ALMOST CONTACT MANIFOLDS, CONNECTIONS WITH TORSION, AND PARALLEL SPINORS

THOMAS FRIEDRICH AND STEFAN IVANOV

Abstract. We classify locally homogeneous quasi-Sasakian manifolds in dimension five that admit a parallel spinor $\psi$ of algebraic type $F \cdot \psi = 0$ with respect to the unique connection $\nabla$ preserving the quasi-Sasakian structure and with totally skew-symmetric torsion. We introduce a certain conformal transformation of almost contact metric manifolds and discuss a link between them and the dilation function in 5-dimensional string theory. We find natural conditions implying conformal invariances of parallel spinors. We present topological obstructions to the existence of parallel spinors in the compact case.

1. Introduction

The basic model in type II string theory is a 6-tuple $(M^6, g, \nabla, T, \Phi, \psi)$ consisting of a Riemannian metric $g$, a metric connection $\nabla$ with totally skew-symmetric torsion form $T$, a dilation function $\Phi$ and a spinor field $\psi$. The string equations can then be written in the following form (see [Stro] and [IP, FI]):

\[
\text{Ric}^\nabla + \frac{1}{2} \delta^\theta(T) + 2 \cdot \nabla^\theta(d\Phi) = 0, \quad \delta^\theta(T) = 2 \cdot (\text{grad}(\Phi) \cdot T),
\]

\[
\nabla \psi = 0, \quad (2 \cdot d\Phi - T) \cdot \psi = 0.
\]

If the dilation function is constant the string equations simplify:

\[
\text{Ric}^\nabla = 0, \quad \delta^\theta(T) = 0, \quad \nabla \psi = 0, \quad T \cdot \psi = 0.
\]

In fact, the bosonic part is taken with one-loop contribution. With two-loop contribution it receives additional terms involving quadratic terms of the curvature (see [HP]). The fermionic part, i.e., the Killing spinor equation, is responsible for the preserved supersymmetries. The number of preserved supersymmetries depends on the number of parallel spinors. In this paper we concentrate our attention mainly on the geometry of the solutions of the Killing spinor equations in dimension five. In this dimension the $\nabla$-parallel spinor field $\psi$ defines, via the formulas

\[\xi \cdot \psi = i \cdot \psi, \quad -2 \cdot \varphi(X) \cdot \psi + \xi \cdot X \cdot \psi = i \cdot X \cdot \psi,\]

an almost contact metric structure $(M^5, g, \xi, \eta, \varphi)$, which is preserved by the connection $\nabla$. A simple algebraic computation yields the equations

\[\varphi(\xi) = 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X) \cdot \eta(Y).\]

However, the contact condition $\eta \wedge (d\eta)^2 \neq 0$ need not be satisfied in general. This suggests that in dimension five the solutions of the type II string equations are related to suitable almost contact metric structures. In fact, not any such geometric structure is admissible. The first condition is that there should exist a connection with totally skew-symmetric torsion and preserving the almost contact metric structure. In the paper [FI] we proved that an almost contact metric structure admits such a connection if and only if the Nijenhuis tensor $N$ is totally

Received by the editors 24th January 2022.

Key words and phrases. Contact structures, string equations.

Supported by the SFB 288 "Differential geometry and quantum physics" of the DFG and the European Human Potential Program EDGE, Research Training Network HPRN-CT-2000-00101. S.Ivanov thanks ICTP for the support and excellent environments.
skew-symmetric and if the vector field $\xi$ is a Killing vector field. In this case the connection is unique and we derived a formula for the torsion form $T$. Suppose now that we fix an almost contact metric structure of that type. Then we have a reduction of the frame bundle $F(M^5)$ to the subgroup $U(2) \subset SO(5)$. But the isotropy group of a spinor in the 5-dimensional spin representation $\Delta_5$ does not coincide with $U(2)$. Therefore the existence of a $\nabla$-parallel spinor field imposes a second condition on the almost contact metric structure under consideration. The almost contact metric structure splits the spinor bundle $\Sigma$ of $M^5$ into two 1-dimensional bundles $\Sigma^\pm$ and into one 2-dimensional bundle $\Sigma^2$. Since the unique connection $\nabla$ preserves the almost contact metric structure, it preserves the decomposition of the spinor bundle, too. Consequently, we obtain two different integrability conditions for the $\nabla$-parallel spinor. The case that the spinor field is in one of the 1-dimensional subbundles $\Sigma^\pm$ means that the connection $\nabla$ has a reduction to the subgroup $SU(2)$. Equivalently, the Ricci tensor of the connection has to satisfy certain algebraic relations. This case was studied in [FI]. In particular, there are compact Sasakian manifolds with $\nabla$-parallel spinor fields of that algebraic type. The aim of this paper is to study the almost contact metric structures in dimension five admitting a $\nabla$-parallel spinor field inside the 2-dimensional bundle $\Sigma^2$. We classify all locally homogeneous quasi-Sasakian structures with a $\nabla$-parallel spinor field in the bundle $\Sigma^2$. Furthermore, we discuss the second Killing equation for spinors in the bundle $\Sigma^2$ involving a dilation function and show that it has to be constant. Concerning solutions of both Killing spinor equations in the bundle $\Sigma^\pm$, we introduce certain transformations depending on a real function (special conformal transformations) of an almost contact metric structure and show that any solution to both Killing equations in the bundle $\Sigma^\pm$ is invariant under these transformations in dimension five. We discuss the close relationship between the dilation function and the Lee form of the structure and show that the dilation function can be interpreted as a conformal factor. In the regular case, we find that the second Killing equation implies that any parallel spinor is projectable. This allows us to list all compact regular solutions to both equations in dimension five. Finally we prove, in any odd dimension, a generalization of Tachibana’s Theorem for harmonic 1-forms on compact quasi-Sasakian manifolds in the presence of some special $\nabla$-parallel spinor.

2. Contact connections with parallel spinors

We start with some basic definitions in contact geometry, on which the book [Blair] or the article [CG] may serve as a general reference. An almost contact metric structure consists of an odd-dimensional manifold $M^{2k+1}$ equipped with a Riemannian metric $g$, a vector field $\xi$ of length one, its dual 1-form $\eta$ as well as an endomorphism $\varphi$ of the tangent bundle such that the algebraic relations

$$\varphi(\xi) = 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad g(\varphi(X), \varphi(Y)) = g(X,Y) - \eta(X) \cdot \eta(Y)$$

are satisfied. If $\eta \wedge (d\eta)^k \neq 0$, we have a contact manifold. The fundamental form $F$ and the Nijenhuis tensor $N$ of an almost contact metric structure are defined by the formulas

$$F(X,Y) := g(X,\varphi(Y)), \quad N(X,Y) := [\varphi,\varphi](X,Y) + d\eta(X,Y) \cdot \xi.$$

It is clear that $\eta \wedge F^k \neq 0$. There are many special types of almost contact metric structures. We introduce those appearing in this paper. An almost contact metric structure is called normal if its Nijenhuis tensor vanishes, $N = 0$. A quasi-Sasakian structure is a normal almost contact metric structure with closed fundamental form, $N = 0$ and $dF = 0$. The vector field $\xi$ of a quasi-Sasakian structure is automatically a Killing vector field. In fact, in Section 3 of the present paper we will prove a more general result. A normal, almost contact metric structure with the property that the derivative $d\eta$ of the contact form is proportional to the fundamental form are called $\alpha$-Sasakian, $N = 0$, $d\eta = \alpha \cdot F$ and $\alpha$ constant. Any $\alpha$-Sasakian structure is quasi-Sasakian. Finally, Sasakian manifolds are characterized by the following two integrability
conditions
\[ N = 0, \quad d\eta = 2 \cdot F. \]

