Perturbation solutions of relativistic viscous hydrodynamics for longitudinally expanding fireballs

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The solutions of relativistic viscous hydrodynamics for longitudinal expanding fireballs is investigated with the Navier-Stokes theory and Israel-Stewart theory. The energy and Euler conservation equations for the viscous fluid are derived in Rindler coordinates with the longitudinal expansion effect is small. Under the perturbation assumption, an analytical perturbation solution for the Navier-Stokes approximation and numerical solutions for the Israel-Stewart approximation are presented. The temperature evolution with both shear viscous effect and longitudinal acceleration effect in the longitudinal expanding framework are presented and specifically temperature profile shows symmetry Gaussian shape in the Rindler coordinates. In addition, in the presence of the longitudinal acceleration expanding effect, the results of the Israel-Stewart approximation are compared to the results from Bjorken and Navier-Stokes approximation, and it gives a good description than the Navier-Stokes theories results at the early stages of evolution.

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I. INTRODUCTION

The Relativistic hydrodynamic theory provides well description of the space-time evolution and many non-equilibrium properties of quark-gluon plasma (QGP) produced in heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) and the Large Hadron Collider (LHC) [1–4].

There has been a lot of excellent progress in solving relativistic viscous hydrodynamics equations analytically with different approximations and special symmetries and numerically in recent decades years [5–36]. Those analytical solutions play a very important role in understanding the evolution dynamics and are good testbeds for numerical solutions.

Recently, a series of interesting analytical solutions for longitudinally expanding relativistic perfect fluid were found by Budapest and Wuhan group [16, 18, 21, 22]. These ideal hydrodynamics solutions combined with Buda-Lund model [37] have been utilized for simulating QGP medium dynamic evolution and readily reproduce the observed final state multiplicity distribution and its dependence on beam energy, collision system, particle mass and freeze-out temperature [14, 16, 19, 21, 23, 38, 39].

However, a lot of comparisons between experimental data and viscous hydrodynamic simulations found that the picture of QGP is a nearly perfect fluid but contains a small specific shear viscosity. The shear viscosity ratio of QGP is very close to the lower bound $1/4\pi$ computed for $N_c = 4$ super-Yang-Mills (SYM) theory in the AdS/CFT correspondence [40–43]. In this paper, we will go beyond both the Csörgő-Nagy-Csanád (CNC) solutions and the Csörgő-Kasza-Csanád-Jiang (CKCJ) solutions of the relativistic perfect fluid for longitudinally expanding fireballs [16, 18, 22] and present a perturbation analytical solution of the longitudinally expanding first order (Navier-Stokes limit) viscous hydrodynamic equations. We furthermore present the numerical results of the second-order (Israel-Stewart limit) viscous hydrodynamics equations as a piece of the longitudinally expanding fireballs theory based on assuming the relaxation time is small [44]. We find that small shear pressure tensor relaxation time $\tau_\pi$ approximation solves the unstable problem of the first order approximation, indicating the stability of the second order numerical results. This study providing us a self-consistent first-order and second-order viscous hydrodynamic with longitudinal expanding dynamics, and lead us to a better understanding of the relationship between viscosity effect and longitudinal acceleration effect for the medium evolution in future phenomenological studies.

The organization of the paper is as follows. In Sec. II, the 2nd viscous hydrodynamic equations are reconstructed in Rindler coordinates according to the Landau-Lifshitz formalism [11], and perturbation solutions are presented. In Sec. III, numerical
results of viscous hydrodynamics for longitudinal expanding fireball are investigated. Brief summary and discussion are given in Sec. IV.

