The BPS states of $\mathcal{N} = 2$ super Yang-Mills theory with gauge group $SU(2)$ are constructed as non-trivial finite-energy solutions of the worldvolume theory of a threebrane probe in $F$ theory. The solutions preserve $1/2$ of $\mathcal{N} = 2$ supersymmetry and provide a worldvolume realization of strings stretching from the probe to a sevenbrane. The positions of the sevenbranes correspond to singularities in the field theory moduli space and to curvature singularities in the supergravity background. We explicitly show how the UV cut off of the effective field theory is mapped into an IR cut off in the supergravity. Finally, we discuss some features of the moduli spaces of these solutions.
1. Introduction

Super Yang-Mills theory with gauge group $SU(N)$ and $\mathcal{N} = 2$ supersymmetry is realized at low energy and weak string coupling, on the worldvolume of $N$ Dirichlet fourbranes stretched between two Neveu-Schwarz fivebranes\[1\]. The singularities where the fourbranes meet the Neveu-Schwarz fivebranes are resolved in the strong string coupling limit where this brane set up is replaced by a single M-theory fivebrane wrapping the Seiberg-Witten curve. In this M-theory description, the fourbranes become tubes, wrapping the eleventh dimension and stretching between the two flat asymptotic sheets of the M-fivebrane. The point at which they meet the fivebrane is resolved - the fourbrane (tube) ending on a fivebrane creates a dimple in the fivebrane. It is a fascinating result that the non-trivial bending of the branes due to these dimples encode the perturbative plus all instanton corrections to the low energy effective action describing the original $SU(N) \mathcal{N} = 2$ super Yang-Mills theory\[1\]. The encoding of quantum effects in the field theory in a bending of the brane geometry is a general feature of gauge theories realized on brane worldvolumes.

The M-theory description of the fivebrane is only valid at large string coupling. Field theory however, is only recovered at weak string coupling\[1\]. Quantities that are protected by supersymmetry are not sensitive to whether they are computed at weak or strong coupling. Consequently, computing these quantities we find the fivebrane and field theory results agree. The fivebrane does not reproduce the field theory result for quantities that are not protected by supersymmetry. In particular, higher derivative corrections to the low energy effective action computed using the fivebrane do not agree with the field theory results\[3\].

An alternative approach to the study of $\mathcal{N} = 2$ super Yang-Mills theory is to realize these theories as the worldvolume theory of a threebrane probe in an F theory background\[5\],\[6\]. The F theory background is set up by a collection of $N$ coincident threebranes, and a number of parallel F theory sevenbranes. The sevenbranes carry R and NS charge. Our notation for the sevenbranes is defined by saying that a $(p,q)$ string can end on a $(p,q)$ sevenbrane. Thus, for example, a $(1,0)$ string is an elementary type IIB string; a $(1,0)$ sevenbrane is a Dirichlet sevenbrane. In this notation, the F theory background contains one $(0,1)$ sevenbrane, one $(2,1)$ sevenbrane and $N_f(1,0)$ sevenbranes. By taking $N$ large and the string coupling small, the curvature of the background geometry becomes

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1 The Seiberg-Witten effective action can also be recovered from a Dirichlet fivebrane of IIB string theory wrapping the Seiberg-Witten curve\[3\].

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small almost everywhere and the supergravity solution can be used reliably\cite{3}. The field theory of interest is realized on the worldvolume of a threebrane probe which explores this geometry\cite{3}. On the supergravity side we are studying the two body interaction between the probe and the \( N \) coincident threebranes. Thus the worldvolume theory of the probe should be compared to the low energy effective action for the theory with gauge group \( SU(2) \). Higher derivative corrections to the low energy effective action have been computed in \cite{7},\cite{8} using the probe worldvolume theory. This work provides strong evidence that the full \( \mathcal{N} = 2 \) low energy Wilsonian action (and not just its leading Seiberg-Witten form) is described by a Born-Infeld world-volume action of a D3 brane in the near horizon geometry of the above F theory background. It was shown in \cite{8} that the Born-Infeld action reproduces exactly the expected structure of non-holomorphic higher derivative corrections to the Seiberg-Witten effective action. Essential to this approach is the construction of a valid background geometry, including a non trivial metric. Even though the leading Seiberg-Witten term is independent of the metric it turns out to be essential for the correct description of non-holomorphic corrections. As is well known, \( \mathcal{N} = 2 \) super Yang-Mills theories have a rich spectrum of BPS states\cite{9},\cite{10}. It is natural to ask what the analog of the field theory BPS states on the threebrane worldvolume are. Sen \cite{11} has identified these states with strings that stretch along geodesics from the probe to a sevenbrane. This description of BPS states provides an elegant description of the selection rules\cite{12} determining the allowed BPS states for the \( N_f \leq 4 \) \( SU(2) \) Seiberg-Witten theory\cite{13}. However, open string junctions can fail to be smooth\cite{14}. In this case, one needs to consider geodesic string junctions to describe the BPS states\cite{15}.  

An assumption implicit in this approach is that the threebrane probe is infinitely heavy as compared to the string. In this approximation, the three brane remains flat and its geometry is unaffected by the string ending on it. Recalling the role that bending plays in the M-theory fivebrane description of field theory, it is natural to identify this approximation with a classical description of the BPS states.

In this article, we will construct a description of BPS states of the Born-Infeld probe world-volume action, which accounts for the bending of the threebrane into a dimple due to the string attached to it\footnote{We thank Oren Bergmann for helpful correspondence on this point.}. The analogy to the M-fivebrane description of field theory

\footnote{An analogous computation for the M-theory fivebrane has been carried out in\cite{16}. However, in this case the BPS states correspond to self-dual strings stretched along the Riemann surface associated with the Seiberg-Witten curve, instead of along the moduli space.}
suggests that we are accounting for quantum effects in the background of these BPS states. We are able to provide a supergravity solution corresponding to a string stretching over a finite interval. On the world-volume of the D3 probe this translates into a cut-off allowing for the existence of finite energy classical solutions of the abelian Born-Infeld action. We show that the long distance properties of the monopole solution to non-abelian gauge theories are reproduced by the abelian low energy effective action. This calculation again requires precise knowledge of the near horizon background geometry. This background geometry pinches the dimple on the D3 brane into an endpoint. By appealing to the holographic principle we are able to interpret the long distance ”cut-off” on the dimple in terms of the correct UV cut-off entering in the definition of the Wilsonian effective action.

In section 2 we reconsider the problem of solving for the background set up by the sevenbranes plus $N$ threebranes. This reduces to the problem of solving the Laplace equation on the background generated by the sevenbranes\cite{6}. We show that the solution to this Laplace equation is duality invariant. By making an explicitly duality invariant ansatz, we are able to accurately construct the background geometry in the large $|a|$ region and close to the sevenbranes. A computation of the square of the Ricci tensor shows that this background has curvature singularities. The interpretation of these singularities in the field theory, are as the points in moduli space where the effective action breaks down due to the appearance of new massless particles.

