1. Introduction

It is known that there exists a quasi-projective coarse moduli space for nonsingular n-dimensional varieties $X$ with ample canonical divisor $K_X$ and given Hilbert function $H(l) = \chi(X, O(lK_X))$ (e.g. see [Vie]). The starting point in the construction of this moduli space is Matsusaka’s big theorem [Mat] which states that the class of these varieties is bounded: there exists an integer $\nu_0 > 0$, independent of $X$, such that $\nu_0 K_X$ is very ample and has no higher cohomology. This condition is equivalent to the existence of a flat family $\mathcal{X} \to B$ over a scheme of finite type whose fibers contain all such varieties.

In order to compactify the moduli space of smooth n-folds, one has to add points at the boundary, corresponding to limits of smooth varieties. The natural way to construct these limits is to take a one-parameter semistable family of n-folds, and find the relative canonical model of this family in the sense of the minimal model program in dimension $n + 1$ (MMP$(n+1)$). All such limits have semi-log-canonical singularities.

We consider the class of stable (almost) smoothable n-folds $X$ with semi-log-canonical singularities and ample canonical divisor $K_X$. Smoothability here means that $X$ admits a deformation to a variety with rational Gorenstein singularities. The moduli functor $\mathcal{M}_H^{sm}$ then assigns to a scheme $S$ the set of isomorphism classes of families of stable smoothable n-folds over $S$ with a given Hilbert function $H$. The main result of this paper is that the minimal model program in dimension $n + 1$ implies boundedness of the functor $\mathcal{M}_H^{sm}$.

Theorem 1.1. Assuming MMP$(n+1)$, there exists a family $\tilde{X} \to B$ over a projective scheme $B$ in $\mathcal{M}_H^{sm}(B)$ whose geometric fibers include all stable smoothable n-folds with Hilbert function $H$.

If, in addition to boundedness, $\mathcal{M}_H^{sm}$ is locally closed, separated, complete, with tame automorphisms, and if the canonical polarization is semi-positive, then Kollár’s theorem [Kol] states that there exists a projective moduli space $M_H^{sm}$ coarsely representing the functor $\mathcal{M}_H^{sm}$. Semi-positivity of the canonical polarization has been proved by Kollár (Theorem 4.12, [Kol]). Since $K_X$ is ample, $X$ has a finite automorphism group by a result of Iitaka (Section 11.7, [Iit]). Separatedness of $\mathcal{M}_H^{sm}$ follows from the uniqueness of log-canonical models (Lemma 2.9). In Lemma 2.6 we prove that semi-log-canonical varieties deform to semi-log-canonical varieties; since the other conditions we require from stable smoothable n-folds are either open or closed, the functor $\mathcal{M}_H^{sm}$ is locally closed. Completeness then follows because $M_H^{sm}$ is the image of the projective scheme $B$. This proves the existence of coarse moduli spaces:

Corollary 1.2. Assuming MMP$(n+1)$, the moduli functor $\mathcal{M}_H^{sm}$ is coarsely represented by a projective scheme $M_H^{sm}$.
The minimal model program MMP(n+1) is known only in dimensions \( n + 1 \leq 3 \). Boundedness for semi-log-canonical surfaces has been proved by Alexeev [Ale1]. With this proof he finished the construction of projective coarse moduli spaces for semi-log-canonical surfaces that was started in [KSB]. The proof of Theorem 1.1 simplifies Alexeev’s result of boundedness for semi-log-canonical surfaces, and generalizes it to n-folds, subject to the minimal model program assumption.

In [Ale2] Alexeev considered the problem of constructing projective coarse moduli spaces for stable pairs \((X, B)\) consisting of an n-fold \( X \) and a divisor \( B \) in \( X \). He showed that the log-minimal model program in dimension \( n + 1 \) together with the boundedness assumption imply the existence of coarse moduli spaces for stable pairs. At the end of the paper we indicate how the proof of Theorem 1.1 can be modified to prove boundedness of stable smoothable pairs.

1.1. Sketch of the proof. We imitate the construction of a stable n-fold \( X \) as a limit of a one-parameter family of (almost) smooth varieties, and try to find all these limits at the same time. By Matsusaka’s theorem [Mat2], the class of normal varieties with rational Gorenstein singularities and ample canonical bundle is bounded. We start with a compactification of the Hilbert scheme parameterizing these varieties, and the universal family over it. To this morphism we apply weak semistable reduction [\( \ell \)-K] and then take the relative canonical model of the weakly semistable family. The crucial step in showing that this canonical model exists is a theorem of Siu and Kawamata on the invariance of plurigenera in families of canonical varieties [Kaw]. If \( f : X \to B \) is a weakly semistable morphism, then for a general nonsingular curve \( C \subset B \) the restriction \( X_C = f^{-1}(C) \) has canonical singularities, so we can find its relative canonical model by the MMP(n + 1) assumption. Now invariance of plurigenera implies that sections of \( f_*\mathcal{O}_{X_C}(K_{X_C}) \) can be lifted to sections of \( f_*\mathcal{O}_X(K_X) \), and it follows from this that the relative canonical ring of \( f : X \to B \) is a finitely generated \( \mathcal{O}_B \)-algebra.

Most results we prove about semi-log-canonical varieties and the moduli functor \( \mathcal{M}_H^{sm} \) are direct generalizations of the corresponding results for surfaces given in [KSB, Ale3], or simplifications of the results for stable pairs [Ale2].

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2. Preliminaries

We work over an algebraically closed field \( k \) of characteristic zero.