II. THE PERTURBATION SOLUTIONS TO THE LONGITUDINALLY EXPANDING FLOW

We work in the so-called Rindler-coordinates for which \( \tau = \sqrt{t^2 - z^2} \) is the proper time and \( \eta_\text{r} = 0.5 \ln[(1 + z/t)/(1 - z/t)] \) is the space-time rapidity [16, 18, 22]. We consider (1+1) dimensional fluid flow in (1+3) dimensions space-time \( (\tau, x, y, \eta_\text{r}) \) since we focus on the perturbation solutions of a longitudinal expanding fireball with shear viscosity. The flow 4-velocity field \( u^\mu \) in this system is

\[
u = (\cosh \Omega, 0, 0, \sinh \Omega),
\]

where flow rapidity \( \Omega \) is a function of space-time rapidity \( \eta_\text{r} \) and is independent of proper time \( \tau \), with the 4-velocity normalized as \( u^\mu u_\mu = 1 \). The second-order hydrodynamic equations without external currents are simply given by

\[
\partial_\mu T^{\mu\nu} = 0,
\]

with the energy-momentum tensor \( T^{\mu\nu} = \varepsilon u^\mu u^\nu - p\Delta^{\mu\nu} + \pi^{\mu\nu} \), where \( \varepsilon \) is the energy density, \( p \) the pressure, \( g^{\mu\nu} = \text{diag}(-1, 1, 1, 1) \) the metric tensor, \( \Delta_{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \) the projection operator which is orthogonal to the fluid velocity. The shear pressure tensor \( \pi^{\mu\nu} \) represents the deviation from ideal hydrodynamics and local equilibrium, and it satisfies \( u^\mu \pi_{\mu\nu} = 0 \) and is traceless \( \pi^\mu_\mu = 0 \) in the Landau Frame.

The energy density and pressure are related to each other by the equation of state (EoS),

\[
\varepsilon = \kappa p,
\]

where \( \kappa \) is usually related to the local temperature [44], in this case we assume \( \kappa \) to be a constant and independent of the temperature.

The fundamental equations of viscous fluid dynamics are established by projecting appropriately the conservation equations of the energy momentum tensor Eq. (2). The conservation equations can be rewritten as,

\[
D \varepsilon = - (\varepsilon + p) \theta + \sigma_{\mu\nu} \pi^{\mu\nu},
\]

and

\[
(\varepsilon + p) Du^\alpha - \nabla^\alpha p + \pi^{\alpha\mu} Du_\mu - \Delta^{\alpha\nu} \nabla^\nu \pi_{\mu\nu},
\]

respectively, where \( D = u^\mu \partial_\mu \) is the comoving derivative and \( \theta = \partial_\mu u^\mu \) is the expansion rate.

In terms of the 14-moment approximation result from [7, 45], \( \partial_\mu u^\mu \geq 0 \) reduce the corresponding thermodynamic forces. The general traceless shear tensor \( \pi^{\mu\nu} \) is [7, 41],

\[
\pi^{\mu\nu} = 2 \eta \sigma^{\mu\nu} - \tau_\pi \left[ \Delta_\alpha^\mu \Delta_\beta^\nu \nabla_\lambda \pi_{\lambda\alpha\beta} + \frac{4}{3} \Delta^{\mu\nu} \pi \right] - \lambda_1 \pi^{(\mu} \pi^{\nu)}, - \lambda_2 \pi^{(\mu} \Omega^{\nu)}, - \lambda_3 \Omega^{(\mu} \pi^{\nu)}, \lambda,
\]

with the symmetric shear tensor \( \sigma^{\mu\nu} \) and the antisymmetric vorticity tensor \( \Omega^{\mu\nu} \) defined as

\[
\sigma^{\mu\nu} \equiv \left( \frac{1}{2} (\Delta_\alpha^\mu \Delta_\beta^\nu + \Delta_\beta^\mu \Delta_\alpha^\nu) - \frac{1}{3} \Delta^{\mu\nu} \pi \right),
\]

\[
\Omega^{\mu\nu} \equiv \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} (\nabla_\alpha u_\beta - \nabla_\beta u_\alpha),
\]

where \( \eta, \tau_\pi, \lambda_1, \lambda_2, \lambda_3 \) are positive transport coefficients in the flat space time. \( \eta \) is the shear viscosity coefficient and \( \tau_\pi \) is the relaxation time for shear pressure tensor corresponding to the dissipative currents, respectively. Shear viscosity ratio \( \eta/s \) of the QGP is very close to the lower bound 1/4π computed for a strongly coupled gauge theory (\( N = 4 \) SYM) in the AdS/CFT correspondence. And relaxation time \( \tau_\pi \) is in fact approximately \( (2 - \ln 2)/(2\pi T) \) [40–43], where \( s \) is the entropy, \( s^\mu \) the entropy four-current, \( T \) the temperature. It is customary to split \( \pi^{\mu\nu} \) order-by-order in terms of \( \sigma^{\mu\nu} \) into a traceless part and the contribution form higher-order term is suppressed by the relaxation time \( \tau_\pi \) and assuming transport coefficients \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) [7, 45], and neglect the contribution from higher order \( \tau_\pi \) terms, one shows that Eqs. (4-5) can be cast into,

