Research Article

Su Hu and Min-Soo Kim*

Euler’s integral, multiple cosine function and zeta values

https://doi.org/10.1515/forum-2023-0426
Received November 24, 2023

Abstract: In 1769, Euler proved the following result:

\[ \int_0^\frac{\pi}{2} \log(\sin \theta) \, d\theta = -\frac{\pi}{2} \log 2. \]

In this paper, as a generalization, we evaluate the definite integrals

\[ \int_0^x \theta^{r-2} \log \left( \cos \left( \frac{\theta}{2} \right) \right) \, d\theta \]

for \( r = 2, 3, 4, \ldots \). We show that it can be expressed by the special values of Kurokawa and Koyama’s multiple cosine functions \( C_r(x) \) or by the special values of alternating zeta and Dirichlet lambda functions. In particular, we get the following explicit expression of the zeta value:

\[ \zeta(3) = \frac{4\pi^2}{21} \log \left( \frac{e^{\psi} C_3(\frac{1}{3})^{16}}{\sqrt{2}} \right), \]

where \( G \) is Catalan’s constant and \( C_3(\frac{1}{3}) \) is the special value of Kurokawa and Koyama’s multiple cosine function \( C_3(x) \) at \( \frac{1}{3} \). Furthermore, we prove several series representations for the logarithm of multiple cosine functions \( \log C_r(\frac{1}{2}) \) by zeta functions, \( L \)-functions or polylogarithms. One of them leads to another expression of \( \zeta(3) \):

\[ \zeta(3) = \frac{72\pi^2}{11} \log \left( \frac{3^\frac{1}{2} C_3(\frac{1}{6})^{16}}{C_2(\frac{1}{2})^{\frac{1}{3}}} \right). \]

Keywords: Euler’s integral, multiple cosine function, zeta value

MSC 2020: 11M06, 11M35

Communicated by: Freydoon Shahidi

1 Introduction

1.1 Zeta functions

The main purpose of this paper is to relate the Euler-type integrals and the multiple cosine functions with the special values of zeta functions. So in this section, to our purpose, firstly we introduce various types of zeta functions.

*Corresponding author: Min-Soo Kim, Department of Mathematics Education, Kyungnam University, Changwon, Gyeongnam 51767, Republic of Korea, e-mail: mskim@kyungnam.ac.kr; https://orcid.org/0000-0002-1867-0817
Su Hu, Department of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, P. R. China, e-mail: mahusu@scut.edu.cn

Open Access. © 2024 the author(s), published by De Gruyter. This work is licensed under the Creative Commons Attribution 4.0 International License.
For $\Re(s) > 1$, the Riemann zeta function is defined by

$$
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
$$  

(1.1)

This function can be analytically continued to a meromorphic function in the complex plane except for a simple pole, with residue 1, at the point $s = 1$. The special number $\zeta(3) = 1.20205 \ldots$ is called Apéry constant. It is named after Apéry, who proved in 1979 that $\zeta(3)$ is irrational (see [4]).

For $\Re(s) > 1$ and $a \neq 0, -1, -2, \ldots$, in 1882, Hurwitz [15] defined the partial zeta function

$$
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s},
$$

which generalized (1.1). As (1.1), this function can also be analytically continued to a meromorphic function in the complex plane except for a simple pole at $s = 1$ with residue 1.

The alternating Hurwitz zeta function is defined by

$$
\zeta_E(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n + a)^s},
$$

where $\Re(s) > 0$ and $a \neq 0, -1, -2, \ldots$ (see [6, 8, 14]). It can be analytically continued to the complex plane without any pole. Sometimes we may use the notation $J(s, a)$ instead of $\zeta_E(s, a)$ (see, e.g., Williams and Zhang [37, p. 36, (1.1)]). There exists the following relationship between $\zeta_E(s, a)$ and $\zeta(s, a)$ (see [37, p. 37, (2.3))]:

$$
\zeta_E(s, a) = 2^{-a} \left( \zeta \left( s, \frac{a}{2} \right) - \zeta \left( s, \frac{a + 1}{2} \right) \right).
$$

Recently, the Fourier expansion and several integral representations, special values and power series expansions, convexity properties of $\zeta_E(s, a)$ have been investigated (see [8, 13, 14]), and it has been found that $\zeta_E(s, a)$ can be used to represent a partial zeta function of cyclotomic fields in one version of Stark’s conjectures in algebraic number theory (see [17, p. 4249, (6.13))].

In particular, setting $a = 1$, the function $\zeta_E(s, a)$ reduces to the alternating zeta function $\zeta_E(s)$ (also known as Dirichlet’s eta or Euler’s eta function),

$$
\zeta_E(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \eta(s).
$$

(1.2)

Obviously,

$$
\zeta_E(s) = (1 - 2^{1-s}) \zeta(s).
$$

From the Taylor expansion of $\log(1 + x)$, we have

$$
\zeta_E(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \log 2
$$

(see [6, 8, 14]). According to Weil’s history [35, pp. 273–276], the function $\zeta_E(s)$ has been used by Euler to “prove”

$$
\frac{\zeta_E(1 - s)}{\zeta_E(s)} = \frac{\Gamma(s)(2^s - 1) \cos \left( \frac{\pi s}{2} \right)}{(2^{s-1} - 1)\pi^{s}},
$$

which leads to the functional equation of $\zeta(s)$. It is also a particular case of Witten’s zeta functions in mathematical physics [28, p. 248, (3.14)], and it has been studied and evaluated at certain positive integers by Sitaramachandra Rao [31] in terms of the Riemann zeta values. See also [10, p. 31, Section 7] and [27, p. 2, (2)].

The Dirichlet lambda function $\lambda(s)$ is defined by

$$
\lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^s}
$$

(1.3)

$$
= \frac{1}{2^s} \zeta \left( s, \frac{1}{2} \right) = (1 - 2^{-s}) \zeta(s)
$$
for $\Re(s) > 1$ (see [14, p. 954, (1.9)]). This function was studied by Euler under the notation $N(s)$ (see [34, p. 70]). Euler also considered its alternating form

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = \frac{1}{2} \zeta(s, \frac{1}{2})$$ (1.4)

for $\Re(s) > 0$, which he denoted by $L(s)$ (see [34, p. 70]). Furthermore, the constant $\beta(2) = G$ is usually named as Catalan’s constant (see [22, 30, 32, 36]). Both functions admit an analytic continuation, $\lambda(s)$ to all $s \neq 1$ and $\beta(s)$ to all $s$. They have been studied in detail by us in [14], in particular, we have obtained a number of infinite families of linear recurrence relations for $\lambda(s)$ at positive even integer arguments $\lambda(2m)$, convolution identities for special values of $\lambda(s)$ at even arguments and special values of $\beta(s)$ at odd arguments.

The Dirichlet $L$-function associated to a Dirichlet character $\chi$ is given by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p} \frac{1}{1 - \chi(p)p^{-s}},$$

which is convergent for $\Re(s) > 1$ and the Euler product is taken over all prime numbers $p$. It was introduced by Dirichlet in 1837 to prove the theorem on primes in arithmetic progressions (see [5, Chapter 7]). For the trivial Dirichlet character $\mathbb{1}$ we have $L(s, \mathbb{1}) = \zeta(s)$. For the principal character $\mathbb{1}_m$ of modulus $m$ induced by $\mathbb{1}$ we have (see [16, p. 255])

$$\zeta(s) = L(s, \mathbb{1}_m) \prod_{p|m} \frac{1}{1 - p^{-s}}.$$ (1.6)

We may also express $L(s, \chi)$ by using the Hurwitz zeta functions. Let $f$ be a positive integer and let $\chi$ be any character modulo $f$. The Dirichlet $L$-function $L(s, \chi)$ is expressed in terms of the Hurwitz zeta function $\zeta(s, a)$ by means of the following formula:

$$L(s, \chi) = f^{-s} \sum_{a=1}^{f-1} \chi(a) \zeta\left(s, \frac{a}{f}\right)$$

for $\Re(s) > 1$, and it can be analytic continued to the whole $s$-plane from the above expression.

### 1.2 Multiple trigonometric functions and the related integrals

Around 1742, Euler successfully calculated the zeta value

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$$

by considering the integration

$$\frac{1}{2} (\arcsin x)^2 = \int_{0}^{x} \frac{\arcsin t}{\sqrt{1-t^2}} dt.$$ 

In concrete, by taking $x = 1$ on the left-hand side, we get $\frac{\pi^2}{4}$, and by expanding $\arcsin t$ as a power series and integrating term-by-term on the right-hand side, we get the sum

$$\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots,$$

where $\lambda(s)$ is the Dirichlet lambda function (see (1.3)). Then by comparing the results on the both sides we arrive at the summation

$$\lambda(2) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}. (1.7)$$

Finally, the identity

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \zeta(2) - \frac{1}{2} \zeta(2) = \frac{3}{4} \zeta(2)$$
leads to
\[ \zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}. \]

More generally, for \( n = 1, 2, 3, \ldots \), Euler obtained
\[ \zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = \frac{(-1)^{n-1}B_{2n}2^{2n}}{2(2n)!}, \quad (1.8) \]
where the \( B_{2n} \) are the Bernoulli numbers defined by the generating function
\[ \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n \]
(see [5, p. 266, Theorem 12.17]). But the explicit formulas for \( \zeta(3) \) and \( \zeta(2n+1) \) are still unknown. For the longstanding history, we refer to a recent book by Nahin [29].

