Lp-LIOUVILLE THEOREMS ON COMPLETE SMOOTH
METRIC MEASURE SPACES

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Abstract. We study some function-theoretic properties on a complete smooth
metric measure space \((M, g, e^{-f} dv)\) with Bakry-Émery Ricci curvature bounded
from below. We derive a Moser’s parabolic Harnack inequality for the \(f\)-heat
equation, which leads to upper and lower Gaussian bounds on the \(f\)-heat
kernel. We also prove \(L^p\)-Liouville theorems in terms of the lower bound of
Bakry-Émery Ricci curvature and the bound of function \(f\), which generalize
the classical Ricci curvature case and the \(N\)-Bakry-Émery Ricci curvature case.

1. Introduction and main results
1.1. Background. Let \((M, g)\) be an \(n\)-dimensional complete Riemannian mani-
ofld and \(f\) be a smooth function on \(M\). We define a symmetric diffusion operator
\(\Delta_f\) (or \(f\)-Laplacian), which is given by
\[
\Delta_f := \Delta - \nabla f \cdot \nabla,
\]
where \(\Delta\) and \(\nabla\) are the Laplacian and covariant derivative of the metric \(g\). The
\(f\)-Laplacian \(\Delta_f\) is the infinitesimal generator of the Dirichlet form
\[
\mathcal{E}(\phi_1, \phi_2) = \int_M (\nabla \phi_1, \nabla \phi_2) d\mu, \quad \forall \phi_1, \phi_2 \in C_0^\infty(M),
\]
where \(\mu\) is an invariant measure of \(\Delta_f\) given by \(d\mu = e^{-f} dv\), and where \(dv\) is the
volume element induced by the metric \(g\). It is well-known that \(\Delta_f\) is self-adjoint
with respect to the weighted measure \(d\mu\). The triple \((M, g, e^{-f} dv)\) is customarily
called a complete smooth metric measure space. On this measure space, we often
consider the \(f\)-heat equation
\[
\left(\frac{\partial}{\partial t} - \Delta_f\right) u = 0
\]
instead of the classical heat equation. If the function \(u\) is independent of time \(t\),
then \(u\) is a \(f\)-harmonic function. In this paper, we denote by \(H(x, y, t)\) the \(f\)-heat
kernel, that is, for each \(x \in M\), \(H(x, y, t) = u(y, t)\) is the minimal solution of the
\(f\)-heat equation with \(u(y, 0) = \delta_x(y)\). Equivalently, \(H(x, y, t)\) is the kernel of the
semigroup \(P_t = e^{t\Delta_f}\) associated to the Dirichlet form \(\mathcal{E}(\phi, \phi)\).

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On the smooth metric measure space \((M,g,e^{-f}\,dv)\), following Bakry and Œmery [1] and [2] (see also [23] and [25]), we define the Bakry-Œmery Ricci curvature
\[
\text{Ric}_f := \text{Ric} + \text{Hess}(f),
\]
where \(\text{Ric}\) denotes the Ricci curvature of the manifold and \(\text{Hess}\) denotes the Hessian with respect to the manifold metric. The Bakry-Œmery Ricci curvature is a natural extension of the Ricci curvature. If \(f\) is constant, \(\text{Ric}_f\) returns to the Ricci curvature \(\text{Ric}\). The Bakry-Œmery Ricci curvature has been extensively studied because it often shares interesting properties with the ordinary Ricci curvature. For example, there exists an interesting Bochner type identity
\[
\Delta_f |\nabla u|^2 = 2|\text{Hess}(u)|^2 + 2\langle \nabla u, \nabla \Delta_f u \rangle + 2\text{Ric}_f(\nabla u, \nabla u).
\]
This identity is parallel to the Bochner identity in the classical Ricci curvature case, and plays an important role in studying comparison theorems (see [42]). For more extended results, the interested reader can consult [3], [4], [8], [14], [24], [25], [26], [38], [39], [41] and [43].

Also, the Bakry-Œmery Ricci curvature has become an important object of study in geometry analysis, in large part due to so-called gradient Ricci solitons. Recall that a complete manifold \((M,g)\) is a gradient Ricci soliton if for some function \(f\) on \(M\) and some constant \(\rho\) we have that
\[
\text{Ric}_f = \rho g.
\]
The soliton is called expanding, steady or shrinking if, respectively, \(\rho < 0\), \(\rho = 0\) and \(\rho > 0\). Ricci solitons possess many interesting geometric and topological properties. See, for example, [6], [7] and [30] for nice explanations on this subject.

Recently, there has been renewed interest in the Bakry-Œmery Ricci curvature and its modified version, the \(N\)-Bakry-Œmery Ricci curvature, defined by
\[
\text{Ric}^N_f := \text{Ric} + \text{Hess}(f) - \frac{df \otimes df}{N},
\]
where \(N\) is a positive constant. For example, Catino, Mantegazza, Mazzieri and Rimoldi [9], Petersen and Wylie [31], and Pigola, Rimoldi and Setti [33] established various Liouville-type or rigidity theorems about these curvatures. Prior to their works, X.-D. Li [23] studied a \(L^1\)-Liouville theorem in case the \(N\)-Bakry-Œmery Ricci curvature is bounded below by a negative quadratic polynomial of the distance function. That is an extension of the classical \(L^1\)-Liouville theorem on Ricci curvatures, proved by P. Li [19]. However, as X.-D. Li said in Subsection 8.6 of [23], we cannot prove a \(L^1\)-Liouville theorem if we only assume a lower bound of the same kind on \(\text{Ric}_f\). Indeed, we can not obtain a Li-Yau type parabolic Harnack inequality under only this curvature assumption. Here it is natural to pose the following problem: What are the optimal geometric or analytic conditions on the smooth metric measure space in order that the Li-Yau parabolic Harnack inequality holds?

In the recent papers [28], [29], Munteanu and Wang partially answered to the above question. In particular, they derived gradient estimates and Liouville properties for positive \(f\)-harmonic functions under suitable growth assumption on \(f\). Their theorems take the form of Yau’s classical result on positive \(f\)-harmonic functions, but the proof they adopt is new and quite different in spirit from Yau’s direct application of the Bochner formula [45]. Their approach essentially relies on the
well-known De Giorgi-Nash-Moser theory. This motivates our proof of Theorem 1.1 in this paper.

1.2. Main results. The purpose of this paper is to further study geometric inequalities for the $f$-heat equation and $L^p$-Liouville theorems for $f$-harmonic functions on complete smooth metric measure spaces. One contribution of this paper is to provide suitable weighted curvature conditions which assure the validity of various well-known geometric inequalities, such as a local $f$-volume doubling property, a local $f$-Neumann Poincaré inequality and a local $f$-mean value inequality, etc. Another contribution of this paper is that we used those geometric inequalities to prove new $L^p$-Liouville theorems on complete smooth metric measure spaces.

This paper can be divided into two parts. In the first part, borrowing the idea of Munteanu and Wang [28, 29], we will derive some geometric inequalities, such as parabolic Harnack inequalities, Hölder continuity estimates and $f$-heat kernel estimates on complete smooth metric measure spaces. We first present a parabolic Harnack inequality on complete smooth metric measure spaces.

**Theorem 1.1.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ for some nonnegative constants $K$ and $A$, then there exist a constant $c(n, A)$ such that, for any $0 < R \leq \infty$ and ball $B_o(r)$, $0 < r < R$ and for any smooth positive solution $u$ of the $f$-heat equation in the cylinder $Q = B_o(r) \times (s-r^2, s)$, we have

$$\sup_{Q_-} u \leq e^{c(n,A)(1+Kr^2)} \cdot \inf_{Q_+} u,$$

where $Q_- := B_o(\frac{1}{2}r) \times (s-\frac{1}{4}r^2, s-\frac{1}{2}r^2)$ and $Q_+ := B_o(\frac{1}{2}r) \times (s-\frac{1}{4}r^2, s)$.

The sketch of the proof of Theorem 1.1 will be given in Section 2. The proof follows by the Moser iteration technique [27], which involves a local Sobolev inequality on a smooth metric measure space. Munteanu and Wang [29] used a similar technique to derive an elliptic Harnack inequality for $f$-harmonic functions. When the metric measure space is a Riemannian manifold, that is, the function $f$ is constant, this result was obtained independently by Saloff-Coste [34] and Grigor'yan [15].

A standard consequence of Theorem 1.1 is a strong Liouville theorem for any $f$-harmonic function (see Corollary 3.2). Theorem 1.1 also implies two-sided $f$-heat kernel bounds. This result is essentially analogous to the case of heat equation on Riemannian manifolds in [36] (see also [17]).

**Theorem 1.2.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f|(x) \leq A$ on the ball $B_o(2R)$ for some nonnegative constants $K$ and $A$, then there exist positive constants $c_i$, $i = 5, 6, 7, 8$, depending only on $n$ and $A$ such that

$$\frac{e^{-c_5(1+Kt)}}{V_f(B_x(\sqrt{t}))} \exp \left( -c_5 \frac{d^2(x,y)}{t} \right) \leq H(x, y, t) \leq \frac{e^{c_8(1+Kt)}}{V_f(B_x(\sqrt{t}))} \exp \left( -c_7 \frac{d^2(x,y)}{t} \right)$$

for any $x, y \in B_o(R/2)$ and $0 < t < R^2/4$, where $V_f(B_x(\sqrt{t}))$ denotes the $f$-volume of the ball $B_x(\sqrt{t})$ with respect to $e^{-f} dv$.

**Remark 1.3.** Theorem 1.2 gives an accurate description of the coefficients of two-sided $f$-heat kernel bounds. It will be crucial in the proof of Theorem 1.6.
The proof strategy of Theorem 1.2 is different from the classical Li-Yau trick \cite{19}. In \cite{19}, two-sided Gaussian bounds on the heat kernel are obtained by the Li-Yau gradient estimate. However, in our case it seems to be impossible to adopt Li-Yau gradient estimate method directly in order to derive upper and lower bounds on the $f$-heat kernel on complete smooth metric measure spaces. In our approach, Gaussian bounds on the $f$-heat kernel rely on the Moser’s parabolic Harnack inequality and the integral estimate of the $f$-heat kernel due to Davies \cite{12}, thus our arguments are similar to the ones of Saloff-Coste \cite{34,35,36} and Grigor’yan \cite{15}. Please see Section 4 for a detailed discussion.

In the second part of this paper, we will investigate various $L^p$-Liouville theorems for $f$-harmonic functions on complete noncompact metric measure space $(M,g,e^{-f}dv)$ under different assumptions on $Ric_f$ and $f$.

We first start recalling a $L^p$-Liouville theorem for positive $f$-subharmonic functions when $1 < p < \infty$, which extends the result in the classical case in \cite{46}. This was originally proved in \cite{32}; see also \cite{33}.

**Theorem 1.4** (Pigola, Rigoli and Setti \cite{32}). Let $(M,g,e^{-f}dv)$ be an $n$-dimensional complete smooth metric measure space. For any $1 < p < \infty$, there does not exist any nonconstant, nonnegative, $L^p(\mu)$-integrable $f$-subharmonic function.

