COMPARISON OF COUNTABILITY CONDITIONS WITHIN THREE FUNDAMENTAL CLASSIFICATIONS OF CONVERGENCES

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Abstract. The interrelations between various classes of convergence spaces defined by countability conditions are studied. Remarkably, they all find characterizations in the usual space of ultrafilters in terms of classical topological properties. This is exploited to produce relevant examples in the realm of convergence spaces from known topological examples.

1. Introduction

Classically, topologies can be described in a variety of ways, including by prescribing the convergence of filters. Continuity from this point of view is simply preservation of limits. Relaxing the resulting axioms of convergence yields the larger category of convergence spaces (1) and continuous maps, which behaves better from the categorical viewpoint (2) and thus allows for constructions and methods unavailable in the realm of topologies. As a result, even if one is only interested in the topological case, there is often much to gain in embedding a topological problem in the larger context of convergence spaces. The method can be compared to using complex numbers to solve problems formulated in the reals. An extensive account of this approach can be found in [7].

In order to introduce convergence structures, let us start with convergence of filters in a topological space, which is then given by

\[(1.1) \quad x \in \lim F \iff F \supseteq \mathcal{N}(x),\]

where \(\mathcal{N}(x)\) is the neighborhood filter of \(x\). Clearly, \(\mathcal{N}(x)\) is then the smallest filter converging to \(x\), and the map \(\mathcal{N}()\) associating with each \(x \in X\) a filter additionally has a filter base composed of open sets around \(x\).

More generally, such a map \(\mathcal{V}()\) associating with each \(x \in X\) a filter \(\mathcal{V}(x)\) (with \(x \in \bigcap \mathcal{V}(x)\)) defines a convergence given by

\[(1.2) \quad x \in \lim F \iff F \supseteq \mathcal{V}(x).\]

\(^1\)It should be emphasized that convergence spaces were not introduced out of an unmotivated quest for generality, but rather because natural phenomenon of convergence failed to be topological. In fact, they were introduced by G. Choquet \(^2\) because topologies turned out to be inadequate in the study of so called upper and lower limits of Kuratowski in spaces of closed subsets of a topological space.

\(^2\)In particular, this category is Cartesian-closed, that is, has canonical function space structures yielding an exponential law, and is extensional, that is, has quotients that are hereditary. In other words, it is a quasitopos.
The filter $\mathcal{V}(x)$, called \textit{vicinity filter}, is the smallest filter convergent to $x$. A convergence defined this way is a \textit{pretopology}. Note that while every topology defines a pretopology, there are non-topological pretopologies \cite{11}.

More generally, a convergence $\xi$ on a set $X$ is a relation between points of $X$ and filters on $X$, denoted $x \in \lim_\xi \mathcal{F}$ (or $x \in \lim \mathcal{F}$ if no confusion can occur) whenever $x$ and $\mathcal{F}$ are related, subject to the following two axioms: for every point $x \in X$ and pair of filters $\mathcal{F}$ and $\mathcal{G}$ on $X$,

\begin{equation}
    x \in \lim \{x\}^+ \quad \mathcal{F} \subseteq \mathcal{G} \implies \lim \mathcal{F} \subseteq \lim \mathcal{G},
\end{equation}

where $\{x\}^+$ is the principal ultrafilter of $x$. Evidently, pretopologies, \textit{a fortiori} topologies, satisfy these axioms.

Convergences have been classified according to various criteria. We focus here on three directions of classification, all introduced by S. Dolecki (See in particular \cite{3}, \cite{4} and \cite{5}):

(1) According to how many filters need to be given at each point to describe the convergence. This leads to the notion of \textit{paving number} $p$ introduced in \cite{5}, and \textit{pseudopaving numbers} $pp$ introduced in \cite{10}. Pretopologies are exactly the convergences of (pseudo) paving number 1. We will focus here on convergences of countable (pseudo) paving number.

(2) According to \textit{depth}, that is, the cardinality of sets of filters for which $\lim$ and $\cap$ commute. Namely, in a pretopology,

\begin{equation}
    \lim \bigcap_{\mathcal{F} \in \mathcal{D}} \mathcal{F} = \bigcap_{\mathcal{F} \in \mathcal{D}} \lim \mathcal{F}
\end{equation}

for any family $\mathcal{D}$ of filters. A convergence is $\kappa$-deep if (1.3) holds for every set $\mathcal{D}$ of filters of cardinality at most $\kappa$. We focus here on \textit{countably deep} convergences, and those \textit{countably deep for ultrafilters}, that is, when $\mathcal{D}$ is restricted to a countable set of ultrafilters.

(3) According to the class of filters whose adherences determine the convergence. \textit{Adherence} of a filter $\mathcal{F}$ is given by $\text{adh} \mathcal{F} = \bigcup_{\mathcal{H} \subseteq \mathcal{F}} \lim \mathcal{H}$. In the case of a principal filter $\{A\}^+$, we write $\text{adh} A$ for $\text{adh} \{A\}^+$. It turns out \cite{3} that the formula

\begin{equation}
    \lim \mathcal{F} = \bigcap \{\text{adh} D : D \in \mathcal{D}, D \# F\},
\end{equation}

where $D \# F$ means that $D \cap F \neq \emptyset$ whenever $D \in \mathcal{D}$ and $F \in \mathcal{F}$, and $\mathcal{D}$ denotes a class of filters, characterizes several important classes of convergences. Namely, if $\mathcal{D}$ is the class $\mathcal{F}_0$ of principal filters, convergences satisfying (1.4) are exactly pretopologies. If $\mathcal{D}$ is the class $\mathcal{F}$ of all filters, they are exactly the \textit{pseudotopologies} introduced by G. Choquet \cite{2}. Considering the intermediate classes $\mathcal{F}_1$ of countably based filters and $\mathcal{F}_{\Lambda 1}$ of filters closed under countable intersections define the classes of \textit{paratopologies} \cite{3}.

\[\text{For instance sequential adherence in a sequential non Fréchet-Urysohn space defines a non-topological pretopology.}\]

While directed graphs cannot be represented as topological spaces unless the relation is transitive, all directed graph can be seen as a pretopology in which $\mathcal{V}(x)$ is the principal filter of $\{x\} \cup \{y : y \rightarrow x\}$.

An easy example on the plane is given by the Féron cross pretopology, in which $\mathcal{V}((x_0, y_0))$ is the filter generated by sets of the form $\{(x_0, y) : |y - y_0| < \epsilon\} \cup \{(x, y_0) : |x - x_0| < \epsilon\}$ for $\epsilon > 0$.\footnote{For instance sequential adherence in a sequential non Fréchet-Urysohn space defines a non-topological pretopology.}
and hypotopologies respectively. Though there are other possibilities as well, we focus here on the latter two classes.

The main goal of this paper is to study the interrelations between the classes of convergences introduced along these three branches of classification. Pretopologies are particular instances of each notion, but they turn out to form exactly the intersection of several pairs of the corresponding classes of convergences. For instance,

**Theorem (Corollary 23).** A convergence is a pretopology if and only if it is a countably deep paratopology.

This applies to hypotopologies because:

**Theorem (Theorem 90).** Every hypotopology is countably deep.

On the other hand,

**Theorem (Theorem 85).** Every countably pseudopaved pseudotopology is a paratopology.

Below is a diagram showing the main classes of convergences considered, with pretopologies as their intersection and examples distinguishing these classes.

![Diagram showing the main classes of convergences](image)

**Figure 1.1.** Examples distinguishing the main classes

It turns out that most of the classes considered are reflective, that is, for each convergence $\xi$ there is the finest convergence $R\xi$ among the convergences coarser...
than $\xi$ that are in the class, and the corresponding operator $R$ is a functor, that is, preserves continuity. Here are the notations we use for the corresponding reflectors:

| class                            | reflector |
|----------------------------------|-----------|
| pseudotopology                   | $S$       |
| paratopology                     | $S_1$     |
| pretopology                      | $S_0$     |
| hypotopology                     | $S_{\Lambda_1}$ |
| countable depth                  | $P_1$     |
| countable depth for ultrafilters | $P_1^U$   |
| pseudotopology of countable depth| $P_1 \triangle S$ |
| pseudotopology of countable depth for ultrafilters | $P_1^U \triangle S = S P_1^U$ |

The understanding emerging from Figure 1.1 is significantly refined by analyzing the reflectors involved. Namely, we establish (Theorem 21) that

$$S_1 P_1^U = S_0,$$

so that in particular $S_1 P_1 = S_0$, but $P_1 S_1 \neq S_0$ (Example 42).

In contrast (Theorem 94)

$$S_{\Lambda_1} S_1 = S_1 S_{\Lambda_1} = S_0.$$

We also analyze when $P_1 = S_0$. This is the case in particular when the pseudopaving number is countable (Proposition 40), but this condition is not necessary (Example 83). Here is a diagram summarizing the main relations:

![Diagram](image)

**Figure 1.2.** A diagram of the main relations between the classes studied. Vertical arrows correspond to an inclusion of classes. For instance the right-bound arrow from $S_0$ to $\rho \leq \omega$ indicates that every pretopology has countable paving number. The dotted arrows correspond to combinations of properties: A pseudotopology whose singletons are cover-Lindelöf is a paratopology, and a paratopology that is countably deep for ultrafilters is a pretopology.
One may note in the diagram above the condition “cover-Lindelöf singletons”. While definitions of compactness and its variants (countable compactness, Lindelöf, etc) in terms of covers or in terms of filters coincide in the topological setting, they become different notions in the general setting of convergences. Remarkably, even singletons do not need to be cover-compact. It turns out that the type of cover-compactness enjoyed by singletons is relevant to our quest, and is thus analyzed in details too.

Finally, a very important aspect of this article is that many results and examples are obtained using topological characterizations of the non-topological notions at hand in spaces of ultrafilters (with their usual topology). It is a striking and somewhat unexpected feature that convergence notions that only make sense for non-topological convergence spaces turn out to find characterizations in the space of ultrafilters in terms of classical topological properties.

More specifically, in [10] countable paving and pseudopaving numbers of a convergence space \((X, \xi)\) were characterized in terms of the topological properties of the subspace 

\[ U_\xi(x) = \{ U \in UX : x \in \text{lim}_\xi U \} \]

of the space \(UX\) of all ultrafilters on \(X\), endowed with its usual Stone topology. It turned out [10] that the same topological properties of the subspace \(UX \setminus U_\xi X\) of non-convergent ultrafilters characterizes completeness and ultra-completeness of \((X, \xi)\), so that known examples of complete non-ultracomplete topological spaces easily yield examples of spaces with countable pseudopaving number, but uncountable paving number.

This approach is systematized in the present paper. As a result, counter-examples to distinguish various classes of convergence spaces are produced thanks to characterizations in terms of topological properties of \(U_\xi(x)\), and by characterizing the same topological property of \(UX \setminus U_\xi X\). In particular, the implications below

\[
\text{compact } \implies \text{hemicompact } \implies \text{\(\sigma\)-compact } \implies \text{Lindelöf } \implies \text{\(\delta\)-closed,}
\]

cannot be reversed, even for a subspace of the form \(UX \setminus U_\xi X\) of a space of ultrafilters \((UX, \beta)\). Each such example provides an \(S \subset \cup^p X\) witnessing properties that allow for the “semi-automatic” production of the corresponding examples for the non-topological local properties of a convergence space studied here. Table 1 at the end of Section 4 gathers the main results in this direction.

2. Preliminaries

We use notations and terminology consistent with the recent book [7]. We refer the reader to [7] for a comprehensive treatment of convergence spaces.

2.1. Set-theoretic conventions and spaces of ultrafilters. If \(X\) is a set, we denote by \(PX\) its powerset, by \([X]^{<\infty}\) the set of finite subsets of \(X\) and by \([X]^\omega\)
the set of countable subsets of $X$. If $\mathcal{A} \subset \mathcal{P}X$, we write

$$\mathcal{A}^\uparrow := \{B \subset X : \exists A \in \mathcal{A}, A \subset B\}$$

$$\mathcal{A}^\cap := \left\{ \bigcap_{S \in \mathcal{F}} S : \mathcal{F} \in [\mathcal{A}]^{<\infty} \right\}$$

$$\mathcal{A}^\cup := \left\{ \bigcup_{S \in \mathcal{F}} S : \mathcal{F} \in [\mathcal{A}]^{<\infty} \right\}$$

$$\mathcal{A}^\# := \{B \subset X : \forall A \in \mathcal{A}, A \cap B \neq \emptyset\}.$$  

We say that two subsets $A$ and $B$ of $\mathcal{P}X$ (henceforth two families of subsets of $X$) mesh, in symbols $A \# B$, if $A \subset B^\#$, equivalently, $B \subset A^\#$.

A family $\mathcal{F}$ of non-empty subsets of $X$ is called a filter if $\mathcal{F} = \mathcal{F}^\cap = \mathcal{F}^\uparrow$ and a filter-base if $\mathcal{F}^\uparrow$ is a filter. Dually, a family $\mathcal{P}$ of proper subsets of $X$ is an ideal if $\mathcal{P} = \mathcal{P}^\cup = \mathcal{P}^\downarrow$ and an ideal-base if $\mathcal{P}^\downarrow$ is an ideal. Of course, $\mathcal{P}$ is an ideal if and only if

$$\mathcal{P}_c := \{X \setminus P : P \in \mathcal{P}\}$$

is a filter. We denote by $\mathcal{F}X$ the set of filters on $X$. Note that $\mathcal{F}X$ is the only family $\mathcal{A}$ satisfying $\mathcal{A} = \mathcal{A}^\uparrow = \mathcal{A}^\cap$ that has an empty element and the only family satisfying $\mathcal{A} = \mathcal{A}^\downarrow = \mathcal{A}^\cup$ that contains $X$. Thus we sometimes call $\{\emptyset\}^\uparrow = \mathcal{P}X = \{X\}^\downarrow$ the degenerate filter on $X$ or the degenerate ideal on $X$. The set $\mathcal{F}X$ is ordered by

$$(2.1) \quad \mathcal{F} \leq \mathcal{G} \iff \forall F \in \mathcal{F} \exists G \in \mathcal{G} \quad G \subset F.$$ 

Infima for this order always exists: if $\mathcal{D} \subset \mathcal{F}X$ then $\bigwedge_{D \in \mathcal{D}} D = \bigcap_{D \in \mathcal{D}} D$. In contrast, the supremum of even a pair of filters may fail to exist. In fact, $\mathcal{F} \cup \mathcal{G}$ exists if and only if $\mathcal{F} \# \mathcal{G}$ and is then generated by sets of the form $F \cap G$ for $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Maximal elements of $\mathcal{F}X$ are called ultrafilters and $\mathcal{F} \in \mathcal{F}X$ is an ultrafilter if and only if $\mathcal{F} = \mathcal{F}^\#$.

Dually to (2.1), we say that a family $\mathcal{R}$ of subsets of $X$ is a refinement of another family $\mathcal{P} \subset \mathcal{P}X$, in symbols, $\mathcal{R} \lt \mathcal{P}$, if for every $R \in \mathcal{R}$ there is $P \in \mathcal{P}$ with $R \subset P$, that is,

$$(2.2) \quad \mathcal{R} \lt \mathcal{P} \iff \forall R \in \mathcal{R} \exists P \in \mathcal{P} \quad R \subset P$$

$$\iff \mathcal{P}_c \geq \mathcal{R}_c.$$  

Given a map $o : \mathcal{P}X \rightarrow \mathcal{P}X$ and $\mathcal{A} \subset \mathcal{P}X$, we write

$$o^\# :\mathcal{A} := \{o(A) : A \in \mathcal{A}\}.$$  

Note that when $o$ is expansive (i.e., $A \subset o(A)$ for all $A \in \mathcal{P}X$) and $\mathcal{F}$ is a filter-base, then $o^\# \mathcal{F}$ is also a filter-base. Hence in the context of filters and expansive operators $o$, we often do not distinguish between $(o^\# \mathcal{F})^\uparrow$ and $o^\# \mathcal{F}$.

We denote by UX the set of ultrafilters on $X$. We call kernel of $\mathcal{F} \in \mathcal{F}X$ the set $\ker \mathcal{F} := \bigcap_{F \in \mathcal{F}} F$.

**Lemma 1.** If $\mathcal{D} \subset \mathcal{F}X$ then

$$\ker \bigwedge_{D \in \mathcal{D}} D = \bigcup_{D \in \mathcal{D}} \ker D.$$  

**Proof.** Suppose $x \notin \bigcup_{D \in \mathcal{D}} \ker D$, that is, $x \notin \ker D$ for every $D \in \mathcal{D}$. In other words, for every $D \in \mathcal{D}$ there is $D_D \in D$ with $x \not\in D_D$. Then $x \notin \bigcup_{D \in \mathcal{D}} D_D$ so that $x \notin \ker \bigwedge_{D \in \mathcal{D}} D$. 


Conversely, if $x \notin \ker \bigwedge_{D \in \mathcal{D}} D$, there is $A \in \bigwedge_{\mathcal{D} \in \mathcal{D}} \mathcal{D}$ with $x \notin A$ so that $x \notin \ker \mathcal{D}$ for all $\mathcal{D} \in \mathcal{D}$, that is, $x \notin \bigcup_{D \in \mathcal{D}} \ker \mathcal{D}$.

We use the convention that $\mathcal{F}$ is the class of all filters, while $\mathcal{F}X$ is the set of all filters on $X$, and similarly, $\mathcal{U}$ is the class of ultrafilters. Thus we write $\mathcal{U} \subset \mathcal{F}$ to indicate that $\mathcal{U}X \subset \mathcal{F}X$ for every set $X$. We will consider other classes of filters with the same conventions: for instance, given a cardinal $\kappa$, $\mathcal{F}_\kappa$ denotes the class of filters with a filter base of cardinality less than $\kappa$. We use $\mathcal{F}_0$ as a shorthand for $\mathcal{F}_{\aleph_0}$, that is, for the class of principal filters (a filter $\mathcal{F}$ is principal if $\ker \mathcal{F} \in \mathcal{F}$). Similarly, $\mathcal{F}_1$ is a shorthand for $\mathcal{F}_{\aleph_1}$ and denotes the class of filters with a countable filter-base, or countably based filters. We will also consider the class $\mathcal{F}_{\Lambda_1}$ of filters closed under countable intersections, and for each class of filter, the exponent $\text{exp}$ denotes the subclass of free filters (a filter $\mathcal{F}$ is free if $\ker \mathcal{F} = \emptyset$). Hence $\mathcal{F}^\circ$ is the class of all free filters, $\mathcal{U}^\circ$ the class of free ultrafilters, $\mathcal{F}_1^\circ$ the class of free countably based filters, and so on. Additionally, if $\mathbb{D}$ is a class of filters, $\mathbb{D} = \{D_c : D \in \mathbb{D}\}$ is the dual class of ideals.

