Forward virtual Compton scattering and the Lamb shift in chiral perturbation theory

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Abstract

We compute the spin-independent structure functions of the forward virtual-photon Compton tensor of the proton at one loop using heavy baryon chiral perturbation theory and dispersion relations. We study the relation between both approaches. We use these results to generalize some sum rules to virtual photon transfer momentum and relate them with sum rules in deep inelastic scattering. We then compute the leading chiral term of the polarizability correction to the Lamb shift of the hydrogen and muonic hydrogen. We obtain $-87.05/n^3 \text{Hz}$ and $-0.148/n^3 \text{meV}$ for the correction to the hydrogen and muonic hydrogen Lamb shift respectively.
1 Introduction

The knowledge of the forward virtual-photon Compton tensor of the proton, \( T^{\mu\nu} \), for energies of the order of the pion mass allows one to check chiral perturbation theory in the (but not deep) Euclidean region \( (Q^2 \equiv -q^2 \sim m_\pi^2 \neq 0) \). This is interesting by itself, since it allows us to test whether chiral perturbation theory computations work equally well in the Euclidean or Minkowski regime. Moreover, the expressions obtained for the structure functions permit one to generalize sum rules typically derived for real photons \( (q^2 = 0) \) to virtual photons in the Euclidean regime \( (q^2 \sim -m_\pi^2 < 0) \). These results provide us with a more direct connection with perturbation theory calculations, which are valid for the very same Euclidean sum rules but at large \( Q^2 \). This idea has already been addressed in the case of the spin-dependent structure functions in Ref. [1]. It is the main motivation of this paper to obtain the spin-independent generalized sum rules. To do so we will compute the spin-independent structure functions at one loop in heavy baryon chiral perturbation theory [2, 3].

These results are also a necessary step in the complete determination of the leading chiral-related polarizability corrections to the Lamb shift of the hydrogen and muonic hydrogen [1]. These corrections are model independent and may yield accurate determinations of the proton radius provided the muonic hydrogen Lamb shift is measured with high enough accuracy [5]. So far only some partial results have been obtained [6]. In this paper we fill this gap by obtaining the complete expression for the leading correction of the polarizability effect to the Lamb shift.

We will then compute \( T^{\mu\nu} \) using Feynman rules and study its relation with dispersion relation computations in depth. We will find this computation to be a useful exercise to illustrate the problems and advantages of working with time-ordered correlators versus its non-ordered counterparts. We will clearly see that the derivation of correct dispersion relations may turn out to be more difficult than expected, and, in particular, that the seagull terms are necessary in order to restore current conservation.

The expressions for the structure functions of \( T^{\mu\nu} \) can be understood as generalized sum rules for \( Q^2 \neq 0 \). By expanding around \( \rho \equiv q \cdot p/m_p = 0 \), we can relate our results with the sum rules in the deep Euclidean region. We will discuss the connection between the chiral and perturbative regime but a quantitative analysis will be relegated to future work.

The structure of the paper is the following. In sec. 2 we define the tensors and form factors we use in our computation. In sec. 3 we compute \( W^{\mu\nu} \) and \( T^{\mu\nu} \) using Feynman rules. In sec. 4 we compute \( T^{\mu\nu} \) using dispersion relations and compare with the previous result. In sec. 5 we analyze the resulting sum rules. In sec. 6 we compute the polarizability corrections to the Lamb shift. In sec. 7 we give our conclusions. In the Appendix some contributions to \( T^{\mu\nu} \) due to the \( \Delta \) particle are computed.

\(^1\)In the same way that the spin-dependent structure functions are necessary for the hyperfine splitting of the hydrogen and muonic hydrogen [4].
2 Definitions

We first define the form factors, which we will understand as pure hadronic quantities, i.e. without electromagnetic corrections.

We define \( J_\mu = \sum_i Q_i \bar{q}_i \gamma^\mu q_i \), where \( i = u, d \) (we will not consider the strange quark in this paper) and \( Q_i \) is the quark charge. The form factors are then defined by the following equation:

\[
\langle p', s | J_\mu | p, s \rangle = \bar{u}(p') \left[ F_1(q^2) \gamma_\mu + i F_2(q^2) \frac{\sigma_\mu_\nu q_\nu}{2m_p} \right] u(p),
\]

where \( q = p' - p \) and \( F_1, F_2 \) are the Dirac and Pauli form factors, respectively. The states are normalized in the following (standard relativistic) way:

\[
\langle p', \lambda' | p, \lambda \rangle = (2\pi)^3 2p^0 \delta^3(p' - p) \delta_{\lambda' \lambda},
\]

and

\[
u(p, s) \bar{u}(p, s) = \frac{1}{\rho} + \frac{\gamma_5 \rho}{2},
\]

where \( s \) is an arbitrary spin four vector obeying \( s^2 = -1 \) and \( p \cdot s = 0 \).

