Geometric Syzygies of Mukai Varieties and General Canonical Curves with Genus \leq 8

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1 Introduction

In this paper we study the syzygies of general canonical curves $C \subset \mathbb{P}^{g-1}$ for $g \leq 8$.

In [GL84] Green and Lazarsfeld construct low-rank-syzygies of $C$ from special linear systems on $C$. More precisely, linear systems of Clifford index $c$ give a $(g - c - 3)$rd syzygy. We call these syzygies geometric syzygies. Green's conjecture [Gre84a] paraphrased in this way is

**Conjecture (Green).** Let $C$ be a canonical curve, then $C$ has no geometric $k$th syzygies $\iff$ $C$ has no $k$th syzygies at all.

This conjecture as received a lot of attention in the last years and it is now known in many cases [Pet23], [Gre84a], [Sch86], [Sch88], [Voi88], [Sch91], [Ehb94], [HR98], [Tei99], [Voi01].

A natural generalization of Green’s conjecture is

**Conjecture (Geometric Syzygy Conjecture).** Let $C$ be a canonical curve, then the geometric $k$th syzygies span the space of all $k$th syzygies.

Both conjectures are equivalent for $k \geq \frac{2g-3}{2}$ since a general canonical curve has no linear systems of Clifford index $c \leq \frac{2g-3}{2}$.

The geometric syzygy conjecture is therefore true, where Green’s conjecture is known. Furthermore the case $k = 0$ (geometric quadrics) was proved by [AM67] for general canonical curves, and by [Gre84b] for all canonical curves. The case $k = 1$ was done for general canonical curves of genus $g \geq 9$ in [vB00].

In this paper we attack the cases $k = 1, g = 6, 7$ and $k = 2, g = 8$ for general canonical curves.
The starting point of our proof is

**Theorem (Mukai).** Every general canonical curve of genus \(7 \leq g \leq 9\) is a general linear section of an embedded rational homogeneous variety \(M_g\). General canonical curves of genus 6 are cut out by a general quadric on a general linear section of a homogeneous variety \(M_6\).

Using this we first consider the schemes of minimal rank first syzygies \((g = 6, 7)\) respectively minimal rank second syzygies \((g = 8)\) of the Mukai varieties \(M_g\) using representation theory. It turns out that all these schemes contain large rational homogeneous varieties.

Passing from Mukai varieties to canonical curves we describe their schemes of geometric syzygies as determinantal loci on the above homogeneous varieties.

Using the resolutions of Eagon-Northcott (for \(g = 6, 8\)) and Lascoux (for \(g = 7\)) we express the cohomology of the corresponding ideal sheafs in terms of the cohomology of homogeneous bundles. The later cohomology is then calculated with the theorem of Bott.

This calculation shows \(h^0(I(1)) = 0\), proving the geometric syzygy conjecture in these cases. More precisely our results are:

**4.2.3 Theorem.** The scheme \(Z\) of last scrollar syzygies of a general canonical curve \(C \subset \mathbb{P}^5\) of genus 6 is a configuration of 5 skew lines in \(\mathbb{P}^4\) that spans the whole \(\mathbb{P}^4\) of first syzygies of \(C\).

**5.2.4 Theorem.** The scheme \(Z\) of last scrollar syzygies of a general canonical curve \(C \subset \mathbb{P}^6\) of genus 7 is a linearly normal ruled surface of degree 84 on a spinor variety \(S_{syz}^+ \subset \mathbb{P}^{15}\). This ruled surface spans the whole \(\mathbb{P}^{15}\) of first syzygies of \(C\).

**6.2.3 Theorem.** The scheme \(Z\) of last scrollar syzygies of a general canonical curve \(C \subset \mathbb{P}^7\) of genus 8 is a configuration of 14 skew conics that lie on a 2-uple embedded \(\mathbb{P}^5 \hookrightarrow \mathbb{P}^{20}\). \(Z\) spans the whole \(\mathbb{P}^{20}\) of second syzygies of \(C\).

The paper is organized as follows:

In section 2 we cover some well known background material on syzygies. In particular we will introduce the rank of a syzygy, the scheme of minimal rank syzygies and the vector bundle of linear forms in subsection 2.1.

In subsection 2.2 we will show, that varieties with very low rank syzygies always lie on certain scrolls. The properties of these scrollar syzygies are studied and their connection with Brill-Noether-Theory is explained.

Subsection 2.3 finally describes what happens to a minimal rank syzygy of a variety \(X\) when \(X\) is intersected with a general linear subspace. It will turn
out that the rank of $s$ can drop and that this rank can be calculated by a morphism of vector bundles $\alpha$ involving the vector bundle of linear forms.

Section 3 fixes the notations for the use of representation theory, and the remaining three sections treat curves of genus 6, 7 and 8 in turn. Starting from the respective Mukai varieties the proof of the geometric syzygy conjecture outlined above is given in full detail.

I would like to thank Kristian Ranestad for the many helpful discussions during my stay at Oslo University. It was there where most of the ideas of this work where born.

I dedicate this paper to my grandmother Lilly-Maria who introduced me to mathematics.

2 Background on Syzygies

In this section we will review some standard facts from the study of linear syzygies.

2.1 Syzygies of Low Rank

Let $X \subset \mathbb{P}(V)$ be a irreducible non degenerate variety, $I_X$ generated by quadrics and

$$I_X \leftrightarrow V_0 \otimes \mathcal{O}(-2) \xleftarrow{\varphi_1} V_1 \otimes \mathcal{O}(-3) \xleftarrow{\varphi_2} \ldots \xleftarrow{\varphi_m} V_m \otimes \mathcal{O}(-m - k)$$

the linear strand of its resolution

2.1.1. Definition. An element $s \in V_k$ is called a $k$-th (linear) syzygy of $X$. $\mathbb{P}(V^*_k)$ is called the space of $k$-th syzygies.

Every linear syzygy $s$ involves a well defined number of linearly independent linear forms. This number is called the rank of $s$. In a more formal way we have:

2.1.2. Definition. Let $s \in V_k$ be a syzygy and

$$\tilde{\varphi}_k : V_k \to V_{k-1} \otimes V$$

the map of vector spaces induced by $\varphi_k$. Then the image of $s$ under this map

$$\tilde{\varphi}_k(s) \in V_{k-1} \otimes V \cong \text{Hom}(V^{*}_{k-1}, V).$$

is a map of vector spaces, and its image

$$\langle s \rangle := \text{Im} \tilde{\varphi}(s) \subset V$$
is called the space of linear forms involved in $s$. Furthermore
\[
\text{rank } s := \text{rank } \tilde{\varphi}(s) = \dim \langle s \rangle
\]
is called the rank of $s$. The zero set $Z_s$ of a syzygy is the linear space subspace of $\mathbb{P}(V)$ where the linear forms involved in $s$ vanish.

2.1.3. Example. Consider the rational normal curve $C \subset \mathbb{P}^3$ with minimal resolution
\[
\begin{array}{ll}
\mathcal{I}_C \coloneqq (yw - z^2, -xw + yz, xz - y^2) & V_0 \otimes \mathcal{O}(-2) \leftarrow V_1 \otimes \mathcal{O}(-2)
\end{array}
\]
where $\dim V_0 = 3$ and $\dim V_1 = 2$. If $s, t$ is a basis of $V_1$ and $q_1, q_2, q_3$ a basis of $V_0$, the map $\tilde{\varphi}_1$ is given by
\[
\begin{array}{ccc}
\tilde{\varphi}_1 & : & V_1 \rightarrow V_0 \otimes V \\
& & s \mapsto x \otimes q_1 + y \otimes q_2 + z \otimes q_3 \\
& & t \mapsto y \otimes q_1 + z \otimes q_2 + w \otimes q_3
\end{array}
\]
With this, the linear forms involved in $s$ are $\langle s \rangle = \langle x, y, z \rangle$, the rank of $s$ is 3 and the Zero locus of $s$ is just the single point $Z_s = (0 : 0 : 0 : 1)$.

To apply geometric methods to the study of low rank syzygies we projectivize the space of $k$th syzygies $\mathbb{P}(V^*_k)$ and give a determinantal description of the space $Y_{\text{min}}$ of minimal rank syzygies. The linear forms involved in these syzygies defines a vector bundle on $Y_{\text{min}}$:

2.1.4. Definition. On the space of $k$th syzygies $\mathbb{P}(V^*_k)$ the map of vector spaces $\tilde{\varphi}_k$ induces a map of vector bundles
\[
\psi : V^*_{k-1} \otimes \mathcal{O}_{\mathbb{P}(V^*_k)}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(V^*_k)}
\]
that satisfies
\[
\psi|_s = \tilde{\varphi}_k(s) \in \text{Hom}(V^*_k, V)
\]
The determinantal loci $Y_r(\psi) \subset \mathbb{P}(V^*_k)$ of $\psi$ are called schemes of rank $r$ syzygies, since the syzygies in their support have rank $\leq r$.

On the scheme of minimal rank syzygies $Y_{\text{min}} := (Y_{r_{\text{min}}}(\psi))_{\text{red}}$ the restricted map $\psi|_Y$ has constant rank $r_{\text{min}}$. Therefore the image $L := \text{Im}(\psi|_Y)$ is a vector bundle. We call it the vector bundle of linear forms, since
\[
L|_s = \langle s \rangle \subset V
\]
for all minimal rank syzygies $s \in Y_{\text{min}}$. 

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2.1.5. Example. For the rational normal curve $C \subset \mathbb{P}^3$ all first syzygies are of rank 3 so $Y_{\text{min}} = Y_3 = \mathbb{P}(V_1^*) = \mathbb{P}^1$. The vector bundle of linear forms on this $\mathbb{P}^1$ is the image of

$$
\psi: V_0 \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^1}
$$

where $\psi$ is given by the flipped syzygy matrix

$$
\begin{pmatrix}
  s & t & 0 & 0 \\
  0 & s & t & 0 \\
  0 & 0 & s & t
\end{pmatrix}.
$$

Since $\psi$ is injective we have $L = 3\mathcal{O}_{\mathbb{P}^1}(-1)$.

2.1.6. Definition. We say that a variety $X$ as above satisfies the \textit{kth minimal rank conjecture}, if the scheme of $k$th minimal rank syzygies $Y_{\text{min}} \subset \mathbb{P}(V^*_k)$ is non degenerate.

This conjecture is easy to verify for certain rational homogeneous varieties:

2.1.7. Proposition. Let $X = G/P \subset \mathbb{P}(V)$ be a linearly normal homogeneous rational variety. If the induced representation

$$\rho_k: G \to GL(V^*_k)$$

is irreducible then the variety of minimal rank $k$th syzygies $Y_{\text{min}}$ of $X$ contains the minimal orbit $G/P_k \subset \mathbb{P}(V^*_k)$ of this representation. In particular $X$ satisfies the $k$-th minimal rank conjecture.

\textbf{Proof}. Since the embedding $X \subset \mathbb{P}(V)$ is linearly normal, it induces a representation

$$\rho: G \to GL(V).$$

Since $X$ is $G$ invariant, this $G$ also acts on the minimal free resolution of $I_X$ and therefore induces a representation

$$\rho_k: G \to GL(V^*_k)$$

Now the rank of a syzygy $s \in \mathbb{P}(V^*_k)$ is invariant under coordinate transformations of $V$ so that the space of minimal syzygies $Y_{\text{min}} \subset \mathbb{P}(V^*_k)$ is $G$ invariant. It is also compact, so it has to contain a minimal orbit $G/P_k$ of $G$ in $\mathbb{P}(V^*_k)$. Since $\rho_k$ is irreducible, there is only one minimal orbit and this minimal orbit is non degenerate. \hfill \square

It will turn out later, that the Mukai varieties for $g = 6, 7, 8$ are of this type.
2.2 Scrollar Syzygies

In this section we want to establish a connection between syzygies of very low rank of a variety $X$, rational scrolls that contain $X$ and pencils of divisors on $X$.

First we describe how to construct the equations of a rational scroll $S$ containing $X$ from a $k$th syzygy of rank $k + 2$. Conversely we find that the minimal rank $k$th syzygies of this scroll are of rank $k + 2$. In fact there is a 1 : 1 correspondence between the fibers of $S$ and its minimal rank $k$th syzygies.

Secondly we observe that we can construct a scroll $S$ containing $X$ from a pencil of divisors on $X$. Conversely the fibers of $S$ will cut out a pencil of divisors on $X$.

It will turn out, that in the case minimal degree complete pencils of a general canonical curve $C \subset \mathbb{P}^{g-1}$ all these correspondences are the same. We will construct an isomorphism from the variety of minimal rank syzygies to the corresponding Brill-Noether locus in this case.

Except for the construction of the above morphism all results of this section are well known.

Let’s start with

2.2.1. Proposition. Let $X \subset \mathbb{P}(V)$ be a variety as above and $s \in V_k$ a $k$th syzygy of rank $k + 2$. Then there exists a rational scroll $S \subset \mathbb{P}(V)$ of degree $k + 2$ and codimension $k + 1$ that contains $X$. Furthermore the vanishing set $Z_s \subset \mathbb{P}(V)$ is a fiber of $S$.

