q-Analog of Gelfand–Graev basis for the noncompact
quantum algebra $U_q(u(n, 1))$*

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Abstract

For the quantum algebra $U_q(gl(n + 1))$ in its reduction on the subalgebra $U_q(gl(n))$ an explicit description of a Mickelsson–Zhelobenko reduction algebra $Z_q(gl(n+1), gl(n))$ is given in terms of the generators and their defining relations. Using this $Z$-algebra we describe Hermitian irreducible representations of a discrete series for the quantum algebra $U_q(u(n, 1))$ which is a real form of $U_q(gl(n))$. Namely, an orthonormal Gelfand–Tsetlin basis is constructed in explicit form.

1 Introduction

In 1950 I.M. Gelfand and M.L. Tsetlin [1] proposed a formal description of finite-dimensional irreducible representations (IR) for the compact Lie algebra $u(n)$. This description is a generalization of results for $u(2)$ and $u(3)$ on the case $u(n)$. It is the following. In the IR space of $u(n)$ there is a orthonormalized basis which is numbered by the following formal schemes

$$
\begin{pmatrix}
m_{1n} & m_{2n} & \cdots & m_{n-1,n} & m_{nn} \\
m_{1,n-1} & m_{2,n-1} & \cdots & m_{n-1,n-1} & \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
m_{12} & \cdots & m_{22} & m_{11}
\end{pmatrix}
$$

(1.1)

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†Deceased.
where all numbers $m_{ij}$ ($1 \leq i < j \leq n$) are nonnegative integers and they satisfy the standard inequalities, ”between conditions”:

$$m_{ij+1} \geq m_{ij} \geq m_{i+1j+1} \quad \text{for} \quad 1 \leq i \leq j \leq n - 1.$$  \hfill (1.2)

The first line of this scheme is defined by the components of the highest weight of $u(n)$ IR, the second line is defined by the components of the highest weight of $u(n-1)$ IR and so on.

Later this basis was constructed in many papers (for example, see [2, 3, 4]) by using one-step lowering and raising operators.

In 1965 I.M. Gelfand and M.I. Graev [5] using analytic continuation of the results for $u(n)$ obtained some results for non-compact Lie algebra $u(n,m)$. They shown that some class of Hermitian IR of $u(n,m)$ is characterized by a ”extremal weight” parametrized by a set of integers $m_N = (m_{1N}, \ldots, m_{NN})$ ($N = n + m$) such that $m_{1N} \geq m_{1N} \geq \cdots \geq m_{NN}$, and by a representation type which is defined by a partition of $n$ in the sum of two nonnegative integers $\alpha$ and $\beta$, $n = \alpha + \beta$ (also see [6]).

For simplicity we consider the case $u(2,1)$. In this case we have three type of scheme

$$\begin{pmatrix} m_{12} & m_{13} & m_{23} & m_{33} \\ m_{11} & m_{22} & m_{23} & m_{33} \\ m_{11} & m_{12} & m_{22} & m_{33} \end{pmatrix}$$  \hfill (1.3)

$$\begin{pmatrix} m_{12} & m_{13} & m_{23} & m_{33} \\ m_{11} & m_{12} & m_{33} & m_{22} \end{pmatrix}$$  \hfill (1.4)

$$\begin{pmatrix} m_{13} & m_{23} & m_{12} & m_{33} \\ m_{11} & m_{22} & m_{33} & m_{22} \end{pmatrix}$$  \hfill (1.5)

The numbers $m_{ij}$ of the first scheme satisfy the following inequalities

$$m_{12} \geq m_{13} + 1 \, , \quad m_{13} + 1 \geq m_{22} \geq m_{23} + 1 \, , \quad m_{12} \geq m_{11} \geq m_{22} .$$  \hfill (1.6)

The numbers $m_{ij}$ of the second scheme satisfy the following inequalities

$$m_{12} \geq m_{13} + 1 \, , \quad m_{13} - 1 \geq m_{22} \, , \quad m_{12} \geq m_{11} \geq m_{22} .$$  \hfill (1.7)

The numbers of the third scheme satisfy the following inequalities

$$m_{23} - 1 \geq m_{12} \geq m_{33} - 1 \, , \quad m_{33} - 1 \geq m_{22} \, , \quad m_{12} \geq m_{11} \geq m_{22} .$$  \hfill (1.8)

Construction of Gelfand–Graev basis for $u(n,m)$ in terms of one-step lowering and raising operators is more complicated then in the compact case $u(n+m)$.

In 1975 T.J. Enright and V.S. Varadarajan [7] obtained some classification of discrete series of non-compact Lie algebras. Later A. Molev [8] shown that for the case $u(n,m)$ Gelfand–Graev modules are some part of Enright–Varadarajan modules and Molev constructed Gelfand–Graev basis for $u(n,m)$ in terms of Mickelsson S-algebra [9].

A goal of this work to obtain analogous results for the non-compact quantum algebra $U_q(u(n,m))$. Because the general case is very complicated we at first consider the case $U_q(u(2,1))$. It should be noted that the special case $U_q(u(2,1))$ was considered in [10, 11].
2 Quantum algebra $U_q(gl(N))$ and its noncompact real forms $U_q(u(n,m))$ \((n+m=N)\)

