LOCALLY INTERACTING DIFFUSIONS AS SPACE-TIME MARKOV RANDOM FIELDS

DANIEL LACKER, KAVITA RAMANAN, AND RUOYU WU

ABSTRACT. We consider a countable system of interacting (possibly non-Markovian) stochastic differential equations driven by independent Brownian motions and indexed by the vertices of a locally finite graph $G = (V, E)$. The drift of the process at each vertex is influenced by the states of that vertex and its neighbors, and the diffusion coefficient depends on the state of only that vertex. Such processes arise in a variety of applications including statistical physics, neuroscience, engineering and math finance. Under general conditions on the coefficients, we show that if the initial conditions form a second-order Markov random field on $d$-dimensional Euclidean space, then at any positive time, the collection of histories of the processes at different vertices forms a second-order Markov random field on path space. We also establish a bijection between (second-order) Gibbs measures on $(\mathbb{R}^d)^V$ (with finite second moments) and a set of space-time (second-order) Gibbs measures on path space, corresponding respectively to the initial law and the law of the solution to the stochastic differential equation. As a corollary, we establish a Gibbs uniqueness property that shows that for infinite graphs the joint distribution of the paths is completely determined by the initial condition and the specifications, namely the family of conditional distributions on finite vertex sets given the configuration on the complement. Along the way, we establish various approximation and projection results for Markov random fields on locally finite graphs that may be of independent interest.

CONTENTS

1. Introduction 1
2. Main results 4
3. Interacting diffusions on a finite graph 8
4. Finite-graph approximations for Markov random fields 14
5. Markov random field property for infinite-dimensional diffusions 19
6. Proof of Gibbs measure properties 23
Appendix A. Proof of pathwise uniqueness under Lipschitz assumptions 28
Appendix B. Justification for applying Girsanov’s theorem 29
References 29

1. INTRODUCTION

1.1. Discussion of results. Given a finite or locally finite infinite graph $G$ with vertex set $V$ and edge set $E$, and a positive integer $d$, consider interacting diffusions that satisfy the following stochastic differential equation (SDE) system:

$$dX_v(t) = b_v(t, X_v(t), X_{N_v(G)}(t)) \, dt + \sigma_v(t, X_v(t)) \, dW_v(t), \quad v \in V, \ t \ge 0,$$

Date: November 5, 2019.
2000 Mathematics Subject Classification. 60H10, 60G60; 82C20.
Key words and phrases. interacting diffusions, infinite-dimensional stochastic differential equations, Gibbs measures, Markov random fields, space-time Gibbs measures, stochastic processes on graphs.
K. Ramanan was supported in part by NSF-DMS Grant 1713032.
where the initial condition $X(0) = (X_v(0))_{v \in V}$ is distributed according to some given probability measure on $(\mathbb{R}^d)^V$. Here, $N_v(G) \subset V$ denotes the neighborhood of $v$ in the graph $G$, $(b_v, \sigma_v)_{v \in V}$ are given drift and diffusion coefficients, and $(W_v)_{v \in V}$ are independent standard $d$-dimensional Brownian motions. Such diffusions arise in a variety of applications, including statistical physics [4,19], neuroscience [14,15], and systemic risk [17]. Under suitable conditions on the coefficients that guarantee the existence of a unique weak solution to the SDE, for any $t > 0$, we study the random field on $C^V_t$ generated by the collection of trajectories $(X_v[t] := (X_v(s))_{s \leq t})_{v \in V}$, where $C_t$ (resp. $C$) denotes the space of $\mathbb{R}^d$-valued continuous functions on $[0,t]$ (resp. $[0,\infty)$).

Our first set of results (Theorems 2.4 and 2.7) show that, under modest conditions on the drift and diffusion coefficients that guarantee a unique weak solution to the SDE system on any locally finite graph, if $(X_v(0))_{v \in V}$ is a second-order Markov random field on $(\mathbb{R}^d)^V$ (as specified in Definition 1.1) then for each $t > 0$, $(X_v[t])_{v \in V}$ is a second-order Markov random field on $C^V_t$. In fact, we establish this result for a more general class of SDEs with possibly non-Markovian dynamics (and potentially infinite memory) defined in Section 2.

Our next set of results relate to an interpretation of the law of the SDE as a space-time Gibbs measure (see Section 2.3 for precise definitions). Specifically, Theorem 2.3 establishes a bijection between (second-order) Gibbs measures on $(\mathbb{R}^d)^V$ (with finite second moments) and a set of space-time (second-order) Gibbs measures on $C^V_t$, corresponding respectively to the initial law and the law of the solution to the SDE. As a consequence, we deduce a Gibbs uniqueness property, which shows that the law of the SDE system is completely determined by its initial condition and its specifications, namely the family of conditional distributions on finite sets given the configuration on the complement. In particular, together these show (see Corollary 2.10) that when the initial distribution is the unique second-order Gibbs measure associated with some specifications on $(\mathbb{R}^d)^V$, then for each $t > 0$, the law of the SDE system is the unique second-order Gibbs measure associated with corresponding specifications on path space $C^V_t$.

Such questions of characterizations of SDEs in terms of space-time Markov random fields are of broad interest. Prior work on similar questions has been mainly restricted to infinite-dimensional Markovian diffusions with drifts that have a local gradient structure [2,5,8,19,21]. The earliest work in this direction is that of Deuschel [8], which considers smooth gradient systems of finite range, that is, where the drift is assumed to be of the form $b_v = \nabla_v h_v$, $v \in V$, for suitable “Hamiltonian” functions $h_v : \mathbb{R}^m \mapsto \mathbb{R}$ defined in terms of smooth finite-range potentials with uniformly bounded second partial derivatives. For such systems, it is shown in [8, Corollary 2.21, Theorem 3.12 and Proposition 4.2] that when the initial condition is a Gibbs measure with respect to a Gibbsian specification with a finite range interaction potential (in the sense of [9, Chapter 2]), the law $P$ of the process $X$ on $C([0,1]; \mathbb{R}^m)$ is a Gibbs state, and is in fact the unique Gibbs state associated with the family of conditional laws $\law(X_v|X_u, u \neq v)$.

The approach in [8] relies crucially on the gradient structure of the drift (see [8, Remark (3.5)i)]) as it uses regularity of the explicit conditional densities $\law(X_v|X_u, u \neq v)$, in order to apply Dobrushin’s uniqueness condition. To show that the Dobrushin contraction coefficient on path space is strictly bounded by 1, a uniform boundedness assumption is imposed on the drifts. This result was later generalized to the case of unbounded drifts, still of locally gradient form, by Catelliaux, Roelly and Zessin [2]. In particular, in [2] Theorems 3.7, 3.22 and 4.9], they establish a bijection between initial Gibbs states on $\mathbb{R}^m$ and Gibbs measures on path space $C([0,1]; \mathbb{R}^d)^\mathbb{Z}^m$. They adopted a different approach from the one we use here, involving Malliavin calculus, a variational characterization and a certain integration-by-parts formula (see also [16,21]), which all exploit the gradient structure of the drift. In [16], a cluster expansion method is also used when the gradient system can be viewed as a small perturbation of a free field.
An additional motivation for our study stems from a recent result of [12], which shows how a second-order Markov property is useful for obtaining an autonomous description of the marginal (local) dynamics of a particle and its neighborhood when the underlying graph \( G \) is a tree. For this purpose, a stronger \textit{global} Markov property is derived in [12, Proposition 7.3] in the setting of an infinite regular tree \( G \) (or, most generally, a unimodular Galton-Watson tree) and homogeneous coefficients, \( (b_v, \sigma_v) = (b, \sigma) \) for all \( v \in V \). Notably, the characterization of the local dynamics in [12] relies on the precise order of the Markov random field (equivalently, range of interaction of the Gibbs state), and not merely the Gibbs property.

We now briefly describe the techniques used to prove our results. First, to establish the Markov random field property for SDEs on finite graphs (Theorem 2.4) we derive and apply a version of the Hammersley-Clifford characterization of Markov random fields on finite graphs. The result for the infinite graph (Theorem 2.7) is obtained by a delicate approximation by finite-dimensional systems in a way that preserves the Markov random field property. Along the way, we establish various approximation results for Markov random fields on locally finite graphs (see Section 4) that may be of independent interest. Thus, our methods are quite different from those of the prior works described above. They do not require any gradient structure of the drift, are not restricted to small perturbations of a free field, and allow for non-Markovian dynamics involving path-dependent coefficients with possibly unbounded memory. More importantly, our contribution clarifies the generic nature of the conditional independence structure in these kinds of models.

Finally, we also provide examples (see Section 3.3) that demonstrate that the Markov random field property we establish cannot in general be significantly strengthened. Precisely, even on a finite graph with gradient drift, in general the collection of histories \( \{X_v(t)\}_{v \in V} \) do not form a first-order Markov random field, nor do the time-\( t \) marginals \( \{X_v(t)\}_{v \in V} \) exhibit any non-trivial conditional independence structure. This highlights the natural problem of identifying special classes of systems for which simpler Markov random field properties are preserved, a problem which we do not address but which has attracted considerable attention in certain contexts. Specifically, in the context of diffusions, the papers [5,6,19,20,26,27] have studied the phenomenon of Gibbs-non-Gibbs transitions and the propagation (or lack thereof) of the Gibbs property at the level of the time-\( t \) marginals, specifically whether the initial law of \( X(0) \) being a Gibbs state on \( (\mathbb{R}^d)^V \) implies that the marginal law of \( X(t) \) is also a Gibbs state on \( (\mathbb{R}^d)^V \). See Remark 3.6 for a more detailed description of these works.

The next section introduces some common notation and basic definitions used throughout the paper. The main results of the paper are stated in Section 2, with their proofs relegated to Sections 3-6.

1.2. Notation and basic definitions. For any vectors \( a, b \in \mathbb{R}^d \), we use \( a \cdot b \) or \( (a, b) \) to denote the inner product. In this paper, unless explicitly stated otherwise, a graph \( G = (V,E) \) always has a finite or countably infinite vertex set, is simple (no self-edges or multi-edges), and is locally finite (i.e., the degree of each vertex is finite). We abuse notation by writing \( v \in G \) to mean \( v \in V \). For a graph \( G = (V,E) \) and a vertex \( v \in V \), we write \( N_v(G) = \{ u \in V : (u, v) \in E \} \) for the set of neighbors of \( v \) in \( G \), noting that this set is empty if \( v \) is an isolated vertex. A \textit{rooted} graph \( G = (V,E,v) \) is a graph equipped with a distinguished vertex \( v \in V \), called the \textit{root}. For two vertices \( u, v \in V \), let \( d(u,v) \) denote the graph distance, i.e., the length of the shortest path from \( u \) to \( v \) (with \( d(u,u) := 0 \)). Also, let \( \text{diam}(A) \) denote the diameter of a set \( A \subset V \); precisely, \( \text{diam}(A) = \sup\{d(u,v) : u, v \in A\} \). For a subset \( A \subset V \), we define the first and second boundaries

\[
\partial_G A = \{ u \in V \setminus A : (u, v) \in E \text{ for some } v \in A \}, \\
\partial_G^2 A = \partial_G A \cup \partial_G(A \cup \partial_G A).
\]  

(1.1)
We will often omit the subscript, writing simply $\partial^2 A$ in place of $\partial^2_G A$, when the underlying graph $G$ is clear. A clique in a graph $G$ is a complete subgraph of $G$, i.e., a set $A \subset V$ such that $(u, v) \in E$ for every $u, v \in A$. Equivalently, a clique is a set $A \subset V$ of diameter at most 1. Define $c_l(G)$ to be the set of all cliques of the graph $G$. Similarly, we will say that any subset $A \subset V$ with diameter at most 2 is a 2-clique of the graph $G$ and let $c_l(G)$ denote the set of 2-cliques of $G$. Moreover, given a graph $G = (V, E)$, $H = (V_H, E_H)$ is said to be an induced subgraph of $G$ if $V_H \subset V$ and $E_H = E \cap \{(u, v) : u, v \in V_H\}$.

For a set $\mathcal{X}$ and a graph $G = (V, E)$, we may write either $\mathcal{X}^V$ or $\mathcal{X}^G$ for the configuration space $\{(x_v)_{v \in V} : x_v \in \mathcal{X} \text{ for every } v \in V\}$. We make use of a standard notation for configurations on subsets of vertices: For $x = (x_v)_{v \in V} \in \mathcal{X}^V$ and $A \subset V$, we write $x_A$ for the element $x_A = (x_v)_{v \in A}$ of $\mathcal{X}^A$. When $\mathcal{X}$ is a Polish space, we write $P(\mathcal{X})$ for the set of Borel probability measures on $\mathcal{X}$, endowed always with the topology of weak convergence. Given any measurable space $\mathcal{X}$, $A \subset V$, and measure $\nu \in P(\mathcal{X}^V)$, $\nu[A]$ represents the restriction of $\nu$ to the set $\mathcal{X}^A$, that is, the image measure under the restriction map $\mathcal{X}^V \ni (x_v)_{v \in V} \mapsto (x_v)_{v \in A} \in \mathcal{X}^A$.

Fixing $d \in \mathbb{N}$, we let $\mathcal{C} = C(\mathbb{R}_+; \mathbb{R}^d)$ denote the path space of $\mathbb{R}^d$-valued continuous functions on $\mathbb{R}_+ = [0, \infty)$, endowed with the topology of uniform convergence on compacts. For $t > 0$, let $\mathcal{C}_t = C([0, t]; \mathbb{R}^d)$ denote its restriction to the time interval $[0, t]$, endowed with the uniform topology. For $x \in \mathcal{C}$ and $t > 0$ we define $\|x\|_{s, t} := \sup_{s \leq t} |x(s)|$, and let $x(t) = (x(s))_{s \leq t} \in \mathcal{C}_t$ denote the restriction of the path $x$ to the time interval $[0, t]$. We assume that $\mathcal{C}$ and $\mathcal{C}_t$ are endowed with their respective Borel $\sigma$-algebras. Also, for any countable set $A$ and probability measure $Q$ on $\mathcal{C}^A$, we write $Q_t$ for the image under $Q$ of the map $\mathcal{C}^A \ni (x_v)_{v \in A} \mapsto (x_v[t])_{v \in A} \in \mathcal{C}^A_t$. The $\sigma$-algebra on a product space will always be the product $\sigma$-algebra, unless explicitly stated otherwise. Given $J, m \in \mathbb{N}$, a measurable function $f : [0, \infty) \times \mathcal{C}^J \mapsto \mathbb{R}^m$, is said to be progressively measurable if for each $t \geq 0$, $f(t, (\hat{x}_u)_{u=1,\ldots,J}) = f(t, (y_u)_{u=1,\ldots,J})$ whenever $x_u[t] = y_u[t]$ for all $u = 1, \ldots, J$.

We end this section by recalling the notion of a (first-order or second-order) Markov random field, which plays a central role in the paper.

**Definition 1.1.** Given a measurable space $\mathcal{X}$, and a (possibly infinite) locally finite graph $G = (V, E)$, let $(Y_v)_{v \in V}$ be a random element of $\mathcal{X}^V$ with some distribution $\nu \in P(\mathcal{X}^V)$. We say that $(Y_v)_{v \in V}$, or equivalently its law $\nu$, is a first-order Markov random field (abbreviated as 1MRF) on $\mathcal{X}^V$ if $Y_A$ is conditionally independent of $Y_{(A, \partial A)^c}$ given $Y_{\partial A}$, for every finite set $A \subset V$. On the other hand, we say that $(Y_v)_{v \in V}$, or equivalently its law $\nu$, is a second-order Markov random field (abbreviated as 2MRF) on $\mathcal{X}^V$ if $Y_A$ is conditionally independent of $Y_{(A, \partial^2 A)^c}$ given $Y_{\partial^2 A}$, for every finite set $A \subset V$. When the space $\mathcal{X}^V$ is clear from the context, we will simply say that $(Y_v)_{v \in V}$, or equivalently its law $\nu$, is a 1MRF or 2MRF.

**Remark 1.2.** In Definition 1.1 it is important to stress that the sets $A$ are required to be finite even when the graph $G$ is infinite. Allowing infinite sets $A$ results in the stronger global Markov property, which we do not study in this paper.

2. Main results

Given a locally finite graph $G = (V, E)$ with a finite or countably infinite vertex set, we are interested in a system of (possibly non-Markovian) interacting stochastic processes, indexed by the vertices of the graph, that satisfy an SDE of the form

$$dX_v(t) = b_v(t, X_v, X_{N_v(G)}) \, dt + \sigma_v(t, X_v) \, dW_v(t), \quad v \in V, \quad (2.1)$$

where $(W_v)_{v \in V}$ are independent Brownian motions, and the initial law $\mu_0 \in P((\mathbb{R}^d)^V)$, of $(X_v(0))_{v \in V}$ and the coefficients $(b_v, \sigma_v)_{v \in V}$ satisfy the conditions stated in Assumption A or
Assumption [B] below, depending on whether the graph is finite or infinite. As mentioned in the introduction, our main results concern the characterization of the law of the solution to the SDE (2.1) as a 2MRF on \( C^V \) (see Definition 1.1).

2.1. The finite graph case. We first consider the case when \( G \) is finite, and the conditions stated in Assumption [A] below are satisfied. Recall, from Section 1.2, the definition of 2-cliques, the notation for trajectories, \( x[t] = (x(s))_{s \leq t} \in C_t \) for \( x \in C \), and the notion of a progressively measurable functional.

