Convergence acceleration of Kaczmarz’s method

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This paper is dedicated to the memory of Bernard Germain–Bonne (1940–2012)

Abstract

The method of alternation projections (MAP) is an iterative procedure for finding the projection of a point on the intersection of closed subspaces of an Hilbert space. The convergence of this method is usually slow, and several methods for its acceleration have already been proposed. In this work, we consider a special MAP, namely Kaczmarz’ method for solving systems of linear equations. The convergence of this method is discussed. After giving its matrix formulation and its projection properties, we consider several procedures for accelerating its convergence. They are based on sequence transformations whose kernels contain sequences of the same form as the sequence of vectors generated by Kaczmarz’ method. Acceleration can be achieved either directly, that is without modifying the sequence obtained by the method, or by restarting it from the vector obtained by acceleration. Numerical examples show the effectiveness of both procedures.

1 Introduction

Let $Q_i$ denote the orthogonal projection on a closed subspace $M_i$ of an Hilbert space $H$, and let $Q_M$ be the composition of the $r$ projecting operators $Q_i$, that is $Q_M = Q_r \cdots Q_1$. Let $P_M$ be the projection on $M$, the intersection of the subspaces $M_i$.

We are looking for the projection of a given point $\bar{x}$ on $M$. It holds

$$\lim_{n \to \infty} Q^n_M \bar{x} = P_M \bar{x}, \quad \forall \bar{x} \in H.$$ 

The method of alternating projections (MAP) consists in the iterations

$$x_{n+1} = Q_M x_n, \quad n = 0, 1, \ldots, \quad x_0 = \bar{x}.$$ 

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This method often converges quite slowly and it needs to be accelerated \cite{22}. For this purpose, the projection operator $Q_M$ can be replaced by another (non necessarily linear) operator $T$ which can depend on $n$.

There are two ways of using $T$

1. Keep the sequence $(x_n)$ as given by MAP unchanged, and consider the new sequence $(y_n)$ built by $y_n = Tx_n$.

2. Iterate the operator $T$, that is consider the new iterations $x_{n+1} = Tx_n$.

There exist many choices for $T$ (the norms are the Euclidean ones). For example, the choice

$$Tx_n = x_n - t_n (Q_M x_n - x_n),$$

with $t_n = (x_n, x_n - Q_M x_n)/\|x_n - Q_M x_n\|^2$ was proposed in \cite{22} p. 44, while $t_n = (r_n, v_n)/(v_n, v_n)$ where $r_n = Q_M x_n - x_n$ and $v_n = Q_M^2 x_n - 2 Q_M x_n + x_n$ was discussed in \cite{27}. More choices are presented in \cite{19}.

Kaczmarz’s method \cite{34} for solving a system of linear equations was proposed on 1937. Later, it was rediscovered by Gordon et al. \cite{29}, and applied in medical imaging. They called it ART (Algebraic Reconstruction Technique), and its original version, or some variants, continue to be used for tomographic imaging. It is a particular case of row projection methods which received much attention (see, for example, \cite{3, 4, 11, 19, 47}), and it enters into the framework of MAP. It is also well suited for parallel computations and large–scale problems, since each step only requires one row of the matrix $A$ (or several rows simultaneously in its block version), and no matrix–vector products. For an impressive list of publications on Kaczmarz’s method, see \cite{18}. Kaczmarz’s method is also often used for solving an overdetermined consistent $M \times N$ linear system with $M \geq N$, but, in this paper, we restrict ourselves to the case of a square regular system. It is easy to extend the algorithms and the theory by properly replacing $N$ by $M$.

A drawback of Kaczmarz’s method, as in general of all projection iterative methods, is its often slow convergence. Thanks to its matrix analysis, we will be able to show how its convergence can be accelerated by some particular choices of the operator $T$ corresponding to the two ways of using it described above.

For definitions and properties of projections, see, for example, \cite{11}. In the sequel, the same notation will be used for a matrix and the projection it represents.

\section{Kaczmarz’s method}

We consider a $N \times N$ linear system $A x = b$. One single step of Kaczmarz’s method consists in

$$p_{n+1} = p_n + \frac{(b - A p_n, e_i)}{(A^T e_i, A^T e_i)} A^T e_i,$$  \hspace{1cm} (1)
where $e_i$ is the $i$th vector of the canonical basis of $\mathbb{R}^N$. There exist several strategies for choosing the index $i$ at each step. The most common one is $i = n \, (\text{mod. } N) + 1$. In this case, the method is called the \textit{cyclic Kaczmarz’s method} or, simply, the \textit{Kaczmarz’s method}, since it was originally proposed by Kaczmarz under this form \cite{Kaczmarz1937}. It corresponds to a restarting from the result obtained after $N$ single steps, that is, in other words, to a renumbering of them or, again in other words, to the extraction of a subsequence. Thus, one iteration of Kaczmarz’s method consists in a complete cycle of steps in their natural order, that is

$$
\begin{align*}
    p_0 &= x_n \\
    p_i &= p_{i-1} + \frac{b - Ap_{i-1}, e_i}{(A^T e_i, A^T e_i)} A^T e_i, \quad i = 1, \ldots, N \\
    x_{n+1} &= p_N 
\end{align*}
$$

(2)

Denote by $a_i = A^T e_i$ the column vector formed by the $i$th row of $A$. Thus, the computation of each vector $p_i$ in (2) does not require any matrix–vector product since $(b - Ap_{i-1}, e_i) = (b, e_i) - (p_{i-1}, A^T e_i)$. Thus, if we denote by $b_i$ the $i$th component of the right hand side $b$, then the computation of $p_i$ is simply given by

$$
    p_i = p_{i-1} + \frac{b_i - (p_{i-1}, a_i)}{||a_i||^2} a_i. \quad (3)
$$

This remark is one of the main advantages of Kaczmarz’ method, and it allows an easy parallel implementation.

Let $M_i = \{ y \mid (b - Ay, e_i) = 0 \}$. Then $p_i$ is the oblique projection of $p_{i-1}$ on $M_i$ along $A^T e_i$.

