A de Finetti representation theorem for infinite dimensional quantum systems and applications to quantum cryptography

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According to the quantum de Finetti theorem, if the state of an $N$-partite system is invariant under permutations of the subsystems then it can be approximated by a state where almost all subsystems are identical copies of each other, provided $N$ is sufficiently large compared to the dimension of the subsystems. The de Finetti theorem has various applications in physics and information theory, where it is for instance used to prove the security of quantum cryptographic schemes. Here, we extend de Finetti’s theorem, showing that the approximation also holds for infinite dimensional systems, as long as the state satisfies certain experimentally verifiable conditions. This is relevant for applications such as quantum key distribution (QKD), where it is often hard—or even impossible—to bound the dimension of the information carriers (which may be corrupted by an adversary). In particular, our result can be applied to prove the security of QKD based on weak coherent states or Gaussian states against general attacks.

I. INTRODUCTION

Systems studied in physics often consist of a large number of identical subsystems. Examples include any type of matter with the individual molecules as subsystems, or a light field consisting of many modes. Similarly, in the context of quantum information processing, one typically considers settings involving a large number of identical information carriers, such as the photons sent over an optical fiber. In all these cases, the state of the overall system is described by a density operator on a product space $\mathcal{H}^\otimes N$.

A main difficulty when studying large composite systems is that their dimension, and hence the number of parameters needed to describe their state, grows exponentially in the number $N$ of subsystems. This is particularly problematic if one wants to prove that a certain statement holds for all possible states of the system. In the context of quantum information processing, the necessity of such proofs arises, for instance, when analyzing the security of cryptographic protocols. Here, an adversary may maliciously manipulate the information carriers, and security must be guaranteed for any resulting state.

The analysis of large composite quantum systems can be vastly simplified under certain symmetry assumptions, using a quantum version of de Finetti’s classical representation theorem [1] proposed recently in [2]. The theorem states that multi-partite density operators which are invariant under permutations of the subsystems are approximated by convex combinations of density operators which have i.i.d. structure $\sigma^\otimes n$ on most subsystems [3]. I.i.d. states can be easily parametrized (they are characterized by the state $\sigma$ of a single subsystem), and a huge variety of tools are available to handle them, particularly in the area of information theory [4].

In information-theoretic applications, permutation symmetry of the states can often be assumed to hold without loss of generality due to inherent symmetries of the underlying problem or the processing scheme. An important example, which we are going to study in more detail, is quantum key distribution (QKD) [5, 6]. Roughly speaking, QKD is the art of establishing a secret key between two distant parties, traditionally called Alice and Bob, connected only by an insecure quantum channel [37]. Most QKD protocols have the property that $N$ signals are exchanged sequentially between Alice and Bob, but the order in which they are transmitted is irrelevant (as long as Alice and Bob coordinate their communication). One can thus equivalently assume that Alice and Bob reorder the signals according to a randomly chosen permutation [38]. Consequently, even if an adversary manipulates the signals in an arbitrarily malicious way, the $N$-partite density operator describing Alice and Bob’s information is permutation invariant.

The quantum de Finetti theorem now implies that, for assessing the security of a QKD protocol, it is sufficient to consider the special case where the state held by Alice and Bob (after communication over the insecure channel) has i.i.d. structure. This, however, exactly corresponds to the situation arising in a collective attack [1, 2], where the adversary is bound to manipulate each of the transmitted signals independently and identically. For a large class of protocols, security against collective attacks is well understood and explicit formulas for the key rate are known (see, e.g., [9] for the rate of key distillation protocols with one-way communication).

The reduction of security proofs to the special case of collective attacks, however, only works for QKD schemes that use low-dimensional signals. This is because the de Finetti representation for states on product spaces $\mathcal{H}^\otimes N$ is subject to the constraint that the dimension $d$ of the subsystems $\mathcal{H}$ be sufficiently smaller than the number $N$ of subsystems. In particular, the de Finetti representa-
tion generally fails if $\mathcal{H}$ is infinite-dimensional. (There exist explicit examples of permutation invariant states $\rho^N$ on $\mathcal{H}^{\otimes N}$, with $\dim(\mathcal{H}) = N$, such that any reduced state $\rho^k$ on $\mathcal{H}^{\otimes k}$, for $k \geq 2$, is highly entangled and, hence, cannot be approximated by a convex combination of i.i.d. states.)

Here, we show that the restriction of the de Finetti representation to low-dimensional spaces $\mathcal{H}$ can be circumvented under certain experimentally verifiable conditions. More precisely, we prove that for any permutation invariant state on a (possibly infinite-dimensional) system $\mathcal{H}^{\otimes N}$, the reduced state on $\mathcal{H}^{\otimes N'}$, for some $N' \approx N$, is approximated by a mixture of density operators with i.i.d. structure, provided that the outcomes of a measurement applied to a few subsystems lie within a given range. As a specific example, we consider measurements with respect to two canonical observables $X$ and $Y$ on $\mathcal{H} = L^2(\mathbb{R})$. The criterion then is that the outcomes of both the $X$ and the $Y$ measurements have small absolute value.

In practical applications, this criterion is often easily verifiable. For example, in continuous variable quantum cryptography [11, 12, 13, 14, 15, 16, 17, 18, 19], which uses signals in $\mathcal{H} = L^2(\mathbb{R})$, measurements with respect to two canonical observables $X$ and $Y$ are usually already part of the protocol. Our extended version of de Finetti’s theorem then implies that these protocols are secure against the most general attacks, provided they are secure against collective attacks. The latter type of security is already proved for many practical continuous variable schemes (see, e.g., [20], which is based on [21], and [22]).

The remainder of this paper is organized as follows. After introducing some notation and terminology, we start in Section III with the proof of the technical lemmas and theorems. These are the building blocks for the derivation of our main claim that permutation invariant states are approximated by almost i.i.d. states, as described in Section IV A (for a first reading, one may skip Section III and directly start with Section IV A where it is shown how the individual technical claims are combined.) Finally, we discuss how our result can be applied to prove the security of QKD schemes (Section IV B).

II. NOTATION AND DEFINITIONS

A. Symmetry and permutation invariance

Let $S_n$ be the set of permutations on $\{1, \ldots, n\}$ and let $\mathcal{H}$ be a Hilbert space. The symmetric subspace of $\mathcal{H}^{\otimes n}$, denoted $\text{Sym}^n(\mathcal{H})$, consists of all vectors $\Phi \in \mathcal{H}^{\otimes n}$ such that $\sigma \Phi = \Phi$ for all $\sigma \in S_n$. The projector on $\text{Sym}^n(\mathcal{H})$ can be written as

$$P_{\text{Sym}^n(\mathcal{H})} = \frac{1}{n!} \sum_{\sigma \in S_n} \pi \ . \quad (1)$$

We denote by $\mathcal{S}(\mathcal{H})$ the set of density operators on the Hilbert space $\mathcal{H}$. An operator $\rho^n \in \mathcal{S}(\mathcal{H}^{\otimes n})$ is said to be permutation invariant if $\pi \rho^n \pi^\dagger = \rho^n$ for all permutations $\pi$.

B. Restricted symmetric subspaces

Let $\mathcal{H}$ be a subspace of $\mathcal{H}$ and let $k, n \in \mathbb{N}$. We define $P_{\mathcal{H}^{\otimes n}}^{k+n}$ as the projector onto the subspace of $\mathcal{H}^{\otimes k+n}$ spanned by all vectors in $\pi(\mathcal{H}^{\otimes k} \otimes \mathcal{H}^{\otimes n})$, for any $\pi \in S_{k+n}$. The projector $P_{\mathcal{H}^{\otimes n}}^{k+n}$ can be decomposed into projectors $P_0 = P_{\mathcal{H}}$ and $P_1 = P_{\mathcal{H}^\perp}$ onto $\mathcal{H}$ and its orthogonal subspace $\mathcal{H}^\perp$, respectively,

$$P_{\mathcal{H}^{\otimes n}}^{k+n} = \sum_{b \in \{0,1\}^{k+n}} P_b \otimes \cdots \otimes P_{b^{k+n}} , \quad (2)$$

where the sum ranges over all bitstrings $b = (b_1, \ldots, b_{k+n}) \in \{0,1\}^{k+n}$ whose relative frequency of 1s, $f_b := \frac{1}{k} \sum_k b_k , \quad (3)$

is not larger than $\frac{k}{k+n}$.

