Restoration of Parity and the Right-Handed Analog of the CKM Matrix

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In a recent Letter we determined analytically the right-handed quark mixing matrix in the minimal Left-Right symmetric theory with generalized Parity. We derived its explicit form as a series expansion in a small parameter that measures the departure from hermiticity of quark mass matrices. Here we analyze carefully the convergence of the series by including higher order terms and by comparing with numerical results. We apply our findings to some phenomenological applications such as the production and decays of the right-handed gauge boson \( W_R \), the neutrinoless double beta decay, the decays of the heavy scalar doublet, the strong CP parameter and the theoretical limits on the new mass scale from the \( K \) and \( B \)-meson physics. In particular, we demonstrate that the relevant coupling for the production of the \( W_R \) gauge boson at hadronic colliders and for the neutrinoless double beta decay equals its left-handed counterpart, within a percent. We also demonstrate that the stability of the theoretical lower limit on the \( W_R \) mass from the \( K \)-meson physics is due to a partial cancellation of the external phases of the right-handed mixing matrix.

I. INTRODUCTION

The Left-Right symmetric theory [1] was put forward forty years ago as an attempt to understand the origin of parity violation in weak interactions through the spontaneous symmetry breaking. The left-right (LR) symmetry was chosen originally as a generalized parity (\( P \)), although it could also be a generalized charge conjugation (\( C \)). Regardless of the nature of LR symmetry, the theory leads to a non-vanishing neutrino mass and to the seesaw mechanism [2–5]. The smallness of neutrino mass is related to the departure from the maximal violation of parity at low energies [2, 3]. Furthermore, the knowledge of neutrino masses and mixings allows to predict their parity at low energies [2, 3]. The knowledge of the RH quark mixing is crucial for a number of phenomenological reasons, both at low energies and at the hadronic colliders.

Once the left-right symmetry is broken, the question is raised regarding the connection between the right-handed (RH) and left-handed (LH) quark and lepton mixing matrices. In the leptonic sector this connection goes away due to the seesaw mechanism which differentiate left and right handed neutrinos, while in the quark sector the situation depends on the nature of LR symmetry.

In the case of \( C \), symmetric Yukawa couplings lead to symmetric quark mass matrices and thus to same left and right mixing angles, same magnitude but opposite signs of the Kobayashi-Maskawa (KM) phase and its right-handed analog, and to arbitrary external phases in the RH sector. The equality of mixing angles imply the same strength for production and hadronic decay rates of left and right-handed charged gauge bosons, while the new RH phases bring in new sources of CP violation.

In the case of \( P \), the situation is not that simple. In spite of having hermitian Yukawas couplings, the spontaneous symmetry breaking of \( P \) introduces a complex phase which spoils the hermiticity of quark mass matrices, and a priori one would imagine no clear relation between LH and RH quark mixings. This soon became a burning issue, and the determination of the RH quark mixing matrix became over the years a kind of Holy Grail of the theory. Its explicit analytic form was only recently achieved [7] in the entire parameter space. In this longer sequel to our Letter we provide the missing technical details and the higher order terms in the series expansion of \( V_R \). We also discuss some important phenomenological applications and their consequences.

A few words regarding the history of the attempts to determine the RH quark mixings. The first step in this direction was made in Ref. [8] through a numerical study in a portion of the parameter space. Some years later, Ref. [9] made an analytical study in the same approximation and found the same hierarchical structure as in the CKM matrix. Finally, this result was established over the entire parameter space by combining analytical and numerical computations in Ref. [10]. What was missing until Ref. [7] was the exact equation for \( V_R \), its analytical solution and a clear demonstration of the approximate equality of mixing angles of the left and right-handed sectors.

A few years ago, a general low energy study of the RH quark mixing matrix was performed in Ref. [11]. More recently, it was argued that the near equality of the RH and LH mixing angles can be studied at the LHC with the b-quark tagging, and probed with a good precision at high luminosity [12], providing yet another motivation for our in-depth study of the RH mixing.

The knowledge of the RH quark mixing is crucial for a number of phenomenological reasons, both at low energies and at the hadronic colliders.

First and foremost, the Majorana nature of left handed and heavy right handed neutrinos leads to neutrinoless double beta decay [13] through both left and right-handed gauge interactions [3, 14]. The RH contribution could easily dominate over the usual LH one, providing a natural example of new physics being responsible for this process [15].

Moreover, one can in principle observe lepton number violation at hadronic colliders in the form of same
sign charged di-leptons, and probe directly the Majorana nature of heavy neutrinos through the so-called Keung-Senjanović (KS) process [16]. Its potential discovery would allow for precise predictions [17] (see also [18]) for neutrinoless double beta decay and lepton flavor violation; see e.g. [19] for a review. Detailed studies [20] support the feasibility of the KS process at the LHC and a possible probe of the LR scale all the way up to 5-6 TeV; for a roadmap, see [21]. Recently the LHC has set the lower limit on the LR scale limit of roughly 3 TeV for a wide range of RH neutrino masses [22]. It should be stressed that it is possible to determine the chirality of $W_R$ gauge boson couplings, as discussed in the KS [23] and the hadronic channels [24].

While the hadronic colliders may be the best for a discovery of $W_R$ due to their high energies, it is important to study also other collider possibilities. For a linear collider analysis we refer the reader to [25]. Recently, a study of the theory for the $ep$ colliders was done in [26].

It was known for a long time that the small $K_L - K_S$ mass difference leads to the lower limit on the LR scale in the minimal model [27], around 3 TeV (for the most recent study, see [28]). In the case of $P$, from the study of the electric dipole moment of the neutron, the limit was raised to about 7 TeV [29], without taking into account the strong CP violation. Meanwhile new studies also improved the chiral perturbation results for this process [30, 31] and recently [31] claimed a limit (which depends on the ultraviolet completion of the theory) of about 20 TeV by studying carefully the strong CP.

It may be worth mentioning that the CMS reported recently [22] a 2.8 sigma excess in the KS process that can be interpreted as an indication of the LR symmetry [32]. It would require however the right-handed gauge coupling appreciably smaller that its left-handed counterpart, which would imply the breaking of LR symmetry at a very high scale. In what follows we work in the minimal model with single scale of symmetry breaking.

The rest of the paper is organized as follows. In section II we discuss our main results, i.e., the derivation of the RH mixing matrix as a perturbation series in a small parameter which measure the hermiticity of the quark mass matrices. We show that mixing angles are well approximated already at the first order; the differences between RH and LH angles is in excellent agreement with the numerical results. The same is true for the KM phase and its RH counterpart, while the external phases in the RH mixing matrix need higher order corrections. We then discuss the issue of the convergence of the external phases.

The phenomenological applications of our results are left for section III, where we discuss the production and decays of the RH charged gauge boson, the neutrinoless double beta decay, the decays of the heavy scalar doublet of the theory, the limits in the LR scale from the K and B meson physics and the computation of the strong CP parameter $\theta$.

Finally, in section IV we summarize our findings and offer an outlook for future work.

