Modeling unbalanced systems in network-like oil and gas processes

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Abstract. In the paper proposes a analog of the method E. Rote (the method of semi-discrete by time variable) to construction converging different schemes when analyzing the mathematical models of network-like oil and gas processes. The proposed method reduce the study of the input initial boundary value problem to study the boundary value problem in a weak setting for elliptical type equations with distributed parameters on the net-work. Thus, there is another possibility, besides the Faedo-Galerkin method, to construction approaches to the desired solution of the initial boundary value problem, to analyze its stability and the way to prove the theorem of the existence of a weak solution to the input problem. The approach is applied to finding sufficient conditions for the existence of weak solutions to other initial boundary value problems with more total boundary conditions -- in which elliptical equations are considered with the boundary conditions of the second or third type. Further analysis is possible on the way to finding the conditions of the Lyapunov stability. The approach can be used to analyze the optimal control problems, as well as the problems of stabilization and stability of differential systems with delay. Presented method of finite difference open new ways of approximation of the states of the parabolic system, analysis of their stability when numerical implementation and algorithmic of optimal control problems.

1. Introduction

The paper provides a fairly sufficiently total approach of using ideas of the method of finite difference and some principles of construction converging different schemes when analyzing initial boundary value problems with distributed parameters on the graph in the class of summable up functions. Below is a analogy of the Rote method [1] which essentially reduce the analysis of the input initial boundary value problem to the study of the boundary value problem for elliptical type equations with distributed parameters on the graph. Thus, there is another possibility [2-4], besides the Faedo-Galerkin method, to construction approaches to the desired solution of the initial boundary value problem, to analyze its stability and the way to prove the theorem of the existence of a weak solution to the input problem. The approach is applied to finding sufficient conditions [5] for the stability of weak solutions to other initial boundary value problems with more total boundary conditions - in which elliptical equations are
considered with the boundary conditions of the second or third type. The solvability of such problems is proved similarly to the reasoning for the problem [6] with the boundary conditions of the first type.

2. Notations, concepts and basic statements. Case \( x \in \mathbb{R}^1 \)

In the represented work uses concepts and notations accepted in the works \([7,8]\): \( \Gamma \) is bounded oriented geometric graph with edges \( \gamma \) parameterized segment \([0,1]\); \( \partial \Gamma \) and \( J(\Gamma) \) are many boundary \( \zeta \) and interior \( \xi \) nodes of the graph, respectively; \( \Gamma_0 \) join all the edges of the graph \( \Gamma \) that do not contain endpoints; \( \Gamma_i = \Gamma_0 \times (0,t) \) \( (\gamma_i = \gamma_0 \times (0,t)) \), \( \partial \Gamma_i = \partial \Gamma \times (0,t) \) \( (t \in (0,T], \ T < \infty \) is arbitrary fixed constant). Necessary spaces and sets: \( C[\Gamma] \) is space of continuous and differentiable on \( \Gamma \) functions (derivative at the endpoints of the ribs is understood as one-sided), \( L_p(\Gamma) \) \( (p = 1, 2) \) is the Banach space \([9]\) of measurable on \( \Gamma_0 \) functions summarized with a \( p \) degree (similar to space \( L_p(\Gamma) \)); \( L_{2,1}(\Gamma) \) is the space of function from \( L_2(\Gamma) \) with the norm, defined by the ratio \( \|u\|_{2,1} = \int_0^T \left( \int_\Gamma u^2(x,t)dx \right)^{1/2} dt \); \( W^{2,1}_1(\Gamma) \) is the space of functions from \( L_2(\Gamma) \) having a generalized first order derivative also from \( L_2(\Gamma) \); \( W^{1,0}_1(\Gamma) \) is the space of functions from \( L_2(\Gamma) \) having a generalized first order derivative by \( x \) belonging \( L_1(\Gamma) \) (similarly entered the space \( W^{1,0}_1(\Gamma) \)). Below is the difference-differential analogue of the parabolic \([10]\) equation

\[
\frac{\partial y(x,t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial y(x,t)}{\partial x} \right) + b(x)y(x,t) = f(x,t), \quad x, t \in \Gamma_t .
\]

with measurable bounded on \( \Gamma_0 \) functions \( a(x), b(x) \) summable with the square:

\[
0 < a, a', b(x) \leq \beta, x \in \Gamma_0 .
\]

The following statement to take place \([11]\).

