Non-Hermitian description of the dynamics of inter-chain pair tunnelling

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We study inter-chain pair tunnelling dynamics based on an exact two-particle solution for a two-leg ladder. We show that the Hermitian Hamiltonian shares a common two-particle eigenstate with a corresponding non-Hermitian Hubbard Hamiltonian in which the non-Hermiticity arises from an on-site interaction of imaginary strength. Our results provide that the dynamic processes of two-particle collision and across-legs tunnelling are well described by the effective non-Hermitian Hubbard Hamiltonian based on the eigenstate equivalence. We also find that any common eigenstate is always associated with the emergence of spectral singularity in the non-Hermitian Hubbard model. This result is valid for both Bose and Fermi systems and provides a clear physical implication of the non-Hermitian Hubbard model.

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I. INTRODUCTION

Complex parameter in a Hamiltonian, such as imaginary potential, has been investigated under the framework of non-Hermitian quantum mechanics [1–12]. The usefulness of the complex parameter can be explored by establishing a correspondence between a non-Hermitian system and a real physical system in an analytically exact manner. The discovery of a parity-time ($\mathcal{PT}$) symmetric non-Hermitian Hamiltonian having an entirely real quantum-mechanical energy spectrum [19] stimulated the efforts of establishing $\mathcal{PT}$ symmetric quantum theory as a complex extension of conventional quantum mechanics [13–15]. This complex extension has profound theoretical and methodological implications in many other subjects, ranging from quantum field theory and mathematical physics [20–23], to solid state [24, 25] and atomic physics [26, 27].

One way of extracting the physical meaning of a pseudo-Hermitian Hamiltonian with a real spectrum is to seek its Hermitian counterparts [8–10]. There exists another Hermitian Hamiltonian that shares the complete or partial spectrum when the spectrum of a pseudo-Hermitian Hamiltonian is real. The metric-operator theory outlined in Ref. [13] provides a mapping between a pseudo-Hermitian Hamiltonian and an equivalent Hermitian counterpart. However, the obtained equivalent Hermitian Hamiltonian is usually quite complicated [13, 30], and it is difficult to determine whether it describes real physics or is just an unrealistic mathematical object. An alternative way to establish the connection between a pseudo-Hermitian Hamiltonian and a physical system is considering the equivalence of eigenstates [31–33]. A Hermitian scattering center at resonant transmission shares the same wave function with the corresponding non-Hermitian tight-binding lattice consisting of the Hermitian scattering center with two additional $\mathcal{PT}$-symmetric on-site complex potentials.

In this paper, we extend this approach to interacting particle systems. In condensed matter physics, inter-chain (inter-layer) pair tunnelling is a popular process, and is an important component for the mechanism of superconductivity [34, 35]. We consider a two-leg system with inter-chain pair tunnelling. Based on the exact two-particle solution, we show that if the two-particle dynamics mainly refers to a specific invariant subspace, then the corresponding two-particle dynamics can be described by an effective non-Hermitian Hubbard system with an imaginary on-site interaction. For a given initial state, the strength of the imaginary on-site interaction is determined by the relative velocity of the two particles. When we consider the two-particle dynamics associated with the probability gain in one leg of the Hermitian system, a set of corresponding non-Hermitian Hamiltonians are related to the spectral singularities. Therefore, the dynamical correspondence is sensitive to the selection of the initial state. The particle-creation dynamics can be realized by considering the time-reversal process of it, which corresponds to the annihilation of two particles. On the other hand, the two-particle tunnelling associated with decrease of the probability in the other leg can be well described by a non-Hermitian Hubbard model with the definite pair dissipation. Especially, when the relative group velocity matches the strength of pair tunnelling, the two-particle probability will exhibit a completely transfer from one leg to the other, which corresponds to pair annihilation in the effective non-Hermitian system. From this point of view, we unveil the connection between the interacting Hermitian and non-Hermitian systems in the context of wavepacket dynamics.

This paper is organized as follows. In Sec. II we introduce the model Hamiltonians and their symmetry. In Sec. III we present the equivalence between the Hermitian Hamiltonian with inter-chain pair tunnelling and non-Hermitian Hamiltonians with an imaginary on-site interaction. Sec. IV and Sec. V are devoted to construct the connection between two types of the systems through wavepacket dynamics. Section VI provides the summary...
and discussion.

II. MODEL HAMILTONIANS

We address a physically meaningful non-Hermitian Hamiltonian by associating pair tunnelling with an imaginary on-site interaction in a non-Hermitian Hubbard model. As an illustration, we consider two simple models described by a Hermitian and a non-Hermitian Hamiltonian.

The Hermitian Hamiltonian can be written as follows

\[ H = H_A + H_B + H_{AB}, \]  

and

\[ H_\rho = -\kappa \sum_{j=1}^{N} \left( a^\dagger_{\rho,j} a^{}_{\rho,j+1} + \text{H.c.} \right), \quad (\rho = A, B), \]  

\[ H_{AB} = -\frac{J}{2} \sum_{j=1}^{N} \left( a^\dagger_{A,j} a^{}_{A,j} a^{}_{B,j} a^\dagger_{B,j} + \text{H.c.} \right). \]

Obviously, it represents a tight-binding system consisting of a two-leg ladder, with each leg \( H_\rho \) \((\rho = A, B)\) having dimension \( N \). The two legs are coupled through a pair tunnelling term \( H_{AB} \), which operates on the motion of multi particles. The Hamiltonian possesses two symmetries. One is the \( P \) symmetry: here \( P \) represents the space-reflection operator (or parity operator), and the effect of the parity operator is \( P a^\dagger_{\rho,j} P^{-1} = a^\dagger_{\rho,j} \). The other is the particle-number symmetry, which ensures probability conservation and leads to the following commutation relation

\[ [\hat{N}_\rho, H] \neq 0, \text{ but } \sum_\rho \hat{N}_\rho, H = 0, \]  

where \( \hat{N}_\rho = \sum_i a^\dagger_{\rho,i} a^{}_{\rho,i} \) \((\rho = A, B)\) are the particle-number operators for the upper and lower legs, respectively. The probability is conserved in the entire system \( H \), but breaks in subsystems \( H_A \) and \( H_B \). The inter-chain pair tunnelling admits a peculiar symmetry,

\[\left( (-1)^{\hat{N}_\rho} , H \right) = 0, \]

i.e., the conservation of particle-number parity.

