The shape of a vertical liquid bridge between arbitrary convex surfaces in axisymmetric case

E V Galaktionov$^1$, N E Galaktionova$^2$ and E A Tropp$^1$

$^1$ Ioffe Institute, Politekhnicheskaya 26, 194021, St Petersburg, Russia
$^2$ Peter the Great St. Petersburg Polytechnic University, Politekhnicheskaya 29, 195251, St Petersburg, Russia

E-mail: evgalakt@mail.ru, nadyavk@mail.ru, tropp@mail.ioffe.ru

Abstract.

The article is devoted to the study of the lateral surface shape of a vertical liquid bridge of small volume between two arbitrary convex solid surfaces in the axisymmetric case. A variational statement of the original problem is given. The solution is found by the iteration method under the assumption that the Bond number is small. An algorithm for the iterative process is proposed, taking into account the action of gravity. It was found that the maximum number of different profiles of the lateral surface of the liquid bridge corresponding to one selected set of parameters is four. As an example, the problem of the lateral surface shape between a sphere and a plane has been solved.

1. Introduction

In recent works [1] and [2], the problem of the vertical liquid bridge shape between two solid horizontal planes was solved. Such a problem arises, in particular, when studying the liquid menisci shape formed during crystal growth according to the Stepanov method [3]. However, the crystallization front in this process, in fact, is not flat, but represents a convex surface of small curvature. As a result, the task of finding the liquid bridge shape between two arbitrary convex surfaces is relevant. Studies of liquid bridges are developing in two directions: study of the shapes of menisci and the study of their stability, moreover both numerical and asymptotic methods are applied. In the work [4], the solution was constructed as a series in powers of a small Bond number, a parametric approach was used, but the subject of research was a horizontal liquid bridge between vertical walls, that is, another geometry. In [5] the vertical liquid bridge between a sphere and a plane in the absence of gravity was studied. The existence of four different profiles of the lateral surface for a given volume, height and contact angles was found. In the work [6], a detailed bibliography of works devoted to solving problems on the shapes of liquid bridges was given. The possible applications of these studies was considered.

In this work, the problem of the lateral surface shape of a vertical liquid bridge of small volume located between two arbitrary convex solid surfaces in the axisymmetric case are solved. The effect of gravity is taken into account. It is assumed that the Bond number is a small parameter of the problem. An algorithm of the iterative process is proposed. As an example, a solution of the liquid bridge shape task between a sphere and a plane is constructed.
2. Formulation of the problem

Let us consider the case of a vertical liquid bridge in contact with two solid convex surfaces (bottom and top) (Figure 1). In view of the assumed axial symmetry, we will solve the problem of finding of the lateral surface liquid bridge profile in a cylindrical coordinate system \((r,z)\). Let the surface tensions between media are \(\sigma_{13}, \sigma_{14}, \sigma_{34}, \sigma_{23}, \sigma_{24}\), respectively. The contact region of the liquid bridge with the surface \(z = f_1(r) (f_1(0) = 0)\) (bottom) is a circle of radius \(r_1\), and with the surface \(z = \hat{h} + f_2(r) (f_2(0) = 0)\) (top) - a circle of radius \(r_2\). We denote the sought functions that describe the profiles of the lower and upper parts of the liquid bridge lateral surface as \(u_1(r)\) and \(u_2(r)\), respectively. The region separating these parts (neck) is the circle of radius \(r_s\) \((r_s \geq 0)\). Since there are no physical reasons for the appearance of sharpened bridge profile, the tangent to profile at the point with abscissa \(r = r_s\) must be vertical: \(u'_1(r_s) = -\infty, u'_2(r_s) = +\infty, u_1(r_s) = u_2(r_s)\). In addition, we assume that the contact angles \(\theta_1, \theta_2\) satisfy the following conditions: \(0 < \theta_i < \pi/2 - \arctan |f'_i(r_i)|, i = 1, 2\), then \(r_s < \min(r_1, r_2)\), and \(u_1(r)\) and \(u_2(r)\) are single-valued functions. The volume of the liquid bridge is considered fixed: \(\{u_1(r), u_2(r)\} = V\). Let us introduce the functional \(\{u_1(r), u_2(r)\}\), which includes the surface energy and the energy of the gravity force [1]. Therefore, we obtain the isoperimetric problem: find the minimum of functional \(\{u_1(r), u_2(r)\}\) provided that functional \(\{u_1(r), u_2(r)\}\) takes the constant value \(V\). In accordance with the Euler theorem on isoperimetric problems, we introduce the extended functional \(\{u_1(r), u_2(r)\} + \Lambda \{u_1(r), u_2(r)\}\) \((\Lambda\) is a Lagrange multiplier). Let us transite to a dimensionless record form of the original problem using the following scaling:

\[
\xi = r/V^{1/3}, \quad \eta_i(\xi) = u_i(r)/V^{1/3}, \quad i = 1, 2,
\]

\[
\varphi_1(\xi) = f_1(r)/V^{1/3}, \quad \varphi_2(\xi) = \hat{h}/V^{1/3} + f_2(r)/V^{1/3} = h + \varphi_2(\xi), \quad \mu = \lambda V^{1/3}/\sigma_{34}.
\]

Here \(g\) is the acceleration of gravity, \(\rho\) is the density of the fluid. Dimensionless constant \(B\) is the Bond number.
2.1. Equations and boundary conditions

By rewriting the extended functional in dimensionless variables and performing its variation, we obtain two Euler equations (for the upper and lower branches, respectively)

\[
\frac{d}{d\xi} \left( \frac{\xi w'_1(\xi)}{\sqrt{1 + (w'_1(\xi))^2}} \right) = B\xi(w_1(\xi) - \varphi_1(\xi)) + \mu\xi, \quad \xi_* < \xi < \xi_1, \tag{1}
\]

\[
\frac{d}{d\xi} \left( \frac{\xi w'_2(\xi)}{\sqrt{1 + (w'_2(\xi))^2}} \right) = B\xi(\varphi_2(\xi) - w_2(\xi)) - \mu\xi, \quad \xi_* < \xi < \xi_2 \tag{2}
\]

and two the transversality conditions

\[
\frac{1 + \varphi'_1(\xi_1)w'_1(\xi_1)}{\sqrt{1 + (w'_1(\xi_1))^2}} = \alpha_{10}\sqrt{1 + (\varphi'_1(\xi_1))^2}, \quad \alpha_{10} = \frac{\alpha_{14} - \alpha_{13}}{\alpha_{34}}, \tag{3}
\]

\[
\frac{1 + \varphi'_2(\xi_2)w'_2(\xi_2)}{\sqrt{1 + (w'_2(\xi_2))^2}} = \alpha_{20}\sqrt{1 + (\varphi'_2(\xi_2))^2}, \quad \alpha_{20} = \frac{\alpha_{24} - \alpha_{23}}{\alpha_{34}}. \tag{4}
\]

In addition to conditions (3)-(4), there are also conditions of the bridge contact with the bottom and the top

\[
w_1(\xi_1) = \varphi_1(\xi_1), \quad w_2(\xi_2) = h + \varphi_2(\xi_2); \tag{5}
\]

the continuity condition of the liquid bridge profile in the neck

\[
w_1(\xi_*) = w_2(\xi_*); \tag{6}
\]

the conditions of verticality of the tangent in the neck

\[
w'_1(\xi_*) = -\infty, \quad w'_2(\xi_*) = +\infty; \tag{7}
\]

the condition of the volume conservation

\[
\int_{\xi_*}^{\xi_1} (w_1(\xi) - \varphi_1(\xi))\xi d\xi + \int_{\xi_*}^{\xi_2} (\varphi_2(\xi) - w_2(\xi))\xi d\xi + \int_{0}^{\xi_*} (\varphi_2(\xi) - \varphi_1(\xi))\xi d\xi = \frac{1}{2\pi}. \tag{8}
\]

