Relativistic Quantum Theory of Cyclotron Resonance in a Medium

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In this paper the relativistic quantum theory of cyclotron resonance in an arbitrary medium is presented. The quantum equation of motion for charged particle in the field of plane electromagnetic wave and uniform magnetic field in a medium is solved in the eikonal approximation. The probabilities of induced multiphoton transitions between Landau levels in strong laser field is calculated.

I. INTRODUCTION

As is known when charged particle moves in the field of transverse electromagnetic wave (EMW) in the presence of uniform magnetic field directed along the wave propagation direction, a resonant effect of the wave on the particle motion is possible. If the interaction takes place in vacuum this is the well known phenomenon of autoresonance when the ratio of the Doppler shifted wave frequency ($\omega'$) to the cyclotron frequency ($\Omega$) is conserved $\omega'/\Omega = const$ and the resonance created at the initial moment is automatically supported throughout interaction. However, if the interaction takes place in a medium where the phase velocity of a EMW larger (plasma like medium) or smaller (dielectric medium) than light speed ($c$) the picture of wave-particle interaction is essentially changed. Particularly, in a medium the autoresonance phenomenon is violated because of nonequidistant Stark-shift of magnetic sublevels of an electron (Landau levels) in the electric field of EMW. As a result the intensity effect of the wave governs the resonance characteristics, and the particle state essentially depends on the initial conditions and the wave field magnitude, at which the nonlinear resonance is achieved. At first the investigation of cyclotron resonance (CR) in the medium in the scope of classical theory was carried out in the papers, where oscillating solutions for the particle energy are obtained. However, such behavior is valid only for EMW intensity less than some critical value. As was shown in, for larger intensities a non-linear resonance phenomenon of so called "Electron Hysteresis" takes place when EMW is turned on adiabatically. If the intensity pick of an actual wave pulse exceeds the mentioned critical value then significant acceleration of charged particles can be achieved (medium may be either plasma or refractive).

Concerning the quantum description of CR the relativistic quantum equation of motion allows exact solution only for CR in vacuum (description of related Quantum Electrodynamic (QED) processes, such as electron-positron pair production, non-linear Compton scattering in the presence of uniform magnetic field and etc. by this wave function see and references therein). It is worthy to mention that the configuration of electromagnetic fields when uniform magnetic field is directed along the propagation of transverse wave is one of the exotic cases when the relativistic quantum equation of motion allows exact solution. In the medium even at the absence of uniform magnetic field the relativistic quantum equation of motion describing the particle-wave interaction reduces to the Mathieu type (in general, Hill type) equation the exact solution of which is unknown. In this case to obtain an approximate analytical solution describing the nonlinear process of interaction is already problematic.

The purpose of this paper is to obtain non-linear over the EMW field approximate solution of relativistic quantum equation of motion for a charged particle in the strong EMW in the medium, in the presence of uniform magnetic field, which will good enough describe the quantum picture of cyclotron resonance, particularly multiphoton stimulated transitions between Landau levels. By this wave function one can treat a large class of non-linear QED processes in strong electromagnetic fields with modifications that the medium brings (e. g., anomalous Doppler effect), taking into account Astrophysical applications as well where CR plays a significant role. One of the advantages of CR in the refractive medium is that for moderate relativistic particle beam one can achieve the cyclotron resonance in optical region (close to Cherenkov resonance) by current lasers and existing uniform magnetic fields ($\sim 10^3 G$), while in the vacuum for the same parameters CR is possible for radio frequencies. The Free Electron Laser version based on the combine scheme of CR and Cherenkov radiation was proposed in.

