ON ANOMALOUS SUBVARIETIES OF HOLONOMY VARIETIES OF HYPERBOLIC 3-MANIFOLDS

BOGWANG JEON

Abstract. Let $M$ be an $n$-cusped hyperbolic 3-manifold having rationally independent cusp shapes and $X$ be its holonomy variety. We first show that every maximal anomalous subvariety of $X$ containing the identity is its subvariety of codimension 1 which arises by having a cusp of $M$ complete. Second, we prove if $X^{oa} = \emptyset$, then $M$ has cusps which are, keeping some other cusps of it complete, strongly geometrically isolated from the rest. Third, we resolve the Zilber-Pink conjecture for holonomy varieties of any 2-cusped hyperbolic 3-manifolds.

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1. Introduction

1.1. Main results

Let $G^n := (\overline{\mathbb{Q}})^n$ or $(\mathbb{C}^*)^n$. By an algebraic subgroup $H$ in $G^n$, we mean the set of solutions of monomial types of equations, and an algebraic coset $K$ is defined to be a translate $gH$ of some algebraic subgroup $H$ by some $g \in G^n$.

In [2], Bombieri-Masser-Zannier defined the following:

**Definition 1.1.** Let $X$ be an algebraic variety in $G^n$. An irreducible subvariety $Y$ of $X$ of positive dimension is called anomalous if it lies in an algebraic coset $K$ in $G^n$ satisfying

$$\dim Y > \dim K + \dim X - n.$$  \hfill (1.1)

In particular, if $K$ is an algebraic subgroup, then $Y$ is called a torsion anomalous subvariety of $X$. Also $Y$ is said to be maximal if it is not contained in a strictly larger anomalous subvariety of $X$.

The expected dimension of $Y$ is

$$\dim X + \dim K - n$$

when both $X$ and $K$ are in general position, and so an anomalous subvariety of $X$ is obtained when $X$ intersects with an algebraic coset of $G^n$ unusually.

The concept arises very commonly in number theory. For instance, the following, which generalizes many classically known results and one of the central conjectures in Diophantine geometry, concerns about the distribution of torsion anomalous subvarieties of $X$.

**Conjecture 1 (Zilber-Pink).** For every irreducible variety $\mathcal{X}(\subset G^n)$ defined over $\overline{\mathbb{Q}}$, there exists a finite set $T$ of proper algebraic subgroups such that, for every algebraic subgroup $H$ and every component $\mathcal{Y}$ of $\mathcal{X} \cap H$ satisfying

$$\dim \mathcal{Y} > \dim H + \dim \mathcal{X} - n,$$

one has $\mathcal{Y} \subset T$ for some $T \in T$.

Note that it is assumed $\dim \mathcal{Y} > 0$ in Definition 1.1, but $\dim \mathcal{Y} = 0$ is allowed in the Zilber-Pink conjecture. If $\dim \mathcal{Y} = 0$, we call it a torsion anomalous point of $X$.

In short, the conjecture says the union of torsion anomalous subvarieties and points are not arbitrarily distributed but instead lying in a finite number of proper algebraic subgroups. In some ways, the conjecture is faithful to the spirit of the Bombieri-Lang conjecture, saying the set of rational points on an algebraic variety of general type is not Zariski dense but contained in its proper algebraic subvarieties. The Zilber-Pink conjecture was proved for the curve case by G. Maurin [9], but is widely open for other cases.

The following theorem is due to Habegger and Bombieri-Masser-Zannier [5]:

**Theorem 1.2 (Bombieri-Habegger-Masser-Zannier).** Let $\mathcal{X}$ be an $r$-dimensional irreducible variety in $G^{r+s}$ defined over $\overline{\mathbb{Q}}$ and $X^{\text{coa}}$ be the remains of $\mathcal{X}$ after removing all anomalous subvarieties of $\mathcal{X}$ of positive dimensions. Then

$$\bigcup_{\dim H = s-1} X^{\text{coa}} \cap H$$

is finite.
The above theorem implies that the set of torsion anomalous points of $X$ lies in its anomalous subvarieties of nontrivial dimensions possibly except for finitely many of them. By the work of Bombieri-Masser-Zannier, it is known that $X^{oa}$ is a Zariski open subset of $X$ (Theorem 2.8). Thus if $X^{oa} \neq \emptyset$ and further $X$ has only finitely many maximal anomalous subvarieties $\{Y_i\}_{1 \leq i \leq n}$, the Zilber-Pink conjecture for $X$ is reduced to the same conjecture over $\{Y_i\}_{1 \leq i \leq n}$. Of course $X^{oa} = \emptyset$ is also possible and Theorem 1.2 tells us nothing in this case.

The goal of this paper is to study the structure of anomalous subvarieties of a special type of algebraic varieties so called holonomy varieties of hyperbolic 3-manifolds. The holonomy variety $X$ of a cusped hyperbolic 3-manifold $M$ is defined as

$$\text{Hom} \left( \pi_1(M) , SL_2\mathbb{C} \right) / \sim$$

with a special choice of coordinates related to the geometric structures of the cusps of $M$. The subject has been studied in great detail, as it provides much topological information of $M$. In the paper, assuming a condition on $M$, we classify the maximal anomalous subvarieties of $X$ containing the identity and find a necessary condition for $M$ to satisfy $X^{oa} = \emptyset$. In particular, we link these classification and condition with well-known geometric concepts in the field.

Our first main result is the following:

**Theorem 1.3.** Let $M$ be an $n$-cusped ($n \geq 2$) hyperbolic 3-manifold having rationally independent cusp shapes and $X$ be its holonomy variety. Then a maximal anomalous subvariety of $X$ containing the identity is its subvariety of codimension 1 obtained by keeping one of the cusps of $M$ complete.

See Definition 2.3 for the precise meaning of the assumption on $M$. If a cusp of $M$ allows the complete hyperbolic metric, two coordinate functions over $X$ associated to the cusp are fixed by 1 and they together determine an anomalous subvariety of $X$ of codimension 1. This is one of the simplest types of the anomalous subvarieties of $X$ naturally arisen. We provide a further detailed account of this in Section 2.1.

The next is our second main theorem:

**Theorem 1.4.** Let $M$ and $X$ be the same as in the above theorem. If $X^{oa} = \emptyset$, then there exist cusps of $M$, keeping some other cusps complete, strongly geometrically isolated (SGI) from the rest cusps of $M$.

Strong geometric isolation \(\text{SGI}\) was first introduced by W. Neumann and A. Reid in [10]. Simply put, it means there exists a set of cusps of a manifold moves independently without affecting the rest cusps. In the case, its holonomy variety $X$ is represented as the product of two varieties of lower dimensions, that is,

$$X = X_1 \times X_2,$$

and so one easily finds $X^{oa} = \emptyset$. (See Theorem 2.5) However, $X^{oa} = \emptyset$ does not necessarily mean SGI but a slightly weaker version of SGI according to Theorem 1.4. See Section 2.2 for the definitions of SGI and its generalization as well as further discussions around them.

Theorem 1.4 is an extension of the following theorem proved by the author in his thesis:

\[\text{For simplicity, let us still denote this by “SGI”}\].
Theorem 1.5. Let \( M \) be a 2-cusped hyperbolic 3-manifold having rationally independent cusp shapes. Then \( X^{oa} = \emptyset \) if and only if two cusps of \( M \) are SGI each other.

Moreover, we also prove the following theorem, which completely classifies \( X^{oa} = \emptyset \) for the 2-cusped case:

**Theorem 1.6.** Let \( M \) be a 2-cusped hyperbolic 3-manifold and \( \mathcal{X} \) be its holonomy variety. If \( \mathcal{X}^{oa} = \emptyset \), then \( \mathcal{X} \) is the product of two algebraic curves.

As a byproduct of Theorem 1.6, we resolve the Zilber-Pink conjecture for the holonomy variety of any 2 cusped hyperbolic 3-manifold. This is our last main result.

**Theorem 1.7.** Let \( M \) be a 2-cusped hyperbolic 3-manifold and \( \mathcal{X} \) be its holonomy variety. Then the Zilber-Pink conjecture is true for \( \mathcal{X} \).

Using the above theorem, in [8], we further prove a weak version of the Zilber-Pink conjecture for \( \mathcal{X} \times \mathcal{X} \) and apply it to classify Dehn fillings of \( M \). Here \( M \) and \( \mathcal{X} \) are the same as in Theorem 1.7.

The general structure theorem of anomalous subvarieties of an arbitrary algebraic variety was given by Bombieri-Masser-Zannier in [2] (see Theorem 2.8). We use this theorem as a key player to attain our main results. Also, instead of holonomy varieties, by taking logarithm to each coordinate, we work in the context of holomorphic functions. In particular, various symmetric properties of Neumann-Zagier potential functions (see Theorem 2.1) play crucial roles throughout the proofs. The basic ideas of the proofs are elementary, primarily based on linear algebra, and many parts of the proofs are relying on computational methods with a variety of interesting aspects.

We finally remark that if \( M \) has cusps SGI from the rest, as mentioned earlier, its holonomy variety \( \mathcal{X} \) satisfies \( \mathcal{X}^{oa} = \emptyset \) and is of the form \( \mathcal{X}_1 \times \mathcal{X}_2 \). Although nothing is obtained from Theorem 1.2 in this case, it is possible to approach the Zilber-Pink conjecture by induction on dimensions of the varieties, and this is exactly how we prove Theorem 1.7. We hope the observation here would help to resolve the Zilber-Pink conjecture fully for holonomy varieties of any hyperbolic 3-manifolds in future work.

1.2. Acknowledgement

The first main result, Theorem 1.3, was announced in [7]. However, as [7] is not intended for publication in a journal, we reproduce it here again. Other two main results, Theorems 1.4 and 1.7, are new and have never appeared in other papers. The author would like to thank Stephan Tillmann and an anonymous referee for many valuable comments on earlier drafts of the paper.

2. Background

2.1. Holonomy variety

Let \( \mathcal{M} \) be an \( n \)-cusped hyperbolic 3-manifold. Let \( T_i \) be a torus cross-section of the \( i \)th-cusp and \( m_i, l_i \) be the chosen meridian-longitude pair of \( T_i \) (\( 1 \leq i \leq n \)). Then the holonomy variety of \( \mathcal{M} \) with respect to \( m_i, l_i \) (\( 1 \leq i \leq n \)) is

\[
\text{Hom} \left( \pi_1(\mathcal{M}), SL_2 \mathbb{C} \right) / \sim,
\]

(2.1)
parametrized by $M_i, L_i$, the derivatives of the holonomies of $m_i, l_i$ respectively ($1 \leq i \leq n$).\footnote{For $n = 1$, this is the so called A-polynomial of $\mathcal{M}$ introduced in \cite{A-polynomial}. See \cite{A-polynomial} for more detailed descriptions of the holonomy variety.} In general, (2.1) has several irreducible components, but we are only interested in a so-called geometric component of it. It is known that a geometric component of (2.1) is an $n$-dimensional algebraic variety in $\mathbb{C}^{2n} := (M_1, L_1, \ldots, M_n, L_n)$ and contains $(1, \ldots, 1)$ which gives rise to the complete hyperbolic metric structure of $\mathcal{M}$. We denote the component by $\mathcal{X}$ and, by abuse of notation, still call it the holonomy variety of $\mathcal{X}$.

Let

$$u_i := \log M_i \quad (1 \leq i \leq n) \quad (2.2)$$

$$v_i := \log L_i \quad (1 \leq i \leq n). \quad (2.3)$$

Then, for each $i$ ($1 \leq i \leq n$), the following statements hold in a neighborhood of the origin in $\mathbb{C}^n$ with $u_1, \ldots, u_n$ as coordinates $\mathbb{C}$:

**Theorem 2.1 (Neumann-Zagier).**

1. $v_i = u_i \cdot \tau_i(u_1, \ldots, u_n)$ where $\tau_i(u_1, \ldots, u_n)$ is a holomorphic function with $\tau_i(0, \ldots, 0) = \tau_i \in \mathbb{C}\setminus\mathbb{R}$ ($1 \leq i \leq n$).

2. There is a holomorphic function $\Phi(u_1, \ldots, u_n)$ such that $v_i = \frac{1}{2} \frac{\partial \Phi}{\partial u_i} (1 \leq i \leq n)$ and $\Phi(0, \ldots, 0) = 0$.

3. $\Phi(u_1, \ldots, u_n)$ is even in each argument and so its Taylor expansion is of the following form:

$$\Phi(u_1, \ldots, u_n) = (\tau_1 u_1^2 + \cdots + \tau_n u_n^2) + (m_{2,0} u_1^4 + \cdots + m_{0,\ldots,4} u_n^4) + \text{(higher order)}.$$ 

We call $\tau_i$ the cusp shape of $T_i$ with respect to $m_i, l_i$ and $\Phi(u_1, \ldots, u_n)$ the Neumann-Zagier potential function of $\mathcal{M}$ with respect to $m_i, l_i$ ($1 \leq i \leq n$).

**Definition 2.2.** The complex manifold defined locally near $(0, \ldots, 0) \in \mathbb{C}^{2n} := (u_1, v_1, \ldots, u_n, v_n)$ via the following holomorphic functions

$$v_i = u_i \cdot \tau_i(u_1, \ldots, u_n) \quad (1 \leq i \leq n) \quad (2.4)$$

is called the analytic holonomy variety of $\mathcal{M}$ and denoted by $\log \mathcal{X}$.

Clearly $\log \mathcal{X}$ is locally biholomorphic to a small neighborhood of $(1, \ldots, 1)$ in $\mathcal{X}$. Let $H$ be an algebraic subgroup in $(\mathbb{C}^*)^{2n}$ defined by\footnote{For convenience of later reference,}

$$M_1^{a_1} L_1^{b_1} \cdots M_n^{a_n} L_n^{b_n} = 1 \quad (1 \leq i \leq m). \quad (2.5)$$

In general, $\mathcal{X} \cap H$ may have several irreducible components but, since we are only concerned about the one containing the identity, we always mean $\mathcal{X} \cap H$ that component unless otherwise stated. Taking logarithm to each coordinate, (2.5) is equivalent to

$$a_{11} u_1 + b_{11} v_1 + \cdots + a_{1n} u_n + b_{1n} v_n = 0 \quad (1 \leq i \leq m),$$

and, locally near the identity, $\mathcal{X} \cap H$ is biholomorphic to the complex manifold defined by

$$a_{11} u_1 + b_{11} u_1 \tau_1(u_1, \ldots, u_n) + \cdots + a_{1n} u_n + b_{1n} u_n \tau_n(u_1, \ldots, u_n) = 0 \quad (1 \leq i \leq m). \quad (2.6)$$

Let $H$ be an algebraic subgroup in $(\mathbb{C}^*)^{2n}$ defined by\footnote{For convenience of later reference,}

$$a_{11} b_{11} \cdots a_{1n} b_{1n} \cdots a_{m1} b_{l1} \cdots a_{mn} b_{mn}$$

is said to be the coefficient matrix of $H$.\footnote{For convenience of later reference,}
The dimension of $\mathcal{X} \cap H$ is obtained by computing the rank of the Jacobian
\[
\begin{pmatrix}
  a_{11} + \tau_1 b_{11} & \cdots & a_{1n} + \tau_n b_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{m1} + \tau_1 b_{l1} & \cdots & a_{mn} + \tau_n b_{mn}
\end{pmatrix}
\] (2.7)
of (2.6) at $(0, \ldots, 0)$. We denote the complex manifold defined in (2.6) by $\log(\mathcal{X} \cap H)$ and call (2.7) simply the Jacobian of $\log(\mathcal{X} \cap H)$.

By Theorem 2.5 (1), for each $i$ ($1 \leq i \leq n$), $u_i = 0$ if and only if $v_i = 0$, and this implies
\[
\mathcal{X} \cap (M_i = L_i = 1) \tag{2.8}
\]
is an anomalous subvariety of $\mathcal{X}$. Geometrically, $M_i = L_i = 1$ corresponds to the complete hyperbolic structure of the $i$-th cusp of $\mathcal{M}$. We call (2.8) the anomalous subvariety of $\mathcal{X}$ obtained by keeping the $i$-th cusp of $\mathcal{M}$ complete.

The following definition appeared in the statements of Theorems 1.3 and 1.4.

**Definition 2.3.** Let $\mathcal{M}$ be an $n$-cusped manifold and $\tau_1, \ldots, \tau_n$ be its cusp shapes. We say $\mathcal{M}$ has rationally independent cusp shapes if the elements in
\[
\{\tau_1 \cdots \tau_i \mid 1 \leq i_1 < \cdots < i_l \leq n\}
\]
are linearly independent over $\mathbb{Q}$.

### 2.2. Geometric isolation

The following is one of the equivalent definitions of SGI given in [10]:

**Definition 2.4.** Let $\mathcal{M}$ be an $n$-cusped hyperbolic 3-manifold. We say cusps $1, \ldots, k$ are strongly geometrically isolated (SGI) from cusps $k + 1, \ldots, n$ if $v_1, \ldots, v_k$ only depend on $u_1, \ldots, u_k$ and not on $u_{k+1}, \ldots, u_n$.

For instance, for a given 2-cusped hyperbolic 3-manifold $\mathcal{M}$ with its holonomy variety $\mathcal{X}$, if $\log \mathcal{X}$ is defined by
\[
v_1 = u_1 \tau_1(u_1), \quad v_2 = u_2 \tau_2(u_2),
\]
then two cusps of $\mathcal{M}$ are SGI each other.

The following theorem is proved easily.

**Theorem 2.5.** Let $\mathcal{M}$ be an $n$-cusped hyperbolic 3-manifold and $\mathcal{X}$ be its holonomy variety. If $\mathcal{M}$ has cusps which are SGI from the rest, then $\mathcal{X}^{\omega\omega} = \emptyset$.

