Supporting Information for

How viscous bubbles collapse: topological and symmetry-breaking instabilities in curvature-driven hydrodynamics

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1. Stokes (non-inertial) dynamics of curved films

In this appendix we describe the derivation of the force balance equations (7a,7b) of the main text. An early version of these equations, which does not include the “extrinsic” charge, \(\rho_{(ext)}\), was obtained by Howell for a thin film of volumetrically-incompressible fluid (Sec. 4.2 of Ref. (1)). Here we exploit a formal similarity to FvK equations that describe the mechanical equilibrium of elastic sheets, to derive Eqs. (7a,7b), highlighting the similarity and difference from Howell’s equations.

We start (Subsec. A) by writing a generic form of the stress tensor of a 2D isotropic fluid model, which does not depend explicitly on the film thickness and is not restricted to a film of incompressible liquid volume. Then we specialize to the case of a thin film of volumetrically-incompressible fluid. In Subsec. B we address the tangential force balance, noting a contribution to the viscous stress tensor (due to rate of change of the film’s thickness), which was not included in Howell’s original derivation of the analogous equations (see also Subsec. D). In Subsec. C we obtain the normal force balance by extending the \textit{viscida} (pointing an analogy to the relation between Euler’s and 1st FvK equations in elasticity theory). In Subsec. E we illustrate the relevance of the two terms on the RHS of Eq. (7) of the main text, through the spherically-symmetric collapse of a depressurized bubble.

A. An invariant form of strain rate and stress. Following Scriven (2), we start with a generic, covariant formulation of the local mechanics and kinematics in a curved, 2D momentum-conserving viscous film. We consider a smooth surface \(\bar{X}(u_1, u_2; t)\) made of viscous fluid and described by two generalized coordinates \(u_{1,2}\), with metric and curvature tensors, \(g_{ij}, k_i, (i,j \in \{1,2\})\), respectively, such that \(|k_i| \ll h^{-1}\). Denoting orthogonal unit vectors locally tangential to the surface by \(t_i\), and a normal vector by \(\hat{n}\), we decompose the fluid velocity \(\vec{v}\) into tangent and normal components, \(\vec{v} = v_t \hat{t}_i + v_n \hat{n}\), respectively. The mechanics of a Newtonian fluid is then expressed through a linear relation between the 2D tensors of stress (force/length) and strain rate (inverse time):

\[
\sigma_{ij} = \mu \det g [\delta_{ij} + \eta_{2d}(\dot{\varepsilon}_{ij} + g^{lm} \dot{\varepsilon}_{lm})],
\]

whereas the strain rate is given by:

\[
\dot{\varepsilon}_{ij} = \frac{1}{2} (\dot{t}_i \cdot \mathcal{D} v_j + \dot{t}_j \cdot \mathcal{D} v_i) + \frac{1}{2} \delta_{ij},
\]

where \(\mathcal{D}\) denotes a covariant derivative. In the above expressions, the viscous part of the stress tensor is proportional to the 2D shear viscosity \(\eta_{2d}\) (pressure-length-time), whereas an explicit dilatational viscosity is ignored. The thermodynamic part of the stress tensor is isotropic, proportional to a “chemical potential” \(\mu\) (energy/area). Note that the strain rate is affected both by gradients of the tangential velocity, as well as by rate of change of the metric that does not involve tangential flow.

For the problem we study here, the 2D surface is actually a mid-surface of a film of thickness \(h(X,t)\) of a volumetrically-incompressible liquid. In this case, temporal variation of the mid-surface metric stems from the rate of change of film’s thickness, \(\partial_t h\), as is evidenced in (6) of the main text. Mass conservation is given by (3):

\[
\partial_t h = -\text{div}(h\vec{v}) = -\text{div}(hv_t) - hv_n (\text{div} \hat{n}),
\]

where \(\text{div} \vec{A} \equiv t_i \cdot (t_i \cdot \mathcal{D}) \vec{A}\)

is the surface covariant divergence of a vector field \(\vec{A}\), and \(\mathcal{H} = \frac{1}{2} (\text{div} \hat{n})\) is the mean curvature of the surface. Equation (9) of the main text is the axisymmetric version of (4).

For a viscous film of finite thickness \(h\), the thermodynamic part of the stress in Eq. (1) is determined by the surface energy of the two free surfaces, hence:

\[
\mu = 2\gamma \cdot (1 + h\nabla^2 h),
\]

where \(\nabla^2\) above is interpreted as Laplace-Beltrami operator. The assumption of nearly-uniform thickness allows us to ignore thickness gradient in the above expression, hence \(\mu = 2\gamma\). More specifically, while temporal variation of the thickness, \(i.e., \partial_t h\), may have a finite \((O(1))\) effect on the surface dynamics and is thus included in our analysis, spatial gradients in the thickness of a film, which we assume to be initially nearly uniform, remain \(O(\epsilon)\) during the flattening process and can therefore be safely neglected. See further discussion of this in Subsec. 3A.4.4 below.

B. Tangential force balance. Since our focus in this paper is on a thin film of volumetrically-incompressible fluid, we choose in this section not to follow the covariant 2D formulation introduced above, but instead the standard method in continuum mechanics, namely starting with the viscous stress of a 3D, volumetrically-incompressible liquid film and performing “dimensional reduction” by integrating over the film’s thickness. This method is analogous to the derivation of FvK equations in classical elasticity theory for a thin solid plate. In this approach, we assume the fluid surface is described as \(\bar{X}(x, y, t) = x\hat{z} + y\hat{y} + z(x, y, t)\hat{z}\), such that \(|\nabla z| \ll 1\), and we can choose, up to corrections of \(O(\nabla z^2)\), the two orthogonal unit tangent vectors as \(t_1 \approx \hat{x} + \partial_x \hat{z}\hat{z}\) and \(t_2 \approx \hat{y} + \partial_y \hat{z}\hat{z}\), and the normal vector is \(\hat{n} \approx \hat{z}\). As we noted throughout the paper, a similar version of Eq. (7) of the main text was obtained by Howell (1). Specifically, Sec. 4.2 of Ref. (1)) employs an asymptotic analysis, where the film’s
thickness and slope of the mid-surface are comparable (i.e. \( h/R \sim |\nabla z| \ll 1 \)). In our approach, both \( h/R \) and \( |\nabla z| \) are assumed small, but not necessarily comparable (i.e. \( h/R \ll |\nabla z| \ll 1 \) is allowed). As we show below (Subsec. 1E), the term \( \nabla^2 (v_n \mathcal{H}) \), obtained in the current derivation, is crucial for recovering a small-slope version of this problem, namely, a spherically-symmetric collapse.

To obtain the 2D strain rate and stress tensors (Eq. (6) of the main text), we decompose the tangential velocity as \( \mathbf{v} = v_x \mathbf{t}_1 + v_y \mathbf{k}_2 \), and construct the corresponding displacement of fluid elements at time interval \( dt \): \( \mathbf{u} = u_x \mathbf{t}_1 + u_y \mathbf{k}_2 \), such that \( \mathbf{v} = \partial_t \mathbf{u} \). We now obtain the strain-rate tensor as the temporal derivative of the strain tensor, employing the small-slope, “geometrical nonlinearity” principle underlying FvK equations of elastic plates. Namely, the strain tensor (and thereby the strain-rate) must include quadratic terms, such that a simple rotation of a fluid element (where \( \partial_x u \neq 0 \)) generates in fact no strain.

\[
\begin{align*}
\dot{\varepsilon}_{xx} &= \partial_t \varepsilon_{xx} \approx \partial_t \left( \partial_x u_x + \frac{1}{2} (\partial_x z)^2 \right) = \partial_x v_x + \frac{1}{2} \partial_t (\partial_x z)^2, \\
\dot{\varepsilon}_{yy} &= \partial_t \varepsilon_{yy} \approx \partial_t \left( \partial_y u_y + \frac{1}{2} (\partial_y z)^2 \right) = \partial_y v_y + \frac{1}{2} \partial_t (\partial_y z)^2, \\
\dot{\varepsilon}_{xy} &= \partial_t \varepsilon_{xy} \approx \frac{1}{2} \left( \partial_y u_x + \partial_x u_y + \partial_z z \partial_y z \right) = \frac{1}{2} \left( \partial_y v_x + \partial_x v_y + \partial_t (\partial_z z \partial_y z) \right). 
\end{align*}
\]

In order to derive analogous expressions for the corresponding components of the thickness-averaged stress in the film, consider one component of the tangential force balance inside the film:

\[
-\partial_x p + 2\eta \partial_x \dot{\varepsilon}_{xx} + 2\eta \partial_y \dot{\varepsilon}_{xy} = 0 
\]

(noting that the term \( 2\eta \partial_x \dot{\varepsilon}_{xx} \) vanishes upon integration over the thickness and applying free surface boundary conditions), where fluid inertia is ignored, and \( \eta \) is the dynamic shear viscosity of the liquid (pressure-time). Note that, similarly to the tangential strain rate components in Eq. (6), the pressure \( p \) is also constant across the film’s thickness, up to \( O(h/L) \). The pressure value is readily obtained by noticing that the free surfaces imply its balancing with a viscous force:

\[
p = 2\eta \partial_n v_n 
\]

whereas the volumetric incompressibility of the liquid implies that (similarly to Eqs. (3) and (4)):

\[
\partial_n v_n = -[v_n (\nabla \cdot \mathbf{n}) + \nabla \cdot \mathbf{v}] = -[2v_n \mathcal{H} + (\partial_x v_x + \partial_y v_y)] 
\]

Plugging the pressure into Eq. (7), and performing the analogous manipulation for the force balance in the \( \hat{y} \) direction, we note that the two tangential force balance equations can be written as:

\[
\nabla \cdot \mathbf{\sigma} = 0
\]

where \( \mathbf{\sigma} \) is a thickness-integrated stress, whose components are:

\[
\begin{align*}
\sigma_{xx} &\approx 2\gamma + 2\eta h (2\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + 2v_n \mathcal{H}) \\
\sigma_{yy} &\approx 2\gamma + 2\eta h (2\dot{\varepsilon}_{yy} + \dot{\varepsilon}_{xx} + 2v_n \mathcal{H}) \\
\sigma_{xy} &\approx 2\eta \dot{\varepsilon}_{xy}.
\end{align*}
\]

