INTERSECTION NUMBERS ON MODULI SPACES AND SYMMETRIES OF A VERLINDE FORMULA

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Abstract. We investigate the geometry and topology of a standard moduli space of stable bundles on a Riemann surface, and use a generalization of the Verlinde formula to derive results on intersection pairings.

1. Introduction

Let $\mathcal{M}_g$ denote the smooth moduli space of stable holomorphic rank 2 vector bundles with fixed determinant of odd degree over a Riemann surface $\Sigma_g$ of genus $g$. The space $\mathcal{M}_g$ has the structure of a complex $(3g-3)$-dimensional Kähler manifold whose anticanonical bundle is the square of an ample line bundle $L$ [24, 21]. The dimensions $h^0(\mathcal{M}_g, \mathcal{O}(L))$ are independent of the complex structure on $\Sigma_g$ and were predicted in [29]. In this paper, we highlight additional calculations that arise from the Desale-Ramanan description [7] of $\mathcal{M}_g$ for the case in which $\Sigma_g$ is hyperelliptic. Our approach is based on the proof of the Verlinde formula by Szendrői [26], and grew out of an attempt to generalize the related twistor geometry studied by the second author in [23].

Universal cohomology classes $\alpha, \beta, \gamma$ were defined on $\mathcal{M}_g$ by Newstead [20] and used to compute the Chern character of the holomorphic tangent bundle $T = T^{1,0} \mathcal{M}_g$. The latter can be expressed in terms of a tautological rank $g-1$ bundle $Q$ with the help of the Adams operator $\psi^2$, and we show in §2 that the appearance of this algebraic gadget leads to quick proofs of both the equation $\beta^g = 0$ and the recurrence relation for the Chern classes of $Q$. Underlying this theory is the fact that $\beta$ coincides with the pullback of the canonical quaternion-Kähler class on a real Grassmannian, providing a link with the index theory in [16]. This approach has a number of simplifying features, though in other ways §2 is a supplement to the papers of Baranovsky [2] and Siebert-Tian [25]. These authors, together with Zagier [32] and King-Newstead [14], have provided a variety of methods for determining the cohomology ring of $\mathcal{M}_g$.

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In §3 we further exploit Adams operators and the fundamental role played by \( Q \) by computing the holomorphic Euler characteristics

\[
V_{g-1}(p, q) = \chi(\mathcal{M}_g, \mathcal{O}(\psi^{p-q}Q \otimes L^{q-1}))
\]

for all \( p, q \in \mathbb{Z} \), thereby extending the Verlinde formula (corresponding to \( p = q \)) into a 2-dimensional array. The symmetries of the title are those enjoyed by the integers \( V_{g-1}(p, q) \) in the \( (p, q) \)-plane. The main purpose of §4 is to show that these symmetries can be used to recover the intersection numbers \( \langle \alpha^m \beta^n \gamma^p, [\mathcal{M}_g] \rangle \), using calculations similar to those of Thaddeus [27] and Donaldson [8]. This formulation in turn provides an alternative route to the Bertram-Szenes proof [4] of the ‘untwisted’ Verlinde formula for the moduli space of semistable rank 2 bundles with fixed determinant of even degree.

We end up encoding the topology of \( \mathcal{M}_g \) into equations involving Chern characters that are particularly easy to write down and remember. For example, it is known that \( \chi(\mathcal{M}_g, \mathcal{O}(T^*)) = g - 1 \), and it follows from the Riemann-Roch theorem that \( \langle \text{ch}(\tilde{T}) \text{td}(T), [\mathcal{M}_g] \rangle = 0 \) where \( \tilde{T} = T^* - g + 1 \). We prove a stronger vanishing theorem, namely that \( \text{ch}(\tilde{T})e^\alpha \) evaluates to zero when paired with any power of \( \beta \). The virtual bundle \( \tilde{T} \) is a natural one to consider since it has virtual rank \( 2g - 2 \) and \( c_i(\tilde{T}) = 0 \) for \( i > 2g - 2 \) by [11]. Although our methods are very special to the rank 2 case, it may be that analogues of these results hold on moduli spaces of higher rank bundles.

2. TANGENT RELATIONS

Let \( \Sigma_g \) be a hyperelliptic curve of genus \( g \), admitting a double-covering \( \Sigma_g \rightarrow \mathbb{C}P^1 \) with distinct branching points \( \omega_1, \ldots, \omega_{2g+2} \). Desale and Ramanan proved in [7] that the manifold \( \mathcal{M}_g \) defined in §1 can then be realized as the variety of \((g-1)\)-dimensional subspaces of \( \mathbb{C}^{2g+2} \) isotropic with respect to the two quadratic forms

\[
\sum_{i=1}^{2g+2} y_i^2, \quad \sum_{i=1}^{2g+2} \omega_i y_i^2.
\]

One therefore obtains a holomorphic embedding of \( \mathcal{M}_g \) into the complex homogeneous space

\[
\mathcal{F}_g = \frac{SO(2g + 2)}{U(g - 1) \times SO(4)}
\]

parametrizing \((g-1)\)-dimensional subspaces which are isotropic with respect to the first quadratic form [26].

