THE GENUS FIELDS OF ARTIN-SCHREIER EXTENSIONS

SU HU AND YAN LI

Abstract. Let $q$ be a power of a prime number $p$. Let $k = \mathbb{F}_q(t)$ be the rational function field with constant field $\mathbb{F}_q$. Let $K = k(\alpha)$ be an Artin-Schreier extension of $k$. In this paper, we explicitly describe the ambiguous ideal classes and the genus field of $K$. Using these results we study the $p$-part of the ideal class group of the integral closure of $\mathbb{F}_q[t]$ in $K$. And we also give an analogy of Rédei-Reichardt’s formulae for $K$.

1. Introduction

In 1951, Hasse [6] introduced genus theory for quadratic number fields which is very important for studying the ideal class groups of quadratic number fields. Later, Fröhlich [3] generalized this theory to arbitrary number fields. In 1996, S.Bae and J.K.Koo [2] defined the genus field for global function fields and developed the analogue of the classical genus theory. In 2000, Guohua Peng [7] explicitly described the genus theory for Kummer function fields.

The genus theory for function fields is also very important for studying the ideal class groups of function fields. Let $l$ be a prime number and $K$ be a cyclic extension of degree $l$ of the rational function field $\mathbb{F}_q(t)$ over a finite field of characteristic $\neq l$. In 2004, Wittmann [12] generalized Guohua Peng’s results to the case $l \nmid q - 1$ and used it to studied the $l$ part of the ideal class group of the integral closure of $\mathbb{F}_q[t]$ in $K$ following an ideal of Gras [4].

Let $q$ be a power of a prime number $p$. Let $k = \mathbb{F}_q(t)$ be the rational function field with constant field $\mathbb{F}_q$. Assume that the polynomial $T^p - T - D \in k(T)$ is irreducible. Let $K = k(\alpha)$ with $\alpha^p - \alpha = D$. $K$ is called an Artin-Schreier extension of $k$ (See [5]). It is well known that every cyclic extension of $\mathbb{F}_q(t)$ of degree $p$ is an Artin-Schreier extension. In this paper, we explicitly describe the genus field of $K$. Using this result we also study the $p$-part of the ideal class group of the integral closure of $\mathbb{F}_q[t]$ in $K$. Our results combined with Wittmann [12]’s results give the complete results for genus theory of cyclic extensions of prime degree over rational function fields.

Let $O_K$ be the integral closure of $\mathbb{F}_q[t]$ in $K$. Let $Cl(K)$ be the ideal class group of the Dedekind domain $O_K$. Let $G(K)$ be the genus field of $K$. Our paper
is organized as follows. In Section 2, we recall the arithmetic of Artin-Schreier extensions. In Section 3, we recall the definition of $G(K)$ and compute the ambiguous ideal classes of $Cl(K)$ using cohomological methods. As a corollary, we obtain the order of $\text{Gal}(G(K)/K)$. In Section 4, we described explicitly $G(K)$. And we also give an analogy of Rédei-Reichardt’s formulae [10] for $K$.

2. The arithmetic of Artin-Schreier extensions

Let $q$ be a power of a prime number $p$. Let $k = \mathbb{F}_q(t)$ be the rational function field. Let $K/k$ be a cyclic extension of degree $p$. Then $K/k$ is an Artin-Schreier extension, that is, $K = k(\alpha)$, where $\alpha^p - \alpha = D$, $D \in \mathbb{F}_q(t)$ and $D$ cannot be written as $x^p - x$ for any $x \in k$. Conversely, for any $D \in \mathbb{F}_q(t)$ and $D$ cannot be written as $x^p - x$ for any $x \in k$, $k(\alpha)/k$ is a cyclic extension of degree $p$, where $\alpha^p - \alpha = D$. Two Artin-Schreier extensions $k(\alpha)$ and $k(\beta)$ such that $\alpha^p - \alpha = D$ and $\beta^p - \beta = D'$ are equal if and only if they satisfy the following relations,

$$
\alpha \rightarrow x\alpha + B_0 = \beta,
$$

$$
D \rightarrow xD + (B_0^p - B_0) = D',
$$

$$
x \in \mathbb{F}_p^*, B_0 \in k.
$$

(See [5] or Artin [1] p.180-181 and p.203-206) Thus we can normalize $D$ to satisfy the following conditions,