Another related system is a non-Hermitian system composed by two independent Hubbard chains, which can be expressed as

\[ \mathcal{H} = \mathcal{H}_A + \mathcal{H}_B, \]  

and

\[ \mathcal{H}_\rho = -\kappa \sum_{i=1}^{N} \left( a^\dagger_{\rho,i} a^{}_{\rho,i+1} + \text{H.c.} \right) + \frac{i U_\rho}{2} \sum_i n_{\rho,i} (n_{\rho,i} - 1), \]

where \( \rho = A, B \). The non-Hermiticity of \( \mathcal{H}_\rho \) arises from the complex on-site interaction \( i U_\rho \).

We note that \( \mathcal{H} \) has the same symmetries as \( H \) does, i.e., \( [\mathcal{H}_\rho, \sum_i n_{\rho,i}] = 0, [\mathcal{H}_\rho, \sum_i n_{\rho,i}] = 0 \), except \( [\mathcal{H}_A, \mathcal{H}_B] = 0 \). This allows us to construct the eigenstates of two models in the same invariant subspaces. For instance, particle-preserving symmetry leads to the two-particle invariant subspace, which can be further decomposed into two invariant subspaces with basis sets \( \{ a^\dagger_{A,i} a^\dagger_{B,j} |0\} \) and \( \{ a^\dagger_{A,i} a^\dagger_{B,j} |0\}, \) respectively. In the next section, we will investigate the connection between the two-particle solutions of these two Hamiltonians. In Fig. 1, we schematically illustrate the system \( H \) and \( \mathcal{H} \).

FIG. 1: (Color online) Schematic illustration of the concerned lattice systems. (a) Two-leg ladder for non-interacting particles. Two particles at the same site can hop simultaneously across two legs with \( J \) being the inter-chain pair tunnelling strength. (b) Two independent Hubbard chains with imaginary on-site interaction \( i U_\rho(\rho = A, B) \).

III. PAIR TUNNELLING AND SPECTRAL SINGULARITY

Now we turn to study the two-particle eigenstates of \( H \) and \( \mathcal{H}_\rho \), from which we expect to establish the connection between two models. We focus on the solutions in the invariant subspace spanned by \( \{ a^\dagger_{A,i} a^\dagger_{B,j} |0\} \), i.e., both particles are either in chain \( A \) or \( B \). The derivation in Appendix VII A shows that for each given \( \{ K, k \} \) with \( K \in [-\pi, \pi], k \in [0, \pi] \), there are two degenerate eigenstates of \( H \) with energy

\[ \varepsilon_K(k) = -4\kappa \cos (K/2) \cos k. \]
And the associated eigenstates can be written as
\[
|\psi_{K,k}^\pm\rangle = \sum_{r \geq 0, \rho = A,B} f_{K,k}^{\rho,\pm}(r) |\phi_{\rho}^\pm(K)\rangle, \tag{9}
\]
and
\[
|\phi_{\rho}^0(K)\rangle = \frac{1}{2\sqrt{N}} \sum_j e^{iKj} a_{\rho,j}^\dagger a_{\rho,j}^\dagger |\text{vac}\rangle, \tag{10}
\]
\[
|\phi_{\rho}^\pm(K)\rangle = \frac{e^{iKr/2}}{\sqrt{N}} \sum_{j} e^{iKj} a_{\rho,j}^\dagger a_{\rho,j+r}^\dagger |\text{vac}\rangle, \tag{11}
\]
where \(|\phi_{\rho}^0(K)\rangle\) and \(|\phi_{\rho}^\pm(K)\rangle\) are translational invariant bases. The corresponding wavefunctions \(f_{K,k}^{\rho,\pm}(r)\) can be expressed explicitly as
\[
f_{K,k}^{A,+}(r) = f_{K,k}^{B,-}(r) = \begin{cases} e^{-ikr} + \eta_{K,k} e^{ikr}, \ r > 0 \\
(1 + \eta_{K,k})/\sqrt{2}, \ r = 0 \end{cases}, \tag{12}
\]
\[
f_{K,k}^{B,+}(r) = f_{K,k}^{A,-}(r) = \begin{cases} \xi_{K,k} e^{ikr}, \ r > 0 \\
(1 + \xi_{K,k})/\sqrt{2}, \ r = 0 \end{cases}, \tag{13}
\]
where
\[
\eta_{K,k} = \frac{\lambda_{K,k}^2 - J^2}{\lambda_{K,k}^2 + J^2}, \quad \xi_{K,k} = -2i\lambda_{K,k} J/\lambda_{K,k}^2 + J^2, \tag{14}
\]
\[
\lambda_{K,k} = 4K \cos(K/2) \sin k. \tag{15}
\]
We note that \(K\) represents the central momentum vector of two particles, while \(k\) represents the relative momentum between the two particles. In this sense, the eigenstates \(|\psi_{K,k}^\pm\rangle\) are associated with the dynamic process in which two particles collide with each other in one leg and then tunnel into the other leg.

Similarly, we can construct the eigenstates of \(\mathcal{H}\) having the same form in Eq. (9) based on the result shown in Appendix VII B
\[
|\chi_{K,k}^\pm\rangle = \sum_{r \geq 0, \rho = A,B} g_{K,k}^{\rho,\pm}(r) |\phi_{\rho}^\pm(K)\rangle, \tag{16}
\]
where
\[
g_{K,k}^{A,+}(r) = g_{K,k}^{A,-}(r) = \begin{cases} e^{-ikr} + \mu_{K,k} e^{ikr}, \ r > 0 \\
(1 + \mu_{K,k})/\sqrt{2}, \ r = 0 \end{cases}, \tag{17}
\]
\[
g_{K,k}^{B,+}(r) = -g_{K,k}^{B,-}(r) = \begin{cases} e^{-ikr} + \nu_{K,k} e^{ikr}, \ r > 0 \\
(1 + \nu_{K,k})/\sqrt{2}, \ r = 0 \end{cases}, \tag{18}
\]
and the parameters are
\[
\mu_{K,k} = \frac{\lambda_{K,k} + U_A}{\lambda_{K,k} - U_A}, \quad \nu_{K,k} = \frac{\lambda_{K,k} + U_B}{\lambda_{K,k} - U_B}. \tag{19}
\]
It is easy to check that when the following conditions are satisfied
\[
U_A = -J^2/\lambda_{K,k}, U_B = \lambda_{K,k}, \tag{20}
\]
we could obtain
\[
|\psi_{K,k}^\pm\rangle = |\chi_{K,k}^\pm\rangle. \tag{21}
\]
Note that the eigenstates \(|\chi_{K,k}^\pm\rangle\) are the functions of \(U_A\) and \(U_B\). The equivalence condition (21) denotes that the \(U_A\) and \(U_B\) are \(|K,k\rangle\) dependent. Thus one requires two indices to label the eigenstate as \(|\chi_{K,k}^\pm(U_A,U_B)\rangle\).

