2D Schrödinger Equation with Singular Even-Power and Inverse-Power Potentials in Non-Commutative Complex space

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Abstract

We obtain exact solutions of the 2D Schrödinger equation with the Singular Even-Power and Inverse-Power Potentials in non-commutative complex space, using the Power-series expansion method. Hence we can say that the Schrödinger equation in non-commutative complex space describes to the particles with spin (1/2) in an external uniform magnetic field. Where the noncommutativity play the role of magnetic field with created the total magnetic moment of particle with spin 1/2, who in turn shifted the spectrum of energy. Such effects are similar to the Zeeman splitting in a commutative space.

Keywords: Central potential, Noncommutative Geometry, complex space,
Pacs numbers: 11.10.Nx, 32.30-r, 03.65-w

1 introduction:

The idea of a noncommutative (NC) space is not new. It can be traced back to Heisenberg, Pauli etc. The Heisenberg algebra:

\[
[x_i, p_j] = i \delta_{ij}, [p_i, p_j] = 0
\]  

is extended with new NC commutation relations between the position coordinate operators themselves:

\[
[\hat{x}_i, \hat{x}_j] = i \theta_{ij}
\]  

This leads to new uncertainty relations:

\[
\Delta \hat{x}^i \Delta \hat{x}^j \geq \frac{1}{2} |\theta^{ij}| \]
which are a good analogy to the Heisenberg uncertainty relations. Where \( \hat{x}^i = x^i - \frac{2}{i} \theta^i_j p_j \) and the parameter \( \theta^{ij} \) is an antisymmetric real matrix of dimension length-square for the noncommutative canonical-type space. Then the space non-commutativity is an extension of quantum mechanics, and a cutoff could provide a solution to the infinities appearing in quantum field theory. An example of a system where space coordinates do not commute is that of a particle in a strong magnetic field. There has recently been a lot of interest in the study of noncommutative canonical-type quantum mechanics [1, 2]. On other hand the solutions of classical dynamical problems of physical systems obtained in terms of complex space variables are well-known. There are also interests in the complex quantum mechanics systems (in two dimensions) [1 − 3], in which we consider a quantum free particle with Singular Even-Power and Invers-Power potentials in noncommutative complex quantum space (so coordinate and momentum operators of this space are written \( \hat{\xi} = \hat{x} + i \hat{y} \) and \( p_\zeta = (p_x - i p_y) / 2 \)).

Furthermore in particular the Singular Even-Power and Invers-Power potentials is useful to study the atomics physics and optical physics [3 − 12]. In this work, we study the effect of the non-commutativity to the free particle with Singular Even-Power and Invers-Power potentials.

This paper is organized as follows. In section 2 we derive the deformed Schrödinger equation for free particle with Singular Even-Power and Invers-Power potentials in noncommutative space, we solve this equation and obtain the non-commutative modification of the energy levels. In section 3 we derive the deformed Schrödinger equation for free particle with Singular Even-Power and Invers-Power potentials in noncommutative complex space, we solve this equation and obtain the non-commutative modification of the energy levels. Finally, in section 4, we draw our conclusions.

## 2 Schrödinger equation with central potential in non-commutative space:

The 2D Schrödinger equation with the central potential \( V(\hat{r}) \) (Singular Even-Power or Invers-Power potentials), in noncommutative space, which have the following form:

\[
\left( -\frac{1}{2m} \Delta + V(\hat{r}) \right) \Psi (r, \varphi) = \hat{E} \Psi (r, \varphi)
\]

where \( \hat{r} \) in noncommutative space as:

\[
\hat{r} = r - \frac{\theta \cdot L}{2r} + O(\Theta^2)
\]

Then eq.(4) will be written to the form:

\[
\left( -\frac{1}{2m} \Delta + V \left( r - \frac{\theta \cdot L}{2r} \right) \right) \Psi (r, \varphi) = \hat{E} \Psi (r, \varphi)
\]
The solution of eq. (7) in polar coordinates \((r, \varphi)\) takes the separable form:

\[
\Psi (r, \varphi) = \frac{1}{\sqrt{2\pi}} \frac{1}{r^{1/2}} R_m (r) \exp (\pm im \varphi)
\]  

(7)

