ON THE ANALYTICITY OF THE GROUP ACTION ON THE LUBIN-TATE SPACE

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ABSTRACT. In this paper we study the analyticity of the group action of the automorphism group \( G \) of a formal module \( \hat{F} \) of height 2 (defined over \( \mathbb{F}_q \)) on the Lubin-Tate deformation space \( X \) of \( \hat{F} \). It is shown that a wide open congruence group of level zero attached to a non-split torus acts analytically on a particular disc in \( X \) on which the period morphism is not injective. For certain other discs with larger radii (defined in terms of quasi-canonical liftings) we find wide open rigid analytic groups which act analytically on these discs.

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1. Introduction

The deformation space. Let \( K \) be a finite extension of \( \mathbb{Q}_p \) with ring of integers \( \mathfrak{o} = \mathfrak{o}_K \), uniformizer \( \pi \), and residue field \( \mathbb{F}_q \). Denote by \( \hat{F} \) a formal \( \mathfrak{o}_K \)-module of \( K \)-height 2 over \( \mathbb{F}_p \). It is well known that \( G = \text{Aut}_A(\hat{F}) \) is isomorphic to the group of units \( \mathfrak{o}_D^* \) of the maximal compact subring \( \mathfrak{o}_D \) of a quaternion division algebra \( D \) with center \( K \), cf. [1, 1.7]. Therefore, \( G \) carries the structure of a locally \( K \)-analytic group. The deformation space \( \mathfrak{X} \) of \( \hat{F} \) is (non-canonically) isomorphic to \( \text{Spf}(\hat{\mathfrak{o}}^{nr}[u]) \), where \( \hat{\mathfrak{o}}^{nr} \) is the completion of the maximal unramified extension of \( \mathfrak{o}_K \) and the group \( G \) acts naturally on \( \mathfrak{X} \) by automorphisms of this formal scheme. In particular, \( G \) acts on the associated
rigid-analytic space \( X = \mathcal{X}^{\text{rig}} \) which we identify (using the chosen coordinate \( u \)) with the wide open unit disc \( \{ u \mid |u| < 1 \} \).

Motivation: locally analytic representations. The motivation for this paper comes from the theory of locally analytic representations of \( p \)-adic groups. Suppose \( \mathcal{V} \) is a \( G \)-equivariant vector bundle on \( X \). The space of global sections \( H^0(X, \mathcal{V}) \) is then a nuclear Fréchet space, and its topological dual space \( H^0(X, \mathcal{V})'_b \), equipped with the strong topology, is a compact inductive limit of Banach spaces. This space carries a \( G \)-action, and the question arises if this representation is locally analytic, and what other properties it may have.

For instance, when \( \mathcal{V} = \mathcal{O}_X \) is the structure sheaf, then the Gross-Hopkins period morphism

\[ \Phi : X \to (\mathbb{P}^1)^{\text{rig}}, \quad u \mapsto [\phi_0(u) : \phi_1(u)] \]

\(^1\)can be used to show that \( H^0(X, \mathcal{O}_X)'_b \) is indeed a locally analytic representation, cf. section \( [3] \). That this action is locally analytic is in fact not very difficult to see in our given situation. However, in order to get a better understanding of \( H^0(X, \mathcal{O}_X)'_b \) as a locally analytic representation, we are interested in the subspaces of vectors which are analytic for certain wide open rigid-analytic groups \( G^o \). In doing so we are following the point of view on locally analytic representations developed by M. Emerton in \([2]\). We are now going to introduce the groups \( G^o \).

The groups \( G^o \). Let \( K_2/K \) be the unramified quadratic extension, and write \( \alpha \mapsto \bar{\alpha} \) be the non-trivial Galois automorphism of \( K_2 \) over \( K \). Then we can represent \( D \) as a \( K \)-subalgebra of \( M_2(K_2) \) as follows:

\[ D = \left\{ \begin{pmatrix} \alpha & \pi \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in K_2 \right\} . \]  

We let \( \mathbb{G} \) be the algebraic group scheme over \( \text{Spec}(\mathfrak{o}_K) \) defined by \( \mathfrak{o}_D^* \), i.e., for every unital commutative \( \mathfrak{o}_K \)-algebra \( R \) one has

\[ \mathbb{G}(R) = (\mathfrak{o}_D \otimes_{\mathfrak{o}_K} R)^* . \]

Let \( \zeta \in K_2^* \) be such that \( \bar{\zeta} = -\zeta \), so that \( \zeta^2 \) is in \( \mathfrak{o}_K^* \). Let \( a_1, a_2, b_1, b_2 \) be indeterminates and put \( \Delta = a_1^2 - \zeta^2 a_2^2 - \pi(b_1^2 - \zeta^2 b_2^2) \). Then

\[ \mathbb{G} = \text{Spec} \left( \mathfrak{o}_K[a_1, a_2, b_1, b_2][\Delta] \right) , \]

\(^1\)That \( H^0(X, \mathcal{O}_X)'_b \) is a locally analytic \( G \)-representation has been shown for more general deformation spaces \( X \) of \( p \)-divisible formal groups and their automorphism groups \( G \) by J. Kohlhaase, cf. \([5]\).
where the co-multiplication is given by

\[
\begin{align*}
  a_1 & \mapsto a_1 a'_1 + \zeta^2 a_2 a'_2 + \varpi b_1 b'_1 - \zeta^2 \varpi b_2 b'_2, \\
  a_2 & \mapsto a_2 a'_1 + a_1 a'_2 + \varpi b_1 b'_2 - \varpi b_2 b'_1, \\
  b_1 & \mapsto a_1 b'_1 - \zeta^2 a_2 b'_2 + b_1 a'_1 + \zeta^2 b_2 a'_2, \\
  b_2 & \mapsto a_1 b'_2 - a_2 b'_1 + b_1 a'_2 + b_2 a'_1.
\end{align*}
\]

Let \( G_K \) be the base change from \( o_K \) to \( K \), and let \( G^\text{rig}_K \) be the associated rigid-analytic group. Its group of \( K \)-valued points is equal to \( D^* \). For an integer \( s \geq 0 \) there is a “wide open” rigid analytic group \( G^o \subset G^\text{rig}_K \) whose group of \( \mathbb{C}_p \)-valued points is given by

\[
\{(a_1, a_2, b_1, b_2) \in G^\text{rig}_K(\mathbb{C}_p) \mid |a_1 - 1| < |\pi|^s, |a_2| < |\pi|^s, |b_1| < |\pi|^{s - \frac{1}{q+1}}, |b_2| < |\pi|^{s - \frac{1}{q+1}}\}.
\]

**Critical radii and critical discs.** In [3] the homogeneous coordinates on \( \mathbb{P}^1 \) are chosen in such a way that the moduli of quasi-canonical lifts which carry an action of an open subgroup of \( o^*_K \) are mapped to the points \([1 : 0]\) and \([0 : 1]\), which are the fixed points of the non-split torus

\[
o^*_K \simeq \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \mid \alpha \in o^*_K \right\} \subset o^*_D.
\]

With respect to the coordinate \( u \) used in [3], the absolute values of the moduli of these quasi-canonical lifts (which carry an action of an open subgroup of \( o^*_K \)) are given by \(|u| = 0\) and

\[
|u| = |\pi|^{\frac{1}{(q+1)q^s}}, \quad s = 0, 1, \ldots.
\]

For \( s \in \mathbb{Z}_{\geq 0} \) we call \( r_s = |\pi|^{\frac{1}{(q+1)q^s}} \) a **critical radius** and consider the affinoid subdomain

\[
\Delta_s = \left\{ u \in X \mid |u| \leq |\pi|^{\frac{1}{(q+1)q^s}} \right\} \subset X,
\]

which we call a **critical disc.** It is easy to see that the action of \( G = o^*_D \) on \( X \) stabilizes any of the discs \( \Delta_s \). Our investigations seem to indicate that the action of \( G \) on \( \Delta_s \) extends to a rigid-analytic action of \( G^o \) on \( \Delta_s \). While, at the moment, we fall short of proving this, we have obtained some partial results in this direction.