An ultrafilter $\mathcal{U}$ on $X$ is either principal (and then of the form $\{x\}^\uparrow$ for some $x \in X$) or free. The map $j : X \to \mathcal{U}X$ defined by $j(x) = \{x\}^\uparrow$ is one-to-one, and we often identify $X$ with $j[X]$, hence writing, $\mathcal{U}^\circ X = \mathcal{U}X \setminus X$.

Given a map $f : X \to Y$ and a filter $\mathcal{F} \subset \mathcal{F}X$, we define the image filter $4$.

$$f[\mathcal{F}] = \{f(F) : F \in \mathcal{F}\}^\uparrow \subset \mathcal{F}Y.$$  

If $\mathcal{F} \subset \mathcal{U}X$ then $f[\mathcal{F}] \subset \mathcal{U}Y$, so that a map $f : X \to Y$ induces a map $\mathcal{U}f : \mathcal{U}X \to \mathcal{U}Y$ defined by $\mathcal{U}f(\mathcal{U}) = f[\mathcal{U}]$.

If $R \subset X \times Y$ is a relation, and $x \in X$, we denote by $R(x) = \{y \in Y : (x, y) \in R\}$, and if $F \subset X$, we write $R[F] = \bigcup_{x \in F} R(x)$. If $\mathcal{F} \subset \mathcal{F}X$ then

$$R[\mathcal{F}] = \{R[F] : F \in \mathcal{F}\}^\uparrow Y$$

is a (possibly degenerate) filter on $Y$. We say that a class $\mathbb{D}$ of filters is $\mathbb{F}_0$-composable if for every $X, Y$ and $R \subset X \times Y$,

$$\mathbb{D} \subset \mathbb{D}X \iff R[\mathbb{D}] \subset \mathbb{D}Y,$$

with the convention that the degenerate filter of a set $Z$ is always in $\mathbb{D}Z$.

If $R \subset X \times Y$ then $R \subset Y \times X$ denotes the inverse relation. If $\mathcal{F}$ is a family of subsets of $X$ and $\mathcal{G}$ is a family of subsets of $Y$ then

$$(2.3) \quad R \in (\mathcal{F} \times \mathcal{G})^\# \iff R[\mathcal{F}] \# \mathcal{G} \iff R^\# \mathcal{G} \# \mathcal{F}.$$  

If $\mathcal{F} \subset \mathcal{F}A$ and $\mathcal{G} : A \to \mathcal{F}(PX)$, we define the contour of $\mathcal{G}$ along $\mathcal{F}$ to be the family of subsets of $X$ given by

$$(2.4) \quad \mathcal{G}(\mathcal{F}) = \bigcup_{a \in \mathcal{F}} \bigcap_{F \in \mathcal{F}} \mathcal{G}(a).$$

Note that if $\mathcal{F} \subset \mathcal{F}A$ and $\mathcal{G} : A \to \mathcal{F}X$, then $\mathcal{G}(\mathcal{F}) \subset \mathcal{F}X$, and if $\mathcal{F} \subset \mathcal{U}X$ and $\mathcal{G} : A \to \mathcal{U}X$, then $\mathcal{G}(\mathcal{F}) \subset \mathcal{U}X$.

We endow $\mathcal{U}X$ with its usual Stone topology, here denoted $\beta$, given by the base

$$\{\beta A : A \subset X\},$$

where

$\beta A = \{U \in \mathcal{U}X : A \subset U\}.$
This topology is zero-dimensional (because the sets $\beta A$ are clopen) and compact Hausdorff. In particular, every ultrafilter $\mathcal{U} \in U(\mathbb{U}X)$ converges for $\beta$ to a unique element of $\mathbb{U}X$:

$$\lim_{\mathcal{U}} \mathcal{U} = \{\mathcal{U}^*\},$$

where

$$\mathcal{U}^* := \bigcup_{A \in \mathcal{U}} \bigcap_{W \in A} W = \mathcal{I}(\mathcal{U})$$

is the contour of the identity map $\mathcal{I}$ of $\mathbb{U}X$ along $\mathcal{U}$.

If $\mathcal{H} \in \mathbb{F}_X$, let $\beta \mathcal{H} = \{U \in \mathbb{U}X : H \leq U\}$.

With this convention, $\beta A$ is a shorthand for $\beta \{A\}^\uparrow$. A non-empty subset $H$ of $\mathbb{U}X$ is closed for $\beta$ if and only if there is a filter $\mathcal{H}$ on $X$ with $H = \beta \mathcal{H}$. If $H \subset \mathbb{U}X$, let $\text{cl}_\beta H = \beta \left( \bigwedge_{U \in H} U \right)$ denote the closure in the Stone topology of $\mathbb{U}X$ and let $\text{cl}_\delta H$ denote the closure in the associated $G_\delta$-topology. Observe that:

**Lemma 2.** The family

$$\{\beta \mathcal{H} : \mathcal{H} \in \mathbb{F}_1\}$$

form a basis for the $G_\delta$-topology of $(\mathbb{U}X, \beta)$.

**Proof.** If $\mathcal{H} \in \mathbb{F}_1$ and $(H_n)_{n \in \omega}$ is a filter-base for $\mathcal{H}$ then $\beta \mathcal{H} = \bigcap_{n \in \omega} \beta H_n$ so that $\beta \mathcal{H}$ is $G_\delta$ (and closed). Let $\mathcal{A}$ be a $G_\delta$-subset of $\mathbb{U}X$. Then there are open subsets $O_i$ of $\mathbb{U}X$ with $A = \bigcap_{i \in \omega} O_i$, and for each $i$, there is a $A_i \subset \mathbb{P}X$ with $O_i = \bigcup_{A \in A_i} \beta A$. Let $U \in A$. Then for each $i \in \omega$, there is $A_i \in A_i$ with $U \in \beta A_i$ so that

$$U \in \bigcap_{i \in \omega} \beta A_i \subset \bigcap_{i \in \omega} O_i \subset A.$$ 

Moreover, $\bigcap_{i \in \omega} \beta A_i$ is a $G_\delta$-subset of $\mathbb{U}X$ and $\bigcap_{i \in \omega} \beta A_i \neq \emptyset$ ensures that $\{A_i : i \in \omega\}$ has the finite intersection property and generates a filter $\mathcal{H} \in \mathbb{F}_1$ with

$$\beta \mathcal{H} = \bigcap_{i \in \omega} \beta A_i.$$ 

Hence every $G_\delta$-set is a union of $G_\delta$-sets of the form $\beta \mathcal{H}$ for $\mathcal{H} \in \mathbb{F}_1$ and thus every $\delta$-open set is also a union of such sets. \hfill \Box

Note also that if $H \subset \mathbb{F}X$ is a family of filters on $X$, then

$$\beta \left( \bigwedge_{\mathcal{H} \in H} \mathcal{H} \right) = \text{cl}_\beta \left( \bigcup_{\mathcal{H} \in H} \beta \mathcal{H} \right),$$

because

$$\beta \left( \bigwedge_{\mathcal{H} \in H} \mathcal{H} \right) = \beta \left( \bigwedge_{\mathcal{H} \in H} \bigwedge_{U \in \beta \mathcal{H}} U \right) = \beta \left( \bigwedge_{U \in \bigcup_{\mathcal{H} \in H} \beta \mathcal{H}} U \right) = \text{cl}_\beta \left( \bigcup_{\mathcal{H} \in H} \beta \mathcal{H} \right).$$

Since for $f : X \to Y$, $B \subset Y$, and $\mathcal{F} \in \mathbb{F}X$,

$$(Uf)^{-1}(\beta B) = \beta \left( f^{-1}[B] \right)$$

and

$$Uf[\beta \mathcal{F}] = \beta (f[\mathcal{F}]),$$

by \cite{7} Corollary II.6.7, we conclude that

**Lemma 3.** If $f : X \to Y$ then $Uf : \mathbb{U}X \to \mathbb{U}Y$ is a continuous perfect map.
2.2. **Convergences.** Recall that a convergence $\xi$ on a set $X$ is a relation between $X$ and the set $\mathcal{F}X$ of filters on $X$ (that is, $\xi \subseteq X \times \mathcal{F}X$), satisfying $(x,\{x\}^\uparrow) \in \xi$ for all $x \in X$ and

$$\mathcal{F} \leq \mathcal{G} \implies \xi^-(\mathcal{F}) \subseteq \xi^-(\mathcal{G})$$

for every filters $\mathcal{F}, \mathcal{G} \in \mathcal{F}X$. We interpret $(x,\mathcal{F}) \in \xi$ as $\mathcal{F}$ converges to $x$ for $\xi$ and hence we use the alternative notations

$$x \in \lim_\xi \mathcal{F} \iff \mathcal{F} \in \lim_\xi \bar{\mathcal{F}}(x)$$

for $(x,\mathcal{F}) \in \xi$. In those terms, the two requirements on $\xi \subseteq X \times \mathcal{F}X$ to be a convergence become

\begin{align}
(2.6) & \quad x \in \lim_\xi \{x\}^\uparrow \\
(2.7) & \quad \mathcal{F} \leq \mathcal{G} \implies \lim_\xi \mathcal{F} \subseteq \lim_\xi \mathcal{G},
\end{align}

for every $x \in X$ and filters $\mathcal{F}, \mathcal{G} \in \mathcal{F}X$.

A map $f$ between two convergence spaces $(X,\xi)$ and $(Y,\sigma)$ is **continuous** if for every $\mathcal{F} \in \mathcal{F}X$ and $x \in X$,

$$x \in \lim_\xi \mathcal{F} \implies f(x) \in \lim_\sigma f[\mathcal{F}].$$

Consistently with [7], we denote by $|\xi|$ the underlying set of a convergence $\xi$, and, if $(X,\xi)$ and $(Y,\sigma)$ are two convergence spaces we often write $f : |\xi| \to |\sigma|$ instead of $f : X \to Y$ even though one may see it as improper since many different convergences have the same underlying set. This allows to talk about the continuity of $f : |\xi| \to |\sigma|$ without having to repeat for what structure.

Every topology can be seen as a convergence. Indeed, if $\tau$ is a topology and $\mathcal{N}_\tau(x)$ denotes the neighborhood filter of $x$ for $\tau$, then

$$x \in \lim_\mathcal{F} \iff \mathcal{F} \geq \mathcal{N}_\tau(x)$$

defines a convergence that completely characterizes $\tau$. Moreover, a function between two topological space is continuous in the topological sense if and only if it is continuous in the sense of convergences. Hence, we do not distinguish between a topology $\tau$ and the convergence it induces, thus embedding the category $\textbf{Top}$ of topological spaces and continuous maps as a full subcategory of the category $\textbf{Conv}$ of convergence spaces and continuous maps.

A point $x$ of a convergence space $(X,\xi)$ is **isolated** if $\lim_{\xi}(x) = \{\{x\}\}^\uparrow$. A prime convergence is a convergence with at most one non-isolated point.

Given two convergences $\xi$ and $\theta$ on the same set $X$, we say that $\xi$ is **finer than** $\theta$ or that $\theta$ is **coarser than** $\xi$, in symbols $\xi \geq \theta$, if the identity map from $(X,\xi)$ to $(X,\theta)$ is continuous, that is, if $\lim_\xi \mathcal{F} \subseteq \lim_\theta \mathcal{F}$ for every $\mathcal{F} \in \mathcal{F}X$. With this order, the set $\mathcal{C}(X)$ of convergences on $X$ is a complete lattice whose greatest element is the discrete topology and least element is the antidiscrete topology, and for which, given $\Xi \subseteq \mathcal{C}(X)$,

$$\lim_{\lor \in \Xi} \mathcal{F} = \bigcap_{\xi \in \Xi} \lim_\xi \mathcal{F} \quad \text{and} \quad \lim_{\land \in \Xi} \mathcal{F} = \bigcup_{\xi \in \Xi} \lim_\xi \mathcal{F}.$$

In fact, $\textbf{Top}$ is a reflective subcategory of $\textbf{Conv}$ and the corresponding reflector $\mathcal{T}$, called **topologizer**, associates to each convergence $\xi$ on $X$ its **topological modification** $\mathcal{T}\xi$, which is the finest topology on $X$ among those coarser than $\xi$ (in $\textbf{Conv}$).
COUNTABILITY CONDITIONS

Concretely, $T_\xi$ is the topology whose closed sets are the subsets of $|\xi|$ that are $\xi$-closed, that is, subsets $C$ satisfying

$$C \in F^\# \implies \lim_\xi F \subseteq C.$$  

A subset $O$ of $|\xi|$ is $\xi$-open if its complement is closed, equivalently if

$$\lim_\xi F \cap O \neq \emptyset \implies O \in F.$$  

Given a map $f : |\xi| \to Y$, there is the finest convergence $\xi f$ on $Y$ making $f$ continuous (from $\xi$), and given $f : X \to |\sigma|$, there is the coarsest convergence $f^-\sigma$ on $X$ making $f$ continuous (to $\sigma$). The convergences $\xi f$ and $f^-\sigma$ are called final convergence for $f$ and $\xi$ and initial convergence for $f$ and $\sigma$ respectively. Note that (2.8)

$$f : |\xi| \to |\sigma| \text{ is continuous} \iff \xi \geq f^-\sigma \iff f\xi \geq \sigma.$$  

If $A \subseteq |\xi|$, the induced convergence by $\xi$ on $A$, or subspace convergence, is $i^-\xi$, where $i : A \to |\xi|$ is the inclusion map. If $\xi$ and $\tau$ are two convergences, the product convergence $\xi \times \tau$ on $|\xi| \times |\tau|$ is the coarsest convergence on $|\xi| \times |\tau|$ making both projections continuous, that is,

$$\xi \times \tau := p_\xi \xi \vee p_\tau \tau,$$

where $p_\xi : |\xi| \times |\tau| \to |\xi|$ and $p_\tau : |\xi| \times |\tau| \to |\tau|$ are the projections defined by $p_\xi(x,y) = x$ and $p_\tau(x,y) = y$ respectively.

The adherence of a filter $H$ on a convergence space $(X, \xi)$ is

$$(2.9) \quad \text{adh}_\xi H = \bigcup_{F \in H} \lim_\xi F = \bigcup_{U \in \beta H} \lim_\xi U,$$

so that $\text{adh}_\xi U = \lim_\xi U$ for every ultrafilter $U$.

Given a class $D$ of filters, a convergence $\xi$ on $X$ is determined by adherences of $D$-filters if

$$\bigcap_{D \in D^\#} \text{adh}_\xi D \subseteq \lim_\xi F,$$

where the reverse inclusion is always true.

**Proposition 4.** [7, Corollary XIV.3.8] If $D$ is an $F_0$-composable class of filters, then the class of convergence spaces determined by adherences of $D$-filters is concretely reflective and the reflector $A_D$ is characterized by

$$\lim_{D \in D^\#} \xi F = \bigcap_{D \in D^\#} \text{adh}_\xi D,$$

and satisfies

$$A_D(f^-\tau) = f^- (A_D \tau)$$

for every map $f$ and convergence $\tau$ on its codomain.

The classes $\mathcal{F}$, $\mathcal{F}_1$, $\mathcal{F}_{\land 1}$ and $\mathcal{F}_0$ are all $F_0$-composable, and we will use the following terminology and notations:

| $D$ | reflector $A_D$ | a convergence $\xi = A_D \xi$ is called |
|-----|----------------|-------------------------------------|
| $\mathcal{F}$ | $S$ | pseudotopology |
| $\mathcal{F}_1$ | $S_1$ | paratopology |
| $\mathcal{F}_0$ | $S_0$ | pretopology |
| $\mathcal{F}_{\land 1}$ | $S_{\land 1}$ | hypotopology |
Of course, every pretopology is a paratopology and a hypotopology, while every paratopology and every hypotopology is a pseudotopology. Note that a convergence \( \xi \) is a topology if and only if it is a pretopology and the adherence operator restricted to principal filters is idempotent (i.e., \( \text{adh}_\xi(\text{adh}_\xi A) = \text{adh}_\xi A \) for all \( A \subset |\xi| \)), in which case it coincides with the topological closure, denoted \( \text{cl}_\xi \).

Accordingly
\[
T \leq S_0 \leq S_1 \leq S \quad \text{and} \quad T \leq S_0 \leq S_{\wedge 1} \leq S.
\]

Given a convergence \( \xi \) and \( x \in |\xi| \), the filter
\[
V_{\xi}(x) := \bigwedge \{ F : x \in \lim_\xi F \}
\]
(2.10)
is called the *vicinity filter of* \( x \) for \( \xi \). In general, \( V_{\xi}(x) \) does not need to converge to \( x \) for \( \xi \). It is a simple exercise to check that \( \xi \) is a pretopology if and only if \( x \in \lim_\xi V_{\xi}(x) \) for each \( x \in |\xi| \), and that
\[
V_{\xi}(x) = V_{S_0 \xi}(x)
\]
for all \( x \in |\xi| \).

Note that \( V_{\xi}(x) : X \to \mathcal{F}X \) so that given \( A \subset |\xi| \) we may consider the contour
\[
V_{\xi}(A) = \bigcap_{A \in A} \bigcap_{x \in A} V_{\xi}(x)
\]
of \( V_{\xi}(x) \) along \( A \), which satisfies
\[
\mathcal{B} \# V_{\xi}(A) \iff \text{adh}_\xi^2 \mathcal{B} \# A.
\]

(2.11)

Given a class \( \mathcal{D} \) of filters, we call a convergence \( \mathcal{D} \)-based if whenever \( x \in \lim_\xi F \), there is \( D \in \mathcal{D} \) such that \( D \leq F \) and \( x \in \lim_\xi D \).

**Proposition 5.** [7, Corollary XIV.4.4] If \( \mathcal{D} \) is an \( \mathcal{F}_0 \)-composable class of filters then the class of \( \mathcal{D} \)-based convergences is (concretely) coreflective and the coreflector is given by
\[
\lim_{\mathcal{B}_0 \xi} \mathcal{F} = \bigcup_{D \in \mathcal{D} \leq \mathcal{F}} \lim_\xi D.
\]

We will be particularly interested in the case of \( \mathcal{F}_1 \)-based convergences, or *first-countable* convergences, in which case we use the notation \( I_1 \) for \( \mathcal{B}_1 \).

