ON THE HILBERT SERIES OF VERTEX COVER ALGEBRAS OF
COHEN-MACAUAY BIPARTITE GRAPHS

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ABSTRACT. We study the Hilbert function and the Hilbert series of the vertex
cover algebra $A(G)$, where $G$ is a Cohen-Macaulay bipartite graph.

MSC: 05E40, 13P10.

Keywords: Cohen-Macaulay bipartite graph, Vertex cover, Hilbert series.

1. Introduction

Let $G = (V, E)$ be a simple (i.e., finite, undirected, loop less and without multiple
edges) graph with the vertex set $V = [n]$ and the edge set $E = E(G)$. A vertex
cover of $G$ is a subset $C \subseteq V$ such that $C \cap \{i, j\} \neq \emptyset$, for any edge $\{i, j\} \in E(G)$.
A vertex cover $C$ of $G$ is called minimal if no proper subset $C' \subset C$ is a vertex
cover of $G$. A graph $G$ is called unmixed if all minimal vertex covers of $G$ have
the same cardinality. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables
over a field $\mathbb{K}$. The edge ideal of $G$ is the monomial ideal $I(G)$ of $R$ generated by
all the quadratic monomials $x_i x_j$ for every $\{i, j\} \in E(G)$. It is said that a graph $G$ is
Cohen–Macaulay (over $\mathbb{K}$) if the quotient ring $R/I(G)$ is Cohen-Macaulay. Every
Cohen-Macaulay graph is unmixed.

A vertex cover $C \subseteq [n]$ can be represented as a $(0, 1)$–vector $c$ that satisfies the
restriction $c(i) + c(j) \geq 1$, for every $\{i, j\} \in E(G)$. For each $k \in \mathbb{N}$, a vertex cover
of $G$ of order $k$, or simply a $k$–vertex cover of $G$, is a vector $c \in \mathbb{N}^n$ such that
c(i) + c(j) $\geq k$, for every $\{i, j\} \in E(G)$. The vertex cover algebra $A(G)$ is defined as
the subalgebra of the one variable polynomial ring $R[t]$ generated by all monomials
$x_1^{c_1} \cdots x_n^{c_n} t^k$, where $c = (c_1, \ldots, c_n) \in \mathbb{N}^n$ is a $k$-vertex cover of $G$. This algebra was
introduced and first studied in [5]. Let $\mathfrak{m}$ be the maximal graded ideal of $R$. The
graded $\mathbb{K}$-algebra $\bar{A}(G) = A(G)/\mathfrak{m} A(G)$ is called the basic cover algebra and it was
introduced and first studied in [4] Section 3.

Our aim in this paper is to study the Hilbert function and series of the vertex
cover algebra $A(G)$ for Cohen-Macaulay bipartite graphs.

Let $P_n = \{p_1, p_2, \ldots, p_n\}$ be a poset with a partial order $\leq$. Let $G = G(P_n)$
be the bipartite graph on the set $V_n = W \cup W'$, where $W = \{x_1, x_2, \ldots, x_n\}$ and
$W' = \{y_1, y_2, \ldots, y_n\}$, whose edge set $E(G)$ consists of all 2-element subsets $\{x_i, y_j\}$
with $p_i \leq p_j$. It is said that a bipartite graph $G$ on $V_n = W \cup W'$ comes from a
poset, if there exists a finite poset $P_n$ on $\{p_1, p_2, \ldots, p_n\}$ such that $p_i \leq p_j$ implies
$i \leq j$, and after relabeling of the vertices of $G$ one has $G = G(P_n)$. Herzog and Hibi


proved in [3] that a bipartite graph $G$ is Cohen-Macaulay if and only if $G$ comes from a poset.

In Section 2, we firstly notice that the Hilbert function and series of the vertex cover algebras $A(G)$ are invariant to poset isomorphisms. We obtain a recurrence relation for the minimal vertex covers of a Cohen-Macaulay graph $G$ and we study the Hilbert function of $A(G)$.

In Section 3, we study the Hilbert series of $A(G)$. For a poset $P_n = \{p_1, p_2, \ldots, p_n\}$ we denote by $\mathcal{J}(P_n)$ the lattice of all poset ideals of $P_n$. For each subset $\emptyset \neq F \subseteq [n]$ we denote by $P_n(F)$ the subposet of $P_n$ induced by the subset $\{p_i | i \in F\}$ and by $G_F$ the bipartite graph that comes from $P_n(F)$. The main result of this paper is given in Theorem 3.4 which shows that one may reduce the computation of the Hilbert series of the vertex cover algebra $A(G)$ to the computation of the Hilbert series of the basic cover algebra $A(G_F)$, for all $F \subseteq [n]$. If $F = \emptyset$, then, by convention, the Hilbert series of $\bar{A}(G_F)$ is equal to $\frac{1}{1-z}$. Namely, we have the following formula:

$$H_{\bar{A}(G_F)}(z) = \frac{1}{(1-z)^n} \sum_{F \subseteq [n]} H_{\bar{A}(G_F)}(z) \left(\frac{z}{1-z}\right)^{n-|F|}.$$

Moreover, we give a combinatorial interpretation for the $h$-vector of $A(G)$ in terms of the poset $P_n$. Using this interpretation we show that the $h$-vector of $A(G)$ is unimodal. We give bounds for its components and derive bounds for $e(A(G))$, the multiplicity of $A(G)$.

We show that both chains and antichains are uniquely determined up to a poset isomorphism by the Hilbert series of their corresponding vertex cover algebras.

