PASSIVE ADVECTION AND THE DEGENERATE ELLIPTIC OPERATORS $M_n$

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Abstract. We prove estimates for the stationary state $n$-point functions at zero molecular diffusivity in the Kraichnan model $[13]$. This is done by proving upper bounds for the heat kernels and Green’s functions of the degenerate elliptic operators $M_n$ that occur in the Hopf equations for the $n$-point functions.

1. Introduction

The Kraichnan model of passive advection is an exactly solvable model that has a very similar phenomenology to the full Navier-Stokes turbulence, but is much simpler in many respects. I’ll only give a very short reminder for the reader. More detailed introductions to the problem we are addressing can be found e.g. in $[9]$ and $[14]$. See also $[7]$, $[15]$ and $[16]$.

Let $T(t, x) \in \mathbb{R}, x \in \mathbb{R}^d$ be a scalar quantity satisfying
\[ \partial_t T = \kappa \Delta T - v \cdot \nabla T + f. \]

In the Kraichnan model we take $v$ and $f$ random, decorrelated in time, independent and Gaussian with mean zero and covariances
\[ \langle v^\alpha(t_1, x_1)v^\beta(t_2, x_2) \rangle = D^{\alpha\beta}(x_1 - x_2)\delta(t_1 - t_2) \]
and
\[ \langle f(t_1, x_1)f(t_2, x_2) \rangle = C(x_1 - x_2)\delta(t_1 - t_2). \]

Here the $v \cdot \nabla T$ should be interpreted in the Stratonovich sense. The incompressibility of the velocity field $v$ is guaranteed by taking
\[ D^{\alpha\beta}(x) = \int e^{-ik \cdot x} D(|k|) \left( \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right) dk \]

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where $D$ is smooth, nonnegative and of compact support in $(0, \infty)$. A
$D$ that mimics turbulent velocities is

\begin{equation}
D(|k|) = |k|^{-(d+2)} \chi \left( |k| \eta + \frac{1}{|k|} \ell \right)
\end{equation}

with $\chi$ smooth, $\chi = 1$ in a neighbourhood of the origin and $\chi(x) = 0$
for $x > 1$. The idea is that $D$ behaves like $|x|^\xi$ in the so-called inertial
range $\eta << |x| << \ell$. The number $\eta$ is called the Kolmogorov scale and
$\ell$ is called the inertial scale. We let $\tilde{C} \in C_0^\infty(\mathbb{R})$ with a nonnegative
Fourier transform and $C := \tilde{C}(\cdot/L)$, with $L > 0$.

One is interested in the statistics of $T(t, x)$ as $t \to \infty$. Let

\begin{equation}
F_n(t, x_1, ..., x_n) := \langle T(t, x_1) ... T(t, x_n) \rangle.
\end{equation}

Given (1.2) and (1.3) the $n$-point functions $F_n$ of the scalar $T$ obey the
so-called Hopf equations (see [16]):

\begin{equation}
\partial_t F_n(t, x_1, ..., x_n) = -M_n F_n(t, x_1, ..., x_n) + 
\sum_{1 \leq i < j \leq n} F_{n-2}(t, x_1, ..., \hat{x_i}, \hat{x_j}, x_n) C(x_i - x_j),
\end{equation}

with

\begin{equation}
M_n := -\sum_{1 \leq i < j \leq n} \sum_{1 \leq \alpha, \beta \leq d} D^{\alpha \beta}(x_i - x_j) \frac{\partial^2}{\partial x_i^\alpha \partial x_j^\beta} - \kappa \sum_{1 \leq i \leq n} \Delta_i.
\end{equation}

The fact that the Hopf equation for $F_n$ does not contain $F_m$ with
$m > n$ makes it easy to solve these equations inductively. The situation
here differs drastically from full Navier-Stokes turbulence, where the
Hopf equation for $F_n$ contains also $F_{n+1}$.

$M_n$ is an elliptic operator and in terms of its heat kernel $F_n$ (with
zero initial condition for simplicity) is given by

\begin{equation}
F_2(t, x) = \int_{t_0}^t ds \int dy \ e^{-(t-s)M_2(x, y)} C(y_1 - y_2)
\end{equation}

\begin{equation}
F_{2n}(t, x) = \sum_{1 \leq i < j \leq 2n} \int_{t_0}^t ds \int dy \ e^{-(t-s)M_{2n}(x, y)} \cdot F_{2n-2}(s, y_1, ..., \hat{y_i}, \hat{y_j}, ..., y_{2n}) C(y_i - y_j) dy
\end{equation}

with vanishing odd correlators.

As $t_0 \to -\infty$ these have the stationary limit

\begin{equation}
F_2 = \int dy \ (M_2)^{-1}(x, y) C(y_1 - y_2)
\end{equation}
(1.12) \( F_{2n} = \sum_{1 \leq i < j \leq 2n} \int (M_{2n})^{-1}(x, y) F_{2n-2}(y_1, \ldots, y_{2n}) C(y_i - y_j) \, dy \).

One is interested in the study of \( F_{2n} \) for \( \eta \) small, \( \ell \) large, \( \kappa \) small and \( L \) large. In this paper we prove bounds for these directly in the limit \( \eta = 0, \ell = \infty \) and \( \kappa = 0 \) with fixed \( L \), say \( L = 1 \). Our methods also allow the study of the limit \( \eta \to 0, \ell \to \infty \) and \( \kappa \to 0 \).

A comment on \( D \) is now in place. While sending \( \eta \to 0 \) and \( \ell \to \infty \) in \( D \), we get into trouble with \( \ell \), since \( D \) diverges as \( \ell \to \infty \). Fortunately it doesn’t matter: Let

(1.13) \( d^{\alpha\beta}(x) := \int dk (1 - e^{ik \cdot x}) D(|k|) \left( \delta^{\alpha\beta} - \frac{k^{\alpha} k^{\beta}}{k^2} \right) \).

Now (1.8) can be written in the following form:

(1.14) \( M_n := \sum_{1 \leq i, j \leq n} \sum_{1 \leq \alpha, \beta \leq d} d^{\alpha\beta}(x_i - x_j) \frac{\partial^2}{\partial x_i^{\alpha} \partial x_j^{\beta}} - \kappa \Delta - D_0 \left( \sum_{1 \leq i \leq n} \sum_{1 \leq \alpha \leq d} \frac{\partial}{\partial x_i^{\alpha}} \right)^2 \).

In (1.7) \( M_n \) acts on translationally invariant functions, so the last term drops out and the rest has a limit as \( \ell \to \infty \).

Finally, here’s our main Theorem, proved directly at \( \eta = 0, \ell = \infty \) and \( \kappa = 0 \):

Theorem 1.1.

(1.15) \( F_{2n}(x) \leq C_n \sum_{\pi} \prod_{\{i, j\} \in \pi} (1 + |x_i - x_j|)^{2-d} \),

where the sum is over pairings of \( \{1, \ldots, 2n\} \).

2. Preliminaries

This section fixes the notation and discusses the results from other papers ([3], [6], [11], [19]) used in this paper. There is an overview of this paper in [3], so the reader might want to start there.

2.1. Degenerate elliptic operators in divergence form. Let \( \Omega \subset \mathbb{R}^n \) be a domain. We shall be interested in second order differential operators in divergence form, i.e. in operators \( H \) of the form \( H = -\nabla \cdot A \nabla \), where \( A \) is a locally square integrable function from \( \Omega \) to real symmetric positive \( n \times n \) matrices with locally square integrable distributional derivative, i.e. \( A \in W_{1,2}^{1,2}(\Omega, \mathbb{M}^n) \). One can make sense of more general operators, but this is not relevant to the results presented in this paper.
Definition 2.1. Let $H$ and $A$ be as above. The matrix $A$ is called the symbol of $H$, and we denote $\sigma(H) := A$. The function $w^H_1(x) := \inf_{v \in \mathbb{R}^{n-1}} \langle v, \sigma(H)(x)v \rangle$ (resp. $w^H_2(x) := \sup_{v \in \mathbb{R}^{n-1}} \langle v, \sigma(H)(x)v \rangle$) is called the greatest lower bound (resp. least upper bound) of the symbol.

We shall also use $\sigma(H)$ to denote the quadratic form $\langle v, A(x)v \rangle$. The usage will be clear from the context. We often speak loosely and forget the attributes “greatest” and “lowest” from the bounds.

If $A$ and $B$ are two symbols and $U \subseteq \mathbb{R}^n$, we shall denote $A \sim^\lambda B$ on $U$, if $\lambda A \leq B \leq \lambda^{-1} A$ a.e. on $U$. If there is $\lambda > 0$ so that $A \sim^\lambda B$ on $U$ we also say $A \sim B$ on $U$. If “on $U$” is dropped, we refer to whole $\mathbb{R}^n$.

We shall use $\mathbb{1}$ to denote the identity matrix. Thus a symbol $A$ is uniformly elliptic iff $A \sim \mathbb{1}$. Moreover, if $A$ and $B$ are symbols on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, then $A \oplus B$ is just the natural symbol on $\mathbb{R}^{n_1+n_2}$.

Definition 2.2. Let $w$ be a nonnegative locally integrable function (a weight) defined on $\mathbb{R}^n$. We denote $w(A) := \int_A w(x) \, dx$. The function $w$ is called a doubling weight (resp. an $A_2$-weight), if there is a constant $C$ such that for every ball $B \subset \mathbb{R}^n$ we have $w(2B) \leq C w(B)$ (resp. $\frac{1}{|B|^q} w(B) w^{-1}(B) \leq C$).

Since by Schwartz inequality $|B|^2 \leq w(B) w^{-1}(B)$, we have $|2B|^2 = 2^{2n} |B|^2 \leq 2^{2n} w(B) w^{-1}(B) \leq 2^{2n} w(B) w^{-1}(2B)$, so we can conclude that an $A_2$-weight is also a doubling weight.

Definition 2.3. Denote $u_B := |B|^{-1} \int_B u(x) \, dx$ and let $w_1, w_2$ be weights on $\mathbb{R}^n$ and let $q > 2$. We say that the Poincaré inequality (resp. Sobolev inequality) holds for $w_1, w_2$ with $q$, if there is $C < \infty$ so that for every ball $B \subset \mathbb{R}^n$ and $u \in W^{1,2}(B)$ (resp. $u \in W^{1,2}_0(B)$) we have

$$\left( w_2(B)^{-1} \int_B |u - u_B|^q w_2 \right)^{1/q} \leq C |B|^{1/n} \left( w_1(B)^{-1} \int_B |\nabla u|^2 w_1 \right)^{1/2} \quad \text{(2.1)}$$

(resp.)

$$\left( w_2(B)^{-1} \int_B |u|^q w_2 \right)^{1/q} \leq C |B|^{1/n} \left( w_1(B)^{-1} \int_B |\nabla u|^2 w_1 \right)^{1/2} \quad \text{(2.2)}$$
Theorem 2.4. (Harnack inequality) Suppose $H := -\nabla \cdot A \nabla$ is a divergence form operator with $w_1 \leq A \leq w_2$ and suppose that the weights $w_1$ and $w_2$ satisfy the following:

1. $w_1$ and $w_2$ are in $A^2$,
2. The Poincaré inequality holds for $w_1$, $w_2$ with some $q > 2$ and
3. The Poincaré inequality holds for $w_1$, $1$ with some $q' > 2$.

