We investigate twisted bilayer graphene near charge neutrality using a generalized Bistritzer-MacDonald continuum model, accounting for corrugation effects. The Fermi velocity vanishes for particular twist angles properly reproducing the physics of the celebrated magic angles. Using group representation theory, we identify all contact interaction potentials compatible with the symmetries of the model. This enables us to identify two classes of quartic interactions leading to either the opening of a gap or to nematic ordering. We then implement a renormalization group analysis to study the competition between these interactions for a twist angle approaching the first magic value. This combined group theory-renormalization study reveals that the proximity to the first magic angle favors the occurrence of a layer-polarized, gapped state with a spatial modulation of interlayer correlations, which we call nematic insulator.

I. INTRODUCTION

Within band theory, a reasonable estimate for the relative strength of the quasiparticles kinetic energy is the ratio between the bandwidth of the conducting bands, and some interaction energy. In normal metals, where the density of states at the Fermi energy is nonzero, the excitations of the Fermi sea largely screen the Coulomb interactions, which renormalizes their strength downwards in a dramatic way. But when the bands near the Fermi energy disperse very little, even small interactions can lead to significant, qualitative consequences.

The prominent example of such systems is provided by Landau levels and the associated fractional quantum Hall effects \[1,2\]. More recently, a different class of materials with vanishing bandwidth was uncovered in twisted bilayer graphene (TBG). When two sheets of graphene are rotated with respect to one another by a small angle of about 1.1°, a large moiré pattern forms with several thousands of atoms per unit cell. Remarkably, for some twist angles—the so-called magic angles—the Fermi velocities of the Dirac cones originating from each layer vanish exactly \[3\]. Other phenomena accompany this effect such as a minimal bandwidth and a sizeable band gap between the conducting and excited bands, under some conditions \[3,4\].

TBG gained considerable momentum after the experimental discovery of correlated insulators at various fillings, with neighboring regions of possibly-unconventional superconductivity \[7,10\]. In addition, scanning tunneling microscopy and transport data point toward “ferromagnetism” and an “anomalous Hall effect” in TBG, but also in thicker van der Waals heterostructures like trilayer graphene \[11,12\]. The tunability of TBG through a rich phase diagram by electronic gating also sparked numerous works in new directions. While the origin of the superconductivity remains unclear \[15,22\], evidence is mounting toward the strongest insulator emerging at charge neutrality—where band theory alone would predict a semimetal—with a charge gap of around 0.86 meV in the most angle-homogeneous devices \[23\]. Other scanning measurements find a three-fold rotation symmetry breaking near the first magic angle at charge neutrality \[24\]. The aim of this article is to unveil the nature of this rotation symmetry breaking insulator at charge neutrality close to the first magic angle and to provide methodology to analyze its occurrence.

The study of interacting phases in systems with vanishing bandwidth is notoriously difficult. Our strategy to tackle this challenge in TBG is based on the combination of an algebraic identification of interactions preserving the symmetries of the low energy description of TBG and a renormalization group approach to select the generically favored interaction as the twist angle approaches its first magic value. In the present case, an additional obstacle lies in the absence of a simple description of the moiré pattern in TBG, thus inhibiting the use of standard field theories.

Our starting point for the non-interacting continuum model of two twisted layers of graphene accounts for several channels of interlayer hoppings. These interlayer hoppings renormalize the Fermi velocity, leading to its vanishing at magic twist angles. We develop a diagrammatic technique to compute the Green’s function and thereby the magic angles to arbitrary order in the interlayer hopping strength \[\alpha\] and non-perturbatively in the imbalance between different hopping channels \[\beta\]. This non-interacting model is then complemented with interactions. By formal group theory considerations, we identify all symmetry-allowed contact or short-ranged interactions. To determine the most favorable one, we develop a renormalization group (RG) technique. The vanishing of the bandwidth at the magic angle leads to a singular behavior of the RG: indeed, any interacting potential, while usually treated in perturbations, now corresponds to a dominant energy scale. Moreover the first magic angle is not determined by a specific value of a parameter of the free field theory, but through a systematic resummation of interlayer hopping terms. To overcome these difficulties, we study the scaling behavior of all interacting potentials as the twist angle is varied. When approaching the first magic angle, we monitor the relevance in the RG sense of all potentials, thereby identifying the dominant interacting instability. We find that as
the twist angle approaches the first magic value, a state
with both a gap opening and a periodic modulation of
interlayer correlations is favored. We call this phase a
nematic insulator.

II. FREE ELECTRON MODEL

Following the seminal work of Ref. [25], we treat TBG
as a periodic moiré superlattice characterized by a twist
angle \( \theta \). The top and bottom Dirac cones of the same
valley, denoted \( K_t \) and \( K_b \), delineate the mini-Brillouin
zone (mBZ) of the superlattice (Fig. 1). Focusing on
the low energy and long wavelength description of TGB,
we restrict ourselves to small momentum transfers that
are diagonal in valley, and thus occur within a single
mBZ [25]. The characteristic kinetic energy scale of the
model, set by the typical difference of kinetic energy of
electrons in different layers, is \( E_c = 2v_0 K \sin(\theta/2) \), where
\( v_0 \) and \( K \) are respectively the Fermi velocity and the
Dirac momentum of monolayer graphene. In addition to
the kinetic energy in each layer, the single-particle Hamil-
tonian involves two different interlayer hopping ampli-
ditudes. First, the amplitude \( w_1 \) of interlayer hopping that is
off-diagonal in graphene sublattice is typically of order
\( w_1 \approx 110 \text{ meV} [25, 27] \). Its strength relative to the kinetic energy
is measured by the dimensionless parameter \( \alpha = w_1 / E_c \). Second, the amplitude \( w_2 = \beta w_1 \) of inter-
layer hopping that is diagonal in graphene sublattice is
measured by the relative strength \( \beta \in [0, 1] \) in compari-
on to off-diagonal hopping. This relative strength is
difficult to determine precisely in experiments, being
affected by corrugation effects, with typical values evalu-
ated as \( \beta \approx 0.82 [28, 29] \). Here we keep \( \beta \) as a free pa-
rameter. Note that our model thus interpolates between
the Bistritzer-MacDonald continuum (BMC) model for

\[ \beta = 1 [3] \] and a chirally symmetric continuum (CSC)
model for \( \beta = 0 [5] \).

Following Ref. [3], we use a rotated basis where the
Dirac cones \( K_{t,b} \) of the two layers have the same \((k_x, k_y) \)
coordinates in the mBZ, and measure all energies in units
of \( E_c \) (see Appendix A for details). The effective Hamilton-
on then reads \( H_0 = H_0 + H_\alpha \) with

\[ H_0 = i (\sigma \cdot \partial) \tau_0, \quad H_\alpha = \alpha \sum_{j=1}^3 e^{-i q_j \tau^+} + \text{h.c.}, \quad (1) \]

where \( \partial = (\partial_x, \partial_y) \) and the hopping matrices \( T_j^+ \) are

\[ T_j^+ = \left( \beta \sigma_0 + e^{i(j-1)\pi/3} \sigma_+ + e^{-i(j-1)\pi/3} \sigma_- \right) \tau_+. \quad (2) \]

Here we introduced two sets of Pauli matrices, \( \sigma \) and \( \tau \),
which describe respectively the sublattice and layer sec-
tors, with \( \sigma_z = \pm 1 = A/B \) and \( \tau_z = \pm 1 = \text{top/bottom} \). The
low-energy physics of this model is nontrivial even
without interactions. Indeed, the interlayer couplings
hibit diagonalizing \( H_0^\prime \). This forbids the use of a simple effective theory valid for all twisting angles \( \theta \) in

\[ \beta = 1 [3] \] and a chirally symmetric continuum (CSC)
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low-energy physics of this model is nontrivial even
without interactions. Indeed, the interlayer couplings
hibit diagonalizing \( H_0^\prime \). This forbids the use of a simple effective theory valid for all twisting angles \( \theta \) in
the vicinity of the magic values. As a result, we resort to a free electron model in which interlayer hopping effects are accounted for by a self-energy which is calculated in a perturbative expansion in $\alpha$. Denoting $G_0$ and $\Sigma$ the translationally-invariant components of the propagator corrected by interlayer hoppings and the self-energy respectively, we have $(G_0^{-1}) = H_0 - \partial_\tau - \Sigma \approx N_\psi [v(\alpha,\beta)(\sigma \cdot \partial_\tau)\partial_\tau - \partial_\tau]$, where $\partial_\tau$ represents the partial derivative with respect to imaginary time, $N_\psi$ is a wavefunction normalization and $v(\alpha,\beta)$ the Fermi velocity renormalized by the hopping processes (see Appendix [B]). An expansion to order 6 in $\alpha$ but exact in $\beta$, diagrammatically represented in Fig. [2] leads to

$$N_\psi v(\alpha,\beta) = 1 - 3\alpha^2 + \alpha^4 (1 - \beta^2)^2$$

$$- \frac{3}{49} \alpha^6 (37 - 112\beta^2 + 119\beta^4 - 70\beta^6).$$

We call $\alpha_0(\beta)$ the lowest value of $\alpha$ for which this Fermi velocity vanishes, which sets the first magic angle value to be approximately 1.1°. As shown in Fig. [3] this first magic value depends weakly on the parameter $\beta$, and thus on corrugation, and ranges from $\alpha_0(1) = 0.598$ for the BMC model to $\alpha_0(0) = 0.585$ for the CSC model. These constitute our first results.