2. Vertex cover algebras of Cohen-Macaulay bipartite graphs

Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ and let $G = G(P_n)$, where $P_n = \{p_1, \ldots, p_n\}$ is a poset with a partial order $\leq$. We recall that, by [5], the vertex cover algebra $A(G)$ is standard graded over $S$ and it is the Rees algebra of the cover ideal $I_G$, which is generated by all monomials $x_1^{c_1} \cdots x_n^{c_n} y_1^{c_{n+1}} \cdots y_n^{c_{2n}}$, where $c = (c_1, \ldots, c_{2n})$ is a 1-vertex cover of $G$. Thus

$$A(G) = S \oplus I_G t \oplus \ldots \oplus I_G^{k} t^k \oplus \ldots$$

Let $\{m_1, m_2, \ldots, m_l\}$ be the minimal system of generators of $I_G$. We view $A(G)$ as a standard graded $K$-algebra by assigning to each $x_i$ and $y_j$, $1 \leq i, j \leq n$ and to each $m_k t$, $1 \leq k \leq l$, the degree 1. Since each monomial $m_k$ corresponds to a minimal vertex cover of $G$ of cardinality $n$, the Hilbert function of $A(G)$ is given by

$$H(A(G), k) = \sum_{j=0}^{k} \dim_K (I_G^j)_{j^n+(k-j)}, \text{ for all } k \geq 0.$$

Remark 2.1. Let $P_n = \{p_1, \ldots, p_n\}$ and $P_n' = \{p'_1, \ldots, p'_n\}$ be two isomorphic finite posets and let $G = G(P_n)$ and $G' = G(P'_n)$. Then the cover ideals $I_G$ and $I_{G'}$ are isomorphic as graded $K$-vector spaces and, consequently, the Hilbert function and series of $A(G)$ and $A(G')$ coincide. Let $f : P_n \to P_n'$ be a poset isomorphism (i.e., $f$ is a bijective map with $p_i \leq p_j$ if and only if $f(p_i) \leq f(p_j)$). Then $f$ induces a
permutation $g$ of $[n]$, $i \to g(i)$, defined by $p'_{g(i)} = f(p_i)$, for every $i \in [n]$. We notice that

$$p_i \leq p_j \iff f(p_i) \leq f(p_j) \iff p'_{g(i)} \leq p'_{g(j)},$$

and we define a map $h : V(G) \to V(G')$ as follows:

$$h(x_i) = x_{g(i)}, \text{ if } i \in [n],$$
$$h(y_j) = y_{g(j)}, \text{ if } j \in [n].$$

Then $h$ induces a $K$-automorphism of $S$ which maps $I_G$ onto $I_{G'}$, hence, $I_G$ and $I_{G'}$ are isomorphic as graded $K$-vector spaces. By (11), we also have

$$H(A(G'), k) = \sum_{j=0}^{k} \dim_K(P^j_{G'})_{j+1} \cdot \dim_K(P^j_{G'})_{j+1}, \text{ for all } k \geq 0.$$  

Since the powers $P^j_G$ and $P^j_{G'}$ are isomorphic as graded $K$-vector spaces as well, for all $j \geq 1$, we get $H(A(G), k) = H(A(G'), k)$, for all $k \geq 0$.

We denote by $\mathcal{M}(G)$ the set of minimal vertex covers of a graph $G$. Vertex covers and stable sets of a graph $G$ are dual concepts, that is, a subset $C \subset V(G)$ is a vertex cover of $G$ if and only if the complement set $V(G) \setminus C$ is a stable set of $G$ (17). Next, inspired by [7, Lemma 2.5], we give a recurrence relation to obtain the set of the minimal vertex covers of a Cohen-Macaulay bipartite graph $G_n$, which comes from a poset $P_n = \{p_1, \ldots, p_n\}$. We denote by $G_{n-1}$ the subgraph of $G_n$ which comes from the poset $P_{n-1} = \{p_1, \ldots, p_{n-1}\}$ and by $V_{n-1}$ the set $\{x_1, \ldots, x_{n-1}\} \cup \{y_1, \ldots, y_{n-1}\}$.

**Proposition 2.2.** Let $G_n = G(P_n)$, where $P_n = \{p_1, \ldots, p_n\}$, $n \geq 2$, is a poset such that $p_i \leq p_j$ implies $i \leq j$. Then a subset $C_n \subset V_n$ is a minimal vertex cover of $G_n$ if and only if either $C_n = C_{n-1} \cup \{y_n\}$, where $C_{n-1} \subset V_{n-1}$ is a minimal vertex cover of $G_{n-1}$ or $C_n = C_{n-1} \cup \{x_n\}$, where $C_{n-1} \subset V_{n-1}$ is a minimal vertex cover of $G_{n-1}$ such that $x_i \in C_{n-1}$ for each $i \in [n-1]$ with $p_i \leq p_n$.

**Proof.** If it is straightforward.

Let us prove 'Only if'. Since $G_n$ is a Cohen-Macaulay graph, it is unmixed and all its minimal vertex covers have the same cardinality, namely $n$, for every $n \geq 2$.

If $n = 2$ the statement obviously holds.

We assume that $n \geq 3$. Let $C_n = \{c_1, \ldots, c_n\}$ be a minimal vertex cover of $G_n$. Put $C_n = \{c_1, \ldots, c_n\}$, $C_{n-1} = C_n \cap V_{n-1}$ and $C'_n = C_n \cap \{x_n, y_n\}$. Obviously, $|C'_n| \leq 2$.

If $|C'_n| = 0$, then $C_n \cap \{x_n, y_n\} = \emptyset$, which is impossible. Now let us suppose that $|C'_n| = 2$, hence $C'_n = \{x_n, y_n\}$ and $|C_{n-1}| = n - 2$. Since $C_n$ is a vertex cover of $G_n$, it follows that the intersection of $C_n$ with every edge $\{x_i, y_j\}$ of the subgraph $G_{n-1}$ ($1 \leq i \leq j \leq n - 1$) is a nonempty subset of $C_{n-1}$, hence $C_{n-1}$ is a vertex cover of $G_{n-1}$ of cardinality $n - 2$. But this is impossible since all minimal vertex covers of $G_{n-1}$ have the cardinality equal to $n - 1$.

It follows that $|C'_n| = 1$, $|C_{n-1}| = n - 1$ and exactly one of the vertices $x_n$ or $y_n$ belongs to $C_n$. We can put, without loss of generality, either $c_n = x_n$ or $c_n = y_n$, and $C_{n-1} = \{c_1, \ldots, c_{n-1}\} \subset V_{n-1}$. Since $C_n$ is a vertex cover of $G_{n-1}$, the intersection of
Let $C_{n}$ with every edge $\{x_{i}, y_{j}\}$ of the subgraph $G_{n-1}$ $(1 \leq i \leq j \leq n-1)$ is a nonempty subset of $C_{n-1}$, hence $C_{n-1}$ is a vertex cover of $G_{n-1}$. Moreover, $C_{n-1}$ is a minimal vertex cover of $G_{n-1}$, since $|C_{n-1}| = n - 1$.

