A GENERALISED $\tau$-INVARIANT FOR THE UNEQUAL PARAMETER CASE

MEINOLF GECK

ABSTRACT. In 1979, Vogan proposed a generalised $\tau$-invariant for characterising primitive ideals in enveloping algebras. Via a known dictionary this translates to an invariant of left cells of finite Weyl groups. Although it is not a complete invariant, it is extremely useful in describing left cells. Here, we propose a general framework for defining such invariants which also applies to Hecke algebras with unequal parameters.

1. Introduction

Let $W$ be a finite Weyl group. Using the corresponding generic Iwahori–Hecke algebra and the "new" basis of this algebra introduced by Kazhdan and Lusztig [16], we obtain partitions of $W$ into left, right and two-sided cells. Analogous notions originally arose in the theory of primitive ideals in enveloping algebras; see Joseph [15]. This is one of the sources for the interest in knowing the cell partitions of $W$. Vogan [23, 24] introduced invariants of left cells which are computable in terms of certain combinatorially defined operators $T_{\alpha\beta}$, $S_{\alpha\beta}$ where $\alpha, \beta$ are adjacent simple roots of $W$. In the case where $W$ is the symmetric group $S_n$, these invariants completely characterise the left cells; see [16 §5], [23 §6]. Although Vogan's invariants are not complete invariants in general, they have turned out to be extremely useful in describing left cells; see, most notably, the work of Garfinkle [6], [7], [8].

Now, the Kazhdan–Lusztig cell partitions are not only defined and interesting for finite Weyl groups, but also for affine Weyl groups and Coxeter groups in general; see, e.g., Lusztig [18, 19]. Furthermore, the original theory was extended by Lusztig [17] to allow the possibility of attaching weights to the simple reflections. The original setting then corresponds to the case where all weights are equal to 1; we will refer to this case as the "equal parameter case". Using ideas from Lusztig [18 §10], our aim here is to propose analogues of Vogan's invariants which work in general, i.e., for arbitrary Coxeter groups and arbitrary (positive) weights.

In Section 2 we briefly recall the basic set-up concerning Iwahori–Hecke algebras and cells in the sense of Kazhdan and Lusztig. As Vogan’s original definition of the generalised $\tau$-invariant relies on the theory of primitive ideals, it only applies to finite Weyl groups. In Section 3 we show how to translate this into the setting of Kazhdan and Lusztig. (A similar translation has also been done by Shi [21 §4.2], who uses a definition slightly

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different from Vogan [23]; our argument seems to be more direct.) Thus, the generalised \( \tau \)-invariant is available for arbitrary Coxeter groups in the equal parameter case. In Section 4, we propose an abstract setting for defining such invariants; this essentially relies on the concept of "induction of cells" [9, 10] and Lusztig’s method of "strings" [18, §10]. In Theorem 4.6 we show that this gives indeed rise to new invariants of left cells. As a by-product of our approach, we obtain new (and less computational) proofs of the results concerning the "star" operations in [16, §4] and the analogous results for "strings" in [18, §10]. We conclude by discussing examples and stating open problems.

2. Weight functions and cells

Let \( W \) be a Coxeter group with generating set \( S \) and corresponding length function \( \ell: W \to \mathbb{Z}_{\geq 0} \). Let \( \pi = \{ p_s \mid s \in S \} \subseteq \mathbb{Z} \) be a set of "weights" where \( p_s = p_t \) whenever \( s, t \in S \) are conjugate in \( W \). This gives rise to a weight function \( p: W \to \mathbb{Z} \) in the sense of Lusztig [19]; for \( w \in W \), we have \( p_w = p_{s_1} + \ldots + p_{s_k} \) where \( w = s_1 \cdots s_k \) \( (s_i \in S) \) is a reduced expression for \( w \). The original setup in [16] corresponds to the case where \( p_s = 1 \) for all \( s \in S \); this will be called the "equal parameter case". We shall assume throughout that \( p_s > 0 \) for all \( s \in S \). (There are standard techniques for reducing the general case to this case; see Bonnafé [3, §2].)

Let \( H = H_A(W, S, \{ p_s \}) \) be the corresponding generic Iwahori–Hecke algebra, where \( A = \mathbb{Z}[v, v^{-1}] \) is the ring of Laurent polynomials in an indeterminate \( v \). This algebra is free over \( A \) with basis \( \{ T_w \mid w \in W \} \), and the multiplication is given by the rule

\[
T_s T_w = \begin{cases} 
T_{sw} & \text{if } sw > w, \\
T_{sw} + (v^{p_s} - v^{-p_s})T_w & \text{if } sw < w,
\end{cases}
\]

where \( s \in S \) and \( w \in W \); here, \( \leq \) denotes the Bruhat–Chevalley order on \( W \).

Let \( \{ C'_w \mid w \in W \} \) be the "new" basis of \( H \) introduced in [16, (1.1.c)], [17, §2]. (These basis elements are denoted \( c_w \) in [19].) For any \( x, y \in W \), we write

\[
C'_x C'_y = \sum_{z \in W} h_{x,y,z} C'_z \quad \text{where } h_{x,y,z} \in A \text{ for all } x, y, z \in W.
\]

We have the following more explicit formula for \( s \in S \), \( y \in W \) (see [17, §6], [19, Chap. 6]):

\[
C'_s C'_y = \begin{cases} 
(v^{p_s} + v^{-p_s})C'_y & \text{if } sy < y, \\
C'_{sy} + \sum_{z \in W : sz < y} M^s_{z,y} C'_z & \text{if } sy > y,
\end{cases}
\]

where \( C'_s = T_s + v^{-p_s}T_1 \) and \( M^s_{z,y} \in A \) is determined as in [17, §3].