The form factors could be (analytically) expanded as follows

\[
F_i(q^2) = F_i + \frac{q^2}{4m_p^2} F'_i + \ldots
\]

for very low momentum. Nevertheless, we will be interested instead in their (non-analytic) behavior in \( q \sim m_\pi \).

We also introduce the Sachs form factors:

\[
G_E(q^2) = F_1(q^2) + \frac{q^2}{4m_p^2} F'_2(q^2), \quad G_M(q^2) = F_1(q^2) + F_2(q^2).
\]

We also define

\[
W^{\mu \nu} = \frac{1}{4\pi} \int d^4x \ e^{iq \cdot x} \langle p, s | [J_\mu(x), J_\nu(0)] | p, s \rangle,
\]

which has the following structure (\( \rho = q \cdot p/m_p \) and we take the sign convention \( \epsilon^{0123} = 1 \)):

\[
W^{\mu \nu} = \left( -g^{\mu \nu} + \frac{q^\mu q^\nu}{q^2} \right) W_1(\rho, q^2) + \frac{1}{m_p^2} \left( p^\mu - \frac{m_\rho q^\mu}{q^2} \right) \left( p^\nu - \frac{m_\rho q^\nu}{q^2} \right) W_2(\rho, q^2)
\]

\[
- \frac{i}{m_p} \epsilon^{\mu \nu \rho \sigma} q_\rho s_\sigma G_1(\rho, q^2) - \frac{i}{m_p^3} \epsilon^{\mu \nu \rho \sigma} q_\rho ((m_\rho s_\sigma - (q \cdot s)p_\sigma) G_2(\rho, q^2) = W^{\mu \nu}_S + W^{\mu \nu}_A,
\]

where \( W^{\mu \nu}_S = W^{\nu \mu}_S \) and \( W^{\mu \nu}_A = -W^{\nu \mu}_A \). Therefore, the first two (spin-independent) terms correspond to \( W^{\mu \nu}_S \) and the last two (spin-dependent) terms correspond to \( W^{\mu \nu}_A \).

We will also need the forward virtual-photon Compton tensor

\[
T^{\mu \nu} = i \int d^4x \ e^{iq \cdot x} \langle p, s | T \{ J_\mu(x), J_\nu(0) \} | p, s \rangle,
\]
which has the following structure \((\rho = q \cdot p/m_p \equiv v \cdot q\) although we will usually work in the rest frame where \(\rho = q^0\)):

\[
T^{\mu\nu} = \left( -g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) S_1(\rho, q^2) + \frac{1}{m_p^2} \left( p^\nu - \frac{m_p \rho}{q^2} q^\mu \right) \left( p^\nu - \frac{m_p \rho}{q^2} q^\nu \right) S_2(\rho, q^2) \\
- \frac{i}{m_p} \epsilon^{\mu\nu\rho\sigma} q_\rho s_\sigma A_1(\rho, q^2) - \frac{i}{m_p^2} \epsilon^{\mu\nu\rho\sigma} q_\rho \left( (m_p \rho) s_\sigma - (q \cdot s) p_\sigma \right) A_2(\rho, q^2),
\]

(2.9)
depending on four scalar functions, which we call structure functions. Similarly to \(W^{\mu\nu}\), we can split the tensor in the symmetric, \(T_\text{sym}^{\mu\nu} = T_\text{sym}^{\nu\mu}\), and antisymmetric piece, \(T_\text{asym}^{\mu\nu} = -T_\text{asym}^{\nu\mu}\). Moreover, it is also usual to split these tensors in two components, which we label \(\text{Born}\) and \(\text{pol.}\), for instance, \(T^{\mu\nu} = T_\text{Born}^{\mu\nu} + T_\text{pol.}^{\mu\nu}\). Each of these terms complies with current conservation\(^2\). The Born term is the contribution coming from the intermediate state being the proton (somewhat the elastic contribution) and the associated structure functions can be written in terms of the form factors. They read

\[
S_1^{\text{Born}}(\rho, q^2) = -2 F_1^2(q^2) - \frac{2(q^2)^2 G_M^2(q^2)}{(2m_p)^2 - (q^2)^2}, \quad (2.10)
\]

\[
S_2^{\text{Born}}(\rho, q^2) = 2 \frac{4m_p^2q^2 F_1^2(q^2) - (q^2)^2 F_2^2(q^2)}{(2m_p)^2 - (q^2)^2}, \quad (2.11)
\]

\[
A_1^{\text{Born}}(\rho, q^2) = -F_2^2(q^2) + \frac{4m_p^2q^2 F_1(q^2) G_M(q^2)}{(2m_p)^2 - (q^2)^2}, \quad (2.12)
\]

\[
A_2^{\text{Born}}(\rho, q^2) = \frac{4m_p^4F_2(q^2)G_M(q^2)}{(2m_p)^2 - (q^2)^2}. \quad (2.13)
\]