Then eq. (7) reduces to the radial equation:

\[
\left( -\frac{1}{2m} \left[ \frac{d^2}{dr^2} - \frac{m^2 - 1/4}{r^2} \right] + V \left( r - \frac{\theta \cdot L}{2r} \right) \right) R(r) = \hat{E} R(r)
\]

(8)

2.1 The Singular Even-Power potential

In the noncommutative spac the Singular Even-Power potential \(V \left( r - \frac{\theta \cdot L}{2r} \right)\) in eq.(8) takes the form:

\[
V \left( r - \frac{\theta \cdot L}{2r} \right) = \hat{V} (r) + V^\theta_{NC} (r)
\]

(9)

where

\[
\hat{V} (r) = ar^2 + br^{-2} + c r^{-4} + d r^{-6} - a \theta \cdot L + \mathcal{O} (\Theta^2)
\]

(10)

and \(V^\theta_{NC} (r)\) is the perturbation term up to \(\mathcal{O} (\Theta^2)\) which takes the form:

\[
V^\theta_{NC} (r) = \frac{d \theta \cdot L}{r^8}.
\]

(11)

and

\[
\hat{c} = c + b \theta \cdot L, \quad \hat{d} = d + 2c \theta \cdot L
\]

(12)

For simplicity, first of all, we choose the coordinate system \((x, y, z)\) so that \(\theta^x = -\theta^y = \theta^z = \theta_z\), such that \(\theta \cdot L = \theta L_z\) and assume that the other components are all zero, then the noncommutative Singular Even-Power potential up to \(\mathcal{O} (\Theta^2)\) as the form:

\[
\hat{V} (r) + V^\theta_{NC} (r) = \hat{V}_O (r) + V^\theta_{pert}
\]

(13)

where

\[
\left\{ \begin{array}{l}
\hat{V}_O (r) = -a \theta m + ar^2 + br^{-2} + c r^{-4} + d r^{-6}, \\
V^\theta_{pert} = V^\theta_{NC} (r) = \frac{d \theta m}{r^8}
\end{array} \right.
\]

(14)

and

\[
\hat{c} = c + b \theta m, \quad \hat{d} = d + 2c \theta m
\]

(15)

To investigate the modification of the energy levels by eq. (14), we use the first-order perturbation theory. The spectrum of \(\hat{H}_0 = -\frac{1}{2m} \Delta + \hat{V}_O (r)\), and the corresponding radial wave functions are well-known and given by [13]:
\[ \Phi (\varphi) = \frac{1}{\sqrt{2\pi}} \exp (\pm i m \varphi) , \quad m = 0, 1, 2... \]
\[ R_{n}^{m} (r) = \exp \left( -\frac{ar^{2} + \sqrt{dr^{2}}}{2} \right) \left( \sum_{p=0}^{n} a_{n} r^{n+p} \right) , \quad (16) \]

where
\[ \hat{\delta} = 3/2 + \hat{c}/ \left( 2\sqrt{d} \right) \]
and the noncommutative energy eigenvalue up to \( O (\Theta^2) \) as:
\[ \hat{E}_{n,m} = \sqrt{a} (2\hat{\mu} + 4 + 4n) - \theta am , \quad \hat{\mu} = \hat{c}/ \left( 2\sqrt{d} \right) \]

the noncommutative correction of the energy levels in the first order of \( \theta \) to the \( n^{th} \) of excitation state:
\[ \Delta E_{\Theta}^{\theta} = d\theta m A \]

where
\[ A = \int \left[ r^{-4} \exp \left( -\frac{ar^{2} + \sqrt{dr^{2}}}{2\sqrt{b}} \right) \left( \sum_{p=0}^{n} a_{n} r^{n+p} \right) \right]^{2} r dr \]

2.2 The Invers-Power potential

In noncommutative space the Invers-Power potential \( V \left( r - \frac{\theta L}{2r} \right) \) in eq.(8) takes the form:
\[ V (\hat{r}) = \hat{V}_{O} (r) + V^{\theta}_{pert} (r) \]

where
\[ \hat{V}_{O} (r) = ar^{-1} + \hat{c}r^{-2} + \hat{d}r^{-3} + \hat{\theta}r^{-4} + O (\Theta^2) \]

and \( V^{\theta}_{NC} (r) \) is the perturbation term up to \( O (\Theta^2) \) which takes the form:
\[ V^{\theta}_{pert} (r) = \left( -\frac{2c}{r^{\mu}} - \frac{3d}{r^{\mu}} \right) \theta m \]

and
\[ \hat{c} = c + a\theta m , \quad \hat{d} = d - b\theta m \]

