Lie symmetries reduction and spectral methods on the fractional two-dimensional heat equation

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Abstract

In this paper, the Lie symmetry analysis is proposed for a space-time convection-diffusion fractional differential equations with the Riemann-Liouville derivative by (2+1) independent variables and one dependent variable. We find a reduction form of our governed fractional differential equation using the similarity solution of our Lie symmetry. One-dimensional optimal system of Lie symmetry algebras is found. We present a computational method via the spectral method based on Bernstein’s operational matrices to solve the two-dimensional fractional heat equation with some initial conditions.

Keywords: Bernstein operational matrices, Fractional derivative, Infinitesimal, Lie symmetry, Optimal system, Prolongation, Similarly solution, Spectral method.

1 Introduction

The diffusion equation is one of the well-known equations with many applications in engineering problems, heat conduction, chemical diffusion, fluid flow, mass transfer, refrigeration, and traffic analysis, and so on, [7,15,30]. Recently, the study of fractional ordinary differential equation and the partial differential equation has attracted much attention due to an exact description of differential equations in fluid mechanics, biology, physics, engineering, and other areas of science [14,42,44]. Lie symmetry analysis method that originally advocated by Sophus Lie plays a powerful tools to obtain some exact solutions of differential equations, [4,27,26]. The fundamental idea of the Lie symmetry analysis is regarding the tangent structural equations under one or several parameters Lie groups of point transformations. One can construct exact solutions including similarity solutions or more general group-invariant solutions by corresponding symmetry reductions, [25,29]. There have been some new generalizations of the classical Lie group analysis for symmetry reductions. For instance, Ovsiannikov, [28], has extended the method of partially invariant solutions. His works are based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space that is called jet space, and preserving the class of given partial differential equations. He has introduced s-dimension optimal system for classification in Lie symmetry algebras of a given differential equation. The efficiency of the Lie symmetry technique for integer-order differential equations, [3,12,20,43], encouraged the scientists for its extension to the fractional differential equation (FDEs) and then many authors have made
their contribution to the applications of symmetry method for the analysis of some FDEs, [6, 8, 10, 11, 18]. In most of the available papers, the symmetries of the FDEs with time-fractional in (1+1) independent variables have been analyzed, [13, 17, 22, 29, 35, 36, 45, 46]. Recently, the symmetry approach has been developed by Singla and Gupta for the complete group classification of space-time fractional partial differential equation with (2+1) independent variables and two dependent variables, [38, 39, 40, 41]. In this article, we consider the following (2+1) fractional convection-diffusion equation

\[
D_\alpha^\alpha t^\alpha u(t,x,y) = D_\beta^\beta x^\beta u(t,x,y) + D_\beta^\beta y^\beta u(t,x,y) + f(x,y,t),
\]

(1)

where \(0 < \alpha \leq 1, 0 < \beta < 1\) and \(D_\alpha^\alpha t^\alpha, D_\beta^\beta x^\beta, D_\beta^\beta y^\beta\) are the Riemann-Liouville fractional derivatives of order \(\alpha\) with respect to the variable \(t, x, y\) and \(f\) is an arbitrary smooth function. The Riemann-Liouville fractional derivatives of order \(\alpha\) with respect to the variable \(x_i\), \(i = 1, \cdots, n\), are, [16],

\[
D_\alpha^\alpha x_i^\alpha u(x_1, \cdots, x_n) = \left\{ \begin{array}{ll}
\frac{\partial^n u}{\partial x_i^n} & , \alpha = n, \\
\frac{1}{\Gamma(n-\alpha)} \frac{\partial^n u}{\partial x_i^n} \int_0^{x_i} (x_i - s)^{n-\alpha-1} u(x_1, \cdots, x_{i-1}, s, x_{i+1}, \cdots, x_n) ds & , 0 \leq n-1 < \alpha < n,
\end{array} \right. 
\]

