RANK OF TROPICAL CURVES AND TROPICAL HYPERSURFACES

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Abstract. This paper is devoted to the bounding and computation of the dimension of deformation spaces of tropical curves and hypersurfaces. This characteristic is interesting in light of the fact that it often coincides with the dimension of equisingular (equigeneric etc.) deformation spaces of algebraic curves and hypersurfaces. In this paper, we obtain a series of precise formulas, upper and lower bounds, and algorithms for computing dimension of deformation spaces of various classes of tropical curves and hypersurfaces.

Part 1. Introduction

The goal of the present work is to study deformation spaces of embedded and parameterized planar and spacial tropical curves as well as deformation spaces of affine tropical hypersurfaces and to either give explicit formulas for their dimension when possible, or provide lower and upper bounds for it together with efficient algorithms for computing the precise values of the dimension.

Our results are related to plane tropical curves (Part II), tropical surfaces in $\mathbb{R}^3$ (Part III), and tropical hypersurfaces in $\mathbb{R}^n$, $n > 3$ (Part IV). We will now shortly describe and comment on the results of the present work leaving a complete formulation for Parts II, III and IV.

(1) Precise formulas. We exhibit two types of precise formulas for the dimension of deformation spaces of tropical curves and hypersurfaces.

One of them equates the actual and the expected dimensions, where the latter means the value obtained by counting the conditions on the parameters imposed by local combinatorial data. The result is that an equisingular family of plane tropical curves is always of expected dimension if the number of higher singularities does not exceed 2 (Corollary 6). Moreover, the actual dimension can be greater than the expected one already for families of tropical curves with three higher singularities (Example 7). It is worth to say that this differs from the algebraic analogue, which states that an equisingular family of plane algebraic curves is of expected dimension when the curves have only nodal singularities [15] (see a similar statement in the tropical approach in [9, 12]), or when the number of higher (non-nodal) singularities does not exceed a bound proportional the degree [5, 13]. We would like to remark that our statement (in the tropical approach) can be used for a tropical enumeration of algebraic curves with any number of nodes and one cusp.

The other precise formula uses some ordering in addition to the local combinatorial data. It holds in a more general situation: for plane tropical curves with at most 3 non-nodal singularities (Theorem 9) and for higher-dimensional tropical hypersurfaces having at most 3 singularities (Theorem 16).

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(2) **Lower and upper bounds.** The lower bounds are usually given by the expected rank (i.e., computed under the condition that all the relations imposed to parameters are independent) as defined below, following [12] and [9]. By attaching additional combinatorial information, we strengthen this lower bound for plane tropical curves (Theorem 8) and generalize it to higher dimensions (Theorem 15).

For the upper bounds we apply another idea. We consider plane tropical curves with their parameterizations and subdivide the parameterizing graph into simple (trivalent) pieces. Then we compute the dimensions of deformation spaces for each component, separately deducing an upper bound for the rank of the original curve (Theorem 4). Furthermore, by analyzing possible dependence of conditions in the latter consideration, we provide a precise formula for the rank of an arbitrary plane tropical curve (Theorem 3) as well as an improved upper bound for a certain class of plane tropical curves (Theorem 4).

At last, we study deformation spaces of spacial tropical curves similar to 1-dimensional skeleta of tropical surfaces (hypersurfaces). These tropical curves rather differ from the simple ones studied by Mikhalkin, since they do not have trivalent vertices. We give an upper bound for their rank in Theorem 13.

(3) **Algorithms.** The results of the theorems are all based on algebraic or combinatorial calculations, like matrix buildings (Theorem 3) and calculations of expected ranks of surfaces (Theorem 10).

The rank of a tropical hypersurface is determined as the dimension of the equisingular deformation space of the hypersurface.

Throughout this article we shall implement two approaches for defining tropical hypersurface:

1. The dual to a Newton polygon with a subdivision. In the case of a tropical curve in \( \mathbb{R}^2 \), we shall denote \( \text{rk}_{\exp}(T) := \# \text{Vert}(S_T) - 1 - \sum_{\delta} (\# \text{Vert}(\delta) - 3) \)

where \( T \subset \mathbb{R}^2 \) is an embedded plane tropical curve, \( \Delta \) is its Newton polygon and \( S_T \) the dual subdivision of \( \Delta \). Here, \( \text{Vert}(S_T) \) is the set of vertices of \( S_T \) and \( \text{Vert}(\delta) \) is the set of vertices of \( \delta \), where \( \delta \) runs over all polygons of \( S_T \). This expected rank is due to [12].

2. As parameterized curves. For any parameterized tropical curve \((\bar{\Gamma}, h)\), we shall denote \( \text{rk}_{\exp}(\bar{\Gamma}, h) := \# \text{End}(\Gamma) + (n - 3)(1 - g) - \sum_{\nu} (\text{mt}(\nu) - 3) \)

where \( \nu \) runs over all vertices of \( \Gamma \) and \( \text{mt}(\nu) \) is the valence of \( \nu \). This expected rank is due to [9].

**End-marked plane tropical curves**

Let \((\bar{\Gamma}, h)\) be an irreducible parameterized plane tropical curve, \(m \leq \# \text{End}(\Gamma)\), and \(\bar{\gamma} = (\gamma_1, ..., \gamma_m) \subset \Gamma\) an ordered configuration of distinct points lying on the interior of noncompact edges of \(\Gamma\), at most one on each edge. We call the tuple \((\bar{\Gamma}, h, \bar{\gamma})\) an *end-marked* tropical curve, and we say that this curve matches an ordered configuration of points \(\bar{p} = p_1, ..., p_m \subset \mathbb{R}^2\), if \(h(\gamma_i) = p_i, \ 1 \leq i \leq m\). For a general parameterized tropical curve \((\bar{\Gamma}, h)\) in \(\mathbb{R}^n\), we define its equiparametric deformation space \(\text{Def}_{\exp}(\bar{\Gamma}, h)\) as the set of parameterized tropical curves \((\bar{\Gamma}', h')\) such that there is a homeomorphism \(\psi: \Gamma \rightarrow \bar{\Gamma}'\) satisfying

\[
\frac{dh'}{|\psi(e)|} = \frac{dh}{|e|}
\]
for each edge $e$ of $\Gamma$. Denote by $Def^p_\rho(\bar{\Gamma}, h)$ the set of those curves $(\bar{\Gamma}', h) \in Def^{\rho^p}(\bar{\Gamma}, h)$ which lift up to end-marked curves $(\bar{\Gamma}', h', \bar{\gamma}')$ matching the given configuration $p$.

**Tropical hypersurfaces and subdivisions of Newton polytopes:**

Given a non-Archimedean field $\mathbb{K}$, define a map $Val : (\mathbb{K}^*)^n$ by

$$Val(z_1, ..., z_n) = (Val(z_1), ..., Val(z_n)).$$

Let $X \subset (\mathbb{K}^*)^n$ be a hypersurface given by a polynomial equation

$$f(z) = \sum_{\omega \in \Delta \cap \mathbb{Z}^n} A_\omega z^\omega = 0,$$

where $\Delta$ is the Newton polytope (the convex hull of the points $\omega \in \mathbb{Z}^n$ such that $A_\omega \neq 0$), $A_\omega \in \mathbb{K}$, $z^\omega = z_1^{\omega_1} ... z_n^{\omega_n}$. The tropical polynomial

$$N_f(x) = \max_{\omega \in \Delta \cap \mathbb{Z}^n} (\langle \omega, x \rangle + c_\omega),$$

where $x \in \mathbb{T}^n$ ($\mathbb{T}$ is the tropical semiring), $c_\omega = Val(A_\omega), \omega \in \Delta \cap \mathbb{Z}^n$, is called the tropicalization of $f$.

