Photon filters in a microwave cavity

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April 1, 2022

Abstract

In an earlier paper we have concluded that time-dependent parameters in atom-mode interaction can be utilized to modify the quantum field in a cavity. When an atom shoots through the cavity field, it is expected to experience a trigonometric time dependence of its coupling constant. We investigate the possibilities this offers to modify the field. As a point of comparison we use the solvable Rosen-Zener model, which has parameter dependencies roughly similar to the ones expected in a real cavity. We do confirm that by repeatedly sending atoms through the cavity, we can obtain filters on the photon states. Highly non-classical states can be obtained. We find that the Rosen-Zener model is more sensitive to the detuning than the case of a trigonometric coupling.

1 Introduction

The Jaynes-Cummings model ([1] and [2]) has been widely used to model the behaviour of atoms in high-quality cavities. Usually it has been used with constant parameters, and the possibility to obtain analytic results for the time evolution has made it a useful tool in analyzing both the theoretical and experimental aspects of cavity QED. In our earlier works ([3] and [4]) we have considered the possibility to tune the cavity parameters during the time of interactions with the individual atoms. In [3], we utilized the fact that the model reduces to sets of uncoupled two-level systems in order to manipulate the quantum state of the cavity. Such modifications of the parameters must take place slowly enough to allow the cavity to follow adiabtically retaining the identity of its individual modes. By introducing exactly solvable models, we were able to explore the possibilities offered by such systems. In particular, we considered the time evolution during the interaction followed by a projective measurement on the emerging atom as a tool to modify the shape of the photon distribution in the cavity; the atoms acted as filters on the photon state.

The Jaynes-Cummings model has been used extensively to model the physics in realistic cavity QED experiments based on microwave radiation [5]. Schemes for preparing various states of the microwave cavity field have been proposed and realized, see e.g. [6]. In such cavities the shape of the electromagnetic modes is mainly given by standing waves approximately described by trigonometric
functions. The atom is supposed to enter the cavity at a node and exit at an opposite node. An atom thus travelling through the cavity sees this mode shape as a time dependence of the coupling constant. The model cannot be solved analytically for this case, except when the detuning is exactly zero. Such a solution was first given by Schlicher [7] in the present context. This work, however, was mainly concerned with the effect of the field on the atom not exploring the effect the atomic observation have on the field. Some other papers [8] and [9] also considers the effects of the time dependence induced by the mode structure traversed by the atom. The present use of the time dependence to shape the quantum state of the field is not considered in these papers.

We also, however, want to consider the exactly solvable Rosen-Zener model, which has a time dependence of the coupling function which approximates the mode shape in a cavity. Here as in the actual cavity, the detuning is taken to be constant. This neglects possible Stark shifts caused by the electromagnetic field. When an atom travels through the standing wave of a real cavity, it thus experiences a time dependent coupling which does not greatly differ from that in the model.

The Rosen-Zener model was not investigated in our previous work [3], but here we use it as a point of comparison for the more realistic trigonometric dependence. We find that the exact shape of the coupling function is not crucial, but also that the Rosen-Zener model is much more sensitive to the detuning than the cavity mode model.

The section 2.1 summarises the results from paper [3] on the time-dependent Jaynes-Cummings model and section 2.2 applies these results to the Rosen-Zener model. In section 3.1 we present the details of our more realistic description of a microwave mode traversed by a sequence of single atoms. Section 3.2 compares the numerically obtained results for the cavity mode with those of the Rosen-Zener and shows how efficient the type of interaction is, if we want to achieve a narrow photon distribution. Section 4 presents some general observations on the results obtained.

2 Time-dependent Jaynes-Cummings model

2.1 Manipulating the cavity field

The Jaynes-Cummings model describes the atom by a two-level state $|\pm\rangle$ and the field consists of a single mode. Further, the cavity losses and spontaneous emission of the atomic levels are neglected, which means that characteristic time-scales in the interacting system are much shorter than the atomic and the cavity loss time. The Jaynes-Cummings Hamiltonian is (with $\hbar = 1$)

$$H = \Omega b^\dagger b + \frac{\omega}{2} \sigma_3 + g \left( b^\dagger \sigma^- + b \sigma^+ \right),$$

where the $\sigma$s are the ordinary Pauli matrices.