**The results of this paper.** Let
\[ \mathbb{T} = \text{Spec} \left( \mathfrak{o}[a_1, a_2] \left[ \frac{1}{a_1^2 - \zeta^2 a_2^2} \right] \right) \subset \mathbb{G} \]

be the subgroup scheme which corresponds to the unramified torus \( \mathfrak{o}^*_K \subset \mathfrak{o}^*_D \). In section 2, we will show that the action of \( \mathfrak{o}^*_K \) on \( \Delta_0 \) extends to an analytic action of the rigid-analytic subgroup

\[
\mathbb{T}^o_0 = \left\{ (a_1, a_2, b_1, b_2) \in \mathbb{G}^{\text{rig}}_K \mid |a_1 - 1| < 1, |a_2| < 1, b_1 = b_2 = 0 \right\}
\]

(1.1.2)

on \( \Delta_0 \), cf. theorem 2.5.1 (2). We prove this by explicitly analyzing the group action of \( \mathfrak{o}^*_K \). For \( g = \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \), with \( \alpha \in \mathfrak{o}^*_K \), we write

\[ g.u = \sum_{n=0}^{\infty} a_n(g)u^n. \]

In section 2.1 we show that each function \( a_n(g) \) is a polynomial in \( E = \frac{\alpha}{\bar{\alpha}} \), and that \( a_n \) vanishes identically if \( n \) is not of the form \( 1 + k(q + 1) \) for \( k \in \mathbb{Z}_{\geq 0} \). Put \( b_k(E) = a_{1+k(q+1)}(E) \). Then, in section 2.2, we show that \( b_k(E) = \frac{1}{q^k} EQ_k(E^{q+1}) \), where \( Q_k(x) \) is a polynomial with coefficients in \( \mathfrak{o}_K \), and \( \deg(Q_k) \leq k \). The key problem is then to estimate \( |Q_k(x)| \) when \( |x - 1| \leq r \) for \( r < 1 \). This requires some fairly delicate arguments which are quite elaborate.

In section 3 we analyze the group action via the derived action of its Lie algebra. In this section we assume eventually that \( K = \mathbb{Q}_p \). For every disc \( \Delta_s \) we show that a certain rigid-analytic subgroup \( \mathbb{H}^o_s \) of \( \mathbb{G}^{\text{rig}}_K \) acts analytically on \( \Delta_s \), cf. theorem 3.2.1 for details. However, \( \mathbb{H}^o_s \) is always strictly contained in the analytic group \( \mathbb{G}^o_s \) defined above.

2. Analyticity of the non-split torus on the first critical disc

2.1. The power series describing the group action.

2.1.1. According to [3 §25], the period map \( \Phi(u) = [\phi_0(u) : \phi_1(u)] \) from the deformation space \( X = \{ u : |u| < 1 \} \) to the rigid-analytic projective space \( (\mathbb{P}^1)^{\text{rig}} \) can be described by power series

\[ \phi_0(u) = \sum_{n=0}^{\infty} c_n u^n, \quad \phi_1(u) = \sum_{n=1}^{\infty} d_n u^n, \]

whose coefficients are given as follows
\[
c_n = \begin{cases} 
1 & \text{if } n = 0 \\
\pi^{-k-1} \frac{q^{2a_0} + q^{2a_1+1} + \cdots + q^{2a_k+k}}{2} & \text{if } n \text{ is of the form } q^{2a_0} + q^{2a_1+1} + \cdots + q^{2a_k+k} \\
0 & \text{otherwise}
\end{cases}
\]

and
\[
d_n = \begin{cases} 
\pi^{-k} & \text{if } n \text{ is of the form } q^{2a_0} + q^{2a_1+1} + \cdots + q^{2a_k+k} \\
0 & \text{otherwise}
\end{cases}
\]

In particular, \(c_n = 0\) if \(q + 1 \nmid n\) and \(d_n = 0\) if \(q + 1 \nmid n - 1\).

The group \(G\) acts on \(\mathbb{P}^1\) by linear transformations. If \(g = \begin{pmatrix} \alpha & \pi \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in G\) and \([x_0 : x_1] \in \mathbb{P}^1\), then
\[
g \cdot [x_0 : x_1] = [\alpha x_0 + \beta x_1 : \pi \bar{\beta} x_0 + \bar{\alpha} x_1],
\]
where \(\Phi\) is the period map. Since \(\Phi\) is \(G\)-equivariant, we have
\[
\left[ \phi_0(g \cdot u), \phi_1(g \cdot u) \right] = g \cdot [\phi_0(u), \phi_1(u)] = [\alpha \phi_0(u), \bar{\alpha} \bar{\phi}_1(u)].
\]

And hence
\[ E \phi_1(u) \phi_0(g \cdot u) = \phi_0(u) \phi_1(g \cdot u) \]

where \( E := \frac{\hat{\alpha}}{\alpha} \). By comparing the coefficients of \( u^n \) in the above equation, we get

\[
(2.1.3) \quad E \sum_{l \leq n} d_l c_m \prod_{k} a_{r_k}^{r_k} = \sum_{m \leq n} c_m d_l \sum_{k} \prod_{k} a_{r_k}^{r_k}.
\]

By induction, we can see that the function \( a_n \) is actually a function of \( E \). So instead of writing \( a_n(g) \), we write \( a_n(E) \) from now on. When \( n = 0 \), equation (2.1.3) becomes

\[
0 = \phi_1(a_0).
\]

Hence \( a_0(E) = 0 \), since \( |a_0(E)| \leq |\pi| \) and \( \phi_1 \) is injective on \( \{ u : |u| \leq |\pi|^{1/q+1} \} \).

When \( n = 1 \), the equation (2.1.3) becomes

\[
Ed_1 c_0 = c_0 d_1 a_1,
\]

hence \( a_1(E) = E \). For \( 2 \leq n \leq q \), we have

\[
0 = c_0 d_1 a_n,
\]

and thus \( a_n(E) = 0 \).

**Lemma 2.1.4.** If \( q + 1 \nmid n - 1 \), then \( a_n(E) = 0 \).

**Proof.** We will prove by induction. Suppose \( a_n(E) = 0 \) for \( n \leq N - 1 \) and \( q + 1 \nmid n - 1 \). Want to check the case \( n = N \).

If \( q + 1 \mid N - 1 \), then we have nothing to show. Now we assume \( q + 1 \nmid N - 1 \). By the induction hypotheses, left hand side of equation (2.1.3) becomes

\[
E \sum_{l \leq N} d_l c_m \sum_{\text{r}+k(q+1)=m \atop \text{r}+(1+k(q+1))=N-l} \prod_{k} a_{1+k(q+1)}^{r_1+k(q+1)} = E \sum_{l \leq N} d_l c_m \sum_{\text{r}+k(q+1)=m \atop \text{r}+(1+k(q+1))=N-l} \prod_{k} a_{1+k(q+1)}^{r_1+k(q+1)}.
\]

Since \( q + 1 \mid m + l - 1 \) if \( c_m d_l \) does not vanish, left hand side become zero as \( \frac{N-m-l}{q+1} \) is not an integer. Similarly, the right hand side becomes

\[
c_0 d_1 a_N.
\]

Hence \( a_N(E) = 0 \) follows. \( \square \)
We remark that J. Kohlhaase has computed the functions $b_n(E) := a_{1+n(q+1)}(E)$ and rewrite equation 2.1.3 as

$$b_n = E \sum_{l \leq n} d_l c_m \sum_{r_k = m}^{0} \prod_{k} b_{k}^{r_k} = \sum_{m \leq n} c_{m} d_{l} \sum_{k}^{\nu} \prod_{k} b_{k}^{r_k}$$

or

$$(2.1.5) \quad b_n = E \sum_{l \leq n} d_l c_m \sum_{r_k = m}^{0} \prod_{k} b_{k}^{r_k} - \sum_{m \leq n} c_{m} d_{l} \sum_{k}^{\nu} \prod_{k} b_{k}^{r_k}$$

Let us consider the first few terms when $n \leq 4$:

$$b_1(E) = \frac{1}{\pi} E \left( \frac{q + 1}{0} \right) b_0^{q+1} - \frac{1}{\pi} b_0 = \frac{1}{\pi} E \left( E^{q+1} - 1 \right) ,$$

$$b_2(E) = \frac{1}{\pi} E \left( \frac{q + 1}{1} \right) b_0^{q} b_1 - \frac{1}{\pi} b_1 = \frac{1}{\pi^2} \left( q + 1 \right) (E^{q+1} - 1)^2 + \frac{1}{\pi^2} q (E^{q+1} - 1) ,$$

$$b_3(E) = \frac{1}{\pi^3} \left( q + 1 \right) (3q + 2) E(E^{q+1} - 1)^3 + \frac{1}{\pi^3} \frac{5q(q + 1)}{2} E(E^{q+1} - 1)^2$$

$$+ \frac{1}{\pi^3} q^2 E(E^{q+1} - 1) ,$$

$$b_4(E) = \frac{1}{\pi^4} \left( q + 1 \right) (2q + 1)(4q + 3) E(E^{q+1} - 1)^4 + \frac{1}{\pi^4} \frac{q(q + 1)(37q + 26)}{6} E(E^{q+1} - 1)^3$$

$$+ \frac{1}{\pi^4} \frac{9q^2(q + 1)}{2} E(E^{q+1} - 1)^2 + \frac{1}{\pi^4} q^3 E(E^{q+1} - 1) .$$

We remark that J. Kohlhaase has computed the functions $b_n$ for $n = 0, 1, 2$, cf. [4] Thm. 1.19] (what is denoted by $\alpha_1$ in loc. cit. coincides with what is here denoted by $E$).
2.2. The coefficients as rational functions on the torus. Here we will present some results about the terms $b_k$ or $a_{1+k(q+1)}$ as functions of $E = \frac{\alpha}{\beta}$.

