CLASSICAL LIMITS OF QUANTUM TOROIDAL AND AFFINE YANGIAN
ALGEBRAS

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Abstract. In this short article, we compute the classical limits of the quantum toroidal and
affine Yangian algebras of \( \mathfrak{sl}_n \), by generalizing our arguments for \( \mathfrak{gl}_1 \) from [T1] (an alternative
proof for \( n > 2 \) is given in [VV]). We also discuss some consequences of these results.

INTRODUCTION

The primary purpose of this note is to provide proofs for the description of the classical limits
of the algebras \( \mathcal{U}_q(n) \) and \( \mathcal{Y}_h(\beta) \) from [FT, TB]. Here \( \mathcal{U}_q(n) \) and \( \mathcal{Y}_h(\beta) \) are the quantum toroidal and
the affine Yangian algebras of \( \mathfrak{sl}_n \) (if \( n \geq 2 \)) or \( \mathfrak{gl}_1 \) (if \( n = 1 \)), while classical limits refer to the
limits of these algebras as \( q \to 1 \) or \( h \to 0 \), respectively. We also discuss the classical limits of
certain constructions for \( \mathcal{U}_q(n) \).

The case \( n = 1 \) has been essentially worked out in [T1]. In this note, we follow the same
approach to prove the \( n > 1 \) generalizations. While writing down this note, we found that the
\( n \geq 3 \) case has been considered in [VV] long time ago (to deduce our Theorems 2.1 and 2.2
one needs to combine [VV] with [BGK]). Hence, the only essentially new case is \( n = 2 \). Meanwhile,
we expect our direct arguments to be applicable in some other situations of interest.

This paper is organized as follows:
• In Section 1, we recall explicit definitions of the Lie algebras \( \mathfrak{u}_d(n) \) and \( \mathfrak{y}_\beta(n) \), whose universal
enveloping algebras coincide with the classical limits of \( \mathcal{U}_q(n) \) and \( \mathcal{Y}_h(\beta) \). We also recall the notion
of \( n \times n \) matrix algebras over the algebras of difference/differential operators on \( \mathbb{C}^\times \) and their
central extensions, denoted by \( \mathfrak{d}_d(n) \) and \( \mathfrak{D}_\beta(n) \), respectively.
• In Section 2, we establish two key isomorphisms relating the classical limit Lie algebras
\( \mathfrak{u}_d(n) \) and \( \mathfrak{y}_\beta(n) \) to the aforementioned Lie algebras \( \mathfrak{d}_d(n) \) and \( \mathfrak{D}_\beta(n) \).
• In Section 3, we discuss the classical limits of the following constructions for \( \mathcal{U}_q(n) \) \((n \geq 2)\):
– the vertical and horizontal copies of a quantum affine algebra \( \mathcal{U}_q(\widehat{\mathfrak{gl}}_n) \) inside \( \mathcal{U}_q(n) \) from [FJMM],
– the Miki’s automorphism \( \omega : \mathcal{U}_q(n) \xrightarrow{\sim} \mathcal{U}_q(n) \) from [M],
– the commutative subalgebras \( \mathcal{A}(s_0, \ldots, s_{n-1}) \) of \( \mathcal{U}_q(n)^+ \) from [FT].

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1. Basic constructions

1.1. The quantum toroidal algebra $U^{(n)}_{q,d}$ and the affine Yangian $U^{(n)}_{h,β}$.

For $n \in \mathbb{N}$, set $[n] := \{0, 1, \ldots, n-1\}$ viewed as a set of mod $n$ residues and $[n]^x := [n] \setminus \{0\}$. For $n \geq 2$, we set $a_{i,j} := 2\delta_{i,j} - \delta_{i+1,j} - \delta_{i,j-1}$ and $m_{i,j} := \delta_{i,j+1} - \delta_{i,j-1}$ for all $i, j \in [n]$.

- Given $h, \beta \in \mathbb{C}$, let $U^{(n)}_{h,\beta}$ be the affine Yangian of $\mathfrak{sl}_n$ (if $n \geq 2$) or $\mathfrak{gl}_1$ (if $n = 1$) as considered in [TB] but without the generators $u^2$ and $e$ (if $n = 1$)

- Given $q, d \in \mathbb{C}^\times$, let $U^{(n)}_{q,d}$ be the quantum toroidal algebra of $\mathfrak{sl}_n$ (if $n \geq 2$) or $\mathfrak{gl}_1$ (if $n = 1$) as considered in [FT] but without the generators $q^{\pm d_1}, q^{\pm d_2}$ and with $\gamma^{\pm 1/2} = q^{\pm c/2}$. These are unital associative $\mathbb{C}$-algebras generated by $\{\bar{e}_{i,k}, \bar{f}_{i,k}, h_{i,k}, e\}_{i,k \in [n]}$ with the defining relations specified in [FT]. We note that algebras $U^{(n)}_{q,d}$ from [TB] Sect. 1.1 are their central quotients.

1.2. The Lie algebra $\bar{u}^{(n)}_d$.

In the $q \to 1$ limit, all the defining relations of $U^{(n)}_{q,d}$ become of Lie type. Therefore, the $q \to 1$ limit of $U^{(n)}_{q,d}$ is isomorphic to the universal enveloping algebra $U(\bar{u}^{(n)}_d)$. The Lie algebra $\bar{u}^{(n)}_d$ is generated by $\{\bar{e}_{i,k}, \bar{f}_{i,k}, h_{i,k}, \bar{c}\}_{i,k \in [n]}$ with $\bar{c}$ being a central element and the rest of the defining relations (u1–u7.2) to be given below in each of the 3 cases of interest: $n > 2$, $n = 2$, and $n = 1$.

- For $n > 2$, the defining relations are

  (u1) $[\bar{h}_{i,k}, \bar{h}_{j,l}] = k a_{i,j} d^{-k m_{i,j}} \delta_{k,-l} \bar{c}$,

  (u2) $[\bar{e}_{i,k+1}, \bar{e}_{j,l}] = d^{-m_{i,j}} [\bar{e}_{i,k}, \bar{e}_{j,l+1}]$,

  (u3) $[\bar{f}_{i,k+1}, \bar{f}_{j,l}] = d^{-m_{i,j}} [\bar{f}_{i,k}, \bar{f}_{j,l+1}]$,

  (u4) $[\bar{e}_{i,k}, \bar{f}_{j,l}] = \delta_{i,j} h_{i,k+l} + k \delta_{i,j} \delta_{k,-l} \bar{c}$,

  (u5) $[\bar{h}_{i,k}, \bar{e}_{j,l}] = a_{i,j} d^{-k m_{i,j}} \bar{e}_{j,l+k}$,

  (u6) $[\bar{h}_{i,k}, \bar{f}_{j,l}] = -a_{i,j} d^{-k m_{i,j}} \bar{f}_{j,l+k}$,

  (u7.1) $\sum_{\pi \in \Sigma_2} [\bar{e}_{i,k_{x(1)}}, [\bar{e}_{i,k_{x(2)}}, \bar{e}_{i,\pm 1,l}]] = 0$ and $[\bar{e}_{i,k}, \bar{e}_{j,l}] = 0$ for $j \neq i, i \pm 1$,

  (u7.2) $\sum_{\pi \in \Sigma_2} [\bar{f}_{i,k_{x(1)}}, [\bar{f}_{i,k_{x(2)}}, \bar{f}_{i,\pm 1,l}]] = 0$ and $[\bar{f}_{i,k}, \bar{f}_{j,l}] = 0$ for $j \neq i, i \pm 1$.

- For $n = 2$, the defining relations are

  (u1) $[\bar{h}_{i,k}, \bar{h}_{i,l}] = 2k \delta_{k,-l} \bar{c}$, $[\bar{h}_{i,k}, \bar{h}_{i+l,1}] = -k (d^k + d^{-k}) \delta_{k,-1} \bar{c}$,

  (u2) $[\bar{e}_{i,k+1}, \bar{e}_{i,l}] = [\bar{e}_{i,k}, \bar{e}_{i+l+1}], [\bar{e}_{i,k+2}, \bar{e}_{i,l+1}] = (d + d^{-1}) [\bar{e}_{i,k+1}, \bar{e}_{i+l+1}] + [\bar{e}_{i,k}, \bar{e}_{i+l+2}] = 0$,

  (u3) $[\bar{f}_{i,k+1}, \bar{f}_{i,l}] = [\bar{f}_{i,k}, \bar{f}_{i+l+1}], [\bar{f}_{i,k+2}, \bar{f}_{i+l+1}] = (d + d^{-1}) [\bar{f}_{i,k+1}, \bar{f}_{i+l+1}] + [\bar{f}_{i,k}, \bar{f}_{i+l+2}] = 0$,

  (u4) $[\bar{e}_{i,k}, \bar{f}_{j,l}] = \delta_{i,j} \bar{h}_{i,k+l} + k \delta_{i,j} \delta_{k,-l} \bar{c}$,