(2)

and the Caputo fractional derivatives are

\[
C_0^\alpha D_\alpha^\alpha x_i^\alpha u(x_1, \cdots, x_n) = \left\{ \begin{array}{ll}
\frac{\partial^n u}{\partial x_i^n} & , \alpha = n, \\
\frac{1}{\Gamma(n-\alpha)} \int_0^{x_i} (x_i - s)^{n-\alpha-1} \frac{\partial^n u}{\partial x_i^n} (x_1, \cdots, x_{i-1}, s, x_{i+1}, \cdots, x_n) ds & , 0 \leq n-1 < \alpha < n,
\end{array} \right. 
\]

(3)

here \(n \in \mathbb{N}\), \(\frac{\partial^n u}{\partial x_i^n}\) is the usual partial derivative of integer order \(n\) with respect to \(x_i\), \(1 \leq i \leq n\). One of the most important numerical methods for solving linear, nonlinear, and fractional equations are Bernstein polynomials which have been used in many papers, [1, 2, 3, 31, 32, 23, 24]. We try to apply the spectral method based on Bernstein’s operational matrices on our given fractional differential equation. Then we compare the exact solution and an approximate solution in an illustrative example for \(f(x,y,t) = 2tx^3y^3 - 6t^2xy^3 - 6t^2x^3y\) with depicting in some different cases. There are some interesting papers for handling (1+1) fractional diffusion equation by different methods, for example, by Sinc-Legendre collocation method, [34], tau approach, [33], differential transform method, [9], and (2+1) fractional heat conduction equation by Legendre polynomials method, [19]. In the present paper, we first compute the Lie symmetry algebra of (1) equation and then one-dimensional optimal system of subalgebras is obtained. A reduction of FDE (1) by its similarity solution is calculated. In the last section we present a numerical result via the spectral method based on Bernstein’s operational matrices to obtain a solution for the 2D fractional heat equation with some initial conditions.

## 2 Symmetry analysis

**Theorem 1** The Lie symmetry algebra of the fractional two-dimensional heat conduction equation [1] is spanned by following vector fields:

\[
\begin{align*}
X_1 &= \frac{\partial}{\partial t}, & X_2 &= \frac{\partial}{\partial x}, & X_3 &= \frac{\partial}{\partial y}, & X_4 &= u \frac{\partial}{\partial u}, \\
X_5 &= \alpha t \frac{\partial}{\partial t} + \beta x \frac{\partial}{\partial x} + \beta y \frac{\partial}{\partial y}, & X_6 &= g(t,x,y) \frac{\partial}{\partial u}.
\end{align*}
\]

(4)
Proof: To find the Lie symmetries of fractional 2D heat equation (1), assume that (1) is invariant under the following one parameter group transformation

\[ \bar{t} = t + \epsilon T(t, x, y, u) + O(\epsilon^2), \]
\[ \bar{x} = x + \epsilon X(t, x, y, u) + O(\epsilon^2), \]
\[ \bar{y} = y + \epsilon Y(t, x, y, u) + O(\epsilon^2), \]
\[ \bar{u} = u + \epsilon U(t, x, y, u) + O(\epsilon^2), \]

(5)

where \( T, X, Y \) and \( U \) are infinitesimals and \( U_\alpha^t, U_\beta^x \) and \( U_\gamma^y \) are prolonged infinitesimals of the order \( \alpha \) and \( \beta \) respectively. Suppose that

\[ \mathbf{X} = T(t, x, y, u) \frac{\partial}{\partial t} + X(t, x, y, u) \frac{\partial}{\partial x} + Y(t, x, y, u) \frac{\partial}{\partial y} + U(t, x, y, u) \frac{\partial}{\partial u} \]

(6)

is the infinitesimal generator corresponding to equation (1) under the one parameter group transformation (5). The \((\alpha, \beta)\)-order prolongation of vector field \( \mathbf{X} \), [21], is defined by

\[ \mathbf{X}^{(\alpha, \beta)} = \mathbf{X} + U_\alpha^t \partial_{D^\alpha_t} + U_\beta^x \partial_{D^\beta_x} + U_\gamma^y \partial_{D^\gamma_y} + \cdots \]

