A minimax fitting algorithm for ultra-precision aspheric surfaces

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Abstract. Aspheric lenses show significant superiority over traditional spherical ones. The peak-to-valley form deviation is an important criterion for surface qualities of optical lenses. The peak-to-valley errors obtained using traditional methods are usually greater than the actual values, which in turn causing unnecessary rejections. In this paper the form errors of aspheric surfaces are evaluated in the sense of minimum zone, i.e. to directly minimize the peak-to-valley deviation of the data points with respect to the nominal surface. A powerful heuristic optimization algorithm, called differential evolution (DE) is adopted. The control parameters are obtained by meta-optimization. Normally the number of data points is very large, which makes the optimization program unacceptably slow. To improve the efficiency, alpha-shapes are employed to decrease the number of data points involved in the DE optimization. Finally numerical examples are presented to validate this minimum zone evaluation method and compare its results with other algorithms.

1. Introduction

Aspheric lenses show notable superiority over conventional spherical lenses in that a multiple-element spherical lens can be replaced by a single aspheric lens. Aspheric surfaces can be represented with [1],

\[ z = f(x, y) = \frac{r^2/R}{1+\sqrt{1-(1+k)r^2/R^2}} + A_4 r^4 + A_6 r^6 + \cdots \] (1)

with \( r = \sqrt{x^2 + y^2} \).

Here \( R \) is the radius of curvature of the underlying sphere, \( k \) is the conic constant and \( \{A_i\} \) are the magnitudes of higher order deviations from sphericity.

The form error of a manufactured lens plays an essential role in its performances. Currently the PV, peak-to-valley deviation, is still a very commonly adopted specification for surface quality [2], despite its recognized drawbacks for characterizing surfaces and lack of link to optical performances. Most current commercial software applies the least squares method to fit the nominal surface and calculates the PV error by the difference between the maximum and minimum residuals. However this approach is likely to overestimate the form tolerance and lead to unnecessary rejections.

Here we attempt to directly minimum the peak-to-valley deviation,

\[ \min(\max_i d_i - \min_i d_i) \] (2)
where $d_i = \pm \| \mathbf{R} \mathbf{p}_i + t - \mathbf{q}_i \|$ is the signed distance from an arbitrary measured point $\mathbf{p}_i$ to its projection $\mathbf{q}_i$ onto the nominal surface. $\mathbf{R}$ is the optimal rotation matrix and $\mathbf{t}$ is the translation vector.

This minimax problem is not continuously differentiable, thus very difficult to be solved. This paper presents a heuristic optimization algorithm, called differential evolution (DE), to conduct minimum zone evaluation of aspheric surfaces. This method shows great superiorities on stability and accuracy, and makes a good balance between exploration and exploitation.

2. A differential evolution algorithm

At each generation, a Donor vector $\mathbf{v}_i$ is generated for each individual of the population (called genome or chromosome) \{$y_i$ | $i = 1, ..., NP$\}. It is the method of creating this Donor vector that demarcates between the various DE schemes. Two mutation schemes 'DE/rand/1/bin' and 'DE/current to best/2/bin' are applied [3,4],

\[
\mathbf{v}_i = \begin{cases} 
\mathbf{y}_i + F(\mathbf{y}_r - \mathbf{y}_s) & \text{rand}[0,1] < p \\
\mathbf{y}_i + F(\mathbf{p}_e - \mathbf{y}_r + \mathbf{y}_s - \mathbf{y}_t) & \text{otherwise}
\end{cases}
\]

where $r$, $s$ and $t$ are integers randomly selected from the range [1, $NP$] (excluding $i$). $F \in (0,2]$ is used to scale the differential vector, $p \in [0,1]$ is a user-set parameter and rand[0, 1] is a random number uniformly generated in the interval [0, 1].

These two strategies are used very commonly in literature and perform well on problems with distinct characteristics. 'DE/rand/1/bin' demonstrates good diversity while 'DE/current to best/2/bin' shows good convergence property.