**Assumption A.** We say that \((G, b, \sigma, \mu_0)\) satisfy Assumption [A] if \( G = (V, E) \) is a finite graph and if \( b = (b_v)_{v \in V}, \sigma = (\sigma_v)_{v \in V}, \mu_0 \in \mathcal{P}(\mathbb{R}^d)^V \) satisfy the following:

(A.1) There exist \( \lambda_v \in \mathcal{P}(\mathbb{R}^d), v \in V \), such that the probability measure \( \mu_0 \) is absolutely continuous with respect to the product measure \( \mu_0^* = \prod_{v \in V} \lambda_v \in \mathcal{P}(\mathbb{R}^d)^V \) and the density satisfies

\[
\frac{d\mu_0}{d\mu_0^*}(x) = \prod_{K \in cl_2(G)} f_K(x_K), \quad x \in (\mathbb{R}^d)^V, \tag{2.2}
\]

for some measurable functions \( f_K : (\mathbb{R}^d)^K \to \mathbb{R}_+ \), \( K \in cl_2(G) \), where \( cl_2(G) \) is the set of 2-cliques of \( G \). In addition, \( \mu_0 \) has a finite second moment.

(A.2) For each \( v \in V \), the drift \( b_v : \mathbb{R}_+ \times C \times C^{N_v(G)} \to \mathbb{R}^d \) is progressively measurable. Moreover, for each \( T \in (0, \infty) \) there exists \( C_T < \infty \) such that

\[
|b_v(t,x,y_{N_v(G)})| \leq C_T \left( 1 + \|x\|_{*t} + \sum_{u \in N_v} \|y_u\|_{*t} \right),
\]

for all \( v \in V \), \( t \in [0,T] \), \( x \in C \), and \( y_{N_v(G)} = (y_u)_{u \in N_v(G)} \in C^{N_v(G)} \).

(A.3) The diffusion matrices \( \sigma_v : \mathbb{R}_+ \times C \to \mathbb{R}^{d \times d} \), \( v \in V \), satisfy the following:

(A.3a) For each \( v \in V \), \( \sigma_v \) is bounded, progressively measurable and invertible, with bounded inverse.

(A.3b) For each \( v \in V \), the following driftless SDE system admits a unique in law weak solution starting from any initial position \( x \in \mathbb{R}^d \):

\[
dX_v(t) = \sigma_v(t,X_v)\ dW_v(t), \quad X_v(0) = x. \tag{2.3}
\]

**Remark 2.1.** A necessary condition for Assumption [A] is that \( \mu_0 \) is a 2MRF and is absolutely continuous with respect to the product measure \( \mu_0^* \); this follows from a form of the Hammersley-Clifford theorem stated in Proposition 3.2 below. If the density \( d\mu_0/d\mu_0^* \) is strictly positive, then it factorizes as in (2.2) if and only if \( \mu_0 \) is a 2MRF.

**Remark 2.2.** If \( \sigma_v(t,x) = \overline{\sigma}_v(t,x(t)) \) depends only on the current state, not the history, and satisfies the additional continuity condition \( \lim_{v \to x} \sup_{0 \leq s \leq T} |\overline{\sigma}_v(s,y) - \overline{\sigma}_v(s,x)| = 0 \) for all \( v \in V \), then Assumption [A] holds as a consequence of Assumption [A] and [B], Chapter 7.

The following proposition shows that, as a simple consequence of Girsanov’s theorem, Assumption [A] guarantees weak existence and uniqueness in law of the SDE system (2.1). Its proof is given in Section 3.3 along the way to proving Theorem 2.4 below.

**Proposition 2.3.** When \((G, b, \sigma, \mu_0)\) satisfy Assumption [A], the SDE system (2.1) has a weak solution that is unique in law.

We now state our main result for the SDE system on finite graphs.
Theorem 2.4. Suppose \((G = (V,E), b, \sigma, \mu_0)\) satisfy Assumption \([A]\) and let \((X_v)_{v \in V}\) be the unique in law solution of the SDE system (2.1) with initial law \(\mu_0\). Then, for each \(t > 0\), \((X_v[t])_{v \in V}\) is a 2MRF on \(C_t^V\). Moreover, \((X_v)_{v \in V}\) is a 2MRF on \(C^V\).

The proof of Theorem 2.4 given in Section 3.2 relies on a certain factorization property (stated in Proposition 3.2) of the density of the law of the SDE on finite graphs with respect to a reference measure. Notice that in Assumption \([A]\) and throughout the paper, we assume there is no interaction in the diffusion coefficients (i.e., no dependence of \(\sigma_v\) on \(X_{N_v}(G)\)), the general case seems out of reach of our approach, because the reference measure in the factorization property must crucially be a product measure. This factorization property is also used in Sections 3.3.1 and 3.3.2 to show that, even when the initial law \(\mu_0\) satisfies Assumption 2.4.

2.2. The infinite graph case. We now consider the SDE system (2.1) in the case when \(G\) is an infinite, though still locally finite, graph. The well-posedness of the SDE system is no longer obvious and in particular does not follow from Girsanov’s theorem as it did when the graph was finite. Indeed, on an infinite graph, when \(b_v \equiv 1\) and \(\sigma_v \equiv 1_d\), \(v \in V\), for instance, it is straightforward to argue, using the law of large numbers, that the law of a weak solution of (2.1) up to some time \(t > 0\) and the law of the corresponding drift-free equation are mutually singular. This necessitates the following additional assumptions compared to Assumption \([A]\).

Recall from Section 1.2 that given a measure \(\nu\) on \(\mathcal{X}^V\) for some Polish space \(\mathcal{X}\), and \(A \subset V\), \(\nu[A]\) denotes the restriction of \(\nu\) to \(A\). Also, we use the notation \(\pi_1 \sim \pi_2\) to denote that the measures \(\pi_1\) and \(\pi_2\) are equivalent, that is, mutually absolutely continuous.

Assumption B. We say that \((G,b,\sigma,\mu_0)\) satisfy Assumption \([B]\) if \(G = (V,E)\) is a countable locally finite connected graph and if \(b = (b_v)_{v \in V}\), \(\sigma = (\sigma_v)_{v \in V}\) and \(\mu_0 \in \mathcal{P}((\mathbb{R}^d)^V)\) satisfy the following:

(B.1) The initial law \(\mu_0\) is a 2MRF on \((\mathbb{R}^d)^V\). Moreover, there exists a product measure \(\mu_0^* = \prod_{v \in V} \lambda_v \in \mathcal{P}((\mathbb{R}^d)^V)\) such that \(\mu_0[A] \sim \mu_0^*[A]\) for each finite set \(A \subset V\). Further, the initial law \(\mu_0\) satisfies

\[
\sup_{v \in V} \int_{(\mathbb{R}^d)^V} |x_v|^2 \mu_0(dx_V) < \infty. \tag{2.4}
\]

(B.2) The drift coefficients \((b_v)_{v \in V}\) satisfy Assumption \([A]\)2, for some constants \((C_T)_{T > 0}\).

(B.3) The diffusion matrices \((\sigma_v)_{v \in V}\) satisfy Assumption \([A]\)3.

(B.4) The SDE system (2.1) is unique in law, and this law is denoted by \(P = P^{\mu_0} \in \mathcal{P}(C^V)\).

Remark 2.5. Using Assumption \([A]\)3b if the graph is finite or Assumption \([B]\)3 if the graph is infinite, we may define for any initial law \(\nu \in \mathcal{P}((\mathbb{R}^d)^V)\) the measure \(P^{\nu} \in \mathcal{P}(C^V)\) to be the law of the unique weak solution of the SDE system

\[
dX_v(t) = \sigma_v(t, X_v) dW_v(t), \quad v \in V, \quad (X_v(0))_{v \in V} \sim \nu. \tag{2.5}
\]

Note in particular that if we take \(\nu = \mu_0^*\), where \(\mu_0^*\) is a product measure as in Assumption \([A]\)1 or \([B]\)1, then \(P^{\nu} = P^{\mu_0^*}\) too is a product measure.

We show in Lemma 5.2 that existence of a solution to (2.1) follows automatically from Assumptions \([B]\)1–3. However, it is worth commenting on the uniqueness condition in Assumption \([B]\)4. The following proposition shows that a suitable Lipschitz condition is enough
to guarantee uniqueness; its proof is standard and hence relegated to Appendix A. Recall in the following that \(\|x\|_{s,t} = \sup_{s \leq t} \|x(s)\|\) for \(x \in \mathcal{C}\).

**Proposition 2.6.** Suppose Assumptions \(\mathcal{B}1–3\) hold, and \((b_v)_{v \in V}\) and \((\sigma_v)_{v \in V}\) are uniformly Lipschitz in the sense that for each \(T > 0\) there exist \(K_T, \bar{K}_T < \infty\) such that

\[
|b_v(t, x, y_{N_v(G)}) - b_v(t, x', y'_{N_v(G)})| \leq K_T \|x - x'\|_{s,t} + \frac{1}{|N_v(G)|} \sum_{u \in N_v(G)} \|y_u - y'_u\|_{s,t},
\]

\[
|\sigma_v(t, x) - \sigma_v(t, x')| \leq \bar{K}_T \|x - x'\|_{s,t},
\]

for all \(v \in V, t \in [0, T], x, x' \in \mathcal{C}\), and \(y_{N_v(G)}, y'_{N_v(G)} \in \mathcal{C}^{N_v(G)}\). Then pathwise uniqueness holds for the SDE system \((2.1)\). In particular, Assumption \(\mathcal{B}4\) holds.

We now state our second main result.

**Theorem 2.7.** Suppose \((G = (V, E), b, \sigma, \mu_0)\) satisfy Assumption \(\mathcal{B}\) and let \((X_v)_{v \in V}\) be the unique in law solution of the SDE system \((2.1)\) with initial law \(\mu_0\). Then, for each \(t > 0\), \((X_v[0, t])_{v \in V}\) is a 2MRF on \(\mathcal{C}^V\). Moreover, \((X_v)_{v \in V}\) is a 2MRF on \(\mathcal{C}^V\).

The proof of Theorem 2.7 is given in Section 5.2 with preparatory results established in Sections 4 and 5.1. The factorization result used in the finite graph case is no longer applicable in the infinite graph case, and thus the proof employs a completely different approach, involving a rather subtle approximation argument, which is outlined in Section 6.1.

### 2.3. Gibbs measures on path space.

Our final results interpret our SDE system in the spirit of Gibbs measures, for which we introduce the following notation. Given a Polish space \(X\), a graph \(G = (V, E)\), a random \(X^V\)-valued element \((Y_v)_{v \in V}\) with law \(\nu \in \mathcal{P}(X^V)\), and two disjoint sets \(A, B \subset V\), we write \(\nu[A | B]\) to denote a version of the regular conditional law of \(Y_A\) given \(Y_B\). Precisely, we view \(\nu[A | B]\) as a measurable map (kernel) from \(\mathcal{X}^B\) to \(\mathcal{P}(\mathcal{X}^A)\). Note that \(\nu\) is a 2MRF if and only if \(\nu[A \upharpoonright V \setminus A](x_{V \setminus A}) = \nu[A \upharpoonright \partial^2 A](x_{\partial^2 A})\) for \(\nu\)-almost every \(x \in \mathcal{X}^V\) and every finite set \(A\). We make use of the following terminology of Gibbs measures (see [9] or [18] for further discussion of this classical framework).

**Definition 2.8.** Given a Polish space \(X\), graph \(G = (V, E)\), and \(\gamma \in \mathcal{P}(X^V)\), define \(G_2(\gamma)\) as the set of 2MRFs \(\nu \in \mathcal{P}(\mathcal{X}^V)\) such that, for each finite set \(A \subset V\), we have \(\nu[A] \sim \gamma[A]\) and also \(\nu[A \upharpoonright \partial^2 A] = \gamma[A \upharpoonright \partial^2 A]\), almost everywhere with respect to \(\gamma[\partial^2 A]\).

In other words, \(G_2(\gamma)\) is the set of (second-order, infinite volume) Gibbs measures corresponding to the specification \(\{\gamma[A \upharpoonright \partial^2 A] : A \subset V\text{ finite}\}\). Note that if \(\gamma\) is itself a 2MRF then \(G_2(\gamma)\) is nonempty, as it contains \(\gamma\) itself. Moreover, it is straightforward to check that, if \(\gamma\) and \(\nu\) are 2MRFs, then \(\nu \in G_2(\gamma)\) if and only if \(\gamma \in G_2(\nu)\). Recall that, by Assumption \(\mathcal{B}4\), the SDE system \((2.1)\) is well-posed starting from any initial distribution. The following bijection result is proved in Section 6.

**Theorem 2.9.** Suppose \((G, b, \sigma, \mu_0)\) satisfy Assumption \(\mathcal{B}\). Let \(P^{\mu_0} \in \mathcal{P}^V\) be the law of the solution of the SDE system \((2.1)\) with initial law \(\mu_0\), and let \(P^{*, \mu_0}\) be the law of the driftless SDE system \((2.5)\) with initial law \(\mu_0\), and define

\[
\mathcal{M}_{\text{init}}(\mu_0) := \left\{ \nu_0 \in G_2(\mu_0) : \sup_{v \in V} \int_{\mathbb{R}^d} |x_v|^2 \nu_0(dx) < \infty \right\},
\]

and

\[
\mathcal{M}_{\text{path}}(\mu_0) := \left\{ Q \in \mathcal{P}(\mathcal{C}^V) : Q_t \in G_2(P^{\mu_0}_t) \forall t \geq 0, \sup_{v \in V} \int_{\mathcal{C}^V} |x_v(0)|^2 Q(dx) < \infty \right\}.
\]
Then it holds that
\[ \mathcal{M}_{\text{init}}(\mu_0) = \left\{ Q \circ (X_V(0))^{-1} : Q \in \mathcal{M}_{\text{path}}(\mu_0) \right\}. \] (2.6)

Moreover, the map \( Q \mapsto Q \circ (X_V(0))^{-1} \) defines a bijection between \( \mathcal{M}_{\text{path}}(\mu_0) \) and \( \mathcal{M}_{\text{init}}(\mu_0) \).

In particular, if \( Q \in \mathcal{P}(\mathcal{C}^V) \) satisfies \( Q_t \in \mathcal{G}_2(P_t^{\mu_0}) \) for all \( t \geq 0 \) and also \( Q \circ (X_V(0))^{-1} = \mu_0 \), then \( Q = P_t^{\mu_0} \).

In fact, we will show in the proof of Theorem 2.9 that the bijection \( Q \mapsto Q \circ (X_V(0))^{-1} \) between the sets \( \mathcal{M}_{\text{path}}(\mu_0) \) and \( \mathcal{M}_{\text{init}}(\mu_0) \) has inverse given by \( \nu_0 \mapsto P_t^{\nu_0} \), where \( P_t^{\nu_0} \) denotes the law of the solution of the SDE (2.1) with initial law \( \nu_0 \), and we note that this SDE is unique in law by Assumption (B.1). Additionally, if \( \mu_0(K^V) = 1 \) for some compact set \( K \subset \mathbb{R}^d \), then (recalling that membership in \( \mathcal{P} \)) requires absolute continuity (2.6) can be rewritten as
\[ \mathcal{G}_2(\mu_0) = \left\{ Q \circ (X_V(0))^{-1} : Q \in \mathcal{P}(\mathcal{C}^V), Q_t \in \mathcal{G}_2(P_t^{\mu_0}) \forall t \geq 0 \right\}. \]

We conclude this section with the following simple corollary of Theorems 2.7 and 2.9.

**Corollary 2.10.** Suppose \((G, b, \sigma, \mu_0)\) satisfy Assumption (B) and \( P_t^{\mu_0} \) represents the unique law of the SDE (2.1). If \( \mathcal{G}_2(\mu_0) \) is a singleton, then the set \( \mathcal{M}_{\text{path}}(\mu_0) \) defined in Theorem 2.9 is equal to the singleton \( \{P_0^{\mu_0}\} \), and hence, \( P_t^{\mu_0} \) is completely characterized by its specifications \( P_t^{\mu_0}[A|\partial^2 A], t \geq 0 \), for finite \( A \subset V \).

**Proof.** Since Assumption (B.1) ensures that \( \mu_0 \) is a 2MRF with finite second moment, the set on the left-hand side of (2.6) always contains \( \mu_0 \). Thus, if \( \mathcal{G}_2(\mu_0) \) is a singleton, then by Theorem 2.9 the set \( \mathcal{M}_{\text{path}}(\mu_0) \) is also a singleton. On the other hand, by Theorem 2.7 for each \( t \geq 0 \) it holds that \( P_t^{\mu_0} \) is a 2MRF and thus \( P_t^{\mu_0} \in \mathcal{G}_2(P_t^{\mu_0}) \). Hence, \( P_t^{\mu_0} \in \mathcal{M}_{\text{path}}(\mu_0) \).

### 3. Interacting diffusions on a finite graph

In Section 3.1 (specifically Proposition 3.2) a useful characterization of a (positive) 2MRF is derived in an abstract setting. This is then used in Section 3.2 to prove Theorem 2.4 along the way Proposition 2.3 is also established. In Sections 3.3.1 and 3.3.2 this characterization is applied to demonstrate via explicit examples that the space-time 2MRF property established in Theorem 2.4 (and hence, Theorem 2.7) cannot in general be substantially improved.

#### 3.1. Clique factorizations

We start by studying the relationship between random fields and factorization properties of their joint density with respect to a given reference measure. Throughout this section, we work with a fixed finite graph \( G = (V, E) \), as well as a fixed Polish space \( \mathcal{X} \), the state space. Recall the definition of the diameter \( \text{diam}(A) \) of a set \( A \subset V \), 1-cliques and 2-cliques of a graph, and 1st- and 2nd-order MRFs given in Section 1.2.

First, we recall a well-known theorem often attributed to Hammersley-Clifford, which can be found in various forms in [9, Theorem 2.30] and [13, Proposition 3.8 and Theorem 3.9], the latter covering our precise setting.

**Proposition 3.1** (Hammersley-Clifford). Assume the graph \( G = (V, E) \) is finite. Assume \( \nu \in \mathcal{P}(\mathcal{X}^V) \) is absolutely continuous with respect to a product measure \( \nu^* = \prod_{v \in V} \theta_v \in \mathcal{P}(\mathcal{X}^V) \) for some \( \theta_v \in \mathcal{P}(\mathcal{X}), v \in V \). Consider the following statements:

1. \( \nu \) is a 1MRF.
2. The density of \( \nu \) with respect to \( \nu^* \) factorizes in the form
   \[
   \frac{d\nu}{d\nu^*}(x) = \prod_{K \in \mathcal{C}_1(G)} f_K(x_K), \quad x \in \mathcal{X}^V,
   \]
   for some measurable functions \( f_K : \mathcal{X}^K \to \mathbb{R}_+ \), for \( K \in \mathcal{C}_1(G) \).
Then (2) implies (1). If also $d\nu/d\nu^*$ is strictly positive, then (1) implies (2).

