ADAPTED COMPLEX TUBES ON THE SYMPLECTIZATION OF
PSEUDO-HERMITIAN MANIFOLDS

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ABSTRACT. Let \((M, \theta)\) be a pseudo-Hermitian space of real dimension
\(2n+1\), that is \(M\) is a CR−manifold of dimension \(2n+1\) and \(\theta\) is a
contact form on \(M\) giving the Levi distribution \(HT(M) \subset TM\). Let \(M^0 \subset T^*M\) be
the canonical symplectization of \((M, \theta)\) and \(M\) be identified with the zero section of \(M^0\). Then \(M^0\)
is a manifold of real dimension \(2(n+1)\) which admit a canonical foliation by surfaces parametrized by
\(\mathbb{C} \ni t + i\sigma \mapsto \phi_p(t + i\sigma) = \sigma g_t(p)\), where \(p \in M\) is arbitrary and \(g_t\) is the
flow generated by the Reeb vector field associated to the contact form \(\theta\).

Let \(J\) be an (integrable) complex structure defined in a neighbourhood
\(U\) of \(M\) in \(M^0\). We say that the pair \((U, J)\) is an adapted complex tube on
\(M^0\) if all the parametrizations \(\phi_p(t + i\sigma)\) defined above are holomorphic
on \(\phi_p^{-1}(U)\).

In this paper we prove that if \((U, J)\) is an adapted complex tube on
\(M^0\), then the real function \(E\) on \(M^0 \subset T^*M\) defined by the condition
\(\alpha = E(\alpha)\theta_{\pi(\alpha)}\), for each \(\alpha \in M^0\), is a canonical equation for \(M\)
which satisfies the homogeneous Monge-Ampère equation \((dd^c E)^{n+1} = 0\).

We also prove that if \(M\) and \(\theta\) are real analytic then the symplectiza-
tion \(M^0\) admits an unique maximal adapted complex tube.

1. ADAPTED COMPLEX TUBES

In this paper we apply some results obtained by the authors in [7] to the
study of suitable “adapted” integrable complex structure on open neigh-
bourhood of a pseudo-Hermitian manifold in its symplectization.

We follow [2] for standard notations in complex and CR−geometry.

Given a smooth real differentiable manifold \(M\) of dimension \(m\) we denote
by \(T^*M\) its cotangent bundle and \(\pi : T^*M \to M\) the canonical projection.

Let \(\theta\) be a smooth 1−form on \(M\) such that \(\theta_p \neq 0\) for each \(p \in M\). Then
the subset of by \(T^*M\)

\[ M^0 = \{ \alpha \in T^*M \mid \alpha \wedge \theta_{\pi(\alpha)} = 0 \}, \]

is the line bundle on \(M \subset T^*M\) of the forms \(\alpha\) which are proportional to the
form \(\theta_{\pi(\alpha)}\). We denote by \(E : M^0 \to \mathbb{R}\) the function which satisfies

\[ \alpha = E(\alpha)\theta_{\pi(\alpha)} \]

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for each $\alpha \in T^*M$.

Denoting by $M_0$ the zero section of $T^*M$ we see that for each $\alpha \in M^\theta$ we have $E(\alpha) = 0$ if, and only if, $\alpha \in M_0$.

We freely identify the manifold $M$ with the zero section $M_0$ in $M^\theta$.

It easy to show that $M^\theta$ is a smooth closed submanifold of $T^*M$ of (real) dimension $m+1$ and $E$ is a smooth function on $M$.

We also denote by $X^\theta$ the vector field on $T^*M$ which is the infinitesimal generator of the one parameter group of transformation defined by

$$h^\theta_t(\alpha) = \alpha + t \theta(\pi(\alpha))$$

for each $\alpha \in T^*M$ and $t \in \mathbb{R}$.

Let $x = (x_1, \ldots, x_m)$ be a local coordinate system on $M$ and let $(x, p) = (x_1, \ldots, x_m, p_1, \ldots, p_m)$ the corresponding local coordinate system on $T^*M$, so that

$$\theta = \sum_{i=1}^m p_i(\theta) dx_i.$$  

Then we have

$$X^\theta = \sum_{i=1}^m p_i(\theta) \frac{\partial}{\partial p_i}.$$  

Clearly the vector field $X^\theta$ is tangent to $M^\theta$ and $X^\theta(E) = 1$. Thus the smooth function $E$ has no critical points on $M^\theta$.