If we choose $c_{n} = x_{n}$, then $y_{n} \notin C_{n}$. Since $C_{n}$ is a vertex cover of $G_{n}$, it follows that $C_{n} \cap \{x_{i}, y_{n}\} = \{x_{i}\}$, for every $\{x_{i}, y_{n}\} \in E(G_{n})$ with $i \in [n-1]$, which implies that $x_{i} \in C_{n}$, for each $i \in [n-1]$ with $\{x_{i}, y_{n}\} \in E(G_{n})$. Hence $x_{i} \in C_{n} \cap V_{n-1} = C_{n-1}$, for each $i \in [n-1]$ with $p_{i} < p_{n}$.

If we choose $c_{n} = y_{n}$, then there is no (other) restriction on the minimal vertex cover $C_{n-1}$ of $G_{n-1}$.

\[\square\]

**Remark 2.3.** Let $G$ be a Cohen-Macaulay bipartite graph which comes from the poset $P_{n}$. By \[4\] Theorem 2.1 there is a one-to-one correspondence between the set $\mathcal{M}(G)$ and the distributive lattice $\mathcal{J}(P_{n})$ of all poset ideals of $P_{n}$. Thus it can be assigned to each minimal vertex cover $C$ of $G$ the poset ideal $\alpha_{C}$ of $P_{n}$ that is defined as $\alpha_{C} = \{p_{i} | x_{i} \in C\}$. Conversely, if $\alpha$ is a poset ideal of $P_{n}$, then the corresponding set $C_{\alpha} = \{x_{i} | p_{i} \in \alpha \} \cup \{y_{j} | p_{j} \notin \alpha \}$ is a minimal vertex cover of $G$. By Proposition 2.2 one may give a recursive procedure to compute the lattice $\mathcal{J}(P_{n})$.

For $C \in \mathcal{M}(G)$ we denote $m_{C} = (\prod_{i \in C} x_{i}) \cdot (\prod_{j \in C} y_{j})$. If $G$ is unmixed, then each $C \in \mathcal{M}(G)$ has exactly $n$ vertices, hence, $\deg m_{C} = n$, for all $C \in \mathcal{M}(G)$. The next result shows a property of monotony of the Hilbert function of an unmixed bipartite graph.

**Proposition 2.4.** Let $G$, $G'$ and $G''$ be unmixed bipartite graphs on $V_{n}$, $n \geq 1$, such that $E(G'') \subset E(G) \subset E(G')$. Then the following inequalities hold:

\[H(A(G'), k) \leq H(A(G), k) \leq H(A(G''), k), \text{ for all } k \geq 0.\]

**Proof.** It is known \([5\, \text{Theorem } 5.1.b]\) that $I_{G} = (m_{C} | C \in \mathcal{M}(G))$. Similarly, we have $I_{G'} = (m_{C} | C \in \mathcal{M}(G'))$ and $I_{G''} = (m_{C} | C \in \mathcal{M}(G''))$. It follows that all the cover ideals are generated in the same degree $n$.

From the inclusions between the edge sets and the hypothesis of unmixedness, we get $\mathcal{M}(G') \subset \mathcal{M}(G) \subset \mathcal{M}(G'')$. Therefore, $I_{G'} \subset I_{G} \subset I_{G''}$. We also have

\[(I_{G'})_{b} \subset (I_{G})_{b} \subset (I_{G''})_{b}, \quad (3)\]

for all integers $a \geq 1$ and $b \geq 0$, which, by \([4]\), implies the desired inequalities. \[\square\]

It is obvious that, for the Cohen-Macaulay bipartite graphs, the chain provides the largest number of edges and the antichain the smallest number of edges.

**Corollary 2.5.** Let $G$ be a Cohen-Macaulay bipartite graph on $V_{n}$, $n \geq 1$. Then the following inequalities hold:

\[H(A(G'), k) \leq H(A(G), k) \leq H(A(G''), k), \text{ for all } k \geq 0, \quad (4)\]

where $G'$ and $G''$ are bipartite graphs on $V_{n}$ that come from a chain, respectively, an antichain with $n$ elements.

**Proof.** Let $G$, $G'$, respectively, $G''$ be graphs that come from a poset $P_{n} = \{p_{1}, \ldots, p_{n}\}$, a chain $P_{n}' = \{p_{1}', \ldots, p_{h}'\}$, respectively, an antichain $P_{n}'' = \{p_{1}'', \ldots, p_{n}'\}$. By Remark 2.1...
we may assume that \( p_i \leq p_j \) and \( p'_i \leq p'_j \) imply \( i \leq j \). It is straightforward to notice that \( E(G'') \subset E(G) \subset E(G') \). Therefore, by applying Proposition 2.4, the desired inequalities follow.

The next result stresses a property of monotonity for the multiplicity of the vertex cover algebra for unmixed bipartite graphs.

**Corollary 2.6.** Let \( G, G' \) and \( G'' \) be unmixed bipartite graphs on \( V_n \) such that \( E(G'') \subset E(G) \subset E(G') \). Then the following inequalities hold:

\[
e(A(G')) \leq e(A(G)) \leq e(A(G'')).\]

**Proof.** By Proposition 2.4 we have \( H(A(G')), k) \leq H(A(G), k) \leq H(A(G''), k) \), for all \( k \geq 0 \). Since \( H(A(G), k) \), \( H(A(G'), k) \), respectively, \( H(A(G''), k) \) are all polynomials of degree \( 2n \) (since \( \text{dim } A(G) = \text{dim } S + 1 = 2n + 1 \) [2] with the leading coefficients \( e(A(G)) \), \( e(A(G')) \), respectively, \( e(A(G'')) \), the conclusion follows.

3. **The Hilbert series of vertex cover algebras of Cohen-Macaulay bipartite graphs**

Let \( S = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \) be the polynomial ring in \( 2n \) variables over a field \( K \) and let \( G = G(P_n) \), where \( P_n = \{p_1, \ldots, p_n\} \) is a poset such that \( p_i \leq p_j \) implies \( i \leq j \).