To extend (1.7) from 2 to 3, Euler got the formula
\[ \lambda(3) = 1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots = \frac{\pi^2}{2} \log 2 + \frac{2}{\pi} \int_0^{\pi/2} \theta \log(\sin \theta) \, d\theta \quad (1.9) \]
(see [34, p. 63]), and in 1769 Euler [9] proved the result
\[ I = \int_0^{\pi/2} \log(\sin \theta) \, d\theta = -\frac{\pi}{2} \log 2, \quad (1.10) \]
which is equal to
\[ I = \int_0^{\pi/2} \log(\cos \theta) \, d\theta \]
(see [25, p. 152]).

Generalizing the above integrals (1.9) and (1.10), for \( 0 \leq x < \pi \) and \( r = 2, 3, 4, \ldots \), Koyama and Kurokawa [18] evaluated the definite integrals
\[ \int_0^{x} \theta^{r-2} \log(\sin \theta) \, d\theta \quad (1.11) \]
and showed that (1.11) is expressed by the multiple sine functions:
\[ \int_0^{x} \theta^{r-2} \log(\sin \theta) \, d\theta = \frac{x^{r-1}}{r-1} \log(\sin x) - \frac{\pi^{r-1}}{r-1} \log S_r \left( \frac{x}{\pi} \right) \quad (1.12) \]
(see [18, Theorem 1]).

It is well known that the definition of sine functions starts from the infinite product representation
\[ \sin x = x \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2 \pi^2} \right) \]
(see [11, p. 44, 1.431 (1)] and [29, p. 28, (1.4.9)]). Denote
\[ S_1(x) = 2 \sin(\pi x) = 2\pi x \prod_{n=1}^{\infty} \left( 1 - \frac{x^2}{n^2 \pi^2} \right). \]

In 1886, as a generalization, Hölder [12] defined the double sine function \( S_2(x) \) from the infinite product
\[ S_2(x) = e^{x} \prod_{n=1}^{\infty} \left( 1 + \frac{x^2}{1 + x^2 n^2} \right)^{n_e}. \]
Then in 1990s, generalizing $S_2(x)$, Kurokawa [19–21] further defined the multiple sine function $S_r(x)$ of order $r = 2, 3, 4, \ldots$ by the Weierstrass product

$$S_r(x) = \exp\left(\frac{x^{r-1}}{r-1} \prod_{n=1}^{\infty} \left\{ P_r\left(\frac{x}{n}\right) P_r\left(\frac{-x}{n}\right)^{-1} \right\}^{n^{-1}} \right),$$

where

$$P_r(x) = (1 - x) \exp\left(x + \frac{x^2}{2} + \cdots + \frac{x^r}{r}\right). \quad (1.13)$$

The cosine function has the infinite product representation

$$\cos x = \prod_{n=1, n \text{ odd}}^{\infty} \left(1 - \frac{x^2}{(n^2)^2}\right) \quad (1.14)$$

(see [11, p. 45, 1.431(3)]). Denote

$$C_1(x) = 2 \cos(\pi x) = 2 \prod_{n=1, n \text{ odd}}^{\infty} \left(1 - \frac{x^2}{(n^2)^2}\right). \quad (1.15)$$

Then, in 2003, Kurokawa and Koyama [23] defined the multiple cosine function $C_r(x)$ from the Weierstrass product

$$C_r(x) = \prod_{n=-\infty, n \text{ odd}}^{\infty} P_r\left(\frac{x}{n}\right) \left\{ P_r\left(\frac{x}{n}\right) P_r\left(\frac{-x}{n}\right)^{-1} \right\}^{(n^2)^{-1}} \quad (1.16)$$

for $r = 2, 3, 4, \ldots$ (see also [24–26]). Letting $r = 2, 3, 4$ in (1.16), we get

$$C_2(x) = \prod_{n=1, n \text{ odd}}^{\infty} \left\{ \left(1 - \frac{x^2}{(n^2)^2}\right)^{\frac{1}{2}} \right\}^{\frac{1}{e^{2x}}},$$

$$C_3(x) = \prod_{n=1, n \text{ odd}}^{\infty} \left\{ \left(1 - \frac{x^2}{(n^2)^2}\right)^{\frac{1}{2}} \right\}^{e^{x^2}}, \quad (1.17)$$

$$C_4(x) = \prod_{n=1, n \text{ odd}}^{\infty} \left\{ \left(1 - \frac{x^2}{(n^2)^2}\right)^{\frac{1}{2}} \right\}^{e^{\frac{1}{2} x^2 + \frac{1}{2} x^4}}$$

(see [23–25]). Then the duplication formulas are expressed as

$$C_r(x)^{r^{-1}} = \frac{S_r(2x)}{S_r(x)^{r^{-1}}} \quad (1.18)$$

for $r \geq 1$ (see [23, p. 848], [24, p. 125], [26, p. 477] and [25, p. 142]). The proof of (1.18) can be found in [24, p. 125, Section 3] and Corollary 3.2 below.

### 1.3 Our results

In this paper, inspiring by Koyama and Kurokawa’s work [18], we evaluate the definite integrals

$$\int_{0}^{x} \theta^{r-2} \log\left(\cos \frac{\theta}{2}\right) d\theta \quad (1.19)$$

for $r = 2, 3, 4, \ldots$. We show that (1.19) can be expressed by the special values of Kurokawa and Koyama’s multiple cosine functions $C_r(x)$ (see Theorem 2.1) or by the special values of alternating zeta and Dirichlet lambda functions (see Theorems 2.3 and 2.4).
In particular, we get the explicit expression of the zeta value
\[
\zeta(3) = \frac{4\pi^2}{21} \log \left( e^{\frac{3\pi}{2}} \frac{C_3(\frac{1}{4})^{16}}{\sqrt{2}} \right),
\]  
\hspace{10cm} (1.20)
where \( G \) is Catalan’s constant and \( C_3(\frac{1}{4}) \) is the special value of Kurokawa and Koyama’s multiple cosine function \( C_3(x) \) at \( \frac{1}{4} \) (see Corollary 2.7). As pointed by Allouche in an email to us, the above identity is equivalent to the following formula by Kurokawa and Wakayama (see [24, p. 123]):
\[
C_3(\frac{1}{4}) = 2^{\frac{1}{6}} \exp \left( \frac{21\zeta(3)}{64\pi^2} - \frac{L(2, \chi_4)}{4\pi} \right),
\]
where \( L(2, \chi_4) \) is equal to the Catalan constant \( G \). Recently, following (1.20), Allouche [3] found a link between the Kurokawa multiple trigonometric functions and two functions introduced respectively by Borwein and Dykshoorn [7] and by Adamchik [2].

Furthermore, we prove several series representations of \( \log C_r(\frac{1}{2}) \) by \( \lambda(2m) \) for \( n = 1, 2, 3, \ldots \) or by \( \zeta_2(r) \) for \( r = 2, 3, 4, \ldots \) and the special values of polylogarithms (see Theorems 2.8 and 2.11). From Theorem 2.11, we express the special values of \( C_2(\frac{1}{2}) \) and \( C_3(\frac{1}{2}) \) by \( L(2, \chi_3), L(2, \chi_5) \), the special values of Dirichlet’s \( L \)-functions, and the zeta value \( \zeta(3) \) (see Corollary 2.15). This leads to another expression of \( \zeta(3) \):
\[
\zeta(3) = \frac{72\pi^2}{11} \log \left( \frac{3\pi C_3(\frac{1}{2})}{C_2(\frac{1}{2})^{\frac{1}{6}}} \right)
\]
(see Remark 2.16).

## 2 Main results

In this section, we state our main results. Their proofs will be given in Section 4. First, we represent (1.19) by the special values of multiple cosine functions.

**Theorem 2.1.** For \( 0 \leq x < \pi \) and \( r = 2, 3, 4, \ldots \), we have
\[
\int_0^x \theta^{r-2} \log \left( \cos \frac{\theta}{2} \right) d\theta = \frac{x^{r-1}}{r-1} \log \left( \cos \frac{x}{2} \right) - \frac{(2\pi)^{r-1}}{r-1} \log C_r \left( \frac{x}{2\pi} \right).
\]

Letting \( x = \frac{\pi}{2} \) in Theorem 2.1, we have:

**Corollary 2.2.** For \( r = 2, 3, 4, \ldots \),
\[
\int_0^{\frac{\pi}{2}} \theta^{r-2} \log \left( \cos \frac{\theta}{2} \right) d\theta = -\frac{\pi^{r-1}}{r-1} \left( \frac{1}{2^r} \log 2 + 2^{r-1} \log C_r \left( \frac{1}{2} \right) \right).
\]

Setting \( r = 2, 3 \) and \( 4 \) in Corollary 2.2 respectively, and by (1.17) with \( x = \frac{1}{4} \), we get the following examples:
\[
\int_0^{\frac{\pi}{2}} \log \left( \cos \frac{\theta}{2} \right) d\theta = \frac{\pi}{2} \log \frac{1}{\sqrt{2}} - 2\pi \log C_3 \left( \frac{1}{4} \right)
\]
\[
= \frac{\pi}{2} \log \frac{1}{\sqrt{2}} - \log \left( \prod_{n=1, n \text{ odd}}^{\infty} \left( \frac{2n - 1}{2n + 1} \right)^{\frac{1}{2}} e^{\frac{3\pi}{2}} \right)^{2\pi},
\]
\[
\int_0^{\frac{\pi}{2}} \theta \log \left( \cos \frac{\theta}{2} \right) d\theta = \frac{\pi^2}{16} \log \frac{1}{\sqrt{2}} - 2\pi^2 \log C_3 \left( \frac{1}{4} \right)
\]
\[
= \frac{\pi^2}{8} \log \frac{1}{\sqrt{2}} - \log \left( \prod_{n=1, n \text{ odd}}^{\infty} \left( 1 - \frac{1}{4n^2} \right)^{\frac{3\pi}{2}} e^{\pi} \right)^{2\pi^2},
\]
and

\[ \int_0^{\pi} \theta^2 \log \left( \cos \frac{\theta}{2} \right) d\theta = \frac{\pi^3}{24} \log \frac{1}{\sqrt{2}} - \frac{8\pi^3}{3} \log \zeta_4 \left( \frac{1}{4} \right) \]

\[ = \frac{\pi^3}{24} \log \frac{1}{\sqrt{2}} - \log \left( \prod_{n=1, n \text{ odd}}^{\infty} \left( \frac{2n-1}{2n+1} \right)^{\frac{n}{\pi^2}} \right) \frac{\pi^2}{4} \]

In the following, we shall employ the usual convention that an empty sum is taken to be zero. For example, if \( n = 0 \), then we understand that \( \sum_{k=1}^{\infty} \). We represent (1.19) with \( x = \frac{\pi}{2} \) by the special values of alternating zeta, lambda and beta functions.

