We define generalized quantum games by introducing the coherent payoff operators and propose a simple scheme to illustrate it. The scheme is implemented with a single spin qubit system and two entangled qubit system. The Nash Equilibrium Theorem is proved for the models.

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**INTRODUCTION**

As a new vehicle to explore the exotic natures of quantum information, quantum games were proposed as the "quantization" of the classical games. Most recently a quantum game was implemented via nuclear magnetic resonance (NMR) system. It was demonstrated that neither of the two players would win the game if they play rationally, but if they adopt quantum strategies both of them would win. A classical game, as is well known, consists of three elements - the players, the strategies taken by the players and the payoff functions. In a gambling process, a player takes a strategy without knowing the strategy adopted by the other players. All players having taken actions simultaneously or successively, the payoff is awarded to each player according to the payoff function, which depends on the actions of all of the players. If a player can maximize his payoff, one says he wins the game.

The early "quantization" of classical game is to replace the classical strategy space with the quantum one consisting of unitary operations on a quantum state. In the ordinary two player quantum game, one takes as the initial quantum state an entangled state (EPR state) correlating two qubits in a distance and the players can take as strategies local unitary operations acting on the two qubits separately. The payoff functions are defined by the probabilities, rather than the probability amplitudes, of projecting the final state to some chosen states. For this reason, we think this definition of quantum game is incoherent. As in this "semi-classical game theory", the payoff function is based on the classical probability to a large extent, there is a fundamental interest in generalizing it to a "fully-quantum game theory".

In this note we make two generalizations. First, we introduce the coherent payoff functions in terms of certain probability amplitudes. Second, we loosen the requirement that the initial state be an entangled qubit state. With these generalizations, we can still prove the Nash Equilibrium Theorem for a special type of quantum games, including single qubit case and two entangled qubit case. Notice that the existence of Nash equilibrium is an essential element in defining an interesting game. But it seems that this point has not been fully realized. Indeed, in some proposed quantum games, while new features have been demonstrated, Nash equilibrium appears to occur accidentally for specifically-chosen parameters. A most recent work concerns the universality of the Nash theorem, but it requires a very large quantum strategy space that consists of both unitary transformations and non-unitary "quantum measurement" operations. Naturally, Nash equilibrium is more likely to occur in a bigger strategy space. However, from the practical point of view, it is desirable to introduce a reasonable quantum game in a limited strategy space with Nash equilibrium whose existence is not the result of carefully choosing parameters. Anyway, it should be possible to implement by practically physical processes.

**GENERALIZED QUANTUM GAMES**

Let us describe the scheme of our generalized N player quantum game in a general framework. Fix a vector space \( V \). Then mathematically the game is a triple \((S, \rho, P)\) where \( \rho \in \text{End}(V) \) being a density matrix, \( S = (S_1, S_2, \cdots, S_N) \), \( S_i \subset \text{End}(V) \) consisting of unitary operators, and \( P = (P_1, P_2, \cdots, P_N) \), \( P_i \in \text{End}(V) \) being an Hermitian operator, called a payoff operator. For a given quantum state described by a density matrix \( \rho \) the players transform it by operations \( U_i \in S_i \) \((i = 1, 2, \cdots, N)\) simultaneously for a static type game or in succession for a
dynamic type game. Based on the obtained final state
\[ \rho_f = (\prod_{k=1}^{N} U_k) \rho (\prod_{k=1}^{N} U_k)^\dagger \]  
(1)
the payoff \( f_i \) for the i'th player is calculated according to the formula
\[ f_i = Tr(P_i \rho_f) \in R \]  
(2)
Notice that in this generalized version, the operations \( U_k \) are not required to be localized and so \( [U_k, U_s] = 0 (s \neq k) \) is not necessarily true. When \( |\rho_f, P_i| \neq 0 \) or \( [U_k, P_i] \neq 0 \), the role of the off-diagonal elements in \( \rho_f \) and \( P_k \) reflect the quantum coherence character of the payoff function. Notice that quantum coherence plays crucial role in quantum computing and quantum information, but it is sensitive to quantum measurement and any environment-couplings with it [10].

