Full colored HOMFLYPT invariants, composite invariants and congruence skein relations

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Abstract
In this paper, we investigate the properties of certain quantum invariants of links by using the HOMFLY skein theory. First, we obtain the limit behavior for the full colored HOMFLYPT invariant which is the natural generalization of the colored HOMFLYPT invariant. Then we focus on the composite invariant which is a certain combination of the full colored HOMFLYPT invariants. Motivated by the study of the Labastida–Mariño–Ooguri–Vafa conjecture for the framed composite invariants of links, we introduce the notion of reformulated composite invariant \( \mathcal{R}_p(L; q, a) \). By using the HOMFLY skein theory, we prove that \( \mathcal{R}_p(L; q, a) \) actually lies in the integral ring \( 2\mathbb{Z}[(q - q^{-1})^2, a^\pm1] \). Finally, we propose a conjectural congruence skein relation for \( \mathcal{R}_p(L; q, a) \) and prove it for certain special cases.

Keywords Colored HOMFLYPT invariants · HOMFLY skein theory · Composite invariants · Congruence skein relations · LMOV conjecture

Mathematics Subject Classification 57K14 · 57K16 · 57K31

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1 Introduction

The HOMFLYPT polynomial is probably the most useful two-variable link invariant which was first discovered by Freyd–Yetter, Lickorish–Millet, Ocneanu, Hoste and Przytychi–Traczyk. Based on the work [29] of Turaev, the HOMFLYPT polynomial can be derived from the quantum group invariant associated with the fundamental representation of the quantum group $\mathcal{U}_{q}(\mathfrak{sl}_N)$ by setting $q^N = a$. More generally, if we consider the quantum group invariants [28] associated with arbitrary irreducible representations of $\mathcal{U}_{q}(\mathfrak{sl}_N)$, by setting $q^N = a$, we will get the colored HOMFLYPT invariants. We refer to [18] for a detailed definition of the colored HOMFLYPT invariants through quantum group invariants of $\mathcal{U}_{q}(\mathfrak{sl}_N)$. On the other hand, the colored HOMFLYPT invariants can also be defined by the satellite construction in the HOMFLY skein theory, see [17] for a nice explanation of the equivalence of the above two definitions.

Indeed, from the view of the HOMFLY skein theory, the colored HOMFLYPT invariant of a link $\mathcal{L}$ with $L$ components labeled by the corresponding partitions $\lambda^1, \ldots, \lambda^{L-1}$ and $\lambda^L$ can be identified with the HOMFLYPT polynomial of the link $\mathcal{L}$ decorated by the skein elements $Q_{\lambda^1}, \ldots, Q_{\lambda^{L-1}}$ and $Q_{\lambda^L}$ in the HOMFLY skein of the annuls $\mathcal{C}$. Given a partition vector $\vec{\lambda} = (\lambda^1, \ldots, \lambda^L)$, the colored HOMFLYPT invariant of the link $\mathcal{L}$ is defined by

$$W_{\vec{\lambda}}(\mathcal{L}; q, a) = q^{-\sum_{\alpha=1}^{L} k_{\lambda^\alpha} w(\mathcal{C}_\alpha)} a^{-\sum_{\alpha=1}^{L} \lambda^\alpha w(\mathcal{C}_\alpha)} \langle \mathcal{L} \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha} \rangle,$$

where $w(\mathcal{C}_\alpha)$ is the writhe number of the $\alpha$-th component $\mathcal{C}_\alpha$ of $\mathcal{L}$, the bracket $\langle \mathcal{L} \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha} \rangle$ is the framed HOMFLYPT polynomial of the satellite link $\mathcal{L} \otimes_{\alpha=1}^{L} Q_{\lambda^\alpha}$. In fact, the basis elements $Q_{\lambda^\alpha}$ used in the above definition of the colored HOMFLYPT invariant lie in the skein $\mathcal{C}_+$, which is the subspace of the full skein of the annulus $\mathcal{C}$. In [8], R. J. Hadji and H. R. Morton constructed the basis elements $Q_{\lambda, \mu}$.
in the full skein $\mathcal{C}$, in particular, $Q_{\lambda,\emptyset} = Q_{\lambda}$ when $\mu$ is the empty partition $\emptyset$. So it is natural to consider the satellite link constructed by using the elements $Q_{\lambda,\mu}$. It is the reason we introduce the notion of the full colored HOMFLYPT invariant for a link $\mathcal{L}$:

$$W_{[\lambda^1,\mu^1], [\lambda^2,\mu^2], \ldots, [\lambda^L,\mu^L]}(\mathcal{C}; q, a) = q^{-\sum_{\alpha=1}^{L}(\kappa_\alpha^\lambda + \kappa_\alpha^\mu)w(K_\alpha)}a^{-\sum_{\alpha=1}^{L}(|\lambda_\alpha^\lambda| + |\mu_\alpha^\mu|)w(K_\alpha)}(\mathcal{L} \otimes_{\alpha=1}^{L} Q_{\lambda_\alpha^\lambda,\mu_\alpha^\mu}).$$

We refer to Sects. 2 and 3 for a review of the HOMFLY skein theory and the definition of the full colored HOMFLYPT invariant for an oriented link.

Then we define the special polynomial for the full colored HOMFLYPT invariant for a link $\mathcal{L}$ with $L$ components as follow

$$H_{[\lambda^1,\mu^1], [\lambda^2,\mu^2], \ldots, [\lambda^L,\mu^L]}(a) = \lim_{q \to 1} W_{[\lambda^1,\mu^1], [\lambda^2,\mu^2], \ldots, [\lambda^L,\mu^L]}(\mathcal{L}; q, a) \prod_{\alpha=1}^{L} P_{K_\alpha}(1, a)|\lambda_\alpha^\lambda| + |\mu_\alpha^\mu|,$$

where $U$ denotes the trivial knot, i.e., the unknot throughout this paper. The special polynomial was first introduced in [9] which is the first term in the genus expansion of the knot invariant from the study of the topological string theory [24].

In this paper, we prove the following exponential growth property which generalizes the corresponding property for ordinary colored HOMFLYPT invariants in [4,33].

**Theorem 1.1** For a link $\mathcal{L}$ with $L$ components $\mathcal{K}_\alpha, \alpha = 1, \ldots, L$, we have

$$H_{[\lambda^1,\mu^1], [\lambda^2,\mu^2], \ldots, [\lambda^L,\mu^L]}(a) = \lim_{q \to 1} W_{[\lambda^1,\mu^1], [\lambda^2,\mu^2], \ldots, [\lambda^L,\mu^L]}(U; q, a),$$

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In this paper, we prove the following exponential growth property which generalizes the corresponding property for ordinary colored HOMFLYPT invariants in [4,33].

$$H_{[\lambda^1,\mu^1], [\lambda^2,\mu^2], \ldots, [\lambda^L,\mu^L]}(a) = \lim_{q \to 1} W_{[\lambda^1,\mu^1], [\lambda^2,\mu^2], \ldots, [\lambda^L,\mu^L]}(U; q, a),$$

where $P_{\mathcal{K}}(q, a)$ is the classical HOMFLYPT polynomial.

The large $N$ duality between Chern-Simons and topological string theory developed by Witten [32] and Gopakumar-Vafa [7] provides a natural relationship between the knot invariants and open Gromov-Witten invariants in mathematics. In this framework, a fantastic example for the framed unknot was constructed by Mariño-Vafa [25]. Later, it was conjectured, in a series of works [14,15], that the generating function of the colored HOMFLYPT invariants can be rewritten in terms of an infinite sequence of integer invariants. This conjecture was referred as to being the Labastida-Mariño-Ooguri-Vafa (LMOV) conjecture in [16]. The LMOV conjecture for (framing independent) colored HOMFLYPT invariants was proved in [16]. But the framed LMOV conjecture for (framing dependent) colored HOMFLYPT invariants with arbitrary framing is still open. An attempt to solve the framed LMOV conjecture was given in [2]. In particular, for the case of a framed unknot $U_\tau$, the framed LMOV conjecture was studied carefully in [19,35,36].

Given the partition vectors $\vec{A} = (A^1, \ldots, A^L)$, $\vec{\lambda} = (\lambda^1, \ldots, \lambda^L)$ and $\vec{\mu} = (\mu^1, \ldots, \mu^L)$, we set $c_{\lambda,\mu}^{A} = \prod_{\alpha=1}^{L} c_{\lambda_\alpha^\lambda,\mu_\alpha^\mu}^{A_\alpha}$, where $c_{\lambda_\alpha^\lambda,\mu_\alpha^\mu}$ denotes the Littlewood-Richardson coefficient determined by the formula (31). The (framing independent)
composite invariant

\[ H_A(\mathcal{L}; q, a) = \sum_{\lambda, \mu} c^A_{\lambda, \mu} W_{[\lambda^1, \mu^1],..,[\lambda^L, \mu^L]}(\mathcal{L}; q, a) \]  

(5)

was introduced by Mariño [21]. He formulated the LMOV conjecture for \( H_A(\mathcal{L}; q, a) \) based on the work [1]. In this paper, we consider the framed composite invariant \( \mathcal{H}_A(\mathcal{L}; q, a) \) (which depends on the framing of the link \( \mathcal{L} \)) and the corresponding framed LMOV conjecture for it. We check that the framed LMOV conjecture for framed composite invariant holds for torus link \( T(2, 2k) \) with small framing \( \tau = (m, n) \).

In the joint work [2] with K. Liu and P. Peng, for a partition \( \mu \in \mathcal{C}^+ \) to define the reformulate colored HOMFLYPT invariant for a link \( \mathcal{L} \) as follow:

\[ Z_{\tilde{\mu}}(\mathcal{L}; q, a) = \langle \mathcal{L} \star \otimes_{a=1}^L P_{\mu^a} \rangle \]  

\[ \check{Z}_{\tilde{\mu}}(\mathcal{L}; q, a) = \{ \tilde{\mu} \} \check{Z}_{\tilde{\mu}}(\mathcal{L}; q, a) \]  

(6)

where \( \tilde{\mu} = (\mu^1, \ldots, \mu^L) \), \( \{ \tilde{\mu} \} = \prod_{a=1}^L \{ \mu^a \} \) and \( \{ \mu \} = \prod_{i=1}^L (q^{\mu_i} - q^{-\mu_i}) \) for a partition \( \mu = (\mu_1, \ldots, \mu_l) \). From the view of the HOMFLY skein theory, the reformulated colored HOMFLYPT invariant \( Z_{\tilde{\mu}}(\mathcal{L}; q, a) \) (or \( \check{Z}_{\tilde{\mu}}(\mathcal{L}; q, a) \)) is simpler than the colored HOMFLYPT invariant \( W_{\tilde{\mu}}(\mathcal{L}; q, a) \) since the expression for \( P_{\tilde{\mu}} \) is simpler than \( Q_{\tilde{\mu}} \) in the skein \( \mathcal{C}^+ \), we refer to [2] for a detailed description of the skein element \( P_{\tilde{\mu}} \). By using the HOMFLY skein theory, we prove in [2] that the reformulated colored HOMFLYPT invariants satisfy the following integrality property.

**Theorem 1.2** For any link \( \mathcal{L} \) with \( L \) components, we have

\[ \check{Z}_{\tilde{\mu}}(\mathcal{L}; q, a) \in \mathbb{Z}[z^2, a^{\pm 1}], \]  

(7)

where we use the notation \( z = q - q^{-1} \) and which will be used throughout this paper for brevity.

In particular, when \( \tilde{\mu} = ((p), \ldots, (p)) \) with \( L \) row partitions \( (p) \) for \( p \in \mathcal{Z}^+ \), we use the notation \( \check{Z}_p(\mathcal{L}; q, a) \) to denote the reformulated colored HOMFLYPT invariant \( \check{Z}_{((p),\ldots,(p))}(\mathcal{L}; q, a) \) for simplicity. We have proposed two congruence skein relations for \( \check{Z}_p(\mathcal{L}; q, a) \) in [2].

In this paper, we introduce an analog reformulated invariant for framed composite invariant. First, given a partition \( \nu \), we associate it a skein element \( R_{\nu} \in \mathcal{C}^+ \) by

\[ R_\nu = \sum_A \chi_A(\nu) \sum_{\lambda, \mu} c^A_{\lambda, \mu} Q_{\lambda, \mu}. \]  

(8)

In particular, when all the \( \mu = \emptyset \) in (8), we have \( R_\nu = P_\nu \in \mathcal{C}^+ \). We define the reformulated composite invariant as follow:

\[ R_{\tilde{\nu}}(\mathcal{L}; q, a) = \langle \mathcal{L} \star \otimes_{\alpha=1} P_{\nu^\alpha} \rangle \]  

\[ \check{R}_{\tilde{\nu}}(\mathcal{L}; q, a) = \{ \tilde{\nu} \} R_{\tilde{\nu}}(\mathcal{L}; q, a). \]  

(9)
Moreover, for \( p \in \mathbb{Z}_+ \), we use the notation \( \tilde{R}_p(\mathcal{L}; q, a) \) to denote \( \tilde{R}_{(p)_\gamma}(\mathcal{L}; q, a) \) for simplicity. We will show that the reformulated composite invariant \( \tilde{R}_p(\mathcal{L}; q, a) \) can be expressed by the original reformulated invariants \( \tilde{Z}_p(\mathcal{L}; q, a) \).

**Theorem 1.3** For a link \( \mathcal{L} \) with \( L \) components, we have

\[
\tilde{R}_p(\mathcal{L}; q, a) = \sum_{k=0}^{L} \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq L} \tilde{Z}_p(\mathcal{L}_{\alpha_1, \alpha_2, \cdots, \alpha_k}; q, a),
\]

where \( \mathcal{L}_{\alpha_1, \alpha_2, \cdots, \alpha_k} \) denotes the link obtained by reversing the orientations of the \( \alpha_1, \cdots, \alpha_k \)-th components of link \( \mathcal{L} \).

Combing Theorem 1.2, we obtain the following integrality result.