By a variety \( X \) we mean a reduced (not necessarily irreducible) separated scheme of finite type over the field \( k \). A birational morphism \( f : X \to Y \) is a morphism that maps every component of \( X \) birationally to a component of \( Y \), and has a birational inverse. Whenever we talk about a Weil divisor \( D \) in \( X \) we assume that the generic point of every component of \( D \) lies in the smooth locus of \( X \). A \( \mathbb{Q} \)-linear combination of Weil divisors \( D = \sum a_iD_i \) in \( X \) is \( \mathbb{Q} \)-Cartier if a multiple \( lD \) is Cartier for some integer \( l > 0 \). If \( f : Y \to X \) is a morphism, we let \( f^*(D) \) be the \( \mathbb{Q} \)-Cartier divisor \( \frac{1}{l}f^*(lD) \); and if \( f \) is birational then \( f_*^{-1}(D) = \frac{1}{l}f_*^{-1}(lD) \) is the strict transform of \( D \).

Suppose that \( X \) contains an open subvariety \( U \subset X \) with complement of codimension at least two such that \( U \) is Gorenstein, i.e. the dualizing sheaf \( \omega_U \) is invertible. We choose a canonical divisor \( K_U \), where \( \mathcal{O}_U(K_U) \cong \omega_U \), and let \( K_X \) be its closure in \( X \). The variety
X is called \( \mathbb{Q} \)-Gorenstein if \( K_X \) is \( \mathbb{Q} \)-Cartier. For a morphism \( f : Y \to X \), we let \( K_{Y/X} \) be such that \( \mathcal{O}_X(K_Y) \cong \mathcal{O}_Y(K_{Y/X}) \otimes f^* \mathcal{O}_X(K_X) \).

**Definition 2.1.** Let \( X \) be a normal \( \mathbb{Q} \)-Gorenstein variety, and let \( D \) be a \( \mathbb{Q} \)-Cartier divisor in \( X \). We say that the pair \( (X, D) \) has canonical (resp. log canonical) singularities if for any birational morphism \( f : Y \to X \) from a normal \( \mathbb{Q} \)-Gorenstein variety \( Y \) we have

\[
K_Y = f^*(K_X + D) - f_*^{-1}(D) + \sum a_i E_i
\]

where all \( a_i \geq 0 \) (resp. \( a_i \geq -1 \)).

To show that a variety has canonical or log-canonical singularities, it suffices to check the numerical condition for a resolution \( f : Y \to X \) of the pair \( (X, D) \), i.e. \( Y \) is nonsingular and the inverse image \( f^*(D) \) is a divisor with components crossing normally.

A flat projective morphism \( f : X \to C \) from a nonsingular variety \( X \) to a nonsingular curve \( C \) is semistable if its fibers are reduced divisors of simple normal crossing. If \( f : X \to C \) is any flat projective morphism to a nonsingular curve, we say that \( X \) admits a semistable resolution if there exists a birational morphism \( g : \tilde{X} \to X \) such that \( f \circ g : \tilde{X} \to C \) is semistable.

To prove Theorem 1.1 we need to assume the existence of relative canonical models (cf. [KMM]) in the following two cases:

**Assumption 2.2.** (MMP(n+1)). Let \( f : X \to Y \) be a morphism of varieties, \( \dim(X) = n + 1 \). Assume that either

1. \( f \) is birational and for some morphism \( g : Y \to C \) to nonsingular curve \( C \), the composition \( g \circ f : X \to C \) is semistable; or
2. \( f \) is a flat projective morphism from a variety \( X \) with canonical singularities to a nonsingular curve \( Y \) with fibers of general type.

Then the relative canonical ring

\[
R_{X/Y} = \bigoplus_{l \geq 0} f_* \mathcal{O}_X(lK_X)
\]

is a finitely generated \( \mathcal{O}_Y \)-algebra. The scheme \( X_{\text{can}} = \text{Proj} R_{X/Y} \) (or the morphism \( f_{\text{can}} : X_{\text{can}} \to Y \)) is called a relative canonical model.

Relative canonical models are unique: if \( X' \to Y \) is another morphism satisfying conditions 1. or 2. above, and if \( X' \) is birational to \( X \) over \( Y \), then \( X'_{\text{can}} \) is isomorphic to \( X_{\text{can}} \) over \( Y \).

**2.1. Semi-log-canonical singularities.**

**Definition 2.3.** (cf. [Ale3] Def. 2.8, [Kol] Def. 4.10) We say that a \( \mathbb{Q} \)-Gorenstein variety \( X \) has semi-log-canonical singularities if

(i) \( X \) satisfies Serre’s condition \( S_2 \);
(ii) \( X \) has normal crossing singularities in codimension 1;
(iii) for any birational morphism \( f : Y \to X \) from a normal \( \mathbb{Q} \)-Gorenstein variety \( Y \) we have

\[
K_Y = f^*(K_X) + \sum a_i E_i
\]

where all \( a_i \geq -1 \).
Remark 2.4. Condition (ii) implies that $X$ is Gorenstein in codimension one, so $K_X$ is well defined. We call the closure of the singular locus in codimension one the double divisor of $X$. Let $\nu : X' \to X$ be the normalization and $\text{cond}(\nu)$ the reduced effective divisor defined by

$$K_{X'} = \nu^*(K_X) - \text{cond}(\nu).$$

Then $\text{cond}(\nu)$ is the inverse image of the double divisor (with coefficients equal to one), and the condition (iii) is equivalent to the pair $(X, \text{cond}(\nu))$ having log-canonical singularities. Again, it suffices to check the numerical condition for a resolution $Y$ of the pair $(X, \text{cond}(\nu))$.

The only semi-log-canonical varieties we are going to consider are fibers of morphisms $Y \to C$ from a variety with canonical singularities to a nonsingular curve. These varieties are Cohen-Macaulay, so the condition (i) is automatically satisfied. If, moreover, the family admits a semistable resolution, then (ii) is also satisfied: localizing at a codimension two point of $Y$ (codimension one in $X = Y_0$) we get a germ of a surface with canonical singularities, admitting a semistable resolution; the fibers of such a surface are curves with normal crossing singularities.