This paper is organized as follows. In Sec. II the wave function of a charged particle moving in a medium in the field of transverse EMW, at the presence of uniform magnetic field directed along the wave propagation direction, is obtained. In Sec. III the CR is considered in the medium and the probabilities of induced multiphoton transitions in the strong circularly polarized EMW are calculated.
II. WAVE FUNCTION OF A PARTICLE IN A PLANE EMW IN THE MEDIUM AT THE PRESENCE OF UNIFORM MAGNETIC FIELD

Let charged particle moves in a medium in the following configuration of EM field

\[ A = A_H + A_w \]  \hspace{1cm} (2.1)

where

\[ A_H = (0, xH_0, 0, 0), \]  \hspace{1cm} (2.2)

is the four-vector potential of uniform magnetic field with the strength \( H_0 \) directed along the \( z \) axis and

\[ A_w = \left\{ A_x \left( t - \frac{\omega}{c} \right), A_y \left( t - \frac{\omega}{c} \right), 0, 0 \right\} \]  \hspace{1cm} (2.3)

is the four-vector potential of transverse EMW propagating along the \( z \) axis. Here for the four-component vectors we have chosen the following metric \( a = (\overline{\alpha}, i \omega) \). In (2.3) \( n \) is the refraction index of the medium (it is assumed quasimonochromatic wave: \( n(\omega) \approx n \)) and \( c \) is the light speed in vacuum. We will assume that the EMW is switched on/off adiabatically so for the four-vector potential we have \( A_w = 0 \) at \( t = \pm \infty \).

In present consideration we will restrict the total energy exchange \( \Delta E \) of a particle with EMW

\[ \Delta E \ll E, \]  \hspace{1cm} (2.4)

where \( E \) is the particle energy.

From the condition (2.4) follows the restriction on the wave frequency \( \omega \):

\[ \hbar \omega \ll E . \]  \hspace{1cm} (2.5)

For this reason in considering case the neglecting of the spin interaction is justified. So we will use the Klein-Gordon equation which for a charged particle in the field (2.1) is

\[ \left\{ \left( i \hbar \partial_t + \frac{e}{c} A_\mu \right)^2 + m^2 c^2 \right\} \Psi = 0, \]  \hspace{1cm} (2.6)

where \( \hbar \) is the Plank constant \( m \) and \( e \) are the particle mass and charge respectively and \( \partial_\mu \equiv \partial/\partial x_\mu \) \((\mu = 1, 2, 3, 4)\) denotes the first derivative of a function with respect to four-component radius vector \( x \).

The charged particle initial state at \( t \rightarrow -\infty \) when \( A_w = 0 \) is well known and has been the topic of numerous studies (see, e.g., [13]). As it is known [13] the motion of the particle in the uniform magnetic field is separated into cyclotron \( (x, y) \) and the longitudinal \( (z) \) degrees of freedom. For longitudinal motion we will assume initial state with momentum \( p_z \), while for cyclotron motion we will assume the state \( \{ s, p_y \} \), where by \( s \) we indicate Landau levels and by \( p_y \) the \( Y \) component of generalized momentum. So the particle initial state when the EMW is adiabatically switched on at \( t \rightarrow -\infty \) is assumed to be

\[ \psi_s = N \Phi_s(x) \exp \left[ \frac{i}{\hbar} \left( p_z z + p_y y - E_s(p_z)t \right) \right], \]  \hspace{1cm} (2.7)

where \( N \) is the normalization constant,

\[ \Phi_s(x) = \frac{1}{\sqrt{2^n s!a \sqrt{\pi}}} \exp \left[ -\frac{(x - cp_z)^2}{2a^2} \right] U_s \left[ \frac{x - cp_z}{\epsilon \hbar H_0} \right] \]  \hspace{1cm} (2.8)

are the wave functions corresponding to Landau levels. Here \( U_s \) are Hermite polynomials with

\[ a = \sqrt{\frac{\epsilon \hbar H_0}{c}} \]

and dispersion law is

\[ E^2_s(p_z) = m^2 c^4 + c^2 p_z^2 + 2ecH_0 \hbar \left( s + \frac{1}{2} \right) . \]  \hspace{1cm} (2.9)

As the EMW field depends only on the \( \tau = t - \frac{\omega}{c} z \) then raising from the symmetry, the particle wave function can be looked for in the following form:

\[ \Psi(r, t) = f(x_\perp, \tau) \exp \left[ \frac{i}{\hbar} (p_z z - E t) \right] \]  \hspace{1cm} (2.10)

