Discretization of quantum pure states and local random unitary channel

Dong Pyo Chi¹ and Kabgyun Jeong²

¹ Department of Mathematical Sciences, Seoul National University, Seoul 151-742, Korea
² Nano Systems Institute (NSI-NCRC), Seoul National University, Seoul 151-742, Korea

(Dated: February 18, 2010)

Abstract

We show that a quantum channel $\mathcal{N}$ constructed by averaging over $O(\log d/\varepsilon^2)$ randomly chosen unitaries gives a local $\varepsilon$-randomizing map with non-negative probability. The idea comes from a small $\varepsilon$-net construction on the higher dimensional unit sphere or quantum pure states. By exploiting the net, we analyze the concentrative phenomenon of an output reduced density matrix of the channel, and this analysis imply that there exists a local random unitary channel, with relatively small unitaries, generically.

PACS numbers: 03.65.Ta 03.67.Hk
I. INTRODUCTION

The probabilistic existence of an \(\varepsilon\)-randomizing map or random unitary channel, with small cardinality of unitaries, has several important implications. The channel can be used to construct almost perfectly secure encryption protocols \([1, 2]\) and give an intuition such as counterexample to additivity conjecture for the classical capacity of quantum channel \([3–5]\).

In the paper, we prove that there exists a quantum channel consisting of unitary matrices with relatively small cardinality \(\mathcal{O}(\log d/\varepsilon^2)\) which is also \(\varepsilon\)-randomizing. This construction deeply relies on the mathematical fact known as general \(\varepsilon\)-net theorem, in special, we consider a higher dimensional unit sphere corresponding to \(d(\gg 1)\) dimensional quantum pure states.

For convenience, we use the following notations throughout the paper. A state can be pure or mixed state on the Hilbert spaces. Especially, a density matrix of the pure state \(|\varphi\rangle\) will be denoted as \(\varphi\), when it is without confusing a mixed state. If \(\varphi_{AB}\) is a composite quantum state of \(\mathbb{C}^{d_Ad_B}\) (\(\equiv \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\)), or simply \(A \otimes B\), the reduced state on \(A\) can be referred to \(\varphi_A\). Given Hilbert space \(\mathcal{H}(\mathbb{C}^d)\), \(\mathcal{B}(\mathbb{C}^d)\) denotes the algebra of complex \(d \times d\) matrices, \(\mathcal{U}(d)\) be the unitary group on the space, and \(\mathbb{I}\) is \(d \times d\) identity matrix. The notation \(\mathbb{P}[X]\) and \(\mathbb{E}[X]\) denote the probability and the expectation value of a given random variable \(X\), respectively. Finally, any functions \(\log\) and \(\exp\) are always taken base 2.

A. Local random unitary channel

A quantum channel is a completely positive trace-preserving (CPT) map \(\mathcal{N} : \mathcal{B}(\mathbb{C}^{d_A}) \rightarrow \mathcal{B}(\mathbb{C}^{d_B})\). Given CPT map \(\mathcal{N}\), it is known that there is a complementary or conjugate channel \(\mathcal{N}^C : \mathcal{B}(\mathbb{C}^{d_A}) \rightarrow \mathcal{B}(\mathbb{C}^{d_B})\). For any input \(\varphi_A\), these two channels \(\mathcal{N}(\varphi_A)\) and \(\mathcal{N}^C(\varphi_A)\)
are related by
\[ \mathcal{N}(\varphi_A) = \text{tr}_E V \varphi_A V^\dagger, \quad \mathcal{N}^C(\varphi_A) = \text{tr}_B V \varphi_A V^\dagger \] (1)
where \( V : \mathbb{C}^{d_A} \to \mathbb{C}^{d_B d_E} \) is a unitary embedding.

We now introduce general notion of the random unitary channel. For any input quantum states \( \varphi_A \), the random unitary channel (RUC) \( \mathcal{N} : \mathcal{B}(\mathbb{C}^{d_A}) \to \mathcal{B}(\mathbb{C}^{d_B}) \) can be described by
\[ \mathcal{N}(\varphi_A) = \sum_{i=1}^{d_E} \omega_i U_i \varphi_A U_i^\dagger, \]
where the weights, \( \omega_1, \ldots, \omega_{d_E} \), are positive values such that \( \sum_{i}^{d_E} \omega_i = 1 \) and the operators \( U_1, \ldots, U_{d_E} \) are some unitary \( d_A \times d_A \) matrices. When the positive weights are all equal to 1/\( d_E \), RUC will be written as \( \mathcal{N}(\varphi_A) = \sum_{i=1}^{d_E} \frac{1}{d_E} U_i \varphi_A U_i^\dagger \). In this place, a map \( \mathcal{N} \) is called \( \varepsilon \)-randomizing if, for all input \( \varphi_A \),
\[ \left\| \mathcal{N}(\varphi_A) - \frac{1}{d_B} \right\|_\infty \leq \frac{\varepsilon}{d_B}, \]
where \( I \) be \( d_B \times d_B \) identity matrix and \( \varepsilon \) is a small positive number upper bounded by 1.

