Associated Derived Invariants for Geometric Mappings of Non-Symmetric Affine Connection Spaces

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Abstract

The invariants of the Thomas and the Weyl type for a mapping between non-symmetric affine connection spaces are obtained with respect to the factored deformation tensor in this paper. Motivated by two invariants of the Weyl type obtained in ([N. O. Vesić, Basic Invariants of Geometric Mappings, 17]), we founded novel invariants of the Weyl type. Invariants for almost geodesic mappings of the third type are searched at the end of this paper.

Key words: curvature tensor, transformation rule, geometric mapping, invariant

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1 Introduction

Many research papers and monographs are devoted to invariants for mappings between affine connection spaces. Some of them are J. Mikeš [1][6–9], I. Hinterleitner [8,9], N. S. Sinyukov [11], M. S. Stanković [10][18][19], Lj. S. Velimirović [19], M. Lj. Zlatanović [20] and many others. The Thomas projective parameter, the Weyl conformal curvature tensor and the Weyl projective tensor [7][9][11] have been studied and generalized by different mathematicians.

Our main purpose in this paper is to obtain some general invariants for geometric mappings. In this research, we will continue the research from [17]. At the end of this paper, we will apply the results from this research to obtain invariants for the almost geodesic mappings of the third type.

1.1 Affine connection spaces

An \(N\)-dimensional manifold \(\mathcal{M}_N\) equipped with the affine connection \(\nabla\) (with torsion) is the affine connection space \(GA_N\) (see [2][5][18][20]).

The affine connection coefficients for the affine connection \(\nabla\) are \(L^i_{jk}\) and it holds \(L^i_{jk} \not\equiv L^i_{kj}\). The symmetric and anti-symmetric part for the coefficients \(L^i_{jk}\) are

\[
L^i_{jk} = \frac{1}{2} (L^i_{jk} + L^i_{kj}) \quad \text{and} \quad L^i_{jk} = \frac{1}{2} (L^i_{jk} - L^i_{kj}). \quad (1.1)
\]

The tensor \(2L^i_{jk}\) is the torsion tensor for the space \(GA_N\).

The symmetric affine connection space whose affine connection coefficients are \(L^i_{jk}\) is the associated space \(\mathbb{A}_N\).

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One kind of covariant differentiation with respect to the affine connection of space \( \mathcal{A}_N \) is defined. This kind of covariant derivative for a tensor \( a_j^i \) of the type \((1,1)\) is

\[
a^i_{jk} = a^i_{j,k} + L^i_{jk}a_j^\alpha - L^\alpha_{jk}a_j^i,
\]

for partial derivative \( \partial/\partial x^k \) denoted by comma.

One identity of Ricci type with respect to the covariant derivative \((1.2)\) is obtained the basic associated invariant of the Weyl type \([17]\) for the mapping \(p,q=1\):

\[
\omega_{j[m|n]} - a^i_{j[m]n} = a^i_{j}R^m_{anm} - a^\alpha_{n}R^\alpha_{jm}m
\]

is searched (see [8,9,11]), for the curvature tensor

\[
R^i_{jmn} = L^i_{jm,n} - L^i_{jn,m} + L^\alpha_{jm}L^i_{\alpha m} - L^\alpha_{jn}L^i_{\alpha m},
\]

of the space \( \mathcal{A}_N \).

### 1.2 Recall to basic invariants

Let the deformation tensor for the mapping \( F : GA_N \rightarrow \mathbb{R}^N \) be \( P^i_{jk} = \omega^i_{(p)jk} - \omega^i_{(p)jk} + \xi^i_{jk}, p,q = 1,2,3, \omega^i_{(1)jk} = L^i_{jk}, \omega^i_{(2)jk} = \omega^i_{jk}, \omega^i_{(3)jk} = -\frac{1}{2}P^i_{jk} \) the corresponding \( \omega^i_{(p)jk} \), and \( \xi^i_{jk} = -\xi^i_{jk} \) as well.

The basic associated invariants of the Thomas type of the first, the second and the third type for the mapping \( F \) are \([17]\)

\[
\bar{T}^i_{(1)jk} = 0, \quad \bar{T}^i_{jk} = \bar{T}^i_{(2)jk} = L^i_{jk} - \omega^i_{jk}, \quad \bar{T}^i_{(3)jk} = \frac{1}{2}(L^i_{jk} + L^i_{jk}).
\]

With respect to the transformation rule of the curvature tensor \( R^i_{jmn} \) of the associated space \( \mathcal{A}_N \), it is obtained the basic associated invariant of the Weyl type \([17]\) for the mapping \( F \)

\[
\bar{W}^i_{jmn} = \bar{W}^i_{(2)jm} = R^i_{jmn} - \omega^i_{jm|n} + \omega^\alpha_{jm}\omega^i_{\alpha n} - \omega^\alpha_{jn}\omega^i_{\alpha m}.
\]

In this paper, we are interested to obtain associated invariants for the mapping \( F \) with respect to the factored geometrical object \( \omega^i_{jk} \), i.e. for \( \omega^i_{jk} = s_1(\delta^i_j r_k + \delta^i_k r_j) + s_2(f^i_j r_k + f^i_k r_j) + s_3\delta^i_{jk}, s_1, s_2, s_3 \in \{0,1\} \).

### 1.3 Motivation

In \([17]\), it is obtained associated invariants for mappings whose deformation tensors are \( P^i_{jk} = \omega^i_{jk} - \omega^i_{jk} + \xi^i_{jk} \). To test the efficiency of the obtained results, the author searched the invariants for a mapping \( F : GA_N \rightarrow \mathbb{R}^N \) with respect to (see Corollary 1 in \([17]\))

\[
\bar{T}^i_{jk} = \delta^i_j u_k + \delta^i_k u_j + \sigma^i_{jk}
\]

