Gravitational blueshift from a collapsing object

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We discuss a counterintuitive phenomenon of classical general relativity, in which a significant fraction of the radiation emitted by a collapsing object and detected by a distant observer may be blueshifted rather than redshifted. The key-point is that when the radiation propagates inside the collapsing body it is blueshifted, and this time interval may be sufficiently long that the effect can be larger than the later redshift due to the propagation in the vacuum exterior, from the surface of the body to the distant observer.

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The phenomenon of photon redshift is well known in cosmology. A photon propagating in an expanding universe is redshifted by the factor \( a(t_2)/a(t_1) \) from the time \( t_1 \) to the time \( t_2 \), where \( a(t) \) is the scale factor of the Friedmann-Robertson-Walker (FRW) metric. If the universe were contracting, the photon would be blueshifted. In this note, we want to discuss the shift experienced by the photons emitted by a collapsing body. In the homogeneous case, the interior metric in comoving coordinates can be written as the time reversal of the FRW metric, therefore it is the same as the one of a contracting universe and photons are thus blueshifted. The crucial point is whether such photons can retain some blueshift when they reach far away observers. For a spherically symmetric system, the exterior vacuum spacetime is described by the Schwarzschild metric, and it is well known that every photon will be redshifted when it propagates from the surface of the object to the distant observer. The counterintuitive result we show here is that a significant fraction of the radiation emitted by a collapsing object and detected by a distant observer may be blueshifted rather than redshifted. The key-point is that when the radiation propagates inside the body to the distant observer. The energy photon will be redshifted when it propagates from the surface of the object to the distant observer. The proper time of the distant observer is described by

\[
\frac{dR}{dT} = \left(1 - \frac{2M_{\text{Sch}}}{R}\right)^{-\frac{1}{2}}
\]

(1)

where \( \lambda \) is an affine parameter. We define the impact parameter as \( b = L/E \) and it corresponds to the radial coordinate on the image plane of the distant observer. For null geodesics, \( g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda) = 0 \) and the motion of the photon in the vacuum spacetime with respect to the proper time of the distant observer is described by

\[
\left(\frac{dR}{dT}\right)^2 = \left(1 - \frac{2M_{\text{Sch}}}{R}\right)^2 - \left(1 - \frac{2M_{\text{Sch}}}{R}\right)^3 \frac{b^2}{R^2}.
\]

(2)

The spacetime inside the collapsing body is described by a dynamical interior solution (see e.g. Ref. [2]). The most general spherically symmetric metric describing a collapsing cloud of matter in comoving coordinates is given by

\[
ds^2 = -e^{2\nu} dt^2 + \frac{\rho^2}{G} dr^2 + \rho^2 d\Omega^2,
\]

(3)

where \( \nu, \rho, \) and \( G \) are functions of \( t \) and \( r \). The energy-momentum tensor is given by

\[
T^\mu_{\nu} = \text{diag}(\varepsilon(r,t), p_r(r,t), p_\theta(r,t), p_\phi(r,t))
\]

(4)

and Einstein’s equations relate the metric functions to the matter content

\[
p_r = \frac{\dot{F}}{\rho^2 \dot{\rho}}, \quad \varepsilon = \frac{F'}{\rho^2 \dot{\rho}},
\]

(5)

\[
\nu' = \frac{2 p_\theta - p_r \rho'}{\varepsilon + p_r} - \frac{\varepsilon'}{\varepsilon + p_r},
\]

(6)

\[
\dot{G} = 2 \frac{\nu'}{\rho'} \dot{\rho} G,
\]

(7)

where the ’ denotes a derivative with respect to \( r \), and the ´ denotes a derivative with respect to \( t \). The function \( F(r,t) \) is called Misner-Sharp mass, and is

\[
F = \rho(1 - G + e^{-2\nu} \rho^2).
\]

(8)

The case of marginally bound collapse of a cloud of dust is particularly simple. Since \( p_r = p_\theta = 0 \), from the...
first of Eq. (6) it follows that $F$ is a function of $r$ only and the matching to the exterior vacuum Schwarzschild spacetime is always possible. Furthermore Eq. (7) reduces to $\nu' = 0$ and we can always choose the time coordinate in such a way that $\nu = 0$. Integration of Eq. (8) is then trivial and gives $G = 1 + f(r)$ and in the marginally bound collapse case we shall take the free integration function $f$ to be zero.