The operator $N = b^\dagger b + \frac{1}{2} \sigma_3$ commutes with $H$ and is therefore a constant of motion. We consider the Hamiltonian $\tilde{H} = H - \Omega N$ and the dynamics of $N$ can be added separately afterwards. In the basis $\{|n, \pm\}$, the transformed Hamiltonian $\tilde{H}$ is in block-diagonal form, and the Schrödinger equation can be
solved within each block

\[
\frac{d}{dt} \begin{bmatrix} a_+(n) \\ a_-(n) \end{bmatrix} = \begin{bmatrix} \frac{\Delta \omega}{2} & g\sqrt{n} \\ g\sqrt{n} & -\frac{\Delta \omega}{2} \end{bmatrix} \begin{bmatrix} a_+(n) \\ a_-(n) \end{bmatrix},
\]

(2)

with the detuning \( \Delta \omega = \omega - \Omega \). As mentioned in the introduction, here the detuning is assumed to be time-independent while the coupling constant \( g \) depends on time.

The state of the whole system can be written as

\[
|\Psi\rangle = c_0 a_-(0)|0,-\rangle + \sum_{n=1}^{\infty} c_n \left[ a_+(n)|n-1, +\rangle + a_-(n)|n,-\rangle \right],
\]

(3)

where the initial state, determined by \( c_n \) and \( a_\pm(0) \), is assumed to be known. The state after the interaction is given by \( a_\pm(\infty) \).

If the atom is found in one of the states \( |\pm\rangle \) after the interaction, the corresponding photon distribution for the cavity mode is then (up to a normalization constant)

\[
P_+^n = |a_\infty^+(n+1)|^2|c_{n+1}|^2
\]

\[
P_-^n = |a_\infty^-(n)|^2|c_{\pm}|^2.
\]

(4)

If we only consider the initial conditions

\[
|a_0^-| = 1, \quad a_0^+ = 0,
\]

(5)

or vice versa, no interference terms need to be considered, see [3]. From (4) it is clear that the initial field distribution \( |c_n|^2 \) is modified by \( |a_\pm(\infty)|^2 \) after the measurement of the atomic state. The shape of these “filter functions” is crucial for what final state the field will be found in.

For zero detuning the Schrödinger equation is analytically solvable [7]. With the initial condition [4], the filter functions will be oscillating with \( n \) according to

\[
|a_\infty^+(n)|^2 = \cos^2(\sqrt{n}A)
\]

\[
|a_\infty^-(n)|^2 = \sin^2(\sqrt{n}A),
\]

(6)

with the area of the coupling given by

\[
A = \int_{-\infty}^{\infty} g(t) dt.
\]

(7)

It follows that the shape of \( g(t) \) is not important in this case, only the area. We note that no transitions occur if \( A \) vanishes. This happens in the trivial situation when the coupling \( g(t) \) is identically equal to zero, but it is also possible if the coupling changes sign. For a small detuning \( \Delta \omega \) we expect solutions similar to (6), but with some corrections of the amplitudes. When the detuning is increased further, the solutions may not necessarily have the oscillating form in (6).

After the passage of one atom, the process can be repeated and, even after several atoms, the state of the cavity mode is fully determined. For example, if
m atoms are injected and all the atoms are found in their lower state $|\rangle$ after the interactions, the field distribution will be

$$P_n^-(m) \propto |a^\infty(n)|^{2m}|c_n|^2.$$  

(8)

This method can then be used to create, for example non-classical states of the field, provided that $|a^\infty(n)|^{2m}$ has a sharp peak where the initial photon distribution differs from zero. Note, however, that if atoms are detected in their upper level, the photon distribution is multiplied by $|a^\infty(n)|^2$ and the asymmetry in (4) makes the analysis more complicated. This situation will be discussed separately.