**Proof.** Without loss of generality, suppose cusps $1, \ldots, k$ ($k < n$) are SGI from the rest. Since each $v_i(u_1, \ldots, u_n)$ ($1 \leq i \leq k$) (resp. $v_j(u_1, \ldots, u_n)$ ($k + 1 \leq j \leq n$)) depends only on $u_1, \ldots, u_k$ (resp. $u_{k+1}, \ldots, u_n$), the holonomy variety $\mathcal{X}$ is of the form $\mathcal{X}_1 \times \mathcal{X}_2$ in $\mathbb{G}^{2k} \times \mathbb{G}^{2n-2k}$. Let $H(\xi_1^m, \ldots, \xi_k^l)$ be an algebraic coset defined by
\[
M_1 = \xi_1^m, \quad L_1 = \xi_1^l, \quad \ldots, \quad M_k = \xi_k^m, \quad L_k = \xi_k^l \tag{2.9}
\]
where $(\xi_1^m, \ldots, \xi_k^l) \in \mathcal{X}_1$. Then clearly $H(\xi_1^m, \ldots, \xi_k^l) \cap \mathcal{X}$ is isomorphic to $\mathcal{X}_2$ for each $(\xi_1^m, \ldots, \xi_k^l) \in \mathcal{X}_1$ and, as
\[
\dim \mathcal{X}_2 = n - k > \dim H(\xi_1^m, \ldots, \xi_k^l) + \dim \mathcal{X} - 2n = (2n - 2k) + (n) - (2n) = n - 2k,
\]
\[4\text{Note that the definition is independent of the choice of } m_i, l_i \ (1 \leq i \leq n).\]
\(H(\xi_1^{\prime}, \ldots, \xi_l^{\prime}) \cap \mathcal{X}\) is an anomalous subvariety of \(\mathcal{X}\). Moreover,

\[
\bigcup_{(\xi_1^{\prime}, \ldots, \xi_l^{\prime}) \in \mathcal{X}_1} (H(\xi_1^{\prime}, \ldots, \xi_l^{\prime}) \cap \mathcal{X}) = \mathcal{X},
\]

implying \(\mathcal{X}^{oa} = \emptyset\).

However, the opposite direction of the above theorem is not true in general. For instance, suppose there exists a 3-cusped hyperbolic 3-manifold \(\mathcal{M}\) whose Neumann-Zagier potential function \(\Phi(u_1, u_2, u_3)\) is given as

\[
\Phi(u_1, u_2, u_3) = \sum_{i, even} a_1 u_i^1 + \sum_{i, even} a_2 u_i^2 + \sum_{i, even} a_3 u_i^3 + \sum_{i, j, even} b_{i,j} u_i^1 u_j^2 + \sum_{k, l, even} c_{k,l} u_k^1 u_l^3,
\]

and thus \(\log \mathcal{X}\) is defined by

\[
\begin{align*}
    v_1 &= \frac{1}{2} \left( \sum_{i, even} i a_1 u_i^1 + \sum_{i,j, even} i b_{i,j} u_i^1 u_j^2 + \sum_{k, l, even} k c_{k,l} u_k^1 u_l^3 \right), \\
    v_2 &= \frac{1}{2} \left( \sum_{i, even} i a_2 u_i^2 + \sum_{i,j, even} j b_{i,j} u_i^1 u_j^2 \right), \\
    v_3 &= \frac{1}{2} \left( \sum_{i, even} i a_3 u_i^3 + \sum_{k, l, even} l c_{k,l} u_k^1 u_l^3 \right),
\end{align*}
\]

For \(\xi_1, \xi_2 \in \mathbb{C}\) sufficiently close to 0,

\[
(u_1 = \xi_1, u_2 = \xi_2, v_2 = v_2(\xi_1, \xi_2)) \cap \log \mathcal{X}
\]

is a 1-dimensional analytic subset of \(\log \mathcal{X}\). Equivalently, if \(H(\xi_1, \xi_2)\) is an algebraic coset defined by

\[
M_1 = e^{i \xi_1}, \quad M_2 = e^{i \xi_2}, \quad L_2 = e^{i v_2(\xi_1, \xi_2)}
\]

then \(\mathcal{X} \cap H(\xi_1, \xi_2)\) is a 1-dimensional anomalous subvariety of \(\mathcal{X}\). Clearly,

\[
\bigcup_{(\xi_1, \xi_2) \in \mathbb{C}^2} \left( (u_1 = \xi_1, u_2 = \xi_2, v_2 = v_2(\xi_1, \xi_2)) \cap \log \mathcal{X} \right) = \log \mathcal{X}
\]

and thus \(\mathcal{X}^{oa} = \emptyset\). However obviously none of the cusps of \(\mathcal{M}\) are SGI from the rest.

Inspired by this, we further refine and generalize (2.4) as below.

**Definition 2.6.** Let \(\mathcal{M}\) be an \(n\)-cusped hyperbolic 3-manifold \((n \geq 3)\). Suppose \(k, l\) are integers such that \(0 < k < l \leq n\). We say that cusps \(1, \ldots, k\) are weakly geometrically isolated (WGI) from cusps \(k + 1, \ldots, l\) if each

\[
v_i(u_1, \ldots, u_i, 0, \ldots, 0) \quad (1 \leq i \leq k)
\]

depends only on \(u_1, \ldots, u_k\) not on \(u_{k+1}, \ldots, u_l\). In other words, keeping cusps \(l + 1, \ldots, n\) complete, if cusps \(1, \ldots, k\) are SGI from cusps \(k + 1, \ldots, l\), then we say cusps \(1, \ldots, k\) are WGI from cusps \(k + 1, \ldots, l\).

For instance, if \(u_1 = 0\) in (2.10), it is reduced to

\[
v_2 = \frac{1}{2} \sum_{i, even} i a_2 u_i^2, \quad v_3 = \frac{1}{2} \sum_{i, even} i a_3 u_i^3.
\]
and therefore the second cusp is WGI from the third cusp in the example.

Using Definition 2.6, now Theorem 1.4 is simply restated as follows:

**Theorem 2.7.** Let $\mathcal{M}$ and $\mathcal{X}$ be the same as in Theorem 1.3. If

$$\mathcal{X}^{oa} = \emptyset,$$

then there exist cusps of $\mathcal{M}$ which are WGI from other cusps of $\mathcal{M}$.

### 2.3. Structure theorem

The following theorem tells us the structure of anomalous subvarieties of an algebraic variety (Theorem 1.4 in [2]).

**Theorem 2.8** (Bombieri-Masser-Zannier). Let $\mathcal{X}$ be an irreducible variety in $\mathbb{G}^n$ of positive dimension defined over $\overline{\mathbb{Q}}$.

(a) For any torus $H$ with

$$1 \leq n - \dim H \leq \dim \mathcal{X},$$

the union $Z_H$ of all subvarieties $\mathcal{Y}$ of $\mathcal{X}$ contained in any coset $K$ of $H$ with

$$\dim H = n - (1 + \dim \mathcal{X}) + \dim \mathcal{Y}$$

is a closed subset of $\mathcal{X}$.

(b) There is a finite collection $\Psi = \Psi_\mathcal{X}$ of such tori $H$ such that every maximal anomalous subvariety $\mathcal{Y}$ of $\mathcal{X}$ is a component of $\mathcal{X} \cap gH$ for some $H$ in $\Psi$ satisfying (2.13) and (2.14) and some $g$ in $Z_H$. Moreover $\mathcal{X}^{oa}$ is obtained from $\mathcal{X}$ by removing the $Z_H$ of all $H$ in $\Psi$, and thus it is Zariski open in $\mathcal{X}$.

Examples explaining the above theorem were already provided in the previous subsection. For $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2(\subset \mathbb{G}^{2k} \times \mathbb{G}^{2n-2k})$ given in the proof of Theorem 2.5, as each $L_i$ (1 ≤ $i$ ≤ $k$) depends only on $M_1, \ldots, M_k$ over $\mathcal{X}_1$, if $H$ is an algebraic subgroup defined by

$$M_1 = \cdots = M_k = L_i = 1 \quad (1 \leq i \leq k),$$

then one can check $Z_H = \mathcal{X}$.

If $Z_H = \mathcal{X}$ (and so $\mathcal{X}^{oa} = \emptyset$), we say $\mathcal{X}$ is foliated by anomalous subvarieties contained in

$$\bigcup_{g \in Z_H} \mathcal{X} \cap gH$$

or algebraic cosets of $H$.

### 3. Maximal anomalous subvarieties of $\mathcal{X}$

#### 3.1. Preliminary lemmas

Before proving Theorem 1.3, we first show a couple of lemmas that will play key roles in the proof of the theorem. The proofs of these lemmas are elementary, mostly based on linear algebra, but they are fairly long. So we would like to recommend a reader to skip ahead the proofs in this section at first reading.

Let us start with the following definition:
Definition 3.1. Let \( \tilde{V} \) be a vector space and \( V = \{v_1, \ldots, v_n\} \) be a basis of \( \tilde{V} \). We say \( v \in \tilde{V} \) is interchangeable with \( v_i \) (in \( \tilde{V} \)) if 
\[
\{v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n\}
\]
is a basis of \( \tilde{V} \). Similarly we say \( A(\subset \tilde{V}) \) is interchangeable with \( B(\subset V) \) if 
\[
(V \setminus B) \cup A
\]
is a basis of \( \tilde{V} \).

For example, if \( \tilde{V} \) is a vector space generated by \( v_1, v_2, v_3 \), then \( v_1 + v_2 \) is interchangeable with either \( v_1 \) or \( v_2 \) in \( \tilde{V} \).

The following lemma is proved easily.

Lemma 3.2. Let \( \tilde{V}_1 \subseteq \cdots \subseteq \tilde{V}_m \) be a sequence of vector spaces. Let 
\[
V_1 := \{v_1, \ldots, v_{h_1}\}
\]
and 
\[
V_i \cup \{v_{h_1+1}, \ldots, v_{h_2}\}
\]
be bases of \( \tilde{V}_i \) and \( \tilde{V}_2 \) respectively. Inductively, we let 
\[
V_{i+1} := \{v_{h_i+1}, \ldots, v_{h_{i+1}}\},
\]
and suppose 
\[
V_1 \cup \cdots \cup V_{i+1}
\]
is a basis of \( \tilde{V}_{i+1} \). For each \( 1 \leq i \leq m \), if there exist \( v_{n_i} \in V_i \) and \( v'_{n_i} \in \tilde{V}_i \) such that \( v_{n_i} \) is interchangeable with \( v'_{n_i} \) (in \( \tilde{V}_i \)), then 
\[
\{v'_{n_1}, \ldots, v'_{n_m}\}
\]
is interchangeable with 
\[
\{v_{n_1}, \ldots, v_{n_m}\}
\]
in \( \tilde{V}_m \).

Proof. Rearranging if necessary, we simply assume 
\[
v_{n_i} = v_{h_i}
\]
for each \( 1 \leq i \leq m \). We also represent each \( v'_{n_i} \) as 
\[
v'_{n_i} = \sum_{j=1}^{h_i} a_{ij}v_j
\]
where \( a_{ih_i} \neq 0 \) for each \( i \). Then the matrix representation of the linear transformation from \( \bigcup_{i=1}^m V_i \) to 
\[
(V_1 \cup \cdots \cup V_m \cup \{v'_{n_1}, \ldots, v'_{n_m}\}) \setminus \{v_{n_1}, \ldots, v_{n_m}\}
\]
with respect to the basis 
\[
V_1 \cup \cdots \cup V_m
\]
is a triangular form with the determinant \( \prod_{i=1}^m a_{ih_i} \neq 0 \). Therefore \( \text{[3.2]} \) is a basis of \( \tilde{V}_m \). □

The following lemma is central and repeatedly used in the proof of Theorem 1.3.
Lemma 3.3. Let
\[ \{v_1, w_1, \ldots, v_n, w_n\} \] (3.3)
be a set of vectors in \( \mathbb{Q}^n \). Suppose, for any subset
\[ \{u_1, \ldots, u_n\} \] (3.4)
of (3.3) where \( u_i = v_i \) or \( w_i \) \((1 \leq i \leq n)\), the vectors in (3.4) are linearly dependent. Then there exists \( \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} \) such that the dimension of the vector space spanned by
\[ \{v_{i_1}, w_{i_1}, \ldots, v_{i_m}, w_{i_m}\} \] (3.5)
is at most \( m \).

Note that \( v_i \) or \( w_i \) could be the zero vector. For instance, if \( v_i = w_i = 0 \), for some \( i \), we get the desired result simply by letting (3.5) be \( \{v_i, w_i\} \).

For each \( i \) \((1 \leq i \leq n)\), we call \( v_i \) (resp. \( w_i \)) the counter vector of \( w_i \) (resp. \( v_i \)).

Proof. Let
\[ U = \{u_{i_1}, \ldots, u_{i_h}\} \] (3.6)
be a subset of (3.3) satisfying
1. \( u_i = v_i \) or \( w_i \) for each \( i \in \{i_1, \ldots, i_h\} \);
2. the vectors in (3.6) are linearly independent over \( \mathbb{Q} \);
3. the cardinality of \( U \) is the biggest among all the subsets of (3.3) satisfying (1) and (2).

Let
\[ U' = \{u'_{j_1}, \ldots, u'_{j_k}\} \] (3.7)
be another subset of (3.3) associated with \( U \) satisfying
1. \( \{j_1, \ldots, j_k\} \subseteq \{i_1, \ldots, i_h\}\);
2. \( u'_j \) := \( w_j \) (resp. \( v_j \)) if \( u'_j := v_j \) (resp. \( w_j \)) for each \( j \in \{j_1, \ldots, j_k\} \);
3. \( u_{i_1}, \ldots, u_{i_h}, u'_{j_1}, \ldots, u'_{j_k} \) are linearly independent over \( \mathbb{Q} \);
4. the cardinality of \( U' \) is the biggest among all the sets satisfying (1), (2), and (3).
(There may be several different choices for \( U \) and \( U' \), but we choose one of them.) Rearranging \( v_i, w_i \) if necessary, we assume
\[ U = \{v_1, \ldots, v_h\} \]
and
\[ U' = \{w_1, \ldots, w_k\} \]
where \( k \leq h < n \). Instead of \( U \) and \( U' \), from now on, let us denote them by \( V_H \) and \( W_K \) respectively. Also the vector spaces spanned by \( V_H \) and \( W_K \) are denoted by \( \tilde{V}_H \) and \( \tilde{W}_K \) respectively.

Claim 3.4. If \( h = k \), then \( v_i = w_i = 0 \) for \( h + 1 \leq i \leq n \).

Proof. Suppose \( v_i \neq 0 \) for some \( i \) \((h + 1 \leq i \leq n)\). By the definition of \( V_H \), \( v_i \in \tilde{V}_H \). Since \( w_1, \ldots, w_h \) are linearly independent vectors not contained in \( \tilde{V}_H \),
\[ w_1, \ldots, w_h, v_i \]
are linearly independent. But this contradicts the assumption on \( h \). Similarly \( w_i = 0 \) for all \( h + 1 \leq i \leq n \). \( \square \)
Thus if $h = k$, by letting (3.5) be
\[ \{ v_{h+1}, w_{h+1}, \ldots, v_n, w_n \} , \]
we get the desired result.

Now suppose $h > k$, and let $V_{H \setminus K} := \{ v_{k+1}, \ldots, v_h \}$
and $\overline{V}_{H \setminus K}$ be the vector space spanned by $V_{H \setminus K}$. We denote $\{ v_{h+1}, \ldots, v_n \}$ and $\{ w_{h+1}, \ldots, w_n \}$
by $V_{N \setminus H}$ and $W_{N \setminus H}$ respectively. Likewise, $\overline{V}_{N \setminus H}$ and $\overline{W}_{N \setminus H}$ represent the vector spaces
spanned by $V_{N \setminus H}$ and $W_{N \setminus H}$ respectively.

**Claim 3.5.** $V_{N \setminus H}, W_{N \setminus H} \subset \overline{V}_{H \setminus K}$.

**Proof.** Suppose there exists $v \in V_{N \setminus H}$ such that $v \notin \overline{V}_{H \setminus K}$.

1. If $v \in \overline{V}_H$, since $v \notin \overline{V}_{H \setminus K}$, the vectors in
   \[ V_{H \setminus K} \cup \{ v \} \]
   are linearly independent vectors in $\overline{V}_H$. Since $\overline{V}_H \cap \overline{W}_K = \{ 0 \}$,
   \[ W_K \cup V_{H \setminus K} \cup \{ v \} \]
   is a set of $(h + 1)$-linearly independent vectors, which contradicts the assumption
   on $h$.
2. If $v \notin \overline{V}_H$, the vectors in
   \[ V_H \cup \{ v \} \]
   are linearly independent, again contradicting the assumption on $h$.

Similarly, one can show $w \in \overline{V}_{H \setminus K}$ for all $w \in W_{N \setminus H}$. \qed

Let $V_1$ be the largest subset\footnote{Since both $V_{N \setminus H}$ and $W_{N \setminus H}$ are in $\overline{V}_{H \setminus K}$ by Claim 3.5, $V_1$ is well-defined.} of $V_{H \setminus K}$ such that every element of $V_1$ is interchangeable
either with a vector of $V_{N \setminus H}$ or $W_{N \setminus H}$ in $\overline{V}_{H \setminus K}$.

If $V_1 = \emptyset$, none of the vectors in $V_{N \setminus H} \cup W_{N \setminus H}$ is interchangeable with any vector of
$V_{H \setminus K}$ in $\overline{V}_{H \setminus K}$, implying
\[ \overline{V}_{N \setminus H} \cup \overline{W}_{N \setminus H} = \{ 0 \} . \]
That is, every vector in $V_{N \setminus H} \cup W_{N \setminus H}$ is the zero vector. Therefore, we get the desired
result by letting (3.5) be $V_{N \setminus H} \cup W_{N \setminus H}$.

