A time-fractional mean field game

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January 23, 2018

Abstract

We consider a Mean Field Games model where the dynamics of the agents is subdiffusive. According to the optimal control interpretation of the problem, we get a system involving fractional time-derivatives for the Hamilton-Jacobi-Bellman and the Fokker-Planck equations. We discuss separately the well-posedness for each of the two equations and then we prove existence and uniqueness of the solution to the Mean Field Games system.

AMS subject classification: 35R11, 60H05, 26A33, 40L20, 49N70.

Keywords: subdiffusion; time change; fractional derivative; fractional Fokker-Planck equation; fractional Hamilton-Jacobi-Bellman equation; Mean Field Games.

1 Introduction

The study of complex systems and the investigation of their structural and dynamical properties is at forefront of the interaction between mathematics and real life sciences such as biology, sociology, epidemiology, etc. Complex systems are characterized by a large number of elementary individuals and strong interactions among these individuals which make their evolution hardly predictable by traditional approaches. Hence in the recent years new techniques have been developed in order to capture specific properties of these systems.

The Mean Field Games (MFG in short) theory introduced in [10], [14] (see [7] for a review) is a new paradigm in the framework of dynamics games with a large number of players to deal with systems composed by rational individuals, i.e. individuals able to choose their behavior on the basis of a set of preferences and to change it in consequence of the interaction with other members of the population. MFG theory has been very successful in applications where an intrinsic rationality is embedded in the complex system such as pedestrian motion, opinion formation, financial market, management of exhaustible resources, etc. From a mathematical point of view a MFG model leads to the study of a strongly coupled system of two partial differential equations: a Hamilton-Jacobi-Bellman (HJB in

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short) equation related to the optimal control problem solved by a single agent, but influenced by the presence of all (or a part of) the other agents; a Fokker-Planck (FP in short) equation governing the evolution of the distribution of the population and driven by the optimal strategies chosen by the agents. We refer to [3] for a nice introduction to the MFG theory.

In the model introduced in [10], [14] the dynamics of the single agent is governed by a (possibly degenerate) Gaussian diffusion process. Hence the underlying environment has no role in the problem or, in other words, is isotropic, an assumption not satisfied in several applications.

The study of anomalous diffusion processes deviating from the classical diffusive behavior has lead to the introduction of a class of subdiffusive processes which displays local motion occasionally interrupted by long sojourns, a trapping effects due to the anisotropy of the medium. A subdiffusive regime is considered to be a better model not only for several transport phenomena in physics, but also, for example, in the study of volatility of financial markets, bacterial motion, bird flight, etc. (see [20] for a review). It is worth noting that the nonmarkovian nature of these processes can be also interpreted as a manifestation of memory effects encountered in the study of the phenomena.

At a microscopic level, the classical construction of a diffusion process as the limit of a Markov chain with finite time average/finite jump variance is replaced, for a subdiffusive process, by the limit of a Continuous Time Random Walk displaying broad spatial jump distribution and/or infinite waiting time between consecutive jumps; at a macroscopic one, the evolution of the probability density function (PDF) associated to a subdiffusive process is governed by a time-fractional FP equation, i.e. a FP equation involving fractional derivatives in time.

In this paper, we introduce a class of MFG problems in which the dynamics of the agent is subdiffusive rather diffusive as in the Lasry-Lions model. The aim is to present a simplified framework in which we take into account only waiting time effects and not jump distribution ones since this model already presents several interesting differences compared to the classical MFG theory. A first important point is to understand the correct formulation of the MFG system in this framework. Time-fractional FP equation governing the evolution of the PDF of a subdiffusive process were first derived for the case of a space-dependent drift and constant diffusion coefficient [18]. Since then, the theory has been progressively generalized to include the case of space-time dependent coefficients which is relevant for our study (see [16], [21]). We will consider weak solution based on a measure theoretic approach, see [4], and we will prove some regularity properties of the measure-valued solution.

On the other hand, a theory of viscosity solution for time-fractional HJB equations has been developed in [6] for the first order case and in [23] for the second order one (including also a nonlocal operator in the space variable). We remark that, in these papers, the connection between the HJB equation and the associated control problem via the dynamic programming principle is not considered. Here we will exploit the Ito’s formula and the corresponding properties developed in [11] to establish this connection; indeed, this is a crucial point to understand the differential game associated to the MFG system and to justify the specific form of the HJB equation considered. In view of the well-posedness of the time-fractional MFG system in Section 5, which requires a regular setting for
the notion of solution to the HJB equation, we consider the theory of mild solutions introduced in [12] which gives, under appropriate assumptions, existence and uniqueness of a classical solution.

After having considered the two equations separately, we tackle the study of the MFG system

\[
\begin{aligned}
-\partial_t v + D_{[t,T]}^{1-\beta}[-\nu \Delta v + \mathcal{H}(t, x, Dv) - G(x, m)] &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d \\
\partial_t m + \left[\nu \Delta \cdot + \text{div}(D_p \mathcal{H}(t, x, Dv) \cdot )\right] (D_{[0,t]}^{1-\beta} m) &= 0, \\
m(0, x) &= m_0(x), \quad v(T, x) = g(x).
\end{aligned}
\]

where $D_{[t,T]}^{1-\beta}$ and $D_{[0,t]}^{1-\beta}$ denote the backward and forward Riemann-Liouville fractional derivatives, $\mathcal{H}$ is a scalar Hamiltonian, convex in the gradient variable, and the term $G$ associates a real valued function $G(x, m)$ to a probability density $m$. We obtain existence of a solution by a fixed point argument; moreover we get uniqueness of the solution adapting a classical argument in MFG theory to this framework and taking advantage of the duality relation between the two equations composing the system.

We mention that the theory of linear and semilinear fractional differential equations is currently an active field of research [1, 15, 25]. For fully nonlinear equations, the theory is at the beginning and, besides the papers [6, 13, 23] dealing with the HJB equation, we also mention [2] where a fractional porous media equation is studied.

The paper is organized as follows. In Section 2, we review some basic definitions of the fractional calculus. In Section 3 we introduce the class of subdiffusive processes and we study the corresponding FP equation. Section 4 is devoted to the time-fractional HJB equation. Finally, in Section 5, we prove the well posedness of the time-fractional MFG system.

2 Fractional calculus

In this section, we remind the reader definitions and some basic properties of fractional operators (we refer to [22] for a complete account of the theory).

Throughout this section, we always assume that $\beta \in (0, 1]$. For $f : (a, b) \to \mathbb{R}$, the forward and backward Riemann-Liouville fractional integrals are defined by

\[
I^{\beta}_{[a,b]} f(t) := \frac{1}{\Gamma(\beta)} \int_a^t f(\tau) \frac{1}{(t-\tau)^1-\beta} d\tau, \\
I^{\beta}_{[t,b]} f(t) := \frac{1}{\Gamma(\beta)} \int_t^b f(\tau) \frac{1}{(\tau-t)^1-\beta} d\tau.
\]