In section 3 we construct solutions corresponding to Dirichlet strings stretching from the probe to the (0,1) sevenbrane. We do this explicitly for the $N_f = 0$ case, but the extension to $N_f \leq 4$ is trivial\cite{4}. These ”magnetic dimple” solutions have the interesting property that they capture the long distance behaviour of the Prasad-Sommerfeld monopole solution\cite{18}. The Prasad-Sommerfeld solution is an exact solution for an $SU(2)$ gauge theory. An important feature of this monopole solution, intimately connected to the non-Abelian structure of the theory, is the way in which both the gauge field and Higgs field need to be excited in order to obtain finite energy solutions. It is interesting that the Abelian worldvolume theory of the probe correctly captures this structure. This is not unexpected since the probe worldvolume theory is describing the low energy limit of a non-Abelian theory. The energy of these solutions is finite due to a cut off which must be imposed. The UV cut off in the field theory maps into an IR cut off in the supergravity, which is a manifestation of the IR/UV correspondence\cite{19}.

\footnote{F-theory backgrounds which correspond to the superconformal limit of the field theories we consider have been constructed in \cite{17}.}
In section four we consider the moduli space of the solutions we have constructed. We are able to compute the metric on the one monopole moduli space exactly. For two monopoles and higher, it does not seem to be possible to do things exactly. However, under the assumption that the monopoles are very widely separated, we are able to construct the asymptotic form of the metric. In this case, we see that the metric receives both perturbative and instanton corrections. The hyper-Kähler structure of these moduli spaces in unaffected by these corrections.

2. Gravitational Interpretation of Singularities in the Field Theory Moduli Space

The model that we consider comprises of a large number $N$ of coincident threebranes and a group of separated but parallel sevenbranes. The sevenbranes have worldvolume coordinates $(x^0, x^1, x^2, x^3, x^4, x^5, x^6, x^7)$; the threebranes have worldvolume coordinates $(x^0, x^1, x^2, x^3)$. The presence of the sevenbranes gives rise to nontrivial monodromies for the complex coupling $\tau = \tau_1 + i\tau_2$ as it is moved in the $(x^8, x^9)$ space transverse to both the sevenbranes and the threebranes. In order to study $\mathcal{N} = 2$ super Yang-Mills theory, one chooses the sevenbrane background in such a way that the complex IIB coupling $\tau$ is equal (up to a factor) to the effective coupling of the low energy limit of the field theory of interest\cite{8}. The spacetime coordinates $(x^8, x^9)$ then play the dual role of coordinates in the supergravity description, and of the (complex) Higgs field $a$ in the field theory\cite{6}. The metric due to the sevenbranes by themselves is given by\cite{21}

$$ds^2 = -(dx^0)^2 + (dx^i)^2 + \tau_2[(dx^8)^2 + (dx^9)^2], \quad (2.1)$$

where $i$ runs over coordinates parallel to the sevenbrane. The addition of $N$ threebranes to this pure sevenbrane background deforms the metric and leads to a non-zero self-dual RR fiveform flux. This flux and the deformed metric are given in terms of $f$

\footnote{The $N$ dependence of $\tau$ has been discussed in\cite{8}. We will not indicate this dependence explicitly.}

\footnote{The precise identification between $(x^8, x^9)$ and $a$ involves a field redefinition as explained in\cite{20}.}
\[ ds^2 = f^{-1/2} dx^2 + f^{1/2} [(dx^i)^2 + \tau_2 ((dx^8)^2 + (dx^9)^2)] = f^{-1/2} dx^2 + f^{1/2} g_{jk} dx^j dx^k, \]

\[ F_{0123j} = -\frac{1}{4} \partial_j f^{-1}, \]  

(2.2)

where \( f \) is a solution to the Laplace equation in the sevenbrane background. \( \tau \)

\[ \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j f) = -(2\pi)^4 N \frac{\delta^6(x - x^0)}{\sqrt{g}}. \]  

(2.3)

The position of the \( N \) source threebranes is \( x^0 \). The index \( i \) in (2.2) runs over coordinates transverse to the threebrane but parallel to the sevenbrane, \( j, k \) runs over coordinates transverse to the threebrane and \( x_\parallel \) denotes the coordinates parallel to the threebrane. For the backgrounds under consideration, (2.3) can be written as

\[ \tau_2 \partial_y^2 + 4 l_s^{-4} \partial_a \bar{a} f = -(2\pi)^4 N \delta^{(4)}(y - y^0) \delta^{(2)}(a) \]  

(2.4)

where \( a, \bar{a} \) and \( \tau_2 \) are the quantities appearing in the Seiberg-Witten solution and \( y \) denotes the coordinates transverse to the threebranes but parallel to the sevenbranes. In \( \tau \), this equation was solved in the large \( |a| \) limit, for \( \tau \) corresponding to \( N = 2 \) field theories with gauge group \( SU(2) \) and \( N_f = 0, 4 \) flavors of matter. It is possible to go beyond this approximation, by noting that the form of (2.4) implies that \( f \) is invariant under a duality transformation. To see this, change from the electric variables \( a \) to the dual magnetic variables \( a_D \). After a little rewriting, we find that (2.4) becomes

\[ \left[ \frac{1}{2i} (\partial_a \bar{a}_D - \partial_a a_D) \partial_y^2 + 4 l_s^{-4} \partial_a \bar{a}_D \partial \bar{a}_D \right] f = -(2\pi)^4 N \delta^{(4)}(y - y^0) \delta^{(2)}(a_D - a^0_D) \]  

(2.5)

If we now identify \( \tau_D = -\frac{1}{\tau} \), we see that \( f \) is also a solution of the Laplace equation written in terms of the dual variables. This result is a consequence of the fact that the Einstein metric is invariant under the classical \( SL(2, R) \) symmetry of IIB supergravity. It is possible to construct an approximate solution \( f \) which is valid in the region corresponding to large \( |a| \) and in the region corresponding to small \( |a_D| \)

\[ f(y, a, \bar{a}) = \frac{l_s^4}{[y^i y^i - \frac{i}{2} l_s^4 (a_D (\bar{a} - \bar{a}_0) - \bar{a}_D (a - a_0))]^2}. \]  

(2.6)
In this last formula, $a_0$ is the constant value of $a$ at $a_D = 0$. For concreteness, we will focus on the pure gauge theory in the discussion which follows, but our results are valid for all $N_f \leq 4$. In the large $a$ limit, from the work of Seiberg and Witten\cite{9}, we know that $a_D$ can be expressed as a function of $a$ as (see for example \cite{22})

$$a_D = \tau_0 a + \frac{2ia}{\pi} \log \left[ \frac{a^2}{\Lambda^2} \right] + \frac{2ia}{\pi} + \frac{a}{2\pi i} \sum_{l=1}^{\infty} c_l (2 - 4l) \left( \frac{\Lambda}{a} \right)^{4l}, \quad (2.7)$$

where $\tau_0$ is the classical coupling and $\Lambda$ is the dynamically generated scale at which the coupling becomes strong. Inserting this into (2.6) and evaluating the expression at $y = 0$ we find the following asymptotic behaviour for $f$

$$f \sim \frac{1}{(a\bar{a} \log |a|)^2}. \quad (2.8)$$

This reproduces the large $|a|$ solution of \cite{8}. In the small $|a_D|$ limit, $a$ can be expressed as\cite{22}

$$a = \tau_0 D a_D - \frac{2ia_D}{4\pi} \log \left[ \frac{a_D}{\Lambda} \right] - \frac{i a_D}{4\pi} - \frac{1}{2\pi i a_D} \Lambda^2 \sum_{l=1}^{\infty} c^D_l \left( \frac{i a_D}{\Lambda} \right)^l. \quad (2.9)$$