It turns out (see [3] or [7, Section XIV.6] for details) that for a topology \( \xi \),
\[
\xi \text{ sequential} \iff \xi \geq T I_1 \xi
\]
\[
\xi \text{ Fréchet-Urysohn} \iff \xi \geq S_0 I_1 \xi
\]
\[
\xi \text{ strongly Fréchet} \iff \xi \geq S_1 I_1 \xi
\]
\[
\xi \text{ bisequential} \iff \xi \geq S I_1 \xi
\]
\[
\xi \text{ first-countable} \iff \xi \geq I_1 \xi,
\]
and we take these functorial inequalities as the definitions of these notions for general convergences.

A convergence is called *Hausdorff* if the cardinality of \( \lim \mathcal{F} \) is at most one, for every filter \( \mathcal{F} \). Of course, a topology is Hausdorff in the usual topological sense if and only if it is in the convergence sense.

Given two convergences \( \xi \) and \( \theta \) on the same set, we say that \( \xi \) is \( \theta \)-*regular* if
\[
\lim_\xi \mathcal{F} \subset \lim_\xi \text{adh}_\theta^2 \mathcal{F}
\]
for every filter $\mathcal{F}$. If $\theta = \xi$, the convergence is simply called regular and if $\theta = T\xi$, then we say that $\xi$ is T-regular. A topology is regular in the usual sense if and only if it is regular (or of course T-regular) in the convergence sense.

2.3. Compactness. A primary source for complements on compactness and its variants in the context of convergence spaces is [4].

2.3.1. Filter compactness. A subset $K$ of a convergence space $(X, \xi)$ is compact if

$$\forall \mathcal{F} \in \mathcal{P}X \; K \in \mathcal{F}^\# \implies \operatorname{adh}_\xi \mathcal{F} \cap K \neq \emptyset.$$  

More generally, given a class $\mathcal{D}$ of filters, a family $\mathcal{A}$ of subsets of $|\xi|$ is $\mathcal{D}$-compact at another family $\mathcal{B}$ if

$$\forall \mathcal{D} \in \mathcal{D} X, \xi \quad \mathcal{D}^\# \mathcal{A} \implies \operatorname{adh}_\xi \mathcal{D} \in \mathcal{B}^\#.$$  

A family of subsets of $(X, \xi)$ that is $\mathcal{D}$-compact at itself is simply called $\mathcal{D}$-compact. It is called $\mathcal{D}$-compactoid if it is $\mathcal{D}$-compact at $\{X\}$. In the case $\mathcal{D} = \mathcal{F}$, we omit the class of filters, so that a subset $K$ of a convergence space is compact(oid) if and only if its principal filter $\{K\}^\uparrow$ is a compact(oid) filter. Let $\mathcal{K}_\mathcal{D}$ denote the class of $\mathcal{D}$-compactoid filters.

A convergence $\xi$ is locally compact(oid) if every convergent filter has a compact(oid) element.

Let $\mathcal{K}(\xi)$ or simply $\mathcal{K}$ denote the set of compactoid subsets of $|\xi|$. Note that unless $\xi$ is compact, the family $\mathcal{K}_c$ of complements of compactoid sets is a filter-base, generating the compactoid filter.

**Lemma 6.** Let $\mathcal{D} \subseteq \mathcal{D} = \mathcal{C}^\mathcal{D} \subseteq \mathcal{P}X$ and let $\xi$ be a non-compact convergence. Then

$$\operatorname{adh}_T \xi \mathcal{D} = \emptyset \iff \mathcal{C}^\mathcal{D} \geq (\mathcal{K}_\mathcal{D})_c.$$  

**Proof.** If

$$\operatorname{adh}_\xi \mathcal{D} \subseteq \operatorname{adh}_T \xi \mathcal{D} = \emptyset = \bigcap_{D \in \mathcal{D}} \mathcal{C} \xi D,$$  

then $\operatorname{adh}_\xi \mathcal{C}^\mathcal{D} = \emptyset$. Thus $K \not\in (\mathcal{C}^\mathcal{D})^\#$ for every $K \in \mathcal{K}_\mathcal{D}(\xi)$ because $\mathcal{C}^\mathcal{D} \in \mathcal{D}$; in other words, $K^c \in \mathcal{C}^\mathcal{D}$, that is, $\mathcal{C}^\mathcal{D} \geq (\mathcal{K}_\mathcal{D})_c$. Assume conversely that $\mathcal{C}^\mathcal{D} \geq (\mathcal{K}_\mathcal{D})_c$. For every $x \in |\xi|$, $\{x\}^c \in \mathcal{C}^\mathcal{D}$ because $\{x\} \in \mathcal{K}$. Hence $\{x\} \not\in (\mathcal{C}^\mathcal{D})^\#$, that is, there is $U \in \mathcal{O}(x)$ with $U \not\in \mathcal{D}^\#$. As a result, $x \notin \operatorname{adh}_T \xi \mathcal{D}$. □

Recall from [7] that a filter $\mathcal{F}$ on a convergence space $(X, \xi)$ is called closed if $\operatorname{adh}_\xi \mathcal{F} = \ker \mathcal{F}$ and that a filter on a Hausdorff convergence space is compact if and only if it is closed and compactoid [7, Corollary IX.7.15].

**Proposition 7.** If $\xi$ is $\theta$-regular and $\mathcal{F}$ is a filter on $|\xi|$ then $\mathcal{F}$ is compactoid if and only if $\operatorname{adh}_\theta^\mathcal{F}$ is compactoid.

**Proof.** If $\operatorname{adh}_\theta^\mathcal{F}$ is compactoid so is any finer filter, in particular $\mathcal{F}$. Assume conversely that $\mathcal{F}$ is compactoid and let $\mathcal{H} = \# \operatorname{adh}_\theta^\mathcal{F}$, equivalently, $\mathcal{V}_\theta(\mathcal{H}) = \# \mathcal{F}$ by [7, Prop. VI.5.2]. Thus $\operatorname{adh}_\xi \mathcal{V}_\theta(\mathcal{H}) = \emptyset$. In view of [2] Prop. VIII.4.1, $\operatorname{adh}_\xi \mathcal{H} = \emptyset$. Thus $\operatorname{adh}_\xi^\mathcal{F}$ is compactoid. □

**Corollary 8.** If $\xi$ is a T-regular pseudotopology and $\mathcal{F}$ is a filter on $|\xi|$, the following are equivalent:

\(^5\)Recall the definition of contours.
(1) $\mathcal{F}$ is compactoid;
(2) $\mathcal{C}_F^\xi$ is compactoid;
(3) $\mathcal{C}_F^\xi$ is compact.

In this case, $\text{adh}_{\xi}(\mathcal{C}_F^\xi) = \ker(\mathcal{C}_F^\xi)$ is a non-empty compact set.

**Proof.** The equivalence of (1) and (2) is Proposition 7 for $\theta = T\xi$. By definition, (3) implies (2), so we only need to prove that (2) implies (3).

To this end, let $\mathcal{U} \in \beta(\mathcal{C}_F^\xi)$. By compactoidness, $\lim\limits_{\xi} \mathcal{U} \neq \emptyset$. Moreover, if $x \in \lim\limits_{\xi} \mathcal{U} \subset \lim\limits_{T\xi} \mathcal{U}$, we have $x \in \lim\limits_{T\xi} N^\xi(\mathcal{U})$. Additionally $N^\xi(\mathcal{U}) \# F$ because $\mathcal{U} \# \mathcal{C}_F^\xi$, so that

$$x \in \bigcap_{F \in \mathcal{F}} \mathcal{C}_F^\xi = \ker(\mathcal{C}_F^\xi).$$

Thus $\lim\limits_{\xi} \mathcal{U} \cap \ker(\mathcal{C}_F^\xi) \neq \emptyset$ and $\mathcal{C}_F^\xi$ is a compact filter. By [7, Cor. IX.7.16], $\text{adh}_{\xi}(\mathcal{C}_F^\xi) = \ker(\mathcal{C}_F^\xi)$ is a non-empty compact set. \qed

Note also that

**Proposition 9.** If $\xi$ is a topology then $\mathcal{D}$ is compact(oid) if and only if $O_{\xi}(\mathcal{D})$ is.

**Proof.** Assume that $\mathcal{D}$ is compact(oid). Let $\mathcal{H} \# O(\mathcal{D})$ equivalently, $\mathcal{C}^\xi \mathcal{H} \# \mathcal{D}$. By compactness of $\mathcal{D}$ (resp. compactoidness) we have

$$\text{adh}_{\xi} \mathcal{C}_F^\xi \mathcal{H} = \text{adh}_{T\xi} \mathcal{H}$$

meshes with $\mathcal{D}$, hence with $O(\mathcal{D})$ (resp. is non-empty).

Conversely assume $O(\mathcal{D})$ is compact. Since $\mathcal{D} \geq O(\mathcal{D})$ if $\mathcal{H} \# \mathcal{D}$ then $\mathcal{H} \# O(\mathcal{D})$ and thus $\text{adh} \mathcal{H} \# O(\mathcal{D})$, equivalently, $\text{cl}(\text{adh} \mathcal{H}) \# \mathcal{D}$ but $\text{cl}(\text{adh} \mathcal{H}) = \text{adh} \mathcal{H}$. \qed

Moreover, [7 Theorem IX.7.17] states that for a Hausdorff convergence $\xi$ if a filter $\mathcal{D} = O_{\xi}(\mathcal{D})^\uparrow$ is compact then $\mathcal{D} = O_{\xi}(K)^\uparrow$ where $K = \text{adh} \mathcal{D}$ is compact.

**Corollary 10.** If $\xi$ is a regular Hausdorff topology and $\mathcal{D}$ is a compact filter then $\text{adh} \mathcal{D}$ is compact and

$$O(\mathcal{D})^\uparrow = O(\text{adh} \mathcal{D})^\uparrow.$$

**Proof.** In view of Proposition 9, $O(\mathcal{D})^\uparrow$ is a compact filter and thus by [7 Theorem IX.7.17], $O(\mathcal{D})^\uparrow = O(\text{adh} O(\mathcal{D})^\uparrow)^\uparrow$ where $\text{adh} O(\mathcal{D})^\uparrow$ is compact. Moreover, as $\xi$ is a regular topology, $\mathcal{H} \# O(\mathcal{D})$ if and only if $\mathcal{C}^\xi \mathcal{H} \# \mathcal{D}$ and $\lim\limits_{\xi} \mathcal{H} = \lim\limits_{\xi} \mathcal{C}^\xi \mathcal{H}$ so that $\text{adh} O(\mathcal{D}) = \text{adh} \mathcal{D}$, and the result follows. \qed

Recall [9] that a topology is bi-k if every convergent ultrafilter there is a compact set $K$ and $\mathcal{H} \in F_1$ with $\ker \mathcal{H} = K$ and

$$U \geq H \geq O(K)^\uparrow.$$

A topology is of pointwise countable type if each point belongs to a compact set of countable character, that is, a compact set $K$ where $O(K)^\uparrow \in F_1$.

**Corollary 11.** [3] A regular Hausdorff topology is bi-k (resp. of pointwise countable type) if and only if every convergent ultrafilter (resp. filter) contains a countably based compactoid filter.
Proof. If \((X, \xi)\) is bi-\(k\) then \(\mathcal{H}\) witnessing the definition is a countably based and compactoid filter. Conversely, if \(\mathcal{U}\) is a convergent filter, then there is a compactoid filter \(\mathcal{H} \in \mathcal{P}_1\) with \(\mathcal{U} \geq \mathcal{H}\). Then \(\text{cl}^2 \mathcal{H}\) is countably based and compact by Corollary \([8]\) and by Corollary \([10]\) \(K = \text{ad} h \mathcal{H} = \ker \text{cl}^2 \mathcal{H}\) is compact and \(\mathcal{O}(\text{cl}^2 \mathcal{H}) = \mathcal{O}(K)\) so that \(\mathcal{U} \geq \mathcal{H} \geq \mathcal{O}(\text{cl}^2 \mathcal{H}) = \mathcal{O}(K)\) and \(\xi\) is bi-\(k\).

Suppose that every convergent filter contains a countably based compactoid filter. In particular for every \(x \in X\) we have \(\mathcal{O}(x) \geq \mathcal{D}\) for some \(\mathcal{D} \in K_{\xi} \cap \mathbb{F}_1\). Moreover, in view of Corollary \([8]\) and Corollary \([10]\) \(K = \text{ad} h \mathcal{D}\) is compact and \(\mathcal{O}(\text{cl}^2 \mathcal{D}) = \mathcal{O}(K)\). Hence
\[
\mathcal{O}(x) \geq \mathcal{D} \geq \text{cl}^2 \mathcal{D} \geq \mathcal{O}(\text{cl}^2 \mathcal{D}) = \mathcal{O}(K),
\]
so that, letting \(\{D_n : n \in \omega\}\) be a decreasing countable filter base of \(\mathcal{D}\), for every \(U \in \mathcal{O}(K)\), there is \(n \in \omega\) with \(K \subset \text{cl}D_n \subset U\) and thus there is \(O_n \in \mathcal{O}(x)\) with \(K \subset O_n \subset U\) and \(K\) is of countable character.

Conversely, assume that \(X\) is a regular Hausdorff topological space of pointwise countable type and let \(\mathcal{F}\) be convergent, that is, \(\mathcal{F} \geq \mathcal{O}(x)\) for some \(x\). Then there is \(K\) of countable character with \(x \in K\) so that \(\mathcal{F} \geq \mathcal{O}(x) \geq \mathcal{O}(K)\) and \(\mathcal{O}(K) \in K_{\xi} \cap \mathbb{F}_1\).

2.3.2. Cover-compactness. A family \(\mathcal{P}\) of subsets of \(\xi\) is a \(\xi\)-cover of \(A \subset |\xi|\) if \(\mathcal{P} \cap \mathcal{F} \neq \emptyset\) for every filter \(\mathcal{F}\) with \(\lim_\mathcal{F} \mathcal{F} \cap A \neq \emptyset\). Note that if \(\xi\) is a topology then \(\mathcal{P} \subset \mathcal{O}_\xi\) is a cover if and only if \(A = \bigcup_{\mathcal{P} \in \mathcal{P}} P\), and moreover, every cover has a refinement that is an open cover.

Dolecki noted \([3]\) that the notion of cover is dual to that of family of empty adherence. More specifically, \(\mathcal{P}\) is a \(\xi\)-cover of \(A\) if and only if
\[
\text{ad} h_\xi \mathcal{P} \cap A = \emptyset,
\]
extending the notion of adherence from filters to general families via
\[
\text{ad} h_\xi A = \bigcup_{\mathcal{H} \# A} \lim_\mathcal{H} \mathcal{H}.
\]
The notion can be extended to families of subsets. We say that \(\mathcal{P}\) is a \(\xi\)-cover of \(A \subset 2^{|\xi|}\) if \(\mathcal{P}\) is a \(\xi\)-cover of some \(A \in A\). With this in mind,

Proposition 12. \([3]\) Theorem 3.1] Let \(\xi\) be a convergence and \(A, \mathcal{P} \subset 2^{|\xi|}\). Then \(\mathcal{P}\) is a \(\xi\)-cover of \(A\) if and only if \(\text{ad} h_\xi \mathcal{P} \notin A^\#\).

Definition 13. \([4]\) Let \(\mathbb{D}\) and \(\mathbb{J}\) be two classes of filters and \(\mathbb{D}_\#\) and \(\mathbb{J}_\#\) the corresponding families of dual ideals. We say that \(A\) is \(\xi\)-cover-\(J^*/\mathbb{D}_\#\)-compact at \(\mathcal{B}\) (for a convergence \(\xi\)) if for every \(\xi\)-cover \(C \in \mathbb{J}_\#\) of \(\mathcal{B}\), there is \(S \subset C\) with \(S \in \mathbb{D}_\#\) such that \(S\) is a \(\xi\)-cover of \(A\), equivalently,
\[
\forall G \in \mathbb{J} \text{ ad} h_\xi G \cap \# B \implies \exists D \in \mathbb{D}_\# \text{ D} \leq G : \text{ad} h_\xi \text{D} \cap \# A,
\]
equivalently
\[
\forall G \in \mathbb{J} \left( \forall D \in \mathbb{D}_\# \text{ D} \leq G \text{ ad} h_\xi D \in A^\# \implies \text{ad} h_\xi G \in B^\# \right).
\]
Remark 14. In view of \([2.2]\) \(A\) is \(\xi\)-cover-\(J^*/\mathbb{D}_\#\)-compact at \(\mathcal{B}\) if and only if every cover in \(\mathbb{J}_\#\) of \(\mathcal{B}\) has a refinement in \(\mathbb{D}_\#\) that is a cover of \(A\).

Proposition 15. \([7]\) Proposition IX.11.13] Let \(\xi\) be a convergence, and let \(A\) and \(\mathcal{B}\) be two families of subsets of \(|\xi|\). \(A\) is \(\xi\)-cover-\(\mathbb{J}_\#\)-compact at \(\mathcal{B}\) if and only if \(\forall_\xi (A)\) is \(\xi\)-compact at \(\mathcal{B}\).
Proposition 16 generalizes to $\mathcal{J}/\mathcal{G}_0$-cover-compactness:

**Proposition 16.** Let $\xi$ be a convergence, let $\mathcal{A}$ and $\mathcal{B}$ be two families of subsets of $|\xi|$ and let $\mathcal{J}$ be a class of filters. $\mathcal{A}$ is $\xi$-cover- $\mathcal{J}/\mathcal{G}_0$-compact at $\mathcal{B}$ if and only if $\mathcal{V}_\xi(\mathcal{A})$ is $\xi$-$\mathcal{J}$-compact at $\mathcal{B}$.

