Conversion of second class constraints by deformation of Lagrangian local symmetries

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Abstract

For a theory with first and second class constraints, we propose a procedure for conversion of second class constraints based on deformation the structure of local symmetries of the Lagrangian formulation. It does not require extension or reduction of configuration space of the theory. We give examples in which the initial formulation implies a non linear realization of some global symmetries, therefore is not convenient. The conversion reveals hidden symmetry presented in the theory. The extra gauge freedom of conversed version is used to search for a parameterization which linearizes the equations of motion. We apply the above procedure to membrane theory (in the formulation with world-volume metric). In the resulting version, all the metric components are gauge degrees of freedom. The above procedure works also in a theory with only second class constraints presented. As an examples, we discuss arbitrary dynamical system of classical mechanics subject to kinematic constraints, $O(N)$-invariant nonlinear sigma-model, and the theory of massive vector field with Maxwell-Proca Lagrangian.

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1 Introduction and outlook

The conventional method for covariant quantization of a theory with second class constraints is to go over to an equivalent formulation where second class constraints are replaced by the first class ones in one or another way. One possibility is to work in extended phase space, where an additional variables can be used for conversion [1]. Another possibility is to search for special deformation of a theory in original phase space, which allows one either to discard half of the constraints ("gauge unfixing method") [2], or to solve the constraints in terms of a redundant number of variables [3]. Then the gauge theory quantization methods can be applied to the resulting formulation.

The above mentioned conversion schemes have been developed in Hamiltonian framework. In this work we propose a conversion scheme based on a Lagrangian formulation. It does not imply a change (extension or reduction) of the number of configuration space variables. Roughly speaking, in Lagrangian theory (with first and second class constraints presented in the Hamiltonian formulation) we search for parameterization of configuration space which results in special deformation of original local symmetries and, in turn, implies conversion of second class constraints.

The work is organized as follows. In the rest of this section we describe our procedure in some details. In sections 2 and 3 we discuss two specific mechanical models where conversion of second class constraints allows one to make manifest hidden global symmetries of the theory. We point also that extra gauge freedom presented in the converted version can be useful for the linearization of equations of motion. In section 4 we convert second class constraints of membrane theory, in the formulation which involves world-volume

1 Green-Schwarz superstring action can be interesting in this respect. While for IIB case fermionic constraints can be covariantly separated into irreducible first and second class subsets [17], type IIA and N = 1 cases remain unsolved problem up to date.
metric. In resulting version all the metric components turn out to be gauge degrees of freedom. Further we explain how the conversion procedure can be applied in a theory with second class constraints only. Here our scheme implies an extension of original space by pure gauge degrees of freedom. For the theory of massive vector field with Maxwell-Proca Lagrangian it simply reduces to introduction of Stuckelberg field (section 6). Arbitrary dynamical system subject to kinematic constraints is considered in section 5, as particular examples we discuss a particle on a sphere and $O(N)$-invariant nonlinear sigma-model.

Here we describe schematically our procedure of conversion. Let $L(q^A, \dot{q}^A)$ be Lagrangian of a theory with first and second class constraints presented in Hamiltonian formulation. In Lagrangian formulation, the first class constraints manifest themselves in invariance of action under some local symmetry transformations [4, 5, 6].

Let

$$\delta q^A = (k) \epsilon R^A(q, \dot{q}) + \ldots ,$$

be infinitesimal form of one of the symmetries, with local parameter $\epsilon(\tau)$ and gauge generator $R^A$. In Eq. (1) $^{(k)} \epsilon \equiv \frac{\partial^{(k)} \epsilon}{\partial \tau^k}$, and the dots stand for all terms with less than $k$-derivatives acting on a parameter. A local symmetry with at most $k$ derivatives acting on a parameter is called $^{(k)}$-symmetry below. $^{(k)}$-symmetry generally implies [7] the appearance of some constraint on the $(k + 1)$-stage of the Dirac procedure (clearly, it means that there is a chain [8] composed of primary, secondary, $\ldots$, $(k + 1)$-stage constraints). This observation will be crucial for our present discussion. Now we describe how the symmetry (1) can be used to convert some pair of second class constraints into a first class constraint.

Let us divide coordinates $q^A$ in two groups: $q^A = (q^i, q^\alpha)$. We change parameterization of the configuration space: $q^A \rightarrow \tilde{q}^A$ ac-
cording to the transformation which involves derivatives of $q^\alpha$

$$q^i = q^i(\tilde{q}^A, \dot{\tilde{q}}^\alpha), \quad q^\alpha = q^\alpha(\tilde{q}^\beta).$$

(2)

We suppose that the transformation is "invertible"

$$\det \frac{\partial q^i}{\partial \tilde{q}^j} \neq 0, \quad \det \frac{\partial q^\alpha}{\partial \tilde{q}^\beta} \neq 0,$$

(3)

which implies that $\tilde{q}^A$ can be determined from (2): $\tilde{q}^i = \tilde{q}^i(q^A, \dot{q}^\alpha), \tilde{q}^\alpha = \tilde{q}^\alpha(q^\beta)$. Owing to the conditions (3), our theory can be equally analyzed in terms of the Lagrangian $\tilde{L} \equiv L(q(\tilde{q}), \dot{q}(\tilde{q}))$. We further suppose that the transformation (2) has been chosen in such a way that $\tilde{L}$ does not involve higher derivatives, modulo to total derivative term (we show below that it is possible in singular theory)

$$\tilde{L}(\tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) = \tilde{L}'(\tilde{q}, \dot{\tilde{q}}) + \frac{dF(\tilde{q}, \dot{\tilde{q}})}{d\tau}.$$  

(4)