Conversely, the components of the strain rate tensor are given by:

\[
\begin{align*}
\dot{\varepsilon}_{xx} &\approx \frac{1}{6\eta h} (2\sigma_{xx} - \sigma_{yy} - 2\gamma) - v_n \mathcal{H}, \\
\dot{\varepsilon}_{yy} &\approx \frac{1}{6\eta h} (2\sigma_{yy} - \sigma_{xx} - 2\gamma) - v_n \mathcal{H}, \\
\dot{\varepsilon}_{xy} &\approx \frac{1}{2\eta} \sigma_{xy}.
\end{align*}
\]

It is well known (e.g. from linear elasticity (4)) that Eq. (9) is automatically satisfied by the celebrated Airy stress potential, \( \Phi^{(0)}(x, y, t) \), such that:

\[
\sigma_{xx} = \partial_y \Phi , \quad \sigma_{yy} = \partial_x \Phi , \quad \sigma_{xy} = -\partial_{xy} \Phi
\]

Substituting (12) for the stresses in (11), adding the second derivatives of the first line with respect to \( y \) and the second line with respect to \( x \) and subtracting the mixed second derivative of the third line, we obtain Eq. (7b) of the main text:

\[
\nabla^4 \Phi = \eta h [-3\partial_t \mathcal{R} + \nabla^2 (v_n \mathcal{H})]
\]

C. Normal force balance. The normal force balance on the film is expressed by Eq. (7a) of the main text. This equation was obtained by Howell (1) as a generalization of the viscosa (5), which is itself analogous to the celebrated Euler’s elastica that describes the normal force balance at mechanical equilibrium of a solid sheet subjected to developable deformation (i.e. \( \mathcal{R} = 0 \)). A reader who is interested in a derivation of Eq. (7a) of the main text through “dimensional reduction” of the Navier-Stokes equation of a non-inertial, volumetrically incompressible liquid film (Trouton’s approach) is referred to Refs. (5) and Secs. 2.4 of Ref. (1). Here we will employ the similarity to the elastica in order to physically motivate this equation.
Elastica: In the absence of external tangential forces, the planar deformation of a naturally flat elastic sheet, \( \vec{X}(x, y) = (x, y, z(x)) \) with \( |z'| \ll 1 \), is characterized by a single non-vanishing stress component that is spatially constant, \( \sigma_{zz}(x, y) = \sigma \), and the normal force balance at mechanical equilibrium is given by:

\[
-\sigma \kappa + B \kappa'' = f_n, \quad \kappa \approx z'',
\]

where \( B \) is the elastic bending modulus and \( f_n \) is an external force/area exerted normally to the sheet. One may readily recognize the first term in this simplified version of the elastica as the resistance of a tensed string (corresponding to \( \sigma > 0 \)) to deflections from flatness, whereas the second term originates from the resistance of a naturally-planar sheet to curvature. Alternatively, Eq. (14) is obtained as the Euler-Lagrange equation that minimizes the elastic energy \( \approx \frac{1}{2}(\sigma \cdot (z')^2 + B \cdot (\kappa')^2) \), with respect to variations of the shape \( z(x) \).

Viscida: The small-slope version of the viscida (5), which describes the normal force balance in planar deformations of a free-surface viscous film, is now obtained from Eq. (14) by expressing the stress \( \sigma \) through its capillary and viscous contributions, \( \sigma = 2\gamma + 4\eta h \dot{z}_{xx} \), where the strain rate \( \dot{z}_{xx} \) is given in Eq. (6), and replacing the elastic bending force by a "viscous bending", namely, \( B \rightarrow \eta h^3 \partial_t \). We thus obtain:

\[
-\sigma \kappa + \eta h^3 \partial_t \kappa'' = f_n,
\]

Physically, the reason that the viscous bending force does not depend on the curvature \( \kappa \) (but only on its gradients) is identical to the reason that the viscous stress depends on gradients of the tangential velocity but not on the velocity itself. In both cases, it is the mutual shearing of liquid layers that generates mechanical resistance rather than net (rigid-body like) motion of the liquid body.

1\textsuperscript{st} FvK equation: Turning back to the case of an elastic sheet, we note that Eq. (14) is readily generalized to non-planar deformations, i.e. \( \vec{X}(x, y) = (x, y, z(x, y)) \) with \( |\nabla z| \ll 1 \), where the stress and curvature are now second rank tensors, rather than scalar functions, defined on a surface. The product \( \sigma \kappa \) becomes a scalar product of two tensors, and the second derivative of the curvature is replaced by the Laplacian of the trace of the curvature tensor. Consequently, normal force balance is:

\[
-\sigma_{ij} \kappa_{ij} + B \nabla^2 \kappa_{ii} = f_n, \quad \text{where: } \kappa_{ij} \approx \partial_{ij} z,
\]

and the stress tensor \( \sigma_{ij} \) is obtained by solving the 2\textsuperscript{nd} FvK equation (that describe in-plane force balance, akin to (13)).

Equations (7a): The formal relation between Eqs. (7a) of the main text and (16) is identical to the relation between the viscida, Eq. (15) and the elastica, Eq. (14). Namely, the elastic stress tensor \( \sigma_{ij} \) is replaced by the liquid stress, which consists of capillary and viscous contributions according to Eq. (6), and is found by solving Eq. (7b) of the main text. Additionally, the elastic bending force is replaced by a viscous bending force, \( \eta h^3 \partial_t \nabla^2 \kappa_{ii} \). The resulting equation can be written as:

\[
-\sigma_{ij} \kappa_{ij} + B \partial_t \nabla^2 \kappa_{ii} = f_n(t), \quad \text{where: } \kappa_{ij} \approx \partial_{ij} z,
\]

Equation (7a) of the main text, where \( f_n(t) = -\Delta P(t) \), is the axisymmetric realization of this equation.

D. Comparison with previous versions. To our knowledge, the first attempt to derive equations of motion for the mid-surface of a thin film of viscous liquid (Stokes) flow, in the presence of Gaussian curvature, was done by Howell (1). The difference between (13) and its counterpart in Ref. (1) is the second term in the square brackets on the RHS. Physically, this difference is related to our inclusion of the effect of the temporal variation of thickness, \( \partial_t h \), implied by mass conservation (Eq. (4) and correspondingly (8)), on the in-plane strain rate tensor.

Another related paper is Ref. (6), which addressed the viscous dynamics of a pressurized bubble slightly perturbed from its equilibrium hemispherical shape. In this case, the shape of the mid-surface is \( z(r, t) = z_{sph}(r) + \delta z(r, \theta, t) \), with \( |\nabla \delta z| \ll |\nabla z_{sph}| \), and the surface dynamics in Ref. (6) is obtained by expanding Eqs. (17,13) to linear order in \( \delta z \).

E. Spherically symmetric solution. Assume now a spherical, free-standing bubble, where the film’s mid-surface has radius \( R_0 \) and thickness \( h_0 \ll R_0 \), and assume that the interior gas’s pressure is suddenly suppressed at \( t = 0 \) from \( P_a + 4\gamma/R_0 \) to the ambient pressure \( P_a \). We consider a non-inertial, volumetrically-incompressible, spherically symmetric solution to this dynamics.

Following Ref. (7), one may solve this problem by considering spherically-symmetric, volumetrically-incompressible flow in the bulk of the film, \( \bar{v} = R_0^{-2} \bar{\rho} \) (where \( \rho \in (R(t) - h(t), R(t) + h(t)] \) is the distance of a fluid element in the film from the bubble’s center) and determine the mid-surface shrinking rate, \( \bar{R} \), by employing free-surface boundary conditions at the internal/external surfaces, \( \rho = R(t) \pm h(t) \). (We thank P. Howell for pointing out this simple solution of spherically symmetric dynamics). For our purpose it suffices to solve this problem within a small-slope description of an axisymmetric mid-surface dynamics, as given by Eqs. (7) of the main text. We can consider a small patch of an evolving spherical surface of radius \( R(t) \) as a paraboloid, \( z(r, t) = a(t) + r^2/2R(t) + O(r/R)^2 \), where \( r \) is now the distance from the axis that connects the sphere center to the center of the small patch (such that \( r \ll R(t) \)). The small-slope approximation \( |\nabla z| \ll 1 \) now amounts to \( r/R(t) \ll 1 \). The Gaussian and mean curvatures, evaluated through the expressions in (8) of the main text, are, respectively, \( R = R^{-2} \), and \( H = R^{-1} \), and the normal velocity is: \( v_n = \partial_t z \sqrt{1 + (\partial_t z)^2} = \dot{a} + \frac{1}{2}(a - \dot{R})(r/R)^2 + O(r/R)^4 \), such that
\[ \rho_{dy} = \rho_d^{(ex)} + \rho_d^{(int)} = R^{-3}(4R + 2\dot{a})/3 + O(r/R)^2. \]

Since a necessary condition for a stress-free dynamics (as implied by normal force balance, see main text), is \( \rho_{dy} = 0 \), we obtain the axisymmetric dynamics of the paraboloid, \( z(r, t) = -2R(t) + r^2/2R(t) \), as the small-slope version of the stress-free spherically-symmetric solution. (We remind the reader that such a “uniformly shrinking” paraboloid does not satisfy the effective clamping at the meniscus.)

### 2. Boundary conditions (BCs) and thermodynamics

The BCs for the surface dynamics, Eqs. (7) of the main text, consist of a subset imposed at \( r \to 0 \), and \( r \to 1 \), in addition to thermodynamic constraint, Eq. (12) of the main text, that determines the disclination charge \( q(t) \) in the non-homogenous BC (13) of the main text. We first give the homogenous BCs, and then explain them, as well the evaluation of the terms in Eq. (12).

Eq. (7) of the main text consists of coupled PDE’s for \( z \) and \( \Phi \), which are are \( 4^{th} \) order in \( r \), thus requiring eight BCs. There is one non-homogenous BC (Eq. (13) of the main text), and seven homogeneous BC’s. Six of them arise from regularity and symmetry considerations (see below)

\[ \begin{align*}
    r & \to 0 : \\
    & \quad \partial_r z \to 0, \quad \partial_r^2 z \to 0, \quad \partial_r \Phi \to 0, \\
    r & \to 1 : \\
    & \quad \Phi \to 0, \quad \partial_r^2 \Phi \to 0, \quad z \to 0.
\end{align*} \]

The seventh homogeneous BC results from the immobility of the meniscus (discussed in the main text), enforcing preservation of its shape at \( r \approx 1 \) while the rest of the film evolves (see below):

\[ r \to 1 : \quad \partial_t \partial_r^2 z \to 0. \]

We turn now to explain these BCs.

