Pre-modular fusion categories of small global dimensions

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Abstract

We first prove an analogue of Lagrange theorem of global dimensions of fusion categories, then we consider classification of pre-modular fusion categories of some small integer global dimensions.

Keywords: Global dimension; pre-modular fusion category; spherical fusion category

Mathematics Subject Classification 2010: 18D10 · 16T05

1 Introduction

Throughout this paper, let $\mathbb{C}$ be the field of complex numbers, $\mathbb{C}^* := \mathbb{C}\setminus\{0\}$, $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ denote the field of rationals and its algebraic closure, respectively. For $r \in \mathbb{N}$, let $\mathbb{Z}_r := \mathbb{Z}/r$. Categories are assumed to be semisimple $\mathbb{C}$-linear finite abelian categories.

For any finite abelian category $\mathcal{C}$, let $\mathcal{O}(\mathcal{C})$ be the set of isomorphism classes of simple objects of $\mathcal{C}$. The cardinal of $\mathcal{O}(\mathcal{C})$ is called rank of $\mathcal{C}$, and will denoted by $\text{rank}(\mathcal{C})$.

A fusion category $\mathcal{C}$ is spherical, if $\mathcal{C}$ admits a pivotal structure $j$, which is a natural isomorphism from identity tensor functor $\text{id}_\mathcal{C}$ to double dual tensor functor $(-)^{**}$, and the pivotal structure satisfies $\text{dim}(X) = \text{dim}(X^*)$ for any object $X$ of $\mathcal{C}$, where $\text{dim}(X)$ is the quantum dimension of $X$ defined by $j$, see section 2 for definition.

We know that all fusion categories of prime FP-dimensions are pointed [9, Corollary 8.30]. However, this is not the case for global dimensions. In [18, Example 5.1.2], Ostrik classified all spherical fusion categories of integer global dimension less than or equal to 5. Specifically, these fusion categories are pointed, or tensor equivalent to an Ising category $\mathcal{I}$ or equivalent to a Deligne tensor product $YL \boxtimes \overline{YL}$, where $YL$ is a Yang-Lee fusion category and $\overline{YL}$ is a Galois conjugate of $YL$. It’s easy to see that $YL \boxtimes \overline{YL}$ is not pointed and $\text{dim}(YL \boxtimes \overline{YL}) = 5$. Therefore, this is a non-trivial task to classify spherical fusion categories of given small global dimensions. Meanwhile, for a given global dimension, it follows from [18, Theorem 1.1.1] that there are finite many tensor equivalence classes of spherical fusion categories, then it is doable to classify spherical fusion categories of small global dimensions.

When classifying spherical fusion categories by global dimensions, one of the main difficulties is to restrict the rank of fusion category for a given dimension, and there have no general methods, see [18, Lemma 4.2.2]. Recall that a spherical fusion category $\mathcal{C}$ is a pre-modular fusion category if $\mathcal{C}$ is braided. There are some classification results of
(super-) modular fusion categories of small ranks, see [2, 3, 16, 17, 19]. So, in this paper, we turn our attentions to pre-modular fusion categories of small integer dimensions.

For fusion categories over field $\mathbb{C}$, it is well-known that Lagrange theorem for FP-dimension is true. That is, for any fusion category $C$ and fusion subcategory $D \subseteq C$, the ratio $\frac{FPdim(C)}{FPdim(D)}$ is an algebraic integer [9, Proposition 8.15]. In Theorem 3.1 we prove an analogue of Lagrange theorem of global dimension. However, $\frac{FPdim(C)}{dim(C)} \neq \frac{dim(D)}{dim(D)}$ in general, see Remark 3.2. Then we can use Theorem 3.1 to restrict global dimensions of pre-modular fusion subcategories of $C$. Together with techniques developed in [17, 18], we can apply some well-known classifications results on pre-modular fusion categories.

The organization of this paper is as follows. In section 2, we recall some basic notions and notations of fusion categories, such as global dimensions, formal codegrees and $d$-numbers introduced in [16]. In subsection 3.1 we give a proof of Lagrange theorem of global dimension of fusion categories in Theorem 3.1. In subsection 3.2 we first prove that pre-modular fusion categories of dimension 7 are pointed in Theorem 3.7, then in Proposition 3.9 we show that pre-modular fusion categories of global dimension 8 are always weakly integral if they are not simple. In subsection 3.3, we show that spherical fusion categories of dimension 6 are weakly integral (Theorem 3.11), and we give a partial result on classification of pre-modular of dimension 10 in Proposition 3.15.

### 2 Preliminaries

#### 2.1 Spherical fusion category

Let $C$ be a fusion category over $\mathbb{C}$. Homomorphism $FPdim(\cdot)$ of $C$ is the unique homomorphism from Grothendick ring $Gr(C)$ to $\mathbb{C}$ such that $FPdim(X) \geq 1$ is an algebraic integer for all objects $X \in O(C)$ [9, Theorem 8.6], and $FPdim(X)$ is called the Frobenius-Perron dimension of object $X$. The Frobenius-Perron dimension $FPdim(C)$ of fusion category $C$ is defined by

$$FPdim(C) := \sum_{X \in O(C)} FPdim(X)^2.$$ 

Fusion category $C$ is weakly integral, if $FPdim(C) \in \mathbb{Z}$; $C$ is integral, if $FPdim(X) \in \mathbb{Z}$ for all $X \in O(C)$. A fusion category $C$ is pointed if and only if all simple objects of $C$ have FP-dimension 1, so $C \cong Vect_G^\omega$ in this case, where $Vect_G^\omega$ is the category of $G$-graded finite-dimension vector space over $\mathbb{C}$, $\omega \in Z^3(G, \mathbb{C}^*)$ is a 3-cocycle. In the following, we use $C_{pt}$ and $C_{int}$ to denote the maximal pointed fusion subcategory and the maximal integral fusion subcategory of $C$, respectively.

Assume $X \in C$ is an object, let $(X^*, ev_X, coev_X)$ be a left dual object of $X$. That is, there exist morphisms $coev_X : I \to X \otimes X^*$ and $ev_X : X^* \otimes X \to I$ satisfying the following equations

$$id_X \otimes ev_X \circ coev_X \otimes id_X = id_X, \quad ev_X \otimes id_{X^*} \circ id_{X^*} \otimes coev_X = id_{X^*}.$$ 

Here, we suppress the associator and unit constraints of $C$. 

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In a fusion category $\mathcal{C}$, since $X \cong X^{**}$ and $\text{Hom}_{\mathcal{C}}(X, X) \cong \mathcal{C}$ for all $X \in \mathcal{O}(\mathcal{C})$, up to scalar, there is a unique isomorphism $\alpha_X : X \rightarrow X^{**}$. Then for any morphism $f : X \rightarrow Y$, we define trace of $f$ as the following scalar
\[
\text{tr}(\alpha_X \circ f) = (\text{ev}_X \circ (\alpha_X \circ f)) \otimes (\text{id}_X \circ \text{coev}_X) : I \rightarrow I.
\]

Following [9][13], then we define square norm of simple object $X$ as
\[
|X|^2 := \text{tr}(\alpha_X) \text{tr}((\alpha_X^{-1}), \text{and global dimension (or, categorical dimension) of fusion category } \mathcal{C} \text{ as}
\[
\text{dim}(\mathcal{C}) := \sum_{X \in \mathcal{O}(\mathcal{C})} |X|^2.
\]

Fusion category $\mathcal{C}$ is said to be pseudo-unitary if $\text{dim}(\mathcal{C}) = FP\text{dim}(\mathcal{C})$. It is easy to see that global dimension $\text{dim}(\mathcal{C})$ is independent of the choice of isomorphisms $\{\alpha_X | X \in \mathcal{O}(\mathcal{C})\}. \text{Moreover, } \text{dim}(\mathcal{C}) \geq 1 \text{ is a positive algebraic integer [9 Theorem 2.3] for an arbitrary fusion category } \mathcal{C}. \text{It follows from [9 Proposition 8.22] that the ratio } \frac{\text{dim}(\mathcal{C})}{\text{FPdim}(\mathcal{C})} \leq 1 \text{ is an algebraic integer. In addition, } \text{dim}(\mathcal{C}) > \frac{3}{2} \text{ if } \mathcal{C} \text{ is a spherical fusion category [13 Theorem 1.1.2]. Definitely, for non-trivial pseudo-unitary fusion categories } \mathcal{C}, \text{ dim}(\mathcal{C}) \leq 2. \text{For more properties of global dimension, we refer the readers to references [7][9][13][18].}

Let $\mathcal{C}$ be a pivotal fusion category with pivotal structure $j$, which is a natural isomorphism from $id_\mathcal{C}$ to the double dual tensor functor $(-)^{**}$. Then we define $\text{dim}_j(X) := \text{tr}(j_X)$, the quantum (or categorical) dimension of $X$ determined by $j$. A direct computation shows that $j_X^{**} = ((j_X)^{**})^{-1}$ [7 Exercise 4.7.9], so $\frac{\text{dim}_j(X)}{\text{dim}_j(X^{**})} = \frac{\text{dim}_j(X)}{\text{dim}_j(X^{**})}$ by [9 Proposition 2.9]. Hence,
\[
\text{dim}(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{dim}_j(X) \text{dim}_j(X^{**}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{dim}_j(X) \text{dim}_j(X^{**}).
\]

It is well-known that $\text{dim}_j(-)$ induces a homomorphism from $Gr(\mathcal{C})$ to $\mathcal{C}$ [7 Proposition 4.7.12]. It was conjectured in [9 Conjecture 2.8] that every fusion category admits a pivotal structure. Pivotal fusion category $\mathcal{C}$ is spherical, if $\text{dim}_j(X) = \text{dim}_j(X^{**})$ for any object $X$ of $\mathcal{C}$. Thus, for spherical fusion category $\mathcal{C},$
\[
\text{dim}(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{dim}_j(X)^2.
\]

We fix a spherical structure $j$ of $\mathcal{C}$, and we use $\text{dim}(X)$ instead of $\text{dim}_j(X)$ to denote the dimension of $X$ below.

