Scalar elliptic equations with a singular drift from a weak Morrey space

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Abstract

In this paper we prove the existence and uniqueness of weak solutions to the Dirichlet problem for an elliptic equation with a drift \( b \) satisfying \( \text{div} \ b \leq 0 \) in \( \Omega \). We assume \( b \) belongs to some weak Morrey class which includes in the 3D case, in particular, drifts having a singularity along the axis \( x_3 \) with the asymptotic \( b(x) \sim c/r \), where \( r = \sqrt{x_1^2 + x_2^2} \).

1 Introduction and Main Results

Assume \( \Omega \subset \mathbb{R}^n \) is a bounded simply connected domain, \( n \geq 3 \), and assume its boundary \( \partial \Omega \) is of class \( C^1 \). We consider the following boundary value problem:

\[
\begin{cases}
-\Delta u + b \cdot \nabla u = -\text{div} \ f & \text{in} \ \Omega, \\
u|_{\partial \Omega} = 0.
\end{cases}
\]  

(1.1)

Here \( u : \Omega \to \mathbb{R} \) is unknown, \( b : \Omega \to \mathbb{R}^n \) and \( f : \Omega \to \mathbb{R}^n \) are given functions. In particular, we are interested in the drift of type (without loss of generality we can assume \( \Omega \) contains the origin)

\( n = 3, \quad b(x) = -\alpha \frac{x'}{|x'|^2}, \quad x' = (x_1, x_2, 0), \quad \alpha \in \mathbb{R} \)  

(1.2)

Motivated by this example, we assume that the drift \( b \) satisfies the condition

\[ b \in L^{2, n-2}_w(\Omega) \]

(1.3)

where \( L^{b, \lambda}_w(\Omega) \) is the weak Morrey space equipped with the quasi-norm

\[ \|b\|_{L^{b, \lambda}_w(\Omega)} := \sup_{x_0 \in \Omega} \sup_{R < \text{diam} \Omega} R^{-\lambda} \|b\|_{L^{\rho, w}(B_R(x_0) \cap \Omega)} \]

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and $L_{p,w}(\Omega)$ is the weak Lebesgue space equipped with the quasi-norm

$$\|b\|_{L_{p,w}(\Omega)} := \sup_{s>0} s \left| \left\{ x \in \Omega : |b(x)| > s \right\} \right|^{\frac{1}{p}}$$ \hspace{1cm} (1.4)

Note that the scale of spaces $L_{w}^{p,n-p}(\Omega)$ is a critical one, i.e. these spaces are invariant under the scaling

$$b^R(x) = R b(Rx), \quad R > 0 \quad \Rightarrow \quad \|b^R\|_{L_{n-p}^{p}(B)} = \|b\|_{L_{w}^{n-p}(B_R)}$$

Concerning further properties of the weak Morrey spaces we refer to [26]. Here we emphasise only that the scale of weak Morrey spaces $L_{w}^{p,n-p}(\Omega)$ is convenient for the description of drifts $b$ which have a certain asymptotic near some singular submanifold $\Sigma \subset \Omega$. For example, for $n = 3$ we have

$$b(x) \sim \frac{1}{x^2}, \quad \Sigma = \text{a plane} \quad \Rightarrow \quad b \in L_{w}^{1,2}(\Omega)$$
$$b(x) \sim \frac{1}{\sqrt{x_1^2+x_2^2}}, \quad \Sigma = \text{a line} \quad \Rightarrow \quad b \in L_{w}^{2,1}(\Omega)$$
$$b(x) \sim \frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \quad \Sigma = \text{a point} \quad \Rightarrow \quad b \in L_{w}^{3,0}(\Omega) = L_{3,w}(\Omega)$$

We define the bilinear form $B[u, \eta]$ by

$$B[u, \eta] := \int_{\Omega} \eta \cdot \nabla u \ dx.$$ \hspace{1cm} (1.5)

Note that for $b$ satisfying (1.3) the bilinear form (1.5) generally speaking is not well-defined for $u \in W_{1}^{p}(\Omega)$, but it is well-defined at least for $u \in W_{1}^{1}(\Omega)$ with $p > 2$ and $\eta \in L_{q}(\Omega)$ with $q > \frac{2p}{p-2}$. So, instead of the standard notion of weak solutions from the energy class $W_{1}^{p}(\Omega)$ we introduce the definition of $p$-weak solutions to the problem (1.1), see also [12], [15], [20]:

**Definition 1.1.** Assume $p > 2$, $f \in L_{p}(\Omega)$ and $b$ satisfy (1.3). We say $u$ is a $p$-weak solution to the problem (1.1) if $u \in W_{1}^{1}(\Omega)$ and $u$ satisfies the identity

$$\int_{\Omega} \nabla u \cdot \nabla \eta \ dx + B[u, \eta] = \int_{\Omega} f \cdot \nabla \eta \ dx, \quad \forall \ \eta \in C_{0}^{\infty}(\Omega).$$ \hspace{1cm} (1.6)

In this paper we always assume that the vector field $b$ has a sign-defined divergence:

$$\text{div} \ b \leq 0 \quad \text{in} \quad \mathcal{D}'(\Omega),$$ \hspace{1cm} (1.7)

In the case (1.2) the condition (1.7) corresponds to $\alpha \geq 0$. The important impact of the condition (1.7) is due to the fact that in this case the quadratic form $B[u, u]$ provides a positive support to the quadratic form of the elliptic operator in (1.1) (see Lemma 1.1 below), while in the opposite case $\text{div} \ b \geq 0$ in $\Omega$ the quadratic form $B[u, u]$ is (formally) non-positive and hence it “shifts” the operator to the “spectral area”. For example, it is well-known that in the case when the condition (1.7) breaks the uniqueness for the problem (1.1) in the class of Hölder continuous (or even smooth) $p$-weak solutions does not hold (see, for instance, [1], [5]).
Lemma 1.1. Assume $b$ satisfies (1.3), (1.7) and $p > 2$. Then for any $u \in L_\infty(\Omega) \cap \dot{W}^1_p(\Omega)$ the quadratic form $B[u, u]$ is non-negative:
\[
B[u, u] \geq 0.
\]

Proof. The result follows from the inclusions $|u|^2 \in \dot{W}^1_p(\Omega)$ and $b \in L_{p'}(\Omega)$, $p' = \frac{p}{p-1}$, and the inequality
\[
B[u, u] = \frac{1}{2} \int_\Omega b \cdot \nabla |u|^2 \, dx \geq 0.
\]

\[\square\]

There are many papers devoted to the investigation of the problem (1.1) in the case of divergence free drift, [3], [4], [5], [6], [11], [23], [24], [25], [27], [28], [29], [31], [32] and references there. Papers devoted to the non-divergence free drifts are not so numerous (see [1], [12], [13], [14], [15], [17], [18], [19], [20], [21], [25] for the references). Our present contribution can be viewed as a multidimensional analogue of the results obtained in [1] in the 2D case for the drift $b$ satisfying (1.7).

The main results of the present paper are the following theorems:

Theorem 1.1. Assume $b$ satisfies (1.3), (1.7). Then there exist $p > 2$ depending only on $n$, $\Omega$ and $\|b\|_{L^2_n - 2(\Omega)}$ such that for any $f \in L_p(\Omega)$ there exists a unique $p$-weak solution $u \in W^1_p(\Omega)$ to the problem (1.1). Moreover, this solution satisfies the estimate
\[
\|u\|_{W^1_p(\Omega)} \leq c \|f\|_{L_p(\Omega)},
\]
with a constant $c > 0$ depending only on $\Omega$ and $\|b\|_{L^2_n - 2(\Omega)}$.

The next result is concerned with the boundedness of weak solutions.

Theorem 1.2. Assume $b$ satisfies (1.3), (1.7), $p > 2$, $q > n$ and $f \in L_q(\Omega)$. Then any $p$-weak solution $u \in W^1_p(\Omega)$ to the problem (1.1) is essentially bounded and satisfies the estimate
\[
\|u\|_{L_\infty(\Omega)} \leq c \|f\|_{L_q(\Omega)},
\]
with a constant $c > 0$ depending only on $\Omega$, $n$ and $q$.

We emphasise that Theorem 1.2 is valid only for $p$-weak solutions of the boundary-value problem (1.1) (or, more generally, under the boundary condition $u|_{\partial \Omega} = \varphi|_{\partial \Omega}$ where the boundary data $\varphi : \Omega \to \mathbb{R}$ are sufficiently smooth, say, $\varphi \in \text{Lip}(\Omega)$). Without any additional requirements on $b$ this theorem has no analogue in the local set-up (or, equivalently, for non-regular boundary data $\varphi \in W^1_p(\Omega)$), see counterexample [5, Example 3.1] with the divergence-free drift $b \in L^{2,n-2}(\Omega)$ in the dimension $n = 4$ and the discussion on this subject.

Our last result is the Hölder continuity of $p$-weak solutions. A priori estimates of the Hölder norm for Lipschitz continuous solutions to the problem (1.1) under the assumptions (1.7), (1.11) were obtained earlier in [25].

Theorem 1.3. Assume $b$ satisfies (1.3), (1.7) and assume additionally
\[
\exists r \in \left(\frac{n}{2}, n\right] : \quad b \in L^{r,n-2}(\Omega).
\]
Assume $q > n$. Then there exist $\mu \in (0, 1)$ depending on $n$, $q$, $\Omega$, and $\|b\|_{L^{r,n-r}(\Omega)}$ such that if $f \in L^q(\Omega)$ then the unique $p$-weak solution $u$ to the problem (1.1) is Hölder continuous on $\bar{\Omega}$ with the exponent $\mu$ and the estimate

$$\|u\|_{C^\mu(\bar{\Omega})} \leq c \|f\|_{L^q(\Omega)},$$

holds with the constant $c > 0$ depending on $q$, $n$, $\Omega$, and $\|b\|_{L^{r,n-r}(\Omega)}$.

Note that the additional requirement (1.11) is superfluous if $n = 3$ as it follows from (1.3). So, in the physical dimension $n = 3$ any $p$-weak solutions are H"older continuous if $b$ satisfies (1.3) and (1.7) only. In particular, $p$-weak solutions are Hölder continuous for the typical drift (1.2).

Here are some comments and possible extensions:

- The assumption $\partial \Omega$ is of class $C^1$ in the theorems above actually is made only for brevity and our proofs can be extended for more general domains. The assumption $\Omega$ is simply connected is needed in Proposition 5.5 below when we construct smooth approximations of the divergence free part of the drift. Actually we used this assumption in [1, Proposition 4.2].

- With minor changes in the proofs our technique allows us to obtain results similar to our Theorems 1.1–1.3 if we replace the Laplace operator in (1.1) by an elliptic operator in the divergence form $-\text{div}(a(x)\nabla u)$ with a uniformly elliptic matrix $a(x) = (a_{jk}(x))$, $a_{jk} \in L^\infty(\Omega)$. We consider this extension to be obvious. In the case of elliptic operators in the non-divergent form with rough coefficients we refer to [17], [18], [19].

