ON RESIDUAL PROPERTIES OF WORD HYPERBOLIC GROUPS

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Abstract. For a fixed word hyperbolic group we compare different residual properties related to quasiconvex subgroups.

1. Introduction

Any group $G$ can be equipped with a profinite topology $\mathcal{P}T(G)$, whose basic open sets are cosets to normal finite index subgroups. It is easy to see that the group operations are continuous in $\mathcal{P}T(G)$. The group is residually finite if and only if the profinite topology is Hausdorff.

A subgroup $H \leq G$ is closed in $\mathcal{P}T(G)$ if and only if it is equal to an intersection of finite index subgroups; equivalently, for any element $g \notin H$ there exists a homomorphism $\varphi$ from $G$ to a finite group $L$ such that $\varphi(g) \notin \varphi(H)$. In this case the subgroup $H$ is called $G$-separable.

The profinite closure of a subgroup $H \leq G$, i.e., the smallest closed subset containing $H$, is equal to the intersection of all finite index subgroups $K$ of $G$ such that $H \leq K$.

A group $G$ is said to be LERF if every finitely generated subgroup is closed in $\mathcal{P}T(G)$. The class of all LERF groups includes free groups [6], surface groups [20] and fundamental groups of certain 3-manifolds [20], [3].

Let $G$ be a (word) hyperbolic group with a finite symmetrized generating set $A$ and let $\Gamma(G,A)$ be the corresponding Cayley graph of $G$. A subset $Q \subseteq G$ is said to be quasiconvex if there exists a constant $\eta \geq 0$ such that for any pair of elements $u, v \in Q$ and any geodesic segment $p$ connecting $u$ and $v$, $p$ belongs to a closed $\eta$-neighborhood of the subset $Q$ in $\Gamma(G,A)$. Quasiconvex subgroups are precisely those finitely generated subgroups which are embedded in $G$ without distortion [10, Lemma 1.6].

As it was noted in [13], in the context of word hyperbolic groups instead of studying LERF-groups it makes sense to study GFERF-groups. A hyperbolic group is called GFERF if each quasiconvex subgroup is closed in $\mathcal{P}T(G)$. Thus, within the class of hyperbolic groups, the notion of GFERF is more general than LERF: every LERF hyperbolic group is GFERF but not vice versa.

Unfortunately, it is absolutely unclear how to decide if a random word hyperbolic group is GFERF (or LERF). With this purpose, D. Long [8] and, later, G. Niblo and B. Williams [16] suggested to utilize the engulfing property. They say that a
subgroup $H \leq G$ is engulfed if it is contained in a proper finite index subgroup of $G$.

The author was mainly interested in the following two theorems established by Niblo and Williams in 2002:

**Theorem A.** ([16, Thm. 4.1]) Let $G$ be a word hyperbolic group and suppose that $G$ engulfs every finitely generated free subgroup with limit set a proper subset of the boundary of $G$. Then the intersection of all finite index subgroups of $G$ is finite. If $G$ is torsion-free then it is residually finite.

**Theorem B.** ([16, Thm. 5.2]) Let $G$ be a word hyperbolic group which engulfs every finitely generated subgroup $K$ such that the limit set $\Lambda(K)$ is a proper subset of the boundary of $G$. Then every quasiconvex subgroup of $G$ has a finite index in its profinite closure in $G$.

The main goal of this paper is to generalize Theorems A and B by weakening their assumptions and, in certain situations, strengthening their conclusions.

In a hyperbolic group $G$ the structure of a distorted subgroup can be very complicated. Thus, the basic idea is to use assumptions which concern only quasiconvex subgroups. We prove the following results in Section 5:

**Theorem 1.** Let $G$ be a hyperbolic group with a generating set of cardinality $s \in \mathbb{N}$. Suppose that each proper free quasiconvex subgroup of rank $s$ is engulfed in $G$. Then the intersection of all finite index subgroups of $G$ is finite. If $G$ is torsion-free then it is residually finite.

**Theorem 2.** Suppose $G$ is a hyperbolic group which engulfs each proper quasiconvex subgroup. Let $H$ be an arbitrary quasiconvex subgroup of $G$. Then $H$ has a finite index in its profinite closure $K$. Moreover, $K = HQ$ where $Q$ is the intersection of all finite index subgroups of $G$.

The assumptions of Theorems 1 and 2 are, indeed, less restrictive than the assumptions of Theorems A and B, because if $H$ is a quasiconvex subgroup of a hyperbolic group $G$ with $[G : H] = \infty$ then the limit set $\Lambda(H)$ is a proper subset of $\partial G$ ([22, Thm. 4], [13, Lemma 8.2]).

In the residually finite case Theorem 2 can be reformulated as follows:

**Theorem 3.** Let $G$ be a residually finite hyperbolic group where every proper quasiconvex subgroup is engulfed. Then $G$ is GFERF.

Combining together the claims of Theorems 1 and 3 one obtains

**Corollary 1.** Let $G$ be a torsion-free hyperbolic group where each proper quasiconvex subgroup is engulfed. Then $G$ is GFERF.

N. Romanovskii [19] and, independently, R. Burns [2] showed that a free product of two LERF groups is again a LERF group. We suggest yet another way for constructing GFERF groups by proving a similar result for them (see Section 6):

**Theorem 4.** Suppose $G_1$ and $G_2$ are GFERF hyperbolic groups. Then the free product $G = G_1 * G_2$ is also a GFERF hyperbolic group.

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2. Preliminaries

Let $G$ be a group with a finite symmetrized generating set $A$. Naturally, this generating set gives rise to a word length function $|y|_G$ for every element $g \in G$. The (left-invariant) word metric $d : G \times G \to \mathbb{N} \cup \{0\}$ is defined by the formula $d(x, y) = |x^{-1}y|_G$ for any $x, y \in G$. This metric can be canonically extended to the Cayley graph $\Gamma(G, A)$ by making each edge isometric to the interval $[0, 1] \subset \mathbb{R}$.

For any three points $x, y, w \in \Gamma(G, A)$, the Gromov product of $x$ and $y$ with respect to $w$ is defined as

$$(x|y)_w \overset{def}{=} \frac{1}{2} \left( d(x, w) + d(y, w) - d(x, y) \right).$$

Since the metric is left-invariant, for arbitrary $x, y, z, w \in \Gamma(G, A)$ their Gromov products satisfy

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta.$$

Equivalently, $G$ is hyperbolic if there exists $\delta \geq 0$ such that each geodesic triangle $\Delta$ in $\Gamma(G, A)$ is $\delta$-slim, i.e., every side of $\Delta$ is contained in a $\delta$-neighborhood of the two other sides (see [1]).

From now on we will assume that $G$ is a hyperbolic group and the constant $\delta$ is large enough so that it satisfies both of the definitions above.

For any two points $x, y \in \Gamma(G, A)$ we fix a geodesic path between them and denote it by $[x, y]$. Let $p$ be a path in the Cayley graph of $G$. Then $p_-, p_+$ will denote the startpoint and the endpoint of $p$, $||p||$ – its length; $\text{lab}(p)$, as usual, will mean the word in the alphabet $A$ written on $p$. $\text{elem}(p) \in G$ will denote the element of the group $G$ represented by the word $\text{lab}(p)$. If $W$ is a word in the $A$, $\text{elem}(W)$ will denote the corresponding element of the group $G$. For a subset $A \subset \Gamma(G, A)$ its closed $\varepsilon$-neighborhood will be denoted by $O_\varepsilon(A)$.

The $\delta$-slimness of geodesic triangles implies $2\delta$-slimness of all geodesic quadrangles $abcd$ in $\Gamma(G, A)$:

$$[a, b] \subset O_{2\delta}([b, c] \cup [c, d] \cup [a, d]).$$

A path $q$ is called $(\lambda, c)$-quasigeodesic if there exist $0 < \lambda \leq 1$, $c \geq 0$, such that for any subpath $p$ of $q$ the inequality $\lambda||p|| - c \leq d(p_-, p_+)$ holds. A word $W$ is said to be $(\lambda, c)$-quasigeodesic if some (equivalently, every) path $q$ in $\Gamma(G, A)$ labelled by $W$ is $(\lambda, c)$-quasigeodesic.