To investigate the modification of the energy levels by eq. (23), we use the first-order perturbation theory. The spectrum of \( \hat{H}_{0} = -\frac{1}{2m} \Delta + \hat{V}_{O} (r) \), and the corresponding radial wave functions are well-known and given by [14]:
\[
\begin{align*}
R^m_n (r) &= N_m r^c \left( \prod_{i=1}^{m} (r - \sigma_i^m) \right) \exp \left( \frac{a + br^2}{r} \right), \\
\hat{E}_m &= - \left( \frac{\hat{\omega}_1 \pm \sqrt{\hat{\omega}_1^2 - 4\hat{D}(2m+\hat{\mu})(\sum_{i=1}^{m} \sigma_i^m + \sqrt{A})}}{4(\sum_{i=1}^{m} \sigma_i^m + \sqrt{A})} \right)^2.
\end{align*}
\]

where

\[
\hat{\omega}_1 = \hat{\lambda} + m(2m + \hat{\mu}) - m(m - 1), \quad \hat{D} = 2 \hat{b}(\hat{c} + m), \quad A = a^2
\]

and

\[
\hat{\lambda} = (1 + \hat{\mu})(2m + \hat{\mu}) + m(m - 1) + 2 \hat{b} \left( \sum_{i=1}^{m} \sigma_i^m + \sqrt{A} \right), \quad \hat{\mu} = \hat{c} - 1
\]

the non-commutative correction of the energy levels in the first order of \( \theta \) to the \( p^{th} \) of excitation state:

\[
\Delta E_{NC}^\theta = -\theta m (2cf(5) + 3df(6))
\]

where

\[
f(5) = \int r^{-4} \left[ N_m r^c \left( \prod_{i=1}^{m} (r - \sigma_i^m) \right) \exp \left( \frac{a + br^2}{r} \right) \right]^2 dr
\]

and

\[
f(6) = \int r^{-5} \left[ N_m r^c \left( \prod_{i=1}^{m} (r - \sigma_i^m) \right) \exp \left( \frac{a + br^2}{r} \right) \right]^2 dr
\]

We have shown that the energy spectrum depends on \( m \). Then we deduced that the non-commutativity plays the role of magnetic field.

3 Schrödinger equation with Singular Even-Power and Invers-Power potentials in non-commutative complex space:

In two dimensional space, the complex coordinates system \((z, \bar{z})\) and momentum \((p_z, p_{\bar{z}})\) as dened by [1, 2]:

\[
\begin{align*}
&z = x + iy \quad \text{and} \quad \bar{z} = x - iy, \\
p_z = \frac{1}{2} (p_x - ip_y) \quad \text{and} \quad p_{\bar{z}} = -\bar{p}_z = \frac{1}{2} (p_x + ip_y)
\end{align*}
\]

5
We are interested to introduce the non-commutative complex operators coordinates and their momentums in a 2D complex space as follows:

\[
\begin{align*}
\hat{z} &= \hat{x} + i\hat{y} = z + \theta \hat{p}_\bar{z} \\
\bar{z} &= \hat{x} - i\hat{y} = \bar{z} - \theta \hat{p}_z \\
\hat{p}_z &= p_z, \quad \bar{p}_\bar{z} &= p_{\bar{z}}
\end{align*}
\]  

The noncommutative algebra (2) can be written as:

\[
[\hat{z}, \bar{z}] = 2\theta, \quad [\hat{z}, p_{\bar{z}}] = [\bar{z}, p_z] = 2\hbar, \quad [p_z, p_{\bar{z}}] = 0
\]  

Then the noncommutative complex operators coordinates as not \( PT \) symmetric

\[ PT \hat{z} PT \neq -\hat{z} \]