Because $P_{\mathcal{H}^{\otimes n}}^{k+n}$ is permutation invariant it commutes with any $\pi \in S_{k+n}$ and, hence, also with the projector $P_{\text{Sym}^{k+n}(\mathcal{H})}$ onto the symmetric subspace of $\mathcal{H}^{\otimes k+n}$ (see [11]). This implies that the product $P_{\mathcal{H}^{\otimes n}}^{k+n} P_{\text{Sym}^{k+n}(\mathcal{H})}$ is a projector. In the following, we denote by $\text{Sym}^{k+n}(\mathcal{H}, \mathcal{H}^{\otimes n})$ the support of this projector. The space $\text{Sym}^{k+n}(\mathcal{H}, \mathcal{H}^{\otimes n})$ thus consists of all symmetric vectors that can be written as superpositions of vectors of the form $\pi(\Phi \otimes \Phi)$, for some $\Phi \in \text{Sym}^k(\mathcal{H})$, $\Phi \in \text{Sym}^n(\mathcal{H})$, and $\pi \in S_{k+n}$.

In the special case where $\mathcal{H} = \text{span}\{\nu\}$ is the vector space spanned by a single vector $\nu \in \mathcal{H}$, we also write $\text{Sym}^{k+n}(\mathcal{H}, \nu^{\otimes n})$ instead of $\text{Sym}^{k+n}(\mathcal{H}, \text{span}\{\nu\}^{\otimes n})$ and call its elements $(k+n)\text{-i.i.d. vectors (along } \nu)$. We also say that a density operator $\rho^{k+n}$ is almost i.i.d. if its support is contained in $\text{Sym}^{k+n}(\mathcal{H}, \nu^{\otimes n})$, for some $k \ll n$.

C. Measurements

Let $U$ and $V$ be nonnegative operators on a Hilbert space $\mathcal{H}$ satisfying $U \leq 1$ and $V \leq 1$. We define the function $\gamma_{U \rightarrow V}$ on $[0,1]$ by

$$\gamma_{U \rightarrow V}(\delta) := \sup\{\text{tr}(V \sigma) : \sigma \in \mathcal{S}(\mathcal{H}); \text{tr}(U \sigma) \leq \delta\} . \quad (4)$$

If $U$ and $V$ are POVM elements then $\gamma_{U \rightarrow V}(\delta)$ corresponds to the maximum probability of obtaining outcome $V$ when measuring a state $\sigma$ for which the probability of outcome $U$ is at most $\delta$. 
III. TECHNICAL STATEMENTS

A. Measurement statistics

Let $\mathcal{U} = \{U_0, U_1\}$ and $\mathcal{V} = \{V_0, V_1\}$ be two binary POVMs on $\mathcal{H}$ with the property that $\gamma_{U_i \rightarrow V_1}(\delta)$ is small for small $\delta$. In other words, for any state $\sigma$, outcome 1 of measurement $\mathcal{V}$ has small probability whenever outcome 1 of measurement $\mathcal{U}$ has small probability. Intuitively, we would then expect that the following holds. If $k$ subsystems of a $(k+n)$-partite permutation invariant state are measured according to $\mathcal{U}$, resulting in a low number of outcomes 1, then the number of outcomes 1 when measuring the $n$ remaining subsystems according to $\mathcal{V}$ is small, too. The following lemma makes this intuition more precise.

Lemma III.1. Let $\mathcal{U} = \{U_0, U_1\}$ and $\mathcal{V} = \{V_0, V_1\}$ be POVMs on $\mathcal{H}$, let $n \geq 2k$, and let $(X_1, \ldots, X_{k+n})$ be the $(k+n)$-partite classical outcome of the measurement $\mathcal{U}^\otimes k \otimes \mathcal{V}^\otimes n$ applied to any permutation invariant $\rho^{k+n} \in \mathcal{S}(\mathcal{H}^\otimes k \cdot \mathcal{V}^\otimes n)$. Then, for any $\delta > 0$,

$$\Pr[f_{X_{k+1}, \ldots, X_{k+n}} > \gamma_{U_1 \rightarrow V_1}(f_{X_1, \ldots, X_k} + \delta) + \delta] \leq 8k^2 e^{-kd^2},$$

where $f_X$ denotes the relative frequency of 1s in $X$ (see (3)).

Qualitatively, the statement of Lemma III.1 is a special case of Lemma 4.1 of [24]. For completeness, we give a proof in the appendix, which also yields tighter bounds for the choice of parameters we are interested in.

B. Bounding the probability of projecting into a low-dimensional subspace

In this section, we derive a bound on the quantity $\gamma_{U_1 \rightarrow V_1}$ for the case where $V_1$ corresponds to the predicate that a measurement of $X^2 + Y^2$, for two canonical observables $X$ and $Y$ on $\mathcal{H} = L^2(\mathbb{R})$, is larger than a threshold $n_0$, and where $U_1$ is the predicate that the outcome of a measurement with respect to either $X^2$ or $Y^2$ is at least $\frac{n_0}{2}$.

For any Hermitian operator $Z$ and $z_0 \in \mathbb{R}$ we define $P^{2 \geq z_0}$ as the projector onto the subspace spanned by the eigenspaces of $Z$ corresponding to (generalized) eigenvalues $z \geq z_0$.

Lemma III.2. Let $X$ and $Y$ be two canonical operators $([X, Y] = i)$, $n_0$ a positive integer, and define

$$U_1 := \frac{1}{2} P^{X^2 \geq n_0/2} + \frac{1}{2} P^{Y^2 \geq n_0/2} \quad \text{and} \quad V_1 := P^{X^2 + Y^2 \geq n_0 + 1}.$$

Then $\gamma_{U_1 \rightarrow V_1}(\delta) \leq 4\delta + \frac{4}{c_0 n_0} e^{-n_0 c_0^2}$, with $c_0 = 1 - \frac{1}{\sqrt{2}}$.

Proof. The proof consists of several steps. First, we define an operator $W_1$ and show that $V_1 \leq 2W_1$. Then we show that, up to a constant, $W_1$ is upper bounded by $2U_1$.

Let us start by defining

$$W_1 := \frac{1}{\pi} \int d\mu_{\alpha} |\alpha\rangle\langle\alpha|,$$

where $|\alpha\rangle$ denotes a coherent state and the integral is extended to the complex plane with $|\alpha|^2 \geq n_0$. By expanding $W_1$ in the Fock basis, $|n\rangle_f \propto \alpha^n$, one obtains that $W_1 = \sum q_n |n\rangle_f \langle n|$ with $q_n = \Gamma(n + 1, n_0)/\Gamma(n + 1, 0)$, where $\Gamma$ is the incomplete Gamma function [24]. Since $q_{n+1} \geq q_n > 0$, we can write $V_1 \leq q_{n+1}^{-1} W_1$, where $q_{n+1}^{-1} = \Gamma(n_0 + 1, 0)/\Gamma(n_0 + 1, n_0) < 2$, which concludes the first part.

For the second part, we first extend our Hilbert space to $\mathcal{H}_1 \otimes \mathcal{H}_2$, and show that we can write

$$W_1 = \int dx dy \int \langle 0|U(|x\rangle X \otimes |y\rangle Y \langle y|U\dagger|0\rangle f,$$

where the integral is defined for $x, y \in \mathbb{R}$ with the restriction $x^2 + y^2 \geq n_0$. Here $|f\rangle \in \mathcal{H}_2$, and $|x\rangle X, |y\rangle Y$ denote generalized eigenstates of $X$ and $Y$, respectively. Furthermore, $U = e^{\frac{\pi}{2} (a_1 \hat{a}_1 - a_2 \hat{a}_2)}$ is the so-called beam splitter operator [24], where $a_{1, 2} := (X_{1, 2} + iY_{1, 2})/\sqrt{2}$ are the annihilation operators acting on the first and second system, respectively. This expression for $W_1$ can be derived by showing that $|f_{x,y}\rangle := f_0(0|U_X x \otimes |y\rangle Y = \pi^{-1/2}(\alpha)$, with $\alpha = x + iy$. This, in turn, can be proved by realizing that it is an eigenstate of the annihilation operator,

$$a_1 |f_{x,y}\rangle = 0, a_2^\dagger |f_{x,y}\rangle = 0,$$

where we have used the fact that $f_0(a_2^\dagger a_2) = 0$ and that $U(a_1 + a_2^\dagger) U = X_1 + iY_2$. The normalization factor can be obtained by noting that the integral over the complex plane of $|\alpha\rangle\langle\alpha|$ is $\pi \mathbb{I}$. By looking at the integration domain in (3) it is clear that $W_1 \leq A + B$, where