- In the case of the Laplace operator in (1.1) (or, more generally, in the case of the operator $-\text{div}(a(x)\nabla u)$ with a sufficiently smooth matrix $a(x)$) our $p$-weak solutions in Theorem 1.1 possess locally integrable second derivatives $u \in W^{2,1}_{s,\text{loc}}(\Omega)$ with some $s > 1$ (if $f$ is sufficiently regular). Hence for $p$-weak solutions the equation (1.1) is valid a.e. in $\Omega$.

Our paper is organized as follows: in Section 2 we introduce some auxiliary results, in Section 3 we prove Theorems 1.1 and 1.2 in Section 4 we present the proofs of Theorems 1.3. In Section 5 we introduce some results concerning smooth approximation of functions from a weak Morrey space $L^{2,n-2}_{2}(\Omega)$. In Section 6 we sketch the proofs of some properties of certain De Giorgi classes which are convenient for the study of elliptic equations with coefficients from Morrey spaces.

In the paper we use the following notation. For any $a, b \in \mathbb{R}^n$ we denote by $a \cdot b = a_k b_k$ their scalar product in $\mathbb{R}^n$. Repeated indexes assume the summation from 1 to $n$. An index after comma means partial derivative with respect to $x_k$, i.e. $f_{,k} := \partial f / \partial x_k$. We denote by $L^p(\Omega)$ and $W^k_p(\Omega)$ the usual Lebesgue and Sobolev spaces. We do not distinguish between functional spaces of scalar and vector functions and omit the target space in notation. $C_0^\infty(\Omega)$ is the space of smooth functions compactly supported in $\Omega$. The space $W^{1}_{0,\text{loc}}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,1}_{\text{loc}}(\Omega)$ norm and $W^{-1}_{p}(\Omega)$ is the dual space for $W^{1}_{p}(\Omega)$ler, $p' = \frac{p}{p-1}$. The space of distributions on $\Omega$ is denoted by $\mathcal{D}'(\Omega)$. By $C^\mu(\bar{\Omega})$, $\mu \in (0, 1)$ we denote the spaces of Hölder continuous functions on $\bar{\Omega}$. The symbols $\rightarrow$ and $\rightharpoonup$ stand for the weak and strong convergence respectively. We denote by $B_R(x_0)$ the ball in $\mathbb{R}^n$ of radius $R$ centered at $x_0$ and
write $B_R$ if $x_0 = 0$. We write $B$ instead of $B_1$ and denote $S_1 := \partial B_1$. For a domain $\Omega \subset \mathbb{R}^n$ we also denote $\Omega_R(x_0) := \Omega \cap B_R(x_0)$. For $u \in L_{\infty}(\omega)$ we denote
\[
\text{osc } u := \text{esssup } u - \text{essinf } u.
\]

We denote by $L^{p,\lambda}(\Omega)$ the Morrey space equipped with the norm
\[
\|u\|_{L^{p,\lambda}(\Omega)} := \sup_{x_0 \in \Omega} \sup_{R < \text{diam } \Omega} R^{-\frac{\lambda}{p}} \|u\|_{L^p(\Omega_R(x_0))}
\]
and we denote by $BMO(\Omega)$ the space of functions with a bounded mean oscillation equipped with the semi-norm
\[
[u]_{BMO(\Omega)} := \sup_{x_0 \in \Omega} \sup_{R < \text{diam } \Omega} R^{-n} \int_{\Omega_R(x_0)} |u(x) - (u)_{\Omega_R(x_0)}| \, dx
\]
where $(f)_{\omega}$ stands for the average of $f$ over the domain $\omega \subset \mathbb{R}^n$. Also we denote by $f * g$ the convolution of functions $f, g : \mathbb{R}^n \to \mathbb{R}$:
\[
(f)_{\omega} := \int_{\omega} f \, dx = \frac{1}{|\omega|} \int f \, dx, \quad (f * g)(x) := \int_{\mathbb{R}^n} f(x - y)g(y) \, dy.
\]

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2 Auxiliary results

In this section we present several auxiliary results. The first result shows that by relaxing an exponent of the integrability of a function we can always switch from a weak Morrey norm to a regular one.

**Proposition 2.1.** For any \( r > 1 \) and \( r_1 \in [1, r) \) we the imbedding \( L^{r,n-r}(\Omega) \hookrightarrow L^{r_1,n-r_1}(\Omega) \) holds. In particular,

\[
\|b\|_{L^{r_1,n-r_1}(\Omega)} \leq c_n \|b\|_{L^{r,n-r}(\Omega)}
\]

**Proof.** The result follows from the Hölder inequality for Lorentz norms, see [7, Section 4.1]. \qed

The next proposition is the well-known extension of the result of C. Fefferman for \( p = 2 \) on the boundedness of a quadratic form on a Sobolev space.

**Proposition 2.2.** Assume \( p \in (1, n) \), \( r \in (1, \frac{n}{p}) \) and \( V \in L^{r,n-rp}(\Omega) \). Then

\[
\int_{\Omega} |V| |u|^p \, dx \leq c_{n,r,p} \|V\|_{L^{r,n-rp}(\Omega)} \|\nabla u\|^p_{L^p(\Omega)}, \quad \forall u \in C_0^\infty(\Omega).
\]

Moreover,

\[
\int_{B_R} |V| |\bar{u}|^p \, dx \leq c_{n,r,p} \|V\|_{L^{r,n-rp}(B_R)} \|\nabla u\|^p_{L^p(B_R)}, \quad \forall u \in W_1^p(B_R),
\]

where \((u)_{B_R} = \frac{1}{|B_R|} \int_{B_R} u \, dx\) and \(\bar{u} := u - (u)_{B_R}\).

**Proof.** The first statements is proved in [2]. The second statement can be obtained from the first one if we apply the standard extension operator for Sobolev functions

\[
T : W_1^p(\Omega) \to \tilde{W}_p^1(\Omega_0), \quad (Tu)|_{\Omega} = u, \quad \|Tu\|_{W_1^p(\Omega_0)} \leq c \|u\|_{W_1^p(\Omega)},
\]

where \(\Omega \subset \subset \Omega_0\). \qed

The next proposition can be viewed as complementary to the previous one because in a certain sense it concerns with the compactness of the bilinear form on a Sobolev space. Actually it explains the role of the additional condition \(1.11\) in Theorem 1.3. Indeed, while the higher integrability of \(\nabla u\) in Theorem 1.1 requires only boundedness of the bilinear form \(1.5\) (cf. Proposition 2.6 below), the Hölder continuity in Theorem 1.3 is based on the compactness (see our proof in Section 4 where we employ essentially the same idea that was used before in [25]). While the boundedness of the bilinear form (in the case of \( p = 2 \)) in Proposition 2.2 holds for \( V = |b|^2 \) for the whole range of Morrey spaces \( V \in L^{r,n-2r}(\Omega) \) with any \( r > 1 \), the compactness in Proposition 2.3 requires the restriction \( r > \frac{n}{2} \). So, we believe that the higher integrability result in a sense is more general and typically can be obtained for less restrictive assumptions on the coefficients of the elliptic operator than the results concerning local boundedness or local Hölder continuity of weak solutions.
Proposition 2.3. Assume \( r \in \left( \frac{n}{2}, +\infty \right) \) and \( b \in L_r(B_R) \) and denote \( m := \frac{2r}{2r-n} \).

Then for any \( \varepsilon > 0 \) there exists \( c_\varepsilon > 0 \) depending only on \( \varepsilon, r \) and \( n \) such that for any \( \zeta \in C^\infty_0(B_R) \) such that \( 0 \leq \zeta \leq 1 \) and for any \( u \in W^1_2(B_R) \) the following inequality holds:

\[
\int_{B_R} |b \cdot \nabla \zeta^m| |u|^2 \, dx \leq \varepsilon \| \zeta^m \nabla u \|_{L^2(B_R)}^2 + c_\varepsilon \left( \| \nabla \zeta \|_{L^\infty(B_R)}^2 + \| b \cdot \nabla \zeta^m \|_{L^2(B_R)} \right) \| u \|_{L^2(B_R)}^2
\]

Proof. By the Hölder inequality we obtain

\[
\int_{B_R} |b \cdot \nabla \zeta^m| |u|^2 \, dx \leq m \| b \cdot \nabla \zeta \|_{L_r(B_R)} \left( \int_{B_R} \zeta^{r(m-1)} |u|^{2r'} \, dx \right)^{1/r'}
\]

Denote \( s := \frac{2}{n-(n-2)r'} > 1 \). For the second integral by the Hölder inequality we obtain

\[
\int_{B_R} \zeta^{r(m-1)} |u|^{2r'} \, dx = \int_{B_R} |u|^{\frac{2}{s} \zeta^{r(m-1)}} |u|^{2r' - \frac{2}{s}} \, dx \leq \|u\|_{L^2(B_R)}^2 \left( \int_{B_R} \zeta^{r(m-1)s'} |u|^{(2r' - \frac{2}{s})s'} \, dx \right)^{1/s'}
\]

Note that \( \frac{2}{s} = n-(n-2)r', \frac{2}{s} - \frac{2}{s'} = n(r'-1), \frac{m}{2 - \frac{2}{s}} = 2^* \equiv \frac{2n}{n-2} \) and \( r'(m-1)s' = \frac{m}{2 - \frac{2}{s}} 2^* \). So, by the Gagliardo-Nirenberg inequality

\[
\left( \int_{B_R} \zeta^{r(m-1)s'} |u|^{(2r' - \frac{2}{s})s'} \, dx \right)^{1/s'} = \left( \int_{B_R} |\zeta^m u|^{2^*} \, dx \right)^{1/s'} \leq \| \nabla (\zeta^m u) \|_{L^2(B_R)}^{2^*}
\]

Then we obtain

\[
\int_{B_R} |b \cdot \nabla \zeta^m| |u|^2 \, dx \leq m \| b \cdot \nabla \zeta \|_{L_r(B_R)} \| u \|_{L^2(B_R)}^{2r'} \| \nabla (\zeta^m u) \|_{L^2(B_R)}^{2^*}
\]

As \( \frac{2^*}{2r'} = \frac{n}{r} < 2 \) we can apply the Young inequality with exponents \( \frac{2r'}{n} \) and \((\frac{2r'}{n})' = m\).

Then taking into account \( \frac{2m}{2r'} = 2 \) we obtain

\[
\int_{B_R} |b \cdot \nabla \zeta^m| |u|^2 \, dx \leq \varepsilon \| \zeta^m \nabla u \|_{L^2(B_R)}^2 + c_\varepsilon \left( 1 + \| b \cdot \nabla \zeta^m \|_{L^2(B_R)} \right) \| u \|_{L^2(B_R)}^2
\]

The next two propositions are just technical tools connecting the behavior of weak Morrey norms of the drift \( b \) with the corresponding norms of the function \( V = |b|^q \).

