\section{Introduction}

Consider \( \mathbb{R}^d, d \geq 3 \), the formal differential operator

\begin{equation}
- \nabla \cdot a \cdot \nabla + b \cdot \nabla \equiv - \sum_{i,j=1}^{d} \nabla_i a_{ij}(x) \nabla_j + \sum_{j=1}^{d} b_j(x) \nabla_j,
\end{equation}

where \( a = a^* : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d \) is \( \mathcal{L}^d \) measurable,

\[ \sigma I \leq a(x) \leq \xi I \quad \text{for } \mathcal{L}^d \ \text{a.e. } x \in \mathbb{R}^d \quad \text{for some } 0 < \sigma < \xi < \infty. \]

By the De Giorgi-Nash theory, the solution \( u \in W^{1,2}(\mathbb{R}^d) \) to the corresponding elliptic equation \( (\mu - \nabla \cdot a \cdot \nabla + b \cdot \nabla) u = f, \mu > 0, f \in L^p \cap L^2, p \in ]d, \infty[, \) is in \( C^{\gamma, \gamma} \), where the Hölder continuity exponent \( \gamma \in \]0, 1[ \) depends only on \( d \) and \( \sigma, \xi, \) provided that \( b : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is in the Nash class \( (\sup \{ [L^p + L^\infty]^d, p > d \}) \), but already the class \( [L^d + L^\infty]^d \) is not admissible (e.g., it is easy to find \( b \in [L^d + L^\infty]^d \) that makes the two-sided Gaussian bounds on the fundamental solution of \( \Box \) invalid). On the other hand, for \( -\Delta + b \cdot \nabla \), the \( C^{0, \gamma} \) regularity of solutions to the corresponding elliptic equations is known to hold for \( b \) having much stronger singularities. Recall that a vector field \( b : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is in the class of form-bounded vector fields \( \mathbf{F}_\delta \equiv \mathbf{F}_\delta(-\Delta), \delta > 0 \) if \( |b| \in L^2_{\text{loc}} \equiv L^2_{\text{loc}}(\mathbb{R}^d) \) and there exist a constant \( \lambda = \lambda_\delta > 0 \) such that

\[ \| b(\lambda - \Delta)^{-\frac{\gamma}{2}} \|_{2 \rightarrow 2} \leq \sqrt{\delta}. \]

The class \( \mathbf{F}_\delta \) contains \( [L^d + L^\infty]^d \) with \( \delta \) arbitrarily small, as follows by the Sobolev Embedding Theorem, as well as vector fields having critical-order singularities such as \( b(x) = \frac{d-2}{2} \sqrt{\delta} |x|^{-2} \) (by Hardy’s inequality) or, more generally, vector fields in \( [L^{d, \infty} + L^{\infty}]^d \), the Campanato-Morrey class or the Chang-Wilson-T. Wolff class, with \( \delta \) depending on the norm of the vector field in these classes, see

\[ \Box \]

\textbf{Abstract.} In \( \mathbb{R}^d, d \geq 3 \), consider the divergence and the non-divergence form operators

\[ - \nabla \cdot a \cdot \nabla + b \cdot \nabla, \quad (i) \]

\[ - a \cdot \nabla^2 + b \cdot \nabla, \quad (ii) \]

where \( a = I + cf \otimes f, \) the vector fields \( \nabla f (i = 1, 2, \ldots, d) \) and \( b \) are form-bounded (this includes the sub-critical class \( [L^d + L^\infty]^d \) as well as vector fields having critical-order singularities). We characterize quantitative dependence on \( c \) and the values of the form-bounds of the \( L^q \rightarrow W^{1,q/(d-2)} \) regularity of the resolvents of the operator realizations of \( \Box \). \( \Box \) in \( L^p \), \( q \geq 2 \lor (d - 2) \) as (minus) generators of positivity preserving \( L^\infty \) contraction \( C_0 \) semigroups. The latter allows to run an iteration procedure \( L^p \rightarrow L^\infty \) that yields associated with \( \Box \), \( \Box \) \( L^q \)-strong Feller semigroups.
e.g. [KIS] for details.) It has been established in [KS] that if $b \in \mathbf{F}_\delta$, $\delta < 1$, then for every $q \in [2, 2/\sqrt{\delta}]$ $-\Delta + b \cdot \nabla$ has an operator realization $\Lambda_q(b)$ on $L^q$ as the generator of a positivity preserving, $L^\infty$ contraction, quasi contraction $C_0$ semigroup $e^{-t\Lambda_q(b)}$ such that $u := (\mu + \Lambda_q(b))^{-1}f$, $f \in L^q$ satisfies

$$\|\nabla u\|_q \leq K_1(\mu - \mu_0)^{-\frac{1}{2}}\|f\|_q, \quad \|\nabla|\nabla u|^{\frac{2}{q}}\|_2 \leq K_2(\mu - \mu_0)^{-\frac{1}{2}}\|f\|_q, \quad \mu > \mu_0,$$

for some constants $\mu_0 \equiv \mu_0(d, q, \delta) > 0$ and $K_i = K_i(d, q, \delta) > 0$, $i = 1, 2$. In particular, if $\delta < 1 \wedge \frac{1}{2}$, there exists $q > 2 \lor (d - 2)$ such that $u \in C^{0, \gamma}$, $\gamma = 1 - \frac{d-2}{q}$. The explicit dependence of the regularity properties of $u$ on $\delta$ (which effectively plays the role of a “coupling constant”) is a crucial feature of the result in [KS].

In the present paper our concern is: to find a class of matrices $a \in (H_u)$ such that the operator \([\text{I}]\) with $b \in \mathbf{F}_\delta$ admits a $W^{1,p}$ and $C^{0, \gamma}$ regularity theory. Below we consider

$$a = I + cf \otimes f, \quad c > -1, \quad f \in [L^\infty \cap W^{1,2}_{\text{loc}}]^d, \quad \|f\|_\infty = 1, \quad \nabla_i f \in \mathbf{F}_{\delta_i}, \quad \delta_i > 0, \quad i = 1, 2, \ldots, d, \quad \delta_f := \sum_{i=1}^d \delta_i. \quad \text{(C}_{\delta_i}\text{)}$$

The model example of such $a$ is the matrix

$$a(x) = I + c|x|^{-2}x \otimes x, \quad x \in \mathbb{R}^d \quad \text{(2)}$$

having critical discontinuity at the origin, see [GS, GrS, KiS2, OGr] and references therein. (Replacing the requirement $\nabla_i f \in \mathbf{F}_{\delta_i}$ by a more restrictive $\nabla_i f \in [L^p + L^\infty]^d$, $p > d$, forces $a$ to be Hölder continuous. On the other hand, a weaker condition $\nabla_i f \in [L^p + L^\infty]^d$, $p < d$, is incompatible with the uniform ellipticity of $a$. The condition \(\text{(C}_{\delta_i}\text{)}\) ($\geq \nabla_i f \in [L^d + L^\infty]^d$) seems to be rather natural. We also note that the operator $-a \cdot \nabla^2$ with $\nabla_k a_{ij} \in L^{d, \infty}$ has been studied earlier in [AT], cf. the discussion below concerning the non-divergence form operators.)

In Theorems \([\text{I}]\) \([\text{II}]\) below we characterize quantitative dependence on $c$, $\delta$, $\delta_f$ of the $L^q \rightarrow W^{1,qd/(d-2)}$ regularity of the resolvent of the operator realization of \([\text{I}]\) as (minus) generator of positivity preserving $L^\infty$ contraction $C_0$ semigroups in $L^q$, $q \geq 2 \lor (d - 2)$.