After the mutation phase, a ‘binominal’ crossover operation is applied,

\[
\mathbf{u}_{ij} = \begin{cases} 
\mathbf{v}_{ij} & \text{if rand}_{j} \in [0,1] < CR \text{ or } j = j_{rand} \\
\mathbf{y}_{ij} & \text{otherwise}
\end{cases}
\]

where $CR \in [0,1)$ is a user-specified crossover constant and $j_{rand}$ is a randomly chosen integer in [1, $NP$] to ensure that the trial vector $\mathbf{u}_i$ will differ from $\mathbf{y}_i$ by at least one component. The subscript $j$ refers to the $j$-th dimension.

Then a selection operation follows,

\[
y^{k+1}_i = \begin{cases} 
\mathbf{u}^{k+1}_i & \text{if } f(\mathbf{u}^{k+1}_i) < f(\mathbf{y}^k_i) \\
\mathbf{y}^k_i & \text{otherwise}
\end{cases}
\]

with $k$ and $k+1$ denoting the individuals in the $k$-th and ($k+1$)-th generations respectively and $f$ representing the objective function to be minimized. In this paper it is the peak-to-valley deviation.

The pseudo-code for optimization program is shown in Algorithm 1.

**Input:** $NP$, $CR$, $F$, $X$, $y_0$

// $X$: data points, $y_0$: initial guess of solution

Initialize population $\mathbf{Y}$ based on $y_0$;

$k = 0$; // generation number

while $k < k_{max}$ do

$k + 1$;

for $i = 1$ to $NP$ do

Evaluate fitness $f(y_i)$;

Update global optimum $\mathbf{p}_g$;

Mutation, crossover and selection of $\mathbf{y}_i$ using Eqs (3)-(5);

end

if termination condition satisfied then

break;

end
The optimal configuration, i.e. the values of the population size \( NP \), the scaling factor \( F \) and the crossover rate \( CR \), is very problem-dependent. According to the 'no-free-lunch theorems', it is not possible to make the optimization program widely applicable whilst maintaining the best performance at every situation [5]. To obtain relatively good performance in different cases, the optimal parameter configuration is individually obtained for each situation. Here meta-optimization is performed off-line using the Local Unimodal Sampling [6]. The running speed of this algorithm is determined by the number of fitness evaluations. Here the number of fitness evaluations is set no greater than 20 000.

The unknown variables in the optimization programme are the five motion parameters (Rotation about \( x \) and \( y \) axes and translation along \( x \), \( y \) and \( z \) directions), the radius \( R \), the conic constant \( k \) and polynomial coefficients \( \{A_i\} \) (if applicable). When the shape parameters are all given, which means only the position of the measured data is to be optimized, this leads to a localization problem, and the dimension of the optimization problem will be 5. The recommended parameter settings for different dimensions are listed in Table 1.

### Table 1. Parameter setting of DE for aspheric surfaces

| \( D \) | \( NP \) | \( CR \) | \( F \) |
|-------|-------|-------|-------|
| 5     | 23    | 0.866 | 0.7549|
| 7     | 29    | 0.8745| 0.747 |
| 8     | 33    | 0.8884| 0.7347|
| 9     | 41    | 0.9046| 0.7223|
| 10    | 45    | 0.9148| 0.7153|
| 11    | 49    | 0.925 | 0.7087|
| 12    | 52    | 0.9331| 0.6972|
| 13    | 55    | 0.9412| 0.6808|
| 14    | 57    | 0.9542| 0.672 |
| 15    | 59    | 0.9636| 0.6631|
| 16    | 61    | 0.9729| 0.6556|

To prevent a too fast decrease of population diversity, the parameters need to satisfy the condition \( 2F^2 - 2/\frac{NP + CR}{NP} \geq 0 \) [7]. Evidently it holds true for all the cases given here.

### 3. Improving the computational efficiency

This optimization problem is highly nonlinear and lots of local minima exist. To get better results for the DE optimization, the space of the unknown variables is narrowed by supplying a good initial guess for the solution. The orthogonal least squares method [8] is used for this purpose.

#### 3.1. Calculating the orthogonal distances

It is straightforward to write the function of an aspheric surface as \( g(\mathbf{x}) = 0 \) with \( \mathbf{x} \) being a point on the surface. Normally the projection point \( \mathbf{q} \) associated with the point \( \mathbf{p} \) is obtained from

\[
\begin{align*}
\nabla g(\mathbf{q}) \times (\mathbf{p} - \mathbf{q}) &= 0 \\
g(\mathbf{q}) &= 0
\end{align*}
\]  

(6)

using the Gauss-Newton or Levenberg-Marquardt algorithm.

