A Note on Homoclinic Orbits for Second Order Hamiltonian Systems

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Abstract

In this paper, we study the existence for the homoclinic orbits for the second order Hamiltonian systems. Under suitable conditions on the potential $V$, we apply the direct method of variations and the Fourier analysis to prove the existence of homoclinic orbits.

Keywords: Homoclinic solutions, Direct method of variations, Fourier analysis.

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1 Introduction and Main Results

Since the pioneer work of Rabinowitz [32] in 1978, many papers used variational methods to study the existence of periodic solutions for Hamiltonian systems. In recent 30 years, variational methods are also widely applied to the existence of homoclinic orbits for Hamiltonian systems(for examples,[1-31,34-48]etc.).specially,in 1990, P.H. Rabinowitz[36] used Mountain Pass Lemma and approximation arguments of periodic so-

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olutions to prove the following theorem about the homoclinic orbits for super-quadratic second order Hamiltonian systems:

**Theorem 1.1.** Suppose the second order Hamiltonian system:

\[ \ddot{q}(t) + V_q(t, q) = 0, \]  

where \( q \in \mathbb{R}^n \) and \( V \) satisfies:

\( \text{(V}_1) \text{: } V(t, q) = -\frac{1}{2}(L(t)q, q) + W(t, q), \text{ where } L \text{ is a continuous } T \text{-periodic matrix valued function and } W \in C^1(R \times \mathbb{R}^n, \mathbb{R}) \text{ is } T \text{-periodic in } t; \)

\( \text{(V}_2) \text{: } L(t) \text{ is positive definite symmetric for all } t \in [0, T]; \)

\( \text{(V}_3) \text{: there is a constant } \mu > 2 \text{ such that } 0 < \mu W(t, q) \leq (q, W_q(t, q)) \text{ for all } q \in \mathbb{R}^n \setminus \{0\}; \)

\( \text{(V}_4) \text{: } W(t, q) = o(|x|) \text{ as } q \to 0 \text{ uniformly for } t \in [0, T]. \)

Then (1.1) possesses a nontrivial homoclinic solution \( q(t) \) emanating from zero such that \( q \in W^{1,2}(R, \mathbb{R}^n) \).

Different from earlier papers, we use Fourier analysis and direct variational method to study the existence of homoclinic orbits for sub-quadratic second order Hamiltonian system, we have the following new theorem:

**Theorem 1.2.** Suppose \( V(t, q) = -\frac{1}{2}|q|^2 + a(t)|q|^{\alpha}, 1 < \alpha < 2 \) and \( a(t) \) satisfies

\( (a_1) a(t) \in C^0(R, \mathbb{R}^+); \)

\( (a_2) a(t) \in L^1(R) \cap L^2(R). \)

Then (1.1) has at least one non-zero homoclinic orbit \( q(t) \) with \( q(\pm \infty) = 0 \) and \( \dot{q}(\pm \infty) = 0 \).

## 2 Some Lemmas

Firstly, let us introduce some notations:

\( H^1(R, \mathbb{R}^n) = W^{1,2}(R, \mathbb{R}^n) \) with the norm \( \|q\|_{H^1} = (\int_{-\infty}^{+\infty} (|q|^2 + |\dot{q}|^2) dt)^{1/2}; \)
\[ \hat{f}(t) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi i xt} dx \] is the Fourier transform of \( f(x) \);
\[ \hat{g}(t) = \int_{-\infty}^{+\infty} g(x)e^{2\pi i xt} dx \] is the Fourier inverse transform of \( g(x) \);
\[ I(q) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q|^2 + |\dot{q}|^2) dt - \int_{-\infty}^{+\infty} a(t)|q|^\alpha dt \] is the functional corresponding to the system (1.1) with \( V(t, q) = -\frac{1}{2}|q|^2 + a(t)|q|^\alpha \), \( 1 < \alpha < 2 \), and \( q \in H^1(R, R^n) \). Now, we list some Lemmas which are necessary for the proof of Theorem 1.2:

**Lemma 2.1.** Under the conditions of Theorem 1.2, for any given \( q \in W^{1,2}(R, R^n) \), we have that \(-\infty < I(q) < +\infty\), that is, \( I(q) \) is well defined.