3. Algorithm for solving the problem

To construct an effective algorithm for solving problem (1)-(8), we introduce the renormalization of all quantities by \( \xi_* \) as follows: a new independent variable \( \eta = \xi/\xi_* \); new sought functions \( v_1(\eta) = w_1(\xi)/\xi_* \); \( i = 1, 2 \); modified functions \( \psi_1(\eta) = \varphi_1(\xi)/\xi_* \); \( \psi_2(\eta) = h/\xi_* + \varphi_2(\xi)/\xi_* = H + \psi_2(\eta) \); modified Bond number \( b = B(\xi_*)^2 \); modified Lagrange multiplier \( M = \mu\xi_* \).

As a result, equations (1) and (2) take the form

\[
\frac{d}{d\eta} \left( \frac{\eta v'_1(\eta)}{\sqrt{1 + (v'_1(\eta))^2}} \right) = b(\psi_1(\eta) - \psi_1(\eta)\eta + M\eta, \quad 1 < \eta < \eta_1, \tag{9}
\]

\[
\frac{d}{d\eta} \left( \frac{\eta v'_2(\eta)}{\sqrt{1 + (v'_2(\eta))^2}} \right) = b(\psi_2(\eta) - v_2(\eta))\eta - M\eta, \quad 1 < \eta < \eta_2. \tag{10}
\]
Integrating these equations taking into account conditions (7), we obtain
\[
\frac{v'_1(\eta)}{\sqrt{1 + (v'_1(\eta))^2}} = \frac{1}{\eta} \left[ b \int_1^{\eta} (v_1(s) - \psi_1(s)) s \, ds + \frac{M(\eta^2 - 1)}{2} - 1 \right] = -\Phi_1(\eta),
\]
(11)
\[
\frac{v'_2(\eta)}{\sqrt{1 + (v'_2(\eta))^2}} = \frac{1}{\eta} \left[ b \int_1^{\eta} (\hat{v}_2(s) - v_2(s)) s \, ds - \frac{M(\eta^2 - 1)}{2} + 1 \right] = \Phi_2(\eta).
\]
(12)
It can be seen that \( \Phi_1(1) = \Phi_2(1) = 1 \). By solving equations (11), (12) with respect to \( v'_1(\eta), v'_2(\eta) \) and integrating the obtained equations taking into account conditions (5), we obtain
\[
v_1(\eta) = \int_1^{\eta} \frac{\Phi_1(s) \, ds}{\sqrt{1 - (\Phi_1(s))^2}}, \quad v_2(\eta) = H + \psi_2(\eta_2) - \int_1^{\eta_2} \frac{\Phi_2(s) \, ds}{\sqrt{1 - (\Phi_2(s))^2}}.
\]
(13)
The condition (6) gives
\[
H = \psi_1(\eta_1) - \psi_2(\eta_2) + \int_1^{\eta_1} \frac{\Phi_1(s) \, ds}{\sqrt{1 - (\Phi_1(s))^2}} + \int_1^{\eta_2} \frac{\Phi_2(s) \, ds}{\sqrt{1 - (\Phi_2(s))^2}}.
\]
(14)
From the transversality conditions follow the equations
\[
\frac{1}{2} M \eta_1^2 + \alpha_1(\eta_1) \eta_1 - \left[ 1 + \frac{1}{2} M - b \int_1^{\eta} (v_1(s) - \psi_1(s)) s \, ds \right] = 0,
\]
(15)
\[
\frac{1}{2} M \eta_2^2 + \alpha_2(\eta_2) \eta_2 - \left[ 1 + \frac{1}{2} M + b \int_1^{\eta_2} (\hat{v}_2(s) - \hat{v}_2(s)) s \, ds \right] = 0,
\]
(16)
where \( \alpha_i(\eta_i) = [\sqrt{1 - (\alpha_{i0})^2} + (-1)^i \alpha_{i0} \psi'_i(\eta_i)] / \sqrt{1 + (\psi'_i(\eta_i))^2}, \ i = 1, 2 \). And finally, the condition of the volume conservation (8) gives an expression for \( \xi_* \),
\[
\xi_* = \left\{ \pi \left[ \int_1^{\eta_1} \frac{\Phi_1(s)^2 \, ds}{\sqrt{1 - (\Phi_1(s))^2}} + \int_1^{\eta_2} \frac{\Phi_2(s)^2 \, ds}{\sqrt{1 - (\Phi_2(s))^2}} + \int_1^{\eta_1} \psi'_1(s)^2 \, ds - \int_0^{\eta_2} \psi'_2(s)^2 \, ds \right] \right\}^{-1/3}.
\]
(17)
Thus, all the relations necessary for solving the problem by the iteration method are obtained. We organize the iterative process by the small parameter \( b \) (main iterative process).