Now we study the spin geometry of the 5-dimensional local model \( \mathbb{R}^5 \) in detail. Let us fix an orthonormal basis \( e_1, \ldots, e_5 \) and consider the almost contact metric structure given by the vector \( \xi := e_5 \) and the skew-symmetric endomorphism \( \varphi := -e_1 \wedge e_2 - e_3 \wedge e_4 \). The fundamental form \( F \) is
\[
F = e_1 \wedge e_2 + e_3 \wedge e_4.
\]
The subgroup of SO(5) preserving the \( (g, \xi, \eta, \varphi) \)-structure is isomorphic to the group U(2). A form \( \sum x_{ij} \cdot e_i \wedge e_j \) belongs to the Lie algebra \( \mathfrak{u}(2) \) if and only if
\[
x_{14} + x_{23} = 0, \quad x_{13} - x_{24} = 0, \quad x_{i5} = 0.
\]
Denote by \( \Delta_5 \) the 5-dimensional spin representation. Vectors act on \( \Delta_5 \) by Clifford multiplication and we will use the following matrix representation (see [Fri2]):
\[
e_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{bmatrix},
\]
\[
e_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad e_5 = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad F = \begin{bmatrix} 2i & 0 & 0 & 0 \\ 0 & -2i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]
The fundamental form acts on \( \Delta_5 \) with eigenvalues \( (2i, -2i, 0, 0) \). Consequently, \( \Delta_5 \) splits as a \( \mathfrak{u}(2) \)-representation into two 1-dimensional representations \( \Delta_5^\pm \) and one 2-representation \( \Delta_5^2 \). These spaces are defined by the conditions
\[
\Delta_5^\pm = \{ \psi \in \Delta_5 : F \cdot \psi = \pm 2i \cdot \psi \}, \quad \Delta_5^2 = \{ \psi \in \Delta_5 : F \cdot \psi = 0 \}.
\]
If we split \( \Delta_5 = \Delta_4^+ \oplus \Delta_4^- \) as a spin(4)-representation, then we will obtain \( \Delta_4^+ = \Delta_5^+ \oplus \Delta_5^- \) and \( \Delta_4^- = \Delta_5^2 \). The isotropy subgroup of a spinor in \( \Delta_5^\pm \) is isomorphic to the group SU(2) and its Lie algebra is given by the formulas:
\[
x_{14} + x_{23} = 0, \quad x_{13} - x_{24} = 0, \quad x_{12} + x_{34} = 0, \quad x_{i5} = 0.
\]
On the other hand, the isotropy subgroup inside U(2) of a spinor in \( \Delta_5^2 \) is the diagonally embedded subgroup U(1):
\[
x_{12} = x_{34}, \quad x_{13} = x_{14} = x_{23} = x_{24} = x_{i5} = 0.
\]
Remark that the diagonally embedded subgroup U(1) ⊂ U(2) induces the trivial homomorphism on the fundamental groups
\[
\pi_1(U(1)) \to \pi_1(SO(4)) = \mathbb{Z}_2
\]
and, consequently, it lifts into the spin group Spin(4). An easy algebraic computation proves the following

**Lemma 2.1.** Let \( \omega \in \mathfrak{u}(2) \) be a 2-form in the Lie algebra of the group U(2). Then the following conditions are equivalent:

1. There exists a nontrivial spinor \( \psi \in \Delta_5^\pm \) such that \( \omega \cdot \psi = 0 \).
2. For any spinor \( \psi \in \Delta_5^\pm \) the Clifford product \( \omega \cdot \psi = 0 \) vanishes.
3. \( \omega \) is proportional to the fundamental form, \( \omega = a \cdot F \).

Let \( (M^5, g, \xi, \eta, \varphi) \) be an almost contact metric manifold and denote by \( \mathcal{F}(M^5) \) its Riemannian frame bundle. The almost contact metric structure defines a U(2)-reduction \( \mathcal{R} \subset \mathcal{F}(M^5) \) of the principal SO(5)-bundle \( \mathcal{F}(M^5) \). We fix a spin structure and we denote by \( \Sigma \) the spin bundle. Again, the fundamental form of the almost contact metric structure splits the spinor bundle into two 1-dimensional subbundles \( \Sigma^\pm \) and one 2-dimensional bundle \( \Sigma^2 \). Consider a metric connection \( \nabla \) preserving the almost contact metric structure, i.e., a connection in \( \mathcal{R} \). Since the
In particular, the two equations $\nabla^2$-dimensional bundle have a solution only in case the connection $\nabla$-spinor in the bundle $\Sigma^2$, the connection reduces to the subgroup $\text{SU}(2)$ and we obtain only one necessary and sufficient condition:

$$R^\nabla_{XY12} + R^\nabla_{XY34} = 0.$$  

Moreover, if the connection $\nabla$ has totally skew-symmetric torsion $T$, then the latter condition becomes equivalent to a certain relation between the Ricci tensor $\text{Ric}^\nabla$ and the exterior derivative $dT$ of the torsion form (see [SSTP], Proposition 9.1). In particular, for Sasakian manifolds and their unique connection with totally skew-symmetric torsion the integrability condition is

$$\text{Ric}^\nabla = 4 \cdot (g - \eta \otimes \eta).$$

Compact examples of this type are known (see [FI], Proposition 9.2 and Remark 9.3).

In this paper we shall study the integrability condition for $\nabla$-parallel spinors $\psi \in \Gamma(\Sigma^2)$ in the 2-dimensional subbundle,

$$\nabla \psi = 0, \quad F \cdot \psi = 0.$$  

Then the connection $\nabla$ reduces to a $\text{U}(1)$-subbundle $\mathcal{R}_0 \subset \mathcal{R}$. Using an adapted frame in $\mathcal{R}_0$ the abelian connection $\nabla$ is given by a 1-form $\lambda$:

$$\nabla e_1 = A \cdot e_2, \quad \nabla e_2 = -A \cdot e_1, \quad \nabla e_3 = A \cdot e_4, \quad \nabla e_4 = -A \cdot e_3, \quad \nabla e_5 = 0.$$  

The curvature form $\Omega^A := dA$ of the $\mathcal{R}_0$-connection $A$ is a well defined 2-form on $M^5$ and the curvature tensor $R^\nabla$ is given by the formulas

$$R^\nabla(X,Y)e_1 = \Omega^A(X,Y)e_2, \quad R^\nabla(X,Y)e_2 = -\Omega^A(X,Y)e_1, \quad R^\nabla(X,Y)e_3 = \Omega^A(X,Y)e_4, \quad R^\nabla(X,Y)e_4 = -\Omega^A(X,Y)e_3, \quad R^\nabla(X,Y)e_5 = 0.$$  

Consequently, we obtain the following condition on the curvature for the existence of $\nabla$-parallel spinors in the 2-dimensional bundle $\Sigma^2$.

**Theorem 2.1.** Let $(M^5, g, \xi, \eta, \varphi)$ be a simply connected, almost contact metric spin manifold and let $\nabla$ be a connection preserving the structure. Then the following conditions are equivalent:

1. There exists a nontrivial, $\nabla$-parallel spinor field in the subbundle $\Sigma^2$,

2. There are two $\nabla$-parallel spinor fields in the subbundle $\Sigma^2$.

3. The curvature tensor $R^\nabla : \Lambda^2(M^5) \to \Lambda^2(M^5)$ is given by the formula

$$R^\nabla(\alpha) = (\Omega^A, \alpha) \cdot F,$$

where $\Omega^A$ is a closed 2-form.

**Remark 2.1.** A connection $\nabla$ with a parallel spinor in one of the bundles $\Sigma^\pm$ and a parallel spinor in the bundle $\Sigma^2$ is flat. Indeed, the condition $R^\nabla_{XY12} + R^\nabla_{XY34} = 0$ yields that the curvature form $\Omega^A$ vanishes. If $M^5$ is compact, then it is a 5-dimensional compact Lie group (see [SSTP]).

A direct computation of the $\nabla$-Ricci tensor yields its components:

$$R^\nabla_{11} = -\Omega^A(e_1, e_2), \quad R^\nabla_{13} = -\Omega^A(e_1, e_3), \quad R^\nabla_{14} = \Omega^A(e_1, e_3),$$

$$R^\nabla_{22} = -\Omega^A(e_1, e_2), \quad R^\nabla_{23} = -\Omega^A(e_2, e_4), \quad R^\nabla_{24} = \Omega^A(e_2, e_3),$$

$$R^\nabla_{31} = \Omega^A(e_2, e_3), \quad R^\nabla_{32} = -\Omega^A(e_1, e_3), \quad R^\nabla_{33} = -\Omega^A(e_3, e_4),$$

$$R^\nabla_{41} = \Omega^A(e_2, e_4), \quad R^\nabla_{42} = -\Omega^A(e_1, e_4), \quad R^\nabla_{44} = -\Omega^A(e_3, e_4),$$

$$R^\nabla_{51} = \Omega^A(e_2, e_5), \quad R^\nabla_{52} = -\Omega^A(e_1, e_5), \quad R^\nabla_{53} = \Omega^A(e_4, e_5), \quad R^\nabla_{54} = -\Omega^A(e_3, e_5).$$

In particular, the two equations $\text{Ric}^\nabla = 0, \nabla \psi = 0$ with a spinor field $\psi \in \Gamma(\Sigma^2)$ in the 2-dimensional bundle have a solution only in case the connection $\nabla$ is flat.
Theorem 2.2. Let \((M^5, g, \xi, \eta, \varphi)\) be an almost contact metric spin manifold and let \(\nabla\) be a connection preserving the structure. If the Ricci tensor of \(\nabla\) vanishes and the connection \(\nabla\) admits a parallel spinor such that \(F \cdot \psi = 0\), then the connection \(\nabla\) is flat.