Let \( Q, W \) denote the duals of the tautological complex vector bundles over \( \mathcal{F}_g \) with fibres \( \mathbb{C}^{g-1}, \mathbb{C}^4 \) and structure groups \( U(g - 1), SO(4) \) respectively. (The notation \( Q \) is consistent with [25], and from now on we shall often drop the subscript \( g \) since the
genus will generally be fixed in our discussion.) The decomposition of the standard representation of $SO(2g + 2)$ on $\mathbb{C}^{2g+2}$ under $U(g - 1) \times SO(4)$ provides the equation
\begin{equation}
Q^* \oplus Q \oplus W = 2g + 2,
\end{equation}
where the right-hand side denotes a trivial vector bundle of rank $2g + 2$. The second form in (1) now determines a non-degenerate section $s$ of the symmetrized tensor product $S^2 Q$, and the zero set of $s$ coincides with $\mathcal{M}$.

The holomorphic tangent bundle $T^{1,0} \mathcal{F}$ of $\mathcal{F}$ is determined by the summand $m$ in the Lie algebra splitting
\[ so(2g + 2)_c \cong (u(g - 1) \oplus so(4))_c \oplus m \oplus \overline{m}, \]
where $c$ denotes complexification. Given that $so(2g + 2)_c \cong \Lambda^2 \mathbb{C}^{2g+2}$, it follows from (2) that we may choose the orientation so that
\[ T^{1,0} \mathcal{F} \cong \Lambda^2 Q \oplus (Q \otimes W). \]
This equation is well known in the context of twistor spaces, since $\mathcal{F}$ is a 3-symmetric twistor space that fibres over the oriented real Grassmannian
\begin{equation}
G_g = Gr_4(\mathbb{R}^{2g+2}) = \frac{SO(2g + 2)}{SO(2g - 2) \times SO(4)}
\end{equation}
for $g \geq 3$ [5, 22]. The term $\Lambda^2 Q$ is simply the holomorphic tangent bundle to the Hermitian symmetric fibres $SO(2g - 2)/U(g - 1)$, and its complement $Q \otimes W$ corresponds to the holomorphic horizontal bundle that plays an important role in the theory of harmonic maps [6].

With the above choices, the normal bundle of $\mathcal{M}$ in $\mathcal{F}$ is isomorphic to $S^2 Q$, and
\[ T^{1,0} \mathcal{F}|_{\mathcal{M}} \cong T^{1,0} \mathcal{M} \oplus S^2 Q|_{\mathcal{M}}. \]
In the notation of K-theory, we may write
\[ T = T^{1,0} \mathcal{M} = \Lambda^2 Q + QW - S^2 Q, \]
where from now on we are using the same symbols to denote bundles pulled back to $\mathcal{M}$. Writing $\psi^2 = S^2 - \Lambda^2$, we have

**Lemma 2.1.** $T = QW - \psi^2 Q$.

The operator $\psi^2$ is one of the series of Adams operators, defined by the formula
\[ \sum_{p \geq 0} (\psi^p E) t^p = r - t \frac{d}{dt} \log \Lambda_{-t} E, \]
where $E \in K(\mathcal{M})$ has virtual rank $r$ and $\Lambda_t E = \sum_{i \geq 0} (\Lambda^i E) t^i$ [10]. Each $\psi^p$ is a ring homomorphism in K-theory, and is characterized by the property that
\begin{equation}
ch_k(\psi^p E) = p^k ch_k(E),
\end{equation}
where \( ch_k \) denotes the term of dimension \( 2k \) in the Chern character. We shall use the operators \( \psi^p \) with \( p \geq 3 \) in \( \S 3 \).

Cohomology classes

\[
(5) \quad \alpha \in H^2(M, \mathbb{Z}), \quad \beta \in H^4(M, \mathbb{Z}), \quad \gamma \in H^6(M, \mathbb{Z})
\]

were introduced by Newstead [19, 1]. They are obtained from the Künneth components of the characteristic class \( c_2(V) \), where \( V \) is a universal \( SO(3) \) bundle over \( M \), and generate the ring \( H^*_I(M) \) of cohomology classes of \( M \) invariant by the action of the mapping class group on \( H^3(M) \). By expressing \( T \) in terms of a push-forward of \( V \) one obtains the following result, which is effectively the definition of (5) for our purposes:

**Theorem 2.1.** [20, Theorem 2]

\[
ch(T) = 3g - 3 + 2\alpha + \sum_{k \geq 2} \frac{s_k}{k!}, \quad \text{where} \quad \left\{ \begin{array}{l}
s_{2k-1} = 2\alpha \beta^{k-1} - 8(k-1)\gamma\beta^{k-2}, \\
s_{2k} = 2(g-1)\beta^k.
\end{array} \right.
\]

As an application of Lemma 2.1 and (2), we see that the complexification of the real tangent bundle of \( M \) is isomorphic to

\[
T + T^* = (Q^* + Q)W - \psi^2(Q^* + Q),
\]

\[
= (2g + 2 - W)W - (2g + 2 - \psi^2W)
\]

\[
= (2g + 2)(W - 1) - W^2 + \psi^2W.
\]