From these expressions one can easily single out the point-like contributions (those due to a point-like particle but with non-trivial anomalous dimension, radius, ...). The remaining contributions, with the one-loop accuracy of our chiral computation, are encoded in the following expression (we split \(G_{E,M}\) into pieces according to its chiral counting: \(G_{E,M}^{(n)} \sim 1/m_p^n \sim 1/\Lambda^n\)):

\[
T_\text{Born}^{\mu\nu} = i\pi \delta(v \cdot q) \times T \left[ u \bar{u} \left( -4p_+ G_E^{(0)} G_E^{(2)} \gamma^\mu \gamma_\nu + \frac{2m_p}{G_E} G_E^{(0)} G_M^{(1)} \left( v_\mu p_+ \left[ s_\nu, s_\sigma \right] q_\rho p_+ - v_\nu p_+ \left[ s_\mu, s_\sigma \right] q_\rho p_+ \right) \right) \right],
\]

(2.14)

where \(p_+ = \frac{1+v \cdot q}{2}\). Note that \(T_\text{Born}^{\mu\nu}\) is proportional to \(\delta(v \cdot q)\). \(G_E^{(0)} = 1\). The expressions for \(G_E^{(2)}\), \(G_M^{(1)}\) can be found in Refs. \([9, 10, 11]\). For the spin-dependent case, the only contribution is the term proportional to \(G_E^{(0)} G_M^{(1)}\), which comes from the \(A_1^{\text{Born}}\) term (this is the only term that contributes to the Zemach contribution to the hyperfine splitting). For the spin-independent case, on which we focus in this paper, we would only need \(G_E^{(2)}\).

Note that some point-like contributions are still encoded in \(G_E^{(2)}\) and \(G_M^{(1)}\). Nevertheless, it should also be stressed that, in many situations, only the non-analytic behavior in the momentum of the structure functions is really relevant. In those cases it is not necessary to

\(^2\) The separation (definition) of the \(\text{Born}\) and \(\text{pol.}\) terms is in general ambiguous, see Refs. \([7, 8]\) for an extensive discussion on this point. In our case, as far as we give an explicit definition for \(T_\text{Born}^{\mu\nu}\), this ambiguity disappears.

\(^3\) In Ref. \([4]\) \(G_M^{(1)}\) was labeled \(G_M^{(2)}\), and is meant to express one loop in chiral perturbation theory.
obtain the complete expressions for the structure functions, and, in particular, counterterms can be avoided. This comment applies to the computation of the non-analytic behavior in the light quark masses and in the splitting between nucleon and the Delta mass (proportional to powers of $1/N_c$ in the large $N_c$ limit) of the Lamb shift (or hyperfine splitting) from the Sachs form factors. This is because only the non-analytic behavior of the structure function gives a non-zero contribution to the dimensional regularized one-loop computation.

In what follows we consider the computation of $T^{\mu\nu}_{S,\text{pol.}}$, which is novel.

### 3 Computation of $T^{\mu\nu}$ and $W^{\mu\nu}$

We will restrict our analysis to the SU(2) flavour case and neglect the $\Delta$ particle. We can obtain a compact expression for the polarizability contribution to $W^{\mu\nu}$ at tree level (for simplicity we set $\nu^\mu = (1,0)$):

$$W^{\mu\nu}_{S,\text{pol.}} = 2m_p \left[ \frac{e g_A}{2f_\pi} \right]^2 \int \frac{d^3p_\pi}{2E_\pi(2\pi)^3} \delta(E_\pi - q^0)$$

\begin{equation}
\times \left\{ g^{\mu i} \left( \frac{g^{\nu 0} p^i_\pi}{q^0} + \frac{k^i (q - 2k)^\mu}{k^2 - m_\pi^2 + i\eta} \right) \right\} \left\{ g^{\nu i} \left( \frac{g^{\mu 0} p^i_\pi}{q^0} + \frac{k^i (q - 2k)^\nu}{k^2 - m_\pi^2 + i\eta} \right) \right\},
\end{equation}

where $E_\pi = \sqrt{p_\pi^2 + m_\pi^2}$, $k = (0, k) = (0, q - p_\pi)$, $k = q - p_\pi$.