We next formulate an analogue for a 2MRF.

**Proposition 3.2** (Second-order Hammersley-Clifford). Assume the graph $G = (V, E)$ is finite. Assume $\nu \in \mathcal{P}(\mathcal{X}^V)$ is absolutely continuous with respect to a product measure $\nu^* = \prod_{v \in V} \theta_v \in \mathcal{P}(\mathcal{X})$ for some $\theta_v \in \mathcal{P}(\mathcal{X}), \ v \in V$. Consider the following statements:

1. $\nu$ is a 2MRF.
2. The density of $\mu$ with respect to $\nu^*$ factorizes in the form
   \[
   \frac{d\nu}{d\nu^*}(x) = \prod_{K \in \text{cl}_2(G)} f_K(x_K), \quad x \in \mathcal{X}^V, \tag{3.1}
   \]
   for some measurable functions $f_K : \mathcal{X}^K \to \mathbb{R}^+, K \in \text{cl}_2(G)$.

Then (2) implies (1). If also $d\nu/d\nu^*$ is strictly positive, then (1) implies (2).

**Proof.** Define the square graph $G^2 = (V, E')$ by connecting any two vertices of distance 2. That is, let
\[
E' := \{(u, v) \in V^2 : 1 \leq d(u, v) \leq 2\},
\]
where $d$ is the graph distance on $G$. It is straightforward to check the following properties:

(i) The 1-cliques of $G^2$ are precisely the 2-cliques of $G$. That is, $\text{cl}_2(G) = \text{cl}_1(G^2)$.
(ii) We have $\partial_{G^2} A = \partial_{G} A$ for any set $A \subseteq V$.

It follows from (ii) that the statement (1) is equivalent to

(1') $\nu$ is a 1MRF relative to the graph $G^2$.

On the other hand, it follows from (i) that (2) is equivalent to

(2') The density of $\nu$ with respect to $\nu^*$ factorizes in the form
\[
\frac{d\nu}{d\nu^*}(x) = \prod_{K \in \text{cl}_1(G^2)} f_K(x_K), \quad x \in \mathcal{X}^V,
\]
for some measurable functions $f_K : \mathcal{X}^K \to \mathbb{R}^+, K \in \text{cl}_1(G^2)$.

The equivalence of (1') and (2') follows from Proposition 3.1. \qed

The 2MRF property is the more intuitive, but the second property of Proposition 3.2 will be quite useful in the analysis as well. Hence, we give it a name.

**Definition 3.3.** We say that $\nu \in \mathcal{P}(\mathcal{X}^V)$ has a 2-clique factorization with respect to $\nu^*$ if the density $d\nu/d\nu^*$ can be written in the form (3.1).

**Remark 3.4.** For a finite graph $G = (V, E)$ and Polish space $\mathcal{X}$, the following cutset characterization of 1MRF’s on $\mathcal{X}^V$ is well known: An $\mathcal{X}^V$-valued random element $(Y_v)_{v \in V}$ is a 1MRF if and only if $Y_A$ is conditionally independent of $Y_B$ given $Y_S$ for any disjoint sets $A, B, S \subseteq V$ with the property that every path starting in $A$ and ending in $B$ contains at least one vertex of $S$. Given the correspondence between a 2MRF on a graph and a 1MRF on the square graph (established in the proof of Proposition 3.2), this is easily seen to imply the following cutset characterization of 2MRFs: An $\mathcal{X}^V$-valued random element $(Y_v)_{v \in V}$ is a 2MRF if and only if $Y_A$ is conditionally independent of $Y_B$ given $Y_S$ for any disjoint sets $A, B, S \subseteq V$ with the property that every path starting in $A$ and ending in $B$ contains at least two adjacent vertices of $S$. 

3.2. Proof of the second-order Markov random field property for a finite graph.
We now present the proofs of Proposition 2.3 and Theorem 2.4. Throughout this section, we work with a fixed finite graph \( G = (V, E) \) and consider the canonical measurable space \( \mathcal{C}^V = (\mathcal{C}^V, \mathcal{B}(\mathcal{C}^V)) \). Let \( (\sigma_v \sigma_v^\top)^{-1}b_v(s, x, y) \) denote the map \( \mathbb{R}^d \times \mathcal{C}^V \ni (s, x) \mapsto \sigma_v(s, x)\sigma_v^\top(s, x) \) and \( b_v(s, x, y) \). Further, observe that \((M_v)_{v \in V}\) are orthogonal under \( P^* \). So Girsanov’s theorem \([10, Corollary 3.5.2]\) implies that under \( P^*_0 \), \( W_v := W_v - \int_0^t \sigma_v(s, x_v)b_v(s, x_v, x_{N_v}(G)) \, ds \), \( v \in V \), are independent \( d \)-dimensional standard Brownian motions on \([0, t]\). From this it follows that under \( P^*_t \), \( W_v \) solves the SDE on \([0, t]\), and the same argument also shows that the restriction to \([0, t]\) of any solution to (2.1) must have law \( P^*_t \) on \( \mathcal{C}_t^V \). Thus, we have uniqueness in law. Weak existence follows from Kolmogorov’s extension theorem on observing that \( \{P_t, t \geq 0\} \) form a consistent family in the sense that \( P_s \) is the restriction of \( P_t \) to \( \mathcal{C}_s^V \) for each \( t > s > 0 \) (due to the martingale property of \( dP_s/dP_t \)). This completes the proof of Proposition 2.3.

On the other hand, the fact that for each \( t > 0 \), \((X_v[t], v \in V)\) is a 2MRF on \( \mathcal{C}_t^V \) follows from (3.5) on applying Proposition 3.2 with \( \mathcal{X} = \mathcal{C}_t \) and \( \mu^* = P^*_t \), noting that \( P^*_t \) is a product measure on \( \mathcal{C}_t \) and that, for each \( v \in V \), \( \{v \} \cup N_v(G) \) is a clique and \( M_v \) of (3.2) is \( X_{\{v\} \cup N_v(G)} \)-measurable. This proves the first assertion of Theorem 2.4. For the second assertion of Theorem 2.4, denote by \( P = P^\mu_0 \in \mathcal{P}(\mathcal{C}^V) \) the law of the unique solution of the SDE system (2.1) with initial law \( \mu_0 \). Fix a finite set \( A \subset V \) and bounded continuous functions \( f, g, h \) on \( \mathcal{C}_A \), \( \mathcal{C}^\beta A \), \( \mathcal{C}^V \setminus (\mathcal{A} \cup \mathcal{B}^2 A) \), respectively. Fix \( t > 0 \) and let \( \mathcal{G}_t := \sigma\{X_{\mathcal{B}^2 A}[t]\} \) and \( \mathcal{G}_\infty := \sigma\{X_{\mathcal{B}^2 A}\} \).
Below, with some abuse of notation, for any \( B \subset V \), we will also interpret elements \( y \in \mathcal{C}_t^B \) as elements of \( \mathcal{C}_t^V \) by simply setting \( y(s) = y(t) \) for \( s \geq t \). Note that with this identification, for any \( x \in \mathcal{C}_t^V \) and \( B \subset V \), \( x_B[t] \to x_B \) in \( \mathcal{C}_t^B \) as \( t \to \infty \). Then, noting that \( \sigma(\cup_{s>0} \mathcal{G}_s) = \mathcal{G}_\infty \), invoking the martingale convergence theorem (in the third equality below), and using the fact that \( P_t = P \circ (X_V[t])^{-1} \) is a 2MRF on \( \mathcal{C}_t^V \) for each \( t \) (in the second equality below), we have

\[
\begin{align*}
\mathbb{E}^P \left[ f(X_A)g(X_{\partial^2 A})h(X_{V \setminus (A\cup \partial^2 A)}) \right] &= \lim_{s \to \infty} \lim_{t \to \infty} \mathbb{E}^P \left[ f(X_A[s])g(X_{\partial^2 A}[t])h(X_{V \setminus (A\cup \partial^2 A)}[s]) \right] \\
&= \lim_{s \to \infty} \lim_{t \to \infty} \mathbb{E}^P \left[ f(X_A[s]) \mid \mathcal{G}_t \right] g(X_{\partial^2 A}[t]) \mathbb{E}^P \left[ h(X_{V \setminus (A\cup \partial^2 A)}[s]) \mid \mathcal{G}_t \right] \\
&= \lim_{s \to \infty} \mathbb{E}^P \left[ f(X_A[s]) \mid \mathcal{G}_\infty \right] g(X_{\partial^2 A}) \mathbb{E}^P \left[ h(X_{V \setminus (A\cup \partial^2 A)}[s]) \mid \mathcal{G}_\infty \right] \\
&= \mathbb{E}^P \left[ f(X_A) \mid \mathcal{G}_\infty \right] g(X_{\partial^2 A}) \mathbb{E}^P \left[ h(X_{V \setminus (A\cup \partial^2 A)}) \mid \mathcal{G}_\infty \right],
\end{align*}
\]

where we have also made repeated use of the boundedness and continuity of \( f, g, h \) and the bounded convergence theorem. This shows that \( X_A \) and \( X_{V \setminus (A\cup \partial^2 A)} \) are conditionally independent given \( X_{\partial^2 A} \) under \( P \), that is, \( P \) is a 2MRF on \( \mathcal{C}_t^V \). This completes the proof.

3.3. Illustrative examples. We now provide examples to show that the 2MRF property cannot in general be strengthened.

3.3.1. The failure of the first-order MRF property for trajectories. In general, \( P_t \) fails to be a first-order Markov random field on \( \mathcal{C}_t^V \) for any \( t > 0 \), even if the initial states are i.i.d. To see why, notice that the density \( dP_t/dP_\ast \) given by (3.5) does not in general admit a clique factorization. Indeed, for \( v \in V \) and \( t > 0 \), we recall the definition of \( M_v \) from (3.2) and \( \mathcal{E}_t(M_v) \) from (3.4), which we write in full as

\[
\mathcal{E}_t(M_v) = \exp \left( \int_0^t (\sigma_v \sigma_v^\top)^{-1} b_v(s, X_v, X_{N_v(G)}) \cdot dX_v(s) \right.
\]

\[
- \frac{1}{2} \int_0^t \left( b_v(s, X_v, X_{N_v(G)}), (\sigma_v \sigma_v^\top)^{-1} b_v(s, X_v, X_{N_v(G)}) \right) ds \bigg).\]

Noting that \( \{ v \} \cup N_v(G) \) is a 2-clique but not a 1-clique, this reveals why one cannot hope for a factorization over 1-cliques. For example, consider the “nice” case where \( \sigma_v \equiv I \) and \( b_v(s, x_v, x_{N_v(G)}) = \sum_{u \in N_v(G)} (x_u(s) - x_v(s)) \). (Equivalently, \( b_v(s, x_v, x_{N_v(G)}) = \nabla_{x_v} h(x) \) is of gradient-type with potential \( h(x) = -\frac{1}{2} \sum_{(u,v) \in E} |x_u - x_v|^2 \), where \( E \) is the edge set of \( G \).) Then the first term in the above exponential splits nicely into a sum of pairwise interactions \( \sum_{u \in N_v(G)} \int_0^t (X_u(s) - X_v(s)) \cdot dX_v(s) \), but the second term becomes

\[
- \frac{1}{2} \sum_{u, w \in N_v(G)} \int_0^t (X_u(s) - X_v(s), X_w(s) - X_v(s)) ds.
\]

It is this term which fails to factorize further over 1-cliques as opposed to 2-cliques and thus precludes the first-order Markov property whenever \( d\mu_0/d\mu'_0 \) is strictly positive due to Proposition 3.1.

To informally provide a different (but arguably more intuitive) perspective on why the first-order Markov property for past histories fails, consider the case when \( G \) is a line segment of length \( \ell = 3 \), labelling the vertices \(-1, 0, 1\). Then, although the driving Brownian motions are all independent and the dynamics of each of the two extreme vertices only depend on its own state and the state of the center vertex, at any time \( t \), conditioning on the past history of the
states of the center vertex, does not make $X_{-1}(t)$ independent of $X_1(t)$ because the conditioning correlates the Brownian motions $W_{-1}$ and $W_1$ on the interval $[0, t]$. This happens because the past history of $X_0$ is influenced by both $W_{-1}$ and $W_1$ via $X_{-1}$ and $X_1$. On the other hand, to see why the 2MRF property nevertheless does hold, note that if $X$ correlates the Brownian motions $W_{-1}$ and $W_1$ via $X_{-1}$ and $X_1$, the dynamics of each of the conditioned vertices depends on a different driving Brownian motion. Thus, although the conditioning changes the distribution of $W_{-1}$ and $W_1$ (for instance, they need no longer be Brownian motions), they remain independent, and hence $X_{-2}(t)$ is conditionally independent of $X_2(t)$ in this case.

**Remark 3.5.** There are certain situations in which $P_t$ is, in fact, a 1MRF for each $t > 0$ (even though we know from the above examples that this is not in general the case). For example, suppose that for every $v \in V$, there exists a clique $K_v$ of $G$ with $v \in K_v \subset N_v(G)$ such that $b_v(t, x_v, x_{N_v(G)}) = \tilde{b}_v(t, x_v, x_{K_v})$ depends on $x_{N_v(G)}$ only through $x_{K_v}$. Suppose also that $d\mu_0/d\mu_0^*$ admits a 1-clique factorization. Then, recalling (3.5), note that for each $v$ the martingale $M_v$ is measurable with respect to $X_{K_v}$, and deduce from Proposition 3.4 that $P_t$ is a first-order Markov random field. For a concrete example that has the above form, consider the case when $G$ is a triangular lattice with $V = \{0, 1, \ldots, m\}$, for some $m \in \mathbb{N}$, with the central vertex $\phi$ having the neighborhood $N_0(G) = \{0, \ldots, m\}$ and for each $v \in V \setminus \{\phi\}$, $N_v(G) := \{\phi, v + 1, v - 1\}$, where the vertices are to be interpreted mod $m + 1$. Further, suppose the initial conditions are i.i.d. and that for some $c \in \mathbb{R}$, $b_v(t, x_v, x_{N_v(G)}) = c(x_v + x_{v+1})$ for $v \in V \setminus \{\phi\}$ and $b_\phi(t, x_\phi, x_{N_\phi(G)}) = cx_\phi$. Then this provides a specific example with $K_v = \{\phi, v + 1\} \subset N_v(G)$ for $v \in V \setminus \{\phi\}$ and $K_\phi = \emptyset$. In a similar spirit, the directed cycle graph model of [2] provides another example.

3.3.2. The failure of MRF properties for time-$t$ marginals. It is natural to wonder if and when the time-$t$ marginals $P_t \circ T(t)^{-1} \in \mathcal{P}(\mathbb{R}^d)^V$ remain a first- or second-order Markov random field, given that this property is true at time 0, or even given i.i.d. initial conditions. This question is related to propagation of Gibbsianness and Gibbs-non-Gibbs transitions that have been studied in the literature, which is discussed in greater detail in Remark 3.5.