Now suppose $V_1 \neq \emptyset$ and, rearranging if necessary, let
\[ V_1 := \{ v_{h_1}, \ldots, v_h \} \quad (h_1 \leq h) \quad (3.8) \]
and $W_1$ be the set of the counter vectors of the vectors in $V_1$. (See Table 1.)

We have the following two claims:

**Claim 3.6.** $\overline{V}_{N \setminus H}, \overline{W}_{N \setminus H} \subset \overline{V}_1$.
Proof. By Claim 3.5, for \( v \in V_{N \setminus H} \), there exist \( a_{k+1}, \ldots, a_h \in \mathbb{Q} \) such that
\[
v = a_{k+1}v_{k+1} + \cdots + a_hv_h.
\]
Note that \( a_j \neq 0 \) \( (k+1 \leq j \leq h) \) if and only if \( v_j \) is interchangeable with \( v \) in \( \widetilde{V}_{H \setminus K} \). By the definition of \( V_1 \), if \( v_j \) is interchangeable with \( v \), then \( v_j \in V_1 \) and so
\[
a_j = 0 \quad (k + 1 \leq j \leq h_1 - 1).
\]
In other words,
\[
v = a_{h_1}v_{h_1} + \cdots + a_hv_h \in \widetilde{V}_1.
\]
Similarly one can show \( w \in \widetilde{V}_1 \) for \( w \in W_{N \setminus H} \).

Claim 3.7. \( W_1 \subset \widetilde{V}_{H \setminus K} \).

Proof. Suppose \( w_1 \notin \widetilde{V}_{H \setminus K} \) for some \( w_1 \in W_1 \) (where \( h_1 \leq i \leq h \)). For \( v_i \in V_1 \), by the definition of \( V_1 \), there exists \( j \) \( (h_1 \leq j \leq n) \) such that either \( v_j \) or \( w_j \) is interchangeable with \( v_i \) in \( \widetilde{V}_{H \setminus K} \). Without loss of generality, we assume \( v_j \) is the one.

1. If \( w_j \in \widetilde{V}_H \), since \( v_j \) is interchangeable with \( v_i \) in \( \widetilde{V}_{H \setminus K} \) and \( w_1 \notin \widetilde{V}_{H \setminus K} \),
\[
(V_{H \setminus K} \setminus \{v_i\}) \cup \{w_1, v_j\}
\]
is a set of linearly independent vectors in \( \widetilde{V}_H \). Since \( \widetilde{V}_H \cap \widetilde{V}_K = \{0\} \), the following \((h+1)\)-vectors in
\[
W_K \cup (V_{H \setminus K} \setminus \{v_i\}) \cup \{w_1, v_j\}
\]
are linearly independent. But this contradicts the assumption on \( h \).

2. Suppose \( w_j \notin \widetilde{V}_H \). In this case,
\[
V_K \cup (V_{H \setminus K} \setminus \{v_i\}) \cup \{w_1, v_j\}
\]
is a set of \((h+1)\)-linearly independent vectors. But this again contradicts the assumption on \( h \).

Let \( V_2 \) be the largest subset of \( V_{H \setminus K} \setminus V_1 \) such that every element in \( V_2 \) is interchangeable with some vector of \( W_1 \) in \( \widetilde{V}_{H \setminus K} \).

If \( V_2 = \emptyset \), none of the vectors in \( W_1 \) is interchangeable with a vector in \( V_{H \setminus K} \setminus V_1 \). Since \( W_1 \subset \widetilde{V}_{H \setminus K} \) (Claim 3.7), this implies \( W_1 \subset \widetilde{V}_1 \). Thus the rank of
\[
V_1 \cup W_1
\]
is at most \( |V_1| = |W_1| \). We get the desired result by letting \( (3.5) \) be \( (3.9) \).

Now assume \( V_2 \neq \emptyset \) and, rearranging if necessary, let
\[
V_2 := \{v_{h_2}, \ldots, v_{h_1-1}\} \quad (h_2 \leq h_1 - 1).
\]
Let \( W_2 \) be the set of the counter vectors of the vectors in \( V_2 \) and \( \widetilde{V}_1 \cup \widetilde{V}_2 \) be the vector space spanned by \( V_1 \cup V_2 \).

Claim 3.8. \( W_1 \subset \widetilde{V}_1 \cup \widetilde{V}_2 \).
are linearly independent. Now we split the problem into two cases.

**Lemma 3.2**, the definition of $V$ and Claim 3.6, there exists $\tilde{w}_1 \in \tilde{V}_{H \setminus K}$ such that

$$ w_i = a_{k+1} v_{k+1} + \cdots + a_h v_h $$

(3.11)

for some $a_{k+1}, \ldots, a_h \in \mathbb{Q}$. Note that $a_j \neq 0$ ($k + 1 \leq j \leq h$) if and only if $v_j$ is interchangeable with $w_i$ in $\tilde{V}_{H \setminus K}$. By the definition of $V_2$, if $v_j \in V_2$ is interchangeable with $w_i$ (in $\tilde{V}_{H \setminus K}$), then either $v_j \in V_1$ or $v_j \in V_2$. Thus $a_j = 0$ for all $k + 1 \leq j \leq h_2 - 1$ in (3.11). In other words, $w_i$ belongs to the vector space spanned by $V_1$ and $V_2$, that is, $\tilde{V}_1 \cup \tilde{V}_2$.

**Claim 3.9.** $W_2 \subset \tilde{V}_{H \setminus K}$.

**Proof.** Suppose $w_i \notin \tilde{V}_{H \setminus K}$ for some $i$ ($h_2 \leq i \leq h_1 - 1$). By the definition of $V_2$, there exists $w_j \in W_1$ ($h_1 \leq j \leq h$) such that $w_j$ is interchangeable with $v_i$ in $\tilde{V}_1 \cup \tilde{V}_2$. By the definition of $V_1$ and Claim 3.6, there exists $v_1$ or $w_1$ ($h + 1 \leq l \leq n$), which is interchangeable with $v_j$ in $\tilde{V}_2$. Without loss of generality, we assume $v_1$ is the one. By Lemma 3.2, \{w_j, v_1\} is interchangeable with \{v_1, v_j\} in $\tilde{V}_{H \setminus K}$ and so

$$(V_{H \setminus K} - \{v_1, v_j\}) \cup \{w_j, v_1\}$$

is a basis of $\tilde{V}_{H \setminus K}$. Since $w_i \notin \tilde{V}_{H \setminus K}$, the following $(h - k + 1)$-vectors in

$$(V_{H \setminus K} - \{v_1, v_j\}) \cup \{w_j, v_1, w_i\}$$

are linearly independent. Now we split the problem into two cases.

1. If $w_i \in \tilde{V}_H$, then the following $(h + 1)$-vectors in

$$(W_K \cup (V_{H \setminus K} - \{v_1, v_j\}) \cup \{w_j, v_1, w_i\})$$

are linearly independent (since $\tilde{W}_K \cap \tilde{V}_H = \{0\}$). But this contradicts the assumption on $h_1$.

2. If $w_i \notin \tilde{V}_H$, then

$$V_K \cup (V_{H \setminus K} - \{v_1, v_j\}) \cup \{w_j, v_1, w_i\}$$

is a set of $(h + 1)$-linearly independent vectors, again contradicting the assumption on $h$.

\[\square\]
We define $V_3$ to be the largest subset of 
$$V_{H \setminus K} - (V_1 \cup V_2)$$
such that every element of $V_2$ is interchangeable with some vector of $W_2$ in $\tilde{V}_{H \setminus K}$.

If $V_3 = \emptyset$, then, similar to the previous cases, one can show
$$W_2 \subset \tilde{V}_1 \cup \tilde{V}_2,$$
and thus
$$W_1 \cup W_2 \subset \tilde{V}_1 \cup \tilde{V}_2$$
by Claim 3.8. Now we get the desired conclusion by letting (3.5) be
$$V_1 \cup V_2 \cup W_1 \cup W_2.$$

If $V_3 \neq \emptyset$, we continue the above process and define $V_4$ analogously. Since $V_m = \emptyset$ for some $m \in \mathbb{N}$, it eventually leads to the desired result. This completes the proof of Lemma 3.3.

3.2. Codimension 1

As a warm-up, we first treat the simplest case of Theorems 1.3-1.4. We prove the theorems under the assumption that the given anomalous subvarieties are all of codimension 1. The proofs in these cases are not only simpler than the general ones but also showing how the ideas of Theorems 2.1 and 2.8 are applied to get the desired results.

We first prove the following lemma using Lemma 3.3.

**Lemma 3.10.** Let
\[
\begin{pmatrix}
  a_1 & b_1 & c_1 & d_1 \\
  a_2 & b_2 & c_2 & d_2
\end{pmatrix}
\]  
(3.12)
be an integer matrix of rank 2, and $\tau_1, \tau_2$ be algebraic numbers such that $1, \tau_1, \tau_2, \tau_1 \tau_2$ are linearly independent over $\mathbb{Q}$. If the rank of the following $(2 \times 2)$-matrix
\[
\begin{pmatrix}
  a_1 + b_1 \tau_1 & c_1 + d_1 \tau_2 \\
  a_2 + b_2 \tau_1 & c_2 + d_2 \tau_2
\end{pmatrix}
\]  
(3.13)
is equal to 1, then either $c_i = d_i = 0$ or $a_i = b_i = 0$ ($i = 1, 2$). That is, (3.12) is given as either
\[
\begin{pmatrix}
  a_1 & b_1 & 0 & 0 \\
  a_2 & b_2 & 0 & 0
\end{pmatrix}
\]  
or\[
\begin{pmatrix}
  0 & 0 & c_1 & d_1 \\
  0 & 0 & c_2 & d_2
\end{pmatrix}.
\]

**Proof.** Let
\[
v_1 := \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \quad w_1 := \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad v_2 := \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad w_2 := \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.
\]
Since the rank of (3.13) is 1, equivalently, we have
\[
\det \begin{pmatrix}
  u_1 & u_2
\end{pmatrix} = 0
\]
where $u_i = v_i$ or $w_i$ for each $1 \leq i \leq 2$. Thus the rank of either
\[
\begin{pmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
 c_1 & d_1 \\
 c_2 & d_2
\end{pmatrix}
\]
(3.14)
is at most 1 by Lemma 3.3. Without loss of generality, we assume the first case and \(a_2 = b_2 = 0\) by applying the Gauss elimination if necessary. Again, as the determinant of (3.13) is 0, it follows that either \(a_1 + \tau_1 b_1 = 0\) or \(c_2 + d_2 \tau_2 = 0\). That is, \(a_1 = b_1 = 0\) or \(c_2 = d_2 = 0\) in (3.13). If \(c_2 = d_2 = 0\), it contradicts the fact that the rank of (3.12) is 2 and so \(a_1 = b_1 = 0\).

Similarly, if the rank of (3.14) is at most 1, then \(c_i = d_i = 0\) for \(i = 1, 2\).

Using the lemma, we prove a special case of Theorem 1.3.

**Theorem 3.11.** Let \(M\) be and \(X\) be the same as in Theorem 1.3. Let \(H\) be an algebraic subgroup of codimension 2 such that \(X \cap H\) is an anomalous subvariety of \(X\) containing \((1, \ldots, 1)\). Then

\[ X \cap H = X \cap (M_i = L_i = 1) \]
(3.15)
for some \(1 \leq i \leq n\).

**Proof.** Let \(H\) be defined by

\[
M_1^{a_{11}} L_1^{b_{11}} \cdots M_n^{a_{1n}} L_n^{b_{1n}} = 1, \\
M_1^{a_{21}} L_1^{b_{21}} \cdots M_n^{a_{2n}} L_n^{b_{2n}} = 1.
\]
(3.16)
As remarked in Section 2.1, \(X \cap H\) is locally biholomorphic to \(\log(X \cap H)\) defined by

\[
a_{11} u_1 + b_{11} (\tau_1 u_1 + \cdots) + \cdots + a_{1n} u_n + b_{1n} (\tau_n u_n + \cdots) = 0, \\
a_{21} u_1 + b_{21} (\tau_1 u_1 + \cdots) + \cdots + a_{2n} u_n + b_{2n} (\tau_n u_n + \cdots) = 0.
\]
(3.17)
Since \(X \cap H\) is an \((n-1)\)-dimensional variety, (3.17) defines an \((n-1)\)-dimensional complex manifold and thus the rank of

\[
\begin{pmatrix}
 a_{11} + b_{11} \tau_1 & \cdots & a_{1n} + b_{1n} \tau_n \\
 a_{21} + b_{21} \tau_1 & \cdots & a_{2n} + b_{2n} \tau_n
\end{pmatrix}
\]
is equal to 1. By Lemma 3.10 for every \(i \neq j\), we have either

\[ a_{1i} = b_{1i} = a_{2i} = b_{2i} = 0 \]
or

\[ a_{1j} = b_{1j} = a_{2j} = b_{2j} = 0. \]

In other words, (3.16) is reduced to

\[
M_i^{a_{1i}} L_i^{b_{1i}} = 1, \\
M_i^{a_{2i}} L_i^{b_{2i}} = 1
\]
for some \(i\). Since \(a_{1i} b_{2i} - a_{2i} b_{1i} \neq 0\) and \((1, \ldots, 1) \in X \cap H\), we get the desired result. \(\square\)

Using the above theorem, we now prove a special case of Theorem 1.4.

**Theorem 3.12.** Let \(M\) and \(X\) be the same as above. Suppose \(X^{\text{oa}} = \emptyset\) and, further, \(X\) has infinitely many maximal anomalous subvarieties of dimension \(n - 1\). Then \(M\) has a cusp which is SGI from the rest.
Proof. By Theorem 2.8, there exists an algebraic subgroup $H$ of codimension 2 such that those anomalous subvarieties are contained in translations of $H$. By the previous theorem, $H$ is

$$M_i = L_i = 1$$

for some $1 \leq i \leq n$ and, without loss of generality, let us assume $i = 1$. If

$$\mathcal{X} \cap (M_1 = \xi_1, \ L_1 = \xi_2) \ (\xi_i \in \mathbb{C})$$

is an $(n - 1)$-dimensional anomalous subvariety of $\mathcal{X}$, equivalently,

$$u_1 = \log \xi_1, \ v_1 = \log \xi_2$$

is an $(n - 1)$-dimensional analytic subset of $\log \mathcal{X}$. But this is possible if and only if $v_1$ depends solely on $u_1$. That is, the first cusp is SGI from the rest. □

3.3. Proof of Theorem 1.3

Throughout this subsection, let $\mathcal{M}$ and $\mathcal{X}$ be the same as in Theorem 1.3. We first prove Theorem 1.3 under the assumption that the codimension of a given algebraic subgroup is less than or equal to $n$, the dimension of $\mathcal{X}$.

Theorem 3.13. Let $H$ be an algebraic subgroup defined by

$$M_i^{a_{i1}} L_1^{b_{i1}} \cdots M_n^{a_{in}} L_n^{b_{in}} = 1 \quad (1 \leq i \leq m), \quad (3.18)$$

where $n \geq m$. Then $\mathcal{X} \cap H$ is an anomalous subvariety of $\mathcal{X}$ if and only if the rank of

$$\begin{pmatrix}
  a_{11} + \tau_1 b_{11} & \cdots & a_{1n} + \tau_n b_{1n} \\
  \vdots & \ddots & \vdots \\
  a_{m1} + \tau_1 b_{m1} & \cdots & a_{mn} + \tau_n b_{mn}
\end{pmatrix} \quad (3.19)$$

is strictly less than $m$. Moreover, every anomalous subvariety $\mathcal{X} \cap H$ of $\mathcal{X}$ satisfies

$$\mathcal{X} \cap H \subset (M_i = L_i = 1)$$

for some $1 \leq i \leq n$.

Proof. “Only if” direction is clear. Indeed, if the rank of (3.19) is $m$, by the implicit function theorem, $\dim (\log(\mathcal{X} \cap H)) = n - m$ and so $\dim(\mathcal{X} \cap H) = n - m$. But this contradicts the fact that $\mathcal{X} \cap H$ is an anomalous subvariety of $\mathcal{X}$ (i.e. $\dim(\mathcal{X} \cap H) > n - m$).

Now we prove “if” direction. For each fixed $n$, we prove the theorem by induction on $m$. Note that the theorem is true for any $n \geq 2$ and $m = 2$ by Theorem 3.11. Now assume $m \geq 3$ and the statement holds for $2, \ldots, m - 1$. We show that the result is true for $m$ as well.

