On the long-time behavior of a continuous duopoly model with constant conjectural variation

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Abstract The paper concerns a continuous model governed by a ODE system originated by a discrete duopoly model with bounded rationality, based on constant conjectural variation. The ultimately boundedness of the solutions (existence of an absorbing set in the phase space) is shown and conditions guaranteeing the nonlinear, global, asymptotic stability of solutions have been found using the Liapunov direct method.

Keyword: Nonlinear duopoly game, Conjectural variation model, Bounded rationality, Continuous models, Nonlinear stability

1 Introduction

The classic model of oligopoly was proposed by the French mathematician A. Cournot [8] and dates back to 1838. The oligopoly market structure showing the action of only two companies is called duopoly. Duopoly is an intermediate situation between monopoly and perfect competition, and analytically is a more complicated case, because an oligopolist must consider not only the behavior of the costumers, but also those of the competitors and their reactions. In duopoly game each duopolist believe that he can calculate the quantity he should produce in order to maximize his profits. In the study of theoretic and realistic problems of duopoly, the Cournot model and successively the conjectural variation model proposed by Bowley in 1924 [5] and later by Frish in 1933 [13] have been generally accepted and become the two important tools, conventional game theory and conjectural variation model, to describe the market behavior. A useful summary of the history and of the debate on the conjectural variations is provided by Giocoli in [14]. Conjectural variation is referred to the beliefs that one firm has about the way its competitor may react if it varies its output. The firm forms a conjecture about the variation in the other firm’s output that will accompany any change in its own output. In the classic Cournot model, it is assumed that each firm treats the output of the other firm as given when it chooses its output. In this case the conjectural variation is zero. Many researchers (see for instance [1]-[4], [10], [15]), have paid a great attention to
the dynamics of games. To make the theory more realistic, people have put forward various bounded rationality behaviors, combined game theory with dynamic system, assumed a delay in the term of bounded rationality, and formed this field of bounded rational dynamic game. Different expectations such as naive expectations, adaptive expectations and bounded rationality have been proposed. Generalizations have been made for the Cournot-Kopel duopoly model [24],[21], by introducing self and cross diffusion terms in the equations [1], when the outputs are in the market in large territories. Given a model, economists need to make predictions on the asymptotic behavior of the system i.e. how the model behaves in the future. Although the mathematical models represent only an approximation of the problem (they consider only some variables that are involved in the phenomenon), they allow to obtain an estimate for the market development.

Recently, in [28], the discrete duopoly model [2] with bounded rationality based on constant conjectural variation has been considered and the local stability of critical points, numerical simulations, Liapunov exponents and fractal dimensions of strange attractors are investigated. In the present paper, we generalize (2) via a continuous ODEs system aimed to investigate on the longtime behavior of the solutions. Precisely, our aim is to

i) show that, in the phase space, there exists an absorbing set of the solutions;

ii) find conditions assuring the nonlinear global asymptotic stability of solutions.

The paper is organized as follows. It begins by recalling the discrete model (2) proposed in [28] (Section 2). Successively, the continuous counterpart of (2) is introduced and the invariance of the first orthant is shown (Section 3). Section 4 is devoted to the existence of an absorbing set and uniqueness. Critical points are determined in Section 5 while linear and nonlinear global asymptotic stability are studied respectively in Section 6 and 7. The paper ends with an Appendix (Section 8) in which the origin of the function (44) is recalled.

2 The discrete model

In the duopoly model, proposed in [28], it is assumed that two duopolists $X$ and $Y$, in the hypothesis of constant conjectural variation, produce similar products for quantity