In Appendix A we give the technical details necessary for the derivation of the analytic expression of the RH mixing matrix at the first order in perturbation theory. We discuss there how to deal with the square roots of matrices needed for this. In Appendix B we derive the the second and third order terms of the same expansion. In Appendix C, for illustrative purposes, we give exact values for the external phases in the case of vanishing mixing angles.

Before we turn to the task of computing the RH mixing matrix, a comment is called regarding the question of potentially dangerous domain walls [33], the product of a spontaneous breaking of discrete symmetries. A natural way of avoiding them is of course inflation, but in a low scale theory it is not available. Another natural way [34] may be non-restoration of symmetries at high temperature [35], but even that may not work [36]. In any case, a tiny (even Planck scale suppressed) explicit breaking [37] suffices to get rid of domain walls.

II. THE RIGHT HANDED QUARK MIXING MATRIX

The LR symmetric theory studied here is based on the $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ gauge group augmented with generalized parity $P$: $q_L \leftrightarrow q_R$ (for recent reviews, see [19]). Quarks and leptons come in LR symmetric representations

$$Q_{L,R} = \begin{pmatrix} u \\ d \end{pmatrix}_{L,R}, \quad \ell_{L,R} = \begin{pmatrix} \nu \\ e \end{pmatrix}_{L,R}.$$  \hspace{1cm} (1)

The Higgs sector consists of the following multiplets [3]: the bi-doublet $\Phi$ and the $SU(2)_L$ triplets $\Delta_L$ and $\Delta_R$, where under generalized parity $P$: $\Phi \leftrightarrow \Phi^\dagger$ and $\Delta_L \leftrightarrow \Delta_R$.

At the first stage of symmetry breaking, the neutral component of $\Delta_R$ develops a vev and breaks the original symmetry down to the SM one. The latter is in turn broken through the vevs of the neutral components of $\Phi$

$$\langle \Phi \rangle = v \text{diag}(\cos \beta, -\sin \beta e^{-ia})$$  \hspace{1cm} (2)

where $v$ is real and positive and $\beta < \pi/4$, $0 < a < 2\pi$.

The quark Yukawa couplings in the minimal theory take the following form

$$-L_Y = \bar{q}_L Y_1 \Phi - Y_2 \sigma_2 \Phi^\dagger \sigma_2 q_R + \text{h.c.}$$  \hspace{1cm} (3)

which gives the following mass matrices for quarks

$$\begin{pmatrix} M_u \\ M_d \end{pmatrix} = v \begin{pmatrix} -c_\beta & -e^{ia}s_\beta \\ e^{-ia}s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$  \hspace{1cm} (4)

where $c_\beta \equiv \cos \beta$ and $s_\beta \equiv \sin \beta$. The underlying generalized parity $P$ implies hermitian Yukawa couplings

$$Y_{1,2}^\dagger = Y_{1,2}$$  \hspace{1cm} (5)
which in turn leads to the following relations between the up and down quark mass matrices

\begin{align}
M_u - M_u^\dagger &= -i s_at_{2\beta} (e^{-i\alpha} t_{\beta} M_u + M_d) \\
M_d - M_d^\dagger &= i s_at_{2\beta} (M_u + e^{i\alpha} t_{\beta} M_d)
\end{align}

(6) (7)

where \( s_a = \sin a \), \( t_{\beta} = \tan \beta \), \( t_{2\beta} = \tan 2\beta \). The amount of the hermiticity of quark mass matrices is measured by the \( s_at_{2\beta} \) parameter. From (7), by focusing on the third generation and ignoring its mixings with the first two, it is straightforward to obtain a rough upper limit \( s_at_{2\beta} \lesssim 2m_d/m_t \). We justify it through an exact result found below.

From the hierarchy of top and bottom quark masses, it is easy to see that \( M_u \) is almost hermitian. Assuming small tan \( \beta \), Ref. [9] worked in the approximation of hermitian \( M_u \) and obtained a semi-analytical form of the RH quark mixing matrix. In what follows we derive an exact equation for \( V_R \) in the full parameter space and find an analytic solution as a series in the small parameter \( s_at_{2\beta} \).

To set notation

\[ M_u = U_L m_u U_L^\dagger, \quad M_d = D_L m_d D_R \]

(8)

where \( m_q \) are diagonal matrices of positive quark masses. The left-handed CKM matrix \( V_L \) and its right-handed analog \( V_R \) are then given by

\[ V_L = U_L^\dagger D_L, \quad V_R = U_R^\dagger D_R \]

(9)

as manifested in LH and RH charged gauge interactions

\[ \mathcal{L}_{\text{gauge}} = -\frac{g}{\sqrt{2}} (\pi_L V_L \bar{W}_L d_L + \pi_R V_R \bar{W}_R d_R) + \text{h.c.} \]

(10)

We refer the reader to Appendix A for the details that go into the following leading order expression [7]

\[(V_R)_{ij} = (V_L)_{ij} - i s_at_{2\beta} \left[ t_{\beta} (V_L)_{ij} + \frac{(V_L m_d V_L^\dagger)_{ik} (V_L)_{kj}}{m_{u_i} + m_{u_k}} + \frac{(V_L)_{ik} (V_L m_u V_L^\dagger)_{kj}}{m_{d_i} + m_{d_j}} \right] + O(s_a^2 t_{2\beta}^2) \]

(11)

This is not the only solution; the others are found through \( V_L \rightarrow S_u V_L S_d \) and \( m_q \rightarrow s_q m_q \), where \( S_u = \text{diag}(s_u) \), \( S_d = \text{diag}(s_d) \) and \( s_q \) are ± signs. This is discussed at the end of the Appendix A. Notice that even when quark mass matrices are hermitian, the LH and RH mixing matrices are not automatically equal due to the sign freedom of quark masses; instead one has the well known result \( V_R = S_u V_L S_d \).

Equation (11) gives the RH mixing matrix \( V_R \) as an expansion in the small parameter \( s_at_{2\beta} \), which, for the sake of convergence, must satisfy

\[ |s_at_{2\beta}| \lesssim \min \left| \frac{m_d_i + m_d_j}{(V_L m_u V_L^\dagger)_{ij}} \right| \approx 2 \frac{m_b}{m_t} \]

(12)

This agrees with what we estimated taking the third generation and should not come as a surprise since the third generation CKM mixing angles are basically negligible in this context.

When \( s_at_{2\beta} \) is close to \( 2m_b/m_t \) one should include higher order terms, which can be found in the Appendix B. This turns out necessary for the phases, while the values of the RH mixing angles obtained from (11) agree very well with the numerical solution of the exact equation (A11).

The first term in the bracket in (11) is quite small, while the third term generally dominates over the second due to large mass ratios of the top quark mass over the down quark masses (basically due to large \( m_t/m_b \)), and to some degree, the ratio of charm quark mass over \( m_d \) and \( m_s \). It is noteworthy that the above results are parametrization independent.

