FREE DENSE SUBGROUPS OF HOLOMORPHIC AUTOMORPHISMS

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Abstract. We show the existence of free dense subgroups, generated by 2 elements, in the holomorphic shear and overshear group of complex-Euclidean space and extend this result to the group of holomorphic automorphisms of Stein manifolds with Density Property, provided there exists a generalized translation. The conjugation operator associated to this generalized translation is hypercyclic on the topological space of holomorphic automorphisms.

1. Introduction

In Functional Analysis and Topological Dynamics, the phenomenon of so-called hypercyclicity of operators and universality of sequences of operators is studied. For a detailed survey, we refer to Grosse-Erdmann [8]. One of the early examples actually comes from Complex Analysis; in 1929 G.D. Birkhoff [4] constructed a holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) such that for a given sequence \( \{a_m\}_{m=1}^{\infty} \subset \mathbb{R} \) with \( \lim_{m \to \infty} a_m = \infty \) the set \( \{ z \mapsto f(z + a_m), m \in \mathbb{N} \} \) is dense (in compact-open topology) in \( \mathcal{O}(\mathbb{C}) \). It follows in particular that for any translation \( \tau : \mathbb{C} \to \mathbb{C}, \tau \not\equiv \text{id} \), the set \( \{ f \circ \tau^m : m \in \mathbb{N} \} \) is dense in \( \mathcal{O}(\mathbb{C}) \). This motivated the following definition:

Definition 1.1. Let \( E \) be a topological space.

A self-map \( T : E \to E \) is called hypercyclic if there exists an \( f \in E \), called hypercyclic element for \( T \), such that the orbit \( \{ T^m(f), m \in \mathbb{N} \} \) is dense in \( E \).

The result of Birkhoff was later generalized to the case of holomorphic functions on \( \mathbb{C}^n \) and more recently by Zajac [17] also to holomorphic functions on pseudo-convex domains of \( \mathbb{C}^n \); he gives (in slightly more general form) the following Theorem which characterizes hypercyclic composition operators:

Theorem 1.2 (Theorem 3.4. in [17]). Let \( \Omega \subset \mathbb{C}^n \) by a pseudo-convex domain and let \( \tau \in \mathcal{O}(\Omega) \) be a holomorphic selfmap. By \( C_\tau : \mathcal{O}(\Omega) \to \mathcal{O}(\Omega) \) we denote the composition operator \( f \mapsto f \circ \tau \). The operator \( C_\tau \) is hypercyclic if and only if \( \tau \) is injective and for every \( \Omega \)-convex compact subset \( K \subset \Omega \) there exists \( m \in \mathbb{N} \) such that \( \tau^m(K) \cap K = \emptyset \) and \( K \cup \tau^m(K) \) is \( \Omega \)-convex.

We now want to consider groups of holomorphic automorphisms and introduce the following notation:

Definition 1.3.

(1) Let \( X \) be a complex manifold. By \( \text{Aut}(X) := \{ f : X \to X \text{ biholomorphic} \} \) we denote the group of holomorphic automorphisms of \( X \), to be understood as topological group with the compact-open topology; and by \( \text{Aut}^*(X) \) its identity component.

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Let X be a complex manifold of dimension $n \in \mathbb{N}$. A holomorphic $n$-form $\omega$ on X which is everywhere non-degenerate is called a holomorphic volume form. By $\text{Aut}_\omega(X) := \{f : X \rightarrow X \text{ biholomorphic} : f^*(\omega) = \omega\}$ we denote the group of volume-preserving holomorphic automorphisms of X, and by $\text{Aut}_\omega^c(X)$ its identity component.

These topological groups are metrizable with the metric of uniform convergence. However, these metric spaces will not be Cauchy-complete, as one can see for example in case of $\text{Aut}(\mathbb{C}^n)$ and a sequence of automorphisms $z \mapsto M_k \cdot z$, where $M_k, k \in \mathbb{N}$, is a complex matrix with $\det M_k \neq 0$ but $\lim_{k \rightarrow \infty} \det M_k = 0$. The phenomenon of hypercyclicity is usually studied on separable Fréchet spaces or at least on complete metric spaces.

