On quantizing semisimple basic algebras, II:
The general case

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November 30, 2001
(Revised March 28, 2022)

Abstract

We prove that there is no consistent polynomial quantization of the coordinate ring of a basic non-nilpotent coadjoint orbit of a semisimple Lie group.

*Supported in part by NSF grant DMS 00-72434.
1 Introduction

In a recent paper (Gotay [2002]) we showed that there do not exist polynomial quantizations of the coordinate ring $P(M)$ of a basic semisimple coadjoint orbit $M \subset \mathfrak{sl}(2, \mathbb{R})^*$. Here we extend this result to any basic non-nilpotent coadjoint orbit of a general semisimple Lie group:

**Theorem 1.1** Let $\mathfrak{b}$ be a finite-dimensional semisimple Lie algebra, and $M$ a basic non-nilpotent coadjoint orbit in $\mathfrak{b}^*$. Then there are no polynomial quantizations of the coordinate ring $P(M)$.

We refer the reader to Gotay [2000] for definitions and discussions of basic coadjoint orbits and quantization.

In particular, a coadjoint orbit $M \subset \mathfrak{b}^*$ is basic provided $\mathfrak{b}$ has no subalgebras which act transitively on $M$ and which globally separate points of $M$. Unfortunately, it is difficult to determine exactly which orbits are basic. From §4 in Gotay, Grabowski, and Grundling [2000] and Prop. 2.1 in Gotay [2002] we know that $M \subset \mathfrak{b}^*$ will be basic for a semisimple Lie algebra $\mathfrak{b}$ whenever

(i) $\mathfrak{b}$ is compact and $M$ is principal,

(ii) $\mathfrak{b}$ is compact and simple, and

(iii) $M$ is principal nilpotent.

Consider the symmetric algebra $S(\mathfrak{b})$, regarded as the ring of polynomials on $\mathfrak{b}^*$. The Lie bracket on $\mathfrak{b}$ may be extended via the Leibniz rule to a Poisson bracket on $S(\mathfrak{b})$, so that the latter becomes a Poisson algebra. Let $I(M)$ be the associative ideal in $S(\mathfrak{b})$ consisting of all polynomials which vanish on $M$ and set $P(M) = S(\mathfrak{b})/I(M)$. Since $M$ is an orbit $I(M)$ is also a Lie ideal, hence a Poisson ideal, so the coordinate ring $P(M)$ of $M$ inherits the structure of a Poisson algebra from $S(\mathfrak{b})$. We denote the Poisson brackets on both $P(M)$ and $S(\mathfrak{b})$ by $\{\cdot, \cdot\}$.

Here we are interested in quantizing the coordinate ring $P(M)$. By a quantization of $P(M)$ we mean a Lie representation $\mathcal{Q}$ thereof by symmetric operators preserving a fixed dense domain $D$ in some separable Hilbert space $\mathcal{H}$, such that $\mathcal{Q} | \mathfrak{b}$ is irreducible, integrable, and faithful. Let $\mathcal{A}$ be the associative operator algebra generated over $\mathbb{C}$ by $I$ and $\{\mathcal{Q}(b) | b \in \mathfrak{b}\}$. We say that a quantization $\mathcal{Q}$ of $P(M)$ is polynomial if $\mathcal{Q}$ is valued in $\mathcal{A}$.
2 Proof of Theorem 1.1

Suppose to the contrary that \( Q \) were a polynomial quantization of \( P(M) \) in a dense invariant domain \( D \) in a Hilbert space \( \mathcal{H} \). By extending \( Q \) to be complex linear, we obtain a Lie representation \( Q_{\mathbb{C}} \) of the Poisson algebra \( P(M, \mathbb{C}) \) of complex-valued polynomials on \( M \) in \( D \).

By assumption the representation of \( b \) in \( D \) provided by \( Q \) may be integrated to a strongly continuous unitary representation \( \Pi \) of the 1-connected Lie group \( B \) with Lie algebra \( b \) in \( \mathcal{H} \). Let \( B_{\mathbb{C}} \) be the universal complexification of \( B \); since \( B \) is simply connected, \( B_{\mathbb{C}} \) can be identified with the 1-connected semisimple complex analytic group with Lie algebra the complexification \( b_{\mathbb{C}} \) of \( b \). (See Varadarajan [1984], pps. 256–258 and 400–404 for background on complexifications of Lie groups.) Since \( B \) is semisimple, \( B \) is a closed subgroup of \( B_{\mathbb{C}} \), and so we may use induction to obtain a strongly continuous unitary representation \( \Pi_{\mathbb{C}} \) of \( B_{\mathbb{C}} \) in a certain infinite-dimensional Hilbert space \( \mathcal{K} \).

