SEP ARA TION THEOREMS FOR COMP ACT HAUSDORFF
FOLIATIONS

WOJCIECH KOZŁOWSKI AND SZYMON M. WALCZAK

ABSTRACT. We investigate compact Hausdorff foliations on compact Riemannian
manifolds in the context of the Gromov-Hausdorff distance theory. We give some
sufficient conditions for such foliations to be separated in the Gromov-Hausdorff
topology (GH-separation theorem).

1. INTRODUCTION

The concept of the Gromov-Hausdorff distance (briefly GH-distance), being a general-
ization of the notion of the Hausdorff distance, was originally introduced by M. Gromov
in the late 1970s. Next, Gromov, Katsuda, Peters and others showed that the GH-
distance theory applied to the Riemannian manifolds leads to remarkable results, e.g.,
the Cheeger’s finiteness theorem follows from the Gromov-Katsuda convergence theorem.

In [7] and [8], the second named author investigated warped compact Hausdorff fo-
liations from the GH-distance theory point of view. He gave a necessary and sufficient
conditions for a sequence of warped compact Hausdorff foliation to be converged to the
space of leaves with quotient metric.

In the light of the results there appears a natural question: Suppose that two compact
metric spaces \( X \) and \( X' \) are GH-close. Are always the compact Hausdorff folia-
tions \((M, \mathcal{F}, g)\) and \((M', \mathcal{F}', g')\) with space of leaves coinciding with \( X \) and \( X' \), respectively,
GH-close?

In this paper we show that for compact Hausdorff foliations the answer is negative
(GH-separation Theorem - Theorem 3 - the main result of the paper).

2. GROMOV-HAUSDORFF DISTANCE

Let \( C, K \subset X \) be compact subsets of a metric space \((X, d)\). The number
\[
d_H(C, K) = \inf \{ \epsilon > 0 : C \subset N(K, \epsilon) \land K \subset N(C, \epsilon) \},
\]
where \( N(A, \epsilon) = \{ x \in X : d(x, A) < \epsilon \} \), is called the Hausdorff distance of \( C \) and \( K \).
Let \((X, d_X)\) and \((Y, d_Y)\) be arbitrary compact metric spaces. Equip the disjoint union \(X \sqcup Y\) with an admissible metric \(d\), i.e., the metric which extends \(d_X\) and \(d_Y\). The Gromov-Hausdorff distance (cf. [1], [3] and [6]) \((\text{GH-distance})\) \(d_{\text{GH}}\) of the spaces \((X, d_X)\) and \((Y, d_Y)\) one can define as
\[
d_{\text{GH}}(X, Y) := \inf\{d_H(X, Y)\},
\]
where the infimum is taken over all admissible metrics on \(X \sqcup Y\). Note that two compact metric spaces are isometric iff their GH-distance equals zero. Consequently, GH-distance in the class of all classes of isometry of compact metric spaces with the GH-distance is a metric.

**Lemma 1.** If there exist \(\epsilon\)-nets \(\{x_1, \ldots, x_k\} \subset X\) and \(\{y_1, \ldots, y_k\} \subset Y\) satisfying for all \(1 \leq i, j \leq k\)
\[
|d_X(x_i, x_j) - d_Y(y_i, y_j)| \leq \epsilon
\]
then \(d_{\text{GH}}(X, Y) \leq 3\epsilon\).

**Proof.** For a proof we refer to [1]. \(\square\)

**Lemma 2.** If \(d_{\text{GH}}(X, Y) \leq \epsilon\) then for every \(\epsilon\)-net \(\{x_1, \ldots, x_k\} \subset X\) there exists a \(3\epsilon\)-net \(\{y_1, \ldots, y_k\} \subset Y\) such that \(|d_X(x_i, x_j) - d_Y(y_i, y_j)| \leq 2\epsilon\) for all \(1 \leq i, j \leq k\).

