DIRAC OPERATORS ON QUANTUM PROJECTIVE SPACES

FRANCESCO D’ANDREA\textsuperscript{1} AND LUDWIK DĄBROWSKI\textsuperscript{2}

\textsuperscript{1}Dép. de Mathématique, U.C. Louvain, Louvain-La-Neuve, B-1348, Belgique
\textsuperscript{2}Scuola Internazionale Superiore di Studi Avanzati, Trieste, I-34127, Italia

Abstract. We construct a family of self-adjoint operators $D_N$, $N \in \mathbb{Z}$, which have compact resolvent and bounded commutators with the coordinate algebra of the quantum projective space $\mathbb{C}P^\ell_q$, for any $\ell \geq 2$ and $0 < q < 1$. They provide $0^+$-dimensional equivariant even spectral triples. If $\ell$ is odd and $N = \frac{1}{2}(\ell + 1)$, the spectral triple is real with KO-dimension $2\ell$ mod 8.

1. Introduction

In recent years several examples of noncommutative riemannian spin manifolds, described in terms of spectral triples [7, 8], have been constructed. Among them there are lowest dimensional quantum groups and their homogeneous spaces (see [9] for references), and $q$-deformed compact simply connected simple Lie groups [23]. An equivariant Dirac operator $D$ satisfying the crucial property of bounded commutators with the coordinates has been constructed on $q$-deformed irreducible flag manifolds in [22] (and shown to yield a finite dimensional differential calculus which coincides with the one of [18]). The other essential property of a spectral triple — that the resolvent of $D$ is compact — though expected, has not yet been demonstrated.

In this paper we analyse a class of $q$-deformed irreducible flag manifolds: namely quantum projective spaces $\mathbb{C}P^\ell_q$ for any $\ell \in \mathbb{N}$. We first give an explicit description of the antiholomorphic part of the differential calculus (the Dolbeault complex) and use it to construct a family (numbered by $N \in \mathbb{Z}$) of self-adjoint operators $D_N$ on $\mathcal{H}_N$, which have bounded commutators with the coordinate algebra $\mathcal{A}(\mathbb{C}P^\ell_q)$. Then, being $\mathbb{C}P^\ell_q$ a homogeneous $SU_q(\ell + 1)$-space, by preserving the equivariance at all steps and by relating $D_N$ to certain Casimir operator of $SU_q(\ell + 1)$, we are able to study the asymptotic behaviour of the spectrum of $D_N$. We find the exponentially growing spectrum, which guarantees the compact resolvent property of $D_N$. Thus $(\mathcal{A}(\mathbb{C}P^\ell_q), \mathcal{H}_N, D_N)$ are bona fide spectral triples on noncommutative homogeneous manifolds $\mathbb{C}P^\ell_q$. This generalizes the simplest case $\mathbb{C}P^1_q$ (that coincides with the standard Podlěš

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sphere) and the case $\mathbb{C}P_2^q$ (that is spin but not spin). The spectral triple with $N = \frac{1}{2}(\ell + 1)$, that exists if $\ell$ is odd, is the analogue of the real spectral triple in [10], and the one with $N = 0$ is the analogue of the spectral triple in [13].

It should be mentioned that the de Rham complex for $\mathbb{C}P^\ell_q$ (on the formal level) appears in [19] and (in local coordinates) in [6]. The relevant differential operator in local coordinates on $\mathbb{C}P^1_q$ appears already in [11], where, in particular, the relation with the $q$-derivative is mentioned.

In the classical limit ($q = 1$), when $\ell$ is odd and $N = \frac{1}{2}(\ell + 1)$, we obtain the canonical Dirac operator (for the Fubini-Study metric) acting on the space of square integrable spinors on $\mathbb{C}P^\ell$, while for $N = 0$ we get the Dolbeault-Dirac operator on the Hilbert space of antiholomorphic forms on $\mathbb{C}P^\ell$. Their spectra agree with the formula in [16], cf. also [4, 27, 1] if $\ell$ is odd.

The plan of the paper is the following. In Sec. 2 we briefly recall what is known about $\mathbb{C}P^1_q$, to prepare the discussion of the general case. In Sec. 3, we describe the basic properties of $U_q(\mathfrak{su}(\ell + 1))$ and — guided by the equivariance condition — introduce a $q$-deformation of the Grassmann algebra and of left invariant vector fields on $\mathbb{C}P^\ell$. The former will be relevant in the construction of the algebra of antiholomorphic forms, the latter in the definition of the exterior derivative. In Sec. 4, we describe the quantum $SU(\ell + 1)$ group and the action of $U_q(\mathfrak{su}(\ell + 1))$ on it, as well as the subalgebras of ‘functions’ on the quantum unitary sphere $S_{2\ell+1}^2$, and on the quantum complex projective space $\mathbb{C}P^\ell_q$. Sec. 5 is dedicated to the differential calculus, and Sec. 6 to spectral triples. General notions on spectral triples are recalled in Appendix A. Finally, in Appendix B we discuss the limit $q \to 1$ and compare our results with the literature.

2. THE ‘EXPOENTIAL’ DIRAC OPERATOR ON $\mathbb{C}P^1_q$

In this section, we briefly recall the geometry of the $q$-deformed $\mathbb{C}P^1$, cf. [10, 26]. We use the notations of [14, 12]: $0 < q < 1$ is a real deformation parameter, $K, K^{-1}, E, F$ are the generators of the Hopf $*$-algebra $U_q(\mathfrak{su}(2))$, $\alpha, \beta$ are the generators of the dual Hopf $*$-algebra $\mathcal{A}(SU_q(2))$, which is an $U_q(\mathfrak{su}(2))$-bimodule $*$-algebra for the left $\triangleright$ and right $\triangleleft$ canonical actions (cf App. A). For each $N \in \mathbb{Z}$, a left $U_q(\mathfrak{su}(2))$-module $\Gamma_N$ is given by

$$\Gamma_N = \{ a \in \mathcal{A}(SU_q(2)) \mid a \triangleleft K = q^{-N}a \} ,$$

and $\mathcal{A}(\mathbb{C}P^1_q) := \Gamma_0$ is a left $U_q(\mathfrak{su}(2))$-module $*$-algebra called the coordinate algebra of the standard Podleś sphere. For each $N \in \mathbb{Z}$, $\Gamma_N$ is also an $\mathcal{A}(\mathbb{C}P^1_q)$-bimodule. As a left $U_q(\mathfrak{su}(2))$-module, we have the following decomposition

$$\Gamma_N \simeq \bigoplus_{n \sim |N|, 2N} V_n ,$$

where $V_n$ is the spin $\frac{1}{2}n$ irreducible $*$-representation of $U_q(\mathfrak{su}(2))$. This is a unitary equivalence if we put on $\Gamma_N$ the inner product coming from the Haar state of $SU_q(2)$ (see [21]). The Casimir element

$$C_q = \left( \frac{q^{\frac{1}{2}}K - q^{-\frac{1}{2}}K^{-1}}{q - q^{-1}} \right)^2 + FE$$
has eigenvalues
\[ C_q|_{V_n} = \left[ \frac{n+1}{2} \right]^2 \cdot id \]
with multiplicity \( \dim V_n = n + 1 \). Here
\[ [x] := \frac{q^x - q^{-x}}{q - q^{-1}} \]
is the \( q \)-analogue of \( x \).

Antiholomorphic 0 and 1-forms are \( \Omega^0 = \mathcal{A}(\mathbb{C}P^1_q) \) and \( \Omega^1 = \Gamma_{-2} \), with the Dolbeault operator and its Hermitian conjugate given by
\[ \bar{\partial} : \Omega^0 \to \Omega^1, \quad a \mapsto \mathcal{L}_F a, \]
\[ \bar{\partial}^\dagger : \Omega^1 \to \Omega^0, \quad a \mapsto \mathcal{L}_E a, \]
where
\[ \mathcal{L}_h a := a \triangleleft S^{-1}(h), \quad \forall \ a \in \mathcal{A}(SU_q(2)), \ h \in U_q(\mathfrak{su}(2)), \]
and \( S \) is the antipode of \( U_q(\mathfrak{su}(2)) \). It is shown already in [11], by using local ‘coordinates’, that \( \bar{\partial} \) is related to the well-known \( q \)-derivative operator; cf. (4.19) therein.

The Dolbeault-Dirac operator \( D \) on \( \Omega^0 \oplus \Omega^1 \) is given by
\[ D(\omega_0, \omega_1) := (\bar{\partial}^\dagger \omega_1, \bar{\partial}\omega_0) = -(q^{-1} \omega_1 \triangleleft E, q \omega_0 \triangleleft F) \]
and satisfies \( D^2 \omega = \omega \triangleleft (C_q - \left[ \frac{1}{2} \right]^2) \). Being an even spectral triple, the spectrum of \( D \) must be symmetric with respect to the origin. It is computed from the spectrum of \( C_q \), by using the above decomposition of \( \Gamma_N \) and the fact that for central elements the left and right canonical actions are equal. It immediately follows that \( D \) has a 1-dimensional kernel, and its non-zero eigenvalues are \( \pm \sqrt{\left[ k \right]\left[ k + 1 \right]} \) with multiplicity \( 2k + 1 \), for all \( k \in \mathbb{N} + 1 \).

To get the Dirac operator (for the Fubini-Study metric) we must tensor \( \Omega^0 \oplus \Omega^1 \) with the square root of the canonical bundle of holomorphic 1-forms, i.e. with \( \Gamma_1 \). We get the space
\[ (\Omega^0 \oplus \Omega^1) \otimes \mathcal{A}(\mathbb{C}P^1_q) \Gamma_1 \simeq \Gamma_1 \oplus \Gamma_{-1}. \]
The Dirac operator \( \mathcal{D} \) is obtained by twisting \( D \) with the Grassmannian connection of \( \Gamma_{-1} \). This goes as follows. Given any \( \mathcal{A}(\mathbb{C}P^1_q) \)-bimodule \( \mathfrak{M} \subset \mathcal{A}(SU_q(2)) \), the map
\[ \phi : \mathfrak{M} \otimes \mathcal{A}(\mathbb{C}P^1_q) \Gamma_1 \to \mathfrak{M}^2 p_B, \quad a \mapsto a(\alpha, \beta), \]
\[ \phi^{-1} : \mathfrak{M}^2 p_B \to \mathfrak{M} \otimes \mathcal{A}(\mathbb{C}P^1_q) \Gamma_1, \quad (a_1, a_2) \mapsto a_1 \alpha^* + a_2 \beta^*, \]
is an isomorphism of left \( \mathcal{A}(\mathbb{C}P^1_q) \)-modules, where \( p_B \) is the \( q \)-analogue of Bott projection
\[ p_B := (\alpha, \beta)^\dagger (\alpha, \beta). \]

The Dirac operator \( \mathcal{D} \) on \( \Gamma_1 \oplus \Gamma_{-1} \) is
\[ \mathcal{D} := \phi^{-1}(D \otimes 1_2) \phi, \]
where \( \phi \) in this case sends \( \Gamma_1 \oplus \Gamma_{-1} \simeq (\Omega^0 \oplus \Omega^1) \otimes A(\mathbb{CP}_1^+) \) \( \Gamma_1 \) to \( (\Omega^0 \oplus \Omega^1)^2 \mathbb{P}_B \). We compute \( \mathcal{D} \) explicitly. For \( v_+ \in \Gamma_1 \) and \( v_- \in \Gamma_{-1} \) we have
\[
\mathcal{D}(v_+, v_-) = \left( q^{-1}v_-(\alpha, \beta) \triangleright E(\frac{q}{\beta}) , q v_+(\alpha, \beta) \triangleright F(\frac{q}{\beta}) \right).
\]
But \( (\alpha, \beta) \triangleright E = 0 \), \( (\alpha, \beta)^\dagger \triangleright F = 0 \) and thus
\[
0 = 1 \triangleright F = (\alpha, \beta)(\frac{q}{\beta}) \triangleright F = q^{-\frac{1}{2}}(\alpha, \beta) \triangleright F(\frac{q}{\beta}) .
\]
Thus \( F, E \) acts non-trivially only on the \( v_+, v_- \) part and using \( (\alpha, \beta) \triangleright K(\alpha, \beta)^\dagger = q^{\frac{1}{2}} \) we get
\[
\mathcal{D}(v_+, v_-) = -q^{\frac{1}{2}}(q^{-1}v_- \triangleright E, q v_+ \triangleright F).
\]
Hence, \( \mathcal{D} \) has an expression similar to \( q^{\frac{1}{2}}D \), although living on a different Hilbert space. This is exactly the Dirac operator of [10] (but for the factor \( q^{\frac{1}{2}} \)), as proved in [26]. Since \( \mathcal{D}^2(v_+, v_-) = (v_+, v_-) \triangleright qC_q \), it immediately follows that \( \mathcal{D} \) has eigenvalues \( \pm q^{\frac{1}{2}}[k] \) with multiplicity \( 2k \), for all \( k \in \mathbb{N} + 1 \).

A crucial difference between \( D \) and \( \mathcal{D} \) is that the latter admits a real structure \( J \). This is the operator
\[
J(v_+, v_-) := K \triangleright (v_-^*, -v_+^*) \triangleright K .
\]
One can show that \( JK\triangleright \) is left \( U_q(\mathfrak{su}(2)) \)-covariant, \( J^2 = -1 \), \( J \) is an isometry, \( J\mathcal{D} = \mathcal{D}J \) and the commutant and first order conditions are satisfied (cf. [26]), meaning that we have a real spectral triple with KO-dimension 2.

The analogous of the operator \( D \) for \( \mathbb{CP}_q^2 \) has been constructed in [13].

3. Preliminaries about \( U_q(\mathfrak{su}(\ell + 1)) \)

For \( 0 < q < 1 \), we denote \( U_q(\mathfrak{su}(\ell + 1)) \) the ‘compact’ real form of the Hopf algebra denoted \( U_q(\mathfrak{sl}(\ell + 1, \mathbb{C})) \) in Sec. 6.1.2 of [21]. As a \( * \)-algebra it is generated by \( \{ K_i = K_i^*, K_i^{-1}, E_i, F_i = E_i^* \}_{i=1,2,...,\ell} \) with relations
\[
[K_i, K_j] = 0 ,
K_i E_i K_i^{-1} = q E_i ,
K_i E_j K_i^{-1} = q^{-1/2} E_j \quad \text{if } |i - j| = 1 ,
K_i E_j K_i^{-1} = E_j \quad \text{if } |i - j| > 1 ,
[E_i, F_j] = \delta_{ij} \frac{K_i^2 - K_i^{-2}}{q - q^{-1}} ,
E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } |i - j| = 1 ,
[E_i, E_j] = 0 \quad \text{if } |i - j| > 1 ,
\]
plus conjugated relations. If we define the \( q \)-commutator as
\[
[a, b]_q := ab - q^{-1}ba ,
\]
the second-last relation can be rewritten in two equivalent forms
\[
[E_i, [E_j, E_i]_q] = 0 \quad \text{or} \quad [[E_i, E_j]_q, E_i]_q = 0 ,
\]
for any $|i - j| = 1$. Coproduct, counit and antipode are given by

$$
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \\
e(\Delta(K_i)) = 1, \quad e(E_i) = 0, \quad S(K_i) = K_i^{-1}, \quad S(E_i) = -qE_i.
$$

Using self-evident notation, we call $U_q(\mathfrak{su}(\ell))$ the Hopf $*$-subalgebra of $U_q(\mathfrak{su}(\ell + 1))$ generated by the elements $\{K_i = K_i^+, K_i^{-1}, E_i, F_i = E_i^*\}_{i=1,2,...,\ell-1}$. Its commutant is the Hopf $*$-subalgebra $U_q(\mathfrak{u}(1)) \subset U_q(\mathfrak{su}(\ell + 1))$ generated by the element $K_1 K_2^* ... K_\ell$ and its inverse. This is a positive operator in all representations we consider. Its positive root of order $\ell$, and its inverse will serve to define a Casimir operator. We enlarge the algebra $U_q(\mathfrak{su}(\ell + 1))$ accordingly.

The element

$$
K_{2\rho} = (K_1^\rho K_2^2(\ell-1) ... K_j^{j(\ell-j+1)} ... K_\ell^j)^2,
$$

(3.2)

implements the square of the antipode:

$$
S^2(h) = K_{2\rho} h K_{2\rho}^{-1}, \quad \forall h \in U_q(\mathfrak{su}(\ell + 1)),
$$

(3.3)

as one can easily check on generators of the Hopf algebra. By [21, Sec. 11.3.4], Ex. 9, we see that the pairing $\langle K_{2\rho}, a \rangle = f_1(a)$ is the character giving the modular automorphism (cf. (11.36) in [21]). The expression $f_1(a,f_1)$ in [21] becomes $K_{2\rho} \odot a \odot K_{2\rho}$ in our notations, and by (11.26) of [21] the Haar state $\varphi : \mathcal{A}(SU_q(\ell + 1)) \to \mathbb{C}$ satisfies

$$
\varphi(ab) = \varphi(b K_{2\rho} \odot a \odot K_{2\rho}),
$$

(3.4)

for all $a, b \in \mathcal{A}(SU_q(\ell + 1))$.

We are interested in highest weight $*$-representations of $U_q(\mathfrak{su}(\ell + 1))$ such that $K_j$ are represented by positive operators. Such irreducible $*$-representations are labeled by $\ell$ non-negative integers $n_1, \ldots, n_\ell$. For $n = (n_1, \ldots, n_\ell) \in \mathbb{N}_\ell$ we denote by $V_n$ the vector space carrying the representation $\rho_n$ with highest weight $n$; the highest weight vector $v$ is annihilated by all the $E_j$'s and satisfies $\rho_n(K_i)v = q^{n_i/2}v$, $i = 1, \ldots, \ell$.

### 3.1. The Casimir operator

Casimir operators for $U_q(\mathfrak{su}(\ell + 1))$ are discussed in [2, 5]. We repeat here the construction from scratch, adapting their notations to ours, to make the paper self-contained. Moreover, some formulæ in the proofs will be useful later on.

For any $j, k \in \{1, \ldots, \ell\}$, with $j < k$, we define the following elements of $U_q(\mathfrak{su}(\ell + 1))$

$$
M_{jk} := [E_j, [E_{j+1}, [E_{j+2}, \ldots [E_{k-1}, E_k]_q]]_q]_q.
$$

That is, if we set $M_{ii} = E_i$ the $M_{jk}$'s are obtained by iteration using

$$
M_{jk} = [E_j, M_{j+1,k}]_q.
$$

(3.5)

(For $q = 1$, $M_{jk}$ and $M_{jk}^*$, together with the Cartan generators, form a basis of the Lie algebra $\mathfrak{su}(\ell + 1)$.) We need the following Lemmas.
Lemma 3.1. The following equalities hold:

\[ [F_i, M_{jk}] = \delta_{ij} M_{j+1,k} K_i^2 - \delta_{ik} K_i^2 M_{j,k-1} - \delta_{ij} \delta_{ik} \frac{K_i^2 - K_q^2}{q-q^{-1}}, \]
\[ (3.6a) \]
\[ [E_i, M_{jk}] = \delta_{ik} M_{j,k-1}^* K_i^2 - \delta_{ij} K_i^2 M_{j+1,k}^* + \delta_{ij} \delta_{ik} \frac{K_i^2 - K_q^2}{q-q^{-1}}, \]
\[ (3.6b) \]

where we set \( M_{jk} := 0 \) when the labels are out of the range (i.e. when \( j > k \)).

Proof. Since (3.6a) implies (3.6b) by adjunction, we have to prove only the former. Since for \( j = k \) (or \( j > k \)) this is trivial, we assume \( j < k \). We notice that \( F_i \) commutes with all \( E_j \)'s but for \( i = j \), and \( K_i \) commutes with all \( E_j \)'s but for \( i = j \pm 1 \). In particular, this means that \( [F_i, M_{jk}] = 0 \) if \( i < j \). For \( i = j \) (so \( i < k \)) using (3.5) we get

\[ [F_i, M_{ik}] = [[F_i, E_i], M_{i+1,k}]_q = [-\frac{K_i^2 - K_q^2}{q-q^{-1}}, M_{i+1,k}]_q = M_{i+1,k} K_i^{-2}, \]

which agrees with (3.6a). If \( i = j + 1 \) and \( i < k \) using again (3.5) and the identity just proved we get

\[ [F_i, M_{i-1,k}] = [E_{i-1}, [F_i, M_{ik}]_q = [E_{i-1}, M_{i+1,k} K_i^{-2}]_q = [E_{i-1}, M_{i+1,k}] K_i^{-2} = 0; \]

this is zero since \( E_{i-1} \) commutes with all \( E_j \) with \( j \geq i + 1 \). Using the equation just proved (which is true for \( i < k \)), by (3.5) we prove by induction on \( j \) that

\[ [F_i, M_{jk}] = [E_j, [F_i, M_{j+1,k}]_q = 0, \]

for all \( j < i < k \). If \( j = i - 1 \) and \( k = i \) we have

\[ [F_i, M_{i-1,i}] = [E_{i-1}, [F_i, E_i]]_q = [E_{i-1}, -\frac{K_i^2 - K_q^2}{q-q^{-1}}]_q = -K_i^2 E_{i-1} = -K_i^2 M_{i-1,i-1}, \]

and again by induction on \( j \) we prove that

\[ [F_i, M_{ji}] = [E_j, [F_i, M_{j+1,i}]]_q = [E_j, -K_i^2 M_{j+1,i-1}]_q = -K_i^2 [E_j, M_{j+1,i-1}]_q = -K_i^2 M_{j,i-1}, \]

for all \( j < i - 1 \). With this (3.6a) is proved for any \( i \leq k \). For \( i > k \) it holds trivially. This concludes the proof.

For all \( j, k \in \{1, \ldots, \ell \} \) let

\[ N_{jk} := (K_i K_{j+1} \ldots K_{\ell}) \cdot (K_{k+1} K_{k+2} \ldots K_{\ell}) \cdot \hat{K}^{-1} \]
\[ (3.7) \]

and notice that

\[ N_{jk}^2 E_i N_{jk}^{-2} = q^{-\delta_{i,j-1} + \delta_{i,j} - \delta_{i,k} + \delta_{i,k+1}} E_i \]
\[ (3.8) \]

for all \( i, j, k \in \{1, \ldots, \ell \} \).

Lemma 3.2. The following equality holds

\[ [E_i, N_{jk}^2 M_{jk}] = \delta_{i,j-1} q N_{jk}^2 M_{ik} - \delta_{i,k+1} N_{jk}^2 M_{ji}, \]
\[ (3.9) \]

where we set \( M_{jk} := 0 \) when the labels are out of the range.
Proof. We assume $j \leq k$, as for $j > k$ the claim is a trivial $0 = 0$. First we notice that if $i < j - 1$ or $i > k + 1$ we have $[E_i, M_{jk}] = 0$ since $E_i$ commutes with any $E_n$ with $|n - i| > 1$; also by (3.8) we have $[E_i, N_{jk}] = 0$, and this proves (3.9) in the cases $i < j - 1$ and $i > k + 1$.

By (3.5) and (3.8) we have the recursive definition of $N_{jk}^2 M_{jk}$,

$$N_{jk}^2 M_{jk} = q^{-1} K_1^2 [E_i, N_{i+1,k}^2 M_{i+1,k}] ,$$

which gives (3.9) in the case $i = j - 1$. On the other hand if $k = i - 1$, since $[E_i, E_i] = 0$ for $i > l + 1$ we have

$$[M_{j,i-1}, E_i]_q = [[E_j, [E_{j+1}, \ldots, [E_{i-2}, E_{i-1} q \ldots q]_q, E_i]_q$$

$$= [E_j, [E_{j+1}, \ldots, [E_{i-2}, E_{i-1} q \ldots q]_q \ldots q]_q = M_{j,i} ,$$

and this together with (3.8) gives (for $j \leq i - 1$)

$$[N_{j,i-1}^2 M_{j,i-1}, E_i] = N_{j,i-1}^2 [M_{j,i-1}, E_i]_q = N_{j,i-1}^2 M_{j,i}$$

which is (3.9) in the case $i = k + 1$. It remains to consider $j \leq i \leq k$, which by (3.8) is equivalent to the following set of equations

$$[E_i, M_{ji}]_q = 0 \quad \text{if } j < i , \quad (3.10a)$$

$$[M_{ik}, E_i]_q = 0 \quad \text{if } i < k , \quad (3.10b)$$

$$[E_i, M_{jk}] = 0 \quad \text{if } i = j = k \text{ or } j < i < k . \quad (3.10c)$$

The case $i = j = k$ is trivial. Furthermore by Serre's relations

$$E_i M_{i-1,i} = E_i [E_{i-1}, E_i]_q = q^{-1} [E_{i-1}, E_i]_q E_i = q^{-1} M_{i-1,i} E_i ,$$

$$E_i M_{i,i+1} = E_i [E_i, E_{i+1}]_q = q [E_i, E_{i+1}]_q E_i = q M_{i,i+1} E_i .$$

Then, for any $j + 1 < i = k$ we prove by induction that

$$E_i M_{j,i} = [E_j, E_i M_{j+1,i}]_q = q^{-1} [E_j, M_{j+1,i}]_q E_i = q^{-1} M_{j+1,i} E_i$$

(3.11a)

which is (3.10a) and for any $j = i < k - 1$ that

$$E_i M_{i,k} = [M_{i,k-1}, E_k]_q = [E_i M_{i,k-1}, E_k]_q = q [M_{i,k-1}, E_i, E_k]_q = q M_{i,k} E_i ,$$

(3.11b)

which is (3.10b).

Consider now $j < n < k$ and notice that for any such $i, j, k$ we can write

$$M_{jk} = [M_{jn}, M_{n+1,k}]_q .$$

Using this equation in the cases $n = i, i - 1$, together with (3.11) and (3.5) we have

$$q E_i M_{jk} - q^{-1} M_{jk} E_i = [M_{ji}, [E_i, M_{i+1,k}]_q] = [M_{ji}, M_{ik}] ,$$

$$q M_{jk} E_i - q^{-1} E_i M_{jk} = [[M_{ji-1}, E_i]_q, M_{ik}] = [M_{ji}, M_{ik}] .$$

The difference of the two lines has to be zero, so

$$0 = q E_i M_{jk} - q^{-1} M_{jk} E_i - q M_{jk} E_i + q^{-1} E_i M_{jk} = [2] [E_i, M_{jk}] .$$

This concludes the proof. \qed
A Casimir operator $C_q$ for $U_q(\mathfrak{su}(\ell + 1))$ is given by the formula

$$C_q = \sum_{i=1}^{\ell} q^{\ell-2} M_{jk}^* N_{jk}^2 + q^{-\ell} \frac{\ell}{(q-q^{-1})^2} \hat{K}^{-2} + \sum_{1 \leq j \leq k \leq \ell} q^{\ell-2j} M_{jk}^* N_{jk}^2 M_{jk} - \frac{[\ell+1]}{(q-q^{-1})^2}. \quad (3.12)$$

**Proposition 3.3.** The operator $C_q$ is real ($C_q = C_q^*$) and central. In the irreducible representation $\rho_n : U_q(\mathfrak{su}(\ell + 1)) \to \text{End}(V_n)$ with highest weight $n = (n_1, \ldots, n_\ell)$, it is proportional to the identity with proportionality constant:

$$\rho_n(C_q) = \frac{1}{2} \sum_{i=1}^{\ell+1} \left[ \sum_{j=}^{\ell+1} q^{(\ell+1-j)n_j} + i - \frac{\ell+2}{2} \right]^2 + \frac{\ell + 1 - [\ell + 1]}{(q-q^{-1})^2}. \quad (3.13)$$

**Proof.** The properties $C_q = C_q^*$ and $K_i C_q = C_q K_i$ are evident. If further

$$[E_i, C_q] = 0 \quad \forall \ i = 1, \ldots, \ell,$$

then by adjunction $C_q$ commutes with all the generators of $U_q(\mathfrak{su}(\ell + 1))$ and so it is central.