Proof. Let $\{x_1, \ldots, x_{k-2}\}$ be a basis of $(s) \subset V$. Consider the Koszul complex

$$
O(-k - 2) \rightarrow (s)^* \otimes O(-k - 1) \rightarrow \cdots \rightarrow \Lambda^k(s)^* \otimes (s)^* O(-2) \rightarrow \Lambda^{k+1}(s)^* \otimes (s)^* O(-1)
$$

where the maps are induced by multiplication with the trace element

$$
\sum x_i^* \otimes x_i
$$

Now $s$ induces a map from $O(-k - 2)$ to $V_k \otimes O(-k - 2)$ that lifts to a map of complexes

\[
\begin{align*}
O(-k - 2) & \rightarrow \cdots \rightarrow \Lambda^k(s)^* \otimes (s)^* O(-2) \rightarrow \Lambda^{k+1}(s)^* \otimes (s)^* O(-1) \\
V_k \otimes O(-k - 2) & \rightarrow \cdots \rightarrow V_0 \otimes (s)^* O(-2) \rightarrow O \rightarrow O_X
\end{align*}
\]
where $\alpha$ is again given by the multiplication with $\sum x_i^* \otimes x_i$ and $\beta$ can be described by the multiplication with $\sum x_i^* \otimes y_i$ where $\{y_1, \ldots, y_{k+2}\} \in V$ are linear forms. We now claim that the image of $\alpha \circ \beta$ is given by the $2 \times 2$-minors of

$$M = \begin{pmatrix} x_1 & \cdots & x_{k-2} \\ y_1 & \cdots & y_{k-2} \end{pmatrix}.$$ 

More precisely we claim

$$(\alpha \circ \beta)(x_{i_1}^* \wedge \cdots \wedge x_{i_k}^*) = \pm x_1^* \wedge \cdots \wedge x_{k+2}^* \otimes \det \begin{pmatrix} x_a & x_b \\ y_a & y_b \end{pmatrix}$$

where $\{i_1, \ldots, i_k\} \cup \{a, b\} = \{1, \ldots, k+2\}$.

Without restriction we can assume $(a, b) = (1, 2)$ and check

$$(\alpha \circ \beta)(x_3^* \wedge \ldots \wedge x_{k-2}^*) = \beta((-1)^k x_1^* \wedge x_3^* \wedge \cdots \wedge x_{k-2}^* \otimes x_1 + x_2^* \wedge x_3^* \wedge \cdots \wedge x_{k-2}^* \otimes x_2)
\begin{align*}
&= (-1)^{2k} x_1^* \wedge \cdots \wedge x_{k-2}^* \otimes x_1 y_2 - x_1^* \wedge \cdots \wedge x_{k-2}^* \otimes x_2 y_1 \\
&= x_1^* \wedge \cdots \wedge x_{k+2}^* \otimes \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}.
\end{align*}$$

So the $2 \times 2$ minors of $M$ are contained in $I_X$. Therefore the variety $S$ cut out by these minors contains $X$.

Furthermore $M$ is 1-generic. Suppose not, then $M$ has after row and column-operations the form

$$M = \begin{pmatrix} x_1 & x_2 & \cdots \\ 0 & y_2 & \cdots \end{pmatrix}$$

and $X \subset Z(x_1 y_2) \iff X \subset Z(x_1)$ or $X \subset Z(y_2)$ since $X$ is irreducible. This is impossible, since $X$ is non degenerate.

Consequently $M$ is 1-generic and $S$ is a scroll as claimed above.

Conversely we have

**2.2.2. Lemma.** Let $S \subset \mathbb{P}(V)$ be a scroll of degree $k+2$ and codimension $k+1$, $k \geq 1$. Then the minimal rank $k$th syzygies of $S$ have rank $k+2$ and the space of minimal rank $k$th syzygies

$$Y_{\text{min}} \subset \mathbb{P}(V^*_k)$$

is isomorphic to the $k$-uple embedding of $\mathbb{P}^1$. Furthermore there is a 1 : 1 correspondence between minimal rank syzygies and fibers of $S$. 

\[\square\]
Proof. $S$ is cut out by the $2 \times 2$-minors of a 1-generic matrix

$$A = \begin{pmatrix} x_1 & \cdots & x_{k+2} \\ y_1 & \cdots & y_{k+2} \end{pmatrix} \quad x_i, y_i \in V.$$

Let

$$\Phi_A : F \otimes G \to V$$

$$f_i \otimes g_1 \mapsto x_i$$

$$f_i \otimes g_2 \mapsto y_i$$

be the corresponding map of vector spaces, where $\dim F = k+2$ and $\dim G = 2$. Then $I_S$ is resolved by the Eagon-Northcott complex

$$I_S \leftarrow V_0 \otimes O(-2) \xleftarrow{\varphi_1} \cdots \xleftarrow{\varphi_k} V_k \otimes O(-k-2) \leftarrow 0$$

with $V_i = \Lambda^{i+2}F \otimes \Lambda^2G \otimes S^iG$.

On the space of $k$th syzygies $\mathbb{P}(V_k^*) \cong \mathbb{P}(\Lambda^{k+2}F \otimes \Lambda^2G \otimes S^kG)^* \cong \mathbb{P}^k$ the group $GL(2)$ acts by coordinate transformation of $G$. The rank of a syzygy is invariant under such operations and therefore the space of minimal rank syzygies $Y_{\text{min}} \subset \mathbb{P}^k$ is invariant under $GL(2)$. Furthermore $Y_{\text{min}}$ is compact since we are considering syzygies of minimal rank. Consequently every component of $Y_{\text{min}}$ must contain the minimal orbit

$$G/P \cong \mathbb{P}^1 \xrightarrow{\text{k-uple}} \mathbb{P}^k.$$

To show $Y = \mathbb{P}^1$ we calculate the tangent space of $Y_{\text{min}}$ in

$$s = f_1 \wedge \cdots \wedge f_{k+2} \otimes g_1 \wedge g_2 \otimes (g_1)^k =: f \otimes g \otimes (g_1)^k.$$

We recall the determinantal description of $Y_{\text{min}}$. Consider the map induced by $\varphi_k$

$$\hat{\varphi} : V_k \to V_{k-1} \otimes V \cong \text{Hom}(V_{k-1}^*, V)$$

$$f \otimes g \otimes g_1^i g_2^{k-i} \mapsto \sum_{i,j} f_i^*(f) \otimes g \otimes g_j^*(g_1^i g_2^{k-i}) \otimes \Phi(f_i, g_j)$$

Since $\Phi(f_i, g_j) \in V$ this induces a map of vector bundles

$$\psi : V_{k-1}^* \otimes \mathcal{O}_{\mathbb{P}^k}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^k}$$

Consider the basis of $V_{k-1}^*$ which is dual to

$$\{ f_1^*(f) \otimes g \otimes g_1^{k-1}, \ldots, f_{k+2}^*(f) \otimes g \otimes g_1^{k-1}, \ldots, f_{k+2}^*(f) \otimes g \otimes g_2^{k-1} \}.$$
In this basis $\psi$ is then given by the matrix

$$
\begin{pmatrix}
  s_0 & s_1 & s_2 & \cdots & s_k \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  s_1 & s_0 & s_2 & \cdots & s_k \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  s_{k-1} & s_k & s_0 & \cdots & s_k \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  s_k & s_{k-1} & s_0 & \cdots & s_k
\end{pmatrix}
$$

Where $s_i = (f \otimes g \otimes g_1^{k-i} g_2^i)^* \in V_k^*$ is a basis of $V_k^*$. In these coordinates we have $s = (1 : 0 : \cdots : 0)$ and therefore

$$
\langle s \rangle = \text{Im}(\Psi|_s) = \langle x_1, \ldots, x_{k+2} \rangle.
$$

Since the matrix $A$ was 1-generic, the above linear forms are linearly independent. This shows that $\text{rank } s = k + 2$ for all $s \in \mathbb{P}^1 \subset \mathbb{P}^k$.

Now consider the tangent vectors

$$
s_\epsilon = (1 : \epsilon_1 : \cdots : \epsilon_k)
$$

Since we can without restriction suppose, that $y_1$ is linearly independent from $x_1, \ldots, x_{k+2}$, and since the syzygies of $Y$ all have rank $k + 2$, all $(k + 3) \times (k + 3)$-minors of the first $k + 3$ columns of the above matrix have to vanish for $s_\epsilon$:
In particular the determinants

\[
\begin{vmatrix}
1 & \epsilon_1 \\
\vdots & \ddots & \vdots \\
\epsilon_i & & 1 \\
\end{vmatrix}
= \epsilon_{i+1}
\]

must vanish, proving \( \epsilon_i = 0 \) for \( i \geq 2 \). Therefore the tangent space of \( Y_{min} \) in \( s \) can be at most 1-dimensional. By applying \( GL(2) \) we get the same result for all points of \( \mathbb{P}^1 \subset Y_{min} \) proving \( \mathbb{P}^1 = Y_{min} \).

The 1 : 1 correspondence is seen as follows.

The fibers of \( S \) are the vanishing sets of the generalized rows \((\lambda, \mu) \cdot A\) of \( A \). Consider the syzygy \( s = f \otimes g \otimes (\lambda g_1 + \mu g_2)^k \in \mathbb{P}^1 \). Then

\[
\langle s \rangle = \text{Im}(\tilde{\varphi}(s)) = \langle \lambda v_{11} + \mu v_{1,2}, \ldots, \lambda v_{k+2,1} + \mu v_{k+2,1} \rangle = (\lambda, \mu) \cdot A.
\]

\[ \square \]

The above propositions suggest the following definition:

2.2.3. **Definition.** Let \( X \) be an irreducible, non-degenerate variety and \( I_X \) generated by quadrics. Then the \( k \)th syzygies of rank \( k+2 \) are called **scrollar syzygies**. The schemes \( Y_{k+2} \) of these syzygies are called **spaces of scrollar syzygies**.

If \( Y_{k+2} \) is nonempty, then this is also the space of minimal rank syzygies \( Y_{min} \) since an irreducible, non-degenerate variety can not have \( k \)th syzygies of rank \( k+1 \).

2.2.4. **Remark.** Scrollar syzygies are the easiest example of the geometric syzygies constructed by Green and Lazarsfeld in [GL84].

We can now make a precise statement of the geometric syzygy conjecture for general canonical curves.

2.2.5. **Conjecture (Geometric Syzygy Conjecture).** Let \( C \subset \mathbb{P}^{g-1} \) be a general canonical curve of genus \( g \). Then all minimal rank syzygies are scrollar, and the spaces of scrollar syzygies are non degenerate.

2.2.6. **Remark.** For special canonical curves it is important to consider the non reduced scheme structure on the space of scrollar syzygies as can be seen in the case of a curve of genus 6 with only one \( g^1_6 \) [AHS], p. 174].
Also there are geometric $k$th-syzygies in the sense of Green and Lazarsfeld \cite{GL84} which are not of rank $k + 2$. These must also be considered in the case of special curves. The easiest example of this phenomenon is exhibited by the plane quintic curve of genus 6 \cite{VBO0}. 

We now turn to the connection between scrolls and pencils. Here we restrict ourselves to the case of a canonical curve $C \subset \mathbb{P}^{g-1}$. Let $|D|$ be a complete pencil of degree $d$ on $C$. Then we can consider the union

$$S_{|D|} = \bigcup_{D' \in |D|} \langle D' \rangle \subset \mathbb{P}^{g-1}$$

where $\langle D' \rangle$ is the linear space spanned by $D'$ in $\mathbb{P}^{g-1}$.

\textbf{2.2.7. Proposition.} $S_{|D|}$ is a rational normal scroll of codimension $g - d$ containing $C$.

\textit{Proof.} Since $C$ is canonically embedded, $\mathbb{P}^{g-1} = \mathbb{P}(H^0(K))$ with $K$ a canonical divisor on $C$. The set of hyperplanes in $\mathbb{P}^{g-1}$ vanishing on $D'$ is therefore $H^0(K - D') = H^1(D')$ and the codimension of $\langle D' \rangle$ correspondingly $h^1(D') = g - d + 1$ by Riemann-Roch. This is the same for all $D' \in |D|$. So $S_{|D|}$ is a rational scroll of codimension $g - d$. Its equations are given by the $2 \times 2$-minors of the $2 \times (g - d + 1)$-matrix obtained from the natural map

$$H^0(D) \otimes H^0(K - D) \to H^0(K)$$

$S_{|D|}$ contains $C$ since $D$ moves in a pencil. 

Conversely consider a scroll containing $C$. Its fibers cut out a pencil of divisors on $C$. These pencils are not always complete:

\textbf{2.2.8. Proposition.} Let $C \subset \mathbb{P}^{g-1}$ be a non hyperelliptic canonical curve of genus $g$ contained in a scroll $S$ of codimension $c$. Let $F$ be a fiber of $S$ and $D = C.F$. Then $|D|$ is a $g_r^c$ with $r \geq 1$ and $d \leq g + r - c - 1$.

\textit{Proof.} The fibers of $S$ cut out a pencil of divisors linearly equivalent to $D$. Therefore $r = \dim |D| \geq 1$.

The codimension of a fiber $F$ is $c + 1$, so $h^0(K - D) \geq c + 1$. Riemann-Roch now gives

$$d = h^0(D) - h^0(K - D) - 1 + g \leq r + 1 - (c + 1) - 1 + g = g + r - c - 1$$

In particular these linear systems have low Clifford index:
2.2.9. Corollary. In the situation above we have \( \text{cliff}(D) \leq g - c - 2 \). If the corresponding complete linear system \(|D|\) is not a pencil, we even have \( \text{cliff}(D) \leq g - c - 3 \).