Thus, in this limit and at $y = 0$ we find

$$f \sim \frac{1}{(a_D a_D \log |a_D|)^2}. \quad (2.10)$$

It is easy to again check that this is the correct leading behaviour for $f$ close to the monopole singularity at $|a_D| = 0$. The corrections to $f$, at $y = 0$ are of order $|a_D|^{-4} (\log |a_D|)^{-4}$. In the large $|a|$ limit, $f \to 0$ and in the small $|a_D|$ limit, $f \to \infty$ signaling potential singularities in both limits. From the field theory point of view, both of these limits correspond to weakly coupled limits of the field theory (or its dual) which leads us to suspect that curvature corrections may not be small in these regions\cite{23}. This is easily confirmed by computing the square of the Ricci tensor in string fame, which for large $|a|$ behaves as $R^{MN} R_{MN} \sim \log |a|$ and for small $|a_D|$ behaves as $R^{MN} R_{MN} \sim \log |a_D|$. It is clear that the point $|a_D| = 0$ corresponds to a curvature singularity in the supergravity background. If we circle this point in the field theory moduli space, the coupling transforms with a nontrivial monodromy, so that this point is to be identified with the position of a sevenbranes. The appearance of these naked singularities in the supergravity background could have been anticipated from no-hair theorems\cite{24}.
Recently a number of interesting type IIB supergravity backgrounds were constructed [25]. These backgrounds exhibit confinement and a running coupling. In addition, a naked singularity appears in spacetime. It is extremely interesting to determine whether this naked singularity can be attributed to a weakly coupled dual description of the theory, since this would provide an example of duality in a non-supersymmetric confining gauge theory.

3. Magnetic Dimples in the Probe Worldvolume

The BPS states of the field theory living on a probe which explores the sevenbrane background, have been identified with strings that stretch along geodesics, from the probe to a sevenbrane [11]. In this section we will look for a description of these BPS states of the probe worldvolume, which accounts for the bending of the threebrane into a dimple, due to the string attached to it. We will focus on the case when a Dirichlet string ends on the probe, corresponding to a "magnetic dimple".

The dynamics of the threebrane probe is given by the Born-Infeld action for a threebrane in the background geometry of the sevenbranes and \( N \) threebranes. This background has non-zero RR five-form flux, axion, dilaton and metric. The worldvolume action is computed using the induced (worldspace) metric

\[
g_{mn} = G_{MN}\partial_mX^M\partial_nX^N + l_s^2 F_{mn} \tag{3.1}
\]

where \( F_{mn} \) is the worldvolume field strength tensor. We will use capital letters to denote spacetime coordinates \( (X^M) \) and lower case letters for worldvolume coordinates \( (x^n) \). We are using the static gauge \( (X^0, X^1, X^2, X^3) = (x^0, x^1, x^2, x^3) \). In addition to the Born-Infeld term the action includes a Wess-Zumino-Witten coupling to the background RR five-form field strength. The probe is taken parallel to the stack of \( N \) threebranes so that no further supersymmetry is broken when the probe is introduced. The worldvolume soliton solutions that we will construct have vanishing instanton number density so that there is no coupling to the axion. The explicit action that we will use is [26].

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As mentioned in the introduction a large number of BPS states of the field theory do not correspond to strings stretched between the threebrane and a sevenbrane, but rather to string junctions (or webs) which connect the threebrane to more than one sevenbrane [14], [15].
\[ S = T_3 \int d^4x \sqrt{-\det(g_{mn} + e^{-\frac{i}{2}\phi}l_s^2 F_{mn})} + T_3 \int d^4x \partial_n X^N_1 \wedge \ldots \wedge \partial_n X^N_4 A_{N_1 N_2 N_3 N_4}, \]

where \( A_{N_1 N_2 N_3 N_4} \) is the potential for the self-dual fiveform and we have omitted the fermions. We work in the Einstein frame so that the threebrane tension \( T_3 = l_s^{-4} \) is independent of the string coupling. In what follows, we will use \( a_D \) to denote the magnetic variable that provides the correct description of the field theory in the small \( |a_D| \) regime. The corresponding low energy effective coupling is denoted \( \tau_D \). The electric variable \( a \) is the correct variable to use in the large \( |a| \) limit and the associated coupling is denoted \( \tau \).

3.1. Spherically Symmetric Solution

In this section we will construct a worldvolume soliton that can be interpreted as a single Dirichlet string stretching from the probe to the "magnetic" (0,1) sevenbrane. We will study the pure gauge theory. The extension to \( 0 < N_f \leq 4 \) is straightforward and amounts the same computation with a different \( \tau \). From the point of view of the worldvolume, the Dirichlet string endpoint behaves as a magnetic monopole\(^2\). If we consider the situation in which the probe and the magnetic sevenbrane have the same \( x^9 \) coordinate and are separated in the \( x^8 \) direction, then the only Higgs field which is excited is \( x^8 \). In addition, because we expect our worldvolume soliton is a magnetic monopole we make the spherically symmetric ansatz \( x^8 = x^8(r) \) and assume that the only non-zero component of the worldvolume field strength tensor in \( F_{\theta\phi} \). With this ansatz, the threebrane action takes the form

\[ S = T_3 \int dt d\theta d\phi \left( \sqrt{f^{-1} + \tau_{2D}(\partial_r x^8)^2} \sqrt{f^{-1} r^4 \sin^2 \theta - \tau_{2D}^1 l_s^4 F_{\theta\phi} F_{\phi\theta} - \frac{1}{f} r^2 \sin \theta} \right). \quad (3.3) \]

We will consider the situation in which the probe is close to the (0,1) sevenbrane corresponding to a region where \( a_D \) is small. For this reason we use the dual coupling in (3.3) and we will identify \( x^8 \) with the dual Higgs \( a_D \). Recall the fact that the Born-Infeld action has the property that a BPS configuration of its Maxwellian truncation satisfies the equation of motion of the full Born-Infeld action\(^2\). In our case, the Maxwellian truncation of the Born-Infeld action is just Seiberg-Witten theory. The supersymmetric variation of the gaugino of the Maxwell theory is
$$\delta \psi = (\Gamma_\theta \phi F_{\theta \phi} + \Gamma_8 \partial_8 x^8) \epsilon.$$ \hfill (3.4)