**Lemma 2.1.** ([1] 5.6.5.11, [1] 3.3] There is a constant $\nu = \nu(\delta, \lambda, c)$ such that for any $(\lambda, c)$-quasigeodesic path $p$ in $\Gamma(G, A)$ and a geodesic $q$ with $p_- = q_-$, $p_+ = q_+$, one has $p \subset O_\nu(q)$ and $q \subset O_\nu(p)$.

**Lemma 2.2.** ([1] Lemma 4.1]) Consider a geodesic quadrangle $x_1x_2x_3x_4$ in the Cayley graph $\Gamma(G, A)$ with $d(x_2, x_3) > d(x_1, x_2) + d(x_3, x_4)$. Then there are points $u, v \in [x_2, x_3]$ such that $d(x_2, u) \leq d(x_1, x_2)$, $d(v, x_3) \leq d(x_3, x_4)$ and the geodesic subsegment $[u, v]$ of $[x_2, x_3]$ lies $2\delta$-close to the side $[x_1, x_4]$.

If $x, g \in G$, we define $x^g = gxg^{-1}$. For a subset $A$ of the group $G$, $A^g = gAg^{-1}$, and the notation $A^G$ will be used to denote the subset $\{gap^{-1} | a \in A, g \in G\} \subset G$. 

Remark 2.1. [10, Remark 2.2] Let $Q \subseteq G$ be quasiconvex, $g \in G$. Then the subsets $gQ, Qg$ and $gQg^{-1}$ are quasiconvex as well.

Thus, a conjugate of a quasiconvex subgroup in a hyperbolic group is quasiconvex itself. Another important property of hyperbolic groups states that any cyclic subgroup is quasiconvex (see [1], for instance).

In this paper we will also use the concept of Gromov boundary of a hyperbolic group $G$ (for a detailed theory the reader is referred to the corresponding chapters in [4] or [1]). A sequence $(x_i)_{i \in \mathbb{N}}$ of elements of the group $G$ is said to be converging to infinity if

$$\lim_{i,j \to \infty} (x_i | x_j)_{1_G} = \infty.$$ 

Two sequences $(x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}$ converging to infinity are said to be equivalent if

$$\lim_{i \to \infty} (x_i | y_i)_{1_G} = \infty.$$ 

The points of the boundary $\partial G$ are identified with the equivalence classes of sequences converging to infinity. It is easy to see that this definition does not depend on the choice of a basepoint: instead of $1_G$ one could use any fixed point $p$ of $\Gamma(G, A)$ (see [1]). If $\alpha$ is the equivalence class of $(x_i)_{i \in \mathbb{N}}$, we will write $\lim_{i \to \infty} x_i = \alpha$.

The space $\partial G$ can be topologized so that it becomes compact, Hausdorff and metrizable (see [1],[4]).

Left multiplication by an element of the group induces a homeomorphic action of $G$ on its boundary: for any $g \in G$ and $[(x_i)_{i \in \mathbb{N}}] \in \partial G$ set

$$g \circ [(x_i)_{i \in \mathbb{N}}] \overset{\text{def}}{=} [(gx_i)_{i \in \mathbb{N}}] \in \partial G.$$ 

If $g \in G$ is an element of infinite order then the sequences $(g^i)_{i \in \mathbb{N}}$ and $(g^{-i})_{i \in \mathbb{N}}$ converge to infinity and we will use the notation

$$\lim_{i \to \infty} g^i = g^\infty \in \partial G, \quad \lim_{i \to \infty} g^{-i} = g^{-\infty} \in \partial G.$$ 

For a subset $A \subseteq G$ the limit set $\Lambda(A)$ of $A$ is the collection of the points $\alpha \in \partial G$ that are limits of sequences (converging to infinity) from $A$.

Let us recall an auxiliary binary relation defined between subsets of an arbitrary group $G$ in [14]: suppose $A, B \subseteq G$, we will write $A \preceq B$ if and only if there exist elements $x_1, \ldots, x_n \in G$ such that

$$A \subseteq Bx_1 \cup Bx_2 \cup \cdots \cup Bx_n.$$ 

It is not difficult to see that the relation "$\preceq$" is transitive and for any $g \in G$, $A \preceq B$ implies $gA \preceq gB$.

Lemma 2.3. ([14, Lemma 2.1]) Let $A, B$ be subgroups of $G$. Then $A \preceq B$ if and only if the index $|A : (A \cap B)|$ is finite.

The basic properties of limit sets are described in the following statement:

Lemma 2.4. ([7, 22, [14, Lemma 6.2]) Suppose $A, B$ are arbitrary subsets of $G$, $g \in G$. Then

(a) $\Lambda(A) = \emptyset$ if and only if $A$ is finite;
(b) $\Lambda(A)$ is a closed subset of the boundary $\partial G$;
(c) $\Lambda(A \cup B) = \Lambda(A) \cup \Lambda(B)$;
(d) $\Lambda(gA) = \Lambda(A), \ g \circ \Lambda(A) = \Lambda(gA)$;
(e) if $A \preceq B$ then $\Lambda(A) \subseteq \Lambda(B)$. 

The following property of limit sets of quasiconvex subgroups was first proved by E. Swenson:

**Lemma 2.5.** (\cite[Thm. 8]{22}, \cite[Lemma 9.1]{11}) Let $A$, $B$ be quasiconvex subgroups of a hyperbolic group $G$. Then $\Lambda(A) \cap \Lambda(B) = \Lambda(A \cap B)$ in $\partial G$.

3. Auxiliary Facts

**Lemma 3.1.** Assume $H$ is an $\eta$-quasiconvex subgroup of a $\delta$-hyperbolic group $G$, $X$ is a word over $A$ representing an element of infinite order in $G$, $0 < \lambda \leq 1$ and $c \geq 0$. Let $\nu = \nu(\delta, \lambda, c)$ be the constant given by Lemma 2.1. There exists a number $N = N(\delta, \eta, \nu, G) \in \mathbb{N}$ such that for any $m \in \mathbb{N}$ the following property holds.

If a word $W \equiv U_1X^nU_2$ is $(\lambda, \epsilon, \nu)$-quasigeodesic and satisfies $\|U_1\|, \|U_2\| > (m + \nu + c)/\lambda$, $n \geq N$ and $\text{elem}(V_1WV_2) \in H$ for some words $V_1, V_2$ with $\|V_1\|, \|V_2\| \leq m$, then there exist $k \in \mathbb{N}$ and $a \in G$ such that $\text{elem}(X^k) \in H^n$ and $|a|_G \leq 2\delta + \nu + \eta$.

**Proof.** Consider a path $q$ in $\Gamma(G, A)$ starting at $1_G$ and labelled by $V_1WV_2$. By our assumptions, $q_+ \in H$. Let $p$ and $r$ be its $(\lambda, \epsilon, \nu)$-quasigeodesic subpaths with $\text{lab}(p) \equiv W$ and $\text{lab}(r) \equiv X^n$ respectively. Choose an arbitrary phase vertex $u \in r$ such that the subpath of $r$ from $r_-$ to $u$ is labelled by some power of $X$.

According to Lemma 2.1, there is $v \in [p_-, p_+]$ satisfying $d(u, v) \leq \nu$. Using the assumptions and the triangle inequality one can achieve

$$d(p_-, v) \geq d(p_-, u) - d(u, v) \geq \lambda\|U_1\| - c - \nu \geq m,$$

and

$$d(p_+, v) \geq d(p_+, u) - d(u, v) \geq \lambda\|U_2\| - c - \nu \geq m.$$

Hence, by Lemma 2.2, $v \in O_{2\delta}(1_G, q_+)$. Thus,

$$u \in O_{2\delta + \nu}(1_G, q_+) \subseteq O_{2\delta + \nu + \eta}(H),$$

i.e., there is an element $a = a(u) \in G$ such that $|a|_G \leq 2\delta + \nu + \eta$ and $u \in Ha$.