The basic associated invariants for a geodesic mapping \( F : GA_N \rightarrow \mathbb{R}^N \) are (because \( \omega^i_{jk} = \frac{1}{N+1}\delta^i_j L^\alpha_{k\alpha} + \frac{1}{N+1}L^\alpha_{jk} \))

\[
\bar{T}^i_{jk} = L^i_{jk} - \frac{1}{N+1}(\delta^i_j L^\alpha_{k\alpha} + \delta^i_k L^\alpha_{k\alpha}),
\]

\[
\bar{W}^i_{jmn} = R^i_{jmn} + \frac{1}{N+1}\delta^i_j L^\alpha_{m\alpha} - \frac{1}{(N+1)^2}\delta^i_j ((N+1)L^\alpha_{j\alpha|n} + L^\alpha_{j\alpha} L^\beta_{m\beta})
\]

\[
+ \frac{1}{(N+1)^2}\delta^i_j ((N+1)L^\alpha_{j\alpha|m} + L^\alpha_{j\alpha} L^\beta_{m\beta})
\]

for \( L^i_{jm|n} = L^i_{jm,n} + L^i_{mn}L^j_{im} - L^\alpha_{jm}L^i_{\alpha m} + L^\alpha_{jm}L^i_{\alpha m} \).
The basic associated invariant \( W_{jmn} \) is different of the Weyl projective tensor \( W_{jmn} = R_{jmn}^i + \frac{1}{N+1} \delta^j_i R_{[mn]} + \frac{N}{N^2 - 1} \delta^i_{[m} R_{jn]} + \frac{1}{N^2 - 1} \delta^i_{[m} R_{n]j} \).

This paper is consisted of the introduction, three sections and conclusion.

1. At the start of the research, we will present the iterative rule for obtaining novel associated invariants of the Weyl type for a mapping \( F : \mathcal{A}_N \rightarrow \mathcal{A}_N \) with respect to the known ones.

2. In the next section of this paper, we will pay attention to the basic invariants of the Thomas and the Weyl types (the invariants \( T_{jk}^i \) and \( W_{jmn}^i \)) for a mapping \( F : \mathcal{A}_N \rightarrow \mathcal{A}_N \). Motivated with the results presented in the Corollary 2.2 in [17], we will obtain associated invariants of the Weyl type with respect to the equality \( \overline{W}_{jmn} - \overline{W}_{jmn}^i = 0 \) and generalize them by applying the iterative process from the first section of this paper.

3. In the fourth section of this paper, we will apply the obtained results to find associated invariants of the Thomas and the Weyl type for an almost geodesic mapping of the third type.

2 Associated derived invariants

With respect to the equations (1.2) (1.3), we get

\[
R_{ij}^a = L_{ij}^a - L_{ij}^a = L_{ij}^a - L_{ij}^a = -R_{ij}.
\]

Let us prove the following theorem.

**Theorem 2.1.** Any invariant \( W_{jmn}^i \) for a geometrical mapping \( F : \mathcal{A}_N \rightarrow \mathcal{A}_N \) obtained with respect to the transformation rule of the curvature tensor \( R_{jmn}^i \) of the associated space \( \mathcal{A}_N \) is anti-symmetric by the indices \( m \) and \( n \).

If the geometrical object

\[
W_{jmn} = R_{jmn}^i + \delta_j^i X_{[mn]}^i + \delta_{[m}^i Y_{jn]}^i + Z_{jmn}^i,
\]

for tensors \( X_{ij} \) and \( Y_{ij} \) of the type \( (0, 2) \), and a tensor \( Z_{jmn} \), \( Z_{jmn} = -Z_{jmn}^i \), of the type \( (1, 3) \), is an invariant for a mapping \( F : \mathcal{A}_N \rightarrow \mathcal{A}_N \), then the geometrical objects

\[
\begin{align*}
W_{jmn}^{(1)i} &= R_{jmn}^i - \frac{1}{N} \delta_j^i (Y_{[mn]} + Z_{[mn]}^i) + \delta_{[m}^i Y_{jn]}^i + Z_{jmn}^i, \\
W_{jmn}^{(2)i} &= R_{jmn}^i - \frac{1}{2} \delta_j^i ((N - 1)Y_{[mn]} - Z_{[mn]}^i + \delta_{[m}^i Y_{jn]}^i + Z_{jmn}^i, \\
W_{jmn}^{(4)i} &= R_{jmn}^i + \frac{1}{N - 1} \delta_j^i R_{[jn]}^i + \delta_j^i X_{[mn]}^i + Z_{jmn}^i \\
&\quad - \frac{1}{N - 1} (\delta_{[m}^i X_{jn]}^i - \delta_{[m}^i X_{nj]}^i) + \frac{1}{N - 1} \delta_{[m}^i Z_{jn]}^i, \\
R_{ij} &= R_{ij} - R_{ij},
\end{align*}
\]

for \( R_{ij} = \frac{1}{2} (R_{ij} + R_{ji}) \), are invariants for this mapping.

**Proof.** Let us consider the transformation rule

\[
\overline{R}_{jmn} = R_{jmn}^i + \Pi_{jmn}^i - \Pi_{jmn}^i,
\]

for the corresponding geometrical object \( \Pi_{jmn}^i \) and its image \( \Pi_{jmn}^i \).
Let also be
\[
\Pi^i_{j(mn)} = \Pi^i_{jmn} + \Pi^i_{jnm}, \quad \Pi^i_{j[mn]} = \Pi^i_{jmn} - \Pi^i_{jnm},
\]
(2.8)

With respect to the equation (2.13), one reads
\[
R^i_{jmn} = -R^i_{jmn}, \quad \text{which means} \quad R^i_{jmn} = \frac{1}{2}(R^i_{jmn} - R^i_{jmn}).
\]
(2.9)

Based on the equations (2.8), (2.12), we get
\[
\overline{R}^i_{jmn} - R^i_{jmn} = \frac{1}{2} (\Pi^i_{j[mn]} - \Pi^i_{j[mn]}) + \frac{1}{2} (\Pi^i_{jmn} - \Pi^i_{jnm}).
\]
(2.10)

After symmetrizing the last equation by the indices \(m\) and \(n\), one obtains
\[
0 = \Pi^i_{j(mn)} - \Pi^i_{j[mn]}.
\]
(2.11)

If substitutes the equation (2.11) into the equation (2.10), one will conclude that the geometrical object
\[
\frac{1}{2}(R^i_{jmn} - R^i_{jnm} - \Pi^i_{j[mn]})
\]

is an invariant for this mapping. This invariant is anti-symmetric by the indices \(m\) and \(n\), which completes the proof of the first part for this theorem.