Moreover, setting

$$
    \lambda_i = \frac{(b - Ap_{i-1}, e_i)}{(A^T e_i, A^T e_i)},
$$

one iteration of Kaczmarz’s method (2) writes

$$
    x_{n+1} = x_n + A^T \Lambda_n, \quad \Lambda_n = (\lambda_1, \ldots, \lambda_N)^T \in \mathbb{R}^N.
$$

\textbf{Remark 1}

Let $y_n$ be the iterates obtained by applying the Gauss–Seidel method to the system $AA^T y = b$. Then $x = A^T y$ and $x_n = A^T y_n$ \cite{Saad1986}.

Let us now analyze each step of (2). Inside one iteration, we have the following orthogonality properties for $i = 1, \ldots, N$ (see \cite{Cullum1969, Stewart1974, Saad1986})

$$
\begin{align*}
    (b - Ap_i, e_i) &= 0, \\
    (p_i - p_{i-1}, x - p_i) &= 0,
\end{align*}
$$

3
and it follows
\[
\|x - p_{i-1}\|^2 = \|x - p_i\|^2 + 2\lambda_i (x - p_i, A^T e_i) + \lambda_i^2 (A^T e_i, A^T e_i),
\]
\[
= \|x - p_i\|^2 + 2\lambda_i (b - Ap_i, e_i) + \|p_i - p_{i-1}\|^2,
\]
since \((b - Ap_i, e_i) = 0\). Thus
\[
\|x - p_{i-1}\|^2 = \|x - p_i\|^2 + \|p_i - p_{i-1}\|^2,
\]
which shows that \(\|x - p_i\| \leq \|x - p_{i-1}\|\). Therefore,
\[
\|x - x_n + 1\| \leq \|x - x_n\|.
\]
Thus, as proved in [26], these inequalities are strict, and Kaczmarz’s method is always converging to the solution of the system.

Obviously \((b - Ap_{i-1}, e_i)^2 \leq (b - Ap_{i-1}, b - Ap_{i-1})\). In the case where \((b - Ap_{i-1}, e_i)^2 \geq (b - Ap_{i-1}, b - Ap_{i-1})/N\), one can prove, by an analysis similar to what is done in [20, p. 122], that the following result holds

**Proposition 1**

If \((b - Ap_{i-1}, e_i)^2 \geq (b - Ap_{i-1}, b - Ap_{i-1})/N\), then
\[
\|x - p_i\|^2 \leq \left(1 - \frac{1}{N\kappa (AA^T)}\right) \|x - p_{i-1}\|^2.
\]

Then, in the general case, this coefficient could be an approximation of the convergence factor of the method. This result is quite similar to a corresponding result proved in [32] for Gastinel’s method [25] and other methods, or to an extension (Thm. 4.27 in [24]) of another result given in [32].

### 2.1 Matrix interpretation

Following [26], where it seems that it first appeared, let us give the matrix interpretation of Kaczmarz’s method [2] (see also [1,20,47]).

We set
\[
\alpha_i = \frac{A^T e_i}{\|A^T e_i\|^2} = \frac{a_i}{\|a_i\|^2},
\]
\[
P_i = I - A\alpha_i e_i^T,
\]
\[
Q_i = A^{-1} P_i A,
\]
\[
\rho_i = b - Ap_i.
\]

We have
\[
Q_i = A^{-1} P_i A = I - \frac{A^T e_i e_i^T A}{\|A^T e_i\|^2} = I - \frac{a_i a_i^T}{\|a_i\|^2} = I - \alpha_i a_i^T.
\]
The matrix $P_i$ represents the oblique projection on $e_i^\perp$ along $AA^T e_i$, while $Q_i$ is the rank $N-1$ orthogonal projection on $(A^T e_i)^\perp$ along $A^T e_i$. Thus, for any vector $y$, $(P_i y, e_i) = 0$ and $(Q_i y, A^T e_i) = 0$.

Inside one iteration, we have, for $i = 1, \ldots, N$,

$$
\begin{align*}
p_i &= Q_i p_{i-1} + (b, e_i) \alpha_i = Q_i p_{i-1} + (I - Q_i)x \\
x - p_i &= x - p_{i-1} - (q_{i-1}, e_i) \alpha_i = Q_i (x - p_{i-1}) \\
q_i &= q_{i-1} - (q_{i-1}, e_i) A \alpha_i = P_i q_{i-1}.
\end{align*}
$$

The first relation can be written

$$p_i = Q_i p_{i-1} + A^{-1}(I - P_i)b.$$  

Notice that $A^{-1}(I - P_i)b = (b, e_i) \alpha_i$.

The second relation in (4) can also be written as

$$x - p_i = x - p_{i-1} - \alpha_i e_i^T q_{i-1},$$

and, replacing $\alpha_i$ by its expression, it follows

$$\|x - p_i\|^2 = \|x - p_{i-1}\|^2 - 2\frac{(q_{i-1}, e_i)}{\|A^T e_i\|^2}(x - p_{i-1}, A^T e_i) + \frac{(q_{i-1}, e_i)^2}{\|A^T e_i\|^2},$$

$$\|x - p_i\|^2 = \|x - p_{i-1}\|^2 - \frac{(q_{i-1}, e_i)^2}{\|A^T e_i\|^2}.$$

Summing up this identity for $i = 1, \ldots, N$, we obtain an expression for the gain of one iteration of Kaczmarz’s method

$$\|x - x_{n+1}\|^2 = \|x - x_n\|^2 - \sum_{i=1}^N \frac{(q_{i-1}, e_i)^2}{\|A^T e_i\|^2}.$$  

Setting

$$P = P_N \cdots P_1$$

$$Q = Q_N \cdots Q_1 = A^{-1}P_N \cdots P_1 A = A^{-1}PA$$

$$r_n = b - Ax_n,$$

it holds

$$\begin{align*}
x - x_{n+1} &= Q(x - x_n) \\
r_{n+1} &= P r_n.
\end{align*}$$

The matrix $Q$ represents an orthogonal projection, but not $P$. It obviously follows

$$x - x_n = Q^n(x - x_0), \quad r_n = P^n r_0, \quad n = 0, 1, \ldots,$$  

$$5$$
and we have

\[ x_{n+1} = Qx_n + A^{-1}(I - P)b. \]  

(7)

