Lehmer’s Problem, McKay’s Correspondence, and 2, 3, 7

Dedicated to the memory of Ruth Michler

Eriko Hironaka

October 22, 2018

1 Introduction

This paper addresses a long standing open problem due to Lehmer in which the triple 2,3,7 plays a notable role. Lehmer’s problem asks whether there is a gap between 1 and the next smallest algebraic integer with respect to Mahler measure. The question has been studied in a wide range of contexts including number theory, ergodic theory, hyperbolic geometry, and knot theory; and relates to basic questions such as describing the distribution of heights of algebraic integers, and of lengths of geodesics on arithmetic surfaces. See, for example, [E-W] and [G-H] for surveys and references. This paper focuses on the role of Coxeter systems in Lehmer’s problem. The analysis also leads to a topological version of McKay’s correspondence.

We review some properties of Coxeter systems in Section 1, and Coxeter links in Section 2. Section 3 covers Lehmer’s problem, and Section 4 contains some remarks on a topological generalization of McKay’s correspondence.

2 Coxeter Systems

A Coxeter system consists of a vector space $V$ with a distinguished ordered basis $B = \{e_1, \ldots, e_n\}$, and an inner product

$$\langle v_i, v_j \rangle = -2 \cos \frac{\pi}{m_{i,j}},$$

where $m_{i,i} = 1$, and if $i \neq j$, $m_{i,j} \in \{2, 3, \ldots, \infty\}$. Associated to the Coxeter system is the Coxeter group $G \subset \text{GL}(V)$ generated by reflections $S = \{s_1, \ldots, s_n\}$ through hyperplanes perpendicular to $e_1, \ldots, e_n$ respectively. The action of $s_i \in S$ is given by

$$s_i(e_j) = e_j - \langle e_i, e_j \rangle e_i.$$

The group $G$ has presentation

$$G = \langle s_1, \ldots, s_n : (s_is_j)^{m_{i,j}} = 1 \rangle.$$
Coxeter systems are typically denoted by \((G, S)\).

A Coxeter system is determined by its Coxeter graph \(\Gamma\). This is the graph with vertices \(\nu_1, \ldots, \nu_n\) corresponding to the elements of \(S\) and edges labeled \(m_{i,j}\) connecting distinct vertices \(\nu_i\) and \(\nu_j\) whenever \(m_{i,j} > 2\).

The Coxeter element of \((G, S)\) is the product of reflections

\[ C = s_1 \ldots s_n, \]

and is an important invariant of the system. For example, the spectral radius of the Coxeter element equals 1 if and only if \(G\) is spherical or affine [A'C1], [How].

2.1 Coxeter Links

A link \(K\) in \(S^3\) is fibered with fiber \(\Sigma\) if \(\partial \Sigma = K\) and \(S^3 \setminus \Sigma = \Sigma \times I\). If \(K\) is a fibered link, there is an associated fibration

\[ \Sigma \longrightarrow S^3 \setminus K \]

\[ S^1 \]

The gluing map \(h: \Sigma \rightarrow \Sigma\) induces a monodromy.

\[ h_*: H_1(\Sigma; \mathbb{R}) \rightarrow H_1(\Sigma; \mathbb{R}). \]

The Alexander polynomial of \(K\) is the characteristic polynomial of \(h_*\).

In this section we will only deal with simply-laced Coxeter systems, where \(m_{i,j} \in \{1, 2, 3\}\). Since in this case all edges on the Coxeter graph are labeled 3, we drop the labeling.

\[ \ell_i \quad \ell_j \]

Figure 1.

An ordered chord system is a collection of oriented chords embedded on a disk \(D\) so that the endpoints lie on the boundary of \(D\). A chord system

\[ \{\ell_1, \ldots, \ell_n\}; \]

is positive if \(\ell_i\) intersects \(\ell_j\) positively whenever \(i > j\). Figure 1 shows two chords segments intersecting positively.

