ON THE SOLUTIONS OF LINEAR VOLterra EQUATIONS OF
THE SECOND KIND WITH SUM KERNELS
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Abstract. We consider a linear Volterra integral equation of the second kind with a sum
kernel \( K(t', t) = \sum_i K_i(t', t) \) and give the solution of the equation in terms of solutions of the
separate equations with kernels \( K_i \), provided these exist. As a corollary, we obtain a novel series
representation for the solution with much improved convergence properties. The error resulting from
truncation of this series can be made as small as desired at any truncation order in a trade-off with
increased computational complexity upon tuning a certain parameter. As a byproduct, we obtain a
novel product integral representation of the solution with respect to a convolution-like product. We
illustrate our results with examples, including the first known Volterra equation solved by Heun’s
confluent functions. This solves a long-standing problem pertaining to the representation of such
functions. The approach presented here has widespread applicability in physics via Volterra equations
with degenerate kernels.

Key words. Integral equations; linear Volterra equations of the second kind; separable kernel;
degenerate kernels; Neumann series; re-summations; product integrals; Heun functions.

AMS subject classifications. 45D05; 45A05

1. Introduction.

1.1. Context and background. Inhomogeneous linear Volterra integral equations of the second kind are equations in an unknown function of two time variables
\( f(t', t) \) given by

\[
 f(t', t) = g(t', t) + \int_{t'}^t K(t', \tau) h(\tau, t) d\tau.
\]

In this expression \( g \) and \( K \) are arbitrary functions of two time variables, \( g \) being termed the inhomogeneity and \( K \) the kernel of the equation, respectively. Such equations are of paramount importance in physics, e.g. in designing quantum computing applications \([9, 1, 20, 14]\). In this case the desirability of analytical expansions for the solution is such that purely numerical approaches are of secondary relevance.

In this work, we focus on a particular type of kernels, which we call sum kernels, which can be expressed as

\[
 K(t', t) := \sum_{i=1}^d K_i(t', t),
\]

and such that the solutions of all the separate Volterra equations with kernels \( K_i \) exist. A widespread example of sum kernels of particular interest are the separable kernels
(also called degenerate kernels), for which all the \( K_i \) satisfy \( K_i(t', t) = a_i(t') b_i(t) \). From an analytical point of view, inhomogeneous linear Volterra equations with general sum kernels are solved indirectly through transformations mapping the equation into a system of coupled linear differential equations with non-constant coefficients \([2, 6, 12, 11]\). The solutions of such systems are in fact themselves non-obvious. To make matter worse, this process is circular, pointless and unsatisfying for (quantum) physics applications for which the Volterra equations originate from reductions of an

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initial system of coupled linear differential equations! To the best of our knowledge, whenever \( i > 1 \) there is no known closed analytical form for the solution of a linear Volterra integral equation of the second kind with arbitrary \(^2\) sum kernel, even in the simpler cases of a separable kernel such that all \( K_i \) are bounded and smooth functions in both \( t \) and \( t' \). In all likeliness, such a form does not in general exist since examples where the solution is a higher transcendental function have already been found \([19]\).

In any case, the best possible analytical approach is therefore to provide a series representation of the solution with ‘good’ convergence and truncation properties. Taylor polynomial expansion have been proposed \([15]\) but the Neumann series obtained via Picard iteration gives a simpler starting point in this quest, as it is amenable to exact, systematic re-summations based on set theory. It is the purpose of this note to present the resulting re-summed series. For the many numerical techniques developed so far to solve Volterra equations as well as theoretic considerations pertaining to the existence uniqueness and smoothness of the solutions, we refer to \([10, 7]\).

1.2. Standard Neumann series formulation of the solutions. It is convenient to introduce first a short hand notation for integrals such as the one appearing in Eq. (1.1). We designate by \(*\)-product a convolution-like product between two functions \( f \) and \( g \) of two time variables

\[
(f * g)(t', t) := \int_{t}^{t'} f(t', \tau) g(\tau, t) d\tau.
\]

The \(*\)-product is also well defined for functions which depend on less than two time variables. Let \( f(t') \) depend on only one time variable and \( g(t', t) \) be as above. Then

\[
(f * g)(t', t) = f(t') \int_{t}^{t'} g(\tau, t) d\tau \quad \text{and} \quad (g * f)(t', t) = \int_{t}^{t'} g(t', \tau) f(\tau) d\tau.
\]

In other terms, the time variable of \( f(t') \) is always treated as the left time variable of a doubly time dependent function. These definitions extend straightforwardly to constants.