Now we state the following result.

**Theorem 2.3.** For \( r = 2, 3, 4, \ldots, \)

\[ \int_0^{\pi} \theta^{r-2} \log \left( \cos \frac{\theta}{2} \right) d\theta = -\frac{\log 2}{r-1} \left( \frac{\pi}{2} \right)^{r-1} + (r - 2)! \sin \left( \frac{r\pi}{2} \right) \zeta_r(r) + \sum_{k=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^{k-1} \left( \frac{r-2}{2k+1} \right) \frac{r-2k-2}{2k+1} \beta(2k+2) \]

\[ + \sum_{k=1}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^{k-1} \left( \frac{r-2}{2k+1} \right) \frac{r-2k-2}{2k+1} \zeta_{2k+1}(2k+1), \]

where \( \left\lfloor x \right\rfloor = \max \{ m \in \mathbb{Z} \mid m \leq x \} \) and \( \left\lceil x \right\rceil = \min \{ m \in \mathbb{Z} \mid m \geq x \} \).

Combining Corollary 2.2 and Theorem 2.3, we arrive at the following theorem.

**Theorem 2.4.** For \( r = 2, 3, 4, \ldots, \)

\[ \log \zeta_r \left( \frac{1}{4} \right) = -\frac{\log 2}{2(r-1)} - \frac{(r-1)!}{(2\pi)^{r-1}} \sin \left( \frac{r\pi}{2} \right) \zeta_r(r) - \frac{r-2}{2^{r-1}} \sum_{k=0}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^{k-1} \left( \frac{r-2}{2k+1} \right) \frac{r-2k-2}{2k+1} \beta(2k+2) \]

\[ - \frac{r-1}{2^{r-1}} \sum_{k=1}^{\left\lfloor \frac{r-1}{2} \right\rfloor} (-1)^{k-1} \left( \frac{r-2}{2k+1} \right) \frac{r-2k-2}{2k+1} \zeta_{2k+1}(2k+1). \]

The Catalan constant

\[ G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.915965594177219015 \ldots \]

is one of famous mysterious constants appearing in many places in mathematics and physics. It can be represented by the special values of Hurwitz zeta functions

\[ G = \beta(2) = \frac{1}{4} \log \left( \frac{1}{2} \right) = \frac{1}{16} \left( \zeta \left( 2, \frac{1}{4} \right) - \zeta \left( 2, \frac{3}{4} \right) \right) \]

(see [22, p. 667, (1.1)] and [32, p. 29, (16)]).

**Example 2.5.** From Theorem 2.3 with \( r = 2, 3, 4, 5 \) and (2.1), we have the following examples:

\[ \int_0^{\pi} \theta^0 \log \left( \cos \frac{\theta}{2} \right) d\theta = -\frac{\pi \log 2}{2} + G, \]

\[ \int_0^{\pi} \theta^1 \log \left( \cos \frac{\theta}{2} \right) d\theta = -\frac{\pi^2 \log 2}{8} + \frac{\pi G}{2} - \frac{7 \zeta(3)}{8}, \]

\[ \int_0^{\pi} \theta^2 \log \left( \cos \frac{\theta}{2} \right) d\theta = -\frac{\pi^3 \log 2}{24} + \frac{\pi^2 G}{4} + \frac{\pi \zeta(3)}{12} - 2 \beta(4), \]

\[ \int_0^{\pi} \theta^3 \log \left( \cos \frac{\theta}{2} \right) d\theta = -\frac{\pi^4 \log 2}{64} + \frac{\pi^3 G}{8} + \frac{3 \pi^2 \zeta(3)}{32} - 3 \pi \beta(4) + \frac{93 \zeta(5)}{16}. \]
Setting $r = 2, 3, 4, 5$ in Theorem 2.4, by (2.1) we get the following corollary.

**Corollary 2.6.** We have
\[
\log C_2\left(\frac{1}{4}\right) = \frac{\log 2}{8} - \frac{G}{2\pi},
\]
\[
\log C_3\left(\frac{1}{4}\right) = \frac{\log 2}{32} - \frac{G}{4\pi} + \frac{7\zeta(3)}{16\pi^2},
\]
\[
\log C_4\left(\frac{1}{4}\right) = \frac{\log 2}{128} - \frac{3G}{32\pi} + \frac{3\zeta(3)}{64\pi^2} + \frac{3\beta(4)}{4\pi^3},
\]
\[
\log C_5\left(\frac{1}{4}\right) = \frac{\log 2}{512} - \frac{G}{32\pi} - \frac{3\zeta(3)}{128\pi^2} + \frac{3\beta(4)}{4\pi^3} - \frac{93\zeta(5)}{64\pi^4}.
\]

From Corollary 2.6 for $r = 3$ and (1.2) we have the following expression for $\zeta(3)$.

**Corollary 2.7.** We have
\[
\zeta(3) = \frac{4\pi^2}{21} \log \left( \frac{e^{\frac{\pi}{4}} C_3\left(\frac{1}{4}\right)^{\frac{1}{16}}}{\sqrt{2}} \right).
\]

We also get the following infinite series representation of $\log C_r\left(\frac{\lambda}{2\pi}\right)$ by $\lambda(2n)$ for $n = 1, 2, 3, \ldots$.

**Theorem 2.8.** For $0 \leq x < \pi$ and $r = 2, 3, 4, \ldots$, we have
\[
\log C_r\left(\frac{x}{2\pi}\right) = \left(\frac{x}{2\pi}\right)^{-r+1} \left( \log \left(\cos \frac{x}{2}\right) + (r-1) \sum_{n=1}^{\infty} \frac{\lambda(2n)}{n(2n+r-1)} \left(\frac{x}{\pi}\right)^{2n} \right).
\]

Setting $x = \frac{\pi}{2}$ in Theorem 2.8, we have:

**Corollary 2.9.** For $r = 2, 3, 4, \ldots$,
\[
\log C_r\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^{-r+1} \left( \log 2 + (r-1) \sum_{n=1}^{\infty} \frac{\lambda(2n)}{n(2n+r-1)} \left(\frac{x}{\pi}\right)^{2n} \right).
\]

This corollary gives the following examples.

**Example 2.10.** Setting $r = 2$, we get
\[
\log C_2\left(\frac{1}{4}\right) = \frac{1}{4} \left( -\frac{1}{2} \log 2 + \sum_{n=1}^{\infty} \frac{\lambda(2n)}{n(2n+1)} \right).
\]

Moreover, by Corollary 2.6 we have
\[
\log C_2\left(\frac{1}{4}\right) = \frac{1}{8} \log 2 - \frac{G}{2\pi}.
\]

Therefore, by [32, p. 244, (694)],
\[
\sum_{n=1}^{\infty} \frac{\lambda(2n)}{n(2n+1)} = \frac{2G}{\pi} - \frac{\zeta(3)}{\pi^2} + \frac{16\beta(4)}{\pi^4},
\]
where we have used (1.3). Similarly, for $r = 3, 4$ and $5$, we find that
\[
\sum_{n=1}^{\infty} \frac{\lambda(2n)}{n(2n+2)} = \frac{1}{2} \log 2 - \frac{2G}{\pi} + \frac{7\zeta(3)}{2\pi^2},
\]
\[
\sum_{n=1}^{\infty} \frac{\lambda(2n)}{n(2n+3)} = \frac{1}{3} \log 2 - \frac{2G}{\pi} - \frac{\zeta(3)}{\pi^2} + \frac{16\beta(4)}{\pi^4},
\]
\[
\sum_{n=1}^{\infty} \frac{\lambda(2n)}{n(2n+4)} = \frac{1}{4} \log 2 - \frac{2G}{\pi} - \frac{3\zeta(3)}{2\pi^2} - \frac{93\zeta(5)}{64\pi^4} + \frac{48\beta(4)}{\pi^4}.
\]

Finally, we state series representations of $\log C_r\left(\frac{\lambda}{2\pi}\right)$ by $\zeta(r)$ and the special values of polylogarithms $\text{Li}_k(x)$.

Recall that the polylogarithm function $\text{Li}_k(x)$ is defined by
\[
\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k},
\]
where $|x| < 1$ and $k = 1, 2, 3, \ldots$ (see [21] and [23]).
Theorem 2.11. Let \( r = 2, 3, 4, \ldots \) Then:

1. We have

\[
\log C_r \left( \frac{x}{2} \right) = -\frac{(r - 1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(-\pi i)^k}{k!} \log \left( \frac{x}{2} \right)^r - \frac{(r - 1)!}{(2\pi i)^{r-1}} \zeta(r)
\]

for \( \text{Im}(x) < 0 \).

