DE BRANGES FUNCTIONS OF SCHROEDINGER EQUATIONS

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Abstract. We characterize the Hermite-Biehler (de Branges) functions $E$ which correspond to Schroedinger operators with $L^2$ potential on the finite interval. From this characterization one can easily deduce a recent theorem by Horvath. We also obtain a result about location of resonances.

1. Introduction

The Krein–de Branges theory of Hilbert spaces of entire functions was created in the mid-twentieth century to treat spectral problems for second order differential equations. The central object of the theory is a Canonical System of differential equations on the real line. The main result of the theory states that there exists a one-to-one correspondence between such systems and de Branges spaces of entire functions. Each de Branges space is generated by a single de Branges entire function $E$ which encodes full information about the space and the differential operator. Via this result, spectral problems for differential operators translate into uniqueness and interpolation problems for spaces of entire functions. After such a translation, they can be viewed in a systematic way and treated using powerful tools of complex analysis. Since its creation, the theory has exceeded its original purpose and now extends to many fields of mathematics. Among them are complex function theory and functional model theory, spectral theory of Jacobi matrices and the theory of orthogonal polynomials, number theory and intriguing relations with the Riemann Hypothesis, see for instance [4, 8, 9, 14, 12].

The Krein–de Branges theory is a classical, yet still developing area of analysis with many important open questions. Among them is a number of 'characterization problems' where a description of de Branges functions corresponding to various important sub-classes of Canonical Systems is required (see e.g., [1, Theorems 1.4, 1.5, 6.1]). Through a standard procedure, see Section 2, many second order equations and systems can be rewritten as Canonical Systems. Among them are Schroedinger operators on an interval

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or half-line, Dirac systems, Krein strings, etc. Via the main theorem of the
Krein–de Branges theory mentioned above, these classes of differential op-
erators can be uniquely identified with corresponding classes of de Branges
entire functions $E$. Descriptions (characterizations) of such classes of entire
functions present a whole set of interesting and challenging problems. Most
of them are still open.

In this paper we solve one of such characterization problems by describing
the set of de Branges functions which correspond to Schrödinger operators
with $L^2$-potentials on the interval. We restrict ourselves to the case $p = 2$
due to the fact that in this case the characterization theorem, see Theorem 2,
takes an especially natural and satisfying form. We obtain our result from
the characterization of spectra for Schrödinger operators by Chelkak [3]
(although, as mentioned in [3], such a spectral characterization in the case
$p = 2$ was known earlier). The difference with $p \neq 2$ is largely due to the fact
that the Fourier transform of an $L^p$-function is not as easy to describe as in
the $p = 2$ case. Nonetheless, our methods allow us to study characterization
theorems for general $L^p$ and other classes of potentials. Such studies will be
presented elsewhere.

To illustrate applications of characterization theorems for de Branges
functions, in Section 4 we obtain a recent theorem by M. Horvath [5] which
is considered by many experts to be the strongest result in that area. As
was demonstrated in [5], it implies a number of classical and recent results
such as Ambarzumian’s theorem, Borg’s two-spectra theorem, the results by
Hoschltadt–Lieberman, Gesztesy–Simon, del Rio–Simon and several others.
Achieving an understanding of connections between Horvath’s methods with
the Krein–de Branges theory served as initial motivation for this paper.

In addition in Section 5 we prove that some logarithmic strip is free of
zeros of $E$. For physicists zeros of $E$ are known as resonances (poles of the
scattering matrix), see [16, 7]. It is easy that this result is essentially sharp
and we give a corresponding example.

Organization of the paper. The paper is organized as follows. In
Section 2 we remind the reader the basics of the theory and give further
references. In Section 3 we formulate and prove the main result of the
paper. Sections 4 and 5 are devoted to applications.

2. Preliminaries

2.1. De Branges spaces. Consider an entire function $E(z)$ satisfying the
inequality

$$|E(z)| > |E(z)|, \quad z \in \mathbb{C}_+,$$

and such that $E \neq 0$ on $\mathbb{R}$. Such functions are usually called de Branges
functions (or Hermite–Biehler functions). The de Branges space $\mathcal{H}(E)$
associated with $E$ is defined to be the space of entire functions $F$ satisfying

$$\frac{F(z)}{E(z)} \in H^2(\mathbb{C}_+), \quad \frac{F^*(z)}{E(z)} \in H^2(\mathbb{C}_+),$$
where \( F^\#(z) = \overline{\overline{F(z)}} \), \( H^2(\mathbb{C}_+) \) is a Hardy class in \( \mathbb{C}_+ \). It is a Hilbert space equipped with the norm \( \|F\|_E = \|F/E\|_{L^2(\mathbb{R})} \). If \( E(z) \) is of exponential type then all the functions in the de Branges space \( \mathcal{H}(E) \) are of exponential type not greater than the type of \( E \) (see, for example, the last part in the proof of Lemma 3.5 in [4]). A de Branges space is called short (or regular) if together with every function \( F(z) \) it contains \( (F(z) - F(a))/(z - a) \) for any \( a \in \mathbb{C} \).

One of the most important features of de Branges spaces is that they admit a second, axiomatic, definition. Let \( \mathcal{H} \) be a Hilbert space of entire functions that satisfies the following axioms:

- (A1) For any \( \lambda \in \mathbb{C} \), point evaluation at \( \lambda \) is a non-zero bounded linear functional on \( \mathcal{H} \);
- (A2) If \( F \in \mathcal{H} \), \( F(\lambda) = 0 \), then \( F(z)\overline{\lambda}/(z - \lambda) \in \mathcal{H} \) with the same norm;
- (A3) If \( F \in \mathcal{H} \) then \( F^\# \in \mathcal{H} \) with the same norm.

Then \( \mathcal{H} = \mathcal{H}(E) \) for a suitable de Branges function \( E \). This is Theorem 23 in [2].