Let us see what one can say about the structure of Hamiltonian constraints of our theory in the new parameterization $\tilde{L}$, in comparison with $L$. One should note that the local symmetry for the set $\tilde{q}$ is generally of $(k+1)$-type: $\delta \tilde{q}^i = (k+1) \frac{\partial \tilde{q}^i}{\partial q^\alpha} \tilde{R}^\alpha(q^A, \dot{q}^A, \ddot{q}^\alpha) + \ldots$. Since the order of the symmetry has been raised by one unit, on $(k+2)$-stage of the Dirac procedure an extra constraint appears. On other hand, the physical sector of $\tilde{L}$ is the same as for $L$. If order of other symmetries (if any) was not lowered, the only possibility\(^2\) is that extra $(k+2)$-stage constraint is of first class, and it replaces some pair of second class constraints of initial formulation. In resume, an appropriate parameterization (2), (3), (4) of the configuration space implies a deformation of local symmetries which, in turn, can result in conversion of second class constraints. Clearly, Eqs. (3), (4) represent only necessary conditions for the conversion.

\(^2\)Here the condition (4) is important. A deformed theory with higher derivatives, being equivalent to the initial one, has more degrees of freedom than the number of variables $q^A$ [5]. So the extra constraints would be responsible for ruling out of these hidden degrees of freedom. Our condition (4) forbids the appearance of the hidden degrees of freedom.
Note that one can consider more general transformations: 
\[ q^i = q^i(\tilde{q}^A, \tilde{q}^\alpha, \ddot{\tilde{q}}^\alpha, \ldots, (s)\tilde{q}^\alpha), \quad q^\alpha = q^\alpha(\tilde{q}^\beta) \]
which involve higher derivatives of \( \tilde{q}^\alpha \). It generally increases the order of symmetry by \( s \) units, and 2s second class constraints can be converted. Example of such a kind is presented in the subsection 2.2.

To illustrate the prescription, let us analyze the following dynamically trivial model defined on configuration space \( x(\tau), y(\tau), z(\tau) \), with the Lagrangian action being

\[ S = \int d\tau \left( \frac{1}{2}(\dot{x} - y)^2 + \frac{1}{2}z^2 \right). \]

(5)

It is invariant under finite local symmetry with the parameter \( \alpha(\tau) \)

\[ \delta x = \alpha, \quad \delta y = \dot{\alpha}, \quad \delta z = 0. \]

(6)

In terms of the variable set \( x, y, z \), the action (5) has \( \dot{\alpha} \)-symmetry. Passing to the Hamiltonian formulation, one obtains the following chains of constraints:

| primary | secondary |
|---------|-----------|
| \( p_y = 0 \) | \( p_x = 0 \) |

(7)

second class chain

\( p_z = 0 \), \( z = 0 \).

(8)

To convert the second class constraints (8), we make the transformation \( z = \tilde{z} + \dot{y} \). In terms of \( x, y, \tilde{z} \) variables, the action acquires the form

\[ S = \int d\tau \left( \frac{1}{2}(\dot{x} - y)^2 + \frac{1}{2}(\dot{\tilde{z}} + \dot{y})^2 \right), \]

and has irreducible \( \ddot{\alpha} \)-symmetry

\[ \delta x = \alpha, \quad \delta y = \dot{\alpha}, \quad \delta \tilde{z} = -\ddot{\alpha}. \]

(10)

The necessary conditions (3), (4) are satisfied, and since order of symmetry has been raised, one expects that second class chain is
replaced now by some tertiary first class constraint. Actually, the action (9) implies the following first class chain

\[ p_{\tilde{z}} = 0, \quad p_{y} = 0, \quad p_{x} = 0. \] (11)

In the gauge \( \tilde{z} = 0 \), the theories (5) and (9) have the same dynamics and thus are equivalent. This example demonstrate also that our procedure is different from the conversion scheme of the work [3] based on a redundant parametrization. In resume, second class constraints have been converted without changing (extension or reduction) of number of variables of the theory.

The condition (4) can be easily satisfied if some variable enters into the action without derivative. In this respect, let us point out that for a singular theory \( L(q, \dot{q}) \), there exists an equivalent formulation \( L'(q', \dot{q}') \) with the desired property. Actually, starting from the singular \( L: \text{rank} \frac{\partial^2 L}{\partial q^A \partial q^B} = [\alpha] < [A] \), one can construct the Hamiltonian \( H = H_0(q^A, p_j) + v^\alpha \Phi_\alpha \), where \( \Phi_\alpha(q^A, p_B) = p_\alpha - f_\alpha(q^A, p_j) \) are primary constraints, and the variables \( q^A \) have been divided in two groups according to the rank condition: \( q^A = (q^1, q^\alpha) \), \( \det \frac{\partial^2 L}{\partial q^A \partial q^B} \neq 0 \). Here \( H_0, f_\alpha \) do not depend on \( p_\alpha \) [9]. We further separate a phase space pair which corresponds to some fixed \( \alpha \), for example \( \alpha = 1: \alpha = 1, \alpha', (q^A, p_A) = (q^1, p_1, z) \). According to [5] (see p. 256), there exists a canonical transformation \( (q^1, p_1, z) \rightarrow (q'^{1}, p'_1, z') \), such that the Hamiltonian acquires the form \( H' = H'_0(q'^{1}, z') + v^1 p'_1 + v^\alpha' \Phi_{\alpha'}(q'^{1}, z') \). One can restore [10] the Lagrangian \( L'(q', \dot{q}') \) which reproduce \( H' \) in the Hamiltonian formalism. By construction, \( L' \) does not depend on \( \dot{q}'^{1} \).

