Non-singlet Q-deformed $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1/2)$ U(1) actions

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Abstract

In this paper we construct $\mathcal{N} = (1, 0)$ and $\mathcal{N} = (1, 1/2)$ non-singlet Q-deformed supersymmetric U(1) actions in components. We obtain an exact expression for the enhanced supersymmetry action by turning off particular degrees of freedom of the deformation tensor. We analyze the behavior of the action upon restoring weekly some of the deformation parameters, obtaining a non trivial interaction term between a scalar and the gauge field, breaking the supersymmetry down to $\mathcal{N} = (1, 0)$. Additionally, we present the corresponding set of unbroken supersymmetry transformations. We work in harmonic superspace in four Euclidean dimensions.

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1 Introduction

Being deformations of field theories and supersymmetry both old ideas deeply analyzed and developed through many decades, it is a natural step to think of extending the Weyl-Moyal product to a deformed algebra of superfields involving the Graßmann sector, thus
leading to non-(anti)commutativity. Recently it has been found that strings in certain backgrounds are related to such deformations of superspace, see for example [1–5]. This has stimulated the study of particular supersymmetric gauge theory deformations, implemented through an associative algebra of superfields whose Moyal product is realized as a function of a bilinear nilpotent Poisson operator. For this reason this formulations are also called Nilpotent deformations. A very interesting feature of such non-(anti)commutative theories is the natural emergence of interactions not present in the corresponding undeformed scenarios. For instance, in our case we will see the apparition of Yukawa-like interactions. The progress made towards the understanding on the renormalizability of non-anticommutative field theories [6–8] is also very motivating.

Nilpotent deformations in extended supersymmetric field theories were first analyzed in superspace [9, 10] and later on in harmonic superspace [11, 12]. In this paper we work in Euclidean harmonic superspace in four dimensions [13], where \((\theta^\alpha)^\ast \neq \bar{\theta}^\alpha\). In general, nilpotent deformations are introduced via Weyl-Moyal product with a bilinear Poisson operator which is constructed either in terms of the supercharges, or in terms of the spinor covariant derivatives [9, 10, 14], leading to \(Q\)- and \(D\)-deformations, respectively. Like in non-commutative field theories, even with there is no unique non-anticommutative generalization of a given supersymmetric theory, a selection scheme can be found based on different physical reasons like symmetry preservation or its relation to string theory. Since \(Q\)-deformations are directly implied by string theory, it seems tempting to continue studying their physical properties, postponing the identification of the specific string backgrounds from which the resulting theories originate. Therefore we concentrate our analysis in \(Q\)-deformations. In this paper we construct the exact \(Q\)-deformed supersymmetric \(\mathcal{N} = (1, 1/2)\) action by dropping consistently some components of the deformation matrix. Afterwards, while weakly restoring some degrees of freedom of the deformation parameters, we break the supersymmetry down to \(\mathcal{N} = (1, 0)\) obtaining a second order action. As we will see in §3 we can also choose these variables to control a certain potentials appearing from the deformation of the \(\mathcal{N} = (1, 1)\) which can not be disentangled by redefinitions of the components fields. We calculate the corresponding expressions for the full set of non-singlet \(Q\)-deformed supersymmetry transformations, together with the Seiberg-Witten-like map which sets a frame where actions are gauge invariant under the canonical undeformed transformations. Though all our actions have partially broken supersymmetry, it has been shown they preserve the so-called twist supersymmetry [19]
by construction.

## 2 Non-singlet Q-deformations and supersymmetry breaking

In general terms the Poisson operator is written as

\[ P = -\tilde{Q}_i^\alpha C_{i\beta}^{\alpha\beta} \tilde{Q}^k_\beta, \quad (2.1) \]

where Greek letters represent Euclidean space-time indexes \(\alpha, \beta = 1, 2\) and \(\dot{\alpha}, \dot{\beta} = \dot{1}, \dot{2}\), whereas Latin indexes stands for SU(2) automorphisms \(i, j = 1, 2\). Both sorts of indices are raised and lowered with the SU(2) metric \(\varepsilon_{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}, \varepsilon_{ik}\), where \(\varepsilon_{12} = 1\). The Moyal product of two superfields is then defined by

\[ A \star B = Ae^P B = AB + APB + \frac{1}{2} AP^2 B + \frac{1}{6} AP^3 B + \frac{1}{24} AP^4 B, \quad P^5 = 0. \quad (2.2) \]

in order to preserve the associativity of the Moyal product, \(2.1\) is chosen to include only undotted supercharges. From \(2.2\) we see that the nilpotent nature of the Poisson operator \(2.1\) advantageously makes the Moyal product polynomial, producing local deformed theories. Q-deformations, in contrast to D-deformations\(^2\), break supersymmetry, but preserve chirality and, in the \(\mathcal{N} = (1, 1)\) harmonic superspace case, also Grassmann harmonic analyticity and the harmonic conditions \(D^{\pm\pm} A = 0\), which are preserved in virtue of the properties \([12]\)

\[ [D^{\pm}_\alpha, P] = 0, \quad [D^{\pm}_{\dot{\alpha}}, P] = 0, \quad [D^{\pm\pm}, P] = 0, \quad (2.3) \]

