On properties of an explicit in time fourth-order vector compact scheme for the multidimensional wave equation

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ABSTRACT
An initial-boundary value problem for the \( n \)-dimensional wave equation is considered. A three-level explicit in time and conditionally stable 4th-order compact scheme constructed recently for \( n = 2 \) and the square mesh is generalized to the case of any \( n \geq 1 \) and the rectangular uniform mesh. Another approach to approximate the solution at the first time level (not exploiting high-order derivatives of the initial functions) is suggested. New stability bounds in the mesh energy norms and the discrete energy conservation laws are given, and the 4th order error bound is rigorously proved. Generalizations to the cases of the non-uniform meshes in space and time as well as of the wave equation with variable coefficients are suggested.

KEYWORDS: wave equation, explicit vector scheme, higher-order compact scheme, stability, error bound

2010 MATHEMATICS SUBJECT CLASSIFICATIONS: 65M06; 65M12; 65M15

1 Introduction

The higher-order compact finite-difference schemes form an important class of numerical methods to solve PDEs due to their much greater computational efficiency compared to more standard 2nd order schemes. In the last decade, the construction and study of higher-order compact schemes for initial-boundary value problems for the 2D and 3D wave or telegraph equations, including the case of variable coefficients, have attracted great interest. Such compact schemes are implicit and most often conditionally stable. Among them, the first type of schemes is constituted by implicit schemes which require application of FFT (for constant coefficients) or iterative methods (for variable coefficients) for their efficient implementation, see, in particular, \cite{2,8,18,22,24}, where additional references are contained.

The second type of compact schemes belongs to the alternating direction implicit (ADI)-like methods (for example, see \cite{15,17}) which implementation is direct, reduces to solving collections of 1D linear algebraic tridiagonal systems and is much faster. The three-level in time compact ADI schemes of several types with the approximation order \( O(h_t^2 + |h|^4) \) were suggested for the 2D wave or telegraph equations with constant coefficients, in particular, in \cite{9,11,14}; for the 3D and multidimensional cases, see also \cite{9,16}. Here \( h_t \) and \( h \) are mesh steps in time and space. More important compact ADI schemes with the 4th approximation order, i.e. \( O(h_t^4 + |h|^4) \), for such 2D and 3D equations were constructed and studied, in particular, in \cite{4,5,12,19,24}. For the variable coefficient 2D wave equation, the compact ADI scheme having the approximation order \( O(h_t^4 + |h|^6) \) has recently been constructed in \cite{3}, and the 4th order compact ADI-like schemes were proposed and studied in the 2D and 3D cases in \cite{10,13,23}.

In the recent paper \cite{7}, a new third type of compact schemes has been suggested for the 2D wave equation in a square, with the non-homogeneous Dirichlet boundary conditions. The

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scheme is three-level explicit in time and conditionally stable; the uniform mesh in time and the square spatial mesh are taken. Its construction begins with applying the usual three-point 1D Numerov scheme in time but then, importantly, not only the solution to the wave equation is approximated as usual but its second spatial derivatives are also approximated independently as auxiliary sought functions. For these auxiliary functions, the three-point 1D Numerov schemes in the corresponding space directions are used. Thus the scheme is not completely explicit (as the classical explicit 2nd order scheme) since the auxiliary functions need to be computed by solving collections of 1D linear algebraic systems with tridiagonal matrices. In this respect, the scheme implementation and its cost are similar to the ADI methods, but the difference is that the auxiliary functions are completely independent and can be computed in parallel. Theoretically, only the spectral stability analysis on harmonics was accomplished in the case of zero free terms in the equations of the scheme, but stability bounds and error bounds were not given. The successful results of numerical experiments were presented. Note that other two-level vector scheme with the first time derivative as the additional sought function are well-known, for example, see \[1,8,25,26\].

In this paper, the scheme from \[7\] is generalized to the case of the \(n\)-dimensional wave equation, \(n \geq 1\), and the uniform rectangular mesh (recall that the square mesh cannot be constructed in any rectangular domain), and the scheme is classified as a vector one more systematically. The new explicit two-level vector equations for the approximate solution at the first time level are constructed too. They are similar to the main equations and exploit only second order finite-differences of the initial functions. Notice that we do not use derivatives of the free term and initial functions in contrast to \[7\] that is essential for applications of the scheme in the non-smooth case, for example, see \[8,10,22,24\]. The scheme is new even in the simplest 1D case.

Next, new results on the conditional stability of the scheme in the mesh energy norm and the discrete energy conservation law are derived from the corresponding general results in \[24,26\] (proved by the energy method) by excluding the auxiliary sought functions. The case of general free terms in the main and auxiliary equations of the scheme is treated that is essential for the completeness of the stability analysis. The stability bound and the energy law contain the operators \(s^{-1}_{kN}\) inverse to the Numerov averages in \(k\)th space direction, \(1 \leq k \leq n\). The imposed stability conditions are shown to be very close to those arising in the spectral analysis, and the sufficient conditions on their validity in the standard terms of the ratios of the time and spatial steps are given. Alternative forms of the stability bound and the energy conservation law not exploiting the operators \(s^{-1}_{kN}\) are presented too. The 4th order error bound in the mesh energy norm is also rigorously proved for the first time based on the derived stability theorem.

Finally, the generalizations of the scheme to the cases of the non-uniform rectangular mesh in space and the non-uniform mesh in time are presented following \[24\], and the important case of the wave equation with variable coefficients is covered too. For general non-uniform meshes, the scheme has the 3th approximation order as other compact schemes. The non-uniform mesh in time is important to accomplish a dynamic choice of the time step. The scheme becomes non-local in time in this case but it still remains explicit in time and simply implemented. In the case of variable coefficients, the algorithm of the scheme implementation and its cost do not change, and the cost is similar to the ADI methods once again.

The paper is organized as follows. Section 2 is devoted to the construction of the scheme. In Section 3 the main theorems are proved. Section 4 deals with the generalizations of the scheme.
2 Construction of an explicit in time 4th-order compact vector scheme for the $n$-dimensional wave equation

We deal with the following initial-boundary value problem (IBVP) for the $n$-dimensional wave equation under the nonhomogeneous Dirichlet boundary condition

\begin{align}
\partial_t^2 u(x,t) - Lu(x,t) &= f(x,t), \quad L := a_{11}^2 \partial_{x_1}^2 + \ldots + a_{nn}^2 \partial_{x_n}^2, \quad \text{in } QT = \Omega \times (0,T); \\
\partial_t u|_{\Gamma_T} &= g(x,t), \quad u|_{t=0} = u_0(x), \quad \partial_t u|_{t=0} = u_1(x), \quad x \in \Omega = (0,X_1) \times \ldots \times (0,X_n). \quad (2.1)-(2.2)
\end{align}

Here $a_1 > 0, \ldots, a_n > 0$ are constants (we take them different for uniformity with $[22, 24]$), $x = (x_1, \ldots, x_n)$, $\Omega = (0,X_1) \times \ldots \times (0,X_n)$, $n \geq 1$. Also $\partial \Omega$ is the boundary of $\Omega$ and $\Gamma_T = \partial \Omega \times (0,T)$ is the lateral surface of $QT$.