2.2. Canonical Systems. Let \( \Omega \) be a symplectic matrix in \( \mathbb{R}^2 \):

\[
\Omega = \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}.
\]

A canonical system of differential equations (CS) is the system

\[
\Omega \dot{X}(t) = zH(t)X(t),
\]

where \( H \) is a real, locally summable, \( 2 \times 2 \)-matrix valued function on an interval \( (a, b) \subset \mathbb{R} \), called a Hamiltonian of the system, and \( X \) is an unknown vector valued function \( X = \begin{pmatrix} u \\
v \end{pmatrix} \). Such systems were considered by M. Krein as a general form of a second order linear differential operator. As was mentioned in the introduction, many standard classes of second order equations can be equivalently rewritten as canonical systems.

Canonical systems and de Branges spaces together constitute the so-called Krein–de Branges theory. The connection between the two is as follows. Let \( X = \begin{pmatrix} A(z, t) \\
B(z, t) \end{pmatrix} \) be a solution to (2.1) satisfying some initial condition at \( a \), for instance \( X(a) = \begin{pmatrix} 0 \\
1 \end{pmatrix} \). Then for any fixed \( t \in (a, b) \), the function \( E_t(z) = A(z, t) + iB(z, t) \) is a de Branges function. Under some minor technical restrictions on the Hamiltonian, the corresponding spaces \( \mathcal{H}(E_t) \) are nested, i.e., \( \mathcal{H}(E_t) \) is isometrically embedded into \( \mathcal{H}(E_s) \) for any \( t < s \). In the opposite direction, any de Branges function \( E \) can be obtained this way from a canonical system (2.1), see Theorem 40 in [2] or Theorems 16-18 [13]. The solution \( X \) can be used as a kernel of an integral operator (Weyl transform) which identifies the space of vector-valued square-summable functions on \( (a, t) \) with \( \mathcal{H}(E_t) \). For more on Krein–de Branges theory see [4, 2, 13].
2.3. **Schroedinger operators.** This paper is devoted to a particular case of a canonical system, the Schroedinger equation on an interval \((a, b)\),

\[-\dddot{u} + qu = z u.\]  \hfill (2.2)

Since this equation can be rewritten as a canonical system (see for instance \([12, 13]\)), it corresponds to a chain of de Branges functions/spaces as described above. Let us present a shorter way to establish this connection, without transforming the equation to the canonical form (2.1), e.g. \([12]\).

Assume that the potential \(q(t)\) is integrable on a finite interval \((a, b)\), i.e. that the operator we consider is regular. We make this restriction for the sake of simplicity, although the theory can be extended to general non-regular cases, see \([12]\). Let \(u_z(t)\) be the solution of (2.1) satisfying the boundary conditions \(u_z(a) = 0\) and \(\dot{u}_z(a) = 1\). The Weyl \(m\)-function is defined as

\[m(z) = \frac{u_z(b)}{\dot{u}_z(b)}.\]

It is well known that the Weyl function is a Herglotz function, i.e., a meromorphic function in \(\mathbb{C}\) with a positive imaginary part in \(\mathbb{C}_+\).

Now let us assume that the operator with Neumann boundary conditions at \(a\) and Dirichlet boundary conditions at \(b\) is positive (otherwise one can add a large positive constant to \(q\)). In this case one may consider the Weyl function after the 'square root transform', i.e. after a change of variables \(m(z) = zm(z^2)\). If the operator is positive then the modified Weyl function \(m(z)\) is also a Herglotz function.

The entire function

\[E(z) = zu_z^2(b) + i\dot{u}_z(b)\]  \hfill (2.3)

is the de Branges function for the corresponding canonical system. In particular, if \(q \equiv 0\), then \(E(z) = ie^{-iz}\), and we get the classical Paley–Wiener space as the corresponding de Branges space \(\mathcal{H}(E)\).

Closely related analytic (meromorphic) function is the Weyl inner function

\[\Theta = \frac{m - i}{m + i} \quad \text{or} \quad \Theta = \frac{E^\#}{E},\]

where \(E^\#(z) = \overline{E(\overline{z})}\). Such functions will also be used in our discussions below. Each of the functions, \(E(z), m(z), m(z)\) or \(\Theta(z)\), determine the operator uniquely, as follows from classical results of Marchenko \([10, 11]\).

3. **A characterization theorem**

3.1. **Spectra of the Schroedinger operators.** Every de Branges function comes from a canonical system, as follows from the main result of Krein–de Branges theory. But which of the de Branges functions can be obtained from Schroedinger operators via the procedure described above? How the properties of the potential translate into the properties of \(E\)?

Once again, let us consider the Schroedinger equation
\[-\ddot{u} + qu = zu\]  
(3.1)
on the interval \([0, 1]\). Denote by \(\sigma_{DD}\) the spectrum of \(L\) with Dirichlet boundary conditions at the endpoints after the square root transform, i.e., \(\sigma_{DD} = \{\lambda_n^2\}\) such that for each \(\lambda_n\) there exists a solution of (3.1) with \(z = \lambda_n^2\) satisfying \(u(0) = u(1) = 0\). Similarly, \(\sigma_{ND}\) will denote the spectrum for the mixed Neumann/Dirichlet conditions, \(\dot{u}(0) = u(1) = 0\). We assume that \(q \in L^1([0, 1])\) is such that both operators with DD and ND boundary conditions are positive (otherwise add a positive constant to \(q\)). In that case both spectra are real.

As defined in Subsections 2.2 and 2.3, \(E(z)\) and \(m(z)\) will denote the de Branges and Weyl functions of the Schroedinger operator. Another standard object associated with \(E(z)\) is the phase function \(\varphi(x) = -\arg E(x)\), a continuous branch of the argument of \(E\) on the real line.