Let us consider the equality \(0 = \overline{W}_{jmn} - W^i_{jmn}\), i.e.
\[
0 = \overline{R}^i_{jmn} - R^i_{jmn} + \delta^i_{jmn} (\overline{X}_{[mn]} - X_{[mn]} + \delta^i_{jmn} (\overline{Y}_{mn} - Y_{mn}) + (\overline{Z}_{[mn]} - Z^i_{jmn}).
\]
(2.12)

After contracting it by the indices \(i\) and \(j\) and applying the equation (2.1), one gets
\[
0 = -R_{[mn]} + R_{[mn]} + N (\overline{X}_{[mn]} - X_{[mn]} + (\overline{Y}_{mn} - Y_{mn}) + (\overline{Z}_{[mn]} - Z^i_{jmn}).
\]
(2.13)

If expresses the summand \(\overline{X}_{[mn]} - X_{[mn]}\) with respect to the equation (2.13) and substitutes it into the equality (2.12), one will confirm the invariance \(\overline{W}^{(1)i}_{jmn} = W^{(1)i}_{jmn}\), for
\[
W^{(1)i}_{jmn} = R^{(1)i}_{jmn} + \frac{1}{N} \delta^i_{jmn} (R_{[mn]} - Y_{mn} - Z^\alpha_{[mn]} + \delta^i_{jmn} Y_{mn} + Z^i_{jmn},
\]
(2.14)

and the corresponding \(\overline{W}^{(1)i}_{jmn}\).

By contracting the equation (2.12) on the indices \(i\) and \(n\), we obtain
\[
0 = \overline{R}_{jm} - R_{jm} - (\overline{X}_{[jm]} - X_{[jm]} + (N - 1) (\overline{Y}_{jm} - Y_{jm}) + (\overline{Z}_{[jm\alpha]} - Z^\alpha_{[jm\alpha]}),
\]
(2.15)

\[
0 = \overline{R}_{jm} - R_{jm} - 2 (\overline{X}_{[jm]} - X_{[jm]} - (N - 1) (\overline{Y}_{[jm]} - Y_{[jm]} + (\overline{Z}_{[jm\alpha]} - Z^\alpha_{[jm\alpha]}).
\]
(2.16)

Based on the equations (2.15), (2.16), we also get
\[
X_{[ij]} - X_{[ij]} = \overline{R}_{ij} - (N - 1) \overline{Y}_{ij} + Z^\alpha_{ij\alpha} - R_{ij} + (N - 1) Y_{ij} - Z^\alpha_{ij\alpha},
\]
(2.17)

\[
\overline{X}_{[ij]} - X_{[ij]} = \frac{1}{2} (\overline{R}_{ij} - (N - 1) \overline{Y}_{ij} + Z^\alpha_{ij\alpha}) - \frac{1}{2} (R_{ij} - (N - 1) Y_{ij} + Z^\alpha_{ij\alpha}),
\]
(2.18)

\[
\overline{Y}_{ij} - Y_{ij} = \frac{1}{N - 1} (\overline{R}_{ij} - \overline{X}_{[ij]} + Z^\alpha_{ij\alpha}) - \frac{1}{N - 1} (R_{ij} - X_{[ij]} + Z^\alpha_{ij\alpha}.
\]
(2.19)
If substitute the expressions \(2.17\), \(2.18\), \(2.19\) into the equation \(2.12\) and use the equality \(\delta^i_m X_{jn} - \delta^i_n X_{jm} = \delta^i_m X_{jn} - \delta^i_m X_{nj}\), we will obtain

\[
\overline{W}^{(2)i}_{jmn} = W^{(2)i}_{jmn}, \quad \overline{W}^{(3)i}_{jmn} = W^{(3)i}_{jmn}, \quad \overline{W}^{(4)i}_{jmn} = W^{(4)i}_{jmn},
\]

for

\[
W^{(2)i}_{jmn} = R^{i}_{jmn} + \delta^i_j (R_{mn} - (N - 1)Y_{mn} + Z^\alpha_{mna}) + \delta^i_m Y_{jn} + Z^i_{jmn}, \quad (2.20)
\]

\[
W^{(3)i}_{jmn} = R^{i}_{jmn} + \frac{1}{2} \delta^i_j (R_{mn} - (N - 1)Y_{mn} + Z^\alpha_{m[n\alpha]} + \delta^i_m Y_{jn} + Z^i_{jmn}, \quad (2.21)
\]

\[
W^{(4)i}_{jmn} = R^{i}_{jmn} + \frac{1}{2} (\frac{1}{N-1} \delta^i_m R_{mn} + \delta^i_j X_{mn}) + \delta^i_m Y_{jn} + Z^i_{jmn}
\]

\[
= \frac{1}{N-1} (\delta^i_m X_{jn} - \delta^i_m X_{nj}) + \frac{1}{N-1} \delta^i_m Z^\alpha_{jmn}, \quad (2.22)
\]

and the corresponding \(\overline{W}^{(2)i}_{jmn}, \overline{W}^{(3)i}_{jmn}, \overline{W}^{(4)i}_{jmn}\).

With respect to the equation \(2.20\) and the first part of this theorem \((W^{(2)i}_{jmn} = -W^{(2)i}_{jmn})\), the invariant \(W^{(2)i}_{jmn}\) satisfies the equality \(W^{(2)i}_{jmn} = \frac{1}{2} (W^{(2)i}_{jmn} - W^{(2)i}_{jmn})\), i.e.

\[
W^{(2)i}_{jmn} = R^{i}_{jmn} + \frac{1}{2} \delta^i_j (2R_{mn} - (N - 1)Y_{mn} + Z^\alpha_{m[n\alpha]} + \delta^i_m Y_{jn} + Z^i_{jmn}. \quad (2.23)
\]

From the equations \(2.20\) and \(2.22\), we find the invariant

\[
W^{(2)i}_{jmn} - W^{(3)i}_{jmn} = \frac{1}{2} \delta^i_j R_{mn}, \quad (2.26)
\]

for the mapping \(\mathcal{F}\), which proves the stated invariance of the geometrical object \(R_{ij}\).

For this reason, the invariants for the mapping \(\mathcal{F}\) obtained in the equations \(2.20\), \(2.21\), \(2.22\), reduce to the geometrical objects \(W^{(1)i}_{jmn}, W^{(2)i}_{jmn}, W^{(4)i}_{jmn}\) given by the equations \(2.23\), \(2.24\), \(2.25\).

The invariants \(W^{(1)i}_{jmn}, W^{(2)i}_{jmn}\) and \(W^{(4)i}_{jmn}\) for the mapping \(\mathcal{F}: GA_N \to GA_N\) given by the equations \(2.23\), \(2.24\), \(2.25\) are the first, the second and the fourth associated derived invariant (with respect to the invariant \(W^{(1)i}_{jmn}\) given by the equation \(2.22\)).