Here, we provide a simple example where both the first-order and second-order Markov property fail for time-$t$ marginals. In fact, in this simple model we will see that there is no non-trivial conditional independence structure. Consider the segment with 5 vertices: $G = (V, E)$ given by $V = \{1, 2, 3, 4, 5\}$ and $E = \{(i, i + 1) : i = 1, 2, 3, 4\}$, and consider the SDE system

$$
\begin{align*}
    dX_1(t) &= (X_2(t) - 2X_1(t)) \, dt + dW_1(t), \\
    dX_i(t) &= (X_{i-1}(t) + X_{i+1}(t) - 2X_i(t)) \, dt + dW_i(t), \quad i = 2, 3, 4, \\
    dX_5(t) &= (X_4(t) - 2X_5(t)) \, dt + dW_5(t),
\end{align*}
$$

with $X_i(0) = 0$ for each $i$. Once again, note that the drift here is of gradient type with potential $h(x) = \sum_{i=1}^4 x_ix_{i+1} - \sum_{i=1}^5 x_i^2$. Letting $X(t)$ denote the column vector $(X_1(t), \ldots, X_5(t))$ and similarly for $W(t)$, we may write this in vector form as

$$
    dX(t) = L X(t) \, dt + dW(t),
$$

where $L = A - 2I$ is the adjacency matrix $A$ of the graph minus twice the identity $I$:

$$
    L = \begin{pmatrix}
        -2 & 1 & 0 & 0 & 0 \\
        1 & -2 & 1 & 0 & 0 \\
        0 & 1 & -2 & 1 & 0 \\
        0 & 0 & 1 & -2 & 1 \\
        0 & 0 & 0 & 1 & -2
    \end{pmatrix}.
$$
The solution of the SDE (3.8) is given by
\[ \mathbf{X}(t) = e^{Lt} \int_0^t e^{-Ls} d\mathbf{W}(s), \quad t > 0. \]
Noting that \( L \) is symmetric and invertible, we deduce that \( \mathbf{X}(t) \) is jointly Gaussian with mean zero and covariance matrix
\[ \mathbb{E}[\mathbf{X}(t)\mathbf{X}(t)^\top] = \int_0^t e^{2Lt} ds = \frac{1}{2} L^{-1}(e^{2Lt} - I). \] (3.9)
This covariance matrix can easily be computed explicitly by noting that the tridiagonal Toeplitz matrix \( A \) is explicitly diagonalizable. To spare the reader any tedium, we provide only some pertinent snapshots. At time \( t = 2 \) the covariance matrix is
\[
\mathbb{E}[\mathbf{X}(2)\mathbf{X}(2)^\top] = \begin{pmatrix}
0.3611 & 0.2388 & 0.1435 & 0.0767 & 0.0324 \\
0.2388 & 0.5046 & 0.3156 & 0.1759 & 0.0767 \\
0.1435 & 0.3156 & 0.5370 & 0.3156 & 0.1435 \\
0.0767 & 0.1759 & 0.3156 & 0.5046 & 0.2388 \\
0.0324 & 0.0767 & 0.1435 & 0.2388 & 0.3611
\end{pmatrix}. \] (3.10)
Using the well known formula for conditional measures of joint Gaussians, we compute from this that
\[ \text{Cov}(X_1(t), X_3(t)|X_2(t)) = \begin{pmatrix}
0.2481 & -0.0058 \\
-0.0058 & 0.3397
\end{pmatrix}, \]
which reveals that \( X_1(t) \) and \( X_3(t) \) are not conditionally independent given \( X_2(t) \). Hence, \( (X_i(t))_{i \in G} \) is not a first-order Markov random field. Similarly, by computing
\[ \text{Cov}(X_1(t), X_4(t)|X_2(t), X_3(t)) = \begin{pmatrix}
0.2480 & -0.0030 \\
-0.0030 & 0.3189
\end{pmatrix}, \]
we see that \( X_1(t) \) and \( X_4(t) \) are not conditionally independent given \( (X_2(t), X_3(t)) \). Hence, \( (X_i(t))_{i \in G} \) is not a second-order Markov random field.
In fact, in this example, there is no non-trivial conditional independence structure, in the sense that there are no two vertices \( i, j \) such that \( X_i(t) \) and \( X_j(t) \) are conditionally independent given \( \{X_k(t) : k \in G \setminus \{i, j\}\} \). This can be read off from the so-called precision matrix, which is simply the inverse of the covariance matrix, \( Q(t) := (\mathbb{E}[\mathbf{X}(t)\mathbf{X}(t)^\top])^{-1} \). As is well known and can easily be seen from the form of the multivariate Gaussian density, the precision matrix reveals the conditional independence structure (see, e.g., [13, Proposition 5.2]), in the following sense: For \( t > 0 \) define the graph \( \tilde{G}(t) = (V, \tilde{E}(t)) \) with the same vertex set \( V \) but with \( (i, j) \in \tilde{E}(t) \) if and only if \( Q_{i,j}(t) \neq 0 \). Then \( \mathbf{X}(t) \) is a (first-order) Markov random field with respect to the graph \( \tilde{G}(t) \). In our example, \( \tilde{G}(t) \) is the complete graph for each \( t > 0 \), and this Markov property is vacuous. (Note, however, that \( Q(t) \to 2L \) as \( t \to \infty \) because \( L \) is negative definite, and the unique invariant measure of this diffusion is a first-order Markov random field with respect to the original graph \( G \).)
A variation on this example gives rise to another interesting phenomenon. Suppose we modify the example by replacing the diagonal entries of \( L \) with zeros, i.e., remove all the \(-2X\) terms from the drifts in (3.7). Then the covariance matrix is again invertible, and now \( Q_{1,4}(t) = Q_{2,5}(t) = 0 \) for all \( t > 0 \), where we continue with the notation of the previous paragraph. That is, \( \tilde{G}(t) \) is not the complete graph, but rather the complete graph with the edges \( (1, 4) \) and \( (2, 5) \) removed, for each \( t > 0 \). In particular, \( X_1(t) \) and \( X_4(t) \) are conditionally independent given \( (X_2(t), X_3(t), X_5(t)) \), for each \( t > 0 \).
Remark 3.6. As mentioned in the introduction, one motivation for studying such conditional independence questions is that (a stronger version of) the MRF structure of interacting SDEs can lead to a more succinct autonomous “local characterization” of the dynamics at a vertex and its neighborhood, as developed in the quite different setting of unimodular Galton-Watson trees in [12]. From this perspective, it would be of interest to investigate if there are non-trivial special cases when the first-order or second-order MRF property for time-$t$ marginals propagates. A different but related question that has been studied in the literature is propagation of Gibbsian-ness for an infinite system of interacting real-valued diffusions indexed by $\mathbb{Z}^d$. Specifically, the work [6] considers a collection of interacting diffusions, indexed by $\mathbb{Z}^d$, with identity covariance and a drift that is the gradient of a Hamiltonian function associated with a certain interaction potential $\Phi$, and with an initial distribution that is also a Gibbs measure (as in Section 2.3) with respect to a Gibbsian specification (in the sense of [9, Chapter 2]) associated with another interaction potential $\Phi_0$, where both interaction potentials $\Phi$ and $\Phi_0$ are assumed to be of finite range and satisfy certain smoothness conditions. It is shown in [6] that when either $t$ or the interaction strength is sufficiently small, the time-$t$ marginals are strongly Gibbsian, that is, associated with Gibbsian specifications that have an absolutely summable, though not necessarily finite range, interaction potential. Extensions of these results to the case of interacting real-valued diffusions on $\mathbb{Z}^d$ with non-Markovian drifts with finite memory (again with finite range interactions and identity covariance) were later obtained in [19] and [20]. The restrictions on the time and interaction strength in these works arise from the fact that perturbative arguments are used. However, in general for moderate interaction strengths and moderate times, the time-$t$ marginals can fail to be Gibbsian (see, e.g., [27], as well as the survey [26], which also discusses related results for spin systems).

4. Finite-graph approximations for Markov random fields

In this section we establish some important preparatory results that are used in the proof of Theorem 2.7, which extends the finite graph results of Theorem 2.4 to the infinite graph setting. Fix $(G, b, \sigma, \mu_0)$ that satisfy Assumption 2.3 and suppose $G = (V, E)$ is countably infinite. Recall that $P = P_{\mu_0}^t \in \mathcal{P}(C^V)$ and $P_{\mu_0}^{s,t} \in \mathcal{P}(C^V)$ denote the unique law of the SDE systems (2.1) and (2.5), respectively, both with initial laws $\mu_0$, which are well-posed by Assumptions (2.4) and (2.3). To show that $P_t = P_{\mu_0}^t$ forms a 2MRF on $C^V$, we can no longer apply the clique factorization arguments used for finite graphs because the formula (3.3) does not extend to infinite graphs. Even worse, the density $dP_t / dP_{\mu_0}^t$ therein does not exist, and it seems impossible to establish directly that by projecting to a finite set $A \subset V$ we have a density $dP_t[A] / dP_{\mu_0}^t[A]$ that admits a 2-clique factorization. Instead, we approximate the measure on the infinite graph by 2MRFs on a growing sequence of finite graphs, arguing that the desired 2MRF property passes to the limit. To highlight some of the subtleties that arise in such an approximation argument, and to better motivate the other results established in this section, we first describe the approximating sequence of measures in Section 4.1. Then in the subsequent two sections we establish some general properties of finite-graph 2MRFs to be used in the proof of Theorem 2.7 in Section 5.2, which are also of independent interest.

4.1. Construction of the approximating sequence of SDEs. We fix $(G, b, \sigma, \mu_0)$ that satisfy Assumption 2.3. As in Section 3.2, we will work with the canonical measure space $C^V = (C^V, \text{Borel}, P_{\mu_0}^t)$, and let $(X_v)_{v \in V} : C^V \to C^V$ again denote the canonical processes. Also, recall from Section 1.2 that given any measurable space $X$, measure $\mu \in \mathcal{P}(\mathcal{X}^V)$ and subset $U \subset V$, $\mu[U] \in \mathcal{P}(\mathcal{X}^U)$ denotes the restriction of $\mu$ to the set $\mathcal{X}^U$.

Let $V_n \subset V, n \in \mathbb{N}$, be such that $\bigcup_n V_n = V$, and let $G_n = (V_n, E_n)$, for some edge set $E_n$ to be specified later. Also, for each $n \in \mathbb{N}$ and $v \in V_n$, let $b_v^n : \mathbb{R}_+ \times \mathcal{C} \times C^{|E_n(G_n)|} \to \mathbb{R}^d$ be any
progressively measurable map that satisfies the same conditions as \( b_v \) in Assumption (B.2). Fix \( t > 0 \) and for each \( n \), define \( P^n_t = P^\mu_{t,n} \in P(C^n_t) \) by

\[
\frac{dP^n_t}{dP^\mu_{t,0}} = \prod_{v \in V_n} \mathcal{E}_t \left( \int_0^t (\sigma_v^{-1} b^n_v(s, X_v, X_{N_v(G_n)})) \cdot dX_v(s) \right) ,
\]

where, as before, \( (\sigma_v, \sigma_v^\top)^{-1} b^n_v(s, x, x_{N_v(G_n)}) \) denotes the map (B.3). We can apply Lemma (B.1) with \( Q = P^\mu_{t,0} \), \( X = (X_v)_{v \in V_n} \) and \( f(t, x) = (\sigma_v^{-1} b^n_v(t, x_v, x_{N_v(G_n)}))_{v \in V_n} \), to conclude that the stochastic exponential in (4.1) is a true \( P^\mu_{t,0} \)-martingale, due to the linear growth, non-degeneracy and boundedness properties of \( b^n_v \) and \( \sigma_v \) in Assumptions (B.2) and (B.3). Hence, the family \( (P^n_t)_{t>0} \) is consistent in the sense that the restriction of \( P^n_t \) to \( C^n_t \) is precisely \( P^n_s \) for each \( t > s > 0 \). Thus, by the Kolmogorov extension theorem, \( (P^n_t)_{t>0} \) uniquely determines a probability measure \( P^n \) on \( C^V \). Now, from (4.1), (2.5) and Girsanov’s theorem [10, Corollary 3.5.2], it follows that under \( P^n \) the canonical process solves the SDE system

\[
\begin{align*}
\frac{dX_v(t)}{dt} &= b^n_v(t, X_v, X_{N_v(G_n)}) \cdot dt + \sigma_v(t, X_v) \cdot dW_v(t), \quad v \in V_n, \\
\frac{dX_v(t)}{dt} &= \sigma_v(t, X_v) \cdot dW_v(t), \quad v \in V \setminus V_n,
\end{align*}
\]

with \((X_v(0))_{v \in V} \sim \mu_0\), where \((W_v)_{v \in V} \) are independent Brownian motions under \( P^n \). Note that for \( v \in V_n \), the third argument of \( b^n_v \) looks only at the states in \( N_v(G_n) \), and thus \( b^n_v \) depends only on the states of vertices in \( G_n \). Thus, \( P^n_{[V_n]} \) is precisely the law of the finite-graph SDE system (2.1) with inputs \((G, (b^n_v)_{v \in V}, (\sigma_v)_{v \in V}, \mu_0[V_n])\).

In order to implement our approximation argument we would like to choose \( G_n \) and \((b^n_v)_{v \in V}\) such that both \( P^n \to P \) and each \( P^n_{[V_n]} \) is a 2MRF. In order to have \( P^n \to P \) we should naturally choose \( V_n \) increasing to \( V \) and \( b^n_v \) to behave like \( b_v \) for most \( v \). But the 2MRF property is more delicate. It would follow from the finite-graph result of Theorem (4.4) that \( P^n_{[V_n]} \) is a 2MRF on \( C^V_n \) only if \( \mu_0[V_n] \) were a 2MRF on \((\mathbb{R}^d)^{V_n} \). But \( \mu_0[V_n] \) is not necessarily a 2MRF for arbitrary \( V_n \) (e.g., with \( G_n \) the induced subgraph), even though \( \mu_0 \) is a 2MRF on the full graph \( G \) by assumption; in other words, the 2MRF property is not in general preserved under projections, as illustrated in Example (4.1) below. However, in Section (4.2) we show that for any Markov random field on an infinite graph \( G = (V, E) \), it is possible to identify a suitable increasing sequence of vertices \((V_n)_{n \in \mathbb{N}} \) and associated graph \( G_n = (V_n, E_n) \) for each \( n \in \mathbb{N} \) that is a slight modification of the induced subgraph on \( V_n \), such that the desired projection property holds. Then, in Section (4.3) we prove some results on preservation of a class of conditional distributions of 2MRFs under restriction to induced subgraphs. The above results are combined with tightness and convergence estimates for the approximating sequence \( \{P^n\} \) obtained in Section (5.1) to complete the proof of Theorem (2.7) in Section (5.2).

4.2. Projections of Markov random fields. We first provide a simple example to illustrate that the restriction of an MRF to an induced subgraph need not remain an MRF.

**Example 4.1.** Suppose \( G \) is a finite two-dimensional lattice, with vertex set \( V \) identified with \((-n, \ldots, n)^2 \) and the usual nearest-neighbor edge set, and let \((Y_v)_{v \in G} \) be a 1MRF on \( \mathbb{R}^V \). Consider the line subgraph \( H = \{(i, 0) : i = -n, \ldots, n\} \subset G \), and consider the restriction \((Y_v)_{v \in H} \) of the 1MRF to \( H \). Note that every path in \( H \) that starts in \( A := \{(0,0)\} \) and ends in \( B := \{(2,0)\} \) must traverse through the vertex of \( S := \{(1,0)\} \). Thus, by the cutset characterization of 1MRFs given in Remark (3.4) for \((Y_v)_{v \in H} \) to be an 1MRF on \( H \), \( Y_{(0,0)} \) must be conditionally independent of \( Y_{(2,0)} \) given \( Y_{(1,0)} \). However, by the same cutset characterization, it is clear that this conditional independence cannot be deduced from the 1MRF property of \((Y_v)_{v \in G} \) on \( G \) since there are paths in \( G \) that start in \( A \) and end in \( B \) that are disjoint from \( S \). Similarly, if we assume \((Y_v)_{v \in G} \) is a 2MRF, the configuration \((Y_v)_{v \in H} \) can fail to be a 2MRF.
This example does suggest, however, that we can restore the MRF property by enlarging the edge set of the induced subgraph to reflect the lost connectivity. The following lemma gives one way to do this which is certainly not the only way, but it serves our purpose. For a random element \((Y_v)_{v \in V}\) of \(\mathcal{X}^V\) with law \(\nu \in \mathcal{P}(\mathcal{X}^V)\), and for a set \(A \subset V\), recall that we write \(\nu[A]\) to denote the law of \(Y_A\), the coordinates in \(A\).

**Lemma 4.2.** Fix a rooted graph \(G = (V, E, \phi)\) and \(n \geq 4\). Define \(V_n := \{v \in V : d(v, \phi) \leq n\}\) and \(U_n := V_n \setminus V_{n-2}\), where \(d\) denotes the graph distance. Define a graph \(G_n = (V_n, E_n)\), where

\[
E_n := \{(u, v) \in V_n \times V_n : (u, v) \in E\} \cup \{(u, v) \in U_n \times U_n, u \neq v\}.
\]

(i) For any \(A \subset V_{n-3}\), it holds that \(\partial_G^2 A = \partial_{G_n} A\). Also, for any \(A' \subset V_{n-2}\), \(\partial_G^2 A' \subset \partial_{G_n} A'\).

(ii) If \(K \in \text{cl}_2(G)\) satisfies \(K \subset V_n\), then \(K \in \text{cl}_2(G_n)\).

(iii) If a \(\mathcal{X}^V\)-valued random variable \((Y_v)_{v \in V}\) is a 2MRF with respect to \(G\), then \((Y_v)_{v \in V_n}\) is a 2MRF with respect to \(G_n\).

(iv) Suppose \(V\) is finite and the law \(\nu\) of \(Y^V\) admits the following 2-clique factorization with respect to a product measure \(\nu^* = \prod_{v \in V} \theta_v \in \mathcal{P}(\mathcal{X}^V)\) for some \(\theta_v \in \mathcal{P}(\mathcal{X})\),

\[
\frac{d\nu}{d\nu^*}(x_V) = \prod_{K \in \text{cl}_2(G)} f_K(x_K),
\]

for some measurable functions \(f_K : \mathcal{X}^K \to \mathbb{R}_+, \text{ for } K \in \text{cl}_2(G)\). Then \(\nu[V_n]\) admits a 2-clique factorization

\[
\frac{d\nu[V_n]}{d\nu^*[V_n]}(x_{V_n}) = \prod_{K \in \text{cl}_2(G_n)} f^0_K(x_K),
\]

for some measurable functions \(f^0_K : \mathcal{X}^K \to \mathbb{R}_+, \text{ for } K \in \text{cl}_2(G_n)\), which additionally satisfy the consistency condition \(f^0_K \equiv f_K\) for \(K \in \text{cl}_2(G)\) such that \(K \subset V_{n-3}\).

**Proof.**

(i) From the definition of \(E_n\) it follows quickly that (a) for \(A' \subset V_{n-2}\), \(\partial_G A' = \partial_{G_n} A'\), and (b) for \(A'' \subset V_{n-1}\), \(\partial_G A'' \subset \partial_{G_n} A''\). Iterate these observations to prove the claims.

(ii) Let \(d_G\) and \(d_{G_n}\) denote the graph distance in \(G\) and \(G_n\), respectively. From the definition of \(E_n\), it is straightforward to argue that \(d_{G_n} \leq d_G\) on \(V_n \times V_n\). Indeed, for any \(u, v \in V_n\) and any path from \(u\) to \(v\) in \(G\), there is a path from \(u\) to \(v\) in \(G_n\) which is not longer. This implies for every \(u, v \in V_n\), \(d_G(u, v) \leq 2\) implies \(d_{G_n}(u, v) \leq 2\), which proves property (ii).

(iii) Let \((Y_v)_{v \in V}\) be a 2MRF with respect to \(G\). Let \(A \subset V_n, B = V_n \setminus (A \cup \partial_G^2 A)\), and \(S := \partial_{G_n}^2 A\). Assuming without loss of generality that \(A\) and \(B\) are nonempty, we must prove that \(Y_A\) and \(Y_B\) are conditionally independent given \(Y_S\). First notice that one cannot have both \(A \cap U_n \neq \emptyset\) and \(B \cap U_n \neq \emptyset\), as this would imply \(d_{G_n}(A, B) \leq 1\), contradicting the definition of \(B\). Therefore we must have either \(A \cap U_n = \emptyset, B \cap U_n = \emptyset\), or both.