Proof.

\[
\text{cliff}(D) = d - 2r \\
\leq g + r - c - 1 - 2r \\
= g - c - 1 - r \\
\leq \begin{cases} 
  g - c - 2 & \text{if } r = 1 \\
  g - c - 3 & \text{if } r > 1
\end{cases}
\]

2.2.10. Corollary. A general canonical curve \( C \in \mathbb{P}^{g-1} \) has scrollar syzygies only up to step \( \lceil \frac{g-5}{2} \rceil \).

Proof. Let \( s \in V_k \) be a scrollar syzygy in step \( k \). Then the corresponding scroll \( S_k \) has codimension \( k + 1 \) by proposition 2.2.1. The divisor \( D \) cut out by a fiber of \( S \) has Clifford index

\[
\text{cliff}(D) \leq g - k - 3
\]

On the other hand it is well known, that on a general canonical curve all divisors have Clifford index at least \( \lceil \frac{g-2}{2} \rceil \). Therefore

\[
k \leq g - 3 - \left\lceil \frac{g - 2}{2} \right\rceil = \left\lceil \frac{g - 5}{2} \right\rceil
\]

2.2.11. Corollary. The scrollar syzygy conjecture implies Green’s conjecture for general canonical curves.

Proof. Assume the scrollar syzygy conjecture. Then all minimal rank syzygies are scrollar. But by the corollary above there are no scrollar syzygies in step \( k > \lceil \frac{g-5}{2} \rceil \). Therefore there can be no syzygies at all in these steps. This is Greens conjecture for the general canonical curve.

We want now to consider the last step in the resolution of a general canonical curve, that still allows syzygies:

2.2.12. Definition. Let \( C \subset \mathbb{P}^{g-1} \) be a general canonical curve. Then the scrollar \( \lceil \frac{g-5}{2} \rceil \)th syzygies of \( C \) are called the last scrollar syzygies of \( C \).
For the last scollar syzygies everything is as nice as possible. First we calculate the degree of the corresponding divisors:

**2.2.13. Lemma.** Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve, a last scollar syzygy, $S$ the corresponding scroll and $D$ the divisor cut out by the fiber $F_s$ corresponding to $s$. Then $|D|$ is a complete pencil of degree $\left\lceil \frac{g+2}{2} \right\rceil$.

*Proof.* Suppose $|D|$ was not a complete pencil. Then by corollary 2.2.9 we would have

$$\text{cliff } D \leq g - \left\lceil \frac{g-5}{2} \right\rceil - 4 = \left\lceil \frac{g-4}{2} \right\rceil,$$

which is impossible for a general canonical curve. Consequently we have $r = 1$ and $\text{cliff } D = \left\lceil \frac{g-2}{2} \right\rceil$ the minimum possible value. In particular

$$d = \text{cliff}(D) + 2r = \left\lceil \frac{g+2}{2} \right\rceil.$$

\[\square\]

This allows us to construct a morphism from the space of last scollar syzygies to the corresponding Brill-Noether-Locus:

**2.2.14. Proposition.** Let $C \subset \mathbb{P}^{g-1}$ be a general canonical curve, and $Y_{\min}$ its scheme of last scollar syzygies. Then there exists an isomorphism

$$\zeta: Y_{\min} \to C^1_{\left\lceil \frac{g+2}{2} \right\rceil}.$$ 

In particular $Y_{\min}$ is a disjoint union of

$$\frac{2}{g+2} \left( \frac{g}{2} \right)$$

classical curves if $g$ is odd, and an irreducible ruled surface over $W^1_{\left\lceil \frac{g+2}{2} \right\rceil}$ if $g$ is odd.

*Proof.* Consider the vector bundle of linear forms $L$ on the variety of $Y_{\min}$ of last scollar syzygies. Let $Q$ be the cokernel of the natural inclusion

$$0 \to L \to V \otimes O_{Y_{\min}} \to Q \to 0.$$

$Q$ is globally generated and has rank $\left\lceil \frac{g}{2} \right\rceil$. It therefore induces a morphism

$$\alpha: Y_{\min} \to G(V, \left\lceil \frac{g}{2} \right\rceil).$$
where $G := G(V, \lceil \frac{g}{2} \rceil)$ is the Grassmannian of $\lceil \frac{g}{2} \rceil$ dimensional quotient spaces of $V$, or equivalently the Grassmannian of $\lceil \frac{g-2}{2} \rceil$ dimensional linear subspaces of $\mathbb{P}^{g-1}$.

Now consider the incidence Variety

$$I = \{(\mathbb{P}^{\lceil \frac{g-2}{2} \rceil}, c) \mid c \in \mathbb{P}^{\lceil \frac{g-2}{2} \rceil} \cap C \subset \mathbb{P}^{g-1} \} \subset G \times C$$

and the diagram

$$\begin{array}{ccc}
D & \rightarrow & I \\
\downarrow & & \downarrow \\
Y_{\text{min}} & \alpha & \rightarrow C \\
\end{array}$$

obtained by base change. $D$ is a family of divisors. The fiber over a scrollar syzygy $s$ is the divisor cut out by the zero locus $Z_s$ of $s$. Lemma 2.2.13 shows that these divisors all have degree $d = \lceil \frac{g+2}{2} \rceil$ and $r = 1$. By the universal property of $C_d^r$ we obtain a morphism

$$\zeta: Y_{\text{min}} \rightarrow C^1_{\lceil \frac{g+2}{2} \rceil}.$$

To prove the surjectivity of $\zeta$ take let $D \in C^1_{\lceil \frac{g+2}{2} \rceil}$ be a divisor. The scroll $S_{\lceil D \rceil}$ spanned by $|D|$ has codimension $g - \lceil \frac{g+2}{2} \rceil$ and the fiber $D$ corresponds to a scrollar $\lceil \frac{g-5}{2} \rceil$th (last) syzygy $s$ with $Z_s = \langle D \rangle$ by lemma 2.2.2. This implies $D \subset Z_s.C$. Equality follows since they have the same degree by lemma 2.2.13.

We are left to prove that $\zeta$ is injective. Assume $s, t$ are two last scrollar syzygies, whose zero sets $Z_s$ and $Z_t$ cut out the same divisor $D = Z_s.C = Z_t.C$. Then the scroll $S_{\lceil D \rceil}$ obtained from the complete pencil $|D|$ is contained in the scrolls $S_s$ and $S_t$ corresponding to $s$ and $t$. Now all these scrolls are of the same dimension, so they have to be equal. Since there is a $1 : 1$ correspondence between divisors $D' \in |D|$, fibers $\langle s \rangle D'$ and scrollar syzygies of $S_{\lceil D \rceil} = S_s = S_t$ we must have $s = t$.

So $\zeta$ is bijective. Now $Y_{\text{min}}$ is reduced by definition, and since $C$ is a general canonical curve, $C^1_{\lceil \frac{g+2}{2} \rceil}$ is normal. So by Zariskis Main Theorem $\zeta$ is an isomorphism.

The description of $Y_{\text{min}} \cong C^1_{\lceil \frac{g+2}{2} \rceil}$ is obtained from Brill-Noether-theory. For a general canonical curve the dimension of $C^1_{\lceil \frac{g+2}{2} \rceil}$ is given by [ACGH85, p.
\[
\dim C_{\left\lceil \frac{g+2}{2} \right\rceil}^1 = \rho + 1 = g - 2 \left( g - \left\lfloor \frac{g+2}{2} \right\rfloor + 1 \right) + 1 = 2 \left\lfloor \frac{g+2}{2} \right\rfloor - 1 - g = \begin{cases} 1 & \text{for } g \text{ even} \\ 2 & \text{for } g \text{ odd} \end{cases}
\]

Now the Abel-Jacobi map
\[\alpha: C_{\left\lceil \frac{g+2}{2} \right\rceil}^1 \to W_{\left\lceil \frac{g+2}{2} \right\rceil}^1 \]

has \(\mathbb{P}^1\)-fibers, so \(C_{\left\lceil \frac{g+2}{2} \right\rceil}^1\) is a disjoint union of finitely many \(\mathbb{P}^1\)’s for \(g\) even and a ruled surface over \(W_{\left\lceil \frac{g+2}{2} \right\rceil}^1\) for \(g\) odd.

In the even case the number of \(\mathbb{P}^1\)’s can be calculated by a formula of Castelnuovo [ACGHS, p. 211]:
\[
\deg W_{\left\lceil \frac{g+2}{2} \right\rceil}^1 = g! \prod_{i=0}^{1} \frac{i!}{(g - \left\lceil \frac{g+2}{2} \right\rceil + 1 + i)!} = \frac{2}{g + 2} \left( g \right)_\frac{g+2}{2}
\]

2.3 Linear Sections

In this section we want consider general linear sections \(X \cap \mathbb{P}(W)\) of \(X\). It is well known that such a general linear section has syzygy spaces of the same dimension as \(X\). So one can consider a syzygy \(s\) of \(X\) also as a syzygy of \(X \cap \mathbb{P}(W)\). The rank of this syzygy can change however, if \(\mathbb{P}(W)\) does not intersect the zero locus \(Z_s\) of \(s\) in the expected codimension.

This will lead to a determinantal description of those minimal rank syzygies of \(X\) that drop rank further when considered as syzygies of \(X \cap \mathbb{P}(W)\).

If \(X\) is a Mukai-Variety and \(C = X \cap \mathbb{P}^{g-1}\) a general canonical curve it will turn out in the remaining sections of this paper, that these determinantal subvarieties are of expected dimension and describe the full space of last scollar syzygies of \(C\).

Let now \(X \subset \mathbb{P}(V)\) be an irreducible, non degenerate variety, \(I_X\) generated by quadrics, \(\mathbb{P}(W) \subset \mathbb{P}(V)\) a linear subspace and \(\pi: V \to W\) the corresponding projection. Consider the intersection \(X \cap \mathbb{P}(W)\) and the linear
strand

\[ I_{X \cap \mathbb{P}(W)/\mathbb{P}(W)} \leftarrow W_0 \otimes \mathcal{O}(-2) \xleftarrow{\varphi_1} W_1 \otimes \mathcal{O}(-3) \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_m} W_m \otimes \mathcal{O}(-m - k) \]

of its resolution.

We start by recalling:

2.3.1. Proposition. The inclusion \( \mathbb{P}(W) \subset \mathbb{P}(V) \) induces linear maps

\[ \pi_k : V_k \to W_k \]

If \( X \) is arithmetically Cohen Macaulay and \( X \cap \mathbb{P}(W) \) is of the expected dimension, then all \( \pi_k \)'s are isomorphisms.

Proof. Since \( X \) is arithmetically Cohen Macaulay, we have in particular \( H^1(X, \mathcal{O}_X(q)) = 0 \) for all \( q \geq 0 \). We can then apply [Gre84a, Thm 3.6.7] to get the result. \( \square \)

If we regard a syzygy \( s \in Y_{\text{min}}(X) \) as a syzygy of \( X \cap \mathbb{P}(W) \) in the way made precise by the preceding proposition, we can calculate the rank of \( s \) there by using the vector bundle of linear forms:

2.3.2. Corollary. Let \( X \) be ACM, \( X \cap \mathbb{P}(W) \) of expected dimension, \( Y_{\text{min}}(X) \subset \mathbb{P}(V^*_k) \cong \mathbb{P}(W^*_k) \) the scheme of \( k \)-th minimal rank syzygies of \( X \) and \( L \) the vector bundle of linear forms on \( Y_{\text{min}}(X) \). Then there exists a map of vector bundles

\[ \alpha : L \to W \otimes \mathcal{O}_{Y_{\text{min}}(X)} \]

such that the rank of a syzygy \( s \in Y_{\text{min}}(X) \) considered as a syzygy of \( X \cap \mathbb{P}(W) \) is

\[ \text{rank}_{X \cap \mathbb{P}(W)} s = \text{rank} \alpha|_s \]

Proof. We have a diagram

\[
\begin{array}{ccc}
V_k & \xrightarrow{\varphi_k} & V_{k-1} \otimes V \\
\pi_k \downarrow & & \downarrow \pi_{k-1} \otimes \pi \\
W_k & \xrightarrow{\varphi_W} & W_{k-1} \otimes W
\end{array}
\]

Since \( \pi_k \) and \( \pi_{k-1} \) are isomorphisms, this induces a diagram.
on $Y_{\min}$ the top map factors over $L$ yielding

$$L \cong \text{Im}(\psi^V)$$

Since $\beta$ is surjective, we have for every syzygy $s \in Y_{\min}$

$$\text{rank}(\psi^W|_s) = \text{rank}(\alpha|_s)$$

as claimed.