The Chern character of this may be read off and then compared with Theorem 2.1 (for this purpose it helps not to replace \(-W^2 + \psi^2W\) by the equivalent \(-2\Lambda^2W\)). An easy calculation gives

\[
(7) \quad ch(T + T^*) = 6g - 6 + 2(g-1)p_1(W) + \frac{1}{6} \left[ (g-1)p_1(W)^2 - 2(g+5)p_2(W) \right] + \cdots
\]

where the Pontrjagin classes are defined by regarding \( W \) as an \( SO(4) \)-bundle. Now (7) must equal twice the sum of the even terms of \( ch(T) \), so \( p_1(W) = \beta \), \( p_2(W) = 0 \) and

\[
ch(W) = 2 + e^{\sqrt{\beta}} + e^{-\sqrt{\beta}}
\]

on the moduli space \( M \).

Since \( Q^* + Q = 2g + 2 - W \) is a genuine complex vector bundle of rank \( 2g - 2 \) with total Chern class

\[
c(W)^{-1} = \sum_{k=0}^{\infty} \beta^k
\]

on \( M \), we get Conjecture (a) of [20]:
Theorem 2.2. $\beta^g = 0$.

This was first proved by Kirwan [15], who established the completeness of the Mumford relations on $H^*_{\mu}(M)$. It also follows from results in [13, 31], and a different proof was given by Weitsman [30] in the more general setting of moduli spaces of flat connections over a Riemann surface with marked points.

From Lemma 2.1 and Theorem 2.1 one may readily compute the Chern character of $Q$ in terms of the classes (5). From this point of view the definition of $M$ as a submanifold of $F$ could not be simpler, as the terms $\bigwedge^2 Q$ and $S^2 Q$ ‘miraculously’ combine into a form specifically adapted for computing characters. The result is

Theorem 2.3.

$$\text{ch}(Q) = g - 1 + \alpha + \sum_{k \geq 1} \frac{s'_k}{k!},$$

where

$$\left\{ \begin{array}{c}
    s'_{2k-1} = \alpha \beta^{k-1} + 2 \gamma \beta^{k-2}, \\
    s'_{2k} = -\beta^k.
\end{array} \right.$$  

Proof. Let $s_k, s'_k$ denote the components of $\text{ch}(T), \text{ch}(Q)$, respectively, in dimension $2k$. Using Lemma 2.1, (4) and (8),

$$3g - 3 + \sum_{k \geq 1} \frac{s_k}{k!} = 2 \left( g - 1 + \sum_{k \geq 1} \frac{s'_k}{k!} \right) \left( 2 + \sum_{k \geq 1} \frac{\beta^k}{(2k)!} \right) - \left( g - 1 + \sum_{k \geq 1} \frac{2^k s'_k}{k!} \right).$$

The result now follows from Theorem 2.1 by induction on $k$.  

An analogue of the last equation can be found in [2], though the authors were led to it by the paper of Siebert and Tian [25], who give an equivalent expression for $\text{ch}(Q)$. Theorem 2.3 leads very quickly to their recurrence relation for the Chern classes of $Q$. Using a standard trick [32], the generating function for the Chern classes of $Q$ is recaptured by the formula

$$c(t) = \exp \left[ \sum_{k \geq 1} \frac{(-1)^{k-1}s_k t^k}{k} \right]$$

$$= \exp \left[ \alpha t + \sum_{n \geq 2} (\alpha \beta^{n-1} + 2 \gamma \beta^{n-2}) \frac{\beta^{2n-1}}{2n - 1} + \sum_{n \geq 1} \beta^n \frac{\beta^{2n}}{2n} \right].$$

The relation [25, Proposition 25], namely

$$(1 - \beta t^2)c'(t) = (\alpha + \beta t + 2 \gamma t^2)c(t)$$

follows immediately, whence

Corollary 2.1. The Chern classes of the rank $g - 1$ bundle $Q$ on $M_g$ satisfy

$$(k + 1)c_{k+1} = \alpha c_k + k\beta c_{k-1} + 2\gamma c_{k-2}.$$
Zagier has shown that the $c_k$ coincide with the Künneth components of the Chern classes in $H^*(\mathcal{M}_g) \otimes H^*(J_g)$ of a higher rank bundle used to define the Mumford relations ($J_g$ is the Jacobian of $\Sigma_g$). The identities in $\alpha, \beta, \gamma$ arising from the equations $c_k = 0$ for $k = g, g+1, g+2$ then provide a minimal set of relations to completely determine the cohomology ring $H^*_T(\mathcal{M})$ [32, 2, 14, 25]. We have set out to show that these equations follow quite directly from Newstead’s own computations, and it is worth pointing out that Corollary 2.1 is analogous, but simpler, to the recurrence relation for the Chern classes of $T$ given at the end of [20].