Let \(A\) be the intersection matrix of the chords in \(\mathcal{L}\) (with diagonal entries equal to zero.) Then \(\mathcal{L}\) is a positive chord system if and only if the lower triangular part of \(A\) is non-negative (and hence the upper triangular part is non-positive.)
From an ordered chord system \( \mathcal{L} \), we can define a fibered link \( K_\mathcal{L} \) with fiber \( \Sigma_\mathcal{L} \) as follows. Consider \( \mathcal{L} \) as an ordered collection of chords on a disk \( D \) embedded as the unit disk in the \( x, y \) plane in \( \mathbb{R}^3 \). Attach bands to \( D \) with one full positive twist as in Figure 2 in the order given by the ordering on the chord system. That is, the twisted band \( \eta_i \) corresponding to the chord \( \ell_i \) lies over the band \( \eta_j \) corresponding to the chord \( \ell_j \) if and only if \( i > j \). This defines a surface \( \Sigma_\mathcal{L} \subset \mathbb{R}^3 \) and its link boundary \( K_\mathcal{L} = \partial \Sigma_\mathcal{L} \). Identifying \( \mathbb{R}^3 \) with the subset of \( S^3 \) considered as the one point compactification of \( \mathbb{R}^3 \) gives a link \( K_\mathcal{L} \) with specific Seifert surface \( \Sigma_\mathcal{L} \).

![Figure 2. Murasugi sum.](image)

Stallings showed (see [Gab]) that the Murasugi sum of two fibered links is fibered. Thus, the links \( K_\mathcal{L} \) are fibered with fiber \( \Sigma_\mathcal{L} \). The oriented chords in \( \mathcal{L} \) extend to give a basis for \( H_1(\Sigma_\mathcal{L}; \mathbb{R}) \). Let \( \Gamma \) be the incidence graph of \( \mathcal{L} \). If \( \mathcal{L} \) is a positive chord system, then we call the pair \((K_\mathcal{L}, \Sigma_\mathcal{L})\) a Coxeter link associated to \((G, S)\). Its Seifert matrix equals \( M = I + A^+ \) and hence \( h_* = M^tM^{-1} \), where \( A \) is the intersection matrix of the chord system defined above. The bilinear form of \((G, S)\) can be written as

\[
B = M + M^t
\]

and the Coxeter element of \((G, S)\) equals (cf [How])

\[
C = -M^tM^{-1} = -h_*.\]

### 2.2 Realizable graphs

A graph \( \Gamma \) is **realizable** if it is the incidence graph of a chord diagram. Here are some examples of realizable graphs:

(i) Complete graphs;

(ii) Cyclic graphs;

(iii) Join of two realizable graphs at one vertex; and

(iv) Trees.
There are, however, obstructions to realizability. Figure 3 gives an example of a non-realizable graph. Let $\Gamma$ be a graph with vertices $S$. A subgraph $\Gamma' \subset \Gamma$ is an \textit{induced subgraph} if for some $S' \subset S$, $\Gamma'$ is the subgraph containing all edges on $\Gamma$ whose endpoints are in $S'$. An induced cycle in $\Gamma$ is a cycle which is an induced subgraph.

\begin{proposition}
\textbf{Proposition 2.1} $\Gamma$ is not realizable if there is a subset $S' \subset S$ such that
(i) $S'$ contains at least three vertices;
(ii) $S'$ is disjoint;
(iii) there is an $s \in S$ so that $s$ is joined by an edge in $\Gamma$ to every vertex in $S'$; and
(iv) there is an induced cycle in $\Gamma$ containing $S'$.
\end{proposition}

\begin{figure}[h]
\centering
\includegraphics[width=0.1\textwidth]{non-realizable-graph.png}
\caption{Non-realizable graph.}
\end{figure}

As was pointed out to me by R. Vogeler, any graph can be realized as an incidence graph if we generalize to higher dimensional diagrams. A higher dimensional chord system is a union of mutually transverse, linear embeddings $(D, \partial D) \rightarrow (B, \partial B)$ where $D$ is an $n$-disk and $B$ is an $n+1$-ball embedded in $\mathbb{R}^{n+2}$.

\begin{lemma}
\textbf{Lemma 2.2} Any finite graph with no self-loops or double edges can be realized as the incidence graph of some higher dimensional chord diagram.
\end{lemma}

An analysis of higher dimensional Coxeter links will be the topic of a future article.