Notice that for an arbitrary spherical fusion category $\mathcal{C}$, we can consider twist $\mathcal{C}^\sigma$ of $\mathcal{C}$, where $\sigma$ belongs to Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Specifically, $\mathcal{C}^\sigma$ is a fusion category with same monoidal functor $\otimes$ as $\mathcal{C}$, but associator of $\mathcal{C}^\sigma$ is obtained by composing the one of $\mathcal{C}$ with automorphism $\sigma$. Moreover, $\text{dim}(\mathcal{C}^\sigma) = \sigma(\text{dim}(\mathcal{C}))$.

In the last, we say a fusion category $\mathcal{C}$ is simple, if $\mathcal{C}$ does not contain any fusion subcategory other than $\mathcal{C}$ and $Vec$. For example, pointed fusion category $Vec_{Z_p}$ is simple for any prime $p$, where $\omega \in Z^3(Z_p, C^*)$ is a 3-cocycle; and fusion category $Rep(G)$ is simple if and only if $G$ is a finite simple group, where $Rep(G)$ is the category of finite-dimensional representations of $G$ over $\mathbb{C}$. 

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2.2 Pre-modular fusion category

Fusion category $\mathcal{C}$ is a braided fusion category, if for any $X, Y, Z \in \mathcal{C}$, there exists a natural isomorphism $c_{X,Y}: X \otimes Y \to Y \otimes X$, and braiding $\theta$ satisfies $c_{X,I} = c_{I,X} = id_X$, $c_{X \otimes Y,Z} = c_{X,Z} \otimes id_Y \circ id_X \otimes c_{Y,Z}, c_{Z,X \otimes Y} = id_X \otimes c_{Z,Y} \circ c_{Z,X} \otimes id_Y$, here we suppress the associativity isomorphism of $\mathcal{C}$.

Let $\mathcal{D}$ be a fusion subcategory of braided fusion category $(\mathcal{C}, c)$, the centralizer $\mathcal{D}'$ of $\mathcal{D}$ is the fusion subcategory generated by all simple objects $X$ of $\mathcal{C}$ such that $c_{Y,X}c_{X,Y} = id_{X \otimes Y}, \forall Y \in \mathcal{D}$. In particular, we call $\mathcal{C}'$ the M"{u}ger center of $\mathcal{C}$.

Braided fusion category $\mathcal{C}$ is a pre-modular (or ribbon) category, if $\mathcal{C}$ is spherical.

For pre-modular fusion category $\mathcal{C}$, we can define S-matrix $S = (s_{X,Y})_{X,Y \in \mathcal{O}(\mathcal{C})}$ and T-matrix $T = (T_{X,Y})_{X,Y \in \mathcal{O}(\mathcal{C})}$ of $\mathcal{C}$ [1]. Specifically, $s_{X,Y} = tr(c_{Y,X}c_{X,Y})$ and $T_{X,Y} = \delta_{X,Y} \theta_X$ for all $X,Y \in \mathcal{O}(\mathcal{C})$, where $\theta$ is the ribbon structure of $\mathcal{C}$.

Consequently, pre-modular fusion category $\mathcal{C}$ is modular if and only if its S-matrix is non-degenerate [1] [6] [7] [12], equivalently M"{u}ger center $\mathcal{C}' = V vec$, where $V vec$ is the category of finite-dimensional vectors spaces over $\mathbb{C}$. We use $\mathcal{C}(G, \eta)$ is the pointed modular fusion category determined by metric group $(G, \eta)$ below, see [6] Appendix A. Meanwhile, a pre-modular fusion category $\mathcal{C}$ is super-modular, if $\mathcal{C}' \cong s V vec$, where $s V vec$ is the category of finite-dimensional super-vectors spaces over $\mathbb{C}$.

Then it follows from [12] Lemma 5.4 or [6] Lemma 3.28] that rank of a supermodular fusion category $\mathcal{C}$ must be even. In fact, given a super-modular fusion category $\mathcal{C}$, let $\mathcal{C}' = \langle \chi \rangle = s V vec$. Based on the partition of set $\mathcal{O}(\mathcal{C}) = \Pi_0 \cup \Pi_1$ of $\mathcal{C}$, we can define the so-called naive fusion rule: for arbitrary simple objects $X, Y, Z \in \Pi_0$,

$$\hat{N}_{X,Y} := dim_\mathbb{C}(\text{Hom}(X \otimes Y, Z)) + dim_\mathbb{C}(\text{Hom}(X \otimes \chi, \chi \otimes Z))$$

Indeed, for any $W \in \Pi_1$, there is a unique $V \in \Pi_0$ such that $W = \chi \otimes V$.

Then there is a non-degenerate symmetric matrix $\hat{S}$ such that the S-matrix of $\mathcal{C}$ is

$$\begin{pmatrix}
\hat{S} & \hat{S} \\
\hat{S} & \hat{S}
\end{pmatrix},$$

and $\hat{S}$ has orthogonal rows. Moreover, as $\Pi_0$ is close under taking dual objects, the naive fusion rule and $\Pi_0$ generate a unital fusion ring $R$ satisfying that all homomorphisms from $R$ to $\mathbb{C}$ have form $\phi(X) = \frac{\delta_{X,Y} \hat{s}}{\delta_{I,Y}}, \forall Y \in \Pi_0$, see [2] Proposition 2.7 for details and [20] for some applications. In particular, we can define homomorphism $FPdim(-)$ of fusion ring $R$ as follows:

$$FPdim(X) := \frac{\hat{s}_{X,Y}}{\delta_{I,Y}}, \forall X \in \Pi_0, \text{ for some } Y \in \Pi_0.$$  

For FP-dimensions of fusion rings, see [7] §3.3. Notice that $FPdim(X)$ is not FP-dimension of simple object $X$ in $\mathcal{C}$, as it is determined by the naive fusion rule $\hat{N}$.

2.3 Formal codegrees and d-numbers

Given a fusion category $\mathcal{C}$, let $\text{Irr}(Gr(\mathcal{C}))$ be the set of isomorphism classes of irreducible representations of $Gr(\mathcal{C})$ over $\mathbb{C}$. For irreducible representations $\chi, \chi' \in \text{Irr}(Gr(\mathcal{C}))$, let $Tr_\chi$ be ordinary trace function on representation $\chi$. Then up to scalar,
by \[10\] there exists a unique central element \( \alpha_x := \sum_{X \in \mathcal{O}(C)} \text{Tr}_X(X)X^* \in \text{Gr}(C) \otimes \mathbb{Z} \mathbb{C} \) such that \( \chi'(\alpha_x) = 0 \) if \( \chi \not\equiv \chi' \), and \( f_x = \chi(\alpha_x) \) is a positive algebraic integer. In fact, it is well-defined for all fusion rings \([10]\).

We call these algebraic integers \( f_x \) \((\forall \chi \in \text{Irr}(\text{Gr}(C)))\) formal codegrees of \( C \) \([10]\).

It is proved in \([17]\) Corollary 2.14 that \( \frac{\dim(f_x)}{\dim(C)} \) are also algebraic integers. And formal codegrees of fusion category \( C \) satisfy the following equation \([17]\) Proposition 2.10]

\[
(1) \quad \sum_{\chi \in \text{Irr}(\text{Gr}(C))} \frac{\chi(1)}{f_x} = 1.
\]

Obviously, we have \( f_x \geq 1 \). Moreover, if spherical fusion category \( C \not\equiv \text{Vec} \), then \( f_x \geq \sqrt{\frac{2 \text{rank}(C)}{\dim(C)}} \geq \sqrt{\frac{2}{3}} \) for all \( \chi \in \text{Irr}(\text{Gr}(C)) \) by \([15]\) Theorem 4.2.1.

Note that if there exists a homomorphisms \( \phi \) from \( \text{Gr}(C) \) to \( \mathbb{C} \), definitely it is an irreducible representation of \( \text{Gr}(C) \). Then the corresponding formal codegrees \( f_\phi \) of \( C \) is given by

\[
\sum_{X \in \mathcal{O}(C)} \phi(X)\phi(X^*) = \sum_{X \in \mathcal{O}(C)} \phi(XX^*) = \sum_{X \in \mathcal{O}(C)} |\phi(X)|^2.
\]

In particular, for arbitrary fusion category \( C \), \( F\text{Pdim}(\_\_\_) \) is an irreducible representation of Grothendieck ring \( \text{Gr}(C) \), so \( F\text{Pdim}(\_\_\_) \) and its Galois conjugates are formal codegrees of \( C \). Similarly, if \( C \) is pivotal then \( \dim(C) \) is also a formal codegree.