Consider the non-divergence form operator

$$-a \cdot \nabla^2 + b \cdot \nabla \equiv -\sum_{i,j=1}^d a_{ij}(x)\nabla_i \nabla_j + \sum_{j=1}^d b_j(x)\nabla_j. \quad \text{(3)}$$

Write $-a \cdot \nabla^2 + b \cdot \nabla \equiv -\nabla \cdot a \cdot \nabla + (\nabla a + b) \cdot \nabla$, where $(\nabla a)_k := \sum_{i=1}^d (\nabla_i a_{ik})$, $k = 1, 2, \ldots, d$. Then

$$\nabla a = c[\text{(divf)}f + f \cdot \nabla f].$$

It is easily seen that the condition \(\text{(C}_{\delta_i}\text{)}\) yields $\nabla a \in \mathbf{F}_{\delta_a}$ with $\delta_a \leq |c|^2(\sqrt{d} + 1)^2 \delta_f$. The latter yields an analogue of Theorem \([\text{II}]\) for \(\text{(3)}\) (Corollary \([\text{I}]\) below).

Theorem \([\text{II}]\) and Corollary \([\text{I}]\) are needed to run an iteration procedure $L^p \rightarrow L^\infty$ that yields associated with \([\text{I}]\), \(\text{(3)}\) Feller semigroups on $C_\infty = C_\infty(\mathbb{R}^d)$ (the space of all continuous functions vanishing at infinity endowed with the sup-norm), see Theorem \([\text{III}]\) and Corollary \([\text{II}]\) below.

In the same manner as it was done in [KIS3] for the operator $-\Delta + b \cdot \nabla$, the Feller process constructed in Corollary \([\text{II}]\) admits a characterization as a weak solution to the stochastic differential equation

$$dX_t = -b(X_t)dt + \sqrt{2a(X_t)}dW_t, \quad X_0 = x_0 \in \mathbb{R}^d.$$
We plan to address this matter in another paper.

All the proofs below work for

\[ a = I + \sum_{j=1}^{\infty} c_j f_j \otimes f_j, \quad \|f_j\|_{\infty} = 1, \]

with \( f_j \) satisfying (C\(_{\delta}\)), and \( c_+ := \sum c_j > 0 c_j < \infty, c_- := \sum c_j < 0 c_j > -1 \). (A decomposition (1) can be obtained from the spectral decomposition of a general uniformly elliptic \( a \).)

2. We now state our results in full.

**Theorem 1** \((-\nabla \cdot a \cdot \nabla)\). Let \( d \geq 3 \). Let \( a = I + cf \otimes f \) be given by (2).

(i) The formal differential expression \(-\nabla \cdot a \cdot \nabla\) has an operator realization \( A_q \) in \( L^q \) for all \( q \in [1, \infty[ \) as the (minus) generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup.

(ii) Assume that (C\(_{\delta}\)) holds with \( \delta_t, c \) and \( q \geq 2 \vee (d - 2) \) satisfying the following constraint:

\[
-(1 + q\sqrt{\delta_t})^{-1} < c < \begin{cases} 
16 [q\sqrt{\delta_t} (8 + q\sqrt{\delta_t})]^{-1} & \text{if } q\sqrt{\delta_t} \leq 4, \\
(q\sqrt{\delta_t} - 1)^{-1} & \text{if } q\sqrt{\delta_t} \geq 4.
\end{cases}
\]

Then, for each \( \mu > 0 \) and \( f \in L^q \), \( u := (\mu + A_q)^{-1} f \) belongs to \( W^{1,q} \cap W^{1,\frac{qd}{d-2}} \). Moreover, there exist constants \( \mu_0 = \mu_0 (d, q, c, \delta_t) > 0 \) and \( K_l = K_l (d, q, c, \delta_t), l = 1, 2 \), such that, for all \( \mu > \mu_0 \),

\[
\|\nabla u\|_{\frac{q}{d-2}} \leq K_1 (\mu - \mu_0)^{-\frac{1}{2}} \|f\|_{q},
\]

\[
\|\nabla u\|_{q} \leq K_2 (\mu - \mu_0)^{-\frac{1}{2}} \|f\|_{q}. \tag{*}
\]

Remarks. 1. \( \delta_t \) effectively estimates from above the “size” of the discontinuities of \( a \).

2. For the matrix (2), the constraints on \( c \) in Theorem 1 (and in other results below) can be substantially relaxed, see [KiS2].

**Theorem 2** \((-\nabla \cdot a \cdot \nabla + b \cdot \nabla)\). Let \( d \geq 3 \). Let \( a = I + cf \otimes f \) be given by (2). Let \( b \in F_\delta \).

(i) If \( \delta_t := [1 + (1 + c)^{-2}] \delta < 4 \), then \(-\nabla \cdot a \cdot \nabla + b \cdot \nabla\) has an operator realization \( \Lambda_q (a, b) \) in \( L^q \) for all \( q \in [\frac{2}{2 - \sqrt{\delta_t}}, \infty[ \) as the (minus) generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup.

(ii) Assume that (C\(_{\delta}\)) holds, \( \nabla a \in F_{\delta_a}, \delta < 1 \wedge (\frac{2}{\alpha - 2})^2, \delta_a, \delta_t, c \) and \( q \geq 2 \vee (d - 2) \) satisfy the constraints:

\[
0 < c < (q - 1 - Q) \begin{cases} 
[(q - 1)\sqrt{\delta_t} + \frac{q^2(\sqrt{\delta} + \sqrt{\delta_t})^2}{16} + (q - 2)\frac{q^2\delta_t}{16}]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta}}{4} - \frac{2\sqrt{\alpha}}{4} \geq 0, \\
(q^2\sqrt{\delta} + (q - 2)\frac{q^2\delta_t}{16} + \frac{q\sqrt{\delta}}{4} - 1)^{-1} & \text{if } 0 \leq 1 - \frac{2\sqrt{\delta}}{4} < \frac{2\sqrt{\alpha}}{4}, \\
((q - 1)(q\sqrt{\delta_t} - 1) + \frac{q\sqrt{\delta}}{4}]^{-1} & \text{if } 1 - \frac{2\sqrt{\delta}}{4} < 0,
\end{cases}
\]

where \( Q := \frac{q\sqrt{\alpha}}{2} [q - 2 + (\sqrt{\delta_a} + \sqrt{\delta}) \frac{d}{2}] \), or

\[
-(q - 1 - Q) [(q - 1)(1 + q\sqrt{\delta_t}) + \frac{q\sqrt{\delta}}{2}]^{-1} < c < 0.
\]

Then there exist constants \( \mu_0 = \mu_0 (d, q, c, \delta, \delta_a, \delta_t) > 0 \) and \( K_l = K_l (d, q, c, \delta, \delta_a, \delta_t), l = 1, 2 \), such that (**) hold for \( u := (\mu + \Lambda_q (a, b))^{-1} f, \mu > \mu_0, f \in L^q \).
Remarks. 1. Taking \( c = 0 \) (then \( \delta_\alpha = 0 \)), we recover in Theorem \([2](ii)\) the result of [KS] Lemma 5: \( \delta < 1 \land \left( \frac{2}{d-2} \right)^2 \).