The above described \( N \) player game includes the original quantum game as a special case. In that case, an \( N \)-multiple entangled pure state \( |\sigma \rangle \in V = V_1 \otimes V_2 \otimes ... \otimes V_N \) is used as an initial state. Here, \( V_k \) is the space under the action of the operation \( U_k \) by the \( k \)'th player. One necessarily has \( [U_k, U_s] = 0 \) and \( |\sigma \rangle \) is transformed by independent operations \( U_k \) simultaneously in different spatial locations. The payoff
\[ f_k(\sigma) = \sum_{j_1, j_2, ..., j} C_{j_1, j_2, ..., j}^{[k]} \langle \sigma_{j_1} \sigma_{j_2} ... \sigma_{j_N} | E \rangle^2 \]  
(3)
for each player is calculated based on the probabilities \( |\langle j_1, j_2, ... j_N | E \rangle|^2 \) for projections of final state \( |E \rangle = U_1 U_2 ... U_N |\sigma \rangle \) onto the basis \( |\sigma_{j_1} \sigma_{j_2} ... \sigma_{j_N} \rangle \), Here, \( C_{j_1, j_2, ..., j}^{[k]} \) are the real parameters assigned for a given game. Obviously, it is a special case of our generalized version with the payoff matrix
\[ P_k = \sum_{j_1, j_2, ..., j} C_{j_1, j_2, ..., j}^{[k]} \langle j_1, j_2, ... j | \sigma_{j_1} \sigma_{j_2} ... \sigma_{j_N} \rangle \]  
(4)
and the initial density matrix \( \rho = |\sigma \rangle \langle \sigma | \).

Next we consider two models of the above generalized quantum game with single qubit and two entangled qubits, in which the Nash theorem holds. For convenience, we start from an abstract classical game called GAME A defined on a subset of \( S^1 \times S^1 : \{ (\theta, \varphi) | 0 \leq \theta, \varphi \leq \frac{\pi}{2} \} \). We will show that the two generalized games are mathematically equivalent to the GAME A. So we need only to see whether the Nash theorem holds universally for the GAME A.

**CLASSICAL ABSTRACT GAME WITH NASH EQUILIBRIUM ON \( S^1 \times S^1 \)**

The GAME A is described as follows. The two players have the strategy spaces \( \{ \theta | 0 \leq \theta \leq \frac{\pi}{2} \} \subset S^1 \) and \( \{ \varphi | 0 \leq \varphi \leq \frac{\pi}{2} \} \subset S^1 \) respectively. The payoff function for the \( i \)-th player can be written in the form
\[ f_i(\theta, \varphi) = p_i + q_i \sin (\theta + \varphi + \Psi_i) \]  
(5)
where \( q_i > 0 \) and \( \Psi_i \in [\frac{\pi}{2}, \frac{\pi}{2}] \).

**Proposition. For Game A, there exists a Nash equilibrium.**

**Proof.** We imitate the proof of Nash Equilibrium Theorem. From the conditions \( 0 \leq \theta, \varphi \leq \frac{\pi}{2} \) and \( \Psi_1 \in [\frac{\pi}{2}, \frac{\pi}{2}] \) we observe that for a fixed \( \varphi \) there exists exactly one \( \psi \), which we denote by \( \chi(\varphi) \), that maximizes the pay off function \( f_1 \). Similarly, for a fixed \( \theta \) there is exactly one \( \varphi \), which we denote by \( \chi(\theta) \), that maximizes the pay off function \( f_2 \). So we can define a map \( g \) from the convex set \( [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \) to itself such that
\[ g(\theta, \varphi) = (\chi(\varphi), \chi(\theta)) \]  
(6)
It is easy to show that \( g \) is a continuous map. Hence, by Brower Fixed Point Theorem \( g \) has a fixed point \( (\theta_0, \varphi_0) \). It is readily verified that \( (\theta_0, \varphi_0) \) is a Nash equilibrium. The proof of the proposition is thus completed.

The Nash equilibrium of GAME A can actually be calculated explicitly. The results, depending on the values of \( \Psi_1 \) and \( \Psi_2 \), are as follows: Suppose \( \Psi_1 = \Psi_2 = \Psi \). Then the Nash equilibrium might not be unique. If \( \Psi \in [\frac{\pi}{2}, 0] \),
then each point \((\theta_0, \varphi_0)\) in \([-\Psi, \frac{\pi}{2}] \times [-\Psi, \frac{\pi}{2}]\) satisfying \(\theta_0 + \varphi_0 + \Psi = \frac{\pi}{2}\) is a Nash equilibrium; If \(\Psi \in [0, \frac{\pi}{2}]\), then each point \((\theta_0, \varphi_0)\) in \([0, \frac{\pi}{2}] \times [0, \frac{\pi}{2} - \Psi]\) satisfying \(\theta_0 + \varphi_0 + \Psi = \frac{\pi}{2}\) is a Nash equilibrium. Suppose \(\Psi_1 \neq \Psi_2\). Then the Nash equilibrium is unique. Precisely, there are the following possibilities.