We now deal with the $L^p$-Liouville theorem in case of $0 < p < 1$. In this case we obtain an analogous result to that obtained by Li and Schoen in \cite{21}. See Subsection 6.1 for a detailed discussion.

**Theorem 1.5.** Let $(M,g,e^{-f}dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. Assume that $f$ is bounded, and there exists a constant $\delta(n) > 0$ depending only on $n$, such that, for some point $o \in M$, the Bakry-Émery Ricci curvature satisfies

$$Ric_f \geq -\delta(n)r^{-2}(x),$$

whenever the distance from $o$ to $x$, $r(x)$, is sufficiently large. Then any nonnegative $L^p(\mu)$-integrable ($0 < p < 1$) $f$-subharmonic function must be identically zero.

Finally, motivated by the P. Li’s work \cite{19} and X.-D. Li’s generalization \cite{23}, we obtain a new $L^1$-Liouville theorem on smooth metric measure spaces.

**Theorem 1.6.** Let $(M,g,e^{-f}dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. Assume that $f$ is bounded, and there exists a constant $C > 0$, such that, for some point $o \in M$, the Bakry-Émery Ricci curvature satisfies

$$Ric_f \geq -C(1 + r^2(x)),$$

where $r(x)$ denotes the distance from $o$ to $x$. Then any nonnegative $L^1(\mu)$-integrable $f$-subharmonic function must be identically constant.

Theorem 1.6 partially answers to a question posed by X.-D. Li (see Subsection 8.6 in \cite{23}). Its proof is similar to the arguments of \cite{19}, where a critical step is the usage of the upper Gaussian bound on the $f$-heat kernel proved in Theorem 1.2. A detailed discussion shall be carried out in Subsection 6.2.

**Remark 1.7.** We remark that the absolute value of a $f$-harmonic function is a nonnegative $f$-subharmonic. Therefore we can conclude that a complete metric measure space does not admit any nonconstant $L^p(\mu)$-integrable $f$-harmonic function under the same hypotheses of Theorems 1.4, 1.5 and 1.6 respectively.
**Remark 1.8.** As many recent authors said in [13], [40] and [44], if the condition on $f$ bounded is replaced by $|\nabla f|$ bounded, then similar results to Theorems 1.5 and 1.6 can be immediately obtained by modifying the arguments of [23]. Indeed, the conditions $\text{Ric}_f \geq -(n-1)K$ and $|\nabla f| \leq a$ imply that

$\text{Ric}_f^N \geq -(n-1) \left( K + \frac{a^2}{N(n-1)} \right)$.

The rest of this paper is organized as follows. In Section 2, we present a local $f$-volume doubling property, a local $f$-Neumann Poincaré inequality and a local Sobolev inequality on complete smooth metric measure spaces. After that, following the arguments of Saloff-Coste [34] or Grigor’yan [15], we establish a Moser’s version of parabolic Harnack inequality. In Section 3, using the parabolic Harnack inequality, we obtain a Hölder continuity estimate for the $f$-heat equation, which implies a strong Liouville theorem. In Section 4, we prove two-sided Gaussian bounds on the $f$-heat kernel on complete smooth metric measure spaces. In Section 5, we derive a $f$-mean value inequality on complete smooth metric measure spaces, which is similar to the case of harmonic functions on a manifold, obtained by Li and Schoen [21]. In Section 6, we establish $L^p$-Liouville theorems on complete smooth metric measure spaces by following the ideas in [19] and [21].

### 2. Poincaré, Sobolev and Harnack inequalities

Let $\Delta_f = \Delta - \nabla f \cdot \nabla$ be the $f$-Laplacian on a complete smooth metric measure space $d\mu = e^{-f}dv$ on a complete Riemannian manifold. For a set $\Omega$, we will denote by $V(\Omega)$ the volume, and by $V_f(\Omega)$ the $f$-volume of $\Omega$. Throughout this section, we will assume

$\text{Ric}_f \geq -(n-1)K$ and $|f(x)| \leq A$

for some nonnegative constants $K$ and $A$. Under these assumptions, we have the validity of the $f$-Laplacian and $f$-volume comparison results.

**Lemma 2.1** (Wei and Wylie [42]). Let $(M, g, e^{-f}dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. If $\text{Ric}_f \geq -(n-1)K$ and $|f(x)| \leq A$ for some nonnegative constants $K$ and $A$, then along any minimizing geodesic starting from $x \in M$ we have

$\Delta_f r(x,y) \leq (n-1+4A)\sqrt{K} \coth \sqrt{Kr}$

for any $0 < r < R$, where $r(x,y) := d(x,y)$ is the distance function. Hence along any minimizing geodesic starting from $x \in M$ we have

$$\frac{V_f(B_x(r_2))}{V_f(B_x(r_1))} \leq \frac{V_{n+4A}^n(K r_2)}{V_{n+4A}^n(K r_1)}$$

(2.1)

for any $0 < r_1 < r_2 < R$. Here $V_{n+4A}^n(K r)$ is the volume of the radius $r$-ball in the model space $M_{n+4A}^n(K)$, the simply connected model space of dimension $n+4A$ with constant curvature $K$.

From (2.1), we easily deduce that

$V_f(B_x(2r)) \leq 2^{n+4A}e^{C(n,A)\sqrt{K}r} \cdot V_f(B_x(r))$

(2.2)

for any $0 < r < R$. This inequality implies that the local $f$-volume doubling property holds. This property will play a crucial role in our paper. We say that
a complete smooth metric measure space \((M, g, e^{-f} dv)\) admits a local \(f\)-volume doubling property if for any fixed \(0 < R < \infty\), there exists a constant \(C(R)\) such that
\[
V_f(B_x(2r)) \leq C(R) \cdot V_f(B_x(r))
\]
for any \(0 < r < R\) and \(x \in M\). Note that, when \(K = 0\), the above inequality holds with \(R = +\infty\), and it called the global \(f\)-volume doubling property.

By Lemma 2.1 following the Buser’s proof [5] or the Saloff-Coste’s alternate proof (Theorem 5.6.5 in [36]), we can easily get a local Neumann Poincaré inequality in the setting of smooth metric measure spaces.

**Lemma 2.2.** Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional complete noncompact smooth metric measure space, and denote by \(r(x)\) the distance function from a fixed origin \(o \in M\). If \(\text{Ric}_f \geq -(n-1)K\) and \(|f|(x) \leq A\) for some nonnegative constants \(K\) and \(A\), then
\[
(2.3) \quad \int_{B_o(r)} |\varphi - \varphi_{B_o(r)}|^2 e^{-f} dv \leq e^{c_1(1+\sqrt{Kr})} \cdot r^2 \int_{B_o(r)} |\nabla \varphi|^2 e^{-f} dv
\]
for any \(x \in M\) such that \(0 < r(x) < R\) and \(\varphi \in C^\infty(B_o(r))\), where \(\varphi_{B_o(r)} := V_f^{-1}(B_o(r)) \int_{B_o(r)} \varphi e^{-f} dv\). The constant \(c_1\) depends only on the dimension \(n\) and \(A\).

**Remark 2.3.** Inequality (2.3) implies that a local \(f\)-Neumann Poincaré inequality holds. In [28], Munteanu and Wang proved a \(f\)-Neumann Poincaré inequality when \(\text{Ric}_f \geq 0\). In [29], they only obtained the validity of a \(f\)-Neumann Poincaré inequality uniformly at small scales. But in our case, the \(f\)-Poincaré inequality can hold on balls of any radius due to a stronger assumption on \(f\), which is a crucial step on the proof of \(L^1\)-Liouville theorem. Because in the course of proof of \(L^1\)-Liouville result, we need to let the radius of balls tend to infinity. Also note that when \(f\) is constant, (2.3) was obtained by Saloff-Coste (see (6) in [35] or Theorem 5.6.5 in [36]).

Combining Lemma 2.1 Lemma 2.2 and the argument in [34], we have a local Sobolev inequality, which is one of the key technical points needed to apply Moser’s iterative technique to derive parabolic Harnack inequalities for the \(f\)-heat equation.

**Lemma 2.4.** Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional complete noncompact smooth metric measure space. If \(\text{Ric}_f \geq -(n-1)K\) and \(|f|(x) \leq A\) for some nonnegative constants \(K\) and \(A\), then for any constant \(p > 2\), there exists a constant \(c_2\), depending on \(n\) and \(A\) such that
\[
(\int_{B_o(r)} |\varphi|^p e^{-f} dv)^{\frac{2}{p}} \leq \frac{e^{c_2(1+\sqrt{Kr})}}{V_f(B_o(r))^\frac{p}{2}} \int_{B_o(r)} (|\nabla \varphi|^2 + r^{-2} |\varphi|^2) e^{-f} dv
\]
for any \(x \in M\) such that \(0 < r(x) < R\) and \(\varphi \in C^\infty(B_o(r))\).

**Sketch proof of Lemma 2.4.** The proof is nearly the same as that of Theorem 2.1 in [34] or Theorem 3.1 in [35] except for our discussion with respect to the weighted measure \(e^{-f} dv\). When \(f\) is constant, this result was confirmed by Saloff-Coste [34] (see also Theorem 3.1 in [35]). We refer the reader to these papers for a nice proof.

□
Remark 2.5. In Lemma 2.4 the local Sobolev inequality is different from Munteanu-Wang’s Neumann Sobolev inequality (Lemma 3.3 in [29]). Here we mainly follow the arguments of Saloff-Coste [34] to derive the local Sobolev inequality, whereas Munteanu and Wang proved their local Neumann Sobolev inequality adapting the same arguments as in [13]. Note also that, while Munteanu and Wang [29] established the local Neumann Sobolev inequality only on the unit balls due to a weaker hypothesis on the oscillation of \( f \) on unit balls, our local Sobolev inequality holds on balls of any radius, due to a stronger assumption on \( f \).

We shall now present a result concerning the Harnack inequality for the \( f \)-heat equation, which is very much similar to the case when \( f \) is constant, obtained by Saloff-Coste [34] and Grigor’yan [15].

**Theorem 2.6.** Let \((M,g,e^{-f}dv)\) be an \( n \)-dimensional complete noncompact smooth metric measure space. Fix \( 0 < R \leq \infty \). Assume that (2.2) and (2.3) are satisfied up to this \( R \). Then there exist constants \( c_3 \) depending on \( n \) and \( A \) such that, for any \( B_o(r), \ o \in M, \ 0 < r < R \) and for any smooth positive solution \( u \) of the \( f \)-heat equation in the cylinder \( Q = B_o(r) \times (s - r^2, s) \), we have

\[
\sup_{Q_-} \{ u \} \leq e^{c_3(1 + Kr^2)} \cdot \inf_{Q_+} \{ u \},
\]

where \( Q_- := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s - \frac{1}{2}r^2) \) and \( Q_+ := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s) \).

**Sketch proof of Theorem 2.6** The proof is the weighted case of the arguments of [34] or [35]. Indeed this result follows from the standard Moser’s technique. Since the conditions of Theorem 2.6 imply a family of local Sobolev inequalities due to Lemma 2.4 combining the local volume doubling property, it is enough to run the Moser’s iteration procedure to prove Theorem 2.6 as explained in [34] or [35].