It is well known that linear Volterra equations of the second kind are solvable using Picard-iterations \([18, 6]\). The underlying principle is simple: noting that, in \(*\)-product notation, the equation takes on the form

\[
f = g + K * f,
\]

we deduce that

\[
f = g + K * f = g + K * (g + K * f) = g + K * g + K * K * f = \cdots
\]

Continuing this process yields the Neumann series representation of the solution

\[
f = \left( \sum_{n \geq 0} K^{*n} \right) * g, \quad (1.2)
\]

with the understanding that \( K^{*0} = \delta(t' - t) \) is the Dirac delta distribution. The same solution holds for matrix-valued Volterra integral equations of the second kind, where the functions \( f, g \) and \( K \) are time-dependent matrices \( F, G \) and \( K \). In physical contexts, this was first uncovered by F. Dyson \([4]\), who called the solution \( H \) the time-

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\(^2\)Special solutions for certain separable kernels have of course been found, we refer the reader to \([11]\) Part I, Chapter 11, for a number of these.
or path-ordered exponential of $G$. These objects are of fundamental importance in the field of quantum dynamics, the time-ordered exponential of the Hamiltonian operator dictating the evolution of a quantum system driven by time-varying forces. There, the question of solving Eq. (1.1) with separable kernels is crucial [5].

The solution presented in Eq. (1.2) can also be cast as a $\ast$-resolvent [10, 18, 6], that is

$$f = (1_\ast - K)^{\ast -1} \ast g,$$

with $1_\ast := \delta(t' - t)$, and the inverse is taken with respect to the $\ast$-product. This follows from the observation that the Neumann series Eq. (1.2) is convergent provided $K$ is a bounded function of time over the time interval of interest. Together with the form of Eq. (1.2), Eq. (1.3) above shows that the whole difficulty in calculating $f \ast R_K$ lies in finding the $\ast$-resolvent of $K$, denoted from now on $R_K := (1_\ast - K)^{\ast -1}$.

A $\ast$-resolvent such as $R_K$ itself satisfies a linear integral Volterra equation of the second kind with kernel $K$ and inhomogeneity $1_\ast = \delta(t' - t)$, as implied by the usual properties of resolvents. Indeed, since $K \ast R_K = R_K - 1_\ast$, we have

$$R_K = 1_\ast + K \ast R_K.$$ 

While the Neumann series representation of $R_K = \delta + K + K^\ast + \cdots$ is guaranteed to exist and converges for bounded kernel $K$, the speed of convergence and quality of the analytical approximations obtained by truncating this series can be very poor. Fortunately, generic properties of $\ast$-resolvents allow exact and systematic re-summations of this series that not only speed-up its convergence but ultimately express $R_K$ in terms of the $\ast$-resolvents $R_{K_i}$ whenever $K = \sum_i K_i(t', t)$. This approach is particularly very suited to separable kernels for which $K_i = a_i(t')b_i(t)$ and thus every $R_{K_i}$ is known exactly.

2. Background.

2.1. Properties of ordinary resolvents. Let us start by coming back to basic properties of resolvents and inverses. For example, considering an ordinary inverse and $u$ and $v$ two formal variables. We have

$$\frac{1}{1 - u - v} = \frac{1}{1 - u} \times \frac{1}{1 - v} + \frac{uv}{(1 - u)(1 - v)} \times \frac{1}{1 - u - v},$$

so that an immediate iterative approach to calculating $1/(1 - u - v)$ is

$$\frac{1}{1 - u - v} = \frac{1}{1 - u} \times \frac{1}{1 - v} + \frac{uv}{(1 - u)(1 - v)} \left( \frac{1}{1 - u} \times \frac{1}{1 - v} + \frac{uv}{(1 - u)(1 - v)} \cdots \right).$$

This iteration leads to the formal series,

$$\frac{1}{1 - u - v} = \sum_{k=0}^{\infty} \frac{(uv)^k}{(1 - u)(1 - v)^k} \frac{1}{1 - u} \frac{1}{1 - v}.$$ 

All of these results hold true for resolvents with respect to the $\ast$-product with the caveat that, lacking commutativity, we must choose an order for the terms. Because no choice is special owing to the $u, v$ exchange symmetry of the resolvent, all choices lead to equally valid results.
2.2. Consequences for ∗-resolvents. Inspired by the observations above, we may express the ∗-resolvent $R_{u+v}$ of a sum of functions $u + v$ in terms of $R_u$ and $R_v$. This directly generalises to any number of functions.