\[
D \varepsilon = - (\varepsilon + p) \theta + 2 \eta \sigma^{\mu\nu} \sigma_{\mu\nu} - 2 \eta \tau_\pi \sigma_{\mu\nu} \left[ \Delta_\alpha^\mu \Delta_\beta^\nu D \sigma_{\alpha\beta} + \frac{4}{3} \sigma^{\mu\nu} \theta \right],
\]
and
\[(\varepsilon + p) Du^\alpha = \nabla^\alpha p + 2\eta \left( \sigma^{\alpha\mu} - \tau_{\pi} \left[ \Delta^{\alpha}_{\mu} \Delta_{\nu}^{\mu} D \sigma^{\nu\mu} + \frac{4}{3} \sigma^{\alpha\mu} \theta \right] \right) Du_{\mu} - 2\eta \Delta^{\alpha\nu} \nabla^\mu \left( \sigma_{\mu\nu} - \tau_{\pi} \left[ \Delta^{\mu}_{\nu} \Delta_{\mu}^{\nu} D \sigma_{\mu\nu} + \frac{4}{3} \sigma_{\mu\nu} \theta \right] \right), \tag{9}\]

The CKCJ solutions [22] and perturbation solutions here are both characterized by the flow velocity field Eq. (1) in the Rindler coordinates. It is straightforward to find that comoving derivative \(D\) and expansion rate \(\theta\) can be expressed as [23],
\[
D = \cosh(\Omega - \eta_0) \frac{\partial}{\partial \tau} + \frac{1}{\tau} \sinh(\Omega - \eta_0) \frac{\partial}{\partial \eta_0}, \tag{10}\]

and
\[
\theta = \sinh(\Omega - \eta_0) \frac{\partial \Omega}{\partial \eta_0} + \frac{1}{\tau} \cosh(\Omega - \eta_0) \frac{\partial \Omega}{\partial \eta_0}, \tag{11}\]

respectively.

With the help of the Gibbs thermodynamic relation and CNC solutions [16], for systems without bulk viscosity and net charge current (net baryon, net electric charge or net strangeness), the hydrodynamic conservation equations Eqs. (8,9) for a longitudinal expanding fireball in the presence of shear viscosity in the Rindler coordinate can be written as,
\[
\frac{\tau}{\kappa} \frac{\partial T}{\partial \tau} + \tanh(\Omega - \eta_0) \frac{\partial T}{\partial \eta_0} + \frac{\Omega'}{\kappa} T = \frac{\Pi_d}{\kappa \tau} \cosh(\Omega - \eta_0) - \frac{\Pi_d \tau_0}{6\kappa T^2} \Omega' \left[ -6 \cosh(2(\Omega - \eta_0))\Omega' \right. \\
- \left. (1 + 7 \cosh(2(\Omega - \eta_0)))\Omega'' + \sinh(2(\Omega - \eta_0))\Omega'' \right], \tag{12}\]

and
\[
\tanh(\Omega - \eta_0) \left[ \frac{\tau}{\kappa} \frac{\partial T}{\partial \tau} + T \Omega' \right] + \frac{\partial T}{\partial \eta_0} = \frac{\Pi_d}{\tau} \left( 2\Omega'/(\Omega' - 1) + \Omega'' \coth(\Omega - \eta_0) \right) \sinh(\Omega - \eta_0) \\
+ \frac{\Pi_d \tau_0}{6\kappa T^2} \left( - \tanh(\Omega - \eta_0) \Omega'(12 + 24 \cosh(2(\Omega - \eta_0))) \right. \\
+ \left. \Omega'(-28 - 46 \cosh(2(\Omega - \eta_0)) + 3(5 + 7 \cosh(2(\Omega - \eta_0)))\Omega') \right) \\
+ (18 \cosh(2(\Omega - \eta_0)) + (1 - 23 \cosh(2(\Omega - \eta_0)))\Omega')\Omega'' \\
- 6\Omega'' - 3 \sinh(2(\Omega - \eta_0))\Omega^{(3)}), \tag{13}\]