Since, after the first iteration of Kaczmarz’s method (which ends with \( p_N \)), we set \( x_1 = p_N \), and we start the second iteration from \( p_0 = x_1 \), it means that we are continuing the original steps \( \Pi \), and that the new vector \( p_1 \) is, in fact, the vector \( p_{N+1} \) of \( \Pi \), that the new vector \( p_2 \) is \( p_{N+2} \), and so on. Thus, Kaczmarz’s method is only a renumbering of the single steps, as previously explained, keeping only those whose index is a multiple of \( N \), that is \( x_{n+1} = p_{(n+1)N} \) and \( r_{n+1} = q_{(n+1)N} \). The main interest of this renumbering lies in the relations \( (3) \), which express the connection between two consecutive iterations by means of the fixed matrix \( Q \), instead of relations using matrices changing at each step of an iteration. We similarly have \( q_{(n+1)N+j-1} = Q(j)q_{nN+j-1}, j = 1, \ldots, N \), with \( Q(j) = P_{j-1} \cdots P_1P_n \cdots P_j \). Notice that \( Q(1) \) is identical to \( Q \).

The matrices \( P \) and \( Q \) are similar, and it holds \( \rho(Q) < 1 \) as proved in \([26]\). Thus, the sequence \( (x_n) \) generated by Kaczmarz’s method converges to the solution \( x \) of the system. Moreover, it follows from standard results on iterations of the form \( (6) \) that

\[ \|x - x_n\| = O(\rho(Q)^n). \]

By slightly improving Thm. 4.4 of \([24]\), which is based on a result by Meany \([35]\) on the norm of a product of orthogonal projections of rank \( N - 1 \), we have the

**Proposition 2**

\[ \|x - x_{n+1}\|^2 \leq \left( 1 - \frac{(\det A)^2}{\prod_{i=1}^{N} \|A^T e_i\|^2} \right) \|x - x_n\|^2. \]

A generalization of this result was recently given in \([2]\) for the case where \( A \) and \( b \) are partitioned into blocks of rows.

Let us give some details about the block version. Assume that the matrix \( A \) is partitioned into the blocks of rows \( A_1^T, A_2^T, \ldots \), where the matrix \( A_i^T \in \mathbb{R}^{N_i \times N} \) contains the rows \( N_1 + \cdots + N_{i-1} + 1 \) up to \( N_1 + \cdots + N_{i-1} + N_i \) of \( A \) (with \( N_0 = 0 \)), and the vector \( b \) is partitioned accordingly. Each single step of the method now consists in the treatment of a block as a whole, and one complete cycle of all the blocks in their natural ordering is called the (cyclic) block Kaczmarz’s method. As for Kaczmarz’s method, we are able to give a matrix interpretation of this extension which remains valid with now \( \alpha_i = (A_i^T)^T = A_i(A_i^T A_i)^{-1} \) and \( P_i = I - A\alpha_i E_i^T \), where \( E_i \in \mathbb{R}^{N \times N_i} \) is the matrix whose columns are the vectors \( e_{N_1 + \cdots + N_{i-1} + 1}, \ldots, e_{N_1 + \cdots + N_{i-1} + N_i} \) of the canonical basis of \( \mathbb{R}^N \), and it holds \( Q_i = I - A_i(A_i^T A_i)^{-1} A_i^T \). With these notations, one step of the block Kaczmarz’s method writes

\[ p_i = p_{i-1} + (A_i^T)E_i^T(b - A p_{i-1}). \]

This relation clearly generalizes \( (3) \).
3 Convergence acceleration

For accelerating the convergence of a sequence, it can be transformed into another one which, under some assumptions, converges faster to the same limit. The idea behind such a sequence transformation is to assume that the sequence to be accelerated behaves as a model sequence, or satisfies some property, depending on unknown parameters. The set of these sequences is called the kernel of the transformation. The unknown parameters are determined by imposing that the sequence interpolates the model sequence from a certain starting index \( n \). Then, the limit of the model sequence, which depends on \( n \) since the parameters depend on it, is taken as an approximation of the limit of the sequence to be accelerated which is thus transformed into a new sequence. Of course, by construction, if the initial sequence belongs to the kernel of the transformation then, for all \( n \), its limit is obtained. Although this observation was never proved rigourously, if the sequence is close in some sense to the kernel of the transformation, there is a good chance that it will be accelerated. This is why the notion of kernel is so important.

In practice, when having to accelerate a given sequence, one can use a known sequence transformation (also called an acceleration algorithm or an extrapolation method), and verify that it can be accelerated by it. Another approach is, starting from some algebraic property of the sequence to be accelerated, to construct a special transformation adapted to it. In both cases, the behavior of the sequence has to be analyzed or has to be characterized by some property.

On convergence acceleration methods for vector sequences, see, for instance, [13].

### 3.1 The sequence generated by Kaczmarz’s method

Consider the sequence of vectors \((x_n)\) obtained by Kaczmarz’s method. Let \(\Pi_\nu\) be the minimal polynomial of the matrix \(Q\) for the vector \(x - x_0\), that is the polynomial of smallest degree \(\nu \leq N\) such that \(\Pi_\nu(Q)(x - x_0) = 0\). Since \(\Pi_\nu(Q)(x - x_0) = A^{-1}\Pi_\nu(P)A(x - x_0) = A^{-1}\Pi_\nu(P)r_0\), \(\Pi_\nu\) is also the minimal polynomial of \(P\) for the vector \(r_0\). Setting \(\Pi_\nu(\xi) = c_0 + c_1\xi + \cdots + c_\nu\xi^\nu\), it holds from (6)

\[
Q^n\Pi_\nu(Q)(x - x_0) = c_0(x - x_n) + c_1(x - x_{n+1}) + \cdots + c_\nu(x - x_{n+\nu}) = 0, \quad n = 0, 1, \ldots
\]

Thus, it follows that the vectors \(x - x_n\) produced by Kaczmarz’s method satisfy such an homogeneous linear difference equation of order \(\nu\), whose solution is well–known.