Theorem 2.1 (Expansion of ∗-resolvents). Let $u$ and $v$ be two functions of two time-variables over $I^2 \subseteq \mathbb{R}^2$ such that $R_u := (1 - u)^{*-1}$, $R_v := (1 - v)^{*-1}$ and $R_{u+v} := (1 - u - v)^{*-1}$ exist. Then

$$R_{u+v} = R_u \ast R_v + u \ast R_u \ast v \ast R_v \ast R_{u+v},$$

(2.2a)

$$= R_u \ast R_v + (R_u - 1_*) \ast (R_v - 1_*) \ast R_{u+v}.$$  

(2.2b)

That is, $R_{u+v}$ is solution of an inhomogeneous linear Volterra integral equation of the second kind with kernel $(R_u - 1_*) \ast (R_v - 1_*)$ and inhomogeneity $R_u \ast R_v$.

Corollary 2.2. Let $u, v$ and $R_u, R_v$ and $R_{u+v}$ be as in Theorem 2.1. Then we have the re-summed Neumann series

$$R_{u+v} = \sum_{k=0}^{\infty} \left( u \ast R_u \ast v \ast R_v \right)^{*k} \ast R_u \ast R_v,$$

(2.3a)

$$= \sum_{k=0}^{\infty} \left( (R_u - 1_*) \ast (R_v - 1_*) \right)^{*k} \ast R_u \ast R_v,$$

(2.3b)

convergence of which is guaranteed whenever $u$ and $v$ are bounded over $I^2 \subseteq \mathbb{R}^2$.

The advantages of the above Theorem for calculating ∗-resolvents are as follows:

i) It extends immediately to any number of functions in the overall resolvent $R_{u+v+w+\ldots} = (1 - u - v - w - \ldots)^{*-1}$;

ii) It necessitates only ∗-resolvents of single functions such as $u$ and $v$, a huge advantage in the case of degenerate kernels;

iii) It provides a fully explicit series which is a re-summed form of the Neumann series;

iv) It yields a dramatic improvement of the convergence speed over the original Neumann series, see §3.1 and the examples of §4.

v) It gives rise to a new formal representation of the solution as a ∗-product-integral, see §3.2.

vi) It remains valid should all functions $u, v, \ldots$ be matrix-valued.\(^3\)

Proof. To establish the Theorem, it suffices to observe that the ∗-resolvent

$$R_{u+v} = \sum_{k=0}^{\infty} (u + v)^{*k},$$

is the characteristic series of the Kleene star of the set $\{u, v\}$. But this Kleene star is given in terms of itself by

$$\{u, v\}^* = \epsilon \cup \{u\}^* \circ \{v\}^* \cup u \circ \{u\}^* \circ v \circ \{v\}^* \circ \{u, v\}^*,$$

where $\epsilon$ is the empty symbol and $\circ$ stands for the concatenation of sets $a \circ \{b, c\} = \{a \circ b, a \circ c\}$. The advantage of this general proof is that it establishes an equivalent

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3In this case, ∗-resolvents such as $(1_\emptyset - V)^{*-1}(t', t)$ should be understood as the time derivative with respect to $t'$ of the ordered exponential of the time-dependent matrix $V(t', t)$.
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of the Theorem for any product, not just the \( \ast \)-product. This includes the ordinary matrix product, for which the Theorem provides a general mean of expressing the resolvent of sums of matrices in terms of the individual matrix resolvents. The equivalency between the two forms presented in Eqs. (2.2) and Eqs. (2.3) follows from the observation that \( u \ast R_u = R_u - 1 \ast u \) and similarly for \( R_v \).

**Remark 2.1.** The formulation of Theorem 2.1 lacks \( u, v \) exchange symmetry contrary to the initial \( \ast \)-resolvent. This is because we chose an explicit order for the terms. A further improvement is possible upon using explicit symmetrisation over all possible orders whenever \( u \ast v \neq v \ast u \).