The fact that the Pontrjagin ring of $\mathcal{M} = \mathcal{M}_g$ is generated by $p_1(\mathcal{M}) = 2(g-1)\beta$ can also be related to the geometry of the real Grassmannian (3). For $Q + Q^*$ and $W$ are (complexifications of) the pullbacks of real vector bundles $\hat{U}, \hat{W}$ over $G = G_g$, and the real tangent bundle of $G$ is isomorphic to $\hat{U} \otimes \hat{W}$. The choice of an orientation of $W$ gives the manifold $G$ a quaternion-Kähler structure. The latter is characterized by a certain non-degenerate closed 4-form $\Omega$ that arises from the curvature of a locally-defined quaternionic line bundle $H$, and represents the integral cohomology class $4u \in H^4(G, \mathbb{Z})$, where $u = -c_2(H)$ [16].

**Proposition 2.1.** $\beta$ is the pullback of the class $4u$ by means of the embedding cum projection $\mathcal{M} \hookrightarrow \mathcal{F} \to \mathcal{G}$.

**Proof.** From above, $\beta$ is the pullback to $\mathcal{M}$ of $\hat{\beta} = p_1(\hat{W})$. A calculation from [23] shows that

$$p_2(\hat{W}) = (\hat{\beta} - 4u)^2 \in H^8(\mathcal{G}, \mathbb{Z}).$$

Assuming that $g \geq 3$, $b_4(\mathcal{M}) = 2$ and so $\hat{\beta} - 4u$ must pull back to $a\alpha^2 + b\beta$ on $\mathcal{M}$ for some $a, b \in \mathbb{Z}$; from (8), $(a\alpha^2 + b\beta)^2 = 0$. However, the remarks after Corollary 2.1 imply that there are no non-trivial relations involving $\alpha^4, \alpha^2\beta, \beta^2$ in $H^8(\mathcal{M})$ except that

$$0 = c_4(Q) - \alpha c_3(Q) = 4 + 2\alpha^2\beta - 3\beta^2$$

in genus 3. (There are actually four distinct quaternion-Kähler structures on $G_3 = SO(8)/(SO(4) \times SO(4))$, and Proposition 2.1 holds for only two of them.) It follows that in all cases $\beta = 4u$ in $H^4(\mathcal{M})$. □

Any quaternion-Kähler manifold $M$ of dimension $4m$ is the base space of a complex manifold $Z$ (the more usual ‘twistor space’) fibred by rational curves. The positive integer

$$v(M) = \left\langle (4u)^m, [M] \right\rangle = \frac{1}{2(m+1)^{2m+1}} c_1(Z)^{2m+1}$$

(10)

determines the ‘quaternionic volume’ of $M$, and can be expressed in terms of dimensions of representations of the isometry group, using techniques from [16]. For
M = G\_g we have m = 2g - 2, Z ∼ SO(2g + 2)/(SO(2g - 2) × U(2)), and one can prove that
\[ v(G\_g) = \frac{2}{g^2} \left( \frac{4g - 3}{2g - 1} \right); \]
by comparison v(\mathbb{HP}^{2g-2}) = 4^{2g-2}. Theorem 2.2 is in contrast to the non-degenerate nature of the 4-form Ω over G, and reflects the failure of M to map onto a quaternionic subvariety of G.

### 3. Character calculations

In this section, we set \( h = g - 1 \) and consider the holomorphic Euler characteristics
\[ V_h(p, q) = \chi(M_{h+1}, \mathcal{O}(\psi^{p-q}Q \otimes L^{q-1})). \]
Given that \( c_1(M) = 2\alpha \) and the canonical bundle of \( M \) is isomorphic to \( L^{-2} \), Serre duality implies that
\[ V_h(p, q) = (-1)^h V_h(-p, -q), \]
with the convention that \( \psi^{-p}Q = \psi^pQ^* \). Following [26], we set \( w = x + x^{-1} - 2 \) and
\[ F(w, p) = \frac{(x^p - x^{-p})(x - x^{-1})}{x + x^{-1} - 2} = \sum_{h \geq 0} \left[ \frac{4}{2h + 1} \left( p + h \right) + \left( p + h - 1 \right) \right] w^h; \]
in order to define
\[ \frac{F(w, p)}{F(w, q)^2} = \sum_{k \geq 0} G_k(p, q) w^k. \]
The next theorem expresses (11) in terms of this generating function.

**Theorem 3.1.** Let \( h \geq 2 \). Then \( V_h(p, 0) = 0 \) for all \( p \), and
\[ V_h(p, q) = 4(-4q)^h (p(h + 1)G_h(q, q) - qG_h(p, q)), \quad p > 0. \]
In particular, \( V_h(p, q) + V_h(-p, q) = 0 \) for all \( p, q \in \mathbb{Z} \).

The resulting symmetries of \( V_h(p, q) \) are illustrated schematically in Figure 1.