$$A = \int dx dx' \int \langle 0|U(|x\rangle X \otimes |x'\rangle X \langle x'|U\dagger|0\rangle f,$$

$$B = \int dx dx' \int \langle 0|U(|x'\rangle Y \otimes |x\rangle Y \langle x'|U\dagger|0\rangle f,$$

where the integral is restricted to $|x|^2 \geq n_0/2$ and $-\infty < x < \infty$, and we have used that the integral of $|x'\rangle X \langle x'|$ is equal to that of $|x'\rangle Y \langle x'|$. Using $U_X x \otimes |x\rangle X = (|x + x'|\sqrt{2}) \otimes |x - x'|/\sqrt{2} X \langle x'|0\rangle x|^2 = e^{-x^2}/\sqrt{4}$, and changing variables in the integrals $|z' = (x + x')/\sqrt{2}, z = \sqrt{2}x$ we obtain

$$A = \frac{1}{2\pi} \int dz e^{-(z - x)^2} = F(X).$$

Analogously, $B = F(Y)$. It is straightforward to show that for all $a > 0$, $F(X) \leq P^{X^2 \geq a^2} + F(a)$, and similarly for $F(Y)$. Noting that $F(a) \leq (1/\sqrt{\pi}) e^{-(\sqrt{\pi}a - a)^2}/(\sqrt{\pi}a - a)$, for $a \in [0, \sqrt{\pi}]$, and choosing $a = \sqrt{n_0}/2$ we conclude the proof. \qed
Lemma III.3 below is a corollary of Lemma III.1 and Lemma III.2. It allows to restrict the support of a $(2k + n)$-partite permutation invariant density operator, provided that measurements of the two canonical operators $X$ and $Y$ on $k$ subsystems only result in small values.

**Lemma III.3.** Let $X$ and $Y$ be two canonical operators on $H$, let $n \geq 2k$, let $\mathcal{H}$ be the support of $P^{X^2 + Y^2 \leq \gamma_0}$, for any $\gamma_0 \geq 12 \ln \frac{2(k+n)}{k}$, and let $\rho_{2k+n}$ be a permutation invariant density operator on $\mathcal{H}^{\otimes 2k+n}$. Let $(Z_1, \ldots, Z_k)$ be the outcomes of measurements of $k$ subsystems of $\rho_{2k+n}$ with respect to $X$ and $Y$ (each chosen with probability $\frac{1}{2}$) and let $F$ be the event that the projection $P_{2k+n}^X$ applied to the remaining $k + n$ subsystems fails. Then

$$\Pr[(\max_{i=1}^k Z_i^2 < \frac{n_0}{2}) \land F] \leq 8k^2 \gamma_0 e^{-\frac{\gamma_0}{49(k+n)^2}}.$$ 

**Proof.** Let $U_1$ and $V_1$ be defined as in Lemma III.2. Furthermore, let $X_1, \ldots, X_k$ be the outcomes of the POVM $U^\otimes k \otimes V^\otimes n$ defined by $U = \{1 - U_1, U_1\}$ and $V = \{1 - V_1, V_1\}$, as in Lemma III.1. The probability we want to bound can then be rewritten as

$$\Pr[(\max_{i=1}^k Z_i^2 < \frac{n_0}{2}) \land F] = \Pr[(f_{X^k} = 0) \land (f_{X^n} > \frac{k}{k+n})].$$

where $f_{X^k}$ and $f_{X^n}$ are the frequencies of 1s in the tuples $X^k = (X_1, \ldots, X_k)$ and $X^n = (X_{k+1}, \ldots, X_{k+n})$, respectively. With $\delta := \frac{k}{\gamma_0(k+n)}$, we have

$$\gamma_{U_1 \to V_1}(\delta) + \delta \leq 5\delta + \frac{8}{\sqrt{n_0}} e^{-\frac{n_0}{49(k+n)^2}} \leq \frac{k}{k+n},$$

and, hence, the probability above can be bounded by

$$\Pr[(f_{X^k} = 0) \land (f_{X^n} > \frac{k}{k+n})] \leq \Pr[(f_{X^k} = 0) \land (f_{X^n} > \gamma_{U_1 \to V_1}(f_{X^k} + \delta) + \delta)] \leq \Pr[f_{X^n} > \gamma_{U_1 \to V_1}(f_{X^k} + \delta) + \delta].$$

The claim then follows from Lemma III.1.

**Remark III.4.** It is straightforward to generalize Lemma III.3 to other measurements, specified by an arbitrary POVM $M = \{M_z\}_{z \in Z}$. The condition $\max_i Z_i^2 < \frac{n_0}{2}$ may then be replaced by the requirement that the outcomes $Z_i$ are contained in a certain set $\bar{Z} \subseteq Z$ such that for any $\delta > 0$

$$\gamma_{U_1 \to P_{\bar{H}}}(\delta) \leq O(\delta),$$

where $U_1 := \sum_{z \in \bar{Z}} M_z$ and where $P_{\bar{H}}$ denotes the projection onto the subspace orthogonal to a finite-dimensional subspace $\mathcal{H}$, which may be chosen depending on $\delta$. For the considerations below (Section IV A), however, the dimension $d$ of $\mathcal{H}$ needs to be bounded by $d \leq O(\delta^{-\frac{2}{7}})$, so that $d \leq O((\frac{n_0}{2})^{\frac{7}{4}})$.

### C. Purification in restricted symmetric subspaces

The de Finetti type statements formulated in Section III.4 below apply to states on the symmetric subspace. The following lemma, which is a generalization of Lemma 4.3 of [3] (see also [10]), allows to extend these statements to general permutation invariant density operators.

**Lemma III.5.** Let $\mathcal{H}$ be a subspace of $H$ and let $\rho_{2k+n} \in \mathcal{S}(\mathcal{H}^{\otimes 2k+n})$ be permutation invariant with support contained in the support of $P_{2k+n}^X$. Then there exists a purification of $\rho_{2k+n}$ on $\text{Sym}^{2k+n}(H \otimes (H \otimes H)^{\otimes n})$. 

**Proof.** Let $\{e_j\}_{j \in J}$ be an orthonormal basis of $\mathcal{H}$ such that $\{e_j\}_{j \in K}$, for some $K \subseteq J$, is a basis of $\mathcal{H}$. We can then define a vector $\Phi \in (\mathcal{H} \otimes \mathcal{H})^{\otimes 2k+n}$ by

$$\Phi = \sum_{j \in J} (\rho_{2k+n} \otimes \mathbb{1}_{\mathcal{H}}^{\otimes 2k+n}) e_j \otimes e_j$$

where, for any $j \in (j_{2k+n}) \in J^{2k+n}$, 

$$e_j = \sum_{j \in J} (\rho_{2k+n} \otimes \mathbb{1}_{\mathcal{H}}^{\otimes 2k+n}) e_j \otimes e_j.$$

The state defined by $\Phi$ is obviously a purification of $\rho_{2k+n}$. Furthermore, because $\rho_{2k+n}$ is permutation invariant, we have for any $\pi \in S_{2k+n}$

$$\pi(\otimes \pi) \Phi = \pi(\otimes \pi) \sum_{j \in J^{2k+n}} (\rho_{2k+n} \otimes \mathbb{1}_{\mathcal{H}}^{\otimes 2k+n}) e_j \otimes e_j$$

and, hence, $\Phi \in \text{Sym}^{2k+n}(H \otimes H)$. It thus remains to verify that $\Phi$ is an element of the support of $P_{2k+n}^{(H \otimes H)^{\otimes n}}$. Since $\rho_{2k+n}$ is contained in the support of $P_{2k+n}^{(H \otimes H)^{\otimes n}}$, the sum in (6) can be restricted to terms such that $e_j$ lies in the support of $P_{2k+n}^{(H \otimes H)^{\otimes n}}$, too (or, equivalently, the tuple $j$ has at most $k$ entries outside $J$). This implies that $\Phi$ lies in the support of $P_{2k+n}^{(H \otimes H)^{\otimes n}} \otimes P_{2k+n}^{(H \otimes H)^{\otimes n}}$. The assertion then follows because this support is contained in the support of $P_{2k+n}^{(H \otimes H)^{\otimes n}}$.