It is evident from (11) that a diagonal \( V_L \) implies a diagonal \( V_R \) at this order. We will see that this fact persists at higher orders and, moreover, it is clearly a solution of the exact equation (A11). The difference between RH and LH mixing angles is controlled by the smallness of the CKM angles and the smallness of the parameter \( s_at_{2\beta} \), as we discuss amply in what follows.

A comment is in order. Our results, such as formula (11), are obtained by using generalized parity \( P \) and are valid at their face value only at the scale of LR symmetry breaking or above. They can be safely applied at high energy; however, in order to use them at low energies one needs to take into account their scale dependence. We are interested principally in a LR scale accessible to the LHC which makes these effects tiny and we will ignore them in the rest of this work.

### A. RH angles and KM phase.

We compute next the difference between the left and right mixing angles and the KM phase and its RH counterpart. To do this we use the following parametrization for a general \( 3 \times 3 \) unitary matrix

\[ V_R \equiv \text{diag}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3}) V(\theta^R_{ij}, \delta^R_{ij}) \text{diag}(e^{i\omega_4}, e^{i\omega_5}, 1) \]

(13)

where \( V(\theta^R_{ij}, \delta^R_{ij}) \equiv V_L \) is the standard form of the left-handed CKM matrix used by the Particle Data Group. Besides the RH analog \( \delta^R_{ij} \) of the KM phase \( \delta_L \) (from now on both called KM phases), \( V_L \) contains five external phases which cannot be rotated away since we used all
the phase freedom in defining the usual CKM matrix in the left sector.

A straightforward computation from (11) gives the leading terms for the differences between mixing angles

\[ \theta_{12}^L - \theta_{12}^R \simeq -s_\alpha t_{2\beta} \frac{m_t}{m_s} s_{12} s_{13} s_{13} \delta \]  
\[ \theta_{23}^L - \theta_{23}^R \simeq -s_\alpha t_{2\beta} \frac{m_t}{m_b} m_s \delta s_{12} s_{13} s_{13} \delta \]  
\[ \theta_{13}^L - \theta_{13}^R \simeq -s_\alpha t_{2\beta} \frac{m_t}{m_b} m_s \delta s_{12} s_{23} s_{23} \delta \]  

and similarly for the KM phases

\[ \delta_R - \delta_L \simeq s_\alpha t_{2\beta} \frac{m_t c_{23}^2 + m_t s_{23}^2}{m_s} \]  

where, for simplicity, we defined \( s_{ij} = \sin \theta_{ij}^L \), \( c_{ij} = \cos \theta_{ij}^L \) and \( s_\delta = \sin \delta_L \). These are leading order terms, more complete expressions for the angles were given in [7], and the exact first order terms can readily be obtained from (11).

It suffices to change the signs of quark masses accordingly to get all the other solutions. As shown below, the absolute values of the mixing angle differences are quite stable under these transformations, while the KM phase difference varies somewhat.

Notice that the angle differences vanish in the limit of CKM phase \( \delta_L \) going to zero. This follows from the fact that in this limit, the first order terms in \( s_\alpha t_{2\beta} \) in (11) are purely imaginary and thus affect only the phases. From the above formulas, it is evident that the angle differences are very small, suppressed by small CKM mixings. Moreover, in the case of 2-3 and especially 1-3 mixing angles there is an additional suppression of a small quark mass ratio \( m_s/m_b \) that compensates for the large \( m_t/m_b \) factor. As if there was a conspiracy in nature to keep the symmetry between LH and RH mixing angles in a world with broken parity.

In Fig. 1 we plot in red lines these first order results, and with blue dots the numerical solutions of the exact equation. The first order is an excellent approximation and the agreement between the two is manifest through the whole physical region of \( s_\alpha t_{2\beta} \). Notice that the phase difference \( \delta_R - \delta_L \) is multiplied with the factor \( \sin \theta_{13}^L \). The reason is that the phases \( \delta_R \) and \( \delta_L \) are always accompanied with \( \sin \theta_{13}^L \) and \( \sin \theta_{13}^R \) (which are practically the same), respectively.

The crucial point, as we noticed, is that the differences of LH and RH mixing angles are always proportional to another small LH mixings, which control their smallness. From Fig. 1 it seems that this holds true in higher orders of perturbation in \( s_\alpha t_{2\beta} \) and the proof comes from a discussion of a non-realistic two generation situation. Indeed, from (11), the difference of left and right mixing angles is zero, since \( V_{12} \) is real. In [7] we showed that the remarkable equality of \( \theta_L \) and \( \theta_R \) is actually exact in the two-generation case, which then guarantees the smallness of the mixing angles differences at all orders.

\[ \omega_1 \simeq -\omega_3 + s_\alpha t_{2\beta} \left( \frac{m_c c_{23} + m_s s_{23}}{m_s} - \frac{m_d c_{12}^2 + m_s s_{12}^2}{2m_u} \right) \]  
\[ \omega_2 \simeq -\omega_3 \simeq s_\alpha t_{2\beta} \frac{m_t}{2m_b} \]  
\[ \omega_4 \simeq \omega_3 - s_\alpha t_{2\beta} \left( \frac{2}{12} m_c c_{23}^2 + m_s s_{23}^2 + s_{12}^2 m_s c_{23}^2 + m_t s_{23}^2 \right) \]  
\[ \omega_5 \simeq \omega_3 - s_\alpha t_{2\beta} \frac{2}{12} m_c c_{23}^2 + m_t s_{23}^2 \]  

The differences between the right and left handed mixing angles and the KM phases. The first order terms are given by red lines, the blue dots denote numerical solutions of the exact equation. The agreement is manifest in the entire physical region \( s_\alpha t_{2\beta} \lesssim 0.03 \), which simply implies the need for higher order terms in (11), as discussed below.

**B. RH external phases.**

First, we compute the external phases from (11) and (13)

\[ \omega_1 \simeq -\omega_3 + s_\alpha t_{2\beta} \left( \frac{m_c c_{23} + m_s s_{23}}{m_s} - \frac{m_d c_{12}^2 + m_s s_{12}^2}{2m_u} \right) \]  
\[ \omega_2 \simeq -\omega_3 \simeq s_\alpha t_{2\beta} \frac{m_t}{2m_b} \]  
\[ \omega_4 \simeq \omega_3 - s_\alpha t_{2\beta} \left( \frac{2}{12} m_c c_{23}^2 + m_s s_{23}^2 + s_{12}^2 m_s c_{23}^2 + m_t s_{23}^2 \right) \]  
\[ \omega_5 \simeq \omega_3 - s_\alpha t_{2\beta} \frac{2}{12} m_c c_{23}^2 + m_t s_{23}^2 \]  

The expressions above are somewhat more precise than what we gave in the previous Letter version of this work. Unlike the expressions for the mixing angles and the KM phases, the external phases depend strongly on the sign transformations that connect different solutions. The above formulas should be taken as an example with all positive signs. It is straightforward to get more precise and complete expressions for all the cases. There is one subtlety to keep in mind: in some cases sign changes make the phases start from \( \pi \) and not from zero, but that is easy to figure out.