The results concerning hypercyclicity of compositions operators on $\mathcal{O}(\mathbb{C}^{n-1})$ can be applied directly to the shear and overshear groups of $\mathbb{C}^n$. The importance of these groups lies in the fact that they are dense subgroups of $\text{Aut}(\mathbb{C}^n)$ resp. $\text{Aut}_\omega(\mathbb{C}^n)$, $\omega = dz_1 \wedge \cdots \wedge dz_n$, as shown by Andersén [2] and Andersén–Lempert [3].

A holomorphic overshear $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ has, after possibly a complex-linear change of coordinates, the form

$$F(z_1, \ldots, z_n) = (z_1, z_2, \ldots, z_{n-1}, \exp(f(z_1, \ldots, z_{n-1})) \cdot z_n + g(z_1, \ldots, z_{n-1}))$$

where $f, g \in \mathcal{O}(\mathbb{C}^{n-1})$. It is called a shear if $f \equiv 0$; in this case $\det(dF) = 1$ which is equivalent to $f^*(\omega) = \omega$. The (over-)shear group is the group generated by all the (over-)shears.

Using Theorem 1.2 for $\Omega = \mathbb{C}^{n-1} \subseteq \mathbb{C}^{n-1}$ with $\tau$ being a translation, and applying this to every holomorphic direction, one obtains immediately that there are $n+1$ (over-)shears which generate a dense subgroup of the (over-)shear group. However, in Theorem 2.6 we show that it is actually possible to generate dense subgroups by only 2 automorphisms of a special form, one of them being a translation. Moreover we are able to show that this dense subgroup is generated freely by those 2 automorphisms.

The conditions for $\tau$ which had been used already for Theorem 1.2 motivate the following definition which we use for the formulation of our main theorem.

**Definition 1.4.** Let X be a complex manifold and $\tau \in \text{Aut}(X)$. We call $\tau$ a generalized translation if for any holomorphically convex compact $K \subseteq X$ there exists an $m \in \mathbb{N}$ such that for the iterate $\tau^m$ it holds that

1. $\tau^m(K) \cap K = \emptyset$ and
2. $\tau^m(K) \cup K$ is $\mathcal{O}(X)$-convex.

An obvious example for $X = \mathbb{C}^n$ is the translation $z \mapsto z + a$, $a \neq 0$.

**Theorem 3.6.** Let X be a Stein manifold with density property and assume there exists a generalized translation $\tau \in \text{Aut}^*(X)$. Then there exists an $F \in \text{Aut}^*(X)$ such that $(\tau, F)$ is a free and dense (in compact-open topology) subgroup of $\text{Aut}^*(X)$.

From the point of view of hypercyclicity we can also state

**Theorem 3.9.** Let $X$ be a Stein manifold with density property and assume there exists a generalized translation $\tau \in \text{Aut}^*(X)$. Then the associated conjugation operator $\tilde{C}_\tau$, defined by $\tilde{C}_\tau(f) = \tau \circ f \circ \tau^{-1}$ is hypercyclic on $\text{Aut}^*(X)$.

The notion of Density Property and so-called Andersén–Lempert Theory is introduced in Section 3. Manifolds with density property in particular include $\mathbb{C}^n$, $n \geq 2$, certain homogeneous spaces and all linear algebraic groups except those with connected components $\mathbb{C}$ or $(\mathbb{C}^*)^m$. The density property of Stein manifolds implies
that the automorphism group is large, in particular infinite dimensional and acts
m-transitively for any $m \in \mathbb{N}$ (Varolin [16]).

2. Complex Euclidean Space

Shears as a special kind of holomorphic automorphisms of $\mathbb{C}^n$ have been intro-
duced by Rosay and Rudin [14]. For a more recent treatment we refer to the
textbook of Forsternič [7].