The fractional integrals are bounded linear operator over $L^p(a, b)$, $p \geq 1$; indeed, by Holder’s inequality, if $f \in L^p(a, b)$, then

\[
\|I^{\beta}_{[a,b]} f\|_{L^p} \leq \frac{|b-a|^\beta/p}{\beta \Gamma(\beta)} \|f\|_{L^p}.
\]
The forward Riemann-Liouville and Caputo derivatives are defined, respectively, by
\[ D_{[a,t]}^\beta f(t) := \frac{d}{dt} \left[ I_{[a,t]}^{1-\beta} f(t) \right] = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t f(\tau) \frac{1}{(t-\tau)\beta} d\tau, \]
(2.1)
\[ \partial_{[a,t]}^\beta f(t) := I_{[a,t]}^{1-\beta} \left[ \frac{df}{dt}(t) \right] = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{df}{d\tau}(\tau) \frac{1}{(t-\tau)\beta} d\tau. \]
(2.2)
while the backward Riemann-Liouville and Caputo derivatives are defined, respectively, by
\[ D_{(t,b]}^\beta f(t) := -\frac{d}{dt} \left[ I_{(t,b]}^{1-\beta} f(t) \right] = -\frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_t^b f(\tau) \frac{1}{(\tau-t)\beta} d\tau, \]
(2.3)
\[ \partial_{(t,b]}^\beta f(t) := -I_{(t,b]}^{1-\beta} \left[ \frac{df}{dt}(t) \right] = -\frac{1}{\Gamma(1-\beta)} \int_t^b \frac{df}{d\tau}(\tau) \frac{1}{(\tau-t)\beta} d\tau. \]
(2.4)
It is easy that for \( \beta \to 1 \) the Riemann-Liouville and Caputo fractional derivatives of a smooth function \( f \) converge to the classical derivative \( \frac{df}{dt} \), i.e. fractional derivatives are an extension of standard derivatives. But, since fractional derivatives are defined by an integral and therefore are nonlocal operators, several properties of differential calculus do not hold. For example, the product formula \( \partial_{[a,t]}^\beta (fg) = (\partial_{[a,t]}^\beta f)g + (\partial_{[a,t]}^\beta g)f \) does not hold and consequently there is no useful formula for the integration by parts. Moreover, if \( \alpha + \beta < 1 \), the identity \( \partial_{[a,t]}^{\alpha+\beta} f = \partial_{[a,t]}^\alpha (\partial_{[a,t]}^\beta f) \) is in general false and it can be recovered only for \( f \) smooth. 

Note that the Riemann-Liouville derivative is defined for a function \( f \in C^0([0,T]) \), while the Caputo derivative, even if it is a derivative of order less than 1, is defined for a function \( f \in C^1([0,T]) \). If \( f \in C^1([0,T]) \), the following formulas, easily obtained by integration by parts, establish the connection between the two types of fractional derivatives
\[ \partial_{[a,t]}^\beta f(t) = D_{[a,t]}^\beta f(t) - \frac{(t-a)^{-\beta}}{\Gamma(1-\beta)} f(a), \]
(2.5)
\[ \partial_{(t,b]}^\beta f(t) = D_{(t,b]}^\beta f(t) - \frac{(b-t)^{-\beta}}{\Gamma(1-\beta)} f(b). \]
(2.6)
The previous identities, also called regularized Caputo derivatives, allow to define Caputo derivatives under less stringent regularity assumptions for \( f \).

3 The time-fractional FP equation

This section is devoted to the study of the time-fractional FP equation, i.e. the FP equation describing the evolution of the PDF for a subdiffusive stochastic process.

A classical way to define a diffusion process is taking the limit in an appropriate sense of a random walk in dimension 1. The corresponding construction for a subdiffusive process is performed through rescaled limits of a continuous-time random walk (CTRW in short). In a CTRW model, a random waiting time \( \gamma_i \) occurs between successive random jumps \( \xi_i \). Jumps and waiting times form a sequence
of i.i.d., mutually independent random variables $(\gamma_i)_{i \in \mathbb{N}} \subset \mathbb{R}^+, (\xi_i)_{i \in \mathbb{N}} \subset \mathbb{R}^d$. Set $s(0) = 0$, $t(0) = 0$ and let $s(n) = \sum_{i=1}^n \xi_i$ and $t(n) = \sum_{i=1}^n \gamma_i$ be the position of the particle after $n$ jumps and the time of $n^{th}$ jump. For $t \geq 0$, define $n(t) = \max\{n \geq 0 : t(n) \leq t\}$, the number of jumps by time $t$, and observe that $n(t)$ and $t(n)$ have the inverse relationship $\{n(t) \geq n\} = \{t(n) \leq t\}$. The CTRW process determines the location reached at time $t$, for a particle performing a random walk in which random particle jumps are separated by random waiting times. Note that the process $x(t)$ is not in general Markovian.

Consider the limit of a CTRW process for the following standard scaling (see [13, 17, 18]): $\xi_i \mapsto \tau^{1/n} \xi_i$, $\gamma_i \mapsto \tau^{1/\beta} \gamma_i$. If the waiting times have finite mean and the jumps have finite variance, then the scaled CTRW converges in distribution to a diffusion process. However, if the waiting times have infinite mean, the limit process is the composition of a $\alpha$-stable Lévy motion $L(t)$ and the inverse of a $\beta$-stable subordinator $E(t)$, where $\alpha \in (0, 2]$ and $\beta \in (0, 1)$. In this paper we only consider the case $\alpha = 2$ (the general case will be considered elsewhere). In this case the limit process is a time-changed diffusion process which is described by the following stochastic differential equation

\[
\begin{cases}
  dX_t = b(t, X_t)dE_t + \sigma(t, X_t)dB_t, \\
  X_0 = x_0
\end{cases}
\]

where $B_t$ is a Brownian motion in $\mathbb{R}^p$ and $E_t$ is the inverse of a $\beta$-stable subordinator $D_t$ with Laplace transform $\mathbb{E}[e^{-\tau D_t}] = e^{-t^{\beta}}$, i.e.

\[
E_t := \inf\{\tau > 0 : D_\tau > t\}, \quad t \geq 0.
\]  

(3.1)

The process $E_t$ is continuous and nondecreasing, moreover for any $t, \gamma > 0$ its $\gamma$-moment is given by

\[
\mathbb{E}(E_t^\gamma) = C(\beta, \gamma)t^{\beta \gamma}
\]

(3.2)

for some positive constant $C(\beta, \gamma)$ (see [18]). Note that the process $E_t$ does not have stationary and independent increments.

To obtain a more explicit description of the process $X_t$, consider the diffusion process $Y_t$ given by

\[
\begin{cases}
  dY_t = b(D_t, Y_t)dt + \sigma(D_t, Y_t)dB_t, \\
  Y_0 = x_0, \quad D_0 = 0
\end{cases}
\]

(3.3)

and assume that $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times p}$ are continuous and satisfy

\[
|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}^d, t \in [0, T]
\]

(3.4)

\[
|b(t, x)| + |\sigma(t, x)| \leq M, \quad \forall x \in \mathbb{R}^d, t \in [0, T],
\]

(3.5)
for constants \( L, M > 0 \). Then, the process \( Y_t \) is well defined and the process \( X_t \) given by (3.3) can be represented in the subordinated form

\[
X_t = Y_{E_t}.
\]  

As explained in [11, 16], the formula (3.6) allows to obtain the following interpretation of the process \( X_t \): the subordinator \( E_t \) is a change of time, hence the standard time \( t \) can be interpreted as an external scale, or the time scale of an external observer, while \( E_t \) as an internal scale which introduces trapping events in the motion. Note that the change of time induced by subordination influences only the dependence of the coefficients \( b \), \( \sigma \) on the time variable. Hence, between two jumps when the particle is not trapped, the process moves according to a standard diffusion process \( Y_t \) since it holds \( D_{E_t} t = t \).

Formula (3.6) allows to deduce the time-fractional FP equation satisfied by the PDF of the process \( X_t \) (see [4, 8, 16, 21]). We have

\[
\partial_t m(t, x) = A \left[ D_{(0, t]}^{1-\beta} m(t, x) \right],
\]  

(3.7)

where \( D_{(0, t]}^{1-\beta} \) is the forward Riemann-Liouville derivate of order \( 1 - \beta \), see (2.1), and

\[
A = -\sum_{i=1}^{d} \partial_{x_i} [b_i(t, x) \cdot] + \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} [a_{ij}(t, x) \cdot]
\]

with \( a = \frac{1}{2} \sigma(t, x) \sigma(t, x)^t \). For \( \beta = 1 \), equation (3.7) reduces to the classical FP equation since in this case \( D_{(0, t]}^{1-\beta} m = m \). Moreover, if the coefficients \( b \), \( \sigma \) are independent of \( t \), exchanging the derivates in time with the ones in space and applying the fractional integral \( I_{(0, t]}^{\beta} \) on both the sides, (3.7) can be rewritten as

\[
\partial_{(0, t]}^{\beta} m(t, x) = Am(t, x),
\]

where \( \partial_{(0, t]}^{\beta} \) is the forward Caputo derivate of order \( \beta \), see (2.2).