\subsection{Comments on ordering and positivity}

In Section 2.1, we saw that a Coxeter graph $\Gamma$ has a corresponding Coxeter link, if it is both realizable as a chord system, and its ordering is compatible with a positive ordering on the chord system.

\begin{lemma}
\textbf{Lemma 2.3} Any chord diagram admits an ordering and orientation which is positive.
\end{lemma}

\begin{proof}
Choose a direction vector $v$ from the center of the disk and orient the chords so that their direction vectors have positive inner product in the usual Euclidean metric on $\mathbb{R}^2$ with $v$. Now order the chords counter-clockwise starting with the chord pointing furthest to the right of $v$.
\end{proof}
Not all orderings on a realizable graph, however, are realizable by a positive chord system, see for example Figure 4.

![Figure 4. Ordered graph which cannot be realized by a positive chord system](image)

As the examples in Section 2.4 show, different orderings on a chord system can give rise to different links. To determine the link, however, it is not necessary to have all the information of the ordering.

Given an ordered graph, there is an associated directed graph, where edges are directed so that they point to the vertex with larger index. As one can see from the construction, we do not need all the information of the ordering on the chord diagram.

**Lemma 2.4** If two orderings on a chord diagram $\mathcal{L}$ have the same directed incidence graph then the resulting fibered links are the same.

Lemma 2.4 is analogous to a result in Coxeter graph theory, which states that Coxeter elements depend only on the directed Coxeter graph [Shi].

**Lemma 2.5** The Coxeter element of Coxeter system $(G, S)$ depends only on the directed Coxeter graph $\Gamma$ of $(G, S)$.

The case when the incidence graph of $\mathcal{L}$ is a tree has been well studied, and the corresponding link has been called an *arborescent link* [Con]. Arborescent links also appear as *slalom links* in [A'C2]. Since any tree is realizable, there exists a Coxeter link associated to any tree. The corresponding Coxeter link will be independent of ordering, but will depend on the realization. Thus, for example, it is possible to find non-equivalent links which are Coxeter links for the same (ordered) Coxeter system.

**Lemma 2.6** If $\Gamma$ is an (unordered) tree with nodes of degrees less than or equal to 3, then the Coxeter link associated to $\Gamma$ is uniquely determined by $\Gamma$.

![Figure 5. Coxeter Link for a star graph.](image)
2.4 Examples

We begin with the classical Dynkin diagrams. using a correspondence between *star-diagrams* shown in Figure 6 and the $K_{p_1,...,p_k}$ pretzel links shown in Figure 5. Since Dynkin diagrams are trees with vertices of degree at most three, the associated link doesn’t depend on their realization as a chord system. The correspondence Dynkin diagrams and their corresponding Coxeter links is shown in Figure 7.

It is possible to cook up examples of non-equivalent links associated to the same ordered Coxeter system using star-diagrams. Take the two realization of the same tree shown in Figure 8. One sees that the link on the left has two knotted components, while the one on
the right has a component which is the unknot, hence the knots are not isotopic equivalent.