**Remark 2.1.** If \(X_{ij} = 0\), the equations \(2.17\), \(2.18\) reduce to

\[
0 = R_{ij} - (N - 1)Y_{ij} + Z^\alpha_{ij\alpha} - R_{ij} + (N - 1)Y_{ij} - Z^\alpha_{ij\alpha}, \quad (2.23)
\]

\[
0 = \frac{1}{2} (R_{ij} - (N - 1)Y_{ij} + Z^\alpha_{ij\alpha}) - \frac{1}{2} (R_{ij} - (N - 1)Y_{ij} + Z^\alpha_{ij\alpha}). \quad (2.24)
\]

By anti-symmetrizing the equation \(2.23\) on the indices \(i\) and \(j\), one will obtain

\[
0 = R_{ij} - (N - 1)Y_{ij} + Z^\alpha_{ij\alpha} - R_{ij} - (N - 1)Y_{ij} - Z^\alpha_{ij\alpha}. \quad (2.23)
\]

From the difference \(2 \cdot (2.23) - (2.23)\), one derives the invariants \(\delta^i_j R_{mn}\) and \(\frac{1}{N} \delta^\alpha_m R_{mn}\) for the mapping \(\mathcal{F}\).
3 Associated invariants for geometric mappings

Let \( F : \mathcal{G}A_N \to \mathcal{G}A_N \) be a mapping which transforms the affine connection coefficients \( L^i_{jk} = L^i_{jk} + L^i_{jk} \) of the space \( \mathcal{G}A_N \) by the rule

\[
\tilde{L}^i_{jk} = L^i_{jk} + s_1 \left[ \delta^i_j (u_k - u_k) + \delta^i_k (u_j - u_j) \right] \\
+ s_2 \left[ (\tilde{T}^i_j \tilde{\sigma}_k + \tilde{T}^i_k \tilde{\sigma}_j) - (f^i_j \sigma_k + f^i_k \sigma_j) \right] + s_3 \left[ \tilde{\phi}^i_{jk} - \phi^i_{jk} \right],
\]

for the corresponding coefficients \( s_1, s_2, s_3 \in \{0, 1\} \), the tensor \( \xi^i_{jk} \) of the type \((1,2)\) anti-symmetric by the indices \( j \) and \( k \), the covariant vectors \( u_j, \tilde{u}_j, \sigma_j, \tilde{\sigma}_j \), the contravariant vectors \( f^i_j, \tilde{f}^i_j \), the affinors \( f^i_j, \tilde{f}^i_j \), and the geometrical objects \( \phi^i_{jk} \) and \( \tilde{\phi}^i_{jk} \) of the type \((1,2)\) such that \( \phi^i_{jk} = \phi^i_{kj} \) and \( \tilde{\phi}^i_{jk} = \tilde{\phi}^i_{kj} \).

After symmetrizing the last equation by the indices \( j \) and \( k \), one gets

\[
\tilde{L}^i_{jk} = L^i_{jk} + s_1 \left[ \delta^i_j (u_k - u_k) + \delta^i_k (u_j - u_j) \right] \\
+ s_2 \left[ (\tilde{T}^i_j \tilde{\sigma}_k + \tilde{T}^i_k \tilde{\sigma}_j) - (f^i_j \sigma_k + f^i_k \sigma_j) \right] + s_3 \left[ \tilde{\phi}^i_{jk} - \phi^i_{jk} \right].
\]

3.1 Basic associated invariant of Thomas type

If contract the equality \((3.2)\) by the indices \( i \) and \( j \) and use the equalities \( \phi^i_{jk} = \phi^i_{kj} \) and \( \tilde{\phi}^i_{jk} = \tilde{\phi}^i_{kj} \), we will obtain

\[
s_1 \psi_k = \frac{1}{N + 1} \left\{ \tilde{L}^\alpha_{\alpha k} - L^\alpha_{\alpha k} - s_2 \left( \tilde{T}^\alpha_j \tilde{\sigma}_k - (f^\alpha_j \sigma_k + f \sigma_k) \right) - s_3 (\tilde{\phi}^\alpha_{\alpha k} - \phi^\alpha_{\alpha k}) \right\},
\]

for \( \psi_i = \tilde{u}_i - u_i \), the scalars \( f^\alpha_i \) and \( \tilde{f} = \tilde{T}^\alpha \).

After substituting the previous expression of \( s_1 \psi \) into the equation \((3.2)\), one obtains

\[
\omega^i_{jk} = s_2 (f^i_j \sigma_k + f^i_k \sigma_j) + s_3 \phi^i_{jk} + \frac{1}{N + 1} \left[ L^\alpha_{\alpha k} - s_2 (f^\alpha_j \sigma_k + f \sigma_k) - s_3 \phi^\alpha_{\alpha k} \right]
+ \frac{1}{N + 1} \left[ \tilde{L}^\alpha_{\alpha k} - s_2 (f^\alpha_j \sigma_k + f \sigma_k) - s_3 \phi^\alpha_{\alpha k} \right].
\]

The basic associated invariant of the Thomas type for the mapping \( F \) is \([17]\)

\[
\tilde{T}^i_{jk} = L^i_{jk} - s_2 (f^i_j \sigma_k + f^i_k \sigma_j) - s_3 \phi^i_{jk} - \frac{1}{N + 1} \left[ \delta^i_j \left[ L^\alpha_{\alpha k} - s_2 (f^\alpha_j \sigma_k + f \sigma_k) - s_3 \phi^\alpha_{\alpha k} \right] \\
+ \delta^i_k \left[ \tilde{L}^\alpha_{\alpha k} - s_2 (f^\alpha_j \sigma_k + f \sigma_k) - s_3 \phi^\alpha_{\alpha k} \right] \right].
\]

It holds the next lemma.

**Lemma 3.1.** Let \( f : \mathcal{G}A_N \to \mathcal{G}A_N \) be a mapping defined on a non-symmetric affine connection space \( \mathcal{G}A_N \). The geometrical object \((3.4)\) is the basic associated invariant of the Thomas type for this mapping.