We denote \( B_G = K[\{x_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq n}, \{u_\alpha\}_{\alpha \in \mathcal{J}(P_n)}] \). The **toric ideal** \( Q_G \) of \( A(G) \) is the kernel of the surjective homomorphism \( \varphi: B_G \to A(G) \) defined by \( \varphi(x_i) = x_i, \varphi(y_j) = y_j, \varphi(u_\alpha) = m_\alpha t \), where \( m_\alpha = (\prod_{p_i \in \alpha} x_i) \cdot (\prod_{j \in \alpha} y_j), \alpha \in \mathcal{J}(P_n) \), are the minimal monomial generators of the cover ideal \( I_G \).

Let \( <_{\text{lex}} \) denote the lexicographic order on \( K[\{x_i\}_{1 \leq i \leq n}, \{y_j\}_{1 \leq j \leq n}] \) induced by the ordering \( x_1 > x_2 > \ldots > x_n > y_1 > \ldots > y_n \) and \( <^\# \) the reverse lexicographic order on \( K[\{u_\alpha\}_{\alpha \in \mathcal{J}(P_n)}] \) induced by an ordering of the variables \( u_\alpha \)'s such that \( u_\alpha > u_\beta \) if \( \beta \subset \alpha \) in \( \mathcal{J}(P_n) \). Let \( <^\#_{\text{lex}} \) be the monomial order on \( B_G \) defined as the product of the monomial orders \( <_{\text{lex}} \) and \( <^\# \) from above. The reduced Gröbner basis \( \mathcal{G} \) of the toric ideal \( Q_G \) of \( A(G) \) with respect to the monomial order \( <^\#_{\text{lex}} \) on \( B_G \) was computed in [3] Theorem 1.1:

\[
\mathcal{G} = \{x_j u_\alpha - y_j u_{\alpha \cup \{p_j\}}, j \in [n], \alpha \in \mathcal{J}(P_n), p_j \not\in \alpha, \alpha \cup \{p_j\} \in \mathcal{J}(P_n), u_\alpha u_\beta - u_{\alpha \cup \beta} u_{\alpha \cap \beta}, \alpha, \beta \in \mathcal{J}(P_n), \alpha \not\subset \beta, \beta \not\subset \alpha, \}
\]

where the initial monomial of each binomial of \( \mathcal{G} \) is the first monomial.

Let \( S_G = K[\{u_\alpha\}_{\alpha \in \mathcal{J}(P_n)}] \) be the polynomial ring in \( |\mathcal{J}(P_n)| \) variables over \( K \), let \( \tilde{A}(G) \) the basic vertex cover algebra and \( \Delta(\mathcal{J}(P_n)) \) the order complex of the lattice \( \mathcal{J}(P_n) \subset \mathbb{Z}^n \) whose vertices are the chains of \( P_n \). (We refer the reader to [1], [4] Section 3 for the definition and properties of the basic cover algebra associated to a graph and [2] §5.1 for the definition and properties of the order complex of a poset.) The **toric ideal** \( \tilde{Q}_G \) of \( \tilde{A}(G) \) is the kernel of the surjective homomorphism \( \pi: S_G \to \tilde{A}(G), \pi(u_\alpha) = m_\alpha \). The reduced Gröbner basis \( \mathcal{G}_0 \) of \( \tilde{Q}_G \) with respect to \( <^\# \) on \( S_G \) was computed in [4] Theorem 3.1:

\[
\mathcal{G}_0 = \{u_\alpha u_\beta - u_{\alpha \cup \beta} u_{\alpha \cap \beta} | \alpha, \beta \in \mathcal{J}(P_n), \alpha \not\subset \beta, \beta \not\subset \alpha, \}
\]
where the initial monomial of each binomial of $G_0$ is the first monomial.

**Proposition 3.1.** The graded $K$-algebra $A(G)$ and the order complex $\Delta(J(P_n))$ have the same $h$-vector.

**Proof.** $Q_G$ is a graded ideal (generated by binomials) and the initial ideal in $\text{in}_{<s}(Q_G)$ of the toric ideal $Q_G$ coincides with the Stanley-Reisner ideal $I_{\Delta(J(P_n))}$, hence $S_G/Q_G$ and $K[\Delta(J(P_n))]$ have the same $h$-vector. Since $S_G/Q_G \simeq A(G)$ as graded $K$-algebras, the conclusion follows. \hfill $\square$

**Remark 3.2.** Since $J(P_n)$ is a full sublattice of the Boolean lattice $L_n$ on the set $\{p_1, p_2, \ldots, p_n\}$ ([3, Theorem 2.2.]), it follows that $\dim \Delta(J(P_n)) = n$. Let $h = (h_0, h_1, \ldots, h_{n+1})$ be the $h$-vector of $J(P_n)$ and $A(G)$. As we noticed above, the basic vertex cover algebra $A(G)$ can be identified with the Hibi ring $S_G/Q_G$, which arises from the distributive lattice $J(P_n)$. The $i$-th component $h_i$ of the $h$-vector of $S_G/Q_G$ and, consequently, of $A(G)$ is equal to the number of linear extensions of $P_n$, which, seen as permutations of $[n]$, have exactly $i$ descents ([6]). In particular,

$$h_i \geq 0, \text{ for all } 0 \leq i \leq n - 1, \quad h_0 = 1, \quad \text{and } h_n = h_{n+1} = 0. \quad (5)$$

For example, if $P_n'' = \{p''_1, \ldots, p''_n\}$ is an antichain, then each permutation of $[n]$ can be seen as a linear extension of $P_n''$, hence, for all $0 \leq i \leq n - 1$, the $i$-th component of the $h$-vector of $\Delta(J(P_n''))$ is equal to the number of all permutations of $[n]$ with exactly $i$ descents, which is the Eulerian number $A(n, i)$.

For each $\emptyset \neq F \subset [n]$ we denote by $P_n(F)$ the subposet of $P_n$ induced by the subset $\{p_i \mid i \in F\}$. The main result of the paper relates the Hilbert series of $A(G)$ to the Hilbert series of $A(G_F)$, for all $F \subset [n]$, where $G_F$ denotes the bipartite graph that comes from the poset $P_n(F)$. If $F = \emptyset$, then, by convention, the Hilbert series of $A(G_F)$ is equal to $\frac{1}{1 - z}$.

In order to prove the main theorem we need a preparatory result.