![Figure 8. Two embeddings of the same tree.](image)

When the incidence graph of a chord system contains cycles different orderings on the graph can give rise to different links. Consider for example, the 5-cycle. Up to isotopy, there is only one chord diagram with this incidence graph, but there are two inequivalent positive orderings as shown in Figure 9. The different orientations give rise to distinct

![Figure 9. Two orientations for the 5-cycle](image)

characteristic polynomials for the Coxeter elements:

\[
\Delta_1(-t) = 1 - t - t^4 + t^5; \quad \text{and} \\
\Delta_2(-t) = 1 - t^2 - t^3 + t^5.
\]
These are computed using the Seifert matrices

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & -1 & -1 & 0 \\
0 & 1 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]

The corresponding links are given in Figure 10. They are distinct iterated torus links.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{link_a.png}
\caption{a)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\includegraphics[width=\textwidth]{link_b.png}
\caption{b)}
\end{subfigure}
\caption{Two Coxeter links for the 5-cycle}
\end{figure}

The simply-laced minimal hyperbolic Coxeter system of smallest dimension is a triangle with a tail. The Coxeter link (see Figure 11) is uniquely determined in this case by the requirement of positivity, and equals the mirror of the $10_{145}$-knot in Rolfsen’s table \cite{rolf} (22, 3, 3 – in Conway’s notation \cite{con}.)

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{triangle.png}
\quad
\includegraphics[width=0.4\textwidth]{coxeter_link.png}
\caption{Coxeter link associated to smallest hyperbolic Coxeter system}
\end{figure}

2.5 Remarks on the Geometry of Coxeter systems and Coxeter links

A Coxeter system is *spherical* if its Coxeter group is a finite Euclidean reflection group (reflections across hyperplanes through the origin). In the simply-laced case, these are the $A$-$D$-$E$ Coxeter systems.

A Coxeter system is *affine* if its Coxeter group is an affine reflection group. The simply laced ones are the $\tilde{A}$-$\tilde{D}$-$\tilde{E}$ Coxeter systems.

The following is a classical fact due to Bourbaki \cite{bour}.
Theorem 2.7 Let \( \Gamma \) be a Coxeter system, and let \( B \) be its associated bilinear form. Then \( \Gamma \) is spherical if and only if \( B \) is positive definite, and affine if and only if \( B \) is positive semi-definite.

A’Campo \([A'C1]\) and Howlett \([How]\) proved the following relation between the geometry of the Coxeter system and the Coxeter element.

Theorem 2.8 Let \( \Gamma \) be a Coxeter system and \( C \) a Coxeter element. Then

- The eigenvalues of \( C \) lie on \( \mathbb{R} \cup S^1 \).
- \( \Gamma \) is spherical if and only if \( C \) has finite order;
- \( \Gamma \) is neither affine or spherical if and only if \( C \) has an eigenvalue greater than one.

Links \( K \) in \( S^3 \) have an analogous classification as torus links and satellite links, which include all algebraic links, and hyperbolic links, whose complements are uniformized by the 3-ball. The monodromy \( h_* \) has finite order and hence its spectral radius is 1 if \( K \) is a fibered iterated torus link, for example, when \( K \) is an algebraic link.

The above discussions bring up the following questions: Let \( K_\mathcal{L} \) be the Coxeter link associated to a simply laced Coxeter system \((G, S)\). Is it true that \( K_\mathcal{L} \) is hyperbolic if and only if the \((G, S)\) has indefinite intersection form? When is the monodromy \( h \) pseudo-Anosov?

3 Lehmer’s Problem

Let \( \alpha \) be an algebraic integer, and define its size to be

\[
\|\alpha\| = \prod_{\beta=\alpha^k, |\beta|>1} \beta.
\]

This is also known as the Mahler measure of the minimal polynomial of \( \alpha \) over \( \mathbb{Q} \). It is well known that \( \|\alpha\| = 1 \) if and only if \( \alpha^N = 1 \) some \( N \). Thus we are interested in algebraic integers which are not roots of unity.

In 1933, Lehmer \([Leh]\) asks whether for each \( \delta > 1 \), there exists an algebraic integer such that

\[1 < \|\alpha\| < 1 + \delta.\]

It is an easy exercise to see that the statement is false if one fixes the degree of the integer.