Since the rank of (3.19) is less than $m$, the determinant of

$$\begin{pmatrix}
  a_{11} + \tau_1 b_{11} & \cdots & a_{1m} + \tau_m b_{1m} \\
  \vdots & \ddots & \vdots \\
  a_{m1} + \tau_1 b_{m1} & \cdots & a_{mm} + \tau_m b_{mm}
\end{pmatrix} \quad (3.20)$$

is 0. Since $\mathcal{M}$ has rationally independent cusp shapes by the assumption, if

$$v_1 := \begin{pmatrix}
  a_{i1} \\
  \vdots \\
  a_{mi}
\end{pmatrix}, \quad w_i := \begin{pmatrix}
  b_{i1} \\
  \vdots \\
  b_{mi}
\end{pmatrix}, \quad 1 \leq i \leq m$$

then...
then
\[
\det \begin{pmatrix}
| & | \\
u_1 & \cdots & u_m \\
| & | \\
\end{pmatrix} = 0
\]

where \(u_i = v_i\) or \(w_i\) for each \(1 \leq i \leq m\). By Lemma 3.3, we get
\[
\begin{align*}
\{i_1, \ldots, i_l\} & \subseteq \{1, \ldots, m\} \quad (l < m)
\end{align*}
\]

such that the rank of
\[
\begin{pmatrix}
a_{i_1 i_1} & b_{i_1 i_1} & \cdots & a_{i_1 i_l} & b_{i_1 i_l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{i_l i_1} & b_{i_l i_1} & \cdots & a_{i_l i_l} & b_{i_l i_l} \\
0 & 0 & \cdots & a_{(l+1)(l+1)} & b_{(l+1)(l+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & a_{m(l+1)} & b_{mn}
\end{pmatrix}
\] (3.22)
is at most \(l\). Let \(l\) be the smallest number having this property and, without loss of
generality, assume \(i_j = j\) for \(1 \leq j \leq l\). Applying Gauss elimination if necessary, we
further suppose the coefficient matrix of \(H'\) and (3.19) are given as
\[
\begin{pmatrix}
a_{11} & \cdots & b_{1l} & a_{1(l+1)} & \cdots & b_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{l1} & \cdots & b_{ll} & a_{(l+1)(l+1)} & \cdots & b_{l(l+1)n} \\
0 & \cdots & 0 & a_{m(l+1)} & \cdots & b_{mn}
\end{pmatrix}
\] (3.23)
and
\[
\begin{pmatrix}
a_{11} + \tau_1 b_{11} & \cdots & a_{1l} + \tau_l b_{1l} & a_{1(l+1)} + \tau_{l+1} b_{1(l+1)} & \cdots & a_{1n} + \tau_n b_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{l1} + \tau_1 b_{ll} & \cdots & a_{ll} + \tau_l b_{ll} & a_{(l+1)(l+1)} + \tau_{l+1} b_{(l+1)(l+1)} & \cdots & a_{ln} + \tau_n b_{ln} \\
0 & \cdots & 0 & a_{(l+1)(l+1)} + \tau_{l+1} b_{(l+1)(l+1)} & \cdots & a_{l(l+1)n} + \tau_n b_{l(l+1)n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & a_{m(l+1)} + \tau_{l+1} b_{m(l+1)} & \cdots & a_{mn} + \tau_n b_{mn}
\end{pmatrix}
\] (3.24)
respectively.

(1) If the rank of
\[
\begin{pmatrix}
a_{11} + \tau_1 b_{11} & \cdots & a_{1l} + \tau_l b_{1l} \\
\vdots & \ddots & \vdots \\
a_{l1} + \tau_1 b_{ll} & \cdots & a_{ll} + \tau_l b_{ll}
\end{pmatrix}
\] (3.24)
is \(m\), then the rank of the following submatrix
\[
\begin{pmatrix}
a_{(l+1)(l+1)} + \tau_{l+1} b_{(l+1)(l+1)} & \cdots & a_{(l+1)n} + \tau_n b_{(l+1)n} \\
\vdots & \ddots & \vdots \\
a_{m(l+1)} + \tau_{l+1} b_{m(l+1)} & \cdots & a_{mn} + \tau_n b_{mn}
\end{pmatrix}
\] of (3.23) is strictly less than \(m - l\) (otherwise, it contradicts the fact that the rank
of (3.23) is strictly less than \(m\)). If \(H'\) is an algebraic subgroup defined by
\[
M_{l+1}^{a_{(l+1)}} L_{l+1}^{b_{(l+1)}} \cdots M_{mn}^{a_{mn}} L_{n}^{b_{mn}} = 1 \quad (l + 1 \leq i \leq m),
\]
by induction, \( X \cap H' \) is an anomalous subvariety of \( X \) containing \( X \cap H \) and contained in
\[
M_i = L_i = 1
\]
for some \( l + 1 \leq i \leq m \).

(2) Suppose the rank of (3.24) is strictly less than \( l \). By Lemma 3.3 there exists
\[
\{i_1, \ldots, i_h\} \subset \{1, \ldots, l\}
\]
such that the rank of
\[
\begin{pmatrix}
a_{i_1} & b_{i_1} & \cdots & a_{i_{i_h}} & b_{i_{i_h}} \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{i_1} & b_{i_1} & \cdots & a_{i_{i_h}} & b_{i_{i_h}}
\end{pmatrix}
\]
is strictly less than \( h \). But this contradicts the assumption on \( l \).

\[\Box\]

Now we complete the proof of Theorem 1.3, which we restate simply as follows:

**Theorem 3.14.** If \( H \) be an algebraic subgroup such that \( X \cap H \) is an anomalous subvariety of \( X \), then
\[
X \cap H \subset (M_i = L_i = 1)
\]
for some \( i \).

**Proof.** Let \( H \) be defined by
\[
M_1^{a_{i_1}} L_1^{b_{i_1}} \cdots M_n^{a_{i_n}} L_n^{b_{i_n}} = 1, \quad (1 \leq i \leq m).
\]
(3.25)
For \( n \geq m \), the theorem was proved in Theorem 3.13, so we assume \( m > n \). If \( H' \) is an algebraic subgroup defined by the first \( n \) equations in (3.25), then \( \log(X \cap H') \) is given as
\[
a_{i_1}u_1 + b_{i_1}(\tau_1 u_1 + \cdots) + \cdots + a_{i_n}u_n + b_{i_n}(\tau_n u_n + \cdots) = 0, \quad (1 \leq i \leq n)
\]
(3.26)
and the Jacobian of (3.26) at \((0, \ldots, 0)\) is
\[
\begin{pmatrix}
a_{11} + \tau_1 b_{11} & \cdots & a_{1n} + \tau_n b_{1n} \\
0 & \ddots & \vdots \\
a_{n1} + \tau_1 b_{n1} & \cdots & a_{nn} + \tau_n b_{nn}
\end{pmatrix}
\]
(3.27)
If the determinant of (3.27) is nonzero, by the inverse function theorem, (3.26) is equivalent to
\[
u_1 = \cdots = u_n = 0,
\]
implying
\[
X \cap H' = X \cap H = X \cap (M_1 = \cdots = M_n = 1).
\]
But this contradicts the fact that the dimension of \( X \cap H \) is positive.

If the determinant of (3.27) is zero, by Theorem 3.13
\[
X \cap H' \subset (M_i = L_i = 1)
\]
for some \( 1 \leq i \leq n \).

\[\Box\]
4. \( \mathcal{X}^\text{an} = \emptyset \)

4.1. Preliminary lemmas

In this subsection, we prove several preliminary lemmas required for the proof of Theorem 1.4.

First, we further refine the definition of an anomalous subvariety in the following:

**Definition 4.1.** Let \( \mathcal{X} \) be an irreducible variety in \( \mathbb{G}^n \) and \( b \geq 0 \) be an integer. We say that \( \mathcal{X} \cap K \) is \( b \)-anomalous if it satisfies

\[
\dim(\mathcal{X} \cap K) = \dim K + \dim \mathcal{X} - n + b.
\]

The above definition is firstly given in [3]. In the original definition, \( b \) is assumed to be positive, but \( b = 0 \) is allowed in our case. (This is for the sake of convenience in the proofs of the lemmas below.) Note that 0-anomalous subvarieties are not anomalous in the sense of Definition 1.1.

For instance, if \( \mathcal{X} \) is the holonomy variety of an \( n \)-cusped hyperbolic 3-manifold, then

\[
\mathcal{X} \cap (M_1 = \cdots = M_m = 1)
\]

is an \( m \)-anomalous subvariety of \( \mathcal{X} \).

In general, let \( H^{(m)} \) be an algebraic subgroup of codimension \( m \) defined by

\[
M_1^{a_{1i}} L_1^{b_{1i}} \cdots M_m^{a_{mi}} L_m^{b_{mi}} = 1 \quad (1 \leq i \leq m)
\]

(4.1)

If the rank of the Jacobian matrix (i.e. (3.20)) associated to \( \log(\mathcal{X} \cap H^{(m)}) \) is \( m \), then \( \mathcal{X} \cap H^{(m)} \) is a 0-anomalous variety of \( \mathcal{X} \), that is,

\[
\mathcal{X} \cap H^{(m)} = \mathcal{X} \cap (M_1 = \cdots = M_m = 1)
\]

by the implicit function theorem (as seen in the previous section). Further, if we add

\[
M_1^{a_{1i}} L_1^{b_{1i}} \cdots M_m^{a_{mi}} L_m^{b_{mi}} = 1 \quad (m + 1 \leq i \leq m + b)
\]

(4.2)

to (4.1) and define \( H^{(m+b)} \) as an algebraic subgroup of codimension \( m + b \) by (4.1)-(4.2), then

\[
\mathcal{X} \cap H^{(m)} \mathcal{X} \cap H^{(m+b)} = \mathcal{X} \cap (M_1 = \cdots = M_m = 1)
\]

and thus \( \mathcal{X} \cap H^{(m+b)} \) is a \( b \)-anomalous subvariety of \( \mathcal{X} \). In the following, we show this is always the case, that is, every \( b \)-anomalous subvariety of \( \mathcal{X} \) always arises in this way.

**Lemma 4.2.** Let \( \mathcal{M} \) and \( \mathcal{X} \) be the same as in Theorem 1.4. If \( H \) be an algebraic subgroup of codimension \( m + b \) such that \( \mathcal{X} \cap H \) is a \( b \)-anomalous subvariety of \( \mathcal{X} \) and

\[
\mathcal{X} \cap H = \mathcal{X} \cap (M_1 = \cdots = M_m = 1),
\]

(4.3)

then \( H \) is defined by equations of the following forms:

\[
M_1^{a_{1i}} L_1^{b_{1i}} \cdots M_m^{a_{mi}} L_m^{b_{mi}} = 1 \quad (1 \leq i \leq m + b).
\]

Further, there exists an algebraic subgroup \( H^{(m)} \) of codimension \( m \) such that

1. \( H \subset H^{(m)} \);
2. \( \mathcal{X} \cap H^{(m)} = \mathcal{X} \cap (M_1 = \cdots = M_m = 1) \).

Further, we state the following lemma, which is analogous to the previous one but more general than that. We first prove Lemma 4.3 and use it to prove Lemma 4.2. Indeed, the first conclusion of Lemma 4.2 follows immediately as a corollary of Lemma 4.3.
Lemma 4.3. Let \(\mathcal{M}\) and \(\mathcal{X}\) be the same as in Theorem 1.4. Let \(H\) be an algebraic subgroup such that \(\mathcal{X} \cap H\) is a \(b\)-anomalous subvariety of \(\mathcal{X}\) and
\[
\mathcal{X} \cap H \subset (M_1 = \cdots = M_m = 1),
\]
\[
\mathcal{X} \cap H \not\subset (M_i = 1)
\]
for \(i \in \{m + 1, \ldots, n\}\). Then \(b \leq m\) and there exists an algebraic subgroup \(H^{(m+b)}\) of codimension \(m + b\) such that
\[
\begin{align*}
(1) & \quad H \subset H^{(m+b)}, \\
(2) & \quad \mathcal{X} \cap H^{(m+b)} = \mathcal{X} \cap (M_1 = \cdots = M_m = 1); \\
(3) & \quad H^{(m+b)} \text{ is defined by equations of the following types:}
\end{align*}
\]
\[
M_1^{a_1} L_1^{b_1} \cdots M_m^{a_m} L_m^{b_m} = 1 \quad (1 \leq i \leq m + b).
\]

Proof. We prove by induction on \(n\) and \(m\). Clearly the claim is true for \(n = m = 1\).

(1) Suppose \(n \geq 2\) and \(m = 1\). It is enough to show either
\[
b = 1, \quad H \subset (M_1 = L_1 = 1)
\]
or
\[
b = 0, \quad H \subset (M_1^d L_1^d = 1)
\]
for some \(c, d \in \mathbb{Z}\). Let
\[
\mathcal{X}_{(1)} := \mathcal{X} \cap (M_1 = L_1 = 1), \quad H_{(1)} := H \cap (M_1 = L_1 = 1).
\]
Then
\[
\dim H - 2 \leq \dim H_{(1)} \leq \dim H
\]
and so
\[
\dim H = \dim H_{(1)} + a
\]
for some \(a \in \{0, 1, 2\}\). We suppose \(\mathcal{X}_{(1)}, H_{(1)}\) are embedded in \(\mathbb{G}^{2(n-1)}( := (M_2, L_2, \ldots, M_n, L_n))\) under the following projection
\[
Pr : (M_1, L_1, \ldots, M_n, L_n) \rightarrow (M_2, L_2, \ldots, M_n, L_n)
\]
and regard \(\mathcal{X}_{(1)}\) as the holonomy variety of an \((n-1)\)-cusped hyperbolic 3-manifold. Note that
\[
\dim \mathcal{X} \cap H = \dim \mathcal{X}_{(1)} \cap H_{(1)} = \dim \mathcal{X} + \dim H - 2n + b
\]
\[
= (\dim \mathcal{X}_{(1)} + 1) + (\dim H_{(1)} + a) - 2n + b = \dim \mathcal{X}_{(1)} + \dim H_{(1)} - 2(n - 1) + a + b - 1.
\]

(a) If \(b \geq 2\), then
\[
\dim \mathcal{X}_{(1)} \cap H_{(1)} \geq \dim \mathcal{X}_{(1)} + \dim H_{(1)} - 2(n - 1) + 1
\]
and so \(\mathcal{X}_{(1)} \cap H_{(1)}\) is an anomalous subvariety of \(\mathcal{X}_{(1)}\) (in \(\mathbb{G}^{2(n-1)}\)). By Theorem 3.14, there exists some \(i \ (2 \leq i \leq n)\) such that
\[
\mathcal{X}_{(1)} \cap H_{(1)} \subset (M_i = L_i = 1).
\]
But this contradicts the assumption on \(m\).

\textsuperscript{6}Equivalently, \(H^{(m+b)}\) satisfies
\[
(H^{(m+b)} \cap (M_1 = L_1 = \cdots = M_m = L_m = 1)) = (M_1 = L_1 = \cdots = M_m = L_m = 1).
\]
(b) For \( b = 1 \) and \( a = 1 \), one gets the same contradiction as above. For \( b = 1 \) and \( a = 0 \), \( \dim H = \dim H(1) \) and this implies
\[
H \subset (M_1 = L_1 = 1).
\]

(c) If \( b = 0 \), then
\[
\dim \mathcal{X}(1) \cap H(1) = \dim \mathcal{X}(1) + \dim H(1) - 2(n - 1) + a - 1.
\]

(i) If \( a = 0 \), then it contradicts the following standard fact (in the intersection theory)
\[
\dim \mathcal{X}(1) \cap H(1) \geq \dim \mathcal{X}(1) + \dim H(1) - 2(n - 1).
\]

(ii) If \( a = 1 \), then
\[
\dim H = 1 + \dim H(1). \tag{4.7}
\]

By the definition of \( H(1) \) given in (4.5), we have
\[
H \subset (M_1, L_1, \ldots, M_1 = 1).
\]

(iii) If \( a = 2 \), then \( \mathcal{X}(1) \cap H(1) \) is an anomalous subvariety of \( \mathcal{X}(1) \), again contradicting the assumption on \( m \).

(2) Now suppose \( n \geq m \geq 2 \) and assume the claim holds for any \( \mathcal{X}, H \) satisfying either \( \dim \mathcal{X} < n \) or
\[
\dim \mathcal{X} = n, \quad \mathcal{X} \cap H \subset (M_{i_1} = \cdots = M_{i_l} = 1), \quad \mathcal{X} \cap H \not\subset (M_j = 1)
\]
where \( l < m \) and \( j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_l\} \). We show the claim holds for \( \mathcal{X}, H \) satisfying
\[
\dim \mathcal{X} = n, \quad \mathcal{X} \cap H \subset (M_1 = \cdots = M_m = 1), \quad \mathcal{X} \cap H \not\subset (M_j = 1)
\]
where \( j \in \{m + 1, \ldots, n\} \). Let \( \mathcal{X}(1), H(1) \) be the same as in (4.5) and consider them as subsets in \( \mathbb{G}^{2n-2} := (M_2, L_2, \ldots, M_n, L_n) \). If \( \dim H = \dim H(1) + a \) (for some \( a \in \{0, 1, 2\} \)), then
\[
\dim \mathcal{X}(1) \cap H(1) = \dim \mathcal{X}(1) + \dim H(1) - 2(n - 1) + a + b - 1
\]
by (4.6). That is, \( \mathcal{X}(1) \cap H(1) \) is an \((a + b - 1)\)-anomalous subvariety of \( \mathcal{X}(1) \) in \( \mathbb{G}^{2n-2} \). Since \( \dim \mathcal{X}(1) = n - 1 \), by induction hypothesis,
\[
b + a - 1 \leq m - 1 \implies b \leq m
\]
and there exists an algebraic subgroup \( H^{(m+a+b-2)}_{(1)} \) of codimension \((m - 1) + (a + b - 1)\) satisfying
\[
H(1) \subset H^{(m+a+b-2)}_{(1)}, \quad \mathcal{X}(1) \cap H^{(m+a+b-2)}_{(1)} = \mathcal{X}(1) \cap (M_2 = \cdots = M_m = 1). \tag{4.8}
\]
Moreover, \( H^{(m+a+b-2)}_{(1)} \) is defined by equations of the following types:
\[
M_2^{a_2} L_2^{b_2} \cdots M_m^{a_m} L_m^{b_m} = 1 \quad (1 \leq i \leq m + a + b - 2).
\]
By the definition of $H^{(1)}$, one further obtains an algebraic subgroup $H^{(m+a+b-2)}$ in $\mathbb{C}^{2n}$ satisfying\(^7\)

$$H \subset H^{(m+a+b-2)}, \quad H^{(m+a+b-2)} \cap (M_1 = L_1 = 1) = H^{(m+a+b-2)} \cap (M_1 = L_1 = 1).$$

(4.10)

(a) If $a = 1$, it means

$$H \subset (M_1^c L_1^d = 1)$$

for some $c, d \in \mathbb{Z}$ and so $H^{(m+b-1)} \cap (M_1^c L_1^d = 1)$ is an algebraic subgroup of codimension $m + b$ containing $H$. Since

$$\mathcal{X} \cap (M_1^c L_1^d = 1) \subset (M_1 = \cdots = M_m = 1),$$

we get

$$\mathcal{X} \cap H^{(m+b-1)} \cap (M_1^c L_1^d = 1) = \mathcal{X} \cap (M_1 = \cdots = M_m = 1)$$

by (4.8) and (4.10). Thus we get the desired result by letting $H^{(m+b)} := H^{(m+b-1)} \cap (M_1^c L_1^d = 1)$.