3. Normal almost contact metric structures with \(\nabla\)-parallel spinors

We suppose that the almost contact metric spin manifold \((M^{2k+1}, g, \xi, \eta, \varphi)\) admits a connection \(\nabla\) with totally skew-symmetric torsion \(T\) and preserving the structure. A connection of this type exists if and only if the vector field \(\xi\) is a Killing vector field and the Nijenhuis tensor \(N\) treated as a \((0,3)\)-tensor is totally skew-symmetric (see [FI]). Moreover, the connection is unique and its torsion form \(T\) is given by the formula
\[
T = \eta \wedge d\eta + d\varphi F + N - \eta \wedge (\xi \lrcorner N),
\]
where \(d\varphi F\) denotes the \(\varphi\)-twisted exterior differential of the fundamental form \(F\). In particular, we have
\[
g(\nabla^g_\xi \eta, Z) = \frac{1}{2} T(\xi, X, Z), \quad \eta = \xi \lrcorner T, \quad \xi \lrcorner d\eta = 0,
\]
where \(\nabla^g\) is the Levi-Civita connection of the Riemannian manifold. If the Nijenhuis tensor \(N\) is totally skew-symmetric then the Killing condition for \(\xi\) becomes equivalent to the equation \(\xi \lrcorner dF = 0\). In fact, we have

Proposition 3.1. Let \((M^{2k+1}, g, \xi, \eta, \varphi)\) be an almost contact metric manifold with totally skew-symmetric Nijenhuis tensor \(N\). The vector field \(\xi\) is Killing if and only if \(\xi \lrcorner dF = 0\).

Proof. In general, the vector field \(\xi\) is a Killing field if and only if
\[
\nabla_X \xi = \frac{1}{2} \cdot X \lrcorner d\eta
\]
holds (see [Blair], page 64). Using the equation (see [Blair], page 53)
\[
2 \cdot g((\nabla_X \varphi) Y, \xi) = -dF(X, Y, \xi) + N(Y, \xi, \varphi(X)) + d\eta(\varphi(Y), X)
\]
we immediately obtain the following expression for \(X \lrcorner d\eta - 2 \cdot \nabla_X \xi\):
\[
g(X \lrcorner d\eta - 2 \cdot \nabla_X \xi, \varphi(Y)) = -dF(X, Y, \xi) + N(Y, \xi, \varphi(X)).
\]
Moreover, in the special case that the Nijenhuis tensor is totally skew-symmetric, we have (see [FI], Lemma 8.3)
\[
N(Y, \xi, \varphi(X)) = -dF(\varphi(X), \varphi(Y), \xi) = -dF(X, Y, \xi),
\]
and the proof of the proposition follows directly. \(\square\)

In dimension five the condition that the Nijenhuis tensor is totally skew-symmetric is equivalent to its vanishing (see [CG]). Therefore we will consider only 5-dimensional, normal almost contact metric structures with a Killing vector field \(\xi\). The integrability condition for the existence of a \(\nabla\)-parallel spinor field in one of the 1-dimensional bundles \(\Sigma^\pm\) was investigated in the paper [FI]. Here we will study the same problem in case that the \(\nabla\)-parallel spinor is in \(\Sigma^2\). First we suppose that the normal almost contact structure is regular, i.e., the orbit space \(N^4 := M^5/\xi\) is a smooth manifold. In this situation the projection
\[
\pi : M^5 \longrightarrow N^4
\]
is a principal \(S^1\)-bundle with curvature form \(\hat{\Omega} := d\eta\). Since the Lie derivative \(L_\xi \varphi = 0\) of the endomorphism \(\varphi\) with respect to \(\xi\) vanishes (see [Blair], p. 49), the manifold admits
\begin{enumerate}
\item a Riemannian metric \(\hat{g}\),
\item an integrable complex structure \(\hat{\varphi}\) corresponding to \(\varphi\),
\item a form \(\hat{F}(X, \hat{Y}) := \hat{g}(\hat{X}, \hat{\varphi}(\hat{Y}))\). In general \((N^4, \hat{g}, \hat{\varphi})\) is not a Kähler manifold.
\item a closed 2-form \(\hat{\Omega}\).
\end{enumerate}
The manifold $M^5$ is oriented by the positive frame $\{e_1, e_2 = -\varphi(e_1), e_3, e_4 = -\varphi(e_3), e_5\}$ and we orient the manifold $N^4$ in the same way. Then $\hat{\psi}$ is a section in the positive twistor bundle $Z^+(N^4)$ of the 4-manifold $N^4$. Suppose that there exists a $\nabla$-parallel spinor $\psi$

$$\nabla_X^2 \psi + \frac{1}{4} \cdot (X \nabla T) \cdot \psi = 0,$$

and let us compute its Lie derivative (see [BG]):

$$\mathcal{L}_\xi \psi = \nabla^2_\xi \psi - \frac{1}{4} \cdot d\eta \cdot \psi = -\frac{1}{2} \cdot d\eta \cdot \psi.$$

Since $d\eta(\varphi(X), \varphi(Y)) = d\eta(X, Y)$ (see Blair, page 51), we can write $d\eta$ in the form

$$d\eta = a \cdot e_1 \wedge e_2 + b \cdot (e_1 \wedge e_3 + e_2 \wedge e_4) + c \cdot (e_1 \wedge e_4 - e_2 \wedge e_3) + d \cdot e_3 \wedge e_4.$$

If the spinor field $\psi$ is a section in the 2-dimensional bundle $\Sigma^2$, then $d\eta \cdot \psi = 0$ holds if and only if $d\eta$ is proportional to $F$. The spinor field $\psi$ defines a parallel spinor field $\hat{\psi}$ on the manifold $N^4$. Since the Spin(4)-representation $\Delta_3^\pm$ is isomorphic to $\Delta_3^\mp$, the parallel spinor field $\hat{\psi} \in \Gamma(N^4; \Sigma^-)$ is a section in the negative spinor bundle and it induces a negative complex structure $\hat{\psi} \in \mathcal{Z}^-(N^4)$. Denote by $\hat{N}$ the manifold $N^4$ equipped with the opposite orientation. Then $(\hat{N}, \hat{g}, \hat{\psi})$ is a Ricci flat, antisedual Kähler manifold and $\hat{\psi} \in \mathcal{Z}^-(\hat{N})$ becomes a negative, integrable complex structure with closed fundamental form, $d\hat{F} = 0$. Therefore $\hat{\psi}$ is a parallel complex structure and, consequently, the negative part $\hat{W}_-$ of the Weyl tensor vanishes, too. In particular, we have proved

**Theorem 3.1.** Let $(M^5, g, \xi, \eta, \varphi)$ be a 5-dimensional, compact $\alpha$-Sasakian spin manifold and denote by $\nabla$ the unique connection with totally skew-symmetric torsion $T = \eta \wedge d\eta$. Moreover, suppose that the contact structure is regular. If there exists a $\nabla$-parallel spinor field $\psi$ such that $F \cdot \psi = 0$, then the $\alpha$-Sasakian structure is an $S^1$-bundle over a flat torus.

If the spinor field $\psi$ is a section in the 1-dimensional bundle $\Sigma^3$, then $d\eta \cdot \psi = 0$ holds if and only if $a = -d$. The projected spinor $\hat{\psi}$ is parallel with respect to a metric connection with torsion, the metric $\hat{g}$ is antisedual and the 2-form $\hat{\Omega}$ is antisedual, too. The space $N^4$ is a hyperhermitian manifold with torsion (or, equivalently, a hypercomplex manifold). In the compact case there are only three possibilities: the flat torus, a $K3$-surface or a Hopf surface (see [DI, H]).