The quantum algebra $U_q(gl(N))$ is generated by the Chevalley elements $q^{e_{ii}}$ \(i=1,\ldots,N\), $e_{i,i+1}$, $e_{i+1,i}$ \(i=1,2,\ldots,N-1\) with the defining relations:

\[
q^{e_{ii}}q^{-e_{ii}} = q^{-e_{ii}}q^{e_{ii}} = 1 , \tag{2.1}
\]

\[
q^{e_{ii}}q^{e_{jj}} = q^{e_{jj}}q^{e_{ii}} , \tag{2.2}
\]

\[
q^{e_{ii}}e_{jk}q^{-e_{ii}} = q^{\delta_{ij}-\delta_{ik}}e_{jk} \quad (|j-k| = 1) , \tag{2.3}
\]

\[
[e_{i,i+1}, e_{j+1,j}] = \delta_{ij} \frac{q^{e_{ii}}-q^{e_{i+1,i+1}} - q^{e_{i+1,i+1}}-e_{ii}}{q - q^{-1}} , \tag{2.4}
\]

\[
[e_{i,i+1}, e_{jj+1}] = 0 \quad \text{for} \ |i-j| \geq 2 , \tag{2.5}
\]

\[
[e_{i+1,i}, e_{j+1,j}] = 0 \quad \text{for} \ |i-j| \geq 2 , \tag{2.6}
\]

\[
[[e_{i,i+1}, e_{jj+1}]_q, e_{jj+1}]_q = 0 \quad \text{for} \ |i-j| = 1 , \tag{2.7}
\]

\[
[[e_{i+1,i}, e_{j+1,j}]_q, e_{j+1,j}]_q = 0 \quad \text{for} \ |i-j| = 1 . \tag{2.8}
\]

where $[e_\beta, e_\gamma]_q$ denotes the $q$-commutator:

\[
[e_\beta, e_\gamma]_q := e_\beta e_\gamma - q^{(\beta,\gamma)} e_\gamma e_\beta . \tag{2.9}
\]

The definition of a quantum algebra also includes operations of a comultiplication $\Delta_q$, an antipode $S_q$, and a co-unit $\varepsilon_q$. Because explicit formulas of these operations will not used in our later calculations, they are not given here.

For construction of the composite root vectors $e_{ij}$ for $|i-j| \geq 2$ we fix the following normal (convex) ordering of the positive root system $\Delta_+$ (see [12])

\[
\epsilon_1 - \epsilon_2 < \epsilon_1 - \epsilon_3 < \epsilon_2 - \epsilon_3 < \cdots < \epsilon_1 - \epsilon_4 < \epsilon_2 - \epsilon_4 < \epsilon_3 - \epsilon_4 < \ldots < \epsilon_1 - \epsilon_k < \epsilon_2 - \epsilon_k < \ldots < \epsilon_{k-1} - \epsilon_k < \epsilon_k < \ldots < \epsilon_1 - \epsilon_N < \epsilon_2 - \epsilon_N < \ldots < \epsilon_{N-1} - \epsilon_N . \tag{2.10}
\]

According to this ordering we set

\[
e_{ij} := [e_{ik}, e_{kj}]_{q^{-1}} , \quad e_{ji} := [e_{jk}, e_{ki}]_q , \tag{2.11}
\]

where $1 \leq i < k < j \leq N$. It should be stressed that the structure of the composite root vectors does not depend on the choice of the index $k$ in the r.h.s. of the definition (2.10). In particular, we have

\[
e_{ij} := [e_{i,i+1}, e_{i+1,j}]_{q^{-1}} = [e_{i,j-1}, e_{j-1,j}]_{q^{-1}} , \tag{2.12}
\]

\[
e_{ji} := [e_{j,i+1}, e_{i+1,i}]_q = [e_{j,j-1}, e_{j-1,j}]_q ,
\]

where $2 \leq i + 1 < j \leq N$. 

3
Using these explicit constructions and the defining relations (2.1)–(2.8) for the Chevalley basis it is not hard to calculate the following relations between the Cartan–Weyl generators \( e_{ij} \) \((i, j = 1, 2, \ldots, N)\):

\[
q^{e_{kk}}e_{ij}q^{-e_{kk}} = q^{\delta_{ki} - \delta_{kj}}e_{ij} \quad (1 \leq i, j, k \leq N),
\]

\[
[e_{ij}, e_{jk}] = \frac{q^{e_{ii}} - q^{e_{jj}} - q^{e_{jj}}e^{e_{ii}}}{q - q^{-1}} \quad (1 \leq i < j \leq N),
\]

\[
[e_{ij}, e_{kl}]q^{-1} = \delta_{jk}e_{il} \quad (1 \leq i < j < k < l \leq N),
\]

\[
[e_{ik}, e_{jl}]q^{-1} = (q - q^{-1})e_{jk}e_{il} \quad (1 \leq i < j < k < l \leq N),
\]

\[
[e_{jk}, e_{il}]q^{-1} = 0 \quad (1 \leq i < j < k < l \leq N),
\]

\[
[e_{kl}, e_{ji}] = 0 \quad (1 \leq i < j < k < l \leq N),
\]

\[
[e_{kl}, e_{ij}] = 0 \quad (1 \leq i < j < k < l \leq N),
\]

\[
[e_{ji}, e_{il}] = e_{ji}q^{e_{ii}} - e_{jj} \quad (1 \leq i < j < l \leq N),
\]

\[
[e_{kl}, e_{li}] = e_{kl}q^{e_{kk}} - e_{ii} \quad (1 \leq i < k < l \leq N),
\]

\[
[e_{jl}, e_{ki}] = (q^{-1} - q)e_{kl}e_{ji}q^{-e_{kk}} \quad (1 \leq i < j < k < l \leq N).
\]

If we apply the Cartan involution \((e_{ij}^* = e_{ji})\) the formulas above, we will get all relations between elements of the Cartan–Weyl basis.

