Relativistic Landau Models and Generation of Fuzzy Spheres

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Abstract

Non-commutative geometry naturally emerges in low energy physics of Landau models as a consequence of level projection. In this work, we proactively utilize the level projection as an effective tool to generate fuzzy geometry. The level projection is specifically applied to the relativistic Landau models. In the first half of the paper, a detail analysis of the relativistic Landau problems on a sphere is presented, where a concise expression of the Dirac-Landau operator eigenstates is obtained based on algebraic methods. We establish $SU(2)$ “gauge” transformation between the relativistic Landau model and the Pauli-Schrödinger non-relativistic quantum mechanics. After the $SU(2)$ transformation, the Dirac operator and the angular momentum operators are found to satisfy the $SO(3,1)$ algebra. In the second half, the fuzzy geometries generated from the relativistic Landau levels are elucidated, where unique properties of the relativistic fuzzy geometries are clarified. We consider mass deformation of the relativistic Landau models and demonstrate its geometrical effects to fuzzy geometry. Super fuzzy geometry is also constructed from a supersymmetric quantum mechanics as the square of the Dirac-Landau operator. Finally, we apply the level projection method to real graphene system to generate valley fuzzy spheres.
C Geometric Quantities of Two-sphere

D Dirac Gauge
1 Introduction

Quantization of the space-time is one of the most fundamental problems in physics. Non-commutative geometry is a promising mathematical framework for the description of quantized space-time [1]. While string theory or matrix theory also suggests appearance of non-commutative geometry [2], the natural energy scale of the non-commutative geometry is considered to be the Planck scale. Interestingly, however, it is well recognized that in low energy physics of some real materials, non-commutative geometry naturally emerges. A well known example is the lowest Landau level physics of the quantum Hall effect, where the electron coordinates effectively satisfy non-commutative algebra due to the presence of strong magnetic field [see [3] and references therein]. More precisely, non-commutative geometry appears in any of the Landau levels as well as the lowest Landau level as a consequence of the level projection. Recently, higher dimensional non-commutative geometry has begun to be applied to studies of topological insulators [4, 5, 6, 7, 8, 9, 10].

Usually, non-commutative geometry is imposed on theories of interest in the beginning, and within the mathematical framework we develop physical theories such as non-commutative quantum field theory. On the other hand, in the set-up of Landau models, non-commutative geometry is not postulated a priori but “generated” as a consequence of level projection. In the work, we proactively utilize the level projection as a tool to derive fuzzy geometries. The merits of this scheme are the following. First, the level projection basically yields a consistent framework of non-commutative geometry. Generally it is far from obvious whether non-commutative geometry can be incorporated in any manifolds, for instance, to curved manifolds, keeping mathematical consistency. However, in the level projection scheme, we have a consistent Hilbert space of the quantum mechanics, and the level projection is just a method to extract a specific subspace of the consistent Hilbert space. Since the whole Hilbert space is well defined, we need not to bother with the mathematical inconsistency in introducing the subspace and the corresponding non-commutative geometry as well. Second, the level projection is rather mechanical, and one can readily introduce fuzzy geometry by following simple instructions to construct effective matrix representation in the subspace. Last, since the level projection scheme is based on physical ideas, mathematics of non-commutative geometry can be understood from a physical point of view, as we shall see in this work.

In the first half of this work, we investigate relativistic Landau models described by Dirac-Landau operator on a sphere. (We shall refer to the Dirac operator in magnetic field as Dirac-Landau operator.) We thus exploit a relativistic counterpart of the Haldane’s sphere [11]. Apart from applications to non-commutative geometry, the relativistic Landau models have increasing importance in recent developments of Dirac matter such as graphene and topological insulator [there are many excellent books and reviews: see [12, 13, 14, 15] for instance]. Theoretical works of Dirac matter with Landau levels on a spherical geometry can be found in Refs.[17, 18] for fullerene, Refs.[19, 20] for the surface of topological insulator, and Refs.[4, 16] for higher dimensional topological insulators. Though the Dirac-Landau equation in flat space has already been intensively investigated in various physical and mathematical contexts [21, 22] and on a sphere as well [23, 24], many studies on a sphere are restricted to zero mode solutions. We present a
full analysis of the relativistic Landau model on a sphere including all relativistic Landau level eigenstates. Our method is based on an algebraic method, which provides a concise way to solve the Dirac-Landau operator and highlights a transparent $SU(2)$ rotational symmetry of the present geometry [Sec.3]. We establish $SU(2)$ transformation between the relativistic Landau model and the Pauli-Schwinger non-relativistic quantum mechanics obtained by Kazama et al. almost forty years ago [20] [Sec.4]. After the $SU(2)$ transformation, the transformed Dirac operator and the angular momentum operators are shown to satisfy the $SO(3,1)$ algebra, which is the “hidden” symmetry of the system. In the second half, we discuss fuzzy geometries generated by the level projection in the relativistic Landau models. In correspondence to each of the relativistic Landau levels, a relativistic fuzzy sphere is derived. We compare behaviors of the relativistic and non-relativistic fuzzy spheres with respect to magnetic field, where particular properties of the relativistic fuzzy spheres are observed [Sec.5]. We also investigate properties of fuzzy spheres under mass deformation [Sec.6]. Interestingly, the relativistic fuzzy spheres for opposite sign Landau levels balance their sizes keeping the sum of their radii invariant. As the square of the Dirac-Landau operator, a supersymmetric quantum mechanics is constructed, where we demonstrate appearance of super fuzzy spheres [Sec.7]. Finally we apply the results to a realistic Dirac material, graphene, to investigate fuzzy geometries with valley degrees of freedom and behaviors under the change of mass parameter [Sec.8]. Sec.2 is a review about the non-relativistic Landau problem and Sec.9 is devoted to summary and discussions.

2 Review of the Non-Relativistic Landau Problem

2.1 Monopole harmonics

As a preliminary, we give a rather detail review of non-relativistic quantum mechanics for a charge-monopole system mainly based on Refs. [22, 27, 28]. We use the standard spherical coordinates,

\[ x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \tag{1} \]

and adopt the Schwinger gauge \[27^2\] [see Appendix D for the Dirac gauge] in which the monopole gauge field is given by

\[ A = ge_{ij3} \frac{z}{r(x^2 + y^2)} x_j dx_i = -g \cos \theta d\phi, \tag{2} \]

or

\[ A_x = g \frac{z}{r(x^2 + y^2)} y = g \frac{1}{r} \cot \theta \cdot \sin \phi, \]

\[ A_y = -g \frac{z}{r(x^2 + y^2)} x = -g \frac{1}{r} \cot \theta \cdot \cos \phi, \]

\[ A_z = 0, \tag{3} \]

where $g$ denotes the monopole charge. In this paper, we consider the case $g \geq 0$. (It is not difficult to expand similar discussions for $g < 0$.) In the Schwinger gauge the gauge field exhibits

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1 The readers may find an analytic method for solving the Dirac-Landau equation in Ref. [25].

2 We utilize terminology, Schwinger gauge, instead of the Schwinger formalism in Ref. [27].
an infinite line singularity on the z-axis, and the direction of the monopole gauge field on the north hemisphere is opposite to that on the south hemisphere (on the equator, the monopole gauge field vanishes). The corresponding field strength is given by

\[ F = dA = g \sin \theta \, d\theta \wedge d\phi, \]  

or

\[ F_i = \epsilon_{ijk} \partial_j A_k = \frac{1}{r^3} x_i. \]  

The covariant derivative is constructed as

\[ D_i = \partial_i - iA_i, \]  

or

\[ -iD_r = -i\partial_r, \quad -iD_\theta = -i\partial_\theta, \quad -iD_\phi = -i\partial_\phi + g \cos \theta, \]  

and the covariant angular momentum is

\[ \Lambda_i^{(g)} = -i\epsilon_{ijk} x_j D_k, \]  

or

\[ \Lambda_x^{(g)} = L_x^{(0)} - g \frac{z^2}{r(x^2 + y^2)} x = L_x^{(0)} - g \frac{\cos^2 \theta}{\sin \theta} \cos \phi, \]

\[ \Lambda_y^{(g)} = L_y^{(0)} - g \frac{z^2}{r(x^2 + y^2)} y = L_y^{(0)} - g \frac{\cos^2 \theta}{\sin \theta} \sin \phi, \]

\[ \Lambda_z^{(g)} = L_z^{(0)} + \frac{1}{r} z = L_z^{(0)} + g \cos \theta. \]  

Here, \( L_i^{(0)} = -i\epsilon_{ijk} x_j \frac{\partial}{\partial x_k} \) represent the free orbital angular momentum operators:

\[ L_x^{(0)} = i(\sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi), \]
\[ L_y^{(0)} = -i(\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi), \]
\[ L_z^{(0)} = -i\partial_\phi. \]  

The total SU(2) angular momentum is constructed as the sum of the covariant and the field angular momenta:

\[ L^{(g)} = \Lambda^{(g)} - r^2 F = \Lambda^{(g)} - \frac{1}{r} x, \]  

or

\[ L_x^{(g)} = i(\sin \phi D_\theta + \cos \phi \cot \theta D_\phi) - g \frac{x}{r}, \]

\[ L_y^{(g)} = -i(\cos \phi D_\theta - \sin \phi \cot \theta D_\phi) - g \frac{y}{r}, \]

\[ L_z^{(g)} = -iD_\phi - g \frac{z}{r}. \]  

3In the Dirac gauge [see Appendix D], the singularity of the gauge field is a semi-infinite string either on the positive z-axis or on the negative z-axis, and the directions of the monopole gauge fields are same on both hemispheres.
With use of (10), they are expressed as
\[ L_x^{(g)} = L_x^{(0)} - g\frac{r}{x^2 + y^2}x = L_x^{(0)} - g\frac{\cos \phi}{\sin \theta}, \]
\[ L_y^{(g)} = L_y^{(0)} - g\frac{r}{x^2 + y^2}y = L_y^{(0)} - g\frac{\sin \phi}{\sin \theta}, \]
\[ L_z^{(g)} = L_z^{(0)}. \] (13)

The square of \(L^{(g)}\) can be represented as
\[ L^{(g^2)} = \frac{1}{\sin \theta} \partial \theta \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial \phi} + ig \cos \theta \right)^2 + g^2 \]
\[ = L^{(0)^2} - 2ig \frac{\cos \theta}{\sin^2 \theta} \frac{\partial}{\partial \phi} + g^2 \frac{1}{\sin^2 \theta} \]
\[ = L^{(0)^2} + 2g \frac{r}{x^2 + y^2} L_z^{(0)} + g^2 \frac{r^2}{x^2 + y^2}, \] (14)

where
\[ L^{(0)^2} = \frac{1}{\sin \theta} \partial_b (\sin \theta \partial_b) - \frac{1}{\sin^2 \theta} \partial_b^2. \] (15)

The monopole harmonics are introduced as the simultaneous eigenstates of \(L^{(g)^2}\) and \(L_z^{(g)}\):
\[ L^{(g)^2} Y_{l,m}^g (\theta, \phi) = l(l + 1) Y_{l,m}^g (\theta, \phi), \]
\[ L_z^{(g)} Y_{l,m}^g (\theta, \phi) = m Y_{l,m}^g (\theta, \phi), \] (16)

where \(l\) and \(m\) take the following values [28]:
\[ l = g + n \quad (n = 0, 1, 2, \cdots), \]
\[ m = -l, -l + 1, \cdots, l - 1, l. \] (17b)

The ladder operators are given by
\[ L_+^{(g)} = L_x^{(g)} + i L_y^{(g)} = \rho \phi + \cot \theta \partial \phi - g \frac{1}{\sin \theta}, \]
\[ L_-^{(g)} = L_x^{(g)} - i L_y^{(g)} = \rho \phi - \cot \theta \partial \phi - g \frac{1}{\sin \theta}, \] (18)

which act to the monopole harmonics as
\[ L_{\pm}^{(g)} Y_{l,m}^g = \sqrt{(l \pm m)(l \pm m + 1)} Y_{l,m \pm 1}^g. \] (19)

The irreducible representation of the monopole harmonics can be obtained by applying the \(SU(2)\) ladder operators to the lowest or highest weight state. The monopole harmonics are explicitly given by [28, 27]
\[ Y_{l,m}^g (\theta, \phi) = 2^m \sqrt{\frac{(2l + 1)(l - m)!(l + m)!}{4\pi(l - g)!(l + g)!}} (1 - x)^{-\frac{m + g}{2}} (1 + x)^{-\frac{m - g}{2}} P_{l+m}^{(m-g,-m+g)}(x) \cdot e^{im\phi} \]
\[ = \sqrt{\frac{(2l + 1)(l - m)!(l + m)!}{4\pi(l - g)!(l + g)!}} \left( \sin \frac{\theta}{2} \right)^{-(m + g)} \left( \cos \frac{\theta}{2} \right)^{-(m - g)} P_{l+m}^{(m-g,-m+g)}(\cos \theta) \cdot e^{im\phi}, \] (20)
where \(P_n^{(\alpha,\beta)}(x)\) denote the Jacobi polynomials [Appendix A]. For uniqueness of the wavefunction, the magnetic quantum number of the azimuthal part of (20) has to take an integer value, \(m = 0, \pm 1, \pm 2, \ldots\). Due to (17b), the monopole charge \(g\) should be quantized as an integer in the Schwinger gauge [27]. Expressing the Jacobi polynomials by the trigonometric function, (20) can be rewritten as [29]

\[
Y_{l,m}^g(\theta, \phi) = (-1)^{l+m} \sqrt{\frac{(2l+1) (l+m)! (l-m)!}{4\pi (l+g)! (l-g)!}} e^{im\phi} 
\sum_n (-1)^n \binom{l-g}{n} \binom{l+g}{g-m+n} (-1)^n u^{g-m+n} v^{l-n+m} u^* n v^* l-g-n,
\]

(21)

or

\[
Y_{l,m}^g(\theta, \phi) = (-1)^{l+m} \sqrt{\frac{(2l+1) (l+m)! (l-m)!}{4\pi (l+g)! (l-g)!}} 
\sum_n \binom{l-g}{n} \binom{l+g}{g-m+n} (-1)^n u^{g-m+n} v^{l-n+m} u^* n v^* l-g-n,
\]

(22)

where \(u\) and \(v\) are the components of the Hopf spinor [3]:

\[
u = \cos \frac{\theta}{2} e^{-i\frac{\phi}{2}}, \quad v = \sin \frac{\theta}{2} e^{i\frac{\phi}{2}},
\]

(23)

and \(u^*\) and \(v^*\) are their complex conjugates. For instance, in the case \(g = 1\) and \(l = 2\), we have

\[
Y_{2,2}^1 = -\sqrt{\frac{5}{\pi}} \sin^3 \frac{\theta}{2} \cos \frac{\theta}{2} e^{i\phi}, \quad Y_{2,1}^1 = \frac{1}{2} \sqrt{\frac{5}{\pi}} (1 + 2 \cos \theta) \sin^2 \frac{\theta}{2} e^{i\phi},
\]

\[
Y_{2,0}^1 = -\frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta, \quad Y_{2,-1}^1 = \frac{1}{2} \sqrt{\frac{5}{\pi}} (-1 + 2 \cos \theta) \cos^2 \frac{\theta}{2} e^{-i\phi},
\]

\[
Y_{2,-2}^1 = \sqrt{\frac{5}{\pi}} \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2} e^{-2i\phi}.
\]

(24)

The non-relativistic Landau Hamiltonian in a monopole background is given by [11]

\[
H = -\frac{1}{2M} \sum_{i=1}^{3} D_i^2
- \frac{1}{r^2} D_r (r^2 D_r) + \frac{1}{r^2 \sin \theta} D_\theta (\sin \theta D_\theta)
+ \frac{1}{r^2 \sin^2 \theta} D_\phi^2
\]

\[
= -\frac{1}{2M} \frac{\partial^2}{\partial r^2} - \frac{1}{M r} \frac{\partial}{\partial r} + \frac{1}{2Mr^2} \Lambda^{(g)^2}.
\]

(25)

which, on a sphere \(r = 1\), reduces to

\[
H^{(g)} = -\frac{1}{2M} \left( \frac{1}{\sin \theta} D_\theta (\sin \theta D_\theta) + \frac{1}{\sin^2 \theta} D_\phi^2 \right) = \frac{1}{2M} \Lambda^{(g)^2} = \frac{1}{2M} (L^{(g)^2} - g^2).
\]

(26)
In the following, we take \( r = 1 \). (We sometimes recover \( r \) to indicate the dimensions of quantities of interest.) Since we have already solved the eigenvalue problem of \( L^{(g)} \), the eigenvalues of \( (26) \) can readily be obtained as

\[
E_{n}^{(g)} = \frac{1}{2M} (n(n + 1) + g(2n + 1)),
\]

where we used (17a), and the degenerate eigenstates of the \( n \)th Landau level are the monopole harmonics (20) with degeneracy,

\[
2l + 1 = 2g + 1 + 2n.
\]

In the lowest Landau level \( n = 0 \) (\( l = g \)), the monopole harmonics are represented as

\[
Y_{g, m}^{(g)}(\theta, \phi) = (-1)^{m+g} \sqrt{\frac{(2g + 1)!}{4\pi(g + m)!(g - m)!}} \left( \frac{\sin \theta}{2} \right)^{m+g} \left( \frac{\cos \theta}{2} \right)^{-m+g} e^{im\phi}
\]

\[
= (-1)^{m+g} \sqrt{\frac{(2g + 1)!}{4\pi(g + m)!(g - m)!}} u^{g-m}v^{g+m}.
\]

The lowest Landau level eigenstates are homogeneous holomorphic polynomials of the Hopf spinor.

### 2.2 Edth operators

The monopole harmonics carry two \( SU(2) \) spin indices, \( m \) and \( g \). (With fixed \( l \), both \( m \) and \( g \) range from \( -l \) to \( l \).) One may expect that ladder operators for \( g \) may exist just like the ladder operators, \( L^{(g)} \), for \( m \). Such operators are known as the edth differential operators [30].

\[
\delta^{(g)}_+ \equiv (\sin \theta)^g (\partial_\theta + i \frac{1}{\sin \theta} \partial_\phi)(\sin \theta)^{-g} = \partial_\theta - g \cot \theta + i \frac{1}{\sin \theta} \partial_\phi
\]

\[
\delta^{(g)}_- \equiv (\sin \theta)^{-g} (\partial_\theta - i \frac{1}{\sin \theta} \partial_\phi)(\sin \theta)^g = \partial_\theta + g \cot \theta - i \frac{1}{\sin \theta} \partial_\phi,
\]

or

\[
\delta^{(g)}_+ = D_\theta + i \frac{1}{\sin \theta} D_\phi,
\]

\[
\delta^{(g)}_- = D_\theta - i \frac{1}{\sin \theta} D_\phi,
\]

\(^4\text{For } g < 0, \text{ the monopole harmonics in the lowest Landau level } (l = |g|) \text{ are given by}

\[
Y_{|g|, m}^{(g)}(\theta, \phi) = \sqrt{\frac{(2|g| + 1)!}{4\pi(|g| + m)!(|g| - m)!}} (u^*)^{|g|+m} (v^*)^{|g|-m}.
\]

\(^5\text{This is the basic observation about the equivalence between the monopole harmonics and spin-weighted spherical harmonics [31, 32].}

\(^6\delta_+ \text{ and } \delta_- \text{ respectively correspond to } \bar{\delta} \text{ and } \bar{\delta} \text{ in Refs. [30, 31, 32].}
where $D_\theta = \partial_\theta$ and $D_\phi = \partial_\phi + ig \cos \theta$ are the covariant derivatives \[7\]. The edth operators indeed act to the monopole harmonics as \[31, 32\] 

\[
\bar{\sigma}^{(l)} Y_{l,m}^{g}(\theta, \phi) = \sqrt{(l-g)(l+g+1)} \ Y_{l,m}^{g+1}(\theta, \phi), \\
\bar{\sigma}^{(l)} Y_{l,m}^{g}(\theta, \phi) = -\sqrt{(l+g)(l-g+1)} \ Y_{l,m}^{g-1}(\theta, \phi). \tag{35}
\]

Notice that, while $\bar{\sigma}^{(l)}_+$ and $\bar{\sigma}^{(l)}_-$ respectively increases and decreases the monopole charge by 1, they are inert with the $SU(2)$ index $l$ (and the magnetic quantum number $m$). Therefore, in the language of Landau level $n$ act to the monopole harmonics as \[31, 32\].