It is easy to check that the above expression satisfies current conservation. From this result one could try to obtain $W_1$ and $W_2$ directly, instead we will obtain them later from the imaginary part of $S_i$.

The expressions we obtain for $S_1$ and $S_2$ using chiral perturbation theory read

$$S^{\text{pol.}}_1 = \frac{1}{\pi} \left( \frac{g_A}{2f_\pi} \right)^2 m_p m_\pi \left\{ \frac{3}{2} \frac{m_\pi^2}{q^2} - \left( 1 + \frac{m_\pi^2}{q^2} \right) \sqrt{1 - z} \right\}$$

\begin{equation}
- \frac{1}{2} \sqrt{\frac{m_\pi^2}{q^2}} \left( 2 + \frac{q^2}{q^2} \right) I_1 (m_\pi^2, q^0, q^2),
\end{equation}

where

$$z = \frac{(q^0)^2}{m_\pi^2}$$

and

$$I_1 (m_\pi^2, q^0, q^2) = \int_0^1 dx \frac{1}{\sqrt{\frac{m_\pi^2}{q^2} - \frac{q^2}{q^2} x - x^2}}$$

\begin{equation}
= - \arctan \left( \frac{q^2}{2m_\pi|q|} \right) + \arctan \left( \frac{2q^2 + q^2}{2|q| \sqrt{m_\pi^2 - q_0^2}} \right)
\end{equation}
\[ S_{2}^{\text{pol.}} = \frac{1}{\pi} \left( \frac{g_{A}}{2f_{\pi}} \right)^{2} m_{\pi} m_{\pi} \frac{q^{2}}{q^{2}} \left\{ -\left( \frac{3}{2} + \left( \frac{1}{2} + \frac{m_{\pi}^{2}}{q^{2}} + \frac{m_{\pi}^{2}}{(q^{0})^{2}} \right) \frac{q^{2}}{q^{2}} \right) \right. \]

\[ - \frac{(q^{0})^{2} q^{2}}{4m_{\pi}^{2}q^{2} + (q^{2})^{2}} \left( \frac{m_{\pi}^{2}}{q^{2}} - \frac{q^{2}}{2q^{2}} \right) \]

\[ + \frac{m_{\pi}^{2}}{q^{2}} \left( 2 - \frac{q^{2}}{(q^{0})^{2}} (1 - z) + \frac{q^{2} (q^{0})^{2}}{4m_{\pi}^{2}q^{2} + (q^{2})^{2}} \right) \sqrt{1 - z} \]

\[ + \frac{1}{2} \sqrt{\frac{m_{\pi}^{2}}{q^{2}}} \left( 2 + 3 \frac{q^{2}}{q^{2}} + \frac{q^{2}}{m_{\pi}^{2}} \right) \mathcal{I}_{1} \left( m_{\pi}^{2}, q^{0}, q^{2} \right) \right\}. \]

We have obtained these results by computing the spin independent part of \( T^{ij} = A\delta^{ij} + Bq^{i}q^{j} \) from the diagrams shown in Fig. 1. We note that only inelastic diagrams contribute to \( T^{ij} \). \( T^{00} \) has been obtained afterwards using current conservation.

\[ T^{00} = \frac{q^{i}q^{j}T^{ij}}{q_{0}^{2}}. \]

Note that this relation may miss singular terms (delta-type) in \( q^{0} \). Those are precisely the ones that appear in the Born term (actually current conservation holds for both the born and polarizability corrections independently). Polarizability terms should not have \( \delta(q^{0}) \) terms. From \( T^{00} \) and \( T_{\mu}^{\mu} \), we can then reconstruct \( S_{i}^{\text{pol.}} \), obtaining the results above.

Figure 1: Diagrams contributing to \( T^{ij} \). Crossed diagrams are not explicitly shown but calculated.

\[ \text{Figure 1: Diagrams contributing to } T^{ij}. \text{ Crossed diagrams are not explicitly shown but calculated.} \]

\[ \text{We have actually checked that this relation also holds by a direct computation of } T^{00} \text{ using Feynman diagrams.} \]
4 Dispersion relations

We now try to reconstruct $T_{\mu\nu}$ from $W_{\mu\nu}$ using dispersion relations. In principle, this procedure could be convenient in chiral computations, since it reduces the number of diagrams to be considered, because $W_{\mu\nu}$ can be obtained with one order less. Nevertheless, in our case this difference is not significant.