Taking into account (2.4) we can consider \( f(x_\perp, \tau) \) as a slowly varying function of \( \tau \) and neglect the second derivative compared with the first order. So from (2.6) for \( f(x_\perp, \tau) \) we will have the following equation:

\[ \left\{ \frac{2i \hbar}{c^2} E \partial_\tau - \left( i \hbar \partial_\mu + \frac{e}{c} A_\mu \right)^2 + \frac{E^2}{c^2} - m^2 c^2 - p_z^2 \right\} f = 0, \]  \hspace{1cm} (2.11)

where

\[ \partial_\mu = \{ \partial_x, \partial_y, 0, 0 \}, \hspace{1cm} x_\perp = \{ x, y, 0, 0 \}, \hspace{0.8cm} \vec{E} = E - \epsilon c p_z \]

In Eq. (2.11) transverse and longitudinal motions are not separated. But after a certain unitarian transformation in the equation for the transformed function the variables are separated [8], that is

\[ \tilde{S} = \exp \left\{ i \mathbf{K}(\tau) \hat{P}_\perp \right\} ; \hspace{0.8cm} \hat{P}_\perp_\mu = -i \hbar \partial_\mu + \frac{e}{c} A_\mu \mu , \]  \hspace{1cm} (2.12)

where \( \mathbf{K}(\tau) \) will be chosen to separate the cyclotron and longitudinal motions and to fulfill the initial condition (2.7):

\[ K_x + i K_y = - \exp \left[ -i \frac{ec}{E} H_0 \tau \right] \]

\[ \times \int_{-\infty}^{\tau} \frac{ec}{\hbar E} (A_x (\tau') + i A_y (\tau')) \exp \left[ \frac{ec}{E} H_0 \tau' \right] d\tau' . \]  \hspace{1cm} (2.13)
For the transformed wave function $\tilde{f} = \hat{S}f(x_\perp, \tau)$ we will have the following equation

$$
\left\{ - \frac{E^2}{c^2} + p_z^2 + m_e c^2 - \frac{2hE}{c^2} \partial_x - \frac{e\hbar^2 E}{c^2} K^{\nu} F_{\mu\nu} \frac{dK^\mu}{d\tau} \right\} \tilde{f}(x_\perp, \tau) = 0, \quad (2.14)
$$

Where $F_{\mu\nu}$ is the tensor of EM field corresponding to uniform magnetic field and $e^\mu = \{1, 1, 0, 0\}$. In Eq. (2.14) the variables are separated and making inverse transformation $f = \hat{S}^* \tilde{f}(x_\perp, \tau)$ gives the solution of the initial equation (taking into account Eq. (2.11)):

$$
\Psi(r, t) = N \exp \left[ \frac{i}{\hbar} (p_z z - E_s(p_z) t) - \frac{i}{\hbar} \int_{-\infty}^\tau Q(\tau') d\tau' \right] 
\times \exp \left[ \frac{e^c}{c} H_0 \kappa (x - \frac{\hbar}{2} K^\kappa) \right] T_s(x_\perp - \hbar K) \quad (2.15)
$$

where

$$
T_s(x_\perp) = \exp \left\{ \frac{i p_y y}{\hbar} \right\} \Phi_s(x) \quad (2.16)
$$

and

$$
Q(\tau) = \frac{\hbar^2}{2E} \left[ \frac{e^c}{c} F_{\mu\nu} K^{\nu} e^\mu + \frac{e^c}{c} A_w \right]^2 - \frac{e\hbar^2 E}{c^2} K^{\nu} F_{\mu\nu} \frac{dK^\mu}{d\tau} \right\] \quad (2.17)
$$