Th e edth and angular momentum operators are mutually commutative:

\[
L^{(g+1)} \bar{\sigma}^{(g)} - \bar{\sigma}^{(g+1)} L^{(g)} = 0, \quad L^{(g-1)} \bar{\sigma}^{(g)} - \bar{\sigma}^{(g-1)} L^{(g)} = 0. \tag{38}
\]

From \[31\], we obtain

\[
\bar{\sigma}^{(g-1)}_+ \bar{\sigma}^{(g)}_- - \bar{\sigma}^{(g+1)}_- \bar{\sigma}^{(g)}_+ = -2g, \tag{36a}
\]

and

\[
\bar{\sigma}^{(g-1)}_+ \bar{\sigma}^{(g)}_- + \bar{\sigma}^{(g+1)}_- \bar{\sigma}^{(g)}_+ = -2(L^{(g)}_z - g^2). \tag{36b}
\]

These relations are essentially the same as of the ladder operators (in the $L_z$ diagonalized basis) with replacement of $m$ with $g$:

\[
L^{(g)}_+ L^{(g)}_- - L^{(g)}_- L^{(g)}_+ = 2m, \tag{37a}
\]

and

\[
L^{(g)}_+ L^{(g)}_- + L^{(g)}_- L^{(g)}_+ = 2(L^{(g)}_z - m^2). \tag{37b}
\]

From the point of view of three-sphere, the analogies between the edth operators and the angular momentum operators are clearly understood [Appendix\[B\]]. The edth and angular momentum operators are mutually commutative:

\[
L^{(g+1)} \bar{\sigma}^{(g)} - \bar{\sigma}^{(g+1)} L^{(g)} = 0, \quad L^{(g-1)} \bar{\sigma}^{(g)} - \bar{\sigma}^{(g-1)} L^{(g)} = 0. \tag{38}
\]

In the Cartesian coordinates, the edth operators are represented as

\[
\bar{\sigma}^{(g)}_+ = \frac{z}{\sqrt{x^2 + y^2}} (x \partial_x + y \partial_y) - \sqrt{x^2 + y^2} \partial_z \pm \frac{r}{\sqrt{x^2 + y^2}} (x \partial_x - y \partial_y) \mp g \frac{z}{\sqrt{x^2 + y^2}} \tag{33}
\]

or, with use of the angular momentum operators \[10\] and \[9\],

\[
\bar{\sigma}^{(g)}_+ = i \frac{1}{\sqrt{x^2 + y^2}} (x L_y^{(s)} - y L_x^{(s)}) \mp g \frac{z}{\sqrt{x^2 + y^2}} \mp \frac{r}{\sqrt{x^2 + y^2}} L_z^{(s)} \\
= i \frac{1}{\sqrt{x^2 + y^2}} (x \Lambda_y^{(s)} - y \Lambda_x^{(s)} \mp r \Lambda_z^{(s)}). \tag{34}
\]
In other words, the edth operators are singlet under the $SU(2)$ angular momentum transformations. Due to the relation (36b), the Landau Hamiltonian (26) can be expressed as

$$H^{(g)} = -\frac{1}{4M}(\hat{\sigma}_+^{(g-1)}\hat{\sigma}_-^{(g)} + \hat{\sigma}_-^{(g+1)}\hat{\sigma}_+^{(g)})$$

$$= -\frac{1}{2M}\hat{\sigma}_-^{(g+1)}\hat{\sigma}_+^{(g)} + \frac{g}{2M}. \quad (40)$$

Eq. (38) implies that the Hamiltonian (40) is invariant under the $SU(2)$ rotations:

$$[H^{(g)}, L^{(g)}] = 0. \quad (41)$$

It is straightforward to confirm that $Y_{l,m}^{g}(\theta, \phi)$ is the eigenstate of the Hamiltonian (40) with the eigenvalues (27) with use of (35). One may find analogies between (40) and the Landau Hamiltonian on a plane, $H_{\text{plane}} = -\frac{1}{2M}(D_x^2 + D_y^2)$ with $[D_x, D_y] = -iB$:

$$H_{\text{plane}} = -\frac{1}{4M}(\bar{D}\bar{D} + \bar{D}D)$$

$$= -\frac{1}{2M}\bar{D}D + \frac{B}{2M}, \quad (42)$$

where $D = D_x + iD_y$, and $\bar{D} = D_x - iD_y$. The covariant derivatives satisfy

$$[D, \bar{D}] = -2i[D_x, D_y] = -2B, \quad (43)$$

which corresponds to (36a). Also from these relations, the edth operators turn out to play the covariant derivatives of the Landau model on the sphere. Furthermore, the center-of-mass coordinates, $X = x - i\frac{1}{2}B D_y$ and $Y = y + i\frac{1}{2}B D_x$, or the magnetic translation operators which commute with the covariant derivatives correspond to the angular momentum operator $L^{(g)}$ on the sphere. Then the correspondences between the plane and sphere cases are summarized as

$$D, \bar{D} \leftrightarrow \hat{\sigma}_+, \hat{\sigma}_-,$$

$$X, Y \leftrightarrow L_x, L_y, L_z. \quad (44)$$

### 3 Relativistic Landau Problem on a Sphere

#### 3.1 Spin connection and the $SU(2)$ angular momentum operator

From the metric on a two-sphere

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad (45)$$

Alternatively using (32), one may explicitly verify

$$-\frac{1}{2M}\hat{\sigma}_-^{(g+1)}\hat{\sigma}_+^{(g)} + \frac{g}{2M} = -\frac{1}{2M}\left(\frac{1}{\sin \theta}D_\theta (\sin \theta D_\theta) + \frac{1}{\sin^2 \theta}D_\phi^2\right) = H^{(g)}. \quad (39)$$
zweibein can be adopted as [see Appendix C for details]
\[ e^1 = d\theta, \quad e^2 = \sin \theta d\phi. \] (46)

The torsion free condition, \( de^a + \omega_{ab} e^b = 0 \), determines the spin connection:
\[ \omega_{12} = -\omega_{21} = -\cos \theta d\phi. \] (47)

We choose the \( SO(2) \) gamma matrices and generator as
\[ \gamma^1 = \sigma_x, \quad \gamma^2 = \sigma_y, \]
\[ \sigma^{12} = -\sigma^{21} = -i\frac{1}{4} [\gamma^a, \gamma^b] = \frac{1}{2} \sigma_z, \] (48)
to have matrix valued spin connection
\[ \omega = \frac{1}{2} \omega_{ab} \sigma^{ab} = -\frac{1}{2} \sigma_z \cos \theta d\phi. \] (49)

Notice that (49) coincides with the monopole gauge field (2) with \( g = -\frac{1}{2} \sigma_z \). This is because that the \( SO(2) \) holonomy of the base-manifold \( S^2 \) is isomorphic to the \( U(1) \) gauge group of the monopole. Consequently, the spin connection effectively modifies the monopole charge by \( \mp \frac{1}{2} \) depending on up and down-components of the spinor. The components of the Dirac-Landau operator are given by
\[ -iD_{\mu} = -i\partial_{\mu} + \omega_{\mu} \otimes 1 - 1 \otimes A_{\mu} = -i\partial_{\mu} - A_{\mu}, \] (50)

where \( A \) denotes a matrix valued gauge field:
\[ A = -g_s \cos \theta d\phi, \] (51)

with
\[ g_s \equiv 1 \otimes g - \frac{1}{2} \sigma_z \otimes 1 = g - \frac{1}{2} \sigma_z. \] (52)

(50) is thus obtained as
\[ -iD_{\theta} = -i\partial_{\theta}, \quad -iD_{\phi} = -i\partial_{\phi} + g_s \cos \theta. \] (53)

It is straightforward to expand similar discussions to Section 2 with replacement:
\[ g \rightarrow g_s. \] (54)

The field strength for \( A \) is derived as
\[ F_{\theta\phi} = -i[D_{\theta}, D_{\phi}] = \partial_{\theta} A_{\phi} - \partial_{\phi} A_{\theta} = g_s \sin \theta, \] (55)
or
\[ F_i = g_s \frac{1}{r^3} x_i. \] (56)
The total angular momentum operator is
\[ \mathbf{J} = \mathbf{L}^{(g_s)} = \begin{pmatrix} L^{(g - \frac{1}{2})} & 0 \\ 0 & L^{(g + \frac{1}{2})} \end{pmatrix}, \]  
(57)
or
\[ J_x = i(\sin \phi \partial_\theta + \cos \phi \cot \theta \partial_\phi) + g_s \frac{1}{r} x, \]
\[ J_y = -i(\cos \phi \partial_\theta - \sin \phi \cot \theta \partial_\phi) + g_s \frac{1}{r} y, \]
\[ J_z = -i \partial_\phi + g_s \frac{1}{r} z, \]
(58)
which satisfy the \( SU(2) \) algebra:
\[ [J_i, J_j] = i \epsilon_{ijk} J_k. \]  
(59)
\( \mathbf{J} \) can be represented as
\[ J_x = L_x^{(0)} - g_s \frac{\cos \phi}{\sin \theta}, \quad J_y = L_y^{(0)} - g_s \frac{\sin \phi}{\sin \theta}, \quad J_z = L_z^{(0)}, \]
(60)
and the \( SU(2) \) Casimir operator is
\[ \mathbf{J}^2 = L^{(0)2} - 2 ig_s \frac{\cos \theta}{\sin^2 \theta} \partial_\phi + g_s^2 \frac{1}{\sin^2 \theta} \]
\[ = L^{(0)2} - 2 ig \frac{\cos \theta}{\sin^2 \theta} \partial_\phi + \frac{1}{4 \sin^2 \theta} (1 + 4g^2) + i \frac{1}{\sin^2 \theta} \sigma_z (\cos \theta \partial_\phi + ig). \]
(61)
Since \( \mathbf{J} \) commutes with the chiral matrix \( \sigma_z \):
\[ [\sigma_z, J_i] = 0, \]
(62)
we can diagonalize \( \mathbf{J}^2 \) in each chiral sector. The eigenvalues of \( \mathbf{J}^2 \) are given by
\[ j(j + 1), \]
(63)
where
\[ j = g - \frac{1}{2} + n \quad (n = 0, 1, 2, \ldots). \]
(64)
For \( j = g - \frac{1}{2} \), the corresponding eigenstates are
\[ \Upsilon^g_{j = g - \frac{1}{2}, m} = \begin{pmatrix} Y^{g - \frac{1}{2}}_{j = g - \frac{1}{2}, m}(\theta, \phi) \\ 0 \end{pmatrix}, \]
(65)
with degeneracy \( 2j + 1 |_{j = g - \frac{1}{2}} = 2g \), while for \( j = g - \frac{1}{2} + n \ (n = 1, 2, \ldots) \), the corresponding eigenstates are
\[ \Upsilon^g_{j = g - \frac{1}{2} + n, m} = \begin{pmatrix} Y^{g - \frac{1}{2}}_{j = g - \frac{1}{2} + n, m}(\theta, \phi) \\ 0 \end{pmatrix}, \quad \Upsilon^g_{j = g - \frac{1}{2} + n, m} = \begin{pmatrix} 0 \\ Y^{g + \frac{1}{2}}_{j = g - \frac{1}{2} + n, m}(\theta, \phi) \end{pmatrix}, \]
(66)
with degeneracy \( 2 \cdot (2j + 1) |_{j = g - \frac{1}{2} + n} = 4(g + n) \).

\[ ^9 \text{Strictly speaking, Eq. (64) holds for non-zero } g. \text{ For } g = 0, \text{ we have } j = \frac{1}{2} + n \ (n = 0, 1, 2, \ldots). \]
3.2 Dirac-Landau operator and eigenvalue problem

Using (53), we construct the Dirac-Landau operator, 

\[-i\mathcal{P} = -ie_\mu \gamma^m D_\mu,\]

as

\[-i\mathcal{P} = -i\sigma_x(\partial_\theta + \frac{1}{2} \cot \theta) - i\sigma_y \frac{1}{\sin \theta}(\partial_\phi + i g \cos \theta)\]

\[= \left( -i \partial_\theta + \frac{1}{\sin \theta}(\partial_\phi + i (g + \frac{1}{2}) \cos \theta) \begin{array}{cc} 0 & -i g \frac{1}{2} \cos \theta \end{array} \right) \begin{array}{cc} 0 \end{array} \right).

(67)

With the Pauli operators (31) or

\[\begin{array}{c} \sigma_x \\ \sigma_y \end{array} = \begin{array}{cc} 0 & -i(\frac{1}{2}(g + \frac{1}{2}) - \frac{1}{2}(g - \frac{1}{2})) \\ -i(\frac{1}{2}(g - \frac{1}{2}) - \frac{1}{2}(g + \frac{1}{2})) & 0 \end{array},

\]

the Dirac-Landau operator is concisely expressed as

\[-i\mathcal{P} = -i\sigma_x \sigma_x^{(g_s)} - i\sigma_y \sigma_y^{(g_s)} = \begin{array}{cc} 0 & -i \sigma_x^{(g_s)} \\ -i \sigma_y^{(g_s)} & 0 \end{array}.

(68)

Note \(\sigma_x^{(g_s)} = D_\theta\) and \(\sigma_y^{(g_s)} = \frac{1}{\sin \theta} D_\phi\). The spin connection term \[^{11}\] induces a difference between monopole charges by 1 in the off-diagonal components, and such “discrepancy” is crucial in the following discussions.

It is not difficult to derive the eigenvalues of the Dirac-Landau operator on a sphere \[^{35}\]. The square of the Dirac-Landau operator gives the SU(2) Casimir of the angular momentum \(J\):

\[(-i\mathcal{P})^2 = -\begin{array}{cc} 0 & -i \sigma_x^{(g_s)} \\ -i \sigma_y^{(g_s)} & 0 \end{array} \begin{array}{cc} 0 & \sigma_x^{(g_s)} \\ \sigma_y^{(g_s)} & 0 \end{array} = \begin{array}{cc} L(g - \frac{1}{2})^2 + \frac{1}{4} - g^2 & 0 \\ 0 & L(g + \frac{1}{2})^2 + \frac{1}{4} - g^2 \end{array} \]

\[= J^2 + \frac{1}{4} - g^2,

(70)

where we used (36). Eq.(70) is consistent with the general formula \[^{35}\]:

\[(-i\mathcal{P})^2 = J^2 - g^2 + \frac{R}{8},\]

(71)

with scalar curvature \(R = 2\) for two-sphere. Therefore the eigenvalues of \((-i\mathcal{P})^2\) are obtained as

\[(-i\mathcal{P})^2 = (j + \frac{1}{2} - g)(j + \frac{1}{2} + g) = n(2g + n),\]

(72)

\[^{10}\] The edth operators are generally given by \(\sigma_m^{(g_s)} = e_\mu D_\mu\) \((m = x, y)\). See Appendix D also.

\[^{11}\] The spin connection term yields the non-hermitian term, \(-i\frac{1}{2} \cot \theta\), in (67). It is well known that on 2D manifolds, the spin connection term vanishes when we modify the Dirac operator to be hermitian [see \[^{33}\] or \[^{34}\] for instance]. Though the present Dirac operator contains the non-hermitian term, its eigenvalues are real numbers.
and those of the Dirac-Landau operator are
\[ \pm \lambda_n = \pm \sqrt{n(2g + n)} \quad (n = 0, 1, 2, \ldots). \] (73)
The eigenstates of the square of the Dirac-Landau operator are exactly the same as of the \( SU(2) \) Casimir \( J^2 \). For \( n = 0 \), the eigenstates of \((-i\hat{P})^2\) are \( \Upsilon^{g}_{j=g-\frac{1}{2}+n,m} \) \((65)\) with degeneracy \( 2g \), and for \( n = 1, 2, \ldots \) the eigenstates are \( \Upsilon^{g}_{j=g-\frac{1}{2}+n,m} \) and \( \Upsilon^{g}_{j=g-\frac{1}{2}+n,m} \) \((66)\) with degeneracy \( 4(g + n) \).

From Eqs. (57) and (69), we can verify that the Dirac-Landau operator itself is invariant under the \( SU(2) \) rotations,
\[ [J, \hat{P}] = \begin{pmatrix} \begin{pmatrix} 0 & \mathcal{O}(g^{+\frac{1}{2}}) - \mathcal{O}(g^{-\frac{1}{2}}) L(g^{+\frac{1}{2}}) + \mathcal{O}(g^{-\frac{1}{2}}) L(g^{-\frac{1}{2}}) \\ L^{(g+\frac{1}{2})} \mathcal{O}(g^{-\frac{1}{2}}) & 0 \end{pmatrix} \end{pmatrix} = 0, \] (74)
where (85) was used. Since the Dirac operator is invariant under the \( SU(2) \) transformation, the relativistic Landau levels have the \( SU(2) \) degeneracy and the eigenstates of the Dirac-Landau operator may be constructed by some linear combination of the eigenstates of \((-i\hat{P})^2\), i.e., \( \Upsilon^{g}_{j,m} \) and \( \Upsilon^{g}_{j,m} \). The Dirac-Landau operator also respects the chiral “symmetry”:
\[ \{-i\hat{P}, \sigma_z\} = 0, \] (75)
and the eigenstates for opposite sign eigenvalues are related by the chiral transformation except for the zero modes.

### 3.2.1 Zero modes \((n = 0)\)

For \( n = 0 \), the relativistic Landau level and the \( SU(2) \) index are respectively given by
\[ \lambda_{n=0} = 0, \quad j = g - \frac{1}{2}, \] (77)
and the corresponding zero modes are\(^{13}\)
\[ \Psi^{g}_{\lambda_0=0,m}(\theta, \phi) = \begin{pmatrix} \Upsilon^{g}_{j=g+\frac{1}{2},m}(\theta, \phi) \\ \mathcal{O}(g^{-\frac{1}{2}}) L(g^{-\frac{1}{2}}) \end{pmatrix} \quad (m = -g + \frac{1}{2}, -g + \frac{3}{2}, \ldots, g - \frac{1}{2}), \] (79)

\(^{12}\)The Dirac operator does not commute with the chiral matrix,
\[ [-i\hat{P}, \sigma_z] \neq 0, \] (76)
and hence there do not exist simultaneous eigenstates of the Dirac-Landau operator and the chiral matrix except for the zero modes \((79)\).