### D. An extended de Finetti-type theorem

The purpose of this section is to derive a de Finetti-type theorem for states on the symmetric subspace of product spaces with possibly infinite-dimensional subsystems (Theorem III.7). We start, however, with a de Finetti-type statement for finite dimensions (Lemma III.6). It can be seen as a strengthened version of the exponential de Finetti theorem proposed in [3]. The claim is that any $(2k + n)$-partite symmetric vector $\Phi$ is approximated by a superposition of vectors that are $(k+n)$-i.i.d. on $k + n$ subsystems. We note that, in contrast, the approximation in [3] has the form of a convex
combination of \((k+n)\)-i.i.d. states. A second difference between Lemma III.6 and the result of [3] is that we use the overlap (i.e., the scalar product between vectors) instead of the trace distance to quantify the quality of the approximation (this slightly simplifies the argument below).

**Lemma III.6.** Let \(\mathcal{H}\) be a d-dimensional Hilbert space and let \(k, n \in \mathbb{N}\). There exists an isometry \(U\) from \(\mathcal{H}^\otimes k\) to a Hilbert space \(\mathcal{H}'\) with orthonormal basis \(\{f_\nu\}_{\nu \in \mathcal{V}}\), where \(\mathcal{V}\) is a finite set of unit vectors \(\nu \in \mathcal{H}\), such that the following holds. For any unit vector \(\Phi \in \text{Sym}^{2k+n}(\mathcal{H})\) there exists a unit vector \(\hat{\Phi} \in \mathcal{H}' \otimes \text{Sym}^{k+n}(\mathcal{H})\) of the form

\[
\hat{\Phi} = \sqrt{\frac{1}{|\mathcal{V}|}} \sum_{\nu \in \mathcal{V}} f_\nu \otimes \hat{\Phi}_\nu
\]

with \(\hat{\Phi}_\nu \in \text{Sym}^{k+n}(\mathcal{H}, \nu^\otimes n)\) such that

\[
\langle \hat{\Phi} | (U \otimes \mathbb{1}^\otimes k) | \Phi \rangle > 1 - k^d e^{-\frac{k(k+1)}{2k+n}}. \tag{8}
\]

Note that (8) can be rewritten in terms of the fidelity \(F(\cdot, \cdot)\) as

\[
F(\hat{\Phi}, (U \otimes \mathbb{1}^\otimes k) | \Phi \rangle > 1 - k^d e^{-\frac{k(k+1)}{2k+n}}. \tag{9}
\]

The de Finetti theorem of [3] (Theorem 4.3.2) can then be obtained by taking the partial trace over \(\mathcal{H}'\) in both arguments of \(F(\cdot, \cdot)\) and converting the fidelity into a trace distance.

**Proof.** The unitary group acts irreducibly on the subspace \(\text{Sym}^k(\mathcal{H})\). Hence, by Schur's lemma,

\[
\int U^\otimes k (|\nu_0\rangle \langle \nu_0|)^\otimes k (U^\otimes k)^\otimes k \omega(U) = \frac{1}{\dim(\text{Sym}^k(\mathcal{H}))} P_{\text{Sym}^k(\mathcal{H})}
\]

where \(\nu_0\) is an arbitrary unit vector in \(\mathcal{H}\) and where \(\omega\) is the Haar measure on the set of unitaries on \(\mathcal{H}\). Note that the integral on the left hand side can be approximated to any accuracy by a sum over a finite set \(\mathcal{V}\) of unit vectors \(\nu \in \mathcal{H}\). That is, for any \(\mu > 0\) there exists a finite set \(\mathcal{V}\) such that

\[
\left\| \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} (|\nu\rangle \langle \nu|)^\otimes k - \frac{1}{\dim(\text{Sym}^k(\mathcal{H}))} P_{\text{Sym}^k(\mathcal{H})} \right\| \leq \mu.
\]

Let \(\mathcal{H}'\) a Hilbert space with orthonormal basis \(\{f_\nu\}_{\nu \in \mathcal{V}}\) and define the linear map \(\tilde{U}\) from \(\mathcal{H}^\otimes k\) to \(\mathcal{H}'\) by

\[
\tilde{U} := \sqrt{\frac{1}{\dim(\text{Sym}^k(\mathcal{H}))}} \sum_{\nu \in \mathcal{V}} f_\nu \langle \nu^\otimes k |.
\]

We then have

\[
\tilde{U}^\dagger \tilde{U} = \frac{\dim(\text{Sym}^k(\mathcal{H}))}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} |\nu^\otimes k\rangle \langle \nu^\otimes k|,
\]

and, consequently,

\[
\left\| \tilde{U}^\dagger \tilde{U} - P_{\text{Sym}^k(\mathcal{H})} \right\| \leq \mu.
\]

In particular, \(\tilde{U}\) can be made arbitrarily close to the isometry \(U := \tilde{U}(\tilde{U}^\dagger \tilde{U})^{-\frac{1}{2}}\) on \(\text{Sym}^k(\mathcal{H})\), i.e., for any \(\mu' > 0\) there exists a finite set \(\mathcal{V}\) such that

\[
\left\| \tilde{U} - U \right\| \leq \mu'.
\]

It thus remains to be shown that inequality (8) holds for \(\tilde{U}\) (because it then also holds for the isometry \(U\), provided \(\mu'\) is sufficiently small).

By the definition of \(\tilde{U}\), the vector \((\tilde{U} \otimes \mathbb{1}^\otimes k) | \Phi \rangle\) can be written as

\[
(\tilde{U} \otimes \mathbb{1}^\otimes k) | \Phi \rangle = \sqrt{\frac{1}{|\mathcal{V}|}} \sum_{\nu \in \mathcal{V}} f_\nu \otimes \Phi_\nu,
\]

where

\[
\Phi_\nu := \sqrt{\text{dim}(\text{Sym}^k(\mathcal{H}))} (\langle \nu^\otimes k \otimes |k+n\rangle) | \Phi \rangle \in \mathcal{H}^\otimes k+n.
\]

We now define the vector \(\hat{\Phi}\) by choosing each \(\hat{\Phi}_\nu\) of the sum (7) as the projection of \(\Phi_\nu\) onto the subspace \(\text{Sym}^{k+n}(\mathcal{H}, \nu^\otimes n)\),

\[
\hat{\Phi}_\nu := P_{\text{Sym}^{k+n}(\mathcal{H}, \nu^\otimes n)} | \Phi \rangle.
\]

Note that the length of the resulting vector \(\hat{\Phi}\) is generally smaller than 1. However, the statement for unit vectors can be obtained by normalizing \(\hat{\Phi}\) (because the normalization can only increase the overlap).

Condition (8) (with \(U\) replaced by \(\tilde{U}\)) can now be rewritten as

\[
\left\| \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} (\Phi_\nu| P_{\text{Sym}^{k+n}(\mathcal{H}, \nu^\otimes n)} | \Phi_\nu \rangle \right\| > 1 - k^d e^{-\frac{k(k+1)}{2k+n}}, \tag{9}
\]

or, equivalently, as

\[
\left\| \frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} (\Phi_\nu| P_{\text{Sym}^{k+n}(\mathcal{H}, \nu^\otimes n)} \right\| < k^d e^{-\frac{k(k+1)}{2k+n}}, \tag{10}
\]

because \(\frac{1}{|\mathcal{V}|} \sum_{\nu \in \mathcal{V}} (\Phi_\nu| \Phi_\nu \rangle \geq 1 - \mu\), for any \(\mu > 0\).

A straightforward calculation (cf. Eq. (4.12) of [3] or the supplementary material of [2]) for a similar but more detailed argument) shows that, for any vector \(\Psi \in \text{Sym}^{2k+n}(\mathcal{H})\),

\[
\langle \Psi | (|\nu\rangle \langle \nu|)^\otimes k \otimes P_{\text{Sym}^{k+n}(\mathcal{H}, \nu^\otimes n)} | \Psi \rangle \leq \frac{(k+n)(k+1)}{2k+n} e^{-\frac{k(k+1)}{2k+n}}
\]

Applying this bound to the individual terms in the sum (10) gives

\[
\langle \Phi_\nu | P_{\text{Sym}^{k+n}(\mathcal{H}, \nu^\otimes n)} | \Phi_\nu \rangle \leq \text{dim}(\text{Sym}^k(\mathcal{H})) e^{-\frac{k(k+1)}{2k+n}} \leq k^d e^{-\frac{k(k+1)}{2k+n}}.
\]

This implies (10) and thus concludes the proof.
Based on Lemma III.6, we now derive a de Finetti-type theorem that applies to states $\rho^{4k+n}$ on the symmetric subspace $\text{Sym}^{4k+n}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^{k+n})$, where $\mathcal{H}$ is a finite-dimensional subspace of a possibly infinite-dimensional Hilbert space $\mathcal{H}$. The claim is that, when tracing out the first $2k$ subsystems, the resulting state $\rho^{2k+n} = \text{tr}_{2k}(\rho^{4k+n})$ is close to a convex combination of $\rho^{2k+n}_\nu$, i.i.d. states $\rho^{2k+n}_\nu$. Here, closeness is measured in terms of the fidelity $F(\cdot, \cdot)$.