  (u5) $[\bar{h}_{i,k}, \bar{e}_{i+l+k}] = 2 \bar{e}_{i+l+k}$, $[\bar{h}_{i,k}, \bar{e}_{i+1,l}] = -(d^k + d^{-k}) \bar{e}_{i+1,l+k}$,
Let us now specify \((y_3–y_6)\) in each of the 3 cases of interest:

\[
[y_2] = [\bar{x}] = (d^k + d^{-k})f_{i+1,t+k},
\]

\[
(\sum_{\pi \in \Sigma_3} [\bar{e}_{i,k}e_{(1)}, [\bar{e}_{i,k}e_{(2)}], [\bar{e}_{i,k}e_{(3)}, \bar{e}_{i+1,l}]] = 0,
\]

\[
(\sum_{\pi \in \Sigma_3} [\bar{f}_{i,k}e_{(1)}, [\bar{f}_{i,k}e_{(2)}], [\bar{f}_{i,k}e_{(3)}, \bar{f}_{i+1,l}]] = 0.
\]

- For \(n = 1\), the defining relations are
  \[
  [\bar{h}_{0,k}, \bar{h}_{0,l}] = k(2 - d^k - d^{-k})\delta_{k,-l}\bar{e},
  \]

\[
(\bar{e}_{0,k+3}, \bar{e}_{0,l} - (1 + d + d^{-1})\bar{e}_{0,k+2}, \bar{e}_{0,l+1}] + (1 + d + d^{-1})\bar{e}_{0,k+1}, \bar{e}_{0,l+2}] - \bar{e}_{0,k}, \bar{e}_{0,l+3} = 0,
\]

\[
(\bar{f}_{0,k+3}, \bar{f}_{0,l} - (1 + d + d^{-1})\bar{f}_{0,k+2}, \bar{f}_{0,l+1}] + (1 + d + d^{-1})\bar{f}_{0,k+1}, \bar{f}_{0,l+2}] - \bar{f}_{0,k}, \bar{f}_{0,l+3} = 0,
\]

\[
(\bar{h}_{0,k}, \bar{f}_{0,l}] = -(2 - d^k - d^{-k})\bar{f}_{0,l+k},
\]

\[
(\sum_{\pi \in \Sigma_3} [\bar{e}_{0,k}, \bar{e}_{0,l}] = 0,
\]

\[
(\sum_{\pi \in \Sigma_3} [\bar{f}_{0,k}, \bar{f}_{0,l}] = 0,
\]

In the above relations \(l, k, k_1, k_2, k_3 \in \mathbb{Z}\) and \(\Sigma_n\) is the symmetric group on \(s\) letters.

1.3. The Lie algebra \(\mathcal{Y}_{\beta}^{(n)}\).

All the defining relations of \(\mathcal{Y}_{h,\beta}^{(n)}\) are well-defined for \(h = 0\) and become of Lie type. Therefore, \(\mathcal{Y}_{\beta}^{(n)} \simeq U(\mathcal{Y}_{\beta}^{(n)})\) where the Lie algebra \(\mathcal{Y}_{\beta}^{(n)}\) is generated by \(\{x_{i,r}^\pm, \xi_{i,r}^\pm\}_{i \in [n]}\) with the defining relations \((y_1–y_6)\) to be given below. The first two of them are independent of \(n \in \mathbb{N}\)

\[
[y_1] = [\xi_{i,r}, \xi_{j,s}] = 0,
\]

\[
[y_2] = [\bar{x}_{i,r}, \bar{x}_{j,s}] = \delta_{i,j}\xi_{i,r+s}.
\]

Let us now specify \((y_3–y_6)\) in each of the 3 cases of interest: \(n > 2, n = 2, \) and \(n = 1\).

- For \(n > 2\), the defining relations are
  \[
  [x_{i,r+1}, x_{j,s}] - [x_{i,r}, x_{j,s+1}] = -m_{i,j}\beta[x_{i,r}, x_{j,s}],
  \]

\[
[\xi_{i,r+1}, x_{j,s}] - [\xi_{i,r}, x_{j,s+1}] = -m_{i,j}\beta[\xi_{i,r}, x_{j,s}],
\]

\[
[\xi_{0}, x_{j,s}] = \pm a_{i,j}x_{j,s},
\]

\[
(\sum_{\pi \in \Sigma_2} [x_{i,r}, x_{j,s}] = 0) \text{ and } [x_{i,r}, x_{j,s}] = 0 \text{ for } j \neq i, i \pm 1.
\]

- For \(n = 2\), the defining relations are
  \[
  [x_{i,r+1}, x_{i,s}] = [x_{i,r}, x_{i,s+1}],
  \]

\[
[x_{i,r+2}, x_{i,s+1}] - 2[x_{i,r+1}, x_{i+1,s+1}] + [x_{i,r}, x_{i+1,s+2}] = \beta^2[x_{i,r}, x_{i+1,s}],
\]
(y4) \[ [\xi_{i,r+1}, x_{i,s}^\pm] = [\xi_{i,r}, x_{i,s+1}^\pm], \]
\[ [\xi_{i,r+2}, x_{i+1,s}^\pm] - 2[\xi_{i,r+1}, x_{i+1,s+1}^\pm] + [\xi_{i,r}, x_{i+1,s+1}^\pm] = \beta^2 [\xi_{i,r}, x_{i+1,s}^\pm], \]
(y5) \[ [\xi_{i,0}, x_{j,s}^\pm] = \pm a_{i,j} x_{j,s}^\pm, \]
\[ [\xi_{i,1}, x_{i+1,s}^\pm] = \mp 2 x_{i+1,s+1}^\pm, \]
(y6) \[ \sum_{\pi \in \Sigma_3} [x_{i,r_{\pi(1)}}, [x_{i,r_{\pi(2)}}, [x_{i,r_{\pi(3)}}, x_{i+1,s}^\pm]]] = 0. \]

For \( n = 1 \), the defining relations are

(y3) \[ [x_{0,r+2}, x_{0,s}^\pm] - 3[x_{0,r+2}, x_{0,s+1}^\pm] + 3[x_{0,r+1}, x_{0,s+2}^\pm] - [x_{0,r}, x_{0,s+3}^\pm] = \beta^2 ([x_{0,r+1}, x_{0,s}^\pm] - [x_{0,r}, x_{0,s+1}^\pm]), \]
(y4) \[ [\xi_{0,r+3}, x_{0,s}^\pm] - 3[\xi_{0,r+2}, x_{0,s+1}^\pm] + 3[\xi_{0,r+1}, x_{0,s+2}^\pm] - [\xi_{0,r}, x_{0,s+3}^\pm] = \beta^2 ([\xi_{0,r+1}, x_{0,s}^\pm] - [\xi_{0,r}, x_{0,s+1}^\pm]), \]
(y5) \[ [\xi_{0,0}, x_{0,s}^\pm] = 0, \quad [\xi_{0,1}, x_{0,s}^\pm] = 0, \quad [\xi_{0,2}, x_{0,s}^\pm] = \mp 2 \beta^2 x_{0,s}^\pm, \]
(y6) \[ \sum_{\pi \in \Sigma_3} [x_{0,r_{\pi(1)}}, [x_{0,r_{\pi(2)}}, [x_{0,r_{\pi(3)}}, x_{0,s}^\pm]]] = 0. \]

In the above relations \( s, r, r_1, r_2, r_3 \in \mathbb{Z}_+ \) and \( i, j \in [n] \).

**Remark 1.1.** For \( \beta \neq 0 \), the assignment \( \bar{x}_{i,r}^\pm \mapsto \beta^r \bar{x}_{i,r}^\pm, \quad \bar{\xi}_{i,r} \mapsto \beta^r \bar{\xi}_{i,r} \) (with \( i \in [n], r \in \mathbb{Z}_+ \)) provides an isomorphism of Lie algebras \( \bar{y}_{i}^{(n)} \sim \gamma_{i}^{(n)} \).

**1.4. Difference operators on \( \mathbb{C}^\times \).**

For \( t \in \mathbb{C}^\times \), define the algebra of \( t\)-difference operators on \( \mathbb{C}^\times \), denoted by \( \mathfrak{d}_t \), to be the unital associative \( \mathbb{C} \)-algebra generated by \( Z^{\pm 1}, D^{\pm 1} \) with the defining relations

\[ Z^{\pm 1} Z^{-1} = 1, \quad D^{\pm 1} D^{-1} = 1, \quad DZ = t \cdot ZD. \]

Define the associative algebra \( \mathfrak{d}_t^{(n)} := \mathbb{M}_n \otimes \mathfrak{d}_t \), where \( \mathbb{M}_n \) stands for the algebra of \( n \times n \) matrices (so that \( \mathfrak{d}_t^{(n)} \) is the algebra of \( n \times n \) matrices with values in \( \mathfrak{d}_t \)). We will view \( \mathfrak{d}_t^{(n)} \) as a Lie algebra with the natural commutator-Lie bracket \( [\cdot, \cdot] \). It is easy to check that the following formulas define two 2-cocycles \( \phi^{(1)}, \phi^{(2)} \in C^2(\mathfrak{d}_t^{(n)}, \mathbb{C}) \):

\[ \phi^{(1)}(M_1 \otimes D^k Z^l, M_2 \otimes D^k Z^l) = l_1 t^{k_1 l_1} \delta_{k_1, k_2} \delta_{l_1, l_2} tr(M_1 M_2), \]
\[ \phi^{(2)}(M_1 \otimes D^k Z^l, M_2 \otimes D^k Z^l) = k_1 t^{k_1 l_1} \delta_{k_1, k_2} \delta_{l_1, l_2} tr(M_1 M_2) \]

for any \( M_1, M_2 \in \mathbb{M}_n \) and \( k_1, k_2, l_1, l_2 \in \mathbb{Z} \).