We first rewrite formally the wave equation (2.1) as a system of equations, with a unique second order partial derivative in $t$ or $x_k$ in each of them:

\begin{align}
\partial_t^2 u(x,t) - (a_{11}^2 u_{11}(x,t) + \ldots + a_{nn}^2 u_{nn}(x,t)) &= f(x,t), \quad \text{in } QT, \quad (2.3) \\
u_{kk}(x,t) := \partial_t^2 u(x,t), \quad 1 \leq k \leq n, \quad \text{in } QT, \quad (2.4)
\end{align}

where $u_{11}, \ldots, u_{nn}$ are the auxiliary sought functions. We consider smooth in $\bar{\Omega}$, solutions to the IBVP (2.1)-(2.2), then (2.3) implies

\begin{align}
\partial_t^2 u = \partial_t^2 (Lu + f) = L\partial_t^2 u + \partial_t^2 f = L(a_{11}^2 u_{11} + \ldots + a_{nn}^2 u_{nn} + f) + \partial_t^2 f. \quad (2.5)
\end{align}

Define the uniform mesh $\omega_{h_t} = \{t_m = mh_t\}_{m=0}^{M}$ on a segment $[0,T]$, with the step $h_t = T/M > 0$ and $M \geq 2$. Let $\omega_{h_x} = \{t_m\}_{m=1}^{M-1}$ be the internal part of $\omega_{h_t}$. Introduce the mesh averages (including the Numerov one) and difference operators in $t$

\begin{align}
s_t y = \frac{1}{2} (\dot{y} + y), \quad s_t y = \frac{1}{2} (y + \dot{y}), \quad s_t N y = \frac{1}{12} (\dot{y} + 10y + \ddot{y}), \\
s_t y = \frac{y - \dot{y}}{h_t}, \quad \delta_t y = \frac{\dot{y} - y}{h_t}, \quad \delta_t y = \frac{\dot{y} - y}{2h_t}, \quad \Lambda_t y = \delta_t \delta_t y = \frac{\dot{y} - 2y + \ddot{y}}{h_t^2}
\end{align}

with $y^m = y(t_m)$, $\dot{y}^m = y^{m-1}$ and $\ddot{y}^m = y^{m+1}$, as well as the operator of summation with the variable upper limit

\begin{align}
I^m_{h_t} y = h_t \sum_{l=1}^{m} y^l \quad \text{for } 1 \leq m \leq M, \quad I^0_{h_t} y = 0.
\end{align}

Due to Taylor’s formula in $t$, the wave equation (2.1) and formula (2.5), we get

\begin{align}
\Lambda_t u = \partial_t^2 u + \frac{1}{12} h_t^2 \partial_t^4 u + O(h_t^4) = a_{11}^2 u_{1111} + \ldots + a_{nn}^2 u_{nnnn} + \frac{1}{12} h_t^2 L(a_{11}^2 u_{1111} + \ldots + a_{nn}^2 u_{nnnn}) \\
+ f + \frac{1}{12} h_t^2 (L + \Lambda_t)f + O(h_t^4) \quad \text{on } \omega_{h_t}, \quad (2.6)
\end{align}

where we prefer to use $\Lambda_t f$ instead of $\partial_t^2 f$ in $[7]$.

Define the uniform mesh $\omega_{h_k} = \{x_{kl} = lh_k, 0 \leq l \leq N_k\}$ in $x_k$ with the step $h_k = \frac{X_k}{N_k}$. Let $\omega_{h_k} = \{x_{kl}, 1 \leq l \leq N_k - 1\}$ be its internal part. Introduce the standard difference approximation to $\partial_t^2 w$ and the Numerov average in $x_k$:

\begin{align}
(\Lambda_k w)_l := \frac{1}{h_k^2} (w_{l+1} - 2w_l + w_{l-1}), \quad s_{kN} w_l := \frac{1}{12} (w_{l+1} + 10w_l + w_{l-1}) = (I + \frac{1}{12} h_k^2 \Lambda_k) w_l \quad (2.7)
\end{align}

on $\omega_{h_k}$, where $w_l = w(x_{kl})$. 

3
Define the uniform rectangular mesh \( \bar{\omega}_h = \bar{\omega}_{h1} \times \ldots \times \bar{\omega}_{hK} \) in \( \bar{\Omega} \) with \( h = (h_1, \ldots, h_n) \). Let \( \omega_h = \omega_{h1} \times \ldots \times \omega_{hK} \) and \( \partial \omega_h = \partial \omega_{h1} \times \ldots \times \partial \omega_{hK} \) be the internal part and boundary of \( \bar{\omega}_h \). Define also the meshes \( \omega_a := \omega_h \times \omega_{h1} \) in \( Q_T \) and \( \partial \omega_a = \partial \omega_h \times \{ t_m \}_{m=1}^M \) on \( \Gamma_T \), where \( h = (h, h_t) \).

Let \( H_h \) be the Euclidean space of functions defined on \( \bar{\omega}_h \) and equal 0 on \( \partial \omega_h \) and endowed with the inner product
\[
(v, w)_h = h_1 \ldots h_n \sum_{x_k \in \omega_h} v(x_k)w(x_k), \quad k = (k_1, \ldots, k_n).
\]
Any operator \( C_h = C^*_h > 0 \) in \( H_h \) generates the norm \( \|w\|_c = (C_h w, w)_h^{1/2} \) in \( H_h \). Recall that
\[
0 < -\Lambda_k \leq \lambda_{\max k} I \quad \text{with} \quad \lambda_{\max k} = \frac{1}{h_k} \sin \frac{(k + 1)\pi}{2} < \frac{1}{h_k}, \quad \frac{2}{3} I < s_{kN} < I \quad \text{in} \quad H_h. \tag{2.8}
\]

Let \( L_h \) be a difference operator such that \( Lw - L_h w = O(|h|^2) \) on \( \omega_h \) for sufficiently smooth \( w \) in \( \bar{\omega}_h \). Various \( L_h \) can be used but below we confine ourselves by the simplest operator
\[
L_h := a_1^2 \Lambda_1 + \ldots + a_n^2 \Lambda_n \quad \text{such that} \quad 0 < -L_h < 4 \left( \frac{a^2}{h_1^2} + \ldots + \frac{a^2}{h_n^2} \right) I \quad \text{in} \quad H_h. \tag{2.9}
\]
Then from expansion (2.6) we immediately get
\[
\Lambda_t u = (I + \frac{1}{12} h_t^2 L_h)(a_1^2 u_{11} + \ldots + a_n^2 u_{nn}) + f_h + O(|h|^4) \quad \text{on} \quad \omega_h, \tag{2.10}
\]
where \( I \) is the identity operator and
\[
f_h := f + \frac{1}{12} h_t^2 (L_h + \Lambda_t) f = (s_{1N} + \frac{1}{12} h_t^2 L_h) f.
\]
Using the Numerov approximation in \( x_k \) for equation (2.4) treated as an ordinary differential equation (ODE) in \( x_k \), we have
\[
s_{kN} u_{kk} - \Lambda_k u = O(h_k^4) \quad \text{on} \quad \omega_h, \quad 1 \leq k \leq n. \tag{2.11}
\]

Omitting the residual terms in the last two expansions (2.10)-(2.11), we pass to the three-level explicit in time vector compact scheme for the wave equation
\[
\Lambda_t v = (I + \frac{1}{12} h_t^2 L_h)(a_1^2 v_{11} + \ldots + a_n^2 v_{nn}) + f_h \quad \text{on} \quad \omega_h, \tag{2.12}
\]
\[
s_{kN} v_{kk} = \Lambda_k v \quad \text{on} \quad \omega_h, \quad 1 \leq k \leq n. \tag{2.13}
\]
Here the sought functions \( v \approx u \) and \( v_{11} \approx u_{11} \), \ldots, \( v_{nn} \approx u_{nn} \) are defined on \( \bar{\omega}_h \times \bar{\omega}_h \) and \( \bar{\omega}_h \times \omega_h \), respectively. We can set the discrete boundary conditions
\[
v|_{\partial \omega_h} = g, \quad a_k^2 v_{kk}|_{\partial \omega_h} = g_k, \quad 1 \leq k \leq n, \tag{2.14}
\]
where in accordance with the wave equation in \( \bar{Q}_T \) and the boundary condition \( u|_{\Gamma_T} = g \) one can calculate \( \partial_t^2 u \) on \( \Gamma_T \) and thus set
\[
g_k := \partial_t^2 g - \sum_{1 \leq l \leq n, l \neq k} a_l^2 \partial_t^2 g - f \quad \text{for} \quad x_k = 0, X_k, \quad g_k := a_k^2 \partial_t^2 g \quad \text{for} \quad x_l = 0, X_l, \quad 1 \leq l \leq n, \ l \neq k,
\]
on \( \Gamma_T \). Recall that equation (2.13) is the usual Numerov scheme for the ODE (2.4) in \( x_k \).