We plot these phases in Fig. 2. Again, the first order results are shown in red, and the numerical results in blue. Notice that in this case the results start diverging for larger values \( s_\alpha t_{2\beta} \gtrsim 0.03 \), which simply implies the need for higher order terms in (11), as discussed below.
mixing angles and KM phases
equivalent to phase changes from zero to \( \pi \) quite sensitively on the signs of quark masses, except for different signs, at least at first order in quantities phenomenologically one can safely ignore differences, very near to each other. When discussing these some dependence on the signs manifested in two smaller spreads of these quantities. The stability of mixing angle differences are really sensitive to the signs that plague (11), due to sign freedom of quark masses. It is easy to see in the first order terms that only the signs of that are included due to their negligible effect. The final corrections to the dominant terms up to third order, one obtains

\[
(V_R)_{33} \approx e^{i\omega_3} \cos \theta_R^{23} \cos \theta_R^{13} \tag{22}
\]

To get \( \omega_3 \), we first expand \( (V_R)_{33} \) up to third order using the expressions given in the Appendix B. Keeping only dominant terms up to third order, one obtains

\[
(V_R)_{33} \approx c_{23} c_{13} \exp \left\{ -i \arcsin \left[ s_a t_{23} \frac{m_t}{2m_b} \left( 1 + s_{23}^2 + s_{13}^2 \right) \right] \right\} \tag{24}
\]

From (22) and (24) one gets

\[
\sin \omega_3 \approx -s_a t_{23} \frac{m_t}{2m_b} \left( 1 + s_{23}^2 + s_{13}^2 \right) \tag{25}
\]

The existence of the solution is then guaranteed by (12).

To get the rest of the phases, it turns out to be sufficient to substitute \( \omega_3 \) in (18)-(21) by the corrected one given above. This takes into account only higher order corrections to the dominant \( m_t/m_b \) term; corrections to the rest of the factors is straightforward but we have not included them due to their negligible effect. The final

C. The impact of sign alternatives

The reader must be worried about the different signs that plague (11), due to sign freedom of quark masses. It is easy to see in the first order terms that only the signs of mixing angle differences are really sensitive to the signs of quark masses. The case of the KM phase difference is more subtle, due to the contribution from both charm and top quarks.

In Fig. 3 we plot the impact of the quark mass signs on the absolute values of mixing angle and KM phase differences computed at the first order in \( s_a t_{23} \). We include all the possible sign variations, which results in bands like spreads of these quantities. The stability of mixing angle differences is remarkable; for the KM phases there is some dependence on the signs manifested in two smaller bands, very near to each other. When discussing these quantities phenomenologically one can safely ignore different signs, at least at first order in \( s_a t_{23} \).

As can be seen in (18)-(21), the external phases depend quite sensitively on the signs of quark masses, except for \( \omega_2 \) and \( \omega_3 \). This is to be expected, for sign changes are equivalent to phase changes from zero to \( \pi \). One has to live with this aspect of the theory when discussing the CP violating processes and it must be taken into account when setting limits on the LR scale.

D. Higher orders: external phases

As one can see from Fig. 2, first order expressions for the external phases (18)-(21) are a good approximation only for small \( s_a t_{23} \). We now address the question of the convergence of the external phases as one approaches to the upper bound of \( s_a t_{23} \).
It is easy to see that agreement with the numerical solution is excellent and covers the whole parameter space. The agreement with the numerical results is now manifest in the whole parameter space. Once again, we confirm that the perturbation expansion in the small $\sin \omega_{3}$ works well already at first order and the convergence of the external phases is under control.

Last but not least, for illustrative purposes, the reader is referred to Appendix C for the exact solution of the RH phases in the non-realistic case of vanishing mixing angles.

III. PHENOMENOLOGICAL APPLICATIONS

A. RH charged gauge boson at colliders

The physics of the $W_R$ charged gauge boson needs the knowledge of $V_R$ as much as one needs to know the CKM matrix in the case $W$. The proximity of LH and RH mixing angles tells us that the $W_R$ production at the hadronic colliders proceeds basically in the same manner as the production of the $W$ boson. This is useful since both CMS and ATLAS [22, 38] in most of their searches for a generic $W'$ assume identical production rates as for the $W$ boson.

More precisely, the production strength of $W_R$ is proportional to

$$|V_{R11}| \simeq c_{12} \left(1 + s_{\alpha} t_{2\beta} \frac{m_t}{m_s} s_{12} s_{23} s_{13} s_{\beta} \right)$$  (26)

It is easy to see that $|V_{R11}|^2$ differs from $|V_{L11}|^2$ by at most a percent; thus to a great precision the production rate of $W_R$ is equal to that of $W_L$. Similarly, the decay rates into top and bottom, or di-jets in general are to an excellent approximation given by assuming the same left and right couplings. Needless to say, it is possible to have LR symmetry broken at large scales; in such a case one has to run down our expressions to the relevant energies. The predictions will then be modified in a calculable manner.

On the other hand, the leptonic decay rates of LH and RH gauge bosons are completely different in general, due to the seesaw mechanism behind neutrino mass. In this case $W_R$ can decay into two jets and two same sign charged leptons [16], an appealing possibility of direct observation of lepton number violation.

B. Neutrinoless double beta decay

The LR symmetric model offers naturally a new contribution [3] to the neutrinoless double beta decay ($0\nu\beta\beta$) through the exchange of $W_R$ and RH neutrinos, in complete analogy with the usual contribution of Majorana light neutrinos. It goes through the RH gauge currents and RH neutrino mass, depending crucially on both lepton and quark RH mixing matrices. The total $0\nu\beta\beta$ rate can then be described by the effective mass parameter [17]

$$|M_{\nu+\nu}| = \sqrt{\left|\left(M_{\nu}\right)_{ee}\right|^2 + \left|\left(V_{R11}\right)_{11}^4 M_{W_R}^8 \left(\frac{k^2}{M_N}\right)_{ee}\right|^2}$$  (27)

where $k$ is a measure of the neutrino virtuality, $M_\nu$ and $M_N$ are the mass matrices of light and heavy neutrinos, respectively. As we mention above, the lepton mixing is not predicted by the theory, and its determination would require the eventual direct observation of lepton number violation along the lines of the KS process. However, it is reassuring to know that the RH quark vertex $(V_L)_{11}$, as shown in (26), can be predicted with arbitrary precision, in the same manner as in the case of the $W_R$ production at hadron colliders.

An additional contribution to $0\nu\beta\beta$ mediated by the RH doubly charged scalar is disfavoured by lepton flavor violating constraints [17]. The rate can also proceed through the Dirac masses of neutrinos but, for RH neutrinos visible at the LHC, it amounts to a sub-leading effect [6]. Nevertheless, these sub-leading contributions also depend on the quark mixing $V_R$ and their precise evaluation is thus equally feasible.