For the sake of convenience, we neglect the \((U_A,U_B)\) of \(|\chi_{K,k}^\pm(U_A,U_B)\rangle\). If we exchange the values of \(U_A\) and \(U_B\)
\[
U_A = \lambda_{K,k}, U_B = -J^2/\lambda_{K,k}, \tag{22}
\]
we have
\[
|\psi_{K,k}^\pm\rangle = |\chi_{K,k}^\pm\rangle. \tag{23}
\]
which arises from the parity symmetry of both $H$ and $\mathcal{H}$. This indicates that the two Hamiltonians have common eigenstates, revealing the connection between a Hermitian and a non-Hermitian Hamiltonian. This connection has the following features: (i) We find that $iU_A$ and $U_B$ are $\{K, k\}$ dependent and for a given $\{K, k\}$, they are all imaginary but with different signs, representing a complementarity pair gain and loss. Further investigation in the next section will show that this ensures the conservation of particles in the whole system. (ii) As an independent non-Hermitian Hubbard chain with on-site strength $iU_{B}$, the derivation in Appendix VII C shows that when $U_{B} = \lambda_{K,k}$ this Hamiltonian has a spectral singularity at point $\{K, k\}$. (iii) Furthermore, we find that in the case of $J^2 = \lambda^2_{K,k}$, two independent non-Hermitian Hubbard chains have a spectral singularity simultaneously at point $\{K, k\}$. The mechanism of the occurrence of the spectral singularity and the corresponding physical implications will be addressed in the next section.

IV. TUNNELLING DYNAMICS

Considering two local particles in one of two legs, which have no overlap with each other, the tunnelling term would have zero effect on the dynamics. But when the two particles meet, particle transfer occurs between two legs. The pair transmission probability depends on many factors as discussed in the following. In this section, we will investigate the dynamics of two-wavepackets collision based on the above formalism. We start our investigation from the time evolution of an initial state as

$$|\Phi(0)\rangle = |\Phi_{A,a}\rangle|\Phi_{A,b}\rangle,$$

which represents two separable boson wavepackets $a$ and $b$. Here

$$|\Phi_{\rho,\gamma}\rangle = \frac{1}{\sqrt{\Omega}} \sum_{j} e^{-\alpha^2(j-N_{\gamma})^2} a_{\rho,j} a_{\rho,j}^\dagger |\text{Vac}\rangle,$$

with $\gamma = a, b$, and $\rho = A, B$ represents a Gaussian wavepacket, which has a width $2\sqrt{\ln 2}/\alpha$, a central position $N_{\gamma}$ in chain $\gamma$ and a group velocity $v_{\gamma} = -2\kappa \sin k_{\gamma}$. The condition that $N_{a} - N_{b} \gg 1/\alpha$ ensures that two initial bosons cannot overlap, and thus having no pair tunnelling. Straightforward derivation shows that

$$|\Phi(0)\rangle = \frac{1}{\sqrt{2}} \sum_{\sigma = \pm} \left(|\Phi_{A,a}\rangle|\Phi_{A,b}\rangle + \sigma|\Phi_{B,a}\rangle|\Phi_{B,b}\rangle\right)\right.$$}

$$= \frac{1}{\sqrt{2\Omega_1}} \sum_{K} e^{-\left(K-2k_{\alpha}\right)^2/4\alpha^2}$$

$$\times e^{-iN_{\gamma}(K-2k_{\alpha})} \left|\psi_{K}^{\pm} (r_{c}, q_{c})\right\rangle,$$

where

$$|\psi_{K}^{\pm} (r_{c}, q_{c})\rangle = \frac{1}{\sqrt{\Omega_2}} \sum_{r} e^{-\alpha^2(r-r_{c})^2/2e^{iq_{c}r/2}} \left|\phi_{r}^{\pm} (K)\right\rangle,$$

and $\Omega_{1,2}$ is the normalized factor. Here we have used the following transformations

$$N_{c} = \frac{1}{2} \left(N_{a} + N_{b}\right), r_{c} = N_{b} - N_{a},$$

$$k_{c} = \frac{1}{2} \left(k_{a} + k_{b}\right), q_{c} = k_{b} - k_{a},$$

and identities

$$2 \left[(j - N_{a})^2 + (l - N_{b})^2\right]$$

$$= \left[(j + l) - (N_{a} + N_{b})\right]^2$$

$$+ \left[(l - j) - (N_{b} - N_{a})\right]^2,$$

$$2 (k_{a}j + k_{b}l)$$

$$= (k_{a} + k_{b}) (j + l) + (k_{b} - k_{a}) (l - j).$$

We note that the component of state $|\Phi(0)\rangle$ on each invariant subspace represents an incident wavepacket along the chains described by $H_{E0}^{\pm}$ with a width $2\sqrt{\ln 2}/\alpha$, a central position $r_{c} = N_{b} - N_{a}$ and a group velocity $v = -4\kappa \cos(K/2) \sin(q_{c}/2)$. It is worth pointing out that as $\alpha \ll 1$, the initial state is distributed mainly in the invariant subspace $K = 2k_{c}$, where the wavepacket moves...
with the group velocity \( v_r = -4k \cos(k) \sin(q_r/2) = v_b - v_a \). Then the time evolution of state \( \Phi(t) \) can be derived by the evolution of wavepacket in two chains \( H_{eq}^{K,\pm} \), which eventually can be obtained from the solution in Eq. (22). Furthermore, according to the solution, the evolved state of \( \phi^\pm_R(r_c, q_c) \) can be expressed approximately in the form of \( e^{i\beta(r_c)} R_{2k_c,q_c/2}^\pm \phi^\pm_R(r_c', q_c) \), which represents a reflected wavepacket in the equivalent semi-infinite chain \( H_{eq}^{K,\pm} \). The expressions of \( R_{2k_c,q_c/2}^\pm \) and \( H_{eq}^{K,\pm} \) are given in the Appendix VIIIB. Here \( \beta(r_c') \), as a function of the position of the reflected wavepacket, is an overall phase and is independent of \( J \). We assume that the collision occurs at instant \( t_0 \), the evolved state at time \( t \gg t_0 \) has the form of