If \(Q\) is nondefective and if we denote by \(\tau_1, \ldots, \tau_\nu\) the distinct zeros of \(\Pi_\nu\), this solution writes

\[
x - x_n = \sum_{i=1}^\nu d_i\tau_i^n v_i, \quad n = 0, 1, \ldots,
\]

where the \(d_i\)'s are scalars and the \(v_i\)'s the eigenvectors corresponding to the \(\tau_i\)'s. If \(Q\) is defective, the expression for \(x - x_n\) still involves powers of the eigenvalues, but it is more complicated.
3.2 Sequence transformations

The vector $x$ can be exactly computed if we are able to build a sequence transformation whose kernel consists of sequences of the form $[3]$. However, since, in practical applications, $\nu$ could be quite large, we will restrict ourselves to building a sequence transformation whose kernel contains all sequences of the form

$$a_0^{(n)}(x - x_n) + a_1^{(n)}(x - x_{n+1}) + \cdots + a_k^{(n)}(x - x_{n+k}) = 0, \quad n = 0, 1, \ldots, \quad (9)$$

with $k \leq \nu$. Thus, solving this equation for the vector $x$, gives us an approximation of it denoted $y_k^{(n)}$ since it depends on $k$ and $n$.

For that purpose, one has to find a procedure for computing the unknown coefficients $a_0^{(n)}, \ldots, a_k^{(n)}$, taking into account that they are numbers while $x - x_{n+i}$ are vectors. All such transformations can be considered as vector generalizations of Shanks’ transformation [40] for sequences of numbers. The common idea behind these transformations is to obtain a system of linear equations whose solution is $a_0^{(n)}, \ldots, a_k^{(n)}$, and then to compute $y_k^{(n)} \simeq x$.

It turns out that several vector sequence transformations based on (or including) such a kernel already exist and have been studied by various authors: the various $\varepsilon$–algorithms [7,18,49], the Minimal Polynomial Extrapolation (MPE) [17], the Modified Minimal Polynomial Extrapolation (MMPE) [7,37,45], the Reduced Rank Extrapolation (RRE) [21,36], Germain–Bonne transformations [28], the $H$–algorithm [16], and the $E$–algorithm [10]. They are described and analyzed, for example, in [5,7,8,10,41,42,44–46].

Writing (9) for the indexes $n$ and $n + 1$, subtracting, and multiplying scalarly by a vector $y$ leads to the scalar equation

$$a_0^{(n)}(y, \Delta x_n) + a_1^{(n)}(y, \Delta x_{n+1}) + \cdots + a_k^{(n)}(y, \Delta x_{n+k}) = 0, \quad (10)$$

where $\Delta$ is the difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

As explained in [11, p. 39], there are several possible strategies for constructing our system which, apart from an additional normalization condition, has to contain $k$ equations

- use only one vector $y$, and write (10) for the indexes $n, \ldots, n + k - 1$,
- write (10) only for the index $n$, and choose $k$ linearly independent vectors $y$,
- write several relations (10), and choose several linearly independent vectors $y$.

Adding to these $k$ equations the condition $a_0^{(n)} + \cdots + a_k^{(n)} = 1$, which does not restrict the generality (and is needed since the sum of the coefficients must be nonzero in order for $x$ in (9) to be uniquely defined), we obtain a system of $k + 1$ equations in the $k + 1$ unknowns $a_0^{(n)}, \ldots, a_k^{(n)}$ (which depend on $n$ and $k$). Its coefficients are denoted $d_{i,j}^{(n)}$, and, for any of the preceding strategies, this system writes

$$\begin{cases}
    a_0^{(n)} + \cdots + a_k^{(n)} = 1 \\
    d_{i,0}^{(n)} a_0^{(n)} + \cdots + d_{i,k}^{(n)} a_k^{(n)} = 0, \quad i = 1, \ldots, k,
\end{cases} \quad (11)$$
where the coefficients \(d_{i,j}^{(n)}\), for \(i = 1, \ldots, k\) and \(j = 0, \ldots, k\), will be given below according to the chosen strategy. Then, for a fixed value of \(k\), our sequence transformation \((x_n) \mapsto (y_k^{(n)})\) is defined by

\[
y_k^{(n)} = a_0^{(n)} x_n + \cdots + a_k^{(n)} x_{n+k}, \quad n = 0, 1, \ldots,
\]

which can be written as

\[
y_k^{(n)} = \begin{vmatrix} x_n & \cdots & x_{n+k} \\ d_{1,0}^{(n)} & \cdots & d_{1,k}^{(n)} \\ \vdots & \vdots & \vdots \\ d_{k,0}^{(n)} & \cdots & d_{k,k}^{(n)} \end{vmatrix} = \begin{vmatrix} x_n & \Delta x_n & \cdots & \Delta x_{n+k-1} \\ d_{1,0}^{(n)} & \delta d_{1,0}^{(n)} & \cdots & \delta d_{1,k-1}^{(n)} \\ \vdots & \vdots & \vdots \\ d_{k,0}^{(n)} & \delta d_{k,0}^{(n)} & \cdots & \delta d_{k,k-1}^{(n)} \end{vmatrix}
\]

(13)

where \(\delta\) is the difference operator defined by \(\delta d_{i,j}^{(n)} = d_{i,j+1}^{(n)} - d_{i,j}^{(n)}\). The second determinantal expression shows that the vector \(y_k^{(n)}\) is the Schur complement \([11]\)

\[
y_k^{(n)} = x_n - [\Delta x_n, \ldots, \Delta x_{n+k-1}] \begin{pmatrix} \delta d_{1,0}^{(n)} & \cdots & \delta d_{1,k-1}^{(n)} \\ \vdots & \vdots & \vdots \\ \delta d_{k,0}^{(n)} & \cdots & \delta d_{k,k-1}^{(n)} \end{pmatrix}^{-1} \begin{pmatrix} d_{1,0}^{(n)} \\ \vdots \\ d_{k,0}^{(n)} \end{pmatrix}.
\]

However, \(y_k^{(n)}\) is not a projection.

The second determinantal formula in \([13]\) means that all our sequences transformations can be also written as

\[
y_k^{(n)} = x_n - \alpha_1^{(n)} \Delta x_n - \cdots - \alpha_k^{(n)} \Delta x_{n+k-1}, \quad k = 0, 1, \ldots,
\]

(14)

where the \(\alpha_i^{(n)}\)'s are solution of the linear system

\[
\delta d_{i,0}^{(n)} \alpha_1^{(n)} + \cdots + \delta d_{i,k-1}^{(n)} \alpha_k^{(n)} = d_{i,0}^{(n)}, \quad i = 1, \ldots, k.
\]

(15)

Let us now specify various choices of the coefficients \(d_{i,j}^{(n)}\), for \(i = 1, \ldots, k\) and \(j = 0, \ldots, k\).