**Theorem 1.4** For any link \( \mathcal{L} \), we have

\[
\tilde{R}_p(\mathcal{L}; q, a) \in 2\mathbb{Z}[z^2, a^{\pm 1}].
\]

More general, we obtain

**Theorem 1.5** For any link \( \mathcal{L} \) with \( L \) components and a partition vector \( \tilde{\mu} = (\mu^1, \ldots, \mu^L) \), we have

\[
\tilde{R}_{\tilde{\mu}}(\mathcal{L}; q, a) \in \mathbb{Z}[z^2, a^{\pm 1}].
\]

Motivated by the study of the framed LMOV conjecture for the framed composite invariants, we propose a congruence skein relation for the reformulated composite invariant \( \tilde{R}_p(\mathcal{L}; q, a) \). When the crossing is the linking between two different components of the link, we have the following skein relation for \( \tilde{R}_1 \) by applying the classical skein relation for HOMFLYPT polynomial:

\[
\tilde{R}_1(\mathcal{L}_+; q, a) - \tilde{R}_1(\mathcal{L}_-; q, a) = z^2 \left( \tilde{R}_1(\mathcal{L}_0; q, a) - \tilde{R}_1(\mathcal{L}_\infty; q, a) \right).
\]

where \( (\mathcal{L}_+, \mathcal{L}_-, \mathcal{L}_0, \mathcal{L}_\infty) \) denotes the quadruple appears in the classical Kauffman skein relation. As to \( \tilde{R}_p(\mathcal{L}; q, a) \), we propose

**Conjecture 1.6** For any prime \( p \), when the crossing is the linking between two different components of the link, we have

\[
\tilde{R}_p(\mathcal{L}_+; q, a) - \tilde{R}_p(\mathcal{L}_-; q, a) \equiv (-1)^{p-1} p[p^2 \left( \tilde{R}_p(\mathcal{L}_0; q, a) - \tilde{R}_p(\mathcal{L}_\infty; q, a) \right) \mod [p]^2 \{p\}^2, \]

where \( \{p\} = q^p - q^{-p} \), \( [p] = \frac{q^p - q^{-p}}{q - q^{-1}} \) and the notation \( A \equiv B \mod C \) denotes \( A - B \in \mathbb{Z}[z^2, a^{\pm 1}] \).
In a recent paper [3], we define the reformulated colored Kauffman invariant \( \widehat{G}_p(L; q, a) \), and we find that \( \widehat{G}_p(L; q, a) \) satisfies the same congruence skein relation as in Conjecture 1.6. It provides a mathematical understanding for the reason why the LMOV conjecture constructed in [21] involves both the composite invariants and the colored Kauffman invariants.

We have tested a lot of examples for the above Conjecture 1.6. In particular, we will prove the following theorem in Sect. 7.

**Theorem 1.7** When \( p = 2 \), the Conjecture 1.6 holds for \( L_+ = T(2, 2k + 2), L_- = T(2, 2k), L_0 = T(2, 2k + 1) \) and \( L_\infty = U(−2k − 1) \), where \( U(−2k − 1) \) denotes the unknot with \( 2k + 1 \) negative kinks.

The rest of this paper is organized as follows. In Sect. 2, we review the basics of HOMFLY skein model. In Sect. 3, we define the full colored HOMFLYPT invariants via HOMFLY skein theory. Then we compute full colored HOMFLYPT invariants for torus links in Sect. 4. Next, we investigate the limit behavior of the full colored HOMFLYPT invariants in Sect. 5. In Sect. 6, we first introduce the composite invariants associated to the full colored HOMFLYPT invariants and review the LMOV conjecture for these composite invariants. Then we formulate a framed version of LMOV conjecture for framed composite invariants. We prove this framed LMOV conjecture in certain special cases. In Sect. 7, we first review the congruence skein relations conjecture formulated in [2] for the reformulated colored HOMFLYPT invariants, and then, we propose a new conjectural congruence skein relation for the reformulated composite invariants. We provide certain examples to support this conjecture. In the final Sect. 8, we provide a detailed computation rule for the skein element \( R_\nu \) and give a proof of the general integrality theorem for the reformulated composite invariants.

## 2 HOMFLY skein theory

Following the descriptions in [8], we define the coefficient ring \( \Lambda = \mathbb{Z}[q^{\pm 1}, a^{\pm 1}] \) with the elements \( q^k - q^{-k} \) admitted as denominators for \( k \geq 1 \). Let \( F \) be a planar surface, the framed HOMFLY skein \( S(F) \) of \( F \) is the \( \Lambda \)-linear combination of the oriented tangles in \( F \), modulo the two local relations as shown in Fig. 1 where \( z = q - q^{-1} \).

It is easy to see that the removal of an unknot is equivalent to multiplication by a scalar \( s = \frac{a-a^{-1}}{q-q^{-1}} \), i.e., we have the relation shown in Fig. 2.

### 2.1 The plane

When \( F = \mathbb{R}^2 \), for a link \( L \) with the link diagram \( D_L \in S(\mathbb{R}^2) \) which gives a scalar in \( \Lambda \) by the definition of HOMFLY skein, we denote this scalar by \( \langle L \rangle \). In particular, as to the unknot \( U \), we have \( \langle U \rangle = \frac{a-a^{-1}}{q-q^{-1}} \).
We call $\langle \mathcal{L} \rangle$ the framed HOMFLYPT polynomial of the link $\mathcal{L}$. The classical HOMFLYPT polynomial is given by

$$P_{\mathcal{L}}(q, a) = a^{-w(\mathcal{L})} \langle \mathcal{L} \rangle / \langle U \rangle,$$

(15)

where $w(\mathcal{L})$ is the writhe number of the link $\mathcal{L}$. Particularly, $P_U(q, a) = 1$.

**Remark 2.1** In some physical literatures, such as [21], the self-writhe $\bar{w}(\mathcal{L})$ instead of $w(\mathcal{L})$ is used in the definition of the HOMFLYPT polynomial (15). The relationship between them is given by

$$w(\mathcal{L}) = \bar{w}(\mathcal{L}) + 2lk(\mathcal{L}),$$

(16)

where $lk(\mathcal{L})$ is the total linking number of the link $\mathcal{L}$. By its definition $\bar{w}(\mathcal{L}) = \sum_{\alpha=1}^{L} w(\mathcal{K}_\alpha)$, if $\mathcal{L}$ is a link with $L$ components $\mathcal{K}_\alpha$, $\alpha = 1, \ldots, L$.

**2.2 The rectangle**

We write $H_{n,m}(q, a)$ (or $H_{n,m}$) for the skein $S(F)$ of $(n, m)$-tangle where $F$ is the rectangle with $n$ inputs and $m$ outputs at the top and matching inputs and outputs at the bottom. There is a natural algebra structure on $H_{n,m}$ by placing tangles one above the another. When $m = 0$, we write $H_n(q, a) = H_{n,0}(q, a)$. 
The algebra $H_{n,m}^N(q)$ is a generalization of the Iwahori-Hecke algebra of type $A$ constructed in [11].

**Definition 2.2** For integers $n, m \geq 0$ and $N \geq n + m$, we define $H_{n,m}^N(q)$ to be the associative $\mathbb{C}(q)$-algebra with unit presented by generators $g_1, g_2, \cdots, g_{n-1}, e$ (if $m = 1$), $g_1^*, g_2^*, \cdots, g_{m-1}^*$ (if $m \geq 2$) and the relations:

1. $g_ig_j = g_jg_i$, $1 \leq i, j \leq n - 1, |i - j| \geq 2$;
2. $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$, $1 \leq i \leq n - 2$;
3. $(g_1 - q)(g_i + q^{-1}) = 0$, $1 \leq i \leq n - 1$;
4. $g_i^*g_j^* = g_j^*g_i^*$, $1 \leq i, j \leq m - 1, |i - j| \geq 2$;
5. $g_i^*g_{i+1}g_i^* = g_{i+1}g_i^*g_i^*$, $1 \leq i \leq m - 2$;
6. $(g_i^* - q)(g_i^* + q^{-1}) = 0$, $1 \leq i \leq m - 1$;
7. $e^2 = [N]e$;
8. $eg_i = g_ie$, $1 \leq i \leq n - 2$;
9. $eg_i^* = g_i^*e$, $2 \leq i \leq m - 1$;
10. $g_ig_j^* = g_j^*g_i$, $1 \leq i \leq n - 1, 1 \leq j \leq m - 1$;
11. $eg_{n-1}e = q^Ne$;
12. $eg_1^*e = q^Ne$;
13. $eg_{n-1}^*e(g_{n-1} - g_1^*) = 0$;
14. $(g_{n-1} - g_1^*)eg_{n-1}^*e = 0$.

If we take $a = q^N$, the skein $H_{n,m}(q, q^N) \cong H_{n,m}^N(q)$.

### 2.3 The annulus

Let $\mathcal{C}$ be the HOMFLY skein of the annulus, i.e., $\mathcal{C} = \mathcal{S}(S^1 \otimes I)$. $\mathcal{C}$ is a commutative algebra with the product induced by placing one annulus outside another. As an algebra, $\mathcal{C}$ is freely generated by the set $\{A_m : m \in \mathbb{Z}\}$, $A_m$ for $m \neq 0$ is the closure of the braid $\sigma_{|m|-1} \cdots \sigma_2 \sigma_1$, and $A_0$ is the empty diagram [30]. The orientation of the curve around the annulus is counter-clockwise for positive $m$ and clockwise for negative $m$. Thus the algebra $\mathcal{C}$ is the product of two subalgebras $\mathcal{C}_+$ and $\mathcal{C}_-$ generated by $\{A_m : m \in \mathbb{Z}, m \geq 0\}$ and $\{A_m : m \in \mathbb{Z}, m \leq 0\}$.

The closure map $\hat{\cdot} : H_{n,m} \rightarrow \mathcal{C}$, induced by taking an $(n, m)$-tangle $T$ to its closure $\hat{T}$ is a $\Lambda$-linear map, whose image is denoted by $\mathcal{C}_{n,m}$. It is clear that every diagram in the annulus presents an element in some $\mathcal{C}_{n,m}$. The algebra $\mathcal{C}_+$ is spanned by the subspace $\mathcal{C}_{n,0}$. There is a good basis $\{Q_{\lambda}\}$ of $\mathcal{C}_+$ consisting of the closures of certain idempotents of Hecke algebra $H_n(q, a)$.

In [8], R. Hadji and H. Morton constructed the basis elements $\{Q_{\lambda,\mu}\}$ explicitly for $\mathcal{C}$. We will review this construction in Sect. 3.

### 2.4 Skein involutions

For every surface $F$, the mirror map in the skein of $F$ is defined as the conjugate linear involution $^\circ$ on the skein of $F$ induced by switching all crossings on diagrams and inverting $q$ and $a$ in $\Lambda$. Thus $\bar{z} = -z$. 

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For the annulus $S^1 \times I$, the rotation of the diagrams in $S^1 \times I$ by $\pi$ about the horizontal axis through the center of annulus induces a map $* : \mathcal{C} \rightarrow \mathcal{C}$. It is easy to see that $(A_m)^* = A_{-m}$, $(C_+)^* = C_-$ and $(C_{n,m})^* = C_{m,n}$.

3 Full colored HOMFLYPT invariants

3.1 Partitions and symmetric functions

We refer to [20] for the basics about the partitions and symmetric functions. A partition $\lambda$ is a finite sequence of positive integers $(\lambda_1, \lambda_2, ..)$ such that

$$\lambda_1 \geq \lambda_2 \geq \cdots$$

(17)

The length of $\lambda$ is the total number of parts in $\lambda$ and denoted by $l(\lambda)$. The degree of $\lambda$ is defined by

$$|\lambda| = \sum_{i=1}^{l(\lambda)} \lambda_i.$$  

(18)

If $|\lambda| = d$, we say that $\lambda$ is a partition of $d$ and denoted as $\lambda \vdash d$. The automorphism group of $\lambda$, denoted by Aut($\lambda$) contains all the permutations that permute parts of $\lambda$ by keeping it as a partition. Obviously, Aut($\lambda$) has order

$$|\text{Aut}(\lambda)| = \prod_{i \geq 1} m_i(\lambda)!$$

(19)

where $m_i(\lambda)$ denotes the number of times that $i$ occurs in $\lambda$. One can also write a partition $\lambda$ as

$$\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots).$$

(20)

The Young diagram of $\lambda$ is a graph with $\lambda_i$ boxes on the $i$-th row for $i = 1, 2, .., l(\lambda)$, where we have enumerate the rows from top to bottom and the columns from left to right.

Given a partition $\lambda$, we define the conjugate partition $\lambda'$ whose Young diagram is the transposed Young diagram of $\lambda$ which is derived from the Young diagram of $\lambda$ by reflection in the main diagonal.

We will use the notation $\mathcal{P}_+$ to denote the set of all the partitions of positive integers. Let $\emptyset$ be the partition of 0, i.e., the empty partition. Define $\mathcal{P} = \mathcal{P}_+ \cup \{\emptyset\}$, and $\mathcal{P}^n$ the $n$ tuple of $\mathcal{P}$. The elements in $\mathcal{P}^n$ denoted by $\vec{\lambda} = (\lambda^1, .., \lambda^n)$ are called partition vectors.
The following numbers associated with a given partition $\lambda$ are used frequently in this paper:

\[ z_\lambda = \prod_{j=1}^{l(\lambda)} j^{m_j(\lambda)} m_j(\lambda)!, \quad (21) \]

\[ k_\lambda = \sum_{j=1}^{l(\lambda)} \lambda_j (\lambda_j - 2j + 1). \quad (22) \]

Obviously, $k_\lambda$ is an even number and $k_\lambda = -k_{\lambda^t}$.