**Proposition 1.** In the above notations, let $(\bar{\Gamma}, h)$ be simple. If $m = \# \rho < \# \text{End}(\Gamma)$, then

$$\text{dim}Def^p_\rho(\bar{\Gamma}, h) = rk(\bar{\Gamma}, h) - m.$$

**Proof.** As $h(\Gamma)$ supports an embedded tropical curve, and as this curve coincides with the corner locus of $N_f([2])$, we may look at the linear domains of $N_f$ and see that restricting $h(\Gamma)$ to $p_i$ means that the parameter of one of the two linear domains around $p_i$ is also restricted, i.e. $a_ip^x + b_ip^y + c_i = a_jp^x + b_jp^y + c_j$ (where $p^x_i$ and $p^y_i$ are the coordinates of $p_i$) and either $c_i$ depends on $c_j$ or vice versa. This means $Def^{\rho^p}(\bar{\Gamma}, h)$ should be reduced by 1 for each $i$. □

This means that the relations imposed by $m < \# \text{End}(\Gamma)$ points on the non-compact edges of $\Gamma$ are always independent. In turn, for $m = \# \text{End}(\Gamma)$ the relations are dependent, and we call this dependence a new balancing condition:

**Proposition 2.** Let $(\bar{\Gamma}, h, \bar{\gamma})$ be an end-marked curve with $m = \# \bar{\gamma} = \# \text{End}(\Gamma), p_i = h(\gamma_i), 1 \leq i \leq m$. Then

$$(1) \sum_{i=1}^m \langle R_{\pi/2}(dh_{\gamma_i}(\tau(e_i))), p_i \rangle = 0,$$

where $e \supset \{\gamma_i\}$ is an edge of $\Gamma$, $\tau(e_i)$ is directed to the univalent vertex, $R_{\pi/2}$ is the rotation by $\pi/2$ in positive direction. Furthermore, if $(\bar{\Gamma}, h)$ is simple, then

$$\text{dim}Def^p_\rho(\bar{\Gamma}, h) = rk(\bar{\Gamma}, h) - m + 1.$$

**Proof.** Again, we shall use the piecewise linear polyhedron $N_f$. Let us order $p_i$ and the linear domains of $N_f$ clockwise. For each $i$ we get

$$\langle R_{\pi/2}(dh_{\gamma_i}(\tau(e_i))), p_i \rangle = c_{i+1} - c_i$$

For $m = \# \bar{\gamma} = \# \text{End}(\Gamma)$ we get $c_{m+1} = c_1$ and thus

$$\sum_{i=1}^m \langle R_{\pi/2}(dh_{\gamma_i}(\tau(e_i))), p_i \rangle = 0.$$
If \((\bar{\Gamma}, h)\) is simple, we can arrange the free parameters of the equations as a matrix, and see that the last condition is dependent on all the others. Therefore only \((m - 1)\) should be reduced from the rank. \(\Box\)

Part 2. Rank of plane tropical curves

**Theorem 3.** Let \(T\) be an embedded plane tropical curve, \(v^{nn}\) the set of its non-nodal vertices, \(h : \Gamma \to T\) a parametrization obtained by resolving nodes of \(T\) and \((\bar{\Gamma}_1, h_1), \ldots, (\bar{\Gamma}_k, h_k)\) the simple parameterizations of the induced (nodal) tropical curves \(T_{v^{nn}}\). Then

\[
\text{rk}(T) \leq \text{rk}_{\text{exp}}(T) + \max \{0, \#(\text{bounded components of } \Gamma \setminus h^{-1}(v^{nn})) - 2\}.
\]

Furthermore, let \(M\) be the matrix of the coefficients of the coordinates of \(v^{nn}\) in the balancing conditions (1) generated by all the bounded components of \(\Gamma \setminus h^{-1}(v^{nn})\). Then

\[
\text{rk}(T) = \text{rk}_{\text{exp}}(T) + \#(\text{bounded components of } \Gamma \setminus h^{-1}(v^{nn})) - \text{rk}M.
\]

**Proof.** For each of the induced tropical curves \(T_{v^{nn}}\), \(\text{rk}(T_{v^{nn}}) = \text{rk}_{\text{exp}}(T_{v^{nn}})\). Together with the coordinates of \(v^{nn}\) they form \(\text{rk}_{\text{exp}}(T)\). But each of the bounded components of \(\Gamma \setminus h^{-1}(v^{nn})\) imposes a condition on the rank and they may depend one on the other. If there are only one or two bounded components, there is no problem (we don’t map two edges of \(\Gamma\), with their vertices, to the same edge in \(T\) where the vertices take the same place). But, if there are more, we should get a correction of \(\#(\text{bounded components of } \Gamma \setminus h^{-1}(v^{nn}))\). It is impossible that all of the conditions are dependent on one condition. There should be at least two independent conditions, and so we finally get

\[
\text{rk}(T) \leq \text{rk}_{\text{exp}}(T) + \max \{0, \#(\text{bounded components of } \Gamma \setminus h^{-1}(v^{nn})) - 2\}.
\]

To be exact, we can determine the dependencies of the conditions by building a matrix \(M\) of the coefficients of the coordinates of \(v^{nn}\) in the balancing conditions (1) generated by all the bounded components of \(\Gamma \setminus h^{-1}(v^{nn})\). The rank of \(M\) is exactly the number of independent conditions. So by extracting the rank of \(M\) from the number of bounded components, we get the number of dependent conditions, which is the exact correction to \(\text{rk}_{\text{exp}}(T)\). \(\Box\)

**Theorem 4.** If the parameterizing graph \(\Gamma\) of an irreducible, trivalent and rational parameterized plane tropical curve \((\bar{\Gamma}, h)\) has precisely \(p\) vertices of valence \(>3\), then

\[
\text{rk}(\bar{\Gamma}, h) \leq \text{rk}_{\text{exp}}(\bar{\Gamma}, h) + \max \{0, p - 2\}.
\]

**Proof.** According to the demands in the theorem, \(\text{rk}_{\text{exp}}(\bar{\Gamma}, h) = \#\text{End}(\Gamma) - 1\). There are two kinds of intersections allowed for the map of the curve:

1. Two vertices intersect
2. A vertex meets the interior of an edge.

We shall now choose an arbitrary vertex of \(\Gamma\), and order the edges of the tree relating to that vertex. As the map "closes" some cycles by the intersections mentioned, we would get the following equations for each kind of intersection: For each vertex in the graph which is made of an intersection of two vertices in \(\Gamma\), we would get the vectorial equation

\[
\sum_i l_i \bar{a}_i = \sum_j l_j \bar{a}_j
\]
where \( a_i \) are the unit vectors of the edges on one side of the cycle, related to the chosen vertex, \( l_i \) are the corresponding lengths of those edges, and the same with \( a_j \) and \( l_j \) for the other side. As this is a vectorial equation, we can make two scalar equations out of it. For each vertex in the graph which is made of an intersection of a vertex and the interior of an edge, we can chose a coordinate set where one of the coordinates is parallel to the interior of the edge in the intersection. By doing that we can have just one scalar equation to describe this event,

\[
\sum_i l_i a_i = \sum_j l_j a_j
\]

where \( a_i \) and \( a_j \) are unit vectors in the perpendicular coordinate, and \( l_i, l_j \) are the lengths of the corresponding edges. As our parameters for the rank calculation are the lengths of the edges, each equation imposes a condition on the expected rank. We can now take the last edge participating in each cycle and order the equations into a matrix according to those last edges, in order to get a block matrix.