2. Theorem \([2](i)\) is an immediate consequence of the following general result. Let \( a \) be an \( L^d \) measurable uniformly elliptic matrix on \( \mathbb{R}^d \). Set \( A \equiv A_2 := [-\nabla \cdot a \cdot \nabla | C_c^{\infty}]_{2 \rightarrow 2} \). A vector field \( b : \mathbb{R}^d \to \mathbb{R}^d \) belongs to \( \mathbf{F}_{\delta_1}(A) \), \( \delta_1 > 0 \), the class of form-bounded vector fields (with respect to \( A \)), if \( b_0^a := b \cdot a^{-1} \cdot b \in L_{\text{loc}}^2 \) and there exists a constant \( \lambda = \lambda_{\delta_1} > 0 \) such that

\[
\| b_0^a (\lambda + A)^{-\frac{1}{2}} \|_{2 \rightarrow 2} \leq \sqrt{\delta_1}.
\]

If \( b \in \mathbf{F}_{\delta_1}(A) \), \( \delta_1 < 4 \), then \(-\nabla \cdot a \cdot \nabla + b \cdot \nabla \) has an operator realization \( \Lambda_q(a, b) \) in \( L^q \) for all \( q \in \left[ \frac{2}{2 - \sqrt{\delta_1}}, \infty \right] \) as the (minus) generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup, see [KS, Theorem 3.2].

**Corollary 1** \((-a \cdot \nabla^2 + b \cdot \nabla \). Let \( d \geq 3 \). Let \( a = I + cf \otimes f \) be given by \([2]\). Let \( b \in \mathbf{F}_{\delta}, \nabla a \in \mathbf{F}_{\delta_\alpha} \). Then \( \nabla a + b \in \mathbf{F}_{\delta_\alpha}, \sqrt{\delta_2} := \sqrt{\delta_\alpha} + \sqrt{\delta} \).

(i) If \( \delta_1 := \left[ 1 \lor (1 + c)^{-2} \right] \delta_2 < 4 \), then \(-a \cdot \nabla^2 + b \cdot \nabla \) has an operator realization \( \Lambda_q(a, \nabla a + b) \) in \( L^q \) for all \( q \in \left[ \frac{2}{2 - \sqrt{\delta_1}}, \infty \right] \) as the (minus) generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup.

(ii) Assume that \([\mathbf{C}_{\delta_\alpha}]\) holds, and \( \delta_2 < 1 \land \left( \frac{2}{d-2} \right)^2 \), \( \delta_\alpha, \delta_t, c, q \geq 2 \lor (d - 2) \) satisfy the constraints:

\[
0 < c < (q - 1 - Q) \begin{cases} \left[ (q - 1) \frac{2}{\sqrt{\delta_2}} + (q - 2) \frac{2}{\sqrt{\delta_2} + 2} \right]^{-1} & \text{if} \quad 1 - 2 \sqrt{\delta_2} - 2 \sqrt{\delta_2} \geq 0, \\
\left( \frac{2}{\sqrt{\delta_2}} + (q - 2) \frac{2}{\sqrt{\delta_2} + 2} + a \frac{2}{\sqrt{\delta_2} + 1} \right) & \text{if} \quad 0 \leq 1 - 2 \sqrt{\delta_2} < 2 \sqrt{\delta_2}, \\
\left( q \frac{2}{\sqrt{\delta_2}} + 2 \frac{2}{\sqrt{\delta_2} + 2} \right) & \text{if} \quad 1 - 2 \sqrt{\delta_2} < 0,
\end{cases}
\]

where \( Q := \frac{2}{\sqrt{\delta_2}} \left[ q - 2 + (\sqrt{\delta_\alpha} + \sqrt{\delta_2}) \right] \), or

\[-(q - 1 - Q) \left( q - 1 \right) \left( 1 + q \sqrt{\delta_t} + \frac{q}{2} \sqrt{\delta_t} \right)^{-1} < c < 0.
\]

Then there exist constants \( \mu_0 = \mu_0(d, q, c, \delta_2, \delta_\alpha, \delta_t) > 0 \) and \( K_l = K_l(d, q, c, \delta_2, \delta_\alpha, \delta_t) \), \( l = 1, 2, \) such that the estimates \([\mathbf{K}_l]\) hold for \( u = (\mu + \Lambda_q(a, \nabla a + b))^{-1} f, \mu > \mu_0, f \in L^q \).

Set \( b_n := e^{\epsilon_n \Delta} (I_1 b), \epsilon_n \downarrow 0, n \geq 1 \), where \( I_1 \) is the indicator of \( \{ x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n \} \). Also, set \( f_n := (f_n^1)_1^d, f_n^1 := e^{\epsilon_n \Delta} (\eta_n f^1), \epsilon_n \downarrow 0, n \geq 1 \), where

\[
\eta_n(x) := \begin{cases} 1, & \text{if} \ |x| < n, \\
0, & \text{if} \ |x| > n + 1.
\end{cases}
\]

**Theorem 3** \((-\nabla \cdot a \cdot \nabla + b \cdot \nabla \). (i) In the assumptions of Theorem \([2](ii)\), the formal differential operator

\(-\nabla \cdot a \cdot \nabla + b \cdot \nabla \) has an operator realization \(-\Lambda_{c_\infty}(a, b) \) as the generator of a positivity preserving contraction \( C_0 \) semigroup in \( C_\infty \) defined by

\[
e^{-t \Lambda_{c_\infty}(a, b)} := s \cdot C_\infty \cdot \lim_{n} e^{-t \Lambda_{c_\infty}(a_n, b_n)} \quad (\text{loc. uniformly in} \; t \geq 0),
\]

where \( a_n := I + c f_n \otimes f_n \in [C_\infty]^{d \times d}, \Lambda_{c_\infty}(a_n, b_n) := -\nabla \cdot a_n \cdot \nabla + b_n \cdot \nabla, D(\Lambda_{c_\infty}(a_n, b_n)) = (1 - \Delta)^{-1} C_\infty.\)
\( (ii) \) \textbf{[The }L^r\text{-strong Feller property]} \((\mu + \Lambda_{C_{\infty}}(a, b))^{-1} : L^r \cap C_{\infty})^{\text{clos}} \in B(L^r, C^{0,1-d})\text{ for some } r > d - 2\text{ and all } \mu > \mu_0.\)

\( (iii) \) The integral kernel of \( e^{-t\Lambda_{C_{\infty}}(a,b)} \) determines the transition probability function of a Feller process.

\[ \text{Corollary 2 } (-a \cdot \nabla^2 + b \cdot \nabla). \text{ (i) In the assumptions of Corollary } [i], \text{ the formal differential operator } -a \cdot \nabla^2 + b \cdot \nabla \text{ has an operator realization } -\Lambda_{C_{\infty}}(a, \nabla a + b) \text{ as the generator of a positivity preserving contraction } C_0 \text{ semigroup in } C_{\infty} \text{ defined by } \]

\[ e^{-t\Lambda_{C_{\infty}}(a, \nabla a + b)} := s-C_{\infty}\text{-lim } n e^{-t\Lambda_{C_{\infty}}(a_n, \nabla a_n + b_n)} \text{ (loc. uniformly in } t \geq 0), \]

where \( a_n = I + cf_n \otimes f_n \subset [C_{\infty}]^{d \times d}, \Lambda_{C_{\infty}}(a_n, \nabla a_n + b_n) := -a_n \cdot \nabla^2 + b_n \cdot \nabla, D(\Lambda_{C_{\infty}}(a_n, \nabla a_n + b_n)) = (1 - \Delta)^{-1}C_{\infty}. \)

\( (ii) \) \textbf{[The }L^r\text{-strong Feller property]} \((\mu + \Lambda_{C_{\infty}}(a, \nabla a + b))^{-1} : L^r \cap C_{\infty})^{\text{clos}} \in B(L^r, C^{0,1-d})\text{ for some } r > d - 2\text{ and all } \mu > \mu_0.\)

\( (iii) \) The integral kernel of \( e^{-t\Lambda_{C_{\infty}}(a, \nabla a + b)} \) determines the transition probability function of a Feller process.