Theorem 2.5. (cf. [KSE] Thm. 5.1.) Let $\pi : X \to C$ be a flat projective morphism from a normal $\mathbb{Q}$-Gorenstein variety to a germ of a nonsingular curve $(C, 0)$.

(i) Suppose that $X$ admits a semistable resolution. Then $X$ has canonical singularities if and only if the special fiber $X_0$ has semi-log-canonical singularities and the general fiber $X_1$ has canonical singularities.

(ii) The product $X \times_CC'$ has canonical singularities for all finite base changes $(C', 0') \to (C, 0)$ if and only if the special fiber $X_0$ has semi-log-canonical singularities and the general fiber $X_1$ has canonical singularities.

Proof. (i) $\Rightarrow$. Let $g : \tilde{X} \to X$ be a semistable resolution, and assume that $X$ has canonical singularities:

$$K_{\tilde{X}} = g^*K_X + \sum_i a_iE_i + \sum_j b_jF_j,$$

where $E_i$ are the exceptional divisors mapping to $0 \in C$, $F_j$ are flat over $C$, and $a_i, b_j \geq 0$ for all $i, j$. If $X'_0 = g^{-1}X_0$ is the strict transform of $X_0$, then by adjunction

$$K_{X'_0} = (K_{\tilde{X}} + X'_0)|_{X'_0} = (K_{\tilde{X}} - \sum E_i)|_{X'_0} = (g^*K_X + \sum_i (a_i - 1)E_i + \sum j b_jF_j)|_{X'_0}.$$

Let $\nu : Y \to X'_0$ be the normalization map. Then $K_Y = \nu^*K_{X'_0} - \text{cond}(\nu)$ where $\text{cond}(\nu)$ is a reduced effective normal crossing divisor, not containing any components supported on $E_i$. This proves that the special fiber $X_0$ is semi-log-canonical because $Y \to X_0$ is a resolution. Since $X$ is canonical, the general fiber $X_1$ is also canonical.

$\Leftarrow$. Let $g : \tilde{X} \to X$ be the relative canonical model over $X$ obtained by applying the MMP($n+1$) assumption to a semistable resolution of $X$. From the assumption that the general fiber of $\pi$ is canonical it follows that $g$ is an isomorphism away from the special fibers; in particular, there are no exceptional divisors $F_j$ flat over $C$ (with notation as above).

Suppose that some $a_i \geq 0$, say $a_0 \geq 0$ and $a_0$ is maximal among the $a_i$. Let $Y \subset E_0$ be a curve mapping to a point in $X$, not lying in any other component of $\tilde{X}_0$. Then

$$K_{\tilde{X}}Y = \sum a_i E_iY = \sum_{i \neq 0} a_iE_iY - \sum_{i \neq 0} a_0E_iY \leq 0.$$
and this contradicts the g-ampleness of $K_{X'}$. So, all $a_i < 0$, and since $X_0$ is semi-log-canonical, the adjunction formula above shows that there are no exceptional divisors $E_i$. Thus, $X$ has canonical singularities.

$(ii) \Rightarrow$. We know that after a finite base change $(C', 0') \to (C, 0)$ the fiber product $X' = X \times_C C'$ admits a semistable resolution. By part $(i)$ the special fiber $X'_{0'} \cong X_0$ is semi-log-canonical and the general fiber $X'_{t} \cong X_t$ is canonical.

$\Leftarrow$. It suffices to prove that $X$ has canonical singularities. For some finite base change $(C', 0') \to (C, 0)$, the fiber product $X' = X \times_C C'$ admits a semistable resolution and has canonical singularities by $(i)$. Let $h : \tilde{X} \to X$ be a resolution of $(X, X_0)$, and let $X' = \tilde{X} \times_X X'$:

$$
\begin{array}{ccc}
X' & \xrightarrow{f} & X \\
g \downarrow & & \downarrow h \\
\tilde{X} & \xrightarrow{j} & \tilde{X}
\end{array}
$$

First assume that $\tilde{X}'$ is normal. Tracing the diagram in two ways we get

$$
K_{\tilde{X}'} = g^*K_{X'} + \sum a_i E_i = g^*f^*K_X + m\tilde{X}'_0 + \sum a_i E_i,
$$

$$
K_{\tilde{X}} = f^*K_{\tilde{X}} + m\tilde{X}_0 = f^*h^*K_X + f^*\sum b_i F_i + m\tilde{X}_0.
$$

Since $X'$ has canonical singularities, all $a_i \geq 0$, hence all $b_i \geq 0$ and $X$ has canonical singularities.

If $\tilde{X}'$ is not normal, let $\nu : \tilde{X}' \to \tilde{X}'$ be the normalization, and $cond(\nu)$ the conductor:

$$
K_{\tilde{X}'} = \nu^*K_{\tilde{X}} - cond(\nu).
$$

Computing $K_{\tilde{X}'}$ in two ways as above, we get

$$
\nu^* \tilde{f}^* \sum_i b_i F_i = \sum a_i E_i + cond(\nu),
$$

where $a_i \geq 0$. Since $cond(\nu)$ is effective, we see that $b_i \geq 0$ and $X$ has canonical singularities.

The proof of the previous theorem can be easily modified to show that semi-log-canonical singularities deform to semi-log-canonical singularities.

**Lemma 2.6.** (cf. [KSB] Cor. 5.5.) Let $\pi : X \to C$ be a flat projective morphism from a $\mathbb{Q}$-Gorenstein variety to a germ of a nonsingular curve $(C, 0)$. If the special fiber $X_0$ has semi-log-canonical singularities, then the general fiber $X_t$ also has semi-log-canonical singularities.