A stronger statement is true for a general intersection with quadric hypersurface. Here the dimension of syzygy spaces and the rank of the syzygies stay the same:

**2.3.3. Proposition.** Let $X \subset \mathbb{P}(V)$ be an irreducible variety, $I_X$ generated by quadrics and $X \cap Q$ the intersection with a general quadric $Q \subset \mathbb{P}(V)$. If

$$I_X \leftarrow V_0 \otimes \mathcal{O}(-2) \leftarrow V_1 \otimes \mathcal{O}(-3) \leftarrow \ldots \leftarrow V_k \otimes \mathcal{O}(-k-2) \leftarrow 0$$

is the linear strand of the resolution of $I_X$, then

$$I_{X \cap Q} \leftarrow (V_0 \oplus Q) \otimes \mathcal{O}(-2) \leftarrow V_1 \otimes \mathcal{O}(-3) \leftarrow \ldots \leftarrow V_k \otimes \mathcal{O}(-k-2) \leftarrow 0$$

is the linear strand of the resolution of $I_{X \cap Q}$. All differentials are the same except for $\varphi_0$ whose matrix has one more column of zeros.

In particular the $k$th syzygies ($k \geq 1$) of $X \cap Q$ are the same as those of $X$ and they have the same ranks.

**Proof.** We prove the proposition on the ring level. Let $R = \mathbb{C}[V]$ be the coordinate ring of $\mathbb{P}(V)$ and

$$C_\bullet: R \leftarrow C_1 \leftarrow \ldots \leftarrow 0$$
the graded minimal free resolution of $R/I_X$. $R/QR$ is resolved by the Koszul complex

$$K_\bullet(Q): R \xleftarrow{Q} R(-2) \leftarrow 0$$

Since $Q$ is a nonzerodivisor in $R/I_X$ the total complex $C_\bullet(Q) := C_\bullet \otimes K_\bullet(Q)$ is a resolution of $R/QI_X = R/I_X \cap Q$ \cite[Thm 16.4]{Mat90}.

For degree reasons the linear strand of $C_\bullet(Q)$ is the same as the one of $C_\bullet$ except for the first step. \hfill \square

Later we will apply the last two propositions to Mukai varieties, as defined by

2.3.4. Theorem (Mukai). Every general canonical curve of genus $7 \leq g \leq 9$ is a general linear section of an embedded rational homogeneous (Mukai) variety $M_g$. General canonical curves of genus 6 are cut out by a general quadric on a general linear section of a homogeneous (Mukai) variety $M_6$.

More explicitly we have

| $g$ | $M_g$ |
|-----|--------|
| 6   | the Grassmannian $\text{Gr}(2,5) \subset \mathbb{P}^9$ |
| 7   | the Spinor Variety $S_{10} \subset \mathbb{P}^{15}$ |
| 8   | the Grassmannian $\text{Gr}(2,6) \subset \mathbb{P}^{14}$ |
| 9   | the symplectic Grassmannian $\text{Gr}(3,6,\eta) \subset \mathbb{P}^{13}$ |

Proof. \cite{Muk92b, Muk92a} \hfill \square

3 Representation Theory

As the Mukai varieties are rational homogeneous, the main tool of our study will we representation theory. We will use the following notations from Fulton/Harris \cite{FH91} and Ottaviani \cite{Ott95}:

3.1. Notation. We will denote by
a semisimple and simply connected Lie group

\( P \subset G \)

a parabolic subgroup

\( p \subset g \)

the corresponding Lie algebras

\( \{ H_i \} \subset \mathfrak{h} \)

a Cartan subalgebra

\( \{ L_i \} \subset \mathfrak{h}^* \)

a basis of \( \mathfrak{h} \)

\( R = \{ \alpha_i \} \subset \mathfrak{h}^* \)

the dual basis to \( \{ H_i \} \)

\( R = R^+ \cup R^- \)

a decomposition into positive and negative roots

\( g = \mathfrak{h} \oplus (\oplus_{\alpha_i \in R^+} \mathfrak{g}_{\alpha_i}) \)

the Cartan decomposition

\( \Delta = \{ \alpha_1, \ldots, \alpha_k \} \subset R^+ \)

the set of simple roots

\( \omega_1, \ldots, \omega_k \)

the corresponding fundamental weights

\( \Sigma \subset \Delta \)

a subset of simple roots

\( R^+(\Sigma) = \{ \alpha \in R^+ | \alpha = \sum_{i} o_i \mathfrak{g} \} \)

the positive roots generated by \( \Delta - \Sigma \)

\( R^-(\Sigma) = \{ \alpha \in R^- | \alpha = \sum_{i} o_i \mathfrak{g} \} \)

the negative roots generated by \( \Delta - \Sigma \)

\( \mathfrak{p}(\Sigma) \)

the subalgebra \( \mathfrak{h} \oplus (\oplus_{\alpha_i \in R^+} \mathfrak{g}_{\alpha_i}) \oplus (\oplus_{\alpha_i \in R^- \setminus \Sigma} \mathfrak{g}_{\alpha_i}) \)

the corresponding parabolic subgroup of \( G \)

\( P(\Sigma) \)

the semisimple part of \( \mathfrak{p}(\Sigma) \)

\( W \)

the Weyl group of \( g \)

We also need the notion of a highest weight vector:

3.2. Definition. Let \( \rho: \mathfrak{g} \to \mathfrak{gl}(V) \) be a representation. A vector \( v \in V \) with

\[ \rho(\mathfrak{g}_\alpha)(v) = 0 \quad \forall \alpha \in R^+ \quad \text{and} \quad \rho(\mathfrak{h})(v) = \lambda(\mathfrak{h})v, \quad \lambda \in \mathfrak{h}^* \]

is called a highest weight vector. \( \lambda \in \mathfrak{h}^* \) is then called a highest weight.

With this we use further notations

3.3. Notation. We denote by

\( \rho_\lambda \)

the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda \)

\( S_\lambda \)

the Schur functor for a partition \( \lambda \)

\( \Lambda_\lambda \)

the Schur functor for the dual partition \( \tilde{\lambda} \)

3.4. Remark. With this notation we have for example

\[ S_{1,1,1} = \Lambda_3 = \Lambda^3 \]

where \( \Lambda^3 \) is the usual exterior product.

Sometimes we represent the Schur functors by young tableaux, in this case

\[ S_{1,1,1} = \Lambda_3 = \begin{array}{|c|c|} \hline & \\ \hline \end{array} \]

If \( g = gl(V) \) we also have the identity \( \rho_{\lambda_1 \mu_1 + \cdots + \lambda_n} = S_{\lambda_1, \ldots, \lambda_n} \).
If $\rho$ is a representation of parabolic subgroup $P \subset G$ with highest weight $\lambda = \lambda_1 L_1 + \cdots + \lambda_n L_n$ we use the notation $E_\rho$, $E(\lambda)$ or $E(\lambda_1, \ldots, \lambda_n)$ for the vector bundle induced by $\rho$ on $G/P$.

3.5. **Theorem (Matsushima).** A vector bundle $E$ of rank $r$ over $G/P$ is homogeneous if and only if there exists a representation $\rho: P \to \text{GL}(r)$ such that $E \cong E_\rho$.

**Proof.** [Ott95, Theorem 9.7]

3.6. **Theorem (Classification of irreducible bundles over $G/P$).**

Let $P(\Sigma) \subset G$ be a parabolic subgroup and $\omega_1, \ldots, \omega_k$ the fundamental weights corresponding to the subset of simple roots $\Sigma \subset \Delta$. Then all irreducible representations of $P(\Sigma)$ are

$$V \otimes L_{\omega_1}^{n_1} \otimes \cdots \otimes L_{\omega_k}^{n_k}$$

where $V$ is a representation of $S_P$ and $n_i \in \mathbb{Z}$. $L_{\omega_i}$ are the one dimensional representations of $S_P$ induced by the fundamental weights.

The weight lattice of $S_P$ is embedded in the weight lattice of $G$. If $\lambda$ is the highest weight of $V$, we will call $\lambda + \sum n_i w_i$ the highest weight of the irreducible representation of $P(\Sigma)$ above.

**Proof.** [Ott95, Proposition 10.9 and remark 10.10]

For the cohomology of homogeneous vector bundles we use

3.7. **Theorem (Bott).** Consider the homogeneous vector bundle $E(\lambda)$ on $X = G/P$ and $\delta$ the sum of fundamental weights of $G$. Then

- $H^i(X, E(\lambda))$ vanishes for all $i$ if there is a root $\alpha$ with $(\alpha, \delta + \lambda) = 0$
- Let $i_0$ be the number of positive roots $\alpha$ with $(\alpha, \delta + \lambda) < 0$. Then $H^i(X, E(\lambda))$ vanishes for $i \neq i_0$ and $H^{i_0}(X, E(\lambda)) = \rho_{w(\delta + \lambda) - \delta}$

where $(,)$ denotes the Killing form on $h^*$, $w(\delta + \lambda)$ is the unique element of the fundamental Weyl chamber which is congruent to $\delta + \lambda$ under the action of the Weyl group, and $\rho_{w(\delta + \lambda) - \delta}$ is the corresponding representation of $G$.

**Proof.** [Ott95, Theorem 11.4]

In several proofs of this paper we need to calculate the decomposition of tensor products of fundamental representations. Formulas for this can be found in [KNS8]. Also we sometimes calculate the dimension of certain representations. This can be done in various combinatoric ways and by the
use of the Weyl character formula. In this paper all calculations of this kind have been checked by the computer program SYMMETRICA [KKL92] which can be used online on the web.

4 Genus 6

The following is well known, but sets the stage for the more involved computations for Mukai varieties of higher genus.

4.1 Syzygies of $M_6$

Let $V$ be a 5-dimensional vector space with basis $\{v_1, \ldots, v_5\}$. In this section we will abbreviate $\Lambda_\lambda V$ by $\Lambda_\lambda$.

The Mukai variety for genus 6 is

$$M_6 = \text{Gr}(V, 2) = \text{Gr}(5, 2) \cong \text{GL}(5)/P \subset \mathbb{P}(\Lambda_2) \cong \mathbb{P}^9$$

The diagonal matrices $H$ form a Cartan subalgebra of $gl_5$ and $p$. The matrices $H_i = E_{i,i}$ are a basis of $h$. Let $\{L_i\}$ be the dual basis of $h^\ast$. The positive roots of $gl_5$ are $L_i - L_j$ with $i > j$ and $\omega_i = \sum_{j=1}^{i-1} L_j$ are the fundamental weights.

Let’s first consider the dimensions of the spaces of linear syzygies. To write these in compact form we use the MACAULAY-notation [GS]:

4.1.1. Proposition. The syzygy-numbers of $M_6 = \text{Gr}(5, 2)$ are

$$
\begin{array}{cccc}
1 & - & - & - \\
- & 5 & 5 & - \\
- & - & - & 1
\end{array}
$$

Proof. The Grassmannian $\text{Gr}(5, 2)$ is cut out by the $4 \times 4$ Pfaffians of a generic $5 \times 5$ matrix $A$. It is therefore a codimension 3 Gorenstein variety and has the resolution

$$I_{\text{Gr}(5, 2)} \leftarrow V^\ast \otimes \mathcal{O}(-2) \leftarrow V \otimes A \otimes \mathcal{O}(-3) \leftarrow \mathcal{O}(-5) \leftarrow 0.$$ 

Since $\dim V = 5$ this gives the above syzygy numbers. $\square$

Since $\text{GL}(5)$ acts on $\text{Gr}(5, 2)$ we also have an action on the syzygies. The corresponding representations are calculated by

4.1.2. Proposition. The linear strand of the resolution of $\text{Gr}(5, 2)$ is

$$I_{\text{Gr}(5, 2)} \leftarrow \Lambda_4 \otimes \mathcal{O}(-2) \leftarrow \Lambda_{51} \otimes \mathcal{O}(-3)$$
Proof. From above we have a linear strand

\[ I_{Gr(5,2)} \leftarrow V_0 \otimes \mathcal{O}(-2) \leftarrow V_1 \otimes \mathcal{O}(-3) \]

with \( \dim V_0 = \dim V_1 = 5 \). \( V_0 \) is a invariant (not necessarily irreducible) subspace of quadrics. This gives

\[ V_0 \subset S_2(A_2) \subset \Lambda_2 \otimes \Lambda_2 = \Lambda_4 \oplus \Lambda_{31} \oplus \Lambda_{22} \]

where the irreducible components have dimensions 5, 45 and 50 respectively. So \( V_0 = \Lambda_4 \). Similarly we have

\[ V_1 \subset \Lambda_4 \otimes \Lambda_2 = \Lambda_{51} \oplus \Lambda_{42} \]

where the irreducible components have dimensions 5 and 45 respectively. This implies \( V_1 = \Lambda_{51} \).

This allows us to describe the minimal rank syzygies \( Y_{\text{min}} \) of \( Gr(5,2) \) and the vector bundle of linear forms on \( Y_{\text{min}} \)

4.1.3. Proposition. The scheme of minimal rank first syzygies of \( Gr(5,2) \) is

\[ Y_{\text{min}} \cong GL(5)/P \cong \mathbb{P}(\Lambda_{51}^\ast) \cong \mathbb{P}^4 \]

The bundle of linear forms on \( Y_{\text{min}} \) is

\[ L|_{Y_{\text{min}}} = E(1,1,0,0,0)^\ast = T_{\mathbb{P}^4}(-2) \]

\( L \) has rank 4.