A BPS background is one for which this variation vanishes. Now, note that if we identify ($\eta^{mn}$ is the flat four dimensional Minkowski metric)

$$\eta^{\theta \phi} \eta^{\theta \phi} F_{\theta \phi} l_s^4 (F_{\theta \phi})^2 = \frac{F_{\theta \phi} l_s^4}{r^4 \sin^2 \theta} = (\partial_r x^8)^2 = \eta^{rr} (\partial_r x^8)^2,$$

we obtain a background invariant under supersymmetries \(3.4\), where $\epsilon$ satisfies $(\epsilon^\theta \Gamma_\theta \epsilon^\phi \Gamma_\phi \pm \epsilon^8 \Gamma_8 \epsilon^r \Gamma_r) \epsilon = 0$. The sign depends on whether we consider a monopole or an anti-monopole background. This condition can be rewritten as $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_8 \epsilon = \pm \epsilon$. Thus, this is a BPS background of Seiberg-Witten theory. We will assume that \(3.5\) provides a valid BPS condition for the full Born-Infeld action. Upon making this ansatz, we find that the determinant factor in the Born-Infeld action can be written as a perfect square, as expected\[29\]. The full Born-Infeld action plus Wess-Zumino-Witten term then simplifies to the Seiberg-Witten low energy effective action \[3\]

$$S = T_3 \int d^4 x \tau_{2D} (\partial_8 x^8)^2.$$ \hfill (3.6)

We have checked that the arguments above can also be made at the level of the equations of motion. The preserved supersymmetries have a natural interpretation. The probe preserves supersymmetries of the form $\epsilon_L Q_L + \epsilon_R Q_R$ where $\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \epsilon_L = \epsilon_R$. The Dirichlet string preserves supersymmetries for which $\Gamma_0 \Gamma_8 \epsilon_L = \pm \epsilon_R$, with the sign depending on whether the string is parallel or anti-parallel to the $x^8$ axis. The two conditions taken together imply that $\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_8 \epsilon_i = \pm \epsilon_i$, $i = L, R$. Thus, the supersymmetries preserved by the ansatz \(3.5\) are exactly the supersymmetries that one would expect to be preserved by a Dirichlet string stretched along the $x^8$ axis. With a suitable choice of the phase of $a_D$ we can choose $a_D l_s^2 = -i x^8$. With this choice, using the formulas for the dual prepotential quoted in \[22\], we find that the dual coupling can be expressed in terms of $x^8$ as

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\[8\] The study of the quantum corrections to solitons by studying the minima of the low energy effective action has been considered in \[30\].

\[9\] This identification is only correct to leading order at low energy. One needs to perform a field redefinition of the field theory Higgs fields before they can be identified with supergravity coordinates \[20\].
\[
\tau_{2D} = \frac{4\pi}{g_{cl,D}^2} - \frac{3}{4\pi} - \frac{1}{2\pi} \log \left[ \frac{x^8}{l_s^2 \Lambda} \right] - \frac{1}{2\pi} \Lambda^2 l_s^4 \sum_{l=1}^{\infty} c^D_l (l-1)(x^8)^{l-2} \frac{1}{(l_s^2 \Lambda)^l}. \tag{3.7}
\]

Note also that \(a\) is real and can be expressed in terms of \(x^8\) and \(\tau_{2D}\) as

\[
a = \int dx^8 \frac{\tau_{2D}}{l_s^2} = \frac{4\pi}{g_{cl,D}^2} x^8 - \frac{1}{4\pi} l_s^2 x^8 - \frac{1}{2\pi} l_s^2 x^8 \log \left[ \frac{x^8}{l_s^2 \Lambda} \right] - \frac{1}{2\pi} \Lambda^2 l_s^2 \sum_{l=1}^{\infty} c^D_l (x^8)^{l-1} \frac{1}{(l_s^2 \Lambda)^l}, \tag{3.8}
\]

The equation of motion which follows from (3.6)

\[
\frac{d}{dr} \left( \tau_{2D} r^2 \frac{dx^8}{dr} \right) = 0, \tag{3.9}
\]

is easily solved to give

\[
\gamma - \frac{\alpha}{r} = a = \frac{4\pi}{g_{cl,D}^2} x^8 - \frac{1}{4\pi} l_s^2 x^8 - \frac{1}{2\pi} l_s^2 x^8 \log \left[ \frac{x^8}{l_s^2 \Lambda} \right] - \frac{1}{2\pi} \Lambda^2 l_s^2 \sum_{l=1}^{\infty} c^D_l (x^8)^{l-1} \frac{1}{(l_s^2 \Lambda)^l}. \tag{3.10}
\]

In principle, this last equation determines the exact profile of our magnetic dimple as a function of \(r\). At large \(r\) our fields have the following behaviour

\[
x^8 = \nu - \frac{\beta}{r}, \quad l_s^4 B_r B_r = l_s^4 F^{\theta \phi} F_{\theta \phi} = (\partial_r x^8)^2 = \frac{\beta^2 l_s^4}{r^4}, \tag{3.11}
\]

where \(\nu\) and \(\beta\) are constants. To interpret these results, it is useful to recall the Prasad-Sommerfeld magnetic monopole solution\cite{18}. This monopole is a solution to the following \(SU(2)\) non-Abelian gauge theory

\[
S = -\frac{1}{e^2} \int d^4x \left( \frac{1}{4} (F_{mn}^a)^2 + \frac{1}{2} (D^a \phi^c)^2 \right), \quad F_{mn}^a = \partial_m A_n^a - \partial_n A_m^a + \epsilon^{abc} A_m^b A_n^c, \quad D_n \phi^a = \partial_n \phi^a + \epsilon^{abc} A_n^b \phi^c. \tag{3.12}
\]

An important feature of the monopole solution, intimately connected to the non-Abelian gauge structure of the theory, is the fact that in order to get a finite energy solution, the monopole solution excites both the gauge field and the Higgs field\cite{31}. The Higgs field component of the Prasad-Sommerfeld monopole is

\[
\phi^a = \hat{\phi}^a \left( \gamma \coth \left( \frac{\gamma r}{\alpha} \right) - \frac{\alpha}{r} \right), \quad \gamma = \frac{\alpha}{r} + 2\gamma e^{-\frac{2\pi}{\alpha}} + 2\gamma e^{-\frac{4\pi}{\alpha}} + ..., \tag{3.13}
\]

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where on the second line we have performed a large $r$ expansion. Clearly, the electric variable $a$ reproduces the large $r$ behaviour of the Higgs field of the Prasad-Sommerfeld solution. Thus, we see that the Abelian worldvolume theory of the threebrane probe catches some of the structure of the non-Abelian field theory whose low energy limit it describes. Comparing $a$ to the asymptotic form of the Higgs field in the Prasad-Sommerfeld solution allows us to interpret the constants of integration $\alpha$ and $\gamma$. The constant $\alpha$ is related to the inverse of the electric charge $1/e$. As $r \to \infty$ $a \to \gamma$ so that $\gamma$ determines the asymptotic Higgs expectation value; i.e. it determines the moduli parameters $a, \bar{a}$ of the field theory or, equivalently, the position of the threebrane in the $(8,9)$ plane. The mass of the $W^\pm$ bosons are given by the ratio $m_W = \gamma/\alpha$. In a similar way, $\nu$ fixes the asymptotic expectation value of the dual Higgs field and $\beta$ is related to the inverse magnetic charge $1/g$. Thus, the mass of the BPS monopole is given by $m_g = \nu/\beta$. We are working in a region of moduli space where $|a_D|$ is small, so that the monopoles are lighter than the $W^\pm$ bosons. We will return to the exponential corrections in (3.13) below.