(7)

where

\[ U_\alpha^t = D^\alpha_t (U - Xu_x - Y u_y - Tu_t) + XD^\alpha_t u_x + Y D^\alpha_t u_y + TD^\alpha_t u_t, \]
\[ = D^\alpha_t U - D^\alpha_t (X u_x) - D^\alpha_t (Y u_y) - D^\alpha_t (T u_t) + XD^\alpha_t u_x + Y D^\alpha_t u_y + TD^\alpha_t u_t \]
\[ U_\beta^x = D^\beta_x (U - Xu_x - Y u_y - Tu_t) + XD^\beta_x u_x + Y D^\beta_x u_y + TD^\beta_x u_t, \]
\[ = D^\beta_x U - D^\beta_x (X u_x) - D^\beta_x (Y u_y) - D^\beta_x (T u_t) + XD^\beta_x u_x + Y D^\beta_x u_y + TD^\beta_x u_t \]
\[ U_\gamma^y = D^\gamma_y (U - Xu_x - Y u_y - Tu_t) + XD^\gamma_y u_x + Y D^\gamma_y u_y + TD^\gamma_y u_t, \]
\[ = D^\gamma_y U - D^\gamma_y (X u_x) - D^\gamma_y (Y u_y) - D^\gamma_y (T u_t) + XD^\gamma_y u_x + Y D^\gamma_y u_y + TD^\gamma_y u_t. \]

(8)

Notice that the symbol \( D_t \) is the total derivative operator with respect to \( t \):

\[ D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u_t} + u_{xt} \frac{\partial}{\partial u_{xt}} + u_{tt} \frac{\partial}{\partial u_{tt}} + \cdots, \]

and the operator \( D^\alpha_t, D^\beta_x \) and \( D^\gamma_y \) are the total fractional derivative operators. By the Leibnitz rules [12]

\[ D^\alpha_t (f(t)g(t)) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) g^{(n)}(t), \quad \alpha > 0, \]
\[ D^\beta_x (h(x)l(x)) = \sum_{n=0}^{\infty} \binom{\beta}{n} D_x^{\beta-n} h(x) l^{(n)}(x), \quad \beta > 0, \]
\[ D^\gamma_y (r(y)z(y)) = \sum_{n=0}^{\infty} \binom{\gamma}{n} D_y^{\gamma-n} r(y) z^{(n)}(y), \quad \beta > 0, \]

(9)

where

\[ \binom{\alpha}{n} = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)\Gamma(n+1)}, \quad \binom{\beta}{n} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-n)\Gamma(n+1)}. \]
Applying (10) into (12), we obtain
\[ U_{i}^\alpha = D_t^\alpha (U) - u D_t^\alpha (U_u) + U_u D_t^\alpha (u) + \sum_{n=1}^{\infty} \left( \frac{\alpha}{n} \right) D_t^\alpha (U_u) D_t^{\alpha-n} (u) - \alpha (u_t T_u + T_t) D_t^\alpha (u) - \sum_{n=1}^{\infty} \left( \frac{\alpha - n}{n+1} \right) D_t^{\alpha-n} (u) D_t^{n+1} (T) + D_t^{\alpha-n} (u_y) D_t^{n} (Y) + D_t^{\alpha-n} (u_x) D_t^{n} (X) \]

By symmetry criteria condition we have
\[ \sum_{n=1}^{\infty} \left( \frac{\beta}{n} \right) \left( \frac{\beta - n}{n+1} \right) D_x^{\beta-n} (u) D_x^{n+1} (X) + D_x^{\beta-n} (u_y) D_x^{n} (Y) + D_x^{\beta-n} (u_t) D_x^{n} (T) \]

By putting (10) into (12), we get the following determining equations:
\[ T_x = T_y = T_u = 0, \quad X_t = Y_t = X_u = 0, \quad U_{tt} = U_{xx} = U_{yy} = U_{uu} = 0 \]

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by solving the (13) we find
\[ T(t) = c_5 \beta t + c_1, \quad X(x) = c_5 \alpha x + c_2, \quad Y(y) = c_5 \alpha y + c_3, U(t, x, y, u) = g(t, x, y) + c_3 u. \]

Therefore the symmetry algebra of the fractional two-dimensional heat conduction equation (1) is spanned by vector fields (4). The commutation relations satisfied by generators (4) are shown in Table 1