3. Main result: application to Volterra equations. We may now simply adapt this general result to the specific purpose of extending \( \ast \)-resolvent of arbitrary separable kernels solely in terms of \( \ast \)-resolvents of singly separable kernels which have the analytical closed form of Proposition 3.2:

**Corollary 3.1.** Let \( I \subseteq \mathbb{R}, t', t \in I^2, d \in \mathbb{N}^\ast \) and \( K(t', t) := \sum_{i=1}^{d} K_i(t', t) \) be a sum resolvent. Let \( g(t', t) \) be a function that is not identically null over \( I^2 \). Then the function \( f \) solution of the linear Volterra integral equation of the second kind with kernel \( K \) and inhomogeneity \( g, f = g + K \ast f \) is given by

\[
f = g + \left( 1 \ast - \frac{d}{i=1} (R_i - 1 \ast) \right) ^{-1} \ast \frac{d}{i=1} R_i \ast g.
\]

Furthermore, if all \( K_i(t', t) \) are bounded over \( I^2 \), then

\[
f = g + \sum_{k=0}^{\infty} \left( \frac{d}{i=1} (R_i - 1 \ast)^{\ast k} \right) \ast \frac{d}{i=1} R_i \ast g.
\]

Let \( f^{(n-1)} \) be the series as above truncated at order \( n - 1 \geq 0 \). The truncation error is exactly

\[
f - f^{(n-1)} = \left( \frac{d}{i=1} (R_i - 1 \ast)^{\ast n} \right) \ast f.
\]

**Proof.** The Proposition is an immediate corollary of Theorem 2.1. Concerning the error term at order \( n \), take the iterative form of Eqs. (2.1). Note that since \( K \) is assumed to be a sum resolvent, all the \( \ast \)-resolvents \( R_{K_i}, 1 \leq i \leq d \) exist by definition of sum resolvents.

Corollary 3.1 is well suited to separable kernels as in such cases all \( R_i \) are exactly available. Recall that a separable kernel takes on the form

\[
K(t', t) = \sum_{i=1}^{d} a_i(t')b_i(t).
\]

For the sake of simplicity, we assume that all \( a_i \) and \( b_i \) are bounded and smooth over \( I \). Although these conditions can be relaxed following [10], they are satisfied in e.g. all quantum physics applications. Now, choosing \( K_i = a_i(t')b_i(t) \), all \( \ast \)-resolvents \( R_{K_i} \) are exactly available. The closed form of \( R_{K_i} \) is almost certainly already known although we could not locate it in the literature. We provide it here with a proof for the sake of completeness.
Proposition 3.2. Let $I \subset \mathbb{R}$ and let $t, t' \in I^2$ be two time variables. Let $f(t', t)$ be a function of $t', t$ over $I^2$ and $K(t', t) := a(t')b(t)$. Let $\alpha := \int K(\tau, t) d\tau = \int a(\tau)b(\tau) d\tau$. Then the solution $f$ of the linear Volterra equation of the second kind $f = g + K \ast f$ with kernel $K$ is

$$f = (1* + Ke^\alpha) \ast g,$$

(3.2a)

or equivalently

$$f(t', t) = g(t', t) + a(t') \int_t^{t'} b(\tau) \exp \left( \int_\tau^{t'} a(\tau')b(\tau') d\tau' \right) g(\tau, t) d\tau. \tag{3.2b}$$

Proof. We proceed by induction on the Neumann series $f = (\sum_n K^{*n}) \ast g$. Convergence of this series is guaranteed whenever $a$ and $b$ are bounded over $I$, however existence of the final form for $f$ is clearly independent from this assumption. In this situation this form can be understood as the analytic continuation of the original Neumann series.

The proposition to be shown is

$$P_n := \left\{ \forall n \in \mathbb{N}^*: K^{*n}(t', t) = a(t')b(t)(\alpha(t') - \alpha(t))^{n-1} \times 1/(n-1)! \right\}.$$

We have $K^{*0} = 1_*$ and obviously $K^{*1} = K$, which provides $P_0$ and $P_1$. Now supposing $P_n$ true,

$$K^{*(n+1)} = \int_t^{t'} K(t', \tau)K^{*n}(\tau, t) d\tau = \frac{1}{(n-1)!} a(t') \int_t^{t'} a(\tau)b(\tau)(\alpha(\tau) - \alpha(t))^{n-1} d\tau,$$

where the $a(\tau)b(\tau)$ term comes from $K^{*n}(\tau, t)$, while the $K(\tau)$ comes from $K(t', \tau)$. Since $a(\tau)b(\tau) = \alpha'(\tau)$, we have

$$K^{*(n+1)} = \frac{1}{(n-1)!} a(t')b(t) \int_t^{t'} \alpha'(\tau)(\alpha(\tau) - \alpha(t))^{n-1} d\tau = \frac{1}{n!} a(t')b(t)(\alpha(\tau) - \alpha(t))^n.$$