### 3.2.1 The vector Shanks' transformations

We first consider sequence transformations which are directly inspired by the scalar sequence transformation of Shanks \([40]\), and can be recursively implemented by Wynn’s \(\varepsilon\)-algorithm \([48]\) or its generalizations, or by the \(E\)-algorithm \([10]\), or the \(H\)-algorithm \([16]\).
• Choosing \(d_{i,j}^{(n)} = (e_p, \Delta x_{n+i+j-1})\), and replacing \(x_j\) in the first row of the numerator of the first ratio in (13) by \((e_p, x_j)\) corresponds to Shanks’ transformation applied componentwise to the vector sequence \((x_n)\), and it gives the \(p\)th component of the vector \(y_k^{(n)}\). This transformation can be recursively implemented, separately for each component \(p = 1, \ldots, N\), by the scalar \(\varepsilon\)-algorithm of Wynn [48].

• The \(\varepsilon\)-algorithm can be also applied directly to a sequence of vectors after defining the inverse \(u^{-1}\) of a vector \(u\) as \(u^{-1} = u/(u, u)\). This vector \(\varepsilon\)-algorithm was also proposed by Wynn [49]. However, for this algorithm, (13) is no longer valid and the determinants have to be replaced either by bigger and more complicated ones [30], or by designants which generalize them in a non–commutative algebra [38, 39]. Thus, in this case, an underlying system of linear equations for \(y_k^{(n)}\) does not exist, and this transformation has to be recursively implemented by the vector \(\varepsilon\)-algorithm whose rules are, for a sequence \((u_n)\) of real (to simplify) vectors in \(\mathbb{R}^N\),

\[
\varepsilon_{-1}^{(n)} = 0 \in \mathbb{R}^N, \quad \varepsilon_0^{(n)} = u_n \in \mathbb{R}^N, \quad n = 0, 1, \ldots \\
\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + (\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)})^{-1}, \quad k, n = 0, 1, \ldots
\]

• The choice \(d_{i,j}^{(n)} = (y, \Delta x_{n+i+j-1})\), where \(y\) is any nonzero vector so that the denominators of (13) differs from zero, leads to the topological Shanks’ transformation introduced in [7]. It can be implemented via the topological \(\varepsilon\)-algorithm. In this algorithm, the inverses are defined in a different way for an even or an odd lower index as

\[
\begin{align*}
\left(\varepsilon_{2k}^{(n+1)} - \varepsilon_{2k}^{(n)}\right)^{-1} &= y/(y, \varepsilon_{2k}^{(n+1)} - \varepsilon_{2k}^{(n)}) \\
\left(\varepsilon_{2k+1}^{(n+1)} - \varepsilon_{2k+1}^{(n)}\right)^{-1} &= (\varepsilon_{2k}^{(n+1)} - \varepsilon_{2k}^{(n)})/(\varepsilon_{2k+1}^{(n+1)} - \varepsilon_{2k+1}^{(n)} - \varepsilon_{2k}^{(n)} - \varepsilon_{2k}^{(n)}).
\end{align*}
\]

In the preceding \(\varepsilon\)-algorithms, only the vectors (or the numbers) with an even lower index \(\varepsilon_{2k}^{(n)}\) are interesting for the purpose of convergence acceleration, those with an odd lower index \(\varepsilon_{2k+1}^{(n)}\) being intermediate computations. Applying any of them to the sequence \((\varepsilon_0^{(n)} = x_n)\) produced by Kaczmarz’s method gives \(\varepsilon_{2k}^{(n)} = y_k^{(n)}\). Notice that the computation of one vector \(\varepsilon_{2k}^{(n)}\) requires the \(2k + 1\) vectors \(x_n, \ldots, x_{n+2k}\). Thus, when \(k\) is increased by 1, the computation of each new vector \(\varepsilon_{2k}^{(n)}\) needs two additional iterates of Kaczmarz’s method, while it only requires one when \(n\) increases.

3.2.2 The MPE, MMPE, and RRE

The following choices also lead to known transformations but, in this case, the computation of \(y_k^{(n)}\) only requires the \(k + 2\) vectors \(x_n, \ldots, x_{n+k+1}\). Algorithms for their recursive implementation also exist [14, 16, 23].
The choice $d_{i,j}^{(n)} = (y_i, \Delta x_{n+j})$, for $i = 1, \ldots, k$ and $j = 0, \ldots, k$ corresponds to the MMPE (Modified Minimal Polynomial Extrapolation) \cite{7,37,45}, where the $y_i$ are linearly independent vectors. This transformation is another generalization of the topological Shanks transformation also given in \cite{7}.

If we take $d_{i,j}^{(n)} = (\Delta x_{n+i-1}, \Delta x_{n+j})$, for $i = 1, \ldots, k$ and $j = 0, \ldots, k$, we obtain the MPE (Minimal Polynomial Extrapolation) \cite{17}. This method is mathematically equivalent to a transformation due to Germain–Bonne \cite{28}. Other choices are proposed in the same reference.

The choice $d_{i,j}^{(n)} = (\Delta^2 x_{n+i-1}, \Delta x_{n+j})$ leads to the RRE (Reduced Rank Extrapolation) \cite{21,36}.

It must be noticed that the vector $y_k^{(n)}$ always exists for the RRE but not for the MPE or the MMPE. Existence conditions are discussed in \cite{33} and \cite{43}.

