PLASMA MODES ALONG THE OPEN FIELD LINES OF A NEUTRON STAR

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ABSTRACT

We consider electrostatic plasma modes along the open field lines of a rotating neutron star. Goldreich-Julian charge density in general relativity is analyzed for the neutron star with zero inclination. It is found that the charge density is maximum at the polar cap and remains almost the same in a certain extended region of the pole. For a steady state Goldreich-Julian charge density we found the usual plasma oscillation along the field lines; plasma frequency resembles the gravitational redshift close to the Schwarzschild radius. We study the nonlinear plasma mode along the field lines. From the system of equations under general relativity, a second-order differential equation is derived. The equation contains a term that describes the growing plasma modes near Schwarzschild radius in a black hole environment. The term vanishes with the distance far away from the gravitating object. For initially zero potential and field on the surface of a neutron star, Goldreich-Julian charge density is found to create the plasma mode, which is enhanced and propagates almost without damping along the open field lines. We briefly outline our plan to extend the work for studying soliton propagation along the open field lines of strongly gravitating objects.

Subject headings: MHD — plasmas — pulsars: general — relativity — stars: neutron

1. INTRODUCTION

Study of plasma modes in the neutron star or black hole environments is related to the investigation of radio emissions coming from these sources (see, e.g., Buzzi & Hines 1995; Mofiz 1997 and references therein). Radio pulsars, which are rotating neutron stars with spin periods ranging from 1.57 ms to 5 s, are characterized by surface magnetic fields of the order of $10^{12}$ G, radii of about 10 km, and central densities in excess of $10^{14}$ g cm$^{-3}$ and so are purely gravitating objects. A spinning magnetized neutron star generates huge potential differences between different parts of its surface (Goldreich & Julian 1969). The cascade generation of electron-positron plasmas in the polar cap region (Sturrock 1971; Ruderman & Sutherland 1975) means that the magnetosphere of a neutron star is filled with plasma, screening the longitudinal electric field. This screening results in the corotation of plasma with a star. Such a rotation is not possible outside the light cylinder, thus it forms essentially different groups of field lines: closed, i.e., those returning the stellar surface, and open, i.e., those crossing the light cylinder and going to infinity. As a result, plasma may leave the neutron star along the open field lines. The charges along the field lines create plasma modes that may be related to the pulsar radiation and its microstructures.

Our study of plasma modes along the field lines is boosted by the pioneering works of Goldreich & Julian (1969), Sturrock (1971), Mestel (1971), Ruderman & Sutherland (1975), and Arons & Scharlemann (1979). The subsequent achievements and some new ideas are reviewed by Arons (1991), Michel (1991), Mestel (1992), and Muslimov & Harding (1997). Although a self-consistent pulsar magnetosphere theory is yet to be developed, the analysis of plasma modes in the pulsar magnetosphere based on the above-mentioned papers provides firm grounds for the construction of such a model.

In this paper, we attempt to extend work done by Muslimov & Harding (1997) by studying plasma modes along the open field lines of a rotating neutron star. In § 2 general relativistic equations describing the electrodynamics of a rotating neutron star are formulated. The equations are rewritten in the frame of reference corotating with the neutron star. We deduce the general system of equations governing the electrostatic modes in the pulsar magnetosphere. A detailed analysis of Goldreich-Julian charge density in general relativity is done in § 3. It is shown that the charge density exponentially decays with the distance away from the surface of the star while it has a periodic dependence on the polar angle along the surface. The field is maximum at the polar cap region and remains almost the same in a certain extended region in the pole. In § 4 we study the linear plasma modes along the open field lines. A general equation governing electrostatic potential is derived. For a steady state Goldreich-Julian charge density, the usual plasma oscillation along the field lines is found. Plasma frequency resembles the gravitational redshift close to the Schwarzschild radius, while at a large distance from the gravitational radius it is the usual plasma oscillation along the field lines. In § 5 we study the nonlinear plasma modes along the field lines. From the system equations under general relativity, a second-order differential equation is derived. The equation contains a term that describes the growing plasma mode near the Schwarzschild radius of a neutron star or a black hole. The term vanishes with the distance far away from the gravitating object. The equation is solved numerically subject to appropriate boundary conditions. It is found that Goldreich-Julian charge density creates the initial field on the surface of the star, which is enhanced near the gravitational radius and propagates almost without damping along the open field lines. In § 6 we conclude our findings and discuss them for further investigations.
2. GENERAL RELATIVISTIC ELECTRODYNAMIC EQUATIONS IN THE COROTATING FRAME OF REFERENCE