$$
D = \sum_{i=1}^{m} \frac{Q_i}{P_i^{e_i}} + f(t),
$$

$$(P_i, Q_i) = 1, \text{ and } p \nmid e_i, \text{ for } 1 \leq i \leq m,$$

$$p \nmid \text{deg}(f(t)), \text{ if } f(t) \notin \mathbb{F}_q,$$

where $P_i(1 \leq i \leq m)$ are monic irreducible polynomials in $\mathbb{F}_q[t]$ and $Q_i(1 \leq i \leq m)$ are polynomials in $\mathbb{F}_q[t]$ such that $\text{deg}(Q_i) < \text{deg}(P_i^{e_i})$. In the rest of this paper, we always assume $D$ has the above normalized forms and denote $\frac{Q_i}{P_i^{e_i}} = D_i$, for $1 \leq i \leq m$. The infinite place $(1/t)$ is splitting, inertial, or ramified in $K$ respectively when $f(t) = 0; f(t)$ is a constant and the equation $x^p - x = f(t)$ has no solutions in $\mathbb{F}_q; f(t)$ is not a constant. Then the field $K$ is called real, inertial imaginary, or ramified imaginary respectively. The finite places of $k$ which are ramified in $K$ are $P_1, \cdots, P_m$ (p.39 of [5]). Let $\Psi_i$ be the place of $K$ lying above $P_i(1 \leq i \leq m)$.

Let $P$ be a finite place of $k$ which is unramified in $K$. Let $(P, K/k)$ be the Artin symbol at $P$. Then

$$
(P, K/k)\alpha = \alpha + \left\lfloor \frac{D}{P} \right\rfloor
$$
and the Hasse symbol \( \{ \frac{D}{P} \} \) is determined by the following equalities:

\[
\{ \frac{D}{P} \} \equiv D + D^p + \cdots D^{N(P)/p} \text{mod } P
\]

\[
\equiv (D + D^p + \cdots D^{N(P)/p})
+ (D + D^q + \cdots D^{N(P)/q})^p
+ \cdots
+ (D + D^q + \cdots D^{N(P)/q})^{q/p} \text{mod } P,
\]

\[
\{ \frac{D}{P} \} = \text{tr}_{\mathbb{F}_q/P} \text{tr}_{(O_K/P)/\mathbb{F}_q}(D) \text{ mod } P
\]

(p.40 of [5]).

3. Ambiguous ideal classes

From this point, we will use the following notations:

- \( q \) – power of a prime number \( p \).
- \( k \) – the rational function field \( \mathbb{F}_q(t) \).
- \( K \) – an Artin-Schreier extension of \( k \) of degree \( p \).
- \( G \) – the Galois group \( \text{Gal}(K/k) \).
- \( \sigma \) – the generator of \( \text{Gal}(K/k) \).
- \( S \) – the set of infinite places of \( K \) (i.e., the primes above \( 1/t \)).
- \( O_K \) – the integral closure of \( \mathbb{F}_q[t] \) in \( K \).
- \( I(K) \) – the group of fractional ideals of \( O_K \).
- \( P(K) \) – the group of principal fractional ideals of \( O_K \).
- \( P(k) \) – the subgroup of \( P(K) \) generated by nonzero elements of \( \mathbb{F}_q(t) \).
- \( Cl(K) \) – the ideal class group of \( O_K \).
- \( H(K) \) – the Hilbert class field of \( K \).
- \( G(K) \) – the genus field of \( K \).
- \( U_K \) – the unit group of \( O_K \).

**Definition 3.1.** (Rosen [8]) The Hilbert class field \( H(K) \) of \( K \) (relative to \( S \)) is the maximal unramified abelian extension of \( K \) such that every infinite places (i.e. \( \in S \)) of \( K \) split completely in \( H(K) \).

**Definition 3.2.** (Bae and Koo [2]) The genus field \( G(K) \) of \( K \) is the maximal abelian extension of \( K \) in \( H(K) \) which is the composite of \( K \) and some abelian extension of \( k \).
For any $G$-module $M$, let $M^G$ be the $G$-module of elements of $M$ fixed by the action of $G$. Without lost of generality, we will assume $K/k$ is a geometric extension in the rest of this paper. We have the following Theorem.

**Theorem 3.3.** The ambiguous ideal classes $\text{Cl}(K)^G$ is a vector space over $\mathbb{F}_p$ generated by by $[\mathfrak{p}_1], [\mathfrak{p}_2], \cdots, [\mathfrak{p}_m]$ with dimension

$$\dim_{\mathbb{F}_p} \text{Cl}(K)^G = \begin{cases} m - 1 & K \text{ is real.} \\ m & K \text{ is imaginary.} \end{cases}$$

Before the proof of the above theorem, we need some lemmas.