Let $t_0, ..., t_4 \in \mathbb{R}$ with $t_0 < ... < t_4$, $\Omega \subseteq \mathbb{R}^n$ open and $K \subseteq \Omega$ compact and connected. Let $u$ be a strictly positive solution to $u_t + Hu = 0$ in $\Omega \times (t_0, t_4)$. Then there is a constant $C < \infty$ depending on $\Omega$, $K$ and $t_0, ..., t_4$, but on $A$ only through the bounds $w_1$ and $w_2$ so that

$$\text{ess sup}_{K \times (t_1, t_2)} u \leq C \text{ess inf}_{K \times (t_3, t_4)} u$$

Proof. This is just Theorem A of [11] supplemented with a covering argument from [17], pages 734-736.

Remark 2.5. For the purposes of Theorem 2.4 the concept of $u$ being a solution of $u_t + Hu = 0$ on $Q := \Omega \times (t_0, t_4)$ means exactly the following:

1. $u \in L^2(Q)$,
2. $u_t \in L^2(Q)$,
3. $|\nabla u|^2 w_2 \in L^1(Q)$ and
4. For all $\phi \in C^1_0(Q)$ we have

$$\int_Q u_t \phi + \langle A \nabla u, \nabla \phi \rangle \, dx \, dt = 0$$

We are going to apply the Harnack inequality only to heat kernels of some degenerate elliptic operators. In particular as long as $t_0 > 0$ all the above items will hold.

Since the heat kernel is a positive distribution, it is a measure and follows from the fact that the heat kernel is a distributional solution of the corresponding degenerate heat equation.

First of all (1) holds because for $t_0 > 0$ the heat kernel is a bounded function on $\Omega \times (t_0, t_4)$ (by Corollary 4.22).

Secondly (2) holds because of the following computation which is justified by Remark 2.8

$$\langle \partial_t K \rangle(s, \cdot, y) = -HK(s, \cdot, y)$$

$$= -e^{-(s-t_0)H}He^{-t_0H/2}K(t_0^{1/2}, \cdot, y).$$

Now since by Remark 2.7 $e^{-tH}$ is a contraction on $L^2$,

$$\sup_{s \in (t_0, t_4)} \|\partial_t K(s, \cdot, y)\| < \infty.$$
Let $A$ be the symbol of $H$. To prove (3) it suffices to show that $|\nabla K|$ is locally in $L^2$, since $w_2$ is locally bounded. Since

$$\int_Q |\nabla K|^2 \, dx \, dt \leq \int_Q w_1^{-1} \langle A\nabla K, \nabla K \rangle \, dx \, dt. \quad (2.7)$$

Since $w_1$ is in $A_2$ (by Lemma A.2), $w_1^{-1}$ is locally integrable, so it suffices to prove that $\langle A\nabla K, \nabla K \rangle$ is essentially bounded on $Q$. We show that for any $0 \leq \phi \in C_0^\infty (Q)$ we have

$$\int_Q \phi \langle A\nabla K, \nabla K \rangle \leq C \int_Q \phi, \quad (2.8)$$

with $C$ not depending on $\phi$.

So we compute using the facts that $K$ and $\nabla \cdot A\nabla K$ are locally bounded:

$$\int_Q \phi \langle A\nabla K, \nabla K \rangle = \int_Q K \nabla \cdot \phi A\nabla K \leq C \left| \int_Q \phi \nabla \cdot A\nabla K \right| \leq 2C \left| \int_Q \phi \nabla \cdot A\nabla K \right| \leq C' \int_Q \phi. \quad (2.9)$$

It follows from the results in §4.2 and Appendix A that this Harnack inequality holds for the operators $M_n$, which will be our main interest and will be defined in §2.3.

2.2. Gaussian upper bounds for heat kernels. The material in this section is mostly taken from [6]. For more information, see sections 1.3, 2.4 and 3.2 there. See also [4] and [19].

Definition 2.6. Let $H \geq 0$ be a real self-adjoint operator on $L^2(\mathbb{R}^n)$. We call the semigroup $e^{-Ht}$ a symmetric Markov semigroup, if it is positivity-preserving and a contraction on $L^\infty(\mathbb{R}^n)$.

Remark 2.7. By saying that $e^{-Ht}$ is a contraction on $L^p$ with $p \neq 2$ we mean that $e^{-Ht}$ is a contraction on $L^p \cap L^2$ and can be extended to a unique contraction on $L^p$. In the case of $L^\infty$ we have to impose the extra condition of weak* continuity to achieve uniqueness since $L^\infty \cap L^2$ is not norm dense in $L^\infty$.

Remark 2.8. A symmetric Markov semigroup is strongly continuous on $L^p$ with $1 \leq p < \infty$, see Theorem 1.4.1 of [6]. This in particular implies that the generator $H$ commutes with the semigroup $e^{-Ht}$ (see [4]).
By Theorem 1.3.5 of [6], any self-adjoint divergence form operator with non-negative symbol and core $C_0^\infty(\mathbb{R}^n)$ gives rise to a symmetric Markov semigroup. The Theorem there is stated for “elliptic” operators, but the proof works for any non-negative symbol. The keywords here are self-adjointness and core $C_\infty^0$. Both follow for $M_n$ from the fact that $\sigma(M_n) \in \mathcal{W}^{1,2}_{\text{loc}}(\mathbb{R}^{n-1})$ (Proposition 4.3). See Theorem 1.2.5 of [6].

**Definition 2.9.** Let $e^{-Ht}$ be a symmetric Markov semigroup on $L^2(\mathbb{R}^n)$. We say that $e^{-Ht}$ is **ultracontractive** if the map $e^{-Ht}$ is bounded from $L^2$ to $L^\infty$ for every $t > 0$.

**Definition 2.10.** Suppose that $C_\infty^0(\mathbb{R}^n) \subseteq \text{Dom}(H)$. Let $e^{-Ht}$ be a symmetric Markov semigroup on $L^2(\mathbb{R}^n)$. We say that $e^{-Ht}$ (or $H$ or $\sigma(H)$) is of dimension $\mu$ if there is $C_2 < \infty$ such that for all $t > 0$ and $f \in L^2(\mathbb{R}^n)$ we have

$$\|e^{-Ht}f\|_\infty \leq C_2 t^{-\mu/4} \|f\|_2.$$  

Note that the dimension of a semigroup need not be unique.

There is a standard method for obtaining global Gaussian upper bounds for heat kernels of divergence form operators with nonnegative symbols using global space-independent bounds. A good reference for this is [4].

**Definition 2.11.** Let $A$ be a symbol on $\mathbb{R}^n$. The function

$$d_A(x, y) := \sup \{ |\phi(x) - \phi(y)| : \phi \text{ is } C^\infty \text{ and bounded with } \langle \nabla \phi, A \nabla \phi \rangle \leq 1 \text{ on } \mathbb{R}^n \}$$

is called the metric associated with $A$ (or $H$, if $H := -\nabla \cdot A \nabla$ or $e^{-tH}$ or the heat kernel of $H$).

The following Theorem was proved by Davies [4].

**Theorem 2.12.** Let $\mu$ be a positive real number. Suppose $H := -\nabla \cdot A \nabla \geq 0$ is a positive self-adjoint divergence form operator with $e^{-Ht}$ a symmetric Markov semigroup of dimension $\mu$. Then for each $\delta > 0$ there is $C_\delta < \infty$ such that the heat kernel $K$ of $e^{-Ht}$ satisfies

$$0 \leq K(t, x, y) \leq C_\delta t^{-\mu/2} \exp\left\{ -\frac{d_A(x, y)^2}{4(1+\delta)t} \right\}$$

for all $0 < t < \infty$ and $x, y \in \mathbb{R}^n$. Besides $\delta$, $C_\delta$ depends only on $\mu$ and the constant $C_2$ of Definition 2.10.

**Proof.** See [4].
We shall use the following Theorem later to get the dimension of $M_n$ in Corollary 4.22. It was proved by Varopoulos [19]. John Nash [18] also proved a similar result.

**Theorem 2.13.** Suppose $C_0^\infty(\mathbb{R}^n) \subseteq \text{Dom}(H)$. Let $e^{-Ht}$ be a symmetric Markov semigroup on $L^2(\mathbb{R}^n)$ and let $\mu > 2$ be given. Then there is $C_1 < \infty$ such that for all $f \in C_0^\infty(\mathbb{R}^n)$ we have

$$\|f\|_{2^\mu/(\mu-2)} \leq C_1 \langle f, Hf \rangle.$$  \hfill (2.13)

if and only if there is $C_2 < \infty$ such that for all $t > 0$ and $f \in L^2(\mathbb{R}^n)$ we have

$$\|e^{-Ht}f\|_{\infty} \leq C_2 t^{-\mu/4}\|f\|_2.$$  \hfill (2.14)

Here the constants $C_1$ and $C_2$ depend only on each other and the number $\mu$.

**Proof.** This is just Theorem 2.4.2 of [3]. \qed

**Remark 2.14.** One can show using the Schwartz Kernel and Radon-Nikodym Theorems that a bounded linear map $L : L^1 \to L^\infty$ has a integral kernel that is a function in $L^\infty$ whose $L^\infty$-norm equals the operator norm of $L$. Since our $e^{-Ht}$ is self-adjoint, boundedness of $e^{-Ht} : L^2 \to L^\infty$ implies boundedness of $e^{-Ht} : L^1 \to L^2$, so in this case we have a heat kernel that is a genuine function.

Finally, we give a nice way to estimate heat kernels of operators $H$ that have symbols satisfying $\sigma(H) \sim A_1 \oplus A_2$.