**Corollary 3.1.** Let \( F : \mathcal{G}A_N \to \mathcal{G}A_N \) be a mapping of a non-symmetric affine connection space \( \mathcal{G}A_N \). In the case of \( s_1 = 0 \), the geometrical object

\[
\tilde{T}^i = L^\alpha_{\alpha k} - s_2 (f^\alpha_i \sigma_k + f \sigma_k) - s_3 \phi^\alpha_{\alpha k}
\]

is an invariant for the mapping \( F \).
In this case, the invariant \(3.4\) reduces to
\[
\hat{T}^i_{jk} = L^i_{jk} - s_2(f^i_j \sigma_k + f^i_k \sigma_j) - s_3 \phi^i_{jk},
\]
for the corresponding coefficients \(s_2, s_3\).

3.2 Basic associated invariant of Weyl type

For the geometrical object \(\omega^i_{jm|n}\) given by the equation \(3.3\), one gets
\[
\omega^i_{jm|n} = s_2(f^i_j \sigma_m + f^i_m \sigma_j + f^i_j \sigma_m + f^i_m \sigma_j) + s_3 \phi^i_{jm|n}
\]
\[
+ \frac{1}{N + 1} \delta^i_j \left[ L^{\alpha}_{n|mn} - s_2(f^\alpha_m \sigma_n + f^\alpha_n \sigma_m + f^\alpha_m \sigma_n + f^\alpha_n \sigma_m - s_3 \phi^\alpha_{n|mn}\right]
\]
\[
+ \frac{1}{N + 1} \delta^i_m \left[ L^{\alpha}_{jn|mn} - s_2(f^\alpha_j \sigma_n + f^\alpha_n \sigma_j + f^\alpha_n \sigma_j + f^\alpha_j \sigma_n - s_3 \phi^\alpha_{jn|mn}\right],
\]
\[
\omega^i_{jm|n} \omega^\alpha_{jn|m} = \frac{2}{(N + 1)^2} \delta^i_j \left[ L^{\alpha}_{n|mn} - s_2(f^\alpha_m \sigma_n + f^\alpha_n \sigma_m) - s_3 \phi^\alpha_{n|mn}\right]
\]
\[
+ \frac{1}{N + 1} \delta^i_m \left[ L^{\alpha}_{jn|mn} - s_2(f^\alpha_j \sigma_m + f^\alpha_m \sigma_j - s_3 \phi^\alpha_{jn|mn}\right]
\]
\[
+ \frac{1}{N + 1} \left[ s_2(f^i_j \sigma_m + f^i_m \sigma_j) + s_3 \phi^i_{jm}\right] + s_3 \phi^i_{jn|n}]
\]
\[
+ \left[ s_2(f^i_j \sigma_m + f^i_m \sigma_j) + s_3 \phi^i_{jm}\right] + s_3 \phi^i_{jn|n}.
\]

If substitute the expressions \(3.7, 3.8\) into the equation \(3.5\), and use the invariance \(R_{[ij]} = R_{[ij]}\) as well, we will obtain that the geometrical object
\[
\tilde{W}^i_{jm|n} = R^i_{jm|n} + A^i_{jm|n} - \frac{1}{N + 1} \left[ \delta^i_j \rho_{[mn]} + \delta^i_m L^\alpha_{jn|n} - \delta^i_j \rho_{jn}\right] - \frac{1}{(N + 1)^2} \delta^i_j S_{jn},
\]
for \(f_i = f, i \equiv f_i\) and
\[
\rho_{ij} = s_2(f^\alpha_i \sigma_j + f^\alpha_j \sigma_i + f^\alpha_i \sigma_j + f^\alpha_j \sigma_i) + s_3 \phi^\alpha_{ij}\]
\[
S_{ij} = (N + 1) \left[ L^\beta_{\alpha\beta} - s_2(f^\alpha \sigma_j + f^\alpha_j \sigma_j) - s_3 \phi^\alpha_{ij}\right]
\]
\[
+ \left[ L^\alpha_{\alpha\alpha} - s_2(f^\alpha \sigma_i + f^\alpha_i \sigma_j + s_3 \phi^\alpha_{ij}\right]
\]
\[
A^i_{jm|n} = -s_2(f^i_j \sigma_m + f^i_m \sigma_j) + f^i_j \sigma_m + f^i_m \sigma_j) - s_3 \phi^i_{jm|n}\]
\[
+ \left[ s_2(f^\alpha_j \sigma_m + f^\alpha_m \sigma_j) + s_3 \phi^\alpha_{jm}\right] + s_3 \phi^\alpha_{jm|n}\]
\[
- \left[ s_2(f^\alpha_j \sigma_m + f^\alpha_m \sigma_j) + s_3 \phi^\alpha_{jm}\right] + s_3 \phi^\alpha_{jm|n},
\]
is the basic associated invariant for the mapping \(F\).
It is satisfied the next equalities

\[ 0 = \overline{R}_{jmn} - \frac{1}{N + 1} \left[ \delta_j \overline{P}_{mn} + \delta_m \overline{L}_{ja[n]} + \delta_a [m \overline{P}_{jn}] \right] - \frac{1}{(N + 1)^2} \delta_{[m} \overline{S}_{jn]} + \overline{A}^i_{jmn} \]

\[ - R^i_{jmn} + \frac{1}{N + 1} \left[ \delta_j \rho_{[mn]} + \delta_m \rho_{a[n]} + \delta_a [m \rho_{jn}] \right] + \frac{1}{(N + 1)^2} \delta_{[m} \overline{S}_{jn]} - A^i_{jmn}, \tag{3.13} \]

\[ \overline{S}_{ij} = \overline{S}_{ji}, \quad \text{so} \quad \overline{S}_{[ij]} = 0, \tag{3.14} \]

\[ \delta^i_m L^\alpha_{a[n]} - \delta^i_m L^\alpha_{an} | j = 0 = \delta^i_m L^\alpha_{a[n]} - \delta^i_m L^\alpha_{an} | j = 0 = \delta^i_m L^\alpha_{a[n]} - \delta^i_m L^\alpha_{an} | j, \tag{3.15} \]

\[ \rho_{[ij]} = s \left( f^\alpha_{[ij]} \sigma_\alpha - f^\alpha_{[ij]} \sigma_\alpha + f^\alpha_{[ij]} \sigma_\alpha | j + f^\alpha_{[ij]} \sigma_\alpha | j \right) + s \phi^\alpha_{[ij]} \tag{3.16} \]

\[ \rho^\alpha_{[ij]} = -s \left( f^\alpha_{[ij]} \sigma_\alpha - f^\alpha_{[ij]} \sigma_\alpha + f^\alpha_{[ij]} \sigma_\alpha | j + f^\alpha_{[ij]} \sigma_\alpha | j \right) - s \phi^\alpha_{[ij]} = -\rho_{[ij]}, \tag{3.17} \]

\[ \rho_{[ij]} = s \left( f^\alpha_{[ij]} \sigma_\alpha + f^\alpha_{[ij]} \sigma_\alpha + f^\alpha_{[ij]} \sigma_\alpha | j + f^\alpha_{[ij]} \sigma_\alpha | j \right) + s \phi^\alpha_{[ij]} = \rho_{[ij]}, \tag{3.18} \]

The equation (3.13) holds based on the invariance $\overline{W}^i_{jmn} - \overline{W}^i_{jmn} = 0$. The equations (3.14) are in fact with respect to the expressions $3.10 \ 3.11 \ 3.12$ of the corresponding geometrical structures.