\(^{13}\)For \( g < 0 \), the zero modes are given by
\[ \begin{pmatrix} 0 \\ \Upsilon^{-|g|+\frac{1}{2},m}(\theta, \phi) \end{pmatrix}. \] (78)
where
\[
Y^{g_{\text{Yg}}_{\frac{1}{2}, m}}(\theta, \phi) = (-1)^{g + m - \frac{1}{2}} \left[ \frac{(2g)!}{4\pi(g + m - \frac{1}{2})!(g - m - \frac{1}{2})!} \langle \sin \frac{\theta}{2} \rangle^{m + g - \frac{1}{2}} \langle \cos \frac{\theta}{2} \rangle^{(m + g - \frac{1}{2})} e^{im\phi} \right. \\
\left. - \sqrt{\frac{\Gamma((2g)!)}{4\pi^2(g + m - \frac{1}{2})!(g - m - \frac{1}{2})!}} u^{-m + g - \frac{1}{2}, m + g - \frac{1}{2}} \right]
\]  
(80)

The zero modes are equal to the lowest Landau level monopole harmonics \( (30) \) with the reduced monopole charge from \( g \) to \( g - \frac{1}{2} \). The degeneracy is
\[
2(g - \frac{1}{2}) + 1 = 2g.
\]  
(81)

It is easy to see that \( \Psi^g_{\lambda_0 = 0, m} \) \( (79) \) are the Dirac operator zero modes with the formula \( (35) \) and \( (69) \). For \( g = 3/2 \), we have three fold degenerate zero modes:
\[
\Psi^{3/2}_{0,1} = \frac{1}{2} \sqrt{\frac{\Gamma((3)!)}{\pi^2}} e^{i\phi} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi^{3/2}_{0,0} = -\frac{1}{2} \sqrt{\frac{\Gamma((3)!)}{\pi^2}} \sin \theta \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Psi^{3/2}_{0,-1} = \frac{1}{2} \sqrt{\frac{\Gamma((3)!)}{\pi^2}} e^{-i\phi} \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]  
(82)

which are in accordance with the results of Ref.\([17]\). The degeneracy of zero modes is expected from the index theorem \([4, 35]\); the 1st Chern number of the monopole gauge field configuration \( (4) \) is given by
\[
c_1 = \frac{1}{2\pi} \int_{S^2} F = 2g,
\]  
(83)

which is equal to \( (81) \).

### 3.2.2 Non-zero modes \( (n = 1, 2, \cdots) \)

We take a linear combination of \( \Upsilon^g_{j,m}(\theta, \phi) \) and \( \Upsilon^g_{j,m}(\theta, \phi) \) so that it becomes the eigenstate of \( -i\mathcal{D} \) with non-zero eigenvalue:
\[
\pm \lambda_n = \pm \sqrt{n(n + 2g)} \quad (n = 1, 2, \cdots).
\]  
(84)

With the aid of \( (35) \), the linear combination is readily obtained by taking a linear combination of \( \Upsilon^g_{j,m}(\theta, \phi) \) and \( \Upsilon^g_{j,m}(\theta, \phi) \) with same weights:
\[
\Psi^g_{\pm \lambda_n, m} = \frac{1}{\sqrt{2}} \left( \Upsilon^g_{j,m}(\theta, \phi) \mp i\Upsilon^g_{j,m}(\theta, \phi) \right)
\]  
(85)

or
\[
\Psi^g_{\pm \lambda_n, m} = \frac{1}{\sqrt{2}} \left( \Upsilon^g_{j=g-\frac{1}{2}+n,m}(\theta, \phi) \mp i\Upsilon^g_{j=g+\frac{1}{2}+(n-1),m}(\theta, \phi) \right),
\]  
(86)

where
\[
j = g - \frac{1}{2} + n, \quad \text{and} \quad m = -j, -j + 1, \cdots, j - 1, j.
\]  
(87a)

(87b)
One may directly check that $\Psi_{g,\lambda_n,m}^{\pm}$ are indeed the eigenstates of the Dirac operator \[ (86) \] of the eigenvalues \[ (84) \] using the formula \[ (35) \]. Notice that, when $g$ is an integer (half-integer), $j$ and $m$ should be half-integers (integers). Both $\Psi_{g,\lambda_n,m}^{+}$ and $\Psi_{g,\lambda_n,m}^{-}$ are $SU(2)$ irreducible representations with the $SU(2)$ index $j$, and the relativistic Landau levels, $+\lambda_n$ and $-\lambda_n$, respectively have the following degeneracy: \[ 2j + 1 = 2(g + n). \] (88)

For $g = 1/2$, three fold degenerate eigenstates at $+\lambda_n = \sqrt{2}$ are given by

$$\psi_{\sqrt{2},1}^{1/2} = -\sqrt{\frac{3}{8\pi}} \left( \sqrt{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) e^{i\phi}, \quad \psi_{\sqrt{2},0}^{1/2} = \frac{1}{4} \sqrt{\frac{3}{\pi}} \left( \sqrt{2} \cos \theta \right),$$

$$\psi_{\sqrt{2},-1}^{1/2} = \sqrt{\frac{3}{8\pi}} \left( \sqrt{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) e^{-i\phi}. \quad (89)$$

We add several comments here. First, \[ (86) \] consists of the monopole harmonics of the $n$th non-relativistic Landau level for monopole charge $g - \frac{1}{2}$ (upper component) and the monopole harmonics of the $(n - 1)$th non-relativistic Landau level for monopole charge $g + \frac{1}{2}$ (lower component). This reminds the eigenstates of the Dirac-Landau Hamiltonian on a plane (see Refs. [36, 37] for instance):

$$\frac{1}{\sqrt{2}} \left( \begin{array}{c} |n\rangle \\ |n - 1\rangle \end{array} \right). \quad (90)$$

In the limit $g \gg n$, the relativistic Landau levels on a plane, $\pm \sqrt{2B \cdot n}$, are reproduced from \[ (84) \] with $B = \frac{g}{r^2}$. Second, for $g = 0$, \[ (86) \] reduces to the free Dirac operator eigenstates with eigenvalues $\pm \lambda_n = \pm n \ (n = 1, 2, \cdots)$:

$$\psi_{g=0}^{\pm,n,m}(\theta, \phi) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} Y_{n-\frac{1}{2},m}^{-\frac{1}{2}}(\theta, \phi) \\ \mp i Y_{n-\frac{1}{2},m}^{\frac{1}{2}}(\theta, \phi) \end{array} \right). \quad (91)$$

This is a concise representation of the Abrikosov’s result [38, 39]. Third, $\psi_{g,\lambda_n,m}^{+}$ and $\psi_{g,\lambda_n,m}^{-}$ are related by the chiral transformation as expected from \[ (75) \]:

$$\psi_{g,\lambda_n,m}^{\mp} = \sigma_z \psi_{g,\lambda_n,m}^{+}. \quad (92)$$

The relativistic Landau levels and corresponding eigenstates are summarized in Fig.1.

4 Relations to the Pauli-Schrödinger Non-Relativistic System

We have discussed the relativistic Landau problem on a sphere. In non-relativistic quantum mechanics, the Landau problem with spin degrees of freedom is described by the Pauli-Schrödinger system.
Figure 1: The Dirac-Landau operator eigenvalues, eigenstates and degeneracy. Eigenstates with opposite sign eigenvalues are related by the chiral transformation.

Hamiltonian in a monopole background. The eigenvalue problem of the Pauli-Schrödinger Hamiltonian was solved by Kazama et al [26], in which the eigenvalues of the parity operator that constitutes the Pauli-Schrödinger Hamiltonian turned out to be

$$\Lambda^{(g)} \cdot \sigma = \pm \lambda_n - 1. \quad (93)$$

Here, $\pm \lambda_n = \pm \sqrt{n(2g + n)}$ are exactly the eigenvalues of the Dirac-Landau operator. This implies a close relation between the relativistic Landau model and the Pauli-Schrödinger system. In this section, we demonstrate that these two systems are indeed related by a simple $SU(2)$ “gauge” transformation. For this goal, we generalize the work of Abrikosov about free Dirac operator [38, 39] to include monopole gauge field.

4.1 The $SU(2)$ “gauge” transformation and $SO(3, 1)$ algebra

Abrikosov showed that the free Dirac operator eigenstates and the spinor spherical harmonics are related by the $SU(2)$ transformation [38, 39]

$$V(\theta, \phi) \equiv e^{-\frac{i}{2} \sigma_z \phi} e^{-\frac{i}{2} \sigma_y \theta} = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i \frac{\phi}{2}} & -\sin \frac{\theta}{2} e^{-i \frac{\phi}{2}} \\ \sin \frac{\theta}{2} e^{i \frac{\phi}{2}} & \cos \frac{\theta}{2} e^{i \frac{\phi}{2}} \end{pmatrix}. \quad (95)$$

16 With use of $D$ functions [Appendix B], $V(\theta, \phi)$ and $V(\theta, \phi)^\dagger$ are represented as

$$V(\theta, \phi) = \begin{pmatrix} D_{\frac{1}{2}, \frac{1}{2}}(\phi, \theta, 0) & D_{\frac{1}{2}, -\frac{1}{2}}(\phi, \theta, 0) \\ D_{\frac{1}{2}, -\frac{1}{2}}(\phi, \theta, 0) & D_{\frac{1}{2}, \frac{1}{2}}(\phi, \theta, 0) \end{pmatrix} = \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}, \quad (94a)$$

$$V(\theta, \phi)^\dagger = \begin{pmatrix} D_{\frac{1}{2}, \frac{1}{2}}(0, -\theta, -\phi) & D_{\frac{1}{2}, -\frac{1}{2}}(0, -\theta, -\phi) \\ D_{\frac{1}{2}, -\frac{1}{2}}(0, -\theta, -\phi) & D_{\frac{1}{2}, \frac{1}{2}}(0, -\theta, -\phi) \end{pmatrix} = \begin{pmatrix} u^* & v \\ -v^* & u \end{pmatrix}, \quad (94b)$$

where $u$ and $v$ are the components of the Hopf spinor [28].
\( V(\theta, \phi) \) is the \( SU(2) \) matrix that induces a spacial rotation of the Pauli matrices:

\[
V(\theta, \phi) \sigma_i V(\theta, \phi) = \sigma_j R_{ji}(\theta, \phi),
\]

(96)

where

\[
R_{ij}(\theta, \phi) = \begin{pmatrix}
\cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0 \\
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta
\end{pmatrix}.
\]

(97)

Notice that \( V \) also generates a \( SU(2) \) pure gauge field \((dW + iW^2 = 0)\) as

\[
W = -iV^\dagger dV = \frac{1}{2} \begin{pmatrix}
-\cos \theta d\phi & id\theta + \sin \theta d\phi \\
-id\theta + \sin \theta d\phi & \cos \theta d\phi
\end{pmatrix},
\]

(98)

and the diagonal part of \( W \) gives the \( U(1) \) monopole gauge field \((2)\):

\[
A = -ig \text{tr}(\sigma_z V^\dagger dV).
\]

(99)

Thus interestingly, the role of \( V(\theta, \phi) \) is two-fold: One is the \( SO(3) \) spacial rotation of the Pauli matrices, and the other is the \( SU(2) \) gauge transformation whose \( U(1) \) part corresponds to the monopole. In the former case, the Pauli matrices of \( V \) are interpreted as the spacial rotation generators, while in the latter they are the gauge group generators.

While both \( J \) and \(-i\mathcal{D}\) are (Pauli) matrix valued differential operators, under the \( V \) transformation they are completely decoupled to a differential operator part and Pauli matrix part:

\[
V(\theta, \phi) J V(\theta, \phi)^\dagger = L^{(g)} + \frac{1}{2} \sigma, \quad (100a)
\]

\[
V(\theta, \phi) (-i\mathcal{D}) V(\theta, \phi)^\dagger = K^{(g)} \cdot \sigma. \quad (100b)
\]

Here, \( L^{(g)} \) is the non-relativistic angular momentum operator \((11)\) while \( K^{(g)} \) represents “boost” operator given by

\[
K^{(g)}_x \equiv -i \cos \theta \cos \phi \frac{\partial}{\partial \theta} + i \frac{1}{\sin \theta} \sin \phi \frac{\partial}{\partial \phi} - g \cot \theta \sin \phi + i \sin \theta \cos \phi,
\]

\[
K^{(g)}_y \equiv -i \cos \theta \sin \phi \frac{\partial}{\partial \theta} - i \frac{1}{\sin \theta} \cos \phi \frac{\partial}{\partial \phi} + g \cot \theta \cos \phi + i \sin \theta \sin \phi,
\]

\[
K^{(g)}_z \equiv i \sin \theta \frac{\partial}{\partial \theta} + i \cos \theta.
\]

(101)

The Dirac-Landau operator is transformed to the “helicity operator”, \( K^{(g)} \cdot \sigma \). Unlike \( D_\mu \) \((53)\), \( K^{(g)}_i \) are simple differential operators (not matrix valued). The role of \( V \) becomes even transparent in the inverse transformation of \((100)\):

\[
V^\dagger L^{(g)} V + V^\dagger \frac{1}{2} \sigma V = J, \quad (102a)
\]

\[
V^\dagger K^{(g)} V \cdot V^\dagger \sigma V = -i\mathcal{D}. \quad (102b)
\]
In (102), \( V \) acts as \( SU(2) \) gauge transformation for \( K(g) \) and \( L(g) \), while acts as \( SO(3) \) spacial rotation for \( \sigma \), as mentioned above.

\( K(g) \) is concisely represented as

\[
K(g) = -iD|_{r=1} + \frac{1}{r} \cdot x,
\]

(103)

where \( D \) represents the Cartesian covariant derivatives in \( flat \) 3D space\(^{17}\)

\[
D_i = \partial_i - iA_i, \quad (i = x, y, z)
\]

(105)

with the gauge field \( \mathbf{3} \). Notice that \( \frac{1}{r} \cdot x \) of (103) is non-hermitian and comes from the spin-connection term of the original Dirac-Landau operator. With the explicit form of \( K(g) \) (101) and \( L(g) \) (13), \( K(g) \) and \( L(g) \) satisfy the \( SO(3,1) \) algebra:

\[
[K_i^{(g)}, K_j^{(g)}] = -i\varepsilon_{ijk}L_k^{(g)},
\]

\[
[L_i^{(g)}, K_j^{(g)}] = i\varepsilon_{ijk}K_k^{(g)},
\]

\[
[L_i^{(g)}, L_j^{(g)}] = i\varepsilon_{ijk}L_k^{(g)},
\]

(106)

and hence we refer to \( K(g) \) as “boost operator”. Eq.(106) holds even if the non-hermitian term \( \frac{1}{r} \cdot x \) was not present in (103). The square of \( K(g) \) is explicitly represented as

\[
K(g)^2 = -\frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) - \frac{1}{\sin^2 \theta} \partial_\phi^2 - 2ig \frac{\cos \theta}{\sin^2 \theta} \partial_\phi + g^2 \frac{\cos^2 \theta}{\sin^2 \theta} + 1,
\]

(107)

which is\(^{18}\)

\[
K(g)^2 = L(g)^2 - g^2 + 1 = \Lambda(g)^2 + 1.
\]

(108)

\( K(g)^2 \) is essentially the non-relativistic Landau Hamiltonian \( \Lambda(g) \):

\[
H = \frac{1}{2M} \Lambda(g)^2 = \frac{1}{2M} (K(g)^2 - 1).
\]

(109)

### 4.2 Relations to spinor monopole harmonics

Here, we give a detail discussion on the helicity operator, \( K(g) \cdot \sigma \). From the algebra (106), it is verified that the square of the helicity operator yields a non-relativistic Hamiltonian,

\[
H' = \frac{1}{2M} ((K(g) \cdot \sigma)^2 - 1) = \frac{1}{2M} (K(g)^2 + L(g) \cdot \sigma - 1).
\]

(110)

---

\(^{17}\) For comparison, we represent the Dirac operator in flat 3D space by spherical coordinates:

\[
-i \sum_{i=1}^3 \frac{\partial}{\partial x_i} = -i\sigma_x \frac{\partial}{\partial x} - i\sigma_y \frac{\partial}{\partial y} - i\sigma_z \frac{\partial}{\partial z}
\]

\[
= -i\sigma_x \left( \frac{\cos \theta \cos \phi \partial}{r} - \frac{\sin \phi}{r \sin \theta} \partial_\phi + \frac{\sin \theta \cos \phi}{r} \partial_r \right)
\]

\[
- i\sigma_y \left( \frac{\cos \theta \sin \phi \partial}{r} + \frac{\cos \phi}{r \sin \theta} \partial_\phi + \frac{\sin \theta \sin \phi}{r} \partial_r \right) - i\sigma_z \left( -\frac{\sin \theta}{r} \partial_\theta + \frac{\cos \theta}{r} \partial_r \right).
\]

(104)

\(^{18}\) The last term 1 of (107) comes from the non-hermitian term (103).
With use of (107), we have
\[ H' = \frac{1}{2M} (\Lambda^{(g)})^2 + L^{(g)} \cdot \sigma = \frac{1}{2M} (\Lambda^{(g)})^2 + \Lambda^{(g)} \cdot \sigma - F \cdot \sigma. \] (111)

Here, \( \frac{1}{2M} \Lambda^{(g)}^2 \) denotes the non-relativistic Landau Hamiltonian (109), \( \Lambda^{(g)} \cdot \sigma \) represents the spin-orbit coupling term known as the Parity operator, and \( F \cdot \sigma \) stands for the Zeeman coupling. \( H' \) is a supersymmetric quantum mechanical Hamiltonian, since it is \( SU(2) \) gauge equivalent to \(-i\mathcal{D})^2 \) [see Sec.7.1 for details] up to a constant. From (106) and (108), we have
\[ (K^{(g)} \cdot \sigma)^2 = L^{(g)}^2 + L^{(g)} \cdot \sigma = (g + 1)(g - 1) = (L^{(g)} + \frac{1}{2} \sigma)^2 - g^2 + \frac{1}{4}, \] (112a)
\[ [K^{(g)} \cdot \sigma, L^{(g)} + \frac{1}{2} \sigma] = [K^{(g)}, L^{(g)}] \cdot \sigma + \frac{1}{2} K^{(g)} \cdot [\sigma, \sigma] = i(K^{(g)} \times \sigma - K^{(g)} \times \sigma) = 0, \] (112b)
which correspond to
\[ (-i\mathcal{D})^2 = J^2 - g^2 + \frac{1}{4}, \] (113a)
\[ [-i\mathcal{D}, J] = 0. \] (113b)

The \( SU(2) \) Casimir eigenvalues for \( L^{(g)} + \frac{1}{2} \sigma \) are
\[ (L^{(g)} + \frac{1}{2} \sigma)^2 = j(j + 1), \] (114)
with \( j = g - \frac{1}{2} + n \) (\( n = 0, 1, 2, \cdots \)), and then from (112a) the eigenvalues of \((K \cdot \sigma)^2\) are
\[ j^2 - g^2 = n(n + 2g), \] (115)
and hence
\[ K^{(g)} \cdot \sigma = \pm \sqrt{n(n + 2g)}, \] (116)
which are identical to the relativistic Landau level (84) as expected. In a similar manner to Sec.3.2 we can derive the eigenstates of the helicity operator \( K^{(g)} \cdot \sigma \). The eigenstates of the \( SU(2) \) Casimir for \( L + \frac{1}{2} \sigma \),
\[ (L^{(g)} + \frac{1}{2} \sigma)^2 \Omega_{j,m} = j(j + 1)\Omega_{j,m}, \] (117a)
\[ (L_z^{(g)} + \frac{1}{2} \sigma_z)\Omega_{j,m} = m\Omega_{j,m}, \quad (m = -j, -j + 1, \cdots, j) \] (117b)
are given by the spinor monopole harmonics:
\[ \Omega_{j,m}^g = \frac{1}{\sqrt{2j}} \left( \sqrt{j + m} Y^{g}_{j - \frac{1}{2}, m - \frac{1}{2}} \right), \quad \Omega_{j,m}^{g*} = \frac{1}{\sqrt{2j + 1}} \left( \frac{-\sqrt{-m + 1} Y^{g}_{j + \frac{1}{2}, m - \frac{1}{2}}}{\sqrt{j + m + 1} Y^{g}_{j + \frac{1}{2}, m + \frac{1}{2}}} \right). \] (118)
The eigenstates of the helicity operator $K^{(g)} \cdot \sigma$ with $\pm \lambda_n = \pm \sqrt{n(n + 2g)}$ ($n = 1, 2, \cdots$) are constructed by their linear combinations:

$$
\Phi_{\pm \lambda_n, m}^g = \frac{1}{\sqrt{2}} (\Omega_{j,m}^g (\theta, \phi) \pm i \Omega_{j,m}^g (\theta, \phi)) \\
= \frac{1}{2} \left( \sqrt{\frac{j+m+1}{j+1}} Y_{j,1/2, m+1/2}^g \pm i \sqrt{\frac{j-m+1}{j+1}} Y_{j,1/2, m-1/2}^g \right) . \quad (j = g - \frac{1}{2} + n) \tag{119}
$$