Recently Muslimov & Harding (1997) derived the general relativistic electrodynamic equations for a neutron star in the corotating frame of reference. It is noted that the effects of general relativity are very important: the dragging of inertial frames of reference significantly affects the electric field generated in the vicinity of a rotating magnetized neutron star, while the static part of the gravitational field results in additional enhancement of electric and magnetic fields near a star.

The metric of an asymptotically flat, stationary, axially symmetric spacetime around a rotating gravitating body (see, e.g., Landau & Lifshitz 1975) is considered. In spherical polar coordinates $x^0 = ct, x^1 = r, x^2 = \theta$, and $x^3 = \phi$, we have

$$ds^2 = A^2(\sqrt{1 - r_g/r^2})^2 - B^2dr^2 - C^2(d\theta)^2 - D^2(d\phi - \omega dt)^2,$$

where $A = B^{-1} = (1 - r_g/r)^{1/2}$ is the gravitational redshift function, $C = r, D = r \sin \theta, r_g = 2GM/c^2$ is the gravitational radius of a body (neutron star) of mass $M$, $J$ is the angular momentum of a neutron star, $c$ is the speed of light, and $G$ is the gravitational constant. The metric in equation (1) is the approximation of a Kerr metric when the ratio $r_g/Mc$ is small. The presence of the non diagonal component in the metric in equation (1) results in the well-known effect of dragging of inertial frames of reference (the Lense-Thirring effect) with the angular velocity

$$\omega = \frac{2GJ}{c^2r^3}.$$

The metric in equation (1) can be transformed to the frame of reference corotating with a neutron star:

$$ds^2 = A^2(\sqrt{1 - r_g/r^2})^2 - B^2dr^2 - C^2(d\theta)^2 - D^2(d\phi - \Omega dt)^2,$$

by transformations $t' = t, r' = r, \theta' = \theta, \phi' = \phi - \Omega_0 t$. Here $\Omega = \omega - \Omega_0$, where $\Omega_0$ is the angular velocity of rotation of the star relative to the distant observer.

The zero angular momentum observer (ZAMO; see, e.g., Thorne, Price, & MacDonald 1986) has the four-velocity

$$e^\gamma\left\{ \frac{1}{\sqrt{1 - r_g/r}}, 0, 0, -\frac{\Omega}{c\sqrt{1 - r_g/r}} \right\}, \quad e^\gamma\left\{ -\sqrt{1 - r_g/r}, 0, 0, 0 \right\}.$$

Then the general relativistic Maxwell equations for observer (eq. [4]) in the metric (eq. [3]) take the form

$$\nabla \cdot B = 0,$$

$$\nabla \times (\varepsilon E - (\omega - \Omega_0) \times B) = -\frac{1}{c} \frac{\partial B}{\partial t},$$

$$\nabla \cdot E = 4\pi \rho,$$

$$\nabla \times (\varepsilon B) + \nabla \times [(\omega - \Omega_0) \times E] - (\omega - \Omega_0)(\nabla \cdot E) = \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} J,$$

where $(\Omega \times B)^\gamma = e^\gamma_{\mu\nu}e^\nu B_\gamma$, and $(\omega - \Omega_0) = \{0, 0, 0, [\omega - \Omega_0]\sin \theta/c(1 - r_g/r)^{1/2}\}$. Similarly, we may write the charge continuity equation in the above mentioned frame as

$$\frac{\partial \rho}{\partial t} + \left\{ \frac{\kappa}{r^3} - 1 \right\} \Omega_0 m \cdot \nabla \rho + \nabla \cdot (\rho \varepsilon J) = 0.$$

Finally, the equation of motion of a charged particle is

$$\frac{1}{\alpha} \frac{d\mathbf{p}}{dt} = m\gamma g + q\left( \frac{E}{c} + \frac{\mathbf{v}}{c} \times B \right) + f.$$