A proper definition of the anticommutators involving the bilinear operator \(P\) is given by

\[ A[\epsilon \cdot G, P] B \equiv -C^{\alpha\beta}_{ij} (\epsilon \cdot \partial^i_{\alpha} A \partial^j_{\beta} B + \partial^i_{\alpha} A [\epsilon \cdot G, \partial^j_{\beta} B]). \quad (2.4) \]

When \(G\) is the generator of a symmetry \(\delta_\epsilon A = -\epsilon^a G_a A\), the commutator above measures to what extent the Moyal product breaks the Leibniz rule for its transformation laws

\[ \delta_\epsilon (A \star B) = \delta_\epsilon A \star B + A \star \delta_\epsilon B. \quad (2.5) \]

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\(^1\)In the sequel, we use all conventions in \([15]\). Note that all operators are left derivations unless explicitly stated. The only right derivation appears on the Poisson operator.

\(^2\)See for example \([16]\)
The deformation parameters \( C^{\alpha \beta}_{ij} = C_{ji}^{\beta \alpha} \) form a constant tensor which can be split in the following way \([9, 12]\)
\[
C^{\alpha \beta}_{ij} = I \varepsilon^{\alpha \beta} \varepsilon_{ik} + \hat{C}^{\alpha \beta}_{ik} .
\] (2.6)

Q-deformations induced by the first term are called singlet or QS-deformations, whereas those associated with the second term can naturally be named non-singlet or QNS-deformations. The singlet term in (2.6) is \( \text{Spin}(4) \times \text{SU}(2) \)-preserving while the non-singlet term involves a \( \text{SU}(2)_L \times \text{SU}(2)_R \) constant tensor which is symmetric under independent permutations of Latin and Greek indices. \( \hat{C}^{\alpha \beta}_{ik} \) in general breaks the space-time and R-symmetry groups \( \text{Spin}(4) \times O(1,1) \times \text{SU}(2) \equiv \text{SU}(2)_L \times \text{SU}(2)_R \times O(1,1) \times \text{SU}(2) \) down to \( \text{SU}(2)_R \). Nevertheless, choosing a particular factorizable form
\[
\hat{C}^{\alpha \beta}_{ij} = c^{(\alpha \beta)} b_{(ij)} ,
\] (2.7)
we are able to recover part of the symmetry group leaving \( \text{U}(1)_L \times \text{SU}(2)_R \times \text{U}(1) \) unbroken. We will see that the bosonic sector of our resulting actions are manifestly invariant under the complete space-time and R-symmetry group. It is clear that, with this matrix decomposition, we discard three degrees of freedom among the nine parameters of the generic non-singlet tensor, leading to the maximal symmetry preserving selection. Observing the structure of the Moyal product in Q-deformations, it is not hard to realize that theories constructed in this frame will have at least 1/4 supersymmetries lost. A simple way to see this is by looking at the only nontrivial commutator \([2.4]\) for supersymmetry charges, namely
\[
A[\bar{\epsilon} \cdot \bar{Q}, P] B = 2i \left( I \varepsilon^{\alpha \beta} \varepsilon_{ij} + \hat{C}^{\alpha \beta}_{ij} \right) \bar{\epsilon}^{\dot{\alpha} i} \left( \partial_{\dot{\alpha} \dot{\alpha}} A \partial^i \dot{J}^j B - \partial^j \dot{A} \partial_{\dot{\alpha} \dot{\alpha}} B \right) .
\] (2.8)

Here we can appreciate that for a generic tensor \( C^{\alpha \beta}_{ij} \) as well as for any value of \( I \) in a singlet deformation, \( \mathcal{N} = (1,1) \) supersymmetry is broken to \( \mathcal{N} = (1,0) \) \([5]\). Only for particular purely non-singlet parameters we are able to enhance the supersymmetry to \( \mathcal{N} = (1,1/2) \) \([17,18]\). For example, from
\[
\hat{C}^{\alpha \beta}_{11} \neq 0 , \quad \hat{C}^{\alpha \beta}_{12} = \hat{C}^{\alpha \beta}_{22} = 0
\] (2.9)
and \([2.9]\), the commutation of \( \varepsilon^{\dot{\alpha} 2} \bar{Q}_{\dot{\alpha} 2} \) with \( P \) obviously follows. Therefore, implementing \([2.9]\), the supersymmetry is broken down to \( \mathcal{N} = (1,1/2) \) recovering the 1/4 fraction generated by \( \bar{Q}_{\dot{2} 2} \). The exact expressions for non-singlet gauge and supersymmetric transformations for the \( \text{U}(1) \) vector multiplet with \([2.9]\) were first constructed in \([18]\), where
the authors also constructed the $\mathcal{N} = (1,1/2)$ invariant action in components to first order in the deformation $C$. In [15] we constructed the bosonic action using the maximally space-time and R-symmetry preserving parameters in (2.7) for the generic case and with $b_{ij}$ restricted to

$$b_{11} \neq 0 \quad b_{12} = b_{22} = 0.$$  \hspace{1cm} (2.10)

which is easily seen to be equivalent to the general solution for vanishing determinant $b^2 = \varepsilon^{ik}\varepsilon^{jl}b_{ij}b_{kl} = 0$. In this case $b_{ij}$ has rank 1 and admits a tensor product decomposition $b_{ij} = b_i b_j$. By means of an appropriate SU(2) rotation one can pick (2.10) without loss of generality.