To introduce a notion of weak solution for (3.7) we need the following result:

**Lemma 3.1.** Let \( u \in L^1([0, T] \times \mathbb{R}^d) \), then

\[
\langle D_{(0, t]}^{1-\beta} u, f \rangle = \int_0^T \int_{\mathbb{R}^d} u(t, x) D_{[t, T]}^{1-\beta} f(t, x) dx dt,
\]

(3.8)

\[
\langle D_{[t, T]}^{1-\beta} u, f \rangle = \int_0^T \int_{\mathbb{R}^d} u(t, x) D_{(0, t]}^{1-\beta} f(t, x) dx dt,
\]

(3.9)

for every \( f \in C_c^\infty((0, T) \times \mathbb{R}^d) \).

**Proof.** Taking into account the identity

\[
\int_0^T k(t) I_{[t, T]}^{\beta} h(t) dt = \int_0^T h(t) I_{(0, t]}^{\beta} k(t) dt,
\]

for constants \( L, M > 0 \). Then, the process \( Y_t \) is well defined and the process \( X_t \) given by (3.3) can be represented in the subordinated form

\[
X_t = Y_{E_t}.
\]  

As explained in [11, 16], the formula (3.6) allows to obtain the following interpretation of the process \( X_t \): the subordinator \( E_t \) is a change of time, hence the standard time \( t \) can be interpreted as an external scale, or the time scale of an external observer, while \( E_t \) as an internal scale which introduces trapping events in the motion. Note that the change of time induced by subordination influences only the dependence of the coefficients \( b \), \( \sigma \) on the time variable. Hence, between two jumps when the particle is not trapped, the process moves according to a standard diffusion process \( Y_t \) since it holds \( D_{E_t} t = t \).

Formula (3.6) allows to deduce the time-fractional FP equation satisfied by the PDF of the process \( X_t \) (see [4, 8, 16, 21]). We have

\[
\partial_t m(t, x) = A \left[ D_{(0, t]}^{1-\beta} m(t, x) \right],
\]  

(3.7)

where \( D_{(0, t]}^{1-\beta} \) is the forward Riemann-Liouville derivate of order \( 1 - \beta \), see (2.1), and

\[
A = -\sum_{i=1}^{d} \partial_{x_i} [b_i(t, x) \cdot] + \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} [a_{ij}(t, x) \cdot]
\]

with \( a = \frac{1}{2} \sigma(t, x) \sigma(t, x)^t \). For \( \beta = 1 \), equation (3.7) reduces to the classical FP equation since in this case \( D_{(0, t]}^{1-\beta} m = m \). Moreover, if the coefficients \( b \), \( \sigma \) are independent of \( t \), exchanging the derivates in time with the ones in space and applying the fractional integral \( I_{(0, t]}^{\beta} \) on both the sides, (3.7) can be rewritten as

\[
\partial_{(0, t]}^{\beta} m(t, x) = Am(t, x),
\]

where \( \partial_{(0, t]}^{\beta} \) is the forward Caputo derivate of order \( \beta \), see (2.2).

To introduce a notion of weak solution for (3.7) we need the following result:

**Lemma 3.1.** Let \( u \in L^1([0, T] \times \mathbb{R}^d) \), then

\[
\langle D_{(0, t]}^{1-\beta} u, f \rangle = \int_0^T \int_{\mathbb{R}^d} u(t, x) D_{[t, T]}^{1-\beta} f(t, x) dx dt,
\]

(3.8)

\[
\langle D_{[t, T]}^{1-\beta} u, f \rangle = \int_0^T \int_{\mathbb{R}^d} u(t, x) D_{(0, t]}^{1-\beta} f(t, x) dx dt,
\]

(3.9)

for every \( f \in C_c^\infty((0, T) \times \mathbb{R}^d) \).

**Proof.** Taking into account the identity

\[
\int_0^T k(t) I_{[t, T]}^{\beta} h(t) dt = \int_0^T h(t) I_{(0, t]}^{\beta} k(t) dt,
\]
we have for \( f \in C^\infty_c((0, T) \times \mathbb{R}^d) \)

\[
\int_0^T \int_{\mathbb{R}^d} D_1^{1-\beta} u(t, x) f(t, x) dx dt = \int_0^T \int_{\mathbb{R}^d} \partial_t (I^\beta_{[0,t]} u)(t, x) f(t, x) dx dt = \\
\int_{\mathbb{R}^d} I^\beta_{[0,t]} u(t, x) f(t, x) dx \left|_0^T \right. - \int_0^T \int_{\mathbb{R}^d} I^\beta_{[0,t]} u(t, x) \partial_t f(t, x) dx dt = \\
- \int_0^T \int_{\mathbb{R}^d} u(t, x) I^\beta_{[t,T]} (\partial_t f(t, x)) dx dt = \int_0^T \int_{\mathbb{R}^d} u(t, x) \partial_1^{1-\beta} f(t, x) dx dt
\]

and the identity (3.8) follows observing that for \( f \in C^\infty_c((0, T) \times \mathbb{R}^d) \), \( \partial_1^{1-\beta} f(t, x) = D_1^{1-\beta} f(t, x) \). The identity (3.9) is proved in a similar way.

The previous lemma, together with the usual distributional rules, justifies the following:

**Definition 3.2.** Given \( m_0 \in \mathcal{P}_1(\mathbb{R}^d) \), \( m \in L^1([0,T], \mathcal{P}_1(\mathbb{R}^d)) \) is said to be a weak solution to (3.7) with the initial condition \( m(x,0) = m_0(x) \) if for any test function \( \phi \in C^\infty_c([0,T) \times \mathbb{R}^d) \), we have

\[
\int_{\mathbb{R}^d} \phi(0, x) dm_0(x) + \int_0^T \int_{\mathbb{R}^d} \left[ \partial_t \phi + D_1^{1-\beta} \sum_{i=1}^d b_i(t, x) \frac{\partial \phi}{\partial x_i} + \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right] dm(t)(x) = 0.
\]

The following result for (3.7) has been proved in [4, 16, 21]

**Theorem 3.3.** If \( m_0 \) is the law of \( X_0 \), then the law \( m_t \) of \( X_t \) is the unique weak solution of the fractional FP equation (3.7) such that \( m(0) = m_0 \).

To conclude this section we provide some regularity properties of the solution of (3.7) (see [3] for a corresponding result in the classical case). Denote with \( \mathcal{P}_1(\mathbb{R}^d) \) the set of the probability measures over \( \mathbb{R}^d \) with finite first moment, endowed with the Kantorovich-Rubinstein distance

\[
d_1(\mu_1, \mu_2) := \inf_{\gamma \in \Pi(\mu_1, \mu_2)} \int_{\mathbb{R}^{2d}} |x-y| d\gamma(x,y),
\]

where \( \Pi(\mu_1, \mu_2) \) is the set of the Borel probability measures on \( \mathbb{R}^{2d} \) with marginal distributions \( \mu_1, \mu_2 \).

**Proposition 3.4.** Given \( m_t \) as in Theorem 3.3, then

- The map \( m: [0, T] \to \mathcal{M}^+(\mathbb{R}^d) \) is \( \frac{\beta}{2} \)-Holder continuous, i.e. there is a constant \( C = C(M, L) \), where \( M, L \) as in (3.4)-(3.5), such that for every \( t, s \in [0, T] \)

\[
d_1(m_t, m_s) \leq C|t - s|^\beta/2,
\]

(3.11)

- There is a constant \( c = c(M) \) such that

\[
\int_{\mathbb{R}^d} |x|^2 dm_t(x) \leq c \left( \mathbb{E}|X_0|^2 + t^{2\beta} + t^\beta \right)
\]