(b) If $a = 0$, then

$$H \subset (M_1 = L_1 = 1).$$

Similar to the previous case, the conclusion follows by letting $H^{(m+b)} := H^{(m+b-2)} \cap (M_1 = L_1 = 1)$.

(c) If $a = 2$, then $H^{(m+b)}$ itself is an algebraic subgroup of codimension $m + b$ containing $H$. So, to complete the proof, it is enough to show

Claim 4.4.

$$\mathcal{X} \cap H^{(m+b)} = \mathcal{X} \cap (M_1 = \cdots = M_m = 1).$$

(4.11)

**Proof of the claim.** As $H^{(m+b)}$ is defined by equations given in (4.9), if (4.11) fails, then the rank of the Jacobian of $\log(\mathcal{X} \cap H^{(m+b)})$ is strictly less than $m$ and it implies $\mathcal{X} \cap H^{(m+b)}$ is an anomalous subvariety of $\mathcal{X}$ (in the sense of Definition 1.1) by Theorem 3.13. Thus

$$\mathcal{X} \cap H^{(m+b)} \subset (M_i = 1)$$

for some $2 \leq i \leq m$ by Theorem 1.3. Let $l$ be the largest number such that

$$\mathcal{X} \cap H^{(m+b)} \subset (M_i = \cdots = M_i = 1)$$

where $\{i_1, \cdots, i_l\} \subseteq \{1, \cdots, m\}$ and, without loss of generality, we simply assume $\{i_1, \cdots, i_l\} = \{m-l+1, \ldots, m\}$. Since $l < m$, by induction hypothesis, there exists an algebraic subgroup $H'$ such that

$$H^{(m+b)} \subset H'$$

and

$$\mathcal{X} \cap H' = \mathcal{X} \cap (M_{m-l+1} = \cdots = M_m = 1).$$

(4.12)

Let

$$\mathcal{X}(i) := \mathcal{X} \cap (M_{m-l+1} = \cdots = M_m = 1),$$

$$H(i) := H \cap (M_{m-l+1} = L_{m-l+1} = \cdots = M_m = L_m = 1),$$

(4.13)

\(^8\)The second equality implies $H^{(m+a+b-2)}$ is defined by equations of the following forms:

$$M_{i_1}^{m+1} L_1^{k_1} \cdots M_{i_l}^{m+1} L_1^{k_l} = 1 \quad (1 \leq i \leq m + a + b - 2).$$

(4.9)

\(^8\)If $i = 1$, then we get the desired result (i.e. (4.11)) by (4.8).
and consider them as varieties embedded in $\mathbb{G}^{2(n-l)} := (M_{m-l+1}, \ldots, L_m))$. Since $\dim X_{(l)} < n$ and

$$X_{(l)} \cap H_{(l)} \subset (M_1 = \cdots = M_{m-l} = 1),$$

by induction hypothesis, there exists an algebraic subgroup $\bar{H}_{(l)}(\subset \mathbb{G}^{2(n-l)})$ satisfying

$$H_{(l)} \subset \bar{H}_{(l)} \quad \text{and} \quad X_{(l)} \cap \bar{H}_{(l)} = X_{(l)} \cap (M_1 = \cdots = M_{m-l} = 1). \quad (4.14)$$

By the definition of $H_{(l)}$ in $(4.13)$, there further exists an algebraic subgroup $H''$ containing $H$ such that

$$\bar{H}_{(l)} \cap (M_{m-l+1} = L_{m-l+1} = \cdots = M_m = L_m = 1) = H'' \cap (M_{m-l+1} = L_{m-l+1} = \cdots = M_m = L_m = 1). \quad (4.15)$$

We argue $H^{(m+b)} = H' \cap H''$. First note that

$$X \cap H' \cap H'' = X \cap (M_1 = \cdots = M_m = 1) \quad (4.16)$$

by $(4.12)$, $(4.14)$ and $(4.15)$, and so

$$X \cap H^{(m+b)} \cap H' \cap H'' = X \cap (M_1 = \cdots = M_m = 1) \quad \text{by the definition of } H^{(m+b)} \text{ and } (4.10).$$

Let

$$X_{(m)} := X \cap (M_1 = \cdots = M_m = 1), \quad H_{(m)} := H \cap (M_1 = L_1 = \cdots = M_m = L_m = 1)$$

and consider $X_{(m)}, H_{(m)}$ as subsets of $\mathbb{G}^{2n-2m} := (M_{m+1}, L_{m+1}, \ldots, M_n, L_n))$. Since

$$\text{codim } H^{(m+b)} = b + m,$$

if $H^{(m+b)} \neq H' \cap H''$, then

$$\text{codim } (H^{(m+b)} \cap H' \cap H'') > b + m.$$

As

$$H \subset H^{(m+b)} \cap H' \cap H''$$

and

$$(H^{(m+b)} \cap H' \cap H'')(M_1 = L_1 = \cdots = M_m = L_m = 1)) = (M_1 = L_1 = \cdots = M_m = L_m = 1),$$

we get

$$\text{codim } H_{(m)} \text{ (in } \mathbb{G}^{2n-2m}) \leq \text{codim } H - \text{codim } (H^{(m+b)} \cap H' \cap H'') < \text{codim } H - b - m,$$

implying

$$\text{dim } H_{(m)} \text{ (in } \mathbb{G}^{2n-2m}) = (2n - 2m) - \text{codim } H_{(m)} > (2n - 2m) - \text{codim } H + b + m$$

$$> (2n - 2m) - (2n - \text{dim } H) + b + m = \text{dim } H - m + b$$

and so

$$\text{dim } X_{(m)} + \text{dim } H_{(m)} - (2n - 2m) > (n - m) + (\text{dim } H - m + b) - (2n - 2m) = \text{dim } H - n + b. \quad (4.17)$$

On the other hand, we have

$$\text{dim } X_{(m)} \cap H_{(m)} = \text{dim } X \cap H = \text{dim } X + \text{dim } H - 2n + b = \text{dim } H - n + b, \quad (4.18)$$
which, combining with (4.17), contradicts the following standard fact (from the intersection theory)

\[ \dim X_{(m)} \cap H_{(m)} \geq \dim X_{(m)} + \dim H_{(m)} - (2n - 2m). \]

Thus \( H^{(m+b)} = H' \cap H'' \) and (4.16) implies (4.11). This completes the proof of the claim. □

Using Lemma 4.3, we now prove Lemma 4.2.

**Proof of Lemma 4.2.** To simplify the proof, let us assume \( b = 1 \) and show the claim only for this case. (Indeed, this is the case that we will need later in the proof of Theorem 1.4.)

For each fixed \( n \), we prove by induction on \( m \). If \( m = 1 \), since \( H \) is defined by \( M_1 = L_1 = 1 \) by Theorem 3.11, we get the desired result by letting \( H(1) \) be defined by either \( M_1 = 1 \) or \( L_1 = 1 \).

Assume \( m \geq 2 \) and the claim is true for \( 1, \ldots, m-1 \). By Lemma 4.3, \( H \) is defined by equations of the following types:

\[ M_{a_1}^{i_1} L_{1}^{b_{i_1}} \cdots M_{m}^{a_m} L_{m}^{b_{m}} = 1 \quad (1 \leq i \leq m + 1). \] (4.19)

Let \( \bar{H}^{(m)} \) be an algebraic subgroup defined by the first \( m \)-equations in (4.19).

1. If \( \dim(\mathcal{X} \cap \bar{H}^{(m)}) = n - m \), the Jacobian matrix (i.e. (3.20)) associated with \( \log(\mathcal{X} \cap \bar{H}^{(m)}) \) is invertible so \( \log(\mathcal{X} \cap \bar{H}^{(m)}) \) is contained in \( u_1 = \cdots = u_m = 0 \) by the implicit function theorem. Hence

\[ \mathcal{X} \cap \bar{H}^{(m)} = \mathcal{X} \cap (M_1 = \cdots = M_m = 1) \]

and the result follows by letting \( H^{(m)} := \bar{H}^{(m)} \).

2. Suppose \( \dim(\mathcal{X} \cap \bar{H}^{(m)}) \geq n - m + 1 \) (i.e. \( \mathcal{X} \cap \bar{H}^{(m)} \) is an anomalous subvariety of \( \mathcal{X} \)). Since \( \text{codim} \ H = \text{codim} \ \bar{H}^{(m)} - 1 \),

\[ n - m = \dim \mathcal{X} \cap H \leq \dim \mathcal{X} \cap \bar{H}^{(m)} - 1 \]

and so \( \dim \mathcal{X} \cap \bar{H}^{(m)} = n - m + 1 \). Let \( l \) be the largest number such that

\[ \mathcal{X} \cap \bar{H}^{(m)} \subset \mathcal{X} \cap (M_{i_1} = \cdots = M_{i_l} = 1) \]

where \( \{i_1, \ldots, i_l\} \subset \{1, \ldots, m\} \) and, without loss of generality, assume \( i_j = j \) (1 \leq j \leq l). By Lemma 4.3, \( \bar{H}^{(m)} \) is contained in an algebraic subgroup \( \bar{H}^{(l+1)} \) defined by equations of the following types

\[ M_{a_1}^{i_1} L_{1}^{b_{i_1}} \cdots M_{l}^{a_l} L_{l}^{b_{l}} = 1 \quad (1 \leq i \leq l + 1) \] (4.20)

and, by induction hypothesis, \( \bar{H}^{(l+1)} \) is further contained in an algebraic subgroup \( \bar{H}^{(l)} \) of codimension \( l \) satisfying

\[ \mathcal{X} \cap \bar{H}^{(l)} = \mathcal{X} \cap \bar{H}^{(l+1)} = \mathcal{X} \cap (M_1 = \cdots = M_l = 1). \] (4.21)

Without loss of generality, if \( \bar{H}^{(l)} \) is defined by the first \( l \) equations in (4.20), then

\[ H \subset \bar{H}^{(m)} \subset \bar{H}^{(l+1)} \subset \bar{H}^{(l)} \]
and the Jacobian matrix of $\log(X \cap H)$ is of the following form:

$$
\begin{pmatrix}
  a_{11} + b_{11} \tau_1 & \ldots & a_{1l} + b_{1l} \tau_l & 0 & \ldots & 0 \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{(l+1)1} + b_{(l+1)1} \tau_1 & \ldots & a_{(l+1)l} + b_{(l+1)l} \tau_l & 0 & \ldots & 0 \\
  a_{(l+2)1} + b_{(l+2)1} \tau_1 & \ldots & a_{(l+2)l} + b_{(l+2)l} \tau_l & a_{(l+2)(l+1)} + b_{(l+2)(l+1)} \tau_{l+1} & \ldots & a_{(l+2)m} + b_{(l+2)m} \tau_m \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
  a_{(m+1)1} + b_{(m+1)1} \tau_1 & \ldots & a_{(m+1)l} + b_{(m+1)l} \tau_l & a_{(m+1)(l+1)} + b_{(m+1)(l+1)} \tau_{l+1} & \ldots & a_{(m+1)m} + b_{(m+1)m} \tau_m 
\end{pmatrix}
$$

By the definition of $H(l)$ and (4.21),

$$
\begin{pmatrix}
  a_{11} + b_{11} \tau_1 & \ldots & a_{1l} + b_{1l} \tau_l \\
  \vdots & \ddots & \vdots \\
  a_{l1} + b_{l1} \tau_1 & \ldots & a_{ll} + b_{ll} \tau_l
\end{pmatrix}
$$

is invertible and, by (4.3),

$$
\begin{pmatrix}
  a_{(l+2)(l+1)} + b_{(l+2)(l+1)} \tau_{l+1} & \ldots & a_{(l+2)m} + b_{(l+2)m} \tau_m \\
  \vdots & \ddots & \vdots \\
  a_{(m+1)(l+1)} + b_{(m+1)(l+1)} \tau_{l+1} & \ldots & a_{(m+1)m} + b_{(m+1)m} \tau_m
\end{pmatrix}
$$

is also invertible. In conclusion, if $H(m)$ is an algebraic subgroup defined by

$$
M_1^{a_{1i}} L_1^{b_{1i}} \cdots M_l^{a_{li}} L_l^{b_{li}} = 1 \quad (1 \leq i \leq l),
$$

$$
M_1^{a_{1j}} L_1^{b_{1j}} \cdots M_m^{a_{mj}} L_m^{b_{mj}} = 1 \quad (l + 2 \leq j \leq m + 1),
$$

then it is the desired one.

\[ \square \]

4.2. Proof of Theorem 1.4

In this subsection, we prove our second main result, Theorem 1.4. We show it by splitting it into several special cases.

First if $X^{oa} = \emptyset$, by Theorem 2.8 there exists an algebraic subgroup $H$ such that $X \cap H$ is a 1-anomalous subvariety of $X$ and $X = Z_H$, i.e., $X$ is foliated by maximal anomalous subvarieties contained in

$$
\bigcup_{g \in Z_H} X \cap gH.
$$

Let $m \geq 1$ be the largest number such that

$$
X \cap H \subset (M_{i_1} = \cdots = M_{i_m} = 1),
$$

$$
X \cap H \not\subset (M_{i_{m+1}} = 1)
$$

for $i_{m+1} \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$. Without loss of generality, we assume

$$
i_j = j \quad (1 \leq j \leq m).
$$

By Lemma 4.3 $H$ is contained in an algebraic subgroup $H^{(m+1)}$ of codimension $m + 1$ defined by equations of the following types

$$
M_1^{a_{1i}} L_1^{b_{1i}} \cdots M_m^{a_{mi}} L_m^{b_{mi}} = 1, \quad (0 \leq i \leq m).
$$
Thus if
\[ \dim(X \cap H) := n - m - l, \quad \codim H := m + l + 1, \quad (l \geq 0), \] (4.27)
one may assume \( H \) is defined by equations of the following forms
\[ M_1^{a_1} L_1^{b_1} \cdots M_m^{a_m} L_m^{b_m} = 1, \quad (1 \leq j \leq l). \] (4.28)

Note that \( H^{(m+1)} \) above satisfies
\[ X \cap H^{(m+1)} = X \cap (M_1 = \cdots = M_m = 1) \]
and, by Lemma 4.2, \( H^{(m+1)} \) is further contained in an algebraic subgroup \( H^{(m)} \) of codimension \( m \) satisfying
\[ X \cap H^{(m)} = X \cap H^{(m+1)} = X \cap (M_1 = \cdots = M_m = 1). \] (4.29)

Without loss of generality, we suppose \( H^{(m)} \) is defined by
\[ M_1^{a_1} L_1^{b_1} \cdots M_m^{a_m} L_m^{b_m} = 1, \quad (1 \leq i \leq m). \] (4.30)

We first claim

**Lemma 4.5.** Having the same notation and assumptions as above, there exists an analytic function \( \Theta(s_1, \ldots, s_m, t_1, \ldots, t_l) \) such that
\[ a_{01} u_1 + b_{01} v_1 + \cdots + a_{0m} u_m + b_{0m} v_m = \Theta(s_1, \ldots, s_m, t_1, \ldots, t_l) \] (4.31)
where
\[ s_i = a_{i1} u_1 + b_{i1} v_1 + \cdots + a_{im} u_m + b_{im} v_m \quad (1 \leq i \leq m), \]
\[ t_j = a_{j1} u_1 + b_{j1} v_1 + \cdots + a_{jm} u_m + b_{jm} v_n \quad (1 \leq j \leq l). \] (4.32)

**Proof.** Since \( X \) is foliated by anomalous subvarieties in \( \log H \), equivalently, \( \log X \) is foliated by elements in \( \bigcup_{g \in G} \log(X \cap gH) \). As each \( \log(X \cap gH) \) is defined by equations of the following types
\[ \zeta_i = a_{i1} u_1 + b_{i1} v_1 + \cdots + a_{im} u_m + b_{im} v_m \quad (0 \leq i \leq m), \]
\[ \zeta_j' = a_{j1} u_1 + b_{j1} v_1 + \cdots + a_{jm} u_m + b_{jm} v_n \quad (1 \leq j \leq l). \] (4.33)
where \( \zeta_j, \zeta_j' \in \mathbb{C} \), if we set
\[ T := \{ (\zeta_0, \ldots, \zeta_l) \in \mathbb{C}^{m+l+1} : \] (4.33) \] is a complex manifold of dimension \( n - m - l \}, \)
then
\[ \dim T = \dim(\log X) - \dim \log(X \cap H) = n - (n - m - l) = m + l. \]
Thus \( T \) is a hypersurface in \( \mathbb{C}^{m+l+1} \) and this implies there exists \( \Theta \) satisfying (4.31). \( \square \)

Later in Section 4.2.3, it will be shown that \( l = 0 \). At the moment, let us assume \( l = 0 \) and consider the following two subcases:

- \( l = 0 \) and there is no \( H' \) such that \( H \subsetneq H' \) and \( \mathcal{X} \cap H' \) is an anomalous subvariety of \( \mathcal{X} \);
- \( l = 0 \) and there exists \( H' \) such that \( H \subsetneq H' \) and \( \mathcal{X} \cap H' \) is an anomalous subvariety of \( \mathcal{X} \).