**Lemma 3.4.** $H^1(G, P(K)) = 1$.

*Proof.* From the following exact sequence

$$1 \longrightarrow U_K \longrightarrow K^* \longrightarrow P(K) \longrightarrow 1,$$

we have

$$1 \longrightarrow H^1(G, P(K)) \longrightarrow H^2(G, U_K) \longrightarrow H^2(G, K^*) \longrightarrow \cdots$$

This is because $K/k$ is a cyclic extension and $H^1(G, K^*) = 1$ (Hilbert Theorem 90). Since

$$H^2(G, U_K) = \frac{U_K^G}{NU_K} = \frac{\mathbb{F}_q^*}{(\mathbb{F}_q^*)^p} = 1,$$

we have $H^1(G, P(K)) = 1$. \hfill \Box

**Lemma 3.5.** If $K$ is imaginary, then $H^1(G, U_K) = 1$.

*Proof.* Since $U_K = \mathbb{F}_q^*$, we have

$$H^1(G, \mathbb{F}_q^*) = \frac{\{ x \in \mathbb{F}_q^* | x^p = 1 \}}{\{ x^{q-1} | x \in \mathbb{F}_q^* \}} = 1.$$

\hfill \Box

**Lemma 3.6.** If $K$ is real, then $H^1(G, U_K) \cong \mathbb{F}_p$.

*Proof.* We denote by $\mathcal{D}$ the group of divisors of $K$, by $\mathcal{P}$ the subgroup of principal divisors. We define $\mathcal{D}(S)$ to be the subgroup of $\mathcal{D}$ generated by the primes in $S$ and $\mathcal{D}^0(S)$ to be the degree zero divisors of $\mathcal{D}(S)$. From Proposition 14.1 of [9], we have the following exact sequence

$$(0) \longrightarrow \mathbb{F}_q^* \longrightarrow U_K \longrightarrow \mathcal{D}^0(S) \longrightarrow \text{Reg} \longrightarrow (0),$$

where the map from $U_K$ to $\mathcal{D}^0(S)$ is given by taken an element of $U_K$ to its divisor and $\text{Reg}$ is a finite group (See Proposition 14.1 and Lemma 14.3 of [9]). By Proposition 7 and Proposition 8 of [11] (p.134), we have $h(U_K) = h(\mathcal{D}^0(S))$,
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where \( h(\ast) \) is the Herbrand Quotient of \( \ast \). By Equation (3.1), we have \( H^2(G, U_K) = 1 \). Thus, we can prove this Lemma by showing \( h(\mathcal{D}^0(S)) = 1/p \).

Let \( \infty \) be any infinite place in \( S \). Thus \( \mathcal{D}^0(S) \) is the free abelian group generated by \((\sigma - 1)\infty, (\sigma^2 - \sigma)\infty, \ldots, (\sigma^{p-1} - \sigma^{p-2})\infty \). And we have

\[
\mathcal{D}^0(S) = \mathbb{Z}[G](\sigma - 1)\infty \approx \frac{\mathbb{Z}[G]}{(1 + \sigma + \cdots \sigma^{p-1})}.
\]

Let \( \zeta_p \) be a \( p \)-th root of unity. We have

\[
(3.3) \quad \frac{\mathbb{Z}[G]}{(1 + \sigma + \cdots \sigma^{p-1})} \cong \mathbb{Z}[\zeta_p],
\]

and the above map is given by taken \( \sigma \) to \( \zeta_p \). From (3.2) and (3.3), we have

\[
H^1(G, \mathcal{D}^0(S)) = \ker N_{\mathcal{D}^0(S)}(I_K) \cong \mathbb{Z}[\zeta_p]/(\zeta_p - 1) \cong \mathbb{F}_p.
\]

Thus \( h(\mathcal{D}^0(S)) = 1/p \).