Here $\alpha = (1 - r_g/r)^{1/2}$ ($\equiv A$, as denoted in the metric in eq. [1]), the parameter $\kappa \equiv (2r_s R^2)/5$, $R$ is the radius of a neutron star, $\rho = \sum n_s q_s J = \sum n_s q_s v_s$, $v_s$ is the velocity, $q_s$ is the charge of the particle, $n_s$ is the particle number density, and summation is over all the species; $\rho = m\gamma \mathbf{v}$ is the momentum of the particle, $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor, $m$ is the rest mass of the particle, $f$ is an external force other than electromagnetic, and $g$ is the gravitational acceleration. All electrodynamic quantities such as magnetic field $B$, electric field $E$, conduction current $J$, and charge density $\rho$ in these equations are measured by ZAMO (eq. [4]). Gradient, curl, and divergence are taken along the curvilinear coordinate

$$e_x = Ae_x = A \frac{\partial}{\partial r}, \quad e_y = e_z = e_\phi = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e_z = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}.$$

$m = r \sin \theta e_\phi$ is the Killing vector, responsible for the axial symmetry.

Assuming $\partial B/\partial t = 0$ in equation (5b) (i.e., considering that the magnetic field of a neutron star is stationary in the corotating frame), from equation (5b) we get

$$\varepsilon E - (\omega - \Omega_0) \times B = -\nabla \Phi,$$
where $\Phi$ is a scalar electrostatic potential. Taking the divergence of equation (8) and making use of equation (5c), we get

$$
\nabla \cdot \left[ \frac{1}{\alpha} \nabla \Phi + \frac{1}{\alpha} (\Omega_0 - \omega) \times \mathbf{B} \right] = -4\pi \rho .
$$

(9)

Equation (9) can be written as (see Muslimov & Tsygan 1986)

$$
\nabla \cdot \left( \frac{1}{\alpha} \nabla \Phi \right) = -4\pi (\rho - \rho_{GJ}) ,
$$

(10)

where

$$
\rho_{GJ} = -\frac{1}{4\pi} \nabla \cdot \left[ \frac{1}{\alpha} (\Omega_0 - \omega) \times \mathbf{B} \right] = -\frac{1}{4\pi} \nabla \cdot \left[ \frac{1}{\alpha} \left( 1 - \frac{\kappa}{r^3} \right) \Omega_0 \times \mathbf{B} \right]
$$

(11)

is the relativistic analog of the Goldreich-Julian (1969) charge density. Finally, the equation of motion of the charged particle is

$$
\left[ \frac{\partial}{\partial t} + \left( \frac{\kappa}{r^3} - 1 \right) \mathbf{\Omega}_0 \cdot \nabla + \alpha v \cdot \nabla \right] p = -q \nabla \Phi ,
$$

(12)

where the gravitational acceleration $\mathbf{g}$ and the nonelectromagnetic force $\mathbf{f}$ are justifiably ignored.

3. GOLDRICH-JULIAN CHARGE DENSITY IN GENERAL RELATIVITY

In a pioneering work, Goldreich & Julian (1969) have shown that a strongly magnetized, highly conducting neutron star, rotating about the magnetic axis, would spontaneously build up a charged magnetosphere. The essence of the argument is that it imposes a charge magnetosphere that is subject to enormous unbalanced electric forces parallel to the magnetic field. Goldreich & Julian (1969) hypothesized that a far better approximation for the magnetosphere would be obtained by shorting out the component of $\mathbf{E}$ along $\mathbf{B}$ by charges originating in the star. The magnetospheric charges that maintain $\mathbf{E} \cdot \mathbf{B} = 0$ are themselves subject to the $\mathbf{E} \times \mathbf{B}$ drift that sets them into corotation with the star. Here we analyze Goldreich-Julian charge density in general relativity.