**Theorem 2.15.** Suppose that for $i = 1, 2$, $A_i$ is a symbol on $\mathbb{R}^{n_i}$ such that $e^{i\nabla \cdot A_i \nabla}$ is a symmetric Markov semigroup on $L^2(\mathbb{R}^{n_i})$ and $B \sim^\lambda A_1 \oplus A_2$. Suppose also that the heat kernels of $A_i$’s satisfy

$$K_{A_i}(t, x, y) \leq C_i t^{-\mu_i/2} \exp\left\{-\frac{d_{A_i}(x, y)^2}{C_i t}\right\}.$$  \hfill (2.15)

Then there is $C < \infty$ depending only on $C_1$, $C_2$, $\mu_1$, $\mu_2$ and $\lambda$ so that the heat kernel of $B$ satisfies

$$K_B(t, x, y) \leq C t^{-\mu_1+\mu_2/2} \exp\left\{-\frac{d_{A_1}(x, y)^2 + d_{A_2}(x, y)^2}{C t}\right\}.$$  \hfill (2.16)

**Proof.** Since $K_{A_1 \oplus A_2}(t, (x_1, x_2), (y_1, y_2)) = K_{A_1}(t, x_1, y_1)K_{A_2}(t, x_2, y_2)$, we can conclude that

$$K_{A_1 \oplus A_2} \leq C_1 C_2 t^{-\mu_1+\mu_2/2},$$  \hfill (2.17)

which by Riesz-Thorin interpolation theorem and the fact that $e^{i\nabla \cdot A_1 \oplus A_2 \nabla}$ is a contraction $L^\infty$ imply (2.14) for $H = -\nabla \cdot A_1 \oplus A_2 \nabla$. Therefore
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by Theorem 2.13

\[(2.18) \quad ||f||^2_{2\mu/(\mu-2)} \leq C_3 \langle \nabla f, (A_1 \oplus A_2) \nabla f \rangle \]

for any \(f \in C^\infty_0(\mathbb{R}^{n_1+n_2})\) with \(C_3\) depending only on \(C_1 C_2 \) and \(\mu_1 + \mu_2\). Since \(A_1 \oplus A_2 \leq \lambda^{-1}B\), we have

\[(2.19) \quad K_B \leq C_4 r^{-\frac{\mu_1+\mu_2}{2}}, \]

with \(C_4\) depending only on \(C_1 C_2 \), \(\mu_1 + \mu_2\) and \(\lambda\). We now apply Theorem 2.12 to conclude the claim. \(\square\)

2.3. The definition of the operators \(M_n\). For the rest of the paper, we fix a constant \(\xi, 0 < \xi < 2\) and an integer \(d \geq 2\). Here \(d\) is the dimension of the “physical” space.

Next, we overload the symbol \(d\) immediately and let \(d\) be the map from \(\mathbb{R}^d\) to \(d \times d\) matrices defined by

\[(2.20) \quad d(x) := C \int_{\mathbb{R}^d} \frac{1 - \cos(k \cdot x)}{|k|^{d+\xi}} (1 - \hat{k} \otimes \hat{k}) \, dk, \]

with

\[(2.21) \quad C := \frac{(4\pi)^{d/2}2^\xi \Gamma((d + \xi + 2)/2)}{(d-1)\Gamma((2-\xi)/2)}. \]

A computation (see e.g. [8]) shows that

\[(2.22) \quad d(x) = |x|^\xi \left(1 + \frac{\xi}{d-1}\right) (1 - \frac{\xi}{d-1} \hat{x} \otimes \hat{x}). \]

In the following definition, we denote vectors in \(\mathbb{R}^{nd}\) by \(\{v_i\}_{i=1}^n\), where each \(v_i\) is a vector in \(\mathbb{R}^d\).

**Definition 2.16.** Let \(n \geq 2\). The operator \(\mathcal{M}_n^{sc} := -\nabla \cdot \sigma(\mathcal{M}_n^{sc}) \nabla\) is the one with the symbol

\[(2.23) \quad \sigma(\mathcal{M}_n^{sc}) := -\sum_{1 \leq i < j \leq n} \langle v_i, d(x_i - x_j) v_j \rangle \]

If \(a \in \mathbb{R}^d\), we denote the vector \((x_i + a)_{i=1}^n\) by \(x + a\). We call a function \(f : \mathbb{R}^{nd} \to \mathbb{R}\) translationally invariant, if for every \(a \in \mathbb{R}^d \) and \(x \in \mathbb{R}^{nd}\) we have \(f(x) = f(x + a)\).

We shall be interested in \(\mathcal{M}_n^{sc}\) acting on translationally invariant functions, so we need to reduce the number of total space dimensions to \((n-1)d\).
In other words, we set $x_i := y_i - y_{i+1}$ for $1 \leq i \leq n - 1$, so
\begin{equation}
\frac{\partial}{\partial y_i^\alpha} = \begin{cases}
\frac{\partial}{\partial x_i^\alpha} & \text{if } i = 1, \\
\frac{\partial}{\partial x_i^\alpha} - \frac{\partial}{\partial x_{i-1}^\alpha} & \text{if } 2 \leq i \leq n - 1 \text{ and} \\
-\frac{\partial}{\partial x_{n-1}^\alpha} & \text{if } i = n.
\end{cases}
\end{equation}

Denote the symbol obtained in this way by $\sigma(M_n)$. A simple calculation shows that $\sigma(M_n)$ equals
\begin{equation}
\sum_{i=1}^{n-1} \sum_{j=i}^{n-1} \langle v_i, (d(\sum_{k=i}^j x_k) - d(\sum_{k=i}^j x_k) + d(\sum_{k=i+1}^j x_k)) v_j \rangle
\end{equation}
In particular,
\begin{equation}
\sigma(M_2) = \langle v_1, d(x_1)v_1 \rangle,
\end{equation}
\begin{equation}
\sigma(M_3) = \langle v_1, d(x_1)v_1 \rangle + \langle v_2, d(x_2)v_2 \rangle + \\
\langle v_1, (d(x_1 + x_2) - d(x_1) - d(x_2))v_2 \rangle
\end{equation}
and
\begin{equation}
\sigma(M_4) = \langle v_1, d(x_1)v_1 \rangle + \langle v_2, d(x_2)v_2 \rangle + \langle v_3, d(x_3)v_3 \rangle + \\
\langle v_1, (d(x_1 + x_2) - d(x_1) - d(x_2))v_2 \rangle + \\
\langle v_2, (d(x_2 + x_3) - d(x_2) - d(x_3))v_3 \rangle + \\
\langle v_1, (d(x_1 + x_2 + x_3) - d(x_1 + x_2) - d(x_2 + x_3) + d(x_2))v_3 \rangle
\end{equation}

3. Overview

Our intent here is to give some intuition on the arguments of this paper and how they lead to the proof of Theorem 1.1. What is obvious at first sight, is that if Theorem 1.1 is to hold, the Green’s functions of the operators $M_{2n}$ should be locally integrable in the sense that for every $n \geq 2$ there is $C < \infty$ so that for every $x \in \mathbb{R}^{(2n-1)d}$ we have
\begin{equation}
\int_{B(x,1)} d(2n-1)d y G_{M_{2n}}(x, y) < C.
\end{equation}

One might hope to get (3.1) to hold using the heat kernel estimate of Theorem 2.12, but unfortunately this direct approach fails. First of all we see that $\sigma(M_2) \sim | \cdot |^{\frac{d}{2}}$. Applying Definition 2.11, Corollary 4.22 and Theorem 2.12 to this, we find a $C < \infty$ such that
\begin{equation}
K_{M_2}(t, x, y) \leq Ct^{-\frac{d}{2}} \exp\{-\frac{|x-y|^2}{Ct}\}
\end{equation}
for $|x| = 1$ and $|x - y| \leq \frac{1}{2}$. Integrating with respect to $t$ from 0 to $\infty$ we get

$$G_{M_2}(x, y) \leq C'|x - y|^{2 - \frac{2d}{2 - \xi}}. \tag{3.3}$$

This estimate yields (3.1) only when $2 - \frac{2d}{2 - \xi} > -d$, that is $\xi < \frac{4}{d + 2}$. We might be satisfied with the fact that (3.1) holds only for small $\xi$, but there is worse to come: For each $\sigma(M_n)$ will have points $x \in \mathbb{S}^{(n-1)d-1}$ so that $\sigma(M_n) \sim 1$ in a neighbourhood $U$ of $x$. A similar argument as above now yields

$$G_{M_{2n}}(x, y) \leq C'|x - y|^{2 - \frac{2(n-1)d}{2 - \xi}} \tag{3.4}$$

for $y \in U$. This yields (3.1) for $M_n$ only when $\xi < \frac{4}{(n-1)d + 2}$, which means trouble: Given $\xi$ with $0 < \xi < 2$, there will always some be $N$ so that our argument above fails to give local integrability for $M_n$ with $n \geq N$.

On the other hand, since $M_2$ is uniformly elliptic in a neighbourhood $U$ of $x$, the heat kernel of $M_2$ should behave like the heat kernel of the Laplacian for small times and small distances from $x$.

Turning this analysis into formulas let’s suppose

$$K_{M_2}(t, x, y) \leq C_2 t^{-\frac{d}{2}} \exp\left\{-\frac{|x - y|^2}{C_2 t}\right\} \tag{3.5}$$

for $|x| = 1$, $|x - y| \leq \epsilon \leq \frac{1}{2}$ and $0 < t \leq t_0$. Since there is $C_3 < \infty$ so that $t^{-\frac{d}{2}} \leq C_3 t^{-\frac{d}{2}}$ for $t \geq t_0$, we can combine (3.2) with (3.5) and conclude that

$$K_{M_2}(t, x, y) \leq C_4 t^{-\frac{d}{2}} \exp\left\{-\frac{|x - y|^2}{C_4 t}\right\} \tag{3.6}$$

for $|x| = 1$, $|x - y| \leq \epsilon$ and $0 < t < \infty$. Now an integration w.r.t. $t$ from 0 to $\infty$ yields

$$G_{M_2}(x, y) \leq C_5 |x - y|^{2 - d} \tag{3.7}$$

for $|x| = 1$ and $|x - y| \leq \epsilon$. The same holds for $M_n$ with $n > 2$. This leads us to a further twist: for $n > 2$, $\sigma(M_n)$ has degeneracies also outside of the origin, but fortunately in the end these turn out not to be problematic.

A few words on the structure of the rest of the paper. In §4 the symbols of $M_n$ are analyzed in detail. The local analysis of the heat kernels is done in §3. Theorem 1.1 is proved in §6. Finally, there are three appendices containing technicalities.
4. The operators \( M_n \)

From now on, we live in \( \mathbb{R}^{(n-1)d} \) and denote vectors of \( \mathbb{R}^{(n-1)d} \) with \( \mathbf{v} = (v_i)_{i=1}^{n-1} \) and \( \mathbf{x} = (x_i)_{i=1}^{n-1} \), where \( v_i, x_i \in \mathbb{R}^d \).

The symbol of \( M_n \) has a bunch of useful symmetries, inherited from \( M_{sc}^n \). For \( L : \mathbb{R}^k \to \mathbb{R}^l \) a surjective linear mapping and \( A \) a symbol on \( \mathbb{R}^k \) which for all \( x \in \mathbb{R}^k \) is constant on \( \{ x \} + \ker L \) denote \( A_L(x) := LA(L^{-1}x)L^T \), where \( L^{-1} \) is some right-inverse of \( L \). Let \( L_n : \mathbb{R}^{nd} \to \mathbb{R}^{(n-1)d} \) be given by the matrix \( (L_n)_{ij} := \delta_{ij} - \delta_{i+d,j} \), so that \( \sigma(M_n) = \sigma(M_{sc}^n)L_n \).