As in [19, §8], we write \( x \leftarrow_L y \) if there exists some \( s \in S \) such that \( h_{s,y,x} \neq 0 \), that is, \( C'_x \) occurs in \( C'_s C'_y \) (when expressed in the \( C' \)-basis). The Kazhdan–Lusztig left pre-order \( \leq_L \) is the transitive closure of \( \leftarrow_L \). The equivalence relation associated with \( \leq_L \) will be denoted by \( \sim_L \) and the corresponding equivalence classes are called the left cells of \( W \).
Similarly, we can define a pre-order $\leq_R$ by considering multiplication by $C'_w$ on the right in the defining relation. The equivalence relation associated with $\leq_R$ will be denoted by $\sim_R$ and the corresponding equivalence classes are called the right cells of $W$. We have

$$x \leq_R y \iff x^{-1} \leq_L y^{-1};$$

see [19, 5.6, 8.1]. Finally, we define a pre-order $\leq_{LR}$ by the condition that $x \leq_{LR} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that, for each $i \in \{1, \ldots, k\}$, we have $x_{i-1} \leq_L x_i$ or $x_{i-1} \leq_R x_i$. The equivalence relation associated with $\leq_{LR}$ will be denoted by $\sim_{LR}$ and the corresponding equivalence classes are called the two-sided cells of $W$.

**Definition 2.1.** A (non-empty) subset $\Gamma$ of $W$ is called ”closed with respect to $\leq_L$” if, for any $x, y \in \Gamma$, we have $\{z \in W \mid x \leq_L z \leq_L y\} \subseteq \Gamma$. Note that any such subset is a union of left cells. A left cell itself is clearly closed with respect to $\leq_L$.

Given a subset $\Gamma \subseteq W$ which is closed with respect to $\leq_L$, we obtain an $\mathcal{H}$-module $[\Gamma]_A := \mathcal{I}_\Gamma/\hat{\mathcal{I}}_\Gamma$, where

$$\mathcal{I}_\Gamma := \langle C'_w \mid w \leq_L z \text{ for some } z \in \Gamma \rangle_A,$$

$$\hat{\mathcal{I}}_\Gamma := \langle C'_w \mid w \not\in \Gamma, w \leq_L z \text{ for some } z \in \Gamma \rangle_A.$$

Note that, by the definition of the pre-order relation $\leq_L$ (and the condition that $\Gamma$ is closed with respect to $\leq_L$), these are left ideals in $\mathcal{H}$. Now denote by $e_x$ ($x \in \Gamma$) the residue class of $C'_x$ in $[\Gamma]_A$. Then the elements $\{e_x \mid x \in \Gamma\}$ form an $A$-basis of $[\Gamma]_A$ and the action of $C'_w$ ($w \in W$) is given by the formula

$$C'_w e_x = \sum_{y \in \Gamma} h_{w,x,y} e_y.$$

A key tool in this work will be the process of ”induction of cells”. Let $I \subseteq S$ and consider the parabolic subgroup $W_I \subseteq W$ generated by $I$. Then

$$X_I := \{w \in W \mid ws > w \text{ for all } s \in I\}$$

is the set of distinguished left coset representatives of $W_I$ in $W$. The map $X_I \times W_I \to W$, $(x, u) \mapsto xu$, is a bijection and we have $\ell(xu) = \ell(x) + \ell(u)$ for all $x \in X_I$ and $u \in W_I$; see [14, §2.1]. Thus, given $w \in W$, we can write uniquely $w = xu$ where $x \in X_I$ and $u \in W_I$. In this case, we denote $\text{pr}_I(w) := u$. Let $\sim_{L,I}$ be the equivalence relation on $W_I$ for which the equivalence classes are the left cells of $W_I$.

**Theorem 2.2** ([14]). Let $I \subseteq S$. If $w, w' \in W$ are such that $w \sim_{L,I} w'$, then $\text{pr}_I(w) \sim_{L,I} \text{pr}_I(w')$. In particular, if $\Gamma$ is a left cell of $W_I$, then $X_I \Gamma$ is a union of left cells of $W$.

**Example 2.3.** Let $\Gamma'$ be a left cell of $W_I$. Then the subset $\Gamma := X_I \Gamma'$ of $W$ is closed with respect to $\leq_L$. (This immediately follows from Theorem 2.2.) Let $\mathcal{H}_I \subseteq \mathcal{H}$ be the parabolic subalgebra spanned by all $T_w$ where $w \in W_I$. Then we obtain the $\mathcal{H}_I$-module
Proposition 2.4 Let $H$ be a symmetric group, we also have $H = W$ for any $x, y \in W$. By 
\[ \langle x, y \rangle = \sum_{x \in X, y \in \Gamma} p_{xy}^* (T_x \otimes e_u), \]
where $p_{xy}^* \in A$ are the relative Kazhdan–Lusztig polynomials of [9 Prop. 3.3] and, for \nany $H$-module $V$, we denote by $\text{Ind}_H^V(V) := H \otimes_{H_I} V$ the \textit{induced module}, \nb \basis{basis}{basis of $\{T_x \otimes e_u | x \in X, y \in \Gamma\}$} (see, for example, [14 §9.1]).

A first invariant of left cells is given as follows. For any $w \in W$, we denote by $\mathcal{R}(w) := \{s \in S | ws < w\}$ the \textit{right descent set} of $w$.

Proposition 2.4 (See [18 2.4] for the equal parameter case and [19 8.6] for the general \nb \case{case}). Let $x, y \in W$. If $x \sim_L y$, then $\mathcal{R}(x) = \mathcal{R}(y)$. Thus, for any $I \subseteq S$, the set $
\{w \in W | \mathcal{R}(w) = I\}$ is an union of left cells of $W$.

We show how this can be deduced from Theorem 2.2. Let $x, y \in W$ be such that $x \sim_L y$. Let $s \in \mathcal{R}(x)$ and set $I = \{s\}$. Then $pr_I(x) = s$ and so $s = pr_I(x) \sim_{L, I} pr_I(y) \in W_I = \{1, s\}$. Since $p_s > 0$, the definitions immediately show that $\{1\}, \{s\}$ are the left cells of $W_I$. Hence, we must have $pr_I(y) = s$ and so $s \in \mathcal{R}(y)$. Thus, we have $\mathcal{R}(x) \subseteq \mathcal{R}(y)$. By \nb \symmetry{symmetry}, we also have $\mathcal{R}(y) \subseteq \mathcal{R}(x)$ and so $\mathcal{R}(x) = \mathcal{R}(y)$, as required.