III. THE PROBABILITIES OF MULTIPHOTON TRANSITIONS

Although the motion of the particle in the uniform magnetic field is separated into cyclotron $(x, y)$ and the longitudinal $(z)$ degrees of freedom (2.7), in energy scale these motions are not separated due to relativistic effects (2.9). However, for not so strong magnetic fields we can separate the energies of longitudinal ($E_\parallel$) and cyclotron motions

$$
E_s(p_z) \simeq E_\parallel + \hbar \Omega \left( s + \frac{1}{2} \right) ; \quad \hbar \Omega s \ll E_\parallel \quad (3.1)
$$

$$
\Omega = ecH_0/E_\parallel, \quad E_\parallel = \sqrt{m_e c^4 + e^2 p_z^2}
$$

Now let us consider the concrete case of circularly polarized EMW

$$
A_w(\tau) = \{- A(\tau) \sin(\omega \tau), gA(\tau) \cos(\omega \tau), 0, 0\} \quad (3.2)
$$

which is in resonance with the particle, i.e. Doppler shifted wave frequency is close to cyclotron one

$$
\omega' \equiv (1 - nv_z/c) \omega \simeq g\Omega \quad (3.3)
$$

where $v_z$ is the particle longitudinal velocity. In (3.2) $g = \pm 1$ correspond to right and left hand circular polarizations of the wave. After the interaction ($\tau \rightarrow +\infty$) from (2.13) at the resonance condition (3.3) we have

$$
K_x = -\frac{eAcT}{\hbar E_\parallel} \cos(\omega \tau) \quad (3.4)
$$

$$
K_y = g\frac{eAcT}{\hbar E_\parallel} \sin(\omega \tau) \quad (3.5)
$$

where $A$ is the average value of $A(\tau)$ and $T$ is the coherent interaction time.

The final state of the particle after the interaction is described by the wave function

$$
\Psi_s = N \exp \left[ \frac{i}{\hbar} (p_z z + p_y y - E_s(p_z) t) + i \frac{eg\Omega Ty}{\hbar c} \sin(\omega \tau) \right] \Phi_s[\rho] 
\times \exp \left[ -i \frac{e^c H_0 \hbar}{c} \left( \frac{eAcT}{2\hbar E_\parallel} \right)^2 \sin(2\omega \tau) \right. \left. + i \frac{eg\hbar eAcT}{\hbar E_\parallel} \sin(\omega \tau) \right] \quad (3.6)
$$

where

$$
\rho = \frac{1}{a} \left( x + \frac{eAcT}{E_\parallel} \cos(\omega \tau) - \frac{a^2 p_y}{\hbar} \right) \quad (3.7)
$$

Expanding the wave function (3.6) in terms of the full basis of the particle eigenstates (2.7)

$$
\Psi_s = \int dp_z dp_y \sum_{s'} C_{ss'}(p_z', p_y') \psi_{s'}(p_z', p_y') \quad (3.8)
$$

we will find the probabilities of the multiphoton induced transitions between the Landau levels.

To calculate the expansion coefficients $C_{ss'}(p_z', p_y')$ we will take into account the result of the following integration

$$
\int \exp(-ikx)\Phi_s(a^{-1}x + ab)\Phi_{s'}(a^{-1}x + ab')
= \exp \{ i\mu + i(s - s')\lambda \} I_{ss'}(\zeta) \quad (3.9)
$$

where $I_{ss'}(\zeta)$ is the Lagger functions and characteristic parameters are determined by the expressions