Lehmer found polynomials with smallest Mahler measure for small degrees and states in \([Leh]\) that the smallest he could find of degree 10 has minimal polynomial

\[P_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.\]

Boyd \([Boyd]\) and Mossinghoff \([Mos]\) have done searches up to degree 40, but so far no one has found a monic noncyclotomic integer polynomial with smaller Mahler measure.
One observes immediately that \( P_L(x) \) is reciprocal, that is
\[
P_L(x) = x^d P_L\left(\frac{1}{x}\right),
\]
where \( d \) is the degree of \( P_L(x) \) (in this case \( d = 10 \).) From Figure 12, one sees also that \( P_L(x) \) has only one root \( \alpha_L = 1.17628 \ldots \) which we will call \( \text{Lehmer’s number} \) outside the unit circle. Thus, \( \alpha_L \) is what is known as a \( \text{Salem number} \), that is, an algebraic integer whose algebraic conjugates lie on or within the unit circle, with at least one conjugate on the unit circle (making the minimal polynomial necessarily reciprocal.) It is not known whether there exist Salem numbers smaller than \( \alpha_L \).

Smyth [Smy] shows that among non-reciprocal polynomials Lehmer’s statement is false, and the smallest Mahler measure \( 1.32472 \ldots \) is attained by
\[
x^3 - x + 1.
\]
Thus, it remains to determine whether there is a similar minimum for Mahler measures of reciprocal monic integer polynomials.

It has been observed in various contexts that Lehmer’s problem is related to the triple \((2, 3, 7)\) and more abstractly to the notion of minimal hyperbolicity. Before going to examples, it is worth remarking that the triple has the simple distinguishing property that, among all \( k \)-tuples of positive integers \((p_1, \ldots, p_k)\), \((2, 3, 7)\) gives the minimal positive value for
\[
k - 2 - \sum_{i=1}^{k} \frac{1}{p_k}.
\]
This property comes into play in the minimality of Lehmer’s number \( \alpha_L \) among the series of Salem numbers and algebraic numbers which we describe in this section.

### 3.1 Growth rates and the \((2, 3, 7)\)-Triangle Group

Consider any pair \((G, S)\), where \( G \) is a group and \( S \) is a set of generators. Let
\[
w_n = \text{number of words of minimal word length } n \text{ in } S
\]
Figure 13. Tiling of the hyperbolic disk by the action of $T_{2,3,7}$.

The growth series of $(G, S)$ is the formal power series

$$f_{(G,S)} = \sum_{n=1}^{\infty} w_n t^n$$

and the growth rate $\alpha$ equals

$$\alpha = \frac{1}{\text{radius of convergence of } f_{(G,S)}}.$$

Another way to say this is that $w_n$ grows like $\alpha^n$ as $n$ gets large.

Let $G = T_{p_1, \ldots, p_k}$ be a polygonal reflection group acting on $S^2$, $\mathbb{E}^2$ or $\mathbb{H}^2$. The group $G$ is the Coxeter group associated to the cyclic Coxeter graph with edges labeled $p_1, \ldots, p_k$, and has presentation

$$G = \langle s_1, \ldots, s_k : (s_is_j)^{m_{ij}} \rangle$$

In this case Floyd and Plotnick [F-P] show the following. (See also, [C-W].)

**Theorem 3.1** The growth series for $(G, S)$ is rational

$$f_{(G,S)} = \frac{R(t)}{\Delta_{p_1, \ldots, p_k}(t)}$$

where $\Delta_{p_1, \ldots, p_k}(t)$ is given by:

$$\Delta_{p_1, \ldots, p_k}(t) = (x - k + 1) \prod_{i=1}^{k} [p_i] + \sum_{i=1}^{k} [p_1] \ldots \widehat{[p_i]} \ldots [p_k],$$

where $[p] = 1 + x + \ldots + x^{p-1}$. Furthermore, $\Delta_{p_1, \ldots, p_k}(t)$ is a reciprocal monic integer polynomial with a root, necessarily a Salem number, outside the unit circle if and only if

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} < k - 2.$$
Thus, the growth rate $\alpha_{p_1,\ldots,p_k}$ of $(G,S)$ is a Salem number if and only if

$$\chi = \frac{1}{p_1} + \cdots + \frac{1}{p_k} - (k - 2) < 0.$$ 

The polynomial $\Delta_{2,3,7}(x)$ equals Lehmer’s polynomial $P_L(x)$, and hence the triangle group $T_{2,3,7}$ has growth rate equal to $\alpha_L$.