**Proof.** Since \(d_{\text{GH}}(X, Y) \leq \epsilon\) there exists an admissible metric \(d\) on \(X \sqcup Y\) such that the Hausdorff distance \(d_{\text{GH}}(X, Y) \leq d_H(X, Y) \leq \epsilon\). Let \(\{x_1, \ldots, x_k\}\) be an \(\epsilon\)-net on \(X\). For every \(i \in \{1, \ldots, k\}\) there exists \(y_i \in Y\), \(y_i \neq y_j\) while \(i \neq j\), such that \(d(x_i, y_i) \leq \epsilon\). Since \(d\) is an extension of metrics \(d_X\) and \(d_Y\), \(\{y_1, \ldots, y_k\}\) is a \(3\epsilon\)-net on \(Y\). Moreover,
\[
d_X(x_i, x_j) \leq d(x_i, y_i) + d(y_i, y_j) + d(y_j, x_j) \leq d_Y(y_i, y_j) + 2\epsilon,
\]
and similarly \(d_Y(y_i, y_j) \leq d_X(x_i, x_j) + 2\epsilon\). \(\square\)

Let \(\text{Cov}_\epsilon(X)\) denote the smallest number of open \(\epsilon\)-balls which covers \(X\), and \(\text{Cap}_\epsilon(X)\) the largest number of disjoint \(\epsilon\)-balls contained in \(X\). Obviously
\[
\text{Cov}_\epsilon(X) \leq \text{Cov}_\epsilon(X).
\]

**Lemma 3.** Let \((X, d)\) be a compact metric space. Then \(\text{Cov}_{2\delta}(M) \leq \text{Cap}_{2\delta}(M)\). More precisely, if \(x_1, \ldots, x_{\text{Cap}_{2\delta}(M)}\) are the centres of disjoint balls \(B(x_i, r)\) then the balls \(B(x_i, 2r)\) cover \(M\).

**Proof.** Let \(k = \text{Cap}_{2\delta}(X)\) and \(B(x_1, \delta), \ldots, B(x_k, \delta)\) be a family of open disjoint balls in \(X\). Let \(x \in X\). Then \(B(x, \delta) \cap B(x_i, \delta) \neq \emptyset\), and \(d(x, y) < \delta\) and \(d(y, x_i) < \delta\). Therefore, \(x \in B(x_i, 2\delta)\) for some \(i \in \{1, \ldots, k\}\). Hence, \(B(x_1, 2\delta), \ldots, B(x_k, 2\delta)\) cover \(X\). \(\square\)
Lemma 4. If $\text{Cov}_\varepsilon(X) < \text{Cap}_{3\varepsilon}(Y)$ then $d_{GH}(X,Y) > \varepsilon$.

Proof. Suppose that $d_{GH}(X,Y) \leq \varepsilon$. Let $\{x_1, \ldots, x_k\}$, $k = \text{Cov}_\varepsilon(X)$, be an $\varepsilon$-net on $X$. Then, by Lemma 2 there exists a $3\varepsilon$-net $\{y_1, \ldots, y_k\}$ on $Y$, so $\text{Cov}_{3\varepsilon}(Y) \leq \text{Cov}_\varepsilon(X)$. Thus

$$\text{Cov}_{3\varepsilon}(Y) \leq \text{Cov}_\varepsilon(X) < \text{Cap}_{3\varepsilon}(Y) \leq \text{Cov}_\varepsilon(Y).$$

Contradiction gives us the statement. □

Let $p \geq 0$. A Borel measure $\mu$ on a metric space $(X,d)$ is called a \textit{p-dimensional Bishop measure on $(X,d)$} if there exist constants $\beta \geq 1$ and $\eta_0 > 0$ such that for all $\eta < \eta_0$ and every $x \in X$

$$(1) \quad \frac{1}{\beta} \eta^p \leq \mu(B(x,\eta)) \leq \beta \eta^p,$$

where $B(x,\eta) = \{ y \in X : d(x,y) < \eta \}$.

Let $(X,d)$ be a length-space, i.e. $d(x,y) = \inf\{l(\gamma)\}$, where $\gamma : [0,1] \to X$ is a curve such that $\gamma(0) = x$, $\gamma(1) = y$, and $l(\gamma)$ denotes the length of $\gamma$.

Lemma 5. If the balls $B(x,\delta)$ and $B(y,\delta)$ are disjoint, then $d(x,y) \geq 2\delta$.