Proposition 2.4. Assume \( b \in L^{2,n-2}_w(\Omega) \) and \( q \in (1,2) \). Denote \( V := |b|^q \). Then

\[
V \in L^{\frac{2}{q},n-2}_w(\Omega), \quad \frac{2}{q} \in (1,2),
\]

and

\[
\| V \|_{L^{\frac{2}{q},n-2}_w(\Omega)} \leq c_n \| b \|^q_{L^{2,n-2}_w(\Omega)}
\]
Proof. The proof is straightforward and we omit it. \qed

**Proposition 2.5.** Assume \( q \in (1, 2) \) and \( V \in L_{w}^{\frac{2}{n-2}}(\Omega) \). Then for any \( r \in (1, \frac{2}{q}) \) we have \( V \in L^{r,n-qr}(\Omega) \) and

\[
\|V\|_{L^{r,n-qr}(\Omega)} \leq c \|V\|_{L_{w}^{\frac{2}{n-2}}(\Omega)}.
\]

Proof. The proof follows from Proposition 2.4. \qed

The next proposition shows that for any \( b \in L_{w}^{2,n-2}(\Omega) \) and any \( p \)-weak solution \( u \) with \( p > 2 \) the drift term \( b \cdot \nabla u \) belongs to \( W_{p}^{-1}(\Omega) \). It is clear that this result can not be extended for \( p = 2 \) as in the case of the drift (1.2) the Hardy inequality fails for \( p = 2 \).

**Proposition 2.6.** Assume \( b \in L_{w}^{2,n-2}(\Omega) \) and \( p \in (2, +\infty) \), \( p' = \frac{p}{p-1} \). Then

\[
|B[u, \eta]| \leq c \|b\|_{L_{w}^{2,n-2}(\Omega)} \|
abla u\|_{L^{p}(\Omega)} \|
abla \eta\|_{L^{p'}(\Omega)}, \quad \forall u, \eta \in C_{0}^{\infty}(\Omega).
\]

where the constant \( c > 0 \) depends only on \( n \) and \( p \).

Proof. Denote \( q := p' \in (1, 2) \), \( V := |b|^q \) and take arbitrary \( r \in (1, \frac{2}{q}) \). Then by Propositions 2.4 and 2.5 we have \( V \in L^{r,n-qr}(\Omega) \) and

\[
\|V\|_{L^{r,n-qr}(\Omega)} \leq c \|b\|_{L_{w}^{2,n-2}(\Omega)}^q.
\]

Using the Hölder inequality and Proposition 2.2 we obtain

\[
|B[u, \eta]| \leq \|
abla u\|_{L^{p}(\Omega)} \left( \int_{\Omega} V|\eta|^q \, dx \right)^{1/q} \leq c \|V\|_{L^{r,n-qr}(\Omega)}^{1/q} \|
abla u\|_{L^{p}(\Omega)} \|
abla \eta\|_{L^{q}(\Omega)} \leq c \|b\|_{L_{w}^{2,n-2}(\Omega)} \|
abla u\|_{L^{p}(\Omega)} \|
abla \eta\|_{L^{q}(\Omega)}.
\]

\qed
3 Proof of Theorems 1.1 and 1.2

For technical reasons we start with the proof of Theorem 1.2. Note that technically it is more convenient to work with a bounded $p$-weak solution $u$ because in this case we can use $u$ as a test function in (1.6).

Proof. For $m > 0$ and we define a truncation $T_m : \mathbb{R} \to \mathbb{R}$ by $T_m(s) := m$ for $s \geq m$, $T_m(s) = -m$ for $s \leq -m$ and $T_m(s) = s$ for $|s| < m$. Now we fix some $m > 0$ and and denote $\bar{u} := T_m(u)$. Then for any $k \geq 0$ we have

$$(\bar{u} - k)_+ \in L_\infty(\Omega) \cap \dot{W}^{1,2}_0(\Omega), \quad \nabla(\bar{u} - k)_+ = \chi_{\Omega \setminus \bar{u} < m} \nabla u$$

Take $\eta = (\bar{u} - k)_+$ in (1.6). For $k \geq m$ we have $\eta \equiv 0$ and hence $B[u, \eta] = 0$. For $k < m$ the condition (1.7) implies

$$B[u, (\bar{u} - k)_+] = \frac{1}{2} \int_{\Omega[k < u < m]} b \cdot \nabla (\bar{u} - k)_+^2 dx + (m - k) \int_{\Omega[u \geq m]} b \cdot \nabla u dx =$$

$$= B[(\bar{u} - k)_+, (\bar{u} - k)_+] + (m - k) \int_{\bar{\Omega}} b \cdot \nabla (u - m)_+ \geq 0$$

Hence for $A_k := \{ x \in \Omega : \bar{u}(x) > k \}$ we obtain

$$\int_{\Omega} |\nabla (\bar{u} - k)_+|^2 dx \leq \|f\|_{L_q(\Omega)}^2 A_k^{1 - \frac{2}{q}}, \quad \forall k \geq 0$$

which implies (see [22 Chapter II, Lemma 5.3])

$$\operatorname{esssup}_\Omega \bar{u} \leq c_{n,q} |\Omega|^\delta \|f\|_{L_q(\Omega)}, \quad \delta := \frac{1}{n} - \frac{1}{q}.$$ 

As this estimate is uniform with respect to $m > 0$ we conclude $u$ is essentially bounded from above and

$$\operatorname{esssup}_\Omega u \leq c_{n,q} |\Omega|^\delta \|f\|_{L_q(\Omega)}.$$ 

Applying the same procedure to $-\bar{u}$ instead of $\bar{u}$ we obtain $u \in L_\infty(\Omega)$ as well as (1.10). □

**Proposition 3.1.** Assume $b$ satisfies (1.7), (1.3). Then for any $f \in L_p(\Omega)$ with $p > 2$ a $p$-weak solution to the problem (1.1), if exists, is unique.

Proof. Assume $u_1$ and $u_2$ are two $p$-weak solutions of (1.1) corresponding to the same $f \in L_p(\Omega)$. Then $u := u_1 - u_2$ is a $p$-weak solution of (1.1) corresponding to $f \equiv 0$. From Theorem 1.2 we obtain $w \equiv 0$ in $\Omega$. □

**Proposition 3.2.** Assume $b$ satisfies (1.7), (1.3), $p > 2$ and $f \in L_q(\Omega)$ with $q > n$. Then for any $p$-weak solution $u \in W^1_p(\Omega)$ to the problem (1.1) the following estimate hold:

$$\|u\|_{W^1_p(\Omega)} \leq c \|f\|_{L_q(\Omega)}$$

with some constant $c > 0$ depending only on $n$ and $\Omega$. 

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Proof. From Theorem 1.2 we obtain \( u \in L_\infty(\Omega) \) and hence the test function \( \eta := u \in L_\infty(\Omega) \cap \tilde{W}_p^1(\Omega) \) is admissible for (1.6). From Lemma 1.1 we obtain \( B[u, u] \geq 0 \) and hence from (1.6) we obtain

\[
\|\nabla u\|_{L_2(\Omega)}^2 \leq \int_{\Omega} f \cdot \nabla u \, dx
\]

\( \square \)

**Proposition 3.3.** Assume \( b \) satisfies (1.7), (1.3) and \( f \in L_q(\Omega) \) with \( q > n \). Then there exists \( p > 2 \) depending only on \( n, \Omega \) and \( \|b\|_{L_2^{2n-2}(\Omega)} \) such that for any \( p \)-weak solution \( u \in W^1_2(\Omega) \) to the problem (1.1) the estimate (1.9) holds with some constant \( c > 0 \) depending only on \( n, p, \Omega \) and \( \|b\|_{L_2^{2n-2}(\Omega)} \).

Proof. Let \( u \) be a \( p \)-weak solution of (1.1). From Theorem 1.2 we obtain \( u \in L_\infty(\Omega) \). First we consider an internal point \( x_0 \in \Omega \). Take \( R > 0 \) such that \( B_{2R}(x_0) \subset \Omega \) and choose a cut-off function \( \zeta \in C_0^\infty(B_{2R}(x_0)) \) so that \( \zeta \equiv 1 \) on \( B_{R}(x_0) \) and \( |\nabla \zeta| \leq c/R \). Denote \( \tilde{u} = u - (u)_{B_{2R}(x_0)} \). Then the function \( \eta = \zeta^2 \tilde{u} \) is admissible for (1.6). It is easy to see that

\[
B[u, \zeta^2 \tilde{u}] = B[\zeta \tilde{u}, \zeta \tilde{u}] - \frac{1}{2} \int_{B_{2R}(x_0)} |\tilde{u}|^2 b \cdot \nabla \zeta^2 \, dx
\]

Taking into account (1.8) we obtain

\[
\frac{1}{2} \|\zeta \nabla u\|^2_{L_2(B_{2R}(x_0))} \leq c \left( \|\tilde{u} \nabla \zeta\|^2_{L_2(B_{2R}(x_0))} + \|f\|^2_{L_2(B_{2R}(x_0))} \right) + \frac{1}{2} \int_{B_{2R}(x_0)} \zeta |\tilde{u}|^2 b \cdot \nabla \zeta \, dx
\]

The last term we estimate as

\[
\int_{B_{2R}(x_0)} \zeta |\tilde{u}|^2 b \cdot \nabla \zeta \, dx \leq \|\nabla \zeta\|_{L_\infty(B_{2R}(x_0))} \|\zeta \tilde{u}\|_{L_{\frac{2n}{n+2}}(B_{2R}(x_0))} \|b \tilde{u}\|_{L_{\frac{2n}{n+2}}(B_{2R}(x_0))}
\]

By imbedding theorem we obtain

\[
\|\zeta \tilde{u}\|_{L_{\frac{2n}{n+2}}(B_{2R}(x_0))} \leq c \left( \|\zeta \nabla u\|_{L_2(B_{2R}(x_0))} + \|u \nabla \zeta\|_{L_2(B_{2R}(x_0))} \right)
\]

Using the Young inequality we get

\[
\frac{1}{4} \|\zeta \nabla u\|^2_{L_2(B_{2R}(x_0))} \leq c \left( \|\tilde{u} \nabla \zeta\|^2_{L_2(B_{2R}(x_0))} + \|f\|^2_{L_2(B_{2R}(x_0))} \right) + c \|\nabla \zeta\|^2_{L_\infty(B_{2R}(x_0))} \|b \tilde{u}\|^2_{L_{\frac{2n}{n+2}}(B_{2R}(x_0))}
\]