3 Non-singlet $\mathcal{N} = (1,0)$ and $\mathcal{N} = (1,1/2)$ Q-deformed actions

We start from the $\mathcal{N} = (1,1)$ Abelian gauge multiplet in four dimensional Euclidean harmonic superspace. As we pointed out in the introduction, the corresponding non-singlet Q-deformed models have some fractions of the original supersymmetry broken. Though valuable effort has been done obtaining the deformed components action in powers on the full set of deformation parameters [18, 20–22], it is clear that obtaining exact expressions for the deformed action in the general case is a very difficult task, even using the matrix decomposition (2.7). For the pure bosonic case [15] we found a closed from of the action using (2.7) and moreover, we were able to redefine the fields in such a way that the Lagrangian took a particular factorized form $\cosh^2(2\phi \sqrt{b^2c^2})L_0$ where $L_0$ is the free undeformed Lagrangian. In the present full supersymmetric case it seems to be not an easy labor to accomplish that kind of simplicity. Nevertheless, it is worthy to analyze different possibilities of the non-singlet deformed action coming from selecting particular structures of the deformation tensor $b_{ij}$. For example, we can interpret $b_{ij}$ as the set of supersymmetry breaking tuning parameters, i.e. specific selections of $b_{ij}$ distinguish between different theories with $\mathcal{N} = (1,0)$, $\mathcal{N} = (1,1/2)$ and $\mathcal{N} = (1/2,1/2)$ supersymmetry, some of them with very simple Lagrangians.

For our purposes, the most appropriate QNS-deformed $\mathcal{N} = (1,1)$ U(1) gauge theory action in harmonic superspace [13], is written in terms of the covariant superfield strength
\[ W = -\frac{1}{4}(\bar{D}^+)V^- \] (3.1)

for which the action takes the form

\[ S = \frac{1}{4} \int d^4x d^4\theta d\bar{\theta} \mathcal{W} \star \mathcal{W} = \frac{1}{4} \int d^4x d^4\theta d\bar{\theta} \mathcal{W}^2. \] (3.2)

Note that \( V^- \) is non-analytic. A general expansion in components reads

\[
V^- = v^- + \bar{\theta}_a^+ v^{(-3)} \dot{\alpha} + \bar{\theta}_a^- v^{- \dot{\alpha}} + (\bar{\theta}^-)^2 A + (\bar{\theta}^+ \bar{\theta}^-) \varphi^- + \bar{\theta}^{- \dot{\alpha}} \bar{\theta}^+ \bar{\theta}^{- \dot{\beta}} \varphi_{\dot{\alpha} \dot{\beta}} + (\bar{\theta}^+)^2 v^{(-4)} + (\bar{\theta}^-)^2 \bar{\theta}_a^+ \tau^{- \dot{\alpha}} + (\bar{\theta}^+)^2 \bar{\theta}_a^- \tau^{(-3)} \dot{\alpha} + (\bar{\theta}^+)^2 (\bar{\theta}^-)^2 \tau^{-}. \] (3.3)

It can be shown [5] that only \( A \), the coefficient of \((\bar{\theta}^-)^2\) in (3.1), contributes to the action

\[ S = \frac{1}{4} \int d^4x d^4\theta d\bar{\theta} A^2. \] (3.4)

This coefficient can be obtained from the flatness equation

\[ D^{++} V^- - D^- V^{++}_{WZ} + [V^{++}_{WZ}, V^-]_* = 0. \] (3.5)

where \( V^{++}_{WZ} \) is the harmonic superfield which carries the \( \mathcal{N} = (1, 1) \) vector multiplet. In chiral coordinates we have

\[ V^{++}_{WZ} = v^{++} + \bar{\theta}_a^+ v^{+ \dot{\alpha}} + (\bar{\theta}^+)^2 v, \] (3.6)

\[ v^{++} = (\bar{\theta}^+)^2 \bar{\phi}, \] (3.7a)

\[ v^{+ \dot{\alpha}} = 2\theta^+ A^\dot{\alpha} + 4(\bar{\theta}^+)^2 \Psi^{\dot{\alpha}} - 2i(\bar{\theta}^+)^2 \theta^{- \alpha} \partial^\alpha \bar{\phi}, \] (3.7b)

\[
v = \phi + 4\theta^+ \Psi^- + 3(\bar{\theta}^+)^2 D^- - i(\theta^+ \bar{\theta}^-) \partial^{\alpha \dot{\alpha}} A_{\alpha \dot{\alpha}} + \theta^{- \alpha} \theta^+ \beta F_{\alpha \beta} - (\theta^+)^2 (\theta^-)^2 \Box \bar{\phi} + 4i(\bar{\theta}^+)^2 \theta^{- \alpha} \partial_{\alpha \dot{\alpha}} \Psi^{\dot{\alpha}}. \] (3.7c)