The explicit formula for the extremal projector for \( U_q(\mathfrak{gl}(N)) \) has the form [13]

\[
p(U_q(\mathfrak{gl}(N))) = p(U_q(\mathfrak{gl}(N - 1)))(p_{11}p_{22}\cdots p_{N-2Np_{N-1N}})
\]

\[
= p_{12}(p_{13}p_{23})\cdots(p_{1i}\cdots p_{ii+1})\cdots(p_{1N}\cdots p_{N-1N}),
\]

where the elements \( p_{ij} \) \((1 \leq i < j \leq N)\) are given by

\[
p_{ij} = \sum_{r=0}^{\infty} \frac{(-1)^s}{r!} \varphi_{ij,r} e_{ij}^r,
\]

\[
\varphi_{ij,r} = q^{(j-i-1)r} \left\{ \prod_{s=1}^{N} [e_{ii} - e_{jj} + j - i + s] \right\}^{-1}.
\]

The extremal projector \( p := p(U_q(\mathfrak{gl}(N)) \) satisfies the relations:

\[
e_{i,i+1}p = p e_{i+1,i} \quad (1 \leq i \leq N - 1), \quad p^2 = p.
\]

The extremal projector \( p \) belongs to the Taylor extension \( TU_q(\mathfrak{gl}(N)) \) of the universal enveloping algebras \( U_q(\mathfrak{gl}(N)) \). The Taylor extension \( TU_q(\mathfrak{gl}(N)) \) is an associative algebra generated by formal Taylor series of the form

\[
\sum_{\{r'\},\{r\}} C_{\{r'\},\{r\}} (q^{e_{11}}, \ldots, q^{e_{NN}}) e_{21}^{r_1} e_{31}^{r_2} \cdots e_{N,N-1}^{r_N} e_{12}^{r_{12}} e_{13}^{r_{13}} \cdots e_{N-1,N}^{r_{N-1,N}}
\]
provided that nonnegative integers \( r'_{12}, r'_{13}, r'_{23}, \ldots, r'_{N-1,N} \) and \( r_{12}, r_{13}, r_{23}, \ldots, r_{N-1,N} \) are subject to the constraints

\[
\left| \sum_{i<j} r'_{ij} - \sum_{i<j} r_{ij} \right| \leq \text{const} \quad (2.27)
\]

for each formal series and the coefficients \( C_{(r'),(r)}(q^{e_{ii}}, \ldots, q^{e_{NN}}) \) are rational functions of the \( q \)-Cartan elements \( q^{e_{ii}} \). The quantum universal enveloping algebra \( U_q(\mathfrak{gl}(N)) \) is a subalgebra of the Taylor extension \( TU_q(\mathfrak{gl}(N)) \), \( U_q(\mathfrak{gl}(N)) \subset TU_q(\mathfrak{gl}(N)) \).

The noncompact quantum algebra \( U_q(\mathfrak{u}(n,m)) \) can be considered as the quantum algebra \( U_q(\mathfrak{gl}(N)) \) \( (N = n + m) \) endowed with the additional Cartan involution (*):

\[
h^*_i = h_i, \quad (i = 1, 2, \ldots, n + m), \quad (2.28)
\]

\[
e^*_{i,i+1} = e_{i+1,i}, \quad e^*_{i+1,i} = e^*_{i+1,i}, \quad \text{for} \ i \neq n, \quad (2.29)
\]

\[
e^*_{n,n+1} = -e_{n+1,n}, \quad e^*_{n+1,n} = -e_{n,n+1}, \quad (2.30)
\]

\[
q^* = q \quad \text{or} \quad q = q^{-1}. \quad (2.31)
\]

Below we will consider the real form \( U_q(\mathfrak{u}(n,1)) \), i.e. the case \( N = n + 1 \).

### 3 The reduction algebra \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \)

In the linear space \( TU_q(\mathfrak{gl}(n + 1)) \) we separate out a subspace of "two-sided highest vectors" with respect to the subalgebra \( U_q(\mathfrak{gl}(n)) \subset U_q(\mathfrak{gl}(n + 1)) \), i.e.

\[
\tilde{Z}_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) = \left\{ x \in TU_q(\mathfrak{gl}(n + 1)) \left| e_{ij}x = xe_{ji} = 0, \ 1 \leq i < j \leq n \right. \right\}. \quad (3.1)
\]

It is evident that if \( x \in \tilde{Z}_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) then

\[
x = p x p, \quad (3.2)
\]

where \( p := p(U_q(\mathfrak{gl}(n)) \left(\mathfrak{gl}(n)\right)) \). Again, using the annihilation properties of the projection operator \( p \) we have that any vector \( x \in \tilde{Z}_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) presents a formal Taylor series on the following monomials

\[
p e_{n+1,1}^{r_1} \cdots e_{n+1,n}^{r_n} e_{n,n+1}^{r_n} \cdots e_{1,n+1}^{r_1} p. \quad (3.3)
\]

It is evident that \( \tilde{Z}_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) is a subalgebra in \( TU_q(\mathfrak{gl}(n + 1)) \). We consider a subspace \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) generated by finite series on the monomials (5.3). It is clear that \( Z_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \) is a subalgebra in \( \tilde{Z}_q(\mathfrak{gl}(n + 1), \mathfrak{gl}(n)) \).

We set

\[
z_i := p e_i p, \quad (i = \pm 1, \pm 2, \ldots, \pm n), \quad (3.4)
\]

where \( e_i = e_{j,n+1}, \ e_{-i} = e_{n+1,j} \ (1 = 1, 2, \ldots, n) \).
Theorem 3.1 The elements \( z_i \) \((i = \pm 1, \pm 2, \ldots, \pm n)\) generates the associative algebra \( Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) and satisfies the following relations

\[
 z_i z_j = A_{ij} z_j z_i \quad \text{for} \quad i + j \neq 0, \quad (i, j = \pm 1, \pm 2, \ldots, \pm n), \quad (3.5)
\]

\[
 z_i z_{-i} = \sum_{j=1}^{n} B_{ij} z_{-i} z_i + \gamma_i \quad \text{for} \quad i = 1, 2, \ldots, n, \quad (3.6)
\]

where

\[
 A_{ij} = 1 \quad \text{for} \quad sgn \, i \neq sgn \, j, \quad (3.7)
\]

\[
 A_{ij} = A_{-j-i} = \frac{[\varphi_{ij} + 1]}{[\varphi_{ij}]} \quad \text{for} \quad 1 \leq i < j \leq n, \quad (3.8)
\]

\[
 B_{ij} = -\frac{b_i^- b_j^+}{[\varphi_{ij} - 1]_1}, \quad \gamma_i = b_i^- [\varphi_{i,n+1} - 1], \quad (3.9)
\]

\[
 b_i^\pm = \prod_{s=i+1}^{n} \frac{[\varphi_{is} \pm 1]}{[\varphi_{is}]}, \quad \varphi_{ik} = e_{ii} - e_{kk} + k - i. \quad (3.10)
\]

A proof of the theorem by direct calculations.