Note, that in terms of \(\varphi\) the spectra of the operator can be identified as

\[\pm \sqrt{\sigma_{DD}} \cup \{0\} = \{x : \varphi(x) = n\pi\}, \quad \pm \sqrt{\sigma_{ND}} = \left\{ x : \varphi(x) = n\pi + \frac{\pi}{2} \right\},\]

Since the phase function is always a growing function on \(\mathbb{R}\), we see that the spectra \(\sigma_{DD}, \sigma_{ND}\) are alternating sequences, which is a well-known fact of spectral theory. If \(E\) is a de Branges function, it can be represented as \(E(z) = A(z) + iB(z)\) where \(A\) and \(B\) are entire functions which are real on the real line. These functions have alternating zero sequences on \(\mathbb{R}\) and can be viewed as analogs of sine and cosine. In terms of these functions, the spectra are seen as the zero sets (note that by the constructions \(A\) is odd and \(B\) is even see (2.3)):

\[\pm \sqrt{\sigma_{DD}} \cup \{0\} = \{x : A(x) = 0\}, \quad \pm \sqrt{\sigma_{ND}} = \{x : B(x) = 0\}.\]

To prove our main results we utilize the following characterization of the spectra of regular Schroedinger operators by D. Chelkak [3].

**Theorem 1.** Two alternating sequences \(\{\lambda_n^2\}\) and \(\{\mu_n^2\}\) on \(\mathbb{R}\) are equal to the spectra, \(\sigma_{DD}\) and \(\sigma_{ND}\) correspondingly, for some Schroedinger operator on the interval \([0, 1]\) with \(q \in L^2([0, 1])\) if and only if they satisfy the asymptotics

\[\lambda_n^2 = \pi^2 n^2 + C + a_n, \quad \mu_n^2 = \pi^2 \left( n - \frac{1}{2} \right)^2 + C + b_n, \quad n \in \mathbb{N}, \quad (3.2)\]

for some real \(C\) and some \(\{a_n\}, \{b_n\} \in l^2\).

To obtain this statement from the main result of [3] one needs to note that DD spectrum for even potentials \(q\) (i.e., \(q(t) \equiv q(1 - t)\)) is an union of DD and ND spectra for the interval \([0, 1/2]\).

Applying the square root transform we get

\[\lambda_n = \pi n + \frac{C}{n} + \frac{a_n}{n}, \quad \mu_n = \pi n - \frac{\pi}{2} + \frac{C}{n} + \frac{b_n}{n}, \quad n \geq 1,\quad (3.3)\]
for some real $C$ and $\{a_n\}, \{b_n\} \in \ell^2$. Put $\lambda_{-n} = -\lambda_n$, $\mu_{-n} = -\mu_n$, $n \geq 1$ and $\lambda_0 = 0$.

**Remark 1.** The constant $C$ in Theorem 1 is equal to $\int_0^1 q(x)dx$, see [3].

In what follows we will use essentially the fact that the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ are complete interpolating sequences in the Paley–Wiener space $PW_1$, the space of all entire functions of exponential type at most 1 which are in $L^2(\mathbb{R})$ or, equivalently, the Fourier image of the space $L^2(-1, 1)$. Recall that $\{\lambda_n\} \subset \mathbb{R}$ is said to be a complete interpolating sequence for $PW_1$ if for any sequence $\{w_n\} \in \ell^2$ there exists a unique function $f \in PW_1$ such that $f(\lambda_n) = w_n$. For a discussion of the interpolation theory and related Riesz bases of reproducing kernels in the Paley–Wiener spaces we refer to [6, 15]. The fact that $\{\lambda_n\}$ and $\{\mu_n\}$ are complete interpolating sequences follows from the basic results of the theory (e.g., Kadets 1/4 theorem).

### 3.2. Main theorem.

The main result of this paper is the following characterization theorem. We say that a de Branges entire function $E(z)$ corresponds to a Schroedinger equation (2.1) if it can be obtained from it following the procedure described in Subsection 2.3. To avoid inessential technicalities from now on we will assume that $q$ with DD and ND boundary conditions generates positive operators.

**Theorem 2.** Let $E = A + iB$ be a de Branges entire function of exponential type 1 and be of Cartwright class.

1. $E$ corresponds to a Schroedinger equation on $[0, 1]$ with $q \in L^2([0, 1])$ if and only if

$$z(A(z) \cos z - B(z) \sin z) = f(z) + C$$

for some real constant $C$ and some even real-valued function $f \in L^2(\mathbb{R})$.

2. Assume that $\tilde{E} = \tilde{A} + i\tilde{B}$ corresponds to a Schroedinger equation on $[0, 1]$ with $q \in L^2([0, 1])$. Then $E = A + iB$ corresponds to a Schroedinger equation on $[0, 1]$ with a square-integrable potential if and only if

$$z(A(z) \tilde{B}(z) - \tilde{A}(z)B(z)) = f(z) + C$$

for some real constant $C$ and some even real-valued function $f \in L^2(\mathbb{R})$.

3. There exists $\varepsilon > 0$ such that for any even function $f \in PW_2$ which is real on $\mathbb{R}$ with $\|f\|_2 < \varepsilon$ there exists a de Branges function $E = A + iB$ which corresponds to a Schroedinger equation on $[0, 1]$ with $q \in L^2([0, 1])$ such that

$$z(A(z) \cos z - B(z) \sin z) = f(z) - f(0).$$

**Remark 2.** It is well-known that the functions $A$ and $B$ in the statement are entire functions of exponential type 1. Hence the function $f$ in statements 1 and 2 is actually a function from the Paley–Wiener space $PW_2$.

**Remark 3.** The property (3.5) may be rewritten as $\text{Re}(zE\tilde{E}) \in \text{Const} + L^2(\mathbb{R})$, where we denote by $\text{Const}$ the class of constant functions.
3.3. **Preliminary estimates.** In the proof of Theorem 2 we will use the following simple lemma about canonical products with zeros satisfying the asymptotics (3.3).