Assuming zero inclination of the rotating star with the magnetic axis, we consider $\mathbf{B} = \{B_r, B_\theta, 0\}$. The components $B_r, B_\theta$ in this case were first derived by Ginzburg & Ozernoy (1964). Later on, similar expressions were derived in a number of papers (see, e.g., Wasserman & Shapiro 1983; Muslimov & Tsygan 1986):

$$
B_r = \frac{2 \cos \theta}{r^3} f(r) \mu ,
$$

(13a)

$$
B_\theta = \frac{\sin \theta}{r^3} \psi(r) \mu ,
$$

(13b)

where

$$
f(r) = -\frac{3r^3}{r^3_g} \left[ \ln \left( 1 - \frac{r^2_g}{r} \right) + \frac{r^2_g}{r} + \frac{1}{2} \left( \frac{r^2_g}{r} \right)^2 \right] ,
$$

(14)

$$
\psi(r) = \frac{3r^2}{r^3_g} \left[ \frac{1}{1 - r^2_g/r} + 2 \frac{r}{r^3_g} \ln \left( 1 - \frac{r^2_g}{r} \right) + 1 \right] \sqrt{1 - \frac{r^2_g}{r}} ,
$$

(15)

and $\mu$ is the magnetic dipole moment of a neutron star.

We perform a detailed calculation of $\rho_{GJ}$ with the magnetic field of a rotating neutron star given by equations (13a) and (13b), respectively. From equation (11) we find

$$
\rho_{GJ} = -\frac{\Omega_0}{4\pi c r^2 \sin \theta} \left\{ \left( 1 - \frac{\kappa}{r^3} \right) \frac{r^3 \sin^2 \theta B_\theta}{\sqrt{1 - r^2_g/r}} - \left( 1 - \frac{\kappa}{r^3} \right) \frac{r^3 \sin^2 \theta B_z}{1 - r^2_g/r} \right\} ,
$$

(16)

The calculation shows that

$$
\rho_{GJ}(r, \theta) = \frac{3\Omega_0 \mu}{4\pi c r^3} \left[ F_1(\bar{r}) \sin^2 \theta - F_2(\bar{r}) (\sin^2 \theta - 2 \cos^2 \theta) \right] ,
$$

(17)

with

$$
F_1(\bar{r}) = \bar{r}^3 \left\{ \left( 1 - \frac{\beta}{\bar{r}^3} \right) \left[ \frac{2}{\bar{r} - 1} - \frac{1}{(\bar{r} - 1)^2} + 2 \ln \left( 1 - \frac{1}{\bar{r}} \right) \right] + \left( 2 + \frac{\beta}{\bar{r}^3} \right) \left[ \frac{1}{\bar{r}^2} + \frac{1}{\bar{r} - 1} + 2 \ln \left( 1 - \frac{1}{\bar{r}} \right) \right] \right\} ,
$$

(18)

$$
F_2(\bar{r}) = \bar{r}^3 \left\{ 2 \frac{(1 - \beta/\bar{r}^3)}{1 - 1/\bar{r}} \left[ \frac{1}{2\bar{r}^2} + \frac{1}{\bar{r}^3} + \ln \left( 1 - \frac{1}{\bar{r}} \right) \right] \right\} .
$$

(19)
Here $\bar{r} = r/r_g$ and $\beta = \kappa/r_g^3$. Asymptotically, as $r/r_g \to \infty$, functions $F_1(\bar{r})$ and $F_2(\bar{r}) \to 1$.

Thus, Goldreich-Julian space charge has two purely general relativistic contributions, one due to the Schwarzschild gravitoelectric parameter $r_g$ and the other due to the gravitomagnetic Kerr parameter $\beta$. They have different dependences on $r$ as $1/r$ and $1/r^3$, respectively. This means that near the surface of the star the gravitomagnetic term is concurrent with the gravitoelectric one. But in the distance from the surface of the star, which is comparable to its radius $R$, the gravitomagnetic term is ignorable small.

We plot $F_1(\bar{r})$ and $F_2(\bar{r})$ for $\beta = 0.1$. The dependence of these functions on $\bar{r}$ is shown in Figures 1 and 2, respectively. The Goldreich-Julian charge density under general relativity is shown in Figure 3.