**Theorem III.7.** Let $\mathcal{H}$ be a d-dimensional subspace of a Hilbert space $\mathcal{H}$, let $n, k \in \mathbb{N}$, and let $\rho^{4k+n}$ be a density operator on $\text{Sym}^{4k+n}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^{k+n})$. Then there exists a probability distribution $p_\nu$ on a finite set $\mathcal{V}$ of unit vectors $\nu \in \mathcal{H}$ and a family $\{\rho^{2k+n}_\nu\}_{\nu \in \mathcal{V}}$ of density operators on $\text{Sym}^{2k+n}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^{k+n})$ such that

$$F(\rho^{2k+n}, \sum_{\nu \in \mathcal{V}} p_\nu \rho^{2k+n}_\nu) > 1 - k^d e^{-\frac{1}{2(2k+n)}}. \quad (11)$$

**Proof.** It suffices to prove the claim for $\rho^{4k+n}$ pure; the statement for general density operators follows by the joint concavity of the fidelity (see, e.g., Chapter 9 of [1]). Let $\Psi \in \text{Sym}^{4k+n}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^{k+n})$. The idea is to write $\Psi$ as a superposition of vectors $\Psi_{j, j'}$ which have at least $2k + n$ subsystems contained in $\mathcal{H}$ (see [15] and [16] below) so that we can apply Lemma III.6 to each of them individually.

Consider the decomposition of $\rho^{2k+n}_{\mathcal{H} \otimes \mathbb{C}^{k+n}}$ according to [2], i.e.,

$$P_{\mathcal{H} \otimes \mathbb{C}^{k+n}}^{2k+n} = \sum_{b \in \{0, 1\}^{2k+n}} P_{b_1} \otimes \cdots \otimes P_{b_{2k+n}} \quad (12)$$

with $P_0 = P_{\mathcal{H}}$ and $P_1 = P_{\mathcal{H}}$.

Furthermore, let $\{e_j\}_{j \in J}$ be a common eigenbasis of the projectors $P_0$ and $P_1$, let $J_0 := J \cup \{0\}$ (assuming that $0 \notin J$), and define the projectors $Q_j$, for $j \in J_0$, by

$$Q_j = \begin{cases} P_0 & \text{if } j = 0 \\ |e_j\rangle \langle e_j| & \text{if } j \in J. \end{cases}$$

Then, starting from (12), it is easy to construct a decomposition of $P_{\mathcal{H} \otimes \mathbb{C}^{k+n}}^{2k+n}$ into mutually orthogonal projectors $Q_{j, j'} = Q_{j_1} \otimes \cdots \otimes Q_{j_{2k+n}}$ for $j' = (j_1, \ldots, j_{2k+n})$,

$$P_{\mathcal{H} \otimes \mathbb{C}^{k+n}}^{2k+n} = \sum_{j' \in \{0, 1\}^{2k+n}} Q_{j'} \quad (13)$$

where $J_{k+n}$ is a subset of $\{0\}^{2k+n}$ containing only tuples $j'$ with exactly $k+n$ indices $\tau$ such that $j'_\tau = 0$. Similarly, we can decompose $P_{\mathcal{H} \otimes \mathbb{C}^{k+n}}^{2k}$ in projectors $Q_j = Q_{j_1} \otimes \cdots \otimes Q_{j_{2k}}$, for $j = (j_1, \ldots, j_{2k})$,

$$P_{\mathcal{H} \otimes \mathbb{C}^{k+n}}^{2k} = \sum_{j \in J_{k,+}} Q_{j_1} \otimes \cdots \otimes Q_{j_{2k}} \quad (14)$$

where $J_{k,+}$ only consists of tuples $j \in J_{k,+}$ with exactly $k$ indices $\tau$ such that $j_\tau = 0$.

By definition, $\text{Sym}^{4k+n}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^{k+n})$ is contained in the support of $P_{\mathcal{H} \otimes \mathbb{C}^{k+n}}^{2k+n}$, which is itself contained in the support of $P_{\mathcal{H} \otimes \mathbb{C}^{k+n}}^{2k}$ $\mathcal{H} \otimes \mathcal{H} \otimes \mathbb{C}^{k+n}$. Hence, any vector $\Psi \in \text{Sym}^{4k+n}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^{k+n})$ can be written as a superposition

$$\Psi = \sum_{j \in J_{k,+}} \Psi_j \quad (15)$$

where, for any $j \in J_{k,+}$,

$$\Psi_j = (Q_j \otimes 1^{\otimes 2k+n}) \Psi = \sum_{j' \in J_{k+n}} \Psi_{j, j'} \quad (16)$$

with probabilities $p_j = \text{tr}(\Psi_j^\dagger \Psi_j)$ and density operators

$$\rho_{j, j'}^{2k+n} = \text{tr}_{2k}(\Psi_j^\dagger \Psi_j)$$

where $\Psi_j$ is a unit vector parallel to $\Psi_j$. Because of the joint concavity of the fidelity, it is thus sufficient to show that (11) holds for all density operators $\rho_{j, j'}^{2k+n}$.

Let $j \in J_{k,+}$ be fixed and let, for any $j' \in J_{k+n}$, $\Psi_{j, j'}$ be a normalization of $\Psi_{j, j'}$. We then have

$$\Psi_{j, j'} = \sum_{j' \in J_{k+n}} a_{j, j'} \Psi_{j, j'}$$

where $a_{j, j'}$ are coefficients satisfying $\sum_{j' \in J_{k+n}} |a_{j, j'}|^2 = 1$. We now apply Lemma III.6 to each of the vectors $\Psi_{j, j'}$ in the sum individually. For this, assume without loss of generality that $j = (j_1, \ldots, j_k, 0, \ldots, 0)$ and $j' = (0, 0, j_{k+1, n+1}, \ldots, j_{2k+n})$ with $j_1, \ldots, j_k, j_{k+1, n+1}, \ldots, j_{2k+n} \in J$ (this form can always be obtained by an appropriate reordering of the subsystems). The vector $\Psi_{j, j'}$ can then be written as

$$\Psi_{j, j'} = e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes \Phi_{j, j'} \otimes e_{j_{k+1, n+1}} \otimes \cdots \otimes e_{j_{2k+n}}$$

where $\Phi_{j, j'} \in \text{Sym}^{2k+n}(\mathcal{H})$. According to Lemma III.6 there exists a vector $\hat{\Phi}_{j, j'}$ of the form

$$\hat{\Phi}_{j, j'} = \sqrt{\frac{1}{|J_{k,+}|}} \sum_{\nu \in \mathcal{V}} f_\nu \otimes \hat{\Phi}_{j, j', \nu} \in \mathcal{H} \otimes \text{Sym}^{2k+n}(\mathcal{H})$$

with $\hat{\Phi}_{j, j', \nu} \in \text{Sym}^{k+n}(\mathcal{H}, \nu \otimes \mathbb{C}^{k+n})$ such that

$$\langle \hat{\Phi}_{j, j', \nu} | (U \otimes 1^{\otimes k+n}) \hat{\Phi}_{j, j'} \rangle > 1 - k^d e^{-\frac{1}{2(2k+n)}}.$$
where $U$ is some fixed isometry (independent of $j'$). With the definition

$$
\hat{\Psi}_{j,j'} = e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes \hat{\Psi}_{j,j'} \otimes e_{j_{k+1}} \otimes \cdots \otimes e_{j_{2k+n}}
$$

this immediately implies

$$
\langle \hat{\Psi}_{j,j'} | (\mathbb{1} \otimes U \otimes \mathbb{1} \otimes^{2k+n}) \hat{\Psi}_{j,j'} \rangle > 1 - k^d e^{-\frac{k(k+1)}{2k+n}}.
$$