\footnote{The reaction-diffusion systems of PDEs are the best candidates for investigating the diffusion processes in spatial domains (see for instance [6]-[7], [9], [12], [19]-[20], [22]-[27]).}
competition. The firms do not have complete information or complete rationality behavior, but they know marginal profit function of constant conjectural variation. It is a model which utilize marginal profit function of static constant conjectural variation model to make output decision. The marginal profit function of conjectural variation of firms $X$ and $Y$ has the following form

$$
\begin{align*}
\Pi_x(x, y) &= \theta_1 - \gamma y - L_1 x \\
\Pi_y(x, y) &= \theta_2 - \gamma x - L_2 y
\end{align*}
$$

with $\theta_i = \alpha_i - c_i > 0$ $(i=1,2)$ where $c_i > 0$ is the marginal cost function, while $\alpha_i$, $L_i$ and $\gamma$ are positive model parameters. Denoted by $x_t$ and $y_t$ the output of firm $X$ and $Y$ respectively, the firms adjust the output of the next period $t + 1$ considering marginal profit function and output of the current period $t$. When marginal profit function is positive they have an action of increasing output; when marginal profit function is negative they have an action of decreasing output. One obtains the following discrete system for the two firms

$$
\begin{align*}
x_{t+1} &= x_t + ax_t(\theta_1 - \gamma y_t - L_1 x_t) \\
y_{t+1} &= y_t + \nu y_t(\theta_2 - \gamma x_t - L_2 y_t)
\end{align*}
$$

where $a > 0$ and $\nu > 0$ represent the speeds of output adjustment of two firms. In [28], the local stability of critical points, numerical simulations, bifurcation diagram, Liapunov exponents and fractal dimensions of strange attractors are investigated. In addition, symmetry model and Bertrand model have been investigated.

### 3 Continuous ODEs model

Assuming continuous time scales, denoting by $u$ and $v$ the outputs of the two firms $X$ and $Y$, respectively, from [2] the nonlinear continuous system for the evolution of $u$ and $v$ is immediately obtained

$$
\begin{align*}
\frac{du}{dt} &= au(\theta_1 - \gamma v - L_1 u) \\
\frac{dv}{dt} &= \nu v(\theta_2 - \gamma u - L_2 v)
\end{align*}
$$

where

$$
\phi: t \in R^+ \rightarrow \phi(t) \in R, \quad \phi \in \{u, v\}.
$$
To (3) we append the initial data
\[ u(0) = u_0, \quad v(0) = v_0 \]
with \( u_0, v_0 \) assigned positive constants.
Further we remark that the first orthant is invariant. In fact, integrating (3), one obtains
\[
\begin{align*}
\begin{cases}
    u &= u_0 \exp \int_0^t a(\theta_1 - \gamma v - L_1 u)dt \\
v &= v_0 \exp \int_0^t \nu(\theta_2 - \gamma u - L_2 v)dt 
\end{cases}
\end{align*}
\]
and hence \( \{u_0 > 0, v_0 > 0\} \) imply \( \{u(t) > 0, v(t) > 0 \ \forall t \geq 0\} \) i.e. the first orthant is invariant.

4 Ultimately boundedness (absorbing sets) and uniqueness

As it is well-known a set \( \mathcal{A} \) of the phase space \((u, v)\), is an absorbing set if, denoting by \( d[(u, v), \mathcal{A}] \) the distance between \([u(t), v(t)]\) and \( \mathcal{A} \) at time \( t \), i.e.
\[
d(t) = \inf_{\mathcal{A}} (|u - \bar{U}|^2 + |v - \bar{V}|^2), \quad (\bar{U}, \bar{V}) \in \mathcal{A},
\]
it follows that

i) \( \mathcal{A} \) is a global attractor, i.e.
\[
\lim_{t \to \infty} d[(u, v), \mathcal{A}] = 0,
\]
with \((u_0, v_0) \in \mathcal{B}\), for any open set \( \mathcal{B} \supset \mathcal{A} \);

ii) \( \mathcal{A} \) is positively invariant, i.e.
\[
(u_0, v_0) \in \mathcal{A} \Rightarrow (u(t), v(t)) \in \mathcal{A}, \quad \forall t > 0.
\]