In the first case, we show that cusps \( 1, \ldots, m \) are SGI from the rest and, in the second, find a proper subset of cusps \( 1, \ldots, m \) which are WGI from cusps \( m + 1, \ldots, n \).
4.2.1. ♠ =⇒ SGI

In this subsection, we prove the following theorem:

**Theorem 4.6.** Let \( \mathcal{M}, \mathcal{X} \) and \( H \) be the same as above. If \( H \) satisfies the assumption in ♠, then cusps \( 1, \ldots, m \) of \( \mathcal{M} \) are SGI from the rest.

The general strategy of the proof is as follows. If cusps \( 1, \ldots, m \) are not SGI from the rest, there exists \( v_i \) (\( 1 \leq i \leq m \)) having a term divisible by some \( u_j \) (\( m + 1 \leq j \leq n \)). We find such a term of the lowest degree, compare two coefficients of the term in (4.31) and get an equality involving \( a_{ij}, b_{ij}, \tau_k \) (\( 0 \leq i, j, k \leq m \)). Under the assumption in ♠, it is shown that there are no nontrivial \( a_{ij}, b_{ij} \) satisfying the equality and thus contradicts the initial assumption.

**Proof.** Let

\[
\Phi(u_1, \ldots, u_n) = \sum_{(i_1, \ldots, i_n) \in \mathbb{Z}^n_+} c_{i_1, \ldots, i_n} u_1^{i_1} \cdots u_n^{i_n}
\]

be the Neumann-Zagier potential function of \( \mathcal{M} \) and \( S \) be the set of all \( u_1^{i_1} \cdots u_n^{i_n} \) satisfying

- \( c_{i_1, \ldots, i_n} \neq 0; \)
- \( (i_1, \ldots, i_m) \neq (0, \ldots, 0); \)
- \( (i_{m+1}, \ldots, i_n) \neq (0, \ldots, 0); \)
- \( i_1 + \cdots + i_m \) is the minimum.

Note that \( S = \emptyset \) if and only if cusps \( 1, \ldots, m \) are SGI from cusps \( m + 1, \ldots, n \). So let us assume \( S \neq \emptyset \) and \( u_1^{i_1} \cdots u_n^{i_n} \in S \). Without loss of generality, it is further assumed \( i_1 \neq 0 \) and

\[
u := \frac{1}{2} i_1 c_{i_1, \ldots, i_n} u_1^{i_1-1} \cdots u_n^{i_n}.
\]

By Lemma 4.5, there exists an analytic function

\[
\Theta(s_1, \ldots, s_m) := e_1 s_1 + \cdots + e_m s_m + \text{higher degrees},
\]

satisfying (4.31). Comparing two coefficients of \( u \) in both the left and right sides of (4.31), we get

\[
b_{01} = \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix}^T \begin{pmatrix} b_{11} \\ \vdots \\ b_{m1} \end{pmatrix}.
\]

(4.34)

If \( b_{01} \neq 0 \), then \( b_{11} \neq 0 \) for some \( i \) (\( 1 \leq i \leq n \)) by (4.34). Hence, by applying Gauss elimination if necessary, it is assumed \( b_{01} = 0 \) in (4.26), (4.31) and (4.34).

Comparing the coefficients of the linear terms of the both sides in (4.31), it follows that

\[
\begin{pmatrix} a_{01} \\ a_{02} + b_{02} \tau_2 \\ \vdots \\ a_{0m} + b_{0m} \tau_m \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} \tau_1 & \cdots & a_{m1} + b_{m1} \tau_1 \\ \vdots & \ddots & \vdots \\ a_{1m} + b_{1m} \tau_m & \cdots & a_{mm} + b_{mm} \tau_m \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_m \end{pmatrix},
\]

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and so

\[
\begin{bmatrix}
e_1 \\
\vdots \\
e_m
\end{bmatrix} = \begin{bmatrix}
a_{11} + b_{11} \tau_1 & \cdots & a_{m1} + b_{m1} \tau_1 \\
\vdots & \ddots & \vdots \\
a_{1m} + b_{1m} \tau_m & \cdots & a_{mm} + b_{mm} \tau_m
\end{bmatrix}^{-1} \begin{bmatrix}
a_{01} \\
a_{02} + b_{02} \tau_2 \\
\vdots \\
a_{0m} + b_{0m} \tau_m
\end{bmatrix}.
\]

Combining it with (4.34), we further get

\[
0(= b_{01}) = \begin{bmatrix}
e_1 \\
\vdots \\
e_m
\end{bmatrix}^T \begin{bmatrix}
b_{11} \\
b_{m1}
\end{bmatrix} = \begin{bmatrix}
a_{01} \\
a_{02} + b_{02} \tau_2 \\
\vdots \\
a_{0m} + b_{0m} \tau_m
\end{bmatrix}^T \begin{bmatrix}
a_{11} + b_{11} \tau_1 & \cdots & a_{1m} + b_{1m} \tau_m \\
\vdots & \ddots & \vdots \\
a_{m1} + b_{m1} \tau_1 & \cdots & a_{mm} + b_{mm} \tau_m
\end{bmatrix}^{-1} \begin{bmatrix}
b_{11} \\
\vdots \\
b_{m1}
\end{bmatrix}.
\]

We show the above equality (4.35) contradicts the condition in ♣. Let

\[
v_i := \begin{bmatrix}
a_{i1} + b_{i1} \tau_1 & \cdots & a_{im} + b_{im} \tau_m
\end{bmatrix} (0 \leq i \leq m).
\]

By the inverse matrix formula, (4.35) is equivalent to

\[
-b_{11} \text{det} \begin{bmatrix}
v_0 \\
v_2 \\
v_3 \\
\vdots \\
v_m
\end{bmatrix} + \cdots + b_{m1} (-1)^m \text{det} \begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
\vdots \\
v_{m-1} \\
v_0
\end{bmatrix} = 0 \implies \sum_{i=1}^{m} b_{i1} (-1)^i \text{det} \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_{m-1} \\
v_0
\end{bmatrix} = 0
\]

(4.37)

where \( \hat{v}_i := v_0 \) for each \( i \). We claim

**Claim 4.7.** \( b_{i1} \neq 0 \) for some \( 1 \leq i \leq m \).

**Proof of Claim 4.7.** On the contrary, suppose \( b_{i1} = 0 \) for all \( 1 \leq i \leq m \). Then the coefficient matrix of \( H \) is

\[
\begin{bmatrix}
a_{01} & 0 & a_{02} & b_{02} & \cdots & a_{0m} & b_{0m} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{m1} & 0 & a_{m2} & b_{m2} & \cdots & a_{mm} & b_{mm}
\end{bmatrix}.
\]

(4.38)

If \( a_{i1} = 0 \) for all \( 0 \leq i \leq m \), then

\[
\mathcal{X} \cap H \not\subset (M_1 = 1),
\]

contradicting our initial assumption. Thus \( a_{i1} \neq 0 \) for some \( i \) and, without loss of generality, we assume \( a_{m1} \neq 0 \). Applying Gauss elimination if necessary, we further assume

\[
a_{01} = \cdots = a_{(m-1)1} = 0
\]

\( ^9 \)Note that

\[
\begin{bmatrix}
a_{11} + b_{11} \tau_1 & \cdots & a_{m1} + b_{m1} \tau_1 \\
\vdots & \ddots & \vdots \\
a_{1m} + b_{1m} \tau_m & \cdots & a_{mm} + b_{mm} \tau_m
\end{bmatrix}
\]

is the Jacobian of \( \log(\mathcal{X} \cap H^{(m)}) \) and it is invertible by the assumptions on \( H^{(m)} \) made in (4.29)-(4.30).
and rewrite (4.38) as
\[
\begin{pmatrix}
0 & 0 & a_0 & b_0 & \ldots & a_m & b_m \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & a_{(m-1)2} & b_{(m-1)2} & \ldots & a_{(m-1)m} & b_{(m-1)m} \\
a_{m1} & 0 & a_{m2} & b_{m2} & \ldots & a_{mm} & b_{mm}
\end{pmatrix}.
\]
Since the rank of
\[
\begin{pmatrix}
0 & a_0 + b_0 \tau_2 & \ldots & a_m + b_m \tau_m \\
\vdots & \vdots & \ddots & \vdots \\
0 & a_{(m-1)2} + b_{(m-1)2} \tau_2 & \ldots & a_{(m-1)m} + b_{(m-1)m} \tau_m \\
a_{m1} & a_{m2} + b_{m2} \tau_2 & \ldots & a_{mm} + b_{mm} \tau_m
\end{pmatrix}
\]
is \(m\) by the assumption, the rank of the following submatrix
\[
\begin{pmatrix}
a_0 + b_0 \tau_2 & \ldots & a_m + b_m \tau_m \\
\vdots & \ddots & \vdots \\
a_{(m-1)2} + b_{(m-1)2} \tau_2 & \ldots & a_{(m-1)m} + b_{(m-1)m} \tau_m
\end{pmatrix}
\]
of (4.39) is \(m - 1\). Let \(H'\) be an algebraic subgroup whose coefficient matrix is
\[
\begin{pmatrix}
a_0 & b_0 & \ldots & a_m & b_m \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{(m-1)} & b_{(m-1)} & \ldots & a_{(m-1)m} & b_{(m-1)m}
\end{pmatrix}.
\]
Then (4.40) is the Jacobian of \(\log(X \cap H')\) and this implies \(X \cap H'\) is an anomalous subvariety of \(X\) by Theorem 3.13. As \(H \subsetneq H'\), it contradicts the assumption on \(H\) made in \(\clubsuit\). In conclusion, \(b_{i1} \neq 0\) for some \(1 \leq i \leq m\).

Without loss of generality, let us assume
\[
\begin{cases}
b_{i1} \neq 0 & (1 \leq i \leq h), \\
b_{i1} = 0 & (h + 1 \leq i \leq m).
\end{cases}
\]
By elementary properties of determinants, (4.37) is equivalent to
\[
\det \begin{pmatrix}
\frac{b_{i1}}{b_{i1}} v_0 \\
\frac{b_{i2}}{b_{i1}} v_1 + v_2 \\
\vdots \\
\frac{b_{ih}}{b_{(h-1)i}} v_{h-1} + v_h \\
v_{h+1} \\
\vdots \\
v_m
\end{pmatrix} = 0
\]
and so
\[
\begin{bmatrix}
  v_0 \\
  b_1 v_1 + b_{11} v_2 \\
  \vdots \\
  b_{h1} v_{h-1} + b_{(h-1)1} v_h \\
  v_{h+1} \\
  \vdots \\
  v_m
\end{bmatrix}
\]
\[
= 0.
\]
(4.41)

Let
\[
\mathbf{w}_1 := (a_{i_1} \ b_{i_1} \ \ldots \ a_{i_m} \ b_{i_m})
\]
and \(H'\) be an algebraic subgroup whose coefficient matrix is
\[
\begin{bmatrix}
  b_{11} \mathbf{w}_0 \\
  b_{21} \mathbf{w}_1 + b_{11} \mathbf{w}_2 \\
  \vdots \\
  b_{h1} \mathbf{w}_{h-1} + b_{(h-1)1} \mathbf{w}_h \\
  \mathbf{w}_{h+1} \\
  \vdots \\
  \mathbf{w}_m
\end{bmatrix}.
\]

Then clearly \(H'\) is an algebraic subgroup satisfying \(H \subsetneq H'\) and, as the matrix in (4.41) is the Jacobian matrix of \(\log(\mathcal{X} \cap H')\), \(\mathcal{X} \cap H'\) is an anomalous subvariety of \(\mathcal{X}\) by Theorem 3.13. However the existence of \(H'\) contradicts the assumption made in \(\clubsuit\). This completes the proof of Theorem 4.6. \(\square\)

4.2.2. \(\clubsuit \implies \text{WGI}\)

Now we consider the second case \(\clubsuit\).

Let \(H'\) be an algebraic subgroup such that \(H \subsetneq H'\) and \(\mathcal{X} \cap H'\) is an anomalous subvariety of \(\mathcal{X}\). We further assume \(H'\) is the largest algebraic subgroup satisfying this property. That is, there is no algebraic subgroup \(H''\) containing \(H'\) properly and \(\mathcal{X} \cap H''\) is an anomalous subvariety of \(\mathcal{X}\). By the assumption,
\[
\mathcal{X} \cap H' = \mathcal{X} \cap (M_{i_1} = \cdots = M_{i_h} = 1)
\]
for some \(\{i_1, \ldots, i_h\} \subset \{1, \ldots, m\}\) and, without loss of generality, we assume \(i_j = j\) \((1 \leq j \leq h)\). By Lemma 4.3, \(H'\) is defined by the following types of equations
\[
M_1^{a_{i_1}} L_1^{b_{i_1}} \cdots M_h^{a_{i_h}} L_h^{b_{i_h}} = 1 \quad (0 \leq i \leq h)
\]
and so \(H\) is defined by
\[
M_1^{a_{i_1}} L_1^{b_{i_1}} \cdots M_h^{a_{i_h}} L_h^{b_{i_h}} = 1 \quad (0 \leq i \leq h),
\]
\[
M_1^{a_{i_1}} L_1^{b_{i_1}} \cdots M_{m}^{a_{i_m}} L_{m}^{b_{i_m}} = 1 \quad (h+1 \leq j \leq m).
\]

**Theorem 4.8.** Let \(\mathcal{M}, \mathcal{X}\) and \(H\) be the same as in Theorem 4.6. Suppose \(H\) satisfies the assumption in \(\clubsuit\) and \(H'\) is an algebraic subgroup containing \(H\) as given above. Then cusps \(1, \ldots, h\) of \(\mathcal{M}\) are WGI from cusps \(m + 1, \ldots, n\) of \(\mathcal{M}\).
Proof. Recall from Lemma 4.5 that there exists an analytic function \( \Theta(s_1, \ldots, s_m) \) satisfying
\[
a_{01}u_1 + b_{01}v_1 + \cdots + a_{0h}u_h + b_{0h}v_h = \Theta(s_1, \ldots, s_m)
\]
where
\[
s_i = a_{i1}u_1 + b_{i1}v_1 + \cdots + a_{ih}u_h + b_{ih}v_h \quad (1 \leq i \leq h),
\]
\[
s_j = a_{j1}u_1 + b_{j1}v_1 + \cdots + a_{jm}u_m + b_{jm}v_m \quad (h + 1 \leq j \leq m).
\]
Also note that the Jacobian of \( \log(X \cap H) \) at \((0, \ldots, 0)\) is
\[
\begin{pmatrix}
a_{01} + b_{01}\tau_1 & \cdots & a_{0h} + b_{0h}\tau_h & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \ddots & \vdots \\
a_{h1} + b_{h1}\tau_1 & \cdots & a_{hh} + b_{hh}\tau_h & 0 & \cdots & 0 \\
a_{(h+1)1} + b_{(h+1)1}\tau_1 & \cdots & a_{(h+1)h} + b_{(h+1)h}\tau_h & a_{(h+1)(h+1)} + b_{(h+1)(h+1)}\tau_{h+1} & \cdots & a_{(h+1)m} + b_{(h+1)m}\tau_m \\
\vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
a_{m1} + b_{m1}\tau_1 & \cdots & a_{mh} + b_{mh}\tau_h & a_{m(h+1)} + b_{m(h+1)}\tau_{h+1} & \cdots & a_{mm} + b_{mm}\tau_m 
\end{pmatrix}
\]
and the following two submatrices
\[
\begin{pmatrix}
a_{11} + b_{11}\tau_1 & \cdots & a_{1h} + b_{1h}\tau_h \\
a_{h1} + b_{h1}\tau_1 & \cdots & a_{hh} + b_{hh}\tau_h 
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
a_{(h+1)(h+1)} + b_{(h+1)(h+1)}\tau_{h+1} & \cdots & a_{(h+1)m} + b_{(h+1)m}\tau_m \\
a_{m(h+1)} + b_{m(h+1)}\tau_{h+1} & \cdots & a_{mm} + b_{mm}\tau_m 
\end{pmatrix}
\]
of (4.42) are invertible\(^{10}\) So if
\[
\Theta(s_1, \ldots, s_m) := e_1s_1 + \cdots + e_ms_m + \text{higher degrees},
\]
then
\[
\begin{pmatrix}
e_1 \\
\vdots \\
e_h
\end{pmatrix}
= \begin{pmatrix}
a_{11} + b_{11}\tau_1 & \cdots & a_{1h} + b_{1h}\tau_h \\
0 & \ddots & \ddots \\
a_{h1} + b_{h1}\tau_1 & \cdots & a_{hh} + b_{hh}\tau_h
\end{pmatrix}^{-1}
\begin{pmatrix}
a_{01} + b_{01}\tau_1 \\
0 & \ddots \\
a_{0h} + b_{0h}\tau_h \\
0 & \ddots \\
a_{0m} + b_{0m}\tau_m
\end{pmatrix}
\]
and
\[
e_{h+1} = \cdots = e_m = 0.
\]
Let
\[
u_{h+1} = \cdots = u_m = 0.
\]
By the assumptions on \( H' \), applying the same methods as in the proof of Theorem 4.6 it is concluded
\[
v_i(u_1, \ldots, u_h, 0, \ldots, 0, u_{m+1}, \ldots, u_n) \quad (1 \leq i \leq h)
\]
depends only on \( u_1, \ldots, u_h \). That is, cusps \( 1, \ldots, h \) of \( M \) are WGI from cusps \( m + 1, \ldots, n \). \( \square \)

\(^{10}\)If the determinant of (4.43) is 0, by Theorem 3.13 there exists an algebraic subgroup \( H'' \) containing \( H' \) such that \( X \cap H'' \) is an anomalous subvariety of \( X \). But it contradicts the assumption on \( H' \). If the determinant of (4.44) is 0, then it contradicts the assumption (4.44).
4.2.3. \( l = 0 \)

Lastly we prove \( l = 0 \) in (4.27). Once \( l = 0 \) is shown, combining with Theorems 4.9, 4.10 it will complete the proof of Theorem 1.4. To prove \( l = 0 \) in (4.27), it is enough to show \( \Theta \) in (4.31) is independent of \( t_j \) \((1 \leq j \leq l)\). Indeed, if \( \Theta \) depends only on \( s_1, \ldots, s_m \), it means an analytic set defined by

\[
\zeta_i = a_{i1}u_1 + b_{i1}v_1 + \cdots + a_{im}u_m + b_{im}v_m
\]

where \( \zeta_i \in \mathbb{C}(1 \leq i \leq m) \) and \( \zeta_0 = \Theta(\zeta_1, \ldots, \zeta_m) \) is an analytic subset of \( \log X \) of dimension \( n - m \). Said differently, a translation of an algebraic subgroup \( H \) of \( \mathbb{C}^n \) is invertible, so if

\[
\begin{pmatrix}
a_{i1} + b_{i1}\tau_1 & \cdots & a_{im} + b_{im}\tau_m \\
\vdots & & \vdots \\
a_{m1} + b_{m1}\tau_1 & \cdots & a_{mm} + b_{mm}\tau_m \\
\end{pmatrix}
\]

contains an anomalous subvariety of \( X \) of dimension \( n - m \). Since \( H \subset H^{(m+1)} \) and each \( X \cap gH \) in (4.27) is a maximal anomalous subvariety of \( X \) of dimension \( n - m - l, \ l = 0 \) follows.