Definition 2.5. For any $w \in W$, the \textit{enhanced right descent set} is defined as $
\mathcal{R}^e(w) := \mathcal{R}(w) \cup \{sts | s, t \in S, st \neq ts, p_s < p_t \text{ and } wsts < w\}$

This provides, at least, a complete invariant for the left cells of dihedral groups, as the following \nb \example{example} shows.

Example 2.6. Let $S = \{s_1, s_2\}$ and assume that $st$ has finite order $m \geq 3$. For $k \geq 0 \nb \text{let } 1_k = s_1 s_2 s_1 \ldots (k \text{ factors}) \text{ and } 2_k = s_2 s_1 s_2 \ldots (k \text{ factors})$. Then the left cells of $W = \langle s_1, s_2 \rangle$ are described as follows; see Lusztig [19 8.7, 8.8]:

(a) If $m$ is odd and $p_{s_1} = p_{s_2} > 0$, then the left cells are
$$\{1_0\}, \{2_1, 1_2, 2_3, \ldots, 1_{m-1}\}, \{1_1, 2_2, 1_3, \ldots, 2_{m-1}\}, \{2_m\}.$$ 

(b) If $m$ is even and $p_{s_1} = p_{s_2} > 0$, then the left cells are
$$\{1_0\}, \{2_1, 1_2, 2_3, \ldots, 1_{m-1}\}, \{1_1, 2_2, 1_3, \ldots, 1_{m-1}\}, \{2_m\}.$$ 

(c) If $m$ is even and $p_{s_2} > p_{s_1} > 0$, then the left cells are
$$\{1_0\}, \{2_1, 1_2, 2_3, \ldots, 1_{m-2}\}, \{2_{m-1}\}, \{1_1\}, \{2_2, 1_3, 2_4, \ldots, 1_{m-1}\}, \{2_m\}.$$ 

By inspection of the three \nb \cases{cases}, we see that two elements $x, y \in W$ lie in the same left cell if and only if $\mathcal{R}^e(x) = \mathcal{R}^e(y)$.

Corollary 2.7. Let $x, y \in W$. If $x \sim_L y$, then $\mathcal{R}^e(x) = \mathcal{R}^e(y)$.
Proof. Assume that \( x \sim_L y \). By Proposition 2.4, we have \( \mathcal{R}(x) = \mathcal{R}(y) \). Let \( s, t \in S \) be such that \( st \neq ts \) and \( p_s < p_t \). Let \( I = \{s, t\} \) and consider the parabolic subgroup \( W_I = \langle s, t \rangle \). By Theorem 2.2, we have \( pr_I(x) \sim_{L,I} pr_I(y) \). As observed in Example 2.6, we have \( xsts < x \) if and only if \( ysts < y \). Consequently, we obtain \( \mathcal{R}^*(x) = \mathcal{R}^*(y) \). □

3. The equal parameter case

We keep the general setting of the previous section. We shall also assume that \( \mathcal{H} \) is bounded in the sense of [19, 13.2]. This is obviously true for all finite Coxeter groups. It also holds, for example, for affine Weyl groups; see the remarks following [19, 13.4].

Definition 3.1 (Vogan [23, 3.10, 3.12]). For any \( s, t \in S \) such that \( st \neq ts \), we set

\[
\mathcal{D}_R(s, t) := \{ w \in W \mid \mathcal{R}(w) \cap \{s, t\} \text{ has exactly one element} \}
\]

and, for any \( w \in \mathcal{D}_R(s, t) \), we set \( \Xi_{s,t}(w) := \{ws, wt\} \cap \mathcal{D}_R(s, t) \). Note that \( \Xi_{s,t}(w) \) consists of one or two elements; in order to have a uniform notation, we consider \( \Xi_{s,t}(w) \) as a multiset with two identical elements if \( \{ws, wt\} \cap \mathcal{D}_R(s, t) \) consists of only one element.

Now let \( n \geq 0 \) and \( y, w \in W \). We define a relation \( y \sim_n w \) inductively as follows. First, let \( n = 0 \). Then \( y \approx_0 w \) if \( \mathcal{R}(y) = \mathcal{R}(w) \). Now let \( n > 0 \) and assume that \( \approx_{n-1} \) has been already defined. Then \( y \approx_n w \) if \( y \approx_{n-1} w \) and if, for any \( s, t \in S \) such that \( y, w \in \mathcal{D}_R(s, t) \) (where \( st \) has order 3 or 4), the following holds. If \( \Xi_{s,t}(y) = \{y_1, y_2\} \) and \( \Xi_{s,t}(w) = \{w_1, w_2\} \), then either \( y_1 \approx_{n-1} w_1, y_2 \approx_{n-1} w_2 \) or \( y_1 \approx_{n-1} w_2, y_2 \approx_{n-1} w_1 \).

If \( y \approx_n w \) for all \( n \geq 0 \), then \( y, w \) are said to have the same generalized \( \tau \)-invariant.

Remark 3.2. Let \( s, t \in S \) be such that \( st \) has finite order \( m \geq 3 \). Let \( I = \{s, t\} \). Then the parabolic subgroup \( W_I \) is a dihedral group of order \( 2m \). For any \( w \in W \), the coset \( wW_I \) can be partitioned into four subsets: one consists of the unique element \( x \) of minimal length, one consists of the unique element of maximal length, one consists of the \( (m-1) \) elements \( xs, xt, xsts, \ldots \) and one consists of the \( (m-1) \) elements \( xt, xts, xst, \ldots \).