**Corollary 3.1.** Let \( c \) be an integer such that \( 0 \leq c \leq 2 + (g - 2)/q \). Then
\[ V_h(cq, q) = c \sum_{j=1}^{q} (-1)^{j+1} \left( h + 1 - (-1)^{j(c+1)} \left( \frac{q}{\sin^2(j\pi/2q)} \right)^h. \right. \]
Setting \( p = q \) in (11) gives
\[
V_h(q, q) = h \dim H^0(\mathcal{M}, \mathcal{O}(L^{q-1})),
\]
since the higher cohomology spaces are zero by Kodaira vanishing and Serre duality. When \( c = 1 \), the right-hand side of (14) does indeed reduce to the Verlinde formula for the dimension of the space of sections of \( L^{q-1} \). This was first deduced from fusion rules [29]; a direct proof was given by Szenes [26], though many other generalizations now exist [18, 13, 28, 3, 9]. Moreover, the right-hand side of (13) is essentially the generating function for the reciprocal of the Verlinde series. Note also that
\[
\frac{1}{h} V_h(1, 1) = \chi(\mathcal{M}, \mathcal{O}) = \left( \text{td}(\mathcal{M}), [\mathcal{M}] \right) = 1
\]
is the Todd genus of \( \mathcal{M} \). Finally, when \( h = 1 \) we have \( Q \cong L \) and it follows that \( V_1(p, q) = V_1(p, p) \) for all \( p, q \).

\[
\text{Figure 1}
\]

\textbf{Proof of Theorem 3.1.} We follow closely Szenes' proof of the Verlinde formula in [26], and his use of the

\textbf{Lemma 3.1.} = [10, Proposition 4.3] Let \( E \) be a vector bundle of rank \( n \) over a smooth projective variety \( X \), and let \( i: M \to X \) be the zero locus of a non-degenerate section of \( E \). Then \( \chi(M, \mathcal{O}(i^*U)) = \chi(X, \mathcal{O}(U \otimes \Lambda_{-1}E^*)) \) for any vector bundle \( U \) on \( X \).
On a homogeneous space, holomorphic Euler characteristics can be computed by means of the Atiyah-Bott fixed point formula. Let $G$ be a reductive Lie group, $P$ a parabolic subgroup of $G$, and $F = G/P$ the corresponding flag manifold. A representation $R$ of $P$ determines both a holomorphic vector bundle $\mathcal{R} = G \times_P R$ over $F$ and a virtual $G$-module

$$\mathcal{I}(R) = \sum_i (-1)^i H^i(F, \mathcal{O}(R)).$$

Let $\mathbb{T}$ be a common maximal torus of $P$ and $G$, let $W_G, W_P$ be the Weyl groups, and $W_r$ the relative Weyl group. The character of the $G$-module $\mathcal{I}_R$ is given by

$$\text{tr}(\mathcal{I}_R) = \sum_{w \in W_r} w \cdot \frac{\text{tr}(R)}{\text{tr}(\Lambda_{-1} A^*)},$$

where $A$ is the $P$-module associated to the holomorphic tangent bundle $T^{1,0}F = A$. The right-hand side of (16) is a function on $\mathbb{T}$, and evaluation at the identity element yields

$$\text{tr}(\mathcal{I}_R)|_e = \chi(F, \mathcal{O}(\mathcal{R})).$$

Returning to the problem at hand, let $H$ denote the subgroup $U(h) \times SO(4)$ of $G = SO(2g + 2)$, where $h = g - 1$. If $B$ denotes the fundamental representation of $U(h)$, then the vector bundles $B^*$ and $\det B^*$ over $F = SO(2g + 2)/H$ pull back to $Q$ and $L$ respectively over $\mathcal{M}$. Lemma 3.1 therefore implies that

$$\chi(\mathcal{M}, \mathcal{O}(\psi^{p-q}Q \otimes L^{q-1})) = \chi(F, \mathcal{O}(\mathcal{R})), $$

where

$$R = R^{p,q} = \psi^{p-q}B^* \otimes (\det B^*)^{q-1} \otimes \Lambda_{-1}(S^2B).$$

Now we proceed to calculate (16). Let $x_1, \ldots, x_{g+1}$ be the characters of the maximal torus of $SO(2g + 2)$ corresponding to the polarisation $\{yj_{j-1} + iy_{j+1} : 1 \leq j \leq g + 1\}$ of $\mathbb{C}^{2g+2}$. The character of the fundamental $SO(2g + 2)$-module is $\sum_{j=1}^{h+2}(x_j + x_j^{-1})$, and that of the fundamental $U(h)$-module $\sum_{j=1}^{h} x_j^{-1}$, Thus,

$$\text{tr}(R^{p,q}) = \prod_{i \leq h} x_i^{-1} \prod_{1 \leq j \leq k \leq h} \left(1 - \frac{1}{x_j x_k}\right) \sum_{\ell=1}^{h} x_{\ell}^{p-q},$$

and from [26],

$$\text{tr}(\Lambda_{-1} A^*) = \prod_{1 \leq i < j \leq h} \left(1 - \frac{1}{x_i x_j}\right) \prod_{1 \leq k \leq h} \left(1 - \frac{1}{x_{h+\varepsilon} x_k}\right) \left(1 - \frac{x_{h+\varepsilon}}{x_k}\right),$$

so it suffices to prove that

$$V_h(p, q) = \lim_{\{x_i \to 1\}} \sum_{w \in W_r} w \cdot \frac{\text{tr}(R^{p,q})}{\text{tr}(\Lambda_{-1} A^*)}. $$