The $m$-th complete symmetric function $h_m$ is defined by its generating function

\[ H(t) = \sum_{m=0}^{\infty} h_m t^m = \prod_{i \geq 1} \frac{1}{1 - x_i t}. \quad (23) \]

The $m$-th elementary symmetric function $e_m$ is defined by its generating function

\[ E(t) = \sum_{m=0}^{\infty} e_m t^m = \prod_{i \geq 1} (1 + x_i t). \quad (24) \]

Obviously,

\[ H(t) E(-t) = 1. \quad (25) \]

The power sum symmetric function of infinite variables $x = (x_1, \ldots, x_N, \ldots)$ is defined by

\[ p_n(x) = \sum_i x_i^n. \quad (26) \]

Given a partition $\lambda$, we define

\[ p_\lambda(x) = \prod_{j=1}^{l(\lambda)} p_{\lambda_j}(x). \quad (27) \]

The Schur function $s_\lambda(x)$ is determined by the Frobenius formula

\[ s_\lambda(x) = \sum_{|\mu| = |\lambda|} \frac{\chi_\lambda(C_\mu)}{z_\mu} p_\mu(x). \quad (28) \]

where $\chi_\lambda$ is the character of the irreducible representation of the symmetric group $S_{|\mu|}$ corresponding to $\lambda$. $C_\mu$ denotes the conjugate class of symmetric group $S_{|\mu|}$.
corresponding to $\mu$. The orthogonality of character formula gives
\begin{equation}
\sum_A \frac{\chi_A(C_\mu)\chi_A(C_\nu)}{z_\mu} = \delta_{\mu\nu}.
\end{equation}

For $\lambda, \mu, \nu \in \mathcal{P}$, we define the Littlewood-Richardson coefficient $c^\nu_{\lambda,\mu}$ as
\begin{equation}
s_\lambda(x)s_\mu(x) = \sum_\nu c^\nu_{\lambda,\mu}s_\nu(x).
\end{equation}

It is easy to see that $c^\nu_{\lambda,\mu}$ can be expressed by the characters of symmetric group by using the Frobenius formula
\begin{equation}
c^\nu_{\lambda,\mu} = \sum_{\rho, \tau} \frac{\chi_\lambda(C_\rho)}{z_\rho} \frac{\chi_\lambda(C_\tau)}{z_\tau} \chi_\nu(C_{\rho\cup\tau}).
\end{equation}

### 3.2 Basic elements in $\mathcal{C}$

Given a permutation $\pi \in S_m$ with the length $l(\pi)$, let $\omega_\pi$ be the positive permutation braid associated to $\pi$, we have $l(\pi) = w(\omega_\pi)$ which is the writhe number of the braid $\omega_\pi$.

We define the quasi-idempotent element
\begin{equation}
a_m = \sum_{\pi \in S_m} q^{l(\pi)} \omega_\pi
\end{equation}
in $H_m$. Let element $h_m \in \mathcal{C}_{m,0}$ be the closure of the elements $\frac{1}{\alpha_m}a_m \in H_m$, i.e.,
\begin{equation}
h_m = \frac{1}{\alpha_m}a_m,
\end{equation}
where $\alpha_m$ is determined by the equation $a_ma_m = \alpha_m a_m$, it gives $\alpha_m = q^{m(m-1)/2} \prod_{i=1}^m \frac{q^i-q^{-i}}{q-q^{-1}}$.

The skein $\mathcal{C}_+$ (resp. $\mathcal{C}_-$) is spanned by the monomials in $\{h_m\}_{m \geq 0}$ (resp. $\{h_k^s\}_{k \geq 0}$). The whole skein $\mathcal{C}$ is spanned by the monomials in $\{h_m, h_k^s\}_{m, k \geq 0}$. $\mathcal{C}_+$ can be regarded as the ring of symmetric functions in variables $x_1, \ldots, x_N$, .. with the coefficient ring $\Lambda$. In this situation, $\mathcal{C}_{m,0}$ consists of the homogeneous functions of degree $m$. The power sum $P_m = \sum x_i^m$ is a symmetric function which can be presented in terms of the complete symmetric functions; hence, it represents a skein element which is also denoted by $P_m \in \mathcal{C}_{m,0}$. Moreover, we have the identity
\begin{equation}
[m]P_m = X_m = \sum_{j=0}^{m-1} A_{m-1-j,j}
\end{equation}
where $[m] = \frac{q^m-q^{-m}}{q-q^{-1}}$ and $A_{i,j}$ is the closure of the braid
\[\sigma_{i+j}\sigma_{i+j-1} \cdots \sigma_j^{-1}\sigma_{i+1}\sigma_i^{-1} \cdots \sigma_1^{-1}.$$
Given a partition $\mu$, we define

$$P_\mu = \prod_{i=1}^{l(\mu)} P_{\mu_i}.$$  \hfill (34)

### 3.3 The meridian maps of $\mathcal{C}$

Take a diagram $X$ in the annulus and link it once with a simple meridian loop, oriented in either direction, to give diagrams $\varphi(X)$ and $\bar{\varphi}(X)$ in the annulus. This induces linear endomorphisms $\varphi, \bar{\varphi}$ of the skein $\mathcal{C}$, called the meridian maps. Each space $\mathcal{C}_{n,m}$ is invariant under $\varphi$ and $\bar{\varphi}$ [22].

It was shown in [17] that the eigenvectors of $\varphi$ on $\mathcal{C}_{n,0}$ are identified with $Q_{\lambda}$, the closure of the idempotents in Hecke algebra $H_n$. Moreover, $Q_{\lambda}$ can be expressed as an explicit integral polynomial in $\{h_m\}_{m \geq 0}$. Then, in [8], Hadji and Morton constructed the eigenvectors of $\varphi$ on the whole skein $\mathcal{C}$ as follow.

### 3.4 Construction of the elements $Q_{\lambda, \mu}$

Given two partitions $\lambda, \mu$ with $l$ and $r$ parts. We first construct a $(l+r) \times (l+r)$-matrix $M_{\lambda, \mu}$ with entries in $\{h_m, h_k^*\}_{m,k \in \mathbb{Z}}$ as follows, where we have let $h_m = 0$ if $m < 0$ and $h_k^* = 0$ if $k < 0$.

$$M_{\lambda, \mu} = \begin{pmatrix} h_{\mu_r}^* & h_{\mu_r-1}^* & \cdots & h_{\mu_r-r-l+1}^* \\ h_{\mu_r-1+1}^* & h_{\mu_r-2}^* & \cdots & h_{\mu_r-1-r-l}^* \\ \vdots & \vdots & \ddots & \vdots \\ h_{\mu_1+(r-1)}^* & h_{\mu_1+(r-2)}^* & \cdots & h_{\mu_1-l}^* \\ h_{\lambda_1-r}^* & h_{\lambda_1-(r-1)}^* & \cdots & h_{\lambda_1-l+1}^* \\ h_{\lambda_1-l-r+1}^* & h_{\lambda_1-s-r+2}^* & \cdots & h_{\lambda_1}^* \end{pmatrix}. \hfill (35)$$

It is easy to see that the subscripts of the diagonal entries in the $h$-rows are the parts $\lambda_1, \lambda_2, \ldots, \lambda_l$ of $\lambda$ in order, while the subscripts of the diagonal entries in the $h^*$-rows are the parts $\mu_1, \mu_2, \ldots, \mu_r$ of $\mu$ in reverse order.

Then, $Q_{\lambda, \mu}$ is defined as the determinant of the matrix $M_{\lambda, \mu}$, i.e.,

$$Q_{\lambda, \mu} = \det M_{\lambda, \mu}.$$  \hfill (36)

**Example 3.1** For two partitions $\lambda = (4, 2, 2)$ and $\mu = (3, 2)$, we have

$$Q_{\lambda, \mu} = \det \begin{pmatrix} h_2^* & h_1^* & 1 & 0 & 0 \\ h_4^* & h_3^* & h_2^* & h_1^* & 1 \\ h_2 & h_3 & h_4 & h_5 & h_6 \\ 0 & 1 & h_1 & h_2 & h_3 \\ 0 & 0 & 1 & h_1 & h_2 \end{pmatrix}. \hfill (37)$$
Given two partitions $\lambda, \mu$, we define

$$k_{\lambda, \mu} = (q - q^{-1}) \left( a \sum_{x \in \lambda} q^{2c(x)} - a^{-1} \sum_{x \in \mu} q^{-2c(x)} \right) + \frac{a - a^{-1}}{q - q^{-1}}$$  \hspace{1cm} (38)$$

where $c(x) = j - i$ is the content of the cell in row $i$ and column $j$ of the diagram.

It was shown in [22] that the set $k_{\lambda, \mu}$ forms a complete set of eigenvalues of the meridian map $\varphi$, each occurring with multiplicity 1. Furthermore, it was proven in [8] that the element $Q_{\lambda, \mu}$ is an eigenvector of the meridian map $\varphi$, with eigenvalue $k_{\lambda, \mu}$. Thus $\{Q_{\lambda, \mu}\}$ forms a basis of $\mathcal{C}$. Moreover, the basis elements $Q_{\lambda, \mu}$ of $\mathcal{C}$ have the property that the product of any two is a nonnegative integral linear combination of basis elements

$$Q_{\lambda, \mu} = \sum_{\sigma, \rho, \nu} (-1)^{|\sigma|} c_{\alpha, \rho}^{\lambda} c_{\rho, \nu}^{\mu} Q_{\rho, \emptyset} Q_{\emptyset, \nu}. \hspace{1cm} (39)$$

### 3.5 Full colored HOMFLYPT invariants

Let $\mathcal{L}$ be a framed link with $L$ components with a fixed numbering. For diagrams $Q_1, \ldots, Q_L$ in the skein model of annulus with the positive oriented core $\mathcal{C}_+$, we define the decoration of $\mathcal{L}$ with $Q_1, \ldots, Q_L$ as the link

$$\mathcal{L} \star \otimes_{\alpha=1}^L Q_{\alpha}$$  \hspace{1cm} (40)$$

which derived from $\mathcal{L}$ by replacing every annulus $\mathcal{L}$ by the annulus with the diagram $Q_{\alpha}$ such that the orientations of the cores match. Each $Q_{\alpha}$ has a small backboard neighborhood in the annulus which makes the decorated link $\mathcal{L} \otimes_{\alpha=1}^L Q_{\alpha}$ into a framed link.

Let $Q_{\lambda^\alpha, \mu^\alpha} \in \mathcal{C}_{d_{\alpha}, t_{\alpha}}$, where $\lambda^\alpha, \mu^\alpha$ are the partitions of positive integers $d_{\alpha}$ and $t_{\alpha}$, respectively, for $\alpha = 1, \ldots, L$, we define the framed full colored HOMFLYPT invariant $\mathcal{H}(\mathcal{L}; \otimes_{\alpha=1}^L Q_{\lambda^\alpha, \mu^\alpha})$ of $\mathcal{L}$ as follow:

$$\mathcal{H}(\mathcal{L}; \otimes_{\alpha=1}^L Q_{\lambda^\alpha, \mu^\alpha}) = \langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{\lambda^\alpha, \mu^\alpha} \rangle. \hspace{1cm} (41)$$

Furthermore, the framing factor for $Q_{\lambda, \mu}$ is $q^{\kappa_{\lambda} + \kappa_{\mu} a^{||\lambda||+||\mu||}}$ [6].

**Definition 3.2** The (framing-independence) full colored HOMFLYPT invariant of $\mathcal{L}$ is defined as follow:

$$W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(\mathcal{L}; q, a) = q^{-\sum_{\alpha=1}^L (\kappa_{\lambda^\alpha} + \kappa_{\mu^\alpha}) w(K_{\alpha}) a - \sum_{\alpha=1}^L (||\lambda^\alpha||+||\mu^\alpha||) w(K_{\alpha})} \langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{\lambda^\alpha, \mu^\alpha} \rangle. \hspace{1cm} (42)$$

In particular, when $\mu^\alpha = \emptyset$ for $\alpha = 1, \ldots, L$, $W_{[\lambda^1], \ldots, [\lambda^L]}(\mathcal{L}; q, a)$ is reduced to the original colored HOMFLYPT invariant $W_{\mathcal{L}}(\mathcal{L}; q, a)$ defined in [33].
Example 3.3  For the unknot $U$, by the formula (39), we have

$$W_{[\lambda, \nu]}(U; q, a) = \langle Q_{\lambda, \mu} \rangle = \sum_{\sigma, \rho, \nu} (-1)^{|\sigma|} c^\lambda_{\sigma, \rho} c^\mu_{\sigma', \nu} s^\#_{\nu}(q, a) s^\#_{\nu}(q, a),$$

where $s^\#_{\mu}(q, a)$ denotes the colored HOMFLYPT invariant $W_{\mu}(U; q, a)$ of $U$.

In the rest of this paper, we will use the notation $s^\#_{\lambda, \nu}(q, a)$ to denote the full colored HOMFLYPT invariant of the unknot $W_{[\lambda, \nu]}(U; q, a)$.

3.6 Symmetric properties

By the definitions of the maps $\overline{\cdot}$ and $\ast$, it is easy to see

$$Q_{\lambda, \mu} = Q_{\lambda, \mu}, \quad Q_{\lambda, \mu} = Q_{\mu, \lambda}.$$  \hfill (44)

For a knot $K$, we have

$$H(K; Q_{\lambda, \mu}) = H(K; Q_{\mu, \lambda}).$$

where the last equality is followed by the fact that the HOMFLYPT polynomial of a knot is independent of its orientation. Similarly, for a link $L$ with $L$-components, we have

$$H(L; \otimes_{\alpha=1}^L Q_{\lambda^\alpha, \mu^\alpha}) = H(L; \otimes_{\alpha=1}^L Q_{\mu^\alpha, \lambda^\alpha}) = H(L; \otimes_{\alpha=1}^L Q_{\mu^\alpha, \lambda^\alpha}).$$

Given a partition $\lambda$ and its conjugate partition $\lambda'$, then in the skein $\mathcal{C}$, we have

$$Q_{\lambda, \mu}|_{q \rightarrow -q^{-1}} = Q_{\lambda', \mu'}.$$  \hfill (47)

Therefore, for a link $L$ with $L$ components, we have

$$H(L; \otimes_{\alpha=1}^L Q_{\lambda^\alpha, \mu^\alpha}; q, a) = H(L; \otimes_{\alpha=1}^L Q_{\lambda^\alpha, \mu^\alpha}; -q^{-1}, a).$$

More symmetric properties for the full colored HOMFLYPT invariants (15) were given in [34] recently.

4 Full colored HOMFLYPT invariants for torus links

Let us consider the $L$-component torus link $T = T_{mL}^{nL}$ which is the closure of the framed $mL$-braid $(\beta_{mL})^{nL}$, where $(m, n) = 1$. The braid $\beta_m$ is shown in Fig. 3.

