Almost order-weakly compact operators on Banach lattices

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Abstract A continuous operator $T$ between two Banach lattices $E$ and $F$ is called almost order-weakly compact, whenever for each almost order bounded subset $A$ of $E$, $T(A)$ is a relatively weakly compact subset of $F$. In Theorem 4 we show that the positive operator $T$ from $E$ into Dedekind complete $F$ is almost order-weakly compact if and only if $T(x_n) \rightarrow \overline{0}$ in $F$ for each disjoint almost order bounded sequence $\{x_n\}$ in $E$. In this manuscript, we study some properties of this class of operators and its relationships with others known operators.

Keywords almost order bounded · weakly compact · order weakly compact · almost order-weakly compact.

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1 Introduction

Since order weakly compact operators play important role in positive operators, our aim in this manuscript is to introduce and study a new class of operators as almost order-weakly compact operators and we establish some

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of its relationships with others known operators. Under some conditions, we show that the adjoint of any almost order-weakly compact operator is so. Every compact and weakly compact operators are almost order-weakly compact operator, but the converse in general not holds.

To state our results, we need to fix some notations and recall some definitions. Let $E$ be a Banach lattice. A subset $A$ is said to be almost order bounded if for any $\epsilon$ there exists $u \in E^+$ such that $A \subseteq [-u, u] + \epsilon B_E$ ($B_E$ is the closed unit ball of $E$). One should observe the following useful fact, which can be easily verified using Riesz decomposition Theorem, that $A \subseteq [-u, u] + \epsilon B_E$ iff \[ \sup_{x \in A} \|([x] - u)^+\| = \sup_{x \in A} \|x - |x| \land u\| \leq \epsilon. \]

By Theorems 4.9 and 3.44 of [1], each almost order bounded subset in order continuous Banach lattice is relatively weakly compact. $A \subseteq L_1(\mu)$ is relatively weakly compact iff it is almost order bounded (see [2]). Recall that a vector $e > 0$ in vector lattice $E$ is an order unit or a strong unit (resp. weak unit) when the ideal $I_e$ (resp. band $B_e$) is equal to $E$; equivalently, for every $x \geq 0$ there exists $n \in \mathbb{N}$ such that $x \leq ne$ (resp. $x \land ne \uparrow x$ for every $x \in E^+$). Suppose that Banach lattice $E$ is an order continuous norm with a weak unit $e$. It is known that $E$ can be represented as a norm and order dense ideal in $L_1(\mu)$ for some finite measure $\mu$ (see [5]). A continuous operator $T$ from a Banach lattice $E$ to a Banach space $X$ is said to be

- **order weakly compact** whenever $T[0, x]$ is a relatively weakly compact subset of $X$ for each $x \in E^+$.
- **$M$-weakly compact** if $T(x_n) \rightharpoonup 0$ holds for every norm bounded disjoint sequence $\{x_n\}$ of $E$.
- **$b$-weakly compact** whenever $T$ carries each $b$-order bounded subset of $E$ into a relatively weakly compact subset of $X$.

A continuous operator $T$ from a Banach space $X$ to a Banach lattice $E$ is said to be

- **$L$-weakly compact** whenever $y_n \rightharpoonup 0$ for every disjoint sequence $\{y_n\}$ in the solid hull of $T(B_X)$.
- **semicom pact** whenever for each $\epsilon \geq 0$ there exists some $u \in E^+$ satisfying \[ \|([T|x| - u]^+)\| \leq \epsilon \] for all $x \in B_X$.

An operator $T : E \to F$ is regular if $T = T_1 - T_2$ where $T_1, T_2 : E \to F$ are positive operators. We denote by $L(E, F) (L'(E, F))$ the space of all operators (regular operators) from $E$ into $F$.

An operator $T : E \to F$ between two vector lattices is said to be lattice homomorphism (resp. preserve disjointness) whenever $T(x \lor y) = T(x) \lor T(y)$ (resp. $x \land y$ in $E$ implies $T(x) \land T(y)$ in $F$).

Recall that $L_0(E, F)$ is the vector space of all order bounded operators from $E$ to $F$.

A Banach space $X$ is said to be Grothendieck space whenever $weak^*$ and weak convergence of sequences in $X'$ (norm dual of $X$) coincide.