**Lemma.** Let fulfill conditions \((2)\) and function \( u(x) \in W^1_2(\Gamma) \) is such that \( \ell(u,\nu) = \int f(x)\eta(x)dx = 0 \) for any \( \eta(x) \in W^1_2(\Gamma) \), \( f(x) \in L_2(\Gamma) \) is fixed function. Then for any edge \( \gamma \subset \Gamma \) narrowing

\[
a(x) \gamma \frac{du(x)}{dx} \quad \text{continuously at the endpoints of the edge } \gamma, \quad \ell(\mu,\nu) = \int \left( a(x) \frac{d\mu}{dx} + b(x)\nu \right) dx .
\]

Let's designate through \( \Omega_0(\Gamma) \) a set of functions \([12]\) \( u(x) \in C[\Gamma] \) that meet the conditions of Lemma \([13]\) and ratios

\[
\sum_{\gamma \in \Delta(\xi)} a(1)_{\gamma} \frac{du(1)_{\gamma}}{dx} = \sum_{\gamma \in \Delta(\xi)} a(0)_{\gamma} \frac{du(0)_{\gamma}}{dx}
\]

in all the nodes \( \xi \in J(\Gamma) \) (here \( R(\xi) \) and \( r(\xi) \) are sets of edges \( \gamma \), respectively oriented "to node \( \xi \)" and "from node \( \xi \")). The closing of the set \( \Omega_0(\Gamma) \) in norm \( W^1_2(\Gamma) \) relabel \( W^1_2(a,\Gamma) \). In addition, if we assume that the functions \( u(x) \in \Omega_0(\Gamma) \) satisfy the boundary condition \( u(x)|_{\partial \Gamma} = 0 \), then we will get space \( W^1_0(a,\Gamma) \). Next, let's designate through \( W^1_{0,1}(a,\Gamma) \) the closure in the norma \( W^1_{2,1}(\Gamma) \) the set of differentiable functions \( \Omega(\Gamma) \), equal to zero near the boundary \( \partial \Gamma \) and satisfying ratios \((3)\) for all nodes \( \xi \in J(\Gamma) \) and for any \( t \in [0,T] \). Analogously let's introduce space \( W^1_{0,1}(a,\Gamma) \) as the closure in the norma \( W^1_{2,1}(\Gamma) \) set of functions \( \Omega(\Gamma) \). The space \( W^1_{0,1}(a,\Gamma) \) describes many states \( y(x,t) \) of the
parabolic system (1), $W^1_0(a, \Gamma)$ -- auxiliary space. For functions $y(x,t) \in W^1_0(a, \Gamma)$ we consider equation (1) with initial and boundary conditions
\[ y \big|_{t=0} = \varphi(x) \in L^2(\Gamma), \quad y \big|_{x=\pm \infty} = 0; \] the first equality in (4) have meaning sense and is understood almost everywhere. The initial boundary value problem (1), (4) in the space $W^1_0(a, \Gamma)$ is a mathematical model of the process of gas and oil transfer on the network $\Gamma$, linear fragments $\Gamma_k$ which have a small section diameter.

3. Main result. Differential-difference system, $x \in \mathbb{R}^1$
In space $W^1_0(a, \Gamma)$ consider the equation (1) and dissect the domain $\Gamma$ planes $t = k\tau$, $k = 0, 1, 2, \ldots, M$, $\tau = T/M$, in addition denote by $\Gamma_k$ section $\Gamma$ the plane $t = k\tau$. Equation (1) will replace differential-difference
\[ \frac{1}{\tau}(u_k(x)-u_{k-1}(x)) - \frac{d}{dx} \left( a(x) \frac{du_k(x)}{dx} \right) + b(x)u_k(x) = f(x,k), \quad k = 1, 2, \ldots, M, \] where $f(x,k) \equiv f(x,k) = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} f(x,t)dt \in L^2(\Gamma)$. Functions $u_k(x)$ ($k = 1, 2, \ldots, M$) will define as a solution to the equation system (5) that meets the conditions
\[ u_k(0) = \varphi(x), \quad u_k(x) \big|_{x=\pm \infty} = 0 \quad (k = 1, 2, \ldots, M). \]