(3.12)
Proof. We first observe that
\[ d_1(m(t), m(s)) \leq \int_{\mathbb{R}^d} |x - y|d\pi(x, y) = \mathbb{E}||X_t - X_s|| \]
where \( \pi \) is the law of the pair \((X_t, X_s)\). Assume w.l.o.g. \( s < t \), then
\[ X_t - X_s = \int_s^t dX_z = \int_s^t b(X_z, z)dE_z + \int_s^t \sigma(X_z, z)dB_{E_z}. \]
By (3.5), we have
\[ \mathbb{E}(|X_t - X_s|) \leq M(\mathbb{E}(|E_t - E_s|) + \mathbb{E}(|B_{E_t} - B_{E_s}|)). \]
Since \( t \geq s \), then \( E_t \geq E_s \) a.s.. Moreover, by (3.2), we have
\[ \mathbb{E}(|E_t - E_s|) = \mathbb{E}(E_t - E_s) = C(\beta, 1)(t^\beta - s^\beta) \leq C(\beta, 1)|t - s|^\beta. \]
On the other hand, denoted by \( h(\cdot, t) \) the PDF of \( E_t \), we have
\[ \mathbb{E}(|B_{E_t} - B_{E_s}|) = \mathbb{E}(\int_{E_s}^{E_t} dB_z) = \int_0^{E_t} \left( \int_0^{E_s} \mathbb{E}(|B_{r_1} - B_{r_2}|) h(r_1, t) h(r_2, s) dr_1 dr_2 \right) \]
\[ = \int_0^{E_t} \left( \int_0^{E_s} \sqrt{r_1 - r_2} h(r_1, t) h(r_2, s) dr_1 dr_2 \right) \]
\[ = \mathbb{E}(\sqrt{E_t - E_s}) \leq \sqrt{\mathbb{E}(E_t - E_s)} \]
\[ \leq C(\beta, 1)^{1/2}|t - s|^{\beta/2}, \]
which gives (3.11). To prove (3.12) we estimate
\[ \int_{\mathbb{R}^d} |x|^2 dm_t(x) = \mathbb{E}(|X_t|^2) \leq 2\mathbb{E} \left[ |X_0|^2 + \int_0^t b(s, X_s)dE_s|^2 + \int_0^t \sigma(s, X_s)dB_{E_s}|^2 \right] \]
\[ \leq 2 \left( \mathbb{E}(|X_0|^2) + MC(\beta, 2)t^{2\beta} + M \int_0^{+\infty} \mathbb{E}(|B_s|^2)h(s, t)ds \right) \]
\[ \leq 2 \left( \mathbb{E}(|X_0|^2) + MC(\beta, 2)t^{2\beta} + M\mathbb{E}(|E_t|) \right) \]
\[ \leq 2(\mathbb{E}(|X_0|^2) + MC(\beta, 2)t^{2\beta} + MC(\beta, 1)t^\beta. \]
The thesis follows taking \( c = 2 \max\{1, MC(\beta, 2), MC(\beta, 1)\}. \)

4 The fractional HJB equation

In this section we introduce an optimal control problem for a class of time-changed diffusion processes and we deduce, at a formal level, the corresponding dynamic programming equation, which turns out
to be a time-fractional HJB equation.

Let \((\Omega, \mathcal{F}, \mathcal{F}^\cup, \mathbb{P})\) be a filtered probability space and let \((B_t)_{t \geq 0}\) be a Brownian motion in \(\mathbb{R}^d\) adapted.

Fixed \(T > 0\), consider the controlled process \((X_s)_{s \geq t}\) given by the solution of the time-changed stochastic differential equation

\[
\begin{aligned}
  dX_s &= f(s, X_s, u_s) dE_s + \sqrt{2\nu} dB_{E_s}, \\
  X_t &= x.
\end{aligned}
\]

In (4.1), \(E_s\) is a non increasing process, defined by

\[
E_s := T - \bar{E}_{T-s}
\]
with \(\bar{E}_s\) the inverse of a \(\beta\)-stable subordinator such that \(\bar{E}_0 = 0\), see (3.1), and \(\nu\) is a positive constant.

The control law \((u_s)_{s \geq 0}\) belongs to \(U\), the class of progressive measurable processes taking values in the compact metric space \(U\). Observe that if the process \((Y_s)_{s \geq t}\) is the controlled diffusion process given by

\[
dY_s = f(D_s, Y_s, \bar{u}_s) ds + \nu dB_s,
\]
then \(X_s = Y_{E_s}\) is a solution of (4.1) for the time-changed control law \(u_s = \bar{u}_{E_s}\).

Given \(L : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}\) and \(g : \mathbb{R}^d \to \mathbb{R}\) representing respectively the running cost and the terminal cost, we consider the cost functional

\[
J(t, x, u) := \mathbb{E}_{x,t} \left[ \int_t^T L(s, X_s, u_s) dE_s + g(X_T) \right],
\]
where \(X_s\) satisfies (4.1). Note that the time in the cost functional (4.4) is rescaled according to the process \(E_t\) such that \(E_T = T\). Indeed, as we explained in the previous section, \(E_s\) is a change of time which represents an inner time scale for the process \(X_s\) and also the cost functional is evaluated according to this scale. Moreover, the agent knows the final cost \(g\) at time \(T\) and, accordingly to this datum, computes the optimal strategy backward in time (see also Remark 4.7).

Define the value function \(v : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) by

\[
v(t, x) = \inf_{u \in U} J(t, x, u). \tag{4.5}
\]

**Proposition 4.1.** Let \((t, x) \in [0, T] \times \mathbb{R}^d\) be given. Then, for every stopping time \(\theta\) valued in \([t, T]\), we have

\[
v(t, x) = \inf_{u \in U} \mathbb{E}_{x,t} \left[ \int_t^\theta L(s, X_s, u_s) dE_s + v(\theta, X_\theta) \right]. \tag{4.6}
\]

The proof is based on standard arguments in control theory (see [24, Thm.3.3]).

Define the Hamiltonian \(\mathcal{H} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\) by

\[
\mathcal{H}(t, x, p) := \sup_{u \in U} \{-f(t, x, u) \cdot p - L(t, x, u)\} \tag{4.7}
\]
and consider the fractional HJB equation

\[
\partial_{(t,T)}^\beta v - \nu \Delta v + \mathcal{H}(t,x,Dv) = 0, \quad (t,x) \in (0,T) \times \mathbb{R}^d
\]

(4.8)

where \(\partial_{(t,T)}^\beta\) is the backward Caputo derivative defined in (2.4). We prove that the value function can be characterized as a solution of (4.8). We need a preliminary lemma.

**Lemma 4.2.** The function \(h(\cdot,r), r \in [s,T]\), is the PDF of the process \(E_s\) defined in (4.2) iff it is a weak solution of

\[
\partial_s h(r,s) = -D_{[s,T]}^{1-\beta}[\partial_r h(r,s)] - \delta_T(r)\delta_T(s).
\]

(4.9)

**Proof.** Recalling that by definition \(E_s = T - \bar{E}_{T-s}\), since the PDF \(\bar{h}(r,s)\) of the process \(\bar{E}_s\) satisfies (see [5, 19])

\[
\partial_{(0,s)}^\beta \bar{h}(r,s) = -\partial_r \bar{h}(r,s),
\]

it follows that

\[
\partial_{(s,T)}^\beta h(r,s) = \partial_r h(r,s).
\]

By (2.6), the previous equality can be equivalently rewritten also as

\[
D_{[s,T]}^\beta h(r,s) = \partial_r h(r,s) + \frac{(T-s)^{-\beta}}{\Gamma(1-\beta)} \delta_T(r).
\]

(4.10)

We prove that (4.9) implies (4.10). Integrating (4.9) from \(s\) to \(T\) we get

\[
h(r,T^-) - h(r,s) = -I_{[s,T]}^\beta \partial_r h(r,s) - \delta_T(r)H(T-s),
\]

where \(H\) is the Heaviside function. Observe that \(h(r,T^-) = h(T-r,0^+) = 0\) (see [19] for more details). Then, applying \(I_{[s,T]}^{1-\beta}\) on both sides, we get

\[
-I_{[s,T]}^{1-\beta} h(r,s) = -\int_s^T \partial_r h(r,\tau)d\tau - \delta_T(r)\frac{(T-s)^{1-\beta}}{\Gamma(2-\beta)}.
\]

Taking the derivative \(\frac{\partial}{\partial s}\), we finally get (4.10).

A similar computation shows that (4.10) implies (4.9): as in [19], applying the fractional derivative in the sense of distribution to (4.10) and using the identity \(D_{[s,T]}^{1-\beta} \partial_{(s,T)}^\beta = \partial_r\), we obtain (4.9).