### 2.2 The Rosen-Zener model

To demonstrate how the method described above can be used to manipulate the state of the field, we look at the Rosen-Zener model which is analytically solvable for the filter functions. In the model the coupling is given by

$$g(t) = g_0 \text{sech}\left(\frac{t}{T}\right).$$  

(9)

and the detuning $\Delta \omega$ is constant. With the initial condition (5), the filter functions becomes

$$|a^\infty(n)|^2 = \sin^2\left(\pi T g_0 \sqrt{n}\right) \text{sech}^2\left(\pi T \Delta \omega / 2\right)$$

$$|a^\infty(n)|^2 = 1 - |a^\infty(n)|^2.$$  

(10)

In the non-adiabatic limit ($\Delta \omega T \approx 0$), the hyperbolic secant is equal to one, and the filter function for a lower level projection is a simple cos-function squared just as the solutions (6). This gives a filter around its maxima

$$n_M = \frac{k^2}{T^2 g_0^2}, \quad k = 0, 1, 2, ...$$  

(11)

and the width of the filter function is

$$\Delta n_a = \frac{\sqrt{n_M}}{T g_0}.$$  

(12)

Thus for an initial field distribution centered around one maximum $n_M$ and with a width $\Delta n_p$ smaller than $\Delta n_a$, a measurement of a lower level atom will decrease the width of the photon distribution. Note that for an initial coherent state we have $\Delta n_p \sim \sqrt{n}$ and the filtering effect is independent of which maximum the field is centered around. From the solutions (10) we also note that as small a detuning as possible is preferable, if we wish to sharpen the photon distribution.

### 3 The microwave cavity

#### 3.1 The model

We consider a microwave cavity with one single mode such that the number of half wavelengths of the mode is $l$. Note that $l$ is not too big, since we assume
to be in the microwave regime. If we now let an atom, moving with velocity $v$, pass through the cavity along the standing wave, the coupling between the mode and the atom is

$$g(t) = \begin{cases} 
  g_0 \cos(kvlt), & -\frac{\pi}{2kv} \leq t \leq \frac{\pi}{2kv} \quad \text{and} \quad l = 1, 3, \ldots \\
  g_0 \sin(kvlt), & -\frac{\pi}{2kv} \leq t \leq \frac{\pi}{2kv} \quad \text{and} \quad l = 2, 4, \ldots \\
  0 & t < -\frac{\pi}{2kv} \quad \text{or} \quad t > \frac{\pi}{2kv}.
\end{cases} \quad (13)$$

The detuning is still assumed to be time-independent. No analytic solution of the Schrödinger equation is known with the coupling given by (13) and a non-zero detuning. However, if the detuning is small, the solution is expected to be approximately given by (6). Also, in the Rosen-Zener model we have seen that the smallest detuning possible will enhance the filtering effect. In this limit, we note that if $l$ is an even number, no transition will take place, since the area $A$ vanishes. However if $l$ is odd, then $A = 2g_0/kvl$ is non-zero and we get oscillating solutions as in (6). Thus it follows, if $l$ is odd, that it is possible to change the state of the field by letting atoms pass through the cavity and detect the states of the emerging atoms. Just as in the Rosen-Zener model, non-classical states of the field can be achieved.

### 3.2 Numerical results

The Schrödinger equation (2) with the coupling given by (13) has been solved numerically using the ordinary Runge-Kutta method. The filter functions obtained are compared with the ones in the Rosen-Zener model. In order to compare them, the pulse-area $A$ must be chosen equal in the two models, which means that $T = \pi/(2kv)$. Further, since the area $A$ is proportional to $l^{-1}$, we rescale it by multiplying $g_0$ with $l$. This just changes the oscillation frequency and is done in order that solutions with different $l$’s may be compared easily.

The function $|a^\infty(n)|^2$ for the microwave case and the Rosen-Zener model is plotted in figure 1 for the dimensionless parameters $g_0 = 5$, $T = 0.1$, $\Delta \omega = 0.5$ and $l = 1, 2, 3$. The figure confirms that, as long as $\Delta \omega$ is small, the form of $g(t)$ is not important. We clearly see that all three curves with non-vanishing coupling area $A$ coincide closely. A difference between the curves can only be seen at the maxima and minima where the two models differ slightly; however the $l = 1$ and $l = 3$ curves are identical. The curve with $l = 2$ has $A = 0$ and no effect is expected to arise as is confirmed in the plot.