2. We have

\[
\log C_r \left( \frac{x}{2} \right) = -\frac{(r - 1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(-\pi i)^k}{k!} \log \left( \frac{x}{2} \right)^r - \frac{(r - 1)!}{(2\pi i)^{r-1}} \zeta(r)
\]

for \( \text{Im}(x) > 0 \).

3. For \( 2 \leq r \in 2\mathbb{Z} \) and \( 0 \leq x < 1 \), we have

\[
C_r \left( \frac{x}{2} \right) = \left( 2 \cos \frac{\pi x}{2} \right)^{\frac{1}{2}r-1} \exp \left( -\frac{1}{2} \frac{(r - 1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(-\pi i)^k}{k!} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^{r-k}} \right)
\]

and

\[
-(-1)^{\frac{1}{2}r} \frac{(r - 1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(-\pi i)^k}{k!} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^{r-k}}.
\]

(4) For \( 3 \leq r \in 1 + 2\mathbb{Z} \) and \( 0 \leq x < 1 \), we have

\[
C_r \left( \frac{x}{2} \right) = \left( 2 \cos \frac{\pi x}{2} \right)^{\frac{1}{2}r-1} \exp \left( -\frac{1}{2} \frac{(r - 1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(-\pi i)^k}{k!} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^{r-k}} \right)
\]

and

\[
-(-1)^{\frac{1}{2}r} \frac{(r - 1)!}{(2\pi i)^{r-1}} \sum_{k=0}^{r-1} \frac{(-\pi i)^k}{k!} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^{r-k}}.
\]

Remark 2.12. The analogue results for multiple sine function was proved by Kurokawa and Koyama [23, p. 849, Theorem 2.8] (also see Kurokawa [21, p. 222, Theorem 2]).

For \( r = 2, 3, 4, 5 \), Theorem 2.11 implies the following results.

Corollary 2.13. For \( 0 \leq x < 1 \), we have

\[
C_4 \left( \frac{x}{2} \right) = \left( 2 \cos \frac{\pi x}{2} \right)^{\frac{1}{2}r} \exp \left( -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^4} + 1 \right) \zeta(3),
\]

\[
C_4 \left( \frac{x}{2} \right) = \left( 2 \cos \frac{\pi x}{2} \right)^{\frac{1}{2}r} \exp \left( -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^4} + 1 \right) \zeta(3),
\]

\[
C_4 \left( \frac{x}{2} \right) = \left( 2 \cos \frac{\pi x}{2} \right)^{\frac{1}{2}r} \exp \left( -\frac{3}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^4} + 3 \frac{\pi^2}{8\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^2} \right),
\]

\[
C_4 \left( \frac{x}{2} \right) = \left( 2 \cos \frac{\pi x}{2} \right)^{\frac{1}{2}r} \exp \left( -\frac{3}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi x)}{n^4} + 3 \frac{\pi^2}{8\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi x)}{n^2} \right) - \frac{3}{\pi^2} \zeta(5),
\]

Remark 2.14. In particular, setting \( x = \frac{1}{2} \) in the above relations and by using the expansions

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \sin(n/2)}{n^s} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = -\beta(s)
\]

and

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n/2)}{n^s} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^s} = -\frac{1}{2^s} \zeta(s),
\]

we recover Corollary 2.6.
By (1.3), (1.5) and (1.6), the Dirichlet series with coefficients \((-1)^n \sin(\frac{\pi n}{3})\) and \((-1)^n \cos(\frac{\pi n}{3})\) can be calculated as follows:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \sin(\frac{\pi n}{3})}{n^s} = \frac{\sqrt{3}}{2} \left( \sum_{n=1,2 \text{ (mod 6)}} (-1)^n \left\{ \frac{1}{n^s} - \frac{1}{(n+3)^s} \right\} + \sum_{n=4,5 \text{ (mod 6)}} (-1)^n \frac{1}{n^s} \right) \\
= \frac{\sqrt{3}}{2} \left( \sum_{n=1 \text{ (mod 6)}} \frac{1}{n^s} - \sum_{n=3 \text{ (mod 6)}} \frac{1}{n^s} \right) \tag{2.3}
\]

and

\[
\sum_{n=1}^{\infty} \frac{(-1)^n \cos(\frac{\pi n}{3})}{n^s} = \frac{1}{2} \left( \sum_{n=1,2,4,5 \text{ (mod 6)}} \frac{(-1)^n}{n^s} + \sum_{n=3 \text{ (mod 6)}} \frac{1}{n^s} \right) \\
= \frac{1}{2} \left( \sum_{n=1,2,4,5 \text{ (mod 6)}} \frac{1}{n^s} + \sum_{n=3 \text{ (mod 6)}} \frac{1}{n^s} \right) \tag{2.4}
\]

Now, setting \(x = \frac{1}{3}\) in Corollary 2.13, by (2.3) and (2.4) we get the following corollary.

**Corollary 2.15.** We have

\[
C_2\left(\frac{1}{6}\right) = 3^{\frac{1}{6}} \exp\left(\frac{\sqrt{3}}{4\pi} \left\{ L(2, \chi_3) - L(2, \chi_6) \right\} \right).
\]

\[
C_3\left(\frac{1}{6}\right) = 3^{\frac{1}{6}} \exp\left(\frac{11}{12\pi^2} \zeta(3) + \frac{\sqrt{3}}{2\pi} \left\{ L(2, \chi_3) - L(2, \chi_6) \right\} \right).
\]

**Remark 2.16.** From Corollary 2.15, we immediately obtain another expression of \(\zeta(3)\):

\[
\zeta(3) = \frac{72\pi^2}{11} \log\left( \frac{3^{\frac{1}{3}} C_3\left(\frac{1}{6}\right)}{C_2\left(\frac{1}{6}\right)} \right).
\]

### 3 Multiple cosine functions

In this section, to our purpose, we state some basic properties of multiple cosine functions. Some of them have been reported in [23, p. 848, Remark 2.7], [24, p. 124] and [25, Propositions 3.1 (4) and 5.1 (4)].

First, we prove the following proposition which is necessary to derive several properties of multiple cosine functions. Note that it has appeared in [23, p. 848] and [24, p. 124] without proof.

**Proposition 3.1.** For \(r = 2, 3, 4, \ldots\), we have \(C_r(0) = 1\) and \(C_r(x)\) is a meromorphic function in \(x \in \mathbb{C}\) satisfying

\[
\frac{C_r'(x)}{C_r(x)} = -\pi x^{r-1} \tan(\pi x).
\]

Thus we have the integral representation

\[
C_r(x) = \exp\left( -\int_0^x \pi t^{r-1} \tan(\pi t) \, dt \right),
\]

where the contour lies in \(\mathbb{C} \setminus \{ \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots \}\).
Proof. The proof goes similarly with [18, Proposition 1] by calculating the logarithmic derivative. When \( r = 1 \), the result follows immediately from (1.15). For \( r = 2, 3, 4, \ldots \), by using

\[
\mathcal{C}_r(x) = \prod_{n=1, \text{n odd}}^{\infty} \left\{ P_r\left( \frac{x}{2} \right) P_r\left( -\frac{x}{2} \right) \right\}^{(2n-1)-1} = \prod_{n=1}^{\infty} \left\{ P_r\left( \frac{x}{2n-1} \right) P_r\left( -\frac{x}{2n-1} \right) \right\}^{(2n-1)-1}
\]

and (1.13), we obtain

\[
\log \mathcal{C}_r(x) = \sum_{n=1}^{\infty} \left( \frac{2n-1}{2} \right)^{r-1} \left\{ \log P_r\left( \frac{x}{2n-1} \right) + (-1)^{r-1} \log P_r\left( -\frac{x}{2n-1} \right) \right\}
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{2n-1}{2} \right)^{r-1} \left\{ \log \left( 1 - \frac{x}{2n-1} \right) + (-1)^{r-1} \log \left( 1 + \frac{x}{2n-1} \right) \right\}
\]

\[
+ \left( \frac{x}{2n-1} + \left( \frac{x}{2n-1} \right)^2 + \cdots + \frac{1}{r} \left( \frac{x}{2n-1} \right)^r \right) \left( 1 - \frac{x}{2n-1} \right) + (-1)^{r-1} \left( \frac{x}{2n-1} + \left( \frac{x}{2n-1} \right)^2 + \cdots + \frac{1}{r} \left( \frac{x}{2n-1} \right)^r \right) \right\}.
\]

Hence

\[
\frac{\mathcal{C}_r(x)}{\mathcal{C}_r(x)} = \sum_{n=1}^{\infty} \left( \frac{2n-1}{2} \right)^{r-1} \left\{ \frac{1}{x - 2n-1} + (-1)^{r-1} \right\} + \left( \frac{x}{2n-1} + \left( \frac{x}{2n-1} \right)^2 + \cdots + \frac{1}{r} \left( \frac{x}{2n-1} \right)^r \right) + (-1)^{r-1} \left( \frac{1}{x + 2n-1} + \left( \frac{x}{2n-1} \right)^2 + \cdots + \frac{1}{r} \left( \frac{x}{2n-1} \right)^r \right).\]