**Proposition 4.3.** Assume that \(f,L: \mathbb{R}^d \times [0,T] \times U \to \mathbb{R}^d\), are continuous functions satisfying

\[
|f(t,x,u) - f(t,y,u)| + |L(t,x,u) - L(t,y,u)| \leq C|x-y| \quad \forall t \in [0,T], x,y \in \mathbb{R}^d, u \in U,
\]

(4.11)

\[
|f(t,x,u)| + |L(t,x,u)| \leq M \quad \forall x \in \mathbb{R}^d, u \in U,
\]

(4.12)

for some positive constant \(C,M > 0\) and that the value function \(v\) defined in (4.5) is smooth. Then it is a classical solution of (4.8).
Proof. Given \((t, x) \in [0, T] \times \mathbb{R}^d\) and \(u \in U\), consider the constant process \(u_t \equiv u\) for all \(t \geq 0\) and let \(X_t\) be the corresponding solution of (4.1). For \(\eta > 0\), define the stopping time

\[
\theta_\eta := \inf\{s > t : (s - t, X_s - x) \notin [0, \eta) \times B_\eta\},
\]

where \(B_\eta\) is the ball in \(\mathbb{R}^d\) with radius \(\eta\). From the dynamic programming principle (4.6), it follows that

\[
\mathbb{E}_{x,t} \left[ v(t, X_t) - v(\theta_\eta, X_{\theta_\eta}) \right] \leq \mathbb{E}_{x,t} \left[ \int_t^{\theta_\eta} L(s, X_s, u) dE_s \right].
\] (4.13)

The right hand side term in (4.13) can be rewritten in the following way

\[
\mathbb{E}_{x,t} \left[ \int_t^{\theta_\eta} L(s, X_s, u) dE_s \right] = \mathbb{E}_{x,t} \left[ \int_{E_t}^{E_{\theta_\eta}} L(D_z, Y_z, u) dz \right]
= - \int_{-\infty}^T \mathbb{E}_{x,t} \left[ \left( \int_T^T L(D_z, Y_z, u) dz \right) (h(r, \theta_\eta) - h(r, t)) \right] dr,
\] (4.14)

where \(D_z\) is the process such that \(E_{D_z} = z\), \(h(\cdot, t)\) is the probability density function of \(E_t\) and \(Y_t\) is given by (4.3). For a fixed \(u \in U\) consider the linear second order operator

\[
\mathcal{L}_u v(t, x) := \nu \Delta v(t, x) + f(t, x, u) Dv(t, x).
\]

By Ito’s formula (see [11])

\[
\mathbb{E}_{x,t} \left[ v(\theta_\eta, X_{\theta_\eta}) - v(t, X_t) \right] = \mathbb{E}_{x,t} \left[ \int_t^{\theta_\eta} dv(s, X_s) \right]
= \mathbb{E}_{x,t} \left[ \int_t^{\theta_\eta} \partial_s v(s, X_s) ds + \int_t^{\theta_\eta} Ds v(s, X_s) dX_s + \frac{1}{2} \int_t^{\theta_\eta} D^2 v(s, X_s) d\langle X_s \rangle \right]
= \mathbb{E}_{x,t} \left[ \int_t^{\theta_\eta} \partial_s v(s, X_s) ds + \int_t^{\theta_\eta} (f(s, X_s, u) \cdot Dv(s, X_s) + \nu \Delta v(s, X_s)) dE_s \right]
+ \int_t^{\theta_\eta} \sqrt{2\nu Dv(s, X_s)} dB_{E_s}
= \mathbb{E}_{x,t} \left[ \int_t^{\theta_\eta} \partial_t v(s, X_s) ds + \int_t^{\theta_\eta} \mathcal{L}_u v(s, X_s) dE_s \right]
= \mathbb{E}_{x,t} \left[ \int_t^{\theta_\eta} \partial_t v(s, X_s) ds \right] - \int_{-\infty}^T \mathbb{E}_{x,t} \left[ \left( \int_r^T \mathcal{L}_u v(D_z, Y_z) dz \right) (h(r, \theta_\eta) - h(r, t)) \right] dr.
\]

Substituting the previous identity and (4.14) in (4.13) and dividing by \(\eta\) on both sides we get

\[
- \mathbb{E}_{x,t} \left[ \frac{1}{\eta} \int_t^{\theta_\eta} \partial_t v(s, X_s) ds \right] + \int_{-\infty}^T \mathbb{E}_{x,t} \left[ \left( \int_r^T \mathcal{L}_u v(D_z, Y_z) dz \right) \frac{h(r, \theta_\eta) - h(r, t)}{\eta} \right] dr \leq
\leq - \int_{-\infty}^T \mathbb{E}_{x,t} \left[ \left( \int_r^T L(D_z, Y_z, u) dz \right) \frac{h(r, \theta_\eta) - h(r, t)}{\eta} \right] dr.
\]
Sending $\eta \to 0^+$, since $\theta_\eta \to t$, by the dominated convergence theorem we have
\[
\int_{\infty}^{T} E_{x,t} \left[ \frac{1}{\eta} \int_{t}^{\theta_\eta} \partial_t v(s, X_s) ds \right] \to \frac{\partial_t v(t, X_t)}{\eta} = \partial_t v(t, x);
\]
\[
\int_{\infty}^{T} E_{x,t} \left[ \left( \int_{r}^{T} (\mathcal{L}_u v(D_z, Y_z) + L(D_z, Y_z, u)) dz \right) \frac{h(r, \theta_\eta) - h(r, t)}{\eta} \right] \to 0^+.
\]

Set $\Phi(r) = \int_{r}^{T} (\mathcal{L}_u v(D_z, Y_z) + L(D_z, Y_z, u)) dz$. Then by (4.9)
\[
\int_{\infty}^{T} E_{x,t} \left[ \Phi(r) \partial_t h(r, t) \right] dr
\]
\[
= - \int_{\infty}^{T} E_{x,t} \left[ \Phi(r) D_{[t,T]}^{1-\beta} \partial_t h(r, t) \right] dr - E_{x,t} [\Phi(T) \delta_T(t)]
\]
\[
= - \int_{\infty}^{T} E_{x,t} \left[ D_{[t,T]}^{1-\beta} (\Phi(r) \partial_t h(r, t)) \right] dr
\]
\[
= -E_{x,t} \left[ D_{[t,T]}^{1-\beta} \left( \Phi(r) h(r, t) \right) |_{\infty}^{T} - \int_{-\infty}^{T} \partial_r \Phi(r) h(r, t) dr \right]
\]
\[
= E_{x,t} \left[ D_{[t,T]}^{1-\beta} \int_{-\infty}^{T} (L(D_r, Y_r, u) + \mathcal{L}_u v(D_r, Y_r)) h(r, t) dr \right]
\]
\[
= D_{[t,T]}^{1-\beta} \left[ L(t, x, u) + \mathcal{L}_u v(t, x) \right].
\]

Then we have
\[
-\partial_t v(t, x) + D_{[t,T]}^{1-\beta} \left[ L(t, x, u) + \mathcal{L}_u v(t, x) \right] \leq 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}^d. \tag{4.15}
\]

Set $F(t, x, u) := L(t, x, u) + \mathcal{L}_u v(t, x)$ and $v_T(x) := v(T, x)$. Recalling that by definition $D_{[t,T]}^{1-\beta} = -\frac{d}{dt} [I_{[t,T]}^{\beta}]$, we can rewrite (4.15) as
\[
-\partial_t v - \frac{d}{dt} [I_{[t,T]}^{\beta} F(t, x, u)] \leq 0.
\]