Let now $(M, \theta)$ be a pseudo-Hermitian manifold, that is $M$ is a orientable CR–manifold of dimension $2n+1$ with non degenerate Levi form and $\theta$ is a non degenerate real 1–form which vanishes on the Levi distribution $HT(M) \subset TM$ and defines the pseudo-Hermitian structure on $M$. Then $(M, \theta)$ is a contact manifold with volume form $\theta \wedge (d\theta)^n$ and there exists a vector field $\xi^\theta_0$, the Reeb vector field, which is characterized by the conditions $\xi^\theta_0 L \theta = 1$ and $\xi^\theta_0 L d\theta = 0$. The pull-back of $d\theta$ is a symplectic form on $M^\theta$ that will be called the symplectization of $M$.

Let $g_t$ be the one parameter (local) group of transformations associated to $\xi^\theta_0$, that is for each $p \in M$ the map

$$g_t(p) : I_p \to M, \quad t \mapsto g_t(p)M, \quad I_p \subset \mathbb{R}$$

is the maximal integral curve of $\xi^\theta_0$ such that $g_0(p) = p$.

For each $p \in M$ let $\tilde{I}_p = \{ z = t + i\sigma \in \mathbb{C} \mid t \in I_p \}$ and the map

$$\phi_p = \phi^\theta_p : \tilde{I}_p \to M^\theta$$

defined by the formula

$$\phi^\theta_p(z) = \phi^\theta(t + i\sigma) = \sigma \theta_{g_t(p)}.$$  

We also denote by $\xi^\theta_0$ the vector field on $M^\theta$ which is the infinitesimal generator of the one parameter local group of transformations defined by

$$g^\theta_t(\alpha) = E(\alpha) \theta_{g_t(p)},$$

where $p = \pi(\alpha)$.  

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Then the vector field $\xi^\theta$ extends $\xi^\theta_0$ and it is easy to show that the one parameter groups $s^\theta_t$ and $h^\theta_t$ commute and generate a two dimensional distribution on $M^\theta$ whose leaves are parametrized by the maps $\phi^\theta_p$. Moreover we have
\[
\begin{align*}
\xi^\theta(E) &= 0, \\
X^\theta(E) &= 1, \\
[\xi^\theta, X^\theta] &= 0.
\end{align*}
\]

We now give the main definition of this paper.

**Definition 1.1.** Let $(M, \theta)$ be a pseudo-Hermitian manifold of real dimension $2n+1$. Let the manifold $M$ be identified with the zero section $M_0$ in $M^\theta$.

An adapted complex tube on $M^\theta$ is a pair $(U, J)$ where $U$ is an open subset of $M^\theta$ containing the zero section $M_0$ and $J$ is an (integrable) complex structure on $U$ which satisfies the following conditions:

1. for each $p \in M$ the set $U_p = \{ \alpha \in U \mid \pi(\alpha) = p \}$ is connected;
2. the restriction of $J$ to the Levi distribution $HT(M)$ is the given CR structure of $HT(M)$;
3. if $p \in M$ the restriction of the map $\phi^\theta_p$ to the open set $\tilde{I}_p \cap (\phi^\theta_p)^{-1}(U)$ is holomorphic.

An adapted complex tube $(U, J)$ with $J$ of class $C^k$, $k = 1, \ldots, \omega$, is said to be $C^k$—maximal, if for any other adapted complex tube on $M^\theta$ such that $J'$ is of class $C^k$, $U \subset U'$ and $J'_{|U} = J$ one has $U = U'$ (and hence $J' = J$).

Adapted tubes of class $C^\infty$ or $C^\omega$ are said to be smooth or real analytic, respectively.

The main results of this paper are the following.

