text{ge}_k \rangle$, with $|r^+_k|^2 + |r^-_k|^2 = 1$, and, by unitarity, $\langle \phi^\pm_k(0) | \psi^\text{ex}_k \rangle = \mp (r^\mp_k)^*$. (In principle, one could also consider $G_{n+\delta t}$, but that requires the knowledge of the full Floquet modes $|\phi^\pm_k(t)\rangle$, i.e. the time-periodic part of the Floquet states, while for $\delta t = 0$, it is enough to know $|\phi^\pm_k(t = 0)\rangle$.) With these basic ingredients, the derivation of a general expression of $G_{n\tau}(u)$ for the uniformly driven Ising chain follows essentially the steps outlined in [37], generalized to an arbitrary finite temperature. The final result can be cast in the form:

$$\ln G_{n\tau}(u) = \sum_{k>0}^{\text{ABC}} \ln \left\{ 1 - \frac{2q_k}{1 + q_k} \sin^2(\mu_k n \tau) \left[ (1 - e^{2iuE_k})(1 - f_k) + (1 - e^{-2iuE_k}) f_k \right] \right\},$$

where $f_k = 1/(e^{2iuE_k} + 1)$ denotes the Fermi occupation function (observe that creating an excitation costs here an energy $2E_k$) and

$$q_k = \frac{2 |r^+_k|^2 |r^-_k|^2}{|r^+_k|^4 + |r^-_k|^4} = \frac{2 |r^+_k|^2 |r^-_k|^2}{1 - 2 |r^+_k|^2 |r^-_k|^2}.$$  

Had we chosen to work with the Laplace transform of the $P(W)$, rather than with the Fourier transform $G(u)$, we would have obtained a similar expression with the formal replacement $u \rightarrow is$. For $T = 0$ we would get

$$\ln G_{n\tau}^{T=0}(is) = \sum_{k>0}^{\text{ABC}} \ln \left\{ 1 - \frac{2q_k}{1 + q_k} \sin^2(\mu_k n \tau) \right\},$$

where $q_k = 2|\gamma_k|^2/[1 + |\gamma_k|^2]$. The outcome of that calculation provides the relevant overlaps $r^\pm_k = \langle \phi^\pm_k(0) | \psi^\text{ge}_k \rangle$, with $|r^+_k|^2 + |r^-_k|^2 = 1$, and, by unitarity, $\langle \phi^\pm_k(0) | \psi^\text{ex}_k \rangle = \mp (r^\mp_k)^*$. (In principle, one could also consider $G_{n+\delta t}$, but that requires the knowledge of the full Floquet modes $|\phi^\pm_k(t)\rangle$, i.e. the time-periodic part of the Floquet states, while for $\delta t = 0$, it is enough to know $|\phi^\pm_k(t = 0)\rangle$.) With these basic ingredients, the derivation of a general expression of $G_{n\tau}(u)$ for the uniformly driven Ising chain follows essentially the steps outlined in [37], generalized to an arbitrary finite temperature. The final result can be cast in the form:

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where $q_k = 2|\gamma_k|^2/[1 + |\gamma_k|^2].$
which actually shows that the expression for $G(i)$ is better behaved at $T = 0$, at the
expense of having to perform an inverse Laplace transform to recover $P(W)$. In the
thermodynamic limit, transforming the sum over $k$ into an integral, we can finally
write:

$$
\lim_{n \to \infty} \frac{\ln G_{nT=0}(is)}{L} \to \int_0^\pi \frac{dk}{2\pi} \ln \left\{ 1 - \frac{2q_k}{1 + q_k} \sin^2(\mu_k n\tau)(1 - e^{-2sE_k}) \right\}.
$$

(12)

This object (see also equation (15)) is the so-called cumulant generating function and
coincides with the expression of Smacchia et al [31]. We observe also that, in the
limit $s \to \infty$, we recover the expression for $g_n$, the logarithm of the dynamical fidelity
$F(\tau) = |\langle \Phi_0 | \Psi(\tau) \rangle|^2 = e^{Lg_n}$ discussed in [37] (see equation (8) there). It is worth
mentioning that $F(n\tau)$ has been found [37] to exhibit sharp non-analyticities as a function of $\tau$ when $s \to 2\pi/\tau \leq 4J$; whether these non-analyticities are connected to the
time-periodic counterpart of the dynamical phase transitions studied in [28] is an open
question at the moment.

4. Steady state of the work probability distribution $P_{n\tau}(W)$ for $n \to \infty$

The question we are now going to address is if the probability distribution of the work
$P_{n\tau}(W)$ tends to ‘synchronize’ with the periodic driving in the asymptotic limit $t_n \to \infty$.
When viewing at the system stroboscopically at the times $t_n = n\tau$ (integer multiples of the period $\tau$), what we want to understand is if $P_{n\tau}(W)$ converges towards a well defined
asymptotic work distribution for $n \to \infty$. The answer is positive, as we are now going
to show. We know from previous work [35] that the quantum average of the work (i.e. the first moment of $P_{n\tau}(W)$) indeed reaches a ‘steady state’ for $n \to \infty$, in the thermo-
dynamic limit. Moreover, as we have recently shown in [37], the $\delta(W)$ Dirac-delta part of the distribution $P_{n\tau}(W)$ (see, for instance, equation (22) below) alias the large-$s$ limit
$g_n = \lim_{s \to \infty} e^{Lg_n}$—which corresponds to the dynamical fidelity—also reaches
a well defined ‘steady state’ for $n \to \infty$: $\lim_{n \to \infty} g_n = g_\infty$. Our claim now is that all the cumulants of $P_{n\tau}(W)$ reach such a ‘steady state’, and therefore so does the whole proba-
bility distribution. To see this, it is enough to show that the whole cumulant generating

---

6 The relevant quantity in [31] translates as follows in terms of our quantities:

$$
|y_0(n\tau)|^2 = \frac{2q_k \sin^2(\mu_k n\tau)}{1 + q_k \cos(2\mu_k n\tau)}.
$$

We also mention that the analysis of [31] identifies the large-$s$ limit of $\ln G_{nT=0}(is)/L$ with a ‘surface’ free-energy
contribution of a 1-dimensional quantum system having a 2-dimensional classical counterpart

$$
-2\gamma_\text{surf} = \int_0^{\tau} \frac{dk}{2\pi} \ln[1 + |y_0(n\tau)|^2],
$$

which coincides with our $g_n$.

