CONCENTRATING BOUNDED STATES FOR FRACTIONAL SCHröDINGER-POISSON SYSTEM

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ABSTRACT. In this paper, we study the following fractional Schrödinger-Poisson system

\[
\begin{align*}
\varepsilon^{2s}(-\Delta)^su + V(x)u + \phi u &= g(u) \quad \text{in } \mathbb{R}^3, \\
\varepsilon^{2t}(-\Delta)^t\phi &= u^2, \ u > 0 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(s, t \in (0, 1), \varepsilon > 0\) is a small parameter. Under some local assumptions on \(V(x)\) and suitable assumptions on the nonlinearity \(g\), we construct a family of positive solutions \(u_\varepsilon \in H_\varepsilon\) which concentrates around the global minima of \(V(x)\) as \(\varepsilon \to 0\).

1. INTRODUCTION

In this paper, we study the following fractional Schrödinger-Poisson system

\[
\begin{align*}
\varepsilon^{2s}(-\Delta)^su + V(x)u + \phi u &= g(u) \quad \text{in } \mathbb{R}^3, \\
\varepsilon^{2t}(-\Delta)^t\phi &= u^2, \ u > 0 \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(s, t \in (0, 1), \varepsilon > 0\) is a small parameter. The potential \(V : \mathbb{R}^3 \to \mathbb{R}\) is a bounded continuous function satisfying

\[
(V_0) \quad \inf_{x \in \mathbb{R}^3} V(x) = V_0 > 0;
\]

\[
(V_1) \quad \text{There is a bounded domain } \Lambda \subset \mathbb{R}^3 \text{ such that } V_0 < \min_{\partial \Lambda} V(x), \ M = \{x \in \Lambda \mid V(x) = V_0\} \neq \emptyset.
\]

Without loss of generality, we may assume that \(0 \in \mathcal{M}\). The nonlinearity \(g : \mathbb{R} \to \mathbb{R}\) is of \(C^1\)-class function satisfying

\[
(g_0) \quad \lim_{\tau \to 0^+} \frac{g(\tau)}{\tau} = 0;
\]

\[
(g_1) \quad \lim_{\tau \to +\infty} \frac{g'(\tau)}{\tau} = 0;
\]

\[
(g_2) \quad \text{there exists } \lambda > 0 \text{ such that } g(\tau) \geq \lambda \tau^{q-1} \text{ for some } \frac{4s+2t}{s+t} < q < 2^*_s \text{ and all } \tau \geq 0;
\]

\[
(g_3) \quad \frac{g'(\tau)}{\tau} \text{ is non-decreasing in } \tau \in (0, +\infty).
\]

Since we are looking for positive solutions, we may assume that \(g(s) = 0\) for \(s < 0\). The non-local operator \((-\Delta)^s\) (\(s \in (0, 1)\)), which is called fractional Laplacian operator, can be defined by

\[
(-\Delta)^s u(x) = C_s \text{ P.V. } \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{3+2s}} \, dy = C_s \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3 \setminus B_{\varepsilon}(x)} \frac{u(x) - u(y)}{|x - y|^{3+2s}} \, dy
\]

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for \( u \in \mathcal{S}(\mathbb{R}^3) \), where \( \mathcal{S}(\mathbb{R}^3) \) is the Schwartz space of rapidly decaying \( C^\infty \) function, \( B_r(x) \) denote an open ball of radius \( r \) centered at \( x \) and the normalization constant \( C_s = \left( \int_{\mathbb{R}^3} \frac{1-\cos(|\zeta|)}{|\zeta|^{3+2s}} \, d\zeta \right)^{-1} \). For \( u \in \mathcal{S}(\mathbb{R}^3) \), the fractional Laplace operator \((-\Delta)^s\) can be expressed as an inverse Fourier transform

\[
(-\Delta)^s u = \mathcal{F}^{-1}\left((2\pi|\xi|)^{2s}\mathcal{F} u(\xi)\right),
\]

where \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) denote the Fourier transform and inverse transform, respectively.

If \( u \) is sufficiently smooth, it is known that (see [30]) it is equivalent to

\[
(-\Delta)^s u(x) = -\frac{1}{2}C_s \int_{\mathbb{R}^3} \frac{u(x+y) + u(x-y) - 2u(x)}{|x-y|^{3+2s}} \, dy.
\]

By a classical solution of (1.1), we mean two continuous functions that \((-\Delta)^s u\) is well defined for all \( x \in \mathbb{R}^3 \) and satisfies (1.1) in pointwise sense.

In recent years, much attention has been given to nonlocal problems driven by the fractional Laplace operator. This operator naturally arises in many physical phenomena, such as: fractional quantum mechanics [24, 25], anomalous diffusion [27], financial [11], obstacle problems [34], conformal geometry and minimal surfaces [9]. It also provides a simple model to describe certain jump Lévy processes in probability theory [7]. A common approach to tackle fractional nonlocal problems, is make use of the extension method due to Caffarelli and Silvestre [10], which allows us to transform a given nonlocal equation into a degenerate elliptic problem in the half-space with a nonlinear Neumann boundary condition, we refer to interesting readers to see the related works [1, 4, 7, 12, 13, 17, 38] and so on. Another approach is that directly investigating the problems in the space \( H^s(\mathbb{R}^3) \), the related works can be referred to see [7, 12, 14, 35, 36, 37, 39, 40, 41, 42] and so on. As we know, if one chooses the second line, due to the presence of the fractional Laplacian \((-\Delta)^s\), which is a nonlocal operator, more accurate estimates are needed usually.

Formally, system (1.1) is regarded as the associated fractional version of the following classical Schrödinger-Poisson system

\[
\begin{cases}
-\varepsilon^2 \Delta u + V(x)u + \phi u = g(x, u) & \text{in } \mathbb{R}^3, \\
-\varepsilon^2 \Delta \phi = u^2 & \text{in } \mathbb{R}^3.
\end{cases}
\]

(1.2)

It is well known that system (1.2) has a strong physical meaning because it appears in semiconductor theory [28]. In particular, systems like (1.2) have been introduced in [5] as a model to describe solitary waves. In (1.2), the first equation is a nonlinear stationary equation (where the nonlinear term simulates the interaction between many particles) that is coupled with a Poisson equation, to be satisfied by \( \phi \), meaning that the potential is determined by the charge of the wave function. For this reason, (1.2) is referred to as a nonlinear Schrödinger-Poisson system.

In recent years, there has been increasing attention to systems like (1.2) when \( 0 < \varepsilon \leq 1 \) on the existence of positive solutions, ground state solutions, multiple solutions and semiclassical states; see for examples [3, 5, 31, 32, 44] and the references therein. Regarding the concentration phenomenon of solutions for Schrödinger-Poisson systems like (1.2), there has been the object of interest for many authors. Ruiz and Vaira [33] proved the existence of multi-bump solutions of system

\[
\begin{cases}
-\varepsilon^2 \Delta u + V(x)u + \phi u = u^p & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 & \text{in } \mathbb{R}^3.
\end{cases}
\]

(1.3)
for \( p \in (1,5) \) and these bumps concentrate around a local minimum of the potential \( V \), through using the singular perturbed methods based on a Lyapunov-Schmitt reduction. Later, by using a Lyapunov-Schmitt reduction method, Ianni and Vaira \[23\] obtained the existence of positive bound state solutions which concentrate on a non-degenerate local minimum or maximum of \( V \). Ianni and Vaira \[22\] also showed the existence of radially symmetric solutions, which concentrate on the spheres. Seok \[34\] considered the existence of the spike solutions for system \( (1.3) \) with \( V \) like \( (1.1) \). For example, when \( \varepsilon > 0 \) sufficiently small, and 
\[
\lambda \neq 0 \text{ is a real parameter, } V(x) \text{ and } b(x) \text{ satisfy some global assumptions, } \]
\[
f \in C(\mathbb{R}^3) \text{ is such that}
\]
\[
\begin{align*}
(i) & \quad f(t) = o(t^3), \\
(ii) & \quad \frac{f(t)}{t} \text{ is increasing on } (0, +\infty), \\
(iii) & \quad |f(t)| \leq c(1 + |t|^p) \quad \text{with } p \in (4,6), \quad \lim_{|t| \to \infty} \frac{F(t)}{t^3} = +\infty,
\end{align*}
\]
the authors proved that problem \( (1.6) \) exists the least energy solution \( u_\varepsilon \in H^1(\mathbb{R}^3) \) for \( \varepsilon > 0 \) sufficiently small, and \( u_\varepsilon \) converges to the least energy solution of the associated limit problem and concentrates to some sets.

In the very recent years, there are much attention to be paid on a similar system like \( (1.1) \). For example, when \( \varepsilon = 1 \) in \( (1.1) \), in \[39\], we established the existence of positive ground state solution for the system \( (1.1) \) with \( g(u) = \mu |u|^{p-1}u + |u|^{2^\ast - 2}u \) for some \( p \in (1, 2^\ast - 1) \) by using the Nehari-Pohozaev manifold combing monotone trick with global compactness Lemma. Using the similar methods, in \[40\], positive ground state solutions for the system \( (1.1) \) with \( g(u) = |u|^{p-1}u \) with \( p \in (2, 2^\ast - 1) \), were established when \( s = t \). In \[45\], the authors studied the existence of radial solutions for the system \( (1.1) \) with the nonlinearity \( g(u) \) satisfying the subcritical or critical assumptions of Berestycki-Lions type. Regarding the semiclassical state of problem \( (1.1) \), there are some results on the existence and multiplicity of solutions. Such as, in \[29\], the authors studied the semiclassical state of the following system

\[
\begin{align*}
\begin{cases}
\varepsilon^{2s}(\Delta)^s u + V(x)u + \phi u = f(u) & \text{in } \mathbb{R}^N, \\
\varepsilon^\theta(\Delta)\phi = \gamma u^2 & \text{in } \mathbb{R}^N,
\end{cases}
\end{align*}
\]
where \( s \in (0, 1), \alpha \in (0, N), \theta \in (0, \alpha), N \in (2s, 2s + \alpha) \), \( \gamma_\alpha \) is a positive constant, \( V(x) \) satisfies (1.3) and \( f(u) \) satisfies the assumptions like (1.5). By using the Ljusternick-Schnirelmann theory of critical point theory, the authors obtained the multiplicity of positive solutions which concentrate on the minima of \( V(x) \) as \( \varepsilon \to 0 \). In [25], by using the methods mentioned before, Liu and Zhang proved the existence and concentration of positive ground state solution for problem (1.1). In [31], we studied the system (1.1) with competing potential, i.e.,

\[
g(u) = K(x)f(u) + Q(x)|u|^{q_2} - 2u,
\]

where \( f \) is a function of \( C^1 \) class, superlinear and subcritical nonlinearity, \( V(x), K(x) \) and \( Q(x) \) are positive continuous functions. Under some suitable assumptions on \( V, K \) and \( Q \), we prove that there is a family of positive ground state solutions which concentrate on the set of minimal points of \( V(x) \) and the sets of maximal points of \( K(x) \) and \( Q(x) \).

In the above mentioned works, the assumption made on potential \( V(x) \) is all global, but for the local assumption like \((V_1)\), there are few works to deal with the fractional Schrödinger-Poisson system (1.1), even for the Schrödinger-Poisson system (1.2). It is well known that the penalization methods developed by del Pino and Felmer [15] is a powerful trick to solve this class of problems, but it requires the arguments of Nehari manifold. Recently this powerful tools have been applied to fractional Schrödinger equations, see [11, 21]. When using the Nehari manifold for the system (1.2), the nonlinearity \( g(x, u) \) has to be sublinear-4 growth. The purpose of this paper is to expand the limit of superlinear-4 growth. Another penalization which was developed by Byeon and Jeanjean [6] is another effective method, but this method is not available for the nonlinear problems involving fractional Laplacian since the fractional operator \(-\Delta\) is nonlocal, this makes the function \( u \) with \( u = 0 \) on \( \mathbb{R}^3 \setminus B_R(0) \), satisfies the equation \( -\Delta u = f(u) \) in \( B_R(0) \), it will not hold on \( \mathbb{R}^3 \setminus B_R(0) \) if \( f(0) = 0 \). But for the local Laplace operator \(-\Delta\), it possesses this properties which \( u \) satisfies the equation \(-\Delta u = f(u) \) with \( f(0) = 0 \) in the whole \( \mathbb{R}^3 \). This property is vital to use the penalization method of Byeon and Jeanjean [6]. The penalization used by Byeon and Jeanjean [6] is defined by

\[
\chi_\varepsilon(x) = \begin{cases} 
0 & x \in \Lambda/\varepsilon, \\
1/\varepsilon & x \notin \Lambda/\varepsilon,
\end{cases} 
\]

\[Q_\varepsilon(u) = \left( \int_{\mathbb{R}^3} \chi_\varepsilon u^2 \, dx - 1 \right)^2_+.
\]

To obtain the \( L^\infty \)-estimates and uniformly decay estimate at infinity, this penalization can not applicable directly because there is no local estimates like Theorem 8.17 in [18]. For tackling these difficulties, we combine the two penalizations mentioned above which has been introduced in Byeon and Wang [8], but a change of second penalization is of the following form

\[Q_\varepsilon(u) = \left( \int_{\mathbb{R}^3 \setminus \Lambda/\varepsilon} u^2 \, dx - \varepsilon \right)^2_+.
\]

In this way, we can achieve the main result as follows.

**Theorem 1.1.** Let \( 2s + 2t > 3, s, t \in (0, 1) \). Suppose that \( V \) satisfies \((V_0)\), \((V_1)\) and \( g \in C^1(\mathbb{R}^+, \mathbb{R}) \) satisfies \((g_0)-(g_3)\). Then there exists an \( \varepsilon_0 > 0 \) such that system (1.1) possesses a positive solution \((u_\varepsilon, \phi_\varepsilon) \in H_\varepsilon \times D^{s,2}(\mathbb{R}^3) \) for all \( \varepsilon \in (0, \varepsilon_0) \). Moreover, there exists a maximum point \( x_\varepsilon \) of \( u_\varepsilon \) such that \( \lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, \mathcal{M}) = 0 \) and

\[
u_\varepsilon(x) \leq \frac{C_{1,s+2s}}{C_0\varepsilon^{3+2s} + |x - x_\varepsilon|^{3+2s}} \quad x \in \mathbb{R}^3, \text{ and } \varepsilon \in (0, \varepsilon_0)
\]
for some constants $C > 0$ and $C_0 \in \mathbb{R}$.

We give some remarks on the above Theorem.

**Remark 1.2.** Observe that if $s = t = 1$, $\frac{4s+2t}{s+t} = 3$, so from (g₂) and (g₃), we see that our assumptions are very weaker than (1.6) and (1.7) in [19] and [44] respectively. On the other hand, we consider the local assumption $(V₁)$ comparing the present works appearing in the literature.

**Remark 1.3.** If a local $L^{∞}$-estimate like Theorem 8.17 in [18] will be established, the assumption $(V₁)$ can be improved as follows

$$\inf_{\Lambda} V(x) < \inf_{\partial \Lambda} V(x).$$

The paper is organized as follows, in Section 2, we give some preliminary results. In Section 3, we prove the existence of positive ground state solutions for "limit problem". In Section 4, we prove the main result Theorem 1.1.

## 2. Variational Setting

In this section, we outline the variational framework for studying problem (1.1) and list some preliminary Lemma which used later. In the sequel, we denote by $\| \cdot \|_p$ the usual norm of the space $L^p(\mathbb{R}^3)$, the letter $c_i$ ($i = 1, 2, \ldots$) or $C$ denote by some positive constants.

### 2.1. Work space stuff.

We define the homogeneous fractional Sobolev space $\mathcal{D}^{α,2}(\mathbb{R}^3)$ as follows

$$\mathcal{D}^{α,2}(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) \mid |ξ|^β (\mathcal{F}u)(ξ) \in L^2(\mathbb{R}^3) \right\}$$

which is the completion of $C_0^∞(\mathbb{R}^3)$ under the norm

$$\|u\|_{\mathcal{D}^{α,2}} = \left( \int_{\mathbb{R}^3} |(-\Delta)^{\frac{α}{2}} u|^2 \, dx \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^3} |ξ|^{2α} |(\mathcal{F}u)(ξ)|^2 \, dξ \right)^{\frac{1}{2}}$$

The fractional Sobolev space $H^α(\mathbb{R}^3)$ can be described by means of the Fourier transform, i.e.

$$H^α(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} (|ξ|^{2α}|(\mathcal{F}u)(ξ)|^2 + |(\mathcal{F}u)(ξ)|^2) \, dξ < +\infty \right\}.$$  

In this case, the inner product and the norm are defined as

$$(u, v) = \int_{\mathbb{R}^3} (|ξ|^{2α}(\mathcal{F}u)(ξ)(\mathcal{F}v)(ξ) + (\mathcal{F}u)(ξ)(\overline{\mathcal{F}v}(ξ))) \, dξ$$

and

$$\|u\|_{H^α} = \left( \int_{\mathbb{R}^3} (|ξ|^{2α}|(\mathcal{F}u)(ξ)|^2 + |(\mathcal{F}u)(ξ)|^2) \, dξ \right)^{\frac{1}{2}}.$$

From Plancherel’s theorem we have $\|u\|_2 = \|\mathcal{F}u\|_2$ and $\||ξ|^α\mathcal{F}u\|_2 = \|(-\Delta)^{\frac{α}{2}} u\|_2$. Hence

$$\|u\|_{H^α} = \left( \int_{\mathbb{R}^3} (|(-\Delta)^{\frac{α}{2}} u(x)|^2 + |u(x)|^2) \, dx \right)^{\frac{1}{2}}, \text{ for all } u \in H^α(\mathbb{R}^3).$$

We denote $\| \cdot \|$ by $\| \cdot \|_{H^α}$ in the sequel for convenience.
In terms of finite differences, the fractional Sobolev space $H^\alpha(\mathbb{R}^3)$ also can be defined as follows

$$H^\alpha(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) \mid D_\alpha u \in L^2(\mathbb{R}^3) \right\},$$

endowed with the natural norm

$$\|u\|_{H^\alpha} = \left( \int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} |D_\alpha u|^2 \, dx \right)^{\frac{1}{2}}.$$

Also, in view of Proposition 3.4 and Proposition 3.6 in [30], we have

$$\|(-\Delta)^{\frac{s}{2}} u\|_2^2 = \int_{\mathbb{R}^3} |\xi|^{2s}|(F u)(\xi)|^2 \, d\xi = \frac{1}{C_{\alpha}} \int_{\mathbb{R}^3} |D_\alpha u|^2 \, dx. \quad (2.1)$$

We define the Sobolev space $H_\varepsilon = \{ u \in H^s(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} V(\varepsilon x) u^2 \, dx < \infty \}$ endowed with the norm

$$\|u\|_{H_\varepsilon} = \left( \int_{\mathbb{R}^3} (|D_\alpha u|^2 + V(\varepsilon x) u^2) \, dx \right)^{\frac{1}{2}}.$$

It is well known that $H^s(\mathbb{R}^3)$ is continuously embedded into $L^r(\mathbb{R}^3)$ for $2 \leq r \leq 2^*_s \left( 2^*_s = \frac{6}{3-2s} \right)$. Obviously, the conclusion also holds for $H_\varepsilon$.