1) \(\Psi_1, \Psi_2 \in [-\frac{\pi}{2}, 0]\), \(\Psi_1 > \Psi_2\), \((\theta_0, \varphi_0) = (-\Psi_1, \frac{\pi}{2})\);
2) \(\Psi_1, \Psi_2 \in [-\frac{\pi}{2}, 0]\), \(\Psi_1 < \Psi_2\), \((\theta_0, \varphi_0) = (-\Psi_2, \frac{\pi}{2})\);
3) \(\Psi_1 \in [-\frac{\pi}{2}, 0], \Psi_2 \in [0, \frac{\pi}{2}]\), \((\theta_0, \varphi_0) = (\frac{\pi}{2}, 0)\);
4) \(\Psi_1 \in [0, \frac{\pi}{2}], \Psi_2 \in [-\frac{\pi}{2}, 0]\), \((\theta_0, \varphi_0) = (0, \frac{\pi}{2})\);
5) \(\Psi_1, \Psi_2 \in [0, \frac{\pi}{2}]\), \(\Psi_1 > \Psi_2\), \((\theta_0, \varphi_0) = (0, \frac{\pi}{2} - \Psi_2)\);
6) \(\Psi_1, \Psi_2 \in [0, \frac{\pi}{2}]\), \(\Psi_1 < \Psi_2\), \((\theta_0, \varphi_0) = (\frac{\pi}{2} - \Psi_1, 0)\).

It is easy to prove these results. For example, when \(\Psi_1, \Psi_2 \in [-\frac{\pi}{2}, 0]\), we have

\[
\chi(\varphi) = \left\{ \begin{array}{ll}
\frac{\pi}{2}, & -\frac{\pi}{2} \leq \varphi + \Psi_1 \leq 0; \\
\frac{\pi}{2} - (\varphi + \Psi_1), & 0 \leq \varphi + \Psi_1 \leq \frac{\pi}{2};
\end{array} \right.
\]

\[
\chi(\theta) = \left\{ \begin{array}{ll}
\frac{\pi}{2}, & -\frac{\pi}{2} \leq \theta + \Psi_2 \leq 0; \\
\frac{\pi}{2} - (\theta + \Psi_2), & 0 \leq \theta + \Psi_2 \leq \frac{\pi}{2}.
\end{array} \right.
\]  

(7)

If \(\Psi_1 > \Psi_2\), it then follows from the fixed point condition \((\chi(\varphi_0), \chi(\theta_0)) = (\theta_0, \varphi_0)\) that \((\theta_0, \varphi_0) = (-\Psi_1, \frac{\pi}{2})\). The other results can be proved in a similar way. We notice that when \(\Psi_1 = \Psi_2\), if the two players do not exchange information, there is little chance of reaching a Nash equilibrium. On the other hand, when \(\Psi_1 \neq \Psi_2\), as the Nash equilibrium is unique it will appear, but at the equilibrium at most only one player will maximize the payoff function.

**ONE QUBIT REALIZATION**

Now we consider the first realization of Game A only using one qubit, which is similar to the classical game of roulette. Given an initial state \(\rho\), the first player operates on it by a rotation

\[U(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},\]

and then the second player operates on the resulted state by another rotation \(U(\varphi)\). Since the two operations \(U(\theta)\) and \(U(\varphi)\) take action in succession, this is a game of dynamic type. Assign two payoff matrices \(P_1\) and \(P_2\) to the two players. By definition they are Hermitian operators. Generally, we can write

\[P_i = \begin{pmatrix} a_i & b_i \\ \overline{b_i} & d_i \end{pmatrix}, \quad i = 1, 2\]

(9)

with respect to the basis \(\{|0\rangle, |1\rangle\}\), where \(a_i, d_i\) are real numbers. The complex elements \(b_i\) characterize the quantum coherence of the payoff functions \(f_i\) \((i = 1, 2)\) for the two players

\[f_i = tr\left(P_i U(\theta + \varphi) \rho U(\theta + \varphi)^\dagger\right),\]

(10)

To acquire the non-trivial nature of quantum coherence it should be required that \([P_1, P_2] \neq 0\). Otherwise one can simultaneously diagonalize \(P_1, P_2, U(\theta)\) and the defined game would be trivial.