7 Strictly speaking, when $L \to \infty$ the $P(W)$ becomes narrower and narrower, on the scale of the average work
$\langle W \rangle \sim L$, with fluctuations which scale as $\sqrt{L}$. Nevertheless, when $L$ is large but not $\infty$ and the approximation of
having set $L \to \infty$ in transforming the sum into an integral holds only until a certain finite time $t^* \sim L$, the question we are asking is meaningful, provided the ‘steady state’ is effectively reached before $t^*$.
function, equation (12), reaches a steady state. With an argument which generalizes that of [37], whose details are given in appendix A, one can show that when \( n \to \infty \) this quantity tends to the stationary value

\[
\xi_k \equiv \left| \tau_k^+ \right|^2 \left| \tau_k^- \right|^2 (1 - e^{-2sE_k}).
\]

The numerical results shown in figure 2 perfectly confirm this analytic prediction: there is a transient—whose details depend on the parameters, for instance on the average field \( h_0 \)—which then leads to an asymptotic result for \( n \to \infty \) given by the simple
analytic expression in equation (13). Figure 2 also illustrates (bottom panel) a case in which the frequency \( \omega_0 \) is such that \( J_0(2A/\omega_0) \approx 0 \), where \( J_0(x) \) is the 0th-order Bessel function, and there is coherent destruction of tunnelling \([39, 48]\); observe that \( \left| \ln \frac{G_\infty(0)}{L} \right| \) has a very small magnitude for this value of \( \omega_0 \).

The cumulants of the asymptotic work distribution are obtained as:

\[
K_m = (-1)^m \frac{d^m}{ds^m} \ln G_\infty = 0 (is) |_{s=0}.
\]

The first cumulant (which is the quantum average of the work performed) is given by:

\[
K_1 = \langle W \rangle_\infty = 4L \int_0^\pi \frac{dk}{2\pi} \left| r_{k+}^0 \right|^2 \left| r_{k-}^0 \right|^2 E_k.
\]

This coincides with the result we would obtain by evaluating

\[
\lim_{n \to \infty} \int_0^\pi \frac{dk}{2\pi} \langle \psi_k(n\tau)| \hat{H}(0)|\psi_k(n\tau) \rangle - E_k^0(0),
\]

which is quite easy to calculate directly. The second cumulant is the variance of the work distribution and is given by:

\[
K_2 = \sigma^2_\infty = L \int_0^\pi \frac{dk}{2\pi} \left[ 4 \left| r_{k+}^0 \right|^2 \left| r_{k-}^0 \right|^2 (1 + 3 \left| r_{k+}^0 \right|^2 \left| r_{k-}^0 \right|^2) E_k^2 \right].
\]

We notice that the \( P(W) \) tends to become narrower and narrower in the thermodynamic limit, as expected, because \( \sigma_\infty / \langle W \rangle_\infty \approx 1/\sqrt{L} \).

5. Universal edge singularity at small \( W \) in \( P_\infty(W) \)

Inspired by the results of \([31]\), we now discuss the behaviour of the asymptotic work probability distribution at small values of \( W \), especially in connection with aspects which are independent of the details of the specific driving protocol. From a technical point of view, the small-\( W \) behaviour of \( P_\infty(W) \) is encoded in the large-\( s \) behaviour of \( G_\infty(is) \), which we can evaluate by means of equation (13). We will show that, indeed, the important ingredients are: (i) the value \( h_1 \) of the initial transverse field \( h(t=0) \), and (ii) the value of the average field \( h_0 \) (and the frequency \( \omega_0 \)), determining if the Floquet spectrum shows a resonance, and is gapless, at \( k = 0 \), or not. Indeed, the value of \( h_1 \) determines the position \( W_\infty \) of the singularity in \( P(W) \) which we observe, while the form of this singularity is determined by the possibility that the Floquet spectrum has a resonance at \( k = 0 \), which is entirely determined by the time-averaged field \( h_0 \) (see equation (6)) hitting a non-equilibrium quantum critical point \( h_{\text{crit}} = h_c + l \omega_0 / 2 \), with \( l \) integer, i.e.

\[
2(h_0 - h_c) = l \omega_0 \quad \text{for some integer } l \in \mathbb{Z},
\]

\[\text{doi:10.1088/1742-5468/2015/08/P08030}\]
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is obeyed. Although these dynamical critical points were already found, with sinusoidal driving, in a high frequency regime within the rotating wave approximation (RWA) [40], here we find them for a generic periodic driving \( h(t) \): details can be found in appendix C. The small-\( W \) universal behaviour of \( P_\infty(W) \) we describe below relies, in the end, only on the properties of the small-\( k \) modes, in particular on the small-\( k \) behaviour of \( |r_k^+|^2 \) in equation (14): we find that if the resonance condition in equation (18) is fulfilled\(^8\), then

\[
|r_k^+|^2 = \frac{1}{2} - \frac{1}{2} \beta k + O(k^2); \\
\text{otherwise}
\]

\[
|r_k^+|^2 = \frac{\alpha^2}{4} k^2 + O(k^4).
\]

The precise values of \( \alpha \) and \( \beta \) depend on the specific form of driving, but the functional form of \( |r_k^+|^2 \) and the Floquet spectrum being gapless at \( k = 0 \) or not depend only on the fulfillment of equation (18). Depending on \( h_i \) and \( h_0 \), we can distinguish essentially three different behaviours of \( P_\infty(W) \) in the small-\( W \) limit:

**Case (a) \( h_i \neq h_c \) and non-resonant Floquet spectrum.** In this case we obtain (see appendix B for the derivation) an approximate analytical formula for \( G_\infty(i s) \), valid when \( s \gg |h_i - h_c| \):

\[
G_\infty(i s) \simeq e^{Lg_\infty} \left( 1 + \frac{a L}{s^{3/2}} e^{-2s|h_i - h_c|} \right).
\]

The inverse Laplace transform predicts that the small-\( W \) behaviour of \( P_\infty(W) \) is given by\(^9\)

\[
P_\infty(W) \simeq e^{Lg_\infty} \left[ \delta(W) + \frac{2aL}{\sqrt{\pi}} \sqrt{W - 2|h_i - h_c|} \theta(W - 2|h_i - h_c|) \right],
\]

which applies whenever \( W \lesssim (2 + \ln 2)|h_i - h_c| \). In both expressions the constant \( a \) is given by:

\[
a \equiv \frac{\alpha^2}{16 \sqrt{\pi}} \left( \frac{|h_i - h_c|}{h_i} \right)^{3/2},
\]

\(^8\) As we show in appendix C there are situations in which equation (20) is valid for \( |r_k^-|^2 \) and not for \( |r_k^+|^2 \). Nevertheless, all the results are unchanged: since \( |r_k^-|^2 + |r_k^+|^2 = 1 \), exchanging the roles of \( |r_k^-|^2 \) and \( |r_k^+|^2 \) has no effect, as equation (13) is symmetric under such an exchange.