**Definition.** A weak solution to a boundary value problem (5), (6) is called functions $u_k(x) = W^1_0(a, \Gamma)$ ($k = 0, 1, 2, \ldots, M$), $u_0(x) = \varphi(x)$ ($\varphi(x) \in L^2(\Gamma)$), satisfying an integral identity
\[ \int_{\Gamma} u_k(x)\eta(x)dx + \ell(u_k(x),\eta) = \int_{\Gamma} f(x,k)\eta(x)dx, \quad k = 1, 2, \ldots, M, \quad \text{for any } \eta(x) \in W^1_0(a, \Gamma); \] equality $u_0(0) = \varphi(x)$ is understood almost everywhere, $u_k(x) = (u_{k}(x) - u_{k-1}(x))/\tau$.

We will establish the correctness of the statements, similar to presented in theorem 1.

**Theorem 1.** A weak solution of the initial boundary value problem (1), (4) is the limit of functions $u_k(x)$, calculated from ratios (5), (6).

4. Notations, concepts and basic statements. Case $x \in \mathbb{R}^n$ ($n \geq 2$)
In the Euclid space $\mathbb{R}^n$ let's look at a network-like bounded area $\mathcal{X}$, comprised of $N$ areas $\mathcal{X}_k$ ($k = 1, N$), pairwise united by means of $M$ nodal place $\omega_j$: $\mathcal{X} = \mathcal{X} \cup \mathcal{X}'$, where $\mathcal{X}' = \bigcup_{k=1}^{N} \mathcal{X}_k$, $\mathcal{X}' = \bigcup_{j=1}^{M} \omega_j$, moreover $\mathcal{X}_k \cap \mathcal{X}_j = \emptyset$ ($k \neq l$), $\omega_j \cap \omega_l = \emptyset$ ($j \neq l$), $\mathcal{X}_k \cap \omega_j = \emptyset$ [14]. Areas $\mathcal{X}_k$ in nodal place $\omega_j$ share common boundaries in the form of adjoining surfaces $S_j$ (meas $S_j > 0$). At each nodal place $\omega_j$ the adjoining surface $S_j$ separating to her $1+m_j$ the areas $\mathcal{X}_{k_0}$ and $\mathcal{X}_{k_j}$ ($1 \leq s \leq m_j \leq N$) has a representation $S_j = \cup_{s=1}^{m_j} S_{js}$. Thus, the nodal place $\omega_j$ is determined by the adjoining surface $S_j$, for which $S_{js}$ are also the adjoining surface $\mathcal{X}_b$ to $\mathcal{X}_{s_j}$, $s = 1, m_j$. The boundary $\partial \mathcal{X}$ of the area $\mathcal{X}$ is called the union of the boundary $\partial \mathcal{X}_j$ of area $\mathcal{X}_j$ ($k = 1, N$), which does not include the adjoining surface of all node places: $\partial \mathcal{X} = \bigcup_{k=1}^{N} \partial \mathcal{X}_k \cup \bigcup_{j=1}^{M} S_j$. The area $\mathcal{X}$ has a network-like structure similar to that of the geometric graph [15], each area $\mathcal{X}_k$ adjoins to one or two node places and has one
or more of the surface adjoining other areas (to compare with the structure [16] of the graph: each edge of the graph has two endpoints, of which one or both are conjugation nodes with the other edges).