The constructed scheme is explicit in time. The reason is that \( \hat{v} \) is found explicitly on \( \omega_h \) from (2.12), since \( \Lambda_t v = d \) means that \( \hat{v} = 2v - \hat{v} + h_t^2 d \), provided that the functions \( v_{11}, \ldots, v_{nn} \) are known on \( \bar{\omega}_h \). The scheme is not completely explicit since \( v_{11}, \ldots, v_{nn} \) are (easily) computed
from 1D difference equations (2.13) with the boundary conditions from (2.14) by solving the collections of corresponding linear algebraic systems with tridiagonal matrices. In this respect, the implementation is similar to the ADI methods. But the difference is that the functions \(v_{11}^m, \ldots, v_{nm}^m\) are independent and can be computed not sequentially but in parallel when \(v^m\) is known. Moreover, there is no need to store all of them simultaneously since only their weighted sum is used to compute \(v^{m+1}\), for \(m = 0, \ldots, M - 1\).

We need also to define \(v^1\) at the first time level \(t = h_t\) with the 4th order of accuracy. This can be done explicitly by using Taylor’s formula and the wave equation, for example, see [25, 27, 18]. But these formulas involve higher-order derivatives (or differences) of the initial functions \(u_0\) and \(u_1\) that is inconvenient in the non-smooth case. We prefer to avoid this and proceed alternatively following [24, 25] and construct the explicit equation for \(v^1\) in the form close to (2.12)-(2.13):

\[
(\delta_t v)^0 = \frac{1}{2} h_t (I + \frac{1}{12} h_t^2 L h) (a_1^2 v_{11}^0 + \ldots + a_n^2 v_{nn}^0) + u_{1h} + \frac{1}{2} h_t f_h^0 \quad \text{on } \omega_h, \tag{2.15}
\]

\[
s_{Kn} v_{kk}^0 = \Lambda_k v^0 \quad \text{on } \omega_h, \quad 1 \leq k \leq n, \tag{2.16}
\]

with suitable \(u_{1h} \approx u_1\) and \(f_h^0 \approx f|_{t=0}\). The values of \(v_{kk}^0\) on \(\partial \omega_h\) can be taken as in (2.14) (or according to the values of \(\partial_t^2 u_0\) on \(\partial \Omega\)). See additional arguments in favor of these equations for \(v^1\) after the proof of Theorem 3.2 below.

**Lemma 2.1.** For the given functions

\[
u_{1h} = (I + \frac{1}{6} h_t^2 L h) u_1 \quad \text{on } \omega_h, \tag{2.17}
\]

\[
f_h^0 = f_{dh}^0 + \frac{1}{12} h_t^2 L_h f_h^0, \quad f_{dh}^0 = f_d^0 + O(h_t^4) \quad \text{on } \omega_h, \tag{2.18}
\]

where \(f_d^0 := f_0 + \frac{1}{2} h_t (\partial_t f_0) + \frac{1}{12} h_t^2 (\partial_t^2 f_0)\) with \(y_0 := y|_{t=0}\) for any \(y\) (see explicit forms of \(f_d^0\) in Remark 2.2 below), the following estimate for the approximation error of equation (2.15) holds

\[
\psi^0 := (\delta_t u)^0 - \frac{1}{2} h_t (I + \frac{1}{12} h_t^2 L h) (a_1^2 u_{110} + \ldots + a_n^2 u_{nn0}) - u_{1h} - \frac{1}{2} h_t f_h^0 = O(|h|^4) \tag{2.19}
\]

on \(\omega_h\).

**Proof.** Using Taylor’s formula at \(t = 0\), the initial conditions (2.2), the wave equation and formula (2.5), we get

\[
(\delta_t u)^0 = u_1 + \frac{1}{2} h_t (L u_0 + f_0) + \frac{1}{6} h_t^2 [L u_1 + (\partial_t f)_0] + \frac{1}{12} h_t^3 [L (L u_0 + f_0) + (\partial_t^2 f)_0] + O(h_t^4).
\]

Since \(L u_0 = a_1^2 u_{110} + \ldots + a_n^2 u_{nn0}\), we derive

\[
\psi^0 = u_1 + \frac{1}{6} h_t^2 L u_1 - u_{1h} + \frac{1}{2} h_t f_d^0 + \frac{1}{12} h_t^2 L f_0 - f_h^0 + O(h_t^4).
\]

For functions (2.17)-(2.18), this implies formula (2.19). \(\Box\)

**Remark 2.1.** If \(u_0\) is smooth and known analytically, then equation (2.15) can be simplified as

\[
(\delta_t v)^0 = \frac{1}{2} h_t (I + \frac{1}{12} h_t^2 L h) L u^0 + u_{1h} + \frac{1}{2} h_t f_h^0 \quad \text{on } \omega_h, \tag{2.20}
\]

with omitting equations (2.16), and formula (2.19) remains valid. But, otherwise, equation (2.20) cannot be used and equations (2.15)-(2.16) become preferable and are more general.

**Remark 2.2.** Let \(0 < h_t \leq \bar{h}_t \leq T\). If \(f\) is sufficiently smooth in \(t\) in \(\bar{Q}_{h_t}\) (or \(\Omega \times [\bar{h}_t, \bar{h}_t]\)), then \(f_{dh}^0 = f_d^0 + O(h_t^3)\) (see (2.18) for the following three- and two-level approximations

\[
f_{dh}^0 = \frac{7}{12} f^0 + \frac{1}{4} f^1 - \frac{1}{12} f^2, \quad f_{dh}^0 = \frac{1}{3} f^0 + \frac{3}{2} f^{1/2} \quad \text{with } f^{1/2} := f|_{t=h_t/2} \tag{2.21}
\]

(or \(f_{dh}^0 = f^0 + \frac{3}{2} h_t \hat{h}_t f^0 + \frac{1}{12} h_t^2 \Lambda f^0 = -\frac{1}{12} f^{-1} + \frac{1}{2} f^0 + \frac{3}{4} f^1 \text{ with } f^{-1} := f|_{t=-h_t}\). See [24, Remark 3]. These formulas are checked easily applying Taylor’s formula at \(t = 0\).
3 Stability and error bounds for the explicit 4th-order compact scheme

To study stability in more detail, we need to take the inhomogeneous version of equations (2.13) and (2.16):

\[ s_{km}v_{kk}^m = \Lambda_k v^m + b_k^m \quad \text{on} \quad \omega_h, \quad 0 \leq m \leq M - 1, \quad 1 \leq k \leq n, \quad (3.1) \]

where \( b_1, \ldots, b_n \) are given functions. Note that in practice these functions are never identically zero due to the round-off errors in computations, so their influence on the solution have to be studied. This is also necessary to derive an error bound below.