**Proof.** Let $\mathcal{A}$ be $\xi$-cover- $\mathcal{J}/\mathcal{G}_0$-compact at $\mathcal{B}$ and let $\mathcal{G} \in \mathcal{J}$ with $\mathcal{G} \# \mathcal{V}_\xi(\mathcal{A})$, equivalently, $\text{adh}_\xi \mathcal{G} \# \mathcal{A}$ by (2.11), so that $\text{adh}_\xi \mathcal{G} \in \mathcal{B}^\#$. Hence $\mathcal{V}_\xi(\mathcal{A})$ is $\xi$-$\mathcal{J}$-compact at $\mathcal{B}$. Assume conversely, that $\mathcal{V}_\xi(\mathcal{A})$ is $\xi$-$\mathcal{J}$-compact at $\mathcal{B}$ and let $\mathcal{G} \in \mathcal{J}$ with $\text{adh}_\xi \mathcal{G} \# \mathcal{A}$, equivalently, $\mathcal{G} \# \mathcal{V}_\xi(\mathcal{A})$, so that $\text{adh}_\xi \mathcal{G} \in \mathcal{B}^\#$, that is, $\mathcal{A}$ is $\xi$-cover- $\mathcal{J}/\mathcal{G}_0$-compact at $\mathcal{B}$. \qed

In that case $\mathcal{A} = \mathcal{B} = \{ \{x\} \}$, we thus obtain

**Corollary 17.** If $\xi$ is a convergence and $\mathcal{J}$ is a class of filters then $S_0 \xi = A_\mathcal{J} \xi$ if and only if every singleton is $\xi$-cover- $\mathcal{J}/\mathcal{G}_0$-compact.

In particular, $S_0 \xi = S_1 \xi = S_0 \xi \{x\}$ if and only if every singleton is cover-compact and $S_1 \xi = S_0 \xi$ if and only if every singleton is cover countably compact.

**Remark 18.** One direction of Corollary 17 is true in general, which was already observed in [4]: If $\mathcal{D} \subseteq \mathcal{J}$ are two classes of filters, then $A_\mathcal{J} \xi = A_\mathcal{D} \xi$ whenever every singleton is $\xi$-cover- $\mathcal{J}/\mathcal{D}_0$-compact. On the other hand, the other implication is not true if $\mathcal{D} \neq \mathcal{D}_0$ (See Example 51).

3. **Countable Depth**

Paratopologies and hypotopologies form natural generalizations of pretopologies, involving a countability condition. We now examine others and how they relate.

Recall that $\xi$ is a pretopology if and only if for each $x \in |\xi|$, the vicinity filter

$$\mathcal{V}_\xi(x) := \bigcap \{ \mathcal{F} : x \in \lim_\xi \mathcal{F} \}$$

converges to $x$. Thus pretopologies are exactly those convergences admitting at each point a smallest convergent filter (i.e., the vicinity filter). Therefore pretopologies can also be characterized as those convergences $\xi$ satisfying

$$(3.1) \quad \lim_\xi \bigcap_{\mathcal{F} \in \mathcal{A}} \mathcal{F} = \bigcap_{\mathcal{F} \in \mathcal{A}} \lim_\xi \mathcal{F}$$

for every subset $\mathcal{A}$ of $\mathcal{F}|\xi|$.

We say that a convergence $\xi$ is **countably deep** or has **countable depth** if (3.1) holds for every countable subset $\mathcal{A}$ of $\mathcal{F}|\xi|$, and **countably deep for ultrafilters** if (3.1) holds for every countable subset $\mathcal{A}$ of $\mathcal{U}|\xi|$.

We prove below (Corollary 23) that convergences that are both countably deep for ultrafilters and paratopologies are exactly pretopologies, so that any non-pretopological paratopology is not countably deep, even for ultrafilters. In view of Corollary 23.

6a property we may call **almost pretopological** by analogy with the established notion **almost topological** for $S_\xi = T_\xi$.

7Indeed, the inequality $A_\mathcal{J} \xi \geq A_\mathcal{D} \xi$ is true in general because $\mathcal{D} \subseteq \mathcal{J}$. On the other hand, if $x \in \lim_\mathcal{A}_\mathcal{D} \xi \mathcal{F}$ and $\mathcal{G} \in \mathcal{J}$ with $\mathcal{G} \# \mathcal{F}$ then $x \in \text{adh}_\xi \mathcal{G}$ and thus $x \in \bigcap_{\mathcal{G} \# \mathcal{F}} \text{adh}_\xi \mathcal{G} = \lim_\mathcal{A}_\mathcal{J} \xi \mathcal{F}$. Otherwise $x \notin \text{adh}_\xi \mathcal{G}$ so that $\mathcal{G}_c \in \mathcal{J}$ is a cover of $\{x\}$ which has then a subcover in $\mathcal{D}_\mathcal{c}$, because $\{x\}$ is $\xi$-cover-$\mathcal{D}_\mathcal{c}$-compact. In other words, there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D} \subseteq \mathcal{G}$ and $x \notin \text{adh}_\xi \mathcal{D}$, which is not possible because $\mathcal{D} \# \mathcal{F}$ and $x \in \lim_\mathcal{A}_\mathcal{D} \xi \mathcal{F}$. 
to produce a countably deep convergence that is not a paratopology, it is enough to produce one that is not a pretopology, which is easily done (e.g., Example 24). We will also give an example of a pseudotopology that is countably deep for ultrafilters but not countably deep (Example 38).

The classes of countably deep convergences and of countably deep for ultrafilters pseudotopologies turn out to be reflective. Namely, given a convergence $\xi$ let $P_1 \xi$ be defined by $x \in \lim_{P_1 \xi} F$ if there is a countable subset $D$ of $\lim_{\xi}^{-1}(x)$ with

$$F \geq \bigwedge_{D \in \mathcal{D}} D,$$

and let $P_1^U \xi$ be defined by $x \in \lim_{P_1^U \xi} F$ if there is a countable subset $D$ of $\lim_{\xi}^{-1}(x) \cap UX$ with

$$F \geq \bigwedge_{U \in \mathcal{D}} U.$$

Remark. We could more generally define $P_{\kappa} \xi$ and $P_{\kappa}^U \xi$ in a similar fashion if $D$ is restricted to sets of cardinality less than $\aleph_\kappa$. In those terms, $P_0$ is the reflector on convergences of finite depth in the sense of [7], where the notation $L$ was used instead. Pretopologies are the convergences $\xi$ such that $\xi = P_\kappa \xi$ for every $\kappa$.

**Proposition 19.** The modifier

1. $P_1$ is a concrete reflector and fix $P_1$ is the class of countably deep convergences;
2. $P_1^U$ is an idempotent concrete functor;
3. $SP_1^U$ is a concrete reflector and fix $SP_1^U$ is the class of pseudotopologies that are countably deep for ultrafilters.

**Proof.** It is plain that $P_1$ is isotone, contractive, and idempotent and that $P_1^U$ is isotope and idempotent.

Moreover,

$$\xi \geq S \xi \geq SP_1^U \xi,$$

for $\lim_{\xi} U \subset \lim_{P_1^U \xi} U$ if $U \in UX$. Therefore, $SP_1^U$ is isotone, contractive and idempotent.

If $f : |\xi| \to |\tau|$ is continuous and $x \in \lim_{P_1 \xi} F$, that is, there is a countable subset $\mathcal{D}$ of $\lim_{\xi}^{-1}(x)$ with $F \geq \bigwedge_{D \in \mathcal{D}} D$, then

$$f[F] \geq f \left[ \bigwedge_{D \in \mathcal{D}} D \right] \geq \bigwedge_{D \in \mathcal{D}} f[D]$$

so that $f(x) \in \lim_{P_1 \xi} f[F]$, for $f(x) \in \lim_{\tau} f[D]$ for every $D \in \mathcal{D}$, by continuity of $f$. A similar argument applies to the effect that $f : |P_1^U \xi| \to |P_1^U \tau|$ is continuous. Therefore $P_1$ and $P_1^U$ are functors, and thus $P_1$ and $SP_1^U$ are concrete reflectors.

Note that a convergence is countably deep if and only if $\xi \leq P_1 \xi$ and thus fix $P_1$ is the class of countably deep convergences.

On the other hand, $SP_1^U \xi$ is a pseudotopology, and is also countably deep for ultrafilters. Indeed, if

$$x \in \bigcap_{i \in \omega} \lim_{SP_1^U \xi} U_i = \bigcap_{i \in \omega} \lim_{P_1^U \xi} U_i$$

there is for each $i$ a sequence $(W_{i,j})_{j \in \omega}$ of $\lim_{\xi}^{-1}(x) \cap UX$ with $U_i \geq \bigwedge_{j \in \omega} W_{i,j}$ so that $\bigwedge_{i \in \omega} U_i \geq \bigwedge_{(i,j) \in \omega^2} W_{i,j}$ with $\{W_{i,j} : (i,j) \in \omega^2\} \subset \lim_{\xi}^{-1}(x) \cap UX$. 

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On other hand, if \( \xi \) is a countably deep for ultrafilters pseudotopology and \( x \in \lim_{S \in \mathcal{P}_1^U} \mathcal{F} \) there for every \( U \in \beta \mathcal{F} \) there is a countable set \( \mathbb{D} \) with \( \mathcal{U} \geq \bigwedge_{D \in \mathbb{D}} D \), so that \( x \in \lim_\xi \mathcal{U} \) because \( \xi \) is countably deep for ultrafilters. Thus \( x \in \lim_\xi \mathcal{F} \) because \( \xi = S \xi \). Thus \( S \mathcal{P}_1^U \xi = \xi \). In other words, \( S \mathcal{P}_1^U \) is the class of pseudotopologies that are countably deep for ultrafilters. □

Note that by definition, \( P_1^U \geq P_1 \), and thus \( S \mathcal{P}_1^U \geq S P_1 \), but while \( S \mathcal{P}_1^U \xi \) is always countably deep for ultrafilters, \( S P_1 \xi \) may fail to be countably deep. Moreover, \( P_1 S \xi \) may fail to be a pseudotopology. See the forthcoming paper [6] for examples and details. However, the class of countably deep pseudotopologies is reflective:

**Proposition 20.** The class of countably deep pseudotopologies is reflective, as an intersection of two reflective classes. Let \( P_1 \triangle S \) denote the corresponding reflector. Then for every \( \xi \) there are ordinals \( \alpha \) and \( \beta \) with \( P_1^\alpha \xi = (S \mathcal{P}_1)^\alpha \xi = (P_1 S)^\beta \xi \).

**Proof.** This is an easy consequence of [7, Proposition XIV.1.18]. □

**Theorem 21.** For every convergence \( \xi \),

\[
S_1 P_1^U \xi = S_0 \xi.
\]

**Proof.** Of course, \( S_1 \xi \geq S_0 \xi \) and \( P_1^U \xi \geq S_0 \xi \) so that

\[
S_1 P_1^U \xi \geq S_1 S_0 \xi = S_0 \xi.
\]

Conversely, if \( x \in \lim_{S_0} \mathcal{F} \) and \( \mathcal{H} \in F_1 \) with decreasing filter-base \( (H_n)_{n \in \omega} \) with \( \mathcal{H} \# \mathcal{F} \), then for every \( n \in \omega \),

\[
x \in \text{adh}_{S_0} \xi H_n = \text{adh}_{\xi} H_n,
\]

so that there is \( \mathcal{U} \in \beta(H_n) \) with \( x \in \lim_\xi \mathcal{U} \). Then

\[
x \in \lim_{P_1^U} \xi \left( \bigwedge_{n \in \omega} \mathcal{U}_n \right)
\]
and \( \bigwedge_{n \in \omega} \mathcal{U} \# \mathcal{H} \) so that \( x \in \text{adh}_{P_1^U} \xi \mathcal{H} \) and thus \( x \in \lim_{S_1} P_1^U \xi \mathcal{F} \). □

In view of Corollary [17] we conclude:

**Corollary 22.** If a convergence \( \xi \) is countably deep for ultrafilters then every singleton is cover-\( \times_1 / \beta \)-compact.

In restriction to paratopologies, we obtain:

**Corollary 23.** The following are equivalent for a paratopology \( \xi \):

1. \( \xi \) is countably deep;
2. \( \xi \) is countably deep for ultrafilters;

---

Here \( F \) is one of the contractive modifiers \( S P_1 \) and \( P_1 S \) and

\[
F^1 = F, \quad F^\alpha = F \left( \bigwedge_{\beta < \alpha} F^\beta \right).
\]
is coarser than the cofinite filter on $X$.

(3) $\xi$ is a pretopology;
(4) every singleton is co-$F_{\omega_1}^\ast$-compact.

Proof. (1) $\implies$ (2) by definition, (2) $\implies$ (3) because if $\xi = S_1 \xi = P_1^Y \xi$ then $\xi = S_0 \xi$ by Theorem 21. (3) $\iff$ (4) is an instance of Corollary 17 and (3) $\implies$ (1) is clear.

We will see (Example 39) that the converse of Corollary 22 is false, that is, there are convergences $\xi$ satisfying $S_1 \xi = S_0 \xi$ that are not countably deep for ultrafilters.

**Example 24** (A pseudotopology of countable depth that is not a paratopology). Let $X$ be an uncountable set and define on $X$ the prime convergence $\xi$ in which the only non-isolated point $x_0$ satisfies

$$x_0 \in \lim_\xi F \iff \ker F \subseteq \{x_0\} \text{ and } [X]^{<\omega} \cap F \neq \emptyset.$$ 

This convergence has countable depth. Indeed, if $x_0 \in \lim_\xi F$ for $\beta F$, then for each $n \in \omega$ there is $C_n \in [X]^{\leq \omega} \cap F_n$ and thus $\bigcup_{n \in \omega} C_n \in [X]^{\leq \omega} \subseteq \bigwedge_{n \in \omega} F_n$. Moreover,

$$\ker \bigwedge_{n \in \omega} F_n \subseteq \bigcup_{n \in \omega} \ker F_n \subseteq \{x_0\},$$

by Lemma 1. Thus $x_0 \in \lim_\xi \bigwedge_{n \in \omega} F_n$.

Moreover, $\xi = S \xi$. Indeed, if $F$ is such that $x_0 \in \lim_\xi U$ for every $U \in \beta F$, then for every $U \in \beta F$ there is $C_U \in U \cap [X]^{\leq \omega}$ and $\ker U \subseteq \{x_0\}$. Thus $\ker F = \ker \bigwedge_{U \in \beta F} U \subseteq \{x_0\}$ and there is a finite set $D$ of $\beta F$ with $\bigcup_{U \in D} C_U \in F$. Clearly, $\bigcup_{U \in D} C_U \in [X]^{\leq \omega}$. Therefore $x_0 \in \lim_\xi F$.

Finally, $\xi$ is not pretopological, so that, in view of Corollary 23 $\xi$ is not paratopological. To see this, note that

$$\mathcal{V}_\xi(x_0) = \bigwedge \{U \in UX : x_0 \in \lim_\xi U\}$$

is coarser than the cofinite filter on $X$, which does not have any countable element, hence does not converge to $x_0$ for $\xi$. Indeed, if $A \in \mathcal{V}_\xi(x_0)$ then, as for every $C \in [X]^{<\omega}$ there is $U \in \beta C$ with $x_0 \in \lim_\xi U$, we have $A \cap C \neq \emptyset$. Hence $A$ can only miss a finite set, that is, $A$ is cofinite.

On the other hand, $P_1$ and $S_1$ do not commute, as we may have

$$P_1 S_1 \xi \geq S P_1 S_1 \xi > S_0 \xi,$$

while $S_1 P_1 \xi = S_0 \xi$ (Example 42). To produce this example, let us develop a little bit more machinery.

4. **Representations of convergence notions in the Stone topology**

We will characterize various properties of a convergence $\xi$ in terms of the topological features of sets of the form

$$\mathbb{U}_\xi(x) := \{U \in UX : x \in \lim_\xi U\}$$

in the Stone space $UX$. We restrict ourselves to pseudotopologies in this section, equivalently, we assume that

$$\beta F \subseteq \mathbb{U}_\xi(x) \implies x \in \lim_\xi F,$$

for all $x \in |\xi|$.
Conversely, a family $\mathcal{B} = \{B_x : x \in X\}$ of subsets of $\mathbb{U}X$ satisfying $\{x\}^\uparrow \in B_x$ for each $x \in X$ determines a unique pseudotopology $\xi_\mathcal{B}$ defined by

$$x \in \lim_{\xi_\mathcal{B}} F \iff \beta F \subset B_x.$$  

4.1. **Closures on $\mathbb{U}X$ and their coincidence on $\mathbb{U}_\xi(x)$**. With a class $\mathbb{D}$ of filters, a closure operator $\cl_{D^*}$ on $\mathbb{U}X$ was associated in [12] by declaring $\{\beta D : D \in \mathbb{D}\}$ a base of open sets, that is,

$$\cl_{D^*} A = \bigcap \{\mathbb{U}X \setminus \beta D : D \in \mathbb{D}, A \subset \mathbb{U}X \setminus \beta D\}.$$  

[12] Prop. 4 clarifies when this closure operator is a topological closure. It is in particular the case for the classes $\mathbb{F}_0$ (yielding the usual topology of $\mathbb{U}X$: $\cl_{\mathbb{F}_0} = \cl_\beta$), $\mathbb{F}_1$ (yielding the $G_\delta$-topology: $\cl_{\mathbb{F}_1} = \cl_\delta$), $\mathbb{F}_{\lambda 1}$ (for which the topology is harder to describe), and $\mathbb{F}$ (yielding the discrete topology). It turns out that:

**Theorem 25.** Let $\xi$ be a pseudotopology. Then

$$\mathbb{U}_{A_\xi}(x) = \cl_{D^*}(\mathbb{U}_\xi(x)).$$

**Proof.**

$$\mathbb{U} \in \mathbb{U}_{A_\xi}(x) \iff \forall D \in \mathbb{D}, D \subset \mathbb{U} \Rightarrow \beta D \cap \mathbb{U}_\xi(x) \neq \emptyset.$$  

Since the sets of the form $\beta D$ for $D \in \mathbb{D}$ and $D \subset \mathbb{U}$ form a basis of neighborhood of $\mathbb{U}$ for the closure $D^*$, the conclusion follows. \[\square\]

| $\mathbb{D}$ | $D^*$ | $\mathbb{U}_{A_\xi}(x)$ |
|-------------|-------|-------------------|
| $\mathbb{F}_0$ | $\beta$ | $\mathbb{U}_{S_\xi}(x) = \cl_\beta(\mathbb{U}_\xi(x))$ |
| $\mathbb{F}_1$ | $\delta$-topology of $\beta$ | $\mathbb{U}_{S_\beta}(x) = \cl_\delta(\mathbb{U}_\xi(x))$ |
| $\mathbb{F}_{\lambda 1}$ | generated by $\beta F$ with $N_{\beta}(\beta F) = N_\delta(\beta F)$ | $\mathbb{U}_{S_{\lambda 1}}(x) = \cl_{(\beta F)}(\mathbb{U}_\xi(x))$ |
| $\mathbb{F}$ | discrete | $\mathbb{U}_{S_\xi}(x) = \mathbb{U}_\xi(x)$ |

In particular, $\xi$ is a pretopology if and only if for every $x \in |\xi|$, the set $\mathbb{U}_\xi(x)$ is closed; a paratopology if and only if for every $x \in |\xi|$, the set $\mathbb{U}_\xi(x)$ is $\delta$-closed (that is, closed for the $G_\delta$ topology of $\beta$).