For construction and study of discrete series of the non-compact quantum algebra \( U_q(\mathfrak{u}(n, 1)) \) we need another relations then (3.6). The system (3.6) expresses the elements \( z_i z_{-i} \) in terms of the elements \( z_{-i} z_i \) \((i = 1, 2, \ldots, n)\) but we would like to express the elements \( z_{-1} z_1, \ldots, z_{-\alpha} z_\alpha, z_{\alpha+1} z_{-\alpha-1}, \ldots, z_{n} z_{-n} \) in terms of the elements \( z_1 z_{-1}, \ldots, z_\alpha z_{-\alpha}, z_{\alpha-1} z_{\alpha+1}, \ldots, z_n z_{-n} \) for \( \alpha = 0, 1, \ldots, n \). These relations are given by the theorem.

**Theorem 3.2** The elements \( z_{-1} z_1, \ldots, z_{-\alpha} z_\alpha, z_{\alpha+1} z_{-\alpha-1}, \ldots, z_{n} z_{-n} \) are expressed in terms of the elements \( z_1 z_{-1}, \ldots, z_\alpha z_{-\alpha}, z_{\alpha-1} z_{\alpha+1}, \ldots, z_n z_{-n} \) by the formulas

\[
 z_{-i} z_i = \sum_{j=1}^{\alpha} B_{ij}^{(a)} z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{il}^{(a)} z_{-i} z_l + \gamma_i^{(a)} \quad (i = 1, 2, \ldots, \alpha), \quad (3.11)
\]

\[
 z_k z_{-k} = \sum_{j=1}^{\alpha} B_{kj}^{(a)} z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{kl}^{(a)} z_{-i} z_l + \gamma_k^{(a)} \quad (k = \alpha + 1, \alpha + 2, \ldots, n), \quad (3.12)
\]

where

\[
 B_{ij}^{(a)} = \frac{b_i^{(a)} + b_j^{(a)}}{[\varphi_{ij} + 1]}, \quad B_{il}^{(a)} = \frac{b_i^{(a)} + b_l^{(a)}}{[\varphi_{il}]}, \quad \gamma_i^{(a)} = -[\varphi_{i,n+1} - \alpha] b_i^{(a)+}, \quad (3.13)
\]

\[
 B_{kj}^{(a)} = -\frac{b_k^{(a)} - b_j^{(a)}}{[\varphi_{kj}]}, \quad B_{kl}^{(a)} = -\frac{b_k^{(a)} - b_l^{(a)}}{[\varphi_{kl} - 1]}, \quad \gamma_k^{(a)} = [\varphi_{i,n+1} - \alpha - 1] b_k^{(a)+}, \quad (3.14)
\]

\[
 b_i^{(a)\pm} = \prod_{s=1}^{i-1} \frac{[\varphi_{is} \pm 1]}{[\varphi_{is}]} \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is}]}{[\varphi_{is} \pm 1]}, \quad b_i^{(a)\pm} = \prod_{s=1}^{\alpha} \frac{[\varphi_{is}]}{[\varphi_{is} \pm 1]} \prod_{s=l+1}^{n-1} \frac{[\varphi_{ls} \pm 1]}{[\varphi_{ls}]} . \quad (3.15)
\]

\(^{1}\text{In the case of} \alpha = 0 \text{we have the relations (3.6) and for} \alpha = n \text{we obtain the system inverse to (3.6)}\)
Moreover the following power relations are valid

\[ z_i^r z_j^s = z_{ij}^r z_{ij}^s \quad \text{for } \forall i, j > 0 \text{ and } r, s \in \mathbb{N} , \quad (3.16) \]

\[ z_i^r z_j^s = z_{ij}^r z_{ij}^s \frac{[\varphi_{ij} + r]! [\varphi_{ij} - s]!}{[\varphi_{ij} + r - s]!} \quad \text{for } 1 \leq i < j \leq n \text{ and } r, s \in \mathbb{N} , \quad (3.17) \]

\[ z_i^r z_j^s = z_{ij}^r z_{ij}^s \frac{[\varphi_{ij} - r + s]! [\varphi_{ij} - r]!}{[\varphi_{ij} + s]!} \quad \text{for } 1 \leq i < j \leq n \text{ and } r, s \in \mathbb{N} , \quad (3.18) \]

\[ z_i^r z_j^s = z_{ij}^r z_{ij}^s \sum_{l=1}^{\alpha} B_{ij}^{(r)}(r) z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{il}^{(r)}(r) z_{-i} z_l + \gamma_i^{(r)}(r) \quad (i = 1, 2, \ldots, \alpha), \quad (3.19) \]

\[ z_k^r z_k^s = z_{kk}^r z_{kk}^s \sum_{j=1}^{\alpha} B_{kj}^{(r)}(r) z_j z_{-j} + \sum_{l=\alpha+1}^{n} B_{kl}^{(r)}(r) z_{-i} z_l + \gamma_k^{(r)}(r) \quad (k = \alpha+1, \ldots, n), \quad (3.20) \]