Let $\emptyset \neq F \subset [n]$ and let $\alpha$ be a poset ideal of $P_n(F)$, where by $\bar{F}$ we mean the complement of $F$ in $[n]$. We denote by $\delta_\alpha$ the maximal subset of $P_n(F)$ such that $\alpha \cup \delta_\alpha \in J(P_n)$. Note that

$$\delta_\alpha = \cup \{\gamma \mid \gamma \subset P_n(F), \alpha \cup \gamma \in J(P_n)\}.$$ 

If we set $\beta = \alpha \cup \delta_\alpha$, then, by the definition of $\delta_\alpha$, $\beta$ has the following property: for any $j \in F$, $p_j \not\in \beta$ implies $\beta \cup \{p_j\} \not\in J(P_n)$.

**Lemma 3.3.** Let $\emptyset \neq F \subset [n]$ and let $S$ be the set of poset ideals $\beta$ of $P_n$ with the property that for any $j \in F$ such that $p_j \not\in \beta$ we have $\beta \cup \{p_j\} \not\in J(P_n)$. Then the map $\varphi: J(P_n(F)) \to S$ defined by $\alpha \mapsto \beta := \alpha \cup \delta_\alpha$, is an isomorphism of posets.

**Proof.** $\varphi$ is invertible. Indeed, the map $\psi: S \to J(P_n(F))$ defined by $\psi(\beta) = \beta \cap P_n(F)$ is the inverse of $\varphi$ since if $\alpha = \beta \cap P_n(F)$, then, by the property of $\beta$, we have $\delta_\alpha = \beta \setminus P_n(F)$. 

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Let \( \alpha_1 \subseteq \alpha_2 \) be poset ideals of \( P_n(F) \) and \( \beta_i = \varphi(\alpha_i) = \alpha_i \cup \delta_i, \ i = 1, 2 \). We only need to show that \( \beta_1 \subseteq \beta_2 \) since the strict inclusion follows from the hypothesis \( \alpha_1 \subseteq \alpha_2 \). Let us assume that \( \beta_1 \nsubseteq \beta_2 \) and let \( p_a, a \in F \), be a minimal element in \( \beta_1 \setminus \beta_2 \). Since \( p_a \notin \beta_2 \), it follows that \( \beta_2 \cup \{p_a\} \) is not a poset ideal of \( P_n \). Therefore there exists \( p_b < p_a \) such that \( p_b \notin \beta_2 \). On the other hand, \( p_b \in \beta_1 \) since \( \beta_1 \in \mathcal{J}(P_n) \), hence, \( p_b \in \beta_1 \setminus \beta_2 \), which leads to a contradiction with the choice of \( p_a \).

Now let \( \beta_1 \nsubseteq \beta_2, \beta_1, \beta_2 \in \mathcal{S} \), and assume that \( \alpha_1 = \alpha_2 \), where \( \alpha_1 = \beta_1 \cap P_n(F) \), and \( \alpha_2 = \beta_2 \cap P_n(F) \). Then \( \delta_1 = \beta_1 \setminus P_n(F) \nsubseteq \delta_2 = \beta_2 \setminus P_n(F) \). But this is impossible since \( \delta_1 \) is maximal among the subsets \( \gamma \) of \( P_n(F) \) such that \( \alpha_1 \cup \gamma \in \mathcal{J}(P_n) \). \( \Box \)

We can state now the main theorem which relates the Hilbert series of the vertex cover algebra \( A(G) \) to the Hilbert series of the basic cover algebras \( \bar{A}(G_F) \) for all \( F \subset [n] \).

**Theorem 3.4.** For \( F \subset [n] \) let \( H_{\bar{A}(G_F)}(z) \) be the Hilbert series of \( \bar{A}(G_F) \) and let \( H_{A(G)}(z) \) be the Hilbert series of \( A(G) \). Then:

\[
H_{A(G)}(z) = \frac{1}{(1-z)^n} \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left( \frac{z}{1-z} \right)^{|F|}.
\]  

(6)

In particular, if \( h(z) = \sum_{j \geq 0} h_j z^j \) and \( h^F(z) = \sum_{j \geq 0} h^F_j z^j \), where \( h = (h_j)_{j \geq 0} \) and \( h^F = (h^F_j)_{j \geq 0} \) are the \( h \)-vectors of \( A(G) \), and, respectively, \( \bar{A}(G_F) \), then

\[
h(z) = \sum_{F \subset [n]} h^F(z) z^{|F|}.
\]  

(7)

**Proof.** Let \( J_G = \text{in}_{\text{lex}}^\#(Q_G) \). It is known that \( B_G/Q_G \) and \( B_G/J_G \) have the same Hilbert series. Let \( B'_G = K[\{x_i\}_{1 \leq i \leq n}, \{u_\alpha\}_{\alpha \in \mathcal{J}(P_n)}] \). By using the following \( K \)-vector space isomorphism

\[ B_G/J_G \cong K[y_1, y_2, ..., y_n] \otimes_K B'_G/(J_G \cap B'_G), \]

we get

\[ H_{A(G)}(z) = H_{B_G/Q_G}(z) = H_{B_G/J_G}(z) = \frac{1}{(1-z)^n} H_{B'_G/(J_G \cap B'_G)}(z). \]

We need to compute the Hilbert series of \( B'_G/(J_G \cap B'_G) \). To this aim we show that we have an isomorphism of \( K \)-vector spaces

\[
B'_G/(J_G \cap B'_G) \cong \bigoplus_{F \subset [n]} \bar{A}(G_F) \otimes_K x_F K[\{x_i\}_{i \in F}].
\]  

(8)

For \( \emptyset \neq F \subset [n] \) let \( J_F \) be the initial ideal with respect to \( <^\# \) of the toric ideal of \( \bar{A}(G_F) \). Then \( J_F = (u_\alpha u_\beta)_{\alpha, \beta \in \mathcal{J}(P_n(F)), \alpha \nsubseteq \beta, \beta \nsubseteq \alpha} \). If \( F = \emptyset \), we put by convention \( J_F = (u_0) \).