(17)
We have that
\[
\frac{\text{tr}(R^{p,q})}{\text{tr}(\Lambda_{-1}A^*)} = \left( \prod_{i \leq h} \frac{x^q_i(x_i - x_i^{-1})}{(x_i + x_i^{-1} - x_{i+1} - x_{i+1}^{-1})(x_i + x_i^{-1} - x_{i+2} - x_{i+2}^{-1})} \right) \sum_{j \leq h} x^{p-q}_j.
\]
To perform the summation in (17) as in [26] we first recall the form of the relative Weyl group \(W_r\) of \(W_{SO(2h+4)}\) with respect to \(W_{U(h)}\) and \(W_{SO(4)}\). It has \(2^h\binom{h+2}{2}\) elements, and

\[W_r = W^\text{signs} \times W^\text{perms}\]

where \(W^\text{signs}\) consists of all the substitutions \(x_i \mapsto x_i^{-1}\) of an even number of variables modulo \(\{x_{h+1} \mapsto x_{h+1}^{-1}, x_{h+2} \mapsto x_{h+2}^{-1}\}\), and \(W^\text{perms}\) consists of all the permutations of two variables. Adding up first with respect to \(W^\text{signs}\) we get

\[
\left( \prod_{i \leq h} \frac{(x_i - x_i^{-1})}{(x_i + x_i^{-1} - x_{i+1} - x_{i+1}^{-1})(x_i + x_i^{-1} - x_{i+2} - x_{i+2}^{-1})} \right) \sum_{j \leq h} (x^p_j - x^q_j) \prod_{i \neq j} (x^q_j - x^q_j),
\]
setting \(x_{h+2} \mapsto 1\) and then adding up with respect to \(W^\text{perms}\) gives a contribution

\[
\sum_{j=1}^{h+1} \left( \prod_{i \neq j} \frac{x_i - x_i^{-1}}{(x_i + x_i^{-1} - 2)(x_i + x_i^{-1} - x_j - x_j^{-1})} \right) \sum_{k \neq j} \left( (x_k^p - x_k^{-p}) \prod_{l \neq k} (x_l^q - x_l^{-q}) \right).
\]

Substituting \(w_i = x_i + x_i^{-1} - 2\), and rearranging the terms converts the above summation into

\[
(18) \quad \left( \sum_{j=1}^{h+1} F(w_j, p) \prod_{i \neq j} F(w_i, q) \right) \sum_{j=1}^{h+1} \frac{(-1)^{j-1} \text{V}m(w_1, \ldots, \hat{w}_j, \ldots, w_{h+1})}{F(w_j, q) \text{V}m(w_1, \ldots, w_{h+1})}
\]

\[
- \left( \prod_{i=1}^{h+1} F(w_i, q) \right) \sum_{j=1}^{h+1} \frac{(-1)^{j-1} F(w_j, p) \text{V}m(w_1, \ldots, \hat{w}_j, \ldots, w_{h+1})}{F(w_j, q)^2 \text{V}m(w_1, \ldots, w_{h+1})},
\]
where \(\text{V}m\) denotes the Vandermonde determinant. Since \(\lim_{x \to 1} F(w, q) = 4q\), the first factors in both summands converge as the \(x_i\) tend to 1. By [26, Lemma 5.3] the first summand tends to \((4q)^h 4p(h+1)G_h(q, q)\), and the second to \((4q)^h 1G_h(p, q)\).

**Proof of Corollary 3.1.** The hypothesis on \(c\) implies that the meromorphic form

\[
F(w, cq) dw
\]

over \(\mathbb{C}P^1\) has no poles at 0 and \(\infty\). The result is then a consequence the residue theorem and [26, Lemma 5.3].
4. Further relations

The identity
\[ V(p, 0) = 0, \quad p \in \mathbb{Z}, \]
(19)
of Theorem 3.1 is also an easy consequence of Theorems 2.2, 2.3 and the Hirzebruch-Riemann-Roch theorem. For the latter implies that
\[ V(p, 0) = \left\langle \text{ch}(\psi^p Q \otimes L^*), [\mathcal{M}] \right\rangle = \left\langle \text{ch}(\psi^p Q)\hat{A}(\mathcal{M}), [\mathcal{M}] \right\rangle, \]
and the \( \hat{A} \) class
\[ \hat{A}(\mathcal{M}) = \left( \frac{1}{2\sqrt{\beta}} \right)^{2g-2} \left( \frac{1}{\sinh \frac{1}{2} \sqrt{\beta}} \right)^{2g-2}, \]
readily computed from (6) and (8), is a polynomial in \( \beta \).

