An arguable addition to the standard Deduction Theorems of first order theories

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We consider an arguable addition to the standard Deduction Theorems of first order theories.

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1. Introduction

We first review, in Meta-theorem 1, the proof of a standard Deduction Theorem - \( ([T], [A]) \vdash_K [B] \) if, and only if, \( [T] \vdash_K [A \Rightarrow B] \) (cf. [Me64], Corollary 2.6, p61) - of classical\(^4\), first-order, theories, where an explicit deduction of \([B]\) from \(([T], [A])\) is known.

We then show, in Corollary 1.2, that, assuming Church’s Thesis, Meta-theorem 1 can be constructively extended to cases where, \( ([T], [A]) \vdash_K [B] \) is established meta-mathematically, assuming the consistency of \(([T], [A])\), but where an explicit deduction of \([B]\) from \(([T], [A])\) is not known.

We finally argue, in Meta-theorem 2, that:

\[ ([T], [A]) \vdash_K [B] \text{ holds if, and only if, } [T] \vdash_K [B] \text{ holds when we assume } [T] \vdash_K [A]. \]

(In other words, that \([B]\) is a deduction from \(([T], [A])\) in \(K\) if, and only if, whenever \([A]\) is a hypothetical deduction from \([T]\) in \(K\), \([B]\) is a deduction from \([T]\) in \(K\).)

2. A standard Deduction Theorem

The following is, essentially, Mendelson’s proof of a standard Deduction Theorem ([Me64], p61, Proposition 2.4 and Corollary 2.6) of an arbitrary first order theory \(K\):

\(^4\) We take Mendelson [Me64] as representative, in the area that it covers, of standard expositions of classical, first order, logic.

\(^5\) We use square brackets to differentiate between a formal expression \([F]\) and its interpretation “\(F\)”, where we follow Mendelson’s definition of an interpretation \(M\) of a formal theory \(K\), and of the interpretation of a formula of \(K\) under \(M\) ([Me64], p49, §2). For instance, we use \([n]\) to denote the numeral in \(K\) whose standard interpretation is the natural number \(n\).

\(^6\) For the purposes of this essay, we assume everywhere that \([T]\) is an abbreviation for a finite set of \(K\)-formulas \(\{[T_1], [T_2], ... [T_l]\}\), whereas \([A]\), \([B]\), ... are closed well-formed formulas of \(K\). We note, also, that “\([A \& B]\)” and “\([A \& B]\)” denote the same \(K\)-formula.
Meta-theorem 1: If \([T]\) is a set of well-formed formulas of an arbitrary first order theory \(K\), and if \([A]\) is a closed well-formed formula of \(K\), and if \((T, A)|\_K [B]\), then \([T]|\_K [A \Rightarrow B]^7\).

Proof: Let \(<[B_1], [B_2], ..., [B_n]>\) be a deduction of \([B]\) from \((T, A)\) in \(K\).

Then, by definition, \([B_n]\) is \([B]\) and, for each \(i\), either \([B_i]\) is an axiom of \(K\), or \([B_i]\) is in \([T]\), or \([B_i]\) is \([A]\), or \([B_i]\) is a direct consequence by some rules of inference of \(K\) of some of the preceding well-formed formulas in the sequence.

We now show, by induction, that \([T]|\_K [A \Rightarrow B_i]\) for each \(i < n\). As inductive hypothesis, we assume that the proposition is true for all deductions of length less than \(n\).

(i) If \([B_i]\) is an axiom, or belongs to \([T]\), then \([T]|\_K [A \Rightarrow B_i]\), since \([B_i \Rightarrow (A \Rightarrow B_i)]\) is an axiom of \(K\).

(ii) If \([B_i]\) is \([A]\), then \([T]|\_K [A \Rightarrow B_i]\), since \([T]|\_K [A \Rightarrow A]\).

(iii) If there exist \(j, k\) less than \(i\) such that \([B_k]\) is \([B_j \Rightarrow B_i]\), then, by the inductive hypothesis, \([T]|\_K [A \Rightarrow B_j]\), and \([T]|\_K [A \Rightarrow (B_j \Rightarrow B_i)]\). Hence, \([T]|\_K [A \Rightarrow B_j]\).