Proof of Theorem 3.3: From the following exact sequence

\[
1 \rightarrow P(K) \rightarrow I(K) \rightarrow Cl(K) \rightarrow 1,
\]

we have

\[
1 \rightarrow P(K)^G \rightarrow I(K)^G \rightarrow Cl(K)^G \rightarrow H^1(G, P(K)) \rightarrow \cdots
\]

Since \( H^1(G, P(K)) = 1 \) by Lemma 3.4, we have

\[
1 \rightarrow P(K)^G \rightarrow I(K)^G \rightarrow Cl(K)^G \rightarrow 1.
\]

Thus

\[
(3.4) \quad 1 \rightarrow \frac{P(K)^G}{P(k)} \rightarrow \frac{I(K)^G}{P(k)} \rightarrow Cl(K)^G \rightarrow 1.
\]

From the following exact sequence

\[
1 \rightarrow U_K \rightarrow K^* \rightarrow P(K) \rightarrow 1,
\]

we have

\[
1 \rightarrow \mathbb{F}_q^* \rightarrow k^* \rightarrow P(K)^G \rightarrow H^1(G, U_K) \rightarrow 1
\]

and

\[
(3.5) \quad H^1(G, U_K) \cong \frac{P(K)^G}{P(k)}.
\]

Since \( \frac{I(K)^G}{P(k)} \) is a vector space over \( \mathbb{F}_p \) with basis \( [\Psi_1], [\Psi_2], \ldots, [\Psi_m] \), by (3.4), (3.5), Lemma 3.5 and Lemma 3.6, we get the desired result.
Remark 3.7. If $K$ is real, it is an interesting question to find explicitly the relation satisfied by $[\Psi_1], [\Psi_2], \cdots, [\Psi_m]$ in $Cl(K)^G$. By Lemma 3.5 if we can find a nontrivial element $\bar{u}$ of $H^1(G, U_K)$, then by Hibiert 90, we have $u = x^{\sigma - 1}$, where $u \in U_K$ and $x \in K$. It is easy to see that
\[
\sum_{i=1}^m \text{ord}_{\Psi_i}(x)[\Psi_i] = 0
\]
in $Cl(K)^G$.

From Proposition 2.4 of [2], we have
\[
(3.6) \quad \text{Gal}(G(K)/K) \cong Cl(K)/(\sigma - 1)Cl(K) \cong Cl(K)^G.
\]
(It should be noted that the last isomorphism is merely an isomorphism of abelian groups but not canonical). Therefore, we get

Corollary 3.8.

\[
\#\text{Gal}(G(K)/K) = \begin{cases} 
p^{m-1} & \text{K is real.} 
p^m & \text{K is imaginary.} 
\end{cases}
\]

4. The genus field $G(K)$

In this section, we prove the following theorem which is the main result of this paper.

Theorem 4.1.

\[
G(K) = \begin{cases} 
k(\alpha_1, \alpha_2, \cdots, \alpha_m) & \text{K is real.} 
k(\beta, \alpha_1, \alpha_2, \cdots, \alpha_m) & \text{K is imaginary.} 
\end{cases}
\]

Where $\alpha_i^p - \alpha_i = D_i = \frac{Q_i}{P_i}(1 \leq i \leq m), \beta^p - \beta = f(t)$, and $D_i, Q_i, P_i, f(t)$ are defined in Section 2.

We only prove the imaginary case. The proof is the same for the real case. Since
\[
(\sum_{i=1}^m \alpha_i + \beta)^p - (\sum_{i=1}^m \alpha_i + \beta) = \sum_{i=1}^m \frac{Q_i}{P_i^p} + f(t) = D,
\]
we can assume $\alpha = \sum_{i=1}^m \alpha_i + \beta$. Before the proof of the above theorem, we need two lemmas.

Lemma 4.2. $E = k(\beta, \alpha_1, \alpha_2, \cdots, \alpha_m)$ is an unramified abelian extension of $K$.

Proof. Let $P$ be a place of $k$ and let $(1/t)$ be the infinite place of $k$. If $P \neq P_1, P_2, \cdots, P_m, (1/t)$, then $P$ is unramified in $k(\beta), k(\alpha_i)(1 \leq i \leq m)$, hence unramified in $E$. Otherwise, without lost of generality, we can suppose $P = P_1$. Since $\alpha = \sum_{i=1}^m \alpha_i + \beta$, we have $E = Kk(\alpha_2, \cdots, \alpha_m, \beta)$. Thus $P = P_1$ is unramified in $k(\alpha_2, \cdots, \alpha_m, \beta)$, hence unramified in $E/K$. \qed
**Lemma 4.3.** The infinite places of $K$ are split completely in $E = k(\beta, \alpha_1, \alpha_2, \ldots, \alpha_m)$.  