**Theorem 3.2.** Let $(M^5, g, \xi, \eta, \varphi)$ be a 5-dimensional, compact, regular normal almost contact metric spin manifold with a Killing vector field $\xi$ and denote by $\nabla$ the unique connection with totally skew-symmetric torsion. If there exists a projectable $\nabla$-parallel spinor field $\psi$ in the bundle $\Sigma^3$, then $M^5$ is a $S^1$-bundle over a flat torus, a $K3$-surface or over a Hopf surface. If the projectable $\nabla$-parallel spinor belongs to the 2-dimensional bundle $\Sigma^2$, then $M^5$ is a $S^1$-bundle over a flat torus or over a Hopf surface.

### 4. The quasi-Sasakian case

We will integrate the structure equations of a quasi-Sasakian manifold admitting a $\nabla$-parallel spinor of type $F \cdot \psi = 0$. Choosing the orthonormal frame $e_1, \ldots, e_5$ in the $\U(1)$-reduction defined by the $\nabla$-parallel spinor, an easy computation yields the proof of the following

**Proposition 4.1.** Let $(M^5, g, \xi, \eta, \varphi)$ be a normal almost contact metric spin manifold with Killing vector field $\xi$. Denote by $\nabla$ the unique connection with totally skew-symmetric torsion $T$ and preserving the structure. Moreover, suppose that there exists a $\nabla$-parallel spinor field of type $F \cdot \psi = 0$. Then, locally, there exists an orthonormal frame as well as a 1-form $A$ such that the structure equations of the Riemannian manifold are given by

$$de_1 = A \wedge e_2 + e_1 \llcorner T, \quad de_2 = -A \wedge e_1 + e_2 \llcorner T, \quad de_3 = A \wedge e_4 + e_3 \llcorner T,$$

$$de_4 = -A \wedge e_3 + e_4 \llcorner T, \quad de_5 = e_5 \llcorner T.$$
Example 4.1. In $\mathbb{R}^5$ we consider the 1-forms
\[
e_1 = \frac{1}{2} dx_1, \quad e_2 = \frac{1}{2} dy_1, \quad e_3 = \frac{1}{2} dx_2, \quad e_4 = \frac{1}{2} dy_2, \quad e_5 = \eta = \frac{1}{2} (dz - y_1 \cdot dx_1 - y_2 \cdot dx_2).
\]
We obtain a Sasakian manifold (see [Blair]) and it is not hard to see that this Sasakian manifold admits $\nabla$-parallel spinors of type $F \cdot \psi = 0$. The Sasakian structure arises from left invariant vector fields on a 5-dimensional Heisenberg group.

First we will prove that Example 4.1 is locally the only Sasakian manifold admitting a $\nabla$-parallel spinor of type $F \cdot \psi = 0$. Partially, we studied this case already in [FI].

Theorem 4.1. Let $(M^5, g, \xi, \eta, \varphi)$ be a 5-dimensional Sasakian spin manifold and denote by $\nabla$ the unique connection with totally skew-symmetric torsion $T = \eta \wedge \mathrm{d} \eta$. If there exists a $\nabla$-parallel spinor field $\psi$ such that $F \cdot \psi = 0$, then the Sasakian structure is locally equivalent to the structure described in Example 4.1.

Proof. In case the almost contact metric structure is Sasakian, the exterior derivative $\mathrm{d} \eta$ is proportional to the fundamental form,
\[
\mathrm{d} \eta = 2 \cdot F, \quad T = 2 \cdot (e_1 \wedge e_2 + e_3 \wedge e_4) \wedge e_5.
\]
The structure equations of the Riemannian manifold can be simplified to:
\[
de_1 = A \wedge e_2 + 2 \cdot e_2 \wedge e_5, \quad de_2 = -A \wedge e_1 - 2 \cdot e_1 \wedge e_5,
\]
\[
de_3 = A \wedge e_4 + 2 \cdot e_3 \wedge e_5, \quad de_4 = -A \wedge e_3 - 2 \cdot e_3 \wedge e_5,
\]
\[
de_5 = 2 \cdot (e_1 \wedge e_2 + e_3 \wedge e_4).
\]
Differentiating the system we see that the 1-form $C := A - 2 \cdot e_5$ satisfies the algebraic equations
\[
dC \wedge e_1 = dC \wedge e_2 = dC \wedge e_3 = dC \wedge e_4 = 0,
\]
i.e., $C$ is a closed form, $dC = 0$. Moreover, we obtain
\[
\Omega^A = dA = 2 \cdot de_5 = 4 \cdot (e_1 \wedge e_2 + e_3 \wedge e_4)
\]
and, in particular, $\mathrm{Ric}^{\nabla} = \mathrm{diag}(-4, -4, -4, -4, 0)$. Locally there exists a function $f$ such that $C$ is the differential of $f$, $C = df$. Using the new frame
\[
e_1^* := \cos(f) \cdot e_1 - \sin(f) \cdot e_2, \quad e_2^* := \sin(f) \cdot e_1 + \cos(f) \cdot e_2,
\]
\[
e_3^* := \cos(f) \cdot e_3 - \sin(f) \cdot e_4, \quad e_4^* := \sin(f) \cdot e_3 + \cos(f) \cdot e_4,
\]
we obtain the equations
\[
de_1^* = de_2^* = de_3^* = de_4^* = 0, \quad de_5^* = 2 \cdot (e_1^* \wedge e_2^* + e_3^* \wedge e_4^*).
\]
We will generalize the argument used for Sasakian manifolds. The aim is the construction of almost contact metric structures admitting $\nabla$-parallel spinors of type $F \cdot \psi = 0$. We assume that the structure under consideration is quasi-Sasakian:
\[
N = 0, \quad dF = 0, \quad \xi \text{ is a Killing field}.
\]
The torsion form is given by $T = \eta \wedge \mathrm{d} \eta$ and $\mathrm{d} \eta$ has the invariance property $\mathrm{d} \eta(\varphi(X), \varphi(Y)) = \mathrm{d} \eta(X, Y)$ (see [Blair], p. 51). Therefore there exist functions $a, b, c, d$ such that
\[
\mathrm{d} \eta = a \cdot e_1 \wedge e_2 + b \cdot (e_1 \wedge e_3 + e_2 \wedge e_4) + c \cdot (e_1 \wedge e_4 - e_2 \wedge e_3) + d \cdot e_3 \wedge e_4
\]
holds. The structure equations yield the following system:
\[
de_1 = A \wedge e_2 + a \cdot e_2 \wedge e_5 + b \cdot e_3 \wedge e_5 + c \cdot e_4 \wedge e_5,
\]
\[
de_2 = -A \wedge e_1 - a \cdot e_1 \wedge e_5 + b \cdot e_4 \wedge e_5 - c \cdot e_3 \wedge e_5,
\]
\[
de_3 = A \wedge e_4 - b \cdot e_1 \wedge e_5 + c \cdot e_2 \wedge e_5 + d \cdot e_4 \wedge e_5,
\]
\[
de_4 = -A \wedge e_3 - c \cdot e_1 \wedge e_5 - b \cdot e_2 \wedge e_5 - d \cdot e_3 \wedge e_5,
\]
\[
de_5 = a \cdot e_1 \wedge e_2 + b \cdot (e_1 \wedge e_3 + e_2 \wedge e_4) + c \cdot (e_1 \wedge e_4 - e_2 \wedge e_3) + d \cdot e_3 \wedge e_4.
\]
We suppose now that \(a, b, c, d\) are constant. For example, this assumption is satisfied if the quasi-Sasakian structure is locally homogeneous. We study the integrability condition of the system \((\ast)\). A straightforward, but lengthy computation yields that there is only one integrability condition.

**Lemma 4.1.** Let \(a, b, c, d\) be constant and \(A\) be a 1-form. The system \((\ast)\) is integrable if and only if the equation

\[
\Omega^A = dA = (ad - b^2 - c^2) \cdot F
\]

holds.