#### A. Homogeneous BCs at \( r \to 0 \). At \( r \to 0 \) we have one non-homogeneous BC (13), on whose rationale we elaborated in the main text, and the 3 homogeneous BCs in Eq. (18) of the main text, which we discuss here.

Consider first the BCs on the shape \( z(r, t) \) at \( r \to 0 \). Regardless of depressurization rate, the absence of an external, localized distribution of normal force anywhere on the film implies that for any finite thickness (i.e. \( \epsilon > 0 \)) the rate of change of the curvature components is finite anywhere. Specifically, integrating Eq. (7a) over a small, vicinity of \( r \approx 0 \) implies that \( z(r, t) \approx z_0(t) + b_2(t)r^2 + b_4(t)r^4 + \cdots \), such that two natural BCs are: \( \partial_r z = 0 \) and \( \partial_{rr} z = 0 \).

Consider now the stress potential, \( \Phi(r, t) \) at the vicinity of \( r \to 0 \), and express it formally as a sum of 4 solutions to the homogeneous (axisymmetric) bi-Laplacian equation \( \Phi \sim \text{cst} + r^2 - r^2 \log r + \log r \) (in addition to possible non-homogeneous contribution from \( \rho_{dy}(r \to 0) \)). Among the 4 homogeneous solution only the last one is forbidden, since \( \Phi \sim \log r \Rightarrow |\sigma_{rr}|, |\sigma_{\theta\theta}| \sim r^{-2} \), which imply an infinite rate of kinetic energy dissipated by viscous flow (see below). The BC \( \partial_r \Phi \to 0 \) allows the 3 other homogeneous solutions and eliminates only the non-physical one.

#### B. BCs at \( r \to 1 \). At \( r \to 1 \) we have 4 homogeneous BC’s, given by Eqs. (19,20) of the main text. Two of them are rather obvious: The condition \( \Phi \to 0 \) at \( r \to 1 \) simply sets an arbitrary constant to the potential (gauge invariance, similarly to electrostatic potential), and the condition \( z \to 0 \) at \( r \to 1 \) derives from the immobility of the meniscus, on which we elaborated in the main text. Next we discuss the remaining two BCs at \( r \to 1 \).

For the thin film of a floating bubble the “boundary” \( r = 1 \) is in fact a meniscus — a nearly symmetric elevation of the liquid bath, of size \( \sim \ell_c \ll 1 \), in both sides of the circle \( r = R_0 = 1 \) (see schematic Fig. 1a), as if the bubble would have been replaced by a thin, perfectly wetting solid wall protruding vertically from the liquid bath at \( r = 1 \). This free-surface structure is dominated by gravity and surface tension, and is thus described by the non-linear Young-Laplace equation (8) (assuming viscous stresses due to flow have negligible effect in this region). Specifically, the height of the ideal Young-Laplace meniscus is \( z_{top} \approx \sqrt{2\sigma_c/\ell_c} \), and its width near the top is characterised by a parabolic profile, \( w(z) \approx (z - z_{top})^2/\ell_c^2 \). Since the real meniscus must terminate at a finite width \( \ell_B \ll \ell_c \), a continuous transition from the film to the meniscus occurs through a boundary layer, of length \( \ell_B \sim \sqrt{\sigma_c/\ell_c} \sim \sqrt{\epsilon/Bo} \). In addition to providing a “pinning ring” to the film that resists changes of its initial radius \( R_0 \), the meniscus acts as a vertically-oriented “funnel”, through which liquid can flow from/into the bath. Even though one may employ methods of singular perturbation theory to carry out a quantitative analysis of the transition between the film and meniscus of large bubbles (i.e. \( Bo \gg 1 \)), following an analogous study for small bubbles (Ref. (9)), we will not pursue this approach here. Instead, we will obtain BCs for the film at the interior vicinity of the meniscus, (i.e. \( |r - \ell_c| \sim \ell_B \)) by assuming that vertically-oriented funneling of liquid into/from the film in the vicinity of the initial meniscus (of radius \( R_0 = 1 \)) persists throughout the depressurization process, even if the meniscus size may deviate from \( \ell_c \) (due to viscous stress) and so does the precise structure of the meniscus profile. Experimental observations clearly support this assumption (see e.g. SI movies 1-3 in Ref. (10)).

As long as liquid can flow freely through the meniscus, the momentum flux at \( r \to 1 \), that is \( 2\pi\sigma_{rr}(r \to 1) \), is not restricted by the liquid bath. Mathematically, we express this physical condition by a homogeneous equation that involves derivatives of the stress potential \( \Phi \) at \( r \to 1 \) and is satisfied by the initial (pressurized) state of the bubble, where the stress is given everywhere by the isotropic surface tension. A general BC of this type is:

\[ \text{at } r \to 1 : \quad c_1 (\partial_r \Phi - \partial_{\theta\theta} \Phi) + c_2 \cdot \partial_{\theta\theta} \Phi = 0, \]
where \(c_1, c_2\) are arbitrary (and may be in principle functions of \(t\), and depend on the dimensionless parameters \(\epsilon\) and/or \(T\)). The BC \(\partial_{\tau r} \Phi \to 0\) in Eq. (19) of the main text is realized by choosing \(c_1 = 0\). We verified that the evolution of the bubble collapse is insensitive to other choices.

The remaining BC at \(r \to 1\) pertains to the flux of angular momentum, that is the torque exerted by the meniscus on the liquid film. Such a torque exists since the steady, vertical orientation of the meniscus implies a sharp transition in the tangent direction of the mid-surface at the vicinity of \(r \approx 1\). We emphasize that this is a real physical effect and not merely a mathematical artifact of the small-slope approximation that we employed extensively for describing the surface dynamics. Indeed, one may view this situation as a viscous analogue of peeling a thin elastic sheet off a rigid adhesive substrate, in which case the torque exerted by the substrate on the sheet gives rise to discontinuity of the curvature, \([\kappa] \propto \sqrt{\sigma B}\), at the peeling front, such that the tangent direction undergoes a finite, \(O(1)\) variation over a "bendo-capillary" distance, \(\sqrt{B/\sigma}\) (11).

Focusing on the normal force balance, Eq. (7a) of the main text, we can pursue further the elastic analogy by noticing that in the vicinity of the meniscus, where the radial component of the curvature tensor is large, force balance must be dominated by terms associated with it. Consequently, at the boundary layer, \(|r - 1| \lesssim \ell_{BL}\), where we expect the stress components to vary smoothly, it is possible to replace Eq. (7a) by the \(\textit{viscosa}\), Eq. (15), where \(\kappa_{rr} \to \kappa(t), \kappa_{r\tau} \to \kappa(s, t) = \partial_s \cos^{-1}([\mathbf{t}_r \cdot \hat{r}])\), and \(s\) is an arc-length parameter of the surface along the tangent direction \(\mathbf{t}_r\). Importantly, the one-dimensional \(\textit{viscosa}\) does not rely on a small-slope approximation and one may thus employ it to interpolate between the bulk of the film (where \(|\mathbf{t}_r - \hat{r}| \sim O(\epsilon^2)\) in the meniscus (where \(|\mathbf{t}_r - \hat{z}|\)). Integrating over the boundary layer, we obtain a relation between the jumps incurred by the curvature and tangent angle upon transitioning from the film to the meniscus:

\[
\epsilon^2 \partial_t (r^{-1} \partial_r z + \partial_{rr} z) + c_3 \ell_{BL} \partial_r \Phi \cdot (\partial_r z + 1) \approx 0. \tag{23}
\]

where the numerical value of \(c_3\) must be determined by carrying out an asymptotic matching of the film and the meniscus.

Going back to the small-slope approximation for the inner part of the bubble, Eq. (22) becomes:

\[
\epsilon^2 \partial_t \left( r^{-1} \partial_r z + \partial_{rr} z \right) + c_3 \ell_{BL} \partial_r \Phi \cdot (\partial_r z + 1) \approx 0. \tag{24}
\]

which replaces the homogeneous BC (21).

**C. Heat production and release of surface energy.** In the main text we invoked the 1st law of thermodynamics through (12), which determines the disclination charge \(q(t)\) in (13) of the main text. Here we express the heat production rate \(P_{\text{vis}}\), and the release of surface energy \(E_{\text{sorb}}\) in terms of the functions \(\Phi(r, t)\) and \(z(r, t)\) that define the axisymmetric dynamics.

Consider a small volume element inside a free-surface film, \(-h/2 < z < h/2\), of viscous, volumetrically-incompressible, axisymmetric flow. Ignoring the heat production due to temperature gradients, the heat production rate in this volume can be written as \(2\eta(\dddot{\epsilon}_{rr}^2 + \dddot{\epsilon}_{\theta\theta}^2 + \dddot{\epsilon}_{zz}^2)\), (see e.g. Eq. 49.6 of Ref. (12)). Substituting \(\dddot{\epsilon}_{zz} = -(\dddot{\epsilon}_{rr} + \dddot{\epsilon}_{\theta\theta})\) and integrating over the thickness we obtain the heat production rate per area of the film:

\[
P_{\text{vis}} = 4\eta \dddot{\epsilon}_{rr}^2 + \dddot{\epsilon}_{\theta\theta}^2 + \dddot{\epsilon}_{rr} \dddot{\epsilon}_{\theta\theta}, \tag{25}
\]

The heat production per area of a liquid film in a free-surface film with curved mid-surface \((z(r, t) \neq 0)\) is readily obtained from (25) by using the corresponding expressions for an axisymmetric strain rate components in (6), namely, \(\dddot{\epsilon}_{rr} = \partial_r v_r + \frac{1}{2} \partial_r (\partial_r z)\) ; \(\dddot{\epsilon}_{\theta\theta} = v_\theta/r\). Incorporating the viscous flow due to temporal variation of the thickness requires us to revise the above calculation by substituting: \(\dddot{\epsilon}_{zz} \to \dddot{\epsilon}_{nn} = -(\dddot{\epsilon}_{rr} + \dddot{\epsilon}_{\theta\theta} + 2v_n \mathcal{H})\), such that we obtain:

\[
P_{\text{vis}} = 4\eta \left( \dddot{\epsilon}_{rr}^2 + \dddot{\epsilon}_{\theta\theta}^2 + \dddot{\epsilon}_{rr} \dddot{\epsilon}_{\theta\theta} + 2(v_n \mathcal{H})^2 + 2v_n \mathcal{H}(\dddot{\epsilon}_{rr} + \dddot{\epsilon}_{\theta\theta}) \right), \tag{26}
\]

where again, \(\dddot{\epsilon}_{rr} = \partial_r v_r + \frac{1}{2} (\partial_r z)\); \(\dddot{\epsilon}_{\theta\theta} = v_\theta/r\). The total heat production rate is obtained upon integrating over the whole surface:

\[
P_{\text{vis}} = 2\pi \int_{r=0}^{1} p_{\text{vis}} r dr. \tag{27}
\]

