The boundary knot method for simulating unsteady incompressible fluid flow

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Abstract

This article presents a numerical algorithm using the Boundary knot (BK) method for the incompressible Navier-Stokes equations. To deal with time derivatives, the forward time differences are employed yielding the Poisson’s equation. The BK method with the general solution of the Helmholtz equation and dual reciprocity (DR) principle is chosen to solve the Poisson’s equation. In numerical examples the reverse principle with thin plate spline (TPS) radial basis functions (RBF) is used to construct the particular solution. It is found that BKM with a classical DR principle gives reasonable results with the advantage of BKM spectral convergence; however the accuracy of the solution is affected by the accuracy of particular solution created using DR.

Keywords: Meshless numerical methods; Boundary knot method; Navier-Stokes equations.

1. Introduction

Incompressible Navier-Stokes flow in two dimensions is one of the several major problems in fluid mechanics that have been extensively studied both theoretically and numerically. In general, the formulation of incompressible Navier-Stokes equations using primitive variables is often used, but it has limitation in approximating the velocity and pressure. The boundary knot method (BKM) is truly meshless method, which requires no elements or global background mesh, for either interpolation or integration purposes. The first article applying BKM method to

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compute convection-diffusion, Helmholtz and Poisson problems was by Chen[1]. The most attractive aspect of BKM is the lack of integration process and spectral convergence. The present paper focuses on primitive variable Navier-Stokes formulation with fractional step method to achieve velocity-pressure decoupling to solve incompressible viscous laminar flow and BKM to solve pressure Poisson equation [2].

2. Governing Equations and Fractional-Step Algorithm

The governing equations for unsteady incompressible viscous fluid flow are Navier-Stokes equations with the continuity equation in the convection term [3]. This equation can be written as

\[
\frac{\partial u}{\partial t} = \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} + f_x, \tag{1}
\]

\[
\frac{\partial v}{\partial t} = \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - \frac{\partial p}{\partial y} + f_y, \tag{2}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}
\]

where \(u\) and \(v\) are the velocities in \(x\) and \(y\) direction respectively, \(p\) is the pressure, \(f_x\) and \(f_y\) are the body forces, \(Re\) is Reynolds number. Eq. 1 and Eq. 2 are the momentum equations and Eq.3 is the continuity equation. A fractional-step algorithm is used to solve this problem (see [3]). The time derivative of the velocity vector in a momentum Eq. 1 and Eq. 2 can be replaced with a difference approximation and following relation is obtained

\[
\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{Re} \left( \frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} \right) - u^n \frac{\partial u^n}{\partial x} - v^n \frac{\partial u^n}{\partial y} - \frac{\partial p^n}{\partial x} + f_x^n, \tag{4}
\]

\[
u^n + \Delta \left[ \frac{1}{Re} \left( \frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} \right) - u^n \frac{\partial u^n}{\partial x} - v^n \frac{\partial u^n}{\partial y} + f_x^n \right] - \Delta \frac{\partial p^n}{\partial x}, \tag{5}
\]

and

\[
\frac{v^{n+1} - v^n}{\Delta t} = \frac{1}{Re} \left( \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} \right) - u^n \frac{\partial v^n}{\partial x} - v^n \frac{\partial v^n}{\partial y} - \frac{\partial p^n}{\partial y} + f_y^n, \tag{6}
\]

\[
v^n + \Delta \left[ \frac{1}{Re} \left( \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} \right) - u^n \frac{\partial v^n}{\partial x} - v^n \frac{\partial v^n}{\partial y} + f_y^n \right] - \Delta \frac{\partial p^n}{\partial y}, \tag{7}
\]

where upper indexes \(n\) and \(n+1\) indicate the time step. Eq.5 and Eq.7 are explicit formula for convection and viscous terms and the implicit one for a pressure term. To simplify Eq.5 and Eq.7 we used the fractional step approximation (see e.g. [3]). According this approximation the intermediate velocities \(\tilde{u}^n\) and \(\tilde{v}^n\) are computed using simplified momentum equation

\[
\tilde{u}^n = u^n + \Delta \left[ \frac{1}{Re} \left( \frac{\partial^2 u^n}{\partial x^2} + \frac{\partial^2 u^n}{\partial y^2} \right) - u^n \frac{\partial u^n}{\partial x} - v^n \frac{\partial u^n}{\partial y} + f_x^n \right], \tag{8}
\]
\[ \ddot{v}^n = v^n + \Delta t \left[ \frac{1}{\text{Re}} \left( \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial^2 v^n}{\partial y^2} \right) - \frac{u^n}{\partial x} \frac{\partial v^n}{\partial x} - \frac{v^n}{\partial y} \frac{\partial v^n}{\partial y} + f_v^n \right]. \] 