### III. SYMMETRY-ALLOWED INTERACTIONS

We now identify all short-ranged interaction potentials allowed by the symmetries of the model. In order to do so we turn to a field theoretic formalism and consider the Euclidean action, $S = S'_0 + S_{\text{int}}$, written as a sum of the free electron term

$$S'_0 = \int d^2r \, d\tau \, \psi^\dagger (H'_0 - \partial_\tau) \psi,$$

and an interaction term $S_{\text{int}}$ which includes generic local quartic couplings between the fermionic fields $\psi^\dagger$ and $\psi$. Using group theoretic methods, detailed in Appendix [C], we identify all couplings allowed by the symmetries of the low energy model [1] [30]. This amounts to identifying scalar invariants built as direct products of irreducible representations of the corresponding symmetry group. We find that the allowed couplings are (i) 8 channels originating from one-dimensional (1d) corepresentations; (ii) 4 channels originating from 2d corepresentations:

$$S_{\text{int}} = - \sum_{i=1}^{8} q_i \int d^2r \, d\tau \, \rho^{(i)}(r) \rho^{(i)}(r)$$

$$- \sum_{j=1}^{4} \lambda_j \int d^2r \, d\tau \, J^{(j)}(r) \cdot J^{(j)}(r),$$

where the densities $\rho^{(i)}(r) = \psi^\dagger R^{(i)}(r) \psi$ and currents $J^{(j)}(r) = \psi^\dagger M^{(j)}(r) \psi$ involve coupling matrices $R^{(i)}(r)$ and $M^{(j)}(r)$. Following our choice of coordinates for the fields in Eq. [1], the coupling matrices in the rotated basis, which enter Eq. [5], are obtained through $R^{(i)} = A(r) R^{(i)} A^\dagger(r)$ and $M^{(j)} = A(r) M^{(j)} A^\dagger(r)$, where $A(r)$ is the transformation matrix of the field (see Appendix [A]). The coupling matrices $R^{(i)}$ and $M^{(j)}$ are provided in Tab. [1]. The couplings $g_i$ and $\lambda_j$ are the amplitudes associated with the corresponding coupling potentials.

| Corep. | $A_1^+$ | $a_1^+$ | $A_2^+$ | $a_2^+$ | $A_1^-$ | $a_1^-$ | $A_2^-$ | $a_2^-$ |
|--------|---------|---------|---------|---------|---------|---------|---------|---------|
| $R^{(i)}$ | $\sigma_0 \tau_0$ | $\sigma_0 \tau_z$ | $\sigma_0 \tau_x$ | $\sigma_0 \tau_y$ | $\sigma_\tau_y$ | $\sigma_\tau_x$ | $\sigma_\tau_z$ | $\sigma_\tau_0$ |
| $C_2$ | ✓ | ✓ | ✓ | ✓ | |
| $P$ | ✓ | ✓ | ✓ | ✓ | ✓ |

TABLE I. One-dimensional (top) and two-dimensional (bottom) corepresentations (corep.) of the magnetic symmetry group of the continuum model, with their associated coupling matrices $R^{(i)}$ and $M^{(j)}$ expressed in terms of the Pauli matrices in sublattice ($\sigma$) and layer ($\tau$) subspaces. These coupling matrices are normalized such that $\text{Tr}[M^{(j)} \cdot (M^{(j)})] = \text{Tr}[R^{(i)} \cdot (R^{(i)})^\dagger] = 4$. Each one-dimensional corep. can either preserve ($✓$) or break the combination of inversion and time reversal symmetries $IT$, the mirror symmetry $C_2$, and the particle-hole antisymmetry $P$, while preserving the three-fold rotational symmetry $C_3$. The ± exponents label the eigenvalue of the IT symmetry.

The moiré pattern is invariant under four discrete “symmetries”: (i) the $2\pi/3$ rotation $C_3 = e^{2\pi i/3}\tau_0$ around the z axis orthogonal to the bilayer together with (ii) the π rotation $C_2 = \sigma_\tau x \tau_x$ around the x axis of Fig. [1] generate the point group $D_3$. (iii) The composition $IT = \sigma_\tau y \mathcal{K}$ of inversion and time reversal is an antiunitary symmetry, where $\mathcal{K}$ denotes complex conjugation, and (iv) the unitary particle-hole antisymmetry $\mathcal{P} = \sigma_\tau x \tau_z$, which satisfies $\{P, H'_0\} = 0$ [32]. The group generated by $D_3$ and $P$ comprises all unitary operations that leave the Hamiltonian invariant up to a sign. We refer to this ensemble as the unitary group $\bar{D}_3$ of the model. It can be decomposed into the semi-direct product $\bar{D}_3 = \{e, P, \bar{e}, \bar{P}\} \times D_3$, where $e = \sigma_0 \tau_0$ is the identity operation, $\bar{e} = (PC_2)^2 = -e$ and $\bar{P} = eP$. The dichromatic magnetic group $\mathcal{M}$ generated by $\bar{D}_3$ and $IT$ can be written as the direct product $\mathcal{M} = \bar{D}_3 \times \{e, IT\}$. Using the Schur-Frobenius criterion [33,38], we determine the corepresentations (corep.) of $\mathcal{M}$ from the irreducible representations of $\bar{D}_3$, which can be constructed from that of $D_3$ by induction and basic properties of linear representation theory (see Appendix [C]).

We combine the resulting coupling matrices of Tab. [1] into three sets. The eight interactions originating from 1d coreps. correspond to the density-density couplings diagonal in sublattice while preserving $C_3$. Out of these
Let us first discuss the nature of the phases induced by these couplings. The interactions of type (i), symmetric in sublattices, neither open a gap at the Dirac point nor induce a density modulation. On the other hand, interactions of type (ii) generate phases with a gap $\Delta_\mu \propto g_\mu \langle \psi^\dagger \sigma_z \tau_\mu \psi \rangle$, reminiscent of the gap opening in Boron Nitride, see Fig. 4(a). These various gapped phases are distinguished by their layer correlations. Current-current interactions of type (iii) lead to radically different phases, in which the Dirac cones of the two layers are shifted with respect to each other by a momentum $2G_\mu \propto \lambda_\mu \langle \psi^\dagger \sigma \tau_\mu \psi \rangle$, as shown in Fig. 4(b). They generate gapless phases with $C_3$-breaking density modulations. These spatial modulations are detected when probing the electronic density from one side of the bilayer, which amounts to coupling the local probe asymmetrically to the top and bottom wavefunctions, thus scanning some interlayer density of the form $|\psi_z + r \psi_b|^2$, where $0 < r < 1$ is the asymmetry parameter. In Fig. 4(c) we compare the corresponding density for an asymmetry $r = 1/2$ in the absence of any instability in Fig. 4(b) with that in the presence of a $C_3$-breaking instability in Fig. 4(d). Stripe-like modulations of the interlayer-correlated density are readily observed in this last case.

To gain further insight into the behavior of these phases close to the first magic angle, we now study their mean-field behavior. As we will show later using a renormalization group analysis, only four out of the twelve couplings are sensitive to the proximity of the magic angle. They correspond to the interaction potentials diagonal in layers, and originate from the $a_1^z$, $a_2^z$ coreps. of set (ii) with respective amplitude $g_z$, $g_0$, and the $E_z^x$, $E_z^y$ coreps. of set (iii) with amplitudes $\lambda_0$, $\lambda_\tau$. The corresponding order parameters satisfy the self-consistency equations

\[
\Delta_{0/z} = -2g_{0/z} \int \frac{d^2q}{(2\pi)^3} \frac{d^2q}{(2\pi)^3} \langle \psi_{q,\omega}^\dagger \sigma_z \tau_0/z \psi_{q,\omega} \rangle, \quad (6a)
\]

\[
\mathcal{G}_{0/z} = -2\lambda_{0/z} \int \frac{d^2q}{(2\pi)^3} \frac{d^2q}{(2\pi)^3} \langle \psi_{q,\omega}^\dagger \sigma \tau_0/z \psi_{q,\omega} \rangle. \quad (6b)
\]

The correlators in Eq. (6) are the translationally-invariant parts of statistical averages computed over the Bloch Hamiltonian density $H_{MF} = H_0 + \sigma \cdot \left( \mathcal{G}_0 \tau_0 + \mathcal{G}_z \tau_z \right) + \sigma_z \left( \Delta_0 \tau_0 + \Delta_z \tau_z \right)$. The corrections by interlayer hoppings of the correlators in Eq. (6) are obtained within a perturbation expansion in $\alpha$. Incorporating the hoppings $H_\alpha$ leads to an enhancement of the order parameters by factors $N_0^{(\mathcal{G}/\Delta)}(\alpha, \beta)$, as is the case for the renormalization of the Fermi velocity; they are calculated diagrammatically to sixth order in $\alpha$ in Appendix D.

The resulting dependence of each separate order parameter on the proximity to the magic angle and for various strengths of the couplings is depicted in Fig. 5. The insulating phases, characterized by a gap $\Delta_0$ or $\Delta_z$, develop at a critical coupling which decreases as the parameter $\alpha$ approaches its magic value $\alpha_0$. Such gapped phases generically occur in some range of twist angles around the magic value, in agreement with the experimental findings of Ref. [41]. At the mean-field level and

IV. NATURE OF THE CORRELATED PHASES

![Image 4](image4.png)

FIG. 4. Schematic dispersion relation of (a) a gapped layer-polarized correlated phase and (b) a density modulated phase. While the gapped phase is characterized by the amplitude of the gap $\Delta_x$ opening at the $K_x$ and $K_b$ Dirac points, the second phase is characterized by a shift of these Dirac points of amplitude $\mathcal{G}_z$. This shift leads to a modulation of the relative amplitude of wavefunctions between the two layers, revealed in the density $|\psi_z + \psi_b/2|^2$ probed e.g. by a STM tip located on the top layer. The behavior of this density is shown (c) without any shift and (d) for a shift $\mathcal{G}_z = 0.3 e_y$. The spatial $C_3$-breaking of this phase is clearly manifested by the appearance of stripes for this density, perpendicular to $\mathcal{G}_z$. 

eight couplings, (i) the four interactions associated with 1d coreps, which preserves $IT$ are those symmetric on the A/B sublattices, of the form $\langle \psi^\dagger \sigma_z \tau_\mu \psi \rangle$ for $\mu = 0, x, y, z$ [39]. They are distinguished by their breaking of $C_2$ or $P$ symmetries. (ii) The four interactions associated with 1d coreps, which break $IT$ are those which are antisymmetric in the A/B sublattices, with couplings of the form $\langle \psi^\dagger \sigma_z \tau_\mu \psi \rangle$. Similarly to set (i), they can break $C_2$ or $P$. Finally, (iii) four interactions originating from 2d coreps. are current-current couplings between the layers, off-diagonal in sublattices, of the form $\langle \psi^\dagger \sigma \tau_\mu \psi \rangle$. They break all symmetries of the free model, and in particular the three-fold rotational symmetry $C_3$. 

IV. NATURE OF THE CORRELATED PHASES

Let us first discuss the nature of the phases induced by these couplings. The interactions of type (i), symmetrical in sublattices, neither open a gap at the Dirac point nor induce a density modulation. On the other hand, interactions of type (ii) generate phases with a gap $\Delta_\mu \propto g_\mu \langle \psi^\dagger \sigma_z \tau_\mu \psi \rangle$, reminiscent of the gap opening in Boron Nitride, see Fig. 4(a). These various gapped phases are distinguished by their layer correlations. Current-current interactions of type (iii) lead to radically different phases, in which the Dirac cones of the two layers are shifted with respect to each other by a momentum $2G_\mu \propto \lambda_\mu \langle \psi^\dagger \sigma \tau_\mu \psi \rangle$, as shown in Fig. 4(b). They generate gapless phases with $C_3$-breaking density modulations. These spatial modulations are detected when probing the electronic density from one side of the bilayer, which amounts to coupling the local probe asymmetrically to the top and bottom wavefunctions, thus scanning some interlayer density of the form $|\psi_z + r \psi_b|^2$, where $0 < r < 1$ is the asymmetry parameter. In Fig. 4(a) we compare the corresponding density for an asymmetry $r = 1/2$ in the absence of any instability in Fig. 4(b) with that in the presence of a $C_3$-breaking instability in Fig. 4(d). Stripe-like modulations of the interlayer-correlated density are readily observed in this last case.