Combining Lemmas 2.1, 2.2 and Theorem 2.6 we immediately have that:

**Corollary 2.7.** Let \((M,g,e^{-f}dv)\) be an \( n \)-dimensional complete noncompact smooth metric measure space. If \( \text{Ric}_f \geq -(n-1)K \) and \( |f(x)| \leq A \) for some nonnegative constants \( K \) and \( A \), then there exist a constant \( c(n,A) \) such that, for any ball \( B_o(r), \ o \in M, \ 0 < r < R \) and for any smooth positive solution \( u \) of the \( f \)-heat equation in the cylinder \( Q = B_o(r) \times (s - r^2, s) \), we have

\[
\sup_{Q_-} \{ u \} \leq e^{c(n,A)(1 + Kr^2)} \cdot \inf_{Q_+} \{ u \},
\]

where \( Q_- := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s - \frac{1}{2}r^2) \) and \( Q_+ := B_o(\frac{1}{2}r) \times (s - \frac{1}{4}r^2, s) \).

**Remark 2.8.** In [37] and [17], Saloff-Coste and Grigor’yan have confirmed that the conjunction of the \( f \)-volume doubling property and the \( f \)-Neumann Poincaré inequality is equivalent to a parabolic Harnack inequality for the \( f \)-heat equation. Our novel feature in this section is that we take into account suitable weighted curvature condition which implies the validity of these inequalities.

3. Liouville theorem

In this section, we will apply the parabolic Harnack inequality to obtain a quantitative Hölder continuity estimate for a solution to the \( f \)-heat equation, and hence derive a strong Liouville property under some suitable assumptions on \( \text{Ric}_f \) and \( f \).

First, we give the Hölder continuity estimate for any solution of the \( f \)-heat equation. When \( f \) is constant this was established in Theorem 5.4.7 of [36].
Theorem 3.1. Under the same assumptions of Theorem 2.6, there exist $\theta \in (0, 1)$, $\alpha \in (0, 1)$ and $A_\kappa = 4 \theta^{-1} (1 - \kappa)^{-\alpha} > 1$, $\kappa \in (0, 1)$, such that any solution $u$ of the $f$-heat equation in $Q = B_0(r) \times (s - r^2, s)$, satisfies

$$\sup_{(y,t), (y',t') \in Q_\kappa} \left\{ \frac{|u(y,t) - u(y',t')|}{|t - t'|^{1/2} + d(y,y')^{\alpha}} \right\} \leq \frac{A_\kappa}{r^\alpha} \sup_{Q} |u|,$$

where $Q_\kappa := B_0(\kappa r) \times (s - \kappa r^2, s)$.

Proof. The proof is nearly the same as in [27] (see also [36]) which uses the parabolic Harnack inequality. For the reader’s convenience, we include a detailed proof of this result. For any non-negative solution $v$ of the $f$-heat equation in $Q$, by Theorem 2.6 we have

$$\frac{1}{V_f(Q_+)} \int_{Q_+} v d\bar{\mu} \leq \max_{Q_+} \{ v \} \leq e^{c(n,A)(1+Kr^2)} \min_{Q_+} \{ v \},$$

where $Q_- := B_0\left(\frac{1}{r^2}\right) \times (s - \frac{3}{2}r^2, s - \frac{1}{2}r^2)$ and $Q_+ := B_0\left(\frac{1}{r^2}\right) \times (s - \frac{1}{2}r^2, s)$, and where $V_f(Q_-)$ denotes the volume of $Q_-$ with respect to the space-time volume form $d\bar{\mu}$. Now we let $u$ be a solution, which is not necessarily non-negative, and let $M_u, m_u$ be the maximum and minimum of $u$ in $Q$. Similarly, let $M_u^+, m_u^+$ be the maximum and minimum of $u$ in $Q_+$. Define

$$\mu_u^- := \frac{1}{\mu(Q_-)} \int_{Q_-} v d\bar{\mu},$$

where $d\bar{\mu}$ denotes the natural product measure on $R \times M$: $d\bar{\mu} = dt \times d\mu$, and where $d\mu = e^{-f} dv$. Applying (3.1) to the non-negative solutions $M_u - u, u - m_u$ yields

$$M_u - \mu_u^- \leq e^{c(n,A)(1+Kr^2)}(M_u - M_u^+)$$

and

$$\mu_u^- - m_u \leq e^{c(n,A)(1+Kr^2)}(m_u^+ - m_u),$$

which imply that

$$(M_u - m_u) \leq e^{c(n,A)(1+Kr^2)}(M_u - m_u) - e^{c(n,A)(1+Kr^2)}(M_u^+ - m_u^+).$$

If we define the oscillations

$$\omega(u, Q) := M_u - m_u \quad \text{and} \quad \omega(u, Q_+) := M_u^+ - m_u^+$$

of $u$ over $Q$ and $Q_+$, then

$$\omega(u, Q) \leq \theta \omega(u, Q),$$

where we assume $e^{c(n,A)(1+Kr^2)} > 1$, and hence $\theta = 1 - e^{-c(n,A)(1+Kr^2)} \in (0, 1)$.

Now we consider $(y, t), (y', t') \in Q_\kappa$. Let

$$\rho = 2 \max\{ d(y,y'), \sqrt{t-t'} \}$$

with $t \geq t'$. Then $(y', t')$ belongs to $Q_0 := B_0(\rho) \times (t - \rho^2, t)$. We also define $\rho_i = 2 \rho_{i-1}, \rho_0 = \rho$ and $Q_i := B_0(\rho_i) \times (t - \rho_i^2, t)$ for all $i \geq 1$. We easily see that

$$(Q_i)_+ = Q_{i-1}.$$

Hence, as long as $Q_i$ is contained in $Q$, (3.2) yields

$$\omega(u, Q_{i-1}) \leq \theta \omega(u, Q_i) \quad \text{and} \quad \omega(u, Q_0) \leq \theta^i \omega(u, Q_i).$$

Below, we consider two cases. If $\rho \leq (1 - \kappa)r$, let $k$ be the integer such that

$$2^k \leq (1 - \kappa)r / \rho < 2^{k+1}.$$
Since \((y,t) \in Q_\kappa\), it follows that
\[
Q_k = B_y(2^k \rho) \times (t - 4^k \rho^2, t) \\
\subset B_y((1 - \kappa)r) \times (t - (1 - \kappa)^2 r^2, t) \\
\subset B_o(r) \times (s - r^2, s) = Q.
\]
Hence we have
\[
\omega(u, Q_0) \leq \theta^k \omega(u, Q) \leq \theta^{-1}(1 - \kappa)^{-\alpha} \left( \frac{2}{\rho} \right)^\alpha \omega(u, Q)
\]
with \(\alpha = -\log_2 \theta\). This implies
\[
\left| u(y,t) - u(y',t') \right| \leq \frac{A_\kappa}{r^\alpha} \sup_{B_0(r)} \{|u|\}
\]
and conclusion follows, where \(A_\kappa := 4\theta^{-1}(1 - \kappa)^{-\alpha}\). The second case is trivial.

Using the Harnack inequality and the Hölder continuity estimate, we immediately derive the following Liouville theorem.

**Corollary 3.2.** Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional complete noncompact smooth metric measure space. Assume that \(\text{Ric}_f \geq 0\) and \(|f(x)| \leq A\) for some nonnegative constant \(A\). Then any solution \(u\) of the \(f\)-harmonic equation which is bounded from below (or above) is constant. Moreover, there exists an \(\alpha \in (0, 1]\) such that any \(f\)-harmonic function \(u\) which satisfies

\[
\lim_{r \to \infty} \left( r^{-\alpha} \cdot \sup_{B_o(r)} \{|u|\} \right) = 0
\]

for some fixed \(o \in M\) is constant.

**Remark 3.3.** Corollary 3.2 was also proved by Munteanu and Wang [28]. We emphasize that this result can be regarded as a direct consequence of Theorem 1.1.

If \(f\) is constant and \(\alpha = 1\), then Corollary 3.2 returns to Cheng’s Liouville property in [11]. If \(f\) is constant, this case appeared in [36] (see also Theorem 4.3 in [34]).

**Proof of Corollary 3.2** We start to prove the first part of corollary. The conditions of corollary imply the parabolic Harnack inequality (Corollary 2.7, \(K = 0\)) and hence the corresponding elliptic Harnack inequality. Assume that \(u\) is a solution of the \(f\)-harmonic equation which is bounded from below (if \(u\) is bounded from above, we then consider \(-u\), which is still bounded from below). Let

\[
m(u) := \inf_{\overline{M}} \{|u|\}.
\]

Applying the elliptic Harnack inequality in the ball \(2B = B_o(2r)\) to the non-negative function \(v = u - m(u)\), we have that

\[
\sup_B \{|u - m(u)|\} \leq C(n, A) \cdot \inf_B \{|u - m(u)|\}.
\]

As the radius of \(B = B_o(r)\) tend to infinity, \(\inf_B \{|u - m(u)|\}\) tends to zero. Therefore we conclude that \(u = m(u)\) is constant.

Below we will prove the second part of corollary. Because \(u\) has sublinear growth by condition \((3.3)\), then \(\alpha\) can be taken in the interval \((0, 1)\). Let \(\alpha\) be as given by Theorem 3.1. Let \(u\) be a function satisfying \(\Delta_f u = 0\) and condition \((3.3)\). Fix
some \( x \in M \) and \( y \) such that \( d(x, y) \leq 1 \). Applying Theorem 3.3 to \( u \) in a ball \( B_R = B_o(R) \) with \( R \) so large that \( x, y \in \frac{1}{2}B_R \), we find that

\[
|u(x) - u(y)| \leq \frac{C}{R^a} \sup_{B_R} \{|u|\},
\]

where constant \( C \) is independent of \( R \). Since the above inequality holds for all \( R \) large enough, we can let \( R \) tend to infinity to obtain that \( |u(x) - u(y)| = 0 \). Since \( x, y \in M \) with \( d(x, y) \leq 1 \) are arbitrary and \( M \) is connected, we conclude that \( u \) must be constant. \( \square \)

4. Two-sided Gaussian bounds on \( f \)-heat kernel

In this section, we shall obtain upper and lower bound estimates for the \( f \)-heat kernel on complete noncompact metric measure space. The proof seems to be different from the classical discussion of Li and Yau in [22]. Our argument is similar to the discussion in Grigor’yan [15] and Saloff-Coste [34].

First, we show that the local \( f \)-Neumann Poincaré inequality and the the local \( f \)-volume doubling property imply a lower bound on the \( f \)-heat kernel. To achieve this, we begin with by the following important lemma.