**Lemma 1.** Let $A$ and $B$ be Cartwright class entire functions which are real on $\mathbb{R}$ and whose zeros $\lambda_n$ and $\mu_n$ satisfy the asymptotics (3.3). Then

$$
\left| \frac{A(z)}{\sin z} \right| \asymp \frac{\text{dist}(z, \{\lambda_n\})}{\text{dist}(z, \mathbb{Z})}, \quad \left| \frac{B(z)}{\cos z} \right| \asymp \frac{\text{dist}(z, \{\mu_n\})}{\text{dist}(z, \mathbb{Z} + \frac{1}{2})},
$$

(3.7)

and, moreover,

$$
A(\pi n) = (-1)^n C_1 \left( \frac{C_n}{n} + \frac{\alpha_n}{n} \right), \quad B(\pi n + \frac{\pi}{2}) = (-1)^{n+1} C_2 \left( \frac{C_n}{n} + \frac{\beta_n}{n} \right),
$$

(3.8)

where $C_1, C_2$ are real nonzero constants and $\{\alpha_n\}, \{\beta_n\} \in \ell^2$.

**Proof.** We prove the formulas (3.7)–(3.8) for the function $A$. The case of the function $B$ is similar. Since $A$ is a Cartwright class functions real on $\mathbb{R}$, it may be represented as principal value products (with obvious modification if $0 \in \{\lambda_n\}$):

$$
A(z) = K z \lim_{R \to \infty} \prod_{0 < |\lambda_n| \leq R} \left( 1 - \frac{z}{\lambda_n} \right),
$$

(3.9)

where $K \in \mathbb{R}$. The formula for $B$ is analogous.

Let $|z|$ be sufficiently large and let $n = n(z)$ be the closest integer to $z$. Then we have

$$
\frac{A(z)}{\sin z} = K \frac{z - \lambda_n}{z - \pi n} \cdot \prod_{k \neq n} \frac{\pi k}{\lambda_k} \cdot \prod_{k \neq n} \left( 1 - \frac{\lambda_k - \pi k}{z - \pi k} \right).
$$

(3.10)

Clearly,

$$
\sum_{k \neq n(z)} \left| \frac{\lambda_k - \pi k}{z - \pi k} \right| \to 0, \quad |z| \to \infty,
$$

which implies the first estimate in (3.8) and even a stronger asymptotics

$$
\frac{A(z)}{\sin z} \sim C_1 \frac{z - \lambda_n(z)}{z - \pi n(z)}, \quad |z| \to \infty, \quad C_1 = K \prod_{k \neq 0} \frac{\pi k}{\lambda_k}.
$$

(3.11)

Moreover, it is easy to see that

$$
\left\{ \sum_{k \neq n} \log \left( 1 - \frac{\lambda_k - \pi k}{z_n - \pi k} \right) \right\} \in \ell^2
$$

(3.12)

as a sequence enumerated by $n$ for any sequence $z_n$ such that $z_n \notin \pi \mathbb{Z}$ and $\{z_n - \pi n\}$ is bounded (one can refer to the boundedness in $\ell^2$ of the discrete Hilbert transform or estimate the sum directly). Thus, when we put $z = \pi n$ in (3.10), we obtain equality (3.8) for $A$ with the same constant $C_1$ as in (3.11).

Let us list several further corollaries of (3.10).
Corollary 1. In the conditions of Lemma 1 we have
\begin{equation}
\{A(\mu_n) - C_1(-1)^n\} \in \ell^2, \quad \{B(\lambda_n) - C_2(-1)^n\} \in \ell^2; \quad (3.13)
\end{equation}
\begin{equation}
\{A'(\lambda_n) - C_1(-1)^n\} \in \ell^2, \quad \{B'(\mu_n) - C_2(-1)^{n+1}\} \in \ell^2; \quad (3.14)
\end{equation}
\begin{equation}
\frac{A(iy)}{C_1 \sin iy} - 1 = \frac{2C}{y} + o\left(\frac{1}{|y|}\right), \quad \frac{B(iy)}{C_2 \cos iy} - 1 = \frac{2C}{y} + o\left(\frac{1}{|y|}\right), \quad |y| \to \infty. \quad (3.15)
\end{equation}

Proof. Inclusions (3.13)–(3.14) follow in a straightforward way from (3.10) and the inclusion (3.12). We omit the details.

Now we prove (3.15). We have
\begin{equation}
\frac{A(z)}{C_1 \sin z} = \exp\left[\sum_{k \neq 0} \log \left(1 - \frac{\lambda_k - \pi k}{z - \pi k}\right)\right]
= \exp\left[\sum_{k \neq 0} \frac{\pi k - \lambda_k}{z - \pi k} + o\left(\frac{1}{\text{dist}(z, \pi \mathbb{Z})^2}\right)\right], \quad (3.16)
\end{equation}
when $|z| \to \infty$. Taking $z = iy$, we get
\begin{align*}
\frac{A(iy)}{C_1 \sin iy} &= 1 + \sum_{k \neq 0} \frac{\pi k - \lambda_k}{iy - \pi k} + o\left(\frac{1}{|y|}\right) = 1 + \sum_{k=1}^{\infty} 2\pi k(\lambda_k - \pi k)\frac{1}{\pi^2 k^2 + y^2} + o\left(\frac{1}{|y|}\right) \\
&= 1 + 2\pi \sum_{k=1}^{\infty} \frac{C + a_k}{\pi^2 k^2 + y^2} + o\left(\frac{1}{|y|}\right) = 1 + \frac{2C}{y} + o\left(\frac{1}{|y|}\right)
\end{align*}
(here we used the fact that $\sum_{k=1}^{\infty} \frac{|y|}{\pi^2 k^2 + y^2} \to \pi^{-1}$ as $|y| \to \infty$).