Consider now the vector

$$
\hat{\Psi}_j := \sum_{j' \in J_{k+n}} \alpha_{j,j'} \hat{\Psi}_{j,j'}.
$$

Since, for any two distinct $j', j'' \in J_{k+n}$, the projectors $Q_j$ and $Q_{j''}$ are mutually orthogonal by definition, we have

$$
\langle \hat{\Psi}_{j,j'} | (\mathbb{1} \otimes U \otimes \mathbb{1} \otimes^{2k+n}) \hat{\Psi}_{j,j'} \rangle = \langle \hat{\Psi}_{j,j''} | (\mathbb{1} \otimes U \otimes \mathbb{1} \otimes^{2k+n}) \hat{\Psi}_{j,j''} \rangle = 0.
$$

Combining this with the above, we find

$$
\langle \hat{\Psi}_j | (\mathbb{1} \otimes U \otimes \mathbb{1} \otimes^{2k+n}) \hat{\Psi}_j \rangle = \sum_j |\alpha_{j,j'}|^2 \langle \hat{\Psi}_{j,j'} | (\mathbb{1} \otimes U \otimes \mathbb{1} \otimes^{2k+n}) \hat{\Psi}_{j,j'} \rangle > 1 - k^d e^{-\frac{k(k+1)}{2k+n}}.
$$

This inequality can be rewritten in terms of the fidelity, which is simply the absolute value of the scalar product. Together with the fact that tracing out subsystems can only increase the fidelity, we obtain

$$
F(\text{tr}_{2k}(\hat{\Psi}_j), \text{tr}_{H',k}(\hat{\Psi}_j)) > 1 - k^d e^{-\frac{k(k+1)}{2k+n}}.
$$

Furthermore, because the density operator $\text{tr}_{2k}(\hat{\Psi}_j)$ is contained in the symmetric subspace $\text{Sym}^{2k+n}(\mathcal{H})$, we can insert a projection onto this subspace without changing the fidelity, i.e.,

$$
F(\text{tr}_{2k}(\hat{\Psi}_j), \rho_j^{2k+n}) > 1 - k^d e^{-\frac{k(k+1)}{2k+n}},
$$

where

$$
\rho_j^{2k+n} := P_{\text{Sym}^{2k+n}(\mathcal{H})} \text{tr}_{k,H'}(\hat{\Psi}_j) P_{\text{Sym}^{2k+n}(\mathcal{H})}.
$$

It remains to verify that the density operator $\rho_j^{2k+n}$ is of the desired form

$$
\rho_j^{2k+n} = \sum_{\nu \in \mathcal{V}} p_{\nu} \rho_{j,\nu}^{2k+n}
$$

(17)

for some appropriately chosen probabilities $p_{\nu}$ and for $\rho_{j,\nu}^{2k+n}$ contained in the subspace $\text{Sym}^{2k+n}(\mathcal{H}, \nu^{\otimes n})$. For this, we define

$$
\rho_{j,\nu}^{2k+n} := P_{\text{Sym}^{2k+n}(\mathcal{H})} \hat{\Psi}_{\nu,\nu} P_{\text{Sym}^{2k+n}(\mathcal{H})}
$$

where $P_{\hat{\Psi}_{\nu,\nu}}$ denotes the projector onto the vector

$$
\hat{\Psi}_{\nu,\nu} := \langle e_{j_1} \otimes \cdots \otimes e_{j_k} \otimes f_{\nu} | \hat{\Psi}_j \rangle.
$$

Identity (17) then follows from the orthogonality of the vectors $f_{\nu}$. Furthermore, by the definition of $\hat{\Psi}_j$ and using the fact that the vectors $\hat{\Psi}_{j,j',\nu}$, for any fixed $\nu$ and arbitrary $j'$, are contained in the support of $P_{\nu^{\otimes n}}^{2k+n}$, one can readily verify that the vector $\rho_{j,\nu}$ is contained in the support of $P_{\nu^{\otimes n}}^{2k+n}$. Consequently, $\rho_j^{2k+n}$ lies in the subspace $\text{Sym}^{2k+n}(\mathcal{H}, \nu^{\otimes n})$.

E. Properties of almost i.i.d. states

Theorem III.7 gives an approximation of permutation invariant states in terms of almost i.i.d. states $\rho_{\nu}$. The significance of this approximation comes from the fact that such states are relatively easy to handle. In particular, their properties very much resemble the properties of (perfect) i.i.d. states $\rho_{\nu}$. For example, the entropy of an almost i.i.d. state $\rho_{\nu}$ is well approximated by the entropy of the corresponding perfect i.i.d. state.

Of particular interest for information-theoretic applications is the smooth min-entropy $\min S[3, 20]$. Let $\rho_{XB}$ be a density operator on $\mathcal{H}_X \otimes \mathcal{H}_B$ which is classical on $\mathcal{H}_X$, i.e.,

$$
\rho_{XB} = \sum_{x \in \mathcal{X}} p_x |e_x\rangle \langle e_x| \otimes \rho_B^x,
$$

for some orthonormal basis $\{e_x\}_{x \in \mathcal{X}}$ of $\mathcal{H}_X$, probabilities $p_x$, and density operators $\rho_B^x$ on $\mathcal{H}_B$. Then, for any $\varepsilon \geq 0$, the $\varepsilon$-smooth entropy of $X$ given $B$, denoted $H_{\min}^\varepsilon(X | B)_\rho$, corresponds to the amount of uniform randomness (relative to $B$) that can be extracted from $X$ by two-universal hashing [21]. (The smoothness parameter $\varepsilon$ quantifies the quality of the resulting randomness in terms of their distance to a random variable which is perfectly uniform and independent of $B$).

For an i.i.d. state $\rho_{XB}^{\otimes N}$, the smooth min-entropy $H_{\min}^\varepsilon$ is asymptotically (for large $N$) equal to the von Neumann entropy $S$, i.e.,

$$
\frac{1}{N} H_{\min}^\varepsilon(X^N | B^N)_{\rho^{\otimes N}} \approx \frac{1}{N} (S(\rho_X^{\otimes N}) - S(\rho_B^{\otimes N})) = S(\rho_B) - S(\rho_B | X) = S(X | B).
$$

The following theorem from [3] extends (one direction of) this relation to almost i.i.d. states.

Theorem III.8. Let $\rho_X^{k+n} \otimes B^{k+n}$ be a density operator on $(\mathcal{H}_X \otimes \mathcal{H}_B)^{\otimes k+n}$ which is classical on the subsystems $\mathcal{H}_X$ and let $\varepsilon > 0$. If there exists a purification of $\rho_X^{k+n} \otimes B^{k+n}$ in $\text{Sym}^{k+n}(\mathcal{H}_X \otimes \mathcal{H}_B \otimes \mathcal{H}_R, \nu^{\otimes n})$, for some $\nu \in \mathcal{V} \otimes \mathcal{H}_B \otimes \mathcal{H}_R$, then

$$
\frac{1}{n} H_{\min}^\varepsilon(X^{k+n} | B^{k+n})_{\rho^{k+n}} \geq S(\sigma_{XB}) - S(\sigma_B) - \delta,
$$

where $\sigma_{XB}$ denotes the reduced density operator on $\mathcal{H}_X \otimes \mathcal{H}_B$.
where $\sigma_{XB} := \text{tr}_R(\ket{\nu}\bra{\nu})$, 
\[
\delta := 5(\ln(\dim \mathcal{H}_X) + 1)\sqrt{\frac{2\ln(4/\varepsilon)}{k+n}} + h\left(\frac{k}{k+n}\right),
\]
and $h(p) \equiv -p \ln p - (1-p) \ln(1-p)$.

Note that the statement depends on the dimension of $\mathcal{H}_X$, but is independent of the dimension of $\mathcal{H}_B$.

IV. IMPLICATIONS

A. Putting things together

The aim of this section is to demonstrate how the technical statements of Section III can be combined to give our main claim, namely that any permutation invariant state $\rho^N$ on $\mathcal{H}^\otimes N$ is approximated by a mixture of states with almost i.i.d. structure, provided the outcomes of certain measurements on a (small) sample of the subsystems lie in a given range. To illustrate this, we assume for concreteness that $\mathcal{H} = L^2(\mathbb{R})$ and that measurements on $k$ subsystems are carried out with respect to two canonical observables $X$ and $Y$, each chosen with probability $\frac{1}{2}$. (According to Remark III.3 the argument below can easily be extended to more general measurements.) Furthermore, we assume that $N = m^4$ and $k = m^3$, for some $m \in \mathbb{N}$.