2.2. Formulation of Problem [111]. It is easily seen that, just performing the change of variables $u(x) \to u(x/\varepsilon)$ and $\phi(x) \to \phi(x/\varepsilon)$, and taking $z = x/\varepsilon$, problem [1.1] can be rewritten as the following equivalent form

$$\begin{cases}
(-\Delta)^{\frac{s}{2}} u + V(\varepsilon z)u + \phi u = g(u) & \text{in } \mathbb{R}^3, \\
(-\Delta)^{\frac{s}{2}} \phi = u^2, & u > 0 \text{ in } \mathbb{R}^3
\end{cases} \quad (2.2)$$

which will be referred from now on. Observe that if $4s + 2t \geq 3$, there holds $2 \leq \frac{12}{3+2t} \leq \frac{6}{3-2s}$, and thus $H_\varepsilon \hookrightarrow L^{\frac{12}{3+2t}}(\mathbb{R}^3)$. Considering $u \in H_\varepsilon$, the linear functional $\mathcal{L}_u : D^{s,2}(\mathbb{R}^3) \to \mathbb{R}$ is defined by $\mathcal{L}_u(v) = \int_{\mathbb{R}^3} u^2 v \, dx$. Similarly, using the Lax-Milgram theorem, there exists a unique $\phi^t_u \in D^{s,2}(\mathbb{R}^3)$ such that

$$C_s \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{(\phi^t_u(z) - \phi^t_u(y)) (v(z) - v(y))}{|z - y|^{3+2s}} \, dy \, dz = \int_{\mathbb{R}^3} (-\Delta)^{\frac{s}{2}} \phi^t_u (-\Delta)^{\frac{s}{2}} v \, dz$$

which is $\phi^t_u$ is a weak solution of $(-\Delta)^{\frac{s}{2}} \phi^t_u = u^2$ and so the representation formula holds

$$\phi^t_u(x) = c_t \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|^{3-2s}} \, dy, \quad x \in \mathbb{R}^3, \quad c_t = \pi^{\frac{s}{2}} 2^{\frac{s}{2}} \frac{\Gamma(\frac{3+2t}{2})}{\Gamma(t)}.$$
Lemma 2.1. For every $u \in H_\varepsilon$ with $4s + 2t \geq 3$, define $\Phi(u) = \phi_u^t \in \mathcal{D}^{t,2}(\mathbb{R}^3)$, where $\phi_u^t$ is the unique solution of equation $(-\Delta)^s \phi = u^2$. Then there hold:

(i) If $u_n \rightharpoonup u$ in $H_\varepsilon$, then $\Phi(u_n) \rightharpoonup \Phi(u)$ in $\mathcal{D}^{t,2}(\mathbb{R}^3)$;

(ii) $\Phi(tu) = t^2\Phi(u)$ for any $t \in \mathbb{R}$;

(iii) For $u \in H_\varepsilon$, one has

$$\|\Phi(u)\|_{\mathcal{D}^{t,2}} \leq C\|u\|^2_{H_\varepsilon}, \quad \int_{\mathbb{R}^3} \Phi(u)u^2 \, dx \leq C\|u\|^4_{H_\varepsilon},$$

where constant $C$ is independent of $u$;

(iv) Let $2s + 2t > 3$, if $u_n \rightharpoonup u$ in $H_\varepsilon$ and $u_n \rightharpoonup u$ a.e. in $\mathbb{R}^3$, then for any $v \in H_\varepsilon$,

$$\int_{\mathbb{R}^3} \phi_{u_n}^t v \, dz \to \int_{\mathbb{R}^3} \phi_u^t v \, dz \quad \text{and} \quad \int_{\mathbb{R}^3} g(u_n)v \, dz \to \int_{\mathbb{R}^3} g(u)v \, dz$$

and thus $u$ is a solution for problem (2.3).

In the end, we recall some regularity results which will be used in the sequel.

Lemma 2.2. ([42]) Assume that $u_n$ are nonnegative weak solution of

\[
\begin{cases}
(-\Delta)^s u + V_n(x)u + \phi u = f_n(x,u) & \text{in } \mathbb{R}^3, \\
(-\Delta)^t \phi = u^2 & \text{in } \mathbb{R}^3,
\end{cases}
\]

where $\{V_n\}$ satisfies $V_n(x) \geq \alpha_0 > 0$ for all $x \in \mathbb{R}^3$ and $f_n(x,\tau)$ is a Carathedory function satisfying that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$|f_n(x,\tau)| \leq \delta|\tau| + C_\delta|\tau|^{2^* - 1}, \quad \forall (x,\tau) \in \mathbb{R}^3 \times \mathbb{R}.$$

Suppose that $u_n$ convergence strongly in $H^s(\mathbb{R}^3)$. Then there exists $C > 0$ such that

$$\|u_n\|_{L^{2^*}} \leq C \quad \text{for all } n.$$

Lemma 2.3. ([34]) Let $w = (-\Delta)^s u$. Assume $w \in L^\infty(\mathbb{R}^n)$ and $u \in L^\infty(\mathbb{R}^n)$ for $s > 0$.

If $2s \leq 1$, then $u \in C^{0,\alpha}(\mathbb{R}^n)$ for any $\alpha \leq 2s$. Moreover

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)$$

for some constant $C$ depending only on $n$, $\alpha$ and $s$.

If $2s > 1$, then $u \in C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s - 1$. Moreover

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{L^\infty(\mathbb{R}^n)}\right)$$

for some constant $C$ depending only on $n$, $\alpha$ and $s$.

Lemma 2.4. ([35]) Assume that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$ and it satisfies

$$\lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2 \, dx = 0$$

where $R > 0$. Then $u_n \rightharpoonup 0$ in $L^r(\mathbb{R}^N)$ for every $2 < r < 2s$.
3. LIMITING PROBLEM

In this section, we consider the "limiting problem" associated with problem (2.2)
\[
\begin{align*}
(-\Delta)^s u + \mu u + \phi u &= g(u) \quad \text{in } \mathbb{R}^3, \\
(-\Delta)^t \phi &= u^2, \quad u > 0 \quad \text{in } \mathbb{R}^3
\end{align*}
\]
for \( \mu > 0 \). We define the energy functional for the limiting problem (3.1) by
\[
I_{\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^3} |D_t u|^2 \, dx + \frac{\mu}{2} \int_{\mathbb{R}^3} |u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_t u^2 \, dx - \int_{\mathbb{R}^3} G(u) \, dx \quad u \in H^s(\mathbb{R}^3).
\]
Let
\[
P_{\mu}(u) = \frac{3 - 2s}{2} \int_{\mathbb{R}^3} |D_t u|^2 \, dx + \frac{3}{2} \int_{\mathbb{R}^3} \mu |u|^2 \, dx + \frac{3 + 2t}{4} \int_{\mathbb{R}^3} \phi_t u^2 \, dx - 3 \int_{\mathbb{R}^3} G(u) \, dx
\]
and
\[
G_{\mu}(u) = (s + t)I_{\mu}(u) - P_{\mu}(u) = \frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} |D_t u|^2 \, dx + \frac{2s + 2t - 3}{2} \mu \int_{\mathbb{R}^3} |u|^2 \, dx
\]
\[+ \frac{4s + 2t - 3}{4} \int_{\mathbb{R}^3} \phi_t u^2 \, dx + \int_{\mathbb{R}^3} \left(3G(u) - (s + t)g(u)\right) \, dx.
\]
We define the Nehari-Pohozaev manifold
\[
M_{\mu} = \{ u \in H^s(\mathbb{R}^3) \setminus \{0\} \mid G_{\mu}(u) = 0 \}
\]
and set \( b_{\mu} = \inf_{u \in M_{\mu}} I_{\mu}(u) \). We list some properties of the manifold \( M_{\mu} \).

**Proposition 3.1.** The set \( M_{\mu} \) possesses the following properties:
(i) \( 0 \not\in \partial M_{\mu} \);
(ii) for any \( u \in H^s(\mathbb{R}^3) \setminus \{0\} \), there exists a unique \( \tau_0 := \tau(u) > 0 \) such that \( u_{\tau_0} \in M_{\mu} \), where \( u_{\tau} = \tau^{s+t} u(\tau x) \). Moreover,
\[
I_{\mu}(u_{\tau_0}) = \max_{\tau \geq 0} I_{\mu}(u_{\tau});
\]
Proof. The proof of (i) and (ii) is standard, it is only to prove the uniqueness of \( \tau \) of (ii). Indeed, if there exist \( \tau_1 > \tau > 0 \) such that \( u_{\tau_1}, u_{\tau} \in M_{\mu} \), then
\[
G_{\mu}(u_{\tau_1}) = 0, \quad G_{\mu}(u_{\tau}) = 0.
\]
By simple computation, we have
\[
\frac{2s + 2t - 3}{2} \left( \frac{1}{\tau_1^{s+t}} - \frac{1}{\tau^{2s}} \right) \int_{\mathbb{R}^3} \mu |u|^2 \, dx
\]
\[= \int_{\mathbb{R}^3} \left( (s + t)g(u_{\tau_1})u_{\tau_1} - 3G(u_{\tau_1}) \tau_1^{4s+2t-3} - \frac{(s + t)g(u_{\tau})u_{\tau} - 3G(u_{\tau})}{\tau^{4s+2t-3}} \right) \, dx
\]
\[+ \frac{2s}{2} \left( \int_{\mathbb{R}^3} (u_{\tau_1})^{2s} \, dx - \int_{\mathbb{R}^3} (u_{\tau})^{2s} \, dx \right)
\]
\[= \int_{\mathbb{R}^3} \left( (s + t)g(\tau_1^{s+t} u_{\tau_1})\tau_1^{s+t} - 3G(\tau_1^{s+t} u_{\tau_1}) \tau_1^{4s+2t} - \frac{(s + t)g(\tau^{s+t} u\tau^{s+t})\tau^{s+t} - 3G(\tau^{s+t} u\tau^{s+t})}{\tau^{4s+2t}} \right) \, dx.
\]
If we show that the function \( \tau \in \mathbb{R}^+ \rightarrow \frac{(s + t)g(\tau^{s+t} u\tau^{s+t})\tau^{s+t} - 3G(\tau^{s+t} u\tau^{s+t})}{\tau^{4s+2t}} \) is non-decreasing, then we get a contradiction and the uniqueness is proved. In fact, by computation and using \((g_3)\), we deduce that
\[
\left( \frac{(s + t)g(\tau^{s+t} u\tau^{s+t})\tau^{s+t} - 3G(\tau^{s+t} u\tau^{s+t})}{\tau^{4s+2t}} \right)' = \frac{1}{\tau^{4s+2t-1}} \left( 3(4s + 2t)G(\tau^{s+t} u) \right).
\[-(s + t)(3s + t + 3)g(\tau^{s+t}u)\tau^{s+t}u + (s + t)^2g'(\tau^{s+t}u)\tau^{2(s+t)}u^2\]
\[\geq \frac{1}{\tau^{4s+2t+1}} \left[ (s + t)^2(q - 1) - (s + t)(3s + t + 3) \right] g(\tau^{s+t}u)\tau^{s+t}u + 3(4s + 2t)G(\tau^{s+t}u) > 0.\]

Lemma 3.2. \(I_\mu\) possesses the mountain pass geometry:

(i) there exist \(\rho_0, \beta_0 > 0\) such that \(I_\mu(u) \geq \beta_0\) for all \(u \in H^s(\mathbb{R}^3)\) with \(\|u\| = \rho_0\);

(ii) there exists \(u_0 \in H^s(\mathbb{R}^3)\) such that \(I_\mu(u_0) < 0\).

Proof. By (g0) and (g1), for any \(\eta > 0\), there exists \(C_\eta > 0\) such that
\[g(t) \leq \eta |t| + C_\eta |t|^{2^* - 1} \quad \text{and} \quad G(t) \leq \frac{\eta}{2} |t|^2 + C_\eta |t|^{2^*} \quad \text{for any } t \in \mathbb{R}. \tag{3.2}\]

Hence, choosing \(\eta = \frac{1}{2}\) and by Sobolev inequality, we have that
\[I_\mu(u) \geq \frac{1}{2} \int_{\mathbb{R}^3} |D_x u|^2 \, dx + \mu \int_{\mathbb{R}^3} |u|^2 \, dx - \frac{\eta}{2} \int_{\mathbb{R}^3} |u|^2 \, dx - C_\eta \int_{\mathbb{R}^3} |u|^{2^*} \, dx \]
\[\geq \frac{1}{4} \|u\|^2 - C_\mu \|u\|^{2^*},\]

thus, there exists \(\rho_0, \beta_0 > 0\) small enough such that \(I_\mu(u) \geq \beta_0\) for \(\|u\| = \rho_0\).

(ii) For any \(u \in H^s(\mathbb{R}^3)\) with \(u \geq 0\), set \(u_\tau(x) = \tau^{(s+t)}u(\tau x)\) with \(\tau > 0\). Thus, by (f3), we deduce that
\[I_\mu(u_\tau) \leq \frac{\tau(4s+2t-3)}{2} \int_{\mathbb{R}^3} |D_x u|^2 \, dx + \frac{\tau(2s+2t-3)}{2} \int_{\mathbb{R}^3} \mu |u|^2 \, dx \]
\[+ \frac{\tau(4s+2t-3)}{4} \int_{\mathbb{R}^3} \phi_\mu^2 u^2 \, dx - C_\tau q(s+t)-3 \int_{\mathbb{R}^3} |u|^q \, dx.\]

Since \(4s + 2t > 3\) and so \(4s + 2t - 3 < q(s + t) - 3\), we obtain that \(I_\mu(u_\tau) \to -\infty\) as \(\tau \to +\infty\). Hence, there exists \(\tau_0 > 0\) large enough such that \(I_\mu(u_0) < 0\), where \(u_0 = u_{\tau_0}\). \(\square\)

From Lemma 3.2, we can define the mountain-pass level of \(I_\mu\) as follows
\[c_\mu = \inf_{\gamma \in \Gamma_\mu} \sup_{t \in [0,1]} I_\mu(\gamma(t))\]
where
\[\Gamma_\mu = \left\{ \gamma \in C([0,1], H^s(\mathbb{R}^3)) \mid \gamma(0) = 0, \ I_\mu(\gamma(1)) < 0 \right\}\]
and \(c_\mu > 0\). By the condition (f3) and using Lemma 3.2, we can show the equivalent characterization of mountain-pass level \(c_\mu\).

Lemma 3.3.
\[c_\mu = b_\mu.\]

Proof. We only need to verify that \(\gamma([0,1]) \cap M_\mu \neq \emptyset\). Indeed, by Lemma 3.2 we see that if \(u \in H^s(\mathbb{R}^3)\setminus \{0\}\), is interior to or on \(M_\mu\), then
\[\frac{4s + 2t - 3}{2} \int_{\mathbb{R}^3} |D_x u|^2 \, dx + \frac{2s + 2t - 3}{2} \int_{\mathbb{R}^3} \mu |u|^2 \, dx + \frac{4s + 2t - 3}{4} \int_{\mathbb{R}^3} \phi_\mu^2 u^2 \, dx \]
\[\geq \int_{\mathbb{R}^3} \left( (s + t)g(u)u - 3G(u) \right) \, dx\]
and 
\[(4s + 2t - 3)I_\mu(u) = G_\mu(u) + s \int_{\mathbb{R}^3} \mu|u|^2 \, dx + \int_{\mathbb{R}^3} \left((s + t)g(u)u - (4s + 2t)G(u)\right) \, dx > 0.\]

Hence, \(\gamma\) crosses \(M_\mu\) since \(\gamma(0) = 0, I_\mu(\gamma(1)) < 0\) which implies that \(G_\mu(\gamma(1)) < 0\), combining with \(G_\mu(\gamma(t)) \geq 0\). Therefore,

\[
\max_{t \in [0,1]} I_\mu(\gamma(t)) \geq \inf_{\mathcal{M}_\mu} I_\mu(w) = b_\mu
\]

and then \(c_\mu \geq b_\mu\). \(\square\)

In order to obtain the boundedness of (PS) sequence, we will construct a (PS) sequence \(\{u_n\}\) for \(I_\mu\) at the level \(c_\mu\) that satisfies \(G_\mu(u_n) \to 0\) as \(n \to +\infty\) i.e.,

**Lemma 3.4.** There exists a sequence \(\{u_n\}\) in \(H^s(\mathbb{R}^3)\) such that as \(n \to +\infty\),

\[
I_\mu(u_n) \to c_\mu, \quad I_\mu'(u_n) \to 0, \quad G_\mu(u_n) \to 0. \quad (3.3)
\]

**Proof.** Define the map \(\Phi : \mathbb{R} \times H^s(\mathbb{R}^3) \to H^s(\mathbb{R}^3)\) for \(\theta \in \mathbb{R}, v \in H^s(\mathbb{R}^3)\) by \(\Phi(\theta, v)(x) = e^{(s+\theta)}v(e^{\theta}x)\). By computation, for every \(\theta \in \mathbb{R}, v \in H^s(\mathbb{R}^3)\), we see that the functional \(I_\mu \circ \Phi\) writes as

\[
(I_\mu \circ \Phi)(\theta, v) = \frac{e^{(4s+2t-3)\theta}}{2} \int_{\mathbb{R}^3} |D_s v|^2 \, dx + \frac{e^{(2s+2t-3)\theta}}{2} \int_{\mathbb{R}^3} |v|^2 \, dx
\]

\[
+ \frac{e^{(4s+2t-3)\theta}}{4} \int_{\mathbb{R}^3} \phi_{\mu} u^2 \, dx - e^{3\theta} \int_{\mathbb{R}^3} G(e^{(s+\theta)}v) \, dx.
\]

Similarly as the proof of (i) of Lemma 3.2, we have that

\[
(I_\mu \circ \Phi)(\theta, v) \geq \frac{1}{4} \|\Phi(\theta, v)\|^2 - C\|\Phi(\theta, v)\|^2^*.
\]

Thus, there exists \(\rho_1, \alpha_1 > 0\) small such that \((I_\mu \circ \Phi)(\theta, v) \geq \alpha_1\) for every \(\theta \in \mathbb{R}\) and \(v \in H^s(\mathbb{R}^3)\) with \(\|\Phi(\theta, v)\| = \rho_1\). Moreover, we have that \((I_\mu \circ \Phi)(0, u_0) < 0\), where \(u_0\) is given in Lemma 3.2. Hence, \(I_\mu \circ \Phi\) possesses the mountain-pass geometry in \(\mathbb{R} \times H^s(\mathbb{R}^3)\). We define the mountain-pass level of \(I_\mu \circ \Phi\) \(\tilde{c}_\mu = \inf_{\tilde{\gamma} \in \Gamma_\mu} \max_{t \in [0,1]} (I_\mu \circ \Phi)(\tilde{\gamma}(t))\),

where \(\Gamma_\mu = \{\tilde{\gamma} \in C([0,1], \mathbb{R} \times H^s(\mathbb{R}^3)) \mid \tilde{\gamma}(0) = 0, (I_\mu \circ \Phi)(\tilde{\gamma}(1)) < 0\}\). Observe that \(\Gamma_\mu = \{\Phi \circ \tilde{\gamma} \mid \tilde{\gamma} \in \Gamma_\mu\}\), the mountain-pass level of \(I_\mu\) coincides with \(I_\mu \circ \Phi\), i.e., \(c_\mu = \tilde{c}_\mu\).