A Banach lattice $E$ is said to be $AM$-space (resp. $AL$-space), if for $x, y \in E$ with $x \land y = 0$, we have $\|x \lor y\| = \max\{\|x\|, \|y\|\}$ (resp. $\|x + y\| = \|x\| + \|y\|$).
A Banach lattice $E$ is said to be $KB$-space whenever every increasing norm bounded sequence of $E^+$ is norm convergent.

Let $E$ be a vector lattice and $x \in E$. A net $\{x_\alpha\} \subseteq E$ is said to be unbounded order convergent to $x$ if $|x_\alpha - x| \wedge u \xrightarrow{uo} 0$ for all $u \in E^+$. We denote this convergence by $x_\alpha \xrightarrow{uo} x$ and write that $\{x_\alpha\}_\alpha$ is $uo$-convergent to $x$.

2 almost order bounded operators

Let $T : E \to F$ be a continuous operator between two Banach lattices. $T$ is said to be almost order-weakly compact operator whenever $T$ maps the almost order bounded subset $A$ of $E$ into an almost order bounded subset of $F$. The vector space of all almost order bounded operators from $E$ to $F$ will be denoted $L_{aob}(E, F)$.

It is obvious that if $T : E \to F$ is a semicompact operator, then it is almost order bounded. If $E$ has an order unit and $T : E \to F$ is order bounded, then it is an almost order bounded operator and if $F$ has an order unit and $T$ is an almost order bounded operator, then it is order bounded.

Here is an example show that the class of order bounded operators differ from the class of almost order bounded.

Example 1 The operator $T : L_1[0, 1] \to c_0\mu$ defined by

$$T(f) = \left( \int_0^1 f(x)\sin x dx, \int_0^1 f(x)\sin 2x dx, \ldots \right)$$

is not order bounded (see page 67 of [1]). Let $A \subseteq L_1[0, 1]$ be an almost order bounded. Hence $A$ is a relatively weakly compact subset of $E$. Because $T$ is continuous, so $T(A)$ also is relatively weakly compact. Since $c_0\mu$ is an AL-space, therefore by Theorem 4.27 of [1], it is lattice isometric by $L_1(\mu)$. Hence $T(A)$ is an almost order bounded subset of $c_0\mu$. So $T$ is an almost order bounded operator.

Here is an example that the operator $T$ is almost order bounded while whose modulus does not exist.

Example 2 Consider the continuous function $g : [0, 1] \to [0, 1]$ defined by $g(x) = x$ if $0 \leq x \leq \frac{1}{2}$ and $g(x) = \frac{1}{2}$ if $\frac{1}{2} < x \leq 1$. Now define the operator $T : C[0, 1] \to C[0, 1]$ by $Tf(x) = f(g(x)) - f(\frac{1}{2})$. $T$ is a regular operator and therefore it is an order bounded operator. Because $C[0, 1]$ is an AM-space with unit, so $T$ is an almost order bounded. Note that the modulus of $T$ does not exist (see Exercise 9 of page 22 of [1]).

We are looking for situations where if $T$ is an almost order bounded, then $|T|$ exist and it is an almost order bounded operator.
Proposition 1 Let \( T : E \to F \) be an almost order bounded operator between two Banach lattices that \( F \) is Dedekind complete and \( E, F \) have an order unit, then the modulus of \( T \) exists and it is almost order bounded.

Proof Let \( T \) be almost order bounded. Since \( F \) has an order unit, therefore \( T \) is an order bounded operator. Because \( F \) is Dedekind complete, so by Theorem 1.18 of [1], \( |T| \) exist and it is an order bounded. Since \( E \) has an order unit, hence \( |T| \) is an almost order bounded.

Proposition 2 If \( T : E \to F \) is an onto lattice homomorphism, then \( T \) is almost order bounded.

Proof Let \( T : E \to F \) be an almost order bounded and \( A \subseteq E \) be an almost order bounded set. It means that for each \( \varepsilon > 0 \) there exists \( u \in E^+ \) that \( \sup_{x \in A} \| (|x| - u)^+ \| \leq \varepsilon \). Since \( T \) is a positive operator therefore it is a continuous operator. Hence we have for each \( \varepsilon > 0 \) there exists \( u \in E^+ \) that \( \sup_{x \in A} \| T(|x| - u)^+ \| \leq \varepsilon \). Since \( T \) is a lattice homomorphism, therefore \( \sup_{x \in A} \| T(|x| - u)^+ \| \leq \varepsilon \). Because \( T \) is onto, so the proof is complete.