We let

\[
\mathcal{L}_n = \{ L_n LL_n^{-1} : L \text{ is a permutation of the coordinate axes of } \mathbb{R}^{nd} \}.
\]

Now \( \sigma(M_n)L = \sigma(M_n) \) for every \( L \in \mathcal{L}_n \).

**Remark 4.1.** Let \( A_1 \) and \( A_2 \) be two symbols on \( \mathbb{R}^k \) and let \( G_1, G_2 \subseteq \text{GL(} \mathbb{R}^k \text{)} \) be their respective symmetry groups, i.e.

\[
G_i := \{ L \in \text{GL(} \mathbb{R}^k \text{)} : A_i^L = A_i \},
\]

for \( i \in \{1, 2\} \). Now if \( A_1 \sim A_2 \) on \( U \), then \( A_1 \sim A_2 \) on \( LU \) for any \( L \in G_1 \cap G_2 \).

**Remark 4.2.** A simple calculation shows that \( M_n \) is degenerate, whenever

\[
\sum_{i=a}^{b} x_i = 0, \text{ where } 1 \leq a \leq b \leq n-1.
\]

In fact these are the only points where \( M_n \) degenerates, as we see in Theorem 4.7. To avoid lengthy statements in the rest of the paper, we denote \( \{ x \in \mathbb{R}^{(n-1)d} : x_i = 0 \} \) by \( \{ x_i = 0 \} \) and similarly for the other sets.

**Proposition 4.3.**

\[
\sigma(M_n) \in W^{1,2}_{\text{loc}}(\mathbb{R}^{(n-1)d})
\]

**Proof.** The case \( 1 < \xi < 2 \) is an easy computation, since then \( \sigma(M_n) \) is continuously differentiable.

In case \( 0 < \xi \leq 1 \), we let

\[
F := \bigcup_{1 \leq a \leq b < n} \{ \sum_{i=a}^{b} x_i = 0 \}.
\]

A relatively simple calculation shows that there is \( C < \infty \) such that

\[
|\nabla(\sigma(M_n))(\mathbf{x})| \leq Cd(\mathbf{x}, F)^{\xi-1}.
\]

Since \( F \) is a finite union of vector subspaces of codimension \( d \geq 2 \), we can conclude that \( d(\mathbf{x}, F)^{\xi-1} \) is a locally square integrable function. \( \square \)
Remark 4.4. It is trivial to get an upper bound for $M_n$:

$$\sigma(M_n) \leq \left( \sup_{|y| = |w| = 1} \langle w, \sigma(M_n)(y)w \rangle \right) |x|^\xi |v|^2.$$  \hfill (4.6)

We obtain a better upper bound in section §4.2.

Proposition 4.5. For any $\epsilon \in (0, 1)$ there is $C < \infty$ such that

$$d\sigma(M_n)(x, y) \leq C |x - y|^{1-\xi/2},$$

when $|x - y| \geq \epsilon |x|$. \hfill (4.7)

Proof. By Definition 2.11 and Remark 4.4 it suffices to show that there is $C < \infty$ such that $d|\cdot|_\xi(x, y) \leq C |x - y|^{1-\xi/2}$, when $|x - y| \geq \epsilon |x|$. Trivial dimensional analysis gives $d|\cdot|_\xi(x, y) = |x|^\xi |v|^2$. Therefore we may assume $|x| = 1$. By rotational symmetry, we may fix $x$. By scaling, there is $C' < \infty$ so that $C'|y|^{1-\xi/2} = d|\cdot|_\xi(0, y)$. Since now

$$d|\cdot|_\xi(x, y) = C' d|\cdot|_\xi(0, x - y)^2,$$\hfill (4.8)

it suffices to show that $f(R) := \sup_{|x - y| = R} d|\cdot|_\xi(x, y)/d|\cdot|_\xi(0, x - y)$ is a bounded function of $R$ for $R \in [\epsilon, \infty)$. Obviously $f$ is continuous. By continuity of $d|\cdot|_\xi$ we have

$$d|\cdot|_\xi(x, y)^2/d|\cdot|_\xi(0, x - y)^2 = d|\cdot|_\xi(x, y)^2/d|\cdot|_\xi(0, x - y)^2 \to d|\cdot|_\xi(0, y)^2/d|\cdot|_\xi(0, y)^2 = 1$$

as $|x - y| \to \infty$. \hfill \Box

4.1. Fourier integral representation and the degeneration set.

Definition 4.6. Let $A$ be a symbol. We call the set

$$(4.10) \quad D\text{gn}(A) := \{ x \in \mathbb{R}^n : A(x) is not invertible \}$$

the degeneration set of $A$.

The following Fourier integral representation of the symbol is crucial for the computation of the degeneration sets of $M_n$ (which then implies corresponding properties for the operators $M_n$ to be introduced later).

Theorem 4.7. The degeneration set of $M_n$ is

$$(4.11) \quad D\text{gn}(M_n) = \bigcup_{1 \leq i \leq j < n} \{ x \in \mathbb{R}^{(n-1)d} : |x_i + ... + x_j| = 0 \}.$$
Proof. By Remark 1.2, it suffices to show that for every \( \mathbf{v} \in \mathbb{R}^{nd} \) with \( \sum_{i=1}^{n} v_i = 0 \) we have \( - \sum_{1 \leq i < j \leq n} \langle v_i, d(x_i - x_j)v_j \rangle > 0 \) whenever \( x_i \neq x_j \) for all \( 1 \leq i < j \leq n \).

We have

\[
- \sum_{1 \leq i < j \leq n} \langle v_i, d(x_i - x_j)v_j \rangle = -\frac{1}{2} \sum_{1 \leq i, j \leq n} \langle v_i, d(x_i - x_j)v_j \rangle
\]

\[
= -\frac{C}{2} \int_{\mathbb{R}^d} \text{Re} \left( \sum_{1 \leq i, j \leq n} \frac{1 - e^{ik(x_i - x_j)}}{|k|^{d+\xi}} \langle v_i, (1 - k \otimes k)v_j \rangle \right) dk
\]

\[
= \frac{C}{2} \int_{\mathbb{R}^d} \text{Re} \left( \sum_{i=1}^{n} v_i e^{ik \cdot x_i} \sum_{i=1}^{n} v_i e^{ik \cdot x_i} \right) dk
\]

The rest goes as in Proposition 1 of [8]: For the integral to be zero, we have to have

\[
\sum_{i=1}^{n} v_i e^{ik \cdot x_i} = \alpha(k)k
\]

almost everywhere for some scalar function \( \alpha \). Taking the exterior product (i.e. the antisymmetric part of the tensor product) with respect to \( k \) and Fourier transforming in the sense of distributions we arrive at

\[
\sum_{i=1}^{n} v_i \wedge \nabla \delta(x - x_n) = 0.
\]

Thus for any smooth test function \( \phi \)

\[
\sum_{i=1}^{n} v_i \wedge \nabla \phi(x_n) = 0.
\]

This is a contradiction since the values of \( \nabla \phi \) can be arbitrarily specified on a discrete set and the \( x_n \)'s are all distinct. \( \square \)

4.2. Estimates for the symbol of \( M_n \). We shall now show that the symbol of \( M_n \) can be estimated using the symbols of \( M_m \), \( m \in \{2, \ldots, n - 1\} \).

**Definition 4.8.** Let \( x \in \mathbb{R}^{(n-1)d} \). The dimension of the zero eigenspace of \( \sigma(M_n) \) at \( x \) divided by \( d \) is called the rank of the point \( x \) and denoted \( \text{rk}(x) \). In particular \( x \) is a degeneration point of \( \sigma(M_n) \) iff \( \text{rk}(x) > 0 \).

Below, for a symbol \( A \) and invertible linear transformation \( L \) we define the symbol \( A^L \) by the formula \( A^L(x) := LA(L^{-1}x)L^T \).
**Theorem 4.9.** Let \( n \geq 2 \) and \( x \in \mathbb{S}^{(n-1)d-1} \). Then either \( M_n \) is uniformly elliptic in some neighbourhood of \( x \) or there is a invertible linear transformation \( L \) of \( \mathbb{R}^{(n-1)d} \), a neighbourhood \( U \) of \( Lx \) so that \( \sigma(M_n)^L \sim \bigoplus_{i=1}^k \sigma(M_{n_i}) \oplus 1 \) on \( U \) with \( k \geq 1 \), each \( n_k \geq 2 \), \( \text{rk}(x) = \sum_{i=1}^k (n_k - 1) < n - 1 \) and \( (Lx)_i = 0 \) for \( 1 \leq i \leq \sum_{j=1}^k (n_k - 1) \).

Let’s introduce some convenient notation at this point. First of all \( \{i, j\} := \{i, \ldots, j\} \). Let \( A \subseteq \{1, n\} \). Then we write

\[
\begin{align*}
    x_A &:= \sum_{i \in A} x_i \\
    \gamma_A &:= \langle v_{\min, A}, (d(x_A) - d(x_{A \setminus \{\min, A\}})) \\
               &\quad - d(x_{A \setminus \{\max, A\}}) + d(x_{A \setminus \{\min A, \max A\}}) \rangle v_{\max, A} \\
    \sigma_A &:= \sum_{i, j \in A : i \leq j} \gamma_{A \{i,j\}}.
\end{align*}
\]

Moreover \( x_{i,j} := x_{\{i,j\}}, \sigma_{i,j} := \sigma_{\{i,j\}}, \gamma_{i,j} := \gamma_{\{i,j\}} \) and \( \sigma_i := \gamma_i = \gamma(i) \).

### 4.3. Two propositions for the proof of Theorem 4.9

Our purpose here is to prove Proposition 4.10 and Proposition 4.17. Let us illustrate what we’re going to do by studying \( \sigma(M_3) \) in some detail.

Let \( x \in \mathbb{S}^{2d-1} \) be such that \( x_1 = 0 \), i.e. \( x = (x_1, x_2) \) with \( x_2 \in \mathbb{S}^{d-1} \). We’ll show that there is a neighbourhood \( U \) of \( x \) and \( C < \infty \) so that for every \( y \in U \) we have

\[
\frac{1}{C}(|y_1|^\xi|v_1|^2 + |y_2|^\xi|v_2|^2) \leq \sigma(M_3)(y) \leq C(|y_1|^\xi|v_1|^2 + |y_2|^\xi|v_2|^2).
\]

Let \( E \) be given by Lemma 4.11 and let \( \epsilon \in (0, \frac{1}{4}) \) be such that

\[
E((2\epsilon)^{1-x/2} + (2\epsilon)^{x/2}) \leq \frac{1}{2}
\]

and let

\[
U := B(0, \epsilon) \times \left\{ \frac{1}{2} < |y_2| < \frac{3}{2} \right\}.
\]

By our choice of \( \epsilon \) we have

\[
|\gamma_{1,2}| \leq \frac{1}{2}(|y_1|^\xi|v_1|^2 + |y_2|^\xi|v_2|^2)
\]

in \( U \). In other words (4.17) holds and thus \( \sigma(M_3) \sim \sigma(M_2) \oplus 1 \) on \( U \).