Proof. From proposition 2.1.7 we know, that \( Y_{\text{min}} \subset \mathbb{P}(\Lambda_{51}^\ast) \cong \mathbb{P}^4 \) must contain the minimal orbit of \( GL(5) \) in \( \mathbb{P}(\Lambda_{51}^\ast) \) under the action

\[ \rho : GL(5) \rightarrow GL(\Lambda_{51}^\ast). \]

Here this orbit \( GL(5)/P \) is the whole \( \mathbb{P}^4 \) such that \( Y_{\text{min}} = \mathbb{P}^4 \).

To describe the vector bundle of linear forms on \( Y_{\text{min}} = GL(5)/P \) we have to determine the action of \( P \) on a fiber of \( L \). We start by considering the dual actions \( \rho^\ast \) of \( GL(5) \) and \( P \) on \( \Lambda_{51} \). The parabolic subgroup \( P \) in its standard representation is then the set of matrices that fix a given \( \mathbb{P}^0 \), i.e matrices of the form

\[
\begin{pmatrix}
* & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & *
\end{pmatrix}
\]
The semisimple part of $P$ is $S_P = \text{GL}(1) \times \text{GL}(4)$ where $\text{GL}(1)$ acts on $\langle v_1 \rangle$ and $\text{GL}(4)$ acts on $\langle v_2, v_3, v_4, v_5 \rangle$ in the standard way.

The maximal weight vector

$$s = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} \in \Lambda_{51}$$

is a syzygy in $Y_{\text{min}}$. To determine the fiber $L|_s$ of $L$ over $s$ we restrict the map

$$\psi: \Lambda_4^* \otimes O_{P(\Lambda_{51}^*)}(-1) \to \Lambda_2 \otimes O_{P(\Lambda_{51}^*)}$$

from definition 2.1.4 to $s$. This gives

$$\psi|_s = \tilde{\varphi}(s) \in \text{Hom}(\Lambda_4^*, \Lambda_2) \cong \Lambda_4 \otimes \Lambda_2$$

where

$$\tilde{\varphi}: \Lambda_{51} \hookrightarrow \Lambda_4 \otimes \Lambda_2.$$

Using Young-diagrams we get

Consequently the fiber $L|_s$ of the line bundle of linear forms is

$$L|_s = \text{Im} \tilde{\varphi}(s) = \left\langle \begin{pmatrix} 1 \\ 5 \\ 1 \\ 4 \\ 1 \\ 3 \\ 1 \\ 2 \end{pmatrix} \right\rangle = \langle v_1 \wedge v_5, v_1 \wedge v_4, v_1 \wedge v_3, v_1 \wedge v_2 \rangle$$

and $L$ is of rank 4. $S_P$ acts irreducibly on this fiber, and $v_1 \wedge v_2$ is the maximal weight vector of weight $L_1 + L_2$. (Notice that the weights with respect to $G$ are the same as the ones with respect to $S_P$ since we can use the same Cartan subalgebra $\mathfrak{h} \subset \text{Lie}S_P \subset \mathfrak{p} \subset \mathfrak{g}$.)
Therefore
\[ E_\rho^* = E(1, 1, 0, 0, 0) \]
and
\[ L = E_\rho = E(1, 1, 0, 0, 0)^* \]
Now the tautological sequence of \( \mathbb{P}^4 \) is
\[ 0 \rightarrow E(0, 1, 0, 0, 0) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^4} \rightarrow E(1, 0, 0, 0, 0) \rightarrow 0 \]
i.e. \( E(1, 0, 0, 0, 0) = \mathcal{O}(1) \), \( E(0, 1, 0, 0, 0) = \Omega(1) \) and \( E(1, 1, 0, 0, 0)^* = \mathcal{T}_{\mathbb{P}^4}(-2) \).

4.2 General Canonical Curves of Genus 6

Let \( C \) be a general canonical curve of genus 6. From Mukai’s theorem we obtain a \( \mathbb{P}^5 \cong \mathbb{P}(W) \subset \mathbb{P}^9 \cong \mathbb{P}(\Lambda_2 V) \) and a quadric in \( \mathbb{P}^5 \) such that
\[ S = \text{Gr}(V, 2) \cap \mathbb{P}^5 \]
is a Del Pezzo surface and
\[ C = S \cap Q \]

4.2.1. Proposition. On \( \mathbb{P}^4 \cong Y_{\text{min}}(\text{Gr}(5, 2)) \) there exists a map of vector bundles
\[ \alpha: \mathcal{T}_{\mathbb{P}^4}(-2) \rightarrow 6\mathcal{O}_{\mathbb{P}^4} \]
such that its rank 3 locus \( Z_3(\alpha) \) is the scheme \( Z \) of last scrollar syzygies of \( C \). \( Z \) is a configuration of 5 skew lines in \( \mathbb{P}^4 \).

Proof. \( \left\lfloor \frac{g-5}{2} \right\rfloor = 1 \) so the first scrollar syzygies of \( C \) are also the last scrollar syzygies. The minimal rank first syzygies of \( \text{Gr}(5, 2) \) are of rank 4 and fill the whole space of first syzygies \( \mathbb{P}(\Lambda^*_5) = \mathbb{P}^4 \) as calculated in proposition 4.1.3.

Since \( S \) is a general linear section of \( \text{Gr}(5, 2) \) we can apply corollary 2.3.2 to obtain a map
\[ \alpha: L \rightarrow W \otimes \mathcal{O}_{Y_{\text{min}}} \]
whose rank calculates the rank of syzygies \( s \in Y_{\text{min}} \) considered as syzygies of \( S \). In our case this is equal to
\[ \alpha: \mathcal{T}_{\mathbb{P}^4}(-2) \rightarrow 6\mathcal{O}_{\mathbb{P}^4} \]
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since $\mathbb{P}(W) = \mathbb{P}^6$ and $L = T_{\mathbb{P}^4}(-2)$. Now $C$ is $S$ intersected with a general quadric and therefore $\alpha$ also calculates the rank of $s$ considered a a syzygies of $C$ by proposition 2.3.3.

The last scrollar syzygies of $C$ are first syzygies of rank 3. The argument above shows that the scheme $Z$ of last scrollar syzygies contains the rank 3 locus $Z_3(\alpha)$ of $\alpha$.

Since $\mathbb{P}(W) \subset \mathbb{P}(\Lambda^2 V)$ is a general subspace, and $L^* = \Omega_{\mathbb{P}^4}(2)$ is globally generated, $Z_3(\alpha)$ is reduced and of expected dimension

$$\dim Z_3(\alpha) = \dim \mathbb{P}^4 - (4 - 3)(6 - 3) = 1.$$

On the other hand we are also in the situation of corollary 2.2.14 which gives an isomorphism

$$\zeta: Z \to C_4^1 = \bigcup_{i=1}^5 \mathbb{P}^1.$$

This shows that $Z_3(\alpha) \subset Z$ is the union of at most 5 disjoint lines.

Since $Z_3(\alpha)$ is of expected dimension and we can calculate its class with Porteous formula [ACGHS][p.86]:

$$z_3(\alpha) = \Delta_{6-3.4-3} \left( \frac{c_t(6O_{\mathbb{P}^4})}{c_t(T_{\mathbb{P}^4}(-2))} \right) = \Delta_{3.1} \left( \frac{c_t(6O_{\mathbb{P}^4}(1))}{c_t(T_{\mathbb{P}^4}(-1))} \right)$$

The Chern polynomials involved are

$$c_t(T_{\mathbb{P}^4}(-1)) = \frac{1}{1 - Ht}$$

as obtained by the Euler-Sequence and

$$c_t(6O_{\mathbb{P}^4}(1)) = (1 + Ht)^6.$$

This yields

$$a = \frac{c_t(6O_{\mathbb{P}^4}(1))}{c_t(T_{\mathbb{P}^4}(-1))} = (1 + Ht)^6(1 - Ht) = 1 + 5Ht + 9H^2t^2 + 5H^3t^3 \pm \ldots$$

and

$$z_3(\alpha) = \Delta_{3.1}(a) = \det(a_3) = 5H^3.$$

Since $Z_3(\alpha)$ is reduced this shows that $Z_3(\alpha)$ contains all 5 lines of $Z$. In particular we have $Z = Z_3(\alpha)$. \qed
4.2.2. Corollary. The ideal sheaf $I_{Z/P^4}$ is resolved by

$$
I_{Z/P^4} \leftarrow 15E(-4,-1,-1,-1,-1)
$$

$$
\leftarrow 6E(-5,-1,-1,-1,-2)
$$

$$
\leftarrow E(-6,-1,-1,-1,-3)
$$

$$
\leftarrow 0
$$

Proof. Since $Z = Z_3(\alpha)$ and $\alpha$ drops rank in the expected dimension, the ideal sheaf is resolved by the corresponding Eagon-Northcott complex

$$
I_{Z/P^4} \leftarrow \Lambda^4W^* \otimes \Lambda^4L \leftarrow \Lambda^5W^* \otimes \Lambda^4L \otimes S_1L \leftarrow \Lambda^6W^* \otimes \Lambda^4L \otimes S_2L \leftarrow 0.
$$

Since $\dim W^* = 6$ we have

$$
\Lambda^iW^* \otimes \mathcal{O} = \binom{6}{i} \mathcal{O}.
$$

This gives the above multiplicities.

Furthermore

$$
\Lambda^4L = \Lambda^4E(1,1,0,0,0)^* = E(4,1,1,1,1)^* = E(-4,-1,-1,-1,-1)
$$

and

$$
S_iL = S_iE(1,1,0,0,0)^* = E(i,i,0,0,0)^* = E(-i,0,0,0,-i).
$$

Applying these equations to the complex above yields the desired resolution. \qed

4.2.3. Theorem. The scheme $Z$ of last scrollar syzygies of a general canonical curve $C \subset \mathbb{P}^5$ of genus 6 is a configuration of 5 skew lines in $\mathbb{P}^4$ that spans the whole $\mathbb{P}^4$ of first syzygies of $C$.

Proof. We have to show, that

$$
Z \subset \mathbb{P}(\Lambda^*_5)
$$

is non degenerate. It is enough to check $h^0(I_{Z/P^4}(1)) = 0$.

Since $\mathcal{O}_{P^4}(1) = E(1,0,0,0,0)$ we obtain

$$
I_{Z/P^4}(1) \leftarrow 15E(-3,-1,-1,-1,-1)
$$

$$
\leftarrow 6E(-4,-1,-1,-1,-2)
$$

$$
\leftarrow E(-5,-1,-1,-1,-3)
$$

$$
\leftarrow 0
$$

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We now calculate the cohomology of the above vector bundles using the theorem of Bott. The fundamental weights of $GL(5)$ are $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_5$. The sum of fundamental weights is therefore $\delta = 5L_1 + 4L_2 + 3L_3 + 2L_4 + 1L_5$. We obtain

| $E(\lambda)$          | $\lambda + \delta$ |
|-----------------------|---------------------|
| $E(-3, -1, -1, -1, -1)$ | $(2, 3, 2, 1, 0)$    |
| $E(-4, -1, -1, -1, -2)$ | $(1, 3, 2, 1, -1)$  |
| $E(-5, -1, -1, -1, -3)$ | $(0, 3, 2, 1, -2)$  |

For the first two rows we find positive roots $L_i - L_j$ with $(L_i - L_j, \lambda + \delta) = 0$. Therefore all cohomology of these bundles vanish. For the last row no such root is found, but there are 3 roots with $(L_i - L_j, \lambda + \delta) < 0$. Therefore the only nonzero cohomology of $E(-5, -1, -1, -1, -3)$ is $H^3$.

Chasing the diagram

\[
\begin{array}{ccccccc}
& & h^0 & h^1 & h^2 & h^3 & h^4 \\
& & 0 & * & 0 & 0 & 0 \\
I_{Z/P^4(1)} & \uparrow & 0 & 0 & 0 & 0 & 0 \\
15 & E(-3, -1, -1, -1, -1) & 0 & 0 & * & 0 & 0 \\
& \uparrow & 0 & 0 & 0 & 0 & 0 \\
6 & E(-4, -1, -1, -1, -2) & 0 & 0 & 0 & * & 0 \\
& \uparrow & 0 & 0 & 0 & 0 & 0 \\
E(-5, -1, -1, -1, -3) & 0 & 0 & 0 & 0 & * & 0 \\
& \uparrow & 0 & 0 & 0 & 0 & 0 \\
& & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

we obtain $h^0(I_{Z/P^4(1)}) = 0$. \qed

4.2.4. Remark. Notice that $h^1(I_{Z/P^4(1)})$ does not vanish. Therefore $Z$ is not linearly normal.

5 Genus 7

5.1 Syzygies of $M_7$

Let $V$ be a 10-dimensional vector space with basis $\{w_1, \ldots, w_5, v_1, \ldots, v_5\}$. And $W$ the subspace spanned by the $w_i$.