If there are no dependencies in the matrix, we get \( rk(\bar{\Gamma}, h) \leq rk_{exp}(\bar{\Gamma}, h) \). If one edge is the last in two cycles created by the second type of intersections, we get two equations ending with the same edge. Those two matrix lines may be linearly dependent, and if all the intersections are alike, we would need a \( \frac{1}{2}p \) correction to the rank, due to the vertices of the second kind. For an intersection of the first kind, we have, naturally, the option that the two rows are dependent, and if all the intersections are alike we get a correction of \( p \) for vertices of the first kind. One more case can be seen when the last edge in a cycle created by the second type is also the last edge in a cycle of the first kind. In such a case, we would get three lines in the matrix ending in the same column, and two of them might be dependent on the third. This would lead us to a correction of 2 to the rank, which may be calculated as 1 for each intersection, and to \( p \) if all the intersections are alike. Therefore, as the graph may have intersections only of the last case, and as we cannot determine the dependencies of the rows in the matrix, we have to take the upper bound for the rank:

\[
 rk(\bar{\Gamma}, h) \leq rk_{exp}(\bar{\Gamma}, h) + \max\{0, p\}.
\]

For \( p = 1 \) we get \( rk(\bar{\Gamma}, h) \leq rk_{exp}(\bar{\Gamma}, h) \leq rk_{exp}(\bar{\Gamma}, h) + \max\{0, p - 2\} \).

For \( p = 2 \) there may be 2 options. The first is no dependencies with no corrections. The second is a matrix with 2 or 3 lines, where at least one is independent and therefore we need a correction of at most \( p - 2 \), what leads to \( rk(\bar{\Gamma}, h) \leq rk_{exp}(\bar{\Gamma}, h) + \max\{0, p - 2\} \).

For \( p > 2 \) we have at least two independent lines in the matrix, as an edge cannot be the last edge in more than 2 cycles, and therefore \( rk(\bar{\Gamma}, h) \leq rk_{exp}(\bar{\Gamma}, h) + \max\{0, p - 2\} \).

For the convenience of the readers, the following lemma is quoted here, with its proof, for later use such as in Cor. 6.

**Lemma 5.** ([12 - Lemma 2.2])

1. If \( T \subset \mathbb{R}^2 \) is non-singular or nodal, then

\[
 rk(T) = rk_{exp}(T)
\]
(2) If $T$ is singular, and $S_T : \Delta = \Delta_1, \ldots, \Delta_N$ is the dual graph of $T$ divided to its polygons, then for $d(S_T) = rk(T) - rk_{exp}(T)$ we get

$$0 \leq 2d(S_T) \leq \sum_{m \geq 2} ((2m - 3)N_{2m} - N'_{2m}) + \sum_{m \geq 2} (2m - 2)N_{2m+1} - 1$$

where $N_m$, $m \geq 3$, is the number of $m$-gons in $S_T$, and $N'_{2m}$, $m \geq 2$, is the number of $2m$-gons in $S$, whose opposite edges are parallel.

Proof. (1) The meaning of non-singular $T$, or nodal $T$ is that its dual graph is built from triangles or parallelograms alone. Therefore, the conditions imposed by the 4-valent vertices of $T$ are independent, as we shall prove. Taking a vector $\bar{a} \in \mathbb{R}^2$ with an irrational slope we can be arranged in order to coorient each edge of any parallelogram in the dual graph so that the normal vector creates an acute angle with $\bar{a}$. Doing that enables us to make a partial ordering of the parallelograms, that can be completed into a linear ordering. Each parallelogram has 2 neighboring edges cooriented outside of the parallelogram, and 2 that are cooriented inside. Thus the coefficients of the linear conditions imposed by the 4-valent vertices of $T$ can be arranged into a triangular matrix, meaning they are independent.

(2) If $S_T$ contains polygons different from triangles and parallelograms, we define a linear ordering on the set of all non-triangles in the same way as in (1). Denote by $e_-(\Delta_i)$ (respectively, $e_+ (\Delta_i)$) the number of edges of a polygon $\Delta_i$ cooriented outside $\Delta_i$ (respectively, inside). Passing inductively over the nontriangular polygons $\Delta_i$, each time we add at least $\min \{e_-(\Delta_i) - 1, |V(\Delta_i)| - 3\}$ new linear conditions independent of all the preceding ones. Thus,

$$d(S_T) \leq \sum_{i=2}^{N} (|V(\Delta_i)| - 3 - \min \{e_-(\Delta_i) - 1, |V(\Delta_i)| - 3\})$$

$$= \sum_{i=2}^{N} \max \{|V(\Delta_i)| - e_-(\Delta_i) - 2, 0\},$$

because, for the initial polygon $\Delta_1$, all $|V(\Delta_1)| - 3$ imposed conditions are independent. Replacing $\bar{a}$ by $-\bar{a}$, we obtain

$$d(S_T) \leq \sum_{i=1}^{N-1} \max \{|V(\Delta_i)| - e_+(\Delta_i) - 2, 0\}.$$

Since

- the relations $1 \leq e_-(\Delta_i) \leq |V(\Delta_i)| - 1$ and $e_-(\Delta_i) + e_+(\Delta_i) = |V(\Delta_i)|$ yield $\max \{|V(\Delta_i)| - e_-(\Delta_i) - 2, 0\} + \max \{|V(\Delta_i)| - e_+(\Delta_i) - 2, 0\} \leq |V(\Delta_i)| - 3$

- for a $2m$-gon with parallel opposite edges we have $e_- = e_+ = m$

$$\Rightarrow \max \{2m - e_- - 2, 0\} + \max \{2m - e_+ - 2, 0\} = 2m - 4,$$

we get

$$2d(S_T) \leq \sum_{m \geq 2} ((2m - 3)N_{2m} - N'_{2m}) + \sum_{m \geq 2} (2m - 2)N_{2m+1}.$$

If among $\Delta_1, \ldots, \Delta_N$ there is a polygon $\Delta_i$ whose number of edges is odd and exceeds 3, or a polygon with an even number of edges and a pair of nonparallel opposite sides, then $\bar{a}$ can be chosen so that $\min \{e_-(\Delta_i), e_+(\Delta_i)\} \geq 2$. Thus, the
contribution of $\Delta_i$ to the bound for $2d(S)$ will be $|V(\Delta_i)| - 4$, which allows us to gain $-1$ on the right-hand side of formula (3), obtaining formula (2).

Finally, assume that all nontriangular polygons in $S$ have an even number of edges, that their opposite sides are parallel and that there is $\Delta_i$ with $|V(\Delta_i)| = 2m \geq 6$. The union of the finite length edges of $T$ is the adjacency graph of $\Delta_1, \ldots, \Delta_N$. We take the vertex corresponding to $\Delta_i$, pick a generic point $O$ in a small neighborhood of this vertex, and orient each finite length edge of $T$ so that it forms an acute angle with the radius vector from $O$ to the middle point of the chosen edge. Equipped with such an orientation, the adjacency graph has no oriented cycles because the terminal point of any edge is farther from $O$ than the initial point. Thus, we obtain a partial ordering on $\Delta_1, \ldots, \Delta_N$ such that, for any $\Delta_k$ with an even number of edges, at least half of the edges are cooriented outside. Then we apply the preceding argument to estimate $d(S_T)$ and notice that the contribution of the initial polygon $\Delta_i$ to such a bound is zero, whereas on the right-hand side of formula (3) it is at least two. This completes the proof of formula (2).