Remark 4.1 In some literatures (such as [18]), the $L$-component torus link $T_{mL}$ is defined to be the closure of the braid $(\sigma_1 \cdots \sigma_{mL-1})^{nL}$. It is clear that $T_{mL}^{nL}$ and $T_{mL}$ represent the same torus link but with different framings.
\[ T = T_{mL}^{nL} \] induces a map \( F_{mL}^{nL} : \bigotimes_{\alpha=1}^{L} C_{d_{\alpha}, r_{\alpha}} \rightarrow C_{m(\sum_{\alpha=1}^{L} d_{\alpha}), m(\sum_{\alpha=1}^{L} r_{\alpha})} \) by taking an element \( \bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha} \) to \( T_{mL}^{nL} \bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha} \). We define \( \tau = F_1 \), then \( \tau \) is the framing change map. Thus if we let 
\[ \tau(\lambda, \mu) = \tau_{\lambda, \mu} Q_{\lambda, \mu}, \] 
then \( \tau_{\lambda, \mu} = q^{(\lambda_1 + \mu_1 + |\lambda| + |\mu|)} \). We define the fractional twist map \( \tau^{n}_{m} : C \rightarrow C \) as the linear map on the basis \( Q_{\lambda, \mu} \) given by 
\[ \tau^{n}_{m}(Q_{\lambda, \mu}) = (\tau_{\lambda, \mu})^{n}_{m} Q_{\lambda, \mu}. \] 

In the following, we give an explicit expression for \( F_{mL}^{nL} \bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha} \). Let \( \Lambda_x \) and \( \Lambda_{x^*} \) be the rings of symmetric functions with variables \( (x_1, x_2, \ldots) \) and \( (x_1^*, x_2^*, \ldots) \) respectively. The Schur functions \( s_\lambda(x)(\lambda \in \mathcal{P}) \) forms a basis of the ring \( \Lambda_x \) \([20]\). It was shown in \([12]\) that the polynomials \( s_{\lambda, \mu}(x; x^*)(\lambda, \mu \in \mathcal{P}) \) (the notation \([\lambda, \mu]_{GL}\) in \([12]\)) forms a \( \mathbb{Z} \) basis of the ring \( \Lambda_x \otimes \Lambda_{x^*} \). We define the \( m \)-th Adams operator \( \Psi_m \) on \( \Lambda_x \) and \( \Lambda_x \otimes \Lambda_{x^*} \) as follow:
\[ \Psi_m(s_\lambda(x)) = s_\lambda(x^m), \quad \Psi_m(s_{\lambda, \mu}(x; x^*)) = s_{\lambda, \mu}(x^m; x^{*m}). \] Since \( \mathcal{C}_+ \) is isomorphic to the ring \( \Lambda_x \) and \( \mathcal{C} \) is isomorphic to the ring \( \Lambda_x \otimes \Lambda_{x^*} \), for any \( Q \in \mathcal{C}_{d, r} \), \( \Psi_m(Q) \) is well-defined. Indeed, \( \Psi_m(Q) \in \mathcal{C}_{md, mr} \).

We have the following formula which is a generalization of the Theorem 13 shown in \([23]\)
\[ F_{mL}^{nL} \bigotimes_{\alpha=1}^{L} Q_{\lambda^\alpha, \mu^\alpha} = \tau^{n}_{m} \left( \prod_{\alpha=1}^{L} \Psi_m(Q_{\lambda^\alpha, \mu^\alpha}) \right). \] Since \( \{Q_{\lambda, \mu} : \lambda, \mu \in \mathcal{P}, |\lambda| = d, |\mu| = r\} \) forms a basis of \( \mathcal{C}_{d, r} \), we have the following expansion 
\[ \prod_{\alpha=1}^{L} \Psi_m(Q_{\lambda^\alpha, \mu^\alpha}) = \sum_{\rho, \nu} c^{[\rho, \nu]}_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L], m} Q_{\rho, \nu}, \]
where $C^{[\rho,\upsilon]}_{[\lambda^1,\mu^1],..,[\lambda^L,\mu^L];m}$ are the coefficients given by the following formula

$$\prod_{\alpha=1}^{L} \Psi_{m}(s_{\lambda^\alpha,\mu^\alpha}(x; x^*)) = \sum_{\rho,\upsilon} C^{[\rho,\upsilon]}_{[\lambda^1,\mu^1],..,[\lambda^L,\mu^L];m} s_{\rho,\upsilon}(x; x^*). \quad (54)$$

By the definition of the fractional twist map of $\tau^{\alpha}_{\frac{m}{m'}}$, we obtain

$$F^{nL}_{mL}(\otimes_{\alpha=1}^{L} Q_{\lambda^\alpha,\mu^\alpha}) = \sum_{\rho,\upsilon} C^{[\rho,\upsilon]}_{[\lambda^1,\mu^1],..,[\lambda^L,\mu^L];m} q^{\frac{m}{m'}}(\chi_{\rho} + \chi_{\upsilon}) a^{\frac{n}{m'}}(|\rho|+|\upsilon|) Q_{\rho,\upsilon}. \quad (55)$$

Therefore, according to the Definition 3.2, the full colored HOMFLYPT invariants of the torus link $T^{nL}_{mL}$ is given by

$$W_{[\lambda^1,\mu^1],..,[\lambda^L,\mu^L]}(T^{nL}_{mL}; q, a) = q^{-m} \sum_{\alpha=1}^{L} (\chi_{\rho} + \chi_{\upsilon}) a^{-n} \sum_{\alpha=1}^{L} q^{\frac{m}{m'}}(\chi_{\rho} + \chi_{\upsilon}) a^{\frac{n}{m'}}(|\rho|+|\upsilon|) Q_{\rho,\upsilon}. \quad (56)$$

where $Q_{\rho,\upsilon} = s^{\#}_{\rho,\upsilon}(q, t)$ is the full colored HOMFLYPT invariant of the unknot $U$.

Now, let us give the explicit expression for the coefficient $C^{[\rho,\upsilon]}_{[\lambda^1,\mu^1],..,[\lambda^L,\mu^L];m}$. We need the following formulas in [12].

$$s_{\xi,\eta}(x; x^*) s_{\rho,\upsilon}(x; x^*) = M^{[\lambda,\mu]}_{[\xi,\eta],[\rho,\upsilon]} s_{\lambda,\mu}(x; x^*), \quad (57)$$

where

$$M^{[\lambda,\mu]}_{[\xi,\eta],[\rho,\upsilon]} = \sum_{\beta,\gamma,\delta} \left( \sum_{\sigma} c_{\alpha,\beta} c_{\sigma,\gamma} \right) \left( \sum_{\epsilon} c_{\sigma,\epsilon} c_{\epsilon,\delta} \right) c^{\lambda}_{\beta,\gamma} c^{\mu}_{\delta,\theta}. \quad (58)$$

$$s_{\lambda,\mu}(x; x^*) = \sum_{\sigma,\rho,\upsilon} (-1)^{[\sigma]} c^{\lambda}_{\sigma,\rho} c^{\mu}_{\sigma,\upsilon} s_{\rho,\upsilon}(x) s_{\upsilon}(x^*). \quad (59)$$

$$s_{\lambda}(x) s_{\mu}(x^*) = \sum_{\epsilon,\rho,\upsilon} c^{\lambda}_{\epsilon,\rho} c^{\mu}_{\epsilon,\upsilon} s_{\rho,\upsilon}(x; x^*). \quad (60)$$

Let $C^{\rho}_{\lambda;m}$ and $C^{[\rho,\upsilon]}_{[\lambda,\mu];m}$ be the coefficients determined by the following formulas:

$$\Psi_{m}(s_{\lambda}(x)) = \sum_{\rho} C^{\rho}_{\lambda;m} s_{\rho}(x), \quad \Psi_{m}(s_{\lambda,\mu}(x; x^*)) = \sum_{\rho,\upsilon} C^{[\rho,\upsilon]}_{[\lambda,\mu];m} s_{\rho,\upsilon}(x; x^*). \quad (61)$$
By formula (60), we have

\[ \Psi_m(s_{\lambda, \mu}(x; x^*)) = \sum_{\sigma, \rho, v} (-1)^{|\sigma|} c_{\sigma, \rho}^{\lambda} c_{\sigma^t, v}^{\mu} \Psi_m(s_{\rho}(x)) \Psi_m(s_{v}(x^*)) \]

\[ = \sum_{\sigma, \rho, v} (-1)^{|\sigma|} c_{\sigma, \rho}^{\lambda} c_{\sigma^t, v}^{\mu} \sum_{\delta, \theta} C_{\rho; m}^{\delta} C_{v; m}^{\theta} s_{\delta}(x)s_{\theta}(x^*) \]

\[ = \sum_{\sigma, \rho, v} (-1)^{|\sigma|} c_{\sigma, \rho}^{\lambda} c_{\sigma^t, v}^{\mu} \sum_{\delta, \theta} C_{\rho; m}^{\delta} C_{v; m}^{\theta} \sum_{\epsilon, \beta, \gamma} c_{\epsilon, \beta}^{\delta} c_{\epsilon, \gamma}^{\theta} s_{\beta, \gamma}(x; x^*). \]

(62)

Hence, we obtain

\[ c_{[\lambda, \mu]; m}^{[\beta; \gamma]} = \sum_{\sigma, \rho, v, \delta, \theta, \epsilon} (-1)^{|\sigma|} c_{\sigma, \rho}^{\lambda} c_{\sigma^t, v}^{\mu} C_{\rho; m}^{\delta} C_{v; m}^{\theta} c_{\epsilon, \beta}^{\delta} c_{\epsilon, \gamma}^{\theta}. \]

(63)

Therefore, as to the torus knot \( T^n_m \), the full HOMFLYPT invariant is given by

\[ W_{[\lambda, \mu]}(T^n_m; q, a) = q^{-m-n(\kappa_2+\kappa_3)} a^{-n-m(|\lambda|+|\mu|)} \sum_{\rho, v} C_{[\lambda, \mu]; m}^{[\rho, v]} q^{|\rho|+|\nu|} a^{\rho} s_{\rho, v}(q, a). \]

(64)

Finally, by formula (57), we have

\[ \prod_{\alpha=1}^{L} \Psi_m(s_{\lambda^\alpha, \mu^\alpha}(x; x^*)) \]

\[ = \sum_{[\lambda_1^1, \mu_1^1]} M_{[\lambda_1^1, \mu_1^1]}^{[\beta_1^1, \gamma_1^1]} \Psi_m(s_{\beta_1^1}(x; x^*)) \prod_{\alpha=3}^{L} \Psi_m(s_{\lambda^\alpha, \mu^\alpha}(x; x^*)) \]

\[ = \cdots \]

\[ = \sum_{[\lambda^1, \mu^1], [\lambda^2, \mu^2]} M_{[\lambda^1, \mu^1], [\lambda^2, \mu^2]}^{[\beta^1, \gamma^1]} M_{[\beta^1, \gamma^1], [\lambda^3, \mu^3]}^{[\beta^2, \gamma^2]} \cdots \]

\[ M_{[\beta^{L-1}, \gamma^{L-1}]}^{[\beta^{L-2}, \gamma^{L-2}], [\lambda^{L}, \mu^{L}]} \Psi_m(s_{\beta^{L-1}, \gamma^{L-1}}(x; x^*)) \]

\[ = \sum_{[\lambda^1, \mu^1], [\lambda^2, \mu^2], \ldots, [\lambda^{L}, \mu^{L}]} c_{[\rho, v]}^{[\beta^{L-1}, \gamma^{L-1}]} s_{\rho, v}(x; x^*). \]

(65)

where we have set

\[ M_{[\lambda^1, \mu^1], [\lambda^2, \mu^2], \ldots, [\lambda^{L}, \mu^{L}]}^{[\beta^1, \gamma^1]} = \sum_{[\lambda^1, \mu^1], [\lambda^2, \mu^2]} M_{[\lambda^1, \mu^1], [\lambda^2, \mu^2]}^{[\beta^1, \gamma^1]} M_{[\beta^1, \gamma^1], [\lambda^3, \mu^3]}^{[\beta^2, \gamma^2]} \cdots M_{[\beta^{L-2}, \gamma^{L-2}], [\lambda^{L}, \mu^{L}]}^{[\beta^{L-1}, \gamma^{L-1}]} . \]

(66)
Thus, we obtain the following formula

\[
C_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]; m}^{[\rho, \nu]} = \sum_{\beta, \gamma} M_{[\lambda^1, \mu^1], [\lambda^2, \mu^2], \ldots, [\lambda^L, \mu^L]}^{[\beta, \gamma]} C_{[\beta, \gamma]; m}^{[\rho, \nu]}, \quad (67)
\]

Combing the formula (64), we get the expression for the full colored HOMFLYPT invariant for torus link \(T_{mL}^{nL}\):

\[
W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]; (T_{mL}^{nL}; q, a)} = q^{-nm} \sum_{m=0}^{L} (k_\alpha + k_\mu) a^{-nm} \sum_{m=0}^{L} (|\lambda^2| + |\mu^3|) \sum_{\beta, \gamma} M_{[\lambda^1, \mu^1], [\lambda^2, \mu^2], \ldots, [\lambda^L, \mu^L]}^{[\beta, \gamma]} q^{nm(k_\beta + k_\gamma)} a^{-mn(|\beta| + |\gamma|)} W_{[\beta, \gamma]; (T_{mL}^{nL}; q, a)}^{(\beta, \gamma)}. \quad (68)
\]

Since \(W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]; (L; q, a)}\) is framing-independent, we also have

\[
W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]; (T (mL, nL); q, a)} = W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]; (T_{mL}^{nL}; q, a)}. \quad (69)
\]

**Remark 4.2** The formula of the full colored HOMFLYPT invariant for the torus link \(W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]; (T (mL, nL))}\) was implied in [21]. An explicit formula for (framed) full colored HOMFLYPT invariant of torus knot \(T (m, n)\) with \(m \cdot n\)-framing appears firstly in [5].

**Example 4.3** As to the torus knot \(T (2, 2k + 1)\), we have

\[
W_{(1, 1), (1)}^{(T (2, 2k + 1))} = a^{-8k-4} \left( 1 + q^{-4k-2} a^{4k+2} s_{(1^2, 1^2)}^{(1^2, 1^2)} \right) \quad (70)
\]

\[
W_{(2, 1)}^{(T (2, 2k + 1))} = a^{-12k-6} \left( -q^{-2k-1} a^{2k+1} s_{(2^2, 1^2)}^{(1^2, 1^2)} + q^{2k+1} a^{2k+1} s_{(2^2, 1^2)}^{(1^2, 1^2)} \right) \quad (71)
\]

\[
W_{(1, 2)}^{(T (2, 2k + 1))} = q^{8k+4} a^{-12k-6} \left( -q^{-2k-1} a^{2k+1} s_{(1^2, 1^2)}^{(1^2, 1^2)} + q^{2k+1} a^{2k+1} s_{(1^2, 1^2)}^{(1^2, 1^2)} \right) \quad (72)
\]
\[-a^{4k+2}s_{(2),(12)} + q^{4k+2}a^{4k+2}s_{(2),(2)} + q^{-24k-12}a^{8k+4}s_{(14),(14)}\]
\[-q^{-16k-8}a^{8k+4}s_{(14),(212)} + q^{-12k-6}a^{8k+4}s_{(14),(2)}\]
\[-q^{-16k-8}a^{8k+4}s_{(212),(14)} + q^{-8k-4}a^{8k+4}s_{(212),(212)} - q^{-4k-2}a^{8k+4}s_{(212),(22)}\]
\[+q^{-12k-6}a^{8k+4}s_{(212),(14)} - q^{-4k-2}a^{8k+4}s_{(212),(212)} + a^{8k+4}s_{(212),(22)}\].