This establishes the implication $P_n \Rightarrow P_{n+1}$ and since $P_1$ holds, $P_n$ is true for all $n \in \mathbb{N}^*$. Then $\sum_{n=0}^{\infty} K^{*n} = 1_* + Ke^\alpha$ and $f$ is obtained upon $\ast$-multiplying by $g$ on the left. \qed

Following Aristotle who said that “for the things we have to learn before we can do, we learn by doing”, it is essential to present examples of applications of the above approach. These are presented in Section 4 in ascending order of difficulty. In the remainder of the present section, we give the convergence analysis of the re-summed series as well as the product integral representation of $h$.

3.1. Convergence analysis. Here we suppose all functions $K_i$ appearing in the kernel $K = \sum_{i=1}^d K_i$ are bounded over $I$ so as to guarantee convergence of the original Neumann series.

Let $f^{(n-1)}_N := \sum_{k=0}^{n-1} K^{*k} \ast g$ be the approximation obtained from the truncating the Neumann series at order $n - 1$, $n \geq 1$. Since

$$f - f^{(n-1)}_N = K^{*n} \ast f,$$

then
defining $C := \sup_{t' \in I^2} K(t', t)$ and $H = \sup_{t' \in I} |f(t', t)|$, we have the immediate bound \[10]\]

$$
\sup_{t, t' \in I^2} |f - f^{(n-1)}_N| \leq \frac{H(C(t' - t))^n}{n!},
$$

which comes from the observation that the error comprises $n$, $*$-products. This bound is saturated by the constant kernel $K(t', t) = C$. For the re-summed series presented here, the error produced by truncating the re-summed series at order $n - 1$, $n \geq 1$, is

$$
f - f^{(n-1)} = \left(\sum_{i=1}^{d} (R_i - 1_*) \right)^* f.
$$

Let $C_i := \sup_{t' \in I^2} K_i(t', t)$ then $C_i \leq C$ and $r_{C_i} := \sup_{t', t \in I^2} |R_i - 1_*|$ exists as $R_i - 1_* = K_i * R_i = \sum_{k>0} K_i^{*k} \leq e^{C_i(t' - t)} - 1$. It follows that $r_{C_i} \leq e^{C_i(t' - t)} - 1$ and for $n > 0$,

$$
\sup_{t, t' \in I^2} |f - f^{(n-1)}| \leq \frac{H e^{C(t' - t)(d-1)(n-1)(t' - t)(d-1)(n-1)+1}}{(d-1)(n-1) + 1!},
$$

(3.3)

since $f - f^{(n)}$ comprises $(d-1)(n-1) + 1$, $*$-products and $e^{C_i(t' - t)} \leq e^{C(t' - t)}$ for all $1 \leq i \leq d$.

Given the product $(d-1)(n-1)$ in the factorial of the denominator, the bound of Eq. (3.3) is a major improvement over that provided by the Neumann series when $d$ is large. This acceleration of convergence is further enhanced if $C/d$ is on the order of 1 or less so that $r_C \sim 1$. But it is always possible to choose $d$ to be large enough for both of these enhancements to take place: it suffices to consider $K$ to be a sum of $d$ pieces (e.g. identical copies $K_i = K/d$). This makes the error as small as desired for any order $n$. In particular, the limit $d \to \infty$ yields exact results even at the 0th order: it is a product integral representation of the solution, presented below. For concrete applications, this procedure will be a trade-off: as $d$ grows, the computational complexity of any order $f^{(n)}$ increases while the convergence to $f$ gets ever better.

### 3.2. Formal representation as product-integral

In this section, we present a formal representation of the solution $h$ as a continuous $*$-product. For a general background on the concept of product integration and the role Vito Volterra played in its inception, we refer to \[16\].

We remark that while representations of the solution of linear Volterra equations of the second kind called product-integration methods do already exist, they should not be confused with the formal approach described here. Rather, they are techniques employed to numerically integrate ill-behaved kernels, rely on time discretisation and use the ordinary product \[10\]. In contrast, our aim is to establish a formal description of the solution as a $*$-product integral.

