Simple method for excitation of a Bose-Einstein condensate

Bogdan Damski\textsuperscript{1}, Zbyszek P. Karkuszewski\textsuperscript{1,2}, Krzysztof Sacha\textsuperscript{1}, and Jakub Zakrzewski\textsuperscript{1},
\textsuperscript{1}Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński, ulica Reymonta 4, PL-30-059 Kraków, Poland
\textsuperscript{2}Theoretical Division, T6, MS B288, Los Alamos National Laboratory, Los Alamos, NM 87545 USA

(Received 1 November 2004; accepted 10 September 2005)

An appropriate, time-dependent modification of the trapping potential may be sufficient to create effectively collective excitations in a cold atom Bose-Einstein condensate. The proposed method is complementary to earlier suggestions and should allow the creation of both dark solitons and vortices.

PACS: 03.75.Fi,05.30.Jp,32.80.Pj

I. INTRODUCTION

Soon after the first spectacular realizations of the Bose-Einstein condensate (BEC) in cooled and trapped atomic gases \cite{1,2}, investigations of possible new effects involving the condensate appeared. The BEC allows us to study several typical quantum mechanical phenomena on a macroscopic level – because the macroscopic sample of atoms is described by a single wavefunction. A standard example is the splitting of the condensate into two spatially separated parts \cite{3} followed by a superposition of the parts. The observation of the interference fringes \cite{4} is a manifestation of the quantum coherence between two macroscopic parts of the condensate. By leaking the atoms from the condensate (typically downwards – due to gravity) one may prepare an “atom laser” \cite{5}.

It has been also realized that collisions between spatially separated condensates may be used to create collective excitations in the condensate \cite{6,7,8}. Assume a standard mean field single particle description of the gas of weakly interacting bosons in the limit of vanishing temperature (for reviews see \cite{9,10,11}). The time-dependent equation governing the state of the BEC is then the celebrated Gross-Pitaevskii equation (GPE). Its solutions may describe either solitary waves or vortices as is typical for the nonlinear equation. For characterization of solitons the language of nonlinear optics \cite{12} is quite useful. One may consider then bright solitons (bell shaped structures propagating without dispersion), dark solitons (with a node in the middle – an analog of the first excited state in the non-interacting particles picture) or the intermediate grey solitons.

Not only collisions may be used to create excitations of the BEC. In fact several schemes have been proposed, some of them being successfully applied in experiments. It has been suggested that a resonant Raman excitation scheme may be utilized to excite vortex states \cite{13}. However, the resonance is modified appreciably during the process of transferring the population from the ground state to the vortex state due to the nonlinearity of the GPE. An apparently more robust approach is the adiabatic scheme of \cite{14} which takes full account of the nonlinearity. It utilizes a controlled laser induced adiabatic transfer, populating solitonic or vortex solutions of GPE, depending on the details of the excitation. The adiabatic transfer uses internal atomic transitions combined with appropriate states of the condensate. A phase imprinting method, originally proposed in \cite{15}, produces a phase shift between two parts of the condensate. This method in fact has been applied experimentally to create dark solitons both in cigar shaped BEC \cite{16} and in the spherically symmetric condensate \cite{17}.

Another technique based on laser stirring of the condensate allows the production of several different vortex states \cite{18}. Recently the method has been applied to create lattices containing over 100 vortices \cite{19}.

The aim of this paper is to discuss in detail yet another method which, in our opinion, may serve to generate collective excitations in a BEC. It allows for the creation of grey (or even dark) solitons as well as vortices. The method which we propose resembles to a certain extent the adiabatic passage scheme of \cite{14}. In the latter, the transfer of population between two internal atomic states is accompanied by an appropriate change of the condensate wavefunction into a dark soliton, two-soliton or vortex solution of the GPE \cite{16}. Our method originally proposed for non-interacting particles \cite{20} and extended to weakly attractive interaction assumes a fast sweep of the laser beam across the trap. In this way the trapping potential varies with time enabling a transfer of population to excited BEC states. Under appropriate conditions this approach allows for an efficient creation of collective excitations in the condensate as explained below.