The zero modes $\lambda_n = 0$ are

$$
\Phi_{0, m}^g = \Omega_{g - \frac{1}{2}, m}^g = \frac{1}{\sqrt{2g+1}} \left( \sqrt{\frac{g+1/2 - m}{g+1/2 + m}} Y_{g, m-1/2}^g \right) \quad (m = -g + \frac{1}{2}, -g + \frac{3}{2}, \cdots, g - \frac{1}{2}). \tag{120}
$$

A bit of calculation shows that linear combinations of $\Omega_{j,m}^g$ and $\Omega_{j,m}^g$ are related to $\Psi_{j,m}^g$ and $\Psi_{j,m}^g$ by the $SU(2)$ transformation:

$$
\Psi_{j,m}^g = V(\theta, \phi)^\dagger (-\cos \alpha \cdot \Omega_{j,m}^g + \sin \alpha \cdot \Omega_{j,m}^g), \quad \tag{124a}
\Psi_{j,m}^g = V(\theta, \phi)^\dagger (\sin \alpha \cdot \Omega_{j,m}^g + \cos \alpha \cdot \Omega_{j,m}^g), \quad \tag{124b}
$$

where

$$
\tan \alpha = \sqrt{\frac{j - g + \frac{1}{2}}{j + g + \frac{1}{2}}}. \tag{125}
$$

Consequently we have

$$
\Psi_{\pm \lambda_n, m}^g = \frac{1}{\sqrt{2}} (\Omega_{j,m}^g \pm i \Omega_{j,m}^g) = -e^{\pm i \alpha} V(\theta, \phi)^\dagger \Phi_{\pm \lambda_n, m}^g. \tag{126}
$$

---

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$$
D_{l,m_1, m_2} \otimes D_{l', m'_1, m'_2} = \sum\left[ D_{l,M_1, M_2} C_{l, m_1, m_2}^{M_1, M_2} \right] D_{l,M_1, M_2} C_{l, m_1, m_2}^{M_1, M_2} \tag{121}
$$

where $C_{l,m_1, m'_1}^{M_1, M_2} = (L, M | l, m; l', m') = (l, m; l', m' | L, M)$ are the Clebsch-Gordan coefficients. Since $D_{l,m_1, m_2}$ have two $SU(2)$ indices, $m_1$ and $m_2$, the $SU(2)$ angular momentum decomposition is respectively applied to two pairs, $(m_1, m'_1)$ and $(m_2, m'_2)$. To derive (121), we used

$$
V(\theta, \phi)^\dagger \Omega_{j,m}^g = \frac{1}{\sqrt{2j+1}} \left( \sqrt{\frac{j + g + \frac{1}{2}}{j - g + \frac{1}{2}}} Y_{j,m}^{g - \frac{1}{2}} \right), \quad V(\theta, \phi)^\dagger \Omega_{j,m}^g = \frac{1}{\sqrt{2j+1}} \left( \sqrt{\frac{j + g + \frac{1}{2}}{j - g + \frac{1}{2}}} Y_{j,m}^{g + \frac{1}{2}} \right), \tag{122}
$$

which is verified by Eq. (119) and (121) with the following Clebsch-Gordan coefficients:

$$
C_{1/2,1/2, 1/2, 1/2}^{L,M} = \frac{1}{2} \delta_{l,l} \delta_{m, m} \sqrt{\frac{l + m + 1}{2l + 1}}, \quad C_{1/2, -1/2, 1/2, 1/2}^{L,M} = \frac{1}{2} \delta_{l,l} \delta_{m, m} \sqrt{\frac{l + m + 1}{2l + 1}}. \tag{123}
$$
Thus up to the irrelevant phase factor, $\Phi_{\pm\lambda,m}^{g}$ is transformed to $\Psi_{\pm\lambda,m}^{g}$ by the $SU(2)$ matrix $V$. For zero modes, 
\[
\Psi_{\lambda_{0}=0,m}^{g} = -V(\theta, \phi) \dagger \Phi_{\lambda_{0}=0,m}^{g}. \tag{127}
\]

### 4.3 Relations to the Pauli-Schödinger eigenstates

Next, we establish relations between the relativistic Landau model and the Pauli-Schödinger non-relativistic system. The Pauli-Schödinger Hamiltonian is given by

\[
H_{P-S} = -\frac{1}{2M} \left( \sum_{i=1}^{3} (\sigma_{i}D_{i})^{2} - \sum_{i=1}^{3} D_{i} - \frac{1}{2M} F \cdot \sigma, \right) \tag{128}
\]

where $D_{i}$ denote the covariant derivative in 3D flat space \[^{6}\] and $F$ represents an external magnetic field, in the present case, the monopole field strength \[^{5}\]. In the spherical coordinates, $H_{P-S}$ is expressed as

\[
H_{P-S} = -\frac{1}{2Mr^{2}} \frac{\partial}{\partial r} r^{2} \frac{\partial}{\partial r} + \frac{1}{2Mr^{2}} (\Lambda^{(g)} \cdot \sigma)(\Lambda^{(g)} \cdot \sigma + 1). \tag{129}
\]

On a sphere, we have

\[
H_{P-S}|_{r=1} = \frac{1}{2M} (\Lambda^{(g)} \cdot \sigma)(\Lambda^{(g)} \cdot \sigma + 1). \tag{130}
\]

Since the Pauli-Schödinger Hamiltonian consists of the Parity operator $\Lambda^{(g)} \cdot \sigma$, the Parity operator eigenstates are automatically the eigenstates of the Pauli-Schödinger Hamiltonian \[^{130}\]. The eigenvalues of $\Lambda^{(g)} \cdot \sigma + 1$ are exactly the same as those of the helicity operator $K^{(g)} \cdot \sigma$, $\pm \lambda_{n} = \pm \sqrt{n(2g + 1)}$, and the corresponding eigenstates of $\Lambda^{(g)} \cdot \sigma + 1$ are \[^{26}\]

\[
\Xi_{\pm\lambda_{n},m}^{g} = \frac{1}{2} \left( \sqrt{1 + \frac{g}{j + \frac{1}{2}}} \pm \sqrt{1 - \frac{g}{j + \frac{1}{2}}} \right) \Omega_{jm}^{g} + \frac{1}{2} \left( \sqrt{1 + \frac{g}{j + \frac{1}{2}}} + \sqrt{1 - \frac{g}{j + \frac{1}{2}}} \right) \Omega_{jm}^{g} \quad (n = 1, 2, \cdots), \tag{131}
\]

where $\Omega_{jm}^{g}$ and $\Omega_{jm}^{g}$ are the spinor monopole harmonics \[^{118}\]. The “coincidence” between the eigenvalues of the parity operator $\Lambda^{(g)} \cdot \sigma + 1$ and the helicity operator $K^{(g)} \cdot \sigma$ is understood by noticing that the relations between the Parity operator and helicity operator:

\[
(\Lambda^{(g)} \cdot \sigma + 1)^{2} = \Lambda^{(g)}^{2} + L^{(g)} \cdot \sigma + 1 = (K^{(g)} \cdot \sigma)^{2}, \tag{132}
\]

where we used the commutation relations of $\Lambda^{(g)}$ \[^{5}\]:

\[
[\Lambda_{i}^{(g)}, \Lambda_{j}^{(g)}] = -i\epsilon_{ijk}(L_{k}^{(g)} - 2\Lambda_{k}^{(g)}). \tag{133}
\]

Therefore, the eigenvalues of $K^{(g)} \cdot \sigma$ and those of $\Lambda^{(g)} \cdot \sigma + 1$ are exactly the same. From \[^{119}\] and \[^{131}\], we can relate $\Xi_{\pm\lambda_{n},m}^{g}$ and $\Phi_{\pm\lambda_{n},m}^{g}$ as

\[
\Xi_{\pm\lambda_{n},m}^{g} = \frac{1}{\sqrt{2}} (e^{-i\beta} \cdot \Phi_{\lambda_{n},m}^{g} + e^{i\beta} \cdot \Phi_{-\lambda_{n},m}^{g}),
\]

\[
\Xi_{-\lambda_{n},m}^{g} = -i\frac{1}{\sqrt{2}} (e^{-i\beta} \cdot \Phi_{\lambda_{n},m}^{g} - e^{i\beta} \cdot \Phi_{-\lambda_{n},m}^{g}), \tag{134}
\]

\[^{20}\]Interestingly, the Pauli-Schödinger Lagrangian enjoys the $OSp(1|2)$ super-conformal symmetry and $(\Lambda^{(g)} \cdot \sigma + 1)$ plays the role of $OSp(1|2)$ Casimir operator \[^{10}\].

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where
\[
\tan \beta = -\frac{j + \frac{1}{2}}{g} \left(1 + \sqrt{1 - \left(\frac{g}{j + \frac{1}{2}}\right)^2}\right). \tag{135}
\]

Consequently, relations between the eigenstates of \(-i\mathcal{D}\) and \(\mathcal{H}_{\text{P-S}}\) are given by
\[
\Psi_{\pm \lambda_n, m}^g = \frac{1}{\sqrt{2}} e^{\pm i\gamma} \cdot V(\theta, \phi) \dagger (\Xi_{\lambda_n, m}^g \mp i\Xi_{-\lambda_n, m}^g) \quad (n = 1, 2, \cdots), \tag{136}
\]
where \(\gamma \equiv \alpha + \beta\), or
\[
\tan \gamma = -\frac{j(j + \frac{1}{2}) \left(\sqrt{j + g + \frac{1}{2}} + \sqrt{j - g + \frac{1}{2}}\right)}{j(j + \frac{1}{2}) \sqrt{j - g + \frac{1}{2}} + g(g - \frac{1}{2}) \sqrt{j + g + \frac{1}{2}}}. \tag{137}
\]

Similarly, the zero modes \((\lambda_0 = 0)\) are given by
\[
\Xi_{\lambda_0=0, m}^g = -\Omega_{\lambda_0=0, m}^g = -\Phi_{\lambda_0=0, m}^g, \tag{138}
\]
and then
\[
\Psi_{\lambda_0=0, m}^g = V(\theta, \phi) \dagger \Xi_{\lambda_0=0, m}^g. \tag{139}
\]

Figure 2 summarizes the mutual relations discussed in this section.

![Diagram](image)

Figure 2: The eigenstates of the Dirac-Landau operator are related to those of the helicity operator by the \(SU(2)\) transformations, (126) and (127). The linear combinations of the eigenstates of the helicity operator give the Parity operator eigenstates, (134) and (138). The Dirac-Landau operator eigenstates are transformed to the Parity operator eigenstates by the \(SU(2)\) transformations and the linear combinations, (136) and (139).
5 Non-Commutative Geometry in Relativistic Landau Levels

5.1 Landau level projection and non-commutative geometry

By diagonalizing the Landau Hamiltonian, we obtain an infinite dimensional Hilbert space spanned by the monopole harmonics. The Hilbert space consists of finite dimensional subspaces labeled by the Landau level index $n$. Sandwiching an operator of interest with the monopole harmonics, we have a matrix representation of the operator. In general, the matrix representation is given by an infinite dimensional matrix made of block matrices. For instance, matrix representation of Cartesian coordinates is given by

$$x_i = \begin{pmatrix}
* & * & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & X_i(n - 1, n) & 0 & 0 & 0 \\
0 & 0 & X_i(n, n - 1) & X_i(n, n) & X_i(n, n + 1) & 0 & 0 \\
0 & 0 & 0 & X_i(n + 1, n) & * & * & 0 \\
0 & 0 & 0 & 0 & * & * & * \\
0 & 0 & 0 & 0 & 0 & * & * 
\end{pmatrix},$$

where $X_i(n_1, n_2)$ denotes $(2g + 2n_1 + 1) \times (2g + 2n_2 + 1)$ block matrix between $n_1$ and $n_2$th Landau levels. (In the case of $x_i$, only the matrix elements of adjacent and intra Landau levels take non-zero values.) The original coordinates $x_i$ are commutative:

$$x_i x_j = x_j x_i.$$  

Let us concentrate the $n$th intra Landau level block of (141); from (140) the left-hand side gives

$$x_i x_j(n, n) = X_i(n, n - 1) X_j(n - 1, n) + X_i(n, n) X_j(n, n) + X_i(n, n + 1) X_j(n + 1, n),$$

while the right-hand side of (141) yields

$$x_j x_i(n, n) = X_j(n, n - 1) X_i(n - 1, n) + X_j(n, n) X_i(n, n) + X_j(n, n + 1) X_i(n + 1, n).$$

Since (142) and (143) are equal, we have

$$[X_i(n, n), X_j(n, n)] = -[X_i(n, n - 1), X_j(n - 1, n)] - [X_i(n + 1, n), X_j(n, n + 1)].$$

Though each of the commutators on the right-hand side of (144) gives both inter and intra Landau level block matrices, the sum of the commutators amounts to be an intra Landau level block matrix only:

$$- [X_i(n, n - 1), X_j(n - 1, n)] - [X_i(n + 1, n), X_j(n, n + 1)] = -i\alpha_n^{(g)} \epsilon_{ijk} X_k(n, n).$$

(Here, $\alpha_n^{(g)}$ denotes a proportional coefficient which will be identified as (152)). It may be a good exercise for readers to check (145) in low dimensional matrices. Consequently, (144) can be rewritten as

$$[X_i(n, n), X_j(n, n)] = -i\alpha_n^{(g)} \epsilon_{ijk} X_k(n, n).$$
where \( \langle \) elements of the coordinates (140) are explicitly given by

\[
\langle x \pm iy \mid Y_{l,m} \rangle = \frac{g}{l(l+1)} \sqrt{(l \pm m)(l \pm m + 1)} \delta_{l,l'} \delta_{m',m \pm 1} \\
\pm \frac{1}{l+1} \sqrt{((l+1)^2 - g^2)(l \pm m + 2)(l \pm m + 1)} \delta_{l,l+1} \delta_{m',m \pm 1} \\
\pm \frac{1}{l} \sqrt{(l^2 - g^2)(l \pm m)(l \pm m - 1)} \delta_{l,l-1} \delta_{m',m \pm 1},
\]

(147a)

\[
\langle x \pm iy \mid Y_{l',m'} \rangle = \frac{g}{l(l+1)} m \delta_{l,l'} \delta_{m',m} \\
\mp \frac{1}{l+1} \sqrt{((l+1)^2 - g^2)((l+1)^2 - m^2)} \delta_{l,l+1} \delta_{m',m} \\
\pm \frac{1}{l} \sqrt{(l^2 - g^2)(l^2 - m^2)} \delta_{l,l-1} \delta_{m',m},
\]

(147b)

where the \( SU(2) \) indices, \( l \) and \( l' \), are related to the Landau level indices, \( n \) and \( n' \), as \( l = g + n \) and \( l' = g + n' \). The first components of the right-hand sides of (147) represent the matrix elements of intra Landau level, \( X_l(n, n) \), while the second and third terms stand for those of the adjacent inter Landau levels, \( X_l(n, n') \) with \( |n - n'| = 1 \). In the limit

\[
g \gg n,
\]

(148)

which we refer to as the non-commutative limit, the diagonal blocks \( X_l(n, n) \) behave as \( O(1) \), while the off-diagonal blocks \( X_l(n, n') \) (\(|n - n'| = 1\)) as \( O(\sqrt{\frac{g}{n}}) \). Thus in the non-commutative limit, the intra Landau level block matrices become dominant compared to inter Landau level block matrices [Fig 3]. The intra Landau level matrix elements can be expressed as

\[
X(n, n)_{m',m} = \langle Y_{l,m'} \mid x \mid Y_{l,m} \rangle = -r \frac{g}{l(l+1)} \langle Y_{l,m'} \mid L^{(g)} \mid Y_{l,m} \rangle, \quad (l = g + n)
\]

(149)

where \( \langle Y_{l,m'} \mid L^{(g)} \mid Y_{l,m} \rangle \) represents the ordinary \( SU(2) \) matrices with spin magnitude \( l \):

\[
\langle Y_{l,m'} \mid L^{(g)}_{x} \mid Y_{l,m} \rangle = \sqrt{(l \pm m)(l \pm m + 1)} \delta_{m',m \pm 1},
\]

(150)
Figure 3: The left-figure shows \((X_1+iX_2)(n+1,n) / (X_1+iX_2)(n,n)\) \((n = 5)\) with respect to the monopole charge \(g\) and the magnetic quantum number \(m\), while the right figure shows \(X_3(n+1,n) / X_3(n,n)\) \((n = 5)\). In the limit \(g \to \infty\), the ratios approach zero, implying that the inter-Landau level components (numerator) are negligible compared to the intra-Landau level components (denominator). (In the right figure, there exists a singularity around \(m = 0\) coming from the small intra-Landau level components of \(X_3\) around \(m = 0\).)

and then \(X(n, n)\) is simply represented as

\[
X(n, n) = -\alpha_n^{(g)} \mathbf{S}_{s=n+g},
\]

(151)

where \(\mathbf{S}_{s=n+g}\) represents the ordinary \((2s + 1) \times (2s + 1)\) \(SU(2)\) matrices with spin magnitude \(s = g + \frac{1}{2}\) and

\[
\alpha_n^{(g)} = r \frac{g}{(g + n)(g + n + 1)}.
\]

(152)

The square of the radius of fuzzy sphere is obtained as

\[
X \cdot X = \alpha_n^{(g)}^2 (g + n)(g + n + 1) \equiv R_n^{(g)}^2,
\]

(153)

where

\[
R_n^{(g)} = \alpha_n^{(g)} \sqrt{(g + n)(g + n + 1)}
\]

\[
= r \frac{g}{\sqrt{(g + n)(g + n + 1)}} \quad (n = 0, 1, 2, \cdots).
\]

(154)

(Hereinafter, we abbreviate the Landau level index \(n\) of \(X(n, n)\) for notational brevity.) One may find that the radius of fuzzy sphere depends on the Landau level index \(n\).

Also based on (III), one can understand the appearance of fuzzy sphere. In the \(n\)th Landau level, the matrix elements of the covariant angular momentum are derived as

\[
\langle Y_{l,m}^g | \mathbf{A}^{(g)} | Y_{l,m}^g \rangle = \left( 1 - \left( \frac{R_n^{(g)}}{r} \right)^2 \right) \langle Y_{l,m}^g | \mathbf{L}^{(g)} | Y_{l,m}^g \rangle.
\]

(155)

\[\text{For instance, } \mathbf{S}_{s=\frac{1}{2}} = \frac{1}{2}\mathbf{\sigma}.\]
Notice that the proportional factor on the right-hand side of (155) does not depend on magnetic angular momenta, \( m \) and \( m' \), as expected by the Wigner-Eckart theorem, so the proportional factor is solely determined by the Landau level index \( n \). Though matrix elements of \( \Lambda^{(g)} \) take non-zero values in each Landau level (155), the matrix elements become negligible compared to those of \( L^{(g)} \) in the non-commutative limit, \( R^{(g)}_n / r \rightarrow \infty \). Indeed, the factor, \( 1 - \frac{(R^{(g)}_n)^2}{(n+g)(n+g+1)} \), monotonically decreases as \( g \) increases [Fig. 4]. Thus in the non-commutative limit, the covariant angular momentum no longer contributes to the total angular momentum in (11) and hence \( x \) can be identified with the operator \( -\frac{2}{g} L^{(g)} \) that satisfy the \( SU(2) \) algebra of fuzzy sphere (146).