The operator norm \( \|\rho\|_\infty \) of any \( \rho \) can be taken to be the square root of the largest eigenvalue of \( \rho^\dagger \rho \). That is, we call a CPT map \( \mathcal{N} \) as random unitary channel, if, for all inputs \( \varphi_A \), the map \( \mathcal{N}(\varphi_A) \) is \( \varepsilon \)-randomizing. In sense of conjugate channel, \( \|\mathcal{N}^C(\varphi_A) - \frac{1}{d_E}\|_\infty \leq \varepsilon/d_E \) also can be defined as an \( \varepsilon \)-randomizing map.

For future works, we need some extended notions concerning to the random unitary channel and \( \varepsilon \)-randomizing.

**Definition 1.** Assume that \( \mathcal{N} : \mathcal{B}(\mathbb{C}^{d_A}) \to \mathcal{B}(\mathbb{C}^{d_B}) \) is a CPT map. For all input states \( \varphi_A \), if
\[ \|\mathcal{N}(\varphi_A)\|_\infty - \frac{1}{d_B} \leq \frac{\varepsilon}{d_B} \quad \text{and} \quad \|\mathcal{N}^C(\varphi_A)\|_\infty - \frac{1}{d_E} \leq \frac{\varepsilon}{d_E}, \] (2)
then \( \mathcal{N} \) is called a local \( \varepsilon \)-randomizing.
For sufficiently large $d_A \gg d_E$, generically the output states of the channel are distributed as

$$\mathcal{N}(\varphi_A) \simeq \begin{pmatrix} \frac{1}{d_B} & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{d_B} \end{pmatrix}$$

and

$$\mathcal{N}^C(\varphi_A) \simeq \begin{pmatrix} \frac{1}{d_E} & 0 \\ 0 & \ddots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \cdots & \frac{1}{d_E} \end{pmatrix},$$

as well as $\frac{1}{d_B}$ is almost equal to $\frac{1}{d_E}$. Naturally, one can take the definition of local random unitary channel from the local $\varepsilon$-randomizing. In this case, note that any output states of the channel $\mathcal{N}(\varphi_A)$ do not need to having full rank; it should be a partially randomized states, for example, $\mathcal{N}(\varphi_A)$ in Eq. (3).

Recently, it was shown that, for all $\varepsilon \in (0, 1]$, $\varepsilon$-randomizing maps exist in sufficiently large dimension $d_A$ such that $d_E$ can be taken to be $O(d_A \log d_A/\varepsilon^2)$ in [2] and $O(d_A/\varepsilon^2)$ in [8] for the Haar distributed $U_i$, respectively. The proof of the theorems is based on a large deviation technique and discretization of quantum pure states via $\varepsilon$-net construction [2, 9].

In this paper, we show that there is a small set of local random unitary channel with cardinality $O(\log d_A/\varepsilon^2)$ only, which is local $\varepsilon$-randomizing.

**Theorem 1.** Let all $\varepsilon \in (0, 1/2]$ and $d_A$ is sufficiently large. Let $\{U_i : 1 \leq i \leq d_E\}$ with $d_E = O(\log d_A/\varepsilon^2)$ be i.i.d. random unitaries distributed according to the Haar measure on $\mathcal{U}(d_A)$. Then the quantum channel $\mathcal{N}(\varphi_A) = \frac{1}{d_E} \sum_{i=1}^{d_E} U_i \varphi_A U_i^\dagger$ is a local $\varepsilon$-randomizing map with non-negative probability.

This construction deeply relies on the fact of discrete geometry known as general $\varepsilon$-net theorem [10], especially we consider an object such as higher dimensional unit sphere
$S^{2d_A-1}$, it is corresponding to $d_A$ dimensional quantum pure states. By some properties of an entangled random subspace, the local $\varepsilon$-randomizing map has its quantum informational meaning.