The strain rate components, \(\dddot{\epsilon}_{rr}\) and \(\dddot{\epsilon}_{\theta\theta}\), along with the mean curvature \(\mathcal{H}\) and normal velocity, \(v_n \approx \partial_t z\), are given in terms of \(\Phi(r, t)\) and \(z(r, t)\) through Eqs. (6,8) of the main text.
The surface area due to the (two faces of) an axisymmetric surface \( z(r,t) \) with surface energy \( \gamma \), is given by
\[
\hat{E}_{surf} = 2\gamma \hat{A} = 2\pi \int_{r=0}^{1} \sqrt{1 + (\partial_r z)^2} r dr \approx \pi \int_{r=0}^{1} (\partial_r z)^2 r dr
\]  

[28]

We note in passing that Eqs. (25-27) rule out the homogeneous mode of the stress potential, \( \Phi \sim \log r \) at \( r \to 0 \), thus justifying the BC \( \partial_r \Phi \to 0 \) at \( r \to 0 \) on which we elaborated above (part A of this section). Indeed, for \( \Phi \sim \log r \) we would have \( |\partial_{rr}|, |\partial_\theta\theta| \sim r^{-2} \), and consequently \( P_{vis} = \infty \).

3. Detailed description of the axisymmetric front dynamics
In this section we provide additional details regarding the front propagation, that is Eqs. (14) of the main text, reproduced here for clarity:

\[
\Phi \approx \Theta \left( r_f(t) - r \right) \Phi_{dis}(r,t),
\]

[29a]

\[
z \approx z_f(t) + \Theta \left( r - r_f(t) \right) z_{per}(r,t).
\]

[29b]

Eq. (29) is valid at the limit \( \epsilon \to 0 \), which we take throughout this section. As we showed in the main text,

\[
\Phi_{dis}(r,t) = \frac{1}{2} q(t) r^2 \left( \log \frac{r}{r_f(t)} - \frac{1}{2} \right),
\]

[30]

such that radial stress, \( \sigma_{rr} = r^{-1} \partial_r \Phi \) is continuous at the front, as implied by in-plane (radial) force balance. Our purpose here is to describe a formal procedure to obtain the front functions, \( z_{per}(r,t), r_f(t), \) and \( z_f(t) \). In order to simplify the analysis, we commence with a generalized model, Eq. (17) of the main text, which we repeat here for clarity:

\[
\rho_{dg} \longrightarrow \rho_{dg}^{(int)} + \delta_{ext} \rho_{dg}^{(ext)}
\]

[31]

where \( \delta_{ext} \) is an auxiliary parameter. We then address in detail the analytic solution of the ‘intrinsic” limit, \( \delta_{ext} = 0 \), which was summarized in the main text, and captures the behavior of the full model very well. Finally, we discuss the numerical solution of the physical case (\( \delta_{ext} = 1 \)) and provide analytic and numerical arguments as to why the intrinsic model captures well the dynamics of the “full” model, with \( \delta_{ext} = 1 \), by analyzing the auxiliary model at short times and small \( \delta_{ext} \).

Formal solution of the front dynamics: First, integrating Eq. (7b) of the main text twice in a disk of radius \( r > r_f \) yields an overall “charge neutrality”: \( 2\pi \int_{r}^{r_f} r' dr' \rho_{dg}(r') = r^{-1} \partial_r \Phi = 0 \), see Eq. (16) of the main text. Since dynamo-geometric charge is localized at \( r = 0 \) and \( r = r_f \), we readily obtain Eq. (2) of the main text. Thus, the “disclination” charge at \( r = 0 \) is screened out by an induced charge at \( r = r_f \).

Second, an additional integration of Eq. (7b) of the main text over a disk of radius \( r > r_f \) yields:

\[
\frac{1}{3} q(t) = -\frac{1}{2} \partial_t (\partial_r z_{per})^2 + \delta_{ext} \frac{r}{6} \partial_r (\partial_r z_{per} \frac{1}{r} \partial_r z_{per} + \partial_{rr} z_{per})
\]

[32]

For a given \( q(t) \), the last equation is a 3\(^{rd}\) order PDE in \( r \) in a time-dependent interval \( r \in (r_f(t), 1) \), whose advancement in time requires 3 BCs at \( r = r_f \) and \( r = 1 \), along with equation for advancing \( r_f(t) \). Two of these 4 equations are obtained by specializing to the vicinity of \( r \approx r_f(t) \). Namely, a second integration of Eq. (7b) of the main text across the front yields two equations at \( r_f \). The first is continuity of the slope, and the second one relates discontinuities \( \Phi_{dis} \) and \( z_{per} \),

\[
0 = \partial_r z_{per}(r \to r_f),
\]

[33]

\[
0 = [[(\partial_r \Phi)(r \to r_f)] + \frac{z_f}{2} \partial_{rr} z]_{r_f},
\]

[34]

where \( [[\cdots]]_{r_f} \) denotes a discontinuity at \( r = r_f(t) \), and the apical height, \( z_f \), is

\[
z_f = -\int_{r_f}^{1} \partial_r z_{per}(r',t) dr'.
\]

[35]
Electrostatic analogy: Consider now the fundamental electrostatic-like analogy of our system, namely the in-plane force balance Eqs. (23) – (24) of the main text, as well as the disclination form of the potential, Eq. (15). For convenience, we rewrite it here,

$$\frac{1}{r} \partial_r r E_R - \rho_{d_g}^{(int)} - \delta_{ext} \rho_{d_g}^{(ext)} = 0,$$

where we recall that $\rho_{d_g}^{(int)} = -(2r)^{-1} \partial_r (\partial_r z)^2, \rho_{d_g}^{(ext)} = (3r)^{-1} \partial_r r \mathcal{H}(\partial z)$. We now integrate Eq. (36) twice to obtain equations for the electric field $E_R$ and electrostatic potential $V_R$, which are dictated by $\Phi$. Integrating once we find,

$$q(t) / 3r = +E_R(r, t) + P_R(r, t),$$

where the polarization obeys,

$$P_R = P_R^{(int)} + P_R^{(ext)} \equiv \frac{1}{2r} \partial_r (\partial_r z)^2 - \frac{\delta_{ext}}{3} \partial_r (\mathcal{H}(\partial z)).$$

Integrating again we obtain,

$$V(t) + \frac{q(t)}{3} \log(r) = V_R(r, t) + V_P(r, t),$$

where $V_P$ is the contribution to the potential from the polarization, such that

$$V_P = \begin{cases} 0 & r \leq r_f \\ V_P^{(int)} + V_P^{(ext)} & r > r_f \end{cases},$$

where

$$V_P^{(int)} = \int_{r_f}^r \frac{\partial_r (\partial_r z(r', t))^2}{2r'} \, dr', \quad V_P^{(ext)} = -\frac{\delta_{ext}}{3} \mathcal{H}(\partial z).$$

Here $r_f$ is situated at $r$ infinitesimally less than $r_f$, i.e. the integral covers the front itself. The constant $V(t)$ is found from the requirement (due to radial force balance at the front) that $\sigma_{rr} = 0$ at $r = r_f$. Eqs. (36)-(40) are akin to the equations of Maxwell electric displacement field and associated potential in continuous media. They must be identically obeyed throughout the entire film, and should be insensitive to the "jump" at the front. Specializing to the front and using Eq. (15) of the main text sets the potential to

$$V(t) = \frac{q(t)}{6} \cdot \{1 - 2 \log (r_f(t))\}.$$  

Furthermore, at the front, $r = r_f$, $V_R$ jumps from a value of $q/6$ to zero, and hence $V_P$ must also jump from zero to $q/6$.

$$[[V_P]]_{r \to r_f} = \frac{q(t)}{6}.$$  

As we shall show, Eqs. (37)-(42) are enough to obtain the dynamics of the front.

A. The intrinsic model.

A.1. Solution of the intrinsic model. For the intrinsic model, $\delta_{ext} = 0$, Eq. (32) is integrable, yielding the curved shape of the mid-surface in the stress-free periphery to be a surface of constant Gaussian curvature, $R=Cs$, which are known as “Sievert surfaces”. Specifically, the family of axisymmetric Sievert surfaces is parameterized by $S>0$:

$$(\partial_r z_{perc})^2 = r^2 - \frac{2}{3} S(t), \quad \text{with} \quad S = -q,$$

Then, integrating Eq. (7b) of the main text twice over a narrow annulus around $r_f$, and using the fact that the front is a travelling wave, such that $\partial_t \approx -\dot{r}_f \partial_r$, we obtain:

$$[[\partial_r \Phi]]_{r_f} = \frac{\dot{r}_f}{6r_f} [[(\partial_r z)^2]]_{r_f} \quad \Rightarrow \quad \dot{r}_f = -\frac{3r_f}{r_f^2 - \frac{2}{3} S} q,$$

where in the last part of Eq. (44) we used Eqs (29,30,43). Physically, Eq. (44) shows that the front is pushed outward by a sharp variation of the hoop stress, $\sigma_{rr} = \partial_r \Phi$ (which becomes discontinuous as $\epsilon \to 0$.)