Proposition 4.10 will be used when we have several (or all) coordinates away from the degeneration set. As might be guessed from our calculation with \( \sigma(M_3) \), the point of Lemmata 4.11,4.14 is that in the
Then there is $C < \infty$ such that if $1 \leq i < n$ and $|x_i| < \frac{1}{2}|x_{i+1}|$, then

$$|\gamma_{i,j}| \leq \text{something} \cdot (|x_i|^\xi |v_i|^2 + |x_j|^\xi |v_j|^2).$$

We have neatly blackboxed all this mess into Proposition 4.17; the Lemmata of this section are not directly used in the proof of Theorem 4.9. The proofs can be found in Appendix B.

**Proposition 4.10.** Suppose $n \geq 1, \epsilon \in (0, 1)$ and let

$$A := \{x \in \mathbb{R}^n : \epsilon \max\{|x_{i,j}| : 1 \leq i \leq j \leq n\} \leq \min\{|x_{i,j}| : 1 \leq i \leq j \leq n\}\}.$$

Then there is $C < \infty$ so that for every $x \in A$ we have

$$\frac{1}{C} \sum_{i=1}^{n} |x_i|^\xi |v_i|^2 \leq \sigma(M_{n+1}) \leq C \sum_{i=1}^{n} |x_i|^\xi |v_i|^2.$$

**Lemma 4.11.** There is a constant $E < \infty$ such that if $1 \leq i < n$ and $|x_i| < \frac{1}{2}|x_{i+1}|$, then

$$|\langle v_i, (d(x_i + x_{i+1}) - d(x_i) - d(x_{i+1}))v_{i+1} \rangle| \leq E\left(\left(\frac{|x_i|}{|x_{i+1}|}\right)^{1-\xi/2} + \left(\frac{|x_i|}{|x_{i+1}|}\right)^{\xi/2}\right)(|x_i|^\xi |v_i|^2 + |x_{i+1}|^\xi |v_{i+1}|^2).$$

**Lemma 4.12.** There is a constant $E < \infty$ such that if $1 \leq i < i+1 < j \leq n, |x_i| < \frac{1}{2} \min\{|x_{i+1,j}|, |x_{i+1,j-1}|\}$ and $|x_j| > 0$, then

$$|\langle v_{i,j}, (d(x_{i,j}) - d(x_{i,j+1}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j \rangle| \leq E\left(\left(\frac{|x_i|}{|x_{i+1,j}|}\right)^{1-\xi/2} \left(\frac{|x_{i+1,j}|}{|x_j|}\right)^{\xi/2} + \left(\frac{|x_i|}{|x_{i+1,j-1}|}\right)^{1-\xi/2} \left(\frac{|x_{i+1,j-1}|}{|x_j|}\right)^{\xi/2}\right) \cdot (|x_i|^\xi |v_i|^2 + |x_j|^\xi |v_j|^2).$$

**Lemma 4.13.** There is a constant $E < \infty$ such that if $1 \leq i < i+1 < j \leq n, \frac{1}{2}|x_{i+1,j-1}| \leq |x_i| < \frac{1}{2}|x_{i+1,j}|$ and $|x_j| > 0$, then

$$|\langle v_i, (d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j \rangle| \leq E\left(\left(\frac{|x_i|}{|x_{i+1,j}|}\right)^{1-\xi/2} \left(\frac{|x_{i+1,j}|}{|x_j|}\right)^{\xi/2} + \left(\frac{|x_i|}{|x_{i+1,j-1}|}\right)^{\xi/2}\right)(|x_i|^\xi |v_i|^2 + |x_j|^\xi |v_j|^2).$$
Lemma 4.14. There is $E < \infty$ so that if $1 \leq i < i + 1 < j \leq n$ and $
abla x_{i} \frac{1}{\max\{|x_{i+1,j-1}|\}}$, we have

\begin{equation}
|v_{i}(d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_{j}| \leq E\left(\frac{|x_{i}|}{|x_{i+1,j-1}|}\right)^{1-\varepsilon/2}\left(\frac{|x_{i}|}{|x_{i+1,j-1}|}\right)^{1-\varepsilon/2}(|x_{i}|^{2}v_{i}^{2} + |x_{j}|^{2}v_{j}^{2}).
\end{equation}

We still have one more Lemma to go before we can start proving Proposition 4.14. We’ll illustrate it with $\sigma(M_{6})$. Let $\mathbf{x} \in \mathbb{S}^{d-1}$ with $|x_{1}| = |x_{3}| = |x_{5}| = 0$ and $|x_{2}|, |x_{4}|, |x_{2,4}| > 0$. By Proposition 4.11, $\sigma_{(2,4)}(y_{2}, y_{4})$ behaves like $|y_{2}|^{\xi}|v_{2}|^{2} + |y_{4}|^{\xi}|v_{4}|^{2}$ in a neighbourhood of $(x_{2}, x_{4})$. Unfortunately the relevant part of $\sigma(M_{6})$ is $\gamma_{2} + \gamma_{4} + \gamma_{2,4}$, but at least we would have some hope, if we could get an estimate of the form

\begin{equation}
|\gamma_{2,4} - \gamma_{(2,4)}| \leq \text{something} \cdot (|y_{2}|^{\xi}|v_{2}|^{2} + |y_{4}|^{\xi}|v_{4}|^{2})
\end{equation}

for $\mathbf{y}$ in a neighbourhood of $\mathbf{x}$.

This is the point of Lemma 4.13. More precisely, let

\begin{equation}
\mu := \min\{|y_{2}|, |y_{4}|, |y_{2,4}|\} \leq \max\{|y_{2}|, |y_{4}|, |y_{2,4}|\} =: \nu
\end{equation}

and let $C < \infty$ be such that if $\frac{\nu}{2} < |y_{2}|, |y_{4}|, |y_{2} + y_{4}| < 2\nu$ we have

\begin{equation}
\frac{1}{C}(|y_{2}|^{\xi}|v_{2}|^{2} + |y_{4}|^{\xi}|v_{4}|^{2}) \leq \sigma_{(2,4)} \leq C(|y_{2}|^{\xi}|v_{2}|^{2} + |y_{4}|^{\xi}|v_{4}|^{2}).
\end{equation}

Let $\epsilon \in (0, \frac{\nu}{6})$ be such that

\begin{equation}
E\left(\frac{2\epsilon}{\mu}\right)^{1-\varepsilon/2}\left(\frac{4\nu}{\mu}\right)^{\varepsilon/2} + \left(\frac{2\epsilon}{\mu}\right)^{\varepsilon/2} \leq \frac{1}{2C}.
\end{equation}

Let

\begin{equation}
U := \{|y_{3}| < \epsilon \text{ and } \frac{\mu}{2} < |y_{2}|, |y_{4}|, |y_{2} + y_{4}| < 2\nu\}.
\end{equation}

By Lemma 4.15 for $\mathbf{y} \in U$ we have

\begin{equation}
|\gamma_{2,4} - \gamma_{(2,4)}| \leq \frac{1}{2C}(|y_{2}|^{\xi}|v_{2}|^{2} + |y_{4}|^{\xi}|v_{4}|^{2}).
\end{equation}

Combining 4.13 with 4.30 we conclude that $\gamma_{2} + \gamma_{4} + \gamma_{2,4}$ behaves like $|y_{2}|^{\xi}|v_{2}|^{2} + |y_{4}|^{\xi}|v_{4}|^{2}$ in $U$.

Again, the proof of the following Lemma can be found in Appendix 3.

Lemma 4.15. There is $E < \infty$ such that if $1 \leq i < j \leq n$ and 
\{i, j\} $\subseteq A$ $\subseteq [i, j]$ and if $\sum_{k \in [i, j]}|x_{k}| \leq \frac{1}{2} \min\{|x_{k}| : k, l \in A, k \leq l\}$

\begin{align*}
&\text{Lemma 4.15. There is } E < \infty \text{ such that if } 1 \leq i < j \leq n \text{ and } \\
&\{i, j\} \subseteq A \subseteq [i, j] \text{ and if } \sum_{k \in [i, j]} |x_{k}| \leq \frac{1}{2} \min\{|x_{k}| : k, l \in A, k \leq l\}.
\end{align*}
Then
\begin{equation}
|\gamma_{i,j} - \gamma_A| \leq E \left( \frac{\sum_{k \in [i,j] \setminus A} |x_k|}{\min \{|x_k| : k, l \in A, k \leq l\}} \right)^{1-\xi/2} \left( \frac{\sum_{k \in \pi} |x_k|}{|x_l|} \right)^{\xi/2}
+ \left( \frac{\sum_{k \in [i,j] \setminus A} |x_k|}{\min \{|x_k| : k, l \in A, k \leq l\}} \right)^{\xi/2} (|x_l|^\xi |v_i|^2 + |x_j|^\xi |v_j|^2).
\end{equation}

If $L \in \text{GL}(\mathbb{R}^{(n-1)d})$, we shall use the following somewhat weird notation: If $x \in \mathbb{R}^{(n-1)d}$, we let $Lx_i := (Lx)_i$ for $1 \leq i \leq n-1$. Similarly, we let $Lx_{i,j} := (Lx)_{i,j}$ for $1 \leq i < j \leq n-1$.

**Remark 4.16.** Let $x$ be a degeneration point of $\sigma(M_n)$. We claim that there is a symmetry $L \in \mathcal{L}_n$ and $A \nsubseteq \{1, ..., n-1\}$ so that $|Lx_i| = 0$ if $i \in A$ and $Lx_{i,j} > 0$ if $\{i, ..., j\} \nsubseteq A$. This is easy to see, if we look at the original symbol $\sigma(M_n^e)$. Then the claim above simply says that if we have points $y_1, ..., y_n \in \mathbb{R}^d$, then there is a permutation $\pi \in S_n$ so that if $y_{\pi(i)} = y_{\pi(j)}$ with $\pi(i) \leq \pi(j)$, then $y_{\pi(i)} = y_k$ with every $k$ with $\pi(i) \leq k \leq \pi(j)$. Still in other words: if we pick $n$ possibly coinciding points from $\mathbb{R}^d$, we can label them with numbers $1, ..., n$ so that the coinciding points get consecutive numbers as labels.

Given $x$ and $A$ as above, write $A$ as
\begin{equation}
\{i_1, ..., j_1\} \cup ... \cup \{i_m, ..., j_m\}
\end{equation}
with $i_1 \leq j_1 < j_1 + 1 < i_2 \leq ... < i_m \leq j_m$ and write $\sigma(M_n)$ as
\begin{equation}
\sigma(M_n) = \sum_{l=1}^{m} \sigma_{i_l,j_l} + \sigma_{A^c} + \sum_{i,j \in A^c} \gamma_{i,j} - \sigma_{A^c} + \text{ the rest}.
\end{equation}

Let $\mu := \min \{|x_{i,j}| : \{i, ..., j\} \nsubseteq A\}$ and $\nu := \max \{|x_{i,j}| : \{i, ..., j\} \nsubseteq A\}$.