**Proof.** First assume that $X$ is normal and admits a semistable resolution. Let $g : \tilde{X} \to X$ be the relative canonical model over $X$ of this resolution. As in the proof of part $(i)$ of the previous theorem, write

$$
K_{\tilde{X}} = g^*K_X + \sum a_i E_i + \sum b_j F_j
$$

where $E_i$ are the exceptional divisors mapping to the special fiber $X_0$, and $F_j$ are the exceptional divisors flat over $C$.

When we restrict to the general fiber $X_t$, it follows from the $g$-ampleness of $K_{\tilde{X}}$ that all $b_j < 0$ (cf. Lemma 2.19, [Kol1]). As before, we get from the adjunction formula that there are no exceptional divisors $E_i$ mapping to $X_0$. Since the special fiber $X_0$
is semi-log-canonical, the coefficients \( b_j \geq -1 \). It follows that the general fiber \( X_t \) is semi-log-canonical because \( \tilde{X}_t \rightarrow X_t \) is a resolution.

If \( X \) is not normal, we let \( \nu : X' \rightarrow X \) be the normalization. We may assume that the components of \( X' \) admit a semistable resolution (with the same base change) \( g : \tilde{X} \rightarrow X' \rightarrow X \). The conductor \( \text{cond}(\nu) \) maps flat onto \( C \), and as in the normal case, the discrepancies \( b_j \) of the general fiber are the same as the discrepancies of the special fiber.

\[ \begin{align*}
\text{2.2. The moduli functor } &M_{H}^{sm}. \\
\text{Definition 2.7.} \text{ (cf. Defs. 5.2, 5.4, [Kol])} \\
1. (a) A stable n-fold \( Y_0 \) is a connected projective n-dimensional variety (not necessarily irreducible) over an algebraically closed field of characteristic zero such that \( Y_0 \) has semi-log-canonical singularities and the canonical divisor \( K_{Y_0} \) is ample. \\
(b) The n-fold \( Y_0 \) is smoothable if there exists a flat projective \( \mathbb{Q} \)-Gorenstein one-parameter family \( \pi : Y \rightarrow C \) of stable n-folds such that the special fiber of \( \pi \) is \( Y_0 \) and the general fiber is a normal n-fold with at most rational Gorenstein singularities. \\
(c) Given a polynomial \( H \), we say that \( Y_0 \) has Hilbert function \( H \) if \( \chi(Y_0, \mathcal{O}(lK_{Y_0})) = H(l) \) for all \( l \geq 0 \).
2. A stable smoothable n-fold with given Hilbert function \( H \) over a scheme \( S \) is a flat projective morphism \( \pi : Y \rightarrow S \) such that \\
(a) every geometric fiber \( Y_s = \pi^{-1}(s) \) is a stable smoothable n-fold with Hilbert function \( H \); and \\
(b) we have a natural isomorphism \( \mathcal{O}_{Y_s}(lK_{Y_s}) \cong \mathcal{O}_Y(lK_Y)|_{Y_s} \) for every \( l > 0 \) and every point \( s \in S \).

\text{Definition 2.8.} We define the moduli functor \( M_{H}^{sm} : \text{(Schemes)} \rightarrow \text{(Sets)} \) by \( M_{H}^{sm}(S) = \{ \text{isomorphism classes of stable smoothable n-folds with given Hilbert function } H \text{ over } S \} \).

Given a smoothing \( \pi : Y \rightarrow C \) of a stable n-fold \( Y_0 \) as in Definition 2.7, the total space \( Y \) has canonical singularities by Theorem 2.5 (ii), and \( K_Y \) is \( \pi \)-ample. So, \( \pi : Y \rightarrow C \) is a relative canonical model. Conversely, if \( \pi : Y \rightarrow C \) is a relative canonical model which admits a semistable resolution, the special fiber \( Y_0 \) is a stable n-fold by Theorem 2.5 (i); and if the generic fiber of \( \pi \) is Gorenstein, then \( \pi : Y \rightarrow C \) is in fact a smoothing of \( Y_0 \). This shows that the moduli functor \( M_{H}^{sm} \) can be defined purely in terms of special fibers of canonical models.

Note that the conditions 1.(b) and 2.(b) in Definition 2.7 are both closed for flat families \( Y \rightarrow S \): the first by deformation theory, and the second by Kollar’s result on push-forward and base change. Lemma 2.6 proves that having semi-log-canonical singularities is an open condition. The condition that \( K_Y \) restricts to an ample divisor on fibers \( Y_s \) is also open (see Remark 3.3 below). It follows that the functor \( M_{H}^{sm} \) is locally closed.

\text{Lemma 2.9.} The functor \( M_{H}^{sm} \) is separated; i.e. any family \( X \rightarrow C \setminus \{0\} \) of stable smoothable n-folds over a punctured curve can be extended to a family over \( C \) in no more than one way.
Proof. (Theorem 3.28, [Ale3].) Suppose that we have two extensions $f_1: X_1 \to C$ and $f_2: X_2 \to C$. Let $\hat{X}$ be a common resolution of $X_1$ and $X_2$, $g_1: \hat{X} \to X_1$, $g_2: \hat{X} \to X_2$. Write

$$K_{\hat{X}} = g_1^*K_{X_1} + \sum a_i E_i + \sum b_j F_j$$

where $E_i$ are the exceptional divisors mapping to $0 \in C$, and $F_j$ are flat over $C$. Taking a finite base change $C' \to C$ if necessary, we may assume that $\hat{X}$ admits a semistable resolution.