The whole system has a gauge degree of freedom that can be fixed by setting the scale at a certain time. The usual prescription is that the area radius $\rho(r, t)$ is set equal to the comoving radius $r$ at the initial time $t_i = 0$, $\rho(r, 0) = r$. We can then introduce a scale function $a$

\[
\rho(r, t) = ra(r, t),
\]

that will go from 1, at the initial time, to 0, at the time of the formation of the singularity. The condition to describe collapse is thus given by $\dot{a} < 0$. The regularity of the energy density at the initial time, as seen from Eq. (6), requires that $F(r) = r^3 M(r)$, with $M(r) = \sum_{n=0}^{\infty} M_n r^n$. The energy density can then be written from the second of Eq. (6) as

\[
\varepsilon = \frac{3M + r M'}{a^2 (a + r^2)}.
\]

The Misner-Sharp mass, Eq. (9), takes the form of an equation of motion

\[
\dot{a} = -\sqrt{\frac{M}{a}},
\]

with the minus sign chosen in order to describe a collapse. The integration of Eq. (12) is straightforward and gives

\[
a(r, t) = \left(1 - \frac{3}{2} \sqrt{M_t}ight)^{2/3}.
\]

On the surface of the collapsing object we have $\rho_b(t) = \rho(t, r_b) = R_b(t)$ and $ds_{in}^2 = ds_{out}^2$, so

\[
dt^2 = \left(1 - \frac{2M_{Sch}}{R_b} \right) dt^2 - \frac{dR_b^2}{\left(1 - \frac{2M_{Sch}}{R_b} \right)^{3/2}}.
\]

Marginally bound collapse describes infalling particles that have zero initial velocity at infinity. The surface of the body thus follows a marginally bound time-like geodesic in the Schwarzschild spacetime

\[
d\rho_b = \frac{dR_b}{dt} = -\sqrt{\frac{2M_{Sch}}{R_b}},
\]

where $t$ is the proper time of the collapsing body and coincides with the comoving time coordinate of the interior solution. From Eqs. (14) and (15), we find the relation between the time coordinates $t$ and $\nu$:

\[
\left(\frac{dT}{dt}\right)^2 = \left(1 - \frac{2M_{Sch}}{R_b} \right)^{-2} \left(1 - \frac{2M_{Sch}}{R_b} + \left(\frac{dR_b}{dt}\right)^2 \right) = \left(1 - \frac{2M_{Sch}}{R_b} \right)^{-2}.
\]

The motion of the surface of the body with respect to the proper time of the distant observer is given by

\[
\frac{dR_b}{dT} = \frac{d\rho_b}{dt} \frac{dt}{dT} = -\sqrt{\frac{2M_{Sch}}{R_b}} \left(1 - \frac{2M_{Sch}}{R_b} \right),
\]

which can be integrated to get the time $T$ at which the radial coordinate of the collapsing object has radius $R_b$

\[
T = T_0 - \frac{2}{3} \sqrt{\frac{R_b}{2M_{Sch}}} (R_b + 6M_{Sch}) + \frac{2M_{Sch} \ln \left(\frac{\sqrt{R_b} + \sqrt{2M_{Sch}}}{\sqrt{R_b} - \sqrt{2M_{Sch}}} \right)}{T_0^2}.
\]

To compute the spectrum of the collapsing body seen by the distant observer, we can proceed as follows. We consider the photons on the image plane of the distant observer, which are characterized by their impact parameter $b$, and we integrate backward in time the trajectory of the photon from the distant observer to the surface of the body with the help of Eqs. (3) and (18). After the photon hits the surface, we follow its propagation backward in time in the interior solution. Inside the collapsing body, the metric is axisymmetric but not stationary, and therefore the photon angular momentum $L$ is conserved but the photon energy $E$ is not. We use again the fact that we are considering light-like trajectories, so $ds^2 = 0$, and we write the equations for the coordinates $t$ and $r$

\[
\frac{dr}{d\lambda} = \sqrt{\frac{\rho^2}{\rho^2} + \frac{L^2}{\rho^2}}.
\]

The spectrum at the time $T$ measured by the distant observer is given by

\[
I(T, \nu_{obs}) = \int 2\pi b^2 \, db \int_\gamma g^3 \, j \, dl,
\]

where $\gamma$ is the photon’s path, $j$ is the emissivity per unit volume in the rest frame of the emitter, $g$ is the gravitational redshift

\[
\frac{v_{obs}}{v_e} = \frac{k_{\mu} v_{obs}^\mu}{k_{\mu} v_e^\mu} = \frac{E}{\Delta},
\]