**Proposition 4.17.** For any $C > 0$ there is a neighbourhood $U$ of $x$ so that
\begin{equation}
|\sum_{i,j \in A^c} \gamma_{i,j} - \sigma_{A^c} + \text{ the rest} | \leq \frac{1}{2C} \sum_{i=1}^{n} |y_i|^\xi |v_i|^2
\end{equation}
for any $y \in U$.

**Proof.** For $\epsilon > 0$ let
\begin{equation}
U^\epsilon := \{y \in \mathbb{R}^{nd} : |y_{i,j}| < \epsilon \text{ if } \{i, ..., j\} \subseteq A \text{ and } \mu/2 < |y_{i,j}| < 2\nu \text{ otherwise}\}.
\end{equation}
Let $N := \frac{n(n-1)}{2}$ be the number of terms in $\sigma(M_n)$. We’ll find $\epsilon > 0$ so that each term in (4.37) is $\leq C < \infty$ and we shall accomplish this by proving in parallel that there is a $(\epsilon)$.

Theorem 4.9. We shall prove this Theorem by induction on $n$.

Proof.

The proof of Theorem 4.9.

(1) Lemma 4.11: $\epsilon < \frac{\mu}{4}$ and $E((\frac{2\mu}{\mu})^{1-\xi/2} + (\frac{2\mu}{\mu})^{\xi/2}) \leq \frac{1}{2NC}$

(2) Lemma 4.12: $\epsilon < \frac{\mu}{4}$ and $2E((\frac{2\mu}{\mu})^{1-\xi/2} + (\frac{4\mu}{\mu})^{\xi/2}) \leq \frac{1}{2NC}$.

(3) Lemma 4.13: $\epsilon < \frac{\mu}{4}$ and $E((\frac{2\mu}{\mu})^{1-\xi/2} + (\frac{2\mu}{\mu})^{\xi/2}) \leq \frac{1}{2NC}$.

(4) Lemma 4.14: $\epsilon < \frac{\mu}{6}$ and $E(2\mu^{2-\xi} \leq \frac{1}{2NC}$.

(5) Lemma 4.15: $\epsilon < \frac{\mu}{4}$ and $E((\frac{2\mu}{\mu})^{1-\xi/2} + (\frac{4\mu}{\mu})^{\xi/2} + (\frac{2\mu}{\mu})^{\xi/2}) \leq \frac{1}{2NC}$.

4.4. The proof of Theorem 4.9.

Proof. (of Theorem 4.9) We shall prove this Theorem by induction on $n$ and we shall accomplish this by proving in parallel that there is a constant $C < \infty$ so that for any $x \in \mathbb{R}^{(n-1)d}$ there is $K \in \mathcal{L}_n$ so that

$$\frac{1}{C} \sum_{i=1}^{n-1} |Kx_i|^\xi |v_i|^2 \leq \sigma(M_n) \leq C \sum_{i=1}^{n-1} |Kx_i|^\xi |v_i|^2.$$

This is trivial for $\sigma(M_2)$. We assume now that the claim above is true for $\sigma(M_m)$, $2 \leq m < n$ and prove it for $\sigma(M_n)$. This is done as follows. For every $x \in S^{n-1}$ we find a neighbourhood $U_x$ of $x$ so that the claim above holds on $U_x$ with a constant $C(x)$ depending on $x$. Since $S^{n-1}$ is compact, there is a finite set $\{x_1, ..., x_k\}$ so that $S^{n-1} \subseteq \bigcup_{i=1}^k U_{x_k}$, so the claim above will then hold with $C = \max_{1 \leq i \leq k} C(x_i)$.

If $x$ is not a degeneration point of $M_{n+1}$, then by Proposition 4.10 the estimate above can be satisfied in a neighbourhood of $x$ with $K = 1$, so we assume $x$ is a degeneration point.

We now apply the symmetry discussed in Remark 4.16, so we can assume there is nonempty $A \subset \{1, ..., n\}$ so that $|x_i| = 0$ if $i \in A$ and $|x_{i,j}| > 0$ if $\{i, ..., j\} \not\subset A$. Write $A$ as $\{i_1, ..., j_1\} \cup ... \cup \{i_m, ..., j_m\}$ with $i_1 \leq j_1 < j_2 < i_2 < ... < i_m \leq j_m$. Denote $A^c := \{1, ..., n\} \setminus A$. We may even assume that $i_1 = 1$ and if $m > 1$, we have $j_m = n$. Note that $\text{rk}(x) = \#(A)$. Let $U'$ be the neighbourhood of $x$ given by Proposition 4.17.

Recall that $\mu$ and $\nu$ were defined as $\mu := \min\{|x_{i,j}| : \{i, ..., j\} \not\subset A\}$ and $\nu := \max\{|x_{i,j}| : \{i, ..., j\} \not\subset A\}$. Let

$$U := U' \cap \{\frac{\mu}{2} < |y_B| < 2\nu : B \not\subset A\}.$$
First of all, let $C < \infty$ be such that our induction hypothesis is satisfied with it for $2 \leq m < n$ and also that $C$ is so large that the conclusion of Proposition 4.10 holds with $\epsilon := \frac{\mu}{4\nu}$. Also we require that

\begin{equation}
\frac{1}{C} \max_{B \subseteq A} |y_B|^\xi \leq 1 \leq C \min_{B \subseteq A} |y_B|^\xi
\end{equation}

holds whenever $y \in U$.

We claim that on $U$ we have

\begin{equation}
\sigma(M_n) \sim \sigma(M_{j_1+1}) \oplus \ldots \oplus \sigma(M_{j_k-1+2}) \oplus \ldots \oplus \sigma(M_{n-j_m+2})
\end{equation}

if $m = 1$ and

\begin{equation}
\sigma(M_n) \sim \sigma(M_{j_1+1}) \oplus \ldots \oplus \sigma(M_{j_k-1+2}) \oplus \ldots \oplus \sigma(M_{n-j_m+2})
\end{equation}

otherwise.

Denote the right-hand sides of these expressions collectively as $\Sigma$. By our induction hypotheses, for any $y' \in U$ and any $k \in \{1, \ldots, m\}$ there is a symmetry $K_k \in L_n$ so that for $1 \leq k \leq m$ we have

\begin{equation}
\frac{1}{C} \sum_{i=1}^{j_k} |K_{y'_i}|^\xi |v_i|^2 \leq \sigma(M_{j_k-1+2})(K_{y'_i}, \ldots, K_{y'_j}) \leq C \sum_{i=1}^{j_k} |K_{y'_i}|^\xi |v_i|^2
\end{equation}

with $C$ not depending on $y'$: Just pick such a symmetry $K_k \in L_n$ for $k \in \{1, \ldots, m\}$ and take any $K \in L_n$ such that the restriction to the $y_{i_k}, \ldots, y_{j_k}$ coordinates is $K_{y_i}$. Here we have been abusing notation with the $K_k$’s so that $K_k$ above operates on coordinates $y_{i_k}, \ldots, y_{j_k}$ and not $y_1, \ldots, y_{j_k-1+1}$. Extend $K_k$ now naturally to whole of $\mathbb{R}^{(n-1)d}$. We can now take $K$ to be say $K = K_{1}K_{2}\ldots K_{m-1}K_{m}$.

Now for every $y' \in U$ fix such a transformation $K_{y'}$ and denote $y := K_{y'}y'$.

By (4.41) and (4.42) we have

\begin{equation}
\frac{1}{C} \sum_{i=1}^{n-1} |y_i|^\xi |v_i|^2 \leq \Sigma(y) \leq C \sum_{i=1}^{n-1} |y_i|^\xi |v_i|^2.
\end{equation}

As before, we write

\begin{equation}
\sigma(M_n) = \sum_{l=1}^{m} \sigma_{i_l,j_l} + \sigma_{A^e} + \sum_{i,j \in A^e} \gamma_{i,j} - \sigma_{A^e} + \text{the rest}.
\end{equation}

The first two terms satisfy

\begin{equation}
\frac{1}{C} \sum_{i=1}^{n-1} |y_i|^\xi |v_i|^2 \leq \sum_{l=1}^{m} \sigma_{i_l,j_l} + \sigma_{A^e} \leq C' \sum_{i=1}^{n-1} |y_i|^\xi |v_i|^2,
\end{equation}

and by Proposition 4.17 we have

\begin{equation}
| \sum_{i,j \in A^e} \gamma_{i,j} - \sigma_{A^e} + \text{the rest} | \leq \frac{1}{2C} \sum_{i=1}^{n} |y_i|^\xi |v_i|^2.
\end{equation}
So we have
\[
\frac{1}{2C} \sum_{i=1}^{n-1} |y_i|^2 v_i^2 \leq \sigma(M_n)(y) \leq (C + \frac{1}{2C}) \sum_{i=1}^{n-1} |y_i|^2 v_i^2.
\]

Let \( U^K := \{ y' \in U : K_{y'} = K \} \). Clearly \( U = \bigcup \{ U^K : K \in \mathcal{K}_n \} \). We just proved that for any \( y' \in U \) we have \( \sigma(M_n) \sim \Sigma \) in \( K_{y'} U^K y' \). Since both \( \Sigma \) and \( \sigma(M_n) \) are invariant under \( K_{y'}^{-1} \) for any \( y' \in U \), we can conclude by Remark 4.1 that \( \sigma(M_n) \sim \Sigma \) on \( U^K y' \). Since \( \mathcal{L}_n \) is finite we can conclude that \( \sigma(M_n) \sim \Sigma \) on \( U \).

Let
\[
\mathcal{L}_n' := \{ L \in GL(\mathbb{R}^{n-1}d) : \exists i_1, j_1, ..., i_{n-1}, j_{n-1} : \forall x_1, ..., x_{n-1} : L((x_1, ..., x_{n-1})) = (x_{i_1,j_1}, ..., x_{i_{n-1},j_{n-1}}) \}.
\]

Obviously, \( \mathcal{L}_n' \) is a finite set. It is also easy to see that it is a group. Note that the \( L \) as constructed in Theorem 4.9 belongs to \( \mathcal{L}_n' \).

Remark 4.18. The following Proposition simply says the following: Suppose we have a symbol of the form
\[
\mathcal{L}_n := \bigoplus_{i=1}^k \sigma(M_{n_i+1}) \oplus 1.
\]

This corresponds to a splitting \( \mathbb{R}^{nd} = \mathbb{R}^{ld} \oplus \mathbb{R}^{(n-l)d} \) with \( l = n_1 + ... + n_k \). Then we can replace \( \mathbb{R}^{(n-l)d} \) with any complementary subspace to \( \mathbb{R}^{ld} \) and the symbol looks the same in these new coordinates as looks the symbol in an neighbourhood of 0 which is bounded in the \( \mathbb{R}^{ld} \)-direction.