Applying $I_{[t,T]}^{1-\beta}$, we get
\[
\partial_t^{\beta} [I_{[t,T]}^{1-\beta} F(t, x, u)] - I_{[t,T]}^{1-\beta} D_{[t,T]}^{1-\beta} F(t, x, u) \leq 0.
\]
Moreover by (2.6)

\[ I_{[t,T]}^{1-\beta} D_{[t,T]}^{1-\beta} F(t,x,u) = I_{[t,T]}^{1-\beta} \left( \partial_{[t,T]}^{1-\beta} F(t,x,u) + \frac{(T-t)^{\beta-1}}{\Gamma(\beta)} F(T,x,u) \right) \]

\[ = -I_{[t,T]}^{1-\beta} \frac{d}{dt} F(t,x,u) + \frac{F(T,x,u)}{\Gamma(\beta)} I_{[t,T]}^{1-\beta} (T-t)^{\beta-1} \]  
(4.16)

\[ = -\int_t^T \partial_s F(s,x,u) ds + \frac{\Gamma(\beta) \Gamma(1-\beta)}{\Gamma(\beta) \Gamma(1-\beta)} F(T,x,u) \]

\[ = -F(T,x,u) + F(t,x,u) + F(T,x,u) = F(t,x,u). \]

Hence

\[ \partial_{[t,T]}^{\beta} v(t,x) - F(t,x,u) \leq 0. \]

Since the previous inequality holds for any \( u \in U \), we finally obtain

\[ \partial_{[t,T]}^{\beta} v - \nu \Delta v + \mathcal{H}(t,x,Dv) \leq 0. \]

To complete the proof, consider \( t_0, t \in (0,T) \) such that \( 0 \leq t_0 < t \leq T \) and such that \( \eta = t - t_0 \) small and, for any \( \epsilon > 0 \), consider an \( \epsilon \)-optimal control \( u^\epsilon \) such that

\[ \mathbb{E}_{x_0,t_0} \left[ \int_{t_0}^t L(s,X_s,u_s^\epsilon) ds + \varphi(t,X_t) \right] \leq \varphi(t_0,X_{t_0}) + \epsilon \eta; \]

due to (4.11) and (4.12), we can apply the previous argument even if \( u^\epsilon_t \) is not constant; then, dividing for \( \eta \), from the Ito’s formula it follows, for \( \eta \to 0^+ \),

\[ -\partial_t v(t,x) + D_{[t,T]}^{1-\beta} [L(t,x,u^\epsilon) + \mathcal{L}_u v(t,x)] \geq -\epsilon, \]

and, applying the same argument used in (4.16), we get the thesis for the arbitrariness of \( \epsilon \). \( \square \)

We now discuss the notion of solution for (4.8) introduced in [13]. Consider the space \( C_0^\infty(\mathbb{R}^d), p \geq 0 \) given by the functions \( f \in C^p(\mathbb{R}^d) \) such that \( f \) and its derivates up to order \( p \) are rapidly decreasing functions on \( \mathbb{R}^d \). To introduce a notion of solution for (4.8), we first consider a linear equation of the form

\[
\begin{aligned}
\partial_{[t,T]}^{\beta} w(t,x) - \nu \Delta w(t,x) &= \ell(t,x), \\
w(T,x) &= w_T(x) 
\end{aligned}
\]  
(4.17)

for given continuous functions \( w_T : \mathbb{R}^d \to \mathbb{R}, \ell : (0,T) \times \mathbb{R}^d \to \mathbb{R} \). If \( w_T \in C_0^\infty(\mathbb{R}^d) \), a solution of (4.17) can be written in the integral form

\[ w(t,x) = \int_{\mathbb{R}^d} S_{\beta,1}(t,x-y) w_T(y) dy + \int_t^T \int_{\mathbb{R}^d} G_{\beta}(s-t,x-y) \ell(s,y) dy ds. \]  
(4.18)

where \( S_{\beta,1}(t,x-x_0) \) and \( G_{\beta}(T-t,x-x_0) \) are the solutions of equation (4.17) with \( w_T(x) = \delta(x-x_0) \), \( \ell(t,x) = 0 \) and, respectively, \( w_T(x) = 0 \), \( \ell(t,x) = \delta(T-t,x-x_0) \) (in the case \( \beta = 1 \), \( G_{\beta} \) and \( S_{\beta,1} \) coincide). Explicit formula for \( G_{\beta} \) and \( S_{\beta,1} \) can be obtained in terms of the Fourier transform of Mittag-Leffler functions (see [13] for details). By formula (4.18), it is natural to introduce the following notion of solution for (4.8)
**Definition 4.4.** Given a continuous function \( g \in C^0_\infty(\mathbb{R}^d) \), we say that \( v \in C^0([0,T], C^1_\infty(\mathbb{R}^d)) \) is a mild solution of (4.8) satisfying the terminal condition \( v(x,T) = g(x) \) if

\[
v(t,x) = \int_{\mathbb{R}^d} S_{\beta,1}(t,x-y)g(y)dy + \int_t^T \int_{\mathbb{R}^d} G_{\beta}(s-t,x-y)\mathcal{H}(s,y,Dv(y))dyds.
\]

We quote from [13] the following existence and uniqueness result

**Theorem 4.5.** Assume that

- \( \mathcal{H}(s,x,p) \) is Lipschitz in \( p \) with Lipschitz constant \( L_1 \), i.e.
  \[
  |\mathcal{H}(s,x,p) - \mathcal{H}(s,x,q)| \leq L_1|p-q|.
  \]  
  (4.19)

- \( \mathcal{H}(s,x,p) \) is Lipschitz in \( x \) with Lipschitz constant \( L_2 \), i.e.
  \[
  |\mathcal{H}(s,x_1,p) - \mathcal{H}(s,x_2,p)| \leq L_2|x_1 - x_2|(1 + |p|).
  \]  
  (4.20)

- \( |\mathcal{H}(s,x,0)| \leq M_1 \), for a constant \( M_1 \) independent of \( x \).

- \( g \in C^2_\infty(\mathbb{R}^d) \).

Then there exists a unique mild solution \( v \) of (4.8). Moreover \( v \in C^0([0,T], C^2_\infty(\mathbb{R}^d)) \) and

\[
\sup_x |D^2v(t,x)| \leq C
\]

with \( C \) depending on \( L_1, L_2, M_1 \).

**Remark 4.6.** Assume that \( f,L \) satisfy (4.11)-(4.12) and the final cost \( g \) is in \( C^2_\infty(\mathbb{R}^N) \). Then the Hamiltonian \( \mathcal{H} \) defined in (4.7) satisfies the assumptions of the previous theorem and therefore there exists a unique mild solution to (4.8) satisfying the terminal condition \( v(T,x) = g(x) \). Moreover a straightforward application of the Verification Theorem, see e.g. [24, Thm. 5.1] allows to conclude that the mild solution coincides with the value function defined in (4.5).

**Remark 4.7.** The choice of the integration with respect to \( E_t \) in (4.4) is justified for modeling purposes, since we want to calculate the cost when the particle is effectively moving. Note that HJB equation so obtained coincides with the one studied in [6, 12, 13, 23]. However, considering integration with respect to the standard time in (4.4), i.e.

\[
J(t,x,u) := E_{x,t}\left[ \int_t^T L(s,X_s,u_s)ds + g(X_T) \right]
\]

and repeating an argument similar to the one in the proof of Prop. 4.3, we find out that in this case the value function \( v \) satisfies the equation

\[
\partial_{[t,T]}^\beta v - \nu \Delta v + \sup_{u \in U} \left\{-f(t,x,u) \cdot Dv - I_{[t,T]}^\beta L(t,x,u) \right\} = 0.
\]

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5 Fractional Mean Field Games

In this section we introduce the MFG system and we prove existence and uniqueness of a classical solution to the problem.

Consider a population of indistinguishable agents distributed at time $t = 0$ according to the density function $m_0$. Each agent moves with a dynamics given by the time-changed SDE

$$dX_s = f(s, X_s, u_s) dE_s + \sqrt{2\nu} dB_s$$

$s \in (t, T]$, and aims to minimize the pay-off functional

$$J(t, x) = E_{x, t} \left[ \int_t^T [L(s, X_s, u_s) + G(X_s, m)] dE_s + g(X_T) \right],$$

(5.1)

where the additional term $G$ represents a cost depending on the distribution of the population at time $s$. Given a distribution $m \in M^+(\{0, T\} \times \mathbb{R}^d)$, by Proposition 4.3 the backward Hamilton-Jacobi equation associated to the previous control problem is

$$\partial_t \beta_{[t, T)} v(t, x) - \nu \Delta v + H(t, x, Dv) - G(x, m) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d,$$

(5.2)

with the terminal condition $v(T, x) = g(x)$, where $H$ is defined as in (4.7).