*Proof.* Since 
\[ |q| = |\hat{q}| \leq \int_{-\infty}^{+\infty} |\hat{q}| dt \leq \left( \int_{-\infty}^{+\infty} \frac{1}{1+t^2} \right)^{\frac{1}{2}} \left( \int_{-\infty}^{+\infty} (1+t^2)|\hat{q}(t)| dt \right)^{\frac{1}{2}} = \pi \|q\|_{H^1}, \]

Hence we have embedding relation \( H^{1,2}(R, R^n) \subset C^0(R, R^n) \). For any given \( q \in H^{1,2}(R, R^n) \), we have
\[ 0 \leq \int_{-\infty}^{+\infty} a(t)|q|^\alpha dt \leq \|q\|_{L^\infty} \int_{-\infty}^{+\infty} a(t) dt < +\infty, \]

which implies
\[ -\infty < I(q) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q|^2 + |\dot{q}|^2) dt - \int_{-\infty}^{+\infty} a(t) |q|^\alpha dt < +\infty. \]

So \( I(q) \) is well-defined. \( \square \)

**Lemma 2.2.** Suppose that the conditions of Theorem 1.2 hold, then we have
\[ I'(q)v = \int_{-\infty}^{+\infty} ((q, v) + (\dot{q}, \dot{v})) dt - \alpha \int_{-\infty}^{+\infty} a(t) |q|^{\alpha-2}(q, v) dt \]

for all \( q, v \in H^1 \), which implies that
\[ I'(q)q = \int_{-\infty}^{+\infty} (|q|^2 + |\dot{q}|^2) dt - \alpha \int_{-\infty}^{+\infty} a(t) |q|^\alpha dt, \]

and \( I'(q) \) is continuous.

*Proof.* Set 
\[ I_1(q) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q|^2 + |\dot{q}|^2) dt \quad \text{and} \quad I_2(q) = \int_{-\infty}^{+\infty} a(t) |q|^\alpha dt. \]
Then we can see that $I(q) = I_1(q) - I_2(q)$. By the property of norm, we conclude that $I_1 \in C^1(H^1, R)$ and 

$$I_1'(q)v = \int_{-\infty}^{+\infty} ((q, v) + (\dot{q}, \dot{v}))dt$$

Now we prove $I_2$ is Frechet differentiable on $H^1$, the proof used some ideas of Coti Zelati-Rabinowitz[26]. Let $W(t, q) = a(t)|q|^\alpha$ Since $|W_q(t, x)| = \alpha a(t)|x|^{\alpha-1}$, so for any $\epsilon > 0$, there exists $\rho > 0$ such that when $|x| \leq \rho$, we have 

$$|W_q(t, x)| \leq \epsilon a(t).$$

It is well known [33] that for any finite $R$,

$$\int_{-R}^{R} W(t, q) dt \in C^1(W^{1,2}([-R, R], R^n); R).$$

So there is $\delta < \min(\frac{\rho}{4}, 1)$ such that for $\phi \in W^{1,2}(R, R^n)$ and $||\phi|| \leq \delta$, we have 

$$| \int_{-R}^{R} \left( W(t, q + \phi) - W(t, q) - W_q(t, q)\phi \right) dt | \leq \frac{\epsilon}{4}||\phi||.$$

Since $q \in W^{1,2}(R, R^n)$, so $q(t) \to 0$ as $t \to \pm\infty$, and we can choose $R$ so large so that for $|t| \geq R$, we have 

$$|q(t)| \leq \frac{\rho}{4}.$$ 

For $\phi \in W^{1,2}(R, R^n)$, by [36] we have 

$$||\phi||_{L^\infty} \leq 2^{1/2}||\phi|| \leq \rho/2.$$ 

By Mean Value Theorem, there exists $\xi = q + \theta \phi$ such that 

$$W(t, q + \phi) - W(t, q) = W_q(t, \xi)\phi$$

Since for $|t| \geq R$, we have 

$$|\xi| \leq \frac{\rho}{4} + \frac{\rho}{2} < \rho.$$ 

So for $|t| \geq R$, we have 

$$|W(t, q + \phi) - W(t, q)| = |W_q(t, \xi)\phi| \leq \epsilon a(t).$$

Hence 

$$\int_{|t| > R} |W(t, q + \phi) - W(t, q)| dt \leq \int_{|t| > R} \epsilon a(t) dt$$