Definition 2.1. A map

$$F(z_1, \ldots, z_n) = (z_1, z_2, \ldots, z_{n-1}, \exp(f(z_1, \ldots, z_{n-1})) \cdot z_n + g(z_1, \ldots, z_{n-1}))$$

where $f, g \in O(\mathbb{C}^{n-1})$, is called an overshear in direction of the $n$-th coordinate
axis (or a shear in direction of the $n$-th coordinate axis if $f \equiv 0$). For $A \in SL(\mathbb{C}^n)$
the conjugates $A^{-1} \circ F \circ A$ are called overshears (or shears if $f \equiv 0$). The group
generated by the shears resp. overshears is called the shear group resp. overshear
group.

Remark 2.2. The group generated by (over-)shears in all coordinate directions
coincides with the (over)-shear group.

For our purposes it will be convenient to introduce the following notion:

Definition 2.3. Set $I : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $n \geq 2$, to be

$$I(z_1, \ldots, z_n) := (z_2, \ldots, z_n, -(−1)^n z_1)$$

A map $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ which can be written as $F = G \circ I$, where $F$ is an (over-)shear
is called a twisted (over-)shear.

Lemma 2.4. The groups generated by (over-)shears and twisted (over-)shears co-
icide.

Proof. By definition, $I$ is a twisted shear, hence every (over-)shear $G$ can be written
as $G = F \circ I^{-1}$ where $F = G \circ I$ is a twisted (over-)shear. It remains to show
that $I$ is a composition of shears: it is sufficient to show that any transposition $t$
of coordinates with a sign change can be written as a composition of shears, for
simplicity $t(z_1, \ldots, z_n) = (z_1, \ldots, z_{n-1}, -z_n)$. Let

$$A(z_1, \ldots, z_{n-1}, z_n) := (z_1, \ldots, z_{n-1}, z_n + z_{n-1})$$
$$B(z_1, \ldots, z_{n-1}, z_n) := (z_1, \ldots, z_{n-1} - z_n, z_n)$$

Then $t = A^{-1} \circ B \circ A$. □

For the following application of hypercyclicity we will need a slightly different
statement than in Theorem 1.2. Despite that the method of proof is essentially the
same and follows the original idea of G.D. Birkhoff [4], we include the proof for a
lack of reference.

Proposition 2.5. Let $X$ be a Stein manifold and let $\tau : X \rightarrow X$ be a generalized
translation. Then there exists a pair of holomorphic functions $f, g \in O(X)$ such
that for any pair of holomorphic functions $\hat{f}, \hat{g} \in O(X)$ there is a subsequence
$\{m_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{N}$ such that

$$f \circ \tau^{m_\ell-1} \rightarrow \hat{f},$$
$$g \circ \tau^{m_\ell-1} \rightarrow 0,$$
$$f \circ \tau^{m_\ell} \rightarrow 0 \text{ and } g \circ \tau^{m_\ell} \rightarrow \hat{g}$$

in compact-open topology.
Theorem 2.6.
Analogous estimates hold for the other three cases.
□

For any translation \( \tau \neq 0 \) of \( \mathbb{C}^n \), \( n \geq 2 \), there exists a holomorphic automorphism \( F \) of \( \mathbb{C}^n \) which is conjugate to a twisted shear map such that \( \langle \tau, F \rangle \) is a free and dense (in compact-open topology) subgroup of the shear group of \( \mathbb{C}^n \).