\(^9\) Remarkably, the form of the singularity in equation (22) for the asymptotic work-distribution function is the same found in [31] in the case of a generic quench starting and ending into the same paramagnetic or ferromagnetic case. This is exactly what we are doing in this periodic driving protocol.
where $\alpha^2$ is such that $|r_k^+|^2 \simeq \alpha^2 k^2/4$ for small $k$ (see equation (20)). Equation (22) predicts an edge singularity in the asymptotic work distribution function at a precise value of $W$ which is totally independent of the details of the periodic protocol (and even of the frequency) but depends only on the initial value $h_i$ of the field. The details of the protocol enter into the strength of the singularity (the coefficient $a$). We notice that the threshold $2|h_i - h_c|$ is the energy which has to be provided to the system to generate an excitation in the $k = 0$ mode; moreover, also the form of the singularity is only determined, through the constant $a$, by the small-$k$ Floquet modes, as detailed in appendix C. So, the behaviour at small $W$ of the work distribution function is dominated by the modes of lowest energy: in retrospective, this is a very reasonable finding. We stress that the previous analytical expressions are approximations valid in a precise range of $s$ or $W$. The only condition for the validity of these approximate formulas, as detailed in appendix B, is that the driving field does not start from the critical point value, i.e. $h_i \neq h_c$ and there are no resonances at $k = 0$ in the Floquet spectrum. We show some instances of the validity of equation (21) in the upper panel of figure 3, where we numerically evaluate $\frac{\ln G_\infty(i\omega)}{L}$ for $h(t) = h_0 + A \cos(\omega_0 t + \phi_0)$ and plot $R(s) = \frac{\ln G_\infty(i\omega)}{L} - g_\infty\sqrt{|e^{-2|h_i - h_c|s}/s^{3/2}|}$ versus $s$ for different values of $h_0 \neq h_c$, $h_i \neq h_c$ and $\omega_0$. We see that, for large $s$, $R(s)$ tends towards a constant, which equation (21) predicts to be $a$, equation (23), denoted by horizontal lines (the values of $\alpha^2$ were obtained by a numerical fitting of $|r_k^+|^2$ for small $k$, according to equation (20)). Overall, we see that there is a good agreement with the asymptotic value of $R(s)$.

Case (b) $h_i \neq h_c$ and resonant Floquet spectrum (equation (18) fulfilled). In this case the Gaussian analysis performed in appendix B fails. We find, nevertheless, an approximate form of $G_\infty(is)$ for large $s$ ($s \gg 1/|h_i - h_c|$) as

$$G_\infty(is) \simeq e^{Lg_\infty}(1 + L a_c s e^{-2|h_i - h_c|} + \cdots),$$

as long as there is no coherent destruction of tunnelling, i.e. for a resonance of order $l$, we have that $J_l(2A/\omega_0) \neq 0$ ($J_l$ being the Bessel function of the first kind of order $l$ [49]). The resulting $P(W)$, after inverse Laplace transform, has a very singular contribution

$$P_\infty(W) \simeq e^{Lg_\infty}(\delta(W) + a_c L \delta'(W - 2|h_i - h_c|) \theta(W - 2|h_i - h_c|) + \cdots).$$

The lower panel of figure 3 shows plots of $R_c(s) \equiv \frac{\ln G_\infty(is)}{L} - g_\infty |se^{-2|h_i - h_c|}|$ for different resonant cases with $l = 0$ and $l = 1$.

Case (c) $h_i = h_c$: When the initial field is critical, $E_k$ is gapless and the behaviour of $G_\infty(is)$ is power-law rather than exponential. The upper panel of figure 4 illustrates several cases with $h_i = h_c$, for a driving of the form $h(t) = h_0 + A \cos(\omega_0 t + \phi_0)$ with $h_0$ such that the Floquet spectrum is resonant with $l = 0$, i.e. $h_0 = h_i = h_c$ (obtained for $\phi_0 = \pm \pi / 2$), showing a clear power-law decay of the form $1/s$:

$$G_\infty(is) \simeq e^{Lg_\infty}\left(1 + \frac{D}{s}\right).$$
Remarkably, this $1/s$ decay is valid also for $\omega_0$ such that there is coherent destruction of tunnelling, for instance when $\omega_0 = 0.3623\ldots$, where $J_0(2A/\omega_0) = 0$.) Correspondingly, we predict a step-singularity in $P_{\infty}(W)$

$$
P_{\infty}(W) \simeq e^{\log(W) + D \theta(W)},$$

(27)

where $\theta$ is the Heaviside function. We notice that in [31] the authors find a very similar formula for $P_t(W)$ (when the time $t$ is finite) in an Ising chain undergoing

**Figure 3.** Asymptotic cumulant generating function for a periodically driven uniform quantum Ising chain. In these figures we assume a driving $h(t) = h_0 + A \cos(\omega_0 t + \phi_0)$, with $A = 1$ and $\phi_0 = 0$. (Top) Driving with $h_0 \neq h_c = 1$ and $h_1 \neq h_c = 1$, plot of $R(s) = \frac{[\ln G_{\infty}(is)]}{[e^{-2|h_1-h_c|s}/s^{3/2}]}$ versus $s$. For large $s$, we see a convergence towards a finite limit ($a$, as defined in equation (23) and calculated by numerically fitting the coefficient $\alpha^2$ appearing in $|r_k|^2 = \alpha^2 k^2/4$ for small $k$). (Bottom) Driving with $h_1 = h_c = 1$ but $h_0$ fulfilling the Floquet resonance condition equation (18) (we take instances with resonances at $l = 0$ and $l = 1$). Plot of $R_c(s) = \frac{[\ln G_{\infty}(is)]}{[s e^{-2|h_1-h_c|s}]}$ versus $s$, showing the convergence towards a finite limit for large $s$, in agreement with equation (24). Notice that cases where there is coherent destruction of tunnelling fail to be described by such a formula.
a generic (non-necessarily periodic) driving which ends at the critical point. In our case, we consider the asymptotic behaviour and, thanks to periodicity, whenever \( h_1 = h_c \) the system not only ends but also starts at the critical point. It turns out that the case \( h_1 = h_c \) is quite rich, and other power-laws are possible if the Floquet spectrum is resonant at \( k = 0 \) with non-zero values of the integer \( l \) in equation (18). For instance, the bottom panel of figure 4 shows cases with \( h_1 = h_c = 1 \) for \( h_0 = 0 \) and \( \phi_0 = 0 \). We see a \( 1/s^b \) decay with \( b = 3 \) or 1, depending on the Floquet spectrum being resonant \( (2 = l \omega_0) \) or not in \( k = 0 \).