The components of (3.5) relevant to determine \( A \) are

\[ \nabla^{++} A = 0, \] (3.8a)

\[ \nabla^{++} v^{- \dot{\alpha}} - v^{+ \dot{\alpha}} = 0, \] (3.8b)

\[ \nabla^{++} \varphi^{-} + 2(A - v) + \frac{1}{2} \{ v^{+ \dot{\alpha}}, v_{\dot{\alpha}}^- \}_* = 0. \] (3.8c)
where
\[ \nabla^{++} = D^{++} + [v^{++}, \cdot] \]  
(3.9)
and \( v^{++}, v^+ \) and \( v \) are defined in (3.7a), (3.7b), (3.7c) respectively. The Q-deformed commutator in (3.9), for a general chiral superfield \( \Phi(x_L, \theta^\pm_\alpha) \) (irrespective of the Grassmann parity of the latter), reads
\[ [v^{++}, \Phi] = -2\partial_{++}v^{++}\partial_{+}\beta\Phi b^{++}c^{\alpha\beta} - 2\partial_{++}v^{++}\partial_{-}\beta\Phi b^{+}c^{\alpha\beta}. \]
(3.10)

Then for the product ansatz (2.7), \( \nabla^{++}\Phi \) becomes
\[ \nabla^{++}\Phi = [\partial^{++} - (\varepsilon^{\alpha\beta} + 4\bar{\phi}b^{++}c^{\alpha\beta}) \theta^+_\alpha \partial_{-}\beta - 4\bar{\phi}b^{++}c^{\alpha\beta}\theta^+_\alpha \partial_{+}\beta] \Phi. \]
(3.11)

It remains to solve the coupled system of equations (3.8), plug the solutions into (3.4) and integrate in the harmonic variables using the list of integrals in Appendix B of [15]. Due to the complexity of this calculation, it was performed with help of a symbolic algebra computer package, resulting into a very lengthy action in components which for our further analysis is not necessary to present here. In fact, to show what happens for particular values of \( b^{ij} \) is much more illustrative.

3.1 \( \mathcal{N} = (1, 1/2) \) supersymmetry action in components

Here, we consider the situation with \( b^2 = 0 \), i.e. for those components of \( b^{ij} \) which are solutions of equation \( \det (b_{ij}) = b_{11}b_{22} - (b_{12})^2 = 0 \). The action in components is
\[
S = \int d^4x_L \left[ -\frac{1}{2} \phi \Box \bar{\phi} - \frac{1}{16} F^{\alpha\beta} F_{\alpha\beta} + \frac{1}{4} D^2 + i\Psi^{k\alpha} \partial_{ad} \bar{\Psi}^\alpha_k + ib_{ij} D^{ij} c^{\alpha\beta} \partial_{(ad)} \bar{\phi} A^\alpha_{b} \\
+ \frac{1}{2} \bar{\phi} b_{ij} D^{ij} c^{\alpha\beta} F_{\alpha\beta} + \frac{4i}{3} b_{ij} c^{\alpha\beta} A_{a\dot{a}} \bar{\Psi}^{i\dot{a}} \partial_{\beta\dot{\beta}} \bar{\Psi}^{j\dot{\beta}} - 4ib_{ij} c^{\beta\alpha} \Psi^{i\beta} \partial_{a\dot{a}} \bar{\phi} \bar{\Psi}^{j\dot{a}} \\
- \frac{4}{3} ib_{ij} c^{\dot{a}\beta} \Psi^{i\dot{a}} \partial_{\dot{a}\beta} \bar{\Psi}^{j\dot{\beta}} + c^{\alpha\beta} F_{a\beta\dot{a}} b_{ij} \bar{\Psi}^i_{\alpha} \bar{\Psi}^{j\dot{a}} - 4c^2 (b_{ij} \bar{\Psi}^i_{\alpha} \bar{\Psi}^{j\dot{a}})^2 \\
- \frac{32}{9} \phi c^2 b_{ij} D^{ij} b_{kl} \bar{\Psi}^k_{\dot{a}} \bar{\Psi}^{l\dot{a}} \right].
\]
(3.12)