(iv) Finally, suppose there is some \(j < i\) such that \([B_i]\) is \([(A\chi)B_j]\), where \(\chi\) is a variable in \(K\). By hypothesis, \([T]|\_K [A \Rightarrow B_j]\). Since \(\chi\) is not a free variable of \([A]\), we have that \([(A\chi)(A \Rightarrow B_j) \Rightarrow (A \Rightarrow (A\chi)B_j)]\) is PA-provable. Since \([T]|\_K [A \Rightarrow B_j]\), it follows by Generalisation that \([T]|\_K [(A\chi)(A \Rightarrow B_j)]\), and so \([T]|\_K [A \Rightarrow (A\chi)B_j]\), i.e. \([T]|\_K [A \Rightarrow B_i]\).

\(^7\) The converse is trivially true (cf. [Sh67], p33).
This completes the induction, and Meta-theorem 1 follows as the special case where \( i = n \). ¶

2.1 A number-theoretic corollary

Now, Gödel has defined ([Go31], p22, Definition 45(6)) a primitive recursive number-theoretic relation \( xB_{(K, \{T\})}y \) that holds if, and only if, \( x \) is the Gödel-number of a deduction from \( T \) of the K-formula whose Gödel-number is \( y \).

We thus have:

**Corollary 1.1**: If the Gödel-number of the well-formed K-formula \([B]\) is \( b \), and that of the well-formed K-formula \([A \Rightarrow B]\) is \( c \), then Meta-theorem 1 holds if, and only if\(^{10}\):

\[
(\exists x)xB_{(K, \{T\}, \{A\})}b \Rightarrow (\exists z)zB_{(K, \{T\})}c
\]

2.2 An extended Deduction Theorem

We next consider the meta-proposition:

**Corollary 1.2**: If we assume Church’s Thesis\(^{11}\), then Meta-theorem 1 holds even if the premise \(([T], [A]) \vdash_{K} [B] \) is established meta-mathematically, assuming the consistency of \(([T], [A])\), but a deduction \(<[B_1], [B_2], ..., [B_n]>\) of \([B]\) from \(([T], [A])\) in K is not known explicitly.

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\(^{8}\) We use the symbol “¶” as an end-of-proof marker.

\(^{9}\) We note that Corollary 1.1 and Corollary 2.2 may be essentially different number-theoretic assertions, which, in the absence of a formal proof, cannot be assumed to be equivalent.

\(^{10}\) We note that this symbolically expresses a meta-equivalence in a recursive arithmetic RA, based on a semantic interpretation of the definition of the primitive recursive relation \( xB_{(K, \{T\})} \); it is not a K-formula.

\(^{11}\) Church’s Thesis: A number-theoretic function is effectively computable if, and only if, it is recursive ([Me64], p147, footnote). We appeal explicitly to Church’s Thesis here to avoid implicitly assuming that every recursive relation is algorithmically decidable.
**Proof:** Since Gödel’s number-theoretic relation $\forall x B_{K, [T]} y$ is primitive recursive, it follows that, if we assume Church’s Thesis - which implies that a number-theoretic relation is decidable if, and only if, it is recursive - we can effectively determine some finite natural number $n$ for which the assertion $n B_{K, [T], [A]} b$ holds, where the Gödel-number of the well-formed K-formula $[B]$ is $b$.

Since $n$ would then, by definition, be the Gödel-number of a deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from $([T], [A])$ in K, we may thus constructively conclude, from the meta-mathematically determined assertion $([T], [A]) \vdash_K [B]$, that some deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from $([T], [A])$ in K can, indeed, be effectively determined. Meta-theorem 1 follows. \[\]

3. **An additional Deduction Theorem**

We, finally, argue that:

**Meta-theorem 2:** If K is an arbitrary first order theory, and if $[A]$ is a closed well-formed formula of K, then $([T], [A]) \vdash_K [B]$ holds if, and only if, $[T] \vdash_K [B]$ holds when we assume that $[T] \vdash_K [A]$ holds.

**Proof:** First, if there is a deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from $([T], [A])$ in K, and there is a deduction, $<[A_1], [A_2], ..., [A_m]>$, of $[A]$ from $[T]$ in K, then $<[A_1], [A_2], ..., [A_m], [B_1], [B_2], ..., [B_n]>$ is a deduction of $[B]$ from $[T]$ in K. Hence, if $([T], [A]) \vdash_K [B]$ holds, then $[T] \vdash_K [B]$ holds when we assume $[T] \vdash_K [A]$.