**Corollary 6.** (1) If the parameterizing graph $\Gamma$ of an irreducible parameterized plane tropical curve $(\Gamma, h)$ has at most two vertices of valence $> 3$, then

$$rk(\Gamma, h) = rk_{\exp}(\Gamma, h)$$

(2) If the dual subdivision $S_T$ of the Newton polygon of an embedded plane tropical curve $T$ contains at most two polygons other than triangles and parallelograms, then

$$rk(T) = rk_{\exp}(T).$$

**Proof.** (1) Between two vertices of valence $> 3$ there can be only one bounded component, and therefore the new balancing condition is independent. (2) According to the proof of lemma 5 if the two polygons do not share an edge, there is no problem. If they do, it is enough to order the polygons in such a way so that each of them will have two neighboring edges cooriented outside.

**Example 7.** Consider a plane tropical curve $T$ and its dual subdivision $S_T$ with 3 quadrangles that are not parallelograms, such that $rk(T) > rk_{\exp}(T)$. In this example (cf. Figure 1), the rank of the curve is 3: 2 for transformations, and 1 for multiplication by a scalar. The expected rank, however, is only 2:

$$rk_{\exp}(T) := \#Vert(S_T) - 1 - \sum_{\delta} (\#Vert(\delta) - 3) = 6 - 1 - 3 = 2$$
Theorem 8. Let $T$ be an embedded plane tropical curve, $S_T$ the dual subdivision of its Newton polygon $\Delta$. If $\delta_1, \ldots, \delta_k$ are all the triangles and parallelograms in $S_T$ and $\delta_{k+1}, \ldots, \delta_l$ are all the polygons in $S_T$ other than triangles and parallelograms then
\[
\text{rk}(T) \leq \#\text{Vert}(S_T) - 1 - \sum_{\delta \in \{\delta_1, \ldots, \delta_k\}} (\#\text{Vert}(\delta) - 3) - \sum_{i=k+1}^{l} \left(\#\text{Vert}(\delta_i) - \max \left\{3, \# \left(\text{Vert}(\delta_i) \cap \bigcup_{j<i} \text{Vert}(\delta_j)\right)\right\}\right),
\]
where $\delta$ runs over the polygons of $S_T$.

Proof. $\#\text{Vert}(S_T) - 1$ is the number of the parameters we deal with. For the triangles there are no conditions imposed. The 4-valent vertices of $S_T$, dual to parallelograms, impose independent conditions. To see these we shall use the method showed in [12] - take a vector $\vec{a} \in \mathbb{R}^2$ with an irrational slope and coorient each edge of any parallelogram so that the normal vector forms an acute angle with $\vec{a}$. This coorientation defines a partial ordering on the set of parallelograms. We can complete this to a linear ordering. Each parallelogram has two edges cooriented outside, which means that the coefficients of the linear conditions imposed by the 4-valent vertices of $S_T$ can be arranged into a triangular matrix, making the conditions independent. This means that each parallelogram imposes one condition, and together with the triangles, each one imposes $\#\text{Vert}(\delta_i) - 3$ conditions. For the rest of the polytopes we can determine only the independence of new conditions, represented by new vertices which were not part of previous polytopes. The number of such vertices is
\[
\#\text{Vert}(\delta_i) - \# \left(\text{Vert}(\delta_i) \cap \bigcup_{j<i} \text{Vert}(\delta_j)\right)
\]
for $i \geq k + 1$.
Some of the "old" vertices of a polytope may have been considered as conditions to a former set of equations, and may as well be considered as conditions in this new set of equations represented by the polytope. These conditions might not be the
same, so an additional reduction should be made. However, we cannot know this by this procedure, and this is why we get only an upper bound for the rank. Changing the partial order may produce other bounds. The minimum of such upper bounds will give the tightest bound.

**Theorem 9.** If the dual subdivision $S_T$ of the Newton polygon of an embedded plane tropical curve $T$ contains precisely three polygons $\delta_1, \delta_2, \delta_3$ other than triangles and parallelograms, then

$$
\text{rk}(T) = \# \text{Vert}(S_T) - 1 - \sum_{\delta \neq \delta_1, \delta_2, \delta_3} (\# \text{Vert}(\delta) - 3) - 3 \sum_{i=1}^{3} \left( \# \text{Vert}(\delta_i) - \max \left\{ 3, \# \left( \text{Vert}(\delta_i) \cap \bigcup_{j<i} \text{Vert}(\delta_j) \right) \right\} \right),
$$

where $\delta$ runs over the polygons of $S_T$.

**Proof.** This theorem is actually a special case of Theorem 15, where $n = 2$. They both reduce the number of new vertices as conditions, unless there are not enough vertices counted as free parameters. □

**Part 3. Rank of tropical surfaces in $\mathbb{R}^3$**

**Theorem 10.** Lower and upper bounds to the rank of a tropical surface in $\mathbb{R}^3$ can be calculated according to a given algorithm.

**Proof.** We shall use the dual graph in this case as well in order to show an algorithm to calculate these bounds. We shall denote $\Delta$ to be the Newton polygon of the tropical surface $A$, and $S_f$ its subdivision.

**step 1:** If $\Delta$ has no subdivision, and $\Delta$ is a polygon of $m$ planes, then $A$ is a $m$-valent vertex with faces between the edges. This means that 4 of the space’s parameters are independent while all the others are linearly dependent on those 4 parameters. This gives a rank of 4 to the surface. This rank is reduced by 1 due to the isotopy moving the whole graph of the piecewise linear function "up" or "down" in the 4-dimensional space, as $A$ is the projection of the corner-locus of the graph.

**step 2:** If $\Delta$ has a subdivision $S_f$, one should choose an arbitrary polygon. This can be done as we deal with locally dependencies. This first polygon, as the one in step 1, contributes 4 to the rank. Now we shall take one of its plane neighbors. As they share a plane, 3 of the degrees of freedom of the second polygon have already been considered, and only one more degree of freedom is left to contribute to the rank. This step sums up to a contribution of 5 freedom degrees to the rank.

**step 3:** Now we shall look for polygons which are plane-neighboring the two previous ones with 4 vertices or more, where not all of those vertices lie in the same plane. Those polygons will not add anything to the rank of the surface, and shall be joined to create a bloc together with the previous two.

**step 4:** To the bloc we created on step 3 we can now add all the polygons which are plane-neighboring the bloc with 4 or more vertices that do not lie in the same plane. Again, there will be no addition to the rank. This step goes on as long as there are polygons which share with the bloc 4 or more vertices, as before. If there are no more polygons of that kind, we go to step 5.

**step 5:** We look for a polygon plane-neighboring the new bloc with 3 vertices. If there is one like that, it shall be added to the bloc, adding along with it 1 to
the rank, as 3 of its freedom-degrees have already been considered. Now we shall return to step 4.

**Step 6**: If there are no more polygons neighboring the bloc, we shall reduce 1 from the rank we got. This is due to the fact that the surface can be considered as the projection to \( \mathbb{R}^3 \) of the corner-locus of the piecewise linear function, which lies in \( \mathbb{R}^4 \). The location of the corner-locus in the fourth axis does not change the projection, and this means that the "first" vertex we chose has no effect on the rank. The only effect on the rank comes from the relationships between the vertices, and not from their absolute value.