\[
\Phi(t) = \sum_{\sigma_1, \sigma_2 = \pm} \Omega^{-1} e^{i\delta(N'_1 - N'_0)} R_{2k_c,q_c/2}^{\sigma_1} \times \sum_{j=1} e^{-\alpha^2(i-N'_0)^2 - \alpha^2(j-N'_0)^2} \times e^{iK_j} e^{ik_c} \times (\tilde{a}_{A,j}^\dagger \tilde{a}_{A,j} + \sigma \tilde{a}_{B,j}^\dagger \tilde{a}_{B,j}^\dagger) |\text{Vac}\rangle
\]

which also represents two separable wavepackets at \( N'_0 \) and \( N'_0 \) respectively. Comparing Eqs. (26) and (33), it is straightforward to figure out that the two-particle wavepackets behave as classical particles, which swap the momenta with each other after collision. For simplicity, we denote an incident single-particle wavepacket as \( \Phi(R) \), where \( \lambda = L, R \) indicates the particle coming from the left or right of the collision zone, and \( p \) is the central momentum. In this context, we give the asymptotic expression for the collision process in the following: at time \( t \ll t_0 \), we have

\[
|L, k_a, A \rangle |R, k_b, A \rangle = \frac{1}{\sqrt{2}} (|F^+ \rangle + |F^- \rangle),
\]

where

\[
|F^\pm \rangle = \frac{1}{\sqrt{2}} (|L, k_a, A \rangle |R, k_b, A \rangle \pm |L, k_a, B \rangle |R, k_b, B \rangle).
\]

and after collision, at time \( t \gg t_0 \), the wavepackets exchange their momenta, which admits

\[
|L, k_a, A \rangle |R, k_b, A \rangle \pm |L, k_a, B \rangle |R, k_b, B \rangle \mapsto R_{2k_c,q_c/2} \times (|L, k_a, A \rangle |R, k_b, A \rangle \\
|L, k_b, A \rangle |R, k_a, B \rangle \pm |L, k_b, B \rangle |R, k_a, B \rangle).
\]

By neglecting the \( J \)-independent overall phase, therefore we have

\[
|L, k_a, A \rangle |R, k_b, A \rangle \mapsto \cos \Delta_{2k_c,q_c/2} |L, k_b, A \rangle |R, k_a, A \rangle \\
+ i \sin \Delta_{2k_c,q_c/2} |L, k_b, B \rangle |R, k_a, B \rangle,
\]

where

\[
\cos \Delta_{2k_c,q_c/2} = e^{i\pm \Delta_{2k_c,q_c/2}},
\]

\[
\sin \Delta_{2k_c,q_c/2} = 2 \tan^{-1} \left( \frac{J}{\lambda_{2k_c,q_c/2}} \right),
\]

as discussed in Appendix VIIIB. Evidently, Eq. (37) shows that after collision, one part of two wavepackets in leg \( A \), which corresponds to the first term in Eq. (37), is reflected as two identical classical particles. Meanwhile another part, which corresponds to the second term in Eq. (37), tunnels into leg \( B \).

Considering a special case with \( v_r = J \), i.e., the pair-tunnelling amplitude is equal to the relative group velocity, we have \( \Delta_{2k_c,q_c/2} = \pi/2 \), and this leads to

\[
|L, k_a, A \rangle |R, k_b, A \rangle \mapsto i |L, k_b, B \rangle |R, k_a, B \rangle.
\]

Clearly, this represents the process that two separable wavepackets on leg \( A \) tunnel into leg \( B \) completely.

V. NON-HERMITIAN DYNAMICS

From the above discussions regarding the dynamics of across-leg tunneling, we see that the two-particle probability transfers from one leg to another. The two-particle probability in one leg is not conserved. Thus, a natural question to ask is whether there exists an effective non-Hermitian Hamiltonian for characterizing such a dynamics. To this end, we first present the connections between Hermitian Hamiltonian \( H \) and \( \mathcal{H} \) in a compact form. There are \( N(N + 1) \) eigenstates of \( H \) in the invariant subspace spanned by \( \{ a_{\rho,i}^\dagger a_{\rho,j}^\dagger |\text{Vac}\rangle \} \). Each of the eigenstates \( \{ b_{K,k}^\pm \} \) corresponds to a specific eigenstate \( b_{K,k}^\pm \) of the non-Hermitian Hubbard chain with the \((K,k)\)-dependent interaction \( iU_{\rho} \) as in Eqs. (21) and (23). We note that the eigenstates of \( H \) and \((K,k)\)-dependent Hamiltonian \( \mathcal{H} \) are related to the index \((K,k)\). In the following, we take a single index \( \eta \) to represent \((K,k)\). For the system with \( 2N \) sites, all possible \((K,k)\) is denoted as \( \eta = 1, 2, ..., N(N + 1)/2 \). The eigenstates of \( H \) is denoted as \( |\tilde{\psi}_\eta \rangle \) \(( \ell \in [1, N(N + 1)]) \) with

\[
|\tilde{\psi}_\eta \rangle \equiv |\psi_{K,k}^\pm \rangle,
\]

\[
|\tilde{\psi}_{\eta + N(N+1)/2} \rangle \equiv |\psi_{K,k}^- \rangle.
\]

Accordingly, the \((K,k)\)-dependent Hamiltonian \( \mathcal{H} \) is denoted as \( \mathcal{H}_\ell \) with

\[
\mathcal{H}_\eta \equiv \mathcal{H}(K,k),
\]

for \( U_A = -J^2/\lambda_{K,k}, U_B = \lambda_{K,k} \),

\[
\mathcal{H}_{\eta + N(N+1)/2} \equiv \mathcal{H}(K,k),
\]

for \( U_A = \lambda_{K,k}, U_B = -J^2/\lambda_{K,k} \).
The eigenstate of $\mathcal{H}_l$ is denoted as $|\bar{\psi}_{l,l'}\rangle$ with

$$|\bar{\psi}_{l,l'}\rangle = |\chi^{+}_{K,k}l\rangle, \quad |\bar{\psi}_{l,l'} + N(N+1)/2\rangle = |\chi_{K,k}^{+}l\rangle,$$

for $\mathcal{H}(K,k)$ with $U_A = -J^2/\lambda_{K,k}$, $U_B = \lambda_{K,k}$,

$$|\bar{\psi}_{\eta} + N(N+1)/2,\eta\rangle \equiv |\chi_{K,k}^{+}l\rangle, \quad |\bar{\psi}_{\eta} + N(N+1)/2,\eta' + N(N+1)/2\rangle \equiv |\chi_{K,k}^{+}l\rangle,$$

for $\mathcal{H}(K,k)$ with $U_A = \lambda_{K,k}$, $U_B = -J^2/\lambda_{K,k}$.