### 3.2.3 The simplest transformations

For $k = 1$, the transformations have the following very simple forms

- the MMPE and the topological Shanks’ transformation write
  \[
  y_1^{(n)} = x_n - \frac{(y_1, \Delta x_n)}{(y_1, \Delta^2 x_n)} \Delta x_n,
  \]

- for the MPE and the transformation due to Germain–Bonne, we have
  \[
  y_1^{(n)} = x_n - \frac{(\Delta x_n, \Delta x_n)}{\Delta x_n, \Delta^2 x_n} \Delta x_n,
  \]

- for the RRE, it holds
  \[
  y_1^{(n)} = x_n - \frac{(\Delta^2 x_n, \Delta x_n)}{\Delta^2 x_n, \Delta^2 x_n} \Delta x_n,
  \]
  which is the same as the one proposed in \cite{27},

- for the vector $\varepsilon$–algorithm, we obtain
  \[
  y_1^{(n)} = x_{n+1} + \frac{\varepsilon_1^{(n+1)} - \varepsilon_1^{(n)}}{(\varepsilon_1^{(n+1)} - \varepsilon_1^{(n)}, \varepsilon_1^{(n+1)} - \varepsilon_1^{(n)})} \quad \text{with} \quad \varepsilon_1^{(n)} = \frac{\Delta x_n}{(\Delta x_n, \Delta x_n)}.
  \]
3.2.4 Orthogonality properties

Using the determinantal formula (13), we have the following orthogonality properties

Proposition 3

The vector \( Q(y^{(n)} - x) - (y^{(n)} - x) \) is orthogonal to

- \( \Delta x_n, \ldots, \Delta x_{n+k-1} \) for the MPE (which is identical to Germain–Bonne transformation),
- \( y \) for the topological Shanks transformation (topological \( \varepsilon \)-algorithm),
- \( \Delta^2 x_n, \ldots, \Delta^2 x_{n+k-1} \) for the RRE,
- \( y_1, \ldots, y_k \) for the MMPE.

For a discussion of the properties of these algorithms and their application to the solution of systems of linear and nonlinear equations, see [33].

4 Acceleration of Kaczmarz’s method

Although they are different, all the acceleration methods presented in Section 3 have the same kernel, namely sequences satisfying (9). Thus, when applied to any sequence of vectors generated by iterations of the form \( x_{n+1} = Bx_n + c \), \( n = 0, 1, \ldots \), we obtain \( y^{(n)}_\nu = (I - B)^{-1}c \) for all \( n \), where \( \nu \) is the degree of the minimal polynomial of \( B \) for the vector \( x - x_0 \). This result means that these methods are direct methods for solving systems of linear equations [7]. Therefore, this property holds for the iterates of Kaczmarz’s method, since \( \Pi\nu(1) \neq 0 \) (otherwise the matrix \( I - Q \) would be singular). Of course, in practice, these algorithms cannot be used for obtaining the exact solution when the dimension of the system is large since, for computing \( y^{(n)}_\nu \), they require the storage of too many vectors.

4.1 The AK and RK algorithms

The preceding algorithms can be used in two different ways for accelerating the iterations (2) produced by Kaczmarz’s method as it will now be explained. Since the computation of \( y^{(n)}_k \) does not require the same number of vectors issued from Kaczmarz’s method, according to the transformation used, in order to have a unified presentation, we will denote by \( \ell + 1 \) the number of vectors \( x_i \) needed to compute \( y^{(n)}_k \) by any of the preceding procedures. Remember that \( \ell \) depends on \( k \) and that \( \ell = 2k \) for the \( \varepsilon \)-algorithms, and \( \ell = k + 1 \) for the other algorithms.

- The first one consists to apply one of the algorithms implementing a sequence transformation to the sequence \( x_0, x_1, \ldots \) given by Kaczmarz’s method, and, after fixing the index \( k \), to build simultaneously the sequence \( z_0 = y^{(0)}_k, z_1 = y^{(1)}_k, \ldots \). The computation of \( y^{(0)}_k \) can only begin after having computed the iterate \( x_\ell \), but the computation of each new transformed vector needs only one new iterate of Kaczmarz’s method. This procedure is called the accelerated Kaczmarz (AK) algorithm. Let us give the general structure for the implementation of the AK algorithm.
Accelerated Kaczmarz (AK) algorithm

Require $A \in \mathbb{R}^{N \times N}$, $b \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$
Choose $k \in \mathbb{N}$, $k \geq 1$
Set $\ell = k + 1$ or $\ell = 2k$
for $n = 0, 1, \ldots$ until convergence do
  $p_0 \leftarrow x_n$
  Compute $p_i$, $i = 1, \ldots, N$
  $x_{n+1} \leftarrow p_N$
  If $n \geq \ell - 1$ then
    Compute $y_k^{(n-\ell+1)}$
    $z_{n-\ell+1} \leftarrow y_k$
  end if
end for $n$

• In the second way, we set $x_0$, and we compute $x_1, \ldots, x_\ell$ by Kaczmarz’s method, we apply one of the algorithms implementing a sequence transformation to them, and we obtain $z_0 = y_k^{(0)}$. Then, we restart Kaczmarz’s method from $z_0$, that is we set $x_0 = z_0 = y_k^{(0)}$, we compute the new $\ell$ vectors $x_1, \ldots, x_\ell$ by Kaczmarz’s method, we apply again the acceleration algorithm to these vectors $x_0, \ldots, x_\ell$, we obtain $z_1 = y_k^{(0)}$, we restart Kaczmarz’s method from $x_0 = z_1$, and so on. This second procedure is called the restarted Kaczmarz (RK) algorithm and it can be implemented as follows.

Restarted Kaczmarz (RK) algorithm

Require $A \in \mathbb{R}^{N \times N}$, $b \in \mathbb{R}^N$, $x_0 \in \mathbb{R}^N$
Choose $k \in \mathbb{N}$, $k \geq 1$
Set $\ell = k + 1$ or $\ell = 2k$
for $n = 0, 1, \ldots$ until convergence do
  for $j = 0, \ldots, \ell - 1$ do
    $p_0 \leftarrow x_j$
    Compute $p_i$, $i = 1, \ldots, N$
    $x_{j+1} \leftarrow p_N$
  end for $j$
  Compute $y_k^{(0)}$
\[ z_n \leftarrow y_k^{(0)} \]
\[ x_0 \leftarrow z_n \]
end for \( n \)

4.2 The case of the \( \varepsilon \)-algorithms

Let us now consider the particular case of the \( \varepsilon \)-algorithms.