The basic vertex cover algebra \( \bar{A}(G_F) \) can be decomposed as a \( K \)-vector space as \( \bar{A}(G_F) \cong \bigoplus_{w \notin J_F} K w \). We notice that \( w \notin J_F \) if and only if \( \text{supp}(w) = \{\alpha_1, ..., \alpha_s\} \),
s ≥ 0, where α_1 ⊆ ... ⊆ α_s is a chain in J(P_n(F)). It follows that for F ⊂ [n] we have
\[ V_F := \bar{A}(G_F) \otimes_K x_F K[x_i]_{i \in F} \simeq \bigoplus Kvw, \]
where the direct sum is taken over all monomials vw with v monomial in the variables x_i such that supp(v) = F and w monomial in the variables u_α such that w /∈ J_F.

As a K-vector space, \( B'_G/(J_G \cap B'_G) \) has the decomposition
\[ B'_G/(J_G \cap B'_G) \simeq \bigoplus_{F \subset [n]} W_F, \]
where \( W_F = \bigoplus Kvw' \) and the direct sum is taken over all monomials v with supp(v) = F and all monomials w' in the variables u_α with α \in J(P_n) such that vw' /\not\in 0 \) modulo \( J_G \cap B'_G \).

In order to prove \( \emptyset \neq F \subseteq [n] \), we only need to show that for each \( F \subset [n] \), the K-vector spaces \( V_F \) and \( W_F \) are isomorphic. This is obvious for \( F = \emptyset \) and \( F = [n] \).

Let us consider now \( \emptyset \neq F \subseteq [n] \). Based on the previous lemma, we are going to show that there exists a bijection between the K-bases of \( V_F \) and \( W_F \).

Let vw be an element of the K-basis of \( V_F \). This means that supp(v) = F and w is of the form \( w = u_{\alpha_1}^{a_1} \cdots u_{\alpha_s}^{a_s} \) for some chain \( \alpha_1 \subseteq ... \subseteq \alpha_s \) in \( J(P_n) \), s ≥ 1.

For each \( 1 \leq i \leq s \), let \( \beta_i = \varphi(\alpha_i) \in J(P_n) \) as it was defined in Lemma 3.3. We map vw to the monomial vw' where \( w' = u_{\beta_1}^{a_1} \cdots u_{\beta_s}^{a_s} \). By Lemma 3.3 we have that \( \beta_1 \subseteq ... \subseteq \beta_s \) is a chain in \( J(P_n) \). Moreover, for any \( j \in F \) and any \( \beta_i \) such that \( p_j \notin \beta_i \), we have \( \beta_i \cup \{p_j\} \notin J(P_n) \). Therefore, vw' is a monomial in the K-basis of \( W_F \).

Conversely, let vw' be a monomial from the K-basis of \( W_F \), where supp(v) = F and \( w' = u_{\beta_1}^{a_1} \cdots u_{\beta_s}^{a_s} \), with \( \beta_1 \subseteq ... \subseteq \beta_s \) a chain in \( J(P_n) \). Let \( \alpha_i = \beta_i \cap P_n(F) \), for \( 1 \leq i \leq s \). Then we associate to vw' the monomial vw in the K-basis of \( V_F \), where \( w = u_{\alpha_1}^{a_1} \cdots u_{\alpha_s}^{a_s} \).

By using again Lemma 3.3 it follows that the above defined maps between the K-bases of \( V_F \) and \( W_F \) are inverse.

By \( \emptyset \neq F \subseteq [n] \) we get
\[ H_{B'_G/(J_G \cap B'_G)}(z) = \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left( \frac{z}{1 - z} \right)^{|F|} = \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left( \frac{z}{1 - z} \right)^{n-|F|}. \]

Hence
\[ H_{\bar{A}(G)}(z) = \frac{1}{(1 - z)^n} \sum_{F \subset [n]} H_{\bar{A}(G_F)}(z) \left( \frac{z}{1 - z} \right)^{n-|F|}. \]

Since \( H_{\bar{A}(G)}(z) = \frac{h(z)}{(1-z)^{n+1}} \) and \( H_{\bar{A}(G_F)} = \frac{h^F(z)}{(1-z)^{n+1}} \), for all \( F \subset [n] \), it follows that \( h(z) = \sum_{F \subset [n]} h^F(z) z^{n-|F|} \).

\[ \square \]

**Corollary 3.5.** For all \( 0 \leq j \leq n - 1 \), the j-th component \( h_j \) of the h-vector of \( A(G) \) is equal to the number of all linear extensions of all \( n - l \)-element subposets of \( P_n \), which, seen as permutations of \([n - l]\), have exactly \( j - l \) descents, for all \( 0 \leq l \leq j \).
Proof. It follows immediately from (7) and Remark 3.2.

Corollary 3.6. The h-vector of \( A(G) \) is unimodal.

Proof. By (7) we get \( h_{n+1} = \sum_{F \subseteq [n]} h_{|F|+1}^F \) and \( h_n = \sum_{F \subseteq [n]} h_{|F|}^F \). By using (5) from Remark 3.2, we have \( h_{|F|}^F = h_{|F|+1}^F = 0 \), for all \( \emptyset \neq F \subseteq [n] \). Hence \( h_{n+1} = h_1^0 = 0 \) and \( h_n = h_0^0 = 1 \). In [5] Corollary 4.4 it is proved that \( A(G) \) is a Gorenstein ring, hence, by [2] Corollary 4.3.8 (b) and Remark 4.3.9 (a)], \( h_i = h_{n-i} \), for all \( 0 \leq i \leq n \).

We denote by \( \nu(l,j) \) the number of all linear extensions of all \( n-l \)-element subposets of \( P_n \) which, seen as permutations of \( [n] \), have exactly \( j \) descents. Furthermore, by Corollary 3.5 \( h_j = \sum_{l=0}^{j} \nu(l,j) \). Let \( 0 \leq j < j + 1 \leq \left\lceil \frac{n}{2} \right\rceil \). Then \( \nu(l,j) \leq \nu(l+1,j+1) \), for all \( 0 \leq l \leq j \), which implies that \( h_{j+1} = \nu(j+1,0) + \sum_{l=0}^{j} \nu(j+1,l+1) \geq \sum_{l=0}^{j} \nu(l,j) = h_j \).

Remark 3.7. The Hilbert series of the vertex cover algebra \( A(G) \) is given by

\[
H_{A(G)}(z) = \frac{h_0 + h_1 z + \ldots + h_{n-1} z^{n-1} + h_n z^n}{(1-z)^{2n+1}},
\]

where \( h = (h_0, \ldots, h_n) \) is the h-vector of \( A(G) \). In particular, we recover the known fact that \( \dim A(G) = 2n + 1 \). It also follows that the a-invariant is \( a = -n - 1 \).