To gain further insight into the behavior of these phases close to the first magic angle, we now study their mean-field behavior. As we will show later using a renormalization group analysis, only four out of the twelve couplings are sensitive to the proximity of the magic angle. They correspond to the interaction potentials diagonal in layers, and originate from the $a_1^z$, $a_2^z$ coreps. of set (ii) with respective amplitude $g_z$, $g_0$, and the $E_z^x$, $E_z^y$ coreps. of set (iii) with amplitudes $\lambda_0$, $\lambda_\tau$. The corresponding order parameters satisfy the self-consistency equations

\[
\Delta_{0/z} = -2g_{0/z} \int \frac{d^2q}{(2\pi)^3} \frac{d^2q}{(2\pi)^3} \langle \psi_{q,\omega}^\dagger \sigma_z \tau_0/z \psi_{q,\omega} \rangle, \quad (6a)
\]

\[
\mathcal{G}_{0/z} = -2\lambda_{0/z} \int \frac{d^2q}{(2\pi)^3} \frac{d^2q}{(2\pi)^3} \langle \psi_{q,\omega}^\dagger \sigma \tau_0/z \psi_{q,\omega} \rangle. \quad (6b)
\]

The correlators in Eq. (6) are the translationally-invariant parts of statistical averages computed over the Bloch Hamiltonian density $H_{MF} = H_0 + \sigma \cdot \left( \mathcal{G}_0 \tau_0 + \mathcal{G}_z \tau_z \right) + \sigma_z \left( \Delta_0 \tau_0 + \Delta_z \tau_z \right)$. The corrections by interlayer hoppings of the correlators in Eq. (6) are obtained within a perturbation expansion in $\alpha$. Incorporating the hoppings $H_\alpha$ leads to an enhancement of the order parameters by factors $N_0^{(\mathcal{G}/\Delta)}(\alpha, \beta)$, as is the case for the renormalization of the Fermi velocity; they are calculated diagrammatically to sixth order in $\alpha$ in Appendix D.

The resulting dependence of each separate order parameter on the proximity to the magic angle and for various strengths of the couplings is depicted in Fig. 5. The insulating phases, characterized by a gap $\Delta_0$ or $\Delta_z$, develop at a critical coupling which decreases as the parameter $\alpha$ approaches its magic value $\alpha_0$. Such gapped phases generically occur in some range of twist angles around the magic value, in agreement with the experimental findings of Ref. [41]. At the mean-field level and
for a fixed $\alpha$, we find that a $\Delta_0$ insulator occurs for weaker couplings $g_0$ than the couplings $g_z$ required for the appearance of the $\Delta_z$ insulator. We will see that this hierarchy is modified when fluctuations are accounted for, demonstrating the necessity to develop the RG approach. The situation for the $C_3$ symmetry breaking phases with periodic modulation is different: while the phase which is antisymmetric in layers, characterized by a $\mathcal{G}_z$ momentum, is also favored by the vanishing bandwidth close to $\alpha_0$, the finite critical strength for the analogous phase symmetric in layer, associated with $\mathcal{G}_0$, does not depend on $\alpha$.

Having identified all interacting instabilities of TBG and established their strong enhancement close to the magic angle, we now study the competition between them by resorting to a renormalization group technique.

V. RENORMALIZATION GROUP PICTURE

As we have seen, the vanishing of the kinetic energy scale set by the renormalized velocity $v$ entails that the four non-trivial interactions are relevant close to the first magic angle. In order to identify the leading instability, we study via a RG approach the competition between them as $\alpha$ approaches the first magic value. Our starting point is the field theory described by the action $S = S_0 + S_{\text{int}}$ where the quadratic action $S_0$ given in Eq. (4) involves both the kinetic energy and the interlayer hoppings. We sum over the latter to capture the vanishing energy scale. Hence, we expand the correlation functions of the interacting theory not only in the coupling constants but also in the amplitude of interlayer hoppings.

Motivated by the experimental observations of an insulating behavior at charge neutrality, we carry out a “particle-hole” Hubbard-Stratonovich transformation suitable to describe gapped phases (as opposed to superconductors). We introduce the scalar bosonic fields $\phi_i$, $i = 1, \ldots, 8$ for each 1d channel and the vector bosonic fields $\varphi_j, j = 1, \ldots, 4$ for each 2d channel, so that the interaction part of the action $S$ becomes:

$$S_{\text{int}} \rightarrow \sum_{i=1}^{8} \int d^{d-1}r d\tau \left( \phi_i^2 + 2 \sqrt{\mu_i} \phi_i \psi^\dagger R^{(i)}(r) \psi \right) + \sum_{j=1}^{4} \int d^{d-1}r d\tau \left( \varphi_j^2 + 2 \sqrt{\lambda_j} \varphi_j \cdot \psi^\dagger M^{(j)}(r) \psi \right). \quad (7)$$

We expand around the lower critical dimension, setting the space-time dimension to $d = 2 + \epsilon$, and renormalize the theory using the minimal subtraction scheme. We introduce the renormalized couplings $g \in \{g_i, \lambda_j\}$ related to the bare ones by $\tilde{g} = V^{-1} N_s^2 Z_\phi^2 Z_\psi^{-1} g$. Here $\mu$ is the momentum scale at which we renormalize the theory and $N_\psi$ is the wavefunction normalization factor generated by interlayer hoppings which was introduced previously. The renormalization constants $Z_\phi$ and $Z_\psi$ for $\phi \in \{\phi_i, \varphi_j\}$ absorb the poles of the three-point vertex and the bosonic self-energy, respectively. We expand these functions to first order in the interaction couplings using the Green’s function corrected by interlayer hoppings $G'_0$ as the fermionic propagator. This expansion is represented by the diagrams shown in Fig. [6] in which interlayer hoppings are treated perturbatively to second order in $\alpha$. We obtain the RG flow equations for the coupling constants as a function of the parameters $\alpha$ and $\beta$. Crucially, we find that only four interactions out of twelve have non-zero divergent corrections. The eight other couplings have a trivial flow, either because the correction has no pole in $\epsilon$—these correspond to the four channels with $\sigma_0$ sublattice structure; or because the correction vanishes at low energy —these correspond to the four channels that are off-diagonal in layer space. We thus restrict our study to the four-dimensional subspace corresponding to the instabilities of Tab. [1] which are all associated with a phase transition toward a correlated phase. These relevant couplings are all diagonal in layer.

We now briefly discuss the essential features of this four-dimensional flow. The gaussian fixed point (FP)
at the origin is always stable in $d = 3$. Besides, we identify four critical points, one for each non-trivial coupling, listed in Table I. They control phase transitions toward the four correlated phases discussed in the mean-field analysis. As $\alpha$ approaches the magic value $\alpha_0$, all four critical FPs collapse towards the gaussian FP. Meanwhile, the (Dirac) semimetallic region, which corresponds to the basin of attraction of the gaussian FP, shrinks and disappears completely. As a result, these four couplings are always relevant close enough to the magic angle, regardless of the value of the bare interaction strength. This scenario provides a natural way of identifying the dominant instabilities near the magic angle: they correspond to the couplings whose critical FPs collapse the fastest towards the origin.

Hence, as follows from Tab. I we discard the couplings $g_0$ and $\lambda_0$ (the amplitudes of interactions which are symmetric in layers) and focus on the competition between the couplings which are antisymmetric in layers, of amplitude $g_z$ associated with the layer-polarized gapped phase and $\lambda_z$ associated with the $C_3$-breaking density-modulated phase. We note that, while the gapped phase is reminiscent of the dynamical mass generation in the Gross-Neveu model [42, 43], the $C_3$-breaking density-modulated phase is specific to TBG. The competition between the two most relevant instabilities is dictated by the following coupled RG flows (for derivation see Appendix E)

$$
-\mu \frac{\partial g_z}{\partial \mu} = -\epsilon g_z + \frac{4g_z^2}{\pi v} + \frac{4g_z\lambda_z}{\pi v} \left[1 - 6\alpha^2 (1 - \beta^2)\right],
$$

$$
-\mu \frac{\partial \lambda_z}{\partial \mu} = -\epsilon \lambda_z + \frac{4\lambda_z^2}{\pi v} \left[1 + 3\alpha^2 (1 + \beta^2)\right] + \frac{2\lambda_z g_z}{\pi v} \left[1 - 6\alpha^2 (1 - \beta^2)\right].
$$

As mentioned above, all FPs collapse to the origin at the magic angle. Thus to explore the competition between the phases, we plot the renormalization flow for the couplings rescaled by the vanishing velocity. The effect of the proximity to the magic angle on this competition is shown in Fig. 7, where we compare the flow close to the first magic angle (b) with that for the case when interlayer hopping is suppressed (a) [44]. This comparison shows that the proximity to the magic angle favors the occurrence of density modulations. The large scale behavior is dominated by the fastest diverging coupling, whether $g_z$ or $\lambda_z$. Within our perturbative RG analysis, a crossover line separates the corresponding regions, whose parametric equation reads

$$
\lambda_z = g_z \frac{[1 + 6\alpha^2 (1 - \beta^2)]/[6\alpha^2 (3 - \beta^2)]}{6\alpha^2 (3 - \beta^2)}.
$$

Around the crossover line, both order parameters coexist over a large range of length scales, corresponding to the appearance of a gapped, periodically modulated state, asymmetric in layers and breaking the $C_3$ and $IT$ symmetries. We call it a nematic insulator by analogy with phases discussed in Ref. [15]. This nematic insulating behavior is characterized by a runaway RG flow of both $\lambda_z, g_z$. It appears for a wide range of coupling parameters, as a consequence of the proximity to the magic angle.

TABLE II. Isolated, non-gaussian critical fixed points (FPs) for the four non-trivial instabilities.

| Channel | Coupling | $\hat{M}_i$ | FP $g_i^*(\alpha, \beta)$ |
|---------|----------|-------------|--------------------------|
| $a_x$   | $g_0$    | $\sigma_\tau_0$ | $\pi v/4 [1 - 12\alpha^2 (1 - \beta^2)]$ |
| $a_y$   | $g_z$    | $\sigma_\tau_z$ | $\pi v/4 [1 - 3\alpha^2 (1 - \beta^2)]$ |
| $E_x^+ $| $\lambda_0$ | $\sigma_\tau_0/\sqrt{2}$ | $\pi v/4 [1 - 3\alpha^2 (1 - \beta^2)]$ |
| $E_y^-$ | $\lambda_z$ | $\sigma_\tau_z/\sqrt{2}$ | $\pi v/4 [1 + 3\alpha^2 (1 + \beta^2)]$ |

FIG. 6. (a) - (b) Polarization (bosonic self-energy) to first order in the couplings, (a) at order $\alpha^0$ and (b) at order $\alpha^2$. (c) - (g) Three-point vertex to first order in the couplings, (c) at order $\alpha^0$; (d) - (e) at order $\alpha^2$ with multiplicity one; (f) - (g) at order $\alpha^2$ with multiplicity two. The double line is the fermionic propagator corrected by interlayer hoppings, while the dashed line is the bosonic propagator. The wavy line represents a pair of opposite hopping processes, summed over all channels with a transfer of momentum $\pm q_j$ for $j = 1, 2, 3$.