We show that the time-fractional FP equation governing the evolution of the distribution $m$ can be obtained by a standard duality argument in MFG theory.

**Proposition 5.1.** The equation (5.2) is equivalent to the equation

$$-\partial_t v(t, x) + D^{1-\beta}_{[t, T)} [-\nu \Delta v + H(t, x, Dv) - G(x, m)] = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d,$$

(5.3)

where $D^{1-\beta}_{[t, T)}$ denotes the backward Riemann-Liouville derivative (2.3).

**Proof.** We get the thesis applying $D^{1-\beta}_{[t, T)}$ to (5.2) and recalling that $D^{1-\beta}_{[t, T]} [\partial^\beta_{[t, T)} c] = -\partial_t c$. \hfill \Box

For $\epsilon > 0$ and $g_w \in C^2_{\infty}(\mathbb{R}^d)$, write the solution of (5.2) with terminal data $g(x) + \epsilon g_w(x)$ as $v + \epsilon w$.

Hence

$$-\partial_t v - \epsilon \partial_t w + D^{1-\beta}_{[t, T)} [-\nu \Delta v - \epsilon \nu \Delta w + H(t, x, Dv + \epsilon Dw) - G(x, m)] = 0.$$

By Taylor’s expansion, the Hamiltonian term can be rewritten as

$$H(t, x, Dv + \epsilon Dw) = H(t, x, Dv) + \epsilon D_p H(t, x, Dv) Dw + o(\epsilon^2).$$

Substituting in (5.2) and isolating the terms with the same order in $\epsilon$, we get the following equation for $w$

$$-\partial_t w + D^{1-\beta}_{[t, T)} [-\nu \Delta w + D_p H(t, x, Dw) Dw] = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^d$$

(5.4)
Assume that \( w \) is a weak solution of (5.4). Integrating with respect to a test function \( m \in C^1_c((0, T); C^2_c(\mathbb{R}^d)) \), we have
\[
\int_0^T \int_{\mathbb{R}^d} \left[ -\partial_t w + D^{1-\beta}_{[t,T]} (-\nu \Delta w + D_p \mathcal{H}(t, x, Dv) Dw) \right] m(x, t) dx dt = 0. \tag{5.5}
\]
Then, taking into account (3.9), we get that (5.5) is equivalent to
\[
\int_0^T \int_{\mathbb{R}^d} w(x, t) \left[ \partial_t m - \nu \Delta(D^{1-\beta}_{[0,t]} m) - \text{div} \left( D_p \mathcal{H}(t, x, Dv) D^{1-\beta}_{[0,t]} m \right) \right] dx dt = 0 \tag{5.6}
\]
By (5.6), we deduce the time-fractional FP problem
\[
\begin{aligned}
\partial_t m(t, x) &= A \left[ D^{1-\beta}_{[0,t]} m(t, x) \right] \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
m(0, x) &= m_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]
where
\[
A \cdot = \nu \Delta \cdot + \text{div}(D_p \mathcal{H}(t, x, Dv) \cdot)
\]
and \(-D_p \mathcal{H}(t, x, Dv)\) is the optimal control obtained by (5.2), is the adjoint of the linearized of the HJB equation, as in the standard MFG theory.

We are now ready to formulate the fractional Mean Field Game system as
\[
\begin{aligned}
-\partial_t v + D^{1-\beta}_{[t,T]} [-\nu \Delta v + \mathcal{H}(t, x, Dv) - G(x, m)] &= 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
\partial_t m - [\nu \Delta \cdot + \text{div}(D_p \mathcal{H}(t, x, Dv) \cdot)](D^{1-\beta}_{[0,t]} m) &= 0, \\
m(0, x) &= m_0(x), \quad v(T, x) = g(x). \tag{5.7}
\end{aligned}
\]

**Remark 5.2.** We stress that (5.7) is the correct form of MFG system preserving the duality relation between the two equations. Indeed, if we consider the HJB equation (5.2) in place of the HJB equation (5.3) in the system (5.7), a computation similar to (5.5)-(5.6) gives the FP equation
\[
D^{\beta}_{[0,t]} m - \nu \Delta m - \text{div}(D_p \mathcal{H}(t, x, Dv)m) = 0
\]
which is not well posed since the corresponding solution \( m \) is not a normalized, non-negative distribution probability function ([9]).

### 5.1 Uniqueness

We assume that \( G : \mathbb{R}^d \times \mathcal{M}^+(\mathbb{R}^d \times [0, T]) \to \mathbb{R} \) is a continuous function whose backward fractional Riemann-Liouville derivative of order \( 1 - \beta \) is monotone with respect to the \( m \)-variable, i.e.
\[
\int_0^T \int_{\mathbb{R}^d} (m_1 - m_2) D^{1-\beta}_{[t,T]} \left( G(x, m_1) - G(x, m_2) \right) dx dt > 0, \tag{5.8}
\]
for every \( m_1, m_2 \in \mathcal{M}^+(0, T] \times \mathbb{R}^d \).

For example, the previous condition is satisfied if \( G(m) = I_{[t,T]}^1 \gamma(m) \) with \( \gamma \) an increasing function. Moreover, if the interaction cost is computed with respect to the external time scale, hence the term \( \mathbb{E}_{x,t} \left[ \int_t^T G(m_s) dE_s \right] \) in (5.1) is replaced by \( \mathbb{E}_{x,t} \left[ \int_t^T G(m_s) ds \right] \), then (5.8) is satisfied if \( G(m) \) is increasing in \( m \), as in the classical monotonicity condition in [14].
Theorem 5.3. There exists a unique classical solution to the MFG system (5.7).

Proof. We apply a standard argument in MFG theory. We assume that there exists two solutions $(v_1, m_1)$ and $(v_2, m_2)$ of (5.7). We set $\bar{v} = v_1 - v_2$, $\bar{m} = m_1 - m_2$ and we write the equations for $\bar{v}$, $\bar{m}$

\[
\begin{cases}
  -\partial_t \bar{v} + D^{1-\beta}_{[t,T]} [-\nu \Delta \bar{v} + \mathcal{H}(t, x, Dv_1) - \mathcal{H}(t, x, Dv_2) - (G(x, m_1) - G(x, m_2))] = 0 \\
  \partial_t \bar{m} - \nu \Delta (D^{1-\beta}_{(0,t]} \bar{m}) - \text{div}(D_p \mathcal{H}(t, x, Dv_1) D^{1-\beta}_{(0,t]} m_1) + \text{div}(D_p \mathcal{H}(t, x, Dv_2) D^{1-\beta}_{(0,t]} m_2) = 0 \\
  \bar{m}(0, x) = 0, \quad \bar{v}(t, x) = 0
\end{cases}
\]

Multiplying the equation for $\bar{m}$ by $\bar{v}$ and integrating we get

\[
\int_0^T \int_{\mathbb{R}^d} \left[ \bar{v} \partial_t \bar{m} + \nu D\bar{v} \cdot D \left( D^{1-\beta}_{[t,T]} \bar{m} \right) + D\bar{v} \cdot \left( D_p \mathcal{H}(t, x, Dv_1) D^{1-\beta}_{(0,t]} m_1 - D_p \mathcal{H}(t, x, Dv_2) D^{1-\beta}_{(0,t]} m_2 \right) \right] dx dt = 0. \tag{5.9}
\]

Multiplying the equation for $\bar{v}$ by $\bar{m}$ and performing a computation similar to the one in (3.10), we have

\[
\int_0^T \int_{\mathbb{R}^d} \left[ \bar{v} \partial_t \bar{m} + D^{1-\beta}_{(0,t]} \bar{m} \left( \mathcal{H}(t, x, Dv_1) - \mathcal{H}(t, x, Dv_2) \right) + \nu D \left( D^{1-\beta}_{(0,t]} \bar{m} \right) \cdot D\bar{v} - \bar{m} D^{1-\beta}_{[t,T]} (G(x, m_1) - G(x, m_2)) \right] dx dt = 0. \tag{5.10}
\]