**Theorem 1** Any set containing the rectangle
\[
S = \left\{(u, v) \in \mathbb{R}_+^2 : 0 < u \leq \frac{\theta_1}{L_1}, \quad 0 < v \leq \frac{\theta_2}{L_2}\right\},
\]
of the phase space is an absorbing set for (3).
Proof. By virtue of (3) it follows that
\[
\frac{du}{dt} = au(\theta_1 - \gamma v - L_1 u) \leq au(\theta_1 - L_1 u) \quad (10)
\]
from which, by setting \( w = \frac{1}{u} \), we obtain
\[
\frac{dw}{dt} \geq -a\theta_1 w + aL_1, \quad (11)
\]
and hence, integrating, it follows that
\[
w \geq w_0 e^{-a\theta_1 t} + \frac{L_1}{\theta_1} (1 - e^{-a\theta_1 t}). \quad (12)
\]
Since (12) implies
\[
\lim_{t \to \infty} w \geq \frac{L_1}{\theta_1}, \quad (13)
\]
one immediately obtains that, \( \forall \varepsilon > 0 \), there exists \( T_1(\varepsilon) > 0 \) such that
\[
u(t) \leq \frac{\theta_1}{L_1} + \varepsilon, \quad \forall t \geq T_1. \quad (14)
\]
Following the same procedure for (3)_2, one obtains the existence of a constant \( T_2(\varepsilon) > 0 \) such that
\[
v(t) < \frac{\theta_2}{L_2} + \varepsilon, \quad \forall t \geq T_2. \quad (15)
\]
For \( \varepsilon \to 0 \) and \( T = \max\{T_1, T_2\} \), from (14)-(13), it follows that
\[
u(t) \leq \frac{\theta_1}{L_1}, \quad \forall t \geq T, \quad (16)
\]
i.e. S is an attractor.
In order to prove that S is positively invariant, let us assume that
\[
0 < u_0 < \frac{\theta_1}{L_1}, \quad 0 < v_0 < \frac{\theta_2}{L_2}, \quad (17)
\]
then, from (12) and (17) it follows that
\[
u(t) \leq \frac{u_0 \theta_1}{u_0 L_1 e^{-a\theta_1 t} + u_0 L_1 (1 - e^{-a\theta_1 t})} = \frac{\theta_1}{L_1} \quad \forall t \geq T. \quad (18)
\]
Following the same steps for \( v(t) \), one obtains
\[
v(t) \leq \frac{\theta_2}{L_2} \quad \forall t \geq T, \quad (19)
\]
i.e. S is invariant.
Remark 1 Since any solution enters in $S$ in a finite time and remains there, without loss of generality, we can confine ourselves to study the dynamical behavior of the system in $S$ assuming $(u_0, v_0) \in S$.

The following uniqueness theorem holds

**Theorem 2** The system $(3) - (4)$ admits a unique solution.

**Proof.** Let $(u_1, v_1)$ and $(u_2, v_2)$ be two solutions of $(3)-(1)$. On setting

$$
\tilde{u} = u_1 - u_2, \quad \tilde{v} = v_1 - v_2
$$

it follows that

$$
\begin{cases}
\frac{d\tilde{u}}{dt} = a\theta_1 \tilde{u} - aL_1(u_1 + u_2)\tilde{u} - a\gamma(u_1v_1 - u_2v_2) \\
\frac{d\tilde{v}}{dt} = \nu\theta_2 \tilde{v} - \nuL_2(v_1 + v_2)\tilde{v} - \nu\gamma(u_1v_1 - u_2v_2)
\end{cases}
$$

with

$$
\tilde{u}(0) = 0, \quad \tilde{v}(0) = 0.
$$

From (21) it follows that

$$
\frac{1}{2} \frac{d}{dt}(\tilde{u}^2 + \tilde{v}^2) = a[\theta_1 - L_1(u_1 + u_2)]\tilde{u}^2 + \nu[\theta_2 - L_2(v_1 + v_2)]\tilde{v}^2 - a\gamma(u_1v_1 - u_2v_2)\tilde{u} - \nu\gamma(u_1v_1 - u_2v_2)\tilde{v}.
$$