To state the stability theorem, we introduce the following self-adjoint operator

\[ A_h := (I + \frac{1}{12}h_t^2L_h)(-a_1^2s_{1N}^{-1} \Lambda_1 - \ldots - a_n^2s_{nN}^{-1} \Lambda_n) \quad \text{in} \quad H_h. \quad (3.2) \]

We impose the following condition between the steps in time and space

\[ \frac{1}{3}h_t^2L_h \leq 1 - \varepsilon \quad \text{for some} \quad 0 \leq \varepsilon < 1. \quad (3.3) \]

The second inequality (2.8) implies \( I < s_{1N}^{-1} < \frac{3}{2}I \) in \( H_h \), and thus inequalities (2.9) and this condition ensure that

\[ \varepsilon I < I + \frac{1}{12}h_t^2L_h < I \Rightarrow 0 \leq -\varepsilon L_h < A_h < (I - \frac{1}{12}h_t^2L_h)(-\frac{3}{2}I) < -\frac{3}{2}I \quad \text{in} \quad H_h. \quad (3.4) \]

We also define the mesh norm \( \| y \|_{L_h^1(H_h)} := \frac{1}{h_t}\| y^0 \|_h. \)

**Theorem 3.1.** Let \( g = 0 \) and \( f|_{\Gamma_R} = 0 \). Then, for scheme (2.12), (2.14), (2.16) and (3.1), under the conditions (3.3) and

\[ \frac{1}{4}h_t^2A_h \leq (1 - \varepsilon_0^2)I \quad \text{for some} \quad 0 < \varepsilon_0 < 1, \quad (3.5) \]

the following stability bound

\[ \max_{1 \leq m \leq M} (\varepsilon_0^2\| \delta_t v^m \|_h^2 + \| \delta_t v^m \|_{A_h}^2) \leq (\| v^0 \|_{A_h}^2 + \varepsilon_0^{-2}\| u_{1h} \|_h^2)^{1/2} + 2\varepsilon_0^{-1}\| f_h + \beta_h \|_{L_h^1(H_h)} \quad (3.6) \]

and the corresponding discrete energy conservation law

\[ \| \tilde{\delta}_tv^m \|_h^2 - \frac{1}{2}h_t^2\| \delta_t v^m \|_{A_h}^2 + \| \tilde{\delta}_tv^m \|_{A_h}^2 = \| \delta_t v^1 \|_h^2 - \frac{1}{2}h_t^2\| \delta_t v^1 \|_{A_h}^2 + 2I_{h_t}^{-1}(f_h + \beta_h, \tilde{\delta}_tv) \]

\[ = (A_h v^0, s_tv^0)_h + (u_1, \delta_t v^0)_h + \frac{1}{2}h_t(f_h + \beta_h, \delta_t v^0)_h + 2I_{h_t}^{-1}(f_h + \beta_h, \tilde{\delta}_tv)_h, \quad (3.7) \]

for \( 1 \leq m \leq M \), are valid. Here

\[ \beta_h := (I + \frac{1}{12}h_t^2L_h)(a_1^2s_{1N}^{-1}b_1 + \ldots + a_n^2s_{nN}^{-1}b_n) \quad \text{in} \quad H_h \quad \text{on} \quad \{ t_m \}_{m=0}^{M-1}, \quad (3.8) \]

and functions \( f_h, b_1, \ldots, b_n: \{ t_m \}_{m=0}^{M-1} \rightarrow H_h \) and \( v^0, u_{1h} \in H_h \) are any \((f_h \text{ and } u_{1h} \text{ are not only those specific defined above). The first equality \( (3.7) \) holds for any } v^1 \in H_h \text{ (not only defined by equations } (2.13),(2.16)). \)
Proof. 1. First, we recall the general three-level scheme

\[ B_h A_h v + A_h v = \varphi \quad \text{in} \ H_h \quad \text{on} \ \omega_{h^-}, \]  
(3.9)

\[ B_h (\delta v)^0 + \frac{1}{2} h_t A_h v^0 = u_1 + \frac{1}{2} h_t \varphi^0 \quad \text{in} \ H_h \]  
(3.10)

(with the weight \( \sigma = 0 \)), in particular, see \[22,24,26\]. Here \( B_h = B_h^* > 0 \) and \( A_h = A_h^* > 0 \) are any operators in \( H_h \) related by the inequality

\[ \frac{1}{4} h_t^2 A_h \leq (1 - \varepsilon_0^2) B_h \quad \text{for some} \quad 0 < \varepsilon_0 < 1. \]  
(3.11)

Then the following stability bound

\[ \max_{1 \leq m \leq M} \left( \varepsilon_0^2 \| \delta_t v^m \|^2_{B_h} + \| \bar{s}_t v^m \|^2_{A_h} \right)^{1/2} \leq \left( \| v^0 \|_{A_h}^2 + \varepsilon_0^{-2} \| B_h^{-1/2} u_{1h} \|_{h}^2 \right)^{1/2} + 2 \varepsilon_0^{-1} \| B_h^{-1/2} f \|_{L^2_h(H_h)} \]  
(3.12)

and the discrete energy conservation law (that entails the stability bound)

\[ \| \delta_t v^m \|^2_{B_h} - \frac{1}{4} h_t^2 \| \delta_t v^m \|^2_{A_h} \leq \| \delta_t v^1 \|^2_{B_h} - \frac{1}{4} h_t^2 \| \delta_t v^1 \|^2_{A_h} + 2 I_{h_1}^{-1} (\varphi, \delta_t v)_h \]

\[ = (A_h v^0, s_t v^0)_h + (u_1, \delta_t v^0)_h + \frac{1}{2} h_t (\varphi^0, \delta_t v^0)_h + 2 I_{h_1}^{-1} (\varphi, \delta_t v)_h, \quad 1 \leq m \leq M, \]  
(3.13)

are valid according to \[24\] Theorem 1 and the proof of Theorem 1 in \[26\] (see also \[17\]). Notice that the first and second equalities (3.13) are valid respectively for any \( v^1 \in H_h \) and \( v^1 \) defined by equation (3.10). Also inequality (3.11) ensures that

\[ \varepsilon_0^2 \| \delta_t v^m \|^2_{B_h} \leq \| \delta_t v^0 \|^2_{B_h} - \frac{1}{4} h_t^2 \| \delta_t v^0 \|^2_{A_h} \forall v^m \in H_h. \]

2. Since \( g = 0 \) and \( f \big|_{1^T} = 0 \), clearly \( v: \mathbb{N}_h \to H_h \) and \( v_{kk}: \{ t_m \}_{m=0}^M \to H_h \), \( 1 \leq k \leq n \). From equation (3.11), we can express \( v_{kk} \) through \( v \):

\[ v_{kk} = s_{kN}^{-1} (\Lambda_k v + b_k) \quad \text{in} \ H_h \quad \text{on} \ \{ t_m \}_{m=0}^{M-1}, \quad 1 \leq k \leq n. \]

Inserting these formulas to equations (2.12) and (2.15), we get the closed equations for \( v \):

\[ \Lambda_t v = \left( I + \frac{1}{12} h_t^2 L_h \right) \left( a_{11}^2 s_{1N}^{-1} \Lambda_1 + \ldots + a_n^2 s_{nN}^{-1} \Lambda_n \right) v + f_h + \beta_h \quad \text{in} \ H_h \quad \text{on} \ \omega_{h^-}, \]  
(3.14)

\[ (\delta_t v)^0 = \frac{1}{2} h_t \left( I + \frac{1}{12} h_t^2 L_h \right) \left( a_{11}^2 s_{1N}^{-1} \Lambda_1 + \ldots + a_n^2 s_{nN}^{-1} \Lambda_n \right) v^0 + u_{1h} + \frac{1}{2} h_t (f_0^0 + \beta_0^0) \quad \text{in} \ H_h, \]  
(3.15)

with \( \beta_h \) defined in (3.8).