If we assume that the eigenvalues of the matrix \( Q \), defined in Section 2.1, are numbered such that \( |\tau_1| > |\tau_2| > \cdots > |\tau_N| > 0 \), then the sequences \( (\varepsilon_{2k}^{(n)}) \), for \( k \leq \nu \) fixed, constructed by these algorithms are such that \( 6, 42, 46 \)

\[ \|x - \varepsilon_{2k}^{(n)}\| = \mathcal{O}(\tau_{k+1}^n), \quad n \to \infty. \]

If the preceding assumptions on the \( \tau_i \)'s are not satisfied (which corresponds to several eigenvalues of \( Q \) having the same modulus or to a defective matrix \( Q \)), the expression for \( x - \varepsilon_{2k} \) is more complicated than given above \[44\] (see \[12\] for the complete expression in the scalar case), but the convergence is still accelerated by the \( \varepsilon \)-algorithms. These results also hold for the topological \( \varepsilon \)-algorithm, independently of the choice of the arbitrary vector \( y \) occurring in this algorithm. Thus, each sequence \( (\varepsilon_{2k}^{(n)}) \) obtained by the accelerated Kaczmarz algorithm converges to \( x \) faster than the preceding sequence \( (\varepsilon_{2k-2}^{(n)}) \) when \( n \) tends to infinity. Quite similar results hold for the RRE and the MPE \[43\].

If the topological \( \varepsilon \)-algorithm is applied to a sequence \( (x_n) \) obtained by any iterative method of the form \( x_{n+1} = Bx_n + c \), with the choice \( y = r_0 = b - Ax_0 \), then the vectors \( \varepsilon_{2k}^{(0)} \), \( k = 0, 1, \ldots \), are identical to those obtained by Lanczos' method (that is, for example, by the biconjugate algorithm or the conjugate gradient algorithm in the symmetric positive definite case) \[2\] Thms. 4.1 and 4.2, p. 186–7. This property comes out from the determinantal expressions of the vectors produced by the topological \( \varepsilon \)-algorithm and by Lanczos’ method. From \[7\], we immediately see that this is the case of the vectors given by Kaczmarz’s method which correspond to \( B = Q \) and \( c = A^{-1}(I - P)b \). Thus, if the topological \( \varepsilon \)-algorithm is applied, with \( y = r_0 \), to the sequence \( (x_n) \) given by Kaczmarz’s method, then the sequence \( (\varepsilon_{2k}^{(0)}) \) is exactly the sequence produced by Lanczos’ method. Using it as explained in the restarted Kaczmarz algorithm corresponds to restarting Kaczmarz’s method with the vector obtained by the Lanczos’ method after \( k \) of its iterates. At each restart, the vector \( y \) has to be taken equal to the corresponding residual.

5 Implementation

The implementation of our convergence acceleration procedures can be realized by three different ways. The first one consists, for a fixed value of \( k \), in solving the linear system \[11\] and using \[12\]. The second way is to employ \[14\] and the system \[15\]. The third possibility is to apply, when
it exists, a recursive algorithm. Since, for the vector $\varepsilon$–algorithm, no underlying system of linear equations is known, its implementation can only be realized through its recursive rules, given in Section 3.2.1.

An important practical problem is the choice of $k$. On one side, the effectiveness of our procedures seems to increase with $k$ but, on the other side, the number of vectors to be stored also increases with it. Thus, if the system is large, the value of $k$ has to be kept quite small since too many vectors will have to be stored. The numerical stability of the procedures has also to be taken into consideration when $k$ increases.

Since Kaczmarz’s method often converges slowly, the quantities $\Delta x_{n+i}$ are small, and the elements of the linear systems to be solved for obtaining the coefficients of the linear combination giving $y_k^{(n)}$ approach zero. Thus, whatever the method used, the linear systems (11) or (15) are ill–conditioned even for small values of $k$, and rounding errors can degrade the results. We tried several ways for computing their solution but the influence on the vectors $y_k^{(n)}$ was small.

Instead of solving the systems, it is possible to use a recursive algorithm for the computation of the vectors $y_k^{(n)}$. Usually, sequence transformations are more efficiently implemented via a recursive algorithm as those mentioned above, and, in our case, this kind of approach seems to give results less sensitive to rounding errors for the topological and the vector $\varepsilon$–algorithms.

The quantities computed by all the recursive algorithms for implementing our transformations can be displayed in a triangular array (or a lozenge one for the $\varepsilon$–algorithms), and $y_k^{(n)}$ corresponds to the lowest rightmost element of the array. Its computation imposes to compute first all the preceding elements in the array. Thus, it could be much too expensive to store all these vectors if $k$ is not quite small and if $N$ is large. Hopefully, it is possible to proceed in the array by ascending diagonals (that is, starting from $y_0^{(n+k)} = x_{n+k}$, to compute $y_1^{(n+k-1)}, \ldots, y_k^{(n)}$), and, thus, storing only one ascending diagonal. For more details on such a technique, see [13].

Another problem is the choice of the arbitrary vector $y$ in the topological Shanks transformation, and of the vectors $y_1, \ldots, y_k$ appearing in the MMPE (see [33] for a discussion about this choice). This is an unsolved problem and further studies will be necessary.

In the literature, the system $Ax = b$ to be solved by Kaczmarz’ method is often replaced by the system $DAx = Db$, where $D = \text{diag}(1/\|a_i\|)$. Thus, each row vector of the new matrix $DA$ will have a norm equal to 1. This preconditioning, of course, simplify the computation of $p_i$ in (11), and we will use it.

As a last comment, notice that, although some transformations are equivalent from the theoretical point of view (for instance the MPE and Germain–Bonne transformation, or the MMPE and the $H$–algorithm), they do not produce exactly the same numerical results. The same is true when we compare the results of a transformation implemented in the different ways described in this paper.
6 Numerical results

Our numerical experiments were performed using Matlab® 7.11 and matrices coming out from the gallery set. The solution was set to \( x = (1, \ldots, 1)^T \) and the right hand side \( b \) was computed accordingly. We took random vectors with components uniformly distributed in \([-1, +1]\) for the arbitrary vector \( y \) in the topological \( \varepsilon \)–algorithm, and for the vectors \( y_i \) in the MMPE.