For pre-modular fusion category \( C \), let \( X, Y \in \mathcal{O}(C) \), then map \( h_X(Y) := \frac{s_{X,Y}}{s_{I,X}} \)
defines a homomorphism from Grothendieck ring \( \text{Gr}(C) \) to \( \mathbb{C} \) \([2]\) Proposition 8.13.11].

In particular, if \( C \) is modular, then Verlinde formula \([12]\) says that \( \{h_X | \forall X \in \mathcal{O}(C)\} \) is a complete set of homomorphisms from Grothendieck ring \( \text{Gr}(C) \) to \( \mathbb{C} \). Therefore, if \( C \) is modular, then formal codegrees are equal to \( \frac{\dim(C)}{\dim(X)^2} \) for objects \( X \in \mathcal{O}(C) \). Indeed, for \( s_{I,X} = \dim(X) \), Verlinde formula implies that

\[
\sum_{Y \in \mathcal{O}(C)} h_X(Y)h_X(Y^*) = \sum_{Y \in \mathcal{O}(C)} \frac{s_{X,Y}}{\dim(X)} \frac{s_{X,Y^*}}{\dim(X)} = \sum_{Y \in \mathcal{O}(C)} \frac{s_{X,Y}s_{X,Y^*}}{\dim(X)^2} = \frac{\dim(C)}{\dim(X)^2}.
\]

Therefore, if \( \dim(X)^2 \in \mathbb{Z} \) for all \( X \in \mathcal{O}(C) \), then \( C \) is weakly integral, as the rational number \( F\text{Pdim}(C) = \frac{\dim(C)}{\dim(Y)^2} \) (for certain \( Y \in \mathcal{O}(C) \)) is an algebraic integer.

Recall that an algebraic integer \( \alpha \) is a \( d \)-number \([16]\) Definition 1.1], if in the algebraic integer ring, the ideal generated by \( \alpha \) is invariant under action of Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Equivalently, there exists a polynomial \( f(x) = x^n + a_1x^{n-1} + \cdots + a_n \) with integer coefficients such that \( f(\alpha) = 0 \) and \( (a_i) \) are positive, for all \( 1 \leq i \leq n \). See \([16]\) Lemma 2.7] for more equivalent conditions about \( d \)-number.

For example, \( \dim(X) \) of simple objects \( X \) of pre-modular fusion category \( C \) are \( d \)-numbers \([16]\) Theorem 1.8]. Moreover, it was proved in \([16]\) Theorem 1.2] that formal codegrees of fusion categories are \( d \)-numbers. However, formal codegrees of fusion rings are not \( d \)-numbers in general, see \([16]\) Example 1.6]. Thus, we can use this property to detect whether a fusion ring is categorifiable, this is called \( d \)-number test.

In addition, following \([7]\) Definition 7.21.13], we say that an algebraic integer \( \alpha \) is totally positive, if \( \alpha \) is still positive under any embedding of algebraic integers into field.
For example, \(\sqrt{5}\) is not a totally positive integer. For any fusion category \(\mathcal{C}\), let \(X \in \mathcal{O}(\mathcal{C})\), \(FPdim(X)^2\) is totally positive, thus \(FPdim(\mathcal{C})\) is totally positive. Indeed, all formal codegrees of \(\mathcal{C}\) are totally positive by [17, Remark 2.12].

We also use the following theorem [9, Corollary 8.53], which is called cyclotomic test in [17, Proposition 2.1].

**Theorem 2.1.** Given a fusion category \(\mathcal{C}\), let \(\chi\) be an irreducible representation of Grothendieck ring \(Gr(\mathcal{C})\). Then \(\chi\) is defined over \(\mathbb{Q}(\xi)\) for some root of unity \(\xi\).

Therefore, \(FPdim(X), FPdim(\mathcal{C}) \in \mathbb{Z}[\xi]\), \(\forall X \in \mathcal{O}(\mathcal{C})\). That is, the Galois groups of minimal polynomials defining FP-dimension of simple objects and \(\mathcal{C}\) have to be abelian. For this purpose, we use program GAP to compute Galois groups of minimal polynomials of \(FPdim(\mathcal{C})\), also we use GAP to do \(d\)-number test.

### 3 Main result

In this section, we first prove the Lagrange theorem for global dimension of fusion categories. Then we consider classifications of pre-modular fusion categories of global dimensions 6, 7, 8 and 10, respectively.

#### 3.1 Lagrange theorem of global dimension

Given a fusion category \(\mathcal{C}\), an algebra \(A \in \mathcal{C}\) is connected, if \(dim(\mathcal{C}(Hom_C(I, A))) = 1\).

In a braided fusion category \(\mathcal{C}\), recall that a commutative algebra \(A\) is said to be an étale algebra, if \(\mathcal{C}_A\) is semisimple [5, Proposition 2.7], where \(\mathcal{C}_A\) is the category of right \(A\)-modules in \(\mathcal{C}\). For a connected étale algebra \(A\) in \(\mathcal{C}\), the subcategory \(\mathcal{C}_A^0 \subseteq \mathcal{C}_A\) of dyslectic (or local) modules of \(A\) in \(\mathcal{C}\) is a braided fusion category. See [5, 11] for details about étale algebras and their dyslectic modules.

Now we are ready to give a proof of the Lagrange theorem for global dimension of fusion categories. For pseudo-unitary fusion categories, this is [9, Proposition 8.15].

**Theorem 3.1.** Let \(\mathcal{C}\) be a fusion category, and let \(\mathcal{D}\) be a fusion subcategory of \(\mathcal{C}\). Then \(\frac{dim(\mathcal{C})}{dim(\mathcal{D})}\) is an algebraic integer.

**Proof.** We can assume that \(\mathcal{C}\) is a spherical fusion category. In fact, if not, it follows from [3, Remark 3.1] and [9, Proposition 5.14] that pivotalization \(\hat{\mathcal{C}}\) of \(\mathcal{C}\) is a spherical fusion category. In addition, we have equality \(dim(\hat{\mathcal{C}}) = 2dim(\mathcal{C})\) [7, Remark 7.21.11]. Consequently, \(\frac{dim(\mathcal{C})}{dim(\mathcal{D})} = \frac{dim(\hat{\mathcal{C}})}{dim(\hat{\mathcal{D}})}\).

For any fusion subcategory \(\mathcal{D} \subseteq \mathcal{C}\), up to isomorphism, [5, Theorem 4.10] says that there exists a unique connected étale subalgebra \(A \subseteq \mathcal{I}(I)\) corresponding to \(\mathcal{D}\), which satisfies \(\frac{FPdim(\mathcal{C})}{FPdim(\mathcal{D})} = FPdim(A)\), where \(\mathcal{I}\) is the right adjoint functor of forgetful tensor functor \(F: \mathcal{Z}(\mathcal{C}) \to \mathcal{C}\). Since \(\mathcal{C}\) is a spherical fusion category, Drinfeld center \(\mathcal{Z}(\mathcal{C})\) is a modular fusion category by [14, Theorem 6.4]. Note that \(A = \oplus_{X \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [A : X]X\), where \([A : X] := \frac{dim(\mathcal{C}(Hom_{\mathcal{Z}(\mathcal{C})}(A, X)))}{dim(X)},\) hence

\[
    dim(A) = \sum_{X \in \mathcal{O}(\mathcal{Z}(\mathcal{C}))} [A : X]dim(X),
\]

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If \([A : X]\) is non-zero, then \(\dim(X) = \frac{\dim(C)}{f_X}\) by [17] Theorem 2.13 for some formal codegree \(f_X\) of \(C\). In particular, since all formal codegrees of \(C\) are positive algebraic integers, we obtain \(\dim(A) > 0\). Thus by [5] Remark 3.4, étale algebra \(A\) is a rigid \(C\)-algebra in sense of [11] Definition 1.11.

Notice that [5] Theorem 4.10] shows that there exists a braided equivalence of non-degenerate fusion categories \(Z(D) \cong Z(C)_A^{\theta}\). Meanwhile, \(\theta_{Z(I)} = id_{Z(I)}\) by [15] Theorem 4.1] or [18] Theorem 2.7], where \(\theta\) is the ribbon structure of \(Z(C)\). Then \(\theta_A = id_A\) as \(A \subseteq I\), and [11] Theorem 4.5] says that \(Z(D) \cong Z(C)_A\) as modular fusion categories. Moreover, \(\dim(Z(D)) = \dim(D)^2\) by [9] Theorem 2.15] or [14] Theorem 1.2]. Then we deduce from [11] Theorem 4.5] again that

\[
\dim(D)^2 = \dim(Z(D)) = \dim(Z(C)_A) = \frac{\dim(Z(C))}{\dim(A)^2} = \frac{\dim(C)^2}{\dim(A)^2},
\]

so the ratio \(\frac{\dim(C)}{\dim(D)} = \dim(A)\) is an algebraic integer, as desired.