Figure 4. Asymptotic cumulant generating function for a periodically driven uniform quantum Ising chain. In these figures we assume a driving \( h(t) = h_0 + A \cos(\omega_0 t + \phi_0) \), with \( A = 1 \). (Top) Instances of the critical case with \( h_1 = h_0 = h_c \) (obtained taking \( \phi_0 = -\pi/2 \)) and different frequencies: a power-law decay like \( 1/s \) is confirmed in agreement with equation (26). (Bottom) The critical case \( h_1 = h_c = 1 \) for \( h_0 = 0 \) and \( \phi_0 = 0 \). We see a \( 1/s^b \) decay with \( b = 3 \) or 1, depending on the Floquet spectrum being resonant \( (2 = l \omega_0) \) or not in \( k = 0 \).
6. Finite temperature results: irreversible entropy generation

One can also calculate the average work performed in the finite temperature case, where the initial state is an equilibrium Gibbs state at a finite temperature \( k_B T = \beta^{-1} \). Using equation (9) for \( \ln G_{n\tau}(u) \), and the fact that the average work performed in a time \( t_f = n\tau \) is given by \(-i\partial \ln G_{n\tau}(u)/\partial u|_{u=0}\), one can readily arrive at the expression:

\[
\langle W \rangle_{irr} = 4L \int_0^\pi \frac{dk}{2\pi} (1 - \cos(2\mu_k n\tau)) |r_k^+|^2 |r_k^-|^2 E_k \tanh(\beta E_k),
\]  

which generalizes equation (16) to finite \( T \) and finite \( n \). When \( n \to \infty \), the rapidly oscillating term \( \cos(2\mu_k n\tau) \) gives a contribution that averages to zero, and we get:

\[
\langle W \rangle_{\infty} = 4L \int_0^\pi \frac{dk}{2\pi} |r_k^+|^2 |r_k^-|^2 E_k \tanh(\beta E_k).
\]  

The statistical nature of the work for a finite system—evidenced by equation (1)—calls for a second law of thermodynamics written as \( \langle W \rangle \geq \Delta F \), which relates the average work performed to the difference in equilibrium free energy \( \Delta F \) corresponding to the initial and final parameter values: here the equality holds only for a quasi-static process. Calling \( \langle W^{irr} \rangle = \langle W \rangle - \Delta F \) the difference between \( \langle W \rangle \) and \( \Delta F \)—and viewing it as the average irreversible work done—we can recast the second law in the form \( \langle W^{irr} \rangle \geq 0 \). The heat transfer between the closed system and the bath being zero, the entire contribution to the entropy generation is in fact due to \( \langle W^{irr} \rangle \), and one can define the irreversible entropy generated as \( \Delta S^{irr} = \beta \langle W^{irr} \rangle \) [22, 24].
Since at stroboscopic times $t_n = n\tau$ the Hamiltonian returns to the original value $\hat{H}(0)$, so that $\Delta F = 0$, one can define an irreversible entropy increase, in the limit $n \to \infty$, as $\Delta S^{\text{irr}} = \beta \langle W \rangle_\infty$. In figure 5 we show that for a driving protocol $h(t) = 1 + \cos(\omega_0 t)$, $\Delta S^{\text{irr}}$ indeed saturates to a steady state value, like the residual energy [35], displaying a sequence of well defined dips and peaks as a function of $\omega_0$: in the small $\omega_0$ regime, there are dips at certain frequencies for which $J_0(2h/h_0) = 0$, a consequence of the coherent destruction of tunnelling [48]. In the intermediate range, on the contrary, one finds peaks at $\omega_0 = 4/p$, with $p$ integer, due to quasi-degeneracies in the Floquet spectrum [35].

7. Conclusion

In conclusion, we have studied a periodically driven transverse-field Ising model and analyzed the behaviour of the stroboscopic characteristic function $G_{nr}(is)$, and hence the stroboscopic work distribution function $P_n(W)$, after $n$ complete periods of driving. Our study establishes that, in the thermodynamic limit, $P_n$ indeed converges for $n \to \infty$ towards a well defined steady state value $P_\infty(W)$ which reproduces the exact asymptotic value of the first cumulant $\langle W \rangle_\infty$ (i.e. the asymptotic value of the average work performed on the system) derived earlier [35]. In the limit $s \to \infty$, on the other hand, $G_{nr}(is)$ reduces to the stroboscopic dynamical fidelity [37].

For large $s$, we are able to provide asymptotic analytical expressions for $G_{\infty}(is)$ and, by means of inverse Laplace transforms, we can derive corresponding expressions describing the small-$W$ behaviour of $P_\infty(W)$. The small-$W$ properties of $P_\infty(W)$ depend strongly on the fact that there is a static critical point $h_c$, and any periodic driving induces further non-equilibrium critical points where the gap in the Floquet spectrum closes up. This finding is in line with the study reported in [40], where, however, the relevant regime was one of large-amplitude driving at large frequencies, and a rotating wave approximation was appropriate. Here, on the contrary, the exact resonances we find reside at low frequencies: for a fixed average field $h_0$, at $\omega = 2(h_0 - h_c)/l$. In any case, the form of the singularity in the work distribution turns out to be a useful detector of such non-equilibrium phase transitions. According to the way the external periodic driving field relates to these critical points we can observe different phenomena. The time-averaged value of the field $h_0$ and its initial value $h_i$ happen to be crucial. Whenever $h_i$ is different from the static critical point $h_c$ and $h_0$ differs from any non-equilibrium critical point (i.e. $2(h_0 - h_c) = \omega_0 \forall l \in \mathbb{Z}$), the asymptotic $P_\infty(W)$ is characterized by a universal edge singularity whose position $W_{\text{th}} = 2|h_i - h_c|l$ depends only on the value of $h_i$. The information about the specific protocol appears only in the strength of the singularity. The low-energy behaviour of $P_\infty(W)$ is essentially determined by the lowest excitation modes and the corresponding energy gap. If $h_i \neq h_c$ but $h_0$ is dynamically critical (i.e. $2(h_0 - h_c) = \omega_0$ for some integer $l \in \mathbb{Z}$), the form of this singularity changes, and becomes a Dirac delta derivative. A completely different behaviour emerges if the initial field is critical, $h_i = h_c$: in this case, the cumulant generating function (more precisely, $\ln \frac{G_{\infty}(is)}{L} - g_{\infty}$) has a power-law decay, as $1/s$, resulting in a step-function contribution $P_\infty(W)$, but other decays, as $1/s^3$, can also be seen if the Floquet spectrum is resonant with $l = 0$. Generalizing our investigation to the finite
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temperature case, we have shown that the irreversible entropy \(\Delta S_{\text{irr}}\), obtained using the first cumulant of the finite temperature characteristic function, also synchronizes with the periodic driving for \(n \to \infty\) and converges to a steady state value for large \(\omega_0\).