Taking into account \( \zeta \equiv 1 \) on \( B_{R}(x_0) \) and \( \|\nabla \zeta\|_{L_\infty(B_{2R}(x_0))} \) \( \leq C/\infty \) we arrive at

\[
\|\nabla u\|_{L_2(B_{R}(x_0))} \leq \frac{c}{R} \|\tilde{u}\|_{L_2(B_{2R}(x_0))} + c \|f\|_{L_2(B_{2R}(x_0))} + \frac{c}{R} \|b \tilde{u}\|_{L_{\frac{2n}{n+2}}(B_{2R}(x_0))}
\]

Denote \( s = \frac{2n}{n+2} \) and \( V := |b|^s \). Then from Proposition 2.4 we get \( V \in L_{w}^{1+\frac{s}{n-2}}(\Omega) \). From Proposition 2.5 we obtain \( V \in L^{r,n-r_s}(\Omega) \) with any \( r \in (1, 1 + \frac{s}{n}) \) and

\[
\|V\|_{L^{r,n-r_s}(\Omega)} \leq c \|b\|^s_{L_{w}^{1+\frac{s}{n-2}}(\Omega)}
\]
From Proposition 2.2 we derive
\[ \| \tilde{u} \|_{L_{n+2}^{2n} (B_{2R}(x_0))} = \left( \int_{B_{2R}(x_0)} V |\tilde{u}|^q \, dx \right)^{1/q} \leq c_{n,r,q} \| V \|_{L_r^n \cap L_\infty} \| \nabla u \|_{L_2(B_{2R}(x_0))} \leq c \| b \|_{L_{n+2}^\infty} \| \nabla u \|_{L_2(B_{2R}(x_0))} \]

Using the imbedding theorem we obtain
\[ \| \tilde{u} \|_{L_2(B_{2R}(x_0))} \leq c \| \nabla u \|_{L_{n+2}^{2n} (B_{2R}(x_0))} \]

So, we arrive at
\[ \| \nabla u \|_{L_2(B_{2R}(x_0))} \leq c \| f \|_{L_2(B_{2R}(x_0))} + \frac{c}{R} \left( 1 + \| b \|_{L_{n+2}^\infty} \right) \| \nabla u \|_{L_{n+2}^{2n} (B_{2R}(x_0))} \]

Now we derive similar estimates near the boundary. Extend \( u \) and \( f \) by zero outside \( \Omega \) and denote this extension by \( \tilde{u} \) and \( \tilde{f} \). Note that as \( \partial \Omega \) is \( C^1 \)-smooth there exists \( R_0 > 0 \) depending on \( \Omega \) such that for any \( R < R_0 \) and any \( x \in \partial \Omega \cap B_R(x) \) \( \| B_R(x) \cap \Omega \| \geq \frac{1}{4} | B_R | \). Take \( x_0 \in \partial \Omega \), \( R < R_0 / 2 \) and denote \( \Omega_R(x_0) := B_R(x_0) \cap \Omega \). Choose a cut-off function \( \zeta \in C_0^\infty (B_{R_0}(x_0)) \) so that \( \zeta \equiv 1 \) on \( B_R(x_0) \). Then the function
\[ \eta := \zeta^2 \in L_\infty (\Omega_{R_0}(x_0)) \]
is admissible for the identity (1.6). After routine computations similar to [1, Theorem 2.1] for any \( x_0 \in \partial \Omega \) and \( R < R_0 \) we obtain
\[ \| \nabla u \|_{L_2(\Omega_{R_0}(x_0))} \leq c \| \tilde{f} \|_{L_2(\Omega_{R_0}(x_0))} + \frac{c}{R} \left( 1 + \| b \|_{L_{n+2}^\infty} \right) \| \nabla u \|_{L_{n+2}^{2n} (\Omega_{R_0}(x_0))} \]

Combining internal and boundary estimates in the standard way and dividing the result by \( R^{n/2} \) for any \( x_0 \in \bar{\Omega} \) and \( R < R_0 \) we obtain the reverse Hölder inequality for \( \nabla u \):
\[ \left( \int_{\Omega_{R_0}(x_0)} |\nabla u|^2 \, dx \right)^{1/2} \leq c_1 \left( \int_{\Omega_{R_0}(x_0)} |f|^2 \, dx \right)^{1/2} + c_2 \left( 1 + \| b \|_{L_{n+2}^\infty} \right) \left( \int_{\Omega_{R_0}(x_0)} |\nabla u|^{\frac{2n}{n+2}} \, dx \right)^{\frac{n+2}{2n}} \]

from which the higher integrability of \( u \) follows via Gehring’s lemma, see, for example, [8, Chapter V]. Namely, there exists \( p \in (2, q] \) and \( c_\star > 0 \) depending only on \( c_1, c_2 \) and \( \| b \|_{L_{n+2}^\infty} \) such that \( \nabla u \in L_p(\Omega) \) and the following estimate holds:
\[ \| \nabla u \|_{L_p(\Omega)} \leq c_\star \left( \| \nabla u \|_{L_2(\Omega)} + \| f \|_{L_p(\Omega)} \right) \]

Combining this estimate with Proposition 3.2 we obtain (1.9). \( \square \)

Now we can prove Theorem 1.1.
Proof. Let $p > 2$ be an exponent defined in Proposition 3.3 and denote $p' = \frac{p}{p-1}$. For $f \in L_p(\Omega)$ we can find $f_\varepsilon \in C_0^\infty(\Omega)$ such that $\|f_\varepsilon - f\|_{L_p(\Omega)} \to 0$. Using Proposition 5.5 from Section 5 we can find $b_\varepsilon \in C^\infty(\Omega)$ satisfying $\text{div } b_\varepsilon \leq 0$ in $\Omega$ such that

$$
\|b_\varepsilon\|_{L_w^{2,n-2}(\Omega)} \leq c \|b\|_{L_w^{2,n-2}(\Omega)}, \quad \|b_\varepsilon - b\|_{L_{p'}(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0.
$$

Let $u_\varepsilon \in C^\infty(\Omega)$ be a smooth solution to the regularized problem

$$
\begin{cases}
-\Delta u_\varepsilon + b_\varepsilon \cdot \nabla u_\varepsilon = -\text{div } f_\varepsilon & \text{in } \Omega, \\
u_\varepsilon|_{\partial \Omega} = 0.
\end{cases}
$$

From Proposition 3.3 we obtain the estimate

$$
\|u_\varepsilon\|_{W^{1,p}_w(\Omega)} \leq c \|f_\varepsilon\|_{L_p(\Omega)}
$$

with a constant $c > 0$ depending only on $n$, $p$, $\Omega$ and $\|b\|_{L_w^{2,n-2}(\Omega)}$. Hence we can take a subsequence $u^\varepsilon$ such that

$$
u_\varepsilon \rightharpoonup u \quad \text{in } W^{1,p}_w(\Omega).
$$

As $b_\varepsilon \to b$ in $L_{p'}(\Omega)$ we can pass to the limit in the integral identity

$$
\int_\Omega \nabla u_\varepsilon \cdot \left(\nabla \eta + b_\varepsilon \eta\right) \, dx = \int_\Omega f_\varepsilon \cdot \nabla \eta \, dx, \quad \forall \, \eta \in C_0^\infty(\Omega).
$$

and obtain (1.6). Hence $u \in \overset{\circ}{W}^{1,p}_w(\Omega)$ is a $p$-weak solution to the problem (1.1). Its uniqueness follows from Proposition 3.1. \hfill \Box
4 Proof of Theorem 1.3

First we introduce the modified De Giorgi classes which are convenient for the study of solutions to the elliptic equations with coefficients from Morrey spaces.

Definition 4.1. We say \( u \in DG(\Omega, k_0) \) if \( u \in W^{1,1}_2(\Omega) \) and there exist constants \( \gamma > 0, F > 0, \alpha \geq 0, q > n \) such that for any \( B_R(x_0) \subset \Omega \), any \( 0 < \rho < R \) and any \( k \geq k_0 \) the following inequality holds

\[
\int_{A_k \cap \Omega \rho(x_0)} |\nabla u|^2 \, dx \leq \frac{\gamma^2}{(R - \rho)^2} \left( 1 + \frac{R^\alpha}{(R - \rho)^\alpha} \right) \int_{A_k \cap \Omega \rho(x_0)} |u - k|^2 \, dx + F^2 |A_k \cap \Omega \rho(x_0)|^{1 - \frac{2}{q}}
\]

(4.1)

where we denote \( A_k := \{ x \in \Omega : u(x) > k \} \). We write \( u \in DG(\Omega) \) if \( u \in DG(\Omega, k_0) \) for any \( k_0 \in \mathbb{R} \).

The second modified De Giorgi class characterizes the behavior of Sobolev functions near the boundary of the domain:

Definition 4.2. We say \( u \in DG(\partial \Omega) \) if \( u \in W^{1,1}_2(\Omega) \) and there exist constants \( \gamma > 0, F > 0, \alpha \geq 0, q > n, R_0 > 0 \) such that for any \( x_0 \in \partial \Omega \), any \( 0 < \rho < R \leq R_0 \) and any \( k \geq 0 \) the following inequality holds

\[
\int_{A_k \cap \Omega \rho(x_0)} |\nabla u|^2 \, dx \leq \frac{\gamma^2}{(R - \rho)^2} \left( 1 + \frac{R^\alpha}{(R - \rho)^\alpha} \right) \int_{A_k \cap \Omega \rho(x_0)} |u - k|^2 \, dx + F^2 |A_k \cap \Omega \rho(x_0)|^{1 - \frac{2}{q}}
\]

(4.2)

where we denote \( \Omega_R(x_0) := \Omega \cap B_R(x_0) \).

The main result of this section is the following theorem:

Theorem 4.1. Assume \( \Omega \subset \mathbb{R}^n \) is a bounded domain and \( \partial \Omega \subset C^1 \).

(i) Assume \( \pm u \in DG(\Omega) \). Then there exists \( \mu \in (0,1) \) depending only on \( n, q, \gamma \) such that \( u \in C^{\mu}_{c,loc}(\Omega) \) and for any \( \Omega' \subset \subset \Omega \)

\[
\|u\|_{C^{\mu}(\Omega')} \leq c(\Omega, \Omega') \left( \|u\|_{L^2(\Omega)} + F \right)
\]

with some constant \( c(\Omega, \Omega') > 0 \) depending only on \( \Omega, \Omega', n, q, \gamma \) and \( \alpha \).

(ii) Assume \( \pm u \in DG(\Omega) \cap DG(\partial \Omega) \). Then there exists \( \mu \in (0,1) \) depending only on \( n, q, \gamma \) such that \( u \in C^{\mu}(\Omega) \) and

\[
\|u\|_{C^{\mu}(\Omega)} \leq c(\Omega) \left( \|u\|_{L^2(\Omega)} + F \right)
\]

with some constant \( c(\Omega) > 0 \) depending only on \( \Omega, R_0, n, q, \gamma \) and \( \alpha \).