**Corollary 26.** Let $\mathbb{D}$ and $\mathbb{J}$ be two classes of filters. A convergence $\xi$ satisfies

$$A_{\mathbb{D}} \xi = A_{\mathbb{J}} \xi$$

if and only if

$$\cl_{D^*}(\mathbb{U}_\xi(x)) = \cl_{J^*}(\mathbb{U}_\xi(x))$$

for every $x \in |\xi|$.

**Proposition 27.** Let $\xi$ be a convergence, $x \in |\xi|$ and let $\kappa < \lambda$ be two cardinals. Then $\mathbb{U}_\xi(x)$ is cover-$\lambda_\xi$/cover-$\kappa_\xi$-compact if and only if for every family $\mathbb{D}$ of filters of cardinality less than $\lambda$

$$x \notin \text{adh}_\xi \bigvee_{D \in \mathbb{D}} D \implies \exists S \subset \mathbb{D}, |S| < \kappa : x \notin \text{adh}_\xi \bigvee_{D \in S} D.$$  

\[\text{Note that then}\]

$$\mathcal{F} \leq \mathcal{G} \implies \lim \mathcal{F} \subset \lim \mathcal{G}$$

rephrases as

$$\beta \mathcal{G} \subset \beta \mathcal{F} \text{ and } \beta \mathcal{F} \subset B_x \implies \beta \mathcal{G} \subset B_x.$$
Proof. Assume $\mathcal{U}_x(x)$ is cover-$\mathcal{F}_{\kappa_\lambda}$-compact and assume that $\mathcal{D}$ is a collection of filters of cardinality less than $\lambda$ with $x \notin \text{adh}_x \bigvee_{D \in \mathcal{D}} D$ equivalently,
\[
\beta \left( \bigvee_{D \in \mathcal{D}} D \right) = \bigcap_{D \in \mathcal{D}} \beta D \subset UX \setminus \mathcal{U}_x(x),
\]
that is,
\[
\mathcal{U}_x(x) \subset \bigcup_{D \in \mathcal{D}} UX \setminus \beta D.
\]
By $\mathcal{F}_{\kappa_\lambda}$-compactness, there is a subset $S$ of $\mathcal{D}$ of cardinality less than $\kappa$ with
\[
\mathcal{U}_x(x) \subset \bigcup_{D \in S} UX \setminus \beta D \iff \beta \left( \bigvee_{D \in S} D \right) \subset UX \setminus \mathcal{U}_x(x),
\]
that is, $x \notin \text{adh}_x \bigvee_{D \in S} D$.

Conversely, assume (4.3), and let $C$ be an open cover of $\mathcal{U}_x(x)$ with $C \in \mathcal{F}_\lambda$. For every $C$ in an ideal base $\mathcal{B}$ of cardinality less than $\lambda$, there is $D_C \in \mathcal{F}_x$ with $C = UX \setminus \beta D_C$. Let $\mathcal{D} = \{ D_C : C \in \mathcal{B} \}$. Then
\[
\mathcal{U}_x(x) \subset \bigcup_{D \in \mathcal{D}} UX \setminus \beta D \iff \beta \left( \bigvee_{D \in \mathcal{D}} D \right) = \bigcap_{D \in \mathcal{D}} \beta D \subset UX \setminus \mathcal{U}_x(x)
\]
so that, in view of (4.1), there is a subset $S$ of $\mathcal{D}$ of cardinality less than $\kappa$ with
\[
x \notin \text{adh}_x \bigvee_{D \in S} D,
\]
so that $\{ UX \setminus \beta D \}$ is a subcover of $C$ of cardinality less than $\kappa$. \hfill \square

We now relate various kinds of compactness of $\mathcal{U}_x(x)$ to the corresponding kind of (cover) compactness of $\{ x \}$.

We say that a class $\mathcal{D}_x$ of ideals is $\beta$-compatible if
\[
\text{(4.2)} \quad C \in \mathcal{D}_x(X) \implies \{ \beta C : C \in \mathcal{C} \} \in \mathcal{D}_x(\mathcal{U}X),
\]
and
\[
\text{(4.3)} \quad \{ \beta (X \setminus D) : D \in \mathcal{D} \} \in \mathcal{D}_x(\mathcal{U}X) \implies \mathcal{D} \in \mathcal{D}(X).
\]

Lemma 28. For every $\kappa$, $\mathcal{F}_{\kappa_\lambda}$ and $\mathcal{F}_\lambda$ are $\beta$-compatible.

Proposition 29. Let $\xi$ be a convergence and $x \in [\xi]$ and let $\mathcal{D} \subseteq \mathcal{J}$ be two classes of filters, where $\mathcal{J}$ is $\beta$-compatible. If $\mathcal{U}_x(x)$ is cover-$\mathcal{J}$-compat. $\mathcal{D}_\kappa$-compact. Moreover, if $\mathcal{J} = \mathcal{F}_\lambda$ and $\mathcal{D}$ is $\beta$-compatible, the converse is true.

Note that
\[
\text{(4.4)} \quad x \notin \text{adh}_x \mathcal{D} \iff \mathcal{U}_x(x) \subset (\beta \mathcal{D})^c = \bigcup_{D \in \mathcal{D}} \beta (X \setminus D).
\]

Proof. Assume that $G \in \mathcal{J}$ with $x \notin \text{adh}_x G$. By (4.1), $\{ \beta (X \setminus G) : G \in \mathcal{G} \}$ is an open cover of $\mathcal{U}_x(x)$, which belongs to $\mathcal{J}$, by $\beta$-compatibility. By (2), it has a subcover in $\mathcal{D}_\kappa$, that is, in view of (4.3), there is $\mathcal{D} \in \mathcal{D}$, $\mathcal{D} \subseteq \mathcal{G}$ with $\mathcal{U}_x(x) \subset \bigcup_{D \in \mathcal{D}} \beta (X \setminus D)$. In view of (4.4), $x \notin \text{adh}_x \mathcal{D}$. Hence $\{ x \}$ is cover-$\mathcal{J}$-compat.
Assume now that \( \{x\} \) is cover-\( \mathcal{F}/\mathcal{D}_\ast \)-compact and let \( C \) be an open cover of \( U_\xi(x) \). It has a refinement of the form \( \{\beta A : A \in \mathcal{A}\} \) for some family \( \mathcal{A} \subset \mathcal{P}X \), which covers \( U_\xi(x) \). As \( \beta(A \cup B) = \beta A \cup \beta B \), we can assume \( \mathcal{A} \) and \( \{\beta A : A \in \mathcal{A}\} \) to be ideals. Hence \( \mathcal{F} = A_x \) is a filter on \( X \) with \( x \notin \text{ad}\_\xi \mathcal{F} \) because of (4.4). Moreover, \( \{x\} \) is cover-\( \mathcal{F}/\mathcal{D}_\ast \)-compact so that there is \( \mathcal{D} \leq \mathcal{F} \) with \( x \notin \text{ad}\_\xi \mathcal{D} \), equivalently, \( \{\beta(X \setminus D) : D \in \mathcal{D}\} \subset \{\beta A : A \in \mathcal{A}\} \) is a cover of \( U_\xi(x) \), which is in \( \mathcal{D}_\ast \) by \( \beta \)-compatibility of \( \mathcal{D} \). Hence \( C \) has a refinement in \( \mathcal{D}_\ast \) that covers \( U_\xi(x) \), which is thus cover-\( \mathcal{F}/\mathcal{D}_\ast \)-compact. \( \square \)

In the case where \( \mathcal{D} = \mathcal{F}_0 \) and \( J = \mathcal{F} \), Proposition 29 and Corollary 27 may be coupled with Corollary 17 to the effect that:

**Corollary 30.** The following are equivalent for a convergence \( \xi \):

1. \( S\xi = S_0\xi \);
2. \( \text{cl}_\beta(U_\xi(x)) = U_\xi(x) \) for every \( x \in |\xi| \);
3. \( \{x\} \) is cover-compact for every \( x \in |\xi| \);
4. \( U_\xi(x) \) is compact for every \( x \in |\xi| \);
5. \( x \notin \text{ad}_\xi \bigvee_{D \in \mathcal{D}} D \implies \exists S \in [\mathcal{D}]^{<\infty} : x \notin \text{ad}_\xi \bigvee_{D \in S} D \).

In case \( J = \mathcal{F} \) and \( \mathcal{D} = \mathcal{F}_1 \), we obtain the following from Proposition 29 and Proposition 27.

**Corollary 31.** Let \( \xi \) be a convergence and \( x \in |\xi| \). The following are equivalent:

1. \( \{x\} \) is cover-Lindelöf;
2. \( U_\xi(x) \) is Lindelöf;
3. \( x \notin \text{ad}_\xi \bigvee_{D \in \mathcal{D}} D \implies \exists S \in [\mathcal{D}]^\omega : x \notin \text{ad}_\xi \bigvee_{D \in S} D \).

For the same classes, Proposition 29 combined with Corollary 26 yields:

**Corollary 32.** Let \( \xi \) be a convergence. Then \( S\xi = S_1\xi \) if and only if \( U_\xi(x) \) is \( \delta \)-closed for every \( x \in |\xi| \).

To summarize

\[
\begin{align*}
\forall x \; U_\xi(x) \text{ is cover-} J/\mathcal{D}_\ast \text{-compact} & \quad \forall x \; \text{cl}_{\mathcal{D}_\ast} (U_\xi(x)) = \text{cl}_{\mathcal{J}_\ast} (U_\xi(x)) \\
J = \mathcal{F} & \quad D = \mathcal{F}_0 \\
\forall x \; \{x\} \text{ is cover-} \mathcal{J}/\mathcal{D}_\ast \text{-compact} & \quad A_D \xi = A_J \xi \quad \text{Example 51}
\end{align*}
\]

I do not know at the moment if the converse of Proposition 29 may fail when \( J \neq \mathcal{F} \). For instance,

**Problem 33.** Is there a convergence \( \xi \) with \( x \in |\xi| \) such that \( \{x\} \) cover-countably compact but \( U_\xi(x) \) is not countably compact?

Note also that:

**Proposition 34.** Let \( \xi \) be a pseudotopology. Then

\[ U_{1_1\xi}(x) = U_{S_{1_1\xi}}(x) = \text{int}_\delta(U_\xi(x)) . \]
Corollary 35. A convergence is bisequential (respectively strongly Fréchet, respectively Fréchet) if and only if $U_\xi(x)$ is $\delta$-open (respectively, $U_\xi(x) = \overline{\text{cl} \beta \text{int}_3 U_\xi(x)}$) for every $x \in |\xi|$.

The interested reader may consult [12] for further details on such characterizations of the classes of bisequential, strongly Fréchet, Fréchet spaces and other similar classes of spaces.

4.2. Countable depth vs countable depth for ultrafilters. Let $K_\sigma(X)$ be the set of $\sigma$-compact subsets of $X$. Note that as compactness is absolute, $\sigma$-compact subsets of $A \subseteq UX$ have the form $\bigcup_{i \in \omega} \beta F_i$ where $F_i \in PX$ for every $i \in \omega$.

Let $\overline{\text{cl}}$ denote the closure in the countably tight modification of $UX$, that is, given $A \subseteq UX$,

$$\overline{\text{cl}}_A = \bigcup_{C \subseteq \overline{\text{cl}}_X} C_{\subseteq \omega}. $$

Proposition 36. Let $\xi$ be a pseudotopology. Then

$$U_{P_1, \xi}(x) = \bigcup_{F \in K_\sigma(U_\xi(x))} \overline{\text{cl}} F,$$

and

$$U_{P_1, \xi}(x) = \overline{\text{cl}}_\beta U_\xi(x).$$

In particular, $\xi$ has countable depth if and only if for every $x \in |\xi|$, $\overline{\text{cl}}_\beta F \subseteq U_\xi(x)$ whenever $F$ is a $\sigma$-compact subset of $U_\xi(x)$.

Proof. As $\xi$ is a pseudotopology, $U \in U_{P_1, \xi}(x)$ if and only if $U \in \beta(\bigwedge_{n \in \omega} H_n)$ for a sequence $(H_n)_{n \in \omega}$ of filters with $\beta H_n \subseteq U_\xi(x)$. The result follows from the observation that sets of the form (2.5) are exactly closures of $\sigma$-compact subsets of $U_\xi(x)$.

As for (4.6), note that $U \in U_{P_1, \xi}(x)$ if and only if there is a sequence $\{W_n : n \in \omega\}$ on $U_\xi(x)$ with $U \geq \bigwedge_{n \in \omega} W_n$, so that $U \in \beta(\bigwedge_{n \in \omega} W_n) = \overline{\text{cl}}_\beta \{W_n : n \in \omega\}$. In other words, $U \in U_{P_1, \xi}(x)$ if and only if $U \in \overline{\text{cl}}_\beta U_\xi(x)$. ∎

Therefore, Corollary 22 can be refined:

Corollary 37. If $\xi$ is countably deep for ultrafilters then $U_\xi(x)$ is countably compact for every $x \in |\xi|$.

Proof. Every countable subset of $U_\xi(x)$ has an accumulation point in $UX$ by compactness. In view of Proposition 36, $U_\xi(x) = \overline{\text{cl}}_\beta U_\xi(x)$ and thus this accumulation point is in $U_\xi(x)$, which is thus countably compact. ∎

Example 38 (A pseudotopology that is countably deep for ultrafilters but not countably deep). Following an idea of [11], take a weak $P$-point $U_0$ of $U^\circ N$ that is not a $P$-point (in $UN$), and take a prime pseudotopology $\xi$ on $N$ with only non-isolated point 1, where

$$U_\xi(1) = \{\{1\}, U^\circ N\} \setminus \{U_0\}.$$ 

Then $U_\xi(1) = \overline{\text{cl}}_\beta U_\xi(1)$ because $U_0$ is a weak $P$-point, so that $\xi = P_1^\beta$, but there is an $F_{\sigma}$ subset $S$ of $U^\circ N$ (hence of $UN$) with $U_0 \in \overline{\text{cl}}_\beta S$, because $U_0$ is not a $P$-point. Thus $\xi$ is not countably deep.
Example 39 (A convergence $\xi$ with $S_1 \xi = S_0 \xi$ that is not countably deep for ultrafilters). Take similarly a free ultrafilter $\mathcal{U}_0$ of $\mathbb{U}^\circ \mathbb{N}$ that is not a weak P-point, and take a prime pseudotopology $\xi$ on $\mathbb{N}$ with only non-isolated point 1, where

$$\mathcal{U}_\xi(1) = \{\{1\}\} \cup \mathbb{U}^\circ \mathbb{N}\setminus\{\mathcal{U}_0\}.$$  

Then $\mathcal{U}_0 \in \text{cl}_\ell(\mathcal{U}_\xi(1)) \setminus \mathcal{U}_\xi(1) \neq P^\dagger_1 \xi$, but

$$\text{cl}_\beta(\mathcal{U}_\xi(1)) = \text{cl}_\delta(\mathcal{U}_\xi(1)) = \{\{1\}\} \cup \mathbb{U}^\circ \mathbb{N},$$

so that $S_1 \xi = S_0 \xi$.

Let us now return to showing that $P_1$ and $S_1$ do not commute. Let’s start by examining when $P_1 \xi$ and $S_0 \xi$ coincide:

**Proposition 40.** The following are equivalent for a pseudotopology $\xi$:

1. $P_1 \xi = S_0 \xi$;
2. For every $x \in |\xi|$, there is $\{D_n : n \in \omega\} \subset \lim^{-\circ}_\xi(x)$ with $\mathcal{V}_\xi(x) = \bigwedge_{n \in \omega} D_n$;
3. there is a $\sigma$-compactoid subset $\mathcal{F}$ of $\mathcal{U}_\xi(x)$ with $\text{cl}_\beta F = \text{cl}_\delta \mathcal{U}_\xi(x)$.

**Proof.**

1. $\implies$ (2): If $P_1 \xi = S_0 \xi$ then $x \in \lim_{P_1} \mathcal{V}_\xi(x)$ for each $x$, that is, (2).

2. $\implies$ (3) because

$$(4.7) \quad \text{cl}_\beta \mathcal{U}_\xi(x) = \beta(\mathcal{V}_\xi(x)) = \beta\left(\bigwedge_{n \in \omega} D_n\right) = \text{cl}_\beta\left(\bigcup_{n \in \omega} \beta D_n\right).$$

3. $\implies$ (1) Since $\mathcal{F} = \bigcup_{n \in \omega} \beta D_n$ for a sequence $\{D_n : n \in \omega\}$ of filters with $\beta D_n \subset \mathcal{U}_\xi(x)$ and $\xi$ is pseudotopological, we conclude that $\{D_n : n \in \omega\} \subset \lim^{-\circ}_\xi(x)$.

*From (4.7), we conclude that $\mathcal{V}_\xi(x)$ converges for $P_1 \xi$, that is, $P_1 \xi = S_0 \xi$. $lacksquare$

**Theorem 41.** There is a set $X$ and a non-closed $\delta$-closed subset $\mathcal{S}$ of $\mathbb{U} \mathcal{X}$ such that $\text{cl}_\beta \mathcal{S} \neq \text{cl}_\beta \mathcal{C}$ whenever $\mathcal{C}$ is an $F_\sigma$ (in $\mathbb{U} \mathcal{X}$) subset of $\mathcal{S}$.

Subsection 4.3 below provides a proof of Theorem 41.

Example 42 (SP). Let $X$ and $\mathcal{S}$ be as in Theorem 41 and let $x_0 \in X$. Let $\xi$ be a prime pseudotopology on $X$ with non-isolated point $x_0$ and $\mathcal{U}_\xi(x_0) = \{\{x_0\}\} \cup \mathcal{S}$. Then $\xi = S_1 \xi > S_0 \xi$ because $\mathcal{U}_\xi(x_0)$ is $\delta$-closed but not closed. On the other hand, in view of Proposition 40, $S_1 \xi > S_0 \xi$ because $\text{cl}_\beta \mathcal{S} \neq \text{cl}_\beta \mathcal{C}$ whenever $\mathcal{C}$ is an $F_\sigma$ subset of $\mathcal{S}$.