\[ \text{Remarks. Since our assumptions on } \delta_t, \delta_a \text{ and } \delta \text{ involve only strict inequalities, we may assume that } \]

\[ \text{[C_6]} \text{ holds for } f_n, \nabla a_n \in F_{\delta_a}, \quad b_n \in F_{\delta} \quad \text{with } \lambda \neq \lambda(n) \quad (5) \]

for appropriate \( \epsilon_n \downarrow 0. \) In fact, the proofs work for any approximations \( \{f_n\}, \{b_n\} \subset [C_{\infty}]^d \) such that \( \|f_n\|_{\infty} = 1, (5) \) holds, and

\[ f_n \to f, \nabla_i f_n \to \nabla_i f \text{ strongly in } [L^2_{\text{loc}}]^d, \quad i = 1, 2, \ldots, d, \]

\[ b_n \to b \text{ strongly in } [L^2_{\text{loc}}]^d. \]

\[ \text{1. Proof of Theorem } 1 \]

\[ \text{Proof of (i). In what follows, we use notation } \]

\[ \langle h \rangle := \int_{\mathbb{R}^d} h(x)dx, \quad \langle h, g \rangle := \langle h \bar{g} \rangle. \]

Define \( t[u, v] := \langle \nabla u \cdot a \cdot \nabla v \rangle, D(t) = W^{1,2}. \) There is a unique self-adjoint operator \( A \equiv A_2 \geq 0 \) on \( L^2 \) associated with the form \( t: D(A) \subset D(t), \langle Au, v \rangle = t[u, v], u \in D(A), v \in D(t). \) \(-A\) is the generator of a positivity preserving \( L^\infty \) contraction \( C_0 \) semigroup \( T^t_2 \equiv e^{-tA}, t \geq 0, \) on \( L^2. \) Then \( T^t_r := [T^t_1 \downarrow L^r \cap L^2]_{L^r \to L^r} \) determines \( C_0 \) semigroup on \( L^r \) for all \( r \in [1, \infty[. \) The generator \(-A_r\) of \( T^t_r \equiv e^{-tA_r}\) is the desired operator realization of \( \nabla \cdot a \cdot \nabla \) in \( L^r, r \in [1, \infty[. \) Moreover, \( (\mu + A_r)^{-1} \) is well defined on \( L^r \) for all \( \mu > 0. \) This completes the proof of the assertion \( (i) \) of the theorem.

\[ \text{Proof of (ii). First, we prove an a priori variant of } (\mathcal{C}). \text{ Set } a_n := I + cf_n \otimes f_n, \text{ where } f_n \text{ have been defined in the beginning of the paper. Since our assumption on } \delta_t \text{ is a strict inequality, we may assume that } \]

\[ \text{[C_6]} \text{ holds for } f_n \text{ for all } n \geq 1 \text{ with } \lambda \neq \lambda(n) \text{ for appropriate } \epsilon_n \downarrow 0. \] We also note that \( \|f_n\|_{\infty} = 1. \)

Set \( u \equiv u_n := (\mu + A^0_2)^{-1}f, 0 \leq f \in C^2 \), where \( A^0_q := -\nabla \cdot a_n \cdot \nabla, D(A^0_q) = W^{2,q}, n \geq 1. \) Clearly, \( 0 \leq u_n \in W^{3,q}. \)
Denote \( w \equiv w_n := \nabla u_n \). For brevity, below we omit the index \( n \): \( f \equiv f_n, a \equiv a_n, A_q \equiv A^n_q \). Set
\[
I_q := \sum_{r=1}^{d} \langle (\nabla_r w)^2 | w|^{q-2} \rangle, \quad J_q := \langle (|\nabla|)| w|^{q-2} \rangle,
\]
\[
\bar{I}_q := \langle (\nabla \cdot w)^2 | w|^{q-2} \rangle, \quad \bar{J}_q := \langle (f \cdot \nabla |w|)^2 | w|^{q-2} \rangle.
\]
Set \([F,G]_\cdot := FG - GF\).

1. We multiply \( \mu u + A_q u = f \) by \( \phi := -\nabla \cdot (|w|^{q-2}) \) and integrate:
\[
\mu \langle |w|^q \rangle + \langle A_q w, w | w|^{q-2} \rangle + \langle |\nabla, A_q u, w| w|^{q-2} \rangle = \langle f, \phi \rangle,
\]
\[
\mu \langle |w|^q \rangle + I_q + cI_q + (q - 2)(J_q + cJ_q) + \langle |\nabla, A_q u, w| w|^{q-2} \rangle = \langle f, \phi \rangle.
\]
The term to evaluate is this:
\[
\langle |\nabla, A_q u, w| w|^{q-2} \rangle = \sum_{r=1}^{d} \langle |\nabla_r A_q - u, w| w|^{q-2} \rangle.
\]
From now on, we omit the summation sign in repeated indices. Note that
\[
|\nabla_r A_q| = -\nabla \cdot (|\nabla r a| \cdot \nabla) = (\nabla_r a)_{il} = c(\nabla_r |f|^2) + cf |\nabla_r f|^2.
\]
Thus,
\[
\langle |\nabla_r A_q| - u, w_r | w|^{q-2} \rangle = \left( \langle (\nabla_r f) | f|^2 + f |\nabla_r f|^2 \rangle \right) = S_1 + S_2,
\]
\[
S_1 = c \langle |\nabla_r f \cdot (|f|^2) \cdot |f| w_r | w|^{q-3} \rangle,
\]
\[
S_2 = c \langle (\nabla_r f) \cdot w, (f \cdot \nabla w_r) | w|^{q-3} \rangle.
\]

By the quadratic estimates and the condition \( \left[C_{\delta 4}\right] \),
\[
S_1 \leq \left| c \right| \left[ \alpha \left( \delta_4 q^2 J_q + \lambda \delta_t \| w \|^q \right) \right] + \frac{1}{4\alpha} I_q \right| + \left| c \right| \langle (q - 2) (1 + \delta_t \| w \|^q) \rangle + \frac{1}{4\alpha} J_q \right] + \left| c \right| \langle (q - 2) \rangle, \quad \alpha, \alpha > 0
\]
\[
S_2 \leq \left| c \right| \left[ \gamma \left( \delta_4 q^2 J_q + \lambda \delta_t \| w \|^q \right) \right] + \frac{1}{4\gamma} I_q \right| + \left| c \right| \langle (q - 2) \rangle, \quad \gamma, \gamma > 0.
\]
Thus, selecting \( \alpha = \alpha_1 = \frac{1}{q\sqrt{\delta_4}} \), we obtain the inequality
\[
\mu \| w \|^q + I_q + \bar{I}_q + (q - 2)(J_q + cJ_q)
\]
\[
\leq \left| c \right| \left[ q \frac{\delta_4}{4} J_q + q \frac{\delta_t}{4} I_q \right] + \left| c \right| \langle (q - 2) \rangle, \quad \mu_0 \equiv \left| c \right| \lambda \sqrt{\delta_4} (q^{-1} + \gamma \delta_t) + \left| c \right| \langle (q - 2) \rangle \lambda \sqrt{\delta_t} (q^{-1} + \gamma_1 \delta_t).
\]

2. Let us prove that there exists constant \( \eta > 0 \) such that
\[
(\mu - \mu_0) \| w \|^q + \eta J_q \leq \langle f, \phi \rangle, \quad (\ast)
\]
Lemma 1. For each \( c > 0 \). First, suppose that \( 1 - \frac{q \sqrt{t}}{4} \geq 0 \). We select \( \gamma = \gamma_1 := \frac{1}{4} \), so the terms \( \bar{I}_q, \bar{J}_q \) are no longer present in \( (\bar{I}^q) \). By the assumption of the theorem \( 1 - c \frac{q \sqrt{t}}{4} \geq 0 \), so using \( J_q \leq I_q \) we obtain

\[
(\mu - \mu_0)\|w\|_q^q + (q - 1) \left[ 1 - c \frac{q \sqrt{t}}{2} - c \frac{q^2 \sqrt{t}}{16} \right] J_q \leq \langle f, \phi \rangle,
\]

where \( \mu_0 = c \lambda \sqrt{\delta t} (q - 1) \left( \frac{1}{q} + \frac{1}{q_0} \right) \) and the coefficient \([\ldots]\) is strictly positive by the assumptions of the theorem.

