Non factorizable $O(\alpha\alpha_s)$ corrections to the process $Z \rightarrow b\bar{b}$

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Abstract

We evaluate non factorizable $O(\alpha\alpha_s)$ corrections to the process $Z \rightarrow b\bar{b}$ due to the virtual t-quark. All two-loop vertex diagrams with $W$'s and charged ghosts $\Phi$'s are included. They are evaluated in the large top-mass expansion up to the 10$^{th}$ order. Gluon Bremsstrahlung is taken into account by integrating over the whole phase space. All calculations, including Bremsstrahlung, are done in dimensional regularization. The expansion coefficients of the large mass expansion are given in closed form. Their expansion in $y = m_Z^2/4m_W^2$ is in agreement with the coefficients up to $O(m_W^6/m_t^6)$ as given by Harlander et al. [1].

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The large statistics of the LEP I experiments yielded precise values for the partial $Z$-decay width into $b$-quarks; expressed in terms of the ratio $R_b = 0.21656 \pm 0.00074$ [2] this is a precision of $\sim 0.3\%$. Therefore precise high order calculations in the Standard Model (SM) are needed.

The partial width of the $Z$-boson decay into a quark-antiquark pair can be parametrized as

$$
\Gamma = \tilde{\Gamma} \left( 1 + \frac{\alpha}{\pi} \delta_{\text{EW}} + \ldots \right) \left( 1 + \frac{\alpha_s}{\pi} + \ldots \right) \left( 1 + \frac{\alpha_s}{\pi} \Delta \right),
$$

where

$$
\tilde{\Gamma} = \frac{\alpha N_c m_Z}{12 s^2 c^2} (v^2 + a^2) \left( 1 + \Pi_Z(m_Z^2) \right)^{-1}.
$$

(1) defines the non factorizable contribution $\Delta$. $s, c$ are the sinus and cosinus of the weak mixing angle; $N_c = 3$ is the color factor; $v$ and $a$ are vector and axial couplings related to isospin $I_3$ and charge $Q$ of the fermion by

$$
v = I_3 - 2Qs^2, \quad a = I_3.
$$

The renormalized self-energy $\Pi_Z$ in (2) accounts for the universal correction due to $Z$-boson renormalization.

The electroweak correction $\delta_{\text{EW}}$ has been calculated long ago [3] (see also [4]). The QCD correction factor $\left( 1 + \frac{\alpha}{\pi} + \ldots \right)$ is known now up to $O(\alpha_s^3)$ in the massless limit [3] and also at $O(m_q^2/s)$ [4]. However, electroweak and QCD corrections do not factorize exactly. Therefore the correction factor $\Delta$ appears. This correction for the quarks of u-, d-, c- and s-type was found in [7], where the masses of quarks were neglected compared to $m_Z$. For the b-quark case it is necessary to consider diagrams with the virtual t-quark. The leading term proportional to $m_t^2$ has been calculated in [8] and the term proportional to $\log m_t^2$ in [9]. See also recent work [10] where leading $m_t$ corrections $O(m_t^2 G_F a_s^2)$ were calculated for $Z \rightarrow b\bar{b}$ decay mode.

In the following we are interested only in contributions with the top quark. However, this alone is neither finite nor gauge invariant (see e.g. the discussion in [1, 7]). Instead we have to consider all diagrams with W-boson exchange. In other words we have the gauge invariant decomposition

$$
\Delta = \Delta^Z + \Delta^W,
$$

where $\Delta^{Z,W}$ denotes contributions due to $Z, W$-boson exchange, respectively.
Recently in [1] the subleading terms for $\Delta_b$ were found. The expansion in $1/m_t^2$ has been made up to $(1/m_t^2)^3$. Each coefficient of the expansion in turn is expanded in the small parameter $y = m_Z^2/4m_t^2W$ by a large mass expansion of subdiagrams. The calculations of [1] were done by cutting the 3-loop two point function of the $Z$-boson.

In a sense our approach is complementary to that of [1]: we calculate directly the amplitude of the $Z \to bb$ process and integrate over the final state phase space. In addition we have to add the gluon Bremsstrahlung to form an IR finite quantity. We also use the standard large mass expansion (LME) technique [11] for both real and virtual gluon processes: $1/m_t^2$ is considered as small parameter and the expansion in $1/m_t^2$ has been performed up to $(1/m_t^2)^{10}$. We do not, however, perform a further expansion of subdiagrams, therefore we obtain closed expressions for the expansion coefficients.