The proof for $B$ is analogous. \qed

3.4. Proof of Theorem 2. In this subsection we prove Theorem 2.

Necessity of (3.4) and (3.5). Assume that $E = A + iB$ corresponds to a Schroedinger equation on $[0,1]$ with $q \in L^2([0,1])$. Then, by Theorem 1, the zeros $\lambda_n$ of $A$ and $\mu_n$ of $B$ satisfy the asymptotics (3.3). By Lemma 1 (see (3.11)), we have
\begin{align*}
\frac{A(iy)}{\sin iy} &\sim C_1, \quad \frac{B(iy)}{\cos iy} \sim C_2, \quad y \to \infty.
\end{align*}
Since $E$ corresponds to a Schroedinger equation, we have $A(iy)/B(iy) \sim \sin iy/\cos iy$ (see [11, Theorem 2.2.1]) and so we conclude that $C_1 = C_2$.

Put $F(z) = z(A(z) \cos z - B(z) \sin z)$. By (3.8), we have
\begin{align*}
F(\pi n) &= \pi n A(\pi n) \cos \pi n = \pi C_1 C + \tilde{\alpha}_n, \quad F(\pi n + \frac{\pi}{2}) = \pi C_1 C + \tilde{\beta}_n,
\end{align*}
for some $\{\tilde{\alpha}_n\}, \{\tilde{\beta}_n\} \in \ell^2$. Hence, $f = F - \pi C_1 C$ is an entire function of exponential type at most 2 such that $\{f(\pi m/2)\}_{m \in \mathbb{Z}} \in \ell^2$ and $f$ is real on the real line and even. Since $\{\pi m/2\}_{m \in \mathbb{Z}}$ is a complete interpolating sequence for
PW_2$, there exists a unique function $g \in PW_2$ such that $f\left( \frac{\pi m}{2} \right) = g\left( \frac{\pi m}{2} \right)$. Therefore $f - g$ vanishes on $\{ \frac{\pi m}{2} \}_{m \in \mathbb{Z}}$ and so $f(z) - g(z) = h(z) \sin 2\pi z$ for some entire function $h$. Since $A, B$ are Cartwright class entire functions of type 1, we conclude that $h$ is of zero exponential type.

Let us show that $h \equiv 0$. Indeed,

$$h(z) = z \left( \frac{A(z)}{2 \sin z} - \frac{B(z)}{2 \cos z} \right) - \frac{g(z) + C}{\sin 2z}.$$  

It follows from (3.15) that

$$y \left( \frac{A(iy)}{\sin iy} - \frac{B(iy)}{\cos iy} \right) = o(1), \quad |y| \to \infty,$$

Since also $|g(iy)/ \sin(2iy)| \to 0$ we conclude that $|h(iy)| \to 0$, $|y| \to \infty$, and finally $h \equiv 0$ by the standard Phragmén–Lindelöf principle.

The proof of necessity of (3.5) in statement 2 is analogous and we omit it.

**Sufficiency of (3.4) and (3.5).** We will prove the more general statement about sufficiency of (3.5). Assume that $E$ corresponds to a Schrödinger equation and let $A$ and $B$ satisfy (3.5) for some $f \in PW_2$ and $C \in \mathbb{R}$. It remains to show that the zeros of $A, B$ satisfy (3.3).

Let us denote the zeros of $\tilde{A}$ and $\tilde{B}$ by $t_n$ and $s_n$ respectively. Then, comparing the values at $t_n$ and $s_n$ we get

$$A(t_n) = \frac{1}{B(t_n)} \left( \frac{C + f(t_n)}{t_n} \right), \quad B(s_n) = -\frac{1}{A(s_n)} \left( \frac{C + f(s_n)}{s_n} \right), \quad n \neq 0.$$

By (3.7) we see that $|\tilde{B}(t_n)| \asymp |\tilde{A}(s_n)| \asymp 1$. Since both $t_n$ and $s_n$ are complete interpolating sequences for $PW_1$, there exist unique functions $g, h \in PW_1$ such that $g(0) = 0$ and

$$g(t_n) = \frac{1}{B(t_n)} \left( \frac{C + f(t_n)}{t_n} \right), \quad h(s_n) = -\frac{1}{A(s_n)} \left( \frac{C + f(s_n)}{s_n} \right), \quad n \neq 0.$$

Since $\tilde{A}$ is odd and $\tilde{B}$ is even we have $g(t_{-n}) = g(-t_n) = -g(t_n)$ and $h(s_{-n}) = h(s_n)$. Hence, $g, h$ are real on $\mathbb{R}$, $g$ is odd and $h$ is even. The function $A - g$ vanishes on $\{t_n\}$ and we conclude that $A - g = \tilde{A}h$ for some entire function $\tilde{g}$. Both $A, \tilde{A}$ are Cartwright class entire functions of type 1 and so $\tilde{g}$ is a constant. Thus, $A = g + \beta \tilde{A}$ and, analogously, $B = h + \beta \tilde{B}$ (equation (3.5) implies that the constant is the same). Without loss of generality assume in what follows that $\beta = 1$. Note that $A$ is odd and $B$ is even.