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\[ \leq ||\phi||_{L^\infty} \varepsilon \int_{-\infty}^{+\infty} a(t)dt \]
\[ \leq 2\varepsilon \int_{-\infty}^{+\infty} a(t)dt||\phi||. \]

Similarly, we also have
\[ \int_{|t|>R} |W_q(t,q)|\phi|dt \leq 2\varepsilon \int_{-\infty}^{+\infty} a(t)dt||\phi||. \]

Hence \( I_2 \) is Frechet differentiable.

Now we prove \( I_2 \) is continuous. Suppose \( q_m \to q \) in \( W^{1,2}(R, R^n) \). Since
\[ \sup_{||\phi||=1} |\int_{-\infty}^{+\infty} (W_q(t,q_m) - W_q(t,q))\phi| \leq \left( \int_{-\infty}^{+\infty} |W_q(t,q_m) - W_q(t,q)|^2 dt \right)^{1/2}. \]

For any given \( \varepsilon > 0 \), we can choose \( R > 0 \) large enough so that \( |t| \geq R \) implies \( |q(t)| \leq \rho, |q_m(t)| \leq \rho \) for \( m \) large and
\[ |W_q(t,q)| \leq \varepsilon a(t), \]
\[ |W_q(t,q_m)| \leq \varepsilon a(t). \]

Hence
\[ \int_{-\infty}^{+\infty} |W_q(t,q_m) - W_q(t,q)|^2 dt \]
\[ \leq \int_{-R}^{R} |W_q(t,q_m) - W_q(t,q)|^2 dt + 2\varepsilon^2 \int_{-\infty}^{+\infty} |a(t)|^2 dt. \]

Hence \( I_2 \) is continuous.

\[ \square \]

**Lemma 2.3.** ([47]) Let \( X \) be a reflexive Banach space, \( M \subset X \) be a weakly closed subset, \( f : M \to R \) be weakly lower semi-continuous. If \( f \) is coercive, that is, \( f(x) \to +\infty \) as \( ||x|| \to +\infty \), then \( f \) attains its infimum on \( M \).

**Lemma 2.4.** \( I(q) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q|^2 + |\dot{q}|^2)dt - \int_{-\infty}^{+\infty} a(t)|q|^\alpha dt \) is coercive for \( q(t) \in H^{1}(R, R^n) \) and \( I(q) \) is bounded from below.

**Proof.**
\[ I(q) = \frac{1}{2} \int_{-\infty}^{+\infty} (|q|^2 + |\dot{q}|^2)dt - \int_{-\infty}^{+\infty} a(t)|q|^\alpha dt \]
\[ \geq \frac{1}{2} ||q||_{H^1}^2 - \pi \frac{\alpha}{\alpha - 2} \int_{-\infty}^{+\infty} a(t)dt ||q||_{H^1}^\alpha. \]
We notice that $1 < \alpha < 2$, so $I(q) \to +\infty$ when $\|q\|_{H^1} \to +\infty$, hence $I(q)$ is coercive on $H^1$.

Let $\phi(x) = \frac{1}{2}x^2 - Cx^\alpha$, $C > 0$, $x \in [0, +\infty)$, then the derivative of $\phi$ is

$$
\phi'(x) = x - C\alpha x^{\alpha - 1}.
$$

when $x = (C\alpha)^{\frac{1}{\alpha - 1}}$, $\phi(x)$ attains its minimum, that is, when $\|q\|_{H^1} = (C\alpha)^{\frac{1}{\alpha - 1}}$, $\frac{1}{2}\|q\|_{H^1}^2 - C\alpha \int_{-\infty}^{+\infty} a(t)dt \|q\|_{H^1}^\alpha$ attains its minimum, where $C = \pi^\frac{\alpha}{2} \int_{-\infty}^{+\infty} a(t)dt$. Then $I(q)$ is bounded from below.