In the noncommutative complex space we notice that \( \hat{z}\bar{z} \neq \bar{z}\hat{z} \). Then we can show that, in the first order of the parameter \( \theta \):

\[
\begin{align*}
\hat{z}\bar{z} &= z\bar{z} - \theta (L_z - 1) \\
\bar{z}\hat{z} &= z\bar{z} - \theta (L_z + 1), \quad s_z = -1/2
\end{align*}
\]

The momentum operator \( \hat{p}^2 \) can be written in 2D non-commutative Complex space as follows:

\[
\hat{p}^2 = 4p_z p_{\bar{z}} = (p_x + ip_y)(p_x - ip_y) = 4p_z p_{\bar{z}} = p^2
\]

In the noncommutative complex coordinate \((\hat{z}, \bar{z})\), the noncommutative Hamiltonian associated by central potential \( V(\hat{r}) \) as:

\[
\hat{H}_{NC} = \begin{pmatrix}
\frac{2}{m}p_z p_{\bar{z}} + V\left(\hat{r} = \sqrt{\hat{z}\bar{z}}\right) & 0 \\
0 & \frac{2}{m}p_z p_{\bar{z}} + V\left(\hat{r} = \sqrt{\bar{z}\hat{z}}\right)
\end{pmatrix}
\]

This Hamiltonian is Hermitian and represents an antiparticle with spin 1/2. The eigenfunction of the system is described by double-component spinor:

\[
\Psi(z, \bar{z}) = \begin{pmatrix}
\Psi^-(z, \bar{z}) \\
\Psi^+(z, \bar{z})
\end{pmatrix}
\]

Where the Schrödinger equation of the system is described by two equations:

\[
\left(-\frac{1}{2m}\Delta + V\left(\sqrt{\hat{z}\bar{z}} - \frac{\theta (L_z + 2s_z)}{2\sqrt{\hat{z}\bar{z}}}\right)\right) \begin{pmatrix}
\Psi^-(z, \bar{z}) \\
\Psi^+(z, \bar{z})
\end{pmatrix} = \hat{E} \begin{pmatrix}
\Psi^-(z, \bar{z}) \\
\Psi^+(z, \bar{z})
\end{pmatrix}, \quad s_z = \mp 1/2
\]

where the sign \((\mp)\) signifies spin down or up.
3.1 The Singular Even-Power potential

In the noncommutative complex space the deformed Hamiltonian operator $\hat{H}_{NC}$ associated by the Singular Even-Power potential or Ivers-Power potential is given by:

$$\hat{H}_{NC} = \frac{2}{m} p_z^2 + V (\hat{r})$$

(41)

where $V (\hat{r})$ is taken to be:

$$V (\hat{r}) = \begin{pmatrix}
  a \hat{z}^2 + \frac{b}{\hat{z}^2} + \frac{c}{\hat{z}^3} + \frac{d}{\hat{z}^4} & 0 \\
  0 & a \hat{z}^2 + \frac{b}{\hat{z}^2} + \frac{c}{\hat{z}^3} + \frac{d}{\hat{z}^4}
\end{pmatrix}$$

(42)

For simplicity we take $\theta_i = \delta_i 3$ and assume that the other components are all zero, then the Singular Even-Power potential as follows:

$$V (\hat{z}, \hat{\bar{z}}) = a r^2 + b r^{-2} + c r^{-4} + d r^{-6} - \theta r (m \mp 1) + V^{\theta \pm}_{NC} (r)$$

(43)

where

$$\theta = c + b \theta (m + 2 s_z), \quad \hat{d} = d + c \theta (m + 2 s_z), \quad s_z = \mp 1/2.$$  

(44)

and

$$\hat{V}^{\pm} (r) = a r^2 + b r^{-2} + c r^{-4} + d r^{-6} - \theta a (m + 2 s_z)$$

(45)

In noncommutative complex space the eq. (40) accepted a solution exact with noncommutative Singular Even-Power potential $V^{\pm}_{O} (r)$ for the wave functions $\Psi^{\pm} (\hat{z}, \hat{\bar{z}})$, as given by:

$$\Psi^{\pm} (\hat{z}, \hat{\bar{z}}) = \Phi (\varphi) R_m (z, \bar{z}) | \pm \rangle$$

(46)