This endows \( \mathfrak{d}_t^{(n)} := \mathfrak{d}_t^{(n)} \oplus \mathbb{C} \cdot c_0^{(1)} \oplus \mathbb{C} \cdot c_0^{(2)} \) with the Lie algebra structure via

\[ [X + \lambda_1 c_0^{(1)} + \lambda_2 c_0^{(2)}, Y + \mu_1 c_0^{(1)} + \mu_2 c_0^{(2)}] = XY - YX + \phi^{(1)}(X, Y) c_0^{(1)} + \phi^{(2)}(X, Y) c_0^{(2)} \]

for any \( X, Y \in \mathfrak{d}_t^{(n)} \) and \( \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C} \). We also define a Lie subalgebra \( \mathfrak{d}_t^{(n),0} \subset \mathfrak{d}_t^{(n)} \) via

\[ \mathfrak{d}_t^{(n),0} := \left\{ \sum A_{k,l} D^k Z^l + \lambda_1 c_0^{(1)} + \lambda_2 c_0^{(2)} \in \mathfrak{d}_t^{(n)} | \lambda_1, \lambda_2 \in \mathbb{C}, A_{k,l} \in \mathbb{M}_n, tr(A_{0,0}) = 0 \right\}. \]
1.5. Differential operators on \( \mathbb{C}^\times \).

For \( s \in \mathbb{C} \), define the algebra of \( s \)-differential operators on \( \mathbb{C}^\times \), denoted by \( \mathcal{D}_s \), to be the unital associative \( \mathbb{C} \)-algebra generated by \( \partial, x^{\pm 1} \) with the defining relations
\[
x^{\pm 1}x^{\mp 1} = 1, \quad \partial x = x(\partial + s).
\]
Define the associative algebra \( \mathcal{D}_s^{(n)} := \mathbb{M}_n \otimes \mathcal{D}_s \) (so that \( \mathcal{D}_s^{(1)} \) is the algebra of \( n \times n \) matrices with values in \( \mathcal{D}_s \)). We will view \( \mathcal{D}_s^{(n)} \) as a Lie algebra with the natural commutator-Lie bracket. Following [BKLY, Formula (2.3)], consider a 2-cocycle \( \phi \in C^2(\mathcal{D}_s^{(n)}, \mathbb{C}) \) given by
\[
\phi(M_1 \otimes f_1(\partial)x^{i_1}, M_2 \otimes f_2(\partial)x^{i_2}) = \begin{cases} 
\text{tr}(M_1M_2) \cdot \sum_{a=0}^{l_1-1} f_1(as)f_2((a - l_1)s) & \text{if } l_1 = -l_2 > 0 \\
- \text{tr}(M_1M_2) \cdot \sum_{a=0}^{l_2-1} f_2(as)f_1((a + l_1)s) & \text{if } l_1 = -l_2 < 0 \\
0 & \text{otherwise}
\end{cases}
\]
for arbitrary polynomials \( f_1, f_2 \) and any \( M_1, M_2 \in \mathbb{M}_n \), \( l_1, l_2 \in \mathbb{Z} \).

This endows \( \mathcal{D}_s^{(n)} := \mathcal{D}_s^{(n)} \oplus \mathbb{C} \cdot c_D \) with the Lie algebra structure via
\[
[X + \lambda c_D, Y + \mu c_D] = XY - YX + \phi(X, Y)c_D
\]
for any \( X, Y \in \mathcal{D}_s^{(n)} \) and \( \lambda, \mu \in \mathbb{C} \).

2. Key isomorphisms

2.1. Main results.

Our first main result establishes a relation between the Lie algebras \( \hat{u}_d^{(n)} \) and \( \delta_d^{(1)} \).

**Theorem 2.1.** For \( d \in \mathbb{C}^\times \) not a root of unity (we will denote this by \( d \neq \sqrt{1} \)), the assignment
\[
\bar{c}_{0,k} \mapsto E_{n,1} \otimes D^k Z, \quad \bar{f}_{0,k} \mapsto E_{n,1,\partial} \otimes Z^{-1} D^k, \quad \bar{h}_{0,k} \mapsto E_{n,n} \otimes D^{2k} - d^{nk} E_{1,1} \otimes D^k + \delta_{0,k}c_D^{(1)}, \quad \bar{e} \mapsto e_{(2)},
\]
\[
\bar{c}_{i,k} \mapsto d^{(n-i)k} E_{i,i+1} \otimes D^k, \quad \bar{f}_{i,k} \mapsto d^{(n-i)k} E_{i+1,i} \otimes D^k, \quad \bar{h}_{i,k} \mapsto d^{(n-i)k} (E_{i,i} - E_{i+i,1}) \otimes D^k
\]
(with \( i \in [n]^\times, k \in \mathbb{Z} \)) provides an isomorphism of Lie algebras \( \theta_d^{(n)} : \hat{u}_d^{(n)} \cong \hat{\delta}_d^{(n,0)} \).

Our second main result establishes a relation between the Lie algebras \( \hat{y}_\beta^{(n)} \) and \( \delta_d^{(n)} \).

**Theorem 2.2.** For \( \beta \neq 0 \), the assignment
\[
\bar{x}_{0,r}^+ \mapsto E_{n,1} \otimes \partial^r x, \quad \bar{x}_{0,r}^- \mapsto E_{n,1} \otimes x^{-1}\partial^r, \quad \bar{\xi}_{0,r} \mapsto E_{n,n} \otimes \partial^r - E_{1,1} \otimes (\partial + n\beta)^r + \delta_{0,r}c_D,
\]
\[
\bar{x}_{i,r}^+ \mapsto E_{i,i+1} \otimes (\partial + (n-i)\beta)^r, \quad \bar{x}_{i,r}^- \mapsto E_{i+1,i} \otimes (\partial + (n-i)\beta)^r, \quad \bar{\xi}_{i,r} \mapsto (E_{i,i} - E_{i+i,1}) \otimes (\partial + (n-i)\beta)^r
\]
(with \( i \in [n]^\times, r \in \mathbb{Z}_+ \)) provides an isomorphism of Lie algebras \( \psi_\beta^{(n)} : \hat{y}_\beta^{(n)} \cong \hat{\delta}_d^{(n)} \).

For \( n = 1 \), these isomorphisms have been essentially established in [TB]. In the rest of this section, we adapt arguments from [TB] to prove the above results for \( n \geq 2 \).

**Remark 2.3.** These two theorems played a crucial role in [TB], while their proofs were missing. In the loc. cit., we considered the quotients \( \hat{u}_d^{(n)}/(\bar{c}) \) and \( \hat{\delta}_d^{(n,0)}/(e_{(2)}) \) and had a different 2-cocycle. Nevertheless, Theorem 2.8 from [TB] is equivalent to the above Theorem 2.1.
2.2. Proof of Theorem 2.1

It is straightforward to see that the assignment from Theorem 2.1 preserves all the defining relations (u1–u7.2), hence, it provides a Lie algebra homomorphism $\theta_d^{(n)} : \bar{u}_d^{(n)} \to \bar{d}_d^{(n),0}$. We also consider the induced homomorphism $\bar{\theta}_d^{(n)} : \bar{u}_d^{(n)} \to \bar{d}_d^{(n),0}$, where $\bar{u}_d^{(n)} := \bar{u}_d^{(n)}/(\bar{e}, \sum_i \bar{h}_i, 0)$ is a central quotient of $\bar{u}_d^{(n)}$. Clearly, it suffices to show that $\bar{\theta}_d^{(n)}$ is an isomorphism.

Let $Q$ be the root lattice of $\hat{\mathfrak{sl}}_n$. The Lie algebras $\bar{u}_d^{(n)}$ and $\bar{d}_d^{(n)}$ are $Q \times \mathbb{Z}$-graded via

$$\deg(e_{i,k}) = (\alpha_i; k), \quad \deg(f_{i,k}) = (-\alpha_i; k), \quad \deg(h_{i,k}) = (0; k),$$

$$\deg(E_{i,j} \otimes D^k \mathbb{Z}^q) = (l\delta + (\alpha_1 + \ldots + \alpha_{j-1}) - (\alpha_1 + \ldots + \alpha_{i-1}); k),$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are the simple positive roots of $\hat{\mathfrak{sl}}_n$, while $\delta = \alpha_0 + \ldots + \alpha_{n-1}$ is the minimal positive imaginary root. Note that $\bar{\theta}_d^{(n)}$ is $Q \times \mathbb{Z}$-graded, and it is easy to see that $\bar{\theta}_d^{(n)}$ is surjective for $d \neq \sqrt{1}$. Therefore, it suffices to prove

$$(\dagger) \quad \dim(\bar{u}_d^{(n)})_{(\alpha; k)} \leq \dim(\bar{d}_d^{(n),0})_{(\alpha; k)}$$

for any $\alpha \in Q$, $k \in \mathbb{Z}$. Note that

$$\dim(\bar{d}_d^{(n),0})_{(\alpha; k)} = \begin{cases} 0 & \text{if } \alpha \text{ is nonzero and is not a root of } \hat{\mathfrak{sl}}_n, \\ 1 & \text{if } \alpha \text{ is a real root of } \hat{\mathfrak{sl}}_n, \\ n & \text{if } \alpha \in \mathbb{Z}\delta \text{ and } (\alpha; k) \neq (0; 0), \\ n-1 & \text{if } (\alpha; k) = (0; 0). \end{cases}$$

For $\alpha \notin \mathbb{Z}\delta$, the inequality $(\dagger)$ can be proved analogously to [MRY] Proposition 3.2 by viewing $\bar{u}_d^{(n)}$ as a module over the horizontal subalgebra generated by $\{\bar{e}_i, \bar{f}_i, 0, \bar{h}_i, 0\}_{i \in [n]}$, which is isomorphic to $\hat{\mathfrak{sl}}_n$. Hence, it remains to handle the case $\alpha = l\delta$. The case $l = 0$ is obvious since $(\bar{u}_d^{(n)})_{(0; k)}$ is spanned by $\{\bar{h}_i\}_{i \in [n]}$. For the rest of the proof, we can assume $l \in \mathbb{N}$.