5.3 Projection to the relativistic Landau levels

With the matrix elements by the monopole harmonics (151), it is easy to derive matrix elements for the relativistic case. The eigenstates of the Dirac-Landau operator are respectively given by

\[
\begin{align*}
\text{n = 0 : } & \psi^{(g)}_{\lambda_0=0,m} = \begin{pmatrix} Y^{g-\frac{1}{2}}_m & 0 \\ g-\frac{1}{2} & 0 \end{pmatrix}, \\
\text{n = 1, 2, } & \cdots : \psi^{(g)}_{\pm\lambda_n,m} = \frac{1}{\sqrt{2}} \begin{pmatrix} Y^{g-\frac{1}{2}}_{j=g-\frac{1}{2}+n,m} \\ -i Y^{g+\frac{1}{2}}_{j=g-\frac{1}{2}+n,m} \end{pmatrix},
\end{align*}
\]

and the matrix elements of \( x \) are derived as

\[
X \equiv \langle \psi^{(g)}_{\pm\lambda_n,m} | x | \psi^{(g)}_{\pm\lambda_n,m} \rangle = -\alpha^{(g)}_n S_{s=n+\frac{g}{2}},
\]

where

\[
\begin{align*}
\text{n = 0 : } & \alpha^{(g)}_0 = \alpha_0^{(g)} = r \frac{1}{g + \frac{1}{2}}, \\
\text{n = 1, 2, } & \cdots : \alpha^{(g)}_n = \frac{1}{2} (a^{(g)}_n + a^{(g+\frac{1}{2})}_n) = r \frac{g}{(g + n - \frac{1}{2})(g + n + \frac{1}{2})},
\end{align*}
\]

\[\text{[157]}\] should be interpreted as the abbreviation form of \( X_{m,m'} \equiv \langle \psi^{(g)}_{\pm\lambda_n,m} | x | \psi^{(g)}_{\pm\lambda_n,m'} \rangle \).
Notice that the matrix elements $X$ are completely identical for positive and negative eigenvalues $\pm \lambda_n$. $X_i$ satisfy the fuzzy sphere algebra:

$$[X_i, X_j] = -i\alpha'_n \epsilon_{ijk} X_k,$$

and the squares of their radii are given by

$$X \cdot X = \alpha'^2_n (n + g - \frac{1}{2})(n + g + \frac{1}{2}) \equiv R'^2_n,$$

where

$$n = 0 : \quad R'_0 = \alpha'_0 \sqrt{(g - \frac{1}{2})(g + \frac{1}{2})} = r \sqrt{\frac{g - \frac{1}{2}}{g + \frac{1}{2}}},$$

$$n = 1, 2, \ldots : \quad R'_n = \alpha'_n \sqrt{(n + g - \frac{1}{2})(n + g + \frac{1}{2})} = r \sqrt{\frac{g}{(g + n - \frac{1}{2})(g + n + \frac{1}{2})}}.$$  \hspace{1cm} (161b)

The sizes of the fuzzy spheres are ordered as [Figs.5]

$$R'^2_0 > R'^2_1 > R'^2_2 > \cdots.$$  \hspace{1cm} (162)

Figure 5: The circles schematically represent the fuzzy spheres on the corresponding relativistic Landau levels. The sizes of two fuzzy spheres on the levels, $+\lambda_n$ and $-\lambda_n$, are identical. The size monotonically decreases as $n$ increases. ($g = 3$ and $M = 2$ are adopted in the figure.)

Here, we compare the sizes of the relativistic and non-relativistic fuzzy spheres. The ratios between the radii are given by

$$n = 0 : \quad \frac{R'_0}{R'^2_0} = \alpha'_0 \sqrt{(g - \frac{1}{2})(g + \frac{1}{2})} = \frac{(g - \frac{1}{2})(g + 1)}{g(g + \frac{1}{2})} < 1,$$

$$n = 1, 2, \ldots : \quad \frac{R'_n}{R'^2_n} = \alpha'_n \sqrt{(n + g - \frac{1}{2})(n + g + \frac{1}{2})} = \frac{(g + n)(g + n + 1)}{(g + n - \frac{1}{2})(g + n + \frac{1}{2})} > 1.$$  \hspace{1cm} (163)
Thus, the radius of the fuzzy sphere for $n = 0$ reduces, while those for $n = 1, 2, \cdots$ enhance. From (163), the ratios are ordered as [Fig 6]

$$\frac{R_1^{(g)}}{R_1^{(g)}} > \frac{R_2^{(g)}}{R_2^{(g)}} > \frac{R_3^{(g)}}{R_3^{(g)}} > \cdots > 1 > \frac{R_0^{(g)}}{R_0^{(g)}}.$$  \hfill (164)

![Figure 6: Plot of the radii (161) with respect to $g$. The solid and dashed curves are respectively for the non-relativistic and relativistic cases, $R_n^{(g)}/r$ and $R'_n^{(g)}/r$, and black, red, green, blue for $n = 0, 1, 2, 3$. Inset depicts the ratios of (163) with same color assignment for $n$.](image)

### 6 Mass Deformation and Balanced Fuzzy Spheres

We consider mass deformation of the relativistic Landau model. In real Dirac matter, mass term is physically induced by Zeeman effect on the surface of topological insulator [44] and sub-lattice asymmetry between A and B sites in graphene [45].

#### 6.1 Mass deformation

Mass term is added to the Dirac-Landau operator as

$$-i\slashed{D} + \sigma_z M = \begin{pmatrix} M & -i\partial_{\pm}^{(g+\frac{1}{2})} \\ -i\partial_{\pm}^{(g-\frac{1}{2})} & -M \end{pmatrix}. \hfill (165)$$

The $SU(2)$ rotational symmetry is still kept exact under the mass deformation

$$[\sigma_z M, J] = 0, \hfill (166)$$

but the chiral symmetry is broken:

$$\{ -i\slashed{D} + \sigma_z M, \sigma_z \} = 2M \neq 0. \hfill (167)$$
The kinetic term $-i\not{D}$ and the mass term $M\sigma_z$ do not commute and hence their simultaneous eigenstates do not exist in general except for the zero modes. Square of the massive Dirac-Landau operator is given by
\[
(-i\not{D} + \sigma_z M)^2 = (-i\not{D})^2 + M^2,
\]
where we used the chiral symmetry of the Dirac-Landau operator, \{-i\not{D}, \sigma_z\} = 0. Therefore, the eigenvalues of $(-i\not{D} + M\sigma_z)^2$ are given by
\[
\Lambda_n^2 \equiv \lambda_n^2 + M^2 = n(n + 2g) + M^2.
\]
The eigenvalues of the mass deformed Dirac-Landau operator are\[23\]
\[
\begin{align*}
    n = 0 & : \Lambda_{n=0} = +M, \\
    n = 1, 2, \cdots & : \pm \Lambda_n = \pm \sqrt{\lambda_n^2 + M^2} = \pm \sqrt{n(n + 2g) + M^2}.
\end{align*}
\]
Notice the absence of $-M$ in the eigenvalues. The zero modes of the (massless) Dirac-Landau operator correspond to those of the massive Dirac-Landau operator with the eigenvalue $+M$. Explicitly, the corresponding eigenstates are given by
\[
\begin{align*}
    n = 0 & : \Psi_{\Lambda_0=M,m}^g = \Psi_{\lambda_0=0,m}^g = \left( Y^{g-\frac{1}{2},m} g^{\frac{1}{2}}, 0 \right), \\
    n = 1, 2, \cdots & : \Psi_{\pm\Lambda_n,m}^g = \sqrt{\frac{\Lambda_n + \lambda_n}{2\Lambda_n}} (\Psi_{\pm\lambda_n,m}^g \pm \frac{M}{\Lambda_n + \lambda_n} \Psi_{\mp\lambda_n,m}^g) \\
    & = \frac{1}{2} \sqrt{\frac{\Lambda_n + \lambda_n}{\lambda_n}} \left( \frac{1 \pm \frac{M}{\lambda_n + \lambda_n}}{\frac{1}{j=g-\frac{1}{2}+n,m}} \right) \left. \left( Y^{g-\frac{1}{2},j=g-\frac{1}{2}+n,m} \right) \pm i(1 \mp \frac{M}{\lambda_n + \lambda_n}) Y^{g+\frac{1}{2},j=g+\frac{1}{2}+\left( n-1 \right),m} \right). \quad (171b)
\end{align*}
\]
Eq. (171) can be chosen as the simultaneous eigenstates of the SU(2) Casimir $J^2$ due to the existence of the SU(2) symmetry. Eq. (171b) shows that the mass term mixes the massless eigenstates with opposite sign eigenvalues of same magnitude. For $\Psi_{+\Lambda_n,m}^g$, the mass term enhances/reduces the weight of the upper/lower component, while for $\Psi_{-\Lambda_n,m}^g$, the opposite. The mass deformed Dirac-Landau operator exhibits the symmetric spectra with respect to the zero energy except for $\Lambda_0 = +M$ [Figs. 7]. The Landau level degeneracies do not change under the mass deformation:
\[
(2j + 1)|_{j=n+g-\frac{1}{2}} = 2(n + g).
\]
It is easy to see that, in the massless limit $M \to 0$, (171) are reduced to (86):
\[
\Psi_{\pm\Lambda_n,m}^g \to \Psi_{\pm\lambda_n,m}^g \quad (n = 1, 2, \cdots). \quad (173)
\]
\[23\]For $g < 0$, we have $\Lambda_{n=0} = -M$ instead of (171a).
The eigenvalues of the Dirac-Landau operator change from the left to the right by the mass deformation \((M > 0)\). The massive Dirac-Landau operator has the eigenvalue, \(+M\), but not \(-M\), which is known as the "parity anomaly".

Also in the limit \(M \to \infty\), we have

\[
\Lambda_n - M \simeq \frac{\lambda_n^2}{2M} = \frac{1}{2M}n(n + 2g),
\]

which reproduce the non-relativistic results, (27) and (20) with replacement of \((n, g)\) by \((n, g - \frac{1}{2})\) or \((n - 1, g + \frac{1}{2})\) up to constant.

Though the massive Dirac-Landau operator does not respect the original chiral symmetry, the spectrum structure suggests the existence of some generalized chiral operator that anti-commutates with the mass deformed Dirac-Landau operator. Such a chiral operator is given by

\[
\mathcal{R} = -i\sigma_z \mathcal{P} = \frac{1}{2}[\sigma_z, -i\mathcal{P}],
\]

or

\[
\mathcal{R} = (\partial_\phi + \frac{1}{2} \cot \theta)\sigma_y - \frac{1}{\sin \theta}(\partial_\phi + ig \cos \theta)\sigma_x.
\]

It is straightforward to demonstrate

\[
\{\mathcal{R}, -i\mathcal{P} + M\sigma_z\} = \frac{1}{2}\{[\sigma_z, -i\mathcal{P} + M\sigma_z], -i\mathcal{P} + M\sigma_z\} = \frac{1}{2}[\sigma_z, (-i\mathcal{P} + M\sigma_z)^2] = \frac{1}{2}[\sigma_z, -\mathcal{P}^2 + M^2] = 0,
\]

and the eigenstates for \(+\Lambda_n\) and \(-\Lambda_n\) are related by \(\mathcal{R}\):

\[
\mathcal{R}\Psi_{\pm\Lambda_n,m} = \pm \lambda_n \Psi_{\pm\Lambda_n,m}.
\]
Since $-i\mathcal{D} \to \pm \lambda_n$ in the massless limit, $\mathcal{R}$ \((175)\) is reduced to the original chiral matrix $\sigma_z$ (times $\pm \lambda_n$).

### 6.2 Balanced fuzzy spheres

Mass deformed Dirac-Landau model introduces fuzzy spheres as

\[
\begin{align*}
  n = 0 : & \quad X_{0-+M} = \langle \psi_{0} \mid \psi_{0} \rangle = \alpha_{0}^{(g)} \cdot S_{g=0} = 1, \\
  n = 1, 2, \ldots : & \quad X_{\pm} = \langle \psi_{\pm} \mid \psi_{\pm} \rangle = \alpha_{\pm}^{(g)}(M) \cdot S_{g=g-\frac{1}{2}},
\end{align*}
\]

where

\[
\begin{align*}
  n = 0 : & \quad \alpha_{0}^{(g)} = \frac{1}{g + \frac{1}{2}}, \\
  n = 1, 2, \ldots : & \quad \alpha_{\pm}^{(g)}(M) \equiv (1 \pm \frac{1}{2g}) \alpha_{n}^{(g)}.
\end{align*}
\]

To derive \((179b)\), we used \((157)\) and \(\langle \psi_{\pm} \mid \psi_{\pm} \rangle = \frac{1}{2g} \alpha_{n}^{(g)} S_{g=g-\frac{1}{2}}\). For $\Lambda_0 = +M$, everything is same as of the fuzzy sphere of the zero modes of the Dirac-Landau operator. \((180b)\) suggests that the mass parameter unevenly affects the non-commutative length scales, $\alpha_{\pm}^{(g)}$ $\,(n = 1, 2, \ldots)\), which have the following properties:

\[
\begin{align*}
  \alpha_{-}^{(g)}(-M) &= \alpha_{+}^{(g)}(M), \\
  \alpha_{+}^{(g)}(M) + \alpha_{-}^{(g)}(M) &= 2\alpha_{n}^{(g)}.
\end{align*}
\]

The radii of the fuzzy spheres are

\[
X_{\pm} = R_{\pm}^{(g)}(M)^2,
\]

where

\[
R_{\pm}^{(g)}(M) \equiv \alpha_{\pm}^{(g)}(M) \cdot \sqrt{(n + g - \frac{1}{2}) (n + g + \frac{1}{2})}.
\]

Sum of the radii of the fuzzy spheres for $+\Lambda_n$ and $-\Lambda_n$ is immune to the mass deformation and same as in the massless case:

\[
R_{+}^{(g)}(M) + R_{-}^{(g)}(M) = 2\alpha_{n}^{(g)} \cdot \sqrt{(n + g - \frac{1}{2}) (n + g + \frac{1}{2})} = 2R_{n}^{(g)}.
\]

To investigate behaviors of the radii under the mass deformation, we define

\[
r_{\pm}^{(g)}(M) \equiv \frac{R_{\pm}^{(g)}(M)}{R_{n}^{(g)}} = \frac{\alpha_{\pm}^{(g)}(M)}{\alpha_{n}^{(g)}} = 1 \pm \frac{1}{2g} M .
\]

$r_{\pm}^{(g)}(M)$ denote the ratios of $R_{\pm}^{(g)}(M)$ with respect to their massless limit, and are depicted in Fig\[8\]. When $M = 0$, there exist two identical fuzzy spheres for $+\lambda_n$ and $-\lambda_n$:

\[
r_{+}^{(g)}(M)|_{M=0} = r_{-}^{(g)}(M)|_{M=0} = 1 .
\]
As the mass parameter is turned, these two fuzzy spheres begin to “correlate” and their radii monotonically change until their sizes reach $1 \pm \frac{1}{2g}$ of their original sizes, which are the radii of the non-relativistic fuzzy spheres of $(n - \frac{1}{2}, g \mp \frac{1}{2})$:

$$R^\prime_{+\Lambda_n}(M) \xrightarrow{M \to \infty} R^\prime_n(g^{\mp \frac{1}{2}}) \quad (< R^\prime_n),$$

$$R^\prime_{-\Lambda_n}(M) \xrightarrow{M \to \infty} R^\prime_n(g^{\pm \frac{1}{2}}) \quad (> R^\prime_n).$$

(187)

It may be visualized as if the fuzzy sphere of $+\lambda_n$ is “absorbed” in the fuzzy sphere of $-\lambda_n$ as $M$ increases [Fig.9]. Thus, we can tune the sizes of the fuzzy spheres (with their radii sum fixed) by changing the mass parameter.

Figure 9: Size change of fuzzy spheres under the mass deformation. The red circle represents the fuzzy sphere for $n = 0$, while the blue circles indicate the sizes of the fuzzy spheres for $\pm \lambda_n = 3$. ($g = 3$ and $M = 2$ are adopted in the figure.)

7  Supersymmetric Landau Model and Super Fuzzy Spheres

A close connection is well known between Dirac operator and supersymmetric quantum mechanics [see Ref.[21] for instance]. Here, we construct supersymmetric quantum mechanical Hamiltonian from the Dirac-Landau operator, and construct super fuzzy spheres by the level projection to supersymmetric Landau models.
7.1 Square of the Dirac-Landau operator

Square of the Dirac operator yields a supersymmetric quantum Hamiltonian:

\[ H^{(g)}_{\text{SUSY}} = \frac{1}{2M} (-i \mathcal{D})^2 = H^{(g_s)} - \frac{1}{2M} g_s \sigma_z, \]  

(190)

or

\[ H^{(g)}_{\text{SUSY}} = \begin{pmatrix} H^{(g - \frac{1}{2})} - \frac{1}{2M} (g - \frac{1}{2}) & 0 \\ 0 & H^{(g + \frac{1}{2})} + \frac{1}{2M} (g + \frac{1}{2}) \end{pmatrix}. \]  

(191)

Here, \( H^{(g_s)} \) is given by

\[ H^{(g_s)} = \frac{1}{2M} \left( \Lambda^{(g - \frac{1}{2})^2} 0 \\ 0 \Lambda^{(g + \frac{1}{2})^2} \right), \]  

(192)

with \( H^{(g)} \). The second term of the right-hand side of (190) represents the Zeeman term.

As partially discussed in Sec.3, the square of the Dirac-Landau operator enjoys both \( SU(2) \) and chiral symmetries:

\[ [H^{(g)}_{\text{SUSY}}, J] = 0, \]  

(193a)

\[ [H^{(g)}_{\text{SUSY}}, \sigma_z] = 0. \]  

(193b)

One may readily verify (193a) and (193b) from \([-i \mathcal{D}, J] = 0 \) and \([-i \mathcal{D}, \sigma_z] = 0 \) using identities \([A^2, B] = [A, [A, B]] \) and \([A^2, B] = [A, \{A, B\}] \) respectively. The energy eigenvalues of the supersymmetric Landau Hamiltonian (192) are given by [Fig.10]

\[ E_n = \frac{1}{2M} (n(n + 2g)) \quad (n = 0, 1, 2, \cdots), \]  

(194)

with degeneracy

\[ n = 0 : 2g, \quad n = 1, 2, \cdots : 4(g + n). \]  

(195a)

(195b)

The corresponding energy eigenstates with definite chiralities are given by

\[ \Upsilon^{ig}_{j=g-\frac{1}{2},m} = \begin{pmatrix} \Upsilon^{g-\frac{1}{2}}_{j=g-\frac{1}{2},m}(\theta, \phi) \\ 0 \end{pmatrix}, \]  

\[ \Upsilon^{ig}_{j=g-\frac{1}{2}+n,m} = \begin{pmatrix} \Upsilon^{g-\frac{1}{2}}_{j=g-\frac{1}{2}+n,m}(\theta, \phi) \\ 0 \end{pmatrix}, \quad \Upsilon^{g}_{j=g+\frac{1}{2}+(n-1),m}(\theta, \phi). \]  

(196a)

(196b)

\[ ^{24} \text{In the thermodynamic limit } g \to \infty \text{ with } g/r^2 \text{ fixed, } H_{\text{SUSY}} \text{ is reduced to the supersymmetric Pauli Hamiltonian on a plane [46]:} \]

\[ H = -\frac{1}{2M} \sum_{i=1,2} D_i^2 - \frac{g}{2M} \sigma_z, \]  

(188)

which is diagonalized as

\[ \frac{g}{M} \begin{pmatrix} n & 0 \\ 0 & n + 1 \end{pmatrix}. \quad (n = 0, 1, 2, \cdots) \]  

(189)
Figure 10: The spectra of the supersymmetric Landau Hamiltonian. The solid and dashed curves respectively correspond to (194) for \( g = 3 \) and 8.