By virtue of the boundedness of solutions and $|\tilde{u}| + |\tilde{v}| < \sqrt{2(\tilde{u}^2 + \tilde{v}^2)}$ (23) leads to

$$
\frac{d}{dt}(\tilde{u}^2 + \tilde{v}^2) \leq \kappa(\tilde{u}^2 + \tilde{v}^2)\frac{1}{2}[1 + (\tilde{u}^2 + \tilde{v}^2)\frac{1}{2}],
$$

with $\kappa$ positive constant. Integrating (24) it immediately follows

$$
\tilde{u}^2 + \tilde{v}^2 \leq \frac{(\tilde{u}_0^2 + \tilde{v}_0^2)e^{\kappa t}}{[1 + (\tilde{u}_0^2 + \tilde{v}_0^2)\frac{1}{2}(1 - e^{\kappa t/2})]^2} \quad \forall t \geq 0.
$$

and in view of (22), one obtains

$$
\tilde{u}^2 + \tilde{v}^2 \equiv 0.
$$
5 Critical points of (3)

The critical points of (3) are the roots of the algebraic system

\[
\begin{align*}
au(\theta_1 - \gamma v - L_1 u) &= 0 \\
\nu v(\theta_2 - \gamma u - L_2 v) &= 0.
\end{align*}
\]  

(26)

Besides the trivial equilibrium \(E_0 = (0, 0)\), system (3) admits the following nontrivial critical points (boundary equilibrium points)

\[
E_1 = (0, \frac{\theta_2}{L_2}), \quad E_2 = (\frac{\theta_1}{L_1}, 0)
\]

(27)

and the only equilibrium of constant conjectural variation

\[
E_3 = \left(\frac{\theta_1 L_2 - \theta_2 \gamma}{L_1 L_2 - \gamma^2}, \frac{\theta_2 L_1 - \theta_1 \gamma}{L_1 L_2 - \gamma^2}\right),
\]

(28)

where the parameters satisfy

\[
\begin{align*}
\theta_1 L_2 - \theta_2 \gamma &> 0 \\
\theta_2 L_1 - \theta_1 \gamma &> 0 \\
L_1 L_2 - \gamma^2 &> 0.
\end{align*}
\]

(29)

**Remark 2** The property of the equilibrium of conjectural variation is that:

i) when firm 1 chooses equilibrium \(\bar{u}\), then \(\bar{v}\) is the optimal selection of firm 2;

ii) when firm 2 chooses \(\bar{v}\), then \(\bar{u}\) is the optimal selection of firm 1;

iii) the realization condition of equilibrium of conjectural variation is that each firm has complete information of market and the ability of complexity rational behavior.

6 Linear stability

Denoting by \((\bar{u}, \bar{v})\) a constant critical point of (3), we set

\[
U = u - \bar{u}, \quad V = v - \bar{v}.
\]

(30)
Then, the equations governing \((U, V)\) are given by

\[
\begin{align*}
\frac{dU}{dt} &= a(\theta_1 - \bar{v}\gamma - 2\bar{u}L_1)U - a\bar{u}\gamma V - a\gamma UV - aL_1U^2 \\
\frac{dV}{dt} &= -\nu\gamma\bar{v}U + \nu(\theta_2 - \gamma\bar{u} - 2\bar{v}L_2)V - \nu\gamma UV - \nu L_2V^2.
\end{align*}
\]

(31)

Setting

\[
\begin{align*}
f(U, V) &= -a\gamma UV - aL_1U^2 \\
g(U, V) &= -\nu\gamma UV - \nu L_2V^2 \\
a_{11} &= a(\theta_1 - \bar{v}\gamma - 2\bar{u}L_1) \\
a_{12} &= -a\bar{u}\gamma \\
a_{21} &= -\nu\gamma\bar{v} \\
a_{22} &= \nu(\theta_2 - \gamma\bar{u} - 2\bar{v}L_2)
\end{align*}
\]

(32)

(31) becomes

\[
\begin{align*}
\frac{dU}{dt} &= a_{11}U + a_{12}V + f(U, V) \\
\frac{dV}{dt} &= a_{21}U + a_{22}V + g(U, V),
\end{align*}
\]

(33)

under the initial data

\[
U(0) = u_0 - \bar{u}, \quad V(0) = v_0 - \bar{v}.
\]