First of all, we remark that this is an exact result for which 3/4 of the original supersymmetries are preserved. The main feature of (3.12) is that we can decouple the interaction between the scalar field \( \bar{\phi} \) and the gauge field and still have a deformed action, contrary to what happens in the singlet case where decoupling the mentioned interaction destroys...
the deformation [5]. Observe also that even in this case, second order terms in the deformation parameters appear. From the corresponding gauge variations we directly propose the minimal Seiberg-Witten like map which take us back to the standard form of the gauge transformations. In [15] we obtained the full set of exact variations, they are

\[ \delta \bar{\phi} = 0, \quad \delta \Psi^k_\alpha = 0, \quad \delta A_{\alpha \bar{\alpha}} = X \coth X \partial_{\alpha \bar{\alpha}} a \]
\[ \delta D_{ij} = 2ib_{ij}c^{\alpha \beta} \partial_{\alpha \bar{\alpha}} \bar{\phi} \partial_{\beta \bar{\beta}} a, \]
\[ \delta \phi = 2\sqrt{c^2 b^2} \left( \frac{1 - X \coth X}{X} \right) A^{\alpha \bar{\alpha}} \partial_{\alpha \bar{\alpha}} a, \]
\[ \delta \Psi^i_\alpha = \left\{ \left[ \frac{4X^2(X \coth X - 1)}{X^2 + \sinh^2 X - X \sinh 2X} \right] b^j c_{\alpha \beta} \right. \]
\[ \left. - \sqrt{c^2 b^2} \left[ \frac{4X \cosh^2 X - 2X^2(\coth X + X) - \sinh 2X}{X^2 + \sinh^2 X - X \sinh 2X} \right] \varepsilon^{ij} \varepsilon_{\alpha \beta} \right\} \bar{\Psi}^j_\alpha \partial_{\beta \bar{\beta}} a. \]

where \( X = 2\bar{\phi} \sqrt{c^{\alpha \beta} c_{\alpha \beta}} b^k b^k \). Imposing \( b^2 = 0 \) we have

\[ \delta A_{\alpha \bar{\alpha}} = \partial_{\alpha \bar{\alpha}} a, \quad \delta \phi = 0, \quad \delta \Psi^i_\alpha = -\frac{4}{3} b^j c_{\beta} \bar{\Psi}^j_\beta \partial_{\alpha \bar{\alpha}} a, \quad \delta D_{ij} = 2ib_{ij}c^{\alpha \beta} \partial_{\alpha \bar{\alpha}} \bar{\phi} \partial_{\beta \bar{\beta}} a \]

Thus the Seiberg-Witten-like map becomes

\[ \Psi^i_\beta = \bar{\Psi}^i_\beta - \frac{4}{3} b^j c_{\beta} \bar{\Psi}^j_\beta A_{\alpha \bar{\alpha}}, \quad D_{ij} = \bar{D}_{ij} - 2ib_{ij}c^{\alpha \beta} A_{\alpha \bar{\alpha}} \bar{\phi} \partial_{\beta \bar{\beta}} a \]

Moreover, we can further redefine \( \bar{\Psi}^k_\alpha \) and \( \bar{D}^{ij} \)

\[ \bar{\Psi}^k_\beta = \psi^{k \beta} - \frac{4}{3} ib^k c_{\beta} \phi \bar{\Psi}^j_\alpha \]
\[ \bar{D}^{ij} = d^{ij} - \phi b^{ij} c^{\alpha \beta}, \quad \bar{D}_{ij} = \frac{64}{9} \phi \bar{c}^2 b^k b^k \bar{\Psi}^k_\alpha \bar{\Psi}^j_\alpha \]

to finally obtain the simple expression

\[ S = \int d^4x L \left[ -\frac{1}{2} \phi \Box \bar{\phi} - \frac{1}{16} F^{\alpha \beta} F_{\alpha \beta} + \frac{1}{4} a^2 \right. \]
\[ + i \psi^{k \alpha} \partial_{\alpha \bar{\alpha}} \bar{\Psi}^k_\alpha - 4ib_{ij}c^{\alpha \beta} \psi^{i \beta} \partial_{\alpha \bar{\alpha}} \bar{\phi} \bar{\Psi}^j_\alpha \]
\[ - c^{\alpha \beta} F_{\alpha \beta} b^i \bar{\Psi}^i_\alpha \bar{\Psi}^{j \alpha} - 4c^2 (b^j \bar{\Psi}^i_\alpha \bar{\Psi}^{j \alpha})^2 \]

The last three terms are not removable under field redefinitions, meaning we are in presence of an interacting theory. Particularly the last two terms are of the same kind as those found in [2], where authors construct a deformed extension of the low energy D3-brane
super Yang-Mills action. Besides, it is very remarkable the occurrence of an additional Yukawa-like interaction potential. This result is also comparable with the first order action found in [18], where the authors brought up the question, whether the exact action has higher order terms or not. It is clear that at least for the product ansatz (2.7) we are able to give an answer: though we already hid almost all second order terms appearing in the action, the last term $4c^2(b_{ij}\overline{\psi}_i^{\dot{\alpha}}\overline{\psi}_j^{\dot{\beta}})^2$ seems to be irremovable.