We consider the Mukai variety

$$M_7 = S_{10}^+ \cong G/P \subset \mathbb{P}(\text{spin}_5^+)$$

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where \( G = \text{Spin}(V) = \text{Spin}(10) \) is the 2 : 1 spin covering of \( \text{SO}(10) \), \( g = \text{Lie}G = \mathfrak{so}_{10} \) its Lie algebra and \( \text{spin}^+_5 \) a irreducible 16-dimensional spinor representation of \( G \). The diagonal matrices \( D(a_1, a_2, a_3, a_4, a_5, a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}, a_5^{-1}) \) with \( a_i \in \mathbb{C} \) form a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{p} \subset \mathfrak{g} \). The matrices \( H_i = E_{i,i} - E_{i+5,i+5} \) form a basis of \( \mathfrak{h} \). Let \( L_i \) be the dual basis of \( \mathfrak{h}^* \). Then the positive roots of \( G \) are \( L_i \pm L_j \) with \( i < j \). The fundamental weight corresponding to \( \text{spin}^+_5 \) is \( \frac{1}{2}(L_1 + \cdots + L_5) \), while the other fundamental weights are \( L_1, L_1 + L_2, L_1 + L_2 + L_3 \) and \( \frac{1}{2}(L_1 + \cdots + L_4 - L_5) \). The corresponding representations are called \( \Lambda_1, \ldots, \Lambda_4 \) and \( \text{spin}^-_5 \) as in [KN88].

\( S_{10}^+ \) parametrizes one set of \( \mathbb{P}^4 \)'s on the smooth Quadric \( Q = v_1w_1 + \cdots + v_5w_5 \). If

\[
\pi: \text{Spin}(10) \to \text{SO}(10)
\]

is the 2 : 1 covering, \( \pi(P) \) is the group of orthogonal matrices that leave a particular \( \mathbb{P}^4 = \mathbb{P}(W) \) invariant, i.e matrices of the form

\[
\begin{pmatrix}
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * & * & * & *
\end{pmatrix}
\]

The semisimple part of \( \pi(P) \) is then

\[
S_{\pi(P)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \text{ with } A \in \text{GL}(W) \right\}
\]

5.1.1. Proposition. The syzygy-numbers of \( M_7 = S_{10}^+ \) are

\[
\begin{array}{cccccccccc}
1 & - & - & - & - & - \\
- & 10 & 16 & - & - & - \\
- & - & - & 16 & 10 & - \\
- & - & - & - & - & 1 \\
\end{array}
\]

Proof. A general linear section \( S_{10}^+ \cap \mathbb{P}^6 \) has the same syzygy numbers as \( S_{10}^+ \) by proposition 2.3.1. The syzygy-numbers of a general canonical curve \( C \subset \mathbb{P}^6 \) of genus 7 are the ones claimed above as shown by Schreyer in [Sch86]. \( \square \)
5.1.2. Proposition. The linear strand of the resolution of \( S_{10}^+ \) is

\[ I_{S_{10}^+} \leftarrow \Lambda_1 \otimes \mathcal{O}(-2) \leftarrow \text{spin}^- \otimes \mathcal{O}(-3) \]

Proof. From above we have the linear strand

\[ I_{S_{10}^+} \leftarrow V_0 \otimes \mathcal{O}(-2) \leftarrow V_1 \otimes \mathcal{O}(-3) \]

with \( \dim V_0 = 10 \) and \( \dim V_1 = 16 \). Now \( V_0 \) is a Spin(10) invariant subset of quadrics in \( \mathbb{P}(\text{spin}^+_5) \):

\[ V_0 \subset S_2(\text{spin}^+_5) \subset \text{spin}^+_5 \otimes \text{spin}^+_5 = \Lambda_5^+ \oplus \Lambda_3 \oplus \Lambda_1, \]

where \( \Lambda_5^+ \) is the irreducible representation corresponding to the maximal weight vector \( L_1 + \cdots + L_5 \). The representations have dimension 126, 120 and 10 respectively. Therefore \( V_0 = \Lambda_1 \). (For the decomposition of the tensor products see [KN88]).

In the next step we know

\[ V_1 \subset \Lambda_1 \otimes \text{spin}^+_5 = \lambda_1 \cdot \text{spin}^+_5 \oplus \text{spin}^-_5 \]

where \( \lambda_1 \cdot \text{spin}^+_5 \) denotes the irreducible representation obtained by adding the maximal weights of \( \Lambda_1 \) and \( \text{spin}^+_5 \). The irreducible summands have dimensions 144 and 16 so that \( V_1 \) must be equal to \( \text{spin}^-_5 \).

5.1.3. Proposition. The scheme \( Y_{\text{min}} \subset \mathbb{P}(\text{spin}^+_5) = \mathbb{P}^{15} \) of minimal rank first syzygies of the Spinor variety \( S_{10}^+ \) contains an isomorphic Spinor variety

\[ S_{syz}^+ = G/P \subset \mathbb{P}(\text{spin}^-_5)^* \cong \mathbb{P}^{15}. \]

The bundle of linear forms on \( S_{syz}^+ \) is

\[ L|_{S_{syz}^+} = E(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^* = \mathcal{B}(-1) \]

where \( \mathcal{B}(-1) \) is the tautological quotient bundle on \( S_{syz}^+ \). \( \mathcal{B}(-1) \) is of rank 5.

Proof. From proposition [2.1.7] we know, that \( Y_{\text{min}} \subset \mathbb{P}(\text{spin}^+_5) \cong \mathbb{P}^{15} \) must contain the minimal orbit \( G/P \) of spin(10) in \( \mathbb{P}(\text{spin}^-_5)^* \) under the action

\[ \rho: \text{Spin}(10) \to GL((\text{spin}^-_5)^*). \]

This is the spinor variety \( S_{syz}^+ \subset \mathbb{P}^{15} \).
To describe the vector bundle of linear forms $L$ on $S^{+}_{syz}$ we have to determine the action of $P$ on a fiber of $L$. We start by considering the dual actions $\rho^*$ of Spin(10), $P$ and $S_{\pi(P)}$ on $\text{spin}^-_{5}$. The lie group of $S_{\pi(P)}$ is

$$\text{Lie } S_{\pi(P)} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \right\} \text{ with } A \in \mathfrak{gl}(W)$$

Let $\mathbb{P}(W) \subset Q$ be the $\mathbb{P}^4$ left invariant by $P$. With this we have the natural representations

$$\Lambda^{\text{even}}W = \Lambda_0 W \oplus \Lambda_2 W \oplus \Lambda_4 W \cong \text{spin}^+_{5}$$

and

$$\Lambda^{\text{odd}}W = \Lambda_1 W \oplus \Lambda_3 W \oplus \Lambda_5 W \cong \text{spin}^-_{5}$$

where $\text{Lie } S_{\pi}$ acts in the natural way. The $w_I = \wedge_i w_I$ are weight vectors of $G$ and $P$ and their weights are $\frac{1}{2}(\sum_{i \in I} L_i - \sum_{i \notin I} L_i)$. [FH91] pp. 305-306.

The maximal weight vector

$$s = w_1 \wedge w_2 \wedge w_3 \wedge w_4 \wedge w_5 \in \text{spin}^-_{5}$$

is a minimal rank syzygy in $S^{+}_{syz} \subset Y_{\text{min}}$. To determine the fiber $L|_s$ we restrict the map

$$\psi: \Lambda^*_1 \otimes \mathcal{O}_{\text{spin}^+_{5}}(-1) \rightarrow \text{spin}^+_{5} \otimes \mathcal{O}_{\text{spin}^+_{5}}$$

from definition 2.1.4 to $s$. This yields

$$\psi|_s = \tilde{\varphi}(s) \in \text{Hom}(\Lambda^*_1, \text{spin}^+_5) \cong \Lambda_1 \otimes S^{+}_{10}$$

where

$$\tilde{\varphi}: S^-_{10} \hookrightarrow \Lambda_1 \otimes S^{+}_{10}$$

is the map defined in 2.1.2. In particular we get

$$\tilde{\varphi}(s) = \tilde{\varphi}(w_1 \wedge w_2 \wedge w_3 \wedge w_4 \wedge w_5) = w_1 \otimes w_2 \wedge w_3 \wedge w_4 \wedge w_5$$

$$- w_2 \otimes w_1 \wedge w_3 \wedge w_4 \wedge w_5 + w_3 \otimes w_1 \wedge w_2 \wedge w_4 \wedge w_5$$

$$- w_4 \otimes w_1 \wedge w_2 \wedge w_3 \wedge w_5 + w_5 \otimes w_1 \wedge w_2 \wedge w_3 \wedge w_4$$
Consequently the fiber of the line bundle of linear forms is

\[ L_s = \text{Im}(\tilde{\varphi}(s)) = \langle w_2 \wedge w_3 \wedge w_4 \wedge w_5, \]
\[ w_1 \wedge w_3 \wedge w_4 \wedge w_5, \]
\[ w_1 \wedge w_2 \wedge w_4 \wedge w_5, \]
\[ w_1 \wedge w_2 \wedge w_3 \wedge w_5, \]
\[ w_1 \wedge w_2 \wedge w_3 \wedge w_4 \rangle \]

and \( L \) is of rank 5. \( S_P \) acts irreducibly on this fiber, and \( w_1 \wedge w_2 \wedge w_3 \wedge w_4 \) is the maximal weight vector with weight \( \frac{1}{2}(L_1 + L_2 + L_3 + L_4 - L_5) \).

Consequently

\[ E_{\rho^*} = \frac{1}{2}E(1,1,1,1,-1) \]

and

\[ L = E_{\rho} = \frac{1}{2}E(1,1,1,1,-1)^* = \frac{1}{2}E(1,-1,-1,-1,-1). \]

Now the tautological sequence on \( S_{10}^+ \) is

\[ 0 \to E(-1,0,0,0,0) \to W \otimes \mathcal{O}_{S_{10}^+} \to E(1,0,0,0,0) \to 0 \]

i.e. \( \mathcal{B} = E(1,0,0,0,0) \). Since \( \mathcal{O}(1) = \frac{1}{2}E(1,1,1,1,1) \) this shows that \( L|_{S_{10}^+} = \mathcal{B}(-1) \).

5.2 General Canonical Curves of Genus 7

Consider a general canonical curve \( C \) of genus 7. Mukai’s Theorem provides us with \( \mathbb{P}^6 = \mathbb{P}(W) \subset \mathbb{P}^{\text{spin}_5^+} \cong \mathbb{P}^{15} \) (a different \( W \) from the last section) such that

\[ C = S_{10}^+ \cap \mathbb{P}^6 \]

5.2.1. Proposition. On the spinor variety \( S_{10}^+ \subset Y_{\text{min}}(S_{10}^+) \) there exists a map of vector bundles

\[ \alpha: \mathcal{B}(-1) \to 7\mathcal{O}_{S_{10}^+} \]

such that its rank 3 locus \( Z_3(\alpha) \) is the scheme \( Z \) of last scrollar syzygies of \( C \). \( Z \) is a ruled surface of degree 84.

Proof. \( \frac{g-5}{2} = 1 \) so the first scrollar syzygies of \( C \) are also the last scrollar syzygies. The minimal rank first syzygies \( s \in S_{syz}^+ \) of \( S_{10}^+ \) are of rank 5 as calculated in proposition 5.1.3.
Since $C$ is a general linear section of $S_{10}^+$ we can apply corollary 2.3.2 to obtain a map

$$\alpha: L \to W \otimes \mathcal{O}_{S_{sz}^+}$$

whose rank calculates the rank of syzygies $s \in S_{sz}^+$ considered as syzygies of $C$. In our case this is equal to

$$\alpha: B(-1) \to 7\mathcal{O}_{S_{sz}^+}.$$  

The last scrollar syzygies of $C$ are first syzygies of rank 3. The argument above shows that the scheme $Z$ of last scrollar syzygies contains the rank 3 locus $Z_3(\alpha)$ of $\alpha$.

Since $\mathbb{P}(W) \subset \mathbb{P}(\text{spin}_5^+)$ is a general subspace, and $L^* = \Omega_{B^*}(1)$ is globally generated, $Z_3(\alpha)$ is reduced and of expected dimension

$$\dim Z_3(\alpha) = \dim S_{sz}^+ - (\text{rank } B - 3)(7 - 3) = 2.$$  

On the other hand we are also in the situation of corollary 2.2.14 which gives an isomorphism

$$\zeta: Z \to C^1_5$$

with $C^1_5$ a ruled surface over $W^1_5$. Since $C^1_5 \cong Z$ is irreducible, this shows that $Z_3(\alpha) \subset Z$ is in fact an equality. In particular $Z$ is a ruled surface.

Since $Z = Z_3(\alpha)$ is of expected dimension and we can calculate its class with Porteous formula [ACGHS85][p.86]:

$$z_3(\alpha) = \Delta_{7-3,5-3} \left( \frac{c_t(7\mathcal{O}_{S_{sz}^+})}{c_t(\mathcal{O}(1))} \right) = \Delta_{4,2} \left( \frac{c_t(7\mathcal{O}_{S_{sz}^+})}{c_t(B(1))} \right)$$

The cohomology ring of $S_{sz}^+$ has been determined by Ranestad and Schreyer in [RS00][p. 30] as

$$H^*(S_{sz}^+, \mathbb{Q}) = \mathbb{Q}[h, b]/(b^2 + 8bh^3 + 8h^6, 6h^5b + 7h^8)$$

where $h$ is the class of a hyperplane section and $b$ is the third Chern class of $B^*$. They also give the Chern polynomial of $B^*$ as

$$c_t(B^*) = 1 - 2ht + 2h^2t^2 + bt^3 + (-2h^4 - 2hb)t^4.$$  

The Chern polynomials needed for Porteous formula above are

$$c_t(B) = \frac{1}{c_t(B^*)}$$
as obtained by the tautological sequence and
\[ c_t(7\mathcal{O}_{\mathbb{P}^4}(1)) = (1 + Ht)^7. \]

This yields
\[
\begin{align*}
a &= \frac{c_t(6\mathcal{O}_{\mathbb{P}^4}(1))}{c_t(7\mathcal{O}_{\mathbb{P}^4})} = (1 + Ht)^7 c_t(\mathcal{B}^*) = \\
&= 1 + 5ht + 9h^2t^2 + (b + 7h^3)t^3 + (5bh + 5h^4)t^4 + (7bh^2 + 7h^5)t^5 + \ldots
\end{align*}
\]
and
\[ z_3(\alpha) = \Delta_{4,2}(a) = \det \begin{pmatrix} a_4 & a_5 \\ a_3 & a_4 \end{pmatrix} = 7H^8. \]

Since the degree of the spinor variety is \(2g - 2 = 12\) the degree of \(Z = Z_3(\alpha)\) is \(7 \cdot 12 = 84\).