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4Our new variables \( \tilde{z}, p_{\tilde{z}} \) do not solve the second class constraints.
2 Conversion in a theory with hidden $SO(1, 4)$
global symmetry

Let’s consider a theory with configuration space variables $x^\mu$, $e$, $g$
(where $\mu = 0, 1, 2, 3$, $\eta_{\mu\nu} = (-, +, +, +)$), and action

$$S = \int d\tau \left( \frac{1}{2e}(\dot{x}^\mu - gx^\mu)^2 + \frac{1}{2e^2}g^2 - ag \right), \quad a = \text{const.} \quad (12)$$

The model has a manifest $SO(1, 3)$ global symmetry. The only local
symmetry is the reparametrization invariance, with form transformations being $\dot{\alpha}$-symmetry

$$\delta \tau = 0, \quad \delta x^\mu = -\alpha \dot{x}^\mu, \quad \delta e = -\dot{\alpha} e, \quad \delta g = -\dot{\alpha} g \quad (13)$$

The model turns out to be interesting in the context of doubly
special relativity [11]. Passing to the Hamiltonian formalism one
obtains the Hamiltonian ($v_i$ denote Lagrangian multipliers for the
corresponding primary constraints)

$$H = \frac{e}{2}p^2 + g(xp) - \frac{g^2}{2e^2} + ag + v_e p_e + v_g p_g, \quad (14)$$
as well as the constraints (the initial constraints have been reorgan-
ized with the aim to separate the first class ones)

$$p_e + (xp + a)p_g = 0, \quad p^2 + (xp + a)^2 + 2ep^2p_g = 0; \quad (15)$$

$$p_g = 0, \quad g - e(xp + a) = 0. \quad (16)$$

The first (second) line represents first (second) class constraints.
The equations of motion of the $(e, x)$-sector can be written as follow

$$\dot{e} = v_e, \quad \dot{p}_e = 0,$$

$$\dot{x}^\mu = e(p^\mu + (xp + a)x^\mu), \quad \dot{p}^\mu = -e(xp + a)p^\mu. \quad (17)$$

In terms of variables

$$\chi^\mu = \frac{ax^\mu}{xp + a}, \quad p^\mu = \frac{ap^\mu}{xp + a}, \quad (18)$$
they acquire a form similar to those of free relativistic particle, namely
\[ \dot{X}^\mu = e P^\mu, \quad \dot{P}^\mu = 0, \quad P^2 = -a^2. \] (19)

The presence of the conserved current \( \dot{P}^\mu = 0 \) indicates a hidden global symmetry related with the homogeneity of the configuration space. As it will be demonstrated, a conversion reveals the symmetry and allows one to find manifestly invariant formulation of the theory.

To convert a pair of second class constraints (16) one needs to raise order of symmetry (13) by unit. From Eq. (13) one notes that it can be achieved by shifting some variable on \( \dot{e} \). Since the variable \( g \) enters into the action without derivative, a shift of the type \( g = \tilde{g} + \dot{e} \) does not lead to higher derivative terms in the action and thus realizes the conversion. It is convenient to accompany the shift by an appropriate change of variables. Namely, let us make the invertible transformation \((x^\mu, e, g) \rightarrow (\tilde{\tilde{x}}^A = (\tilde{\tilde{x}}^\mu, \tilde{\tilde{x}}^4), \tilde{g})\), where
\[ \tilde{\tilde{x}}^\mu = e^{-\frac{1}{2}} x^\mu, \quad \tilde{\tilde{x}}^4 = e^{-\frac{1}{2}}, \quad \tilde{\tilde{g}} = g - \frac{\dot{\tilde{g}}}{2e}. \] (20)

In terms of these variables the action (12) acquires the form
\[ \tilde{S} = \int d\tau \left( \frac{1}{2} (\tilde{\tilde{x}}^A - \tilde{\tilde{g}} \tilde{\tilde{x}}^A)^2 - a \tilde{\tilde{g}} \right), \quad \eta_{AB} = (-, +, +, +, +), \] (21)
where the einbein \( e \) was combined with \( \tilde{\tilde{x}}^\mu \) to form a 5-vector. The resulting action has a manifest \( SO(1, 4) \) global symmetry. The conserved current \( P^\mu \) then corresponds to the symmetry under rotations in \((\tilde{\tilde{x}}^\mu, \tilde{\tilde{x}}^4)\)-planes. THE Local symmetry of the action (21) can be obtained from Eqs. (13), (20), and is of \( \ddot{\alpha} \)-type
\[ \delta \tau = 0, \quad \delta \tilde{\tilde{x}}^A = \frac{1}{2} \ddot{\alpha} \tilde{\tilde{x}}^A - \alpha \ddot{\tilde{\tilde{x}}}^A, \quad \delta \tilde{\tilde{g}} = \frac{1}{2} \ddot{\alpha} - \dot{\alpha} \tilde{\tilde{g}} - \alpha \ddot{\tilde{\tilde{g}}}. \] (22)

Passing to the Hamiltonian formulation one obtains the Hamiltonian
\[ H = \frac{1}{2} \tilde{p}^2 + \tilde{\tilde{g}} \tilde{x}^A \tilde{p}_A + a \tilde{\tilde{g}} + v \tilde{\tilde{g}} \tilde{p}, \] (23)
and the constraints
\[ \ddot{p}_g = 0, \quad \dot{x}^A \dot{p}_A + a = 0, \quad \dot{p}_4 \dot{p}_A = 0, \] (24)
all of them being the first class. Thus \( \tilde{S} \) represents the converted version of the action (12). Let us write equations of motion for \( x^A \)-sector
\[ \dot{x}^A = \ddot{p}_A + \ddot{g} \dot{x}^A, \quad \dot{p}^A = -\ddot{g} \dot{p}^A. \] (25)
In the gauge \( \ddot{g} = \ddot{x}^\mu \ddot{p}_\mu + a, \ \ddot{p}_4 = \ddot{x}^\mu \ddot{p}_\mu + a \) for the theory (21) one reproduces the initial dynamics (17) (taken in the gauge \( e = 1 \)). Going over to the gauge \( \ddot{g} = 0, \ \ddot{p}_4 = a \), one obtains the free equations (19). Hence the extra gauge freedom, resulting from the conversion of second class constraints, can be used for search for parametrization which linearises equations of motion.