Let $K(t', t)$ be a bounded kernel. Let $d \in \mathbb{N}^*$, $\epsilon := 1/d$ and $K_\epsilon(t', t) := \epsilon K(t', t)$. Seeing $K(t', t)$ as comprising a sum of $d$ kernels, the 0th order of the re-summed series for the solution $f$ of $f = g + K * f$ is

$$
f^{(0)}(t', t) = \bigotimes_{i=1}^{d} (1_* - \epsilon K_i)^* g,
$$

(3.4)
The ordinary Neumann series for this solution is the Taylor expansion
\[
\lim_{\epsilon \to 0} f^{(0)}(t) = \lim_{\epsilon \to 0} \left[ (1 \times - K_\epsilon) * g \right] = \lim_{\epsilon \to 0} \left\{ g + dK_\epsilon * g + \left( \frac{d}{2} \right) K_\epsilon^2 * g + \left( \frac{d}{3} \right) K_\epsilon^3 * g + \cdots \right\},
\]
(3.5)

The last equality follows from the observation that \( g + K * g + K^2 * g + \cdots \) is the Neumann series representation of \( f \).

In the limit \( d \to \infty \), the \( * \)-product of Eq. (3.4) is a product-integral in the sense originally developed by Volterra [16]. Formally we have:

**Proposition 3.3.** Let \( g(t', t) \) and \( f(t', t) \) be two functions of two time variables \( t, t' \in I \). Then the solution of the inhomogeneous linear Volterra integral equation of the second kind \( h = f + g * h \) is

\[
f = \frac{t'}{t} \left( 1* - K(\tau', \tau) \right) * g.
\]

The proof is already given by Eqs. (3.4, 3.5).

**4. Illustrative Examples.** We provide three examples of application of our main result, Corollary 3.1, in ascending order of difficulty: i) solution to the linear Volterra equation of the second kind with constant kernel; ii) solution for a double separable kernel \( (d = 2) \); iii) a new representation of Heun’s confluent functions as solution of a triple separable kernel \( (d = 3) \).

**Example 4.1 (Constant kernel).**

Let \( a \) and \( b \) be two constants and \( K(t', t) := a + b \) be the kernel of the homogenous linear Volterra equation of the second kind \( f = 1* + K * f \), where we recall that \( 1* = \delta(t' - t) \). The exact solution is of course

\[
f(t', t) = \delta(t', t) + (a + b) e^{(a+b)(t' - t)}.
\]

The ordinary Neumann series for this solution is the Taylor expansion

\[
f(t', t) = \delta(t', t) + a + b + (a + b)(t' - t) + (a + b) \frac{(t' - t)^2}{2} + \cdots
\]

In contrast, Corollary 3.1 with \( d = 2 \) gives at order 0,

\[
f^{(0)}(t', t) := (1* - a) * (1* - b) \ast^{-1},
\]

\[
= (1* + ac(t' - t)) * (1* + bc(t' - t)),
\]

\[
= 1* + ac(t' - t) + bc(t' - t) + \int_t^{t'} ae^{a(t' - \tau)} be^{b(\tau - t)} d\tau,
\]

\[
= \delta(t' - t) + ae^{a(t' - t)} + be^{b(t' - t)} + \frac{ab}{a - b} \left( e^{a(t' - t)} - e^{b(t' - t)} \right).
\]
Note that this is actually well defined in the limit $b \to a$ where the last term yields $a^2(t' - t)e^{a(t' - t)}$, while in the limits $a \to 0$ or $b \to 0$ the 0th order becomes exact as expected. At order 1, we get

$$f^{(1)}(t', t) := \delta(t', t) + a e^{a(t'^2 - t)} + b e^{b(t'^2 - t)} + \frac{a b}{a - b} \left( e^{a(t'^2 - t)} - e^{b(t'^2 - t)} \right) + \frac{a b e^{-(a+b)t}}{(a - b)^2} \times$$

$$\left( b e^{a(t'^2 - t)} \left( a e^{a(t'^2 - t)} - b e^{b(t'^2 - t)} \right) - a e^{a(t'^2 + b t)} \left( a e^{a(t'^2 - t)} + b e^{b(t'^2 - t)} \right) \right) .$$

**Example 4.2 (Heun functions).**

Heun confluent functions are special transcendental functions known from general relativity and astrophysics [8] as well as quantum optics [19]. Heun’s functions are known only from the context of differential equations [13, 3], being defined as the solution to Heuns differential equation. No integral satisfied by these functions is known so far. To quote a recent review [8] on Heun’s functions, the current state of knowledge is as follows:

“No example has been given of a solution of Heuns equation expressed in the form of a definite integral or contour integral involving only functions which are, in some sense, simpler [...] This statement does not exclude the possibility of having an infinite series of integrals with ‘simpler’ integrands”.