Remark 1 If \( T : E \to F \) is onto lattice homomorphism and \( F \) is Archimedean, then \( |T| \) exists and it is an almost order bounded.

Proof Since \( T \) is lattice homomorphism, therefore it is an order bounded and disjointness preserving. Hence by Theorem 2.40 of [1], \( |T| \) exists. It is obvious that \( |T| \) is a lattice homomorphism. By Proposition 2, \( |T| \) is an almost order bounded.

3 almost order-weakly compact operators

Let \( T : E \to F \) be a continuous operator between two Banach lattices. \( T \) is said to be almost order-weakly compact operator (for short, \( ao-wc \) operator) whenever \( T \) maps the almost order bounded subset \( A \) of \( E \) into a relatively weakly compact subset of \( F \).

By Theorem 3.40 of [1], \( T \) is \( ao-wc \) operator iff for every almost order bounded sequence \( \{x_n\} \) of \( E \) the sequence \( \{T(x_n)\} \) has a weak convergent subsequence in \( F \).

The collection of all \( ao-wc \) operators between two Banach lattices \( E \) and \( F \) will be show by \( K_{ao-wc}(E, F) \).

It is obvious that each compact and weakly compact operator is \( ao-wc \) and each \( ao-wc \) operator is an order weakly compact operator.

By Theorem 5.23 and 5.27 of [1], we have the following result.

Theorem 1 1. Each continuous operator \( T \) from a Grothendieck Banach lattice \( E \) into a Banach lattice \( F \) is an \( ao-wc \) operator.
2. Let \( T \) be a positive operator from a Banach lattice \( E \) into a Banach lattice \( F \) and \( E^* \) has order continuous norm. If \( F \) is a KB-space, then \( T \) is \( ao-wc \).
In the following we have some examples of ao-wc operators.

**Example 3** 1. Since $C[0,1]$ is a Grothendieck space, therefore by Theorem [1](1), the continuous operator $T : C[0,1] \to c_0$, given by

$$T(f) = \left( \int_0^1 f(x) \sin xdx, \int_0^1 f(x) \sin 2xdx, \ldots \right),$$

is an ao-wc operator.

2. Since $c'$ has order continuous norm and $\mathbb{R}$ is a KB-space, therefore by Theorem [1](2), the functional $f : c \to \mathbb{R}$ defined by

$$f(x_1, x_2, \ldots) = \lim_{n \to \infty} x_n$$

is an ao-wc operator.

**Proposition 3** Let $E$, $F$ and $G$ be three Banach lattices, $T : E \to F$ and $S : F \to G$ be two ao-wc operators. By one of the following conditions, $S \circ T$ is an ao-wc operator.

1. $F$ is an AL-space.
2. $F$ has order continuous norm with a weak unit.

**Proof** Let $A \subseteq E$ be almost order bounded. By assumption, $T(A)$ is relatively weakly compact subset of $F$. If $F$ is an AL-space, then by Theorem 4.27 of [1], $F$ is lattice isometric to some concrete $L_1(\mu)$ and if $F$ has order continuous norm with a weak unit, then $F$ is norm and order dense ideal in $L_1(\mu)$. Therefore $T(A)$ is an almost order bounded subset of $F$. So by assumption, $S(T(A))$ is relatively weakly compact subset of $G$. Hence $S \circ T$ is an ao-wc operator.

As following example the adjoint of ao-wc operator in general is not ao-wc operator.

**Example 4** Let $A \subseteq \ell^1$ be an almost order bounded set. Since $\ell^1$ has order continuous norm, therefore $A$ is relatively weakly compact. Thus the identity operator $I : \ell^1 \to \ell^1$ is an ao-wc operator. Since the identity operator $I : \ell^\infty \to \ell^\infty$ is not order weakly compact, therefore it is not ao-wc.

In the following theorem, under some conditions, we show that the adjoint of ao-wc operator is so.

**Theorem 2** Let $T : E \to F$ be an ao-wc operator between two Banach lattices. If any of the following conditions are met, then $T'$ is ao-wc.