$\nu_{obs}$ is the photon frequency as measured by the distant observer, $\nu_e$ is the photon frequency with respect to the emitter, $v_{obs} = (1, 0, 0, 0)$ is the 4-velocity of the distant observer, $v_e = (1, 0, 0, 0)$ is the 4-velocity of the emitter, and $k^\mu$ is the 4-momentum of the photon. For simplicity, here we assume that the emission is monochromatic with rest-frame frequency $\nu_*$ and proportional to the square of the energy density $\varepsilon$ (as we may expect in a two-body collision)

\[
j = \begin{cases} 
0 & \text{if } R > R_b, \\
\varepsilon^2 \delta (\nu_e - \nu_*) & \text{if } R < R_b.
\end{cases}
\]
Note that $dl$ is the proper length in the rest-frame of the emitter and in our model it turns out to be equal to $dt$

$$dl = \sqrt{3g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}} \, d\lambda = dt.$$  \hspace{1cm} (24)

If we use

$$\frac{dR_b}{d\lambda} = \dot{R}_b \frac{dt}{d\lambda} + R_b \frac{dr}{d\lambda}$$  \hspace{1cm} (25)

and

$$\frac{L^2}{R_b^2} = \left(\frac{dt}{d\lambda}\right)^2 + R_b^2 \left(\frac{dr}{d\lambda}\right)^2$$ \hspace{1cm} (26)

in the equation $(ds^2_{\text{out}})_b = 0$, that is

$$\left(\frac{dT}{d\lambda}\right)^2 = \left(1 - \frac{2M_{\text{Sch}}}{R_b}\right)^{-2} \left[\left(\frac{dR_b}{d\lambda}\right)^2 + \left(1 - \frac{2M_{\text{Sch}}}{R_b}\right) \frac{L^2}{R_b^2}\right],$$  \hspace{1cm} (27)

we find

$$\frac{dT}{d\lambda} = \left(1 - \frac{2M_{\text{Sch}}}{R_b}\right)^{-1} \left(\frac{dt}{d\lambda} - \frac{2M_{\text{Sch}}}{R_b} \frac{dr}{d\lambda}\right) = \frac{dt}{d\lambda} - \left(1 - \frac{2M_{\text{Sch}}}{R_b}\right)^{-1} \sqrt{\frac{2M_{\text{Sch}}}{R_b}} \dot{R}_b,$$ \hspace{1cm} (28)

and eventually we can write $dt/d\lambda$ at the boundary

$$\left(\frac{dt}{d\lambda}\right)_b = \frac{E}{\left(1 - \frac{2M_{\text{Sch}}}{R_b}\right)} \left(1 + \sqrt{\frac{2M_{\text{Sch}}}{R_b}} \sqrt{1 - \left(1 - \frac{2M_{\text{Sch}}}{R_b}\right) \frac{b^2}{R_b^2}}\right).$$ \hspace{1cm} (29)

Here we have the usual gravitational redshift of the Schwarzschild metric, the term $E \left(1 - \frac{2M_{\text{Sch}}}{R_b}\right)^{-1}$, plus a correction due to the non-vanishing velocity of the surface of the collapsing body (Doppler redshift).

Fig. 1 shows the spectra of the collapsing body for different times in the case of a homogeneous and inhomogeneous cloud of dust. In the homogeneous case, we have $M(r) = M_0$, where $M_0$ is a constant determined by the condition $r_b^3 M_0 = 2M_{\text{Sch}}$. In the inhomogeneous case, we have

$$M(r) = M_0 + M_2 r^2,$$ \hspace{1cm} (30)

and we have chosen $M_0 = 0.01$ and $M_2 = -0.00015$ ($M_2 < 0$ to have a higher density at the center of the body and $M_0 + M_2 r_b^2 > 0$ to have positive density). Especially at the beginning of the collapse, when the boundary is larger, we have a quite unexpected fraction of radiation with frequency $\nu_{\text{obs}} > \nu_*$; that is, the radiation has been blueshifted. The blueshift is experienced by the photons that propagate for a sufficiently long time inside the collapsing body. Indeed, if we compute the curves of $g(\lambda)$ and $r(\lambda)$ for the photons with $b = 0$, we find the picture in Fig. 2.