Remark 2.4. For $n > 2$, this step is different from the argument in [YY] Sect. 13, where the authors prove that $u_d^{(n)}$ is the universal central extension of $d_d^{(n),0}$ by showing that the former does not admit non-split central extensions.

Let $\bar{u}_d^{(n)\geq}$ be the subalgebra of $\bar{u}_d^{(n)}$ generated by $\{\bar{e}_{i,k}, \bar{h}_{i,k}\}_{i \in [n], k \geq 0}$. It is isomorphic to an abstract Lie algebra generated by $\{\bar{e}_{i,k}, \bar{h}_{i,k}\}_{i \in [n], k \geq 0}$ subject to the defining relations (u1,u2,u5,u7.1) with $\bar{c} = 0$ and $\sum_i \bar{h}_{i,0} = 0$. It suffices to show that $\dim(\bar{u}_d^{(n)\geq})_{(l\delta; k)} \leq n$ for any $l \in \mathbb{N}, k \in \mathbb{Z}$.

Introduce the length $N$ commutator: $[a_1; a_2; \ldots; a_{N-1}; a_N]_N := [a_1, [a_2, \ldots, [a_{N-1}, a_N] \ldots]]$.

We say that this commutator starts from $a_1$. The degree $(l\delta; k)$ subspace of $\bar{u}_d^{(n)\geq}$ is spanned by length $ln$ commutators $[\bar{e}_{i_1,k_1}; \ldots; \bar{e}_{i_n,k_n}]_n$ such that $k_1 + \ldots + k_n = k$ and $\alpha_{i_1} + \ldots + \alpha_{i_n} = l\delta$. Define $v_{a,b}^{(i,l)} := [\bar{e}_{i_1,a_1}, \bar{e}_{i_1+1,b}; \ldots; \bar{e}_{i-1,b}]_{n-1}$. Note that $v_{a,b}^{(i,l)} \in (\bar{u}_d^{(n)\geq})_{(l\delta; a+b)}$ and $v_{a,b}^{(i,l)} \neq 0$ since $\bar{\theta}_d^{(n)}(v_{a,b}^{(i,l)}) \neq 0$. Combining this with $\dim(v_{a,b}^{(i,l)})_{(l\delta-a-\alpha_{k-k_1})} \leq 1$, we see that $(\bar{u}_d^{(n)\geq})_{(l\delta; k)}$ is spanned by $v_{a,k-a}^{(i,l)}$. It remains to show that the rank of this system is at most $n$.

- Case $k = 0$.

Define $v_1 := v_{0,(1)}^{(1,l)}$, $\ldots$, $v_{n-1} := v_{0,(n-1)}^{(1,l)}$, $v_n := v_{1,0}^{(0,1)}$ and set $V(l; 0) := \text{span}_\mathbb{C}(v_1, \ldots, v_n)$. We prove $v_{a,b}^{(i,l)} \in V(l; 0)$ for all $i \in [n], a \in \mathbb{Z}$ by induction on $|a|$. The case $a = 0$ follows from

$$(\diamond) \quad v_{0,0}^{(0,1)} + v_{0,0}^{(1,1)} + \ldots + v_{0,0}^{(n-1,1)} = 0,$$

1 The argument in the loc. cit. used the extra relation $[\bar{e}_{j,a}, \bar{e}_{j,b}] = 0$ for any $j \in [n], a, b \in \mathbb{Z}$ (with $n > 1$). However, this relation is a simple consequence of (u2).
which is obvious once the horizontal subalgebra of $\mathfrak{l}_d^{(n)}$ is identified with $\mathfrak{sl}_n[Z, Z^{-1}]$.

To proceed further, we need the following technical result based on non-degeneracy of the matrices $(a_{i,j}d^{km,j})_{i}^{n} (n > 2)$ and $(2\delta_{i,j} - (d^k + d^{-k})\delta_{i,j+1})_{i}^{n}$ for any $d \neq \sqrt{1}, k \neq 0$.

**Lemma 2.5.** For any fixed $i \in [n], k \neq 0$, there exists an element $\tilde{h}_{i,k}^{l} \in \text{span}_{\mathbb{C}}(\tilde{h}_{0,k}, \ldots, \tilde{h}_{n-1,k})$ such that $[\tilde{h}_{i,k}^{l}, \tilde{e}_{j,l}] = \delta_{i,j} \tilde{e}_{j,l+1}$ for all $j \in [n], l \in \mathbb{Z}$.

First, we prove $v_{-1,1}^{(l)} \in V(l; 0)$. Applying $\text{ad}(\tilde{h}_{i-1}^{l}) \text{ad}(\tilde{h}_{0,1}^{l})$ to the equality (2), we get a sum of $l^2 n$ length $ln$ commutators being zero. Among those, $l^2 n - l + \delta_{i,0}$ belong to $V(l; 0)$ as they start either from $\tilde{e}_{i',0}$ ($i' \in [n]$) or $\tilde{e}_{0,1}$. The remaining $l - \delta_{i,0}$ commutators start from $\tilde{e}_{i-1}$ and therefore are multiples of $v_{-1,1}^{(l)}$. It remains to show that the sum of these $l - \delta_{i,0}$ terms is nonzero. For the latter, it suffices to verify that the image of this sum under $\tilde{g}_{d}^{(n)}$ is nonzero, which is a straightforward computation based on the assumption $d \neq \sqrt{1}$. To prove $v_{1,1}^{(l)} \in V(l; 0)$, we apply $\text{ad}(\tilde{h}_{i,1}^{l}) \text{ad}(\tilde{h}_{i+1,1}^{l})$ to (2) and follow the same arguments.

To perform the inductive step, we assume that $v_{i,a}^{(l)} \in V(l; 0)$ for all $i \in [n], |a| \leq N$ and we shall prove $v_{i,a}^{(l)} \in \text{span}_{\mathbb{C}}(\tilde{h}_{i,a}^{l} + 2, \ldots, \tilde{h}_{i,a}^{l+N}) \in V(l; 0)$. Applying $\text{ad}(\tilde{h}_{i,a}^{l+N}) \text{ad}(\tilde{h}_{i-1,a}^{l+N}) \text{ad}(\tilde{h}_{i,a}^{l+1}) \text{ad}(\tilde{h}_{i,a}^{l})$ to (2), we get a sum of $l^3 n$ length $ln$ commutators being zero. By the induction hypothesis, all of them, except for those starting from $\tilde{e}_{i,\pm(N+1)}$, belong to $V(l; 0)$. The remaining $l - \delta_{i,0}$ terms are multiples of $v_{i,a}^{(l)} \in \text{span}_{\mathbb{C}}(\tilde{h}_{i,a}^{l+N})$. For $(n, l) \neq (2, 1)$, it is easy to see that the sum of their images under $\tilde{g}_{d}^{(n)}$ is nonzero, implying $v_{i,a}^{(l)} \in V(l; 0)$. In the remaining case $(n, l) = (2, 1)$, the inclusion $v_{i,a}^{(l)} \in V(l; 0)$ follows from the relation (n2).

This completes our induction step. Hence, $(\tilde{g}_{d}^{(n)}; \tau)_{(l;0)} = V(l; 0) \Rightarrow \dim(\tilde{g}_{d}^{(n)}; \tau)_{(l;0)} \leq n$.

**Case 0 < k < l.**

Define $v_{\mathfrak{a}} := v_{\mathfrak{a},k}^{(l)}$, , $v_{n-1} := v_{0,k}^{(l)}$, $v_{n} := v_{0,k}^{(0,l)}$ and set $V(l; k) := \text{span}_{\mathbb{C}}(v_{1}, \ldots, v_{n})$. We claim that $v_{i,a}^{(l)} \in V(l; k)$ for any $i \in [n], a \in \mathbb{Z}$. We will prove this in three steps.

**Step 1:** Proof of $v_{k,0}^{(l)} \in V(l; k)$ for any $i \in [n]$.

Applying $\text{ad}(\tilde{h}_{k}^{l})$ to the equality (2), we immediately get $v_{k,0}^{(l)} \in V(l; k)$.

**Step 2:** Proof of $v_{a,k}^{(l)} \in V(l; k)$ for any $i \in [n], 0 < a < k$.

It is known that any degree $k$ symmetric polynomial in $\{x_j\}_{j=1}^{l}$ is a polynomial in $\{x_j^{k}\}_{j=1}^{l}$.