1. $E$ has an order unit.
2. $E'$ is a KB-space and $F'$ has an order unit.

**Proof** 1. Let $E$ has an order unit and $T : E \to F$ be ao-wc. If $A \subseteq E$ is norm bounded, then $A$ is an order bounded and therefore almost order bounded. Hence by assumption, $T(A)$ is a relatively weakly compact subset of $F$. It means that $T$ is a weakly compact operator. Therefore by Theorem 5.5 of [4], $T'$ is weakly compact and hence it is an ao-wc operator.
2. Let $T : E \to F$ be an ao wc operator. Therefore $T$ is an order weakly compact operator. Since $E'$ is a KB-space, by Theorem 3.3 of [2], $T'$ also is an order weakly compact operator. Since $F'$ has an order unit, it is clear that $T'$ is ao wc.

We know that each compact and weakly compact operator is an ao wc operator, but by following example the converse in general not holds.

Example 5 The identity operator $I : \ell^1 \to \ell^1$ is an ao wc operator but is not compact or weakly compact operator.

Corollary 1 Under conditions of Theorem 2, an operator $T : E \to F$ is weakly compact iff it is ao wc.

Proof Let $E$ has an order unit and operator $T : E \to F$ be ao wc, then it is a weakly compact operator.

Let $E'$ be a KB-space, $F'$ has an order unit and operator $T : E \to F$ is ao wc. By Theorem 2, $T'$ is ao wc. Because $F'$ has an order unit, $T'$ is weakly compact. By Theorem 5.5 of [3], $T$ is weakly compact.

Remark 2 Let $E$ be a Banach lattice with an order unit. Then a subset $A$ of $E$ is norm bounded iff is order bounded iff it is almost order bounded. Therefore an operator $T : E \to F$ is weakly compact if and only if is order weakly compact if and only if is ao wc.

Remark 3 Under conditions of Theorem 2, if $T : E \to F$ is ao wc, then by Corollary 1 and Theorem 5.44 of [1], there exist a reflexive Banach lattice $G$, lattice homomorphism $Q : E \to G$ and positive operator $S : G \to F$ that $T = S \circ Q$.

Note that the identity operator $I : \ell^\infty \to \ell^\infty$ is not ao wc, however its adjoint $I : (\ell^\infty)' \to (\ell^\infty)'$ is ao wc.

Let $T : E \to F$ be an operator between two Banach lattices. If $T' : F' \to E'$ is ao wc and $F'$ has an order unit, then $T'$ is weakly compact and therefore $T$ is weakly compact. Hence $T$ is ao wc. If $T$ is $M$-weakly compact or $L$-weakly compact, then by Theorem 5.61 of [1], $T$ is weakly compact and therefore $T$ is an ao wc operator. Thus we have the following result.

Theorem 3 Let $T : E \to F$ be an operator between two Banach lattices. By one of the following conditions $T$ is an ao wc operator.

1. $T$ is $M$-weakly compact,
2. $T$ is $L$-weakly compact,

Moreover if $F$ has order continuous norm and $T : E \to F$ is ao wc.

If $T : E \to F$ is semicompact operator, or dominated by a semicompact operator, then $T$ is an ao wc. Let $A$ be an almost order bounded subset of $E$. Then $A$ is norm bounded. Therefore if $T$ is a semicompact operator, $T(A)$ is an almost order bounded in $F$. Since $F$ has order continuous norm, $T(A)$ is relatively weakly compact subset of $F$. Hence $T$ is an ao wc operator. If $T$ is dominated by a semicompact operator, then by Theorem 5.72 of [1], $T$ is semicompact operator and so is an ao wc operator.
Remark 4 1. An ao-wc operator need not be an M-weakly or L-weakly compact operator. For instance, the identity operator $I : L_1[0,1] \to L_1[0,1]$ is ao-wc, but is not M-weakly or L-weakly compact operator.

2. Note that if $F$ has not order continuous norm, then each semicompact operator $T : E \to F$ is not necessarily ao-wc. For example, the identity operator $I : \ell^\infty \to \ell^\infty$ is semicompact and $\ell^\infty$ has not order continuous norm. Thus $I$ is not ao-wc.