Let $d = m^\frac{3}{2}$ and let $\mathcal{H}$ be the support of $p^{X^2 + Y^2} \leq n_0$ for some $n_0 \in \mathbb{N}$ such that $12\ln(7m) \leq n_0 \leq d$. We first apply Lemma III.3 to infer that, if all $k$ measurement outcomes $z_1, \ldots, z_k$ satisfy $z_i^2 \leq \frac{d}{k}$ then the state $\rho^{(m-1)k}$ on the remaining $(m-1)k$ subsystems is almost certainly contained in the support of $\rho_{\mathcal{H}^\otimes (m-2)k}$. Hence, according to Lemma III.5 there exists a purification $\tilde{\rho}^{(m-1)k}$ of $\rho^{(m-1)k}$ on $\text{Sym}^{(m-1)k}(\mathcal{H} \otimes \mathcal{H}, (\mathcal{H} \otimes \mathcal{H})^{\otimes (m-3)k})$. Theorem III.7 now provides an approximation of the reduced state $\tilde{\rho}^{(m-5)k}$ in terms of a mixture of almost i.i.d. states $\rho_{\nu}^{(m-5)k}$, parametrized by $\nu \in \mathcal{H} \otimes \mathcal{H}$. More precisely, each density operator $\rho_{\nu}^{(m-5)k}$ is contained in $\text{Sym}^{(m-5)k}(\mathcal{H} \otimes \mathcal{H}, \nu^{\otimes (m-9)k})$, and their convex combination is exponentially (in $m$) close to $\tilde{\rho}^{(m-5)k}$. In particular, by taking the trace over the purifying systems, we conclude that the reduced state $\rho^{(1-\mu)N}$ is approximated by a mixture of states that have i.i.d. structure on $(1 - \mu - \mu')N$ subsystems, where $\mu = 5N^{-\frac{3}{4}}$ and $\mu' = 4N^{-\frac{1}{2}}$.

B. Application to QKD

A main application of de Finetti’s representation theorem is in the area of quantum information theory. As explained in the introduction, the theorem can be employed for the analysis of schemes involving a large number of information carriers, whose joint state may be difficult to describe in general. A typical and practically relevant example is QKD, where the challenge is to find security proofs that take into account all possible attacks of an adversary.

Most QKD protocols can be subdivided into two parts. In the first part, also known as distribution phase, the two legitimate parties, Alice and Bob, use an (insecure) quantum communication channel in order to distribute correlated information. (Alternatively, in an entanglement-based scheme [4], Alice and Bob receive this correlated information as an input from an external source, which may be controlled by an adversary.) In the second part, the distillation phase, Alice and Bob process this information to extract a pair of secret keys. This process usually only involves classical communication (over an authentic channel).

The analysis based on de Finetti’s theorem sketched below applies to a large class of QKD schemes, which includes almost all protocols proposed in the literature [39]. More concretely, the following conditions must hold.

1. We assume that the information held by Alice and Bob after the distribution phase consists of $N$ parts, for some sufficiently large $N$. The protocol should be invariant under permutations of these parts. This requirement is usually satisfied because each of the $N$ signals is prepared, sent, and received independently of the other signals.

2. In the last step of the distillation phase, the final key is computed in a classical post-processing procedure consisting of information reconciliation (error correction) and privacy amplification by two-universal hashing [27]. As yet, no alternative method for distilling the final key is known, so this criterion is not restrictive [40].

3. The protocol must perform a measurement $\mathcal{M} = \{M_z\}_{z \in \mathcal{Z}}$ on a sample of the received signals and only continue if all outcomes $z$ are contained in a given set $\mathcal{Z} \subset \mathcal{Z}$ that allows to conclude that the dimension of the relevant Hilbert space $\mathcal{H}$ is finite (cf. Remark III.3). Note that this requirement is trivial if the signal space already has small dimension.

A concrete example in $\mathcal{H} = L^2(\mathbb{R})$ are measurements $\mathcal{M}$ with respect to two canonical observables $X$ and $Y$, each of them chosen with probability $\frac{1}{2}$. The set $\mathcal{Z}$ can then be defined as the set of all outcomes $z$ such that $z^2 \leq \frac{d}{k}$ and $\mathcal{H}$ is the space spanned by the eigenvectors of $X^2 + Y^2$ corresponding to eigenvalues larger than $n_0$, for some appropriately chosen $n_0$ (see Lemma III.2 and Lemma III.3).

According to Property III if a key distilled from $N$ signals in state $\rho^N$ is secure then the same is true for the key distilled from a permuted state $\pi \rho^N \pi^\dagger$, for any permutation $\pi \in S_N$. We can thus assume without loss of generality that the $N$ signals are permuted at random and,
hence, their state $\rho^N$ is permutation invariant. Now, according to the argument in Section IV-A and using Property 3 we conclude that $\rho^N$, for some $N' \approx N$, is approximated by a mixture of almost i.i.d. states $\rho_\nu$ (see previous section for explicit parameters). Finally, we use Property 2 which implies that the only relevant quantity is the smooth min-entropy of the measured data $X^N$ conditioned on the adversary’s information $E^N$ (see [3] for a detailed argument). By Theorem III.8, the smooth min-entropy of almost i.i.d. states is approximated by the corresponding entropy of i.i.d. states [11]. Hence, we can without loss of generality assume that $\rho^N$ is an i.i.d. state, which could equivalently be the result of a collective attack. Summarizing, we have thus proved that any QKD protocol satisfying the above three conditions is secure against general attacks whenever it is secure against collective attacks.

V. CONCLUSIONS

We have shown that permutation invariant states on large $N$-partite systems are approximated by a convex combination of almost i.i.d. states, provided measurements on a few subsystems with respect to certain observables only give bounded values. In particular, under this condition, a permutation invariant state can be considered equal to an unknown i.i.d. state, except an arbitrarily small fraction of the subsystems. This has various implications. Of particular interest to experimental physics is that state tomography can be employed without the need for i.i.d. assumptions, as discussed in [2] for the special case of low-dimensional systems.

Applied to quantum cryptography, our result enables full security proofs for QKD schemes in the (practically relevant case) where the dimension of the signal space may be unbounded. This is an intrinsic property of continuous variable protocols, but the necessity of taking into account infinite-dimensional systems may also arise in the analysis of discrete variable schemes, for instance when they are implemented using weak coherent pulses (see, e.g., [28]). The security of these schemes has been investigated intensively, but most proofs are only valid under the assumption of collective attacks (see Introduction for references and [12]). The de Finetti representation theorem derived here allows to drop this assumption, implying that security holds against all possible attacks. The main requirement is that certain tests are carried out on a sample of the transmitted signals. For continuous variable protocols with signal states on $H = L^2(\mathbb{R})$, one possibility is to check that measurements with respect to two canonical observables only result in small outcomes. By modifying Lemma III.2 one may replace this requirement by a criterion based on alternative measurable quantities such as the photon number [24].

VI. ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF LEMMA III.2

The proof is based on the following lemma, which states that the statistics obtained from the observation of $k$ out of $k+n$ binary values $X_1, X_2, \ldots, X_{k+n}$ gives a good estimate for the probability distribution of any of the remaining values, provided the overall distribution is permutation invariant.

**Lemma A.1.** Let $n \geq k$ and let $P_{X_1,\ldots,X_{k+n}}$ be a permutation invariant probability distribution over $\{0,1\}^{k+n}$. Then

$$\Pr[|p_{i}|_{X_1,\ldots,X_k} - f_{X_1,\ldots,X_k} | \geq \delta] \leq 2k^2e^{-k\delta^2},$$

where, for any $x = (x_1,\ldots,x_k)$, $p_{x}$ denotes the probability that $X_{k+1} = 1$ conditioned on $(X_1,\ldots,X_k) = x$.