Using Lemmas 3.1-3.2 we get

$$[E_i, \sum_{j \leq k} q^{\ell-2j} M_{jk}^* N_{jk}^2 M_{jk}] = \sum_{j \leq k} q^{\ell-2j} [E_i, M_{jk}^* N_{jk}^2 M_{jk}] + \sum_{j \leq k} q^{\ell-2j} M_{jk}^*[E_i, N_{jk}^2 M_{jk}]$$

$$= \sum_{j \leq i} q^{\ell-2j} M_{jk}^* K_{i}^2 N_{jk}^2 M_{jk} - \sum_{k \geq i} q^{\ell-2j} K_{i}^{-2} M_{i+1,k}^* N_{jk}^2 M_{jk}$$

$$+ \sum_{k \geq i+1} q^{\ell-2i} M_{i+1,k}^* N_{i+1,k}^2 M_{i,k} - \sum_{j \leq i} q^{\ell-2j} M_{ji}^* N_{j,i}^2 M_{ji} + \frac{K_{i}^2 - K_{i}^{-2}}{q-q^{-1}} q^{\ell-2i} N_{i}^2 E_i.$$

Since $K_{i}^2 M_{i+1,k}^* = q M_{i+1,k}^* K_{i}^2$, $N_{i,k} = K_{i} N_{i+1,k}$ and $N_{j,i} = K_{i} N_{j,i}$, all the terms cancel but the framed one. Using (3.8) we get

$$[E_i, \sum_{j=1}^{\ell} q^{\ell-2j} N_{j,i}^2] = \sum_{j=1}^{\ell} q^{\ell+2j} (q^{2(\delta_{i,j,-1})} - 1) N_{j,i}^2 E_i$$

$$= \begin{cases} 
q^{\ell+2} N_{j,i}^2 E_i & \text{if } i < \ell, \\
q^{-2} N_{j,i}^2 E_i & \text{if } i = \ell.
\end{cases}$$

Thus if $i < \ell$ last commutator cancel with the framed equation and we get $[E_i, C_q] = [E_i, A]$, with $A := q^{-\ell}(q-q^{-1})^{-2} \hat{K}^{-2}$, while if $i = \ell$ we have

$$[E_i, C_q] = [E_i, A] - K_{i}^{-2}(q-q^{-1})^{-1} q^{-\ell+1} N_{i}^2 E_i.$$

Observe that

$$[E_i, K_{i}^{-2}] = \delta_{i\ell} q(q-q^{-1}) K_{i}^{-2} E_i,$$

which implies that $[E_i, C_q] = 0$ for all $i$. This concludes the first part of the proof.

By Schur’s Lemma $\rho_n(C_q)$ is proportional to the identity. We can compute the proportionality constant by applying it to the highest weight vector $v_n$. By construction $\rho_n(M_{jk})v_n = 0$,
being annihilated by all $E_i$'s, and $\rho_n(K_i)v_n = q^{n_i/2}v_n$, thus
\[
\rho_n(C_q) = \sum_{i=1}^{\ell+1} q^{i-2-2i} - \frac{2}{q-1} \left( \sum_{j=i}^{\ell} n_j - \sum_{j=\ell+1-j}^{\ell} (\ell+1-j)n_j \right) - \frac{[\ell+1]}{(q-q^{-1})^2},
\]
times the identity operator on $V_n$. If we call $i' = \ell + 2 - i$ and $j' = \ell + 1 - j$, last equation can be rewritten as
\[
\rho_n(C_q) = \sum_{i'=1}^{\ell+1} q^{i'-\ell-2} - \frac{2}{q-1} \left( \sum_{j'=i'}^{\ell} n_{j'} - \sum_{j'=\ell+1-j'}^{\ell} j'n_{j'} \right) - \frac{[\ell+1]}{(q-q^{-1})^2},
\]
and the sum of last two equations gives
\[
2\rho_n(C_q) = \sum_{i=1}^{\ell+1} q^{i-2-2i} - \frac{2}{q-1} \left( \sum_{j=i}^{\ell} n_j - \sum_{j=\ell+1-j}^{\ell} (\ell+1-j)n_j \right) + q^{-\ell-2+2i} - \frac{2}{q-1} \left( \sum_{j=i}^{\ell} n_j - \sum_{j=\ell+1-j}^{\ell} (\ell+1-j)n_j \right) - \frac{2}{(q-q^{-1})^2}
\]
\[
= \sum_{i=1}^{\ell+1} \left[ -\ell^2 + i + \sum_{j=1}^{\ell} jn_j - \sum_{j=1}^{\ell} (\ell+1-j)n_j \right] + 2 \frac{\ell + 1 - [\ell+1]}{(q-q^{-1})^2}.
\]
This concludes the proof.

From Weyl’s character formula [20] we know that the multiplicity of the eigenvalue (3.13) is
\[
\dim V_n = \frac{\prod_{1 \leq r \leq \ell} (s - r + 1 + \sum_{i=r}^{n} n_i)}{\prod_{r=1}^{\ell} r!}.
\]

We shall need later certain class of $V_n$, with $n = (n_1,0,0,\ldots,0,n_\ell) + \xi_k$, where $\xi_k$ is the $\ell$-tuple with $k$-th component equal to one and all the others equal to zero. That is, $n$ has components $n_i = n_1 \delta_{i,1} + n_2 \delta_{i,\ell} + \delta_{i,k}$, for $i = 1,\ldots, \ell$ and $k$ is a fixed number in $\{1,2,\ldots, \ell\}$.

**Lemma 3.4.** For any $1 \leq k \leq \ell$, the dimension of the irreducible representation $V_n$ with highest weight $n_i = n_1 \delta_{i,1} + n_2 \delta_{i,\ell} + \delta_{i,k}$ is
\[
\dim V_n = \frac{k(n_1 + n_\ell + \ell + 1)}{(n_1 + k)(n_\ell + \ell + 1 - k)} \left( \begin{array}{c} n_1 + \ell \\ \ell \end{array} \right) \left( \begin{array}{c} n_\ell + \ell \\ \ell \end{array} \right) \left( \begin{array}{c} \ell \\ k \end{array} \right).
\]

The eigenvalue $\lambda_{n_1,n_\ell,N}$ of $C_q$ in such a representation is given by
\[
2\lambda_{n_1,n_\ell,N} = [n_1 + k][n_1 - \frac{2N}{\ell+1} + \ell + 2 - k] + [n_\ell][n_\ell + \frac{2N}{\ell+1} + \ell] + [\ell+1]\frac{N}{\ell+1}^2,
\]
where we call $N := n_1 - n_\ell + k$.

**Proof.** We divide the product in the numerator of (3.15) in the following cases
\[
\{1 \leq r \leq s \leq \ell\} = \{1 < r \leq s < k\} \cup \{k < r \leq s < \ell\} \cup \{1 < r \leq k \leq s < \ell\} \cup \{1 < r \leq \ell\} \cup \{1 < r \leq s = \ell\} \cup \{r = 1, s = \ell\}.
\]
With a simple computation we get
\[
\prod_{1 < r \leq k} \left( s - r + 1 + \sum_{i=r}^{s} n_i \right) = k^{-2} r!, \quad \prod_{k < r \leq \ell} \left( s - r + 1 + \sum_{i=r}^{s} n_i \right) = \prod_{r=1}^{\ell-k-1} r!,
\]
\[
\prod_{1 < r \leq k \leq \ell} \left( s - r + 1 + \sum_{i=r}^{s} n_i \right) = \frac{r!}{\prod_{i=1}^{r-1} i!}.
\]
and plugging these in Weyl’s character formula we get (3.16).

The eigenvalue is given by (3.13):
\[
2\lambda_{n_1,n_\ell,N} = \left[ n_1 - \frac{1}{\ell+1}(n_1 - n_\ell + k) + \frac{\ell + 2}{2} \right]^2 + \sum_{i=1}^{\ell} \left[ \frac{1}{\ell+1}(n_1 - n_\ell + k) + i - \frac{\ell + 2}{2} \right]^2 + 2 \frac{\ell + 1 - [\ell + 1]}{(q - q^{-1})^2}
\]
\[
= \left[ n_1 - \frac{1}{\ell+1}(n_1 - n_\ell + k) + \frac{\ell + 2}{2} \right]^2 - \left[ \frac{1}{\ell+1}(n_1 - n_\ell + k) + k - \frac{\ell + 2}{2} \right]^2
\]
\[
+ \left[ n_\ell + \frac{1}{\ell+1}(n_1 - n_\ell + k) + \frac{\ell}{2} \right]^2 - \left[ \frac{1}{\ell+1}(n_1 - n_\ell + k) + \frac{\ell}{2} \right]^2
\]
\[
+ \sum_{i=1}^{\ell+1} \left[ \frac{1}{\ell+1}(n_1 - n_\ell + k) + i - \frac{\ell + 2}{2} \right]^2 + 2 \frac{\ell + 1 - [\ell + 1]}{(q - q^{-1})^2}.
\]
But for all \( t \),
\[
\sum_{i=1}^{\ell+1} \left[ i - \frac{\ell + 2}{2} + t \right]^2 = \frac{(q^{2t} + q^{-2t})[\ell + 1] - 2(\ell + 1)}{(q - q^{-1})^2} = [\ell + 1][t]^2 - 2 \frac{\ell + 1 - [\ell + 1]}{(q - q^{-1})^2}.
\]
Thus, with the substitution \( n_1 - n_\ell + k = N \), we get
\[
2\lambda_{n_1,n_\ell,N} = \left[ n_1 - \frac{N}{\ell+1} + \frac{\ell + 2}{2} \right]^2 - \left[ \frac{N}{\ell+1} + k - \frac{\ell + 2}{2} \right]^2
\]
\[
+ \left[ n_\ell + \frac{N}{\ell+1} + \frac{\ell}{2} \right]^2 - \left[ \frac{N}{\ell+1} + \frac{\ell}{2} \right]^2
\]
\[
+ [\ell + 1]\left[ \frac{N}{\ell+1} \right]^2.
\]
Using in the first two lines the algebraic identity \(|x|^2 - |y|^2 = |x-y||x+y|\), we prove (3.17).

**Lemma 3.5.** A Casimir operator \( C_q^\ell \) for \( U_q(\mathfrak{su}(\ell)) \) is defined by
\[
\hat{K'}^2 C_q^\ell = \sum_{i=1}^{\ell} \frac{q^{\ell+i-2}}{(q^{-1}-q)^2} N_{i-i-1}^2 + \sum_{1 \leq j \leq k \leq \ell-1} q^{\ell-2j} M_{j,k} N_{j,k} M_{j,k} - \frac{[\ell]}{(q^{-1}-q)^2} \hat{K'}^2.
\]
Its spectrum is given by (3.13), with a replacement \( \ell \rightarrow \ell - 1 \).

**Proof.** Take (3.12), replace \( \ell \) with \( \ell - 1 \), \( K' \) with \( K = (K_1 K_2 \ldots K_{\ell-1})^2 = \hat{K'}^{\ell+1} \hat{K'}^{\ell-2} \),
\[
N_{j,k} = (K_j K_{j+1} \ldots K_{\ell-1}) \cdot (K_{k+1} K_{k+2} \ldots K_{\ell-1}) \cdot \hat{K'}^{-1} = \hat{K'}^{-\frac{1}{2}} N_{j,k}.
\]
$M'_{jk}$ with $M'_{jk} \equiv M''_{jk}$ (for $k \leq \ell - 1$), obtaining the Casimir
\[
\mathcal{C}'_q = \sum_{i=1}^{\ell-1} q^{i+1+2i} N_{i,i-1} + \frac{q^{i+1}}{(q-q^{-1})^2} \hat{K}^{-2} + \sum_{1 \leq j \leq k \leq \ell-1} q^{i-2j} M'_{jk} N_{j,k} M'_{jk} - \frac{|e|}{(q-q^{-1})^2}
\]
\[
= \hat{K}^{-2} \left( \sum_{i=1}^{\ell-1} q^{i+1+2i} N_{i,i-1} + \frac{q^{i+1}}{(q-q^{-1})^2} \hat{K}^{-2} + \sum_{1 \leq j \leq k \leq \ell-1} q^{i-2j} M'_{jk} N_{j,k} M'_{jk} \right) - \frac{|e|}{(q-q^{-1})^2}.
\]
Since $K'_{\ell} \hat{K}^{-2} = N_{\ell,\ell-1}$, the last equation is exactly (3.18).

The relation between the Casimir (3.12) of $U_q(\mathfrak{su}(\ell + 1))$ and the Casimir (3.18) of $U_q(\mathfrak{su}(\ell))$ is
\[
\mathcal{C}_q = q^{\hat{K}^{-2}} \left( \mathcal{C}'_q + \frac{|e|}{(q-q^{-1})^2} \hat{K}^{-2} + \sum_{i=1}^{\ell} q^{i+1+2i} M'_{i,i} N_{i,i-1} M'_{i,i-1} - \frac{|e|}{(q-q^{-1})^2} \right). \tag{3.19}
\]

3.2. The quantum Grassmann algebra. Irreducible representations of $U_q(\mathfrak{su}(\ell))$ are labeled by $n = (n_1, \ldots, n_{\ell-1})$, with $q^{\ell n_i}$ the eigenvalue of $K_i$ corresponding to the highest weight vector (the vector that is annihilated by all $E_i$'s, $i = 1, \ldots, \ell - 1$). These are all the finite-dimensional irreducible highest weight representations such that $K_i$ are positive operators (thus, having a well-defined $q \to 1$ limit). Before discussing representations, we need some preliminaries.

For a fixed $k = 1, \ldots, \ell - 1$, we denote $\Lambda_k$ the following set of multi-indices (the set of $k$-partitions of $\ell$):
\[
\Lambda_k := \left\{ \underline{i} = (i_1, i_2, \ldots, i_k) \in \mathbb{Z}^k \mid 1 \leq i_1 < i_2 < \ldots < i_k \leq \ell \right\}. \tag{3.20}
\]
Let
\[
j \# \underline{i} := \sum_{h=1}^{k} (\delta_{i_h,j} - \delta_{i_h,j+1}) \tag{3.21}
\]
be the number of times $j$ appears in the string $\underline{i}$ minus the number of times $j + 1$ appears in it. Due to the inequality in (3.20), this is either 0 or ±1.

Given two multi-indices $\underline{i}' \in \Lambda_{k'}$ and $\underline{i}'' \in \Lambda_{k''}$ with empty intersection, $\underline{i}' \cap \underline{i}'' = \emptyset$ (that is $i'_r \neq i''_r \forall r, s$), we denote $\underline{i}' \cup \underline{i}'' \in \Lambda_{k'+k''}$ the ordered set with elements $\{i'_r, i''_s\}_{r,s}$. In this case, for all $j$,
\[
j \# (\underline{i}' \cup \underline{i}'') = j \# \underline{i}' + j \# \underline{i}''. \tag{3.22}
\]
For any $\underline{i} \in \Lambda_k$ there is only one $(\ell - k)$-tuple $\underline{i}^c = (i^c_1, \ldots, i^c_{\ell-k}) \in \Lambda_{\ell-k}$, such that $i_r \neq i^c_s \forall r, s$. This is given by $\underline{i}^c = (1, 2, \ldots, \ell) \smallsetminus \underline{i}$ (as ordered sets), and since
\[
(\underline{i}^c)^c = \underline{i}, \tag{3.23}
\]
the map $\Lambda_k \to \Lambda_{\ell-k}$, $\underline{i} \mapsto \underline{i}^c$ is a bijection (in fact, an involution). As $\underline{i} \cap \underline{i}^c = \emptyset$ and $\underline{i} \cup \underline{i}^c = \{1, 2, \ldots, \ell\}$, by (3.22) we have
\[
j \# \underline{i} + j \# \underline{i}^c = 0. \tag{3.24}
\]
If $j \# \underline{i} = +1$ it means that there is $r$ such that $i_r = j$ and $i_{r+1} > j + 1$; in this case, we denote
\[
\underline{i}^{j+} = (i_1, \ldots, i_{r-1}, j+1, i_{r+1}, \ldots, i_k). \tag{3.25a}
\]
If \( j \# i = -1 \) it means that there is \( r \) such that \( i_r = j + 1 \) and \( i_{r-1} < j \); in this case, we denote
\[
\dot{i}^{j,-} = (i_1, \ldots, i_{r-1}, j, i_{r+1}, \ldots, i_k).
\]
(3.25b)

Both \( \dot{i}^{j,+} \) and \( \dot{i}^{j,-} \) satisfy the inequality in (3.20), so \( \dot{i}^{j,+}, \dot{i}^{j,-} \in \Lambda_k \).

**Lemma 3.6.** For all \( j, l, \dot{i} \) the following identities hold:
\[
l\#_l \dot{i}^{j,+} = l\#_l - 2\delta_{j,l} + \delta_{j,l+1} + \delta_{j,l-1} \quad \text{whenever } j \# \dot{i} = 1 ,
\]
(3.26a)
\[
\delta_{j\#_l,1} \dot{i}^{l,+,1} + \delta_{l\#_j,1} \dot{i}^{j,-,1} = \delta_{j,l} \dot{i} \# \dot{i} ,
\]
(3.26b)
\[
(\dot{i}^{j,+})^c = (\dot{i}^c)^{j,-} \quad \text{whenever } j \# \dot{i} = 1 .
\]
(3.26c)

**Proof.** If \( j \# \dot{i} = +1 \) (resp. \(-1\)) and \( r \) is the integer such that \( i_r = j \) (resp. \( j + 1 \)), then
\[
l\#_l - l\#_l \dot{i}^{j,+} = \sum_{h=1}^k (\delta_{i_h - i_{h+1}}, \delta_{i_{h+1} - i_h} + \delta_{i_{h+1}, i_h - i_{h+1}}) = 2\delta_{j,l} - \delta_{j,l+1} - \delta_{j,l-1} ,
\]
\[
l\#_l - l\#_l \dot{i}^{j,-} = \sum_{h=1}^k (\delta_{i_h - i_{h+1}}, \delta_{i_{h+1} - i_h} + \delta_{i_{h+1}, i_h - i_{h+1}}) = -2\delta_{j,l} + \delta_{j,l+1} + \delta_{j,l-1} .
\]
The former equation is just (3.26a). Next, if \( |j - l| = 1 \) we have
\[
\delta_{j\#_l,1} \dot{i}^{l,+,1} + \delta_{l\#_j,1} \dot{i}^{j,-,1} = \delta_{j,1} \dot{i} \# \dot{i} ,
\]
and this is zero since \( l\#_l \dot{i} \) can never be \(-2\). Similarly \( \delta_{j\#_l,1} \dot{i}^{l,+,1} + \delta_{l\#_j,1} \dot{i}^{j,-,1} = 0 \).

If \( |j - l| > 1 \), then \( l\#_l \dot{i}^{j,+} = l\#_l, j\#_j \dot{i}^{l,-} = j\#_j \), and
\[
\delta_{j\#_l,1} \dot{i}^{l,+,1} + \delta_{l\#_j,1} \dot{i}^{j,-,1} = \delta_{j,1} \dot{i}^{l,+,1} + \delta_{l,1} \dot{i}^{j,-,1} .
\]
If \( j = l \) then \( \delta_{j\#_l,1} \dot{i}^{l,+,1} + \delta_{l\#_j,1} \dot{i}^{j,-,1} = \delta_{j,1} \dot{i}^{l,+,1} + \delta_{l,1} \dot{i}^{j,-,1} = 0 \), and their difference is
\[
\delta_{j\#_l,1} \dot{i}^{l,+,1} - \delta_{l\#_j,1} \dot{i}^{j,-,1} = j \# \dot{i} .
\]

This proves (3.26b).

We pass to (3.26c). We assume \( j \# \dot{i} = 1 \), that by (3.24) is equivalent to the condition \( j \# \dot{i}^c = -1 \). We prove the equation by induction on the length \( k \) of \( \dot{i} \). For \( k = 1 \) we have:
\[
\dot{i} = (j), \quad \dot{i}^{j,+} = (j + 1), \quad \dot{i}^c = (1, 2, \ldots, j - 1, j + 1, j + 2, \ldots, \ell) ,
\]
and
\[
(\dot{i}^c)^{j,-} = (1, \ldots, j - 1, j, j + 2, \ldots, \ell) = (\dot{i}^{j,+})^c .
\]
Suppose now \( (\dot{i}^{j,+})^c = (\dot{i}^c)^{j,-} \) is true for any \( \dot{i} \in \Lambda_k \) with \( j \# \dot{i} = 1 \) (for a fixed \( k \geq 1 \)). Let \( \dot{i}' \in \Lambda_{k+1} \) with \( j \# \dot{i}' = 1 \). We write \( \dot{i}' = (\dot{i}'_{k+1}) \) if \( \dot{i}'_{k+1} \neq j \) and \( \dot{i}' = (\dot{i}', \dot{i}) \) if \( \dot{i}' \neq j \). In both cases \( \dot{i} \in \Lambda_k \) has \( j \# \dot{i} = 1 \). We consider the former case, the latter being symmetric. We have
\[
\dot{i}^{j,+} = (\dot{i}^{j,+}, \dot{i}'_{k+1}) ,
\]
and (as ordered sets)
\[
\dot{i}^c = (1, 2, \ldots, \ell) \setminus (\dot{i}, \dot{i}'_{k+1}) = \dot{i}^c \setminus (\dot{i}'_{k+1}) ,
\]
and being \( \dot{i}'_{k+1} \neq j + 1 \) (as \( j \# \dot{i} = 1 \)), by inductive hypothesis
\[
(\dot{i}', \dot{i})^{j,-} = (\dot{i}^{j,-})^c \setminus (\dot{i}'_{k+1}) = (\dot{i}^{j,+})^c \setminus (\dot{i}'_{k+1})^c = (\dot{i}^{j,+})^c .
\]
This concludes the proof.

We denote by $W_k \simeq \mathbb{C}^{|\lambda|}$ the linear space of vectors $w = (w_j)_{j \in \Lambda_k}$ with components $w_j \in \mathbb{C}$ labeled by the above multi-index. A transformation $T \in \text{End}(W_k)$ sends a vector $w$ into the vector $Tw$ with components $(Tw)_j$.

**Proposition 3.7.** Let $1 \leq k \leq \ell - 1$. The irreducible *-representation of $U_q(\mathfrak{su}(\ell))$ with highest weight

$$\delta^k = (0, \ldots, 0, 1, 0, \ldots, 0),$$

is given explicitly by the map $\sigma_k : U_q(\mathfrak{su}(\ell)) \to \text{End}(W_k)$ defined on generators by

$$\{\sigma_k(K_j)w\}_j = q^{\frac{1}{2}j\#i}w_j,$$
$$\{\sigma_k(E_j)w\}_j = \delta_{j\#i+1}w_{j^+,},$$
$$\{\sigma_k(F_j)w\}_j = \delta_{j\#i-1}w_{j^-},$$

where $j\#i$ is defined in (3.21) and $i^{\pm} \pm$ are defined in (3.25).

For $k = 0$ or $\ell$, we denote $W_0 = W_\ell = \mathbb{C}$ the vector space underlying the trivial representation (i.e. $\sigma_0 = \sigma_\ell = \epsilon$ is the counit of $U_q(\mathfrak{su}(\ell))$).

**Proof.** The vector $w$ with $w_{(1,2,\ldots,k)} = 1$ and all other components equal to zero is annihilated by all the $E_i$’s. Since $\sigma_k(K_j)w = q^{\delta_{j,k}}w$, the vector $w$ is the highest weight vector of the irreducible representation with highest weight $\delta^k$, that is then a subrepresentation of the (would-be) representation $\sigma_k$. Having both the same dimension $\binom{l}{k}$ (by Weyl’s character formula), they coincide if $\sigma_k$ is a representation. Since $\sigma_k(K_j)$ is Hermitian and the Hermitian conjugated of $\sigma_k(E_j)$ is $\sigma_k(F_j)$, the would-be representation is unitary. We now prove that the defining relations of $U_q(\mathfrak{su}(\ell))$ are satisfied (being $\sigma_k(h)^* = \sigma_k(h^*)$, if a relation holds, its conjugated holds too). We omit the representation symbol $\sigma_k$.

That $K_j$’s commute each other is trivial, being all diagonal. Further, by (3.26a):

$$\{K_iE_jK_i^{-1}w\}_j = \delta_{j\#i+1}q^{\frac{1}{2}(j\#i-1\#i^+)}w_{j^+,} = q^{\delta_{j,i^-} - \frac{1}{2}(\delta_{j,i+1} + \delta_{j,i-1})}w_{j^+,} = \{q^{\delta_{j,i^-} - \frac{1}{2}(\delta_{j,i+1} + \delta_{j,i-1})}E_jw\}_j.$$

This means $K_jE_jK_j^{-1} = qE_j$, $K_iE_jK_i^{-1} = q^{-\frac{1}{2}}E_j$ if $|j - l| = 1$ and $K_iE_jK_i^{-1} = E_j$ otherwise.

Next, by (3.26b):

$$\{[E_j,F_l]w\}_j = \delta_{j\#i+1}\delta_{l\#j^+,} - \delta_{l\#i^-1}\delta_{j\#i^+_1}w_{j},$$

where we used the fact that being $j\#i \in \{0, \pm 1\}$, it is always $[j\#i] = j\#i$.

Concerning Serre’s relation, it is enough to show that $E_j^2 = E_jE_jE_j = 0$ for all $j$. We have

$$\{E_j^2w\}_j = \delta_{j\#i+1}\delta_{j\#i^+,}w_{(j^+,)j^+,}.$$
and this is zero since by (3.26a) \( j \# i^{\pm} = j \# i - 2 = -1 \) whenever \( j \# i = 1 \). Again by (3.26a), for \( l = j \pm 1 \) we have
\[
j \# i = 1 \land l \# i^{\pm} = 1 \Rightarrow j \# (i^{\pm})^{l} = j \# i^{\pm} + 1 = j \# i - 1 = 0.
\]
So for \( l = j \pm 1 \),
\[
\{ E_{j}E_{l}E_{j}w \}_{i} = \delta_{j \# i + 1} \delta_{l \# i^{\pm} + 1} \delta_{j \# (i^{\pm})^{l} + 1} w((i^{\pm})^{l})_{i} = 0.
\]
This concludes the proof.

For \( q = 1 \), we have \( W_{k} \cong \wedge^{k} W_{1} \) for all \( 0 \leq k \leq \ell \). We are going to define a deformation \( \wedge_{q} \) of the wedge product such that \( \wedge_{q} : W_{h} \otimes W_{k} \to W_{h+k} \) intertwines the Hopf tensor product of the representations \( \sigma_{h} \) and \( \sigma_{k} \) with the representation \( \sigma_{h+k} \), where \( W_{h+k} := 0 \) if \( h + k \leq \ell \).

Before that, we need a brief interlude on permutations. We take notations from [3].

A representation \( \sigma \) of the wedge product such that \( \wedge \sigma \) intertwines the Hopf tensor product of the representations \( \sigma_{h} \) and \( \sigma_{k} \) with the representation \( \sigma_{h+k} \), where \( W_{h+k} := 0 \) if \( h + k \leq \ell \).

Before that, we need a brief interlude on permutations. We take notations from [3].

Let \( S_{n} \) be the group of permutations of the set with \( n \) elements \( \{1, 2, \ldots, n\} \); if \( p \in S_{n} \), we denote \( p(i) \) the image of \( i \in \{1, \ldots, n\} \) through \( p \), and with \( p^{i} \) the position of \( i \) after the permutation (we use the superscript to avoid confusion, since in [3] they call \( p_{i} = p(i) \)). For example, let \( n = 3 \); if \( p \) is the permutation
\[
p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}
\]
then \((p(1), p(2), p(3)) = (3, 1, 2)\), while \( p^{1} = 2, p^{2} = 3 \) and \( p^{3} = 1 \). A generic permutation is
\[
p = \begin{pmatrix} 1 & p(1) & \ldots & n \\ 2 & p(2) & \ldots & p(n) \end{pmatrix}
\]
and the relation between \( p^{i} \) and \( p(i) \) is the following: \( p^{i} = p^{-1}(i) \) where \( p^{-1} \) is the inverse of \( p \) in \( S_{n} \) (\( p^{i} \) is the new position of \( i \), \( p(p^{i}) \) is the number in position \( p^{i} \), then \( p(p^{i}) = i \)). In the above example, one can easily check that
\[
\begin{pmatrix} 1 & 2 & 3 \\ p^{1} & p^{2} & p^{3} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}
\]
is the inverse of \( p \). The product of two permutations \( p \cdot p^{i} \) is their composition, where as usual the permutation on the right act first: thus \( pp^{i} \) sends \( i \) into \( p(p^{i}(i)) \). For example if
\[
p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad p^{i} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},
\]
then \( p^{i} \) sends \( 1 \mapsto 1 \), \( p \) sends \( 1 \mapsto 2 \), so \( pp^{i} \) sends \( 1 \mapsto 2 \), etc.; on the other hand, \( p \) sends \( 1 \mapsto 2 \), \( p^{i} \) sends \( 2 \mapsto 3 \), so \( p^{i}p \) sends \( 1 \mapsto 3 \), etc.: the result is that
\[
pp^{i} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad p^{i}p = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.
\]

A representation \( \pi : S_{n} \to \text{End}(\mathbb{Z}^{n}) \) is given by \( \pi(p)(\hat{\imath}) := (i_{p1}, i_{p2}, \ldots, i_{pn}) \) for all \( p \in S_{n} \) and any \( n \)-tuple \( \hat{\imath} = (i_{1}, i_{2}, \ldots, i_{n}) \in \mathbb{Z}^{n} \). Notice that \( \pi(p) \) sends \( \Lambda_{n} \subset \mathbb{Z}^{n} \) into \( \Lambda_{n} \) only if \( p \) is the trivial permutation. We call
\[
\hat{\imath}_{\pi(p)} := (i_{p1}, i_{p2}, \ldots, i_{pn}) = \pi(p^{-1})(\hat{\imath}).
\]
With this notation \( (\hat{\imath}_{\pi(p)})^{p} = \hat{\imath}_{\pi(pp^{i})} \).

\[14\]
Recall that $S_n$ is generated by simple transpositions, that are the transformations that exchange two consecutive elements and keep all others fixed. The length $|p|$ of a permutation $p$ is the minimal number of simple transpositions needed to express $p$. The length of $p$ coincides with the number of inversions in $p$, that is the number of pairs $(i, j)$ such that $1 \leq i < j \leq n$ and $p(i) > p(j)$ (cf. [3, Proposition 1.5.2]):

$$||p|| = \text{card}\{ (i, j) : i < j \text{ and } p(i) > p(j) \}.$$ 

This can be written also as

$$||p|| = \text{card}\{ (i, j) : i < j \text{ and } p^i > p^j \},$$

since $p$ and $p^{-1}$ have always the same length (if $p = ss' \ldots s''$ is a reduced expression of $p$, then $p^{-1} = s'' \ldots s' s$ is a reduced expression of $p^{-1}$).