Following Lusztig [18, 10.2], the last two subsets (ordered as above) are called strings. (Note that Lusztig considers the coset \( W_I w \) but, by taking inverses, the two versions are clearly equivalent.) Thus, if \( w \in \mathcal{D}_R(s, t) \), then \( w \) belongs to a unique string which we denote by \( \lambda_w \). Then we certainly have

\[
\Xi_{s,t}(w) \subseteq \lambda_w \subseteq \mathcal{D}_R(s, t) \quad \text{for all } w \in \mathcal{D}_R(s, t).
\]

As in [18, 10.6], we set

\[
\Gamma^* := \left( \bigcup_{w \in \Gamma} \lambda_w \right) \setminus \Gamma \quad \text{for any subset } \Gamma \subseteq \mathcal{D}_R(s, t).
\]

Now assume that we are in the equal parameter case and that \( \Gamma \) is a left cell of \( W \) such that \( \Gamma \subseteq \mathcal{D}_R(s, t) \). Then the following two results are known to hold.

(a) If \( m = 3 \), then \( \Gamma^* \) also is a left cell; see Kazhdan–Lusztig [14, Cor. 4.3]. (In this case, we have \( \Gamma^* = \{w^* \mid w \in \Gamma\} \) where \( w^* \) is the unique element of \( \Xi_{s,t}(w) \).)

(b) If \( m > 3 \), then \( \Gamma^* \) is a union of at most \( (m-2) \) left cells of \( W \); see [18, Prop. 10.7].
(For the proof of (b), it is assumed in [loc. cit.] that $W$ is crystallographic in order to guarantee certain positivity properties, but this assumption is now superfluous thanks to Elias–Williamson [5].)

With these preparations, we can now state the following result which was originally formulated and proved by Vogan in the language of primitive ideals in enveloping algebras.

**Proposition 3.3** (Kazhdan–Lusztig [16 §4], Lusztig [18 §10], Vogan [23 §3]). Assume that we are in the equal parameter case. Let $\Gamma$ be a left cell of $W$. Then all elements in $\Gamma$ have the same generalised $\tau$-invariant.

**Proof.** If $W$ is a finite Weyl group, this follows from the results in [23 §3], using the known dictionary (see, e.g., Barbasch–Vogan [11 §2]) between cells as defined in Section 2 and the corresponding notions in the theory of primitive ideals. In the general case, one cannot appeal to the theory of primitive ideals or other geometric arguments. Instead we argue as follows, using results from [16 §4] and [18 §10].

We will prove by induction on $n$ that, if $y, w \in W$ are such that $y \sim_L w$, then $y \approx_n w$. For $n = 0$, this holds by Proposition 2.3. Now let $n > 0$. By induction, we already know that $y \approx_{n-1} w$. Then it remains to consider $s, t \in S$ such that $st \neq ts$ and $y, w \in D_R(s, t).$ If $st$ has order 3, then Remark 3.2(a) shows that $\Sigma_{s,t}(y) = \{y^s, y^s\}$ and $\Sigma_{s,t}(w) = \{w^s, w^s\}$; furthermore, $y^s \sim_L w^s$ and so $y^s \approx_{n-1} w^s$, by induction. Now assume that $st$ has order 4. In this case, the argument is more complicated (as it is also in the setting of [23 §3]). Let $I = \{s, t\}$ and $\Gamma$ be the left cell containing $y, w$. Since all elements in $\Gamma$ have the same right descent set, we can choose the notation such that $xs < x$ and $xt > x$ for all $x \in \Gamma$. Then, for $x \in \Gamma$, we have $x = x's, x = x'ts$ or $x = x'sts$ where $x' \in X_I$. This yields that

\[
\Sigma_{s,t}(x) = \begin{cases} 
\{x's, x't\} & \text{if } x = x's, \\
\{x't, x'tst\} & \text{if } x = x'ts, \\
\{x's, x't\} & \text{if } x = x'sts.
\end{cases}
\]

Now we distinguish two cases.

**Case 1.** Assume that there exists some $x \in \Gamma$ such that $x = x's$ or $x = x'sts$. Then $\lambda_x = (x's, x's, x'sts)$ and so $\Gamma^*$ contains elements with different right descent sets. Hence, by Remark 3.2(b), $\Gamma^*$ is the union of two distinct left cells $\Gamma_1$ and $\Gamma_2$, where we choose the notation such that:

- all elements in $\Gamma_1$ have $s$ in their right descent set, but not $t$;
- all elements in $\Gamma_2$ have $t$ in their right descent set, but not $s$.

Now consider $y, w \in \Gamma$; we write $\Sigma_{s,t}(y) = \{y_1, y_2\} \subseteq \Gamma^*$ and $\Sigma_{s,t}(w) = \{w_1, w_2\} \subseteq \Gamma^*$. By (†), all the elements $y_1, y_1, w_1, w_2$ belong to $\Gamma_2$. In particular, $y_1 \sim_L w_1, y_2 \sim_L w_2$ and so, by induction, $y_1 \approx_{n-1} w_1, y_2 \approx_{n-1} w_2$.

**Case 2.** We are not in Case 1, that is, all elements $x \in \Gamma$ have the form $x = x'ts$ where $x' \in X_I$. Then $\lambda_x = (x't, x'ts, x'tst)$ for each $x \in \Gamma$. Let us label the elements in such a string as $x_1, x_2, x_3$. Then $x = x_2$ and $\Sigma_{s,t}(x) = \{x't, x'tst\} = \{x_1, x_3\}$. 