These equations present a particular case of equations (3.9)-(3.10) with the operators \( B_h := I \) and \( A_h := A_h \) given in (3.29). Then the stability condition (3.11) takes the form (3.5), and the stability bound (3.6) and energy conservation law (3.7) follow from respective general relations (3.12) and (3.13). In addition, the following inequality holds

\[ \varepsilon_0^2 \| \delta_t v^m \|^2_{B_h} \leq \| \delta_t v^m \|^2_{B_h} - \frac{1}{4} h_t^2 \| \delta_t v^m \|^2_{A_h} \]

that is essential on the left in (3.7).

\[ \square \]
Note that the norms of \( f_h + \beta_h \) and \( u_{1h} \) in (3.6) are bounded uniformly in \( h \) by the same norms of \( f, b_1, \ldots, b_n \) and \( u_1 \) due to the stability condition (3.3).

Let us compare the above stability conditions with the corresponding conditions arising in the frequently used spectral method (it was used in [7] as well). In this method, one can consider the system of eigenvectors of the operator \(-L_h\) in \( H_h\):

\[-L_h e_\ell = \lambda_\ell (-L_h)e_\ell \text{ on } \omega_h, \quad e_\ell = \sin \frac{\pi l_1}{X_1} \cdots \sin \frac{\pi l_n}{X_n}, \quad 1 \leq l_1 \leq N_1 - 1, \ldots, 1 \leq l_n \leq N_n - 1.\]

with \( \ell = (l_1, \ldots, l_n) \). Hereafter \( \lambda_\ell (A_h) \) are the corresponding eigenvalues of the operator \( A_h \).

Inserting the solutions in the form \( v^m = e_\ell y^m \), \( m \geq 0 \) into the homogeneous equation (3.14) with \( f_h + \beta_h = 0 \), we get the difference equation for \( y \):

\[\Lambda_I y + \lambda_\ell y = 0 \iff \hat{y} - (2 - h^2 \lambda_\ell) y + \tilde{y} = 0 \text{ on } \omega_h,\]

where

\[A_h e_\ell = \lambda_\ell e_\ell \text{ on } \omega_h, \quad \lambda_\ell = \left[1 - \frac{1}{12} h^2 \lambda \right] \lambda \ell \left(-\alpha^2_{11} - \ldots - \alpha^2_{nn} \right) + \lambda (\ell) - \omega^2_{11} \ell_1 - \ldots - \omega^2_{nn} \ell_n].\]

The corresponding characteristic equation is

\[q^2 - (2 - h^2 \lambda_\ell)q + 1 = 0,\]

and the inequality \(|q_{1,2}| \leq 1\) for its roots is the standard spectral stability condition. But we prefer to exclude the cases \( q_1 = q_2 = \pm 1 \) since they allows \( y^m \) to be unbounded in \( m \) in contrast to the stability bound (3.6). This leads to the condition \((2 - h^2 \lambda_\ell) - 4 < 0\) for the discriminant of the quadratic equation, i.e., the condition \(0 < \frac{1}{2} h^2 \lambda_\ell < 1\) for any \( \ell \) that is clearly equivalent to the operator inequalities

\[0 < I + \frac{1}{12} h^2 L_h, \quad \frac{1}{4} h^2 A_h < I \text{ in } H_h.\]

They are respectively equivalent to the left inequality (3.4) for \( \varepsilon = 0 \) and close to condition (3.5).

But the last operator inequalities ensure only a weakened form of stability. Due to the second inequality (3.4), the norm \( \| \cdot \|_{A_h} \) can be bounded from above by \( \| \cdot \|_{-L_h} \) and from below as well provided that \( \varepsilon > 0 \); so it is desirable that \( 0 < \varepsilon < 1 \). Also the case \( 0 < \varepsilon_0 < 1 \) is essential to ensure stability in the full mesh energy norm.

The second inequality (3.4) implies that condition (3.5) follows from the number inequality

\[
\frac{1}{4} h^2 (1 - \frac{1}{12} h^2 \lambda) \leq 1 - \varepsilon_0^2 \quad \forall \lambda = \lambda (\ell) (-L_h). \tag{3.16}
\]

This inequality follows from the quadratic inequality

\[
\frac{3}{2} (1 - \frac{p}{3}) p \leq 1 - \varepsilon_0^2 \quad \forall 0 < p \leq h^2 (\frac{\alpha^2_{11}}{h^2} + \ldots + \frac{\alpha^2_{nn}}{h^2}). \tag{3.17}
\]

Solving it, we obtain that both conditions (3.3) and (3.5) are valid provided that

\[
h^2 (\frac{\alpha^2_{11}}{h^2} + \ldots + \frac{\alpha^2_{nn}}{h^2}) \leq \min \left\{ 3(1 - \varepsilon), \frac{1}{2} [3 - (1 + 8 \varepsilon_0^{1/2})] \right\};
\]

note that here \( \frac{3}{2} (1 - \varepsilon_0^2) < \frac{1}{2} [3 - (1 + 8 \varepsilon_0^{1/2})]. \]
Remark 3.1. Inequality (3.16) is not precise since it used the inequality $s_{kN}^{-1} \leq \frac{\bar{s}}{2} I$ in $H_h$, $1 \leq k \leq n$. Involving eigenvalues of $s_{kN}$ leads to the multivariate inequality

$$[1 - \frac{1}{2} (p_1 \sigma_1 + \ldots + p_n \sigma_n)] \left( \frac{p_1 \sigma_1}{1-\frac{1}{2} \sigma_1} + \ldots + \frac{p_n \sigma_n}{1-\frac{1}{2} \sigma_n} \right) \leq 1 - \frac{\varepsilon_0^2}{2} \forall \sigma_1, \ldots, \sigma_n \in [0, 1],$$

with $p_1 = h_t^2 \frac{\sigma_1}{h_i^2}, \ldots, p_n = h_t^2 \frac{\sigma_n}{h_i^2}$, more accurate than (3.16). Its solving is possible but rather cumbersome even for $n = 1$, and here we will not come into details (see also [7] for $p_1 = p_2$, $\varepsilon_0 = 0$ and $n = 2$).

Equations (3.14) and (3.19) that appeared in the last proof together with (3.8) can be written without inverse operators which is more standard for other compact schemes. Applying the operator $\tilde{s}_N := s_1N \ldots s_nN$ to them, we obtain

$$\tilde{s}_N \Lambda_t v + (I + \frac{1}{12} h_t^2 L_h) \tilde{A}_N v = \tilde{s}_N f_h + \tilde{s}_N \beta_h \text{ in } H_h \text{ on } \omega_{ht},$$

$$\tilde{s}_N (\tilde{u}^L t)^0 + \frac{1}{2} h_t (I + \frac{1}{12} h_t^2 L_h) \tilde{A}_N v^0 = \tilde{s}_N u_{1h} + \frac{1}{2} h_t (\tilde{s}_N f_h^0 + \tilde{s}_N \beta_h^0) \text{ in } H_h,$$

$$\tilde{s}_N \beta_h = (I + \frac{1}{12} h_t^2 L_h) (\tilde{s}_1 b_1 + \ldots + \tilde{s}_n b_n) \text{ in } H_h \text{ on } \{t_m\}_{m=0}^{M-1}$$

since $L_h$ commutes with $\tilde{s}_N$. Here we use the operators

$$\tilde{A}_N := - (a_1^2 \tilde{s}_N \Lambda_1 + \ldots + a_n^2 \tilde{s}_N \Lambda_n), \quad \tilde{s}_N := \prod_{1 \leq k \leq n, k \neq l} s_{kN}$$

such that $\tilde{A}_N = \tilde{A}_N > 0$ and $0 < \tilde{s}_{kN} = \tilde{s}_{kN} < I$ in $H_h$, $1 \leq k \leq n$. According to relations (3.12)-(3.13), these equations imply other forms of the stability bound