However, after reexpansion we fully agree on the first few coefficients given in [1]. We also agree on the numerical estimates following from [1] and from our work. A detailed comparison was given in [12]. However in the present work the results are given in the parametrization (1) which seems to be very natural.

In a recent paper [13] some scalar 2-loop vertex diagrams relevant for $\Delta_b$ were analysed. There it was found that the series in $1/m_t^2$ has a rather bad behaviour (expansion coefficients grow like $4^n$). Therefore one would assume that higher terms of the expansion are needed. However, for the complete physical quantity this is not the case. The `dangerous' terms cancel as well as the auxiliary structures like the polylogarithms $L_2, L_3$ which enter separate diagrams but not the sum of all contributions.

In the course of our calculation we used a program written in FORM [14]. The input for the FORM procedures was generated by the C program DIANA [15].

The amplitude of the process $Z(q) \to \bar{b}(p_1) + b(p_2)$ is given by

$$\mathcal{M} = \epsilon^\mu(q) T_\mu(q, p_1, p_2),$$  \hspace{1cm} (5)

where $\epsilon^\mu(q)$ is the $Z$-boson wave function and the amplitude $T_\mu$ reads

$$T_\mu = -i\frac{e}{2sc} \bar{b}(p_1) \left[ \gamma_\mu v(q^2) - \gamma_\mu \gamma_5 a(q^2) \right] b(p_2)$$  \hspace{1cm} (6)

where $e$ is the electric charge; $\bar{b}, b$ are wave functions of b-quarks. At the tree
level the couplings $v_b$ and $a_b$ are given by

$$v_b = -\frac{1}{2} + \frac{2}{3} \sin^2 \Theta_W, \quad a_b = -\frac{1}{2},$$  \hspace{1cm} (7)

while in higher orders (7) gets corrections to both real and imaginary parts.

Evaluating the width from (8), using the standard rules, we get in $d = 4 - 2\varepsilon$ dimensional space-time

$$\Gamma = \left(\frac{\mu^2}{m_Z^2}\right)^{(4-d)/2} e^{\varepsilon \gamma_E} \frac{\alpha N_c m_Z}{12 s^2 c^2} \frac{\Gamma(d/2)}{\Gamma(d-2)} (|v|^2 + |a|^2),$$  \hspace{1cm} (8)

where $\mu$ is an arbitrary parameter with the dimension of a mass. It is related to the parameter $\mu$ of dimensional regularization by $\mu^2 = 4\pi e^{-\varepsilon \gamma_E} \mu^2$.

At 1-loop level the diagrams contributing to this process are shown in Fig.1. They were evaluated in [3, 4].

Figure 1: 1-loop diagrams with the t-quark contributing to the $Zb\bar{b}$ process of order $O(\alpha)$. Diagrams of order $O(\alpha \alpha_s)$ are obtained by adding gluon lines.

Mixed $O(\alpha \alpha_s)$ corrections are obtained from the diagrams of Fig.1 by adding in all possible ways a gluon line. In this way we obtain 2-loop vertex diagrams as well as 1-loop Bremstrahlung diagrams.

To obtain the renormalized expressions, for our purpose it is enough to renormalize the mass of the t-quark to order $O(\alpha_s)$ and the wave function of the b-quark up to $O(\alpha \alpha_s)$.  

3
The renormalization constant of $m_t$ in the on-shell scheme is given by

$$Z_{m_t} = 1 - \frac{\alpha_s}{\pi} C_F \left( \frac{3}{4\epsilon} + 1 - \frac{3}{4} \log \frac{m_t^2}{\mu^2} \right). \quad (9)$$

The effect of the $b$-quark renormalization is effectively performed as

$$(v - a)_R = Z_b (v - a)_0, \quad (10)$$

with the $b$-quark renormalization constant $Z_b$ (see appendix), while the right handed combination $(v + a)$ remains unrenormalized.

