Generalized Hermite processes, discrete chaos and limit theorems

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Generalized Hermite processes, discrete chaos and limit theorems

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Abstract

We introduce a broad class of self-similar processes \( \{ Z(t), t \geq 0 \} \) called generalized Hermite processes. They have stationary increments, are defined on a Wiener chaos with Hurst index \( H \in (1/2, 1) \), and include Hermite processes as a special case. They are defined through a homogeneous kernel \( g \), called “generalized Hermite kernel”, which replaces the product of power functions in the definition of Hermite processes. The generalized Hermite kernels \( g \) can also be used to generate long-range dependent stationary sequences forming a discrete chaos process \( \{ X(n) \} \). In addition, we consider a fractionally-filtered version \( Z^{\beta}(t) \) of \( Z(t) \), which allows \( H \in (0, 1/2) \). Corresponding non-central limit theorems are established. We also give a multivariate limit theorem which mixes central and non-central limit theorems.

1 Introduction

A stochastic process \( \{ X(t), t \geq 0 \} \) with finite variance taking values in \( \mathbb{R} \) is said to be self-similar if there is a constant called Hurst coefficient \( H > 0 \), such that for any scaling factor \( a > 0 \), \( X(at) \overset{f.d.d.}{\sim} a^H X(t) \), where \( f.d.d. \) means equality in finite-dimensional distributions. If a self-similar process \( \{ X(t), t \geq 0 \} \) has also stationary increments, namely, if for any \( h \geq 0 \), \( \{ Y(t) := X(t + h) - X(t), t \geq 0 \} \) is a stationary process, then we say that \( \{ X(t), t \geq 0 \} \) is \( H \)-sssi. The natural range of \( H \) is \((0, 1)\), which implies \( \mathbb{E}X(t) = 0 \) for all \( t \geq 0 \). We refer the reader to Chapter 3 of Embrechts and Maejima [7] for details.

The fundamental theorem of Lamperti (Lamperti [12]) states that \( H \)-sssi processes are the only possible limit laws of normalized partial sum of stationary sequences, that is, if

\[
\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \overset{f.d.d.}{\rightarrow} Y(t)
\]

and \( A(N) \rightarrow \infty \) as \( N \rightarrow \infty \), where \( \{ X(n) \} \) is stationary, then \( \{ Y(t), t \geq 0 \} \) has to be \( H \)-sssi for some \( H > 0 \), and \( A(N) \) has to be regularly varying with exponent \( H \). The notation \( f.d.d. \) stands for convergence in finite-dimensional distributions (f.d.d.).

The best known example of Lamperti’s fundamental theorem is when \( \{ X(n) \} \) is i.i.d. or a short-range dependent (SRD) sequence, then the limit \( Y(t) \) is Brownian motion which is \( \frac{1}{2} \)-sssi. If \( \{ X(n) \} \) has long-range dependence (LRD), the limit \( Y(t) \) is often \( H \)-sssi with \( H > 1/2 \). The most typical \( H \)-sssi process is fractional Brownian motion \( B_H(t) \), but there are also non-Gaussian processes, e.g., Hermite processes (Taqqu [29], Dobrushin and Major [6]). The Hermite process of order 1 is fractional Brownian motion, but when the order is greater than or equal to 2, its law belongs to higher-order Wiener chaos (see, e.g., Peccati and Taqqu [23]) and is thus non-Gaussian.

The Hermite processes have attracted a lot of attention. The first-order Hermite process, namely fractional Brownian motion, has been studied intensively by numerous researchers since its popularization by Mandelbrot and Van Ness [19], and we refer the reader to a recent monograph Nourdin [21] and the references

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of the Wiener-Itô integral.

\[ I_k(x_1, \ldots, x_k) := \int_0^t g(s - x_1, \ldots, s - x_k)1_{\{s > x_1, \ldots, s > x_k\}} \, ds, \]

with \( g \) being some suitable homogeneous function on \( \mathbb{R}^k_+ \) called \textit{generalized Hermite kernel}. For example,

\[ g(x_1, \ldots, x_k) = \max \left( \frac{x_1 \ldots x_k}{x_1^{-\alpha} + \ldots + x_k^{-\alpha}}, x_1^{\alpha/k} \ldots x_k^{\alpha/k} \right), \quad x \in \mathbb{R}^k_+, \quad \alpha \in (-k/2 - 1/2, -k/2). \quad (1) \]

We call the corresponding H-sssi process \( Z(t) \) a \textit{generalized Hermite process}. We then construct a class of \textit{discrete chaos processes} as

\[ X(n) = \sum_{(i_1, \ldots, i_k) \in \mathbb{Z}^+_k} g(i_1, \ldots, i_k)\epsilon_{n-i_1} \ldots \epsilon_{n-i_k}, \]

where \( \{\epsilon_i\} \) are i.i.d. noise, and the prime ‘ exclusion of the diagonals \( i_p = i_q, p \neq q \). We show that the normalized partial sum of \( X(n) \) converges to the generalized Hermite process \( Z(t) \) defined by the same \( g \). We also obtain processes with \( H \in (0, 1/2) \) by applying an additional fractional filter. The increments of these processes have negative dependence. Finally, we state a multivariate limit theorem which mixes central and non-central limits, including cases where there is an additional fractional filter.

The paper is organized as follows. In Section 2, we review the Hermite processes. In Section 3, the generalized Hermite processes are introduced. In Section 4, we consider the discrete chaos processes. In Section 5, we prove a hypercontractivity relation for infinite discrete chaos. In Section 6, we show that the discrete chaos processes converge weakly to the generalized Hermite processes, including situations where \( H < 1/2 \).

## 2 Brief review of Hermite processes

The Hermite processes are defined with the aid of a multiple stochastic integral called \textit{Wiener-Itô integral}. We give here a brief introduction to this integral. For the proofs of our statements and additional details, we refer the reader to Major \cite{17} and Nualart \cite{22}, for example. The Wiener-Itô integral is defined for any \( f \in L^2(\mathbb{R}^k) \) as

\[ I_k(f) := \int_{\mathbb{R}^k} f(x_1, \ldots, x_k)W(dx_1) \ldots W(dx_k), \]

where \( W(\cdot) \) is Brownian motion viewed as a random integrator, and the prime ‘ indicates that we don’t integrate on the diagonals \( x_p = x_q, p \neq q \). The integral \( I_k(\cdot) \) can be defined first for elementary functions \( f = \sum_{n=1}^{\infty} a_n 1_{A_n} \), where \( A_n \)'s are off-diagonal cubes in \( \mathbb{R}^k \). This results in a linear combination of \( k \)-fold product of independent centered Gaussian random variables. One then extends this in the usual way to any \( f \in L^2(\mathbb{R}^k) \). The random variable \( I_k(f) \) is also said to belong to the \( k \)-th Wiener chaos \( \mathcal{H}_k \), which is the Hilbert space generated by \( I_k(f) \) when \( f \) varies in \( L^2(\mathbb{R}^k) \). Here we state the following important properties of the Wiener-Itô integral \( I_k(\cdot) \):
1. $I_k(\cdot)$ is a linear mapping from $L^2(\mathbb{R}^k)$ to $L^2(\Omega)$.

2. If $f_\sigma(x_1, \ldots, x_k) := f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$, where $\sigma$ is any permutation of $(1, \ldots, k)$, then $I_k(f_\sigma) = I_k(f)$.

   It hence suffices to focus on symmetric integrands (symmetrize $f$ as
   \[
   \hat{f}(x_1, \ldots, x_k) := \frac{1}{k!} \sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(k)})
   \]
   when necessary).

3. Suppose $f \in L^2(\mathbb{R}^p)$ and $g \in L^2(\mathbb{R}^q)$, and both are symmetric. Then
   \[
   \mathbb{E}I_p(f)I_q(g) = \begin{cases} 
   k! \int_{\mathbb{R}^k} f(x)g(x)dx, & \text{if } p = q = k; \\
   0, & \text{if } p \neq q.
   \end{cases}
   \]

   If $f \in L^2(\mathbb{R}^k)$ is not symmetric, one gets
   \[
   \mathbb{E}I_p(f)^2 = \|f\|_{L^2(\mathbb{R}^k)}^2 \leq k!\|f\|_{L^2(\mathbb{R}^k)}^2.
   \]

An Hermite process of order $k$ is an $H$-sssi process with $1/2 < H < 1$, which is represented by the following Wiener-Itô integral:

\[
Z_H^{(k)}(t) = a_{k,d} \int_{\mathbb{R}^k} t \prod_{p=1}^k \int_0^t (s - x_j)^{d-1} ds \ W(dx_1) \ldots W(dx_k),
\]

where $a_{k,d}$ is some positive constant that makes $\text{Var}(Z_H^{(k)}(1)) = 1$. We call (2) the time-domain representation. It is known that Hermite processes admit other representations in terms of Wiener-Itô integrals (see Pipiras and Taqqu [24], among which we note the spectral-domain representation:

\[
Z_H^{(k)}(t) = b_{k,d} \int_{\mathbb{R}^k} \frac{e^{i(u_1 + \ldots + u_k)t - 1}}{i(u_1 + \ldots + u_k)} |u_1|^{-d} \ldots |u_k|^{-d} \hat{W}(du_1) \ldots \hat{W}(du_k),
\]

where $\hat{W}(\cdot)$ is a complex-valued Brownian motion (with real and imaginary parts being independent) viewed as a random integrator (see, e.g., p.22 of Embrechts and Maejima [7]), the double prime $''$ indicates the exclusion of the hyper-diagonals $u_p = \pm u_q$, $p \neq q$, and $b_{k,d}$ is some positive constant that makes $\text{Var}(Z_H^{(k)}(1)) = 1$. In the sequel, we use $\hat{I}_k(\cdot)$ to denote a $k$-tuple Wiener-Itô integral with respect to the complex-valued Brownian motion $\hat{W}(\cdot)$. In fact, the kernel inside the Wiener-Itô integral in (3) is the Fourier transform of the kernel in (2) up to some unimportant factors. The connection between the time-domain and spectral-domain representation is through the following general result:

**Proposition 2.1.** (Proposition 9.3.1 of Peccati and Taqqu [24]) Let $g_j(\mathbf{x})$ be a real-valued function in $L^2(\mathbb{R}^k)$, $j = 1, \ldots, J$. Let

\[
\hat{g}_j(\mathbf{u}) = \int_{\mathbb{R}^k} g_j(\mathbf{x})e^{i(\mathbf{u}, \mathbf{x})}d\mathbf{x}
\]

be the Fourier transform. Then

\[
\left( I_{k_1}(g_1), \ldots, I_{k_J}(g_J) \right) \overset{d}{=} \left( (2\pi)^{-k_1/2}\hat{I}_{k_1}(\hat{g}_1w_1^{\otimes k_1}), \ldots, (2\pi)^{-k_J/2}\hat{I}_{k_J}(\hat{g}_Jw_J^{\otimes k_J}) \right),
\]

for any $|w_j(u)| = 1$ and $w_j(u) = w_j(-u)$, $j = 1, \ldots, J$, where $w_1^{\otimes k_1}u_1 \ldots u_k := w(u_1) \ldots w(u_k)$.

The factors $w_j^{\otimes k_j}$, $j = 1, \ldots, J$ do not change the distributions due to the change-of-variable formula of Wiener-Itô integrals (see, e.g., Proposition 4.2 of Dobrushin [3]).
The Hermite process of order \( k = 1 \) is fractional Brownian motion \( B_H(t) \), and that of order \( k = 2 \) is called Rosenblatt process whose marginal distribution was discovered by Rosenblatt [26]. We note that all \( H \)-sssi processes with unit variance at \( t = 1 \) have covariance

\[
R(s, t) = \frac{1}{2} (s^{2H} + t^{2H} - |s - t|^{2H}),
\]
as is the case for Hermite process of arbitrary order.

Hermite processes arise as limits of partial sum of nonlinear LRD sequences. In the following two theorems, \( A(N) \) is a normalization factor guaranteeing unit asymptotic variance for the partial sum process at \( t = 1 \). We use \( \Rightarrow \) to denote weak convergence in the Skorohod space \( D[0, 1] \) with the uniform metric.