Proposition 4.19. Let \( \sigma \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}) \oplus 1 \) on a set \( U \subseteq B \times \mathbb{R}^{(n-l)d} \) with \( B \) bounded and \( l := \text{rk}(0) = \sum_{i=1}^k n_i \). Let \( L \in GL(\mathbb{R}^{nd}) \) be such that
\[
\begin{align*}
(1) & \quad L : \{0\} \times \mathbb{R}^{(n-l)d} = \{0\} \times \mathbb{R}^{(n-l)d} \\
(2) & \quad \text{Let } P : \mathbb{R}^{nd} \to \mathbb{R}^{ld} \text{ be the natural projection onto the first } ld \text{ coordinates and let } L' := L \upharpoonright \mathbb{R}^{ld} \times \{0\}. \quad \text{Then}
\end{align*}
\]

\[
L' \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}).
\]

With these assumptions
\[
\sigma^L \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}) \oplus 1
\]
on \( LU \).
Proof. Without loss of generality we may assume that

\[(4.52) \quad L := \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix},\]

with \(M\) an \(\mathbb{R}^{(n-l)d} \times \mathbb{R}^{ld}\)-matrix.

Also without loss of generality we may assume \(U = B(0, 1) \times \mathbb{R}^{(n-l)d}\).

Let \(A := \bigoplus_{i=1}^{k} \sigma(M_{n_i+1})\). Denote \(v := (v_1, v_2)\) and \(x := (x_1, x_2)\) where \(v_1, x_1 \in \mathbb{R}^{ld}\) and \(v_2, x_2 \in \mathbb{R}^{(n-l)d}\). Then

\[(4.53) \quad \langle v, (A \oplus \mathbb{1})^L(x)v \rangle = \langle v_1, A((L^{-1}x_1)v_1) \rangle + \langle v_1, A((L^{-1}x_1)M^Tv_2) \rangle + \langle v_2, A((L^{-1}x_1)v_1) + v_2 \rangle^2 =: (*)\]

Since \(A(x)\) is a symmetric matrix for every \(x\) the two middle terms are equal. Moreover, \((L^{-1}x_1) = x_1\) and thus

\[(4.54) \quad (*) = \langle v_1, A(x_1)v_1 \rangle + 2\langle v_1, A(x_1)M^Tv_2 \rangle + |v_2|^2 =: (**)\]

Next, we use induction on \(rk_0 = n_1 + \ldots + n_{k}\). If \(rk_0 = 1\), i.e. \(A = \sigma(M_2)\) we have

\[(4.55) \quad \frac{1}{C}(|v_1|^2 + |v_2|^2) \leq (**) \leq C(|v_1|^2 + |v_2|^2)\]

for some \(C < \infty\) when \((x_1, x_2) \in S^{d-1} \times \mathbb{R}^{(n-1)d}\). Adding \((|x_1|^{-\xi} - 1)|v_2|^2\) and multiplying by \(|x_1|^\xi\) yields

\[(4.56) \quad \frac{1}{C}(|x_1|^\xi |v_1|^2 + |v_2|^2) \leq (**) \leq C(|x_1|^\xi |v_1|^2 + |v_2|^2)\]

when \((x_1, x_2) \in B(0, 1) \times \mathbb{R}^{(n-1)d}\). Since \(\sigma(M_2) \sim |\cdot|^\xi\) we can conclude our claim.

Next, suppose our Proposition is true for configurations of rank \(< l\) and we prove our claim when \(rk_0 = l\). Now cover \(S^{ld-1}\) by finitely many open sets \(B_1, \ldots, B_m\) so that

\[(4.57) \quad \bigoplus_{i=1}^{k} \sigma(M_{n_i+1})^{L_j} \sim \bigoplus_{i=1}^{k_j} \sigma(M_{n_{j,i}+1}) \oplus \mathbb{1}\]

on \(B_j\) with some linear transformation \(L_j\) and with \(\sum_{i=1}^{k_j} n_{j,i} < l\).

Letting \(L'_j := L(L_j \oplus \mathbb{1})\), and applying this Theorem on \(U_j := B_j \times \mathbb{R}^{(n-l)d}\) we see that

\[(4.58) \quad \sigma(M_{n+1})^{L'_j} \sim \bigoplus_{i=1}^{k_j} \sigma(M_{n_{j,i}+1}) \oplus \mathbb{1}\]

on \(U_j\).
Now a similar argument as above for rank 0 yields the desired conclusion. The reader may fill in the details.

The following is an immediate corollary to this proposition.

**Corollary 4.20.** Let $L \in \mathcal{L}$ be such that for some neighbourhood $U$ of $x$ we have

$$\sigma(M_{n+1})^L \sim \bigoplus_{i=1}^{k} \sigma(M_{n+1}) \oplus \mathbb{1}$$

on $LU$. Then for every $L' \in \mathcal{L}$ such that

$$L^{-1} = L'^{-1} \text{ on } \{|x_i| = 0 : 1 \leq i \leq \text{rk } x\}$$

we have

$$\sigma(M_{n+1})^{L'} \sim \bigoplus_{i=1}^{k} \sigma(M_{n+1}) \oplus \mathbb{1}$$

on $L'U$.

4.5. Some Corollaries.

**Corollary 4.21.** For every $n \geq 2$ there is $C > 0$ such that

$$Cd(x, D\text{gn}(M_n))^\xi \leq \sigma(M_n).$$

The proof of this fact is easy and thus omitted. The assumptions of Theorem 2.4 are now satisfied (by Corollary 4.21, Theorem A.1 and Proposition A.3) for $M_n$. Moreover, we can directly calculate the dimension of $M_n$:

**Corollary 4.22.** There is $C < \infty$ such that for any $f \in L^2(\mathbb{R}^{(n-1)d})$ we have

$$\|e^{-M_n t} f\|_\infty \leq Ct^{-\frac{(n-1)d}{4-2\xi}} \|f\|_2.$$ 

Moreover, $C$ depends only on the lower bound for $\sigma(M_n)$.

**Proof.** By Proposition A.3 there is $C < \infty$ so that

$$\|f\|_q \leq C \|d(x, D\text{gn}(M_n))^{\xi/2} \nabla f\|_2 =: (*)$$

for any $f \in C_0^\infty(\mathbb{R}^{(n-1)d})$ with $q := \frac{2n}{n+\xi-2}$.

By Corollary 4.21 we have

$$(* \leq C'(f, M_n f).$$

Finally, by Theorem 2.13 we can conclude that (4.63) holds. \qed
Corollary 4.23. For any \( \rho \in (0, 1) \) there is \( C < \infty \) such that for any \( x \in \mathbb{R}^{(n-1)d} \) and any \( y \not\in B(x, \rho|x|) \) we have

\[
K_{M_n}(t, x, y) \leq Ct^{-\frac{(n-1)d}{2-\xi}} \exp\left\{-\frac{|x-y|^{2-\xi}}{Ct}\right\}
\]

and

\[
G_{M_n}(x, y) \leq C|x-y|^{2-\xi-(n-1)d}.
\]

Proof. This is a direct consequence of Proposition 4.5, Theorem 2.12 and Corollary 4.22. \( \square \)

Corollary 4.24. Suppose \( A \sim_{\lambda} \sigma(M_{n_1+1}) \oplus \ldots \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1} \) on \( \mathbb{R}^{ld} \times \mathbb{R}^{(n-l)d} \) with \( l := n_1 + \ldots + n_k < n \) and let \( \epsilon > 0 \) be given. Then there is \( C < \infty \) such that if \( z \not\in B(y_1, \epsilon|y_1|) \times B(y_2, \epsilon|y_1|^{1-\xi/2}) \) (here \( y := (y_1, y_2) \in \mathbb{R}^{ld} \times \mathbb{R}^{(n-l)d} \), we have

\[
K_A(t, y, z) \leq Ct^{-\frac{ld}{2-\xi}} \exp\left\{-\frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{Ct}\right\}.
\]

Moreover \( C \) depends on \( A \) only through \( \lambda, n_1, \ldots, n_k \) and \( n \).

Proof. The proof is straightforward using Theorem 2.13, Proposition 4.5 and Corollary 4.22, and we leave the details for the reader. The only finesse is the appearance of \( B(y_2, \epsilon|y_1|^{1-\xi/2}) \) above. This is due to the fact that if \( z_1 \in B(y_1, \epsilon|y_1|) \) and \( z_2 \not\in B(y_2, \epsilon|y_1|^{1-\xi/2}) \), we have

\[
|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2 \leq (\epsilon|y_1|)^{2-\xi} + |y_2 - z_2|^2 \leq \epsilon^{-\xi}|y_2 - z_2|^2 + |y_2 - z_2|^2.
\]

\( \square \)

5. Local estimates for the heat kernel

The main result in this section is Theorem 5.12. Superficially it is very similar to Corollary 4.24, but there is a very important difference: in Corollary 4.24 one assumes that

\[
A \sim \sigma(M_{n_1+1}) \oplus \ldots \oplus \sigma(M_{n_k+1}) \oplus \mathbb{1}
\]

in \( \mathbb{R}^{nd} \) but in Theorem 5.12 \( A = \sigma(M_{n+1}) \) and (5.1) holds only in a relatively compact neighbourhood of a point \( x \). The point of this section is to close the gap between these two results. We start with some technicalities and prove a uniform version of the Harnack inequality adapted to our case.

Remark 5.1. In a few places we use the somewhat terse assumption “\( A \) has a heat kernel”. In these places we assume that \( A \) has a heat kernel...
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$K$ such that both $K(\cdot, x, \cdot)$ and $K(\cdot, \cdot, x)$ are solutions to $u_t + Au = 0$ in the sense of Remark 2.5 and that for every $t$ and $x$ we have both

\[(5.2) \quad \int dy K(t, x, y) \leq 1 \quad \text{and} \quad \int dy K(t, y, x) \leq 1.\]

In the cases that are of interest to us (see Remark 2.14) this is the case and moreover our heat kernels are symmetric in the spatial coordinates.

A well-known argument (see for example [20], section I.3, page 5) yields the following: Suppose $A$ is a divergence-form operator on $\mathbb{R}^n$ with a nonnegative symbol. Suppose also that $A$ is uniformly elliptic on some ball $B$ and that $A$ has a heat kernel. Then for any ball $B' \subset B$ there is $C < \infty$ such that we have

\[(5.3) \quad K(t, x, y) \leq Ct^{-n/2}\]

whenever $t \in (0, 1]$, $x \in B'$ and $y \in \mathbb{R}^d$. We shall now make a generalization (Corollary 5.5) of this result.