where \(\Pi_d = \frac{4\eta}{3\tau}\) is related to the shear viscosity ratio, \(\Omega'\) approximately characterizes the longitudinal acceleration of flow element in the medium, \(\Omega',\ \Omega'',\ \Omega^{(3)}\) are derivative function of flow rapidity, \(\Omega' = \frac{\partial \Omega}{\partial \tau},\ \Omega'' = \frac{\partial^2 \Omega}{\partial \tau^2},\ \Omega^{(3)} = \frac{\partial^3 \Omega}{\partial \tau^3}\), respectively.

**Case A. Perturbation solution with Navier-Stokes approximation**

For a perfect fluid with longitudinal acceleration, \(\Omega \neq \eta_0\), shear viscosity \(\Pi_d = 0\), the first exact solutions are presented by Cs"orgö, Nagy, and Csanád or CNC family of solutions of Refs. [16, 18] with condition \(\Omega = \lambda \eta_0\), where \(\lambda = 1 + \lambda^*\), and \(\lambda^*\) is the longitudinal acceleration parameter to describe the dynamic of longitudinal expanding fireballs. Under the same assumption with CNC solutions, a finite and accelerating, realistic 1+1 dimensional solution of relativistic hydrodynamics was recently given by Csörgö, Kasza, Csanád and Jiang (CKCJ) [22].

For a relativistic hydrodynamic in the Navier-Stokes (first order) approximation, fluid flow rapidity \(\Omega = (1 + \lambda^*)\eta_0\), shear viscosity tensor \(\pi^{\mu\nu} = 2\eta^{\mu\nu}\), shear viscosity ratio \(\Pi_d = 4\eta/3\tau\), the relaxation time \(\tau_{\pi} = 0\). The last terms in the right of Eqs. (12, 13) disappear automatically. However, the reduced conservation equations Eqs. (12, 13) including first order approximation are still a set of nonlinear differential equations, which are notoriously hard to solve analytically. Fortunately, based on the results from the ideal hydro [21, 38], we found that the longitudinal acceleration parameter \(\lambda^*\) extracted from the experimental data is pretty small (0 < \(\lambda^* \ll 1\)), which resulting in a simply perturbation solution.

We assume the \(\lambda^*\) is a small number here, up to the leading order \(O(\lambda^*)\), Eqs. (12, 13) yields a partial differential equation depending on \(\tau\) only, and the perturbation temperature solution \(T(\tau, \eta_0)\) is
\[
T(\tau, \eta_0) = T_1(\eta_0) \left( \frac{\tau_0}{\tau} \right)^{1 + \lambda^*} + \frac{2(\lambda^* + 1)\Pi_d}{(\kappa - 1)\tau_0} \left( \frac{\tau_0}{\tau} \right)^{1 + \lambda^*} \left[ 1 - \left( \frac{\tau_0}{\tau} \right)^{1 - \frac{\lambda^*}{\kappa}} \right], \tag{14}\]

where \(\tau_0\) is the value of proper time, \(T_1(\eta_0)\) is an unfixed function.
Putting Eq. (14) into the Euler equation Eq. (13), one gets

\[ T_1(\eta_s) = T_0 \exp\left[ -\frac{1}{2} \lambda^* (1 - \frac{1}{\kappa}) \eta_s^2 \right] - \frac{1 - \exp\left[ -\frac{1}{2} \lambda^* (1 - \frac{1}{\kappa}) \eta_s^2 \right]\Pi_d}{(\kappa - 1) \tau_0}, \tag{15} \]

where \( T_0 \) define the values for temperature at the proper time \( \tau_0 \) and coordinate rapidity \( \eta_s = 0 \).