**Theorem 2.2.** (Dobrushin and Major [6]; Taqqu [29].) Suppose that \( \{X(n)\} \) is a Gaussian stationary sequence with autocovariance

\[
\gamma(n) \sim cn^{2d-1}
\]

as \( n \to \infty \) for some constant \( c > 0 \) and

\[
1/2(1 - 1/k) < d < 1/2.
\]

Let

\[
H_k(x) := (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}
\]

be the \( k \)-th Hermite polynomial, \( k \geq 1 \). Then

\[
\frac{1}{A(N)} \sum_{n=1}^{[Nt]} H_k(X(n)) \Rightarrow Z_d^{(k)}(t).
\]

**Theorem 2.3.** (Surgailis [28], see also Giraitis et al. [8] Chapter 4.8.) Let \( \{\epsilon_i\} \) be an i.i.d. sequence with mean 0 variance 1,

\[
a_n \sim cn^{d-1}
\]

as \( n \to \infty \) for some constant \( c > 0 \) and

\[
1/2(1 - 1/k) < d < 1/2.
\]

Let

\[
X(n) = \sum_{0 < i_1, \ldots, i_k < \infty} a_{i_1} \cdots a_{i_k} \epsilon_{n-i_1} \cdots \epsilon_{n-i_k},
\]

where the prime \( ' \) indicates that one doesn’t sum on the diagonals \( i_p = i_q p \neq q \). Then

\[
\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \Rightarrow Z_d^{(k)}(t).
\]

**Remark 2.4.** The Hermite polynomial in Theorem 2.2 can be replaced by a general function \( G(\cdot) \) such that \( \mathbb{E}G(X_n) = 0, \mathbb{E}G(X_n)^2 < \infty \), due to the orthogonal expansion of \( G(x) \) with respect to Hermite polynomials, and the fact that only the leading term in the expansion contributes to the limit law. Similarly, the off-diagonal multilinear polynomial-form process \( X(n) \) in Theorem 2.3 can be replaced by a suitable function of the linear process \( Y(n) := \sum_{i \geq 1} a_i \epsilon_{n-i} \). In both of the above theorems \( \overset{f.d.d.}{\to} \) can be strengthened to weak convergence \( \Rightarrow \) (Proposition 4.4.2 of Giraitis et al. [8]).

**Remark 2.5.** The range of the parameter \( d \) in both of the theorems guarantees that the summand is LRD in the sense that the autocovariance decays as a power function with an exponent in the range \((-1, 0)\). We note also that the constant \( c > 0 \) appearing in both theorems can be replaced by a slowly varying function.
3 Generalized Hermite Processes

We introduce first some notation, which will be used throughout. \( \mathbb{R}_+ = (0, \infty), \mathbb{Z}_+ = \{1, 2, \ldots\} \). \( x = (x_1, \ldots, x_k) \in \mathbb{R}^k, i = (i_1, \ldots, i_k) \in \mathbb{Z}^k, 0 = (0, \ldots, 0), 1 = (1, \ldots, 1) \). For any real number \( x \), \( \lfloor x \rfloor = \sup\{n \in \mathbb{Z}, n \leq x\} \), and \( \lceil x \rceil = \lfloor x \rfloor + 1 \). We write \( x > y \) (or \( \geq \)) if \( x_j > y_j \) (or \( \geq \)), \( j = 1, \ldots, k \).

Condition 3 guarantees stationary increments. Self-similarity and stationary increments can be rigorously checked by the change-of-variable formula of Wiener-Itô integrals (Proposition 4.2 of Dobrushin [5]).

The Hermite process, for instance, which is defined in (2) can be obtained following the scheme of Proposition 3.1 by letting

\[ h_{\lambda}(x) = \lambda^{H+k\beta/2}h_t(\lambda^\beta x) \quad \text{for a.e.} \quad x \in \mathbb{R}^k \quad \text{and all} \ t > 0; \]

\[ h_{t+s}(x) - h_t(x) = h_s(x+ta) \quad \text{for a.e.} \quad x \in \mathbb{R}^k \quad \text{and all} \ t > 0. \]

Then \( Z(t) := I_k(h_t) \) is an \( H \)-sssi process.

Condition 1 guarantees that the Wiener-Itô integral is well defined. Condition 2 yields self-similarity, where the term \( k\beta/2 \) in the exponent compensates for the scaling of the \( k \)-tuple Brownian motion integrators. Condition 3 guarantees stationary increments. Self-similarity and stationary increments can be rigorously checked by the change-of-variable formula of Wiener-Itô integrals (Proposition 4.2 of Dobrushin [5]).

The Hermite process, for instance, which is defined in 2 can be obtained following the scheme of Proposition 3.1 by letting

\[ h_t(x) = \int_0^t g(s1-x)1_{\{s1>x\}}(s)ds, \]

and

\[ g(x) = \prod_{j=1}^k x_j^{d-1}, \ x_j > 0. \]

(4)

It is easy to check that the conditions on \( h_t \) in Proposition 3.1 are all satisfied with \( \beta = -1 \) in condition 2 and \( H = kd - k/2 + 1 \). One can also check that the integrand in the spectral-domain representation in 3 also satisfies the first two conditions in Proposition 3.1 but with \( \beta = 1 \) in Condition 2 instead. The third condition, however, must be replaced by \( \hat{h}_{t+s}(u) - \hat{h}_t(u) = e^{-it\langle a, u \rangle} \hat{h}_s(u) \) due to the Fourier-transform relation.

Our first goal is to extend the kernel \( g \) in 4 to some general class of functions. To do so, we define the following class of functions on \( \mathbb{R}_+^k \), which first appeared in Mori and Oodaira [20] to study the law of iterated logarithm:

**Definition 3.2.** We say that a nonzero measurable function \( g(x) \) defined on \( \mathbb{R}_+^k \) is a *generalized Hermite kernel*, if it satisfies

A. \( g(\lambda x) = \lambda^\alpha g(x), \ \forall \lambda > 0 \), where \( \alpha \in (-\frac{k+1}{2}, -\frac{k}{2}) \);

B. \( \int_{\mathbb{R}_+^k} g(x)g(1+x)d\mathbf{x} < \infty. \)

One can check that the Hermite kernel \( g \) in 4 satisfies the above assumptions.
Remark 3.3. The range of \( \alpha \) in Condition \( \text{A} \) is non-overlapping for different \( k \), and extends from \(-1/2\) to \(-\infty\) with all the multiples of \(-1/2\) excluded.

Remark 3.4. Suppose \( g_1 \) and \( g_2 \) are generalized Hermite kernels having order \( k_1 \), \( k_2 \) and homogeneity exponent \( \alpha_1 \), \( \alpha_2 \) respectively. If in addition, \( \alpha_1 + \alpha_2 > -(k_1 + k_2 + 1)/2 \), then \( g_1 \otimes g_2(x_1, x_2) := g_1(x_1)g_2(x_2) \) is a generalized Hermite kernel having order \( k_1 + k_2 \) and homogeneity exponent \( \alpha_1 + \alpha_2 \).

Theorem 3.5. Let \( g(x) \) be a generalized Hermite kernel defined in Definition 3.2. Then

\[
h_t(x) = \int_0^t g(s1 - x)1_{\{s1 > x\}} ds
\]

is well-defined in \( L^2(\mathbb{R}^k) \), \( \forall t > 0 \), and the process defined by \( Z_t := I_k(h_t) \) is an \( H \)-sssi process with

\[
H = \alpha + k/2 + 1 \in (1/2, 1).
\]

Proof. To check that \( h_t \in L^2(\mathbb{R}^k) \), we write

\[
\int_{\mathbb{R}^k} h_t(x)^2 dx = \int_{\mathbb{R}^k} dx \int_0^t ds_1 ds_2 g(s_1 x - x)g(s_2 x - x)1_{\{s_1 > x\}} 1_{\{s_2 > x\}}.
\]

We want to change the integration order by integrating on \( x \) first. By Fubini, we need to check that the absolute value of the integrand is integrable, that is,

\[
2 \int_0^t ds_1 \int_0^{s_1} ds_2 \int_{\mathbb{R}^k} dx \left| g(s_1 x - x)g(s_2 x - x)1_{\{s_1 > x\}} \right| (\text{by symmetry of } s_1 < s_2 \text{ and } s_1 > s_2)
\]

\[
= 2 \int_0^t ds \int_0^{s} du \int_{\mathbb{R}^k} dw \left| g(w)g(u1 + w) \right| (s = s_1, \ u = s_2 - s_1, \ w = s_1 x - x)
\]

\[
= 2 \int_0^t ds \int_0^{s} du \int_{\mathbb{R}^k} u^k dy \left| g(uy)g(u + uy) \right| (\text{by Condition } \text{A} \text{ of Definition } 3.2),
\]

where the last expression is finite by \( 2\alpha + k + 1 > 0 \) and Condition \( \text{B} \). Hence by the same calculation, but without absolute values,

\[
\int_{\mathbb{R}^k} h_t(x)^2 dx = 2 \int_0^t ds \int_0^{s} u^{2\alpha + k} du \int_{\mathbb{R}^k} dy \left| g(y)g(1 + y) \right|
\]

\[
= \frac{t^{2\alpha + k + 2}}{(\alpha + k/2 + 1)(2\alpha + k + 2)} \int_{\mathbb{R}^k} dy \left| g(y)g(1 + y) \right|.
\]

To check self-similarity (Condition \( \text{B} \) of Proposition 3.1 with \( \beta = -1 \)),

\[
h_{\lambda t}(x) = \int_0^{\lambda t} g(s1 - x)1_{\{s1 > x\}} ds = \lambda^{\alpha + 1} \int_0^t g(r1 - \lambda^{-1} x)1_{\{r1 > \lambda^{-1} x\}} dr = \lambda^{\alpha + 1} h_t(\lambda^{-1} x),
\]

where the second equality uses Condition \( \text{A} \) of Definition 3.2. The Hurst coefficient \( H \) of \( I_k(h_t) \) is obtained from \( \alpha + 1 = H - k/2 \). To check stationary increments (Condition \( \text{B} \) of Proposition 3.1), for any \( t, r > 0 \),

\[
h_{t+r}(x) - h_t(x) = \int_t^{t+r} g(s1 - x)1_{\{s1 > x\}} ds = \int_0^r g(u1 + t1 - x)1_{\{u1 + t1 > x\}} du = h_r(x - t1).
\]
Remark 3.6. As a byproduct of the above proof, we obtain that under the conditions of Definition 3.2 one has \( \int_0^t |g(s1 - x)|1_{\{s1 > x\}}(s)ds < \infty \) for a.e. \( x \in \mathbb{R}^k \), and
\[
\mathbb{E}Z(t)^2(k!)^{-1} \leq \|h_t\|^2_{L^2(\mathbb{R}^k)} = \frac{t^{2H}}{H(2H - 1)}C_g,
\]
where \( C_g := \int_{\mathbb{R}^k} g(x)g(1 + x)dx \), and the first inequality becomes equality if \( g \) and hence \( h_t \) is symmetric.
Note that \( C_g > 0 \) must hold, otherwise \( h_t(x) = \int_0^t g(s1 - x)1_{\{s1 > x\}}ds = 0 \) for a.e. \( x \in \mathbb{R}^k \) and any \( t > 0 \), which implies that \( g \) is zero a.e., and thus contradicts the assumption.
Remark 3.7. Since \( \forall f \in L^2(\mathbb{R}^k) \), \( I_k(f) = I_k(\hat{f}) \), where \( \hat{f} \) is the symmetrization of \( f \) (Nualart [22] p.9), it suffices to focus on symmetric generalized Hermite kernels \( g \) only. In the sequel, we will not always assume that \( g \) is symmetric for convenience, while being aware that \( g \) can always be symmetrized.

Definition 3.8. The process
\[
Z(t) := \int_{\mathbb{R}^k} \int_0^t g(s - x_1, \ldots, s - x_k)1_{\{s > s_1, \ldots, s > s_k\}}ds W(dx_1) \ldots W(dx_k)
\]
which we simply write \( Z(t) = I_k(h_t) \) with \( h_t(x) = \int_0^t g(s1 - x)1_{\{s1 > x\}}ds \), where \( g \) is a generalized Hermite kernel defined in Definition 3.2, is called a \textit{generalized Hermite process}.

Remark 3.9. It is known (see, e.g., Janson [10] Theorem 6.12) that if a random variable \( X \) belongs to the \( k \)-th Wiener chaos, then there \( \exists a, b, t_0 > 0 \) such that for \( t \geq t_0 \),
\[
\exp(-at^{2/k}) \leq P(|X| > t) \leq \exp(-bt^{2/k}).
\]
This shows that the generalized Hermite processes of different orders must necessarily have different laws, and the higher the order gets, the heavier the tail of the marginal distribution becomes, while they all have moments of any order.

The generalized Hermite process \( Z(t) \) admits a continuous version, which follows from the following general result:

Proposition 3.10. If \( \{Z(t), t \geq 0\} \) is an \( H \)-sssi process whose marginal distribution satisfies \( \mathbb{E}|Z(1)|^\gamma < \infty \) for some \( \gamma > H^{-1} \), then \( Z(t) \) admits a continuous version.

Proof. Using stationary increments and self-similarity, we have
\[
\mathbb{E}|Z(t) - Z(s)|^\gamma = \mathbb{E}|Z(t - s)|^\gamma = |t - s|^{H\gamma} \mathbb{E}|Z(1)|^\gamma.
\]
Since \( H\gamma > 1 \), Kolmogorov’s criterion applies.