VI. DISCUSSION AND OUTLOOK

Applying group theory supplemented by a renormalization group approach, we found that a gapped nematic state with $C_3$ breaking modulation of density is favored at charge neutrality in TBG when the twist angle approaches its first magic value. A gap was observed at charge neutrality in TBG both in scanning tunneling microscopy and spectroscopy studies [14, 24, 46, 47] as well as in four-terminal transport measurements [23]. Three-fold symmetry breaking and nematic ordering were also reported [22, 24, 47]. Both of these experimental observations strongly support the occurrence of a nematic insulating state at charge neutrality in TBG, as we obtain within our RG scenario. We also find that such a
state persists even when the strength of interactions is weakened by screening as was experimentally observed in Ref. [49]. Let us stress that our RG approach identifies a gapped nematic behavior in the perturbative scaling regime, but does not rule out that other types of correlations develop at larger length scales, including those of intervalley-coherent and generalized ferromagnetic insulating states recently discussed in Refs. [50–52]. It is worth noting that the energies of these different ground states seem to be very close to each other, suggesting a strong sensitivity to experimental conditions: indeed, h-BN encapsulation, which induces chiral symmetry breaking, favors the layer-polarized insulators such as the nematic insulator discussed in this paper as opposed to intervalley-coherent or generalized ferromagnetic insulating states [52, 53]. Finally, let us note that the occurrence of an analogous nematic insulating state close to quantum spin Hall phase transitions raises the questions of its relation with the topological nature of the underlying semimetal [32, 54, 55].

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APPENDIX A: Change of basis

The Hamiltonian describing the low-energy physics near the two twisted Dirac cones at $K_{t,b}$ originating from a single valley of graphene can be written as [54]

$$\hat{H} = \int d^2r \, \hat{\psi}^\dagger \left( v_0 \sigma \cdot (i\partial + q_0) \frac{\hat{T}(r)}{T(r)} v_0 \sigma \cdot (i\partial - q_0) \right) \hat{\psi}. $$

(A1)

The momentum $q_1 = K_t - K_b$ gives the relative displacement of the Dirac momentum $K$ of each layer due to the twist, while $v_0$ is the Fermi velocity of graphene. Notice that there are three equivalent $K$ points in monolayer graphene, each leading to one copy of the Hamiltonian $\hat{H}$ with a relative displacement $q_j, j = 1, 2, 3$, where the momenta $q_2$ and $q_3$ are obtained through a rotation of $q_1$ by an angle of $2\pi/3$ and $4\pi/3$ respectively. The interlayer hopping matrix $\hat{T}(r)$ reads

$$\begin{pmatrix} T(r) & 3 \\ J_j = \frac{t_{AA}}{3} \sigma_0 + \frac{t_{AB}}{3} (\sigma_+ e^{-2i(j-1)\pi/3} + \text{h.c.}) \end{pmatrix}, $$

(A2)

where $t_{AA}$ and $t_{AB}$ are the hopping amplitudes in the AA and AB/BA regions, respectively.

Hamiltonian (A1) is simplified by rotating the basis [3]

$$\hat{\psi}(r, \tau) = A_1(r) \psi(r, \tau), \quad A_j(r) = e^{-i(q_j \cdot r/2)\tau_z}, $$

(A3)

which brings the Dirac cones to the same momentum:

$$H = \int d^2r \, \hat{\psi}^\dagger \left( v_0 \sigma \cdot i\partial - T(r) \right) \hat{\psi}, $$

(A4)

where $T(r) = \sum_{j=1}^{3} e^{-i q_j \cdot r} T_j^+$. Applying the same change of basis to quartic terms in the action, e.g. corresponding to a density-density interaction of the form

$$S_{int} = g \int d^2r \, d\tau \, \hat{\psi}^\dagger \hat{\psi}(r, \tau) R \hat{\psi}(r, \tau), $$

(A5)

we arrive at

$$S_{int} = g \int d^2r \, d\tau \, \hat{\psi}^\dagger \hat{\psi}(r, \tau) R(r) \psi(r, \tau) \psi(r, \tau) $$

(A6)

with the rotated interaction matrix

$$R(r) = \frac{1}{3} \sum_{j=1}^3 A_j^\dagger(r) \hat{R} A_j(r). $$

(A7)

Though both $\hat{R}$ and $R(r)$ describe contact interactions, while $\hat{R}$ is space-independent, $R(r)$ depends in general on the position as a consequence of Eq. (A7).
(i) If $\hat{R}$ is diagonal in layer, i.e., proportional to $\tau_{0/z}$, it commutes with $A_j(r)$ so that $R(r) = \hat{R}$.

(ii) If $\hat{R}$ is not diagonal in layer, i.e., proportional to $\tau_{x/y}$, it does not commute with $A_j(r)$ so that $R(r)$ differs from $\hat{R}$. In that case, $R(r)$ is modulated periodically over a distance of the order of the moire lattice constant. Indeed, we have

$$
\begin{aligned}
\frac{1}{3} \sum_{j=1}^{3} A^+_j(r) \tau_x A_j(r) &= f_1(r) \tau_x + f_2(r) \tau_y \\
\frac{1}{3} \sum_{j=1}^{3} A^+_j(r) \tau_y A_j(r) &= f_2(r) \tau_x + f_1(r) \tau_y
\end{aligned}
$$

(A8)

with $f_1(r) = \frac{1}{3} \sum_{j=1}^{3} \cos(q_j \cdot r)$, $f_2(r) = \frac{1}{3} \sum_{j=1}^{3} \sin(q_j \cdot r)$. These results also apply to current-current quartic interactions, where $\hat{R}$ is replaced by a vector of matrices $M$.

Appendix B: Diagrammatic technique for the non-interacting theory

To expand any observable in interlayer hoppings in the absence of interactions, it is not mandatory to resort to a field theoretical approach. We do so however, because it is useful for applying RG when interactions are included. To that end we need to introduce the Feynman rules specific to this unusual field theory. The free fermionic propagator associated to the action of the decoupled bilayer $S_0 = \int d^2r \, d\tau \, \psi^\dagger (H_0 - \partial_\tau) \psi$ reads

$$
G_0(k, \Omega) = (\sigma \cdot k - i\Omega)^{-1}
$$

(B1)

in Fourier space, where $k$ is the momentum, $\Omega$ the Matsubara frequency, and we omitted the identity matrices $\sigma_0$ and $\tau_0$ for simplicity. When drawing Feynman diagrams, we represent the free propagator (B1) with a solid line. We reserve $q$ and $\omega$ for the internal momentum and Matsubara frequency and use $k$ and $\Omega$ for external ones. Any correlation function can be written as an ensemble average $\langle \ldots \rangle_0$ over $S_0$. In particular for the time-ordered two-point function we have

$$
\langle T \psi_0^{\dagger} \psi_0^\dagger \rangle_0 = \frac{\langle T \psi^{\dagger} e^{-S_0} \rangle_0}{\langle e^{-S_0} \rangle_0},
$$

(B2)

where $\langle \ldots \rangle_0$ denotes the ensemble average over the quadratic action $S_0 = S_0 + S_\alpha$, which includes the hopping action $S_\alpha = \int d^2r \, d\tau \, \psi^\dagger H_\alpha \psi$, from which an expansion order by order in $\alpha$ can be carried out. The two-point function (B2) is non-diagonal in momentum space, since $S_\alpha$ reduces the continuous translational symmetry to the discrete translational symmetry over the reciprocal lattice $\mathcal{R}$, which is the $\mathbb{Z}$-module generated by the (linearly dependent) family of vectors $\{q_j, j = 1, 2, 3\}$. For every vector $b$ in $\mathcal{R}$, we define the component $G_0^b(k, \Omega)$ of the two-point function such that

$$
\langle T \psi_0^{\dagger} \psi_0^\dagger \rangle_0 = \sum_{b \in \mathcal{R}} G_0^b(b, k, \Omega) \delta(b - q),
$$

(B3)

for all momenta $k$, $q$ and frequency $\Omega$. We focus on how interlayer hoppings renormalize the dispersion relation, so that we are mainly interested in the translational invariant part $G_0^b(k, \Omega) = G_0^b(0, k, \Omega)$ of the fermionic propagator, represented in Fig. 8(f) as a double line. Successive interlayer hoppings that transfer the momenta $(\eta_1 q_{j_1}, \ldots, \eta_m q_{j_m})$ in this precise order – where $\eta_1, \ldots, \eta_m = \pm$, with the plus sign for a hopping to the top layer, and a minus sign to the bottom layer – give a non-zero contribution to $G_0^b(k, \Omega)$ if the following conditions are met.

(i) Total momentum is conserved, i.e., $\sum_{r=1}^{m} \eta_r q_{j_r} = 0$.

(ii) Consecutive hopping processes affect different layers, i.e., $\eta_2r = -\eta_{2r-1}$ for all $r = 1, \ldots, n/2$.

In particular, condition (ii) forbids odd numbers of insertions, so that all correlation functions can be expanded in $\alpha^2$, and entails that a hopping sequence is determined by the momenta and the sign of only the first hopping process $\eta = \eta_1$. Joined with condition (i), it also yields that the transfer of a momentum at one point of the diagram must be followed by the transfer of the opposite momentum at another point. Thus we can join insertions of opposite momenta by a wavy line like in Figs. 8(a), 8(d).