First assume that $X_1$ is normal. As in the proof of Theorem 2.3, we get that $a_i \geq 0$ and $b_j \geq -1$. Since $K_{X_1}$ is $f_1$-ample, $X_1$ is the log-canonical model of the divisor $K_{\hat{X}} + \sum F_j$ and is given as

$$X_1 \cong \text{Proj} \bigoplus_{l \geq 0} (f_1 \circ g_1)_* \mathcal{O}_{\hat{X}}(l(K_{\hat{X}} + \sum F_j)).$$

The same is true for $X_2$, and since $X_1$ and $X_2$ are isomorphic away from the special fibers, the divisors $F_j$ are the same in both cases. Hence $X_1$ and $X_2$ are isomorphic.

If $X_1$ is not normal, we can recover its normalization $X_1'$ as the log-canonical model of the divisor $K_{\hat{X}} + \sum F_j$, where now $\sum F_j$ also includes the components of the inverse image of $\text{cond}(\nu)$ flat over $C$. This proves that $X_1$ and $X_2$ are isomorphic in codimension one. Both $X_1$ and $X_2$ satisfy Serre’s condition $S_2$, so they are in fact isomorphic. \(\square\)

2.3. Weak semistable reduction. Recall from [KKMS] that $U_X \subset X$ is a toroidal embedding if at every closed point $x \in X$ it is formally isomorphic to a torus embedding $T_x \subset Y_x$; such $T_x \subset Y_x$ is called a local model at $x$. A surjective morphism $f: (U_X \subset X) \to (U_B \subset B)$ is toroidal if it is equivariant in local models at every point. We only consider toroidal morphisms without horizontal divisors; i.e. no component of $X \setminus U_X$ dominates a component of $B$.

Definition 2.10. A toroidal morphism $f: (U_X \subset X) \to (U_B \subset B)$ without horizontal divisors is weakly semistable if

1. $f$ is equidimensional;
2. $f$ has reduced fibers; and
3. $B$ is nonsingular.

The morphism $f$ is semistable if also $X$ is nonsingular.

The following weak semistable reduction theorem was proved in [N-K];

Theorem 2.11. Let $X \to B$ be a surjective morphism of projective varieties with geometrically integral generic fiber. There exist a generically finite proper surjective morphism $B' \to B$ and a proper birational morphism $X' \to X \times_B B'$ such that the induced morphism $f': X' \to B'$ is weakly semistable. \(\square\)

Using the toroidal structure of a weakly semistable morphism it is not difficult to prove the following properties:

Lemma 2.12. Let $f: X \to B$ be weakly semistable. Then

1. $X$ has rational Gorenstein (hence canonical) singularities.
2. If $g: C \to B$ is a morphism from a nonsingular curve such that $g(C) \not\subset B \setminus U_B$ then $C$ and $X_C = X \times_B C$ can be given toroidal structures such that $X_C \to C$ is again toroidal and weakly semistable; in particular, $X_C$ has canonical singularities.
Proof. Part 1. is proved in [K-K], Lemma 6.1. Part 2. is proved in [K-K], Lemma 6.2 for the case when \( g : C \to B \) is a dominant morphism; the proof for a curve \( C \) is the same word-by-word. \( \square \)

Lemma 2.13. Let \( f : X \to C \) be a weakly semistable morphism from a variety \( X \) with \( \dim X = n + 1 \) to a curve \( C \). Assume that the fibers of \( f \) are of general type, and let \( f_{can} : X_{can} \to C \) be the relative canonical model. Then the special fiber \( X_{can,0} = f_{can}^{-1}(0) \) is a stable \( n \)-fold.

Proof. By Lemma 2.12, if \( g : (C',0') \to (C,0) \) is a finite base change, then \( X' = X \times_C C' \) is again canonical. Since \( K_{X'} = g^*K_X + mX'_0 \), we see that finding the canonical model commutes with taking a finite base change:
\[
X_{can} \times_C C' \cong (X \times_C C')_{can}
\]
and we can apply Theorem 2.5 (ii) to conclude that \( X_{can,0} \) has semi-log-canonical singularities. \( \square \)

It can be shown directly that the fibers of a weakly semistable morphism have semi-log-canonical singularities. We may assume that the base is a curve. Then in a local model the special fiber is the complement of the big torus in a toric variety. Alexeev has shown [Ale2] that this complement is semi-log-canonical.

Lemma 2.14. Let \( f : X \to C \) be as in the previous lemma, and assume that the genus \( g(C) \geq 3 \). Then the sheaf \( f_*\mathcal{O}_X(lK_X) \) is generated by global sections for all \( l \geq 2 \).

Proof. (Kollár [Kol]) By Theorem 4.12 in [Kol], the vector bundle \( E_l = f_*\mathcal{O}_X(lK_X/C) \) is semipositive for all \( l \geq 0 \); that means, any quotient line bundle of \( E_l \) has non-negative degree on \( C \). If \( L \) is a line bundle with \( \deg(L) > 2g(C) - 2 \) then \( \mathcal{O}_C(K_C) \) cannot be a quotient of \( E_l \otimes L \). Hence \( H^1(C,E_l \otimes L) = 0 \). Now if \( \deg(L) > 2g(C) \) then \( E_l \otimes L \) is generated by global sections. In particular, \( f_*\mathcal{O}_X(lK_X) = E_l \otimes \mathcal{O}_C(lK_C) \) is generated by global sections for all \( l \geq 2 \). \( \square \)

2.4. Deformations of canonical singularities and invariance of plurigenera. The following two theorems were proved by Kawamata [Kaw], generalizing results of Siu [Siu].

