Noncommutative corrections to the holographic entanglement entropy of the pure AdS spacetime and Schwarzschild-AdS black hole

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Abstract

We compute the noncommutative corrections to the holographic entanglement entropy of the pure AdS spacetime and Schwarzschild-AdS black hole, where the noncommutative background is suitably constructed that matches the Poincaré coordinate system. In particular, we find a reasonable tetrad with subtlety, which not only matches the metrics of the pure AdS spacetime and Schwarzschild-AdS black hole in the commutative case, but also makes the noncommutative corrections real rather than complex. Based on the Ryu-Takayanagi formula and the perturbation expansion method, we give the noncommutative effects that are the logarithmic contribution to the holographic entanglement entropy for the pure AdS spacetime and the logarithmic contribution plus a both mass and noncommutative parameter related term for the Schwarzschild-AdS black hole, respectively.

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1 Introduction

One way to understand quantum gravity is based on the holographic correspondence. Inspired by the idea largely from the open/closed string duality [1], a lot of attentions have been paid to the correspondence. Though it is hard to give a general proof for this kind of dualities, there are many evidences to support such a duality. Importantly, some evidences can give us specific ideas on how to connect the classical gravity with quantum theory. The Ryu-Takayanagi (RT) formula shows [2] that the entanglement entropy of conformal fields is proportional to the minimal surface area in the Einstein gravity, and the proofs of the formula shed [3,4] light on understanding the AdS/CFT duality in the cosmic brane or in the topological black hole background. In addition, the time-dependent entanglement entropy has also been studied [5], where one can get much general boundary information through bulk structures, and vice versa. However, there are still many mysteries to be disclosed before completing a holographic dictionary. Here we list some of mysteries and the literature in which some efforts have been made to try to reveal these mysteries.

- How to code [6] the bulk information to the fields on the boundary whose information seems to be nonlocal?
- How to extend [7,8] the RT formula beyond the classical gravity? Is there any requirement that the formula should meet?
- Whether can the AdS/CFT duality be extended [9–12] to the de Sitter space or flat space besides the consideration from symmetry?
- What happens [13,14] in the dual theory if the quantum field on the boundary is taken into account in the bulk?
Though the questions mentioned above have been demonstrated on the side of gravity, the explanations to them on the side of quantum field remain unsolved. Moreover, the entanglement entropy does not always follow the area law, see, for instance, an exception [15]. In order to complement the holographic dictionary, the noncommutativity of spacetime should be considered. Here, we give a very brief review of developments about the holographic entanglement entropy (HEE) associated with the noncommutativity. The noncommutative spacetimes, early dubbed quantized spacetimes, could be traced back to Snyder’s pioneering work [16]. The revival of noncommutative theories about half a century after Snyder’s work originated [17] from the low energy effective field theory of string theory. Then, the combination of HEE and noncommutative geometry was proposed [18], where the noncommutative effect contributes a divergent term to entanglement entropy. Moreover, based on a smeared matter source induced noncommutative geometry [19], the gravitational collapse of a shell of dust in the 4-dimensional noncommutative spacetime was studied [20], where the noncommutative parameter is responsible for the delay of thermalization time. Besides, the noncommutative contribution to the HEE from the scalar field in the noncommutative BTZ black hole was considered [21] through the action correspondence, where the structure of the noncommutative HEE is similar to that of thermal entropy of the BTZ black hole. However, a clear clarification of noncommutative contributions on a boundary is still lacking.

Besides the divergent area term, non-leading contributions to the holographic dictionary are of significance. In particular, it is meaningful to investigate the relationship between non-leading contributions and non-locality of spacetime, which may provide a new source for such contributions. On the one hand, the non-leading contributions can extend the dictionary beyond the area term. For example, the background entanglement contributes [13, 14] the one-loop quantum correction to entanglement entropy. On the other hand, the non-leading contributions play [22] an important role in the quantum phase transition.

In the present paper, we generalize the RT formula by adding a non-leading term to the surface area, where this term is induced by the noncommutative structure of spacetimes. We find a specific tetrad and propose a noncommutative relation that is suitable to the Poincaré coordinate system. Then, the corresponding Moyal product can be given, which is different from those forms already constructed [23–25]. Our treatment is a physically straightforward modification to the classical theory of gravity. As it stems from the spacetime structure itself, the noncommutative geometry motivates us to analyze the significance of this emergent structure to the HEE.