Finally, inputting Eq. (15) into Eq. (14), a perturbation solution of the \( 1 + 1 \) D embedding \( 1 + 3 \) D relativistic viscous hydrodynamics can be written as,

\[ T(\tau, \eta_s) = T_0 \left( \frac{\tau_0}{\tau} \right)^{1+\lambda^*} \left[ \exp\left[ -\frac{1}{2} \lambda^* (1 - \frac{1}{\kappa}) \eta_s^2 \right] + \frac{R_0^{-1}}{\kappa - 1} \left( 2\lambda^* + \exp\left[ -\frac{1}{2} \lambda^* (1 - \frac{1}{\kappa}) \eta_s^2 \right] - (2\lambda^* + 1) \left( \frac{\tau_0}{\tau} \right)^{\frac{\lambda^*}{\kappa - 1}} \right) \right], \tag{16} \]

where the Reynolds number is \( R_0^{-1} = \frac{\Pi_d}{\eta_s \kappa_0} \) \([7, 46]\).

This conditional perturbation solution is very nontrivial since it involves two different transport coefficients and many nonvanishing components of the longitudinal expanding properties. It is also implies that for a non-vanishing longitudinal acceleration parameter \( \lambda^* \), the cooling rate is larger than for the ideal case. Meanwhile, a non-zero viscosity makes the cooling rate smaller than for the ideal case \([38]\).

The profile of \( T(\tau, \eta_s) \) is a \( (1+1) \) dimensional scaling solution in \( (1+3) \) dimensions and the \( \eta_s \) dependence of temperature density is of the Gaussian form, see Fig. 1 right panel. Note that when \( \lambda^* = 0 \) and \( R_0^{-1} = 0 \), one obtains the same solutions as same as the ideal hydrodynamic Bjorken solution \([13]\), when \( \lambda^* = 0 \) and \( R_0^{-1} \neq 0 \), one obtains the first order Bjorken solutions \([7, 45]\), if \( \lambda^* \neq 0 \) and \( R_0^{-1} = 0 \), one obtains a special solution which is consistent with the CNC solutions’ case (c) in \([16, 18]\), and when one solve the Eqs. (12, 13) directly with \( R_0^{-1} = 0 \) and \( \lambda^* \neq 0 \), one obtains the CKCJ solutions \([22]\).

**Case B. Perturbation equations with Israel-Stewart approximation**

The temperature profile Eq. (16) shows a peak at earlier proper time \( \tau \) in the Navier Stokes approximation, see Fig. 1. The source of this acausality can be understood from the constitutive relations satisfied by the dissipative currents \( \pi \eta = 2\eta\xi_\mu\nu \). The linear relationship between dissipative currents and gradients of the primary fluid-dynamical variables imply that any inhomogeneity of \( \nu^\mu \), immediately results in dissipative currents. This instantaneous effect causes the first order theory to be unstable at earlier times. Fortunately, people found that the Israel-Stewart (second order) approximation are suitable in describing the physical process happening at earlier times, and it describes the counteract of acceleration effect and viscosity effect well. However, it’s hard to solve the the differential equations Eqs. (12, 13) analytically with the Israel-Stewart approximation. So we numerically solve the temperature time dependence Eq. (17) first at \( \eta_s = 0 \) with the initial condition \( T(0, 0, 0) = 0.65 \text{ GeV} \) first, here the grid length of \( \tau = 0.05 \text{ fm} \). Then, for each \( \eta_s \), we solve the temperature rapidity dependence Eq. (13) step by step with the results from the Eq. (12), and solve these equation th together, the grid length of \( \eta_s \) is 0.05, too. The temperature distribution of thermodynamic quantities (\( \epsilon, T, p \)) in whole \( (\tau, \eta_s) \) coordinates with initial condition \( T(0, \eta_s) \) now is a Gaussian shape, see Fig. 2. Furthermore, in order to compare with the perturbation results from the first order approximation, the Eqs. (12, 13) can be rewritten up to the leading order \( \mathcal{O}(\lambda^*) \) as follow,

\[ \frac{\partial T}{\partial \tau} = \frac{2\lambda^* + 1}{3} \Pi_d \frac{\lambda^*}{\kappa - 1} \frac{T}{3} - \frac{(2 - \ln 2) \Pi_d}{9\pi \tau^2} (1 + 6\lambda^*) \]

Above differential equations (17, 18) can not be solved analytically, we using the same numerically method as for the Eqs. (12, 13), solve the above second-order viscous hydrodynamic equations (17, 18) with the conformal equation of state \( \epsilon = 3p \) and relaxation time \( \tau_\eta = \frac{2 - \ln 3}{2\pi \tau} \) \([40-43]\) directly in the Rindler coordinates, the numerical results are presented in Fig. 3.