3 Conversion in a theory with hidden conformal symmetry

Here we discuss a conversion of a chain with four second class constraints presented. Let us consider an action with manifest \( SO(1,4) \) global symmetry
\[ S = \int d\tau \left( \frac{1}{2e}(\dot{x}^A)^2 - \frac{e}{2}m^2 + g \left((x^A)^2 - a^2\right) \right), \] (26)
where \( A, B = 0, 1, 2, 3, 4, \ \eta_{AB} = (-, +, +, +), \ m, a = \text{const}, a \neq 0 \). It is a reparametrization invariant, with the form transformations being \( \dot{\alpha} \)-symmetry, see (13). In the Hamiltonian formulation one finds the following system of constraints
\[ p_e + \frac{m^2}{2a^2} p_g = 0, \quad (p^A)^2 + m^2 - \frac{m^2}{a^2}((x^A)^2 - a^2) = 0; \] (27)
\[ p_g = 0, \quad (x^A)^2 - a^2 = 0, \quad x^A p_A = 0, \quad g - \frac{m^2}{2a^2} e = 0. \] (28)
The first (second) line represents first (second) class constraints. The chain of four second class constraints can be converted by raising of order of the local symmetry by two units. To this end, let us make invertible transformation \((x^A, e, g) \rightarrow (\tilde{x}^M = (\tilde{x}^A, \tilde{x}^5), \tilde{g})\), where
\[
\tilde{x}^A = e^{-\frac{1}{2}}x^A, \quad \tilde{x}^5 = ae^{-\frac{1}{2}}, \quad \tilde{g} = eg + \frac{3e^2}{8e^2} - \frac{\tilde{e}}{4e}.
\] (29)
For this set of variables, the action (26) acquires the form (note that there are no of higher derivative terms)
\[
\tilde{S} = \int d\tau \left( \frac{1}{2}(\dot{\tilde{x}}^M)^2 + \tilde{g}(\tilde{x}^M)^2 - \frac{1}{2}a^2m^2(\tilde{x}^5)^{-2} \right),
\] \(\eta_{MN} = (-, +, +, +, -)\). (30)
Local symmetry of (30) can be obtained from Eqs. (13), (29), and is of \((\alpha)\)-type
\[
\delta \tau = 0, \quad \delta \tilde{x}^M = \frac{1}{2}\alpha \tilde{x}^M - \alpha \dot{x}^M, \quad \delta \tilde{g} = \frac{1}{4}(\alpha - 2\dot{\alpha} - \tilde{g} + \alpha \dot{\tilde{g}}). \] (31)
In the Hamiltonian formulation one obtains the constraints
\[
\tilde{p}_g = 0, \quad (\tilde{x}^M)^2 = 0, \quad \tilde{x}^M \tilde{p}_M = 0, \quad (\tilde{p}_M)^2 + c^2 m^2(\tilde{x}^5)^{-2} = 0, \] (32)
all of them being the first class. Thus the transformation (29) turn out \(\alpha\)-symmetry of the initial action into \((\alpha)\)-symmetry, which results in replacement of four second class constraints (28) by a pair of first class ones.

For completeness, let us compare equations of motion for the action \(S\)
\[
\dot{e} = v_e, \quad \dot{\tilde{p}}_e = 0; \quad \dot{x}^A = ep^A, \quad \dot{p}^A = m^2a^{-2}e x^A, \] (33)
with the corresponding equations for the action \(\tilde{S}\)
\[
\dot{\tilde{x}}^5 = \tilde{p}^5, \quad \dot{\tilde{p}}^5 = 2\tilde{g}\tilde{x}^5 - a^2m^2(\tilde{x}^5)^{-3};
\]
\begin{equation}
\dot{x}^A = \dot{p}^A, \quad \ddot{p}^A = 2\tilde{g}\dot{x}^A. \tag{34}
\end{equation}

In the gauge \( e = 1 \) for the first theory and \( \ddot{x}^5 = a, \) \( \ddot{p}^5 = 0, \) \( \tilde{g} = \frac{a^2}{2m^2} \) for the second theory the equations (as well as the remaining constraints) coincide. The constraints \( (\dot{x}^M)^2 = 0, \) \( \ddot{x}^M\dddot{p}_M = 0 \) can also be linearized, see [12].

The action (26) with \( m = 0 \) implies conservation of \( p^A: \) \( \dot{p}^A = 0, \) the latter equation appears as one of equations of motion. It indicates on hidden global symmetry responsible for the current. The conversion of second class constraints made by transition to the action (30) reveals the symmetry: the action (30) with \( m = 0 \) is \( SO(2,4) \)-invariant. The current \( p^A \) corresponds to rotations in \( (\dot{x}^A, \dot{x}^5) \)-planes.

4 Conversion of second class constraints in the membrane action

Here we consider a membrane in terms of variables \( x^\mu(\sigma^i), \) \( g^{ij}(\sigma^i), \) where \( \sigma^i, \) \( i = 0,1,2 \) are coordinates parametrizing world-volume, \( x^\mu, \) \( \mu = 0,1,2, \ldots, D - 1 \) gives embedding of the world-volume in a Minkowski space-time, \( g^{ij} \) represent metric on the world-volume. The membrane action [13]

\begin{equation}
S = \frac{T}{2} \int d^3\sigma (-\text{det} g^{ij}) \frac{1}{2} (-g^{ij}\partial_i x^\mu \partial_j x^\mu + 1), \tag{35}
\end{equation}

is invariant under reparametrizations on the world-volume, where \( x^\mu \) are scalar functions and \( g^{ij} \) is a second rank tensor. The corresponding infinitesimal transformations of the form are

\begin{align*}
\delta \sigma^i &= 0, \quad \delta x^\mu = -\xi^i \partial_i x^\mu, \\
\delta g^{ij} &= g^{ik}\partial_k \xi^j + g^{jk}\partial_k \xi^i - \xi^k \partial_k g^{ij} = g^{i0}\dot{\xi}^j + g^{j0}\dot{\xi}^i + \ldots, \tag{36}
\end{align*}

where in the second line we have omitted those terms which do not involve time derivative of parameters. Owing to \( \dot{\xi} \)-symmetry (36),
six first class constraints appear in the Hamiltonian formulation. Besides (note that the metric obeys algebraic equations), more six constraints of second class are presented. We demonstrate below, how the second class constraints can be converted into first class ones by deformation of local symmetry (36).