(44)

(45)

Note that the eigenstate $|\bar{\psi}_{l,l'}\rangle$ possesses two subscripts. The first one indicates the $(K,k)$-dependent on-site interactions $U_A$ and $U_B$, and the second one denotes the center and relative momenta $(K,k)$ of the eigenstate for a given $U_A$ and $U_B$. In Fig. 3(a), we illustrate the $|\bar{\psi}_{l,l'}\rangle$ via ket matrix. The states in $l$th row represents the complete set of eigenstates of $\mathcal{H}_l$. Based on this notation, the Schrodinger equations become compact

$$H |\bar{\psi}_{l}\rangle = E_l |\bar{\psi}_{l}\rangle, \quad (46)$$

$$\bar{\mathcal{H}}_l |\bar{\psi}_{l,l'}\rangle = \varepsilon_{l,l'} |\bar{\psi}_{l,l'}\rangle. \quad (47)$$

Note that $\varepsilon_{l,l'}$ is related to the scattering solution of two particles, which possesses the form of $\varepsilon_{l,l'} = -4\kappa \cos(K'/2) \cos k'$, where $(K',k')$ denotes possible center and relative momenta. These eigenstates have simple relations

$$|\bar{\psi}_{l}\rangle = |\chi_{l,l}\rangle, \quad E_l = \varepsilon_{l,l} = -4\kappa \cos(K/2) \cos k, \quad (48)$$

which indicate that the diagonal states $\{|\chi_{l,l}\rangle\}$ of Fig. 3(a) is the complete set of eigenstates of $H$. Here, $|\psi_{\eta}\rangle$ ($|\psi_{\eta} + N(N+1)/2\rangle$) represents that the two particles collide with each other in leg $A$ ($B$) and then tunnel into leg $B$ ($A$).

When considering the dynamical correspondence in the non-Hermitian Hubbard system, there exists two kinds of dynamical processes corresponding to $|\psi_{\eta}\rangle$ and $|\psi_{\eta} + N(N+1)/2\rangle$: (i) $\chi^A_{\eta,\eta}$ denotes the two-particle collision process in leg $A$ accompanied by the decrease of the two-particle probability while $\chi^B_{\eta,\eta}$ represents a process related to the increase of two-particle probability in leg $B$. (ii) $\chi^B_{\eta,\eta} + N(N+1)/2,\eta\rangle = \left(|\chi_{\eta} + N(N+1)/2,\eta\rangle - |\chi_{\eta} + N(N+1)/2,\eta\rangle\right) / \sqrt{2}$ denotes the two-particle collision process in leg $B$ accompanied by the decrease of the two-particle probability, and $\chi^A_{\eta,\eta} + N(N+1)/2,\eta\rangle = \left(|\chi_{\eta} + N(N+1)/2,\eta\rangle + |\chi_{\eta} + N(N+1)/2,\eta\rangle\right) / \sqrt{2}$ denotes a process associated with the increase of two-particle probability in leg $A$. For a collision process along leg $\rho$ ($\rho = A,B$) in Hermitian systems, there are $N(N+1)/2$ related non-Hermitian Hamiltonians. Therefore one cannot obtain a Hubbard chain with a certain value of $iU_{\rho}$ to describe the dynamics along one of two legs. However, for an initial state distributed mainly in the vicinity of $|\bar{\psi}_{l}\rangle$, the correspondence of dynamics can be characterized by the eigenstates around $l$th row in which the value of $l_0$ is determined by the central and relative momenta $(K_0,k_0)$ of the considered initial state. This corresponds to a block around certain $\left|\bar{\psi}_{l_0,l_0}\right\rangle$, which can be shown in Fig. 3(b). For the sake of simplicity and convenience, we confine the discussions to the case of $l_0 \in [1,(N+1)/2]$. The conclusion still holds for the case of $l_0 \in [(N+1)/2 + 1,(N+1)/N]$, in which the $U_A$ and $U_B$ exchange their values. To seek an effective non-Hermitian Hamiltonian to characterize such a dynamics, we first consider the collision dynamics in leg $A$, which is accompanied by the decrease of two-particle probability. If the involved wavefunctions changes slowly around $\left(|\chi_{l_0,l_0}\rangle + |\bar{\psi}_{l_0,l_0 + N(N+1)/2}\rangle\right) / \sqrt{2}$, then one can use an effective non-Hermitian Hamiltonian $\mathcal{H}_A(K_0,k_0)$ with a definite $U_A = -J^2/\lambda_{K_0,k_0}$ as an approximation to describe such a dynamics of leg $A$ in the Hermitian system. To this end, we take the derivative of the function $\mu_{K,k}$ with respect to $K$ and $k$,

$$\frac{\partial \mu_{K,k}}{\partial K}_{K_0,k_0} = -\Lambda \sin (K_0/2) \sin k_0, \quad (49)$$

$$\frac{\partial \mu_{K,k}}{\partial k}_{K_0,k_0} = 2\Lambda \cos (K_0/2) \cos k_0, \quad (50)$$

where $\Lambda = 8\kappa J^2 \lambda_{K_0,k_0}\left|\la \bar{\psi}_{l}\rangle\right|^2$. The optimal condition can be achieved when $(\partial \mu_{K,k} / \partial k)_{K_0,k_0} = 0$, and $(\partial \mu_{K,k} / \partial K)_{K_0,k_0} = 0$. This can be realized through adjusting the relative group velocity of the initial two wavepackets. The condition also indicates that all the rows in such a block are identical approximately as shown in Fig. 3(b). Then the diagonal states of the block can be replaced by that in a row with green shadowed. On the other hand, for the dynamics along leg $B$, each of the eigenstates in the vicinity of $|\bar{\psi}_{l_0,l_0}\rangle = \left(|\chi_{l_0,l_0}\rangle - |\chi_{l_0,l_0 + N(N+1)/2}\rangle\right) / \sqrt{2}$ corresponds to a spectral singularity of the non-Hermitian Hamiltonian $\mathcal{H}_B(K,k)$. This leads to the coefficients $\lambda_{K,k} = U_B (\nu_{K,k} = \infty)$. In order to avoid this divergence, we can rewrite the expression of Eq. (18) in the form