\begin{lemma}
(Sobolev-Rellich-Kondrachov [47]) Let $\Omega$ is bounded domain, then

$$W^{1,2}(\Omega, \mathbb{R}) \subset C(\Omega, \mathbb{R})$$

and the embedding is compact.
\end{lemma}

\begin{lemma}
([47]) Let $X$ be a Banach space, then the norm $\| \cdot \|$ is weakly lower semi-continuous.
\end{lemma}

\begin{lemma}
Assume $V(t, q)$ is defined as in Theorem 1.2, then $I(q)$ is weakly lower semi-continuous.
\end{lemma}

\textbf{Proof:} We define

$$I_k(q) = \int_{-k}^{k} (|q|^2 + |\dot{q}|^2)dt - \int_{-k}^{k} a(t)|q|^\alpha dt,$$

$$\tilde{q}_l(t) = q_l(t)\chi_{[-k,k]},$$

$$\tilde{q}(t) = q(t)\chi_{[-k,k]}.$$  

We divide the proof into three steps:

\textbf{Step 1:} if $q_l(t) \rightharpoonup q(t)$ in $H^1(R, \mathbb{R}^n)$, then $\tilde{q}_l(t) \rightharpoonup \tilde{q}(t)$ in $H^1([-k,k], \mathbb{R}^n) = H^1[-k,k].$

For any $\tilde{f}(t) \in H^1([-k,k], \mathbb{R}^n)$, define

$$f(x) = \begin{cases} 
\tilde{f}(t), & t \in [-k,k]; \\
0, & x \notin [-k,k].
\end{cases}$$

Then

$$\int_{-k}^{k} \tilde{f}(t)\tilde{q}_l(t)dt = \int_{-\infty}^{+\infty} f(t)q_l(t)dt,$$

$$\int_{-k}^{k} \tilde{f}(t)\tilde{q}(t)dt = \int_{-\infty}^{+\infty} f(t)q_l(t)dt;$$

\textbf{Step 2:}
\[
\int_{-k}^{k} \tilde{f}(t) \tilde{q}(t) dt = \int_{-\infty}^{+\infty} f(t) q(t) dt,
\]
\[
\int_{-k}^{k} \tilde{f}(t) \tilde{q}(t) dt = \int_{-\infty}^{+\infty} \tilde{f}(t) \tilde{q}(t) dt.
\]
Since \( q_l(t) \rightarrow q(t) \) in \( H^1(R,R^n) \), then by Riesz representation Theorem, we have that
\[
\int_{-\infty}^{+\infty} f(t) q_l(t) dt + \int_{-\infty}^{+\infty} \tilde{f}(t) \tilde{q}_l(t) dt \rightarrow \int_{-\infty}^{+\infty} f(t) q(t) dt + \int_{-\infty}^{+\infty} \tilde{f}(t) \tilde{q}(t) dt,
\]
then
\[
\int_{-k}^{k} \tilde{f}(t) \tilde{q}_l(t) dt + \int_{-k}^{k} \tilde{f}(t) \tilde{q}_l(t) dt \rightarrow \int_{-k}^{k} \tilde{f}(t) \tilde{q}(t) dt + \int_{-k}^{k} \tilde{f}(t) \tilde{q}(t) dt,
\]
that is, \( \tilde{q}_l(t) \rightarrow \tilde{q}(t) \) in \( H^1[-k, k] \).