First, we take as the initial state the pure state

\[|\psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)\]

(11)

of a qubit. Then we have

\[U(\theta) |0\rangle = \cos \theta |0\rangle + \sin \theta |1\rangle,\]
\[U(\theta) |1\rangle = -\sin \theta |0\rangle + \cos \theta |1\rangle.\]

(12)

After simple calculation we obtain

\[f_i = \frac{1}{2}[\langle a_i + d_i \rangle + \langle a_i - d_i \rangle \sin 2(\theta + \varphi) + \langle b_i + \overline{b_i} \rangle \cos 2(\theta + \varphi)].\]

(13)
If \( a_i - d_i \geq 0 \) then \( f_i \) can be rewritten as
\[
 f_i = p_i + q_i \sin(2(\theta + \varphi) + \Psi_i),
\]
where
\[
 p_i = \frac{1}{2} (a_i + d_i),
 q_i = \frac{1}{2} \sqrt{(a_i - d_i)^2 + (b_i + \overline{b_i})^2},
 \Psi_i = \arctan \left( \frac{b_i + \overline{b_i}}{a_i - d_i} \right).
\]
Clearly, if we put the restrictions \( a_i - d_i \geq 0 \) and \( 0 \leq \theta, \varphi \leq \frac{\pi}{4} \) then this quantum game is a realization of Game A. Thus in this case it has a Nash equilibrium.

Next, we take the mixed state
\[
 \rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}, 0 \leq p \leq 1
\]
as the initial state. In this case we have
\[
 f_i = \frac{1}{2} \left[ (a_i + d_i) + (1 - 2p) (b_i + \overline{b_i}) \sin(2(\theta + \varphi)) + (1 - p) (d_i - a_i) \cos(2(\theta + \varphi)) \right].
\]
It does not harm to assume \( p \leq \frac{1}{2} \). Then if \( b_i + \overline{b_i} \geq 0 \), \( f_i \) can be rewritten as
\[
 f_i = p_i + q_i \sin(2(\theta + \varphi) + \Psi_i),
\]
where
\[
 p_i = \frac{1}{2} (a_i + d_i),
 q_i = \frac{1}{2} \sqrt{(d_i - a_i)^2 (1-p)^2 + (b_i + \overline{b_i})^2 (1-2p)^2},
 \Psi_i = \arctan \left( \frac{(1-p)(d_i - a_i)}{(1-2p)(b_i + \overline{b_i})} \right).
\]
Thus this game with the additional restriction \( 0 \leq \theta, \varphi \leq \frac{\pi}{4} \) is also a realization of GAME A.

TWO QUBIT REALIZATION

Our next model is a static quantum game with two qubits. We take the quantum entangled state
\[
 |\phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)
\]
for a simple two qubit system. The two players independently operate on the first and the second qubits by the above defined \( U(\theta) \) and \( U(\varphi) \) respectively. Notice that, since the two local operations \( U_1(\theta) = U(\theta) \otimes 1 \) and \( U_2(\varphi) = 1 \otimes U(\varphi) \) can take action simultaneously or in succession, we can realize this two-bit game both in the static and dynamic ways. As in the first model, two Hermitian payoff operators \( P_1 \) and \( P_2 \) are assigned to the two players and the two payoff functions \( f_i \) \( (i = 1, 2) \) are defined as
\[
 f_i = \text{tr} \left( P_i (U(\theta) \otimes U(\varphi)) |\phi\rangle \langle \phi| (U(\theta) \otimes U(\varphi))^\dagger \right).
\]
Suppose that \( P_i \) has the matrix representation
\[
 (P_i)_{kl} = x_{kl}^i, x_{kl}^i = x_{lk}^i, k, l = 1, 2, 3, 4,
\]
with respect to the basis
\[ \{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle \}. \] (21)

Then by direct calculation we obtain
\[
4f_i = \left( \sum_{j=1}^{4} x_{jj}^i + 2 \text{Re} \, x_{23}^i - 2 \text{Re} \, x_{14}^i \right) \\
+ (-x_{11}^i + x_{22}^i + x_{33}^i - x_{44}^i) \\
+ 2 \text{Re} \, x_{23}^i + 2 \text{Re} \, x_{14}^i \cos 2(\theta + \varphi) \\
- 2(\text{Re} \, x_{12}^i + \text{Re} \, x_{13}^i + \text{Re} \, x_{24}^i) \\
+ \text{Re} \, x_{34}^i \sin 2(\theta + \varphi). \] (22)