B. Entangled random states

As mentioned above, let’s consider the unitary embedding $V : \mathbb{C}^{d_A} \rightarrow \mathbb{C}^{d_Bd_E}$. Let’s define $\varphi_{BE} \in \mathbb{C}^{d_Bd_E}$ be a higher dimensional bipartite state. Especially, a set $\mathcal{P}(\mathbb{C}^{d_Bd_E})$ denotes the set of all bipartite pure states lying on $\mathbb{C}^{d_Bd_E}$. For the pure state $|\varphi\rangle_{BE}$, it is known that there exists a unique, and unitarily invariant, uniform distribution $\mu_h$, which is given by the Haar measure on the unitary group $U(d_Bd_E)$. Also there is a uniform measure for its subspaces $\mathbb{C}^s \subset \mathbb{C}^{d_Bd_E}$ that is unitarily invariant. A random pure state $|\varphi\rangle_{BE}$ is defined as $|\varphi\rangle_{BE}$ is any random variable drawn by $\mu_h$ on $\mathcal{P}(\mathbb{C}^{d_Bd_E})$. Similarly a random pure substates $|\varphi\rangle_S$ with dimension $s$ is any pure states induced by unitarily invariant measure on $\mathcal{P}(\mathbb{C}^s) \subset \mathbb{C}^{d_Bd_E}$.

Page’s conjecture [12, 13] states that the average von Neumann entropy of $\text{tr}_E(\varphi_{BE})$ has almost all maximum value, that is, any random pure states are near-maximally entangled state on a bipartite higher dimensional space [14–16]. For $|\varphi\rangle_S \in \mathcal{P}(\mathbb{C}^s)$, it is also near-maximally entangled state [11]:

**Theorem 2 (Entangled Subspaces).** Let $\mathbb{C}^{d_Bd_E}$ be a bipartite system with dimension $d_Bd_E$ ($d_B \geq d_E \geq 3$) and $0 < \alpha < \log d_B$. Then, with high probability, there exists a subspace $\mathcal{P}(\mathbb{C}^s) \subset \mathbb{C}^{d_Bd_E}$ of dimension $s = \mathcal{O}\left(d_Bd_E\left(\frac{\alpha}{\log d_B}\right)^{5/2}\right)$ s.t. all states $|\varphi\rangle_S \in \mathbb{C}^s$ have entanglement at least

$$E(\varphi_S) = S(\varphi_B) \geq \log d_B - \alpha - \frac{1}{\ln 2} \frac{d_B}{d_E},$$

(4)
where $S(\varphi_B)$ is von Neumann entropy of $\varphi_B$.

Note that the dimension $s$ of $S$ is surely less than the total dimension $d_Bd_E$ of $\mathbb{C}^{d_Bd_E}$ as well as $\varphi_S \in \mathcal{P}(\mathbb{C}^s)$ is almost entangled state. Recall that all bipartite quantum pure state can be written in Schmidt decomposition form, $|\varphi\rangle_{BE} = \sum_{i=1}^{\min\{d_B,d_E\}} \sqrt{\lambda_i} |e_i\rangle_B |f_i\rangle_E$, where $B\langle e_i|e_j\rangle_B = \delta_{ij} = E\langle f_i|f_j\rangle_E$ and $\sqrt{\lambda_i}$ is the Schmidt coefficients, furthermore, if $\lambda_i$ are all equal, then it is maximally entangled state.

As mentioned above, a local random unitary channel may induce a partially randomized quantum state $\varphi'_E$ with some low-rank less than $\varphi_B$ (having full-rank) of RUC’s output. Imagine a purification of near-maximally mixed state $\varphi_B$, resulting state $|\varphi\rangle_{BE}$ will be a maximally entangled state on $\mathbb{C}^{d_Bd_E}$. For the same reason, any purification of $\varphi'_E$ also can be considered as a maximally entangled state, $|\varphi\rangle_S$, on the random subspace $S$ with the dimension $s$.

II. SMALL $\varepsilon$-NET ON UNIT SPHERE

In this section, we define several important notions and investigate some of their mathematical facts concerned to an $\varepsilon$-net on the unit sphere. Especially $S^{2d-1}$ denote a higher dimensional unit sphere on $\mathbb{R}^{2d}$, which is generally corresponding to all quantum pure states on $\mathbb{C}^d$. Now we show that there exists a small $\varepsilon$-net $N$ for $S^{2d-1}$ with cardinality $|N| = \mathcal{O}(d \log(\varepsilon^{-1})/\varepsilon)$.

Let $(X, \mathcal{F})$ be a $\mu$-measurable set system and $\mathcal{F} \subseteq X$, here $\mu$ be a natural probability measure on $X$. For every $\varepsilon \in [0, 1]$, an $N \subseteq X$ is called an $\varepsilon$-net for the system $X$ with respect to $\mu$ if $N \cap F_i \neq \emptyset$ for all $F_i \in \mathcal{F}$ with $\mu(F_i) \geq \varepsilon |10]$. To describe the $\varepsilon$-net above, we need to a new parameter $VC-dim(\mathcal{F})$ of $X$, which is called Vapnik-Chervonenkis or just
simply VC dimension of $\mathcal{F}$.