By the general minimax principle (139, Theorem 2.8), there exists a sequence \(\{(\theta_n, v_n)\} \subset \mathbb{R} \times H^s(\mathbb{R}^3)\) such that

\[
(I_\mu \circ \Phi)(\theta_n, v_n) \to c_\mu, \quad (I_\mu \circ \Phi)'(\theta_n, v_n) \to 0, \quad \theta_n \to 0. \quad (3.4)
\]

The detailed proof refer the readers to see Proposition 3.4 in [20]. For every \((h, \phi) \in \mathbb{R} \times H^s(\mathbb{R}^3)\), we deduce that

\[
(I_\mu \circ \Phi)(\theta_n, v_n)'(h, \phi) = (I_\mu'(\Phi(\theta_n, v_n)), \Phi(\theta_n, \phi)) + G_\mu(\Phi(\theta_n, v_n))h.
\]

Taking \(h = 1, \phi = 0\) in (3.5), we get

\[
G_\mu(\Phi(\theta_n, v_n)) \to 0.
\]
For every $\phi \in H^s(\mathbb{R}^3)$, set $\varphi(x) = e^{-(s+t)\theta_n}\phi(e^{-\theta_n}x)$, $h = 0$ in \eqref{eq:3.1}, by \eqref{assumption:S2}, we get
\[
\langle I'_\mu(\Phi(\theta_n, v_n)), \phi \rangle = o_n(1)\|e^{-(s+t)\theta_n}\phi(e^{-\theta_n}x)\| = o_n(1)\|\phi\|.
\]
Denoting $u_n = \Phi(\theta_n, v_n)$, combining with \eqref{assumption:S2}, the conclusion follows. □

**Lemma 3.5.** Every sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ satisfying \eqref{eq:3.3} is bounded in $H^s(\mathbb{R}^3)$.

**Proof.** By \eqref{eq:3.3}, we deduce that
\[
cu + o_n(1) = I_\mu(u_n) - \frac{1}{q(s+t) - 3}G_\mu(u_n)
\]
\[
= \frac{(q-4)s + (q-2)t}{2(q(s+t) - 3)} \int_{\mathbb{R}^3} |D_s u_n|^2 \, dx + \frac{(q-2)(s+t)}{2(q(s+t) - 3)} \mu \int_{\mathbb{R}^3} |u_n|^2 \, dx
\]
\[
+ \frac{(q-4)s + (q-2)t}{4(q(s+t) - 3)} \int_{\mathbb{R}^3} \phi^\prime u_n |u_n|^2 \, dx + \frac{s + t}{q(s+t) - 3} \int_{\mathbb{R}^3} \left(g(u_n)u_n - qG(u_n)\right) \, dx
\]
which implies the boundedness of the sequence $\{u_n\}$ in $H^s(\mathbb{R}^3)$ due to $q > \frac{s+2t}{s+t}$.

By using the Vanishing Lemma \[\ref{assumption:S2}\] it is not difficult to deduce that the bounded sequence $\{u_n\} \subset H^s(\mathbb{R}^3)$ given in \eqref{eq:3.3} is non-vanishing. That is,

**Lemma 3.6.** There exists a sequence $\{x_n\} \subset \mathbb{R}^3$ and $R > 0$, $\beta > 0$ such that $\int_{B_R(x_n)} |u_n|^2 \, dx \geq \beta$.

Combining Lemma \[\ref{lem:3.2}\] with Lemma \[\ref{assumption:S2}\] and Lemma \[\ref{lem:3.6}\] we can show the existence of positive ground state solution for the limiting problem \[\eqref{eq:3.1}\].

**Proposition 3.7.** Problem \[\eqref{eq:3.1}\] possesses a positive ground state solution $u \in H^s(\mathbb{R}^3)$.

**Proof.** Let $\{u_n\}$ be the sequence given in \eqref{eq:3.3}. Set $\tilde{u}_n(x) = u_n(x + x_n)$, where $\{x_n\}$ is the sequence obtained in Lemma \[\ref{lem:3.6}\]. Thus $\{\tilde{u}_n\}$ is still bounded in $H^s(\mathbb{R}^3)$ and so up to a subsequence, still denoted by $\{\tilde{u}_n\}$, we may assume that there exists $\tilde{u} \in H^s(\mathbb{R}^3)$ such that
\[
\begin{cases}
\tilde{u}_n \rightharpoonup \tilde{u} & \text{in } H^s(\mathbb{R}^3),
\tilde{u}_n \to \tilde{u} & \text{in } L^p_{\text{loc}}(\mathbb{R}^3) \text{ for all } 1 \leq p < 2^*_s, \\
\tilde{u}_n \to \tilde{u} & \text{a.e. } \mathbb{R}^3.
\end{cases}
\]

It follows from Lemma \[\ref{lem:3.1}\] that $\tilde{u}$ is nontrivial. Moreover, $\tilde{u}$ is a nontrivial solution of problem \[\eqref{eq:3.1}\], and so $G_\mu(\tilde{u}) = 0$. By Fatou’s Lemma and \eqref{eq:3.3}, we have
\[
cu = b_\mu \leq I_\mu(\tilde{u}) = I_\mu(\tilde{u}) - \frac{1}{4s+2t-3}G_\mu(\tilde{u}) = \frac{s}{4s+2t-3} \int_{\mathbb{R}^3} |\tilde{u}|^2 \, dx
\]
\[
+ \frac{s + t}{4s + 2t - 3} \int_{\mathbb{R}^3} \left(f(\tilde{u})\tilde{u} - \frac{4s + 2t}{s + t} F(\tilde{u})\right) \, dx
\]
\[
\leq \liminf_{n \to \infty} \left[ \frac{s + t}{4s + 2t - 3} \int_{\mathbb{R}^3} \left(g(\tilde{u}_n)\tilde{u}_n - \frac{4s + 2t}{s + t} G(\tilde{u}_n)\right) \, dx + \frac{s}{4s + 2t - 3} \int_{\mathbb{R}^3} |\tilde{u}_n|^2 \, dx \right]
\]
\[
= \liminf_{n \to \infty} \left[ I_\mu(\tilde{u}_n) - \frac{1}{4s+2t-3}G_\mu(\tilde{u}_n) \right] = \liminf_{n \to \infty} \left[ I_\mu(u_n) - \frac{1}{4s+2t-3}G_\mu(u_n) \right] = cu
\]
which implies that \( \tilde{u}_n \to \tilde{u} \) in \( H^s(\mathbb{R}^3) \). Indeed, from the above inequality, we get
\[
\int_{\mathbb{R}^3} \tilde{u}_n^2 \, dx \to \int_{\mathbb{R}^3} \tilde{u}^2 \, dx.
\]
By virtue of the Brezis-Lieb Lemma and interpolation argument, we conclude that
\( \tilde{u}_n \to \tilde{u} \) in \( L^r(\mathbb{R}^3) \) for all \( 2 \leq r < 2_s^* \).

Hence, from the standard arguments, it follows that \( \tilde{u}_n \to \tilde{u} \) in \( H^s(\mathbb{R}^3) \). Therefore, by Lemma 3.3, we conclude that \( I_\mu(\tilde{u}) = c_\mu \) and \( I_\mu'(\tilde{u}) = 0 \).

Next, we show that the ground state solution of (3.1) is positive. Indeed, by standard argument to the proof Proposition 4.4 in [41], using Lemma 2.3 two times and the hypothesis \((q_1)\), we have that \( \tilde{u} \in C^{2,\alpha}(\mathbb{R}^3) \) for any \( \alpha \in (0,1) \) for \( s > \frac{1}{2} \).

Using \( -\tilde{u} \) as a testing function, it is easy to see that \( \tilde{u} \geq 0 \). Since \( \tilde{u} \in C^{2,\alpha}(\mathbb{R}^3) \), by Lemma 3.2 in [30], we have that
\[
(\Delta)^s \tilde{u}(x) = -\frac{1}{2} C(s) \int_{\mathbb{R}^3} \frac{\tilde{u}(x+y) + \tilde{u}(x-y) - 2\tilde{u}(x)}{|x-y|^{3+2s}} \, dx \, dy, \quad \forall \, x \in \mathbb{R}^3.
\]
Assume that there exists \( x_0 \in \mathbb{R}^3 \) such that \( \tilde{u}(x_0) = 0 \), then from \( \tilde{u} \geq 0 \) and \( \tilde{u} \neq 0 \), we get
\[
(\Delta)^s \tilde{u}(x_0) = -\frac{1}{2} C(s) \int_{\mathbb{R}^3} \frac{\tilde{u}(x_0+y) + \tilde{u}(x_0-y)}{|x_0-y|^{3+2s}} \, dx \, dy < 0.
\]
However, observe that \( (\Delta)^s \tilde{u}(x_0) = -\mu \tilde{u}(x_0) - (\phi_{\tilde{u}}(x_0)+f(\tilde{u}(x_0)))+\tilde{u}(x_0))^{2s-1} = 0 \), a contradiction. Hence, \( \tilde{u}(x) > 0 \), for every \( x \in \mathbb{R}^3 \). The proof is completed.

Let \( L_\mu \) be the set of ground state solutions \( W \) of (3.1) satisfying \( W(0) = \max_{\mathbb{R}^3} W(x) \). Then we obtain the following compactness of \( L_\mu \).

**Proposition 3.8.** (i) For each \( \mu > 0 \), \( L_\mu \) is compact in \( H^s(\mathbb{R}^3) \).

(ii) \( 0 < W(x) \leq \frac{C}{1+|x|^{s+2}} \) for any \( x \in \mathbb{R}^3 \).

**Proof.**  (i) For any \( W \in H^s(\mathbb{R}^3) \), we have
\[
c_\mu = I_\mu(W) - \frac{1}{q(s+t)-3} G_\mu(W)
= \frac{(q-4)s+(q-2)t}{2(q(s+t)-3)} \int_{\mathbb{R}^3} |D_s W|^2 \, dx + \frac{(q-2)(s+t)}{2(q(s+t)-3)} \mu \int_{\mathbb{R}^3} W^2 \, dx
+ \frac{(q-4)s+(q-2)t}{4(q(s+t)-3)} \int_{\mathbb{R}^3} \phi_{\tilde{u}}W^2 \, dx + \frac{s+t}{q(s+t)-3} \int_{\mathbb{R}^3} (g(W)W - qG(W)) \, dx
\]
which yields the boundedness of \( L_\mu \) in \( H^s(\mathbb{R}^3) \).

Similar to the proof of Lemma 3.9 and Proposition 3.7, we verify that for any bounded \( \{W_n\} \subset L_\mu \), up to a subsequence, there exist \( \{x_n\} \subset \mathbb{R}^3 \) and \( W_0 \in H^s(\mathbb{R}^3) \) such that \( W_n(x) := W_n(x+x_n) \to W_0 \) in \( H^s(\mathbb{R}^3) \). By Lemma 2.2, we see that
\[
\|W_n\|_\infty = \|W_n\|_\infty \leq C,
\]
where \( C \) is independent on \( n \).

On the other hand, from the boundedness of \( \{W_n\} \) in \( H^s(\mathbb{R}^3) \), up to a subsequence, we may assume that there exists \( W_0 \in H^s(\mathbb{R}^3) \) such that \( W_n \to W_0 \) in \( H^s(\mathbb{R}^3) \) and \( W_n \to W_0 \) in \( L^r_{loc}(\mathbb{R}^3) \) for \( 1 \leq r < 2_s^* \) and \( W_n \to W_0 \) a.e. \( \mathbb{R}^3 \). Since \( W_n \) is a solution of (3.1), in view of Lemma 2.3 and (3.6), we see that \( \|W_n\|_{C^{1,\alpha}(\mathbb{R}^3)} \leq C \).
for some $\alpha \in (0, 1)$, where $C$ depending only on $\alpha$ and $s$. The Arzela-Ascoli’s Theorem shows that $W_n(0) \to W_0(0)$ as $n \to \infty$. Since $W_n(0)$ is a global maximum for $W_n(x)$, then we have that

$$0 \leq (-\Delta)^s W_n(0) = -\mu W_n(0) - \phi_n^{W_n}(0) W_n(0) + g(W_n(0))$$

which leads to $W_n(0) \geq C_0 > 0$. Hence, $W_0(0) \geq C_0 > 0$, this means that $W_0$ is nontrivial.

Finally, similar arguments as in the proof of Proposition 3.7 we can show that $W_n \to W_0$ in $H^s(\mathbb{R}^3)$. This completes the proof that $\mathcal{L}_\mu$ is compact in $H^s(\mathbb{R}^3)$.

(ii) By Lemma 4.2 and Lemma 4.3 in [16], by scaling, there exists a continuous function $U$ such that

$$0 < U(x) \leq \frac{C}{1 + |x|^{3+2s}}$$

and

$$(-\Delta)^s U + \frac{\mu}{2} U = 0 \quad \text{on } \mathbb{R}^3 \setminus B_R(0),$$

for some suitable $R > 0$. By standard argument, using the fact that $W \in L^p(\mathbb{R}^3) \cap C^{1,\alpha}(\mathbb{R}^3)$ for all $2 \leq p \leq \infty$, we infer that $\lim_{|x| \to \infty} W(x) = 0$. Thus there exists $R_1 > 0$ (we can choose $R_1 > R$) large enough such that

$$(-\Delta)^s W + \frac{\mu}{2} W = (-\Delta)^s W + \mu W - \frac{\mu}{2} W = g(W) - \phi W - \frac{\mu}{2} W$$

$$\leq g(W) - \frac{\mu}{2} W \leq 0$$

for any $x \in \mathbb{R}^3 \setminus B_{R_1}(0)$. Therefore, we have obtained that

$$(-\Delta)^s U + \frac{\mu}{2} U \geq (-\Delta)^s W + \frac{\mu}{2} W \quad \text{on } \mathbb{R}^3 \setminus B_{R_1}(0). \quad (3.7)$$

Let $A = \inf_{B_{R_1}(0)} U > 0$, $Z(x) = (B + 1) U - AW$, where $B = \|W\|_{\infty} \leq C < \infty$. We claim that $Z(x) \geq 0$ for all $x \in \mathbb{R}^3$. If the claim is true, we have that

$$0 < W(x) \leq \frac{B + 1}{A} U(x) \leq \frac{C}{1 + |x|^{3+2s}} \quad \text{for all } x \in \mathbb{R}^3$$

and the conclusion is proved.

Suppose by contradiction that there exists $\{x_n\} \subset \mathbb{R}^3$ such that

$$\inf_{x \in \mathbb{R}^3} Z(x) = \lim_{n \to \infty} Z(x_n) < 0. \quad (3.8)$$

Since $\lim_{|x| \to \infty} U(x) = \lim_{|x| \to \infty} W(x) = 0$ by virtue of (3.7), then $\lim_{|x| \to \infty} Z(x) = 0$. Hence, sequence $\{x_n\}$ must be bounded and then up to a subsequence, we may assume that $x_n \to x_0 \in \mathbb{R}^3$. From (3.8) and the continuity of $Z(x)$, we have that

$$\inf_{x \in \mathbb{R}^3} Z(x) = Z(x_0) < 0$$

which yields

$$(-\Delta)^s Z(x_0) + \frac{\mu}{2} Z(x_0) = \frac{\mu}{2} Z(x_0) - \frac{1}{2} C s \int_{\mathbb{R}^3} \frac{Z(x_0 + y) + Z(x_0 - y) - 2Z(x_0)}{|x - y|^{3+2s}} \, dy$$

$$< 0.$$
Note that $Z(x) \geq A \mathbb{B} + U - A \mathbb{B} > 0$ on $B_{R_1}(0)$, this leads to $x_0 \in \mathbb{R}^3 \setminus B_{R_1}(0) \subset \mathbb{R}^3 \setminus B_R(0)$. From (3.6), we have that

$$(-\Delta)^s Z(x_0) + \frac{\mu}{2} Z(x_0) = \left((\mathbb{B} + 1)\left((-\Delta)^s U + \frac{\mu}{2} U\right) - A\left((-\Delta)^s W + \frac{\mu}{2} W\right)\right)|_{x = x_0} \geq 0$$

which is a contradiction. Thus, the claim holds true and the proof is completed.

\[\Box\]

4. The penalization scheme

For the bounded domain $\Lambda$ given in (V1), $k > 2$, $a > 0$ such that $g(a) = \frac{V_0}{k} a$ where $V_0$ is mentioned in (V0), we consider a new problem

$$(-\Delta)^s u + V(\varepsilon) u + \phi^i u = f(\varepsilon, u) \quad \text{in } \mathbb{R}^3$$

(4.1)

where $f(\varepsilon, \tau) = \chi_{\Lambda, \varepsilon}(\varepsilon \tau) g(\tau) + (1 - \chi_{\Lambda, \varepsilon}(\varepsilon \tau)) \tilde{g}(\tau)$ with

$$\tilde{g}(\tau) = \begin{cases} f(\tau) & \text{if } \tau \leq a, \\ \frac{V_0}{k} \tau & \text{if } \tau > a \end{cases}$$

and $\chi_{\Lambda, \varepsilon}(\varepsilon \tau) = 1$ if $\varepsilon \in \Lambda, \varepsilon(\tau) = 0$ if $\varepsilon \notin \Lambda, \varepsilon$, where $\Lambda_{\varepsilon} = \Lambda / \varepsilon$. It is easy to see that under the assumptions (g0)-(g3), $f(\varepsilon, \tau)$ is a Caratheodory function and satisfies the following assumptions:

(1) $f(\varepsilon, \tau) = o(\tau)$ as $\tau \to 0$ uniformly on $\varepsilon \in \mathbb{R}^3$;

(2) $f(\varepsilon, \tau) \leq g(\tau)$ for all $\tau \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^3$, $f(\varepsilon, \tau) = 0$ for all $\varepsilon \in \mathbb{R}^3$ and $\tau < 0$, $f(\varepsilon, \tau) = G(\tau)$ for $\varepsilon \in \mathbb{R}^3, \tau \in [0, a]$;

(3) $0 < 2 \tilde{G}(\tau) \leq \tilde{g}(\tau) \tau \leq \frac{V_0}{k} \tau^2 \leq \frac{V(\varepsilon)}{k} \tau^2$ for all $\tau \geq 0$ with the number $k > 2$, where $\tilde{G}(\tau)$ is a prime function of $\tilde{g}$;

(4) $\frac{f(\varepsilon, \tau)}{\tau}$ is nondecreasing in $\tau \in \mathbb{R}^+$ uniformly for $\varepsilon \in \mathbb{R}^3, \frac{f(\varepsilon, \tau)}{\tau^2}$ is nondecreasing in $\tau \in \mathbb{R}^+$ and $\varepsilon \in \Lambda, \frac{f(\varepsilon, \tau)}{\tau^3}$ is nondecreasing in $\tau \in (0, a)$ and $\varepsilon \in \mathbb{R}^3 \setminus \Lambda$.