(2) For any translation \( \tau \neq 0 \) of \( \mathbb{C}^n \), \( n \geq 2 \), there exists a holomorphic automorphism \( f \) of \( \mathbb{C}^n \) which is conjugate to a twisted overshear map such that \( \langle \tau, F \rangle \) is a free and dense (in compact-open topology) subgroup of the overshear group of \( \mathbb{C}^n \).
Proof. After conjugating with a special complex-linear map, we may assume that
\[ \tau(z) = \tau(z_1, \ldots, z_n) = (z_1 + b, \ldots, z_n + b), \quad b \in \mathbb{R}_{>0} \]
We then define the following elements of the overshear group of \( \mathbb{C}^n \):

\[ I(z) := (z_2, \ldots, z_n, -(1)^n z_1) \]
\[ F_{f,g}(z) := (z_2, \ldots, z_n, -(-(1)^n\exp(f(z_2, \ldots, z_n)) \cdot z_1 + g(z_2, \ldots, z_n) + (1 - (-1)^n)z_n) \]

where \( f, g \in \mathcal{O}(\mathbb{C}^{n-1}) \). Note that for \( f \equiv 0 \), these two elements actually belong to the shear group. The different signs depending on the dimension ensure that the Jacobian is equal to 1.

1. We first treat the shear group, where \( f \equiv 0 \). By Theorem 1.2 we may choose \( g \) to be a hypercyclic element for the composition operator associated to the translation \((z_2, \ldots, z_n) \mapsto (z_2 + b, \ldots, z_n + b)\) of \( \mathbb{C}^{n-1} \). For the conjugates of \( F_{0,g} \) by the \( m \)-th iteration of \( \tau \) we obtain:

\[ (\tau^{-m} \circ F_{0,g} \circ \tau^m)(z_1, \ldots, z_n) = (z_1, \ldots, z_n, -(1)^n z_1 + g(z_2 + mb, \ldots, z_n + mb) + 2z_n) \]

Thus, by hypercyclicity of the composition operator we obtain that the set \( \{ \tau^{-m} \circ F_{0,g} \circ \tau^m, m \in \mathbb{N} \} \) is dense in \( \{ F_{0,h}, h \in \mathcal{O}(\mathbb{C}^{n-1}) \} \). In particular it is possible to approximate the map \( I \) by compositions of \( F_{0,g} \) and \( \tau \), hence also \( I^n = (-1)^n 0 \) and \( I^{-1}(z) = I^{2n-1}(z) = (-1)^n z_1, \ldots, z_{n-1} \).

Conjugating with \( I^1, \ldots, I^{n-1} \) will give twisted shears in all directions, and \( F_{0,h} \circ I^{-1} \) is an ordinary shear. Therefore all shears and their compositions can be approximated this way.

2. For the overshear group we proceed in exactly the same way. By Proposition 2.3 using an ordinary translation \( \tau : \mathbb{C}^n \to \mathbb{C}^n \) it is possible to approximate all twisted overshears of the form \( F_{0,\hat{g}} \) and \( F_{f,\hat{g}} \) and hence also \( F_{f,\hat{g}} = F_{0,\hat{g}} \circ I^{-1} \circ F_{f,\hat{g}} \).

To ensure that the generated group is actually a free group, we need to show that no reduced word formed by \( F_{f,\hat{g}} \) and \( \tau \) can equal the identity. This follows from an application of Nevanlinna theory similar to a degree argument as in Ahern and Rudin [1]. \( \square \)

Corollary 2.7.

1. For any translation \( \tau \neq 0 \) of \( \mathbb{C}^n, n \geq 2 \), there exists an \( F \in \text{Aut}_\omega (\mathbb{C}^n) \) such that \( (\tau, F) \) is a free and dense (in compact-open topology) subgroup of \( \text{Aut}_\omega (\mathbb{C}^n) \), where \( \omega = d z_1 \wedge \cdots \wedge d z_n \).

2. For any translation \( \tau \neq 0 \) of \( \mathbb{C}^n, n \geq 2 \), there exists an \( F \in \text{Aut}(\mathbb{C}^n) \) such that \( (\tau, F) \) is a free and dense (in compact-open topology) subgroup of \( \text{Aut}(\mathbb{C}^n) \).