By least-squares fitting of the curve at $\theta = 0$, we find that Goldreich-Julian charge density decays with the distance away from the star as follows:

$$\rho_{GJ} = \frac{10.5053}{r^3} - \frac{5.05692}{r^2} + \frac{1.06093}{r} - 0.084179.$$  

The charge density is maximum at the polar cap region and remains almost the same in a certain extended region in the pole, but it falls down away from the polar cap. It is to be noted that the expression for $\rho_{GJ}$ obtained by Muslimov & Harding (1997) shows similar results.
4. LINEAR PLASMA MODES ALONG THE OPEN FIELD LINES

The theory of cascade generation of electron-positron plasma at the polar cap region of a rotating plasma is developed by Ruderman & Sutherland (1975). According to the theoretical model, because of the escape of charge particles along the open field lines, a polar potential gap is produced that continuously breaks down by forming an electron-positron pair on a timescale of a few microseconds. A photon of energy greater than $2mc^2$ produces an electron-positron pair. The electric field of the gap accelerates the positron out of the gap and accelerates the electron toward the stellar surface. The electron moves along a curved magnetic field line and radiates an energetic photon that goes on to produce a pair as it has a sufficient component of momentum perpendicular to the magnetic field. Very recently, Zhu & Ruderman (1997) explained the $e^{-}e^{+}$ pair production from a Crab-like pulsar. Electrons and positrons are accelerated in opposite directions to extremely high energies.

The Lorentz factor $\gamma$ of the "primary" electron and positron is given by

$$eE \cdot \dot{B}c \approx \frac{e^2}{c^3} \gamma^4 \left(\frac{c^2}{r_s}\right)^2,$$

where $r_s$ is the curvature radius of the local magnetic field lines and $E \cdot B$ is the pulsar magnetospheric electric field component along $B$. For the Crab pulsar, the above equation gives $\gamma$ of the order of $10^7$. Curvature photons radiated by the primary electrons and positrons have energy $\approx h\gamma^3(c/r_s)$. Each primary electron and positron would produce about $N_s \approx (\gamma e^2/\hbar c) \approx 10^5$ curvature photons. To sustain the primary current flow and account for the observed X-ray and $\gamma$-ray luminosity from the Crab pulsar, the needed primary particle flux is around $10^{33} \text{s}^{-1}$. This cascade of pair production, acceleration of electrons and positrons along curved field lines, curvature radiation, and pair production results in a "spark" breakdown of the gap.

Assuming a steady state thermodynamical equilibrium plasma state in the polar cap region, we study the linear plasma modes along the field lines. From the system of equations (6), (10), and (12), we derive the following linearized equations:

$$\mathbf{V} \cdot \left(\frac{1}{\sqrt{1 - r_g/r}} \nabla \Phi\right) = -4\pi(\sum \sigma_i \delta n_x - \rho_{GJ}),$$

$$\frac{\partial}{\partial t'} \left(\frac{\delta n_x}{n_0}\right) + \mathbf{V} \cdot \left(\sqrt{1 - r_g/r} \mathbf{v}_s\right) = 0,$$

$$\frac{\partial \mathbf{v}_s}{\partial t'} = -\frac{q_s}{m} \nabla \Phi,$$

where $\partial/\partial t' = \partial/\partial t + (\kappa/r^3 - 1)\Omega_0 \mathbf{v} \cdot \nabla$ is the global time derivative along ZAMO trajectories, $s = e, e^+$ is the plasma species, $\delta n_x$ is the density fluctuation of the plasma species, $n_0$ is the equilibrium plasma density, and $\rho_{GJ}$ is the Goldreich-Julian charge density as defined by equation (17). The system of equations (20)–(22) is equivalent to the following equation:

$$\frac{\partial}{\partial t^2} \left[\mathbf{V} \cdot \left(\frac{1}{\sqrt{1 - r_g/r}} \nabla \Phi\right) - 4\pi \rho_{GJ}\right] + \mathbf{V} \cdot \left(\omega_{p0} \sqrt{1 - r_g/r} \nabla \Phi\right) = 0.$$
where $\omega_{p_0}^2 = (8\pi n_0 e^2)/m$. Now, by defining the electric field arising from charge separation and the corotational electric field, which is the source of $\rho_{GJ}$,

$$E \equiv -\frac{1}{\sqrt{1 - r_g/r}} \nabla \Phi, \quad \rho_{GJ} = \frac{1}{4\pi} \nabla \cdot E_c,$$

(24)

and from equation (23) we find

$$\frac{\partial}{\partial t^2} [\nabla \cdot (E + E_c)] + \nabla \cdot \left[ \omega_{p_0}^2 \left( 1 - \frac{r_g}{r} \right) E \right] = 0,$$

(25)

which gives

$$\frac{\partial}{\partial t^2} E + \omega_{p_0}^2 \left( 1 - \frac{r_g}{r} \right) E = -\frac{\partial E_c}{\partial t^2}.$$

(26)

For $\partial E_c/\partial t^2 = 0$ we may write the solution of equation (26) as

$$E = E_0 \exp \left( -i\omega_{p_0} \sqrt{1 - \frac{r_g}{r}} t' \right).$$

(27)

From the above solution, we find that the plasma frequency in general relativity is now defined as

$$\omega_p^2 = \omega_{p_0}^2 \left( 1 - \frac{r_g}{r} \right),$$

(28)

which is equivalent to the gravitational redshift of the oscillation. Figure 4 shows the dependence of plasma oscillation on the distance away from the gravitational radius of the star.