Second, if there is a deduction $<[B_1], [B_2], ..., [B_n]>$ of $[B]$ from $[T]$ in K, then we have, trivially, that, if $[T] \vdash_K [B]$ holds when we assume $[T] \vdash_K [A]$, then $([T], [A]) \vdash_K [B]$ holds.

Last, we assume that there is no deduction, $<[B_1], [B_2], ..., [B_n]>$, of $[B]$ from $[T]$ in K. If, now, $[T] \vdash_K [B]$ holds when we assume that $[T] \vdash_K [A]$ holds in any extension K’ of K,
then, if we assume that there is a sequence \([A_1], [A_2], ..., [A_m]\) of well-formed \(K'\)-formulas such that \([A_m]\) is \([A]\) and, for each \(m \geq i \geq 1\), either \([A_i]\) is an axiom of \(K'\), or \([A_i]\) is in \([T]\), or \([A_i]\) is a direct consequence by some rules of inference of \(K'\) of some of the preceding well-formed formulas in the sequence, then it follows from our hypothesis\(^{12}\) that we can show, by induction on the deduction length \(n\), that there is a sequence \([B_1], [B_2], ..., [B_n]\) of well-formed \(K\)-formulas such that \([B_1]\) is \([A]\)\(^{13}\), \([B_n]\) is \([B]\) and, for each \(i > 1\), either \([B_i]\) is an axiom of \(K\), or \([B_i]\) is in \([T]\), or \([B_i]\) is a direct consequence by some rules of inference of \(K\) of some of the preceding well-formed formulas in the sequence.

In other words, if there is a deduction, \([A_1], [A_2], ..., [A_m]\) of \([A]\) from \([T]\) in \(K'\), then, by our hypothesis, \([A], [B_2], ..., [B_n]\) is a deduction of \([B]\) from \(([T], [A])\) in \(K'\). By definition, it follows that \([A], [B_2], ..., [B_n]\) is, then, a deduction of \([B]\) from \(([T], [A])\) in \(K\). We thus have that, if \([T]\mid\vdash_{K} [B]\) holds when we assume \([T]\mid\vdash_{K} [A]\), then \(([T], [A])\mid\vdash_{K} [B]\) holds. This completes the proof. \(\|\)

In view of Corollary 1.2, we thus have:

**Corollary 2.1**: If we assume Church’s Thesis, and if \([A]\) is a closed well-formed formula of \(K\), then we may conclude that \([T]\mid\vdash_{K} ([A] \Rightarrow [B])\) holds if \([T]\mid\vdash_{K} [B]\) holds when we assume \([T]\mid\vdash_{K} [A]\).\(^{14}\)

We note that, in the notation of Corollary 1.1, if the Gödel-number of the well-formed \(K\)-formula \([A]\) is \(a\), then Corollary 2.1 holds if, and only if\(^{15}\):

\(^{12}\) That \([T]\mid\vdash_{K} [A]\) holds.

\(^{13}\) \([A]\) is, thus, the hypothesis in the sequence; it is the only well-formed \(K\)-formula in the sequence that is not an axiom of \(K\), not in \([T]\), and not a direct consequence of the axioms of \(K\) by any rules of inference of \(K\).

\(^{14}\) We give a model-theoretic proof of Corollary 2.1 in the Appendix.

\(^{15}\) We note that this, too, is not a \(K\)-formula, but a semantic meta-equivalence, based on the definition of the primitive recursive relation \(x\langle B_{(K', [T])}\rangle y\).
Corollary 2.2: \(((\exists x)xB_{(K,[T])}a \Rightarrow (Ex)uB_{(K,[T])}b) \Rightarrow (Ez)zB_{(K,[T])}c\).

4. Conclusion

We have argued that Meta-theorem 2 is a valid Deduction Theorem of any first order theory. However, standard interpretations of Gödel’s reasoning and conclusions are inconsistent with the consequences of this Meta-theorem in an arbitrary first order theory\(^{16}\). Hence, in the absence of constructive, and intuitionistically unobjectionable, reasons for denying the applicability of the Meta-theorem, and of its Corollary 2.1, to a first order theory, such interpretations ought not to be considered as definitive.

Appendix 1: A model-theoretic proof of Corollary 2.1

We note that there is a model-theoretic proof of Corollary 2.1. The case \([T]\models_K [B] \models [A]\) is straightforward.