**Lemma 2.2.1.**  
(1) $b_k(E)$ is of the form $\pi^{-k}E Q_k(E^{q+1})$ where $Q_k \in \mathfrak{o}_K[x]$.

(2) With $Q_k$ as in (1) we have $\deg_x(Q_k) \leq k$.

(3) With $Q_k$ as in (1) we have $Q_k(0) \equiv (-1)^k \mod \pi$.

(4) $|b_k| = |\pi^{-k}|$ where the supremum norm is taken over $|E| \leq 1$.

**Proof.** Part (4) follows from part (1) and part (3).

We are now going to prove (1), (2) and (3) at the same time by induction on $k$. It is clear that $b_0$ and $b_1$ satisfy the statements.

Suppose the statements are true for $k \leq N - 1$, where $N \geq 2$. Then equation 2.1.5 can be rewritten as

\[ b_N = C_1 + C_2 + C_3 + C_4, \]

where

\[ C_1 = -\pi^{-1}b_{N-1}, \]

\[ C_2 = Ed_1c_{q+1} \sum_{\substack{|r| = q+1 \backslash r_k \geq q, \exists k \geq 0 \sum kr_k = N-1}} \left( \frac{q+1}{r} \right) \frac{b_r}{r}, \]

\[ C_3 = Ed_1c_{q+1} \sum_{\substack{|r| = q+1 \backslash r_k \leq q-1, \forall k \geq 0 \sum kr_k = N-1}} \left( \frac{q+1}{r} \right) \frac{b_r}{r}, \]

\[ C_4 = E \sum_{m \geq 1, q+2 < m+l} d_m c_l \sum_{\substack{|r| = m \backslash kr_k = N-\frac{m+l-1}{q+1}}} \left( \frac{m}{r} \right) \frac{b_r}{r} - \sum_{q+2 < m+l} d_m c_l \sum_{\substack{|r| = l \backslash kr_k = N-\frac{m+l-1}{q+1}}} \left( \frac{l}{r} \right) \frac{b_r}{r}. \]

In the above expression, $r$ denotes the multi-index $(r_0, r_1, r_2, \ldots)$ and $|r|$ denotes $\sum r_k$. If $n = |r|$, $\binom{n}{l}$ denotes $\frac{n!}{l!(n-l)!}$. Finally, $b_r^-$ denotes $\prod b_{r,k}^-$. Part (1) and part (2) follows from directly from the induction hypothesis.

To prove part (3), multiply $b_N$ by $\pi^N$ and modulo $\pi$. In particular, $\pi^N C_3 \equiv 0$ as $\frac{m+l-1}{q+1} + \nu(c_m d_l) > 0$ if $m + l > q + 2$. Hence $\pi^N \frac{b_N}{E} \equiv -\pi^{N-1} \frac{b_{N-1}}{E} + \pi^N C_4 \mod \pi$. Put $E = 0$ and the result follows. \[ \square \]
The goal of this section is to find estimates for \( b_n \) when \(|E - 1| < 1\). To obtain these estimates we need to describe \( b_{n,k} \) more precisely. We will use the recursive formula \(2.1.5\) to define polynomials \( b_{n,k}(E) \) which are of the form \( \pi^{-k}E Q_{n,k}(E^{q+1}) \) with \( Q_{n,k}(x) \in \mathfrak{o}_K[x] \) for \( 0 \leq k \leq n \) such that \( b_n = \sum b_{n,k} \) and with good control on the order of \((x - 1)\) in \( Q_{n,k} \). In particular, \( \pi^{-n}Q_n = \sum \pi^{-k}Q_{n,k} \) and \( ||b_{n,k}|| \leq |\pi|^{-k} \).

First of all, \( b_{0,0}(E) := b_0(E) = E \). Suppose we have already defined \( b_{n,k} \) for \( 0 \leq k \leq n < N \). Then \(2.2.2\) suggests the following definition for \( s < N \):

\[
(2.2.2) \quad b_{N,s} = \sum_{m+l>1 \atop m<l} c_m d_l \left( E - E^{q \left\lfloor \log_q(l) \right\rfloor} \right) \sum_{|r,i|=m} \left( \frac{m}{r,i} b_{r,i} \right) + \sum_{m+l>1 \atop m<l} c_m d_l E^{q \left\lfloor \log_q(l) \right\rfloor} \sum_{|r,i|=m} \left( \frac{m + q \left\lfloor \log_q(l) \right\rfloor}{r,i} b_{r,i} \right) - \sum_{m+l>1 \atop m<l} c_m d_l \sum_{|r,i|=l, r_0,0 < q \left\lfloor \log_q(l) \right\rfloor, p \left( \frac{l}{r,i} \right)} \left( \frac{l}{r,i} b_{r,i} \right) - \sum_{m+l>1 \atop m<l} c_m d_l \sum_{|r,i|=l, r_0,0 < q \left\lfloor \log_q(l) \right\rfloor, p \left( \frac{l}{r,i} \right)} \left( \frac{l}{r,i} b_{r,i} \right) + \sum_{m+l>1 \atop l < m} \left( \pi^1 + q \left\lfloor \log_q(m) \right\rfloor \right) \sum_{|r,i|=l} \left( \frac{l}{r,i} b_{r,i} \right) - \sum_{m+l>1 \atop l < m} \left( \pi^1 + q \left\lfloor \log_q(m) \right\rfloor \right) \sum_{|r,i|=l} \left( \frac{l}{r,i} b_{r,i} \right) + \sum_{m+l>1 \atop m<l} c_m d_l E^{q \left\lfloor \log_q(m) \right\rfloor} \sum_{|r,i|=l} \left( \frac{l + q \left\lfloor \log_q(m) \right\rfloor}{r,i} b_{r,i} \right) - \sum_{m+l>1 \atop l < m} c_m d_l E^{q \left\lfloor \log_q(m) \right\rfloor} \sum_{|r,i|=l} \left( \frac{l + q \left\lfloor \log_q(m) \right\rfloor}{r,i} b_{r,i} \right).
\]
2.3. The function \( R \). To express the sought-for estimates of the functions \( b_{n,k} \), we need to define some auxiliary functions and study their basic properties.

**Definition 2.3.1.**

(1) For \( r \geq 0 \), define \( T_r \) by the
\[ T_r = 1 + q + q^2 + \cdots + q^r. \]

(2) Suppose \( n > 0 \) is an integer such that \( T_r \leq n < T_{r+1} \). For, \( l \in \mathbb{Z}_{\geq 0} \), define \( n_l \) backward inductively by

\[
n_l = \begin{cases} 
0, & \text{if } l \geq r + 1 \\
\left\lfloor \frac{n-n_{r-1}T_{r-1}-\cdots-n_{l+1}T_{l+1}}{T_l} \right\rfloor, & \text{if } 0 \leq l \leq r.
\end{cases}
\]

Define a mapping \( \sigma : \mathbb{Z} \to \bigoplus_{\mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \) by

\[
\sigma(n) = \begin{cases} 
\{ n_l \}_{l \in \mathbb{Z}_{\geq 0}}, & \text{if } n \geq 0 \\
\{ 0 \}_{l \in \mathbb{Z}_{\geq 0}}, & \text{if } n < 0.
\end{cases}
\]

(3) Define functions \( R', P : \bigoplus_{\mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \) by

\[
R'(\{ n_l \}_{l \in \mathbb{Z}_{\geq 0}}) = \sum_l n_l q^l
\]

and

\[
P(\{ n_l \}_{l \in \mathbb{Z}_{\geq 0}}) = \sum_l n_l T_l.
\]

(4) Finally, define \( R \) by

\[
R(n) := R'(\sigma(n)).
\]

**Remark 2.3.2.**

(1) In the above definition, the sequences \( \{ n_l \} \) for some non-negative integer \( n \) is characterized by:

- \( n_l \leq q \) for all \( l \)
- \( n_l = q \) for at most one \( l \). If there exist such \( l \), \( n_k = 0 \) for all \( k < l \).