Let $E, F$ be two normed vector lattices. Recall from [8], a continuous operator $T : E \to F$ is said to be $\sigma$-uom-continuous, if for each norm bounded $uo$-null sequence $\{x_n\} \subseteq E$ implies $T(x_n) \xrightarrow{\|\|} 0$.

Theorem 4 Let $E$ and $F$ be two Banach lattices that $F$ is Dedekind complete. The positive operator $T : E \to F$ is ao-wc iff for each disjoint almost order bounded sequence $\{x_n\}$ in $E$ implies $T(x_n) \xrightarrow{\|[\|} 0$ in $F$.

Proof Let the operator $T : E \to F$ be ao-wc. This means that for every $\epsilon$ there exists $u \in E^+$ such that $T([-u, u] + \epsilon B_E)$ is relatively weakly compact.

Let $I_z$ be the ideal generated by $z \in [-u, u] + \epsilon B_E$ in $E$. The operator $T|I_z : I_z \to F$ is weakly compact. Since $I_z$ is an AM-space with order unit, therefore $T|I_z : I_z \to F$ is M-weakly and hence by Remark 2.8 of [8], is $\sigma$-uom-continuous. It is clear that the extension of operator $T|I_z, T : E \to F$ is $\sigma$-uom-continuous. If $\{x_n\} \subseteq E$ is an almost order bounded and disjoint, hence it is norm bounded and $uo$-null. So we have $T(x_n) \xrightarrow{\|\|} 0$. Conversely, let $A \subseteq E$ be an almost order bounded set. Then for each $\epsilon$ there exists $u \in E^+$ such that $A \subseteq [-u, u] + \epsilon B_E$. Let $I_u$ be the ideal generated by $u$ in $E$ and $\{x_n\} \subseteq A$ be a disjoint sequence. It is clear that $\{x_n\}$ is norm bounded. By assumption, we have $T(x_n) \xrightarrow{\|\|} 0$ in $F$. Therefore $T : I_u \oplus E \to F$ is M-weakly compact, and so by Theorem 3 $T : I_u \oplus E \to F$ is an ao-wc operator. Thus $T : E \to F$ is ao-wc.

Corollary 2 1. Let $T : E \to F$, $S : F \to G$ be two ao-wc operators where $F, G$ are Dedekind complete and $\{x_n\} \subseteq E$ is a disjoint almost order bounded sequence. By Theorem 4 we have $T(x_n) \xrightarrow{\|\|} 0$. Since $S$ is a continuous operator, $S(T(x_n)) \xrightarrow{\|\|} 0$. Therefore $S \circ T$ is ao-wc operator.

2. By Theorem 5.60 of [11], obviously that if $T : E \to F$ is an ao-wc operator, then for each $\epsilon > 0$ there exists some $u \in E^+$ such that $\|T((|x| - u)^+))\| < \epsilon$ holds for all $x \in A$ that $A$ is an almost order bounded subset of $E$.

Recall that a Banach lattice $E$ is said to have the dual positive Schur property if every positive $w^*$-null sequence in $E^*$ is norm null.

Theorem 5 The following statements are true.

1. Let $E$ be a Banach lattice Dedekind complete. $E$ has order continuous norm iff each positive operator $T$ from $E$ into each Banach lattice $F$ is an ao-wc operator.
2. Let $E$ be a Banach lattice Dedekind complete. $E$ has order continuous norm iff each almost order bounded disjoint sequence $\{x_n\} \subseteq E$ is norm null.

3. If $E$ has the property (b) and each operator $T^2 : E \to E$ is ao-uc, then $E$ has order continuous norm.

4. Let $T : E \to F$ be a continuous operator between two Banach lattices $E, F$ that $F$ is Dedekind complete. If $|T|$ exists and it is ao-uc, then $T$ is also ao-uc.

5. If $E$ has the dual positive Schur property, $F$ has order continuous norm and Dedekind complete, then adjoint of each positive operator $T : E \to F$ is an ao-uc operator.

**Proof**

1. Let $E$ has order continuous norm and $\{x_n\}$ be an almost order bounded disjoint sequence in $E$. Therefore $x_n \xrightarrow{uo} 0$ in $E$. By Proposition 3.7 of [7], $x_n \xrightarrow{wc} 0$. By continuity of $T$, it follows that $Tx_n \xrightarrow{wc} 0$ in $F$.