The process is over and we got the upper bound of the rank.

If according to the full order we defined on the polygons, we get to a polygon whose 5 vertices or more have been already determined, it means we might have encountered a double condition. In such case, \( \text{rk}(A) \) should be decreased by 1 for every vertex of that kind. As this reduction is just of suspicious vertices, we get to a lower bound. Changing the first two polygons in step 2 may produce other bounds. Finally, the minimum upper bound and the maximum lower bound give the tightest bounds this procedure allows.

The next section deals with tropical curves in \( \mathbb{R}^3 \) whose cycles are planar with no edge going through them, and their vertices' valence is 4 or more. The motivation to study these curves is the 1-dimensional skeletons of tropical hypersurfaces in \( \mathbb{R}^3 \).

**Lemma 11.** The deformation space of a tropical hypersurface, and the deformation space of the curve which is derived from the hypersurface as a 1-skeleton, are the same.

**Proof.** The deformations of a face of the tropical hypersurface are done by deforming its edges. The deformations of 3-dimensional polytopes are done by deforming their faces, and so on. Therefore, by induction, the curve which is determined by the intersections of the polytopes in the tropical hypersurface, actually defines the "structure" of the hypersurface and shares the same deformation space with it.

**Definition 12.** Let \( C \) be a tropical curve in \( \mathbb{R}^3 \) whose minimal cycles are planar and no edge goes through them. Close those cycles into faces and take the complement. A bounded connectivity component of the complement is adjacent to some edges of the curve. For each such component, we shall call the adjacent edges of the curve "closed volume".

**Theorem 13.** Let \( T \) be a tropical curve with valencies of 4 or more, with no minimal cycles which are not planar, and no edge goes through them. The overvalence of the curve is defined by the valencies higher than 4. Let us assume that this curve can be achieved from one of its maximal trees in the following way. For each closed volume, we have to close at least one minimal cycle in the inductive order of building the curve from the tree. In other words, there is no closed volume created only by other closed volumes in the building procedure. This curve's rank is bounded from below:

\[
\text{rk}(T) \geq \frac{\#\text{End}(T)}{2} + 1 - \frac{1}{2}\text{ov}(T) + N_{cl}
\]

where \( N_{cl} \) is the number of "closed volumes".

**Proof.** First, we shall explain \( \frac{\#\text{End}(T)}{2} + 1 - \frac{1}{2}\text{ov}(T) \). The number of bounded edges in such a curve is \( \frac{4V - \#\text{End}(T) - \text{ov}(T)}{2} \). In order to get a maximal tree out of this
curve, we have to take out \( g \) edges. Then we get, by the Euler Characteristic for trees \((E = V - 1)\), that \( 2V - \frac{1}{2} \# End(T) + \frac{1}{2} ov(T) - g = V - 1 \), which implies that
\[
V = \frac{\# End(T) - ov(T) + 2g - 2}{2}
\]
and therefore the curve has \( \frac{\# End(T)}{2} + 2g - 2 - \frac{1}{2} ov(T) \) bounded edges. This means the rank of the maximal tree is \( \frac{\# End(T)}{2} + g - 2 - \frac{1}{2} ov(T) + n \), and the \( n \) came from the \( n \) dimensional space in which the tree can be shifted. Closing each cycle of the curve from the tree sets a condition on the rank. Since the tree needs \( g \) edges to reconstruct the curve, and since we deal with \( \mathbb{R}^3 \), we get that
\[
\text{rk}(T) \geq \frac{\# End(T)}{2} + 1 - \frac{1}{2} ov(T).
\]
Regarding \( N_{cl} \) - we shall use induction on \( N_{cl} \):
\( N_{cl} = 1 \): Each of the first \( g - 1 \) minimal cycles closed from the maximal tree dictates a condition upon the rank. The last edge missing, though, does not have any effect on the rank. Therefore, we shouldn’t have subtracted \( g \) from the rank of the maximal tree in order to get to the rank of the curve. Instead, we should have subtracted only \( g - 1 \), and therefore a correction of 1 can be added to the lower bound.

Now let us assume this is true for such curve with \( N_{cl} \) closed volumes. Let’s take such curve \( T \) with \( N_{cl} + 1 \) closed volumes. From this curve, according to the assumption, we can choose one closed volume with an edge adjacent to it alone (the last closed volume built from the tree, for example), and take off all the edges and rays adjacent only to this closed volume. We shall add rays to the remaining of the curve in order to make it tropical. This new tropical curve \( T' \) implements all the conditions of the theorem and has \( N_{cl} \) closed volumes, and therefore \( \text{rk}(T') \geq \frac{\# End(T)}{2} + 1 - \frac{\text{ov}(T)}{2} + N_{cl} \). Now let us take the maximal tree of \( T' \) (without the rays), and extend it to a maximal tree of \( T \). Closing this tree to \( T \) in a way that the \( N_{cl} \) closed volumes of \( T' \) are closed first, gives a "compensation" of \( N_{cl} \) to the rank. Now we have to complete the last closed volume. Some of its minimal cycles are already closed, but as it has one edge adjacent to it alone, we still have to close some other minimal cycles. This is analogous to the closing of the one closed volume in the first step of the induction. This means that due to this last closed volume, the lower bound can be higher by 1, and
\[
\text{rk}(T) \geq \frac{\# End(T)}{2} + 1 - \frac{\text{ov}(T)}{2} + (N_{cl} + 1).
\]

We should notice that the minimal cycles of the 1-skeleton curve of a tropical hypersurface in \( \mathbb{R}^3 \) are always like the minimal cycles in the above definition. We can use this fact in order to bound the rank of this curve by the above theorem. We should also notice that different hypersurfaces in \( \mathbb{R}^3 \) may have the same 1-skeleton, and therefore their ranks are bounded by the same number.

**Remark 14.** The lower bound for higher dimensional hypersurfaces can be built in the same manner. For example, 1-skeletons of tropical hypersurfaces in \( \mathbb{R}^4 \) are at least 5-valent. Taking a hypersurface in \( \mathbb{R}^4 \) leads to a lower bound according to those considerations: the number of bounded edges is \( \frac{5V - \# End(T)}{4} \). The number of vertices is calculated, inductively, by: \( \# End(T) + 2g = 3V + 2 + ov(T) \), which implies that \( V = \frac{\# End(T) + 2g - 2 - ov(T)}{4} \). Combining the two equations leads to
\[
\text{rk}(T) \geq \frac{\frac{\# End(T)}{3} + \frac{5}{3} g - \frac{5}{3} - \frac{\text{ov}(T)}{3}}{3}
\]
bounded edges in the curve and therefore to a maximal tree with rank \( \text{rk}(Tree) = \frac{\# End(T)}{3} + \frac{5}{3} g - \frac{5}{3} - \frac{\text{ov}(T)}{3} + n \) where \( n = 4 \). Completing the maximal tree into the curve \( T \), with planar cycles, leads to
\[
\text{rk}(T) \geq \frac{\# End(T)}{3} - \frac{g}{4} + 2 \frac{1}{3} - \frac{\text{ov}(T)}{3}.
\]
Part 4. Rank of tropical hypersurfaces in $\mathbb{R}^n$, $n > 3$