In Section II we present the method for a non-interacting particles model. We discuss both the excitation of solitons in the one dimensional (1D) case \cite{15} as well as the possibility of vortex creation in the effective two-dimensional (2D) example. In the next sections we extend the approach to interacting particles. In Section III we apply it to particles with repulsive atom-atom interactions (positive scattering length $a_0$) in 1D model case. In this way we complement our earlier study for attractive interactions \cite{21} for the case most often met in experiments with a BEC \cite{14}.

Extending the treat-
ment to 2D, we also consider the excitation of vortices for interacting particles. In Section IV we show some related observations on behavior of GPE energy levels while changing a parameter of the potential.

II. NON-INTERACTING PARTICLES

Let us first consider the simplest situation: The case of non-interacting particles. A BEC of non-interacting particles is not realized in nature, yet it may serve as a good model to describe the basic idea underlying the scheme we proposed. For an excitation of solitons propagating along a given direction 1D model is clearly sufficient. For excitation of vortices, though, at least a 2D model is required. Such low-dimensionality models may be fully justified for non-interacting particles, assuming separability of the trapping potential. For interacting particles such simplified models may be of value for appropriately prepared (cigar shaped or flat disc shaped, respectively) condensates [11–13,23–27].

Let us discuss the excitation of solitons first. Consider the condensate which occupies the ground state of the harmonic trap. Let us now sweep the region where the condensate is located with an additional laser beam, whose frequency is appropriately tuned close to the resonance of some internal atomic transition. By an adiabatic elimination of the upper atomic state (possible if the laser is detuned from the exact resonance) one may show [28] that such a laser beam creates an effective additional potential well (or a barrier, depending on the sign of the detuning with respect to the atomic transition) for the motion in the external atomic degree of freedom. The laser intensity is typically Gaussian-shaped in the direction perpendicular to the direction of propagation. The atoms experience a potential proportional to the intensity which may be represented as

$$V(x) = \frac{x^2}{2} + U_0 \text{arctan}(x_0) \exp\left(\frac{(x - x_0)^2}{2\sigma^2}\right).$$

We use the trapping harmonic oscillator units, i.e. $\omega t$ for time and $\sqrt{\hbar/m\omega}$ for length, where $\omega$ is harmonic oscillator frequency while $m$ stands for atomic mass. (These units are also used through the rest of the paper). A similar modification of the potential has been used already in experiments to split the condensate into two parts [3]. We propose modifying the trapping potential in a different way. Assume that the local Gaussian well is created on the very edge of the harmonic potential well thus not affecting the condensate. Then we change the laser beam direction slowly, in effect sweeping the well across the trapping potential. At the same time we gradually decrease the laser intensity, thereby decreasing the depth of the well. Such a procedure is equivalent to a change of $x_0$ from some negative value to zero in (1) assuming $U_0$ positive.

The procedure proposed has a clear quantum mechanical meaning. For a sufficiently slow change of the potential the system adiabatically follows its quantum energy levels except in the vicinity of avoided crossings. The energy gap of crossings may be controlled by choosing appropriate values of $U_0$ and $\sigma$ in (1). In particular it is easy to arrange that the avoided crossing between the ground and the first excited state of the potential occurs during the sweep of the local potential well (see Fig. 1). Moreover such an avoided crossing may be made sufficiently narrow to be passed diabatically during the potential sweep. Then, when the local potential well disappears the particle, originally in the ground state of the harmonic trap, is left with a high probability, $p_1$, in a first excited state (the Landau-Zener transition). The efficiency of the process depends on the size of the avoided crossing (which should be much smaller than the mean splitting between levels) and how quickly the potential is modified.

As discussed in the earlier report [22], numerical simulations fully confirm the proposed scheme. Without any special optimization attempt, choosing the parameters of the potential (1) as $U_0 = 13.4$, $\sigma = 0.2$ and changing $x_0$ from $-7$ to $0$ with the velocity $\dot{x}_0 = 0.02$ yields $p_1 = 0.99$. Similar values of $p_1$ are obtained for different values of $U_0$ and $\sigma$. The method is also robust with respect to the functional form of the potential well. We have checked that similar $p_1$ values are obtained if instead of the $\text{arctan}(x_0)$ we use other smooth monotonically changing functions of $x_0$.