3.2 $\mathcal{N} = (1, 1/2) \to \mathcal{N} = (1, 0)$ supersymmetry breaking

Turning on $b^2 \neq 0$ contributions and applying the corresponding Seiberg-Witten map, we obtain the following action up to first order in $b^2$

$$S = \int d^4x_L \left[ -\frac{1}{2} \phi \Box \overline{\phi} - \frac{1}{16} \tilde{F}^2 + \frac{1}{4} \tilde{D}^2 + i\overline{\psi}^{\kappa \alpha} \partial_{\alpha} \overline{\psi}_k^{\dot{\alpha}} - 4i b_{ij} c_{\alpha}^{\dot{\beta}} \overline{\psi}_j^{\dot{\beta}} \partial_{\alpha} \overline{\psi}_i^{\dot{\alpha}} - 4i \overline{\phi} b_{ij} c_{\alpha}^{\dot{\beta}} \overline{\psi}_j^{\dot{\beta}} \partial_{\alpha} \overline{\psi}_i^{\dot{\alpha}} \right] + O(b^3).$$

The most important feature of this action is the non trivial interaction term

$$\frac{b^2 c^2}{6} \overline{\phi}^2 \tilde{F}^2 + b^2 c^2 \overline{\phi}^2 \tilde{D}^2 + c^{\alpha \beta} \tilde{F}_{\alpha \beta} b_{ij} \overline{\psi}_i^{\dot{\alpha}} \overline{\psi}_j^{\dot{\beta}} + 4 c^2 (b_{ij} \overline{\psi}_i^{\dot{\alpha}} \overline{\psi}_j^{\dot{\beta}})^2$$

These kind of interactions appearing here and in [5, 15] can not be disentangled by a redefinition of the fields. In order to give an interpretation of parameter $b^{ij}$ one can for example consider the limit

$$b_{11} = 1, \quad b_{12} = 0, \quad b_{22} \ll 1.$$ 

Action (3.19) can be interpreted as the weak coupling limit of an interacting theory for $\overline{\phi}$ and the gauge field, with $b_{22}$ as the coupling parameter. Another interpretation of selection (3.21) (see [15]) comes from taking $\theta_1^{\alpha}$ as the left Graßmann coordinate of some $\mathcal{N} = (1/2, 1/2)$ subspace of $\mathcal{N} = (1, 1)$ superspace, i.e. $\theta_1^{\alpha} \equiv \theta^{\alpha}$. Assuming the pseudoconjugation for all involved quantities as in [12], and selecting the relevant broken automorphism U(1) and O(1,1) symmetries of $\mathcal{N} = (1, 1)$ superalgebra in such a
way that $b_{ik} \equiv (b_{11}, b_{22}, b_{12}) = (1, b_{22}, 0)$, the deformation operator \((2.1)\) for the choice \((2.7)\) and $I = 0$ is reduced to $P = -\partial_\alpha c^{\alpha \beta} \partial_\beta - b_{22} \partial_\alpha c^{\alpha \beta} \partial_\beta$. In other words, it can be expressed as a sum of the mutually commuting chiral Poisson operators on two different $\mathcal{N} = (1/2, 1/2)$ subspaces of $\mathcal{N} = (1, 1)$ superspace, with $b_{22}$ being the “ratio” of two Seiberg deformation matrices. When $b_{22} = 0$, we fall back into the case with only one $\mathcal{N} = (0, 1/2)$ supersymmetry broken. For $b_{22} \neq 0$, both $\mathcal{N} = (0, 1/2)$ supersymmetries are broken. The parameter $b_{22}$ measures the breakdown of the second $\mathcal{N} = (0, 1/2)$ supersymmetry which is implicit in the $\mathcal{N} = (1/2, 1/2)$ superfield formulation based on the superspace $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$. Recall that within the standard complex conjugation the reduction to $\mathcal{N} = (1/2, 1/2)$ superspace makes no sense since the latter is not closed under such conjugation [12].

4 Non-singlet unbroken supersymmetry transformations

In [15] we presented detailed procedures involved in the calculation of supersymmetry transformations and we gave a subalgebra as an example. Here, we give the corresponding set of transformations to each case presented in the former section. We start by discussing the unbroken $\mathcal{N} = (1, 1/2)$ supersymmetry transformations corresponding to the action \((3.12)\). We recall that this transformations were already calculated in [18] by choosing the particular matrix $\hat{C}_{11}^{\alpha \beta}$ and they are in fact equivalent to our results when $\hat{C}_{11}^{\alpha \beta} = c^{\alpha \beta} b_{11}$. Nevertheless, as we pointed out before, once we have done this factorization it is equivalent to choose any solution of $\det (b_{ij}) = 0$ recovering the manifest R-invariant symmetry of the expressions, a worthy reason to show the $\mathcal{N} = (1, 0)$ sector as an example

$$\delta \bar{\phi} = 0,$$  \hspace{1cm} \((4.1)\)

$$\delta A_{\alpha \dot{\alpha}} = \left[2 \varepsilon_{\alpha \beta} \varepsilon_{ij} + 8 \bar{\phi} c^{\alpha \beta} b_{ij} \right] e^{i \beta} \Psi_{\dot{\alpha}}^j,$$  \hspace{1cm} \((4.2)\)