**Case 1:** Suppose \(A \cap U_n = \emptyset\). This means \(A \subset V_{n-2}\) and hence \(\partial_G^2 A \subset \partial_{G_n} A\) by (i). Since \(Y_A\) and \(Y_{V_n \setminus (A \cup \partial_G^2 A)}\) are conditionally independent given \(Y_{\partial_G^2 A}\), we then have conditional independence of \(Y_A\) and \(Y_B\) given \(Y_S\). Indeed, this uses the elementary fact that if \((Z_1, Z_2, Z_3, Z_4)\) are random variables with \(Z_1\) conditionally independent of \((Z_2, Z_3)\) given \(Z_4\), then \(Z_1\) is conditionally independent of \(Z_2\) given \((Z_3, Z_4)\).

**Case 2:** Suppose \(B \cap U_n = \emptyset\). This means \(B \subset V_{n-2}\) and hence, again by (i), \(\partial_G^2 B \subset \partial_{G_n}^2 B\). Also note that \(\partial_G^2 B \subset S\) (since otherwise \(A \cap \partial_{G_n}^2 B \neq \emptyset\), which contradicts the definition of \(B\)). Since the 2MRF property with respect to \(G\) implies \(Y_B\) and \(Y_{V_n \setminus (B \cup \partial_G^2 B)}\) are independent conditioned on \(Y_{\partial_G^2 B}\), we then have conditional independence of \(Y_B\) and
4.3. **Conditional distributions of second-order Markov random fields.** First, in Lemma 4.3, given a 2MRF with respect to a graph, and another 2MRF on a subgraph, or more generally given MRFs on two overlapping graphs, we identify conditions under which the conditional distributions of a subset in the intersection (given its complement) coincide for both 2MRFs. This will be used to establish, for a suitable choice of \( b^n \), a certain consistency condition for the sequence of approximating measures \( \{\mathcal{P}^n\} \) used in the proof of Theorem 2.7. Let us briefly recall a notation we introduced more carefully just before Theorem 2.9. For \( \nu \in \mathcal{P}(\mathcal{X}^V) \) and \( A, B \subset V \) we write \( \nu[A \mid B] \) for the conditional law of the \( A \)-coordinates given the \( B \)-coordinates.

**Lemma 4.3.** Let \( G = (V_G, E_G) \) and \( H = (V_H, E_H) \) be finite graphs, and assume \( V^* \subset V_G \cap V_H \) satisfies

\[
E_G \cap (V^* \times V^*) = E_H \cap (V^* \times V^*).
\]

Moreover, let \( A \subset V^* \) satisfy \( \partial^2_G A \subset V^* \) and \( \partial^2_H A \subset V^* \). Then \( \partial^2_H A = \partial^2_G A =: \partial^2 A \), and it holds that

\[
\{K \in \text{cl}_2(G) : K \cap A \neq \emptyset\} = \{K \in \text{cl}_2(H) : K \cap A \neq \emptyset\} =: K_A.
\]

Next, let \( \nu^H \in \mathcal{P}(\mathcal{X}^V_H) \) and \( \nu^G \in \mathcal{P}(\mathcal{X}^V_G) \), and suppose there exists a product measure \( \nu^* = \prod_{v \in V_G \cup V_H} \theta_v \in \mathcal{P}(\mathcal{X}^V) \), \( v \in V_G \cup V_H \), such that the densities factorize as

\[
\frac{d\nu^H}{d\nu^*}[V_H](x_{V_H}) = \prod_{K \in \text{cl}_2(H)} f^H_K(x_K), \quad \frac{d\nu^G}{d\nu^*}[V_G](x_{V_G}) = \prod_{K \in \text{cl}_2(G)} f^G_K(x_K),
\]

where \( f^G_K(x_K) \) and \( f^H_K(x_K) \) are densities depending on \( x_K \) for \( K \in \text{cl}_2(G) \) and \( K \in \text{cl}_2(H) \), respectively.
for measurable functions \((f^H_K : \mathcal{X}^K \to \mathbb{R}_+)_{K \in \mathcal{C}_2(H)}\) and \((f^G_K : \mathcal{X}^K \to \mathbb{R}_+)_{K \in \mathcal{C}_2(G)}\) satisfying \(f^H_K \equiv f^G_K\) for all \(K \in \mathcal{K}_A\). Then \(\nu^H[A | \partial^2 A] = \nu^G[A | \partial^2 A]\), almost surely with respect to \(\nu^*[\partial^2 A]\).

**Proof.** Let \(A \subset V^*\) satisfy \(\partial^2_A \subset V^*\) and \(\partial^2_A \subset V^*\). It is immediate from \((4.3)\) that \(\partial^2_H \subset \partial^2_A\), and we write simply \(\partial^2 A\) for this set. To check \((4.4)\), note that if \(K \in \mathcal{C}_2(G)\) intersects \(A\), then \(K \subset A \cup \partial^2 A \subset V^*\). By \((4.3)\) the edge sets of \(G\) and \(H\) agree when restricted to \(V^*\), and we deduce that \(K \in \mathcal{C}_2(H)\); this proves \(\subset\) in \((4.4)\), but the reverse inclusion follows by the same argument. Note that, with \(\mathcal{K}_A\) as defined in \((4.4)\), we have also shown that

\[
K \in \mathcal{K}_A \Rightarrow K \subset V^*. \tag{4.5}
\]

Let us work in the rest of the proof on the canonical probability space \((\mathcal{X}^V_{G \cup V_H}, \text{Borel}, \nu^*)\), with \(\mathbb{E}\) denoting expectation on this space, and all equations are understood to hold \(\nu^*\)-almost surely. Let \(I = (I_v)_{v \in V_G \cup V_H}\) denote the identity map on \(\mathcal{X}^V_{G \cup V_H}\). By Proposition \(3.2\) \(\nu^H\) is a 2MRF, and so \(\nu^H[A | \partial^2 A](I_{\partial^2 A}) = \nu^H[A | V_H \setminus A](I_{V_H \setminus A})\) a.s. Hence,

\[
\frac{d\nu^H[A | \partial^2 A](I_{\partial^2 A})}{d\nu^*[A]}(I_A) = \frac{\frac{d\nu^H}{d\nu^*[V_H]}(I_{V_H})}{\frac{d\nu^H}{d\nu^*[V_H]}(I_{V_H})} = \frac{\mathbb{E}[\frac{d\nu^H}{d\nu^*[V_H]}(I_{V_H}) | I_{V_H \setminus A}]}{\mathbb{E}[\frac{d\nu^H}{d\nu^*[V_H]}(I_{V_H}) | I_{V_H \setminus A}]} = \frac{\prod_{K \in \mathcal{C}_2(H)} f^H_K(I_K)}{\prod_{K \in \mathcal{C}_2(H)} f^H_K(I_K)} = \mathbb{E}[\prod_{K \in \mathcal{C}_2(H)} f^H_K(I_K) | I_{V_H \setminus A}],
\]

where we emphasize that the expectation in the denominator is with respect to independent random variables \((I_v)_{v \in V_G \cup V_H}\).

The key observation is that if \(K \in \mathcal{C}_2(H)\) does not intersect \(A\), then the term \(f^H_K(I_K)\) factors out of the conditional expectation and cancels. Hence, with \(\mathcal{K}_A\) as in \((4.4)\), we see that

\[
\frac{d\nu^H[A | \partial^2 A](I_{\partial^2 A})}{d\nu^*[A]}(I_A) = \prod_{K \in \mathcal{K}_A} f^H_K(I_K) \mathbb{E}[\prod_{K \in \mathcal{K}_A} f^H_K(I_K) | I_{V_H \setminus A}]. \tag{4.6}
\]

Since \(I_{V_H \setminus V^*}\) is independent of \(I_{V^*}\), in view of \((4.5)\), we may equivalently condition on \(I_{V^*} \setminus A\) in the denominator of the term on the right-hand side of \((4.6)\) to obtain

\[
\frac{d\nu^H[A | \partial^2 A](I_{\partial^2 A})}{d\nu^*[A]}(I_A) = \prod_{K \in \mathcal{K}_A} f^H_K(I_K) \mathbb{E}[\prod_{K \in \mathcal{K}_A} f^H_K(I_K) | I_{V^*} \setminus A].
\]

Repeating the same arguments that led us to this point, we also find that

\[
\frac{d\nu^G[A | \partial^2 A](I_{\partial^2 A})}{d\nu^*[A]}(I_A) = \prod_{K \in \mathcal{K}_A} f^G_K(I_K) \mathbb{E}[\prod_{K \in \mathcal{K}_A} f^G_K(I_K) | I_{V^*} \setminus A].
\]

Recalling that \(f^H_K \equiv f^G_K\) for \(K \in \mathcal{K}_A\) by assumption, the proof is complete. \(\square\)

The last lemma allows us to deduce the following insensitivity result that shows that given a finite graph \(G = (V, E)\) and associated SDE \((2.1)\), the conditional law of trajectories of particles in a set \(A \subset V\) given the trajectories of particles at the double-boundary \(\partial^2 A\) of the set does not depend on the graph structure outside of \(A \cup \partial^2 A\).

**Proposition 4.4.** Let \(G = (V_G, E_G)\) and \(H = (V_H, E_H)\) be finite graphs, and assume \(V^* \subset V_G \cap V_H\) satisfies \((4.3)\). Let \(A \subset V^*\) satisfy \(\partial^2_A \subset V^*\) and \(\partial^2_A \subset V^*\), so that Lemma \(4.3\) ensures that \(\partial^2_H \subset \partial^2_A\) and that \((4.4)\) holds (defining \(\mathcal{K}_A\) as therein). Suppose \((G, \delta^{G}, \hat{\sigma}^G, \mu^G_H)\) and \((H, \nu^{H}, \sigma^{H}, \mu^{H}_H)\) both satisfy Assumption \(\mathcal{A}\) and let \(P^G \in \mathcal{P}(\mathcal{C}^{V_G})\) and \(P^H \in \mathcal{P}(\mathcal{C}^{V_H})\) be the corresponding unique laws of the SDE described in \((2.1)\). Further, suppose the following consistivity conditions hold:
(i) We have
\[ b^H_v \equiv b^G_v, \quad \text{for } v \in A \cup \partial^2 A, \]
\[ \sigma^H_v \equiv \sigma^G_v, \quad \text{for } v \in V_G \cap V_H. \]

(ii) There is a product measure \( \mu^*_0 = \prod_{v \in V_G \cup V_H} \lambda_v \in \mathcal{P}(\mathbb{R}^d) \) for some \( \lambda_v \in \mathcal{P}(\mathbb{R}^d), v \in V_G \cup V_H \), such that \( \mu^*_0 \) and \( \mu^0 \) admit 2-clique factorizations:
\[
\frac{d\mu^G_0}{d\mu^*_0}(x_{V_G}) = \prod_{K \in \text{cl}_2(G)} f^G_K(x_K),
\]
\[
\frac{d\mu^H_0}{d\mu^*_0}(x_{V_H}) = \prod_{K \in \text{cl}_2(H)} f^H_K(x_K),
\]
for some measurable functions \( f^G_K : (\mathbb{R}^d)^K \to \mathbb{R}_+ \) and \( f^H_K : (\mathbb{R}^d)^K \to \mathbb{R}_+ \) that satisfy the consistency condition \( f^G_K \equiv f^H_K \) for every \( K \in \mathcal{K}_A \). As usual, let \( P^* \in \mathcal{P}(\mathbb{C}^{V_G \cup V_H}) \) be the unique law of the SDE (2.1) and its projection, \( P^*_t[A|\partial^2 A] = P^*_t[A|\partial^2 A] \) for each \( t > 0 \), both in the sense of \( P^*_t[\partial^2 A] \)-almost sure and \( P^*_t[\partial^2 A] \)-almost sure equality.

**Proof.** As in (2.5), let \( P^* \in \mathcal{P}(\mathbb{C}^{V_G \cup V_H}) \) be the unique law of the solution \( X = (X_v)_{v \in V_G \cup V_H} \) of the driftless SDE
\[
dX_v(t) = \sigma^G_v(t, X_v) dW_v(t), \quad v \in V_G, \\
dX_v(t) = \sigma^H_v(t, X_v) dW_v(t), \quad v \in V_H \\
\]
initialized with \( X(0) \sim \mu^*_0 \). Again working on the canonical probability space \( \mathcal{C}^{V_G \cup V_H}, \text{Borel}, P^* \), define the martingales \( M^H = (M^H_v)_{v \in V_H} \) by
\[
M^H(t) := \int_0^t (\sigma^H_v(t, X_v) - \mu^H_v(s, X_v, X_{N_v(H)})) \cdot dX_v(s), \quad v \in V_H,
\]
with \( M^G = (M^G_v)_{v \in V_G} \), defined analogously, as in (3.2). Using (3.3) and (4.9) we can write
\[
\frac{dP^*_t}{dP^*_0}[V_H] = \prod_{K \in \text{cl}_2(H)} f^H_K(X_K(0)) \prod_{v \in V_H} \mathcal{E}_t(M^H_v),
\]
\[
\frac{dP^*_t}{dP^*_0}[V_G] = \prod_{K \in \text{cl}_2(G)} f^G_K(X_K(0)) \prod_{v \in V_G} \mathcal{E}_t(M^G_v).
\]
with \( \mathcal{E}_t \) defined as in (3.4). Note that if \( v \in A \cup \partial A \), then we have \( N_v(H) = N_v(G) \); indeed, this is due to (4.7) and the inclusions \( \partial_G A \subset V^* \) and \( \partial_H A \subset V^* \). Thus, by the consistency conditions (4.8) along with the expressions above for \( M^H \) and \( M^G \), we have \( \mathcal{E}_t(M^H_v) = \mathcal{E}_t(M^G_v) \) for \( v \in A \cup \partial A \). Applying Lemma 4.3 with \( \mathcal{X} = C_t, \mathcal{U} = \mathcal{V}, \mathcal{X}^* = \mathcal{Y}, \mathcal{Y}^* = \mathcal{Z} \), \( \nu^* = \mathcal{P}^* \), \( \nu^H = \mathcal{P}^*_t \) and \( \nu^G = \mathcal{P}^*_t \), it follows from the consistency conditions (i) and (ii) that \( P^*_t[A|\partial^2 A] = P^*_t[A|\partial^2 A] \) holds in the sense of \( P^*_t[\partial^2 A] \)-almost sure equality. Since both \( P^*_t[\partial^2 A] \) and \( P^*_t[\partial^2 A] \) are absolutely continuous with respect to \( P^*_t[\partial^2 A] \), the claim follows. \( \square \)

5. **Markov random field property for infinite-dimensional diffusions**

Fix a countably infinite connected graph \( G = (V, E) \), and let \( (G, b, \sigma, \mu_0) \) be as in Assumption 13. As usual, let \( P = P^{\mu_0} \) and \( P_t = P^{\mu_0} \) denote the unique law of the SDE (2.1) and its projection, and let \( P^{*, \mu_0} \) be the law of the canonical SDE system (2.5) started from initial law \( \mu_0 \). In this section we will prove Theorem 2.7, that is, the 2MRF property for \( P_t \) and \( P \). We will also use the same canonical space \( (\mathbb{C}^V, \text{Borel}, P^{*, \mu_0}) \) and canonical processes \((X_v)_{v \in V}\) as in Section 1.1.

Throughout, choose an arbitrary vertex \( \rho \in V \) to be the root, and let \( G_n = (V_n, E_n) \) and \( U_n = V_n \setminus V_{n-2} \) be as defined in Lemma 4.2. Also, set
\[
b^n_v = b_v, \quad v \in V_{n-2}, \quad b^n_v = 0, \quad v \in U_n,
\]
(5.1)
5.1. **Tightness and convergence results.** In the following, let \( H(\cdot|\cdot) \) denote relative entropy, defined for \( \nu \ll \mu \) by

\[
H(\nu|\mu) = \int \frac{d\nu}{d\mu} \log\frac{d\nu}{d\mu} \, d\mu,
\]

and \( H(\nu|\mu) = \infty \) for \( \nu \not\ll \mu \). Recall also our notation \(|x|_{*,t} := \sup_{0 \leq s \leq t} |x_s|\) for the truncated supremum norm.

**Lemma 5.1.** Suppose Assumption \( \mathcal{B} \) holds. For each \( t > 0 \) and each finite set \( A \subset V \), we have

\[
\sup_n \sup_{v \in V_n} \mathbb{E}^n \left[ \|X_v\|_{*,t}^2 \right] < \infty,
\]

(5.2)

\[
\sup_n H \left( P^n_t[A] | P^n_t^{\mu}[A] \right) < \infty,
\]

(5.3)

\[
\sup_n H \left( P^n_t^{*}[A] | P^n_t^{\mu}[A] \right) < \infty.
\]

(5.4)

**Proof.** Fix \( t > 0 \). We begin with a standard estimate. Recall the definition of \( b^n_v \) from (5.1), apply Itô’s formula to the SDE (4.2), and use the linear growth of \( b_v \), where we have used the inclusion \( V \subset Q \). Lemma B.1 with (5.1) to conclude that, for each \( n \in \mathbb{N} \) and \( v \in V_n \),

\[
\mathbb{E}^n \left[ \|X_v\|_{*,t}^2 \right] \leq C \mathbb{E}^n \left[ |X_v(0)|^2 + \int_0^t |b^n_v(s, X_v, X_{N_v}(G))|^2 \, ds + \int_0^t |\sigma_v(s, X_v(s))|^2 \, ds \right]
\]

\[
\leq C \mathbb{E}^n \left[ 1 + |X_v(0)|^2 + \int_0^t \left( |X_v|_{*,s}^2 + \frac{1}{|N_v(G)|} \sum_{u \in N_v(G)} |X_u|_{*,s}^2 \right) \, ds \right],
\]

where \( C < \infty \) is a constant that can change from line to line but does not depend on \( n \) or \( v \). This implies that

\[
\sup_{v \in V_n} \mathbb{E}^n \left[ \|X_v\|_{*,t}^2 \right] \leq C \left( 1 + \sup_{v \in V} \mathbb{E}^n \left[ |X_v(0)|^2 \right] + \int_0^t \sup_{v \in V_n} \mathbb{E}^n \left[ \|X_v\|_{*,s}^2 \right] \, ds \right),
\]

where we have used the inclusion \( V_n \subset V \). Apply Gronwall’s inequality to find

\[
\sup_{v \in V_n} \mathbb{E}^n \left[ \|X_v\|_{*,t}^2 \right] \leq C \left( 1 + \sup_{v \in V} \int_{\mathbb{R}^d} |x_v|^2 \mu_0(dx_v) \right).
\]

(5.5)

The right-hand side is finite by Assumption (B1), and so (5.2) follows.