(34)

Linearizing, one obtains

\[
\frac{d}{dt} \begin{pmatrix} U \\ V \end{pmatrix} = \mathcal{L} \begin{pmatrix} U \\ V \end{pmatrix}
\]

(35)

with \(\mathcal{L}\) given by

\[
\mathcal{L} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.
\]

(36)

Denoting by \(\lambda_i\) \((i = 1, 2)\) the eigenvalues of \(\mathcal{L}\), it is well-known that

\[
\begin{align*}
I_0 &= a_{11} + a_{22} = \lambda_1 + \lambda_2 \\
A_0 &= a_{11}a_{22} - a_{12}a_{21} = \lambda_1\lambda_2
\end{align*}
\]

(37)

and hence, if and only if,

\[
\begin{align*}
I_0(\bar{u}, \bar{v}) < 0 \\
A_0(\bar{u}, \bar{v}) > 0
\end{align*}
\]

(38)

the critical point \((\bar{u}, \bar{v})\) is linearly stable.

**Theorem 3** By virtue of (38), it follows that
i) $E_3$ is linearly stable;

ii) $E_0, E_1, E_2$ are unstable;

iii) $E_0$ is an unstable node; $E_1, E_2$ are saddle point; $E_3$ is a stable node;

iv) the spectral equation is given by $\lambda^2 - I_0 \lambda + A_0 = 0$;

v) $(I_0^2 - 4A_0)_{(u,v)} = 0$, $\forall i = \{0, 1, 2, 3\}$ i.e. the eigenvalues are real.

Proof. By virtue of (38), one obtains

\[
\begin{align*}
I_0(E_3) &= -\frac{aL_1(\theta_1L_2 - \theta_2\gamma)}{L_1L_2 - \gamma^2} - \frac{\nu L_2(\theta_2L_1 - \theta_1\gamma)}{L_1L_2 - \gamma^2} < 0 \\
A_0(E_3) &= \frac{a\nu(\theta_1L_2 - \theta_2\gamma)(\theta_2L_1 - \theta_1\gamma)}{L_1L_2 - \gamma^2} > 0, \\
I_0(E_0) &= a\theta_1 + \nu \theta_2 > 0 \\
A_0(E_0) &= a\theta_1\theta_2\nu > 0, \\
I_0(E_1) &= a(\theta_1L_2 - \theta_2\gamma) - \nu \theta_2 \\
A_0(E_1) &= \frac{a(\theta_1L_2 - \theta_2\gamma)}{L_2}(-\nu \theta_2) < 0, \\
I_0(E_2) &= \frac{\nu(\theta_2L_1 - \theta_1\gamma)}{L_1} - a\theta_1 \\
A_0(E_2) &= \frac{\nu(\theta_2L_1 - \theta_1\gamma)}{L_1}(-a\theta_1) < 0,
\end{align*}
\]

and hence, i)- ii) immediately follow.

In order to prove iii), iv), v), from

\[
\det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0
\]

it follows that, the spectral equation is given by $\lambda^2 - I_0 \lambda + A_0 = 0$.

In addition, since

\[
I_0^2 - 4A_0 = (a_{11} - a_{22})^2 + 4a_{12}a_{21} > 0 \quad \forall E_i \ (i = 0, 1, 2, 3)
\]

the eigenvalues are real and v) holds.

Further, it immediately follows that
$E_0$ is an unstable node since $\{\lambda_1 = a\theta_1 > 0, \lambda_2 = \nu\theta_2 > 0\}$;

$E_1$ is a saddle point, since $\{\lambda_1 = \frac{a(\theta_1 L_2 - \theta_2 \gamma)}{L_2} > 0, \lambda_2 = -\nu\theta_2 < 0\}$;

$E_2$ is a saddle point, since $\{\lambda_1 = -a\theta_1 < 0, \lambda_2 = \frac{\nu(\theta_2 L_1 - \theta_1 \gamma)}{L_1} > 0\}$;