Conversely, fix real parameters \(a, b, c, d\) and suppose that \(b^2 + c^2 - ad \neq 0\). Then there exist \((\text{see } \text{[FI]}\text{, chapter 7.6})\) locally 1-forms \(e_1, \ldots, e_5\), \(A\) solving the system \((\ast)\) and the equation

\[
dA = (ad - b^2 - c^2) \cdot F.
\]

We define a Riemannian metric as well as an almost contact metric structure by the condition that \(e_1, \ldots, e_5 = \eta\) is an orthonormal frame. We denote the corresponding almost contact metric manifold by \(M^5(a, b, c, d)\). Its connection forms \(\omega_{ij}\) of the Levi-Civita connection are:

\[
\begin{align*}
2 \cdot \omega_{15} &= (+a \cdot e_2 + b \cdot e_3 + c \cdot e_4), \\
2 \cdot \omega_{25} &= (-a \cdot e_1 - c \cdot e_3 + b \cdot e_4), \\
2 \cdot \omega_{25} &= (-b \cdot e_1 + c \cdot e_2 + d \cdot e_4), \\
2 \cdot \omega_{45} &= (-c \cdot e_1 - b \cdot e_2 - d \cdot e_3), \\
2 \cdot \omega_{12} &= 2 \cdot A - a \cdot e_5, \\
2 \cdot \omega_{13} &= -b \cdot e_5, \\
2 \cdot \omega_{14} &= -c \cdot e_5, \\
2 \cdot \omega_{34} &= 2 \cdot A - d \cdot e_5, \\
2 \cdot \omega_{23} &= c \cdot e_5, \\
2 \cdot \omega_{24} &= -b \cdot e_5.
\end{align*}
\]

We verify now by a direct computation that \(\xi\) is a Killing vector field, the Nijenhuis tensor vanishes and the fundamental form \(F\) is closed, \(dF = 0\). Moreover, the torsion form \(T = \eta \wedge d\eta\) is coclosed, \(\delta(T) = 0\). Let us summarize the result.

**Theorem 4.2.** The almost contact metric manifold \(M^5(a, b, c, d)\) has the following properties:

1. \(\xi\) is a Killing vector field.
2. The Nijenhuis tensor vanishes, \(N = 0\).
3. The fundamental form is closed, \(dF = 0\).
4. \(d\eta \wedge d\eta = (ad - b^2 - c^2) \cdot F \wedge F\).
5. The torsion form \(T = \eta \wedge d\eta\) is coclosed, \(\delta(T) = 0\).
6. There exist two \(\nabla\)-parallel spinor fields with \(F \cdot \psi = 0\).
7. The Ricci tensor of the connection \(\nabla\) is

\[
\text{Ric}_\nabla = (b^2 + c^2 - ad) \cdot \text{diag}(1, 1, 1, 1, 0).
\]

Conversely, if \(M^5\) is a locally homogeneous, quasi-Sasakian spin manifold with a \(\nabla\)-parallel spinor of type \(F \cdot \psi = 0\), then \(M^5\) is locally equivalent to one of the manifolds \(M^5(a, b, c, d)\).

For any metric connection with totally skew-symmetric torsion the action of the 4-form

\[
2 \cdot \sigma^T := \sum_{i=1}^{5} (e_i \lrcorner T) \wedge (e_i \lrcorner T)
\]

on spinors plays an important role (see \([\text{FI}]\)). For the quasi-Sasakian structures \(M^5(a, b, c, d)\) we immediately obtain the formula

\[
dT + 2 \cdot \sigma^T + \text{Scal}_\nabla = 4 \cdot (ad - b^2 - c^2) \cdot (e_1 \wedge e_2 \wedge e_3 \wedge e_4 - 1).
\]

Moreover, on \(\Sigma^2\) the relation \(e_1 \cdot e_2 = -e_3 \cdot e_4\) holds and then we obtain

\[
e_1 \cdot e_2 \cdot e_3 \cdot e_4 - 1 = 0.
\]

Finally we apply a general integral formula (see \([\text{FI}]\)).

**Theorem 4.3.** Let \((M^5, g, \xi, \eta, \varphi)\) be a compact, locally homogeneous quasi-Sasakian spin manifold. Denote by \(D^\nabla\) the Dirac operator defined by its unique connection \(\nabla\) with totally skew-symmetric torsion. Then any \(\nabla\)-harmonic spinor of type \(F \cdot \psi = 0\) is \(\nabla\)-parallel,

\[
\{D^\nabla \psi = 0, \ F \cdot \psi = 0\} = \{\nabla \psi = 0, \ F \cdot \psi = 0\}.
\]
Remark that $dT + 2 \cdot \sigma T + \text{Scal} \nabla$ acts on the bundles $\Sigma^\pm$ by a multiplication by $\pm 8 \cdot (ad - b^2 - c^2)$. Hence, in one of these 1-dimensional subbundles of the spinor bundle there are no $\nabla$-harmonic spinors at all.

We will interpret the quasi-Sasakian structures $M^5(a, b, c, d)$. The system

$$
\begin{align*}
\frac{dA}{\omega} &= (ad - b^2 - c^2) \cdot (e_1 \wedge e_2 + e_3 \wedge e_4), \\
\frac{de_1}{\omega} &= A \wedge e_2 + a \cdot e_1 \wedge e_5 + b \cdot e_3 \wedge e_5 + c \cdot e_4 \wedge e_5, \\
\frac{de_2}{\omega} &= -A \wedge e_1 - a \cdot e_1 \wedge e_5 + b \cdot e_4 \wedge e_5 - c \cdot e_3 \wedge e_5, \\
\frac{de_3}{\omega} &= A \wedge e_4 - b \cdot e_1 \wedge e_5 + c \cdot e_2 \wedge e_5 + d \cdot e_4 \wedge e_5, \\
\frac{de_4}{\omega} &= -A \wedge e_3 - c \cdot e_1 \wedge e_5 - b \cdot e_2 \wedge e_5 - d \cdot e_3 \wedge e_5,
\end{align*}
$$

has a solution in $\mathbb{R}^6$ (see [Fla3], Lie’s third Theorem). Moreover, there exists a 6-dimensional Lie group $G(a, b, c, d)$ such that the restricted forms $e_1, \ldots, e_5$ are independent.

Any submanifold $M^5 \subset G(a, b, c, d)$ such that the restricted forms $e_1, \ldots, e_5$ are independent is a model for the quasi-Sasakian structure $M^5(a, b, c, d)$. In the Sasakian case $(a = d = 2, c = d = 0)$ the Lie group $G(1, 0, 0, 1)$ is the product of the 5-dimensional Heisenberg group by $\mathbb{R}^1$. Let us discuss the case of $a = c = d = 0$, $b = 1$. Introducing the 1-forms

$$
\sqrt{2} \cdot e_1^* := e_1 + e_4, \quad \sqrt{2} \cdot e_2^* := e_2 - e_3, \quad \sqrt{2} \cdot e_3^* := e_1 - e_4, \quad \sqrt{2} \cdot e_4^* := e_2 + e_3,
$$

we have the following system

$$
\begin{align*}
&\text{de}_1^* = C_1^* \wedge e_2^*, \quad \text{de}_2^* = -C_1^* \wedge e_1^*, \quad \text{de}_3^* = C_2^* \wedge e_4^*, \quad \text{de}_4^* = -C_2^* \wedge e_3^*, \\
&\text{dc}_1^* = -2 \cdot e_1^* \wedge e_2^*, \quad \text{dc}_2^* = -2 \cdot e_3^* \wedge e_4^*.
\end{align*}
$$

Consequently, in case of $a = c = d = 0$, $b = 1$, the group $G$ is isomorphic to the group $S^3 \times S^3$ and a generic submanifold of codimension one admits a quasi-Sasakian structure of that type. An interesting example is the 5-dimensional Stiefel manifold $V_{4,2}$ defined as the set of orthogonal pairs $(x, y)$ of vectors in $\mathbb{R}^4$ of length one,

$$
V_{4,2} = \{(x, y) \in \mathbb{R}^4 : ||x|| = ||y|| = 1, x(x, y) = 0\} \subset S^3 \times S^3.
$$

The Stiefel manifold is a naturally reductive space, it admits an Einstein metric constructed first by J. Jensen (see [Men]) as well as real Killing spinors (see [GII]). The Killing spinor induces a Sasakian structure on $V_{4,2}$, but the spinor does not belong to the subbundle $\Sigma^\perp$ with respect to this Sasakian structure. However, there is a second quasi-Sasakian structure realizing our parameters $a = c = d = 0$ and $b \neq 0$ and it corresponds to the embedding of $V_{4,2}$ into $S^3 \times S^3$. Finally one can construct a third, nonquasi-Sasakian structure on $V_{4,2}$ with

$$
N = 0, \quad dF \neq 0, \quad d^2 F = 0.
$$

Its torsion is thus again given by $T = \eta \wedge d\eta$, it admits a $\nabla$-parallel spinor, but this spinor does not belong to the bundle $\Sigma^\perp$. In a forthcoming paper (see [Agr]) this example and more general connections on it will be discussed from the point of view of naturally reductive spaces. In particular, it turns out that for all three structures the corresponding connection coincides with the canonical connection of the reductive space.