(73)

\[W_{[(1^2),(2)]}(T(2, 2k + 1)) \]
\[= a^{-16k-8}\left(q^{-4k-2} + q^{4k+2}s_{(12),(12)} - a^{-4k+2}s_{(12),(2)} - a^{-4k+2}s_{(2),(12)}\right)\]
\[+q^{4k+2}a^{4k+2}s_{(2),(2)} + q^{-12k-6}a^{8k+4}s_{(14),(2)} - q^{-8k-4}a^{8k+4}s_{(14),(31)}\]
\[+a^{8k+4}s_{(4),(14)} - q^{-4k-2}a^{8k+4}s_{(212),(212)} + a^{8k+4}s_{(212),(31)}\]
\[+q^{-16k+6}a^{8k+4}s_{(212),(4)}\].

(74)

\[W_{[(2),(2)]}(T(2, 2k + 1)) \]
\[= q^{-16k-8} - a^{-16k-8}\left(1 + q^{-4k-2} + a^{-4k+2}s_{(12),(12)} - a^{-4k+2}s_{(12),(2)}\right)\]
\[-a^{4k+2}s_{(2),(12)} + q^{4k+2}a^{4k+2}s_{(2),(2)} + a^{8k+4}s_{(22),(22)}\]
\[-q^{4k+2}a^{8k+4}s_{(22),(31)} + q^{12k+6}a^{8k+4}s_{(22),(4)} - q^{4k+2}a^{8k+4}s_{(31),(22)}\]
\[+q^{8k+4}a^{8k+4}s_{(31),(31)} - q^{16k+8}a^{8k+4}s_{(31),(4)} + q^{12k+6}a^{8k+4}s_{(4),(22)}\]
\[+q^{-16k+8}a^{8k+4}s_{(4),(31)} + q^{24k+12}a^{8k+4}s_{(4),(4)}\].

(75)

See Appendix C in [5] for the similar expressions for these formulas with \(k = 1\).

**Example 4.4** As to the torus link \(T(2, 2k)\), we have

\[W_{[(2),(1)],[((1),[0])]}(T(2, 2k)) \]
\[= q^{-2k}a^{-4k}\left(q^{2k}a^{2k}s_{(2),(2)} + a^{4k}s_{(21),(1)} + q^{6k}a^{4k}s_{(3),(1)}\right)\]

(76)

\[W_{[(2),(1)],[((2),[0])]}(T(2, 2k)) \]
\[= q^{-4k}a^{-5k}\left(a^{3k}s_{(21),(1)} + q^{6k}a^{3k}s_{(3),(1)}\right)\]
\[+a^{5k}s_{(22),(1)} + q^{4k}a^{5k}s_{(31),(1)} + q^{12k}a^{5k}s_{(4),(1)}\].

(77)

\[W_{[(2),(1)],[((1),[0])]}(T(2, 2k)) \]
\[= a^{-5k}\left(q^{2k}a^{3k}s_{(2),(1)} + q^{-2k}a^{5k}s_{(21),(12)} + q^{4k}a^{5k}s_{(3),(13)}\right)\]

(78)

\[W_{[(2),(1)],[((1),[1])]}(T(2, 2k)) \]
\[= q^{-2k}a^{-5k}\left(a^{k}s_{(1),(1)} + q^{-2k}a^{3k}s_{(21),(12)} + 2q^{2k}a^{3k}s_{(2),(1)}\right)\]
\[+q^{-2k}a^{5k}s_{(21),(12)} + q^{2k}a^{5k}s_{(21),(2)} + q^{4k}a^{5k}s_{(3),(12)} + q^{8k}a^{5k}s_{(3),(2)}\]

(79)
\begin{align}
W_{[(2),(1^2)],[(1),(1)]}((T(2, 2k))) &= a^{-6k} \left( a^{2k} s^\#_{(1),(1)} + q^{-4k} a^{4k} s^\#_{(1^2),(1^2)} 
+ 2a^{4k} s^\#_{(2),(1^2)} + q^{4k} a^{4k} s^\#_{(2),(2)} + q^{-6k} a^{6k} s^\#_{(2),(1^3)}
+ a^{6k} s^\#_{(3),(21)} + a^{6k} s^\#_{(3),(1^3)} + q^{6k} a^{6k} s^\#_{(3),(21)} \right).
\end{align}

\section{5 Special polynomials}

For a knot $K$ and a partition $\lambda \in \mathcal{P}$, Dunin-Barkowski et al. \cite{4} defined the following special polynomial for colored HOMFLY invariant of the knot $K$

\[ H^K_\lambda(a) = \lim_{q \to 1} W_\lambda(K; q, a) W_\lambda(U; q, a). \]

In particular, when $\lambda = (1)$, we have

\[ H^K_{(1)}(a) = \lim_{q \to 1} \frac{W_{(1)}(K; q, a)}{W_{(1)}(U; q, a)} = P_K(1, a) \]

where $P_K(q, a)$ is the HOMFLYPT polynomial whose definition is given by formula (15).

After testing many examples \cite{4,9,10}, they proposed the following conjectural formula:

\[ H^K_\lambda(a) = H^K_{(1)}(a)^{[\lambda]} . \]

A rigid mathematical proof of the formula (83) was given in \cite{16} and \cite{33} with different methods. In fact, they have proved that the formula (83) holds for any link $L$. The special polynomial for a link $L$ with $L$ components is defined as follow:

\[ H^L_\lambda(a) = \lim_{q \to 1} \frac{W^L_\lambda(L; q, a)}{W^L_\lambda(U \otimes L; q, a)}. \]

\textbf{Theorem 5.1} \cite{16} and \cite{33} \textit{Given $\tilde{\lambda} = (\lambda^1, ..., \lambda^L) \in \mathcal{P}^L$ and a link $L$ with $L$ components $\mathcal{K}_{\alpha}, \alpha = 1, .., L$, we have}

\[ H^L_\lambda(a) = \prod_{\alpha=1}^{L} H^{K\alpha}_{(1)}(a)^{[\lambda^\alpha]} . \]

Motivated by the above results, we can also define the special polynomial for the full colored HOMFLYPT invariant for a link $L$ with $L$ components similarly:

\[ H^{[\lambda^1, \mu^1],..,[\lambda^L, \mu^L]}_\lambda(a) = \lim_{q \to 1} \frac{W^{[\lambda^1, \mu^1],..,[\lambda^L, \mu^L]}(L; q, a)}{\prod_{\alpha=1}^{L} W^{[\lambda^\alpha, \mu^\alpha]}(U; q, a)}. \]
Theorem 5.2 For a link $L$ with $L$ components $K_\alpha$, $\alpha = 1, \ldots, L$, we have

$$H^L_{[\lambda_1, \mu_1], \ldots, [\lambda_L, \mu_L]}(a) = \prod_{\alpha=1}^{L} P^{K_\alpha}_{[\lambda^\alpha, \mu^\alpha]}(1, a)^{|\lambda^\alpha| + |\mu^\alpha|}. \quad (87)$$

In order to prove the Theorem 5.2, we need to introduce a classical result due to Lichorish and Millet [14] which showed that for a given link $L$ with $L$ components, the lowest power of $q - q^{-1}$ in the HOMFLYPT polynomial $P_L(q, a)$ is $1 - L$.

Theorem 5.3 (Lickorish–Millett [14]) For a link $L$ with $L$ components $K_\alpha$, $\alpha = 1, \ldots, L$, the HOMFLYPT polynomial of $L$ can be written in the following form

$$P_L(q, a) = \sum_{g \geq 0} p^{L}_{2g+1-L}(a)(q - q^{-1})^{2g+1-L}. \quad (88)$$

Moreover,

$$P^{L}_{1-L}(a) = a^{-2lk(L)}(a - a^{-1})^{L-1} \prod_{\alpha=1}^{L} P^{K_\alpha}_0(a) \quad (89)$$

where $p^{K_\alpha}_0(a)$ is the HOMFLYPT polynomial of the $\alpha$-th component $K_\alpha$ of the link $L$ with $q = 1$, i.e., $p^{K_\alpha}_0(a) = P^{K_\alpha}_0(1, a)$.

By the definition of the HOMFLY polynomial (15), we have

$$\langle L \rangle = \sum_{g \geq 0} \hat{p}^{L}_{2g+1-L}(a)(q - q^{-1})^{2g-L} \quad (90)$$

where $\hat{p}^{L}_{2g+1-L}(a) = a^{-\varpi(L)}p^{L}_{2g+1-L}(a)(a - a^{-1})$. Hence

$$\hat{p}^{L}_{1-L}(a) = a^{\varpi(L)}(a - a^{-1})^{L} \prod_{\alpha=1}^{L} P^{K_\alpha}_0(a) \quad (91)$$

by the formula (16).

We now prove the Theorem 5.2.

Proof We only give the proof for the case of a knot $K$. It is easy to generalize this proof to the case of any link $\mathcal{L}$. Given two partitions $\lambda$ and $\mu$ with $|\lambda| = n$ and $|\mu| = m$, by the Frobenius formula (28), we obtain

$$Q_{\lambda, \mu} = Q_{\lambda} Q_{\mu}^* + \sum_{\sigma \neq \emptyset} (-1)^{|\sigma|} \chi_{\sigma, \rho}^{\lambda} e_{\sigma, \rho}^{\mu} Q_{\rho} Q_{\nu}^* \quad (92)$$

where $\chi_{\lambda}(C^{(1)^n}) \chi_{\mu}(C^{(1)^m}) = \sum_{s} LT_s$. \hspace{1cm} $\Box$
The main observation is that the leading term $\frac{\chi_\lambda(C_1^{(m)})X_\mu(C_1^{(m)})}{z_1^{(m)}z_1^{(m)}}P_{(1^n)}^*(P_{(1^m)}^*)$ contains $(m+n)$-components in the skein $C$, while the remaining terms $LT_s$ have the components less than $(n+m)$.

By the Definition 3.2, we have

$$W_{[\lambda, \mu]}(K; q, a) = q^{-(\kappa^\lambda + \kappa^\mu)w(K)}a^{-(n+m)w(K)}(KQ, \lambda, \mu)$$

$$= q^{-(\kappa^\lambda + \kappa^\mu)w(K)}a^{-(n+m)w(K)} \left( \frac{\chi_\lambda(C_1^{(m)})X_\mu(C_1^{(m)})}{z_1^{(m)}z_1^{(m)}} \right) \langle K \star P_{(1^n)}^*P_{(1^m)}^* \rangle + \sum_s \langle K \star LT_s \rangle \right)$$

(93)

and

$$s_{\lambda, \mu}^\#(q, a) = \left( \frac{\chi_\lambda(C_1^{(m)})X_\mu(C_1^{(m)})}{z_1^{(m)}z_1^{(m)}} \right) \left( \frac{a - a^{-1}}{q - q^{-1}} \right)^{n+m} \sum_s \langle LT_s \rangle.$$

(94)

Since $K \star P_{(1^n)}^*P_{(1^m)}^*$ is a link with $n + m$ components, according to the expansion formula (90), we have

$$\langle K \star P_{(1^n)}^*P_{(1^m)}^* \rangle = \sum_{g=0}^{n+m} \hat{p}_{2g+1}^K \hat{p}_{2g+1}^{P_{(1^n)}^*P_{(1^m)}^*} (q - q^{-1})^{2g-(n+m)}.$$

(95)

For the link $K \star LT_s$ with the number of components $L(K \star LT_s) \leq n + m - 1$, we also have

$$\langle K \star LT_s \rangle = \sum_{g=0}^{n+m-1} \hat{p}_{2g+1}^K \hat{p}_{2g+1}^{LT_s} (q - q^{-1})^{2g-(n+m)}.$$

(96)

Since $\frac{\chi_\lambda(C_1^{(m)})X_\mu(C_1^{(m)})}{z_1^{(m)}z_1^{(m)}} \neq 0$, by a direct calculation, we obtain

$$\lim_{q \to 1} \frac{W_{[\lambda, \mu]}(K; q, a)}{s_{[\lambda, \mu]}^\#(q, a) \rangle} = \frac{a^{-(n+m)w(K)}\hat{p}_1^{K \star P_{(1^n)}^*P_{(1^m)}^*}(a)}{a - a^{-1}^{n+m}}$$

(97)

Moreover, the formula (91) implies

$$\hat{p}_1^{K \star P_{(1^n)}^*P_{(1^m)}^*}(a) = a^{\bar{w}(K \star P_{(1^n)}^*P_{(1^m)}^*)} \left( a - a^{-1} \right)^{n+m} \left( p_0^K(a) \right)^{n+m},$$

(98)

and it is clear that $\bar{w}(K \star P_{(1^n)}^*P_{(1^m)}^*) = (n + m)w(K)$, thus we have

$$\lim_{q \to 1} \frac{W_{[\lambda, \mu]}(K; q, a)}{s_{[\lambda, \mu]}^\#(q, a)} = p_0^K(a)^{n+m} = P_K(1, a)^{n+m}.$$

(99)
6 Composite invariants and integrality property

6.1 LMOV conjecture for composite invariants

For a link \( L \) with \( L \) components, and \( \vec{A} = (A^1, \ldots, A^L), \vec{\lambda} = (\lambda^1, \ldots, \lambda^L), \vec{\mu} = (\mu^1, \ldots, \mu^L) \in \mathcal{P}^L \), set \( c_{\vec{\lambda}, \vec{\mu}}^A = \prod_{\alpha = 1}^{L} c_{\lambda^\alpha, \mu^\alpha}^{A^\alpha} \), where \( c_{\lambda^\alpha, \mu^\alpha}^{A^\alpha} \) is the Littlewood-Richardson coefficient, we define the composite invariant for \( L \) as follow

\[
H_{\vec{A}}(L; q, a) = \sum_{\vec{\lambda}, \vec{\mu}} c_{\vec{\lambda}, \vec{\mu}}^A W_{[\lambda^1, \mu^1], \ldots, [\lambda^L, \mu^L]}(L; q, a).
\] (100)

The Chern-Simons partition function for the composite invariant is the generating function given by

\[
Z_{CS}(L; q, a) = \sum_{\vec{A}} H_{\vec{A}}(L; q, a) s_{\vec{A}}(x).
\] (101)

The functions \( h_{\vec{A}}(L; q, a) \) are determined by the following formula

\[
F_{CS} = \log Z_{CS} = \sum_{d=1}^{\infty} \frac{1}{d} \sum_{\vec{A}} h_{\vec{A}}(L; q^d, a^d) s_{\vec{A}}(x^d).
\] (102)

For convenience, we introduce the notation

\[
T_{AB}(x) = \sum_{\mu} \frac{\chi_A(C_{\mu}) \chi_B(C_{\mu})}{z_\mu} p_\mu(x).
\] (103)

By the orthogonal relation for the characters of symmetric group, we obtain

\[
T_{AB}^{-1}(x) = \sum_{\mu} \frac{\chi_A(C_{\mu}) \chi_B(C_{\mu})}{z_\mu} \frac{1}{p_\mu(x)}.
\] (104)

In particularly, we have

\[
T_{AB}(q^\rho) = \sum_{\mu} \frac{\chi_A(C_{\mu}) \chi_B(C_{\mu})}{z_\mu} \prod_{i=1}^{l(\mu)} \frac{1}{q^{\mu_i} - q^{-\mu_i}},
\] (105)

if we let \( q^\rho = (q^{-1}, q^{-3}, q^{-5}, \ldots) \).