Conversely, let $E$ has not order continuous norm. By Theorem 2.7 of [3], there exists an operator $T$ from $E$ into $\ell^\infty$ such that $T$ is not order weakly compact and therefore is not ao-uc.

2. Let $E$ has order continuous norm, therefore the identity operator $I : E \to E$ is ao-uc. Then $x_n = Ix_n \xrightarrow{wc} 0$ where $\{x_n\} \subseteq E$ is almost order bounded disjoint sequence.

Conversely, let $\{x_n\}$ be an order bounded disjoint sequence in $E$. Therefore $\{x_n\}$ is almost order bounded disjoint in $E$. Hence by assumption $x_n = Ix_n \xrightarrow{wc} 0$. By Theorem 4.14 of [4], $E$ has order continuous norm.

3. By contradiction, assume that $E$ has not order continuous norm, it follows from the proof of Theorem 2 of [11], that $E$ contains a closed order copy of $c_0$ and there exists a positive projection $P : E \to c_0$. Let $i : c_0 \to E$ be the canonical injection. Obviously that $T = i \circ P : E \to E$ is not $b$-weakly compact. Since $E$ has property (b), therefore $T$ is not order weakly compact, and so $T^2$ is not ao-uc.

4. Let $0 \leq T \leq S$ and $S$ be an ao-uc operator. If $\{x_n\}$ is an almost order bounded and disjoint sequence in $E$, then by Theorem [3] $S(x_n) \xrightarrow{wc} 0$. Therefore $T(x_n) \xrightarrow{wc} 0$. We have $-|T| \leq T \leq |T|$ and so $0 \leq T + |T| \leq 2|T|$. It follows that $T$ is an ao-uc operator whenever $|T|$ is ao-uc.

5. Let $\{f_n\}$ be an almost order bounded disjoint sequence in $F'$. Then $f_n \xrightarrow{uo} 0$ in $F'$. Without loss of generality, assume $0 \leq f_n$. Note that $0 \leq T'f_n$. Now since $F$ has order continuous norm, by Theorem 2.1 from [6], $f_n \xrightarrow{wc} 0$ in $F'$. Since $T'$ is $w^*-w^*$ continuous, hence $T'f_n \xrightarrow{wc} 0$ in $F'$. Since $E$ has the dual positive Schur property, hence $T'f_n \xrightarrow{wc} 0$ in $E'$.

**Proposition 4** If $E$ has an order unit, then $T : E \to F$ is $\sigma$-uon-continuous if and only if it is an ao-uc operator.

**Proof** Let $T : E \to F$ be an ao-uc operator, then $T$ is order weakly compact. Let $\{x_n\} \subseteq E$ be a norm bounded disjoint sequence. Since $E$ has an order unit,
then \( \{x_n\} \) is order bounded disjoint sequence. By assumption and Theorem 5.57 of [1], \( T(x_n) \xrightarrow{\|\cdot\|} 0 \). So \( T \) is \( M \)-weakly compact and therefore by remark 2.8 of [8], \( T \) is \( \sigma \)-uon-continuous.

In the following, we establish some relationships between the class of \( ao \)-wc operators and the class of semicompact operators. By Remark 2, we know that the class of \( ao \)-wc operators different with the class of semicompact operators, but as following we see some relations.

**Theorem 6** Let \( T : E \to F \) be an \( ao \)-wc operator between two Banach lattices. Then \( T \) is semicompact operator.

**Proof** Let \( T : E \to F \) be \( ao \)-wc. Let \( A \) be an almost order bounded subset of \( E \). Without loss of generality we assume that for each \( \epsilon \) there exists \( u \in E^+ \) such that \( A = [-u,u] + \epsilon B_E \). Let \( p(x) = \|x\| \). Then \( \lim p(Tx_n) = 0 \) holds for each disjoint sequence \( \{x_n\} \) in \( A \). By Theorem 4.36 of [1], there exists some \( v \in E^+ \) satisfying \( \|T(|x| - v)^+\| \leq \epsilon \) for all \( x \in A \). Put \( w = Tv \in F^+ \), and note that

\[
\begin{align*}
(|Tx| - w)^+ &= \\
(|Tx| - Tv)^+ &\leq \\
(T|x| - Tv)^+ &= \\
(T(|x| - v))^+ &\leq \\
T((|x| - v)^+).
\end{align*}
\]

Therefore \( T \) is a semicompact operator.