Now, suppose that \( 1 - \frac{q \sqrt{t}}{4} < 0 \). We select \( \gamma = \gamma_1 := \frac{1}{q \sqrt{t}} \) and replace \( \bar{J}_q, \bar{I}_q \) by \( J_q, I_q \). Then, since \( 1 - c \left( \frac{q \sqrt{t}}{2} - 1 \right) \geq 0 \) by the assumptions of the theorem, we apply \( J_q \leq I_q \) to obtain

\[
(\mu - \mu_0)\|w\|_q^q + (q - 1) \left[ 1 - c(q \sqrt{\delta t} - 1) \right] J_q \leq \langle f, \phi \rangle,
\]

where \( \mu_0 = c \lambda \sqrt{\delta t} (q - 1) \left( \frac{1}{q} + \frac{q \sqrt{t}}{4} \right) \) and the coefficient \([\ldots]\) is strictly positive by the assumption of the theorem. We have proved \( (\bar{I}^q) \) with \( \mu_0 = c \lambda \sqrt{\delta t} (q - 1) \left( \frac{1}{q} + \frac{q \sqrt{t}}{4} \right) \).

Remark. Elementary considerations show that the above choice of \( \alpha, \alpha_1, \gamma, \gamma_1 \) is the best possible.

Case \( c < 0 \). We select \( \gamma = \gamma_1 := \frac{1}{q \sqrt{t}} \), so that

\[
\mu \|w\|_q^q + \left( 1 - |c| \frac{q \sqrt{t}}{4} \right) I_q + \left[ q - 2 - |c|(q - 1) \frac{q \sqrt{t}}{2} - |c|(q - 2) \frac{q \sqrt{t}}{4} \right] J_q \leq |c| \left( 1 + \frac{q \sqrt{t}}{4} \right) \bar{I}_q + |c|(q - 2) \left( 1 + \frac{q \sqrt{t}}{4} \right) \bar{J}_q + \mu_0 \|w\|_q^q + \langle f, \phi \rangle,
\]

where \( \mu_0 = 2c \lambda \sqrt{\delta t} \frac{q - 1}{q} \). Next, using \( \bar{I}_q \leq I_q, \bar{J}_q \leq J_q \), we obtain

\[
(\mu - \mu_0)\|w\|_q^q + \left( 1 - |c| - |c| \frac{q \sqrt{t}}{2} \right) I_q + \left[ q - 2 - |c|(q - 1) \frac{q \sqrt{t}}{2} - |c|(q - 2) \frac{q \sqrt{t}}{4} - |c|(q - 2) \left( 1 + \frac{q \sqrt{t}}{4} \right) \right] J_q \leq \langle f, \phi \rangle.
\]

By the assumptions of the theorem, \( 1 - |c| - |c| \frac{2q \sqrt{t}}{2} \geq 0 \). Therefore, by \( I_q \geq J_q \),

\[
(\mu - \mu_0)\|w\|_q^q + \left[ q - 1 - |c|(q - 1) - |c|q^2 \sqrt{\delta t} \right] J_q \leq \langle f, \phi \rangle,
\]

and hence the coefficient \([\ldots]\) is strictly positive. We have proved \( (\bar{I}^q) \).

3. We estimate the term \( \langle f, \phi \rangle \) as follows.

Lemma 1. For each \( \varepsilon_0 > 0 \) there exists a constant \( C = C(\varepsilon_0) < \infty \) such that

\[
\langle f, \phi \rangle \leq \varepsilon_0 I_q + C \|w\|_q^{q-2} \|f\|_q^2.
\]

Proof of Lemma 1. We have:

\[
\langle f, \phi \rangle = \langle -\Delta u, |w|^{q-2} f \rangle + (q - 2) \langle |w|^{q-3} w \cdot \nabla |w|, f \rangle =: F_1 + F_2.
\]
Due to $|\Delta u|^2 \leq d|\nabla w|^2$ and $(|w|^{q-2} f) \leq \|w\|^{q-2} \|f\|_q^2,$

$$F_1 \leq \sqrt{dI_q^2} \|w\|^{\frac{q-2}{q}} \|f\|_q, \quad F_2 \leq (q-2)J_q^2 \|w\|^{\frac{q-2}{q}} \|f\|_q,$$

Now the standard quadratic estimates yield the lemma. □

We choose $\varepsilon_0 > 0$ in Lemma [1] so small that in the estimates below we can ignore $\varepsilon_0 I_q.$

4. Clearly, (ii) yields the inequalities

$$\|\nabla u_n\|_q \leq K_1(\mu - \mu_0)^{-\frac{1}{2}} \|f\|_q, \quad K_1 := C_1^\frac{1}{2},$$

$$\|\nabla u_n\|_{qj} \leq K_2(\mu - \mu_0)^{-\frac{1}{2} + \frac{1}{q}} \|f\|_q, \quad K_2 := C_2\eta^{-\frac{1}{2}} (q^2/4)^{\frac{1}{2} - \frac{1}{q}},$$

where $C_S$ is the constant in the Sobolev Embedding Theorem. So, [KIS Theorem 3.5] ($(\mu + A_q)^{-1} = s-L_q^q$-lim$_n (\mu + A_q^n)^{-1}$) yields (tw). The proof of Theorem [4] is completed.

2. PROOF OF THEOREM [2]

Proof of (i). Recall that a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $F_{r_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2 := [-\nabla \cdot a \cdot \nabla \mid C_c^{\infty} \text{loc}]$, if $b^2_a := b \cdot a^{-1} \cdot b \in L^1_{\text{loc}}$ and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that

$$\|b_a(\lambda + A)^{-\frac{1}{2}}\|_{2-2} \leq \sqrt{\delta_1}.$$

It is easily seen that if $b \in F_{r_1}(A),$ with $\delta_1 := [1 \lor (1+c)^{-2}] \delta$. By the assumptions of the theorem, $\delta_1 < 4$. Therefore, by [KIS Theorem 3.2], $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ has an operator realization $\Lambda_q(a,b)$ in $L^q,$ $q \in [\frac{2}{2-\sqrt{\delta_1}}, \infty[, as the (minus) generator of a positivity preserving $L^\infty$ contraction quasi contraction $C_0$ semigroup. Moreover, $(\mu + \Lambda_q(a,b))^{-1}$ is well defined on $L^q$ for all $\mu > \frac{\lambda_\delta}{2(q-1)}.$ This completes the proof of (i).