Now let \( C \) be a compact real form of \( B_{\mathbb{C}} \), and denote by \( \Gamma \) the restriction of \( \Pi_{\mathbb{C}} \) to \( C \). As every strongly continuous unitary representation of a compact Lie group is completely reducible, we may decompose

\[
\mathcal{K} = \bigoplus_{i \in I} \mathcal{K}_i
\]

for some index set \( I \subset \mathbb{Z} \), where the finite-dimensional invariant subspaces \( \mathcal{K}_i \) are the carriers of the irreducible constituents \( \Gamma_i \) of \( \Gamma \). Let \( c \) be the Lie algebra of \( C \); then for each \( i \in I \), we have the derived representation \( d\Gamma_i \) of \( c \) in \( \mathcal{K}_i \). Set \( d\Gamma = \bigoplus_{i \in I} d\Gamma_i \); this gives a representation of \( c \) in the dense subspace

\[
D_{\mathbb{C}} = \bigoplus_{i \in I} \mathcal{K}_i.
\]

Choose a basis \( \{c_1, \ldots, c_r\} \) of \( c \). Since \( c_{\mathbb{C}} = b_{\mathbb{C}} \) and as by assumption \( Q \) is valued in \( A \), for every \( f \in P(M, \mathbb{C}) \) we may expand

\[
Q_{\mathbb{C}}(f) = \sum_{n_1, \ldots, n_r} a_{n_1, \ldots, n_r}^f Q_{\mathbb{C}}(c_1)^{n_1} \cdots Q_{\mathbb{C}}(c_r)^{n_r}
\]

for some coefficients \( a_{n_1, \ldots, n_r}^f \). By means of this formula we can extend the representation \( d\Gamma \) of \( c \) to a Lie representation \( \gamma \) of \( P(M, \mathbb{C}) \) in \( D_{\mathbb{C}} \):

\[
\gamma(f) = \sum_{n_1, \ldots, n_r} a_{n_1, \ldots, n_r}^f d\Gamma(c_1)^{n_1} \cdots d\Gamma(c_r)^{n_r}
\]
with the same coefficients. As each subspace $K_i$ is invariant, $\gamma$ restricts to a representation $\gamma_i$ of $P(M, \mathbb{C})$ in $K_i$. We will show that the existence of these representations $\gamma_i$ leads to a contradiction.

To this end we recall the following algebraic fact, the proof of which is given in Gotay, Grabowski, and Grundling [2000].

**Lemma 2.1** If $L$ is a finite-codimensional Lie ideal of an infinite-dimensional Poisson algebra $P$ with identity, then either $L$ contains the derived ideal $\{ P, P \}$ or there is a maximal finite-codimensional associative ideal $J$ of $P$ such that $\{ P, P \} \subset J$.

We apply Lemma 2.1 to each $L_i = \ker \gamma_i$ which, as $K_i$ is finite-dimensional, has finite codimension in $P = P(M, \mathbb{C})$. First suppose there is an $i$ for which $\{ P, P \} \not\subset L_i$. Then there must exist a maximal finite-codimensional associative ideal $J_i$ in $P$ with $\{ P, P \} \subset J_i$. If $\rho$ is the projection $S(b_C) \to P$, then $I_i = \rho^{-1}(J_i)$ is a maximal finite-codimensional associative ideal in $S(b_C)$ with $\{ S(b_C), S(b_C) \} \subset I_i$. Since by semisimplicity

$$b_C = \{ b_C, b_C \} \subset \{ S(b_C), S(b_C) \} \subset I_i,$$

and since $1 \not\in I_i$ (as $I_i$ is proper), it follows that $I_i$ is the associative ideal generated by $b_C$. (Actually, this shows that $S(b_C) = \mathbb{C} \oplus I_i$.)

Since the orbit $M$ is not nilpotent, there is a nonzero Casimir $\Omega \in S(b_C)$, i.e. $\rho(\Omega) = \omega$ for some constant $\omega \neq 0$. Since $b_C$ is semisimple it follows from the above observations that $\Omega \in I_i$. But then $\Omega - \omega \not\in I_i$, which is a contradiction since $\Omega - \omega \in \ker \rho \subset I_i$.

Thus for every $i$ it must be the case that $\{ P, P \} \subset L_i$. Again semisimplicity gives $b_C = \{ b_C, b_C \} \subset L_i$, and so $\gamma \mid b_C = 0$. In particular, then, $d\Gamma = 0$. Since $c$ is a compact real form of $b_C$, the Cartan decomposition of $b_C$ implies that $d\Pi_C = 0$. It follows from the induction construction that the original derived representation $d\Pi$ of $b$ in the domain $D$ must be zero as well. But then $\mathcal{Q} \mid b = 0$, which contradicts the requirement that a quantization represent $b$ faithfully. This concludes the proof of Theorem 1.1. $\blacksquare$

We remark that Theorem 1.1 was already known when $b$ is compact (Gotay, Grabowski and Grundling [2000]), in which case the proof above simplifies greatly and provides an alternate means of establishing Theorem 2 *ibid*. Notice also that when $b$ is compact every quantization of $P(M)$ is necessarily polynomial; this follows from the observation that since $\mathcal{Q} \mid b$ is irreducible the representation space $\mathcal{H}$ must be finite-dimensional together with a well known fact about enveloping algebras (Prop. 2.6.5 in Dixmier [1976]).
3 Discussion

The key observation underlying Theorem 1.1 is that as \( M \subset b^* \) is non-nilpotent, its ideal \( I(M) \) is nonhomogeneous. If \( M \) is a basic nilpotent orbit, on the other hand, then \( I(M) \) is homogeneous, and from Theorem 1.1 in Gotay [2002] we know that there do exist polynomial quantizations of \( P(M) \). (Although it is not clear to what extent these are “nontrivial” in general.) Taken together, these two results serve to establish a conjecture of Gotay [2000] when \( b \) is semisimple: \textit{There exists a consistent polynomial quantization of } \( P(M) \) \textit{if and only if } \( I(M) \) \textit{is homogeneous.}

Finally, we remark that the restriction to basic coadjoint orbits is for physical reasons, cf. Gotay [2000]. For our purposes here, we may delete the adjective “basic” and consider arbitrary orbits. Both Theorem 1.1 and the conjecture above will remain valid in this extended context.

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