For the family of polynomials $\Delta_{p_1,\ldots,p_k}(x)$ Lehmer’s problem is solved [Hir].

**Theorem 3.2** Among the polynomials $\Delta_{p_1,\ldots,p_k}(x)$, the one with smallest Mahler measure is Lehmer’s polynomial $\Delta_{2,3,7}(x)$.

This result is suggestive since among hyperbolic orbifold spheres $(S^2; p_1,\ldots,p_k)$, the one with maximal orbifold Euler characteristic $\chi$ and minimal hyperbolic area is $(S^2; 2,3,7)$.

### 3.2 Alexander polynomials and the $(2, 3, 7, -1)$ Pretzel Knot

Reidemeister [Reid] remarked that the $(-2, 3, 7)$-pretzel knot shown in Figure 14 has Alexander polynomial $P_L(-x)$.

![Figure 14](image-url)

The $(-2, 3, 7)$ pretzel knot is equivalent to the $(2, 3, 7, -1)$ pretzel knot. Let $K_{p_1,\ldots,p_k}$ be the $(p_1,\ldots,p_k,-1,\ldots,-1)$-pretzel link, where the number of “-1”s is $k - 2$. It turns out that the Alexander polynomial of $K_{p_1,\ldots,p_k}$ is related to the denominator of the growth series of $T_{p_1,\ldots,p_k}$ [Hir].

**Theorem 3.3** The pretzel link $K_{p_1,\ldots,p_k}$ is fibered and has Alexander polynomial $\Delta_{p_1,\ldots,p_k}(-x)$.

Thus, Theorem 3.2 implies the following.

**Corollary 3.4** Among pretzel links $K_{p_1,\ldots,p_k}$, the Mahler measure of the Alexander polynomial is minimized by $K_{2,3,7}$.

Although the relation between the Alexander polynomial and the denominator of the growth series has not been fully explained there is a natural relation between the pretzel links and the polygonal reflection groups which we describe in Section 4.
3.3 \( E_{10} \) diagram

The Coxeter \( E_{10} \) diagram can be thought of as the \((2,3,7)\)-star Coxeter graph as can be seen by comparing Figure 15 with Figure 6.

![Figure 15. \( E_{10} \)-Coxeter graph](image)

McMullen observes that the characteristic polynomial of the Coxeter element is Lehmer’s polynomial \( p_L(x) \), and its leading eigenvalue is \( \alpha_L \). Furthermore, he shows the following.

**Theorem 3.5** Let \( C \) be the Coxeter element of a Coxeter system, and let \( \lambda(C) \) be the spectral radius of \( C \). Then among non-spherical and non-affine Coxeter systems, \( \lambda(C) \) achieves its minimum when \( (G, S) \) is the Coxeter system corresponding to the \( E_{10} \) diagram.

This solves Lehmer’s problem for Coxeter elements of Coxeter systems, and as a consequence also for the monodromy of Coxeter links, generalizing Corollary 3.4.

4 Correspondence between stars and polygons

The results in the previous sections show that the growth rate of \((G, S)\) corresponding to the Coxeter system \( T_{p_1, \ldots, p_k} \) and the spectral radius of the Coxeter system \( \text{Star}(p_1, \ldots, p_k) \) are equal.

![Figure 16. \( \text{Star}(p_1, \ldots, p_k) \) and \( T_{p_1, \ldots, p_k} \)](image)

Here we give a topological relation between the polygonal- and star-Coxeter systems. Let \( P \) be the sphere \( S^2 \), the Euclidean plane \( \mathbb{E}^2 \), or the hyperbolic plane \( \mathbb{H}^2 \). Let \( T(P) \) be the unit tangent bundle of \( P \). Let \( G \) be the \( p_1, \ldots, p_k \)-polygonal reflection group acting on \( P \).