The structure functions of $W_{\mu\nu}$ are the ones that appear in $ep$ inelastic scattering. We can relate them to the structure functions of $T_{\mu\nu}$ by using its spectral decomposition, which relates the imaginary term of $T_{\mu\nu}$ to $W_{\mu\nu}$:

$$W_{\mu\nu}(q)|_{q^0>0} = \frac{1}{2\pi}\Im T_{\mu\nu}(q)|_{q^0>0}.$$  

We can use this equality to obtain $W_1$ and $W_2$. They read

$$W_1^{\text{pol.}}(q^0, q^2) = \frac{1}{2\pi}\Im S_1^{\text{pol.}} = \theta(q_0^2 - m^2) \frac{1}{2\pi^2} \left( \frac{g_A}{2f_{\pi}} \right)^2 m_p m_\pi \left\{ \left( 1 + \frac{m^2}{q^2} \right) \sqrt{q_0^2 - m^2} - \frac{1}{4} \sqrt{\frac{m^2}{q^2}} \left( 2 + \frac{q^2}{q_0^2} \right) \ln \left[ \frac{q_0^2 + q^2 + \sqrt{4q^2(q_0^2 - m^2)}}{q_0^2 + q^2 - \sqrt{4q^2(q_0^2 - m^2)}} \right] \right\},$$

$$W_2^{\text{pol.}}(q^0, q^2) = \frac{1}{2\pi}\Im S_2^{\text{pol.}} = \theta(q_0^2 - m^2) \frac{1}{2\pi^2} \left( \frac{g_A}{2f_{\pi}} \right)^2 m_p m_\pi \frac{q^2}{q^2}$$

$$\times \left\{ -\frac{m^2}{q^2} \left( 2 - \frac{q^2}{q_0^2} \left( 1 - \frac{q_0^2}{m^2} \right) + \frac{q^2 q_0^2}{4m^2 q^2 + (q^2)^2} \right) \sqrt{q_0^2 - m^2} + \frac{1}{4} \sqrt{\frac{m^2}{q^2}} \left( 2 + \frac{q^2}{q_0^2} + \frac{q^2}{m^2} \right) \ln \left[ \frac{q_0^2 + q^2 + \sqrt{4q^2(q_0^2 - m^2)}}{q_0^2 + q^2 - \sqrt{4q^2(q_0^2 - m^2)}} \right] \right\}. \tag{4.22}$$

On the other hand, by using the spectral decomposition (naively, without caring about convergence), we obtain the following equalities:\footnote{The expressions for $T_{\mu\nu}$ obtained from Feynman rules and dispersion relations may differ. Therefore, we label those obtained with dispersion relations with the label $DR$.}

$$T_{S,DR}^{0\mu}(q) = \frac{1}{2}(T_{DR}^{0\mu}(q) + T_{DR}^{0\mu}(q)) = 4 \int_{q_{0,th}}^{\infty} dq'_0 q'_0 W_{S,DR}^{0\mu}(q', q^2) \frac{W_{S,DR}^{0\mu}(q', q^2)}{q'_0 - q_0 - i\epsilon},$$ \tag{4.24}$$

and

$$T_{S,DR}^{\mu\nu}(q) = \frac{1}{2}(T_{DR}^{\mu\nu}(q) + T_{DR}^{\mu\nu}(q)) = 2 \int_{q_{0,th}}^{\infty} dq'_0 q'_0 \frac{W_{S,DR}^{\mu\nu}(q', q^2)}{q'_0 - q_0 - i\epsilon}. \tag{4.25}$$

otherwise. It should be stressed that the equalities are derived at $q^2 = \text{constant}$. This is quite different from the standard dispersion relations that are derived at $Q^2 = -q^2 = \text{constant}$. The latter are derived using the Cauchy theorem and the assumed analytic properties of the structure functions. In principle, the same results should be obtained in both approaches (up to, maybe, some polynomials in $q^0$).

Using Eq. (4.25), the first four (plus the crossed ones) diagrams in fig. 1 can be obtained using dispersion relation techniques on a diagram-by-diagram basis. If we want to obtain the
complete result using dispersion techniques the last diagram in fig. [1] is missing. This could be expected because, in general, the use of dispersion relations reproduces the full result up to, may be, some local terms, polynomials in \( q^0 \), which are usually called seagull terms. The last diagram can be obtained by direct computation (being a constant, it can also be fixed from the normalization constant of \( A \) (which is momentum independent) in the limit of real photons and \( q^0 \to 0 \), since this limit is known.). Overall we find

\[
T_{S}^{ij} = T_{S,DR}^{ij} + T_{\text{seagull}}^{ij},
\]

(4.26)

where

\[
T_{\text{seagull}}^{\mu\nu} = -g^{\mu\nu} \frac{3}{2\pi} \left( \frac{g_A}{2f_\pi} \right)^2 m_p m_\pi.
\]

(4.27)

For the "00" component we find (we do not consider \( \delta(q^0) \) contributions, which are encoded in the Born term)

\[
T_{S}^{00} = T_{S,DR}^{00}.
\]

(4.28)

In this case there is not a one-to-one correspondence between the diagrams obtained from Feynman rules and the dispersion relation analysis. Note also that current conservation is lost if we only consider the result obtained from the dispersion analysis. The seagull term is needed for the "ij" component, whereas for the "00" component the seagull term cancels the no one-to-one correspondence between the Feynman diagrams and the dispersion relation computation.