Proof. By a result of Andersén [2], the group of holomorphic shears is dense in \( \text{Aut}_\omega (\mathbb{C}^n), \omega = d z_1 \wedge \cdots \wedge d z_n \), and by the corresponding result of Andersén–Lempert [3], the group of holomorphic overshears is dense in \( \text{Aut}(\mathbb{C}^n) \). \( \square \)

3. Density Property

The density property was introduced in Complex Geometry by Varolin [13, 16]. For a survey about the current state of research related to density property and Andersén–Lempert theory, we refer to Kaliman and Kutzschebauch [9].
Definition 3.1. A complex manifold $X$ has the density property if in the compact-open topology the Lie algebra $\text{Lie}_{\text{hol}}(X)$ generated by completely integrable holomorphic vector fields on $X$ is dense in the Lie algebra $\text{VF}_{\text{hol}}(X)$ of all holomorphic vector fields on $X$.

Definition 3.2. Let a complex manifold $X$ be equipped with a holomorphic volume form $\omega$. We say that $X$ has the volume density property with respect to $\omega$ if in the compact-open topology the Lie algebra $\text{Lie}_{\omega,\text{hol}}(X)$ generated by completely integrable holomorphic vector fields $\nu$ such that $\nu(\omega) = 0$, is dense in the Lie algebra $\text{VF}_{\omega,\text{hol}}(X)$ of all holomorphic vector fields that annihilate $\omega$.

The following theorem is the central result of Andersén–Lempert theory (originating from works of Andersén and Lempert [2], [3]), has been stated by Varolin [15, 16] after introducing the density property and is given in the following form in [9] by Kaliman and Kutzschebauch, but essentially (for $\mathbb{C}^n$) proven already in [6] by Forstnerič and Rosay.

Theorem 3.3 (Theorem 2 in [9]). Let $X$ be a Stein manifold with the density (resp. volume density) property and let $\Omega$ be an open subset of $X$. In case of volume density property further assume that in de-Rham cohomology $H^{n-1}(\Omega, \mathbb{C}) = 0$. Suppose that $\Phi : [0, 1] \times \Omega \to X$ is a $C^1$-smooth map such that

1. $\Phi_t : \Omega \to X$ is holomorphic and injective (and resp. volume preserving) for every $t \in [0, 1]$.
2. $\Phi_0 : \Omega \to X$ is the natural embedding of $\Omega$ into $X$.
3. $\Phi_t(\Omega)$ is a Runge subset of $X$ for every $t \in [0, 1]$.

Then for each $\varepsilon > 0$ and every compact subset $K \subset \Omega$ there is a continuous family $\alpha : [0, 1] \to \text{Aut}(X)$ of holomorphic (and resp. volume preserving) automorphisms of $X$ such that

$$\alpha_0 = \text{id} \quad \text{and} \quad \sup_K d(\alpha_t, \Phi_t) < \varepsilon$$

for every $t \in [0, 1]$.

Examples 3.4.

1. $\mathbb{C}^n$, $n \geq 2$, have the density property.
2. $\mathbb{C}^* \times \mathbb{C}^*$ has the volume density property for the holomorphic volume form $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$. Whether it has the density property is not clear.
3. Homogeneous Stein manifolds $X = G/K$, where $G$ is a semi-simple Lie group have the density property. (see [5]).
4. Linear algebraic groups except those with $\mathbb{C}$ and $(\mathbb{C}^*)^n$ as a connected component have the density property, and all linear algebraic groups have the volume density property with respect to the Haar form (see [12], [10]). Note that examples 1 and 2 are special cases of linear algebraic groups.
5. A hypersurface $H \subset \mathbb{C}^{n+2}$ of the form $H = \{(x, u, v) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C} : f(x) = u \cdot v\}$ where $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial with smooth zero fiber, has both the density property and the volume density property with respect to a unique algebraic volume form (see [12], [11]).

We will also need the following lemma which makes the distinction between connected and path-connected component superfluous for groups of holomorphic automorphisms.