\(\square\)

**Remark 3.2.** Let \(C\) be a fusion category. For fusion subcategory \(D \subseteq C\), Theorem 3.1 also says that \(\frac{\dim(C)}{\dim(D)} = \frac{FPdim(C)}{FPdim(D)}\) if and only if \(FPdim(A) = \dim(A)\). This equality fails in general, however. For example, let \(D = YL\) be the Yang-Lee fusion category of global dimension \(\frac{5+\sqrt{5}}{2}\), its conjugate \(Y\overline{L}\) has global dimension \(\frac{5-\sqrt{5}}{2}\). Assume \(C = YL \bigotimes Y\overline{L}\), so \(\dim(C) = 5\) and \(FPdim(C) = (\frac{5+\sqrt{5}}{2})^2 = \frac{15+5\sqrt{5}}{2}\) since \(FPdim(YL) = \frac{5+\sqrt{5}}{2}\). These two ratios are not equal obviously.

Given two tensor categories \(C\) and \(D\), recall that a tensor functor \(F : C \to D\) is said to be surjective, if every simple object of \(D\) is a subobject of \(F(X)\) for some object \(X \in C\); \(F\) is injective if \(F\) is bijective on sets of morphisms [7] Definition 1.8.3].

Same as [9] Corollary 8.11], we have the following corollary:

**Corollary 3.3.** Let \(C\) and \(D\) be fusion categories, assume \(F : C \to D\) is a surjective tensor functor. Then \(\frac{\dim(C)}{\dim(D)}\) is an algebraic integer.

**Proof.** Since \(F\) is a surjective tensor functor, we can regard \(D\) as an indecomposable left \(C\)-module category via tensor functor \(F\). Let \(C_D^*\) and \(D_D^*\) be the tensor categories of left module functors of \(C\) and \(D\) with respect to module category \(D\), respectively. This is the so-called exact pair \((F, D)\), see [7] §7.17] and [9] for details.

Therefore, we deduce from [7] Corollary 7.12.13] that both \(C_D^*\) and \(D_D^*\) are fusion categories. Then we obtain an injective tensor functor \(F^* : D_D^* \to C_D^*\) by [9] Proposition 5.3]. It follows from [7] Proposition 9.3.9] that we have the following equation

\[
\frac{\dim(C)}{\dim(D)} = \frac{\dim(C_D^*)}{\dim(D_D^*)},
\]

which is an algebraic integer by Theorem 3.1.

\(\square\)

In Corollary 3.3 it is easy to see that \(\frac{\dim(C)}{\dim(D)} \neq \frac{FPdim(C)}{FPdim(D)}\) in general. For example, let \(D\) be a fusion category such that \(\dim(D) \neq FPdim(D)\), and \(C = Z(D)\). Let \(F : Z(D) \to D\) be the forgetful tensor functor, so \(F\) is surjective. Then [9] Theorem 2.15] and [9] Proposition 8.12] say that \(\frac{\dim(C)}{\dim(D)} = \dim(D) \neq FPdim(D) = \frac{FPdim(C)}{FPdim(D)}\).
3.2 Pre-modular fusion categories of dimensions 7

Spherical fusion categories $\mathcal{C}$ of integer dimension with $\text{dim}(\mathcal{C}) \leq 5$ were classified completely in [18] Example 5.1.2. Explicitly, $\mathcal{C}$ is pointed, or $\mathcal{C}$ is equivalent to an Ising category $\mathcal{I}$, or $\mathcal{C}$ is equivalent to the Deligne tensor product $Y \otimes Y L$. We first classify pre-modular fusion categories of dimension 6.

**Proposition 3.4.** Let $\mathcal{C}$ be a pre-modular fusion category of global dimension 6, then $\mathcal{C}$ is an integral fusion category.

**Proof.** We consider M"uger center $\mathcal{C}'$ of $\mathcal{C}$. Obviously, $\mathcal{C}$ is integral if $\mathcal{C}$ is symmetric. If $\mathcal{C}'$ is a proper subcategory of $\mathcal{C}$, then $\text{dim}(\mathcal{C}') = 1, 2, 3$ by Theorem 3.1. If $\mathcal{C}' = \text{Rep}(G)$ is Tannakian, then $\mathcal{C} \cong \mathcal{D}'$ is integral by [18] Example 5.1.2. If $\mathcal{C}' \cong s\text{Vec}$, then $\text{rank}(\mathcal{C}) = 4, 6$. [18] Remark 4.2.3 shows that $\text{rank}(\mathcal{C}) = 6$ if and only if $\mathcal{C}$ is pointed. If $\text{rank}(\mathcal{C}) = 4$, then for any simple object $X \in \mathcal{C}$, $\text{dim}(X)^2 \in \{1, 2\}$. Meanwhile, [20] Corollary 3.4 shows that $\frac{6}{\text{dim}(X)}$ is an algebraic integer, this is impossible. If $\mathcal{C}$ is modular, when $\text{rank}(\mathcal{C}) < 6$, results of [3] [19] imply that there is no such modular fusion category. Therefore, $\mathcal{C}$ is an integral fusion category. \qed

For any modular fusion category $\mathcal{A}$, let $\mathcal{E}$ be an arbitrary symmetric fusion subcategory of $\mathcal{A}$, then $\mathcal{E} \subseteq \mathcal{E}'$ by definition. [6] Theorem 3.10 and Theorem 3.1 show that $\text{dim}(\mathcal{E}') = \text{FPdim}(\mathcal{E}')$ is an integer. Assume $p$ is a prime. Let $\mathcal{C}$ be a pre-modular fusion category of global dimension $p$. Since $\mathcal{C}'$ is a symmetric fusion subcategory of $\mathcal{C}$, and $\text{dim}(\mathcal{C}') = \text{FPdim}(\mathcal{C}')$ is an integer, Theorem 3.1 shows that rational number $\frac{p}{\text{dim}(\mathcal{C}')}$ is an algebraic integer. Then $\mathcal{C}$ is symmetric if $p = \text{dim}(\mathcal{C}')$, otherwise $\mathcal{C}$ is modular. If $\mathcal{C}$ is symmetric, obviously $\mathcal{C} \cong \text{Rep}(\mathbb{Z}_p)$ or $\mathcal{C} \cong s\text{Vec}$, so $\mathcal{C}$ is a pointed fusion category.

We assume that pre-modular fusion category $\mathcal{C}$ is not pointed below. In particular, $\mathcal{C}$ is modular and $\mathcal{C}_{\text{int}} = C_{\text{pt}} = \text{Vec}$ by Theorem 3.1. Moreover, when prime $p > 5$, it follows from [3] [19] that dimensions of modular fusion categories of rank less than 6 do not equal to $p$, then [18] Lemma 4.2.2 shows that $6 \leq \text{rank}(\mathcal{C}) \leq p - 1$. Let $\mathcal{D} \subseteq \mathcal{C}$ be an arbitrary fusion subcategory, note that $\mathcal{D} \cap \mathcal{D}'$ is symmetric, where $\mathcal{D}'$ is the centralizer of $\mathcal{D}$, so $\mathcal{D} \cap \mathcal{D}' = \text{Vec}$. Therefore, $\mathcal{D}$ is also a modular fusion category. Consequently, it follows from [3] Theorem 3.13] that $\mathcal{C} \cong \mathcal{A}_1 \boxtimes \cdots \boxtimes \mathcal{A}_s$, where $\mathcal{A}_i$ are simple modular fusion subcategories of $\mathcal{C}$, $1 \leq i \leq s < \text{rank}(\mathcal{C}) \leq p$.

Note that $\text{rank}(\mathcal{C}) = \prod_{i=1}^s \text{rank}(\mathcal{A}_i)$. Hence, in order to classify modular fusion categories of prime global dimensions, first we have to consider classification of simple non-pointed modular fusion categories of small ranks. When $\text{dim}(\mathcal{C}) = 7, 11, 13$, classification results of [3] [19] say that modular fusion category $\mathcal{C}$ must be simple.

To classify modular fusion categories of global dimension 7, we need to prove the following two lemmas.

**Lemma 3.5.** If there exists a fusion category $\mathcal{C}$ of rank 6 and $\text{FPdim}(\mathcal{C}) = \frac{31 + 7\sqrt{5}}{2}$, moreover $\mathcal{C}_{\text{int}} = \text{Vec}$, and $\text{FPdim}(X) \in \mathbb{Q}(\sqrt{5})$ for all $X \in \mathcal{O}(\mathcal{C})$. Then

$$\text{FPdim}(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{FPdim}(X)^2 = 1 + 4 \times \left(\frac{1 + \sqrt{5}}{2}\right)^2 + \left(\frac{3 + \sqrt{5}}{2}\right)^2.$$
is the unique decomposition into sum of squares of FP-dimensions of simple objects.