We now introduce the translational part $\Sigma_\alpha(k, \Omega)$ of the self-energy as

$$
\Sigma_\alpha^b(k, \Omega) = G_0^b(k, \Omega)^{-1} - \Sigma_\alpha(k, \Omega).
$$

(B4)

The contributions to $\Sigma_\alpha(k, \Omega)$ come from the connected two-point diagrams that conserve total momentum and that cannot be cut by one stroke into two subdiagrams that conserve themselves total momentum. Expanding Eq. (B2) to sixth order in $\alpha$, we can decompose it as $\Sigma_\alpha(k, \Omega) = \Sigma_\alpha^2(k, \Omega) + \Sigma_\alpha^4(k, \Omega) + \Sigma_\alpha^6_{\text{rem}}(k, \Omega) + \Sigma_\alpha^6_{\text{rem}}(k, \Omega) + \Sigma_\alpha^6_{\text{rem}}(k, \Omega)$. In the following we use the shortcut $\tilde{\eta} = -\eta$ with $\eta = \pm$, and $j, l, k = 1, 2, 3$. The second order contribution (Fig. 8(a)) reads

$$
\Sigma_\alpha^2(k, \Omega) = \alpha^2 \sum_{\eta, j} T_{j}^\eta G_0(k + \eta q_j, \Omega) T_{j}^\eta.
$$

(B5)

The fourth order contribution (Fig. 8(b)) reads

$$
\Sigma_\alpha^4(k, \Omega) = \alpha^4 \sum_{\eta, j : \eta \not= l} T_{j}^\eta G_0(k + \eta q_j, \Omega) T_{i}^\eta G_0(k + \eta q_j - \eta q_l, \Omega) T_{j}^\eta G_0(k + \eta q_j, \Omega) T_{j}^\eta.
$$

(B6)
The sixth order contribution splits into three terms. The first diagram hosts three nested hopping lines (Fig. S1(c)),
\[
\Sigma^6_{\alpha,\text{nes}}(k, \Omega) = \alpha^6 \sum_{n,l \neq (j,k)} T^\eta_j G_0(k + \eta q_j, \Omega) T^\eta_l G_0(k + \eta q_j - \eta q_l, \Omega) T^\eta_l G_0(k + \eta q_j - \eta q_l - \eta q_{l'}, \Omega).
\]

The second diagram hosts two hopping lines in a row, embedded in a third one (Fig. S1(d)),
\[
\Sigma^6_{\alpha,\text{row}}(k, \Omega) = \alpha^6 \sum_{n,l \neq (j,k)} T^\eta_j G_0(k + \eta q_j, \Omega) T^\eta_l G_0(k + \eta q_j - \eta q_l, \Omega) T^\eta_l G_0(k + \eta q_j - \eta q_l - \eta q_{l'}, \Omega).
\]

The third diagram consists in three crossing hopping lines (Fig. S1(e)),
\[
\Sigma^6_{\alpha,\text{cro}}(k, \Omega) = \alpha^6 \sum_{n,l \neq (j,k)} T^\eta_l G_0(k + \eta q_{l'}, \Omega) T^\eta_l G_0(k + \eta q_{l'} - \eta q_l, \Omega) T^\eta_l G_0(k + \eta q_{l'} - \eta q_l - \eta q_{l''}, \Omega).
\]

Within a low-energy theory where \(k, \Omega \ll 1\), we can further expand to order two in momentum \(k\), and one in Matsubara frequency \(\Omega\), which results in
\[
\Sigma_\alpha(k, \Omega) = \left[3\alpha^2 - \alpha^2(1 - \beta^2)^2 + \frac{3\alpha^6}{49} (37 - 112\beta^2 + 119\beta^4 - 70\beta^6)\right] \sigma \cdot k \tau_0
\]
\[
+ \left[3\alpha^2 - \alpha^2(1 - \beta^2)^2 \right] \left( \begin{array}{c} 0 \\ -ik^2 \\ 0 \end{array} \right) \tau_2
\]
\[
+ \left[3\alpha^2(1 + \beta^2) + 2\alpha^4(1 + 7\beta^2 + 4\beta^4) + \frac{3\alpha^6}{28} (8 + 16\beta^2 + 376\beta^4 + 187\beta^6)\right] i\Omega \sigma_0 \tau_0.
\]

where \(k = k_x + ik_y\). If one keeps only the correction to the linear dispersion, the translational part of the fermionic propagator corrected by interlayer hoppings can be massaged into
\[
G^\alpha_0(k, \Omega) = N^{-1}_\psi (v \sigma \cdot k - i\Omega)^{-1}
\]
with the normalisation of the wave function \(N_\psi = 1 + 3\alpha^2(1 + \beta^2)^2 + 2\alpha^4(1 + 7\beta^2 + 4\beta^4) + \frac{3\alpha^6}{28} (8 + 16\beta^2 + 376\beta^4 + 187\beta^6)\), and the Fermi velocity dressed by interlayer hoppings
\[
v = \frac{1 - 3\alpha^2 + \alpha^4(1 - \beta^2)^2 - \frac{3\alpha^6}{28} (37 - 112\beta^2 + 119\beta^4 - 70\beta^6)}{1 + 3\alpha^2(1 + \beta^2) + 2\alpha^4(1 + 7\beta^2 + 4\beta^4) + \frac{3\alpha^6}{28} (8 + 16\beta^2 + 376\beta^4 + 187\beta^6)},
\]
as given in Eq. [3]. We remind that \(v\) is expressed in units of the Fermi velocity \(v_0\) of monolayer graphene.

**Appendix C: Symmetries of the model**

by the interlayer hopping term. Indeed, the Hamiltonian

1. Complete symmetries

The symmetries of the single-particle Hamiltonian \(H'_0 = H_0 + H_\alpha\) of Eq. [4] are highly constrained...
The composition of inversion around the layers. Both sublattices and layers are flipped, so that noted as $\mathcal{IT}$. The operation $\mathcal{H}$ breaks Lorentz invariance, continuous space translations and layer plus pseudospin rotational symmetry. The symmetry group is thus reduced to the symmmorphic space-time group $\mathbb{R} \times (a_1Z + a_2Z) \times D_3$, composed of time translation $\mathbb{R}$, discrete translations on the moiré lattice $a_1Z + a_2Z$, where $a_1$ and $a_2$ are the superlattice vectors, and the point group $D_3$ generated by the rotation $C_3$ around the $z$ axis and the rotation $C_2$ around the $x$ axis,

$$D_3 = \{ e, C_3, C_3^2, C_2, C_2C_3, C_2C_3^2 \}. \quad (C1)$$

Henceforth we disregard the translational symmetries and focus on the magnetic group generated by the point group $D_3$ and the two special “symmetries” $\mathcal{IT}$ and $\mathcal{P}$. These operations act by conjugation on the single-particle Hamiltonian $\mathcal{H}^0$ of $D_3$ in the four-dimensional representation (4d rep.), denoted as $\Gamma$, whose unitary matrix representation we now define.

The operation $C_3$ rotates the bilayer by an angle $2\pi/3$ around the $z$ axis perpendicular to the bilayer. Only the sublattice pseudospin is rotated, while the layer pseudospin is unaffected, so that

$$\Gamma(C_3) = e^{2i\pi/3\sigma_z \tau_0}. \quad (C2)$$

The operation $C_2$ rotates the bilayer by an angle $\pi$ around the $x$ axis of Fig. 1 at mid-distance between the layers. Both sublattices and layers are flipped, so that

$$\Gamma(C_2) = \sigma_x \tau_x. \quad (C3)$$

The composition of inversion $\mathcal{I}$ and time reversal $\mathcal{T}$, denoted as $\mathcal{IT}$, is antunitary and represented by

$$\Gamma(\mathcal{IT}) = \sigma_x \tau_0 \mathcal{K}, \quad (C4)$$

where $\mathcal{K}$ denotes the complex conjugation of the matrix elements. The operations $\mathcal{R}$ defined in Eq. (C2) to (C4) are pure symmetries of the Hamiltonian, which means that $\Gamma(R)^{-1} \mathcal{H}^0(\mathcal{R} \mathcal{R}_T) \Gamma(R) = \mathcal{H}^0(r, t)$ for $R = C_3, C_2$, and $\Gamma(R)^{-1} \mathcal{H}^0(\mathcal{R} \mathcal{R}_T) \Gamma(R) = \mathcal{H}^0(r, t)$ for the antunitary element $R = \mathcal{IT}$. Finally, the unitary particle-hole operation $P$ reverses the energy, and acts in real space as a reflection $x \mapsto -x$. Its matrix representation reads

$$\Gamma(P) = \sigma_x \tau_z. \quad (C5)$$

Following Ref. [32] we define $\mathcal{P}$ as a unitary operation in order to have a single antiunitary generator ($\mathcal{IT}$), unlike the convention of Ref. [57]. This operation is an antisymmetry of the Hamiltonian, which means that $\Gamma(\mathcal{P})^{-1} \mathcal{H}^0(\mathcal{P} \mathcal{R}, \mathcal{P}t) \Gamma(\mathcal{P}) = -\mathcal{H}^0(r, t)$. This antisymmetry is lost when the angular dependence of the kinetic terms $\sigma_{\pm \theta/2} \cdot \mathbf{k}$ is kept, when terms quadratic in momentum are included in the single-particle Hamiltonian, or when intervalley scattering is permitted [57]. Since $\Gamma$ is faithful rep., we can infer the multiplication table of the magnetic group from that of the matrix representation of the four generators.

## 2. Unitary group

The unitary group $\hat{D}_3$ generated by $D_3$ and $P$ can be cast into the semi-direct product

$$\hat{D}_3 = \{ e, \mathcal{P}, \bar{e}, \bar{P} \} \ltimes D_3, \quad (C6)$$

where $\bar{e} = (PC_2)^2$ and a barred operator represents the product of this operator by $\bar{e}$. The classes of conjugation of $D_3$ are given in Tab. III. The double group $D_3^D = D_3 \times \{ e, \mathcal{P}, \bar{e}, \bar{P} \}$ is a normal subgroup of $\hat{D}_3$, while $D_3$ is not. To find the irreducible representation (irrep.) of $\hat{D}_3$—listed in Tab. XIV—we can either find them from scratch using the composite operator method [33], or construct them by induction and other tricks from that of $D_3$. Let us illustrate the second method.