3 The optimal system of one dimension subalgebras

Let $G$ be a Lie group with corresponded Lie algebra $\mathfrak{g}$, i.e., $\mathfrak{g} = T_e G$. There is an inner automorphism $\tau_\alpha \rightarrow \tau^2 \alpha \tau^{-1}$ of the group $G$ for every arbitrary element $\tau \in G$. Every an automorphism of the group $G$
induces an automorphism of \( g \). The set of all these automorphisms forms a Lie group called the adjoint group \( G^A \). For arbitrary infinitesimal generators \( X \) and \( Y \) in \( g \), the linear mapping \( \text{Ad} X(Y) : Y \rightarrow [X, Y] \) is an automorphism of \( g \), called the inner derivation of \( g \). The set of all these inner derivations equippd by the Lie bracket \( [\text{Ad} X, \text{Ad} Y] = \text{Ad}[X, Y] \) is a Lie algebra \( g^A \) called the adjoint algebra of \( g \). Two subalgebras in \( g \) are conjugate if there is a transformation of \( G^A \) which takes one subalgebra into the other. The collection of pairwise non-conjugate \( s \)-dimensional subalgebras is called the optimal system of subalgebras of order \( s \) that was introduced by Ovsiannikov [28]. Actually solving the optimal system problem is to determine the conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation i.e. \( \mathfrak{g} \text{Ad}(\tau) \mathfrak{h} \) for some \( \tau \) in a given Lie group. The adjoint action is given by the Lie series

\[
\text{Ad}(\exp(s X_i))X_j = X_j - s [X_i, X_j] + \frac{s^2}{2} [X_i, [X_i, X_j]] - \cdots,
\]

where \( s \) is a parameter and \( i, j = 1, \ldots, n \). We can simplify a given arbitrary element,

\[
X = \sum_{i=1}^{5} a_i X_i,
\]

of the Lie algebra \( g = \langle X_1, \ldots, X_5 \rangle \). Note that the elements of \( g \) can be represented by vectors \( (a_1, \ldots, a_5) \in \mathbb{R}^5 \) since each of them can be written in the form \( \langle X_i \rangle \) for some constants \( a_1, \ldots, a_5 \). Hence, the adjoint action can be regarded as (in fact is) a group of linear transformations of the vectors \( (a_1, \ldots, a_5) \).

**Theorem 2** Let \( g = \langle X_1, \ldots, X_5 \rangle \) be the finite dimension Lie symmetry algebras of fractional heat equation [7]. The optimal system of subalgebras of order one is generated by

1. \( g_1 = \langle X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 \rangle \)
2. \( g_2 = \langle X_3 + b_4 X_4 \rangle \)
3. \( g_3 = \langle X_2 + c_4 X_4 \rangle \)
4. \( g_4 = \langle d_4 X_4 + d_5 X_5 \rangle \)
5. \( g_5 = \langle X_2 + e_3 X_3 + e_4 X_4 + e_5 X_5 \rangle, \; e_3 \neq 0 \)
6. \( g_6 = \langle X_1 + f_4 X_4 + f_5 X_5 \rangle, \; f_3 \neq 0 \)
7. \( g_7 = \langle X_1 + g_4 X_4 + g_5 X_5 \rangle, \; g_2 \neq 0 \)
8. \( g_8 = \langle X_1 + h_2 X_2 + h_3 X_3 + h_4 X_4 + h_5 X_5 \rangle, \; h_2, h_3 \neq 0 \)

where \( a_i, b_4, c_4, d_i \)'s, \( e_i \)'s, \( f_i \)'s, \( g_i \)'s and \( h_i \)'s belong \( \mathbb{R} \) are arbitrary constants.
Proof: The function $F^s_i: g \to g$ defined by $X \mapsto \text{Ad}(\exp(s_iX_i).X)$ is a linear map, for $i = 1, \cdots, 5$. The matrices $M^s_i$ of $F^s_i$ with respect to basis $\{X_1, \cdots, X_5\}$ are