If we contract the equation (3.13) by the invariants $i$ and $j$, use the symmetry (3.14) and the expression (3.17), and with respect to the invariance $-\overline{T}^a_{[ij]} = \overline{R}_{[ij]} = R_{[ij]} = -L^\alpha_{[ij]}$ as well, we will obtain

\[ 0 = - \frac{N}{N + 1} \rho_{[mn]} + \frac{N}{N + 1} \rho_{[mn]} - \rho_{[mn]} - \frac{N}{N + 1} \rho_{[mn]} - \frac{1}{(N + 1)^2} \delta_{[m} \overline{S}_{jn]} - A^a_{[mn]} \rho_{[mn]}, \tag{3.19} \]

i.e.

\[ \rho_{[mn]} = \rho_{[mn]}. \tag{3.19} \]

Based on the equations (3.16) (3.17) (3.19), one gets

\[ \overline{A}^{\alpha}_{[ij]} = A^{\alpha}_{[a[i]}, \quad \overline{A}_{[ij]}^\alpha = A^{\alpha}_{[i]a}. \tag{3.20} \]

Hence, the basic invariant $\overline{W}^i_{jmn}$ for the mapping $F$ given by the equation (3.19) reduces to

\[ \overline{W}^i_{jmn} = R^i_{jmn} + A^i_{jmn} - \frac{1}{N + 1} \left[ \delta_j \overline{P}_{mn} + \delta_m \overline{L}_{ja[n]} + \delta_a [m \overline{P}_{jn}] \right] - \frac{1}{(N + 1)^2} \delta_{[m} \overline{S}_{jn]} \tag{3.19} \]

The next lemma holds.

**Lemma 3.2.** Let $F : G\mathcal{K}_N \rightarrow G\mathcal{K}_N$ be a mapping whose deformation tensor is $P^i_{jk}$, $P^i_{jk} = \overline{w}^i_{jk} - \omega^i_{jk}$, for the geometrical object $\omega^i_{jk}$ given by the equation (3.3).

The geometrical object $\overline{W}^i_{jmn}$ given by the equation (3.19) is the associated basic invariant of the Weyl type for the mapping $F$. \hfill \square

### 3.3 Associated derived invariants with respect to basic invariants

If we contract the equality $\overline{T}^i_{jk} = \overline{T}^i_{jk} = 0$ by any pair of indices $(i, j)$ or $(i, k)$ as well, we will express the transformation rule for terms in brackets from the equation (3.14). This transformation rule will not change the form of the invariant $\overline{T}^i_{jk}$.

With respect to the invariant $\overline{W}^i_{jmn}$ for the mapping $F$ given by the equation (3.19), we get

\[ X_{ij} = 0, \quad Y_{ij} = -\frac{1}{(N + 1)^2} \left[ (N + 1) \left( L^\alpha_{a[i]} - \rho_{ij} \right) + \overline{S}_{ij} \right], \quad Z^i_{jmn} = A^i_{jmn}. \tag{3.21} \]
After substituting the terms (3.21) into the equations (2.3, 2.4, 2.5), and applying the invariance \( A^\alpha_{[ij]\alpha} = A^\alpha_{[ij]\alpha} \), one confirms the validity of the next theorem.

**Theorem 3.1.** Let \( F : \mathbb{G}_N \rightarrow \mathbb{G}_N \) be a mapping whose deformation tensor is \( P^i_{jk}, P^i_{jk} = \omega^i_{jk} - \omega^i_{jk} \), for the geometrical object \( \omega^i_{jk} \) given by the equation (3.3).

The geometrical objects \( \tilde{W}^{(1)i}_{jmn} = \tilde{W}^i_{jmn}, \tilde{W}^{(2)i}_{jmn} = \tilde{W}^{(1)i}_{jmn}, \)

\[
\tilde{W}^{(4)i}_{jmn} = R^i_{jmn} + \frac{1}{N - 1} \delta^i_{[m} R_{n]} + A^i_{jmn} + \frac{1}{N - 1} \delta^i_{[m} A^\alpha_{n] \alpha},
\]

(3.22)

for the geometrical objects \( \rho_{ij}, \tilde{S}_{ij}, A^i_{jmn} \) given by the equations (3.10, 3.11, 3.12), are invariants for the mapping \( F \).

**Corollary 3.2.** The geometrical object

\[
\frac{1}{N(N + 1)} W^i_{jmn} = R^i_{jmn} - \frac{1}{(N + 1)^2} \left( (N + 1) \left( \delta^i_{[m} R^\alpha_{n]} + \delta^i_{[m} A^\alpha_{n] \alpha} \right) + \delta^i_{[m} A^\alpha_{n] \alpha} \right) + A^i_{jmn},
\]

(3.23)

is an invariant for the mapping \( F \).

All associated invariants of the Weyl type for the mapping \( F \) are linear combinations of the invariants \( \tilde{W}^{(1)i}_{jmn}, \tilde{W}^{(4)i}_{jmn}, \) and the compositions of the invariants of the Thomas type for the mapping \( F \) given by the corresponding of the equations (3.3, 3.4).

**Proof.** With respect to the equations (2.2, 3.9), we get

\[
X_{ij} = 0 \quad Y_{ij} = -\frac{1}{N + 1} L^\alpha_{ij} + \frac{1}{N + 1} \rho_{ij} - \frac{1}{(N + 1)^2} \tilde{S}_{ij}, \quad Z^i_{jmn} = A^i_{jmn}.
\]

(3.24)

If substitute the geometrical objects \( X_{ij}, Y_{ij}, Z^i_{jmn} \) given by the equation (3.24) into the equations (2.3, 2.4, 2.5), we will obtain the invariants \( \tilde{W}^i_{jmn} \) and \( \tilde{W}^{(4)i}_{jmn} \) given by the equations (3.23) and (3.22) respectively.