Let us study the zero asymptotics for $A = \tilde{A} + g$, the case of $B = \tilde{B} + h$ is analogous. Recall that $\{t_n\}$ is a complete interpolating sequence for $PW_1$ and so the functions $\frac{A(z)}{A(t_n)(z-t_n)}$ form a Riesz basis in $PW_1$. Hence,
expanding $g$ with respect to this Riesz basis, we get

$$g(z) = \sum_{n \in \mathbb{Z}} \frac{\tilde{A}(z)}{A'(t_n)B(t_n)(z - t_n)} \cdot \left( \frac{C}{t_n} + \frac{f(t_n)}{t_n} \right).$$

Consider the equation $\tilde{A} + g_0 = 0$ or, equivalently,

$$1 + \sum_{n \in \mathbb{Z}} \frac{1}{A'(t_n)B(t_n)(x - t_n)} \cdot \left( \frac{C}{t_n} + \frac{f(t_n)}{t_n} \right) = 0. \quad (3.17)$$

It is easy to see the sum in (3.17) tends to zero as $x \to \infty$ and $x$ is separated from $\{t_n\}$. Thus, for any $\delta > 0$ the expression in the left-hand side of (3.17) is positive on $(t_n + \delta, t_{n+1} - \delta)$ for sufficiently large $|t_n|$, and it has a unique zero in the interval $(t_n - \delta, t_n + \delta)$. This zero (denote it by $\lambda_n$) satisfies

$$\lambda_n = t_n - \frac{1}{\tilde{A}'(t_n)B(t_n)(\lambda_n - t_n)} \cdot \left( 1 + \sum_{k \neq n} \frac{1}{\tilde{A}'(t_k)B(t_k)(\lambda_n - t_k)} \left( \frac{C}{t_k} + \frac{f(t_k)}{t_k} \right) \right)^{-1}.$$

Again, it is easy to see that

$$\left\{ \sum_{k \neq n} \frac{1}{\tilde{A}'(t_k)B(t_k)(\lambda_n - t_k)} \cdot \left( \frac{C}{t_k} + \frac{f(t_k)}{t_k} \right) \right\}_{n \in \mathbb{Z}} \subset \ell^2$$

for any choice of $\lambda_n \notin \{t_k\}$ such that $\lambda_n \in (n - 1/3, n + 1/3)$ for sufficiently large $|n|$. Combining this with the asymptotics (3.13)–(3.14) of $B(t_n)$ and $\tilde{A}'(t_n)$ we conclude that

$$\lambda_n = t_n - \frac{C}{t_n} + \frac{\alpha_n}{t_n} = \pi n + \frac{\tilde{C}}{n} + \frac{\tilde{\alpha}_n}{n}, \quad (3.18)$$

where $\tilde{C}$ is some constant and $\{\alpha_n\}, \{\tilde{\alpha}_n\} \subset \ell^2$. Thus, we have shown that the zeros $\lambda_n$ of $A + g$ have the required asymptotics.

**Proof of Statement 3 of Theorem 2.** This statement essentially is already proved. Let $f \in \text{PW}_2$ be given. We need to find the functions $A, B$ such that (3.6) is satisfied. Comparing the values at $\pi n$, and $\pi n + 1/2$ we get

$$A(\pi n) = (-1)^n \frac{f(\pi n) - f(0)}{\pi n}, \quad n \neq 0, \quad A(0) = f'(0),$$

$$B \left( \pi n + \frac{\pi}{2} \right) = (-1)^{n+1} \frac{f(\pi n + \pi/2) - f(0)}{\pi n + \pi/2}.$$

There exist unique functions $g, h \in \text{PW}_1$ (which are real on $\mathbb{R}$) such that

$$g(0) = f'(0), \quad g(\pi n) = (-1)^n \frac{f(\pi n) - f(0)}{\pi n}, \quad n \neq 0,$$

$$h \left( \pi n + \frac{\pi}{2} \right) = (-1)^{n+1} \frac{f(\pi n + \pi/2) - f(0)}{\pi n + \pi/2}.$$

Now put $A = \sin z + g$, $B = \cos z + h$. It is clear that the function

$$Q(z) = z(A(z) \cos z - B(z) \sin z) = z(g(z) \cos z - h(z) \sin z)$$

satisfies (3.6).
coincides with \( f - f(0) \) at the points \( \{ \frac{\pi m}{2} \}_{m \in \mathbb{Z}} \). Hence, \( Q(z) = f(z) - f(0) + P(z) \sin 2z \) for some entire function \( P \). Since \( f \in PW_2 \) and \( Q \in zPW_2 \), a standard Phragmén–Lindelöf principle shows that \( P \) is at most constant.

Since additionally \( Q'(0) = f'(0) \) we conclude that \( P \equiv 0 \). Thus, the constructed functions \( A \) and \( B \) satisfy equality (3.6). Now arguing as in the proof of sufficiency of (3.5) we may conclude that the zeros \( \lambda_n \) and \( \mu_n \) of \( A \) and \( B \) have asymptotics (3.3). Moreover, \( A \) and \( B \) will be Cartwright class functions given by the infinite products of the form (3.9).

The only difference with the above argument is that in statements 1 and 2 we assumed from the very beginning that \( E = A + iB \) is a de Branges function, while in statement 3 we must prove that \( E \) is a de Branges function. It is well known that if \( A \) and \( B \) are zero genus canonical products of the form (3.9) (with the constants \( K > 0 \)) then \( E = A + iB \) is a de Branges function if and only if the zeros of \( A \) and \( B \) are interlacing. However, in general we can only conclude from the representations (3.4)–(3.5) that the zeros of \( A \) and \( B \) are interlacing with possible exception of a finite number of zeros.

To prove the interlacing property we need to use the fact that the norm of \( f \) is sufficiently small. Indeed, the norms of functions \( g \) and \( h \) depend on the norm of \( f \) and if \( \| f \|_2 \) is sufficiently small, then it is easily seen from the previous arguments that both \( C = -f(0) \) and \( \tilde{\alpha}_n \) in (3.18) are sufficiently small and we can get \( |\lambda_n - \pi n| < 1/2 \) for all \( n \in \mathbb{Z} \). Analogously, we get \( |\mu_n - \pi n - \pi/2| < 1/2 \) when \( \| f \|_2 \) is sufficiently small, whence \( \lambda_n \) and \( \mu_n \) are interlacing.

\[\Box\]

## 4. Horvath’ theorem

In this section we illustrate applications of our results by obtaining a recent theorem by M. Horvath [5].