Here we give two Volterra equations of the second kind for which Heun’s confluent function is the solution, implying two infinite series of integrals converging to Heun’s function via Corollary 3.1.

We start with the work of Xie and Hai [19], who considered the system of coupled linear differential equations with non-constant coefficients,

$$2i \omega \frac{d}{dt} a(t) = \nu b(t) + f(t) a(t), \quad (4.1a)$$

$$2i \omega \frac{d}{dt} b(t) = \nu a(t) - f(t) b(t), \quad (4.1b)$$

where $f(t) := f_1 \sin(\omega t)$ and $f_1$, $\nu$ and $\omega$ are real parameters originating from a quantum Hamiltonian. The authors showed that this system is equivalent to Heun’s equation, hence is solved by certain Heun confluent functions, here denoted $H_c$. For example, for $a(t)$ we have [19],

$$a(t) = c_1 e^{i(f_1/\omega) \sin^2(\omega t/2)} H_c(\nu, \alpha, \beta, \gamma, \delta, \eta) + c_2 e^{i(f_1/\omega) \sin^2(\omega t/2)} \sin(\omega t/2) H_c(\nu, -\beta, \gamma, \delta, \eta),$$

where $c_1$ and $c_2$ are constants determined from the initial conditions and $\alpha := 2i f_1/\omega$, $\beta = \gamma = -1/2$, $\delta = i f_1/\omega$ and $\eta = -i f_1/(2\omega) + 3/8 - \nu^2/(4\omega^2)$.

The system of Eq. (4.1) is easily mapped into a homogenous linear Volterra equation of the second kind via the method of path-sum [5]. This gives, for $c_1 = 1$ and $c_2 = 0$,

$$a(t)|_{c_1=1,c_2=0} = \int_0^t \left( 1 + (i/2) f - \nu^2/4 \right) R^{(8)}(\sigma, 0) \, d\sigma,$$
where \( R := 1 * F * 1 = \int_{t_1}^{t_2} \int_{t_2}^{t} F(\tau_2, \tau_1) d\tau_2 d\tau_1 \) and

\[
F(t', t) := (1_\ast - (i/2)f)^{*-1} = \delta(t' - t) + \frac{i}{2} \sin(\omega t') e^{-i(2\omega)(\cos(\omega t') - \cos(\omega t))},
\]

as per Proposition 3.2. The part of the solution with \( c_1 = 0 \) and \( c_2 = 1 \) is obtained from the above by \( \ast \)-multiplication of \( \hat{\alpha}(t) \) with \( q(t', t) := (\nu/2) \int_0^t F(t', \tau) d\tau \), as dictated again by path-sum [5].

The integral equation of interest here is that satisfied by \( \hat{\alpha} \), namely

\[
\hat{\alpha} = 1_\ast + (-i/2)f + (\nu^2/4)R \ast \hat{\alpha}
\]

This is precisely a homogeneous linear Volterra integral equation of the second kind with separable kernel (kernel separability is guaranteed by path-sum)

\[
g(t', t) = -\frac{i}{2} f(t') + \frac{\nu^2}{4} s(t') - \frac{\nu^2}{4} s(t).
\]

with \( s := \int_{t_1}^{t} F(\tau_2, \tau_1) d\tau_2 d\tau_1 \) so that \( R(t', t) = s(t') - s(t) \). The formulation of Heun’s confluent functions via the solution to the Volterra Eq. (4.2) gives, by Corollary 3.1 and with \( d = 3 \), a new series representation of these functions, so far expressed through a power series expansion around the origin [17, 13]:

\[
e^{i(f_1/\omega) \sin^2(\omega t/2)} \mathcal{H}_c(\alpha, \beta, \gamma, \delta, \eta, \sin^2(\omega t/2)) = \sum_{k \geq 0} \left( (f \ast F \ast (\nu^2 s(t')/4) \ast S_1 \ast (-\nu^2 s(t)/4) \ast S_2)^{\ast k} \ast F \ast S_1 \ast S_2 \right) = \left( (1_\ast + f \ast F \ast (\nu^2 s(t')/4) \ast S_1 \ast (-\nu^2 s(t)/4) \ast S_2 + \cdots) \ast F \ast S_1 \ast S_2 \right).
\]