$E_3$ is a stable node, since

$$
\begin{cases}
\lambda_1 = -|I_0| - \sqrt{I_0^2 - 4A_0} < 0, \\
\lambda_2 = -|I_0| + \sqrt{I_0^2 - 4A_0} < 0
\end{cases}
$$

7 Nonlinear stability of $E_3$

The nonlinear stability analysis is based on the Rionero’s function [11], [16]-[18],

$$
\mathcal{V} = \frac{1}{2}[A_0(U^2 + V^2) + (a_{11}V - a_{21}U)^2 + (a_{12}V - a_{22}U)^2]
$$

(44)

whose time derivative along the solutions of (33) is given by

$$
\dot{\mathcal{V}} = -A_0|I_0|(U^2 + V^2) + \Psi
$$

(45)

with $A_0$, $I_0$ given by (37),

$$
\Psi = (\alpha_1 U - \alpha_3 V)f(U, V) + (\alpha_2 V - \alpha_3 U)g(U, V)
$$

(46)

and

$$
\alpha_1 = A_0 + a_{21}^2 + a_{22}^2 > 0, \quad \alpha_2 = A_0 + a_{11}^2 + a_{12}^2 > 0, \\
\alpha_3 = a_{11}a_{21} + a_{12}a_{22} > 0.
$$

(47)

**Theorem 4** The equilibrium of conjectural variation $E_3$ is nonlinearly asymptotically exponentially stable, according to

$$
\mathcal{V} \leq \mathcal{V}(0)e^{-(1-\eta)h_1 t} \quad \forall t > 0
$$

(48)

with $\eta \in ]0, 1[$ and $h_1$ positive constant, under the local condition

$$
U_0^2 + V_0^2 \leq \frac{(A_0|I_0|\delta_1)^2}{2M^2\delta_2^2}
$$

(49)

with $\delta_1, \delta_2$ positive constants and globally nonlinearly asymptotically exponentially stable under the condition

$$
\frac{\theta_1^2}{L_1^2} + \frac{\theta_2^2}{L_2^2} \leq \frac{(A_0|I_0|\delta_1)^2}{2M^2\delta_2^2}.
$$

(50)
Proof. From (46) immediately one obtains
\[
\begin{align*}
\alpha_3 (aL_1 + \nu \gamma) &- \alpha_1 a \gamma \right U^2 V - \alpha_1 a L_1 U^3 \\
+ \alpha_3 (\nu L_2 + a \gamma) &- \alpha_2 \nu \gamma \right U V^2 - \alpha_2 \nu L_2 V^3 \\
\leq & M(U^2 + V^2) (|U| + |V|) \leq \sqrt{2} M(U^2 + V^2)^{3/2},
\end{align*}
\]
where
\[
M = \max(m_1, m_2, m_3, m_4) \quad m_1 = |\alpha_3 (aL_1 + \nu \gamma) - \alpha_1 a \gamma| \\
m_2 = |\alpha_3 (\nu L_2 + a \gamma) - \alpha_2 \nu \gamma| \quad m_3 = \alpha_1 a L_1, \quad m_4 = \alpha_2 \nu L_2.
\]
Setting \( E = (U^2 + V^2) \) and on taking into account that there exist two positive constants \( \delta_1, \delta_2 \) such that \( \delta_1 E \leq V \leq \delta_2 E \) (52), in view of (51), the (45) leads to
\[
\dot{V} \leq - \frac{A_0 |I_0| \delta_2}{\delta_2} V + \frac{\sqrt{2} M}{\delta_1^{3/2}} V^{3/2}
\]
and hence
\[
\dot{V} \leq -(h_1 - h_2 V^{3/2}) V,
\]
with \( h_1 = \frac{A_0 |I_0|}{\delta_2}, \quad h_2 = \frac{\sqrt{2} M}{\delta_1^{3/2}}. \) Choosing
\[
\begin{align*}
h_2 V_1^{3/2}(0) &< h_1, \quad \text{i.e.} \quad h_2 V_2^{3/2}(0) = \eta h_1, \quad \eta \in \{0, 1\}
\end{align*}
\]
by recursive argument, (48) immediately follows.
On the other hand, in view of Theorem 1, one has
\[
U_0^2 + V_0^2 \leq \frac{\theta_1^2}{L_1^2} + \frac{\theta_2^2}{L_2^2}
\]
and hence (50) immediately follows.