Summarizing, we see a strong relationship between the features of \(P_\infty(W)\), the existence of nonequilibrium quantum critical points and the way the driving field relates to them. This work is a first step towards the application of time-periodic probes to understand the existence of a quantum phase transition by looking at the work distribution function. In this sense, we are generalizing the very interesting works in [26–28, 50], which refer to the case of a sudden quench. In this perspective, it is also interesting to see if it is possible to induce non-equilibrium phase transitions in systems without static transitions and how this influences the work statistics. Another possible direction is to see how the quantum driven system being regular or ergodic ([51–54]) influences the work statistics. A lot of work still remains to be done, starting from the consideration of other tractable cases, like the Dicke model [50].

Acknowledgments

We acknowledge discussions with E G Dalla Torre. Research was supported by the Coleman–Soref foundation, by MIUR, through PRIN-2010LLKJBX_001, by SNSF, through SINERGIA Project CRSII2 136287/1, by ISF under grant agreement n. 203617 and by the EU FP7 under grant agreement n. 280555, and the ERC Advanced Research Grant N. 320796 MODPHYSFRICT. SS acknowledges CSIR, India and AD acknowledges SERB, DST, India for financial support. AD and SS acknowledges Abdus Salam ICTP, Trieste, for hospitality.

Appendix A

In this appendix we show how the stroboscopic cumulant generating function tends towards the stationary value given by equation (13). Our first step is to expand the logarithm in equation (12), which we report here for the reader’s convenience

\[
\ln \frac{G^{T=0}_{\text{тир}}(is)}{L} = \int_0^\pi \frac{dk}{2\pi} \ln \left\{ 1 - \frac{2q_k}{1 + q_k} \sin^2(\mu_k n\tau)(1 - e^{-2sE_k}) \right\}. \quad (A.1)
\]

To that purpose, we first show that the second term inside the logarithm is <1. We know that \(\sin^2(n\mu_k \tau) \leq 1\) and \(|1 - e^{-2sE_k}| < 1\) whenever \(s > 0\) and \(s \neq \infty\). As for the overall prefactor, we notice that

\[
\xi_k \equiv \frac{2q_k}{1 + q_k} = 4 |r_k^+|^2 |r_k^-|^2 = 4 |r_k^+|^2 (1 - |r_k^+|^2) \leq 1,
\]

where the value \(\xi_k = 1\) is obtained for \(|r_k^+|^2 = 1/2\). Hence, the expansion of the logarithm is certainly possible for all \(s < \infty\). Defining

doi:10.1088/1742-5468/2015/08/P08030
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\[ \xi_k(s) = \frac{2q_k}{1 + q_k}(1 - e^{-2sE_k}) = \xi_k(1 - e^{-2sE_k}), \quad (A.3) \]

we have:

\[ \ln \frac{G_{nr}^{T=0}(is)}{L} = -\sum_{m=1}^{\infty} \int_{0}^{\pi} \frac{dk}{2\pi} \frac{\xi_k^m(s)}{m} \sin^{2m}(\mu_k n \tau), \quad (A.4) \]

where we have exchanged the integral and the sum over \( m \), due to the dominated convergence theorem. We then write a binomial expansion of the sine term in terms of exponentials:

\[ \sin^{2m}(\mu_k n \tau) = (-1)^m \sum_{j=0}^{2m} \binom{2m}{j} (-1)^j e^{2i(m-j)\mu_k n \tau}. \quad (A.5) \]

The sum over \( j \), for each \( m \), has a finite number of terms: there is no problem in exchanging the integral over \( k \) with this sum. We now observe that the \( j \neq m \) terms contain rapidly oscillating factors and vanish in the limit \( n \to \infty \), thanks to the Riemann–Lebesgue lemma and the smoothness of the factors \( \xi_k^m \). Hence, in the limit \( n \to \infty \), we retain only the \( j = m \) terms and write

\[ \lim_{n \to \infty} \ln \frac{G_{nr}^{T=0}(is)}{L} = -\int_{0}^{\pi} \frac{dk}{2\pi} \sum_{m=1}^{\infty} \frac{1}{4^m} \binom{2m}{m} \xi_k^m(s) \frac{\xi_k^m}{m}. \quad (A.6) \]

We can write this expression in a closed form, by defining

\[ f(\xi) \equiv \sum_{m=1}^{\infty} \frac{1}{4^m} \binom{2m}{m} \frac{\xi_k^m}{m} \]

and noticing that:

\[ \frac{d}{d\xi} f(\xi) = \frac{1}{\xi} \left[ 2\sqrt{\xi} \frac{d}{d\xi} \arcsin(\sqrt{\xi}) - 1 \right], \quad (A.7) \]

which can be integrated to give

\[ f(\xi) = \int_{0}^{\xi} \frac{1}{\xi'} \left[ \frac{1}{\sqrt{1-\xi'}} - 1 \right] d\xi' = 2 \int_{0}^{\arcsin(\sqrt{\xi})} \tan(\frac{\eta}{2}) d\eta = -4 \ln(\cos(\eta')) \frac{1}{2} \arcsin(\sqrt{\xi}) = -2 \ln \left[ \frac{1}{2} \left( 1 + \sqrt{1-\xi} \right) \right]. \quad (A.8) \]

(In the last steps we have substituted \( \xi = \sin^2(\eta) \) and \( \eta' = \eta/2 \).) In conclusion, we arrive at the desired result:

\[ \lim_{n \to \infty} \ln \frac{G_{nr}^{T=0}(is)}{L} = 2 \int_{0}^{\pi} \frac{dk}{2\pi} \ln \left[ \frac{1 + \sqrt{1 - \xi_k(s)}}{2} \right] \quad (A.9) \]

where \( \xi_k(s) \) is given in equation (A.3).