The identity (20) was used by Thaddeus to show that the Verlinde formula (Corollary 3.1 with \( c = 1 \)) determines the intersection form on \( \mathcal{M} \). In [27, Equation (30)], he gives the intersection numbers
\[ \left\langle \alpha^m \beta^n \gamma^p, [\mathcal{M}] \right\rangle = (-1)^{p-g} \frac{g!m!}{(g-p)!q!} 2^{2g-2-p}(2^q - 2)B_q, \]
(21)
where \( m + 2n + 3p = 3g - 3 \), \( q = m + p - g + 1 \), and \( B_q \) is the qth Bernoulli number (equal to \( q! \) times the coefficient of \( x^q \) in \( x/(e^x - 1) \)). Another key point in the argument is [27, Proposition (26)], namely that \( \gamma \) is Poincaré dual to \( 2g \) copies of \( \mathcal{M}_{g-1} \), so that
\[ \left\langle \alpha^m \beta^n \gamma^p, [\mathcal{M}] \right\rangle = 2g\left\langle \alpha^m \beta^n \gamma^{p-1}, [\mathcal{M}_{g-1}] \right\rangle, \quad m + 2n + 3p = 3g - 3. \]
(22)

In this section, we shall show that the intersection numbers (21) are in fact determined by (19) and the identities
\[ V_h(0, p) = 0, \]
(23)
\[ V_h(p, p) + V(-p, p) = 0, \quad p \in \mathbb{Z} \]
(24)
that follow from Theorem 3.1, or rather its proof. Following closely the notation of Donaldson [8, §5], we set
\[ I_k^{(g)} = \frac{1}{(g-1+2k)!}\left\langle \alpha^{g-1+2k} \beta^{g-1-k}, [\mathcal{M}] \right\rangle, \]
and also
\[ I^{(g)}(t) = \sum_{k=0}^{g-1} I_k^{(g)} t^{2k}. \]

**Theorem 4.1.** \( I^{(g)}(t) = (-4)^{g-1} \frac{t}{\sinh t} \).
Proof. Interpreting (19) as a polynomial identity in \( p \), using the Hirzebruch-Riemann-Roch theorem and (22), yields the equation

\[
 t^2 \frac{d}{dt} \left( \frac{I^g(t) \sinh t}{t} \right) = g(t - \sinh t)(I^g(t) + 4I^{(g-1)}(t)) \tag{25}
\]

modulo \( t^{2g} \). Similarly, (24) gives

\[
 t^2 \frac{d}{dt} \left( \frac{I^g(t) \sinh t}{t} \right) + tI^g(t)(1 - \cosh 2t) = g(2t - \sinh 2t)(I^g(t) + 4I^{(g-1)}(t)),
\]

which can be simplified into

\[
 2t^2 \frac{d}{dt} \left( \frac{I^g(t) \sinh t}{t} \right) = g(2t - \sinh(2t))(I^g(t) + 4I^{(g-1)}(t)). \tag{26}
\]

Both sides of (25) and (26) must now be identically zero, and ignoring the modulo \( t^{2g} \),
\[ I^g(t) = \frac{C(g)}{\sinh t} \]
where \( C(g) \) is a function of \( g \) such that \( C(g) + 4C(g-1) = 0 \).

But, using the description (1) of \( M_2 \) as the intersection of two quadrics in \( \mathbb{CP}^5 \), we find that \( C(2) = -\langle \alpha^3, [\mathcal{M}_2] \rangle = -4 \). It also follows now that \( I^g(t) = (-4)^{g-1}K(t^2) \), where \( K \) is the generating function of the intersection pairings described in [8].

Let \( \mathcal{N}_g \) denote the moduli space of semistable rank 2 vector bundles with fixed even degree determinant over a compact Riemann surface \( \Sigma_g \) of genus \( g \), and let \( L_0 \) be the generator of \( \text{Pic}(\mathcal{N}_g) \). An easy application of the above knowledge of \( I^g(t) \) now yields

**Theorem 4.2.** [4, Theorem 1.1] \( \dim H^0(\mathcal{N}_g, L_0^{k-2}) = \sum_{j=1}^{k-1} \left( \frac{k}{1 - \cos(2j\pi/k)} \right)^{g-1} \).

Proof. Let \( \mathcal{U} \) be a universal rank 2 bundle over \( \mathcal{M}_g \times \Sigma_g \), so that if \( m \in \mathcal{M} \) represents the bundle \( E \to \Sigma \) then \( \mathcal{U}|_{\mathcal{M} \times \{m\} \times \Sigma} \cong E \); this may be chosen such that if \( U_x = \mathcal{U}|_{\mathcal{M} \times \{x\}} \) then \( \det(U_x) \cong L \) over \( \mathcal{M} \). Bertram and Szenes [4] use a Hecke correspondence to prove that

\[ \dim H^0(\mathcal{N}_g, L_0^k) = \chi(\mathcal{M}_g, S^kU_x). \]

Note that \( c(U_x) = 1 + \alpha + \frac{1}{4} (\alpha^2 - \beta) \), so that \( c(U_x \otimes L^{-1/2}) = 1 - \beta/4 \), and

\[ \mathrm{ch}(S^k(U_x \otimes L^{-1/2})) = \frac{\sinh((k + 1)\sqrt{\beta}/2)}{\sinh(\sqrt{\beta}/2)}. \]