Proof. Suppose that $d(x,y) < 2\delta$. Let $\gamma : [0,1] \to X$ be a curve from $x = \gamma(0)$ to $y = \gamma(1)$ with its length $l(\gamma) < 2\delta$. Let $t_0 \in [0,1]$ be such that

$$l(\gamma|[0,t_0]) = l(\gamma|[t_0,1]) = \frac{1}{2}l(\gamma) \leq \delta.$$

If $z = \gamma(1/2)$ then $d(x,z) < \delta$ and $d(z,y) < \delta$. Thus, $z \in B(x,\delta) \cap B(y,\delta)$. Contradiction ends our proof. □

Lemma 6. Let $(X,d)$ be a compact length space, $p \geq 1$, and let $\mu$ be a \textit{p-dimensional Bishop measure on $(X,d)$} with constants $\beta > 1$, $\eta_0 > 0$. There exist positive constants $C \geq 1$ and $\theta > 0$ such that for every $0 < r < \theta$ and $x \in M$,

$$\frac{1}{C r^p} \mu(X) \leq \text{Cap}_r(X) \leq \frac{C}{r^p} \mu(X).$$

Proof. Let $0 < r < \eta_0$, $k = \text{Cap}_r(X)$, and let $B(x_1,\delta), \ldots, B(x_k,\delta)$ be a family of open disjoint balls in $X$. By (1), $\mu(B(x_i,r)) \geq \beta^{-1} r^p$, and

$$(2) \quad \mu(X) \geq \sum_{i=1}^{k} \mu(B(x_i,r)) \geq k \cdot \frac{1}{\beta} r^p.$$

Let $0 < r < \eta_0/2$. By Lemma 3 we have

$$(3) \quad \mu(X) \leq \sum_{i=1}^{k} \mu(B(x_i,2r)) \leq k \cdot \beta (2r)^p.$$

Putting $C = \beta 2^p$ and $\theta = \eta_0/2$, (2) and (3) give us the statement. □
Corollary 1. Let $0 < r < \theta$ and $\alpha > 0$ be such that $\alpha r < \theta$. Then

$$\alpha^{-p}C^{-2}\text{Cap}_r(X) \leq \text{Cap}_{\alpha r}(X) \leq \alpha^{-p}C^2\text{Cap}_r(X).$$

Remark 1. Note that the volume form on a $n$-dimensional compact Riemannian manifold defines an $n$-dimensional Bishop measure.

3. Compact Hausdorff foliations

A foliation with all leaves compact is called a compact foliation. Let us consider any compact foliation $\mathcal{F}$ on a manifold $M$, and let $\pi : M \to \mathcal{L}$ denote a quotient map onto the space of leaves $\mathcal{L}$, this means that $\pi$ identifies each leaf to a point. The space of leaves often is non-Hausdorff. Due to the results by D.B.A. Epstein [2], we recall theorems that describe the topology of such foliation:

Theorem 1. The following conditions are equivalent.

(i) $\pi$ is a closed map.
(ii) $\pi$ maps compact sets onto closed sets.
(iii) Each leaf has arbitrarily small saturated neighbourhoods.
(iv) $\mathcal{L}$ with quotient topology is Hausdorff.
(v) If $K \subseteq M$ is compact, then the saturation of $K$ is also compact.

Proof. For a proof we refer to [2], Theorem 4.1.

Let $M$ be a Riemannian manifold and $N$ a submanifold on $M$. One can consider the induced Riemannian structure on $N$ and introduce a volume of $N$ as it’s volume $\text{vol}_N$ in the induced Riemannian structure. The next theorem describes the relation between the volume of the leaves defined above (briefly the volume function), the holonomy group of a leaf, and the topology of the space of leaves of a foliation $\mathcal{F}$ on a Riemannian manifold $(M, g)$.

Theorem 2. If $(M, \mathcal{F}, g)$ is a foliated Riemannian manifold and $L$ is a compact leaf of $\mathcal{F}$, then the following conditions are equivalent.

(i) There exists a saturated neighbourhood $N$ of the leaf $L$ such that the volume function is bounded on $N$.
(ii) The holonomy group of $L$ is finite.

Proof. For a proof we refer to [2], Theorem 4.2.

The conditions of Theorem 2 imply that some saturated neighbourhood $U$ of a compact leaf $L$ consists of compact leaves, and in $U$ the conditions of Theorem 1 are satisfied in $U$. 
Moreover, by Reeb Stability Theorem, on a foliated manifold the conditions of Theorem 1 imply the conditions of Theorem 2.

A compact foliation which space of leaves is Hausdorff is called compact Hausdorff foliation. As an easy corollary of the above theorems we have:

**Corollary 2.** Let \((M, \mathcal{F}, g)\) be a compact Riemannian manifold carrying compact Hausdorff foliation. Then \(\sup_{L \in \mathcal{F}} \text{vol}(L) < \infty\).