C. The theoretical limits from K and B physics

Another nice example of the relevance of $V_R$ are the low energy limits on the LR scale from the kaon and B-meson physics. They depend crucially on the RH mixings, and we will illustrate it on a few examples.
K meson physics. Let us discuss the $K - \bar{K}$ mass difference. As is well known, in this case the dominant effect comes from the charm quark in the box diagram with the $W_L$ and $W_R$ charged gauge bosons. Only the real part of the amplitude contributes and it is proportional to the following form of the RH mixing matrix

$$\text{Re}[A_{LR}(K \to \bar{K})] \propto \text{Re}[(V_L)_{21}(V_R^*)_{21}(V_R^*)_{22}]
\approx s_\omega s_\delta e^2 s_{12}^2 s_{23}^2
\times \left[ 1 - \frac{1}{8} (s_\omega s_\delta)^2 \left( \frac{c_{12}^2}{m_s} + \frac{s_{12}^2}{m_d} \right)^2 (m_c c_{23}^2 + m_t s_{23}^2)^2 \right]$$

(28)

where we have used the formula for $V_R$ up to second order (for higher orders see Appendix B). Alternatively, neglecting the small difference between left and right mixing angles one gets $c_{12}^2 s_{12}^2$ cos$(\omega_4 - \omega_5)$. Using (20) and (21) one arrives again at (28). Due to different sign options in $V_R$, the cosine can be either $-1$ (and remain stable for any $s_\omega s_\delta$ because of the cancellation of the first and second term in the rest of the difference $\omega_4 - \omega_5$) or lie between 1 and $\sim 0.7$, in this case depending also on the value of $s_\omega s_\delta$.

Notice an interesting fact that helps the stability of the limit on the $W_R$ mass. The difference $\omega_4 - \omega_5$ is substantially smaller that the individual $\omega_4$ and $\omega_5$ phases as seen from (20)-(21), and we can see clearly how the limit on the scale actually emerges.

In a similar manner one can compute the imaginary part of the amplitude, which we leave as an exercise to the reader.

B meson physics. In this case the relevant quantity to compute to the leading order, due to the top quark dominance, is (see e.g. [10])

$$r_d \propto (V_R^*)_{23}(V_R^*)_{31}$$

(29)

where $d_i$ stands for strange ($i = 2$) or down ($i = 1$) quark. Neglecting the tiny difference between the left and right mixing angles, one gets the leading contribution

$$r_d \propto c_{23}c_{13}\left(s_{12}s_{23} - c_{12}c_{23}s_{13} e^{i\delta_R}\right) e^{i\omega_4}$$

(30)

$$r_s \propto -c_{23}c_{13}s_{12}s_{23} e^{i\omega_5}$$

(31)

with $\delta_R$, $\omega_4$ and $\omega_5$ given approximately by (17), (20) and (21), respectively at first order (to account for higher orders corrections one should use also (25)). Again these analytical results facilitate numerical studies.

D. The physics of the heavy scalar doublet from the bi-doublet

It has been known for a long time that the second doublet in the bi-doublet, due to the large amount of flavor violation it induces in neutral currents, has to have a large mass on the order 10-15 TeV or so (see for example [10], [11]). This is not a problem since it gets its mass [39] from the large triplet vev, responsible for the masses of heavy gauge boson. Its interactions can be deduced from the Yukawa in (3)

$$-L_Y = -\overline{q}_L \left[ \frac{M_u}{v} h - \frac{M_d + e^{-i\beta}s_\delta M_u}{vc_{2\beta}} V^\dagger \right] u_R
+ \overline{q}_L \left[ \frac{M_d}{v} - i\sigma_2 h^* - \frac{M_u + e^{i\beta}s_\delta M_d}{vc_{2\beta}} i\sigma_2 H^* \right] d_R + h.c.$$  

(32)

where

$$h = c_\beta \phi_1 + e^{-i\beta}s_\beta \phi_2, \quad H = -e^{i\beta}s_\beta \phi_1 + c_\beta \phi_2$$

(33)

$h$ and $H$ are the doublets with and without a vev, respectively, and $\phi_1$ and $\phi_2$ are SU(2) doublets with $Y = -1$, forming the bi-doublet $\Phi = (\phi_1, i\sigma_2 \phi_2^*)$. The light doublet, $h$, is effectively the SM one, while $H$ stands for the heavy doublet. Since $H$ has to weigh more than 10 TeV, they can be taken with an excellent precision to be mass eigenstates. The Yukawa interaction of $H$ in the physical basis becomes

$$-L_H = H_0^{\dagger} u_L \left[ V_L m_d V_R^\dagger + e^{-i\beta}s_\beta m_u \right] u_R$$

$$+ H_0^{\dagger} d_L \left[ V_L^\dagger m_u V_R + e^{i\beta}s_\beta m_d \right] d_R$$

$$+ H_0^{\dagger} \overline{d}_L \left[ m_d V_R^\dagger + e^{-i\beta}s_\beta V_L^\dagger m_u \right] u_R$$

$$- H_0^{\dagger} \overline{u}_L \left[ m_u V_R + e^{i\beta}s_\beta V_L m_d \right] d_R + h.c.$$  

(34)

The above interactions depend crucially on $V_R$ just like the gauge interactions of the heavy gauge bosons. This is relevant for the low energy processes such as $K_L - K_S$ mass difference and for the production and identification of the heavy doublet $H$ at future hadronic colliders.

Regarding $K - \bar{K}$ mass difference. The neutral component of the heavy doublet adds coherently to the box diagram and will not affect the stability of the bound on the $W_R$ mass. The tree level amplitude is proportional to

$$\text{Re}[A_H(K \to \bar{K})] \propto \text{Re}[(V_L^* m_u V_R)_{21}(V_L^* m_u V_L)_{21}]$$

(35)

which, due to gauge invariance [40], has the same flavor dependence as in the usual $W_L - W_R$ box diagram. Using as before higher orders of $V_R$ one gets the same leading term as in (28).

Similarly, the production and decays of the heavy doublet depend strongly on the quark mixings $V_R$ and, for small values of $\beta$, it may dominate the interaction. Notice an important fact: $H^0$ decays principally in $b\bar{b}$, a very clear signature. The other decay rates, such as say, $t\bar{t}$ depend on $\beta$. 
E. The strong CP parameter

As argued originally in [41], generalized parity guarantees the vanishing of the QCD strong CP parameter $\theta$ before the symmetry breaking. This makes the effective $\tilde{\theta}$ calculable through the knowledge of $V_R$, as done recently approximately in [31]. As a particularly nice illustration of the power of our results, in what follows we calculate exactly the first order of the series expansion of $\tilde{\theta}$ in terms of the small parameter $s_a t_{23}$.