doi:10.1088/1742-5468/2015/08/P08030
Appendix B

We analyze here the asymptotic large-\(s\) behaviour of \(G^{T=0}_\infty(is)\). We start by assuming that the initial transverse field is not critical, \(h_1 \neq h_c = 1\), so that there is a gap in the spectrum of the initial Hamiltonian, \(E_k \geq |h_1 - h_c| > 0\). We need this condition because we would like to expand equation (A.9) to lowest order in \(e^{-sE_k}\) and this is possible, provided \(E_k > 0\) and \(s \gg |h_1 - h_c|/h_1\). With the previously defined shorthand \(\xi_k \equiv \xi_k(s \to \infty) = 2q_s/(1 + q_s)\) (which is a positive quantity in \([0, 1]\)) we can rewrite equation (A.9) as

\[
\log G^{T=0}_\infty(is) = 2 \int_0^\pi \frac{dk}{2\pi} \ln \left[ \frac{1 + \sqrt{1 - \xi_k + \xi_k e^{-2sE_k}}}{2} \right]. \tag{B.1}
\]

Expanding this to first order in \(e^{-2sE_k}\) (with the assumption \(\xi_k \neq 1\), that is \(\tau_k^2 \neq 1/2\), see comments below) we find:

\[
\log G^{T=0}_\infty(is) \approx \log G_{\infty}(is) + \int_0^\pi \frac{dk}{2\pi} \frac{\xi_k}{\sqrt{1 - \xi_k}} \left( 1 + \sqrt{1 - \xi_k} - e^{-2sE_k} \right). \tag{B.2}
\]

The expansion is questionable whenever there are \(k\)-points such that \(\xi_k = 1\). We will see below that this is indeed the case at \(k = 0\) whenever the field oscillates around a non-equilibrium critical value given by equation (18) and the Floquet spectrum is resonant in \(k = 0\). Even restricting ourselves to non-resonant cases, we would possibly find \(k\)-points where \(\xi_k = 1\), but this time with \(k_0 > 0\). This would still seem to be an issue at first glance: indeed, near these points we would expand \(\xi_k\) quadratically, \(\xi_k = 1 - \lambda^2(k - k_0)^2 + \mathcal{O}((k - k_0)^4)\), and we would therefore have, in the integrand of equation (B.2), some logarithmic singularities originated by terms of the form \(1/(k - k_0)\). In the neighborhood of these points where \(\xi_k = 1\), however, a different expansion is more appropriate. Considering one such \(k_0\) where \(\xi_k = 1\), we can write the contribution to \(\log G^{T=0}_\infty(is)/L\) from the neighborhood of such a point, see equation (B.1), as

\[
-2 \int_{k_0 - \epsilon}^{k_0 + \epsilon} \frac{dk}{2\pi} \ln \left[ \frac{1 + e^{-sE_k}}{2} \right] - 2 \ln \frac{2}{\epsilon} + e^{-2sE_0} \int_{-\epsilon}^{\epsilon} \frac{dk'}{2\pi} e^{-2sE_0} \tag{B.3}
\]

where we have approximated \(\xi_k \approx 1\), expanded the logarithm and introduced the group velocity \(v_{k_0} = \partial E_k/\partial k\). This is indeed a convergent expression. More importantly, when \(k_0 = 0\) the singularities needing such a special treatment are in a region where the integrand is exponentially smaller, due to the prefactor \(e^{-2sE_0}\), then the main contribution comes from the \(k = 0\) region, which we are now going to analyze.

To proceed with the case \(h_1 \neq h_c = 1\), we expand \(E_k\) up to second order in \(k\)

\[
E_k = |h_1 - h_c| + \frac{h_1}{2|h_1 - h_c|} k^2 + \mathcal{O}(k^4). \tag{B.3}
\]

Because we are assuming \(|h_1 - h_c| > 0\) and \(s \gg 1/|h_1 - h_c|\), only those values of \(k\) such that \(k < \sqrt{|h_1 - h_c|}/h_1\) contribute significantly to the integral in equation (B.2): we would be willing, therefore, to approximate the integral with a Gaussian one, extending the
upper integration limit to $\infty$ and expanding the factor multiplying $e^{-2sE_i}$ to the lowest order in $k$. But here the $k = 0$ point plays a tricky role. For a generic periodic driving where $h(t)$ oscillates around the average value $h_0$ one quickly realizes, by focusing on the evolution operator $U(\tau, 0)$ for modes with a small $k$ (as detailed in appendix C) that there are two cases:

(i) The Floquet spectrum is resonant in $k = 0$ whenever

$$2(h_0 - h_c) = l \omega_0$$

for some integer $l \in \mathbb{Z}$

and then $\mu_k^\pm \propto \pm k$ and

$$|r_k^\pm|^2 = \frac{1}{2} - \frac{\beta k}{2} + O(k^2),$$

hence $\xi_0 = 1$ and the expansion in equation (B.2) is inappropriate. As detailed in appendix C, for a sinusoidal driving of the form $h(t) = h_0 + A \cos(\omega_0 t + \phi_0)$ there are special situations where there is coherent destruction of tunnelling (CDT) [48, 55]: if $J_l(2A/\omega_0) = 0$, for the value of $l$ that realizes the resonance, then a gap is opened in the Floquet spectrum and $\xi_0 = 0$, posing no problem with equation (B.2).

(ii) If, on the contrary, the Floquet spectrum is not resonant in $k = 0$, then

$$|r_k^\pm|^2 = \frac{\alpha^2 k^2}{4} + O(k^4).$$

hence $\xi_0 = 0$, again posing no problem with equation (B.2).