$$M^s_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-s_1 & 0 & 0 & 0 & 1
\end{bmatrix}, \quad M^s_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & -s_2 & 0 & 0 & 1
\end{bmatrix}, \quad M^s_3 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$M^s_4 = I_5, \quad M^s_5 = \begin{bmatrix}
\exp(s_5) & 0 & 0 & 0 & 0 \\
0 & \exp(s_5) & 0 & 0 & 0 \\
0 & 0 & \exp(s_5) & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

where $I_5$ is $5 \times 5$ identity matrix. Let $X = \sum_{i=1}^5 a_iX_i$ then it is seen that

$$F^{s_5} \circ F^{s_4} \circ \cdots \circ F^{s_1} : X \mapsto [a_1 \exp(s_5)]X_1 + [a_2 \exp(s_5)]X_2 + [a_3 \exp(s_5)]X_3 + a_4X_4 + [a_5 - a_1s_1 \exp(s_5) - a_2s_2 \exp(s_5) - a_3s_3 \exp(s_5)]X_5.$$  

If $a_1 \neq 0$ then we can omit the coefficient of $X_5$ by setting $s_1 = \frac{a_5}{a_1}$ and $s_2 = s_3 = s_5 = 0$. Scaling $X$, we can assume that $a_1 = 1$. So, $X$ reduces to the case (1). If $a_1 = a_2 = 0$ and $a_3 \neq 0$ then by putting $s_3 = \frac{a_5}{a_3}$ and $s_5 = 0$, we can assume $a_3 = 1$ by scaling $X$ and therefore we find the case (2). If $a_1 = a_3 = 0$ and $a_2 \neq 0$ then one can vanish the coefficients of $X_1, X_3$ and $X_5$ by setting $s_2 = \frac{a_5}{a_2}, s_5 = 0$. Scaling $X$, we can assume that $a_2 = 1$. So, $X$ is reduced to the case (3). If $a_1 = a_2 = a_3 = 0$ then $X$ is reduced to the case (4). If $a_1 = 0$ and $a_2, a_3 \neq 0$ then one can vanish the coefficients of $X_1$ by setting $s_2 = \frac{a_5}{a_2}, s_3 = -\frac{a_5}{a_3}$ and $s_5 = 0$. Scaling $X$, we can assume that $a_2 = 1$. So, $X$ is reduced to the case (5). If $a_2 = 0$ and $a_1, a_3 \neq 0$ then one can vanish the coefficients of $X_2$ by setting $s_1 = \frac{a_5}{a_1}, s_3 = -\frac{a_5}{a_3}$ and $s_5 = 0$. Scaling $X$, we can assume that $a_1 = 1$. So, $X$ is reduced to the case (6). If $a_3 = 0$ and $a_1, a_2 \neq 0$ then one can vanish the coefficients of $X_3$ by setting $s_1 = \frac{a_5}{a_1}, s_2 = \frac{a_5}{a_2}$ and $s_5 = 0$. Scaling $X$, we can assume that $a_1 = 1$. Then $X$ changes to the case (7). If $a_1, a_2, a_3 \neq 0$ then one can vanish the coefficients of $X_3$ by setting $s_1 = \frac{a_5}{a_1}, s_2 = -\frac{a_5}{a_2}$ and $s_3 = s_5 = 0$. Scaling $X$, we can assume that $a_1 = 1$ and thus $X$ maps to the case (8). \qed

4 Symmetry reduction

The infinitesimal generators $X_1, \cdots, X_4$ provide trivial invariant solutions and hence we focus to deduce the corresponded characteristic equation of vector field $X_5$ for getting the reduction equation. The corresponded characteristic equation of $X_5 = \alpha \frac{\partial}{\partial t} + \beta x \frac{\partial}{\partial t} + \beta y \frac{\partial}{\partial y}$ is

$$\frac{dt}{\alpha t} = \frac{dx}{\beta x} = \frac{dy}{\beta y}.$$ 

Solving above equation leads to the following similarity transformation

$$u = \omega(\xi_1, \xi_2), \quad \xi_1 = xt^{-\frac{\alpha}{\beta}}, \quad \xi_2 = yt^{-\frac{\beta}{\beta}}.$$  