In terms of the matrices that diagonalize the up and down quark mass matrices, one has

$$\tilde{\theta} = \arg \det U_L^1 U_R D_L^1 D_R$$  \hspace{1cm} (36)

Now defining, as in Appendix A, $U_u = U_L^1 U_R$, $U_d = D_L^1 D_R$, and using $V_L U_d = U_u V_R$ one gets (noting also that $\arg \det V_L = 0$)

$$\tilde{\theta} = \arg \det U_R^2 V_R$$  \hspace{1cm} (37)

Using the explicit form of $U_u$ as a matrix square root from (A9), it gives

$$\tilde{\theta} = \arg \det \left[ V_R + i s_a t_{23} (t_\beta e^{-i a} V_R + m_u^{-1} V_L m_d) \right]$$  \hspace{1cm} (38)

This is an tree-level expression, depending mainly on quark masses and mixing matrices. It could serve to compute $\tilde{\theta}$ with an arbitrary precision. We can expand (38) in the small parameter $s_a t_{23}$ and obtain the leading term

$$\tilde{\theta} = s_a t_{23} \frac{1}{2} \Re \{ m_u^{-1} V_L m_d V_L^\dagger - m_d^{-1} V_L V_R \}$$  \hspace{1cm} (39)

This expression and the exact one above, disagrees with the approximate form presented in [31]. The dominant term, though, is the same and is given by $\tilde{\theta} \simeq s_a t_{23} m_t/2 m_s$, and could have been readily guessed by ignoring the small third generation CKM mixing angles. It is obviously huge unless $s_a t_{23}$ is vanishingly small. This forces a large bound on the LR scale as argued in [31].

How solid is this argument? In order to answer this, let us digress for a moment and discuss the question of strong CP in the SM. The fact that, experimentally, the strong CP parameter must be small $\tilde{\theta} \lesssim 10^{-10}$ has been coined the strong CP problem. But is this really a problem? In other words, is the perturbative contribution to $\tilde{\theta}$ much bigger than the experimental upper bound? The answer is no, in fact it is much smaller as we now remind the reader.

It is true that the SM does not predict the value of $\tilde{\theta}$ and it is a puzzle why it is so small, in view of the fact that the weak CP violation parameter is much bigger. It is also true that $\tilde{\theta}$ appears to be divergent in perturbation theory, however, not before the sixth loop [42] and for any reasonable value of the cut-off it is negligibly small. For example, for $\Lambda_{cutoff} \simeq M_{Planck}$ one gets $\tilde{\theta} \simeq 10^{-19}$ [42], orders of magnitude below the experimental limit. In other words, the question of the strong CP violation is a question of high energy physics, not a question of the SM itself. More precisely, the large scale that sets the value of $\tilde{\theta}$ decouples effectively from low energy processes, in the same way the scale of grand unification decouples from low energies, except from fixing the weak mixing angle [43] (of course, there is the new possibility of inducing a proton decay, but this is consistent with the decoupling in usual interactions).

A nice example of new high energy physics is the Peccei-Quinn (PQ) symmetry [44]. It fixes $\tilde{\theta}$ no matter how large its scale is, and for practical purposes it decouples from low energy phenomena, the exception being a highly sensitive physics of the axion, just as proton decay in grand unification. The fact that the PQ symmetry does the job in the LR theory without affecting low energy physics has been emphasised already in [31]. The message from this is simple: one should not worry about $\tilde{\theta}$ in theories at low, or relatively low, energies, such as the SM or the LR symmetric theory. This is a philosophy we will succumb to, and simply imagine a new physics such as PQ symmetry that takes care of the strong CP issue. In other words, we will allow $s_a t_{23}$ to take any value below its strict phenomenological bound in (12). Independently of how this is settled, and independently of the precise value of $\tilde{\theta}$ one could obtain, however, a lower limit on the LR scale by considering the electric dipole moments of atoms, such as mercury. Although well measured, its theoretical value is plagued by large uncertainties, so setting a reliable limit will have to wait some more.

IV. SUMMARY AND OUTLOOK

In a recent Letter we were able to elucidate the long awaited form of the RH quark mixing matrix $V_R$ in the minimal LR symmetric model augmented with generalized parity. We found exact equations, valid in the entire parameter space, which allow for its numerical determination. We gave there an approximate form for $V_R$ as a leading term in the expansion of a small parameter which measures the departure from the hermiticity of the quark mass matrices. Moreover, we argued in favor of the proximity between left and right mixing angles, using the important fact that in the two generation case $\theta_R = \theta_L$. The small CKM mixing angles, together with a small ratio between strange and bottom quark masses, then guarantee practically equal mixing angles in the realistic three generation case.

In this companion paper we provided a more detailed and complete discussion, paying special attention to the convergence of the employed expansion by calculating the second and third order of the series. In the case of mixing angles, we find an excellent agreement with the numerical
results already at the first order, while in the case of the phases in general one needs higher orders.

We also discussed here a number of phenomenological applications that depend strongly on the form of the RH quark mixing. Our study shows the importance of the explicit analytical and numerical knowledge of $V_{\text{R}}$. In the case of $W_{\text{R}}$ we demonstrate that the collider production and signature in di-jet decays is perfectly well described with the usual SM couplings of the $W$ boson, except the opposite chirality. Similarly, we determined the strength of the right-handed gauge currents which control the rates of neutrinoless double beta decay. We also show how the decay rates of the heavy scalar doublet in the bi-doublet can be studied with great precision now that $V_{\text{R}}$ is made transparent. As a nice example of the application of our results we give an exact and a leading term expression for the strong CP parameter $\bar{\theta}$. More important, we gave an explicit expression for the dependence of $K_L - K_S$ mass difference on $V_{\text{R}}$ and showed that the stability of the resulting limit on the $W_{\text{R}}$ mass is due to a partial cancellation of external phases.

One last comment. The determination of $V_{\text{R}}$, as we showed here, requires only the minimal quark Yukawa sector and it does not depend, at least not at the tree level, on the situation in the leptonic sector or on the details of the parity breaking at the high scale. In other words, it does need the Majorana seesaw picture of neutrinos and could as well apply to the original LR model [1]. After all, neutrinos could be Dirac particles [48], no matter how appealing we find the seesaw picture. All that matters here is that our findings hold true in any LR symmetric model with the single bi-doublet $\Phi$. While we prefer the seesaw picture which makes the LR theory complete and predictive when it comes to neutrino mass, it is up to experiment to decide.

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Appendix A: Leading order

In order to determine $V_{\text{R}}$ one has to deal with a matrix equation involving square roots of matrices [7]. For simplicity and illustration we begin with the following square root

$$\sqrt{m^2 + i\epsilon A}$$  \hspace{1cm} (A1)

where $m$ is a diagonal matrix, $A$ a matrix, and $\epsilon$ an expansion parameter. The problem is to express the above square root in terms of the elements of $m$ and $A$ for small $\epsilon$. There are different ways of attacking this problem; we will work in 2 dimensions, where a root of matrix itself has a simple analytical form. The resulting expression turns out to be valid for any number of dimension.