In the degenerate case, i.e., $a = 1, b = c = d = 0$, we have to solve

$$
\begin{align*}
&\text{de}_1 = A \wedge e_2 + e_2 \wedge e_5, \quad \text{de}_2 = -A \wedge e_1 - e_1 \wedge e_5, \quad \text{de}_3 = A \wedge e_4, \quad \text{de}_4 = -A \wedge e_3, \\
&\text{de}_5 = e_1 \wedge e_2, \quad \text{dA} = 0.
\end{align*}
$$

Locally the 1-form $A$ is the differential of some function, $A = df$. We introduce again a new frame $e_1^*, \ldots, e_5^*$ by the same formulas as in the proof of Theorem 4.1. The equations now read

$$
\begin{align*}
&\text{de}_1^* = e_2^* \wedge e_5^*, \quad \text{de}_2^* = e_5^* \wedge e_1^*, \quad \text{de}_3^* = e_1^* \wedge e_2^*, \quad \text{de}_4^* = 0, \quad \text{de}_5^* = 0.
\end{align*}
$$
and, consequently, the degenerate quasi-Sasakian structure $M^5(1,0,0,0)$ is realized on the Lie group $S^3 \times \mathbb{R}^2$.

5. The dilation function and conformal transformations

Consider an almost contact metric structure $(M^{2k+1}, g, \xi, \eta, \varphi)$ with skew-symmetric Nijenhuis tensor $N$ and Killing vector field $\xi$. The torsion of its unique connection $\nabla$ is given by

$$T = \eta \wedge d\eta + d^2F + N - \eta \wedge (\xi \lrcorner N).$$

Let us introduce the Lee form

$$\theta(X) := -\frac{1}{2} \sum_{i=1}^{2k+1} T(\varphi(X), e_i, \varphi(e_i)).$$

Remark that all vectors in the tuple $(\varphi(X), e_i, \varphi(e_i))$ are orthogonal to $\xi$ except in case $e_i = \xi$. Consequently, we have

$$\eta \wedge d\eta(\varphi(X), e_i, \varphi(e_i)) = 0, \quad \eta \wedge (\xi \lrcorner N)(\varphi(X), e_i, \varphi(e_i)) = 0.$$  

Moreover, $N$ is skew-symmetric, we obtain the formula

$$N(\varphi(X), e_i, \varphi(e_i)) = g(N(e_i, \varphi(e_i)), \varphi(X)) = d\eta(e_i, \varphi(e_i)) \cdot g(\xi, \varphi(X)) = 0.$$  

To summarize, the Lee form $\theta$ depends only on the exterior derivative of the fundamental form,

$$\theta(X) := -\frac{1}{2} \sum_{i=1}^{2k+1} dF(X, e_i, \varphi(e_i)).$$

We define a new almost contact metric structure by the formulas

$$\varphi' := \varphi, \quad \xi' := \xi, \quad \eta' := \eta, \quad g' := e^{2f} \cdot g + (1 - e^{2f}) \cdot \eta \otimes \eta.$$  

If the function $f$ is constant along the integral curve of $\xi$, $df(\xi) = 0$, we call the transformation special conformal. Moreover, computing the Lie derivative $L_\xi g'$ of the new metric $g'$ with respect to the vector field, we obtain the following

**Proposition 5.1.** The class of all almost contact metric manifolds with skew-symmetric Nijenhuis tensor and Killing vector field $\xi$ is invariant under special conformal transformations.

In particular, if $(M^{2k+1}, g, \xi, \eta, \varphi)$ admits a connection with totally skew-symmetric torsion and preserving the structure, then the transformed manifold admits again such a connection. Let us compute the new torsion form $T'$ as well as the new Lee form $\theta'$. Since the fundamental form and the Nijenhuis tensor transform as

$$F' = e^{2f} \cdot F, \quad N' = N$$

and the new torsion form is given by

$$T' = T + (e^{2f} - 1) \cdot d^2F + 2 \cdot e^{2f} \cdot (df \circ \varphi) \wedge F, \quad \theta' = \theta + 2 \cdot df.$$  

We compute the relation between the spinorial covariant derivatives $\nabla^g$, $\nabla^{g'}$ corresponding to the Levi-Civita connections of $g$ and $g'$ as well as between $\nabla$ and $\nabla'$. Let us summarize the result:

**Proposition 5.2.** Let $(M^{2k+1}, g, \xi, \eta, \varphi)$ be an almost contact metric manifold with skew-symmetric Nijenhuis tensor and Killing vector field $\xi$. The torsion form $T'$ and the Lee form $\theta'$ of a structure obtained by a special conformal transformation are given by the formulas:

$$T' = T + (e^{2f} - 1) \cdot d^2F + 2 \cdot e^{2f} \cdot (df \circ \varphi) \wedge F, \quad \theta' = \theta + 2 \cdot df.$$  

We compute the relation between the spinorial covariant derivatives $\nabla^g$, $\nabla^{g'}$ corresponding to the Levi-Civita connections of $g$ and $g'$ as well as between $\nabla$ and $\nabla'$. Let us summarize the result:

**Proposition 5.3.** Let $X$ be a vector field orthogonal to $\xi$ and $\psi$ be an arbitrary spinor field. Then the following formulas hold:

$$\nabla^g_X \psi = \nabla_X^g \psi + \frac{1}{4} \cdot (df \cdot X \cdot \psi) + \frac{1}{4} \cdot (X \lrcorner d\eta) \cdot \xi \cdot \psi,$$

$$\nabla^{g'}_\xi \psi = \nabla^g_\xi \psi + \frac{e^{-2f} - 1}{4} \cdot d\eta \cdot \psi.$$  

If, in addition, the Nijenhuis tensor $N = 0$ vanishes, one has

$$
\nabla_X^\prime \psi = \nabla_X \psi + \frac{1}{4} \cdot (df \cdot X \cdot X \cdot df) \cdot \psi - \frac{1}{2} \cdot X \cdot (df \circ \varphi) \wedge F \cdot \psi,
$$

$$
\nabla_{\xi}^\prime \psi = \nabla_{\xi} \psi + \frac{e^{-2f} - 1}{2} \cdot d\eta \cdot \psi.
$$

Proof. The first two formulas are direct consequences of the transformation of the Levi-Civita connection. For example, let us compute the third formula. First remark that the new torsion form $T'$ can be expressed as

$$
T' = e^{2f} \cdot T + (1 - e^{2f}) \cdot \eta \wedge d\eta - 2 \cdot e^{2f} \cdot (df \circ \varphi) \wedge F.
$$

In a fixed $g'$-orthonormal frame $e'_1 := e^{-f} \cdot e_1, \ldots, e'_{2k} := e^{-f} \cdot e_{2k}, e'_{2k+1} := e_{2k+1} = \xi$ we obtain

$$
\nabla_X^\prime \psi = \nabla_X g' \psi + \frac{1}{8} \sum_{i,j=1}^{2k} T'(X, e'_i, e'_j) \cdot e_i \cdot e_j \cdot \psi + \frac{1}{4} \sum_{i=1}^{2k} T'(X, e'_i, \xi) \cdot e_i \cdot \xi \cdot \psi
$$

$$
= \nabla_X g' \psi + \frac{1}{8} \sum_{i,j=1}^{2k} T(X, e_i, e_j) \cdot e_i \cdot e_j \cdot \psi - 2 \cdot (X \cdot (df \circ \varphi) \wedge F) \cdot \psi + \frac{e^{-f}}{4} \cdot (X \cdot d\eta) \cdot \xi \cdot \psi.
$$

We used that $\eta \wedge d\eta(X, e_i, e_j) = 0$ vanishes for $X$ orthogonal to $\xi$ and $i, j \leq 2k$ as well as the formula

$$
T'(X, e'_i, \xi) = (\xi \wedge T')(X, e'_i) = d\eta(X, e'_i) = e^{-f} \cdot d\eta(X, e_i).
$$

We apply now the formula for the relation between the spinorial connections $\nabla^g'$ and $\nabla^g$ and combine the latter formula with the definition of the connection $\nabla$. Then the term involving $(X \cdot d\eta) \cdot \xi \cdot \psi$ cancels and finally we get the required formula.