Where the angular function $\Phi (\varphi)$ and radial function $R_m (z, \bar{z})$, as follows, respectively [8, 13]:

$$\left\{ \begin{array}{l}
\Phi (\varphi) = \frac{1}{\sqrt{2\pi}} \exp (\pm i m \varphi) \quad \text{where} \quad m = 0; 1, 2,... \\
R_m (z, \bar{z}) = \exp (p_m (z, \bar{z})) \sum_{n=0} a_n (z \bar{z})^{2n+1} 
\end{array} \right.$$  

(47)

and $p_m (z, \bar{z})$ is given by:

$$p_m (z, \bar{z}) = \frac{1}{2} \sqrt{a z \bar{z}} + \frac{1}{2} \sqrt{d (z \bar{z})^{-1}}$$

(48)

For $n = 0$ and $n = 1$ we have, the radial functions and the energies corresponding the stationary state and first excited states, respectively:
\[
\begin{align*}
R_m^{(0)} &= a_0 (z \bar{z})^{\delta/2} \exp \left( -\frac{\sqrt{\sigma} (z \bar{z})^2 + \sqrt{\delta}}{2z \bar{z}} \right) \\
E_0 &= \sqrt{\sigma} (4 + 2\hat{\mu}) - \theta a (m \mp 1)
\end{align*}
\]

and
\[
\begin{align*}
R_m^{(1)} &= (a_0 + a_1 z \bar{z}) (z \bar{z})^{\delta/2} \exp \left( -\frac{\sqrt{\sigma} (z \bar{z})^2 + \sqrt{\delta}}{2z \bar{z}} \right) \\
E_1 &= \sqrt{\sigma} (8 + 2\hat{\mu}) - \theta a (m \mp 1)
\end{align*}
\]

and the second term \(V_{NC}^{\theta \pm} \) in eq. (43), is the perturbation term up to the second order of \(\theta\) which takes the form:

\[
V_{NC}^{\theta \pm} (r) = V_{pert}^{\theta \pm} (r) = \theta f (r) (m - 2s_z)
\]

where

\[
f (r) = \frac{3d}{r^8}
\]

and \(s_z = \mp \frac{1}{2}\), described a particle of spin \(1/2\).

Hence we can say that the Schrödinger equation in non-commutative complex space describes to the particles with spin \((1/2)\) in an external uniform magnetic field. To investigate the modification of the energy levels by eq. (50), we use the rst-order perturbation theory. The spectrum of \(ar^2 + br^{-2} + c r^{-4} + dr^{-6} - \theta a (m + 2s_z)\) and the corresponding wave functions are well-known and given by:

\[
R_m = (a_0 + a_1 r^2 + ... a_p r^{2p}) r^{\delta} \exp \left( -\frac{\sqrt{\sigma} r^2}{2r^2} - \frac{\sqrt{\delta}}{2} r^{-2} \right)
\]

where the noncommutative energy levels are given by:

\[
\hat{E}_n = \sqrt{\sigma} (2\hat{\mu} + 4 + 4n) - \theta a (m \mp 1)
\]

where

\[
\hat{\mu} = \frac{\hat{c}}{2\sqrt{\delta}}
\]

Now to obtain the the correction of the energy levels \(E_{NC}^{\theta \pm}\), associate with the spin up and \(E_{NC}^{\theta \pm}\) associate with the spin down to first order in \(\theta\), for the perturbation potential \(V_{NC}^{\theta \pm} (r)\) are given by:

\[
E_{NC}^{\theta \pm} = \langle n | V_{NC}^{\theta \pm} (r) | n \rangle = -\theta (m \pm 1) \int \Psi^{\pm}(p)^* (z, \bar{z}) f (z, \bar{z}) \Psi^{\pm}(p) (z, \bar{z}) r dr
\]

\[
= -\theta (m \pm 1) \int R^{(p)*} (r) f (r) R^{(p)} (r) r dr
\]
Now to obtain the modification to the energy levels for the \( n = 0 \) as a result of the noncommutative term in eqs. (50), we use the first-order perturbation theory. The expectation value of \( r^{-8} \) with respect to the exact solution, are given by:

\[
D = 3 da_0 \int_0^{+\infty} r^{3-\frac{7}{2}} \exp \left( -\sqrt{ad} - \sqrt{dr} - 2 \right) dr
\]