Using the formula for square root in the $2 \times 2$ case we have (taking all signs positive to ease the presentation)

$$\sqrt{m^2 + i\epsilon A} = \frac{(m^2 + i\epsilon A) + \sqrt{d}I_{2 \times 2}}{\sqrt{t + 2\sqrt{d}}}$$  \hspace{1cm} (A2)

where $d = \det(m^2 + i\epsilon A)$ and $t = \text{tr}(m^2 + i\epsilon A)$. Expanding first

$$\sqrt{d} = m_1 m_2 \left[ 1 + i \frac{\epsilon}{2} \text{tr}(m^{-2}A) \right]$$  \hspace{1cm} (A3)

$$\frac{1}{\sqrt{t + 2\sqrt{d}}} = \frac{1}{m_1 + m_2} \left( 1 - \frac{i \epsilon \text{tr}A + m_1 m_2 \text{tr}(m^{-2}A)}{2 (m_1 + m_2)^2} \right)$$  \hspace{1cm} (A4)

and using them in (A2) one arrives at

$$\left(\sqrt{m^2 + i\epsilon A}\right)_{ij} = m_i \delta_{ij} + i\epsilon \frac{A_{ij}}{m_i + m_j} + O(\epsilon^2)$$  \hspace{1cm} (A5)

It is easy to check that this expansion is valid regardless the number of dimensions. Still, here we outline the general steps needed to find the square root of a matrix $M$ of arbitrary dimensions. This is useful also when solving the equation numerically. The first step is to decompose $M$ into

$$M = XJX^{-1}$$  \hspace{1cm} (A6)

where $J$ is called the normal form of $M$. Once $J$ and $X$ are found (see [47] for the details), one finds the root as

$$\sqrt{M} = XY\sqrt{J}Y^{-1}X^{-1}$$  \hspace{1cm} (A7)
Here, the matrix $Y$ parametrizes a set of continuous solutions and it is only present in some very special cases.

Coming back to the square root in (A1), now working in any number of dimensions, the above decomposition allows a straightforward expansion in terms of the small parameter. The result being once again formula (A5).

In what follows we outline the steps needed to determine $V_R$. It will be useful to introduce unitary matrices $U_u$ and $U_d$ which become diagonal sign matrices when the corresponding mass matrices are hermitian. In the notation of (8)

$$U_u = U^*_L U_R, \quad U_d = D^*_L D_R$$  \hspace{1cm} (A8)

Then from (6) and (7) one finds

$$U_u = \frac{1}{m_u} \sqrt{m_u^2 + i s_a t_{2\beta}} \left( t_\beta e^{-i s_a m_u^2} + m_u V_L m_d V_R^\dagger \right)$$  \hspace{1cm} (A9)

$$U_d = \frac{1}{m_d} \sqrt{m_d^2 - i s_a t_{2\beta}} \left( t_\beta e^{i s_a m_d^2} + m_d V_L^\dagger m_u V_R \right)$$  \hspace{1cm} (A10)

One has an additional relation which arises from the definition of the mixing matrices

$$V_L U_d = U_u V_R$$  \hspace{1cm} (A11)

Together with (A9) and (A10) it allows the determination of $V_R$ in terms of $V_L, m_u, m_d, t_{2\beta}$ and $\beta$. Indeed using (A5) one can expand the square roots in $U_u$ and $U_d$ in (A9) and (A10) in powers of $s_a t_{2\beta}$

$$(U_u)_{ij} = (S_u)_{ij} + i s_a t_{2\beta} \left( \frac{t_\beta}{2} (S_u)_{ij} + \frac{(V_L m_d V_R^\dagger)_{ij}}{m_u + \bar{m}_u} \right) + O(s_a^2 t_{2\beta}^2)$$  \hspace{1cm} (A12)

$$(U_d)_{ij} = (S_d)_{ij} - i s_a t_{2\beta} \left( \frac{t_\beta}{2} (S_d)_{ij} + \frac{(V_L^\dagger m_u V_R)_{ij}}{m_d + \bar{m}_d} \right) + O(s_a^2 t_{2\beta}^2)$$  \hspace{1cm} (A13)

where $S_{q_i} = \text{diag}(s_{q_i})$, $\bar{m}_{q_i} = s_{q_i} m_{q_i}$, and $s_{q_i}$ are $\pm$ signs. Using these expressions in (A11) one can find $V_R$ as a power series in $s_a t_{2\beta}$

$$V_R = V_R^{(0)} - i s_a t_{2\beta} V_R^{(1)} + \cdots$$  \hspace{1cm} (A14)

The explicit form of $V_R^{(0)}$ and $V_R^{(1)}$ can be read-off from (11), with $V_R^{(0)} = V_L$ up to sign matrices, and $V_R^{(1)}$ the term in brackets. There are $2^{2n-1}$ independent solutions for $n$ generations, due to the square root nature of (A9) and (A10). The rest is found through $V_L \rightarrow S_u V_L S_d$ and $m_{q_i} \rightarrow s_{q_i} m_{q_i}$, where $S_u = \text{diag}(s_{u_i})$, $S_d = \text{diag}(s_{d_i})$ with, as before, $s_{q_i}$ being $\pm$ signs.

**Appendix B: Higher order terms**

Let us turn again to the two-dimensional prototype case. From (A2) it is straightforward to get higher order terms

$$\left( \frac{1}{m} \sqrt{m^2 + i \epsilon m A} \right)_{ij} \approx \delta_{ij} + i \epsilon \frac{A_{ij}}{m_i + m_j} + \epsilon^2 \frac{A_{ik} m_k A_{kj}}{(m_i + m_k)(m_k + m_j)(m_i + m_j)}$$

$$- i \epsilon^3 \frac{A_{ik} m_k A_{ik} A_{kj} m_k A_{kj}}{(m_i + m_k)(m_k + m_j)(m_i + m_j)(m_i + m_j)} \left( \frac{1}{m_i + m_k} + \frac{1}{m_j + m_k} \right) + O(\epsilon^4)$$  \hspace{1cm} (B1)

This can be shown to holds in general for matrices of any dimension.

We can now proceed to determination of the second and third order of $V_R$ in terms of $s_a t_{2\beta}$. For the sake of space and clarity, we present only contributions of the type $m_u/m_d$ due to the large top to bottom quark masses ratio, since these are the only ones that can compensate for the smallness of $s_a t_{2\beta}$. As we have seen already at the first order, all other terms are sub-leading at this level. For this purpose it is enough we take $U_u \simeq S_u$ and compute only higher orders of $U_d$. 