**III. RESULTS AND DISCUSSION**

The temperature profiles obtained in the previous section are now applied to study the longitudinal expanding dynamics, the initial condition \( T(\tau_0, \eta_{s0}) \) can be arbitrarily chosen. Following the result from \([7]\), the initial proper time \( \tau_0 = 0.2 \text{ fm}/c \), and initial temperature \( T(0.2, 0.0) = 0.65 \text{ GeV} \) are used in the calculation.

Fig. 1 show the longitudinal expanding effect dependence of temperature evolution in the Navier-Stokes approximation. In the left panel of Fig. 1 shows the time-dependence of the temperature for different viscosity and the longitudinal acceleration parameter \( \lambda^* \). The black curve is the ideal Bjorken flow. It is seen that the larger the longitudinal acceleration parameter \( \lambda^* \), the
faster the medium cool down. However, the viscosity effect slow down the medium cooling. It is important to note that there is a peak at early time in $T$ in the case of first order approximation. In the right panel of Fig. 1 shows the space-time rapidity dependence of the temperature at $\tau = 2$ fm/c. The temperature distribution of ideal Bjorken flow (black curve) and the Bjorken flow under Navier-Stokes limit (blue curve) show a flat-plateau shape. The effect of the longitudinal accelerating expanding, however, make temperature distribution to a Gaussian shape (red and orange curve).

![Diagram showing temperature evolution](image1)

**FIG. 1:** (Color online) Temperature profile in the Navier-Stokes approximation for different longitudinal acceleration parameters $\lambda^*$. Equation of state $\varepsilon = 3\rho_s$ shear viscosity ratio $\eta/s = 1/4\pi$. Black solid curve is the ideal Bjorken flow for reference, blue solid curve is the 1st order Bjorken flow. Left panel: The proper time $\tau$ evolution of temperature for $\eta_s = 0$. Right panel: The space rapidity $\eta_s$ evolution of temperature for $\tau = 2$ fm/c. Results from the perturbation results Eq.(16).

Fig. 2 show the completely temperature evolution for different longitudinal acceleration parameter $\lambda^*$ in the Israel-Stewart approximation. In the left panel of Fig. 2 shows the time-dependence of the temperature, one finds no peak at the early time of $T$, the first order theory significantly underpredicts the work done during the expansion relative to the Israel-Stewart approximation. One also finds the effect of viscous compensates the effect from longitudinal acceleration when $\eta/s = 1/4\pi$ and $\lambda^* = 0.05$ at larger proper time of evolution, the viscous curve (red dashed) almost overlaps with the Bjorken flow (black solid). The longitudinal expanding effect make the medium cool down fast and there is no peak at early time in $T$. In the right panel of Fig. 2 shows the temperature distribution in $(\tau, \eta_s)$ coordinates with $\lambda^* = 0.1$.

![Diagram showing temperature evolution](image2)

**FIG. 2:** (Color online) Temperature profile in the Israel-Stewart approximation. Left panel: The proper time $\tau$ evolution of temperature for $\eta_s = 0$. Black solid curve is the ideal Bjorken flow for reference. Right panel: The space-time evolution of temperature in $(\tau, \eta_s)$ coordinates, the longitudinal acceleration parameter $\lambda^* = 0.1$. Numerical results from Eqs. (12, 13).

So far our focus has been on study the temperature evolution of perturbation solutions through the Navier-Stokes theories and Israel-Stewart theories independently. Now we analyze the difference between these two theories under the same longitudinal acceleration effect. We numerically solve the differential equations Eqs. (17, 18) together with the initial condition
$T_0(0.2, 0.0) = 0.65$ GeV first. In the left panel of Fig. 3 show the comparison of the second order perturbation solutions, the first order perturbation solutions and Bjorken solution. In the right panel of Fig. 3 show the comparison between the second order perturbation solutions and the completely numerical results. For small $\lambda^*$, we find that the perturbation solutions are stable and show good agreement with completely numerical results at large time.