If \( \Lambda \) is a sequence of real points, we denote by \( \sqrt{\Lambda} \) the set \( \{ z | z^2 \in \Lambda \} \). The notation \( \sqrt{\Lambda} \cup \{ \ast, \ast \} \) stands for the set obtained from \( \sqrt{\Lambda} \) by addition of any two real numbers (not from \( \Lambda \)).

We say that \( \Lambda \in \mathbb{R} \) is a defining set in the class \( \text{Schr}(L^2, D) \) of Schrödinger operators on \([0,1]\) with \( L^2 \)-potential and Dirichlet boundary condition at 0 if there do not exist two different operators \( L, \tilde{L} \) from this class whose Weyl functions \( m \) and \( \tilde{m} \) are equal on \( \Lambda \).

A version of the following theorem is proved in [5] for all \( 1 \leq p \leq \infty \). In this paper we treat only the case \( p = 2 \), although a similar argument can be applied to other \( p \).

**Theorem 3.** A set \( \Lambda \in \mathbb{R} \) is a defining set in the class \( \text{Schr}(L^2, D) \) if and only if \( \sqrt{\Lambda} \cup \{ \ast, \ast \} \) is a uniqueness set in the Paley–Wiener space \( PW_2 \).

**Proof.** A simple proof of the 'if' part was given in [12]. Here we present a version of it for reader's convenience. The 'only if' part follows from
Theorem 2.

If: Suppose that \( \mathbf{m} \) and \( \tilde{\mathbf{m}} \) are equal on \( \Lambda \) for some \( L, \tilde{L} \). Once again, without loss of generality we can assume that both operators are positive. Otherwise, we may add a large positive constant \( a \) to both potentials, and using the transformation

\[
F(z) \mapsto F(\sqrt{z^2 + a^2})
\]

for even entire functions we observe that \( \sqrt{\Lambda} \) is a uniqueness set if \( \sqrt{\Lambda} + a \) is.

Then, after the square root transform, \( m \) and \( \tilde{m} \) are equal on the set \( \sqrt{\Lambda} \). Also, by our definitions, \( m(0) = \tilde{m}(0) \). Hence \( \tilde{\Theta} = \Theta \) on \( \sqrt{\Lambda} \cup \{0\} \), i.e. the function \( (z - a)(\Theta - \tilde{\Theta}) = 0 \) on \( \sqrt{\Lambda} \cup \{0, a\} \), where \( a \) is any point not in \( \sqrt{\Lambda} \cup \{0\} \). By the definition of Weyl inner functions, the last equation translates into \( (z-a)(\tilde{E} - E)^\# \tilde{E} - E^\# E) = 0 \) or equivalently \( (z-a)(\tilde{A}\tilde{B} - AB) = 0 \). Since by statement 2 of Theorem 2 the function \( (z-a)(\tilde{A}\tilde{B} - AB) \) belongs to \( PW_2 \), we obtain a contradiction.

Only if: Without loss of generality, \( 0, 1 \notin \sqrt{\Lambda} \). Suppose that \( \sqrt{\Lambda} \cup \{0, 1\} \) is not a uniqueness set for \( PW_2 \) and let \( f \in PW_2 \) be a non-trivial function which vanishes on that set and real on \( \mathbb{R} \). At least one of the functions \( f(z) \) and \( \frac{f(z)}{z-1} \) is not odd. Assume that \( f \) is not odd. Put \( \tilde{f}(z) = f(z) + f(-z) \). Clearly \( \tilde{f} \) is a non-trivial even function.

By Theorem 2, \( \tilde{f} = z(A \cos z - B \sin z) \) for some \( E = A+iB \) corresponding to a Schrödinger operator \( L \). It is left to notice that then \( \mathbf{m} = \mathbf{m}_0 \) on \( \Lambda \), where \( \mathbf{m} \) corresponds to \( L \) and \( \mathbf{m}_0 \) corresponds to the free operator. \( \Box \)

Remark 4. In the second part of our proof we could obtain the following statement. If \( \sqrt{\Lambda} \cup \{*, *\} \) is not a uniqueness set in the Paley–Wiener class \( PW_2 \), then for any operator from \( Schr(L^2, D) \) there exists another operator from the same class whose \( \mathbf{m} \)-function takes the same values on \( \Lambda \).

5. Distribution of zeros of \( E \)

In this section we study the distribution of zeros of an entire function \( E \) corresponding to the Schrödinger equation.

Recall that all zeros of a de Branges function belong to \( \mathbb{C}^- \). We will show that in any logarithmic strip there exists only finite number of zeros of \( E \) corresponding to a Schrödinger equation with \( L^2 \) potential.

Theorem 4. Let \( E \) be a de Branges function which corresponds to a Schrödinger equation with \( L^2 \) potential. Then there exists \( C > 0 \) such that the logarithmic strip

\[
\left\{ z \in \mathbb{C}^- : -\frac{1}{2} \log(|\text{Re} z| + 2) + C \leq \text{Im} z < 0 \right\}
\]

contains only finite number of zeros of \( E \).
We will need two lemmas.

**Lemma 2.** For any $\delta > 0$ there exist two constants $c_\delta, C_\delta > 0$ such that

\[
c_\delta \leq \left| \frac{A(z)}{\sin \pi z} \right| \leq C_\delta, \quad c_\delta \leq \left| \frac{B(z)}{\cos \pi z} \right| \leq C_\delta,
\]

whenever $|\text{Im } z| \geq \delta$.