Proof. Let \( FPdim(C) = 1 + \sum_{i=1}^{5} \frac{a_i + b_i \sqrt{5}}{2} \), where \( a_i, b_i \) are positive rational numbers, and \( FPdim(X_i)^2 = \frac{a_i + b_i \sqrt{5}}{2} \), \( 1 \leq i \leq 5 \). Indeed, by assumption if \( b_i \neq 0 \). If \( b_i < 0 \), since \( \frac{a_i + b_i \sqrt{5}}{2} \) is square of FP-dimension of simple objects, while \( \frac{a_i - b_i \sqrt{5}}{2} > \frac{a_i + b_i \sqrt{5}}{2} \), this contradicts to property of FP-dimension of simple object \([2]\) Proposition 3.3.4]. If \( a_i \leq 0 \), then by definition \( \frac{a_i + b_i \sqrt{5}}{2} \) can not totally positive algebraic integers.

Next, we show \( a_i \) and \( b_i \) are integers. Let \( \alpha_i = \frac{a_i + b_i \sqrt{5}}{2} \), then \( \alpha_i \) are roots of equations \( 4x^2 - 4a_i x + a_i^2 - 5b_i^2 = 0 \). Hence, \( a_i \in \mathbb{Z} \). Since \( \frac{a_i + b_i \sqrt{5}}{2} \) is a totally positive algebraic integer, \( \frac{a_i - b_i \sqrt{5}}{2} \) is also positive. Let \( 4m = a_i^2 - 5b_i^2 \), where \( m \) is a positive integer, then \( b_i = \sqrt{\frac{a_i^2 - 4m}{5}} \) is a rational number, which means \( b_i \) is an integer.

Assume \( b_i \leq b_j \) whenever \( i \leq j \). Therefore, \( (b_1, b_2, b_3, b_4, b_5) = (1, 1, 1, 2, 2) \) or \( (1, 1, 1, 1, 3) \). In the first case, if \( X_5 \in O(C) \), and

\[
FPdim(X_5)^2 = \frac{a_5 + 2\sqrt{5}}{2} = \left( \frac{a_5 + \beta_5 \sqrt{5}}{2} \right)^2 = \frac{a_5^2 + 4\beta_5^2}{4} + \frac{a_5 \beta_5 \sqrt{5}}{2},
\]

then \( \begin{cases} a_5 = 2 \quad \text{or} \quad \alpha_5 = 1 \\ \beta_5 = 1 \end{cases} \). This is impossible, as both \( \frac{2i + \sqrt{5}}{2} \) and \( \frac{1 + i \sqrt{5}}{2} \) are not an algebraic integers. In the second case, similarly, we have \( \begin{cases} a_5 = 1 \quad \text{or} \quad \alpha_5 = 3 \\ \beta_5 = 2 \end{cases} \).

If \( FPdim(X_5) = \frac{1 + i \sqrt{5}}{2} \), then \( FPdim(X_5)^2 = \frac{2 + 2\sqrt{5}}{2} \), this is impossible. Then we obtain that \( FPdim(X_5) = \frac{1 + i \sqrt{5}}{2} \) and \( FPdim(X_j) = \frac{1 + i \sqrt{5}}{2} \), for \( 1 \leq j \leq 4 \).

Lemma 3.6. Let \( C \) be a fusion category of rank 6 and \( FPdim(C) = \frac{2 + \sqrt{5}}{2} \), moreover \( C_{int} = Vec \), and \( FPdim(X) \in \mathbb{Q}(\sqrt{5}) \) for all \( X \in O(C) \). Then exists exactly two self-dual simple objects in \( O(C) \).

Proof. Assume \( O(C) = \{ I, V, W, X, Y, Z \} \). By Lemma 3.5 let \( FPdim(Z) = \frac{2 + \sqrt{5}}{2} \), FP-dimensions of \( V, W, X, Y \) are all equal to \( \frac{1 + \sqrt{5}}{2} \). Obviously, \( Z \) is self-dual.

If \( V, W, X, Y \) are self-dual simple objects. Let \( A \not\cong X \) be an arbitrary simple object of FP-dimension \( \frac{1 + \sqrt{5}}{2} \), then by computing FP-dimension of \( X \otimes A, X \otimes A = Z \).

Meanwhile, \( X \otimes X = I \oplus B \), while \( B \) is another simple object. We claim \( B \not\cong X \).

Indeed, if \( B \cong X \), then \( C \) contains a Yang-Lee fusion category \( YL = \langle X \rangle \). However, the ratio \( \frac{FPdim(B)}{FPdim(X)} \) is not an algebraic integer, this contradicts to \([2]\) Proposition 8.15].

Again, \( B \otimes X \cong Z \). While, \( (B \otimes X) \otimes X = Z \otimes X \), and

\[
(B \otimes X) \otimes X \cong B \otimes (X \otimes X) \cong B \otimes (I \oplus B) \supseteq I,
\]

which implies that \( X \) is a dual object of \( Z \), this is impossible.

If \( X, Y \) are self-dual simple objects, and \( V^* = W \). Then \( X \otimes X = I \oplus A \) and \( Y \otimes Y = I \oplus B \), where \( A, B \) are self-dual simple objects. Since \( C \) does not contain a Yang-Lee fusion category, we must have \( A \cong Y \) and \( B \cong X \). In addition, we have \( Y \otimes X \cong Z \). However,

\[
Z \otimes X \cong (Y \otimes X) \otimes X \cong Y \otimes (X \otimes X) \cong Y \otimes (I \oplus Y) \supseteq I,
\]

which again implies that \( X \) is a dual object of \( Z \), this is impossible.
Theorem 3.7. Let $\mathcal{C}$ be a modular fusion category of global dimension 7. Then is pointed. That is, $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_7, \eta)$.

Proof. On the contrary, assume that $\mathcal{C}$ is not pointed, then previously argument says that $\mathcal{C}$ is a simple modular fusion category of rank 6. In particular, homomorphisms $\text{dim}(-)$ and $FPdim(-)$ are not in the same orbits under the action of Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and each orbit of $\text{dim}(-)$ and $FPdim(-)$ has at least two homomorphisms. Hence, homomorphisms from $Gr(\mathcal{C})$ to $\mathcal{C}$ are divided into two or three or four orbits.

(i). There are two orbits of homomorphisms from $Gr(\mathcal{C})$ to $\mathcal{C}$.

(ii). If orbit of $\text{dim}(-)$ contains 4 homomorphisms. Let $f$ be a formal codegree of $\mathcal{C}$, which is a Galois conjugate of $FPdim(\mathcal{C})$. Then \cite{17} Proposition 2.10 (or equation \[1\]) means that

$$\frac{4}{7} + \frac{1}{f} + \frac{1}{FPdim(\mathcal{C})} = 1$$

While all formal codegrees of $\mathcal{C}$ are bigger than 1 and $FPdim(\mathcal{C}) > 7$, moreover $FPdim(\mathcal{C}) \cdot f \neq 7^2$ by \cite{17} Corollary 2.14. Therefore, $FPdim(\mathcal{C})$ and $f$ are roots of equation $x^2 - 21x + 49 = 0$, so $FPdim(\mathcal{C}) = \frac{21 + 7\sqrt{5}}{2}$ and $f = \frac{21 - 7\sqrt{5}}{2}$. Since $\mathcal{C}$ is modular, all formal codegrees are like $\sum_{X \in \mathcal{O}(\mathcal{C})} \text{h}_Y(X) \text{h}_Y(X^*)$ for simple objects $X$, then

$$\text{dim}(X)^2 \in \{1, \frac{3 + \sqrt{5}}{2}, \frac{3 - \sqrt{5}}{2}\}, \forall X \in \mathcal{O}(\mathcal{C}).$$

Note that there are four simple objects $X$ of $\mathcal{C}$ which satisfy $\text{dim}(X)^2 = 1$. Moreover, simple object $X$ is self-dual if $\text{dim}(X)^2 = \frac{3 + \sqrt{5}}{2}$. Since $\mathcal{C}$ is spherical, $\text{dim}(X^*) = \text{dim}(X)$ for any simple object $A$ of $\mathcal{C}$. Then we see that there are at least four simple objects are self-dual, by Lemma \[3.6\] such fusion category $\mathcal{C}$ does not exist.

(ii). If the orbit of $\text{dim}(-)$ has 3 homomorphisms. Let $f_1 \leq f_2 \leq f_3 = FPdim(\mathcal{C})$ be distinct Galois conjugates of $FPdim(\mathcal{C})$. By equation \[1\] we have

$$\frac{3}{7} + \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} = 1,$$

so, $f_1 > \frac{7}{3}$, $f_2 > \frac{7}{2}$, and $f_3 > 7$. Again, $f_1f_2f_3 \neq 7^3$, so $f_1f_2f_3 = 49$ or $f_1f_2f_3 = 343$. If $f_1f_2f_3 = 49$, then $d$-number test \cite{16} Lemma 2.7 shows that $f_1, f_2, f_3$ are roots of equation $x^3 - 14x^2 + 28x - 49 = 0$, which fails to satisfy cyclotomic test in Theorem \[2.1\] and similarly, if $f_1f_2f_3 = 343$, then $f_1, f_2, f_3$ are roots of equation $x^3 - \alpha x^2 + 196x - 343 = 0$, where $\alpha = f_1 + f_2 + f_3$. Obviously, $d$-number test \cite{16} Lemma 2.7 shows that $7|\alpha$, then $\alpha \geq 14$.