The supersymmetric structure becomes obvious when we express \( H_{\text{SUSY}}^{(g)} \) as

\[
H_{\text{SUSY}}^{(g)} = -\frac{1}{2M} \begin{pmatrix}
\delta^{(g+\frac{1}{2})}_{-} & 0 \\
0 & \delta^{(g-\frac{1}{2})}_{+} \\
\end{pmatrix} = \{ Q^{(g)}, \bar{Q}^{(g)} \},
\]

(197)

where \( Q^{(g)} \) and \( \bar{Q}^{(g)} \) are nilpotent super-charges:

\[
(Q^{(g)})^2 = (\bar{Q}^{(g)})^2 = 0,
\]

(198)

as given by

\[
Q^{(g)} = -\frac{1}{\sqrt{2M}} \sigma^+ \delta^{(g+\frac{1}{2})}_{-} = \frac{1}{\sqrt{2M}} \begin{pmatrix}
0 & \delta^{(g+\frac{1}{2})}_{-} \\
0 & 0 \\
\end{pmatrix},
\]

(199)

\[
\bar{Q}^{(g)} = \frac{1}{\sqrt{2M}} \sigma^- \delta^{(g-\frac{1}{2})}_{+} = \frac{1}{\sqrt{2M}} \begin{pmatrix}
0 & 0 \\
\delta^{(g-\frac{1}{2})}_{+} & 0 \\
\end{pmatrix}.
\]

(200)

From the nilpotency of the supercharges [198], it is obvious that the supersymmetric Landau Hamiltonian respects the supersymmetry:

\[
[H_{\text{SUSY}}^{(g)}, Q^{(g)}] = [H_{\text{SUSY}}^{(g)}, \bar{Q}^{(g)}] = 0.
\]

(201)

The supercharges are also \( SU(2) \) singlet operators,

\[
[J, Q^{(g)}] = [J, \bar{Q}^{(g)}] = 0,
\]

(202)

which anticommute with the chirality matrix:

\[
\{ Q^{(g)}, \sigma_z \} = \{ \bar{Q}^{(g)}, \sigma_z \} = 0.
\]

(203)

\( Q^{(g)} \) and \( \bar{Q}^{(g)} \) act to the opposite chirality eigenstates of the \( n \)th Landau level as

\[
Q^{(g)} \gamma^{g}_{j=g+n+\frac{1}{2},m} = \sqrt{\frac{(n+2g+1)(n+1)}{2M}} \gamma^{g}_{j=g+n+\frac{1}{2},m}, \quad \bar{Q}^{(g)} \gamma^{g}_{j=g+n+\frac{1}{2},m} = 0,
\]

\[
\bar{Q}^{(g)} \gamma^{g}_{j=g+n-\frac{1}{2},m} = \sqrt{\frac{(n+2g)n}{2M}} \gamma^{g}_{j=g+n-\frac{1}{2},m}, \quad Q^{(g)} \gamma^{g}_{j=g+n-\frac{1}{2},m} = 0.
\]

(204)
7.2 Super fuzzy spheres

For each supersymmetric Landau level of \( n \neq 0 \), we introduce two fuzzy spheres from the opposite chirality states, \( \Upsilon_{g,j,m}^{\sigma} \) and \( \Upsilon_{g,j,m}^{\sigma'} \):

\[
X^{(-)} \equiv \langle \Upsilon_{g,j,m}^{\sigma} | \Upsilon_{g,j,m}^{\sigma'} \rangle = -\alpha_n^{(g-\frac{1}{2})} S_{s=n+g-\frac{1}{2}},
\]

\[
X^{(+)} \equiv \langle \Upsilon_{g,j,m}^{\sigma} | \Upsilon_{g,j,m}^{\sigma'} \rangle = -\alpha_n^{(g+\frac{1}{2})} S_{s=n-g+\frac{1}{2}}.
\]

(Eigenstates of the supersymmetric \( n = 0 \) Landau level are same as of the zero modes of the relativistic Landau model as discussed in Sec. 5.3.) These two fuzzy spheres may be considered as super partners, since \( \Upsilon_{g,j,m}^{\sigma} \) and \( \Upsilon_{g,j,m}^{\sigma'} \) are related by the supersymmetric transformation \( 204 \). We shall refer to these two fuzzy spheres as super fuzzy spheres \(^{25}\). The radii of the super fuzzy spheres \( 205 \) slightly differ as

\[
R_{n}^{(g-\frac{1}{2})} = \alpha_n^{(g-\frac{1}{2})} \sqrt{j(j+1)} \bigg|_{j=g+n-\frac{1}{2}} = r \frac{g-\frac{1}{2}}{\sqrt{(g+n-\frac{1}{2})(g+n+\frac{1}{2})}},
\]

\[
R_{n-1}^{(g+\frac{1}{2})} = \alpha_{n-1}^{(g+\frac{1}{2})} \sqrt{j(j+1)} \bigg|_{j=g+n-\frac{1}{2}} = r \frac{g+\frac{1}{2}}{\sqrt{(g+n-\frac{1}{2})(g+n+\frac{1}{2})}}.
\]

Their behaviors with respect to \( g \) are plotted in Fig. 11. As \( g \) increases, \( R_{n}^{(g-\frac{1}{2})} \) and \( R_{n-1}^{(g+\frac{1}{2})} \) asymptotically approach to same value, \( r \frac{g}{g+n} \). The radius of the relativistic fuzzy sphere \( 161 \) is

![Figure 11: \( R_{n-1}^{(g+\frac{1}{2})} / r \) (\( n = 1, 2, 3, 4 \)) correspond to the solid curves (red, orange, green, blue), while \( R_{n}^{(g-\frac{1}{2})} / r \) the dashed curves.]

\(^{25}\)We adopt a terminology, super fuzzy sphere, instead of fuzzy supersphere since fuzzy supersphere usually means a fuzzy sphere made of graded Lie algebra [see Ref. \([17]\) for instance.] Fuzzy superspheres appear in the Landau levels of the \( UOSp(1|2) \) invariant Landau model \([15] [19] \).
the average of the radii of the super fuzzy spheres:

\[ R'_n^{(g)} = \frac{1}{2} \left( R_{n-1}^{(g+\frac{1}{2})} + R_{n}^{(g-\frac{1}{2})} \right). \]  

(207)

The mass deformation just brings a constant shift to the supersymmetric Landau Hamiltonian:

\[ \frac{1}{2M}(-i\mathcal{P} + M\sigma_z)^2 = \frac{1}{2M}(-i\mathcal{P})^2 + \frac{1}{2}M = H_{\text{SUSY}}^{(g)} + \frac{1}{2}M, \]  

(208)

and does not affect the supersymmetric eigenstates \([196]\) and so the super fuzzy spheres either.

8 Valley Fuzzy Spheres from Graphene

In this section, we apply the above analysis to the realistic graphene system.

8.1 Graphene spectrum

In graphene, the spinor components of the Dirac operator indicate A and B sub-lattice degrees of freedom. In addition to the sub-lattice degrees of freedom, graphene accommodates the valley degrees of freedom of \(K\) and \(K'\) points, in which low energy physics is described by

\[ -i\mathcal{P}_{K\oplus K'} \equiv \begin{pmatrix} -i\mathcal{P}_K & 0 \\ 0 & -i\mathcal{P}_{K'} \end{pmatrix}, \]  

(209)

where

\[ -i\mathcal{P}_K \equiv -i\sigma_x D_\theta - i\frac{1}{\sin \theta} \sigma_y D_\phi, \quad -i\mathcal{P}_{K'} \equiv -i\sigma_x D_\theta + i\frac{1}{\sin \theta} \sigma_y D_\phi, \]  

(210)

with \(D_\theta\) and \(D_\phi\) \([53]\). These are related as

\[ -i\mathcal{P}_K = \sigma_x(-i\mathcal{P}_{K'})\sigma_x. \]  

(211)

The SU(2) operator that commutes with \(-i\mathcal{P}_{K\oplus K'}\) is given by

\[ J = \begin{pmatrix} L^{(g_s)} & 0 \\ 0 & L^{(\bar{g}_s)} \end{pmatrix} = \begin{pmatrix} L^{(g-\frac{1}{2})} & 0 & 0 & 0 \\ 0 & L^{(g+\frac{1}{2})} & 0 & 0 \\ 0 & 0 & L^{(g+\frac{1}{2})} & 0 \\ 0 & 0 & 0 & L^{(g-\frac{1}{2})} \end{pmatrix}, \]  

(212)

where \(g_s = g - \frac{1}{2}\sigma_z\) and \(\bar{g}_s = g + \frac{1}{2}\sigma_z\). \(J\) satisfies

\[ [J_i, J_j] = i\epsilon_{ijk} J_k, \]  

(213)

and

\[ J^2 = \begin{pmatrix} j_K(j_K + 1) & 1_{2j_K+1} & 0 \\ 1_{2j_K+1} & j_{K'}(j_{K'} + 1) & 1_{2j_{K'}+1} \\ 0 & 1_{2j_{K'}+1} & j_K(j_K + 1) \end{pmatrix}, \]  

(214)
where $1_{2j+1}$ denotes $(2j + 1) \times (2j + 1)$ unit matrix and

$$j_K = g - \frac{1}{2} + n_K, \quad j_{K'} = g - \frac{1}{2} + n_{K'}. \quad (n_K, n_{K'} = 0, 1, 2, \cdots) \quad (215)$$

Square of the graphene Hamiltonian $[209]$ is given by

$$(-i\hat{D}_K \oplus K')^2 = J^2 - g^2 + \frac{1}{4}, \quad (216)$$

$-i\hat{D}_K$ and $-i\hat{D}_{K'}$ have the same spectrum, and so the spectrum of $-i\hat{D}_K \oplus K'$ is equally given by

$$\pm \lambda_n = \pm \sqrt{n(2g + n)} \quad (n = 0, 1, 2, \cdots), \quad (217)$$

and the corresponding degeneracy for each of $+\lambda_n$ and $-\lambda_n$ is

$$2 \times (2j + 1) = 4(g + n) \quad (n = 0, 1, 2, \cdots). \quad (218)$$

Obviously, $2 \times$ comes from the valley degrees of freedom. The eigenstates are denoted as

- $n = 0$ : $\Psi^g_{\lambda_0=0,m; K} = \begin{pmatrix} Y_{j=g-\frac{1}{2},m}^g \\ 0 \end{pmatrix}$, $\Psi^g_{\lambda_0=0,m; K'} = \begin{pmatrix} 0 \\ Y_{j=g-\frac{1}{2},m}^g \end{pmatrix}$, $\quad (219a)$
- $n = 1, 2, \cdots$ : $\Psi^g_{\pm\lambda_n,m; K} = \begin{pmatrix} Y_{j,m}^g \pm \frac{i}{2} \\ \mp i Y_{j,m}^g + \frac{1}{2} \end{pmatrix}$, $\Psi^g_{\pm\lambda_n,m; K'} = \begin{pmatrix} \mp i Y_{j,m}^g + \frac{1}{2} \\ Y_{j,m}^g - \frac{i}{2} \end{pmatrix}$, $\quad (219b)$

which are related as

$$\Psi^g_{\pm\lambda_n,m; K} = \sigma_x \Psi^g_{\pm\lambda_n,m; K'}. \quad (220)$$

### 8.2 Mass deformation and valley fuzzy spheres

We consider mass deformation of the Dirac-Landau operators at $K$ and $K'$ points:

$$-i\hat{D}_K + M\sigma_z, \quad -i\hat{D}_{K'} + M\sigma_z, \quad (221)$$

to have

$$(-i\hat{D} + M\sigma_z)_{K \oplus K'} = \begin{pmatrix} -i\hat{D}_K + M\sigma_z & 0 \\ 0 & -i\hat{D}_{K'} + M\sigma_z \end{pmatrix}. \quad (222)$$

In each valley, the mass deformed Dirac-Landau operator is readily diagonalized:

- $K$ : $+\Lambda_0 = +M$ \quad $(n = 0)$, $\pm \Lambda_n$ \quad $(n = 1, 2, \cdots)$, $\quad (223a)$
- $K'$ : $-\Lambda_0 = -M$ \quad $(n = 0)$, $\pm \Lambda_n$ \quad $(n = 1, 2, \cdots)$, $\quad (223b)$
with $2(n + g)$ $(n = 0, 1, 2, \ldots)$ degeneracy each. The corresponding eigenstates are

$$\Psi_{\pm \lambda_n, m; K}(M) = \sqrt{\frac{\lambda_n + \lambda_n}{2\Lambda_n}} \left( \Psi_{\pm \lambda_n, m; K} \pm \frac{M}{\lambda_n + \lambda_n} \Psi_{\mp \lambda_n, m; K} \right)$$

$$= \frac{1}{2} \sqrt{\frac{\lambda_n + \lambda_n}{2\Lambda_n}} \begin{pmatrix} (1 \pm \frac{M}{\lambda_n + \lambda_n}) Y_{j = g - \frac{1}{2} + n, m}^{g-\frac{1}{2}} Y_{j = g + \frac{1}{2} + (n-1), m}^{g+\frac{1}{2}} \mp i(1 \mp \frac{M}{\lambda_n + \lambda_n}) Y_{j = g + \frac{1}{2} + (n-1), m}^{g+\frac{1}{2}} \end{pmatrix},$$

(225a)

$$\Psi_{\pm \lambda_n, m; K'}(M) = \sqrt{\frac{\lambda_n + \lambda_n}{2\Lambda_n}} \left( \Psi_{\pm \lambda_n, m; K'} \mp \frac{M}{\lambda_n + \lambda_n} \Psi_{\mp \lambda_n, m; K'} \right)$$

$$= \frac{1}{2} \sqrt{\frac{\lambda_n + \lambda_n}{2\Lambda_n}} \begin{pmatrix} \mp i(1 \pm \frac{M}{\lambda_n + \lambda_n}) Y_{j = g + \frac{1}{2} + (n-1), m}^{g+\frac{1}{2}} \pm \mp i(1 \mp \frac{M}{\lambda_n + \lambda_n}) Y_{j = g - \frac{1}{2} + n, m}^{g-\frac{1}{2}} \end{pmatrix}.$$  (225b)

The mass deformed graphene spectrum is given by

$$\pm \Lambda_n = \pm \sqrt{n(n + 2g) + M^2} \quad (n = 0, 1, 2, \ldots),$$

(226)

with degeneracy [Fig.12]

$$\pm \Lambda_0 = +M : 2g \quad (n = 0),$$

$$\pm \Lambda_n = -M : 4(g + n) \quad (n = 1, 2, \ldots).$$

(227)

The reflection symmetry of the spectra with respect to the zero energy still exists under the mass deformation, though either of the mass deformed Dirac-Landau operators at $K$ and $K'$ points does not respect the chiral symmetry. The reflection symmetry is guaranteed by

$$\{R, (-i\mathcal{D} + M\sigma_z)_{K \sqcup K'} \} = 0,$$

(228)

with

$$R = i \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}.$$  (229)

Eq. (228) can readily be verified from the relation:

$$\sigma_y (-i\mathcal{D}_K + M\sigma_z) + (-i\mathcal{D}_{K'} + M\sigma_z)\sigma_y = 0.$$  (230)

$R$ relates the eigenstates with opposite sign eigenvalues on $K$ and $K'$ points:

$$\Psi_{\pm \lambda_n, m; K}(M) = i\sigma_y \Psi_{\pm \lambda_n, m; K'}(M).$$  (231)

With use of the eigenstates of $K$ and $K'$ valleys ([224]), valley fuzzy spheres are introduced as

$$X^K_{\pm \lambda_n} = \langle \Psi_{\pm \lambda_n, m; K}(M) | x | \Psi_{\pm \lambda_n, m; K}(M) \rangle = -\alpha'_{\pm \lambda_n}(M) \cdot S_{s = g + n - \frac{1}{2}},$$

(232a)

$$X^{K'}_{\pm \lambda_n} = \langle \Psi_{\pm \lambda_n, m; K'}(M) | x | \Psi_{\pm \lambda_n, m; K'}(M) \rangle = -\alpha'_{\pm \lambda_n}(M) \cdot S_{s = g + n - \frac{1}{2}},$$

(232b)

In the massless limit $M \to 0$, they are reduced to

$$\Psi_{\pm \lambda_n, m; K}(M) \to \Psi_{\pm \lambda_n, m; K'}, \quad \Psi_{\pm \lambda_n, m; K'}(M) \to \Psi_{\pm \lambda_n, m; K'},$$

(224)
Figure 12: The blue and green blobs respectively correspond to the eigenstates of the Dirac-Landau operators at $K$ and $K'$ points. The degeneracy of zero modes is lifted to $M$ and $-M$ when the mass term is added.

where (231) was used. Thus, we have

$$X^K_{\Lambda_n} \cdot X^K_{\Lambda_n} = X^{K'}_{-\Lambda_n} \cdot X^{K'}_{-\Lambda_n} = R'^{(g)}(M)^2,$$

(233a)

$$X^K_{-\Lambda_n} \cdot X^K_{-\Lambda_n} = X^{K'}_{\Lambda_n} \cdot X^{K'}_{\Lambda_n} = R'^{(g)}(-M)^2,$$

(233b)

where $R'^{(g)}(M)$ are given by (183). As the mass parameter is turned on and increases, the four fuzzy spheres for $n(\neq 0)$th Landau level change their sizes, two of which expand and the other two shrink, while the two fuzzy spheres for $n = 0$ do not vary their sizes [Fig.13].

Figure 13: The circles represent the sizes of the fuzzy spheres on the corresponding Landau levels. ($g = 3$ and $M = 2$ are adopted in the figure.)
9 Summary

We gave a thorough study of the relativistic Landau models and derived non-commutative geometry by applying the level projection method to the relativistic Landau models. We obtained a concise expression of the eigenstates of the Dirac-Landau operator on a sphere, which turned out to be related to non-relativistic Pauli-Schödinger eigenstates by the $SU(2)$ gauge transformation. After the $SU(2)$ transformation, the Dirac-Landau operator acts as the boost operator of the Lorentz group. We constructed the relativistic fuzzy spheres with use of the relativistic Landau level eigenstates and found that the fuzzy sphere of zero modes reduces its size while fuzzy spheres of non-zero Landau levels enhance their sizes compared to their non-relativistic counterparts. Under the mass deformation, two fuzzy spheres of positive and negative relativistic Landau levels vary their sizes keeping the sum of their radii constant, while the size of the fuzzy sphere of zero modes does not vary. We also constructed super fuzzy spheres from a supersymmetric Landau model as the square of the Dirac-Landau operator, and discussed their behaviors with respect to the monopole charge. Finally, we investigated graphene system. Due to the valley degrees of freedom, each Landau level is two-fold degenerate compared to the single Dirac-Landau case, and there appear valley fuzzy spheres. We discussed the reflection symmetry of the graphene spectrum and clarified the particular properties of the valley spheres under the mass deformation.