Now, since the alphabet $A$ is finite, there are only finitely many elements in $G$ having length at most $(2\delta + \nu + \eta)$. Hence, if $n$ is large enough, there should be two different phase vertices $u_1, u_2 \in r$ with $a(u_1) = a(u_2) = a$. By the construction,

$$u_1^{-1}u_2 = \text{elem}(X^k) \in a^{-1}Ha$$

for some positive integer $k$ ($X^k$ is a label of the segment of $r$ from $u_1$ to $u_2$, provided these points are chosen in a correct order). Q.e.d.

**Lemma 3.2.** Assume $G$ is a hyperbolic group and $H \leq G$ is a quasiconvex subgroup. If $g \in G$ and $gH \leq H$, then $H \leq gH$.

**Proof.** If $H$ is finite, the statement is trivial. Our assumptions immediately imply

$$g^{k-1}H \leq g^{k-2}H \leq \cdots \leq gH \leq H$$

for all $k \in \mathbb{N}$. Hence,

$$g^{k-1}H \leq H.$$

If $g$ has order $k \in \mathbb{N}$ in the group $G$, then to achieve the desired result it is enough to multiply both sides of the above formula by $g$.

Thus, we can further assume that $H$ is infinite and the element $g$ has infinite order. Therefore, $H$ has at least one limit point $\alpha \in \Lambda(H)$. Observe that (1) implies

$$g^n \circ \Lambda(H) = \Lambda(g^nH) \subseteq \Lambda(H),$$
thus \( g^n \circ \alpha \in \Lambda(H) \) for all \( n \in \mathbb{N} \).

If \( \alpha \neq g^{-\infty} \) in \( \partial G \), it is well known (see, for instance, [4, 8.16]) that the sequence
\[
(g^n \circ \alpha)_{n \in \mathbb{N}}
\]
converges to \( g^{-\infty} \). Since \( \Lambda(H) \) is a closed subset of the boundary \( \partial G \), in either case we achieve
\[
\Lambda(\langle g \rangle_\infty) = \{ g^\infty, g^{-\infty} \} \cap \Lambda(H) \neq \emptyset.
\]
By Lemma 2.5, the latter implies \( g^k \in H \) for some \( k \in \mathbb{N} \). Combining this fact with (1) we get \( H = g^k H \leq gH \), which concludes the proof. \( \square \)

The previous lemma has an interesting corollary:

**Lemma 3.3.** Suppose \( H, K \) are subgroups of a hyperbolic group \( G \) such that \( H \leq K \), \( |K : H| = \infty \) and \( H \) is quasiconvex. Then \( |K : (K \cap H^g)| = \infty \) for any \( g \in G \).

**Proof.** If \( |K : (K \cap H^g)| < \infty \) for some \( g \in G \) then \( H \leq K \leq gHg^{-1} \leq gH \) according to Lemma 2.5. Consequently \( g^{-1}H \leq H \). Hence \( H \leq g^{-1}H \) by Lemma 3.2 implying \( gHg^{-1} \leq gH \leq H \). But the latter leads to \( K \leq H \) which contradicts the condition \( |K : H| = \infty \) (see Lemma 2.6). \( \square \)

If \( G \) is a hyperbolic group, each element \( g \in G \) of infinite order belongs to a unique maximal elementary subgroup \( E(g) \). By [17] Lemmas 1.16, 1.17 it has the following description:

\[
(2) \quad E(g) = \{ x \in G \mid xg^k x^{-1} = g^l \text{ for some } k, l \in \mathbb{Z} \setminus \{0\} \} = \\
\{ x \in G \mid xg^nx^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{N} \}.
\]

Note that the subgroup \( E^+(g) \) defined as \( \{ x \in G \mid xg^nx^{-1} = g^n \text{ for some } n \in \mathbb{N} \} \) has index at most 2 in \( E(g) \).

Let \( W_1, W_2, \ldots, W_l \) be words in \( \mathcal{A} \) representing elements \( g_1, g_2, \ldots, g_l \) of infinite order, where \( E(g_i) \neq E(g_j) \) for \( i \neq j \). The following lemma will be useful:

**Lemma 3.4.** ([17] Lemma 2.3) There exist constants \( \lambda = \lambda(W_1, W_2, \ldots, W_l) > 0 \), \( c = c(W_1, W_2, \ldots, W_l) \geq 0 \) and \( N = N(W_1, W_2, \ldots, W_l) > 0 \) such that any path \( p \) in the Cayley graph \( \Gamma(G, \mathcal{A}) \) with label \( W_{i_1}^m W_{i_2}^m \ldots W_{i_s}^m \) is \((\lambda, c)\)-quasigeodesic if \( i_k \neq i_{k+1} \) for \( k = 1, 2, \ldots, s-1 \), and \( |m_k| > N \) for \( k = 2, 3, \ldots, s-1 \) (each \( i_k \) belongs to \( \{1, \ldots, l\} \)).

For a subgroup \( H \) of \( G \) denote by \( H^0 \) the set of elements of infinite order in \( H \); if \( A \subseteq G \), the subgroup \( C_H(A) = \{ h \in H \mid g^h = g, \forall g \in A \} \) is the centralizer of \( A \) in \( H \).

Set \( E(H) = \bigcap_{x \in H^0} E(x) \). If \( H \) is a non-elementary subgroup of \( G \), then \( E(H) \) is the unique maximal finite subgroup of \( G \) normalized by \( H \) ([17] Prop. 1)).

If \( g \in G^0 \), \( T(g) \) will be used to denote the set of elements of finite order in the subgroup \( E(g) \).

Let \( G \) be a hyperbolic group and \( H \) be its non-elementary subgroup. Recalling the definition from [17] (and using the terminology from [12]), an element \( g \in H^0 \) will be called \( H \)-suitable if \( E(H) = T(g) \) and
\[
E(g) = E^+(g) = C_H(g) = T(g) \times \langle g \rangle.
\]
In particular, if the element \( g \) is \( H \)-suitable then \( g \in C_H(E(H)) \).

Two elements \( g, h \in G \) of infinite order are called commensurable if \( g^k = ah^l a^{-1} \) for some non-zero integers \( k, l \) and some \( a \in G \). The following important statement was proved by A. Ol’shanskii in 1993:
Lemma 3.5. ([17] Lemma 3.8]) Every non-elementary subgroup $H$ of a hyperbolic group $G$ contains an infinite set of pairwise non-commensurable $H$-suitable elements.

Suitable elements can be modified in a natural way:

Lemma 3.6. ([12] Lemma 4.3]) Let $H$ be a non-elementary subgroup of a hyperbolic group $G$, and $g$ be an $H$-suitable element. If $y \in C_H(E(H)) \setminus E(g)$ then there exists $N \in \mathbb{N}$ such that the element $yg^n$ has infinite order in $H$ and is $H$-suitable for every $n \geq N$.

In [14] the author studied properties of quasiconvex subgroups of infinite index and showed

Lemma 3.7. ([14] Prop. 1]) Suppose $H$ is a quasiconvex subgroup of a hyperbolic group $G$ and $K$ is any subgroup of $G$ that satisfies $|K : (K \cap H^g)| = \infty$ for all $g \in G$. Then there exists an element $x \in K$ having infinite order, such that $\langle x \rangle, H^G = \{1_G\}$.

Later we will utilize a stronger fact:

Lemma 3.8. Assume $H, K$ are subgroups of a $\delta$-hyperbolic group $G$, $H$ is $\eta$-quasiconvex, $K$ is non-elementary and $|K : (K \cap H^g)| = \infty$ for every $g \in G$. Then there exists a $K$-suitable element $y \in K$ such that $\langle y \rangle, H^G = \{1_G\}$.