But this method is time consuming, especially when there are many data points. Consequently at the first tens of iterations of the optimization progress, the distance is approximated with [9]

\[
d = \pm ||\mathbf{p} - \mathbf{q}|| \approx \frac{g(\mathbf{p})}{||\nabla g(\mathbf{p})||}
\]  

(7)
As the motion and shape parameters have been approximately identified using least squares, the distance $d$ will not be very large, i.e. $p$ is reasonably near to the associated surface. Thus this approximation is acceptable. At the final iterations, the orthogonal distance will be calculated from Equation (6).

3.2. Reducing the number of data points

In practice, the number of measured data points may be up to millions, but actually only dozens of ‘significant points’ determine the width of the error band (the point number is related with the number of unknown variables). If removing some 'unnecessary' points from the data set, the optimization process can be greatly accelerated. Fortunately, the $\alpha$-shape technique meets this requirement.

An $\alpha$-shape is a well-defined polytope, derived from the Delaunay triangulation of a point set, with a parameter $\alpha \in \mathbb{R}$ controlling the desired level of detail [10]. In order to improve the discrepancy, we replace the $z$ coordinates of the data points by the signed residuals resulted from the least squares fitting and then scale the points into a unit cubic. If implementing the 3D Delaunay triangulation, i.e. organizing the discrete points into a set of tetrahedra, the real ‘key points’ are likely located at some tetrahedra with large circumscribed spheres, thus some tetrahedra with small circumscribed spheres can be omitted. But many ‘boundary points’ will be retained unnecessarily, hence the boundary and interior points are handled separately. Viewing from the $z$ direction, the boundary points are recognized using the modified Graham scan method [11]. In the programme, the tetrahedra are sorted by their radii of circumscribed spheres (in descending order). The set for the points to be kept is initialized as null, and then the vertices of each tetrahedron are checked successively. The checking procedure is presented in Algorithm 2.

\begin{verbatim}
Input: point set \{x_i\}_{i=1}^N, points to be retained \(Y = \emptyset\), Delaunay tetrahedra \(T \in \mathbb{N}^{M \times 4}\);
Find the boundary points \(B\);
for \(j = 1\) to \(M\) do
    for \(i = 1\) to \(4\) do
        if \(T_{ji} \in B\) then
            if \(T_{ji}\) has the greatest positive or the smallest negative \(z\) coordinate among the four vertices of \(T_j\) then
                Put \(T_{ji}\) into \(Y\);
            end
        else
            Put \(T_{ji}\) into \(Y\);
        end
    end
    if The number of points in \(Y\) satisfies the pre-set limit then
        break;
    end
end
Output: \(Y\);
\end{verbatim}

Algorithm 2. The checking process for the vertices to be retained

Figure 1 shows a 2D example. Given 300 points, 30 points on the envelop are sampled using the $\alpha$-shape technique. The number of points to be retained is directly related with the number of unknown variables. For example, in the evaluation of flatness (resp. sphericity), there are three (resp. four) variables and four (resp. five) extreme points are needed to calculate the minimum zone error. For the same reason, if $n$ polynomial terms are involved in the aspheric function, at least $n+8$ points are needed. To avoid removing extreme points by mistake, $2n+14$ points will be kept in the DE optimization.
When a set of significant points are selected, DE optimization is undertaken. Then the residuals of all the data points with respect to the fitted surface are calculated. If some data points are not contained within the tolerance zone, these exterior points are added into the significant point set and the minimum zone evaluation program is run again. This procedure is repeated until all the data points are contained within the tolerance zone.

4. Experimental validation and discussion

The validity of the proposed algorithm is verified with a data set of 8100 points, as shown in Figure 2. The unit of length here is mm if not specified otherwise. The nominal shape parameters are given as \( R = 520 \), \( k = -0.7 \), \( A_4 = 5.2e-5 \), \( A_6 = -6.5e-6 \), \( A_8 = 3.11e-8 \), \( A_{10} = 3.222e-9 \). Noise with its amplitude \( \sigma = 1 \) \( \mu \)m is introduced using the fractal Brownian function [12]. The dimensionality of the optimization problem is 11 and the control parameters in the DE program is adopted as \( NP = 49 \), \( CR = 0.925 \) and \( F = 0.7087 \), in accordance with Table 1.