5 Generalized Sum Rules

We have checked that in the limit of real photons \( (q^2 = 0) \) our expressions for \( S_{i}^{\text{pol.}} \) agree with those obtained in Ref. [9]. Obviously in the limit of \( q^0 = |q| \to 0 \), we also obtain the polarizabilities, \( \alpha_E \) and \( \beta_E \), which correspond to some specific sum rules.

We can now take the opposite limit: \( \rho \to 0 \) with \( Q^2 \) constant. This limit allows us a closer connection with the sum rules that appear in deep inelastic scattering. If we define the moments of deep inelastic sum rules as (we recall that \( x = Q^2/(2p \cdot q) \))

\[
M_N^{(i)}(Q^2) = \int_0^1 dx x^{N-1} W_i(x, Q^2),
\]

(5.29)

then the structure functions read (due to the symmetry of the structure functions only moments with even \( N \) appear in our case)

\[
S_i(\rho, Q^2) = \sum_{N=0,2,4,...} \rho^N S_i^{(N)}(Q^2),
\]

(5.30)

where

\[
S_i^{(N)}(Q^2) = 4 \int_{Q^2/(2m_p)}^\infty \frac{d\rho}{\rho^{N+1}} W_i(\rho, Q^2) = \left( \frac{2m_p}{Q^2} \right)^N M_N^{(i)}(Q^2).
\]

(5.31)

Let us note that in the standard derivation of Eq. (5.31) one assumes that one can neglect the behavior at infinity. In our case we have an expression for \( S_i(\rho, Q^2) \), which we can Taylor
expand around $\rho = 0$ to obtain expressions for $M_N^{(i)}(Q^2)$ at $Q^2 \sim m_n^2$. Therefore, $S_i(\rho, Q^2)$ by itself can be understood as a generalized sum rule, where the behavior at $\rho \sim Q^2 \sim m_n^2$ of $S_i(\rho, Q^2)$ is dictated by the integral of $W_i(\rho', Q^2)$ over all values of $\rho'$ weighted by the function $1/(\rho^2 - \rho'^2)$. Actually, it is argued in Ref. [1] that, in general, it is more convenient to directly work with $S_i(\rho, Q^2)$ rather than with the moments because of the existence of the inelastic thresholds. Either way our predictions can be tested with experiment through dispersion relations. Note as well that this is not necessarily equivalent to checking with experiment their imaginary parts at low energies, because the sum rules involve the integration over an arbitrary large momentum. Therefore, the naive use of Eq. (5.31) with the chiral expressions obtained in Eqs. (4.22) and (4.23) for $W_i$ may yield wrong results. This is because of the not fast enough convergence of $W_i(\rho, Q^2)$ at infinity, which does not allow us to neglect its contribution there (nevertheless one can do so if enough derivatives are taken, but then one has to fix the low energy unknown constants).

Eq. (5.29) allows us to relate in a single quantity perturbation theory and the chiral expansion. At large $Q^2$, one can use perturbation theory to know their scaling with $Q^2$ (for a recent state of the art see [12], for instance). At low $Q^2$ one can use the chiral expressions obtained here (the combination of the elastic and inelastic terms).

Note that in Eq. (5.31) both the elastic and inelastic contribution have to be added by definition. At large $Q^2$ the dominant contribution comes from the inelastic term. Therefore, this one should match with the perturbative behavior at large $Q^2$. On the other hand, our computation holds at small $Q^2 \sim m_n^2 \ll \Lambda_\chi$. In this situation the largest contribution may come from the elastic term. Either way one may consider sum rules for elastic or inelastic terms and compare them with the chiral predictions. One example could be the Baldin sum rule, which corresponds to the sum of the polarizabilities: $\alpha_E + \beta_E$, and can be obtained from dispersion relations. The generalized Baldin sum rule would correspond (up to normalization) to

$$M_2^{(1)}(Q^2) - (\text{elastic term}).$$

(5.32)

For a discussion on the generalized Baldin sum rule see, for instance, Refs. [13, 14]. In this paper we stop here, and relegate the quantitative analysis of the sum rules, including their known behavior at low (chiral) and high (perturbation theory) energies, as well as a comparison with experiment, to future work.