5.2.2. Remark. On can also calculate the degree of \(Z\) via Brill-Noether-Theory. See \cite{vB00} for a formula.

5.2.3. Corollary. The ideal sheaf \(I_{Z/S^+_{xy}}\) is resolved by
\[
I_{Z/S^+_{xy}} \leftarrow \Lambda_4 \otimes S_{1111}
\]
\[
\leftarrow \Lambda_5 \otimes S_{2111} + \Lambda_{41} \otimes S_{11111}
\]
\[
\leftarrow \Lambda_6 \otimes S_{3111} + \Lambda_{51} \otimes S_{21111}
\]
\[
\leftarrow \Lambda_7 \otimes S_{4111} + \Lambda_{61} \otimes S_{31111} + \Lambda_{55} \otimes S_{22222}
\]
\[
\leftarrow \Lambda_{71} \otimes S_{41111} + \Lambda_{65} \otimes S_{32222}
\]
\[
\leftarrow \Lambda_{75} \otimes S_{42222} + \Lambda_{66} \otimes S_{33222}
\]
\[
\leftarrow \Lambda_{76} \otimes S_{43222}
\]
\[
\leftarrow \Lambda_{77} \otimes S_{44222}
\]
\[
\leftarrow 0
\]

where \(\Lambda_\lambda = \Lambda_\lambda W^*\) and \(S_\mu = S_\mu L\).

Proof. Since \(\alpha : L \to 7\mathcal{O}_Y\) drops rank in the expected dimension
\[ \dim Z_3(\alpha) = 10 - (5 - 3)(7 - 3) = 2 \]
the resolution of \(I_{Z/S^+_{xy}}\) can be calculated using the methods of Lascoux. In the notation of \cite[Thm 3.3]{Las78} the above resolution is \(k(\alpha, 2, 0)\) since \(\alpha\) drops rank by two on \(Z_3(\alpha)\).

This gives
5.2.4. Theorem. The scheme $Z$ of last scrollar syzygies of a general canonical curve $C \subset \mathbb{P}^6$ of genus $7$ is a linearly normal ruled surface of degree $84$ on a spinor variety $S^+_{syz} \subset \mathbb{P}^{15}$. This ruled surface spans the whole $\mathbb{P}^{15}$ of first syzygies of $C$.

Proof. We have to show, that

$$Z \subset S^+_{10} \subset \mathbb{P}(\text{spin}_5)^*$$

is non degenerate. It is enough to check $h^0(I_{Z/S^+_{syz}}(1)) = 0$. We tensor the resolution above by $\mathcal{O}_Y(1) = \frac{1}{7}E(1,1,1,1,1)$. We calculate the cohomology of the resulting complex using the theorem of Bott. Notice that

$$S_\mu L = S_\mu \frac{1}{7}E(1,-1,-1,-1,-1)$$

$$= S_\mu \left( \frac{1}{7}E(2,0,0,0) \otimes \mathcal{O}_Y(-1) \right)$$

$$= (S_\mu E(1,0,0,0)) \otimes \mathcal{O}_Y(-|\mu|)$$

$$= E(\mu) \otimes \mathcal{O}_Y(-|\mu|)$$

$$= E(\mu - \frac{1}{7}|\mu|(1,1,1,1,1))$$

$$= \frac{1}{7}E(2\mu - |\mu|(1,1,1,1,1))$$

With the sum of fundamental weights $\delta = 4 \cdot L_1 + 3 \cdot L_2 + 2 \cdot L_3 + 1 \cdot L_4 + 0 \cdot L_5$ we have

| $|\mu| - 1$ | Line bundle | $\delta + \lambda$ |
|---|---|---|
| $\mathcal{O}(1)$ | $\frac{1}{7}E(1,1,1,1,1)$ | $(9, 7, 5, 3, 1)$ |
| 3 | $S_{1111}L(1)$ | $\frac{1}{7}E(-1,-1,-1,-1,-3)$ | $(7, 5, 3, 1,-3)$ |
| 4 | $S_{2111}L(1)$ | $\frac{1}{7}E(0,-2,-2,-2,-4)$ | $(8, 4, 2, 0,-4)$ |
| 4 | $S_{11111}L(1)$ | $\frac{1}{7}E(-2,-2,-2,-2,-2)$ | $(6, 4, 2, 0,-2)$ |
| 5 | $S_{3111}L(1)$ | $\frac{1}{7}E(1,-3,-3,-3,-5)$ | $(9, 3, 1,-1,-5)$ |
| 5 | $S_{21111}L(1)$ | $\frac{1}{7}E(-1,-3,-3,-3,-3)$ | $(7, 3, 1,-1,-3)$ |
| 6 | $S_{4111}L(1)$ | $\frac{1}{7}E(2,-4,-4,-4,-6)$ | $(10, 2, 0,-2,-6)$ |
| 6 | $S_{31111}L(1)$ | $\frac{1}{7}E(0,-4,-4,-4,-4)$ | $(8, 2, 0,-2,-4)$ |
| 9 | $S_{22222}L(1)$ | $\frac{1}{7}E(-5,-5,-5,-5,-5)$ | $(3, 1,-1,-3,-5)$ |
| 7 | $S_{41111}L(1)$ | $\frac{1}{7}E(1,-5,-5,-5,-5)$ | $(9, 1,-1,-3,-5)$ |
| 10 | $S_{32222}L(1)$ | $\frac{1}{7}E(-4,-6,-6,-6,-6)$ | $(4, 0,-2,-4,-6)$ |
| 11 | $S_{42222}L(1)$ | $\frac{1}{7}E(-3,-7,-7,-7,-7)$ | $(5,-1,-3,-5,-7)$ |
| 11 | $S_{33222}L(1)$ | $\frac{1}{7}E(-5,-5,-7,-7,-7)$ | $(3, 1,-3,-5,-7)$ |
| 12 | $S_{34222}L(1)$ | $\frac{1}{7}E(-4,-6,-8,-8,-8)$ | $(4, 0,-4,-6,-8)$ |
| 13 | $S_{44222}L(1)$ | $\frac{1}{7}E(-5,-5,-9,-9,-9)$ | $(3, 1,-5,-7,-9)$ |

where we use the shorthand notation $(d_1,d_2,d_3,d_4,d_5) := \frac{1}{7}(d_1L_1 + \cdots + d_5L_5)$ for the last column.
For the first bundle we have no vanishing root, and all positive roots $\alpha$ satisfy $(\alpha, \lambda + \delta) > 0$. Therefore $i_0 = 0$ and all higher cohomology vanishes for $O(1)$. For the last bundle there are 9 positive roots $\alpha$ with $(\alpha, \lambda + \delta) < 0$. Consequently the only non vanishing cohomology is $H^9(S_{44222} L(1))$. For all remaining bundles there is at least one integer that appears with opposite signs. Therefore we have a root $\alpha = L_i + L_j$ with $(\alpha, \lambda + \delta) = 0$, and all cohomology groups of these bundles vanish.

Chasing the diagram

\[
\begin{array}{cccccccccc}
0 & h^0 & h^1 & h^2 & h^3 & h^4 & h^5 & h^6 & h^7 & h^8 & h^9 & h^{10} \\
\uparrow & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{Z/S_{nys}} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_4 \otimes S_{1111} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_5 \otimes S_{2111} + \Lambda_4 \otimes S_{11111} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_6 \otimes S_{3111} + \Lambda_5 \otimes S_{21111} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_7 \otimes S_{41111} + \Lambda_6 \otimes S_{31111} + \Lambda_5 \otimes S_{22222} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_7 \otimes S_{411111} + \Lambda_6 \otimes S_{311111} + \Lambda_5 \otimes S_{222222} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_7 \otimes S_{42222} + \Lambda_6 \otimes S_{32222} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_7 \otimes S_{43222} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_7 \otimes S_{44222} & & & & & & & & & & & \\
\uparrow & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & & & & & & & & & & \\
\end{array}
\]

yields in particular

\[h^0(I_{Z/S_{nys}^+}(1)) = h^1(I_{Z/S_{nys}^+}(1)) = 0.\]

This shows that the space of scrollar syzygies of $C$ is non degenerate and linearly normal. \hfill \Box

### 6 Genus 8

This case is very similar to the genus 6 case, since the Mukai variety is also a Grassmannian $Gr(V, 2)$ but with dim $V = 6$. Let $\{v_1, \ldots, v_6\}$ be a basis of $V$. 

36
6.1 Syzygies of $M_8$

The Mukai variety for genus 8 is

$$M_8 = \text{Gr}(V, 2) = \text{Gr}(6, 2) \cong GL(6)/P \subset \mathbb{P}(\Lambda_2) \cong \mathbb{P}^{14}$$

The diagonal matrices $h$ form a Cartan subalgebra of $gl_6$ and $p$. The matrices $H_i = E_{i,i}$ are a basis of $h$. Let $\{L_i\}$ be the dual basis of $h^\ast$. The positive roots of $gl_6$ are $L_i - L_j$ with $i > j$ and $\omega_i = \sum_{j=0}^{i-1} L_j$ are the fundamental weights.

6.1.1. Proposition. The syzygy-numbers of $M_8 = \text{Gr}(6, 2)$ are

$$
\begin{array}{cccccc}
1 & - & - & - & - & - \\
- & 15 & 35 & 21 & - & - \\
- & - & - & 21 & 35 & 15 \\
- & - & - & - & - & 1
\end{array}
$$

Proof. A general linear section $M_8 \cap \mathbb{P}^7$ has the same syzygy numbers as $M_8$ by proposition 2.3.1. The syzygy-numbers of a general canonical curve $C \subset \mathbb{P}^7$ of genus 8 are the ones claimed above as shown by Schreyer in [Sch86].

6.1.2. Proposition. The linear strand of the resolution of $\text{Gr}(6, 2)$ is

$$I_{\text{Gr}(6, 2)} \leftarrow \Lambda_4 \otimes \mathcal{O}(-2) \leftarrow \Lambda_{51} \otimes \mathcal{O}(-3) \leftarrow \Lambda_{611} \otimes \mathcal{O}(-4)$$

Proof. From above we have a linear strand

$$I_{\text{Gr}(6, 2)} \leftarrow V_0 \otimes \mathcal{O}(-2) \leftarrow V_1 \otimes \mathcal{O}(-3) \leftarrow V_2 \otimes \mathcal{O}(-4)$$

with $\dim V_0 = 15$, $\dim V_1 = 35$ and $\dim V_2 = 21$. $V_0$ is a invariant (not necessarily irreducible) subspace of quadrics. This gives

$$V_0 \subset S_2(\Lambda_2) \subset \Lambda_2 \otimes \Lambda_2 = \Lambda_4 \oplus \Lambda_{31} \oplus \Lambda_{22}$$

where the irreducible components have dimensions 15, 105 and 105 respectively. So $V_0 = \Lambda_4$. Similarly we have

$$V_1 \subset \Lambda_4 \otimes \Lambda_2 = \Lambda_6 \oplus \Lambda_{51} \oplus \Lambda_{42}$$

where the irreducible components have dimensions 1, 35 and 175 respectively. This implies $V_1 = \Lambda_{51}$.

Finally we observe

$$V_2 \subset \Lambda_{51} \otimes \Lambda_2 = \Lambda_{611} \oplus \Lambda_{62} \oplus \Lambda_{521} \oplus \Lambda_{53}$$

where the irreducible components have dimensions 21, 15, 384 and 105 respectively. Consequently we have $V_2 = \Lambda_{611}$. 

37
6.1.3. Proposition. The scheme of minimal rank second syzygies of \( \text{Gr}(6, 2) \) contains the minimal orbit

\[
\text{GL}(6)/P \cong \mathbb{P}^5 \xrightarrow{\text{2-uple}} \mathbb{P}(\Lambda^*_{611}) \cong \mathbb{P}^{20}
\]

The bundle of linear forms on \( \mathbb{P}^5 \) is

\[
L = E(1, 1, 0, 0, 0, 0)^* = T_{\mathbb{P}^5}(-2)
\]

\( L \) has rank 5.

Proof. From proposition 2.1.7 we know, that \( Y_{\text{min}} \subset \mathbb{P}(\Lambda^*_{611}) \cong \mathbb{P}^{20} \) must contain the minimal orbit of \( \text{GL}(6) \) in \( \mathbb{P}(\Lambda^*_{611}) \) under the action

\[
\rho: \text{GL}(6) \to \text{GL}(\Lambda^*_{611}).
\]

Here this orbit \( \text{GL}(5)/P \) is the 2-uple embedded \( \mathbb{P}^5 \).