$$g_{K,k}^{B,+}(r) = -g_{K,k}^{-}(r) = \left\{ \begin{array}{ll} \zeta_k e^{-ikr} + \zeta_k e^{ikr}, & r > 0 \\ 2\Lambda_{K,k} / \sqrt{2}, & r = 0 \end{array} \right., \quad (51)$$
where $\zeta_k = \lambda_{K,k} - U_B$, and $\zeta_k = \lambda_{K,k} + U_B$. Here we want to point out that the relative magnitude between the amplitudes of right-going wave $e^{ikr}$ and left-going wave $e^{-ikr}$ is meaningful, since we focus on the scattering solution in the limit of $N \to \infty$. In this sense, the form of the wavefunctions $g_{K,k}^{A,\sigma}(r)$ and $g_{K,k}^{B,\sigma}(r)$ ($\sigma = \pm$) are not unique. After multiplying $K - k$ dependent constant, the renormalized scattering solutions are still the corresponding eigenstates of $H_A$ and $H_B$. In the definition of Eq. (18), the existence of the spectral singularity in system can be determined by either $\mu_{K,k} = 0$ ($\nu_{K,k} = 0$) or $\mu_{K,k} = \infty$ ($\nu_{K,k} = \infty$), which is associated with the pair-annihilation or pair-creation process. This corresponds to the case of $\zeta_k = 0$ or $\zeta_k = 0$ in Eq. (51).

To obtain the effective non-Hermitian Hamiltonian, we focus on the variation of $\zeta_k$ in the vicinity of $|\chi_{l_0,t_0}^{B}|$ in the ket matrix. As we have done in leg $A$, we take the partial derivative of the $\zeta_k$ with respect to $K$ and $k$, respectively, which yields

$$\frac{\partial \zeta_k}{\partial K} |_{K_0,k_0} = 0, \tag{52}$$

$$\frac{\partial \zeta_k}{\partial k} |_{K_0,k_0} = 0. \tag{53}$$

This indicates that one can replace the diagonal states with row states for any given momentum $(K_0, k_0)$ in the ket matrix as shown in Fig. 3(b). Thus, an effective Hamiltonian with definite $U_B = \lambda_{K_0,k_0}$ can be employed to simulate the dynamics of leg $B$ in the Hermitian system. Here we want to stress that there is no
tunneling between non-Hermitian Hamiltonians $\mathcal{H}_A$ and $\mathcal{H}_B$. Thus, we cannot employ an effective non-Hermitian Hamiltonian $\mathcal{H}$ with definite strengths of the pair dissipation and gain to describe the tunneling dynamics between two legs. The dynamical correspondence of leg $B$ can be obtained through another method, which will be detailed in the following.

In parallel, we can investigate the dynamics of a two-wavepacket collision by analyzing the time evolution of the initial state $|\Phi (0)\rangle$ in the effective non-Hermitian system $\mathcal{H}$ with $U_A = -J^2/\nu_r$. Similarly, we can obtain the asymptotic expression for the collision process as

$$\begin{align}
|L, k_b, A \rangle |R, k_b, A \rangle &\rightarrow \cos \Delta_{2k_c,q/2} (|L, k_b, A \rangle |R, k_b, A \rangle ,
\end{align}$$

which has the same form as the wave function in leg $A$ of Eq. (57). This indicates that the effective non-Hermitian Hamiltonian $\mathcal{H}$ can describe the wavepacket dynamics in subsystem (leg $A$) of a Hermitian system $\mathcal{H}$. Naturally, when the strength of the imaginary on-site interaction $U_A$ is equal to the relative group velocity $\nu_r$, the two particles will exhibit a behaviour of pair annihilation in leg $A$ and will never tunnel into leg $B$. This process is schematically illustrated in Fig. 2(b). Note that $\mathcal{H}$ cannot describe the wavepacket dynamics in leg $B$, because there is no tunneling between leg $A$ and $B$ in $\mathcal{H}$. However, the final state $|L, k_B, B \rangle |R, k_B, B \rangle$ in leg $B$ as in Eq. (57) can be prepared by using non-Hermitian Hamiltonian $\mathcal{H}_B$ in another way. To this end, we require an initial state to simulate the creation of a pair of particles. Moreover, the modulus of the initial state should tend to be 0, owing to the fact that no one can create a pair of particles out of nothing. Then the initial state driven by the $\mathcal{H}_B$ will evolve to $|L, k_B, B \rangle |R, k_B, B \rangle$ accompanied by the increase of two-particle probability. However, the selection of such an initial state is too cumbersome. There is a lot of states with near-zero-modulus value. The different types of the initial states will exhibit distinct dynamical behaviors. In other words, the dynamics of the system is sensitive to the initial state. Therefore, the elaborately selection of the initial state is a crucial step to successfully mimic the dynamics of leg $B$ in Hermitian system. Fortunately, we can chose the initial state by considering the time-reversal process of the dynamics of leg $B$, which corresponds to the annihilation of two wavepackets $|L, k_B, B \rangle |R, k_B, B \rangle$. This can be realized through adjusting the on-site interaction $U_B$ and relative group velocity $\nu_r$ based on the result obtained in leg $A$. In this sense, the final near-zero-modulus state can be selected as an initial state of the particles creation process. And the corresponding driven Hamiltonian can also be obtained by taking the time-reversal operation on the related non-Hermitian Hubbard Hamiltonian with pair annihilation.

In order to further validate the conclusion obtained above, we compare the local-state dynamics in two such systems by numerical simulation. To do this, we introduce the quantity $|\langle \Phi (t) | n_j | \Phi (t) \rangle|^2$ to characterize the shape and probability distribution of the two wavepacket-sets in Fig. 4. For the Hermitian case as shown in Fig. 4(a) and 4(b), one can see that when the two wavepackets enter into the interaction region, the probability of the two wavepacket transfers from leg $A$ to $B$ due to the pair tunnelling $J$. The process of the decrease of the two-wavepacket probability in leg $A$ can also be approximately described through the two-wavepacket dynamics in an effective non-Hermitian Hubbard Hamiltonian $\mathcal{H}$ with $U_A = -J^2/\nu_r$, as is shown in Fig. 4(c).