Choose $P_{i,l}$ such that $\text{Sym}(x_1x_2 \cdots x_k) = P_{i,l}(\sum_j x_j, \ldots, \sum_k x_j^k)$. Define $L_{i,k,l} \in \text{End}(\tilde{g}_{d}^{(n)}; \tau)$ via $L_{i,k,l} = P_{i,l}(\text{ad}(\tilde{h}_{i,k}^{l}), \ldots, \text{ad}(\tilde{h}_{i,k}^{l}))$. Applying $L_{i,k,l}$ to the equality (2), we get a sum of $l^3 n$ length $ln$ commutators being zero. Each of these terms starts either from $\tilde{e}_{i',0}$ ($i' \in [n]$) or $\tilde{e}_{i,1}$. In the former case the commutator belongs to $V(l; k)$, while in the latter case the commutator is a multiple of $v_{i,k-1}^{(l)}$. There are $\binom{l}{k-1}$ terms starting from $\tilde{e}_{i,1}$ and the sum of their images under $\tilde{g}_{d}^{(n)}$ is nonzero. Therefore, $v_{i,k-1}^{(l)} \in V(l; k)$.

Applying the same arguments to the symmetric function $\text{Sym}(x_1^a x_2 \cdots x_{k-a+1})$, we analogously get $v_{i,a,k}^{(l)} \in V(l; k)$ for any $i \in [n], 0 < a < k$.

**Step 3:** Proof of $v_{a,k}^{(l)} \in V(l; k)$ for any $i \in [n]$ and $a \notin \{0, 1, \ldots, k\}$.

We prove $v_{-N,k+N}^{(l)} \in V(l; k)$ for all $i \in [n], N \in \mathbb{Z}$ by induction on $N$. The case $N = 0$ is clear. Assume $v_{a,k}^{(l)} \in V(l; k)$ for any $i \in [n], -N < a < k + N$. Applying $\text{ad}(\tilde{h}_{a-N}^{l}) \text{ad}(\tilde{h}_{a+1,k+N})$ to (2), we get a sum of $l^3 n$ length $ln$ commutators being zero. Each of these terms either belongs to $V(l; k)$ by the induction hypothesis or is a multiple
of \(v^{(1,l)}_{-1,k+N+1}\). There are \(l(l - \delta_{n,2})\) summands of the latter form and the sum of their images under \(\tilde{u}_{d}^{(n)}\) is nonzero. This implies \(v^{(1,l)}_{-1,k+N+1} \in V(l;k)\).

To prove \(v^{(1,l)}_{-1,N+1,1} \in V(l;k)\), we apply \(\text{ad}(\hat{h}'_{k,N+1})\ \text{ad}(\hat{h}'_{k+2,1})\ \text{ad}(\hat{h}'_{1,N})\) to \(\bigotimes\) and follow the same arguments.

- **Case of an arbitrary** \(k\).

It is clear that \(L_{\kappa,l,l}^{}\) induces an isomorphism \((\tilde{u}_{d}^{(n),\geq})_{(\delta;\kappa')} \rightarrow (\tilde{u}_{d}^{(n),\geq})_{(\delta;\kappa'+1)}\) for any \(\kappa' \in \mathbb{Z}\). In particular, \(\dim((\tilde{u}_{d}^{(n),\geq})_{(\delta;\kappa)}) = \dim((\tilde{u}_{d}^{(n),\geq})_{(\delta;\kappa+k+l)}) \leq n\), due to the previous two cases.

### 2.3. Proof of Lemma \([2,2]\)

It is straightforward to see that the assignment from Theorem \([2,2]\) preserves all the defining relations \((y_1 - y_6)\), hence, it provides a Lie algebra homomorphism \(\tilde{\varphi}_{\beta}^{(n)} : \tilde{y}_{\beta}^{(n)} \rightarrow \tilde{\delta}_{\beta}^{(n)}\). We also consider the induced homomorphism \(\tilde{\varphi}_{\beta}^{(n)} : \tilde{y}_{\beta}^{(n)} \rightarrow \tilde{\delta}_{\beta}^{(n)}\), where \(\tilde{y}_{\beta}^{(n)} := \tilde{y}_{\beta}^{(n)}/(\sum_{i} \xi_{i,0})\) is a central quotient of \(\tilde{y}_{\beta}^{(n)}\). Clearly, it suffices to show that \(\tilde{y}_{\beta}^{(n)}\) is an isomorphism.

The Lie algebra \(\tilde{\delta}_{\beta}^{(n)}\) is \(Q\)-graded via \(\deg_{1}(\tilde{x}_{i,r}^{+}) = \pm \alpha_{i}\), \(\deg_{1}(\tilde{x}_{i,r}^{-}) = 0\) and \(\mathbb{Z}_{+}\)-filtered as a quotient of the free Lie algebra on \(\{\tilde{x}_{i,r}^{+} : \xi_{i,r} \in [n]\}\) graded via \(\deg_{2}(\tilde{x}_{i,r}^{+}) = r\), \(\deg_{2}(\tilde{x}_{i,r}^{-}) = r\). The Lie algebra \(\tilde{\delta}_{\beta}^{(n)}\) is also \(Q\)-graded via \(\deg_{1}(E_{i,j} \otimes \partial^{x}) = l\delta + (\alpha_{1} + \ldots + \alpha_{j-1}) - (\alpha_{1} + \ldots + \alpha_{i-1})\) and \(\mathbb{Z}_{+}\)-filtered with the filtration \(\leq k\) subspace consisting of the finite sums \(\sum_{d \in \mathbb{Z}} A_{i,j} \partial^{x_{d}}\), where \(A_{i,j} \in \mathcal{M}_{n}\) and \(tr(A_{i,j}) = 0\) for any \(j \in \mathbb{Z}\). Let \((\tilde{u}_{d}^{(n)})_{(\alpha;\leq k)}\) and \((\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k)}\) denote the subspaces of \(\tilde{u}_{d}^{(n)}\) and \(\tilde{\delta}_{\beta}^{(n)}\), respectively, consisting of the degree \(\alpha\) and filtration \(\leq k\) elements.

Note that \((\tilde{u}_{d}^{(n)})_{(\alpha;\leq k)} \subset (\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k)}\) for any \(\alpha \in Q, k \in \mathbb{Z}_{+}\). Hence, we get linear maps \(\tilde{u}_{d;\alpha,k}^{(n)} : (\tilde{u}_{d}^{(n)})_{(\alpha;\leq k)}/(\tilde{u}_{d}^{(n)})_{(\alpha;\leq k-1)} \rightarrow (\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k)}/(\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k-1)}\). We claim that all the maps \(\tilde{u}_{d;\alpha,k}^{(n)}\) are isomorphisms. To prove this, it suffices to show that \(\tilde{u}_{d;\alpha,k}^{(n)}\) is surjective and

\[
\dim((\tilde{u}_{d}^{(n)})_{(\alpha;\leq k)}) - \dim((\tilde{u}_{d}^{(n)})_{(\alpha;\leq k-1)}) \leq \dim((\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k)}) - \dim((\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k-1)})
\]

for any \(\alpha \in Q, k \in \mathbb{Z}_{+}\). The right-hand side of (1) can be simplified as follows:

\[
\dim((\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k)}) - \dim((\tilde{\delta}_{\beta}^{(n)})_{(\alpha;\leq k-1)}) = \begin{cases} 
0 & \text{if } \alpha \text{ is nonzero and is not a root of } \tilde{u}_{\alpha} \\
1 & \text{if } \alpha \text{ is a real root of } \tilde{u}_{\alpha} \\
-\delta_{k,0} & \text{if } \alpha \text{ is an imaginary root or zero}
\end{cases}
\]

For \(\alpha \notin \mathbb{Z} \delta\), the inequality (1) and the surjectivity of \(\tilde{u}_{d;\alpha,k}^{(n)}\) can be deduced in the same way as (1). Hence, it remains to handle the case \(\alpha = l \delta\). The \(l = 0\) case is obvious since the degree 0 subspace of \(\tilde{u}_{d}^{(n)}\) is spanned by \(\xi_{i,r}\). For the rest of the proof, we can assume \(l \in \mathbb{N}\).

Let \(\tilde{u}_{d}^{(n),\geq}\) be the subalgebra of \(\tilde{u}_{d}^{(n)}\) generated by \(\{\tilde{x}_{i,r}^{+}, \xi_{i,r}\}_{r \in \mathbb{Z}_{+}}\). It is isomorphic to an abstract Lie algebra generated by \(\{\tilde{x}_{i,r}, \xi_{i,r}\}_{r \in \mathbb{Z}_{+}}\) subject to the defining relations \((y_{1}, y_{3}, y_{4}, y_{5}, y_{6})\) and \(\sum_{i} \xi_{i,0} = 0\). It suffices to show that \(\dim((\tilde{u}_{d}^{(n),\geq})_{(\delta;\leq k)}) - \dim((\tilde{u}_{d}^{(n),\geq})_{(\delta;\leq k-1)}) \leq n - \delta_{k,0}\) and \(\tilde{u}_{d;\alpha,k}^{(n)}\) is surjective for any \(l \in \mathbb{N}, k \in \mathbb{Z}_{+}\).