The two 1d irrep. of the quotient group $\hat{D}_3/D_3^D = \{ e, \mathcal{P} \}$ generate the irreps $A_1$ and $A_1^\dagger$. The irrep. $A_1$ and $A_2$ of $D_3$ induce the reps $A_1 \uparrow \hat{D}_3 \sim A_1 \oplus A_1 \oplus E_3$ and $A_2 \uparrow \hat{D}_3 \sim A_2 \oplus a_2 \oplus E_3$ respectively, where $\oplus$ denotes a direct sum and $\sim$ the equivalence of rep. The commutator subgroup $[\hat{D}_3, \hat{D}_3]$ is isomorphic to $D_3$, whose index $|\hat{D}_3/D_3|$ = 4 gives the number of 1d irrep. Hence we have found all 1d irrep. The cardinal of the group being $|\hat{D}_3| = 24$, the remaining irrep. are five 2d irrep., including $E_3$. We can decompose the 4d rep. $\Gamma$ defined by Eq. (C2) to (C4) and (C5) as $\Gamma \sim E_1 \oplus E_2$. The 2d irrep. $E$ of $D_3$ induces the rep. $E \uparrow \hat{D}_3 \sim E_1 \oplus$
For any irrep \( \rho \) the Schur-Frobenius criterion \([34–38]\) to each irrep. of 3. Magnetic group

The group generated by 93 and IT is the dichromatic magnetic group

\[
\mathcal{M} = \tilde{D}_3 \times \{e, IT\}. \tag{C7}
\]

To find the corepresentations (corep.) of \( \mathcal{M} \), we apply the Schur-Frobenius criterion \([34,35]\) to each irrep. of \( \tilde{D}_3 \). The Schur-Frobenius criterion states the following. For any irrep \( \rho \) of \( \tilde{D}_3 \), let us define the rep. \( \rho' : \tilde{D}_3 \to M_\rho(C), R \mapsto \rho(\text{IT} R \text{IT}^{-1})^* \). We are necessarily in one of the three following scenarios. (i) Either \( \rho \) is equivalent to its primed counterpart, in which case there exists an invertible matrix \( U \) such that \( \rho' = U \rho U^{-1} \). (ii) If \( \rho U^* = -\rho(\text{IT})^2 \), there is no Kramer degeneracy: the corep. issued from \( \rho \) has the same dimension as \( \rho \) and satisfies \( \rho(\text{IT}) = \pm U \). (iii) If \( \rho U^* = -\rho(\text{IT})^2 \), the corep. issued from \( \rho \) has twice the dimension. (iii) Or \( \rho \) is not equivalent to its primed counterpart, in which case \( \rho' \) is necessarily equivalent to another irrep. of \( \tilde{D}_3 \), and the corep. has again twice the dimension of \( \rho \), and coincides with \( \rho \otimes \rho' \) on \( \tilde{D}_3 \).

It turns out that all irrep. of \( \tilde{D}_3 \) pertain to case (i), except \( E_1 \) and \( E_5 \), which fall into case (iii). In the former case, each irrep. leads to two coreps., with \( \rho(\text{IT}) = \pm 1 \) for the 1d irrep. or \( \rho(\text{IT}) = \pm \sigma_2 \) for the 2d irrep., where \( \sigma_2 \) represents here a generic Pauli matrix, but has nothing to do with the pseudospin. In the latter case, the one corep. formed by \( E_1 \) and \( E_5 \) is equivalent to the 4d rep. \( \Gamma \). In the following, we write each corep. with an exponent \( \pm \) to indicate whether the eigenvalues of \( \rho(\text{IT}) \) are +1 or -1.

\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
Irrep. & \( e \) & \( \bar{e} \) & \( 2C_3 \) & \( 2\bar{C}_3 \) & \( 2PC_2 \) & \( 2PC_2\bar{C}_3 \) & \( 2PC_2\bar{C}_3^2 \) & \( 6P \) & \( 6C_2 \) \\
\hline
\( A_1 \) & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\( A_2 \) & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\
a & 1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 \\
a & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\
e & 2 & -2 & -1 & 1 & 0 & \sqrt{3} & -\sqrt{3} & 0 & 0 \\
e & 2 & -2 & -1 & 2 & -1 & 1 & -1 & 0 & 0 \\
e & 2 & -2 & 2 & -2 & 0 & 0 & 0 & 0 & 0 \\
e & 2 & 2 & -1 & -1 & -2 & 1 & 1 & 0 & 0 \\
e & 2 & -2 & -1 & 1 & 0 & -\sqrt{3} & \sqrt{3} & 0 & 0 \\
\hline
\end{tabular}
\caption{Table of characters of the unitary group \( \tilde{D}_3 \). Each column corresponds to a class of conjugation, and each line to an irrep. We use the symbols \( A \) and \( E \) prescribed by Mulliken's notation; the symbol \( a \) denotes a 1d irrep. whose character differs than one on antisymmetric operators.}
\end{table}

Table IV. Table of characters of the unitary group \( \tilde{D}_3 \). Each column corresponds to a class of conjugation, and each line to an irrep. We use the symbols \( A \) and \( E \) prescribed by Mulliken's notation; the symbol \( a \) denotes a 1d irrep. whose character differs than one on antisymmetric operators.

\[
\rho \otimes \rho = A_1^+.
\]

Thus the quartic interaction corresponding to one of these irreps is of the form \( M \otimes \tilde{M} \), where \( \tilde{M} \) is found applying Eqs. (C8) and (C10). The interaction matrices

4. Quartic interactions

The direct product \( \Gamma^\dagger \otimes \Gamma \) dictates how the bilinear \( \psi_i^\dagger \psi_j \) with \( i, j = 1, \ldots, 4 \), transforms under the magnetic group \( \mathcal{M} \). The Clebsch-Gordan series reads

\[
X(\Gamma^\dagger \otimes \Gamma)X^{-1} = A_1^+ + a_1^+ \oplus A_2^+ + a_2^+ \oplus a_1 \oplus a_1^- \oplus a_2 \oplus a_2^\dagger \oplus E_4 \oplus E_4^\dagger \oplus E_2 \oplus E_2^\dagger, \tag{C8}
\]

where the transformation matrix \( X \) contains the Clebsch-Gordan coefficients, which can be found using the formula (C3),

\[
X_{ik,\gamma m}X_{jl,\gamma n}^* = \frac{n_\gamma}{|\mathcal{M}|} \sum_{R \in \mathcal{M}} \rho_\gamma(R)_{mn}^* \Gamma^\dagger(R)_{ij} \Gamma(R)_{kl}, \tag{C9}
\]

where \( \rho_\gamma \) is the \( \gamma \)-th irrep. in the series (C8), with dimension \( n_\gamma \) and \(|\mathcal{M}| = 48 \) is the cardinal of the magnetic group. The coefficients of the matrices \( M_\gamma \) of Eq. (9) that transforms by conjugation according to the irrep. \( \rho_\gamma \) are listed in the \( \gamma \)-th column of \( X \), i.e. for \( a, b = 1, \ldots, 4 \), the two components of the vector \( M_\gamma \), read

\[
(M_\gamma^{(1)})_{ab} = X_{ab,\gamma 1}, \quad (M_\gamma^{(2)})_{ab} = X_{ab,\gamma 2}. \tag{C10}
\]

To find the quartic interaction that preserve the magnetic group \( \mathcal{M} \) we must find all copies of the trivial irrep. \( A_1^+ \) into the product \( (\Gamma^\dagger \otimes \Gamma) \otimes (\Gamma^\dagger \otimes \Gamma) \). By inspecting the characters, it is clear that only products of the same irrep. decompose themselves into a copy of \( A_1^+ \). For the 1d irrep. \( \rho = A_1^+ + a_1^+ A_2^+ + a_2^+ A_3^+ + a_1^- A_3^- + a_2^- \), the decomposition is simply

\[
\rho \otimes \rho = A_1^+.
\]

Thus the quartic interaction corresponding to one of these irreps is of the form \( M \otimes M \), where \( M \) is found applying Eqs. (C8) and (C10). The interaction matrices
for these eight 1d irreps are listed in Tab. I. For the 2d irreps, we have
\[ E_2^\eta \otimes E_2^\eta \sim A_1^\eta \oplus A_2^\eta \oplus E_1^\eta, \quad (C12) \]
\[ E_1^\eta \otimes E_2^\eta \sim A_1^\eta \oplus A_2^\eta \oplus E_2^\eta, \quad (C13) \]
for \( \eta = \pm \). For each of these irreps, the invariant combination transforming as \( A_1^\pm \) is \( M \otimes N \), where \( (M, N) \) is the basis of the two-dimensional space on which the irrep acts. The interaction matrices for these four 2d irreps are listed in Tab. I.

The interaction matrices for these four 2d irreps are

\[ \left[ \begin{array}{c}
\Delta_0/\Lambda \\
\Delta_3/\Lambda \\
\Delta_5/\Lambda \\
\Delta_7/\Lambda
\end{array} \right] \]

\[ \left( \begin{array}{c}
\tau_0/\Lambda \\
\tau_3/\Lambda \\
\tau_5/\Lambda \\
\tau_7/\Lambda
\end{array} \right) \]

\[ \Delta_0/\Lambda = -2g_{0/z} \int d\omega \int_{\Lambda} \frac{d^2q}{(2\pi)^2} (\psi_{q,\omega}^\dagger \sigma z \tau_0/\Lambda \psi_{q,\omega}), \quad (D1a) \]
\[ \mathcal{G}_{0/z} = -2\lambda_{0/z} \int d\omega \int_{\Lambda} \frac{d^2q}{(2\pi)^2} (\psi_{q,\omega}^\dagger \sigma \tau_0/\Lambda \psi_{q,\omega}), \quad (D1b) \]

where the momentum integral runs over a square of side \( \Lambda \). The correlators in Eq. (D1) are the translationally-invariant parts of statistical averages computed over the Bloch Hamiltonian density \( H'_{MF} = H^0 + \sigma \cdot (\mathcal{G}_0 \tau_0 + \mathcal{G}_z \tau_z) + \mathcal{G}_x \tau_x + \mathcal{G}_y \tau_y + \mathcal{G}_z \tau_z. \) The perturbative expansion of the correlators in \( \alpha \) can be done along the lines of computing the self-energy. We have previously found that the propagator corrected by interlayer hoppings \( G'_0 \) is of the form \( (B11) \); similarly, the mean-field Hamiltonian becomes

\[ H'_{MF} \mapsto N_\psi \left[ v N_\psi \left( \left( k + N_0^{(G)} \mathcal{G}_0 \right) \tau_0 + N_z^{(G)} \mathcal{G}_z \tau_z \right) \right. \]
\[ \left. + \sigma \left( N_0^{(\Delta)} \Delta_0 \tau_0 + N_z^{(\Delta)} \Delta_0 \tau_0 \right) \right] \]. \quad (D2)