Lemma 3.5. The connected component of the group of holomorphic automorphisms resp. volume-preserving holomorphic automorphisms of a complex manifold resp. complex manifold with a holomorphic volume form is $C^1$-path-connected.

Proof. See Lind [13, Remark 6.6]."
Theorem 3.6. Let \( X \) be a Stein manifold with density property and assume there exists a generalized translation \( \tau \in \text{Aut}^*(X) \). Then there exists an \( F \in \text{Aut}^*(X) \) such that \( (\tau, F) \) is a free and dense (in compact-open topology) subgroup of \( \text{Aut}^*(X) \).

Proof. Because \( X \) is Stein, there exists an exhaustion \( \{ L_j \}_{j \in \mathbb{N}} \) of \( X \) with \( \mathcal{O}(X) \)-convex compacts.

We choose a countable dense subset (in compact-open topology) \( \{ g_j \}_{j \in \mathbb{N}} \subset \text{Aut}^*(X) \). Such a set exists, since we can think of \( \text{Aut}^*(X) \subset \mathcal{O}(X, X) \subset \mathcal{O}(\mathbb{C}^n, \mathbb{C}^n) \) as metric spaces, where we take \( X \) to be properly embedded into \( \mathbb{C}^n \), and \( \mathcal{O}(\mathbb{C}^n, \mathbb{C}^n) \) is known to be a separable Fréchet space.

We now construct \( F \) inductively as a sequence of maps \( F_j, j \in \mathbb{N} \), together with sequences \( \{ \varepsilon_j \}_{j \in \mathbb{N}} \subset \mathbb{R}_{>0} \) with \( \sum_{j=1}^{\infty} \varepsilon_j < \infty \), \( \{ k(j) \}_{j \in \mathbb{N}}, k(j) \in \mathbb{N}, k(j) < k(j+1) \), and \( m_j \in \mathbb{N} \), such that the following hold at step \( j \):

\[
(a_j) \sup_{L_{k(j)}} d(\tau^{-m_j} \circ F_j \circ \tau^{m_j}, g_j) < \varepsilon_i, \forall i \leq j,
\]

\[
(b_j) \sup_{L_{k(j)}} (d(F_j, F_{j-1}) + d(F_{j-1}^{-1}, F_{j-1}^{-1})) < \varepsilon_j.
\]

We set \( F_1 = \text{Id}, \varepsilon_1 = k(1) = m(1) = 1 \). Assuming that \( g_1 = \text{id} \) we have \((a_1)\) and \((a_2)\) is void.

Assume now that we have constructed \( F_{j-1} \) along with the integers for some \( j \geq 2 \).

Choose \( k(j) \) so large that

\[
\tau^{m_j}(L_{k(j)}) \subset L_{k(j)}^2 \text{ for all } i < j.
\]

By Lemma 5.3 there exist \( C^1 \)-paths \( [0, 1] \ni t \to \varphi_t^j, \psi_t^j \in \text{Aut}^*(X) \) connecting \( F_{j-1} \) and \( g_j \) respectively to the identity. Let \( K_j \) be a large enough holomorphically convex compact set containing both the complete \( \varphi_t^j \)-orbit of \( L_{k(j)} \cup F_{j-1}^{-1}(L_{k(j)}) \) and the complete \( \psi_t^j \)-orbit of \( L_{k(j)} \) in its interior. Choose \( m_j \) large enough such that \( K_j \) and \( \tau^{m_j}(K_j) \) are disjoint and such that their union is \( \mathcal{O}(X) \)-convex. Let

\[
C_j := [L_{k(j)} \cup F_{j-1}^{-1}(L_{k(j)})].
\]

We define an isotopy \( \Phi_t^j \) of maps near \( C_j \cup \tau^{m_j}(L_{k(j)}) \) by setting it equal to \( \varphi_t^j \) near \( C_j \) and \( \tau^{m_j} \circ \psi_t^j \circ \tau^{-m_j} \) near \( \tau^{m_j}(L_{k(j)}) \).