(9)

When we compare Eq. 5, 7 and Eq. 8, 9 we get

\[ u^{n+1} = \bar{u}^n - \Delta t \frac{\partial p^n}{\partial x}, \quad \quad (10) \]

\[ v^{n+1} = \bar{v}^n - \Delta t \frac{\partial p^n}{\partial y}. \quad \quad (11) \]

The intermediate velocities \( \bar{u}^n \) and \( \bar{v}^n \) does not satisfy the continuity equation (Eq. 3). The velocities \( u^{n+1} \) and \( v^{n+1} \) must satisfy the continuity equation which implies

\[ \frac{\partial^2 p^n}{\partial x^2} + \frac{\partial^2 p^n}{\partial y^2} = \frac{1}{\Delta t} \left( \frac{\partial \bar{u}^n}{\partial x} + \frac{\partial \bar{v}^n}{\partial y} \right) \quad \quad (12) \]

Eq. 12 is the Poisson’s equation with non-zero source term [4]. The Eq. 10 and Eq. 11 are solved explicitly by updating nodal values for velocities. The pressure Eq. 12 is solved using BKM over problem domain with boundary conditions \( p^n|_{\Gamma} = \bar{p}^n \) and \( \partial p^n/\partial n = \bar{q}^n \).

3. The BKM method formulation

Among typical meshfree boundary-type numerical schemes are the local boundary integral equation (LBIEM) method and the method of fundamental solutions (MFS) [1]. The meshfree LBIEM is in fact a combination of the radial basis function (RBF) technique with the boundary element scheme, whereas the MFS is a boundary-type radial basis function (RBF) collocation scheme. The LBIEM involve singular integration and hence are mathematically more complicated in comparing with the commonly used finite element method (FEM). In addition, their low-order approximations also downgrade computational efficiency and are not easily used for engineers. On the other hand, the MFS possesses integration-free, spectral convergence, easy-to-use, and inherently meshfree merits. In recent years, the MFS, also known as the regular boundary element method, revives partly thanks to its combination with the dual reciprocity method (DRM) for handling inhomogeneous problems. In the use of a singular fundamental solution, which can be considered a RBF, the MFS, however, requires a controversial fictitious boundary outside physical domains, which largely impedes its practical use for complex geometry problems. As an alternative RBF approach, Chen and Tanaka [1] recently developed the boundary knot method (BKM), where the perplexing artificial boundary in the MFS is eliminated via the nonsingular general solution instead of the singular fundamental solution. Just like the MFS and the dual reciprocity BEM (DR-BEM) [2], the BKM also uses the DRM to evaluate the particular solution. The method can be considered a new type of the Trefftz method, which combines the DRM, RBF, and nonsingular general solution.

Here, we introduce the BKM with a Poisson problem described by Equ. 12. The governing equation (12) can be restated as

\[ \left( \frac{\partial^2 p^n}{\partial x^2} + \frac{\partial^2 p^n}{\partial y^2} \right) + \delta p = \frac{1}{\Delta t} \left( \frac{\partial \bar{u}^n}{\partial x} + \frac{\partial \bar{v}^n}{\partial y} \right) + \delta p \quad \quad (13) \]

where \( \delta \) is an artificial parameter. Eq. 13 is, Helmholtz equation. This strategy can be understood that the use of nonsingular general solutions of Helmholtz-like equation with a small characteristic parameter \( \delta \) approximates the
constant general solution of the Laplace equation. For example, the general solution of the 2D Helmholtz operator (13) is the Bessel function of the first kind of the zero-order and can be expanded as

$$J_0(\delta r) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2k} k! k!} (\delta r)^{2k},$$

(14)

where \( r \) denotes the Euclidean distance. As the parameter \( \delta \) goes to zero, the \( J_0(\delta r) \) approaches constant 1. In the limiting process, the general solution of the Helmholtz operator is the general solution of the Laplacian. The solution of the Poisson problem can be split as the homogeneous and particular solutions

$$u = u_h + u_p,$$

(15)

The particular solution satisfies the governing equation but not the boundary conditions. To evaluate the particular solution, the inhomogeneous term is approximated by

$$\frac{1}{\Delta t} \left( \frac{\partial u^n}{\partial x} + \frac{\partial u^n}{\partial y} \right) + \delta p = \sum_{j=1}^{N+L} \beta_j \phi(r_j),$$