So, restricting our consideration to non-critical initial fields, $h_i \neq h_c$, and non-resonant Floquet spectrum, and performing the appropriate Gaussian integral emerging from equation (B.2), we finally arrive at:

$$\ln \frac{G_T^{T=0}(is)}{L} \simeq g_{\infty} + \frac{\alpha^2}{16\sqrt{\pi}} \left( \frac{|h_i - h_c|}{h_i s} \right)^{3/2} e^{-2s|h_i - h_c|} + \cdots,$$

where we have assumed $s \gg 1/|h_i - h_c|$, while $\alpha$ is the constant appearing in the quadratic expansion of $|r_k^\pm|^2$, see equation (B.6). Since $s$ is large, we can equivalently recast this equation as

$$G_T^{T=0}(is) \simeq e^{Lg_{\infty}} \left( 1 + \frac{a L}{s^{3/2}} e^{-2s|h_i - h_c|} + \cdots \right),$$

where we have defined

$$a \equiv \frac{\alpha^2}{16\sqrt{\pi}} \left( \frac{|h_i - h_c|}{h_i} \right)^{3/2}.$$

Performing the inverse Laplace transform, it is not difficult to show that the $P(W)$ associated to equation (B.8) is:

$$\text{doi}:10.1088/1742-5468/2015/08/P08030$$
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\[ P_\infty(W) \simeq e^{Lg_\infty} \left( \delta(W) + \frac{2aL}{\sqrt{\pi}} \sqrt{W - 2|h_1 - h_c|} \theta(W - 2|h_1 - h_c|) + \cdots \right). \quad (B.10) \]

So, our theory predicts a square-root edge singularity in the asymptotic work distribution at a precise value \( W_{01} = 2|h_1 - h_c| \) of \( W \), i.e. simply the initial gap of the system, a value which is totally independent of the details of the periodic protocol (and even of the frequency), which only enter into the prefactor \( a \).

Finally, let us briefly consider the remaining cases where the previous Gaussian analysis fails. There are, essentially, two cases left:

(i) **Case \( h_1 \neq h_c \) and resonant Floquet spectrum.** The starting point of this case is equation (B.1), with \( E_k \) given by equation (B.3) but with the complication that \( \xi_k \rightarrow 1 \) in the vicinity of \( k = 0 \). An analysis of the singularity emerging shows that the leading term for large \( s \) is now given by:

\[ \ln \frac{G_{T=0}^{\infty}(is)}{L} \simeq g_\infty + a_c \, s \, e^{-2s|h_1 - h_c|} + \cdots. \quad (B.11) \]

Exponentiating, we can equivalently rewrite:

\[ G_{T=0}^{\infty}(is) \simeq e^{Lg_\infty}(1 + a_c \, Ls \, e^{-2s|h_1 - h_c|} + \cdots). \quad (B.12) \]

The resulting \( P(W) \), after inverse Laplace transform, has a very singular contribution proportional to a Dirac delta derivative:

\[ P_\infty(W) \simeq e^{Lg_\infty}(\delta(W) + a_c \, L \, \delta'(W - 2|h_1 - h_c|) \, \theta(W - 2|h_1 - h_c|) + \cdots) \quad (B.13) \]

(ii) **Cases with \( h_1 = h_c \).** Whenever the initial field \( h_1 \) coincides with the critical field \( h_c \), the initial spectrum \( E_k \) is gapless, and this changes completely the large-\( s \) behaviour of \( G_{T=0}^{\infty}(is) \), from exponential to power-law. The scenario is quite rich. For instance, when \( h_0 = h_c \) the behaviour is of the form

\[ G_{\infty}(is) \simeq e^{Lg_\infty} \left( 1 + \frac{D}{s} \right), \quad (B.14) \]

leading to a small-\( W \) probability distribution

\[ P_\infty(W) \simeq e^{Lg_\infty}(\delta(W) + D \, \theta(W)), \quad (B.15) \]

where \( \theta \) is the Heaviside function. But this does not exhaust all the possibilities: when \( h_0 = 0 \) we find that the leading asymptotics of \( G_{\infty}(is) \) is \( 1/s^3 \) rather than \( 1/s \) whenever the Floquet-resonant condition equation (18) is fulfilled. A thorough study of the gapless scenario is left to future studies.

**Appendix C**

In this appendix we prove the statements leading to the resonance condition equation (18) and the related small-\( k \) expansions equations (B.5) and (B.6). Let us start with the resonance condition. The argument is very similar to the one reported in

doi:10.1088/1742-5468/2015/08/P08030
the Supplemental Material of [35]. Let us consider equation (8) with a generic time-periodic \( h(t) \) with period \( \tau = 2\pi/\omega_0 \) and apply to it the time-dependent rotation \( V_k(t) = \exp(-i f(t) \sigma_k) \), where \( f(t) \equiv \int_0^t \! dt' (h(t') - h_0) \). The matrix \( \hat{H}_k(t) \) in the new representation has the form

\[
\hat{H}_k(t) = V_k^* (t) \hat{H}_k(t) V_k(t) - i V_k^* (t) \dot{V}_k(t) \equiv \begin{bmatrix} h_0 - \cos k & -i \sin k e^{2i f(t)} \\ i \sin k e^{-2i f(t)} & -h_0 + \cos k \end{bmatrix}. \tag{C.1}
\]

The unitary matrix \( V_k(t) \) has the remarkable property that \( V_k(\tau) = V_k(0) = 1 \). Because of that, we can immediately write the time-evolution operator over one period \( \tau \) in the original frame as

\[
\mathcal{U}_k(\tau, 0) = \overline{\mathcal{T}} \ e^{-i \int_0^\tau \! dt \ \hat{H}_k(t)}, \tag{C.2}
\]

where \( \overline{\mathcal{T}} \) stands for time ordering. In essence, the Hamiltonian in the rotated frame directly determines the form of the Floquet modes and quasi-energies, obtained by diagonalizing \( \mathcal{U}_k(\tau, 0) \). With a vanishing driving, the Floquet spectrum is given by the eigenenergies of the Hamiltonian folded in the first Brillouin zone [35], therefore we have a resonance whenever \( 2E_k = \omega_0 \) for some integer \( l \). For a finite external field \( h(t) \), most of the resonances turn into avoided crossings, but for the modes with \( k = 0 \) and \( k = \pi \). Indeed, we can see from equation (C.1) that the field couples to \( \sin k \), which vanishes at \( k = 0, \pi \). If we diagonalize equation (C.2) at those values of \( k \), we easily find the single-particle Floquet quasi-energies as \( \mu_0^\pm = \pm |h_0 - h_c| \) and \( \mu_\pi^\pm = \pm |h_0 + h_c| \), where \( h_c = 1 \). The resonances at \( k = 0 \) are particularly important since, for \( s \to \infty \) only the smallest values of \( E_k \) matter (see equations (13) and (14)), and these occur near \( k = 0 \).