Based on the equations (2.2, 3.22) we read

\[
X_{ij} = 0 \quad Y_{ij} = \frac{1}{N} R^i_{ij} + \frac{1}{N + 1} A^\alpha_{ij \alpha}, \quad Z^i_{jmn} = A^i_{jmn}.
\]

(3.25)

From these \( X_{ij}, Y_{ij}, Z^i_{jmn} \) involved into the equations (2.3, 2.4, 2.5), we get the invariant \( \tilde{W}^{(4)i}_{jmn} \) for the mapping \( F \) given by the equation (3.22).

From the invariant \( \tilde{W}^{i}_{jmn} \), one gets

\[
X_{ij} = 0 \quad Y_{ij} = -\frac{1}{N + 1} L^\alpha_{ij} + \frac{1}{N + 1} \rho_{ij} - \frac{1}{(N + 1)^2} A^\alpha_{ij \alpha}, \quad Z^i_{jmn} = A^i_{jmn}.
\]

(3.26)

After substituting the expressions (3.25) of the geometrical objects \( X_{ij}, Y_{ij}, Z^i_{jmn} \) into the equations (2.3, 2.4, 2.5), one obtains the invariants \( \tilde{W}^{i}_{jmn}, \tilde{W}^{(4)i}_{jmn} \).

In this way, we completed the recursion for obtaining invariants of the Weyl type with respect to the Theorem 2.1. For this reason, any other associated invariant of the Weyl type is the linear combination of the previously obtained invariants \( \tilde{W}^{(1)i}_{jmn}, \tilde{W}^{(4)i}_{jmn}, \) \( \tilde{W}^{i}_{jmn} \), and the corresponding compositions of the invariants of the Thomas type given by the equations (3.3, 3.4).
Corollary 3.3. If the geometrical object $A_{jmn}^i$ is
\[
A_{jmn}^i = \delta_j^i P_{[mn]} + \delta_j^i \mathcal{Q}_{jn} + \mathcal{N}_{jmn}^i, \tag{3.27}
\]
then the invariants $\tilde{W}_{jmn}^i$, $\tilde{W}_{jmn}^{(4)i}$, $W_{jmn}^i$, transform to
\[
\begin{align*}
\tilde{W}_{jmn}^i &= R_{jmn}^i - \frac{1}{N+1} (\delta_j^i L_{[m]n}^\alpha - \delta_j^i m_{jn}) - \frac{1}{(N+1)^2} \delta_j^i \tilde{S}_{jn} \\
\tilde{W}_{jmn}^{(4)i} &= R_{jmn}^i + \frac{1}{N+1} \delta_j^i R_{jn} + \delta_j^i P_{[mn]} + \mathcal{N}_{jmn}^i + \frac{1}{N+1} \delta_j^i \mathcal{N}_{jmn}^{[\alpha}, \\
W_{jmn}^i &= R_{jmn}^i - \frac{1}{N+1} \delta_j^i L_{[m]n}^\alpha - \frac{1}{(N+1)^2} \delta_j^i \mathcal{N}_{jmn}^{[\alpha}. \tag{3.28, 3.29, 3.30}
\end{align*}
\]
\[
A_{ij\alpha} = -(N-1)Q_{ij} + \mathcal{N}_{ij\alpha}, \tag{3.31}
\]
Proof. If contracts the equality (3.27) by the indices $i$ and $n$, one obtains
\[
A_{ij\alpha} = -(N-1)Q_{ij} + \mathcal{N}_{ij\alpha}, \tag{3.31}
\]
After substituting the equations (3.27, 3.29), into the equations (3.31), one completes the proof for this corollary.

4 Invariants for almost geodesic mappings of third type

In attempt to generalize the concept of geodesics, N. S. Sinyukov started the research about almost geodesic mappings [11]. J. Mikeš and his research group have continued the study about almost geodesic mappings of symmetric affine connection spaces [11].

M. S. Stanković [12–14] generalized the theory of almost geodesic mappings of symmetric affine connection spaces. Many authors have continued this research [10, 15, 16, 18, 19] and many others.

There are three kinds and two types of almost geodesic mappings $F : \tilde{GA}_N \rightarrow \tilde{GA}_N$.

We are interested to obtain the invariants for equitorsion almost geodesic mappings $F : \tilde{GA}_N \rightarrow \tilde{GA}_N$ of both first and the second kind.

The basic equations of the equitorsion almost geodesic mapping $F : \tilde{GA}_N \rightarrow \tilde{GA}_N$ of a $p$-th kind, $p = 1, 2$, are
\[
\begin{align*}
L_{jk}^i &= L_{jk}^i + \delta_j^i \psi_k + \delta_k^i \psi_j + \sigma_{jk} \varphi^i, \\
\varphi_{ij}^p &= \nu_j \varphi^i + \mu \delta_j^i, \tag{4.1}
\end{align*}
\]
for $\varphi_{ij}^1 = \varphi_{ij}^i + L_{ij}^i \varphi^\alpha \equiv \varphi_{ij}^i + L_{ij}^i \varphi^\alpha$ and $\varphi_{ij}^2 = \varphi_{ij}^i + L_{ij}^i \varphi^\alpha$.

To complete this research, we need the next equalities
\[
\begin{align*}
s_1 = 1, s_2 = 0, s_3 = 1, \sigma_{jk}^i &= -\frac{1}{2} \sigma_{jk} \varphi^i, \tag{4.2} \\
\rho_{ij} &= -\frac{1}{2} \sigma_{io} \varphi^\alpha - \frac{1}{2} \sigma_{io} \varphi_{ij}^\alpha - \frac{1}{2} \sigma_{io} \nu_j \varphi^\alpha - \frac{1}{2} \mu \sigma_{ij} - \frac{(-1)^p}{2} \sigma_{io} L_{io}^j \varphi^\alpha; \tag{4.3} \\
\tilde{S}_{ij} &= -\frac{N+1}{2} \sigma_{ij} \varphi^\alpha \left[ \frac{\varphi_{ij}^\beta}{2} \sigma_{ij} \right] + \left[ L_{io}^j + \frac{1}{2} \sigma_{io} \varphi^\alpha \right] \left[ L_{io}^j + \frac{1}{2} \sigma_{i\beta} \varphi^\beta \right], \tag{4.4}
\end{align*}
\]
\[ A_{jmn}^i = \frac{1}{2} (\sigma_{jm|n} \phi^i + \sigma_{jm} \phi^i_{|n}) + \frac{1}{4} (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \phi^\alpha \phi^i \\
= -\frac{1}{4} \mu \omega^i_{[m|n]} + \frac{1}{4} (\sigma_{jm} \sigma_{|n} - \sigma_{jn} \sigma_{|m}) \phi^i + \frac{1}{4} (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \phi^\alpha \phi^i \\
+ \frac{1}{4} \left[ \sigma_{jm} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) - \sigma_{jn} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) \right]. \tag{4.5} \]