The global time derivative along ZAMO trajectories is defined as

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \left( \frac{\kappa}{r^3} - 1 \right) \Omega_0 \frac{\partial}{\partial \phi}.$$

(29)

Thus, we may define

$$t' = t + \frac{\phi}{(\kappa/r^3 - 1)\Omega_0},$$

(30)

and hence the solution of linear plasma mode is

$$E(t, r, \phi) = E_0 \exp \left\{ -i\omega_{p_0} \sqrt{1 - \frac{r_g}{r}} \left[ t + \frac{\phi}{(\kappa/r^3 - 1)\Omega_0} \right] \right\}.$$

(31)

Introducing the dimensionless quantities $\epsilon = E/E_0$, $\tau = \omega_{p_0} t$, $\tilde{r} = r/r_g$, $\chi = \omega_{p_0}/\Omega_0$, and $\delta = \kappa/r^3$ from equation (31), we find

![Plot](image-url)
We do some analysis of the linear electrostatic modes around a rotating neutron star. First, we consider the Schwarzschild radius equal to half the radius of the neutron star, i.e., \( r_g = R/2 \). Then we find \( \delta = 8/5 \). Considering relatively dense plasma, we put \( \chi = 10^8 \). We consider a fixed azimuthal angle \( \phi = \pi/4 \). For \( \tau = 0 \) we plot the field \( \epsilon(\tau) \), which is shown in Figure 5. We find that the electrostatic field generated by Goldreich-Julian charge density is maximum near the star surface and falls quickly from the star.

5. NONLINEAR PLASMA MODES ALONG THE FIELD LINES

Now, we consider nonlinear plasma modes along the open field lines around a rotating neutron star. The system of equations governing the nonlinear modes can be written as

\[
\frac{\partial}{\partial t} (n_s) + \mathbf{V} \cdot \left( \sqrt{1 - \frac{r_g}{r}} n_s \mathbf{v}_s \right) = 0 ,
\]

\[
\left( \frac{\partial}{\partial \tau} + \sqrt{1 - \frac{r_g}{r}} \mathbf{v}_s \cdot \mathbf{V} \right) \mathbf{v}_s = - \frac{q_s}{m} \mathbf{V} \Phi ,
\]

\[
\mathbf{V} \cdot \left[ \frac{1}{(1-r_g/r)^{1/2}} \mathbf{V} \Phi \right] = -4\pi (\sum n_s q_s - \rho_{GJ}) .
\]

For simplicity, we consider \( \mathbf{v} \cdot \mathbf{V} \approx v_{sr} \cdot \mathbf{V} \partial / \partial r \) (i.e., one-dimensional wave propagation along \( r \)) and introduce a moving frame \( \eta = r - V\tau \), where \( V \) is a constant. In the considered moving frame, from equations (33) and (34) we get

\[
n_s = \frac{n_0 V}{V - (1-r_g/\eta)^{1/2} v_{sr}} ,
\]

\[
v_{sr} = \frac{1}{mV} \left( q_s \Phi + \sqrt{1 - \frac{r_g}{\eta} \frac{e^2}{2mV^2} \Phi^2} \right) .
\]