For any \( \hat{\varepsilon}_j \) we may now apply the Andersén–Lempert Theorem and obtain an \( F_j \in \text{Aut}^*(X) \) such that

\[
\sup_{L_{k(j)}} d(F_j, F_{j-1}) + d(F_{j-1}^{-1}, F_{j-1}^{-1}) < \hat{\varepsilon}_j
\]

and

\[
\sup_{\tau^{m_j}(L_{k(j)})} d(F_j, \tau^{m_j} \circ g_j \circ \tau^{-m_j}) < \varepsilon_j
\]

We see that we may choose \( \hat{\varepsilon}_j < 2^{-j} \) small enough such that \((a_j)\) and \((b_j)\) are satisfied.

To ensure that the group generated by \( \tau \) and \( F \) is free, we need to avoid that there is a non-trivial reduced word (formed by \( \tau \) and \( F \)) equal to the identity. For induction step \( j \) there are finitely many words of length \( \leq j \). By changing the choice of \( F_j \) within the required bounds we can make sure that on the compact \( L_{k(j)} \) no word of length \( \leq j \) equals the identity. Set \( \delta_j := \min \{ \sup_{L_{k(j)}} d(w, \text{id}) : w \neq \text{id} \text{ reduced word of length } \leq j, \text{ consisting of } \tau \text{ and } F_j \} \) and choose all subsequent \( \varepsilon_j, k \geq j + 1, \text{ small enough such that } \sum_{k=j+1}^{\infty} C_k \varepsilon_k < \delta_j \) where the constants \( C_k \) take into account the number of possible words.

We have now obtained a sequence \( F_j \) of holomorphic automorphisms of \( X \) and it follows from the conditions \((b_j)\) the \( F_j \rightarrow F \in \text{Aut}^*(X) \). It is immediate from
the condition \((a_j)\) that \(\{\tau^{-m_j} \circ F_j \circ \tau^{m_j}\}_{j \in \mathbb{N}}\) is dense in \(\text{Aut}^*(X)\). It follows that \(\tau^{-m_j} \circ F \circ \tau^{m_j} = (\tau^{-m_j} \circ F_j \circ \tau^{m_j}) \circ (\tau^{-m_j} \circ (F_j^{-1} \circ F) \circ \tau^{m_j}), j \in \mathbb{N}\), a dense sequence in \(\text{Aut}^*(X)\), since \(\tau^{-m_j} \circ (F_j^{-1} \circ F) \circ \tau^{m_j} \to \text{id}\) provided \(\varepsilon_j\) is decreasing fast enough. \(\square\)

**Remark 3.7.** In case \((X, \omega)\) would be a Stein manifold of dimension \(n \in \mathbb{N}\) with volume density property and an exhaustion of compacts \(\{L_j\}_{j \in \mathbb{N}}\) with de Rham cohomology \(H^{n-1}(L_j, \mathbb{C}) = 0\), one has an similar statement as in Theorem 3.6. Actually, a careful investigation of the proof of the Andersén–Lempert Theorem shows that the condition \(H^{n-1}(L_j, \mathbb{C}) = 0\) can be dropped for components of \(L_j\) where one wants to approximate the identity and for components where one wants to approximate an already globally defined automorphism, provided that \(H^{n-1}(X, \mathbb{C}) = 0\) holds.