Proof. Set $K' = C_K(E(K))$. Since $E(K)$ is a finite subgroup normalized by $K$, we have $|K : K'| < \infty$. Therefore $|K' : (K' \cap H^g)| = \infty$ for all $g \in G$. Applying Lemma 3.7 we can find an element of infinite order $x \in K'$ such that $\langle x \rangle, H^G = \{1_G\}$. By Lemma 3.6 there is a $K$-suitable element $z \in K$ which is non-commensurable with $x$, hence $\langle x \rangle, E(z) = \{1_G\}$.

Choose some words $X$ and $Z$ in the alphabet $A$ representing $x$ and $z$ respectively. Then one is able to find the numbers $\lambda = \lambda(X, Z)$, $c = c(X, Z)$ and $N_1 = N_1(X, Z)$ from the claim of Lemma 3.3.

Define $\nu = \nu(\delta, \lambda, c)$ and $N_2 = N_2(\delta, \eta, \nu, G)$ as in Lemmas 2.1 and 3.1. Denote $N = \max\{N_1, N_2\}$ and apply Lemma 3.6 to obtain $n \geq N_1$ such that the element $y = x^nz^n \in K$ is $K$-suitable.

It remains to check that $\langle y \rangle, H^G = \{1_G\}$. Assume the contrary, i.e., there exist $t \in \mathbb{N}$ and $g \in G$ such that $y^t \in H^g$. Then for each $l \in \mathbb{N}$, the element $y^t \in H^g$ will be represented by the $(\lambda, c)$-quasigeodesic word $W \equiv (X^N Z^N)^t$. And if the number $l$ is chosen sufficiently large (compared to $m = |g^{-1}|_G = |g|_G$), it should be possible to find a subword of the form $X^N$ in the “middle” of $W$ which satisfies all the assumptions of Lemma 3.1. Hence, $x^k = \text{elem}(X^k) \in H^G$ for some $k \in \mathbb{N}$. The latter contradicts to the construction of $x$. Thus the lemma is proved. \hfill \square

As usual, let $G$ be a $\delta$-hyperbolic group and $H$– its $\eta$-quasiconvex subgroup.

Lemma 3.9. Suppose the elements $x_1, x_2 \in G$ have infinite order, $E(x_1) \neq E(x_2)$, and for each $i = 1, 2$, satisfy $\langle x_i \rangle, H^G = \{1_G\}$. Then there exists a number $N \in \mathbb{N}$ such that for any $m, n \geq N$ the elements $x_1^m$, $x_2^n$ freely generate a free subgroup $F$ of rank 2 in $G$. Moreover, $F$ is quasiconvex and $F \cap H^G = \{1_G\}$.

Proof. Choose some words $X_1$ and $X_2$ in the alphabet $A$ representing the elements $x_1$ and $x_2$. Apply Lemma 3.3 to find the corresponding $\lambda = \lambda(X_1, X_2)$, $c = c(X_1, X_2)$ and $N_1 = N_1(X_1, X_2)$. Then one can find the constant $\nu = \nu(\delta, \lambda, c)$
from the claim of Lemma 2.1 and define \( N_2 = N_2(\delta, \eta, \nu, G) \) in accordance with Lemma 3.1

Set \( N = \max\{N_1, N_2, |e/\lambda| + 1\} \). Consider arbitrary integers \( m, n \geq N \) and the subgroup \( F = \langle x_1^m, x_2^n \rangle \leq G \). By Lemma 3.3 any non-empty (freely) reduced word \( W \) in the generators \( \{X_1^m, X_2^n\} \) is \((\lambda, c)\)-quasigeodesic. Hence

\[
|\text{elem}(W)|_G \geq \lambda \|W\| - c \geq \lambda N - c > 0.
\]

Consequently, \( \text{elem}(W) \neq 1_G \) in \( G \), implying that \( F \) is free with the free generating set \( \{x_1^m, x_2^n\} \). By the construction of \( \nu \), \( F \) will be \( \varepsilon \)-quasiconvex where

\[
\varepsilon = \nu + \frac{1}{2} \max\{m\|X_1\|, n\|X_2\|\}.
\]

Consider a non-empty cyclically reduced word \( W \) in the generators \( \{X_1^m, X_2^n\} \). For establishing the last claim, it is sufficient to demonstrate that \( \text{elem}(W) \notin H^G \). Arguing as in Lemma 3.8 suppose \( \text{elem}(W) \in H^g \) for some \( g \in G \). Then \((\text{elem}(W))^l \in H^g \) for every \( l \in \mathbb{N} \). By choosing \( l \) sufficiently large and applying Lemma 4.1 one will obtain a contradiction with the assumption \( \langle x_i \rangle_\infty \cap H^G = \{1_G\} \), \( i = 1, 2 \), as in Lemma 3.8 Therefore \( F \cap H^G = \{1_G\} \). Q.e.d.

**Corollary 2.** With the assumptions of Lemma 3.8 \( K \) has a free subgroup \( F \) of rank 2 which is quasiconvex in \( G \), \( E(F) = E(K) \) and \( F \cap H^G = \{1_G\} \).

**Proof.** Choose a \( K \)-suitable element \( x_1 \in K \) according to Lemma 3.8 Since \( K \) is non-elementary, there exists \( y \in K \setminus E(x_1) \). Therefore, \( x_2 \text{ def } yx_1y^{-1} \in K^0 \) and \( E(x_2) \neq E(x_1) \) (as it can be seen from [2]). By the construction, \( \langle x_i \rangle_\infty \cap H^G = \{1_G\} \), \( i = 1, 2 \). Hence the subgroup \( F \) can be found by applying Lemma 3.9 Evidently \( E(K) \leq E(F) \), and \( E(F) \subseteq T(x_1) = E(K) \). Thus \( E(F) = E(K) \). Q.e.d.

### 4. Free products of quasiconvex subgroups

Below we will assume that \( G \) is a \( \delta \)-hyperbolic group generated by a finite set \( \mathcal{A} \).

First let us recall some properties of the hyperbolic boundary.

**Lemma 4.1.** ([12] Lemma 2.14) Suppose \( A \) and \( B \) are arbitrary subsets of \( G \) and \( \Lambda(A) \cap \Lambda(B) = \emptyset \). Then \( \sup_{a \in A, b \in B} \{(a|b)_{1_G}\} < \infty \).

**Remark 4.1.** Suppose \( g, x \in G \) and \( g \) has infinite order. If \( (x \circ \{g^{\pm \infty}\}) \cap \{g^{\pm \infty}\} \neq \emptyset \) in the boundary \( \partial G \), then \( x \in E(g) \). If \( E(g) = E^+(g) \) then \( g^{\infty} \notin G \circ \{g^{-\infty}\} \).

Note that \( x \circ \{g^{\pm \infty}\} = \{(xgx^{-1})^{\pm \infty}\} = \Lambda((xgx^{-1})) \subseteq \partial G \). Since any cyclic subgroup in a hyperbolic group is quasiconvex, we can apply Lemma 2.3 to show that \( (g) \cap (xgx^{-1}) \neq \{1_G\} \). Hence \( x \in E(g) \).

Let \( E(g) = E^+(g) \) and suppose that \( g^{\infty} = x \circ g^{-\infty} \) for some \( x \in G \). Then, as we showed above, \( x \in E(g) = E^+(g) \). Hence

\[
x \circ g^{-\infty} = \lim_{n \to -\infty} (xgx^{-1})^n = \lim_{n \to -\infty} g^n = g^{-\infty}.
\]

Thus we achieve a contradiction with the inequality \( g^{\infty} \neq g^{-\infty} \).

**Lemma 4.2.** Let \( g, x \in G \) where \( g \) has infinite order and \( E(g) = E^+(g) \). Then there is \( N_1 \in \mathbb{N} \) such that for every \( n \geq N_1 \) the element \( xg^n \in G \) has infinite order.
Proof. If \( x \notin E(g) \) then the claim follows by [11] Lemma 9.14.