This is because \( qT_{r+1} = T_{r+1} \) and \( (q-1)T_r + (q-1)T_{r-1} + \cdots + (q-1)T_{s+1} + qT_s = T_{r+1} - (r - s + 1) < T_{r+1} \) if \( r \geq s \).

(2) \( P \) is the left inverse of \( \sigma|_{\mathbb{Z}_{\geq 0}} \).

(3) We can give \( \bigoplus_{\mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \) a lexicographical order: \( \{ n_l \} > \{ m_l \} \) if there exists \( N \geq 0 \) s.t. \( n_N > m_N \) and \( n_l = m_l \) for all \( l > N \). Moreover, if \( \{ n_l \} = \sigma(n) \), then \( \{ n_l \} \geq \{ m_l \} \) for all \( \{ m_l \} \) s.t. \( P(m_l) = n \).
Here are some basic properties of the map $R$.

**Lemma 2.3.3.**

1. $R(n)$ is non-decreasing and $R(n+1) - R(n) \leq 1$.
2. $R(i+j) \leq R(i) + R(j)$.
3. If $n > 0$, then $qR(n) \geq R(qn+1)$.
4. $R(n) \geq \frac{2n-1}{q}n$ and the equality only holds at $n = 0$.

**Proof.** Fix $n \in \mathbb{Z}_{\geq 0}$, Set $\{n_t\} = \sigma(n)$. Suppose $\{N_t\} \in \oplus_{\mathbb{Z}\geq 0} \mathbb{Z}_{\geq 0}$ s.t. $P(\{N_t\})$ also equals to $n$.

Claim: $R'(\{N_t\}) \geq R'(\{n_t\}) = R(n)$. Equivalently, $R'$ attains minimum at the biggest element $\{n_t\}$ among $\{N_t\}$ with $P(\{N_t\}) = n$.

Proof of the claim. Suppose $n < T_{r+1}$. Then for all $\{N_t\}$ with $P(\{N_t\}) = n$, $N_t = 0$ for all $l \geq r+1$.

Case 1) $\exists l \geq 0$ s.t. $N_t \geq q$. Choose $i_0$ be the largest such index.

Case 1A) $\exists l < i_0$ s.t. $N_t \neq 0$. Choose $j_0$ be the smallest such index. Define $\tilde{N}_t$ by

$$
\tilde{N}_t = \begin{cases} 
N_t, & \text{if } l \neq i_0 + 1, i_0, j_0 - 1 \\
N_{i_0 + 1}, & \text{if } l = i_0 + 1 \\
N_{i_0} - q, & \text{if } l = i_0 \\
N_{j_0} - 1, & \text{if } l = j_0 \\
q, & \text{if } l = j_0 - 1 \geq 0 
\end{cases}
$$

Then $P(\{\tilde{N}_t\}) = n$ and $\{\tilde{N}_t\} \succ \{N_t\}$ in lexicographical order. Also, $R'(\{\tilde{N}_t\}) = R'(\{N_t\})$ if $j_0 \geq 1$ and $R'(\{\tilde{N}_t\}) = R'(\{N_t\}) - 1$ if $j_0 = 0$.

Case 1B) $N_t = 0$ for all $l < i_0$ and $N_{i_0} \geq q + 1$. Define $\tilde{N}_t$ by

$$
\tilde{N}_t = \begin{cases} 
N_t, & \text{if } l \neq i_0 + 1, i_0, i_0 - 1 \\
N_{i_0 + 1}, & \text{if } l = i_0 + 1 \\
N_{i_0} - q - 1, & \text{if } l = i_0 \\
q, & \text{if } l = i_0 - 1 \geq 0 
\end{cases}
$$

Again, $P(\{\tilde{N}_t\}) = n$ and $\{\tilde{N}_t\} \succ \{N_t\}$. In this situation, we have $R'(\{\tilde{N}_t\}) = R'(\{N_t\})$ if $i_0 \geq 1$ and $R'(\{\tilde{N}_t\}) = R'(\{N_t\}) - 1$ if $i_0 = 0$.

Case 1C) $N_t = 0$ for all $l < i_0$ and $N_{i_0} = q$. Then $\{\tilde{N}_t\} = \{n_t\}$ by the remark [2.3.2].

Case 2) $N_t \leq p - 1$ for all $l$. Then $\{\tilde{N}_t\} = \{n_t\}$ as in Case 1C.
The conclusion we can draw here is that we can increase the lexicographical order of \( \{ N_l \} \) successively while the \( R' \) value does not increase at the same time until we get \( \{ n_l \} \). Hence the claim follows.

We will first focus on \( n, i, j \geq 0 \). Now for part (2) of the Lemma, we define \( \{ i_l \} := \sigma(i) \) and \( \{ j_l \} := \sigma(j) \). Define \( N_l := i_l + j_l \) for all \( l \). Observe that \( P(\{ N_l \}) = i + j \), so \( R(i) + R(j) = R'(\{ N_l \}) \geq R(i + j) \).

For part (1) of the lemma, define \( N_l \) by:

\[
N_l = \begin{cases} 
  n_l, & \text{if } l > 0 \\
  n_0 + 1, & \text{if } l = 0
\end{cases}
\]

where \( \{ n_l \} = \sigma(n) \). In particular, \( P(\{ N_l \}) = n + 1 \). Hence,

\[
R(n) + 1 = R'(\{ N_l \}) \geq R(n + 1) .
\]

Since \( \{ n_l \} = \sigma \), so either \( n_l \leq q - 1 \) for all \( l \), or \( n_l \leq q - 1 \) except \( n_i = q \) for some \( i \geq 0 \) and \( n_l = 0 \) for all \( l < i \). (Case 1C and 2 in the proof of previous claim.) In the first case, \( \{ N_l \} \) is in Case 1C or 2, hence \( \{ N_l \} = \sigma(n + 1) \) and \( R(n + 1) = R(n) + 1 \). In the second case, \( \{ N_l \} \) is in Case 1A or 1B and it is not hard to see \( R'(\{ N_l \}) = R(n + 1) + 1 \) and hence \( R(n + 1) = R(n) \) in this case. In particular, \( R(n + 1) \geq R(n) \) in all cases.

For part (3), define \( N_l \) by

\[
N_l = \begin{cases} 
  qn_l, & \text{if } l > 0 \\
  qn_0 + 1, & \text{if } l = 0
\end{cases}
\]

where \( \{ n_l \} = \sigma(n) \). In particular, \( P(\{ N_l \}) = qn + 1 \) and \( R'(\{ N_l \}) = qR(n + 1) + 1 \). Since \( n > 0 \), \( \{ n_l \} \neq 0 \) and \( \{ N_l \} \) is in Case 1A or 1B. Hence \( qR(n) + 1 = R'(\{ N_l \}) > R(qn + 1) \) and thus, \( qR(n) \geq R(qn + 1) \).

It is clear that when \( n < 0 \), \( R(n) = 0 \leq R(n + 1) \leq 1 \). So part 1 follows. If \( i \geq 0 > j \), then \( i + j < i \) and hence \( R(i + j) \leq R(i) = R(i) + R(j) \). Similarly for \( i, j < 0 \). Hence part 2 follows.

For part (4), Observe that \( T_l(q - 1) = q \cdot q^l - 1 \) for all \( l > 0 \) and hence \( (q - 1) \sum n_l T_l < q \sum n_l q^l \) unless \( \{ n_l \} = \{ 0 \} \) in which the equality holds. Therefore \( R(n) \geq \frac{q - 1}{q}n \) when \( n \geq 0 \). And it is also clear that the strict inequality holds for \( n < 0 \).

2.4. Estimates for the norms of the coefficients. We start by estimating the order of vanishing of the polynomial \( Q_{n,n} \) at \( x = 1 \).
Proposition 2.4.1. \( \text{ord}_{x-1}(Q_{n,n}) \geq R(n) \). Furthermore, the equality holds when \( n = T_l \) for some \( l \geq 0 \). In particular, \( Q_n(x) \equiv (x - 1)^{R(n)}h(x) \mod \pi \) for some \( h \in \mathbb{Z}[[x]] \).