Theorem 2.15. Let \( \pi : X \to S \) be a flat morphism from a germ of a variety \( (X,x_0) \) to a germ of a smooth curve \( (S,s_0) \) whose special fiber \( X_0 = \pi^{-1}(s_0) \) has only canonical singularities. Then \( X \) has only canonical singularities. In particular, the fibers \( X_s = \pi^{-1}(s) \) have only canonical singularities. \( \square \)

Theorem 2.16. Let \( \pi : X \to S \) be a projective flat morphism from a normal variety to a germ of a smooth curve \( (S,s_0) \). Assume that the fibers \( X_s = \pi^{-1}(s) \) have only canonical singularities and are of general type for all \( s \in S \). Then the plurigenus \( P_m(X_s) = \dim H^0(X_s,mK_{X_s}) \) is constant as a function on \( s \in S \) for any \( m > 0 \). \( \square \)

The two theorems for families of \( n \)-folds follow easily from the \( MMP(n+1) \) assumption. However, we are going to apply them for families of \( (n+1) \)-dimensional varieties, and we do not want to assume minimal model program in dimension \( n+2 \).
3. Proof of boundedness

By Matsusaka’s theorem (Theorem 2.4, [Mat2]) there exists an integer $\nu_0 > 0$ such that if $X$ is a normal variety with rational Gorenstein singularities, with ample canonical divisor $K_X$, and with given Hilbert function $H(l) = \chi(X, lK_X)$, then $\nu_0 K_X$ is very ample and has no higher cohomology. Thus, the Hilbert scheme parameterizing embeddings $X \in \mathbb{P}^{H(\nu_0) - 1}$ with Hilbert function $H(\nu_0 l)$ has a locally closed subscheme parameterizing $\nu_0$-canonical embeddings of varieties $X$ with rational Gorenstein singularities. We give this subscheme the induced reduced structure and call its closure $B_0$. Let $f_0 : X_0 \to B_0$ be the (closure of the) universal family over $B_0$. We apply weak semistable reduction (Theorem 2.11) to the morphism $f_0$ to get a weakly semistable morphism

$$f : X \to B.$$ 

By Lemma 2.12 we know that for a general nonsingular curve $C \subset B$ the inverse image $X_C = f^{-1}(C)$ has canonical singularities. We can then apply the MMP(n+1) assumption and find a canonical model $X_{C,\text{can}}$ for $X_C$. Since the general fiber of $X_{C,\text{can}} \to C$ has rational Gorenstein singularities, we see that the special fiber is a stable smoothable n-fold with Hilbert function $H$.

The idea of the proof is to construct a relative canonical model for the whole family $f : X \to B$ and to show that the restriction of this canonical model to a curve $C$ is the canonical model $X_{C,\text{can}}$.

**Lemma 3.1.** The relative canonical ring

$$R_{X/B} = \bigoplus_{l \geq 0} f_* \mathcal{O}_X(lK_X)$$

is a finitely generated $\mathcal{O}_B$-algebra.

To prove the lemma we sweep $B$ locally with nonsingular curves. By Bertini’s theorem a general hyperplane section through $b \in B$ is nonsingular, and the same is true for nearby hyperplane sections. Applying this dim $B - 1$ times we get the following diagram:

$$
\begin{array}{ccc}
X_1 = B_1 \times_B X & \longrightarrow & X \\
\downarrow & & \downarrow f \\
B_1 & \phi \longrightarrow & B \\
g \downarrow & & \\
S & & 
\end{array}
$$

where

1. $(g : B_1 \to S, \phi)$ is a family of nonsingular curves in $B$ parameterized by $S$, no curve lying in $B \setminus U_B$;
2. there exist $b_1 \in B_1$, $\phi(b_1) = b$, and open neighborhoods $b_1 \in U_1$, $b \in U$ such that $\phi : U_1 \to U$ is an isomorphism;
3. we may assume that the hyperplane sections have high degree, so the fibers of $g$ will have high genus ($\geq 3$).

Since Lemma 3.1 is local in $B$ we may replace $B$ by $B_1$ and $X$ by $X_1$. 
**Lemma 3.2.** Consider the Cartesian diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow f & & \downarrow f \\
C & \longrightarrow & B \\
\downarrow g & & \downarrow g \\
\{s\} & \longrightarrow & S
\end{array}
\]

where \(s \in S\) is a closed point, \(C = g^{-1}(s)\), and \(Y = f^{-1}(C)\). Then the natural morphism

\[f_*\mathcal{O}_X(lK_X) \otimes_{\mathcal{O}_B} \mathcal{O}_C \rightarrow f_*\mathcal{O}_Y(lK_Y)
\]

is an isomorphism for every \(s \in S\) and every \(l \geq 2\).

**Proof.** By “cohomology and base change” [Har] it suffices to prove the surjectivity of this morphism. Lemma 2.14 shows that the sheaf \(f_*\mathcal{O}_Y(lK_Y)\) is generated by global sections \(H^0(Y, \mathcal{O}_Y(lK_Y))\) for every \(l \geq 2\). The family \(g \circ f : X \rightarrow S\) has fibers of general type (by sub-additivity of Kodaira dimension [Vie1], since the fibers of both \(f\) and \(g\) are of general type) and with canonical singularities (by Lemma 2.12). Theorem 2.16 now implies that a global section in \(H^0(Y, \mathcal{O}_Y(lK_Y))\) can be lifted to a local section of \((g \circ f)_*\mathcal{O}_X(lK_X)\), which gives a section of \(f_*\mathcal{O}_X(lK_X)\). \(\square\)