Now, let us consider the space of leaves \(L\) of an arbitrary compact Hausdorff foliation on a compact Riemannian manifold. Let us introduce on \(L\) a metric \(\rho\) defined by

\[
\rho(L, L') = \inf \left\{ \sum_{i=1}^{n-1} \text{dist}(L_i, L_{i+1}) \right\},
\]

where \(L_1 = L, L_n = L'\), and the infimum is taken over all finite sequences of leaves. One can see that for a compact Riemannian foliation \(\mathcal{F}\) the distance \(\rho\) coincides with Hausdorff distance of leaves of \(\mathcal{F}\).

**Remark 2.** Let \(g, g'\) be two Riemannian metrics on a compact foliated manifold \((M, \text{mathcalF})\), where \(\mathcal{F}\) is a compact Hausdorff foliation. Denote by \(\rho\) and \(\rho'\) two metrics on the space of leaves constructed using \(g\) and \(g'\), respectively. Since \(M\) is compact, then \(\frac{1}{C}g \leq g' \leq Cg\) for some constant \(C \geq 1\). One can check that

\[
\frac{1}{C}\rho \leq \rho' \leq C\rho.
\]

In further considerations we will need the following:

**Lemma 7.** For every compact Hausdorff foliation \(\mathcal{F}\) on a compact Riemannian manifold \((M, g)\) there exists Riemannian structure \(\tilde{g}\) on \(M\) such that \(\mathcal{F}\) becomes a Riemannian foliation, and for any leaf \(L \in \mathcal{F}\) we have \(\tilde{g}|L = g|L\).

*Proof. Obvious. See [5].* □

4. **Separation Theorem**

We say that a compact metric space \((X', d')\) is broader than a metric space \((X, d)\), and we briefly write \(X' \succeq X\), if \(\text{Cap}_\delta(X') \geq \text{Cap}_\delta(X)\) for all \(\delta > 0\).

Let \(d > 0\) be a real number. Let us denote by \(\mathcal{M}(d, C, p, n)\) the class of all \(n\)-dimensional compact foliated Riemannian manifolds \((M, \mathcal{F}, g)\) carrying a compact Hausdorff foliation of dimension \(p\) satisfying:

1. For any leaf \(L \in \mathcal{F}\), \(\epsilon < d\), and any two balls \(B_L(x, \epsilon), B_L(y, \epsilon)\) that are disjoint in \(L\), the balls \(B(x, \epsilon)\) and \(B(y, \epsilon)\) are disjoint in \(M\);
(2) \( \max_{L \in \mathcal{F}} C_L \leq C, \min_{L \in \mathcal{F}} \theta_L \leq C, \) \( C_M \leq C, \theta_M \leq C, \) where \( C_L, \theta_L, C_M, \theta_M \) are the constants mentioned in Lemma [3] for a leaf \( L \) of \( \mathcal{F} \) and for the manifold \( M \), respectively;
(3) \( \frac{1}{C} \leq \text{vol}(L) \leq C \) for all \( L \in \mathcal{F} \);
(4) There exists a Riemannian structure \( \tilde{g} \) on \( M \) satisfying \( \frac{1}{C} g \leq \tilde{g} \leq C g \) such that on \( (M, \tilde{g}) \) the foliation \( \mathcal{F} \) becomes a compact Riemannian foliation.

Let \( d > 0, C \geq 1, \) and let \( p, p', n, n' \in \mathbb{N} \) be such that \( p' > p \) and \( n' \geq n \).

**Theorem 3.** [GH-separation Theorem] There exists \( \epsilon > 0 \) such that for any \( (M, \mathcal{F}, g) \in \mathcal{M}(d_0, C, p, n) \) and \( (M', \mathcal{F}', g') \in \mathcal{M}(d_0, C, p', n') \) such that \( (M'/\mathcal{F}', p') \succeq (M/\mathcal{F}, \rho) \) we have \( d_{\text{GH}}(M, M') > \epsilon \).

**Proof.** Let \( \mathcal{L} = M/\mathcal{F}, \mathcal{L}' = M'/\mathcal{F}', \) and let \( \pi : M \to \mathcal{L} \) and \( \pi' : M' \to \mathcal{L}' \) denote the natural projections. Let \( r < \min\{d, C\}/3C \), and let \( B(x_1, r/2), \ldots, B(x_k, r/2), k = \text{Cap}_{r/2}(\mathcal{L}), \) be a family of open disjoint balls in \( \mathcal{L} \). Since \( \mathcal{L}' \succeq \mathcal{L} \) we can choose points \( x_1', \ldots, x_k' \) in \( \mathcal{L}' \) such that the balls \( B(x_1', r/2), \ldots, B(x_k', r/2) \) are also disjoint.