Proof. We will use induction on \( n \). When \( s = n > 0 \), equation (2.2.2) becomes

\[
\frac{E^{q+1} - 1}{\pi} b_{n-1,n-1} + \pi^{-1} E \sum_{|r,i|=q+1 \atop r_k,i \geq q, \exists k \text{ with } k > 0} (q+1) b_i r_i \sum_{|r,i|=q+1 \atop r_k,i \geq q, \exists k \text{ with } k > 0} (q+1) b_i r_i
\]

as \( i \leq k \) and \( \nu(c_m) + \nu(d_l) \geq -\frac{m+l-1}{q+1} \) for \( m + l > q + 2 \).

Here \( \text{ord}_{q^{q+1}} \frac{E^{q+1} - 1}{\pi} b_{n-1,n-1} \geq 1 + R(n - 1) \geq R(n) \) by induction.

Also, \( \text{ord}_{q^{q+1}} \frac{E^{q+1} - 1}{\pi} b_{n-1,n-1} \geq 1 + R(n - 1) \geq R(n) \) by lemma (2.3.3) as \( r_k,i \geq q \) for some \( k > 0 \).

Therefore, we get \( \text{ord}_{x-1}(Q_{n,n}) \geq R(n) \).

\[ b_{r_0,0} = b_0 = E \text{ and } b_{T_l,1} = b_1 = E \frac{E^{q+1} - 1}{\pi}, \text{ so } b_{T_l} = R(T_l) \text{ for } l = 0, 1. \]

And we proceed with induction.

Suppose now \( n = T_{l+1} \) with \( l > 0 \), then

\[
\frac{E^{q+1} - 1}{\pi} b_{n-1,n-1} + \pi^{-1} E b_{T_l,1} b_0 + \pi^{-1} E \sum_{|r,i|=q+1 \atop r_k,i \geq q, \exists k \text{ with } k \neq 0, T_l} (q+1) b_i r_i
\]

Here \( \text{ord}_{q^{q+1}} \frac{E^{q+1} - 1}{\pi} b_{n-1,n-1} \geq 1 + R(n - 1) = 1 + R(n) \) by induction as \( R(n - 1) = R(qT_l) = q^{l+1} = R(T_{l+1}) = R(n) \).

Also, for some \( k \neq 0, T_l \) and \( r_{k,k} > q \), \( \text{ord}_{q^{q+1}} \frac{E^{q+1} - 1}{\pi} b_{n-1,n-1} \geq \sum_k r_{k,k} R(k) > R(\sum k r_{k,k}) = R(qT_l) = R(T_{l+1}) \) because of the following:

It is clear that \( \sum r_{k,k} \sigma(k) < \sigma(qT_l) =: \{ n_r \}_{r \geq 0} \) and \( n_r = 0 \) for all \( r \geq n_l = q \). Then there finite many steps as in the proof of Lemma (2.3.3) \( \sum r_{k,k} \sigma(k) < \{ N_r^1 \} < \{ N_r^2 \} < \cdots < \{ N_r^k \} < \sigma(qT_l) \) with non-increasing \( R' \) values. Since \( n_r = 0 \) for \( r \neq l \), we see that \( N_0^1 \) can only be 1, and so \( R'(\{ N_r^k \}) = 1 + R(qT_l) \).

Finally, \( \text{ord}_{q^{q+1}} b_{T_l,1} b_0 = q R(T_l) = R(T_{l+1}) \) and hence \( \text{ord}_{q^{q+1}} b_{T_{l+1},T_{l+1}} = R(T_{l+1}) \). \( \square \)

Remark 2.4.2. The above proposition show that the lower bound \( R(n) \) is actually sharp for infinitely many \( n \).
Now we consider the vanishing order of $Q_{n,s}$ at $x = 1$.

**Proposition 2.4.3.**

1. $\text{ord}_{x-1} Q_{n,s} \geq R \left( s - 2 \left\lfloor \frac{n-s}{q-1} \right\rfloor \right)$ for all $0 \leq s \leq n$.

2. When $s = 2 \left\lfloor \frac{n-s}{q-1} \right\rfloor$ and $n > 0$, $\text{ord}_{x-1} Q_{n,s} \geq 1$.

**Proof.** We will prove (1) and (2) by induction on $n$.

First look at the case $1 \leq n \leq q - 2$. By definition,

$$b_{1,1} = \frac{E^{q+1} - 1}{\pi} E$$

and $b_{1,0} = 0$. For $2 \leq n \leq q - 2$ and $0 < s \leq n$,

$$b_{n,s} = \frac{E^{q+1} - 1}{\pi} b_{n-1,s-1} + \frac{q}{\pi} E^{q+1} b_{n-1,s} + \pi^{-1} E \sum_{|r,i| = q-1, \forall (k,i)} \left( q + 1 \right) b_{r,i} r^{*i}.$$

And $b_{n,0} = 0$ for $2 \leq n \leq q - 2$. Notice that $0 = 2 \left\lfloor \frac{n-0}{q-1} \right\rfloor$ iff $0 \leq n \leq q - 2$.

By induction, we can show $\text{ord}_{x-1} Q_{n,s} \geq s$ when $0 \leq s \leq n \leq q - 2$.

Now suppose the statement for $0 \leq s \leq n < N$ where $N \geq q - 1$.

We will check each term of $b_{N,s}$ in equation 2.2.2

For terms $(E^{1+q^{\left\lfloor \log_q (m) \right\rfloor}} - 1) \left( \frac{l}{r,i} \right) b_{r,i} r^{*i}$ with $l < m$, $|r,i| = l$, $\sum k,i r_{k,i} = N - \frac{m+l-1}{q+1}$ and $\sum k,i i r_{k,i} = s + \nu(c_m d_i)$,

$$\text{ord}_{E^{q+1} - 1} \geq 1 + \sum r_{k,i} R \left( i - 2 \left\lfloor \frac{k-i}{q-1} \right\rfloor \right)$$

$$\geq 1 + R \left( \sum i r_{k,i} - 2 \sum r_{k,i} \left\lfloor \frac{k-i}{q-1} \right\rfloor \right)$$

$$\geq 1 + R \left( s + \nu(c_m d_i) - 2 \left[ \frac{N - \frac{m+l-1}{q+1} - s - \nu(c_m d_i)}{q-1} \right] \right).$$

Observe that if $m + l > q + 2$,

$$s + \nu(c_m d_i) - 2 \left[ \frac{N - \frac{m+l-1}{q+1} - s - \nu(c_m d_i)}{q-1} \right] \geq s - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor.$$
To show this, we can separate into two cases: $q+2 < m+l < q^3+2$ and $m+l \geq q^3+2$. For the first case, the only non zero $d_i c_m$ is at $m+l = q^2$, $q^2+q+1$, $q^2+2q+2$. And the inequality can be check directly. For the second case, notice that $-\lfloor \log_q (m+l) \rfloor \leq \nu(c_m d_l) \leq 0$ and hence $\frac{m+l-1}{q+1} \geq (q-1)q^{\lfloor \log_q (m+l) \rfloor} - (\nu(c_m d_l))$ and $2q^{\lfloor \log_q (m+l) \rfloor} - \nu(c_m d_l) > 0$. So we actually have $s + \nu(c_m d_l) - 2 \left\lfloor \frac{N-m+l-1}{q+1} - s - \nu(c_m d_l) \right\rfloor > s - 2 \left\lfloor \frac{N-s}{q+1} \right\rfloor$ in the second case.

Hence,

$$\text{ord}_{x-1} \geq \begin{cases} 1 + R \left( s - 1 - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor \right), & \text{if } m+l = q+2 \\ 1 + R \left( s - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor \right), & \text{if } m+l > q+2 \end{cases}$$

which is at least $R \left( s - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor \right)$. And for any $N$ and $s$, it is at least 1.

Similarly for the respective term when $m < l$.