Given a maximal parabolic subgroup $S_k \times S_{n-k}$ of $S_n$ we can consider the left cosets

$$S_n^{(k)} := S_n / (S_k \times S_{n-k}).$$

The quotient $S_n^{(k)}$ coincides with the set of permutations (cf. [3], Sec. 2.4)

$$S_n^{(k)} = \{ p \in S_n | p(1) < p(2) < \ldots < p(k) \text{ and } p(k+1) < p(k+2) < \ldots < p(n) \}.$$

The map $p \mapsto p^{-1}$ gives a bijection between $S_n^{(k)}$ and the right cosets $S_{k,n-k} := (S_k \times S_{n-k}) \setminus S_n$, that is then given by those permutations $p$ that satisfy $p^1 < p^2 \ldots < p^k$ and $p^{k+1} < p^{k+2} < \ldots < p^n$. Elements of $S_{k,n-k}$ are called $(k, n-k)$-shuffles (see e.g. [21], Sec. 13.2).

We’ll need the following lemma (cf. [3], Proposition 2.4.4, Cor. 2.4.5 and 2.12).

**Lemma 3.8.** Any $p \in S_n$ can be uniquely factorized $p = p'p''$ with $p' \in S_n^{(k)}$ and $p'' \in S_k \times S_{n-k}$, or as $p = \tilde{p}'\tilde{p}''$ with $\tilde{p}' \in S_k \times S_{n-k}$ and $\tilde{p}'' \in S_{k,n-k}$, and in both cases

$$||p|| = ||p'|| + ||p''|| = ||\tilde{p}'|| + ||\tilde{p}''||.$$

Elements of these two quotients correspond to maps $\Lambda_{h+k} \rightarrow \Lambda_h \times \Lambda_k$: $\forall \ i \in \Lambda_{h+k}$ we have $\{ \pi(p)(i) \in \Lambda_h \times \Lambda_k \iff p \in S_{h,k} \}$ and $\{ \tilde{i}_{\pi(p)} \in \Lambda_h \times \Lambda_k \iff p \in S_{h,k}^{(h)} \}$. With this preparation, we define $\land_q : W_h \otimes W_k \rightarrow W_{h+k}$ by the formula

$$(v \land_q w)_{\tilde{i}} = \sum_{p \in S_{h+k}^{(h)}} (-q^{-1})^{||p||} v_{i_{p(1)}, \ldots, i_{p(h)}} w_{i_{p(h+1)}, \ldots, i_{p(h+k)}} (3.29)$$

for all $v = (v_{ij}) \in W_h$ and $w = (w_{ij}) \in W_k$, and for all $h, k = 0, \ldots, \ell$ with $h+k \leq \ell$. Moreover, we set $v \land_q w := 0$ if $h + k > \ell$. By definition of $S_{h,k}^{(h)}$, this map is well defined.

Remark: when there is no risk of confusion, we write $i_1, \ldots, i_k$ without parenthesis instead of $(i_1, \ldots, i_k)$. For example in the equation above, $v_{i_{p(1)}, \ldots, i_{p(h)}}$ means $v_{(i_{p(1)}, \ldots, i_{p(h)})}$; also $\tilde{i} \setminus j$ means $\tilde{i} \setminus (j)$.

**Proposition 3.9.** The above product satisfies the following properties:

1. $\land_q$ is surjective.
2. $\land_q$ is associative.
We have equal to

\(\sigma_{k+1}(x)(v \wedge w) = \{\sigma_h(x(1))v\} \wedge q \{\sigma_k(x(2))w\}\)  

(3.30)

for all \(v \in W_h\), \(w \in W_k\) and for all \(x \in U_q(\mathfrak{su}(\ell))\). Here we use Sweedler-type notation, \(\Delta(x) = x_{(1)} \otimes x_{(2)}\), for the coproduct.

**Proof.** The map \(\wedge_q\) is clearly surjective, since for all \(i \in \Lambda_n\), \(n = h + k\), we can find \(v \in W_h\) and \(w \in W_k\) such that \((v \wedge w)_i \neq 0\). For instance, take as \(v\) the vector with \(v_{i_1, \ldots, i_h} = 1\) and all other components zero, and as \(w\) the vector with \(w_{i_{h+1}, \ldots, i_{h+k}} = 1\) and all other components zero; this yields \((v \wedge w)_i = v_{i_1, \ldots, i_h} w_{i_{h+1}, \ldots, i_{h+k}} = 1\), which shows point 1.

Concerning point 2, we need to prove that for any \(k \in \{3, 4, \ldots, \ell\}\), no matter how we parenthesize the product \(v^1 \wedge_q v^2 \wedge_q \ldots \wedge_q v^k\) of vectors in \(W_1\), the result will be the same, and is given by the formula

\[
(v^1 \wedge_q v^2 \wedge_q \ldots \wedge_q v^k)_i = \sum_{p \in S_k} (-q^{-1})||p|| v^1_{p(1)} v^2_{p(2)} \ldots v^k_{p(k)}.
\]

(3.31)

We prove (3.31) by induction on \(k\). It’s easy to check when \(k = 3\):

\[
\{(v^1 \wedge_q v^2) \wedge_q v^3\}_{i_1, i_2, i_3} = \{(v^1 \wedge_q (v^2 \wedge_q v^3))_{i_1, i_2, i_3}
\]

\[
= v^1 v^2 v^3 - q^{-1} v^2 v^1 v^3 + q^{-2} v^1 v^3 v^2 + q^{-2} v^1 v^2 v^3 - q^{-3} v^1 v^2 v^3.
\]

Now we assume the claim is true for a generic \(k \geq 3\), and prove that \((v^1 \wedge_q \ldots \wedge_q v^k) \wedge_q v^{k+1}\) is equal to \(v^1 \wedge_q (v^2 \wedge_q \ldots \wedge_q v^{k+1})\) and given by the expression (3.31) (with \(k\) replaced by \(k + 1\)). We have

\[
\{(v^1 \wedge_q \ldots \wedge_q v^k) \wedge_q v^{k+1}\}_i = \sum_{p' \in S_{k+1}} (-q^{-1})||p'||(v^1 \wedge_q \ldots \wedge_q v^k)_{p'(1), \ldots, p'(k)} v^{k+1}_{p'(k+1)}
\]

\[
= \sum_{p' \in S_{k+1}, p'' \in S_k} (-q^{-1})||p'||||p''|| v^1_{p'(1)} v^2_{p'(2)} \ldots v^k_{p'(k)} v^{k+1}_{p''(k+1)}.
\]

The composition \(p = p' \circ (p'' \times id)\) is clearly an element of \(S_{k+1}\). But also the converse is true: by Lemma 3.8 for any \(p \in S_{k+1}\) there is a unique decomposition \(p = p' \circ (p'' \times id)\) with \(p' \in S_{k+1}^{(k)}\) and \(p'' \times id \in S_k \times S_1\) (\(S_1 = \{id\}\) is the trivial group), and satisfies \(||p|| = ||p'|| + ||p''||\). Thus,

\[
\{(v^1 \wedge_q \ldots \wedge_q v^k) \wedge_q v^{k+1}\}_i = \sum_{p \in S_{k+1}} (-1)^||p'|| v^1_{p(1)} v^2_{p(2)} \ldots v^{k+1}_{p(k+1)}.
\]

The proof can be mirrored: we get the analogous claim for \((v^1 \wedge_q (v^2 \wedge_q \ldots \wedge_q v^{k+1}))\) by using the decomposition \(S_{k+1} = S_{k+1}^{(1)} \times (S_1 \times S_k)\). This proves the inductive step.

Now, the map \(\wedge_q\) being surjective, any \(w \in W_k\), \(w' \in W_{k'}\) and \(w'' \in W_{k''}\) can be written as

\[
w = v^1 \wedge_q \ldots \wedge_q v^k, \quad w' = v^{k+1} \wedge_q \ldots \wedge_q v^{k+k'}, \quad w'' = v^{k+k'+1} \wedge_q \ldots \wedge_q v^{k+k'+k''},
\]

with \(v^j \in W_1\). Since the product of any number of vectors in \(W_1\) is associative, we have

\[
(w \wedge q w') \wedge q w'' = v^1 \wedge q \ldots \wedge q v^{k+k'+k''} = w \wedge q (w' \wedge q w'')
\]
and this concludes the proof of associativity.

We pass to point 3, the claim that $\wedge_q : W_h \otimes W_k \to W_{h+k}$ intertwines the Hopf tensor product of the representations $\sigma_h$ and $\sigma_k$ with the representation $\sigma_{h+k}$. It is enough to prove that

$$\sigma_n(x)(v^1 \wedge_q v^2 \wedge \ldots \wedge_q v^n) = \{\sigma_1(x(1))v^1\} \wedge_q \{\sigma_1(x(2))v^2\} \wedge_q \ldots \wedge_q \{\sigma_1(x(n))v^n\},$$  \hspace{1cm} (3.32)

for any number $n$ of vectors $v^j \in W_1$. Equation (3.32), together with associativity and surjectivity of $\wedge_q$, implies (3.30). Indeed, any $v \in W_h$ (resp. $w \in W_k$) can be written as products $v = v^1 \wedge_q \ldots \wedge_q v^h$ (resp. $w = v^{h+1} \wedge_q \ldots \wedge_q v^{h+k}$) of vectors $v^j \in W_1$, and using (3.32) and coassociativity of the coproduct we get

$$\sigma_{h+k}(x)(v \wedge_q w) = \sigma_{h+k}(x)(v^1 \wedge_q \ldots \wedge_q v^{h+k})$$

$$= \{\sigma_1(x(1))v^1\} \wedge_q \{\sigma_1(x(2))v^2\} \wedge_q \ldots \wedge_q \{\sigma_1(x(n))v^n\}$$

$$= \{\sigma_h(x(1))(v^1 \wedge_q \ldots \wedge_q v^h)\} \wedge_q \{\sigma_k(x(2))(v^{h+1} \wedge_q \ldots \wedge_q v^{h+k})\}$$

$$= \{\sigma_h(x(1))v\} \wedge_q \{\sigma_k(x(2))w\}.$$  

Last step is to prove (3.32). Dealing with *-representations, it is enough to do the check for $x = K_j, E_j$ (and for all $j = 1, \ldots, \ell - 1$). For $x = K_j$, the property (3.32) follows from the simple observation that, by (3.21), $j\# \hat{i} = \sum_{r=1}^n j\#(i_r)$ for all $i \in \Lambda_n$. For $x = E_j$, the $n$th power of the coproduct is

$$\Delta_n(E_j) = \sum_{r=1}^n (K_j^{-1})^\otimes(r-1) \otimes E_j \otimes K_j^\otimes(n-r)$$

and the right hand side of (3.32) becomes

$$\text{rhs} = \sum_{r=1}^n \sigma_1(K_j^{-1})v^1 \wedge_q \ldots \wedge_q \sigma_1(K_j^{-1})v^{r-1} \wedge_q \sigma_1(E_j)v^r \wedge_q \sigma_1(K_j)v^{r+1} \wedge_q \ldots \wedge_q \sigma_1(K_j)v^n.$$  

By (3.27a) and (3.27b) we have $\{\sigma_1(K_j)v\}_i = q^{\frac{i}{2}(\delta_{i,j} - \delta_{i,j-1})}v_i$ and $\{\sigma_1(E_j)v\}_i = \delta_{i,j}v_{i+1}$ for all $v \in W_1$. Thus, using (3.31) we rewrite the $i$th component of previous equation as

$$(\text{rhs})^i = \sum_{r=1}^n \sum_{p \in S_n | p(r) = j} (-q^{-1})^{|p|} q^{\frac{i}{2}(\sum_{s<r} - \sum_{s>r})} \delta_{i,p(s)}v_{p(1)}^{r-1} \ldots v_{p(r-1)}^{r-1} v_{p(r+1)}^r \ldots v_{p(n)}^n.$$  

Being $p$ a permutation, $j \in i_{\pi(p)}$ if and only if $j \in i$. We have three cases: 1) $j \notin i$, 2) $j, j+1 \in i$ and 3) $j \in i$ but $j+1 \notin i$: the first two cases correspond to $j\# \hat{i} \neq 1$, the third to $j\# \hat{i} = 1$.

If $j \notin i$ the equation is trivially zero (the sum is empty). In the second case we get zero too, but to prove it requires some work. Firstly, notice that if $r, s$ are the integer such that $i_r = j$ and $i_s = j+1$, then $s = r + 1$ (by contradiction, assume there exists $h$ such that $r < h < s$, then there is also $i_h$ such that $j < i_h < j + 1$, that is a contradiction being $i_h$ an integer). Assume then that $\hat{i} = (i_1, \ldots, i_{r-1}, j, j+1, j_{r+2}, \ldots, i_n)$, and call

$$A = \{p \in S_n | p^r < p^{r+1}\}, \quad B = \{p \in S_n | p^r > p^{r+1}\}.$$
We have
\[(\text{rhs})_q = \sum_{p \in A} (-q)^{-1} |p| \sum_{p \in B} (-q)^{1} \sum_{p \in B} (-q)^{x-1} v_{p(1)} \ldots v_{p(x-1)} v_{p(x)} v_{p(x+1)} \ldots v_{p(n)} + \sum_{p' \in B} (-q)^{x-1} v_{p'(1)} \ldots v_{p'(x-1)} v_{p'(x)} v_{p'(x+1)} \ldots v_{p'(n)} \]

The effect of the composition $p^{-1} \rightarrow p^{-1} \circ s_r$, with $s_r$ the simple transposition exchanging $r$ with $r + 1$, is to exchange $p^{-1}(r)$ with $p^{-1}(r + 1)$ in the complete expression of $p^{-1}$ (cf. [3], Pag. 20, with $x = p^{-1}$). But this is equivalent to the transformation $p \mapsto s_r \circ p$ (as $s_r^2 = 1$), whose effect is then to exchange $p^r$ with $p^{r+1}$ (as $p^j = p^{-1}(j)$), thus giving a bijection $A \rightarrow B$.

The change of variable $p' = s_r \circ p$ turns the second sum in last equation into the first, but for a global sign (by [3, 1.26], $|p^{-1}| = |p^{-1}| + 1$, and then $|p'| = |p| + 1$, for all $p' \in B$).

Hence, the two sums cancel and the result is zero. Therefore, $(\text{rhs})_q$ is zero unless $j \#_q = 1$.

We have
\[
(\text{rhs})_q = \delta_j \sum_{p \in S_n} (-q)^{1} \sum_{p \in S_n} (-q)^{x-1} v_{p(1)} \ldots v_{p(x-1)} v_{p(x)} v_{p(x+1)} \ldots v_{p(n)} x: t(p) = j
\]

that by (3.27b) is exactly the $j$th component of the left hand side of (3.32). This concludes the proof.

We set $\mathbf{Gr}^\ell_q := \otimes_{k=0}^\ell W_k$, equipped with $\wedge_q$, that by Proposition 3.9 is a graded associative algebra - generated by $W_1$ - and a left $U_q(\mathfrak{su}(\ell))$-module algebra. This is a $q$-analogue of the $2^\ell$ dimensional Grassmann algebra. Indeed, for its dimension we have $\dim \mathbf{Gr}_q^\ell = \sum_{k=0}^\ell \dim W_k = \sum_{k=0}^\ell \binom{\ell}{k} = 2^\ell$. We list explicitly the wedge product between elements with degree 0 and 1. If $a \in W_0$ and $v \in W_1$ then $a \wedge_q v = av$ and $v \wedge_q a = va$; if $v, w \in W_1$:

\[(v \wedge_q w)_{i_1,i_2} = v_{i_1} w_{i_2} - q^{-1} v_{i_2} w_{i_1}, \quad \forall 1 \leq i_1 < i_2 \leq \ell;\]

if $v \in W_1$, $w \in W_2$:

\[(v \wedge_q w)_{i_1,i_2,i_3} = v_{i_1} w_{i_2,i_3} - q^{-1} v_{i_2} w_{i_1,i_3} + q^{-2} v_{i_3} w_{i_1,i_2}, \quad \forall 1 \leq i_1 < i_2 < i_3 \leq \ell.\]

Also, the formula for the product of $v \in W_1$ and $w \in W_2$ will be useful later. Any $p \in S^{(1)}_{k+1}$ has the form $p : (i_1, \ldots, i_{k+1}) \rightarrow (i_r) \times (i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{k+1})$ for some $1 \leq r \leq k + 1$, and $|p| = r - 1$. Thus

\[
(v \wedge_q w)_q = \sum_{r=1}^{k+1} (-q)^{1-r} v_{i_r} w_{i_{r-1} \cup i_r} ;
\]

and similarly

\[
(w \wedge_q v)_q = \sum_{r=1}^{k+1} (-q)^{r-k-1} w_{i_{r-1} \cup i_r} v_{i_r} .
\]

To discuss the first order condition, we'll need the following antilinear map $J : W_k \rightarrow W_{\ell-k}$,

\[
(Jw)_q = (-q)^{|\bar{u}|} q^{\frac{1}{2}((\ell+1) |w|)},
\]

where $|\bar{u}| := i_1 + i_2 + \ldots + i_{\ell-k}$ and $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$.\]
**Proposition 3.10.** Let $0 \leq k \leq \ell$. The map $J : W_k \rightarrow W_{\ell-k}$ has square
\[ J^2 = (-1)^{\lfloor \frac{\ell+1}{2} \rfloor}, \]
with $[t]$ the integer part of $t$. It is equivariant, in the following sense:
\[ \sigma_{\ell-k}(x^*)J = J\sigma_k(S(x)) \quad (3.34) \]
for all $x \in U_q(\mathfrak{su}(\ell))$.

**Proof.** Clearly, for $w \in W_k$,
\[ (J^2w)_i = (-q^{-1})^{i_1+i_2+\ldots+i_k+i'_1+i'_2+\ldots+i'_{\ell-k}}q^{\frac{1}{2}\ell(\ell+1)}w_{(\ell^c)^c}, \]
but $(\ell^c)^c = i$ and $\{i_s,i'_s\}_{s,i}$ is the set as all integers between 1 and $\ell$, so their sum is $\frac{1}{2}\ell(\ell+1)$, and
\[ (J^2w)_i = (-q^{-1})^{\frac{1}{2}\ell(\ell+1)}q^{\frac{1}{2}\ell(\ell+1)}w_i = (-1)^{\frac{1}{2}\ell(\ell+1)}w_i. \]

Note that $\frac{1}{2}\ell(\ell+1)$ has the same parity of $\frac{1}{2}(\ell+1)$ if $\ell$ is odd, and it has the same parity of $\frac{1}{2}\ell$ if $\ell$ is even. In both cases it has the same parity as $\lfloor \frac{\ell+1}{2} \rfloor$. This proves the claim about $J^2$.

We pass to (3.34). Let $c_{i,\ell} := (-q^{-1})^{i_1+i_2+\ldots+i_k+i'_1+i'_2+\ldots+i'_{\ell-k}}q^{\frac{1}{2}\ell(\ell+1)}$. Firstly, by (3.24) (we omit the representation symbols)
\[ (K_jw)_i = q^{\frac{1}{2}(j\#i)}(Jw)_i = q^{-\frac{1}{2}(j\#i)}c_{i,\ell}^{-1}w_{(\ell^c)^c} = c_{i,\ell}(K_j^{-1}w)_i, \]
that is $K_j^*Jw = J S(K_j)w$.

Now we use (3.24), (3.26c), and the observation that $-q^{-1}c_{i,\ell} = c_{i'+1,\ell}$, to compute
\[ \{J(-q^{-1}F_jw)\}_i = c_{i,\ell}(-q^{-1}F_jw)_i = -q^{-1}c_{i,\ell}d_{j\#i}^c,-1w_{(\ell^c)^c}, \]
\[ = c_{i',1}d_{j\#i}^c1w_{(\ell^c)^c} = d_{j\#i}^c(\ell^c)^c = (E_jw)_{i',1} = \{E_jw\}_i. \]

Since $J^2 = \pm 1$, we have also $-q^{-1}F_jJ = JE_j$. Hence, we have $x^*J = J S(x)$ for arbitrary generator $x = K_j, E_j, F_j$ of $U_q(\mathfrak{su}(\ell))$, and this concludes the proof. \[ \square \]

To any $x \in W_1$, we associate an operation of left ‘exterior product’ $\mathbf{e}_x^L : W_k \rightarrow W_{k+1}$ (resp. right ‘exterior product’ $\mathbf{e}_x^R : W_k \rightarrow W_{k+1}$) via the rule
\[ \mathbf{e}_x^Lw = x \wedge_q w, \quad \mathbf{e}_x^Rw = (-q)^k w \wedge_q x. \quad (3.35) \]

We define the left (resp. right) ‘contraction’ as the adjoint $i_x^L$ of $\mathbf{e}_x^L$ (resp. $i_x^R$ of $\mathbf{e}_x^R$) with respect to the inner product on $W_k$ given by
\[ \langle v, w \rangle := \sum_{i \in \Lambda_k} \overline{w_i} v_i, \]
for all $v, w \in W_k$.

**Proposition 3.11.** We have
\[ J\mathbf{e}_x^LJ^{-1} = -q^R x, \]
for all $x \in W_1$. As a consequence, denoting $L(j, \overset{c}{j})$ the position of $j$ inside the string $\overset{c}{j}$, we have
\[
(i^R_x v)_1 = \sum_{j \notin \overset{c}{j}} (-q)^{L(j, j') - 1} x_j v_{\overset{c}{j}, j}, \tag{3.36}
\]
for all $v \in W_{k+1}$ and $\overset{c}{j} \in \Lambda_k$.

Proof. For any $v \in W_{k+1}$ and $\overset{c}{j} \in \Lambda_k$,
\[
(J_{x}^{L} J_{x}^{-1} v)_{\overset{c}{j}} = (-q^{-1})^{\ell} q^{1/2} \ell (\ell+1) (e_{x}^{L} J_{x}^{-1} v)_{\overset{c}{j}}
= (-q^{-1})^{\ell} q^{1/2} \ell (\ell+1) \sum_{r=1}^{\ell-k} (-q^{-1})^{r-1} (J_{x}^{-1} v)_{\overset{c}{j}, r} x_{\overset{c}{j}, r}
= \sum_{r=1}^{\ell-k} (-q)^{r-1} v_{\overset{c}{j}, r} x_{\overset{c}{j}, r}. \tag{3.37}
\]
Thus,
\[
\langle J_{x}^{L} J_{x}^{-1} v, w \rangle = \sum_{\overset{c}{j} \in \Lambda_k} \sum_{r=1}^{\ell-k} (-q)^{r-1} v_{\overset{c}{j}, r} x_{\overset{c}{j}, r} w_{\overset{c}{j}, r} = \sum_{\overset{c}{j} \in \Lambda_k} \sum_{j \notin \overset{c}{j}} (-q)^{j-L(j, \overset{c}{j})} v_{\overset{c}{j}, j} x_{j} w_{\overset{c}{j}, j},
\]
\[
\langle v, -q e_{x}^{R} w \rangle = \sum_{\overset{c}{j} \in \Lambda_{k+1}} \sum_{r=1}^{k+1} (-q)^{r} v_{\overset{c}{j}, r} x_{\overset{c}{j}, r} w_{\overset{c}{j}, r-1} = \sum_{\overset{c}{j} \in \Lambda_{k+1}} \sum_{j \notin \overset{c}{j}} (-q)^{L(j, \overset{c}{j})} v_{\overset{c}{j}, j} x_{j} w_{\overset{c}{j}, j-1},
\]
where $L(j, \overset{c}{j})$ (resp. $L(j, \overset{c}{j})$) is the position of $j$ inside $\overset{c}{j}$ (resp. $\overset{c}{j}$). We have
\[
\sum_{\overset{c}{j} \in \Lambda_k} \sum_{j \notin \overset{c}{j}} f_{\overset{c}{j}, j} = \sum_{\overset{c}{j} \in \Lambda_{k+1}} \sum_{j \notin \overset{c}{j}} f_{\overset{c}{j}, j},
\]
for any $f$ and modulo a proportionality constant (any $\overset{c}{j} \in \Lambda_{k+1}$ can be written as a union $\overset{c}{j} \cup j$, but the decomposition is not unique). To check that the normalization is correct we take $f$ with all components equal to 1, and get
\[
\sum_{\overset{c}{j} \in \Lambda_k} \sum_{j \notin \overset{c}{j}} 1 = (\ell - k) \sum_{\overset{c}{j} \in \Lambda_k} 1 = (\ell - k)(\ell),
\]
\[
\sum_{\overset{c}{j} \in \Lambda_{k+1}} \sum_{j \notin \overset{c}{j}} 1 = (k+1) \sum_{\overset{c}{j} \in \Lambda_{k+1}} 1 = (k+1)(\ell),
\]
and the two quantities above coincide. It remains to show that
\[
L(j, \overset{c}{j}) + L(j, \overset{c}{j} \cup j) = j + 1 \tag{3.38}
\]
for all $\overset{c}{j} \in \Lambda_k$ and $j \notin \overset{c}{j}$. From this it follows immediately $\langle J_{x}^{L} J_{x}^{-1} v, w \rangle = \langle v, -q e_{x}^{R} w \rangle$, so that the adjoint $-q e_{x}^{R}$ of $-q e_{x}^{R}$ is exactly $J_{x}^{L} J_{x}^{-1}$.

We now prove (3.38) by induction. Let $k = 1$ and $\overset{c}{j} = (i_1)$. We have $L(j, \overset{c}{j} \cup j) = 1$ if $j < i_1$ and $L(j, \overset{c}{j} \cup j) = 2$ if $j > i_1$. Concerning the left hand side, $\overset{c}{j} = \{1, 2, \ldots, i_1 - 1, i_1 + 1, \ldots, \ell\}$ and the position of $j$ inside $\overset{c}{j}$ is $j$ itself if $j < i_1$, and $j - 1$ (for the empty position corresponding to $i_1$) if $j > i_1$. In both cases, the sum is $j + 1$.