In 2009, M. Mariño [21] proposed the following LMOV conjecture for the composite invariants:
**Conjecture 6.1** Let \( z = q - q^{-1} \), we have

\[
\hat{h}_{\vec{A}}(L; q, a) = \sum_{\vec{A}} h_{\vec{A}}(L; q, a) T_{\vec{A}}(q^0) \in z^{-2} \mathbb{Z}[z^2, a^{\pm 1}].
\] (106)

In other words, the conjecture implies that there exist a series of integer invariants \( N_{\vec{B}, g, Q} \) such that

\[
\hat{h}_{\vec{B}}(L; q, a) = \sum_{g \geq 0} \sum_{Q \in \mathbb{Z}} N_{\vec{B}, g, Q} z^{2g-2} a^Q.
\] (107)

The Conjecture 6.1 was checked for a lot of torus knots and links in [21,31].

**6.2 Framed LMOV conjecture for the framed composite invariants**

In this subsection, we introduce the framed version of LMOV conjecture for the framed composite invariants. Then the Conjecture 6.1 can be viewed as a particular case of this framed LMOV conjecture with framing zero.

Given a link \( L \) with \( L \) components \( K_\alpha \), \( \alpha = 1, \ldots, L \), we define the framed Chern-Simons partition function as follow

\[
Z_{CS}(L; q, a) = \sum_{\vec{A}} \prod_{\alpha=1}^{L} w(K_\alpha) s_{\vec{A}_\alpha}(x^\alpha) s_{\vec{A}}(\vec{x}).
\] (108)

where \( \mathcal{H}_{\vec{A}}(L; q, a) \) is the framed composite invariant given by

\[
\mathcal{H}_{\vec{A}}(L; q, a) = \sum_{\vec{\lambda}, \vec{\mu} \in \mathcal{P}^L} c_{\vec{\lambda}, \vec{\mu}} \mathcal{H}(L; \otimes_{\alpha=1}^{L} Q^{\lambda_\alpha, \mu_\alpha}).
\] (109)

There also exist functions \( h_{\vec{A}}(L; q, a) \) such that:

\[
\mathcal{F}_{CS} = \log Z_{CS} = \sum_{d=1}^{\infty} \frac{1}{d} \sum_{\vec{A} \in \mathcal{P}^L, \vec{A} \neq 0} h_{\vec{A}}(L; q^d, a^d)s_{\vec{A}}(\vec{x}^d).
\] (110)

**Conjecture 6.2** (Framed LMOV conjecture for the framed composite invariants) For a link \( L \) with \( L \) components, we have
\[
\hat{h}_B(\mathcal{L}; q, a) = \sum_{A} h_A^{-}(\mathcal{L}; q, a) \prod_{a=1}^{L} T_{A^a B^a}(q^a)
\]
\[
= \sum_{g=0}^{\infty} \sum_{Q \in \mathbb{Z}} N_{B^g Q}^{\mathcal{C}} (q - q^{-1})^{2g-2} a^Q \in \mathbb{Z}[\mathbb{Z}, a^\pm 1].
\]

(111)

In other words, there exist the integral invariants \(N_{B^g Q}^{\mathcal{C}} \in \mathbb{Z}\), and \(N_{B^g Q}^{\mathcal{C}} \) vanishes for large \(g, Q\).

The Conjecture 6.2 was studied first in [27], and was only checked for torus knots. In this paper, we check a lot of examples for Hopf links. In the following, we provide an example for Hopf link with different framings.

**Example 6.3** As to the Hopf link \(T(2, 2)\), it has two components \(K_1\) and \(K_2\). In fact, \(K_1 = K_2 = U\). We use the notation \(T(2, 2)(m, n)\) to denote the link obtained by adding \(m\) and \(n\) kinks to \(K_1\) and \(K_2\), respectively. Thus, the link \(T(2, 2)(m, n)\) has the framing \(\tau = (\tau_1, \tau_2) = (m, n)\).

We have computed \(\hat{h}_B(T(2, 2)(m, n); q, a)\) for small \(m, n\) and \(\tilde{B}\).

1. For \(T(2, 2)(0, 0)\):

   \[
   \begin{align*}
   \hat{h}(2, 2) &= (a^2 - 1)((a^2 - 7 + 6a^2)z^{-2} + 2a^2) . \\
   \hat{h}(2, 12) &= (a^2 - 1)(-2a^{-4} + 3a^{-2} - 3 + 2a^2)z^{-2} . \\
   \hat{h}(12, 2) &= (a^2 - 1)(-2a^{-4} + 3a^{-2} - 3 + 2a^2)z^{-2} . \\
   \hat{h}(12, 12) &= (a^2 - 1)((-6a^{-4} + 7a^{-2} - 1)z^{-2} - 2a^{-4}).
   \end{align*}
   \]

2. For \(T(2, 2)(1, -1)\):

   \[
   \begin{align*}
   \hat{h}(2, 2) &= (a^2 - 1)((7a^2 - 11 + 4a^2)z^{-2} + (-2 + 2a^2)) . \\
   \hat{h}(2, 12) &= (a^2 - 1)((-2a^{-4} + 19a^{-2} - 19 + 2a^2)z^{-2} + (4a^{-2} - 4)) . \\
   \hat{h}(12, 2) &= (a^2 - 1)(-2a^{-4} + 3a^{-2} - 3 + 2a^2)z^{-2} . \\
   \hat{h}(12, 12) &= (a^2 - 1)((-4a^{-4} + 11a^{-2} - 7)z^{-2} + (-2a^{-4} + 2a^{-2})).
   \end{align*}
   \]

3. For \(T(2, 2)(1, 0)\):

   \[
   \begin{align*}
   \hat{h}(2, 2) &= (a^2 - 1)((3 - 17a^2 + 14a^4)z^{-2} + (-4a^2 + 10a^4) + 2a^4z^2) . \\
   \hat{h}(2, 12) &= (a^2 - 1)((7 - 11a^2 + 4a^4)z^{-2} + (-2a^2 + 2a^4)) . \\
   \hat{h}(12, 2) &= (a^2 - 1)((1 - 7a^2 + 6a^4)z^{-2} + 2a^4) . \\
   \hat{h}(12, 12) &= (a^2 - 1)(-2a^{-2} - 3 + 3a^2 + 2a^4)z^{-2}.
   \end{align*}
   \]
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(4) For $T(2, 2)(-1, 0)$:

\[
\hat{h}_{(2)}(a^2 - 1)(-2a^6 + 3a^4 - 3a^2 + 2)z^{-2}
\]

\[
\hat{h}_{(2)(1^2)} = (a^2 - 1)((-6a^6 + 7a^4 - a^2)z^{-2} - 2a^6)
\]

\[
\hat{h}_{(1^2)(1^2)} = (a^2 - 1)((-4a^6 + 11a^4 - 7a^2)z^{-2} + (-2a^6 + 2a^4))
\]

\[
\hat{h}_{(1^2)(1^2)} = (a^2 - 1)((-14a^6 + 17a^4 - 3a^2)z^{-2}
\]

\[+ (-10a^6 + 4a^4) - 2a^6z^2)\]

(5) For $T(2, 2)(1, 1)$:

\[
\hat{h}_{(2)}(a^2 - 1)((9a^2 - 39a^4 + 30a^6)z^{-2} + (-16a^4 + 34a^6)
\]

\[+ (-2a^4 + 14a^6)z^2 + 2a^6z^4).\]

\[
\hat{h}_{(2)(1^2)} = (a^2 - 1)((3a^2 - 17a^4 + 14a^6)z^{-2} + (-4a^4 + 10a^6) + 2a^6z^2).
\]

\[
\hat{h}_{(1^2)(1^2)} = (a^2 - 1)((3a^2 - 17a^4 + 14a^6)z^{-2} + (-4a^4 + 10a^6) + 2a^6z^2).
\]

\[
\hat{h}_{(1^2)(1^2)} = (a^2 - 1)((a^2 - 7a^4 + 6a^6)z^{-2} + 2a^6).
\]

7 Reformulated composite invariants and congruence skein relation

7.1 Review of the previous work

In the joint work [2] with K. Liu and P. Peng, for $\mu \in \mathcal{P}$, we use the skein element $P_{\mu} \subset \mathcal{C}_{[\mu], 0}$ to introduce the reformulated colored HOMFLYPT invariant for a link $L$. For any link $L$, $\hat{\mathcal{Z}}_{\mu}(L; q, a)$ defined by $Z_{\mu}(L; q, a)$

\[
\mathcal{Z}_{\mu}(L; q, a) = \langle L^* \otimes_{a=1}^{L} P_{\mu^0} \rangle, \quad \hat{\mathcal{Z}}_{\mu}(L; q, a) = \{\hat{\mu}\} \hat{\mathcal{Z}}_{\mu}(L; q, a). \quad (112)
\]

The framed LMOV conjecture is reduced to the study of the properties of these reformulated colored HOMFLYPT invariants. From the view of the HOMFLY skein theory, the reformulated colored HOMFLYPT invariant $\mathcal{Z}_{\mu}(L; q, a)$ or $\hat{\mathcal{Z}}_{\mu}(L; q, a)$ is simpler than the colored HOMFLYPT invariant $W_{\mu}(L; q, a)$, since the expression for $P_{\mu}$ is simpler than $Q_{\mu}$ in the skein $\mathcal{C}$, see [2] for a detailed descriptions. By using the HOMFLY skein theory, we prove in [2] that the reformulated colored HOMFLYPT invariants satisfy the following integrality property

**Theorem 7.1** [2] For any link $L$ with $L$ components,

\[
\hat{\mathcal{Z}}_{\mu}(L; q, a) \in \mathbb{Z}[a^\pm 1]. \quad (113)
\]

In particular, when $\mu = ((p), \ldots, (p))$ with $L$ row partitions $(p)$ for $p \in \mathbb{Z}_+$, we use the notation $\mathcal{Z}_{p}(L; q, a)$ to denote the reformulated colored HOMFLYPT invariant $\hat{\mathcal{Z}}_{((p), \ldots, (p))}(L; q, a)$ for simplicity. We have proposed the following congruence skein relations for the reformulated colored HOMFLY-PT invariant $\hat{\mathcal{Z}}_{p}(L; q, a)$ in [2]:

\[\hat{\mathcal{Z}}_{p}(L; q, a)\]
Conjecture 7.2 For any link $L$ and a prime number $p$, we have

$$
\hat{Z}_p(L_+; q, a) - \hat{Z}_p(L_-; q, a) \equiv (-1)^{p-1} \hat{Z}_p(L_0; q, a) \mod [p]^2,
$$

(114)

when the crossing is the self-crossing of a knot, and

$$
\hat{Z}_p(L_+; q, a) - \hat{Z}_p(L_-; q, a) \equiv (-1)^{p-1} p(p^2 - 1) \hat{Z}_p(L_0; q, a) \mod [p]^2.
$$

(115)

when the crossing is the linking of two different components of the link $L$. Where the notation $A \equiv B \mod C$ denotes $A - B \in \mathbb{Z}[z^2, a^\pm 1]$. And $\{p\} = q^p - q^{-p}$, $[p] = (q^p - q^{-p})/(q - q^{-1})$.

The Conjecture 7.2 has been tested by a lot of examples in [2]. As an application, we obtain the following result for any link $L$.

Corollary 7.3 (Assuming the Conjecture 7.2 is right) Let $L$ be a link with $L$ components $K_\alpha$, $\alpha = 1, \ldots, L$. Define $\hat{w}(L) = \sum_{\alpha=1}^L w(K_\alpha)$, $w(K)$ denotes the writhe number of the knot $K$. For any prime number $p$, we have

$$
\hat{Z}_p(L; q, a) \equiv (-1)^{p-1} \hat{w}(L) \hat{Z}_p(L; q^p, a^p) \mod [p]^2.
$$

(116)

In fact, Corollary 7.3 is a nontrivial consequence of the framed LMOV conjecture.

In conclusion, these interesting structures for the reformulated colored HOMFLYPT invariant convince us that it is useful to study the reformulated invariant $\hat{Z}_p(L; q, a)$ or $\hat{\tilde{Z}}_p(L; q, a)$ instead of $W_\mu(L; q, a)$ in HOMFLY skein theory.