Using equations (36) and (37), in equation (35) we derive the nonlinear equation for the plasma mode along the field line of the rotating neutron star:

\[
\frac{d^2 \Phi}{d\eta^2} - \frac{r_g}{2\eta^2(1-r_g/\eta)} \frac{d\Phi}{d\eta} + \frac{\omega_p^2(1-r_g/\eta)}{V^2} \left( \frac{\Phi}{1 - 2(1-r_g/\eta)(e\Phi/mV^2)^2 + (1/4)(1-r_g/\eta)^2(e\Phi/mV^2)^2} \right) = 4\pi \sqrt{1 - \frac{r_g}{\eta} \rho_{GJ}} .
\]
Now, introducing dimensionless quantities
\[ \Phi = \frac{e\Phi}{mc^2}, \quad \hat{\eta} = \frac{\eta}{r_\theta}, \quad \hat{\omega}_{p0} = \frac{\omega_{p0} r_\theta}{V}, \]  (39)
we write equation (38) in dimensionless form:
\[ \frac{d^2\Phi}{d\hat{\eta}^2} - \frac{1}{2\hat{\eta}(\hat{\eta} - 1)} \frac{d\Phi}{d\hat{\eta}} + \hat{\omega}_{p0}^2 \left( 1 - \frac{1}{\hat{\eta}} \right) \frac{\Phi}{1 - 2(1 - 1/\hat{\eta})\Phi^2 + (1/4)(1 - 1/\hat{\eta})^2\Phi^4} = F_\epsilon, \]  (40)
where
\[ F_\epsilon = -\frac{3\Omega_\epsilon \Omega_* r_\theta^2}{2V^2} \left( \frac{R}{\eta} \right)^3 \sqrt{1 - \frac{1}{\hat{\eta}} [F_1(\hat{\eta}) \sin^2 \theta - F_2(\hat{\eta})(\sin^2 \theta - 2 \cos^2 \theta)]}. \]  (41)
Here \( \Omega_* = eB_0/mc; F_1(\hat{\eta}) \) and \( F_2(\hat{\eta}) \) are determined by equations (18) and (19), respectively.

We numerically solve equation (40) in the polar cap region (\( \theta \approx 0 \)) of a neutron star subject to the appropriate boundary conditions. Following Goldreich & Julian (1969) and Muslimov & Harding (1997), we assume that the surface of a polar cap and that formed by the last open field lines can be treated as electric equipotentials. We therefore adopt the condition \( \Phi(r = R) = 0 \). Second, we require that the steady state component of the electric field parallel to the magnetic field vanishes at the polar cap surface, i.e., \( d\Phi(r = R)/d\eta = 0 \). By considering \( r_\theta = R/2 \) for a neutron star, we write the boundary conditions as \( \Phi(2) = 0 \) and \( \Phi'(2) = 0 \). The solution of equation (40) with the mentioned boundary condition is shown graphically in Figure 6. We find that the Goldreich-Julian charge density creates the initial potential on the surface. Near the radius the potential is enhanced, and it propagates almost without damping along the field lines.

6. DISCUSSION AND CONCLUSION

We study the electrostatic plasma modes along the open field lines of a rotating neutron star. The dragging of inertial frame and the effect of general relativity are fully considered in this study. We perform a detailed analysis of Goldreich-Julian charge density in general relativity. Since pulsars having smaller obliquity have larger accelerating drops and thus are favored for \( \gamma \)-ray pulsar emissions (Muslimov 1995) and support the single pole \( \gamma \)-ray pulsar models (Daugherty & Harding 1994, 1996; Dermer & Sterner 1994), we confine our analysis to the zero inclination of the rotating neutron star. As pulsar radiation takes place in the plasma environment or the radiation passes through a plasma medium, we consider the electrostatic plasma modes along the open field lines. We study both the linear and nonlinear modes in the neutron star or black hole plasma environment. Our general conclusion from the above analysis may be summarized as follows:

1. Goldreich-Julian charge density is maximum in the polar cap region and remains almost the same in a certain extended region of the pole. The charge density decays with the distance away from the surface of the star.
2. Plasma oscillation along the field lines resembles the gravitational redshift near Schwarzschild radius.
3. Plasma modes grow near the gravitational radius in the black hole environment.
4. For initially zero field on the surface of a rotating neutron star, Goldreich-Julian charge density, which is enhanced near the surface and propagates almost without damping along the open field lines, creates the plasma modes.

For further study of plasma dynamics in the neutron star or black hole environment, we plan to extend our earlier investigations on soliton (see Moñiz 1989a, 1989b, 1990, 1997; Moñiz, de Angelis, & Fortani 1985; Moñiz, Tsintsadze, & Tsintsadze 1995) propagation along the open field lines of strongly gravitating objects.

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