If
\[
\text{Re} \, x_{12}^i + \text{Re} \, x_{13}^i + \text{Re} \, x_{24}^i + \text{Re} \, x_{34}^i \leq 0, \] (23) 

the pay off functions can be rewritten as
\[
f_i = p_i + q_i \sin (2(\theta + \varphi) + \Psi_i), \] (24)

where
\[
p_i = \frac{1}{4} \left( \sum_{j=1}^{4} x_{jj}^i + 2 \text{Re} \, x_{23}^i - 2 \text{Re} \, x_{14}^i \right), \\
q_i = \frac{1}{4} \left( -x_{11}^i + x_{22}^i + x_{33}^i - x_{44}^i \\
+ 2 \text{Re} \, x_{23}^i + 2 \text{Re} \, x_{14}^i \right)^2 + \\
4 \left( \text{Re} \, x_{12}^i + \text{Re} \, x_{13}^i + \text{Re} \, x_{24}^i + \text{Re} \, x_{34}^i \right)^2 \right]^{1/2} \\
\Psi_i = -\arctan \frac{A}{B} : \\
A = -x_{11}^i + x_{22}^i + x_{33}^i - x_{44}^i + 2 \text{Re} \, x_{23}^i + 2 \text{Re} \, x_{14}^i \\
B = \text{Re} \, x_{12}^i + \text{Re} \, x_{13}^i + \text{Re} \, x_{24}^i + \text{Re} \, x_{34}^i \] (25)

Thus we conclude that this quantum game with the restrictions
\[
\text{Re} \, x_{12}^i + \text{Re} \, x_{13}^i + \text{Re} \, x_{24}^i + \text{Re} \, x_{34}^i \leq 0 \] (26) 

and \(0 \leq \theta, \varphi \leq \frac{\pi}{4}\) also realizes Game A.

**REMARKS**

In sum, we introduce in this paper a new type of quantum game and prove the Nash Equilibrium Theorem. We also calculate the equilibrium explicitly. In the two models though "coherent" pay-off functions of quantum nature are introduced, the universal existence of Nash equilibrium follows from the simple mathematical structure of the classical Game A. In this sense, the game is not really quantized. On the other hand, this suggests the possibility of studying quantum games from an abstract point of view, transcending concrete examples.

Finally we point out that the interesting examples introduced in this letter can easily be implemented in experiments. For the above one-spin qubit model, the operation \(U(\theta)\) can be realized as a Rabi rotation of angle \(\theta\) around z-axe in a spin procession experiment. If we take the two payoff matrices to be the Pauli matrices \(\sigma_z\) and \(\sigma_y\) respectively, then the measurement of the payoff function is to see the average polarization of spin along \(x = axe\) and \(z = axe\). Thus this
quantum game is implementable by a qubit NMR quantum computing system\[12\], described as one nucleous spin in a magnetic field driven by a radio frequency (rf) field. This system realizes a single qubit quantum logic gate.

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[1] For a review see, e.g., C. H. Bennett and D. P. DiVincenzo, Nature 404, 247 (2000); M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[2] D.A. Meyer, Phys. Rev. Lett. 82, 1052 (1999).
[3] J. Eisert, M. Wilkens and M. Lewenstein, Phys. Rev. Lett. 83, 3077 (1999).
[4] L. Goldenberg, L. Vaidman, and S. Wiesner, Phys. Rev. Lett. 82, 3356 (1999); S. C. Benjamin and P. M. Hayden, Phys. Rev. A 64, 030301 (2001); N. F. Johnson, Phys. Rev. A 63, 020302 (2001).
[5] J. Du, H. Li, X. Xu, M. Shi, J. Wu, X. Zhou and R. Han, Phys. Rev. Lett. 88, 137902 (2002).
[6] J. von Neumann and O. Morgenstern, The Theory of Games and Economic Behaviour (Princeton University Press, Princeton, NJ, 1947); For an introduction, see, e.g., E. Rasmusen, Games and Information (Blackwell, Oxford, UK, 1995).
[7] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 77-78 (1935).
[8] J. F. Nash, Adv. Math. 54, 286 (1951).
[9] C.F Lee, N.Johnson,Quantum Game Theory, LANL e-print quant-ph/0207012.
[10] C.P. Sun, Phys. Rev. A 48, 878 (1993); C.P. Sun et al, Fortschr. Phys. 43, 585 (1995).
[11] C.P. Sun, H. Zhan, and X.F. Liu, Phys. Rev. A 58, 1810 (1998).
[12] M. A. Nielsen, E. Knill, and R. Laflamme, Nature (London) 396, 52 (1998); D. G. Cory et al., Phys. Rev. Lett. 81, 2152 (1998); E. Knill et al., Phys. Rev. Lett. 86, 5811 (2001); J. A. Jones, M. Mosca, and R. H. Hansen, Nature (London) 393, 344 (1998); X. Fang, X. Zhu, M. Feng, X. Mao, and F. Du, Phys. Rev. A 61, 022307 (2000); J. Du et al., Phys. Rev. A 63, 042302 (2001); 64, 042306 (2001).