For terms $(\frac{l}{r,i})b_i l i^\nu$ with $l < m$, $|r, i| = l$ or $m$, $\sum k, i k r_k, i = N - \frac{m+l-1}{q+1}$ and $\sum k, i i r_k, i = s + \nu(c_m d_l) + 1$,

$$\text{ord}_{E_{q+1-1}} \geq R \left( s + \nu(c_m d_l) + 1 - 2 \left\lfloor \frac{N-m+l-1}{q+1} - s - \nu(c_m d_l) - 1 \right\rfloor \right)$$

$$\geq \begin{cases} R \left( s - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor \right), & \text{if } m+l = q+2 \\ R \left( s + 1 - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor \right), & \text{if } m+l > q+2 \end{cases}$$

$$\geq R \left( s - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor \right).$$

Suppose $s - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor = 0$ and $s \geq 2$, then there must exist $k \geq i$ with $k > 0$ s.t. $r_{k, i} > 0$ and $i - 2 \left\lfloor \frac{k-i}{q-1} \right\rfloor \geq 0$. Otherwise, we have all $k, i$ with $r_{k, i} > 0$, $i + 1 - 2 \left\lfloor \frac{k-i}{q-1} \right\rfloor \leq 0$ or $k = 0 = s$. If $m+l < q^2$, then $s + \nu(c_m d_l) + 1 > 0$ and so $r_{0,0} < l$. Therefore,

$$0 > \sum r_{k, i} \left( i - 2 \left\lfloor \frac{k-i}{q-1} \right\rfloor \right) \geq s + \nu(c_m d_l) + 1 - 2 \left\lfloor \frac{N-m+l-1}{q+1} - s - \nu(c_m d_l) - 1 \right\rfloor$$

$$\geq s - 2 \left\lfloor \frac{N-s}{q-1} \right\rfloor \geq 0$$
and leads to a contraction. If $m + l \geq q^3$, then

$$0 \geq \sum r_{k,i} \left( i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor \right) \geq s + \nu(c_md_l) + 1 - 2 \left\lfloor \frac{N - m + l - 1}{q+1} - s - \nu(c_md_l) - 1 \right\rfloor$$

$$> s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor \geq 0$$

and again leads to contradiction. So $E^{q+1} - 1$ divides some $b_{k,i}$ with $r_{k,i} > 0$ and thus divides $b_{i^r,i^i}$.

For terms $(\frac{l}{r,i}) b_{k,i^r,i^i}$ with $m < l$, $|r,i| = l$, $r_{0,0,0} < q^{\lfloor \log_q(l) \rfloor}$, $p \nmid (\frac{l}{r,i})$, $\sum_k r_{k,i} = N - \frac{m + l - 1}{q+1}$ and $\sum_k i r_{k,i} = s + \nu(c_md_l)$. In particular, there exists $(k_0,i_0) \neq (0,0)$ s.t. $r_{k_0,i_0} \geq q^{\lfloor \log_q(l) \rfloor}$.

Case 1) $i_0 - 2 \left\lfloor \frac{k_0 - i_0}{q - 1} \right\rfloor > 0$. Then

$$\text{ord}_{E^{q+1} - 1} \geq \sum r_{k,i} R \left( i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor \right)$$

$$\geq R \left( 1 + \sum i r_{k,i} - 2 \sum r_{k,i} \left\lfloor \frac{k - i}{q - 1} \right\rfloor \right)$$

$$\geq R \left( 1 + s + \nu(c_md_l) - 2 \left\lfloor \frac{N - \frac{m + l - 1}{q+1} - s - \nu(c_md_l)}{q - 1} \right\rfloor \right)$$

$$\geq \begin{cases} R \left( s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor \right), & \text{if } m + l = q + 2 \\ R \left( 1 + s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor \right), & \text{if } m + l > q + 2 \end{cases}$$

And $b_{k_0,i_0}$ is divisible by $E^{q+1} - 1$ implies the same holds for $b_{i^r,i^i}$.

Case 2) $i_0 - 2 \left\lfloor \frac{k_0 - i_0}{q - 1} \right\rfloor = 0$. Then $b_{k_0,i_0}$ is divisible by $E^{q+1} - 1$ and so is $b_{i^r,i^i}$.
ord_{E^{q+1}-1} \geq r_{k_0,i_0} + \sum r_{k,i} R \left( i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor \right)
\geq R \left( r_{k_0,i_0} + s + \nu(c_md_l) - 2 \left\lfloor \frac{N - m + l - 1}{q + 1} - s - \nu(c_md_l) \right\rfloor \right)
\geq \left\{ \begin{array}{ll}
r_{k_0,i_0} - 1 + s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor, & \text{if } m + l = q + 2 \\
r_{k_0,i_0} + s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor, & \text{if } m + l > q + 2 .
\end{array} \right.

Case 3) \ i_o - 2 \left\lfloor \frac{k_0 - i_o}{q - 1} \right\rfloor \leq -1. \text{ Then }
\sum_{(k,i) \neq (k_0,i_0)} r_{k,i}(i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor) \geq r_{k_0,i_0} + \sum r_{k,i}(i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor)
\geq r_{k_0,i_0} + s + \nu(c_md_l) - 2 \left\lfloor \frac{N - m + l - 1}{q + 1} - s - \nu(c_md_l) \right\rfloor
\geq \left\{ \begin{array}{ll}
r_{k_0,i_0} - 1 + s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor, & \text{if } m + l = q + 2 \\
r_{k_0,i_0} + s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor, & \text{if } m + l > q + 2 .
\end{array} \right.

So
ord_{E^{q+1}-1} \geq \sum_{(k,i) \neq (k_0,i_0)} r_{k,i} R \left( i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor \right)
\geq R \left( s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor \right).

Suppose s - 2 \left\lfloor \frac{N - s}{q - 1} \right\rfloor = 0, \text{ then } \sum_{(k,i) \neq (k_0,i_0)} r_{k,i}(i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor) > 0. \text{ Hence there is } k \geq i
\text{ with } r_{k,i} > 0 \text{ and } i - 2 \left\lfloor \frac{k - i}{q - 1} \right\rfloor > 0. \text{ Hence } E^{q+1}\mid b_i \bar{\alpha}_i.

The argument is similar for the respective terms with l < m. This completes the proof of
the proposition. \qed

2.5. Radius of convergence for the group action on }\Delta_0. \text{ As a consequence of 2.4.3
we obtain the following

Theorem 2.5.1. \quad (1) Suppose } \left| x - 1 \right| \leq \left| \pi \right|. \text{ Then } \left| \pi^{-n}Q_n \right| \leq \left| \pi \right|^{\frac{2}{n}}. \text{ Hence } \left| \alpha - 1 \right| \leq \left| \pi \right| \text{ implies that } \left| b_n\left( \frac{\alpha}{n} \right) \right| \leq \left| \pi \right|^{\frac{2}{n}}.
(2) Suppose \(|\pi|^\frac{r}{s+1} \leq r < 1\) and \(|x - 1| \leq r\). Then \(|Q_n(x)| \leq r^{\frac{n(q-1)}{q}}\). In particular, the action of

\[
\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix} \right| \alpha \in \mathcal{K}_2 \right\} \subset \mathcal{O}_D^* = G
\]

on \(\Delta_0 = \{|u| \leq |\pi|\}\) extends to a rigid-analytic action of the rigid-analytic group \(T\), cf. 1.1.2, on \(\Delta_0\).

Proof. Write \(\pi^{-n}Q_n = \sum \pi^{-s}Q_{n,s} = \pi^{-n} \sum \pi^{n-s}Q_{n,s}\).

(1) Suppose \(|x - 1| \leq |\pi|\), then

\[
|\pi^{n-s}Q_{n,s}(x)| \leq |\pi|^{(n-s)+R(s-2\left\lfloor \frac{n-s}{q-1} \right\rfloor)} \\
\leq |\pi|^{(n-s)+\frac{2-1}{q}(s-2\frac{n-s}{q-1})} \\
= |\pi|^{\frac{(q-2)n+2s}{q}} \\
\leq |\pi|^{\frac{n}{q}}.
\]

Hence \(|\pi^{-n}Q_n| \leq |\pi|^{\frac{n}{q}}\).