Proof of (ii). First, we prove an a priori variant of (tw). Set $a_n := I + c_{\varepsilon_n} \otimes f_n,$ where $f_n$ have been defined in the beginning of the paper. Since our assumptions on $\delta_1, \delta_2,$ and $\delta_3$ involve only strict inequalities, we may assume that (C$_{\delta_1}$) holds for $f_n, \nabla a_n \in F_{r_1}, b_n \in F_{\delta}$ with $\lambda \neq \lambda(n)$ for appropriate $\varepsilon_n > 0.$ We also note that $\|f_n\|_\infty = 1.$

Denote $A_q^n := -\nabla \cdot a_n \cdot \nabla, D(A_q^n) = W^{2,q}.$ Set $u \equiv u_n := (\mu + \Lambda_q(a_n,b_n))^{-1} f,$ $0 \leq f \in C_c^1, n \geq 1,$ where $\Lambda_q(a_n,b_n) = A_q^n + b_n \cdot \nabla, D(\Lambda_q(a_n,b_n)) = D(A_q^n).$ Clearly, $0 \leq u_n \in W^{3,q}.$ It is easily seen that $b_n \in F_{\delta_1}(A^n)$ with $\lambda \neq \lambda(n),$ so $(\mu + \Lambda_q(a_n,b_n))^{-1}$ are well defined on $L^q$ for all $n \geq 1, \mu > \frac{\lambda_\delta}{2(q-1)}.$

1. Denote $w \equiv w_n := \nabla u_n.$ Below we omit the index $n$: $f \equiv f_n, a \equiv a_n, b \equiv b_n, A_q \equiv A_q^n.$ Set

$$I_q := \langle (\nabla w)^2 \rangle, \quad J_q := \langle (|w|) \rangle, \quad I_q := \langle (\nabla w)^2 \rangle, \quad J_q := \langle (|w|) \rangle.$$

Arguing as in the proof of Theorem [4] we arrive at
Next, we bound $F_1$ as follows.

\[ \langle b \cdot w, f, \mu u \rangle + B_q \langle w, \phi \rangle \]

where $\mu_0 := |c|\sqrt{\delta_1(q^{-1} + \gamma_1 \sqrt{\delta_1})}$, and $\gamma, \gamma_1 > 0$ are to be chosen.

2. We estimate the term $\langle b \cdot w, \phi \rangle$ as follows.

**Lemma 2.** There exist constants $C_i \ (i = 0, 1)$ such that

\[ \langle b \cdot w, \phi \rangle \leq \left[ (\sqrt{\delta_1 - \delta}) \frac{q^2}{2} \right]_2 (q - 2) \langle w, \nabla |w|, -b \cdot w \rangle \]

Proof. We have:

\[ \langle b \cdot w, \phi \rangle = \langle -\Delta u, |w|^q-2(-b \cdot w) \rangle + (q - 2) \langle |w|^q-3w \cdot \nabla |w|, -b \cdot w \rangle \]

Set $B_q := \langle |b \cdot w|^2 |w|^{q-2} \rangle$. We have

\[ F_2 \leq (q - 2)B_q^\frac{1}{2}J_q^\frac{1}{2}. \]

Next, we bound $F_1$. Recall that $\nabla a = c[(\text{div} f) + f \cdot \nabla f]$. We represent $-\Delta u = \nabla \cdot (a - 1) \cdot w - \mu u - b \cdot w + f$, and evaluate:

\[ F_1 = \langle \nabla \cdot (a - 1) \cdot w, |w|^{q-2}(-b \cdot w) \rangle + \langle (-\mu u - b \cdot w + f), |w|^{q-2}(-b \cdot w) \rangle \]

Set $P_q := \langle |\nabla a \cdot w|^2 |w|^{q-2} \rangle$. We bound $F_1$ from above by applying consecutively the following estimates:

1°) $\langle \nabla a \cdot w, |w|^{q-2}(-b \cdot w) \rangle \leq P_q^\frac{1}{2}B_q^\frac{1}{2}.$

2°) $\langle f \cdot (f \cdot \nabla w), |w|^{q-2}(-b \cdot w) \rangle \leq \bar{I}_q^\frac{1}{2}B_q^\frac{1}{2}.$

3°) $\langle \mu u, |w|^{q-2}(-b \cdot w) \rangle \leq \frac{\mu}{\mu - \omega_q}B_q^\frac{1}{2} ||w||^{2q-2} ||f||_q \quad \text{(here } \frac{2}{2-q} < q \Rightarrow ||u|| \leq (\mu - \omega_q)^{-1} ||f||_q).$

4°) $\langle b \cdot w, |w|^{q-2}(-b \cdot w) \rangle = B_q.$

5°) $\langle f, |w|^{q-2}(-b \cdot w) \rangle \leq B_q^\frac{1}{2} ||w||^{2q-2} ||f||_q.$

In 3°) and 5°) we estimate $B_q^\frac{1}{2} ||w||^{2q-2} ||f||_q \leq \epsilon_0 B_q + \frac{1}{4\epsilon_0} ||w||^{q-2} ||f||_q^2 \quad (\epsilon_0 > 0).$

Therefore,

\[ \langle b \cdot w, \phi \rangle \leq P_q^\frac{1}{2}B_q^\frac{1}{2} + |c|I_q^\frac{1}{2}B_q^\frac{1}{2} + B_q + (q - 2)B_q^\frac{1}{2}J_q^\frac{1}{2} + \epsilon_0 B_q + C_1(\epsilon_0) ||w||^{q-2} ||f||_q^2. \]
It is easily seen that \( b \in F_\delta \) is equivalent to the inequality
\[
\langle b^2 | \varphi \rangle \leq \delta (|\nabla \varphi|^2) + \lambda \delta (|\varphi|^2), \quad \varphi \in W^{1,2}.
\]

Thus,
\[
B_q \leq \|b|w|^q_2 \|_2 \leq \delta \|\nabla |w|^q_2 \|_2 + \lambda \delta \|w|_q^q = \frac{q^2 \delta}{4} J_q + \lambda \delta \|w|_q^q.
\]

Similarly, using that \( \nabla a \in F_{\delta_a} \), we obtain
\[
P_q \leq \|\nabla \|w|^q_2 \|_2 \leq \delta_a \|\nabla |w|^q_2 \|_2 + \lambda \delta_a \|w|_q^q = \frac{q^2 \delta_a}{4} J_q + \lambda \delta_a \|w|_q^q.
\]

Then selecting \( \varepsilon_0 > 0 \) sufficiently small, and noticing that the assumption on \( \delta, \delta_a \) in the theorem are strict inequalities, we can and will ignore below the terms multiplied by \( \varepsilon_0 \). The proof of Lemma 2 is completed.

In (7), we apply Lemma 2 where the inequality \( \frac{q^2 \sqrt{\delta}}{2} J_q^\frac{1}{2} I_q^\frac{1}{2} \leq \gamma_2 q^4 \frac{\delta}{4} J_q + \frac{1}{4 \gamma_2} I_q \), \( \gamma_2 > 0 \), is used. Thus, we have
\[
\mu \|w|_q^q + I_q + c I_q + (q - 2)(J_q + c J_q)
\]
\[
\leq |c| \left[ \frac{q \sqrt{\delta q}}{4} J_q + \frac{q \sqrt{\delta q}}{4} I_q \right] + |c| (q - 2) \frac{q \sqrt{\delta q}}{2} J_q
\]
\[
+ |c| \left[ (\gamma q + \gamma_2) \frac{q^2}{4} J_q + \left( \frac{1}{4 \gamma} + \frac{1}{4 \gamma_2} \right) I_q \right] + |c| (q - 2) \left[ \gamma q J_q + \frac{1}{4 \gamma} J_q \right]
\]
\[
+ \left[ (\sqrt{\delta q} + \sqrt{\delta q} + \gamma_2) \frac{q^2}{4} J_q + (q - 2) \frac{q \sqrt{\delta}}{2} J_q + \mu_0 \|w|_q^q + C_1 \|w|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle, \right.
\]

where \( \mu_0 := |c| \lambda \sqrt{\delta q} (q^{-1} + \gamma \sqrt{\delta q}) + |c| (q - 2) \lambda \sqrt{\delta q} (q^{-1} + \gamma_1 \sqrt{\delta q}) + C_0. \)

3. Let us prove that there exists constant \( \eta > 0 \) such that
\[
(\mu - \mu_0) \|w|_q^q + \eta J_q \leq C_1 \|w|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle. \quad (\ast)
\]