The procedure goes as follows. We establish a noncommutative construction by using the Moyal product in the Poincaré coordinate system. Then, according to this construction, we compute the noncommutative HEE of the pure AdS spacetime and Schwarzschild-AdS black hole. As the noncommutative parameter is much smaller than one, it is reasonable [26, 27] to expand perturbatively a geodesic curve with respect to this parameter. That is, it is
explicit to study the noncommutative contributions to the HEE order by order. We find that 
the noncommutative geometry contributes a logarithmic divergent term to the entanglement 
entropy of the pure AdS spacetime, where this term is proportional to the noncommutative 
parameter. Furthermore, a similar discussion is made for the Schwarzschild-AdS black hole 
by the additional consideration of the black hole mass, and the result contains a both mass 
and noncommutative parameter related term besides the logarithmic contribution.

The rest of the paper is organized as follows. In section 2, we compute the holographic 
entanglement entropy of the pure AdS spacetime with noncommutativity. Then, we turn to 
the Schwarzschild-AdS black hole in section 3, where the perturbation of the mass parameter 
should be performed before the consideration of the noncommutative correction. Finally, we 
summarize our results in section 4.

\section{Minimal surface area of the pure AdS spacetime with 
noncommutatvity}

The metric of the 4-dimensional AdS spacetime in the Poincaré coordinate system takes the 
following form,

\[ ds^2 = R^2 \left( -dt^2 + dx^2 + dy^2 + dz^2 \right) / z^2, \]  

(1)

where \( R \) represents the AdS radius. Due to the rotational symmetry in the two models studied 
in this and next sections, it is convenient to adopt the polar coordinates, \((\rho, \phi)\), in the \((x, y)\) plane, and to rewrite the above metric to be

\[ ds^2 = R^2 \left( -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2 \right) / z^2. \]  

(2)

If one only wants to match the above metric, there exist many equivalent formulations 
of tetrad, i.e. any of them gives the same metric. However, we find that the formulations of 
tetrad are of subtlety when the metric is generalized to noncommutative spacetimes. That is, 
we can write many equivalent formulations of tetrad in the commutative case, but most of 
them lead to a complex correction of holographic entanglement entropy in the noncommutative 
ppase. Fortunately, we find out such a tetrad that meets the requirement in the commutative 
ppase and simultaneously gives real corrections of holographic entanglement entropy for our 
wo models. This tetrad can be set to be

\[ (k^a)_\mu = (l^a, n^a, m^a, w^a), \]  

(3)

where the four components are defined with the Kronecker delta as follows,

\[ l^a = R \left( \frac{i\delta^a_0}{2z^2} + i\delta^a_2 \right), \quad n^a = R \delta^a_1, \quad m^a = R \left( -i\rho \frac{\delta^a_0}{2z^2} + i\rho \delta^a_2 \right), \quad w^a = R \delta^a_3. \]  

(4)
Therefore, the metric can be constructed in terms of the tetrad,
\[ g_{\mu\nu} = \eta_{ab}k^a_{\mu}k^b_{\nu}, \]
where \( \eta_{ab} \) is defined by
\[ \eta_{ab} = \begin{pmatrix} 0 & 0 & 1 + i & 0 \\ 0 & 1 & 0 & 0 \\ 1 - i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

As \( z \to 0 \) corresponds to the infinity of the AdS spacetime, we propose the following noncommutative relation in the Poincaré coordinate system by using the commutator between the operator of \( 1/z \) and that of \( \rho \),
\[ \left[ \frac{1}{z}, \hat{\rho} \right] = ih, \]
where \( h \) is the noncommutative parameter with no dimensions. We note that this commutation relation is in agreement with the components of the tetrad, see eq. (4), and that such a relation naturally leads to a dimensionless noncommutative parameter which is quite special.