Subtracting (5.9) to (5.10), we get

\[
\begin{align*}
  &\int_0^T \int_{\mathbb{R}^d} \left( m_1 - m_2 \right) D^{1-\beta}_{[t,T]} (G(x, m_1) - G(x, m_2)) dx dt + \\
  &\int_0^T \int_{\mathbb{R}^d} D^{1-\beta}_{(0,t]} m_1 \left( \mathcal{H}(t, x, Dv_2) - \mathcal{H}(t, x, Dv_1) - D_p \mathcal{H}(t, x, Dv_1) D(v_2 - v_1) \right) dx dt + \\
  &\int_0^T \int_{\mathbb{R}^d} D^{1-\beta}_{(0,t]} m_2 \left( \mathcal{H}(t, x, Dv_1) - \mathcal{H}(t, x, Dv_2) - D_p \mathcal{H}(t, x, Dv_2) D(v_1 - v_2) \right) dx dt = 0.
\end{align*}
\]

Since each of the three terms in the previous identity is nonnegative, in view of assumption (5.8) it follows that $m_1 = m_2$. By the uniqueness of the solution to (5.3), we finally get $v_1 = v_2$. \qed

5.2 Existence

We now prove existence of a classical solution to the MFG system (5.7). In this section we assume that the Hamiltonian $\mathcal{H} \in C^1([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ and satisfies (4.19)-(4.20), $g \in C^2_{\infty}(\mathbb{R}^d)$, $m_0 \in \mathcal{P}_1(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d} |x|^2 dm_0 < \infty$ and $G$ is uniformly bounded and Lipschitz continuous, i.e.

\[
\begin{align*}
  &|G(x_1, m_1) - G(x_2, m_2)| \leq C_1 ||x_1 - x_2|| + d_1(m_1, m_2), \quad \forall (x_1, m_1), (x_2, m_2) \in \mathbb{R}^d \times \mathcal{M}_1^+(\mathbb{R}^d), \\
  &|G(x, m)| \leq C_2, \quad \forall (x, m) \in \mathbb{R}^d \times \mathcal{M}_1^+(\mathbb{R}^d),
\end{align*}
\]

for some positive constant $C_1, C_2 > 0$.  

Theorem 5.4. There exists a solution \((v, m) \in C([0, T]; C^1_{\infty}(\mathbb{R}^d)) \times C([0, T]; \mathcal{P}_1(\mathbb{R}^d))\) to (5.7), where \(v\) is a mild solution of the HJB equation and \(m\) is a weak solution of the FP equation.

Proof. We prove existence of a solution by a fixed point argument. Let \(\mathcal{C}\) be the subset of \(C([0, T], \mathcal{P}_1(\mathbb{R}^d))\) given by distribution functions which are \(\frac{\beta}{2}\)-Holder continuous with constant \(M_1\) (to be fixed later) and such that

\[
\sup_{t \in [0, T]} \int_{\mathbb{R}^d} |x|^2 \, d\mu_t(x) \leq M_1. \tag{5.11}
\]

Since \(\mathcal{P}_1\) is a convex set, closed with respect to \(d_1\), then \(\mathcal{C}\) is convex and closed with respect to the distance \(\sup_{t \in [0, T]} d_1(\mu_t, \nu_t)\). It is also compact due to (5.11). We define a map \(\Phi : \mathcal{C} \to C([0, T], \mathcal{P}_1(\mathbb{R}^d))\) in the following way:

(i) Given \(\mu \in \mathcal{C}\), consider the HJB equation

\[
\begin{cases}
-\partial_t v(t, x) + D^{1-\beta}_{[t,T]}(-\nu \Delta v + \mathcal{H}(t, x, Dv) - G(x, \mu)) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d \\
v(T, x) = g(x).
\end{cases}
\]

By Theorem 4.5 and Proposition 5.1, there exist a unique mild solution \(v = v(\mu) \in C^0([0, T]; C^1_{\infty}(\mathbb{R}^d))\).

(ii) Given \(v\) by the previous step, consider the time-fractional FP equation

\[
\begin{cases}
\partial_t m - |\nu \Delta \cdot + \text{div}(D_{\nu} \mathcal{H}(t, x, Dv) \cdot D^{1-\beta}_{[0, t]} m) = 0, \\
m(0, x) = m_0(x).
\end{cases}
\]

By Theorem 3.3, there exists a unique weak solution \(m \in C([0, T], \mathcal{P}_1(\mathbb{R}^d))\).

Hence the map \(m := \Phi(\mu)\) defined by steps (i)-(ii) is well defined. Moreover, for

\[
M_1 := \max\{c(\beta, M)(\mathbb{E}(|X_0|^2) + T^{\beta} + T^{3/2}), C(\beta, M)\},
\]

see (3.11)-(3.12), we have that \(m \in \mathcal{C}\), hence \(\Phi\) maps \(\mathcal{C}\) into itself. We show that the map \(\Phi\) is continuous. Consider a sequence \(\mu_n \in \mathcal{C}\) converging to some \(\mu \in \mathcal{C}\) and let \((v_n, m_n)\), \((v, m)\) the corresponding functions defined in steps (i)-(ii).

In [12, Thm.11], it is shown that there exists a positive constant \(C_0\) such that the solution of the HJB equation is bounded in \(C^2(\mathbb{R}^d)\) for every \(t \in [0, T]\). Since the constant \(C_0\) depends only on the bounds (4.19)-(4.20), it follows that

\[
\sup_{t \in [0, T]} \|v_n\|_{C^2(\mathbb{R}^d)} \leq C_0,
\]

uniformly in \(n \in \mathbb{N}\). Hence we conclude that, up to a subsequence, \(v_n\) and \(Dv_n\) locally uniformly converge, respectively, to \(v\) and \(Dv\).

It is easily seen that any converging subsequence of the relatively compact sequence \(m_n\) is a weak solution of the FP equation associated to \(v\). Since the solution of this equation is unique, we get that all the sequence \(m_n\) converges to \(m\) and therefore \(\Phi\) is continuous.

By the Schauder fixed point Theorem, the map \(\Phi\) admits a fixed point \(m = \Phi(m)\) in \(\mathcal{C}\). It follows that the corresponding couple \((v, m)\) defined in steps (i)-(ii) is a solution of (5.7). \(\square\)
Remark 5.5. We consider steady solution of (5.7), i.e. solutions such that \( v(x,t) = v(x,T), \) \( m(x,t) = m(x,0), \) for all \( t \in [0,T]. \) If \( (v,m) \) is a steady solution, then (5.7) reduces to

\[
D^{1-\beta}_{[0,T]}(\nu \Delta v + \mathcal{H}(t,x,Dv) - G(x,m)) = 0,
\]

\[
[\nu \Delta + \text{div}(D_p \mathcal{H}(t,x,Dv))] (D^{1-\beta}_{[0,T]} m) = 0. \tag{5.12}
\]

The first equation in (5.12) is equivalent to

\[-\nu \Delta v + \mathcal{H}(t,x,Dv) - G(x,m) = c(x)(T-t)^{1-\beta}.\]

If \( \mathcal{H} \) does not depend on \( t, \) then the previous equation is satisfied if and only if \( c(x) \equiv 0 \) and

\[-\nu \Delta v + \mathcal{H}(x,Dv) - G(x,m) = 0, \quad x \in \mathbb{R}^d.\]

On the other hand, due to the independence of \( m \) on the \( t-\)variable, it can be easily verified that

\[D^{1-\beta}_{[0,T]} m = C_1(\beta)t^{\beta-1}m(x),\]

for some constant \( C_1 \neq 0. \) Then, the second equation in (5.12) is satisfied if

\[\nu \Delta m + \text{div}(D_p \mathcal{H}(x,Dv)m) = 0.\]

We conclude that \( (v,m) \) is a steady solution of (5.7) if it solves

\[
\begin{cases}
-\nu \Delta v + \mathcal{H}(x,Dv) = G(x,m), & x \in \mathbb{R}^d, \\
\nu \Delta m + \text{div}(D_p \mathcal{H}(x,Dv)m) = 0.
\end{cases}
\]

Hence steady solutions of (5.7) coincide with the ones of the classical MFG system.

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