where

\[ B_{ij}^{(r)}(r) = \frac{[r] b_j^{(r)}(r) b_i^{(r)} - 1}{[\varphi_{ij} + r]}, \quad B_{il}^{(r)}(r) = \frac{[r] b_i^{(r)}(r) b_l^{(r)} + 1}{[\varphi_{il} + r - 1]}, \quad (3.21) \]

\[ \gamma_i^{(r)}(r) = -[r] [\varphi_{i,n+1} - \alpha + r - 1] b_i^{(r)}(r) \quad (1 \leq i, j \leq \alpha < l \leq n), \quad (3.22) \]

\[ B_{kj}^{(r)}(r) = -\frac{[r] b_k^{(r)}(r) b_j^{(r)} - 1}{[\varphi_{kj} - r + 1]}, \quad B_{kl}^{(r)}(r) = -\frac{[r] b_k^{(r)}(r) b_l^{(r)} + 1}{[\varphi_{kl} - r]}, \quad (3.23) \]

Here

\[ b_i^{(r)}(r) = \left( \prod_{s=1}^{i-1} \frac{[\varphi_{is} + r]}{[\varphi_{is} + r - 1]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is} + r - 1]}{[\varphi_{is} + r]} \right) \quad (1 \leq i \leq \alpha), \quad (3.24) \]

\[ b_i^{(r)}(r) = \left( \prod_{s=1}^{i-1} \frac{[\varphi_{is} - 1]}{[\varphi_{is}]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is}]}{[\varphi_{is} - 1]} \right) \quad (1 \leq i \leq \alpha), \quad (3.25) \]

\[ b_i^{(r)}(r) = \left( \prod_{s=1}^{\alpha} \frac{[\varphi_{is} - r + 1]}{[\varphi_{is} - r]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is} - r]}{[\varphi_{is} - r + 1]} \right) \quad (\alpha + 1 \leq l \leq n), \quad (3.26) \]

\[ b_i^{(r)}(r) = \left( \prod_{s=1}^{\alpha} \frac{[\varphi_{is}]}{[\varphi_{is} + 1]} \right) \left( \prod_{s=\alpha+1}^{n-1} \frac{[\varphi_{is} + 1]}{[\varphi_{is}]} \right) \quad (\alpha + 1 \leq l \leq n), \quad (3.27) \]
4 Shapovalov’s forms on $\mathcal{Z}_q^*(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$

We consider on the $\mathcal{Z}$-algebra $\mathcal{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ two real forms: compact and noncompact.

The compact real form on $\mathcal{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ is defined by the involution ($^*$) which is given as follows

$$z_{\pm i}^* = z_{\mp i} \quad (i = 1, 2, \ldots, n),$$

$$e_{ii}^* = e_{ii} \quad (i = 1, 2, \ldots, n+1).$$

(4.1) (4.2)

This involution can be considered as generalization of the Cartan involution in $\mathcal{U}_q(\mathfrak{gl}(n+1))$ to the Taylor extension, $TU_q(\mathfrak{gl}(n+1))$. The $\mathcal{Z}$-algebra $\mathcal{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ with this involution is called the compact real form and denoted by the symbol $\mathcal{Z}_q^{(c)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

The noncompact real form on $\mathcal{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ is defined by the involution $^*$ which is given as follows

$$z_{\pm i}^* = -z_{\pm i} \quad (i = 1, 2, \ldots, n),$$

$$e_{ii}^* = e_{ii} \quad (i = 1, 2, \ldots, n+1).$$

(4.3) (4.4)

This involution can be considered as generalization of the noncompact involution in $\mathcal{U}_q(\mathfrak{gl}(n+1))$ to the Taylor extension, $TU_q(\mathfrak{gl}(n+1))$. The $\mathcal{Z}$-algebra $\mathcal{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ with this involution is called the noncompact real form and denoted by the symbol $\mathcal{Z}_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

Let $p^{(\alpha)}$ be the extremal projector for $\mathcal{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ satisfying the relations

$$z_{-i}p^{(\alpha)} = p^{(\alpha)}z_i \quad \text{for } i = 1, 2, \ldots, \alpha,$$

$$z_kp^{(\alpha)} = p^{(\alpha)}z_{-k} \quad \text{for } k = \alpha + 1, \alpha + 2, \ldots, n,$$

$$[e_{ii}, p^{(\alpha)}] = 0 \quad \text{for } i = 1, 2, \ldots, n.$$

(4.5) (4.6) (4.7)

This extremal projector depends on the index $\alpha$, which defines what elements are considered as ”raising” and what elements are considered as ”lowering”, i.e. in our case the elements $z_{-1}, z_{-2}, \ldots, z_{-\alpha}, z_{\alpha+1}, \ldots, z_n$ are raising and the elements $z_1, z_2, \ldots, z_{\alpha}, z_{-\alpha-1}, \ldots, z_{-n}$ are lowering. It should be noted that the ”raising” and ”lowering” subsets generate disjoint subalgebras in $\mathcal{Z}_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$. The operator $p^{(\alpha)}$ can be constructed in explicit form.