It is not hard to see that the action of \( G \) on \( T(P) \) makes \( T(P) \) a branched cover over \( S^3 \) with branching index 2 on a link \( K \).
Let $X$ be a manifold and $Y$ a formal sum of codimension one disjoint submanifolds of $X$ with integer coefficients. We denote by $(X; Y)$ the orbifold $X$ with orbifold singularities at $Y$ of the appropriate multiplicity.

The covering $T(P) \to S^3$ factors through an unbranched covering of a manifold $M$ which double covers $S^3$ as in the following diagram:

$$
\begin{array}{c}
T(P) \\
\downarrow G^{(2)} \\
M \\
\downarrow \mathbb{Z}_2 \\
(S^3; 2[K])
\end{array}
$$

A double covering of $S^3$ branched along any Montesinos link $K$ is Seifert fibered over $S^2$ (see, for example, [B-Z]). The $p_1, \ldots, p_k$-star Coxeter link is only a particular example of a Montesinos link for which the double cover is Seifert fibered over $(S^2; p_1[x_1] + \cdots + p_k[x_k])$, but there are many others with different Alexander polynomials. For the particular 3-manifold $M$ defined above, however, one can show the following is true.

**Proposition 4.1** The link $K$ coming from the quotient of the homogeneous space $T(P)$ by the $p_1, \ldots, p_k$-polygonal Coxeter group is the Coxeter link associated to the $p_1, \ldots, p_k$-star Coxeter system.

### 4.1 Example: Klein Singularities

The above discussion generalizes a phenomenon encountered in the theory of isolated hypersurface singularities.

Let $\tilde{G}$ be a finite subgroup of $SU(2)$ acting in the usual way on $\mathbb{C}^2$. The quotient $X$ is a hypersurface in $\mathbb{C}^3$ with isolated singularity called a Klein singularity. The resolution diagrams of these singularities are exactly the Coxeter graphs (or Dynkin diagrams) of the simply laced spherical Coxeter systems. The correspondence between finite subgroups of $SU(2)$ and the Dynkin diagrams is known as McKay’s Correspondence [McK] (see also [G-V] and [Slo]).

Coxeter links can be used to give a topological description of the correspondence. Let $\tilde{G}$ be a finite subgroup of $SU(2)$. Then $\tilde{G}$ is the binary extension of a finite subgroup $G^{(2)}$ of $SO(3)$ which in turn (with the exceptional case of $G^{(2)} = \mathbb{Z}_n$) is an index 2 subgroup of a $(p, q, r)$-triangle group $G$. The group $G^{(2)} = \mathbb{Z}_n$ corresponds to the $(2, n)$-triangle group. Relating each group with the $(p, q, r)$-star diagram gives the McKay correspondence, as seen in the following diagram.
For these triples $p, q, r$ in the above diagram, the $(p, q, r)$-star Coxeter link $K$ is algebraic, and the induced branched covering of the link $M$ of the singularity $X$ over $S^3$ is induced by a generic projection of $X$ to $\mathbb{C}^2$. Thus, we obtain the following commutative diagram.

\[
\begin{array}{ccc}
\mathbb{R}P^3 = T(S^2) & \xleftarrow{\ell} & S^3 \subset \mathbb{C}^2 \\
G(2) & \downarrow & \tilde{G} \\
M & \subset & X \\
\mathbb{Z}_2 & \downarrow & \rho \\
S^3 & \subset & \mathbb{C}^2
\end{array}
\]

A minimal desingularization of $X$ can be obtained by blowing up $\mathbb{C}^2$ to desingularize the branch curve of the map $\rho$, and pulling back over $X$.

\[
\begin{array}{ccc}
X & \xleftarrow{\bar{\alpha}} & \bar{X} \\
\rho & \downarrow & \tilde{\rho} \\
\mathbb{C}^2 & \xleftarrow{\alpha} & \tilde{\mathbb{C}}^2
\end{array}
\]

One way to verify that the resolution diagram for $X$ must equal the Dynkin diagram corresponding to the link, is to note that the effect of desingularizing the branch curve is the same as unknotting the branch link by a sequence of simple Dehn twists on suitable closed curves in the complement.
References

[A’C1] N. A’Campo. Sur les valeurs propres de la transformation de Coxeter. *Invent. Math.* **33**(1976), 61–67.