To make the width IR finite, in addition to the nonradiating process we have to add the soft contribution from the Bremsstrahlung process

$$Z \rightarrow b + \bar{b} + g. \quad (11)$$

Actually we will include the hard gluon emission as well by integrating over the full gluon phase space. Of particular interest is the calculation of the Bremsstrahlung in terms of the LME. The kinematics of the (gluon-) Bremsstrahlung for the process (11) is given by

$$q \rightarrow p_1 + p_2 + p_3, \quad p_1^2 = p_2^2 = p_3^2 = 0. \quad (12)$$

Thus we have 3 invariants

$$p_1 p_2, \quad p_2 p_3, \quad p_1 p_3. \quad (13)$$

We are interested in the integrated Bremsstrahlung. The corresponding width in $d$ dimensions can be written as

$$\Gamma_{Br} = \left( \frac{\mu^2}{m_Z^2} \right)^{4-d} e^{2\gamma_E} \frac{m_Z}{768\pi^3 \Gamma(d-2)} \frac{1}{(d-2)} \int_0^1 dx \, dy \, x^{d-3}(1-x)^{d/2-2} y^{d/2-2}(1-y)^{d/2-2} |M_{Br}|^2, \quad (14)$$

where $M_{Br}$ is the Bremsstrahlung amplitude and the invariants can be expressed in terms of $x$ and $y$ as

$$p_1 p_2 = \frac{m_Z^2}{2} x (1 - y),$$

$$p_2 p_3 = \frac{m_Z^2}{2} (1 - x),$$

$$p_1 p_3 = \frac{m_Z^2}{2} xy.$$
It should be noted that neither (8) nor (14) are finite separately after the renormalization. They both have IR divergences up to $O(1/\varepsilon^2)$. Therefore it is quite important to use general $d$-dimensional expressions and take the limit $d \to 4$ only after adding up (8) and (14). Otherwise some finite contributions will be lost.

We have checked by explicit calculation that indeed the result is finite in the sum of $\Gamma$ and $\Gamma_B$, and our result for $\Delta^W_b$ reads

$$\Delta^W_b = \frac{C_F}{(v_b^2 + a_b^2)s^2} \left[ \frac{m^2_t}{m^2_W} \left\{ \zeta_2 \left( \frac{1}{16} + \frac{1}{32y} \right) \right\} + \left\{ -\frac{3245}{11664}y - \frac{7499}{46656} - \frac{1009}{93312y} + \zeta_2 \left( \frac{53}{324y} + \frac{173}{1296} + \frac{67}{2592y} \right) \right. \right. $$

$$+ L_c \left( -\frac{7}{1944}y - \frac{7}{1944} - \frac{7}{7776y} \right) + L_t \left( -\frac{7}{1944}y - \frac{7}{1944} - \frac{7}{7776y} \right) \} 

$$+ \frac{m^2_W}{m^2_t} \left\{ -\frac{23939}{21600}y^2 - \frac{453539}{777600}y - \frac{262937}{1555200} - \frac{89}{1152y} \right. $$

$$+ \zeta_2 \left( \frac{11}{18} y^2 + \frac{13}{144y} + \frac{257}{864y} + \frac{175}{864y} \right) \right. $$

$$+ I_0 \left( \frac{13}{216} y^2 + \frac{13}{48y} - \frac{299}{96} - \frac{13}{576y^2} \right) $$

$$+ L_c \left( \frac{17}{38880} y^2 - \frac{823}{38880y} - \frac{3343}{155520} - \frac{7}{1296y} \right) $$

$$+ L_t \left( \frac{2357}{38880} y^2 + \frac{8537}{38880y} - \frac{45823}{155520} - \frac{1009}{5184y} \right) \} \right)$$

$$+ \frac{m^4_W}{m^4_t} \left\{ -\frac{652232029}{178605000} y^3 - \frac{938540803}{158760000} y^2 + \frac{362957621}{317520000} y + \frac{1198673}{1166400} - \frac{1099}{3888y} \right. $$

$$+ \zeta_2 \left( \frac{4838}{2025} y^3 + \frac{4769}{1080} y^2 - \frac{14539}{10800} y - \frac{239}{324} + \frac{10}{27y^2} \right)$$

$$+ I_0 \left( \frac{91}{540} y^3 - \frac{383}{540} y^2 + \frac{4139}{4320}y + \frac{1021}{4320} - \frac{533}{1920y} - \frac{5}{128y^2} \right) $$