we using the following standard integral [15]:

\[
\int_0^{+\infty} r^{v-1} \exp \left( - \left( \frac{\lambda_2}{r} + \lambda_1 r \right) \right) dr = 2 \left( \frac{\lambda_2}{\lambda_1} \right)^{\frac{v}{2}} K_v \left( 2 \sqrt{\lambda_1 \lambda_2} \right)
\]

where \( K_v \) the modified Bessel function of second kind and order \( v \) and \( \lambda_1 \) and \( \lambda_2 \) are positive and \( |2\sqrt{\lambda_1 \lambda_2}| \left( \frac{\pi}{2} \right) \), after the explicit calculation, the term of eq.(56) take the form:

\[
D = \frac{3}{2} da_0 \left( \sqrt{\frac{d}{a}} \right)^{\frac{3-a}{2}} K_{(\delta-3)} \left( 2 \left( \frac{ad}{a} \right)^{\frac{3}{2}} \right)
\]

Hence the modification to the energy levels \( E_{\theta,0,m}^{(\pm)} \) is given by:

\[
E_{\theta,0,m}^{(\pm)} = \frac{3}{2} da_0 (m \mp 1) \theta \left( \sqrt{\frac{d}{a}} \right)^{\frac{3-a}{2}} K_{(\delta-3)} \left( 2 \left( \frac{ad}{a} \right)^{\frac{3}{2}} \right).
\]

and the non-commutative correction of the energy levels, corresponding the first excited states \( E_{\theta,1,m}^{(\pm)} \), in the first order of \( \theta \):

\[
E_{\theta,1,m}^{(\pm)} = (m \mp 1) \theta \int_0^{+\infty} \left( a_0 + a_1 r^2 \right) r^{\frac{3-a}{2}} \exp \left( -\sqrt{ad} + \sqrt{dr} - 2 \right) \left( \frac{3d}{r^8} \right) rdr
\]

\[
= (m \mp 1) \theta \left( \sum_{i=1}^{3} A_i \right)
\]

where

\[ 9 \]
\[ A^1 = \frac{3a^2}{2} \left( \sqrt{\frac{\delta}{\alpha}} \right)^{2/3} K_{3-3} \left( 2 \left( \alpha \delta \right)^{\frac{1}{2}} \right), \]
\[ A^2 = 3a_1 a_0 \left( \sqrt{\frac{\delta}{\alpha}} \right)^{1/2} K_{3-2} \left( 2 \left( \alpha \delta \right)^{\frac{1}{2}} \right), \]
\[ A^3 = \frac{3d^2}{2} a_1^2 \left( \sqrt{\frac{\delta}{\alpha}} \right)^{1/2} K_{3-1} \left( 2 \left( \alpha \delta \right)^{\frac{1}{2}} \right). \]

Know by the same method, the non-commutative modification of the energy levels \( E_{\theta, p, m} \) to the \( p^{th} \) of excitation state, up in the first order of \( \theta \):

\[ E_{\theta, p, m} = (m \mp 1) \frac{\theta}{\pi} \int_0^{+\infty} \left( \left( a_0 + a_1 r^2 + ... a_pr^{2p} \right) \right)^2 r^{2\delta} \exp \left( -\frac{\sqrt{\alpha} r^2}{2} - \frac{\sqrt{d} r^{-2}}{2} \right) \times \]
\[ \left( 3d \right) \left( \frac{r^2}{\sqrt{\pi}} \right) dr \]
\[ = (m \mp 1) \theta A \]  \hspace{1cm} (62)

where

\[ A = \int_0^{+\infty} \left( \left( a_0 + a_1 r^2 + ... a_pr^{2p} \right) \right)^2 r^{2\delta} \exp \left( -\frac{\sqrt{\alpha} r^2}{2} - \frac{\sqrt{d} r^{-2}}{2} \right) \times \]
\[ \left( 3d \right) \left( \frac{r^2}{\sqrt{\pi}} \right) r dr \]  \hspace{1cm} (63)

The energy levels to the \( p^{th} \) of excitation state in noncommutative complex space as:

\[ \hat{E} = \hat{E}_{p, m} + (m \mp 1) \theta A \]  \hspace{1cm} (64)

We have shown that the non-commutativity induces the lamb shift where the energy spectrum depends on \( m \) and is splitting of two levels. Then we deduced that the non-commutativity plays the role of magnetic field, it also creates a spin of particle.