**Lemma 4.1.** Under the same assumptions of Theorem 2.6, there exists a constant \( c_4 := c_4(n, A) \) such that, for any \( x, y \in B_o(\frac{1}{2}R) \), and any \( 0 < s < t < \infty \) and any non-negative solution \( u \) of the \( f \)-heat equation in \( M \times (0, \infty) \),

\[
\ln \left( \frac{u(x, s)}{u(y, t)} \right) \leq c_4 \left[ K + \frac{1}{R^2} + \frac{1}{s} \right] (t-s) + \frac{d^2(x, y)}{t-s}.
\]

**Sketch proof of Lemma 4.1.** Since Theorem 2.6 implies a parabolic Harnack inequality, it is sufficient to prove the above inequalities by carefully choosing different space-time solutions. Please see Corollary 5.4.4 in [36] or Corollary 5.4 in [35] for a detailed proof. \( \square \)

Using Lemma 4.1, we can get a lower bound on the \( f \)-heat kernel on complete metric measure spaces.

**Proposition 4.2.** Under the same assumptions of Theorem 2.6, there exists a constant \( c_5 := c_5(n, A) \) such that, for any \( x, y \in B_o(\frac{1}{2}R) \) and any \( 0 < t < \infty \), the \( f \)-heat kernel \( H(x, y, t) \) satisfies

\[
H(x, y, t) \geq H(x, x, t) \exp \left[ -c_5 \left( 1 + \frac{t}{R^2} + Kt + \frac{d^2(x, y)}{t} \right) \right].
\]

Moreover, there exists a constant \( c_6 := c_6(n, A) \) such that, for any \( x, y \in B_o(\frac{1}{2}R) \) and any \( 0 < t < R^2 \)

\[
H(x, y, t) \geq e^{-c_6(1+Kt)} \frac{V_f(B_x(\sqrt{t}))}{V_f(B_{\sqrt{t}}(\sqrt{t}))} \exp \left( -c_5 \frac{d^2(x, y)}{t} \right).
\]

**Proof.** The proof follows from that of Theorem 5.4.11 in [36] with minor modifications. In fact using Lemma 4.1, we let \( u(y, t) = H(x, y, t) \) with \( x \) fixed and \( s = t/2 \) and then we get (4.1), where we used the fact that \( H(x, x, t) \) is non-increasing.

Below we prove (4.2). Note that the conditions of the proposition imply a parabolic Harnack inequality, which leads to the on-diagonal \( f \)-heat kernel lower bound
for all $x \in M$ and $0 < t < R^2$. Indeed we fix $0 < t < R$ and consider $\phi$ be a smooth function such that $0 \leq \phi \leq 1$, $\phi = 1$ on $B := B_2(\sqrt{t})$ and $\phi = 0$ on $M \setminus 2B$. Define

$$u(y,t) = \begin{cases} P_t \phi(y) & \text{if } t > 0 \\ \phi(y) & \text{if } t \leq 0, \end{cases}$$

$P_t = e^{t \Delta_f}$ be the heat semigroup of $\Delta_f$ on $L^2(M,\mu)$. Obviously, $u(y,t)$ satisfies $(\partial_t - \Delta_f)u = 0$ on $B \times (-\infty, \infty)$. Applying the parabolic Harnack inequality, first to $u$, and then to the $f$-heat kernel $(y,s) \to H(x,y,s)$, we have

$$1 = u(x,0) \leq e^{c(1+Kt)} u(x,t/2)$$

$$= e^{c(1+Kt)} \int_{B(x,\sqrt{t})} H(x,y,t/2) \phi(y) d\mu(y)$$

$$\leq e^{c(1+Kt)} \int_{B(x,2\sqrt{t})} H(x,y,t/2) d\mu(y)$$

$$\leq e^{2c(1+Kt)} V_f(B_x(2\sqrt{t})) H(x,x,t)$$

$$\leq e^{2c(1+Kt)} V_f(B_x(\sqrt{t})) 2^{n+4A} e^{C(n,A) \sqrt{Kt}} H(x,x,t),$$

where in the last inequality we used (2.2). This gives (4.3) as desired. Hence (4.2) then easily follows by (4.1) and (4.3).

Secondly, we can show that the local $f$-Neumann Poincaré inequality and the local $f$-volume doubling property also imply an upper bound on the $f$-heat kernel. To achieve this, the following integral estimate is critically useful due to Davies [12].

**Lemma 4.3** (Davies [12]). Let $(M,g,e^{-f} dv)$ be an $n$-dimensional complete smooth metric measure space. Let $\lambda_1 > 0$ be the bottom of the $L^2$-spectrum of the $f$-Laplacian. Assume that $B_1$ and $B_2$ are bounded subsets of $M$. Then

$$\int_{B_1} \int_{B_2} H(x,y,t) d\mu(y) d\mu(x) \leq e^{-\lambda_1 t} V_f(B_1)^{1/2} V_f(B_2)^{1/2} \exp \left( -\frac{d^2(B_1,B_2)}{4t} \right),$$

where $d(B_1,B_2)$ denotes the distance between the sets $B_1$ and $B_2$.

We now give an upper bound on the fundamental solution of the $f$-heat equation.

**Proposition 4.4.** Under the same assumptions of Theorem 2.6 there exist constants $c_7$ and $c_8$ such that, for any $x,y \in B_o(\frac{1}{2} R)$ and $0 < t < R^2/4$, the $f$-heat kernel $H(x,y,t)$ satisfies

$$H(x,y,t) \leq e^{c_8(1+Kt)} V_f(B_x(\sqrt{t})) \exp \left( -c_7 \frac{d^2(x,y)}{t} \right).$$

**Proof.** Fix a fixed $y \in B_o(x)$ and $\delta > 0$, applying Lemma 4.1 to the positive solution $u(x,t) = H(x,y,t)$ by taking $s = t$ and $t = (1+\delta)t$,

$$H(x,y,t) \leq H(x',y,(1+\delta)t) \cdot \exp \left\{ c_4 \left[ \left( K + \frac{1}{R^2} + \frac{1}{t} \right) \delta t + \frac{d^2(x,x')}{\delta t} \right] \right\}. $$
Integrating over \( x' \in B_x(\sqrt{t}) \) gives
\[
H(x, y, t) \leq \exp \left[ c_4 \left( (K + R^{-2}) \delta t + \delta + \frac{1}{\delta} \right) \right] V_f^{-1}(B_x(\sqrt{t}))
\]
\[
\times \int_{B_x(\sqrt{t})} H(x', y, (1 + \delta)t) d\mu(x').
\]
(4.5)

Applying Lemma 4.1 and the same argument to the positive solution
\[
u(y, t) = \int_{B_x(\sqrt{t})} H(x', y, t) d\mu(x'),
\]
by taking \( s = (1 + \delta)t \) and \( t = (1 + 2\delta)t \), we obtain
\[
\int_{B_x(\sqrt{t})} H(x', y, (1 + \delta)t) d\mu(x') \leq \exp \left[ c_4 \left( (K + R^{-2}) \delta t + \delta + \frac{1}{\delta} \right) \right] V_f^{-1}(B_y(\sqrt{t}))
\]
\[
\times \int_{B_y(\sqrt{t})} \int_{B_x(\sqrt{t})} H(x', y', (1 + 2\delta)t) d\mu(x') d\mu(y').
\]

Substituting this into (4.5) yields
\[
H(x, y, t) \leq \exp \left[ 2c_4 \left( (K + R^{-2}) \delta t + \delta + \frac{1}{\delta} \right) \right] V_f^{-1}(B_x(\sqrt{t})) V_f^{-1}(B_y(\sqrt{t}))
\]
\[
\times \int_{B_y(\sqrt{t})} \int_{B_x(\sqrt{t})} H(x', y', (1 + 2\delta)t) d\mu(x') d\mu(y').
\]

Combining this with Lemma 4.3 we have
\[
H(x, y, t) \leq \exp \left[ 2c_4 \left( (K + R^{-2}) \delta t + \delta + \frac{1}{\delta} \right) \right] - \lambda_1 t
\]
\[
\times V_f^{-1/2}(B_x(\sqrt{t})) V_f^{-1/2}(B_y(\sqrt{t}))) \exp \left[ -\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4(1 + 2\delta)t} \right].
\]
(4.6)

Notice that if \( d(x, y) \leq 2\sqrt{t} \), then \( d(B_x(\sqrt{t}), B_y(\sqrt{t})) = 0 \) and hence
\[
-d^2(B_x(\sqrt{t}), B_y(\sqrt{t})) = 0 \leq 1 - \frac{d^2(x, y)}{4(1 + 2\delta)t}
\]
and if \( d(x, y) > 2\sqrt{t} \), then \( d(B_x(\sqrt{t}), B_y(\sqrt{t})) = d(x, y) - 2\sqrt{t} \) hence
\[
-\frac{d^2(B_x(\sqrt{t}), B_y(\sqrt{t}))}{4(1 + 2\delta)t} = -\frac{(d(x, y) - 2\sqrt{t})^2}{4(1 + 2\delta)t} \leq -\frac{d^2(x, y)}{4(1 + 2\delta)t} + \frac{1}{2\delta}
\]
Therefore in any case, (4.6) becomes
\[
H(x, y, t) \leq \exp \left[ 1 + 2 \left( c_4 + \frac{1}{4} \right) \left( (K + R^{-2}) \delta t + \delta + \frac{1}{\delta} \right) \right] - \lambda_1 t
\]
\[
\times V_f^{-1/2}(B_x(\sqrt{t})) V_f^{-1/2}(B_y(\sqrt{t}))) \exp \left( -\frac{d^2(x, y)}{4(1 + 2\delta)t} \right).
\]
(4.7)

Now we want to estimate \( (K + R^{-2}) \delta t + \delta + \frac{1}{\delta} \) in (4.7). Let
\[
\delta = \min \left\{ \epsilon, \left( (K + R^{-2})t \right)^{-1/2} \right\}.
\]
If \( \left( (K + R^{-2})t \right)^{-1/2} \leq \epsilon \), then
\[
(K + R^{-2}) \delta t + \delta + \frac{1}{\delta} \leq 2 \left( (K + R^{-2})t \right)^{1/2} + \epsilon.
\]
If \([ (K + R^{-2})t ]^{-1/2} > \epsilon\), then we have
\[
(K + R^{-2})\delta t + \delta + \frac{1}{\delta} \leq \left[ (K + R^{-2})t \right] \epsilon + \epsilon + \frac{1}{\epsilon}.
\]
\[
\leq \left[ (K + R^{-2})t \right]^{1/2} + \epsilon + \frac{1}{\epsilon}.
\]
Hence, in either case, the right hand side of (4.7) can be estimate by
\[
H(x, y, t) \leq \exp \left[ 1 + 2 \left( c_4 + \frac{1}{4} \right) \left( 2 \left[ (K + R^{-2})t \right]^{1/2} + \epsilon + \frac{1}{\epsilon} \right) - \lambda_1 t \right]
\times V_f^{-1/2}(B_x(\sqrt{t}))V_f^{-1/2}(B_y(\sqrt{t}))\exp \left( -\frac{d^2(x, y)}{4(1 + 2\epsilon)t} \right).
\]
Moreover the volume doubling property implies (see, e.g., Lemma 5.2.7 in [36]) that
\[
V_f(x, \sqrt{t}) \leq C(n, A) \exp \left( C(n, A)\sqrt{Kt} \cdot \frac{d(x, y)}{\sqrt{t}} \right) V_f(y, \sqrt{t})
\leq C(n, A) \exp \left( \bar{C}(n, A, \epsilon)Kt + \frac{d^2(x, y)}{8(1 + 2\epsilon)t} \right) V_f(y, \sqrt{t}).
\]
Substituting this into (4.8) and using \(0 < t < R^2/4\), then the theorem follows. \(\Box\)

Combining Lemmas [21, 22] and Propositions [12, 14] immediately yields two-sided \(f\)-heat kernel bounds on complete noncompact metric measure spaces.