Under the above noncommutative relation, the multiplication of functions in noncommutative geometry can be realized through the Moyal product, see, for instance, the Moyal product in two variables,
\[ f(u, v) \ast g(u, v) = f(u, v)e^{ih\left(\hat{\partial}_u \hat{\partial}_v - \hat{\partial}_v \hat{\partial}_u\right)}g(u, v), \]
where \( u = \frac{1}{z} \) and \( v = \rho \). As a result, we can write the noncommutative metric from the ordinary (commutative) one, eq. (5), as follows,
\[ \bar{g}_{\mu\nu} = \eta_{ab}k^a_{\mu} \ast k^b_{\nu}, \]
from which we derive the square of the line element with the noncommutativity up to the first order of \( h \),
\[ d\bar{s}^2 = R^2 \frac{-dt^2 + hzdtd\phi + d\rho^2 + (\rho^2 - h\rho z)d\phi^2 + dz^2}{z^2}. \]

The entanglement entropy can be expressed through the RT formula,
\[ S_{ee} = S_{\text{HEE}} = \frac{A}{4G_N}, \]
where \( A \) denotes the minimal surface area in a bulk. For the convenience in the following discussions, we introduce a new polar coordinate system, \((r, \theta)\), to replace \((\rho, z)\),
\[ \rho = r \cos \theta, \quad z = r \sin \theta, \]
where \( \theta \in [\epsilon, \pi/2] \), \( \epsilon \) is a regularization factor which is close to zero and associated with a lattice length of fields on boundary, and \( r \) represents the spacetime scale. Thus, we just calculate the minimal surface area in the following contexts and regard it as the HEE.
Now we can easily write the minimal surface area with the noncommutative deformation by using eqs. (10) and (12) as follows,

\[
A = 2\pi R^2 \int_\epsilon^{\pi/2} \sqrt{(\dot{r}^2 + r^2)(\cos^2 \theta - h \sin \theta \cos \theta)} \frac{d\theta}{r \sin^2 \theta},
\]

(13)

where a dot stands for the differentiation with respect to \( \theta \). As the noncommutative parameter \( h \) is much smaller than one, we expand the geodesic curve \( r(\theta) \) with respect to it,

\[
r(\theta) = r_0 + h \bar{r}(\theta) + O(h^2),
\]

(14)

where \( r_0 \) is the initial constant corresponding to the space scale and \( \bar{r}(\theta) \) represents the first order noncommutative modification to the curve. Because the term related to \( \dot{r}^2 \) is proportional to \( h^2 \), see eq. (14), the minimal surface area, eq. (13), reduces approximately to the following form if only the first order of \( h \) is considered,

\[
A = 2\pi R^2 \int_\epsilon^{\pi/2} \sqrt{\cos^2 \theta - h \sin \theta \cos \theta} \frac{\sin^2 \theta}{\sin^2 \theta} d\theta \equiv A_0 + \bar{A},
\]

(15)

where \( A_0 \) is the ordinary (commutative) minimal surface area and \( \bar{A} \) is the first order noncommutative correction to \( A_0 \). Making the Taylor expansion of eq. (15) with respect to \( h \), we can easily obtain \( A_0 \) and \( \bar{A} \), respectively,

\[
A_0 = 2\pi R^2 \left( \frac{1}{\sin \epsilon} - 1 \right),
\]

(16)

and

\[
\bar{A} = \pi R^2 h \ln \frac{\epsilon}{2},
\]

(17)

where the latter can also be understood as the nonlocal contribution from the noncommutative spacetime.

From eq. (17) we see that the noncommutative effect gives a logarithmic divergent term with a suppression factor \( h \). Because \( \bar{A} \) is minus, the noncommutative effect decreases the HEE of the pure AdS spacetime.

3 Minimal surface area of the Schwarzschild-AdS black hole with noncommutativity

We deal with this model in two steps. In the first step, we compute the minimal surface area of the Schwarzschild-AdS black hole by regarding \([26,28]\) the mass of the Schwarzschild black
hole, $M$, as a perturbative parameter in the pure AdS spacetime. Then, following the method we applied in the above section, we derive in the second step the noncommutative correction to the minimal surface area of the Schwarzschild-AdS black hole. That is, based on the pure AdS spacetime, our result contains the contribution from the pure AdS spacetime together with the mass correction and the noncommutative parameter correction.

The metric of the 4-dimensional Schwarzschild-AdS black hole takes the form [28],

$$ds^2 = \frac{R^2}{z^2} \left(-f(z)dt^2 + \frac{dz^2}{f(z)} + d\rho^2 + \rho^2 d\phi^2\right),$$

where $f(z)$ that is associated with the black hole mass $M$ reads

$$f(z) = 1 - Mz^3. \quad (19)$$

As just mentioned above, $M$ is treated as a perturbative parameter under the condition $Mz^3 \ll 1$.