**Theorem 15.** Let $X$ be a tropical hypersurface in $\mathbb{R}^n$, $n > 3$, with the dual subdivision $S_X$ of its Newton polytope $\Delta$. Let $\delta_1, ..., \delta_k$ be all the $n$-polytopes of $S_X$ other than simplices (and somehow ordered). Then

\[
\text{rk}(X) \leq \# \text{Vert}(S_X) - 1 - \sum_{i=1}^{k} \left( \# \left( \text{Vert}(\delta_i) \setminus \bigcup_{j<i} \text{Vert}(\delta_j) \right) \right) - n + \text{dimConvHull} \left( \text{Vert}(\delta_i) \cap \bigcup_{j<i} \text{Vert}(\delta_j) \right)
\]

**Proof.** $\# \text{Vert}(S_X) - 1$ is the number of parameters we deal with. Along the order we have defined, we get to a polytope other than a simplex, and reduce the number of its new vertices as new conditions. This reduction is noted by

\[
\# \left( \text{Vert}(\delta_i) \setminus \bigcup_{j<i} \text{Vert}(\delta_j) \right)
\]

If the vertices this polytope shares with its predecessors are positioned in a space with a dimension less than $n$, some of the new vertices of the polytope do not represent a condition, and therefore should be added again. This case is noted by

\[
-n + \text{dimConvHull} \left( \text{Vert}(\delta_i) \cap \bigcup_{j<i} \text{Vert}(\delta_j) \right)
\]

Some of the "old" vertices of a polytope may have been considered as conditions to former set of equations, and may be considered as conditions on this new set of equations represented by the polytope. These conditions might not be the same, and therefore, an additional reduction should be made. However, we cannot know this by the procedure itself, which is why we get only an upper bound for the rank. Changing the order may produce other bounds. The minimum upper bound gives the tightest bound. $\square$

**Theorem 16.** Let $X$ be a tropical hypersurface in $\mathbb{R}^n$, $n \geq 3$, with the dual subdivision $S_X$ of its Newton polytope $\Delta$. If $S_X$ contains at most three $n$-polytopes $\delta_i$, $0 \leq i \leq i_0 \leq 3$, other than simplices, then

\[
\text{rk}(X) = \# \text{Vert}(S_X) - 1 - \sum_{i=1}^{i_0} \left( \# \left( \text{Vert}(\delta_i) \setminus \bigcup_{j<i} \text{Vert}(\delta_j) \right) \right) - n + \text{dimConvHull} \left( \text{Vert}(\delta_i) \cap \bigcup_{j<i} \text{Vert}(\delta_j) \right)
\]

**Proof.** First we shall prove the theorem for $n = 3$. Each of the higher valence vertices adjacent to a polytope with 5 or more vertices in the dual subdivision of $X$. Such polytopes shall be later on referred to as "OV polytopes". If no two OV polytopes share vertices, the rank can be calculated exactly. If only two share some vertices, the rank can also be calculated exactly. Furthermore, we shall see that even if any two of the OV polytopes share vertices, the rank can be calculated exactly. If one such pair shares only one vertex, all the conditions are independent.
The problem arises when each pair shares an edge or a face. This is due to the fact that there might appear "double conditions", i.e.: conditions that suit to one of the OV polytopes, but not to its edge/face neighbor. Let us assume, for start, that each pair of the OV polytopes shares an edge with each of its neighbors, i.e.: we have 3 edges where each edge belongs to two OV polytopes. Let us refer to those edges as $R = \{r_1, r_2\}$, $P = \{p_1, p_2\}$, $Q = \{q_1, q_2\}$. If one of the three edges does not lie in the same plane with one of the other two edges, then the OV polytopes adjacent to these two edges do not have a double condition, simply because none of the 4 vertices represents a condition.

If all the 3 edges are parallel, we have:

\[
\begin{align*}
(4) & \quad <q_1, x> + d_1 = <q_2, x> + d_2 = <p_1, x> + c_1 = <p_2, x> + c_2 \\
& \quad \quad \text{for some } x \in \mathbb{R}^3. \\
(5) & \quad <q_1, y> + d_1 = <q_2, y> + d_2 = <r_1, y> + e_1 = <r_2, y> + e_2 \\
& \quad \quad \text{for some } y \in \mathbb{R}^3.
\end{align*}
\]

We shall now prove that the following equation:

\[
(6) \quad <p_1, z> + c_1 = <p_2, z> + c_2 = <r_1, z> + e_1
\]

for some $z \in \mathbb{R}^3$, can be extended to the following equation:

\[
(7) \quad <p_1, z> + c_1 = <p_2, z> + c_2 = <r_1, z> + e_1 = <r_2, z> + e_2
\]

for some $z$.

Reducing (4) from (6) and (4) from (5) gives, respectively:

\[
\begin{align*}
p_1(z - x) &= p_2(z - x) \Rightarrow (p_1 - p_2)(z - x) = 0 \\
q_1(y - x) &= q_2(y - x) \Rightarrow (q_1 - q_2)(y - x) = 0
\end{align*}
\]

\[
\Rightarrow (r_1 - r_2)(z - y) = 0, \text{ because } P||Q||R. \text{ Adding (5) to the last equation leads to (7).}
\]

Now we shall look at the case where each two of the edges lie in a plane, but are not parallel (cf. Figure 2).

Therefore, the constellation of the edges is such that they can be extrapolated into lines which intersect at a point that we shall call "a". So we have:
\[ p_1 = a + \overline{p}, \quad p_2 = a + a'\overline{p}, \quad 1 \neq a' \in \mathbb{R} \]
\[ r_1 = a + \overline{r}, \quad r_2 = a + b'\overline{r}, \quad 1 \neq b' \in \mathbb{R} \]
\[ q_1 = a + \overline{q}, \quad q_2 = a + c'\overline{q}, \quad 1 \neq c' \in \mathbb{R} \]

And we have (4), (5), (6) as before, where we need to prove (7).

As we may shift our curve \( X \) freely in \( \mathbb{R}^3 \), we may choose \( x = (0,0,0) \) and so we get:

\[ < p_1, x > + c_1 = < a + \overline{p}, (0,0,0) > + c_1 = c_1 = c_2 = d_1 = d_2 \]

If we add a constant \( k \) to all of the monomials defining the hypersurface which defines the curve \( X \), we shall get the same curve \( X \). Therefore, we may choose \( c_1 = c_1 = c_2 = d_1 = d_2 = 0 \). So far with \( x \).

Now we shall look at \( y \). We know that

\[ < q_1, y > = < q_2, y > = < q_1 - q_2, y > = 0 \Rightarrow < \overline{q}, y > = 0 \]

(the \( d \)'s equal to 0). But we also know that \( < q_1, y > = < a + \overline{q}, y > = < a, y > + < \overline{q}, y > = < a, y > \) and therefore \( < a, y > = < r_1, y > + e_1 = < a + \overline{r}, y > + e_1 = < a, y > + < \overline{r}, y > + e_1 \), which implies that

\[ e_1 = - < \overline{r}, y > \]

and doing the same with \( r_2 \) and \( e_2 \) leads us to

\[ e_2 = - b' < \overline{r}, y > . \]

So far with \( y \).