While we focused on the fuzzy geometry in the relativistic Landau models, the level projection itself is a versatile method to introduce fuzzy geometry from physical models. It may be interesting to apply the level projection to other manifolds to generate a variety of fuzzy geometry and investigate their geometrical behaviors controlled by physical parameters. We have not discussed many-body physics of the relativistic Landau system. The present analysis has an advantage for numerical calculations because of its rotational symmetry. We will report applications of the present spherical formalism to relativistic quantum Hall effect in a future publication [50].

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A Jacobi Polynomials

Jacobi polynomials are defined by
\[
P^{(\alpha,\beta)}_n(x) = \frac{(-1)^n}{2^n n!} (1 - x)^{-\alpha} (1 + x)^{-\beta} \frac{d^n}{dx^n} (1 - x)^{n+\alpha} (1 + x)^{n+\beta},
\]
where $x \in [-1, 1]$. Normalization is the following:
\[
\int_{-1}^{1} dx \ (1 - x)^{-\alpha} (1 + x)^{-\beta} P^{(\alpha,\beta)}_n(x)^* P^{(\alpha,\beta)}_m(x) = \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} \frac{\Gamma(n + \alpha + a)\Gamma(n + \beta + 1)}{n! \Gamma(n + \alpha + \beta + 1)} \delta_{n,m}.
\]
The Jacobi polynomial is a solution of a second-order differential equation:

\[(1 - x^2) \frac{d^2 P_n^{(\alpha, \beta)}(x)}{dx^2} - ((\alpha + \beta + 2)x + \alpha - \beta) \frac{dP_n^{(\alpha, \beta)}(x)}{dx} + n(n + \alpha + \beta + 1)P_n^{(\alpha, \beta)}(x) = 0. \tag{236}\]

For \(n = 0, 1, 2\), the Jacobi polynomials are given by

\[P_0^{(\alpha, \beta)} = 1,\]
\[P_1^{(\alpha, \beta)}(x) = \frac{1}{2}((\alpha + \beta + 2)x + (\alpha - \beta)),\]
\[P_2^{(\alpha, \beta)}(x) = \frac{1}{8}(((\alpha + \beta)^2 + 7(\alpha + \beta) + 12)x^2 + 2(\alpha + \beta + 3)(\alpha - \beta)x + (\alpha - \beta)^2 - (\alpha + \beta) - 4). \tag{237}\]

## B From Three-sphere Point of View

The Dirac monopole set-up is mathematically equivalent to the 1st Hopf map [see Ref.\[4\] for instance]:

\[S^3 \xrightarrow{S^1 \rightarrow} S^2. \tag{238}\]

The total manifold is \(S^3\) and the algebras of the Landau problem on the two-sphere is naturally understood from the perspective of \(S^3\). The symmetry of the present system is the rotational symmetry of the total manifold \(S^3\):

\[SO(4) \simeq SU(2)_L \otimes SU(2)_R. \tag{239}\]

### B.1 D functions

With the Euler angle parametrization, the \(SU(2)\) element is expressed as

\[g(\chi, \theta, \phi) = e^{i\frac{\chi}{2}S_y}e^{i\theta S_y}e^{i\phi S_z} = g^\dagger(-\phi, -\theta, -\chi) \quad (0 \leq \chi \leq 4\pi, \ 0 \leq \theta \leq \pi, \ 0 \leq \phi \leq 2\pi). \tag{240}\]

The Wigner’s \(D\) function, \(D_{l,m,g}(\phi, \theta, \chi)\), is introduced as a generalization of \(\text{[240]}\) for arbitrary representation with \(SU(2)\) Casimir index \(l\):

\[D_{l,m,g}(\phi, \theta, \chi) \equiv \langle l, m|e^{-i\phi S_z}e^{-i\theta S_y}e^{-i\chi S_z}|l, g\rangle = D_{l,g,m}(-\chi, -\theta, -\phi)^* \quad (m, g = l, l - 1, \ldots, -l). \tag{241}\]

For the fundamental representation, \(D_{l=m/2,m,g}(-\chi, -\theta, -\phi) = g(\chi, \theta, \phi)_{m,g}\). The \(D\) function can be expressed as a simple product of functions of each angular coordinate:

\[D_{l,m,g}(\phi, \theta, \chi) = e^{-i(m\phi + g\chi)}d_{l,m,g}(\theta), \tag{242}\]

with

\[d_{l,m,g}(\theta) \equiv \langle l, m|e^{-i\theta S_y}|l, g\rangle, \tag{243}\]

\[42\]
which has a symmetry under the interchange of two magnetic quantum numbers;

\[ d_{l,m,g}(\theta) = (-1)^{m-g}d_{l,g,m}(\theta). \]  

(244)

The explicit form of \( D \) function is

\[ D_{l,m,g}(\phi, \theta, \chi) = \frac{(-1)^{m-g}}{2m} \sqrt{\frac{(l+m)!}{(l+g)!}} \sqrt{\frac{(l-m)!}{(l-g)!}} (1-x)^{m-g} \left( 1 + x \right)^{m+g} P_{l-m}^{(m-g,m+g)}(x) \cdot e^{i(m\phi + g\chi)}, \]

(245)

where \( x = \cos \theta \) and \( P_n^{(a,b)} \) denote the Jacobi polynomials (234). The monopole harmonics (20) are related to the Wigner’s \( D \) functions as [51]:

\[ Y^g_{l,m}(\theta, \phi) = (-1)^{m+g} \sqrt{\frac{2l+1}{4\pi}} D_{l,-m,g}(\phi, \theta, 0) \]

\[ = (-1)^{m+g} \sqrt{\frac{2l+1}{4\pi}} d_{l,-m,g}(\theta) e^{im\phi} \]

(246)

or

\[ D_{l,m,g}(\phi, \theta, \chi) = (-1)^{m-g} \sqrt{\frac{4\pi}{2l+1}} Y^g_{l,-m}(\theta, \phi) e^{-ig\chi}. \]

(247)

B.2 Maurer-Cartan 1 form and left and right actions

The \( D \) function carries the \( SU(2) \) Casimir index \( l \) and two magnetic quantum numbers, \( m \) and \( g \). We construct two independent sets of \( SU(2) \) algebras whose simultaneous eigenstates to be \( D \) function by using the Maurer-Cartan formulation.

The left Maurer-Cartan 1 form is given by the formula

\[ -ig^1 dg = e^1(L)\sigma_i, \]

(248)

and from (240)

\[ g(\chi, \theta, \phi) = \begin{pmatrix} e^{i \frac{\phi}{2}} & 0 & e^{-i \frac{\chi}{2}} \\ 0 & e^{-i \frac{\phi}{2}} & 0 \\ -e^{-i \frac{\chi}{2}} & -e^{i \frac{\phi}{2}} & e^{-i \frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos \frac{\chi}{2} & \sin \frac{\chi}{2} & 0 \\ -\sin \frac{\chi}{2} & \cos \frac{\chi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i \frac{\phi}{2}} & 0 & e^{-i \frac{\chi}{2}} \\ 0 & e^{-i \frac{\phi}{2}} & 0 \\ -e^{-i \frac{\chi}{2}} & -e^{i \frac{\phi}{2}} & e^{-i \frac{\phi}{2}} \end{pmatrix}, \]

(249)

we have

\[ e^1(L) = \sin \theta \cos \phi d\chi - \sin \phi d\theta, \]
\[ e^2(L) = \sin \theta \sin \phi d\chi + \cos \phi d\theta, \]
\[ e^3(L) = \cos \theta d\chi + d\phi. \]

(250)

Similarly, the right Maurer-Cartan 1 form is given by

\[ idg \cdot g^1 = e^1(R)\sigma_i, \]

(251)
and

\[ e^1(R) = \sin \theta \cos \chi d\phi - \sin \chi d\theta, \]
\[ e^2(R) = -\sin \theta \sin \chi d\phi - \cos \chi d\theta, \]
\[ e^3(R) = -\cos \theta d\phi - d\chi. \]  \hspace{1cm} (252)

It is easy to check the \( e^i(L) \) and \( e^i(R) \) satisfy the Maurer-Cartan equations:

\[ de^i(L) - \frac{1}{2} e^{ijk} e^j(L) \wedge e^k(L) = 0, \]
\[ de^i(R) - \frac{1}{2} e^{ijk} e^j(R) \wedge e^k(R) = 0. \]  \hspace{1cm} (253)

The metric is read off from

\[ ds^2 = e^i(L)e^i(L) = e^i(R)e^i(R) = g_{\mu\nu} dx^\mu dx^\nu, \quad (x^\mu = \chi, \theta, \phi) \]  \hspace{1cm} (254)

as

\[
 g_{\mu\nu} = \begin{pmatrix}
 g_{\chi\chi} & g_{\chi\theta} & g_{\chi\phi} \\
 g_{\theta\chi} & g_{\theta\theta} & g_{\theta\phi} \\
 g_{\phi\chi} & g_{\phi\theta} & g_{\phi\phi}
\end{pmatrix}
 = \begin{pmatrix}
 1 & 0 & \cos \theta \\
 0 & 1 & 0 \\
 \cos \theta & 0 & 1
\end{pmatrix},
\]

and then

\[
 g^{\mu\nu} = \frac{1}{\sin^2 \theta} \begin{pmatrix}
 1 & 0 & -\cos \theta \\
 0 & \sin^2 \theta & 0 \\
 -\cos \theta & 0 & 1
\end{pmatrix}.
\]  \hspace{1cm} (256)

\( e^i \) are derived from \( e^i = e^i_{\mu} dx^\mu \) and the dual Killing spinor \( e_{\mu}^i \) are introduced to satisfy \( e^i_{\mu} e_{\nu}^i = \delta^\mu_\nu \). The Killing vectors dual to the left and right Maurer-Cartan 1 form are respectively given by

\[ L_i = -ie_{\mu}^i(L) \partial_\mu, \]
\[ R_i = -ie_{\mu}^i(R) \partial_\mu, \]  \hspace{1cm} (257)

or

\[ L_x = -i(-\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi + \frac{\cos \phi}{\sin \theta} \partial_\chi), \]
\[ L_y = -i(\cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi + \frac{\sin \phi}{\sin \theta} \partial_\chi), \]
\[ L_z = -i \partial_\phi, \]  \hspace{1cm} (258)

and

\[ R_x = i(\sin \chi \partial_\theta + \cot \theta \cos \chi \partial_\phi - \frac{\cos \chi}{\sin \theta} \partial_\phi), \]
\[ R_y = i(\cos \chi \partial_\theta - \cot \theta \sin \chi \partial_\phi + \frac{\sin \chi}{\sin \theta} \partial_\phi), \]
\[ R_z = i \partial_\chi. \]  \hspace{1cm} (259)
and (259) are mutually transformed by the interchange:

\[ \phi \leftrightarrow -\chi, \quad \theta \leftrightarrow -\theta. \]  

(260)

They satisfy the two independent \( SU(2) \) algebras:

\[ [L_i, L_j] = i\epsilon_{ijk} L_k, \quad [R_i, R_j] = i\epsilon_{ijk} R_k, \quad [L_i, R_j] = 0. \]  

(261)

(261) is a direct consequence of the Maurer-Cartan equation (253). The ladder operators for the two \( SU(2) \) algebras are respectively constructed as

\[ L_+ = L_x + iL_y = e^{i\phi}(\partial_\theta + i \cot \theta \partial_\phi - i \frac{1}{\sin \theta} \partial_\chi), \]
\[ L_- = L_x - iL_y = -e^{-i\phi}(\partial_\theta - i \cot \theta \partial_\phi + i \frac{1}{\sin \theta} \partial_\chi), \]  

(262)

and

\[ R_+ = R_x + iR_y = -e^{-i\chi}(\partial_\theta - i \cot \theta \partial_\chi + i \frac{1}{\sin \theta} \partial_\phi), \]
\[ R_- = R_x - iR_y = e^{i\chi}(\partial_\theta + i \cot \theta \partial_\chi - i \frac{1}{\sin \theta} \partial_\phi). \]  

(263)

They act to the \( D \) functions as

\[ L_+ D_{l,-m,g}(\phi, \theta, \chi) = -\sqrt{(l-m)(l+m+1)} \ D_{l,-m-1,g}(\theta, \phi, \chi), \]
\[ L_- D_{l,-m,g}(\phi, \theta, \chi) = -\sqrt{(l+m)(l-m+1)} \ D_{l,-m+1,g}(\theta, \phi, \chi), \]  

(264)

and

\[ R_+ D_{l,-m,g}(\phi, \theta, \chi) = \sqrt{(l-g)(l+g+1)} \ D_{l,-m+1,g}(\phi, \theta, \chi), \]
\[ R_- D_{l,-m,g}(\phi, \theta, \chi) = \sqrt{(l+g)(l-g+1)} \ D_{l,-m-1,g}(\phi, \theta, \chi). \]  

(265)

Thus, \( L \) and \( R \) are respectively the left- and right-actions to \( D \) functions. The \( SU(2) \) Casimirs of \( R \) and \( L \) are equally given by

\[ L^2 = R^2 = -\frac{1}{\sin \theta} \partial_\theta(\sin \theta \partial_\theta) - \frac{1}{\sin^2 \theta} \left( \partial_\phi^2 + \partial_\chi^2 - 2\cos \theta \partial_\theta \partial_\chi \right). \]  

(266)

Therefore, \( D_{l,-m,g}(\phi, \theta, \chi) \) is the simultaneous eigenstate of two independent \( SU(2) \) algebras and the corresponding eigenvalues are

\[ L^2 = R^2 = l(l+1), \]
\[ L_z = m, \]
\[ R_z = g. \]  

(267)

By replacing \( R_z = i\partial_\chi \) with \( g \), we obtain the angular momentum operator \( L^{(g)} \) from the left Killing vector (258):

\[ L^{(g)} = L|_{i\partial_\chi \rightarrow g}, \]  

(268)

and the edth operators (31) from the right Killing vector (263):

\[ \partial_+^{(g)} = -R_+|_{i\partial_\chi \rightarrow g}, \]
\[ \partial_-^{(g)} = R_-|_{i\partial_\chi \rightarrow g}. \]  

(269)
B.3 (2+2) spherical coordinate representation and SO(4) spherical harmonics

The above argument based on the Maurer-Cartan 1-form is mathematically elegant and the calculations are easy, but rather abstract. Here, we derive some results from a simple quantum mechanical argument. Calculations are rather laborious but straightforward and familiar to any physicist.

From the SU(2) group element \( g (g^\dagger g = 1_2, \det g = 1) \), \( S^3 \) coordinates \( X_{\mu=1,2,3,4} \) \((\sum_{\mu=1}^4 X_{\mu}X_{\mu} = 1)\) are extracted as

\[
g = \begin{pmatrix} X_4 - iX_3 & -X_2 - iX_1 \\ X_2 - iX_1 & X_4 + iX_3 \end{pmatrix}.
\]  (270)

In the case of (240), we have

\[
X_1 = \sin \frac{\theta}{2} \sin \frac{1}{2}(\phi - \chi), \\
X_2 = -\sin \frac{\theta}{2} \cos \frac{1}{2}(\phi - \chi), \\
X_3 = -\cos \frac{\theta}{2} \sin \frac{1}{2}(\phi + \chi), \\
X_4 = \cos \frac{\theta}{2} \cos \frac{1}{2}(\phi + \chi),
\]  (271)

which is known as the (2+2) spherical coordinate representation. The metric on \( S^3 \) is derived as

\[
\sum_{\mu=1}^4 dX_{\mu}dX_{\mu} = \frac{1}{4} (d\chi^2 + d\theta^2 + d\phi^2 + 2 \cos \theta d\chi d\phi),
\]  (272)

which is equal to (256) up to the unimportant proportional factor. The SO(4) free angular momentum operators are given by

\[
L_{\mu\nu} = -iX_{\mu} \frac{\partial}{\partial X_{\nu}} + iX_{\nu} \frac{\partial}{\partial X_{\mu}},
\]  (273)

and the corresponding \( SU(2)_L \oplus SU(2)_R \) operators are constructed as

\[
L_i = \frac{1}{4} \eta^i_{\mu\nu} L_{\mu\nu} = \frac{1}{4} \epsilon_{ijk} L_{ij} + \frac{1}{2} L_{i4}, \tag{274a}
\]
\[
R_i = \frac{1}{4} \bar{\eta}^i_{\mu\nu} L_{\mu\nu} = \frac{1}{4} \epsilon_{ijk} L_{ij} - \frac{1}{2} L_{i4}, \tag{274b}
\]

where \( \eta^i_{\mu\nu} \) and \( \bar{\eta}^i_{\mu\nu} \) are the 'tHooft symbols:

\[
\eta^i_{\mu\nu} = \epsilon_{\mu\nu4} + \delta_{\mu i} \delta_{\nu4} - \delta_{\nu i} \delta_{\mu4}, \\
\bar{\eta}^i_{\mu\nu} = \epsilon_{\mu\nu4} - \delta_{\mu i} \delta_{\nu4} + \delta_{\nu i} \delta_{\mu4}.
\]  (275)

A bit of calculation shows that, in the (2+2) spherical coordinate representation, (274a) and (274b) are exactly identical with the left and right dual Killing vectors, (258) and (259). Therefore, the
left, right dual Killing vectors are understood as the two independent \( SU(2) \) sets of the free \( SO(4) \) angular momentum. The \( SO(4) \) Casimir is given as

\[
\sum_{\mu<\nu=1}^{4} L_{\mu\nu}^2 = 2(L^2 + R^2) = k(k + 2) \quad (k \equiv 2l = 0, 1, 2, \ldots), \tag{276}
\]

and from the existence of two magnetic quantum numbers of the \( D \) function, \( m, s = -l, -l + 1, \ldots, l \), the degeneracy of the irreducible representation of the \( SO(4) \) Casimir index \( k \) is\(^{27}\)

\[
(2l + 1)^2 = (k + 1)^2. \tag{277}
\]

The \( D \) functions, \( D_{l-m,s}(\phi, \theta, \chi) \), which are the simultaneous irreducible representation of two \( SU(2) \) algebras, constitutes the basis states of the \( SO(4) \) spherical harmonics. In other words, the \( D \) function is simply the \( SO(4) \) spherical harmonics in the \((2+2)\) spherical coordinate representation.