By Theorems 3 and 6, we have the following result.

**Corollary 3** 1. Each operator \( T : E \to F \) that it is \( ao \)-wc is an almost order bounded operator.

2. Let \( F \) be a Banach lattice with order continuous norm. Then \( T : E \to F \) is \( ao \)-wc if and only if it is a semicompact operator.

If \( T \) is \( ao \)-wc, in general \( |T| \) is not exist, see the following example.

**Example 6** The operator \( T : L_1[0,1] \to c_0 \) defined by

\[
T(f) = \left( \int_0^1 f(x) \sin x dx, \int_0^1 f(x) \sin 2x dx, \cdots \right),
\]

is an \( ao \)-wc operator. Note that by Exercise 10 of page 289 of [1], its modulus does not exist.

In the following theorem, under some conditions, we show that \( |T| \) exist and is \( ao \)-wc whenever \( T \) is \( ao \)-wc.

Recall that a Banach lattice \( E \) is said to have the property \( (P) \) if there exists a positive contractive projection \( P : E'' \to E \) where \( E \) is identified with a sublattice of its topological bidual \( E'' \).
Theorem 7 Let $T : E \to F$ be an ao-wc operator. By one of the following conditions, the modulus of $T$ exists and it is an ao-wc operator.

1. $E$ is an AL-space and $F$ has the property (P).
2. $E$ and $F$ have order unit.
3. $F$ is Archimedean Dedekind complete and $T$ is an order bounded preserves disjointness.

Proof 1. By Theorem 1.7 of [10], we have $L^r(E, F) = L(E, F)$. Therefore $|T|$ exists. Since $E$ has order continuous norm, by Theorem 5, $|T| : E \to F$ is an ao-wc operator.

2. Since $E$ has an order unit, $T$ is a weakly compact operator. Since $F$ has an order unit, therefore by Theorem 2.3 of [9], the modulus of $T$ exists and it is a weakly compact operator. It is obvious that $|T|$ is an ao-wc operator.

3. By Theorem 2.40 of [11], $|T|$ exists and for all $x$, we have $|T|(x) = |T|(x)| = |T(x)|$. If $x_n \subseteq E$ is an almost order bounded disjoint sequence, then by assumption $T(x_n) \rightarrow 0$. For each $n$, we have $|T|(x_n) = |T(x_n)| = |T(x_n)| \rightarrow 0$ in $F$. The inequality $||(T(x_n))|| \leq ||T||x_n||$, implies that $|T|(x_n) \rightarrow 0$.

Hence $|T|$ is an ao-wc operator.

Theorem 8 Let $E$ and $F$ have order unit with $F$ Dedekind complete. Then $K_{ao-wc}(E, F) \cap L_b(E, F)$ is a band in $L_b(E, F)$.

Proof It is obvious that if $T, S \in K_{ao-wc}(E, F) \cap L_b(E, F)$ and $\alpha \in \mathbb{R}$, then $T + \alpha S \in K_{ao-wc}(E, F) \cap L_b(E, F)$.

Let $|S| \leq |T|$ where $T \in K_{ao-wc}(E, F) \cap L_b(E, F)$, $S \in L_b(E, F)$ and $\{x_n\} \subseteq E$ be almost order bounded disjoint sequence. Without loss of generality, assume that $x_n \geq 0$ for all $n$. By Theorem 7 $|T|(x_n) \rightarrow 0$. The inequalities $|S(x_n)| \leq |S|(x_n) \leq |T|(x_n)$ implies that $S(x_n) \rightarrow 0$. Therefore $S \in K_{ao-wc}(E, F) \cap L_b(E, F)$, and so $K_{ao-wc}(E, F) \cap L_b(E, F)$ is an ideal of $L_b(E, F)$.

Now let $0 \leq T_\alpha \vdash T$ in $L_b(E, F)$ with $\{T_\alpha\} \subseteq K_{ao-wc}(E, F) \cap L_b(E, F)$. Since $T$ is positive, therefore $T$ is order bounded and since $E$ has an order unit, then by Example 8 $T$ is ao-wc. Hence $T \in K_{ao-wc}(E, F) \cap L_b(E, F)$.

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