VI. SUMMARY

In summary, we have studied the inter-chain pair tunneling dynamics based on the exact two-particle solution of a two-leg ladder. It is shown that the Hermitian Hamiltonian shares a common two-particle eigenstate with a corresponding non-Hermitian Hubbard model, in which the non-Hermiticity arises from an imaginary on-site interaction. Such a common state is associated with the spectral singularity of the equivalent non-Hermitian system. The dynamical correspondence is dependent on the selection of the initial state. For the dynamics accompanied with the increase of the two-particle probability, such an initial state can be obtained through a time-reversal process of the annihilation of two wavepackets. On the other hand, the reduction of the two-particle probability in the other leg of the Hermitian system can be well characterized by the effective non-Hermitian Hubbard model with the definite strength of pair dissipation, which is also determined by the relative and center momenta of the initial state. In addition, we have also found that the two particles display perfect transfer from one leg to the other when $\nu_r = J$, which corresponds to the pair annihilation in the effective non-Hermitian Hubbard system with the strength of the imaginary on-site interaction $\nu_r = U_p$. This result is valid for both Bose and Fermi systems and provides a clear physical implication of the non-Hermitian Hubbard model.

VII. APPENDIX

A. Solution of the two-leg ladder

In this section, we derive the solution of the Hamiltonian shown in Eq. (1) in a two-particle invariant subspace. Here, we take the periodic boundary condition that $a_{p,j} = a_{p,j+N}$. Due to the symmetry in Eq. (1), which preserves the parity of particle number in each leg, the $\mathcal{P}$ symmetry, and the translational symmetry,
the basis spanning the subspace can be constructed as
\[ |\varphi^+_{0}(K,r)\rangle = \frac{1}{\sqrt{2N}} \sum_{j} e^{ikj} \left(a_{A,j}^\dagger a_{A,j}^\dagger \right) |\text{vac}\rangle , \]
\[ \pm a_{B,j}^\dagger a_{B,j}^\dagger |\text{vac}\rangle \]  
(55)
\[ |\varphi^-_{r}(K,r)\rangle = \frac{1}{\sqrt{2N}} \sum_{j} e^{ikj/2} \left(a_{A,j}^\dagger a_{A,j}^\dagger \right) |\text{vac}\rangle , \]
\[ \pm a_{B,j}^\dagger a_{B,j+r}^\dagger |\text{vac}\rangle , \]  
(56)
where \( K = 2n\pi/N, n \in [-N/2,N/2] \) is the momentum vector, and \( \pm \) denote two degenerate subspaces originating from the \( P \) symmetry. A two-particle eigenstate has the form of
\[ |\psi_{K,k}^\pm\rangle = \sum_{r} F_{K,k}^\pm (r) |\varphi_{r}^\pm (K)\rangle , \]
(57)
with the condition \( F_{K,k}^\pm (-1) = 0 \), where the two degenerate wave functions \( F_{K,k}^\pm (r) \) satisfy the Schrödinger equations
\[ Q^K_r F_{K,k}^\pm (r+1) + Q^{K^\dagger}_r (-1)^n Q^K_r |\delta_{r,N_0} - \varepsilon_K| F_{K,k}^\pm (r) = 0, \]
(58)
with the eigen energy \( \varepsilon_K \) in the invariant subspace indexed by \( K \). Here the factors are \( Q^K_r = -2\sqrt{2}\kappa \cos (K/2) \) for \( r = 0 \) and \( -2\kappa \cos (K/2) \) for \( r \neq 0 \), respectively. It indicates that the eigen problem of two-particle matrix can be reduced to a single-particle governed by the equivalent Hamiltonians
\[ H_{eq}^{K,\pm} = \pm J |0\rangle \langle 0| + \sum_{i=1}^{\infty} (Q^K_r |i\rangle \langle i+1| + \text{H.c.}) , \]
(59)
which clearly represents a semi-infinite chain with the ending on-site potential \( J \). We are concerned with only the scattering solution by the 0th end. The Bethe ansatz solutions have the form
\[ F_{K,k}^\pm (r) = e^{-ikr} + R^\pm e^{ikr} \]
(60)
Substituting \( F_{K,k}^\pm (r) \) into Eq. \( 58 \), we have
\[ \varepsilon_K (k) = -4\kappa \cos (K/2) \cos k, k \in [0,\pi] , \]
(61)
and
\[ R^\pm_{K,k} = \frac{i\lambda_{K,k} \pm J}{i\lambda_{K,k} \mp J} = e^{\pm i\Delta_{K,k}}, \]
(62)
with
\[ \lambda_{K,k} = 4\kappa \cos (K/2) \sin k, \]
\[ \Delta_{K,k} = 2\tan^{-1} \left(-\frac{J}{\lambda_{K,k}}\right) . \]
(63)
(64)
For convenience in the application of wavepacket dynamics, we rewrite the solutions in the form
\[ |\psi_{K,k}^\pm\rangle = \sum_{r,\rho} f^{\rho,\pm}_{K,k} (r) |\varphi_{r}^\rho (K)\rangle , \]
(65)
where \( \rho = A, B \) and
\[ |\varphi_{0}^\rho (K)\rangle = \frac{1}{\sqrt{2N}} \sum_{j} e^{ikj} a_{p,j}^\dagger a_{p,j}^\dagger |\text{vac}\rangle , \]
(66)
\[ |\varphi_{r}^\rho (K)\rangle = \frac{1}{\sqrt{2N}} \sum_{j} e^{ikj/2} a_{p,j}^\dagger a_{p,j+r}^\dagger |\text{vac}\rangle , \]
(67)
\( (r > 1) \).

The corresponding wavefunctions \( f^{\rho,\pm}_{K,k} (r) \) can be expressed as
\[ f^{A,+}_{K,k} (r) = f^{B,-}_{K,k} (r) \]
\[ = \begin{cases} e^{-ikr} + \frac{\lambda_{K,k}^2 + J^2}{\lambda_{K,k}^2 + J^2} e^{ikr}, r > 0 \\ \left(1 + \frac{\lambda_{K,k}^2 + J^2}{\lambda_{K,k}^2 + J^2}\right) / \sqrt{2}, r = 0 \end{cases} , \]
(68)
and
\[ f^{B,+}_{K,k} (r) = f^{A,-}_{K,k} (r) \]
\[ = \begin{cases} -\frac{\lambda_{K,k}^2 + J^2}{\lambda_{K,k}^2 + J^2} e^{ikr}, r > 0 \\ -\sqrt{2}\lambda_{K,k}^2 / \lambda_{K,k}^2, r = 0 \end{cases} . \]
(69)