Here \( S_1(t', t) := \delta(t' - t) + (\nu^2/4)s(t') e^{i(t')s(\tau)} d\tau \), \( S_2(t', t) := \delta(t' - t) - (\nu^2/4)s(t) e^{i(t')s(\tau)} d\tau \).
Giscard, On Volterra equations with sum kernels

[2] J. Cerha, A note on Volterra integral equations with degenerate kernel, Commentationes Mathematicae Universitatis Carolinae, 13 (1972), pp. 659–672.

[3] NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.20 of 2018-09-15, 2018, http://dlmf.nist.gov/. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds. For integral equations satisfied by Heun’s functions see §31.10 by B. D. Sleeman and V. B. Kuznetsov.

[4] F. J. Dyson, Divergence of Perturbation Theory in Quantum Electrodynamics, Physical Review, 85 (1952), pp. 631–632, https://doi.org/10.1103/PhysRev.85.631.

[5] P.-L. Giscard, K. Lui, S. J. Thwaite, and D. Jaksch, An exact formulation of the time-ordered exponential using path-sums, Journal of Mathematical Physics, 56 (2015), p. 053503, https://doi.org/10.1063/1.4920925, https://arxiv.org/abs/https://doi.org/10.1063/1.4920925.

[6] G. Gripenberg, S. O. Londen, and O. Staffans, Volterra Integral and Functional Equations, Oxford Mathematical Monographs, Cambridge University Press, Cambridge, 1990.

[7] W. Hackbusch, Integral Equations: Theory and Numerical Treatment, no. 120 in International Series of Numerical Mathematics, Birkhäuser, Basel, 1. softcover ed.]; [reprint of the orig. 1st ed. 1995 ed.], 1995. OCLC: 934739095.

[8] M. Hortacsu, Heun functions and some of their applications in physics, Advances in High Energy Physics, 2018 (2018), p. 8621573, https://doi.org/10.1155/2018/8621573.

[9] Y. Kayanuma, Role of phase coherence in the transition dynamics of a periodically driven two-level system, Phys. Rev. A, 50 (1994), pp. 843–845, https://doi.org/10.1103/PhysRevA.50.843, https://link.aps.org/doi/10.1103/PhysRevA.50.843.

[10] P. Linz, Analytical and Numerical Methods for Volterra Equations, Society for Industrial and Applied Mathematics, 1985, https://doi.org/10.1137/1.9781611970852.

[11] A. D. Polyanin and A. V. Manzhirov, Handbook of Integral Equations, Chapman and Hall / CRC, second ed., 2008.

[12] L. Razdolsky, Integral Volterra Equations, in Probability Based High Temperature Engineering, Springer International Publishing, Cham, 2017, pp. 55–100, https://doi.org/10.1007/978-3-319-41909-1_2.

[13] A. Ronveaux, F. M. Arscott, S. Y. Slavyanov, S. D., W. G., M. P., and D. A., Heun’s Differential Equations, Oxford University Press, 1995.

[14] H. Schmidt, The floquet theory of the two-level system revisited, Zeitschrift für Naturforschung A, 73 (2018), pp. 705 – 731, https://doi.org/10.1515/zna-2018-0211.

[15] M. Sezer, Taylor polynomial solutions of Volterra integral equations, International Journal of Mathematical Education in Science and Technology, 25 (1994), pp. 625–633, https://doi.org/10.1080/002073994025050501.

[16] A. Slavík, Product Integration, Its History and Applications, no. 29 in History of Mathematics, Matfyzpress, Praha, 2007. OCLC: 254426623.

[17] S. Y. Slavyanov and W. Lay, Special Functions: A Unified Theory Based on Singularities, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[18] F. G. Tricomi, Integral Equations, Dover Publications, 1985.

[19] Q. Xie and W. Hui, Analytical results for a monochromatically driven two-level system, Physical Review A, 82 (2010), p. 032117, https://doi.org/10.1103/PhysRevA.82.032117.

[20] D. Zeuch, F. Hassler, J. Slim, and D. P. DiVincenzo, Exact Rotating Wave Approximation, ArXiv e-prints, (2018), arXiv:1807.02858, p. arXiv:1807.02858, https://arxiv.org/abs/1807.02858.