**Definition 2.** Let $\mathcal{F}$ be a subset on $X$. Assume that another $A \subseteq X$ is shattered by $\mathcal{F}$ if $\mathcal{F}|_A = 2^A$, i.e., the restriction of $\mathcal{F}$ on $A$ gives a power set of $A$. Then the VC dimension of $\mathcal{F}$ is defined:

$$VC\text{-}dim(\mathcal{F}) = \sup_{A \subseteq X} \{|A| : \mathcal{F}|_A = 2^A\}.$$ (5)

The restriction of $\mathcal{F}$ on $A$ is defined by $\mathcal{F}|_A = \{F_i \cap A : F_i \in \mathcal{F}\}$. It is well known that a system $\mathcal{F}$ of all half-planes in the plane $\mathbb{R}^2$ have $VC\text{-}dim(\mathcal{F}) = 3$ in [10]. If an $m$-point subset $A$ lies in $X$, then the shatter function of $\mathcal{F}$ is defined by

$$\sigma_\mathcal{F}(m) = \max_{A \subseteq X, |A| = m} |\mathcal{F}|_A|.$$ In other words, $\sigma_\mathcal{F}(m)$ is the maximum possible value of distinct intersections of the sets of $\mathcal{F}$ with $A \subseteq X$. For $VC\text{-}dim(\mathcal{F}) \leq d$, the shatter function satisfies that $\sigma_\mathcal{F}(m) \leq \sum_{j=1}^{d} mC_j$. (This bound is known as shatter function lemma [10].)

In this paper, we substitute $X$ and $\mathcal{F}$ to $S^{2d-1}$ and a cap, $\mathcal{C}$, respectively. Formally, $S^{2d-1} := \{|x| \in \mathbb{C}^d : \|x\|_2 = 1\}$. Let’s consider a uniform probability measure $\mu$ on $S^{2d-1}$. For any measurable subset $S \subset S^{2d-1}$,

$$\mu(S) = \frac{\text{vol}(S)}{\text{vol}(S^{2d-1})} = \text{vol}(S),$$

where the second equality follows from $\mu(S^{2d-1}) = 1$ by definition. A cap on $S^{2d-1}$ is defined:

$$\mathcal{C} = S^{2d-1} \cap \{|x| : \langle u|x \rangle \geq 1 - h\}$$ (6)

for some unit vector $|u| \in S^{2d-1}$ (exactly, $|u|$ is the center of $\mathcal{C}$), we refer to $h$ as the height of the cap. Note that $\mathcal{C}$ can be considered as a (geodesic) convex set on $S^{2d-1}$ with $\mu(\mathcal{C}) > 0$. In such a cap, we know that, for all $h \leq \frac{1}{2}$, the asymptotic radius and their $(2d-1)$-dimensional
The volume of $\mathcal{C}$ are bounded by $\Theta(h^{1/2})$ and $\Theta(h^{(2d-1)/2})$ as $h \to 0$, respectively. Next lemma states $VC-dim(\mathcal{C})$ on the higher dimensional unit sphere.

**Lemma 3.** The VC dimension of all closed cap $\mathcal{C}$ on $S^{2d-1}$ is equal to $2d + 1$.

**Proof.** By Randon’s lemma (This lemma states that any $(d+1)$-point set on $\mathbb{R}^d$ can be shattered by the system of all closed half-space.), any set of $2d$ affinely independent points on the $(2d-1)$-dimensional unit sphere can be shattered (See e.g. Lemma 10.3.1 in [10]), and then $VC-dim(\mathcal{C})$ on $S^{2d-1}$ is equivalent to $2d$. For every $h \in (0, 1)$, all closed cap on $S^{2d-1}$ allow a factor of the additional 1 dimension. 

For example, the system of all closed half-space $\mathcal{F}$ on $S^2$ has $VC-dim(\mathcal{F}) = 3$, but the cap on a sphere enlarging $VC-dim(\mathcal{C}) = 4$. Note that $S^2$ corresponds to exactly $S^3$ in the above arguments; the difference comes from the convenience $\mathbb{C}^d \cong \mathbb{R}^{2d}$ instead of $\mathbb{C}^d \cong \mathbb{R}^{2d-1}$. Here we need an additional lemma for the proof of following theorem (Theorem 5), and see also details of the proof of Lemma 10.2.6 in [10].