Given a finite subset \( A \subset V \), define \( Q^n_t \in \mathcal{P}(\mathcal{C}^V_t) \) by

\[
\frac{dQ^n_t}{dP^n_t^{\mu_0}} = \prod_{v \in V_n \setminus A} \mathcal{E}_t(M^n_v), \quad \text{where} \quad M^n_v := \int_0^t (\sigma_v \sigma_v^T)^{-1} b^n_v(s, X_v, X_{N_v}(G)) \cdot dX_v(s), \ v \in V.
\]

Due to Assumptions (B2) and (B3), and the definition of \( P^{*,\mu_0} \) from Remark 2.3, we can apply Lemma [B.1] with \( Q = P^{*,\mu_0}, \ X = (X_v)_{v \in V_n} \) and \( f(t, x) = (1_{\{v \in V_n \setminus A\}} \sigma_v^{-1} b^n_v(t, x_v, X_{N_v}(G)))_{v \in V_n} \), to conclude that \( \frac{dQ^n_t}{dP^n_t^{\mu_0}} \) is a true \( P^{*,\mu_0} \)-martingale. It then follows from Girsanov’s theorem...
Corollary 3.5.2] and the uniqueness in law of the driftless SDE (2.3) that \( Q^n_t[A] = P^n_{s,t} \). By a similar argument,

\[
\frac{dP^n_t}{dP^n_{s,t}} = \prod_{v \in V_n} \mathcal{E}_t(M_v^n), \quad \text{and thus} \quad \frac{dP^n_t}{dQ^n_t} = \prod_{v \in \mathcal{A} \cap V_n} \mathcal{E}_t(M_v^n),
\]

where, in case \( A \cap V_n = \emptyset \), we interpret the empty product as 1. Once again invoking the linear growth of \( b \), the boundedness of \( \sigma_v \), (5.3), the fact that \( Q^n_t[A] = P^n_{s,t}[A] \) and Remark 2.5 note that Girsanov’s theorem also shows that for every \( v \in A \cap V_n \), under \( P^n \), \( M_v^n - [M_v^n] \) is a martingale and \( [M_v^n](t) = \int_0^t |\sigma_v^{-1}b^n_v(s,X_v,X_{N_v(G_n)})|^2 \, ds \). Then, use the data processing inequality of relative entropy to obtain

\[
H(P^n_t[A] | P^n_{s,t}[A]) = H(P^n_t[A] | Q^n_t[A]) \leq H(P^n_t | Q^n_t)
\]

\[
= \sum_{v \in \mathcal{A} \cap V_n} \mathbb{E}^{P^n} \left[ M_v^n(t) - \frac{1}{2}[M_v^n](t) \right]
\]

\[
= \frac{1}{2} \sum_{v \in \mathcal{A} \cap V_n} \mathbb{E}^{P^n} \left[ \int_0^t |\sigma_v^{-1}b^n_v(s,X_v,X_{N_v(G_n)})|^2 \, ds \right]
\]

\[
\leq C \sum_{v \in \mathcal{A} \cap V_n} \mathbb{E}^{P^n} \left[ 1 + \|X_v\|^2_{V^*_x,t} + \frac{1}{|N_v(G_n)|} \sum_{u \in N_v(G_n)} \|X_u\|^2_{V^*_x,t} \right]
\]

\[
\leq C|A| \left( 1 + 2 \sup_{v \in V_n} \mathbb{E}^{P^n} \left[ \|X_v\|^2_{V^*_x,t} \right] \right).
\]

Therefore (5.3) follows from (5.2).

Noting that due to the identity \( Q^n_t[A] = P^n_{s,t}[A] \) and Remark 2.5 under \( Q^n \), \( (X_v)_{v \in A} \) is driftless and \( M_v^n \) is a martingale. Therefore, noting that

\[
\frac{dQ^n_t}{dP^n_t} = \prod_{v \in \mathcal{A} \cap V_n} \exp \left( -M_v^n(t) + \frac{1}{2} \int_0^t |\sigma_v^{-1}b^n_v(s,X_v,X_{N_v(G_n)})|^2 \, ds \right),
\]

another application of the the data processing inequality of relative entropy yields

\[
H(P^n_{s,t}[A] | P^n_t[A]) = H(Q^n_t[A] | P^n_t[A]) \leq H(Q^n_t | P^n_t)
\]

\[
= \frac{1}{2} \sum_{v \in \mathcal{A} \cap V_n} \mathbb{E}^{Q^n} \left[ \int_0^t |\sigma_v^{-1}b^n_v(s,X_v,X_{N_v(G_n)})|^2 \, ds \right]
\]

\[
\leq C \sum_{v \in \mathcal{A} \cap V_n} \mathbb{E}^{Q^n} \left[ 1 + \|X_v\|^2_{V^*_x,t} + \frac{1}{|N_v(G_n)|} \sum_{u \in N_v(G_n)} \|X_u\|^2_{V^*_x,t} \right]
\]

\[
\leq C|A| \left( 1 + 2 \sup_{v \in V_n} \mathbb{E}^{Q^n} \left[ \|X_v\|^2_{V^*_x,t} \right] \right).
\]

The same argument that was used to obtain (5.3) can also be used to show that (5.3) holds with \( P^n \) replaced by \( Q^n \). Therefore \( \sup_n \sup_{v \in V_n} \mathbb{E}^{Q^n} \left[ \|X_v\|^2_{V^*_x,t} \right] < \infty \) by Assumption (B1), and hence the last display implies (5.4). \( \square \)
The next lemma will be used to show both that the existence of a weak solution to the infinite SDE system (2.1) holds automatically and also that it arises as the limit of finite-graph systems. Recall that $P \in \mathcal{P}(\mathcal{C})$ denotes the law of the solution of (2.1).

**Lemma 5.2.** Suppose Assumption (B) holds. Then $P^n \to P$ weakly on $\mathcal{C}$. Moreover, for any finite set $A' \subset V$, any $t > 0$, and any bounded measurable function $\psi : \mathcal{C} \to \mathbb{R}$, we have

$$\lim_{n \to \infty} \mathbb{E}^{P^n}[\psi(X_{A'}[t])] = \mathbb{E}^P[\psi(X_{A'}[t])].$$

**Proof.** Fix $t > 0$. The entropy bound of (5.3) shows that $(P^n[A'])_{n \in \mathbb{N}}$ are precompact in the weak* topology induced on $\mathcal{P}(\mathcal{C})$ by the bounded measurable functions on $\mathcal{C}$ [3, Lemma 6.2.16]. In particular, this sequence is tight, and since this holds for every finite set $A'$ and every $t > 0$ we deduce that the entire sequence $(P^n)_{n \in \mathbb{N}}$ is tight in $\mathcal{C}$. Note also that for sufficiently large $n$ it holds under $P^n$ that the processes

$$\int_0^s \sigma^{-1}_v(r, X_v) dX_v(r) - \int_0^s \sigma^{-1}_v(r, X_v) b_v(r, X_v, X_{N_v(G)}) dr, \quad s \geq 0, \quad v \in V_{n-2}, \quad (5.6)$$

are independent standard Wiener processes, due to the consistency condition for the $b_v$’s and the identity $N_v(G_n) = N_v(G)$ valid for $v \in V_{n-2}$.

Now let $Q \in \mathcal{P}^\infty$ be any weak (in the usual sense) subsequential limit of $(P^n)_{n \in \mathbb{N}}$, with $P^m \to Q$ weakly. The aforementioned precompactness in the weak* topology implies that

$$\lim_{k \to \infty} \mathbb{E}^{P^m_k}[\psi(X_{A'}[t])] = \mathbb{E}^Q[\psi(X_{A'}[t])],$$

for any finite set $A' \subset V$, any $t > 0$, and any bounded measurable function $\psi$ on $\mathcal{C}$. We conclude that, under $Q$, the processes in (5.6) are independent Wiener processes, for $v \in V$. This shows that $Q$ is the law of a weak solution of the SDE system (2.1), which we know to be unique by assumption (B5). Hence, $Q = P$. □

5.2. **Proof of the second-order Markov random field property on the infinite graph.**

**Proof of Theorem 2.7.** Fix $(G = (V, E), b, \sigma, \mu_0)$ and $X = (X_v)_{v \in V}$ as in the statement of the theorem. For $n \geq 4$, consider the sequence of graphs $G_n = (V_n, E_n), n \in \mathbb{N}$ constructed from $G$ as in Lemma 4.2. We first note that due to the fact that $\mu_0$ is a 2MRF by Assumption (B1), part (iii) of Lemma 4.2, with $\mathcal{X} = \mathbb{R}^d, \nu = \mu_0, \nu^* = \mu_0^*$, ensures that $\mu_0[V_n]$ is a 2MRF with respect to the graph $G_n$. Moreover, since $d\mu_0[V_n]/d\mu_0^*[V_n]$ is strictly positive by Assumption (B1), Proposition 5.2 shows that $\mu_0[V_n]$ admits a 2-clique factorization with respect to the product measure $\mu_0^*[V_n]$ for each $n$. Hence, $\mu_0[V_n]$ satisfies Assumption (A1), which when combined with the definition of $b_v^n = (b_v^n)_{v \in V}$ in (5.1) and the fact that $b, \sigma$ satisfy Assumptions (B2) and (B3), shows that $(G_n, b^n, \sigma_{v})_{v \in V_n}, \mu_0[V_n])$ satisfy Assumption (A). Since $P^n[V_n]$ is the law of the SDE (1.2) on the finite graph $G_n$, it is a 2MRF by Theorem 2.4.

Now, fix two finite sets $A, B \subset V$ with $B$ disjoint of $A \cup \partial^2 A$, where throughout, we use $\partial^2$ to denote $\partial^2 G$. Let $n_0$ denote the smallest integer greater than or equal to 4 for which $A \cup \partial^2 A \cup B \subset V_{n_0-3}$, and let $n \geq n_0$. Then, part (iv) of Lemma 4.2 again with $\mathcal{X} = \mathbb{R}^d, \nu = \mu_0$, and $\nu^* = \mu_0^*$, ensures that $\mu_0[V_n]$ and $\mu_0[V_{n_0}]$ admit 2-clique factorizations which are consistent in the sense that the corresponding measurable functions $f_{G_n}^K$ and $f_{G_{n_0}}^K$ agree for every $K \in \text{cl}_2(G_{n_0})$ that intersects $A$ (equivalently, for every $K \in \text{cl}_2(G)$ that intersects $A$). Since $b_v^n = b_v^{n_0} = b_v$ for all $v \in A \cup \partial^2 A$ by (5.2), and since $A \cup \partial^2 A \subset V_{n_0-3}$, we may apply Proposition 4.4 with $G = G_n, H = G_{n_0}, V_v = V_{n_0-3}, \mu_0^G = \mu_0[V_k], \mu_0^{G_{n_0}} = \mu_0[V_{n_0}]$, and $(b_v^{G_{n_0}}, \sigma_v^G) = (b_v, \sigma_v)$ for $v \in G_k$ and $k \in \{n_0, n\}$, to deduce that $P^n_t[A | \partial^2 A] = P^{n_0}_t[A | \partial^2 A]$ for all $n \geq n_0$. In
other words, this implies that given a bounded continuous function $f$, there exists a measurable function $\varphi$ (that does not depend on $n$) such that

$$\varphi(X_{\partial^2 A}[t]) = \mathbb{E}^{P^n}[f(X_A[t]) | X_{\partial^2 A}[t]], \quad P^n - a.s., \text{ for } n \geq n_0. \tag{5.7}$$

Now, fix additional bounded continuous functions $g, h$. For $t > 0$, taking the conditional expectation with respect to $X_{V_n \setminus A}[t]$ inside the expectation on the left-hand side below and using the 2MRF property of $P^n$ we have

$$\mathbb{E}^{P^n}[f(X_A[t])g(X_{\partial^2 A}[t])h(X_B[t])] = \mathbb{E}^{P^n}[\mathbb{E}^{P^n}[f(X_A[t]) | X_{\partial^2 A}[t]]g(X_{\partial^2 A}[t])h(X_B[t])].$$

When combined with (5.7), this implies

$$\mathbb{E}^{P^n}[f(X_A[t])g(X_{\partial^2 A}[t])h(X_B[t])] = \mathbb{E}^{P^n}[\varphi(X_{\partial^2 A}[t])g(X_{\partial^2 A}[t])h(X_B[t])].$$

Using the second part of Lemma 5.2 for the finite set $A' = A \cup \partial^2 A \cup B$, and for both $\psi(y_{A'}) := f(y_A)g(y_{\partial^2 A})h(y_B)$, and $\psi(y_{A'}) = \varphi(y_{\partial^2 A})g(y_{\partial^2 A})h(y_B)$ for $y_{A'} \in C_t$, we may pass to the limit $n \to \infty$ and denote $P = P^{\mu_0}$ to get

$$\mathbb{E}^{P}[f(X_A[t])g(X_{\partial^2 A}[t])h(X_B[t])] = \mathbb{E}^{P}[\varphi(X_{\partial^2 A}[t])g(X_{\partial^2 A}[t])h(X_B[t])].$$

This at once shows both that

$$\mathbb{E}^{P}[f(X_A[t]) | X_{\partial^2 A}[t]] = \varphi(X_{\partial^2 A}[t]) = \mathbb{E}^{P^n}[f(X_A[t]) | X_{\partial^2 A}[t]],$$

for all bounded continuous $f$ and $n \geq n_0$, which proves Proposition 5.4 below, and also that $X_A[t]$ and $X_B[t]$ are conditionally independent given $X_{\partial^2 A}[t]$ under $P$. The latter proves the first statement in Theorem 2.9, except for the fact that we have only proven this conditional independence when $A$ and $B$ are finite. Because $B \subset V \setminus (A \cup \partial^2 A)$ was an arbitrary finite set and $\{X_B[t] : B \subset A \cup \partial^2 A\}$ generates the same $\sigma$-field as $X_{A \cup \partial^2 A}[t]$, we deduce that $P_t$ is a 2MRF. The second statement follows from the same argument as in (5.6) using the 2MRF property of $P_t = P^{\mu_0}_t$. This completes the proof. \hfill $\Box$

We recapitulate two results that were established in the course of the proof, which may be of independent interest, and which are used in the proof of Theorem 2.9 in the next section.

**Remark 5.3.** Note that the first paragraph of the proof above shows that if $(G, b, \sigma, \mu_0)$ satisfy Assumption B and for $G_n = (V_n, E_n), n \in \mathbb{N}$, is as in Lemma 4.2 and $b^n, P^n, n \in \mathbb{N}, t > 0$, are as defined at the beginning of Section 5, then $P_t^n[V_n]$ is a 2MRF for each $n \in \mathbb{N}$ and $t > 0$.

**Proposition 5.4.** Suppose $(G, b, \sigma, \mu_0)$ satisfy Assumption B and let $G_n = (V_n, E_n), n \in \mathbb{N}$, be the sequence of graphs constructed from $G$ as in Lemma 4.2. Fix $t > 0$, and let $P_t = P^{\mu_0}_t$ be the law of the unique weak solution to the SDE (2.1) with initial law $\mu_0$. Then, for $n \geq 3$, and $A \subset V_{n-3}$, $\partial^2 \cup A \subset V_n$ and, $P_t$-almost surely,

$$P_t[A | \partial^2 G, A] = P^n[A | \partial^2 G, A].$$

6. Proof of Gibbs measure properties

In this section we prove the Gibbs uniqueness property of Theorem 2.9. Recall the definition of $P^{*, \mu_0}$ as the law of the solution of (2.5) initialized at $\mu_0$.

**Proof of Theorem 2.9.** Let $(G, b, \sigma, \mu_0)$ satisfy Assumption B and let $P^{\mu_0}$ be the unique solution of the SDE system (2.1) with initial law $\mu_0$. We work again on the canonical space $(C^V, \text{Borel}, P^{*, \mu_0})$, with $X_V = (X_v)_{v \in V}$ denoting the canonical process. Define the sets $\mathcal{M}_{\text{init}} = \mathcal{M}_{\text{init}}(\mu_0)$ and $\mathcal{M}_{\text{path}} = \mathcal{M}_{\text{path}}(\mu_0)$ as in the statement of the theorem. For any $v_0 \in \mathcal{M}_{\text{init}}$, the SDE system (2.1) is well-posed starting from $v_0$, and we let $P^{v_0} \in \mathcal{P}(C^V)$ denote the law of this
solution. The proof of the theorem is broken down into five claims.

Claim 1. Suppose \((G, b, \sigma, \mu_0)\) satisfies Assumption [3] If \(\nu_0 \in \mathcal{M}_{\text{init}}\), then \((G, b, \sigma, \nu_0)\) also satisfies Assumption [3] and for every finite set \(A \subset V\) and \(t > 0\) we have \(P^{\nu_0}_t[A] \sim P^{\mu_0}_t[A]\).

Proof of Claim 1. First, suppose \(\nu_0 \in \mathcal{M}_{\text{init}}\). By the definition of \(\mathcal{M}_{\text{init}}\), \(\nu_0\) has a finite second moment. Moreover, for each finite set \(A \subset V\), we have \(\nu_0[A] \sim \mu_0[A]\) since \(\nu_0 \in \mathcal{G}_2(\mu_0)\) implies (by Definition [2.3]) that \(\nu_0[A] \sim \mu_0[A]\), and Assumption \((\mathbf{2.1})\) ensures that \(\mu_0[A] \sim \mu^*_0[A]\). Thus \(\nu_0\) satisfies Assumption \((\mathbf{3.1})\).