Some figures show the Euclidean norms of the errors and, in order to compare the acceleration brought by each procedure, we also give the ratios of the norms of the errors between the iterate \( z_n \) obtained by the AK or the RK algorithm and the iterate of Kaczmarz’ method with the highest index used in its construction (for AK), or the iterate with the highest index which would have been used if we have led the method continue without restarting it (for RK), that is

\[
\frac{\|z_n - x\|}{\|x_{n+\ell} - x\|} \quad \text{(AK)} \quad \text{and} \quad \frac{\|z_n - x\|}{\|x_{(n+1)(\ell+1)} - x\|} \quad \text{(RK)}.
\]

We consider the parter matrix \( A, N = 1000, \kappa(A) \approx 4.2306 \), a Toeplitz matrix with singular values near \( \pi \). In the Figures 1 and 2, we compare Kaczmarz’ method with the MMPE, MPE, RRE and the topological \( \varepsilon \)–algorithm, implemented by solving the system (11), and the vector \( \varepsilon \)–algorithm, respectively for the AK and RK algorithms, with \( k = 5 \).

![Figure 1: AK algorithm: errors and ratios for parter matrix, N = 1000, k = 5.](image)

In these figures we see that all methods achieve a good precision with an advantage for the vector \( \varepsilon \)–algorithm. Moreover, its convergence is smoother. The ratios grow up because all methods almost stagnate when a good precision is attained while the error of Kaczmarz’ method continues to decrease slowly. In particular, for RK algorithm, the vector \( \varepsilon \)–algorithm attains its full precision after 4 iterations while the AK algorithm needs more iterations. For this example, the dominant eigenvalue of \( A \) is 0.8732178, and the second one is 0.3170877. Thus, according to the theoretical results of Section 4.2, a good acceleration is observed with \( k = 1 \) for all procedures.
Figure 2: RK algorithm: errors and ratios for parter matrix, \( N = 1000, k = 5 \).

The Figure 3 shows the ratios for the clement matrix, \( \kappa(A) \approx 1.13145 \times 10^{83} \), a tridiagonal matrix with zero diagonal entries, again with \( N = 1000 \) and \( k = 5 \). For this example, the MPE and the RRE coincide. Again the vector \( \varepsilon \)-algorithm is the best.

For toeppen, a pentadiagonal Toeplitz matrix, with \( N = 1000, k = 5 \), we have the Figure 4. Notice that the MMPE and the topological \( \varepsilon \)-algorithm do not work well and that this behavior could be due to the choice of the vectors \( y \) and \( y_i \). In fact, these choices can affect the results in a quite serious way. For instance, these methods sometimes exhibit better convergence and acceleration with \( y = (1, \ldots, 1)^T \) and \( y_i = e_i \). With the RK algorithm, considering the first 50 iterations for \( k = 8 \), the vector \( \varepsilon \)-algorithm attains a ratio of \( 10^{-9} \) at iteration 20 and after 24 iterations a division by zero occurs. The ratios for the RRE and the MPE have a minimum of \( 10^{-10} \) at iteration 25. The topological \( \varepsilon \)-algorithm diverges from the beginning. The MMPE exhibits an erratic convergence and the ratio goes down to \( 10^{-7} \) at the iteration 42, and then increases.

We also tried our procedures on a bigger matrix. The results for lesp, a tridiagonal matrix with real, sensitive eigenvalues, \( N = 10000, k = 5 \), \( \kappa(A) \approx 6.9553 \times 10^3 \), with the AK algorithm are given in Figure 5. We see that Kaczmarz’ method and the acceleration procedures all attain full precision after 90 iterations. Thus all ratios will grow up. However, the vector \( \varepsilon \)-algorithm attains an error of less than \( 10^{-11} \) after about 20 iterations while Kaczmarz’s method has only an error of order \( 10^{-3} \) at the same iteration.

An important point in any iterative method is to have a quite reliable stopping criterion. Usually such iterations are stopped by using the residuals. However such a computation will need a matrix–vector product and, in our case, one of the interests of Kaczmarz’ method will be lost. Thus, since the results given by our acceleration procedures often stagnate when some precision is attained while those of Kaczmarz’ continue to decrease, the iterations can be stopped as soon as the following ratios grow up significantly.
An example with the vector $\varepsilon$–algorithm is given in the Figure 6 (left: AK algorithm, right: RK algorithm). The solid line corresponds to the norm of the error and the dashed one to the ratio for the stopping criterion.

However, it must be noticed that, in the case of the RK algorithm, this stopping criterion involves iterates of Kaczmarz’ method that have not been computed and used in the acceleration procedures. Thus, it is not usable in practice. Since the quantities $\|\Delta z_n\|$ usually decrease rapidly, the iterates can be stopped when it is small enough and begins to stagnate or even to grow up.

In our examples, we also add a white noise between $10^{-2}$ and $10^{-8}$ to the vector $b$. The norm
of the error of the results obtained by our acceleration procedures attains the level of the noise in most cases.

For the matrix $\text{baart}$ of the Matlab Regularization toolbox \cite{31} of dimension 120 whose condition number is $2.28705 \times 10^{18}$, an error of the form $e = \delta \|b\|u/\sqrt{N}$ where $\delta$ is between $10^{-2}$ and $10^{-8}$ and $u$ is a vector whose components are random variables from a normal distribution with mean 0 and standard deviation 1, was added to $b$. The norm of the error achieved with the vector $\varepsilon$–algorithm goes down to $10^{-0.75}$ for $k = 1, 2$, and 3, which is a little bit better than the results obtained in \cite{15}.

Let us mention that the stopping criterion given above only works correctly for small noises. Maybe, it because the vector $b$ is not involved in it.
7 Conclusions

From our numerical results, it seems that the vector ε–algorithm is the best procedure for accelerating Kaczmarz’ method. However, the recursive implementation of the other procedures has to be tested numerically to see if it leads to better and more stable results. In our numerical examples, we tried several values of k. Although, when the dominant eigenvalue of A is well separated, k = 1 leads to quite a good acceleration, it seems that higher values produce better results in general. Anyway, the choice of k and of the vectors y and y_i are important points which need to be studied more deeply. Other recursive algorithms, such as those developed by Germain–Bonne [28], or the VTT and the BVTT [14], not considered in this work, have also to be tried.

The acceleration of the Symmetric Kaczmarz’ and the Randomized Kaczmarz’ methods, which are also sequential row–action methods that update the solution using one row of A at each step, of other methods of the MAP class, of the SIRT (Simultaneous Iterative Reconstruction Techniques) methods, has to be considered. Finally, applications to tomography and, in general, to image reconstruction have to be considered.

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