Now, in every leaf \( L_i' = (\pi')^{-1}(x_i') \) let us choose points
\[ \xi_{i_1}, \ldots, \xi_{i_l'}, \]
where \( i' = \min_{L \in \mathcal{F}'} \text{Cap}_{r/2}(L) > 0, \) such that the balls \( B_{L_i'}(x_1', r/2), \ldots, B_{L_i'}(x_k', r/2) \) are disjoint. Since \( r < d \), the balls \( B(\xi_{i_1}', r/2) \) are disjoint in \( M' \). Consequently, by Lemma [6] we have
\[ \text{Cap}_{2r}(M') \geq k \cdot l' \geq k \cdot \frac{2^{p'}}{\max_{L \in \mathcal{F}'} C_{L'}^{p'}} \min_{L \in \mathcal{F}} \text{vol}'(L) \geq \frac{k}{C^2} \frac{2^{p'}}{r^{p'}} \]
where \( \text{vol}'(L) \) denote the volume of a leaf in the induced Riemannian structure.

Now, let \( \tilde{g} \) be a Riemannian structure on \( M \) mentioned in Lemma [7] such that \( \mathcal{F} \) becomes a Riemannian foliation and such that
\[ \frac{1}{C} g \leq \tilde{g} \leq C g. \]
Let us choose in \( L_i = \pi^{-1}(x_i) \) points \( \xi_{i_1}, \ldots, \xi_{i_l}, \) \( l_i = \text{Cap}_{r/2}(L_i) \) such the balls \( B_{L_i}(\xi_{i,j}, r/2) \) are pairwise disjoint on \( (L_i, \tilde{g}|_{L_i}) \). By [5] and Remark [2] the balls \( B(\xi_{i,j}, Cr) \) covers \( (M, g), \) and
\[ \text{Cov}_{Cr}(M) \leq k \cdot l, \]
where \( l = \max_{i \in \{1, \ldots, k\}} \{l_i\} \). Moreover, by [5] and Lemma [6]
\[ l \leq \max_{L \in \mathcal{F}} \text{Cap}_{C}(L) \leq \max_{L \in \mathcal{F}} \frac{(2C)^p}{r^p} \cdot \text{vol}(L) \leq \frac{2^p C^{p+2}}{r^p}, \]
and
\[ \text{Cov}_{Cr}(M) \leq k \cdot \frac{2^p C^{p+2}}{r^p} =: B(r). \]
By Corollary 1,
\[ \text{Cap}_{3C \cdot r}(M') \geq \frac{1}{(6C)^{n'} \cdot (C)^2} \text{Cap}_{2}(M'). \]

Next, by (4),
\[ \text{Cap}_{3C \cdot r}(M') \geq \frac{k}{(6C)^{n'} \cdot C^4} \cdot \frac{2p'}{r^{p'}} =: A(r) \]

It follows that
\[ \frac{A(r)}{B(r)} = \beta \cdot r^{p-p'}. \]

where \( \beta \) depends only on \( C, p', \) and \( n' \). Since \( p < p' \) then \( \lim_{r \to 0} \frac{A(r)}{B(r)} = +\infty. \) Hence, there exists \( \epsilon_0 < r \) such that \( \text{Cap}_{3C \epsilon_0}(M') \geq A(r) > B(r) \geq \text{Cov}_{C \epsilon_0}(M). \) By Lemma 2, we obtain \( d_{GH}(M, M') > C \epsilon_0. \)

**References**

[1] D.-P. Chi and G. Yun, Gromov-Hausdorff Topology and its Applications to Riemannian Manifolds, Seoul National University, (1998).
[2] D.B.A. Epstein, Foliations with all leaves compact, Ann. Inst. Fourier Grenoble 26 (1976), 265-2822.
[3] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser, (1999).
[4] A. Katsuda, Gromov's convergence theorem and its application, Nagoya Math. J. Vol. 100 (1985), pp. 11-48.
[5] I. Moerdijk and J. Mrcun, Introduction to Foliations and Lie Groupoids, Cambridge University Press, (2003).
[6] P. Petersen, Riemannian Geometry, Springer, (1997).
[7] Sz. M. Walczak, Collapse of warped foliations, Diff. Geom. App., 25/6 (2007), 649-654.
[8] Sz. M. Walczak, Warped compact foliations, Preprint of Faculty of Mathematics, University of Łódź, 2006/07.