Now we assume (3.38) is true for $k \geq 1$ generic, and prove that it is true for $k + 1$. Let $\overset{c}{j} \in \Lambda_{k+1}$. Call $\overset{c}{j}'' = (i_1, \ldots, i_k)$, $\overset{c}{j} = \overset{c}{j}'' \cup i_{k+1}$ and $\overset{c}{j} = \overset{c}{j}'' \cup i_{k+1}$. If $i_{k+1} > j$ the inductive step follows from
\[
L(j, \overset{c}{j} \cup j) = L(j, \overset{c}{j}'' \cup j), \quad L(j, \overset{c}{j}) = L(j, \overset{c}{j}'' \cup i_{k+1}).
\]
If \( i_{k+1} < j \) then \( i_r < j \) for all \( r \), and
\[
L(j, i \cup j) = k + 2, \quad L(j, i^c) = j - k - 1.
\]
The sum is again \( j + 1 \). To conclude, if we take (3.37) and use (3.38), we get (3.36). \( \square \)

Remark: for \( q = 1 \), \( \epsilon_x^L = \epsilon_x^R =: \epsilon_x \), and for all \( x, y \in W_1 \), we have \( \epsilon_x \epsilon_y + \epsilon_y \epsilon_x = 0 \) and \( \epsilon_x \epsilon_y + \epsilon_y \epsilon_x = (x, y) \cdot id_{\Delta q} \). From this it follows that the map \( x \mapsto \epsilon_x + \epsilon_x \) gives a representation of the Clifford algebra generated by \( W_1 \). For \( q \neq 1 \) this is no more true (for example, \( (\epsilon_x^L)^2 = \epsilon_x^L \epsilon_x^L \), and \( x \wedge q x \) is not always zero). Fortunately, we don’t need this property in the sequel.

We conclude with a lemma on the quantum dimension \( \dim_q W_k \) of \( W_k \), that is defined as the trace of
\[
\prod_{j=1}^{\ell-1} K_j^{2j(\ell - j)} = K_{2p} \hat{K}^{\ell-1},
\]
which is the analogue of the element in (3.2) for the Hopf algebra \( U_q(\mathfrak{su}(\ell)) \) (cf. Sec. 7.1.6 of [21]). Recall that \( \hat{K} \) is defined in (3.1). The geometrical meaning of \( \dim_q W_k \) is the value of the \( U_q(\mathfrak{su}(\ell)) \) invariant of the unknotted coloured by the representation \( W_k \) (cf. [24]).

**Lemma 3.12.** The quantum dimension of \( W_k \) is given by
\[
\dim_q W_k = \sum_{j, i} q^{k(\ell+1)-2|\bar{j}|}.
\]
It is symmetric under the exchange \( q \to q^{-1} \), and its explicit value is
\[
\dim_q W_k = \frac{[\ell]!}{[k]! [\ell - k]!},
\]
where \( [n]! := [n][n-1] \ldots [1] \) if \( n \geq 1 \), and \( [0]! := 1 \).

**Proof.** The general matrix element of \( K_{2p} \hat{K}^{\ell-1} \) along the diagonal is
\[
\sigma_k (K_{2p} \hat{K}^{\ell-1})_{j, i} = q^{\sum_{j=1}^{\ell-1} j(\ell - j) \cdot (j \# i)}.
\]
But
\[
\sum_{j=1}^{\ell-1} j(\ell - j) \cdot (j \# i) = \sum_{j=1}^{\ell-1} j(\ell - j) \sum_{h=1}^{k} (\delta_{j,i_h} - \delta_{j+1,i_h})
\]
\[
= \sum_{h=1}^{k} \sum_{j=1}^{\ell-1} j(\ell - j) (\delta_{j,i_h} - \delta_{j+1,i_h})
\]
\[
= \sum_{h=1}^{k} \left( \sum_{j=1}^{\ell-1} j(\ell - j) \delta_{j,i_h} - \sum_{j=2}^{\ell} (j - 1)(\ell - j + 1) \delta_{j,i_h} \right)
\]
\[
= \sum_{h=1}^{k} \left( \sum_{j=1}^{\ell} j(\ell - j) \delta_{j,i_h} - \sum_{j=1}^{\ell} (j - 1)(\ell - j + 1) \delta_{j,i_h} \right)
\]
\[
= \sum_{h=1}^{k} \sum_{j=1}^{\ell} \left( j(\ell - j) - (j - 1)(\ell - j + 1) \right) \delta_{j,i_h}
\]
\[
= \sum_{h=1}^{k} \sum_{j=1}^{\ell} (\ell + 1 - 2j) \delta_{j,i_h} = \sum_{h=1}^{k} (\ell + 1 - 2i_h)
\]
\[
= k(\ell + 1) - 2|\bar{j}|.
\]
Thus:
\[
\sigma_k (K_{2p} \hat{K}^{\ell-1})_{j, i} = q^{k(\ell+1)-2|\bar{j}|}.
\]  (3.39)
This proves the first formula for \( \dim_q W_k \).

A \( q \)-analogue of the hook formula for the quantum dimension of a general irreducible representation of \( U_q(\mathfrak{su}(\ell)) \) is discussed in the unpublished paper [24]. We derive a simpler formula for the representations we are interested in. Let

\[
c_k := \dim_q W_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq \ell} q^{k(\ell+1)-2(i_1+i_2+\ldots+i_k)},
\]

where the dependence on \( \ell \) is put in evidence. We decompose \( \Lambda_k \) as

\[
\Lambda_k = \{1 \leq i_1 < i_2 < \ldots < i_k \leq \ell\} = \{1 \leq i_1 < i_2 < \ldots < i_k \leq \ell - 1\} \cup \{1 \leq i_1 < i_2 < \ldots < i_k-1 \leq \ell - 1; i_k = \ell\}.
\]

This gives the recursive equation

\[
c_k = q^k \sum_{1 \leq i_1 < \ldots < i_k \leq \ell - 1} q^{k - 2(i_1 + i_2 + \ldots + i_k)} + q^{-\ell - k} \sum_{1 \leq i_1 < \ldots < i_{k-1} \leq \ell - 1} q^{(k-1)\ell - 2(i_1 + i_2 + \ldots + i_{k-1})} = q^k c_{k-1}^\ell + q^{-(\ell - k)} c_k^{\ell-1}.
\]

But since \( q^k [\ell - k] + q^{-(\ell - k)} [k] = [\ell] \), we have also

\[
\frac{[\ell]}{[k][\ell-k]} = q^k \frac{[\ell-1]}{[k][\ell-1-k]} + q^{-(\ell - k)} \frac{[\ell-1]}{[k-1][\ell-k-1]}.
\]

Thus, \( c_k^\ell = c_1 \frac{[\ell]}{[k][\ell-k]} \). But \( c_1^1 = 1 \), and this concludes the proof.

3.3. **Left invariant vector fields over \( \mathbb{C}P_q^\ell \).** Classically (for \( q = 1 \)), left invariant vector fields on \( \mathbb{C}P^\ell \cong SU(\ell + 1)/U(\ell) \) are given by the right action of elements of \( u^+(\ell) \), the orthogonal complement of \( u(\ell) \) inside \( \mathfrak{su}(\ell + 1) \). That is, any left invariant vector field is a linear combination of the operators \( M_{i,\ell} \) and \( M_{i,\ell}^* \) (see Sec. 3.1), acting via the right canonical action. In particular, the former gives the Dolbeault operator \( \partial \). We are interested only in the latter one. The vector \( v = (v_i) \) with components \( v_i = M_{i,\ell}^*|_{q=1} \) can be thought of as an element of \( \mathfrak{su}(\ell + 1) \otimes \mathbb{C} W_1 \) that is right \( u(\ell) \)-invariant with respect to the tensor product of the right adjoint action – ‘dressed’ with \( S^{-1} \) – and the action via \( \sigma_1 \). It is then clear how to generalize it to \( q \neq 1 \).

We denote by

\[
x \overset{\text{ad}}{\circ} h := S(h_{(1)}x)h_{(2)}
\]

the right adjoint action of \( U_q(\mathfrak{su}(\ell + 1)) \) on itself, and set

\[
\mathfrak{X} := \{v \in U_q(\mathfrak{su}(\ell + 1)) \otimes \mathbb{C} W_1 \mid v \overset{\text{ad}}{\circ} \hat{K} = qv, \sigma_1(h_{(2)})v \overset{\text{ad}}{\circ} S^{-1}(h_{(1)}) = \epsilon(h)v, \forall h \in U_q(\mathfrak{su}(\ell))\}.
\]

Classically, primitive elements of \( \mathfrak{X} \) (i.e. such that \( \Delta(x) = x \otimes 1 + 1 \otimes x \)) are proportional to the vector with components \( M_{i,\ell}^* \). We look for a natural \( q \)-deformation of these elements belonging to \( \mathfrak{X} \) (the subset of primitive elements is empty for \( q \neq 1 \)).

**Lemma 3.13.** Let \( \{e^i\}_{i=1,\ldots,\ell} \) be the canonical basis of \( W_1 \cong \mathbb{C}^\ell \) and

\[
X_i := N_{i,\ell} M_{i,\ell}^*, \quad \forall i = 1, \ldots, \ell,
\]

...
where $N_{i\ell}$ are defined in (3.7). Then the vector

$$X = \sum_i e^i X_i \in U_q(\mathfrak{su}(\ell + 1)) \otimes \mathbb{C} W_1$$

belongs to $\mathfrak{X}$.

**Proof.** We have

$$\hat{K}^{-1} X_I \hat{K} = \hat{K} K^{-1}_{\ell} (\hat{K}^{-1} F_{\ell} \hat{K})$$

and

$$\hat{K}^{-1} F_{\ell} \hat{K} = (K_{\ell-1}^{\ell} K_{\ell}^{\ell})^{-1/\ell} F_{\ell} (K_{\ell-1}^{\ell} K_{\ell}^{\ell})^{2/\ell} = q^{(\ell-\frac{1}{2}(\ell-1)) \frac{2}{\ell}} F_{\ell} = q F_{\ell} .$$

Since $\hat{K}$ commutes with $F_i$ for all $i = 1, \ldots, \ell - 1$, we have $X_i \stackrel{ad}{\rightarrow} \hat{K} = \hat{K}^{-1} X_i \hat{K} = q X_i$ for all $i = 1, \ldots, \ell$. Next, to show the right $U_q(\mathfrak{su}(\ell))$-invariance, it is enough to work with generators, that means to verify

$$X \stackrel{ad}{\rightarrow} K_j = \sigma_1(K_j) X , \quad X \stackrel{ad}{\rightarrow} E_j = \sigma_1(E_j) X , \quad X \stackrel{ad}{\rightarrow} F_j = \sigma_1(F_j) X ,$$

for all $j = 1, \ldots, \ell - 1$. We prove by induction that

$$X_i \stackrel{ad}{\rightarrow} K_j := K_j^{-1} X_i K_j = q^{\frac{1}{2}(\delta_{i,j} - \delta_{i,j+1})} X_i \equiv \{\sigma_1(K_j) X\}_i ,$$

for all $i = 1, \ldots, \ell$ and for all $j = 1, \ldots, \ell - 1$. It is true for $i = \ell$, since $F_{\ell}$ commutes with $K_j$ for all $j < \ell - 1$, and $K_{\ell-1}^{-1} F_{\ell} K_{\ell-1} = q^{-\frac{1}{2}} F_{\ell}$. The inductive step comes from the recursive relation

$$X_i = N_{i,\ell} [N_{i+1,\ell}^{-1} X_{i+1}, F_{i}]_q$$

that together with $F_i \stackrel{ad}{\rightarrow} K_j = q^{\delta_{i,j} - \frac{1}{2}(\delta_{i,j+1} + \delta_{i,j-1})}$ gives

$$X_i \stackrel{ad}{\rightarrow} K_j = N_{i,\ell} [N_{i+1,\ell}^{-1} X_{i+1} \stackrel{ad}{\rightarrow} K_j, F_i \stackrel{ad}{\rightarrow} K_j]_q$$

$$= q^{\frac{1}{2}(\delta_{i,j+1} - \delta_{i,j+1+1})} q^{\delta_{i,j} - \frac{1}{2}(\delta_{i,j+1} + \delta_{i,j-1})} N_{i,\ell} [N_{i+1,\ell}^{-1} X_{i+1}, F_i]_q$$

$$= q^{\frac{1}{2}(\delta_{i,j+1} - \delta_{i,j+1})} X_i ,$$

for all $i = 1, \ldots, \ell - 1$. Since $K_j M_{i,\ell}^* K_j^{-1} = q^{-\frac{1}{2}(\delta_{i,j} - \delta_{i,j+1})} M_{i,\ell}^*$, $N_{i,\ell} E_j N_{i,\ell}^{-1} = q^{\frac{1}{2}(\delta_{i,j} - \delta_{i,j+1})}$ and $K_j E_j K_j^{-1} = q E_j$, we have

$$X_i \stackrel{ad}{\rightarrow} E_j = K_j X_i E_j - q E_j X_i K_j$$

$$= K_j N_{i,\ell} M_{i,\ell}^* E_j - q E_j N_{i,\ell} M_{i,\ell}^* K_j$$

$$= K_j N_{i,\ell} \{ M_{i,\ell}^* E_j - q(K_j^{-1} N_{i,\ell}^{-1} E_j N_{i,\ell} K_j)(K_j^{-1} M_{i,\ell}^* K_j) \}$$

$$= K_j N_{i,\ell} [M_{i,\ell}^*, E_j]$$

and by (3.6b) and $N_{i,\ell} = K_i N_{i+1,\ell}$, this is

$$= K_j N_{i,\ell} \delta_{i,j} K_i^{-2} M_{i+1,\ell}^*$$

$$= \delta_{i,j} N_{i+1,\ell} M_{i+1,\ell}^*$$

$$= \delta_{i,j} X_{i+1} \equiv \{\sigma_1(E_j) X\}_i .$$
Finally,
\[ X_i \overset{\text{ad}}{\preceq} F_j = K_j N_{i,\ell} M_{i,\ell}^* F_j - q^{-1} F_j N_{i,\ell} M_{i,\ell}^* K_j \]
\[ = K_j N_{i,\ell}^{-1} \{ N_{i,\ell}^2 M_{i,\ell}^* F_j - q^{-1} (K_j^{-1} N_{i,\ell} F_j N_{i,\ell}^{-1} K_j) N_{i,\ell}^2 (K_j^{-1} M_{i,\ell}^* K_j) \} \]
\[ = K_j N_{i,\ell}^{-1} [N_{i,\ell}^2 M_{i,\ell}^*, F_j] \]
and by the adjoint of (3.9) this is
\[ = K_j N_{i,\ell}^{-1} \delta_{i,j+1} N_{i,\ell}^2 M_{i-1,\ell}^* \]
\[ = \delta_{i,j+1} N_{i-1,\ell} M_{i-1,\ell}^* \]
\[ = \delta_{i,j+1} X_{i-1} = \{ \sigma_1(F_j) X \}_i . \]

This concludes the proof. □

The following lemmas will be useful later.

**Lemma 3.14.** We have
\[ X_i X_j = q^{-1} X_j X_i , \] (3.40)
for all 1 ≤ i < j ≤ ℓ.

**Proof.** From \( X_i = N_{i,\ell} M_{i,\ell}^* \), and (3.8), which implies \( N_{j,\ell} M_{j,\ell}^* N_{j,\ell}^{-1} = q^{1/2} \delta_{j,j} M_{j,\ell}^* \), we deduce
\[ [X_i, X_j]_q = q^{1/2} \delta_{j,j}^{-1} N_{j,\ell} M_{j,\ell}^* [M_{j,\ell}, M_{i,\ell}]_q . \]
Now we prove, by induction on \( j \), that \([M_{j,\ell}, M_{i,\ell}]_q = 0 \) for all \( j > i \). For \( j = \ell \) we have
\[ [E_\ell, M_{i,\ell}]_q = 0 \]
by (3.10a). Then, if assume the claim true for a generic \( j, i+1 < j \leq \ell \), using the identity
\[ [[a, b]_q, c]_q = [a, [b, c]_q]_q + q^{-1} [[a, c], b] \]
we get
\[ [M_{j-1,\ell}, M_{i,\ell}]_q = [[E_{j-1}, M_{j,\ell}]_q, M_{i,\ell}]_q = [E_{j-1}, [M_{j,\ell}, M_{i,\ell}]_q]_q + q^{-1} [[E_{j-1}, M_{i,\ell}], M_{j,\ell}]_q . \]
The first term is zero by inductive hypothesis, and the second is zero since \([E_{j-1}, M_{i,\ell}]_q = 0 \) for all \( i < j - 1 < \ell \) by (3.10c). Thus, \([M_{j,\ell}, M_{i,\ell}]_q = 0 \) implies \([M_{j-1,\ell}, M_{i,\ell}]_q = 0 \) for all \( i+1 < j \leq \ell \), and this concludes the proof. □

**Lemma 3.15.** We have
\[ \Delta(X_i) = X_i \otimes N_{i,i-1} + \tilde{K}^{-1} \otimes X_i + q^{-1/2} (q - q^{-1}) \sum_{j=i}^{\ell-1} X_{j+1} \otimes N_{i,j} M_{i,j}^* . \] (3.41)
for all 1 ≤ i ≤ ℓ.
Suppose (3.42) gives the correct value of $\Delta(M)$. This immediately implies the claim for $X_i$. For $i = \ell$ the equality (3.42) holds

$$\Delta(E_\ell) = E_\ell \otimes K_\ell + K_\ell^{-1} \otimes E_\ell .$$

Suppose (3.42) gives the correct value of $\Delta(M_{i+1,\ell})$ for some $i + 1 \leq \ell$. Then

$$\Delta(M_{i,\ell}) = [\Delta(E_i), \Delta(M_{i+1,\ell})]_q$$

$$= [E_i \otimes K_\ell, M_{i+1,\ell} \otimes K_{i+1} \ldots K_\ell]_q + [K_i^{-1} \otimes E_i, M_{i+1,\ell} \otimes K_{i+1} \ldots K_\ell]_q$$

$$+ c \sum_{j=i+1}^{\ell-1} [E_i \otimes K_j, M_{j+1,\ell} (K_{i+1} \ldots K_j)^{-1} \otimes M_{i+1,\ell}]_q$$

$$+ c \sum_{j=i+1}^{\ell-1} [K_i^{-1} \otimes E_i, M_{j+1,\ell} (K_{i+1} \ldots K_j)^{-1} \otimes M_{i+1,\ell}]_q$$

$$= M_{i,\ell} \otimes K_\ell + \frac{1}{2} (q - q^{-1}) M_{i+1,\ell} K_\ell^{-1} \otimes E_i (K_{i+1} \ldots K_\ell)$$

$$+ 0 + (K_i \ldots K_\ell)^{-1} \otimes M_{i,\ell}$$

$$+ c \sum_{j=i+1}^{\ell-1} q^{-\frac{1}{2}} [E_i, M_{j+1,\ell}] (K_{i+1} \ldots K_j)^{-1} \otimes M_{i+1,j}, K_j (K_{i+1} \ldots K_\ell)$$

$$+ c \sum_{j=i+1}^{\ell-1} M_{j+1,\ell} (K_i \ldots K_j)^{-1} \otimes [E_i, M_{i+1,j}] q (K_{j+1} \ldots K_\ell)$$

$$= M_{i,\ell} \otimes K_\ell + (K_i \ldots K_\ell)^{-1} \otimes M_{i,\ell}$$

$$+ c \sum_{j=i}^{\ell-1} M_{j+1,\ell} (K_i \ldots K_j)^{-1} \otimes M_{i,j} (K_{j+1} \ldots K_\ell) ,$$

and last equation is exactly the right hand side of (3.42). \hfill \Box

**Lemma 3.16.** We have

$$[X_i^*, S^{-1}(X_j)] = q^{\frac{1}{2}} (K N_{j,i-1})^{-1} S^{-1}(M_{j,i-1})$$

for all $\ell \geq i > j \geq 1$ , \hspace{1cm} (3.43a)

$$[X_i^*, S^{-1}(X_i)] = -q \frac{(K_i \ldots K_\ell)^2 - (K_i \ldots K_\ell)^{-2}}{q - q^{-1}}$$

for all $i = 1, \ldots, \ell$ . \hspace{1cm} (3.43b)

**Proof.** Since $K_j X_j K_i^{-1} = q^{\frac{1}{2}} (\delta_{ij} - \delta_{j,i-1}) X_j$, we have

$$N_{i,\ell} X_j N_{i,\ell}^{-1} = q^{\frac{1}{2} (1 - \delta_{i,j})} X_j ,$$

and

$$[X_i^*, S^{-1}(X_j)] = q^{\frac{1}{2} (\delta_{ij} - 1)} (K_j \ldots K_{i-1})^{-1} [M_{i,\ell}, S^{-1}(M_{j,\ell})] ,$$

for all $i \geq j$. Thus, it is equivalent to prove the equalities

$$[M_{i,\ell}, S^{-1}(M_{j,\ell})] = q (K_i \ldots K_\ell)^{-2} S^{-1}(M_{j,\ell}^*)$$

for all $\ell \geq i > j \geq 1$ , \hspace{1cm} (3.44a)

$$[M_{i,\ell}, S^{-1}(M_{j,\ell})] = -q \frac{(K_i \ldots K_\ell)^2 - (K_i \ldots K_\ell)^{-2}}{q - q^{-1}}$$

for all $i = 1, \ldots, \ell$ . \hspace{1cm} (3.44b)
Let us start with (3.44a). We prove it by induction on $i$. For $i = \ell$ ($1 \leq j < i$), (3.6b) gives

$$[E_{\ell}, S^{-1}(M_{j,\ell}^*)] = qS^{-1}([E_{\ell}, M_{j,\ell}^*]) = qS^{-1}(M_{j,\ell-1}^* K_{\ell}^2) = qK_{\ell}^{-2} S^{-1}(M_{j,\ell-1}^*) .$$

Suppose, as claimed in (3.44a), that $[M_{i+1,\ell}, S^{-1}(M_{j,\ell}^*)] = q(K_{i+1} \ldots K_{\ell})^{-2} S^{-1}(M_{j,i}^*)$ for some $i + 1 \leq \ell$. Since $M_{i,\ell} = [E_{i}, M_{i+1,\ell}]q$ we have

$$[M_{i,\ell}, S^{-1}(M_{j,\ell}^*)] = [[E_{i}, S^{-1}(M_{j,\ell}^*)], M_{i+1,\ell}]q + [E_{i}, [M_{i+1,\ell}, S^{-1}(M_{j,\ell}^*)]]q .$$

But the first term is zero by (3.6b), since $j < i < \ell$ are all distinct, while in the second term we use the inductive hypothesis. Thus

$$[M_{i,\ell}, S^{-1}(M_{j,\ell}^*)] = q[E_{i}, (K_{i+1} \ldots K_{\ell})^{-2} S^{-1}(M_{j,i}^*)] = q(K_{i+1} \ldots K_{\ell})^{-2} S^{-1}([E_{i}, M_{j,\ell}^*]) .$$

Using again (3.6b) we find

$$[M_{i,\ell}, S^{-1}(M_{j,\ell}^*)] = q(K_{i+1} \ldots K_{\ell})^{-2} S^{-1}(M_{j,i-1}^* K_{\ell}^2) = q(K_{i} \ldots K_{\ell})^{-2} S^{-1}(M_{j,i-1}^*) ,$$

and this proves the inductive step, and then (3.44a).

We pass to (3.44b), and prove it by induction on $i$. Since $[M_{i,\ell}, S^{-1}(M_{j,\ell}^*)] = -q[E_{i}, F_{\ell}]$, the claim is true for $i = \ell$. We now show that the claim (3.44b) for $[M_{i+1,\ell}, S^{-1}(M_{j,i+1,\ell}^*)]$ implies the claim for $[M_{i,\ell}, S^{-1}(M_{j,i}^*)]$, for any $1 < i + 1 \leq \ell$. Since $M_{i,\ell} = [E_{i}, M_{i+1,\ell}]q$ we have

$$[M_{i,\ell}, S^{-1}(M_{j,\ell}^*)] = -q[[E_{i}, M_{i+1,\ell}]q, [F_{i}, S^{-1}(M_{j,i}^*)]]q .$$

But $E_{i}$ commutes with $S^{-1}(M_{j,i+1,\ell}^*)$ and $F_{i}$ commutes with $M_{i+1,\ell}$. We then use the following identity

$$[[a, b], c, [d, q]] = [[[a, b], c, q] + [c, [a, b], d], q] ,$$

that is valid whenever $[a, b] = [b, c] = 0$, to write

$$-q^{-1}[M_{i,\ell}, S^{-1}(M_{j,\ell}^*)] = [[[a, c], b, q] + [c, [a, b], d], q] ,$$

where in our case $a = E_{i}, b = M_{i+1,\ell}, c = F_{i}$ and $d = S^{-1}(M_{j,i+1,\ell}^*)$. By inductive hypothesis,

$$[b, d] = [M_{i+1,\ell}, S^{-1}(M_{j,i+1,\ell}^*)] = -q \frac{(K_{i+1} \ldots K_{\ell})^2 - (K_{i} \ldots K_{\ell})^2}{q - q^{-1}} ,$$

so

$$[a, [b, d]]q = -E_{i}(K_{i+1} \ldots K_{\ell})^2 ,$$

and

$$[c, [a, b], d]q] = [E_{i}, F_{i}](K_{i+1} \ldots K_{\ell})^2 = \frac{(K_{i+1} \ldots K_{\ell})^2 - K_{i}^{-2}(K_{i+1} \ldots K_{\ell})^2}{q - q^{-1}} .$$

Similarly,

$$[a, c] = \frac{K_{i}^2 - K_{i}^{-2}}{q - q^{-1}} ,$$

so

$$[[a, c], b]q = -M_{i+1,\ell}K_{i}^{-2} ,$$

and

$$[[a, c], b, q, d]q] = q^{-1}[S^{-1}(M_{i+1,\ell}^*), M_{i+1,\ell}K_{i}^{-2} .$$
that again by inductive hypothesis gives
\[ [[[a, c], b]_q, d]_q = \frac{K_i^{-2}(K_{i+1} \ldots K_L)^2 - (K_i \ldots K_L)^{-2}}{q - q^{-1}}. \]
The sum is
\[ [[[a, c], b]_q, d]_q + [c, [a, [b, d]]]_q = \frac{(K_i \ldots K_L)^2 - (K_i \ldots K_L)^{-2}}{q - q^{-1}}, \]
and this proves the inductive step.

\[ \square \]

4. THE QUANTUM PROJECTIVE SPACE $\mathbb{CP}_q^\ell$ AND EQUIVARIANT MODULES

We recall the definition of the Hopf $\ast$-algebra $\mathcal{A}(SU_q(\ell + 1))$, deformation of the algebra of representative functions on $SU(\ell + 1)$ (cf. [21], Sec. 9.2). As a $\ast$-algebra it is generated by elements $u_{i,j}^k$ ($i, j = 1, \ldots, \ell + 1$) with commutation relations
\[
\begin{align*}
    u_{i,j}^k u_{k,l}^j &= q u_{i,j}^k u_{k,l}^j, &\forall i < j, \\
    [u_{i,j}^k, u_{k,l}^j] &= 0 &u_{i,j}^k u_{k,j}^j &= (q - q^{-1}) u_{i,j}^k u_{k,j}^j &\forall i < j, k < l,
\end{align*}
\]
and with determinant relation
\[
\sum_{\pi \in S_{\ell+1}} (-q)^{|\pi|} u_{p(1)}^1 u_{p(2)}^2 \ldots u_{p(\ell+1)}^{\ell+1} = 1,
\]
where the sum is over all permutations $\pi$ of the set $\{1, 2, \ldots, \ell + 1\}$ and $|\pi|$ is the number of inversions in $\pi$. The $\ast$-structure is given by
\[
(u_{i,j}^k)^* = (-q)^{j-i} \sum_{\pi \in S_{\ell}} (-q)^{|\pi|} u_{p(1)}^{k_1} u_{p(2)}^{k_2} \ldots u_{p(n_\ell)}^{k_{n_\ell}}
\]
with $\{k_1, \ldots, k_\ell\} = \{1, \ldots, \ell + 1\} \setminus \{i\}$, $\{n_1, \ldots, n_\ell\} = \{1, \ldots, \ell + 1\} \setminus \{j\}$ (as ordered sets) and the sum is over all permutations $\pi$ of the set $\{n_1, \ldots, n_\ell\}$. Coproduct, counit and antipode are of ‘matrix’ type:
\[
\Delta(u_{i,j}^k) = \sum_k u_{i,k}^j \otimes u_{k,j}^j, \quad \epsilon(u_{i,j}^k) = \delta_{i,j}^{k,j}, \quad S(u_{i,j}^k) = (u_{i,j}^k)^*.
\]

The basic representation $\pi : U_q(\mathfrak{su}(\ell + 1)) \to \text{Mat}_{\ell+1}(\mathbb{C})$ of $U_q(\mathfrak{su}(\ell + 1))$, i.e. the one with highest weight $(n_1, \ldots, n_\ell) = (0, 0, \ldots, 0, 1)$, is given in matrix form by
\[
\pi_{i,j}^k(1) = \delta_{j}^{i,j}, \quad \pi_{i,j}^k(K_i) = \delta_{j}^{i,j} q^{\frac{1}{2}(\delta_{i+j,1} - \delta_{i,j})}, \quad \pi_{i,j}^k(E_i) = \delta_{i+1,j}^{i,j}, \quad (4.1)
\]
where $i \in \{1, \ldots, \ell\}$ and $j, k \in \{1, \ldots, \ell + 1\}$. Since $\pi(\hat{K}^{(\ell+1)/2})$ is a positive operator, $\pi(\hat{K}) := \pi(\hat{K}^{(\ell+1)/2})^{2/(\ell+1)}$ is well defined and the representation can be extended to the extension of $U_q(\mathfrak{su}(\ell + 1))$ by $\hat{K}$.