**Lemma 4.** Let $X = X_1 + \cdots + X_t$, where the $X_i$ are independent random variables,

\[
X_i = \begin{cases} 
1 & \text{with probability } \varepsilon, \\
0 & \text{otherwise.}
\end{cases}
\]

Then $\mathbb{P} [X \geq \frac{1}{2}t\varepsilon] \geq \frac{1}{2}$, when $t\varepsilon \geq 8$.

For the higher dimensional unit sphere $S^{2d-1}$, we can construct a small $\varepsilon$-net which may be almost optimal.

**Theorem 5.** Let $\mu$ be a uniform probability measure on $S^{2d-1}$, $\mathcal{C} \subset S^{2d-1}$ be a cap of $\mu$-measurable subsets with $VC-dim(\mathcal{C}) \leq 2d + 1$. If $d \geq 1$, and $\varepsilon \leq \frac{1}{2}$, then there exists an $\varepsilon$-net
for the set system \((S^{2d-1}, C)\) w.r.t. \(\mu\) of cardinality

\[
|N| = \mathcal{O}\left(\frac{d^1 \log \frac{1}{\varepsilon}}{\varepsilon}ight).
\]  

(7)

Proof. The proof is almost equivalent to the proof of Theorem 10.2.4 in [10], on the other hand our proof has an essential difference by using the cap \(C\), Eq. (6) above, on \(S^{2d-1}\).

First of all, let’s define three random samples \(\Sigma_1, \Sigma_2\) and \(\Sigma_3\). Assume that \(t = \lceil Cd\varepsilon^1 \log \frac{1}{\varepsilon} \rceil\), and \(\Sigma_1\) be a random sample drawn from \(t\) independent random draw on \(S^{2d-1}\), where each elements satisfy the probability measure \(\mu\). W.l.o.g., all \(K_i \in C\) hold \(\mu(K_i) \geq \varepsilon\). By \(t\) more independent random draw (of another purpose), we pick some random sample \(\Sigma_2 \subset S^{2d-1}\) and fix an integer \(k = t\varepsilon/2\). Finally, let’s define a fixed \(\Sigma_3\), which is a random sample picked by \(2t\) independent random draw from \(S^{2d-1}\) and fix a set \(K^* \in C\).

Now we consider two events \(E_1\) and \(E_2\). Let \(E_1\) be the event so that the random sample \(\Sigma_1\) fails to be an \(\varepsilon\)-net, i.e., \(\Sigma_1 \cap K_i = \emptyset\) for all \(\mu(K_i) \geq \varepsilon\). Similarly, \(E_2\) be the event such that there exists an \(K_i \in C\) with \(\Sigma_1 \cap K_i = \emptyset\) and \(|\Sigma_2 \cap K_i| \geq k\). Clearly \(E_2\) needs \(E_1\) plus something more condition, so \(\mathbb{P}[E_2] \leq \mathbb{P}[E_1]\). We need another probabilistic condition such that \(\mathbb{P}[E_2] \geq \mathbb{P}[E_1]/2\). Suppose that there is \(K_i\) with \(\Sigma_1 \cap K_i = \emptyset\), and let’s fix one of them \(K^*\). Then \(\mathbb{P}[E_2|\Sigma_1] \geq \mathbb{P}[|\Sigma_2 \cap K^*| \geq k] \geq \frac{1}{2}\). The value of \(|\Sigma_2 \cap K^*|\) behaves like the random variable \(X = X_1 + \cdots + X_t\). By using Lemma 4 above second inequality holds. So \(2\mathbb{P}[E_2|\Sigma_1] \geq \mathbb{P}[E_1|\Sigma_1]\) for all \(\Sigma_1\), and thus \(2\mathbb{P}[E_2] \geq \mathbb{P}[E_1]\).

Next, we must bound the distribution \(\mathbb{P}[E_2]\). If we define a conditional probability \(P_{K^*} = \mathbb{P}[\Sigma_1 \cap K^* = \emptyset, \Sigma_2 \cap K \geq k|\Sigma_3]\), then

\[
P_{K^*} \leq \mathbb{P}[\Sigma_1 \cap K^* = \emptyset|\Sigma_3] = \frac{2t-kC_t}{2tC_t} \leq \left(1 - \frac{k}{2t}\right)^t \leq e^{-(k/2)t} = e^{-Cd\log(1/\varepsilon)/4} = \varepsilon^{Cd/4}.
\]