Now, let us recall (6) for \( z \): \( < p_1, z > = < p_2, z > = < r_1, z > + e_1 \). And so \( < p_1, z > = < a + \overline{p}, z > = < a, z > + < \overline{p}, z > \) which equals \( < p_2, z > = < a + a'\overline{p}, z > = < a, z > + a' < \overline{p}, z > \) and therefore

\[ (1 - a') < \overline{p}, z > = 0 \Rightarrow < \overline{r}, z > = 0 \Rightarrow < p_1, z > = < a, z > . \]

This leads to \( < a, z > = < p_1, z > = < r_1, z > + e_1 = < a + \overline{r}, z > - < \overline{r}, y > = < a, z > + < \overline{r}, z - y > \), which implies that \( < \overline{r}, z - y > = 0 \)

It is now left to prove is that \( < r_2, z > + e_2 \) equals to (6). In other words, we need to prove \( < r_2, z > + e_2 = < a, z > \). The prove is as follows.

\[ < r_2, z > + e_2 = < a + b'\overline{r}, z > - b' < \overline{r}, y > = < a, z > + b' < \overline{r}, z - y > = < a, z > . \]

The last case is where all the 3 edges lie in the same plane, but are not parallel (it is impossible in that case that only 2 are parallel, because then it is impossible to have polytopes which do not intersect each other).

Let \( p_1 = a + \overline{p}, p_2 = a + \alpha \overline{p} \)
\( q_1 = a + \overline{q}, q_2 = a + \beta \overline{q} \)
\( r_1 = a + \gamma \overline{q} + \overline{r}, r_2 = a + \gamma \overline{q} + \delta \overline{r} \)

Again, we shall set \( x = 0 \) and \( c_i = e_i = 0 \) and therefore \( < \overline{q}, y > = < \overline{p}, z > = 0 \).

We know \( < r_1, y > + e_1 = < q_1, y > = < a, y > + < \overline{q}, y > \) and also \( < r_1, y > + e_1 = < a, y > + \gamma < \overline{q}, y > + < \overline{r}, y > + e_1 \), and therefore \( e_1 = - < \overline{r}, y > \). In the same manner we get \( e_2 = - \delta < \overline{r}, y > \). As well, we know \( < p_1, z > = < r_1, z > + e_1 = < a, z > + \gamma < \overline{q}, z > + < \overline{r}, z - y > \) and \( < p_1, z > = < a, z > + < \overline{p}, z > = < a, z > \), what leads to \( \gamma < \overline{q}, z > = - < \overline{r}, z - y > \). As we are dealing with the deformation space of a given hypersurface with three vertices of high valency, we know that for some \( y \) and \( z \),

\[ < \overline{r}, z - y > = 0 \] due to the characteristics of the duality. As \( < \overline{q}, z > \neq 0 \) we get
\( \gamma = 0 \), and so \( \langle r_1, z \rangle + e_1 = \langle a, z \rangle \). Now we shall look at \( \langle r_2, z \rangle + e_2 = \langle a, z \rangle + \gamma \langle \bar{q}, z \rangle + \delta \langle \bar{r}, z - y \rangle = \langle a, z \rangle = \langle r_1, z \rangle + e_1 \). This is the proof for the case where each pair of our three OV polytopes shares an edge (The last case, where all the edges lie in the same plane, is actually also a proof that a curve with only three vertices of valency higher than 3 in \( \mathbb{R}^2 \) gets an exact rank as well).

Now we shall look at the case where two of the OV polytopes share a face \( P = \{ p, p + \bar{p}, p + \bar{p} \} \), and each other pair shares an edge \( Q = \{ q_1, q_2 \}, R = \{ r_1, r_2 \} \). We now have a new set of equations:

\[
(8) \quad \langle p, x \rangle + c_1 = \langle p + \bar{p}, x \rangle + e_2 = \langle q_1, x \rangle + d_1 = \langle q_2, x \rangle + d_2
\]

for some \( x \in \mathbb{R}^3 \).

\[
(9) \quad \langle q_1, y \rangle + d_1 = \langle q_2, y \rangle + q_2 = \langle r_1, y \rangle + e_1 = \langle r_2, y \rangle + e_2
\]

for some \( y \in \mathbb{R}^3 \).

We shall now prove that the following equation

\[
(10) \quad \langle p, z \rangle + c_1 = \langle p + \bar{p}, z \rangle + c_2 = \langle p + \bar{p}, z \rangle + c_3 = \langle r_1, z \rangle + e_1
\]

for some \( z \in \mathbb{R}^3 \), can be expanded to the following equation:

\[
(11) \quad \langle p, z \rangle + c_1 = \langle p + \bar{p}, z \rangle + c_2 = \langle p + \bar{p}, z \rangle + c_3 = \langle r_1, z \rangle + e_1 = \langle r_2, z \rangle + e_2
\]

for the same \( z \).

Let us define \( \bar{r} \) to be a vector parallel to \( r_2 - r_1 \) and the same with \( \bar{q} \). There are several cases possible: the edges are not in the same plane (cf. Figure 3), the edges lie in the same plane but are not parallel (cf. Figure 4), the edges are parallel to each other and to the face (cf. Figure 5), the edges are parallel but not to the face (cf. Figure 6).

Figure 3: \( \bar{q} \) and \( \bar{r} \) are not in the same plane

Figure 4: \( \bar{q} \) and \( \bar{r} \) are in the same plane, but are not parallel
For all cases we should set \( x = \bar{0}, e_3 = d_4 = 0 \), which implies that \( < \bar{q}, y > = < \bar{p}, z > = < \bar{p}, z > = 0 \). The first case is obvious, because we can find an order to the polytopes in which the polytope dual to \( y \) is the last polytope, and no double conditions can occur. The second case, where \( R \) and \( Q \) lie in the same plane but are not parallel - let us assume \( R \)'s extrapolation intersects the extrapolation of \( P \). Then:

\[
\begin{align*}
r_1 &= p + a\bar{p} + b\bar{p} + c\bar{r}, r_2 = p + a\bar{p} + b\bar{p} + d\bar{r} \\
r_1 &= q_1 + e\bar{q} + c\bar{r}, r_2 = q_1 + d\bar{q} + (c' + d - c)\bar{r}.
\end{align*}
\]

\( < q_1, y > = < r_1, y > + e_1 = < q_1, y > + c' < \bar{r}, y > + e_1 \), which implies that \( e_1 = -c' < \bar{r}, y > \). In the same manner \( e_2 = -(c' + d - c) < \bar{r}, y > \)

\( < p, z > = < r_1, z > + e_1 = < p, z > + c < \bar{r}, z > - c' < \bar{r}, y > \) which implies that \( c < \bar{r}, z > = c' < \bar{r}, y > \). Again, as we know there are some \( y_1 \) and \( z_1 \) which fulfill

\( < \bar{r}, z_1 - y_1 > = 0 \) we get \( c = c' \) and so, for all \( y \) and \( z \) we have \( < \bar{r}, z - y > = 0 \). Therefore \( < r_2, z > + e_2 = < p, z > + d < \bar{r}, y > - (c' + d - c) < \bar{r}, y > = < p, z > = < r_1, z > + e_1 \). The meaning of \( c = c' \) is that this case is possible only if \( R \)'s extrapolation meets \( Q \)'s extrapolation on \( P \)'s extrapolation. Otherwise, this dual division does not represent an hypersurface with 3 vertices with valencies greater than 4.

The second case is that of \( R \parallel Q \parallel P \). In that case \( \bar{r} = a\bar{p} + b\bar{p} \) and \( \bar{q} = a'\bar{p} + b'\bar{p} \) if

\( < \bar{r}, z > = < \bar{q}, z > = 0 \). We should also recall that \( < \bar{q}, y > = 0 \) and as \( R \parallel Q \) we also get \( < \bar{r}, y > = 0 \). Therefore, the question if \( < r_1, z > + e_1 = < r_2, z > + e_2 \) is equivalent to the question whether \( e_1 = e_2 \), but we know \( < r_1, y > + e_1 = < r_2, y > + e_2 \) and \( < \bar{r}, y > = 0 \), and so we get \( e_1 = e_2 \).