In our notation (that agrees with [21]), the action of a matrix $T = ((T_{i,j}^k))$ on a vector $v = (v^1, \ldots, v^{\ell+1})^t \in \mathbb{C}^{\ell+1}$ gives the vector $Tv$ with components
\[
(Tv)^j = \sum_{k=1}^{\ell+1} T_{j,k}^j v^k.
\]
That is, in $M_{\ell}^j$ the label $j$ indicates the row while the label $k$ indicates the column, and $v = (v^j)$ is a column vector. Modulo a replacement $\ell \to \ell - 1$, this representation is equivalent to the representation $W_\ell$ of $U_q(\mathfrak{su}(\ell))$ given in Sec. 3.2: the intertwiner is the map sending $v = (v^1, \ldots, v^{\ell})^t \in \mathbb{C}^\ell$ to the vector $w \in W_{\ell-1}$ with components given by $w_1, w_{j-1, \ldots, j+1, \ldots, \ell} = v^j$. 

With the representation (4.1) one defines a pairing between $U_q(\frak{su}(\ell+1))$ and $\mathcal{A}(SU_q(\ell+1))$ through the formulæ (cf. [21], Sec. 9.4)

$$\langle h, 1 \rangle := \epsilon(h), \quad \langle h, u^i_k \rangle := \pi^i_k(h), \quad \forall h \in U_q(\frak{su}(\ell + 1)), \quad (4.2)$$

and with this pairing one constructs the left and right canonical actions, $h \triangleright a = a_{(1)} \langle h, a_{(2)} \rangle$ and $a \triangleleft h = \langle h, a_{(1)} \rangle a_{(2)}$, which on generators are then given by

$$h \triangleright u^i_j = \sum_k u^i_k \pi^i_j(h), \quad u^i_j \triangleleft h = \sum_k \pi^i_k(h) u^i_j.$$  

Note that the Casimir $C_q$, being central, satisfies $C_q \triangleright a = a \triangleleft C_q$ for all $a \in \mathcal{A}(SU_q(\ell + 1))$.

Remark: by definition of left canonical action, the pairing between $U_q(\frak{su}(\ell + 1))$ and $\mathcal{A}(SU_q(\ell + 1))$ can be written as $\langle h, a \rangle = \epsilon(h \triangleright a)$. Since $\hat{K}^{(\ell+1)/2}$ is a positive diagonalizable (invertible) operator, it is immediate to extend the left canonical action to its (positive) root $\hat{K}$, and to extend the pairing to the extension of $U_q(\frak{su}(\ell + 1))$ by $\hat{K}$ and $\hat{K}^{-1}$.

Consider the following left action of $U_q(\frak{su}(\ell + 1))$ on $\mathcal{A}(SU_q(\ell + 1))$:

$$\mathcal{L}_h a := a \triangleleft S^{-1}(h). \quad (4.3)$$

Notice that the map $h \mapsto \mathcal{L}_h$ is a *-representation of $U_q(\frak{su}(\ell + 1))$ on $\mathcal{A}(SU_q(\ell + 1))$, for the inner product $(a, b) := \varphi(a^* b)$ on $\mathcal{A}(SU_q(\ell + 1))$ coming from the Haar state. Indeed, using the right invariance of $\varphi$, we get

$$(\mathcal{L}_h \cdot a, b) = \varphi(\{a \triangleleft S^{-1}(h^*)\}^* b) = \varphi(\{a^* \triangleleft h\} b)$$

$$= \varphi(\{a^* \triangleleft h_{(1)}\} \epsilon(h_{(2)}) b) = \varphi(\{a^* \triangleleft h_{(1)}\} \{b \triangleleft S^{-1}(h_{(3)}) h_{(2)}\})$$

$$= \varphi(\{a^* (b \triangleleft S^{-1}(h_{(2)}))\} \triangleleft h_{(1)}) = \epsilon(h_{(1)}) \varphi(\{a^* \triangleleft S^{-1}(h_{(2)})\})$$

$$= \varphi(a^* \{b \triangleleft S^{-1}(h)\}) = (a, \mathcal{L}_h b),$$

that means $\mathcal{L}_h^* = (\mathcal{L}_h)^\dagger$ for all $h \in U_q(\frak{su}(\ell + 1))$.

The algebra $\mathcal{A}(S^{2\ell+1}_q)$ of ‘functions’ on the unitary quantum sphere $S^{2\ell+1}_q$ is defined as

$$\mathcal{A}(S^{2\ell+1}_q) := \{a \in \mathcal{A}(SU_q(\ell + 1)) \mid \mathcal{L}_h a = \epsilon(h) a \forall h \in U_q(\frak{su}(\ell))\}.$$  

There is an isomorphism with the *-algebra generated by $\{z_i, z^*_i\}_{i=1,\ldots,\ell+1}$ with relations

$$z_iz_j = qz_jz_i \quad \forall i < j,$$

$$z^*_iz_j = qz_jz^*_i \quad \forall i \neq j,$$

$$[z^*_1, z_1] = 0,$$

$$[z^*_i, z_{i+1}] = (1 - q^2) \sum_{j=1}^{i} z_jz^*_j \quad \forall i = 1, \ldots, \ell,$$

$$z_1z^*_1 + z_2z^*_2 + \ldots + z_{\ell+1}z^*_{\ell+1} = 1,$$

given on generators by the identification $z_i = u^i_{i+1}$ [29].

The algebra $\mathcal{A}(\mathbb{C}P^\ell_q)$ of ‘functions’ on the quantum projective space $\mathbb{C}P^\ell_q$ is defined as

$$\mathcal{A}(\mathbb{C}P^\ell_q) := \{a \in \mathcal{A}(S^{2\ell+1}_q) \mid \mathcal{L}_K a = a\}.$$
Let $U_q(u(\ell))$ be the Hopf algebra generated by $U_q(u(\ell))$, $\hat{K}$ and $\hat{K}^{-1}$. If $\lambda$ is an $N$-dimensional $*$-representation of $U_q(u(\ell))$, the tensor product (over $\mathbb{C}$) $\mathcal{A}(SU_q(\ell+1)) \otimes \mathbb{C}^N \simeq \mathcal{A}(SU_q(\ell+1))^N$ carries a natural action of $U_q(u(\ell))$, that is the Hopf tensor product of $\mathcal{L}$ and $\lambda$. We call $\mathfrak{M}(\lambda) = \mathcal{A}(SU_q(\ell + 1)) \boxtimes \lambda \mathbb{C}^N$ the subset of $U_q(u(\ell))$-invariant elements:

$$\mathfrak{M}(\lambda) := \left\{ v \in \mathcal{A}(SU_q(\ell + 1))^N \mid \{ \mathcal{L}_{h(1)} \otimes \lambda(h(2)) \} v = \epsilon(h) v, \forall h \in U_q(u(\ell)) \right\},$$

where $v = (v_1, \ldots, v_N)^t$ is a column vector and row by column multiplication is understood. This is an $\mathcal{A}(\mathbb{C}P_q^\ell)$-bimodule and a left $\mathcal{A}(\mathbb{C}P_q^\ell) \times U_q(u(\ell + 1))$-module.

**Lemma 4.1.** Let $v \in \mathcal{A}(SU_q(\ell + 1))^N$. The following are equivalent:

(i) $\{ \mathcal{L}_{h(1)} \otimes \lambda(h(2)) \} v = \epsilon(h) v$ for all $h \in U_q(u(\ell))$;
(ii) $\lambda(S(h(1))) v \triangleleft h(2) = \epsilon(h) v$ for all $h \in U_q(u(\ell))$;
(iii) $v \triangleleft h = \lambda(h) v$ for all $h \in U_q(u(\ell))$.

**Proof.** Equivalence between (i) and (ii) is straightforward: just replace $h$ with $S(h)$.

Last condition can be rewritten as $\mathcal{L}_h v = \lambda(S^{-1}(h)) v$. Since $S^{-1}(h(2)) h(1) = \epsilon(h)$,

$$\mathcal{L}_h \otimes id_{\mathbb{C}^N} = (id_{\mathcal{A}(SU_q(\ell+1))} \otimes \lambda(S^{-1}(h(3))))(\mathcal{L}_{h(1)} \otimes \lambda(h(2))).$$

Thus (i) implies (iii): if $v$ is invariant,

$$\mathcal{L}_h v = \lambda(S^{-1}(h(3)))(\mathcal{L}_{h(1)} \otimes \lambda(h(2))) v = \lambda(S^{-1}(h(2))) \epsilon(h(1)) v = \lambda(S^{-1}(h)) v.$$

Vice versa, if $v \triangleleft h = \lambda(h) v$, we have

$$(\mathcal{L}_{h(1)} \otimes \lambda(h(2))) v = \lambda(h(2)) v \triangleleft S^{-1}(h(1)) = \lambda(h(2) S^{-1}(h(1))) v = \epsilon(h) v.$$

Hence, (iii) implies (i). \qed

We introduce an $\mathcal{A}(\mathbb{C}P_q^\ell)$-valued sesquilinear map:

$$\mathfrak{M}(\lambda) \times \mathfrak{M}(\lambda) \to \mathcal{A}(\mathbb{C}P_q^\ell), \quad (v, v') \mapsto v^\dagger v',$$

where $v^\dagger$ is the conjugate transpose of $v$ and row by column multiplication is understood. Indeed, if $v, v' \in \mathfrak{M}(\lambda)$, then $v^\dagger v' \in \mathcal{A}(\mathbb{C}P_q^\ell)$:

$$v^\dagger v' \triangleleft h = (v^\dagger \triangleleft h(1))(v' \triangleleft h(2)) = (v^\dagger \triangleleft h(1)) \lambda(h(2)) \lambda(S(h(3)))(v' \triangleleft h(4))$$

$$= (v^\dagger \triangleleft h(1)) \lambda(h(2)) \epsilon(h(3)) v' = \{ \lambda(h^*_t(2)) v \triangleleft S(h(1))^* \}^* v'$$

$$= \{ \lambda(S(t(1))) v \triangleleft t(2) \}^* v' = \epsilon(t)^* v^\dagger v' = \epsilon(h) v^\dagger v',$$

for all $h \in U_q(u(\ell))$, where we denoted $t := S(h)^*$.

Composing this map with the Haar functional $\varphi : \mathcal{A}(SU_q(\ell + 1)) \to \mathbb{C}$, we get a non-degenerate inner product on $\mathfrak{M}(\lambda)$, $(v, v') := \varphi(v^\dagger v')$. Thus $\mathfrak{M}(\lambda)$ are noncommutative ‘homogeneous Hermitian vector bundles’ over $\mathbb{C}P_q^\ell$. 

4.1. Line bundles on $\mathbb{C}P_q^\ell$. Let
\[
[n]! := [n][n-1] \ldots [1],
\]
for any $n \geq 1$ (and $[0]! := 1$). For $j_1, \ldots, j_{\ell+1} \in \mathbb{N}$, we define the $q$-multinomial coefficients as
\[
[j_1, \ldots, j_{\ell+1}] := \frac{[j_1 + \ldots + j_{\ell+1}]!}{[j_1]! \ldots [j_{\ell+1}]!} q^{-\sum r<s j_r j_s}.
\]
The following lemma is a generalization of a similar lemma for $\mathbb{C}P^2_q$ (cf. [15]).

**Lemma 4.2.** The generators $z_i$ of $A(S^\ell_q)$ satisfy
\[
\sum_{j_1 + \ldots + j_{\ell+1} = N} [j_1, \ldots, j_{\ell+1}]! z_1^{j_1} \ldots z_{\ell+1}^{j_{\ell+1}} = 1,
\]
for all $N \in \mathbb{N}$.

**Proof.** The equality
\[
[j_1 + \ldots + j_{\ell+1}] = \sum_{i=1}^{\ell+1} [j_i] q^\sum_{k=1}^{i-1} j_k - \sum_{k=i+1}^{\ell+1} j_k
\]
implies
\[
[j_1, \ldots, j_{\ell+1}]! = \sum_{i=1}^{\ell+1} [j_1, \ldots, j_i - 1, \ldots, j_{\ell+1}]! q^{-2 \sum_{k=i+1}^{\ell+1} j_k}.
\]
(4.4)

Let $c_N$ be the polynomial we want to compute,
\[
c_N := \sum_{j_1 + \ldots + j_{\ell+1} = N} [j_1, \ldots, j_{\ell+1}]! z_1^{j_1} \ldots z_{\ell+1}^{j_{\ell+1}}.
\]
We prove the lemma by induction on $N$. For $N = 1$, we get the spherical relation of the algebra $c_1 = \sum_{i=1}^{\ell+1} z_i^{j_i} = 1$. From this, it follows that $c_N$ can be rewritten as
\[
c_N = \sum_{j_1 + \ldots + j_{\ell+1} = N} [j_1, \ldots, j_{\ell+1}]! z_1^{j_1} \ldots z_{\ell+1}^{j_{\ell+1}} \left( \sum_{i=1}^{\ell+1} z_i^{j_i} \right) \left( z_1^{j_1} \ldots z_{\ell+1}^{j_{\ell+1}} \right)^{*},
\]
and using the commutation rules of the algebra ($z_i z_j = q z_j z_i$ for all $i < j$) we get
\[
c_N = \sum_{i=1}^{\ell+1} \sum_{j_1 + \ldots + j_{\ell+1} = N} [j_1, \ldots, j_{\ell+1}]! q^{-2 \sum_{k=i+1}^{\ell+1} j_k} z_1^{j_1} \ldots z_i^{j_i} \ldots z_{\ell+1}^{j_{\ell+1}} \left( z_1^{j_1} \ldots z_i^{j_i} \ldots z_{\ell+1}^{j_{\ell+1}} \right)^{*}
\]
\[
= \sum_{i=1}^{\ell+1} \sum_{j_1 + \ldots + j_{\ell+1} = N+1} [j_1, \ldots, j_i - 1, \ldots, j_{\ell+1}]! q^{-2 \sum_{k=i+1}^{\ell+1} j_k} z_i^{j_i} \ldots z_{\ell+1}^{j_{\ell+1}} \left( z_1^{j_1} \ldots z_i^{j_i} \ldots z_{\ell+1}^{j_{\ell+1}} \right)^{*}
\]
and using (4.4) we get
\[
= \sum_{j_1 + \ldots + j_{\ell+1} = N+1} [j_1, \ldots, j_{\ell+1}]! z_1^{j_1} \ldots z_{\ell+1}^{j_{\ell+1}} \left( z_1^{j_1} \ldots z_i^{j_i} \ldots z_{\ell+1}^{j_{\ell+1}} \right)^{*}
\]
\[
= c_{N+1}.
\]
This proves the inductive step. \qed
As a consequence, the vector $\Psi_N = (\psi^N_{j_1, \ldots, j_{\ell+1}})$ with components

$$\psi^N_{j_1, \ldots, j_{\ell+1}} := [j_1, \ldots, j_{\ell+1}]^{1/2} (z^1_{j_1} \cdots z^\ell_{j_{\ell+1}})^*,$$ \quad $\forall j_1 + \ldots + j_{\ell+1} = N,$ \hspace{0.5cm} (4.5)

satisfies $\Psi_N^\dagger \Psi_N = 1$. Thus $P_N := \Psi_N \Psi_N^\dagger$ is a projection.

We observe that the constraint (4.5), $j_1, \ldots, j_{\ell+1} \in \mathbb{N}$ and $j_1 + j_2 + \ldots + j_{\ell+1} = N$, holds if and only if

$$1 \leq j_1 + 1 < j_1 + j_2 + 2 < \ldots < j_1 + \ldots + j_{\ell} + \ell \leq j_1 + \ldots + j_{\ell+1} + \ell = N + \ell.$$ Therefore, the size of $P_N$, that is the number of $j_1, \ldots, j_{\ell+1}$ which sum up to $N$, equals the number of $\ell$-partitions of $N + \ell$, which is $\binom{N+\ell}{\ell}$.

Since $\Psi_N \triangleleft K_i = \Psi_N$ for $i \neq \ell$, and $\Psi_N \triangleleft K_\ell = q^{-N/2} \Psi_N$, we have

$$\mathcal{L}_K \Psi_N = \Psi_N \triangleleft \hat{K}^{-1} = q^{\ell/\ell+1} \Psi_N.$$ Let

$$\Gamma_N := \{a \in \mathcal{A}(S_q^{d+1}) \, | \, \mathcal{L}_K a = q^{\ell/\ell+1} a\}.$$ The map

$$\phi : \Gamma_N \rightarrow \mathcal{A}(\mathbb{CP}^\ell_q) \{^N_{\ell+1}\} P_N, \quad a \mapsto a \Psi_N^\dagger,$$

$$\phi^{-1} : \mathcal{A}(\mathbb{CP}^\ell_q) \{^N_{\ell+1}\} P_N \rightarrow \Gamma_N, \quad v \mapsto v \cdot \Psi_N,$$

with $v = (v^N_{j_1, \ldots, j_{\ell}})$ a row vector and $v \cdot \Psi_N$ the scalar product, is an isomorphism of left $\mathcal{A}(\mathbb{CP}^\ell_q)$-modules. The above discussion can be mirrored, exchanging the role of $z_i$ and $z_i^\ast$, to get a projective module description of the modules $\Gamma_N$ with $N < 0$. Notice that for $q = 1$, the projection giving $\Gamma_N$ is simply the transpose of the projection giving $\Gamma_{-N}$. The special case $S := \Gamma_{\frac{N}{2}}$, that exists only if $\ell$ is odd, is the $q$-analogue of the module of sections on the square root of the canonical bundle over $\mathbb{CP}^\ell$. In the case $\ell = 1$ (i.e. for the standard Podleś sphere), such a module has been discussed in Sec. 2.

5. Antiholomorphic forms on $\mathbb{CP}^\ell_q$

Recall that $U_q(\mathfrak{u}(\ell))$ is generated by $U_q(\mathfrak{su}(\ell))$, $\hat{K}$ and its inverse. Since it is a central extension of $U_q(\mathfrak{su}(\ell))$, any representation of $\lambda : U_q(\mathfrak{su}(\ell)) \rightarrow \text{End}(V)$ can be lifted to a representation $\hat{\lambda} : U_q(\mathfrak{u}(\ell)) \rightarrow \text{End}(V)$ by defining $\hat{\lambda}(\hat{K})$ to be any non-zero (and non-negative) multiple of the identity endomorphism. For the representation $\sigma_k : U_q(\mathfrak{su}(\ell)) \rightarrow \text{End}(W_k)$ studied in Sec. 3.2, and for any $N \in \mathbb{N}$, we define a lift $\sigma_k^N$ of $\sigma_k$ as

$$\sigma_k^N(\hat{K}) = q^{k-\ell/\ell+1} \cdot id_{W_k}.$$
5.1. **The algebra of forms.** We call \( \Omega^k_N := \mathfrak{M}(\sigma^N_k) \) and notice that \( \Omega^k := \Omega^k_0 \) is the \( q \)-analogue of the module antiholomorphic \( k \)-forms, and
\[
\Omega^k_N \simeq \Omega^k \otimes_{\mathcal{A}(\mathbb{C}P^\ell_q)} \Gamma_N
\]
consists of \( k \)-forms twisted with the ‘line bundle’ \( \Gamma_N \). For odd \( \ell \), chiral spinors are given by
\[
S_k := \Omega^k_{\frac{3}{2}(\ell+1)} \simeq \Omega^k \otimes_{\mathcal{A}(\mathbb{C}P^\ell_q)} \mathcal{S}.
\]
When \( q = 1 \) we have \( \Omega^k = \Lambda^k \Omega^1 \), where the antisymmetric tensor product is over the algebra \( \mathcal{A}(\mathbb{C}P^\ell) \). Being \( \Omega^1 \) the module of antiholomorphic 1-forms (classically, \( \partial a = (a \lhd X_1, \ldots, a \lhd X_\ell)^t \in \Omega^1 \) for all \( a \in \mathcal{A}(\mathbb{C}P^\ell) \), where \( X_i \) are defined in Lemma 3.13), \( \Omega^k \) are antiholomorphic \( k \)-forms, and \( S_k \) are chiral spinors. This justifies the our terminology. The geometrical meaning of \( \llcorner K \rceil \) acting on forms is also clear: eigenspaces consist of homogeneous forms, and the eigenvalue is \( q \)-degree of the forms.

Notice that \( \Omega^0_0 = \Omega^0_{\ell+1} = \mathcal{A}(\mathbb{C}P^\ell_q) \), and that \( \Omega^k_N \) is an \( \mathcal{A}(\mathbb{C}P^\ell_q) \)-bimodule and a left \( \mathcal{A}(\mathbb{C}P^\ell_q) \rtimes U_q(\mathfrak{su}(\ell+1)) \)-module (for all \( k, N \)). An associative product \( \cdot \in \Omega^k_N \times \Omega^k_N \rightarrow \Omega^{k+k'}_{N+N'} \) is obtained by composing the product in the Grassmann algebra \( \mathfrak{Gr}_q^\ell \) and the product in \( \mathcal{A}(SU_q(\ell+1)) \) (and extended linearly). For \( \omega = a\cdot v \in \Omega^k_N \) and \( \omega' = a'\cdot v' \in \Omega^k_{N'} \), with \( a, a' \in \mathcal{A}(SU_q(\ell+1)) \), \( v \in W_k, \ v' \in W_{k'} \), the product is given by
\[
\omega \cdot \omega' := a \cdot a' (v \cdot v').
\]
In particular when \( N = N' = 0 \), we have that \( \Omega^* := \bigoplus_{k=0}^\ell \Omega^k \) is a graded associative algebra.

Let us check that the above product is really an element of \( \Omega^{k+k'}_{N+N'} \). By (3.30),
\[
\sigma_{k+k'}(S(h_{(3)}))((\omega \cdot \omega')) \cdot h_{(2)} = (a \lhd h_{(3)}) \cdot (a' \lhd h_{(4)}) \sigma_k(S(h_{(2)}))v \cdot \sigma_{k'}(S(h_{(1)}))v'
\]
and since by hypothesis \( \omega \in \Omega^k_N \), i.e. \( \sigma_k(S(h_{(2)})) \cdot h_{(3)} = \epsilon(h_{(2)}) \cdot \omega \), this is
\[
= \omega \cdot \sigma_{k'}(S(h_{(1)}))\cdot h_{(2)}
\]
that using \( \omega' \in \Omega^k_{N'} \) gives
\[
= \epsilon(h) \omega \cdot \omega'.
\]
Since we have also \( (\omega \cdot \omega') \cdot \llcorner K \rceil = (\omega \cdot \llcorner K \rceil) \cdot (\omega' \cdot \llcorner K \rceil) = q^{k+k'} - \sigma_{k+k'}(N+N') \), we conclude that \( \omega \cdot \omega' \in \Omega^{k+k'}_{N+N'} \) for all \( \omega \in \Omega^k_N \) and \( \omega' \in \Omega^k_{N'} \). Associativity follows from associativity of both the products in \( \mathcal{A}(SU_q(\ell+1)) \) and \( \mathfrak{Gr}_q^\ell \).

We need few more lemmas.

**Lemma 5.1.** We have
\[
\omega_{1,2,\ldots,k,r} \lhd M_{r,s-1} = \omega_{1,2,\ldots,k,s} \quad (5.1)
\]
for all \( \omega \in \Omega^k_N \) and for all \( 0 \leq k < r < s \leq \ell \).
Proof. By definition of $\Omega^k_N$, $\omega_j \triangledown E_j = \{\sigma_k(E_j)\omega\}_{\hat{\omega}}$ for all $\omega \in \Omega^k_N$ and all $1 \leq j \leq \ell - 1$, and by (3.27b) we have:

$$\omega_{1,2,\ldots,k,r} \triangledown E_j = \delta_{r,j} \omega_{1,2,\ldots,k,r+1},$$

for all $j > k$. Using $M_{r,s-1} = [E_r, M_{r+1,s-1}]_q$ we get

$$\omega_{1,2,\ldots,k,r} \triangledown M_{r,s-1} = \omega_{1,2,\ldots,k,r+1} \triangledown M_{r+1,s-1},$$

which by iteration gives (5.1).

Lemma 5.2. For all $\omega \in \Omega^k_N$ and for all $1 \leq r \leq k < s \leq \ell$, we have

$$\omega_{1,\ldots,\hat{r},\ldots,k-1,k,s} \triangledown S^{-1}(M^*_{r,s-1}) = (-q)^{k-r+1}\omega_{1,\ldots,k}.$$

Here $\hat{r}$ means that $r$ is omitted from the list.

Proof. By definition of $\Omega^k_N$, $\omega_j \triangledown F_j = \{\sigma_k(F_j)\omega\}_{\hat{\omega}}$ for all $\omega \in \Omega^k_N$ and all $1 \leq j \leq \ell - 1$, and by (3.27c) we have

$$\omega_{1,\ldots,\hat{r},\ldots,k+1} \triangledown F_j = \delta_{r,j} \omega_{1,\ldots,\hat{r},\ldots,k+1,1}.$$

The equation $M_{r,s-1} = [E_r, M_{r+1,s-1}]_q$ implies $S^{-1}(M^*_{r,s-1}) = -q[F_r, S^{-1}(M^*_{r+1,s-1})]_q$. Using this, we get

$$\omega_{1,\ldots,\hat{r},\ldots,k+1} \triangledown S^{-1}(M^*_{r,k}) = -q\omega_{1,\ldots,\hat{r},\ldots,k+1} \triangledown S^{-1}(M^*_{r+1,k})$$

and by iteration

$$= (-q)^{k-r}\omega_{1,\ldots,k-1,k+1} \triangledown S^{-1}(M^*_{k,k})$$

$$= (-q)^{k-r+1}\omega_{1,\ldots,k+1} \triangledown F_k$$

$$= (-q)^{k-r+1}\omega_{1,\ldots,k}.$$

This proves the case $s = k + 1$. In the same way from

$$\omega_{1,\ldots,\hat{r},\ldots,k-1,k,s} \triangledown F_n = (1 - \delta_{n,k})\left(\delta_{n,r}\omega_{1,\ldots,\hat{r},\ldots,k-1,k,s} + \delta_{n,s-1}\omega_{1,\ldots,\hat{r},\ldots,k-1,k,s}\right)$$

we get

$$\omega_{1,\ldots,\hat{r},\ldots,k,s} \triangledown S^{-1}(M^*_{r,k}) = -q\omega_{1,\ldots,\hat{r},\ldots,k+1} \triangledown S^{-1}(M^*_{r+1,k})$$

and by iteration

$$= (-q)^{k-r+1}\omega_{1,\ldots,k-1,s} \triangledown F_k = 0,$$

and also

$$\omega_{1,\ldots,\hat{r},\ldots,k,s} \triangledown S^{-1}(M^*_{k+1,s-1}) = \omega_{1,\ldots,\hat{r},\ldots,k,s} \triangledown F_{s-1}S^{-1}(M^*_{k+1,s-2})$$

$$= \omega_{1,\ldots,\hat{r},\ldots,k,s-1} \triangledown S^{-1}(M^*_{k+1,s-2})$$

and by iteration

$$= -q\omega_{1,\ldots,\hat{r},\ldots,k+1}. $$
for all $s > k + 1$. For all $1 \leq r \leq k$ and $k + 1 < s \leq \ell$ we have $M_{r,s-1} = [M_{r,k} , M_{k + 1 , s-1}]_q$, thus $S^{-1}(M_{r,s-1}) = [S^{-1}(M_{r,k}) , S^{-1}(M_{k + 1 , s-1})]_q$ and

$$\omega_{1,\ldots,r,k,s} \in \omega_{1,\ldots,r-1,k,s} \omega_{r+1,\ldots,k,s} = \omega_{1,\ldots,r,k,s} \omega_{r+1,\ldots,k,s} \in \omega_{1,\ldots,r-1,k,s} \omega_{r+1,\ldots,k,s}.$$

This concludes the proof. \hfill \square 

**Lemma 5.3.** We have

$$\eta_{i+1} \omega_{1,\ldots,i,k} = -q \delta_{i,k} \eta_j \quad (5.2)$$

for all $\eta \in \Omega^1_N$ and for all $0 \leq i \leq \ell - 1$ and $1 \leq j \leq k \leq \ell - 1$.