To describe the vector bundle of linear forms on \( \mathbb{P}^5 = \text{GL}(6)/P \) we have to determine the action of \( P \) on a fiber of \( L \). We start by considering the dual actions \( \rho^* \) of \( \text{GL}(6) \) and \( P \) on \( \Lambda^*_{611} \). The parabolic subgroup in its standard representation is then the set of matrices that fix a given \( \mathbb{P}^0 \), i.e matrices of the form

\[
\begin{pmatrix}
* & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & * \\
0 & * & * & * & *
\end{pmatrix}
\]

The semisimple part of \( P \) is \( S_P = \text{GL}(1) \times \text{GL}(5) \) where \( \text{GL}(1) \) acts on \( \langle v_1 \rangle \) and \( \text{GL}(5) \) acts on \( \langle v_2, v_3, v_4, v_5, v_6 \rangle \) in the standard way.

The maximal weight vector

\[
s = \begin{pmatrix} 1 & 1 & 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} \in \Lambda_{611}
\]

is a syzygy of minimal rank in \( \mathbb{P}^5 \to \mathbb{P}^{20} \). To determine the fiber \( L|_s \) of the bundle of linear forms over \( s \) we restrict the map

\[
\psi: \Lambda^*_{51} \otimes \mathcal{O}_{\mathbb{P}(\Lambda^*_{611})}(-1) \to \Lambda_2 \otimes \mathcal{O}_{\mathbb{P}(\Lambda^*_{611})}
\]
from definition 2.1.4 to $s$. This gives
\[ \psi|_s = \tilde{\varphi}(s) \in \text{Hom}(\Lambda^*_s, \Lambda_2) \cong \Lambda_5 \otimes \Lambda_2 \]
where
\[ \tilde{\varphi} : \Lambda_6 \hookrightarrow \Lambda_5 \otimes \Lambda_2. \]

Using Young diagrams we get
\[ \tilde{\varphi} : \Lambda_6 \hookrightarrow \Lambda_5 \otimes \Lambda_2. \]

Consequently the fiber of the line bundle of linear forms is
\[ L|_s = \text{Im} \tilde{\varphi}(s) = \langle v_1 \wedge v_6, v_1 \wedge v_5, v_1 \wedge v_4, v_1 \wedge v_3, v_1 \wedge v_2 \rangle \]
and $L$ is of rank 5. $S_P$ acts irreducibly on this fiber, and $v_1 \wedge v_2$ is the maximal weight vector of weight $L_1 + L_2$.

Therefore
\[ E_{\rho^*} = E(1, 1, 0, 0, 0, 0) \]
and
\[ L = E_{\rho} = E(1, 1, 0, 0, 0, 0)^* \]
Now the tautological sequence of $\mathbb{P}^5$ is
\[ 0 \to E(0, 1, 0, 0, 0, 0) \to V \otimes O_{\mathbb{P}^4} \to E(1, 0, 0, 0, 0, 0) \to 0 \]
i.e. $E(1, 0, 0, 0, 0, 0) = O(1)$, $E(0, 1, 0, 0, 0, 0) = \Omega(1)$ and $E(1, 1, 0, 0, 0, 0)^* = T_{\mathbb{P}^5}(-2)$.  

39
6.2 General Canonical Curves of Genus 8

Let now $C$ be a general canonical curve of genus 8. From Mukai’s Theorem we obtain a $\mathbb{P}^7 \cong \mathbb{P}(W) \subset \mathbb{P}(\Lambda^2 V) \cong \mathbb{P}^{14}$ such that

$$C = \text{Gr}(6, 2) \cap \mathbb{P}^7$$

### 6.2.1. Proposition

On $\mathbb{P}^5 \hookrightarrow \mathbb{P}^{20}$ there exists a map of vector bundles

$$\alpha: \mathcal{T}_{\mathbb{P}^5}(-2) \to 8\mathcal{O}_{\mathbb{P}^5}$$

such that its rank 4 locus $Z_4(\alpha)$ is the scheme $Z$ of last scrollar syzygies of $C$. $Z$ is a configuration of 14 skew conics on the 2-uple embedding $\mathbb{P}^5 \hookrightarrow \mathbb{P}^{20}$.

**Proof.** $\left\lfloor \frac{g-5}{2} \right\rfloor = 2$ so the second scrollar syzygies of $C$ are the last scrollar syzygies. The minimal rank second syzygies of $\text{Gr}(6, 2)$ are of rank 5 and fill at least a 2-uple embedded $\mathbb{P}^5 \hookrightarrow \mathbb{P}^{20}$ as calculated in proposition 6.1.3.

Since $C$ is a general linear section of $\text{Gr}(6, 2)$ we can apply corollary 2.3.2 to obtain a map

$$\alpha: L \to W \otimes \mathcal{O}_{Y_{\text{min}}}$$

whose rank calculates the rank of syzygies $s \in Y_{\text{min}}$ considered as syzygies of $C$. In our case this restricts to

$$\alpha: \mathcal{T}_{\mathbb{P}^5}(-2) \to 8\mathcal{O}_{\mathbb{P}^5}$$

on our $\mathbb{P}^5$.

The last scrollar syzygies of $C$ are second syzygies of rank 4. The argument above shows that the scheme $Z$ of last scrollar syzygies contains the rank 4 locus $Z_4(\alpha)$ of $\alpha$.

Since $\mathbb{P}(W) \subset \mathbb{P}(\Lambda^2 V)$ is a general subspace, and $L^* = \Omega_{\mathbb{P}^5}(2)$ is globally generated, $Z_4(\alpha)$ is reduced and of expected dimension

$$\dim Z_4(\alpha) = \dim \mathbb{P}^5 - (5 - 4)(8 - 4) = 1.$$ 

On the other hand we are also in the situation of corollary 2.2.14 which gives an isomorphism

$$\zeta: Z \to C^1_5 = \bigcup_{i=1}^{14} \mathbb{P}^1$$

This shows that $Z_4(\alpha)$ is the union of at most 14 disjoint $\mathbb{P}^1$'s. Each of these $\mathbb{P}^1$ is the scheme of second minimal rank syzygies of a scroll. These schemes are rational normal curves of degree 2 as calculated in proposition 2.2.2.
Since they lie on the 2-ple embedding of $\mathbb{P}^5$ in $\mathbb{P}^{20}$ they are the images of lines in $\mathbb{P}^5$.

Since $Z_4(\alpha)$ is of expected dimension and we can calculate its class with Porteous formula [ACGH85][p.86]:

$$z_4(\alpha) = \Delta_{8-4,5-4} \left( \frac{c_t(8\mathcal{O}_{\mathbb{P}^5})}{c_t(\mathcal{T}_{\mathbb{P}^5}(-2))} \right) = \Delta_{4,1} \left( \frac{c_t(8\mathcal{O}_{\mathbb{P}^5}(1))}{c_t(\mathcal{T}_{\mathbb{P}^5}(-1))} \right)$$

The Chern polynomials involved are

$$c_t(\mathcal{T}_{\mathbb{P}^5}(-1)) = \frac{1}{1-Ht}$$

as obtained by the Euler-Sequençe and

$$c_t(8\mathcal{O}_{\mathbb{P}^5}(1)) = (1 + Ht)^8.$$ 

This yields

$$a = \frac{c_t(8\mathcal{O}_{\mathbb{P}^5}(1))}{c_t(\mathcal{T}_{\mathbb{P}^5}(-1))} = (1 + Ht)^8(1 - Ht) = 1 + 7Ht + 20H^2t^2 + 28H^3t^3 + 14H^4t^4 \pm \ldots$$

and

$$z_4(\alpha) = \Delta_{4,1}(a) = \det(a_4) = 14H^3.$$ 

Since $Z_4(\alpha)$ is reduced this shows that $Z_4(\alpha)$ contains all 14 conics of $Z \subset \mathbb{P}^5 \xrightarrow{2-ple} \mathbb{P}^{20}$. In particular we have $Z = Z_4(\alpha)$.

6.2.2. Corollary. The ideal sheaf $I_{Z/\mathbb{P}^5}$ is resolved by

$$I_{Z/\mathbb{P}^5} \leftarrow 56 \ E(-5,-1,-1,-1,-1,-1)$$

$$\leftarrow 28 \ E(-6,-1,-1,-1,-1,-2)$$

$$\leftarrow 8 \ E(-7,-1,-1,-1,-1,-3)$$

$$\leftarrow E(-8,-1,-1,-1,-1,-4)$$

$$\leftarrow 0$$

Proof. Since $\alpha$ drops rank in the expected dimension

$$1 = 5 - (5 - 4)(8 - 4),$$

the ideal sheaf is resolved by the corresponding Eagon-Northcott complex

$$I_{Z/\mathbb{P}^5} \leftarrow \Lambda^5W^* \otimes \Lambda^5L \leftarrow \Lambda^6W^* \otimes \Lambda^5L \otimes S_1L \leftarrow \Lambda^7W^* \otimes \Lambda^5L \otimes S_2L \leftarrow \Lambda^8W^* \otimes \Lambda^5L \otimes S_3L \leftarrow 0.$$
Since \( \dim W^* = 8 \) we have

\[
\Lambda^i W^* \otimes \mathcal{O} = \binom{8}{i} \mathcal{O}.
\]

This gives the above multiplicities.

Furthermore

\[
\Lambda^5 L = \Lambda^5 E(1,1,0,0,0) = E(5,1,1,1,1) = E(-5,-1,-1,-1,-1)
\]

and

\[
S_i L = S_i E(1,1,0,0,0) = E(i,i,0,0,0) = E(-i,0,0,0,-i).
\]

Applying these equations to the complex above yields the desired resolution.

6.2.3. Theorem. The scheme \( Z \) of last scrollar syzygies of a general canonical curve \( C \subset \mathbb{P}^7 \) of genus 8 is a configuration of 14 skew conics that lie on a 2-uple embedded \( \mathbb{P}^5 \hookrightarrow \mathbb{P}^{20} \). \( Z \) spans the whole \( \mathbb{P}^{20} \) of second syzygies of \( C \).

Proof. We have to show, that

\[
Z \subset \mathbb{P}(\Lambda^*_{6,11}) = \mathbb{P}^{20}
\]

is non degenerate. It is enough to check \( h^0(I_{Z/\mathbb{P}^5}(2)) = 0 \).

Since \( \mathcal{O}_{\mathbb{P}^5}(2) = E(2,0,0,0,0,0) \), this yields

\[
I_{Z/\mathbb{P}^5}(2) \leftarrow 56 E(-3,-1,-1,-1,-1,-1)
\]

\[
\leftarrow 28 E(-4,-1,-1,-1,-1,-2)
\]

\[
\leftarrow 8 E(-5,-1,-1,-1,-1,-3)
\]

\[
\leftarrow E(-6,-1,-1,-1,-1,-4)
\]

\[
\leftarrow 0
\]

We now calculate the cohomology of the above vector bundles using the theorem of Bott. The fundamental weights of \( GL(6) \) are \( L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_6 \). The sum of fundamental weights is therefore \( \delta = 6L_1 + 5L_2 + 4L_3 + 3L_4 + 2L_5 + L_6 \). We obtain

| \( E(\lambda) \)          | \( \lambda + \delta \) |
|--------------------------|------------------------|
| \( E(-3,-1,-1,-1,-1,-1)  \) | (3, 4, 3, 2, 1, 0)     |
| \( E(-4,-1,-1,-1,-1,-2)  \) | (2, 4, 3, 2, 1, -1)    |
| \( E(-5,-1,-1,-1,-1,-3)  \) | (1, 4, 3, 2, 1, -2)    |
| \( E(-6,-1,-1,-1,-1,-4)  \) | (0, 4, 3, 2, 1, -3)    |
For the first three rows we find positive roots \( L_i - L_j \) with \((L_i - L_j, \lambda + \delta) = 0\). Therefore all cohomology of these bundles vanish. For the last row no such root is found, but there are 4 roots with \((L_i - L_j, \lambda + \delta) < 0\). Therefore the only nonzero cohomology of \( E(-6, -1, -1, -1, -1, -1) \) is \( H^4 \).

Chasing the diagram

\[
\begin{array}{ccccccc}
  & h^0 & h^1 & h^2 & h^3 & h^4 & h^5 \\
0 & \uparrow & 0 & * & 0 & 0 & 0 \\
I_{\mathbb{Z}/5^5}(2) & \uparrow & 56 E(-3, -1, -1, -1, -1, -1) & 0 & 0 & 0 & 0 & 0 \\
& \uparrow & 28 E(-4, -1, -1, -1, -1, -2) & 0 & 0 & 0 & 0 & 0 \\
& \uparrow & 8 E(-5, -1, -1, -1, -1, -3) & 0 & 0 & 0 & 0 & 0 \\
& \uparrow & E(-5, -1, -1, -1, -1, -3) & 0 & 0 & 0 & * & 0 \\
& \uparrow & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

we obtain \( h^0(I_{\mathbb{Z}/5^5}(2)) = 0 \). □

### 6.2.4. Remark

Notice that \( h^1(I_{\mathbb{Z}/5^5}(2)) \) does not vanish. Therefore \( Z \) is not linearly normal.

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