Set \( Q := \left( \sqrt{\delta q} + \delta \right) \frac{q^2}{4} + (q - 2) \frac{q \sqrt{\delta q}}{2}. \)

Case \( c > 0 \). First, suppose that \( 1 - \frac{q \sqrt{\delta q}}{4} - \frac{q \sqrt{\delta}}{4} \geq 0 \). We select \( \gamma, \gamma_2 > 0 \) such that \( \frac{1}{4 \gamma} + \frac{1}{4 \gamma_2} = 1 \) while \( \gamma q + \gamma_2 q \) attains its minimal value. It is easily seen that \( \gamma = \frac{1}{4} (1 + \sqrt{\frac{q}{\delta}}), \gamma_2 = \frac{1}{4} (1 + \sqrt{\frac{q}{\delta}}). \) We have \( 1 - \frac{q \sqrt{\delta q}}{4} \leq 0 \), and select \( \gamma_1 = \frac{1}{4} \). Thus, the terms \( \bar{I}_q, \bar{J}_q \) are no longer present in (\( \ast \)):
\[
\mu \|w|_q^q + \left( 1 - c \frac{q \sqrt{\delta q}}{4} \right) J_q
\]
\[
+ \left[ q - 2 - c \frac{q \sqrt{\delta q}}{4} - c(q - 2) \frac{q \sqrt{\delta q}}{2} - c(\delta q + 2 \sqrt{\delta q} + \delta) \frac{q^2}{16} - c(q - 2) \frac{q^2 \delta q}{16} - Q \right] J_q
\]
\[
\leq \mu_0 \|w|_q^q + C_1 \|w|_q^{q-2} \|f\|_q^2 + \langle f, \phi \rangle.
\]
By the assumptions of the theorem, $1 - c \frac{q \sqrt{\delta t}}{4} \geq 0$, so by $J_q \leq I_q$ we obtain
\[
\mu \|w\|_q^q + \left[ q - 1 - c(q - 1) \frac{q \sqrt{\delta t}}{2} - c(\delta t + 2 \sqrt{\delta t} + \delta) \frac{q^2}{16} - c(q - 2) \frac{q^2 \delta t}{16} - Q \right] J_q
\leq \mu_0 \|w\|_q^q + C_1 \|w\|^q \|f\|^2_q + \langle f, \phi \rangle.
\]

Next, suppose that $1 - \frac{q \sqrt{\delta}}{4} < 0$, but $1 - \frac{q \sqrt{\delta}}{4} \geq 0$. We select $\gamma = \frac{1}{q \sqrt{\delta}}$, $\gamma_2 = \frac{1}{q \sqrt{\delta}}$, and $\gamma_1 = \frac{1}{4}$. Then the term $\bar{I}_q$ is no longer present, so using $\bar{I}_q \leq I_q$ we obtain
\[
\mu \|w\|_q^q + \left[ 1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \right] I_q
+ \left[ q - 2 - c \frac{q \sqrt{\delta t}}{4} - c(q - 2) \frac{q \sqrt{\delta t}}{2} - c \frac{q \sqrt{\delta t}}{2} + q \sqrt{\delta} - c(q - 2) \frac{q^2 \delta t}{16} - Q \right] J_q
\leq \mu_0 \|w\|_q^q + C_1 \|w\|_q^q \|f\|^2_q + \langle f, \phi \rangle.
\]

Thus, since $1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \geq 0$ by the assumptions of the theorem, we have using $J_q \leq I_q$
\[
\mu \|w\|_q^q + \left[ q - 1 + c - c \frac{q \sqrt{\delta t}}{2} - c \frac{q \sqrt{\delta t}}{2} - c(q - 2) \frac{q^2 \delta t}{16} - Q \right] J_q
\leq \mu_0 \|w\|_q^q + C_1 \|w\|_q^q \|f\|^2_q + \langle f, \phi \rangle.
\]

Finally, suppose that $1 - \frac{q \sqrt{\delta}}{4} < 0$. We select $\gamma = \gamma_1 = \frac{1}{q \sqrt{\delta}}$, $\gamma_2 = \frac{1}{q \sqrt{\delta}}$. Then using $\bar{I}_q \leq I_q$, $\bar{I}_q \leq J_q$ we obtain
\[
\mu \|w\|_q^q + \left[ 1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \right] I_q + \left[ q - 2 + c(q - 2) \left( 1 - \frac{q \sqrt{\delta t}}{4} \right) \right. 
- c \frac{q \sqrt{\delta t}}{4} - c(q - 2) \frac{q \sqrt{\delta t}}{2} - c \frac{q \sqrt{\delta t}}{2} + q \sqrt{\delta} - c(q - 2) \frac{q \sqrt{\delta t}}{4} - Q \right] J_q
\leq \mu_0 \|w\|_q^q + C_1 \|w\|_q^q \|f\|^2_q + \langle f, \phi \rangle.
\]

Since $1 + c \left( 1 - \frac{q \sqrt{\delta t}}{2} - \frac{q \sqrt{\delta}}{4} \right) \geq 0$ by the assumptions of the theorem, we have using $J_q \leq I_q$
\[
\mu \|w\|_q^q + \left[ q - 1 + c(q - 1) - c \frac{q \sqrt{\delta t}}{2} - c(q - 1) q \sqrt{\delta t} - Q \right] J_q
\leq \mu_0 \|w\|_q^q + C_1 \|w\|_q^q \|f\|^2_q + \langle f, \phi \rangle.
\]

In all three cases, the coefficient of $J_q$ is positive. We have proved (7).

Case $c < 0$. In (6), select $\gamma = \gamma_1 = \frac{1}{q \sqrt{\delta t}}$, $\gamma_2 = \frac{1}{q \sqrt{\delta}}$:
\[
\mu \|w\|_q^q + \left( 1 - |c| \frac{q \sqrt{\delta t}}{4} \right) I_q
+ \left[ q - 2 - |c|(q - 1) \frac{q \sqrt{\delta t}}{2} - |c|(q - 2) \frac{q \sqrt{\delta t}}{4} - |c| \frac{q \sqrt{\delta}}{4} - Q \right] J_q
- |c| \left( 1 + \frac{q \sqrt{\delta t}}{4} + \frac{q \sqrt{\delta t}}{4} \right) \bar{I}_q - |c|(q - 2) \left( 1 + \frac{q \sqrt{\delta t}}{4} \right) \bar{I}_q \leq \mu_0 \|w\|_q^q + C_1 \|w\|_q^q \|f\|^2_q + \langle f, \phi \rangle.
\]
Using $I_q \geq \bar{I}_q$, $J_q \geq \bar{J}_q$, we obtain

$$\mu \|w\|^q_q + \left(1 - |c| \left(1 + \frac{q\sqrt{\delta_t}}{2} + \frac{q\sqrt{\delta}}{4}\right)\right) I_q$$

$$+ \left[ q - 2 - |c|(q - 1)\frac{q\sqrt{\delta_t}}{2} - |c|(q - 2)\frac{q\sqrt{\delta_t}}{2} - |c|\frac{q\sqrt{\delta}}{4} - |c|(q - 2)\left(1 + \frac{q\sqrt{\delta_t}}{4}\right) - Q \right] J_q$$

$$\leq \mu_0 \|w\|^q_q + C_1 \|w\|^q_q - 2\|f\|^2_q + \langle f, \phi \rangle.$$  

By the assumptions of the theorem, $1 - |c|(1 + \frac{q\sqrt{\delta_t}}{2} + \frac{q\sqrt{\delta}}{4}) \geq 0$. Therefore, by $I_q \geq J_q$,

$$\mu \|w\|^q_q + \left[ q - 1 - |c| \left(1 + \frac{q\sqrt{\delta_t}}{2} + \frac{q\sqrt{\delta}}{4}\right) \right]$$

$$- |c|(q - 1)\frac{q\sqrt{\delta_t}}{2} - |c|(q - 2)\frac{q\sqrt{\delta_t}}{2} - |c|\frac{q\sqrt{\delta}}{4} - |c|(q - 2)\left(1 + \frac{q\sqrt{\delta_t}}{4}\right) - Q \right] J_q$$

$$\leq \mu_0 \|w\|^q_q + C_1 \|w\|^q_q - 2\|f\|^2_q + \langle f, \phi \rangle,$$

where the coefficient of $J_q$ is strictly positive by the assumptions of the theorem. We have proved (1).