4.3. **Topological properties of $\mathcal{U}_\xi X$.** We observe an interplay between coincidence of different closures for $\mathcal{U}_\xi X$ and various kind of compactness of $\mathcal{U}_\xi X$ as we did for $\mathcal{U}_\xi(x)$ in Section 4.1.

**Proposition 43.** $\mathcal{U} \in \text{cl}_\delta(\mathcal{U}_\xi X)$ if and only if $\mathcal{U}$ is $\mathcal{D}$-compactoid.

**Proof.** $\mathcal{U} \in \text{cl}_\delta(\mathcal{U}_\xi X)$ if for every $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D} \not\subseteq \mathcal{U}$ we have $\beta \mathcal{D} \cap \mathcal{U}_\xi X \neq \emptyset$, that is,

$$\mathcal{D} \ni \mathcal{D} \not\subseteq \mathcal{U} \implies \text{adh}_\xi \mathcal{D} \neq \emptyset.$$ 

In other words, $\mathcal{U} \in \text{cl}_\delta(\mathcal{U}_\xi X)$ if and only if $\mathcal{U}$ is $\mathcal{D}$-compactoid. $lacksquare$

Recall that $\mathcal{K}_\mathcal{D}$ denote the classes of all $\mathcal{D}$-compactoid filters (for a given convergence).

**Proposition 44.** Let $\xi$ be a convergence and $\mathcal{D} \subset \mathcal{J}$ be two classes of filters. The following are equivalent:
Proof. The inclusions $\text{cl}_J(\bigcup \xi X) \subseteq \text{cl}_I(\bigcup \xi X)$ and $K_J(\xi) \subseteq K_I(\xi)$ are always true because $\mathbb{D} \subseteq \mathbb{J}$.

In view of Proposition 43, by Proposition 44, $\text{cl}_J(\bigcup \xi X) = \text{cl}_I(\bigcup \xi X)$ if and only if $K_J(\xi) = K_I(\xi)$. 

Note that $K_{\mathbb{F}_0} = \mathbb{F}$ because every filter is $\mathbb{F}_0$-compactoid. Hence if $\mathbb{D} = \mathbb{F}_0$ we obtain:

**Corollary 45.** Let $\mathbb{J}$ be a class of filters and $\xi$ be a convergence. Then $\text{cl}_J(\bigcup \xi X) = \text{cl}_I(\bigcup \xi X)$ if and only if $\xi$ is $\mathbb{J}$-compact.

**Proof.** By Proposition 44,

$$\text{cl}_J(\bigcup \xi X) = \text{cl}_I(\bigcup \xi X) \iff K_{\mathbb{F}_0}(\xi) = \mathbb{F}(\xi) = K_J(\xi)$$

so that every filter is $\mathbb{J}$-compactoid. In particular, every $\mathbb{G} \in \mathbb{J}$ is $\mathbb{J}$-compactoid and $\text{adh}_J \mathbb{G} \neq \emptyset$. Conversely, if $\xi$ is $\mathbb{J}$-compact then every filter in $\mathbb{J}$ has non-empty adherence and thus every filter is $\mathbb{J}$-compactoid. 

Now let's consider the case where $\mathbb{J} = \mathbb{F}$ in Proposition 44. Then $\text{cl}_\mathbb{F}$ is the identity because $\mathbb{F}^*$ is the discrete topology.

An ideal cover $\mathbb{C}$ of $(X, \xi)$ is proper if $X \not\subseteq \mathbb{C}$. Another cover $\mathbb{S}$ is compatible with $\mathbb{C}$ if the ideal generated by $\mathbb{C} \cup \mathbb{S}$ is proper.

Of course two ideal covers $\mathbb{C}$ and $\mathbb{S}$ (ad$\mathbb{C}$ = ad$\mathbb{S}$ = $\emptyset$) are compatible if and only if $\mathbb{C} \cdot \mathbb{S}$.

Given a class of filters $\mathbb{D}$, we say that a cover $\mathbb{C}$ is of class $\mathbb{D}_x$ if $(\mathbb{C}^x)_c \in \mathbb{D}$.

**Corollary 46.** The following are equivalent:

1. $\bigcup \xi X$ is $\mathbb{D}_x$-closed;
2. $K_\mathbb{D}(\xi) = K_I(\xi)$, that is, every $\mathbb{D}$-compactoid filter on $\xi$ is compactoid;
3. $\xi$ is $\mathbb{D}$-compact;
4. $\xi$ is ($\mathbb{D} \cap \bigcup$)-compactoid;
5. For every proper ideal cover, there is a compatible proper cover of class $\mathbb{D}_x$.

**Proof.** (1) $\iff$ (2) is Proposition 44. Assume (2). If $\mathbb{F} \in K_\mathbb{D}$ then $\mathbb{F}$ is compactoid and in particular $\text{adh}_\mathbb{D} \mathbb{F} \neq \emptyset$. Thus $\xi$ is $\mathbb{D}$-compactoid. That (3) $\implies$ (4) is obvious. On the other hand, (4) $\implies$ (2) because if $\mathbb{F} \in K_\mathbb{D}$ then $\mathbb{U} \in K_\mathbb{D} \cap \bigcup$ for every $\mathbb{U} \in \beta \mathbb{F}$ and thus $\lim_\chi \mathbb{U} \neq \emptyset$, that is, $\mathbb{F} \in K_I$.

Finally, (3) $\iff$ (5) because (3) can be rephrased in terms of covers:

$$(\mathbb{F} \in K_\mathbb{D} \iff \text{adh}_\mathbb{D} \mathbb{F} \neq \emptyset) \iff (\text{adh}_\mathbb{D} \mathbb{F} = \emptyset \iff \mathbb{F} \not\subseteq K_\mathbb{D})$$

and the latest condition rephrases as (5) because a proper ideal cover $\mathbb{C}$ satisfies $\mathbb{F} = C_c \in \mathbb{F}$ with $\text{adh}_\mathbb{D} \mathbb{F}$, and if $\mathbb{D} \in \mathbb{D}$ with $\mathbb{D} \# \mathbb{F}$ and $\text{adh}_\mathbb{D} \mathbb{F} = \emptyset$ then $\mathbb{S} = D_c$ is a compatible cover in $\mathbb{D}_x$. 

Recall that $K_{\mathbb{F}_0} = \mathbb{F}$, so that $\bigcup \xi X$ is closed (hence compact) if and only if $\xi$ is compact.

**Theorem 47.** Let $\mathbb{D} \subseteq \mathbb{J}$ be two classes of filters and let $\xi$ be a convergence. If $\bigcup \xi X$ is cover-$\mathbb{J}$/$\mathbb{D}_x$-compact and $\mathbb{J}$ is $\beta$-compatible then $\xi$ is cover-$\mathbb{J}$/$\mathbb{D}_x$-compact. If $\mathbb{J} = \mathbb{F}$ and $\mathbb{D}$ is $\beta$-compatible, the converse is true.
Proof. Let $\mathcal{G} \in \mathcal{I}$ with $\text{adh}_\mathcal{I}\mathcal{G} = \emptyset$. Then $\mathcal{U}_\xi X \subset (\beta\mathcal{G})^c = \bigcup_{\mathcal{G} \in \mathcal{G}} \beta(X \setminus \mathcal{G})$ and thus $\{\beta(X \setminus \mathcal{G}) : \mathcal{G} \in \mathcal{G}\}$ is a cover of $\mathcal{U}_\xi X$ in $\mathcal{I}_\beta$ by $\beta$-compatibility of $\mathcal{I}$. As $\mathcal{U}_\xi X$ is cover-$\mathcal{I}_\beta$-compact, there is $\mathcal{S} \subset \{\beta(X \setminus \mathcal{G}) : \mathcal{G} \in \mathcal{G}\}$ which is a cover of $\mathcal{U}_\xi X$ of class $\mathcal{D}_\beta$, so that $\{\mathcal{G} \in \mathcal{G} : \beta(X \setminus \mathcal{G}) \in \mathcal{S}\}$ belongs to $\mathcal{D}_\beta$ and satisfies $\text{adh}_\mathcal{I}\mathcal{D} = \emptyset$. Thus $\xi$ is cover-$\mathcal{I}_\beta$-compact.

Assume now that $\xi$ is cover-$\mathcal{I}_\beta$/$\mathcal{D}_\beta$-compact and let $\mathcal{C}$ be an open cover of $\mathcal{U}_\xi X$. It has a refinement of the form $\{\beta A : A \in \mathcal{A}\}$ for some family $\mathcal{A} \subset \mathcal{P}X$, which covers $\mathcal{U}_\xi X$. As $\beta(A \cup B) = \beta A \cup \beta B$, we can assume $\mathcal{A}$ and $\{\beta A : A \in \mathcal{A}\}$ to be ideals. Hence $\mathcal{F} = \mathcal{A}_\xi$ is a filter on $X$ with $\text{adh}_\mathcal{I}\mathcal{F} = \emptyset$. Moreover, $\xi$ is cover-$\mathcal{I}_\beta$/$\mathcal{D}_\beta$-compact so that there is $\mathcal{D} \leq \mathcal{F}$ with $\text{adh}_\mathcal{I}\mathcal{D} = \emptyset$, equivalently, $\{\beta(X \setminus \mathcal{D}) : \mathcal{D} \in \mathcal{D}_\beta\} \subset \{\beta A : A \in \mathcal{A}\}$ is a cover of $\mathcal{U}_\xi X$, which is in $\mathcal{D}_\beta$ by $\beta$-compatibility of $\mathcal{D}$. Hence $\mathcal{C}$ has a refinement in $\mathcal{D}_\beta$ that covers $\mathcal{U}_\xi X$, which is thus cover-$\mathcal{I}_\beta$/$\mathcal{D}_\beta$-compact. $\square$

In particular,

**Corollary 48.** $\mathcal{U}_\xi X$ is Lindelöf if and only if $\xi$ is cover-Lindelöf.

As a Lindelöf subset of a topological space is also $\delta$-closed, we obtain:

**Corollary 49.** If $\xi$ is cover-Lindelöf then $\mathcal{U}_\xi X$ is $\delta$-closed.

The converse of Corollary 49 is false:

**Example 50** (A topology $\xi$ such that $\mathcal{U}_\xi X$ is $\delta$-closed but not Lindelöf). Let $X$ be an uncountable set and let $\xi$ be a prime topology on $X$ with non-isolated point $\infty$ and $\mathcal{N}_\xi(\infty) = \{\infty\} \cup \{x_n : n \in \omega\}$, where $\{x_n : n \in \omega\}$ is a free sequence. Then $\xi$ is not Lindelöf and thus $\mathcal{U}_\xi X$ is not Lindelöf by Corollary 18. On the other hand, if $\mathcal{U} \in \mathcal{K}_\xi \cap \mathcal{U}^\circ X$ then $\mathcal{H}(x_n)_{n \in \omega}$ for every $\mathcal{H} \in \mathcal{F}_1$ with $\mathcal{H} \leq \mathcal{U}$. If there was $k \in \omega$ with $\{x_n : n \geq k\} \notin \mathcal{U}$, then $X \setminus \{x_n : n \geq k\} \notin \mathcal{U}$ and, given $\mathcal{H} \in \mathcal{F}_1$, $\mathcal{H} \leq \mathcal{U}$, the filter $\mathcal{H}' = \mathcal{H} \vee (X \setminus \{x_n : n \geq k\})$ would be a countably based subfilter of $\mathcal{U}$ that does not mesh with $(x_n)_{n \in \omega}$. Hence $\mathcal{U} \geq (x_n)_{n \in \omega}$ and $\mathcal{U}$ converges. In view of Proposition 46 for $\mathcal{D} = \mathcal{F}_1$, $\mathcal{U}_\xi X$ is $\delta$-closed.

We can transpose this simple example to $\mathcal{U}_\xi(x)$ to the effect that there is:

**Example 51** (A convergence $\sigma$ with $S \sigma = S_1 \sigma$ and a non cover-Lindelöf singleton). Consider the set $S = \mathcal{U}_\xi X \subset \mathcal{U}X$ as in Example 50 and consider on $X$ the prime pseudotopology $\sigma$ defined by

$$\mathcal{U}_\sigma(\infty) = \{\infty\}^\uparrow \cup S.$$  

Then $\sigma = S \sigma = S_1 \sigma$ because $S$ is $\delta$-closed. On other hand, $\mathcal{U}_\sigma(\infty)$ is not Lindelöf (because $S$ is not) and thus $\{\infty\}$ is not cover-Lindelöf by Corollary 31.

**Proposition 52.** The following are equivalent:

1. there is an $\mathcal{F}_\sigma$ subset $S$ of $\mathcal{U}_\xi X$ such that $\text{cl}_\beta S = \text{cl}_\beta (\mathcal{U}_\xi X) = \mathcal{U}X$;
2. There is a sequence $(\mathcal{D}_n)_{n \in \omega}$ of compactoid filters such that $\beta(\bigwedge_{n \in \omega} \mathcal{D}_n) = \mathcal{U}X$, equivalently,

$$\bigwedge_{n \in \omega} \mathcal{D}_n = \{X\}.$$  

Proof. Note that since $\{x\}^\uparrow : x \in X \} \subset \mathcal{U}_\xi X$, the set $\mathcal{U}_\xi X$ is dense in $(\mathcal{U}X, \beta)$, that is, $\text{cl}_\beta (\mathcal{U}_\xi X) = \mathcal{U}X$. 


By definition, there is an $F_{\sigma}$ subset $S$ of $U_{\xi}X$ such that $cl_{\beta}S = cl_{\beta}(U_{\xi}X)$ if and only if there is a sequence $(D_n)_{n \in \omega}$ of filters with $\beta D_n \subset U_{\xi}X$ and $cl_{\beta}(\bigcup_{n \in \omega} \beta D_n) = cl_{\beta}U_{\xi}X$. The conclusion follows from $[41]$. □

**Corollary 53.** Let $(X, \xi)$ be a $T$-regular pseudotopological space. Then the following are equivalent:

1. $\xi$ is $\sigma$-compact;
2. $U_{\xi}X$ is $\sigma$-compact;
3. there is an $F_{\sigma}$ subset $S$ of $U_{\xi}X$ with $cl_{\beta}S = cl_{\beta}(U_{\xi}X) = UX$.

**Proof.** (1) $\implies$ (2): If $X = \bigcup_{n \in \omega} K_n$ where each $K_n$ is compact. Let $D_n = O(K_n)^\uparrow$. Then $U_{\xi}X = \bigcup_{n \in \omega} \beta D_n$ is $\sigma$-compact. Indeed, if $U \in U_{\xi}X$, there is $x \in \lim_{\xi} U$ and thus there is $n \in \omega$ with $x \in \lim_{\xi} U \cap K_n$. Then $U \geq O(K_n) = D_n$ belongs to $\beta D_n$.

That (2) $\implies$ (3) is clear. To see that (3) $\implies$ (1), assume there is an $S \subset U_{\xi}X$ as in (3). In view of Proposition $[52]$ there is a sequence $(D_n)_{n \in \omega}$ of compactoid filters such that $\bigwedge_{n \in \omega} D_n = \{X\}$. In view of Corollary $[53]$ each $cl_{\xi}^2 D_n$ is a compact filter with compact kernel $K_n = ker cl_{\xi}^2 D_n$. Moreover, $\bigwedge_{n \in \omega} cl_{\xi}^2 D_n = \{X\}$ because $cl_{\xi}^2 D_n \leq D_n$ for each $n \in \omega$, so that, in view of Lemma $[1]$

$$X = ker \bigwedge_{n \in \omega} cl_{\xi}^2 D_n = \bigcup_{n \in \omega} ker cl_{\xi}^2 D_n = \bigcup_{n \in \omega} K_n.$$ 

Therefore, $\xi$ is $\sigma$-compact. □

Theorem $[41]$ now follows at once, as there are regular Lindelöf topological spaces that are not $\sigma$-compact (e.g., $\mathbb{R}^\omega$). For such a space $(X, \xi)$, the set $U_{\xi}X$ is $\delta$-closed (in fact, Lindelöf) by Corollary $[40]$ but not closed because $\xi$ is not compact. Moreover, by Corollary $[53]$ there is no $F_{\sigma}$ subset $S$ of $U_{\xi}X$ with $cl_{\beta}S = cl_{\beta}(U_{\xi}X)$ because $\xi$ is not $\sigma$-compact.

Recall that a topological space is *hemicom pact* if there is a sequence of compact subsets $(K_n)_{n \in \omega}$ such that for every compact subset $K$, there is $n \in \omega$ with $K \subset K_n$. Of course every hemicompact topology is $\sigma$-compact but not conversely.

**Proposition 54.** Let $\xi$ be a convergence. The following are equivalent:

1. $U_{\xi}X$ is hemicompact;
2. There is a sequence $(D_n)_{n \in \omega}$ of compactoid filter such that $F \in \mathcal{K}_{\xi} \implies \exists n : F \geq D_n$.

**Proof.** $U_{\xi}X$ is hemicompact if there is a sequence of compact subsets, that is, a sequence of filters $D_n$ with $\beta D_n \subset U_{\xi}X$ such that any compact subset $\beta F \subset U_{\xi}X$ for $F \in FX$, there is $n \in \omega$ with $\beta F \subset \beta D_n$, equivalently, $D_n \leq F$. As $\beta F \subset U_{\xi}X$ if and only if $F \in \mathcal{K}_{\xi}$, the conclusion follows. □

**Corollary 55.** If $\xi$ is a regular Hausdorff topology then $\xi$ is hemicompact if and only if $U_{\xi}X$ is hemicompact.

**Proof.** Assume that $\xi$ is hemicompact and that $(K_n)_{n \in \omega}$ witnesses the definition. Let $D_n = O(K_n)^\uparrow$. Clearly, $D_n$ is compact(oid) because $K_n$ is compact. If now $F$ is a compactoid filter, then in view of Corollary $[53]$ and Corollary $[10]$ $adl(cl_{\xi}^2 F) = adl F$ is compact, so that there is $n \in \omega$ with $adl F \subset K_n$ and thus $O(K_n) \subset O(adl F) = O(cl_{\xi}^2 F) \leq F$.
and thus $D_n \leq F$. In view of Proposition 54, $U_\xi X$ is hemicompact.