3.2 The Invers-Power potential

In the noncommutative complex space the deformed Invers-Power potential is given by:

\[ V (\vec{r}) = \begin{pmatrix}
\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} + \frac{c}{\sqrt{2}} + \frac{d}{\sqrt{2}} & 0 \\
0 & \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} + \frac{c}{\sqrt{2}} + \frac{d}{\sqrt{2}}
\end{pmatrix} \]  \hspace{1cm} (65)
For simplicity we take $\theta_i = \delta_{3i}$ and assume that the other components are all zero, then the Ivers-Power potential $V(\hat{z}, \bar{z})$ as follows:

$$V(\vec{r}) = ar^{-1} + br^{-2} + cr^{-3} + dr^{-4} + V_{NC}^{\theta \pm}(r)$$  \hspace{1cm} (66)

where

$$\hat{c} = c + a\theta (m + 2s_z), \quad \hat{d} = d - b\theta (m + 2s_z)$$  \hspace{1cm} (67)

and the second term $V_{NC}^{\theta \pm}(r)$ is the perturbation term up to the second order of $\theta$ which takes the form:

$$V_{NC}^{\theta \pm}(r) = \theta \left( \frac{2c}{r^5} + \frac{3d}{r^6} \right) (L_z \mp 2s_z)$$  \hspace{1cm} (68)

where

$$g(r) = \frac{2c}{r^5} + \frac{3d}{r^6}$$  \hspace{1cm} (69)

Now the radial functions and the energies corresponding the stationary state and first excited states, respectively with exact solution, as follows[14] :

$$R_0 = N_0 (z\bar{z})^{c_0/2} \exp \left( \frac{a + b\bar{z}}{\sqrt{z\bar{z}}} \right),$$  \hspace{1cm} (70)

$$\hat{E}_0^{\pm} = \frac{1}{16a} \left[ \hat{\lambda}_0 \pm \sqrt{\hat{\lambda}_0^2 - 2b\hat{d}} \right]^2$$  \hspace{1cm} (71)

where

$$\hat{\lambda}_0 = \mu (1 + \mu) + \frac{\hat{d}\sqrt{a}}{1 + \mu}, \quad \mu = \frac{b}{2\sqrt{a}}; \quad c_0 = \lambda_0 + \frac{1}{4}$$  \hspace{1cm} (72)

and

$$R_1^{(1)} = N_1 \left( \sqrt{z\bar{z}} - \sigma_1^{(1)} \right) z\bar{z}^{c_1} \exp \left( \frac{a + b\bar{z}}{\sqrt{z\bar{z}}} \right),$$  \hspace{1cm} (73)

$$\hat{E}_1^{\pm} = - \left( \frac{\hat{\lambda}_1 \pm \left[ \hat{\lambda}_1^2 - 4\hat{d} \left( \sigma_1^{(1)} + \sqrt{a} \right) (1 + \mu) \right]^{1/2}}{4 \left( \sigma_1^{(1)} + \sqrt{a} \right)} \right)^2$$  \hspace{1cm} (74)

where

$$c_1 = 1 + \mu; \quad \hat{\lambda}_1 = (1 + \mu) (2 + \mu) + \frac{\hat{d}\sqrt{a}}{2 + \mu} \left( \sigma_1^{(1)} + \sqrt{a} \right)$$  \hspace{1cm} (75)
Then the correction of the energy levels for the Ivers-Power potential are given by:

\[ E_{NC}^{\pm} = \langle n | V_{NC}^{\pm} (r) | n \rangle = \theta (m \pm 1) \int \Psi_{\pm}^{\pm}(z, \bar{z}) \ g (z, \bar{z}) \ \Psi_{\pm}^{\pm}(z, \bar{z}) \ r^2 \ dr \]

Now to obtain the modification to the energy levels for the \( n = 0 \) as a result of the noncommutative terms in eqs. (69), we use the first-order perturbation theory. The expectation value of \( r^{-5} \) and \( r^{-6} \) with respect to the exact solution, are given by:

\[ \langle r^{-k} \rangle = N_0^2 \int drr^{2c+1-k} \exp 2 (ar^{-1} + br) \]

Putting these results together one gets the modifications of the energy:

\[ E_{NC}^{\pm} = \theta (m \pm 1) (f (5) + f (6)) \]

The eigenvalues are \( E_{NC}^{\pm} \), Correspond to the energy values for the charged particule with the spin 1/2 in magnetic field, where the noncommutativity play the role of magnetic field with the created the total magnetic moment of particle with spin 1/2, who in turn shifted the spectrum of energy, therefore degeneracy is removed.