Our modified De Giorgi classes in Definitions 4.1 and 4.2 are very similar to ones introduced earlier in [22]. The proof of Theorem 4.1 is absolutely standard and follows to the general line in [22]. But due to the minor difference in the proof and for the reader convenience we give some comments to the proof of Theorem 4.1 in Section 6 to this paper. Note that for the drift satisfying (1.7), (1.11) an a priori estimate (1.12) for Lipschitz continuous solutions was obtained earlier in [25].

Now we turn to the proof of Theorem 1.3.
Proof. First we consider the case $B_R(x_0) \subset \Omega$. Denote for brevity $B_R := B_R(x_0)$. Take a cut-off function $\zeta \in C_0^\infty(B_R)$ such that

$$0 \leq \zeta \leq 1, \quad \zeta \equiv 1 \quad \text{on} \quad B_\rho, \quad |\nabla \zeta| \leq \frac{c}{R - \rho}. \quad (4.3)$$

Define $m := \frac{2r}{2r - n}$ and note that for $r \in (\frac{n}{2}, n]$ we have $m \geq 2$ and hence $\zeta^{m-1} \leq \zeta^2$. Assume $k \in \mathbb{R}$ is arbitrary and take $\eta = \zeta^m (u - k)_+$ in (1.6). It is easy to see that

$$B[u, \eta] = B[\zeta^m (u - k)_+, \zeta^m (u - k)_+] - \frac{1}{2} \int_\Omega b \cdot \nabla \zeta^m |(u - k)_+|^2 \, dx$$

Taking into account (1.8) we obtain

$$\|\zeta^m \nabla (u - k)_+\|_{L^2(B_R)}^2 \leq c \|u - k)_+\|_{L^2(\Omega_R)}^2 + \frac{1}{2} \int_\Omega b \cdot \nabla \zeta^m |(u - k)_+|^2 \, dx + \|f\|_{L^q(\Omega)}^2 |A_k \cap B_R|^{1 - \frac{2}{q}}$$

We estimate the drift term using Proposition 2.3

$$\int_\Omega b \cdot \nabla \zeta^m |(u - k)_+|^2 \, dx \leq \varepsilon \|\zeta^m \nabla (u - k)_+\|_{L^2(B_R)}^2 + c \left(\|\nabla \zeta\|_{L^\infty(B_R)}^2 + \|b \cdot \nabla \zeta^m\|_{L^r(B_R)}^2\right) |(u - k)_+|_{L^2(B_R)}^2$$

Denote $\alpha := m - 2 \geq 0$. Taking use of $b \in L^{r-n-r}(\Omega)$ we obtain

$$\|b \cdot \nabla \zeta^m\|_{L^r(B_R)} \leq c \|b\|_{L^{r-n-r}(\Omega)}^m \frac{R^{(n-r)m}}{(R - \rho)^m} = c \|b\|_{L^{r-n-r}(\Omega)}^m \left(\frac{R^\alpha}{(R - \rho)^{2+\alpha}} \right)$$

So, if we fix sufficiently small $\varepsilon > 0$ for any $k \in \mathbb{R}$ we obtain

$$\frac{1}{2} \|\nabla (u - k)_+\|_{L^2(B_\rho)}^2 \leq \frac{c}{(R - \rho)^2} \left(1 + \|b\|_{L^{r-n-r}(\Omega)}^m \left(\frac{R^\alpha}{(R - \rho)^\alpha}\right)\right) |(u - k)_+|_{L^2(B_R)}^2 + \|f\|_{L^q(\Omega)}^2 |A_k \cap B_R|^{1 - \frac{2}{q}}$$

Hence we obtain that $u \in DG(\Omega)$. Applying the same arguments to $-u$ instead of $u$ we also obtain $-u \in DG(\Omega)$.

Now we turn to the estimates near the boundary. Assume $x_0 \in \partial \Omega$ and denote $\Omega_R := \Omega_R(x_0) \equiv \Omega \cap B_R(x_0)$. Take a cut-off function $\zeta \in C_0^\infty(B_R)$ satisfying (4.3). Then for any $k \geq 0$ the function $\eta = \zeta^m (u - k)_+$ vanishes on $\partial \Omega$ and hence it is admissible for the identity (1.7). Repeating all previous arguments we arrive at

$$\frac{1}{2} \|\nabla (u - k)_+\|_{L^2(\Omega_\rho)}^2 \leq \frac{c}{(R - \rho)^2} \left(1 + \|b\|_{L^{r-n-r}(\Omega)}^m \left(\frac{R^\alpha}{(R - \rho)^\alpha}\right)\right) |(u - k)_+|_{L^2(\Omega_R)}^2 + \|f\|_{L^q(\Omega)}^2 |A_k \cap \Omega_R|^{1 - \frac{2}{q}}$$

which holds for any $k \geq 0$. Hence $u \in DG(\partial \Omega)$. Applying the same arguments to $-u$ instead of $u$ we also obtain $-u \in DG(\partial \Omega)$. Now Theorem 1.3 follows directly from Theorem 1.1. \hfill $\blacksquare$
5 Appendix 1

In this section we prove some results about smooth approximations of functions from weak Morrey spaces. First of all, we split the drift $b$ satisfying (1.3), (1.7) onto divergence free and potential parts

$$b = b_0 + \nabla h \quad \text{a.e. in } \Omega,$$

where

$$\text{div} b_0 = 0 \quad \text{in } \mathcal{D}'(\Omega) \quad \text{(5.2)}$$

and $h$ is a weak solution to the Dirichlet problem

$$\begin{cases} 
\Delta h = \text{div} b & \text{in } \mathcal{D}'(\Omega) \\
|h|_{\partial \Omega} = 0
\end{cases} \quad \text{(5.3)}$$

As $b$ satisfies (1.3) we conclude $h \in \dot{W}^1_q(\Omega)$ and $b_0 \in L_q(\Omega)$ for any $q \in [1,2)$. Now we prove an estimate of the weak Morrey norm for $\nabla h$:

**Proposition 5.1.** Assume $b \in L^{2,\lambda}_w(\Omega)$, $\lambda \in [0, n)$, and assume $h \in \dot{W}^1_q(\Omega)$ is a weak solution to the problem (5.3). Then $\nabla h \in L^{2,\lambda}_w(\Omega)$ and $\nabla h$ satisfies the estimate

$$\|\nabla h\|_{L^{2,\lambda}_w(\Omega)} \leq c \|b\|_{L^{2,\lambda}_w(\Omega)} \quad \text{(5.4)}$$

where the constant $c > 0$ depends only on $n$, $\lambda$ and $\Omega$.

**Proof.** For $\lambda = 0$ the result follows from the general interpolation theory for Lorentz spaces, see [7, Theorem 1.4.19]. So, we have

$$\|\nabla h\|_{L^{2,\lambda}_w(\Omega)} \leq c \|b\|_{L^{2,\lambda}_w(\Omega)} \quad \text{(5.5)}$$

with some constant $c > 0$ depending only on $n$, $\lambda$ and $\Omega$. To prove (5.4) for $\lambda \in (0, n)$ we take arbitrary $B_R := B_R(x_0) \subset \Omega$ and $\rho < R$. Let $w$ be harmonic in $B_R$ and satisfies $w|_{\partial B_R} = h|_{\partial B_R}$. Then

$$\|\nabla h\|_{L^{2,\lambda}_w(B_R)} \leq c \left( \frac{\rho}{R} \right)^n \|\nabla h\|_{L^{2,\lambda}_w(B_R)} + c \|\nabla h - \nabla w\|_{L^{2,\lambda}_w(B_R)}$$

As we already know the result for $\lambda = 0$ we have the inequality

$$\|\nabla h - \nabla w\|_{L^{2,\lambda}_w(B_R)} \leq c_n \|b\|_{L^{2,\lambda}_w(B_R)}$$

with a constant $c_n > 0$ depending only on $n$. So, for any $0 < \rho < R$ we obtain the estimate

$$\|\nabla h\|_{L^{2,\lambda}_w(B_R)} \leq c \left( \frac{\rho}{R} \right)^n \|\nabla h\|_{L^{2,\lambda}_w(B_R)} + c \|b\|_{L^{2,\lambda}_w(\Omega)} R^\lambda$$

Then using [8, Lemma 5.13] we arrive at

$$\|\nabla h\|_{L^{2,\lambda}_w(B_R)} \leq c \rho^\lambda \left( \|\nabla h\|_{L^{2,\lambda}_w(\Omega)} + \|b\|_{L^{2,\lambda}_w(\Omega)} \right)$$

with $c > 0$ depending only on $n$, $\lambda$ and $\Omega$. Thanks to the Dirichlet boundary conditions a similar estimate also holds for boundary points $x_0 \in \partial \Omega$ with $B_R(x_0)$ replaced by $\Omega_{\rho}(x_0) := \Omega \cap B_R(x_0)$. Hence $\nabla h \in L^{2,\lambda}_w(\Omega)$ and

$$\|\nabla h\|_{L^{2,\lambda}_w(B_R)} \leq c \left( \|\nabla h\|_{L^{2,\lambda}_w(\Omega)} + \|b\|_{L^{2,\lambda}_w(\Omega)} \right)$$

with $c > 0$ depending only on $n$, $\lambda$ and $\Omega$. Now (5.4) follows from (5.5). \qed
Now we prove some properties of a mollification of potential vector fields from critical weak Morrey spaces:

**Proposition 5.2.** Assume \( h \in \dot{W}^{1,1}_1(\Omega) \), and \( \nabla h \in L^{2,n-2}_w(\Omega) \). For \( \varepsilon > 0 \) we denote

\[
h_\varepsilon(x) := \int_{\mathbb{R}^n} \omega_\varepsilon(x - y) h(y) \, dy, \quad x \in \mathbb{R}^n
\]

(we extend \( h \) by zero onto \( \mathbb{R}^n \) and \( \omega_\varepsilon(x) = \varepsilon^{-n} \omega(x/\varepsilon) \) is a standard Sobolev kernel).

Then for any \( x_0 \in \Omega \), any \( 0 < R < \text{diam } \Omega \) and any \( \varepsilon > 0 \) we have the estimate

\[
\| \nabla h_\varepsilon \|_{L^{2,w}_2(B_R(x_0))} \leq c \| \nabla h \|_{L^{2,w}_2(\Omega)}^2 R^{n-2}
\]

(5.6)

where the constant \( c > 0 \) depends only on \( n \) and \( \Omega \).