$$\max_{1 \leq m \leq M} (\varepsilon_0^2 \| \tilde{u}^v m \|^2 \tilde{s}_N + \| \tilde{v}^m \|^2 \tilde{A}_N) \leq \left( \| v^0 \| \tilde{A}_N + \varepsilon_0^2 \| u_{1h} \|^2 \tilde{s}_N \right)^{1/2} + 2 \varepsilon_0^{-1} \| \tilde{s}_N^{1/2} (f_h + \beta_h) \|_{L_h^2(H_h)}$$

and the corresponding discrete energy conservation law

$$\| \tilde{v}^m \|^2 \tilde{s}_N - \frac{1}{4} h_t^2 \| \tilde{u}^v m \|^2 \tilde{A}_N + \| \tilde{v}^m \|^2 \tilde{A}_N = \left( \tilde{A}_N v^0, s_t v^0 \right)_h + (\tilde{s}_N u_{1h}, \tilde{v}^0)_h + \frac{1}{2} h_t (\tilde{s}_N (f_h^0 + \beta_h^0), \tilde{v}^0)_h$$

for $1 \leq m \leq M$. Here the operators $\tilde{s}_N$ and $\tilde{A}_N$ satisfy the inequalities

$$\left( \frac{\bar{s}}{2} \right)^n I < \tilde{s}_N < I, \quad \varepsilon (\frac{\bar{s}}{2})^{-1} (-L_h) < \varepsilon \tilde{A}_N < \tilde{A}_N := (I + \frac{1}{12} h_t^2 L_h) \tilde{A}_N < \tilde{A}_N < -L_h \text{ in } H_h.$$ 

Consequently, in the stability bound (3.21), the norms $\| \cdot \|_{\tilde{s}_N}$ and $\| \cdot \|_{\tilde{A}_N}$ can be bounded from above and below by respectively $\| \cdot \|_h$ and $\| \cdot \|_{\tilde{A}_N}$ (or $\| \cdot \|_{-L_h}$) for $0 < \varepsilon < 1$, and $\tilde{s}_N^{1/2}$ can be omitted in front of $f_h + \beta_h$.

Note that the scheme in the form (3.18)-(3.20) is not compact since it involves the product $L_h \tilde{A}_N$. The related implicit conditionally stable 4th-order compact scheme

$$\tilde{s}_N + \frac{1}{12} h_t^2 \tilde{A}_N \Lambda_t v + \tilde{A}_N v = s_N f + \frac{1}{12} h_t^2 \Lambda_t f, \quad \text{in } H_h \text{ on } \omega_{ht},$$

$$\tilde{s}_N + \frac{1}{12} h_t^2 \tilde{A}_N (\tilde{u}^L t)^0 + \frac{1}{2} h_t \tilde{A}_N v^0 = (s_N + \frac{1}{12} h_t^2 L_h) u_1 + \frac{1}{2} h_t \left[ f^0_{\tilde{u}^L t} + (s_N - I) f^0 \right] \quad \text{in } H_h,$$

together with its ADI version (which is not reproduced here) with the same properties have recently been studied in detail in [24]. Here $s_N := I + \frac{1}{12} h_t^2 \Lambda_t f + \ldots + \frac{1}{12} h_t^2 \Lambda_n$.

The next main result concerns the 4th order error bound and is based on Theorem 3.1. Notice that $g$ and $f|_{\Gamma_T}$ can be general (not only zero) in it.
Theorem 3.2. Under the stability conditions (3.3) and (3.5), the following 4th order error bound in the mesh energy norm for scheme (2.12)-(2.16) with \( v^0 = u^0 \) on \( \bar{\omega}_h \) holds

\[
\max_{1 \leq m \leq M} \left( \varepsilon_0 \| \bar{s}_t (u - v)^m \|_h + \sqrt{\varepsilon} \| \bar{s}_t (u - v)^m \|_{-L_h} \right) = O(|h|^4). \tag{3.23}
\]

Proof. We define the approximation errors of equations (2.12)-(2.13) as follows

\[
\psi := \Lambda_t u - \left( I + \frac{1}{12} h_t^2 L_h \right) (a^0_t a_{11} + \ldots + a^0_n a_{nn}) - f_h \quad \text{on} \quad \omega_h, \tag{3.24}
\]

\[
\psi_{kk} := s_k u_{kk} - \Lambda_k u \quad \text{on} \quad \omega_h, \quad 1 \leq k \leq n. \tag{3.25}
\]

Formulas (2.10)-(2.13) together with (2.19) mean that the approximation errors of equations (2.12), (2.13) and (2.15)-(2.16) has the 4th order:

\[
\max_{\omega_h} |\psi| + \max_{\omega_h} |\psi|^0 + \max_{0 \leq m \leq M-1} \max_{\omega_h} \left( |\psi|^1 + \ldots + |\psi|^m \right) = O(|h|^4). \tag{3.26}
\]

Due to the equations for \( v, v_{11}, \ldots, v_{nn} \) as well as the definitions of \( \psi, \psi_{kk} \) and \( \psi^0 \), the errors \( r := u - v, r_{11} := u_{11} - v_{11}, \ldots, r_{nn} := u_{nn} - v_{nn} \) satisfy the following equations

\[
\Lambda_t r - \left( I + \frac{1}{12} h_t^2 L_h \right) (a^0_t a_{11} + \ldots + a^0_n a_{nn}) = \psi \quad \text{on} \quad \omega_h,
\]

\[
s_k r_{kk} - \Lambda_k r = \psi_{kk} \quad \text{on} \quad \omega_h, \quad 1 \leq k \leq n,
\]

\[
(\delta_t r)^0 - \frac{1}{2} h_t \left( I + \frac{1}{12} h_t^2 L_h \right) (a^0_t a_{11} + \ldots + a^0_n a_{nn}) = \psi^0 \quad \text{on} \quad \omega_h,
\]

\[
s_k r_{kk}^0 - \Lambda_k r^0 = \psi_{kk}^0 \quad \text{on} \quad \omega_h, \quad 1 \leq k \leq n,
\]

with the approximation errors on the right, and

\[
r |\omega_h = 0, \quad r_{kk} |\omega_h = 0, \quad 1 \leq k \leq n, \quad r^0 = 0.
\]

The stability bound (3.6) applied to these equations and estimate (3.26) imply the error bound

\[
\max_{1 \leq m \leq M} \left( \varepsilon_0 \| \bar{s}_t (u - v)^m \|_h + \sqrt{\varepsilon} \| \bar{s}_t (u - v)^m \|_{-L_h} \right) \leq 2 \varepsilon_0^{-1} \left\{ \| \psi^0 \|_h + \frac{3}{2} a_{\max}^2 \left( \| \psi^0_1 \|_h + \ldots + \| \psi^0_n \|_h \right) \right\} + 2 \varepsilon_0^{-1} \left\{ \| \psi^m_1 \|_h + \ldots + \| \psi^m_n \|_h \right\} = O(|h|^4)
\]

with \( a_{\max} := \max_{1 \leq i \leq n} \). It suffices to apply on the left the second inequality (3.4).

In the case where \( v^1 \) is defined alternatively to (2.15)-(2.16), according to the first equality in the energy conservation law valid for any \( v^1 \), one should guarantee the 4th order approximation of \( v^1 \approx u(\cdot, h_t) \) in the more complicated form

\[
\| \bar{s}_t (u - v)^m \|_h + \| \bar{s}_t (u - v)^m \|_{A_h} = O(|h|^4) \quad \text{for} \quad m = 1
\]

to prove error bound like (3.23), cf. [5].