Now consider \( y, w \in \Gamma \). By definition, there is a chain of elements which connect \( y \) to \( w \) via the elementary relations \( \leftarrow_L \), and vice versa. Assume first that \( y, w \) are directly connected as \( y \leftarrow_L w \). Using the labelling \( y = y_2, w = w_2 \) and the notation of [18, 10.4], this means that \( a_{22} \neq 0 \). Hence, the identities "\( a_{11} = a_{33} \)", \( a_{13} = a_{31} \), \( a_{22} = a_{11} + a_{13} \) in [18, 10.4.2] imply that

\[
(y_1 \leftarrow_L w_1 \text{ and } y_2 \leftarrow_L w_3) \text{ or } (y_1 \leftarrow_L w_3 \text{ and } y_2 \leftarrow_L w_1).
\]

(See also [21, Prop. 4.6].) We shall write this as \( \mathfrak{I}_{s,t}(y) \leftarrow_L \mathfrak{I}_{s,t}(w) \). Now, in general, there is a sequence of elements \( y = y^{(0)}, y^{(1)}, \ldots, y^{(k)} = w \) in \( \Gamma \) such that \( y^{(i-1)} \leftarrow_L y^{(i)} \) for \( 1 \leq i \leq k \). At each step, we have \( \mathfrak{I}_{s,t}(y^{(i-1)}) \leftarrow_L \mathfrak{I}_{s,t}(y^{(i)}) \) by the previous argument. Combining these steps, we conclude that either \( y_1 \leq_L w_1, y_3 \leq_L w_3 \) or \( y_1 \leq_L w_3, y_3 \leq_L w_1 \). Now, all elements in a string belong to the same right cell (see [18, 10.5]); in particular, all the elements \( y_i, w_j \) belong to the same two-sided cell. Hence, [18, Cor. 6.3] implies that either \( y_1 \sim_L w_1, y_3 \sim_L w_3 \) or \( y_1 \sim_L w_3, y_3 \sim_L w_1 \). (Once again, the assumption in [loc. cit.] that \( W \) is crystallographic is now superfluous thanks to [5].) Consequently, by induction, we have either \( y_1 \approx_{n-1} w_1, y_3 \approx_{n-1} w_3 \) or \( y_1 \approx_{n-1} w_3, y_3 \approx_{n-1} w_1 \). \( \square \)

One of the most striking results about this invariant has been obtained by Garfinkle [8, Theorem 3.5.9]: two elements of a Weyl group of type \( B_n \) belong to the same left cell (equal parameter case) if and only if the elements have the same generalised \( \tau \)-invariant. This fails in general; a counter-example is given by \( W \) of type \( D_n \) for \( n \geq 6 \) (as mentioned in the introduction of [4]).

**Remark 3.4.** Note that, if \( st \) has order \( m = 4 \), then the set \( \mathfrak{I}_{s,t}(w) \) may contain two distinct elements. In order to obtain a single-valued operator, Vogan [21, §4] (for the case \( m = 4 \)) and Lusztig [18, §10] (for any \( m \geq 4 \)) propose an alternative construction, as follows.

Let \( s, t \in S \) be such that \( st \) has finite order \( m \geq 3 \). As in [18, 10.6], we define an involution

\[
D_R(s, t) \to D_R(s, t), \quad w \mapsto \bar{w},
\]
as follows. Let \( w \in D_R(s, t) \). Then \( w \) is contained in a unique string \( \lambda_w \) with respect to \( s, t \); see Remark 3.2. Let \( i \in \{1, \ldots, m-1\} \) be the index such that \( w \) is the \( i \)th element of \( \lambda_w \). Then \( \bar{w} \) is defined to be the \( (m-i) \)th element of \( \lambda_w \). Now let \( \Gamma \subseteq D_R(s, t) \) be a left cell. Then \( \bar{\Gamma} = \{ \bar{w} \mid w \in \Gamma \} \) also is a left cell by [18, Prop. 10.7]. (Again, it is assumed in [loc. cit.] that \( W \) is crystallographic, but this is now superfluous thanks to [5].)

Hence, setting \( \bar{\mathfrak{I}}_{s,t}(w) := \{ \bar{w} \} \) for any \( w \in D_R(s, t) \), we obtain a new "generalised \( \bar{\tau} \)-invariant" by exactly the same procedure as in Definition 3.1 using \( \bar{\mathfrak{I}}_{s,t} \) instead of \( \mathfrak{I}_{s,t} \) and allowing any \( s, t \in S \) such that \( st \) has finite order at least 3.

The above procedure is the model for the more general construction of invariants below. As we shall see in Example 4.7, this even provides a new proof—which does not rely on [5]—for the fact that the map \( w \mapsto \bar{w} \) preserves left cells.
4. An abstract setting for generalised \( \tau \)-invariants

We keep the general setting of Section 2, where \( \pi = \{ p_s \mid s \in S \} \) are positive weights for \( W \).

**Definition 4.1.** A pair \((I, \delta)\) consisting of a subset \( I \subseteq S \) and a map \( \delta : W_I \to W_I \) is called **admissible** if the following conditions are satisfied for every left cell \( \Gamma' \subseteq W_I \) (with respect to the weights \( \{ p_s \mid s \in I \} \)):

1. The restriction of \( \delta \) to \( \Gamma' \) is injective and \( \delta(\Gamma') \) also is a left cell.
2. The map \( \delta \) induces an \( \mathcal{H}_I \)-module isomorphism \( \left[ \Gamma' \right]_A \cong \left[ \delta(\Gamma') \right]_A \).

We say that \((I, \delta)\) is **strongly admissible** if, in addition to (1) and (2), the following condition is satisfied:

3. We have \( w \sim_{R,I} \delta(u) \) for all \( u \in W_I \).

The map \( \delta \) has a canonical extension to a map \( \tilde{\delta} : W \to W \): Given \( w \in W_I \), we write \( w = xu \) where \( x \in X_I \) and \( u \in W_I \); then we set \( \tilde{\delta}(w) := x\delta(u) \).

The situation considered by Kazhdan–Lusztig [16, §4] fits into this setting as follows.