**Examples 3.8.** In the following examples of Stein manifolds with volume density property, the method proof of the preceding theorem fails for various reasons:

- The manifold \(\mathbb{C}^*\) with volume form \(\omega = \frac{dz}{z}\) has the volume density property; \(\text{Aut}(\mathbb{C}^*) = \text{Aut}_\omega(\mathbb{C}^*) = \{z \mapsto a \cdot z^{\pm 1} : a \in \mathbb{C}^*\}\). Therefore it is obvious that no generalized translation exists on \(\mathbb{C}^*\). In addition there is due to dimensional reasons \((n-1 = 0)\) the obstruction \(\text{dim} H^0(\mathbb{C}^*, \mathbb{C}) = 1\).
- The manifold \(\mathbb{C}\) with volume form \(\omega = dz\) has the density property as well, and actually all volume-preserving automorphisms consist of translations. However again due to dimensional reasons there is the obstruction \(\text{dim} H^0(\mathbb{C}, \mathbb{C}) = 1\).
- The manifolds \(X := \mathbb{C}^* \times \mathbb{C}^*\) and \(Y := \mathbb{C} \times \mathbb{C}^*\) with volume forms \(\omega_X = \frac{dz}{z} \wedge \frac{dw}{w}\) resp. \(\omega_Y = dz \wedge \frac{dw}{w}\) are known to have the the volume density property. Generalized translations, e.g. \(\tau_X(z, w) = (2z, \frac{1}{2}w)\), \(\tau_Y(z, w) = (z + 1, w)\), exist, but in these cases \((n-1 = 1)\) there is a topological obstruction \(\text{dim} H^1(X, \mathbb{C}) = 2\) resp. \(\text{dim} H^1(Y, \mathbb{C}) = 1\).

The proof of Theorem 3.6 reveals as well that:

**Theorem 3.9.** Let \(X\) be a Stein manifold with density property and assume there exists a generalized translation \(\tau \in \text{Aut}^*(X)\). Then the associated conjugation operator \(\hat{C}_\tau\), defined by \(\hat{C}_\tau(f) = \tau \circ f \circ \tau^{-1}\), is hypercyclic on \(\text{Aut}^*(X)\).

4. Existence of Generalized Translations

In the last section we give a couple of examples of manifolds with Density Property with a generalized translation.

**Example 4.1.** Let \(Y\) be a Stein manifold, then there exists a generalized translation on \(X = \mathbb{C} \times Y\), given by \((z, y) \mapsto (z + a, y)\) for \(a \in \mathbb{C}^*\). In case \(Y\) is a complex Lie group, \(Y\) has the Density Property (Varolin [15]).

**Example 4.2.** Let \(Y\) be a Stein manifold, then there exists a generalized translation on \(X = \mathbb{C}^* \times \mathbb{C}^* \times Y\), given by \((z, w, y) \mapsto (az, w/a, y)\). In case \(Y\) is a complex Lie group with Volume Density Property, \(Y\) has the Volume Density Property (Varolin [15]).

**Example 4.3.** Let \(p(z) \in \mathbb{C}[z]\) be a polynomial with simple roots. Then the so-called Danielewski surface

\[
D_p := \{(x, y, z) \in \mathbb{C}^3 : x \cdot y = p(z)\}
\]
has the Density Property (Kaliman and Kutzschebauch [11]). The following map is a generalized translation \( \tau_a, a \in \mathbb{C}^* \), on \( D_p \):

\[
(x, y, z) \mapsto \left( x, \frac{p(z + a \cdot x)}{x}, z + a \cdot x \right)
\]

\[
= \left( x, y + \frac{p(z + a \cdot x) - p(z)}{x}, z + a \cdot x \right)
\]

\[
= \left( x, y + \frac{p(z + a \cdot x) - p(z)}{x}, z + a \cdot x \right)
\]

\[
= \left( x, y + p'(z) \cdot a + \sum_{k=2}^{\infty} p^{(k)}(z) \cdot a^k \cdot x^{k-1}, z + a \cdot x \right)
\]

Note that any compact of \( D_p \) can be moved as far away as possible by iterations of \( \tau_a^m = \tau_{am} \). The only crucial case one has to check is for \( x = 0 \): this implies \( p(z) = 0 \); but since \( p \) has only simple and finitely many zeros, \( p'(z) \) is bounded away from 0, and so is the shift in \( y \)-direction.

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