Obviously, if $u_\varepsilon$ is a solution of (4.1) satisfying $u_\varepsilon(z) \leq a$ for $\varepsilon \in \mathbb{R}^3$, then $u_\varepsilon$ is indeed a solution of the original problem (2.3).

For $u \in H_\varepsilon$, let $P_\varepsilon(u) = \int_{\mathbb{R}^3} (|D_\varepsilon u|^2 + V(\varepsilon) u^2) \, dz + \frac{1}{4} \int_{\mathbb{R}^3} \phi^i u^2 \, dz - \int_{\mathbb{R}^3} F(\varepsilon, u) \, dz$.

We define

$$Q_\varepsilon(v) = \left(\int_{\mathbb{R}^3 \setminus \Lambda_{\varepsilon}} v^2 \, dz - \varepsilon\right)_+^2.$$

This type of penalization was firstly introduced in [8, 9], which will act as a penalization to force the concentration phenomena to occur inside $\Lambda$. Let us define the functional $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ as follows

$$J_\varepsilon(u) = P_\varepsilon(u) + Q_\varepsilon(u).$$

Clearly, $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$. To find solutions of (4.1) which concentrates in $\Lambda$ as $\varepsilon \to 0$, we shall search critical points of $J_\varepsilon$ such that $Q_\varepsilon$ is zero.

Now, we construct a set of approximate solutions of (4.1). Set

$$\delta_0 = \frac{1}{10} \text{dist}(\mathcal{M}, \mathbb{R}^3 \setminus \Lambda), \quad \beta \in (0, \delta_0).$$
we fix a cut-off function \( \varphi \in C_c^\infty (\mathbb{R}^3) \) such that \( 0 \leq \varphi \leq 1 \), \( \varphi = 1 \) for \( |z| \leq \beta \), \( \varphi = 0 \) for \( |z| \geq 2\beta \) and \( |\nabla \varphi| \leq C/\beta \). Set \( \varphi_\varepsilon (z) = \varphi(\varepsilon z) \), for any \( W \in \mathcal{L}_V \) and any point \( y \in \mathcal{M}^\beta = \{ y \in \mathbb{R}^3 \mid \inf_{z \in \mathcal{M}} |y - z| \leq \beta \} \), we define

\[
W^\varepsilon_y (z) = \varphi_\varepsilon (z - \frac{y}{\varepsilon}) W(z - \frac{y}{\varepsilon}).
\]

Similarly, for \( A \subset H_\varepsilon \), we use the notation

\[
A^a = \{ u \in H_\varepsilon \mid \inf_{v \in A} \| u - v \|_{H_\varepsilon} \leq a \}.
\]

We want to find a solution near the set

\[
N_\varepsilon = \{ W^\varepsilon_y (z) \mid y \in \mathcal{M}^\beta, W \in \mathcal{L}_V \}
\]

for \( \varepsilon > 0 \) sufficiently small.

**Lemma 4.1.** \( N_\varepsilon \) is uniformly bounded in \( H_\varepsilon \) and it is compact in \( H_\varepsilon \) for any \( \varepsilon > 0 \).

**Proof.** For any \( W^\varepsilon_y \in N_\varepsilon \), by Hölder’s inequality, we have

\[
\| W^\varepsilon_y \|^2_{H_\varepsilon} = \int_{\mathbb{R}^3} |D_s(\varphi_\varepsilon W)|^2 \, dz + \int_{\mathbb{R}^3} V(\varepsilon z + y) \varphi_\varepsilon^2(z) W^2(z) \, dz
\]

\[
\leq 2 \int_{\mathbb{R}^3} \varphi_\varepsilon^2 |D_s W|^2 \, dz + 2 \int_{\mathbb{R}^3} W^2 |D_s \varphi_\varepsilon|^2 \, dz
\]

\[
+ \sup_{y \in \mathcal{M}^\beta, z \in B_{2\beta/\varepsilon}(0)} V(\varepsilon z + y) \int_{B_{2\beta/\varepsilon}(0)} \varphi_\varepsilon^2(z) W^2(z) \, dz
\]

\[
\leq 2 \int_{\mathbb{R}^3} |D_s W|^2 \, dz + C \int_{\mathbb{R}^3} W^2 \, dz + 2 \left( \int_{\mathbb{R}^3} W^2 \, dz \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} |D_s \varphi_\varepsilon|^\frac{2}{3} \, dz \right)^{\frac{3}{2}}
\]

and directly computations, we get

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi_\varepsilon(z) - \varphi_\varepsilon(y)|^2}{|z - y|^{3+2s}} \, dy \, dz = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\varphi(z) - \varphi(y)|^2}{|z - y|^{3+2s}} \, dy \, dz
\]

\[
= \int_{\mathbb{R}^3 \setminus B_{2\beta}(0)} \int_{\mathbb{R}^3} \frac{|\varphi(z) - \varphi(y)|^2}{|z - y|^{3+2s}} \, dy \, dz + \int_{B_{2\beta}(0)} \int_{\mathbb{R}^3} \frac{|\varphi(z) - \varphi(y)|^2}{|z - y|^{3+2s}} \, dy \, dz
\]

\[
= \int_{\mathbb{R}^3 \setminus B_{2\beta}(0)} \int_{|z - y| \leq \beta} \frac{1}{|z - y|^{1+2s}} \, dy \, dz + \int_{B_{2\beta}(0)} \int_{|z - y| > \beta} \frac{|\varphi(z) - \varphi(y)|^2}{|z - y|^{3+2s}} \, dy \, dz
\]

\[
\leq C \left( \frac{1}{\beta^2} \int_{B_{2\beta}(0)} \int_{|z - y| \leq \beta} \frac{1}{|z - y|^{1+2s}} \, dy \, dz + \int_{B_{2\beta}(0)} \int_{|z - y| > \beta} \frac{|\varphi(z) - \varphi(y)|^2}{|z - y|^{3+2s}} \, dy \, dz \right)
\]

\[
+ \frac{1}{\beta^2} \int_{B_{2\beta}(0)} \int_{|z - y| \leq \beta} \frac{1}{|z - y|^{1+2s}} \, dy \, dz + \int_{B_{2\beta}(0)} \int_{|z - y| > \beta} \frac{1}{|z - y|^{3+2s}} \, dy \, dz
\]

\[
\leq C \left( 1 + \int_{\mathbb{R}^3 \setminus B_{2\beta}(0)} \int_{|z - y| > \beta} \frac{|\varphi(z) - \varphi(y)|^2}{|z - y|^{3+2s}} \, dy \, dz \right)
\]

\[
= C \left( 1 + \int_{\mathbb{R}^3 \setminus B_{2\beta}(0)} \int_{|z - y| > \beta} \frac{1}{|z - y|^{1+2s}} \, dy \, dz \right)
\]

\[
+ \frac{1}{\beta^2} \int_{B_{2\beta}(0)} \int_{|z - y| \leq \beta} \frac{1}{|z - y|^{3+2s}} \, dy \, dz
\]

\[
= C \left( 1 + \int_{\mathbb{R}^3 \setminus B_{2\beta}(0)} \int_{|z - y| > \beta} \frac{1}{|z - y|^{1+2s}} \, dy \, dz \right)
\]

\[
+ \frac{1}{\beta^2} \int_{B_{2\beta}(0)} \int_{|z - y| \leq \beta} \frac{1}{|z - y|^{3+2s}} \, dy \, dz
\]
Thus, we obtain
\[
\|W^y_w\|_{H_\varepsilon} \leq C\|W\|^2 \quad (4.2)
\]
for all \(y \in \mathcal{M}^\beta, W \in \mathcal{L}_{V_0}\) and \(\varepsilon\). From the boundedness of \(\mathcal{L}_{V_0}\), we see that \(\mathcal{N}_\varepsilon\) is uniformly bounded in \(H_\varepsilon\).

Now let \(\{W_n\}\) be a sequence in \(\mathcal{N}_\varepsilon\), then there exists \(\{U_n\} \subset \mathcal{L}_{V_0}\) and \(\{x_n\} \subset \mathcal{M}^\beta\) satisfying \(W_n(z) = \varphi_\varepsilon(z - \frac{x_n}{\varepsilon})U_n(z - \frac{x_n}{\varepsilon})\). The compactness of \(\mathcal{L}_{V_0}\) and \(\mathcal{M}^\beta\) imply that the existence of \(U_0 \in \mathcal{L}_{V_0}\) and \(x_0 \in \mathcal{M}^\beta\) such that \(U_n \to U\) in \(H^s(\mathbb{R}^3)\) and \(x_n \to x_0\) in \(\mathbb{R}^3\), up to subsequences.

Define \(W_0(z) = \varphi_\varepsilon(z - \frac{x_0}{\varepsilon})U_0(z - \frac{x_0}{\varepsilon})\), we have \(W_0 \in \mathcal{N}_\varepsilon\). From (4.2), it is easy to know that \(W_n \to W_0\) in \(H_\varepsilon\).

For \(W^* \in \mathcal{L}_{V_0}\) arbitrary but fixed, we define
\[
W_{\varepsilon,\tau}(z) := \varphi(\varepsilon z)W^*_\tau(z) = \tau^{s+t}\varphi(\varepsilon z)W^*(\tau z),
\]
we will show that \(J_\varepsilon\) possesses the mountain-pass geometry.

Similar to the proof of Lemma 4.2, we can conclude that \(J_\varepsilon(u) > 0\) for \(\|u\|_{H_\varepsilon}\) small and there exists \(\tau_0 > 0\) such that \(I_{V_0}(W^*_\tau) < -3\), where \(W^*_\tau(z) = \tau^{s+t}W^*(\tau z)\).

**Lemma 4.2.**
\[
\sup_{\tau \in [0, \tau_0]} |J_\varepsilon(W_{\varepsilon,\tau}) - I_{V_0}(W^*_\tau(z))| \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]

**Proof.** Since \(\text{supp}(W_{\varepsilon,\tau}) \subset A_{\varepsilon}\), we have \(Q_\varepsilon(W_{\varepsilon,\tau}) \equiv 0\) and so \(J_\varepsilon(W_{\varepsilon,\tau}) = P_\varepsilon(W_{\varepsilon,\tau})\).

Then for any \(\tau \in [0, \tau_0]\), we get
\[
\left|P_\varepsilon(W_{\varepsilon,\tau}) - I_{V_0}(W^*_\tau(z))\right| \leq \frac{1}{2} \int_{\mathbb{R}^3} \left|D_s W_{\varepsilon,\tau}|^2 - |D_s W^*_\tau|^2\right| dz + \frac{1}{2} \int_{\mathbb{R}^3} (V(\varepsilon z)W_{\varepsilon,\tau} - V_0(W^*_\tau)^2) dz
\]
\[
+ \frac{1}{4} \int_{\mathbb{R}^3} (\phi_{W_{\varepsilon,\tau}} W_{\varepsilon,\tau} - \phi_{W^*_\tau}(W^*_\tau)^2) dz + \left|\int_{\mathbb{R}^3} (G(\varepsilon z, W_{\varepsilon,\tau})) dz\right|
\]
\[
:= I_1 + I_2 + \frac{1}{2} I_3 + I_4.
\]

In order to estimate \(I_i (i = 1, 2, 3, 4)\), we set \(h(\tau) = \frac{\tau^{s+t}}{(3+t-s)|z|^{3+t}}\) for \(\tau \in [0, +\infty)\) and \(|z| > 0\). Directly computations, we see that \(h(\tau)\) attains its maximum at \(\tau_{\max} = \left(\frac{s+t}{(3+t-s)|z|^{3+t}}\right)^{\frac{3+t}{s+t}}\) and
\[
\sup_{\tau \in [0, +\infty)} h(\tau) = h(\tau_{\max}) = \frac{(3 + t - s)}{3 + 2s}\left(\frac{3 + t - s}{3 + 2t - s}\right)^{\frac{3+t}{s+t}} \frac{1}{|z|^{3+t}}.
\]

Observe that \(|z| \geq \left(\frac{s+t}{3+t-s}\right)^{\frac{1}{s+t}} \frac{1}{\tau_0}\), i.e., \(\tau_{\max} \leq \tau_0\), we have that
\[
\sup_{\tau \in [0, \tau_0]} h(\tau) = h(\tau_{\max}).
\]
If $|z| < \left( \frac{t_0 + t}{3 + t + s} \right)^{\frac{1}{\alpha}}$, i.e., $\tau_{\max} > \tau_0$, we have that

$$\sup_{\tau \in [0, \tau_0]} h(\tau) = h(\tau_0).$$

Now, by (ii) of Proposition 3.5, Fubini's Theorem and $W \in C^{1, \alpha}(\mathbb{R}^3)$, we have that

$$A_1 = \tau^{2(s+t)} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{|z - y|^{1+2s}} \left( (\varphi^2(z) - 1) |W(\tau z) - W(\tau y)|^2 + |\varphi_z(z) - \varphi_z(y)|^2 W^2(\tau y) \right. \right.$$  

$$+ 2\varphi_z(z)(\varphi_z(z) - \varphi_z(y)) (W(\tau z) - W(\tau y)) W(\tau y) \left. \right) dy \: dz \right|$$  

$$\leq \tau^{2(s+t)} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\varphi^2(z) - 1| |W(\tau z) - W(\tau y)|^2 \frac{dy \: dz}{|z - y|^{1+2s}} \right. + 2(1 + \int_{\mathbb{R}^3} \varphi^2 \|\partial_s W(\tau z)\|^2 dz)$$  

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W^2(\tau z) \frac{|\varphi(z) - \varphi(y)|^2}{|z - y|^{1+2s}} dy \: dz \right)$$  

$$\leq \tau^{2(s+t)} \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\varphi^2(z) - 1| \left( \chi_{\{|z - y|<1\}} |\tau_0^{2(s+t+1)} |z - y|^{1+2s} + \chi_{\{|z - y|>1\}} |\tau_0^{2(s+t+1)} |z - y|^{1+2s} \right) dy \: dz \right.$$  

$$+ C(\tau_0^2 + 1) \left( \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \max \left\{ h^2(\tau_{\max}), h^2(\tau_0) \right\} \left| \varphi(z) - \varphi(y) \right|^2 \frac{dy \: dz}{|z - y|^{1+2s}} \right).$$

Thus, the Lebesgue Dominated Convergence Theorem implies that $\sup_{\tau \in [0, \tau_0]} A_1 \to 0$ as $\varepsilon \to 0$.

For $A_2$, since

$$A_2 = \tau^{2(s+t)} \left| \int_{\mathbb{R}^3} (V(z \varepsilon - V_0) \varphi_e^2(z) W^2(\tau z) dz + V_0 \int_{\mathbb{R}^3} (\varphi_e^2(z) - 1) W^2(\tau z) dz \right|$$  

$$\leq \int_{\mathbb{R}^3} (V(z \varepsilon - V_0) \varphi_e^2(z) \max \left\{ h^2(\tau_{\max}), h^2(\tau_0) \right\} dz$$  

$$+ V_0 \int_{\mathbb{R}^3} |\varphi_e^2(z) - 1| \max \left\{ h^2(\tau_{\max}), h^2(\tau_0) \right\} dz,$$

by the Lebesgue Dominated Convergence Theorem, we obtain that $\sup_{\tau \in [0, \tau_0]} A_2 \to 0$ as $\varepsilon \to 0$.

For $A_3$, similarly arguments as above proof of $A_2$, we have that

$$A_3 \leq \tau^{4(s+t)} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left| \varphi_e^2(z) - \varphi_e^2(y) - 1 \right| |W^2(\tau y) W^2(\tau z)|}{|z - y|^{3-2t}} dy \: dz$$  

$$\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left| \varphi_e^2(z) - \varphi_e^2(y) - 1 \right|}{|z - y|^{3-2t}} \max \left\{ h^2(\tau_{\max}), h^2(\tau_0) \right\} \left| \varphi_e(z) - \varphi_e(y) \right| \frac{dy \: dz}{|z - y|^{1+2s}}.$$  

Using the Lebesgue Dominated Convergence Theorem, we get that $\sup_{\tau \in [0, \tau_0]} A_3 \to 0$ as $\varepsilon \to 0$.

For $A_4$, From $W \in L^\infty(\mathbb{R}^3)$ and (3.2), we deduce that

$$A_4 \leq \int_{\mathbb{R}^3} \left| G(\tau^{s+t} \varphi_e W(\tau z)) - G(\tau^{s+t} W(\tau z)) \right| dz \leq C \tau^{2(s+t)} \int_{\mathbb{R}^3} (W^2(\tau z) + W^{2s}(\tau z)) |\varphi_z(z) - 1| dz$$  

$$\leq C \int_{\mathbb{R}^3} \tau^{2(s+t)} W^2(\tau z) |\varphi_z(z) - 1| dz \leq C \int_{\mathbb{R}^3} \max \left\{ h^2(\tau_{\max}), h^2(\tau_0) \right\} |\varphi(z) - 1| dz.$$
Thus, \( A_4 \to 0 \) as \( \varepsilon \to 0 \). Therefore, \( \mathcal{J}_\varepsilon(W_{\varepsilon, \tau}) \to \mathcal{I}_{V_0}(W^*) \) as \( \varepsilon \to 0 \), uniformly on \( \tau \in [0, \tau_0] \). \( \square \)

Since \( 0 \in \mathcal{M} \) and \( \Lambda \) is an open set, there exists \( R > 0 \) such that \( B_R(0) \subset \Lambda \), and by Proposition 4.3 (ii), we have

\[
\int_{\mathbb{R}^3 \setminus A_\varepsilon} W^2_{\varepsilon, \tau_0} \, dz \leq \tau_0^{2(s+1)} \int_{\mathbb{R}^3 \setminus B_{R/\varepsilon}(0)} (W^*(\tau_0 z))^2 \, dz \leq C\varepsilon^{4s+3} \leq C\varepsilon^{4s+3}
\]

which implies that \( Q_\varepsilon(W_{\varepsilon, \tau_0}) \equiv 0 \) for \( \varepsilon > 0 \) small. Thus, by Lemma 4.2 we have

\[
\mathcal{J}_\varepsilon(W_{\varepsilon, \tau_0}) = P_\varepsilon(W_{\varepsilon, \tau_0}) = \mathcal{I}_{V_0}(W^*_n) + o(1) < -2 \quad \text{for } \varepsilon > 0 \text{ small.}
\]

Therefore, we can define the Mountain-Pass level of \( \mathcal{J}_\varepsilon \) given by

\[
C_\varepsilon := \inf_{\gamma \in \mathcal{A}_\varepsilon} \max_{\tau \geq 0} \mathcal{J}_\varepsilon(\gamma(\tau)),
\]

where \( \mathcal{A}_\varepsilon = \{ \gamma \in C([0, 1], H_\varepsilon) \mid \gamma(0) = 0, \gamma(1) = W_{\varepsilon, \tau_0} \} \). Furthermore, by well-known arguments (see for instance [6, 20] for a proof in a local setting that extends smoothly to our case) it is possible to prove the following Lemma.