(16)

where \( \beta_j \) are the unknown coefficients. \( N \) and \( L \) are, respectively, the numbers of knots on the domain and boundary. The use of interior points is usually necessary to guarantee the accuracy and convergence of the BKM solution of inhomogeneous problems. The \( r_j = ||x-x_j|| \) quantity represents the Euclidean distance and \( \phi \) is the multi quadrics (MQ) radial basis function (RBF). By forcing approximation Eq. 16 to exactly satisfy governing equations at all nodes, we can uniquely determine

$$\beta = A_\phi^{-1} \left\{ \frac{1}{\Delta t} \left( \frac{\partial u^n}{\partial x} + \frac{\partial u^n}{\partial y} \right) \bigg|_{x_i} + \delta p \bigg|_{x_i} \right\},$$

(17)

where \( A_\phi^{-1} \) is the nonsingular MQ RBF interpolation matrix. The homogeneous solution \( u_h \) has to satisfy both governing equation and boundary conditions. By means of the nonsingular general solution, the BKM expressions are given by

$$\sum_{j=1}^{L} \alpha_j J_0(\delta r_j) = D(x_i) - u_p(x_i),$$

(18)

$$\sum_{j=1}^{L} \alpha_j \frac{\partial J_0(\delta r_j)}{\partial n} = N(x_i) - \frac{\partial u_p(x_i)}{\partial n},$$

(19)

where \( i \) is the index of source points on boundary, \( \alpha_j \) are the desired coefficients, \( n \) is the unit outward normal and \( i \) and \( m \) are, respectively, the numbers of knots on the Dirichlet and Neumann boundary surfaces. In the case if inhomogeneous term in Eq.13 involves the dependent unknown \( p \), we need to constitute a set of supplementary equations for unknowns at the inner nodes as follow

$$\sum_{j=1}^{L} \alpha_j J_0(\delta r_j) = u_n - u_p(x_i)$$

(20)

where \( n \) denotes the index of each internal response knot. We have obtained a total \( N+L \) simultaneous algebraic equations for the unknown coefficients \( \alpha \). It is remained to calculate the solution \( u \) value at any inner node by
The fractional step algorithm can be summarized as follows [5]
- Start with initial velocity and pressure field
- Compute intermediate velocity field using Eq.5 and Eq.7 for each node
- Solve the Poisson equation (Eq.12) using BKM to obtain the pressure
- Update velocities using Eq.10 and Eq.11

4. Numerical example - lid-driven cavity flow

Lid-driven cavity flow is used as a standard test problem for the validation of numerical solutions of incompressible Navier–Stokes flow. The top wall of the cavity moves with a velocity $u_0=1$, and no-slip impermeable boundary conditions are assumed along the other three walls [6]. The geometry and boundary conditions are presented in Fig.1a.

![Fig. 1. (a) Lid-driven cavity, model dimensions and boundary conditions (b) velocity field vectors for Re=100.](image)

The difficulty of this problem lies in the presence of singularities of pressure and velocity at the two upper corners of the cavity. The point "meshes" tested here had 51x51 points uniformly distributed over 2D unity domain. The Reynolds number for the simulation was Re=100. The pressure patterns shows spurious shapes around corners and boundaries of cavity (Fig.2b).
Fig. 2. (a) Lid-driven cavity, streamlines pattern and (b) pressure contours for Re=100.

The pressure anomalies are caused by the process of construction of particular solution with equally spaced internal nodes which is unable to match high pressure gradients. On the other side the velocities (Fig.1b) and streamline patterns (Fig.2a) agrees very well with lid-driven cavity problem solution obtained using FVM (Tab.1).

Table 1. L₂ errors of lid-driven cavity flow at steady state.

| Profile | x=0.25 | x=0.5 | x=0.75 | y=0.25 | y=0.5 | y=0.75 |
|---------|--------|--------|--------|--------|--------|--------|
|         | 2.897 x 10⁻² | 4.789 x 10⁻² | 7.567 x 10⁻² | 3.745 x 10⁻³ | 7.134 x 10⁻⁴ | 1.484 x 10⁻⁴ |

5. Conclusions

In this article, a numerical algorithm using the meshless boundary knot (BKM) method for the incompressible Navier-Stokes equations is demonstrated. To deal with convection term, the fractional step method was adopted and the set of recurrent equations was derived for time stepping procedure. The ability of the BKM code to solve fluid dynamics problems was presented by solving lid-driven cavity flow problem with reasonable accuracy when compared to solution performed using finite volume method (FVM).

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