To check that the phenomenon is real and not the product of a numerical error, it is convenient to do the analytical calculation in a simple example, namely the Oppenheimer-Snyder (OS) homogeneous case [4]. We consider in this context a photon emitted on the surface of the collapsing body towards the interior with pure radial negative velocity. Such a photon exits the body from the opposite side and reaches the distant observer with impact parameter $b = 0$. Let $t_1$ and $t_2$ be, respectively, the time of emission and of departure from the interior. In our example the photon trajectory $r(t)$ will have a starting point at $r(t_1) = -r_b$ and it will reach the boundary at $r(t_2) = r_b$. We can then exploit the fact that $ds^2_{\text{in}} = 0$ and find a relation between the scale factor at the time $t_1$ and $t_2$

$$2r_b = \int_{t_1}^{t_2} \frac{dr}{a(\tau)} = \frac{2}{\sqrt{M_0}} \left[\sqrt{a(t_1)} - \sqrt{a(t_2)}\right].$$ \hspace{1cm} (31)

If we call $R_i = a(t_i) r_b$, we have

$$R_1 = \left(\sqrt{R_2} + \sqrt{2M_{\text{Sch}}}\right)^2.$$ \hspace{1cm} (32)

In this simple case, the redshift factor $g$ is given by

$$g = \frac{R_2}{R_1} \left(\frac{dt}{d\lambda}\right)_b = \frac{(R_2 - 2M_{\text{Sch}}) \left(\sqrt{R_2} + \sqrt{2M_{\text{Sch}}}\right)}{R_2^{1/2}},$$ \hspace{1cm} (33)

where the factor $R_2/R_1 > 1$ is the blueshift produced inside the collapsing object and the factor $(dt/d\lambda)_b$ is the redshift experienced during the propagation in the Schwarzschild spacetime, which includes the Doppler redshift caused by the motion of the surface of the collapsing body. The plot of $g(R_2)$ is shown in Fig. 3 and we see that the photon can indeed be blueshifted.
Conclusions — In this note, we have considered the gravitational collapse of a spherically symmetric cloud of dust and we have computed the spectrum of the radiation emitted by this body for a simple emissivity function. The problem is interesting if one wishes to test the observable nature of the “naked singularities” that arise in some theoretical models [5] (see [6] and references therein for a review on gravitational collapse and [7] for issues related to photons emitted from the singularities). The exterior metric is described by the Schwarzschild solution, while the geometry of the interior spacetime is described by the LTB metric. The photons are blueshifted when they propagate inside the collapsing object, and redshifted when they propagate in the vacuum, from the surface of the body to the detector of the distant observer. The counterintuitive phenomenon is that a non-negligible fraction of the radiation reaching far away observers may be blueshifted. The key-point is that some radiation can propagate in the interior spacetime for a sufficiently long time to be significantly blueshifted, so that the subsequent redshift in the vacuum may allow the photon to reach the distant observer with an energy larger than the one at the time of the emission. This phenomenon might be observed in some astrophysical scenarios, for instance in the neutrino spectrum of a supernova.

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FIG. 1. Spectra of a homogeneous (red solid line) and inhomogeneous (green dashed line) collapsing dust clouds at the time $T = 0$ (top left panel), $T = 20$ (top right panel), $T = 40$ (central left panel), $T = 50$ (central right panel), $T = 53$ (bottom left panel), and $T = 55$ (bottom right panel). Time $T$ in units $2M_{\text{Sch}} = 1$; intensity $I$ in arbitrary units. See the text for more details.
FIG. 2. Comoving radial coordinate of the point of the emission of the photon inside the collapsing object (red solid line) and the photon redshift factor $g$ (green dashed line) as a function of the affine parameter $\lambda$ for photons with impact parameter $b = 0$ and reaching the distant observer at the time $T = 0$ (left panel) and $T = 55$ (right panel). Photons emitted at $\lambda = 0$ have $r = r_b$ and they are affected only by the gravitational redshift in the Schwarzschild spacetime and by the Doppler redshift due to the non-vanishing velocity of the surface of the collapsing body. Photons emitted at $r = r_b$ and finite $\lambda$ ($\lambda \approx 35$ for $T = 0$ and 2 for $T = 55$) cross the whole body and exit from the opposite side: they experience the maximum gravitational blueshift and arrive at the distant observer with the highest energies. Time $T$ in units $2M_{\text{Sch}} = 1$. Homogeneous collapse model.

FIG. 3. Gravitational redshift $g$ as a function of $R_2$, for a photon emitted on the surface of the collapsing body with pure radial negative velocity. Such a photon crosses the body and exits from the surface on the opposite side when the radius of the object is $R_2$. As the time of propagation inside the collapsing body can be long, the photon may be detected by the distant observer with a frequency $\nu_{\text{obs}} > \nu_*$, i.e. the photon is blueshifted. $R_2$ in units $2M_{\text{Sch}} = 1$. See the text for more details.