With respect to the equations \(3.27, 4.5\), one gets

\[ \mathcal{P}_{ij} = 0, \quad \mathcal{Q}_{ij} = -\frac{1}{4} \mu \sigma_{ij}, \]

\[ N_{jmn}^i = \frac{1}{4} (\sigma_{jm} \sigma_{|n} - \sigma_{jn} \sigma_{|m}) \phi^i + \frac{1}{4} (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \phi^\alpha \phi^i \]

\[ + \frac{1}{4} \left[ \sigma_{jm} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) - \sigma_{jn} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) \right]. \tag{4.6} \]

The basic invariant for the almost geodesic mapping \( \mathcal{F} \) is

\[ \bar{W}_{jmn}^i = R_{jmn}^i - \frac{N + 3}{4(N + 1)} \delta_{[m|n]}^i \mu \sigma_{j|m} + \frac{1}{4} (\sigma_{jm} \sigma_{|n} - \sigma_{jn} \sigma_{|m}) \phi^i + \frac{1}{4} (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \phi^\alpha \phi^i \]

\[ + \frac{1}{4} \left[ \sigma_{jm} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) - \sigma_{jn} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) \right] \]

\[ - \frac{1}{N + 1} \left[ \delta_{[m|n]}^i \mathcal{N}_{\alpha}^{\beta} + \frac{1}{2} \left[ \delta_{[m|n]}^i \sigma_{\alpha} + \delta_{[m|n]}^i \sigma_{\beta} \right] \mathcal{L}_{\alpha}^i \phi^\alpha \phi^\beta + \frac{1}{2} \left[ \delta_{[m|n]}^i \sigma_{\alpha} + \delta_{[m|n]}^i \sigma_{\beta} \right] \mathcal{L}_{\alpha}^i \phi^\alpha \phi^\beta \right]. \tag{4.7} \]

The fourth associated derived invariant for the mapping \( \mathcal{F} \) is

\[ \bar{W}_{jmn}^{(4)i} = R_{jmn}^{(4)i} + \frac{1}{N - 1} \delta_{[m|n]}^i R_{j|m} + \frac{1}{4} (\sigma_{jm} \sigma_{|n} - \sigma_{jn} \sigma_{|m}) \phi^i + \frac{1}{4} (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \phi^\alpha \phi^i \]

\[ + \frac{1}{4} \left[ \sigma_{jm} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) - \sigma_{jn} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) \right] \]

\[ + \frac{1}{4(N - 1)} \left[ \delta_{[m|n]}^i \sigma_{\alpha} + \delta_{[m|n]}^i \sigma_{\beta} \right] \mathcal{L}_{\alpha}^i \phi^\alpha \phi^\beta + \frac{1}{4(N - 1)} \delta_{[m|n]}^i \sigma_{\alpha} + \delta_{[m|n]}^i \sigma_{\beta} \mathcal{L}_{\alpha}^i \phi^\alpha \phi^\beta \right]. \tag{4.8} \]

The invariant \( \bar{W}_{jmn} \) for the almost geodesic mapping \( \mathcal{F} \) is

\[ \bar{W}_{jmn} = R_{jmn} - \frac{1}{2(N - 1)} \left[ 2 \delta_{[m|n]}^i \mathcal{L}_{\alpha}^{i\beta} + \delta_{[m|n]}^i \sigma_{\alpha} + \delta_{[m|n]}^i \sigma_{\beta} \right] \mathcal{L}_{\alpha}^{i\beta} \phi^\alpha \phi^\beta \]

\[ - \frac{(N + 2)^2}{4(N + 1)^2} \delta_{[m|n]}^i \mu \sigma_{j|m} + \frac{1}{4} (\sigma_{jm} \sigma_{|n} - \sigma_{jn} \sigma_{|m}) \phi^i + \frac{1}{4} (\sigma_{jm} \sigma_{an} - \sigma_{jn} \sigma_{am}) \phi^\alpha \phi^i \]

\[ + \frac{1}{4} \left[ \sigma_{jm} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) - \sigma_{jn} (\nu_m \phi^i + (-1)^p \mathcal{L}_{\alpha}^i \phi^\alpha) \right] \]

\[ - \frac{1}{4(N + 1)^2} \left[ \delta_{[m|n]}^i \sigma_{\alpha} + \delta_{[m|n]}^i \sigma_{\beta} \right] \mathcal{L}_{\alpha}^{i\beta} \phi^\alpha \phi^\beta + \frac{1}{4(N + 1)^2} \delta_{[m|n]}^i \sigma_{\alpha} + \delta_{[m|n]}^i \sigma_{\beta} \mathcal{L}_{\alpha}^{i\beta} \phi^\alpha \phi^\beta \right]. \tag{4.9} \]

## 5 Conclusion

We developed the methodology for obtaining associated invariants for mappings between non-symmetric affine connection spaces in this paper.
In the Section 2, it was continued the research from about derived invariants for geometrical mappings (see [17]). We obtained three invariants $W^{(1)}_{jmn}$, $W^{(2)}_{jmn}$, $W^{(4)}_{jmn}$ from the invariant $W^{(3)}_{jmn}$ (see the Theorem 2.1 Eqs. (2.2, 2.3, 2.4, 2.5)). In this section, we founded the auxiliary invariant for any mapping which will simplify some of the invariants obtained until now.

In the Section 3 we obtained the associated basic and associated derived invariants for a mapping $F: GA_N \rightarrow GA_N$.

In the Section 4 it were applied the results obtained in the Section 3 for finding invariants for almost geodesic mappings of the third type.

In the future researches, we will discuss about the space of invariants with respect to the results obtained in the Section 2. The results obtained in this paper will be generalized with respect to the transformation rules of torsion tensors under different mappings.

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