6 Lamb shift

An important application of our result is the evaluation of the chiral-related polarizability effects to the Lamb shift of the hydrogen and muonic hydrogen. As we have already mentioned in the introduction, this result is of special importance in the case of the muonic hydrogen, since the measurement of the muonic hydrogen Lamb shift may yield a very precise determination of the proton radius. At this respect polarizability effects appear as one of the main sources of uncertainty. Therefore, a precise (and model independent) determination of their value can significantly pin down the errors of the proton radius. Here we provide this value for the purely chiral-related effects.

We refer to [6] (see also [4]) for further details, here we only quote the main formulas we need for the computation. In that reference, the polarizability effects were encoded in a
matching coefficient named $c_{3,NR}^{pl}$. The expression for $c_{3,NR}^{pl}$ can be passed to the Euclidean, which then reads

$$
c_{3,NR}^{pl} = -e^4 m_pm_l \int \frac{dk_E}{(2\pi)^4} \frac{1}{k_E^4} \frac{1}{k^4} + 4m_i^2 k_{0,E}^2 \times \left\{ (3k_{0,E}^2 + k^2)S_1(ik_{0,E}, -k_E^2) - k^2 S_2(ik_{0,E}, -k_E^2) \right\},
$$

(6.33)

where $m_l$ is the mass of the lepton.

Using the explicit expressions for $S_i^{pol}$ obtained in Eqs. (3.16) and (3.19) one could obtain the chiral contribution to $c_{3,NR}^{pl}$ due to the polarizability effects. In particular, it is convenient to rewrite $c_{3,NR}^{pl}$ in the following manner, which is amenable for a numerical analysis:

$$
c_{3,NR}^{pl} = -e^4 m_2^2 m_l \left( \frac{g_A}{f_\pi} \right)^2 \int \frac{d^{D-1}k_E}{(2\pi)^{D-1}} \frac{1}{(1 + k^2)^4} \int_0^\infty \frac{dw}{\pi} w^{D-5} \frac{1}{w^2 + 4m_i^2 m_\pi^2 (1 + k^2)^2} \times \left\{ (2 + (1 + k^2)^2)A_E(w^2, k^2) + (1 + k^2)^2 k^2 w^2 B_E(w^2, k^2) \right\},
$$

(6.34)

where (for $D = 4$)

$$
A_E = -\frac{1}{4\pi} \left[ -\frac{3}{2} + \sqrt{1 + w^2} + \int_0^1 dx \frac{1 - x}{\sqrt{1 + x^2 w^2 + x(1 - x) w^2 k^2}} \right],
$$

(6.35)

$$
B_E = \frac{1}{8\pi} \left[ \int_0^1 dx \frac{1 - 2x}{\sqrt{1 + x^2 w^2 + x(1 - x) w^2 k^2}} - \frac{1}{2} \int_0^1 dx \frac{(1 - x)(1 - 2x)^2}{(1 + x^2 w^2 + x(1 - x) w^2 k^2)^{3/2}} \right].
$$

The functions $A_E$ and $B_E$ correspond to the Euclidean version of the functions $A$ and $B$ that appear in $T^{ij} = A\delta^{ij} + Bk^i k^j$ up to a normalization factor and some momentum rescaling.

Note that Eq. (6.34) is finite. Nevertheless, one has to be careful in the computation, since the contributions proportional to $A_E$ and $B_E$ are not finite by themselves and a cancellation of infinities needs to take place between both terms.

The limit of small lepton mass agrees (with logarithmic accuracy) with the result of Ref. [6] in Eq. (51), which could be rewritten in terms of the polarizabilities [15, 16].

Following again Ref. [6], the contribution to the potential coming from the polarizability correction reads

$$
\delta V = -\frac{e^4 m_2^2 m_l}{m_\pi^2} (3)(r),
$$

(6.36)

and its contribution to the Lamb shift reads ($\langle \rangle$ stands for the average over polarizations)

$$
\delta E(n) = \langle E(n, l = 0) - E(n, l = 1) \rangle = -\frac{e^4 m_2^2 m_l}{m_\pi^2} \delta_{00} \frac{1}{n} \left( \frac{\mu_{l,p} \alpha}{n} \right)^3,
$$

(6.37)

where $\mu_{l,p} = m_l m_p / (m_l + m_p)$.
We can now put some numbers. For the case of Hydrogen we obtain

$$
\delta E_{ep}^{\text{pol.}}|_{\Delta \to \infty} = -\frac{87.0488}{n^3} \text{ Hz}.
$$

(6.38)

This number should be compared with the number obtained in the logarithmic approximation in Ref. [6]:

$$
\delta E_{ep,\text{log.}}^{\text{pol.}}|_{\Delta \to \infty} = -\frac{64.4841}{n^3} \text{ Hz}.
$$

We can see that the logarithmic approximation works reasonably well. This could be expected since the ratio $m_e/m_\pi$ is a small number, yet the finite piece is sizeable. In any case, at present, the polarizability correction is beyond the experimental accuracy of the hydrogen Lamb shift due to the $m_e/m_\pi$ suppression factor.