![FIG. 1. Energy levels for a single particle in the potential (1) for $U_0 = 13.4$ and $\sigma = 0.2$ as a function of $x_0$. Note the narrow avoided crossing between the ground (the lowest solid line) and first excited (the lowest dashed line) states around $x_0 = -4.5$. Similar avoided crossings occur between the first and second excited states, between the second and third ones and so on.](image-url)
of the condensate. It is sufficient to sweep the local potential well twice in order to get a highly efficient transfer of population from the ground state to the second excited state of the potential. This extension is easily tested numerically in the model with non-interacting particles. Keeping the same parameters as in the single excitation case but merely repeating the potential sweep the second time we get $p_2 = 0.99$ as the squared overlap between the final wavefunction and the second excited state of the trap. A third consecutive sweep yields a “triply” exited state with the probability $p_3 = 0.99$ again without any modification of the parameters of the sweeping potential.

All these tests of single as well as multiple excitations indicate that, while we propose to excite the condensate by sweeping the trapping potential using the local potential well, the excitation may in fact be realized in a number of different ways. The key feature necessary for our method is the presence a narrow isolated avoided crossing between the ground and excited states.

$$U(x, y) = \frac{x^2 + y^2}{2} - U_0 \sqrt{\arctan(|x_0|)} \exp \left( \frac{-(x-x_0)^2 + y^2}{2\sigma^2} \right). \quad (3)$$

For different distances $|x_0|$ of the laser beam from the center of the trapping potential we can calculate energy levels of the Hamiltonian (3). For appropriate values of $U_0$, $\sigma$ and $\Omega$, when changing $x_0$ from some negative value to zero, one can observe a narrow avoided crossing between the ground and first excited states (see Fig. 3). The latter corresponds, for vanishing laser beam, to the first excited state of the harmonic potential with $L_z = 1$.

For the same parameters as used in Fig. 3 we perform a time dependent numerical simulation. Starting from a ground state of the condensate and changing $x_0$ from $-5$ to $0$ with a velocity $\dot{x}_0 = 0.036$, we obtain the first excited state with $L_z = 1$ with more than 99% accuracy. One may envision that after a single sweep we illuminate a condensate second time with a similar sweep. This, via a second avoided crossing (compare Fig. 3) yields $L_z = 2$ state with high efficiency (more than 99%).

Provided, therefore, that the resulting picture is not modified strongly by the interaction of particles constituting a BEC the proposed method should be able to produce both solitons and vortices with quite high efficiency. The next sections describe the effect of the interactions on the proposed mechanism.

III. COLLECTIVE EXCITATIONS IN BEC OF INTERACTING PARTICLES

While the proposed method seems to be quite robust for non-interacting particles model its applicability to a BEC of interacting particles is far from clear. After all, the particle interaction necessarily changes the energy levels of the system. The proposed method relies on narrow avoided crossings between levels when changing the parameter of the system – it is thus sensitive to the details of level dynamics. It is not obvious whether the presence of the interaction between the particles will not destroy the proposed mechanism of the excitation.

The partial preliminary answer has been given already in [13], where we considered the effect of attractive atom-atom interactions present in a BEC of Li atoms [3] on the creation of solitons in a 1D model. For such a condensate the number of particles is not too big and the effect of atom-atom interactions on the behavior of the system is rather small. By a direct integration of the time-dependent GPE in 1D

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + g|\psi|^2\psi, \quad (4)$$

(starting with the ground state of the condensate in the harmonic trap for $g = -5$) we were able to get a 97.5%
population transfer into a collective state corresponding to the first excited state in the independent particle model. In (4) \( g \) is a measure of nonlinearity and is proportional to number of particles \( N \) in the BEC. The value \( g = -5 \) taken for numerical simulation corresponds, for a trap used in [3], to \( N = 900 \) atoms in the condensate fraction – a typical number in Li condensate [3]. Clearly this is not the full story. The 1D approach for interacting atoms is not exact, since nonlinearity couples different degrees of freedom. Still, a one-dimensional approach based on the GPE is often used and may be justified for asymmetric traps [11–13,23–27]. More importantly most of the condensates realized in laboratories consist of particles with repulsive atom-atom interactions. Such condensates can easily hold about \( N = 10^5 \) particles [11–13]. Consequently the nonlinear term in (4) becomes much more important than for the attractive interaction. Thus the true test of our method requires a simulation for large positive \( g \) values.