[13] Then we need to evaluate

\[ \left\langle \frac{\sinh((k + 1)\sqrt{\beta}/2)}{\sqrt{\beta}/2}e^{(k+2)\alpha/2} \left( \frac{\sqrt{\beta}/2}{\sinh(\sqrt{\beta}/2)} \right)^{2g-1}, [\mathcal{M}] \right\rangle. \]
Using Theorem 4.1, this is readily seen to equal
\[
\text{Res}_{t=0} \left[ \left( \frac{k+2}{-2\sinh^2(t/(k+1))} \right)^{g-1} \frac{k+1}{k+2} \frac{\sinh t}{\sinh((k+2)t/(k+1)) \sinh(t/(k+1))} \right] dt \\
= \sum_{j=1}^{k+1} \left( \frac{k+2}{1 - \cos(2\pi j/(k+2))} \right)^{g-1} \sqrt{-1} \text{Res}_{u=\pi j} \left[ \frac{\sin((k+1)u/(k+2))}{\sin u \sin(u/(k+2))} \right] du \\
= \sum_{j=1}^{k+1} \left( \frac{k+2}{1 - \cos(2\pi j/(k+2))} \right)^{g-1}.
\]

In the last part of this paper, we shall combine separate uses of the Adams operators, namely Lemma 2.1 and the identities (23),(24) to express the intersection form of the smooth moduli space \(M = M_g\) in an alternative form. The bundle \(Q\) was defined geometrically only in the hyperelliptic case, and to de-emphasise its role at this stage we consider in addition
\[
\tilde{T} = T^* - g + 1 \in K(M).
\]
Note that this has virtual rank \(2g - 2\) and vanishing higher Chern classes \(c_i(\tilde{T}) = 0\) for \(i > 2g - 2\) by Gieseker’s theorem [11, 32].

Let us say that a cohomology class \(\delta \in H^*(M)\) is saturated if \(\langle \delta \beta^j, [M] \rangle = 0\) for all \(j \geq 0\). Thus, any polynomial in \(\beta\) is itself saturated. With this terminology,

**Proposition 4.1.** \(\text{ch}(Q^* \otimes L)\) and \(\text{ch}(\tilde{T} \otimes L)\) are both saturated, i.e.
\[(a) \quad \langle \text{ch}(Q^*) e^\alpha \beta^j, [M] \rangle = 0, \quad j \geq 0; \quad (b) \quad \langle \text{ch}(\tilde{T}) e^\alpha \beta^j, [M] \rangle = 0, \quad j \geq 0.\]

**Proof.** Since \(\psi^p\) is a ring homomorphism in K-theory, (23) implies
\[0 = \langle \text{ch}(\psi^p Q^* \otimes L^{-1}) \text{td}(M), [M] \rangle = \langle \text{ch}(\psi^p (Q^* \otimes L)) \hat{A}(M), [M] \rangle.\]
Equation (a) follows from the fact that the identity above is true for all \(p \in \mathbb{Z}\). Now consider the decomposition
\[(T^* - g + 1) \otimes L = Q^* \otimes L \otimes W - (\psi^2 Q^* + g - 1) \otimes L.\]
(8) and part (a) imply that \(\text{ch}(Q^* \otimes L \otimes W)\) is saturated. It therefore suffices to show that \(\text{ch}((\psi^2 Q^* + g - 1) \otimes L)\) is saturated, but given that
\[\psi^p((\psi^2 Q + g - 1) \otimes L^*) = \psi^{2p} Q \otimes L^{-p} + (g - 1)L^{-p},\]
this follows from (24). \[\square\]
The equations of Proposition 4.1 for \( j \geq g - 1 \) follow immediately from Theorems 2.2 and 2.3, though taking \( j = 0, \ldots, g - 2 \) gives independent relations. For example, expanding (a) shows that

\[
2kI_k^{(g)} + (g + 2k) \sum_{i=1}^{k} \frac{f_{k-i}^{(g)}}{(2r+1)!}
\]

is a linear combination of pairings \( \langle \alpha^m \beta^n \gamma, [\mathcal{M}] \rangle \) for each \( k \) with \( 1 \leq k \leq g - 1 \). The intersection numbers (21) can then be determined by the equations of Proposition 4.1(a) (with Theorem 2.3), (15) and (22) by induction on the genus \( g \).

In view of the Riemann-Roch equation

\[
\chi(\mathcal{M}, T^*) - g + 1 = \left\langle \text{ch}(T) e^\alpha \hat{A} \mathcal{M}, [\mathcal{M}] \right\rangle
\]

that follows from (15), Proposition 4.1(b) correctly predicts the coefficient

\[
\chi(\mathcal{M}, T^*) = g - 1
\]

of \( t \) in the polynomial \( \chi_t = (1 - t)^{g-1} (1 + t)^{2g-2} \) computed in [17]. This observation suggests that there should exist more direct proofs of Proposition 4.1.

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