The effect of interlayer hoppings is to enhance the order parameters by factors \( N_0^{(G)/(\Delta)}(\alpha, \beta) \), which are the counterparts of the renormalized Fermi velocity for the matrix structures corresponding to those order parameters. They are calculated diagrammatically to sixth order in \( \alpha \) and satisfy

\[ \begin{align}
N_\psi N_0^{(\Delta)} &= 1 + 3\alpha^2 (1 - \beta^2) + 2\alpha^4 (1 - \beta^2) (1 + 2\beta^2) + \frac{1}{28} \alpha^6 (24 - 80\beta^2 + 352\beta^4 - 233\beta^6), \\
N_\psi N_z^{(\Delta)} &= 1 - 3\alpha^2 (1 - \beta^2) + 2\alpha^4 (1 - \beta^2) (1 - 4\beta^2) - \frac{1}{28} \alpha^6 (56 - 304\beta^2 + 872\beta^4 - 561\beta^6), \\
vN_\psi N_0^{(G)} &= 1 - 3\alpha^2 + 2\alpha^4 (1 - \beta^2)^2 - \frac{3}{49} \alpha^6 (37 - 112\beta^2 + 119\beta^4 - 70\beta^6), \\
vN_\psi N_z^{(G)} &= 1 + 3\alpha^2 + 2\alpha^4 (1 + 10\beta^2 + \beta^4) + \frac{3}{49} \alpha^6 (9 + 441\beta^4 + 70\beta^6),
\end{align} \]

\[ \begin{align}
\Delta_0/\Lambda &= g_{0/z} N_0^{(\Delta)} \Lambda^2, \\
\mathcal{G}_{0/z} &= \lambda_{0/z} N_z^{(\Delta)} \Lambda^2.
\end{align} \]

\[ \begin{align}
\Delta_0/\Lambda &= g_{0/z} N_0^{(\Delta)} \Lambda^2, \\
\mathcal{G}_{0/z} &= \lambda_{0/z} N_z^{(\Delta)} \Lambda^2.
\end{align} \]
where for simplicity we assume the shift momenta to be aligned along a crystallographic axis of the moiré pattern, here along the y axis. The dimensionless functions $F$ and $F_{0/\pi}$ read

$$F(x) = \frac{2x}{\pi^2} \left[ -\log \left( \sqrt{x^2 + 2} - 1 \right) + \log \left( \sqrt{x^2 + 2} + 1 \right) - 2x \cot^{-1} \left( \sqrt{x^2 + 2} \right) + 2 \coth^{-1} \left( \sqrt{x^2 + 2} \right) \right],$$

$$F_0(x) = \frac{1}{\pi^2} \left[ -\sqrt{y_-} + \sqrt{y_+} + \tanh^{-1} \left( \sqrt{y_-} \right) - \tanh^{-1} \left( \sqrt{y_+} \right) - y_- \coth^{-1} \left( \sqrt{y_-} \right) + y_+ \coth^{-1} \left( \sqrt{y_+} \right) \right],$$

$$F_z(x) = \frac{1}{\pi^2} \left[ x^2 \log \left( \frac{1 - x}{1 + x} \right) - 2x^2 \tanh^{-1}(x) + (1 - 2x) \log \left( \sqrt{y_-} - 1 \right) - (1 + 2x) \log \left( \sqrt{y_+} - 1 \right) + z_+ \log \left( \sqrt{y_+} + 1 \right) - z_- \log \left( \sqrt{y_-} + 1 \right) + 2 \left( \sqrt{y_+} - \sqrt{y_-} \right) \right],$$

where $y_\pm = 2 + x(x \pm 2)$ and $z_\pm = 1 + 2x(x \pm 1)$.

**Appendix E: Renormalization**

1. **Hubbard-Stratonovitch decoupling**

We aim at finding the most relevant insulating state near charge neutrality. It is therefore practical to decouple the interactions in the direct, particle-hole channel, to evince order parameters of the form $\langle \psi^\dagger M \psi \rangle$ for $M \in \{ R, M \}$, where the bracket $\langle ... \rangle$ denotes the ensemble average over the complete action $S = S_0 + S_{\text{int}}$.

Using Hubbard-Stratonovitch transformations, we introduce one auxiliary bosonic field for each interaction, whose ground state value in the correlated phase is a constant solution of the classical equation of motion. We must distinguish between the 1d corep., for which a scalar field $\phi_i$ for $i = 1, \ldots, 8$, is sufficient, and the 2d corep., for which a two-component field $\varphi_j = \{ \varphi_{j,1}, \varphi_{j,2} \}$ must be introduced, for $j = 9, \ldots, 12$. Such transformation enables to recast the action for quartic fermion interactions [5] into

$$S_{\text{int}}[\psi^\dagger, \psi] \rightarrow S_{\text{Hub}}[\psi^\dagger, \psi, \phi]$$

$$= \sum_{i=1}^{8} \int d^2r \, d\tau \left( \phi_i^2 + 2\sqrt{\lambda_i} \psi^\dagger \phi_i R_i \psi \right) + \sum_{j=9}^{12} \int d^2r \, d\tau \left( \varphi_j^2 + 2\sqrt{\lambda_j} \psi^\dagger \varphi_j \cdot M_j \psi \right),$$

where the sum runs over both 1d and 2d irreps.. For simplicity we dropped the spatial and time dependences of the fields in Eq. (E1). The action for quartic fermion interactions thus splits into a bosonic quadratic action (the first term in the parenthesis), and a three-point vertex which takes the form of a Yukawa coupling (the second term in the parenthesis).

2. **Renormalization procedure**

The field theory described by the sum of action $S_0$ and action $S_{\text{Hub}}$ has critical dimension $d_c = 2$, which entails that in $d = 3$ space-time dimensions, we expect that all interactions lead to quantum critical points that are perturbative in the small parameter $\epsilon = d - 2$. From now onwards, we work in Fourier space and express all fields and integrals in terms of the momentum $q$ and the Matsubara frequency $\omega$. A general Fourier-transformed field is written $\phi_{q,\omega}$ for the bosonic case, or $\{ \psi_{q,\omega}, \psi_{q,\omega}^\dagger \}$ for the fermionic case, where $\psi_{q,\omega} = (\psi_{q,\omega})^\dagger$ denotes the conjugate of the Fourier-transformed field $\psi_{q,\omega}$. To renormalize the field theory, we assume that the complete action $S$ is actually expressed in terms of bare fields $\{ \phi, \psi \}$ for $\phi \in \{ \phi_i, \varphi_j \}$ and couplings $\hat{g}$ for $g \in \{ g_i, \lambda_j \}$, which are ill-defined in the interacting theory. The physical parameters and fields—written without the $\hat{}$ symbol—are connected to their bare counterparts through the so-called $Z$ constants.

We define define the Z constants for the fields such that

$$\hat{\phi} = Z_\phi^{1/2} \phi, \quad \hat{\psi} = Z_\psi^{1/2} \psi.$$

To regularize the theory, we work in an isotropic space-time of dimension $d = 2 + \epsilon$, and introduce a mass scale $\mu$ to make the regularized couplings dimensionless. A renormalized coupling $\hat{g}$ is linked to its bare value $\hat{g}$ by

$$\hat{g} = \mu^{-\epsilon} N^2 Z_\phi^2 Z_\psi^{-1} \hat{g}.$$

We included the normalization of the wavefunction $N_\psi$ in the redefinition of the couplings in order to compensate at all loop orders those arising from the corrected fermionic propagator $G_0$, given in Eq. (B11). Owing to dimensional regularization, we must promote the Pauli matrices in $S_0$ to a Clifford algebra in arbitrary dimension $d$, satisfying the anticommutation rules $\{ \sigma_i, \sigma_j \} = 2\delta_{ij}$ for $i, j = 1, \ldots, d$. Using Eqs. (E2) and (E3), we find the renormalized action $S_R = S_{R,0} + S_{R,\alpha} + S_{R,\phi} + S_{R,\psi}$, where the quadratic, decoupled action reads

$$S_{R,0} = \int_{q,\omega} \psi_{q,\omega}^\dagger \sigma_i \cdot \sigma_j \tau_0 - i\omega \sigma_0 \tau_0 \psi_{q,\omega}.$$
The renormalized bosonic part of the action is given by
\[
S_{\text{R,\phi}} = \sum_{i=1}^{8} Z_{\phi_i} \int_{q,\omega} (\phi_i^2)_{q} + \sum_{j=1}^{12} Z_{\varphi_j} \int_{q,\omega} (\varphi_j^2)_{q}.
\] (E6)

Finally, the renormalized interaction between fermionic and bosonic fields is
\[
S_{\text{R,\text{int}}} = 2\mu^{-\epsilon/2} N_\psi \left[ \sum_{i=1}^{8} Z_{g_i} \sqrt{g_i} \int_{q,\omega} q_{-p} R_i \psi_p + \sum_{j=1}^{12} Z_{\lambda_j} \sqrt{\lambda_j} \int_{q,\omega} q_{-p} M_j \psi_p \right].
\] (E7)

In Eqs. (E6) - (E7), we have omitted dependence of the fields on frequency and used the shorthand
\[
\int_{q,\omega} = \int_{\mathbb{R}^d} \frac{d^{d-1}q \, d\omega}{(2\pi)^d}.
\] (E8)

To fix values of the \(Z\) constants we use the minimal subtraction (MS) scheme, i.e. we absorb in them only the divergent parts of the diagrams. Inspecting Eqs. (E6) and (E7), we see that the constant \(Z_{\phi_i}\) can be found from the divergences of the polarization, i.e. the bosonic self-energy, while the constant \(Z_{g_i}\) can be determined by absorbing the divergences of the three-point vertices. Before computing explicitly the diagrams for the polarization and vertices let us outline the general strategy.

### 3. Preliminary mathematical remarks

Taking into account the interlayer hopping is done into two steps. We first draw the diagrams without insertion the hopping matrices, and then replace all solid lines by double lines. This corresponds to replacing free propagators by the propagators dressed by interlayer hoppings, like in the polarization shown in Fig. 9(a). In second step we include wavy lines, i.e. hopping matrices, connecting different propagators (see Fig. 9(b)). This splitting allows us to explicitly extract factors of \(v^{-1}\), where \(v\) is the Fermi velocity corrected by interlayer hoppings and which vanishes at the first magic angle. We restrict our computation to order \(\alpha^2\), which is the first non-trivial order.

When expanding product of matrices and integrating the trace, useful relations can be found in Refs. [58–60]. The Feynman trick
\[
\frac{1}{AB} = \int_0^1 \frac{dx}{Ax + B(1-x)},
\] (E9)
valid for any expressions \(A\) and \(B\), enable to linearize products of denominators. For the four relevant interactions we consider here, the space-time integrals are isotropic and can be computed in arbitrary dimension \(d\) using
\[
\int \frac{d^dQ}{(2\pi)^d} \frac{Q^{2a}}{(Q^2 + m^2)^b} = \frac{\Gamma(b - a - d/2)\Gamma(a + d/2)}{(4\pi)^d/2\Gamma(b)\Gamma(d/2)} m^{2(b-a-d/2)},
\] (E10)
for any reals \(a\) and \(b\), and where \(Q = (q,\omega)\) is the relativistic \(d\)-momentum. The dummy mass \(m \to 0\) plays the role of an infrared regulator and \(\Gamma\) denotes Euler’s Gamma function, which satisfies
\[
\Gamma(-n + x) = \frac{(-1)^n}{n!} \left[ \frac{1}{x} + \Psi(n + 1) + \mathcal{O}(x) \right]
\] (E11)
for all real \(x\) and integer \(n\); this relation is usually used with \(x = \epsilon\). In Eq. (E11), \(\Psi = (\ln \Gamma)’\) is Euler’s Digamma function, which does not intervene at one loop, since we discard all finite quantities in the MS scheme.