From bound (3.23), we see that it is important to take \( \varepsilon > 0 \) and \( \varepsilon_0 > 0 \) in stability conditions (3.3) and (3.5) to ensure the error bound in the full mesh energy norm.
4 Some generalizations of the explicit in time compact vector scheme

4.1. The case of the non-uniform mesh in space.

Define the general non-uniform mesh \( \bar{\omega}_{hk} \) in \( x_k \) with the nodes \( 0 = x_{k0} < x_{k1} < \ldots < x_{kN_k} = X_k \) and the steps \( h_{kl} = x_{kl} - x_{k(l-1)} \), for \( l = 1, \ldots, N_k \) and \( 1 \leq k \leq n \). Let

\[
\omega_{hk} = \{ x_{kl} \}_{l=1}^{N_k-1}, \quad h_{kl} = \frac{1}{2}(h_{kl} + h_{k(l+1)}), \quad h_{kmax} = \max_{1 \leq l \leq N_k} h_{kl}, \quad h_{max} = \max_{1 \leq k \leq n} h_{kmax}.
\]

Let now \( \bar{\omega}_h = \bar{\omega}_{h1} \times \ldots \times \bar{\omega}_{hN} \), \( \omega_h = \omega_{h1} \times \ldots \times \omega_{hn} \) and \( \partial \omega_h = \bar{\omega}_h \setminus \omega_h \).

We generalize the difference operators (2.12)-(2.16) in the standard way

\[
\Lambda_k w_l = \frac{1}{h_{kl}} (w_{l+1} - w_{l}) - \frac{w_{l} - w_{l-1}}{h_{kl}}, \quad s_{kN} w_l = \frac{1}{12} (\alpha_{kl}w_{l-1} + 10\gamma_{kl}w_{l} + \beta_{kl}w_{l+1}),
\]

on \( \omega_{hk} \), where \( w_l = w(x_{kl}) \) and \( 1 \leq k \leq n \), with the coefficients

\[
\alpha_{kl} = 2 - \frac{h_{kl}^2}{h_{kl}h_{k+1}}, \quad \beta_{kl} = 2 - \frac{h_{k+1}^2}{h_{k+1}h_{kl}}, \quad \gamma_{kl} = 1 + \frac{(h_{k+1} - h_{kl})^2}{5h_{kl}h_{k+1}} \cdot \alpha_{kl} + 10\gamma_{kl} + \beta_{kl} = 12.
\]

Concerning \( s_{kN} \) and its approximation error, for example, see [3,20,24].

Scheme (2.12)-(2.16) is automatically generalized to the case of the non-uniform spatial mesh \( \bar{\omega}_h \), with these generalized \( \Lambda_k \) and \( s_{kN} \). But now the approximation error orders in (2.10), (2.11) and (2.19) are reduced down to respectively \( O(h_t^4 + h_t^2 h_{max}) \), \( O(h_{kmax}^3) \) and \( O(h_t^4 + h_{kmax}^2) \). For smoothly varying steps \( h_{kl} \) in each spatial direction, here \( h_{max} \) and \( h_{kmax}^2 \) can be replaced with \( h_{max}^2 \) and \( h_{kmax}^4 \), and the approximation errors increase up to the 4th order.

The operator \( s_{kN} \) is not self-adjoint in \( H_h \) and do not commute with \( \Lambda_k \), therefore the above proof of Theorem 3.1 (and thus Theorem 3.2) is not applicable any more, and another more cumbersome stability study like in [20] is required. This leads to stronger conditions on \( h_t \), and, moreover, there arise some unpleasant pitfalls in this respect [22].

4.2. The case of the non-uniform mesh in time.

Define also the general non-uniform mesh \( \bar{\omega}_t \) in \( t \) with the nodes \( 0 = t_0 < t_1 < \ldots < t_M = T \) and the steps \( h_{tm} = t_{m} - t_{m-1} \). This case is important, in particular, to accomplish a dynamic choice of the time step, and a new moment will arise to cover it.

Let \( h_{t+m} = h_{t(m+1)} \), \( h_{t} = \frac{1}{2}(h_{t} + h_{t+}) \) and \( h_{max} = \max_{1 \leq m \leq M} h_{tm} \). We define the generalized difference operators in \( t \) quite similar to those in space given in the previous item

\[
\Lambda_t y^m = \frac{1}{h_{stm}} (y_{m+1}^{t(m+1)} - y_{m}^{t(m+1)} - y_{m}^{t} + y_{m+1}^{t}), \quad s_{tNm} y^m = \frac{1}{12} (\alpha_{tm}y^{m-1} + 10\gamma_{tm}y^m + \beta_{tm}y^{m+1}),
\]

for \( 1 \leq m \leq M - 1 \), with the coefficients

\[
\alpha_{tm} = 2 - \frac{h_{t(m+1)}^2}{h_{tm}h_{stm}}, \quad \beta_{tm} = 2 - \frac{h_{t}^2}{h_{t(m+1)}h_{stm}}, \quad \gamma_{tm} = 1 + \frac{(h_{t(m+1)} - h_{tm})^2}{5h_{tm}h_{t(m+1)}} \cdot \alpha_{tm} + 10\gamma_{tm} + \beta_{tm} = 12.
\]

We define the average in \( t \) related to the linear finite elements

\[
(q_t y)^m = \frac{1}{h_{stm}} \int_{t_{m-1}}^{t_{m+1}} y(t) e_m(t) \, dt, \quad e_m(t) = \frac{t_m - t_{m-1}}{h_{tm}} \text{ on } [t_{m-1}, t_m], \quad e_m(t) = \frac{t_{m+1} - t}{h_{t(m+1)}} \text{ on } [t_m, t_{m+1}],
\]
for \(1 \leq m \leq M - 1\). The following two expansions follow from Taylor’s formula

\[
q_t y = y + \frac{1}{3}(h_{t+} - h_t)\partial_t y + \frac{1}{12}(h_{t+}^2 - h_{t+}h_t + h_t^2)\partial_t^2 y + q_t \rho_3(\partial_t^3 y), \tag{4.1}
\]

\[
q_t y = s_{tN} y + \tilde{\rho}_3(\partial_t^3 y), \tag{4.2}
\]

\[
|\rho_3(\partial_t^3 y)| + |\tilde{\rho}_3(\partial_t^3 y)| \leq c_3 h_{t|m}^3 |\partial_t^3 y||C_{u_{m-1},t_{m+1}}| \tag{4.3}
\]
on \(\omega_{h_t}\), for example, see \cite{24}.

Similarly to \cite{24}, we apply the operator \(q_t\) to the wave equation \((2.11)\). Due to the known formula \(q_t \partial_t^2 u = \Lambda_t u\) and expansion \((4.1)\) together with bound \((4.3)\), we obtain

\[
\Lambda_t u = q_t \partial_t^2 u = \partial_t(Lu + f) = L\partial_t u + \partial_t f, \quad \partial_t u = I_t \partial_t^2 u + u_1 = I_t(Lu + f) + u_1, 
\]

where \((I_t a)(t) := \int_0^t a(\theta) \, d\theta\). Applying these formulas together with \((2.5)\) once again, we pass from expansion \((4.4)\) to the following one

\[
\Lambda_t u = \left[ I + \frac{1}{12}(h_{t+}^2 - h_{t+}h_t + h_t^2) L \right] (Lu + f) + \frac{1}{3}(h_{t+} - h_t)L[I_t(Lu + f) + u_1] + f_{h_t} + \mathcal{O}(h_{t|\text{max}}^3). \tag{4.4}
\]