Fig. 3 shows the final wavefunction obtained for the \( g = 50 \) case by a numerical integration of Eq. (4), taking as the initial state the ground state of the condensate for the same value of \( g \). The parameters of the potential are \( U_0 = 13.4 \) and \( \sigma = 0.2 \), and the velocity of the sweep is \( \dot{x}_0 = 0.6 \). The overlap of the wavefunction depicted in Fig. 3a with the ideal “excited” state of the condensate is \( p_1 = 0.98 \) at the end of the potential sweeping.

Similarly successful is a double application of the potential sweep in order to obtain a “two-node” collective state of the condensate. We use the same parameters as above. In fact for a second sweep we start from the wavefunction shown in Fig. 3a and make the second sweep identical to the first one. The results are depicted in Fig. 3b. The overlap of the wavefunction at the end of the sweeping with the exact solution of the time-independent GPE is \( p_2 = 0.82 \).

Consider now the excitation of vortices in the 2D model. The time dependent GPE in rotating \( x, y \) coordinates reads

\[
\frac{i\partial\psi}{\partial t} = H\psi + g|\psi|^2\psi,
\]

where \( H \) is given by Eq. (2). We get the time independent version by substituting \( \psi(x, y, t) = \exp(-i\mu t)\varphi(x, y) \) with \( \mu \) being the chemical potential. The resulting time-independent equation

\[
H\varphi + g|\varphi|^2\varphi = \mu\varphi
\]

can be solved by the method of self-consistent field or by using the imaginary time propagation approach. The former is quite robust for the 1D GPE but for 2D case it does not work efficiently. The latter is extremely effective for the “ground state” of the condensate, however, its application to “excited” states becomes difficult (especially for a trap without rotational symmetry) since for \( g \neq 0 \) the stationary solutions are no longer orthogonal. The third possibility is to find the solution of Eq. (5) by minimizing \( \langle \phi|\phi \rangle \), where

\[
\phi = H\hat{\varphi} + g|\hat{\varphi}|^2\hat{\varphi} - \hat{\mu}\hat{\varphi},
\]
with
\[ \mu = \langle \tilde{\varphi} | H + g | \tilde{\varphi} \rangle^2 | \tilde{\varphi} \rangle. \] (8)

Indeed, starting with some initial function \( \tilde{\varphi} \) decomposed in a given basis, standard procedures of minimizing of multidimensional functions can lead to a desired solution provided the initial guess function \( \tilde{\varphi} \) is sufficiently close to the exact solution \( \varphi \).

With solutions of the time-independent GPE at hand, we may prepare a BEC in its ground state, integrate the time-dependent GPE with the spiral-like potential sweep (given by (3) in the rotating frame), and compare the final wavefunction with the excited vortex-like solutions of the time-independent GPE. The resulting wavefunctions are presented in Fig. 4 for \( g = 100 \) and \( g = 500 \). Their overlaps with the excited state corresponding to one quantum of the angular momentum (per particle) are 0.99 and 0.98 respectively. The non-ideal population transfer is responsible for a slightly asymmetric shape of the final wavefunctions. However, the method may be quite effective in generating vortex-like excitations in a BEC. Let us note that our approach resembles to some extent the stirring approach to vortex creation [20]. In that method the potential as a whole is rotated with a certain frequency \( \Omega \). In our approach a static, cylindrically symmetric potential is supplemented, by an additional laser beam, with a narrow (with respect to the BEC dimension) structure moving along the spiral. Thus the physical picture of transferring the angular momentum to the condensate is a bit different in both cases. Naturally it differs also from the phase imprinting technique [17].

IV. BEHAVIOR OF LEVELS FOR INTERACTING PARTICLES

The original physical picture of the excitation scheme, as discussed in Section II for the non-interacting particles model, is based on the diabatic transitions between levels via narrow avoided crossings. The actual implementation for interacting particles has been tested by numerical integration of the time-dependent GPE without resorting to the explicit changes of GPE levels with respect to a modification of the potential. To see whether the same picture may be invoked for the interacting particles we have solved the 1D time-independent GPE not only for the harmonic potential but also in the presence of the laser beam. Obtained energy levels [29] are shown in Fig. 5 as a function of the position of the center of the laser beam. Observe the presence of loop-like structures. Such a behavior is impossible in the case of a linear Schrödinger equation. Actually, it can be shown that the changes of levels’ energies may be considered as a true Hamiltonian classical dynamics where the energies of levels play the role of positions of fictitious particles while the changing parameter corresponds to a fictitious time [30].