For larger values of $\Delta \omega$, the shape of the coupling $g(t)$ becomes more important. In the Rosen-Zener model we still have an oscillating function, but the minima of $|a^\infty(n)|^2$ will now differ from zero since $\text{sech}^2(\pi T \Delta \omega) < 1$. It is interesting to look for this behavior in the microwave model. Figure 2 is again showing $|a^\infty(n)|^2$ for the two models with the same parameters as in figure 1, except for the detuning $\Delta \omega = 5$. The filter functions for the microwave model seem to depend more weakly on $\Delta \omega$ than in the Rosen-Zener model. We also note that the value of the minima is not constant, but is approaching zero for larger $n$’s. The weak $\Delta \omega$-dependence in the cavity model means that the filtering effect is improved. On the other hand, if the atoms are used to probe the state of the cavity field, then it may be preferable to have a large $\Delta \omega$-dependence, see [10]. We also see that, in the $l = 2$ case, this does not
necessarily imply that $|a^\infty(n)|^2 = 1$, even though the area $A$ is zero. It is also clear from the figure that the $l = 1$ and $l = 3$ curves differs in this case.

From the plots of the filter functions and the discussion above, we conclude that it is possible to create a sharpened photon distribution by letting atoms pass the microwave cavity. To check this, we assume the initial photon distribution in the cavity to be a Poissonian

$$|c_n|^2 = \exp(-\bar{n})\frac{\bar{n}^n}{n!},$$

centered around the second maximum of $|a^\infty(n)|^2$ at $n = 16$. If we measure $m$ atoms in their lower state after the interactions, the photon distribution will be given by (5). The distribution $P_n^-(m)$ is shown in figure 3 with $l = 1$ and the other parameters are as in figure 1, the number of atoms is $m = 1, 5$ and 25 and initially $\bar{n} = 16$. It is manifest that the width of the distribution is decreasing when $m$ is increased.

Figure 3 shows clearly that a sharpened distribution is achieved. Another way to investigate the state of the field is to study the Mandel $Q$-parameter, which is defined as

$$Q = \frac{\langle n^2 \rangle - \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle}.$$  

For a Poissonian distribution, the $Q$-parameter is zero, for a super-Poissonian state it is greater than zero and for a non-classical sub-Poissonian state we have $-1 < Q < 0$. In figure 4 we have plotted the $Q$-parameter as a function of the coupling-strength $g_0$ when twenty-five ($m = 25$) atoms have passed the cavity, and all of them have been recorded in their lower state. The dimensionless parameters are $\Delta \omega = 0.5$, $T = 0.1$ and $l = 1$ and the initial field was in a coherent state (14) with $\bar{n} = 20$. The plot indicates that a non-classical state is achieved for a large range of values of $g_0$. If the initial state of the field is known, the $Q$-parameter can be used to find the optimal parameters $g_0$ and $T$ needed to get a sharpened photon distribution.

### 4 Conclusion

In this paper we have discussed the filtering action of a model where the time dependence of the coupling constant is provided by the field experienced by an atom traversing a cavity eigenmode. Numerically we have investigated a trigonometric dependence on the position in the cavity. This model neglects possible complications arising from edge effects at the entrance or exit of the cavity. The resulting effect is expected to be small, but when the situation is such that the onset of the interaction switches between sudden and adiabatic, there are interesting questions to explore. This was, in fact, investigated in the case of zero detuning in [9].

For the method presented above to work, it is crucial that the atom-field interaction-time is considerably smaller than the cavity loss time $\tau_{loss}$. Typical experimental values are $v \sim 300$ m/s and $\nu \sim 50$ GHz, which gives the interaction time

$$\tau_{int} = \frac{\pi}{kv} \sim 10\mu s.$$  

For a high-$Q$ microwave cavity one can achieve cavity loss times as large as $\tau_{loss} \sim 0.3$ s, from which we conclude that it is possible to manipulate the field
before it decays. Suitable choice of the atomic transition makes the atomic decay times sufficiently large too.