The wave equation implies

\[
\partial_t^3 u = \partial_t(Lu + f) = L\partial_t u + \partial_t f, \quad \partial_t u = I_t \partial_t^2 u + u_1 = I_t(Lu + f) + u_1,
\]

where \((I_t a)(t) := \int_0^t a(\theta) \, d\theta\). Applying these formulas together with \((2.5)\) once again, we pass from expansion \((4.4)\) to the following one

\[
\Lambda_t u = \left[ I + \frac{1}{12}(h_{t+}^2 - h_{t+}h_t + h_t^2) L \right] (Lu + f) + \frac{1}{3}(h_{t+} - h_t)L[I_t(Lu + f) + u_1] + f_{h_t} + \mathcal{O}(h_{t|\text{max}}^3) 
\]

with

\[
f_{h_t} := f + \frac{1}{3}(h_{t+} - h_t)\partial_t f + \frac{1}{12}(h_{t+}^2 - h_{t+}h_t + h_t^2)\partial_t^2 f.
\]

We replace \(f_{h_t}\) with simpler \(s_{tN} f\) since \(f_{h_t} - s_{tN} f = \mathcal{O}(h_{t|\text{max}}^3)\) due to formulas \((4.1)-(4.3)\). In addition, the well-known bound for the error of the compound trapezoid formula holds

\[
\max_{1 \leq m \leq M} |(I_t y)(t_m) - I_{h_t}^m \bar{s}_t y| = \mathcal{O}(h_{t|\text{max}}^2).
\]

Consequently, using the auxiliary functions \((2.4)\), finally we can write

\[
\Lambda_t u = \left[ I + \frac{1}{12}(h_{t+}^2 - h_{t+}h_t + h_t^2) L \right] (u_{11} + \ldots + u_{mn}) 
+ \frac{1}{3}(h_{t+} - h_t)L_h[I_{h_t} \tilde{s}_t(u_{11} + \ldots + u_{mn} + f) + Lu_1] + f_{h} + \mathcal{O}(h_{t|\text{max}}^3 + h_{t|\text{max}}|h^2|) \tag{4.5}
\]

with

\[
f_h := s_{tN} f + \frac{1}{12}(h_{t+}^2 - h_{t+}h_t + h_t^2) L_h f.
\]

Omitting the residual term, we pass to the following scheme

\[
\Lambda_t v = \left[ I + \frac{1}{12}(h_{t+}^2 - h_{t+}h_t + h_t^2) L_h \right] (a_1^2 v_{11} + \ldots + a_m^2 v_{nn}) 
+ \frac{1}{3}(h_{t+} - h_t)L_h[I_{h_t} \tilde{s}_t(a_1^2 v_{11} + \ldots + a_m^2 v_{nn} + f) + Lu_1] + f_{h} \quad \text{on } \omega_h, \tag{4.6}
\]

\[
s_{kN} v_{kk} \Lambda_k v \quad \text{on } \omega_h, \quad 1 \leq k \leq n, \tag{4.7}
\]

Together with equations \((2.15)-(2.16)\), where the step \(h_t = h_{t1}\) has to be taken. In contrast to the above schemes, this scheme is non-local in time but nevertheless its implementation is easy based on the elementary recurrent formula

\[
I_{h_t}^m \tilde{s}_t d = I_{h_t}^{m-1} d + h_{tm} \tilde{s}_t d^m, \quad m = 1, \ldots, M - 1, \quad d := a_1^2 v_{11} + \ldots + a_m^2 v_{nn} + f,
\]

though one more array is required to store this function at \(m\)th time level.
For a smoothly varying time step \( h_{tm} \), the approximation error order increases up to \( O(h_{t_{\text{max}}}^4 + h_{t_{\text{max}}}^2 h_{m_{\text{max}}}^2 |h|^2) \) in (4.5). Moreover, in this case, the non-uniform mesh in space can be taken similarly to the previous Item 4.1 ensuring the approximation error order \( O(h_{t_{\text{max}}}^4 + h_{t_{\text{max}}}^2 h_{m_{\text{max}}}^2) \) in general or even \( O(h_{t_{\text{max}}}^4 + h_{t_{\text{max}}}^2 h_{m_{\text{max}}}^2) \) for smoothly varying mesh steps \( h_{kl} \) in all spatial directions.

**4.3. The case of the wave equation with variable coefficients**

We also consider the generalized wave equation

\[
\rho(x)\partial_t^2 u(x,t) - Lu(x,t) = f(x,t) \quad \text{in} \quad Q_T, \quad (4.8)
\]

with the variable coefficients \( 0 < \rho \leq \rho(x) \) and \( a_1(x), a_2(x), \ldots, a_n(x) > 0 \) smooth in \( \bar{\Omega} \). Since now

\[
\rho \partial_t^2 u = \partial_t^2 (Lu + f) = L\left[\frac{1}{\rho}(Lu + f)\right] + \partial_t^2 f,
\]

we have (for the uniform meshes in time and space)

\[
\rho \Lambda_t u = \rho \partial_t^2 u + \frac{1}{12} \rho I_h^2 \partial_t^4 u + O(h_t^4) = \left(\rho I + \frac{1}{12} \rho I_h^2 L\right)\left(\frac{1}{\rho}(Lu + f)\right) + f + \frac{1}{12} \rho h_t^2 \left(\partial_t^2 f + Lh_f^t\right) + O(h_t^4)
\]

\[
= \left(\rho I + \frac{1}{12} \rho h_t^2 L_h\right)\left(\frac{\partial_t^4 u_{11} + \ldots + \partial_t^2 u_{nn}}{\rho}\right) + f + O(|h|^4),
\]

where

\[
f_h := f + \frac{1}{12} \rho h_t^2 (\Lambda_t f + Lh_f^t).
\]

In a standard manner, this leads us to the following generalized equation (2.12):

\[
\rho \Lambda_t v = \left(\rho I + \frac{1}{12} \rho h_t^2 L_h\right)\left(\frac{\partial_t^4 u_{11} + \ldots + \partial_t^2 u_{nn}}{\rho}\right) + f + O(|h|^4),
\]

with the same equation (2.13). The boundary conditions (2.14) are the same but with the slightly more general functions

\[
g_k := \rho \partial_t^2 g - \sum_{1 \leq l \leq n, l \neq k} a_{ii}^2 \partial_t^2 g - f \quad \text{for} \quad x_k = 0, X_k \quad \text{on} \quad \Gamma_T.
\]

It can be also checked that equation (2.15) is generalized as follows

\[
\rho(\delta_t u) = \frac{1}{2} h_t \left(\rho I + \frac{1}{12} h_t^2 L_h\right)\left(\frac{\partial_t^4 u_{00}^{11} + \ldots + \partial_t^2 u_{00}^{nn}}{\rho}\right) + u_{1h} + \frac{1}{2} h_t f_h^0
\]

with

\[
u_{1h} := \left(\rho I + \frac{1}{6} h_t^2 L_h\right)u_{11}, \quad f_h^0 := f^{(0)} + \frac{1}{12} h_t^2 L_h^0 f_h^0
\]

on \( \omega_h \), together with the same equation (2.16). These formulas ensure the following estimate for the approximation error

\[
\rho(\delta_t u) - \frac{1}{2} h_t \left(\rho I + \frac{1}{12} h_t^2 L_h\right)\left(\frac{\partial_t^4 u_{00}^{11} + \ldots + \partial_t^2 u_{00}^{nn}}{\rho}\right) - u_{1h} - \frac{1}{2} h_t f_h^0 = O(|h|^4) \quad \text{on} \quad \omega_h.
\]

For the derived scheme, the stability and error bounds need another study. The scheme can be generalized for the case of non-uniform meshes in space and time similarly as described above.

**Disclosure statement**

No potential conflict of interests was reported by the author.

**Funding**

This paper was supported by the Russian Science Foundation under Grant no. 19-11-00169.

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