Remark 3.11. In Mori and Oodaira [20], the following laws of iterated logarithm are obtained for the generalized Hermite process \( Z(t) \):
\[
\limsup_{n \to \infty} \frac{Z(n)}{n^H(2 \log_2 n)^{k/2}} = l_1, \quad \liminf_{n \to \infty} \frac{Z(n)}{n^H(2 \log_2 n)^{k/2}} = l_2 \ \text{a.s.,}
\]
where \( l_1 = \sup K_h \) and \( l_2 = \inf K_h \) with the set
\[
K_h := \left\{ \int_{\mathbb{R}^k} h_1(x)\xi(x_1) \ldots \xi(x_k)dx : \|\xi\|_{L^2(\mathbb{R})} \leq 1 \right\}.
\]
In the spirit of [3], we can consider the spectral-domain representation of the generalized Hermite processes. Since \( h_t(x) = \int_0^t g(s1 - x)1_{\{s1 > x\}}(s)ds \in L^2(\mathbb{R}) \), it always has an \( L^2 \)-sense Fourier transform \( \hat{h}_t \). We give an explicit way to calculate \( \hat{h}_t \) when \( g \) is integrable in a neighborhood of the origin. Note that since \( g \) is homogeneous, it suffices to assume integrability on the unit cube \((0, 1]^k\).
Proposition 3.12. Suppose that
\[
\int_{[0,1]^k} |g(x)| < \infty.
\]
Let \( g_n(x) = g(x)1_{[0,n]^k}(x) \), and \( \hat{g}_n(u) := \int_{\mathbb{R}^k} g_n(x)e^{i(u,x)}dx \) be its Fourier transform. Set
\[
\hat{h}_{t,n} := \frac{e^{it(u,1)} - 1}{i(u,1)} \hat{g}_n(-u),
\]
then \( \hat{h}_{t,n} \) converges in \( L^2(\mathbb{R}^k) \) to \( \hat{h}_t \). Moreover, there is a function \( \hat{g}(u) \) defined for a.e. \( u \in \mathbb{R}^k \), such that,
\[
\hat{h}_t(u) = \frac{e^{it(u,1)} - 1}{i(u,1)}\hat{g}(-u).
\]

Proof. Due to (6), the Fourier transform of \( g_n \) is well-defined pointwise as
\[
\hat{g}_n(u) = \int_{\mathbb{R}^k} g(x)1_{[0,n]^k}(x)e^{i(u,x)}dx.
\]
Let
\[
h_{t,n}(x) = \int_0^t g_n(s1 - x)1_{\{s1 \geq x\}}(s)ds = \int_0^t g(s1 - x)1_{\{s1 \leq x\}}(s)ds.
\]
Note that \( |g_n(x)| \leq |g(x)| \), by the proof of Theorem 3.5, \( h_{t,n}(x) \) is \( L^2(\mathbb{R}^k) \), and by the Dominated Convergence Theorem, \( h_{t,n} \) converges to \( h_t \) pointwise as \( n \to \infty \). Since \( |h_{t,n}| \leq \int_0^1 |g(s1 - x)|1_{\{s1 \geq x\}}(s)ds \), by the Dominated Convergence Theorem in \( L^2(\mathbb{R}^k) \), \( h_{t,n} \) converges to \( h_t \) in \( L^2(\mathbb{R}^k) \). By Plancherel’s isometry, \( \hat{h}_{t,n} \), the Fourier transform of \( h_{t,n} \), converges in \( L^2(\mathbb{R}^k) \) to \( \hat{h}_t \). But
\[
\hat{h}_{t,n}(u) := \int_{\mathbb{R}^k} \int_0^t g(s1 - x)1_{\{s1 \leq x\}}(s)ds e^{i(u,x)}dx
\]
\[
= \int_0^t \int_{\mathbb{R}^k} e^{i(u,s1)}g(s1 - x)e^{i(-u,s1 - x)}1_{\{0 < s1 \leq x \leq 1\}}(x)dxds
\]
\[
= \int_0^t e^{i(u,s1)}ds \int_{\mathbb{R}^k} g(y)1_{\{0 < y \leq 1\}} e^{i(-u,y)}dy
\]
\[
= \frac{e^{it(u,1)} - 1}{i(u,1)} \hat{g}_n(-u),
\]
where the change of integration order is valid because by (6).
\[
\int_0^t ds \int_{\mathbb{R}^k} dx |g(s1 - x)|1_{\{s1 \leq x \}} = \int_0^t ds \int_{\mathbb{R}^k} |g(y)|1_{\{0 < y \leq 1\}}dy < \infty.
\]

We now prove (7). The fact that \( \hat{h}_{t,n} \) converges in \( L^2(\mathbb{R}^k) \) to \( \hat{h}_t \) implies that \( \hat{g}_n \) is a Cauchy sequence in \( L^2(\mathbb{R}^k, \mu_t) \), where \( \mu_t \) is the measure given by
\[
\mu_t(A) = \int_A \left| \frac{e^{it(u,1)} - 1}{i(u,1)} \right|^2 du = \int_A \frac{2 - 2 \cos(t(u,1))}{(u,1)^2} du
\]
for any measurable set \( A \subset \mathbb{R}^k \). Hence there exists a \( \hat{g} \in L^2(\mathbb{R}^k, \mu_t) \) which is the limit of \( \hat{g}_n \) in \( L^2(\mathbb{R}^k, \mu_t) \). Since \( \mu_t \) is equivalent to Lebesgue measure, \( \hat{g} \) is determined a.e. on \( \mathbb{R}^k \), and there exists a subsequence of \( \hat{g}_n \) that converges a.e. to \( \hat{g} \). So (7) holds.

\[
\square
\]

Remark 3.13. Note that \( \hat{g} \) is not the \( L^2 \)-sense Fourier transform of \( g1_{\mathbb{R}^k} \), since \( g \notin L^2(\mathbb{R}^k) \). One can, however, evaluate the limit of \( \hat{g}_n \) pointwise as an improper integral, as is done in the Hermite kernel case (4) (see Lemma 6.2 of Taqqu [29]).
Definition 3.16. According to Lemma 7.1 of Mori and Oodaira [20], Class (B) forms a dense subclass of the class of generalized Hermite kernels in the sense that for any generalized Hermite kernel $g$ and any $\epsilon > 0$, there exists $g_\epsilon$ in Class (B), such that $\|h - h_\epsilon\|_{L^2(\mathbb{R}^k)} < \epsilon$, where $h(x) = \int_0^1 g(s1 - x)1_{\{s1 > x\}} ds$ and $h_\epsilon(x) = \int_0^1 g_\epsilon(s1 - x)1_{\{s1 > x\}} ds$.

Note that Class (B) does not include the original Hermite kernel in (4). We now introduce a class of generalized Hermite kernels, called Class (L), which includes generalized Hermite kernels of the form:

$$g(x) = \prod_{j=1}^k x_j^{\gamma_j}, \quad (10)$$

where each $-1 < \gamma_j < -1/2$ and $-k/2 - 1/2 < \sum_j \gamma_j < -k/2$. These particular kernels with $k = 2$ has been considered in Maejima and Tudor [14] where the resulting process is called non-symmetric Rosenblatt process. We hence call the kernel in (10) a non-symmetric Hermite kernel. Note that despite the name, one can always symmetrize these kernels. Class (L) will appear in the discrete chaos processes and the limit theorems considered later.

Definition 3.18. We say that a generalized Hermite kernel $g$ on $\mathbb{R}^k_+$ having homogeneity exponent $\alpha$ is of Class (L) (L stands for “limit” as in “limit theorems”), if

1. $g$ is continuous a.e. on $\mathbb{R}^k_+$;
2. $|g(x)| \leq g^*(x)$ a.e. $x \in \mathbb{R}^k$, where $g^*$ is a finite linear combination of non-symmetric Hermite kernels:

$$\prod_{j=1}^k x_j^{\gamma_j},$$

where $\gamma_j \in (-1, -1/2)$, $j = 1, \ldots, k$, and $\sum_{j=1}^k \gamma_j = \alpha \in (-k/2 - 1/2, -k/2)$. For example, $g^*(x)$ could be $x_1^{-3/4} x_2^{-5/8} + x_1^{-9/16} x_2^{-13/16}$ if $k = 2$. In this case, $\alpha = -11/8$. 

The limit $\hat{g}$ in (4) is also a homogeneous function:

Proposition 3.14. The function $\hat{g}$ defined in Remark 3.12 satisfies for any $\lambda > 0$, $g(\lambda u) = \lambda^{-\alpha-k} \hat{g}(u)$ for a.e. $u \in \mathbb{R}^k$.

Proof. Following (3) and using Condition A of Definition 3.2 and noting that $(\lambda u, x) = (u, \lambda x)$, we have

$$\hat{g}_n(\lambda u) = \lambda^{-\alpha} \int_{\mathbb{R}^k} g(\lambda x)1_{(0,n)^k}(x)e^{i(u, \lambda x)} dx = \lambda^{-\alpha-k} \int_{\mathbb{R}^k} g(y)1_{(0,\lambda n)^k}(y)e^{i(u,y)} dy = \lambda^{-\alpha-k} \hat{g}_n(u).$$

Then let $n \to \infty$ through a subsequence so that both sides converge a.e.. \qed

Remark 3.17. The spectral-domain representation of the Hermite process in (3) is indeed obtained as $\hat{g}(u) = c \prod_{j=1}^k |u_k|^{-\alpha} w(u)$ for some constant $c > 0$, where the function $w(u) = \prod_{j=1}^k e^{-\alpha(u_j)^{1/2}}$ can be omitted (see Proposition 2.1).

3.2 Special kernels and examples

We introduce now some subclasses of the generalized Hermite kernels $g$ defined in Definition 3.2, which will be of interest later when dealing with limit theorems. Note that the kernel $g$ is determined by its value on the positive unit sphere $S^k_+ := \{x \in \mathbb{R}^k_+, |x| = 1\}$. Because it is homogeneous, $g$ is always radially continuous and it is decreasing since $\alpha < 0$ in Definition 3.2. Thus assuming that $g$ is continuous on $S^k_+$, a.e. (with respect to the uniform measure on the $S^k_+$ is the same as assuming $g$ is continuous a.e. on $\mathbb{R}^k_+$.

Definition 3.16. We say that a generalized Hermite kernel $g$ is of Class (B) (B stands for “boundedness”), if on $S^k_+$, it is continuous a.e. and bounded. Consequently,

$$|g(x)| \leq ||x||^\alpha g(x/||x||) \leq c||x||^\alpha$$

for some $c > 0$.

Remark 3.17. According to Lemma 7.1 of Mori and Oodaira [20], Class (B) forms a dense subclass of the class of generalized Hermite kernels in the sense that for any generalized Hermite kernel $g$ and any $\epsilon > 0$, there exists $g_\epsilon$ in Class (B), such that $\|h - h_\epsilon\|_{L^2(\mathbb{R}^k)} < \epsilon$, where $h(x) = \int_0^1 g(s1 - x)1_{\{s1 > x\}} ds$ and $h_\epsilon(x) = \int_0^1 g_\epsilon(s1 - x)1_{\{s1 > x\}} ds$.

Note that Class (B) does not include the original Hermite kernel in (4). We now introduce a class of generalized Hermite kernels, called Class (L), which includes generalized Hermite kernels of the form:

$$g(x) = \prod_{j=1}^k x_j^{\gamma_j},$$

where each $-1 < \gamma_j < -1/2$ and $-k/2 - 1/2 < \sum_j \gamma_j < -k/2$. These particular kernels with $k = 2$ has been considered in Maejima and Tudor [14] where the resulting process is called non-symmetric Rosenblatt process. We hence call the kernel in (10) a non-symmetric Hermite kernel. Note that despite the name, one can always symmetrize these kernels. Class (L) will appear in the discrete chaos processes and the limit theorems considered later.
Remark 3.19. If two functions $g_1$ and $g_2$ on $\mathbb{R}_+^k$ satisfy Condition 2 of Definition 3.18, then $\int_{\mathbb{R}_+^k} |g_1(x)g_2(1+x)|dx < \infty$ automatically holds, which can be seen by using the following identity: for any $\gamma, \delta \in (-1, -1/2)$,
\[
\int_0^\infty x^{\gamma}(1+x)^{\delta}dx = B(\gamma+1, -\gamma-\delta-1),
\]
where $B(\cdot, \cdot)$ is the beta function. In addition, $\int_{(0,1]^k} |g_1(x)|dx < \infty$ also holds.

Proposition 3.20. Class (L) contains Class (B).

Proof. Suppose $g$ is a generalized Hermite kernel of Class (B). Then there exist constants $C_1, C_2 > 0$, such that
\[
|g(x)| \leq C_1 \|x\|^\alpha C_2 \prod_{j=1}^k x_j^{\alpha/k},
\]
where we have used the arithmetic-geometric mean inequality $k^{-1} \sum_{j=1}^k y_j \geq \left( \prod_{j=1}^k y_j \right)^{1/k}$ and $\alpha < 0$. So Condition 2 of Definition 3.18 is satisfied with $g^*$ being a single term where $\gamma_1 = \ldots = \gamma_k = \alpha/k$.