**Theorem 1.1.** Let $(M, \theta)$ be a pseudo-Hermitian manifold of real dimension $2n+1$ with $M$ and $\theta$ of class $C^\infty$. Let $(U, J)$ be any smooth adapted complex tube on $M^\theta$. Then the function $E$ satisfies
\[
\begin{align*}
(dd^c E)^n &+ 1 = 0 \text{ on } U, \\
dE \wedge d^c E \wedge (dd^c E)^n &\neq 0 \text{ near } M, \\
E_{|M} &= 0, \\
d^c E_{|T(M)} &= -\theta.
\end{align*}
\]

**Theorem 1.2.** Let $(M, \theta)$ be a pseudo-Hermitian manifold of real dimension $2n+1$ with $M$ and $\theta$ real analytic. Then there exists a unique real analytic maximal adapted complex tube $(U, J)$ on $M^\theta$.

**Remark 1.1.** Adapted complex structure on the tangent (and cotangent) bundle a Riemannian manifolds are studied in [5], [6] and, independently, in [4].
2. PROOF OF THEOREM 1.1

Let \((M, \theta)\) be a fixed pseudo-Hermitian manifold.

**Lemma 2.1.** If \((U, J)\) is any smooth adapted complex tube on \(M^\theta\) then

\[
J(\xi^\theta) = X^\theta.
\]

**Proof.** Let \(\alpha \in U \subset M^\theta\) and \(p = \pi(\alpha)\). Then \(\alpha = \sigma_0 \theta_p\), where \(\sigma_0 = E(\alpha)\).

By definition of \(\xi^\theta\)

\[
\xi^\theta(\alpha) = \frac{d}{dt} g^\theta_t(\alpha) \bigg|_{t=0} = \frac{d}{dt} (\sigma_0 \theta_{\alpha(t)}) \bigg|_{t=0}.
\]

Since the map \(\phi_p(t + i\sigma)\) is holomorphic we obtain

\[
J(\xi^\theta(\alpha)) = \frac{d}{d\sigma} \phi_p(i(\sigma_0 + \sigma)) \bigg|_{\sigma=0} = \frac{d}{d\sigma} (\alpha + \sigma \theta_p) \bigg|_{\sigma=0}
\]

\[
= \frac{d}{d\sigma} h^\theta_{\sigma}(\alpha) \bigg|_{\sigma=0} = X^\theta(\alpha),
\]

as desired. \(\Box\)

Using the terminology of [7] we observe that by (1) and (3) the pair \((\xi^\theta, -E)\) is a smooth calibrated foliation, that is

\[
[\xi^\theta, J(\xi^\theta)] = 0,
\]

\[
dE(\xi^\theta) = 0,
\]

\[
d^cE(\xi^\theta) = -1.
\]

Moreover the function \(E\) vanishes on \(M\) and the vector field \(\xi^\theta\) extends the Reeb vector field \(\xi^\theta_0\) on \(M\). Since the Reeb vector field \(\xi^\theta_0\) is an infinitesimal symmetry for the contact distribution \(HT(M)\) from [7, Theorem 3.1] it follows that the form \(\xi^\theta \mathbf{L} d^cE = L_{\xi^\theta} d^cE\) vanishes identically on \(U\). Here \(L_{\xi^\theta} d^cE\) stands for the Lie derivative of the form \(d^cE\) with respect to the vector field \(\xi^\theta\). Since \(\xi^\theta \neq 0\) on \(U\) it follows that the form \(d^cE\) satisfies the Monge-Ampère equation \((d^cE)^{n+1} = 0\) on \(U\).

Clearly, the function \(E\) vanishes exactly on \(M\) and hence the restriction of \(d^cE\) to \(TM\) is \(\lambda \theta\) for some (smooth) function \(\lambda : M \rightarrow \mathbb{R}\). For each \(p \in M\) we have

\[
\lambda(p) = \lambda(p) \theta_p(\xi^\theta(p)) = d^cE(\xi^\theta(p)) = d^cE(\xi^\theta(p)) = -dE(X^\theta(p)) = -1
\]

and hence \(d^cE|_{T(M)} = -\theta\).
Since $\theta \wedge (d\theta)^n$ is a volume form on $M$ we have $dE \wedge d^c E \wedge (dd^c E)^n \neq 0$ in a neighbourhood of $M$.

This completes the proof of Theorem 1.1.

3. PROOF OF THEOREM 1.2

Let $(M, \theta)$ be a fixed pseudo-Hermitian manifold.