We begin with making convenient parametrization of the world-volume metric. Namely, let’s consider the following change of variables

$$g^{ij} \rightarrow (N, N^a, \gamma^{ab}), a, b = 1, 2,$$

where

$$g^{ij} = \begin{pmatrix} -(\det \gamma^{ab})^{-1}N^2 & (\det \gamma^{ab})^{-1}NN^a \\ (\det \gamma^{ab})^{-1}NN^b & (\det \gamma^{ab})^{-1}(\gamma^{ab} - N^a N^b) \end{pmatrix}. \quad (37)$$

It is invertible, with the inverse transformation being

$$N = g^{00}(-\det g^{ij})^{-\frac{1}{2}}, \quad N^a = g^{0a}(-\det g^{ij})^{-\frac{1}{2}}, \quad \gamma^{ab} = (\det g^{ij})^{-1}(g^{ab}g^{00} - g^{0a}g^{0b}). \quad (38)$$

Now the action acquires a polynomial form for all variables except $N$

$$S = \frac{T}{2} \int d^3\sigma (N(\dot{x}^\mu - N^{-1}N^a \partial_a x^\mu)^2 - N^{-1}\gamma^{ab} \partial_a x^\mu \partial_b x^\mu + N^{-1} \det \gamma^{ab}) \quad (39)$$

Moreover, the symmetry (36) acquires more transparent form for the new variables, in particular, $\delta \gamma^{ab}$ do not involves time derivative of the parameters

$$\delta N = N\dot{\xi}^0 + \ldots, \quad \delta N^a = N\dot{\xi}^a + \ldots, \quad \delta \gamma^{ab} = 0 + \ldots \quad (40)$$

In the Hamiltonian formalism the action (39) implies the constraints

$$p_N = 0, \quad \frac{P^2}{T^2} + \det(\partial_a x \partial_b x) = 0, \quad p_{N^a} = 0, \quad p\partial_a x = 0, \quad (41)$$

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4they are related with conventional ADM variables $g^{00} = -\tilde{N}^{-2}, \ g^{0a} = \tilde{N}^{-2} \tilde{N}^a, \ g^{ab} = \tilde{\gamma}^{ab} - \tilde{N}^{-2} \tilde{N}^a \tilde{N}^b$ as follows: $\tilde{N} = (\det \gamma^{ab})^{\frac{1}{2}} N^{-1}, \ \tilde{N}^a = N^a N^{-1}, \ \tilde{\gamma}^{ab} = (\det \gamma^{ab})^{-1} \gamma^{ab}$.

5An additional transformation $N^{-1}N^a = \tilde{N}^a, \ N\gamma^{ab} = \tilde{\gamma}^{ab}$ implies polynomial form of action. But the hatted variables have more complicated transformation law.
\[ \pi_{ab} = 0, \quad (\det \gamma_{cd})^{-1} \gamma_{ab} = \partial_a x \partial_b x, \quad (42) \]

where the last line represents six second class constraints. Here \( \pi_{ab} \) are conjugated momenta for \( \gamma_{ab} \), and \( \gamma_{ab} \) is inverse matrix for \( \gamma^{ab} \).

The last expression in (42) is equivalent to \( \gamma_{22} = \partial_1 x \partial_1 x \), \( \gamma_{12} = -\partial_1 x \partial_2 x \), \( \gamma_{11} = \partial_2 x \partial_2 x \). Eq. (40) suggests that conversion can be performed by the following shift in Eq. (39)

\[ \gamma_{ab} = \begin{pmatrix} h_{11} + \dot{N}_1 & h_{12} + \dot{N} \\ h_{12} + \dot{N} & h_{22} + \dot{N}_2 \end{pmatrix}, \quad (43) \]

In comparison with the initial action (39), one has now kinetic terms for \( N \)-fields. So the only three primary constraints appear: \( \pi_{ab} = 0 \), where \( \pi_{ab} \) are conjugated momenta for \( h_{ab} \). On the other hand, the modified action has three \( \ddot{\alpha} \)-symmetries, see Eqs. (40), (43). Thus one expects appearance of three tertiary first class constraints, the latter replace six second class constraints (42) of the initial formulation. In some details, for the modified action

\[ \tilde{S} = \frac{T}{2} \int d^3 \sigma \left[ N(\dot{x}^\mu - N^{-1} N^a \partial_a x^\mu)^2 - N^{-1} ((h^{aa} + \dot{N}^a) \partial_a x \partial_a x + 2(h_{12} + \dot{N}) \partial_1 x \partial_2 x - (h_{11} + \dot{N}^1) (h_{22} + \dot{N}^2) - (h_{12} + \dot{N})^2) \right], \quad (44) \]

one obtains the Hamiltonian

\[ H = \int d^2 \sigma \left( \frac{1}{2TN} \dot{p}^2 + \frac{\dot{N}^a}{N} p \partial_a x - \frac{N}{2T} p_N^2 + \frac{2N}{T} p_{N1} p_{N2} - p_N h_{12} + p_{N2} h^{aa} - p_N \partial_1 x \partial_2 x + p_{N2} \partial_1 x \partial_1 x + p_{N1} \partial_2 x \partial_2 x + \frac{T}{2N} \det(\partial_a x \partial_b x) + v_{h}^{ab} \pi_{ab} \right), \quad (45) \]

as well as the following three chains of first class constraints

\[ \pi_{12} = 0, \quad p_N = 0, \quad \frac{p^2}{T^2} + \det(\partial_a x \partial_b x) = 0, \]
\[ \pi_{aa} = 0, \quad p_{N_a} = 0, \quad p \partial_a x = 0. \quad (46) \]