**Proof.** From $\eta_{i+1} \omega_{n} = \delta_{i,n} \eta_i$, we get

$$\eta_{i+1} \omega_{j,k} = 0 \quad \text{if } i < j \text{ or } i > k.$$

We have $M_{j,k} = [M_{j,n} , M_{n+1,k}]_q$, i.e. $S^{-1}(M_{j,k}) = [S^{-1}(M_{j,n}) , S^{-1}(M_{n+1,k})]_q$ for all $j \leq n < k$. In particular $S^{-1}(M_{j,i}) = -q[S^{-1}(M_{j,i-1}) , F_i]_q$ applied to $\eta_{i+1}$ gives

$$\eta_{i+1} \omega_{1,\ldots,i,k} = \eta_{i+1} \omega_{1,\ldots,i,j} \omega_{j+1,\ldots,i,k} = \eta_{i+1} \omega_{1,\ldots,i,j} \omega_{j+1,\ldots,i,k} = \eta_{i} \omega_{1,\ldots,i,j} \omega_{j+1,\ldots,i,k} = \eta_{i} \omega_{1,\ldots,i,j} \omega_{j+1,\ldots,i,k} = \eta_{j+1} \omega_{1,\ldots,i+1,k} \omega_{i+1,\ldots,j}$$

that iterated gives

$$\eta_{i+1} \omega_{1,\ldots,i,k} = \eta_{j+1} \omega_{1,\ldots,i,j} \omega_{i+1,\ldots,j} = -q \eta_{j+1} \omega_{1,\ldots,j} \omega_{j+1,\ldots,j} = -q \eta_{j} \omega_{1,\ldots,j} \omega_{j+1,\ldots,j} = -q \eta_{j} \omega_{1,\ldots,j} \omega_{j+1,\ldots,j}.$$

Finally, for $j \leq i < k$ we have

$$\eta_{i+1} \omega_{1,\ldots,i,k} = \eta_{i+1} \omega_{1,\ldots,i,j} \omega_{j+1,\ldots,i,k} = \eta_{i+1} \omega_{1,\ldots,i,j} \omega_{j+1,\ldots,i,k} = \eta_{i} \omega_{1,\ldots,i,j} \omega_{j+1,\ldots,i,k} = \eta_{j+1} \omega_{1,\ldots,j} \omega_{j+1,\ldots,j} = -q \eta_{j} \omega_{1,\ldots,j} \omega_{j+1,\ldots,j}.$$

This concludes the proof. \hfill \square 

The following lemma simplifies considerably the computations with forms.

**Lemma 5.4.** Two elements $\omega , \omega' \in \Omega^k_N$ are equal if and only if $\omega_{1,2,\ldots,k} = \omega'_{1,2,\ldots,k}$. Moreover, for all $\omega , \omega' \in \Omega^k_N$ and $i \in \Lambda_k$, we have

$$\varphi(\omega' \omega) = \frac{q^{2(1-k)(1+i)}}{\dim_q W_k} \langle \omega , \omega' \rangle \quad (5.3)$$

where the quantum dimension $\dim_q W_k$ is given explicitly in Lemma 3.12.

**Proof.** Since

$$\omega_{i} = \omega_{1,2,\ldots,k} \omega(E_k E_{k+1} \ldots E_{i-1}) \ldots (E_2 E_3 \ldots E_{i-1}) \omega(E_1 E_2 \ldots E_{i-1}),$$

for all $\omega \in \Omega^k_N$ and all $i \in \Lambda_k$, if $\omega_{1,2,\ldots,k}$ is zero, all components of $\omega$ are zero. By linearity, this proves the first claim.
By inverting the previous transformation we get

$$\omega_{1,2,\ldots,k} = \omega_1^j \circ (F_1 \cdot 1 \cdot F_1 \cdot 2 \cdot \ldots \cdot F_1) (F_2 \cdot 1 \cdot F_2 \cdot 2 \cdot \ldots \cdot F_2) \ldots (F_k \cdot 1 \cdot F_k \cdot 2 \cdot \ldots \cdot F_k). \quad (5.4)$$

Assume $j \# i = -1$, so that $\omega_1^j \cdot K_j = q^{1/2} \omega_1^j$, $\omega_1^i \cdot K_j^{-1} = q^{-1/2} \omega_1^i$,

$$\omega_1^j \cdot F_j E_j = \{ \sigma_k(F_j) \omega_1^j \}_i \cdot E_j = \omega_{1,j-} \cdot E_j = \{ \sigma_k(E_j) \omega_1^j \}_i \cdot E_j = \omega_{(j),j,+} = \omega_1^i,$$

and by covariance of the action

$$q^{\frac{1}{2}} \{ \omega_1^j (\omega_1^i \cdot F_j) \} \cdot E_j = \omega_1^i \{ \omega_1^j \cdot F_j E_j \} - q^2 (\omega_1^i \cdot F_j) \{ \omega_1^j \cdot F_j \}$$

$$= \omega_1^i \omega_1^j - q^2 (\omega_1^i \cdot F_j)^* (\omega_1^j \cdot F_j).$$

Applying the Haar state $\varphi$ to both sides of this identity we get

$$0 = \varphi(\omega_1^i \omega_1^j) - q^2 \varphi(\omega_1^i \cdot F_j)^* (\omega_1^j \cdot F_j),$$

where the left hand side vanishes due to invariance of $\varphi$. By iterated use of the last equation, together with (5.4), we get

$$\varphi(\omega_1^i \omega_1^j) = q^{2(i_1+1+i_2+2+\ldots+i_k-k)} \varphi(\omega_{1,2,\ldots,k} \omega_{1,2,\ldots,k}^*) = q^{2|q|^{-k}} \varphi(\omega_{1,2,\ldots,k} \omega_{1,2,\ldots,k}^*).$$

Sum over $i$, using Lemma 3.12, gives

$$\langle \omega, \omega' \rangle = q^{k(k-1)} \dim_q W_k \varphi(\omega_{1,2,\ldots,k} \omega_{1,2,\ldots,k}^*),$$

and $\varphi(\omega_1^i \omega_1^j) = q^{k(k-1)} (\dim_q W_k)^{-1} \langle \omega, \omega' \rangle$. This concludes the proof. \hfill \square

5.2. Harmonic decomposition of $\Omega^k_N$. Since left and right canonical actions of $U_q(\mathfrak{su}(\ell+1))$ are mutually commuting, the space $\bigoplus_{k=0}^\ell \Omega^k_N$, $N \in \mathbb{Z}$, carries a left action of $U_q(\mathfrak{su}(\ell+1))$. We decompose it into irreducible representations using the Gelfand-Tsetlin basis.

**Proposition 5.5.** As left $U_q(\mathfrak{su}(\ell+1))$-modules we have the following equivalences:

$$\Omega^0_N \simeq \begin{cases} \bigoplus_{m \in \mathbb{N}} V_{m+N,0,\ldots,0,m}, & \text{if } N \geq 0, \\
\bigoplus_{m \in \mathbb{N}} V_{m,0,\ldots,0,m-N}, & \text{if } N < 0 \end{cases},$$

$$\Omega^k_N \simeq \begin{cases} \bigoplus_{m \in \mathbb{N}} V_{m+N-k,0,\ldots,0,m} + \varepsilon_k \bigoplus V_{m+N-k-1,0,\ldots,0,m} + \varepsilon_{k+1}, & \text{if } 1 \leq k \leq \min(N,\ell) - 1, \\
\bigoplus_{m \in \mathbb{N}} V_{m,0,\ldots,0,m-N+k} + \varepsilon_k \bigoplus V_{m,0,\ldots,0,m-N+k+1} + \varepsilon_{k+1}, & \text{if } \max(1,N) \leq k \leq \ell - 1 \end{cases},$$

$$\Omega^\ell_N \simeq \begin{cases} \bigoplus_{m \in \mathbb{N}} V_{m,0,\ldots,0,m-N+\ell+1}, & \text{if } N \leq \ell, \\
\bigoplus_{m \in \mathbb{N}} V_{m+N-\ell,0,\ldots,0,m}, & \text{if } N > \ell, \end{cases},$$

where $\varepsilon_k = (e_k^1, \ldots, e_k^\ell)$ is the $\ell$-tuple with components $e_k^j = \delta_{j,k}$ and $V_m$ is the vector space carrying the irreducible representation $\rho_n$. The trivial representation $V_{(0,\ldots,0)}$ appears (with multiplicity 1) in just two cases: as a subrepresentation of $\Omega^0_N$ for $N = 0$, and as a subrepresentation of $\Omega^\ell_N$ for $N = \ell + 1$. 

Proof. By Peter-Weyl theorem, the algebra $\mathcal{A}(SU_q(\ell + 1))$ is a multiplicity free direct sum of representations $\rho_n \otimes \rho_n^*$ of $U_q(\mathfrak{su}(\ell + 1)) \otimes U_q(\mathfrak{su}(\ell + 1))$, where $n = (n_1, \ldots, n_\ell)$ runs in $\mathbb{N}^\ell$ and $\rho_n^*$ is the representation dual to $\rho_n$. We have

$$\Omega_N^k \simeq \bigoplus_n (\sigma_k^N, \rho_n^*) V_n$$

where $(\sigma_k^N, \rho_n^*)$ is the multiplicity of the irreducible representation $\sigma_k^N$ of $U_q(\mathfrak{u}(\ell))$ inside the representation $\rho_n^*$ of $U_q(\mathfrak{su}(\ell + 1))$. We have $\rho_n^* \simeq \rho_{n'}$, where $n' := (n_\ell, \ldots, n_1)$. A basis of $V_{n'}$ is given by Gelfand-Tsetlin tableaux (GT tableaux, for short), which are arrays of integers of the form

$$\begin{pmatrix}
  r_{1,1} & r_{1,2} & \ldots & r_{1,\ell} & 0 \\
  r_{2,1} & r_{2,2} & \ldots & r_{2,\ell} \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{\ell+1,1} & r_{\ell+1,2} \\
\end{pmatrix}$$

(5.5)

with $r_{i,j} \geq r_{i+1,j} \geq r_{i,j+1}$ for all $i,j$ (from now on it is understood that $r_{ij} = 0$ if $i,j$ are out of range, i.e. if $i < 1$, $j < 1$, or $i + j > \ell + 2$). The highest weight $n'$ has entries $n'_{i,j} = n_{1,i} - r_{1,i+1}$. Usually GT tableaux are defined modulo a global rescaling (two arrays are equivalent if they differ by a constant). Here for each equivalence class of GT tableaux, we choose the representative which has zero in the top-right corner, $r_{1,\ell+1} = 0$. If we remove the first row from (5.5) we obtain a GT tableau of $U_q(\mathfrak{su}(\ell))$ with highest weight $m = (m_1, \ldots, m_{\ell-1})$ given by $m_i = r_{2,i} - r_{2,i+1}$. In particular, $\sigma_k$ appears (with multiplicity 1) in $\rho_{n'}$ if and only if

$$n' = (n'_1, 0, \ldots, 0, n'_\ell) + \xi_{\ell-k} \quad \text{or} \quad n' = (n'_1, 0, \ldots, 0, n'_\ell) + \xi_{\ell-k+1},$$

(5.6)

for $1 \leq k \leq \ell - 1$, and if and only if $n' = (n'_1, 0, \ldots, 0, n'_\ell)$ for $k = 0, \ell$.

In Gelfand-Tsetlin notations [17], $K_i = q^{\frac{1}{2}H_i}$ is represented by $q^{\frac{1}{2}(E_{i+1,i+1} - E_{i,i})}$, thus

$$\hat{K}^{\ell+1} = q^{-\ell E_{1,1} + \sum_{i=1}^\ell E_{i,i}}.$$  

The eigenvalue of $E_{i,i}$ when applied to the generic GT tableau is

$$\sum_{j=1}^i r_{\ell+2-i,j} - \sum_{j=1}^{i-1} r_{\ell+3-i,j},$$

and then the eigenvalue of $\hat{K}$ is

$$q^{\sum_{j=1}^i (r_{\ell+1-j} - r_{2,j})}.$$  

So, for $k = 0, \ell$ we have $\hat{K}^{\ell+1} = q^{(n'_1 - n'_2)} \cdot \text{id}$, and for $1 \leq k \leq \ell - 1$ we have, in the two cases listed in (5.6), $\hat{K}^{\ell+1} = q^{(n'_1 - n'_2) - \ell + k} \cdot \text{id}$ resp. $\hat{K}^{\ell+1} = q^{(n'_1 - n'_2) + k} \cdot \text{id}$. In each case, the eigenvalue of $\hat{K}^{\ell+1}$ must be equal to $q^{(\ell+1)k - \ell N}$, that means

- if $k = 0 \Rightarrow n'_1 = n'_2 - N$,
- if $1 \leq k \leq \ell - 1 \Rightarrow n'_1 = n'_2 + k + 1 - N$ or $n'_1 = n'_2 + k - N$,
- if $k = \ell \Rightarrow n'_1 = n'_2 + \ell + 1 - N$.

Recalling that $n_j = n'_{\ell+1-j}$, we get Proposition 5.5. \qed
5.3. The Dolbeault operator. We take as a $q$-analogue of the Dolbeault operator
\[
\bar{\partial} := \sum_{i=1}^{\ell} \mathcal{L}_{KX_i} \otimes e_i^L,
\] (5.7)
where $X_i$ are the operator in Lemma 3.13 and $e_i^L$ is the exterior product in (3.35). Explicitly, for any $\omega \in \Omega_N^k$,
\[
(\bar{\partial} \omega)_k = \sum_{r=1}^{k+1} (-q)^{1-r} \omega_{k-r} \wedge S^{-1}(KX_r).
\]
Being $h \mapsto \mathcal{L}_h$ a $*$-representation, the Hermitian conjugate $\bar{\partial}^\dagger$ of $\bar{\partial}$ is
\[
\bar{\partial}^\dagger = \sum_{i=1}^{\ell} \mathcal{L}_{X_i^* K} \otimes e_i^L.
\] (5.8)

**Proposition 5.6.** The operator $\bar{\partial}$ maps $\Omega_N^k$ in $\Omega_N^{k+1}$ and satisfies $\bar{\partial}^2 = 0$. The operator $\bar{\partial}^\dagger$ maps $\Omega_N^{k+1}$ in $\Omega_N^k$ and satisfies $(\bar{\partial}^\dagger)^2 = 0$.

**Proof.** Since $h \mapsto \mathcal{L}_h$ is a representation, the algebraic identity $ht = (t \overset{ad}{\to} S^{-1}(h(2)))h(1)$ implies
\[
\mathcal{L}_h \mathcal{L}_t = \mathcal{L}_{S^{-1}(h(2))}^{ad} \mathcal{L}_{h(1)}
\]
for all $h, t \in U_q(\mathfrak{su}(\ell+1))$. By (3.30) we have also
\[
\sigma_{k+1}^N(h)(x \wedge_q v) = \{\sigma_{1}^0(h(1))x\} \wedge_q \{\sigma_{k}^N(h(2))v\}
\]
for all $h \in U_q(\mathfrak{u}(\ell))$, $x \in W_1$ and $v \in W_k$. This means $\sigma_{k+1}^N(h)e_i^L = e_i^L \sigma_{1}^0(h(1))x \cdot \sigma_{k}^N(h(2))$. Thus, for all $h \in U_q(\mathfrak{u}(\ell))$,
\[
\{\mathcal{L}_{h(1)} \otimes \sigma_{k+1}^N(h(2))\} \bar{\partial} = \sum_{i=1}^{\ell} \mathcal{L}_{h(1)} \mathcal{L}_{KX_i} \otimes \sigma_{k+1}^N(h(2))e_i^L
\]
\[
= \sum_{i=1}^{\ell} \mathcal{L}_{KX_i} \ominus \wedge S^{-1}(h(2)) \mathcal{L}_{h(1)} \otimes e_i^L \sigma_{1}^0(h(1))x \cdot \sigma_{k}^N(h(2)).
\]
But $K$ commutes with all $h \in U_q(\mathfrak{u}(\ell))$, and $^q_S(h(2)) \otimes \sigma_{1}^0(h(3))e_i^L = (h(2)) \sum_{i} X_i \otimes e_i$ by Lemma 3.13. We conclude that
\[
\{\mathcal{L}_{h(1)} \otimes \sigma_{k+1}^N(h(2))\} \bar{\partial} = \bar{\partial} \{\mathcal{L}_{h(1)} \otimes \sigma_{k}^N(h(2))\},
\]
for all $h \in U_q(\mathfrak{u}(\ell))$. Hence, $\bar{\partial}$ maps invariant elements into invariant elements, and $\bar{\partial}(\Omega_N^k) \subset \Omega_N^{k+1}$. By adjunction, being all representations unitary,
\[
\bar{\partial}^\dagger \{\mathcal{L}_{h(1)}^* \otimes \sigma_{k+1}^N(h^*_2)\} = \{\mathcal{L}_{h(1)}^* \otimes \sigma_{k}^N(h^*_2)\} \bar{\partial}^\dagger.
\]
Hence, $\bar{\partial}^\dagger$ maps invariant elements into invariant elements, and $\bar{\partial}^\dagger(\Omega_N^{k+1}) \subset \Omega_N^k$.

Now we prove that $\bar{\partial}^2 = 0$ (and by adjunction $(\bar{\partial}^\dagger)^2 = 0$). Using the associativity of the wedge-product, and $X_iK = qKX_i$ we get
\[
\bar{\partial}^2 \omega = q \omega \wedge_q \wedge S^{-1}(X \wedge_q X) K^{-2},
\]
where $X = \sum_i e^i X_i$. But from (3.40) it follows that
\[
(X \wedge_q X)_{i_1, i_2} = X_{i_1} X_{i_2} - q^{-1} X_{i_2} X_{i_1} = 0
\]
for all $i_1 < i_2$, thus $X \wedge_q X = 0$ and $\bar{\partial}^2 = 0$. 
\[\Box\]
Thus for any \( N \), we have a left \( U_q(\mathfrak{su}(\ell+1)) \)-covariant cohomology complex \( (\Omega^*_N, \partial) \) over \( \Omega^0_N \). In particular if \( N = 0 \), this complex gives a differential calculus over \( A(\mathbb{C}P^k_q) \) (if \( N \neq 0 \), \( \Omega^*_N \) is not closed under the wedge product). In fact, we now prove two different Leibniz properties of \( \partial \).

**Lemma 5.7.** We have
\[
\partial(a\omega) = a(\partial\omega) + (\partial a) \wedge_q \omega ,
\]
for all \( a \in A(\mathbb{C}P^k_q) \) and \( \omega \in \Omega^k_N \).

**Proof.** Since \( S^{-1} \) is anticommutative, by applying \( S^{-1} \) to (3.41) we get
\[
\Delta(S^{-1}(\hat{K}X_i)) = S^{-1}(\hat{K}X_i) \otimes 1 + N^{-1}_{x,y} \hat{K}^{-1} \otimes S^{-1}(\hat{K}X_i)
\]
\[
+ q^{-\frac{1}{2}}(q - q^{-1}) \sum_{j=i}^{t-1} \bar{K}^{-1}N^{-1}_{ij} S^{-1}(M^*_i) \otimes S^{-1}(\hat{K}X_j + 1) .
\]

With this, by using the covariance of the right action and the right \( U_q(\mathfrak{u}(\ell)) \)-invariance of \( a \), we prove the identity in (5.9):
\[
\{\partial(a\omega)\}_2 = \sum_{r=1}^{k+1} (-q)^{1-r}(a\omega)_2^{-i-r} S^{-1}(\hat{K}X_i)
\]
\[
= \sum_{r=1}^{k+1} (-q)^{1-r} \left\{ (a \circ S^{-1}(\hat{K}X_i))_{\omega_2^{-i-r}} + (a \circ N^{-1}_{i,j} \hat{K}^{-1})(\omega_2^{-i-r} \circ S^{-1}(\hat{K}X_i)) \right\}
\]
\[
+ q^{-\frac{1}{2}}(q - q^{-1}) \sum_{r=1}^{k+1} (-q)^{1-r} \sum_{j=i}^{t-1} (a \circ \hat{K}^{-1}N^{-1}_{ij} S^{-1}(M^*_i))_{(\omega_2^{-i-r} \circ S^{-1}(\hat{K}X_j + 1))}
\]
\[
= \sum_{r=1}^{k+1} (-q)^{1-r} \left\{ (a \circ S^{-1}(\hat{K}X_i))_{\omega_2^{-i-r}} + a(\omega_2^{-i-r} \circ S^{-1}(\hat{K}X_i)) \right\}
\]
\[
= \{(\partial a) \wedge_q \omega + a(\partial \omega)\}_2 .
\]

Hence, the commutator of \( \partial \) with the operator of left multiplication by a ‘function’ \( a \) gives the left multiplication by the differential of \( a \), and this will be useful to construct spectral triples. The second Leibniz property – that is more difficult to prove – tells us that \( \partial \) is a graded derivation on \( \Omega^*_N \).

**Proposition 5.8.** The datum \( (\Omega^*_N, \partial) \) is a left \( U_q(\mathfrak{su}(\ell+1)) \)-covariant differential calculus over \( A(\mathbb{C}P^k_q) \). That is, for all \( \omega, \omega' \in \Omega^*_N \):
\[
\partial(\omega \wedge_q \omega') = (\partial \omega) \wedge_q \omega' + (-1)^k \omega \wedge_q (\partial \omega') ,
\]
where \( k \) is the degree of \( \omega \).

**Proof.** If \( k = 0 \), (5.11) is a particular case of (5.9).

Now we consider the case \( k = 1 \). Let \( \eta \in \Omega^0_0 \) and \( \omega' \in \Omega^k_0 \). By Lemma 5.4, in order to prove the equation
\[
\partial(\eta \wedge_q \omega') = (\partial \eta) \wedge_q \omega' - \eta \wedge_q (\partial \omega')
\]
(5.12) it is enough to show that
\[
\{\partial(\eta \wedge_q \omega')\}_{1,2,\ldots,k'+2} = \{(\partial \eta) \wedge_q \omega'\}_{1,2,\ldots,k'+2} - \{\eta \wedge_q (\partial \omega')\}_{1,2,\ldots,k'+2} .
\]
We have

\[
\{ \partial (\eta \wedge \omega') \}_{1,2,\ldots,k'+2} = \sum_{1 \leq s < r \leq k'+2} (-q)^{2-r-s} (\eta_s \omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r)
+ \sum_{1 \leq r < s \leq k'+2} (-q)^{3-r-s} (\eta_s \omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r),
\]

and by (5.10):

\[
\{ \partial (\eta \wedge \omega') - (\partial \eta) \wedge \omega' \}_{1,2,\ldots,k'+2} = \sum_{1 \leq s < r \leq k'+2} (-q)^{2-r-s} (\eta_s \triangleq N^{-1}_{r,r-1} \hat{K}^{-1}(\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r))
+ q^{-\frac{1}{2}}(q^{-1}) \sum_{1 \leq s < r \leq k'+2} \sum_{j=r}^{\ell-1} (-q)^{2-r-s} (\eta_s \triangleq \hat{K}^{-1}N^{-1}_{r,j} S^{-1}(M^*_r)) (\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_{j+1})
+ \sum_{1 \leq r < s \leq k'+2} (-q)^{3-r-s} (\eta_s \triangleq N^{-1}_{r,r-1} \hat{K}^{-1}(\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r))
+ q^{-\frac{1}{2}}(q^{-1}) \sum_{1 \leq r < s \leq k'+2} \sum_{j=r}^{\ell-1} (-q)^{3-r-s} (\eta_s \triangleq \hat{K}^{-1}N^{-1}_{r,j} S^{-1}(M^*_r)) (\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_{j+1}).
\]

By (5.2), all the terms with \(j \neq s - 1\) are zero. Thus

\[
\{ \partial (\eta \wedge \omega') - (\partial \eta) \wedge \omega' \}_{1,2,\ldots,k'+2} = \sum_{1 \leq s < r \leq k'+2} (-q)^{2-r-s} (\eta_s \triangleq N^{-1}_{r,r-1} \hat{K}^{-1}(\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r))
+ \sum_{1 \leq r < s \leq k'+2} (-q)^{3-r-s} (\eta_s \triangleq N^{-1}_{r,r-1} \hat{K}^{-1}(\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r))
+ q^{-\frac{1}{2}}(q^{-1}) \sum_{1 \leq r < s \leq k'+2} (-q)^{3-r-s} (\eta_s \triangleq \hat{K}^{-1}N^{-1}_{r,s-1} S^{-1}(M^*_r)) (\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_s).
\]

We have

\[
\eta_s \triangleq \hat{K}^{-1}N^{-1}_{r,r-1} = q^{-1} \eta_s \quad \text{if } r > s ,
\eta_s \triangleq \hat{K}^{-1}N^{-1}_{r,r-1} = q^{-1} \eta_s \quad \text{if } r < s ,
\eta_s \triangleq \hat{K}^{-1}N^{-1}_{r,r-1} = q^{-3} \eta_s \quad \text{if } r < s .
\]

We have also \(\eta_s \triangleq S^{-1}(M^*_{r,s-1}) = -q \eta_r\) by (5.2). Thus,

\[
\{ \partial (\eta \wedge \omega') - (\partial \eta) \wedge \omega' \}_{1,2,\ldots,k'+2} = -q^{-2} \sum_{1 \leq s < r \leq k'+2} (-q)^{3-r-s} \eta_s (\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r)
- \sum_{1 \leq r < s \leq k'+2} (-q)^{2-r-s} \eta_s (\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_r)
+ (q^{-2} - 1) \sum_{1 \leq r < s \leq k'+2} (-q)^{3-r-s} \eta_r (\omega'_{(1,\ldots,k'+2)\setminus\{r,s\}}) \triangleq S^{-1}(\hat{K}X_s).\]
If we exchange $r$ with $s$ in last summation, we get
\[
\{ \tilde{\partial}(\eta \wedge \omega') - (\tilde{\partial}\eta) \wedge \omega' \}_{1, \ldots, k'+2} = - \sum_{1 \leq r < s \leq k'+2} (-q)^{3-r-s} \eta_s (\omega'_{(1, \ldots, k'+2) \setminus \{r, s\}} \lhd S^{-1}( \hat{K} X_r ) ) - \sum_{1 \leq r < s \leq k'+2} (-q)^{2-r-s} \eta_s (\omega'_{(1, \ldots, k'+2) \setminus \{r, s\}} \lhd S^{-1}( \hat{K} X_r ) ) = - \{ \eta \wedge \eta (\tilde{\partial}\omega') \}_{1, \ldots, k'+2} .
\]

This proves (5.12).

Now let $k \geq 1$. Since $\Omega^k_N$ is generated by 1-forms, we can write any $k$-form as a sum of elements $\omega = \eta^1 \wedge \eta^2 \wedge \ldots \wedge \eta^k$, with $\eta^j \in \Omega^1_N$. By iterated use of (5.12) we get
\[
\tilde{\partial}(\omega \wedge \omega') = \tilde{\partial}(\eta^1 \wedge \eta^2 \wedge \ldots \wedge \eta^k \wedge \omega') = (\tilde{\partial}\eta^1) \wedge \eta^2 \wedge \ldots \wedge \eta^k \wedge \omega' - \eta^1 \wedge \eta^2 \wedge \ldots \wedge \eta^k \wedge (\tilde{\partial}\omega') = \sum_{j=1}^{k} (-1)^{j-1} \eta^1 \wedge \ldots \wedge (\partial \eta^j) \wedge \eta^j \wedge \ldots \wedge \eta^k \wedge \omega' + (-1)^{k} \eta^1 \wedge \ldots \wedge \eta^k \wedge (\tilde{\partial}\omega') = (\tilde{\partial}\omega) \wedge \omega' + (-1)^{k} \omega \wedge (\tilde{\partial}\omega') .
\]

This concludes the proof. \qed

5.4. **Reality and the first order condition.** If we tensor the operator $J$ in (3.33) with the involution ‘*’ on $\mathcal{A}(SU_q(\ell + 1))$ we get an antilinear map $\ast \otimes J : \Omega^k_N \rightarrow \mathcal{A}(SU_q(\ell + 1)) \otimes \mathfrak{g}_q^\ell$. One can check the the image of $\Omega^k_N = \mathcal{A}(SU_q(\ell + 1)) \boxtimes_{\mathbb{C}^N} W_k$ through $\ast \otimes J$ is in the space $W_{\ell-k} \boxtimes_{\mathbb{C}^N_{\ell-1}} \mathcal{A}(SU_q(\ell + 1))$, that is defined like $\Omega^\ell_{\ell+1-N}$ but for the order of the factors in the Hopf tensor product. Thus, due to anticommutativity of $U_q(\mathfrak{su}(\ell))$, the image of $\ast \otimes J$ is in general not an element of $\Omega^\ast_N$, for any value of $N'$. To cure this problem one could compose $\ast \otimes J$ with the $R$-matrix of $U_q(\mathfrak{su}(\ell))$, that is trivial if $q = 1$ or if $\ell = 1$ (so, in the commutative case and also on the standard Podleś sphere we don’t have this problem). We prefer to proceed in the following simpler way.