Let $\Phi : M \times \mathbb{R} \to M^\theta$ be the map defined by $\Phi(p, \sigma) = \sigma \theta_p$ for each $p \in M$ and each $\sigma \in \mathbb{R}$. Then $\Phi$ is a diffeomorphism between $M \times \mathbb{R}$ and $M^\theta$.

By [1] there exists a pair $(\tilde{M}, j)$ where $\tilde{M}$ is a complex manifold and $j$ is a real analytic embedding of $M$ in $\tilde{M}$. Then, by analytic extension, there exist an open neighbourhood $V$ of $M \times \{0\}$ in $M \times \mathbb{C}$ and a map $\Psi^c : V \to \tilde{M}$ satisfying

1. for each $p \in M$ the set $I_p = \{ t \in \mathbb{R} \mid (p, t) \in V \}$ is an open interval in $\mathbb{R}$ and $\Psi^c(p, t) = g_t(p)$ is the maximal integral curve of the vector field $\xi^\theta_0$ such that $g_0(p) = 0$, i.e. $\Psi^c(p, 0) = 0$ for each $p \in M$;
2. for each $(p, t) \in V \cap M \times \mathbb{R}$ the set $V_{(p, t)} = \{ \sigma \in \mathbb{R} \mid (p, t + i\sigma) \in V \}$ is an open interval in $\mathbb{R}$ containing the origin $0 \in \mathbb{R}$;
3. for each $p \in M$ the map $z = (t + i\sigma) \mapsto \Psi^c(p, z)$ is holomorphic when defined.

Then, set $W = \{ (p, \sigma) \in M \times \mathbb{R} \mid (p, i\sigma) \in V \}$ and define $\Psi : W \to \tilde{M}$ by the formula $\Psi(p, \sigma) = \Psi^c(p, i\sigma)$. Since $X^\theta(p) = J(\xi^\theta_0(p))$ is transversal to $M$ for each $p \in M$ it follows that, shrinking $V$ (and hence $W$) and $\tilde{M}$ if necessary, the map $\Psi$ is a real analytic diffeomorphism between $W$ and $\tilde{M}$.

Let $U = \Phi(V)$ and $J$ be the pullback under $\Phi \circ \Psi^{-1}$ of the complex structure of $M$. It is then easy to show that $(U, J)$ is a real analytic adapted complex tube on $M^\theta$.

This proves the existence of real analytic adapted complex tubes.

Let now $(U_1, J_1)$ and $(U_2, J_2)$ be two real analytic adapted complex tubes on $M^\theta$. We claim that $J_1$ and $J_2$ agree on the intersection $U_1 \cap U_2$. By an analytic continuation argument it suffices to prove that $J_1$ and $J_2$ agree on a neighbourhood of $M$ in $U_1 \cap U_2$.

Again by [1], there exist open neighbourhoods $V_1 \subset U_1$ and $V_2 \subset U_2$ of $M$ and a real analytic diffeomorphism $F : V_1 \to V_2$ which is the identity map on $M$ and which is holomorphic when $V_1$ and $V_2$ are endowed by the complex structures $J_1$ and $J_2$ respectively.

We are going to prove that $V_1 = V_2$ and the map $F$ actually is the identity map. Indeed, let $\alpha \in V_1$ be arbitrary, $p = \pi(\alpha)$ and $\sigma_0 = E(\alpha)$. Then $\alpha = \sigma_0 \theta_p$. The maps $f_1(z) = \phi_p(z)$ and $f_2(z) = F(\phi_p(z))$ are both holomorphic and coincide when $z$ is real. By analytic continuation $f_1 = f_2$. In particular, we have

$F(\alpha) = F(\phi_p(i\sigma_0)) = \phi_p(i\sigma_0) = \alpha$,

that is the map $F$ is the identity map on $V_1$, as claimed.
Now a standard argument implies that there exist a unique real analytic complex structure $J$ on the set $U$ defined as the union of all the open neighbourhood $U'$ of $M$ which admit a complex structure $J'$ which make the pair $(U',J')$ an adapted complex tube on $M^g$.

The pair $(U,J)$ is then the required unique real analytic maximal adapted complex tube for $M^g$.

This concludes the proof of Theorem 1.2.

**Remark 3.1.** The construction of the map $\Psi$ is similar to a one given in [3].

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