The trigonometric mode shape has also been investigated by Schlicher \[7\], but he considers mainly the ability of the atom to follow the changing field adiabatically. He introduces no explicit measurement on the atom and does not discuss the change of the cavity field effected by the interaction with the atom.

In our work we find that the Rosen-Zener model is much more sensitive to the detuning between the atoms and the radiation than the simple trigonometric model. This may be understood to derive from the different interaction times; in the trigonometric model, the total interaction time is determined by the period of the mode function. In the Rosen-Zener model, on the other hand, the interaction acts over an infinite interval. Because the detuning defines the rate of oscillation of the two levels, the effect of it may show up more dramatically when it acts over the longer time interval. A smooth switch on/off will enhance the possibility for the atom to follow the field.

In order to verify this conjecture, we have investigated a variety of pulse shapes with different sharpness of switching on and off the interaction. The trigonometric function disappears linearly and the Rosen-Zener model exponentially. Faster switching is achieved by a square-wave pulse or a Gaussian. A slower switching can be achieved by using a Lorentzian. We thus have the switch on and off rate occurring in the ordered sequence

\[
\text{square} - \text{wave} > \text{trigonometric} > \text{Gaussian} > \text{Rosen} - \text{Zener} > \text{Lorentzian}.
\]

(17)

In order to check the dependence on the detuning $\Delta \omega$ for these cases, we choose the minimum at $n = 49$ in figure 2. The shift of the value at this minimum is shown in figure 5. As we can see the sensitivity is indeed given by the sequence in (17). In the comparison the pulse area has been chosen the same in each case; the other details of the pulses are given in the Appendix.

The two-level problem with Lorentzian and Gaussian pulse shapes has been investigated before \[11\] and \[12\]. However, they did not discuss the dependence on the detuning or how the pulse shape affect the possibility for adiabatic following. As the models in their works are semiclassical, the photon statistics discussed here is, of course, not relevant in their papers.

### Appendix

The pulse area is chosen to be

\[
A = \int_{-\infty}^{\infty} g(t) dt = \pi T g_0 = \frac{g_0 \pi}{10},
\]

(18)

so that setting the time-scale $T = 0.1$ in the Rosen-Zener model determines the normalization for the different cases. Table 1 gives the different shapes of the pulses and parameters used in figure 5.
| Pulse shape          | Parameters                       |
|---------------------|----------------------------------|
| Square-wave: $g(t) = g_0 \frac{1}{2} \left[ \tanh \left( \frac{t + \tau}{t_s} \right) - \tanh \left( \frac{t - \tau}{t_s} \right) \right]$ | $\tau = \pi T$, $t_s = 0.02$ |
| Trigonometric: $g(t) = g_0 \cos(kv t)$ | $kv = \frac{2}{\pi T}$ |
| Gaussian: $g(t) = \frac{g_0}{\sqrt{\pi} 2\sigma} \exp \left[ -\frac{(t/\sigma)^2}{4\sigma^2} \right]$ | $\tau = \pi T$, $\sigma = 0.3$ |
| Rosen-Zener: $g(t) = g_0 \text{sech} \left( \frac{t}{T} \right)$ | $T = 0.1$ |
| Lorentzian: $g(t) = g_0 \frac{(\gamma/\pi)^2}{(t/\tau)^2 + \gamma^2}$ | $\tau = \pi T$, $\gamma = 0.3$ |

The parameter $\tau = \pi T$ gives the chosen pulse area and $t_s$, $\sigma$ and $\gamma$ are measures of the different widths/steepnesses in the models. The widths are chosen such that $g(0) \approx 1$ for all pulses. The coupling strength is as in figure 2, $g_0 = 5$, and is the same for all models. Note that we have used hyperbolic tangent functions for simulation of the square-wave pulse. However, in the limit that $t_s$ goes to zero, we obtain a step-function pulse which is analytically solvable.
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