(17)
Theorem 3  The similarity solution \([17]\) transforms the fractional heat equation \([1]\) to
\[
(P^{1,n-a}_{\frac{1}{3},\frac{3}{2}} \omega)(\xi_1, \xi_2) = t^\alpha (x^{-\beta} + y^{-\beta}) \left( P^{1,n-\beta}_{1,\infty} \omega \right)(\xi_1, \xi_2),
\]
(18)
here \(P^{\tau,\alpha}_{\gamma_1,\gamma_2} \) denotes the extended left-hand sided Erdelyi-Kober fractional derivative operator, \([17, 22]\), with following definition
\[
(P^{\tau,\alpha}_{\gamma_1,\gamma_2} \omega)(z_1, z_2) = n^{-1} \sum_{j=0}^{n-1} \left( \tau + j - \frac{1}{\gamma_1} z_1 \frac{\partial}{\partial z_1} - \frac{1}{\gamma_2} z_2 \frac{\partial}{\partial z_2} \right) \times \left( K^{\tau,\alpha}_{\gamma_1,\gamma_2} \omega \right)(z_1, z_2),
\]
(19)
and here
\[
(K^{\tau,\alpha}_{\gamma_1,\gamma_2} \omega)(z_1, z_2) = \begin{cases} 
\frac{1}{\Gamma(\alpha)} \int_0^\infty (\theta - 1)^{n-1} \theta^{-(\tau+\alpha)} \omega(z_1 \theta^{\frac{1}{\gamma_1}}, z_2 \theta^{\frac{1}{\gamma_2}}) d\theta, & \alpha > 0, \\
\omega(z_1, z_2), & \alpha = 0, 
\end{cases}
\]
(20)
and
\[
n = \begin{cases} 
[\alpha] + 1, & \alpha \notin \mathbb{N}, \\
\alpha, & \alpha \in \mathbb{N}.
\end{cases}
\]

Proof: Substituting transformation \([17]\) into \([1]\) leads to
\[
D_t^\alpha \omega(\xi_1, \xi_2) = D_x^\beta \omega(\xi_1, \xi_2) + D_y^\beta \omega(\xi_1, \xi_2),
\]
(21)
For \(n-1 < \alpha < n\) and \(n-1 < \beta < n\), \(n = 1, 2, 3, \cdots\), then the similarity transformation \([21]\) becomes
\[
D_t^\alpha \omega(\xi_1, \xi_2) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} \omega(xs^{-\frac{\alpha}{n}}, ys^{-\frac{\alpha}{n}}) ds,
\]
(22)
\[
D_x^\beta \omega(\xi_1, \xi_2) = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial x^n} \int_0^x (x-s)^{n-\beta-1} \omega(st^{-\frac{\beta}{n}}, yt^{-\frac{\beta}{n}}) ds,
\]
(23)
\[
D_y^\beta \omega(\xi_1, \xi_2) = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial y^n} \int_0^y (y-s)^{n-\beta-1} \omega(xt^{-\frac{\beta}{n}}, st^{-\frac{\beta}{n}}) ds.
\]
(24)
By change of a variable \(\theta = \frac{t}{s}\) then \(ds = -\frac{t}{\theta^2} d\theta\), the equation \([22]\) can be written as
\[
D_t^\alpha \omega(\xi_1, \xi_2) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_1^\infty (t-\frac{\theta}{s})^{n-\alpha-1} \omega(xs^{-\frac{\alpha}{n}}, ys^{-\frac{\alpha}{n}}) t^\frac{\alpha}{\theta^2} d\theta
\]
\[
= \frac{\partial^n}{\partial t^n} \left[ t^{n-\alpha} \frac{1}{\Gamma(n-\alpha)} \int_1^\infty (\theta - 1)^{n-\alpha-1} \theta^{-(n-\alpha+1)} \omega(\xi_1 \theta^{\frac{\alpha}{n}}, \xi_2 \theta^{\frac{\alpha}{n}}) d\theta \right]
\]
\[
= \frac{\partial^n}{\partial t^n} \left[ t^{n-\alpha} \left( K^{1,n-\alpha}_{\frac{1}{n},\frac{1}{n}} \omega \right)(\xi_1, \xi_2) \right].
\]
By the chain rule of differentiation yields
\[
\frac{\partial^n}{\partial t^n} \left[ t^{n-\alpha} \left( K^{1,n-\alpha}_{\frac{1}{n},\frac{1}{n}} \omega \right)(\xi_1, \xi_2) \right] = \frac{\partial^{n-1}}{\partial t^{n-1}} \left[ \frac{\partial}{\partial t} \left( t^{n-\alpha} \left( K^{1,n-\alpha}_{\frac{1}{n},\frac{1}{n}} \omega \right)(\xi_1, \xi_2) \right) \right]
\]
\[
= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[ t^{n-\alpha-1} \frac{\beta}{\alpha} \xi_1 \frac{\partial}{\partial \xi_1} - \frac{\beta}{\alpha} \xi_2 \frac{\partial}{\partial \xi_2} \right] \times \left( K^{1,n-\alpha}_{\frac{1}{n},\frac{1}{n}} \omega \right)(\xi_1, \xi_2)
\]
\[
= t^{-\alpha} \sum_{j=0}^{n-1} \left( 1 - \alpha + j - \frac{\beta}{\alpha} \xi_1 \frac{\partial}{\partial \xi_1} - \frac{\beta}{\alpha} \xi_2 \frac{\partial}{\partial \xi_2} \right) \times \left( K^{1,n-\alpha}_{\frac{1}{n},\frac{1}{n}} \omega \right)(\xi_1, \xi_2)
\]
\[
= t^{-\alpha} \left( P^{1,n-\alpha}_{\frac{1}{n},\frac{1}{n}} \omega \right)(\xi_1, \xi_2),
\]
Figure 1: (a) The absolute error at $t = 0.5, x, y \in [0, 1]$ for $\alpha = 1, \beta = 2$ and $M = 4$, (b) The absolute error at $x = 0.5, t, y \in [0, 1]$ for $\alpha = 1, \beta = 2$ and $M = 4$, (c) The absolute error at $y = 0.5, t, x \in [0, 1]$ for $\alpha = 1, \beta = 2$ and $M = 4$.