### 4. Polarization

The one-loop polarization \(\Pi_i\) is the self-energy of the auxiliary field \(\phi_i\) (for \(i = 1, \ldots, 12\)). If \(\Delta_i(k, \Omega)\) denotes the corrected propagator of the bosonic field, we have \(\Delta_i^{-1}(k, \Omega) = Z_{\phi_i} - \Pi_i(k, \Omega)\). The one-loop diagrams contributing to the polarization at zero external momentum \(k = 0\) and fixed frequency \(\Omega\) are drawn in Fig. 9. The polarization at order \(\alpha^0\) reads
\[
\Pi_i^0 = -4g_i N_\psi^2 \int_{q,\omega} \text{Tr}[M_i G_0'(q,\omega) \cdot M_i G_0'(q,\omega)].
\] (E12)

Notice that for the sake of generality, we will write all interaction matrices as the vectors \(M_i\), which can either denote a single matrix \(R_i\) for \(i = 1, \ldots, 8\), or a two-component vector \(M_i\) for \(i = 8, \ldots, 12\). The polarization at order \(\alpha^2\) reads
\[
\Pi_i = -4g_i N_\psi^2 \alpha^2 \sum_{n,j} \int_{q,\omega} \text{Tr}[M_i G_0'(q,\omega) T^n_j G_0'(q + \eta\mathbf{q}_j,\omega) \cdot M_i G_0'(q + \eta\mathbf{q}_j,\omega)].
\] (E13)

The pole of the integral in Eq. (E12) per number of fermion flavors \(n\) (equal to four in our case), is given by \(\Pi_i^n = -4g_i I_i/n\) where
\[
I_i = \lim_{\epsilon \to 0} \frac{\nu N_\psi^2}{n} \int_{q,\omega} \text{Tr}[M_i G_0'(q,\omega) \cdot M_i G_0'(q,\omega)].
\] (E14)

In Eq. (E14) we can use again the separation of energy scales: the theory is meaningful only at low energy, i.e. for \(q,\omega \ll 1\), so that \(G_0'(q + \eta\mathbf{q}_j,\omega)\) can be replaced by
G'_0(\eta q_j, 0). This results in \( \Pi_i^1 = -3\alpha^2 \chi h_i(\beta) \Pi_i^0 \) where \( \chi \) equals either +1 for the interaction matrices \( \sigma z \tau_0 \) and \( -\sigma_1 \) or -1 for the interaction matrices \( \sigma z \tau_z \) and \( -\sigma_1 \); and the corrugation-dependent function \( h_i(\beta) \) equals either \( 1 - \beta^2 \) or 1 if the interaction matrix matches \( \sigma_0 \) or \( \sigma_z \) in the pseudospinor, respectively. This fixes the renormalization constant to

\[
Z_{\phi_i} = 1 - \frac{4nq_i I_i[1 + 3\alpha^2 \chi h_i(\beta)]}{\nu \epsilon}.
\]

(E15)

5. Vertices

We denote the one-loop contribution to the three-point vertex of interaction \( i \) renormalised by interaction \( l \) by \( V_{il} \). The one-loop vertices at zero external momentum \( k = 0 \) and fixed frequency \( \Omega \) are drawn in Fig. 9(c) to 9(g) and computed in Eq. (E16) to (E20).

For the vertices correcting an interaction \( i \) associated to a 2d channel, we write the vertex for only one component of the matrix \( M_i \), simply denoted as \( M_i \). The three-point vertex at order \( \alpha^0 \), given by diagram shown in Fig. 9(c) reads

\[
V_{il}^0 = N_{il}^3(2/\sqrt{g_i})(4g_i) \int_{q, \omega} M_i G_0(q, \omega) M_i G_0(q, \omega) \cdot M_i.
\]

(E16)

The three-point vertex at order \( \alpha^2 \) (mixed diagram with two interlayer hopping) have either multiplicity one, or two. Those with multiplicity one nest either an internal hopping line, as in Fig. 9(d)

\[
V_{il}^{1,\text{int}} = N_{il}^3(2/\sqrt{g_i})(4g_i) \alpha^2 \int_{q, \omega} M_i G_0(q, \omega) T_i^0 G_0(q + \eta q_j, \omega) M_i G_0(q + \eta q_j, \omega) T_i^0 G_0(q, \omega) \cdot M_i.
\]

(E17)

or an external hopping line, as in Fig. 9(e)

\[
V_{il}^{1,\text{ext}} = N_{il}^3(2/\sqrt{g_i})(4g_i) \alpha^2 \int_{q, \omega} T_i^0 G_0(\eta q_j, \omega) M_i G_0(q, \omega) M_i G_0(q, \omega) \cdot M_i G_0(\eta q_j, \omega) T_i^0.
\]

(E18)

The mixed diagrams with multiplicity two nest either an isolated hopping line, as in Fig. 9(f)

\[
V_{il}^{1,\text{iso}} = 2N_{il}^3(2/\sqrt{g_i})(4g_i) \alpha^2 \int_{q, \omega} T_i^0 G_0(\eta q_j, \omega) M_i G_0(q + \eta q_j, \omega) T_i^0 G_0(q, \omega) M_i G_0(q, \omega) \cdot M_i.
\]

(E19)

or a hopping line that crosses the interaction line, shown in Fig. 9(g)

\[
V_{il}^{1,\text{cro}} = 2N_{il}^3(2/\sqrt{g_i})(4g_i) \alpha^2 \int_{q, \omega} M_i G_0(\eta q_j, \omega) T_i^0 G_0(q, \omega) M_i G_0(q, \omega) \cdot M_i G_0(\eta q_j, \omega) T_i^0.
\]

(E20)
| $\alpha^2$ | $a_{2}^2 (\sigma_{z} \tau_{0})$ | $a_{1}^{-} (\sigma_{z} \tau_{3})$ | $E_{2}^{+} (\sigma_{\tau_{0}} / \sqrt{2})$ | $E_{4}^{-} (\sigma_{\tau_{3}} / \sqrt{2})$ |
|---|---|---|---|---|
| $a_{2}^{-}$ | $\frac{4}{7} [1 - 12 \alpha^{2} (1 - \beta^{2})]$ | $- \frac{4}{7} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ | $\frac{4}{7} [1 - 6 \alpha^{2} (1 + \beta^{2})]$ | $\frac{4}{7} [1 + 6 \alpha^{2} (3 - \beta^{2})]$ |
| $a_{1}^{-}$ | $- \frac{4}{7} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ | $\frac{4}{7} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ | $\frac{4}{7} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ | $\frac{4}{7} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ |
| $E_{2}^{+}$ | $2 \frac{\pi}{\alpha} [1 - 6 \alpha^{2} (1 + \beta^{2})]$ | $\frac{2}{\alpha} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ | $\frac{4}{\alpha} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ | $- \frac{12}{\alpha} \alpha^{2} \beta^{2}$ |
| $E_{4}^{-}$ | $2 \frac{\pi}{\alpha} [1 + 6 \alpha^{2} (3 - \beta^{2})]$ | $\frac{2}{\alpha} [1 - 6 \alpha^{2} (1 - \beta^{2})]$ | $- \frac{12}{\alpha} \alpha^{2} \beta^{2}$ | $\frac{4}{\alpha} [1 + 3 \alpha^{2} (1 + \beta^{2})]$ |

TABLE V. List of the functions $f_{il}(\alpha, \beta)$ appearing in the RG flows of the four non-trivial channels $a_{2}^{-}$, $a_{1}^{-}$, $E_{2}^{+}$, and $E_{4}^{-}$. The interaction matrices associated to each of these channels are indicated in the first line of the table.

Similarly, we can define the pole of the integral appearing in Eq. (E16) as

$$J_{il} = \lim_{\epsilon \to 0} \frac{\nu \epsilon N_{W}^{2}}{n} \int_{\eta,\omega} \frac{v_{\epsilon} N_{W}^{2}}{n} \int_{\eta,\omega} \text{Tr} [M_{il} G_{0}^{\prime}(q, \omega) M_{i} M_{il}] = \begin{cases} 0 & \text{if } (M_{i}, M_{il}) \text{ match } (\sigma, \sigma), \\
\frac{1}{4 \pi} & \text{if } (\sigma, \sigma_{z}), \\
\frac{1}{4 \pi} & \text{if } (\sigma_{z}, \sigma_{z}), \\
\frac{1}{4 \pi} & \text{if } (\sigma_{z}, \sigma_{z}). \end{cases}$$

(E21)

such that $V_{il}^{1,\text{in}} = V_{il}^{1,\text{ext}} = 3 \alpha^{2} \chi_{i} h_{i}(\beta) V_{il}^{0}$ with $V_{il}^{0} = N_{\psi}(2i \sqrt{g_{i}})(4g_{il} J_{il}/v_{\epsilon})$. We also define the pole appearing in the sum of the diagrams with multiplicity two as

$$K_{il}(\beta) = \lim_{\epsilon \to 0} \frac{\nu \epsilon N_{W}^{4}}{n} \sum_{\eta, \omega} \int_{\eta, \omega} \text{Tr} [M_{il} G_{0}^{\prime}(\eta q_{j}, \omega) T^{\prime}_{j} G_{0}^{\prime}(q, \omega) M_{i} G_{0}^{\prime}(\eta q_{j}, \omega) \cdot M_{i} (M_{il}, T^{\prime}_{j} G_{0}^{\prime}(\eta q_{j}, \omega) + G_{0}^{\prime}(\eta q_{j}, \omega) T^{\prime}_{j} M_{il})]$$

(E22)

such that $V_{il}^{1,\text{iso}} + V_{il}^{1,\text{cro}} = N_{\psi}(2i \sqrt{g_{i}})(8 \alpha^{2} g_{il} K_{il}(\beta)/v_{\epsilon})$. The integrals $J_{il}$ are numerical constants, dependent neither of the number of fermion flavors $n$ nor the correlation parameter $\beta$, while $K_{il}(\beta)$ depends on the correlation parameter. Using commutation relations between interaction and hopping matrices, we can express all vertices (E16) – (E20) in terms of $J_{il}$ and $K_{il}$ only. We then find the vertex renormalization constant to be

$$Z_{g_{i}} = 1 - \frac{4}{v_{\epsilon}} \sum_{l} g_{l} [(1 + 6 \alpha^{2} h_{i}(\beta) \chi_{i}) J_{il} + 2 \alpha^{2} K_{il}(\beta)].$$

(E23)

6. RG flow equations

We express $Z_{i} = Z_{g_{i}} Z_{\phi_{i}}^{-1}$ to first order in the coupling constants as

$$Z_{i} = 1 + \frac{12}{v_{\epsilon}} \sum_{l=1}^{12} f_{il}(\alpha, \beta) g_{l}.$$

(E24)
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