**Lemma 4.3.**

\[
\lim_{\varepsilon \to 0} C_\varepsilon = \lim_{\varepsilon \to 0} D_\varepsilon := \lim_{\varepsilon \to 0} \max_{\tau \in [0, 1]} \mathcal{J}_\varepsilon(\gamma_\varepsilon(\tau)) = c_{V_0} \quad (4.3)
\]

where \( \gamma(\tau) = W_{\varepsilon, \tau_0} \) for \( \tau \in [0, 1] \) and \( c_{V_0} = \mathcal{I}_{V_0}(W^*) \) for \( W^* \in \mathcal{L}_{V_0} \).

**Proof.** First we will prove that \( \limsup C_\varepsilon \leq c_{V_0} \). Setting \( \gamma(\tau) = W_{\varepsilon, \tau_0} \) for \( \tau \in [0, 1] \), we get \( \gamma_\varepsilon \in \Gamma_\varepsilon \) and from Lemma 4.2, we have

\[
\limsup_{\varepsilon \to 0} C_\varepsilon \leq \limsup_{\varepsilon \to 0} \max_{\tau \in [0, 1]} \mathcal{J}_\varepsilon(\gamma_\varepsilon(\tau)) \leq \limsup_{\varepsilon \to 0} \max_{\tau \in [0, 1]} \mathcal{J}_\varepsilon(W_{\varepsilon, \tau}) \leq \max_{\tau \in [0, \tau_0]} \mathcal{I}_{V_0}(W^*_n) = \mathcal{I}_{V_0}(W) = c_{V_0}.
\]

which we conclude the first part of the proof. Next we shall prove that \( \liminf C_\varepsilon \geq c_{V_0} \). Assume the contrary that \( \liminf C_\varepsilon < c_{V_0} \). Then there exist \( \delta_0 > 0, \varepsilon_0 \to 0 \) and \( \gamma_n := \gamma_{\varepsilon_0, n} \in \mathcal{A}_{\varepsilon_0, n} \) satisfying \( \mathcal{J}_\varepsilon(\gamma_n(\tau)) < c_{V_0} - \delta_0 \) for \( \tau \in [0, 1] \). Since \( P_{\varepsilon_n}(\gamma_n(0)) = 0 \) and \( P_{\varepsilon_n}(\gamma_n(1)) \leq \mathcal{J}_{\varepsilon_n}(W_{\varepsilon_n, \tau_0}) < -2 \), we can find \( \tau_0 \in (0, 1) \) such that \( P_{\varepsilon_n}(\gamma_n(\tau)) \geq -1 \) for \( \tau \in (0, \tau_0) \) and \( P_{\varepsilon_n}(\gamma_n(\tau)) = -1 \). Since

\[
P_{\varepsilon_n}(\gamma_n(\tau)) = \mathcal{I}_{V_0}(\gamma_n(\tau)) + \frac{1}{2} \int_{\mathbb{R}^3} (V(\varepsilon_n z) - V_0) \gamma_n^2(\tau) \, dz + \int_{\mathbb{R}^3} [G(\gamma_n(\tau)) - F(\varepsilon_n z, \gamma_n(\tau))] \, dz
\]

\[
\geq \mathcal{I}_{V_0}(\gamma_n(\tau)) + \frac{1}{2} \int_{\mathbb{R}^3} (V(\varepsilon_n z) - V_0) \gamma_n^2(\tau) \, dz \geq \mathcal{I}_{V_0}(\gamma_n(\tau)), \quad \forall \tau \in (0, \tau_0),
\]

then

\[
\mathcal{I}_{V_0}(\gamma_n(\tau_0)) \leq P_{\varepsilon_n}(\gamma_n(\tau_0)) = -1 < 0.
\]

Recalling that the mountain pass level for \( \mathcal{I}_{V_0} \) corresponds to the least energy level, we have \( \max_{\tau \in [0, \tau_0]} \mathcal{I}_{V_0}(\gamma_n(\tau)) \geq c_{V_0} \). Since \( Q_{\varepsilon_n}(\gamma_n(\tau)) \geq 0 \), by the estimates above we obtain

\[
c_{V_0} - \delta_0 > \max_{\tau \in [0, 1]} \mathcal{J}_{\varepsilon_n}(\gamma_n(\tau)) \geq \max_{\tau \in [0, 1]} P_{\varepsilon_n}(\gamma_n(\tau)) \geq \max_{\tau \in [0, \tau_0]} P_{\varepsilon_n}(\gamma_n(\tau)) \geq \max_{\tau \in [0, \tau_0]} \mathcal{I}_{V_0}(\gamma_n(\tau)) \geq c_{V_0}.
\]
This contradiction completes the proof. □

**Lemma 4.4.** There exists a small $d_0 > 0$ such that for any $\{\varepsilon_i\}, \{u_{\varepsilon_i}\}$ satisfying $\lim_{i \to \infty} \varepsilon_i \to 0$, $u_{\varepsilon_i} \in N_{\varepsilon_i}^{d_0}$ and
\[
\lim_{i \to \infty} J_{\varepsilon_i}(u_{\varepsilon}) \leq c_{V_0} \quad \text{and} \quad \lim_{i \to \infty} J'_{\varepsilon_i}(u_{\varepsilon_i}) = 0,
\]
there exist, up to a subsequence, $x_i \in \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $W \in \mathcal{L}_{V_0}$ such that
\[
\lim_{i \to \infty} \varepsilon_i |x_i - x_0| = 0 \quad \text{and} \quad \lim_{i \to \infty} \|u_{\varepsilon_i} - \varphi_{\varepsilon}(\cdot - x_i)W(\cdot - x_i)\|_{H_\varepsilon} = 0.
\]

**Proof.** In the proof we will drop the index $i$ and write $\varepsilon$ instead of $\varepsilon_i$ for simplicity, and we still use $\varepsilon$ after taking a subsequence. By the definition of $N_{\varepsilon_i}^{d_0}$, there exist $\{W_{\varepsilon}\} \subset \mathcal{L}_{V_0}$ and $\{x_{\varepsilon}\} \subset \mathcal{M}$ such that for $\varepsilon$ small,
\[
\|u_{\varepsilon} - \varphi_{\varepsilon}(-x_{\varepsilon}/\varepsilon)W(\cdot - x_{\varepsilon}/\varepsilon)\|_{H_\varepsilon} \leq \frac{3}{2}d_0.
\]
Since $\mathcal{L}_{V_0}$ and $\mathcal{M}$ are compact, there exist $W_0 \in \mathcal{L}_{V_0}$, $x_0 \in \mathcal{M}$ such that $W_{\varepsilon} \to W_0$ in $H^s(\mathbb{R}^3)$ and $x_{\varepsilon} \to x_0$ as $\varepsilon \to 0$. Thus, for $\varepsilon > 0$ small,
\[
\|u_{\varepsilon} - \varphi_{\varepsilon}(-x_{\varepsilon}/\varepsilon)W_0(\cdot - x_{\varepsilon}/\varepsilon)\|_{H_\varepsilon} \leq 2d_0. \quad (4.4)
\]

**Step 1.** We claim that
\[
\lim_{\varepsilon \to 0} \sup_{y \in A_\varepsilon} \int_{B_1(y)} |u_{\varepsilon}|^2 \, dz = 0,
\]
where $A_\varepsilon = B_{3\beta/\varepsilon}(x_{\varepsilon}/\varepsilon) \setminus B_{\beta/2\varepsilon}(x_{\varepsilon}/\varepsilon)$. Suppose by contradiction that
\[
\lim_{\varepsilon \to 0} \inf \sup_{y \in A_\varepsilon} \int_{B_1(y)} |u_{\varepsilon}|^2 \, dz > 0.
\]
Thus, there exists $y_{\varepsilon} \in A_\varepsilon$ such that $\int_{B_1(y_{\varepsilon})} |u_{\varepsilon}|^2 \, dz > 0$ for $\varepsilon > 0$ small. Since $y_{\varepsilon} \in A_\varepsilon$, there exists $y^* \in \mathcal{M}^\beta \subset \Lambda$ such that $\varepsilon y_{\varepsilon} \to y^*$ as $\varepsilon \to 0$. Set $v_{\varepsilon}(z) = u_{\varepsilon}(z + y_{\varepsilon})$, then for $\varepsilon > 0$ small,
\[
\int_{B_1(0)} |v_{\varepsilon}|^2 \, dz > 0. \quad (4.6)
\]
Thus, up to a subsequence, we may assume that there exists $v \in H^s(\mathbb{R}^3)$ such that $v_{\varepsilon} \to v$ in $H^s(\mathbb{R}^3)$, $v_{\varepsilon} \to v$ in $L^p_{loc}(\mathbb{R}^3)$ for $1 \leq p < 2^*_s$ and $v_{\varepsilon} \to v$ a.e. in $\mathbb{R}^3$. By (4.6), we see that $v \neq 0$ and $v$ satisfies
\[
(-\Delta)^s v + V(y^*) v + \phi^* v = g(v) \quad z \in \mathbb{R}^3. \quad (4.7)
\]
Indeed, by the definition of weakly convergence, we have
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(v_{\varepsilon}(z) - v_{\varepsilon}(y))(\varphi(z) - \varphi(y))}{|z - y|^{3+2s}} \, dy \, dz + \int_{\mathbb{R}^3} V(y^*) v_{\varepsilon} \varphi \, dz \to \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(v(z) - v(y))(\varphi(z) - \varphi(y))}{|z - y|^{3+2s}} \, dy \, dz + \int_{\mathbb{R}^3} V(y^*) v \varphi \, dz
\]
for any $\varphi \in C^1_0(\mathbb{R}^3)$. Now given $\varphi \in C^\infty(\mathbb{R}^3)$, we have $\|\varphi(\cdot - y_{\varepsilon})\|_{H_\varepsilon} \leq C$ and so $\langle J'_{\varepsilon}(u_{\varepsilon}), \varphi(\cdot - y_{\varepsilon}) \rangle \to 0$ as $\varepsilon \to 0$. Using the fact that $v_{\varepsilon} \to v$ in $L^p_{loc}(\mathbb{R}^3)$ for
1 \leq p < 2^*_\alpha$, the Lebesgue dominated convergence Theorem, the boundedness of $\text{supp}(\varphi)$ and $(g_0)-(g_1)$, it follows that

$$
\int_{\mathbb{R}^3} (V(\varepsilon z + \varepsilon y_\varepsilon) - V(y^*))v_\varepsilon \varphi \, dz \to 0,
$$

$$
\int_{\mathbb{R}^3} (\phi^\varepsilon_v v_\varepsilon - \phi^\varepsilon_v v) \varphi \, dz \to 0,
$$

$$
\int_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} u_\varepsilon(z) \varphi(z - y_\varepsilon) \, dz = \int_{\mathbb{R}^3 \setminus \Lambda_+ + y_\varepsilon} v_\varepsilon(z) \varphi(z) \, dz \to 0
$$

and

$$
\int_{\mathbb{R}^3} (f(\varepsilon z + \varepsilon y_\varepsilon, v_\varepsilon) - g(v)) \varphi \, dz \to 0
$$

for any $\varphi \in C^\infty_0(\mathbb{R}^3)$. Therefore, we get that

$$
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(v(z) - v(y))(\varphi(z) - \varphi(y))}{|z - y|^3} \, dy \, dz + \int_{\mathbb{R}^3} V(y^*)v \varphi \, dz + \int_{\mathbb{R}^3} \phi^\varepsilon_v v \varphi \, dz - \int_{\mathbb{R}^3} g(v) \varphi \, dz = 0
$$

for any $\varphi \in C^\infty_0(\mathbb{R}^3)$. Since $\varphi$ is arbitrary and $C^\infty_0(\mathbb{R}^3)$ is dense in $H_\varepsilon$, it follows that $v$ satisfies (4.7).

Thus, we have

$$
cv(y^*) \leq \mathcal{I}_V(y^*)(v) = \mathcal{I}_V(y^*)(v) - \frac{1}{4s + 2t - 3} G_{V(y^*)}(v)
$$

$$
= s \int_{\mathbb{R}^3} V(y^*)v^2 \, dz + \frac{s + t}{4s + 2t - 3} \int_{\mathbb{R}^3} [g(v)v - \frac{4s + 2t}{s + t} G(v)] \, dz
$$

$$
\leq s \|v\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |v|^2 \, dz + \frac{s + t}{4s + 2t - 3} \int_{\mathbb{R}^3} [g(v)v - \frac{4s + 2t}{s + t} G(v)] \, dz.
$$

Hence, for sufficiently large $r > 0$, by Fatou's Lemma, we have that

$$
\lim_{\varepsilon \to 0} \inf \frac{s}{r} \int_{B_r(y_\varepsilon)} |u_\varepsilon|^2 \, dz + \frac{s + t}{4s + 2t - 3} \int_{B_r(y_\varepsilon)} [g(u_\varepsilon)u_\varepsilon - \frac{4s + 2t}{s + t} G(u_\varepsilon)] \, dz
$$

$$
= \lim_{\varepsilon \to 0} \inf \frac{s}{r} \int_{B_r(y_\varepsilon)} |v_\varepsilon|^2 \, dz + \frac{s + t}{4s + 2t - 3} \int_{B_r(y_\varepsilon)} [g(v_\varepsilon)v_\varepsilon - \frac{4s + 2t}{s + t} G(v_\varepsilon)] \, dz
$$

$$
\geq \left[ \frac{s}{r} \|v\|_{L^\infty(\mathbb{R}^3)} \int_{B_r(0)} |v|^2 \, dz + \frac{s + t}{4s + 2t - 3} \int_{B_r(0)} [g(v)v - \frac{4s + 2t}{s + t} G(v)] \, dz \right]
$$

$$
\geq \frac{1}{2} \left[ \frac{s}{r} \|v\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |v|^2 \, dz + \frac{s + t}{4s + 2t - 3} \int_{\mathbb{R}^3} [g(v)v - \frac{4s + 2t}{s + t} G(v)] \, dz \right]
$$

$$
\geq \frac{1}{2} cv(x^*) > 0.
$$

On the other hand, by the Sobolev embedding theorem, (3.2) and (4.4), one has

$$
s \|v\|_{L^\infty(\mathbb{R}^3)} \int_{B_r(y_\varepsilon)} |u_\varepsilon|^2 \, dz + \frac{s + t}{4s + 2t - 3} \int_{B_r(y_\varepsilon)} [g(u_\varepsilon)u_\varepsilon - \frac{4s + 2t}{s + t} G(u_\varepsilon)] \, dz
$$

$$
\leq Cd_0 + C \int_{B_r(y_\varepsilon)} \varphi(\varepsilon z - x_\varepsilon) W_0(z - \frac{x_\varepsilon}{\varepsilon})^2 \, dz \leq Cd_0 + C \int_{B_r(y_\varepsilon - \frac{x_\varepsilon}{\varepsilon})} |W_0|^2 \, dz
$$

Observing that $y_\varepsilon \in A_\varepsilon$, implies that $|y_\varepsilon - \frac{x_\varepsilon}{\varepsilon}| \geq \frac{\varepsilon}{2r}$, then for $\varepsilon > 0$ small enough, there hold

$$
\int_{B_r(y_\varepsilon - \frac{x_\varepsilon}{\varepsilon})} |W_0|^2 \, dz = o(1),
$$
where \( o(1) \to 0 \) as \( \varepsilon \to 0 \). Thus, we have proved that
\[
\frac{1}{2} c_{V(y^* \cdot)} \leq s\|V\|_{L^\infty(\mathbb{R}^3)} \int_{B_r(y)} |u_\varepsilon|^2 dz + \frac{s + t}{4s + 2t - 3} \int_{B_r(y)} [g(u_\varepsilon)u_\varepsilon - \frac{4s + 2t}{s + t} G(u_\varepsilon)] dz \\
\leq Cd_0 + o(1).
\]
This leads to a contradiction if \( d_0 \) is small enough.

From (4.5) and the Vanishing Lemma 2.4 we conclude that
\[
\lim_{\varepsilon \to 0} \int_{A^1_\varepsilon} |u_\varepsilon|^p dz = 0 \quad p \in (2, 2^*_s),
\]
where \( A^1_\varepsilon = B_{2\beta/\varepsilon}(\frac{x}{\varepsilon}) \setminus B_{\beta/\varepsilon}(\frac{x}{\varepsilon}) \). Indeed, taking a smooth cut-off function \( \psi_\varepsilon \in C_0^\infty(\mathbb{R}^3) \) such that \( \psi_\varepsilon = 1 \) on \( B_{2\beta/\varepsilon}(\frac{x}{\varepsilon}) \), \( \psi_\varepsilon = 0 \) on \( A^2_\varepsilon = B_{3\beta/\varepsilon - 1}(\frac{x}{\varepsilon}) \). Since \( u_\varepsilon \in H_\varepsilon \) and using \( (V_0) \), it is easy to check that \( u_\varepsilon \psi_\varepsilon \in H^s(\mathbb{R}^3) \). Moreover,
\[
\sup_{y \in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^2 dz \geq \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_\varepsilon \psi_\varepsilon|^2 dz.
\]
By Vanishing Lemma 2.4 we have that for \( p \in (2, 2^*_s) \),
\[
\int_{\mathbb{R}^3} |u_\varepsilon \psi_\varepsilon|^p dz \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
Since \( A^1_\varepsilon \subset A^2_\varepsilon \) for \( \varepsilon > 0 \) small, so (4.8) holds.