Finally, we exploit the assumption of the VC-dim(C), which any set of \(C\) have at most
\[ \sum_{j=0}^{2d+1} 2^j C_j \text{ distinct intersections with } \Sigma_3, \text{ via the shatter function lemma. For all fixed } \Sigma_3, \]
\[
\mathbb{P}[\mathcal{E}_2 | \Sigma_3] \leq (2C_0 + \cdots + 2C_{2d+1}) \times \varepsilon^{Cd/4}
\leq \left( \frac{2te}{2d+1} \right)^{2d+1} \times \varepsilon^{Cd/4} = \left( \frac{2te}{2d+1} \right)^{2d+1} \times \left( \varepsilon^{C'/4} \right)^{2d+1}
= \left( 2e(1/\varepsilon) \log(1/\varepsilon) \times \varepsilon^{C'/4} \right)^{2d+1} < \frac{1}{2},
\]
if \( d \geq 1, \varepsilon \leq 1/2 \) and some constant \( C' \) is sufficiently large. So \( \mathbb{P}[\Sigma_1] \leq 2\mathbb{P}[\Sigma_2] < 1 \), which completes the proof. \( \square \)

If we define \( d = d_B \), and for all \( \varepsilon \leq 1/2 \), then there exists an \( \varepsilon \)-net \( N \) for \( S^{2d_B-1} \) with cardinality \( |N| = O\left(d_B^{1/2} \log \frac{1}{\varepsilon} \right) \). For the proof of Theorem \( \Pi \) not only the above \( \varepsilon \)-net construction but also the following Lemma 6 of a large deviation estimate are crucial, and see details of the proof in \([2, 9]\). Note that in their proof they use the equal dimension of input and output, i.e., \( N : \mathcal{B}(\mathbb{C}^{d_A}) \rightarrow \mathcal{B}(\mathbb{C}^{d_B}) \) s.t. \( d_A = d_B = d \).

**Lemma 6.** Let \( \varphi_A \) be a pure state, and \( \Pi \) be a rank \( p \) projector. Let \( \{U_i : 1 \leq i \leq d_E\} \) be a sequence of \( \mathcal{U}(d_A) \)-valued i.i.d. random variable, distributed according to Haar measure. Then, for all \( \varepsilon \in (0, 1) \),
\[
\mathbb{P}\left[ \left| \frac{1}{d_E} \sum_{i=1}^{d_E} \text{tr} \left( U_i \varphi_A U_i^\dagger \right) - \frac{p}{d_B} \right| \geq \frac{\varepsilon p}{d_B} \right] \leq 2e^{-d_E p^2 \varepsilon^2 / 8d_B^2}.
\]

Unfortunately, the lemma directly cannot be applied to constructing the local random unitary channel, because the operator norm concern to a different output parameters. On this account, let’s consider a concentrated phenomenon of the output reduced density matrices.
III. CONCENTRATION OF REDUCED STATES

We have already mentioned that a random pure state as well as its random pure substates are almost surely maximally entangled in Section I B. In special we take into account a concentration of reduced density matrices of $\hat{P}(S) \subset \mathbb{C}^{d_B \times d_E}$, and improve the Theorem 2 by using the $\varepsilon$-net theorem (see Theorem 5) and large deviation technique (Lemma 6).

Recall the definition of local $\varepsilon$-randomizing of a channel $N$: $\|N(\varphi_A)\|_{\infty} - \frac{1}{d_B} \leq \frac{\varepsilon}{d_B}$. Note that the image of $\varphi_A$ under the unitary embedding $V : \mathbb{C}^{d_A} \rightarrow \mathbb{C}^{d_B \times d_E}$ can be considered as a subspace $S$ of dimension $s$ in $\mathbb{C}^{d_B \times d_E}$, and it is highly entangled.

Lemma 7. Let $|\varphi_S\rangle$ be a random pure state on $\mathbb{C}^{d_B \times d_E}$, and $\varepsilon \in (0, 1]$. Then

$$\mathbb{P}\left[\|N(\varphi_A)\|_{\infty} - \frac{1}{d_B} \geq \varepsilon \frac{d_B}{d_B}\right] \leq \left(\frac{C}{\varepsilon} \log \frac{d_B}{\varepsilon}\right) e^{-\frac{d_E \varepsilon^2}{14 \ln 2}}. \quad (9)$$

Proof. By using the Cremér’s rule and for a squared Gaussian random random variable, we can obtain the following bound:

$$\mathbb{P}\left[\frac{1}{d_E} \sum_{i=1}^{d_E} X_i \geq (1 + \varepsilon)\sigma^2\right] \leq e^{-d_E \frac{\varepsilon^2 - \ln(1+\varepsilon)}{2 \ln 2}} \leq e^{-\frac{(d_E)\varepsilon^2}{14 \ln 2}},$$

where $\{X_i\}$ are some real-valued i.i.d. random variables and $\sigma$ denotes a standard deviation of the distribution. Let’s substitute the parameters from $\sigma^2$ and $X_i$ to $\frac{1}{d_B}$ and $\text{tr}(\varphi_B \text{tr}_E(U_i \psi_{BE} U_i^\dagger))$, respectively. Here, $\psi_{BE}$ is a random pure state on $\mathcal{P}(\mathbb{C}^{d_B \times d_E})$, but $\varphi_B$ in $\mathcal{P}(\mathbb{C}^{d_B})$. Then we obtain a new Cremér’s bound such that

$$\mathbb{P}\left[\text{tr}(\varphi_B \text{tr}_E(U \psi_{BE} U^\dagger)) - \frac{1}{d_B} \geq \varepsilon \frac{d_B}{d_B}\right] \leq e^{-\frac{(d_E)\varepsilon^2}{14 \ln 2}}. \quad (10)$$