7.2 Reformulated composite invariants

In the following, we introduce an analog reformulated invariant for the framed composite invariant. First, for any partition $\nu \in \mathcal{P}$, we associate it a skein element $R_\nu \in \mathcal{C}$ by

$$
R_\nu = \sum_A \chi_A(\nu) \sum_{\lambda, \mu} c^A_{\lambda, \mu} Q_{\lambda, \mu}.
$$

(117)

In particular, if we take $\mu = \emptyset$ in (117), then we have $R_\nu = P_\nu \in \mathcal{C}_+.$

Definition 7.4 For a link $L$ with $L$ components and a partition vector $\vec{\nu} = (\nu^1, \ldots, \mu^L)$, we define the reformulated composite invariants of $L$ as follow

$$
\mathcal{R}_{\vec{\nu}}(L; q, a) = \langle L \ast \otimes_{\alpha=1} R_{\nu^\alpha} \rangle, \quad \tilde{\mathcal{R}}_{\vec{\nu}}(L; q, a) = \{\vec{\nu}\} \mathcal{R}_{\vec{\nu}}(L; q, a).
$$

(118)

Moreover, for $p \in \mathbb{Z}_+$, we use the notation $\mathcal{R}_p(L; q, a)$ to denote the $\mathcal{R}_{(p, \ldots, p)}(L; q, a)$ for simplicity.
By this definition, the framed Chern-Simons partition $Z_{CS}(\mathcal{L}; q, a)$ defined by the formula (108) can be rewritten as

$$Z_{CS}(\mathcal{L}; q, a) = \sum_{\hat{v} \in \mathcal{P}^L} (-1)^{\sum_{\alpha=1}^L w(K_\alpha)|\nu^\alpha|} R_{\hat{v}}(\mathcal{L}; q, a) p_{\nu}(x).$$  \hspace{1cm} (119)$$

As in [2], in order to study the framed LMOV conjecture, it is natural to investigate the properties of $R_{\nu}(\mathcal{L}; q, a)$. The detailed calculations shown in Sect. 8 lead to the following expression for $R_{\nu}$ in the full skein $\mathcal{C}$:

$$R_{\nu} = \sum_A \chi_A(\nu) \sum_{\lambda, \mu} c^A_{\lambda, \mu} Q_{\lambda, \mu}$$

$$= \sum_{B \cup C = v} \frac{z_v}{z_B z_C} P_B P_C^* + \sum_{B \cup C = v} \sum_{\tau \cup \eta = B} \sum_{\tau \cup \pi = C} \frac{z_v}{z_\eta z_\pi} (-1)^{l(\tau)} P_\eta P^*_\pi.$$  \hspace{1cm} (120)$$

Thus, in particular, for the partition $\nu = (p)$, we have

$$R_{(p)} = P_p + P^*_p.$$  \hspace{1cm} (121)$$

For an oriented knot $K$, we reverse the orientation of $K$ which gives the knot $K^*$. In other words, $K$ and $K^*$ are two knots with the opposite orientation. Because for a knot, the HOMFLY skein relation is independent of the orientation of knot, we obtain $\langle K \rangle = \langle K^* \rangle$. Furthermore, for $Q \in \mathcal{C}$, we have

$$\langle K^* Q^* \rangle = \langle K^* Q \rangle = \langle (K^* Q)^* \rangle = \langle K^* Q \rangle.$$  \hspace{1cm} (122)$$

Hence for a knot $K$, we have

$$\tilde{R}_p(K; q, a) = \langle p \rangle \langle K^* Q \rangle$$

$$= \langle p \rangle (\langle K^* P_p \rangle + \langle K^* P^*_p \rangle)$$

$$= 2\langle p \rangle \langle K^* P_p \rangle$$

$$= 2 \tilde{Z}_p(K; q, a).$$  \hspace{1cm} (123)$$

Now we consider the case of link, let $\mathcal{L}$ be an oriented link with $L$ components $K_\alpha$ with $\alpha = 1, 2, ..., L$. For convenience, we also write $\mathcal{L} = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_L$. We use the notation $\mathcal{L}^*$ to denote the new link obtained by reversing the orientations of all components, i.e., $\mathcal{L}^* = K_1^* \sqcup K_2^* \sqcup \cdots \sqcup K_L^*$. Similarly, we have $\langle \mathcal{L} \rangle = \langle \mathcal{L}^* \rangle$. Furthermore, given $Q_\alpha \in \mathcal{C}$, for $\alpha = 1, 2, ..., L$, we also have

$$\langle \mathcal{L}^* \otimes^L Q^*_\alpha \rangle = \langle \mathcal{L}^* \otimes^L Q_\alpha \rangle = \langle (\mathcal{L} \otimes^L Q_\alpha)^* \rangle = \langle \mathcal{L} \otimes^L Q_\alpha \rangle.$$  \hspace{1cm} (124)$$
Let \(1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq L\) be the indices in \(\{1, 2, \ldots, L\}\). By reversing the orientations of the components \(K_{\alpha_1}, \ldots, K_{\alpha_k}\), we obtain a new link denoted by

\[
L_{\alpha_1, \alpha_2, \ldots, \alpha_k} = K_1 \sqcup K_2 \sqcup \cdots \sqcup K_{\alpha_1} \sqcup \cdots \sqcup K_{\alpha_k} \sqcup \cdots \sqcup K_L.
\]  

(125)

It is obvious that

\[
L^*_{\alpha_1, \alpha_2, \ldots, \alpha_k} = L_{1, 2, \ldots, \hat{\alpha}_1, \ldots, \hat{\alpha}_2, \ldots, \hat{\alpha}_k, \ldots, L}
\]  

(126)

where \(\hat{\alpha}_i\) denotes the index \(\alpha_i\) is omitted.

Combing the above notations, applying the formula (121), we finally obtain

**Theorem 7.5**

\[
\tilde{\mathcal{R}}_p(L; q, a) = \sum_{k=0}^{L} \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_k \leq L} \tilde{Z}_p(L_{\alpha_1, \alpha_2, \ldots, \alpha_k}; q, a).
\]  

(127)

Then Theorem 7.1 implies the following integrality result:

**Theorem 7.6** For any link \(L\), we have

\[
\tilde{\mathcal{R}}_p(L; q, a) \in \mathbb{Z}[z^2, a^{\pm1}].
\]  

(128)

**Remark 7.7** In general, we will prove that for any \(\tilde{\mu} \in P^L\), \(\tilde{\mathcal{R}}_{\tilde{\mu}}(L; q, a) \in \mathbb{Z}[z^2, a^{\pm1}]\) in Sect. 8, see Theorem 8.1.

**Remark 7.8** In fact, \(\tilde{\mathcal{R}}_p(L; q, a) \in 2\mathbb{Z}[z^2, a^{\pm1}]\). Since

\[
\tilde{Z}_p(L_{\alpha_1, \alpha_2, \ldots, \alpha_k}; q, a) = \tilde{Z}_p(L_{1, 2, \ldots, \hat{\alpha}_1, \ldots, \hat{\alpha}_2, \ldots, \hat{\alpha}_k, \ldots, L}; q, a)
\]  

(129)

then by (124) and (126), the \(2^L\) terms in the summation of the formula (127) can be recollected as the summation \(2 \times (2^L - 1\) different terms).

**Example 7.9** When \(L = 2\), \(L = K_1 \sqcup K_2\). We have

\[
\tilde{\mathcal{R}}_p(K_1 \sqcup K_2) = \{p\}^2 (Z_{(p),(p)}(K_1 \sqcup K_2)) + Z_{(p),(p)}(K_1^* \sqcup K_2^*)
\]

\[
+ Z_{(p),(p)}(K_1^* \sqcup K_2) + Z_{(p),(p)}(K_1 \sqcup K_2^*)
\]

\[
= 2\{p\}^2 (Z_{(p),(p)}(K_1 \sqcup K_2) + Z_{(p),(p)}(K_1 \sqcup K_2^*)) \in 2\mathbb{Z}[z^2, a^{\pm1}].
\]  

(130)

where the second “=” is from \(Z_{(p),(p)}(K_1 \sqcup K_2) = Z_{(p),(p)}(K_1^* \sqcup K_2^*)\) and \(Z_{(p),(p)}(K_1 \sqcup K_2) = Z_{(p),(p)}(K_1 \sqcup K_2^*)\), since changing all the orientations of the components of a link does not change the HOMFLYPT invariant.
7.3 Congruence skein relation

When the crossing is the linking between two different components of the link, we have the following skein relation for \( \tilde{\mathcal{R}}_1 \) by the classical skein relation for HOMFLYPT polynomial

\[
\tilde{\mathcal{R}}_1(L_+; q, a) - \tilde{\mathcal{R}}_1(L_-; q, a) = \varepsilon^2 \left( \tilde{\mathcal{R}}_1(L_0; q, a) - \tilde{\mathcal{R}}_1(L_\infty; q, a) \right).
\]

(131)

where \((L_+, L_-, L_0, L_\infty)\) denotes the quadruple appears in the classical Kauffman skein relation. So \( \tilde{\mathcal{R}}_1(L; q, a) \) satisfies the classical Kauffman type skein relation.

As to the reformulated composite invariants \( \tilde{\mathcal{R}}_p(L; q, a) \), we propose the following

**Conjecture 7.10** (Congruence skein relation for reformulated composite invariants)

*For any prime \( p \), when the crossing is the linking between two different components of the link, we have*

\[
\tilde{\mathcal{R}}_p(L_+; q, a) - \tilde{\mathcal{R}}_p(L_-; q, a) \\
\equiv (-1)^{p-1} p[p]^2 \left( \tilde{\mathcal{R}}_p(L_0; q, a) - \tilde{\mathcal{R}}_p(L_\infty; q, a) \right) \mod [p]^2[p]^2.
\]

(132)

The Conjecture 7.10 has been tested by a lot of examples. In particular, we have the following result

**Theorem 7.11** When \( p = 2 \), the conjecture holds for \( L_+ = T(2, 2k + 2), L_- = T(2, 2k), L_0 = T(2, 2k + 1) \) and \( L_\infty = U(-2k - 1) \), where \( U(-2k - 1) \) denotes the unknot with \( 2k + 1 \) negative kinks.

**Proof** We need to prove the following identity:

\[
\tilde{\mathcal{R}}_2(T(2, 2k + 2); q, a) - \tilde{\mathcal{R}}_2(T(2, 2k); q, a) \\
\equiv -2[2]^2 \left( \tilde{\mathcal{R}}_2(T(2, 2k + 1); q, a) - \tilde{\mathcal{R}}_2(U(-2k - 1); q, a) \right) \mod [2]^2[2]^2.
\]

(133)

By formula (127), we have

\[
\tilde{\mathcal{R}}_2(T(2, 2k); q, a) = 2[2]^2 (\mathcal{Z}_{(2)}(T(2, 2k); q, a) + \mathcal{Z}_{(2)}((T(2, 2k))^*; q, a))
\]

(134)

where \((T(2, 2k))^*\) denotes the link obtained by reversing the orientation of the second component of \( T(2, 2k) \). Then

\[
\mathcal{Z}_{(2)}((T(2, 2k))^*; q, a) \\
= W_{(2),(0),(0),(2)}(T(2, 2k); q, a) - 2W_{(2),(0),(0),(2)}(T(2, 2k); q, a) \\
+ W_{(1^2),(0),(0),(1^2)}(T(2, 2k); q, a)
\]

(135)
and

\[ \tilde{R}_2(T(2, 2k + 1); q, a) = 2[2] \tilde{Z}_2(T(2, 2k + 1); q, a), \]
\[ \tilde{R}_2(U(-2k - 1); q, a) = 2[2] \tilde{Z}_2(U(-2k - 1); q, a). \]  

(136)  

(137)

In [2], we have proved the following formula (see Theorem 4.4 in [2]):

\[ \tilde{Z}_2(T(2, 2k + 2); q, a) - \tilde{Z}_2(T(2, 2k); q, a) \equiv -2[2]^2 \tilde{Z}_2(T(2, 2k + 1); q, a) \mod [2]^2[2]^2. \]  

(138)

So in order to prove the formula (133), we only need to show

\[ \tilde{Z}_2((T(2, 2k + 2))^*; q, a) - \tilde{Z}_2((T(2, 2k))^*; q, a) \equiv 2[2]^2 \tilde{Z}_2(U(-2k - 1); q, a) \mod [2]^2[2]^2. \]  

(139)

By the formula (56), we have

\[ W_{1221}(T(2, 2k)) = s_{1121}(q, a) + q^{-4k} q^{-2k} s_{1111}(q, a) + q^{-4k} a^{-4k} \]  

(140)

\[ W_{1221}(T(2, 2k)) = s_{1121}(q, a) + a^{-2k} s_{1111}(q, a) \]  

(141)

\[ W_{1221}(T(2, 2k)) = s_{1121}(q, a) + q^{4k} a^{-2k} s_{1111}(q, a) + q^{4k} a^{-4k}. \]  

(142)

Thus

\[ \tilde{Z}_2((T(2, 2k))^*; q, a) = [2]^2 \left( s_{1122}(q, a) - 2s_{1112}(q, a) + s_{1111}(q, a) \right) \]  

\[ + (q^{-4k} - 2 + q^{4k} a^{-2k} s_{1111}(q, a) + (q^{-4k} + q^{4k}) a^{-2k}). \]  

(143)

By the formula (43), we get

\[ \tilde{Z}_2((T(2, 2k))^*; q, a) \equiv (a^2 - a^{-2})^2 + 2(a^{-4k} - 2)(q^2 - q^{-2})^2 \mod [2]^2[2]^2. \]  

(144)

By using the framing change formula in [2](see Theorem 3.15 in [2]), we obtain

\[ \tilde{Z}_2(U(-2k - 1); q, a) \equiv -a^{-4k-2} \tilde{Z}_2(U; q, a) \mod [2]^2 \]  

\[ = -a^{-4k-2}(a^2 - a^{-2}) \mod [2]^2. \]  

(145)

Therefore, we have
\[
\tilde{Z}_{(2)(2)}((T(2, 2k + 2))^*; q, a) - \tilde{Z}_{(2)(2)}((T(2, 2k))^*; q, a) \\
- 2[2]^2 \tilde{Z}_2(U(-2k - 1); q, a) \\
\equiv (2a^{-4k-4} - 2a^{-4k})(q^2 - q^{-2})^2 + 2[2]^2 a^{-4k-2}(a^2 - a^{-2}) \\
= (2a^{-4k-4} - 2a^{-4k})(q^2 - q^{-2})^2 + 2(a^{-4k} - a^{-4k-4})(q^2 - q^{-2})^2 \\
= 0 \mod [2]^2[2]^2.
\]

The proof is completed. \(\square\)

**Remark 7.12** In a recent paper [3], we define the reformulated colored Kauffman invariant \(\tilde{G}_p(L; q, a)\), and we find that \(\tilde{G}_p(L; q, a)\) satisfies the same congruence skein relation as in Conjecture 7.10. It provides a mathematical understanding for the reason why the LMOV conjecture constructed in [21] involves both the composite invariants and colored Kauffman invariants.