**Theorem 4.5.** Let \((M, g, e^{-f}dv)\) be an \(n\)-dimensional complete noncompact smooth metric measure space. If \(\text{Ric}_f \geq -(n-1)K\) and \(|f(x)| \leq A\) on \(B_o(2R)\) for some nonnegative constants \(K\) and \(A\), then there exist positive constants \(c_i\), \(i = 5, 6, 7, 8\), depending on \(n\) and \(A\) such that the \(f\)-heat kernel \(H(x, y, t)\) satisfies
\[
e^{-c_6(1+Kt)} \frac{d^2(x, y)}{t} \leq H(x, y, t) \leq e^{c_7(1+Kt)} \frac{d^2(x, y)}{t} \exp \left( -c_8d^2(x, y) \right)
\]
for any \(x, y \in B_o(R/2)\) and \(0 < t < R^2/4\).

**Remark 4.6.** In [37] and [17], Saloff-Coste and Grigor’yan have proved that the conjunction of the \(f\)-volume doubling property and the \(f\)-Neumann Poincaré inequality is equivalent to the two-sided \(f\)-heat kernel bounds, whereas we give concrete weighted curvature condition to achieve these estimates.

### 5. \(f\)-Mean Value Inequality

In this section, the main objective is to derive a mean value inequality on complete noncompact metric measure space, which is a natural generalization of the Li-Schoen’s result in [21]. First, we give the following Poincaré inequality.

**Theorem 5.1.** Let \((M, g, e^{-f}dv)\) be a complete noncompact smooth metric measure space. Let \(o \in M\) and \(R > 0\). If \(\text{Ric}_f \geq -(n-1)K\) and \(|f(x)| \leq A\) for some nonnegative constants \(K\) and \(A\), then for any \(\alpha \geq 1\), there exists constants \(C_3\) and \(C_4\) depending only on \(\alpha\), \(n\) and \(A\) such that
\[
\int_{B_o(R)} |\phi|^{\alpha} d\mu \leq C_3 \left( \frac{R}{1 + \sqrt{K}R} \right)^{\alpha} e^{C_4(1+\sqrt{K}R)} \int_{B_o(R)} |\nabla \phi|^{\alpha} d\mu
\]
for any compactly supported function $\phi$ on $B_o(R)$. In particular, the first Dirichlet eigenvalue $\mu_1$ of $f$-Laplacian on $B_o(R)$ satisfies

$$\mu_1 \geq C_3^{-1} \left( \frac{R}{1 + \sqrt{KR}} \right)^{-2} e^{-C_4(1 + \sqrt{KR})}.$$  

**Sketch proof of Lemma 5.1.** The proof is exactly the same as that of Corollary 1.1 proved by Li-Schoen [21] except that the classical Laplacian comparison is replaced by the generalized Laplacian comparison (see Lemma 2.1)

$$\Delta_f r(x) \leq (n - 1 + 4A) \sqrt{K} \coth \sqrt{Kr}$$

$$\leq \frac{n - 1 + 4A}{r} + (n - 1 + 4A) \sqrt{K}.$$

Besides this, all the integration calculations should be done with respect to the new measure $\mu$. To save the length of paper, we omit details of the proof. □

We now proceed to derive the $L^2$ $f$-mean value inequality by Theorem 5.1, which is a weighted version of Li-Schoen’s result in [21].

**Theorem 5.2.** Let $(M, g, e^{-f} dv)$ be a complete noncompact smooth metric measure space. Assume that $\text{Ric}_f \geq -(n - 1)K$ with $|f(x)| \leq A$ for some nonnegative constants $K$ and $A$. Let $o \in M$ and $R > 0$, and let $u$ be a nonnegative $f$-subharmonic function defined on $B_o(R)$. There exists a constant $C_5$, depending only on $n$ and $A$ such that for any $\tau \in (0, 1/2)$ we have

$$\sup_{B_o((1 - \tau)R)} u^2 \leq e^{c(n,A)(1 + \sqrt{KR})V_f^{-1}(B_o(R))} \int_{B_o((1 - \tau)R)} u^2 d\mu.$$

**Proof.** The proof is similar to the Li-Schoen’s proof of Theorem 1.2 in [21]. We include it here for the reader’s convenience. Let $h$ be a harmonic function on $B_o((1 - 2^{-1}\tau)R)$ obtained by the solving the Dirichlet boundary problem

$$\Delta_f h = 0 \quad \text{on} \quad B_o((1 - \tau/2)R),$$

and

$$h = u \quad \text{on} \quad \partial B_o((1 - \tau/2)R).$$

Since $u$ is nonnegative, by the maximum principle, the function $h$ is positive on the ball $B_o((1 - 2^{-1}\tau)R)$. Moreover,

$$u \leq h \quad \text{on} \quad B_o((1 - \tau/2)R).$$

Using Lemmas 2.1, 2.2 and 2.4 by the Moser iteration argument as in [28], we have the following elliptic Harnack inequality

$$\sup_{B_o((1 - \tau)R)} h \leq e^{c(n,A)(1 + \sqrt{KR})} \inf_{B_o((1 - \tau)R)} h,$$

where $c$ depends only on $n$ and $A$. In particular,

$$\sup_{B_o((1 - \tau)R)} u^2 \leq \sup_{B_o((1 - \tau)R)} h^2$$

$$\leq e^{c(n,A)(1 + \sqrt{KR})} \inf_{B_o((1 - \tau)R)} h^2$$

$$\leq e^{c(n,A)(1 + \sqrt{KR})V_f^{-1}(B_o((1 - \tau)R))} \int_{B_o((1 - \tau)R)} h^2 d\mu.$$
Below we will estimate the $L^2(\mu)$-norm of $h$ in terms of the $L^2(\mu)$-norm of $u$. By the triangle inequality, we have

\begin{equation}
\int_{B_o((1-\tau)R)} h^2 d\mu \leq 2 \int_{B_o((1-\tau)R)} (h-u)^2 d\mu + 2 \int_{B_o((1-\tau)R)} u^2 d\mu \\
\leq 2 \int_{B_o((1-\tau/2)R)} (h-u)^2 d\mu + 2 \int_{B_o(R)} u^2 d\mu.
\end{equation}

(5.2)

Since $(h-u)$ vanishes on $\partial B_o((1-\tau/2)R)$ we can apply Theorem 5.1 to show that

\begin{equation}
\int_{B_o((1-\tau/2)R)} (h-u)^2 d\mu \leq C_3 R^2 e^{C_4(1+\sqrt{KR})} \int_{B_o((1-\tau/2)R)} |\nabla (h-u)|^2 d\mu
\end{equation}

where we have used the triangle inequality again. Since the Dirichlet integral of $h$ is least among all functions which coincide with $h$ on the boundary, from above we conclude that

\begin{equation}
\int_{B_o((1-\tau/2)R)} (h-u)^2 d\mu \leq \frac{4C_3 R^2 e^{C_4(1+\sqrt{KR})}}{(1+\sqrt{KR})^2} \int_{B_o((1-\tau/2)R)} |\nabla u|^2 d\mu.
\end{equation}

(5.3)

Now we use the fact that $u$ is $f$-subharmonic to estimate the Dirichlet integral of $u$ in terms of the $L^2$-norm of $u$. We have for any $\phi$ with compact support in $B_o(R)$

\[
0 \leq \int_{B_o(R)} \phi^2 u \Delta f u d\mu \\
= -\int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu + 2 \int_{B_o(R)} \phi u \langle \nabla \phi, \nabla u \rangle d\mu \\
\leq -\int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu + 2 \left( \int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu \right)^{1/2} \left( \int_{B_o(R)} u^2 |\nabla \phi|^2 d\mu \right)^{1/2}.
\]

Thus

\[
\int_{B_o(R)} \phi^2 |\nabla u|^2 d\mu \leq 4 \int_{B_o(R)} u^2 |\nabla \phi|^2 d\mu.
\]

We let $\phi(r(x))$ be a cut-off function given by a function of $r(x) = r(o, x)$ alone, such that $\phi(r) = 1$ on $B_o((1-\tau/2)R)$, $\phi(r) = 0$ on $\partial B_o(R)$, and satisfying

\[
|\nabla \phi| \leq \frac{c}{rR}.
\]

Then the above inequality becomes

\[
\int_{B_o((1-\tau/2)R)} |\nabla u|^2 d\mu \leq \frac{4c^2}{r^2 R^2} \int_{B_o(R)} u^2 d\mu.
\]
Combining this with (5.1), (5.2) and (5.3) yields (5.4)

\[
\sup_{B_o((1-\tau)R)} u^2 \leq C \left( \frac{32e^2C_3\tau^{-2}eC_3(1+\sqrt{R})}{(1+\sqrt{R})^2} + 2 \right) V_f^{-1}(B_o((1-\tau)R)) \int_{B_o(R)} u^2 d\mu \\
\leq C\theta^{-C_3(1+\sqrt{R})eC_3(1+\sqrt{R})} V_f^{-1}(B_o((1-\tau)R)) \int_{B_o(R)} u^2 d\mu
\]

for some new constants \( C_i = C_i(n, A), i = 6, 7, 8 \). To finish the proof, we also need to estimate the \( f \)-volume of \( B_o(R) \) in terms of the volume of \( B_o((1-\tau)R) \). Recall the bound for \( \Delta f r^2 \):

\[
\Delta f r^2 \leq 2(n+4A) + 2\sqrt{K} (n-1+4A)r,
\]

and hence

\[
\int_{B_o(t)} \Delta f r^2 d\mu \leq 2(n+4A)V_f(t) + 2\sqrt{K} (n-1+4A) \int_{B_o(t)} rd\mu,
\]

where \( V_f(t) = Vol_f(B_o(t)) \). By Green formula, since

\[
\int_{B_o(t)} \Delta f r^2 d\mu = \int_{\partial B_o(t)} \frac{\partial r^2}{\partial r} d\sigma = 2t \frac{\partial V_f(B_o(t))}{\partial t},
\]

then

\[
tV_f(t) \leq (n+4A)V_f(t) + \sqrt{K} (n-1+4A)tV_f(t).
\]

Hence the function \( t^{-(n+4A)}e^{-\sqrt{K}(n-1+4A)t}V_f(t) \) is decreasing in \( t \geq 0 \). Therefore

(5.5) \( V_f^{-1}(B_o((1-\tau)R)) \leq V_f^{-1}(B_o(R))\left(\frac{1}{1-\tau}\right)^{n+4A} \cdot e^{\sqrt{K}R\tau(n-1+4A)} \),

where \( 0 < \tau < 1/2 \). Combining this with (5.2) completes the proof of theorem. \( \square \)

In the following, we show that the \( L^p \) \( f \)-mean value inequality for any \( p \in (0, 2] \) is a formal consequence of that given in Theorem 5.2

**Theorem 5.3.** Under the same assumption of Theorem 5.2, for any \( p \in (0, 2] \), there exists a constant \( c \) depending only on \( n, p, \) and \( A \) such that

\[
\sup_{B_o((1-\tau)R)} u^p \leq \tau^{-c(1+\sqrt{R})} V_f^{-1}(R) \int_{B_o(R)} u^p d\mu
\]

for any \( \tau \in (0, 1/2) \), where \( V_f^{-1}(R) := V_f^{-1}(B_o(R)) \).