Conversely, let $(D_n)_{n \in \omega}$ be as in Proposition 54 and let

$$K_n = \text{adh}_\xi D_n = \text{adh}_\xi (\text{cl}_\xi D_n).$$

If $K$ is a compact subset of $|\xi|$ then $\{K\}^\uparrow$ is a compact filter, and thus $\{K\} \geq D_n \geq \text{cl}_\xi D_n$ for some $n \in \omega$, that is, $K \subset \ker \text{cl}_\xi D_n = \text{adh}_\xi D_n = K_n$. □

The following two propositions are straightforward and characterize the closure of $U_\xi X$ under the closures involved in the definitions of countable depth and countable depth for ultrafilters:

**Proposition 56.** Let $\xi$ be a convergence. The following are equivalent:

1. $\text{cl}_\beta S \subseteq U_\xi X$ whenever $S$ is an $F_\sigma$-subset of $U_\xi X$;
2. If $(D_n)_{n \in \omega}$ is a sequence of compactoid filters, then $\bigwedge_{n \in \omega} D_n$ is compactoid.

**Proposition 57.** Let $\xi$ be a convergence. The following are equivalent:

1. $U_\xi X = \text{cl}_t (U_\xi X)$;
2. If $(U_n)_{n \in \omega}$ is a sequence of convergent ultrafilters, then $\bigwedge_{n \in \omega} U_n$ is compactoid.

### 4.4. Topological properties of $U_X \setminus U_\xi X$.

Note that $U \in \text{cl}_\beta (U_X \setminus U_\xi X) \iff \forall U \in U \exists W \in \beta U : \lim_\xi W = \emptyset$,

that is, $U \in \text{cl}_\beta (U_X \setminus U_\xi X)$ if and only if $U$ has no compactoid element, equivalently, every co-compactoid set is in $U$. In other words,

$$\lim_\xi U \neq \emptyset \implies \exists D \in \mathbb{D} : D \leq U \text{ and } D \text{ is compactoid.}$$

As a result:

**Proposition 58.** $U_X \setminus U_\xi X$ is $\mathbb{D}^*$-closed if and only if

$$\lim_\xi U \neq \emptyset \implies \exists D \in \mathbb{D} : D \leq U \text{ and } D \text{ is compactoid.}$$

More generally,

**Proposition 59.** Let $\mathbb{D}, \mathbb{J}$ be two classes of filters. The following are equivalent:

1. $\text{cl}_\beta (U_X \setminus U_\xi X) = \text{cl}_{\beta J} (U_X \setminus U_\xi X)$;
2. For every $U \in U_X$,

$$\exists G \in \mathbb{J} \cap \mathbb{K}_F, G \leq U \implies \exists D \in \mathbb{D} \cap \mathbb{K}_F, D \leq U.$$

Recall that $F_0^*$ is the usual topology $\beta$ of $U_X$.

**Proposition 60.** Let $\mathbb{J}$ be a class of filters. The following are equivalent:

1. $\text{cl}_\beta (U_X \setminus U_\xi X) = \text{cl}_\beta (U_X \setminus U_\xi X)$;
2. For every $U \in U_X$,

$$\exists G \in \mathbb{J} \cap \mathbb{K}_F, G \leq U \implies \exists K \in F_0 \cap \mathbb{K}_F, K \in U;$$
(3) Every compactoid filter in \( \mathcal{J} \) has a compactoid element.

Proof. (1) \( \iff \) (2) is Proposition \[58\] for \( D = F_0 \) and (3) \( \implies \) (2) is clear. For (2) \( \implies \) (3), let \( G \in \mathcal{J} \cap K_\varepsilon \). For every \( U \in \beta G \), there is by (2) a compactoid set \( K_U \in \mathcal{U} \). By compactness of \( \beta G \), there is a finite subset \( F \) of \( \beta G \) with \( \bigcup_{U \in F} K_U \in G \), and \( \bigcup_{U \in F} K_U \) is compactoid as a finite union of compactoid sets. \( \square \)

In particular, \( cl_\beta(\mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X}) \subset cl_\delta(\mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X}) \) if and only if every countably based compactoid filter has a compactoid element. In this case, countably based convergent filter contain a compactoid sets, and thus:

**Corollary 61.** If \( cl_\beta(\mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X}) \subset cl_\delta(\mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X}) \) then \( I_1 \xi \) is locally \( \xi \)-compactoid.

As a result of Proposition \[58\] for \( D = F_0 \), we obtain \[10\] (10):

**Corollary 62.** The following are equivalent:

1. \( \mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X} \) is \( \beta \)-closed;
2. \( \mathcal{U}_\xi \mathcal{X} \) is \( \beta \)-open;
3. \( \xi \) is locally compactoid;
4. \( adh_\xi K_c = \emptyset \).

Proof. That (1) \( \iff \) (2) is obvious.

1) \( \implies \) (3): Let \( \mathcal{F} \) be a convergent filter. For every \( U \in \beta \mathcal{F} \), there is, by Proposition \[58\], a compactoid set \( K_U \in \mathcal{U} \). By compactness of \( \beta \mathcal{F} \), there is a finite subset \( S \) of \( \beta \mathcal{F} \) with \( \bigcup_{U \in S} K_U \in \mathcal{F} \). As \( \bigcup_{U \in S} K_U \) is a finite union of compactoid sets, it is compactoid.

3) \( \implies \) (4): If \( \xi \) is locally compactoid, a convergent filter cannot mesh with \( K_c \).

4) \( \implies \) (1): If \( U \in \mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X} \) and \( \lim_\xi U \neq \emptyset \) and \( adh_\xi K_c = \emptyset \) then there is \( H \in K_c \) with \( H \notin \mathcal{U}_c \setminus \mathcal{U} \), so that \( H^c \), which is compactoid, belongs to \( \mathcal{U} \). In view of Proposition \[58\] we obtain (1). \( \square \)

**Corollary 63.** Let \( \xi \) be a non-compact topology. Then \( \xi \) is locally compactoid if and only if

\[
cl_\xi^2 K_c = K_c.
\]

Proof. If \( \xi \) is locally compactoid, then \( adh_\xi K_c = \emptyset \) by Corollary \[62\], so that, in view of Lemma \[6\], \( cl_\xi^2 K_c \geq K_c \). The reverse inequality is always true and thus \( cl_\xi^2 K_c = K_c \). Conversely, if \( cl_\xi^2 K_c = K_c \) then \( adh_\xi K_c = \emptyset \) by Lemma \[6\] and thus \( \xi \) is locally compactoid by Corollary \[62\]. \( \square \)

Recall that \( F_1^* \) is the \( G_\delta \)-topology of \( (\mathcal{U}, \beta) \).

As a result of Proposition \[58\] and Corollary \[11\],

**Corollary 64.** If \( \xi \) is a regular Hausdorff topology, then \( \mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X} \) is \( \delta \)-closed if and only if \( \xi \) is bi-k.

**Corollary 65.** If \( \xi \) is bisequential then \( \mathcal{U} \setminus \mathcal{U}_\xi \mathcal{X} \) is \( \delta \)-closed.

As there are bi-k spaces that are not bisequential, the converse of Corollary \[65\] is not true but in the case of a prime convergence on a countable set, it is:

**Theorem 66.** Let \( \xi \) be a countable prime convergence. The following are equivalent:

---

\[^{10}\text{This result can be found in [19] and was revisited in [10]. A proof is included here for completeness and to illustrate an instance of Proposition [58]}.\]
Proof. That (2) \iff (3) is obvious and (1) \implies (2) is Corollary \ref{corollary}. To see that (3) \implies (1), let $x_0$ denote the non-isolated point of $\xi$. If $x_0 \in \lim_\xi U$ for some free ultrafilter $U$, then $U \in \mathcal{U}_\xi X$, which is $\delta$-open so that, in view of Lemma \ref{lemma} there is $D \in F_1$ with

$$U \in \beta D = \beta D^* \cup \beta D^\circ \subset \mathcal{U}_\xi X.$$  

If $U \geq D^\circ$, we have $D^\circ \in F_1$ and $x_0 \in \lim_\xi D^\circ$ because convergent free ultrafilters can only converge to $x_0$. Hence $x \in \lim_\xi \ker D \cup \lim_\xi U$.

If $U \geq D^*$ then $\ker D \in U$. As $U$ is free, $\ker D$ is infinite, and all free ultrafilters on $\ker D$ converge to $x_0$ so that $x_0 \in \lim_\xi (\ker D)$. Because $|\xi|$ is countable, $(\ker D)_0 \in F_1$ and thus $x_0 \in \lim_\xi \xi U$. \hfill \Box

\textbf{Theorem 67.} Let $D \subset J$ be two classes of filters, where $J$ is $\beta$-compatible. If $\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X$ is $\exists/-/\ast$-compact then

(4.9) $$G \in J \cap K_F \implies \exists D \in D \cap K_F : D \leq G.$$  

Moreover, if $J = F$ and $D$ is $\beta$-compatible, the converse is true.

Proof. Suppose $\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X$ is $\exists/-/\ast$-compact and let $G \in K_F \cap J$. Then $\beta G \cap (\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X) = \emptyset$, that is,

$$\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X \subset \mathcal{U}_\xi X \setminus \beta G = \bigcup_{G \in G} \beta (X \setminus G).$$  

By $\beta$-compatibility of $J$, $\{\beta (X \setminus G) : G \in G\}$ is a cover of $\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X$ in $J_\ast$. By $\exists/-/\ast$-compactness, there is $D \in D$ with $D \leq G$ and

$$\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X \subset \mathcal{U}_\xi X \setminus \beta D = \bigcup_{D \in D} \beta (X \setminus D),$$  

so that $D \in K_F \cap D$. Assume now (4.9) in the case $J = F$. Any open cover $G$ of $\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X$ has a refinement of the form $\{\beta A : A \in A\}$ where $A$ is an ideal, that is also a cover. Let $F$ be the filter $A_c$. Then

$$\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X \subset \bigcup_{A \in A} \beta A = \mathcal{U}_\xi X \setminus \beta F,$$  

so that $F \in K_F$. By (4.9), there is $D \in D \cap K_F$ with $D \leq F$. Hence

$$\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X \subset \mathcal{U}_\xi X \setminus \beta D,$$  

and by $\beta$-compatibility of $D$, $\{\beta (X \setminus D) : D \in D\}$ is a subcover of $\{\beta A : A \in A\}$ of class $D_\ast$. \hfill \Box

\textbf{Corollary 68.} Let $(X, \xi)$ be a convergence space. Then $\mathcal{U}_\xi X \setminus \mathcal{U}_\xi X$ is Lindelöf if and only if every compactoid filter contains a countably based compactoid filter.

As there are bi-$k$ regular Hausdorff topological spaces that are not of pointwise countable type (see [2]), we obtain:

\textbf{Corollary 69.} There is $X$ and a subset $S$ of $\mathcal{U}_\xi X$ which is $\delta$-closed but not Lindelöf.
Proof. For such a topological space \((X, \xi)\), \(S = UX \setminus U_{\xi}X\) is \(\delta\)-closed because \(\xi\) is bi-
\(k\) by Corollary 11. As \(\xi\) is not of pointwise countable type, there is by Corollary 64 a convergent, hence compactoid, filter that does not contain any countably based compactoid filter. In view of Corollary 68, \(S\) is not Lindelöf. \(\Box\)

Corollary 70. If \((X, \xi)\) is of countable type \(\text{[11]}\) then \(UX \setminus U_{\xi}X\) is Lindelöf.

Proof. If \(F\) is compactoid then in view of Corollary 64 and Corollary 10, \(F \supseteq O(\text{cl}^F) = O(\text{adh} F)\) and \(\text{adh} F\) is compact. As \(\xi\) is of countable type, there is a compact set \(K\) with \(O(\text{adh} F) \supseteq O(K)\) and \(O(K)\) is a countably based compact filter. \(\Box\)

Finally note that \(\sigma\)-compactness of \(UX \setminus U_{\xi}X\) has been characterized in terms of completeness. A family \(\mathbb{D}\) of non-adherent filters on \((X, \xi)\) is cocomplete if for every \(G \in F X\)
\[\text{adh}_\xi G = \emptyset \implies \exists D \in \mathbb{D} \; G \# D.\]

A convergence \(\xi\) is countably complete if it admits a countable cocomplete family of filters. A completely regular topological space is countably complete if and only if it is Čech-complete. See \([5, 10]\) for details.

Proposition 71. \([10, 5, \text{Corollary 7}]\) \(UX \setminus U_{\xi}X\) is \(\sigma\)-compact if and only if \(\xi\) is countably complete.

Corollary 72. There is a set \(X\) and a subset \(S\) of \(U X\) that is Lindelöf but not \(\sigma\)-compact.

Proof. Let \((X, \tau)\) be a non complete metric space (such as \(\mathbb{Q}\) with its usual topology). Then, in view of Corollary 70 and Proposition 71 \(S = UX \setminus U_{\tau}X\) is as desired. \(\Box\)

Similarly, hemicompactness of \(UX \setminus U_{\xi}X\) has been characterized in terms of ultracompleteness. A family \(\mathbb{D}\) of non-adherent filters on \((X, \xi)\) is ultracocomplete if for every \(G \in F X\)
\[\text{adh}_\xi G = \emptyset \implies \exists D \in \mathbb{D} \; G \geq D.\]

A convergence \(\xi\) is countably ultracomplete if it admits a countable ultracomplete family of filters. See \([10]\) for details.

Proposition 73. \([10, \text{Corollary 9}]\) A convergence \(\xi\) is countably ultracomplete if and only if \(UX \setminus U_{\xi}X\) is hemicompact.

Corollary 74. There is a set \(X\) and a subset \(S\) of \(U X\) that is hemicompact but not \(\beta\)-closed.

Proof. If \((X, \tau)\) is countably ultracomplete but not locally compactoid (e.g., \([1, \text{Example 3.1}]\)), then in view of Corollary 62 and Proposition 73 \(S = UX \setminus U_{\tau}X\) is as desired. \(\Box\)

Finally, let us note that similarly to Corollary 53 \(\sigma\)-compactness of \(UX \setminus U_{\xi}X\) is equivalent to having a \(\sigma\)-compact dense subset.

Proposition 75. The following are equivalent:

\[\text{[11]}\] that is, every compact set is contained in a compact set of countable character. Clearly, metrizable spaces are of countable type.
(1) there is an $F_\sigma$ subset $S$ of $\bigcup X \setminus \bigcup \xi X$ such that

$$\text{cl}_\beta S = \text{cl}_\beta (\bigcup X \setminus \bigcup \xi X);$$

(2) there is a countable family $(D_i)_{i \in \omega}$ of non-adherent filters such that

$$\bigcap_{i \in \omega} D_i = \mathcal{K}_c;$$

(3) there is a family $(C_i)_{i \in \omega}$ of ideal covers such that whenever one selects $C_i \in C_i$ for every $i \in \omega$, the set $\bigcap_{i \in \omega} C_i$ is a compactoid set.

Proof. If (1) there is a sequence $(D_i)_{i \in \omega}$ of non-adherent filters such that

$$\text{cl}_\beta (\bigcup X \setminus \bigcup \xi X) \subset \text{cl}_\beta \{D_n : n \in \omega\},$$

that is, in view of (4.8),

$$\beta \mathcal{K}_c \subset \beta \bigcap_{i \in \omega} D_i.$$

Since $\text{adh} D_i = \emptyset$, $D_i \geq \mathcal{K}_c$ for every $i$, hence $\bigcap_{i \in \omega} D_i \geq \mathcal{K}_c$ and thus $\beta \mathcal{K}_c = \beta (\bigcap_{i \in \omega} D_i)$, equivalently, $\bigcap_{i \in \omega} D_i = \mathcal{K}_c$.

(2) $\implies$ (3) : Take $C_i = (D_i)_{x}$. If one selects $C_i \in C_i$, equivalently, $D_i = C^c_i \in D_i$ then $\bigcup_{i \in \omega} D_i \in \mathcal{K}_c$ so that there is a compactoid set $K^c \subset \bigcup_{i \in \omega} D_i$ equivalently, $\bigcap_{i \in \omega} C_i \subset K$ so that $\bigcap_{i \in \omega} C_i$ is compactoid.

(3) $\implies$ (1) Let $D_i = (C_i)_{x} \in \mathcal{F}X$, where $(C_i)_{i \in \omega}$ is as in (3). We show that

$$\text{cl}_\beta \bigcup_{i \in \omega} \beta D_i = \beta \bigcap_{i \in \omega} D_i = \bigcup X \setminus \bigcup \xi X.$$

Indeed, each $\beta D_i \subset \bigcup X \setminus \bigcup \xi X$ because each $C_i$ is a cover of $(X, \xi)$. If $U \notin \beta (\bigcap_{i \in \omega} D_i)$, there is a selection $D_i \in D_i$ with $\bigcup_{i \in \omega} D_i \notin U$, equivalently a selection $C_i = X \setminus D_i \in C_i$ with $\bigcap_{i \in \omega} C_i \notin U$. Because $\bigcap_{i \in \omega} C_i$ is compactoid we conclude that $\lim_\xi U \neq \emptyset$, that is, $U \in \bigcup_{i \in \omega} X$, that is, $U \notin \bigcup X \setminus \bigcup \xi X$. □

4.5. **Summary of results in this section.** The table below summarizes the main results of the section, together with a few more easily verified characterizations that may prove useful for future reference.
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| property | \(U_1(x)\) | \(U_1 X\) | \(UX \setminus U_1 X\) |
|----------|-------------|-------------|----------------|
| \(\delta\)-closed (contr) | \(S_\delta = S_0 \xi\) | \(\xi\) compact | \(\xi\) locally compact |
| \(\delta\)-open | finitely generated | \(\xi\) locally compact | \(\xi\) compact |
| \(\delta\)-closed | \(S_\delta = S_1 \xi\) | \(\xi\) is \(K_p\) compact | \(\exists D \in K_p\) \(\lim_{U} \neq \emptyset \implies \exists D \in F_1 \cap K_p, D \subseteq U\) \(\iff\) \(\xi\) is \(\delta\)-compact (reg. H. top.) |
| \(\beta\)-open | bisontential | \(\lim_{U} \neq \emptyset \implies \exists D \in F_{1,1} \cap K_p, D \subseteq U\) | \(\xi\) is \(K_p\) compact |
| \(\beta\)-closed | \(S_\beta = S_{1,1} \xi\) | \(\xi\) is \(K_p\) compact | \(\exists D \in K_p\) \(\lim_{U} \neq \emptyset \implies \exists D \in F_{1,1} \cap K_p, D \subseteq U\) |
| \((F_{1,1})^*\)-closed | \(P\) space | \(\lim_{U} \neq \emptyset \implies \exists D \in F_{1,1} \cap K_p, D \subseteq U\) | \(\xi\) is \(K_p\) compact |
| \((F_{1,1})^*\)-open | \(\exists D \subseteq \xi, \text{for every filter } F \cap |D| \subseteq U\) | \(\forall x, F_1 \cap |D| \subseteq U\) | \(\xi\) is \(\omega\)-\(F_1\)-complete |

\(\epsilon\)-compact
\(\epsilon\)-open
\(\epsilon\)-closed
\(\epsilon\)-subdivision
\(\epsilon\)-subdivision
\(\epsilon\)-subdivision
\(\epsilon\)-subdivision
\(\epsilon\)-subdivision

Table 1. Summary of characterizations in UX

5. PAVING AND PSEUDOPAVING NUMBERS

A family \(D\) of filters on a convergence space \((X, \xi)\) is a *pavement at x* if every \(D \in D\) converges to \(x\) and, for every filter \(F\) converging to \(x\), there is \(D \in D\) with \(D \leq F\). The family \(D\) is a *pseudopavement at x* if every \(D \in D\) converges to \(x\) and, for every ultrafilter \(U\) converging to \(x\), there is \(D \in D\) with \(U \in D\).