Thus all the metric components turn out to be gauge degrees of freedom in the theory (44). Starting from the Hamiltonian (45),
one obtains the well known membrane equations of motion [14] in
the gauge $N = 1$, $N^a = 0$, $(\text{det} h_{ab})^{-1} h_{ab} = \partial_a x \partial_b x$ (they can be linearized for half-rigid membrane [15]).

In resume, we have found a special representation (37), (43) for
the membrane world-volume metric. The reparametrization invar-
iance for the new variables turns out to be a symmetry of $\bar{\alpha}$-type,
which implies conversion of second class constraints presented in
the initial action. In the modified action (44), all the metric com-
nents are gauge degrees of freedom. It would be interesting to find
manifestly $\bar{\alpha}$-symmetry covariant formulation for the action (44).

5 Classical mechanics subject to kinematic con-
straints as a gauge theory

Our conversion trick can be realized also in a theory with second
class constraints only (i. e. in a theory without local symme-
tries presented in the initial formulation). To proceed with, one
notes that arbitrary theory without local symmetry can be treated
as a gauge theory on appropriately extended configuration space.
Namely, given theory with the action $S(q^A)$ on configuration space
$q^A$ can be equally considered as a theory on the space $q^A, a$, with
local transformations defined by $q'^A = q^A, a' = a + \alpha$, where $a$ is one
more configuration space variable. Since $a$ does not enter into the
action, the latter is invariant under the local transformations. The
trivial gauge symmetry of the extended formulation can be further
used for the conversion of second class constraints according to our
procedure.\footnote{\textit{It is general situation: for an arbitrary locally invariant theory one can chose special
variables such that the action does not depend on some of them [5].}}

\footnote{\textit{There are other possibilities to create trivial local symmetries. For example, in a given
Lagrangian action with one of variables being $q$, let us make the substitution $q = ab$, where
$a, b$ represent new configuration space variables. The resulting action is equivalent to the
initial one, an auxiliary character of one of new degrees of freedom is guaranteed by the trivial}}
Let us see how it works on example of classical mechanics with kinematic constraints. Let $L_0(q^a, \dot{q}^b)$ be Lagrangian of some system of classical mechanics in terms of generalized coordinates $q^a$. The Lagrangian is supposed to be nondegenerate

$$\det \frac{\partial^2 L_0}{\partial \dot{q}^a \partial \dot{q}^b} \neq 0. \tag{47}$$

A motion restricted to lie on some hypersurface defined by nondegenerate system of equations $G_i(q^a) = 0$, $\text{rank} \frac{\partial G_i}{\partial q^a} = [i] < [a]$ can be described by the well known action with Lagrangian multipliers $\lambda^i(\tau)$

$$S = \int d\tau (L_0(q, \dot{q}) + \lambda^i G_i(q)). \tag{48}$$

Here the variables $\lambda^i(\tau)$ are considered on equal footing with original variables $q^a(\tau)$. Let us construct a Hamiltonian description of the system. Due to the rank condition (47), equations for the momenta:

$$p_a = \frac{\partial L_0}{\partial \dot{q}^a} |_{\dot{q} = f(q,p)} \equiv p_a, \quad \det \frac{\partial f^a}{\partial p_b} \neq 0. \tag{49}$$

Conjugated momenta for $\lambda^i$ represent $[i]$ primary constraints of the theory: $p_{\lambda i} = 0$. Then one obtains the Hamiltonian

$$H = H' - \lambda^i G_i(q) + \nu^i p_{\lambda i}, \quad H' \equiv p_a f^a - L(q,f). \tag{50}$$

Conservation in time of the primary constraints: $\dot{p}_{\lambda i} = \{p_{\lambda i}, H\} = 0$ implies secondary constraints $G_i(q) = 0$. In turn, conservation of $G$ gives tertiary constraints $F_i \equiv G_{ia}(q)f^a(q,p) = 0$, where $G_{ia} \equiv \frac{\partial G_i}{\partial q^a}$ and Eq. (49) was used. Poisson brackets of the constraints are

$$\{G_i, F_j\} = G_{ia} \frac{\partial f^c}{\partial p_a} G_{jc} \equiv \Delta_{ij}. \tag{51}$$

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gauge symmetry: $a \rightarrow a' = aa$, $b \rightarrow b' = a^{-1}b$. Another simple possibility is to write $q = a + b$, which implies the symmetry $a \rightarrow a' = a + a$, $b \rightarrow b' = b - a$. The well known examples of such a kind transformation are einbein formulation in gravity theory: $g_{\mu\nu} = e^a_{\mu} e^a_{\nu}$ (which implies local Lorentz invariance), as well as duality transformations in some specific models [16].
Owing to Eq.(49) and the condition \( \text{rank} G_{ia} = [i] \) one concludes \( \det \Delta_{ij} \neq 0 \). The inverse matrix for \( \Delta \) is denoted as \( \tilde{\Delta}_{ij} \). Further, the condition \( \dot{F}_i = 0 \) implies fourth stage constraints \( \lambda^i - \tilde{\Delta}^{ij}\{F_j, H'\} = 0 \). At last, conservation in time of these constraints determines all the remaining velocities: \( v^i_\lambda = \{\tilde{\Delta}^{ij}\{F_j, H'\}, H' - \lambda^k G_k\} \). Thus we have a theory with \( 4[i] \) second class constraints

\[
p_{\lambda i} = 0, \quad G_i = 0, \quad f^a G_{ia} = 0, \quad \lambda^i - \tilde{\Delta}^{ij}\{F_j, H'\} = 0. \quad (52)
\]