(2) Suppose \(|x - 1| \leq r\) where \(|\pi|^\frac{s}{s+1} \leq r < 1\). Then

\[
|\pi^{n-s}Q_{n,s}(x)| \leq r^{\frac{q+1}{q}(n-s)+R(s-2\left\lfloor \frac{n-s}{q-1} \right\rfloor)} \\
\leq r^{\frac{q+1}{q}(n-s)+\frac{2-1}{q}(s-2\frac{n-s}{q-1})} \\
= r^{\frac{(q-1)n}{q}}.
\]

In particular, when \(|\alpha - 1| \leq r\) and \(|u| \leq |\pi|\frac{s}{s+1}\),

\[
|b_n(\alpha)u^{1+n(q+1)}| \leq r^{\frac{q-1}{q}|\pi|} \to 0
\]
as \(n \to \infty\). \(\square\)

3. Analyticity on critical discs of larger radius

In this section, put \(r_s = |\pi|^{\frac{1}{s+q^2}}\) for \(s \in \mathbb{Z}_{\geq 0}\), which we sometimes call a critical radius. The function \(\phi_1\) vanishes at \(u = 0\), and all its other zeros are located on the annuli

\[
\mathcal{A}_s = \{u \in X \mid |u| = r_s\},
\]

where \(\mathbb{Z}_{\geq 0}\) is the set of non-negative integers.
for odd $s = 1, 3, 5, \ldots$, and $\phi_1(u)$ has precisely $(1 + q)q^s$ zeros on $A_s$ for odd $s$. The zeros of $\phi_0(u)$ are located on the annuli $A_s$ for even $s = 0, 2, 4, \ldots$, and $\phi_0(u)$ has precisely $(1 + q)q^s$ zeros on $A_s$ for even $s$. Let

$$\Delta_s = \{ u \in X \mid |u| \leq r_s \} ,$$

be the disc of critical radius $r_s$ centered at zero.

### 3.1. Estimates for the action of the Lie algebra.

**Lemma 3.1.1.**

1. Let $s \geq 0$ be even. Then

$$||\phi_0||_{\Delta_s} = |\pi|^{-\frac{s}{2} + \frac{q^s - 1}{(q^2 - 1)q^s}} , \quad ||\phi_1||_{\Delta_s} = |\pi|^{-\frac{s}{2} + \frac{q^{s+1} - 1}{(q^2 - 1)q^s}} .$$

2. Let $s \geq 1$ be odd. Then

$$||\phi_0||_{\Delta_s} = |\pi|^{-\frac{s}{2} + \frac{q^s - 1}{(q^2 - 1)q^s}} , \quad ||\phi_1||_{\Delta_s} = |\pi|^{-\frac{s}{2} + \frac{q^{s+1} - 1}{(q^2 - 1)q^s}} .$$

3. For all $s \geq 0$ one has

$$||\phi_0\phi_1||_{\Delta_s} = |\pi|^{-\frac{s}{2} + \frac{q^s - 1}{(q^2 - 1)q^s}} .$$

**Proof.** (1) It is not hard to check that $\pi^{-\frac{s}{2}}u^{1+q+q^2+\cdots+q^s-1}$ and $\pi^{-\frac{s+1}{2}}u^{1+q+q^2+\cdots+q^{s+1}}$ are the dominating terms of $\phi_0$ when $|u| = r_s$ for $s > 0$. 1 and $\pi^{-1}u^{1+q}$ are the dominating terms of $\phi_0$ when $s = 0$. Therefore,

$$||\phi_0||_{\Delta_s} = |\pi|^{-\frac{s}{2} + \frac{q^s - 1}{(q^2 - 1)q^s}} = |\pi|^{-\frac{s}{2} + \frac{q^s - 1}{(q^2 - 1)q^s}} .$$

Similarly, $\pi^{-\frac{s}{2}}u^{1+q+q^2+\cdots+q^s}$ is the dominating term of $\phi_1$. Hence

$$||\phi_1||_{\Delta_s} = |\pi|^{-\frac{s}{2} + \frac{q^s - 1}{(q^2 - 1)q^s}} = |\pi|^{-\frac{s}{2} + \frac{q^s - 1}{(q^2 - 1)q^s}} .$$

The treatment for part (2) is the same and part (3) follows from (1), (2). \qed

**3.1.2. The action of the Lie algebra.** Let $\zeta \in \mathfrak{o}_{K_2}^*$ be such that $\bar{\zeta} = -\zeta$. Let $G$ be the group scheme over $\mathfrak{o} = \mathfrak{o}_K$ whose $\mathfrak{o}$-valued points are $\mathfrak{o}_D^*$, cf. section 1. Denote by $g$ the relative Lie algebra of $G$ over $\mathfrak{o}$. Consider the following $\mathfrak{o}$-basis of $g$:

$$\mathfrak{r}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \mathfrak{r}_2 = \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix} , \quad \mathfrak{v}_1 = \begin{pmatrix} 0 & \pi \\ 1 & 0 \end{pmatrix} , \quad \mathfrak{v}_2 = \begin{pmatrix} 0 & -\pi \zeta \\ \zeta & 0 \end{pmatrix} .$$

We note that
We write elements of $\mathbb{P}^1$ as $[x_0 : x_1]$ and put $w = \frac{x_1}{x_0}$. Elements $\mathfrak{z}$ in the Lie algebra $\mathfrak{g}$ act on rational functions $f = f(w)$ on $\mathbb{P}^1$ as follows:

$$(\mathfrak{z}.f)(w) = \frac{d}{dt}f(e^{t\mathfrak{z}}w)\big|_{t=0}.$$ 

Note that

$$\exp(t\mathfrak{z}_1) = \begin{pmatrix} \cosh(t\sqrt{\pi}) & \sqrt{\pi}\sinh(t\sqrt{\pi}) \\ \sinh(t\sqrt{\pi})/\sqrt{\pi} & \cosh(t\sqrt{\pi}) \end{pmatrix},$$

and

$$\exp(t\mathfrak{z}_2) = \begin{pmatrix} \cos(t\zeta\sqrt{\pi}) & -\sqrt{\pi}\sin(t\zeta\sqrt{\pi}) \\ \sin(t\zeta\sqrt{\pi})/\sqrt{\pi} & \cos(t\zeta\sqrt{\pi}) \end{pmatrix}.$$ 

According to the formula 2.1.2 for the group action we compute

$$(\mathfrak{x}_2.f)(w) = \left. \frac{d}{dt}f(e^{-2\zeta\sqrt{\pi}}w) \right|_{t=0} = -2\zeta w f'(w),$$

$$(\mathfrak{y}_1.f)(w) = \left. \frac{d}{dt}f \left( \frac{\cosh(t\sqrt{\pi})w + \sqrt{\pi}\sinh(t\sqrt{\pi})}{\sinh(t\sqrt{\pi})/\sqrt{\pi}w + \cosh(t\sqrt{\pi})} \right) \right|_{t=0} = \left. \frac{\sinh(t\sqrt{\pi})\sqrt{\pi}w + \pi\cosh(t\sqrt{\pi}) - w(\cosh(t\sqrt{\pi})w + \sqrt{\pi}\sinh(t\sqrt{\pi}))}{(\sinh(t\sqrt{\pi})/\sqrt{\pi}w + \cosh(t\sqrt{\pi}))^2} \right|_{t=0} \cdot f'(w) = (\pi - w^2)f'(w),$$

$$(\mathfrak{y}_2.f)(w) = \left. \frac{d}{dt}f \left( \frac{\cos(t\zeta\sqrt{\pi})w - \zeta\pi\cos(t\zeta\sqrt{\pi}) - \pi\zeta(\cos(t\sqrt{\pi})w + \zeta\sqrt{\pi}\sin(t\sqrt{\pi}))}{(\sin(t\sqrt{\pi})/\sqrt{\pi}w + \cos(t\sqrt{\pi}))^2} \right) \right|_{t=0} = -\zeta(\pi + w^2)f'(w).$$

Therefore, under the map from $\mathfrak{g}_0$ to the ring of differential operators on $P$ we have

$$(3.1.3) \quad \mathfrak{x}_2 \mapsto -2\zeta w \partial_w, \quad \mathfrak{y}_1 \mapsto (\pi - w^2)\partial_w, \quad \mathfrak{y}_2 \mapsto -\zeta(\pi + w^2)\partial_w.$$ 

Setting $w = \frac{\phi_1(u)}{\phi_0(u)}$, we find
\[ dw = \frac{\phi_1'\phi_0 - \phi_1\phi_0'}{\phi_0^2} du = \frac{\varepsilon}{\phi_0^2} du , \]

where \( \varepsilon = \phi_1'\phi_0 - \phi_1\phi_0' \). As \( \partial_w \) is dual to \( dw \) we get thus

\[ 1 = \langle \partial_w, dw \rangle = \frac{\varepsilon}{\phi_0^2} \langle \partial_w, du \rangle \]

and hence \( \partial_w = \frac{\phi_0^2}{\varepsilon} \partial_u \). From (3.1.3) we then deduce (3.1.4)

\[ \eta_2 \mapsto -2\zeta \frac{\phi_0\phi_1}{\varepsilon} \pi - \frac{1}{(q+1)q^s} \partial_u , \]

\[ \eta_1 \mapsto \frac{\pi\phi_0^2 - \phi_1^2}{\varepsilon} \pi - \frac{1}{(q+1)q^s} \partial_u , \]

\[ \eta_2 \mapsto -\zeta \frac{\phi_0^2 + \phi_1^2}{\varepsilon} \pi - \frac{1}{(q+1)q^s} \partial_u . \]