**Proof.** Let $X = (X_1,\ldots,X_k)$. We show that

$$E[e^{k(p_{x} - f_{x})^2}] \leq 2k^2,$$  \hspace{1cm} (A1)

where $E[\cdot]$ denotes the expectation value. The claim then follows because, by Markov’s inequality,

$$\Pr[|p_{x} - f_{x} | \geq \delta] = \Pr[e^{k(p_{x} - f_{x})^2} \geq e^{k\delta^2}] \leq E[e^{k(p_{x} - f_{x})^2}] e^{-k\delta^2} \leq 2k^2 e^{-k\delta^2}.$$

To show (A1) we use the observation that, for any permutation invariant distribution $P_{Z_1,\ldots,Z_k}$ of binary values, the distribution of any individual value $Z_i$ equals the expectation of the frequency distribution of the whole tuple $(Z_1,\ldots,Z_k)$, i.e.,

$$Pr[Z_i = 1] = E[f_{Z_1,\ldots,Z_k}].$$

In particular, we have for any $x = (x_1,\ldots,x_k)$,

$$p_{x} = E[f_{X_{k+1} \ldots X_{k+n}} | X = x].$$

Using convexity of the function $x \mapsto e^x$, we get

$$e^{k(p_{x} - f_{x})^2} = e^{kE[f_{X_{k+1} \ldots X_{k+n}} - f_{X} | X = x]^2} \leq E[e^{k(f_{X_{k+1} \ldots X_{k+n}} - f_{X})^2} | X = x]$$

and, hence,

$$E[e^{k(p_{x} - f_{x})^2}] \leq E[e^{k(f_{X_{k+1} \ldots X_{k+n}} - f_{X})^2}].$$

It thus remains to be shown that

$$E[e^{k(f_{X_{k+1} \ldots X_{k+n}} - f_{X_1,\ldots,X_k})^2}] \leq 2k^2.$$  \hspace{1cm} (A2)
for any permutation invariant distribution $P_{X_1 \cdots X_{2k}}$. Because any permutation invariant distribution can be written as a convex combination of permutation invariant distributions with fixed frequency distribution, we can without loss of generality assume that $f_{X_1 \cdots X_{2k}} = \frac{1}{2k}$ holds with certainty for any fixed $r \in \{0, \ldots, 2k\}$. The expectation value on the left hand side of (A2) is then given explicitly as

$$E\left[e^{k(f_{X_{k+1}} \cdots X_{2k} - f_{X_1 \cdots X_k})^2}\right] = \min_{s=\max(0,r-k)}^{(k)} (\frac{k}{r})^2 e^{k(r - \frac{s}{k})^2}$$

where $r_{k,r,s} = \frac{1}{k} \ln(\frac{2^k}{r})$.

To bound the term $r_{k,r,s}$ we use an approximation of the binomial coefficient by Wozencraft and Reifenn (see also Lemma 17.5.1 of [32])

$$\frac{e^{Nh(p)}}{\sqrt{8Ng(p)}} \leq \left(\frac{N}{pN}\right) \leq \frac{e^{Nh(p)}}{\sqrt{2N}}$$

where $h(p) \equiv -p \ln p - (1-p) \ln (1-p)$ is the binary entropy function (written with respect to the basis $e$) and where $g(p) \equiv p(1-p)$. The approximation holds for any $N \in \mathbb{N}$ and $0 < p < 1$ such that $pN \in \mathbb{N}$. Because $g(p) \leq \frac{1}{4}$ for any $p$, the first inequality implies

$$\left(\frac{N}{pN}\right) \geq \frac{e^{Nh(p)}}{\sqrt{2N}}$$

Furthermore, since $g(p) \geq \frac{1}{2k}$ for any $N > 1$ and $\frac{1}{N} \leq p \leq 1 - \frac{1}{N}$, the second inequality implies the well known upper bound

$$\left(\frac{N}{pN}\right) \leq e^{Nh(p)}$$

which also holds for $N = 1$, $p = 0$, and $p = 1$. Inserting these bounds into (A1), we find

$$r_{k,r,s} \geq 2h(\frac{r}{2k}) - h(\frac{\alpha}{k}) - h(\frac{r - s}{k}) - \frac{1}{2k} \ln(4k). \quad (A5)$$

Using some standard analysis, one finds that

$$2h(\frac{\alpha + \beta}{2}) - h(\alpha) - h(\beta) \geq (\alpha - \beta)^2$$

for any $\alpha, \beta \in [0,1]$. Combining this with (A5) and inserting in (A3) yields (A2) and thus concludes the proof.

The following lemma is an immediate corollary of Lemma [A1] applied to the sequence of values obtained from measurements of a permutation invariant state.

**Lemma A.2.** Let $n \geq k$, let $U = \{U_0, U_1\}$ be a binary POVM on $\mathcal{H}$, let $\rho^{k+n} \in \mathcal{S}(\mathcal{H}^{k+n})$ be permutation invariant, and let $(X_1, \ldots, X_n)$ be the outcome of the measurement $U^{\otimes k}$ applied to the first $k$ subsystems of $\rho^{k+n}$. Then

$$\Pr[\|\text{tr}(U_1 \rho_{X_1 \cdots X_k}) - f_{X_1 \cdots X_k}\| \geq \delta] \leq 2k^4 e^{-k \delta^2},$$

where, for any $x = (x_1, \ldots, x_k)$, $\rho_x^k$ is the reduced state on a single subsystem conditioned on the measurement outcome $(X_1, \ldots, X_k) = x$.

We are now ready to prove Lemma [III.1]

**Proof of Lemma [III.1].** Let $X^k := (X_1, \ldots, X_k)$ and $X^{n/2} := (X_{k+1}, \ldots, X_{k+n/2})$ (where, for simplicity, we assume that $n$ is even). Applying Lemma A.2 to the density operator $\rho_{X^{n/2}}$ describing the state conditioned on the outcomes of measurement $V^\otimes n/2$ applied to $n/2$ subsystems of $\rho^{k+n}$, we get

$$\Pr[\|\text{tr}(U_1 \rho_{X^k X^{n/2}}) - f_X^k\| \geq \delta] \leq 2k^4 e^{-k \delta^2}. \quad (A6)$$

Similarly and using $n/2 \geq k$ we find

$$\Pr[\|\text{tr}(V_1 \rho_{X^k X^{n/2}}) - f_X^{n/2}\| \geq \delta] \leq 2k^4 e^{-k \delta^2}. \quad (A7)$$

By the definition of the quantity $\gamma_{U_1 - V_1}$, we have

$$\Pr[f_{X^{n/2}} > \gamma_{U_1 - V_1}(f_X^k + \delta)] = \Pr[\|\text{tr}(U_1 \sigma) - f_X^k + \delta\| \leq \|\text{tr}(V_1 \sigma) - f_X^{n/2} - \delta\|] \leq \Pr[\|\text{tr}(U_1 \rho_{X^k X^{n/2}}) > f_X^k + \delta\| \text{tr}(V_1 \rho_{X^k X^{n/2}}) < f_X^{n/2} - \delta\|] \leq 4k^4 e^{-k \delta^2},$$

where the last inequality follows from (A6) and (A7), and the union bound.

To conclude the proof, we use the observation that

$$f_{X_{k+1} \cdots X_{k+n}} = \frac{1}{2} f_{X_{k+1} \cdots X_{k+n/2} + 1} f_{X_{k+n/2+1} \cdots X_{k+n}}$$

which implies

$$\Pr[f_{X_{k+1} \cdots X_{k+n}} > \gamma_{U_1 - V_1}(f_X^k + \delta) + \delta] \leq \Pr[f_{X_{k+1} \cdots X_{k+n/2} > \gamma_{U_1 - V_1}(f_X^k + \delta)] + \Pr[f_{X_{k+n/2+1} \cdots X_{k+n}} > \gamma_{U_1 - V_1}(f_X^k + \delta) + \delta] \leq 8k^4 e^{-k \delta^2}. \quad \Box$$
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[36] The term i.i.d. (for independent and identically distributed) is traditionally used in probability and information theory for random variables $X_1, \ldots, X_N$ whose probability mass (or density) function is of the form $P_{X_1,\ldots,X_N} = P_X \times \cdots \times P_X$. We use it here for multipartite quantum states of the form $\sigma^\otimes N$.
[37] In addition, Alice and Bob need to be able to exchange classical messages authentically or, alternatively, share a short initial key.
[38] We emphasize here that this random permutation is only used in the theoretical analysis, but need not be implemented in the actual protocol, as shown in [33].
[39] Among the few exceptions are the Differential Phase Shift (DPS) [34] and the Coherent One-Way (COW) Protocol [35]. Both rely on measurements involving two subsequent signals at the same time, so that the order in which the signals are received is relevant.
[40] Note that the distillation phase may involve other steps such as sifting or advantage distillation, but these need to be carried out on single signals (or small blocks of signals) independently.
[41] Theorem 11.3 can be applied because $X^N$ usually is a sequence of digitally represented values, so that $\mathcal{H}_X$ has finite dimension.
[42] A remarkable exception are QKD schemes using Gaussian states, for which security against general attacks can also be proved using a result on the extremality of Gaussian states [21], as shown in work done in parallel to ours [30].