Now we state

**Theorem 4.9.** \( \Theta \) in (4.31) is independent of \( t_j \) \((1 \leq j \leq l)\). That is, \( \Theta \) depends only on \( s_i \) \((1 \leq i \leq m)\).

By (4.32), the Jacobian matrix of \( s_1, \ldots, s_m, t_1, \ldots, t_l \) at \((u_1, \ldots, u_n) = (0, \ldots, 0)\) is

\[
\begin{pmatrix}
a_{i1} + b_{i1}\tau_1 & \cdots & a_{im} + b_{im}\tau_m \\
\vdots & & \vdots \\
a_{m1} + b_{m1}\tau_1 & \cdots & a_{mm} + b_{mm}\tau_m \\
a_{i1}' + b_{i1}'\tau_1 & \cdots & a_{im}' + b_{im}'\tau_m \\
\vdots & & \vdots \\
a_{m1}' + b_{m1}'\tau_1 & \cdots & a_{mm}' + b_{mm}'\tau_m \\
\end{pmatrix}
\]

To simplify the proof, we further reduce (4.45) to a row-echelon form using changes of variables as follows. By the assumption on \( H^{(m)} \) in (4.29)-(4.30),

\[
A := \begin{pmatrix}
a_{i1} + b_{i1}\tau_1 & \cdots & a_{im} + b_{im}\tau_m \\
\vdots & & \vdots \\
a_{m1} + b_{m1}\tau_1 & \cdots & a_{mm} + b_{mm}\tau_m \\
\end{pmatrix}
\]

is invertible, so if

\[
\begin{pmatrix}
x_1 \\
\vdots \\
x_m \\
\end{pmatrix} := A^{-1} \begin{pmatrix}
s_1 \\
\vdots \\
s_m \\
\end{pmatrix},
\]

then each \( x_i \) \((1 \leq i \leq m)\) is of the form

\[
x_i = u_i + \text{higher degrees}.
\]

Adding linear combinations of \( x_1, \ldots, x_m \) to each \( t_j \) \((1 \leq j \leq l)\) if necessary, we further suppose

\[
a_{i1}' = b_{i1}' = \cdots = a_{im}' = b_{im}' = 0 \quad (1 \leq j \leq l)
\]

in (4.32) and (4.45). Since

\[
X \cap H \not\subset (M_j = 1)
\]

remains anomalous, we prove

\[
\Theta(\zeta_1, \ldots, \zeta_m) = \Theta(0, \ldots, 0) = \Theta_0
\]

in (4.27), concluding the proof of Theorem 1.4.
for \( m + 1 \leq j \leq n \), equivalently, it implies
\[
e_j \notin R(A')
\] (4.49)
where \( R(A') \) is the row vector space of
\[
A' := \begin{pmatrix}
ad'_{1(m+1)} + b'_{1(m+1)} \tau_{m+1} & \cdots & ad'_{1n} + b'_{1n} \tau_n \\
\vdots & \ddots & \vdots \\
ad'_{l(m+1)} + b'_{l(m+1)} \tau_{m+1} & \cdots & ad'_{ln} + b'_{ln} \tau_n
\end{pmatrix}
\]
and \( e_j \) is a unit \((n-m)\times 1\) matrix whose \( j \)-th entry is 1. By (4.49), we therefore find an invertible \((l \times l)\)-matrix \( L \) such that, for
\[
\begin{pmatrix}
y_1 \\
\vdots \\
y_l
\end{pmatrix} = L \begin{pmatrix}
t_1 \\
\vdots \\
t_l
\end{pmatrix},
\] (4.50)
\[
\begin{pmatrix}
y_1 \\
\vdots \\
y_l
\end{pmatrix}
\]
is given as
\[
\begin{pmatrix}
\sum_{k=m_1}^{n_1} c_{1k} u_j + \text{higher degrees} \\
\vdots \\
\sum_{k=m_1}^{n_1} c_{lk} u_j + \text{higher degrees}
\end{pmatrix}
\] (4.51)
where the coefficients \( c_{jk} \) satisfy the following
- \( m_j < n_j \ (1 \leq j \leq l) \);
- \( c_{jm_j}, c_{jn_j} \neq 0 \ (1 \leq j \leq l) \);
- \( m + 1 \leq m_1 < \cdots < m_l \);
- \( n_1 < \cdots < n_l \leq n \).

In conclusion, by changing variables from \( s_1, \ldots, s_m, t_1, \ldots, t_l \) to \( x_1, \ldots, x_m, y_1, \ldots, y_l \) via (4.46), (4.48) and (4.50), the matrix in (4.45) is transformed into the following \((m + l) \times n\) row-echelon form matrix
\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0 & 0 & 0 \\
0 & \cdots & 0 & c_{1m_1} & \cdots & c_{1n_l} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & c_{lm_1} & \cdots & c_{ln_l}
\end{pmatrix}
\]
which is the Jacobian of \( x_1, \ldots, x_m, y_1, \ldots, y_l \) at \((u_1, \ldots, u_n) = (0, \ldots, 0)\). Clearly, \( \Theta \) is a function of \( x_1, \ldots, x_m, y_1, \ldots, y_l \) if and only if it is a function of \( x_1, \ldots, x_m, y_1, \ldots, y_l \). By abuse of notation, we rewrite (4.31) as
\[
a_{01} u_1 + b_{01} v_1 + \cdots + a_{0m} u_m + b_{0m} v_m = \Theta(x_1, \ldots, x_m, y_1, \ldots, y_l).
\] (4.53)

Now Theorem 4.9 is equivalent to the following theorem:

**Theorem 4.10.** \( \Theta \) in (4.53) is independent of \( y_1, \ldots, y_l \). That is, \( \Theta \) in (4.53) depends only on \( x_i \ (1 \leq i \leq m) \).
We prove the theorem by showing that, if \( \Theta \) depends on \( x_i \) (1 \( \leq \) \( i \) \( \leq \) \( m \)), then \( \Theta \), as a function of \( u_1, \ldots, u_n \) (by (4.47) and (4.51)), contains a term such that the exponent of some \( u_i \) (\( m + 1 \leq i \leq n \)) in the term is odd. However, according to Theorem 2.1, if some \( u_i \) (\( m + 1 \leq i \leq n \)) in each term of \( a_0u_1 + b_0v_1 + \cdots + a_mu_m + b_mv_m \) must be even and so it is a contradiction.

**Proof of Theorem 4.10** Let

\[
\Theta(x_1, \ldots, x_m, y_1, \ldots, y_l) := \sum_{(i_1, \ldots, i_m,j_1,\ldots,j_l) \in \mathbb{Z}^{m+l}} c_{i_1,\ldots,i_m,j_1,\ldots,j_l} x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_l^{j_l}, \tag{4.54}
\]

and \( S \) be the set of all monomials \( x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_l^{j_l} \) in (4.54) satisfying

1. \( c_{i_1,\ldots,i_m,j_1,\ldots,j_l} \neq 0 \);
2. \( (i_1, \ldots, i_m) \neq (0, \ldots, 0) \);
3. \( (j_1, \ldots, j_l) \neq (0, \ldots, 0) \);
4. \( i_1 + \cdots + i_m \) is the minimum.

We fix \( (i_1, \ldots, i_m) \) and define \( T(i_1, \ldots, i_m) \) to be the set of monomials \( x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_l^{j_l} \) in \( S \) further satisfying

1. each element in \( T(i_1, \ldots, i_m) \) is divisible by \( x_1^{i_1} \cdots x_m^{i_m} \);
2. \( j_1 + \cdots + j_l \) is the minimum.

Let \( \Theta_{T(i_1, \ldots, i_m)} \) be the following subseries of \( \Theta_{[1]}^\text{lead} \)

\[
\sum_{x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_l^{j_l} \in T(i_1, \ldots, i_m)} c_{i_1,\ldots,i_m,j_1,\ldots,j_l} x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_l^{j_l}.
\]

By (4.47) and (4.51), if \( \Theta_{T(i_1, \ldots, i_m)} \) is represented as a function of \( u_1, \ldots, u_n \), then

\[
\Theta_{T(i_1, \ldots, i_m)}^{\text{lead}} := \sum_{x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_l^{j_l} \in T(i_1, \ldots, i_m)} c_{i_1,\ldots,i_m,j_1,\ldots,j_l} x_1^{i_1} \cdots x_m^{i_m} y_1^{j_1} \cdots y_l^{j_l} u_1^{j_1} \cdots u_n^{j_l}
\]

is the leading terms of it. Since the degree of \( u_j \) (\( m + 1 \leq j \leq n \)) in each term of

\[
a_0u_1 + b_0v_1 + \cdots + a_mu_m + b_mv_m
\]

is even by Theorem 2.1, the same property must be true for \( \Theta \), \( \Theta_{T(i_1, \ldots, i_m)} \), \( \Theta_{T(i_1, \ldots, i_m)}^{\text{lead}} \).

However, we show this is impossible in the following claim.

For simplicity, we say a monomial \( u_1^{i_1} \cdots u_n^{i_n} \) is **odd** if some \( i_j \) (1 \( \leq \) \( j \) \( \leq \) \( n \)) is odd.

**Claim 4.11.** Let

\[
Z_i := \sum_{j=m_1}^{m_i} c_{1j} u_j, \quad \ldots, \quad Z_l := \sum_{j=m_l}^{n_l} c_{1j} u_j.
\]

For \( r \in \mathbb{N} \), any linear combination of elements in

\[
\{z_1^{j_1} \cdots z_l^{j_l} \mid j_1 + \cdots + j_l = r\}, \tag{4.55}
\]

is a contradiction.

---

11By the definition of \( T(i_1, \ldots, i_m) \), \( \Theta_{T(i_1, \ldots, i_m)} \) contains all the terms of \( \Theta \) of the smallest degree and divisible both by \( u_1^{i_1} \cdots u_m^{i_m} \) and some \( u_j \) where \( m + 1 \leq j \leq n \).
when considered as a polynomial in $u_j$ ($m + 1 \leq j \leq n$), contains an odd term.

Proof of the claim. We prove by induction on $l$. For $l = 1$, clearly

$$z_1^r = \left( \sum_{j=m_1}^{n_1} c_{1j} u_j \right)^r$$

contains an odd term as $z_1$ has at least two non-trivial terms (i.e. $u_{m_1}$ and $u_{n_1}$) by the assumptions given in (4.52).

Suppose $l \geq 2$ and the claim is true for $1, \ldots, l - 1$. Let

$$\sum_{j_1 + \cdots + j_l = r} c_{j_1 \ldots j_l} z_1^{j_1} \cdots z_l^{j_l} \quad (l \geq 2) \quad (4.56)$$

be any linear sum of elements in (4.55).

(1) First, assume $\tilde{c}_{r,0,\ldots,0} \neq 0$ in (4.56).

(a) If $r$ is odd, by (4.52), $u_{m_1}$ is a non-trivial odd term appearing only in $z_1^r$, thus it appears in (4.56) as well.

(b) For $r$ even, we split it into the following two cases.

(i) If the coefficient of $z_1^{r-1} z_j$ in (4.56) is non-zero for some $j$ ($2 \leq j \leq l$), let $h$ be the largest such $j$. Then $u_{m_1}^{-1} u_{n_h}$ is odd and appears only in $z_1^{r-1} z_h$ (again by (4.52)), hence (4.56) possesses the desired property.

(ii) If the coefficient of $z_1^{r-1} z_j$ in (4.56) is zero for every $j$ ($2 \leq j \leq l$), then an odd monomial $u_{m_1}^{-1} u_{n_1}$ appears only in $z_1^r$, so it does in (4.56) as well.

(2) Now suppose $\tilde{c}_{r,0,\ldots,0} = 0$ in (4.56). By induction, for each $k$ ($0 \leq k < r$), any linear combination of elements in $\{ z_2^{j_2} \cdots z_l^{j_l} \mid j_2 + \cdots + j_l = r - k \}$, when represented as a polynomial of $u_j$ ($m + 1 \leq j \leq n$), has an odd term. Hence, for each $k$ ($0 \leq k < r$), a linear sum of any elements in

$$Z_k := \{ z_1^{j_1} \cdots z_l^{j_l} \mid j_1 + \cdots + j_l = r, \quad j_1 = k \}$$

contains an odd term as well divisible by $u_{m_1}^k$. This further implies a linear combination of elements in $\bigcup_{k=0}^{r-1} Z_k$, when expressed as a polynomial in $u_j$ ($m + 1 \leq j \leq n$), contains a non-trivial odd term.

This completes the proofs of Claim 4.11 as well as Theorems 4.9, 4.10.

5. 2-cusped case

In this section, we prove Theorems 1.6 and 1.7.

5.1. Classification of $\mathcal{X}^{oa} = \emptyset$ (Proof of Theorem 1.6)

A more detailed description of Theorem 1.6 is stated as

Theorem 5.1. Let $M$ be a 2-cusped hyperbolic 3-manifold and $\mathcal{X}$ be its holonomy variety. If $\mathcal{X}^{oa} = \emptyset$, then $\mathcal{X}$ is the product of two algebraic curves. More precisely, if $\mathcal{X}^{oa} = \emptyset$, then either one of the following holds:
Then define a 1-dimensional complex manifold for infinitely many $\xi$ following equations

\[ f(M_1^aL_1^bM_2^aL_2^b, M_1^mM_2^m) = 0, \quad f(M_1^aL_1^bM_2^aL_2^b, M_1^mM_2^m) = 0 \]  

(5.1)

for some $a, b, c, d \in \mathbb{Z}$ and $m \in \mathbb{Q}$ satisfying $mbd \neq 0$.

The proof of the above theorem is based on Theorem 2.8 as well as various symmetric properties of $\log X$ given in Theorem 2.4.

\textbf{Proof.} By Theorem 1.3, it is enough to show that if $M$ has rationally dependent cusp shapes and the two cusps of $M$ are not SGI each other, then $X$ is defined by equations given in (5.1).

Since $X^{\text{an}} = \emptyset$, by Theorem 2.8, $X$ is foliated by maximal anomalous subvarieties contained in

\[ \bigcup_{g \in \mathbb{Z}_H} X \cap gH. \]

Let $H$ be defined by

\[ M_1^{a_1}L_1^{b_1}M_2^{a_1}L_2^{d_1} = 1, \quad M_1^{a_2}L_1^{b_2}M_2^{a_2}L_2^{d_2} = 1. \]

(5.2)

By changing basis if necessary, we assume (5.2) is of the following form

\[ M_1^{a_1}L_1^{b_1}M_2^{a_1}L_2^{d_1} = 1, \quad M_1^{a_2}M_2^{b_2} = 1. \]

Then $X \cap H$ is locally biholomorphic to

\[ a_1u_1 + b_1v_1 + c_1u_2 + d_1v_2 = 0, \quad a_2u_1 + c_2u_2 = 0. \]

If $X \cap gH$ is an anomalous subvariety of $X$ for infinitely many $g \in \mathbb{Z}_H$, equivalently, the following equations

\[ a_1u_1 + b_1v_1 + c_1u_2 + d_1v_2 = \xi_1, \quad a_2u_1 + c_2u_2 = \xi_2 \]

define a 1-dimensional complex manifold for infinitely many $\xi_1, \xi_2 \in \mathbb{C}$. Thus there exists a holomorphic function $h$ such that

\[ a_1u_1 + b_1v_1 + c_1u_2 + d_1v_2 = h(a_2u_1 + c_2u_2). \]

(5.3)

If $b_1 = 0$ (resp. $d_1 = 0$), then one can easily check that $d_1 = 0$ (resp. $b_1 = 0$) and

\[ a_1u_1 + c_1u_2 = m(a_2u_1 + c_2u_2) \]

for some $m \in \mathbb{Q} \setminus \{0\}$. But this contradicts the fact that $H$ is an algebraic subgroup of dimension 2. Without loss of generality, we assume $b_1, d_1 \neq 0$. Now we claim

**Claim 5.2.**

\[ (c_2, a_2) = m(b_1, d_1) \]

for some $m \in \mathbb{Q} \setminus \{0\}$.