B. Solution of the non-Hermitian Hubbard model

Similarly, considering the Hamiltonian \( H \), we find that it admits all the symmetries we used for solving the eigen problem of \( H \). Then a two-particle state for \( H_\rho \) is written as
\[ |\kappa_{K,k}^\rho\rangle = \sum_{r} G_{K,k}^\rho (r) |\varphi_{r}^\rho (K)\rangle , \]
(70)
\( (G_{K,k}^\rho (1) = 0) \)
where wave functions \( G_{K,k}^\rho (r) \) satisfy the Schrödinger equations
\[ Q^K_r G_{K,k}^\rho (r+1) + Q^{K^\dagger}_r (-1)^n Q^K_r |\delta_{r,N_0} - \varepsilon_K| G_{K,k}^\rho (r) = 0, \]
(71)
with the eigen energy \( \varepsilon_K \) in the invariant subspace indexed by \( K \). We are concerned with only the scattering solution by the 0th end. In this sense, \( G_{K,k}^\rho \) can be obtained from the two equivalent Hamiltonians in two subspaces
\[ H_{eq}^{K,\rho} = iU_\rho |0\rangle \langle 0| + \sum_{i=0}^{\infty} (Q^K_r |i\rangle \langle i+1| + \text{H.c.}) , \]
(72)
By the same procedures, we have
\[ G_{K,k}^\rho (r) = \begin{cases} e^{-ikr} + \frac{\lambda_{K,k}^2 + U_\rho^2}{\lambda_{K,k}^2 + U_\rho^2} e^{ikr}, r > 0 \\ \left(1 + \frac{\lambda_{K,k}^2 + U_\rho^2}{\lambda_{K,k}^2 + U_\rho^2}\right) / \sqrt{2}, r = 0 \end{cases} . \]
(73)
with eigen energy $\epsilon_K(k) = -4\kappa \cos(K/2) \cos k$, $k \in [0, \pi]$. Furthermore, we can rewrite the solution in the form

$$g_{K,k}^\pm(r) = \left[ G_{K,k}^A(r) \pm G_{K,k}^B(r) \right] / \sqrt{2}. \quad (74)$$

### C. Spectral singularity of Hubbard chain

We note that wave function $G_{K,k}^\rho(r)$ only depends on $\rho$ via $U_\rho$. This is because the two chains $A$ and $B$ are independent. Then $G_{K,k}^\rho(r)$ actually represents the two-particle solution of a non-Hermitian Hubbard Hamiltonian on a single chain $\rho$ with on-site imaginary interaction strength $iU_\rho$. We find that $G_{K,k}^\rho(r) \to \infty$ as $U_\rho = \lambda_{K,k}$, which indicates a spectral singularity at $\{K,k\}$ [11, 36].

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[1] S. Klaiman, and L. S. Cederbaum, Phys. Rev. A 78, 062113 (2008).
[2] M. Znojil, Phys. Rev. D 78, 025026 (2008).
[3] K. G. Makris, R. El-Ganainy, D. N. Christodoulides and Z. H. Musslimani, Phys. Rev. Lett. 100, 103904 (2008).
[4] Z. H. Musslimani, K. G. Makris, R. El-Ganainy and D. N. Christodoulides, Phys. Rev. Lett. 100, 030402 (2008).
[5] C. M. Bender, and P. D. Mannheim, Phys. Rev. Lett. 100, 110402 (2008).
[6] U. D. Jentschura, A. Surzhykov, and J. Zinn-Justin, Phys. Rev. Lett. 102, 011601 (2009).
[7] J. T. Shen, and S. Fan, Phys. Rev. A 79, 023837 (2009).
[8] A. Mostafazadeh, J. Phys. A: Math. Gen. 38, 6557 (2005).
[9] A. Mostafazadeh, J. Phys. A: Math. Gen. 39, 10171 (2006).
[10] A. Mostafazadeh, J. Phys. A: Math. Gen. 39, 13495 (2006).
[11] X. Z. Zhang, L. Jin, and Z. Song, Phys. Rev. A 87, 042118 (2013).
[12] G. R. Li, X. Z. Zhang, and Z. Song, Ann. Phys. (NY) 349, 288 (2014).
[13] A. Mostafazadeh and A. Batal, J. Phys. A: Math. Gen. 37, 11645 (2004).
[14] A. Mostafazadeh, J. Phys. A: Math. Gen. 36, 7081 (2003).
[15] H. F. Jones, J. Phys. A: Math. Gen. 38, 1741 (2005).
[16] C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. 40, 2201 (1999).
[17] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Gen. 34, L391 (2001); P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Gen. 34, 5679 (2001).
[18] A. Mostafazadeh, J. Math. Phys. 43, 3944 (2002).
[19] C. M. Bender, and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[20] M. Znojil, Phys. Lett. A 285, 7 (2001).
[21] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. 89, 270401 (2002).
[22] H. F. Jones, J. Phys. A: Math. Theor. 42, 135303 (2009).
[23] C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, Phys. Rev. Lett. 98, 040403 (2007).
[24] O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, Phys. Rev. Lett. 103, 030402 (2009).
[25] C. T. West, T. Kottos, and T. Prosen, Phys. Rev. Lett. 104, 054102 (2010).
[26] E. M. Graefe, H. J. Korsch, and A. E. Niederle, Phys. Rev. Lett. 101, 150408 (2008).
[27] E. M. Graefe, H. J. Korsch, and A. E. Niederle, Phys. Rev A 82, 013629 (2010).
[28] E. M. Graefe, U. Günther, H. J. Korsch, and A. E. Niederle, J. Phys. A: Math. Theor. 41, 255206 (2008).
[29] E. M. Graefe, C. Liverani, J. Phys. A: Math. Theor. 45, 444015 (2013).
[30] L. Jin and Z. Song, Phys. Rev. A 80, 052107 (2009).
[31] L. Jin, and Z. Song, Phys. Rev. A 84, 042116 (2011).
[32] L. Jin, and Z. Song, J. Phys. A: Math. Theor. 44, 375304 (2011).
[33] L. Jin, and Z. Song, Phys. Rev. A 85, 012111 (2012).
[34] J. M. Wheatley, T. C. Hsu, and P. W. Anderson, Phys. Rev. B 37, 5897 (1988).
[35] J. M. Wheatley, T. C. Hsu, and P. W. Anderson, Nature 333, 121 (1988).
[36] A. Mostafazadeh, Phys. Rev. A 80, 032711 (2009).