Remark 3 In view of i) and ii) of Theorem 3, it follows that, in the passage from the discrete model to the continuous ODEs one (with constant conjectural variation), the constant steady states do not vary. Their stability-instability don’t vary, except the fact that in the continuous model, \( E_3 \) is reached exponentially when \( t \to \infty \), from any initial state. In fact, we remark that, since in the case \( E \in \{E_0, E_1, E_2\} \), generally \( I_0(E) A_0(E) < 0 \) does not continues to hold, therefore does not appear a stabilizing effect of the nonlinearities.
8 Appendix

We define *peculiar* the Liapunov function able to give for the nonlinear stability-instability, exactly the same conditions of the linear stability. We, following [18], here recall the construction of a such Liapunov function for a binary system of O.D.Es, since this function that appears in Section 7.

Let us consider the stability of the zero solution of the system

\[
\begin{align*}
\frac{dx}{dt} &= ax + by + f(x, y), \\
\frac{dy}{dt} &= cx + dy + g(x, y),
\end{align*}
\]

\[\text{(57)}\]

\((f, g\) being nonlinear and such that \(f(0, 0) = g(0, 0) = 0\) and introduce the function

\[W = \frac{1}{2} I [A(x^2 + y^2) + (ay - cx)^2 + (by - dx)^2],\]

\[\text{(58)}\]

with

\[I = a + d = \lambda_1 + \lambda_2, \quad A = ad - bc = \lambda_1 \cdot \lambda_2,\]

\[\text{(59)}\]

\(\lambda_1, \lambda_2\) eigenvalues of \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\). Since

\[
\begin{align*}
\dot{x} &= ax^2 + bxy + xf, \\
\dot{y} &= cxy + dy^2 + yg,
\end{align*}
\]

\[\text{(60)}\]

by straightforward calculations it follows that

\[
\frac{dW}{dt} = IA(x^2 + y^2) + \Psi,
\]

\[\text{(61)}\]

with

\[
\begin{align*}
\Psi &= \varepsilon I [(\alpha_1 x - \alpha_3 y)f + (\alpha_2 y - \alpha_3 x)g], \\
\alpha_1 &= A + c^2 + d^2, \quad \alpha_2 = A + a^2 + b^2, \quad \alpha_3 = ac + bd.
\end{align*}
\]

\[\text{(62)}\]

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References

[1] H. N. Agiza, On the analysis of stability, bifurcation, chaos, and chaos control of Kopel map, *Chaos Solitons Fractals*, 10 (1999) 1909–1916.

[2] H.N. Agiza, A.A. Elsadany and M. Kopel, Nonlinear dynamics in the Cournot duopoly game with heterogeneous players, *Physica A* 320 (2003) 512–524.

[3] G.I., Bischi, L. Gardini, and M. Kopel, Noninvertible maps and Complex basin boundaries in dynamic economic models with coexisting attractors, *Chaos and Complexity Letters*, 2 (1) (2006) 43–74.

[4] G. I., Bischi and A. Naimzada, Global analysis of a dynamic duopoly game with bounded rationality, *Advanced in Dynamic Games and Application*, 5 Chapter 20, Birkhouser, 1999.

[5] A.L. Bowley, *The Mathematical Groundwork of Economics*. Oxdord University Press. 1924.

[6] F. Capone, R. De Luca and I. Torcicollo, Longtime behavior of vertical throughflows for binary mixtures in porous layers, *Int. Journal of Non-Linear Mechanics* 52 (2013) 1–7.

[7] F. Capone, V. De Cataldis, R. De Luca and I. Torcicollo, On the stability of vertical constant throughflows for binary mixtures in porous layers, *Int. Journal of Non-Linear Mechanics* 59 (2014) 1–8.

[8] A. Cournot, Recherches sur les Principes Mathematiques de la Theorie des Richesses Hachette, Paris. 1838.