Consider next the small $r$ limit. It is clear that in this limit $a \to \infty$. To correctly interpret this divergence in $a$, recall that the Seiberg-Witten effective action is a Wilsonian effective action, obtained by integrating out all fluctuations above the scale set by the mass of the lightest BPS state in the theory. In our case, the lightest BPS states are the monopoles and this scale is $m_g$. The largest fluctuations left in the effective low energy theory all have energies less than $m_g$. By Heisenberg’s uncertainty relation, the effective theory must be cut off at a smallest distance of $1/m_g$. Clearly then, the divergence in the $r \to 0$ limit is unphysical and it occurs at length scales below which the effective theory is valid. What is the interpretation of this short distance (UV) field theory cut off in the supergravity description? To answer this question, we will need a better understanding of the small $r$ behaviour of $x^8$. Towards this end, note that with our choice $a_D$, $\tau$ is pure imaginary so that

$$\tau = -\frac{1}{\tau_D} = \frac{i}{\tau_{2D}} = i\tau_2.$$  

(3.14)

Thus, the action (3.6) can be written as

$$S = T_3 \int d^4x \tau_2 (\partial_r a)^2 = T_3 \int d^4x \frac{(\partial_r a)^2}{\tau_{2D}}.$$  

(3.15)

\[10\] It is the electric variable $a$ of Seiberg-Witten theory that is related to the Higgs field appearing in the original microscopic $SU(2)$ theory.
Noting that $\tau_2 = \partial a/\partial x^8$, the equation of motion for $a$ implies

$$
\frac{r^2}{\tau_2} \frac{\partial a}{\partial r} = \frac{r^2}{\partial x^8} \frac{\partial x^8}{\partial r}
$$

is a constant. Thus, the expressions in (3.11) are exact. At $r = 1/m_g = \nu/\beta$ we find that

$$
a_D = -i \left( \nu - \frac{\beta}{r} \right) = 0.
$$

Thus, the magnetic dimple is cut off at $a_D = 0$. Intuitively this is pleasing: the magnetic dimple (Dirichlet string) should end on the magnetic $(0,1)$ sevenbrane which is indeed located at $a_D = 0$. To understand the geometry of the dimple close to $a_D = 0$, note that the induced metric is

$$
\begin{align*}
\text{ds}^2 &= f^{-1/2}(-dt^2 + d\Omega_2) + (f^{-1/2} + f^{1/2}\tau_2 \partial_r x^8 \partial_r x^8) dr^2 \\
&= l_s^2 a_D a_D^* \log |a_D| (-dt^2 + d\Omega_2) + \\
&\quad + (l_s^2 a_D a_D^* \log |a_D| + \frac{\tau_2 D}{l_s^2 a_D a_D^* \log |a_D|} \partial_r x^8 \partial_r x^8) dr^2. \\
\end{align*}
$$

(3.17)

In the above expression we have used the approximate solution (2.6) for the metric, obtained in the last section. This is a valid approximation since we are interested in the geometry close to $a_D = 0$ where our expression for $f$ becomes exact. The proper length to $a_D = 0$ is clearly infinite. It is also clear from (3.17) that the dimple ends at a single point at $a_D = 0$, which is again satisfying. Thus, although we motivated the need to introduce a cut off from the point of view of the low energy effective action realized on the probe, one could equally well argue for \textit{exactly the same} cut off from the dual gravity description. In the super Yang-Mills theory one has a short distance (UV) cut off; on the gravity side one has a long distance (IR) cut off. The connection between these cut offs is expected as a consequence of the UV/IR correspondence.

A direct consequence of the short distance cut off in field theory, is that the monopole appears as a sphere of radius $1/m_g$. From the point of view of the effective field theory, it is not possible to localize the monopole any further. The more a particle in quantum field theory is localized, the higher the energy of the cloud of virtual particle fluctuations surrounding it will be. If one localizes the monopole in the effective field theory any further, the energy of the fluctuations becomes high enough to excite virtual monopole-antimonopole pairs, and hence we would leave the domain of validity of the effective field theory. The origin of the exponential corrections in (3.13) can be traced back to virtual
$W^\pm$ bosons by noting that the factors multiplying $r$ in the exponent are proportional to the boson mass $m_W$. The Higgs field of the Prasad-Sommerfeld monopole goes smoothly to zero as $r \to 0$ so that these bosons resolve the singular monopole core. The fact that these fluctuations have been integrated out of the effective theory naturally explains why $a$ does not receive any exponential corrections. Although these corrections are crucial for the description of the monopole core, they are not needed by the effective field theory which describes only low energy (large distance) phenomena. In a similar way, virtual monopole-antimonopole pairs will resolve the singular behaviour of $x^8$ as $r \to 0$.

The cut-off that we employ in our work has already been anticipated in a completely different context in [32]. In [32] the effective field theory realisation of BPS states was studied in the field theory limit of IIB string theory compactified on a Calabi-Yau manifold. In this description, the BPS states arise as limits of the attractor "black holes" in $\mathcal{N} = 2$ supergravity. To analyse the BPS equations around $a_D = 0$ (a singular point in the field theory moduli space), it is safer to employ the "attractor-like" formulation of the BPS equations. The solution with a finite $a_D = 0$ core then arises as a solution of the BPS equations of motion. Our results are in agreement with those of [32].

We will now comment on the relation between our study and the results presented in [16]. Above we argued that there was a need for a cut-off because the world-volume theory of the probe is a Wilsonian effective action, obtained by integrating all fluctuations above some energy scale out of the theory. However, this argument could also be made directly in the supergravity description by examining the induced metric given above.

The form of this induced metric relies crucially on the fact that the probe is moving in the background of sevenbranes and $N$ threebranes. In particular, by taking $N \to \infty$ the authors of [3] argued that the background geometry can be trusted in the field theory limit. By accounting for the deformation of the background due to these $N$ threebranes our analysis goes beyond the low energy approximation and, in particular, can be trusted when computing quantities not protected by supersymmetry. Note that the induced metric is non-holomorphic. In the case of the M-theory fivebrane, one does not expect to reproduce quantities that are non-holomorphic [3]. Indeed, the background geometry for the M-theory fivebrane analysis is flat, so it is difficult to see how an analog of the induced metric could be realised. It is an open question as to whether a cut-off can consistently be used in this case.