![Graph showing perturbation and Bjorken solutions](image)

**FIG. 3:** (Color online) The proper time evolution of temperature density for given primary initial conditions. Left panel: Perturbation results of temperature profile in the Navier-Stokes approximation (1st) and in the Israel-Stewart theory (2nd). The longitudinal acceleration parameter $\lambda^* = 0.10$. Black solid curve is the ideal Bjorken flow for reference. Right panel: Temperature profile comparison between the completely solution (solid curve) with perturbation solutions (dashed curve) in the Israel-Stewart theory for different $\lambda^*$, the grid of $\tau$ is $\Delta \tau = 0.05$ in the numerical code.

**IV. SUMMARY**

We have investigated the relativistic viscous hydrodynamics for longitudinal expanding fireballs in terms of the Navier-Stokes theory and Israel-Stewart theory by embedding 1+1 D fluid into a 1+3 D space-time. The results obtained in this paper are summarized as follows.

1. We expand the current knowledge of accelerating hydrodynamics [16, 18, 22] by including the second-order viscous corrections in the relativistic hydrodynamics fluid with longitudinal expanding fireballs and general equation of state. The effect of longitudinal acceleration accelerates the thermodynamics evolution of medium while the viscosity effect decelerates the evolution in the Minkowski space-time.

2. The perturbation solution from the Navier-Stokes approximation is explicit and simple in mathematical structure, and it is consistent with the results from Refs.[38] and valid in leading order accuracy of the longitudinal acceleration parameter $\lambda^*$. The temperature distribution here indicates a Gaussian shape in the $\eta$ direction.

3. For small perturbations along the longitudinal directions, as we presented in Fig. 1, the perturbation solution from the Navier-Stokes approximation is stable in region $\tau \geq \tau_0$ while it is unstable in region $\tau \leq \tau_0$.

4. The numerical results from the Israel-Stewart approximation in longitudinal expansion relativistic viscous hydrodynamics solve the causal problem and the temperature profile in the Rindler coordinate are presented.

There are still many open questions about such perturbation solutions and results.

1. For consistency and stability, the perturbation solution is meaningful when $\lambda^* \ll 1$, for arbitrary longitudinal acceleration parameter $\lambda^*$, e.g. $|\lambda^*| \gg 1$, such perturbation approximation become unsuitable and we need to solve the differential equations completely by other numerical method, such as 3+1D CLvisc [33]. (2) The shear pressure tensor relaxation time $\tau_\pi$ assumed above for the second theory is definitely oversimplified, and it is only valid for smaller values of $\tau_\pi$. While physically motivated, we acknowledge that this method is imperfect. (3) Transverse expansion cannot be neglected, especially during the later stages of the fireball, significantly changing the observables at RHIC and LHC. In reality the expansion of the system will not be purely longitudinal, the system will also expand transversally [25, 28]. (4) It is important to note that the QGP bulk viscosity ratio $\zeta/s$ is not zero from the lattice QCD calculation, the effect of bulk viscosity property play a curial role when temperature is larger than $3T_c$ [48, 49]. Recently, new solutions of first order viscous hydrodynamics for Hubble-type flow are presented to study the bulk viscosity [47], however, the second order theory of such fluid is still unknown. (5) In principle, the second order approximation should depend on a larger number of independent transport coefficient, e.g. $\tau_\eta$, $\tau_\pi$, $\lambda_1$, $\lambda_2$, $\lambda_3$, and, the direction extension results Eqs. (8, 9) from the $\partial_\mu s^\mu \geq 0$ are, in fact, incomplete and ad-hoc. In order to determine these transport coefficients, microscopic
theories, such as kinetic theory should be studied [50]. (6) Chapman-Enskog expansion and completely Grid’s 14-moment methods [51] could be used to study the higher order correction. (7) Recently, Duke group presented a novel method to study the effective viscosities [44], which points a new way to study the shear viscosity and bulk viscosity for QGP. As a next step, we try to study above parts in more accurate studies in the future.

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