**Proof.** The proof is similar to the proof of Theorem 2.1 in [21]. However, for the sake of completeness, we include the details here. By Theorem 5.2 for any \( \delta \in (0, 1/2] \), \( \theta \in [1/2, 1-\delta] \), we have

\[
\sup_{B_o(\theta R)} u^2 \leq \delta^{-C_3(1+\sqrt{R})} V_f^{-1}((\theta + \delta)R) \int_{B_o((\theta+\delta)R)} u^2 d\mu.
\]

Since \( \theta + \delta \geq 1/2 \), this inequality implies

\[
\sup_{B_o(\theta R)} u^2 \leq \delta^{-C_3(1+\sqrt{R})} V_f^{-1}(2^{-1}R) \int_{B_o((\theta+\delta)R)} u^2 d\mu.
\]
We also note that
\[
\int_{B_\delta((\theta+\delta)R)} u^2 \, d\mu \leq \left( \sup_{B_\delta((\theta+\delta)R)} u^2 \right)^{1-p/2} \int_{B_\delta((\theta+\delta)R)} u^p \, d\mu.
\]
Hence
\[
\sup_{B_\delta(\theta R)} u^2 \leq \delta^{-C_5 (1+\sqrt{KR})} V_f^{-1} (2^{-1} R) \left( \sup_{B_\delta((\theta+\delta)R)} u^2 \right)^{1-p/2} \int_{B_\delta((\theta+\delta)R)} u^p \, d\mu.
\]
If we set
\[
M(\theta) := \sup_{B_\delta(\theta R)} u^2
\]
and
\[
N := V_f^{-1} (2^{-1} R) \int_{B_\delta(R)} u^p \, d\mu,
\]
we have shown
\[
M(\theta) \leq N \delta^{-C_5 (1+\sqrt{KR})} M(\theta + \delta)^{1-p/2}
\]
for any \(\delta \in (0, 1/2]\) and \(\theta \in [1/2, 1-\delta]\). Choosing
\[
\theta_0 = 1 - \tau \quad \text{and} \quad \theta_i = \theta_{i-1} + 2^{-i} \tau
\]
for \(i = 1, 2, 3, \ldots\), we have that
\[
M(\theta_{i-1}) \leq N \delta^{-C_5 (1+\sqrt{KR})} M(\theta_i)^\lambda,
\]
where \(\lambda = 1 - p/2\) and \(N_1 = N \delta^{-C_5 (1+\sqrt{KR})}\). Iterating yields
\[
M(\theta_0) \leq K \sum_{i=1}^{\infty} 2^{\lambda i} \delta^{C_5 (1+\sqrt{KR})} \lambda^{i-1} M(\theta_j)^\lambda
\]
for any \(j \geq 1\). Letting \(j\) tend to infinity yields
\[
M(\theta_0) \leq (\tau - C_9 (1+\sqrt{KR})) \left[ V_f^{-1} (2^{-1} R) \int_{B_\delta(R)} u^p \, d\mu \right]^{2/p},
\]
where \(C_9\) depends only on \(n, p\) and \(A\). By the definition of \(M(\theta_0)\), we have
\[
\sup_{B_\delta((1-\tau)R)} u^p \leq (\tau^{-2^{-1}p} C_9 (1+\sqrt{KR}) V_f^{-1} (2^{-1} R) \int_{B_\delta(R)} u^p \, d\mu.
\]
Finally, by the relation (5.5), i.e.,
\[
V_f^{-1} (2^{-1} R) \leq C(n, A) e^{C (1+\sqrt{KR})} V_f^{-1} (R),
\]
the theorem follows. \(\square\)

6. \(L^p\)-Liouville Theorem

In this section, we will study various \(L^p\)-Liouville theorems on complete noncompact smooth metric measure spaces. Our results extend the classical \(L^p\)-Liouville theorems obtained by Li and Schoen in [21] and P. Li [19] and their weighted versions proved by X.-D. Li in [23].
6. The $0 < p < 1$ case. For $0 < p < 1$, we have a new weighted version of Li-Schoen’s $L^p$-Liouville theorem in [21].

**Theorem 6.1.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional complete noncompact smooth metric measure space. Assume that $f$ is bounded, and there exists a constant $\delta(n) > 0$ depending only on $n$, such that, for some point $o \in M$, the Bakry-Émery Ricci curvature satisfies

$$\text{Ric}_f \geq -\delta(n)r^{-2}(x),$$

whenever the distance from $o$ to $x$, $r(x)$, is sufficiently large. Then any nonnegative $L^p(\mu)$-integrable ($0 < p < 1$) $f$-subharmonic function must be identically zero.

**Proof of Theorem 6.1.** The proof is similar to the arguments of Li and Schoen (see Theorem 2.5 in [21]). Since the arguments leading to Theorem 5.3 are local, by choosing more or less $\beta > 0$, the $\Delta_o$-mean value inequality

$$\int_{B(x,R)} u^p d\mu \leq 2c(1+\sqrt{K(x,5R/R)}V_{f}^{-1}(R))\int_{B_o(R)} u^p d\mu$$

for nonnegative $f$-subharmonic functions $u$ on $B_x(5R)$, where $\text{Ric}_f \geq -(n-1)K(x,5R)$ and $|f|(x) \leq A$ for some nonnegative constants $K$ and $A$ on $B_x(5R)$. Here the constant $c$ depends on $u$, $p$ and $A$. In the following, we will use (6.1) to show that $u$ must vanish at infinity if the nonnegative function $u$ is $f$-subharmonic on $M$ with $u \in L^p(\mu)$ ($0 < p < 1$). In fact, by the volume comparison theorem mentioned above, under the hypothesis on $\text{Ric}_f$ and $f$, $M$ must be of $f$-infinite volume and $u$ must be identically zero.

Let $x \in M$ and consider a minimal geodesic $\gamma$ joining $o$ to $x$ such that $\gamma(0) = o$ and $\gamma(T) = x$, where $T = r(o, x)$. We then define a set of values $\{t_i \in [0, T] \}_{i=0}^k$ satisfying

$$t_0 = 0, \quad t_1 = 1 + \beta, \quad \ldots, \quad t_i = 2 \sum_{j=0}^{i} \beta^j - 1 - \beta^i,$$

where $\beta > 1$ to be chosen later, and $t_k = 2 \sum_{j=0}^{k} \beta^j - 1 - \beta^k$ is the largest such value with $t_k < T$. We denote the points $x_i = \gamma(t_i)$ and they obviously satisfy

$$r(x_i, x_{i+1}) = \beta^i + \beta^{i+1}, \quad r(o, x_i) = t_i \quad \text{and} \quad r(x_k, x) < \beta^k + \beta^{k+1}.$$

Moreover, the set of geodesic balls $B_{x_i}(\beta^i)$ cover $\gamma([0, 2 \sum_{j=0}^{k} \beta^j - 1])$ and they have disjoint interiors. We now claim that

$$V_f(B_{x_i}(\beta^i)) \geq C \left( \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} \right) V_f(B_o(1))$$

for a fixed $\beta > 2/(2^{1/n} - 1)^{-1} > 1$. The proof of this claim essentially follows the arguments of Cheeger-Gromov-Taylor in [10]. For the sake of completeness, we will outline the proof of this claim again.

For each $1 \leq i \leq k$, a relative comparison theorem (see (4.10) in [12]) argument shows that

$$V_f(B_{x_i}(\beta^i)) \geq D_i \left[ V_f(B_{x_i}(\beta^i + 2\beta^{i-1})) - V_f(B_{x_i}(\beta^i)) \right] \geq D_i V_f(B_{x_{i-1}}(\beta^{i-1})), $$
where
\[ D_i = \frac{\int_0^{\beta^i} e^{K(x_i, \beta^i + 2\beta^{i-1})} \sinh^{n-1+4A} t dt}{\int_0^{(\beta^i+2\beta^{i-1})} e^{K(x_i, \beta^i + 2\beta^{i-1})} \sinh^{n-1+4A} t dt}, \]

since \( \text{Ric}_f \geq -(n - 1)K(x_i, \beta^i + 2\beta^{i-1}) \) and \( |f(x) \leq A \) for some nonnegative constants \( K \) and \( A \) on \( B_{x_i}(\beta^i + 2\beta^{i-1}) \). Iterating this inequality, we conclude that

\[ V_f(B_{x_n}(\beta^k)) \geq V_f(B_{o}(1)) \prod_{i=1}^{k} D_i. \]

Since \( r(o, x_i) = 2 \sum_{j=0}^{i} \beta^j - 1 - \beta^i \), the curvature assumption implies that

\[ \sqrt{K(x_i, \beta^i + 2\beta^{i-1})} \leq \sqrt{\delta(n)} \cdot \left( \frac{\beta - 1}{2\beta^i - \beta - 1} \right) \]

for sufficiently large \( i \). Hence

\[ \beta^i \sqrt{K(x_i, \beta^i + 2\beta^{i-1})} \leq \sqrt{\delta(n)} \cdot \frac{(\beta - 1)^{\beta^i}}{2\beta^i - \beta - 1} = \sqrt{\delta(n)} \cdot \frac{(\beta - 1)^{\beta}}{2 - \beta^i - \beta - 1} \]

which can be made arbitrarily small for a fixed \( \beta > 2/(2^{1/n} - 1)^{-1} > 1 \) by choosing \( \delta(n) \) to be sufficiently small. Hence \( D_i \) has the following approximation

\[ D_i \sim \frac{(\beta^i)^{n+4A}}{(\beta^i + 2\beta^{i-1})^{n+4A} - (\beta^i)^{n+4A}} = \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} \]

by simply approximating \( \sinh t \) with \( t \). Hence (6.2) follows by combining (6.3).

In the following, we shall estimate \( V_f(B_x(\beta^{k+1})) \). We achieve it by two cases.