$$\delta \bar{\Psi}_{\dot{\alpha}}^i = -i \left[ \varepsilon^{\alpha \beta} \varepsilon_{ij} - 4 \bar{\phi} c^{\alpha \beta} b_{ij} \right] \epsilon_{j \beta} \partial_{\alpha \dot{\alpha}} \bar{\phi},$$  \hspace{1cm} \((4.3)\)

$$\delta \phi = \Psi_{\alpha}^{i j} \left[ 2 \varepsilon^{\alpha \beta} \varepsilon_{ij} + \frac{16}{3} \bar{\phi} c^{\alpha \beta} b_{ij} \right] + A_{\alpha}^{i} \bar{\Psi}_{\dot{\alpha}}^j \left[ \frac{40}{3} c^{\alpha \beta} b_{ij} \right].$$  \hspace{1cm} \((4.4)\)
\[ \delta D^{ij} = 2i \partial^{\alpha \dot{\alpha}} \left[ \epsilon_{\alpha}^{(i} \bar{\Psi}^{j)} + 4i \phi e^{(k \beta} \bar{\Psi}^{i)} c_{\alpha \beta} b_{k}^{j} - 4i \bar{\phi}^{2} c^{2} b^{ij} e_{\alpha}^{k} \bar{\Psi}^{l} b_{kl} \right] \] (4.5)

\[ \delta \bar{\Psi}^{\alpha \dot{\alpha}} = \left( -D^{ij} \epsilon^{\alpha \beta} + \left\{ \frac{1}{2} F^{\alpha \beta} + \frac{8}{3} \left[ (b \cdot \bar{\Psi} \bar{\Psi}) - \bar{\phi} (b \cdot D) \right] c^{\alpha \beta} \right\} \right) \epsilon^{ij} \]

\[ + \left\{ \frac{2}{3} F^{\alpha \beta} + \left\{ \frac{40}{3} c^{2} \bar{\phi} (b \cdot \bar{\Psi} \bar{\Psi}) - \frac{28}{3} c^{2} \bar{\phi}^{2} (b \cdot D) + \frac{\sqrt{2} c^{2}}{2} i (A \cdot \partial \bar{\phi}) - 2i \bar{\phi} (c \cdot G) \right\} \epsilon^{\alpha \beta} \]

\[ - \frac{2}{3} i \left[ 2 (A \cdot \partial \bar{\phi}) + \bar{\phi} (\partial \cdot A) \right] c^{\alpha \beta} \right\} \epsilon^{ij} \] (4.6)

For the \( \mathcal{N} = (1, 1/2) \) supersymmetry, we could also calculate the \( \mathcal{N} = (0, 1/2) \) unbroken sector generated by \( \bar{Q}_{1\dot{\alpha}} \) which would be absolutely equivalent to the exact result presented in [18]. Finally, we display the full unbroken \( \mathcal{N} = (1, 0) \) transformation laws which leaves \( \bar{8} \) invariant. They are

\[ \delta \bar{\phi} = 0, \] (4.7a)

\[ \delta A_{\alpha \dot{\alpha}} = \left[ 2 \left( 1 + \frac{4}{3} b^{2} c^{2} \bar{\phi}^{2} \right) \epsilon_{\alpha \beta} \epsilon_{ij} + 8 \bar{\phi} c_{\alpha \beta} b_{ij} \right] \epsilon^{ij} \bar{\Psi}^{\dot{\alpha}}, \] (4.7b)

\[ \delta \bar{\Psi}^{i}_{\dot{\alpha}} = -i \left[ (1 + 4 b^{2} c^{2} \bar{\phi}^{2}) \epsilon^{\alpha \beta} \epsilon_{ij} - 4 \bar{\phi} c^{\alpha \beta} b^{ij} \right] \epsilon_{j \beta} \partial_{\alpha \dot{\alpha}} \bar{\phi} + O(b^{3}), \] (4.7c)

\[ \delta \phi = \Psi_{\alpha}^{i} \epsilon_{\beta} \left[ 2 \epsilon_{\alpha \beta} \epsilon_{ij} + \frac{16}{3} \bar{\phi} c^{\alpha \beta} b_{ij} \right] + A^{\dot{\alpha}} \bar{\Psi}^{i}_{\dot{\alpha}} \epsilon_{\beta} \left[ \frac{40}{3} c^{\alpha \beta} b_{ij} \right] + O(b^{3}), \] (4.7d)

\[ \delta D^{ij} = 2i \partial^{\alpha \dot{\alpha}} \left[ \left( 1 + \frac{1}{3} b^{2} c^{2} \bar{\phi}^{2} \right) \epsilon_{\alpha}^{(i} \bar{\Psi}^{j)} + 4i \phi e^{(k \beta} \bar{\Psi}^{i)} c_{\alpha \beta} b_{k}^{j} - 4i \bar{\phi}^{2} c^{2} b^{ij} e_{\alpha}^{k} \bar{\Psi}^{l} b_{kl} \right] + O(b^{3}), \] (4.7e)