Let now \( \pi^{-\frac{1}{(q+1)q^s}} \) be any element of absolute value equal to \( |\pi|^{-\frac{1}{(q+1)q^s}} \), and put \( u_s = \pi^{-\frac{1}{(q+1)q^s}} u \), which is a coordinate function on \( \Delta_s \) with supremum norm 1. Then \( \partial_u = \pi^{-\frac{1}{(q+1)q^s}} \partial_{u_s} \) and the formulas in (3.1.4) become (3.1.5)

\[ \eta_2 \mapsto -2\zeta \frac{\phi_0\phi_1}{\varepsilon} \pi - \frac{1}{(q+1)q^s} \partial_{u_s} , \]

\[ \eta_1 \mapsto \frac{\pi\phi_0^2 - \phi_1^2}{\varepsilon} \pi - \frac{1}{(q+1)q^s} \partial_{u_s} , \]

\[ \eta_2 \mapsto -\zeta \frac{\phi_0^2 + \phi_1^2}{\varepsilon} \pi - \frac{1}{(q+1)q^s} \partial_{u_s} . \]

**Proposition 3.1.6.**  

(1) Let \( s \geq 0 \) be even. Then

\[ \| \pi^{s+1} \phi_0^2 \pi - \frac{1}{(q+1)q^s} \|_{\Delta_s} = \| \pi \|^{1 + \frac{2s+1}{(q^2-1)q^s} - \frac{1}{(q+1)q^s}} = \| \pi \|^{1 + \frac{2}{q^2-1} - \frac{1}{(q+1)q^s}} = \| \pi \|^{\frac{2s+1}{q^2-1} - \frac{1}{(q+1)q^s}} , \]

and

\[ \| \pi^s \phi_1^2 \pi - \frac{1}{(q+1)q^s} \|_{\Delta_s} = \| \pi \|^{\frac{2s+1}{(q^2-1)q^s} - \frac{1}{(q+1)q^s}} = \| \pi \|^{\frac{2s+1}{q^2-1} - \frac{1}{(q+1)q^s}} . \]

In particular, when \( s = 0 \):

\[ \| \pi^1 \phi_0^2 \pi - \frac{1}{(q+1)q^s} \|_{\Delta_0} = \| \pi \|_{\frac{q}{q^2+1}} , \quad \| \pi^1 \phi_1^2 \pi - \frac{1}{(q+1)q^s} \|_{\Delta_0} = \| \pi \|_{\frac{1}{q^2+1}} . \]

(2) Let \( s \geq 1 \) be odd. Then

\[ \| \pi^{s+1} \phi_0^2 \pi - \frac{1}{(q+1)q^s} \|_{\Delta_s} = \| \pi \|^{\frac{2s+1}{(q^2-1)q^s} - \frac{1}{(q+1)q^s}} = \| \pi \|^{\frac{2s+1}{q^2-1} - \frac{1}{(q+1)q^s}} , \]
and
\[
|| \pi^s \phi_1 \pi^{-1} ||_{\Delta_s} = |\pi|^{1 + \frac{2q^s - 2}{(q^s - 1)p^r} - \frac{1}{(q+1)p^r}} = |\pi|^{1 - \frac{2}{q^s - 1} - \frac{1}{q+1}} = |\pi|^{\frac{q^s + 1}{q^s - 1} - \frac{1}{q+1}}.
\]

(3) For all \( s \geq 0 \) one has
\[
|| \pi^s \phi_0 \phi_1 \pi^{-1} ||_{\Delta_s} = |\pi|^{\frac{1}{q-1} - \frac{2}{q^s - 1} - \frac{1}{q+1}p^r} = |\pi|^{\frac{1}{q-1} - \frac{1}{q+1}}.
\]

3.2. Groups acting analytically on larger critical discs. As an immediate consequence of the previous proposition we obtain the following result.

**Theorem 3.2.1.** Let \( K = \mathbb{Q}_p \) (hence \( q = p \) and \( \pi = p \)), \( K_s = \mathbb{Q}_p(p^{\frac{1}{(p-1)p^r}}) \), and put
\[
\mathfrak{h}_s = \mathfrak{o} \cdot p^s \mathfrak{r}_1 \oplus \mathfrak{o} \cdot p^{s+ \frac{1}{p^r - 1} + \frac{1}{p^r}} \mathfrak{r}_2 \oplus \mathfrak{o} \cdot p^{s+\frac{1}{p^r - 1} + \frac{1}{p^r}} \mathfrak{y}_1 \oplus \mathfrak{o} \cdot p^{s+\frac{1}{p^r - 1} + \frac{1}{p^r}} \mathfrak{y}_2.
\]
This is a Lie algebra over the ring of integers \( \mathfrak{o}_{K_s} \) in \( K_s \). There is a group scheme \( \mathbb{H}_s \) over \( \mathfrak{o}_{K_s} \) with Lie algebra \( \mathfrak{h}_s \). Denote by \( \mathbb{H}_s^0 \) the completion of this group scheme along the unit section, and let \( \mathbb{H}_s^0 \) be the associated rigid-analytic group. Then \( \mathbb{H}_s^0 \) acts analytically on \( \Delta_s \).

**Remark 3.2.2.** Suppose \( K = \mathbb{Q}_p \) and consider the case \( s = 0 \). Then theorem (2.5.1) implies that, in the formula for \( \mathfrak{h}_0 \) above, we can replace \( \mathfrak{o} \cdot p^{\frac{1}{p^r - 1}} \mathfrak{r}_2 \) by \( \mathfrak{o} \cdot \mathfrak{r}_2 \). Hence, for \( s = 0 \), we can replace the Lie algebra \( \mathfrak{h}_0 \) in the theorem above by
\[
\mathfrak{h}_0' = \mathfrak{o} \cdot \mathfrak{r}_1 \oplus \mathfrak{o} \cdot \mathfrak{r}_2 \oplus \mathfrak{o} \cdot p^{-\frac{1}{p^r - 1} + \frac{1}{p^r}} \mathfrak{y}_1 \oplus \mathfrak{o} \cdot p^{-\frac{1}{p^r - 1} + \frac{1}{p^r}} \mathfrak{y}_2.
\]

3.2.3. Let \( s \geq 0 \). Suppose \( u_0 \in \Delta_s \) and let \( B^-(u_0, r) \) be the largest wide open disc such that \( \Phi \) is injective on \( B^-(u_0, r) \). In [6] we call \( B^-(u_0, r) \) a disc of injectivity around \( u_0 \), and we have shown that \( r = |\pi u_0^2|^{\frac{1}{q-1}} \). In [6, sec. 3] we describe the image of \( B^-(u_0, r_0) \) under \( \Phi \) which is again a wide open disc and whose radius we determine.

Suppose \( g = \left( \begin{array}{c} \alpha \\ \beta \\ \bar{\alpha} \end{array} \right) \in G \). Then \( g \cdot \Phi(u_0) \in \Phi(B^-(u_0, r_0)) \) for all \( u_0 \in \Delta_s \) if and only if \( |\alpha - \bar{\alpha}| < |\pi|^s \) and \( |\beta| < |\pi|^s - \frac{1}{p^r} \). Therefore, we would expect that we could actually replace the Lie algebra \( \mathfrak{h}_s \) in theorem by the larger Lie algebra
\[
\mathfrak{g}_s = \mathfrak{o} \cdot p^s \mathfrak{r}_1 \oplus \mathfrak{o} \cdot p^s \mathfrak{r}_2 \oplus \mathfrak{o} \cdot p^{s+\frac{1}{p^r + 1}} \mathfrak{y}_1 \oplus \mathfrak{o} \cdot p^{s+\frac{1}{p^r + 1}} \mathfrak{y}_2
\]
which differs from \( \mathfrak{h}_s \) by the factor \( p^{\frac{1}{(p-1)p^r}} \) in front of the generators \( \mathfrak{r}_2, \mathfrak{y}_1, \) and \( \mathfrak{y}_2 \).
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