Now let \(t > 0\), and let \(A \subset V\) be finite. From Lemma 5.2, it follows that \(P^\nu_t \to P^{\mu_0}\) weakly. It then follows from (5.3), (5.4) and the lower semicontinuity of relative entropy that \(P^\nu_t[A] \ll P^{\mu_0}_t[A]\) and \(P^\nu_t[A] \ll P^{\mu_0}_t[A]\). Therefore \(P^\nu_t[A] \sim P^{\mu_0}_t[A]\) and similarly \(P^\nu_t[A] \sim P^{\mu_0}_t[A]\). Finally, note that \(P^{\mu_0}_t[A]\) (resp. \(P^{\nu}_t[A]\)) is the law of the solution of the SDE system

\[
dX(t) = \sigma(t, X(t)) dW_v(t),
\]

with initial law \(\mu_0[A]\) (resp. \(\nu_0[A]\)), and it follows from \(\nu_0[A] \sim \mu_0[A]\) that \(P^{\nu}_t[A] \sim P^{\mu_0}_t[A]\). Putting it together, we have \(P^{\nu}_t[A] \sim P^{\mu_0}_t[A] \sim P^{\mu_0}_t[A] \sim P^{\mu_0}_t[A]\). \(\Box\)

Claim 2. For any \(Q \in \mathcal{M}_{\text{path}}\), we have \(Q_0 := Q \circ (X_V(0))^{-1} \in \mathcal{M}_{\text{init}}\).

Proof of Claim 2. The proof of this claim is straightforward: fix \(Q \in \mathcal{M}_{\text{path}}\) and set \(Q_0 := Q \circ (X_V(0))^{-1}\). Then by the definition of \(\mathcal{M}_{\text{path}}\), we have \(Q_t \in \mathcal{G}_2(P^{\mu_0}_t)\) for all \(t \geq 0\) and \(\sup_{v \in V} \int_{\mathbb{R}^d} |x_v|^2 Q_0(dx) < \infty\). Taking \(t = 0\) gives \(Q_0 \in \mathcal{G}_2(\mu_0)\), where we have used the elementary fact that \(P^{\mu_0}_0 = P^{\mu_0} \circ (X_V(0))^{-1} = \mu_0\). Thus \(Q_0\) belongs to \(\mathcal{M}_{\text{init}}\). \(\Box\)

Claim 3. If \(\nu_0 \in \mathcal{M}_{\text{init}}\) then \(P^{\nu_0} \in \mathcal{M}_{\text{path}}\).

Proof of Claim 3. Fix \(\nu_0 \in \mathcal{M}_{\text{init}}\). Then by the first assertion of Claim 1, for every finite set \(A \subset V\) and \(t \geq 0\), \(P^{\nu_0}_t[A] \sim P^{\mu_0}_t[A]\). So it only remains to show that for every \(t > 0\),

\[
P^{\nu_0}_t[A|\partial^2 A] = P^{\mu_0}_t[A|\partial^2 A], \quad \text{for finite } A \subset V.
\]

First, recall that Claim 1 also shows that \((G, b, \sigma, \nu_0)\) satisfies Assumption [3] Next, let \(G_n = (V_n, E_n)\) be the increasing sequence of finite graphs defined in Lemma [4.2] and let \(P^{\mu_0,n}, P^{\nu_0,n} \in \mathcal{P}(\mathcal{C}^V)\) denote the law of the solution of the corresponding SDE system (4.2) with initial laws \(\mu_0[V_n]\) and \(\nu_0[V_n]\), respectively. Throughout this proof, the boundary operator \(\partial\) is always with respect to the infinite graph \(G\). Fix \(A \subset V\) finite, and fix \(n\) large enough that \(A \subset V_{n-3}\), recalling that \(V_n\) was defined in Section 4.1. By Proposition 4.4, we have both

\[
P^{\mu_0}_t[A|\partial^2 A] = P^{\nu_0,n}_t[A|\partial^2 A],
\]

\[
P^{\mu_0}_t[A|\partial^2 A] = P^{\nu_0,n}_t[A|\partial^2 A].
\]

By Lemma 4.2(iii), \(\nu_0[V_n]\) is a 2MRF. Also since \(\nu_0[V_n] \sim \mu_0[V_n]\) implies \(d\nu_0[V_n]/d\mu_0[V_n] > 0\), by Proposition 3.2 there is a 2-clique factorization

\[
\frac{d\nu_0[V_n]}{d\mu_0[V_n]}(x_{V_n}) = \prod_{K \in \mathcal{C}_2(G_n)} g^n_K(x_K),
\]

for some measurable functions \(g^n_K : (\mathbb{R}^d)^K \to \mathbb{R}_+\). Similarly \(\mu_0[V_n]\) admits a 2-clique factorization,

\[
\frac{d\mu_0[V_n]}{d\nu_0[V_n]}(x_{V_n}) = \prod_{K \in \mathcal{C}_2(G_n)} f^n_K(x_K),
\]
for some measurable functions $f^\alpha_K : (\mathbb{R}^d)_K \to \mathbb{R}_+$. We claim (and justify below) that $f^\alpha_K$ and $g^\alpha_K$ can be chosen to be consistent, i.e., so that
\begin{equation}
  f^\alpha_K \equiv g^\alpha_K, \quad \text{for all } K \in \mathcal{C}_2(G_n) \text{ with } K \cap A \neq \emptyset. \tag{6.5}
\end{equation}
To see this, let $I_{V_n} = (I_v)_{v \in V_n}$ denote the canonical random variable on the probability space $(\mathbb{R}^d)^{V_n}$. Define
\begin{equation}
  \hat{f}^\alpha_n(I_{U_n}) = \mathbb{E}^{\mu_0^\alpha} \left[ \frac{d\mu_0[V_n]}{d\mu_0^\alpha[V_n]} \right] I_{U_n}, \quad \hat{g}^\alpha_n(I_{U_n}) = \mathbb{E}^{\mu_0^\alpha} \left[ \frac{d\nu_0[V_n]}{d\mu_0^\alpha[V_n]} \right] I_{U_n}.
\end{equation}
Recalling that $U_n = V_n \setminus V_{n-2}$, and using (6.3), we have
\begin{equation}
  \frac{d\nu_0[V_{n-2} \mid U_n]}{d\mu_0^\alpha[V_{n-2} \mid U_n]} = \frac{1}{\hat{g}^\alpha_n(I_{U_n})} \prod_{K \in \mathcal{C}_2(G_n)} g^\alpha_K(I_K).
\end{equation}
Applying the same argument to $\mu_0$ rather than $\nu_0$ and using (6.4), we also obtain
\begin{equation}
  \frac{d\mu_0[V_{n-2} \mid U_n]}{d\mu_0^\alpha[V_{n-2} \mid U_n]} = \frac{d\mu_0[V_n]}{d\mu_0^\alpha[V_n]} \hat{f}^\alpha_n(I_{U_n}) \prod_{K \in \mathcal{C}_2(G_n)} g^\alpha_K(I_K).
\end{equation}
Further, recognizing that $U_n = \partial^2 V_{n-2}$, since $\nu_0 \in \mathcal{G}_2(\mu_0)$ we see that
\begin{equation}
  \nu_0[V_{n-2} \mid U_n] = \mu_0[V_{n-2} \mid U_n].
\end{equation}
Combining the last three displays, we find
\begin{equation}
  \frac{d\mu_0[V_n]}{d\mu_0^\alpha[V_n]}(I_{U_n}) = \frac{d\nu_0[V_{n-2} \mid U_n]}{d\mu_0^\alpha[V_{n-2} \mid U_n]} \hat{f}^\alpha_n(I_{U_n}) = \frac{\hat{f}^\alpha_n(I_{U_n})}{\hat{g}^\alpha_n(I_{U_n})} \prod_{K \in \mathcal{C}_2(G_n)} g^\alpha_K(I_K).
\end{equation}
Comparing this with (6.4) and noting that $U_n \in \mathcal{C}_2(G_n)$, we can thus take $f^\alpha_K \equiv g^\alpha_K$ in (6.4) for $K \in \mathcal{C}_2(G_n) \setminus \{U_n\}$ and $f^\alpha_K \equiv g^\alpha_K \hat{f}^\alpha_n / \hat{g}^\alpha_n$. This proves the above consistency claim; indeed, since $A \subset V_{n-3}$, we know that $U_n$ does not intersect $A \cup \partial A$.

Let $\mathcal{K} = \{K \in \mathcal{C}_2(G_n) : K \cap A \neq \emptyset\}$. Using the consistency property (6.5), we can finally conclude from Proposition 4.4 (with $V_G = V_H = V_n$, $b^G_v = b^H_v = b_v$, $\sigma^G_v = \sigma^H_v = \sigma_v$, $\mu^G_0 = \mu_0$ and $\mu^H_0 = \nu_0$) that $P_{t_0}^{\mu_0^n}[A \mid \partial^2 A] = P_t^{\mu_0^n}[A \mid \partial^2 A]$. Recalling (6.2), this completes the proof of (6.1).

Together, claims 2 and 3 prove (2.6). We now prove the last assertion of the theorem.

Claim 4. If $Q \in \mathcal{P}(\mathcal{C}^V)$ satisfies $Q_t \in \mathcal{G}_2(\mathcal{P}^{\mu_0^n}_t)$ for all $t \geq 0$ and also $Q \circ (X_V(0))^{-1} = \mu_0$, then $Q = P^{\mu_0}$.

Proof of Claim 4. As in the proof of Claim 3, we let $G_n = (V_n, E_n)$ be the increasing sequence of finite graphs defined in Lemma 4.2 and let the boundary operator $\partial$ always be with respect to the infinite graph $G$. Also, let $P^n = P^{\mu_0^n} \in \mathcal{P}(\mathcal{C}^V)$ denote the law of the solution of the corresponding SDE system (1.2) with initial law $\mu_0[V_n]$. Now, fix a finite set $A \subset V$ and $T \in (0, \infty)$. Let $n_0$ denote the smallest integer such that $A \cup \partial^2 A \subset V_{n_0-3}$. Define the martingales
\begin{equation}
  M^n_v(t) = \int_0^t (\sigma_\nu \sigma^\top_v)^{-1} b^n_v(s, X_v, X_{N_v(G_n)}) \cdot dX_v(s), \quad n \in \mathbb{N}, v \in V_n.
\end{equation}
Due to Assumptions (B2) and (B3), it follows from Lemma 4.1 (with $Q = P^{*, \mu_0}$, $X = (X_v)_{v \in V_n}$ and $f(t, x) = (\sigma_\nu^{-1} b^*_v(t, x_v, x_{N_v(G_n)}) \nu_{v_0})$) that $\mathcal{E}(M^n_v)$ is a $P^{*, \mu_0}$-martingale. Thus, by Girsanov’s
Now, by applying Remark 5.3 to \((G, b, \sigma, \mu_0)\) and \((G, 0, \sigma, \mu_0)\), respectively, it follows that the measures \(P^n_t[\{V_n\}]\) and \(P^{*, \mu_0}_t[\{V_n\}]\) are 2MRFs with respect to \(G_n\). Hence, for \(n \geq n_0\),
\[
\frac{dP^n_t[A | \partial^2 A]}{dP^{*, \mu_0}_t[A | \partial^2 A]} = \prod_{v \in V_n} \mathcal{E}_t(M^n_v) \left( \prod_{v \in V_n} \mathcal{E}_t(M^n_v) \right) \frac{dP^n_t[\{V_n\}]}{dP^{*, \mu_0}_t[\{V_n\}]} \left( \prod_{v \in V_n} \mathcal{E}_t(M^n_v) \right) \left( X_{V_n \setminus A}[t] \right).
\]

For \(v \in V_n \setminus (A \cup \partial A)\), \(\mathcal{E}_t(M^n_v)\) is measurable with respect to \(X_{V_n \setminus A}[t]\) and thus factors out of the conditional expectation and cancels. Thus,
\[
\frac{dP^n_t[A | \partial^2 A]}{dP^{*, \mu_0}_t[A | \partial^2 A]} = \prod_{v \in A \cup \partial A} \mathcal{E}_t(M^n_v) \left( \prod_{v \in A \cup \partial A} \mathcal{E}_t(M^n_v) \right) \left( X_{V_n \setminus A}[t] \right).
\]

Because \(Q_t \in \mathcal{G}_2(P^{*, \mu_0})\) by assumption, we have \(Q_t[A | \partial^2 A] = P^{*, \mu_0}_t[A | \partial^2 A]\). By Proposition 5.3.4, we have \(P^{*, \mu_0}_t[A | \partial^2 A] = P^{*, \mu_0}_t[A | \partial^2 A]\), and it follows that the density \(dQ_t[A | \partial^2 A]/dP^{*, \mu_0}_t[A | \partial^2 A]\) is given by the same expression (6.7).

Now take \(A = V_{n-2}\), and note that \(U_n := V_n \setminus V_{n-2} = \partial^2 V_{n-2}\). Because \(Q_t[U_n] \sim P^{*, \mu_0}_t[U_n]\) by assumption, and because both \(Q\) and \(P^{*, \mu_0}\) start from the same initial state distribution \(\mu_0\), we may use the martingale representation theorem (specifically, apply Remark 6.7 below with \(\xi = dQ_T[U_n]/dP^{*, \mu_0}_T[U_n]\), which clearly satisfies \(E^{P^{*, \mu_0}}[\xi] = 1\)) to find progressively measurable functions \(r_v : [0, T] \times \mathcal{C}_U \to \mathbb{R}^d, v \in U_n\), which are \(dt \otimes dP^{*, \mu_0}\)-square-integrable such that in terms of the associated \(X_{U_n}\)-adapted continuous martingales \(R^n_v(t) = \int_0^t r_v(s, X_{U_n}) \cdot dX_v(s), t \in [0, T], v \in U_n\), we can write for \(t \in [0, T]\),
\[
\frac{dQ_t[U_n]}{dP^{*, \mu_0}_t[U_n]} = \prod_{v \in U_n} \mathcal{E}_t(R^n_v).
\]

Note that these martingales are orthogonal, that is, the covariation process \([R^n_v, R^n_u]\) is identically zero for \(v \neq u\). Thus, since \(U_n \cap V_{n-2} = \emptyset, U_n \cup V_{n-2} = V_n\) and \(V_{n-1} = V_{n-2} \cup \partial V_{n-2}\), applying (6.7) with \(A = V_{n-2}\) we have
\[
\frac{dQ_t[V_n]}{dP^{*, \mu_0}_t[V_n]} = \frac{dQ_t[V_{n-2} | U_n]}{dP^{*, \mu_0}_t[V_{n-2} | U_n]} \frac{dQ_t[U_n]}{dP^{*, \mu_0}_t[U_n]} = \prod_{v \in U_n} \mathcal{E}_t(R^n_v) \prod_{v \in V_{n-1}} \mathcal{E}_t(M^n_v) \left( \prod_{v \in V_{n-1}} \mathcal{E}_t(M^n_v) \right) \left( X_{U_n}[t] \right).
\]

The process in the denominator is a positive martingale (as the optional projection of a martingale) adapted to \(X_{U_n}\) and thus, again using the martingale representation theorem (this time applying Remark 6.7 below with \(\xi = E^{P^{*, \mu_0}}[\prod_{v \in V_{n-1}} \mathcal{E}_T(M^n_v) \left( X_{U_n}[T] \right)]\) and invoking (6.6) to conclude that \(E^{P^{*, \mu_0}}[\xi] = E^{P^{*, \mu_0}}[dP^n_t[V_{n-1}]/dP^{*, \mu_0}(V_{n-1})] = 1\), there exist square integrable,
progressively measurable functions \( \tilde{r}_v^n \) (as above) and associated \( X_{U_n} \)-adapted continuous martingales \( \tilde{R}_v^n(t) = \int_0^t \tilde{r}_v^n(s, X_{U_n}) \cdot dX_v(s), v \in U_n \), such that

\[
\mathbb{E}^{P^*,\mu_0} \left[ \prod_{v \in V_{n-1}} \mathcal{E}_t(M_v^n) \left| X_{U_n}[t] \right. \right] = \prod_{v \in U_n} \mathcal{E}_t(\tilde{R}_v^n),
\]

Now note that, for any continuous martingales \( Z \) and \( (Z_i)_{i \in I} \), with \( I \) a finite index set, we have the identities \( 1/\mathcal{E}(Z) = \mathcal{E}(-Z) e^{|Z|} \) and

\[
\prod_{i \in I} \mathcal{E}(Z_i) = \exp \left( \sum_{i \in I} Z_i - \frac{1}{2} \sum_{i \in I} [Z_i] \right) = \exp \left( \sum_{i \in I} Z_i - \frac{1}{2} \left[ \sum_{i \in I} Z_i \right] + \frac{1}{2} \sum_{i,j \in I, i \neq j} [Z_i, Z_j] \right) = \mathcal{E} \left( \sum_{i \in I} Z_i \right) \exp \left( \frac{1}{2} \sum_{i,j \in I, i \neq j} [Z_i, Z_j] \right),
\]

where \([Z_i, Z_j] \) denotes the covariation process. Hence,

\[
\frac{dQ_t[V_n]}{dP^{*\mu_0}_t[V_n]} = \prod_{v \in U_n} \mathcal{E}_t(R_v^n) \mathcal{E}_t(-\tilde{R}_v^n) \exp([\tilde{R}_v^n](t)) \prod_{v \in V_{n-1}} \mathcal{E}_t(M_v^n)
\]

\[
= \prod_{v \in V_{n-2}} \mathcal{E}_t(M_v^n) \prod_{v \in V_{n-1} \setminus V_{n-2}} \mathcal{E}_t(M_v^n + R_v^n - \tilde{R}_v^n) \exp\left([M_v^n - \tilde{R}_v^n, R_v^n - \tilde{R}_v^n](t)\right)
\]

\[
= \prod_{v \in V_{n-1} \setminus V_{n-2}} \mathcal{E}_t(R_v^n - \tilde{R}_v^n) \exp\left([\tilde{R}_v^n, \tilde{R}_v^n - R_v^n](t)\right).
\]

Recalling the orthogonality properties of \( R_v^n \) and \( \tilde{R}_v^n \) mentioned above, we see that we can write \( Z_t := dQ_t[V_n]/dP^{*\mu_0}_t[V_n] \) in the form \( Z(t) = \mathcal{E}_t(N) e^{A(t)} \), where \( N \) is a continuous square-integrable martingale and \( A(t) \) is square-integrable and a.s. absolutely continuous with \( A(0) = 0 \). Since \( Z \) is a martingale, we necessarily have \( A \equiv 0 \); indeed, Itô’s formula gives \( dZ(t) = Z(t)(dN(t) + dA(t)) \), and for \( Z \) to be a martingale we must have \( dA(t) = 0 \). It follows that

\[
\frac{dQ_t[V_n]}{dP^{*\mu_0}_t[V_n]} = \prod_{v \in V_{n-2}} \mathcal{E}_t(M_v^n) \prod_{v \in V_{n-1} \setminus V_{n-2}} \mathcal{E}_t(M_v^n + R_v^n - \tilde{R}_v^n) \prod_{v \in V_n \setminus V_{n-1}} \mathcal{E}_t(R_v^n - \tilde{R}_v^n).
\]

Since \( Z \) is a \( P^{*\mu_0} \)-martingale, Girsanov’s theorem \[10\] Corollary 3.5.2] can be applied, using the definition of \( M_v^n \), to deduce that \( Q_t[V_n] \) is the law of a solution \( (X_v^n[t])_{v \in V_n} \) of the SDE system (perhaps on an auxiliary the probability space)

\[
dX_v^n(s) = b_v(s, X_v^n, X_{N_v(G_v)}) ds + \sigma(s, X_v^n) dB_v(s), \quad \text{for } v \in V_{n-2},
\]

\[
dX_v^n(s) = \left( (r_v^n - \tilde{r}_v^n)(s, X_v^n) + b_v^n(s, X_v^n, X_{N_v(G_v)}) \right) ds + \sigma(s, X_v^n) dB_v(s), \quad \text{for } v \in V_{n-1} \setminus V_{n-2},
\]

\[
dX_v^n(s) = (r_v^n - \tilde{r}_v^n)(s, X_v^n) ds + \sigma(s, X_v^n) dB_v(s), \quad \text{for } v \in V_n \setminus V_{n-1},
\]

where \( (B_v)_{v \in V_n} \) are independent Brownian motions.