**Example 4.2.** Let \( I = \{s, t\} \) with \( s \neq t \) and \( st \) of order 3; then \( W_I \) is isomorphic to the symmetric group \( S_3 \). The left cells of \( W_I \) are easily determined; they are

\[
\Gamma'_1 := \{1\}, \quad \Gamma' := \{s, ts\}, \quad \Gamma'_t := \{t, st\}, \quad \Gamma'_0 := \{sts\}.
\]

The matrix representation of \( \mathcal{H}_I \) afforded by \( \left[ \Gamma'_s \right]_A \) with respect to the basis \( \{ e_s, e_{ts} \} \) is given by (where we set \( p := p_s = p_t > 0 \)):

\[
C'_s \mapsto \begin{bmatrix} v^p + v^{-p} & 1 \\ 0 & 0 \end{bmatrix}, \quad C'_t \mapsto \begin{bmatrix} 0 & 0 \\ 1 & v^p + v^{-p} \end{bmatrix},
\]

and we obtain exactly the same matrices when we consider the matrix representation afforded by \( \left[ \Gamma'_t \right]_A \) with respect to the basis \( \{ e_{st}, e_t \} \). (See [16, 7.2, 7.3, 8.7] where dihedral groups in general are considered.) Thus, the conditions (1), (2), (3) in Definition 4.1 hold for \((I, \delta)\), if we define \( \delta : W_I \to W_I \) as follows:

\[
\delta : \begin{array}{cccccc}
1 & s & t & st & ts & sts \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & st & ts & s & t & sts
\end{array}
\]

We notice that, if \( w \in W \) is such that \( w \in \mathcal{D}_R(s, t) \) (see Definition 3.1), then \( \{ \tilde{\delta}(w) \} = \{ ws, wt \} \cap \mathcal{D}_R(s, t) \), hence \( \tilde{\delta}(w) = w^* \) with the notation of [16, §4].

**Proposition 4.3.** Let \((I, \delta)\) be an admissible pair. Then the following hold.

(a) If \( \Gamma \) is a left cell of \( W \), then so is \( \tilde{\delta}(\Gamma) \) (where \( \tilde{\delta} \) is the canonical extension of \( \delta \) to \( W \)) and \( \tilde{\delta} \) induces an \( \mathcal{H} \)-module isomorphism \( \left[ \Gamma \right]_A \cong \left[ \tilde{\delta}(\Gamma) \right]_A \).

(b) If \((I, \delta)\) is strongly admissible, then we have \( w \sim_{R} \tilde{\delta}(w) \) for all \( w \in W \).
Proof. (a) By Theorem 2.2 there is a left cell $\Gamma'$ of $W_I$ such that $\Gamma \subseteq X_I \Gamma'$. By condition (1) in Definition 4.1 the set $\Gamma'_I := \delta(\Gamma')$ also is a left cell of $W_I$ and, by condition (2), the map $\delta$ induces an $H_I$-module isomorphism $[\Gamma']_A \cong [\Gamma'_I]_A$. By Example 2.3 the subsets $X_I \Gamma'$ and $X_I \Gamma'_I$ of $W$ are closed with respect to $\leq_L$ and, hence, we have corresponding $H$-modules $[X_I \Gamma']_A$ and $[X_I \Gamma'_I]_A$. These two $H$-modules are isomorphic to the induced modules $\text{Ind}^S_{L}([\Gamma'])$ and $\text{Ind}^S_{L}([\Gamma'_I])$, respectively, where explicit isomorphisms are given by the formula in Example 2.3. Now, by [10, Lemma 3.8], we have

\[ p^*_{xu,yv} = p^*_{xu_1,yv_1} \quad \text{for all } x, y \in X_I \text{ and } u, v \in \Gamma', \]

where we set $u_1 = \delta(u)$ and $v_1 = \delta(v)$ for $u, v \in \Gamma'$. By [10, Prop. 3.9], this implies that $\tilde{\delta}$ maps the partition of $X_I \Gamma'$ into left cells of $W$ onto the analogous partition of $X_I \Gamma'_I$. In particular, since $\Gamma \subseteq X_I \Gamma'$, the set $\tilde{\delta}(\Gamma) \subseteq X_I \Gamma'_I$ is a left cell of $W$; furthermore, [10, Prop. 3.9] also shows that $\tilde{\delta}$ induces an $H$-module isomorphism $[\Gamma]_A \cong [\tilde{\delta}(\Gamma)]_A$.

(b) Since condition (3) in Definition 4.1 is assumed to hold, this is just a restatement of [19, Prop. 9.11(b)]. \qed

As a first consequence, we can now show that [16, Cor. 4.3] (concerning the Kazhdan–Lusztig star operations) holds for general weight functions. (Partial results in this direction are obtained in [22, Cor. 3.5(4)].) Note that some work has to be done to obtain this generalisation since, in the setting of [16, §4], the polynomials $M^*_y,w$ are constant, which is no longer true in the general case and so some new arguments are required.

**Corollary 4.4.** Let $s, t \in S$ be such that $st$ has order 3. Then, for any $w \in D_R(s, t)$, there is a unique $w^* \in D_R(s, t)$ such that $\Sigma_{s,t}(w) = \{w^*, w^*\}$ (as in [16, §4] and Definition 3.7). Let $\Gamma \subseteq D_R(s, t)$ be a left cell (with respect to the given weights $\{p_s \mid s \in S\}$). Then $\Gamma^* := \{w^* \mid w \in \Gamma\}$ also is a left cell. Furthermore, the map $w \mapsto w^*$ induces an $H$-module isomorphism $[\Gamma]_A \to [\Gamma^*]_A$ and we have $w \sim_R w^*$ for all $w \in \Gamma$.

Proof. Let $I = \{s, t\}$ and define $\delta: W_I \to W_I$ as in Example 3.2. We already noted that then $\tilde{\delta}(w) = w^*$ for all $w \in D_R(s, t)$. Hence, the assertions follow from Proposition 4.3. \qed

In analogy to Definition 3.1 we can now introduce an invariant of left cells as follows.

**Definition 4.5.** Let $\Delta$ be a collection of admissible pairs $(I, \delta)$ as in Definition 4.1. For each $\Delta$ which occurs as a first component of a pair in $\Delta$, we assume that we are given a relation $\Lambda_I \subseteq W_I \times W_I$ which contains the relation defined by $\sim_L$. (For example, $\Lambda_I = \{(u, u') \in W_I \times W_I \mid R(u) = R(u')\}$; see Proposition 2.4.)