Let us introduce on $\mathcal{Z}_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ the following sesquilinear Shapovalov form. For any elements $x, y \in \mathcal{Z}_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ we set

$$B^{(\alpha)}(x, y) = p^{(\alpha)}y^*xp^{(\alpha)}.$$

(4.8)

Therefore the Shapovalov form also depends on the index $\alpha (\alpha = 0, 1, 2, \ldots, n)$. We fix $\alpha (\alpha = 0, 1, 2, \ldots, n)$ and for each set of nonnegative integers $\{r\} = (r_1, r_2, \ldots, r_n)$ introduce a vector in the space $\mathcal{Z}_q^{(nc)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$ by the formula

$$v^{(\alpha)}_{\{r\}} = z_{r_\alpha} \cdots z_{r_1}^* z_{-\alpha-1}^* \cdots z_{-n}^*.$$

(4.9)
Theorem 4.1 For each fixed $\alpha$ $(\alpha = 0, 1, 2, \ldots, n)$ the vectors \( \{ v_{(r)}^{(\alpha)} \} \) are pairwise orthogonal with respect to the Shapovalov form (4.8)

\[
B^{(\alpha)}(v_{(r)}^{(\alpha)}, v_{(r')}^{(\alpha)}) = \delta_{(r),(r')}B^{(\alpha)}(v_{(r)}^{(\alpha)}, v_{(r)}^{(\alpha)}) .
\]

and

\[
B^{(\alpha)}(v_{(r)}^{(\alpha)}, v_{(\{r\})}) = \left( \prod_{i=1}^{\alpha} \frac{[\varphi_{in+1} - \alpha + r_i - 1]!}{[\varphi_{in+1} - \alpha - 1]!} \right) \left( \prod_{l=\alpha+1}^{n} \frac{[\varphi_{nl+1} + \alpha + r_l]!}{[\varphi_{nl+1} - \alpha + r_l]!} \right) \times \prod_{1 \leq i < j \leq \alpha} \frac{[\varphi_{ij} + r_i - r_j]![\varphi_{ij} - 1]!}{[\varphi_{ij} + r_j]![\varphi_{ij} - r_j - 1]!} \times \prod_{\alpha+1 \leq k < l \leq n} \frac{[\varphi_{kl} - r_k + r_l]![\varphi_{kl} - 1]!}{[\varphi_{kl} - r_k - 1]![\varphi_{kl} + r_l]!} \times \prod_{1 \leq \alpha < l \leq n} \frac{[\varphi_{il} + r_i - 1]![\varphi_{il} + r_l - 1]![\varphi_{il} - 1]!}{[\varphi_{il} + r_i + r_l]![\varphi_{il} - r_l]!} p^{(\alpha)} .
\]

As a consequence of this theorem we obtain that the Shapovalov form is not degenerate on \( Z_q^{(\alpha)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \).

In the case of the compact \( Z \)-algebra \( Z_q^{(c)}(\mathfrak{gl}(n+1), \mathfrak{gl}(n)) \) the Shapovalov form \( B(x, y) \) is defined by the formula (4.8) where \( \alpha = 0 \), \( p^{(0)} \) is the standard extremal projector of the subalgebra \( \mathfrak{gl}(n) \) and the involution (*) is given by the formulas (4.1). It is not difficult to see that

\[
B(v_{(r)}, v_{(r')}) = \delta_{(r),(r')}B(v_{(r)}, v_{(r)}) .
\]

where \( v_{(r)} := v_{(r)}^{(0)} \), and

\[
B(v_{(r)}, v_{(\{r\})}) = (-1)^{\sum_{i=1}^{\alpha} r_i} B^{(0)}(v_{(r)}^{(0)}, v_{(r)}^{(0)})
\]

\[
= \prod_{l=1}^{n} \frac{[\varphi_{nl+1} - r_l]!}{[\varphi_{nl+1}]!} \prod_{1 \leq k < l \leq n} \frac{[\varphi_{kl} - r_k + r_l]![\varphi_{kl} - 1]!}{[\varphi_{kl} - r_k - 1]![\varphi_{kl} + r_l]!} .
\]

5 Discrete series of representations for \( U_q(u(n, 1)) \)

As well as in the classical case [9] each Hermitian irreducible representation of the discrete series for the noncompact quantum algebra \( U_q(u(n, 1)) \) is defined uniquely by some extremal vector \( |xw\rangle \), the vector of extremal weight\(^2\). This vector should be a highest vector with respect to the compact subalgebra \( U_q(u(n)) \oplus U_q(u(1)) \). Since the quantum algebra \( U_q(u(1)) \) is generated only by one Cartan element \( q^{n+1} \cdot n+1 \) the vector \( |xw\rangle \) should be annihilated by the raising generators \( e_{ij} \) \((1 \leq i < j \leq n)\) of the compact subalgebra \( U_q(u(n)) \). So the vector \( |xw\rangle \) satisfies the relations

\[
e_{ii}|xw\rangle = \mu_i|xw\rangle \quad (i = 1, 2, \ldots, n+1) ,
\]

\[
e_{ij}|xw\rangle = 0 \quad (1 \leq i < j \leq n) ,
\]

\(^2\)We assume that the vector \( |xw\rangle \) is orthonormalized, \( \langle xw|xw\rangle = 1 \)
where the weight components $\mu_i$ ($i = 1, 2, \ldots, n$) are integers subjected to the condition $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$.

Such weights can be compared with respect to standard lexicographic ordering. Namely, $\mu > \mu'$, where $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\mu' = (\mu'_1, \mu'_2, \ldots, \mu'_n)$, if a first nonvanishing component of the difference $\mu - \mu'$ is positive.

The component $\mu_{n+1}$ is also an integer. In a general case for finite-dimensional irreducible representations of the compact quantum algebra $U_q(u(n)) \oplus U_q(u(1))$ the weights $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\mu_{n+1}$ are not ordering. If we choose some ordering for these weights, for example, as follows $(\mu_1, \ldots, \mu_\alpha, \mu_{n+1}, \mu_{\alpha+1}, \ldots, \mu_n)$, then such $n + 1$-components weights can be compared.

The extremal vector $|xw\rangle$ has minimal such weight $\Lambda^{(\alpha)}_{n+1} := (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$ where $\lambda_{i,n+1} := \mu_i$ ($i = 1, 2, \ldots, \alpha$), $\lambda_{\alpha+1,n+1} := \mu_{n+1}$, $\lambda_{l+1,n+1} := \mu_l$ ($l = \alpha + 1, \ldots, n$).