In the first step, we at first give the minimal surface area of the Schwarzschild-AdS black hole in the polar coordinate $(r, \theta)$,

$$A_0 = 2\pi R^2 \int_{\epsilon}^{\bar{r}} \left[r^2 + \dot{r}^2 + Mr^3 \sin^3 \theta \left(1 + Mr^3 \sin^3 \theta \right) \left(r^2 \cos^2 \theta + r\dot{r} \sin 2\theta + \dot{r}^2 \sin^2 \theta\right)\right]^{1/2} \frac{\cos \theta}{r \sin^2 \theta} \, d\theta. \quad (20)$$

By expanding $r(\theta)$ with respect to $M$, we have

$$r(\theta) = l + Ma(\theta) + O(M^2), \quad (21)$$

where $l$ is a constant which is associated with a space scale and satisfies the inequality $Mi^3 \ll 1$. Substituting eq. (21) into eq. (20), we then derive the leading contribution up to the first order of mass $M$,

$$A_0 = 2\pi R^2 \int_{\epsilon}^{\bar{r}} \left[1 + \frac{1}{2} Mi^3 \sin^3 \theta \cos^2 \theta \right] \frac{\cos \theta}{\sin^2 \theta} \, d\theta$$

$$A_0 = A_0 + A_M,$$

where $A_0$ is the minimal surface area of the pure AdS spacetime, see eq. (16), and $A_M$ represents the mass correction to $A_0$,

$$A_M = \frac{1}{4} \pi R^2 Mi^3. \quad (23)$$

As a result, we work out the minimal surface area of the Schwarzschild-AdS black hole, $A_0$, in the approximation of the first order of $M$. 

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Now we calculate the noncommutative correction to $A_0$ in the second step by following the way utilized in the above section. Combining eq. (20) together with eq. (10), we write the minimal surface area of the Schwarzschild-AdS black hole with noncommutativity,

$$A = 2\pi R^2 \int_{\epsilon}^{\pi/2} \left[ r^2 + \dot{r}^2 + M r^3 \sin^2 \theta \left( 1 + M r^3 \sin^3 \theta \right) \left( r^2 \cos^2 \theta + \dot{r} r \sin 2\theta + \dot{r}^2 \sin^2 \theta \right) \right]^{1/2} \times \left( \cos^2 \theta - h \sin \theta \cos \theta \right)^{1/2} \frac{d\theta}{r \sin^2 \theta}. \quad (24)$$

Defining

$$A \equiv A_0 + \bar{A}, \quad (25)$$

and substituting both eq. (14) and eq. (21) into eq. (24), we derive the contribution from the noncommutative modification up to the first order of $h$,

$$\bar{A} = -\pi R^2 h \int_{\epsilon}^{\pi/2} \left( 1 + \frac{1}{2} M r^3 \sin^3 \theta \cos^2 \theta \right) \frac{d\theta}{\sin \theta} = \pi R^2 h \ln \frac{\epsilon}{2} - \frac{\pi}{32} \pi R^2 M l^3 h. \quad (26)$$

When the mass parameter is set to be zero, the above result gets back to that of the pure AdS situation, see the first term of eq. (26) or eq. (17). On the other hand, besides the noncommutative correction from the pure AdS spacetime, the noncommutativity also modifies the contribution from the mass term of the black hole, see the second term of eq. (26).

4 Summary

The significance of our work is to notice that the noncommutative generalization of the RT formula is nontrivial. On the commutative level, there exist many equivalent formulations of tetrad that give rise to the same metric. However, on the noncommutative level, most of them lead to complex noncommutative corrections of the HEE, which is unphysical. Therefore, the construction of a specific tetrad is of subtlety. Fortunately, we have found such a tetrad and obtained the real noncommutative corrections to the HEE for the pure AdS spacetime and Schwarzschild-AdS black hole.

As the RT formula connects the classical gravity to the quantum field, this connection gives us insight into the duality when the quantization of spacetime is taken into account. Considering that the nonlocal field theory has been used \cite{12} to explain the volume law of the flat HEE, we may expect a similar structure in field theories that corresponds to the noncommutative modification in gravity theories.
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