\[ \delta \Psi^{\alpha \dot{\alpha}} = \left( -D^{ij} \epsilon^{\alpha \beta} + \left\{ \left( \frac{1}{2} + \frac{10}{9} b^{2} c^{2} \bar{\phi}^{2} \right) F^{\alpha \beta} + 2ib^{2} c^{2} \bar{\phi} G^{\alpha \beta} \right\} \right) \epsilon^{ij} \]

\[ + \frac{2}{3} i b^{2} c^{2} \bar{\phi}^{2} \left[ 8 (A \cdot \partial \bar{\phi}) + \bar{\phi} (\partial \cdot A) \right] \epsilon^{\alpha \beta} + \frac{2}{3} \left\{ 4 (b \cdot \bar{\Psi} \bar{\Psi}) - 4 \bar{\phi} (b \cdot D) \right\} \epsilon^{ij} \]

\[ - \frac{\sqrt{2} c^{2}}{3} i b^{2} \bar{\phi} (A \cdot \partial \bar{\phi}) - \frac{5}{3} b^{2} \bar{\phi}^{2} (c \cdot F) - \frac{2}{3} i b^{2} \bar{\phi}^{2} (c \cdot G) \epsilon^{ij} \]

\[ + \left\{ \frac{2}{3} F^{\alpha \beta} + \left\{ \frac{40}{3} c^{2} \bar{\phi} (b \cdot \bar{\Psi} \bar{\Psi}) - \frac{28}{3} c^{2} \bar{\phi}^{2} (b \cdot D) + \frac{\sqrt{2} c^{2}}{2} i (A \cdot \partial \bar{\phi}) - 2i \bar{\phi} (c \cdot G) \right\} \epsilon^{\alpha \beta} \]

\[ - \frac{2}{3} i \left[ 2 (A \cdot \partial \bar{\phi}) + \bar{\phi} (\partial \cdot A) \right] \epsilon^{\alpha \beta} \right\} \epsilon^{ij} + O(b^{3}) \] (4.7f)
where we have defined the following shorthands

\[ G_{\alpha\beta} \equiv A_{(\alpha\dot{\alpha}} \partial_{\beta)} \], \quad \mathcal{F}^{\alpha\beta} \equiv c^{(\alpha\gamma} F_{\gamma}\beta) \], \quad (b \cdot \bar{\Psi} \Psi) \equiv b_{ij} \bar{\Psi}^i \bar{\Psi}^{i\dot{\alpha}} \]  

We would like to comment that these expressions were not calculated using series expansions on the deformation parameters. Implementing algorithms given in [15] and using a computer program, we actually obtained the corresponding extremely lengthy exact results and took the appropriate limit afterwards.

5 Conclusions

We have studied non-singlet Q-deformations of \( \mathcal{N} = (1, 1) \) gauge theories in harmonic superspace in four Euclidean dimensions, using the decomposition matrix \( \check{C}_{ij} = c^{\alpha\beta}b_{ij} \) which preserves space-time and R-symmetry group \( U(1)_L \times SU(2)_R \times U(1) \). Imposing the condition \( b^2 = 0 \) we built the exact expression of the \( \mathcal{N} = (1, 1/2) \) action. This Lagrangian is characterized by the presence of interaction terms comparable with the \( \mathcal{N} = 1 + \frac{1}{2} \) deformed low energy action of a D3-brane constructed in [2]. It is also worth notice that the interaction potential found has a Yukawa-like term. It is notable that there are second order terms in the deformation parameters which can not be removed by redefinition of the fields. It is also remarkable that despite the complete removal of the interaction between the scalar field \( \tilde{\phi} \) and the gauge field, we still have a deformed action, contrary to what happens in the singlet case where decoupling the mentioned interaction implies the complete disappearing of deformation [5]. We can say that this is an exclusive feature of non-singlet deformations.

Additionally we study the behavior of the action upon restoring the degrees of freedom in \( b^{ij} \), by analyzing the structure of the first terms with non trivial \( b^2 \). We recall that this action has \( \mathcal{N} = (1, 0) \), thus we conclude that we have broken the supersymmetry by turning on the \( b^2 \neq 0 \) parameters. The most remarkable feature of this result is the non trivial interaction term

\[ \frac{b^2 c^2}{6} \bar{\phi}^2 \check{F}^{\alpha\beta} \check{F}_{\alpha\beta} \]  

This striking interaction, which in general characterizes Q-deformations of \( \mathcal{N} = (1, 1) \) gauge multiplet (see for example [5, 15]), is not possible to disentangle via redefinition of the fields. In general terms we can interpret components of \( b^{ij} \) as supersymmetry breaking tuning parameters. Turning on some of \( b^{ij} \) degrees of freedom we expose the non
trivial interactions between $\bar{\phi}$ and the gauge field, allowing its interpretation as coupling constants.

It would be interesting to study the renormalizability properties of these actions, and to find their non Abelian extensions, as well as possible instanton solutions emerging from these theories. Another attractive topic is to study non-singlet Q-deformed Hypermultiplets with $b^2 = 0$.

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