Here $P_{1}^{1, n - \alpha}$ and $K_{1}^{1, \infty}$ denote the Erdelyi-Kober fractional derivative operators, \cite{19, 21}, by definitions \cite{19} and \cite{20}. Therefore

$$D_{t}^{\alpha} \omega(\xi_1, \xi_2) = t^{-\alpha} \left( P_{1, \infty}^{1, n - \alpha} \omega \right) (\xi_1, \xi_2). \tag{25}$$

With similar calculations we obtain

$$D_{x}^{\beta} \omega(\xi_1, \xi_2) = x^{-\beta} \left( P_{1}^{1, \infty, \beta} \omega \right) (\xi_1, \xi_2), \quad D_{y}^{\beta} \omega(\xi_1, \xi_2) = y^{-\beta} \left( P_{1, \infty}^{1, n - \beta} \omega \right) (\xi_1, \xi_2). \tag{26}$$

Putting \cite{23} and \cite{26} into \cite{21}, our governed equation reduces to \cite{18}.

\section{5 Spectral method for \cite{1} equation}

Now we apply the spectral method based on Bernstein operational matrices \cite{1, 2, 3, 31, 32} to solve the two-dimensional fractional heat equation \cite{1} with following conditions:

$$u(0, x, y) = 0, \quad u(x, 0, y) = 0, \quad u(t, x, 0) = 0,$$
$$\frac{\partial u(t, 0, y)}{\partial x} = 0, \quad \frac{\partial u(t, x, 0)}{\partial y} = 0.$$

In this method, $M$ is the number of members in Bernstein basis for any axes direction ($t, x$ and $y$). First, we change the problem \cite{1} to the Caputo sense by the following relation.