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**8 Appendix**

**8.1 The expression for \(R_\nu\)**

We use the notations \(A, B, C, \ldots\) and \(\lambda, \mu, \nu, \rho, \sigma, \delta, \xi, \eta, \tau, \ldots\) to denote the partitions. Given two partitions \(\lambda = (\lambda_1, \ldots, \lambda_l(\lambda))\) and \(\mu = (\mu_1, \ldots, \mu_l(\mu))\), we use \(\lambda \cup \mu\) to denote the new partition with all its parts are given by \(\lambda_1, \ldots, \lambda_l(\lambda), \mu_1, \ldots, \mu_l(\mu)\). Moreover, the summing notation \(\sum_{B \cup C = \nu}^{\circ} \) denotes the sum of all the partitions \(B\) and \(C\) (including the case \(B, C = \emptyset\)) such that \(B \cup C = \nu\). And the summing notation \(\sum_{B \cup C = \nu}^{\circ} \) denotes the sum of all the partitions \(B\) and \(C\) with \(B \neq \emptyset\) and \(C \neq \emptyset\) such that \(B \cup C = \nu\).

Recall the expression for the Littlewood-Richardson coefficient \(c_{\lambda, \mu}^A\),

\[
c_{\lambda, \mu}^A = \sum_{B, C} \frac{\chi_A(B) \chi_A(C)}{z_B z_C} \chi_A(B \cup C),
\]

(147)
and the orthogonality of character formula

\[ \sum_{A} \frac{\chi_A(\mu)\chi_A(v)}{z_{\mu}} = \delta_{\mu v}, \quad (148) \]

we obtain

\[ \sum_{A} \chi_A(v)c_{\lambda, \mu}^A = \sum_{B \cup C = v} \frac{z_{\nu}}{z_{B}z_{C}} \chi_{\lambda}(B)\chi_{\mu}(C). \quad (149) \]

Since

\[ Q_{\lambda, \mu} = \sum_{\sigma, \rho, \delta} (-1)^{|\sigma|} c_{\sigma, \rho}^{\lambda} c_{\sigma, \delta}^{\mu} Q_{\rho} Q_{\delta}^* \]

\[ = Q_{\lambda} Q_{\mu}^* + \sum_{\sigma, \rho, \delta \neq \emptyset} (-1)^{|\sigma|} c_{\sigma, \rho}^{\lambda} c_{\sigma, \delta}^{\mu} Q_{\rho} Q_{\delta}^*, \quad (150) \]

we have

\[ R_{\nu} = \sum_{A} \chi_A(v) \sum_{\lambda, \mu} c_{\lambda, \mu}^A Q_{\lambda, \mu} \]

\[ = \sum_{B \cup C = v} \frac{z_{\nu}}{z_{B}z_{C}} \sum_{\lambda, \mu} \chi_{\lambda}(B)\chi_{\mu}(C) Q_{\lambda} Q_{\mu}^* \]

\[ + \sum_{B \cup C = v} \frac{z_{\nu}}{z_{B}z_{C}} \sum_{\lambda, \mu} \chi_{\lambda}(B)\chi_{\mu}(C) \sum_{\sigma, \rho, \delta \neq \emptyset} (-1)^{|\sigma|} c_{\sigma, \rho}^{\lambda} c_{\sigma, \delta}^{\mu} Q_{\rho} Q_{\delta}^*. \quad (151) \]

The first term of in the right-hand side of the formula (151) denoted by \( I \) is given by

\[ I = \sum_{B \cup C = v} \frac{z_{\nu}}{z_{B}z_{C}} \sum_{\lambda, \mu} \chi_{\lambda}(B)\chi_{\mu}(C) Q_{\lambda} Q_{\mu}^* \]

\[ = \sum_{B \cup C = v} \frac{z_{\nu}}{z_{B}z_{C}} P_{B} P_{C}^*. \quad (152) \]

Next, we compute the second term \( II \) as follow. We write

\[ c_{\sigma, \rho}^{\lambda} = \sum_{\xi, \eta} \frac{\chi_{\sigma}(\xi)\chi_{\rho}(\eta)}{z_{\xi}z_{\eta}} \chi_{\lambda}(\xi \cup \eta), \quad c_{\sigma, \delta}^{\mu} = \sum_{\tau, \pi} \frac{\chi_{\sigma}(\tau)\chi_{\delta}(\pi)}{z_{\tau}z_{\pi}} \chi_{\mu}(\tau \cup \pi) \quad (153) \]

By using the orthogonality relation (148) twice, we obtain
\[ II = \sum_{B \cup C = v} \frac{z_v}{Z_B Z_C} \sum_{\sigma, \rho, \delta \neq \emptyset} (-1)^{\sigma} \sum_{\xi \cup \eta = B} \frac{z_{\xi \xi \eta \eta}}{Z_{\xi \xi \eta \eta}} \chi_\sigma (\xi) \chi_\rho (\eta) \]

\[ = \sum_{B \cup C = v} \sum_{\xi \cup \eta = B} \sum_{\tau \cup \pi = C} \frac{z_v}{Z_{\xi \xi \eta \eta} Z_{\tau \tau \pi \pi}} (-1)^{\sigma} \chi_\sigma (\xi) \chi_\sigma (\tau) \chi_\rho (\eta) \chi_\delta (\pi) Q_\rho Q_\delta^* \]

(154)

Since \( \chi_\sigma (\tau) = (-1)^{|\tau|-l(\tau)} \chi_\sigma (\tau) \), by using the orthogonality relation (148) again, we obtain

\[ II = \sum_{B \cup C = v} \sum_{\xi \cup \eta = B} \sum_{\tau \cup \pi = C} \frac{z_v}{Z_{\xi \xi \eta \eta} Z_{\tau \tau \pi \pi}} (-1)^{\sigma} \chi_\sigma (\xi) \chi_\sigma (\tau) \chi_\rho (\eta) \chi_\delta (\pi) Q_\rho Q_\delta^* \]

(155)

Thus, we have

\[ R_v = \sum_{B \cup C = v} \frac{z_v}{Z_B Z_C} P_B P_C^* + \sum_{B \cup C = v} \sum_{\tau\cup \pi = C} \sum_{\xi \cup \eta = B} \frac{z_v}{Z_{\xi \xi \eta \eta} Z_{\tau \tau \pi \pi}} (-1)^{l(\tau)} P_\eta P_\pi^* \]  

(156)

### 8.2 Integrality of the reformulated composite invariants \( \tilde{\mathcal{R}}_{\tilde{\mu}} (\mathcal{L}; q, a) \)

By using the expression of \( R_v \) shown in the formula (156), we can prove the following integrality Theorem.

**Theorem 8.1** For any link \( \mathcal{L} \) with \( L \) components, and \( \tilde{\mu} = (\mu^1, \ldots, \mu^L) \in \mathcal{P}^L \), we have

\[ \tilde{\mathcal{R}}_{\tilde{\mu}} (\mathcal{L}; q, a) \in \mathbb{Z}[z^2, a^{\pm 1}] \]

(157)

**Proof** For brevity, we only give the proof for the case of a knot. It is easy to generalize this proof the general case for any links.

For a partition \( v = B \cup C \), we define the integer

\[ n_{BC}^v = \frac{z_v}{Z_B Z_C} = \left( \frac{|\text{Aut}(v)|}{|\text{Aut}(B)|} \right) \]

(158)
By the expression for $R_\nu$ given by formula (156), we have

$$R_\nu = \sum_{B \cup C = \nu} n_{BC}^\nu P_B P_C^* + \sum_{B \cup C = \nu} \sum_\tau \sum_\eta \sum_\pi n_{BC}^\nu h_{\tau, \eta} h_{\tau, \pi} z_\tau (-1)^{l(\tau)} P_\eta P_\pi^*. \tag{159}$$

Then

$$\tilde{R}_\nu = \{\nu\} R_\nu$$

$$= \sum_{B \cup C = \nu} n_{BC}^\nu \tilde{P}_B \tilde{P}_C^* + \sum_{B \cup C = \nu} \sum_\tau \sum_\eta \sum_\pi n_{BC}^\nu h_{\tau, \eta} h_{\tau, \pi} z_\tau (-1)^{l(\tau)} \{\tau\}^2 \tilde{P}_\eta \tilde{P}_\pi^*. \tag{160}$$

For an orientation knot $K$, we construct a new knot $K_{(l, r)}$ with $l + r$ parallel components of $K$, while the first $l$ components have the same orientation with $K$ and the rest $r$ components have the reversed orientation. For any two partitions $\lambda, \mu$ with $l(\lambda) = l$ and $l(\mu) = r$, we have

$$\mathcal{H}(K \star \tilde{P}_\lambda \tilde{P}_\mu^*) = z \mathcal{H}(K_{(l, r)} \star \otimes_{i=1}^l \otimes_{j=1}^r X_{\lambda_i} X_{\mu_j})$$

$$= z \mathcal{Z}(K_{(l, r)} \star \otimes_{i=1}^l \otimes_{j=1}^r X_{\lambda_i} X_{\mu_j}) \in \mathbb{Z}[z^2, a^{\pm 1}], \tag{161}$$

where we have used Lemma 3.8 in [2].

By the expression of $\tilde{R}_\nu$, and for any partition $\tau$, it is clear $\{\tau\}^2 \in \mathbb{Z}[z^2]$, thus we obtain

$$\tilde{R}_\nu(K; q, a) \in \mathbb{Z}[z^2, a^{\pm 1}]. \tag{162}$$

\[\square\]

References

1. Bouchard, V., Florea, B., Mariño, M.: Topological open string amplitudes on orientifolds. JHEP 0502, 002 (2005)
2. Chen, Q., Liu, K., Peng, P., Zhu, S.: Congruent skein relations for colored HOMFLY-PT invariants and colored Jones polynomials. arXiv:1402.3571
3. Chen, Q., Zhu, S.: New structures for colored Kauffman invariants, preprint
4. Dunin-Barkowski, P., Mironov, A., Morozov, A., Sleptsov, A., Smirnov, A.: Superpolynomials for toric knots from evolution induced by cut-and-join operators. arXiv:1106.4305
5. Gu, J., Jockers, H., Klemm, A., Sorouh, M.: knot invariants from topological recursion on augmentation varieties. arXiv:1401.5095
6. Gross, D., Taylor, W.: Two-dimensional QCD is a string theory. Nucl. Phys. B 400, 181 (1993)
7. Gopakumar, R., Vafa, C.: On the gauge theory/geometry correspondence. Adv. Theor. Math. Phys. 3(5), 1415–1443 (1999)
8. Hadji, R.J., Morton, H.R.: A basis for the full Homfly skein of the annulus. Math. Proc. Camb. Philos. Soc. 141(1), 81–100 (2006)
9. Itoyama, H., Mironov, A., Morozov, A., Morozov, An.: HOMFLY and superpolynomials for figure eight knot in all symmetric and antisymmetric representations. arXiv:1203.5978
10. Itoyama, H., Mironov, A., Morozov, A., Morozov, A.: Character expansion for HOMFLY polynomials. III. All 3-Strand braids in the first symmetric representation. arXiv:1204.4785
11. Kosuda, M., Murakami, J.: Centralizer algebras of the mixed tensor representations of quantum group $U_q(\mathfrak{gl}(n, C))$. Osaka J. Math. 30, 475–507 (1993)
12. Kioke, K.: On the decomposition of tensor products of the representations of the classical groups: by means of the universal character. Adv. Math. 74, 57 (1989)
13. Liu, C.-C., Liu, K., Zhou, J.: A proof of a conjecture of Mariño-Vafa on Hodge integrals. J. Differential Geom. 65 (2003)
14. Lickorish, W.B.R., Millett, K.C.: A polynomial invariant of oriented links. Topology 26, 107 (1987)
15. Labastida, J.M.F., Mariño, M.V.C.: Knots, links and branes at large N. J. High Energy Phys. (11):Paper 7–42 (2000)
16. Liu, K., Peng, P.: Proof of the Labastida–Mariño–Ooguri–Vafa conjecture. J. Differ. Geom. 85(3), 479–525 (2010)
17. Lukac, S.G.: Homfly skeins and the Hopf link. PhD. thesis, University of Liverpool (2001)
18. Lin, X.-S., Zheng, H.: On the Hecke algebra and the colored HOMFLY polynomial. arXiv:math.QA/0601267
19. Luo, W., Zhu, S.: Integrality of the LMOV invariants for framed unknot. Commun. Number Theory Phys. 13(1), 81–100 (2019)
20. MacDolnald, I.G.: Symmetric Functions and Hall Polynomials, 2nd edn. Charendon Press, Oxford (1995)
21. Mariño, M.: String theory and the Kauffman polynomial. arXiv:0904.1088
22. Morton, H.R., Hadji, R.J.: HOMFLY polynomials of generalized Hopf links. Algebr. Geom. Topol. 2, 11–32 (2002)
23. Morton, H.R., Manchon, P.M.G.: Geometrical relations and plethysms in the Homfly skein of the annulus. J. Lond. Math. Soc. 78, 305–328 (2008)
24. Mironov, A., Morozov, A., Sleptsov, A.: Genus expansion of HOMFLY polynomials. arXiv:1303.1015
25. Mariño, M., Vafa, C.: Framed knots at large N. In: Orbifolds in mathematics and physics (Madison, WI, 2001), volume 310 of Contemp. Math., pp. 185–204. Amer. Math. Soc., Providence, RI (2002)
26. Ooguri, H., Vafa, C.: Knot invariants and topological strings. Nucl. Phys. B 577(3), 419–438 (2000)
27. Paul, C., Borhade, P., Ramadevi, P.: Composite invariants and unoriented topological string amplitudes. arxiv:1003.5282
28. Reshetikhin, N.Y., Turaev, V.G.: Invariants of 3-manifolds via link polynomials and quantum groups. Invent. Math. 103(1), 547–597 (1991)
29. Turaev, V.G.: The Yang–Baxter equation and invariants of links. Invent. Math. 92, 527–553 (1988)
30. Turaev, V.G.: The Conway and Kauffman modules of a solid torus. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 167 (1988), Issled. Topol. 6, 79–89
31. Stevan, S.: Chern–Simons invariants of TorusKnots and links. arxiv:1003.2861
32. Witten, E.: Chern–Simons gauge theory as a string theory. In: The Floer memorial volume, volume 133 of Progr. Math., pp. 637–678. Birkhäuser, Basel (1995)
33. Zhu, S.: Colored HOMFLY polynomials via skein theory. J. High Energy Phys. 10, 229 (2013)
34. Zhu, S.: A simple proof of the strong integrality for full colored HOMFLYPT invariants. J. Knot Theory Ramifications. 28 (2019), no. 7, 1950046, 16 p
35. Zhu, S.: Topological strings, quiver varieties, and Rogers–Ramanujan identities. Ramanujan J 48, 399 (2019)
36. Zhu, S.: On explicit formulae of LMOV invariants. J. High Energy Phys. 10, 076 (2019)