Corollary 3.8. Let \( e(A(G)) \) be the multiplicity of \( A(G) \) and let \( e(\overline{A}(G_F)) \) the multiplicity of \( \overline{A}(G_F) \) for \( F \subseteq [n] \). Then

\[
e(A(G)) = \sum_{F \subseteq [n]} e(\overline{A}(G_F)).
\]

Proof. It follows immediately from (7).

Let \( P_3 = \{p_1, p_2, p_3\} \) be the poset with \( p_1 \leq p_2 \) and \( p_1 \leq p_3 \) and \( G_3 = G(P_3) \). Then

\[
H_{A(G_0)}(z) = \frac{1}{1-z}, \quad H_{A(G_{(1)})}(z) = H_{A(G_{(2)})}(z) = H_{A(G_{(3)})}(z) = \frac{1}{(1-z)^2}, \quad H_{A(G_{(1,2)})}(z) = \frac{1}{(1-z)^3}, \quad H_{A(G_{(1,3)})}(z) = \frac{1+z}{(1-z)^4}, \quad H_{A(G_{(2,3)})}(z) = \frac{1+z}{(1-z)^4},
\]

and the Hilbert series of \( A(G_3) \) is:

\[
H_{A(G_3)}(z) = \frac{1}{(1-z)^3} \sum_{F \subseteq [3]} H_{A(G_F)}(z) \left( \frac{z}{1-z} \right)^{3-|F|} = \frac{z^3 + 4z^2 + 4z + 1}{(1-z)^7}.
\]

Hence \( h_0 = h_3 = 1, h_1 = h_2 = 4, h_4 = 0, e(A(G_3)) = 10 \). We can also compute the h-vector of \( A(G_3) \) by using Corollary 3.5. The poset \( P_3 \) has two linear extensions, which, seen as permutation of \( [3] \), are equal to \( id_3 \) and \( (23) \). Hence \( h_0 = 1 \), since there exists only one linear extension of \( P_3 \), which, seen as a permutation of \( [3] \), has exactly 0 descents. Furthermore, \( P_3 \) has three 2-element subposets, the chains \( P_3(\{1,2\}) \) and \( P_3(\{1,3\}) \) with a linear extension corresponding to \( id_2 \), and the antichain \( P_3(\{2,3\}) \) with two linear extensions corresponding to \( id_2 \) and \( (23) \). Thus \( h_1 = 4 \), since there exists only one linear extension of \( P_3 \), which, seen as a permutation of \( [3] \), has exactly 1 descent and each of the subposets \( P_3(\{1,2\}), P_3(\{1,3\}) \)
and $P_3(\{2,3\})$ has one linear extension, which, seen as a permutation of $[2]$, has exactly 0 descents.

Let $L_n$ be the Boolean lattice on $\{p_1, p_2, ..., p_n\}$, $n \geq 1$, and $A(p, q)$ be the Eulerian number for $1 \leq q \leq n$ and $0 \leq p < q$. By convention, we put $A(0,0) = 1$ and $A(q,q) = 0$, for all $1 \leq q \leq n$.

We compute the Hilbert series of the vertex cover algebra of the Cohen-Macaulay bipartite graphs that come from a chain and an antichain.

**Proposition 3.9.** Let $G'$ be a bipartite graph that comes from a chain and $G''$ a bipartite graph that comes from an antichain with $n$ elements, $n \geq 1$. Then we have

(i) $H_{A(G')}(z) = \frac{(1+z)^n}{(1-z)^{n+1}}$. In particular, $e(A(G')) = 2^n$.

(ii) $H_{A(G'')}(z) = \sum_{j=0}^{n} \frac{n!}{j!} A(n-l,j) A^{l} z^j$. In particular, $e(A(G'')) = n! \sum_{l=0}^{n} \frac{1}{l!}$.

**Proof.** (i) We may assume that $G' = G(P_n')$, where $P_n' = \{p_1', p_2', ..., p_n'\}$ is the chain with $p_1' \leq p_2' \leq ... \leq p_n'$. $P_n'$ as well as all its subposets have a unique linear extension. Therefore, the $h$-vector of $G'$ is $(\binom{n}{0}, \binom{n}{1}, ... , \binom{n}{n})$.

(ii) Let $G'' = G(P_n'')$, where $P_n'' = \{p_1'', p_2'', ..., p_n''\}$ is an antichain. If $F = [n]$, then, by convention, $A(0,0) = 1 = h^F_0$. If $F \not\subseteq [n]$, then $\mathcal{J}(P_n''(F))$ is a Boolean lattice on the set $P_n''(F)$, which implies that $\mathcal{J}(P_n''(F))$ is isomorphic to $L_{n-l}$, where $l = |F|$. Therefore, by Remark 3.2, $h^F_I = A(n-l,i)$, for all $0 \leq i \leq n-l-1$. If $i = n-l$, then $A(n-l,i) = 0$ (by convention) and $h^F_I = 0$ (by Remark 3.2), which implies that $A(n-l,i) = h^F_I$. By (ii) we have $h^F_I = \sum_{j=0}^{n} h_{j-1}^F$, hence $h^F_I = \sum_{j=0}^{n} \frac{n!}{j!} A(n-l,j)$, for all $0 \leq j \leq n$.

We get $e(A(G'')) = \sum_{j=0}^{n} h^F_I = \sum_{j=0}^{n} h^F_I + 1 = \sum_{j=0}^{n-1} \frac{n!}{j!} A(n-l,j) + 1 = \sum_{l=0}^{n-1} \frac{n!}{l!} A(n-l,j) + 1$. We obviously have $\sum_{j=0}^{n-l-1} A(n-l,j) = (n-l)!$, for all $0 \leq l \leq n-1$. Therefore, $e(A(G'')) = \sum_{l=0}^{n} \frac{n!}{l!} \cdot (n-l)! + 1 = n! \sum_{l=0}^{n} \frac{1}{l!}$. □

**Remark 3.10.** The reduced Gröbner basis $\mathcal{G}'$ of the toric ideal $Q_{G'}$ of $A(G')$ with respect to the monomial order $<_{\text{lex}}$ on the polynomial ring $B_{G'}$ is:

$\mathcal{G}' = \{ x_j u_{p_1'...p_{j-1}'} - y_j u_{p_1'...p_{j-1}'} | j \in [n] \}$,

where the initial monomial of each binomial of $\mathcal{G}'$ is the first monomial.