**Remark 3.3.** For the next proof we note that ampleness is an open condition: given a flat projective morphism \(f : X \rightarrow B\) and a \(\mathbb{Q}\)-Cartier divisor \(D\) on \(X\) such that \(D|_{X_b}\) is ample for some \(b \in B\), then \(D\) is \(f\)-ample over a neighborhood of \(b\) in \(B\). This follows from Kleiman’s numerical criterion for ampleness [Kle]. If \(Z_1(X/B)\) is the free group generated by reduced irreducible curves in \(X\) mapping to a point in \(B\), and \(N_1(X/B)\) is \(Z_1(X/B) \otimes \mathbb{R}\) modulo numerical equivalence (see [KMM] for notations), then we have natural linear maps \(Z_1(X_b) \rightarrow N_1(X/B)\) and \(Z_1(X_\eta) \rightarrow N_1(X/B)\) where \(\eta\) is the generic point of \(B\). Now the second map factors through the first: for a curve over the generic point of \(B\) we take its closure and restrict to the special fiber. Also, the maps restricted to effective cones factor. If \(D\) is ample on \(X_b\) then \(D\) is positive on the effective cone \(\overline{NE}(X_b) \setminus \{0\}\), hence it is positive on \(\overline{NE}(X_\eta) \setminus \{0\}\), and so \(D\) is ample on \(X_\eta\).

**Proof of Lemma 3.1.** The statement is local in \(B\), so we may replace \(B\) by a smaller open neighborhood of \(b \in B\) if necessary. We use the same notation as in Lemma 3.2.

By the minimal model program assumption MMP(n+1) applied to \(f : Y \rightarrow C\), the sheaf \(R_{Y/C}\) is a finitely generated \(\mathcal{O}_C\)-algebra, and by Lemma 3.2, \(R_{X/B} \otimes \mathcal{O}_C \cong R_{Y/C}\), at least in high degrees. Since the graded pieces \((R_{X/B})_l = f_*\mathcal{O}_X(lK_X)\) are finite \(\mathcal{O}_B\)-modules we can apply Nakayama’s lemma and conclude that \(R_{X/B} \otimes B_b\) is a finitely generated \(\mathcal{O}_{B,b}\)-algebra. Consider

\[\text{Proj} \ (R_{X/B} \otimes B_b) \rightarrow \text{Spec} \ (B_b).
\]

Since this morphism is projective, defined by finitely many equations, we get a projective scheme \(\tilde{X}\) over an open neighborhood \(U \subset B\) of \(b\), and after replacing \(B\) by \(U\):

\[\tilde{f} : \tilde{X} \rightarrow B.
\]

By construction, \(\tilde{Y} = \tilde{f}^{-1}(C)\) is a relative canonical model of \(f : Y \rightarrow C\); in particular, it has canonical singularities. By Theorem 2.15 canonical singularities deform to canonical singularities, hence \(\tilde{X}\) has canonical singularities.
The canonical divisor $K_X$ is $\tilde{f}$-ample when restricted to $\tilde{Y}$. Using the numerical criterion for ampleness, $K_X$ is $\tilde{f}$-ample over a neighborhood of $b$. Again, we may replace $B$ by a smaller open set and assume that $K_X$ is $\tilde{f}$-ample.

Now $\tilde{f} : \tilde{X} \to B$ is a relative canonical model, birational to $f : X \to B$ over $B$, hence the relative canonical rings are the same

$$R_{X/B} \cong R_{\tilde{X}/B}.$$ 

Since the latter is finitely generated as an $O_B$-algebra, so is the former. \hfill \Box

**Proof of Theorem 1.1** We show that $\tilde{f} : \tilde{X} = \text{Proj} R_{X/B} \to B$ is the required family.

It is clear from the construction that every closed fiber $\tilde{f}^{-1}(b)$ is a stable smoothable n-fold with Hilbert function $H$. Indeed, we have constructed through every closed point $b \in B$ a nonsingular curve $C$ such that $\tilde{f}^{-1}(C)$ is the relative canonical model of a weakly semistable family over $C$ with Gorenstein generic fiber, so the special fiber is stable and smoothable by Lemma 2.13.

To prove the converse, namely that the fibers of $\tilde{f}$ contain all stable smoothable n-folds with Hilbert function $H$, we go back to the construction of $\tilde{f} : \tilde{X} \to B$ from the universal family over the Hilbert scheme. We have the commutative diagram

$$\begin{array}{ccc}
\tilde{X} & \leftarrow & X \\
\downarrow \tilde{f} & & \downarrow f \\
B & \leftarrow & B \\
\end{array}$$

where the morphisms $B \to B_0$ and $X \to X_1$ come from weak semistable reduction and the map $X \to \tilde{X}$ is only rational.

Let $V_{B_0} \subset B_0$ be the open set parameterizing $\nu_0$-canonical embeddings of normal varieties with rational Gorenstein singularities; and let $V_B \subset B, V_{X_1} \subset X_1$, and $V_{\tilde{X}} \subset \tilde{X}$ be the inverse images of $V_{B_0}$. Now $f_1 : V_{X_1} \to V_B$ is a family of normal varieties with rational Gorenstein singularities and ample canonical bundle over a nonsingular base. It follows that $V_{X_1}$ is normal with canonical singularities and $f_1 : V_{X_1} \to V_B$ is a relative canonical model, birational to $\tilde{f} : V_{\tilde{X}} \to V_B$ over $V_B$. By uniqueness of the relative canonical model, the two are isomorphic $V_{\tilde{X}} \cong V_{X_1}$ over $V_B$.