If \([T]\models_K [B]\) does not hold, then, as noted in Meta-theorem 2, if \([T]\models_K [B]\) holds when we assume \([T]\models_K [A]\), then there is a sequence \(<[B_1], [B_2], ..., [B_n]>\) of well-formed K-formulas such that \([B_1]\) is \([A]\), \([B_n]\) is \([B]\) and, for each \(i > 1\), either \([B_i]\) is an axiom of K, or \([B_i]\) is in \([T]\), or \([B_i]\) is a direct consequence by some rules of inference of K of some of the preceding well-formed formulas in the sequence.

(We note that if \([T]\) is the set of well-formed K-formulas \([T_1], [T_2], ..., [T_i]\) then \((T & [A])\) denotes the well-formed K-formula \([T_1 & T_2 & ..., T_i & A]\), and, \((T & A)\) denotes its interpretation in M, i.e., \(T_1 & T_2 & ..., T_i & A\).)

If, now, any well-formed formula in \(([T], [A])\) is false under an interpretation M of K, then \((T & A) \Rightarrow B\) is vacuously true in M.

\(^{16}\) See Appendix 2.
If, however, all the well-formed formulas in \((T, A)\) are true under interpretation in \(M\), then the sequence \(<[B_1], [B_2], ..., [B_n]\>\) interprets as a deduction in \(M\), since the interpretation preserves the axioms and rules of inference of \(K\) (cf. [Me64], p57). Thus \([B]\) is true in \(M\), and so is \((T \& A) \Rightarrow B\).

In other words, we cannot have \((T, A)\) true, and \([B]\) false, under interpretation in \(M\), as this would imply that there is some extension \(K'\) of \(K\) in which \([T]\models_K [A]\), but not \([T]\models_K [B]\); this would contradict our hypothesis, which implies that, in any extension \(K'\) of \(K\) in which we have \([T]\models_K [A]\), we also have \([T]\models_K [B]\).

Hence, \((T \& A) \Rightarrow B\) is true in all models of \(K\). By a consequence of Gödel’s Completeness Theorem for an arbitrary first order theory ([Me64], p68, Corollary 2.15(a)), it follows that \(\models_K ([T] \& [A]) \Rightarrow [B]\); and, ipso facto, that \([T]\models_K ([A] \Rightarrow [B])\).

**Appendix 2: Gödel’s reasoning and Corollary 2.1**

In his seminal 1931 paper [Go31], Gödel meta-mathematically argues that, assuming any formal system of Peano Arithmetic, \(PA\), is simply consistent, we can define an “undecidable” \(PA\)-proposition, \([(Ax)R(x)]\), such that (cf. [Go31], #1, p25):

If \([(Ax)R(x)]\) is \(PA\)-provable, then \(\neg(Ax)R(x)\) is \(PA\)-provable.\(^{17}\)

Now, by Corollary 2.1, it should follow that:

\([(Ax)R(x) \Rightarrow \neg(Ax)R(x)]\) is \(PA\)-provable,

and, therefore, that:

\(^{17}\)Gödel essentially argues, number-theoretically, that, if the Gödel-number of \([(Ax)R(x)]\) is 17Gen\(r\), and if this formula is \(PA\)-provable, then the \(PA\)-formula whose Gödel-number is Neg(17Gen\(r\)), i.e., \(\neg(Ax)R(x)\), is also \(PA\)-provable if \(PA\) is assumed simply consistent.
\((\neg (Ax)R(x))\) is PA-provable.

Since Gödel also proved that, if PA is assumed simply consistent, then \([R(n)]\) is PA-provable for any, given, natural number \(n\), Corollary 2.1 implies that PA is omega-inconsistent.

(We note that Gödel defined a first order theory K as omega-consistent if, and only if, for every well-formed formula \([F(x)]\) of K, if \(\vdash_K [F(n)]\) for every numeral \([n]\), then it is not the case that \(\vdash_K (Ex)F(x)\) (cf. [Me64], p142; see also [Go31], p23-24).

However, this conclusion is inconsistent with standard interpretations of Gödel’s reasoning, which, first, assert both \([(Ax)R(x)]\) and \((\neg (Ax)R(x))\) as PA-unprovable, and, second, assume that PA can be omega-consistent\(^{18}\). Such interpretations, therefore, implicitly deny that the PA-provability of \((\neg (Ax)R(x))\) can be inferred from the above meta-argument; ipso facto, they imply that Corollary 2.1 is false.

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\(^{18}\) We note that Gödel’s Incompleteness Theorems assume significance only if we presume that the arithmetic, in which they are derived, can be omega-consistent (cf. [Go31], p23-24)