Case 1: \( r(x, x_k) \leq \beta^k(\beta - 1) \). In this case, we see that

\[ B_{x_k}(\beta^k) \subset B_x(\beta^{k+1}) \]

and hence

\[ V_f(B_{x_k}(\beta^k)) \leq V_f(B_x(\beta^{k+1})). \]

Combining this with (6.2), we conclude that

\[ V_f(B_x(\beta^{k+1})) \geq C \left( \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} \right)^k V_f(B_{o}(1)). \]

Case 2: \( r(x, x_k) > \beta^k(\beta - 1) \). In this setting, we see that

\[ B_{x_k}(\beta^k) \subset B_x(r(x, x_k) + \beta^k) \setminus B_x(r(x, x_k) - \beta^k). \]

Using a relative comparison theorem, we have that

\[ V_f(B_x(\beta^k)) \geq D \left[ V_f(B_x(r(x, x_k) + \beta^k)) - V_f(B_x(r(x, x_k) - \beta^k)) \right] \geq D \cdot V_f(B_{x_k}(\beta^k)), \]
where

\[
D = \frac{\int_0^{\beta_k} \sqrt{K(x,r(x,x_k) + \beta^k)} \sinh^{n-1+4A} t \, dt}{\int (r(x,x_k) + \beta^k) \sqrt{K(x,r(x,x_k) + \beta^k)} \sinh^{n-1+4A} t \, dt}
\]

Argument as above, since

\[
(r(x,x_k) + \beta^k) \sqrt{K(x,r(x,x_k) + \beta^k)} \leq (\beta^{k+1} + 2\beta^k) \sqrt{K(x,r(x,x_k) + \beta^k)}
\]

\[
\leq \frac{\sqrt{\delta(n)}}{2} : \beta (\beta - 1)
\]

can be made sufficiently small, we can approximate \(D\) by

\[
D \sim \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A}}.
\]

Combining this with (6.2) yields

\[
V_f(B_x(\beta^{k+1})) \geq 
C \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A}} \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}}^{k+1} V_f(B_o(1))
\]

\[
\geq \tilde{C} \frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}}^k V_f(B_o(1)),
\]

where \(\tilde{C}\) depends on \(n, A\) and \(\beta\). In any case, (6.4) is valid.

If we let \(x \to \infty\), the value \(k \to \infty\). Note that the choice of \(\beta\) ensures that

\[
\frac{\beta^{n+4A}}{(\beta + 2)^{n+4A} - \beta^{n+4A}} > 1,
\]

and hence the right hand side of (6.4) tends to infinity.

On the other hand, let us now apply to any point \(x\) sufficiently far from \(o\). The assumption of theorem asserts that \(R \sqrt{K(x,5R)}\) is bounded from above. Combining this fact with (6.1), we have

\[
u^p(x) \leq CV^{-1}_f(B_o(R)),
\]

where \(C\) also depends on the \(L^p\)-norm of \(u\). Using the value \(R = \beta^{k+1}\) in (6.5), the right hand side of (6.5) vanishes as \(x \to \infty\), thus proving that \(u(x) \to 0\) as \(x \to \infty\) and Theorem 6.1 follows by the maximum principle. \(\square\)

6.2. The \(p = 1\) case. The proof of this case is more complex. We shall follow the arguments of Li [19] (see also [20] and [23]) to derive the following result.

**Theorem 6.2.** Let \((M,g,e^{-f} dv)\) be an \(n\)-dimensional complete noncompact smooth metric measure space. Assume that \(f\) is bounded, and there exists a constant \(C > 0\), such that, for some point \(o \in M\), the Bakry-Émery Ricci curvature satisfies

\[\text{Ric}_f \geq -C(1 + r^2(x)),\]

where \(r(x)\) denotes the distance from \(o\) to \(x\). Then any nonnegative \(L^1(\mu)\)-integrable \(f\)-subharmonic function must be identically constant.

Following the trick of P. Li [19] (see also [23]) to prove Theorem 6.2, at first, we need the following integration by parts formula.
Theorem 6.3. Under the same assumptions of Theorem 6.2, for any nonnegative $L^1(\mu)$-integrable $f$-subharmonic function $g$, we have

$$\int_M \Delta_f g H(x, y, t) g(y) d\mu(y) = \int_M H(x, y, t) \Delta_f g(y) d\mu(y).$$

Proof of Theorem 6.3. Similar to the proof of Theorem 1 in [19] (see also Theorem 6.1 in [23]), applying the Green formula on $B_o(R)$, we have

$$\left| \int_{B_o(R)} \Delta_f g H(x, y, t) g(y) d\mu(y) - \int_{B_o(R)} H(x, y, t) \Delta_f g(y) d\mu(y) \right|$$

$$= \left| \int_{\partial B_o(R)} \frac{\partial}{\partial r} H(x, y, t) g(y) d\mu_{\sigma, R}(y) - \int_{\partial B_o(R)} H(x, y, t) \frac{\partial}{\partial r} g(y) d\mu_{\sigma, R}(y) \right|$$

$$\leq \int_{\partial B_o(R)} |\nabla H|(x, y, t) g(y) d\mu_{\sigma, R}(y) + \int_{\partial B_o(R)} H(x, y, t) |\nabla g|(y) d\mu_{\sigma, R}(y),$$

where $\mu_{\sigma, R}$ denotes the weighted area measure induced by $\mu$ on $\partial B_o(R)$. In the following we shall prove that the above two boundary integrals vanish as $R \to \infty$, which can be achieved by five steps.

Step 1. In Theorem 6.3 we show that any nonnegative subharmonic function $g(x)$ must satisfy

$$\sup_{B_o(R)} g(x) \leq e^{c(1+R \sqrt{K(R)})} V_f^{-1}(2R) \int_{B_o(2R)} g(y) d\mu(y)$$

for some constant $c = c(n, A)$, where $-(n-1)K(R)$ is the lower bound of the Bakry-Émery Ricci curvature on $B_o(4R)$ and $|f| \leq A$. Applying our theorem assumption, we have the estimate

$$\sup_{B_o(R)} g(x) \leq C e^{cR^2} V_f^{-1}(2R) \|g\|_{L^1(\mu)}$$

for some constants $\alpha := \alpha(n, A)$ and $C := C(n, A)$. Consider $\phi(y) = \phi(r(y))$ to be a nonnegative cut-off function such that $0 \leq \phi \leq 1$, $|\nabla \phi| \leq \sqrt{3}$ and

$$\phi(r(y)) = \begin{cases} 
1 & \text{on } B_o(R + 1) \setminus B_o(R), \\
0 & \text{on } B_o(R - 1) \cup (M \setminus B_o(R + 2)).
\end{cases}$$

Since $g$ is $f$-subharmonic function, by the Schwarz inequality we have

$$0 \leq \int_M \phi^2 g \Delta_f g d\mu = - \int_M \nabla(\phi^2 g) \nabla g d\mu$$

$$= -2 \int_M \phi g \nabla \phi \nabla g d\mu - \int_M \phi^2 |\nabla g|^2 d\mu$$

$$\leq 2 \int_M |\nabla \phi|^2 g^2 d\mu - \frac{1}{2} \int_M \phi^2 |\nabla g|^2 d\mu.$$
Then using the definition of $\phi$ and \ref{6.6}, we have that
\[
\int_{B_o(R+1)\backslash B_o(R)} |\nabla g|^2 d\mu \leq 4 \int_M |\nabla \phi|^2 g^2 d\mu \leq 12 \int_{B_o(R+2)} g^2 d\mu
\]
\[
\leq 12 \sup_{B_o(R+2)} g \cdot \|g\|_{L^1(\mu)}
\]
\[
\leq \frac{C e^{\alpha(R+2)^2}}{V_f(2R+4)} \cdot \|g\|^2_{L^1(\mu)}.
\]

On the other hand, using the Schwarz inequality, we get
\[
\int_{B_o(R+1)\backslash B_o(R)} |\nabla g| d\mu \leq \left( \int_{B_o(R+1)\backslash B_o(R)} |\nabla g|^2 d\mu \right)^{1/2} \cdot [V_f(R+1)\backslash V_f(R)]^{1/2}
\]
\[
\leq \left( \int_{B_o(R+1)\backslash B_o(R)} |\nabla g|^2 d\mu \right)^{1/2} \cdot V_f(2R+4)^{1/2}.
\]

Combining the above two inequalities, we have
\[
(6.7) \quad \int_{B_o(R+1)\backslash B_o(R)} |\nabla g| d\mu \leq C_{10} e^{\alpha R^2} \cdot \|g\|_{L^1(\mu)},
\]
where $C_{10} = C_{10}(n, A)$.

**Step 2.** We first estimate the $f$-heat kernel $H(x, y, t)$. Recall that, by Theorem \ref{4.5}, the $f$-heat kernel $H(x, y, t)$ satisfies
\[
H(x, y, t) \leq \frac{e^{cs(1+K(R)t)}}{V_f(B_x(\sqrt{t}))} \exp \left(-c_7 \frac{d^2(x, y)}{t} \right)
\]
for all $x, y \in B_o(R)$ and $0 < t < R^2/8$, where $-(n-1)K(R)$ is the lower bound of the Bakry-Émery Ricci curvature on $B_o(2R)$. Here the constants $c_7$ and $c_8$ depending on $n$ and $A$. Combining this with the assumption of our theorem, we deduce that
\[
(6.8) \quad H(x, y, t) \leq \frac{C}{V_f(B_x(\sqrt{t}))} \exp \left(-c_7 \frac{d^2(x, y)}{t} + \alpha R^2 t \right)
\]
for all $x, y \in B_o(R)$ and $0 < t < R^2/8$. Then combining \ref{6.7} with \ref{6.8} gives
\[
J_1 := \int_{B_o(R+1)\backslash B_o(R)} H(x, y, t)|\nabla g|(y) d\mu(y)
\]
\[
\leq \left( \sup_{y \in B_o(R+1)\backslash B_o(R)} H(x, y, t) \right) \int_{B_o(R+1)\backslash B_o(R)} |\nabla g| d\mu
\]
\[
\leq \frac{C_{11} \|g\|_{L^1(\mu)}}{V_f(B_x(\sqrt{t}))} \cdot \exp \left[-c_7 \frac{(R - d(o, x))^2}{t} + \alpha R^2 t + \alpha R^2 \right],
\]
where $C_1 = C_1(n, A)$. Note that
\[
-c \left( R - d(o, x) \right)^2 + \alpha R^2 t + \alpha R^2 \\
= \left(\alpha t + \alpha - \frac{ct}{t}\right) R^2 + 2\alpha t R d(o, x) - c \frac{d^2(o, x)}{t} \\
\leq \left(\alpha t + \alpha - \frac{ct}{t}\right) R^2 + \frac{ct}{2t} R^2 + c \frac{d^2(o, x)}{t} \\
= \left(\alpha t + \alpha - \frac{ct}{2t}\right) R^2 + c \frac{d^2(o, x)}{t}.
\]
Thus, for $T$ sufficiently small and for all $t \in (0, T)$ there exists some fixed constant $\beta > 0$ such that
\[
J_1 \leq C_1 \|g\|_{L^1(\mu)} V_f^{-1}(B_x(\sqrt{t})) \cdot \exp \left( -\beta R^2 t + c \frac{d^2(o, x)}{t} \right).
\]
Hence, for all $t \in (0, T)$ and all $x \in M$, $J_1$ tends to zero as $R$ tends to infinity.