In dimension five one verifies directly that the following endomorphisms acting on the different parts of the spinor coincide:

$$
2 \cdot X \cdot (df \circ \varphi) \wedge F = df \cdot X \cdot X \cdot df \quad \text{on the bundle } \Sigma^\pm,
$$

$$
2 \cdot X \cdot (df \circ \varphi) \wedge F = -df \cdot X + X \cdot df \quad \text{on the bundle } \Sigma^2.
$$

Consequently, the formulas for the transformation of the spinorial covariant derivative in directions $X$ orthogonal to $\xi$ can be simplified,

$$
\nabla_X^\prime \psi = \nabla_X \psi \quad \text{for } \psi \in \Gamma(\Sigma^\pm),
$$

$$
\nabla_X^\prime \psi = \nabla_X \psi + \frac{1}{2} (df \cdot X \cdot X \cdot df) \cdot \psi \quad \text{for } \psi \in \Gamma(\Sigma^2).
$$

We see that a $\nabla$-parallel spinor in the bundle $\Sigma^2$ never transforms into a $\nabla'$-parallel spinor excepted if $f$ is constant. In the bundle $\Sigma^\pm$ the situation is different, it may occur.

**Theorem 5.1.** Let $(M^5, g, \xi, \eta, \varphi, \nabla)$ and $(M^5, g', \xi, \eta, \varphi, \nabla')$ be two 5-dimensional almost contact metric normal manifolds with Killing vector field $\xi$ which are connected by a special conformal transformation. If $d\eta$ is antiself-dual then a spinor is $\nabla$-parallel if and only if it is $\nabla'$-parallel.

In dimension five the Lee form determines completely the exterior derivative $dF$ of the fundamental form, provided the vector field $\xi$ is Killing:

$$
dF = \theta \wedge F, \quad d^2 F = - (\theta \circ \varphi) \wedge F, \quad T = \eta \wedge d\eta - (\theta \circ \varphi) \wedge F.
$$

Consider a structure with closed Lee form $\theta$ and solve locally the equation $2 \cdot df + \theta = 0$. Since $\theta(\xi) = 0$, the function $f$ does not depend on $\xi$ and we obtain

$$
d(F') = d(e^{2f} \cdot F) = e^{2f} \cdot (2 \cdot df \wedge F + dF) = e^{2f} \cdot (2 \cdot df + \theta) \wedge F = 0.
$$

**Proposition 5.4.** Let $(M^5, g, \xi, \eta, \varphi)$ be a 5-dimensional, normal almost contact metric manifold with closed Lee form and Killing vector field $\xi$. Then the space is locally special conformal to a quasi-Sasakian manifold.
We investigate the algebraic equation \((2 \cdot d\Phi - T) \cdot \psi = 0\) for the differential of a dilation function \(\Phi\) on a 5-dimensional manifold. If the almost contact structure is regular this second Killing equation means that the spinor is projectable. Again, there are two cases depending on the algebraic type of the spinor. The corresponding relations between the 1-form \(d\Phi\) and the 3-form \(T\) were computed in the paper [FL] (Lemma 7.2 and lemma 7.5). In order to formulate these relations in an invariant way, let us introduce the Hodge operator \(*_4\) acting in the 4-dimensional orthogonal complement of the vector field \(\xi\).

**Proposition 5.5.** The equation \((2 \cdot d\Phi - T) \cdot \psi = 0\)

1. admits a solution \(0 \neq \psi \in \Sigma^\pm\) if and only if \(2 \cdot d\Phi = -\theta, \quad *_4 d\eta = -d\eta\).
2. admits a solution \(0 \neq \psi \in \Sigma^2\) if and only if \(2 \cdot d\Phi = \theta, \quad *_4 d\eta = d\eta\).

**Theorem 5.2.** Let \((M^5, g, \xi, \eta, \varphi)\) be a normal almost contact metric manifold with Killing vector field \(\xi\). If there exist a function \(\Phi\) and an arbitrary, nontrivial spinor field \(\psi\) such that

\[(2 \cdot d\Phi - T) \cdot \psi = 0,\]

then the Lee form is closed and \(M^5\) is locally special conformal to a quasi-Sasakian manifold.

We discuss the case that \(\psi\) is a function \(\Phi\) and let us introduce the Hodge operator \(*_4\) acting in the 4-dimensional orthogonal complement of the vector field \(\xi\).

**Theorem 5.3.** Let \((M^5, g, \xi, \eta, \varphi)\) be a normal almost contact metric spin manifold with Killing vector field \(\xi\) and let \(\Phi\) be a smooth function. If there exists an arbitrary, nontrivial spinor field \(\psi\) such that

\[(2 \cdot d\Phi - T) \cdot \psi = 0,\]

then \(M^5\) is specially conformal to a Sasakian manifold \((M^5, g', \xi, \eta, \varphi)\). The metric \(g'\) as well as the torsion \(T'\) of the connection \(\nabla'\) are given by the formulas

\[g' = e^{-2\Phi} \cdot g + (1 - e^{-2\Phi}) \cdot \eta \otimes \eta, \quad F' = C \cdot a \cdot F = C \cdot d\eta',\]

i.e., the manifold \((M^5, g', \xi, \eta, \varphi)\) is \(\alpha\)-Sasakian.

**Theorem 5.4.** Let \((M^5, g', \xi, \eta, \varphi)\) be a normal almost contact metric spin manifold with Killing vector field \(\xi\) and let \(\Phi\) be a smooth function. If there exists an arbitrary, nontrivial spinor field \(\psi\) such that

\[(2 \cdot d\Phi - T) \cdot \psi = 0,\]

then \(M^5\) is specially conformal to a Sasakian manifold \((M^5, g', \xi, \eta, \varphi)\). The metric \(g'\) as well as the torsion \(T'\) of the connection \(\nabla'\) are given by the formulas

\[g' = e^{-2\Phi} \cdot g + (1 - e^{-2\Phi}) \cdot \eta \otimes \eta, \quad T' = T + 2 \cdot (d\Phi \circ \varphi) \wedge F.\]

The spinor field \(\psi\) is a solution of the algebraic equation \(T' \cdot \psi = 0\) and for any spinor field in \(\Sigma^2\) we have

\[\nabla'_{\xi} \psi = \nabla_{\xi} \psi + \frac{1}{2} \cdot (X \cdot d\Phi - d\Phi \cdot X) \cdot \psi, \quad \nabla'_{\xi} \psi = \nabla_{\xi} \psi.\]

We did not assume that the spinor field \(\psi\) is \(\nabla\)-parallel. But now let us consider a \(\nabla\)-parallel spinor field \(\psi\) in \(\Sigma^2\) on a quasi-Sasakian spin manifold,

\[\nabla \psi = 0, \quad (2 \cdot d\Phi - T) \cdot \psi = 0, \quad dF = 0.\]

The relation \(d\eta = a \cdot F\) yields that the function \(a\) is constant and, consequently, the dilation function \(\Phi\) is constant, too.
Theorem 5.4. Let \((M^5, g, \xi, \eta, \varphi)\) be a quasi-Sasakian spin manifold with a Killing vector \(\xi\) and let \(\Phi\) be a smooth function. If there exists a nontrivial \(\nabla\)-parallel spinor field \(\psi \in \Gamma(\Sigma)\) such that

\[
(2 \cdot d\Phi - T) \cdot \psi = 0, \quad \nabla \psi = 0,
\]

then the dilation function \(\Phi\) is constant and \(M^5\) is special conformal homothetic to a Sasakian manifold of Example 4.1.

Example 5.1. Consider the manifolds \(M^5(a, b, c, d)\). Using the formula for the 2-form \(d\eta\) as well as the matrix representation of the 5-dimensional Clifford algebra we compute that the endomorphism \(T = \eta \wedge d\eta\) acts on \(\Sigma^2\) as a symmetric endomorphism with two eigenvalues \(\pm \sqrt{(a-d)^2 + 4b^2 + 4c^2}\). In particular, a \(\nabla\)-parallel spinor satisfying the algebraic equation \(T \cdot \psi = 0\) exists only in case that \(a = d\) and \(b = 0, c = 0\).