Then, by observing the expressions

\[
\frac{1}{2n-1} + \frac{x}{(2n-1)^2} + \cdots + \frac{x^{r-1}}{(2n-1)^r} = \frac{\left( \frac{x}{2n-1} \right)^r - 1}{x - 2n-1}
\]

and

\[
-\frac{1}{2n-1} + \frac{x}{(2n-1)^2} + \cdots + (-1)^r \frac{x^{r-1}}{(2n-1)^r} = \frac{(-1)^r \left( \frac{x}{2n-1} \right)^r - 1}{x + 2n-1},
\]

we see that

\[
\frac{\mathcal{C}_r(x)}{\mathcal{C}_r(x)} = \sum_{n=1}^{\infty} \left( \frac{x}{x - 2n-1} - \frac{x}{x + 2n-1} \right) = \sum_{n=1}^{\infty} 2x^r x^r = 8x^r \sum_{n=1}^{\infty} \frac{1}{(2n)^2 - (2n-1)^2} = -\pi x^{r-1} \tan(\pi x),
\]

where we have used the expansion [11, p. 43, 1.421 (1)]

\[
\tan\left( \frac{\pi x}{2} \right) = \frac{4x}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 - x^2}.
\]

This completes the proof of Proposition 3.1. \(\square\)

From Proposition 3.1, we get a new proof for the following result by Kurokawa and Wakayama [24, p. 125].
Corollary 3.2 (Duplication formulas). For \( r = 1, 2, 3, \ldots \),
\[
\mathcal{C}_r(x)^{2^{r-1}} = \frac{S_r(2x)}{S_r(x)^{2^{r-1}}},
\]

**Proof.** By Proposition 3.1 and the trigonometric identity
\[
\cot(x + y) = \frac{\cot x \cot y - 1}{\cot x + \cot y},
\]
we have
\[
2^{r-1} \log \mathcal{C}_r(x) = -2^{r-1} \int_0^x \pi t^{r-1} \tan(\pi t) \, dt
\]
\[
= 2^{r-1} \int_0^x \pi t^{r-1} \left( \frac{\cot^2(\pi t) - 1}{\cot(\pi t)} - \cot(\pi t) \right) \, dt
\]
\[
= 2^{r-1} \int_0^x \pi t^{r-1} \left( 2 \cot(2\pi t) - \cot(\pi t) \right) \, dt
\]
\[
= \int_0^x \pi (2t)^{r-1} \cot(2\pi t) \, dt - 2^{r-1} \int_0^x \pi t^{r-1} \cot(\pi t) \, dt
\]
\[
= \log S_r(2x) - 2^{r-1} \log S_r(x)
\]
\[
= \log \frac{S_r(2x)}{S_r(x)^{2^{r-1}}},
\]
where we have used the identity
\[
\log S_r(x) = \int_0^x \pi t^{r-1} \cot(\pi t) \, dt
\]
(see [18, Proposition 2]). Thus the corollary follows. \(\square\)

**Proposition 3.3.** For \( 0 \leq x < 1 \) and \( r = 2, 3, 4, \ldots \),
\[
\log \mathcal{C}_r\left(\frac{x}{2}\right) = -\frac{1}{2^r} \int_0^x \pi t^{r-1} \cot\left(\frac{\pi t}{2}\right) \, dt.
\]

**Proof.** Since \( \mathcal{C}_r(0) = 1 \), both sides of the above equation are 0 at the boundary point \( x = 0 \). So it only needs to show that the logarithmic differentiations of both sides are equal, but this immediately follows from Proposition 3.1 by replacing \( x \) with \( \frac{x}{2} \). \(\square\)

**Proposition 3.4.** Let \( r = 2, 3, 4, \ldots \). Then:

1. We have
\[
\mathcal{C}_r(x + 1) = \frac{\mathcal{C}_r(1)}{2} \prod_{k=1}^r \mathcal{C}_k(x)^{(-1)^{k-1}}.
\]
2. For \( 3 \leq N \in 1 + 2\mathbb{Z} \), we have
\[
\mathcal{C}_r(Nx) = A_r(N) \prod_{a=0}^{N-1} \mathcal{C}_r\left(x + \frac{2a}{N}\right)^{N-1} \prod_{k=1}^r \prod_{a=1}^{N-1} \mathcal{C}_r\left(x + \frac{2a}{N}\right)^{(-1)^{r-k}(\frac{k-1}{2})(2a)^{r-1}N^{k-1}},
\]
where
\[
A_r(N)^{-1} = \left(\mathcal{C}_r\left(\frac{2}{N}\right) \cdots \mathcal{C}_r\left(\frac{2(N-1)}{N}\right)\right)^{N-1} \prod_{k=1}^r \prod_{a=1}^{N-1} \mathcal{C}_r\left(\frac{2a}{N}\right)^{(-1)^{r-k}(\frac{k-1}{2})(2a)^{r-1}N^{k-1}}.
\]
Proof. We employ the method in [23, Theorem 2.10 (a) and (b)]. By Proposition 3.1, we have

\[
\frac{C_r'(x + 1)}{C_r(x + 1)} = -\pi (x + 1)^{r-1} \tan(\pi x)
\]

\[
= -\pi \sum_{k=1}^{r} \binom{r-1}{k-1} x^{k-1} \tan(\pi x)
\]

\[
= \sum_{k=1}^{r} \binom{r-1}{k-1} C_k'(x) C_k(x),
\]

thus

\[
\frac{d}{dx} \log C_r(x + 1) = \sum_{k=1}^{r} \binom{r-1}{k-1} \frac{d}{dx} \log C_k(x)
\]

\[
= \frac{d}{dx} \log \prod_{k=1}^{r} C_k(x)^{\binom{r-1}{k-1}}.
\]

Therefore

\[
C_r(x + 1) = C \prod_{k=1}^{r} C_k(x)^{\binom{r-1}{k-1}} (3.1)
\]

with some constant C. Now put

\[
F(x) = \frac{C_r(x + 1)}{\prod_{k=1}^{r} C_k(x)^{\binom{r-1}{k-1}}}.
\]

Since $C_1(0) = 2$ and $C_2(0) = \cdots = C_r(0) = 1$, by (3.1) and (3.2) we have

\[
C = F(0) = \lim_{x \to 0} \frac{C_r(x + 1)}{\prod_{k=1}^{r} C_k(x)^{\binom{r-1}{k-1}}}
\]

\[
= \frac{C_r(1)}{2}
\]

and (1) follows.

For (2), let $3 \leq N \in 1 + 2\mathbb{Z}$. By

\[
2 \cos(Nx) = \frac{N-1}{a=0} \left\{ 2 \cos \left( x + \frac{2\pi a}{N} \right) \right\}
\]

(see [11, p. 41, 1.393 (1)]), we have

\[
N \tan(\pi x) = -\frac{1}{\pi} \frac{d}{dx} \log 2 \cos(\pi x)
\]

\[
= -\sum_{a=0}^{N-1} \frac{1}{\pi} \frac{d}{dx} \log 2 \cos \left( \pi x + \frac{2\pi a}{N} \right)
\]

\[
= \sum_{a=0}^{N-1} \tan \left( x + \frac{2a}{N} \right).
\]

Then combing Proposition 3.1 and (3.3), we get

\[
\frac{d}{dx} \log C_r(Nx) = -\pi N(Nx)^{r-1} \tan(\pi N x)
\]

\[
= -\pi N^{r-1} x^{r-1} \sum_{a=0}^{N-1} \tan \left( x + \frac{2a}{N} \right).
\]

Since

\[
x^{r-1} = \left( x + \frac{2a}{N} \right)^{r-1} + \sum_{k=1}^{r-1} \binom{r-1}{k-1} \left( \frac{2a}{N} \right)^{r-k} \left( x + \frac{2a}{N} \right)^{k-1},
\]
by applying Proposition 3.1 again we obtain
\[
\frac{d}{dx} \log C_r(Nx) = -\pi N^{r-1} \left( x + \frac{2a}{N} \right)^{r-1} \sum_{k=0}^{N-1} \tan \left( x + \frac{2a}{N} \right)
\]
\[
- \pi N^{r-1} \sum_{k=1}^{r-1} \frac{r-1}{k-1} \left( \frac{-2a}{N} \right)^{r-k} \left( x + \frac{2a}{N} \right)^{k-1} \sum_{a=0}^{N-1} \tan \left( x + \frac{2a}{N} \right)
\]
\[
= N^{r-1} \sum_{a=0}^{N-1} \frac{d}{dx} \left( \log C_r(x + \frac{2a}{N}) \right) + \sum_{a=0}^{N-1} \sum_{k=1}^{r-1} (-1)^{r-k} \left( \frac{2a}{N} \right)^{r-k} \frac{r-k}{(k-1)!} \frac{d}{dx} \left( \log C_r(x + \frac{2a}{N}) \right)
\]
\[
= N^{r-1} \sum_{a=0}^{N-1} \frac{d}{dx} \left( \prod_{a=1}^{N-1} C_r(x + \frac{2a}{N}) \right) + N^{r-1} \sum_{a=0}^{N-1} \frac{d}{dx} \left( \prod_{a=1}^{N-1} C_r(x + \frac{2a}{N}) \right)^{-1} (-1)^{r-k}(\frac{-2a}{N})^{r-k-1},
\]
which leads to (2). □

**Proposition 3.5.** For \( r = 2, 3, 4, \ldots \), the multiple cosine function \( C_r(x) \) satisfies the following second order algebraic differential equation

\[
C_r''(x) = (1 - x^{1-r}) \frac{C_r'(x)^2}{C_r(x)} + (r - 1) \frac{C_r'(x)}{x} - \pi^2 x^{r-1} C_r(x)
\]

with \( C_r(0) = 1 \) and \( C_r'(0) = 0 \).