It is useful to repeat the derivation of Eqs. (43,44) by employing the electrostatic analogy. Eq. (38) simplifies to the following,

$$\frac{q(t)}{3} = -\frac{1}{2} \partial_t (\partial_r z_{perc})^2.$$

In addition, the jump in $V_P$ implies a jump in $(\partial_r z)^2$. This immediately gives the solution of Eq. (37) with

$$\partial_r z_{perc} = -\sqrt{r^2 - S(t)}, \quad z_{perc}(r_f) = 0,$$

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which describes a Seivert surface of constant Gaussian curvature. For convenience, we define:

\[ \dot{S} = f_0 S, \quad f_0 = \frac{2}{3}, \]

and find that Eqs. (37,38) imply that the surface evolves according to

\[ \dot{S} = -q. \]

Next, consider Eq. (42). Using the piecewise form of \( z \), Eq. (29), and evaluating the jump condition, Eq. (42) in an infinitesimal region around the front, we obtain,

\[ \frac{q}{\delta} = \frac{r_f}{2r_f} (r_f^2 - f_0 S), \]

which recovers Eq. (44). Thus, the jump condition completely determines the dynamics. Fig. S2 depicts the evolution of the intrinsic model. It shows how the slope of \( z_{perc} \) jumps at the front, and evolves exactly according to the analytic expression, Eq. (29).

Inspection of Eqs. (43,44) shows that the dynamics is completely dictated by \( q(t) \), and some algebraic manipulation yield:

\[ r_f(t) = \sqrt{\frac{4}{3} S(t)} \quad ; \quad z_f(t) = z(0, t) = r_f^2 \left( \mathcal{F}(r_f^{-1}) - \mathcal{F}(1) \right), \]

where \( \mathcal{F}(x) = \frac{1}{2} \left[ 2x \sqrt{x^2 - \frac{1}{2}} - \log \left( 2x + 2 \sqrt{x^2 - \frac{1}{2}} \right) \right] \) and \( S(t) = \int_0^t q(t') dt' \). At each \( t \), the physical value of \( q(t) \) is then determined by Eq. (12) of the main text.

### A.2. The disclination charge.

In the above calculation for the intrinsic model, we considered the disclination charge \( q(t) \) as a given temporal function, and determined the surface dynamics, Eqs. (46,47) through the two 1st order ODE’s (48,49). Now we turn to determine the disclination charge \( q(t) \), and thereby fully determine the dynamics.

In the main text, we noted that \( q(t) \) is determined by the thermodynamic Eq. (12), where the rates of heat production rate, \( P_{vis} \), and surface energy release, \( \dot{E}_{surf} \), are given in Subsec. C of the SI. As we pointed out in the main text, in the intrinsic model the effect of temporal variation of the film thickness on the mechanics is ignored altogether, and therefore we use \( p_{vis} \), Eq. (25), to compute the total heat production rate, Eq. (27). The strain rates \( \dot{\varepsilon}_{rr}, \dot{\varepsilon}_{\theta\theta} \), are computed from \( \Phi_{dis} \), Eq. (15) of the main text, and the stress-strain rate relation, Eq. (6) of the main text, with \( \sigma_{rr} = r^{-1} \partial_r \Phi \) and \( \sigma_{\theta\theta} = \partial_{\theta} \Phi \), yielding:

\[
\dot{\varepsilon}_{rr} = \frac{1}{6} \left\{ -2 + \Theta(r - r_f) (\log \frac{r}{r_f} - 1) \right\}; \quad \dot{\varepsilon}_{\theta\theta} = \frac{1}{6} \left\{ -2 + \Theta(r - r_f) (\log \frac{r}{r_f} + 2) \right\},
\]

where \( \Theta(x) \) is the Heaviside function. For the variation rate of surface energy \( \dot{E}_{surf} = 2\gamma \dot{A} \), Eq. (28), we find using Eq. (46):

\[
\dot{A} \approx \pi \left( \frac{3}{2} S - r_f^2 \right) r_f \dot{r}_f + \frac{1}{2} \left( 1 - r_f^2 \right) q.
\]

Substituting the above expressions in Eqs. (27,28) and Eq. (12) of the main text, we obtain:

\[
q(t) = (r_f)^{-2} \left( \sqrt{1 - 4r_f^2} - 1 \right)
\]

At short times, where \( r_f \ll 1 \), we obtain \( q \approx -2 \).

For the full model, we do not have an analytic expression, but the similarity of the dynamics (see below) suggests that Eq. (52) remains a reasonable approximation.

As we discussed in the main text, a direct calculation of Eq. (12) is complicated due to its non-local nature, hence it is useful to replace it with the local, non-homogeneous BC, Eq. (13). The physical value of \( q(t) \) may be determined by requiring consistency with Eq. (12) of the main text. However, one may re-scale time by \( q(t) \), such that the evolution of the surface is not affected by the actual value of \( q(t) \).

To implement in our numerical analysis the BC, Eq. (13) of the main text, we require:

\[
\sigma_{rr}(r = l_{core}) = 1 \Rightarrow q(t) = -2 \log(r_f/l_{core}).
\]

Numerically, we find \( l_{core} = 3a_0 \), where \( a_0 \) is the numerical mesh constant (see Sec. 6). Clearly, Eq. (53) and (52) yield essentially the same dynamics, as long as \( r_f \) is small but larger than \( l_{core} \) (which condition is always satisfied). The figures in the manuscript all use the local constraint above.
A.3. The velocity field. It is useful to employ the analytic solution we obtained for the intrinsic model, $\delta_{ext} = 0$, to address also corresponding velocity field.

Inspecting the strain rate components that we evaluated above, Eq. (51), we note that the presence of a disclination-like singularity at $r \to 0$ must come in tandem with a generalized version of the kinematic relation between velocity and strain rate, Eq. (6c) of the main text, which includes a velocity independent contribution to the hoop strain rate:

$$\dot{e}_{rr} = \partial_r v_r + \frac{1}{2} \partial_k (\partial_r z)^2 \;;
\dot{e}_{t\theta} = v_r / r + D \cdot q$$

where: $D = \begin{cases} 8/3 & r < r_f \\ 1/3 & r > r_f \end{cases}$

such that the fluid velocity is:

$$v_r = -\frac{1}{3} r \cdot \left\{ \begin{array}{ll} 1 - 2q(\log \frac{r}{r_f} - 2) & r < r_f \\ (1 + q) & r > r_f \end{array} \right.$$  

Notice that the velocity field is discontinuous at $r_f$ (i.e. undergoes an $O(1)$ variation over the scale $\ell_{BL} \sim \sqrt{\epsilon}$, such that its average value $\langle v_r \rangle_r f$ and jump $\|[v_r]\|_r f$ are:

$$\langle v_r \rangle_r f = r_f \left( -\frac{1}{3} + \frac{1}{2} q \right) \;;
\|[v_r]\|_r f = qr f.$$  

While the above expressions are specific to the analytically-tractable intrinsic model, $\delta_{ext} = 0$, we note that a discontinuity of $v_r$ across the front, $r = r_f$, does not stem from the discontinuity of the slope $\partial_r z$ (which is smoothed out for any $\delta_{ext} > 0$), but rather from the discontinuity of $\partial_{t\theta} = \partial_r \Phi$, and consequently $\dot{e}_{t\theta}$, which characterizes also the physical model, $\delta_{ext} = 1$. Consequently, we expect that a sharp variation in the tangential fluid velocity across the front is a real effect.

We note that the “revision”, Eq. (54), of the common relationship between velocity and strain rate in a free-surface fluid film is analogous to the revision of the displacement-strain relationship implied by disclination in an elastic sheet. The elastic analogue of a disclination is realized, for instance, by inserting an azimuthal sector of opening angle $\gamma$ into a solid disk (Volterra construction (4)), and features an analogous contribution to the hoop strain, proportional to the disclination charge $q$, similarly to Eq. (54). In our surface dynamics, the dynamo-geometrical charge $q(t)$ reflects an axisymmetric excess strain rate of hoops that is not captured by the 2D velocity field $v_r$.

A.4. Spatial variation of thickness. Gradients in the film thickness $h$ imply corrections to Eq. (7b) of the main text. These corrections stem from two sources. The first type, related to Eq. (5), originates in the thermodynamic (capillary) stress, in Eq. (10), $2\gamma \to 2\gamma(1 + h \nabla^2 h)$, where $\nabla^2 h$ is a small-slope approximation to the curvature of the thickness profile. Including this correction in the above analysis yields through the double differentiation of (11), an additional term, $2\gamma \nabla^2 h$ in Eq. (13). The second type of thickness correction is the viscous stress in (10). Upon double differentiation of Eq. (11), one finds additional terms in Eq. (13) that are proportional to $h^{-1} \nabla^2 h \nabla^2 \Phi$ and $h^{-1} \nabla h \cdot \nabla \nabla^2 h$. Employing our solution of the intrinsic model, we will show below that, as long as the initial thickness profile is smooth, with $|\nabla h| \ll 1$, these thickness gradient terms have negligible effect on the surface dynamics. Strictly speaking, variation of the film’s thickness from its initial state occurs only in the true physical model, $\delta_{ext} = 1$, which incorporates fluid mass conservation through Eq. (9) of the main text. Nevertheless, even for the intrinsic model, $\delta_{ext} = 0$, for which analytic expressions of the radial velocity $v_r(r, t)$ and mid-surface shape $z(r, t)$ are given in the preceding sections, we can study the evolution of the thickness $h(r, t)$ through Eq. (9) of the main text as a totally passive scalar function. We argue that the upper bound estimate we obtain below through this analysis remain valid also for the physical model, $\delta_{ext} = 1$.

Consider then the evolution of the thickness field $h(r, t)$, assuming the film thickness is initially uniform, $h(r, t=0) = \epsilon$ in the dimensionless convention used in the main text, or more generally, $h(r, t=0) = \epsilon + \Delta h_0(r)$, where $\epsilon \nabla^2 \Delta h_0 \ll 1$. Substituting in Eq. (9) of the main text analytic the expressions for $v_r$ and $\partial_t z$ we obtained above, we note that $\partial_t h \sim O(\epsilon)$, exhibits a logarithmic divergence $\sim \log r$, which terminates at the disclination core, $r \sim \ell_{core}$, but is otherwise continuous everywhere except at the vicinity of the moving front, $r = r_f(t)$, such that $\nabla^2 h \sim O(\epsilon)$ both at $r < r_f$ and $r > r_f$. The turning to the vicinity of the front, we note that the mid-surface’s slope, $\partial_t z$, and fluid velocity $v_r$, undergo $O(1)$ jump across the front, as indicated by Eqs. (46, 57). The continuity equation, Eq. (9) of the main text, thus yields $\partial_t \|[h]\|_{r f} \sim -\frac{4}{3} q \epsilon$ and consequently a jump in the thickness across the front, $\|[h]\|_{r f} \sim \epsilon t$. This implies that $\nabla^2 h \sim \epsilon t / \ell_{BL} \sim O(\epsilon t)$ at the front, where we again used the front width $\ell_{BL} \sim \sqrt{\epsilon}$.