**Proof.** For \( R < \varepsilon \) we have

\[
\nabla h_\varepsilon(x) = \int_{B_\varepsilon(x)} \nabla \omega_\varepsilon(x - y) \left( h(y) - h_{x,\varepsilon} \right) \, dy
\]

where \( h_{x,\varepsilon} := (h)_{B_\varepsilon(x)} \). Taking into account \( \| \nabla \omega_\varepsilon \|_{L^{\infty}(\mathbb{R}^n)} \leq \frac{c_n}{\varepsilon} \| h \|_{BMO(\Omega)} \)

we obtain

\[
\| \nabla h_\varepsilon \|_{L^{\infty}(\mathbb{R}^n)} \leq \frac{c_n}{\varepsilon} [h]_{BMO(\Omega)}
\]

where

\[
[h]_{BMO(\Omega)} := \sup_{x \in \Omega} \sup_{R < \text{diam } \Omega} \int_{B_\varepsilon(x)} |h - h_{x,\varepsilon}| \, dy
\]

The last inequality gives also

\[
\| \nabla h_\varepsilon \|_{L^2(B_R(x_0))}^2 \leq \frac{c_n}{\varepsilon^2} [h]_{BMO(\Omega)}^2 R^n
\]

and taking into account \( R < \varepsilon \) we obtain

\[
\| \nabla h_\varepsilon \|_{L^2(B_R(x_0))}^2 \leq c_n [h]_{BMO(\Omega)}^2 R^{n-2}
\]

Using [30] (see also [26, Theorem 391 (2)]) we obtain

\[
[h]_{BMO(\Omega)} \leq c_n \| \nabla h \|_{L^{1,n-1}(\Omega)}
\]

Using the Hölder inequality for weak Lorentz spaces we obtain

\[
\| \nabla h \|_{L^{1,n-1}(\Omega)} \leq c \| \nabla h \|_{L^{2,n-2}_w(\Omega)}
\]

Using also a trivial estimate

\[
\| \nabla h_\varepsilon \|_{L^{2,w}_2(B_R(x_0))} \leq \| \nabla h_\varepsilon \|_{L^2(B_R(x_0))}
\]

we arrive at (5.6).

Now consider the case \( R > \varepsilon \). Let us fix some \( x_0 \in \mathbb{R}^n \) and consider the operator

\[
T_\varepsilon f(x) = \int_{B_\varepsilon(x)} \omega_\varepsilon(x - y) f(y) \, dy, \quad x \in B_R(x_0)
\]
Let us show $T_\varepsilon : L_p(B_{2R}(x_0)) \to L_p(B_R(x_0))$ is a bounded linear operator for any $1 \leq p \leq +\infty$ with
$$\|T_\varepsilon\|_{L_p(B_{2R}(x_0)) \to L_p(B_R(x_0))} = 1.$$  
\hfill (5.7)
Indeed for any $1 \leq p < +\infty$ we have
$$|T_\varepsilon f(x)|^p \leq \int_{B_r(x)} \omega_\varepsilon(x-y)|f(y)|^p \, dy$$
and hence
$$\|T_\varepsilon f\|^p_{L_p(B_R(x_0))} \leq \int_{B_{2r}(x_0)} \int_{B_r(x)} \omega_\varepsilon(x-y)|f(y)|^p \, dy$$
From $\varepsilon < R$ for any $x \in B_R(x_0)$ we obtain $B_r(x) \subset B_{2R}(x_0)$ and hence by Fubini’s theorem
$$\|T_\varepsilon f\|^p_{L_p(B_R(x_0))} \leq \int_{B_{2r}(x_0)} \int_{B_{2R}(x_0)} \omega_\varepsilon(x-y)|f(y)|^p \, dy \leq \int_{B_{2r}(x_0)} |f(y)|^p \, dy \int_{B_{2R}(x_0)} \omega_\varepsilon(x-y) \, dx \leq \|f\|^p_{L_p(B_{2R}(x_0))}$$
So, in the case of $R > \varepsilon$ for any $1 \leq p \leq +\infty$ we obtain
$$\|T_\varepsilon f\|_{L_p(B_R(x_0))} \leq \|f\|_{L_p(B_{2R}(x_0))}$$
which implies (5.7). From the Marcinkiewicz interpolation theorem for Lorentz spaces (see [7, Theorem 1.4.19]) we conclude that $T_\varepsilon$ is also bounded as an operator from $L_{2,w}(B_{2R}(x_0))$ into $L_{2,w}(B_R(x_0))$ and its norm is independent of $\varepsilon > 0$, $R > 0$ and $x_0 \in \Omega$, i.e.
$$\|T_\varepsilon f\|_{L_{2,w}(B_R(x_0))} \leq c \|f\|_{L_{2,w}(B_{2R}(x_0))}$$
with some constant $c > 0$ depending only on $n$. Applying the obtained result to the function
$$\nabla h_\varepsilon(x) = \int_{B_r(x)} \omega_\varepsilon(x-y) \nabla h(y) \, dy$$
we obtain for any $x_0 \in \Omega$ and any $R > \varepsilon$
$$\|\nabla h_\varepsilon\|_{L_{2,w}(B_R(x_0))} \leq c \|\nabla h\|_{L_{2,w}(B_{2R}(x_0))}$$
\hfill (5.8)
which gives (5.6) in the case $R \geq \varepsilon$. Combining results for $R < \varepsilon$ and $R \geq \varepsilon$ we obtain (5.6) for all $x_0 \in \Omega$ and $R > 0$. \hfill \box

Now we show that the extension operator for divergence-free vector fields preserves a weak Morrey norm.

**Proposition 5.3.** Let $\Omega$ and $\Omega_0 \subset \mathbb{R}^n$ be bounded simply connected domains, $\partial\Omega$ is of class $C^1$, and assume $\Omega \Subset \Omega_0$. Assume $b \in L_w^{2,\lambda}(\Omega)$ with $\lambda \in [0,n)$ and $\text{div} \, b = 0$ in $\mathcal{D}^\prime(\Omega)$. Then there exists a bounded linear operator $T : L_w^{2,\lambda}(\Omega) \to L_w^{2,\lambda}(\mathbb{R}^n)$ such that the function $\tilde{b} := Tb$ satisfies $\text{supp} \, \tilde{b} \subset \bar{\Omega}_0$, $\tilde{b}|_{\Omega} = b$ and the estimate
$$\|\tilde{b}\|_{L_w^{2,\lambda}(\mathbb{R}^n)} \leq c \|b\|_{L_w^{2,\lambda}(\Omega)}$$
\hfill (5.9)
where the constant $c > 0$ depends only on $n$, $\lambda$, $\Omega$ and $\Omega_0$. 

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Proof. The existence of the extension operator $T$ is well-known, see, for example, \cite[§3.1, Theorem 3.1]{7}. So, we prove only the estimate \eqref{5.9}. Assume $p \in (1, +\infty)$ and denote
\[ J_p(\Omega) := \{ u \in L_p(\Omega) : \text{div} u = 0 \text{ in } \mathcal{D}'(\Omega) \}. \]
Denote by $E : W^1_p(\Omega_0 \setminus \Omega) \to W^1_p(\Omega_0)$ any bounded linear extension operator, i.e. $(E\eta)|_{\Omega_0 \setminus \Omega} = \eta$. Then the formula
\[ l(\eta) := -\int_{\Omega} b \cdot \nabla \tilde{\eta} \, dx, \quad \tilde{\eta} := E\eta, \quad \eta \in W^1_p(\Omega_0 \setminus \Omega) \]
determines a bounded linear functional on the Banach space $W^1_p(\Omega_0 \setminus \Omega)$. Moreover, the condition $\text{div} b = 0$ in $\mathcal{D}'(\Omega)$ implies that the functional $l$ is independent of the extension operator $E$. Hence there exists a unique solution $\varphi \in W^1_p(\Omega_0 \setminus \Omega)$ of the Neumann problem
\[ \Delta \varphi = 0 \text{ in } \Omega_0 \setminus \Omega, \quad \frac{\partial \varphi}{\partial \nu}|_{\partial \Omega_0} = 0, \quad \frac{\partial \varphi}{\partial \nu}|_{\partial \Omega} = b \cdot \tilde{\nu} \]
where $\tilde{\nu}$ is the outer normal to $\Omega_0 \setminus \Omega$. Formally we define $\varphi$ satisfying the relations
\[ \int_{\Omega_0 \setminus \Omega} \nabla \varphi \cdot \nabla \eta \, dx = l(\eta), \quad \forall \eta \in W^1_p(\Omega_0 \setminus \Omega), \]
\[ \int_{\Omega_0 \setminus \Omega} \varphi \, dx = 0, \quad \|\nabla \varphi\|_{L_p(\Omega_0 \setminus \Omega)} \leq c \|b\|_{L_p(\Omega)}, \]
Now we take $\tilde{b} := b$ in $\Omega$ and $\tilde{\varphi} := \nabla \varphi$ in $\Omega_0 \setminus \Omega$ and then extend $\tilde{b}$ by zero to the whole $\mathbb{R}^n$. It is easy to see that $\tilde{b} \in J_p(\mathbb{R}^n)$ and $Tb := \tilde{b}$ satisfies all necessary properties.