**Proof.** From Proposition 3.1, we obtain
\[
\frac{d}{dx} \left( \frac{1}{\pi x^{r-1} C_r'(x)} \right) = -\pi \sec^2(\pi x)
\]
\[
= -\pi (\tan^2(\pi x) + 1)
\]
\[
= -\pi \left( \frac{1}{\pi x^{r-1} C_r(x)} \right)^2 + 1
\]

(3.4)

On the other hand, by applying the derivative formula in calculus directly, we have
\[
\frac{d}{dx} \left( \frac{1}{\pi x^{r-1} C_r(x)} \right) = -\frac{r-1}{x} \frac{C_r''(x)}{C_r(x)} + \frac{1}{\pi x^{r-1}} \left( \frac{C_r''(x)}{C_r(x)} - \frac{C_r'(x)^2}{C_r(x)^2} \right).
\]

(3.5)

Then, by comparing (3.4) and (3.5), we get
\[
-\frac{r-1}{x} \frac{C_r''(x)}{C_r(x)} + \frac{C_r'(x)^2}{C_r(x)} = -\frac{1}{\pi x^{r-1}} \frac{C_r'(x)^2}{C_r(x)^2} - \pi^2 x^{r-1} C_r(x),
\]
which is equivalent to the statement of the proposition. □

**Remark 3.6.** Propositions 3.5 is an analogy of Painlevé’s differential equation of type III.

### 4 Proofs of the results

In this section, we prove the results stated in Section 2.

**The first proof of Theorem 2.1**

As remarked by Allouche in an email to us, this result can be implied by (1.12) if using
\[
\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}
\]
to write
\[ \log \left( \cos \frac{\theta}{2} \right) = \log(\sin \theta) - \log \left( \sin \frac{\theta}{2} \right) - \log 2 \] (4.1)

and noticing the relation between \( C_r(x) \) and \( S_r(x) \) (see (1.18)). Following his idea, we give a detailed proof as follows.

From (4.1) and (1.12) we have
\[
\int_0^x \theta^{r-2} \log \left( \cos \frac{\theta}{2} \right) d\theta = \int_0^x \theta^{r-2} \log(\sin \theta) d\theta - 2^{r-1} \int_0^x \theta^{r-2} \log(\sin \theta) d\theta - \log 2 \int_0^x \theta^{r-2} d\theta
\]
\[
= \frac{x^{r-1}}{r-1} \log(\sin x) - \frac{\pi^{r-1}}{r-1} \log s_r \left( \frac{x}{\pi r} \right)
\]
\[
- 2^{r-1} \left( \frac{x}{2} \right)^{r-1} \log \left( \sin \frac{x}{2} \right) - \frac{\pi^{r-1}}{r-1} \log s_r \left( \frac{x}{2 \pi} \right) - \frac{x^{r-1}}{r-1} \log 2
\] (4.2)

On the other hand, the logarithmic of (1.18) yields
\[ 2^{r-1} \log C_r \left( \frac{x}{2 \pi} \right) = \log s_r \left( \frac{x}{\pi r} \right) - 2^{r-1} \log s_r \left( \frac{x}{2 \pi} \right). \] (4.3)

Then substituting (4.1) and (4.3) into (4.2), we get our result.

**The second proof of Theorem 2.1**

Here we also derive this result directly. From Proposition 3.3 and the integration by parts, we have
\[
\log C_r \left( \frac{x}{2} \right) = \frac{1}{2^{r-1}} \left( \left[ t^{r-1} \log \left( \cos \frac{\pi t}{2} \right) \right]_0^x - \left( r-1 \right) \int_0^x t^{r-2} \log \left( \cos \frac{\pi t}{2} \right) dt \right)
\]
\[
= \frac{1}{2^{r-1}} \left( x^{r-1} \log \left( \cos \frac{\pi x}{2} \right) - (r-1) \int_0^x t^{r-2} \log \left( \cos \frac{\pi t}{2} \right) dt \right).
\]

Then changing the variable to \( \theta = \pi t \) in this integral, we have
\[
\log C_r \left( \frac{x}{2} \right) = \frac{1}{2^{r-1}} \left( x^{r-1} \log \left( \cos \frac{\pi x}{2} \right) - \frac{r-1}{r^{r-1}} \int_0^{\pi x} \theta^{r-2} \log \left( \cos \frac{\theta}{2} \right) d\theta \right).
\]

Now, letting \( x \to \frac{x}{\pi r} \), the assertion follows.

**Proof of Theorem 2.3**

To prove this, we need the following two lemmas.

**Lemma 4.1.** For \( r = 0, 1, 2, \ldots \), we have
\[
\int_0^x \theta^r \cos(n \theta) d\theta = \sum_{k=0}^{r} \frac{r!}{k!} \frac{k!}{n^{k+1}} \sin \left( \frac{n \pi}{2} + \frac{k \pi}{2} \right) x^{r-k} - \frac{r!}{n^{r+1}} \sin \left( \frac{r \pi}{2} \right).
\]

**Proof.** Recall the indefinite integral (see [11, p. 226, 2.633 (2)])
\[
\int \theta^r \cos(n \theta) d\theta = \sum_{k=0}^{r} \frac{k!}{k!} \frac{r!}{n^{k+1}} \sin \left( \frac{n \theta + k \pi}{2} \right) + C.
\]
Proof. Setting \( F \) for the calculation of the right-hand side, we split the summation into three parts \( I \).

** Lemma 4.2.** For \( r = 0, 1, 2, \ldots \), we have

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_{0}^{\pi} \theta^r \cos(n\theta) \, d\theta = \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k (2k)! \left( \frac{\pi}{2} \right)^{r-2k} \frac{r}{2k+1} \beta(2k+2) + \sum_{k=0}^{\lfloor r/2 \rfloor} (-1)^k (2k-1)! \left( \frac{r}{2k-1} \right) \left( \frac{\pi}{2} \right)^{r-2k+1} \zeta(2k+1) - r! \left( \frac{\pi}{2} \right) \zeta(r+2),
\]

where \(|x| = \max\{m \in \mathbb{Z}: m \leq x\}\) and \(|x| = \min\{m \in \mathbb{Z}: m \geq x\}\).

** Proof.** Setting \( x = \frac{r}{2} \) in Lemma 4.1, by the fundamental formula of angle addition for the sine function, we obtain

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_{0}^{\pi} \theta^r \cos(n\theta) \, d\theta = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} \frac{\pi}{2} \right)^{r-k} \frac{1}{n^{k+1}} \left( \sin \left( \frac{n\pi}{2} \right) \cos \left( \frac{k\pi}{2} \right) + \sin \left( \frac{k\pi}{2} \right) \cos \left( \frac{n\pi}{2} \right) \right) - r! \frac{1}{n^{r+1}} \sin \left( \frac{r\pi}{2} \right).
\]

For the calculation of the right-hand side, we split the summation into three parts \( I_1, I_2, I_3 \) according to the terms \( \sin \left( \frac{n\pi}{2} \right) \cos \left( \frac{k\pi}{2} \right) \), \( \sin \left( \frac{k\pi}{2} \right) \cos \left( \frac{n\pi}{2} \right) \) and \( \sin \left( \frac{r\pi}{2} \right) \).

First we calculate the sum \( I_1 \). From (1.4), we have

\[
I_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} \frac{\pi}{2} \frac{1}{n^{k+1}} \sin \left( \frac{n\pi}{2} \right) \cos \left( \frac{k\pi}{2} \right) - r! \frac{1}{n^{r+1}} \sin \left( \frac{r\pi}{2} \right).
\]

Then we calculate the sum \( I_2 \). From (1.2), we have

\[
I_2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} \frac{\pi}{2} \frac{1}{n^{k+1}} \sin \left( \frac{k\pi}{2} \right) \cos \left( \frac{n\pi}{2} \right) - r! \frac{1}{n^{r+1}} \sin \left( \frac{r\pi}{2} \right).
\]

which completes the proof of Lemma 4.1. \( \square \)
Comparing (4.8) and (4.9) we get
\[ x \text{ result from 0 to } \pi. \]

Since
\[ \int_0^{\pi/2} \log(\cos \theta) \, d\theta = -2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(2\theta)}{n}, \]
we have
\[ \int_0^{\pi/2} \theta^{-2} \log(\cos \theta) \, d\theta = \int_0^{\pi/2} \left( -\log 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(2\theta)}{n} \right) \, d\theta = -\frac{\log 2}{r-1} \left( \frac{\pi}{2} \right)^{r-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cos(2\theta)}{n} \int_0^{\pi/2} \theta^{r-2} \cos(2\theta) \, d\theta \]
for \( r = 2, 3, 4, \ldots \). Then replacing \( r \) by \( r - 2 \) in Lemma 4.2 and substituting the result into the right-hand side of the above equation, after some elementary calculations, we obtain the desired result.