$$+ L_c L_t \left( \frac{1}{36}y + \frac{1}{36} + \frac{1}{144y} \right) + L_t^2 \left( \frac{1}{36}y + \frac{1}{36} + \frac{1}{144y} \right) \} \right.$$
\[ + L_c \left( -31 \frac{y^3}{283500} + 283 \frac{y^2}{212625} - 105443 \frac{y}{3402000} - 623 \frac{1}{19440} - \frac{7}{864y} \right) \]
\[ + L_t \left( -23903 \frac{y^3}{141750} - 1060861 \frac{y^2}{1701000} + 3447757 \frac{y}{3402000} - 11551 \frac{1}{38880} - \frac{811}{1728y} \right) \]
\[ + O \left( \frac{m_t^6}{m_W^6} \right) \]  

(15)

where \( L_t = \log(m_t^2/m_W^2) \), \( L_c = \log c^2 \) and \( y = m_Z^2/4m_W^2 \). The only function that enters the answer is

\[ I_0 = -\frac{1}{2} \int_0^1 \frac{\log(1-ty)}{\sqrt{1-t}} \, dt. \]  

(16)

We do not give here higher coefficients in analytic form because of their complexity. Instead we give below 10 coefficients of the \( 1/m_t^2 \) expansion numerically. For \( m_t = 175\text{GeV}, m_W = 80.33\text{GeV} \) and \( m_Z = 91.187\text{GeV} \) we obtain

\[ \Delta_W^b = \frac{4.1878}{t} + 2.3057 - 8.0270 t - 28.0471 t^2 \]
\[ - 39.5864 t^3 - 32.7842 t^4 - 8.7501 t^5 + 20.0429 t^6 \]
\[ + 36.7551 t^7 + 37.3991 t^8 + 65.1923 t^9 + 307.874 t^{10}, \]  

(17)

where \( t = m_t^2/m_W^2 \sim 0.21 \).

Separating the leading term from the higher order ones, we may write this as

\[ \Delta_W^b = 19.8764 - 1.0654 = 18.8109, \]  

(18)

which tells us that the terms of higher order amount to only \( \sim 5\% \) of the leading term. Obviously also the series in \( 1/m_t^2 \) converges quite rapidly. The reason, however, for the smallness of the correction is the alternation in sign of the various higher order contributions.

For completeness and for comparison we also present the numbers for the non factorizable contributions of the lighter quarks and the \( Z \)-contribution of the \( b \)-quark. The result for these contributions are taken from [7]. To slightly improve them, we performed a Padé approximation for the expansions in \( x = 1 \) for \( \Delta_Z \) and \( x = m_Z^2/m_W^2 \) for \( \Delta_W^b \) (however this gives only minor changes of order of few percents in formulae (19) and (20)). For \( \Delta_Z \) we have

\[ \Delta_Z = \begin{cases} 
-0.489 & \text{for } u,c \\
-0.796 & \text{for } d,s,b 
\end{cases} \]  

(19)
Taking into account (18), we have for $\Delta^W$

$$\Delta^W = \begin{cases} 
-3.652 & \text{for } u,c \\
-3.745 & \text{for } d,s \\
+18.811 & \text{for } b 
\end{cases} \quad (20)$$

The above numbers demonstrate again that due to the heavy top quark the $\Delta^W_b$ contribution is larger by almost an order of magnitude than all the other contributions. The smallness of the subleading terms in $1/m_t^2$ compared is surprising, however. This is a highly nontrivial result. It is, e.g., in contrast to the results found in Ref. [16], where an $O(\alpha^2 m_t^2/m_W^2)$ calculation of $\Delta r$ was performed and it was found that this correction is of the same order as the leading $O(\alpha^2 m_t^4/m_W^4)$ correction.

A detailed comparison of our results with those of Ref. [1] is given in [12].

A Renormalization of the b-quark wave function

In this section we evaluate the virtual top quark contribution to the wave function renormalization constant $Z_b$ for the b-quark field in the on-shell scheme. We need this constant up to order $O(\alpha \alpha_s)$. At 1-loop order two diagrams contribute. They are shown in Fig.2. Adding in all possible ways one gluon line we get 8 diagrams of order $O(\alpha \alpha_s)$.