We introduce
\[
\mathcal{J}_0 := (\ast \otimes J)(\mathcal{L}_{K_{2\rho}}^{-1} \otimes id) ,
\]
where $K_{2\rho}$ is defined in (3.2).

**Proposition 5.9.** The operator $\mathcal{J}_0$ maps $\Omega^k_N$ into $\Omega^{\ell-k}_{\ell+1-N}$.

*Proof.* Let $h \in U_q(\mathfrak{su}(\ell))$. By (3.34) we have $\sigma_{\ell-k}(h)J = J\sigma_k(S(h^\ast))$, and $\mathcal{L}_h \ast = \ast \mathcal{L}_S(h^\ast)$ being $\mathcal{A}(SU_q(\ell + 1))$ a right $U_q(\mathfrak{su}(\ell + 1))$-module $\ast$-algebra. Thus, for all $h \in U_q(\mathfrak{su}(\ell))$,
\[
\{ \mathcal{L}_{h(1)} \otimes \sigma_{\ell-k}(h(2)) \} \mathcal{J}_0 = (\ast \otimes J) \{ \mathcal{L}_{S(h(1))} \otimes \sigma_k(S(h(2))) \} (\mathcal{L}_{K_{2\rho}}^{-1} \otimes id)
\]
\[
= \mathcal{J}_0 \{ \mathcal{L}_{K_{2\rho}t(2)K_{2\rho}^{-1}} \otimes \sigma_k(t(1)) \} = \mathcal{J}_0 \{ \mathcal{L}_{S^{2}(t(2))} \otimes \sigma_k(t(1)) \} ,
\]
where we used (3.3) and called $t = S(h^\ast)$. But
\[
\{ \mathcal{L}_{S^{2}(t(2))} \otimes \sigma_k(t(1)) \} \omega = \sigma_k(t(1)) \omega \lhd S(t(2)) = \sigma_k(t(1)) S(t(2)) \omega = \epsilon(t) \omega .
\]
for all $\omega \in \Omega^k_N$ and $t \in U_q(\mathfrak{su}(\ell))$, where in the second equality we used Lemma 4.1. Thus, $J_0$ maps $\Omega^k_N$ into $\bigoplus_{N' \in \mathbb{Z}} \Omega^k_{N'}$, the elements of $\mathcal{A}(SU_q(\ell+1)) \otimes W_{\ell-k}$ that are invariant under $L_{h(2)} \otimes \sigma_{\ell-k}(h(2))$, $h \in U_q(\mathfrak{su}(\ell))$. Moreover, since $a^* \triangleleft \hat{K} = (a \triangleleft \hat{K}^{-1})^*$, we have:

$$q^{-k-\ell+k\ell N'}J_0\omega = (J_0\omega) \triangleleft \hat{K} = J_0(\omega \triangleleft \hat{K}^{-1}) = q^{-k-\ell+k\ell N'}J_0\omega .$$

It follows that $J_0(\Omega^k_N) \subset \Omega^k_{N'}$ with $N' = \ell + 1 - N$. □

The map $J_0$ is equivariant:

$$h \triangleright (J_0 \omega) = J_0(S(h)^* \triangleright \omega) , \quad \forall h \in U_q(\mathfrak{su}(\ell + 1)) .$$

(5.14)

This follows immediately from the fact that $h \triangleright$ commutes with any endomorphism of $W_k$ (it acts diagonally on $\Omega^k_N$), while $h \triangleright a^* = \{S(h)^* \triangleright a\}^*$ since $\mathcal{A}(SU_q(\ell + 1))$ is a left $U_q(\mathfrak{su}(\ell + 1))$-module $*$-algebra. The antiunitary part of $J_0$ is a natural candidate for the real structure. We are going to compute it explicitly.

**Lemma 5.10.** We have

$$(J_0 \omega)_1 = (-1)^{N + \frac{\ell(\ell + 1)}{2}} q^{\frac{\ell(\ell - 1)}{2}(\omega_1^c \triangleright K_2^\frac{1}{2} \hat{K}^{\ell + 1})^*} ,$$

for all $\omega \in \Omega^k_N$.

**Proof.** By (3.39) we have

$$\omega_1^c \triangleright K_2^\frac{1}{2} \hat{K}^{-\ell - 1} = q^{k(\ell + 1) - 2\ell} \omega_1^c .$$

Thus

$$\omega_1^c \triangleright K_2^\frac{1}{2} \hat{K}^{\ell + 1} = \omega_1^c \triangleright (K_2^\frac{1}{2} \hat{K}^{-\ell - 1})^{-\frac{1}{2}} \hat{K}^{\frac{\ell + 1}{2}} = q^{\frac{\ell(\ell - 1)}{2} N} \omega_1^c = q^{\ell + \frac{\ell(\ell + 1)}{2}} \omega_1^c ,$$

(5.15)

for all $\omega \in \Omega^k_N$. With this, we can compute $J_0 \omega$. By (5.13) and (3.33), we have

$$(J_0 \omega)_1 = (-q^{-1})^{\ell} q^{\frac{\ell(\ell + 1)}{2}} (\omega_1^c \triangleright K_2^\frac{1}{2})^*$$

and by (5.15)

$$= (-q^{-1})^{\ell} q^{\frac{\ell(\ell + 1)}{2}} q^{\ell + \frac{\ell(\ell + 1)}{2}} (\omega_1^c \triangleright K_2^\frac{1}{2} K_2^\frac{1}{2} \hat{K}^{\ell + 1})^*$$

$$= (-1)^{\ell} q^{\frac{\ell(\ell + 1)}{2}} (\omega_1^c \triangleright K_2^\frac{1}{2} \hat{K}^{\ell + 1})^* .$$

This concludes the proof. □

**Lemma 5.11.** For all $\omega, \omega' \in \Omega^k_N$,

$$\langle J_0(K_2^\frac{1}{2} \triangleright \omega \triangleleft \hat{K}^{-2(\ell + 1)}), J_0(K_2^\frac{1}{2} \triangleright \omega' \triangleleft \hat{K}^{-2(\ell + 1)}) \rangle = q^{(N - \frac{\ell(\ell + 1)}{2})} \langle \omega, \omega' \rangle .$$

**Proof.** From Lemma 5.10 and equation (3.4) we get

$$\langle J_0(K_2^\frac{1}{2} \triangleright \omega), J_0(K_2^\frac{1}{2} \triangleright \omega') \rangle = \sum_{t \in L_{-k}} \varphi((J_0K_2^\frac{1}{2} \triangleright \omega)_{1}) (J_0K_2^\frac{1}{2} \triangleright \omega')_{1}$$

$$= \sum_{t \in L_{-k}} q^{(N - \frac{\ell(\ell + 1)}{2})} \varphi(\{K_2^\frac{1}{2} \triangleright \omega_1^c \triangleleft K_2^\frac{1}{2} \hat{K}^{\ell + 1}\} \{K_2^\frac{1}{2} \triangleright (\omega_1^c)^* \triangleleft K_2^\frac{1}{2} \hat{K}^{-\ell - 1}\})$$
Concerning point (ii), we have Using Proposition 3.11, and the identities 

(by the modular property of the Haar state, see (3.4))

(by invariance of the Haar state)

(using (3.39))

(using (5.3))

This concludes the proof. □

As a consequence of Lemma 5.11, the antiunitary part of $\mathcal{J}_0$ is the map $\mathcal{J} : \Omega^k_N \to \Omega^{k-1}_N$ given by

$$\mathcal{J} \omega = q^{(\frac{\ell+1}{2})-k} K_{2p}^{\frac{1}{2}} \rho ((\mathcal{J}_0 \omega) \circ \hat{K}^{2(\ell+1)})$$

for all $\omega \in \Omega^k_N$.

**Proposition 5.12.** The operator $\mathcal{J}$ satisfies

(i) $\mathcal{J}^2 = (-1)^{\frac{\ell+1}{2}}$;

(ii) $\mathcal{J} a \mathcal{J}^{-1} \omega = \omega \cdot (K_{2p}^{\frac{1}{2}} \gg a^*)$ for all $\omega \in \Omega^k_N$ and $a \in A(\mathbb{CP}^\ell_i)$;

(iii) $\mathcal{J} \partial \mathcal{J}^{-1}|_{\Omega^k_N} = q^{N-\frac{\ell}{2}} \partial^\dagger$.

In particular, we find the following alternative expression for the operator $\partial^\dagger$ in (5.8):

$$\partial^\dagger|_{\Omega^k_N} = -q^{-\frac{k-1}{2}} \sum_{i=1}^\ell q^{-2i} \mathcal{L}_{S(X_i^*; \hat{K})} \otimes \epsilon_i^R.$$  

**Proof.** Point (i) follows from Proposition 3.10 and the observation that $\mathcal{J}^2 = \mathcal{J}_0^2 = (\ast \otimes J)^2$.

Concerning point (ii), we have

$$\mathcal{J} a \mathcal{J}^{-1} \omega = K_{2p}^{\frac{1}{2}} \rho \left( a K_{2p}^{\frac{1}{2}} \gg \omega^* \circ \hat{K}^{2(\ell+1)} \right)^* \circ \hat{K}^{2(\ell+1)} = \left( (K_{2p}^{\frac{1}{2}} \rho a)^* \right)^* \circ \omega \left( K_{2p}^{\frac{1}{2}} \rho a^* \right).$$

It remains point (iii). By definition $\partial = \sum_{i=1}^\ell \mathcal{L}_{K X_i} \otimes \epsilon_i^R$, and

$$\mathcal{J} \partial \mathcal{J}^{-1} = \sum_{i=1}^\ell \mathcal{L}_{K_{2p} \hat{K}^{2(\ell+1)}} S(X_i^*; \hat{K}) \mathcal{L}_{K_{2p} \hat{K}^{2(\ell+1)}} \otimes J \epsilon_i^R.$$  

Using Proposition 3.11, and the identities $K_{2p} \rho K_{2p}^{-1} = S^2(h)$ and $K X_i^* \hat{K}^{-1} = q X_i^*$, we get

$$\mathcal{J} \partial \mathcal{J}^{-1} = -q^{-2(\ell+1)} \sum_{i=1}^\ell \mathcal{L}_{S^2(X_i^*; \hat{K})} \otimes \epsilon_i^R.$$
Notice that $S^2(\hat{K}) = \hat{K}$. Moreover, $S^2$ is an algebra morphism, $S^2(E_j) = q^2E_j$, and $X^*_i$ is the product of $\ell + 1 - i$ operators $E_j$’s. Thus $S^2(X^*_i) = q^{2(\ell + 1 - i)}X^*_i$ and

$$\mathcal{J}\partial\mathcal{J}^{-1} = -q\sum_{i=1}^\ell q^{-2i}L_{S(X^*_i\hat{K})} \otimes i^R.$$  

We want to prove that this operator is just $q^{k-N}\partial\dagger$; this will give as a corollary (5.17). By Lemma 5.4, it is enough to prove that for all $\omega \in \Omega_N^{k+1}$ we have

$$(\partial\dagger\omega)_{1,2,...,k} = (q^{N-k}\mathcal{J}\partial\mathcal{J}^{-1}\omega)_{1,2,...,k}.$$  

Using (3.36) we find the explicit formulæ:

$$(\partial\dagger\omega)_{1,...,k} = \sum_{i=k+1}^\ell (-q)^{-k}\omega_{1,2,...,k,i} \triangleleft S^{-1}(X^*_i\hat{K}),$$  

$$(\mathcal{J}\partial\mathcal{J}^{-1}\omega)_{1,...,k} = \sum_{i=k+1}^\ell (-q)^{k+1-2i}\omega_{1,2,...,k,i} \triangleleft X^*_i\hat{K}.$$  

From the equation $h_{(2)}S^{-1}(h_{(1)}) = \epsilon(h)$, applied to $h = X^*_s\hat{K}$, and using (3.41) for the coproduct, we get:

$$-X^*_s\hat{K} = K\omega r_{r-1} S^{-1}(X^*_s\hat{K}) + q^{-\frac{1}{2}}(q - q^{-1}) \sum_{s=r+1}^\ell \hat{K}\omega r_{s-1} M r_{s-1} S^{-1}(X^*_s\hat{K}).$$  

Plugging this into previous formula we get

$$(-\mathcal{J}\partial\mathcal{J}^{-1}\omega)_{1,2,...,k} = \sum_{r=k+1}^\ell (-q)^{k+1-2r}\omega_{1,2,...,k,r} \triangleleft \hat{K}\omega r_{r-1} S^{-1}(X^*_r\hat{K})$$  

$$+ q^{-\frac{1}{2}}(q - q^{-1}) \sum_{r=k+1}^\ell (-q)^{k+1-2r} \sum_{s=r+1}^\ell \omega_{1,2,...,k,r} \triangleleft \hat{K}\omega r_{s-1} M r_{s-1} S^{-1}(X^*_s\hat{K}).$$  

From the definition of $\Omega_N^{k+1}$ and $\sigma_N^{k+1}$ we deduce that $\omega \triangleleft \hat{K} = q^{k+1-\frac{1}{2}\ell+\sigma_N}\omega$, and since

$$N_{r,s-1}^{(2)} = (K_r \ldots K_{\ell-1})^{(2)}(K_s \ldots K_{\ell-1})^{(2)}(K_1 K_2 \ldots K_{\ell-1})^{-2}\hat{K},$$  

we have also

$$\omega_{1,...,k,r} \triangleleft \hat{K}\omega r_{s-1} = q^{\frac{1}{2}(1+\delta_{r,s})+k-N}\omega_{1,...,k,r}.$$  

Thus,

$$(-q^{N-k-1}\mathcal{J}\partial\mathcal{J}^{-1}\omega)_{1,2,...,k} = \sum_{r=k+1}^\ell (-q)^{k+1-2r}\omega_{1,2,...,k,r} \triangleleft S^{-1}(X^*_r\hat{K})$$  

$$+ (1-q^{-2}) \sum_{r=k+1}^\ell (-q)^{k+1-2r} \sum_{s=r+1}^\ell \omega_{1,2,...,k,r} \triangleleft M r_{s-1} S^{-1}(X^*_s\hat{K}).$$  

\[\text{DIrac Operators on Quantum Projective Spaces 43}\]
Now we use (5.1) and get:

\[-q^{N-k-1} \mathcal{F} \tilde{\partial} \mathcal{J}^{-1} \omega)_{1, \ldots, k} = \sum_{r=k+1}^{\ell} (-q)^{k+1-2r} \omega_{1, \ldots, k, r} \cdot S^{-1}(X_r^s \hat{K})
\]

+ \left(1 - q^{-2}\right) \sum_{r=k+1}^{\ell} (-q)^{k+1-2r} \sum_{s=r+1}^{\ell} \omega_{1, \ldots, k, s} \cdot S^{-1}(X_s^r \hat{K})

We use \(\sum_{r=k+1}^{\ell} \sum_{s=r+1}^{\ell} = \sum_{s=k+2}^{\ell} \sum_{r=k+1}^{s-1}\) to change the order of the summations, and use

\[\sum_{r=k+1}^{s-1} q^{-2r} = \frac{q^{-2(k+1)} - q^{-2s}}{1 - q^{-2}}\]

to get

\[-q^{N-k-1} \mathcal{F} \tilde{\partial} \mathcal{J}^{-1} \omega)_{1, \ldots, k} = \sum_{r=k+1}^{\ell} (-q)^{k+1-2r} \omega_{1, \ldots, k, r} \cdot S^{-1}(X_r^s \hat{K})
\]

+ \sum_{s=k+2}^{\ell} (-q)^{k+1}(q^{-2(k+1)} - q^{-2s})\omega_{1, \ldots, k, s} \cdot S^{-1}(X_s^r \hat{K})

= \sum_{s=k+1}^{\ell} (-q)^{-k-1}\omega_{1, \ldots, k, s} \cdot S^{-1}(X_s^r \hat{K})

= -q^{-1}(\tilde{\partial} \omega)_{1, \ldots, k}.

This concludes the proof. \(\square\)

6. A FAMILY OF SPECTRAL TRIPLES FOR \(\mathbb{CP}_q^\ell\)

In this section we present spectral triples for \(\mathbb{CP}_q^\ell\). Let \(\mathcal{H}_N\) be the Hilbert space completion of \(\Omega_N^*\), and let \(\gamma_N\) be +1 on even forms and −1 on odd forms. A bounded ∗-representation of \(\mathcal{A}(\mathbb{CP}_q^\ell)\) is given by (the completion of) left multiplication on \(\Omega_N^*\), and a densely defined ∗-representation of \(U_q(\mathfrak{su}(\ell))\) is given by the left action on \(\Omega_N^*\). To construct Dirac operators, there are basically two possibilities. The first is to use the operator \(\tilde{\partial}|_{\Omega_N^*}\) and its Hermitian conjugate. Since \(\Omega_N^k \simeq \Omega^k \otimes_{\mathcal{A}(\mathbb{CP}_q^\ell)} \Gamma_N\), another possibility is to use the natural connection \(\nabla_N : \Omega_N^k \rightarrow \Omega_N^{k+1}\) defined by twisting the flat connection on \(\Omega^k\) with the Grassmannian connection of \(\Gamma_N\). That is, we define \(\nabla_N\) as a composition:

\[
\begin{array}{c}
\Omega_N^k \\
\nabla_N \downarrow \\
\Omega_N^{k+1}
\end{array}
\xrightarrow{\cdot \Psi_N}
\begin{array}{c}
(\Omega_0^k)^{\otimes r} \\
\tilde{\partial} \otimes 1_r \\
(\Omega_0^{k+1})^{\otimes r}
\end{array}
\]

where \(\Psi_N\) is the vector given in (4.5), \(r := [N+\ell]_\ell\) is its size, and \(1_r\) is the identity matrix of size \(r\). Explicitly

\[\nabla_N \omega := \{\tilde{\partial} (\omega \Psi_N^*)\} \cdot \Psi_N^*\]
for all $\omega \in \Omega^k_N$. The two choices coincide, that is $\nabla_N = \partial|_{\Omega^k_N}$ as shown in the next lemma.

**Lemma 6.1.** We have

$$\nabla_N \omega = \partial \omega,$$

for all $\omega \in \Omega^k_N$.

**Proof.** We give the proof for $N \geq 0$. By definition, the components of $\Psi^\dagger_N$ are monomials in $z_i$. As $\pi^I_{1+1}(F_j) = 0$, $z_i \circ F_j = 0$ and so $\Psi^\dagger_N \circ F_j = 0$. This means that $\Psi^\dagger_N \circ S^{-1}(X_i) = 0$. Then by (3.41)

$$\{\partial(\omega \Psi^\dagger_N)\}_{1} = \sum_{r=1}^{k+1} (-q)^{1-r} (\omega_{J^r+1} \circ S^{-1}(KX_i)) \Psi^\dagger_N = (\partial \omega)_{1} \Psi^\dagger_N$$

and $\nabla_N \omega = \partial \omega$. \hfill $\square$

Thus, for $c_{\ell,N,k} \in \mathbb{C}$ we define the following Dirac-type operator

$$D(c)|_{\Omega^k_N} := c_{\ell,N,k} \partial + \bar{c}_{\ell,N,k-1} \bar{\partial}^\dagger.$$

The antilinear map $J$ given by (5.16) extends to an isometry $\mathcal{H}_N \rightarrow \mathcal{H}_{-1-N}$. In particular, for odd $\ell$ it is an automorphism of $\mathcal{H}_{\frac{1}{2}}(\ell+1)$. A preferred Dirac operator $D_N$ on $\mathcal{H}_N$ – belonging to the class $D(c)$ – is given by

$$D_N := q^{\frac{1}{2}(k-N)} \partial + q^{\frac{1}{2}(k-N-1)} \bar{\partial}^\dagger.$$

By Proposition 5.12, we have $JD_N = D_{\ell+1-N}J$ on $\Omega^k_N$, that for odd $\ell$ and $N = \frac{1}{2}(\ell + 1)$ means that $D_N$ commutes with $J$.

Remark: the operator $D_0$ is a $q$-analogue of the Dolbeault-Dirac operator of $\mathbb{C}P^\ell$, while $D_N$ is the twist of $D_0$ with the Grassmannian connection of the line bundle $\Gamma_N$. If $\ell$ is odd and $N = \frac{1}{2}(\ell + 1)$, $D_N$ is a $q$-analogue of the Dirac operator of the Fubini-Study metric.

The main result of this section is the following theorem.

**Theorem 6.2.** The datum $(A(\mathbb{C}P^\ell_q), \mathcal{H}_N, D_N, \gamma_N)$ is a $0^+\text{-dimensional equivariant even spectral triple. If } \ell \text{ is odd and } N = \frac{1}{2}(\ell + 1), \text{ the spectral triple } (A(\mathbb{C}P^\ell_q), \mathcal{H}_N, D_N, \gamma_N, J) \text{ is real with } KO\text{-dimension } 2\ell \text{ mod } 8.$

The rest of this section is devoted to the proof of this theorem. Let us check the conditions of a spectral triple, as recalled in Appendix A. Equivariance holds by construction: $\Omega^k_N$ is dense in $\mathcal{H}_N$, it is a left $A(\mathbb{C}P^\ell_q) \times U_q(\text{su}(\ell + 1))$-module, the Dirac-type operators (symmetric by construction since $\bar{\partial}^\dagger$ is the Hermitian conjugate of $\bar{\partial}$) are defined on the dense domain $\Omega^k_N$ and commute with the action of $U_q(\text{su}(\ell + 1))$ (since $\bar{\partial}$ does), and $J$ is the antiunitary part of $J_0$ – given by (5.13) – that is equivariant due to (5.14).

Next, for any $a \in A(\mathbb{C}P^\ell_q)$,

$$[D(c), a] = c_{\ell,N,k} [\partial, a] - \bar{c}_{\ell,N,k-1} [\bar{\partial}, a^*] \dagger.$$

But by (5.9), $[\bar{\partial}, a]$ is the operator of left multiplication by $\bar{\partial} a$, and this shows the bounded commutator condition.
As far as the grading is concerned, the action of $\mathcal{A}(\mathbb{CP}_q^\ell) \rtimes U_q(\mathfrak{su}(\ell + 1))$ does not change the parity of forms, while the Dirac operator does. The operator $\mathcal{J}$ sends $\Omega_N^k$ to $\Omega_{N+1-k}^{\ell+1-N}$, that is it exchanges the parity of forms exactly when $\ell$ is odd, that shows the last condition in (A.2). The remaining two conditions in (A.2), $D_N \mathcal{J} = \mathcal{J} D_N$ and $\mathcal{J}^2 = (-1)^{\frac{1}{2}(\ell+1)}$, follow from Proposition 5.12 for any odd $\ell$ and $N = \frac{1}{2}(\ell + 1)$.

We now pass to the commutant and first order condition, cf. (A.1). Proposition 5.12 tells us that for odd $\ell$ and $N = \frac{1}{2}(\ell + 1)$, $\mathcal{J} a \mathcal{J}^{-1}$ is the operator of right multiplication by $K_{2p}^\frac{1}{2} a^*$. Since left and right $\mathcal{A}(\mathbb{CP}_q^\ell)$-module structure of $\Omega_N^k$ commute, this proves the commutant condition. Moreover, due to the modular property of the Haar state ((3.4)),

$$\langle \omega, \omega' (K_{2p}^\frac{1}{2} a^*) \rangle = \varphi (\omega, \omega' (K_{2p}^\frac{1}{2} a^*) \omega') = \varphi (\{ (K_{2p}^\frac{1}{2} a)^\dagger \omega' \} = \langle \omega (K_{2p}^\frac{1}{2} a), \omega' \rangle = \langle \mathcal{J} a^* \mathcal{J}^{-1} \omega, \omega' \rangle.$$

Hence $(\mathcal{J} a \mathcal{J}^{-1})^\dagger = \mathcal{J} a^* \mathcal{J}^{-1}$. Since $[\tilde{\partial}, a] = (\tilde{\partial} a) \wedge_q$, from the associativity of $\wedge_q$ we deduce the first order condition for $\tilde{\partial}$. This condition for $D_N$ follows from the identity

$$[[D_N, a], \mathcal{J} b \mathcal{J}^{-1}] = c_{\ell,N,k} [[\tilde{\partial}, a], \mathcal{J} b \mathcal{J}^{-1}] + c_{\ell,N,k-1} [[\tilde{\partial}, a^*], \mathcal{J} b^* \mathcal{J}^{-1}]^\dagger,$$

that is valid for all $a, b \in \mathcal{A}(\mathbb{CP}_q^\ell)$.

Next, we show that $D_N$ is diagonalizable by relating it to the Casimir of $U_q(\mathfrak{su}(\ell + 1))$ (then, being a symmetric operator, $D_N$ has a canonical selfadjoint extension). We shall also prove that the eigenvalues of $D_N$ diverge exponentially (while their multiplicities are only polynomially divergent). This implies that $(D_N^2 + 1)^{-1}$ is compact and that $D_N$ is $0^+$-summable, which will complete the proof of Theorem 6.2. We need first two technical lemmas.

**Lemma 6.3.** We have

$$(\bar{\partial} \bar{\partial}^i + \bar{\partial} \bar{\partial}^i) \omega = \omega \circ \left( q^{2N} \sum_{i=1}^\ell q^{-2i} X_i X_i^* + q^{N+\ell-k} k(\ell + 1 - N) \right)$$

for all $\omega \in \Omega_N^k$.