$$
\mu = \frac{ka^2(b + b')}{2}; \quad \lambda = \tan^{-1} \frac{k}{b' - b}
$$

$$
\zeta = a^2 k^2 + (b - b')^2 \quad .
$$
Then we get the following transition amplitudes,
\[ C_{ss'}(p_y', p_z') = \delta(p_y - p_y')\delta(p_z - p_z') - (s - s')g\omega n\hbar e^{-1} \]
\[ \times \exp \left\{ \frac{i}{\hbar} (E - E' - (s - s')g\omega)t \right\} I_{ss'} [\zeta] \]  
(3.10)
where \( \delta(p) \)'s are the Dirac \( \delta \)-functions expressing the momentum conservation laws and the argument of the Lagrange function is
\[ \zeta = \frac{e^2 A^2 T^2 \Omega}{2\hbar E_\parallel} \]  
(3.11)
According to the expression (3.10), transition of the particle from an initial state \( \{s, p_y, p_z\} \) to a final state \( \{s', p_y', p_z'\} \) is accompanied by emission or absorption of \( s - s' \) photons. Consequently, substituting (3.10) into (3.8) and integrating by momentum we can write the particle wave function in another form:
\[ \Psi_s = N \sum_{s'} s' \exp \left\{ -\frac{i}{\hbar} (E - (s - s')g\omega)t \right\} \Phi_{s'}(x) \]  
(3.12)
The probability of the induced transition \( s \to s' \) between the Landau levels ultimately is defined from the formula (3.12):
\[ w_{ss'} = I_{ss'}^2 \left[ \frac{e^2 A^2 T^2 \Omega}{2\hbar E_\parallel} \right] \]  
(3.13)
As is seen from (3.12), when \( 1 - n\nu / c > 0 \) and \( q = 1 \) in the field of strong EMW the Landau levels are exited at the absorption of the wave quanta - normal Doppler effect, while in the case \( 1 - n\nu / c < 0 \) and \( q = -1 \), which is possible in the refractive medium \( (n > 1) \), Landau levels are exited at the emission of the wave quanta, i.e. takes place anomal Doppler effect.
Let us estimate the average number of emitted (absorbed) photons in quasiclassical limit \( (s >> 1) \) when multiphoton processes dominate and the process has a classical nature. The argument of the Lagrange function can be represented as
\[ \zeta = \frac{1}{4s} \left( \frac{\Delta \varepsilon_{cl}}{\hbar \omega} \right)^2 , \]  
(3.14)
where \( \Delta \varepsilon_{cl} \simeq eE \nu r, T \) (E is the EMW field strength, \( \nu r \simeq c\sqrt{2\hbar s T / E_\parallel} \) is the particle mean transverse velocity) is the energy change of the particle according to classical theory. At high intensities of the EMW: \( \Delta \varepsilon_{cl} >> \hbar \omega \), the Lagrange function is maximal at \( \zeta \to \zeta_0 = (\sqrt{s'} - \sqrt{s})^2 \), exponentially falling beyond \( \zeta_0 \). For the transition \( s \to s' \) and when \( |s - s'| << s \) we have \( \zeta_0 \sim (s' - s)^2 / 4s \). Comparison of this expression with (3.13) shows that the most probable transitions are
\[ |s - s'| \sim \frac{\Delta \varepsilon_{cl}}{\hbar \omega} \]  
in accordance with the correspondence principle.

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[1] A.A. Kolomensky, A.N.Lebedev, Doct. AN SSSR 145, 1259 (1962).
[2] V.Ya. Davidovsky, Zh. Eksp. Teor. Fiz. 43, 886 (1962).
[3] A.A. Kolomensky, A.N. Lebedev, Zh. Eksp. Teor. Fiz. 44, 261 (1963).
[4] V.M. Haroutunian, H.K. Avetissian, Izv. AN Arm SSR, Fizika 9, 110 (1975).
[5] C.S. Roberts, S.J. Buchsbaum, Phys. Rev. A 135, 381 (1965).
[6] H.K. Avetissian, Abstracts of the ICOMP V, Paris, 1990.
[7] P.J. Redmond, Journ. Math. Phys. 7, 1163 (1965).
[8] I.M. Ternov, V.R. Khalilov, V.N. Radionov, *Interaction of Garged Particles with Strong Electromagnetic field*, Moscow (1982) [in Russian].
[9] H.K. Avetissian et al., Phys.Lett. A 244, 25 (1998).
[10] H.K. Avetissian et al., Phys.Lett. A 246, 16 (1998).
[11] V.L. Ginsburg, *Theoretical Physics and Astrophysics*, Nauka, Moscow (1957) [in Russian].
[12] H.K. Avetissian, K.Z. Hatsogortsian, Zh.Tekh.Fiz. 54, 2347 (1984).
[13] A.E. Akhiezer, V.B. Berestetski, *Quantum Electrodynamics*, Nauka, Moscow (1969).