**Proof of Theorem 2.8**

From Euler’s infinite product representation of the cosine function (1.14), we have
\[ \log(\cos \theta) = \sum_{m=0}^{\infty} \log(1 - \frac{4\theta^2}{(2m+1)^2 \pi^2}). \]
Since
\[ \log(1 - \theta) = -\sum_{n=1}^{\infty} \frac{\theta^n}{n} \]
for \( |\theta| < 1 \), we see that
\[ \log(\cos \theta) = -\sum_{n=1}^{\infty} \left( \frac{2\theta}{\pi} \right)^{2n} \frac{1}{n} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2 n^{2m}} \]
\[ = -\sum_{n=1}^{\infty} \frac{2^{2n} \lambda(2n)}{\pi^{2n} n} \theta^{2n} \]
for \( |\theta| < \frac{\pi}{2} \). In the above equation, replacing \( \theta \) by \( \frac{\theta}{2} \), then multiplying both sides by \( \theta^{r-2} \) and integrating the result from 0 to \( x \), we have
\[ \int_0^{x} \theta^{-2} \log(\cos \frac{\theta}{2}) \, d\theta = -\sum_{n=1}^{\infty} \frac{\lambda(2n)}{\pi^{2n} n} \frac{\theta^{2n+r-1}}{2n + r - 1}, \]
where \( 0 < x < \pi \). On the other hand, by Theorem 2.1 we have
\[ \int_0^{x} \theta^{-2} \log(\cos \frac{\theta}{2}) \, d\theta = \frac{x^{r-1}}{r-1} \log(\cos \frac{x}{2}) - \frac{(2\pi)^{r-1}}{r-1} \log C_r(\frac{x}{2\pi}). \]
Comparing (4.8) and (4.9) we get
\[ \frac{(2\pi)^{r-1}}{r-1} \log C_r(\frac{x}{2\pi}) = \frac{x^{r-1}}{r-1} \log(\cos \frac{x}{2}) + \sum_{n=1}^{\infty} \frac{\lambda(2n)}{\pi^{2n} n} \frac{x^{2n+r-1}}{2n + r - 1}. \]
The result is now easily established.
Proof of Theorem 2.11

(1) By Proposition 3.3, we have
\[
\log C_r \left(\frac{x}{2}\right) = -\frac{\pi}{2r} \int_0^x t^{r-1} \tan \left(\frac{\pi t}{2}\right) \, dt.
\]
If \(\text{Im}(x) < 0\), then by changing the variable \(t = x\theta\) (\(0 \leq \theta \leq 1\)) and using the following formulas (cf. [21, p. 223] and [23, p. 849])
\[
\tan \left(\frac{\pi x\theta}{2}\right) = \frac{1}{1 + e^{-\pi x\theta} - i} = -i \left(1 + \frac{2}{1 + e^{-\pi x\theta}}\right)
\]
\[
= i \left(1 + 2 \sum_{n=0}^\infty (-1)^{n+1} e^{-\pi n x\theta}\right)
\]
for \(\theta > 0\) and
\[
\theta^{r-1} e^\alpha d\theta = (-1)^{r-1} (r-1)! \frac{e^\alpha}{\alpha^r} \left(\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \alpha^k - e^{-\alpha}\right)
\]
for \(\alpha \in \mathbb{C} \setminus \{0\}\) (see [21, p. 223] and [23, p. 850]), we get
\[
\log C_r \left(\frac{x}{2}\right) = -\frac{i\pi x^r}{2r} \int_0^1 \theta^{r-1} \left(-1 + 2 \sum_{n=1}^\infty (-1)^{n-1} e^{-\pi n x\theta}\right) \, d\theta
\]
\[
= \frac{\pi x^r}{r^2} \frac{i\pi x^r}{2r-1} \sum_{n=1}^\infty (-1)^{n+1} \int_0^1 \theta^{r-1} e^{-\pi n x\theta} \, d\theta
\]
\[
= \frac{(r-1)!}{(2\pi i)^{r-1}} \sum_{k=0}^\infty \frac{(\pi i)^k}{k!} \text{Li}_{r-k}(-e^{-\pi i x}) x^k + \frac{\pi i}{r^2} x^r - \frac{(r-1)!}{(2\pi i)^{r-1}} \xi(r),
\]
and (1) is proved.

(2) When \(\text{Im}(x) > 0\), the proof for (2) similar.

(3)–(4) From (4.10) we have
\[
\log C_r \left(\frac{x}{2}\right) = -\frac{(r-1)!}{(2\pi i)^{r-1}} \sum_{k=0}^\infty \frac{(\pi i)^k}{k!} \text{Li}_{r-k}(-e^{-\pi i x}) x^k - \left(\frac{x}{2}\right)^{r-1} \text{Li}_1(-e^{-\pi i x}) + \frac{\pi i}{r^2} x^r - \frac{(r-1)!}{(2\pi i)^{r-1}} \xi(r).
\]
Since
\[
\text{Li}_1(-e^{-\pi i x}) = -\log(1 + e^{-\pi i x}) = \log \left(2 e^{-\frac{\pi x}{2}} \cos \frac{\pi x}{2}\right),
\]
we get
\[
\log C_r \left(\frac{x}{2}\right) = -\frac{(r-1)!}{(2\pi i)^{r-1}} \sum_{k=0}^\infty \frac{(\pi i)^k}{k!} \text{Li}_{r-k}(-e^{-\pi i x}) x^k - i\pi \left(\frac{x}{2}\right)^{r-1} \text{Li}_1(-e^{-\pi i x}) + \frac{\pi i}{r^2} x^r - \frac{(r-1)!}{(2\pi i)^{r-1}} \xi(r).
\]
Then by taking the exponential on the both sides of the above equation, we get
\[
C_r \left(\frac{x}{2}\right) = \left(2 \cos \frac{\pi x}{2}\right)^{r-1} \exp \left(-\frac{(r-1)!}{(2\pi i)^{r-1}} \sum_{k=0}^\infty \frac{(\pi i)^k}{k!} \text{Li}_{r-k}(-e^{-\pi i x}) - i\pi \left(\frac{x}{2}\right)^{r-1} \text{Li}_1(-e^{-\pi i x}) + \frac{\pi i}{r^2} x^r - \frac{(r-1)!}{(2\pi i)^{r-1}} \xi(r)\right).
\]
Finally, by taking the real part in the above expression for \(2 \leq r \in 2\mathbb{Z}\) and \(3 \leq r \in 1 + 2\mathbb{Z}\) respectively, also notice that (see (2.2))
\[
\text{Li}_{r-k}(-e^{-\pi i x}) = \sum_{n=1}^\infty \frac{(-1)^n e^{-\pi i n x}}{n^r} = \sum_{n=1}^\infty \frac{(-1)^n \cos(\pi i n x) - i \sin(\pi i n x)}{n^r},
\]
we obtain (3) and (4).
5 Miscellaneous results

In this section, we present several new representations for $\log C_r(x)$ and some series involving $\lambda(2k)$, the special values of Dirichlet’s lambda function at positive even integer arguments.

Let $Cl_2(\theta)$ be the Clausen function defined by

$$Cl_2(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2}.$$ 

The Clausen function $Cl_2$ is related to the following expression (see for instance [32, p. 106, (2)])

$$Cl_2(\theta) = \theta \log \pi - \theta \log \left(\sin \frac{\theta}{2}\right) + 2\pi \log \frac{G\left(\frac{1}{2} - \frac{\theta}{\pi}\right)}{G\left(\frac{1}{2} + \frac{\theta}{\pi}\right)},$$

(5.1)

where $G(x)$ is the Barnes $G$-function. From Corollary 2.13 and (5.1), we obtain

$$C_2\left(\frac{x}{2}\right) = \left(2 \cos \frac{\pi x}{2}\right)^\frac{i}{2} \exp \left(\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\sin(\pi n(x+1))}{n^2}\right)$$

$$= \left(2 \cos \frac{\pi x}{2}\right)^\frac{i}{2} \exp \left(\frac{1}{2\pi} Cl_2(\pi(x+1))\right),$$

(5.2)

since $(-1)^n \sin(\pi nx) = \sin(\pi n(x+1))$ for $n = 1, 2, 3, \ldots$. Taking logarithm on both sides of (5.2) and using (5.1) with $\theta = \pi(x+1)$, we get

$$\log C_2\left(\frac{x}{2}\right) = \frac{x}{2} \log(2\pi) + \log \sqrt{\pi} - \frac{1}{2} \log \left(\cos \frac{\pi x}{2}\right) + \log \frac{G\left(\frac{1}{2} - \frac{x}{4}\right)}{G\left(\frac{1}{2} + \frac{x}{4}\right)}.$$ 

(5.3)

Then, by using Proposition 3.3, we have

$$\int_0^x n t \tan\left(\frac{\pi t}{2}\right) dt = -2x \log(2\pi) - 2 \log \pi + 2 \log \left(\cos \frac{\pi x}{2}\right) - 4 \log \frac{G\left(\frac{1}{2} - \frac{x}{4}\right)}{G\left(\frac{1}{2} + \frac{x}{4}\right)}.$$ 

As an application, setting $x = \frac{1}{2}$ in (5.3), we get

$$\log C_2\left(\frac{1}{4}\right) = \log(4\pi) - 4 \log \sqrt{\pi} + \log \frac{G\left(\frac{1}{4}\right)}{G\left(\frac{3}{4}\right)},$$

(5.4)

By using Corollary 2.6 and (5.4), we see that

$$\log \frac{G\left(\frac{1}{4}\right)}{G\left(\frac{3}{4}\right)} = -3 \log 2 - \frac{3 \log \pi}{4} - \frac{G}{2\pi},$$

which is equivalent to

$$G\left(\frac{7}{4}\right) = 2^{\frac{3}{2}} \pi^\frac{3}{2} e^{\frac{\pi}{2}} G\left(\frac{1}{4}\right).$$

Then, by considering the following expression due to Choi and Srivastava [32, p. 30, (23)]:

$$\log G\left(\frac{1}{4}\right) = -\frac{G}{4\pi} - \frac{3}{4} \log \Gamma\left(\frac{1}{4}\right) - \frac{9 \log A}{8} + \frac{3}{32},$$

we get

$$G\left(\frac{7}{4}\right) = 2^{\frac{3}{2}} \pi^\frac{3}{2} e^{\frac{\pi}{2}} A^{-\frac{3}{2}} \Gamma\left(\frac{1}{4}\right)^{-\frac{1}{2}}.$$ 