We conclude that for an initially smooth profile of the film thickness, the corrections to due thickness gradients in the capillary and viscous terms in the stress tensor (10) remain small throughout the flattening process ($t \sim O(1)$) and do not affect the surface dynamics. We note though that thickness gradients must be incorporated into Eqs. (7), at least locally, into order to describe several aspects of the surface dynamics that are not addressed in this manuscript. An important example, is rupture-induced depressurization, where the thickness gradient at the vicinity of the hole’s edge must be considered, as was done for a planar film in Refs. (13, 14). Another important example is in the experiment of Oratis et al. (10), where it was shown in (15) that the initial thickness profile is nonuniform due to the slow drainage that occurs when the gas bubble rises to the interface.
B. The auxiliary model. We now generalize to the case \( \delta_{ext} > 0 \). In contrast to the intrinsic model (\( \delta_{ext} = 0 \)), we do not have explicit analytic expressions for \( z_{per} \) in this case, nevertheless we are able to show that the qualitative behavior of the front remains unchanged. Specifically, we find that the external terms (associated with mean curvature \( H \)) simply act to broaden the jump in \( \partial_r z_{per} \) at \( r \approx r_f \) into a boundary layer of width \( \sim \delta_{ext} \), such that the form of Eq. (46) remains valid, but only in a "far-field" region of the interval \( r > r_f \). This smoothing of the jump in \( \partial_r z_{per} \) means that the jump in \( V_P \), which is dictated by Eq. (42) and is insensitive to the value of \( \delta_{ext} \), is now taken up by the external contribution, \( V_{P}^{(ext)} \) in Eq. (40).

To see this, let us consider a front dynamics, Eq. (29), for a given \( q(t) \), with \( \Phi_{dis} \) given by Eq. (15) of the main text, and the the following ansatz for \( z_{per} \):

\[
(\partial_r z_{per})^2 = (r^2 - S)\Theta(r - r_0^f) + rG_\delta \left( \frac{r - r_0^f}{\delta_{ext}} \right),
\]

where \( r_0^f \) the location of the front in the intrinsic model (defined formally through Eq. (49)), the first term on the RHS is formally identical to the solution of the intrinsic model (Eq. (46)), and the second term on the RHS describes a localized boundary layer function \( rG_\delta \) of width \( \delta_{ext} \), which vanishes at the front, \( G_\delta(r_f - r_0^f) = 0 \). In addition to \( G_\delta \) the ansatz (29, 58) is characterized by two temporal functions, \( r_f(t) \) and \( S(t) \). We will show now that for a given value of \( \delta_{ext} \) there exist \( \{r_f(t), S(t)\} \) for which the surface dynamics is properly described by the ansatz (58). Furthermore, we will show that \( \{r_f(t), S(t)\} \) can be computed perturbatively by treating \( \delta_{ext} \) as an (artificial) expansion parameter around the analytic solution of the intrinsic model \( \delta_{ext} = 0 \). Note that as defined, \( G_\delta \) has a discontinuous jump as it crosses \( r_0^f \), rather than \( r_f \).

Employing the ansatz (29, 58), we now analyze the problem at short times, but still well after the formation of the front, \( T \ll t \ll 1 \), where Eq. (58) implies that \( S \ll 1 \). We assume, and confirm in a self-consistently manner, that for \( S \ll 1 \) the vicinity of the perimeter, \( r \lesssim 1 \), is described by the first ("bare") term on the RHS of Eq. (58). To see this, begin with the jump condition at the front for \( V_P \), noticing that Eqs. (40) and (42) imply,

\[
[V_P]_{r \rightarrow r_f} = -\frac{\delta_{ext}}{3} \tilde{z}_f [\partial_{rr} z]_{r \rightarrow r_f},
\]

yielding,

\[
(\partial_r z_{per})_{r \rightarrow r_f} = -\frac{q}{2\tilde{z}_f \delta_{ext}}.
\]

In Eq. (59) we kept only the highest order spatial derivative, which appears only in \( V_P^{(ext)} \). This implies that, in contrast to the intrinsic model (\( \delta_{ext} = 0 \)), the jump no longer determines the evolution of \( r_f \). Instead, Eq. (60) implies a second BC for \( G_\delta \) (in addition to the aforementioned condition, \( G_\delta(r_f - r_0^f) = 0 \)).

Next we consider the polarization in the vicinity of \( r \lesssim 1 \), i.e. far away from the perimeter. By our assumption, \( G_\delta \) is negligible there, hence we take the intrinsic contribution only. Recalling Eq. (37) with \( E_R = 0 \) at \( r > r_f \), and with \( P_R^{(ext)} \) of similar form as in the intrinsic model, and

\[
P_R^{(ext)} = -\frac{\delta_{ext}}{3} \left[ (\partial_r z) \partial_r H + H \partial_t (\partial_r z) \right]
\approx +\frac{\delta_{ext}}{3} \partial_t (\partial_r z),
\]

where we used \( H \approx -1 \), expanding to leading order in \( S(t) \). This leads again to Eq. (48), with

\[
\tilde{S} = f_3 S, \quad f_3 = \frac{2}{3 - \delta_{ext}},
\]

thereby self-consistently confirming our assumption, that for \( S \ll 1 \), \( G_\delta \) vanishes near \( r = 1 \).

Eq. (62) defines \( S \) for any \( \delta \geq 0 \), so that we need only address now \( r_f \). To this end, we consider again the electrostatic-like potential. Plugging in our ansatz, Eq. (58), into Eq. (40), and integrating up to the vicinity of \( r = 1 \), we find that the smooth \( \log r \) dependence in Eq. (39) is provided by the terms in \( V_P \) that depend explicitly on \( S \), leaving an equation that involves only \( q(t), r_f(t), r_0^f(t) \) (and an additional unknown time-dependent constant from \( G_\delta \)). Explicitly, the equation for \( r_f \) is given by:

\[
\frac{q(t)}{6} + \frac{q(t)}{3} \log \frac{1}{r_f} = \frac{q(t)}{2} f_3 \log \frac{1}{r_f} - \frac{r_0^f}{2r_f^3} \left( \frac{r_0^f}{r_f} \right)^2 - f_3 S + \delta_{ext} \int G_\delta(x) dx,
\]

where in the last term we used the \( \delta_{ext} \) dependence of \( G_\delta \) and its localized nature to bound its integral.

Clearly, Eq. (63) determines the dynamics of \( r_f \). At zeroth order in \( \delta_{ext} \), it reduces to the intrinsic model's equation of motion for \( r_0^f \). Expanding the equation in powers of \( \delta \) (where we ignore a possible weak dependence of \( q(t) \) on \( \delta_{ext} \)), we obtain a formal expression for \( r_f \) as successive orders of \( \delta_{ext} \):

\[
r_f = r_0^f + \delta_{ext} r_1^f + \cdots
\]

One may readily verify that the next order term sets \( r_1^f = O(1) \), implying an \( O(\delta_{ext}) \) correction to \( r_f \) as expected.

Fig. S3 provides numerical confirmation of the arguments we gave above. Fig. S3a shows \( \partial_r z_{per} \) for different \( \delta_{ext} \) at a specific time. The formation of the boundary layer is clear, and remarkably, all the lines cross at a single point, confirming that \( r_0^f \) is independent of \( \delta_{ext} \). Figs. S3b and S3c compare \( S(t) \) and \( r_f(t) \) for different \( \delta_{ext} \), confirming the form of Eq. (62) and showing that the boundary layer merely shifts \( r_f \) relative to \( r_0^f \) by approximately a constant of order \( \delta_{ext} \).
4. Singular perturbation theory for the microscopic quantities $T, \epsilon$

In this section we address two elements in the axisymmetric surface dynamics of a rapidly depressurized bubble, where the viscous bending term is pronounced – the radius $\ell_{\text{core}}$ of the flattened nucleus, and the width $\ell_{\text{BL}}$ of the moving front that separates the flattened zone from the curved periphery, Eq. (71). In both calculations we employ the analytic solution of the intrinsic model ($\delta_{\text{ext}} = 0$).

A. Adiabatically depressurized core. The size $\ell_{\text{core}}$ of the initial “nucleus”, which marks the zone flattened while the pressure drops, and thereby defines the initial value of $r_f$, is set by the two small parameters, $\epsilon$ and $T$ (Eqs. (4,5)). To see this, let us start by following our “membrane limit” approach of the main text, setting $\epsilon \to 0$ in Eq. (7a) of the main text. Assuming that at the core zone, $r < \ell_{\text{core}}$, the film remains close to its initial, uniformly-tensed state (in contrast to the periphery, which remains curved and hence becomes stress-free when the pressure drops), we expand Eqs. (7) around the uniformly-tensed initial condition, Eq. (11) of the main text, to leading order in the deviation, $r^2 \delta(r,t)$, from the initial stress potential. This expansion yields:

$$\partial_t \Phi \approx r \left(1 + 2\delta(r,t)\right), \quad \partial_r z \approx -e^{-\frac{\ell}{T}} r \left(1 + \delta(r,t)\right),$$

where $\delta(r,t) = \frac{3}{16} r^2 e^{-\frac{2\epsilon}{T}}$.

Eq. (65) shows that the perturbation to the stress remains small (i.e. $|\delta| \ll 1$) only for $r \ll \ell_{\text{core}} \sim \sqrt{T}$, yielding the initial core size for sufficiently small $\epsilon$.