Now let $n \geq 0$ and $y, w \in W$. Then we define a relation $y \equiv_n w$ inductively as follows.

(i) For $n = 0$, we have $y \equiv_0 w$ if $(\text{pr}_I(y), \text{pr}_I(w)) \in \Lambda_I$ for all $(I, \delta) \in \Delta$.

(ii) Now let $n > 0$ and assume that $\equiv_{n-1}$ has been already defined. Then $y \equiv_n w$ if $y \equiv_{n-1} w$ and $\tilde{\delta}(y) \equiv_{n-1} \tilde{\delta}(w)$ for all $(I, \delta) \in \Delta$.

If $y \equiv_n w$ for all $n \geq 0$, then $y, w$ are said to have the same generalized $\tilde{\tau}^\Delta$-invariant.

**Corollary 4.6.** In the setting of Definition 4.1, all elements in a left cell $\Gamma$ of $W$ have the same generalized $\tilde{\tau}^\Delta$-invariant.
Proof. We prove by induction on \( n \) that, if \( y, w \in W \) are such that \( y \sim_L w \), then \( y \equiv_n w \). For \( n = 0 \), this holds by Theorem 2.2. Now assume that \( n > 0 \). By induction, we already know that \( y \equiv_{n-1} w \). Then it remains to consider a pair \((I, \delta) \in \Delta\). By Proposition 4.3(a), we have \( \tilde{\delta}(y) \sim_L \tilde{\delta}(w) \) and, by induction, we have \( \delta(y) \equiv_{n-1} \tilde{\delta}(w) \).

The situation considered by Lusztig [18 §10] (see also Vogan [24 §4] and McGovern [20 §4] for the case \( m = 4 \)) fits into this setting as follows.

Example 4.7. Let \( \mathcal{J}_{\geq 3} \) be the set of all subsets \( I \subseteq S \) such that \( I = \{s, t\} \), where \( s \neq t \), \( p_s = p_t \) and \( st \) has finite order \( m \geq 3 \). For any \( I \in \mathcal{J}_{\geq 3} \), the group \( W_I \) is a dihedral group of order \( 2m \). For \( k \geq 0 \) let \( 1_k = sts \ldots (k \text{ factors}) \) and \( 2_k = tst \ldots (k \text{ factors}) \). Then the left cells of \( W_I \) are described as follows (see Example 2.6):

\[
\{1_0\}, \{2_1, 1_2, 2_3, \ldots, 1_{m-1}\}, \{1_1, 2_2, 1_3, \ldots, 2_{m-1}\}, \{2_m\} \quad (m \text{ odd}), \\
\{1_0\}, \{2_1, 1_2, 2_3, \ldots, 2_{m-1}\}, \{1_1, 2_2, 1_3, \ldots, 1_{m-1}\}, \{2_m\} \quad (m \text{ even}).
\]

We define an involution \( \delta : W_I \to W_I \) as follows:

\[
\delta(1_0) = 1_0, \quad \delta(2_m) = 2_m, \quad \delta(1_k) = 1_{m-k}, \quad \delta(2_k) = 2_{m-k} \quad \text{for } 1 \leq k \leq m - 1.
\]

If \( m \) is odd, then \( \delta \) preserves each of the left cells \( \{1_0\}, \{2_m\} \) and interchanges the two left cells with \( m - 1 \) elements (reversing the order in which the elements are listed). If \( m \) is even, then \( \delta \) preserves each of the four left cells of \( W_I \), where in each of the two left cells with \( m - 1 \) elements, the order of the elements is reversed. Thus, conditions (1) and (3) in Definition 4.1 hold for all pairs in the collection

\[
\Delta_{\geq 3} := \{(I, \delta) \mid I \in \mathcal{J}_{\geq 3}\}.
\]

Using the formulæ in [19 7.2, 7.3], it is straightforward to check that condition (2) also holds. For \( m = 3 \), this has been done explicitly in Example 4.2. Let us also show explicitly how this works for \( m = 4, 5 \).

First let \( m = 4 \). Consider the two left cells \( \Gamma'_s = \{s, ts, sts\} \) and \( \Gamma'_t = \{t, st, tst\} \). The matrix representation afforded by \([\Gamma'_s]_A\) with respect to the basis \( \{e_s, e_{ts}, e_{sts}\} \) is given by:

\[
C'_s \mapsto \begin{bmatrix} v^p + v^{-p} & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & v^p + v^{-p} \end{bmatrix}, \quad C'_t \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 1 & v^p + v^{-p} & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

(\( p := p_s = p_t \)). The matrix representation afforded by \([\Gamma'_t]_A\) with respect to the basis \( \{e_t, e_{st}, e_{tst}\} \) is given by:

\[
C''_s \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 1 & v^p + v^{-p} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C''_t \mapsto \begin{bmatrix} v^p + v^{-p} & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & v^p + v^{-p} \end{bmatrix}.
\]

Thus, there is no bijection \( \Gamma'_s \to \Gamma'_t \) which induces an \( \mathcal{H}_I \)-module isomorphism \([\Gamma'_s]_A \cong [\Gamma'_t]_A\). However, we have \( \delta(\Gamma'_s) = \Gamma'_t \) where \( s \mapsto st, ts \mapsto ts, sts \mapsto s \), and this map yields a non-trivial \( \mathcal{H}_I \)-module automorphism of \([\Gamma'_s]_A\); a similar remark applies to \([\Gamma'_t]_A\).
Now let $m = 5$. We have the two left cells $\Gamma'_1 = \{s, ts, sts, tsts\}$ and $\Gamma'_1 = \{t, st, tst, stst\}$. The matrix representation afforded by $[\Gamma'_1]_A$ with respect to the basis $\{e_s, e_{ts}, e_{sts}, e_{tsts}\}$ is given by:

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & v^p + v^{-p} & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

and we obtain exactly the same matrices when we consider the matrix representation afforded by $[\Gamma'_1]_A$ with respect to the basis $\{e_{sts}, e_{st}, e_{t}, e_t\}$.