6. Harmonic 1-forms in the presence of \(\nabla\)-parallel spinors

A classical Theorem of Tachibana states that on a compact \(K\)-contact manifold \((M^{2k+1}, g, \xi, \eta, \varphi)\) any harmonic 1-form \(\omega\) is orthogonal to \(\eta, \xi \neq \omega = 0\) (see [Blair], page 69). The \(K\)-contact condition means by definition that \(\xi\) is a Killing vector field and \(d\eta\) is proportional to the fundamental form \(F\), \(d\eta = 2 \cdot F\). In particular, in Tachibana’s Theorem there is no assumption about the Nijenhuis tensor \(N\). We will prove a similar result, but in a quite different situation.

Theorem 6.1. Let \((M^{2k+1}, g, \xi, \eta, \varphi)\) be a compact, almost contact metric manifold with totally skew-symmetric Nijenhuis tensor \(N\) and Killing vector field \(\xi\). Denote by \(T\) the torsion form of its contact unique connection \(\nabla\) and suppose that

\[
\xi \not\subset \delta(T - d^2 F) = 0
\]

holds. Then any harmonic 1-form is orthogonal to \(\eta\).

Proof. Basically, we follow the proof of Tachibana’s Theorem. A harmonic 1-form \(\omega\) on a compact Riemannian manifold is invariant under any Killing vector field, \(\mathcal{L}_\xi \omega = 0\). Since

\[
0 = \mathcal{L}_\xi \omega = \xi \not\subset d\omega + d(\xi \not\subset \omega) = d(\xi \not\subset \omega)
\]

the function \(\omega(\xi)\) is constant. We introduce the 1-form \(\beta := \omega - \omega(\xi) \cdot \eta\) and compute

\[
\Delta(\beta) = -\omega(\xi) \cdot \Delta(\eta) = -\omega(\xi) \cdot (\delta d(\eta) + d\delta(\eta)) = -\omega(\xi) \cdot \delta d(\eta).
\]

On the other hand, since \(\xi\) is a Killing vector field, we have

\[
\delta d\eta(X) - \frac{1}{2} \cdot \eta(X) \cdot |d\eta|^2 = \delta d\eta(X) - 2 \cdot \eta(X) \cdot |\nabla \eta|^2 = \delta(\eta \wedge d\eta)(X, \xi) = \delta(T - d^2 F)(X, \xi)
\]

and the assumption yields the equality

\[
2 \cdot \delta d(\eta) = |d\eta|^2 \cdot \eta.
\]

Using that \(\beta\) and \(\eta\) are orthogonal, \((\beta, \eta) = 0\), we obtain

\[
2 \cdot \int_{M^{2k+1}} \Delta(\beta) (\beta) = - \int_{M^{2k+1}} \omega(\xi) \cdot |d\eta|^2 \cdot (\beta, \eta) = 0
\]

and, consequently, \(\beta\) is a harmonic 1-form. Since \(d\eta\) is a nonvanishing 2-form we conclude that \(\omega(\xi) = 0\).

Remark 6.1. We proved in fact that any harmonic 1-form satisfying the algebraic condition \(\delta(T - d^2 F)(\omega, \xi) = 0\) is orthogonal to \(\eta\).

Example 6.1. The latter Theorem applies for quasi-Sasakian manifolds \((N = 0, dF = 0\) and \(\xi\) Killing) with divergence free torsion form \(T\). Locally homogeneous examples of this type have been discussed in section 4.
Example 6.2. Consider a quasi-Sasakian spin manifold. The spinor bundle $\Sigma$ of $\mathcal{M}^{2k+1}$ splits into several parts and precisely two subbundle $\Sigma^\pm$ are 1-dimensional. If there exists a $\nabla$-parallel spinor in one of these bundles, then the condition of the Theorem is satisfied. Indeed, using Proposition 9.1 of [FI], the integrability condition of the parallel spinor yields the equation

$$\delta T(X, \xi) = -\text{Ric}^\nabla(\xi, X) = 0.$$ 

Let us again discuss the 5-dimensional quasi-Sasaki case with a $\nabla$-parallel spinor in more details. If the spinor is a section in one of the bundles $\Sigma^\pm$, then any harmonic 1-form is orthogonal to $\eta$ (see Example 6.2). Suppose next that the $\nabla$-parallel spinor is in $\Sigma^2$. The Ricci tensor of the connection $\nabla$ is given by the curvature form $\Omega^A$ of the connection $A$ (see Theorem 2.1):

$$\text{Ric}^\nabla(Y, Z) = \Omega^A(Y, \varphi(Z)).$$

From the formula (see [FI], section 2)

$$\delta(T)(X, \xi) = \text{Ric}^\nabla(\xi, X) - \text{Ric}^\nabla(X, \xi) = \Omega^A(\xi, \varphi(X)) - \Omega^A(X, \varphi(\xi)) = \Omega^A(\xi, \varphi(X))$$

we see that $\xi \lln \delta(T)$ coincides with $(\xi \lln \Omega^A) \circ \varphi$. We compute $\Omega^A \wedge F$. In fact, we have the general formula (see [FI])

$$\Omega^A \wedge F = \delta(R^n) = \frac{3}{2} \cdot dt - \sigma^T, \quad \sigma^T := \frac{1}{2} \sum_{i=1}^5 (e_i \lln T) \wedge (e_i \lln T)$$

and the structure equations for a quasi-Sasakian manifold with parallel spinor field in $\Sigma^2$ yield

$$\xi \lln dt = \xi \lln \sigma^T = 0.$$ 

Finally, we conclude that $\xi \lln (\Omega^A \wedge F) = 0$ and the form $\xi \lln \delta(T)$ vanishes.

Theorem 6.2. Let $(\mathcal{M}^5, g, \xi, \eta, \varphi)$ be a compact, 5-dimensional quasi-Sasakian manifold with a $\nabla$-parallel spinor. Then any harmonic 1-form is orthogonal to $\eta$.

References

[Agri] I. Agricola, Connections on naturally reductive spaces, their Dirac operator and homogeneous models in string theory, to appear in Comm. Math. Phys.

[Blair] D. Blair, Contact manifolds in Riemannian geometry, Lect. Notes No. 509, Springer-Verlag 1976.

[BG] J. P. Bourguignon et P. Gauduchon, Spineurs, operateurs de Dirac et variations de metriques, Comm. Math. Phys. 144 (1992), 581-599.

[CG] D. Chinea and J. C. Marrero, Classifications of almost contact metric structures, Rev. Roumaine Math. Pures Appl. 37 (1992), 199-212.

[DI] P. Dalakov and S. Ivanov, Harmonic spinors of Dirac operators of connections with torsion in dimension 4, Class. Quant. Grav. 18 (2001), 253-265.

[Fla] H. Flanders, Differential forms with applications to the physical sciences, Academic Press, New-York-London 1963.

[Fri1] Th. Friedrich, Der erste Eigenwert des Dirac Operators einer kompakten, Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung, Math. Nachr. 97 (1980), 117-146.

[Fri2] Th. Friedrich, Dirac operators in Riemannian geometry, Graduate Studies in Mathematics, vol. 25, American Mathematical Society, Prov., 2000.

[FI] Th. Friedrich and S. Ivanov, Parallel spinors and connections with skew-symmetric torsion in string theory, Asian Journ. Math. 6 (2002), 303-336.

[HP] P. S. Howe and G. Papadopoulos, Finitness and anomalies in (4,0) supersymmetric sigma models, Nucl. Phys. B 381 (1992), 360.

[IP] S. Ivanov and G. Papadopoulos, Vanishing theorems and string background, Class. Quant. Grav. 18 (2001), 1089-1110.

[Jen] G. R. Jensen, Imbeddings of Stiefel manifolds into Grassmannians, Duke Math. Journ. 42 (1975), 397-407.

[SSTP] P. Spindel, A. Sevrin, W. Troost, A. van Proeyen, Extended supersymmetric σ-models on group manifolds, Nucl. Phys. B 308 (1988), 662-698.

[Stro] A. Strominger, Superstrings with torsion, Nucl. Phys. B 274 (1986), 253-284.
Thomas Friedrich
Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Sitz: WBC Adlershof
D-10099 Berlin, Germany
friedric@mathematik.hu-berlin.de

Stefan Ivanov
Faculty of Mathematics and Informatics
University of Sofia “St. Kl. Ohridski”
bvd. James Bourchier 5
1164 Sofia, Bulgaria
ivanovsp@fmi.uni-sofia.bg