Considering depressurization rate, $T \ll \epsilon \ll 1$, the viscous bending term in Eq. (7a) of the main text, which underlies Eq. (65) is justified. Estimating the time derivative during the depressurization period as $\partial_t \sim T^{-1}$, and the spatial derivative and slope in the adiabatic region by $\partial_{rr} \sim \ell_{\text{core}}^{-3}$ and $\partial_r z \sim \ell_{\text{core}}$, respectively, with $\ell_{\text{core}} \sim \sqrt{T}$, we find that viscous bending is negligible in comparison to pressure (which is $O(1)$ at $t < O(T)$), as long as:

$$\epsilon^2 |\partial_r \partial_{r\text{core}} z| \ll |\Delta P| \implies \epsilon \ll T \ll 1.$$  

In the complementary parameter regime, $T \ll \epsilon \ll 1$, the resistance of viscous bending to the formation of a flattened disk in the spherically-shaped film is dominant, and assuming this process occurs over a time longer than $T$ (which assumption must be verified self-consistently) we may neglect now the pressure. Assuming again that the stress within the initial flattening core of the film remains close to the original (uniformly tensed) state, $\sigma_{rr} \approx 1$, Eq. (7a) of the main text now becomes:

$$\frac{1}{r} \partial_r (rg) \approx \epsilon^2 \partial_r \partial_t (r \partial_r (r^{-1} \partial_r (rg)))$$

$$\Rightarrow g \approx \epsilon^2 \partial_t \partial_r (r^{-1} \partial_r (rg)),$$

where we denote $g = \partial_r z$. Interestingly, the formal structure of this equation is an “inverse diffusion” (that is, $\partial_t \to \int dt$). This observation suggests we can approximate the flattening disk via a similarity solution, in analogy to diffusion of a scalar field:

$$g(r,t) = -t^{-1/2} F(\xi), \quad \xi \equiv \epsilon t^{1/2} r^{-1},$$

such that Eq. (67) reduces to an ODE for the similarity function $F(\xi)$:

$$(\xi^{-1} - 2\xi) F - F' + 2\xi F'' + \xi^2 F''' = 0.$$  

Eq. (69) has analytic solutions in terms of hypergeometric functions. We identify the physical solution as having $F(0) = 0$, corresponding to $g(r = 0, t)$ and a finite $F(\xi \to \infty)$, corresponding to a flattening region, see Eq. (68). Fig. S4 depicts the result. To obtain the scaling of $\ell_{\text{core}}, T_{\text{core}}$ in this parameter regime (complementary to (66)) we note that the above similarity solution of Eq. (7a) of the main text implies one scaling relation: $\ell_{\text{core}} T_{\text{core}}^{-1} \epsilon^{-1} \sim 1$. A second scaling relation is obtained by requiring compatibility with a solution of Eq. (7b) of the main text. For such a solution the LHS of Eq. (7b) describes an $O(1)$ spatial variation of the stress (between the tensed core and the stress-free periphery) over a length scale $\ell_{\text{core}}$, whereas the RHS describes $O(1)$ temporal variation of the Gaussian curvature $R$ (from a spherical shape to flat disk) over a time scale $T_{\text{core}}$. Matching the RHS with the LHS (and noting that the stress itself is a second derivative of the stress potential), this consideration yields the scaling relation: $\ell_{\text{core}}^2 \sim T_{\text{core}}^{1/3}$. Taken together, these two conditions show that $\ell_{\text{core}} \sim \sqrt{T}, T_{\text{core}} \sim \epsilon$, and combining with the scaling obtained above for the parameter regime (66), we obtain:

$$r_f(T_{\text{core}}) \sim \ell_{\text{core}} \sim T_{\text{core}} \sim \max(\sqrt{T}, \sqrt{\epsilon}).$$
8. Inner structure of the moving front. As we noted in the main text, the viscous bending term becomes dominant at the propagating front, $r \approx r_f(t)$, where the large gradient in $\partial_z z(r,t)$ enables a balance between the two sides of Eq. (7a) of the main text, thus allowing both $\partial_t \Phi$ and $\partial_t z$ to attain finite values (even through $\Delta P = 0$) that interpolate across the Heaviside functions, Eq. (14) of the main text. Correspondingly, we write the front solution (Eq. (14)) at the vicinity of $r \approx r_f(t)$:

$$z \approx z_f + z_{perc} \xi (\xi) , \Phi \approx \Phi_{BL} g_{BL}(\xi)$$

where $z_{perc} = z_{perc}(r_f(t), t)$, $\Phi_{dis} = \Phi_{dis}(r_f(t), t)$, $\xi = (r - r_f(t))/\ell_{BL}$ and $\ell_{BL} \sim \sqrt{r}$.

The method of asymptotic matching posits that, when analyzed in terms of the “inner” variable $\xi = [r - r_f(t)]/\ell_{BL}$, the “inner” functions, $\Phi_{BL}$ and $z_{BL}$, must match the respective “outer” functions, $\Phi_{dis}$ and $z_{perc}$, at $\xi \to \pm \infty$. Consequently, Eqs (14,15) of the main text indicate that $\partial_t \Phi = \sqrt{\ell_{BL}}$, where $\Psi_{BL}$ and $z_{BL}$ are both $\sim O(1)$. Changing variables from $(r,t) \to (\xi)$, and recalling that $\ell_{BL} \ll 1$, we readily note that a balance between the two sides of Eqs. (7) of the main text implies that $\ell_{BL} \sim \sqrt{r}$. This scaling is confirmed by the numerical solution, see e.g. Figs. S2 and S5, and the discussion regarding them in Sec. 6. In the front vicinity, the PDEs (7) of the main text reduce to a coupled set of ODEs for the functions $\Psi_{BL}(\xi, t)$ and $g_{BL}(\xi, t)$, for the explicit time $t$ acts merely as a parameter (and $t'$ indicates derivation w.r.t. $\xi$):

$$r_f^{-1} \cdot (\Psi_{BL} g_{BL})' = -\hat{r}_f \cdot g_{BL}'$$

$\Psi_{BL}' = \frac{3}{2} \hat{r}_f (g_{BL}')'$

The BCs for these equations, obtained by matching with $\Phi_{dis}$ and $g_{perc}$, Eq. (14) of the main text, are:

$$\xi \to \infty : \ g_{BL} \to -\sqrt{r_f^2 - \hat{S}} , \ \Psi_{BL} \to 0$$

$$\xi \to -\infty : \ g_{BL} \to 0 , \ \Psi_{BL}' \to q/2$$

where $r_f, \hat{r}_f, \hat{S}$ are totally determined by the “outer” analysis that was described in the previous section of the SI. Thus, for a given set of outer parameters, $r_f, \hat{r}_f, S, q$, a solution of ODEs (72), which one may obtain numerically, fully determines the inner functions, $\Phi_{BL}$ and $z_{BL}$. For the sake of brevity, and since the exact solution of these equations is of marginal importance for the purpose of this manuscript, we will not describe it here.

5. Dynamics of the wrinkled state

Relaxing the assumption of axial symmetry (but retaining a small slope approximation), the force balance Eqs. (7) of the main text acquire azimuthal dependence upon replacing $\hat{L}_r^2 \to \hat{L}_r^2 = \hat{L}_r^2 + 2 \zeta \hat{r}_2$, and $\hat{r}_2 (\hat{r}_2 \Phi \partial_\theta z) \to \sigma_{ij} \partial_i \partial_j z$, where the stress components $\sigma_{ij}$ are given in terms of the stress potential $\Phi$ and the Gaussian and mean curvatures are expressed through the determinant of the curvature tensor. Below we write the equations (written here for the full model, i.e. $\delta_{ext} = 1$).

Substituting the single-mode ansatz Eq. (21) of the main text in this generalized version of the force balance equations, and retaining only $\theta$-independent terms and terms proportional to $e^{im\theta}$, one obtains from Eq. (7a) of the main text two equations for the normal force balance:

$$r^{-1} \partial_r \left( (\partial_r z_0)(\partial_r \Phi_0) \right) + \frac{1}{2r} \partial_r (\partial_r \varphi \partial_r \zeta) - \frac{m^2}{r^2} \left[ \partial_r^2 (\varphi \zeta) - 2 \partial_r (r^{-1} \varphi \zeta) \right] = -\Delta P(t) + \epsilon^2 \hat{L}_r^2 (\partial_r z_0)$$

$$r^{-1} \partial_r \left( (\partial_r z_0)(\partial_r \varphi) \right) + r^{-1} \partial_r \left( (\partial_r \varphi)(\partial_r \Phi_0) \right) - \frac{m^2}{r^2} \left( \partial_r^2 z_0 \varphi + \zeta \partial_r^2 \Phi_0 \right) = -\delta P_m(t) + \epsilon^2 \hat{L}_r (\partial_r \zeta)$$

Here,

$$\hat{L}_m[f] = \left[ \hat{L}_r^2 - \frac{m^2}{r^2} \left( 4 - 2r \partial_r + 2r^2 \partial_r^2 \right) + \frac{m^4}{r^4} \right] f$$

and $\delta P_m(t)$ is an “angular” force, which breaks the axial symmetry explicitly, and represents some microscopic fluctuations, e.g. thermal fluctuations. The equations for tangential force balance are:

$$\hat{L}_r^2 \Phi_0 = \frac{1}{r} \partial_r \left\{ -\frac{3}{2} \partial_\theta \left( (\partial_\theta z_0)^2 + \frac{1}{2} (\partial_\theta \zeta)^2 - \frac{m^2}{r^2} \zeta \partial_r \zeta + \frac{1}{2} \partial_\theta \zeta \left( \frac{1}{r} \partial_r (r \zeta) - \frac{m^2}{r^2} \right) \right) \right\}$$

$$\hat{L}_r \varphi = \frac{3}{r} \partial_\theta \left\{ \partial_\theta z_0 \partial_\theta \zeta + \partial_\theta \zeta \partial_\theta z_0 - \frac{m^2}{r^2} \zeta \partial_\theta \zeta + \frac{1}{2} \partial_\theta \zeta \left( \frac{1}{r} \partial_r (r \zeta) - \frac{m^2}{r^2} \right) \right\}$$

Analogous set of equations have been analyzed for elastic sheets (16, 17) and shells (18) that undergo wrinkling instabilities above a threshold level of confinement. Even though such equations are derived through an uncontrolled truncation of higher-order harmonics (i.e. terms $\propto e^{ikm\theta}$ with $k > 1$), it has been shown in these studies that this method captures quantitatively the non-perturbative effect of radial wrinkles on the stress in the confined body and the consequent surface dynamics.

6. Numerical methods

In this section we describe the numerical algorithm we used in this work. We also provide details of the analysis and parameters used to generate the quantitative figures in this paper.

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The numerical algorithm. We obtained all the numerical results reported in this paper by implementing an exact numerical solution of the axisymmetric equations, Eqs. (7a) and (7b). The equations form a set of nonlinear coupled differential-algebraic equations (DAEs) for the fields \( z(r,t), \Phi(r,t) \).