**Case** \(n = 2\).

The degree \(l \delta\) subspace of \(\tilde{u}_{d}^{(2),\geq}\) is spanned by all length 2l commutators \([\tilde{x}_{i,a}^{+}, \tilde{x}_{i+1,b}^{+}]_{2l}\) such that \(\alpha_{i_1} + \ldots + \alpha_{i_2} = l \delta\). Define \(w_{a,b}^{(l)} := [\tilde{x}_{i,a}^{+}, \tilde{x}_{i+1,0}^{+}, \ldots, \tilde{x}_{i,0}^{+}, \tilde{x}_{i+1,b}^{+}]_{2l}\) for any \(i \in [2]\)
and $a,b \in \mathbb{Z}_+$. Due to our description of the degree $l\delta - \alpha_i$ subspace of $\overline{\mathfrak{g}}^{(2)}_{\beta}$, we see that $(\overline{\mathfrak{g}}^{(2)}_{\beta})_{l\delta}$ is spanned by $\{u_{a,b}^{(i)}\}_{a,b \in \mathbb{Z}_+}$. Moreover, $(\overline{\mathfrak{g}}^{(2)}_{\beta})_{(l\delta \leq k)}$ is spanned by $\{u_{a,b}^{(i)}\}_{a,b \in \mathbb{Z}_+}$ with $a + b \leq k$. Therefore, the inequality $\dim(\overline{\mathfrak{g}}^{(2)}_{\beta})_{(l\delta \leq k)} - \dim(\overline{\mathfrak{g}}^{(2)}_{\beta})_{(l\delta \leq k-1)} \leq 2 - \delta_{k,0}$ and the surjectivity of $\overline{\beta}_{3,l;0,k}$ follow from our next result:

**Proposition 2.6.** Define $W(l; N) := \text{span}_{\mathbb{C}}\{w_{0,M}^{(i)}\}_{i \in [2], l \in \mathbb{N}, N \in \mathbb{Z}_+}$.

(a) We have $w_{a,b}^{(i)} \in W(l; a + b)$ for any $i \in [2], l \in \mathbb{N}, a, b \in \mathbb{Z}_+$.

(b) The images of $\{\overline{\mathfrak{g}}^{(2)}_{\beta}(w_{0,M}^{(i)})\}_{i \in [2]}$ in the quotient space $(\mathfrak{g}^{(2)}_{\beta})_{(l\delta \leq N)} / (\mathfrak{g}^{(2)}_{\beta})_{(l\delta \leq N-1)}$ are linearly independent for any $l, N \in \mathbb{N}$.

**Proof of Proposition 2.6**

(a) Our proof is based on the following simple equalities:

\[
\sum_{i \in [2]} w_{0,0}^{(i)} = 0, \tag{1}
\]

\[
[H_3, \bar{x}_{i,r}^+] = \bar{x}_{i,r+1}^+, \quad [H_4, \bar{x}_{i,r}^+] = \bar{x}_{i,r+2}^+, \tag{2}
\]

where $H_3 := \frac{1}{\beta} \sum_{i \in [2]} \tilde{c}_{i,3}$, $H_4 := \frac{1}{\beta} \sum_{i \in [2]} \tilde{c}_{i,4} + \frac{1}{2} \sum_{i \in [2]} \tilde{c}_{i,2}$.

\[\circ \text{ Proof of } w_{1,b}^{(i)} \in W(l; 1 + b)\]

We prove this by induction on $b$. Applying $\text{ad}(H_3)$ or $\text{ad}(\bar{\xi}_{i,1})$ to $[1]$, we get $w_{0,0}^{(i)} + w_{1,0}^{(i)} \in W(l; 1)$ and $w_{0,0}^{(i)} + w_{1,0}^{(i)+1} \in W(l; 1)$, respectively. Hence, $w_{1,0}^{(i)} \in W(l; 1)$, which is the basis of induction. To perform the inductive step, we assume $w_{1,b}^{(i)} \in W(l; 1 + b)$ for any $0 \leq b \leq M$.

In particular, $w_{1,M}^{(i)} = \sum_{N \leq M+1} c_{j,N} w_{0,N}^{(j,i)}$ for some $c_{j,N} \in \mathbb{C}$. Applying $\overline{\beta}^{(2)}_{\beta}$ to this equality, we find $c_{l,M+1} = \frac{M+1}{M+1} - c_{l+1,M+1} = \frac{1}{M+1}$. Hence $w_{1,M}^{(i)} = \frac{M+1}{M+1} - w_{0,M+1}^{(i)} + \frac{1}{M+1} w_{0,M+1}^{(i)+1} \in W(l; M)$.

Applying $\text{ad}(H_3 + \frac{1}{\beta} \xi_{i,1})$ to this inclusion, we get $lw_{0,M+1}^{(i)} + \frac{1}{M+1} w_{0,M+1}^{(i)+1} \in W(l; M+2)$. For $M > 0$, this yields $w_{1,M+1}^{(i)} \in W(l; M+2)$ as $w_{1,M+1}^{(i)} \in \text{span}_{\mathbb{C}}\{w_{0,M+1}^{(j,i)}, w_{0,M+1}^{(j+1,i)}\}_{i \in [2]}$.

It remains to treat separately the case $M = 0$. We can assume $l > 1$ as the case $l = 1$ is simple. Applying $\text{ad}(H_3 + \frac{1}{\beta} \xi_{i,1})$ to the inclusion $w_{1,0}^{(i)} + (l - 1) w_{0,0}^{(i)} + lw_{0,1}^{(i)} \in W(l; 0)$, we get $2(l - 1) w_{1,1}^{(i)} + (l - 1) w_{2,0}^{(i)} \in W(l; 2)$. On the other hand, applying $\text{ad}(H_4 + \frac{1}{\beta} \xi_{i,2})$ to $[1]$, we find $w_{2,0}^{(i)} \in W(l; 2)$. This implies $w_{1,1}^{(i)} \in W(l; 2)$.

\[\circ \text{ Proof of } w_{a,b}^{(i)} \in W(l; a + b) \text{ for } a > 1\]

We prove this by induction on $a$. The base cases of induction $a = 0, 1$ have already been treated. To perform the inductive step, we assume $w_{a,b}^{(i)} \in W(l; a + b)$ for all $0 \leq a \leq M$ and $b \in \mathbb{Z}_+$. In particular, $w_{a,b}^{(i)} = \sum_{j \in [2]} d_{j,N} w_{0,N}^{(j,i)}$ for some $d_{j,N} \in \mathbb{C}$. Applying $\text{ad}(H_3)$ to this equality and using the induction hypothesis, we immediately get $w_{a,b}^{(i)} \in W(l; M + b + 1)$.

(b) Straightforward computations yield $\overline{\beta}^{(2)}_{\beta}(w_{0,N}^{(1)}) = 2^{N-1}(E_{1,1} - E_{2,2}) \otimes \partial^N x^l + \text{l.o.t., } \overline{\beta}^{(2)}_{\beta}(w_{0,N}^{(0)}) + w_{0,N}^{(1)}) = -2^{l-1}N \beta \cdot (E_{1,1} + E_{2,2}) \otimes \partial^{N-1} x^l + \text{l.o.t.}$, where l.o.t. denote summands with lower power of $\partial$. The result follows.

This completes our proof of Theorem [2.2] for $n = 2$.

Case $n > 2$.

The proof for $n > 2$ is completely analogous and crucially uses the same equalities [1] and [2]: we leave details to the interested reader. ■
3. Consequences

3.1. Classical limits of the vertical and horizontal quantum affine \( \mathfrak{gl}_n \).

For \( n \geq 2 \), the algebra \( \mathcal{U}_q^{(n)} \) contains two subalgebras \( \hat{U}_q^v \), \( \hat{U}_q^h \) isomorphic to the quantum affine \( \hat{U}_q(\mathfrak{sl}_n) \). Here \( \hat{U}_q^v \) is generated by \( \{ c_{i,k}, f_{i,k}, b_{i,k}, c \}_{k \in \mathbb{Z}}, \) while \( \hat{U}_q^h \) is generated by \( \{ e_{i,0}, f_{i,0}, h_{i,0} \}_{i \in [n]} \). The following result is obvious.

**Lemma 3.1.** For \( d \neq \sqrt{1} \), the isomorphism \( \theta^{(n)}_d \) identifies the \( q \to 1 \) limits of the subalgebras \( \hat{U}_q^v \) and \( \hat{U}_q^h \) with the universal enveloping algebras of \( \mathfrak{sl}_n[D, D^{-1}] \oplus \mathbb{C} \cdot c^{(2)} \) and \( \mathfrak{sl}_n[Z, Z^{-1}] \oplus \mathbb{C} \cdot c^{(1)} \), respectively.

According to [FJMM], the algebra \( \mathcal{U}_q^{(n)} \) also contains two Heisenberg subalgebras \( \mathfrak{h}^v \) and \( \mathfrak{h}^h \), which commute with \( \hat{U}_q^v \) and \( \hat{U}_q^h \), respectively. This yields two copies of the quantum affine \( U_q(\mathfrak{gl}_n) \) inside \( \mathcal{U}_q^{(n)} \), which will be denoted by \( \hat{U}_q^v \) and \( \hat{U}_q^h \), respectively.