Meanwhile, $\alpha < (f_1 - \frac{3}{7})f_2 + (f_3 - 6)f_1 + (f_2 - \frac{5}{2})f_3 < 196 - \frac{3}{7}196$, then $\alpha \leq 78$. However, for these integers, a direct computation shows that there is no solution for cyclotomic test in Theorem \[2.1\] if $\alpha \neq 28$. While when
\( \alpha = 28 \), roots of equation \( x^3 - 28x^2 + 196x - 343 = 0 \) are equal to 7, \( \frac{21 + \sqrt{3}}{2} \). By assumption the orbit of \( FPdim(\cdot) \) also have three homomorphisms, equation \( x^3 - 28x^2 + 196x - 343 = 0 \) must not have integer root, this is a contradiction.

(iii). If the orbit of \( dim(\cdot) \) contains 2 homomorphisms. Then \( \mathcal{C} \) contains a unique self-dual non-trivial simple object \( X \) such that \( dim(X)^2 = 1 \). Let \( f_1 \leq f_2 \leq f_3 \leq f_4 \) be conjugated formal codegrees of \( \mathcal{C} \). Assume they are roots of equation \( x^4 - \beta_1 x^3 + \beta_2 x^2 - \beta_3 x + \beta_4 = 0 \) (**), where \( \beta_i \) are positive integers, \( 1 \leq i \leq 4 \). Hence equation (**) shows that

\[
\frac{2}{7} + \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} = 1.
\]

Then \( f_1 > \frac{7}{5} \), \( f_2 > \frac{44}{15} \), \( f_3 > \frac{44}{15} \) and \( f_4 > 7 \). Hence \( \beta_1 \geq 21 \) and \( \beta_4 > 10 \beta_1 \).

Note that \( \beta_2 \geq 6 \sqrt{\beta_1} \), and \( \beta_2 = 6 \sqrt{\beta_1} \) if and only if \( f_1 = f_2 = f_3 = f_4 \), so \( \beta_2 > 6 \sqrt{\beta_1} \) and \( \beta_1 > 4 \sqrt{\beta_4} \). We know that \( \beta_4 \beta_1^4 \), then \( \beta_4 = 343, 2401 \).

Meanwhile, let \( \beta_2 = 7^n \), where \( n \) is a positive integer such that \( (7, n) = 1 \), \( d \)-number test \cite{16} Lemma 2.7] shows \( \frac{\beta_2}{\beta_1^4} \), so \( y \geq 2 \). Also,

\[
\beta_2 = \sum_{1 \leq i < j \leq 4} f_if_j < f_1(f_2 + f_3) + (f_2 - \frac{9}{5})f_1f_4 + (f_1 - \frac{2}{5})f_2f_3 + (f_3 - \frac{16}{5})f_2f_4 + (f_1 - \frac{2}{5})f_3f_4 < \beta_3 - f_4(\frac{16}{5}f_2 + \frac{2}{5}f_3 + \frac{1}{5}f_1).
\]

When \( \beta_3 = 343 \), then \( \beta_1 = 21, \beta_3 = 245 \), and \( 98 \leq \beta_2 \leq 81 - 77 = 168 \). So \( \beta_2 = 98, 147 \) for \( 49 | \beta_2 \). In these two cases, they do not satisfy cyclotomic test in Theorem 2.1. When \( \beta_1 = 2401 \), then \( \beta_3 = 1715, 35 \leq \beta_1 \leq 168 \). Since \( 1715 = \beta_3 > 21f_4, \beta_2 \leq \beta_3 - 890 = 825 \). While, except when \( \beta_2 = 392 \) and \( \beta_1 = 35 \), either all roots of equation (**2) are not real or they don’t satisfy cyclotomic test in Theorem 2.1. When \( \beta_2 = 392 \) and \( \beta_1 = 35 \), roots of equation \( x^4 - 35x^3 + 392x^2 - 1715x + 2401 = 0 \) are \( 7, 7, \frac{21 + \sqrt{3}}{2} \), this is a contradiction by argument of subcase (ii).

(2). There are three orbits of homomorphisms under Galois group’s action. If each orbit has two homomorphisms. Let \( f_1 \leq f_2 \) and \( f_3 \leq f_4 \) be conjugated formal codegrees of \( \mathcal{C} \). Hence, by equation (**)

\[
\frac{2}{7} + \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} = 1.
\]

Assume \( f_2 = FPdim(\mathcal{C}) \), then \( f_1f_2 = 49 \), in addition, \( \frac{1}{f_1} + \frac{1}{f_2} < \frac{2}{7} \) and \( f_1 + f_2 \geq 2\sqrt{f_1f_2} = 14 \). While \( FPdim(\mathcal{C}) > 7 \), so \( f_1 + f_2 = 21, 28 \). If \( f_1 + f_2 = 28 \), then \( \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{4} \), however, there is no real solutions in equation \( x^2 - 7x + 49 = 0 \). Therefore, \( f_1 + f_2 = 21 \) and \( f_3 = f_4 = 7 \), but subcase (i) shows that modular fusion category of dimension 7 has to be pointed, this is impossible. If there exists a homomorphism \( \phi \) from \( Gr(\mathcal{C}) \) to \( \mathcal{C} \) fixed by group \( Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \), then formal codegrees \( \sum_{X \in \mathcal{C}(\cdot)} [\phi(X)]^2 = 7 \) by \cite{15} Theorem 1.1.2] since it is an integer and divides 7. Hence, arguments of subcases (ii) and (iii) say that there is a contradiction.
Proposition 3.9. Let \( C \) be a pre-modular fusion category of global dimension 8. If \( C \) is not simple, then \( C \) is weakly integral.

Proof. If \( C \) contains a non-trivial Tannakian subcategory \( \text{Rep}(G) \), then \( C \cong D^G \) with \( \dim(D) = \frac{|G|}{|C|} \) by [6] Proposition 4.26. Then \( D \) is weakly integral by [18] Example 5.1.2, so is \( C \) by [6] Corollary 4.27. We always assume \( C \) does not contain non-trivial Tannakian subcategory below. Then it suffices to show that \( C \) is weakly integral when \( \mathcal{C}' = sVec \) or \( \mathcal{C}' = Vec \).

If \( C \) is a super-modular fusion category, then [6] Lemma 3.28] implies that \( \text{rank}(C) \) is even, and \( \text{rank}(C) \leq 8 \) by [18] Remark 4.2.3. We deduce from [2] and [18] Remark 4.2.3 that \( C \cong sVec \boxtimes A \), where \( A \) is a modular fusion category of global dimension 4. Consequently, \( C \) is a weakly integral fusion category.

If \( C \) is a modular fusion category. Let \( A \) be an arbitrary proper fusion subcategory of \( C \). Then \( \dim(A') \dim(A) = \dim(C) = 8 \) by [6] Theorem 3.10], where \( A' \) is the centralizer of \( A \) in \( C \). Since symmetric fusion category \( A \cap A' \subseteq A \) and \( A \cap A' \subseteq A' \), it follows from Theorem 3.11 that \( \dim(A \cap A')^2 \leq 8 \). So \( \dim(A \cap A') = 1 \) or 2.

Since \( C \) does not contain non-trivial Tannakian subcategory, \( A \) and \( A' \) are both modular or both super-modular. If \( A \cap A' = Vec \), then \( C \cong A \boxtimes A' \) by [6] Theorem 3.13, and \( \text{rank}(A) = 2, 3, 4 \). Hence \( A \) and \( A' \) are modular fusion categories of global dimension 2 or 4 by [19], so \( C \) is weakly integral. If \( A \cap A' = sVec \), then \( C \) contains a super-modular fusion category \( B \) of dimension 4 by [6] Theorem 3.10. By [18] Example 5.1.2 and [6] Corollary A.19] \( B \cong sVec \boxtimes C(Z_2, \eta) \). Then \( C \cong C(Z_2, \eta) \boxtimes C(Z_2, \eta)' \) by [6] Theorem 3.13, so centralizer \( C(Z_2, \eta)' \in C \) is either an Ising category or pointed by [18] Example 5.1.2, again \( C \) is weakly integral.

Remark 3.10. Let \( B \) be a pre-modular fusion category of global dimension 9, using same methods as Proposition 3.7] it can be proved that \( B \) is pointed if \( B \) is not simple.

Let \( A := YL \boxtimes YL \), then modular fusion category \( A \boxtimes A \) has global dimension 25 and \( \text{rank}(A \boxtimes A) = 16 \). Moreover, there exist modular fusion categories \( C \) of rank 3 [17], whose global dimensions are roots of equation \( x^3 - 14x^2 + 49x - 49 = 0 \). Assume \( C_i \) \((1 \leq i \leq 3) \) are three twists of \( C \) such that the corresponding global dimensions...
are exactly conjugated roots of previous equation. Let \( D := C_1 \boxtimes C_2 \boxtimes C_3 \), then \( D \) has global dimension 49 with \( \text{rank}(D) = 27 \). Thus, it is much more complicated to classify pre-modular fusion categories of global dimension \( p^2 \) when prime \( p \) is larger than 3.