![FIG. 5. Energy (i.e. chemical potential [29]) levels of the condensate in the potential (1) for \( g = 50, U_0 = 13.4 \) and \( \sigma = 0.2 \) versus a value of the parameter \( x_0 \), i.e. the position of the laser beam.](image1)

To explain the appearance of the loops let us consider solutions of the time-independent GPE for weak attractive particle interactions. Figure 6 shows the lowest energy levels for both the non-interacting and interacting case with \( g = -1 \) [see (6)]. In the non-interacting case avoided crossings result from changing the shape of the double well potential. To approximately predict energy levels in a double well one may consider each well separately. Then for an asymmetric potential, the resulting energies in each well are usually considerably different. However, for certain shapes,
the left. A little, i.e. to shift the center of the laser beam back to (9). The only way to restore balance is to lift the left well because of the density term in the effective potential (9) separately. The resulting energy in the right well creates a laser. At the point of the avoided crossing both wells are equally populated i.e. the ground state consists of equally weighted symmetric superposition of eigenstates of the right and left well.

Transfer of particles from the harmonic trap to the dip accompanied by shifting of the laser to the left can be carried on until nothing is left in the former. When this stage is reached (point B in Fig. 5), all particles are localized in the left dip. Now this dip can be lowered without populating the right well. This will decrease the energy of the condensate which at some point matches the value of the starting energy with all particles in the right well. This is indicated by C in Fig. 5. The situation where it is possible to have the same values of a chemical potential for states with all particles in the left or right well is in fact a necessary condition for loop-like structure to be present. Recall that in the non-interacting case, matching of corresponding energy levels is responsible for an avoided crossing.

For \( g > 0 \) it is possible to have a similar scenario to the attractive particles case. The only exception is that the ground energy level does not reveal any loop (see Fig. 5). Indeed, to transfer the probability density of the ground state from the harmonic well to the well created by the laser beam we have to move the beam further and further towards the center because, contrary to the attractive particles case, taking the density from the right well leads to a deepening of the effective potential (9) at that place.

Analyzing the interacting particle case we may consider a solution of the GPE as if it is a solution of the Schrödinger equation with an effective potential

\[
V_{eff}(x) = V(x) + g|\varphi(x)|^2. \tag{9}
\]

For \( g < 0 \), the interaction term deepens a potential well in which the probability density is localized (see Fig. 5). Consider the ground state of the harmonic trap and suppose that the laser beam approaches the center of this trap. At each position of the beam \( x_0 \), we can calculate approximate energies considering each well of the effective potential (9) separately. The resulting energy in the left well is higher than the one in the right well when the probability density is situated in the right well (compare Fig. 7). This holds up to the point indicated as A in Fig. 5, where the lowest eigenenergy of left well treated alone approaches the energy of the condensate in the harmonic trap. At that point we can start populating the left well. However, transferring particles from the right to the left well breaks the energy balance between the two because of the density term in the effective potential (9). The only way to restore balance is to lift the left well a little, i.e. to shift the center of the laser beam back to the left.
V. SUMMARY AND CONCLUSIONS

The proposed method of efficient collective excitation of a BEC seems to be quite robust and allows preparing both solitonic and vortex-like excitations of the condensate. Although the method is complementary to other schemes, some of which have been already implemented experimentally, still it may be advantageous in some cases. As we have discussed already in [16] our approach is somehow closest in spirit to the adiabatic scheme proposed in [16]. That approach seems also to be quite robust and allows various excitations to be created. The method of [16] effectively uses two atomic internal states, and thus involves a “two-component” condensate. Our approach does not entangle internal and external excitations – this may be advantageous in some applications. More importantly our diabatic (or rather “generalized diabatic”) method takes necessarily a short time – typically of the order of a few periods of the trap in our runs. This has to be compared with several hundreds of periods necessary for the excitation using the adiabatic process [16].

Needless to say while one may argue about the advantages of the proposed scheme a most straightforward way to test it would be an experimental approach. It seems that both the linear potential sweep for cigar shaped BEC or the spiral like excitation for disc shaped quasi 2D condensates require relatively minor changes in the already existing laboratory set-ups.