So, assume \( x \in E(g) \). Since \( E(g) = E^+ (g) \), the center of \( E(g) \) has finite index in it, thus \( E(g) \) is an FC-group. By B.H. Neumann’s Theorem [15] the elements of finite order form a subgroup \( T(g) \leq E(g) \). Therefore the cardinality of the intersection \( \{ x g^k \mid k \in \mathbb{Z} \} \cap T(g) \) can be at most 1. Thus \( x g^n \notin T(g) \) for each sufficiently large \( n \).

\[ \square \]

The main result of this paper is based on the following statement concerning broken lines in a \( \delta \)-hyperbolic metric space:

**Lemma 4.3.** ([18] Lemma 21, [13] Lemma 3.5) Let \( p = [y_0, y_1, \ldots, y_n] \) be a broken line in \( \Gamma (G,A) \) such that \( ||[y_{i-1}, y_i]|| > C_1 \forall i = 1, \ldots, n \), and \( (y_{i-1} y_{i+1})_i < C_0 \forall i = 1, \ldots, n - 1 \), where \( C_0 \geq 14 \delta \), \( C_1 > 12 (C_0 + \delta) \). Then the geodesic segment \( \{ y_0, y_n \} \) is contained in the closed \( 14 \delta \)-neighborhood of \( p \) and \( ||[y_0, y_n]|| \geq ||p||/2 \).

Suppose \( a, b, c \) and \( d \) are arbitrary points in \( \Gamma (G,A) \). Considering the geodesic triangle with the vertices \( 1_G, a \) and \( ab \), one can observe that

\[
(a|ab)_{1_G} = |a|_G - (1_G|ab)_a = |a|_G - (a^{-1}|b)_{1_G}.
\]

Now, since \( \Gamma (G,A) \) is \( \delta \)-hyperbolic,

\[ (a|c)_{1_G} \geq \min \{ (a|ab)_{1_G}, (ab|c)_{1_G} \} - \delta = \min \{ |a|_G - (a^{-1}|b)_{1_G}, (ab|c)_{1_G} \} - \delta. \]

Replacing \( c \) with \( cd \) in the above formula, we get

\[ (a|cd)_{1_G} \geq \min \{ |a|_G - (a^{-1}|b)_{1_G}, (ab|cd)_{1_G} \} - \delta. \]

The Gromov product is symmetric, therefore we are able to combine formulas (3) and (4) to achieve

\[ (a|c)_{1_G} \geq \min \{ |c|_G - (c^{-1}|d)_{1_G}, (a|cd)_{1_G} \} - \delta \geq \min \{ |a|_G - (a^{-1}|b)_{1_G}, |c|_G - (c^{-1}|d)_{1_G}, (ab|cd)_{1_G} \} - 2 \delta. \]

**Theorem 5.** Consider some elements \( g_1, x_1, g_2, x_2, \ldots, g_s, x_s \in G \) and an \( \eta \)-quasiconvex subgroup \( H \leq G \). Suppose the following three conditions are satisfied:

- \( g_1, \ldots, g_s \) have infinite order and are pairwise non-commensurable;
- \( E(g_i) = E^+(g_i) \) for each \( i = 1, 2, \ldots, s \);
- \( E(g_i) \cap H = E(g_i) \cap x_i^{-1} H x_i = \{ 1_G \} \) for each \( i = 1, 2, \ldots, s \).

Then there exists a number \( N \in \mathbb{N} \) such that for every \( n \geq N \) the elements \( x_i g_i^n \in G \), \( i = 1, 2, \ldots, s \), have infinite order, and the subgroup

\[ M \overset{\text{def}}{=} \langle H, x_1 g_1^n, \ldots, x_s g_s^n \rangle \leq G \]

is quasiconvex in \( G \) and isomorphic (in the canonical way) to the free product

\[ H * \langle x_1 g_1^n \rangle * \cdots * \langle x_s g_s^n \rangle. \]

**Proof.** Choose arbitrary elements \( w_1, w_2 \in M \) and define \( w = w_1^{-1} w_2 \in M \). Then this element has a presentation

\[ w = h_0 (x_{1i_1} g_{i_1}^{e_1})^{i_1} h_1 (x_{2i_2} g_{i_2}^{e_2})^{i_2} \cdots h_{l-1} (x_{li_l} g_{i_l}^{e_l})^{i_l} h_l, \]

where \( h_j \in H, i_j \in \{ 1, \ldots, s \}, e_j \in \{ 1, -1 \}, j = 1, 2, \ldots, l, l \in \mathbb{N} \cup \{ 0 \} \).

Moreover, we can assume that the presentation (6) is reduced in the following sense: if \( 1 \leq j \leq l - 1 \), \( i_j = i_{j+1} \) and \( e_{j+1} = -e_j \) then \( h_j \neq 1_G \).
Consider a geodesic broken line \([y_0, y_1, \ldots, y_{l+1}]\) in \(\Gamma(G,A)\) with \(y_0 = w_1\) and \(\text{elem}([y_k, y_{k+1}]) = h_k(x_{i_k}, g_{i_{k+1}}^n)^{r_k}, \ k = 0, 1, \ldots, l-1, \ \text{elem}([y_l, y_{l+1}]) = h_l.\) Therefore \(\text{elem}([y_0, y_{l+1}]) = w\) and \(y_{l+1} = w_2.\)

Now we are going to find upper bounds for the Gromov products

\[
(y_{k-1}y_{k+1})_y = (y_{k-1}y_{k-1}y_{k-1}y_{k+1})_G, \ k = 1, \ldots, l.
\]

By the assumptions of the theorem, Lemma 2.3 implies that

\[
x_i \circ g_i^{\infty} = (x_i g_i x_i^{-1})^{-1} \notin \Lambda(H) \text{ and } g_i^{-\infty} \notin \Lambda(H), \ i = 1, \ldots, s.
\]

Since

\[
\Lambda \left( \left\{ (x_i g_i^m, g_i^{-m} x_i^{-1} \mid m \in \mathbb{N}, 1 \leq i \leq s \right\} \right) = \{ x_i \circ g_i^{\infty}, g_i^{-\infty} \mid 1 \leq i \leq s \} \subset \partial G,
\]

we are able to apply Lemma 4.1 to define

\[
\alpha \overset{\text{def}}{=} \max \left\{ \left| (h x_i g_i^m)_G, (h g_i^{-m} x_i^{-1})_G \right| \ h \in H, 1 \leq i \leq s, m \in \mathbb{N} \right\} < \infty.
\]

Similarly, since \(g_i\) and \(g_j\) are non-commensurable if \(i \neq j\) and \(E(g_i) = E^+(g_i),\) we have (according to Lemma 2.5 and Remark 4.1)

\[
G \circ \{ g_i^+ \} \cap G \circ \{ g_j^+ \} = \emptyset, \ G \circ \{ g_i^\infty \} \cap G \circ \{ g_j^\infty \} = \emptyset, \ i, j \in \{ 1, \ldots, s \}, i \neq j.
\]

Hence, the following numbers are also finite:

\[
\beta_1 \overset{\text{def}}{=} \max \left\{ \left| (x_i g_i^m)_G h x_j g_j^k \right| \ h \in H, \ |h|_G \leq 2 \alpha + 2\delta, 1 \leq i, j \leq s, m, t \in \mathbb{N} \right\},
\]

\[
\beta_2 \overset{\text{def}}{=} \max \left\{ \left| (x_i g_i^m)_G h x_j g_j^k \right| \ h \in H, \ |h|_G \leq 2 \alpha + 2\delta, 1 \leq i, j \leq s, i \neq j, m, t \in \mathbb{N} \right\}.
\]

Note that if \(h \in H \setminus \{1_G\}\) then, according to our assumptions, \(x_i^{-1} h x_i \notin E(g_i).\) Therefore, \(\{ g_i^{+\infty} \} \cap (x_i^{-1} h x_i) \circ \{ g_j^{+\infty} \} = \emptyset\) (Remark 4.1), implying \(x_i \circ g_i^{\infty} \neq (h x_i) \circ g_i^{\infty}, i = 1, \ldots, s.\) We can also use Remark 4.1 to show that \(g_i^{\infty} \neq h \circ g_i^{\infty}\) for each \(i.\) Consequently, by Lemma 4.1

\[
\beta_3 \overset{\text{def}}{=} \max \left\{ \left| (x_i g_i^m)_G h x_j g_j^k \right| \ h \in H, \ |h|_G \leq 2 \alpha + 2\delta, h \neq 1_G, 1 \leq i \leq s, m, t \in \mathbb{N} \right\} < \infty.
\]