**Proof.** Let $C_1$ be the constant defined by (3.11). Recall the estimate (3.16) from the proof of Corollary 1:

\[
\frac{A(z)}{C_1 \sin z} = \exp \left[ \sum_{k \neq 0} \frac{\pi k - \lambda_k}{z - \pi k} + o\left( \frac{1}{\text{dist}(z, \pi Z)^2} \right) \right]
\]

\[
= \exp \left[ - \sum_{k \neq 0} \frac{c}{k(z - \pi k)} - \sum_{k \neq 0} \frac{a_k}{z - \pi k} + o(1) \right],
\]

when $|z| \to \infty$, $|\text{Im } z| \geq \delta$. Clearly, $\left| \sum_{k \neq 0} \frac{a_k}{z - \pi k} \right| = O(1)$, $|\text{Im } z| \geq \delta$, and

\[
\sum_{k \neq 0} \frac{1}{k(z - \pi k)} = \frac{\pi}{z} \left( \cot z - \frac{1}{z} \right).
\]

Hence the expression in the exponent is bounded by a constant depending only on $\delta$, but not on $z$. \(\square\)

**Lemma 3.** There exists $\delta > 0$ such that the strip $\{-\delta \leq \text{Im } z < 0\}$ contains no zeros of $E$.

**Proof.** Let $\varphi$ be a phase function for $E$. It is well known that

\[
\varphi'(t) = \sum_n \frac{|\text{Im } z_n|}{|t - z_n|^2} + a,
\]

where $z_n$ are zeros of $E$ and $a$ is some non-negative real constant. In particular, $\varphi'(z_n) \geq \frac{1}{|\text{Im } z_n|}$. It is sufficient to show that for a function $E$ which corresponds to a Schroedinger equation we have $\varphi' \in L^\infty(\mathbb{R})$.

Note that $\varphi' = |\Theta'|/2$, where $\Theta = \frac{m+i}{m+i}$ is the Weyl inner function. Clearly, the modified Weyl function $m$ is given by

\[
m(z) = \frac{A(z)}{B(z)} = \sum_n \frac{A(\mu_n)}{B'(\mu_n)} \left( \frac{1}{z - \mu_n} + \frac{1}{\mu_n} \right).
\]

Put $v_n = -\frac{A(\mu_n)}{B'(\mu_n)}$. By Corollary 1, we have $v_n = 1 + w_n \in \ell^2$. Then

\[
\varphi'(t) = \frac{m'(t)}{|m(t) + i|^2} = \left| i + \sum_n v_n \left( \frac{1}{t - \mu_n} + \frac{1}{\mu_n} \right) \right|^{-2} \times \sum_n \frac{v_n}{|t - \mu_n|^2}.
\]

Clearly, for any $\varepsilon > 0$ there exists $C_\varepsilon$ such that

\[
\varphi'(t) \leq \sum_n \frac{v_n}{|t - \mu_n|^2} \leq C_\varepsilon.
\]
whenever \( \text{dist}\{t, \{\mu_n\}\} \geq \varepsilon \).

On the other hand, if \( \varepsilon \) is sufficiently small and \(|t - \mu_k| < \varepsilon \) and \( k \) is sufficiently large, we have

\[
\left| \frac{1}{\mu_k} + \frac{1}{t - \mu_k} + \sum_{n \neq k} v_n \left( \frac{1}{t - \mu_n} + \frac{1}{\mu_n} \right) + i \right| \asymp \frac{1}{|t - \mu_k|} \tag{5.1}
\]

and \( \sum_n \frac{v_n}{|t - \mu_n|^2} \asymp \frac{1}{|t - \mu_k|} \) whence \( \varphi'(t) \leq C\varepsilon \) for some constant \( C\varepsilon \) independent of \( t \).

In the estimate (5.1) we used that, by the asymptotics of \( \mu_n \) and \( v_n \),

\[
\left| \sum_{n \neq k} v_n \left( \frac{1}{t - \mu_n} + \frac{1}{\mu_n} \right) - \sum_n \left( \frac{1}{t - (\pi n + \pi/2)} + \frac{1}{\pi n + \pi/2} \right) \right| \lesssim 1,
\]

when \(|t - \mu_k| \leq \varepsilon \). \( \square \)

**Proof of Theorem 4.** Let \( \{z_n = x_n + iy_n\} \) denote the zeros of \( E \). By Lemma 3 there exists \( \delta > 0 \) such that \( y_n \leq -\delta \). By Theorem 3 there exists \( f \in PW_2 \) and \( C \in \mathbb{R} \) such that

\[ z(A(z) \cos z - B(z) \sin z) = f(z) + C. \]

Since \( E(z_n) = 0 \) we have \( B(z_n) = iA(z_n) \). Hence, \( z_n A(z_n) e^{-iz_n} = f(z_n) + C \).

By Lemma 2, \(|A(z_n)| \asymp |\sin z_n|\). Thus,

\[ |z_n e^{iz_n}| \lesssim \frac{|f(z_n) + C|}{|\sin z_n|} \lesssim e^{|y_n|}, \]

that is, \(|x_n| + |y_n| \leq M e^{2|y_n|} \), for some \( M > 0 \). Hence, the logarithmic strip \(|y| \leq \frac{1}{2} \log \frac{1}{M} \) contains no zeros of \( E \) if \( M \) is sufficiently big. \( \square \)

**Remark 5.** By Theorem 3 the function

\[ E(z) = \frac{[z^2 - 9\pi^2/16] \sin z}{z^2 - \pi^2} + i \cos z = \frac{(32z^2 - 25\pi^2) ie^{-iz} - 7\pi^2 e^{iz}}{32(z^2 - \pi^2)}. \]

corresponds to a Schroedinger operator with \( L^2 \) potential and all zeros of \( E \) belong to a logarithmic strip \( \{ -\log(|Re z|) - M < \text{Im} z < 0 \} \). Thus, the logarithmic form of the strip is optimal. However, in this example the coefficient at \(|\text{Re } z|\) in the definition of the zero-free strip is 1, while in Theorem 4 it is 1/2.

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