$$\frac{C_{0}^{\alpha} D_{t}^{\alpha} f(t)}{\alpha} = \alpha D_{t}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k - \alpha + 1)} t^{k - \alpha}, \tag{27}$$
so we have
\[
\frac{C^\alpha}{0} D_t^\alpha u(t, x, y) = \frac{C^{\beta}}{0} D_x^\beta u(t, x, y) + \frac{C^{\beta}}{0} D_y^\beta u(t, x, y) + f(t, x, y) .
\] (28)

Now, we apply the Bernstein operational matrices and propose the approximations as below:
\[
\frac{C^\alpha}{0} D_t^\alpha u(t, x, y) \approx \psi^T(t) \hat{K} \hat{\psi}(x, y) .
\] (29)

Applying Riemann–Liouville fractional integration of order \( \alpha \) with respect to \( t \) on (1) equation we have
\[
I^\alpha \frac{C^\alpha}{0} D_t^\alpha u(t, x, y) \approx \psi^T(t) P^\alpha \hat{K} \hat{\psi}(x, y) ,
\] (30)

where \( P^\alpha \) is fractional integral operational matrix based on the one dimensional Bernstein basis \( \psi(t) \). Then we get
\[
u(t, x, y) \approx \psi^T(t) P^\alpha \hat{K} \hat{\psi}(x, y)
\] (31)

Taking \( \beta \) order derivative of \( u(t, x, y) \) we get
\[
\frac{C^\alpha}{0} D_x^\beta u(t, x, y) \approx \psi^T(t) P^\alpha \hat{K} H^{(\beta,x)} \hat{\psi}(x, y) ,
\] (32)
and
\[
\frac{C^\alpha}{0} D_y^\beta u(t, x, y) \approx \psi^T(t) P^\alpha \hat{K} H^{(\beta,y)} \hat{\psi}(x, y) ,
\] (33)

where \( H^{(\beta,x)} \) and \( H^{(\beta,y)} \) are the Caputo fractional operational matrices with respect to \( x \) and \( y \), respectively based on the two dimensional Bernstein basis \( \hat{\psi}(x, y) \). Also, we can obtain
\[
f(t, x, y) \approx \psi^T(t) F \hat{\psi}(x, y) ,
\] (34)
Using (34) we get

$$\psi^T(t) K \hat{\psi}(x,y) = \psi^T(t) P^{\alpha T} K H^{(\beta,x)} \hat{\psi}(x,y) + \psi^T(t) P^{\alpha T} K H^{(\beta,y)} \hat{\psi}(x,y) + \psi^T(t) F \hat{\psi}(x,y),$$

which can be rewritten as

$$\psi^T(t) \left( K - P^{\alpha T} K H^{(\beta,x)} - P^{\alpha T} K H^{(\beta,y)} - F \right) \hat{\psi}(x,y) = 0.$$  

Hence it follows that

$$K - P^{\alpha T} K H^{(\beta,x)} - P^{\alpha T} K H^{(\beta,y)} - F = 0,$$

Using the value of $K$ in (37) we can get the approximate solution of the problem (1) from (34).

**Example 1** Consider the following two-dimensional fractional heat conduction equation

$$D^\alpha_t u(t,x,y) = D^\beta_x u(t,x,y) + D^\beta_y u(t,x,y) + 2tx^3y^3 - 6t^2xy^3 - 6t^2x^3y,$$

where $0 < \alpha \leq 1$, $1 < \beta \leq 2$ and $t,x,y \in [0,1]$. The exact solution for $\alpha = 1$ and $\beta = 2$ is

$$u(t,x,y) = t^2x^3y^3.$$

In Figure 1, we calculate the absolute error at $t = 0.5$, $x,y \in [0,1]$ and then the approximate solutions are in good agreement with exact solutions. Also in Figure 2 we depict the approximate solutions for the fractional values of $\alpha$ and $\beta = 2$ at $t = 0.5$, $x,y \in [0,1]$ and $x = 0.5$, $t \in [0,1]$ and $y = 0.5$, $x \in [0,1]$, respectively. Therefore, as $\alpha \to 1$ the approximate solutions equal to the exact solutions as expected.

It is notable that the used PC is Intel(R) Core(TM) i7-7700K CPU 4.20GHz. Also we apply version 13 of the Mathematica software for obtaining the results.

**Data Availability**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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