From the Marcinkiewicz interpolation theorem (see \cite[Theorem 1.4.19]{7}) we conclude that $T$ is also bounded as an operator from $L_{2,w}(\Omega)$ into $L_{2,w}(\mathbb{R}^n)$. Namely, for any $b \in L_{2,w}(\Omega)$ such that $\text{div} b = 0$ in $\mathcal{D}'(\Omega)$ we have $\tilde{b} \in L_{2,w}(\Omega_0)$ and
\[ \|\tilde{b}\|_{L_{2,w}(\mathbb{R}^n)} \leq c \|b\|_{L_{2,w}(\Omega)} \] (5.10)
with $c > 0$ depending only on $n$, $\lambda$, $\Omega$ and $\Omega_0$. Now let us prove the estimate \eqref{5.9}. Assume first $x_0 \in \partial \Omega$ and denote $B_R := B_R(x_0)$, $\Omega_R := B_R(x_0) \cap \Omega$, $\Omega_R^c := B_R(x_0) \setminus \Omega$. Let $w$ be harmonic in $\Omega_R^c$ with zero average and satisfies
\[ \int_{\Omega_R^c} \nabla w \cdot \nabla \eta \, dx = -\int_{\Omega_R} b \cdot \nabla \tilde{\eta} \, dx, \quad \tilde{\eta} := E\eta, \quad \forall \eta \in C^1(\Omega_R^c) \]
In the other words we assume $w$ is a weak solution to the Neumann problem
\[ \Delta w = 0 \text{ in } \Omega_R^c, \quad \frac{\partial w}{\partial \nu}|_{\partial B_R \setminus \Omega} = 0, \quad \frac{\partial w}{\partial \nu}|_{\partial \Omega \cap B_R} = b \cdot \nu|_{\partial \Omega \cap B_R} \]
It is well-known that the linear operator $b \in L_p(\Omega_R) \mapsto w \in W^1_p(\Omega_R^c)$ is bounded for any $1 < p < +\infty$ and from the interpolation theory for Lorentz spaces (see \cite[Theorem 1.4.19]{7}) we obtain the estimate for weak Lebesgue norm
\[ \|\nabla w\|_{L_{2,w}(\Omega_R^c)} \leq c \|b\|_{L_{2,w}(\Omega_R)} \]
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with some constant \( c > 0 \) depending only on \( n \) and \( \Omega \). We emphasize here that \( c \) does not depend on \( x_0 \in \partial \Omega \) and \( R < R_0 \) for some \( R_0 > 0 \) which is determined by \( C^1 \)-smooth \( \partial \Omega \). The last inequality implies the estimate
\[
\| \nabla w \|^2_{L^2,w(\Omega_R^c)} \leq c \| b \|^2_{L^2,w(\Omega)} R^\lambda
\]
On the other hand, the function \( \varphi - w \) is harmonic in \( \Omega_R^c \) and satisfies the homogeneous Neumann conditions on the part of the boundary \( \partial \Omega \cap \partial B_R \). Hence it is smooth up to \( \partial \Omega \) and the following estimate holds:
\[
\| \nabla (\varphi - w) \|^2_{L^\infty(\Omega_R^c)} \leq \frac{c}{R^n} \| \nabla (\varphi - w) \|^2_{L^2,w(\Omega_R^c)}
\]
with \( c > 0 \) depending only on \( n \) and \( \Omega \). For any \( 0 < \rho < R \) the last inequality gives the estimate
\[
\| \nabla (\varphi - w) \|^2_{L^2,w(\Omega_R^c)} \leq c \left( \frac{\rho}{R} \right)^n \| \nabla (\varphi - w) \|^2_{L^2,w(\Omega_R^c)}
\]
Then for any \( 0 < \rho < R \) we obtain
\[
\| \nabla \varphi \|^2_{L^2,w(\Omega_R^c)} \leq c \| \nabla (\varphi - w) \|^2_{L^2,w(\Omega_R^c)} + c \| \nabla w \|^2_{L^2,w(\Omega_R^c)} \leq c \left( \frac{\rho}{R} \right)^n \| \nabla \varphi \|^2_{L^2,w(\Omega_R^c)} + c \| \nabla \varphi \|^2_{L^2,w(\Omega_R^c)} \leq c \left( \frac{\rho}{R} \right)^n \| \nabla \varphi \|^2_{L^2,w(\Omega_R^c)} + c \| b \|^2_{L^2,w(\Omega)} R^\lambda
\]
Then using [S] Lemma 5.13] we arrive at
\[
\| \nabla \varphi \|^2_{L^2,w(B_\rho(x_0))} \leq c \rho^\lambda \left( \| \nabla \varphi \|^2_{L^2,w(\Omega_0 \setminus \Omega)} + \| b \|^2_{L^2,w(\Omega)} \right)
\]
with some constant \( c > 0 \) depending on \( n \) and \( \Omega \). As \( \varphi \) is harmonic in \( \Omega_0 \setminus \Omega \) for any \( B_R(x_0) \subset \Omega_0 \setminus \Omega \) and any \( 0 < \rho < R \) it satisfies also the estimate
\[
\| \nabla \varphi \|^2_{L^2,w(B_\rho(x_0))} \leq c \left( \frac{\rho}{R} \right)^n \| \nabla \varphi \|^2_{L^2,w(B_\rho(x_0))}
\]
As \( \varphi \) satisfies the homogeneous Neumann condition on \( \partial \Omega_0 \) for any \( x_0 \in \partial \Omega_0 \) we have also
\[
\| \nabla \varphi \|^2_{L^2,w(B_\rho(x_0)) \cap \Omega_0} \leq c \left( \frac{\rho}{R} \right)^n \| \nabla \varphi \|^2_{L^2,w(B_\rho(x_0)) \cap \Omega_0}
\]
Combining the internal and boundary estimates in the standard way we arrive at
\[
\| \nabla \varphi \|^2_{L^2,w(\Omega_R^c(x_0))} \leq c \rho^\lambda \left( \| \nabla \varphi \|^2_{L^2,w(\Omega_0 \setminus \Omega)} + \| b \|^2_{L^2,w(\Omega)} \right)
\]
for any \( x_0 \in \overline{\Omega_0 \setminus \Omega} \) with a constant \( c > 0 \) depending on \( n \), \( \Omega \) and \( \Omega_0 \). So, we obtain \( \nabla \varphi \in L^2_w(\Omega_0 \setminus \Omega) \) and
\[
\| \nabla \varphi \|^2_{L^2_w(\Omega_0 \setminus \Omega)} \leq c \left( \| \nabla \varphi \|^2_{L^2,w(\Omega_0 \setminus \Omega)} + \| b \|^2_{L^2,w(\Omega)} \right)
\]
Taking into account [5.10] we obtain
\[
\| \nabla \varphi \|^2_{L^2_w(\Omega_0 \setminus \Omega)} \leq c \| b \|^2_{L^2,w(\Omega)}
\]
From the definition of \( \tilde{b} \) we obtain
\[
\| \tilde{b} \|^2_{L^2,w(B_\rho(x_0))} \leq c \left( \| \nabla \varphi \|^2_{L^2,w(\Omega_R^c(x_0))} + \| b \|^2_{L^2,w(\Omega_\rho(x_0))} \right)
\]
which gives (5.9).
Now we prove an analogue of Proposition 5.2 for the divergence-free vector fields.

**Proposition 5.4.** Assume $b \in L_w^{2, n-2}(\mathbb{R}^n)$ is compactly supported with $\text{supp} b \subset \Omega$, and assume $\text{div} b = 0$ in $\mathcal{D}'(\mathbb{R}^n)$. For $\varepsilon > 0$ we denote

$$b_\varepsilon(x) := \int_{\mathbb{R}^n} \omega_\varepsilon(x-y) b(y) \, dy, \quad x \in \mathbb{R}^n$$

Then

$$\text{div} b_\varepsilon = 0 \quad \text{in} \quad \mathbb{R}^n,$$

$$\|b_\varepsilon - b\|_{L_q(\Omega)} \to 0 \quad \forall q \in [1, 2) \quad (5.11)$$

and for any $x_0 \in \Omega$, any $0 < R < \text{diam} \Omega$ and any $\varepsilon > 0$ we have the estimate

$$\|b_\varepsilon\|_{L^2_{2, w}(B_R(x_0))} \leq c \|b\|_{L^2_{2, w}(\Omega)} R^{n-2} \quad (5.12)$$

where the constant $c > 0$ depends only on $n$ and $\Omega$.

**Proof.** First we introduce the vector potential $A$ for $b$. Let $A = (A_{jk})$ be an $n \times n$ skew-symmetric matrix whose components determined by

$$A_{jk}(x) = \int_{\mathbb{R}^n} \left( K_j(x-y)b_k(y) - K_k(x-y)b_j(y) \right) dy, \quad x \in \mathbb{R}^n$$

where

$$K_j(x) := \frac{\partial E}{\partial x_j}(x)$$

and $E(x) = \frac{1}{(n-2)|S_1| |x|^{n-2}}$ is the fundamental solution for $-\Delta$ operator in $\mathbb{R}^n$.

Formally, we interpret $b = b_j dx^j$ as a 1-form on $\mathbb{R}^n$, introduce $A$ as a 2-form satisfying the Hodge system $dA = 0$, $d^* A = b$ and solve the equation for 2-forms

$$\Box A = db$$

in $\mathbb{R}^n$ where $\Box = d^* d + dd^*$ is the Laplace operator. In the case $n = 3$ we can simply write $b = \text{rot} a$ for some vector field $a : \mathbb{R}^3 \to \mathbb{R}^3$.

As the kernel $K(x) = (K_j(x))$ satisfies the estimate

$$|K(x)| \leq \frac{c_n}{|x|^{n-1}}, \quad x \in \mathbb{R}^n$$

using (31) (see also [26, Theorem 391 (2)]) we obtain $A \in BMO(\Omega)$ and

$$[A]_{BMO(\Omega)} \leq c \|b\|_{L^{1, n-1}(\Omega)} \quad (5.13)$$

Moreover, from the theory of singular integrals one can see that

$$\text{div} A = b \quad \text{a.e. in} \quad \mathbb{R}^n$$

where we denote $\text{div} A := (A_{jk,k})$ (in terms of forms this relation is equivalent to $d^* A = b$ in $\mathbb{R}^n$). Now we introduce the mollification

$$A_\varepsilon(x) := \int_{\mathbb{R}^n} \omega_\varepsilon(x-y) A(y) \, dy, \quad x \in \mathbb{R}^n$$

where $\omega_\varepsilon(x) := \varepsilon^{-n} \omega(x/\varepsilon)$ and $\omega \in C_0^\infty(B)$ is a standard Sobolev kernel, and take

$$b_\varepsilon := \text{div} A_\varepsilon = \omega_\varepsilon * b \quad \text{in} \quad \mathbb{R}^n$$
Clearly, we have $A_\varepsilon \to A$ in $W^{1, q}_{q, \text{loc}}(\mathbb{R}^n)$ for any $q \in [1, 2)$ and hence we obtain (5.11). So, all we need to show is the estimate (5.12).

The rest of the proof goes similar to Proposition 5.2. For $R < \varepsilon$ we use the representation

$$ (b_j)_\varepsilon(x) = \int_{B_\varepsilon(x)} \frac{\partial \omega_\varepsilon(x-y)}{\partial x_k} \left(A_{jk}(y) - (A_{jk})_{B_\varepsilon(x)}\right) dy $$

and obtain

$$ \|b_\varepsilon\|_{L^2(B_R(x_0))} \leq c_n [A]^2_{BMO(\Omega)} R^{n-2} $$

from which (5.12) follows with the help of (5.13). And for $R \geq \varepsilon$ we use the representation

$$ b_\varepsilon(x) = \int_{B_\varepsilon(x)} \omega_\varepsilon(x-y)b(y) dy, \quad x \in \mathbb{R}^n $$

and similar to (5.8) we obtain

$$ \|b_\varepsilon\|_{L^2,w(B_R(x_0))} \leq c \|b\|_{L^2,w(B_2R(x_0))} $$

from which (5.12) follows in the case of $R \geq \varepsilon$. Combining results for $R < \varepsilon$ and $R \geq \varepsilon$ we obtain (5.12) for all $x_0 \in \Omega$ and $R > 0$.

We conclude this section with an approximation result for weak Morrey spaces.