Remark 3.21. In view of Remark 3.6 and Remark 3.19, one can check that Class (B) or Class (L) if adding in the a.e. 0-valued function, with fixed order $k$ and fixed homogeneity component $\alpha \in (-k/2 - 1/2, -k/2)$, forms an inner product space, with the inner product specified as
\[
\langle g_1, g_2 \rangle := \left\langle \int_0^1 g_1(s1-\cdot)ds, \int_0^1 g_2(s1-\cdot)ds \right\rangle_{L^2(\mathbb{R}^k)} = \frac{1}{2H(2H-1)} \int_{\mathbb{R}_+^k} g_1(x)g_2(1+x) + g_1(1+x)g_2(x)dx,
\]
where $H = \alpha + k/2 + 1$, which yields the norm
\[
\|g\| := \left\| \int_0^1 g(s1-\cdot)ds \right\|_{L^2(\mathbb{R}^k)} = \left( \frac{1}{H(2H-1)} \int_{\mathbb{R}_+^k} g(x)g(1+x)dx \right)^{1/2}.
\]

Here are several examples.

Example 3.22. Suppose $g(x) = \|x\|^\alpha$, where $\alpha \in (-1/2 - k/2, -k/2)$. This $g$ belongs to Class (B) and thus also Class (L). The pseudo-Fourier transform (Proposition 3.12) of $g$ is $\hat{g}(u) = c\|u\|^{-\alpha-k}$ ((25.25) of Samko et al. [27]) for some constant $c > 0$, which provides the spectral representation by (4).

Example 3.23. Another example of Class (B):
\[
g(x) = \prod_{j=1}^k x_j^{a_j},
\]
where $a_j > 0$ and $b > 0$, yielding a homogeneity exponent $\alpha = \sum_{j=1}^k a_j - b \in (-1/2 - k/2, -k/2)$.

Example 3.24. We give yet another example of Class (L) but not (B):
\[
g(x) = g_0(x) \vee \left( \prod_{j=1}^k x_j^{\alpha/k} \right),
\]
where $g_0(x) > 0$ is any generalized Hermite kernel of Class (B) on $\mathbb{R}_+^k$ with homogeneity exponent $\alpha$.  

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3.3 Fractionally filtered kernels

According to Theorem 3.5, the generalized Hermite process introduced above admits a Hurst coefficient $H > 1/2$ only. To obtain an $H$-fractional process with $0 < H < 1/2$, we consider the following fractionally filtered kernel:

$$h^\beta_t(x) = \int_{\mathbb{R}^k} l^\beta_t(s)g(s1 - x)1_{\{ s1 > x \}}ds,$$

where $g$ is a generalized Hermite kernel defined in Definition 3.2 with homogeneity exponent

$$\alpha \in (-k/2 - 1/2, -k/2),$$

and

$$l^\beta_t(s) = \frac{1}{\beta} \left[ (t-s)^\beta_+ - (-s)^\beta_+ \right], \beta \neq 0. \tag{12}$$

One can extend it to $\beta = 0$ by writing $l^0_t(s) = 1_{(0, \beta]}(s)$, but this would lead us back to the generalized Hermite process case. We hence assume throughout that $\beta \neq 0$. The following proposition gives the range of $\beta$ for which $I_k(h^\beta_t)$ is well-defined.

**Proposition 3.25.** If

$$-1 < -\alpha - \frac{k}{2} - 1 < \beta < -\alpha - \frac{k}{2} < \frac{1}{2}, \quad \beta \neq 0$$

then $h^\beta_t \in L^2(\mathbb{R}^k)$.

**Proof.**

$$\int_{\mathbb{R}^k} h^\beta_t(x)^2dx \leq 2\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{\infty} ds_2 \int_{\mathbb{R}^k} dx \ l_s(s_1)l_t(s_2)|g(s_11 - x)g(s_21 - x)|1_{\{ s_11 > x \}}$$

$$= 2\int_{-\infty}^{\infty} ds \int_{0}^{\infty} du \int_{\mathbb{R}^k_+} dw \ l^\beta_t(s)l^\beta_t(s + u)|g(w)g(u1 + w)| \quad (s = s_1, u = s_2 - s_1, w = s_11 - x)$$

$$= 2\int_{-\infty}^{\infty} ds \ l^\beta_t(s) \int_{0}^{\infty} l^\beta_t(s + u)u^{2\alpha + k}du \int_{\mathbb{R}^k_+} dy \ |g(y)g(1 + y)|.$$

We thus focus on showing $\int_{-\infty}^{\infty} ds \ l^\beta_t(s) \int_{0}^{\infty} l^\beta_t(s + u)u^{2\alpha + k}du < \infty$. Recall that for any $c > 0$, we have

$$\int_{0}^{c} (c - s)^{\gamma_1}s^{\gamma_2}ds = c^{\gamma_1 + \gamma_2 + 1} \int_{0}^{1} (1 - s)^{\gamma_1}s^{\gamma_2}ds = c^{\gamma_1 + \gamma_2 + 1}B(\gamma_1 + 1, \gamma_2 + 1), \forall \gamma_1, \gamma_2 > -1.$$

So by noting that $\beta > -1$ and $2\alpha + k > -1$, we have

$$\int_{0}^{\infty} l^\beta_t(s + u)u^{2\alpha + k}du = \frac{1}{\beta} \int_{0}^{\infty} \left[ (t-s-u)^\beta_+ - (-s-u)^\beta_+ \right] u^{2\alpha + k}du$$

$$= \frac{1}{\beta} \int_{0}^{t-s} (t-s-u)^\beta_+u^{2\alpha + k}du + \int_{0}^{t} (-s-u)^\beta_+u^{2\alpha + k}du$$

$$= \frac{B(\beta + 1, 2\alpha + k + 1)}{\beta} \left[ (t-s)^{\beta + \delta}_+ - (-s)^{\beta + \delta}_+ \right],$$

where

$$\delta = 2\alpha + k + 1 \in (0, 1). \tag{15}$$

We thus want to determine when the following holds:

$$\int_{\mathbb{R}} \left( (t-s)^{\beta}_+ - (-s)^{\beta}_+ \right) \left( (t-s)^{\beta + \delta}_+ - (-s)^{\beta + \delta}_+ \right) ds < \infty.$$

Suppose $t > 0$. The potential integrability problems appear near $s = -\infty, 0, t$. Near $s = -\infty$, the integrand behaves like $|s|^{2\beta + \delta - 2}$, and thus we need $2\beta + \delta - 2 < -1$; near $s = 0$, the integrand behaves like $|s|^{2\beta + \delta}$, and thus $2\beta + \delta > -1$; near $s = t$, the integrand behaves like $|t-s|^{2\beta + \delta}$, and thus again $2\beta + \delta > -1$. In view of (15), these requirements are satisfied by (13).
Proposition 3.30. Suppose that (6) holds. Then the function \( f_{x,t}(s) := l_t(s)g(s1-x)|1_{s1>x} \) is in \( L^1(\mathbb{R}) \) for any \( t > 0 \) and a.e. \( x \in \mathbb{R}^k \).

Theorem 3.27. The process defined by \( Z^\beta(t) := I_k(h_t^\beta) \) with \( h_t^\beta \) given in (12), namely,

\[
Z^\beta(t) = \int_{\mathbb{R}^k} \frac{1}{\beta}[(t-s)^\beta_h - (-s)^\beta_h]g(s-x_1, \ldots, s-x_k)1_{s1>x}dsW(dx_1) \ldots W(dx_k), \tag{16}
\]

is an \( H \)-process with \( H = \alpha + \beta + k/2 + 1 \in (0,1) \).

Proof. By (12), one has for any \( \lambda > 0, l_{t}^\beta(s) = \lambda^\beta l_{t\lambda}(s) \), and for any \( t, h > 0, l_{t+h}(s) = l_{t}^\beta(s-h) \). In addition, \( g \) is homogeneous with exponent \( \alpha \). The conclusion then follows by Proposition 3.31.

Remark 3.28. In the case \( \beta > 0 \), one is able to write \( l_{t}^\beta(s) = \int_0^t(r-s)^\beta -1 dr \), and thus by Fubini

\[
h_t^\beta(x) = \int_0^t ds \int_\mathbb{R} (r-s)^\beta -1 g(s1-x)1_{s1>x}. \tag{17}
\]

Remark 3.29. To get the anti-persistent case \( H < 1/2 \), choose \( \beta \in (-\alpha-k/2-1,-\alpha-k/2-1/2) \).

We now state an analog of (7) for the spectral representation of the process \( Z^\beta(t) \):

Proposition 3.30. Suppose that (6) holds. Then the \( L^2 \)-sense Fourier transform of \( h_t^\beta \) is

\[
\hat{h}_t^\beta(u) = (e^{i\beta(u,1)} - 1)(i(u,1))^{-\beta -1}g(-u)\Gamma(\beta), \quad \text{a.e. } u \in \mathbb{R}^k, \tag{18}
\]

where \( \hat{g} \) is defined in Proposition 3.12.

Proof. Let \( g_n(x) = g(x)1_{(0,n)^k}(x) \), and \( l_{t,n}^\beta = \beta^{-1}[(t-s)^\beta_11_{t-s<n} - (-s)^\beta_11_{s<n}] \). Set

\[
h_{t,n}^\beta(x) = \int_\mathbb{R} l_{t,n}(s)g_n(s1-x)ds.
\]

Similar to the proof of Proposition 3.12 one can show that \( h_{t,n}^\beta \) converges in \( L^2(\mathbb{R}^k) \) to \( h_t^\beta \) as \( n \to \infty \) through the Dominated Convergence Theorem by noting that \( |g_n| \leq |g| \) and \( |l_{t,n}^\beta| \leq l_t^\beta \).

Since the truncated \( l_{t,n} \) and \( g_n \) admit \( L^1 \)-Fourier transforms \( \hat{l}_{t,n} \) and \( \hat{g}_n \) respectively, one can write the Fourier transform of \( h_{t,n}^\beta \) as :

\[
\hat{h}_{t,n}^\beta(u) = \hat{l}_{t,n}(u)\hat{g}_n(-u),
\]

(compare with (1)). Since \( h_{t,n}^\beta \) converges in \( L^2(\mathbb{R}) \) to \( h_t^\beta \) as \( n \to \infty \), by Plancherel’s isometry, \( \hat{h}_{t,n}^\beta \) converges in \( L^2(\mathbb{R}^k) \) to \( \hat{h}_t^\beta \). One now needs to identify (18) with the limit of \( \hat{h}_{t,n}^\beta \).

We first compute \( \hat{l}_{t,n}^\beta \). When \( \beta < 0 \), one has by change of variable that

\[
l_{t,n}^\beta(u) = \beta^{-1} \left( \int_\mathbb{R} e^{iux}(x-u)^\beta_11_{x<u}dx - \int_\mathbb{R} e^{iux}(-x)^\beta_11_{-x<n}dx \right)
\]

\[
= \beta^{-1}(e^{iut} - 1) \int_0^n e^{-iux}s^\beta ds. \tag{19}
\]

When \( \beta > 0 \), one has

\[
l_{t,n}^\beta(u) = \int_\mathbb{R} 1_{(0,t]}(x)(x-u)^\beta_11_{x<u}dx = (1_{(0,t]} * b_n)(u),
\]
where \( b_n(x) = (-x)^{\beta-1} \mathbb{1}_{\{-x<\epsilon\}} \). We have the Fourier transforms \( \hat{b}_n(u) = \frac{e^{iut} - 1}{iu} \), and
\[
\hat{b}_n(u) = \int_{\mathbb{R}} e^{-ixu} (-x)^{\beta-1} \mathbb{1}_{\{-x<\epsilon\}} dx = \int_{0}^{n} e^{-ius\beta} ds.
\]
So
\[
\hat{b}_{n,\epsilon}(u) = \frac{e^{iut} - 1}{iu} \int_{0}^{n} e^{-ius\beta} ds.
\]
By Gradshteyn and Ryzhik [9] Formula 3.761.4 and 3.761.9, for \( \mu \in (0,1) \),
\[
\lim_{n \to \infty} \int_{0}^{n} e^{-ius\mu} ds = |u|^{-\mu} \Gamma(\mu) \cos\left(\frac{\mu\pi}{2}\right) - \operatorname{sign}(u) |u|^{-\mu} \Gamma(\mu) \sin\left(\frac{\mu\pi}{2}\right).
\]
Combining the foregoing limit with (19) and (20), we deduce
\[
\lim_{n \to \infty} \hat{b}_{n,\epsilon}(u) = (e^{iut} - 1)(iu)^{-\beta} \Gamma(\beta).
\]
Recall that there exists a subsequence \( \hat{g}_{n,\epsilon} \) converges a.e. to the pseudo-Fourier transform \( \hat{g} \) as \( k \to \infty \) (Proposition 3.12). So \( \hat{b}_{n,\epsilon}(u) \hat{g}_{n,\epsilon}(u) \) converges to \( \hat{b}(u) \hat{g}(u) \) for a.e. \( u \in \mathbb{R}^k \). But at the same time \( \hat{b}_{n,\epsilon}(u) \hat{g}_{n,\epsilon}(u) \) converges in \( L^2(\mathbb{R}^k) \) to \( \hat{h}_\beta \). So we identify \( \hat{b}_\beta \) with the expression in (18) \( \square \)

**Remark 3.31.** By Proposition 3.1 we get a spectral representation \( Z^\beta(t) \overset{f.d.d.}{=} \hat{I}(\hat{g}_\beta) \). The kernel (18) in the spectral-domain has been considered by Major [18] in the special case where \( \hat{g}(u) = e \prod_{j=1}^{k} |u_j|^{-d} \) is the kernel for the spectral representation of Hermite process.