### 3.3 Spherical fusion categories of other dimensions

In this subsection, we show that spherical fusion category \( C \) is weakly integral when \( \text{dim}(C) = 6 \). Obviously, there exists pre-modular fusion category \( C \) which are not weakly integral when \( \text{dim}(C) = 10, C \cong YL \boxtimes \mathcal{Y} \boxtimes \mathcal{sVcc} \), for example. In the last, we classify pre-modular fusion categories of dimension 10 which are not simple.

Now, we classify spherical fusion categories of dimension 6, which generalizes result of Proposition 3.4. The proof of the following theorem is same as Theorem 3.7.

**Theorem 3.11.** Spherical fusion categories of dimension 6 are weakly integral.

**Proof.** Let \( C \) be spherical fusion category of dimension 6. It follows from \([13] \) Remark 4.2.3 that \( C \) is pointed if and only if \( C \) has rank 6. Moreover, if \( \text{rank}(C) = 3 \), this is a direct result of [17] Theorem 1.1. Below we assume that \( C \) is not weakly integral, in particular, homomorphisms \( \text{dim}(\cdot) \) and \( \text{FPdim}(\cdot) \) are not same. Hence, \( \text{Gr}(C) \) is commutative and \( \text{rank}(C) = 4, 5 \). Note that homomorphisms from \( \text{Gr}(C) \) to \( C \) are divided into two or three orbits under the action of Galois group \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \). Moreover, \( \text{dim}(C_{\text{int}}) = \text{FPdim}(C_{\text{int}}) \) equals to 1, 2, 3 by Theorem 3.4, so \( C_{\text{int}} = C_{\text{pt}} \). Here, we only give a proof when \( \text{rank}(C) = 4 \), the other is same.

We know that orbits of \( \text{dim}(\cdot) \) and \( \text{FPdim}(\cdot) \) contain 2 homomorphisms, respectively. Let \( f_2 \) be a formal codegree of \( C \), which is a Galois conjugate of \( f_1 = \text{FPdim}(C) \). Then equation (1) says that

\[
\frac{1}{3} + \frac{1}{f_1} + \frac{1}{f_2} = 1,
\]

in addition, \( f_1 f_2 \geq 36 \) by [9] Proposition 8.22 and \( f_1 + f_2 > 6 \). Hence, \( f_1 f_2 = 12, 18, 36 \). If \( f_1 f_2 = 12 \), then \( f_1 = 6 \) and \( f_2 = 2 \), this is impossible.

If \( f_1 f_2 = 18 \), then \( f_1 = 6 + 3\sqrt{2} \) and \( f_1 = 6 - 3\sqrt{2} \). If \( \text{FPdim}(C_{\text{int}}) = 3 \), then there is a unique simple object \( X \) such that \( \text{FPdim}(X)^2 = 3 + 3\sqrt{2} \). Let \( G(C) = \mathbb{Z}_2 = \langle g \rangle \), then \( g \otimes X = X \). However, \( X \otimes X = I \oplus g \oplus g^2 \oplus nX \), so \( n \text{FPdim}(X) = 3\sqrt{2} \), this is impossible. If \( \text{FPdim}(C_{\text{int}}) = 2 \), then as in Lemma 3.5 there exist positive integers \( \alpha_i, \beta_i \) such that

\[
6 + 3\sqrt{2} = 2 + \frac{a_1 + b_1\sqrt{2}}{2} + \frac{a_2 + b_2\sqrt{2}}{2} = 2 + \left( \frac{\alpha_1 + \beta_1\sqrt{2}}{2} \right)^2 + \left( \frac{\alpha_2 + \beta_2\sqrt{2}}{2} \right)^2.
\]

So, \( (b_1, b_2) = (5, 1), (4, 2), (3, 3) \). These cases can be excluded through a direct computation. The remaining cases can be proved similarly, we omit the proof.

Hence, spherical fusion categories \( C \) of dimension 6 are weakly integral. Then the classification of spherical fusion categories \( C \) follows from \([8] \) Theorem 1.1. \( \square \)

Next, we classify pre-modular fusion categories of dimension 10. We begin with classification of super-modular fusion categories of dimension 10.
In [4], some techniques for classifying super-modular $C$ are established, these are similar to that of $[3]$. Recall that Galois group $Gal(C) := Gal(\mathbb{Q}(\bar{S})/\mathbb{Q})$ is an abelian group, where $\mathbb{Q}(\bar{S})$ is the extension field of $\mathbb{Q}$ by coefficients of $\bar{S}$, see subsection 2.2 for definition of matrix $\bar{S}$. Moreover, for any $\sigma \in Gal(C)$, let $\hat{\sigma}$ be the induced morphism of $\sigma$ on set $\Pi_0$, then we have

$$\sigma(\frac{s_{X,Y}}{s_{I,Y}}) = \frac{s_{X,\hat{\sigma}(Y)}}{s_{I,\hat{\sigma}(Y)}}, \forall X,Y \in \Pi_0.$$ 

**Lemma 3.12.** Let $C$ be a super-modular fusion category of global dimension 10. Then $C \cong sVec \boxtimes C(\mathbb{Z}_5, \eta)$ or $C \cong (YL \boxtimes YL) \boxtimes sVec$.

**Proof.** It follows from $[2]$ that super-modular fusion categories $C$ of dimension 10 have rank 8 or 10. If $rank(C) = 10$, then $C$ is pointed by $[18]$ Remark 4.2.3, and $C \cong sVec \boxtimes C(\mathbb{Z}_8, \eta)$ by $[6]$ Corollary A.19. If $rank(C) = 8$, we show that $C$ contains a proper fusion subcategory $A$, which is not equivalent to $C$.

On the contrary, assume all fusion subcategories of $C$ are equivalent to $Vec$, $C' \cong sVec$ or $C$. By $[4]$ Lemma 4.1 we find that super-modular fusion category $C$ of rank 8 is pointed if $C$ is not self-dual. So, when $dim(C) = 10$ and $rank(C) = 8$, super-modular fusion category $C$ must be self-dual. Hence, coefficients of $\bar{S}$ belong to the field of real numbers. Let $\Pi_0 = \{X_0, X_1, X_2, X_3\}$, $X_0 = I$. By $[4]$ Theorem 3.1, it suffices to show that such super-modular fusion category does not exist when $Gal(C) = \mathbb{Z}_4 = \langle (0123) \rangle$, $\mathbb{Z}_3 = \langle (012) \rangle$ or $\mathbb{Z}_2 = \langle (01)(23) \rangle$.

If $Gal(C) = \mathbb{Z}_4 = \langle (0123) \rangle$, then there exists $\sigma \in Gal(C)$ such that

$$FPdim(X_i) = \frac{s_{X_i,Y}}{s_{I,Y}} = \frac{s_{X_i,\hat{\sigma}(I)}}{s_{I,\hat{\sigma}(I)}} = \sigma(\frac{s_{X_i,I}}{s_{I,I}}) = \sigma(dim(X_i)).$$

Hence, $\sum_{i=0}^{3} FPdim(X_i)^2 = 5$. Since $FPdim(X_i) \geq 1$, $FPdim(X_i) < 2$ for all $i$. By $[7]$ Corollary 3.3.16, $FPdim(X_i) = 2cos(\frac{\sigma}{n_i})$, where $n_i \geq 3$ are positive integers, this is impossible by a direct computation.

If $Gal(C) = \mathbb{Z}_3 = \langle (012) \rangle$. For fusion ring $R$, the orbit of $dim(-)$ under Galois group $Gal(\mathbb{Q}/\mathbb{Q})$ has at least three homomorphisms. In fact, given an arbitrary $\sigma \in Gal(C)$, $[4]$ Equation (2) shows that

$$\sigma(\frac{s_{X,X_i}}{s_{I,X_i}}) = \frac{s_{X,\hat{\sigma}(X_i)}}{s_{I,\hat{\sigma}(X_i)}} = \frac{s_{X,X_{\sigma(i)}}}{s_{I,X_{\sigma(i)}}}, \forall X, X_i \in \Pi_0.$$ 

and $dim(X) = \frac{s_{X,X_i}}{s_{I,X_i}}$. Therefore, the orbit of $FPdim(-)$ belongs to orbit of $dim(-)$ or it is fixed by $Gal(\mathbb{Q}/\mathbb{Q})$. In the first case, above calculation shows that it is impossible. In the second case, it means that $FPdim(X_i)$ takes values in the ring of integers and $FPdim(X_i) = \frac{s_{X_i,X_3}}{s_{I,X_3}} = \frac{s_{X_i,X_3}}{dim(X_3)}$, then property of $\bar{S}$ $[2]$ Proposition 2.7 shows that

$$\sum_{i=0}^{3} s_{X_i,X_3}s_{X_3,X_i} = \sum_{i=0}^{3} s_{X_i,X_3}s_{X_3,X_i} = dim(X_3)^2 \sum_{i=0}^{3} FPdim(X_i)^2 = 5.$$
Since $\dim(X_3)^2 = \sum_{i=0}^3 (\text{FPdim}(X_i))^2$ is a rational number, which is also an algebraic integer, it follows that $\dim(X_3)^2 \in \mathbb{Z}$. So, the integer sum $\sum_{i=0}^3 \text{FPdim}(X_i)^2$ is equal to 1 or 5, this is a contradiction for $\text{FPdim}(X_i)$ are positive integers, $0 \leq i \leq 3$.