**Lemma 3.2.** For \( d \neq \sqrt{1} \), the isomorphism \( \theta^{(n)}_d \) identifies the \( q \to 1 \) limits of the subalgebras \( \hat{U}_q^v \) and \( \hat{U}_q^h \) with the universal enveloping algebras of \( \mathfrak{gl}_n[D, D^{-1}] \oplus \mathbb{C} \cdot c^{(2)} \) and \( \mathfrak{gl}_n[Z, Z^{-1}] \oplus \mathbb{C} \cdot c^{(1)} \), where \( \mathfrak{gl}_n[D, D^{-1}] := \mathfrak{sl}_n \otimes 1 \oplus \bigoplus_{k \neq 0} \mathfrak{gl}_n \otimes D^k \) and \( \mathfrak{gl}_n[Z, Z^{-1}] := \mathfrak{sl}_n \otimes 1 \oplus \bigoplus_{k \neq 0} \mathfrak{gl}_n \otimes Z^k \).

**Proof of Lemma 3.2.**

(i) First, we recall the construction of \( \mathfrak{h}^v \) from [FJMM] Sect. 2.2. For any \( k \neq 0 \) and \( i,j \in [n] \), define the constants \( b_n(i,j;k) := \left\{ \begin{array}{ll} d^{-k}\frac{d^{-km_{i,j}}2^{k_{ai,j}}-q^{k_{ai,j}}}{k_{ai,j}(q^k-1)} & \text{if } n > 2 \vspace{1mm} \\
\delta_{i,j} \cdot a_{i,j}d^{-k_{ai,j}} & \text{if } n = 2 \end{array} \right. \) so that their \( q \to 1 \) limits are equal to \( \tilde{b}_n(i,j;k) = \left\{ \begin{array}{ll} d^{-k}\frac{d^{-km_{i,j}}2^{k_{ai,j}}-q^{k_{ai,j}}}{k_{ai,j}(q^k-1)} & \text{if } n > 2 \vspace{1mm} \\
\delta_{i,j} \cdot a_{i,j}d^{-k_{ai,j}} & \text{if } n = 2 \end{array} \right. \) for any fixed \( k \neq 0 \), let \( \{ c_{i,k} \}_{i \in [n]} \) be a unique solution of the system \( \sum_{i \in [n]} b_n(i,j;k) c_{i,k} = 0 \) for all \( j \in [n]^\times \) with \( c_{0,k} = 1 \). By construction, the subalgebra \( \mathfrak{h}^v \) is generated by \( q^{1/2} \) and the elements \( \{ h^v_k := \sum_{j \in [n]} c_{i,k} h_{i,j} \}_{k \neq 0} \). The image of the \( q \to 1 \) limit of \( h^v_k \) under \( \theta^{(n)}_d \) equals

\[
H^v_k = \left( \tilde{c}_{0,k}(E_{i,j} - E_{i+1,j+1}) + \sum_{i=1}^{n-1} \tilde{c}_{i,k} d^{(n-i)k} (E_{i,i} - E_{i+1,i+1}) \right) \otimes D^k,
\]

where the constants \( \{ \tilde{c}_{i,k} \} \) satisfy \( \sum_{i \in [n]} \tilde{b}_n(i,j;k) \tilde{c}_{i,k} = 0 \) for all \( j \in [n]^\times \) and \( \tilde{c}_{0,k} = 1 \). Hence, \( H^v_k = \frac{1 - q^{kn}}{n} \cdot I_n \otimes D^k \) with \( I_n = \sum_{j=1}^n E_{j,j} \). It remains to notice that the Lie subalgebra of \( \hat{U}_q^{(n),0} \) generated by \( \mathfrak{sl}_n[D, D^{-1}] \oplus \mathbb{C} \cdot c^{(2)} \) and \( \{ I_n \otimes D^k \}_{k \neq 0} \) is exactly \( \mathfrak{gl}_n[D, D^{-1}] \oplus \mathbb{C} \cdot c^{(2)} \).

(ii) According to [FJMM], \( \hat{U}_q^h \) is a preimage of \( \hat{U}_q^v \) under the Miki’s automorphism \( \varpi \). Combining (i) with Lemma 3.4 below, we get the description of \( \theta^{(n)}_d(q \to 1 \text{ limit of } \hat{U}_q^h) \). \( \square \)

3.2. Classical limit of the Miki’s automorphism.

The natural ‘90 degree rotation’ automorphism of \( \mathcal{U}_q^{(1)} \) (due to Burban–Schiffmann) admits a generalization to the case of \( \mathcal{U}_q^{(n)} \) with \( n \geq 2 \) (due to Miki).

**Theorem 3.3.** [M] For \( n \geq 2 \), there exists an automorphism \( \varpi \) of \( \mathcal{U}_q^{(n)} \) such that

\[
\varpi(U_q^v) = U_q^h, \quad \varpi(U_q^h) = U_q^v, \quad \varpi(c) = - \sum_{i \in [n]} h_{i,0}, \quad \varpi(\sum_{i \in [n]} h_{i,0}) = c.
\]
Our next result provides a description of the $q \to 1$ limit of $\varpi$, denoted by $\varpi$, viewed as an automorphism of the universal enveloping algebra $U(\mathfrak{d}_{d_n}^{(n),0})$.

**Lemma 3.4.** $\varpi$ is induced by an automorphism of the Lie algebra $\mathfrak{d}_{d_n}^{(n),0}$ defined via

\[(\ast) \quad c_0^{(1)} \mapsto c_0^{(2)}, \quad c_0^{(2)} \mapsto -c_0^{(1)}, \quad A \otimes D^k Z^l \mapsto d^{-nk}(-d)^n A \otimes Z^{-k} D^l, \quad \forall A \in \mathbb{M}_n, \; k, l \in \mathbb{Z}.
\]

**Proof of Lemma 3.4.** It is easy to see that the formulas $(\ast)$ define a Lie algebra automorphism; we denote its restriction to $\mathfrak{d}_{d_n}^{(n),0}$ by $\varpi$. On the other hand, the action of $\varpi$ on the generators $\{e_i, 0, f_i, 0, h_i, \pm 1\}_{i \in [n]}$ was computed in [12] Proposition 1.4. Taking the $q \to 1$ limit in these formulas, we get

\[
\varpi : E_{i,i+1} \to E_{i,i+1} \otimes 1, \quad E_{i+1,i} \otimes 1 \to E_{i+1,i} \otimes 1,
\]

\[
\varpi : E_{n,1} \otimes Z \to (-d)^n E_{n,1} \otimes D, \quad E_{1,n} \otimes Z^{-1} \to (-d)^n E_{1,n} \otimes D^{-1},
\]

\[
\varpi : (E_{i,i} - E_{i+1,i+1}) \otimes D^\pm \mapsto d^{\mp n}(E_{i,i} - E_{i+1,i+1}) \otimes Z^{\mp 1}
\]

for all $1 \leq i \leq n - 1$. Therefore, images of the elements

\[
E_{i,i+1} \otimes 1, \quad E_{i+1,i} \otimes 1, \quad E_{n,1} \otimes Z, \quad E_{1,n} \otimes Z^{-1}, \quad (E_{i,i} - E_{i+1,i+1}) \otimes D^\pm, \quad c_0^{(1)}, \quad c_0^{(2)}
\]

under $\varpi$ and $\varpi$ coincide. This completes our proof, since these elements generate $\mathfrak{d}_{d_n}^{(n),0}$.

### 3.3. Classical limit of the commutative subalgebras $A(\bar{s})$

Let $\mathcal{U}_{d_n}^{(n),+}$ be the subalgebra of $\mathcal{U}_{d_n}^{(n),0}$ generated by $\{e_{i,k}\}_{i \in [n]}$. In [FT], we introduced certain \"large\" commutative subalgebras $A(\bar{s})$ of $\mathcal{U}_{d_n}^{(n),+}$ via the shuffle realization $\Psi : \mathcal{U}_{d_n}^{(n),+} \to \mathcal{S}$. We refer the interested reader to [FT] for a definition of the shuffle algebra $\mathcal{S}$ and its subalgebras $A(\bar{s})$, where $\bar{s} = (s_0, s_1, \ldots, s_{n-1}) \in (\mathbb{C}^\times)^{[n]}$ satisfy $s_0 s_1 \cdots s_{n-1} = 1$ and are generic. Let $\text{diag}_n \subset \mathbb{M}_n$ be the subspace of diagonal matrices.

**Proposition 3.5.** For $d \neq \sqrt{T}$ and a generic $\bar{s} = (s_0, \ldots, s_{n-1})$ satisfying $s_0 \cdots s_{n-1} = 1$, the isomorphism $\theta_d^{(n)}$ identifies the $q \to 1$ limit of $A(\bar{s})$ with the universal enveloping algebra of the commutative Lie subalgebra $\bigoplus_{k>0} \text{diag}_n \otimes Z^k$ of $\mathfrak{d}_{d_n}^{(n),0}$.