## 4 Discrete chaos processes

In this section, we introduce a class of stationary sequence which converges to a generalized Hermite process of Class (L) as defined in Definition 3.13.

First we define the discrete chaos, or the discrete multiple stochastic integral, \( Q_k(\cdot; \epsilon) \) with respect to the i.i.d. noise \( \epsilon := (\epsilon_i, i \in \mathbb{Z}) \).

Let \( h \) be a function defined in \( \mathbb{Z}^k \) such that \( \sum'_{i \in \mathbb{Z}^k} h(i)^2 < \infty \), where \( ' \) indicate the exclusion of the diagonals \( i_p = i_q, p \neq q \). The following sum
\[
Q_k(h) = Q_k(h, \epsilon) = \sum_{(i_1, \ldots, i_k) \in \mathbb{Z}^k}^{'} h(i_1, \ldots, i_k) \epsilon_{i_1} \cdots \epsilon_{i_k} = \sum_{i \in \mathbb{Z}^k} h(i) \prod_{p=1}^{k} \epsilon_{i_p},
\]

is called the discrete chaos of order \( k \). It is easy to see that switching the arguments, say \( i_p \) and \( i_q \), \( p \neq q \), of \( h(i_1, \ldots, i_k) \), does not change \( Q_k(h) \). So if \( h \) is the symmetrization \( h \), then \( Q_k(h) = Q_k(\hat{h}) \).

The discrete chaos is related to Wiener chaos by a limit theorem. Suppose now we have a sequence of function vectors \( h_n = (h_{1,n}, \ldots, h_{j,n}) \) where each \( h_{j,n} \in L^2(\mathbb{Z}^k) \), \( j = 1, \ldots, J \). The following proposition concerns the convergence of the discrete chaos to the Wiener chaos:

**Proposition 4.1.** Let \( \hat{h}_{j,n}(x) = n^{k_j/2} h_{j,n} ((nx) + c_j) \), \( j = 1, \ldots, J \), where \( c_j \in \mathbb{Z}^k \). Suppose that there exists \( h_j \in L^2(\mathbb{R}^k) \), such that
\[
\| \hat{h}_{j,n} - h_j \|_{L^2(\mathbb{R}^k)} \to 0
\]
as \( n \to \infty \). Then, as \( n \to \infty \),
\[
Q := \left( Q_{k_1}(h_{1,n}), \ldots, Q_{k_J}(h_{J,n}) \right) \overset{d}{\to} I := \left( I_{k_1}(h_1), \ldots, I_{k_J}(h_J) \right),
\]
where each \( I_{k_j}(\cdot), j = 1, \ldots, J \), denotes the \( k_j \)-tuple Wiener-Itô integral with respect to the same standard Brownian motion \( W \).
For a proof, we refer the reader to the proof of Proposition 14.3.2 of Giraitis et al. on the univariate case. The proof for the multivariate case (corresponding to Proposition 14.3.3 of Giraitis et al.) is similar once the Crámer-Wald Device is applied. The difference between Proposition 4.1 and Proposition 14.3.3 of Giraitis et al. is that we add the shift $c_j$ for more flexibility. This extension requires only an easy modification to the proof.

The causal discrete chaos process of order $k \geq 1$ is a stationary sequence $\{X(n), n \in \mathbb{Z}\}$ defined by:

$$X(n) = \sum_{0 < i_1, \ldots, i_k < \infty} a(i_1, \ldots, i_k) \epsilon_{n - i_1} \ldots \epsilon_{n - i_k} = \sum_{-\infty < i_1, \ldots, i_k < n} a(n - i_1, \ldots, n - i_k) \epsilon_{i_1} \ldots \epsilon_{i_k},$$  \hspace{1cm} (22)  

where $'$ indicates that the sum excludes the diagonals $i_p = i_q, p \neq q, \{\epsilon_n\}$ is an i.i.d. sequence with mean 0 and variance 1, $a(i)$ is a function on $\mathbb{Z}^k$, and we require that it satisfies $\sum_{i > 0} |a(i)^2| < \infty$, so that $X(n)$ is well-defined in the $L^2(\Omega)$-sense. Note that when $k = 1$, $X(n)$ is plainly a linear process.

Due to the off-diagonality, the autocovariance of $\{X(n)\}$ is given by the simple formula

$$\gamma(n) := \text{Cov}(X(n), X(0)) = k! \sum_{i > 0} \tilde{a}(i) \tilde{a}(i + |n|1),$$  \hspace{1cm} (23)  

where $\tilde{a}(\cdot)$ is the symmetrization of $a(\cdot)$.

We now focus on the following case:

$$a(i) = g(i)L(i),$$  \hspace{1cm} (24)  

where $g$ is a generalized Hermite kernel of Class (L) defined in Definition 3.18 and $L$ is a bounded function on $\mathbb{Z}^k$ which satisfies the following: for any $x \in \mathbb{R}^k_+$ and for any bounded $\mathbb{Z}^k$-valued function $B(\cdot)$ defined on $\mathbb{Z}_+$, we have

$$L([nx] + B(n)) \to 1, \text{ as } n \to \infty.$$  \hspace{1cm} (25)  

Note that $X(n)$ is well-defined in $L^2(\Omega)$ since $\sum_{i \in \mathbb{Z}^k} g^*(i)^2 < \infty$, where $g^*$ is a linear combination of terms of the form $\prod_{j=1}^k x_j^{\gamma_j}$ with every $\gamma_j < -1/2$.

**Remark 4.2.** Note that the boundedness of $L$ and (25) are strictly weaker than assuming that $L(i) \to 1$ as $||i|| \to \infty$ for some norm $\| \cdot \|$ on $\mathbb{R}^k$ (recall that norms are equivalent in the finite-dimensional space). Indeed, consider

$$L(i_1, i_2) = \begin{cases} 2 & \text{if } i_2 = 1; \\ 1 & \text{otherwise.} \end{cases}$$

Suppose that $B$ is bounded by $M$. Then $L([nx] + B(n)) = 1$ for large $n$. On the other hand, consider $||i|| = \max(i_1, i_2)$. Then if $(i_1, i_2) = (i_1, 1)$, $i_1 \to \infty$, we have $||i|| = i_1 \to \infty$ but $L(i_1, i_2) = L(i_1, 1) = 2$.

**Remark 4.3.** In practice, Relation (25) implies that for any fixed $x \in \mathbb{R}^k_+$ and $c \in \mathbb{Z}^k_+$, $L([nx] + c) \to 1$ as $n \to \infty$.

The following Proposition shows that one can get long-range dependence if $g$ is of Class (L).

**Proposition 4.4.** If $a(i)$ is as given in (24), where $g$ has homogeneity exponent $\alpha \in (-1/2 - k/2, -k/2)$ (or $2\alpha + k \in (-1, 0)$), then the autocovariance of the discrete chaos process $\{X(n)\}$ satisfies

$$\gamma(n) \sim k! Cg_n^{2H-2}, \text{ as } n \to \infty,$$  \hspace{1cm} (26)  

where $Cg = \int_{\mathbb{R}^k_+} g(x) \tilde{g}(1 + x) > 0$, $H = \alpha + k/2 + 1 \in (1/2, 1)$, with $\tilde{g}$ being the symmetrization of $g$. In addition, as $N \to \infty$,

$$\text{Var}\left[\sum_{n=1}^N X(n)\right] \sim \frac{k! Cg}{H(2H - 1)} N^{2H}. \hspace{1cm} (27)$$
Proof. Assume without loss of generality that \( g \) is already symmetric.

\[
(k!)^{-1} \gamma(n) = \sum_{i=0}^{t} g(i)g(n1 + i)L(n1 + i)L(i)
\]

\[= n^{2\alpha+k} \sum_{i=0}^{t} g \left( \frac{i}{n} \right) g \left( 1 + \frac{i}{n} \right) L(\frac{i}{n})L(1 + \frac{i}{n}) \]

\[= n^{2\alpha+k} \int_{\mathbb{R}_+^k} 1_{D^c}(x)g_n(x)g_n(1 + x)dx,
\]

where \( g_n(x) = g(\frac{n|x|+1}{n})L(n|x| + 1) \), \( D^c_n = \{ x \in \mathbb{R}_+^k, [nx_p] \neq [nx_q], p \neq q \in \{1, \ldots, k\} \} \). Note that \( 1_{D_n}(x) = 1 \) as \( n \) becomes large enough, for any \( x \in D^c := \{ x \in \mathbb{R}_+^k, x_p \neq x_q, p \neq q \in \{1, \ldots, k\} \} \), and that the diagonal set \( D := \mathbb{R}_+^k \setminus D^c \) has measure 0. Since \( g \) belongs to Class (L), \( g \) is continuous a.e., so \( g_n(x) \rightarrow g(x) \) a.e. as \( n \rightarrow \infty \). Furthermore, there exists \( g^*(x) \) which is a linear combination of the form \( \prod_{j=1}^{k} x_j^q \) (Condition 2 of Definition 3.18), so that for a.e. \( x \in \mathbb{R}_+^k \),

\[
|g_n(x)| \leq g^* \left( \frac{nx}{n} + 1 \right) \leq g^*(x),
\]

since \( L \) is bounded and \( g^* \) is decreasing in its every variable. Note that \( \int_{\mathbb{R}_+^k} g^*(x)g^*(1 + x)dx < \infty \), and \( g \) is a.e. continuous. So it remains to apply the Dominated Convergence Theorem.

Finally, (27) follows by first noting that

\[
\text{Var}[\sum_{n=1}^{N} X(n)] = \sum_{n}(N - |n|)\gamma(n) = N \sum_{|n|<N} \gamma(n) - \sum_{|n|<N} |n|\gamma(n),
\]

and then using the asymptotics of \( \gamma(n) \) just derived.

\[
\square
\]

5 Hypercontractivity for infinite discrete chaos

Let \( X_M \) be a finite discrete chaos defined as

\[
X_M = \sum_{-M1 \leq i \leq M1} h(i)\epsilon_{i_1} \ldots \epsilon_{i_k},
\]

(28)

where \( h(i) = h(i_1, \ldots, i_k) \) is a function on \( \mathbb{Z}^k \), \( M \in \mathbb{Z}_+ \), and we assume that \( \{ \epsilon_i \} \) is a sequence of i.i.d. variables with \( \mathbb{E}\epsilon_i = 0, \mathbb{E}\epsilon_i^2 = 1 \). Then we have the following moment-comparison inequality, also called “hypercontractivity inequality”:

Proposition 5.1. Suppose that \( \mathbb{E}|\epsilon_i|^p < \infty \) with \( p \geq 2 \). Then

\[
\mathbb{E}[|X_M|^p]^{1/p} \leq d_{p,k}\mathbb{E}[|X_M|^2]^{1/2},
\]

(29)

where \( d_{p,k} \) is a constant depending only on \( p \) and \( k \).

For a proof of (29), where \( M \) is finite, see Lemma 4.3 of Krakowiak and Szulga [11], where the so-called MPZ(\( p \)) condition (Definition 1.5 of Krakowiak and Szulga [11]) is trivially satisfied since the \( \epsilon_i \)'s are identically distributed.

Now we extend (29) to the case \( M = \infty \). The result is used in Theorem 6.3, 6.11 and 6.14 below for proving tightness in \( D[0,1] \).
Proposition 5.2. Suppose that \( \sum_{i \in \mathbb{Z}^k} h(i)^2 < \infty \). Let \( X = \sum_{i \in \mathbb{Z}^k} h(i) \prod_{p=1}^{k} \epsilon_{i_p} \). If for some \( p' > p > 2 \), \( \mathbb{E}[|\epsilon_i|^{p'}] < \infty \), then one has
\[
\mathbb{E}[|X|^{p}]^{1/p} \leq d_{p,k} \mathbb{E}[|X|^2]^{1/2}
\]
Proof. Let \( X_M \) be the truncated finite chaos as in (28). The condition on \( h \) implies that \( X_M \to X \) in \( L^2(\Omega) \). Moreover, one has by (29),
\[
\mathbb{E}[|X_M|^{p'}] \leq d_{p,k} \mathbb{E}[|X_M|^2]^{p'/2} \leq d_{p,k} \left( \sum_{i \in \mathbb{Z}^k} h(i)^2 \right)^{p'/2}
\]
This implies that \( \{|X_M|^{p}, M \geq 1\} \) and \( \{|X_M|^2, M \geq 1\} \) are uniformly integrable, implying convergence of the corresponding moments. So one can then let \( M \to \infty \) on both sides of (29) and obtain (30). \( \square \)

6 Joint convergence of the discrete chaoses

Our goal here is to obtain non-central limit theorems for the discrete chaos process introduced in Section 4. We shall, in fact, prove both a central limit theorem for the SRD case (getting Brownian motion as limit) and a non-central limit theorem for the LRD case (getting the generalized Hermite process introduced in Section 3 as limit). We also consider non-central limit theorems leading to the fractionally filtered generalized Hermite process introduced in Section 3. Finally, we derive a multivariate limit theorem which mixes central and non-central limit theorems.