**Step 3.** Below we shall estimate the gradient of $H$. Here we adapt the Li’s proof trick (see Section 18 in [20]). Consider the integral with respect to $d\mu$:
\[
\int_M \phi^2(y)|\nabla H|^2(x, y, t) = -2 \int_M \langle H(x, y, t)\nabla \phi(y), \phi(y)\nabla H(x, y, t) \rangle \\
- \int_M \phi^2(y)H(x, y, t)\Delta f H(x, y, t) \\
\leq 2 \int_M |\nabla \phi|^2(y) H^2(x, y, t) + \frac{1}{2} \int_M \phi^2(y)|\nabla H|^2(x, y, t) \\
- \int_M \phi^2(y)H(x, y, t)\Delta f H(x, y, t).
\]
This implies
\[
\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 \leq \int_M \phi^2(y)|\nabla H|^2(x, y, t) \\
\leq 4 \int_M |\nabla \phi|^2 H^2 - 2 \int_M \phi^2 H \Delta f H \leq 12 \int_{B_o(R+2) \setminus B_o(R-1)} H^2 + 2 \int_{B_o(R+2) \setminus B_o(R-1)} H |\Delta f H| \\
\leq 12 \int_{B_o(R+2) \setminus B_o(R-1)} H^2 + 2 \left( \int_{B_o(R+2) \setminus B_o(R-1)} H^2 \right)^{\frac{1}{2}} \left( \int_M (\Delta f H)^2 \right)^{\frac{1}{2}}.
\]
It is known that the heat semi-group is contractive in $L^1(\mu)$, hence
\[
\int_M H(x, y, t)d\mu(y) \leq 1.
\]
Using this and (6.8), we can estimate
\[
\int_{B_o(R+2) \setminus B_o(R-1)} H^2(x, y, t)d\mu \leq \sup_{y \in B_o(R+2) \setminus B_o(R-1)} H(x, y, t) \\
\leq \frac{C_{12}}{V_f(B_x(\sqrt{t}))} \times \exp \left[ -c \left( R - 1 - d(o, x) \right)^2 \right] + \alpha R^2 t.
\]
Also, we claim that there exists a constant $C_{13} > 0$ such that

$$
(6.11) \quad \int_M (\Delta_f H)^2(x, y, t) d\mu \leq \frac{C_{13}}{t^2} H(x, x, t).
$$

To prove this inequality, we first derive the inequality for any Dirichlet $f$-heat kernel $H$ defined on a compact subdomain of $M$. Using the fact that $f$-heat kernel on $M$ can be obtained by taking limits of Dirichlet $f$-heat kernels on a compact exhaustion of $M$, then (6.11) follows. Indeed, if $H(x, y, t)$ is a Dirichlet $f$-heat kernel on a compact subdomain $\Omega \subset M$, using the eigenfunction expansion, then $H(x, y, t)$ can be written as the form

$$
H(x, y, t) = \sum_{i} e^{-\lambda_i t} \psi_i(x) \psi_i(y),
$$

where $\{\psi_i\}$ are orthonormal basis of the space of $L^2(\mu)$ functions with Dirichlet boundary value satisfying the equation

$$
\Delta_f \psi_i = -\lambda_i \psi_i.
$$

Differentiating with respect to the variable $y$, we have

$$
\Delta_f H(x, y, t) = -\sum_{i} \lambda_i e^{-\lambda_i t} \psi_i(x) \psi_i(y).
$$

Noticing that $s^2 e^{-2s} \leq C_{13} e^{-s}$ for all $0 \leq s < \infty$, therefore

$$
\int_M (\Delta_f H)^2 d\mu(y) \leq C_{13} t^{-2} \sum_{i} e^{-\lambda_i t} \psi_i^2(x) = C_{13} t^{-2} H(x, x, t)
$$

and claim (6.11) follows. Now combining (6.9), (6.10) and (6.11), we obtain

$$
\int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 d\mu \leq C_{14} \left[ V_f^{-1} + t^{-1} V_f^{-1/2} H^+(x, x, t) \right] \times \exp \left[ -c_7 \frac{(R - 1 - d(o, x))^2}{2t} + \alpha R^2 t \right],
$$

where $V_f := V_f(B_x(\sqrt{t}))$. Applying Schwarz inequality, we have

$$
(6.12) \quad \int_{B_o(R+1) \setminus B_o(R)} |\nabla H| d\mu \leq [V_f(B_o(R + 1)) \setminus V_f(B_o(R))]^{1/2}
\times \left[ \int_{B_o(R+1) \setminus B_o(R)} |\nabla H|^2 d\mu \right]^{1/2}
\leq V_f^{1/2} (B_o(R + 1)) \left[ V_f^{-1} + t^{-1} V_f^{-1/2} H^+(x, x, t) \right]^{1/2}
\times \exp \left[ -c_7 \frac{(R - 1 - d(o, x))^2}{4t} + \alpha R^2 t \right].
$$
Therefore, by \((6.6)\), \((6.12)\) and Schwarz inequality we see that
\[
J_2 := \int_{B_o(R+1) \setminus B_o(R)} |\nabla H(x, y, t)| g(y) d\mu(y)
\]
\[
\leq \sup_{y \in B_o(R+1) \setminus B_o(R)} g(y) \cdot \int_{B_o(R+1) \setminus B_o(R)} |\nabla H(x, y, t)| d\mu(y)
\]
\[
\leq \frac{C \epsilon^2 (R+1)^2 \|g\|_{L^1(\mu)}}{V_f(B_o(2R+2))} \cdot V_f^{1/2}(B_o(R+1)) \left[ V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} \tilde{H}^{\frac{1}{2}}(x, x, t) \right]^{1/2}
\]
\[
\times \exp \left[ -c_7 \frac{(R - 1 - d(o, x))^2}{4t} + \frac{\alpha}{2} R^2 t \right]
\]
\[
\leq \frac{C \epsilon^2 (R+1)^2 \|g\|_{L^1(\mu)}}{V_f(B_o(2R+2))} \cdot \left[ V_f^{-1} + t^{-1} V_f^{-\frac{1}{2}} \tilde{H}^{\frac{1}{2}}(x, x, t) \right]^{1/2}
\]
\[
\times \exp \left[ -c_7 \frac{(R - 1 - d(o, x))^2}{4t} + \frac{\alpha}{2} R^2 t + 2\alpha R^2 \right],
\]
where \(V_f := V_f(B_o(\sqrt{t}))\). Similar to the discussion of \(J_1\), by choosing \(T\) sufficiently small, for all \(t \in (0, T)\) and all \(x \in M\), \(J_2\) also tends to zero when \(R\) tends to infinity.

**Step 4.** Recall that the co-area formula states that for all \(h \in \mathcal{C}_0^\infty(M)\),
\[
\int_{B_o(R+1) \setminus B_o(R)} h(y) d\mu(y) = \int_R^{R+1} \left[ \int_{\partial B_o(r)} h(y) d\mu_{\sigma, r}(y) \right] dr,
\]
where \(\mu_{\sigma, r}\) denotes the weight area-measure induced by \(\mu\) on \(\partial B(o, r)\). By the mean value theorem, for any \(R > 0\) there exists \(\bar{R} \in (R, R+1)\) such that
\[
J := \int_{B_o(R)} [H(x, y, t)|\nabla g|(y) + |\nabla H|(x, y, t)g(y)] d\mu_o(\bar{R})(y)
\]
\[
= \int_{B_o(R+1) \setminus B_o(R)} [H(x, y, t)|\nabla g|(y) + |\nabla H|(x, y, t)g(y)] d\mu(y)
\]
\[
= J_1 + J_2.
\]
By step 2 and step 3, we know that by choosing \(T\) sufficiently small, for all \(t \in (0, T)\) and all \(x \in M\), \(J\) tends to zero as \(\bar{R}\) (and hence \(R\)) tends to infinity. Therefore we complete Theorem 6.3 for \(T\) sufficiently small.

**Step 5.** At last, using the semigroup property of the \(f\)-heat equation, we have
\[
\frac{\partial}{\partial(s+t)} \left( e^{(s+t)\Delta_f} g \right) = \frac{\partial}{\partial t} \left( e^{s\Delta_f} e^{t\Delta_f} g \right) = e^{s\Delta_f} \frac{\partial}{\partial t} \left( e^{t\Delta_f} g \right)
\]
\[
eq e^{s\Delta_f} e^{t\Delta_f} (\Delta_f g) = e^{(s+t)\Delta_f} (\Delta_f g)
\]
which implies Theorem 6.3 for all time \(t\).

\(\square\)

Now we can finish the proof of Theorem 6.2 following the idea in [19].

**Proof of Theorem 6.2.** Let \(g(x)\) be a nonnegative, \(L^1\)-integrable and \(f\)-subharmonic function defined on \(M\). Now we define
\[
g(x, t) := \int_M H(x, y, t) g(y) d\mu(y)
\]
with \( g(x, 0) = g(x) \). By Theorem 6.3

\[
\frac{\partial}{\partial t} g(x, t) = \int_M \frac{\partial}{\partial t} H(x, y, t)g(y) d\mu(y)
\]

\[
= \int_M \Delta f_y H(x, y, t)g(y) d\mu(y)
\]

\[
= \int_M H(x, y, t)\Delta f_y g(y) d\mu(y) \geq 0.
\]

Therefore we confirmed \( g(x, t) \) is increasing for all \( t \). On the other hand, under the assumption of our theorem, by Lemma 2.1 in [29] we conclude that

\[
V_f(B_o(R)) \leq C e^{c(n)R^2}
\]

holds for all \( R > 0 \), where \( C > 0 \) is a constant depending on \( A \) and \( B_o(1) \). Hence

\[
\int_1^\infty \frac{R}{\log V_f(B_o(R))} dR = \infty.
\]

By Grigor’yan’s result in [16] (see also Theorem 3.13 in [17]), this implies

\[
\int_M H(x, y, t) d\mu(y) = 1
\]

for all \( y \in M \) and \( t > 0 \). To finish our theorem, this equality implies

\[
\int_M g(x, t) d\mu(x) = \int_M \int_M H(x, y, t)g(y) d\mu(y) d\mu(x) = \int_M g(y) d\mu(y).
\]

Since \( g(x, t) \) is increasing in \( t \), we conclude that \( g(x, t) = g(x) \) and hence \( \Delta f g(x) = 0 \). On the other hand, for any positive constant \( a \), let us define a new function

\[ h(x) := \min\{g(x), a\}. \]

Then \( h \) satisfies

\[ 0 \leq h(x) \leq g(x), \quad |\nabla h| \leq |\nabla g| \quad \text{and} \quad \Delta f h(x) \leq 0. \]

In particular, it will satisfy the same estimates, (6.6) and (6.7), as \( g \). Hence we can show that

\[
\frac{\partial}{\partial t} \int_M H(x, y, t)h(y) d\mu(y) = \int_M H(x, y, t)\Delta f y h(y) d\mu(y) \leq 0.
\]

Note that \( h \) is still \( L^1 \), following the same argument as before, we have \( \Delta f h(x) = 0 \). By the regularity theory of \( f \)-harmonic functions, this is impossible unless \( h = g \) or \( h = a \). Since \( a \) is arbitrary and \( g \) is nonnegative, this implies \( g \) must be identically constant. \( \square \)

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