4. We estimate the term $\langle f, \phi \rangle$ by Lemma [1]. For each $\varepsilon_0 > 0$ there exists a constant $C = C(\varepsilon_0) < \infty$ such that

$$\langle f, \phi \rangle \leq \varepsilon_0 I_q + C\|w\|^q_q - 2\|f\|^2_q.$$  

We choose $\varepsilon_0 > 0$ so small that in the estimates below we can ignore $\varepsilon_0 I_q$.

Then (1) yields the inequalities

$$\|\nabla u_n\|_q \leq K_1(\mu - \mu_0)^{-\frac{1}{2}}\|f\|_q, \quad K_1 := (C + C_1)^{\frac{1}{2}},$$

$$\|\nabla u_n\|_{q_j} \leq K_2(\mu - \mu_0)^{\frac{1}{2} - \frac{1}{q}}\|f\|_q, \quad K_2 := C_S^\eta - \eta (q^2/4)^\eta (C + C_1)^{\frac{1}{2} - \frac{1}{q}},$$

where $C_S$ is the constant in the Sobolev Embedding Theorem.

If $c > 0$ then $\delta_1 = \delta < 1$. If $c < 0$ then elementary arguments show that, by the assumptions of the theorem, $\delta_1 = (1 - |c|)^{-2}\delta < 1$. Therefore, [KS] Theorem 3.5] $(\mu + \Lambda_q(a,b))^{-1} = s-L^q\text{-lim}_{n}(\mu + \Lambda_q(a_n,b_n))^{-1}$ yields (2). The proof of Theorem 2 is completed.

3. The iteration procedure

The following is a direct extension of the iteration procedure in [KS]. Let $a \in (H_u)$.

Recall that a vector field $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ belongs to $F_{\delta_1}(A)$, $\delta_1 > 0$, the class of form-bounded vector fields (with respect to $A \equiv A_2 := [-\nabla \cdot a \cdot \nabla \mid C_{C}^\{\text{loc}\} \leq 2]$, if $b_n^\delta := b \cdot a^{-1} \cdot b \in L^1_{\text{loc}}$ and there exists a constant $\lambda = \lambda_{\delta_1} > 0$ such that $\|b_n(\lambda + A)^{-\frac{1}{2}}\|_{2\rightarrow 2} \leq \sqrt{\delta_1}$.

Consider

$$\{a_n\}_{n=1}^\infty \subset [C_1]^d \cap (H_u, a, \delta)$$

and

$$\{b_n\}_{n=1}^\infty \subset [C_1]^d \cap \bigcap_{m \geq 1} F_{\delta_1}(A^m), \quad \delta_1 < 4, \quad \lambda \neq \lambda(n, m).$$

Here $A^m \equiv A(a_m)$. 
By [KiS, Theorem 3.2], $-\Lambda_r(a_n, b_n) := \nabla \cdot a_n \cdot \nabla - b_n \cdot \nabla$, $D(\Lambda_r(a_n, b_n)) = W^{2,r}$, is the generator of a positivity preserving $L^\infty$ contraction quasi contraction $C_0$ semigroup on $L^r$, $r \in \left[ \frac{2}{2-\sqrt{\alpha_1}}, \infty \right[$, with the resolvent set of $-\Lambda_r(a_n, b_n)$ containing $\mu > \omega_r := \frac{\lambda_1}{2(r-1)}$ for all $n \geq 1$.

Set $u_n := (\mu + \Lambda_r(a_n, b_n))^{-1} f$, $f \in L^1 \cap L^\infty$ and $g := u_m - u_n$.

**Lemma 3.** There are positive constants $C = C(d), k = k(\delta_1)$ such that

$$\|g\|_{r,j} \leq (C\sigma^{-1}(\delta_1 + 2\xi\sigma^{-1}))(1 + 2\xi)\|\nabla u_m\|_{q,j}^2 \left( r^{2k} \right)^{\frac{1}{2}} \|\nabla^\gamma g\|_{r'(r-2)},$$

where $q \in \left[ \frac{2}{2-\sqrt{\alpha_1}} \vee (d-2), \frac{2}{\sqrt{\alpha_1}} \right]$, $2x = qj$, $j = \frac{d}{2-2}$, $x' := \frac{x}{x}$ and $x'(r-2) > \frac{2}{2-\sqrt{\alpha_1}}$, $\mu > \lambda_{\delta_1}$.

The proof follows closely [KiS, proof of Lemma 3.12] or [KS, proof of Lemma 6].

Iterating the inequality of Lemma 3, we arrive at

**Lemma 4.** In the notation of Lemma 3, assume that $\sup_m \|\nabla u_m\|_{q,j}^2 < \infty$, $\mu > \mu_0$. Then for any $r_0 > \frac{2}{2-\sqrt{\alpha_1}}$

$$\|g\|_\infty \leq B\|g\|_{r_0}^{\gamma}, \quad \mu \geq 1 + \mu_0 \vee \lambda_{\delta_1},$$

where $\gamma = \left( 1 - \frac{x'}{j} \left( 1 - \frac{x'}{r} + \frac{2x'}{j} \right) \right)^{-1} > 0$, and $B = B(d, \delta_1) < \infty$.

The proof repeats [KiS, proof of Lemma 3.13] or [KS, proof of Lemma 7].

**Remark.** The assumption $\sup_m \|\nabla u_m\|_{q,j}^2 < \infty$ in Lemma 4 is crucial and holds e.g. in the assumptions of Theorem 2 (ii).

4. PROOF OF THEOREM 3

By Lemma 4 and the second inequality in (xx), we have for all $r_0 > \frac{2}{2-\sqrt{\alpha_1}}$

$$\|u_n - u_m\|_\infty \leq B\|u_n - u_m\|_{r_0}^{\gamma}, \quad \mu \geq 1 + \mu_0 \vee \lambda_{\delta_1},$$

where $\gamma > 0$, $B < \infty$, and $u_n := (\mu + \Lambda_{r_0}(a_n, b_n))^{-1} f$, $f \in L^1 \cap L^\infty$. By [KiS, Theorem 3.5],

$$(\mu + \Lambda_{r_0}(a, b))^{-1} = s.C_{\infty} \lim_n (\mu + \Lambda_{r_0}(a_n, b_n))^{-1},$$

so $\{u_n\}$ is fundamental in $C_\infty$.

**Lemma 5.** $s.C_{\infty} \lim_{\mu \uparrow \infty} \mu (\mu + \Lambda_{C_{\infty}}(a_n, b_n))^{-1} = 1$ uniformly in $n$.

The proof follows closely [KiS, proof of Lemma 3.16].

We are in position to complete the proof of Theorem 3. The assertion (i) follows from the fact that $\{u_n\}$ is fundamental in $C_\infty$ and Lemma 5 by applying the Trotter Approximation Theorem. (ii) is Theorem 2 (xx). The proof of (iii) is standard. The proof of Theorem 3 is completed.

**Remark.** The arguments of the present paper extend more or less directly to the time-dependent case $\partial_t - \nabla \cdot a(t, x) \cdot \nabla + b(t, x) \cdot \nabla$, cf. [Ki].
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