We first define here precisely what SRD and LRD stand for in the context of discrete chaos process. Recall that \( \tilde{a}(\cdot) \) denotes the symmetrization of \( a(\cdot) \).

Definition 6.1. We say a discrete chaos process \( \{X(n)\} \) given in (22) is

- SRD, if \( \sum_{n=\infty}^{\infty} \sum_{i \in \mathbb{Z}^k} |\tilde{a}(i)\tilde{a}(i + |n|1)| < \infty \) and \( \sum_{n=\infty}^{\infty} |\gamma(n)| > 0 \);
- LRD, if \( a(i) = g(i)L(i) \) as given in (23). In particular, \( g \) is a generalized Hermite kernel of Class (L).

Remark 6.2. The definitions of SRD and LRD in Definition 6.1 are distinct. Indeed, the SRD condition implies that \( \sum_n |\gamma(n)| < \infty \), while LRD yields \( \sum_n |\gamma(n)| = \infty \) by Proposition 4.4.

6.1 Central limit theorem

Theorem 6.3. If a discrete chaos process \( \{X(n)\} \) given in (22) is SRD in the sense of Definition 6.1, then
\[
\frac{1}{N_{1/2}} \sum_{n=1}^{[N]} X(n) \xrightarrow{d,d.} \sigma B(t)
\]
where \( B(t) \) is a standard Brownian motion, and \( \sigma^2 = \sum_{n=\infty}^{\infty} |\gamma(n)| \).

Proof. Assume without loss of generality that \( a(\cdot) \) is symmetric. The proof is similar to the proof of Theorem 2.3 found on p.108 of Giraitis et al. [8], so we give only a sketch. The central idea is to introduce the \( m \)-truncation of \( X(n) \), namely, \( X^{(m)}(n) := \sum_{0 \leq i \leq m} a(i) \prod_{j=1}^{k} \epsilon_{n-j} \), and then let \( m \to \infty \). The sequence \( \{X^{(m)}(n), n \in \mathbb{Z}\} \) is \( m \)-dependent, so the classical invariance principle applies (Billingsley [3] Theorem 5.2). The long-run variance \( \sigma^2 = \sum_n |\gamma(n)| \) is a standard result. We now check that the \( L^2(\Omega) \) approximation is valid as \( m \to \infty \), that is,
\[
\lim_{m \to \infty} \sup_{N \in \mathbb{Z}^+} \text{Var}[Y^{(m)}_N(t) - Y_N(t)] = 0, \quad t > 0,
\]
where \( Y^{(m)}_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} X^{(m)}(n) \) and \( Y_N(t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} X(n) \), which is similar to (4.8.7) of Giraitis et al. [8]. Indeed,
\[
\text{Var}[Y^{(m)}_N(t) - Y_N(t)] = \frac{1}{N} \text{Var} \left[ \sum_{n=1}^{[N]} (X^{(m)}_n - X_n) \right] = \frac{[N]}{N} \sum_{n=1}^{[N]} \gamma_m(n) \left( 1 - \frac{|n|}{[N]} \right) \leq t \sum_{n=-\infty}^{\infty} |\gamma_m(n)|.
\]
where
\[ \gamma_m(n) := E(X_n - X_n^{(m)})(X_0 - X_0^{(m)}) = k! \sum_{i \geq 1} a(i)a(n1 + i). \]

For a fixed \( n \in \mathbb{Z} \), \( \gamma_m(n) \to 0 \) as \( m \to \infty \), and \( |\gamma_m(n)| \leq \rho(n) \), where \( \rho(n) = k! \sum_{i \geq 0} |a(i)a(i + n1)| \), which satisfies \( \sum_{n} \rho(n) < \infty \) by the SRD assumption in Definition 6.1. Since the bound in (33) does not depend on \( N \), the Dominated Convergence Theorem applies and thus (32) holds.

To strengthen the conclusion of Theorem 6.3 to weak convergence, we have to make some additional assumptions to prove tightness.

**Theorem 6.4.** Theorem 6.3 holds with \( \mathcal{d.d.} \) replaced by weak convergence \( \Rightarrow \) in \( D[0, 1] \), if either of the following holds:

1. There exists \( \delta > 0 \), such that \( E(|\epsilon_i|^{2+\delta}) < \infty \);
2. There exists an \( M > 0 \) such that \( a(i) = 0 \) whenever \( i > M1 \).

**Proof.** Look first at case 1. Let
\[ Y_N(t) := \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} X(n) \]
Select \( p \in (2, 2 + \delta) \). By Proposition 5.2 one has
\[ E[|Y_N(t) - Y_N(s)|^p] \leq cE[|Y_N(t) - Y_N(s)|^2]^{p/2}, \]
where \( c \) is some constant which doesn’t depend on \( s, t \) or \( N \). Note that \( \sum_{n} |\gamma(n)| < \infty \) due to SRD assumption, we have
\[ E \left[ |Y_N(t) - Y_N(s)|^2 \right] = \frac{1}{N} E\left[ \sum_{n=1}^{[Nt]-[Ns]} X(n)^2 \right] \]
\[ \leq \frac{[Nt] - [Ns]}{N} \sum_{|n| < [Nt] - [Ns]} \left( 1 - \frac{|n|}{[Nt] - [Ns]} \right) \gamma(n) \leq \frac{[Nt] - [Ns]}{N} \sum_{n=-\infty}^{\infty} |\gamma(n)|. \]
Combining (34) and (33), we have for some constant \( C > 0 \) that
\[ E[|Y_N(t) - Y_N(s)|^p] \leq cE[|Y_N(t) - Y_N(s)|^2]^{p/2} \leq C|F_N(t) - F_N(s)|^{p/2}, \]
where \( F_N(t) = [Nt]/N \). Now by applying Lemma 4.4.1 and Theorem 4.4.1 of Giraitis et al. [6], noting that \( p/2 > 1 \), we conclude that tightness holds.

For case 2, \( X(n) \) is \( M \)-dependent, so by Theorem 5.2 of Billingsley [3] tightness holds as well. \( \square \)

### 6.2 Non-central limit theorem

The following theorem shows that in the LRD case, the discrete chaos process converges weakly to a generalized Hermite process.

**Theorem 6.5.** If a discrete chaos process \( \{X(n)\} \) given in (22) is LRD in the sense of Definition 6.1 then
\[ \frac{1}{N^H} \sum_{n=1}^{[Nt]} X(n) \Rightarrow Z(t), \]
in \( D[0, 1] \), where \( Z(t) \) is the generalized Hermite process in [3], and
\[ H = \alpha + k/2 + 1 \in \left( \frac{1}{2}, 1 \right), \]
where \( \alpha \in (-1/2 - k/2, -k/2) \) is the homogeneity exponent of \( g \) and \( k \) is the order of \( \{X(n)\} \).
Proof. Tightness in $D[0,1]$ is standard since $H > 1/2$. We only need to show convergence in finite-dimensional distributions. Assume for simplicity that $a(i) = g(i)$ or equivalently $L(i) = 1$. The inclusion of a general $L$ can be done as in the proof of Proposition 4.4. We want to show that

$$
\frac{1}{N^H} \sum_{n=1}^{[Nt]} X(n) = \sum_{(i_1, \ldots, i_k) \in \mathbb{Z}^k} \frac{1}{N^{\alpha+k/2+1}} \sum_{n=1}^{[Nt]} g(n1 - i)1_{\{n1 > i\}} \epsilon_{i_1} \cdots \epsilon_{i_k} \overset{f.d.d.}{\to} Z(t),
$$

(37)

where $Q_k(\cdot)$ is defined in (21). Now in view of Proposition 4.1, we only need to check that

$$
\|\tilde{h}_{t,N}(x) - h_t(x)\|_{L^2(\mathbb{R}^k)} \to 0,
$$

(38)

where

$$
\tilde{h}_{t,N}(x) = \int_0^t g(s1 - x)1_{\{s1 > x\}} ds,
$$

and

$$
\tilde{h}_{t,N}(x) := N^{k/2} h_{t,N}([Nx] + 1) = \frac{1}{N^{\alpha+1}} \sum_{n=1}^{[Nt]} g(n1 - [Nx] - 1)1_{\{n1 > [Nx]+1\}}
$$

$$
= \sum_{n=1}^{[Nt]} g \left( \frac{n1 - [Nx] - 1}{N} \right) 1_{\{n1 > [Nx]+1\}} = \int_0^t g \left( \frac{[Ns1] - [Nx]}{N} \right) 1_{\{[Ns1] > [Nx]\}} ds - R_N(t,x).
$$

where

$$
R_N(t,x) = \frac{Nt - [Nt]}{N} g \left( \frac{[Ns1] - [Nx]}{N} \right) 1_{\{[Ns1] > [Nx]\}}.
$$

Note that we have replaced $i$ by $[Nx] + 1$ and $n$ by $[Ns] + 1$. By Condition 2 in Definition 3.18 there exists a positive generalized Hermite kernel $g^*(x)$ which is a linear combination of the form $\prod_{j=1}^k x_j^{\gamma_j}$, such that $|g(x)| \leq g^*(x)$ for a.e. $x \in \mathbb{R}^k$. We assume without loss of generality that $g^*(x) = \prod_{j=1}^k x_j^{\gamma_j}$. Since $[Ns1] > [Nx]$ implies $s1 > x$, we have

$$
\left| g \left( \frac{[Ns1] - [Nx]}{N} \right) \right| 1_{\{[Ns1] > [Nx]\}} \leq \left( \prod_{j=1}^k \left( \frac{[Ns] - [Nx]}{N} \right)^{\gamma_j} \right) 1_{\{[Ns1] > [Nx]\}} a.e.
$$

(39)

Moreover, if $0 < [Ns] - [Nx] = k \in \mathbb{Z}_+$, then $Ns - 1 - Nx \leq k$, and hence $s - x \leq \frac{k+1}{N}$. So we have for any $\gamma < 0$ that

$$
\sup_{N \geq 1, [Ns] > [Nx]} \left( \frac{[Ns] - [Nx]}{N} \right)^{\gamma} (s - x)^{-\gamma} \leq \sup_{N \geq 1, [Ns] - [Nx] = k \geq 1} \left( \frac{k}{N} \right)^{\gamma} (s - x)^{-\gamma}
$$

$$
\leq \sup_{N \geq 1, k \geq 1} \left( \frac{k+1}{N} \right)^{-\gamma} = 2^{-\gamma}.
$$

(40)

So we have for some constant $C > 0$,

$$
\left| g \left( \frac{[Ns1] - [Nx]}{N} \right) \right| 1_{\{[Ns1] > [Nx]\}} \leq C g^*(s1 - x)1_{\{s1 > x\}}.
$$

(41)

Since $g(x)$ by assumption of Class (L) is continuous a.e., $g \left( \frac{[Ns1] - [Nx]}{N} \right) 1_{\{[Ns1] > [Nx]\}}$ converges a.e. to $g(s1 - x)1_{\{s1 > x\}}$ as $N \to \infty$. In view of (41), and noting that $\int_{\mathbb{R}^k} ds \left( \int_0^t g^*(s1 - x)1_{\{s1 > x\}} ds \right)^2 < \infty$ because $g^*$ is a generalized Hermite kernel, one then applies the Dominated Convergence Theorem to conclude the $L^2$ convergence of $\int_0^t g \left( \frac{[Ns1] - [Nx]}{N} \right) 1_{\{[Ns1] > [Nx]\}} ds$ to $h_t(x)$. For the remainder term $R_{N,t}(x)$, one has

$$
\|R_{N,t}(x)\|_{L^2(\mathbb{R}^k)}^2 = N^{-2H} ([Nt] - [Nt])^2 \sum_{i>0} g(i)^2 \to 0
$$

as $N \to \infty$. The proof is thus complete. \qed
Example 6.6. Consider the kernel $g(x)$ defined in [1]. It belongs to Class (L) by Example 3.24. Hence by Theorem 6.5 we have the following weak convergence in $D[0, 1]$:

\[
\frac{1}{N^n} \sum_{n=1}^{[N^n]} \sum_{(i_1, \ldots, i_k) \in Z^k_+} \left( \frac{\prod_{j=1}^{k} i_j}{2^\alpha \prod_{j=1}^{k} i_j} \right)^{\epsilon_{n-i_1} \cdots \epsilon_{n-i_k} \Rightarrow} \int_{\mathbb{R}^k} \int_0^t \left( \frac{\prod_{j=1}^{k} (s - x_j)^{1/2}}{\prod_{j=1}^{k} (s - x_j)^{1/2}} \right)^{\epsilon_{m-1} \cdots \epsilon_{m-k}} ds W(dx_1) \cdots W(dx_k),
\]

where $H = \alpha + k/2 + 1$.