We have found that “dynamics” of chemical potential in the time-independent GPE reveals interesting loop-like structures. This prevents us from interpreting the changes of the chemical potential levels, with respect to the parameter, as some form of level dynamics known from the linear Schrödinger equation. The observed “hysteresis-like” behavior has its origin in the nonlinearity of the GPE as explained in the text. Its relation to the linear quantum mechanics of multiparticle theory is being currently investigated.

To summarize, we have proposed a simple scheme which enables us to create collective excitations (both solitons and vortices) of the Bose-Einstein condensate. This scheme may serve, we believe, as an alternative to other proposed methods already utilized experimentally.

Note added in proof: The scheme considered by us, similarly to other works on this subject [3, 4, 5], is based on the applicability of the GPE to the description of collectively excited states of the BEC. That has been questioned very recently [6].

VI. ACKNOWLEDGMENT

We are grateful to Maciek Lewenstein and Kazik Rzążewski for encouragement. Special thanks are due to Mariusz Gajda for suggesting the imaginary time method to get stationary solutions of the GPE, and to Andrzej Ostruszka for the help in solving computer problems. Support of KBN under project 5 P03B 088 21 is acknowledged.

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science 269, 198 (1995).
[2] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).
[3] C. C. Bradley, C. A. Sackett, and R. G. Hulet, Phys. Rev. Lett. 78, 985 (1997); see also C. C. Bradley, C. A. Sackett, J. J. Tollet, and R. G. Hulet, Phys. Rev. Lett. 75, 1687 (1995); C. A. Sackett, C. C. Bradley, M. Welling, and R. G. Hulet, Appl. Phys. B65, 433 (1997); J. M. Gerton, D. Strekalov, I. Prodan, and R. G. Hulet, Nature 408, 692 (2000).
[4] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science 275, 637 (1997).
[5] N. Naraschewski, H. Wallis, and A. Schenzle, J. I. Cirac, and P. Zoller, Phys. Rev. A 54, 2185 (1996).
[6] A. Röhr, M. Naraschewski, A. Schenzle, and H. Wallis, Phys. Rev. Lett. 78, 4143 (1997).
[7] M. Holland, K. Burnett, C. Gardiner, J. I. Cirac, and P. Zoller, Phys. Rev. A 54, R1757 (1996).
[8] M. O. Mewes, M. R. Andrews, D. M. Kurn, D. S. Durfee, C. G. Townsend, and W. Ketterle, Phys. Rev. Lett. 78, 582 (1997).
[9] W. P. Reinhardt and C. W. Clark, J. Phys. B30, L785 (1997).
[10] T. F. Scott, R. J. Ballagh, and K. Burnett, J. Phys. B31, L329 (1998).
[11] A. S. Parkins and D. Walls, Phys. Rep. 303, 1 (1998).
[12] F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. 71, 463 (1999).
[13] A. J. Leggett, Rev. Mod. Phys. 73, 307 (2001).
[14] J. K. Taylor, Ed., Optical Solitons Theory & Experiment (Cambridge Univ. Press, New York, 1992).
[15] K. P. Marzlin and W. Zhang, Phys. Rev. Lett. 79, 4728 (1997).
[16] R. Dum, J. I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. Lett. 80, 2972 (1998).
[17] L. Dobrek, M. Gajda, M. Lewenstein, K. Sengstock, G. Birkl, and W. Ertmer, Phys. Rev. A 60, R3381 (1999).
[18] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, K. Sengstock, A. Sanpera, G. V. Shlyapnikov, and M. Lewenstein, Phys. Rev. Lett. 83, 5198 (1999).
[19] J. Denschlag, J. E. Simsarian, D. L. Feder, C. W. Clark, L. A. Collins, J. Cubizolles, L. Deng, E. W. Hagley, K. Helmerson, W. P. Reinhardt, S. L. Rolston, B. I. Schneider, and W. D. Phillips, Science 287, 97 (2000).
[20] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. 84, 806 (2000).
[21] J. R. Abo-Shaeer, C. Raman, J. M. Vogels, W. Ketterle, Science 292, 476 (2001).
For simplicity we call by energy the value of $\mu$ obtained for $\varphi(x)$ – a time-independent solution of GPE. Strictly speaking $\mu$ should be referred to as a chemical potential while the energy per particle $\epsilon = \mu - g/2 \int |\varphi(x)|^4 dx$. The plots of $\epsilon$ are similar to that of $\mu$ and are, therefore, not shown.