Let \(p(\xi, x)\) denote the *paving number* of \(\xi\) at \(x\), that is, the smallest cardinality of a pavement at \(x\) for \(\xi\), and let \(pp(\xi, x)\) denote the *pseudopaving number* of \(\xi\) at \(x\), that is, the smallest cardinality of a pseudopavement at \(x\) for \(\xi\). Accordingly, the *paving number* and *pseudopaving number* of \(\xi\) are given by \(p(\xi) = \sup_{x \in \xi} p(\xi, x)\) and \(pp(\xi) = \sup_{x \in \xi} pp(\xi, x)\).

Remark. Since in a convergence the filter \(\{x\}^\uparrow\) always converges to \(x\), the paving and pseudopaving numbers are always at least 1 (and pretopologies are exactly the 1-paved convergences, equivalently, the 1-pseudopaved convergences).

Thus [10, Theorem 26] and [10, Theorem 28] relating paving and pseudopaving numbers of a dual convergence to the ultracompleteness and completeness numbers of the base convergence are only valid for cardinality greater or equal to 1, though
ultracompleteness and completeness numbers can meaningfully be 0. This was not properly spelled out in [10].

**Lemma 76.** The following are equivalent for a family $\mathcal{D} \subset \lim^{-}\xi(x)$.

1. $\mathcal{D}$ is a pseudopavement at $x$;
2. For every filter $\mathcal{F}$ with $x \in \text{adh}_\xi \mathcal{F}$, there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D} \# \mathcal{F}$;
3. For every filter $\mathcal{F}$ with $x \in \lim_\xi \mathcal{F}$, there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D} \# \mathcal{F}$;
4. $U_\xi(x) = \bigcup_{\mathcal{D} \in \mathcal{D}} \beta \mathcal{D}$.

**Proof.** (1) $\iff$ (3) $\iff$ (4) is [10, Theorem 17]. Of course, (2) $\Rightarrow$ (3). To see that (1) $\implies$ (2), let $x \in \text{adh}_\xi \mathcal{F}$. Then there is $\mathcal{U} \in \beta \mathcal{F}$ with $x \in \lim_\xi \mathcal{U}$. Since $\mathcal{D}$ is a pseudopavement at $x$, there is $\mathcal{D} \in \mathcal{D}$ with $U \geq \mathcal{D}$. Thus $\mathcal{F} \# \mathcal{D}$. $\square$

**Corollary 77.** [10, Corollary 18] Let $\xi$ be a convergence. Then $pp(\xi, x) \leq \omega$ if and only if $U_\xi(x)$ is $\sigma$-compact.

**Corollary 78.** If $\xi$ is a convergence with countable pseudopaving number then every singleton is cover-Lindelöf.

**Proof.** If $pp(\xi, x) \leq \omega$ then $U_\xi(x)$ is $\sigma$-compact, hence Lindelöf, and the conclusion follows from Corollary 31. $\square$

The converse is false:

**Example 79** (A convergence of uncountable pseudopaving number with cover-Lindelöf singletons). Take $X$ and $S$ as in Corollary 72. Let $\xi$ be a prime convergence on $X$ with non-isolated point $x_0$, defined by

$$x_0 \in \lim_\xi \mathcal{F} \iff \beta \mathcal{F} \subset S \cup \{ \{x_0\}^\uparrow \}.$$

Since $S$ is Lindelöf, so is $S \cup \{ \{x_0\}^\uparrow \} = U_\xi(x_0)$ and thus $\{x_0\}$ is cover-Lindelöf by Corollary 31. On other hand, $U_\xi(x_0) = S \cup \{ \{x_0\}^\uparrow \}$ is not $\sigma$-compact, so that $pp(\xi) > \omega$ by Corollary 77.

Note also that

**Proposition 80.** [10, Corollary 20] Let $\xi$ be a convergence. Then $p(\xi, x) \leq \omega$ if and only if $U_\xi(x)$ is hemicompact.

**Lemma 81.** If $\mathcal{D}$ is a pseudopavement of $\xi$ at $x$ then

$$V_\xi(x) = \bigwedge_{\mathcal{D} \in \mathcal{D}} \mathcal{D}.$$ 

**Proof.** For every $\mathcal{U} \in \lim_\xi(x)$, there is $\mathcal{D} \in \mathcal{D}$ with $\mathcal{D} \leq \mathcal{U}$ so that

$$\bigwedge_{\mathcal{D} \in \mathcal{D}} \mathcal{D} \leq \bigwedge_{\mathcal{U} \in \lim_\xi(x)} \mathcal{U} = V_\xi(x).$$

The reverse inequality follows from $\mathcal{D} \subset \lim_\xi(x)$. $\square$

**Proposition 82.** If $\xi$ has countable pseudopaving number then $P_1 \xi = S_0 \xi$. In particular, a convergence that is countably deep and has countable pseudopaving number is a pretopology.
Proof. If $D$ is a countable pseudopavement at $x$, then $x \in \lim_{\mathcal{P}\xi} \bigwedge_{D \in \mathcal{D}} D$ and $\bigwedge_{D \in \mathcal{D}} D = \mathcal{V}_\xi(x)$ by Lemma [8]. Thus, $x \in \lim_{\mathcal{P}\xi} \mathcal{V}_\xi(x)$ so that $S_0 \xi \geq P_1 \xi$. The reverse inequality is always true. \hfill \square

Maybe somewhat surprisingly in view of Corollary [53] and Proposition [76] the converse is not true:

**Example 83** (A convergence $\xi$ of uncountable pseudopaving number with $P_1 \xi = S_0 \xi$). Let $\{D_n : n \in \omega\}$ be a partition of $\omega$ into infinite subsets and for each $n \in \omega$ let $D(n)$ be an ultrafilter with $D_n \in D(n)$. Define on $\omega \cup \{\infty\}$ the prime convergence $\xi$ given by

$$\infty \in \lim_\xi \mathcal{F} \iff \exists \mathcal{U} \in \mathcal{U}_\omega : \mathcal{F} = D(\mathcal{U}) = \bigcup_{\mathcal{U} \in \mathcal{U}} \bigcap_{n \in \mathcal{U}} D(n).$$

As a contour of ultrafilters along an ultrafilter is an ultrafilter, $\lim^-_\xi (\infty) \subset \mathcal{U}_\omega$. Therefore, the only pseudopavement of $\xi$ at $\infty$ is $\lim^-_\xi (\infty)$. Moreover, using the notion of types of Frolík [8], we see by [8] Theorem C that for each convergent $\mathcal{F}$, its type is produced by at most $2^n$ ultrafilters $\mathcal{U}$, and there are $2^{2^n}$ ultrafilters, so convergent filters have $2^{2^n}$ different types. In particular, card ($\lim^-_\xi (\infty)$) = $2^{2^n}$ and $\mathsf{pp}(\xi) = 2^{2^n}$. On the other hand, $D(n) = D(\{\infty\})$ converges to $\infty$, so that $\mathcal{V}_\xi(\infty) \leq \bigwedge_{n \in \omega} D(n)$. Conversely, suppose there is $S \in \bigwedge_{n \in \omega} D(n)$ with $S \notin \mathcal{V}_\xi(\infty)$, equivalently, $S^c \in \mathcal{V}_\xi(\infty)^\#$. Because $S \in \bigwedge_{n \in \omega} D(n)$ for each $n \in \omega$ there is $B_n \in D(n)$ with $\bigcup_{n \in \omega} B_n \subset S$. On the other hand, because $S^c \in \mathcal{V}_\xi(\infty)^\#$, there is $\mathcal{U} \in \mathcal{U}_\omega$ with $S^c \in D(\mathcal{U})$, that is, there is $\mathcal{U} \in \mathcal{U}$ with $S^c \in \bigwedge_{n \in \mathcal{U}} D(n)$, which is not possible. Indeed, this means that for every $n \in \mathcal{U}$, there is $A_n \in D(n)$ with $\bigcup_{n \in \mathcal{U}} \bigcap_{n \notin \mathcal{U}} B_n \subset S_c$, but then

$$\bigcup_{n \notin \mathcal{U}} B_n \cup \bigcup_{n \in \mathcal{U}} (B_n \cap A_n)$$

is a subset of $S$ that has non-empty intersection with $S_c^c$. Hence $\mathcal{V}_\xi(\infty) = \bigwedge_{n \in \omega} D(n)$ with $\infty \in \lim_\xi D(n)$ for all $n$, so that $\infty \in \lim_{\mathcal{P}\xi} \mathcal{V}_\xi(\infty)$, that is, $P_1 \xi = S_0 \xi$.

**Corollary 84.** There is a set $X$ and a subset $S$ of $\mathcal{U}X$ which is not $F_\alpha$ but has a dense $F_\alpha$-subset.

There are convergences with countable pseudopaving number that are not pretopologies (e.g., the sequential modification of the countable sequential fan), hence not countably deep, and, as we have already seen, countably deep convergences that are not pretopologies, hence not of countable pseudopaving number. On the other hand, a pseudotopology of countable pseudopaving number is a paratopology:

**Theorem 85.** If $\xi$ is a convergence with cover-Lindelöf singletons (in particular if $\mathsf{pp}(\xi) \leq \omega$), then

$$S \xi = S_1 \xi.$$  

In particular, a pseudotopology of countable pseudopaving number is a paratopology.

Proof. Though we could easily give a direct proof, we know by Corollary [31] that singletons are cover-Lindelöf if and only if $\mathcal{U}_\xi(x)$ is Lindelöf for each $x$, hence closed. In view of Corollary [32] we conclude that $S \xi = S_1 \xi$. Note that in view of Corollary [78] the condition is satisfied in particular if $\mathsf{pp}(\xi) \leq \omega$. \hfill \square
On the other hand, a paratopology may have non-cover-Lindelöf singletons (hence uncountable pseudopaving number), or may have countable paving number without being a pretopology.

Example 86 (A paratopology with a non cover-Lindelöf singleton). Let $X$ and $S$ be as in Corollary 69. Let $\xi$ be a prime convergence on $X$ with non-isolated point $x_0$, defined by

$$x_0 \in \lim_\xi F \iff \beta F \subset S \cup \{\{x_0\}^\uparrow\}.$$ 

Since $S$ is $\delta$-closed, so is $S \cup \{\{x_0\}^\uparrow\}$ and thus $\xi = S_1 \xi$ by Theorem 25. On other hand, $S \cup \{\{x_0\}^\uparrow\}$ is not Lindelöf, so that $\{x_0\}$ is not cover-Lindelöf by Corollary 31.

Example 87 (A paratopology of countable paving number that is not a pretopology). Take $X$ and $S$ as in Corollary 74 and define on $X$ the prime pseudotopology with non-isolated point $x_0$ for which

$$x_0 \in \lim_\xi F \iff \beta F \subset S \cup \{\{x_0\}^\uparrow\}.$$ 

In view of Proposition 80, this a pseudotopology with countable paving number, hence a paratopology with countable paving number by Theorem 85. As $S \cup \{\{x_0\}^\uparrow\}$ is not closed, $\xi$ is not a pretopology.

6. HYPOTOPOLOGIES

Recall that a convergence $\xi$ is an hypotopology if $\xi = S_{\lambda_1} \xi$. The notion was introduced in 4 and is useful in the study of the Lindelöf property.

Lemma 88. If $X$ is a countable set, then $F_{\lambda_1} X = F_0 X$.

Corollary 89. For every convergence $\xi$ on a countable set, $S_{\lambda_1} \xi = S_0 \xi$.

It turns out that

Theorem 90. Every hypotopology is of countable depth:

$$P_1 \geq P_1 \triangle S \geq S_{\lambda_1}.$$ 

Proof. If $x \in \lim_\xi F$, then there is a sequence $\{F_n\}_n$ of filters with $x \in \lim_\xi F_n$ for every $n \in \mathbb{N}$ and $F \geq \bigwedge_{n \in \mathbb{N}} F_n$. Let $\mathcal{H} \in F_{\lambda_1}$ with $\mathcal{H} \# F$. Then $\mathcal{H} \# \bigwedge_{n \in \mathbb{N}} F_n$ so that, there is $n \in \mathbb{N}$ with $\mathcal{H} \# F_n$. Indeed, assume to the contrary that for every $n \in \mathbb{N}$, there is $F_n \in \mathcal{F},$ with $F_n \notin \mathcal{H} \#,$ that is, $F_n \in \mathcal{H}$. Then $\bigcap F_n = \mathcal{H}$ because $\mathcal{H} \in F_{\lambda_1}$ and the complement of $\bigcap F_n$ belongs to $\bigwedge_{n \in \mathbb{N}} F_n$ in contradiction to $\mathcal{H} \# \bigwedge_{n \in \mathbb{N}} F_n$. Thus $x \in \text{adh}_\xi \mathcal{H}$ and we conclude that $x \in \lim_\xi F$. Thus $P_1 \geq S_{\lambda_1}$.

Thus hypotopologies are countably deep pseudotopologies and thus the finest countably deep pseudotopology coarser than a convergence is finer than the finest hypotopology coarser than that convergence. Hence $P_1 \triangle S \geq S_{\lambda_1}$. \hfill \Box

Remark 91. Note that Example 24 is a hypotopology. Indeed, $x_0 \in \lim_\xi F$ if $\ker F \subset \{x_0\}$ and $F \cap X = \emptyset$, that is, $\beta F \cap \beta(X_1) = \emptyset$. Moreover, $F_{\lambda_1} = \{\mathcal{H} : \mathcal{H} \geq (X_1)\}$. In particular, if $\mathcal{H} \in F_{\lambda_1}$ then $\text{adh}_\xi \mathcal{H} = \emptyset$. Let $x_0 \in \lim S_{\lambda_1} \xi$. Then the only countably deep filters meshing with $F$ are not free. If $x_0 \notin \lim_\xi F$ then $F \cap X = \emptyset$ which is not the case, or $\ker F \notin \{x_0\}$. Let $x \neq x_0 \in \ker F$. Then $\{x\}^\uparrow \in F_{\lambda_1}$ and $\{x\}^\uparrow \# F$ so $x_0 \in \text{adh}_\xi \{x\}^\uparrow = \lim_\xi \{x\}^\uparrow$ and thus $x = x_0$. So $x_0 \in \lim_\xi F$, that is, $\xi = S_{\lambda_1} \xi$. 

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Example 92 (A countably deep pseudotopology that is not a hypotopology (assuming that there is a $P$-point in $U^\omega$)). Suppose $\mathcal{U}_0$ is a $P$-point. Let $\xi$ be the prime pseudotopology on $\omega \cup \{\infty\}$ defined by
\[
\infty \in \lim_{\mathcal{F}} \mathcal{F} \iff \beta \mathcal{F} \subset (U^\omega \setminus \{U_0\}) \cup \{\infty\}.
\]
It is not pretopological because $U^\omega \setminus \{U_0\}$ is not closed, hence not hypotopological by Corollary 89. On the other hand, $\xi$ is countably deep because $U^\omega \setminus \{U_0\}$ contains the closure of its every $F_\sigma$-subsets.

In particular, we may have
\[
SP_1 \xi \geq P_1 \triangle S \xi > S_{\Lambda_1} \xi.
\]

Corollary 93.

\[
S_1 S_{\Lambda_1} = S_0.
\]

Proof. Of course $S_1 S_{\Lambda_1} \geq S_0$ because $S_1 \geq S_0$ and $S_{\Lambda_1} \geq S_0$. In view of Theorem 90,
\[
S_0 = S_1 P_1 \geq S_1 S_{\Lambda_1},
\]
where the first equality follows from Proposition 21.

Remarkably, while $P_1 S_1 \neq S_1 P_1$ as observed in Example 42, we have in contrast

Theorem 94.

\[
S_{\Lambda_1} S_1 = S_1 S_{\Lambda_1} = S_0.
\]

Proof. In view of Corollary 93, we only need to show that $S_0 \geq S_{\Lambda_1} S_1$. To this end, we need to show that $x \in \lim_{\mathcal{F}, \mathcal{U}} \mathcal{U}_1 \mathcal{V}_1(x)$, equivalently that
\[
\beta(\mathcal{V}_1(x)) = \text{cl}_\beta(\mathcal{U}_1(x)) \subset U_{S_{\Lambda_1} S_1}(x) = \text{cl}_\beta(\mathcal{U}_1(x)).
\]
Suppose that $\mathcal{U} \notin \text{cl}(F_{\Lambda_1}), \text{cl}_\beta \mathcal{U}_1(x)$, equivalently, there is $\mathcal{H} \in F_{\Lambda_1}$ with $\mathcal{U} \in \beta \mathcal{H}$ and $\beta \mathcal{H} \cap \text{cl}_\beta \mathcal{U}_1(x) = \emptyset$, equivalently, $\mathcal{U}_1(x) \notin (N_\beta(\beta \mathcal{H}))^\#$. Since $\mathcal{H} \in F_{\Lambda_1}$, $N_\beta(\beta \mathcal{H}) = N_\beta^#(\beta \mathcal{H})$ (See, e.g., [12]). Therefore, $\mathcal{U}_1(x) \notin (N_\beta(\beta \mathcal{H}))^\#$ equivalently, $\beta \mathcal{H} \cap \text{cl}_\beta \mathcal{U}_1(x) = \emptyset$. In particular, $\mathcal{U} \notin \text{cl}_\beta(\mathcal{U}_1(x))$. 

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