---

11 We would like to thank Frederik Denef for bringing this to our attention.
We now compute the energy of the magnetic dimple. Since the dimple is at rest, this energy should be proportional to the mass of the monopole. As discussed above, \(1/\alpha\) is playing the role of the electric charge \(e\). The Dirac quantization condition is \(eg = 4\pi\), so that we can identify the magnetic charge \(g = 4\pi\alpha\). The magnetic dimple is an excitation of the flat threebrane located at \(a = \gamma\). We will denote the corresponding value of the dual variable \(a_D\) by \(a_D^0\). Thus, the mass of a magnetic monopole of this theory is \(m = g|a_D^0| = 4\pi\alpha|a_D^0|\). To compute the mass of the magnetic dimple note that (we have set the \(\theta\)-angle to zero)

\[
\tau_{2D} = -i \frac{\partial a}{\partial a_D} = -i \frac{\partial a}{\partial r} \frac{\partial r}{\partial a_D} = i \frac{\alpha}{r^2} \frac{\partial r}{\partial a_D}.
\]

Since the kinetic energy vanishes, the energy of the soliton is proportional to the Lagrangian

\[
E = -L = -i s^{-4} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin \theta r^2 \tau_{2D}(\partial r x^8)^2
\]

\[
= +i 4\pi \alpha \int_0^{a_D^0} da_D = 4\pi \alpha |a_D^0|.
\]

The mass of the dimple lends further support to its interpretation as a magnetic monopole.

### 3.2. Multimonopole Solutions

In this section we will describe solutions containing an arbitrary number of separated static dimples. We will look for static solutions, that have \(x^8\) and \(F_{ij} i, j = 1, 2, 3\) excited. It is useful to again consider the Maxwellian truncation of the full Born-Infeld action to motivate an ansatz. The variation of the gaugino of the Maxwellian theory, assuming the most general static magnetic field, is given by

\[
\delta \psi = (\Gamma_{ij} F^{ij} + \eta^{kk} \Gamma_{8k} \partial_k x^8) \epsilon.
\]

Thus, a solution which satisfies

\[
\frac{1}{2} \eta_{il} \epsilon^{jk} F_{jk} = \pm \partial_l x^8,
\]

will be invariant under supersymmetries \((3.18)\) as long as \(\epsilon\) satisfies \(\Gamma_1 \Gamma_2 \Gamma_3 \Gamma_8 \epsilon = \pm \epsilon\). The choice of sign again depends on whether one is describing a monopole or an anti-monopole background. As explained above, these are exactly the supersymmetries that one would expect to be preserved by a Dirichlet string stretched along the \(x^8\) axis. Upon making
this ansatz, we again find that the determinant appearing in the Born-Infeld action can
be written as a perfect square and that once again the dynamics for the field theory of
the threebrane probe worldvolume is described by the Seiberg-Witten low energy effective
action. After noting that the identification $a_D l_s^2 = -ix^8$ implies that $\tau_{2D} = l_s^2 \partial a / \partial x^8$, the
equation of motion following from the Seiberg-Witten effective action can be written as

$$\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} a = 0. \quad (3.20)$$

This is just the free Laplace equation which is easily solved

$$a = \gamma + \sum_{i=1}^{n} \frac{\alpha \eta}{\left[ (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 \right]^{1/2}}, \quad (3.21)$$

where $\eta = \pm 1$. The number of dimples $n$ and their location $(x_i, y_i, z_i)$ is completely
arbitrary. This is to be expected - BPS states do not experience a static force. In the
above, we have set each of the numerators equal to $\alpha$, because as we have seen above
this factor is related to the electric charge $e$. The classical solution would of course allow
arbitrary coefficients. The coefficients of these terms can be positive or negative. All
strings must have the same orientation for this to be a BPS state. Thus the interpretation
of terms with a positive coefficient is that they correspond to strings that end on the
threebrane; terms with a negative coefficient correspond to strings that start from the
threebrane. The finite energy solutions that we consider only allow for strings that start
from the probe and end on the sevenbrane, and consequently we fix the sign of all terms
to be negative. Each of the dimples above ends in a point at the sevenbranes at $a_D = 0$.
The electric Higgs field $a$ again reproduces the known asymptotic behaviour of the Higgs
field component of multi-monopole solutions in non-Abelian ($SU(2)$) gauge theories[33].

We will now consider the mass of this multi-monopole solution. In the limit that the
dimples are very widely separated, the mass of each dimple can be computed separately.
In this widely separated dimple limit, the total energy is simply the sum of the energy
for each dimple, so that the energy of the $n$-dimple solution is indeed consistent with its
interpretation as an $n$-monopole state. Of course the total energy of the $n$-dimple state is
independent of the locations of each dimple so that once the energy is known for widely
separated dimples, it is known for all possible positions of the dimples.
4. Metric of the Monopole Moduli Space

In the previous section we constructed smooth finite energy solutions to the equations of motion. The solution describing $n$ BPS monopoles is known to be a function of $4n$ moduli parameters\[34\]. The position of each monopole accounts for $3n$ of these parameters. The remaining $n$ parameters correspond to phases of the particles. By allowing these moduli to become time dependent it is possible to construct the low energy equations of motion for the solitons\[35\]. The resulting low energy monopole dynamics is given by finding the geodesics on the monopole moduli space. The low energy dynamics of $\mathcal{N} = 2$ supersymmetric monopoles has been studied in \[36\]. In this section we would like to see if the metric on monopole moduli space can be extracted from the solutions we constructed above.

4.1. One Monopole Moduli Space

A single monopole has four moduli parameters. Three of these parameters may be identified with the center of mass position of the monopole, $x^i$, whilst the fourth moduli parameter corresponds to the phase of the monopole, $\chi$. We will denote these moduli as $z^\alpha = (\chi, x^1, x^2, x^3)$. After allowing the moduli to pick up a time dependence, the action picks up the following additional kinetic term

$$S = \int d^4x \tau_2 \dot{a}_D \dot{\bar{a}}_D = \int dt \mathcal{G}_{\alpha\beta} \dot{z}^\alpha \dot{z}^\beta,$$

(4.1)

where we have introduced the metric on the one monopole moduli space

$$\mathcal{G}_{\alpha\beta} \equiv \int d^3x \tau_2 \frac{\partial a_D}{\partial z^\alpha} \frac{\partial \bar{a}_D}{\partial z^\beta}.$$  

(4.2)

Consider first the center of mass of the monopole. To construct the moduli space dependence on these coordinates, replace $x^i_0 \rightarrow x^i_0 + x^i(t)$, where $x^i_0$ denotes the initial (time independent) monopole position. The action picks up the following kinetic terms

$$S = \int d^4x \tau_2 \dot{a}_D \dot{\bar{a}}_D = \int d^4x \tau_2 \frac{\partial a_D}{\partial x^i(t)} \frac{\partial \bar{a}_D}{\partial x^j(t)} \dot{x}^i(t) \dot{x}^j(t) = \int dt \dot{x}^i(t) \dot{x}^j(t) \mathcal{G}_{ij},$$

(4.3)