### B.4 Effective representation of the \( SO(4) \) operators

The \( SU(2) \) group element \(^{240}\) can be written as

\[
g(\chi, \theta, \phi) = \begin{pmatrix} u(\theta, \phi) e^{i \frac{1}{2} \chi} & v(\theta, \phi) e^{i \frac{1}{2} \chi} \\ -v(\theta, \phi)^* e^{-i \frac{1}{2} \chi} & u(\theta, \phi)^* e^{-i \frac{1}{2} \chi} \end{pmatrix} \tag{278}
\]

where \( v \) and \( u \) are the components of the Hopf spinor,

\[
\psi = \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} \sin \frac{\theta}{2} e^{-i \frac{1}{2} \phi} \\ \cos \frac{\theta}{2} e^{i \frac{1}{2} \phi} \end{pmatrix}. \tag{279}
\]

Since the monopole harmonics \(^{22}\) are the homogeneous polynomials of the components of the Hopf spinor, the angular momentum and edth operators can effectively be expressed by the Hopf spinor in each of the Landau levels. The angular momentum and edth operators act to the Hopf spinor as

\[
L^\frac{1}{2} \psi = -\frac{1}{2} \sigma \psi, \quad L^{(-\frac{1}{2})} \psi^* = \frac{1}{2} \sigma^t \psi^*, \tag{280a}
\]

\[
\partial^\frac{(-\frac{1}{2})}{} \psi = -i \sigma_y \psi^*, \quad \partial^\frac{(\frac{1}{2})}{} \psi = i \sigma_y \psi^*, \quad \partial^\frac{(\frac{1}{2})}{} \psi = \partial^\frac{(-\frac{1}{2})}{} \psi^* = 0, \tag{280b}
\]

and are effectively expressed as

\[
L = -\frac{1}{2} \psi^t \sigma \frac{\partial}{\partial \psi} + \frac{1}{2} \psi^t \sigma \frac{\partial}{\partial \psi^*}, \tag{281a}
\]

\[
\partial^+ = \psi^t i \sigma_y \frac{\partial}{\partial \psi^*}, \quad \partial^- = \psi^t i \sigma_y \frac{\partial}{\partial \psi}, \tag{281b}
\]

\(^{27}\)This result is consistent with the general formula of the \( SO(D) \) spherical harmonics, whose Casimir eigenvalue is \( \sum_{\mu<\nu=1}^{D} L_{\mu\nu}^2 = k(D - 2 + 1) \) \( (k = 0, 1, 2, \ldots) \) with degeneracy \( \frac{(k+D-3)(k+D-2)}{D(D-2)} \).
which satisfy

\[ [\mathcal{O}_+, \mathcal{O}_-] = 2i\mathcal{O}_z, \]
\[ [-i\mathcal{O}_z, \mathcal{O}_\pm] = \pm\mathcal{O}_\pm. \]  

Here, the operator

\[-i\mathcal{O}_z \equiv \frac{1}{2} (\psi^\dagger \frac{\partial}{\partial \psi} - \psi \frac{\partial}{\partial \psi^\dagger}) \]

represents the monopole charge operator since its eigenvalue is \( g \) \[\text{[see (22)]}\). Obviously, \( \{L_x, L_y, L_z\} \) and \( \{R_x, R_y, R_z\} \equiv \{-i\mathcal{O}_x, -i\mathcal{O}_y, -i\mathcal{O}_z\} \) with

\[ \mathcal{O}_x \equiv \frac{1}{2} (\mathcal{O}_+ + \mathcal{O}_-) = i\frac{1}{2} \psi^t \sigma_y \frac{\partial}{\partial \psi^\dagger} + i\frac{1}{2} \psi^\dagger \sigma_y \frac{\partial}{\partial \psi}, \]
\[ \mathcal{O}_y \equiv -i\frac{1}{2} (\mathcal{O}_+ - \mathcal{O}_-) = \frac{1}{2} \psi^t \sigma_y \frac{\partial}{\partial \psi^\dagger} - \frac{1}{2} \psi^\dagger \sigma_y \frac{\partial}{\partial \psi}, \]

satisfy two independent \( SU(2) \) algebras;

\[ [L_i, L_j] = i\epsilon_{ijk} L_k, \quad [R_i, R_j] = i\epsilon_{ijk} R_k, \quad [L_i, R_j] = 0. \]  

These results are consistent with Ref.\[52\].

### C Geometric Quantities of Two-sphere

With the local coordinates, \( \mu = \theta, \phi \), \( S^2 \) metric is expressed as

\[ ds^2 = d\theta^2 + \sin^2 \theta \, d\phi^2. \]  

From the formula

\[ ds^2 = \delta_{mn} e^m_\mu e^n_\nu dx^\mu dx^\nu, \]

the zweibein of two-sphere is derived as \[28\]

\[ e^m_\mu = \begin{pmatrix} 1 & 0 \\ 0 & \sin \theta \end{pmatrix} \quad (m = 1, 2, \quad \mu = \theta, \phi) \]

and its inverse that satisfy \( e^m_\mu e^n_\mu = \delta^m_n \) and \( e^m_\mu e^n_\nu = \delta^\mu_\nu \) is

\[ e_m^\mu = \begin{pmatrix} 1 & 0 \\ 0 & \sin^{-1} \theta \end{pmatrix}. \]

\[\text{Choice of zweibein is not unique. For instance, we can adopt zweibein as} \]

\[ e^1 = \cos \phi d\theta - \sin \theta \sin \phi d\phi, \]
\[ e^2 = \sin \phi d\theta + \sin \theta \cos \phi d\phi, \]

and consequently the spin connection is

\[ \omega_{12} = (1 - \cos \theta) d\phi, \]

which corresponds to the Dirac gauge \[303\].
Non-zero components of Christoffel symbol, $\Gamma^\mu_{\nu\rho} = \Gamma^\mu_{\rho\nu}$, are given by
\[
\Gamma^\theta_{\phi\phi} = -\sin \theta \cos \theta, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot \theta,
\]  
and from the formula,
\[
\omega_{mn\mu} = -e_m^\nu (\partial_\mu e_n^\nu + \Gamma^\nu_{\mu\rho} e_n^\rho),
\]
we have
\[
\omega_{12\mu} = (\omega_{12\theta}, \omega_{12\phi}) = (0, -\cos \theta).
\]
We adopt the $SO(2)$ gamma matrices $\gamma^1 = \sigma_x$, $\gamma^2 = \sigma_y$, to have
\[
\sigma^{12} = -\sigma^{21} = -\frac{i}{4} [\gamma^1, \gamma^2] = \frac{1}{2} \sigma_z,
\]
and then the spin connection, $\omega_\mu = \sum_{m<n=1,2} \omega_{mn\mu} \sigma^{mn}$, is constructed as
\[
\omega_\theta = 0, \quad \omega_\phi = -\frac{1}{2} \cos \theta \sigma_z.
\]
Consequently, the Dirac operator, $-i\nabla_\mu = -i\partial_\mu + \omega_\mu$, is obtained as
\[
- i\nabla_\theta = -i\partial_\theta, \quad - i\nabla_\phi = -i\partial_\phi - \frac{1}{2} \cos \theta \sigma_z,
\]
or
\[
\nabla = \gamma^m e_\mu^m \nabla_\mu = \sigma_x \nabla_\theta + \frac{1}{\sin \theta} \sigma_y \nabla_\phi = \sigma_x (\partial_\theta + \frac{1}{2} \cot \theta) + \frac{1}{\sin \theta} \sigma_y \partial_\phi.
\]
Square of the Dirac operator yields the Laplacian and the scalar curvature:
\[
\nabla^2 = \Delta - \frac{1}{4} R,
\]
where
\[
\Delta = \frac{1}{\sqrt{g}} \nabla_\mu (g^\mu\nu \sqrt{g} \nabla_\nu) = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta) + \frac{1}{\sin^2 \theta} (\partial_\phi - i \frac{1}{2} \cos \theta \sigma_z)^2,
\]
and
\[
R = -4ie_\mu^m e_\nu^n \sigma^{mn} [\nabla_\mu, \nabla_\nu] = 2.
\]
There are a number of works about the Dirac operator on a two-sphere [38, 39, 53, 54, 55].

**D Dirac Gauge**

In the Dirac gauge, monopole gauge field is represented as
\[
A = -g \frac{1}{r(r+z)} \epsilon_{ij3} x_j dx_i = g (1 - \cos \theta) d\phi.
\]
The singularity lies on a semi-infinite line of the negative $z$ axis. The field strength is
\[
F = dA = g \sin \theta d\theta \wedge d\phi.
\]
In the vector notation, the gauge field is given by

\[ A = \tan \frac{\theta}{2} e^{i \phi} \]  

(305)

The covariant and total angular momentum operators are respectively expressed as

\[ \Lambda_x = L_x^{(0)} + g \cos \theta \tan \frac{\theta}{2} \cos \phi, \]
\[ \Lambda_y = L_y^{(0)} + g \cos \theta \tan \frac{\theta}{2} \sin \phi, \]
\[ \Lambda_z = L_z^{(0)} - g(1 - \cos \theta), \]  

(306)

and

\[ L_x^{(g)} = \Lambda_x - g \frac{1}{r} x = i(\sin \phi \frac{\partial}{\partial \theta} + \cos \phi \cot \theta \frac{\partial}{\partial \phi}) - g \cos \phi \tan \frac{\theta}{2}, \]
\[ L_y^{(g)} = \Lambda_y - g \frac{1}{r} y = -i(\cos \phi \frac{\partial}{\partial \theta} - \sin \phi \cot \theta \frac{\partial}{\partial \phi}) - g \sin \phi \tan \frac{\theta}{2}, \]
\[ L_z^{(g)} = \Lambda_z - g \frac{1}{r} z = -i \frac{\partial}{\partial \phi} - g. \]  

(307)

Square of \( L^{(g)} \) is

\[ (L^{(g)})^2 = - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial}{\partial \theta}) - \frac{1}{\sin^2 \theta} (\frac{\partial}{\partial \phi} - ig(1 - \cos \theta))^2 + g^2 \]
\[ = -(1 - x^2) \frac{\partial^2}{\partial x^2} + 2x \frac{\partial}{\partial x} + \frac{1}{1 - x^2} (i \frac{\partial}{\partial \phi} + g(1 - x))^2 + g^2, \]  

(308)

with \( x = \cos \theta \).

The Dirac gauge is related the Schwinger gauge by \( U(1) \) transformation:

\[ A_S \rightarrow A_D = A_S + i(e^{ig\phi})d(e^{-ig\phi}) = A_D + g d \phi, \]  

(309)

where \( A_D \) denotes \( [303] \) and \( A_S \) represents \([2]\), and then the monopole harmonics of the Dirac gauge are given by

\[ \mathcal{Y}_{l,m}^g(\theta, \phi) = Y_{l,m}^g(\theta, \phi) \cdot e^{ig\phi} = (-1)^{m+g} \sqrt{\frac{2l+1}{4\pi}} D_{l,-m,g}(\phi, \theta, -\phi), \]  

(310)

where \( Y_{l,m}^g \) represent the monopole harmonics in the Schwinger gauge \([246]\). \([310]\) can be expressed as

\[ \mathcal{Y}_{l,m}^g(\theta, \phi) = \sqrt{(2l+1)(l-m)!(l+m)!} \left( \frac{1}{\sin \frac{\theta}{2}} \right)^{(m+g)} \left( \frac{1}{\cos \frac{\theta}{2}} \right)^{(m-g)} P_{l+m,-m-g}^{(l-m-g,-l+m)}(\cos \theta) \cdot e^{i(l+m)\phi}, \]  

(311)

with \( x = \cos \theta \), and are related to the \( D \) functions as

\[ \mathcal{Y}_{l,m}^g(\theta, \phi) = (-1)^{m+g} \sqrt{\frac{2l+1}{4\pi}} D_{l,-m,g}(\phi, \theta, -\phi). \]  

(312)
Due to the uniqueness of wavefunction, \( m + g \) of the azimuthal angle part of (311) should be an integer \( 27 \).

We can readily obtain the eigenstates of the Dirac-Landau operator in the Dirac gauge by simply multiplying the phase factor \( e^{ig\phi} \) to those of the Schwinger gauge:

\[
\begin{align*}
  n = 0 & : \Psi^q_{\lambda=0,m}(\theta, \phi) = Y_{j=g-\frac{1}{2},m}(\theta, \phi) \cdot \sqrt{\frac{(2g)!}{4\pi(g + m - \frac{1}{2})!(g - m - \frac{1}{2})!}} \cdot e^{i(m+g)\phi} \cdot \sin^j \theta \cos^m \frac{\theta}{2}, \\
  n = 1, 2, \ldots & : \Psi_{\pm \lambda, m}(\theta, \phi) = \frac{1}{\sqrt{2}} \left( Y_{j,m}^{-\frac{1}{2}}(\theta, \phi) + \mp i Y_{j,m}^{\frac{1}{2}}(\theta, \phi) \right) \cdot \sqrt{\frac{(2l+1)(l-m)!(l+m)!}{2\pi}} \cdot e^{i(m+g)\phi} \cdot \sin^j \theta \cos^m \frac{\theta}{2}.
\end{align*}
\]

or

\[
\begin{align*}
  n = 0 & : \Psi^q_{\lambda=0,m}(\theta, \phi) = (-1)^g \cdot \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{(2l+1)(l-m)!(l+m)!}{2\pi}} \cdot e^{i(m+g)\phi} \cdot \frac{1}{\sqrt{(l-g+\frac{1}{2})!(l+g-\frac{1}{2})!}} \cdot \frac{1}{\sqrt{(l-g-\frac{1}{2})!(l+g+\frac{1}{2})!}} \cdot \sin^j \theta \cos^m \frac{\theta}{2} \cdot \sin^j \theta \sin \phi \pm i \sin \phi \cdot \sin^j \theta \cos \phi \cdot P_{l+m}^{(m-g+\frac{1}{2},-m-g-\frac{1}{2})}(\cos \theta) \cdot P_{l+m}^{(m-g-\frac{1}{2},-m-g+\frac{1}{2})}(\cos \theta), \\
  n = 1, 2, \ldots & : \Psi_{\pm \lambda, m}(\theta, \phi) = \frac{1}{\sqrt{2}} \left( Y_{j,m}^{-\frac{1}{2}}(\theta, \phi) + \mp i Y_{j,m}^{\frac{1}{2}}(\theta, \phi) \right) \cdot \frac{1}{\sqrt{(l-g+\frac{1}{2})!(l+g-\frac{1}{2})!}} \cdot \frac{1}{\sqrt{(l-g-\frac{1}{2})!(l+g+\frac{1}{2})!}} \cdot \sin^j \theta \cos^m \frac{\theta}{2} \cdot \sin^j \theta \sin \phi \pm i \sin \phi \cdot \sin^j \theta \cos \phi \cdot P_{l+m}^{(m-g+\frac{1}{2},-m-g-\frac{1}{2})}(\cos \theta) \cdot P_{l+m}^{(m-g-\frac{1}{2},-m-g+\frac{1}{2})}(\cos \theta),
\end{align*}
\]

where \( j = n + g - \frac{1}{2} \). The eigenvalues are the same as of the Schwinger gauge: \( \pm \lambda_n = \pm \sqrt{n(n + 2g)} \) with \( n = 0, 1, 2, \ldots \). Notice when \( g \) is an integer (half-integer), \( j \) should be a half-integer (integer) and so \( m \). Consequently, \( m + g \) of the azimuthal phase factor of (314) is always a half-integer.

In the Dirac gauge, the edth operators and the “boost” operators corresponding to (311) and (101) are respectively represented as

\[
\begin{align*}
  \hat{\sigma}_+^{(g)} &= e^{i\phi}(\partial_\phi + g \tan \frac{\theta}{2} + i \frac{1}{\sin \theta} \partial_\phi), \\
  \hat{\sigma}_-^{(g)} &= e^{-i\phi}(\partial_\phi - g \tan \frac{\theta}{2} - i \frac{1}{\sin \theta} \partial_\phi),
\end{align*}
\]

and

\[
\begin{align*}
  K_x^{(g)} &= -i(\cos \theta \cos \phi \frac{\partial}{\partial \phi} - \frac{1}{\sin \theta} \sin \phi \frac{\partial}{\partial \phi} - \sin \theta \cos \phi + ig \tan \frac{\theta}{2} \sin \phi), \\
  K_y^{(g)} &= -i(\cos \theta \sin \phi \frac{\partial}{\partial \phi} + \frac{1}{\sin \theta} \cos \phi \frac{\partial}{\partial \phi} - \sin \theta \sin \phi - ig \tan \frac{\theta}{2} \cos \phi), \\
  K_z^{(g)} &= i(\sin \theta \frac{\partial}{\partial \theta} + \cos \theta).
\end{align*}
\]

(316)
Derivation of the edth operators may need some explanation. From (289) the zweibein in the Dirac gauge is given by

\[ e^m_\mu = \begin{pmatrix} \cos \phi & -\sin \theta \sin \phi \\ \sin \phi & \sin \theta \cos \phi \end{pmatrix} \quad (m = x, y, \mu = \theta, \phi) \] (317)

and its inverse that satisfies \( e^m_\mu e^n_\mu = \delta^m_n \) and \( e^m_\mu e^n_\nu = \delta^m_\nu \) is

\[ e^m_\mu = \frac{1}{\sin \theta} \begin{pmatrix} \sin \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \phi \end{pmatrix}. \] (318)

The edth operators

\[ \bar{\sigma}_m = e^m_\mu D_\mu \] (319)

are given by

\[ \bar{\sigma}^{(g)}_x = \cos \phi D_\theta - \frac{\sin \phi}{\sin \theta} D_\phi, \]
\[ \bar{\sigma}^{(g)}_y = \sin \phi D_\theta + \frac{\cos \phi}{\sin \theta} D_\phi, \] (320)

where \( D_\mu \) are the covariant derivatives in the Dirac gauge:

\[ D_\theta = \partial_\theta, \quad D_\phi = \partial_\phi - ig(1 - \cos \theta), \] (321)

and then we obtain

\[ \bar{\sigma}^{(g)}_+ = \bar{\sigma}^{(g)}_x + i\bar{\sigma}^{(g)}_y = e^{i\phi}(D_\theta + \frac{1}{\sin \theta} D_\phi), \]
\[ \bar{\sigma}^{(g)}_- = \bar{\sigma}^{(g)}_x - i\bar{\sigma}^{(g)}_y = e^{-i\phi}(D_\theta - \frac{1}{\sin \theta} D_\phi), \] (322)

which yield (315). The zweibeins in the Schwinger gauge \((e^S)_m^\mu = (292)\) and the Dirac gauge \((e^D)_m^\mu = (317)\) are related by the \(SO(2)\) transformation,

\[ R_m^n(\phi) \equiv (e^S)_m^\mu (e^D)^n_\mu = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}. \] (323)

Therefore, the edth operators in the Schwinger gauge \((\bar{\sigma}^S)_m = (e^S)_m^\mu (D^S)_\mu ((D^S)_\mu \equiv (7))\) and the Dirac gauge \((\bar{\sigma}^D)_m = (e^D)_m^\mu (D^D)_\mu ((D^D)_\mu \equiv (321))\) are related as

\[ (\bar{\sigma}^S)_m^{(g)} = R_m^n(\theta) e^{-ig\phi} (\bar{\sigma}^D)_n^{(g)} e^{ig\phi} \quad (m, n = x, y), \] (324)

so \( \bar{\sigma}^{(g)}_\pm = \bar{\sigma}^{(g)}_x \pm i\bar{\sigma}^{(g)}_y \) are

\[ (\bar{\sigma}^S)_+^{(g)} = e^{-i(g+1)\phi} (\bar{\sigma}^D)_+^{(g)} e^{ig\phi}, \]
\[ (\bar{\sigma}^S)_-^{(g)} = e^{-i(g-1)\phi} (\bar{\sigma}^D)_-^{(g)} e^{ig\phi}, \] (325)
or

\begin{align}
(\bar{\partial}^D)^{(g)}_+ &= e^{ig\phi} (\bar{\partial}^S)^{(g)}_+ e^{-ig\phi}, \\
(\bar{\partial}^D)^{(g)}_- &= e^{ig\phi} (\bar{\partial}^S)^{(g)}_- e^{-ig\phi},
\end{align}

(326) gives (322) through (32). Using (326), one may readily verify the relations associated with the edth operators, such as (36) and (38), in the Dirac gauge.

The Dirac operator is constructed as

\begin{align}
-i\mathcal{D} = -i\sigma^m\sigma_m &= -i\begin{pmatrix} 0 & \partial^{(g+\frac{1}{2})} \\ \partial^{(g-\frac{1}{2})} & 0 \end{pmatrix} \\
&= -i\begin{pmatrix} 0 & e^{-i\phi}(D^g_{\theta} + \frac{1}{\sin\theta}D^g_{\phi}) \\ e^{i\phi}(D^g_{\theta} - \frac{1}{\sin\theta}D^g_{\phi}) & 0 \end{pmatrix},
\end{align}

(327)

where \( D^g_{\mu} \equiv (321) \).

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