Our equations of motion are prescribed in terms of the scalars \( \Phi, z \) rather than the currents or velocities. In addition, Eqs. (73)-(77) only involve even powers of spatial derivatives. Hence it is natural to define the numerical difference equations on the numerical lattice sites, rather than on the bonds as is common in hydrodynamic difference schemes. We discretized all differential operators symmetrically, balancing forward and backwards difference operators. For example, the operator \( \hat{L}_r^2 \) is implemented as,

\[
\hat{L}_r^2[f_n] \rightarrow r_n^{-1}D_f[r_{n-1/2}D_b(r_n^{-1}D_b(r_{n+1/2}D_f f_n))],
\]

where

\[
D_f(X_n) = \frac{X_{n+1} - X_n}{a_0}, \quad D_b(X_n) = \frac{X_n - X_{n-1}}{a_0}
\]

are the usual forward/backward differences, \( X_{n+1/2} = 0.5(X_n + X_{n+1}) \) is the averaged field value on the bond position, and

\[
r_n = a_0 n, \quad n = 0 \ldots N - 1, \quad a_0 = (N - 1)^{-1}
\]

defines the numerical lattice from 0 to 1, where \( N \) is the mesh size.

Because we are treating a system of DAEs, it is not possible to integrate the equations directly (either explicitly or implicitly). We solved the equations using the method of lines via Mathematica’s DAE solver implemented in the NDSolve function. As a general rule, the equations were stable as long as \( \sqrt{\epsilon/T} \leq 1 \), as can be expected from the \( \epsilon^2 \partial \xi \) structure in Eqs. (73)-(74). We did not systematically investigate the stability of the equations.

Boundary conditions. As discussed in the main text, the development of the front-disclination pair and the following dynamics of the bubble collapse require non-homogeneous boundary conditions for the Airy stress function. Table S1 details the numerical boundary conditions we used in our simulations.

The Neumann boundary condition for \( \Phi_0 \) at \( r = 0 \) is denoted as \( n_0(t) \), given by

\[
n_0(t) = \frac{a_0}{2} e^{-t/T}.
\]

The reason for this choice is to interpolate smoothly from the initial conditions, which do not obey Neumann boundary conditions on the discrete lattice, to the final one which does. Leaving the original \( O(a_0) \) condition does not change the numerics qualitatively but did affect the stability of the numerics and the behavior near \( r = 0 \). This is true for other boundary conditions as well: changing the boundary conditions did not qualitatively change our results, with the exception that, as discussed in the main text, changing the \( r = 0 \) boundary conditions for \( \Phi_0 \) to fully homogeneous ones leads to formation of a phantom bubble, rather than a propagating front. As a general rule, changing the conditions by e.g. modifying the constants \( 1, -1 \) to other numbers, or by changing the second derivative constraint on \( z(r = 1) \) to a first order one, just resulted in the formation of a boundary layer of order \( \ell_{DL} \).

Detailed description of the figures and the analysis used to generate them. We now provide details of our figure production.

For each figure, we first give a sketch of the technique used (if relevant), and then provide the relevant parameters.

Figure 1. In this figure, only panel (b) is a quantitative solution of our equations. The parameters used to generate it are:

\[
\epsilon = \frac{1}{3} \times 10^{-4}, \quad T = 0.01, \quad N = 2^{10}, \quad \delta_{ext} = 1
\]

and it is a snapshot of the solution at \( t = 0.8 \).

Figure 2. This figure does not use quantitative data, except for the bubble profile in panel (c), which is a snapshot from the same solution as Fig. 1b, at \( t = 0.35 \).

Figure 3. In order to generate this figure, we solved the equations numerically for the set of parameters given below. Panel (a) depicts the solution for the full model, \( \delta_{ext} = 1 \). To generate panel (b), we solved the auxiliary model with \( \delta_{ext} = 10^{-4} \), i.e. a solution very close to the intrinsic one. Then, we extracted the positions of \( r_f(t), S(t) \) by fitting the numerical \( (\partial_z)^2(r,t) \) to the relevant Sievert surface expression in Eqs. (29) and (46) of the SI and (15) of the main text (we regularized the Heaviside functions by a boundary layer of width \( \sqrt{\epsilon} \sim \ell_{DL} \)). Fig. S2 depicts some typical results of the fitting process, showing excellent agreement with the analytic expressions and the resulting high accuracy of the fit.

The analytic expressions are just those appearing in the text. They do not depend on any fitting parameter except for \( \ell_{core} \). In our fitting we took

\[
\ell_{core} = 3a_0,
\]

which is exactly the size of the numerical core defined by the boundary conditions above. This choice also appeared, in a qualitative analysis, to give a best fit to the data.

The parameters used to generate this figure are:

\[
\epsilon = 1 \times (1/3) \times 10^{-4}, \quad T = 0.01, \quad N = 2^{10}.
\]
Figure 4. The data in this figure was received directly from the authors of Ref. (10). The values of $R, a_0, \gamma, \eta, \rho$ necessary for the scaling analysis were all taken from the experimental data and can be obtained either from the work itself or from the authors. The four colors blue, yellow, green and red represent different viscosities $\eta = 10, 100, 800, 300$ Pa·s, respectively. We obtained each panel in the figure by rescaling the horizontal axis differently; the scaling is given in the label of each panel. We then fit the data to a linear equation using Mathematica’s FindFit procedure.

Figure S1. The parameters used to generate this figure are:

$$\epsilon = 1 \times 10^{-5}, T = 0.01, N = 2^9, \delta_{ext} = 0.$$ \[84]\]

Figure S2. The parameters used to generate panel (a) of this figure are:

$$\epsilon = 1 \times 10^{-5}, T = 0.01, N = 2^9, \delta_{ext} = 0.$$ \[85]\]

The parameters used to generate panels (b,c) of this figure are:

$$\epsilon = (1/3) \times 10^{-4}, T = 0.01, N = 2^{10}, \delta_{ext} = 10^{-4}.$$ \[86]\]

The reasons for using two different datasets were purely technical.

Figure S3. In order to generate this figure, we solved the equations numerically for the set of parameters given below, using the $\delta_{ext}$ values shown in the figure panels. Panel (a) shows snapshots at $t = 0.6$. To find the parameters $r_f, S$ shown in panels (b,c) we first fit $\sigma_{\theta\theta}$ to the analytic expression, as discussed above for Fig. 3. Then we fit $\langle \partial_z \rangle^2$ to the intrinsic analytic expression, broadened by a boundary layer with a width obtain as an additional fitting parameter. Fig. S5 shows an example of the fitting process, which is clearly very successful, proving the accuracy of our analysis in Sec. 3.

The parameters used to generate this figure are:

$$\epsilon = 1 \times (1/3) \times 10^{-4}, T = 0.01, N = 2^{10}, \delta_{ext} = 10^{-4}, 0.5, 1.$$ \[87]\]
Fig. S1. Insensitivity of the bubble collapse to details of the BCs (21) and (23) at $r \to 1$ (using the intrinsic model for clarity). (a) The traces depict the evolution of $\sigma_{\theta\theta}(t)$ for various times in a solution of the axisymmetric equations. The dashed blue lines correspond to $c_1 = 0, c_2 = 1$, which is the choice we made for the numerics in this paper. The solid black line corresponds to $c_1 = 1, c_2 = 0$. The two sets of data are indistinguishable within the numerical accuracy of the solution. (b) The traces depict the evolution of $(\partial r z)^2$ for various times in a solution of the axisymmetric equations. The dashed blue lines correspond to $\partial_{r} z = 0$, equivalent to choosing $c_3 = 0$, which is the choice we made for the numerics in this paper. The solid green line corresponds to $\partial_{r} z = -1$, equivalent to choosing $1 \ll c_3$. The two sets of data are indistinguishable within the numerical accuracy of the solution, except for a small boundary layer near $r = 1$. 
Fig. S2. Evolution of the intrinsic solution. (a,b) Evolution of $(\partial_z \alpha)^2$ and $\sigma_{\theta\theta}$. The dots are the numerical solution at different times, and the thin lines are the analytic expression, Eq. (29). (c) A snapshot of the intrinsic model evolution. Notice the abrupt change of slope at the front, compared to Fig. 1b.
Fig. S3. Solution of the auxiliary model for various $\delta_{ext}$. The blue traces depict an extremely small $\delta_{ext}$, approximately conforming to the intrinsic model predictions as show in the main text. The yellow and green traces depict finite $\delta_{ext}$, with the green begin the full model $\delta_{ext} = 1$. (a) A snapshot of $(\partial_t z_{per})^2$ for different $\delta_{ext}$, showing the formation of a boundary layer around the jump in $\partial_t z_{per}$ in the intrinsic model. The centerline of the boundary layer, which is the front in the intrinsic model, $r_f^{(0)}$, remains approximately unchanged, as seen by the crossing of the lines. (b) $S(t)$ for different $\delta_{ext}$. The independence on $\delta_{ext}$ at short times confirms the validity of Eq. (62). (c) $r_f(t)$ for different $\delta_{ext}$, showing that increasing $\delta_{ext}$ merely forms an almost constant boundary layer.
Fig. S4. Similarity solution of the “inverse diffusion” equations (67) and (69).
Fig. S5. Depiction of the fitting process for $\delta_{ext} = 1$. The figure depicts $(\partial_r z)^2$ (a) and $\sigma_{\theta \theta}$ (b). The dots are the numerical solution at different times, and the thin solid lines are the analytic expression, Eq. (29). For panel (a), the analytic expression is broadened by a boundary layer as discussed in the text regarding Fig. S3.
Table S1. Numerical implementation of boundary conditions for $\Phi_0$ and $z_0$

| Boundary       | $\Phi_0$                | $z_0$                  |
|---------------|-------------------------|------------------------|
| $r_0 = 0$     | $D_f(\Phi_0)_0 = n_0(t)$| $D_f(z_0)_0 = -\frac{1}{2}h$ |
|               | $r^{-\frac{1}{2}} D_f(\Phi_0)_1 = 1$ | $r^{-\frac{1}{2}} D_f(z_0)_1 = 1$ |
| $r_{N-1} = 1$ | $D^2_f(\Phi_0)_{N-1} = 0$ | $D^2_f(z_0)_{N-1} = -1$ |
|               | $(\Phi_0)_{N-1} = 0$    | $(z_0)_{N-1} = 0$     |
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