[A’C2] N. A’Campo. Planar trees, slalom curves and hyperbolic knots. *Inst. Hautes Études Sci. Publ. Math.* **88**(1998), 171–180.

[Bour] N. Bourbaki. *Groupes et algèbres de Lie*. Hermann, Paris, 1968.

[Boyd] D.W. Boyd. Reciprocal polynomials having small measure. II. *Math. Comp.* **53**(1989), 355–357, S1–S5.

[B-Z] G. Burde and H. Zieschang. *Knots*. Walter de Gruyter, Berlin, 1985.

[C-W] J. Cannon and P. Wagreich. Growth functions of surface groups. *Math. Ann.* **293**(1992), 239–257.

[Con] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 329–358. Pergamon, Oxford, 1970.

[E-W] G. Everest and T. Ward. *Heights of polynomials and entropy in algebraic dynamics*. Universitext. Springer-Verlag London, Ltd., London, 1999.

[F-P] W. J. Floyd and S. P. Plotnick. Symmetries of planar growth functions of Coxeter groups. *Invent. Math.* **93**(1988), 501–543.

[G-V] J. L. Verdier G. Gonzalez-Sprinberg. Construction géométrique de la correspondance de McKay. *Ann. Sci. École Norm. Sup. (4)* **16**(1983), 409–449 (1984).

[Gab] D. Gabai. The Murasugi Sum is a Natural Geometric Operation. In *Low Dimensional Topology*, volume 20 of *Cont. Math.*, pages 131–144. A.M.S, 1983.

[G-H] E. Ghate and E. Hironaka. The Geometry of Salem numbers. *Bulletin of Amer. Math. Soc.* **38**(2001), 293–314.

[Hir] E. Hironaka. The Lehmer Polynomial and Pretzel Knots. *Bulletin of Canadian Math. Soc.* **44**(2001), 440–451.

[How] R. Howlett. Coxeter groups and $M$-matrices. *Bull. London Math. Soc.* **14**(1982), 137–141.

[Leh] D. H. Lehmer. Factorization of certain cyclotomic functions. *Ann. of Math.* **34**(1933), 461–469.

[McK] J. McKay. Graphs, singularities, and finite groups. *Proc. Symp. Math* **37**(1980), 183–186.
[McM] C. McMullen. Coxeter systems, Salem numbers, and the Hilbert metric. \textit{preprint} (2001).

[Mos] M. Mossinghoff. Polynomials with small Mahler measure. \textit{Mathematics of Computation} \textbf{67}(1998), 1697–1705.

[Reid] K. Reidemeister. \textit{Knotentheorie}. Springer, Berlin, 1932.

[Rolf] D. Rolfsen. \textit{Knots and Links}. Publish or Perish, Inc, Berkeley, 1976.

[Shi] J.-Y. Shi. The enumeration of Coxeter elements. \textit{J. Alg. Comb.} \textbf{6}(1997), 161–171.

[Slo] P. Slodowy. Platonic solids, Kleinian singularities and Lie groups. In \textit{Algebraic Geometry}, volume 1008 of \textit{Lecture Notes in Mathematics}, pages 102–138. Springer, Berlin, 1983.

[Smy] C. J. Smyth. On the product of the conjugates outside the unit circle of an algebraic integer. \textit{Bull. London Math. Soc.} \textbf{3}(1971), 169–175.

Eriko Hironaka  
Department of Mathematics  
Florida State University  
Tallahassee, FL 32306  
Email: hironaka@math.fsu.edu