Using (B1) one has up to the third order
\[
(U_d)_{ij} \simeq (S_d)_{ij} - i s a t_{23} \left( \frac{V^j_i m_u V_R}{\bar{m}_{d_i} + \bar{m}_{d_j}} + \frac{s^2 a t_{23}}{s^2 a t_{23} + \bar{m}_{d_i} + \bar{m}_{d_j}} \left( \frac{V^j_i m_u V_R}{\bar{m}_{d_i} + \bar{m}_{d_j}} \right) \bar{m}_{d_i} + \bar{m}_{d_j} \right) + \frac{s^3 a t_{23}}{s^3 a t_{23} + \bar{m}_{d_i} + \bar{m}_{d_j}} \left( \frac{V^j_i m_u V_R}{\bar{m}_{d_i} + \bar{m}_{d_j}} \right) \bar{m}_{d_i} + \bar{m}_{d_j} \right)
\]  
where \(S_d = \text{diag}(s_{d_i}, \bar{m}_{d_i} = s_{d_i} \bar{m}_{d_i} \text{ and } s_{d_i} \text{ are } \pm \text{ signs. Expressing also } V_R \text{ as a series of powers of } s a t_{23}\)
\[
V_R = V_R^{(0)} - i s a t_{23} V_R^{(1)} - (s a t_{23})^2 V_R^{(2)} + i(s a t_{23})^3 V_R^{(3)} + O(s a t_{23})^4
\]

and using it together with (B2) in (A11) one gets, after equating each order of \(s a t_{23}\), the following recurrence relations
\[
(V_R^{(0)})_{ij} = (S_u V_L S_d)_{ij}
\]  
(B4)

\[
(V_R^{(1)})_{ij} \simeq \frac{(V^j_i m_u V_R^{(0)})_{kj}}{m_{d_k} + \bar{m}_{d_j}}
\]  
(B5)

\[
(V_R^{(2)})_{ij} \simeq \frac{(V^j_i m_u V_R^{(1)})_{kj}}{m_{d_k} + \bar{m}_{d_j}} - \frac{(V^j_i m_u V_R^{(0)})_{kj}}{m_{d_k} + \bar{m}_{d_j}} \left( \frac{m_{d_k} + \bar{m}_{d_j}}{m_{d_k} + \bar{m}_{d_j} + m_{d_j}} \right)
\]  
(B6)

\[
(V_R^{(3)})_{ij} \simeq \frac{(V^j_i m_u V_R^{(2)})_{kj}}{m_{d_k} + \bar{m}_{d_j}} - \frac{(V^j_i m_u V_R^{(1)})_{kj}}{m_{d_k} + \bar{m}_{d_j}} \left( \frac{m_{d_k} + \bar{m}_{d_j}}{m_{d_k} + \bar{m}_{d_j} + m_{d_j}} \right)
\]  
(B7)

After solving these equations one arrives at the final result
\[
(V_R^{(0)})_{ij} = (V_L)_{ij}
\]  
(B8)

\[
(V_R^{(1)})_{ij} \simeq (V_L)_{ik} \left( \frac{V^j_i m_u V_L}{m_{d_i} + \bar{m}_{d_j}} \right) + \frac{m_{d_k}}{m_{d_k} + \bar{m}_{d_j}}
\]  
(B9)

\[
(V_R^{(2)})_{ij} \simeq (V_L)_{ik} \left( \frac{V^j_i m_u V_L}{m_{d_i} + \bar{m}_{d_j}} \right) \left( \frac{m_{d_k}}{m_{d_k} + \bar{m}_{d_j}} \right) \left( \frac{m_{d_k} + \bar{m}_{d_j}}{m_{d_k} + \bar{m}_{d_j} + m_{d_j}} \right)
\]  
(B10)

\[
(V_R^{(3)})_{ij} \simeq (V_L)_{ik} \left( \frac{V^j_i m_u V_L}{m_{d_i} + \bar{m}_{d_j}} \right) \left( \frac{m_{d_k}}{m_{d_k} + \bar{m}_{d_j}} \right) \left( \frac{m_{d_k} + \bar{m}_{d_j}}{m_{d_k} + \bar{m}_{d_j} + m_{d_j}} \right) \left( \frac{m_{d_k} + \bar{m}_{d_j} + m_{d_k}}{m_{d_k} + \bar{m}_{d_j} + m_{d_j}} \right)
\]  
(B11)

The other solutions can be found by changing, in the above expressions, \(V_L \to S_u V_L S_d\) and \(m_{q_i} \to s_{q_i} m_{q_i}\) with \(s_{q_i}\) being \(\pm\) signs. We see once again that a diagonal \(V_L\) implies a diagonal for \(V_R\), as was found at the first order. This confirms that small differences between RH and LH mixing angles are protected by small CKM angles. This fact is supported by the exact result of the equality of mixing angles in the two-generation case. We emphasize the parametrization independence of the above expressions, which should simplify the task of applying our results to physical processes; one can opt for a preferred parametrization.

A few words regarding the convergence of the expansion. We have already given the necessary condition in (12) when discussing the first order terms, but we wish to reassure the reader that the higher order terms respect it. In the above equations there are potentially large terms, that however by inspection are seen to be at most of order one
\[
\max \left| \frac{m_{d_k}}{m_{d_k} + m_{d_j}} \right| \simeq 1
\]  
(B12)

\[
\max \left| \frac{m_{d_k} (m_{d_k} m_{d_j} - m_{d_j} m_{d_k})}{m_{d_k} + m_{d_j}} \right| \simeq 1
\]  
(B13)

and thus the convergence is guaranteed when (12) is satisfied. The convergence of the series was also checked by the comparison with the numerical results.
Appendix C: Zero mixing angles

For the sake of illustration and simplicity, we go to the limit of vanishing LH mixing angles, which imply also diagonal $V_R$. This is useful for the case of small mixings, such as $\theta_{23}$ and $\theta_{13}$ and gives an idea of what happens in the general case with non-zero angles. From (A11), one then finds $V_R$ diagonal with

$$ (V_R)_{ii} = \pm \sqrt{1 - \left( s_{a2\beta} \frac{m_{u_i}^2 - m_{d_i}^2}{2m_{d_i}m_{u_i}} \right)^2 - is_{a2\beta} \frac{m_{u_i}^2 + m_{d_i}^2}{2m_{d_i}m_{u_i}}} \frac{1 + is_{a2\beta} \frac{m_{u_i}^2 + m_{d_i}^2}{2m_{d_i}m_{u_i}}}{1 + is_{a2\beta} \frac{m_{u_i}^2 - m_{d_i}^2}{2m_{d_i}m_{u_i}}} e^{-ia} \quad (C1) $$

The solution is valid when

$$ |s_{a2\beta}| < 2 \frac{|m_{d_i}m_{u_i}|}{|m_{u_i}^2 - m_{d_i}^2|} \quad (C2) $$

where one should take clearly the smallest value. Applied, for example, to the 2-3 generation case, this would give again roughly the limit $2m_b/m_t$ found before. In this case, by expanding in $s_{a2\beta}$ one gets (ignoring corrections up to $m_b/m_t$)

$$ (V_R)_{33} \simeq \pm \sqrt{1 - \left( s_{a2\beta} \frac{1}{2} \frac{m_t}{m_b} \right)^2 - is_{a2\beta} \frac{1}{2} \frac{m_t}{m_b}} \quad (C3) $$

Since in our convention $(V_R)_{33} = e^{i\omega_3}$, this leads to

$$ \sin \omega_3 \simeq -s_{a2\beta} \frac{1}{2} \frac{m_t}{m_b} \quad (C4) $$

As we show in (25), one gets basically the same result in the limit when the mixing angles are small.
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