The conversion can be carried out by making of the following transformation in the action (48)

\[
\lambda^i = \tilde{\lambda}^i + \ddot{e}^i, \quad (53)
\]

where auxiliary configuration space variable \( e^i(\tau) \) has been introduced. The modified action

\[
\tilde{S} = \int d\tau (L_0(q, \dot{q}) - \dot{e}^i G_{ia} \dot{q}^a + \tilde{\lambda}^i G_i(q)), \quad (54)
\]

does not contain higher derivative terms and is invariant under local transformations \( \tilde{\lambda}^i \rightarrow \tilde{\lambda}^i + \ddot{\alpha}^i, \quad e^i \rightarrow e^i = e^i - \alpha^i \). Due to this \( \ddot{\alpha} \)-symmetry one expects an appearance of \( 3[i] \) first class constraints in the Hamiltonian formulation for the theory (54). To confirm this, let us write defining equations for conjugated momenta

\[
p_a \equiv \frac{\partial L}{\partial \dot{q}^a} = \frac{\partial L_0}{\partial \dot{q}^a} - \dot{e}^i G_{ia}, \quad p_{ei} \equiv \frac{\partial L}{\partial \dot{e}^i} = -G_{ia} \dot{q}^a, \quad p_{\lambda i} = 0. \quad (55)
\]

The last equation represents \( [i] \) primary constraints. Remaining equations can be resolved in relation of the velocities \( \dot{q}^a, \dot{e}^i \), since the corresponding block of Hessian matrix is non degenerate. It can be easily seen in special coordinates chosen as follows. The initial coordinates \( q^a \) can be reordered in such a way that rank minor of the matrix \( \frac{\partial G_i}{\partial q^a} \) is placed on the right: \( q^a = (q^\alpha, \ q^i) \), \( \det \frac{\partial G_i}{\partial q^i} \neq 0 \). Now, let us make invertible change of variables \( q^a \rightarrow \bar{q}^a \), where \( \bar{q}^a = q^\alpha, \ \bar{q}^i = G_i(q^a) \). In this variables our Lagrangian is

\[
\bar{L} = L_0(\bar{q}, \dot{\bar{q}}) - \dot{e}^i \bar{q}^i + \tilde{\lambda}^i \bar{q}^i. \quad (56)
\]
From this expression one immediately finds the determinant of the Hessian matrix being \( \det \frac{\partial^2 L}{\partial \dot{q} \partial \dot{e}} = \det \frac{\partial^2 L_0}{\partial \dot{q} \partial \dot{q}} \). It is nonzero since in classical mechanics the quadratic form \( \frac{\partial^2 L_0}{\partial \dot{q} \partial \dot{q}} \) is positive defined.

Let us return to analysis of the action (54). The corresponding Hamiltonian is

\[
H = p_0 \dot{q}^a + p_e \dot{e}^i - L_0(q, \dot{q}) + \dot{e}^i G_{ia} q^a - \tilde{\lambda}^i G_i(q) + v_i^i p_{\tilde{\lambda} i},
\]

where \( \dot{q}^a, \dot{e}^i \) are solutions of equations (55). As before, secondary constraints turn out to be \( G_i(q) = 0 \). Their conservation in time can be easily computed by using of Eq.(55): \( \dot{G}_i = \{G_i, H\} = -p_{ei} \) which gives tertiary constraints \( p_{ei} = 0 \). Then the complete constraint system is composed by \( 3[i] \) first class constraints

\[
p_{\tilde{\lambda} i} = 0, \quad G_i = 0, \quad p_{ei} = 0.
\]

First class constraint \( p_{ei} = 0 \) simply states that variables \( e^i \) are pure gauge degrees of freedom, as it was expected. They can be removed from the formulation if one chooses the gauge \( e^i = 0 \). The remaining \( 2[i] \) first class constraints in Eq.(58) replace \( 4[i] \) second class constraints of the initial formulation.

As a particular example, let us consider a motion of a particle on 2-sphere of radius \( c \), with the action being

\[
S = \int d^3 x \left( \frac{1}{2} m (\dot{x}^i)^2 + \lambda ((x^i)^2 - c^2) \right).
\]

It implies the following chain of 4 second class constraints

\[
p_\lambda = 0, \quad x^2 - c^2 = 0, \quad xp = 0, \quad p^2 + 2mc^2 \lambda = 0.
\]

The conversion is achieved by the transformation \( \lambda = \tilde{\lambda} + \frac{1}{2} m \dot{e} \), which generates the symmetry \( \tilde{\lambda} \to \tilde{\lambda}' = \tilde{\lambda} + \frac{1}{2} m \dot{\alpha} \), \( e \to e' = e - \alpha \). The modified action

\[
\tilde{S} = \int d^3 x \left( \frac{1}{2} m \dot{x}^2 - m \dot{e} \dot{x} + \tilde{\lambda} (x^2 - c^2) \right).
\]
implies first class constraints only, namely

\[ p_\lambda = 0, \quad x^2 - c^2 = 0, \quad p_e = 0. \] (62)

**O(N)-invariant nonlinear sigma-model**

\[ S = \int d^D x \left( \frac{1}{2} (\partial_\mu \phi^a)^2 + \lambda ((\phi^a)^2 - 1) \right), \] (63)

represents example of field theory with similar structure of second class constraints. Hence the transformation \( \lambda = \tilde{\lambda} + \partial_\mu \partial^\mu e \) gives formulation with first class constraints only

\[ \tilde{S} = \int d^D x \left( \frac{1}{2} (\partial_\mu \phi^a)^2 - 2 \partial_\mu e \partial^\mu \phi^a + \tilde{\lambda} ((\phi^a)^2 - 1) \right). \] (64)