\textbf{Proof of the claim.} Let $h(t)$ in (5.3) be defined by $\sum_{i=1}^{\infty} e_it^{2i-1}$ and so

\[ a_1u_1 + b_1v_1 + c_1u_2 + d_1v_2 = \sum_{i=1}^{\infty} e_1(a_2u_1 + c_2u_2)2i-1 = \sum_{i=1}^{\infty} e_1 \left( \sum_{j=0}^{2i-1} \binom{2i-1}{j} a_2^{2i-1-j} c_2^{2i-1-j} u_1^{j} u_2^{2i-1-j} \right). \]

(5.4)
By Theorem 2.1 since the degree of \( u_i \) (resp. \( u_{i+1} \)) in every term of \( v_i \) is odd (resp. even), we split (5.4) as follows:

\[
a_1 u_1 + b_1 v_1 = \sum_{i=1}^{\infty} e_i \left( \sum_{j=0,\text{even}}^{2i-2} \binom{2i-1}{j} a_2^{2i-1-j} c_j u_1^{2i-1-j} w_2^j \right),
\]
\[
c_1 u_2 + d_1 v_2 = \sum_{i=1}^{\infty} e_i \left( \sum_{j=0,\text{even}}^{2i-2} \binom{2i-1}{j} c_2^{2i-1-j} a_j u_1^{2i-1-j} w_1^j \right). \tag{5.5}
\]

Since \( \frac{1}{2} \frac{\partial \Phi}{\partial u_1} = v_1 \) and \( \frac{1}{2} \frac{\partial \Phi}{\partial u_2} = v_2 \) by Theorem 2.1 we get

\[
\frac{1}{b_1} \frac{1}{2i - j} e_i \binom{2i-1}{j} a_2^{2i-1-j} c_j u_1^{2i-1-j} w_2^j = \frac{1}{d_1} \frac{1}{2i - l} e_i \binom{2i-1}{l} c_2^{2i-1-l} a_l u_2^{2i-1-l} w_1^l \tag{5.6}
\]

for all \( i, j, l \) such that \( j + l = 2i (\geq 4) \) from (5.5). Now (5.6) implies

\[
\frac{1}{b_1} \frac{1}{2i - j} \binom{2i-1}{j} a_2^{2i-1-j} c_j = \frac{1}{d_1} \frac{1}{2i - l} \binom{2i-1}{l} c_2^{2i-1-l} a_l \quad \Rightarrow \quad \frac{1}{l} \binom{2i-1}{j} d_1 c_2 = \frac{1}{j} \binom{2i-1}{l} b_1 a_2
\]

\[
\Rightarrow \quad \frac{1}{l} \frac{1}{j!} (2i - 1) (2i - 1 - j)! b_1 a_2 = \frac{1}{1} \frac{1}{j!} (2i - 1) (2i - 1 - j)! b_1 a_2
\]

\[
\Rightarrow \quad \frac{1}{l} \frac{1}{j!} (2i - 1) (2i - 1 - j)! b_1 a_2 = \frac{1}{1} \frac{1}{j!} (2i - 1) (2i - 1 - j)! b_1 a_2
\]

This completes the proof of the claim. \( \square \)

By Theorem 2.1 since the degree of \( u_1 \) (resp. \( u_2 \)) in each term of \( v_2 \) is even (resp. odd), (5.4) implies

\[
a_1 u_1 + b_1 v_1 - c_1 u_2 - d_1 v_2 = \sum_{i=1}^{\infty} e_i (a_2 u_1 - c_2 u_2)^{2i-1} = h(a_2 u_1 - c_2 u_2). \tag{5.7}
\]

Let \( \mathcal{C} := \mathcal{X} \cap (M_2 = L_2 = 1) \) and \( \mathcal{C}' \) be the image of \( \mathcal{C} \) under the following transformation:

\[
M_1' := M_1^{a_1} L_1^{b_1}, \quad L_1' := M_1^{a_2}.
\]

By projecting onto the first two coordinates if necessary, we consider \( \mathcal{C}' \) as an algebraic curve in \( \mathbb{C}^2 := (M_1', L_1') \). Let \( f(M_1', L_1') = 0 \) be the defining equation of \( \mathcal{C}' \), which is, near \( (1, 1) \), locally biholomorphic to

\[
v_1' = h(u_1')
\]

where \( u_1' := \log M_1', v_1' := \log L_1' \). Then (5.3) (resp. (5.7)) is equivalent to

\[
f(M_1^{a_1} L_1^{b_1} M_2^{a_2} L_2^{d_1}, M_1^{a_1} M_2^{c_2}) = 0 \quad \text{(resp. } f(M_1^{a_1} L_1^{b_1} M_2^{c_1} L_2^{-d_1}, M_1^{a_2} M_2^{c_2}) = 0). \tag{5.8}
\]

Since \( v_1 \) and \( v_2 \) are determined by (5.3) and (5.7), \( \mathcal{X} \) is defined by the two equations in (5.8). \( \square \)
5.2. Zilber-Pink conjecture (Proof of Theorem 1.7)

Finally we prove our last main result Theorem 1.7 in this subsection. Before proving it, we first quote a couple of theorems needed in the proof.

**Theorem 5.3.** Let $M$ be a 1-cusped hyperbolic 3-manifold and $X$ be its holonomy variety. Then $X^{oa} \neq \emptyset$ and the height of $X \cap H$ is uniformly bounded for any algebraic subgroup $H$ of dimension 1. Moreover, $X$ contains only finitely many torsion points.

**Proof.** See Theorems 3.10-11 in [6] for the first statement. The last one follows from Theorem 1.2. □

The following is Lemma 8.1 in [1].

**Theorem 5.4 (Bombieri-Masser-Zannier).** Let $X$ be an algebraic variety in $G^n$ of dimension $k \leq n-1$ defined over $\overline{\mathbb{Q}}$ and $X^{ta}$ be the complement of torsion anomalous subvarieties of $X$. Then for any $B \geq 0$ there are at most finitely many points $P$ in $X^{ta}(\overline{\mathbb{Q}}) \cap H_{n-k-1}$ with $h(P) \leq B$ where $H_{n-k-1}$ is the set of algebraic subgroups of dimension $n-k-1$ and $h(P)$ is the height of $P$.

Also recall the following theorem of Maurin [9] mentioned earlier in Section 1.1.

**Theorem 5.5 (Maurin).** Let $C$ be an algebraic curve defined over $\mathbb{Q}$ and $H_2$ be the set of all the algebraic subgroups of codimension 2. If $H_2 \cap C$ is not finite, then $C$ is contained in an algebraic subgroup.

Now we prove Theorem 1.7.

**Theorem 1.7** Let $M$ be a 2-cusped hyperbolic 3-manifold and $X$ be its holonomy variety. Then the Zilber-Pink conjecture is true for $X$.

**Proof.** First note that $X$ is not contained in an algebraic subgroup by Theorem 3.10 in [6]. To prove the theorem, it is enough to show that $X$ has only finitely many 1-dimensional torsion anomalous subvarieties and, possibly except for finitely many, most torsion anomalous points of $X$ are lying over those 1-dimensional torsion anomalous subvarieties.

1. Suppose $X^{oa} \neq \emptyset$  
   We first assume $X$ contains infinitely many torsion anomalous points. More precisely, let $\{H_i\}_{i \in \mathcal{I}}$ be a family of infinitely many different algebraic subgroups of dimension 1 such that $X \cap H_i \neq \emptyset$ for each $i \in \mathcal{I}$. Then

   $$\bigcup_{i \in \mathcal{I}} X^{oa} \cap H_i$$

   is a finite set by Theorem 1.2 and thus, except for those finitely many, almost all of

   $$\bigcup_{i \in \mathcal{I}} X \cap H_i \quad (5.9)$$

12Since $X$ is an algebraic surface, $X^{oa} \neq \emptyset$ implies $X$ has only finitely many 1-dimensional anomalous subvarieties and they are all maximal.
are all contained in anomalous subvarieties of $\mathcal{X}$.

Let $K$ be an algebraic coset such that $\mathcal{X} \cap K$ is an anomalous subvariety of $\mathcal{X}$ containing infinitely many points in (5.9). We claim $\mathcal{X} \cap K$ is a torsion anomalous subvariety of $\mathcal{X}$. Since $\mathcal{X} \cap K$ is an algebraic curve in $G^4$ intersecting with infinitely many 1-dimensional algebraic subgroups $H_i$, there exists an algebraic subgroup $H$ such that

$$\mathcal{X} \cap K \subset H$$

(5.10)

by Maurin’s theorem. Let $H$ be an algebraic subgroup of the smallest dimension satisfying (5.10). If there exists $H_i$ ($i \in I$) such that $\dim(H_i \cap H) = 0$ and $\mathcal{X} \cap H_i \subset K$, then $\mathcal{X} \cap H_i (= H \cap H_i)$ is torsion and thus $K$ is a torsion algebraic coset. Now suppose $\dim(H_i \cap H) = 1$ for all $i \in I$. Projecting onto $H$, we regard $\mathcal{X} \cap K \cap H$ as an algebraic curve in $G^{\dim H}$ ($\cong H$). Since (5.11) intersects with infinitely many $H_i \cap H$, if

$$\dim H_i + \dim(\mathcal{X} \cap K) < \dim H,$$

there exists an algebraic subgroup $H'$ ($\subset H$) satisfying

$$\mathcal{X} \cap K \subset H'$$

by Maurin’s theorem again. However this contradicts the assumption on $H$. Therefore

$$2 = \dim H_i + \dim(\mathcal{X} \cap K) \geq \dim H$$

and so either $\dim H = 1$ or 2.

(a) Suppose $\dim H = 1$.

(i) If $\dim(\mathcal{X} \cap H) = 1$, then

$$\mathcal{X} \cap K = \mathcal{X} \cap H$$

and thus $\mathcal{X} \cap K$ is a torsion anomalous subvariety of $\mathcal{X}$.

(ii) If $\dim(\mathcal{X} \cap H) = 0$, then it $\mathcal{X} \cap H$ is a set of finite points, contradicting the fact that it contains infinitely many $\mathcal{X} \cap H_i$.

(b) If $\dim H = 2$, as $\mathcal{X} \not\subset H$ (by Theorem 3.10 in [6]) and $\mathcal{X} \cap H_1 \subset \mathcal{X} \cap H$ for infinitely many $i$, we get $\dim (\mathcal{X} \cap H) = 1$. That is,

$$\mathcal{X} \cap K = \mathcal{X} \cap H,$$

implying $\mathcal{X} \cap K$ is a torsion anomalous subvariety of $\mathcal{X}$.

(2) Now we assume $\mathcal{X}^{oa} = \emptyset$.

(a) First suppose two cusp of $\mathcal{M}$ are SGI each other and so $\mathcal{X}$ is defined by

$$f_1(M_1, L_1) = 0, \quad f_2(M_2, L_2) = 0$$

where $f_i(M_i, L_i) = 0$ is the defining equation of $\mathcal{X} \cap (M_i = L_i = 1)$. Then every anomalous subvariety of $\mathcal{X}$ is either

$$(f_1(M_1, L_1) = 0) \cap (M_2 = \xi_1, L_2 = \xi_2)$$

or

$$(M_1 = \xi_1, L_1 = \xi_2) \cap (f_2(M_2, L_2) = 0)$$

for some $\xi_1, \xi_2 \in \mathbb{C}$. 39
Let \( \{ H_i \}_{i \in I} \) be a family of infinitely many 2-dimensional algebraic subgroups such that \( \{ X \cap H_i \}_{i \in I} \) are anomalous subvarieties of \( X \). For each \( i \), there exist \( \xi_{i1}, \xi_{i2} \in \mathbb{C} \) such that

\[
X \cap H_i = (f_1(M_1, L_1) = 0) \cap (M_2 = \xi_{i1}, L_2 = \xi_{i2})
\]

(5.12)

or

\[
X \cap H_i = (M_1 = \xi_{i1}, L_1 = \xi_{i2}) \cap (f_2(M_2, L_2) = 0).
\]

Without loss of generality, we assume the first case and \( H_i \) is defined by

\[
M_1^{a_{1i} b_{1i}} M_2^{c_{1i} d_{1i}} L_1^{e_{1i}} L_2^{f_{1i}} = 1,
\]

\[
M_1^{a_{2i} b_{2i}} M_2^{c_{2i} d_{2i}} L_1^{e_{2i}} L_2^{f_{2i}} = 1.
\]

(5.13)

(A) Suppose \( c_{1i} d_{2i} - c_{2i} d_{1i} \neq 0 \).

- If \( a_{1i} b_{2i} - a_{2i} b_{1i} \neq 0 \), then \( H_i \cap (M_2 = \xi_{i1}, L_2 = \xi_{i2}) \) is

\[
M_1^{a_{1i} b_{1i}} \xi_{i1}^e \xi_{i2}^f = 1,
\]

\[
M_1^{a_{2i} b_{2i}} \xi_{i1}^e \xi_{i2}^f = 1.
\]

(5.14)

and so \( M_1, L_1 \) are also constants, which contradicts the fact \( \dim(X \cap H_i) = 1 \).

- If \( a_{1i} b_{2i} - a_{2i} b_{1i} = 0 \), without loss of generality, we further assume \( c_{1i} d_{2i} - c_{2i} d_{1i} = 0 \), and so \( X \cap H_i \) is of the form:

\[
M_1^{a_{1i} b_{1i}} M_2^{c_{1i} d_{1i}} L_1^{e_{1i}} L_2^{f_{1i}} = 1,
\]

\[
M_2^{c_{2i} d_{2i}} L_2^{f_{2i}} = 1.
\]

(5.15)

If \( a_{1i} \neq 0 \) or \( b_{1i} \neq 0 \), then \( H_i \cap (M_2 = \xi_{i1}, L_2 = \xi_{i2}) \) is

\[
M_1^{a_{1i} b_{1i}} \xi_{i1}^e \xi_{i2}^f = 1,
\]

\[
\xi_{i1}^e \xi_{i2}^f = 1.
\]

(5.16)

and so

\[
X \cap H_i \cap (M_2 = \xi_{i1}, L_2 = \xi_{i2})
\]

is a set of points, again contradicting the assumption that it is 1-dimensional. Thus \( a_{1i} = b_{1i} = 0 \) and \( \xi_{i1}, \xi_{i2} \) in (5.12) are torsion numbers. By Theorem (5.3), as there are only finitely many torsion points in \( f_2(M_2, L_2) = 0 \), it is concluded that

\[
\bigcup_{i \in I} X \cap H_i
\]

is a family of finitely many torsion anomalous subvarieties of \( X \).

(B) If \( c_{1i} d_{2i} - c_{2i} d_{1i} = 0 \), we simply assume (5.13) is of the following form:

\[
M_1^{a_{1i} b_{1i}} M_2^{c_{1i} d_{1i}} L_1^{e_{1i}} L_2^{f_{1i}} = 1,
\]

\[
M_1^{a_{2i} b_{2i}} L_1^{e_{2i}} = 1.
\]

(5.17)

Since \( M_1, L_1 \) are constants on

\[
(f_1(M_1, L_1) = 0) \cap (M_1^{a_{2i} b_{2i}} L_1^{e_{2i}} = 1),
\]

(40)
we conclude
\[ \mathcal{X} \cap H_i \cap (M_2 = \xi_{1i}, L_2 = \xi_{2i}) \]
is 0-dimensional, which is a contradiction.

(ii) Suppose \( \{ H_i \}_{i \in \mathcal{I}} \) are infinitely many algebraic subgroups of dimension 1 such that
\[ \mathcal{X} \cap H_i \neq \emptyset \]
for each \( i \in \mathcal{I} \). Without loss of generality, by applying Gauss elimination if necessary, we further assume \( H_i \) is defined by
\[
\begin{align*}
M_1^{a_{1i}}L_1^{b_{1i}}M_2^{c_{1i}}L_2^{d_{1i}} &= 1, \\
L_1^{b_{2i}}M_2^{c_{2i}}L_2^{d_{2i}} &= 1, \\
M_2^{c_{3i}}L_2^{d_{3i}} &= 1.
\end{align*}
\]
If
\[ P_i := (\xi_{1i}, \xi_{2i}, \xi_{3i}, \xi_{4i}) \in \mathcal{X} \cap H_i, \]
since
\[ (\xi_{3i}, \xi_{4i}) \in (f_2(M_2, L_2) = 0) \cap (M_2^{c_{3i}}L_2^{d_{3i}} = 1), \]
the height of \((\xi_{3i}, \xi_{4i})\) is uniformly bounded by Theorem 5.3. Similarly, the height of \((\xi_{1i}, \xi_{2i})\) is also uniformly bounded. By Theorem 5.4, possibly except for finitely many, \( \{ P_i \}_{i \in \mathcal{I}} \) are contained in torsion anomalous subvarieties of \( \mathcal{X} \). As shown above, \( \mathcal{X} \) has only finitely many torsion anomalous subvarieties, so it is finally concluded
\[
\bigcup_{i \in \mathcal{I}} \mathcal{X} \cap H_i
\]
is contained in finitely many torsion anomalous subvarieties of \( \mathcal{X} \) except for finitely many elements of it.

(b) If two cusps of \( \mathcal{M} \) are not SGI each other, by Theorem 5.1, there exists a polynomial \( f(x, y) = 0 \) such that \( \mathcal{X} \) is defined by
\[
\begin{align*}
f(M_1^aL_1^bM_2^cL_2^d, M_1^{md}M_2^{mb}) &= 0, \\
f(M_1^aL_1^bM_2^{-c}L_2^{-d}, M_1^{md}M_2^{-mb}) &= 0. \quad (5.18)
\end{align*}
\]
Let
\[
M'_1 := M_1^aL_1^bM_2^dL_2^d, \quad L'_1 := M_1^{md}M_2^{mb}, \quad M'_2 := M_1^aL_1^bM_2^{-c}L_2^{-d}, \quad L'_2 := M_1^{md}M_2^{-mb}.
\]
Then (5.18) becomes
\[
\begin{align*}
f(M'_1, L'_1) &= 0, \\
f(M'_2, L'_2) &= 0. \quad (5.19)
\end{align*}
\]
and so the problem is reduced to the previous case.

\[ \square \]
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Department of Mathematics, POSTECH
77 Cheong-Am Ro, Pohang, South Korea

Email Address: bogwang.jeon@postech.ac.kr