**Proof of Proposition 3.5.**

According to the main result [FT] Theorem 3.3.], the algebra $A(\bar{s})$ is a polynomial algebra in the generators $\{F_{i,k} \}_{0 \leq i \leq n-1, \; k \in \mathbb{Z}}$, where $F_{i,k}$ is the coefficient of $(-\mu)^{n-i}$ in $F^K_k(\bar{s})$ defined via

\[
F^K_k(\bar{s}) := \prod_{i \in [n]} \prod_{1 \leq i \neq j \leq k} (x_{i,j} - q^{-2} x_{i,j}) \cdot \prod_{i \in [n]} (s_0 \cdots s_i x_{i,j} - \mu \sum_{j=1}^k x_{i,j} - x_{i+1,j})' \in S_k \bar{s}.
\]

First, we compute the $q \to 1$ limit of $A(\bar{s})$. Choose $\beta_1 \in \mathbb{Z}$ such that the $q \to 1$ limit of $(q-1)^{\beta_1} F_{0,1}^\mu$ is well-defined and is non-zero. Define $F_{i,1} := (q-1)^{\beta_1} F_{i,1}^\mu$ and let $F_{i,1}$ denote the $q \to 1$ limit of $F_{i,1}$ (if it exists). According to [FT] Corollary 3.12, the element $F_{0,1}$ is a non-zero multiple of the first generator $h_i^{(1)}$ of the Heisenberg subalgebra $h^{(1)}$. Combining this with Lemmas 3.2 and 3.3, we see that $\theta_d^{(n)}(F_{0,1}) = \mu_1 : I_n \otimes Z$ for some $\mu_1 \in \mathbb{C}^\times$.

For $1 \leq i \leq n$, define $a_i := s_0 \cdots s_{i-1} \in \mathbb{C}^\times$, $A_i(d) := \sum_{j=1}^n d^{1-n d_j} E_{j,j} \in \mathbb{M}_n$, and let $e_i(y_1, \ldots, y_n)$ be the $i$th elementary symmetric function in the variables $\{y_j\}_{j=1}^n$.

**Lemma 3.6.** (a) The limit $\bar{F}_{i,1}$ is well-defined and $\bar{F}_{i,1} := \mu_1 e_i(a_1 A_1(d), \ldots, a_n A_n(d)) \otimes Z$.
(b) The limits $\{\bar{F}_{i,1} \}_{i=0}^{n-1}$ are linearly independent and $\{\theta_d^{(n)}(\bar{F}_{i,1})\}_{i=0}^{n-1}$ span $\text{diag}_n \otimes Z$.

\[\text{According to [FT] Lemma 3.4, we have } \beta_1 = n - 1.\]
Proof of Lemma 3.7

(a) It suffices to show that the image of the $q \rightarrow 1$ limit of $\frac{\prod_{i,j=1}^{k} F_{0,k}}{\prod_{i,j=1}^{k} x_{i,j}} F_{0,1}$ under $\theta_{d}^{(n)}$ equals $\mu_{k} A_{i}(d) \otimes Z$. Recall the elements $\bar{h}_{i,\pm 1} \in \text{span}_{C}(h_{0,\pm 1}, \ldots, h_{n-1,\pm 1})$ from Lemma 2.5 such that $[\bar{h}_{i,1}, \bar{e}_{j,l}] = \delta_{i,j} \bar{e}_{j,\pm 1}$ for any $j \in [n], l \in \mathbb{Z}$. Since $\Psi[(\bar{e}_{j,l})] = x_{j,l+1}$, we see that the $q \rightarrow 1$ limit of $\frac{\prod_{i,j=1}^{k} F_{0,1}}{\prod_{i,j=1}^{k} x_{i,j}}$ equals ad($\bar{h}_{i,-1}$) ad($\bar{h}_{i,0}$). Combining the equality

$$
\theta_{d}^{(n)}(\bar{h}_{i,\pm 1}) = \left(\frac{d^{\pm(2n-i)} - d^{\pm(n-i)}}{d^{\pm n} - 1}(E_{1,1} + \cdots + E_{i,i}) + d^{\pm(n-i)} \right) \otimes D^{\pm 1}
$$

with $\theta_{d}^{(n)}(F_{0,1}) = \mu_{1} I_{n} \otimes Z$, we find $\theta_{d}^{(n)}(\text{ad}(\bar{h}_{i,-1}) \text{ad}(\bar{h}_{i,0})(F_{0,1})) = \mu_{1} A_{i}(d) \otimes Z$ as claimed.

(b) Let $C(d)$ be an $n \times n$ matrix whose rows are the diagonals of $\{e_{i}(a_{1}A_{1}(d), \ldots, a_{n}A_{n}(d))\}_{i=1}^{n}$. If $d \neq \sqrt{T}$ and $a_{i} \neq a_{j}$ for $i \neq j$ (which is the case for generic $s$), then det($C(d)$) $\neq 0$ due to the Vandermonde determinant. The result follows. □

Let us generalize the above result to $k > 1$. According to [T2], Theorems 3.2, 3.5], we have

$$
\Psi \left( \exp \left( \sum_{r=1}^{\infty} a_{r}(d, q) \varphi(h_{0,r}) \xi^{-r} \right) \right) = \sum_{k=0}^{\infty} (q - 1)^{kn} b_{k}(d, q) F_{0,k} \xi^{-k},
$$

where $c$ is a formal variable, the $q \rightarrow 1$ limits $\bar{a}_{r}(d)$ and $\bar{b}_{k}(d, q)$ of the constants $a_{r}(d, q)$ and $b_{k}(d, q)$ are nonzero for $d \neq 0$, $h_{0,r}$ are spans of $h_{0,\pm}, \ldots, h_{n-1,\pm})$ are defined via $\varphi(h_{0,r}, h_{i,r}) = \delta_{i,0}$ with the bilinear form $\varphi$ given by $\varphi(h_{i,r}, h_{j,s}) = \delta_{r,s} \cdot \frac{h_{r,s}(s)}{q - q^{-1}}$. Following our proof of Lemma 3.2 we see that $h_{0,r} = (q - 1)^{k} \lambda_{r}(d, q) h_{0,r}$ and the $q \rightarrow 1$ limit of $\lambda_{r}(d, q)$ is nonzero. Combining this with Lemmas 4.2 and 4.4 we find $\theta_{d}^{(n)}(q \rightarrow 1$ limit of $(q - 1)^{kn} F_{0,k}) = \bar{c}_{r}(d) \cdot I_{n} \otimes Z^{r}$, where $\bar{c}_{r}(d) \neq 0$ for $d \neq 0, \sqrt{T}$. Define $F_{i,k} := (q - 1)^{kn} F_{i,k}$ and let $\bar{F}_{i,k}$ denote the $q \rightarrow 1$ limit of $F_{i,k}$ (if it exists). We also set $\mu_{r} := \bar{a}_{r}(d) \bar{c}_{r}(d) / \bar{b}_{r}(d) \in \mathbb{C}^{\times}$.

The above discussion implies that $\theta_{d}^{(n)}(F_{0,k}) = \mu_{k} \cdot I_{n} \otimes Z^{k}$ for any $k \in \mathbb{N}$.

Lemma 3.7. The limit $\bar{F}_{i,k}$ is well-defined and $\theta_{d}^{(n)}(\bar{F}_{i,k}) = \mu_{k} e_{i}(a_{1}A_{1}(d), \ldots, a_{n}A_{n}(d)) \otimes Z^{k}$. Moreover, the elements $\{\theta_{d}^{(n)}(\bar{F}_{i,k})\}_{i=1}^{n}$ are linearly independent and span $\text{diag}_{n} \otimes Z^{k}$.

Proof of Lemma 3.7

To prove the first statement, it suffices to show

$$
(3) \quad \theta_{d}^{(n)}(q \rightarrow 1 \lim \frac{\prod_{j=1}^{k} x_{i,j}^{1-j} F_{0,k}}{\prod_{j=1}^{k} x_{i,j}} F_{0,1}) = \mu_{k} A_{i}(d^{k}) \otimes Z^{k} \text{ for any } 1 \leq i \leq n.
$$

Recall the elements $\bar{h}_{i,\pm k} \in \text{span}_{C}(h_{0,\pm k}, \ldots, h_{n-1,\pm k})$ from Lemma 2.5 such that $[\bar{h}_{i,\pm k}, \bar{e}_{j,l}] = \delta_{i,j} \bar{e}_{j,\pm 1}$ for any $j \in [n], l \in \mathbb{Z}$ and the polynomials $P_{k}(d)$ introduced in our proof of Theorem 2.1. Define $L_{i,\pm k} \in \text{End}_{C}(\bar{u}_{d})$ via $L_{i,\pm k} = P_{k}(d) \text{ad}(\bar{h}_{i,\pm k})$. Then, the $q \rightarrow 1$ limit of $\prod_{i=1}^{k} \frac{x_{i,j}^{1-j} F_{0,k}}{x_{i,j}} F_{0,1} = L_{i-1,k} F_{i,k}(F_{0,k})$. To derive (3), one needs to apply the formula

$$
\theta_{d}^{(n)}(\bar{h}_{i,\pm k}) = \left(\frac{d^{\pm(2n-1)k}}{d^{\pm n k} - 1}(E_{1,1} + \cdots + E_{i,i}) + d^{\pm(n-i)k} \right) \otimes D^{\pm k}
$$

together with the identity $P_{k}(d) = d^{2k} - d^{k}$.

The linear independence of $\theta_{d}^{(n)}(\bar{F}_{i,k})$ is proved completely analogously to Lemma 3.0 (b). □

It remains to note that Proposition 3.5 follows from Lemma 3.7 by induction on $k$. □
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