If $\text{Gal}(C) = \mathbb{Z}_2 = \langle (01)(23) \rangle$. We see that the orbit of $\dim(-)$ under Galois group $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ has at least two homomorphisms. It follows from [4, Theorem 3.9] that $\mathbb{Q}(S) = \mathbb{Q}(\dim(X_1))$ is a non-trivial $\mathbb{Z}_2$-extension of $\mathbb{Q}$. Let $\sigma$ be the non-trivial element of Galois group $\text{Gal}(C)$, then by [4, Proposition 2.7] and [4, Equation (2)],

$$\sigma(5) = 5 = \sum_{i=0}^3 \frac{\hat{s}_{X_i} - X_0}{\hat{s}_{X_0} - X_1}^2 = \sum_{i=0}^3 \frac{\hat{s}_{X_i} - X_1}{\hat{s}_{X_0} - X_1}^2 = \frac{5}{\dim(X_1)^2},$$

which shows that $\dim(X_1) = \pm 1$, this is a contradiction.

Let $\mathcal{A}$ be an arbitrary proper fusion subcategory of $\mathcal{C}$ such that $\mathcal{A} \not\cong \mathcal{C}'$, then by [6, Theorem 3.10], we have equation

$$\dim(\mathcal{A})\dim(\mathcal{A}') = \dim(\mathcal{C})\dim(\mathcal{C}' \cap \mathcal{A}).$$

Because $\mathcal{A} \cap \mathcal{A}'$ is symmetric, $\dim(\mathcal{A} \cap \mathcal{A}')$ is an integer, and Theorem [5,1] imply that $\frac{\dim(\mathcal{A} \cap \mathcal{A}')}{\dim(\mathcal{A})\dim(\mathcal{A}')}^2$ is a positive integer. If $\mathcal{C}' \subseteq \mathcal{A}$, then $\mathcal{A} \cap \mathcal{A}' = \text{Vec}$. That is, $\mathcal{A}$ is a super-modular fusion category of rank less than 8. Since $\frac{\dim(\mathcal{C})}{\dim(\mathcal{A})}$ is an algebraic integer and $\mathcal{C}$ is not pointed, it follows from [2, section 3] that $\mathcal{A} \cong \text{Vec} \boxtimes \mathcal{B}$, where $\mathcal{B}$ is a Yang-Lee fusion category. If $\mathcal{C}' \cap \mathcal{A} = \text{Vec}$, then $\mathcal{A} \cap \mathcal{A}' = \text{Vec}$, so $\mathcal{A}$ is modular and $\mathcal{A}'$ is super-modular. By [6, Theorem 3.13], $\mathcal{C} \cong \mathcal{A} \boxtimes \mathcal{A}'$, and $\text{rank}(\mathcal{A}) = 2$ or 4. Then [19] or [2] show that $\mathcal{C} \cong (\mathbb{Y} \boxtimes \mathbb{Y}) \boxtimes \text{Vec}$ as $\dim(\mathcal{C}) = 10$.

**Remark 3.13.** The proof of Lemma 3.12 also shows that there does not exit super-modular fusion categories $\mathcal{C}$ of dimension 14 with $\text{rank}(\mathcal{C}) = 8$.

The following lemma is a direct result of Theorem 3.1.

**Lemma 3.14.** Let $\mathcal{C}$ be a pre-modular fusion category of global dimension $2p$, where $p$ is an odd prime. If $\mathcal{A} \subseteq \mathcal{C}$ is a non-trivial symmetric fusion subcategory, then $\mathcal{A} \subseteq \mathcal{C}'$.

**Proof.** If $\mathcal{C}$ is symmetric, this is trivial. We assume $\mathcal{C}$ is not symmetric below, indeed we claim that $\mathcal{A} = \mathcal{C}'$. By [3, Theorem 3.10], we have equation

$$\dim(\mathcal{A})\dim(\mathcal{A}') = \dim(\mathcal{C})\dim(\mathcal{C}' \cap \mathcal{A}).$$

If $\mathcal{C}' \cap \mathcal{A} = \text{Vec}$, while $\mathcal{A} \subseteq \mathcal{A}'$, where $\mathcal{A}'$ is the centralizer of $\mathcal{A}$ in $\mathcal{C}$, then Theorem 3.1 implies that $\dim(\mathcal{A})\dim(\mathcal{A}')$, so $\dim(\mathcal{A})^2|2p$, this is a contradiction. So, $\mathcal{A} \subseteq \mathcal{C}'$. Since $\dim(\mathcal{A}) | \dim(\mathcal{C}')$, $\dim(\mathcal{C}) = 2p$ and $\mathcal{C}$ is not symmetric, $\mathcal{A} = \mathcal{C}'$ as claimed.

**Proposition 3.15.** Let $\mathcal{C}$ be a pre-modular fusion category of global dimension 10. If $\mathcal{C}$ is not simple, then $\mathcal{C}$ is pointed, or $\mathcal{C} \cong \text{Rep}(D_{10})$, or $\mathcal{C} \cong (\mathbb{Y} \boxtimes \mathbb{Y}) \boxtimes \mathcal{D}$, where $\mathcal{D}$ is a pointed pre-modular fusion category of dimension 2, $D_{10}$ is the dihedral group of order 10.
Proof. Let \( \mathcal{A} \) be a non-trivial fusion subcategory of \( \mathcal{C} \) with \( \dim(\mathcal{A}) \in \mathbb{Z} \). It follows from Theorem 3.1 that \( \dim(\mathcal{A}) = 2 \) or \( 5 \), then \( \mathcal{A} \) is symmetric or \( \mathcal{A} \) is modular. If \( \mathcal{A} \) is modular, then \( \mathcal{C} \cong \mathcal{A} \boxtimes \mathcal{D} \) by [6, Theorem 3.13], where \( \mathcal{D} \) is the centralizer of \( \mathcal{A} \) in \( \mathcal{C} \). Therefore, [18, Example 5.1.2] shows that \( \mathcal{A} \cong \mathcal{C}(\mathbb{Z}_2, \eta) \) or \( \mathcal{A} \cong \mathcal{Y}L \boxtimes \mathcal{Y}L \). Moreover, note that the structure of \( \mathcal{D} \) can also be deduced from [18, Example 5.1.2].

If \( \mathcal{A} \) is symmetric, then \( \mathcal{A} = \mathcal{C}' \) by Lemma 3.14. When \( \dim(\mathcal{A}) = 5 \), then \( \mathcal{A} \) is a Tannakian fusion category. Hence \( \mathcal{C} \cong \mathcal{D}^{\mathbb{Z}_5} \cong \text{Rep}(\mathbb{Z}_5) \boxtimes \mathcal{C}(\mathbb{Z}_2, \eta) \), where \( \mathcal{D} = \mathcal{C}_{\mathbb{Z}_5} \) is a modular fusion category of dimension 2 by [3, Proposition 4.30]. When \( \dim(\mathcal{A}) = 2 \), then \( \mathcal{A} \) is braided equivalent to \( \text{Rep}(\mathbb{Z}_2) \) or \( s\text{Vec} \). In the first case, \( \mathcal{C} \cong B^{\mathbb{Z}_2} \), where \( B \) is a modular fusion category of dimension 5. Hence [18, Example 5.1.2] says that \( \mathcal{C} \cong (\mathcal{Y}L \boxtimes \mathcal{Y}L)^{\mathbb{Z}_2} \), or \( \mathcal{C} \cong (\mathcal{Y}L \boxtimes \mathcal{Y}L) \boxtimes \text{Rep}(\mathbb{Z}_2) \), or \( \mathcal{C} \) is pointed, or \( \mathcal{C} \cong \text{Rep}(D_{10}) \). If \( \mathcal{A} \cong s\text{Vec} \), then this is exactly classification of Lemma 3.12.

Now let \( \mathcal{A} \) be a non-trivial fusion subcategory such that \( \dim(\mathcal{A}) \notin \mathbb{Z} \). Then it suffices to classify \( \mathcal{C} \) when \( \mathcal{A}' = \text{Vec} \). Notice that \( \mathcal{A} \cap \mathcal{A}' \) is a symmetric fusion subcategory of integer global dimension. If \( \mathcal{A} \cap \mathcal{A}' \) is non-trivial, we are done. If \( \mathcal{A} \cap \mathcal{A}' = \text{Vec} \), then \( \mathcal{A} \) and \( \mathcal{A}' \) are modular and \( \mathcal{C} \cong \mathcal{A} \boxtimes \mathcal{A}' \) by [6, Theorem 3.13]. Note that \( \text{rank}(\mathcal{A}) \cdot \text{rank}(\mathcal{A}') = \text{rank}(\mathcal{C}) \) and \( \text{rank}(\mathcal{C}) \leq 10 \) by [18, Lemma 4.2.2], then classification follows from [3, 19].

Acknowledgements

The author is grateful to Prof. Ostrik for insightful conversations on étale algebras and totally positive algebraic integers, particularly for providing reference [11]. Part of this paper was written during a visit of the author at University of Oregon supported by China Scholarship Council (grant No. 201806140143), he appreciates the Department of Mathematics for their warm hospitality.

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