In the subsequent, we will show that Corollary 2.15 in fact implies a relation between the special values of Barnes’ $G$-function and the Dirichlet $L$-function. Putting $x = \frac{1}{2}$ in (5.3) and by simplifying, we get

$$\log C_2\left(\frac{1}{6}\right) = \frac{2 \log(2\pi)}{3} - \frac{3}{4} \log \frac{G\left(\frac{1}{4}\right)}{G\left(\frac{3}{4}\right)},$$

(5.5)
Then, by substituting the identity (see Corollary 2.15)

\[ C_{3e}^{(1)} \left( \frac{1}{6} \right) = 3^{\frac{1}{6}} \exp \left( \frac{\sqrt{3}}{4\pi} \left( \frac{1}{4} L(2, \chi_3) - L(2, \chi_6) \right) \right) \]

into (5.5), we obtain

\[
\log \frac{G(\frac{1}{3})}{G(\frac{5}{3})} = \frac{\log 3}{3} - \frac{2 \log(2m)}{3} + \frac{\sqrt{3}}{4\pi} \left( \frac{1}{4} L(2, \chi_3) - L(2, \chi_6) \right). 
\] (5.6)

If going on substituting the formula (see [1, p. 16])

\[
\log G(\frac{1}{3}) = \frac{\log 3}{72} + \frac{\pi}{18\sqrt{3}} \log \left( \frac{1}{3} \right) - \frac{4 \log A}{12\pi \sqrt{3}} + \frac{\psi^{(1)}(\frac{1}{3})}{12} + \frac{1}{9} 
\]

into (5.6), we further get

\[
\log G(\frac{5}{3}) = -\frac{23}{72} \log 3 + \frac{\pi}{18\sqrt{3}} \log \left( \frac{1}{3} \right) - \frac{4 \log A}{12\pi \sqrt{3}} + \frac{\psi^{(1)}(\frac{1}{3})}{12} + \frac{1}{9} + \frac{2 \log(2m)}{3} - \frac{\sqrt{3}}{4\pi} \left( \frac{1}{4} L(2, \chi_3) - L(2, \chi_6) \right), 
\]

where

\[
\psi^{(1)}(x) = \frac{d^2 \log \Gamma(x)}{dx^2} 
\]

is the polygamma function.

From the definition of the triple cosine function (1.17) and the power series expansion

\[
\log(1 + x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} 
\]

for \(|x| < 1\), we may represent \( \log C_3(x) \) as

\[
\log C_3(x) = \sum_{n=1}^{\infty} \left( \frac{2n - 1}{2} \right)^2 \log \left( 1 - \frac{4x^2}{(2n - 1)^2} \right) + x^2 
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{(2n - 1)^2}{4} \right) \sum_{k=1}^{\infty} \frac{4k}{(2n - 1)2k} \frac{x^{2k}}{k} + x^2 
\]

\[
= - \sum_{k=2}^{\infty} 4^{k-1} \lambda(2k - 2) \frac{x^{2k}}{k}, 
\]

which is equivalent to

\[
\log C_3(x) = - \sum_{k=1}^{\infty} 4^{2k} \lambda(2k) \frac{x^{2k+2}}{k+1}. 
\] (5.7)

Then, by combining (1.3) and (5.7), and using (331) in [32, p. 221], \( \log C_3(x) \) has the following representation:

\[
\log C_3(x) = - \log(2^{\frac{1}{2}} \pi \cdot e^{-\frac{1}{2}} \cdot A^\frac{1}{2}) - (1 - \log(2\pi)) \frac{x^2}{2} - \frac{1}{4} \log \left( \frac{1}{2} + x \right) \log \left( \frac{1}{2} - x \right) 
\]

\[
- \left( \frac{1}{2} + x \right) \log G \left( \frac{1}{2} + x \right) - \left( \frac{1}{2} - x \right) \log G \left( \frac{1}{2} - x \right) + \int_0^x \log G \left( t + \frac{1}{2} \right) dt + \int_0^{-x} \log G \left( t + \frac{1}{2} \right) dt 
\]

for \(|x| < \frac{1}{2}\), where \( A \) is the Glaisher–Kinkelin constant (also see [32, p. 25]).

Note that (5.7) can be generalized from \( r = 3 \) to \( r = 2, 3, 4, \ldots \) as follows. Recall that (see Proposition 3.1)

\[
\log C_r(x) = - \int_0^x \pi r^{r-1} \tan(\pi t) \, dt 
\] (5.8)

and notice the well-known identity

\[
t \tan(t) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k} - 1)B_{2k}}{(2k)!} t^{2k} 
\] (5.9)
for $|t| < \frac{\pi}{2}$. By combining (5.9) with (1.8), and setting $t \to \pi t$, we have

$$\pi t \tan(\pi t) = 2 \sum_{k=1}^{\infty} 2^{2k} \lambda(2k) t^{2k}$$

(5.10)

for $|t| < \frac{1}{2}$, where we have used the relation $\lambda(s) = (1 - 2^{-s}) \zeta(s)$ (see (1.3)). Then, for $r = 2, 3, 4, \ldots$, by multiplying (5.10) with $t^{r-2}$ and integrating the result equation, from (5.8) we get

$$\log \zeta_r(x) = -2 \sum_{k=1}^{\infty} 2^{2k} \lambda(2k) \frac{x^{2k+r-1}}{2k + r - 1}.$$  

(5.11)

This representation is valid for $|x| < \frac{1}{2}$.

Setting $x = \frac{1}{4}$ in (5.11), and using Corollary 2.6 for $r = 2, 3, 4$ and 5, it is readily to obtain

$$\sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 1)2^{2k}} = -\frac{\log 2}{4} + \frac{G}{\pi}$$  

(see [32, p. 241, (666)]),

$$\sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 2)2^{2k}} = -\frac{\log 2}{4} + \frac{2G}{\pi} - \frac{7\zeta_E(3)}{2\pi^2},$$

$$\sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 3)2^{2k}} = -\frac{\log 2}{4} + \frac{3G}{\pi} + \frac{3\zeta_E(3)}{\pi^2} - \frac{24\beta(4)}{\pi^3},$$

$$\sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 4)2^{2k}} = -\frac{\log 2}{4} + \frac{4G}{\pi} + \frac{3\zeta_E(3)}{\pi^2} - \frac{96\beta(4)}{\pi^3} + \frac{186\zeta_E(5)}{\pi^4}.$$  

Furthermore, by subtracting the above series, we get

$$\sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 1)(2k + 2)2^{2k}} = \frac{G}{\pi} + \frac{7\zeta_E(3)}{2\pi^2},$$

$$\sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 2)(2k + 3)2^{2k}} = \frac{G}{\pi} - \frac{5\zeta_E(3)}{\pi^2} + \frac{24\beta(4)}{\pi^3},$$

$$\sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 3)(2k + 4)2^{2k}} = -\frac{3\zeta_E(3)}{\pi^2} + \frac{72\beta(4)}{\pi^3} - \frac{186\zeta_E(5)}{\pi^4}.$$  

From this, we have the following series representation for $\zeta_E(3)$:

$$\zeta_E(3) = \frac{2\pi^2}{7} \left( \frac{G}{\pi} + \sum_{k=1}^{\infty} \frac{\lambda(2k)}{(2k + 1)(2k + 2)2^{2k}} \right).$$

Acknowledgment: We are grateful to Professor Jean-Paul Allouche for his interested in this work and for his many helpful comments and suggestions.

Funding: This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2022R1F1A1065551).

References

[1] V. S. Adamchik, On the Barnes function, in: Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation, ACM, New York (2001), 15–20.
[2] V. S. Adamchik, The multiple gamma function and its application to computation of series, Ramanujan J. 9 (2005), no. 3, 271–288.
[3] J.-P. Allouche, Hölder and Kurokawa meet Borwein–Dykshoorn and Adamchik, J. Ramanujan Math. Soc. 38 (2023), no. 3, 265–273.
[4] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, in: Journées Arithmétiques de Luminy, Astérisque 61, Société Mathématique de France, Paris (1979), 11–13.
[5] T. M. Apostol, Introduction to Analytic Number Theory, Undergrad. Texts Math., Springer, New York, 1976.
[6] R. Ayoub, Euler and the zeta function, *Amer. Math. Monthly* **81** (1974), 1067–1086.

[7] P. Borwein and D. Dykshoorn, An interesting infinite product, *J. Math. Anal. Appl.* **179** (1993), no. 1, 203–207.

[8] D. Cvijović, A note on convexity properties of functions related to the Hurwitz zeta and alternating Hurwitz zeta function, *J. Math. Anal. Appl.* **487** (2020), no. 1, Article ID 123972.

[9] L. Euler, De summis serierum numeros Bernoullianos involventium, *Novi Comm. Acad. Sci. Petropolitanae* **14** (1769), 129–167.

[10] P. Flajolet and B. Salvy, Euler sums and contour integral representations, *Exp. Math.* **7** (1998), no. 1, 15–35.

[11] P. Borwein and E.W. Zudilin, Zeta functions and Galois representations, *Adv. Math.* **156** (2001), no. 2, 155–193.

[12] A. Weil, Number Theory: An Approach Through History, from Hammurapi to Legendre, Birkhäuser, Boston, 1984.

[13] G. T. Williams, A new method of evaluating (z^2n), *Amer. Math. Monthly* **60** (1953), 19–25.

[14] K. S. Williams and N. Y. Zhang, Special values of the Lerch zeta function and the evaluation of certain integrals, *Proc. Amer. Math. Soc.* **119** (1993), no. 1, 35–49.