The Differential Evolution optimization program was coded in MATLAB R2009a and run on a PC with Intel(R) Core (TM) 2 Duo CPU8500 3.16GHz, 3.24GB RAM. The program was run 100 times and the obtained PV errors and running time (including ODF least squares initial fitting, alpha-shape point reduction and DE parameter optimization) are listed in Table 2. For comparison, the direct DE optimization without point reduction was run 100 times and the corresponding results are also presented.

| Table 2. Fitted results of the aspheric surface |
|-----------------------------------------------|
| method           | LS ODF max | DE mean |
|                  |         |          |
| PV/\( \mu \)m    | 3.79     | 3.32     |
|                  | 3.28     | 3.21     |
|                   | 3.22     | 3.22     |
| Time/s           | 0.14     | 5.40     |
|                  | 194.93   |          |

In the table it can be seen that the proposed DE optimization can obtain a much smaller PV form error compared with the least squares ODF, meanwhile, at a cost of much longer running time. But the
average time for the whole fitting process is less than six seconds, which is acceptable in practical applications. The point reduction technique using $\alpha$-shapes reduced the running time by 97%.

5. Conclusions
This paper proposes an optimization method using differential evolution to evaluate the peak-to-valley form errors of aspheric surfaces. To make the optimization program specifically works well for different dimensions of unknown variables, the parameter configuration is obtained using meta-optimization. Additionally, the alpha shapes is adopted to get the 'envelop points' which potentially determine the PV form error, so that the points involved in the optimization program are greatly reduced. Experimental results prove the running time can be reduced by 97% without influencing the optimization results. This program can also be utilized in aspheric surface fitting to calculate the shape parameters from the given measured data points or surface matching to determine the optimal relative position between the measured data and the nominal surface.

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References
[1] ISO 10110-12:2007 Optics and Photonics—Preparation of Drawings for Optical Elements and Systems—Part 12: Aspheric Surfaces
[2] ISO 10110-5:2007 Optics and Photonics—Preparation of Drawings for Optical Elements and Systems—Part 5: Surface Form Tolerances
[3] Price K and Storn R 1995 Differential evolution— a simple and efficient adaptive scheme for global optimization over continuous spaces, Technical Report, Intern. Computer Science Inst., Berkley
[4] Qin A K and Suganthan P N 2005 Self-adaptive differential evolution algorithm for numerical optimization, in: 2005 IEEE Congress on Evolutionary Computation 2 1785–91
[5] Wolpert D H and Macready W G 1997 No free lunch theorems for optimization, IEEE Trans. Evol. Comput 1 67–82
[6] Pedersen S M E H 2010 Tuning and Simplifying Heuristical Optimization, PhD thesis, University of Southampton, UK
[7] Zaharie D 2002 Critical values for the control parameters of differential evolution algorithms, in: R. Matoušk, P. Ösmera (Eds.), Proc. 8th Intern. Conf. Soft Computing Mendel, Brno, Czech Republic 62–7
[8] Zhang X 2009 Freeform Surface Fitting for Precision Coordinate Metrology, PhD thesis, University of Huddersfield, Huddersfield, UK
[9] Taubin G 1991 Estimation of planar curves, surfaces and nonplanar spaces curves defined by implicit equations with applications to edge and range image segmentation, IEEE Trans. Patt. Anal. Mach. Intell. 13 1115–38
[10] Edelsbrunner H and Mücke E P 1994 Three-dimensional alpha shapes, ACM Trans. on Graphics 13 43–72
[11] Zhang X, Jiang X and Scott P J 2008 Orthogonal distance fitting of precision free-form surfaces based on $l_1$ norm, in: F. Pavese, M. Bär, A. B. Forbes, et al. (Eds.), Adv. Math. and Comput. Tools in Metrology and Testing VIII, World Scientific 385–90
[12] Mandelbrot B B and van Ness J W 1968 Fractional brownian motions, fractional noises and applications, SIAM Review 10 422–37