Since the velocity of the monopole is small, we keep only terms which are quadratic in the monopole velocity. Thus, when we compute

$$\mathcal{G}_{ij} = -\int d^3x \tau_2 \frac{\partial a_D}{\partial x^i} \frac{\partial a_D}{\partial x^j} = -\int dr d\theta d\phi r^2 \sin \theta \tau_2 \left( \frac{\partial a_D}{\partial r} \right)^2 \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j},$$

(4.4)
we evaluate the integrand at the static monopole solution. Using the result

\[ \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin(\theta) \frac{\partial r}{\partial x^i} \frac{\partial r}{\partial x^j} = \frac{1}{2} 4\pi \delta_{ij}, \]

we find

\[ G_{ij} = -\frac{1}{2} \delta_{ij} \int drr^2 \tau_{2D} \left( \frac{\partial a_D}{\partial r} \right)^2 = \frac{1}{2} 4\pi \alpha |a_D^0| \delta_{ij}. \quad (4.5) \]

Consider now the fourth moduli parameter. Under the large gauge transformation \( g = e^{\chi(t)a_D} \) one finds that

\[ \delta A_i = \partial_i(\chi(t)a_D), \quad \delta A_0 = \partial_0(\chi(t)a_D), \quad \delta a_D = 0. \quad (4.6) \]

Note that since the gauge group \( U(1) \) is compact, the parameter \( \chi(t) \) is a periodic coordinate. It is possible to modify this transformation so that the potential energy remains constant and a small electric field is turned on. The modified transformation is

\[ \delta A_i = \partial_i(\chi(t)a_D), \quad \delta A_0 = \partial_0(\chi(t)a_D) - \dot{\chi} a_D, \quad \delta a_D = 0. \quad (4.7) \]

After the transformation, the electric field is \( E_i = F_{0i} = i\dot{\chi} \partial_i a_D = \dot{\chi} B_i \). The fourth parameter is \( \chi \) and its velocity controls the magnitude of the electric field which is switched on. The extra kinetic contribution to the action is

\[ S = \frac{1}{2} \int d^4xF_{0r}^2 = \frac{1}{2} \int dt \dot{\chi}^2 \int d^3xB_rB^r = \frac{1}{2} \int dt \dot{\chi}^2 4\pi \alpha |a_D^0|. \quad (4.8) \]

Thus, we find that the quantum mechanics for the collective coordinates on the one monopole moduli space is described by the action

\[ S = \int dt \frac{1}{2} 4\pi \alpha |a_D^0| \delta_{\alpha\beta} z^\alpha z^\beta. \quad (4.9) \]

This is the correct result. Thus, the monopole moduli space is topologically \( R^3 \times S^1 \). The induced metric on the moduli space is simply the flat metric. It is well known that this metric is hyper-Kähler. This fits nicely with the structure of the moduli space quantum mechanics: the dimple preserves \( \mathcal{N} = 1 \) supersymmetry in the four dimensional field theory. As a result, we would expect an action with \( \mathcal{N} = 4 \) worldline supersymmetry. In one dimension there are two types of multiplets with four supercharges. The dimensional reduction of two dimensional \( (2,2) \) supersymmetry leads to \( \mathcal{N} = 4A \) supersymmetry. The
presence of this supersymmetry requires that the moduli space be a Kähler manifold. The second supersymmetry, \( \mathcal{N} = 4B \) is obtained by reducing two dimensional \((4,0)\) supersymmetry. The presence of this supersymmetry requires that the moduli space is a hyper-Kähler manifold. The only fermion zero modes in the monopole background in \( \mathcal{N} = 2 \) super Yang-Mills theory is chiral in the sense \( \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_8 \epsilon = \pm \epsilon \) as explained above. Thus, the supersymmetry on the worldline is \( \mathcal{N} = 4B \) supersymmetry. The fact that the moduli space is hyper-Kähler is a direct consequence of supersymmetry.

4.2. Multi Monopole Moduli Space

The asymptotic metric on the moduli space of two widely separated BPS monopoles has been constructed by Manton. In this approach, one considers the dynamics of two dyons. After constructing the Lagrangian that describes the dyons motion in \( \mathbb{R}^3 \), with constant electric charges as parameters, one identifies the electric charges as arising from the motion on circles associated with the fourth moduli parameter of the monopoles. The explicit form of the metric for the relative collective coordinates is

\[
ds^2 = U(r) d\xi^i dr^i + \frac{g^2}{2\pi M U(r)} (d\chi + \omega^i dr^i)^2, \quad U(r) = 1 - \frac{g^2}{2\pi M (r^i r^i)^{1/2}},
\]

where \( r^i \) is the relative coordinate of the two monopoles, \( \chi \) is the relative phase, \( M \) the monopole mass, \( g \) the magnetic charge and \( \omega^i \) the Dirac monopole potential which satisfies \( \epsilon^{ijk} \partial^j \omega^k = r^i / (r^i r^j)^{3/2} \). This is just the Taub-NUT metric with negative mass. In this section, we will show that it is possible to reproduce the first term in this metric from the dimple solutions. The second term could presumably be reproduced by studying dyonic dimples.

We will consider the following two-dimple solution

\[
a = \gamma - \sum_{i=1}^2 \frac{\alpha}{|\vec{x} - \vec{x}_i|}.
\]

In the case of the two dimple solution, it is no longer possible to compute the moduli space metric exactly, and we have to resort to approximate techniques. To reproduce the first term in (4.10) we need to evaluate

\[
\mathcal{G}_{\alpha\beta} = - \int d^3x \tau_2 D^\alpha \frac{\partial a_D}{\partial x^\alpha} \frac{\partial a_D}{\partial x^\beta} = i \int d^3x \frac{\partial a_D}{\partial x^\alpha} \frac{\partial a_D}{\partial x^\beta},
\]

where

\[
\tau_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and \( a_D \) is the Dirac monopole field.
where \( \alpha, \beta \) can take any one of six values corresponding to any of the three spatial coordinates of either monopole. We do not know \( a_D \) as a function of the \( x^\alpha \); we will write (4.11) as \( a = \gamma - \Delta \). Setting \( a_D = a_D^0 + \beta_1 \Delta + O(\Delta^2) \) in the expression for \( a(a_D) \) we find

\[
a(a_D) = a(a_D^0) + \beta_1 \Delta \frac{\partial a}{\partial a_D} \bigg|_{a_D=a_D^0} = \gamma - \Delta. \tag{4.13}
\]

This approximation is excellent at large \( r \) where \( \Delta << 1 \). Thus, we find that

\[
a_D = a_D^0 - \Delta \left( \frac{\partial a}{\partial a_D} \bigg|_{a_D=a_D^0} \right)^{-1} = a_D^0 + i \frac{\Delta}{\tau_2 D(a_D^0)}. \tag{4.14}
\]