Let’s denote $\varphi_B$ equal to $\varphi$ on the sphere, and $\tilde{\varphi}_B$ just to $\tilde{\varphi}$ on the net, for shortly. By exploiting the definition of conjugate channel and operator norm induced by some pure
state, we obtain a relation that
\[
\| \mathcal{N}(\varphi_A) \|_\infty = \| \text{tr}_E(U \psi_{BE} U^\dagger) \|_\infty = \sup_{\varphi \in B} \text{tr}(\varphi \text{tr}_E(U \psi_{BE} U^\dagger)) \\
= \sup_{\varphi \in B} [\text{tr}(\varphi \text{tr}_E(U \psi_{BE} U^\dagger)) - \text{tr}(\hat{\varphi} \text{tr}_E(U \psi_{BE} U^\dagger))] + \sup_{\varphi \in B} \text{tr}(\hat{\varphi} \text{tr}_E(U \psi_{BE} U^\dagger)) \\
= \sup_{\varphi \in \mathcal{N}_B} \text{tr}(\varphi - \hat{\varphi}) \text{tr}_E(U \psi_{BE} U^\dagger) + \sup_{\varphi \in \mathcal{N}_B} \text{tr}(\hat{\varphi} \text{tr}_E(U \psi_{BE} U^\dagger)).
\] (11)

Above Eq. (11) followed by the definition of induced operator norm, and the supremum in the last equality run over all points on the net \( \mathcal{N}_B \). Now we fix \( \varepsilon \in \mathbb{R} \), then \( | \mathcal{N}_B | = \left( C \frac{d_B^2}{\varepsilon} \log \frac{2d_B}{\varepsilon} \right) \), \( C \) be an universal constat. Furthermore we use the fact which if \( ||\varphi\rangle - |\tilde{\varphi}\rangle||_1 \leq \varepsilon \), then \( \text{tr}(\varphi - \tilde{\varphi})\Pi \leq \frac{\varepsilon}{2} \), where \( |\varphi\rangle \) and \( |\tilde{\varphi}\rangle \) are points on the unit sphere and on the net, respectively. \( \Pi \) is a projector such that \( \Pi \in [0, I] \) \[18]. Thus
\[
\| \mathcal{N}(\varphi_A) \|_\infty = \sup_{\tilde{\varphi} \in \mathcal{N}_B} \text{tr}(\varphi - \tilde{\varphi}) \text{tr}_E(U \psi_{BE} U^\dagger) + \sup_{\tilde{\varphi} \in \mathcal{N}_B} \text{tr}(\tilde{\varphi} \text{tr}_E(U \psi_{BE} U^\dagger)) \\
\leq \sup_{\tilde{\varphi} \in \mathcal{N}_B} \text{tr}(\tilde{\varphi} \text{tr}_E(U \psi_{BE} U^\dagger)) + \frac{\varepsilon}{4d_B}.
\]

Now, we use the union bound and Lemma \[6\]. Then, for some constant \( C \),
\[
\mathbb{P} \left[ \frac{1}{d_B} \geq \frac{\varepsilon}{d_B} \right] \leq \mathbb{P} \left[ \sup_{\tilde{\varphi} \in \mathcal{N}_B} \text{tr}(\tilde{\varphi} \text{tr}_E(U \psi_{BE} U^\dagger)) - \frac{1}{d_B} \geq \frac{3\varepsilon}{4d_B} \right] \\
\leq \left( C \frac{d_B^2}{\varepsilon} \log \frac{d_B}{\varepsilon} \right) e^{-\frac{9\varepsilon^2}{16d_B}}.
\]

Let’s briefly summarize the previous results for finishing proof of our main theorem: Theorem \[1\] Theorem \[5\] states that there exists an \( \varepsilon \)-net of cardinality \( | \mathcal{N}_B | = O \left( \frac{1}{d_B^2} \log \frac{1}{\varepsilon} \right) \) for a higher dimensional unit sphere \( S^{2d_B - 1} \) and its cap \( \mathcal{C} \subset S^{2d_B - 1} \), constrained by \( VC-dim(\mathcal{C}) \leq 2d_B + 1 \). By using the net, we have investigated that the concentration of reduced density matrix which is almost maximally mixed state with high probability. The equation \( (9) \) in Lemma \[7\] describes the concentration phenomenon of density matrix, furthermore a bound of the inequality imply the proof of the main result.