*Proof.* Since $\alpha = \sum_{i=1}^{m} \alpha_i + \beta$, we have $E = Kk(\alpha_1, \alpha_2, \ldots, \alpha_m)$. Since the infinite place $(1/t)$ of $k$ splits completely in $k(\alpha_1, \alpha_2, \ldots, \alpha_m)$, hence $(1/t)$ also splits completely in $E/K$.\qed

Proof of Theorem 4.1 From Lemma 4.2 and 4.3 we have

\[(4.1) \quad k(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta) \subset G(K).\]

Comparing ramifications, $k(\beta), k(\alpha_i)(1 \leq i \leq m)$ are linearly disjoint over $k$, so

\[ [k(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta) : k] = p^{m+1} \]

and

\[ [k(\alpha_1, \alpha_2, \ldots, \alpha_m, \beta) : K] = p^m. \]

Thus from Corollary 3.8 and (4.1), we get the result.

5. The $p$-part of $\text{Cl}(K)$

If $l$ is a prime number, $K$ is a cyclic extension of $k$ of degree $l$, and $\mathbb{Z}_l$ is the ring of $l$-adic integers, then $\text{Cl}(K)_l$ is a finite module over the discrete valuation ring $\mathbb{Z}_l[\sigma]/(1 + \sigma + \cdots + \sigma^{l-1})$. Thus its Galois module structure is given by the dimensions:

\[ \lambda_i = \dim(\text{Cl}(K)_l^{(\sigma - 1)i^j}/\text{Cl}(K)_l^{(\sigma - 1)^i}) \]

for $i \geq 1$, these quotients being $\mathbb{F}_l$ vector spaces in a natural way. In number field situations, the dimensions $\lambda_i$ have been investigated by Rédei [10] for $l = 2$ and Gras [4] for arbitrary $l$. In function field situations, these dimensions $\lambda_i$ have been investigated by Wittmann for $l \neq p$. In this section, we give a formulae to compute $\lambda_2$ for $l = p$. This is an analogy of Rédei-Reichardt’s formulae [10] for Artin-Schreier extensions.

If $K$ is imaginary, as in the proof of Theorem 4.1, we suppose that $K = k(\alpha)$, where $\alpha = \sum_{i=1}^{m} \alpha_i + \beta$. We have the following sequence of maps

\[ \text{Cl}(K)^G \longrightarrow \text{Cl}(K)/(\sigma - 1)\text{Cl}(K) \cong \text{Gal}(G(K)/K) \hookrightarrow \text{Gal}(G(K)/k) \]

\[ \cong \text{Gal}(k(\alpha_1)/k) \times \cdots \times \text{Gal}(k(\alpha_m)/k) \times \text{Gal}(k(\beta)/k). \]

Considering $[\Psi_i] \in \text{Cl}(K)^G(1 \leq i \leq m)$ under these maps, we have

\[ [\Psi_i] \longmapsto [\tilde{\Psi}_i] \longmapsto ([\Psi_i], G(K)/K) \longmapsto (\Psi_i, G(K)/K) \]

\[ \longmapsto ((P_i, k(\alpha_1)/k), \ldots, (P_i, k(\alpha_m)/k), (P_i, k(\beta)/k)), \]

where the $i$-th component is $(\Psi_i, G(K)/K)_{k(\alpha_i)}$.

We define the Rédei matrix $R = (R_{ij}) \in M_{m \times m}(\mathbb{F}_p)$ as following:

\[ R_{ij} = \frac{D_j}{P_i}, \text{ for } 1 \leq i, j \leq m, i \neq j, \]
and $R_{ii}$ is defined to satisfy the equality:

$$\sum_{j=1}^{m} R_{ij} + \{ \frac{f}{P_i} \} = 0.$$ 

From the discussions in section 2, we have

$$(\Psi, G(K)/K)\alpha = \alpha,$$

$$(\Psi, G(K)/K)\alpha_j = \alpha_j + \{ \frac{D_j}{P_i} \}, \text{for } i \neq j$$

so it is easy to see the image of $\text{Cl}(K)^G \to \text{Cl}(K)/(\sigma - 1)\text{Cl}(K)$ is isomorphic to the vector space generated by the row vectors $(R_{11}, R_{12}, \cdots, R_{im}, \{ \frac{f}{P_i} \}) (1 \leq i \leq m)$.