We notice that the initial ideal in $<_{\text{lex}}$ $(Q_{G'}) = (x_j u_{p_1'...p_{j-1}'} | j \in [n])$ is a complete intersection, which implies that the toric ideal $Q_{G'}$ is a complete intersection. Thus $A(G')$ has a pure resolution given by the Koszul complex.

**Proposition 3.11.** Let $G$ be a Cohen-Macaulay bipartite graph on $V_n$, $n \geq 1$. Then the following assertions hold:
(i) \( G \) comes from a chain if and only if \( H_{A(G)}(z) = \frac{(1+z)^n}{(1-z)^{n+1}}; \)
(ii) \( G \) comes from an antichain if and only if \( H_{A(G)}(z) = \frac{h''_n z^n + h''_{n-1} z^{n-1} + \cdots + h''_1 z + h''_0}{(1-z)^{n+1}} \),

where \( h'' = (h''_n, h''_1, \ldots, h''_n) \) is the \( h \)-vector of the vertex cover algebra \( A(G'') \) of the bipartite graph \( G'' \) that comes from an antichain \( P''_n = \{p''_1, p''_2, \ldots, p''_n\} \).

Proof. Let us suppose that \( G \) comes from a poset \( P_n = \{p_1, p_2, \ldots, p_n\}, n \geq 1 \), and let \( h = (h_0, h_1, \ldots, h_n) \) be the \( h \)-vector of \( A(G) \). In the first place we need to compute the component \( h_1 \). By using (7), we get \( h_1 = h_1^n + n \). But \( h_1^n \) is the component of rank 1 in the \( h \)-vector of \( A(G) \). By using the formula which relates the \( h \)-vector to the \( f \)-vector for the order complex \( \Delta(J(P_n)) \), we immediately get \( h_1^n = |J(P_n)| - n - 1 \), which implies that \( h_1 = |J(P_n)| - 1 \).

(i) Let \( h_1 = n \). Then \( |J(P_n)| = n + 1 \), which implies that \( P_n \) is a chain.
(ii) Let \( h_1 = h_1^n = |J(P''_n)| - 1 = 2^n - 1 \). Then \( |J(P_n)| = 2^n \), which implies that \( P_n \) is an antichain.

In both cases the converse follows from Proposition 3.9. \( \square \)

Proposition 3.12. Let \( G \) be a Cohen-Macaulay bipartite graph on \( V_n, n \geq 1 \). If \( h = (h_0, h_1, \ldots, h_n) \) is the \( h \)-vector of \( A(G) \), then \( \binom{n}{j} \leq h_j \leq h''_j \), for all \( 0 \leq j \leq n \), where \( G'' \) comes from an antichain with \( n \) elements and \( h'' = (h''_0, h''_1, \ldots, h''_n) \) is the \( h \)-vector of \( A(G'') \).

Proof. We may assume without loss of generality that \( G = G(P_n) \), where \( P_n = \{p_1, \ldots, p_n\} \) is a poset such that \( p_i \leq p_j \) implies \( i \leq j \). Let \( P'_n = \{p'_1, p'_2, \ldots, p'_n\} \) the chain with \( p'_1 \leq p'_2 \leq \cdots \leq p'_n \) and \( P''_n = \{p''_1, p''_2, \ldots, p''_n\} \) an antichain. By using (7) and (5), we get \( h_0 = 1 = h''_0 \) and \( h_n = 1 = h''_n \). Let \( 1 \leq j \leq n - 1 \). By Corollary 3.5, \( h_j \) is equal to the number of all linear extensions of all \( n-l \)-element subposets, which, seen as permutations of \([n-l]\), have exactly \( j-l \) descents, for all \( 0 \leq l \leq j \). Each \( n-l \)-element subposet of \( P'_n \), respectively, \( P''_n \) is a chain, respectively, an antichain, hence it has only one linear extension which corresponds to \( id_{n-l} \), respectively, it has \( (n-l)! \) linear extensions which correspond to all permutations of \([n-l]\). Therefore \( \binom{n}{j} \leq h_j \leq h''_j \), for all \( 1 \leq j \leq n - 1 \). \( \square \)

Corollary 3.13. Let \( G \) be a bipartite graph that comes from a poset with \( n \) elements, \( n \geq 1 \). Then \( 2^n \leq e(A(G)) \leq n! \sum_{i=0}^{n} \frac{1}{i!} \). The left equality holds if and only if the poset is a chain and the right equality holds if and only if the poset is an antichain.

Proof. Let \( G' = G(P''_n) \) and \( G'' = G(P''_n) \), where \( P'_n = \{p'_1, p'_2, \ldots, p'_n\} \) is a chain and \( P''_n = \{p''_1, p''_2, \ldots, p''_n\} \) is an antichain. We may assume without loss of generality that \( p'_1 \leq p'_2 \leq \cdots \leq p'_n \) and \( G = G(P_n) \), where \( P_n = \{p_1, p_2, \ldots, p_n\} \) is a poset such that \( p_i \leq p_j \) implies \( i \leq j \). Let \( h, h', h'' \) be the \( h \)-vector of \( A(G) \), \( A(G') \), respectively, \( A(G'') \). By summing up the inequalities \( h'_j \leq h_j \leq h''_j \) from Proposition 3.12 or by applying Corollary 2.6, we obtain \( e(A(G')) \leq e(A(G)) \leq e(A(G'')) \). Next, from Proposition 3.9, we get the desired inequalities. The left equality, respectively, the right equality holds if and only if \( h'_j = h_j \), respectively, \( h_j = h''_j \), for all \( 0 \leq j \leq n \), therefore, by using Proposition 3.11, this is equivalent to \( P_n = P'_n \), respectively, \( P_n = P''_n \). \( \square \)
Acknowledgment

I would like to thank Professor Jürgen Herzog for very useful suggestions and discussions on the subject of this paper. I am also very grateful to Professor Volkmar Welker who explained to me the combinatorial significance of the $h$-vector of a Hibi ring.

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