**Proposition 5.5.** Assume $b$ satisfies (1.3), (1.7) and $q \in [1, 2)$. Then there exists $b_\varepsilon \in C^\infty(\Omega)$ such that $\text{div } b_\varepsilon \leq 0$ in $\Omega$ and

$$ \|b_\varepsilon\|_{L^2_w, n-2(\Omega)} \leq c \|b\|_{L^2_w, n-2(\Omega)} \quad \text{and} \quad \|b_\varepsilon - b\|_{L^q(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad (5.14) $$

**Proof.** Let $h$ and $b_0$ satisfy (5.1), (5.2), (5.3) and let $\tilde{b}_0$ be an extension of $b_0$ onto $\mathbb{R}^n$ constructed in Proposition 5.3. Then from Propositions 5.1 and 5.3 we obtain

$$ \|\tilde{b}_0\|_{L^2_w, n-2(\mathbb{R}^n)} + \|\nabla h\|_{L^2_w, n-2(\mathbb{R}^n)} \leq c \|b\|_{L^2_w, n-2(\mathbb{R}^n)} $$

Denote by $h_\varepsilon := \omega_\varepsilon \ast h$ and $\tilde{b}_0^\varepsilon := \omega_\varepsilon \ast \tilde{b}_0$ the mollification of $h$ and $\tilde{b}_0$ defined in Propositions 5.2 and 5.4. Then by Propositions 5.2 and 5.4 we obtain

$$ \|\tilde{b}_0^\varepsilon\|_{L^2_w, n-2(\mathbb{R}^n)} + \|\nabla h_\varepsilon\|_{L^2_w, n-2(\mathbb{R}^n)} \leq c \|b\|_{L^2_w, n-2(\mathbb{R}^n)} $$

$$ \|\nabla h_\varepsilon - \nabla h\|_{L^q(\Omega)} \to 0, \quad \|\tilde{b}_0^\varepsilon - b_0\|_{L^q(\Omega)} \to 0 $$

From $\text{div } \tilde{b}_0 = 0$ in $\mathcal{D}'(\mathbb{R}^n)$ we obtain $\text{div } \tilde{b}_0^\varepsilon = 0$ in $\mathbb{R}^n$. From (1.7) we conclude $h$ satisfies $\Delta h \leq 0$ in $\mathcal{D}'(\mathbb{R}^n)$. It is well-known (see, for example, [10] Theorem 3.2.11) that in this case functions $h_\varepsilon$ are also superharmonic, i.e. $\Delta h_\varepsilon \leq 0$ in $\mathbb{R}^n$. Denote now $b_\varepsilon := \tilde{b}_0^\varepsilon + \nabla h_\varepsilon$. Then $b_\varepsilon \in C^\infty(\Omega)$ satisfies $\text{div } b_\varepsilon \leq 0$ in $\Omega$ as well as (5.14). □
6 Appendix 2

In this section we sketch the proof of Theorem 4.1. We start with the local maximum estimate:

**Lemma 6.1.** Assume \( u \in DG(\Omega, k_0) \). Then for any \( B_R := B_R(x_0) \subset \Omega \)

\[
\sup_{B_{R/2}} (u - k_0)_+ \leq c_* \left[ \left( \int_{B_R} |(u - k_0)_+|^2 \, dx \right)^{1/2} + FR^{1-\frac{2}{q}} \right]
\]  

(6.1)

where \( (u - k_0)_+ := \max\{u - k_0, 0\} \) and \( c_* > 0 \) depends only on \( n, \gamma \) and \( \alpha \).

**Proof.** To simplify the presentation we assume \( R = 1 \) and \( k_0 = 0 \). Denote

\[
H := c_* \left[ \left( \int_{B} u^2 \, dx \right)^{1/2} + F \right]
\]

where the precise value of the constant \( c_* = c_*(n, \gamma, \alpha) > 0 \) will be fixed later.

Define

\[
\rho_m := \frac{1}{2} + \frac{1}{2^{m+1}}, \quad k_m := H - \frac{H}{2^m}, \quad J_m := \int_{B_{\rho_m}} (u - k_m)_+^2 \, dx,
\]

\[
\mu_m := |\{x \in B_{\rho_m} : u(x) > k_m+1\}|
\]

From \( J_m \geq (k_m+1 - k_m)^2 \mu_m \) we obtain

\[
\mu_m \leq \frac{2m+2}{H^2} J_m
\]  

(6.2)

Define \( \rho'_m := \frac{1}{2}(\rho_{m+1} + \rho_m) \) and define cut-off functions

\[
\zeta_m \in C_0^\infty(B_{\rho'_m}), \quad \zeta_m \equiv 1 \text{ on } B_{\rho_{m+1}}, \quad |\nabla \zeta_m| \leq \frac{c}{\rho_m - \rho_{m+1}}.
\]

From the Hölder inequality and the imbedding \( W^{1,2}_2(B_{\rho'_m}) \hookrightarrow L^{2n/(n-2)}(B_{\rho'_m}) \) we obtain

\[
J_{m+1} \leq C(n) \frac{2}{\rho_m} \left( \|\nabla(u - k_{m+1})_+\|_{L^2(B_{\rho'_m})}^2 + \frac{C}{(\rho_m - \rho_{m+1})^2} J_m \right)
\]  

(6.3)

Using \( u \in DG(B) \) we estimate

\[
\|\nabla(u - k_{m+1})_+\|_{L^2(B_{\rho'_m})}^2 \leq \left( \frac{\gamma^2}{(\rho_m - \rho_{m+1})^2} \right) \left( 1 + \frac{\rho_m^\alpha}{(\rho_m - \rho_{m+1})^\alpha} \right) J_m + F^2 \mu_m^{1-\frac{2}{q}}
\]  

(6.4)

Denote \( Y_m := J_m/H^2 \). Taking into account \( \rho_m - \rho_{m+1} = \frac{1}{2^{m+1}}, \rho_m \leq 1 \) and \( F^2 \leq 1 \) from (6.2), (6.3), (6.4) we obtain

\[
Y_{m+1} \leq c_0 b^{m} Y_m^{1+\varepsilon}(1 + Y_m^{\varepsilon_1})
\]  

(6.5)

where \( b := 2^{2+\alpha+\frac{4}{q}}, \varepsilon = \frac{n}{q} - \frac{2}{q}, \varepsilon_1 = \frac{2}{q} \) and \( c_0 \geq 1 \) depending only on \( n, \gamma, \alpha \). Now we use the following lemma (its proof follows easily by induction):
Lemma 6.2. Assume \( \{Y_m\}_{m=1}^{\infty} \subset \mathbb{R} \) are nonnegative and there exist \( \varepsilon > 0, \varepsilon_1 \geq 0, b \geq 1 \) and \( c_0 \geq 1 \) such that (6.5) holds. Assume \( Y_0 \leq \theta \) where \( \theta := (2c_0)^{-1/\varepsilon} b^{-1/\varepsilon^2} \). Then
\[
Y_m \leq \theta b^{-m/\varepsilon}, \quad \forall \ m \in \mathbb{N}.
\]

To finish the proof of Lemma 6.1 we fix \( c_* > 1 \sqrt{\theta} \) where \( \theta \) is defined in Lemma 6.2. Then \( Y_0 < \theta \) holds and hence by Lemma 6.2 we obtain \( Y_m \to 0 \) which gives (6.1). 

Now we formulate so-called “Thin set lemma” (or “Density lemma”):

Lemma 6.3. Assume \( u \in DG(\Omega, k_0) \). Then there exists \( \theta_0 \in (0,1) \) such that for any \( B_{2\rho} := B_{2\rho}(x_0) \in \Omega \)
\[
|B_{2\rho} \cap A_{k_0}| \leq \theta_0 |B_{2\rho}|
\]
then either
\[
\sup_{B_{2\rho}} (u - k_0)_+ \leq \frac{1}{2} \sup_{B_{2\rho}} (u - k_0)_+ \quad (6.6)
\]
or
\[
\sup_{B_{2\rho}} (u - k_0)_+ \leq 2c_* F \rho^{1-\frac{\alpha}{q}} \quad (6.7)
\]
where \( c_* > 0 \) is a constant from (6.1).

Proof. Take \( \theta_0 \in (0,1) \) such that \( 2c_* \theta_0^{1/2} = \frac{1}{2} \). Assume (6.7) does not hold. Then from (6.1) we obtain (6.6). 

Now we formulate a result called “How to find a thin set”:

Lemma 6.4. Assume \( u \in DG(\Omega, k_0) \). Then for any \( \delta \in (0,1) \) and \( \forall \ \theta \in (0,1) \)
there exists \( s \in \mathbb{N}, \ s = s(\delta, \theta, n, \gamma) \) such that if for some \( B_{4\rho} \in \Omega \) the following estimate is valid
\[
|B_{2\rho} \setminus A_{k_0}| \geq \delta \ |B_{2\rho}|
\]
then either
\[
|B_{2\rho} \cap A_{\bar{k}}| \leq \theta \ |B_{2\rho}|, 
\]
or
\[
M(4\rho) - k_0 \leq F \rho^{1-\frac{\alpha}{q}}
\]
Here we denote \( \bar{k} = M(4\rho) - \frac{1}{2^s} (M(4\rho) - k_0) \) and \( M(4\rho) := \sup_{B_{4\rho}} u \).

The statement of Lemma 6.4 is concerned only with the concentric balls of radiiuses \( 2\rho \) and \( 4\rho \) with a fixed ratio, for which estimates (4.1) and (4.2) coincide with the corresponding estimates for the standard De Giorgi classes. Hence the proof of Lemma 6.4 follows by the standard method (see, for example, [22, Lemma 6.3]) and we omit it here.

From Lemmas 6.1, 6.3 and 6.4 by the standard method [22, Lemma 6.4] we obtain a local estimate of the oscillation:
Lemma 6.5. Assume $\theta_0 \in (0, 1)$ is a constant from Lemma 6.3 and assume $\pm u \in DG(\Omega)$. Then for any $B_{4\rho} \equiv B_{4\rho}(x_0)$ such that $B_{4\rho}(x_0) \subset \Omega$ either

$$\text{osc}_{B_{\rho}} u \leq \left(1 - \frac{1}{2^{n+2}}\right) \text{osc}_{B_{4\rho}} u,$$

or

$$\text{osc}_{B_{\rho}} u \leq 4c_s \rho^{1 - \frac{n}{q}}$$

Here $s_0 = s\left(\frac{1}{2}, \theta_0, n, \gamma\right)$, and $s \in \mathbb{N}$, $s = s(\delta, \theta, n, \gamma)$ is from Lemma 6.4.

Now Theorem 4.1 part (i) follows from our Lemma 6.5 with the help of [22, Lemma 4.8].

Moreover, if $u \in DG(\Omega) \cap DG(\partial \Omega)$ then Lemmas 6.1, 6.3 and 6.4 have straightforward analogues for points $x_0 \in \partial \Omega$, balls $B_R(x_0)$ replaces by $\Omega_R(x_0) = \Omega \cap B_R(x_0)$ and $k_0 = 0$. If we have $\pm u \in DG(\Omega) \cap DG(\partial \Omega)$ then Lemma 6.5 is valid for $B_{4\rho}(x_0)$ replaced by $\Omega_{4\rho}(x_0)$. Then Theorem 4.1 part (ii) follows by the standard combination of the internal and boundary estimates of the oscillation of $u$. 


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