Step 2: if \( q_l(t) \rightarrow q(t) \) in \( H^1(R, R^n) \), then by Step 1, we have that \( \tilde{q}_l(t) \rightarrow \tilde{q}(t) \) in \( H^1([-k, k], R^n) = H^1[-k, k] \). By compact embedding theorem (Lemma 2.3) and functional analysis, \( \tilde{q}_l(t) \rightarrow \tilde{q}(t) \) uniformly in \( C([-k, k], R^n) \), then we have
\[
\lim_{l \rightarrow +\infty} \left( \int_{-k}^{k} a(t) |\tilde{q}_l|^\alpha dt \right) \rightarrow \int_{-k}^{+k} a(t) |\tilde{q}(t)|^\alpha dt.
\]
We have known that for \( q, q_l \in W^{1,2}(R, R^n) \), when \( t \rightarrow \pm \infty \),
\[
q(t) \rightarrow 0, \quad q_l(t) \rightarrow 0.
\]
So by \( a(t) \in L^1(R, R) \) we have
\[
\int_{|t| \geq k} a(t) |q|^\alpha dt \leq \sup_{|t| \geq k} |q(t)|^\alpha \int_{|t| \geq k} a(t) dt \rightarrow 0, \quad k \rightarrow +\infty.
\]
\[
\int_{|t| \geq k} a(t) |q_l|^\alpha dt \leq \sup_{|t| \geq k} |q_l(t)|^\alpha \int_{|t| \geq k} a(t) dt \rightarrow 0, \quad k \rightarrow +\infty.
\]
Hence
\[
\lim_{l \rightarrow +\infty} \lim_{k \rightarrow +\infty} \left( -\int_{-k}^{k} a(t) |\tilde{q}_l|^\alpha dt \right) = \lim_{l \rightarrow +\infty} -\int_{-\infty}^{+\infty} a(t) |q_l|^\alpha dt = -\int_{-\infty}^{+\infty} a(t) |q(t)|^\alpha dt.
\]
Since the norm is weakly lower semi-continuous (w.l.s.c.), so we have \( \lim \| q_l \|_{H^1(R, R^n)} \geq \| q \|_{H^1(R, R^n)} \). So \( \lim \inf_{l \rightarrow +\infty} I(q_l) \geq I(q) \). Then we apply the Lemma 2.1 to get a minimizer for \( I(q) \) on \( H^{1,2}(R, R^n) \).
Step 3: Now we need to prove the minimizer is non-zero and $q(\pm \infty) = 0$ and $\dot{q}(\pm \infty) = 0$. For any given $q_0 \in H^1(R, R^n), \text{since } 1 < \alpha < 2, \text{ so for } r > 0 \text{ small enough, we have}$

$$I(rq_0) = \frac{1}{2} \int_{-\infty}^{+\infty} r^2(|q_0|^2 + |\dot{q}_0|^2)dt - r^\alpha \int_{-\infty}^{+\infty} a(t)|q_0|^\alpha dt < 0.$$ 

So

$$\min_{q \in H^1(R, R^n)} I(q) < 0,$$

and the minimum value must be negative, hence the minimizer must be non-zero. Similar to Rabinowitz [36] P.37, we can prove the minimizer $q(t)$ satisfies that

$$q(\pm \infty) = 0, \dot{q}(\pm \infty) = 0.$$ 

By $q \in H^1(R, R^n)$, we have

$$\int_{|t| \geq A} (|q(t)|^2 + |\dot{q}(t)|^2)dt \to 0.$$ 

By [36], we have

$$|q(t)| \leq 2\left[ \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|\dot{q}(s)|^2 + |q(s)|^2)ds \right]^{1/2}.$$ 

Hence $q(\pm \infty) = 0$.

By [36], we have

$$|\dot{q}(t)| \leq 2\left[ \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|\ddot{q}(s)|^2 + |\dot{q}(s)|^2)ds \right]^{1/2}.$$ 

So if $\int_A^{A+1} |\ddot{q}(t)|^2 dt \to 0$, as $A \to +\infty$, we have $\dot{q}(\pm \infty) = 0$. Since $q(t)$ is the solution of (1.1), so we have

$$|\ddot{q}(t)|^2 = | - V_q(t, q)|^2 \leq 2(|\dot{q}(t)|^2 + \alpha^2 |a(t)|^2 |q(t)|^{2(\alpha-1)})$$ 

Since we have proved $q(t) \to 0$ as $t \to \pm \infty$, and by $a(t) \in L^2$, we can have

$$\int_A^{A+1} (|\ddot{q}(t)|^2 + \alpha^2 |a(t)|^2 |q(t)|^{2(\alpha-1)}) dt \to 0.$$ 

$\ddot{q}(\pm \infty) = 0$. 

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