3.1. First iteration (construction of a zeroth approximation)
At the first iteration, we believe that Bond number is zero (we construct an approximate solution in zero gravity conditions). Let us set the value of the parameter \( M \) from the admissible values range (thereby determining the functions \( \Phi_1(\eta), \Phi_2(\eta) \)). The admissible values of the parameter \( M \) we will consider the values for which each of equations (15), (16) has at least the one positive root greater than unity. For positive admissible values, there is only one such root for each of the equations, and for negative admissible values, there are two such roots for each of the equations. In the latter case, one value of the parameter \( M \) correspond to four solutions, which are determined by the choice of the roots of equations (15), (16). The roots choice variants: variant (+ +) - maximal roots of equations (15), (16); variant (+ -) - maximal root of equation...
(15) and minimal root of equation (16): variant (- +) - minimal - (15) and maximum - (16) and finally variant (- -) - both roots are minimal.

We find the roots of equations (15), (16) by the method of successive approximations (the auxiliary iteration process necessary at the first iteration to start the main iteration process and take into account the shape of the given surfaces). Namely, we believe that \( f_1'(r) = 0 \), \( f_2'(r) = 0 \), then \( \alpha_1(\eta) = \sqrt{1 - (\alpha_0)^2} \), \( i = 1, 2 \) (liquid bridge between planes) and we find the positive roots of quadratic equations (15), (16), which larger then unit. We determine the value \( \xi_* \) according to formula (17). We use these found values of roots to calculate \( \alpha_i(\eta) \), \( i = 1, 2 \), further we solve the quadratic equations (15), (16) and again we find \( \xi_* \). We continue this process until the difference in the values of the roots at two successive approximations becomes less of the given value. We find the value \( H \) by formula (14), the functions \( v_1(\eta), v_2(\eta) \) by formulas (13), and finally \( \xi_1, \xi_2, h, w_1(\xi), w_2(\xi) \). This is the process of constructing the zeroth approximation for given values of \( \alpha_0 \), \( i = 1, 2 \) and the selected value of the parameter \( M \) ends.

3.2. Second and subsequent iterations

If we are not limiting to considering the liquid bridge in zero gravity, then we continue the calculation process for the same selected value of the parameter \( M \). Namely, we find the value of the parameter \( b = B(\xi_*)^2 \), define the new auxiliary functions \( \Phi_1(\eta), \Phi_2(\eta) \) (in accordance with formulas (11), (12)) using the functions \( v_1, v_2, \psi_1, \psi_2 \) obtained at the first iteration. We solve equations (15), (16). Here they already are quadratic equations with constant coefficients. We find \( \xi_*, H, v_1, v_2 \), then a new modified Bond number \( b = B(\xi_*)^2 \) and do the third iteration in the same way. And so on, until the differences in solutions become less of the value that determines the accuracy of the calculations. Now we carry out all the described calculations for each of permissible values of the parameter \( M \), build the dependence of the height of the liquid bridge \( h \) on the parameter \( M \), and we find the problem solution to corresponding to the set value of the height.