Let $\pi : Y \to C$ be a smoothing of a stable n-fold $Y_0$, and let $Y_\eta = \pi^{-1}(\eta)$ be the fiber over the generic point $\eta$ of $C$. Since $Y_\eta$ is a projective rational Gorenstein scheme with ample canonical bundle and Hilbert function $H$, we can find a morphism $\eta \to V_{B_0}$ such that the universal family $V_{X_0}$ pulls back to $Y_\eta$. There exists a finite morphism $(C', \eta') \to (C, \eta)$ such that we can lift $\eta \to V_{B_0}$ to $\eta' \to V_B$. By completeness of $B$ we get a morphism $C' \to B$. Let $\hat{Y} = C' \times_B \tilde{X}$, and let $Y' = C' \times_C Y$. The two families $\hat{Y} \to C'$ and $Y' \to C'$ are isomorphic over the generic point, so their special fibers must be isomorphic by separatedness of the functor $\mathcal{M}_H^{sm}$ (Lemma 2.3). \hfill \Box

### 3.1. Boundedness for stable smoothable pairs

With the notations of [Ale2], we consider stable pairs $(Y_0, D_0)$ where $Y_0$ is a connected projective variety and $D_0$ is a reduced effective Weil divisor in $Y_0$ such that the pair $(Y_0, D_0)$ has semi-log-canonical singularities and $K_{Y_0} + D_0$ is an ample $\mathbb{Q}$-Cartier divisor. We say that $(Y_0, D_0)$ is smoothable if it admits a deformation to a stable pair $(Y_t, D_t)$ where $Y_t$ has rational singularities and $K_{Y_t} + D_t$ is Cartier. Further, we fix the Euler characteristics $\chi(Y_0, l(K_{Y_0} + D_0))$ and $\chi(D_0, l(K_{Y_0} + D_0)|_{D_0})$ for all $l$. 


By Matsusaka’s theorem, for some \( \nu_0 > 0 \) the divisor \( \nu_0(K_Y + D_t) \) is very ample and without higher cohomology for all pairs \((Y_t, D_t)\) as above. There exists a Hilbert scheme parameterizing embeddings \( Y_t \subset \mathbb{P}^N \) with the given Hilbert function. Since \( D_t \subset Y_t \subset \mathbb{P}^N \) also has a fixed Hilbert function, there is a Hilbert scheme parameterizing such embeddings. In the product of the two Hilbert schemes we can find a locally closed subscheme parameterizing the required embeddings \((Y_t, B_t) \subset \mathbb{P}^N \) via \( \nu_0(K_Y + D_t) \); there is also a universal family \((Y, D)\) over this subscheme.

As before, we complete the universal family and apply weak semistable reduction to it. We can make sure that the proper transform \( D \) of the universal divisor \( D \) in the weakly semistable family \( f : X \to B \) is the union of horizontal toroidal divisors. The proof that \( \bigoplus \mathcal{O}_X(l(K_X + D)) \) is a finitely generated \( \mathcal{O}_B \)-algebra goes as before, replacing the \( \text{MMP}(n + 1) \) with \( \log \text{MMP}(n + 1) \) assumption, if Kollár’s semipositivity result and Siu-Kawamata theorems on invariance of plurigenera and deformations of canonical singularities also hold for pairs. I do not know if these are true. Kollár’s semipositivity theorem can be avoided in the proof of boundedness by considering ramified cyclic covers of the base variety \( S \); but to construct the moduli space we would still need it. As remarked above, the theorems on invariance of plurigenera and deformations of canonical singularities can be replaced by the \( \text{MMP}(n + 2) \) assumption; this is, however, a very strong assumption, considering for example the case of surfaces \( n = 2 \).

References

[N-K] D. Abramovich and K. Karu, Weak semistable reduction in characteristic 0, preprint. [alg-geom/9707012]

[Ale1] V. Alexeev, Boundedness and \( K^2 \) for log surfaces, Int. J. Math 5 (1994), no. 6, 779-810.

[Ale2] V. Alexeev, Log canonical singularities and complete moduli of stable pairs, preprint. [alg-geom/9608013]

[Ale3] V. Alexeev, Moduli spaces \( M_{g,n}(W) \) for surfaces, preprint. [alg-geom/9410003]

[Har] R. Hartshorne, Algebraic Geometry., Graduate Texts in Mathematics, vol. 52, Springer-Verlag New York, Inc., 1977.

[Iit] S. Iitaka, Algebraic Geometry. An introduction to birational geometry of algebraic varieties, Graduate Texts in Mathematics, vol. 76, Springer-Verlag New York, Inc., 1982.

[Kaw] Y. Kawamata, Deformations of Canonical singularities, preprint. [alg-geom/9712018]

[KMM] Y. Kawamata, K. Matsuda, K. Matsuki Introduction to the minimal model problem, Advanced Studies in Pure Mathematics 10 (1987) 283-360.

[KKMS] G. Kempf, F. Knudsen, D. Mumford and B. Saint-Donat, Toroidal Embeddings I, Springer, LNM 339, 1973.

[Kle] S. Kleiman, Toward a numerical theory of ampleness, Ann. Math. 84 (1966) 293-344.

[Kol] J. Kollár, Projectivity of complete moduli, J. Diff. Geom. 32 (1990), 235-268.

[Kol1] J. Kollár et al., Flips and abundance for algebraic threefolds, Astérisque 211 (1992), 1-258.

[KSB] J. Kollár and N. Shepherd-Barron, Threefolds and deformations of surface singularities, Invent. Math 91 (1988), 299-338.

[Mat1] T. Matsusaka, Polarized varieties with given Hilbert polynomial, Amer. J. Math. 9 (1972) 1027-1077.

[Mat2] T. Matsusaka, On polarized normal varieties, I, Nagoya Math. J. Vol 104 (1986), 175-211.

[Siu] Y.-T. Siu, Invariance of plurigenera, preprint. [alg-geom/9712016]

[vie] E. Viehweg, Quasi-projective moduli for polarized manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete Band 30. Springer, 1995.

[vie1] E. Viehweg, Die Additivität der Kodaira Dimension für projektive Fasserräume über Varietäten des allgemeinen Typs, Jour. reine und angew. Math. 330 (1982), 132-142.

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