6.3 Non-central limit theorem with fractional filter

In the spirit of Rosenblatt [25] and Major [18], we consider here the non-central limit theorem for the fractionally filtered generalized Hermite process introduced in Section 3.3. Assume throughout that the generalized Hermite kernel $g$ is of Class (L) (Definition 3.18).

Definition 6.7. Let $X(n) = \sum_{i\leq n} a(n-1) \prod_{j=1}^{k} \epsilon_{ij}$ be the same discrete chaos process as in Theorem 6.5. We say that a discrete process $U(n)$ is fLRD (fractionally-filtered LRD discrete chaos process) if

\[
U(n) = \sum_{m=1}^{\infty} C_m X(n-m) = \sum_{m=-\infty}^{n-1} C_{n-m} \sum_{1\leq m_1 < m} a(m-1) \prod_{j=1}^{k} \epsilon_{ij},
\]

(42)

where $a(i) = g(i)L(i)$ as in (24) with $g$ being a generalized Hermite kernel in Class (L),

\[
C_n \sim c n^{\beta - 1}
\]

as $n \to \infty$, and where, as in Proposition 3.25

\[
\beta \in \left( -\frac{2\alpha + k + 2}{2}, -\frac{2\alpha + k}{2} \right).
\]

(43)

$U(n)$ is well-defined in the $L^2(\Omega)$ sense. Indeed, we have the following:

Lemma 6.8. We have

\[
\sum_{i \in \mathbb{Z}^k} \left( \sum_{m < n} |C_{n-m} a(m-1) 1_{\{m_1 > i\}} | \right)^2 < \infty.
\]

Proof. Note that $a(\cdot) = g(\cdot)L(\cdot)$, where $g$ is of Class (L). So by Definition 3.18 there exists a $g^*(x) > 0$ which is a finite linear combination of the form $\prod_{j=1}^{k} x_j^{\gamma_j}$, such that $|g(x)| < g^*(x)$. Note that $L$ is bounded and $|C_n| \leq c n^{\beta - 1}$. Set $n = -1$ without loss of generality due to stationarity. We hence need to show that

\[
\sum_{i \in \mathbb{Z}^k} \left( \sum_{m < -1} (-m)^{\beta - 1} g^*(m-1) 1_{\{m_1 > i\}} \right)^2 < \infty.
\]

(44)

It suffices to show this when $\beta > 0$, since for any $\beta' \leq 0$ and $\beta > 0$, $(-m)^{\beta' - 1} \leq (-m)^{\beta - 1}$ for all $m < -1$. The preceding sum can be rewritten as an integral by replacing $m$ by $|s|$ and $i$ by $|x|.

\[
\int_{\mathbb{R}^k} 1_{D=x} ds \left( \int_{-\infty}^{-1} ds (-|s|)^{\beta - 1} g^*(|s|) 1_{\{|s_1 > |x|\}} \right)^2,
\]

(45)
where \( D^c = \{ x \in \mathbb{R}^k : [x_p] \neq [x_q], p \neq q \} \). By \([s] \leq s, \beta - 1 < 0, \) and \(41, 45\) is bounded by (up to a constant)

\[
\int_{\mathbb{R}^k} dx \left( \int_{-\infty}^{1} ds (-s)^{\beta-1} g^* (s \mathbf{1} - x) 1_{\{s > x\}} \right)^2 = \int_{-\infty}^{-1} ds (-s)^{\beta-1} \int_{0}^{-s} du (-s - u)^{\beta-1} u^{2\alpha+k} \int_{\mathbb{R}^k} dy g^*(y) g^*(1 + y) = \int_{1}^{\infty} s^{2\alpha+2\beta+k-1} ds B(\beta, 2\alpha + k + 1) C_{g^*} < \infty,
\]

where we have used a change of variable similar to the lines below \([14]\), and in addition the assumptions \( \beta > 0, 2\alpha + k > -1, 2\alpha + 2\beta + k < 0, \) and \( g^* \) is a generalized Hermite kernel.

\[
\square
\]

**Remark 6.9.** Lemma \([6.8]\) not only shows that \( U(n) \) is well-defined in \( L^2(\Omega) \), it also allows changing the order of summations, which will be used in proving the non-central limit theorem below.

Next we want to obtain non-centrally limit theorems, that is, to show that the suitably normalized partial sum of \( U(n) \) defined in \([42]\) converges to the fractionally-filtered generalized Hermite process introduced in Section \([3.3]\). We need to distinguish two cases: \( \beta > 0 \) (which increases \( H \)) and \( \beta < 0 \) (which decreases \( H \)).

We first consider \( \beta > 0 \):

**Theorem 6.10.** Let \( U(n) \) be as in \([42]\) with \( \beta \in (0, -\alpha - k/2) \). Then

\[
\frac{1}{NH} \sum_{n=1}^{\lfloor Nt \rfloor} U(n) \Rightarrow Z^\beta(t),
\]

where

\[
1/2 < \alpha + k/2 + 1 < H = \alpha + \beta + k/2 + 1 < 1,
\]

and \( Z^\beta(t) \) is the fractionally-filtered generalized Hermite process defined in Theorem \([3.27]\). It is defined using the same \( g \) and \( \beta \) as \( U(n) \).

**Proof.** Since \( H > 1/2 \), tightness in \( D[0,1] \) is standard. We now show convergence in finite-dimensional distributions. Assume for simplicity that \( C_m = m^{\beta-1} \) and \( L(i) = 1 \). By Lemma \([6.8]\), we are able to change the order of the summations to write:

\[
\frac{1}{NH} \sum_{n=1}^{\lfloor Nt \rfloor} U(n) = \sum_{i \in \mathbb{Z}^k} \frac{1}{NH} \sum_{n=1}^{\lfloor Nt \rfloor} \sum_{m < n} (n - m)^{\beta-1} g(m1 - i) 1_{\{m1 > i\}} \prod_{j=1}^{k} \epsilon_{i_j} = \sum_{i \in \mathbb{Z}^k} h^\beta_{t,N}(i) \prod_{j=1}^{k} \epsilon_{i_j} = Q_k(h^\beta_{t,N}),
\]

and by setting \( \tilde{h}^\beta_{t,N}(x) = N^{k/2} h^\beta_{t,N}([Nx] + 1) \), we have

\[
\tilde{h}^\beta_{t,N}(x) = \frac{1}{N^{\alpha + \beta + 1}} \sum_{n=1}^{\lfloor Nt \rfloor} \sum_{m < n} (n - m)^{\beta-1} g(m1 - [Nx] - 1) 1_{\{m1 > [Nx] - 1\}} = \sum_{n=1}^{\lfloor Nt \rfloor} \sum_{m < n} \left( \frac{n - m}{N} \right)^{\beta-1} g \left( m1 - \frac{[Nx] - 1}{N} \right) 1_{\{m1 > [Nx] - 1\}} \frac{1}{N^2} = \int_0^t ds \int dx \left( \frac{[Ns] - [Nr]}{N} \right)^{\beta-1} g \left( \frac{[Nr1] - [Nx]}{N} \right) 1_{\{[Nr1] > [Nx]\}} - R_{N,t}(x)
\]

\[
= : \int_0^t ds \int dx G_N(s, r, x) 1_{K_N} - R_{N,t}(x)
\]

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where we associate $i$ with $\lfloor Nx \rfloor + 1$, $n$ with $\lfloor Ns \rfloor + 1$, and $m$ with $\lfloor Nr \rfloor + 1$,

$$G_N(s, r, x) := \left( \frac{\lfloor Ns \rfloor - \lfloor Nr \rfloor}{N} \right)^{\beta-1} g \left( \frac{\lfloor Nr1 \rfloor - \lfloor Nx \rfloor}{N} \right),$$

$$K_N = \{\lfloor Ns \rfloor > \lfloor Nr \rfloor, \lfloor Nr1 \rfloor > \lfloor Nx \rfloor\} \subset \{s > r, r1 > x\},$$

and

$$R_{N,t}(x) = \frac{Nt - [Nt]}{N} \int_\mathbb{R} dr \left( \frac{[Nt] - [Nr]}{N} \right)^{\beta-1} g \left( \frac{[Nr1] - [Nx]}{N} \right) 1_{\{[Nr1] > [Nx]\}}.$$

In view of Proposition 4.11 we need to show that $h_{t,N}^\beta \rightarrow h_t^\beta$ and $R_{N,t} \rightarrow 0$ in $L^2(\mathbb{R}^k)$, where

$$h_t^\beta(x) := \int_0^t ds \int_\mathbb{R} dr (s - r)^{\beta-1} g(r1 - x) 1_{\{r1 > x\}}.$$

Using \([39]\) and \([40]\) (note that $\beta - 1 < 0$) as in the proof of Theorem 6.5 we can bound the integrand as

$$|G_N(s, r, x)|_{K_N} \leq C(s - r)^{\beta-1} g^*(r1 - x) 1_{\{r1 > x\}}$$

for some $C > 0$, where $g^*(x)$ is a generalized Hermite kernel from Definition 3.18. Because

$$h^*(x) := (s - r)^{\beta-1} g^*(r1 - x) 1_{\{r1 > x\}} \in L^2(\mathbb{R}^k)$$

by \([17]\) and Proposition 3.22 and $g$ is a.e. continuous, it remains to apply the Dominated Convergence Theorem to conclude $h_{t,N}^\beta \rightarrow h_t^\beta$. For the remainder term $R_{N,t}(x)$, one has

$$\|R_{N,t}(x)\|_{L^2(\mathbb{R}^k)}^2 = N^{-2H}(Nt - [Nt]) \sum_{i \in \mathbb{Z}^k} \left( \sum_{m < [Nt]} ([Nt] - m)^{\beta-1} g(m1 - i) 1_{\{m1 > i\}} \right)^2,$$

which, in view of \([44]\), converges to 0 as $N \rightarrow \infty$. The proof is thus complete.

We now treat the case $\beta < 0$. This case is more delicate than the case $\beta > 0$ in two ways: a) an additional assumption on the linear-filter response \(\{C_n\}\) has to be made; b) if $\beta$ is chosen such that $H < 1/2$, then tightness of the normalized partial sum process needs also additional assumptions.

When $\beta < 0$, we have

$$\sum_{n=1}^\infty |C_n| < \infty.$$ 

If $f_X$ is the spectral density of \(\{X(n)\}\), then the spectral density of \(\{U(n)\}\) is

$$f_U(\lambda) = |C(e^{i\lambda})|^2 f_X(\lambda),$$

where $C(z) := \sum_{n \in \mathbb{Z}} C_n z^n$, and the transfer function $H(\lambda) := |C(e^{i\lambda})|^2$ is continuous. Since $X(n)$ is LRD (see Proposition 4.14), its spectral density blows up at the origin. To dampen it we need to multiply it by an $H(\lambda)$ which converges to 0 as $\lambda \rightarrow 0$. This means that $H(0) = |\sum_{n=1}^\infty C_n|^2 = 0$, and hence we need to assume $\sum_{n=1}^\infty C_n = 0$.

**Theorem 6.11.** Let $U(n)$ be as in \([44]\) with $\beta \in (-\alpha - k/2 - 1, 0)$, and assume in addition that

$$\sum_{n=1}^\infty C_n = 0.$$ 

(46)

Then

$$\frac{1}{N^H} \sum_{n=1}^{\lfloor Nt \rfloor} U(n) \overset{f.d.d.}{\rightarrow} Z(\beta)(t),$$

where $Z(\beta)(t)$ is a Gaussian process with spectral density $Z(\beta)(\lambda)$.
where
\[ 0 < H = \alpha + \beta + k/2 + 1 < \alpha + k/2 + 1 < 1, \]
\[ Z^\beta(t) \text{ is the fractionally-filtered generalized Hermite process defined in Theorem 3.27. It is defined using the same } g \text{ and } \beta \text{ as } U(n). \]

If in addition, either a) \( H > 1/2 \), or b) \( H < 1/2 \) and for some \( p > 1/H \), \( \mathbb{E}|\epsilon_i|^p < \infty \), then the above \( \overset{f.d.d.}{\to} \) can be replaced with weak convergence in \( D[0,1] \).