Finally, define \(\beta = \max\{\beta_1, \beta_2, \beta_3\} < \infty,\)

\[
(7) \quad C_0 = \max\{\alpha + 2\delta, \beta + \delta, 14\delta\} + 1 \text{ and } C_1 = 12(C_0 + \delta) + 1.
\]

Since each \(g_i, i = 1, \ldots, s,\) has infinite order in \(G\) there exists \(N \in \mathbb{N}\) such that for any \(i \in \{1, \ldots, s\}\) one has

\[
(8) \quad |g_i|^G > \max\{\alpha, \beta, 2C_1\} + \alpha + |x_i|^G + 2\delta, \ \forall \ n \geq N.
\]

By Lemma 4.2, without loss of generality, we can assume that the order of \(x_i g_i^n, \ i = 1, 2, \ldots, s,\) is infinite for every \(n \geq N.\) Fix an integer \(n \geq N\) and choose any \(k \in \{1, \ldots, l-1\}.\) Then

\[
(y_{k-1}y_{k+1})_y = ((x_i g_i^n)^{-k} h_{k-1}^{-1} h_k(x_{i_{k+1}} g_{i_{k+1}}^n)^{r_{k+1}})_G.
\]
To simplify the notation, let us denote $a = (x_i g^n_{i_k})^{-c_k}$, $b = h_k^{-1}$, $c = h_k$ and $d = (x_{k+1} g^n_{i_{k+1}})^{r_{k+1}}$. By construction

$$(a|c)_{1_G}, (a^{-1}|b)_{1_G}, (c^{-1}|d)_{1_G} \leq \alpha.$$  

Now we need to consider two separate cases.

**Case 1:** $|h_k|_G = |c|_G \leq 2\alpha + 2\delta$. Then, due to the definition of the number $\beta$, the inequality $(a|cd)_{1_G} \leq \beta$ holds. Therefore, applying formulas (11) and (9) one obtains

$$\beta \geq (a|cd)_{1_G} \geq \min \{ |a|_G - (a^{-1}|b)_{1_G}, (ab|cd)_{1_G} \} - \delta \geq 
\min \{ |g^n_{i_k}|_G - |x_{i_k}|_G - |c|_G - \alpha, (ab|cd)_{1_G} \} - \delta.$$  

By (8), $|g^n_{i_k}|_G - |x_{i_k}|_G - \alpha > \beta + \delta$, hence the above inequality implies

$$(yk-1)|yk+1)_{yk} = (ab|cd)_{1_G} \leq \beta + \delta < C_0.$$  

**Case 2:** $|h_k|_G = |c|_G > 2\alpha + 2\delta$. Apply formulas (5) and (9) to achieve

$$\alpha \geq (a|c)_{1_G} \geq \min \{ |a|_G - (a^{-1}|b)_{1_G}, (ab|cd)_{1_G} \} - 2\delta.$$  

Observing $|a|_G - \alpha \geq |g^n_{i_k}|_G - |x_{i_k}|_G - \alpha > \alpha + 2\delta$ and $|c|_G - \alpha > \alpha + 2\delta$, we can conclude that

$$(yk-1)|yk+1)_{yk} = (ab|cd)_{1_G} \leq \alpha + 2\delta < C_0.$$  

At last, let us estimate the product $(y_{l-1}|y_{l+1})_{y_l} = ((x_i g^n_{i_k})^{-c_k} h_k^{-1}|h_l)_{1_G}$. Denote $a = (x_i g^n_{i_k})^{-c_k}$, $b = h_k^{-1}$ and $c = h_l$. Using formula (3) and the definition of $\alpha$ one can obtain

$$\alpha \geq (a|c)_{1_G} \geq \min \{ |x_{i_k} g^n_{i_k}|_G - \alpha, (ab|c)_{1_G} \} - \delta.$$  

As before, the latter implies that

$$(y_{l-1}|y_{l+1})_{y_l} = (ab|c)_{1_G} \leq \alpha + \delta < C_0.$$  

Thus, we have shown that

$$(yk-1)|yk+1)_{yk} < C_0 \text{ for each } k = 1, 2, \ldots, l.$$  

Now we need to find a lower bound for the lengths of the sides in the broken line $[y_0, y_1, \ldots, y_{l+1}]$.

Let $0 \leq k \leq l - 1$. Note that $|ab|_G \geq |b|_G - (a^{-1}|b)_{1_G}$ for any $a, b \in G$. Hence

$$\|[y_k, y_{k+1}]\| = |h_k(x_{k+1} g^n_{i_{k+1}})^{r_{k+1}}|_G \geq 
|((x_{k+1} g^n_{i_{k+1}})^{r_{k+1}} - h_k^{-1}) | (x_{k+1} g^n_{i_{k+1}})^{r_{k+1}} |_G \geq |g^n_{i_{k+1}}|_G - |x_{i_{k+1}}|_G - \alpha.$$  

Applying inequality (8) we obtain

$$(y_{l-1}|y_{l+1})_{y_l} > 2C_1 \text{ if } 0 \leq k \leq l - 1.$$  

Depending on the length of the last side, $\|[y_l, y_{l+1}]\| = |h_l|_G$, there can occur two different situations.

**Case 1:** $\|[y_l, y_{l+1}]\| = |h_l|_G \leq C_1$. Then we can use inequalities (10) and (11) to apply Lemma 4.3 to the geodesic broken line $p' = [y_0, \ldots, y_l]$. Hence $[y_0, y_l] \in \mathcal{O}_{14\delta}(p')$ and $d(y_0, y_l) \geq \|p'\|/2$.

Since geodesic triangles in $\Gamma(G, \mathcal{A})$ are $\delta$-slim, we have

$$[y_0, y_{l+1}] \subset \mathcal{O}_{\delta}([y_0, y_l]) \cup [y_l, y_{l+1}] \subset \mathcal{O}_{\delta+C_1}([y_0, y_l]) \subset \mathcal{O}_{15\delta+C_1}(p').$$
Now, if \( l \geq 1 \) in the presentation (6), one can use (11) to obtain
\[
d(y_0, y_{l+1}) \geq d(y_l, y_{l+1}) - d(y_l, y_0) \geq \|y_l\|^2/2 - C_1 \geq \|y_0, y_1\|^2/2 - C_1 > 0.
\]

**Case 2:** \( \|y_l, y_{l+1}\| = |h_l|_G > C_1 \). Then we can apply Lemma 4.3 to the broken line \( p = [y_0, \ldots, y_l, y_{l+1}] \), thus achieving
\[
[y_0, y_{l+1}] \subset O_{\delta + \zeta}(p).
\]

As before, if \( l \geq 1 \) one has
\[
d(y_0, y_{l+1}) \geq \|p\|/2 > 0.
\]

Thus, in either case we have established the following properties:
(12) \( [y_0, y_{l+1}] \subset O_{15\delta + C_1}(p) \) and
(13) \( d(y_0, y_{l+1}) > 0 \).

The inequality (13) implies that \( w \neq 1_G \) in the group \( G \) for any element \( w \in M \) having a "reduced" presentation (6) with \( l \geq 1 \). Therefore
\[
M \cong H * \langle x_1 g_1^n \rangle * \cdots * \langle x_s g_s^n \rangle.
\]

As \( n \geq N \) is fixed, one is able to define the constants
\[
\zeta = \max_{1 \leq i \leq s} \{|x_i g_i^n|_G\} < \infty \quad \text{and} \quad \varepsilon = 16\delta + C_1 + \eta + \zeta.
\]

We will finish the proof by showing that \( [y_0, y_{l+1}] \subset O_{\varepsilon}(M) \) which implies that \( M \) is \( \varepsilon \)-quasiconvex.