12
Proof of Theorem. Recall Eq. (9) in Lemma. We want to bound the right hand side of the inequality above by 1, depending on $d_E$. Let’s take $d_E \geq C'' \log \frac{d_B}{\varepsilon^2}$ where $C''$ be a suitable constant, then $P\left[\|\mathcal{N}(\varphi_A)\|_\infty - \frac{1}{d_B} \geq \frac{1}{d_B}\right] \leq 1$. This bound straightforwardly means that our claim is true.

Finally, if we choose the dimension of input space of the channel $\mathcal{N}$ equal to its output ($d_A = d_B$), then the proof of Theorem will be completed; For all $\varepsilon \in (0, \frac{1}{2}]$ and $d_A$ is sufficiently large. Let’s $\{U_i\}_{i=1}^{d_E}$ with $d_E = O(\log d_A/\varepsilon^2)$ be an i.i.d. random unitary matrices distributed according to the Haar measure on $\mathcal{U}(d_A)$. Then the channel $\mathcal{N}(\varphi_A) = \frac{1}{d_E} \sum_{i=1}^{d_E} U_i \varphi_A U_i^\dagger$ is a local $\varepsilon$-randomizing map with positive probability.

IV. CONCLUSIONS

In conclusion, we have shown that a quantum channel such that local random unitary channel, $\mathcal{N}$, constructed by averaging over $O(\log d/\varepsilon^2)$ randomly chosen unitaries gives a local $\varepsilon$-randomizing map with positive probability. The whole idea begins from not only the small $\varepsilon$-net construction on the higher dimensional unit sphere corresponding to the sufficiently larger dimensional quantum pure states, but also the analyzing a phenomenon of concentration of output reduced density matrix of the quantum channel. Generically, a higher dimensional bipartite quantum pure states (or random pure states) are almost all maximally entangled, and its entropy of the reduced density matrices are maximally mixed.

By using the result, one could attempt to improving the bound on a communication resources for the private quantum channel, quantum superdense coding and quantum data hiding etc. Furthermore, one may investigate the minimal dimensions of the violation of additivity conjecture for the minimum output entropy.
Acknowledgments

This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant No. 2009-0072627).

[1] A. Ambainis, M. Mosca, A. Tapp, and R. de Wolf, In: 41st Ann. Symp. on Found. of Comp. Sci., New York: John Wiley/ IEEE Comput. Soc. Press, 547–553, (2000).

[2] P. Hayden, D. Leung, P. W. Shor, and A. Winter, Commun. Math. Phys. 250, 371 (2004).

[3] P. Hayden and A. Winter, Commun. Math. Phys. 284, 263 (2008).

[4] T. Cubitt, A. S. Harrow, D. Leung, A. Montanaro, and A. Winter, Commun. Math. Phys. 284, 281 (2008).

[5] M. B. Hastings, Nature Physics 5, 255 (2009).

[6] A. S. Holevo, Probab. Theory and Appl. 51, 133 (2005).

[7] C. King, K. Matsumoto, M. Nathanson, and M. B. Ruskai, Markov Processes and Related Fields 13, 391 (2007).

[8] G. Aubrun, Commun. Math. Phys. 288, 1103 (2009).

[9] C. H. Bennett, P. Hayden, D. Leung, P. W. Shor, and A. Winter, IEEE Trans. Inf. Theory, 51, 56 (2005).

[10] J. Matoušek, Lectures on Discrete Geometry (Springer-Verlag, New York, 2002).

[11] P. Hayden, D. Leung, and A. Winter, Commun. Math. Phys. 265, 95 (2006).

[12] S. Lloyd and H. Pages, Annals of Physics 188, 186 (1988).

[13] D. Page, Phys. Rev. Lett. 71, 1291 (1993).
[14] S. K. Foong and S. Kanno, Phys. Rev. Lett. 72, 1148 (1994).

[15] J. Sanchez-Ruiz, Phys. Rev. E 52, 5653 (1995).

[16] S. Sen, Phys. Rev. Lett. 77, 1 (1996).

[17] A. Dembo and O. Zeitouni, Large deviations techniques and applications (Springer-Verlag, New York, 1993).

[18] A. Harrow, P. Hayden, and D. Leung, Phys. Rev. Lett. 92 187901, (2004).