We notice that, if $w \in W$ is any element such that $w \in D_R(s, t)$ (see Definition 2.6), then $\tilde{\delta}(w) = \tilde{w}$, with $\tilde{w}$ as defined in Remark 3.4. Thus, Proposition 4.3 provides a new proof of the part of [16, Prop. 10.7] concerning the tilde construction; this new proof does not rely on the positivity properties used in [loc. cit.].

Finally, we consider a genuine case of unequal parameters.

**Example 4.8.** Let $\mathcal{J}_\pi$ be the set of all subsets $I \subseteq S$ such that $I = \{s, t\}$, where $s \neq t$, $p_s < p_t$ and $st$ has finite even order $m \geq 4$. For any $I \in \mathcal{J}_\pi$, the group $W_I$ is a dihedral group of order $2m$. For $k \geq 0$ let $1_k = sts \ldots (k \text{ factors})$ and $2_k = tst \ldots (k \text{ factors})$. Then the left cells of $W_I$ are described as follows (see Example 2.6):

$$
\{1_0\}, \{2_1, 1_2, 2_3, \ldots, 1_{m-2}\}, \{2_{m-1}\}, \{1_1\}, \{2_2, 1_3, 2_4, \ldots, 1_{m-1}\}, \{2_m\}.
$$

We define an involution $\delta: W_I \to W_I$ as follows: $\delta(w) = w$ for $w \in \{1_0, 1_1, 2_{m-1}, 2_m\}$ and

$$
\delta: \begin{array}{cccc}
2_1 & 1_2 & 2_3 & \ldots & 1_{m-2} \\
\uparrow & \uparrow & \uparrow & \ldots & \uparrow \\
2_2 & 1_3 & 2_4 & \ldots & 1_{m-1}
\end{array}
$$

Thus, $\delta$ preserves each of the left cells $\{1_0\}, \{2_{m-1}\}, \{1_1\}, \{2_m\}$ and interchanges the two left cells with $m - 2$ elements (preserving the order in which the elements are listed). So, conditions (1) and (3) in Definition 4.1 hold for all pairs in the collection

$$
\Delta_\pi := \{(I, \delta) \mid I \in \mathcal{J}_\pi\}.
$$

Using the knowledge of the polynomials $M^s_{y,w}$ (see [14] Exc. 11.4 or [19] 7.5, 7.6), it is straightforward to check that condition (2) also holds. Let us show explicitly how this works for $m = 4, 6$.

First let $m = 4$. We have to consider the two left cells $\Gamma'_1 = \{t, st\}$ and $\Gamma'_1 = \{ts, sts\}$. The matrix representation afforded by $[\Gamma'_1]_A$ with respect to the basis $\{e_t, e_{st}\}$ is given by:

$$
\begin{bmatrix}
0 & 0 \\
v^{ps} + v^{-ps} & 0
\end{bmatrix},
\begin{bmatrix}
v^{pt} + v^{-pt} & v^{pt- ps} + v^{ps- pt} \\
v^{pt- ps} + v^{ps- pt} & 0
\end{bmatrix}
$$

and we obtain exactly the same matrices when we consider the matrix representation afforded by $[\Gamma'_2]_A$ with respect to the basis $\{e_{ts}, e_{sts}\}$. 
Next consider the case $m = 6$. We have the two left cells $\Gamma'_1 = \{t, st, tst, stst\}$ and $\Gamma'_2 = \{ts, sts, tst, ststs\}$. The two matrices describing the action of $C'_s$ and $C'_t$ on $[\Gamma'_1]_A$ with respect to the basis $\{e_t, e_{st}, e_{tst}, e_{stst}\}$ are given by

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & v^p_s + v^{-p_s} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & v^p_s + v^{-p_s}
\end{bmatrix}
\quad \quad \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & v^{pt} + v^{-pt} & v^{pt-p_s} + v^{p_t-p_s} & 0 \\
0 & 0 & 1 & v^{pt} + v^{-pt} \\
0 & 0 & 0 & 0
\end{bmatrix},
$$

respectively, and we obtain exactly the same matrices when we consider the matrix representation afforded by $[\Gamma'_2]_A$ with respect to the basis $\{e_t, e_{st}, e_{tst}, e_{stst}\}$.

For any subset $I = \{s, t\} \subseteq S$ where $s \neq t$ and $st$ has finite order $m \geq 3$, we set $\Lambda_I = \{(u, u') \in W_I \times W_I \mid R^\pi(u) = R^\pi(u')\}$; see Definition [2.5]. With this convention, we would now like to state the following conjecture.

**Conjecture 4.9.** Two elements $y, w \in W$ belong to the same left cell (with respect to the given weights) if and only if $y, w$ belong to the same two-sided cell and $y, w$ have the same generalised $\tau^\Delta$-invariant where $\Delta = \Delta_{\geq 3} \cup \Delta_{\pi}$ (see Examples [4.7] and [4.8]).

If $W$ is finite and we are in the equal parameter case, then Conjecture 4.9 is known to hold except possibly in type $B_n, D_n$; see the remarks at the end of [13, §6]. We have checked that the conjecture also holds for $F_4, B_n (n \leq 7)$ and all possible weights, using PyCox [12].

By considering collections $\Delta$ with subsets $I \subseteq S$ of size bigger than 2, one can obtain further refinements of the above invariants. In particular, it is likely that the results of Bonnafé and Iancu [2, 4] can be interpreted in terms of generalised $\tau^\Delta$-invariants for suitable collections $\Delta$. This will be discussed elsewhere.

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Fachbereich Mathematik, IAZ - Lehrstuhl für Algebra, Universität Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany

E-mail address: meinolf.geck@mathematik.uni-stuttgart.de