![Figure 2: 1-loop diagrams with the t-quark contributing to the b-quark wave function renormalization in $O(\alpha)$. Diagrams in $O(\alpha \alpha_s)$ are obtained by adding one gluon line.](image-url)
The self-energy of the $b$-quark reads

$$\hat{\Sigma}(p) = \hat{p} \frac{1 - \gamma_5}{2} \Sigma_L(p) + \hat{p} \frac{1 + \gamma_5}{2} \Sigma_R(p).$$

By explicit calculation we find $\Sigma_R = 0$. Therefore the renormalization constant $Z_b$ in the on-shell scheme ($m_b = 0$) reads

$$Z_b = 1 + \frac{1 - \gamma_5}{2} \Sigma_L(0) = 1 + \frac{1 - \gamma_5}{2} \frac{\alpha}{4\pi s^2} \left( z_1 + \frac{\alpha_s}{4\pi} C_F z_2 \right).$$

For the sake of completeness we give below the first 4 coefficients of the expansion of $z_{1,2}$ in $m_W^2/m_t^2$. Note that we have to keep terms up to $O(\varepsilon)$ in the 1-loop part and terms up to $O(1)$ in the 2-loop part.

\[ z_1 = \frac{m_t^2}{m_W^2} \left\{ \frac{1}{4\varepsilon} + \frac{L_t}{4} + \frac{L_\mu}{4} - \frac{3}{8} \right. \]
\[ + \varepsilon \left( -\frac{1}{8} \zeta_2 - \frac{1}{4} L_t L_\mu + \frac{3}{8} L_t - \frac{1}{8} L_\mu^2 + \frac{3}{8} L_\mu - \frac{1}{8} L_\mu^2 - \frac{7}{16} \right) \}
\[ + \left\{ \frac{1}{2\varepsilon} + \frac{L_t}{2} + \frac{L_\mu}{2} - \frac{1}{2} \right. \]
\[ + \varepsilon \left( -\frac{1}{4} \zeta_2 - L_t L_\mu + L_t - \frac{1}{2} L_\mu^2 + \frac{1}{2} L_\mu - \frac{1}{4} L_\mu^2 - \frac{1}{2} \right) \}
\[ + \frac{m_W^2}{m_t^2} \left\{ \frac{7}{4} L_t - \frac{3}{4} + \varepsilon \left( -\frac{7}{4} L_t L_\mu + \frac{13}{8} L_t - \frac{7}{8} L_\mu^2 + \frac{3}{4} L_\mu - \frac{5}{8} \right) \right\} \]
\[ + \frac{m_W^4}{m_t^4} \left\{ \frac{5}{2} L_t - \frac{3}{4} + \varepsilon \left( -\frac{5}{2} L_t L_\mu + \frac{9}{4} L_t - \frac{5}{4} L_\mu^2 + \frac{3}{4} L_\mu - \frac{5}{8} \right) \right\} \]
\[ + O\left( \frac{m_W^6}{m_t^6} \right), \]

\[ z_2 = \frac{m_t^2}{m_W^2} \left\{ \frac{3}{4\varepsilon^2} + \frac{1}{\varepsilon} \left( -\frac{3}{2} L_t - \frac{3}{2} L_\mu + \frac{5}{2} \right) \right. \]
\[ + 4 + \frac{3}{4} \zeta_2 + 3 L_t L_\mu - 5 L_t + \frac{3}{2} L_\mu^2 - 5 L_\mu + \frac{3}{2} L_\mu^2 \right\} \]
\[ + \left\{ \frac{3}{4 \epsilon} - \frac{1}{8} - 6 \zeta_2 - \frac{3}{2} L_t - \frac{3}{2} L_\mu \right\} + \frac{m_W^2}{m_t^2} \left\{ \frac{27}{4} - \frac{21}{2} \zeta_2 + \frac{33}{4} L_t \right\} \\
+ \frac{m_W^4}{m_t^4} \left\{ \frac{51}{4} - 15 \zeta_2 + \frac{39}{2} L_t \right\} + O(\frac{m_W^6}{m_t^6}), \]

where \( L_t = \log(m_t^2/m_W^2) \) and \( L_\mu = \log(\mu^2/m_W^2) \).

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