**Proof.** Note that by Lemma 6.8 we can change the order of summations to write:
\[ Y_N(t) := \frac{1}{N^H} \sum_{n=1}^{\lfloor Nt \rfloor} U(n) = \frac{1}{N^H} \sum_{i \in \mathbb{Z}} \sum_{n=1}^{\lfloor Nt \rfloor} C_{n-m} \sum_{i=1}^{\lfloor Nt \rfloor} a(m1-i)^1 \prod_{j=1}^{k} \epsilon_{ij} = \frac{1}{N^H} \sum_{i \in \mathbb{Z}} a(m1-i)^1 \sum_{n=1}^{\lfloor Nt \rfloor} \prod_{j=1}^{k} \epsilon_{ij} = Q_k(h_{t,N}^\beta), \]
where
\[ h_{t,N}^\beta(i) = \frac{1}{N^H} \sum_{m \in \mathbb{Z}} a(m1-i)^1 (m1+i)^1 \sum_{n=1}^{\lfloor Nt \rfloor} C_{n-m}. \]

Making use of (46), and using \( l \) to denote a generic function such that \( l(i) \to 1 \) as \( i \to \infty \), we have if \( m \geq 1 \),
\[ \sum_{n=1}^{\lfloor Nt \rfloor} C_{n-m} = \sum_{n=1}^{\lfloor Nt \rfloor} C_n = \beta^{-1} l(\lfloor Nt \rfloor - m + 1) \lfloor Nt \rfloor - m + 1 \beta; \]
and if \( m \leq 0 \),
\[ \sum_{n=1}^{\lfloor Nt \rfloor} C_{n-m} = \sum_{n=1}^{\lfloor Nt \rfloor} C_n = \sum_{n=1}^{\lfloor Nt \rfloor} C_n = \sum_{n=1}^{\lfloor Nt \rfloor} C_n = - \sum_{n=1}^{\lfloor Nt \rfloor} C_n = \beta^{-1} \left[ l(\lfloor Nt \rfloor - m + 1) \lfloor Nt \rfloor - m + 1 \beta - l(-m)(-m)^\beta \right]. \]
So by letting \( i \) correspond to \( \lfloor Nx \rfloor + 1 \) and \( m \) to \( \lfloor Ns \rfloor + 1 \) (omitting \( L \) and \( l \) for simplicity),
\[ h_{t,N}^\beta(x) = N^{k/2} h_{t,N}^\beta(\lfloor Nx \rfloor + 1) \]
\[ = \beta^{-1} \int \mathbb{R} g \left( \frac{\lfloor Ns \rfloor - \lfloor Nx \rfloor}{N} \right) \mathbf{1}_{\{ \lfloor Ns \rfloor 1 > \lfloor Ns \rfloor \}} \left( \left( \frac{\lfloor Nt \rfloor - \lfloor Ns \rfloor}{N} \right)^\beta - \left( -\frac{\lfloor Ns \rfloor - 1}{N} \right)^\beta \right) ds. \]

Using similar arguments as in the proof of Theorem 6.5 we can bound the absolute value of the integrand above by \( C g^\ast(s1 - x)1_{\{s1 > x\}} \left( (t - s)^\beta - (-s)^\beta \right) \) for some \( C > 0 \), where \( g^\ast \) is a generalized Hermite kernel from Definition 3.13 (for the last term, we use \( \lfloor Ns \rfloor + 1 \geq Ns \)). Note that \( \beta < 0 \) in this case. By applying the Dominated Convergence Theorem, we get the desired f.d.d. convergence using Proposition 1.4.

Now we turn to the weak convergence. When \( H > 1/2 \), the tightness is standard. To show tightness under condition \( H < 1/2 \) and \( \mathbb{E}|\epsilon_i|^p < \infty \), Proposition 6.2 and the above f.d.d. convergence imply that for some constant \( c, C > 0 \) free from \( s, t, N \),
\[ \mathbb{E}|Y_N(t) - Y_N(s)|^{p'} \leq c \mathbb{E}[|Y_N(t) - Y_N(s)|^2]^{p'/2} \leq C|F_N(t) - F_N(s)|^pH, \]
where \( F_N(t) = \lfloor Nt \rfloor / N, p' < p \) and \( p'H > 1 \). Now by Lemma 4.4.1 and Theorem 4.4.1 of Giraitis et al. 8, we conclude that tightness holds.
6.4 Mixed multivariate limit theorem

In Bai and Taqqu [1], a multivariate version of Theorem 2.3 is obtained, where both central and non-central convergence appear simultaneously. We will state here a similar theorem.

Suppose that \( \mathbf{X}(n) = (X_1(n), \ldots, X_J(n)) \) is a vector of discrete chaos process defined on the same noise but with different coefficients, that is,

\[
X_j(n) = \sum_{0 < i_1, \ldots, i_{k_j} < \infty} a_j(i_1, \ldots, i_{k_j}) \epsilon_{n-i_1} \cdots \epsilon_{n-i_{k_j}} = \sum_{i \geq 0} a_j(i) \prod_{p=1}^{k_j} \epsilon_{n-i_p}, \tag{47}
\]

where we assume \( \{\epsilon_i\} \) is an i.i.d. random sequence with mean 0 and variance 1. For convenience we let \( a_j(i_1, \ldots, i_{k_j}) = a_j(i) = a_j(i)1_{i \geq 0} \), and \( \tilde{a}_j(\cdot) \) denotes the symmetrization of \( a_j(\cdot) \).

**Definition 6.12.** We say that the vector sequence of discrete chaos processes \( \{\mathbf{X}(n)\} \) is

- **SRD**, if every component \( X_j(n) \) is SRD in the sense of Definition 6.1, and in addition, for any \( p \neq q \in \{1, \ldots, J\} \),

\[
\sum_{n=-\infty}^{\infty} \sum_{i > 0} |\tilde{a}_p(i)\tilde{a}_q(n1 + i)| < \infty; \tag{48}
\]

- **LRD**, if every component \( X_j(n) \) is LRD in the sense of Definition 6.1.

- **fLRD**, if every component \( X_j(n) \) is a fractionally-filtered LRD discrete chaos process in the sense of Definition 6.7. Note: these components were denoted \( U(n) \) in that definition.

**Remark 6.13.** If the vector sequence is SRD, then [18] guarantees that the cross-covariance \( \gamma_{p,q}(n) := \text{Cov}(X_p(n), X_q(0)) \) satisfies \( \sum_{n} |\gamma_{p,q}(n)| < \infty \). As in Proposition 2.5 of [1], we have that as \( N \to \infty \),

\[
\text{Cov} \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt_1]} X_p(n), \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt_2]} X_q(n) \right) \to (t_1 \wedge t_2) \sum_{n=-\infty}^{\infty} \gamma_{p,q}(n). \tag{49}
\]

Note that \( \gamma_{p,q}(n) = 0 \) always if the orders \( k_p \neq k_q \).

We will now consider a general case where SRD and LRD and fLRD vectors can all be present in \( \mathbf{X}(n) \). We divide \( \mathbf{X}(n) \) into four parts

\[
\mathbf{X}(n) = (\mathbf{X}_{S_1}(n), \mathbf{X}_{S_2}(n), \mathbf{X}_{L}(n), \mathbf{X}_{F}(n))
\]

of dimension \( J_{S_1}, J_{S_2}, J_L, J_F \) respectively, which are defined as follows:

(i) all the components of \( \mathbf{X}_{S_1}(n) = (X_{1,S_1}(n), \ldots, X_{J_{S_1},S_1}(n)) \) have order \( k = 1 \), namely, are all linear processes;

(ii) every component of \( \mathbf{X}_{S_2}(n) = (X_{1,S_2}(n), \ldots, X_{J_{S_2},S_2}(n)) \) has order \( k \geq 2 \), and the combined vector

\[
\mathbf{X}_{S}(n) = (\mathbf{X}_{S_1}(n), \mathbf{X}_{S_2}(n)) = (X_{1,S}(n), \ldots, X_{J_{S},S}(n)); \quad J_S = J_{S_1} + J_{S_2},
\]

is SRD in the sense of Definition 6.12;

(iii) the vector \( \mathbf{X}_{L}(n) = (X_{1,L}(n), \ldots, X_{J_{L},L}(n)) \) is LRD in the sense of Definition 6.12 with correspondingly generalized Hermite kernels \( \mathbf{g} = (g_{1,L}, \ldots, g_{J_{L},L}) \);

(iv) the vector \( \mathbf{X}_{F}(n) = (X_{1,F}(n), \ldots, X_{J_{F},F}(n)) \) is fLRD in the sense of Definition 6.12 with correspondingly generalized Hermite kernels \( \mathbf{g} = (g_{1,F}, \ldots, g_{J_{F},F}) \) and fractional exponent \( \beta = (\beta_1, \ldots, \beta_{J_F}) \).

We now state the multivariate limit theorem. We use \( Y_N \) (with subscript \( S_1, S_2, L \) or \( F \)) to denote the corresponding normalized sum \( Y_N(t) := N^{-H} \sum_{n=1}^{[Nt]} X(n) \), where \( X(n) \) is a component of \( \mathbf{X}(n) \), \( H \) is such that \( \text{Var}(Y_N(1)) \) converges to some constant \( c > 0 \) as \( N \to \infty \).
Theorem 6.14. Following the notation defined above, one has

\[(Y_{N,S_1}(t), Y_{N,S_2}(t), Y_{N,L}(t), Y_{N,F}(t)) \xrightarrow{f.d.d.} (B_1(t), B_2(t), Z(t), Z^\beta(t)),\]

where

(i) \(B_1(t) = W(t) := (\sigma_1 W(t), \ldots, \sigma_{J_{S_1}} W(t))\) defined by the same standard Brownian motion \(W(t)\), and

\[\sigma_p = \sum_{n=-\infty}^{\infty} \sum_{i>0} a_{p,s_1}(n)a_{p,s_1}(n+i), \quad p = 1, \ldots, J_{S_1}.\]

(ii) \(B_2(t)\) is a multivariate Brownian motion with the covariance given by \(\Omega\);

(iii) \(Z(t)\) is a multivariate generalized Hermite process defined as in \(\mathcal{B}\) by the kernels \((g_{1,L}, \ldots, g_{J_{L},L})\) and using the \(W(t)\) in Point (i) as Brownian motion integrator.

(iv) \(Z^\beta(t)\) is a multivariate fractionally-filtered generalized Hermite process defined as in \(\mathcal{B}\) by the kernels \((g_{1,F}, \ldots, g_{J_{F},F})\), fractional exponent \(\beta = (\beta_1, \ldots, \beta_{J_F})\) and using the \(W(t)\) in Point (i) as Brownian motion integrator.

Moreover, \(B_2(t)\) is always independent of \((B_1(t), Z(t), Z^\beta(t))\).

In addition, \(f.d.d.\) in \(\mathcal{B}\) can be replaced with weak convergence in \(D[0,1]^d\), if every component of \(X_{S_1}\) and \(X_{S_2}\) satisfies the assumption in Theorem 6.4 and every component of \(X_F\) satisfies the assumption given at the end of Theorem 6.17.

The proof is similar to that of Theorem 3.5 of Bai and Taqqu [1]. We only provide some heuristics. The processes \(B_2(t), Z(t)\) and \(Z^\beta(t)\) involve the same integrator \(W(\cdot)\) because they are defined in terms of the same \(\epsilon_i\)'s. To understand the independence statement, note that the independence between \(B_2\) and \(W\) stems from the uncorrelatedness between \(X_{S_2}\) and \(X_{S_1}\), since \(X_{S_2}\) belongs to a discrete chaos of order \(k \geq 2\), while \(X_{S_1}\) belongs to a discrete chaos of order \(k = 1\). \(B_2\) is therefore independent of \(B_1\). \(B_2\) is also independent of \(Z\) and \(Z^\beta\), because \(Z\) and \(Z^\beta\) have \(W\) as integrators.

Remark 6.15. The pairwise dependence between components of \(Z\), of \(Z^\beta\), and between cross components in Theorem 6.14 can be checked using the criterion due to Ustunel and Zakai [31], that is, if \(f \in L^2(\mathbb{R}^p)\) and \(g \in L^2(\mathbb{R}^q)\), and both are symmetric, then the multiple Wiener-Itô integrals \(I_p(f)\) and \(I_q(g)\) are independent, if and only if

\[f \otimes g(x_1, \ldots, x_{p+q-2}) := \int_{\mathbb{R}} f(x_1, \ldots, x_{p-1}, y)g(x_p, \ldots, x_{p+q-2}, y)dy = 0 \quad a.e..\]

For example, suppose that two generalized Hermite kernels \(g_1\) and \(g_2\) on \(\mathbb{R}^p_+\) and \(\mathbb{R}^q_+\) are symmetric, then the corresponding two generalized Hermite processes are independent if and only if

\[\int_{\mathbb{R}} \int_0^t g_1(s-x_1, \ldots, s-x_{p-1}, s-y) ds \int_0^t g_2(s-x_p, \ldots, s-x_{p+q-2}, s-y) ds dy = 0 \quad a.e. ,\]

where we use the abbreviation \(g_j(x) = g_j(x)1_{(x>0)}, \ j = 1, 2\). Obviously, if \(g_1\) and \(g_2\) are both positive, then the dependence always holds. This is true, for example, for the symmetrized version of the kernels in (III).

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