[9] M. De Angelis, Asymptotic Estimates Related to an integro Differential Equation, *Nonlinear Dynamics and Systems Theory* 13 (3) (2013) 217–228.

[10] A.A. Elsadany, Dynamics of a delayed duopoly game with bounded rationality, *Mathematical and Computer Modelling* 52 (2010), 1497–1489.
[11] J. N., Flavin and S. Rionero, Cross-diffusion influence on the nonlinear $l^2$-stability analysis for a Lotka-Volterra reaction-diffusion system of PDEs, *IMA J. of Applied Mathematics* 72 (2007) 540–555.

[12] J.N. Flavin and S. Rionero, *Qualitative estimates for partial differential equations: an introduction*, CRC Press, Boca Raton (FL), 1996.

[13] R. Frish, Monopoly-polipoly - the concept of force in the economy, *International Economic Papers*, 1 ([1933] 1951) 23–26.

[14] N. Giocoli, The Escape from conjectural variations: the consistency condition in duopoly theory from Bowley to Fellner, *Cambridge Journal of Economics*, 29 (2005) 601–618.

[15] M. Kopel, Simple and complex adjustment dynamics in Cournot duopoly models, *Chaos Solitons Fractals* 12 (1996) 2031–2048.

[16] S. Rionero, $L^2$-energy stability via new dependent variables for circumventing strongly nonlinear reaction terms, *Nonlinear Analysis* 70 (2009), 2530–2541.

[17] S. Rionero, $L^2$-energy decay of convective nonlinear PDEs reaction-diffusion systems via auxiliary ODEs systems, *Ricerche di Matematica*, DOI 10.1007/s11587-015-0231-2 (2015).

[18] S. Rionero, Stability of ternary reaction-diffusion dynamical systems, *Rend. Lincei Mat. Appl.* 22 (2011) 245–268.

[19] S. Rionero, A rigorous reduction of the $L^2$-stability of the solutions to a nonlinear binary reaction-diffusion system of PDE’s to the stability of the solutions to a linear system of ODE’s, *Journal of Mathematical Analysis and Applications*, 319 (2006) 377–397.

[20] S. Rionero, A nonlinear $L^2$-stability analysis for two species dynamics with dispersal, *Math. Biosc. Eng.* 3 (1) (2006) 189–204.

[21] S. Rionero and I. Torcicollo, Stability of a continuous reaction-diffusion Cournot-Kopel Duopoly Game Model, *Acta Applicandae Mathematicae* 132 (1), (2014) 505–513.

[22] S. Rionero and I. Torcicollo, On an ill-posed problem in nonlinear heat conduction, *Transport Theory and Statistical Physics* 29 (1&2) (2000) 173–186.
[23] S. Rionero and I. Torcicollo, On the pointwise continuous dependence of an approximate solution of a nonlinear heat conduction ill-posed problem, *Rend. Acc. Sci. Fis. Mat.*, Napoli, **LXVII** - Serie IV, (2000) 169–179.

[24] I. Torcicollo, On the dynamics of a non-linear Duopoly game model, *Int. Journal of Non-Linear Mechanics* **57** (2013) 799–805.

[25] I. Torcicollo, Su alcuni problemi di diffusione non lineare, *Bollettino della Unione Matematica Italiana* A3 (3) (2000) 407-410.

[26] I. Torcicollo and M. Vitiello, A note on the nonlinear pointwise stability for the equation $u_t = \Delta F(u)$ in the exterior of a sphere, *Rend. Acc. Sc. Fis. Mat.* Napoli, **LXX** (2003) 111–117.

[27] I. Torcicollo, M. Vitiello, On the nonlinear diffusion in the exterior of a sphere, *Proceedings 11th International Conference on Wave and Stability in Continuous Media*, Porto Ercole, Italy. Book Editor(s): Monaco, R; Bianchi, MP; Rionero, S; (2002) 563–568.

[28] W. Yu and Y. Yu, A dynamic duopoly model with bounded rationality based on constant conjectural variation, *Economic Modelling*, **37** (1) (2014) 103–112.