**Step 2.** Set \( u_{\varepsilon,1}(z) = \varphi(\varepsilon z - x_\varepsilon)u_\varepsilon(z) \), \( u_{\varepsilon,2}(z) = (1 - \varphi(\varepsilon z - x_\varepsilon))u_\varepsilon(z) \). Direct computation, we have
\[
\int_{\mathbb{R}^3} |D_s u_{\varepsilon,1}|^2 dz = \int_{\mathbb{R}^3} |D_s u_{\varepsilon,1}|^2 dz + \int_{\mathbb{R}^3} |D_s u_{\varepsilon,2}|^2 dz \\
+ 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_{\varepsilon,1}(x) - u_{\varepsilon,1}(y))(u_{\varepsilon,2}(x) - u_{\varepsilon,2}(y))}{|x - y|^{3 + 2s}} dy dx \\
\geq \int_{\mathbb{R}^3} |D_s u_{\varepsilon,1}|^2 dz + \int_{\mathbb{R}^3} |D_s u_{\varepsilon,2}|^2 dz + o(1) \quad (4.9)
\]
Indeed,
\[
(u_{\varepsilon,1}(z) - u_{\varepsilon,1}(y))(u_{\varepsilon,2}(z) - u_{\varepsilon,2}(y)) \\
= \varphi(\varepsilon z - x_\varepsilon)(1 - \varphi(\varepsilon z - x_\varepsilon))|D_s u_\varepsilon|^2 + \varphi(\varepsilon z - x_\varepsilon)(\varphi(\varepsilon z - x_\varepsilon) - \varphi(\varepsilon y - x_\varepsilon)) \\
(u_{\varepsilon}(z) - u_{\varepsilon}(y))u_\varepsilon(y) + (1 - \varphi(\varepsilon z - x_\varepsilon))(\varphi(\varepsilon z - x_\varepsilon) - \varphi(\varepsilon y - x_\varepsilon))(u_\varepsilon(z) - u_\varepsilon(y))u_\varepsilon(y) \\
- (\varphi(\varepsilon z - x_\varepsilon) - \varphi(\varepsilon y - x_\varepsilon))|u_\varepsilon(y)|^2 \\
:= \varphi(\varepsilon z - x_\varepsilon)(1 - \varphi(\varepsilon z - x_\varepsilon))|D_s u_\varepsilon|^2 + B_1 + B_2 - B_3.
\]
Next we show that \( \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} B_i dz = 0 \), \( i = 1, 2, 3 \). If these are proved, we get
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_{\varepsilon,1}(x) - u_{\varepsilon,1}(y))(u_{\varepsilon,2}(x) - u_{\varepsilon,2}(y))}{|x - y|^{3 + 2s}} dy dx \geq \int_{\mathbb{R}^3} \varphi(\varepsilon z - x_\varepsilon)(1 - \varphi(\varepsilon z - x_\varepsilon))|D_s u_\varepsilon|^2 + o(1) \quad (4.10)
\]
and so (4.9) follows. Here \( o(1) \to 0 \) as \( \varepsilon \to 0 \).

Observe that
\[
\int_{\mathbb{R}^3} B_1 dz \leq \left( \int_{\mathbb{R}^3} (\varphi(\varepsilon z - x_\varepsilon))^2 |D_s u_\varepsilon|^2 dz \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |D_s \varphi(\varepsilon z - x_\varepsilon)|^2 u_\varepsilon^2 dz \right)^{\frac{1}{2}} \\
\leq C \left( \int_{\mathbb{R}^3} |D_s \varphi(\varepsilon z - x_\varepsilon)|^2 u_\varepsilon^2 dz \right)^{\frac{1}{2}} = C \left( \int_{\mathbb{R}^3} B_3 dz \right)^{\frac{1}{2}}
\]
and similarly, we have
\[ \int_{\mathbb{R}^3} B_3 \, dz \leq \left( \int_{\mathbb{R}^3} |D_{\phi}(\varepsilon \cdot -x_\varepsilon)|^2 u_\varepsilon^2 \, dz \right)^{\frac{3}{4}} = C \left( \int_{\mathbb{R}^3} B_3 \, dz \right)^{\frac{3}{4}}. \]
Hence, it is sufficient to prove that
\[ \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} B_3 \, dz = 0. \] (4.11)
In fact, direct computations, we deduce that
\[
\int_{\mathbb{R}^3} B_3 \, dz = \int_{\mathbb{R}^3} u_\varepsilon^2 \int_{\mathbb{R}^3} \frac{|\phi(z) - \phi(y)|^2}{|z - y|^{3/2}} \, dy \, dz \]
\[
= \varepsilon^{2s-3} \int_{\mathbb{R}^3} u_\varepsilon^2 \int_{\mathbb{R}^3} \frac{|\phi(z) - \phi(y)|^2}{|z - y|^{3/2}} \, dy \, dz \]
\[
\leq \varepsilon^{2s-3} \int_{\mathbb{R}^3} u_\varepsilon^2 \int_{|z-y|\leq 1} \frac{1}{|z-y|^{1+2s}} \, dy + \int_{|z-y|\geq 1} \frac{1}{|z-y|^{3/2}} \, dy \, dz \]
\[
\leq \frac{C_\varepsilon^{2s-3}}{\beta^{2s}} \int_{\mathbb{R}^3} u_\varepsilon^2 \, dz = \frac{C_\varepsilon^{2s}}{\beta^{2s}} \int_{\mathbb{R}^3} u_\varepsilon^2 \, dz \leq \frac{C_\varepsilon^{2s}}{\beta^{2s}} \varepsilon^{2s}. \]
From the estimate above, we conclude that (4.11) follows. Thus (4.10) holds.

By (4.8), we deduce that
\[
\int_{\mathbb{R}^3} V(\varepsilon \phi) |u_{\varepsilon}|^2 \, dz \geq \int_{\mathbb{R}^3} V(\varepsilon \phi) |u_{\varepsilon,1}|^2 \, dz + \int_{\mathbb{R}^3} V(\varepsilon \phi) |u_{\varepsilon,2}|^2 \, dz \]
\[
\int_{\mathbb{R}^3} \phi_t^{1} |u_{\varepsilon}|^2 \, dz \geq \int_{\mathbb{R}^3} \phi_t^{1} |u_{\varepsilon,1}|^2 \, dz + \int_{\mathbb{R}^3} \phi_t^{1} |u_{\varepsilon,2}|^2 \, dz \]
\[
\int_{\mathbb{R}^3} F(\varepsilon \phi, u_{\varepsilon}) \, dz = \int_{\mathbb{R}^3} F(\varepsilon \phi, u_{\varepsilon,1}) \, dz + \int_{\mathbb{R}^3} F(\varepsilon \phi, u_{\varepsilon,2}) \, dz + o(1) \text{ as } \varepsilon \to 0 \]
and
\[ Q_\varepsilon(u_{\varepsilon,1}) = 0, \quad Q_\varepsilon(u_{\varepsilon,2}) = Q_\varepsilon(u_{\varepsilon}) \geq 0. \]
Hence, we get
\[ \mathcal{J}_\varepsilon(u_{\varepsilon}) \geq P_\varepsilon(u_{\varepsilon,1}) + P_\varepsilon(u_{\varepsilon,2}) + o(1), \] (4.12)
where \( o(1) \to 0 \) as \( \varepsilon \to 0 \).

We now estimate \( P_\varepsilon(u_{\varepsilon,2}) \). It follows from (4.4) that
\[ \|u_{\varepsilon,2}\|_{H_\varepsilon} \leq 6d_0 + o(1), \]
where \( o(1) \to 0 \) as \( \varepsilon \to 0 \) and the above inequality implies that
\[ \limsup_{\varepsilon \to 0} \|u_{\varepsilon,2}\|_{H_\varepsilon} \leq 6d_0. \] (4.13)
Then, by (3.2), we get
\[ P_\varepsilon(u_{\varepsilon,2}) \geq \frac{1}{2} \|u_{\varepsilon,2}\|_{H_\varepsilon}^2 - \int_{\mathbb{R}^3} F(\varepsilon \phi, u_{\varepsilon,2}) \, dz \geq \frac{1}{4} \|u_{\varepsilon,2}\|_{H_\varepsilon}^2 - C \|u_{\varepsilon,2}\|_{H_\varepsilon}^2, \]
\[ = \|u_{\varepsilon,2}\|_{H_\varepsilon}^2 \left( \frac{1}{4} - C \|u_{\varepsilon,2}\|_{H_\varepsilon}^2 - 2 \right) \geq \|u_{\varepsilon,2}\|_{H_\varepsilon}^2 \left( \frac{1}{4} - C(6d_0)^2 - 2 \right). \] (4.14)
In particular, taking \( d_0 > 0 \) small enough, we can assume that \( P_\varepsilon(u_{\varepsilon,2}) \geq 0 \). Hence, from (4.12), it holds
\[ \mathcal{J}_\varepsilon(u_{\varepsilon}) \geq P_\varepsilon(u_{\varepsilon,1}) + o(1). \] (4.15)
Furthermore, by \( \text{(1.S)} \) and \( \text{(1.10)} \), it is easy to check that
\[
\int_{\mathbb{R}^3} \phi_{u_\varepsilon}^1 u_{\varepsilon,1} u_{\varepsilon,2} \, dz \leq \int_{A_1} \phi_{u_\varepsilon}^1 |u_{\varepsilon}|^2 \, dz \leq \|\phi_{u_\varepsilon}^1\|_{L^{\infty}(A_1^c)} \to 0
\]
and
\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_{\varepsilon,1}(z) - u_{\varepsilon,1}(y))(u_{\varepsilon,2}(z) - u_{\varepsilon,2}(y))}{|z - y|^{3+2s}} \, dy \, dz \geq o(1).
\]
Hence, using the facts that \( \langle \mathcal{J}'_{\varepsilon}(u_\varepsilon), u_{\varepsilon,2} \rangle \to 0 \) as \( \varepsilon \to 0 \), \( \langle Q'_\varepsilon(u_\varepsilon), u_{\varepsilon,2} \rangle \geq 0 \) and \( \text{(3.8)} \), we have that
\[
\|u_{\varepsilon,2}\|_{H^s}^2 + o(1)
\]
\[
\leq \|u_{\varepsilon,2}\|_{H^s}^2 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_{\varepsilon,1}(z) - u_{\varepsilon,1}(y))(u_{\varepsilon,2}(z) - u_{\varepsilon,2}(y))}{|z - y|^{3+2s}} \, dy \, dz + \int_{\mathbb{R}^3} V(\varepsilon z) u_{\varepsilon,1} u_{\varepsilon,2} \, dz
\]
\[
+ \int_{\mathbb{R}^3} \phi_{u_\varepsilon}^1 u_{\varepsilon,1} u_{\varepsilon,2} \, dz
\]
\[
\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_{\varepsilon}(z) - u_{\varepsilon}(y))(u_{\varepsilon,2}(z) - u_{\varepsilon,2}(y))}{|z - y|^{3+2s}} \, dy \, dz + \int_{\mathbb{R}^3} V(\varepsilon z) u_{\varepsilon,2} \, dz + \langle Q_\varepsilon(u_\varepsilon), u_{\varepsilon,2} \rangle
\]
\[
+ \int_{\mathbb{R}^3} \phi_{u_\varepsilon}^1 u_{\varepsilon,2} \, dz + o(1) = \int_{\mathbb{R}^3} f(\varepsilon z, u_{\varepsilon,2}) \, dz + o(1)
\]
\[
\leq \eta \|u_{\varepsilon,2}\|_{L^2}^2 + C \int_{\mathbb{R}^3} |u_{\varepsilon}|^{2^*_s - 1}|u_{\varepsilon,2}| \, dz + o(1)
\]
\[
\leq \eta \|u_{\varepsilon,2}\|_{L^2}^2 + C \int_{\mathbb{R}^3} \left( |u_{\varepsilon}|^{2^*_s} + |u_{\varepsilon,1}|^{2^*_s - 1}|u_{\varepsilon,2}| \right) \, dz + o(1) \leq \eta \|u_{\varepsilon,2}\|_{H^s}^2 + C \|u_{\varepsilon,2}\|_{H^s}^{2^*_s} + o(1).
\]
Combining with \( \text{(1.13)} \), we get that
\[
\left( \frac{1}{2} - C d_0^{2^*_s - 2} \right) \|u_{\varepsilon,2}\|_{H^s}^2 \leq \left( \frac{1}{2} - C \|u_{\varepsilon,2}\|_{H^s}^{2^*_s - 2} \right) \|u_{\varepsilon,2}\|_{H^s}^2 + o(1) \leq o(1).
\]
Thus, taking \( d_0 > 0 \) sufficiently small, we have
\[
\lim_{\varepsilon \to 0} \|u_{\varepsilon,2}\|_{H^s} = 0. \tag{4.16}
\]

We next estimate \( P_\varepsilon(u_{\varepsilon,1}) \). Denote \( \tilde{u}_\varepsilon(z) = u_{\varepsilon,1}(z + \frac{z}{\varepsilon}) = \varphi(\varepsilon z) u_{\varepsilon}(z + \frac{z}{\varepsilon}) \), then \( \{\tilde{u}_\varepsilon\} \) is bounded in \( H^s(\mathbb{R}^3) \) by virtue of \( (V_0) \). Thus, up to a subsequence, we may assume that there exists a \( \tilde{u} \in H^s(\mathbb{R}^3) \) such that \( \tilde{u}_\varepsilon \to \tilde{u} \) in \( H^s(\mathbb{R}^3) \), \( \tilde{u}_\varepsilon \to \tilde{u} \) in \( L^p_{\text{loc}}(\mathbb{R}^3) \) for \( 1 \leq p < 2^*_s \), \( \tilde{u}_\varepsilon \to \tilde{u} \) a.e. in \( \mathbb{R}^3 \) and \( \tilde{u} \) satisfies
\[
(-\Delta)^s v + V(x_0) v + \phi_{\varepsilon}^1 v = g(v) \quad z \in \mathbb{R}^3. \tag{4.17}
\]

We now claim that
\[
\lim_{\varepsilon \to 0} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |\tilde{u}_\varepsilon - \tilde{u}|^2 \, dz = 0. \tag{4.18}
\]
Suppose the contrary that there exists \( \hat{y}_\varepsilon \in \mathbb{R}^3 \) such that
\[
\lim_{\varepsilon \to 0} \int_{B_1(\hat{y}_\varepsilon)} |\tilde{u}_\varepsilon - \tilde{u}|^2 \, dz > 0. \tag{4.19}
\]
Since \( \tilde{u}_\varepsilon \to \tilde{u} \) in \( L^p_{\text{loc}}(\mathbb{R}^3) \) for \( 1 \leq p < 2^*_s \), we have \( \{\hat{y}_\varepsilon\} \subset \mathbb{R}^3 \) must be unbounded. Thus, up to a subsequence, still denoted by \( \{\hat{y}_\varepsilon\} \), we may assume that \( |\hat{y}_\varepsilon| \to +\infty \).
as $\varepsilon \to 0$. Therefore,

$$
\lim_{\varepsilon \to 0} \int_{B_1(\hat{y}_\varepsilon)} |\hat{u}|^2 \, dz = 0, \quad \lim_{\varepsilon \to 0} \int_{B_1(\hat{y}_\varepsilon)} |\hat{u}_\varepsilon|^2 \, dz > 0. \tag{4.20}
$$

Since $\varphi(z) = 0$ for $|z| \geq 2\beta$, so $|\hat{y}_\varepsilon| \leq \frac{3\beta}{\varepsilon}$ for $\varepsilon$ small. If $|\hat{y}_\varepsilon| \geq \frac{\beta}{2\varepsilon}$, then $\hat{y}_\varepsilon \in B_{3\beta/\varepsilon}(0) \setminus B_{\beta/\varepsilon}(0)$, and by (4.5), we get

$$
\liminf_{\varepsilon \to 0} \int_{B_1(\hat{y}_\varepsilon)} |\hat{u}_\varepsilon|^2 \, dz \leq \liminf_{\varepsilon \to 0} \sup_{y \in B_{3\beta/\varepsilon}(0) \setminus B_{\beta/\varepsilon}(0)} \int_{B_1(y)} |u_\varepsilon(z + \frac{x}{\varepsilon})|^2 \, dz 
\leq \liminf_{\varepsilon \to 0} \sup_{y \in A_\varepsilon} \int_{B_1(y)} |\hat{u}_\varepsilon|^2 \, dz = 0
$$

which contradicts with (4.20). Thus $|\hat{y}_\varepsilon| \leq \frac{\beta}{2\varepsilon}$ for $\varepsilon > 0$ small. Without loss of generality, we may assume that $\varepsilon \hat{y}_\varepsilon \to Z_0 = \hat{B}_{\beta/2}(0)$ and $\hat{u}_\varepsilon \to \hat{u}$ in $H^s(\mathbb{R}^3)$, where $\hat{u}_\varepsilon(z) := \hat{u}_\varepsilon(z + \hat{y}_\varepsilon)$. Obviously, $\hat{u} \neq 0$. It is easy to check that $\hat{u}$ satisfies that

$$
(-\Delta)^s v + V(x_0 + z_0) v + \phi_\varepsilon^s v = g(v) \quad \text{in} \quad \mathbb{R}^3.
$$

Similarly as in the proof of the case $u \neq 0$ of the claim (4.5), we can get a contradiction for $d_0$ sufficient small. Hence, the claim (4.18) holds and so using the Vanishing Lemma 2.4 we see that

$$
\hat{u}_\varepsilon \to \hat{u} \quad \text{in} \quad L^p(\mathbb{R}^3), \quad p \in (2, 2_\ast^\ast). \tag{4.21}
$$

By (4.15), recalling that $u_\varepsilon(z) = u_{\varepsilon,1}(z + \frac{x}{\varepsilon})$, we have

$$
P_\varepsilon(\hat{u}_{\varepsilon}) \leq C\varepsilon + o(1).
$$

Letting $\varepsilon \to 0$, and using (4.21), (V0), we get

$$
\mathcal{I}_{V(x_0)}(\hat{u}) \leq C\varepsilon.
$$

On the other hand, in view of $\langle J'_s(u_{\varepsilon,1}), u_{\varepsilon,1} \rangle \to 0$ and (4.16), and $\langle Q'_s(u_{\varepsilon,1}), u_{\varepsilon,1} \rangle = 0$, we deduce that

$$
\int_{\mathbb{R}^3} |D_s \hat{u}|^2 \, dz = \int_{\mathbb{R}^3} V(\varepsilon z + x_\varepsilon) |\hat{u}_\varepsilon|^2 \, dz + \int_{\mathbb{R}^3} \phi_{\varepsilon}^s |\hat{u}_\varepsilon|^2 \, dz = \int_{\mathbb{R}^3} f(\varepsilon z, \hat{u}_\varepsilon) \hat{u}_\varepsilon \, dz + o(1),
$$

then by Fatou’s Lemma, (4.21) and (4.17), we have that

$$
\int_{\mathbb{R}^3} |D_s \hat{u}|^2 \, dz + \int_{\mathbb{R}^3} V(x_0) |\hat{u}|^2 \, dz + \int_{\mathbb{R}^3} \phi_{\varepsilon}^s |\hat{u}|^2 \, dz \leq \liminf_{\varepsilon \to 0} \left( \int_{\mathbb{R}^3} |D_s \hat{u}_\varepsilon|^2 \, dz + \int_{\mathbb{R}^3} V(\varepsilon z + x_\varepsilon) |\hat{u}_\varepsilon|^2 \, dz + \int_{\mathbb{R}^3} \phi_{\varepsilon}^s |\hat{u}_\varepsilon|^2 \, dz \right) 
= \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^3} f(\varepsilon z, \hat{u}_\varepsilon) \hat{u}_\varepsilon \, dz = \int_{\mathbb{R}^3} g(\hat{u}) \hat{u} \, dz
$$

$$
= \int_{\mathbb{R}^3} |D_s \hat{u}|^2 \, dz + \int_{\mathbb{R}^3} V(x_0) |\hat{u}|^2 \, dz + \int_{\mathbb{R}^3} \phi_{\varepsilon}^s |\hat{u}|^2 \, dz,
$$

which implies that

$$
\int_{\mathbb{R}^3} |D_s \hat{u}_\varepsilon|^2 \, dz \to \int_{\mathbb{R}^3} |D_s \hat{u}|^2 \, dz,
$$

and

$$
\int_{\mathbb{R}^3} V(\varepsilon z + x_\varepsilon) |\hat{u}_\varepsilon|^2 \, dz \to \int_{\mathbb{R}^3} V(x_0) |\hat{u}|^2 \, dz.
$$
Hence, by $(V_0)$, we can deduce that
\[
\hat{u}_\varepsilon \to \hat{u} \text{ in } H^s(\mathbb{R}^3). \tag{4.22}
\]
By (4.13), (4.23), it is easy to check that $\hat{u} \neq 0$. By (4.17), we have $\mathcal{I}_{V(x_0)}(\hat{u}) \geq c_{V(x_0)}$. Hence, $\mathcal{I}_{V(x_0)}(\hat{u}) = c_{V(x_0)}$ is proved. In view of $x_0 \in M^\delta \subset \Lambda$, we have that $V(x_0) = V_0$ and $x_0 \in \mathcal{M}$. As a consequence, $\hat{u}$ is, up to a translation in the $x$-variable, an element of $\mathcal{L}_{V_0}$, namely there exists $W \in \mathcal{L}_{V_0}$ and $z_0 \in \mathbb{R}^3$ such that $\hat{u}(z) = W(z - z_0)$. Consequently, from (4.23), (4.16) and (4.22), we have that
\[
\|u_\varepsilon - \varphi_\varepsilon (\cdot - \frac{x_\varepsilon}{\varepsilon} - z_0)W(\cdot - \frac{x_\varepsilon}{\varepsilon} - z_0)\|_{H^s} \to 0 \quad \text{as } \varepsilon \to 0.
\]
Observing that $\varepsilon(\frac{x_\varepsilon}{\varepsilon} + z_0) \to x_0 \in \mathcal{M}$ as $\varepsilon \to 0$, so the proof is completed. \hfill \Box

For $a \in \mathbb{R}$ we define the sublevel set of $\mathcal{J}_\varepsilon$ as follows
\[
\mathcal{J}_\varepsilon^a = \{ u \in H_\varepsilon \mid \mathcal{J}_\varepsilon(u) \leq a \}.
\]

We observe that the result of Lemma 4.4 holds for $d_0 > 0$ sufficiently small independently of the sequences satisfying the assumptions.