We conclude that

$$\lambda_2 = \dim_{\mathbb{F}_p}(\text{Cl}(K)^G_l/(\sigma - 1)\text{Cl}(K)^G_l) = \dim_{\mathbb{F}_p}(\text{Cl}(K)^{(\sigma - 1)^2})$$

$$= \dim_{\mathbb{F}_p}(\ker(\text{Cl}(K)^G \to \text{Cl}(K)/(\sigma - 1)\text{Cl}(K)))$$

$$= \dim_{\mathbb{F}_p}(\text{Cl}(K)^G) - \dim_{\mathbb{F}_p}(\text{Im}(\text{Cl}(K)^G \to \text{Cl}(K)/(\sigma - 1)\text{Cl}(K)))$$

$$= m - \text{rank}(R).$$

Since the proof of real case is similar, we only give the results and sketch the proof.

If $K$ is real, from the discussions in section 2, we have $f(t) = 0$, so

$$D = \sum_{i=1}^{m} D_i.$$ 

We define the Rédei matrix $R = (R_{ij}) \in M_{m \times m}(\mathbb{F}_p)$ as following:

$$R_{ij} = \{ \frac{D_j}{P_i} \}, \text{ for } 1 \leq i, j \leq m, i \neq j,$$

and $R_{ii}$ is defined to satisfy the equality:

$$\sum_{j=1}^{m} R_{ij} = 0.$$ 

The same procedure as the imaginary case shows that the image of $\text{Cl}(K)^G \to \text{Cl}(K)/(\sigma - 1)\text{Cl}(K)$ is isomorphic to the vector spaces generated by the row vectors of Rédei matrix. Thus

$$\lambda_2 = \dim_{\mathbb{F}_p}(\text{Cl}(K)^G) - \dim_{\mathbb{F}_p}(\text{Im}(\text{Cl}(K)^G \to \text{Cl}(K)/(\sigma - 1)\text{Cl}(K)))$$

$$= m - 1 - \text{rank}(R).$$

**Theorem 5.1.** If $K$ is imaginary, then $\lambda_2 = m - \text{rank}(R)$; if $K$ is real, then $\lambda_2 = m - 1 - \text{rank}(R)$, where $R$ is the Rédei matrix defined above.
If $p = 2$, then $\sigma$ acting on $Cl(K)$ equal to $-1$. So $\lambda_1, \lambda_2$ equal to the 2-rank, 4-rank of ideal class group $Cl(K)$, respectively. In particular, the above theorem tells us the 4-rank of ideal class group $Cl(K)$ which is an analogue of classical Rédei-Reichardt’s 4-rank formulae for narrow ideal class group of quadratic number fields.

REFERENCES

[1] E.Artin, Algebraic numbers and algebraic functions, AMS CHELSEA PUBLISHING, 2005.
[2] S.Bae and J.K.Koo, Genus theory for function fields, J.Austral.Math.Soc.(Series A)60(1996),301-310.
[3] A.Fröhlich, Central extensions, Galois groups, and ideal classes of number fields, Contemp.Math.24(Amer.Math.Soc.Providence,1983).
[4] G.Gras, Sur les l-classes d'idéaux dans les extensions cycliques relatives de degré premier 11,II,Ann.Inst.Fourier 23(3)(1973)1-48;Ann.Inst.Fourier 23(4)(1973)45-64.
[5] H.Hasse, Theorie der relativ zyklischen algebraischen Funktionenkörper, insbesondere bei endlichem Konstantenkörper. J.Reine Angew.Math. 172(1934),37-54.
[6] H.Hasse, Zur Geschlecht Theorie in quadratische Zahlkörpern, J.Math.Soc.Japan 3(1951),45-51.
[7] G. Peng, The genus fields of Kummer function fields, J.Number Theory 98(2003), 221-227.
[8] M.Rosen, The Hilbert class field in function field, Exp.Math.5(1987), 365-378.
[9] M.Rosen, Number Theory in Function Fields, Springer-verlag, New York,2002.
[10] L.Rédei, Arithmetischer Beweis des Satzes über die Anzahl der durch vier teilbaren Invarianten der absoluten Klassengruppe im quadratischen Zahlkörper, J.Reine Angew Math.171(1935) 55-60.
[11] J.P.Serre, Local fields, Springer-verlag, New York, 1979.
[12] C.Wittmann, $l$-class groups of cyclic function fields of degree $l$, Finite Fields Appl.13(2007), 327-347.

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
E-mail address: hus04@mails.tsinghua.edu.cn, liyan_00@mails.tsinghua.edu.cn