Define \( X_v^n = 0 \) for \( v \notin V_n \). Since the sets \( V_n \) increase to \( V \), it is easily shown as in Lemma \[5,2\] that, as \( n \to \infty \), \( (X_v^n[T])_{v \in V} \) converges in law in \( C_T^V \) to a solution of the infinite SDE system \([2,1]\) with initial distribution \( \mu_0 \), restricted to the interval \( [0, T] \). Recalling that \( (X_v^n(0))_{v \in V} \sim \mu_0 \) and that the solution to the infinite SDE system is unique in law by Assumption \([13]\), we conclude
that \((X_v^n[T])_{v \in V}\) converges in law to \(P_{T}^{\mu_0}\). But \(X_v^n[T]\) has law \(Q[T|V_0]\) by construction, which implies \((X_v^n[T])_{v \in V}\) converges in law to \(Q[T]\). Therefore \(Q_T = P_T^{\mu_0}\). Since \(T \in (0, \infty)\) was arbitrary, \(Q = P^{\mu_0}\), which completes the proof of Claim 4.

To complete the proof of the theorem, it only remains to establish the bijection between the two sets in (2.6). However, we now show that this is a simple consequence of the last claim.

Claim 5. The map \(Q \mapsto Q \circ (X_V(0))^{-1}\) defines a bijection between the sets \(\mathcal{M}_{\text{path}}\) and \(\mathcal{M}_{\text{init}}\).

Proof of Claim 5. Let \(Q \in \mathcal{M}_{\text{path}}\), and set \(\nu_0 := Q \circ (X_V(0))^{-1}\). By Claim 2, \(\nu_0\) belongs to \(\mathcal{M}_{\text{init}}\), and by Claim 3, \(P^{\nu_0}\) lies in \(\mathcal{M}_{\text{path}}\). Since trivially \(P^{\nu_0} \circ (X_V(0))^{-1} = \nu_0\), to prove the claim it suffices to prove that \(Q = P^{\nu_0}\). By Claim 1, \((G, b, \sigma, \nu_0)\) satisfies Assumption \(\mathcal{B}\) and thus Claim 4 applies with \(\nu_0\) in place of \(\mu_0\). That is, by applying Claim 4 to \(\nu_0\) instead of \(\mu_0\), we deduce that if \(Q \in \mathcal{P}(C^V)\) satisfies \(Q_t \in \mathcal{G}_2(P^{\nu_0})\) for all \(t \geq 0\) and also \(Q \circ (X_V(0))^{-1} = \nu_0\), then \(Q = P^{\nu_0}\). By definition of \(\mathcal{M}_{\text{path}}\) we have \(Q_t \in \mathcal{G}_2(P^{\nu_0})\) for all \(t \geq 0\), and it follows from (6.1), which was established in the proof of Claim 3, that \(\mathcal{G}_2(P^{\nu_0}) = \mathcal{G}_2(P^{\mu_0})\). This proves Claim 5, which completes the proof of Theorem 2.9.

Remark 6.1. We sketch here the argument behind the use of the martingale representation theorem in the proof of Theorem 2.9 above. Recall that by Assumption (A3b) the SDE system \(dX_v(t) = \sigma_v(t, X_v) dW_v(t), \ v \in U_n,\) with initial law \(\mu_0\) is unique in law, with the law of the solution \(X = (X_v)_{v \in U_n}\) given by \(P^{\ast, \mu_0}[U_n]\). This implies uniqueness of the associated martingale problem (cf. [10] Corollary 5.4.9), which is known to imply that the solution has the predictable representation property (cf. [22] Theorem V.25.1] or [24] Theorem 2.7], in the following sense: For \(T < \infty\) and an \(\mathcal{F}_T^X\)-measurable random variable \(\xi > 0\) with \(\mathbb{E}[\xi] = 1\), the martingale \(Z(t) = \mathbb{E}[\xi | \mathcal{F}_t^X] > 0, t \in [0, T]\), where \(\mathcal{F}_t^X = \sigma(X(s) : s \leq t)\), can be represented as \(Z(t) = 1 + \int_0^t \varphi(s, X) \cdot dX(s)\) for some predictable process \(\varphi : [0, T] \times \mathbb{R} \to \mathbb{R}\) satisfying \(\int_0^T |\varphi(t, X)|^2 dt < \infty\) a.s., recalling that \(\sigma_v\) is uniformly bounded and nondegenerate. Then, for \(t \in [0, T]\), setting \(\psi(t, X) = \varphi(t, X)/Z(t)\), by Itô’s formula, we have \(d \log Z(t) = \psi(t, X) \cdot dX(t) - \frac{1}{2} \psi(t, X)^T d[X](t) \psi(t, X)\). Hence, \(Z(t) = E_t(\int_0^t \psi(s, X) \cdot dX(s)), t \in [0, T]\).

Appendix A. Proof of pathwise uniqueness under Lipschitz assumptions

Proof of Proposition 2.4. Let \((X_v)_{v \in V}\) and \((\bar{X}_v)_{v \in V}\) denote two solutions driven by the same Wiener processes and starting from the same initial states. Fix \(T < \infty\). For each \(v \in V\) and \(t \in [0, T]\), by Itô’s formula, the boundedness of \(\sigma\) (see Assumption (B3)), the assumed Lipschitz condition on the drift and diffusion coefficients we have

\[
\mathbb{E} \left[ \|X_v - \bar{X}_v\|^2_{s,t} \right] \leq 2t \mathbb{E} \left[ \int_0^t \left| b_v(s, X_v(s), X_{N_v(G)}(s)) - b_v(s, \bar{X}_v(s), \bar{X}_{N_v(G)}(s)) \right|^2 ds \right] \\
+ 8 \mathbb{E} \left[ \int_0^t \left| \sigma_v(s, X_v(s)) - \sigma_v(s, \bar{X}_v(s)) \right|^2 ds \right] \\
\leq 4t K_T^2 \mathbb{E} \left[ \int_0^t \left( \|X_v - \bar{X}_v\|^2_{s,s} + \frac{1}{|N_v(G)|} \sum_{u \in N_v(G)} \|X_u - \bar{X}_u\|^2_{s,s} \right) ds \right] \\
+ 8 K_T^2 \mathbb{E} \left[ \int_0^t \|X_v - \bar{X}_v\|^2_{s,s} ds \right].
\]
Hence,
\[
\sup_{v \in V} \mathbb{E} \left[ \|X_v - \bar{X}_v\|^2_{s,t} \right] \leq 8(tK_f^2 + \bar{K}_f^2) \int_0^t \sup_{v \in V} \mathbb{E} \left[ \|X_v - \bar{X}_v\|^2_{s,s} \right] ds.
\]
Complete the proof using Gronwall’s inequality. □

Appendix B. Justification for applying Girsanov’s theorem

In this section we state a result that justifies our repeated application of Girsanov’s theorem under the condition that the drift is progressively measurable and has linear growth. Lemma B.1 below is in fact a path-dependent multi-dimensional version of [11, Theorems 5.1 and 8.1]. A simpler proof is provided here for completeness.

Let \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{Q})\) be a filtered probability space supporting a \(\mathcal{F}\)-Brownian motion \(W\) of dimension \(m\) as well as an \(\mathcal{F}\)-adapted process \(X\) of dimension \(d\) such that \(X\) satisfies the SDE
\[
dX(t) = \sigma(t, X) \, dW(t), \quad X(0) \sim \mu,
\]
where \(\mu \in \mathcal{P}(\mathbb{R}^d)\) and \(\sigma : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^{d \times m}\) is bounded and progressively measurable. Also, let \(\mathbb{E}\) denote expectation with respect to \(\mathbb{Q}\). Fix a progressively measurable \(f : \mathbb{R}_+ \times \mathcal{C} \to \mathbb{R}^m\), and define the stochastic integral
\[
M_t := \int_0^t f(s, X) \cdot dW_s, \quad t \in [0, \infty),
\]
which is well defined (and a local martingale) due to the linear growth condition \(\text{(B.2)}\) imposed on \(f\) in the lemma below. Recall in what follows that \(\|x\|_{s,t} = \sup_{s \leq u \leq t} |x(s)|\).

Lemma B.1. Under the above setting, suppose for each \(T \in (0, \infty)\) there exists \(C_T < \infty\) such that
\[
|f(t, x)| \leq C_T (1 + \|x\|_{s,t}),
\]
for all \(t \in [0, T]\) and \(x \in \mathcal{C}\). Then the Doleans exponential \(\{\mathcal{E}_t(M)\}_{t \geq 0}\) defined in \(\text{(3.3)}\) is a true \(\mathbb{Q}\)-martingale.

Proof. Since \(\{\mathcal{E}_t(M)\}_{t \geq 0}\) is always a \(\mathbb{Q}\)-supermartingale, it suffices to show that \(\mathbb{E}[\mathcal{E}_T(M)|X(0) = x] = 1\) for each \(x \in \mathbb{R}^d\) and \(T \in (0, \infty)\). So fix \(T \in (0, \infty)\) and assume without loss of generality that \(X(0) = x \in \mathbb{R}^d\) in \(\text{(B.1)}\). Since \(\sigma\) is bounded, \(X\) is a martingale and, from standard concentration inequalities for martingales (see, e.g., [25, Lemma 2.1]), we can find some \(C > 0\) such that \(\mathbb{Q}(\|X - x\|_{s,T} \geq a) \leq \exp(-Ca^2)\) for each \(a > 0\). It then follows from the equivalence between sub-Gaussian tails and finite square-exponential moments (see, e.g., [1, Section 2.3]) that there exists \(c > 0\) such that \(\mathbb{E}[\exp(c\|X\|_{s,T}^2)] < \infty\). Now taking \(0 = t_0 < t_1 < \cdots < t_{n(T)} = T\) with \(t_n - t_{n-1} \leq c/C_T^2\), and using the linear growth condition \(\text{(B.2)}\) on \(f\), we have
\[
\mathbb{E} \left[ \exp \left( \frac{1}{2} \int_{t_{n-1}}^{t_n} |f(s, X)|^2 \, ds \right) \right] \leq \mathbb{E} \left[ \exp \left( (t_n - t_{n-1})C_T^2 (1 + \|X\|_{s,T}^2) \right) \right] < \infty.
\]
It then follows from [10, Corollary 3.5.14] that \(\{\mathcal{E}_t(M)\}_{t \geq 0}\) is a true \(\mathbb{Q}\)-martingale. □

References

[1] S. Boucheron, G. Lugosi, and P. Massart, Concentration inequalities: A nonasymptotic theory of independence, Oxford university press, 2013.
[2] P. Cattiaux, S. Roelly, and H. Zessin, Une approche Gibbsienne des diffusions Browniennes infinitédimensionnelles, Probability Theory and Related Fields 104 (1996), 147–179.
[3] A. Dembo and O. Zeitouni, Large deviations techniques and applications, Vol. 38, Springer Science & Business Media, 2009.
[4] D. Dereudre, *Interacting Brownian particles and Gibbs fields on pathspaces*, ESAIM:Probability and Statistics 7 (2003), 251–277.

[5] D. Dereudre and S. Roelly, *On Gibbsianness of infinite-dimensional diffusions*, Markov Processes and Related Fields 10 (2004), 395–410.

[6] ______, *Propagation of Gibbsianness for infinite-dimensional gradient Brownian diffusions*, Journal of Statistical Physics 121 (2005), no. 3-4, 511–551.

[7] N. Detering, J.-P. Fouque, and T. Ichiba, *Directed chain stochastic differential equations*, arXiv preprint arXiv:1805.01962v2 (2018).

[8] J.D. Deuschel, *Infinite-dimensional diffusion processes as Gibbs measures on $C[0,1]$*, Probability Theory and Related Fields 76 (1987), 325–340.

[9] H.-O. Georgii, *Gibbs measures and phase transitions*, Vol. 9, Walter de Gruyter, 2011.

[10] I. Karatzas and S.E. Shreve, *Brownian motion and stochastic calculus*, Graduate Texts in Mathematics, Springer New York, 1991.

[11] F. Klebaner and R. Liptser, *When a stochastic exponential is a true martingale. extension of the bené method*, Theory of Probability & Its Applications 58 (2014), no. 1, 38–62, available at https://doi.org/10.1137/S0040585X97986382.

[12] D. Lacker, K. Ramanan, and R. Wu, *Large sparse networks of interacting diffusions*, 2019. preprint, https://arxiv.org/abs/1904.02585.

[13] S.L. Lauritzen, *Graphical models*, Vol. 17, Clarendon Press, 1996.

[14] E. Luçon and W. Stannat, *Mean field limit for disordered diffusions with singular interactions*, Annals of Applied Probability 24 (2014), no. 5, 1946–1993.

[15] G.S. Medvedev, *The continuum limit of the Kuramoto model on sparse directed graphs*, arXiv preprint arXiv:1802.03787 (2018).

[16] R.A. Minlos, S. Roelly, and H. Zessin, *Gibbs states on space-time*, Potential Analysis 13 (2000), 367–408.

[17] S. Nadtochiy and M. Shkolnikov, *Mean field systems on networks, with singular interaction through hitting times*, arXiv preprint arXiv:1807.02015 (2018).

[18] F. Rassoul-Agha and T. Seppäläinen, *A course on large deviations with an introduction to Gibbs measures*, Vol. 162, American Mathematical Soc., 2015.

[19] F. Redig, S. Roelly, and W. Ruszel, *Short-time Gibbsianness for infinite-dimensional diffusions with space-time interaction*, Journal of Statistical Physics 138 (2010), 1124–1144.

[20] S. Roelly and W.M. Ruszel, *Propagation of Gibbsianness for infinite-dimensional diffusions with space-time interaction*, Markov Proc. Rel. Fields 20 (2014), 653–574.

[21] S. Roelly and H. Zessin, *Une caractérisation des mesures de Gibbs sur $c(0,1)^d$ par le calcul des variations stochastiques*, Annales de l'I. H. P., section B 29 (1993), 327–338.

[22] L.C.G. Rogers and D. Williams, *Diffusions, Markov processes and martingales: Volume 2, Itô calculus*, Vol. 2, Cambridge University Press, 2000.

[23] D.W. Stroock and S.R.S. Varadhan, *Multidimensional diffusion processes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 233, Springer-Verlag, Berlin-New York, 1979. MR532498

[24] D.W. Stroock and M. Yor, *On extremal solutions of martingale problems*, Annales scientifiques de l’école normale supérieure, 1980, pp. 95–164.

[25] S. van de Geer, *Exponential inequalities for martingales, with application to maximum likelihood estimation for counting processes*, Ann. Statist. 23 (1995), no. 5, 1779–1801.

[26] A.C.D. Van Enter, C. Külske, A.A. Opoku, and W.M. Ruszel, *Gibbs–non-Gibbs properties for n-vector lattice and mean-field models*, Brazilian Journal of Probability and Statistics 24 (2010), no. 2, 226–255.

[27] A.C.D. Van Enter and W.M. Ruszel, *Gibbsianness versus Non-Gibbsianness of time-evolved planar rotor models*, Stochastic Processes and their Applications 119 (2009), no. 6, 1866–1888.

COLUMBIA UNIVERSITY, NEW YORK, NEW YORK

DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY

UNIVERSITY OF MICHIGAN, ANN ARBOR

E-mail address: daniel.lacker@columbia.edu, kavita_ramanan@brown.edu, ruoyu@umich.edu