Consider an arbitrary \( k \in \{0, 1, \ldots, l\} \). Using the construction of \( y_k \in M \) and \( \zeta \), and \( \delta \)-hyperbolicity of the Cayley graph we achieve
\[
[y_k, y_{k+1}] \subset O_{\delta + \zeta}([y_k, y_kh_k]).
\]

\( H \) is \( \eta \)-quasiconvex, therefore \([1_G, h_k] \subset O_{\eta}(H)\). The metric in \( \Gamma(G, A) \) is invariant under the action of \( G \) by left translations, consequently
\[
[y_k, y_kh_k] \subset O_{\eta}(y_k H) \subset O_{\eta}(M).
\]

Combining the latter formula with (14) leads us to
\[
[y_k, y_{k+1}] \subset O_{\delta + \eta + \zeta}(M) \quad \text{for each \( k \).}
\]

Finally, an application of (12) yields
\[
[y_0, y_{l+1}] \subset O_{15\delta + C_1 + \delta + \eta + \zeta}(M) = O_{\varepsilon}(M),
\]
as desired. \( \square \)

5. Hyperbolic groups with engulfing

**Proof of Theorem 7.** Since every elementary group is residually finite, it is sufficient to consider the case when \( G \) is non-elementary. Let \( x_1, \ldots, x_s \) be the generators of \( G \).

Define \( K = \bigcap_{L \leq G, [G:L] < \infty} L \); then \( K \) is normal in \( G \). Suppose \( K \) is infinite. If the subgroup \( K \) were elementary, then it would be quasiconvex (this follows directly from Lemmas 3.4 and 2.1). Hence, according to a result proved by Mihalik and Towle [19] (see also [13] Cor. 2)), it would have a finite index in \( G \), thus the group \( G \) would also have to be elementary. Therefore \( K \) can not be elementary.
Now we can apply Lemma 5.3 to find pairwise non-commensurable $K$-suitable elements $g_1, \ldots, g_{s+1} \in K$. Since the trivial subgroup $H = \{1_G\} \leq G$ is quasiconvex, we can use Theorem 5 to show that the subgroups $M = \langle x_1g_1^n, \ldots, x_sg_s^n \rangle$ and $M' = \langle x_1g_1^n, \ldots, x_sg_s^n, g_{s+1}^n \rangle$ are free (of ranks $s$ and $(s + 1)$ respectively) and quasiconvex in $G$ for some sufficiently large $n \in \mathbb{N}$.

Note that $M$ is a proper (infinite index) subgroup of $G$ because $|M' : M| = \infty$. According to our assumptions, there exists a proper finite index subgroup $L \leq G$ with $M \leq L$. By the construction, $K \leq L$, thus $x_i g_i^n, g_i \in L$ for each $i = 1, \ldots, s$. Consequently $x_i \in L$ for each $i = 1, \ldots, s$, contradicting with properness of $L$.

Therefore $K$ is finite. If $G$ is torsion-free then $K$ is trivial, and, thus, $G$ is residually finite.

As $E(G)$ is the maximal finite normal subgroup in $G$, we obtain the following statement right away:

**Corollary 3.** With the assumptions of Theorem 4 suppose, in addition, that $E(G) = \{1_G\}$. Then $G$ is residually finite.

Below it will be convenient to use the following equivalence relation between subsets of a group $G$ defined in [14]: for any $A, B \subseteq G$ such that $A \preceq B$ and $B \preceq A$ we will write $A \approx B$.

**Remark 5.1.** ([14] Remark 3) If $A, B \subseteq G$, $A \approx B$ and $A$ is quasiconvex, then $B$ is also quasiconvex.

In particular, if $A \preceq B$ are subgroups of $G$ and $A$ has finite index in $B$ then $A \approx B$. Hence $A$ is quasiconvex if and only if $B$ is quasiconvex.

**Lemma 5.1.** Let $G$ be a residually finite hyperbolic group and let $H \leq G$ be a quasiconvex subgroup. Suppose that every proper quasiconvex subgroup of $G$ is engulfed. Then $H$ has a finite index in its profinite closure $K$ in $G$.

**Proof.** Arguing by the contrary, assume that $|K : H| = \infty$. Since the group $G$ is residually finite, any finite subset is closed in the profinite topology. Thus $H$ is infinite; hence $K$ is non-elementary.

Choose some generating set $x_1, \ldots, x_s$ of the group $G$. Since $G$ is residually finite, it has a finite index subgroup $G_1$ satisfying

\[
G_1 \cap \left( E(K) \cup \bigcup_{i=1}^{s} x_i E(K) x_i^{-1} \right) = \{1_G\}.
\]

Define $H_1 = H \cap G_1$; then $|H : H_1| < \infty$ and, according to Remark 5.1, $H_1$ is quasiconvex. The profinite closure $K_1$ of $H_1$ in $G$ has a finite index in $K$, therefore $K_1$ is non-elementary and $|K_1 : H_1| = \infty$. The definition of a finite index subgroup implies that there is $l \in \mathbb{N}$ such that $g^l \in K_1$ for each $g \in K$. Since $E(g) = E(g^l)$ for any $g \in K^0$ we have

\[
E(K) \subseteq E(K_1) = \bigcap_{g \in (K_1)^0} E(g) \subseteq \bigcap_{g \in K^0} E(g^l) = \bigcap_{g \in K^0} E(g) = E(K).
\]

Thus $E(K_1) = E(K)$.

By Lemma 3.3 $|K_1 : (K_1 \cap H_1^g)| = \infty$ for every $g \in G$ and we can use Corollary 2 to obtain a free subgroup $F \leq K_1$ of rank 2 satisfying $F \cap H_1^G = \{1_G\}$ and $E(F) = E(K_1) = E(K)$. 

According to Lemma 3.8 there exist elements \(g_1, \ldots, g_{s+1} \in F\) which are pairwise non-commensurable and \(F\)-suitable. Consequently \(E(g_i) = \langle g_i \rangle_\infty \times E(K)\) and, since \(E(K)\) is finite, formula (15) implies that

\[
E(g_i) \cap H_1 = E(K) \cap H_1 = \{1_G\}, \quad i = 1, 2, \ldots, s + 1,\]

and

\[
E(g_i) \cap x_i^{-1} H_1 x_i = E(K) \cap x_i^{-1} H_1 x_i = \{1_G\}, \quad i = 1, 2, \ldots, s.
\]

Now we apply Theorem 5 to find \(n \in \mathbb{N}\) such that the subgroups

\[
M = \langle H_1, x_1 g_1^n, \ldots, x_s g_s^n \rangle \quad \text{and} \quad M' = \langle H_1, x_1 g_1^n, \ldots, x_s g_s^n, g_{s+1} \rangle
\]

are quasiconvex in \(G\) and \(M' = M * \langle g_{s+1} \rangle_\infty \leq G\). Thus, \(|M' : M| = \infty\) and \(M\) is a proper subgroup of \(G\).

The subgroup \(M\) is engulfed by our assumptions, therefore \(G\) has a proper finite index subgroup \(L\) containing \(M\). Observe that \(K_1 \leq L\) because \(H_1 \leq M \leq L\), and, since \(x_i g_i^n, g_i \in M \cup K_1 \subset L\), we get \(x_i \in L\) for each \(i = 1, \ldots, s\). The latter implies \(G = L – \) a contradiction.

We will now prove Theorem 3 which strengthens the the statement of the previous lemma.

**Proof of Theorem 3.** We can assume that \(G\) is non-elementary because any elementary group is LERF. Since \(G\) is residually finite, there is a finite index subgroup \(G_1 \leq G\) with \(G_1 \cap E(G) = \{1_G\}\).

Take an arbitrary quasiconvex subgroup \(H \leq G\) and set \(H_1 = H \cap G_1\). As it follows from Remark 5.1, \(H_1\) is quasiconvex in \(G\). Therefore, according to Lemma 5.1, \(H_1\) has a finite index in its profinite closure \(K_1\) in \(G\).