The dual theory is weakly coupled, so that \( \tau_2 D(a_D^0) \) is large. Thus, the correction to \( a_D^0 \) in the expression for \( a_D \) is indeed small and the approximation that we are using is valid. Using this expression for \( a_D \) in (4.12) we find

\[
G_{ij} = \frac{\alpha^2}{\tau_2 D(a_D^0)} \left( \frac{\partial}{\partial x_1^i} \frac{1}{|\vec{x} - \vec{x}_1|} \right) \left( \frac{\partial}{\partial x_1^j} \frac{1}{|\vec{x} - \vec{x}_1|} \right).
\tag{4.15}
\]

To evaluate this integral it is useful to change coordinates to a spherical coordinate system centered about monopole 1. In these coordinates

\[
G_{ij} = \delta_{ij} \frac{1}{2} \frac{4\pi \alpha^2}{\tau_2 D(a_D^0)} \int dr \frac{1}{r^2}. \tag{4.16}
\]

The integral must again be cut off at the lower limit\(^1\) where \( a_D = 0 \). The value of the cut off is given by

\[
0 = a_D^0 + \frac{i \alpha}{\tau_2 D(a_D^0)r} + \frac{i \alpha}{\tau_2 D(a_D^0)|\vec{x} + \vec{x}_1 - \vec{x}_2|} = a_D^0 + \frac{i \alpha}{\tau_2 D(a_D^0)r} + \frac{i \alpha}{\tau_2 D(a_D^0)r_{12}} + O\left(\frac{r}{r_{12}^2}\right)
\]

\[
\frac{1}{r} = \frac{\tau_2 D(a_D^0)|a_D^0|}{\alpha} \frac{1}{r_{12}}.
\tag{4.17}
\]

where \( r_{12} \), the magnitude of the relative coordinate, is assumed to be large. Thus, we find that

\(^1\) Strictly speaking we should also exclude a circular region centered around \( \vec{x} = \vec{x}_1 - \vec{x}_2 \). However, the integrand is of order \( |\vec{x}_1 - \vec{x}_2|^{-2} \) in this region and in addition the area of the region to be excluded is \( \pi/m_g^2 \), so that this is a negligible effect.
\[ G_{ij} = \frac{1}{2} \delta_{ij} 4\pi \frac{\alpha^2}{\tau_{2D}(a_D^0)} \left( \frac{\tau_{2D}(a_D^0)|a_D^0|}{\alpha} - \frac{1}{r_{12}} \right) = \frac{1}{2} \delta_{ij} \left( 4\pi \alpha|a_D^0| - \frac{g^2}{4\pi \tau_{2D}(a_D^0)r_{12}} \right). \] (4.18)

In a similar way, we compute

\[ G_{i+3,j+3} = \int d^3x \frac{\alpha^2}{\tau_{2D}(a_D^0)} \left( \frac{\partial}{\partial x_2^j} \frac{1}{|\vec{x} - \vec{x}_2|} \right) \left( \frac{\partial}{\partial x_2^j} \frac{1}{|\vec{x} - \vec{x}_2|} \right) = \frac{1}{2} \delta_{ij} \left( 4\pi \alpha|a_D^0| - \frac{g^2}{4\pi \tau_{2D}(a_D^0)r_{12}} \right). \] (4.19)

To finish the calculation of the metric on the two monopole moduli space, we need to compute

\[ G_{i+3,j} = \int d^3x \frac{\alpha^2}{\tau_{2D}(a_D^0)} \left( \frac{\partial}{\partial x_2^j} \frac{1}{|\vec{x} - \vec{x}_2|} \right) \left( \frac{\partial}{\partial x_1^i} \frac{1}{|\vec{x} - \vec{x}_1|} \right). \] (4.20)

We do not need to evaluate (4.20) directly. If we introduce center of mass and relative coordinates as

\[ \vec{r}_{cm} = \frac{1}{2} (\vec{x}_1 + \vec{x}_2), \quad \vec{r}_{12} = \vec{x}_1 - \vec{x}_2, \] (4.21)

it is a simple exercise to compute the center of mass contribution to the action

\[ S = \int dt \vec{r}_{cm}^i \dot{\vec{r}}_{cm}^i 4\pi \alpha|a_D^0|. \] (4.22)

The integral that had to be performed to obtain this result was proportional to the action itself. This result fixes

\[ G_{i+3,j} = G_{i,j+3} = \frac{1}{2} \delta_{ij} \frac{g^2}{4\pi \tau_{2D}(a_D^0)r_{12}}. \] (4.23)

Putting the above results together, we find the action which governs the relative motion of the two monopoles is

\[ S_{rel} = \int dt \left( \frac{4\pi \alpha|a_D^0|}{4} - \frac{g^2}{8\pi \tau_{2D}(a_D^0)r_{12}} \right) \frac{dr_{12}^i}{dt} \frac{dr_{12}^i}{dt}. \] (4.24)

Thus, the low energy relative motion is geodesic motion for a metric on \( R^3 \) given by \( ds^2 = U(r)dr^idr^i \), with \( U(r) = 1 - g^2/(8\pi^2 \alpha|a_D^0|\tau_{2D}(a_D^0)r) \). This reproduces the first term in (4.10). Notice however that the magnetic coupling comes with a factor of \( 1/\tau_{2D}(a_D^0) \).
This factor has its origin in the loop plus instanton corrections that were summed to obtain the low energy effective action. These corrections do not change the fact that the two monopole moduli space is hyper-Kähler. Thus, the dimple solutions on the probe reproduces the quantum corrected metric. This non-trivial metric has its origin in the fact that the forces due to dilaton and photon exchange no longer cancel at non-zero velocity due to the different retardation effects for spin zero and spin one exchange[31].

The exact two monopole metric has been determined by Atiyah and Hitchin[39]. The fact that it has an \( SO(3) \) isometry arising from rotational invariance, that in four dimensions hyper-Kähler implies self-dual curvature and the fact that the metric is known to be complete determines it exactly. Expanding the Atiyah-Hitchin metric and neglecting exponential corrections, one recovers the Taub-NUT metric[34]. If we again interpret the origin of the exponential corrections as having to do with virtual \( W^{\pm} \) boson effects, it is natural to expect that the exact treatment of the dimples in the probe worldvolume theory will recover the (quantum corrected) Taub-NUT metric and not the Atiyah-Hitchin metric. The metric on the moduli space of \( n \) well separated monopoles has been computed by Gibbons and Manton[37] by studying the dynamics of \( n \) well separated dyons. The exact monopole metric computed for a tetrahedrally symmetric charge 4 monopole was found to be exponentially close to the Gibbons-Manton metric[40]. This result was extended in [41] where it was shown that the Gibbons-Manton metric is exponentially close to the exact metric for the general \( n \) monopole solution. In view of these results it is natural to conjecture that an exact treatment of the \( n \)-dimple solution in the probe worldvolume theory will recover the (quantum corrected) Gibbons-Manton metric.

Note Added: When this work was near completion, we received [42]. In this work finite energy BPS states corresponding to open strings that start and end on threebranes were constructed. These correspond to BPS states of the \( \mathcal{N} = 4 \) super Yang-Mills theory. These authors argue for the same cut off employed in our work.

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