This result is important because it rect the existence of Lamb shift, which is induced by the non-commutativity of the space. Obviously, when \( \theta = 0 \); then \( E_{NC}^{\pm} = 0 \), which is exactly the result of the space-space commuting case, where the energy-levels are not shifted.

Now we can written the noncommutative Hamiltonian (38) in the noncommutative complex space as:

\[ \hat{H} = \left( \begin{array}{cc} \hat{H}_{\pm \bar{z}} & 0 \\ 0 & \hat{H}_{\bar{z} \bar{z}} \end{array} \right) \]

where \( \hat{H}_{\pm \bar{z}} \) and \( \hat{H}_{\bar{z} \bar{z}} \) are given by:

\[ \hat{H}_{\pm \bar{z}} = \frac{2}{m_0} p_{\pm \bar{z}} + V (r) - \theta f (r) (L_z + 2s_z) \equiv \hat{H}^-, \text{ where } s_z = -\frac{1}{2} \]

and

\[ \hat{H}_{\bar{z} \bar{z}} = \frac{2}{m_0} p_{\bar{z} \bar{z}} + V (r) - \theta h (r) (L_z + 2s_z) \equiv \hat{H}^+, \text{ where } s_z = +\frac{1}{2} \]

Then the noncommutative Hamiltonian (79), as follows:
The Hamiltonian in (82) takes the form:

\[ \hat{H} = \left( \begin{array}{cc} \frac{2}{m_0} p_z^2 + V(r) - \theta h(r) (L_z + 2s_z) & 0 \\ 0 & \frac{2}{m_0} p_z^2 + V(r) - \theta h(r) (L_z + 2s_z) \end{array} \right) \]  

(82)

The Hamiltonian in (82) takes the form:

\[ \hat{H} = H_{or} + H_{NC}^\theta \]  

(83)

where \( H_{or} \) is the ordinary Hamiltonian given by:

\[ H_{or} = \left( \frac{2}{m_0} p_z^2 + V(r) \right) I_{2 \times 2} \]  

(84)

where \( I_{2 \times 2} \) is the identity matrix united in 2D space, and \( H_{NC}^\theta \) is given by:

\[ H_{NC}^\theta = h(r) g_j J \cdot \theta I_{2 \times 2} \]  

(85)

where \( h(r) \) is the radial function \( f(r) \) or the radial function \( g(r) \), \( g_j \) Lande factor and \( J = L + s \), is the total angular momentum. Thus is similar to the Zeeman effect, this proofs that the non-commutativity has an effect similar to the Zeeman effects which induced by the magnetic field [14], where the non-commutativity leads the role of the magnetic field. This represents Lamb shift corrections for \( l = 0 \). This result is very important: as a possible means of introducing electron spin we replace \( l = \pm (j + 1/2) \) and \( n \rightarrow n - j - 1 - 1/2 \), where \( j \) is the quantum number associated to the total angular momentum. Then the \( l = 0 \) state has the same total quantum number \( j = 1/2 \). In this case the non-commutative value of the energy levels indicates the splitting of 1s states.

4 Conclusion

In this paper we started from quantum particule with the central potentials, the singular even-power and Invers-Power potentials in a canonical non-commutative complex space, using the Moyal product method, we have derived the deformed Schrodinger equation. Using the power series expansion method to solving and we found that the noncommutative energy is shifted to \((2j + 1)\) levels, it acts here like a Lamb shift in Dirac theory. This proofs that the non-commutativity has an effect similar to the Zeeman effects which induced by the magnetic field [16], where the non-commutativity leads the role of the magnetic field and it also creates a spin of particle.
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