The last case is where \( R \parallel Q \), but \( R \)'s extrapolation intersects \( P \)'s extrapolation.

\( < \bar{q}, y > = 0 \) and \( R \parallel Q \), which implies that \( < \bar{r}, y > = 0 \). \( r_1 = p + a\bar{p} + b\bar{p} + c\bar{r} = q_1 + a'\bar{p} + b'\bar{p} + c'\bar{r}, r_2 = p + a\bar{p} + b\bar{p} + d\bar{r} = q_1 + a'\bar{p} + b'\bar{p} + (c' + d - c)\bar{r} \).

Let us mark \( \bar{p} \equiv a'\bar{p} + b'\bar{p} \). Now, \( < q_1, y > = < r_1, y > + e_1 = < q_1, y > + < p, y > + e_1 \), which implies that \( e_1 = -< \bar{p}, y > \), and in the same manner \( e_2 = -< \bar{p}, y > \) (as \( < \bar{r}, y > = 0 \)).

\[
\begin{align*}
< p, z > = < r_1, z > + e_1 = < p, z > + c < \bar{r}, z > - < p, y > \\
< r_2, z > + e_2 = < p, z > + d < \bar{r}, z > - < \bar{p}, y >
\end{align*}
\]

We know \( c \neq d \) and therefore an equality between \( < r_1, z > + e_1 \) and \( < r_2, z > + e_2 \) can only be achieved if \( < \bar{r}, z > = 0 \). However, we know this cannot be since \( R \)'s and \( P \)'s extrapolations intersect, and \( P \)'s extrapolation is perpendicular to \( z \). This means there cannot be a hypersurface with three vertices with OV which is dual to such dual subdivision. This close the section of a face and two edges.

The case of 3 OV polytopes where two of them share an edge \( R = \{ r_1, r_2 \} \) and each
of the other pairs shares a face $P = \{p, p + \bar{p}, p + \tilde{p}\}$ and $Q = \{q, q + \bar{q}, q + \tilde{q}\}$. We now have a new set of equations:

\[
\begin{align*}
(12) & \quad <p, x> + c_1 = <p + \bar{p}, x> + c_2 = <p + \tilde{p}, x> + c_3 = <q, x> + d_1 = \\
& \quad <q + \bar{q}, x> + d_2 = <q + \tilde{q}, x> + d_3
\end{align*}
\]

for some $x \in \mathbb{R}^3$.

\[
\begin{align*}
(13) & \quad <q, y> + d_1 = <q + \bar{q}, y> + d_2 = <q + \tilde{q}, y> + d_3 = <r_1, y> + e_1 = <r_2, y> + e_2
\end{align*}
\]

for some $y \in \mathbb{R}^3$.
We shall now prove that the following equation

\[
\begin{align*}
(14) & \quad <p, z> + c_1 = <p + \bar{p}, z> + c_2 = <p + \tilde{p}, z> + c_3 = <r_1, z> + e_1
\end{align*}
\]

for some $z \in \mathbb{R}^3$, can be expanded to the following equation:

\[
\begin{align*}
(15) & \quad <p, z> + c_1 = <p + \bar{p}, z> + c_2 = <p + \tilde{p}, z> + c_3 = <r_1, z> + e_1 = <r_2, z> + e_2
\end{align*}
\]

for the same $z$.
Let us define $\bar{r}$ to be a vector which is parallel to $r_2 - r_1$. The possibilities are: The edge and the two faces are all parallel (cf. Figure 7), the extrapolation of the edge intersects the extrapolation of the two parallel faces (cf. Figure 8), the extrapolation of the edge is parallel to the intersection of the extrapolations of the two faces (which are not parallel) (cf. Figure 9), the edge is parallel to one face but not to the other (cf. Figure 10).

Figure 7: The edge $\bar{r}$ and the two planes are all parallel

Figure 8: The extrapolation of $\bar{r}$ intersects the extrapolation of the two parallel planes
For all the cases we shall take $x = 0, c_i = d_i = 0$, which implies that $<\bar{q}, y> = <\bar{q}, z> = <\bar{p}, z> = 0$. The first case cannot be, because one cannot create non intersecting convex polytopes from this constellation. In the second case we have $r_1 = q + a\bar{q} + b\bar{q} + c\bar{r}, r_2 = q + a\bar{q} + b\bar{q} + d\bar{r}$ and so $<q, y> = <r_1, y> + e_1 = <q, y> + a <\bar{q}, y> + b <\bar{q}, y> + c <\bar{r}, y> + e_1 = <q, y> + c <\bar{r}, y> + e_1$, which implies that $e_1 = -c <\bar{r}, y>$ and in the same manner $e_2 = -d <\bar{r}, y>$. We can also represent $r_1 = p + a'\bar{p} + b'\bar{p} + c'\bar{r}, r_2 = p + a'\bar{p} + b'\bar{p} + (c' + d - c)\bar{r}$ and so $<p, z> = <r_1, z> + e_1 = <p, z> + a' <\bar{p}, z> + b' <\bar{p}, z> + c' <\bar{r}, z> - c <\bar{r}, y> = <p, z> + c' <\bar{r}, z> - c <\bar{r}, y>$ and as we know, there are $y_1$ and $z_1$ for which $<\bar{r}, z_1 - y_1> = 0$ due to the duality, and therefore $c = c'$. That leads to $<\bar{r}, z - y> = 0$ for all relevant $y$ and $z$. That, in turn, leads to $<r_2, z> + e_2 = <p, z> + d <\bar{r}, z> - d <\bar{r}, y>$ and since $<\bar{r}, z - y> = 0$ we get $<r_2, z> + e_2 = <p, z> = <r_1, z> + e_1$. The fact that $c = c'$ means that the only option for that case to describe the wanted hypersurface is if the extrapolation of $R$ intersects the intersection of the two faces.

In the third case we have $\bar{r} = a\bar{p} + b\bar{p}$, which implies that $<\bar{r}, z> = 0$. Since we know $<r_1, y> + e_1 = <r_2, y> + e_2$ we get $e_1 = e_2$. And that is why $<r_1, z> + e_1 = <r_2, z> + e_2$.

The last case is the one where $R$ is parallel to $P$, without loss of generality, and $R$'s extrapolation intersects $Q$'s. Then we have $\bar{r} = a\bar{p} + b\bar{p}$, which implies $<\bar{r}, y> = 0$. Since $r_1 = q + a\bar{q} + b\bar{q} + c\bar{r}, r_2 = q + a\bar{q} + b\bar{q} + d\bar{r}$ we get $<q, y> = <r_1, y> + e_1 = <q, y> + a <\bar{q}, y> + b <\bar{q}, y> + c <\bar{r}, y> + e_1 = <q, y> + e_1$, which implies that $e_1 = -c <\bar{r}, y>$ and the same way leads to $e_2 = -d <\bar{r}, y>$. Since we know $<\bar{r}, z> = 0$ the question whether $<r_1, z> + e_1 = <r_2, z> + e_2$ is equivalent to the question whether $e_1 = e_2$ for which we know the answer is "no" since $c \neq d$ and since $<\bar{r}, y> \neq 0$. The last inequality arises from the given fact that $R$ is not parallel to $Q$, and from the fact that $y$ is perpendicular to $Q$.

The last option, where each two polytopes intersect in a face, can be understood from the previous option - a hypersurface as needed can be created only when all the extrapolations to the faces intersect in the same line. In this case, there are no double conditions, and the counting in the order suggested is accurate. □

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