**Lemma 4.5.** Let $d_0$ be the number given in Lemma 4.4. Then for any $d \in (0, d_0)$, there exist positive constants $\varepsilon_d > 0$, $\rho_d > 0$ and $\alpha_d > 0$ such that
\[
\|J'_\varepsilon^d(u)\|_{(H^s)^\prime} \geq \alpha_d > 0 \quad \text{for every } u \in J_{\varepsilon}^{c_{V_0} + \rho_d} \cap (N^d_{\varepsilon} \setminus N^d_{\varepsilon}) \text{ and } \varepsilon \in (0, \varepsilon_d).
\]

**Proof.** By contradiction we suppose that for some $d \in (0, d_0)$, there exists $\{\varepsilon_i\}$, $\{\rho_i\}$ and $u_i \in J_{\varepsilon_i}^{c_{V_0} + \rho_i} \cap (N^d_{\varepsilon_i} \setminus N^d_{\varepsilon_i})$ such that
\[
\|J'_\varepsilon^d(u_i)\|_{(H^s)^\prime} \to 0 \quad \text{as } i \to \infty.
\]
By Lemma 4.3 we can find $\{y_i\} \subset \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $W \in \mathcal{L}_{V_0}$ such that
\[
\lim_{i \to \infty} \|\varepsilon_i y_i - x_0\| = 0 \quad \lim_{i \to \infty} \|u_i - \varphi_{\varepsilon_i}(\cdot - y_i)W(\cdot - y_i)\|_{H^s} = 0.
\]
Thus, $\varepsilon_i y_i \in \mathcal{M}^d$ for sufficiently large $i$ and then by the definition of $N_{\varepsilon_i}$ and $N^d_{\varepsilon_i}$, we obtain that $\varphi_{\varepsilon_i}(\cdot - y_i)W(\cdot - y_i) \in N_{\varepsilon_i}$ and $u_i \in N^d_{\varepsilon_i}$ for sufficiently large $i$. This contradicts with $u_i \notin N^d_{\varepsilon_i}$ and completes the proof. \hfill \Box

We recall the definition (4.3) of $\gamma_\varepsilon(\tau)$. The following Lemma holds.

**Lemma 4.6.** There exists $M_0 > 0$ such that for any $\delta > 0$ small, there exists $\alpha_\delta > 0$ and $\varepsilon_\delta > 0$ such that if $J_\varepsilon(\gamma_\varepsilon(\tau)) \geq c_{V_0} - \alpha_\delta$ and $\varepsilon \in (0, \varepsilon_\delta)$, then $\gamma_\varepsilon(\tau) \in N^d_{\varepsilon M_0 \delta}$.

**Proof.** First, for any $u \in H^s(\mathbb{R}^3)$, we have that
\[
\int_{\mathbb{R}^3} |D_s(\varphi u)|^2 |u|^2 dz \leq \int_{\mathbb{R}^3} |\varphi^2| D_s u^2 dz + 2 \int_{\mathbb{R}^3} u^2 |D_s \varphi u|^2 dz
\]
\[
\leq 2 \int_{\mathbb{R}^3} |D_s u|^2 dz + \left( \int_{\mathbb{R}^3} |u|^2 dz \right)^{\frac{\delta}{2}} \left( \int_{\mathbb{R}^3} |D_s \varphi u|^2 dz \right)^{\frac{2}{\delta}}
\]
\[
\leq 2 \int_{\mathbb{R}^3} |D_s u|^2 dz + C \left( \int_{\mathbb{R}^3} |u|^2 dz \right)^{\frac{\delta}{2}} \left( \int_{\mathbb{R}^3} \frac{|D_s \varphi u|^2}{|u|^2} dz \right)^{\frac{2}{\delta}}.
\]
Thus, there exists $M_0 > 0$ such that
\[
\|\varphi u\|_{H^s} \leq M_0 \|u\|. \tag{4.23}
\]
The remain proof is similar to the proof of Lemma 4.5 in [20], we omit its proof. \hfill \Box
We are now ready to show that the penalized functional $J_{\varepsilon}$ possesses a critical point for every $\varepsilon > 0$ sufficiently small. Choose $\delta_1 > 0$ such that $M_0\delta_1 < \frac{d\varepsilon}{2}$ in Lemma 4.6 and fixing $d = \frac{d\varepsilon}{2} := d_1$ in Lemma 4.5. Similar to the proof of Lemma 4.6 in [20], we can prove the following result.

**Lemma 4.7.** There exists $\varepsilon > 0$ such that for each $\varepsilon \in (0, \varepsilon)$, there exists a sequence $\{u_{\varepsilon,n}\} \subset J_{\varepsilon}^{\varepsilon} \cap N_{\varepsilon}^{d_0}$ such that $J_{\varepsilon}^{\varepsilon}(u_{\varepsilon,n}) \to 0$ in $(H_{\varepsilon})'$ as $n \to \infty$.

**Lemma 4.8.** $J_{\varepsilon}$ possesses a nontrivial critical point $u_{\varepsilon} \in N_{\varepsilon}^{d_0} \cap J_{\varepsilon}^{D_1+\varepsilon}$ for $\varepsilon \in (0, \varepsilon]$.

**Proof.** By Lemma 4.7, there exists $\varepsilon > 0$ such that for each $\varepsilon \in (0, \varepsilon)$, there exists a sequence $\{u_{\varepsilon,n}\} \subset J_{\varepsilon}^{\varepsilon} \cap N_{\varepsilon}^{d_0}$ such that $J_{\varepsilon}^{\varepsilon}(u_{\varepsilon,n}) \to 0$ as $n \to \infty$ in $(H_{\varepsilon})'$. Since $N_{\varepsilon}^{d_0}$ is bounded, then $\{u_{\varepsilon,n}\}$ is bounded in $H_{\varepsilon}$ and up to a subsequence, we may assume that there exists $u_{\varepsilon,n} \rightharpoonup u_{\varepsilon} \in H_{\varepsilon}$ such that $u_{\varepsilon,n} \to u_{\varepsilon}$ in $H_{\varepsilon}$, $u_{\varepsilon,n} \to u_{\varepsilon}$ in $L^p_{\text{loc}}(\mathbb{R}^3)$ for $1 \leq p \leq 2^\ast$ and $u_{\varepsilon,n} \to u_{\varepsilon}$ a.e. in $\mathbb{R}^3$.

We claim that

$$\limsup_{R \to \infty} \sup_{n \geq 1} \int_{|x| \geq R} (|D_x u_{\varepsilon,n}|^2 + V(\varepsilon z)|u_{\varepsilon,n}|^2) \, dz = 0. \quad (4.24)$$

Indeed, Choosing a cutoff function $\psi_\rho \in C^\infty(\mathbb{R}^3)$ such that $\psi_\rho(z) = 1$ on $\mathbb{R}^3 \setminus B_\rho(0)$, $\psi_\rho(z) = 0$ on $B_\rho(0)$, $0 \leq \psi_\rho \leq 1$ and $|\nabla \psi_\rho| \leq \frac{C}{\rho}$. Since $\psi_\rho u_{\varepsilon,n} \in H_{\varepsilon}$, then $J_{\varepsilon}^{\varepsilon}(u_{\varepsilon,n}), \psi_\rho u_{\varepsilon,n} \to 0$ as $n \to \infty$. Thus, for sufficiently large $\rho$ such that $\Lambda \subset B_\rho(0)$, we have

$$\int_{\mathbb{R}^3} f(z, u_{\varepsilon,n}) \, dz - \int_{\mathbb{R}^3} \phi_{u_{\varepsilon,n}}^2 \psi_\rho \, dz \leq \int_{\mathbb{R}^3} |\psi_\rho|^2 \, dz \leq \frac{V_0}{k} \int_{\mathbb{R}^3} |u_{\varepsilon,n}|^2 \, dz.$$  

In view of the fact that $|D_x \psi_\rho|^2 \leq \frac{C}{\rho^2}$ for any $z \in \mathbb{R}^3$ and Hölder’s inequality, we deduce that

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(u_{\varepsilon,n}(z) - u_{\varepsilon,n}(y))(\psi_\rho(z) - \psi_\rho(y))u_{\varepsilon,n}(y)}{|z - y|^{3+2s}} \, dy \, dz \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |D_x \psi_\rho|^2 |u_{\varepsilon,n}|^2 \, dz \leq \frac{C}{\rho^s}.$$  

Therefore, from the estimates above, we obtain

$$\int_{\mathbb{R}^3 \setminus B_\rho(0)} (|D_x u_{\varepsilon,n}|^2 + V(\varepsilon z)|u_{\varepsilon,n}|^2) \, dz \leq \frac{C}{\rho^s}. $$

Thus, the claim follows. From (4.24), we see that $u_{\varepsilon,n} \to u_{\varepsilon}$ in $L^2(\mathbb{R}^3)$. By use of interpolation inequality, we conclude that $u_{\varepsilon,n} \to u_{\varepsilon}$ in $L^p(\mathbb{R}^3)$ for $2 \leq p < 2^\ast$. It follows from standard arguments that $u_{\varepsilon,n} \to u_{\varepsilon}$ in $H_{\varepsilon}$. Since $0 \not\in N_{\varepsilon}^{d_0}$, $u_{\varepsilon} \not= 0$ and $u_{\varepsilon} \in N_{\varepsilon}^{d_0} \cap J_{\varepsilon}^{D_1+\varepsilon}$. The proof is completed. \[\square\]
5. Proof of Theorem 1.1

From Lemma 4.8, we see that there exists $\varepsilon > 0$ and $d_0 > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$, $\{u_\varepsilon\} \in \mathcal{N}_{d_0}^{\varepsilon} \cap \mathcal{J}_{P, \varepsilon + \varepsilon}$ is a nontrivial solution of the problem

$$(-\Delta)^s u + V(\varepsilon z)u + \phi'_0 u + 4\left(\int_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} u^2 dz - \varepsilon\right) + \chi_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} u = f(\varepsilon z, u) \quad \text{in } \mathbb{R}^3,$$

(5.1)

where $\chi_{\mathbb{R}^3 \setminus \Lambda_\varepsilon}$ is the characteristic function of the set $\mathbb{R}^3 \setminus \Lambda_\varepsilon$. Taking $-u_\varepsilon^{-}$ as a test function in (5.1), we can conclude that $u_\varepsilon \geq 0$. Since $u_\varepsilon \in \mathcal{N}_{d_0}^{\varepsilon} \cap \mathcal{J}_{P, \varepsilon + \varepsilon}$, by Lemma 4.4, we get that $\{u_\varepsilon\}$ is uniformly bounded in $\varepsilon \in (0, \bar{\varepsilon}]$ and $\{J_\varepsilon(u_\varepsilon)\}$ is uniformly bounded from above for all $\varepsilon > 0$ small. Thus, it is easy to check that $\{Q_\varepsilon(u_\varepsilon)\}$ uniformly bounded for all $\varepsilon > 0$ small.

**Step 1.** We claim that there exists $\tilde{\varepsilon} > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}]$

$$\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} \leq C. \quad (5.2)$$

Otherwise, there exists $\varepsilon_i \to 0$ such that $\|u_{\varepsilon_i}\|_{L^\infty(\mathbb{R}^3)} \to \infty$ as $i \to \infty$. By Lemma 4.4, there exist $x_0 \in \mathcal{M}$ and $W_0 \in \mathcal{L}_{V_0}$ satisfying

$$\varepsilon_i x_{\varepsilon_i} \to x_0 \quad \text{and} \quad \|u_{\varepsilon_i} - \varphi_{\varepsilon_i}(\cdot - x_{\varepsilon_i})W_0(\cdot - x_{\varepsilon_i})\|_{H_{\varepsilon_i}} \to 0$$

as $i \to \infty$. Thus

$$\lim_{i \to \infty} \|u_{\varepsilon_i}(\cdot + x_{\varepsilon_i}) - W_0\| \leq \lim_{i \to \infty} \|u_{\varepsilon_i} - \varphi_{\varepsilon_i}(\cdot - x_{\varepsilon_i})W_0(\cdot - x_{\varepsilon_i})\|_{H_{\varepsilon_i}} + \lim_{i \to \infty} \|1 - \varphi_{\varepsilon_i}\|W_0\| = 0.$$

By Lemma 2.2, we conclude that $\|u_{\varepsilon_i}(\cdot + x_{\varepsilon_i})\|_{L^\infty(\mathbb{R}^3)} \leq C$ which leads to a contradiction.

**Step 2.** For any sequence $\{\varepsilon_i\}$ with $\varepsilon_i \to 0$, by Lemma 4.4 there exist, up to a subsequence, $\{x_{\varepsilon_i}\} \subset \mathbb{R}^3$, $x_0 \in \mathcal{M}$, $W_0 \in \mathcal{L}_{V_0}$ such that

$$\varepsilon_i x_{\varepsilon_i} \to x_0 \quad \text{and} \quad \|u_{\varepsilon_i} - \varphi_{\varepsilon_i}(\cdot - x_{\varepsilon_i})W_0(\cdot - x_{\varepsilon_i})\|_{H_{\varepsilon_i}} \to 0$$

which implies that

$$w_{\varepsilon_i}(z) := u_{\varepsilon_i}(z + x_{\varepsilon_i}) \to W_0 \quad \text{in } H^s(\mathbb{R}^3). \quad (5.3)$$

By (5.2), we see that $w_{\varepsilon_i} \to W_0$ in $L^p(\mathbb{R}^3)$ for $1 \leq p < \infty$.

Now, setting

$$h_{\varepsilon_i}(z) = w_{\varepsilon_i}(z) + f(\varepsilon_i z + \varepsilon_i x_{\varepsilon_i}, w_{\varepsilon_i}(z)) - \left[V(\varepsilon_i z + \varepsilon_i x_{\varepsilon_i})w_{\varepsilon_i}(z) + \phi'_{w_{\varepsilon_i}}(z)\right]w_{\varepsilon_i}(z) + 4\left(\int_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} w_{\varepsilon_i}^2 dz - \varepsilon_i\right) + \chi_{\mathbb{R}^3 \setminus \Lambda_\varepsilon} - x_{\varepsilon_i}(z)w_{\varepsilon_i}(z)\left].

Clearly, in view of the uniformly boundedness of $Q_{\varepsilon_i}(w_{\varepsilon_i})$ and (5.2), thus there exists $C > 0$ such that $|h_{\varepsilon_i}(z)| \leq C$ for any $z \in \mathbb{R}^3$ and $i \in \mathbb{N}$. By (5.2) and (5.3), we have that

$$\int_{\mathbb{R}^3} \chi_{\mathbb{R}^3 \setminus A_{\varepsilon_i} - x_{\varepsilon_i}}(z)w_{\varepsilon_i}(z) dz \leq \int_{\mathbb{R}^3} |w_{\varepsilon_i} - W_0|dz + \int_{\mathbb{R}^3} \chi_{\mathbb{R}^3 \setminus A_{\varepsilon_i} - x_{\varepsilon_i}}(z)W_0(z) dz$$

$$= \int_{\mathbb{R}^3 \setminus A_{\varepsilon_i} - x_{\varepsilon_i}} W_0(z) dz + o(1)$$

$$\leq \int_{\mathbb{R}^3 \setminus B_{\varepsilon_i/3}(0)} W_0(z) dz + o(1)$$

$$\to 0 \quad \text{as } i \to \infty.$$
Therefore, \( h_{\varepsilon_i} \rightarrow h \) in \( L^q(\mathbb{R}^3) \) for \( 1 \leq q < \infty \), where \( h(z) = W_0(z) + g(W_0) - V(x_0)W_0 - \phi'_W W_0 \). We rewrite the equation (5.1) as

\[
(-\Delta)^s w_{\varepsilon_i} + w_{\varepsilon_i} = h_{\varepsilon_i}, \quad z \in \mathbb{R}^3.
\]

According to the arguments in [10], we see that

\[
w_{\varepsilon_i}(z) = \int_{\mathbb{R}^3} K(z - y) h_{\varepsilon_i}(y) \, dy, \quad z \in \mathbb{R}^3,
\]

where \( K \) is a Bessel potential, which possesses the following properties:

- \((K_1)\) \( K \) is positive, radially symmetric and smooth in \( \mathbb{R}^3 \setminus \{0\} \);
- \((K_2)\) there exists a constant \( C > 0 \) such that \( K(x) \leq \frac{C}{|x|^{3+2s}} \) for all \( x \in \mathbb{R}^3 \setminus \{0\} \);
- \((K_3)\) \( K \in L^r(\mathbb{R}^3) \) for \( r \in [1, \frac{3}{3-2s}) \).