**Proof.** Using (5.7) for $\bar{\partial}$ and (5.17) for $\bar{\partial}^i$, we compute

$$(\bar{\partial} \bar{\partial}^i + \bar{\partial} \bar{\partial}^i) |_{\Omega_N^k} = -q^{N-k} \sum_{i,j=1}^\ell q^{-2j} (q \mathcal{L}_{KX_i} \mathcal{L}_{S(X_j^*)} \otimes \epsilon^{L_R}_{i,j} + \mathcal{L}_{S(X_j^*)} \mathcal{L}_{KX_i} \otimes \epsilon^{R_L}_{i,j}).$$

Since $X_i \mathcal{K} = q \mathcal{K} X_i$, we have $X_i^* \mathcal{K} = q^{-1} \mathcal{K} X_i^*$ and

$$\mathcal{L}_{KX_i} \mathcal{L}_{S(X_j^*)} = \mathcal{L}_{KX_i} \mathcal{L}_{S(X_j^*)} = \mathcal{L}_{KX_i} \mathcal{L}_{S(X_j^*)} = q^{-1} \mathcal{L}_{S(X_j^*)} \mathcal{L}_{KX_i} = q^{-1} \mathcal{L}_{S(X_j^*)} X_i.$$

Thus

$$(\bar{\partial} \bar{\partial}^i + \bar{\partial} \bar{\partial}^i) |_{\Omega_N^k} = -q^{N-k} \sum_{i,j=1}^\ell q^{-2j} (q \mathcal{L}_{S(X_j^*)} \otimes \epsilon^{L_R}_{i,j} + q^{-1} \mathcal{L}_{S(X_j^*)} X_i \otimes \epsilon^{R_L}_{i,j}).$$
Using (3.36) we get

$$\sum_{i,j=1}^{\ell} q^{-2j} (L_{X_i} S(X_j^*) \otimes e_{i}^{L} e_{j}^{R} \omega)_{\underline{r}} = \sum_{r \in \underline{r}, s \notin (\underline{r}, r)} (-q)^{-L(\underline{r}, \underline{r})+L(\underline{r}, s)} \omega_{(\underline{r}, s)} \otimes q^{-2s} X_r^* S^{-1}(X_r)$$

$$= \sum_{r \in \underline{r}} \omega_{\underline{r}} \otimes q^{-2r} X_r^* S^{-1}(X_r) + \sum_{r \in \underline{r}, s \notin (\underline{r}, r)} (-q)^{-L(\underline{r}, \underline{r})+L(\underline{r}, s)} \omega_{(\underline{r}, s)} \otimes q^{-2s} X_r^* S^{-1}(X_r)$$

$$+ \sum_{r \in \underline{r}, s \notin (\underline{r}, r), r<s} (-q)^{-L(\underline{r}, \underline{r})+L(\underline{r}, s)} \omega_{(\underline{r}, s)} \otimes q^{-2s} X_r^* S^{-1}(X_r)$$

and

$$\sum_{i,j=1}^{\ell} q^{-2j} (L_{S(X_i^*)} X_j \otimes e_{j}^{L} e_{i}^{R} \omega)_{\underline{r}} = \sum_{s \notin \underline{r}, r \in (\underline{r}, s)} (-q)^{L(s, \underline{r})-L(r, \underline{r})} \omega_{(\underline{r}, s)} \otimes q^{-2s} S^{-1}(X_r) X_s^*$$

$$= \sum_{s \notin \underline{r}} \omega_{\underline{r}} \otimes q^{-2s} S^{-1}(X_s) X_s^* + \sum_{s \notin \underline{r}, r \in (\underline{r}, s)} (-q)^{L(s, \underline{r})-L(r, \underline{r})} \omega_{(\underline{r}, s)} \otimes q^{-2s} S^{-1}(X_r) X_s^*$$

$$+ \sum_{s \notin \underline{r}, r \notin (\underline{r}, s), r<s} (-q)^{L(s, \underline{r})-L(r, \underline{r})} \omega_{(\underline{r}, s)} \otimes q^{-2s} S^{-1}(X_r) X_s^*.$$ 

In particular, we read off the component $i = (1, 2, \ldots, k)$,

$$-q^{k-N} \{(\vec{\partial}_0 \vec{\partial}) + \vec{\partial}_d \vec{\partial}_d \omega\}_{1, \ldots, k} = \sum_{r=1}^{k} \omega_{1, \ldots, k} \otimes q^{-2r} X_r^* S^{-1}(X_r) + q^{-1} \sum_{s=k+1}^{\ell} \omega_{1, \ldots, k} \otimes q^{-2s} S^{-1}(X_s) X_s^*$$

$$+ \sum_{r=1}^{k} \sum_{s=k+1}^{\ell} (-q)^{k-r-2s} \omega_{1, \ldots, r, \ldots, k} \otimes X_s^* S^{-1}(X_r) .$$

We use (3.43a) and Lemma 5.2 to rewrite last term:

$$\sum_{r=1}^{k} \sum_{s=k+1}^{\ell} (-q)^{k-r-2s} \omega_{1, \ldots, r, \ldots, k} \otimes X_s^* S^{-1}(X_r) = -q^2 \sum_{r=1}^{k} \sum_{s=k+1}^{\ell} q^{2(k-r-s)} \omega_{1, \ldots, k} \otimes (\hat{K}N_{r,s-1})^{-1}$$

$$= -q^{N+1-k} \omega_{1, \ldots, k} \sum_{r=1}^{k} \sum_{s=k+1}^{\ell} q^{2(k-r-s)} = -q^{N-\ell-k-1} [\ell - k] \omega_{1, \ldots, k} .$$

Thus,

$$-q^{k-N} \{(\vec{\partial}_0 \vec{\partial}) + \vec{\partial}_d \vec{\partial}_d \omega\}_{1, \ldots, k} = \sum_{r=1}^{k} \omega_{1, \ldots, k} \otimes q^{-2r} X_r^* S^{-1}(X_r) + q^{-1} \sum_{s=k+1}^{\ell} \omega_{1, \ldots, k} \otimes q^{-2s} S^{-1}(X_s) X_s^*$$

$$- q^{N-\ell-k-1} [\ell - k] \omega_{1, \ldots, k} .$$

Now, we rewrite the first summation using (3.43b) and get:

$$-q^{k-N} \{(\vec{\partial}_0 \vec{\partial}) + \vec{\partial}_d \vec{\partial}_d \omega\}_{1, \ldots, k} = \sum_{r=1}^{k} \omega_{1, \ldots, k} \otimes q^{-2r} S^{-1}(X_r) X_r^* + q^{-1} \sum_{s=k+1}^{\ell} \omega_{1, \ldots, k} \otimes q^{-2s} S^{-1}(X_s) X_s^*$$

$$- q \sum_{r=1}^{k} q^{-2r} \omega_{1, \ldots, k} \otimes (K_{r,s} K_{s,r})^2 = q^{-q-1} - q^{N-\ell-k-1} [\ell - k] \omega_{1, \ldots, k} .$$
The last sum is
\[\sum_{r=1}^{k} q^{-2r} \omega_{1,\ldots,k} \prec (K_{r_{-1},K_{r_{+1}}})^2 \prec q^{-4} = [k - N + 1] \omega_{1,\ldots,k} \sum_{r=1}^{k} q^{-2r} = q^{k-1}[k][k - N + 1] \omega_{1,\ldots,k}.\]

Thus
\[-q^{k-N} \{(\bar{\partial}q\bar{\partial}) + \bar{\partial}q\bar{\partial})\omega\}_{1,\ldots,k} = \sum_{r=1}^{k} \omega_{1,\ldots,k} \prec q^{-2r} S^{-1}(X_{r})X_{r}^* + q^{-1} \sum_{r=1}^{k} \omega_{1,\ldots,k} \prec q^{-2r} S^{-1}(X_{r})X_{r}^* - q^{-[k][\ell + 1 - N]} \omega_{1,\ldots,k}.\]

Applying the identity \(S^{-1}(h_{(2)})h_{(1)} = \epsilon(h)\) to \(h = X_{r}\), and using (3.41) for \(\Delta(X_{r})\), we get
\[-S^{-1}(X_{r}) = qN_{r_{-1},r_{+1}} \hat{K}X_{r} + q^2(q - q^{-1}) \sum_{s=r+1}^{\ell} N_{r_{-1},s_{-1}} \hat{K}S^{-1}(M_{s_{-1}}X_{s}).\]

Since \(\omega_{1,\ldots,k} \prec F_{n} = 0\) for all \(1 \leq n \leq \ell - 1\), we have \(\omega_{1,\ldots,k} \prec S^{-1}(M_{s_{-1}}X_{s})\) for all \(s \leq \ell\). Thus
\[\omega_{1,\ldots,k} \prec S^{-1}(X_{r}) = -q\omega_{1,\ldots,k} \prec N_{r_{-1},r_{+1}} \hat{K}X_{r} = \begin{cases} -q^{k-N+2N} \omega_{1,\ldots,k} \prec X_{r}, & \text{if } 1 \leq r \leq k, \\ -q^{k-N+2N} \omega_{1,\ldots,k} \prec X_{r}, & \text{if } k + 1 \leq r \leq \ell. \end{cases}\]

With this, we obtain
\[\{(\partialq\partial) + \bar{\partial}q\partial)(\omega)\}_{1,2,\ldots,k} = q^{2N} \sum_{r=1}^{\ell} \omega_{1,2,\ldots,k} \prec q^{-2r} X_{r}X_{r}^* + q^{N-\ell-k}[k][\ell + 1 - N] \omega_{1,\ldots,k}.\]

By Lemma 5.4, the last equality implies
\[\{(\partialq\partial) + \bar{\partial}q\partial)(\omega)\}_{1,2,\ldots,k} = q^{2N} \sum_{r=1}^{\ell} \omega_{1,2,\ldots,k} \prec q^{-2r} X_{r}X_{r}^* + q^{N-\ell-k}[k][\ell + 1 - N] \omega_{2,\ldots,k},\]

and this concludes the proof. \(\square\)

**Lemma 6.4.** We have
\[q^{2(k-N)}(\partialq\partial) + q^2\partialq\partial)\omega = \omega \prec q^{2N} \sum_{i=1}^{\ell} q^{-2i} S^{-1}(X_{i}X_{i}^*) + q^{-N-\ell+k+1}[\ell - k][N] \]  
(6.3)
for all \(\omega \in \Omega_{N}^{k} \).

**Proof.** We substitute \(k \to \ell - k\) and \(N \to \ell + 1 - N\) in (6.2) and apply conjugation by \(\mathcal{J}\). By Proposition 5.12 the left hand side becomes the left hand side of (6.3). Since for any \(h \in U_{q}(su(\ell+1))\), \((\mathcal{J}^{-1})_{\omega}h = \mathcal{J}^{-1}(\omega \hat{K}^{-2(\ell+1)}S^{-1}(h)\hat{K}^{2(\ell+1)}),\) and since \(X_{i}X_{i}^*\) commutes with \(\hat{K}\), the right hand side of (6.2) becomes the right hand side of (6.3). \(\square\)

Using the identity
\[D_{N}^{2}_{|\Omega_{N}^{k}} = q^{k-N-1}(\partialq\partial) + q^2\partialq\partial) = q^{k-N-1}(q\partialq\partial + \bar{\partial}q\partial + q^2\partialq\partial) + q^{-1}q^{-2i} \]  
from (6.2) and (6.3) we get the following expression for the square of the Dirac operator
\[(1 + q)D_{N}^{2}_{|\Omega_{N}^{k}} = q^{k-N-1} \sum_{i=1}^{\ell} q^{-2i}(\cdot) \prec X_{i}X_{i}^* + q^{k-N-1} \sum_{i=1}^{\ell} q^{-2i}(\cdot) \prec S^{-1}(X_{i}X_{i}^*) + q^{-[\ell + 1 - N]} + q^{-[\ell - k][N]}.\]  
(6.4)
Since $D_N^2$ leaves the degree of forms invariant, it is the direct sum of finitely many $D_N^2|_{\Omega_N^k}$. Therefore, it suffices to prove that the eigenvalues of $D_N^2|_{\Omega_N^k}$ diverge exponentially and the multiplicities diverge polynomially. For that, we now study the first term in (6.4).

**Lemma 6.5.** We have
\[
\langle C_q - \sum_{i=1}^{\ell} q^{\ell+1-2i} X_i X_i^\dagger \rangle |_{\Omega_N^k} = q^{1+\frac{q^k-2}{q-1}} \left( \frac{1}{2} [k][\ell+1-\frac{\ell+2}{q}k] + \frac{1}{2} [\ell][1/\ell]^2 + \frac{[\ell]}{(q-q^{-1})^2} \right) + \frac{1}{(q-q^{-1})^2} q^{-2k+\ell(q^{-1}+1)} - \frac{[\ell]}{(q-q^{-1})^2}.
\]

**Proof.** From (3.19) we get
\[
C_q - \sum_{i=1}^{\ell} q^{\ell+1-2i} X_i X_i^\dagger = q^{1+\frac{q^k-2}{q-1}} \left( C_q' + \frac{[\ell]}{(q-q^{-1})^2} \right) + \frac{q^{-\ell}}{(q-q^{-1})^2} K^{-2} - \frac{[\ell]}{(q-q^{-1})^2},
\]
where $C_q$ is the Casimir of $U_q(\mathfrak{su}(\ell))$ as in (3.12) and $C_q'$ is the Casimir of $U_q(\mathfrak{su}(\ell))$ as in (3.18). The representation $\sigma_N^k$ in the definition of $\Omega_N^k$ has highest weight $n_1 = \delta_{1,k}$. By Lemma 3.4 (with the replacement $\ell \to \ell - 1$ and $n_1 = n_\ell = 0$) the eigenvalue of $C_q'$ in this representation is
\[
\langle 2C_q' |_{\Omega_N^k} = [k][\ell+1-\frac{\ell+2}{q}k] + [\ell][1/\ell]^2. \tag{6.5}
\]
Moreover $\langle K |_{\Omega_N^k} = q^{k-\frac{2}{q+1}} N$. Combining these equations concludes the proof. \[\Box\]

We are interested in the asymptotic behaviour of the eigenvalues of $D_N$. From (6.4) and the last lemma we deduce that
\[
(1 + q) D_N^2 |_{\Omega_N^k} \simeq q^{-k-\frac{\ell+1}{q-1}} N^{-\ell-1} C_q + q^{\frac{2\ell+1}{q+1}} N^{-k-\ell} S^{-1}(C_q),
\]
where ‘$\simeq$’ means equality modulo constant multiples of the identity operator on $\Omega_N^k$. Since the coefficients in front of $C_q$ and $S^{-1}(C_q)$ in $(1 + q) D_N^2|_{\Omega_N^k}$ are positive (so that it is not possible that the dominating contributions cancel), it is enough to show that $C_q|_{\Omega_N^k}$ and $S^{-1}(C_q)|_{\Omega_N^k}$ separately have eigenvalues that diverge exponentially.

Recall that for central elements, like $C_q$ and $S^{-1}(C_q)$, the left and right canonical actions coincide. The decomposition of $\Omega_N^k$ with respect of the left action of $U_q(\mathfrak{su}(\ell))$ was given in Proposition 5.5. We set $n_\ell = n$ and $n_1 = n + N - k$, where $n \gg 1$. We use (3.17) and, for $1 \leq k \leq \ell$, we find two series of eigenvalues for $C_q|_{\Omega_N^k}$ given by
\[
q^{-2n} q^{-\frac{2\ell}{q+1} N-\ell+2k} + q^{2\ell+1 N-\ell} + O(1), \quad q^{-2n} q^{-\frac{2\ell}{q+1} N-\ell+2k} + q^{2\ell+1 N-\ell} + O(1),
\]
where by $O(1)$ we mean terms that remain bounded when $n \to \infty$. These eigenvalues diverge like $q^{-2n}$ and have multiplicity that, according to (3.16), is polynomial in $n$. If $k = 0$ or $k = \ell$, the same holds true except one of the two series is absent.

A similar reasoning applies to $S^{-1}(C_q)|_{\Omega_N^k}$. The eigenvalue of $S^{-1}(C_q)$ in the representation with highest weight $(n_1, n_2, \ldots, n_\ell)$ is the eigenvalue of $C_q$ in the representation with highest weight $(n_\ell, \ldots, n_2, n_1)$. Thus the two series of eigenvalues for $S^{-1}(C_q)|_{\Omega_N^k}$ are given asymptotically by
\[
q^{-2n} q^{-\frac{2\ell}{q+1} N-\ell-2k} + q^{2\ell+1 N-\ell} + O(1), \quad q^{-2n} q^{-\frac{2\ell}{q+1} N-\ell-2k} + q^{2\ell+1 N-\ell} + O(1),
\]
and one of the two is absent if \( k = 0 \) or \( k = \ell \). With this, the proof of Theorem 6.2 is completed.

Our asymptotic analysis can be extended to the explicit computation of the spectrum of \( D_N \), but we are not giving it since the formulæ are complicated and not particularly illuminating.

We conclude with few remarks on the case \( \ell = 2 \). The combination of Lemma 6.3 and Lemma 6.5 gives (after a tedious check in all the three cases \( k = 0, 1, 2 \)) the simple formula

\[
(\bar{\partial} + \bar{\partial}^\dagger)^2|_{\Omega^N} = q^{ \frac{2}{3} N - 3} C_q + \frac{1}{1 - q}(q^N[N] - q^{\frac{3}{2}} + 2[3][\frac{N}{3}]) .
\]

From this, the explicit expression of the spectrum of the Dirac operator \( \bar{\partial} + \bar{\partial}^\dagger \) follows, which extends the case \( N = 0 \) studied in [13]. For \( N = 0 \) we get

\[
(\bar{\partial} + \bar{\partial}^\dagger)^2|_{\Omega^0} = q^{-3} C_q ,
\]

which is the square of the Dirac operator considered in [13], although in different conventions which explains the \( q^{-3} \) factor.

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Appendix A. Some general definitions

We recall the notion of equivariant (unital) spectral triple, see e.g. [7, 28]. Let \( \mathcal{A} \) be a (complex, associative) involutive algebra with unity, \( (\mathcal{U}, \epsilon, \Delta, S) \) a Hopf \( * \)-algebra and suppose \( \mathcal{A} \) is a left \( \mathcal{U} \)-module \( * \)-algebra, which means that the left action ‘\( \triangleright \)’ of \( \mathcal{U} \) on \( \mathcal{A} \) satisfies

\[
h \triangleright ab = (h(1) \triangleright a)(h(2) \triangleright b) , \quad h \triangleright 1 = \epsilon(h)1 , \quad h \triangleright a^* = \{S(h)^* \triangleright a\}^* ,
\]

for all \( h \in \mathcal{U} \) and \( a, b \in \mathcal{A} \). As usual we use Sweedler notation for the coproduct, \( \Delta(h) = h(1) \otimes h(2) \) with summation understood. The left crossed product \( \mathcal{A} \rtimes \mathcal{U} \) is the \( * \)-algebra generated by \( \mathcal{A} \) and \( \mathcal{U} \) with crossed commutation relations

\[
ha = (h(1) \triangleright a)h(2) , \quad \forall h \in \mathcal{U}, a \in \mathcal{A} .
\]

The data \( (\mathcal{A}, \mathcal{H}, D) \) is called an \( \mathcal{U} \)-equivariant spectral triple if (i) there is a dense subspace \( \mathcal{M} \) of \( \mathcal{H} \) carrying a \( * \)-representation \( \pi \) of \( \mathcal{A} \rtimes \mathcal{U} \), (ii) \( D \) is a (unbounded) selfadjoint operator with compact resolvent and with domain containing \( \mathcal{M} \), (iii) \( \pi(a) \) and \([D, \pi(a)]\) extend to bounded operators on \( \mathcal{H} \) for all \( a \in \mathcal{A} \), (iv) \([D, \pi(h)]\) = 0 on \( \mathcal{M} \) for any \( h \in \mathcal{U} \). As usual we refer to \( D \) as the ‘Dirac operator’, in analogy with the commutative situation where spectral triples are canonically associated to spin structures. The representation symbol \( \pi \) will be omitted.

An equivariant spectral triple is called even if there exists a grading \( \gamma \) on \( \mathcal{H} \) (i.e. a bounded operator satisfying \( \gamma^2 = 1 \)) such that the Dirac operator is odd and the crossed product algebra is even:

\[
\gamma D + D \gamma = 0 , \quad t \gamma = \gamma t \quad \forall t \in \mathcal{A} \rtimes \mathcal{U} .
\]
An even spectral triple is called real if there exists an antilinear isometry $J$ on $\mathcal{H}$ such that
\[
J^2 = \pm 1, \quad JD = \pm DJ, \quad J\gamma = \pm \gamma J,
\]
and such that for all $a, b \in \mathcal{A}$
\[
[a, JbJ^{-1}] = 0, \quad [[D, a], JbJ^{-1}] = 0. \quad \text{(A.1)}
\]
The signs ‘±’ in (A.2) are determined by the dimension of the geometry [8]. For a 2$\ell$-dimensional space, with $\ell$ odd, we have
\[
J^2 = (-1)^{\ell(\ell+1)/2}, \quad JD = DJ, \quad J\gamma = -J\gamma. \quad \text{(A.2)}
\]

Finally, we call $(\mathcal{A}, \mathcal{H}, D, \gamma, J)$ a ‘real equivariant even spectral triple’ if $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is an equivariant even spectral triple, $J$ is a real structure and there is an (unbounded) antilinear operator $T$ on $\mathcal{H}$ whose antiunitary part is $J$ and satisfying
\[
Th = S(h)^*T,
\]
on the joint domain of $h$ and $T$, and for all $h \in \mathcal{U}$.

**Appendix B. Twisted Dirac operators on $\mathbb{CP}^\ell$**

In this section we consider the limit $q = 1$. If we work with formal power series in $h := \log q$, the Cartan generators $H_i$ are obtained from $K_1$ through the rescaling $K_1 = q^{H_1/2}$. In the $q \to 1$ limit, the elements $\{H_i, E_i, F_i\}_{i=1,\ldots,\ell}$ satisfy the well known Chevalley-Serre relations (see [25], Sec. VI.4), and the elements $\{H_i, M_{jk}^+, M_{jk}^*\}_{i,j,k=1,\ldots,\ell}$ form a linear basis of the Lie algebra $\mathfrak{su}(\ell + 1)$ called Cartan-Weyl basis (see [21], Sec. 6.1.1).

The $q \to 1$ limit of the Casimir (3.12) is
\[
C_{q=1} = \frac{1}{2} A + \sum_{1 \leq i < j \leq \ell} M_{ij}^* M_{ij}
\]
where
\[
A = \sum_{i=1}^{\ell+1} \left( \frac{\sum_{j=1}^{i-1} jH_j - \sum_{j=1}^{\ell} (\ell+1-j)H_j}{\ell+1} \right)^2 - \sum_{i=1}^{\ell+1} \left( i - \frac{\ell+2}{2} \right)^2 - \sum_{i=1}^{\ell+1} \left( i - \frac{\ell+2}{2} \right)^2.
\]

The term $-2\frac{\ell+2}{2} \sum_{i=1}^{\ell+1} \frac{\sum_{j=1}^{i-1} jH_j - \sum_{j=1}^{\ell} (\ell+1-j)H_j}{\ell+1}$ is zero, as one can see inverting the order in the summation. Furthermore
\[
\frac{1}{\ell+1} \sum_{i=1}^{\ell+1} \left( \sum_{j=1}^{i-1} jH_j - \sum_{j=i}^{\ell} (\ell+1-j)H_j \right) = \frac{2}{\ell+1} \sum_{j=1}^{\ell} \left( \sum_{i=1}^{\ell} ij - \sum_{i=1}^{\ell} i(\ell+1-j) \right) H_j
\]
\[
= \sum_{j=1}^{\ell} j(\ell+1-j)H_j.
\]

Now we develop the square in $A$ using the formula
\[
\left( \sum_{i=1}^{n} a_i \right)^2 = \sum_{i=1}^{n} a_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j
\]
and get
\[ A = \sum_{i=1}^{\ell} \frac{i(i+1)}{\ell+1} H_i(\ell + 1 - H_i) + 2 \sum_{1 \leq i < j \leq \ell} \frac{i(i+1-j)}{\ell+1} H_iH_j. \]

Hence
\[ C_{q=1} = \frac{1}{2} \sum_{i=1}^{\ell} \frac{i(i+1)}{\ell+1} H_i(\ell + 1 - H_i) + \sum_{1 \leq i < j \leq \ell} \frac{i(i+1-j)}{\ell+1} H_iH_j + M_i^*M_{ij}. \]  \hspace{1cm} (B.1)

The relation between the Casimir of \( U(\mathfrak{su}(\ell + 1)) \) and the Casimir of \( U(\mathfrak{su}(\ell)) \) is:
\[ \sum_{i=1}^{\ell} X_i X_i^* = C_{q=1} - C'_{q=1} - \frac{\ell+1}{8\ell} \hat{H}(\hat{H} + \ell) \]

where \( \hat{H} = \frac{2}{\ell+1} \sum_{i=1}^{\ell} iH_i \). This can be obtained both as \( q \to 1 \) limit of (3.19) or using (B.1).

The limit of Dirac operator in (6.1) is \( D_N = \hat{\partial} + \hat{\partial}^\dagger \), and the formulæ (6.2) gives
\[ D_N^2|_{\Omega_N^k} = C_{q=1} - C'_{q=1} - \frac{\ell+1}{8\ell} \hat{H}(\hat{H} + \ell) + k(\ell + 1 - N), \]
where the right action is understood. The operator \( \langle \hat{H} \rangle^k_{\Omega_N^k} \) is given by \( 2k - \frac{2\ell}{\ell+1} N \) times the identity operator. The operator \( \langle C'_{q=1} \rangle \) is constant on \( \Omega_N^k \) and given by (cf. (6.5))
\[ \langle C'_{q=1} \rangle = \frac{\ell+1}{2\ell} k(\ell - k). \]

Thus
\[ D_N^2|_{\Omega_N^k} = C_{q=1} + \frac{\ell}{2\ell+1} N(\ell + 1 - N). \]

Notice that the constant on the right hand side does not depend on \( k \). If we forget Cumulatively, the results of Proposition 5.5 can be can be simplified as follows. If \( N \leq 0 \),
\[ \Omega_N^* \simeq V(0,0,\ldots,0,-N) \oplus 2 \bigoplus_{k=1}^{\ell} \bigoplus_{m \in \mathbb{N}} V(m,0,\ldots,0,m-N+k) + \xi_k, \]
if \( N > \ell \),
\[ \Omega_N^* \simeq V(N-\ell-1,0,\ldots,0,0) \oplus 2 \bigoplus_{k=1}^{\ell} \bigoplus_{m \in \mathbb{N}} V(m+N-k,0,\ldots,0,m) + \xi_k, \]
and if \( 1 \leq N \leq \ell \),
\[ \Omega_N^* \simeq 2 \bigoplus_{k=1}^{N-1} \bigoplus_{m \in \mathbb{N}} V(m+N-k,0,\ldots,0,m) + \xi_k \]
\[ \oplus 2 \bigoplus_{m \in \mathbb{N}} V(m,0,\ldots,0,m) + \xi_N \]
\[ \oplus 2 \bigoplus_{k=N+1}^{\ell} \bigoplus_{m \in \mathbb{N}} V(m,0,\ldots,0,m-N+k) + \xi_k. \]

Eigenvalues of \( D_N \) are computed using the \( q \to 1 \) limit of Lemma 3.4. Let
\[ \lambda_{m,k}^N := \sqrt{(m+N)(m+k)}, \] \hspace{1cm} (B.2a)
\[ \mu_{m,k}^N := \frac{k(2m+k+N)}{(m+N)(m+k)} \binom{m+\ell}{m} \binom{m+k+N-1}{\ell} \binom{\ell}{k}. \] \hspace{1cm} (B.2b)

If \( N \leq 0 \), \( D_N \) has a kernel of dimension \( \binom{-N+\ell}{\ell} \) while non-zero eigenvalues \( \lambda \)'s and their multiplicities \( \mu \)'s are given by
\[ \text{Sp}(D_N) \setminus \ker(D_N) = \{ (\pm \lambda_{m,k}^N, \mu_{m,k}^N) ; 1 \leq k \leq \ell, m \in \mathbb{N} \}. \]
If \( N > \ell \), \( D_N \) has a kernel of dimension \( \binom{N-1}{\ell} \) and
\[
\text{Sp}(D_N) \setminus \ker(D_N) = \left\{ (\pm \lambda_{m,k}^N, \mu_{m,k}^N) ; 1 \leq k \leq \ell, m \in \mathbb{N} \right\}.
\]
Notice that in the cases above \( \text{Sp}(D_N) = \text{Sp}(D_{\ell+1-N}) \). If \( 1 \leq N \leq \ell \), \( D_N \) is invertible with spectrum
\[
\text{Sp}(D_N) = \left\{ (\pm \lambda_{m,\ell+1-k}^N, \mu_{m,\ell+1-k}^N) ; 1 \leq k < N, m \in \mathbb{N} \right\} \cup \left\{ (\pm \lambda_{m,k}^{\ell+1-N}, \mu_{m,k}^{\ell+1-N}) ; N \leq k \leq \ell, m \in \mathbb{N} \right\}.
\]
In particular, if \( \ell \) is odd and \( N = \frac{1}{2}(\ell + 1) \), the Dirac operator of the Fubini-Study metric \( D = D_{\frac{1}{2}(\ell+1)} \) has eigenvalues:
\[
\pm \sqrt{(m + \frac{\ell+1}{2})(m + k)}, \quad \frac{\ell+1}{2} \leq k \leq \ell, m \in \mathbb{N}.
\]

Let us compare these results with the literature.

For odd \( \ell \), the spectrum of the Dirac operator of the Fubini-Study metric on \( \mathbb{C}P^\ell \) (i.e. \( D_N \) when \( N = \frac{1}{2}(\ell + 1) \)) has been computed in [4] (cf. also [27]). The same spectrum has been computed with a different method in [1], and by Theorem 4.6 in [1] it coincides with the one in [4, 27] after the right parameter substitutions. Finally, in [16] the spectrum of \( D_N \) for arbitrary \( N \) is computed (for both odd and even \( \ell \)); when \( \ell \) is odd and \( N = \frac{1}{2}(\ell + 1) \), coincides with the spectrum in [1].

We obtain the same result. The comparison with the notations of [16, App. B] is as follows: \( n := \ell \) is the dimension (over \( \mathbb{C} \)) of the space, \( q := \ell + 1 - N \) is the ‘charge’ of the bundle used to construct the Dirac operator, \( l := m + k + N - \ell - 1 \) and \( k' := k - 1 \) is the integers that label the eigenvalues. Modulo a misprint in last line of [16, App. B] (the constraint is \( l + q - k' \geq 1 \), with a \(-\) sign, cf. (105) of their paper), their spectrum coincide with our.

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