Folding the quasi-energy into the first Brillouin zone, we can see that the Floquet spectrum is resonant at \( k = 0 \) when:

\[
2(h_0 - h_c) = \omega_0 \quad \text{for some integer } l \in \mathbb{Z}. \tag{C.3}
\]

We now move to establishing equations (B.5) and (B.6). For this purpose we have to consider that the overlap factors \( |\tau_k^\pm|^2 = |\langle \phi_k^\pm (0) | \psi_k^\pm \rangle|^2 \) are obtained from the Floquet modes \( |\phi_k^\pm (0) \rangle \), which are the eigenstates of an Hermitian operator, the Floquet Hamiltonian \( \hat{H}_k^F \), defined as

\[
e^{-i r \hat{H}_k^F} \equiv \overline{\mathcal{T}} \ e^{-i \int_0^\tau \! dt \ \hat{H}_k(t)}. \tag{C.4}
\]

(The Floquet quasi-energies are the eigenvalues of \( \hat{H}_k^F \).) From the above discussion, see equations (C.1) and (C.2), we immediately see that

\[
\hat{H}_k^F |_{k = 0} = \begin{bmatrix} h_0 - h_c & 0 \\ 0 & h_c - h_0 \end{bmatrix}.
\]

Because the Floquet quasi-energies are defined up to translations of an integer number of \( \omega_0 \), this Hamiltonian is equivalent to
where $\tilde{h} = (h_0 - h_c) - \omega_0 / 2$ is $h_0 - h_c$ folded in the first Brillouin zone $[-\omega_0 / 2, \omega_0 / 2]$. We see that $\tilde{h} = 0$ whenever there is a resonance, i.e. equation (C.3) is valid. Equation (C.4) shows that $H_k^F$ must have the same symmetry properties of $H_k(t)$. We see from equation (7), that the symmetry relation

$$H_k^F = -H_k^F$$

is valid; therefore we must also have $H_k^F = (H_k^F)^*$. Because of equation (C.5), and the relations $\text{Tr}[H_k(t)] = 0$, $\sigma_x = \sigma_x^*$, $\sigma_y = \sigma_y^*$, we can write the following second-order-in-$k$ expansion\(^{10}\) of $H_k^F$,

$$H_k^F_{\text{small } k} = \begin{pmatrix} \tilde{h} + a_x k^2 - a_y k^2 & a_x k^2 - i a_y k^2 \\ a_x k^2 + i a_y k^2 & -\tilde{h} - a_x k^2 \end{pmatrix} + \mathcal{O}(k^3).$$

In general the coefficients $a_x$, $a_y$ and $a_z$ are non-vanishing; they can vanish in some cases, giving rise to interesting phenomena which we will discuss later in some detail. Whenever the resonance condition equation (C.3) is not fulfilled (hence $\tilde{h} \neq h_0$), the second-order-in-$k$ expansion of Floquet modes (expressed in the basis $B = \{|0\rangle, \hat{c}_k^\dagger \hat{e}_{-k}^\dagger |0\rangle\}$) is

$$[\phi_{\text{small } k}^+]_B = \begin{pmatrix} 1 - \frac{1}{8} \left( \frac{a_y}{\tilde{h}} \right)^2 k^2 \\ \frac{i a_y}{2 \tilde{h}} k \end{pmatrix}$$

and

$$[\phi_{\text{small } k}^-]_B = \begin{pmatrix} \frac{i a_y}{2 \tilde{h}} k \\ 1 - \frac{1}{8} \left( \frac{a_y}{\tilde{h}} \right)^2 k^2 \end{pmatrix},$$

which applies to the case $\tilde{h} > 0$; these two states should be exchanged if $\tilde{h} < 0$. Notice that $a_x$ and $a_z$ do not enter at this order of approximation. Moving now to the initial Hamiltonian ground state $|\psi_k^{gs}\rangle$, the diagonalization of equation (7) immediately gives (for $h_i > h_c = 1$)

$$[\psi_{\text{small } k}^{gs}]_B = \begin{pmatrix} \frac{ik}{2 (h_i - h_c)} k^2 \\ 1 - \frac{8 (h_i - h_c)^2}{k^2} \end{pmatrix}.$$  

Hence, for the overlap $|r_k^+|^2$ we find

$$|r_k^+|^2 = \langle \phi_k^+ | \psi_k^{gs} \rangle^2 = \frac{1}{4} \alpha^2 k^2 + \mathcal{O}(k^3) \quad \text{with} \quad \alpha^2 \equiv \frac{1}{|h_i - h_c|} - \frac{a_y}{\tilde{h}}.$$ 

\(^{10}\) The vanishing of the trace of $H_k^F$ at any $k$ comes from the vanishing of the trace of $H_k(t)$ and the formula $\text{Tr}[H_k^F] = \frac{1}{2} \int_0^\tau \text{Tr}[H_k(t)] d\tau$, which is a corollary of the Liouville’s theorem [56].
which is indeed equation (B.6). If $\bar{h} < 0$ and $h_i > h_c$, or $\bar{h} > 0$ and $h_i < h_c$, then we find

$$|r_k^+|^2 = \frac{\alpha^2}{4} k^2 + O(k^3),$$

but the crucial ingredient determining equation (A.9) is identical, since $\xi_k \simeq \alpha^2 k^2$ in both cases, see equation (A.2).

For the resonant case $\bar{h} = 0$ we find

$$|\phi_{\text{small } k}^+ \rangle_{BG} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \frac{1}{2} \frac{a_x}{a_y} k \\ i \left( 1 - \frac{1}{2} \frac{a_x}{a_y} k \right) \end{pmatrix}$$

which (if $h_i > h_c$) gives rise to

$$|r_k^+|^2 = \frac{1}{2} \left( 1 - \frac{a_x}{a_y} k \right) + O(k^2).$$

This is indeed equation (B.5) with $\beta = a_x/a_y$. Notice that $a_x$ does not enter in this expression; on the other hand, if $a_x = 0$ then $\beta = 0$: this would not alter the previous analysis, which relies on the zero-order expansion of $|r_k^+|^2$ in the resonant case. It is important, however, to discuss the case in which $a_y$ vanishes: our formula is valid only if $a_y \neq 0$. This is generally true, up to special cases where there is coherent destruction of tunnelling (CDT) [55], which leads to $a_x = 0$ also, and we fall back to equation (B.6). In the supplemental material of [35] it is shown that, for a sinusoidal driving $h(t) = h_0 + A \cos(\omega_0 t + \phi_0)$, CDT occurs if $h_0 = h_c$ and $J_0(2A/\omega_0) = 0$. More in general, if the resonance condition equation (C.3) is valid for $l \neq 0$, we can show, exactly with the same arguments used in [35], that there is CDT whenever $J_l(2A/\omega_0) = 0$.

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