We define two sets \( A_\delta = \{ y \in \mathbb{R}^3 \mid |z - y| \geq \frac{1}{\delta} \} \) and \( B_\delta = \{ y \in \mathbb{R}^3 \mid |z - y| < \frac{1}{\delta} \} \). Hence,

\[
0 \leq w_{\varepsilon_i}(z) \leq \int_{\mathbb{R}^3} K(z - y) |h_{\varepsilon_i}(y)| \, dy = \int_{A_\delta} K(z - y) |h_{\varepsilon_i}(y)| \, dy + \int_{B_\delta} K(z - y) |h_{\varepsilon_i}(y)| \, dy.
\]

From the definition of \( A_\delta \) and \((K_2)\), we have that for all \( n \in \mathbb{N} \),

\[
\int_{A_\delta} K(z - y) |h_{\varepsilon_i}(y)| \, dy \leq C \delta^n |h_{\varepsilon_i}|_{\infty} \int_{A_\delta} \frac{1}{|z - y|^{3+s}} \, dy \leq C \delta^n \int_{A_\delta} \frac{1}{|z - y|^{3+s}} \, dy := C \delta^{2s}.
\]

On the other hand, by Hölder’s inequality and \((K_3)\), we deduce that

\[
\int_{B_\delta} K(z - y) |h_{\varepsilon_i}(y)| \, dy \leq \int_{B_\delta} K(z - y) |h_{\varepsilon_i} - h| \, dy + \int_{B_\delta} K(z - y) |h| \, dy
\]

\[
\leq \left( \int_{B_\delta} K \frac{6}{3+2s} \, dy \right)^{\frac{3+2s}{6}} \left( \int_{B_\delta} |h_{\varepsilon_i} - h| \frac{6}{3+2s} \, dy \right)^{\frac{3-2s}{6}} + \left( \int_{B_\delta} K \frac{6}{3+2s} \, dy \right)^{\frac{3+2s}{6}} \left( \int_{B_\delta} |h| \frac{6}{3+2s} \, dy \right)^{\frac{3-2s}{6}}
\]

\[
\leq \left( \int_{\mathbb{R}^3} K \frac{6}{3+2s} \, dy \right)^{\frac{3+2s}{6}} \left( \int_{B_\delta} |h_{\varepsilon_i} - h| \frac{6}{3+2s} \, dy \right)^{\frac{3-2s}{6}} + \left( \int_{\mathbb{R}^3} K \frac{6}{3+2s} \, dy \right)^{\frac{3+2s}{6}} \left( \int_{B_\delta} |h| \frac{6}{3+2s} \, dy \right)^{\frac{3-2s}{6}},
\]

where we have used the fact that \( \frac{6}{3+2s} < \frac{3}{3-2s} \).

Since \( \int_{B_{R_0}} |h| \frac{6}{3+2s} \, dy \xrightarrow{R_0 \to 0} 0 \) as \( |z| \to +\infty \), thus, we deduce that there exist \( i_0 \in \mathbb{N} \) and \( R_0 > 0 \) independence of \( \delta > 0 \) such that

\[
\int_{B_{R_0}} K(z - y) |h_{\varepsilon_i}(y)| \, dy \leq \delta, \quad \forall i \geq i_0 \quad \text{and} \quad |z| \geq R_0.
\]

Hence,

\[
\int_{\mathbb{R}^3} K(z - y) |h_{\varepsilon_i}(y)| \, dy \leq C \delta^{2s} + \delta, \quad \forall i \geq i_0 \quad \text{and} \quad |z| \geq R_0.
\]

For each \( i \in \{1, 2, \cdots, i_0-1\} \), there exists \( R_i > 0 \) such that \( \left( \int_{B_{R_i}} |h_{\varepsilon_i}| \frac{6}{3+2s} \, dy \right)^{\frac{3-2s}{6}} < \delta \) as \( |z| \geq R_i \). Thus, for \( |z| \geq R_i \), we have that

\[
\int_{\mathbb{R}^3} K(z - y) |h_{\varepsilon_i}(y)| \, dy \leq C \delta^{2s} + \int_{B_{R_i}} K(z - y) |h_{\varepsilon_i}(y)| \, dy
\]

\[
\leq C \delta^{2s} + |K| \frac{6}{3+2s} \int_{B_{R_i}} \frac{1}{|z|^{3+2s}} \, dy \leq C(\delta^{2s} + \delta)
\]
for each $i \in \{1, 2, \cdots, i_0 - 1\}$. Therefore, taking $R = \max\{R_0, R_1, \cdots, R_{i_0-1}\}$, we infer that for any $i \in \mathbb{N}$, there holds
\[
0 \leq w_{\varepsilon_i}(z) \leq \int_{\mathbb{R}^3} K(z - y)|h_{\varepsilon_i}(y)| \, dy \leq C\delta^{2s} + \delta, \quad \text{for all } |z| \geq R
\]
which implies that $\lim_{|z| \to \infty} w_{\varepsilon_i}(z) = 0$ uniformly in $i \in \mathbb{N}$.

Similar arguments to the proof of (ii) of Proposition 3.8 we see that for any $i \in \mathbb{N}$ but fixed, there exists $C > 0$ independent of $\varepsilon_i > 0$ such that
\[
0 \leq w_{\varepsilon_i}(z) \leq \frac{C}{1 + |z|^{\gamma+2s}} \quad \text{for any } z \in \mathbb{R}^3.
\]
Therefore,
\[
\int_{\mathbb{R}^3 \setminus \Lambda_{\varepsilon_i}} w_{\varepsilon_i}^2(\varepsilon_i) \, dz \leq C \int_{\mathbb{R}^3 \setminus B_{R_{\varepsilon_i}}(0)} \frac{1}{(1 + |z|^{\gamma+2s})^2} \, dz \leq C\varepsilon_i^{4s+3}
\]
which implies that $Q_{\varepsilon_i}(z) = 0$ for $\varepsilon_i > 0$ small. Hence $w_{\varepsilon_i}$ is a solution of the following problem
\[
(-\Delta)^s u + V(\varepsilon z)u + \phi v u = f(\varepsilon z, u) \quad \text{in } \mathbb{R}^3.
\]

**Step 3.** For the $w_{\varepsilon}$ above in Step 2, $w_{\varepsilon}(z) = w_{\varepsilon}(z - x_\varepsilon) < a$, for all $z \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon}$. Noting that $\varepsilon x_\varepsilon \to x_0$ and $x_0 \in \Lambda$. Thus, there exists $R' > 0$ such that $B_{R'}(\varepsilon x_\varepsilon) \subset \Lambda$ for $\varepsilon > 0$ small. Hence, $B_{R'/\varepsilon}(x_\varepsilon) \subset \Lambda_{\varepsilon}$ for $\varepsilon > 0$ small. Moreover, by Step 2, there is $R_1 > 0$ such that $w_{\varepsilon}(z) < a$ for $|z| \geq R_1$. Thus,
\[
0 \leq w_{\varepsilon}(z) = w_{\varepsilon}(z - x_\varepsilon) < a, \quad \text{for all } z \in \mathbb{R}^3 \setminus B_{R_1}(x_\varepsilon) \quad \text{and } \varepsilon > 0 \text{ small}.
\]
Since
\[
\mathbb{R}^3 \setminus \Lambda_{\varepsilon} \subset \mathbb{R}^3 \setminus B_{R'/\varepsilon}(x_\varepsilon) \subset \mathbb{R}^3 \setminus B_{R_1}(x_\varepsilon) \quad \text{and } \varepsilon > 0 \text{ small}
\]
and then
\[
w_{\varepsilon}(z) = w_{\varepsilon}(z - x_\varepsilon) < a \quad \forall z \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon} \quad \text{and } \varepsilon > 0 \text{ small}.
\]

**Step 4.** By Lemma 4.8, we see that problem (5.1) has a nonnegative solution $v_{\varepsilon}$ for all $\varepsilon \in (0, \overline{\varepsilon}]$. From Step 3, there exists $\varepsilon_0 > 0$ such that
\[
v_{\varepsilon}(z) < a \quad \forall z \in \mathbb{R}^3 \setminus \Lambda_{\varepsilon} \quad \text{and } \varepsilon \in (0, \varepsilon_0)
\]
which implies that $f(\varepsilon z, v_{\varepsilon}) = g(v_{\varepsilon})$. Thus, $v_{\varepsilon}$ is a solution of problem
\[
(-\Delta)^s v + V(\varepsilon) v + \phi v v = g(v) \quad z \in \mathbb{R}^3.
\]
for all $\varepsilon \in (0, \varepsilon_0)$. Let $u_{\varepsilon}(x) = v_{\varepsilon}(x/\varepsilon)$ for every $\varepsilon \in (0, \varepsilon_0)$, it follows that $u_{\varepsilon}$ must be a solution to original problem (1.1) for $\varepsilon \in (0, \varepsilon_0)$.

If $y_{\varepsilon}$ denotes a global maximum point of $v_{\varepsilon}$, then
\[
v_{\varepsilon}(y_{\varepsilon}) \geq a \quad \forall \varepsilon \in (0, \varepsilon_0).
\]
Suppose that $v_{\varepsilon}(y_{\varepsilon}) < a$, taking $v_{\varepsilon}$ as a text function for (5.4), we get
\[
V_0 \int_{\mathbb{R}^3} v_{\varepsilon}^2 \, dz \leq \int_{\mathbb{R}^3} V(\varepsilon z) v_{\varepsilon}^2 \, dz \leq \int_{\mathbb{R}^3} g(v_{\varepsilon}) v_{\varepsilon} \, dz \leq \int_{\mathbb{R}^3} \frac{v_{\varepsilon}^2 g(v_{\varepsilon})}{a} \, dz = \frac{V_0}{k} \int_{\mathbb{R}^3} v_{\varepsilon}^2 \, dz
\]
which we have used the hypothesis \((q_1)\) and \(q - 2 > 0\). Hence we get a contradiction owing to the choosing \(k > 2\). In view of Step 3, we see that \(\{y_\varepsilon\}\) is bounded for \(\varepsilon \in (0, \varepsilon_0)\).

In what follows, setting \(z_\varepsilon = \varepsilon y_\varepsilon + \varepsilon x_\varepsilon\), where \(\{x_\varepsilon\}\) is given in Step 2. Since \(u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon} - x_\varepsilon)\), then \(z_\varepsilon\) is a global maximum point of \(u_\varepsilon\) and \(u_\varepsilon(z_\varepsilon) \geq a\) for all \(\varepsilon \in (0, \varepsilon_0)\).

Now, we claim that \(\lim_{\varepsilon \to 0^+} V(z_\varepsilon) = V_0\). Indeed, if the above limit does not hold, there is \(\varepsilon_\eta \to 0^+\) and \(\gamma_0 > 0\) such that
\[
V(z_{\varepsilon_n}) \geq V_0 + \gamma_0 \quad \forall n \in \mathbb{N}.
\] (5.6)

By Step 2, we know that \(\lim_{|z| \to \infty} v_\varepsilon_n(z) = 0\) uniformly in \(n \in \mathbb{N}\). From \([5, 6]\), thus \(\{z_{\varepsilon_n}\}\) is a bounded sequence. Using Lemma \([4, 4]\) we know that there is \(x_0 \in \mathcal{M}\) such that \(V(x_0) = V_0\) and \(\varepsilon_n x_{\varepsilon_n} \to x_0\). Hence, \(z_{\varepsilon_n} = \varepsilon_n x_{\varepsilon_n} + \varepsilon_n y_{\varepsilon_n} \to x_0\) which implies that \(V(z_{\varepsilon_n}) \to V(x_0) = V_0\) contradicting with \([5.6]\).

To complete the proof, we only need to prove the decay properties of \(u_\varepsilon\). Similar argument to the proof of Proposition \([6, 8]\) we can obtain that
\[
0 < v_\varepsilon(z) \leq \frac{C}{1 + |z|^{3+2s}}.
\]

Thus, by the boundedness of \(\{y_\varepsilon\}\), i.e., there exists \(C_0 > 0\) such that \(|y_\varepsilon| \leq C_0\), we have
\[
u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon} - x_\varepsilon) \leq \frac{C}{1 + |\frac{x - x_\varepsilon}{\varepsilon}|^{3+2s}} \leq \frac{C\varepsilon^{3+2s}}{1 - \varepsilon^{3+2s}} \frac{C_0^{3+2s}}{1 - C_0^{3+2s}} + |x - z_\varepsilon|^{3+2s}
\]
\[
:= \frac{C\varepsilon^{3+2s}}{\varepsilon^{3+2s}C_1 + |x - z_\varepsilon|^{3+2s}}.
\]

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References
[1] C. O. Alves and O. H. Miyagaki, Existence and concentration of solution for a class of fractional elliptic equation in \(\mathbb{R}^N\) via penalization method, Calc. Var. Partial Differential Equations, 55 (2016) 1–19.
[2] V. Ambrosi, Multiplicity of positive solutions for a class of fractional Schrödinger equations via penalization method, Annali di Matematica Pura ed Applicata, DOI 10.1007/s10231-017-0652-5.
[3] A. Ambrosetti and D. Ruiz, Multiple bound states for the Schrödinger-Poisson problem, Comm. Contemp. Math. 10 (2008) 391–404.
[4] C. Brandle, E. Colorado, A. de Pablo and U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian, Proc. Roy. Soc. Edinburgh Sect. A 143 (2013) 39–71.
[5] V. Benci and D. Fortunato, An eigenvalue problem for the Schrödinger-Maxwell equations, Top. Methods. Nonlinear Anal. 11 (1998) 283–293.
[6] J. Byeon and L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, Arch. Ration. Mech. Anal. 185 (2007) 185–200.
[7] C. Bucur and E. Valdinoci, Nonlocal diffusion and applications, Lecture Notes of the Unione Matematica Italiana, vol.20, Springer, International Publishing, ISBN978-3-319-28738-6, 2016, xii, 155pp.
[8] J. Byeon and Z. Q. Wang, Standing waves with a critical frequency for nonlinear Schrödinger equations II, Calc. Var. Partial Differential Equations, 18 (2003) 207–219.
[9] S. Y. A. Chang and M. del Moral Gonzalez, Fractional Laplacian in conformal geometry, Adv. Math. 226 (2011) 1410–1432.
[10] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.
[11] R. Cont and P. Tankov, Financial modeling with jump processes, Chapman Hall/CRC Financial Mathematics Series, Boca Raton, 2004.

[12] X. Chang and Z. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, Nonlinearity 26 (2013) 479–494.

[13] J. Dávila, M. Del Pino and J. C. Wei, Concentrating standing waves for fractional nonlinear Schrödinger equation, J. Differential Equations, 256 (2014) 858-892.

[14] J. Davila, M. del Pino, S. Dipierro and E. Valdinoci, Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum, Anal. PDE, 8 (2015) 1165-1235.

[15] M. del Pino and P. L. Felmer, Local mountain pass for semilinear elliptic problems in unbounded domains, Calc. Var. Partial Differential Equations 4 (1996) 121–137.

[16] P. Felmer, A. Quaas and J. Tan, Positive solutions of nonlinear Schrödinger equation with the fractional Laplacian, Proc. Royal Soc. Edinburgh A 142 (2012) 1237-1262.

[17] R. L. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, Comm. Pure Appl. Math. 69 (2016) 671–1726.

[18] X. M. He, Multiplicity and concentration of positive solutions for the Schrödinger-Poisson equations, Z. Angew. Math. Phys. 5 (2011) 869–889.

[19] Y. He and G. B. Li, Standing waves for a class of Schrödinger-Poisson equations in $\mathbb{R}^3$ involving critical Sobolev exponents, Ann. Acad. Sci. Fenn. Math. 40 (2015) 729–766.

[20] X. M. He and W. M. Zou, Existence and concentration result for the fractional Schrödinger equations with critical nonlinearities, Calc. Var. Partial Differential Equations (2016) 55:91.

[21] I. Ianni and G. Vaira, Solutions of the Schrödinger-Poisson problem concentrating on spheres, Part I: Necessary conditions, Math. Models Meth. Appl. Sci. 19 (2009) 707–720.

[22] I. Ianni and G. Vaira, On concentration of positive bound states for the Schrödinger-Poisson problem with potentials, Adv. Nonlinear Stud. 8 (2008) 573–595.

[23] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Physics Letters A 268 (2000) 298–305.

[24] Z. S. Liu and J. J. Zhang, Multiplicity and concentration of positive solutions for the fractional Schrödinger-Poisson systems with critical growth, ESAIM : Control, Optim. Calc. Var., DOI: 10.1051/cocv/2016063, (2016).

[25] R. Metzler and J. Klafter, The random walks guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.

[26] P. Markowich, C. Ringhofer and C. Schmeiser, Semiconductor Equations, Springer-Verlag, Vienna, 1990.

[27] E. G. Murcia and G. Siciliano, Positive semiclassical states for a fractional Schrödinger-Poisson system, arXiv:1601.00455v1.

[28] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional sobolev spaces, Bulletin des Sciences Mathematiques 136 (2012) 521–573.

[29] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Func. Anal. 237 (2006) 655–674.

[30] D. Ruiz, Semicalssical states for coupled Schrödinger-Maxwell equations: Concentration around a sphere, Math. Models Methods Appl. Sci. 15 (2005) 141–164.

[31] D. Ruiz and G. Vaira, Cluster solutions for the Schrödinger-Poisson-Slater problem around a local minimum of potential, Rev. Mat. Iberoamericana 27 (2011) 253–271.

[32] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, Comm. Pure Appl. Math. 60 (2007) 67–112.

[33] S. Secchi, Ground state solutions for nonlinear fractional Schrödinger equations in $\mathbb{R}^N$, J. Math. Phys. 54 (2013) 031501.

[34] X. D. Shang and J. H. Zhang, Ground states for fractional Schrödinger equations with critical growth, Nonlinearity 27 (2014) 187–207.

[35] K. M. Teng, Multiple solutions for a class of fractional Schrödinger equation in $\mathbb{R}^N$, Nonlinear Anal. Real World Appl. 21 (2015) 76–86.

[36] K. M. Teng and X. M. He, Ground state solution for fractional Schrödinger equations with critical Sobolev exponent, Comm. Pure Appl. Anal. 15 (2016) 991–1008.
[39] K. M. Teng, Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent, J. Differential Equations 261 (2016) 3061–3106.

[40] K. M. Teng, Ground state solutions for the nonlinear fractional Schrödinger-Poisson system, arXiv:1605.06732.

[41] K. M. Teng and R. P. Agarwal, Existence and concentration of positive ground state solutions for nonlinear fractional Schrödinger-Poisson system with critical growth, arXiv:1702.05387.

[42] K. M. Teng, Concentrating bounded states for fractional Schrödinger-Poisson system involving critical Sobolev exponent, submitted.

[43] M. Willem, Minimax theorems, Progress in Nonlinear Differential Equations and their Applications 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

[44] J. Wang, L. X. Tian, J. X. Xu and F. B. Zhang, Existence and concentration of positive solutions for semilinear Schrödinger-Poisson systems in $\mathbb{R}^3$, Calc. Var. Partial Differential Equations, 48 (2013) 243–273.

[45] J. Zhang, J. M. DO Ó and M. Squassina, Fractional Schrödinger-Poisson system with a general subcritical or critical nonlinearity, Adv. Nonlinear Stud. 16 (2016) 15–30.

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