Let \( C^0(\mathfrak{F}) \) is a set of functions \( u(x) \in C(\mathfrak{F}) \cap C^1(\mathfrak{F}) \) satisfying the the condition of agreement

\[
\int_{S_j} a_j(x) \frac{\partial u_j(x)}{\partial n_j} ds + \sum_{i=1}^{m_j} \int_{S_{ij}} a_{ij}(x) u_{ij}(x) \frac{\partial n_{ij}}{\partial n_j} ds = 0, \quad x \in S_{ij}, i = 1, m_j, \quad \text{for each node } \nu_j \text{ on surfaces } S_j = \bigcup_{r=1}^{m_j} S_{ij}, \quad j = 1, M; \quad \text{here } u_j(x) \text{ and } u_{ij}(x) \text{ are narrowing the function } u(x)\) [17] on \( S_j \) and \( S_{ij} \), vectors \( n_j \) and \( n_{ij} \) are outer normals to \( S_j \) and \( S_{ij} \), respectively, \( i = 1, m_j, j = 1, M \) [18].

Let \( C^0(\mathfrak{F}) \) is a set of functions from \( C^0(\mathfrak{F}) \) having a compact carrier lying in \( \mathfrak{F} \) (in other words, \( C^0(\mathfrak{F}) \) is a set of functions that are zero near \( \partial \mathfrak{F} \)). Closing the set \( C^0(\mathfrak{F}) \) in norm \( \mathbb{P} = \sqrt{(u, u)^2} \), where \( (u, v)^2 = \sum_{k=1}^{n} (uv)_{2k} \), let's call space \( W^0(\mathfrak{F}) \). Let the next \( \Omega_0(\mathfrak{F}) \) is the set of functions \( u(x, t) \in W^1(\mathfrak{F}) \), whose traces are defined in sections of the domain \( \mathfrak{F} \) the plane \( t = t_0 (t_0 \in [0, T]) \) as a function of class \( W^0(\mathfrak{F}) \). The closing of the set \( \Omega_0(\mathfrak{F}) \) in norm \( W^1_0(\mathfrak{F}, \mathfrak{F}) \) relabel \( W^1_0(a, \mathfrak{F}) \). In space \( W^1_0(a, \mathfrak{F}) \) consider the initial boundary value problem [19,20]:

\[
\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x} \left( a(x) \frac{\partial u(x,t)}{\partial x} \right) + b(x)u(x,t) = f(x,t), \quad u\big|_{x=0}=\varphi(x), \quad \varphi(x) \in L_2(\mathfrak{F}). \tag{7}
\]

Function \( f(x,t) \in L_{2,1}(\mathfrak{F}) \) is the space of summable on domain \( \mathfrak{F} \) functions,

\[
P_{L_{2,1}}(\mathfrak{F}) = \int_0^T \left( \int_\mathfrak{F} |f(x,t)|^2 dx \right)^{\frac{1}{2}} dt; \quad \frac{\partial}{\partial x} \left( a(x) \frac{\partial u(x,t)}{\partial x} \right) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u(x,t)}{\partial x_i} \right), \quad a(x), b(x) \in L_2(\mathfrak{F}),
\]

\( 0 < a \leq a^* < \infty, \quad |b(x)| \leq \beta < \infty \). The initial boundary value problem (7) in the space \( W^1_0(a, \mathfrak{F}) \) is a mathematical model [21-23] of the process of gas and oil transfer on the network \( \mathfrak{F} \), linear fragments of \( \mathfrak{F} \) which have a large enough diameter of the section, disproportionate to the length of the linear fragment [24] of the network [25].

5. Conclusions

The work outlines an approach to the analysis of the differential system with distributed parameters on the graph, which, not using the Faedo-Galerkin method, establishes the theorem of the existence of a solution to the initial boundary value problem (1), (4) and at the same time gives you the opportunity to obtain the conditions of stability (countably stability) of the investigated problem. The proposed method can be used for solve other initial boundary value problems. In this case, the boundary conditions of the second or third types is added to the elliptical equations (9). Note also, the used approach it is not difficult to extend to the case when \( \Gamma \) is a netlike domain of Euclidean space \( \mathbb{R}^n \) \( (n \geq 2) \). Further analysis is possible on the way to finding the conditions of the Lyapunov stability of problem (1), (4). The approach can be used to analyze the optimal control problems of [26, 27], as well as the problems of stabilization and stability of differential systems [28] with delay[29, 30].

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