For the case of the muonic hydrogen we have $m_\mu/m_\pi$, which is not suppressed. Numerically, we obtain

$$
\delta E_{\mu p}^{\text{pol.}}|_{\Delta \to \infty} = -\frac{0.147614}{n^3} \text{ meV}.
$$

(6.39)

In this case the logarithmic approximation does not work. On the other hand, for the muon case, the approximation in which the mass of the muon is equal to the pion mass in $C_3^{pl, NR}$ is a good approximation:

$$
\delta E_{\mu p}^{\text{pol.}}|_{\Delta \to \infty} (m_\mu = m_\pi) = -\frac{0.132339}{n^3} \text{ meV}.
$$

(6.40)

If we compare these numbers with the Zemach and vacuum polarization hadronic contributions obtained in Ref. [6], we can see that they are more or less of the same size. This should be contrasted with the hyperfine case [4], where the Zemach correction was the largest contribution (in this last case there is not vacuum polarization correction).

Our numbers are quite similar (within the expected uncertainties) to those obtained from models or from dispersion relations with experimental data for the form factors [16, 17, 18, 19, 20].

Finally, let us stress that our results give the leading term (in the chiral counting) to the Zemach and polarizability correction. Note as well that our result is parameter free, no new counterterm is needed. Therefore, it is model independent. The correction to our expression is $O(m_\pi/\Lambda_{QCD})$ suppressed. The scale of $\Lambda_{QCD}$ is typically dictated by the next resonance that has not been integrated out, in our case the $\Delta$, or, being more specific, the proton-$\Delta$ mass difference: $m_\Delta - m_p$. This produces the largest uncertainty to our result. We expect to compute these corrections elsewhere.

7 Conclusions

We have computed the spin-independent structure functions of the forward virtual-photon Compton tensor of the proton at one loop using heavy baryon chiral perturbation theory and dispersion relations. We have studied the relation between both approaches, and used our results to generalize some sum rules to virtual photon transfer momentum and relate them with sum rules in deep inelastic scattering. We have then computed the leading
chiral corrections to the polarizability effect of the Lamb shift of the hydrogen and muonic hydrogen. We have obtained $-87.05/n^3 \text{Hz}$ and $-0.148/n^3 \text{meV}$ for the correction to the hydrogen and muonic hydrogen Lamb shift respectively.

Extensions of this work include the computation of the $\Delta$ effects, as well as repeating the computation in SU(3) (including the strange quark). On the other hand we expect to perform a quantitative application of these results to the muonic hydrogen, as well as to the generalized sum rules.

Acknowledgments
A.P. acknowledges discussions with Santi Peris and Juan Rojo. This work is partially supported by the network Flavianet MRTN-CT-2006-035482, by the spanish grant FPA2007-60275, and by the catalan grant SGR2005-00916.

A $\Delta$ contribution
It is not the aim of this paper to obtain the full set of $\Delta$-related contributions to the structure functions, which we relegate to future work, yet we can not avoid displaying a few terms (those which appear at tree level) that are easy to obtain.

For the elastic $\Delta$-dependent terms we obtain

$$W_1(q_0, q^2) = \frac{4}{9} \left( \frac{b_{1,F}}{2m_p} \right)^2 m_p \delta(q_0 - \Delta) k^2,$$

$$W_2(q_0, q^2) = -\frac{4}{9} \left( \frac{b_{1,F}}{2m_p} \right)^2 m_p \delta(q_0 - \Delta) k^2. \quad (A.41)$$

$$S_1(q_0, q^2) = \frac{16}{9} \left( \frac{b_{1,F}}{2m_p} \right)^2 q^2 \frac{m_p \Delta}{\Delta^2 - q_0^2 - i\epsilon},$$

$$S_2(q_0, q^2) = \frac{16}{9} \left( \frac{b_{1,F}}{2m_p} \right)^2 (-\Delta^2 + q^2) \frac{m_p \Delta}{\Delta^2 - q_0^2 - i\epsilon}. \quad (A.42)$$

$\Delta = m_\Delta - m_p$, $b_{1,F}$ is the $\Delta$–photon coupling, for extra details see [4].

We obtain the same result for these structure functions, either if we compute them with dispersion relations, or with Feynman rules.

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