6 **Conversion in Maxwell-Proca Lagrangian for massive vector field**

As one more example of the conversion in a theory with second class constraints only, we consider massive vector field \( A^\mu(x^\nu) \) in Minkowski space (with the signature being \((-+,+++)\)) . It is described by the following action:

\[ S = \int d^4 x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu \right), \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu. \] (65)

Passing to the Hamiltonian formulation one finds the Hamiltonian

\[ H = \int d^3 x \left( \frac{1}{2} p_i^2 - p_i \partial_i A^0 + \frac{1}{4} F^2 - \frac{1}{2} m^2 A^\mu A_\mu + v_0 p_0 \right), \] (66)

as well as primary and secondary constraints

\[ p_0 = 0, \quad \partial_i p_i + m^2 A^0 = 0. \] (67)

The system is of second class, with the Poisson bracket algebra being

\[ \{ \partial_i p_i + m^2 A^0, p_0 \} = m^2 \delta^3(x - y). \] (68)
Conservation in time of the secondary constraint determines the velocity \( v_0 = -\partial_k A_k \). The equations of motion for propagating modes turn out to be

\[
\partial_0 \dot{A}^i = p^j - \partial_j A^0, \quad \partial_0 p_i = \partial_k F_{ki} + m^2 A_i, \quad (69)
\]
while the modes \( A^0, p_0 \) are determined by the algebraic equations (67). In a converted version these modes turn into gauge degrees of freedom. For the case, a transformation which creates desirable \( \dot{\alpha} \)-symmetry consist in introduction of Stuckelberg field \( \phi(x^\mu) \)

\[
A_\mu = \tilde{A}_\mu - \partial_\mu \phi. \quad (70)
\]

According to our philosophy, one can think that, from the beginning, we have a theory on configuration space \( A_\mu, \phi \), with the local symmetry being \( A'^\mu = A^\mu \), \( \phi' = \phi + \alpha \), and the action given by Eq. (65). That is \( \phi \) does not enter into the action. Then one introduces the new parametrization (70) of the configuration space: \( A_\mu, \phi \rightarrow \tilde{A}_\mu, \phi \). The modified action

\[
\tilde{S} = \int d^4x \left( -\frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} m^2 (\tilde{A}^\mu - \partial^\mu \phi)(\tilde{A}_\mu - \partial_\mu \phi) \right), \quad (71)
\]
is invariant under the local transformations

\[
\phi \rightarrow \phi' = \phi + \alpha, \quad \tilde{A}_\mu \rightarrow \tilde{A}'_\mu = \tilde{A}_\mu + \partial_\mu \alpha, \quad (72)
\]
where \( \tilde{A}_\mu \) transforms as electromagnetic field. Due to this \( \dot{\alpha} \)-symmetry, one expects appearance of two first class constraints in the modified formulation. Actually, primary constraint of the theory (71) is the same as before: \( \tilde{p}_0 = 0 \). Then the Hamiltonian turns out to be

\[
H = \int d^3x \left( \frac{1}{2} \tilde{p}_i^2 - \tilde{p}_i \partial_i \tilde{A}^0 + \frac{1}{4} \tilde{F}_{ij}^2 - \frac{1}{2m^2} p^2_\phi - p_\phi \tilde{A}^0 - \frac{1}{2} m^2 (\tilde{A}^i - \partial_i \phi)^2 + v_0 \tilde{p}_0 \right), \quad (73)
\]
and implies secondary constraint \( \partial_i \tilde{p}_i + p_\phi = 0 \). Complete constraint system

\[
\tilde{p}_0 = 0, \quad \partial_i \tilde{p}_i + p_\phi = 0, \quad (74)
\]
is of first class. The last constraint in Eq. (74) states that \( \phi \) is an auxiliary degree of freedom. It can be removed by the gauge \( \phi = 0 \). The first class constraint \( \tilde{p}_0 = 0 \) replaces two second class constraints (67) of initial formulation, and states that \( A^0 \) is a gauge degree of freedom in the modified formulation (71). Equations of motion for propagating modes in the modified theory are slightly different

\[
\partial_0 \tilde{A}^i = \tilde{p}^i - \partial_i \tilde{A}^0, \quad \partial_0 \tilde{p}_i = \partial_k \tilde{F}_{ki} + m^2 (\tilde{A}_i - \partial_i \phi). \tag{75}
\]

Nevertheless, in the gauge \( \phi = 0 \) they coincide with corresponding equations (69) of initial formulation.

7 Conclusion

In this work we have proposed scheme for conversion of second class constraints which is mainly deal with the Lagrangian formulation of a theory. For the Lagrangian theory with first and second class constraints presented in the Hamiltonian formulation, it does not require neither increase nor decrease of number of initial variables. The scheme was developed on a base of the following observations.

a) One can change parameterization of configuration space by making use of transformation which involves derivatives of variables, see Eq. (2). The condition (3) then guarantees that the theory can be equally analyzed in terms of the new variables.

b) Such a kind transformation increase order of local symmetry: transformation law for the new variables involves more derivatives acting on the local symmetry parameters as compare with the initial formulation.

c) In turn, it generally implies appearance [7] of higher-stage first class constraints, the latter replace second class constraints of the original formulation.
While we have formulated only necessary conditions (3), (4) for our conversion scheme, its efficacy has been demonstrated on a number of examples. In particular, conversion of second class constraints for the membrane action (35) as well as for an arbitrary dynamical system subject to kinematic constraints (48) was not realized by the methods developed in [1-3].

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