The vector $|\Lambda^{(\alpha)}_{n+1}\rangle := |xw\rangle$ with such weight $\Lambda^{(\alpha)}_{n+1}$ satisfies the relations

$$z_i|\Lambda^{(\alpha)}_{n+1}\rangle = 0, \quad \text{for } i = 1, 2, \ldots, \alpha, \quad (5.3)$$

$$z_k|\Lambda^{(\alpha)}_{n+1}\rangle = 0, \quad \text{for } k = \alpha + 1, \alpha + 2, \ldots, n, \quad (5.4)$$

It is evident that any highest weight vector $|\Lambda^{(\alpha)}_{n+1}; \Lambda_n\rangle$ with respect to the compact subalgebra $U_q(u(n))$ has the form

$$|\Lambda^{(\alpha)}_{n+1}; \Lambda_n\rangle = z_1^{r_1} \cdots z_{\alpha-1}^{r_{\alpha-1}} \cdots z_{n+1}^{r_{n+1}} |\Lambda^{(\alpha)}_{n+1}\rangle. \quad (5.5)$$

Here the integers $\{r\}$ are defined the weights $\Lambda^{(\alpha)}_{n+1} = (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n+1,n+1})$, where $\lambda_{i,n+1} \geq \lambda_{i+1,n+1}$ ($i = 1, 2, \ldots, n$), and $\Lambda_n = (\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{nn})$, where $\lambda_{m} \geq \lambda_{i,n+1}$ ($i = 1, 2, \ldots, n - 1$). Namely,

$$r_i = \lambda_{in} - \lambda_{i,n+1} \quad (i = 1, \ldots, \alpha), \quad (5.6)$$

$$r_l = \lambda_{l+1,n+1} - \lambda_{ln} \quad (l = \alpha + 1, \ldots, n). \quad (5.7)$$

If we would like to calculate the scalar product two such vectors (5.5) then using results for the Shapovalov form we obtain

$$\langle \Lambda_n; \Lambda^{(\alpha)}_{n+1} | \Lambda^{(\alpha)}_{n+1}; \Lambda'_n \rangle = \delta_{\Lambda_n, \Lambda'_n} \langle \Lambda_n; \Lambda^{(\alpha)}_{n+1} | \Lambda^{(\alpha)}_{n+1}; \Lambda_n \rangle, \quad (5.8)$$

$$\langle \Lambda_n; \Lambda^{(\alpha)}_{n+1} | \Lambda^{(\alpha)}_{n+1}; \Lambda_n \rangle = B^{(\alpha)}(v^{(\alpha)}_{\{r\}}, v^{(\alpha)}_{\{r\}}) \big|_{\Lambda^{(\alpha)}_{n+1}}. \quad (5.9)$$

where symbol $|_{\Lambda^{(\alpha)}_{n+1}}$ means that we specialize the Shapovalov form for the extremal weight $\Lambda_{n+1}$, that is we replace the Cartan elements $e_{ii}, e_{jj}$ in the functions $\varphi_{ij}$ by corresponding components $\lambda_{i,n+1}$ and $\lambda_{j,n+1}$. From the condition that

$$\langle \Lambda_n; \Lambda^{(\alpha)}_{n+1} | \Lambda^{(\alpha)}_{n+1}; \Lambda_n \rangle > 0 \quad (5.10)$$

we find all admissible highest weights $\Lambda_n$ of the compact subalgebra $U_q(u(n))$. We formulate this result as the theorem.
Theorem 5.1 Every Hermitian irreducible representation of the discrete series for the noncompact quantum algebra $U_q(u(n,1))$ with the extremal weight $\Lambda^{(a)}_{n+1} = (\lambda_{1,n+1}, \lambda_{2,n+1}, \ldots, \lambda_{n,n+1})$, where integer $\lambda_{i,n+1}$ satisfy the inequalities $\lambda_{i,n+1} \geq \lambda_{i+1,n+1}$ ($i = 1, 2, \ldots, n$), under the restriction $U_q(u(n,1)) \downarrow U_q(u(n))$ contains all multiplicity free irreducible representations of the compact subalgebra $U_q(u(n))$ with the highest weights $\Lambda_n = (\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n})$ satisfying the conditions:

$$\lambda_{1,n} \geq \lambda_{1,n+1} \geq \lambda_{2,n} \geq \cdots \geq \lambda_{\alpha,n} \geq \lambda_{\alpha,n+1},$$

$$\lambda_{\alpha+2,n+1} \geq \lambda_{\alpha+1,n} \geq \lambda_{\alpha+3,n+1} \geq \cdots \geq \lambda_{n+1,n+1} \geq \lambda_{n,n}.$$  

(5.11)

The vectors

$$|\Lambda^{(a)}_{n+1}; \Lambda_n\rangle = F^{(a)}(\Lambda_n; \Lambda^{(a)}_{n+1})|\Lambda^{(a)}_{n+1}\rangle,$$  

(5.12)

where the "lowering" operator $F^{(a)}(\Lambda_n; \Lambda^{(a)}_{n+1})$ is given by

$$F^{(a)}(\Lambda_n; \Lambda^{(a)}_{n+1}) = N^{(a)}(\Lambda_n; \Lambda^{(a)}_{n+1}) z_\alpha^{-\lambda_{\alpha,n}+\lambda_{\alpha,n+1}} \cdots z_1^{-\lambda_{1,n}+\lambda_{1,n+1}} \times$$

$$z_\alpha^{\lambda_{\alpha+2,n+1}+\lambda_{\alpha+1,n}+\lambda_{\alpha+3,n+1}} \cdots z_{n-1}^{\lambda_{n+1,n+1}+\lambda_{n,n}},$$