If \(H_1 = K_1\), i.e., \(H_1\) is closed in the profinite topology on \(G\), then so is \(H\). Thus, we can suppose that \(H_1 \neq K_1\). Consequently \(|G : H_1| = \infty\), and thus \(|G : K_1| = \infty\).

The subgroup \(K_1\) is quasiconvex according to Remark 5.1 hence we can apply Lemma 3.8 to find a \(G\)-suitable element \(g \in G\) such that \(\langle g \rangle_\infty \cap K_1 = \{1_G\}\). Since \(E(g) = \langle g \rangle \times E(G)\), \(E(G)\) is finite and \(K_1 \leq G_1\), we have

\[
E(g) \cap K_1 = E(G) \cap K_1 = \{1_G\}.
\]

Now we can apply Theorem 5 to find a number \(n \in \mathbb{N}\) such that the subgroups

\[
M = \langle H_1, g^n \rangle \quad \text{and} \quad M' = \langle K_1, g^n \rangle
\]

are quasiconvex in \(G\) and \(M' = K_1 * \langle g^n \rangle_\infty\).

Using properties of free products, we observe that \(M \leq M'\) and \(|M' : M| = \infty\) because \(H_1 \leq K_1\). On the other hand, \(M'\) is contained inside of the profinite closure of \(M\) in \(G\). Thus we achieve a contradiction with the claim of Lemma 5.1.

Before proceeding with the next statement, we need to recall some facts concerning quasiisometries of metric spaces. Let \(\mathcal{X}\) and \(\mathcal{Y}\) be geodesic metric spaces with metrics \(d(\cdot, \cdot)\) and \(e(\cdot, \cdot)\) respectively. A map \(f : \mathcal{X} \to \mathcal{Y}\) is called a quasiisometry if there are constants \(D_1 > 0\) and \(D_2 \geq 0\) such that

\[
D_1^{-1} d(a, b) - D_2 \leq e(f(a), f(b)) \leq D_1 d(a, b) + D_2 \quad \forall a, b \in \mathcal{X}.
\]

The spaces \(\mathcal{X}\) and \(\mathcal{Y}\) are said to be quasiisometric if there exists a quasiisometry \(f : \mathcal{X} \to \mathcal{Y}\) whose image is quasidense in \(\mathcal{Y}\), i.e., there exists \(\varepsilon \geq 0\) such that for each \(y \in \mathcal{Y}\) there is \(x \in \mathcal{X}\) with \(e(y, f(x)) \leq \varepsilon\).

M. Gromov [5] showed that if \(\mathcal{X}\) is hyperbolic and quasiisometric to \(\mathcal{Y}\) (through some map \(f : \mathcal{X} \to \mathcal{Y}\)) then the space \(\mathcal{Y}\) is hyperbolic too. He also noted that in this case the image \(f(Q)\) of any quasiconvex subset \(Q \subseteq \mathcal{X}\) will be quasiconvex in \(\mathcal{Y}\).
Proof of Theorem 3. Note that $Q$ is a finite normal subgroup of $G$ by Theorem 1.

Consider the quotient $G_1 = G/Q$ together with the natural homomorphism $\psi : G \to G_1$. Since $Q$ is finite, $\psi$ is a quasimorphism between $G$ and $G_1$ ($G_1$ is equipped with the word metric induced by the image of the finite generating set of $G$). Therefore, $G_1$ is also hyperbolic and any preimage map $\bar{\psi}^{-1} : G_1 \to G$ (which maps an element of $G_1$ to some element of $G$ belonging to the corresponding left coset modulo $Q$) is a quasimorphism as well.

Choose an arbitrary proper quasiconvex subgroup $H_1 \leq G_1$. Then $\bar{\psi}^{-1}(H_1)$ is a quasiconvex subset of $G$ and
\[
\bar{\psi}^{-1}(H_1) \subseteq \psi^{-1}(H_1) \subseteq \bar{\psi}^{-1}(H_1) \cdot Q,
\]
where $\psi^{-1}(H_1)$ is the full preimage of $H_1$ in $G$.

As $Q$ is finite, the above formula implies $\bar{\psi}^{-1}(H_1) \approx \psi^{-1}(H_1)$. Therefore $\psi^{-1}(H_1)$ is quasiconvex in $G$ by Remark 5.1. According to our assumptions, there is a proper finite index subgroup $L \leq G$ containing $\psi^{-1}(H_1)$. By definition, $Q \leq L$, hence $\psi(L)$ is a proper finite index subgroup of $G_1$ with $H_1 \leq \psi(L)$.

Thus, we have shown that $G_1$ also engulfs each proper quasiconvex subgroup. By the construction, $G_1$ is residually finite and, therefore, GFERF (Theorem 3).

Consider any quasiconvex subgroup $H \leq G$. Then $\psi(H)$ is quasiconvex in $G_1$ and, thus, it is closed in the profinite topology on $G_1$. The homomorphism $\psi$ is a continuous map if $G$ and $G_1$ are equipped with their profinite topologies, thus the full preimage $\psi^{-1}(\psi(H)) = H \cdot Q$ is closed in $G$. Obviously $H, Q \leq K$ (where $K$ is the profinite closure of $H$ in $G$), hence $K = HQ$. Q.e.d.

### 6. Free products of GFERF groups

In the previous section we considered hyperbolic groups which engulf every proper quasiconvex subgroup. Let us name them QE-groups, for brevity.

As it can be seen from Theorem 2 any QE-group $G$ is very close to being GFERF. In fact, $G$ is quasimetric to the quotient $G/E(G)$ which is GFERF by Corollary 3 and Theorem 3. Nevertheless, the answer to the question whether each QE-group is GFERF is still unclear. Theorem 2 would yield a positive answer if a free product of any two QE-groups were a QE-group itself. Unfortunately, the author is unable to prove this; actually, he doubts if this is true in general.

However, the following statement, proved by R. Burns, can be used to show that a free product of GFERF-groups is, again, GFERF:

**Lemma 6.1.** ([2] Thm. 1.1) Suppose $G$ is a free product of its subgroups $G_i$ indexed by some set $I$, and let $H$ be a finitely generated subgroup. If for each $i \in I$, $g \in G$, the subgroup $(H^0 \cap G_i)$ is $G_i$-separable, then $H$ is $G$-separable.

Let $H$ be a subgroup of a group $G$. Suppose $A$ and $B$ are finite generating sets for $G$ and $H$ respectively and $| \cdot |_G$, $| \cdot |_H$ are the corresponding length functions. Set $\hat{c} = \max \{|b|_G : b \in B\}$. Evidently, $|h|_G \leq \hat{c}|h|_H$ for all $h \in H$.

$H$ is called undistorted in $G$ if there exists a constant $c \geq 0$ such that $|h|_H \leq c|h|_G$ for every $h \in H$. In a hyperbolic group $G$, a finitely generated subgroup is undistorted if and only if it is quasiconvex ([2] Lemma 1.6).

**Proof of Theorem 4.** It is well known that a free product of hyperbolic groups is a hyperbolic group (see, for instance, [3] 1.34]). Thus, $G$ is hyperbolic. Clearly, the subgroups $G_1$ and $G_2$ are undistorted in $G$; consequently, they are quasiconvex.
Choose an arbitrary quasiconvex subgroup $H \leq G$, an element $g \in G$ and $i \in \{1, 2\}$. The subgroup $H^g$ is quasiconvex by Remark 2.1. Since the intersection of two quasiconvex subgroups is quasiconvex (Prop. 3), $(H^g \cap G_i)$ is quasiconvex in $G$. Consequently, $(H^g \cap G_i)$ is undistorted in $G$, and, hence, it is undistorted in $G_i$. Thus $(H^g \cap G_i)$ is $G_i$-separable because $G_i$ is GFERF.

According to Lemma 6.1, $H$ is $G$-separable. Q.e.d.

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