4. Calculation example: sphere and plane

We apply that algorithm to calculate a special case: the top is the sphere \( \varphi_2(\xi) = 4 - \sqrt{16 - \xi^2} \), the bottom is the plane \( \varphi_1(\xi) = 0 \). We restrict ourselves to considering the zeroth approximation \( (B = 0) \). The calculation is carried out for the case \( \alpha_{10} = 0.4 \), \( \alpha_{20} = 0.6 \). Three steps of an auxiliary iterative process provide an accuracy of 0.5 percent. We introduce the notation \( h_0 = H \xi_* \), where the values of \( H \) and \( \xi_* \) are taken after three steps. Figure 2 shows the dependence of the value \( h_0 \), which determines the distance between a sphere and a plane, on the parameter \( M \). For positive values of the parameter \( M \) we have one curve, and for permissible negative values of \( M \) we get four different curves corresponding to four variants of the choosing of the roots pairs of quadratic equations (15), (16). Unlike the case of a liquid bridge between two parallel hard planes [1], here there is a finite positive value of the parameter \( M \) corresponding to the point of tangency of the sphere and plane and there are no solutions for values \( M \) that exceed this value. There are intervals of height changes corresponding to the presence of one, two, three and four solutions. The latter case is illustrated by figure 3. Here are shown four different profiles of the lateral surface of a liquid bridge between a sphere and a plane, corresponding to one value of the bridge height \( h_0 = 0.9 \) and four different values of the parameter \( M \). Profile (a) (variant (+ +), \( M = -0.305 \)) will be implemented on practice, since it is this profile that gives the smallest of the four values of the energy functional \( J(u_1(r), u_2(r)) \).

5. Conclusions

A variational statement of the original problem is given. The solution was found under the assumption that the Bond number is small. The method of iteration over a small parameter of the problem was used, and the start was taken from the case of a liquid bridge between parallel
Figure 2. The dependence of the liquid bridge height $h_0$ on the parameter $M$ for the case $\alpha_{10} = 0.4$, $\alpha_{20} = 0.6$. Graph 1 corresponds to the variant $(++)$; graph 2 - $(+-)$; graph 3 - $(-+)$; graph 4 - $(--)$.

Figure 3. Four different profiles of the liquid bridge, corresponding to a single value of its height $h_0 = 0.9$, for the case $\alpha_{10} = 0.4$, $\alpha_{20} = 0.6$. (a) - variant $(++)$, $M = -0.305$; (b) - $(+-)$, $M = -0.22$; (c) - $(-+)$, $M = -0.13$; (d) - $(--)$, $M = -0.01$. 
planes in weightlessness. The introduction of the $M$ parameter made it possible to significantly simplify the solution algorithm and, based on the analysis of the dependence of the bridge height on this parameter, to indicate the regions of existence of one, two, three, and four solutions for a given bridge volume, its height and contact angles. It was shown that the maximum number of different solutions for the same boundary conditions and in this general case will be equal to four. As an example, a zero approximation for the problem of a liquid bridge between a sphere and a plane was constructed.

Thus, an effective algorithm for solving the problem of finding the shape of the lateral surface of a vertical liquid bridge between two arbitrary solid convex surfaces of small curvature in the axisymmetric case, taking into account the action of gravity, was proposed.

References
[1] Galaktionov E V, Galaktionova N E and Tropp E A 2017 Tech. Phys. 62 1482
[2] Galaktionov E V, Galaktionova N E and Tropp E A 2019 J. Phys.: Conf. Ser. 1400 044025
[3] Antonov P I, Nikanorov S P and Tatarchenko V A 1977 J. Cryst. Growth 42 447
[4] Haynes M, O’Brien S B G and Benilov E S 2016 Phys. Fluids 28 042107
[5] Rubinstein B Y and Fel L G 2014 J. Colloid Interface Sci. 417 37
[6] Teixeira P I C and Teixeira M A C 2020 J. Phys.: Condens. Matter 32 034002