(5.13)

for all highest weights $\Lambda_n = (\lambda_{1,n}, \lambda_{2,n}, \ldots, \lambda_{n,n})$ constrained by the conditions (5.11) form the orthonormal basis in the space of the highest vectors with respect to the compact subalgebra $U_q(u(n))$. Here in (5.13) the normalized factor $N^{(a)}(\Lambda_n; \Lambda^{(a)}_{n+1})$ is given as follows

$$N^{(a)}(\Lambda_n; \Lambda^{(a)}_{n+1}) = (\Lambda_n; \Lambda^{(a)}_{n+1})|\Lambda^{(a)}_{n+1}; \Lambda_n\rangle^{-\frac{1}{2}}$$

$$= \left\{ \prod_{i=1}^{\alpha} \frac{[l_{i,n+1} - l_{\alpha+1,n+1} - 2\alpha + n - 1]!}{[l_{\alpha+1,n+1} - l_{i,n+1}]! [l_{i,n+1} - l_{\alpha+1,n+1} - 2\alpha + n - 1]!} \right. \times$$

$$\left. \prod_{l=\alpha+1}^{n} \frac{[l_{i,n+1} - l_{\alpha+1,n+1} + 2\alpha - n - 1]!}{[l_{\alpha+1,n+1} - l_{i,n+1}]! [l_{i,n+1} - l_{\alpha+1,n+1} + 2\alpha - n]!} \right. \times$$

$$\left. \prod_{1 \leq i < j \leq \alpha} \frac{[l_{i,n} - l_{j,n+1}]! [l_{i,n+1} - l_{j,n}]!}{[l_{i,n} - l_{j,n}]! [l_{i,n+1} - l_{j,n+1}]!} \right. \times$$

$$\left. \prod_{\alpha+1 \leq k < l \leq n} \frac{[l_{kn} - l_{k+1,n+1} - 2]! [l_{k+1,n+1} - l_{kn} - 1]!}{[l_{kn} - l_{kn}]! [l_{k+1,n+1} - l_{kn} - 1]!} \right. \times$$

$$\left. \prod_{1 \leq i < \alpha \leq l \leq n} \frac{[l_{in} - l_{ln}]! [l_{i,n+1} - l_{ln} - 2]! [l_{i,n+1} - l_{ln} - 1]!}{[l_{in} - l_{ln}]! [l_{i,n+1} - l_{ln} - 2]! [l_{i,n+1} - l_{ln} - 1]!} \right\}^{\frac{1}{2}},$$

$(l_{sp} := \lambda_{sp} - s)$.

This result coincides with the classical Gelfand–Graev case [5, 6]. Using analogous construction of the Gelfand–Tsetlin basis for the compact quantum algebra $U_q(u(n))[12]$ we obtain a q-analog of the Gelfand–Graev–Tsetlin basis for $U_q(u(n,1))$. Namely, in the $U_q(u(n,1))$-module
with the extremal weight $\Lambda^{(\alpha)}_{n+1}$ there is the orthogonal Gelfand–Graev–Tsetlin basis consisting of all vectors of the form

$$\left| \Lambda \right> := \left( \begin{array}{c} \Lambda^{(\alpha)}_{n+1} \\ \Lambda_n \\ \vdots \\ \Lambda_2 \\ \Lambda_1 \end{array} \right) = F_{-}(\Lambda_1; \Lambda_2)F_{-}(\Lambda_2; \Lambda_3)\cdots F_{-}(\Lambda_{n-1}; \Lambda_n)|\Lambda^{(\alpha)}_n; \Lambda_n\rangle,$$

(5.15)

where $\Lambda_j = (\lambda_{1j}, \lambda_{2j}, \ldots, \lambda_{jj})$ ($j = 1, 2, \ldots, n$) and the numbers $\lambda_{ij}$ satisfy the standard ”between conditions” for the quantum algebra $U_q(u(n))$, i.e.

$$\lambda_{i,j+1} \geq \lambda_{ij} \geq \lambda_{i+1,j+1} \quad \text{for} \quad 1 \leq i \leq j \leq n - 1.$$  

(5.16)

The lowering operators $F_{-}(\Lambda_k; \Lambda_{k+1})$, ($k = 1, 2, \ldots, n - 1$), are given by

$$F_{-}(\Lambda_k; \Lambda_{k+1}) = N(\Lambda_k; \Lambda_{k+1}) p(U_q(u(k))) \prod_{i=1}^{k} (e_{k+1 i})^{\lambda_{k+1} - \lambda_{ik}},$$  

(5.17)

$$N(\Lambda_k; \Lambda_{k+1}) = \left\{ \prod_{i=1}^{k} \frac{[l_{ik} - l_{k+1,k+1} - 1]!}{[l_{i,k+1} - l_{ik}] ![l_{i,k+1} - l_{k+1,k+1} - 1]!} \right\} \times \prod_{1 \leq i < j \leq k} \frac{[l_{i,k+1} - l_{jk}] ![l_{ik} - l_{jk} - 1]!}{[l_{ik} - l_{jk}] ![l_{i,k+1} - l_{jk} - 1]!} \right\}^{1/2} \right.$$  

(5.18)

where $l_{ij} := \lambda_{ij} - i$. This explicit construction allows to obtain formulas for the action of $U_q(u(n,1))$-generators. These results will be presented in another article.

6 Summary

Thus we obtain the explicit description of the Hermitian irreducible representations for the noncompact quantum algebra $U_q(u(n,1))$ by the reduction $Z$-algebras for description of which we used the standard extremal projectors.

Next step: to obtain an analogous results for the case $U_q(u(n,2))$. For this aim we need construct extremal projector $p^{(\alpha)}$ which is expressed in terms of the $Z$-algebra $Z_q(\mathfrak{gl}(n+1), \mathfrak{gl}(n))$.

Final aim: to consider the general case $U_q(u(n,m))$. In this case extremal projectors of new type will be used.

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