So for the rest of the section we fix a symbol $A$ on $\mathbb{R}^{nd}$ and suppose that

\[(5.4) \quad A \sim^\lambda \sigma(M_{n_1+1}) \oplus \cdots \oplus \sigma(M_{n_k+1}) \oplus 1\]

on $B(0, 2) \times B(0, 2) \subseteq \mathbb{R}^{ld} \times \mathbb{R}^{(n-l)d}$, where $l := n_1 + \cdots + n_k$. Let's denote

\[(5.5) \quad Q := \overline{B}(0, 1) \times \overline{B}(0, 1) \quad \text{and} \quad D := S^{nd-1} \times \overline{B}(0, 1).\]

**Proposition 5.2.** For each $t \in (0, 1]$ there is an open covering $\{U^t_y\}_{y \in Q}$ of $Q$ with the following properties:

1. $y \in U^t_y$ for every $y \in Q$ and $t \in (0, 1]$.
2. There is $\epsilon > 0$ not depending on $t$ such that $B(y_1, ct^{1/2-\epsilon}) \times B(y_2, \epsilon \sqrt{t}) \subseteq U^t_y$.
3. For every $t \in (0, 1]$, every $y \in Q$ and every positive solution $u$ of $u_t = \nabla \cdot A\nabla u$ on $(0, 3) \times U^t_y$ we have

\[(5.6) \quad \sup_{y' \in U^t_y} u(t, y') \leq C \inf_{y' \in U^t_y} u(2t, y').\]

Moreover, $C$ depends on $A$ only through $\lambda$, $n_1, \ldots, n_k$ and $n$.

**Remark 5.3.** Strictly speaking in (5.3) we only assume $u$ is a solution of $u_t = \nabla \cdot A\nabla u$ in the sense of Remark 2.3 on $(\epsilon, 3) \times U^t_y$ for every $\epsilon \in (0, 3)$.

**Corollary 5.4.** Proposition 5.2 holds with obvious modifications for any affine transform $A^K$ of $A$ with possibly different $\epsilon$ and $C$. 
To give some intuition to the reader we first give a Corollary to this Proposition.

**Corollary 5.5.** There is $C < \infty$ such that

\[(5.7) \quad K_A(t, y, y') \leq Ct^{-\frac{ld}{2} - \frac{(n-l)d}{2}} \]

for any $y \in Q$, $y' \in \mathbb{R}^{nd}$ and $t \in (0, 1]$. 

**Proof.** By Proposition 5.2 for any $y \in Q$ and $y' \in \mathbb{R}^{nd}$ we have

\[
t \frac{ld}{2} + \frac{(n-l)d}{2} K_A(t, y, y') \leq C' |U_y^t| \sup_{y'' \in U_y^t} K_A(t, y'', y')
\]

\[
\leq CC' |U_y^t| \inf_{y'' \in U_y^t} K_A(2t, y'', y')
\]

\[
\leq CC' \int_{U_y^t} K_A(2t, y'', y') \, dy''
\]

\[
\leq CC'.
\]

(5.8)

Next we prove a small Lemma used in the proof of Proposition 5.2.

The setup here is the following. Let $y \in D$. In our proof of Proposition 5.2 we use induction on rank. By Theorem 4.9 there is an invertible affine transformation $K_y$ of $\mathbb{R}^{nd}$ sending $y$ to 0 so that

\[(5.9) \quad A^{K_y} \sim \sigma(M_{n_1'+1}) \oplus \ldots \oplus \sigma(M_{n_k'+1}) \oplus 1\]

on $B(0, 2) \times B(0, 2)$ with $l' := n_1' + \ldots + n_k' < l$. Now Lemma 5.6 allows us to conclude that if (3) of Proposition 5.2 holds for the covering associated with $y$ in $K_y$-coordinates with some $\epsilon$ (for convenience, we have put this $\epsilon$ equal to 1 in the statement of Lemma 5.6), then it holds in the usual coordinates of $\mathbb{R}^{nd}$ with some other $\epsilon$.

Here is our choice of the subspaces for Lemma 5.6:

1. $S_1 := K_y^{-1}[\mathbb{R}^{l' d} \times \{0\}] - \{y\}$ and
2. $S_2 := K_y^{-1}[\{0\} \times \mathbb{R}^{(n-l')d}] - \{y\}$.

In other words $S_2$ is the degeneration subspace associated with $y$. The fact that $y \in Q$ guarantees that $\{0\} \times \mathbb{R}^{(n-l')d} \subseteq S_2$. Note that the $-\{y\}$ in the definition of $S_2$ is redundant, since $y \in S_2$, but we didn’t want to confuse the reader a few lines ago, did we?

**Lemma 5.6.** Let $S_1, S_2$ be a splitting of $\mathbb{R}^{nd}$ into complementary subspaces so that $\{0\} \times \mathbb{R}^{(n-l')d} \subseteq S_2$. Assume also that each of them is equipped with a norm and denote the balls with respect to these norms with $B_i(x, r)$ with $i = 1, 2$. Then there is $\epsilon > 0$ so that

\[(5.10) \quad B(0, ct^{1/(2-\ell)}) \times B(0, \epsilon \sqrt{t}) \subseteq B_1(0, t^{1/(2-\ell)}) \times B_2(0, \sqrt{t})\]
for any $t \in (0, 1]$.

**Proof.** Obviously there is $\epsilon > 0$ so that

$$B(0, \epsilon) \times B(0, \epsilon) \subseteq B_1(0, 1) \times B_2(0, 1)$$

Let us write $B(0, \epsilon t^{1/(2-\xi)}) \times B(0, \epsilon \sqrt{t})$ as

$$B(0, \epsilon t^{1/(2-\xi)}) \times \mathbb{R}^{(n-l)d} \cap B(0, \epsilon \sqrt{t}) \times B(0, \epsilon \sqrt{t})$$

and similarly for $B_1(0, t^{1/(2-\xi)}) \times B_2(0, \sqrt{t})$ (we used the fact that $t^{1/(2-\xi)} \leq \sqrt{t}$ for $t \in (0, 1)$).

Now since $\{0\} \times \mathbb{R}^{(n-l)d} \subseteq S_2$, we conclude by scaling that

$$B(0, \epsilon t^{1/(2-\xi)}) \times \mathbb{R}^{(n-l)d} \subseteq B_1(0, t^{1/(2-\xi)}) \times S_2.$$ 

for any $t > 0$.

Also by scaling we get

$$B(0, \epsilon \sqrt{t}) \times B(0, \epsilon \sqrt{t}) \subseteq B_1(0, \sqrt{t}) \times B_2(0, \sqrt{t}).$$ 

for any $t > 0$. □

**Proof.** (of Proposition 5.2)

If $l = 0$, then we just choose $U_y^t := B(y, \sqrt{t})$. Obviously, these sets satisfy (2) above and by classical results (see again [20], section I.3, page 5) they satisfy (3) too.

Next we assume that the cases $< l$ have been handled and prove the Proposition for $l$. This is done in three phases:

1. Phase 1: Use our induction hypothesis (i.e. that the cases $< l$ have been handled) to handle points in $D$.
2. Phase 2: Use scaling to handle points $z \in Q$ with $0 < |z_1| < 1$ and times $t \in (0, |z_1|^{2-\xi})$. And finally
3. Phase 3: Do something creative for points $z \in Q$ and times $t \in (|z_1|^{2-\xi}, 1]$. Note that this includes defining the sets $U_z^t$ when $|z_1| = 0$.

First, phase 1: By compactness, there is $\{y_1, ..., y_k\} \subseteq D$ so that $\{K_{y_i}^{-1}[B(0, 1) \times B(0, 1)]\}_{i=1}^k$ cover $D$. Obviously each $y_i$ is of rank $< l$.

For each $t \in (0, 1]$ and $z \in D$ pick $U_z^t$ to be one of the $U_y^t$’s associated with some of the $y_1, ..., y_k$ (this is possible by induction hypothesis and Corollary 5.4). Now these $U_z^t$’s satisfy (2) and (3), where (3) satisfied by induction and (2) is satisfied by Lemma 5.3 (and the discussion before it) and finiteness of the set $\{y_1, ..., y_k\}$.

Next, phase 2: We define the sets $U_z^t$ for $z$’s with $0 < |z_1| < 1$ and $t \in (0, |z_1|^{2-\xi}]$. This is achieved by scaling $A$ outwards so that in this scaling $z$ travels to $D$. Then the symbol $A_z$ obtained this way has the same upper and lower bounds as $A$ on $B(0, 2) \times B(0, 2)$, so we can use
our sets $U'_y$ defined above for $y \in D$. After this we just scale things back.

So, let $z \in Q$ with $0 < |z_1| < 1$ and let

\begin{equation}
 y^z := (y_1/|z_1|, z_2 + (y_2 - z_2)/|z_1|^{1-\xi/2}).
\end{equation}

Let $A^z$ be defined by

\begin{equation}
 A^z_{ij}(y) := \begin{cases} 
 |z_1|^\xi A_{ij}(y^z) & \text{if } 1 \le i, j \le ld \\
 |z_1|^{\xi/2} A_{ij}(y^z) & \text{if } 1 \le i \le ld < j \le nd \text{ or } \\
 \sigma(A_{ij}(y^z)) & \text{if } 1 \le j \le ld < i \le nd \\
 1 & \text{if } ld < i, j \le nd
\end{cases}
\end{equation}

Similarly define $u^z$ by $u^z(t, y) := u(|z_1|^{\xi-2}, y^z)$. Now if $u$ satisfies $u_t = \nabla A \cdot \nabla u$ on $(0, 3) \times B(0, 2) \times B(0, 2)$, then $u^z$ satisfies $u^z_t = \nabla \cdot A^z \nabla u^z$ on this same set. Since now if $A = \sigma(M_{n+1}) \oplus \ldots \sigma(M_{n+1}) \oplus \mathbb{1}$ on $B(0, 2) \times B(0, 2)$, then the same is true of $A^z$ we can conclude that (2) and (3) hold for $A^z$ with the same constants as for $A$. So if we scale back and let

\begin{equation}
 U^z_1 = \{(z_1|y_1, z_2 + |z_1|^{1-\xi/2}(y_2 - z_2)) : (y_1, y_2) \in U^z_2 \}
\end{equation}

then (2) and (3) hold for these whenever defined.

Finally, phase 3: To finish the argument, we set for $t \ge |z_1|^{2-\xi}

\begin{equation}
 U^t_z = B(0, \frac{3}{2} t^{1/(2-\xi)}) \times B(z_2, \frac{1}{2} \sqrt{t}).
\end{equation}

Now (2) holds for these sets. To prove (3) we may assume without loss of generality that $z_2 = 0$ and let $A^t$ be defined as follows:

\begin{equation}
 A^t_{ij}(y_1, y_2) := \begin{cases} 
 t^{-\frac{\xi}{2}} A_{ij}(y_1 t^{1/(2-\xi)}, y_2 \sqrt{t}) & \text{if } 1 \le i, j \le ld \\
 t^{-\frac{\xi}{2}} A_{ij}(y_1 t^{1/(2-\xi)}, y_2 \sqrt{t}) & \text{if } 1 \le i \le ld < j \le nd \text{ or } \\
 A_{ij}(y_1 t^{1/(2-\xi)}, y_2 \sqrt{t}) & \text{if } 1 \le j \le ld < i \le nd \\
 A_{ij}(y_1 t^{1/(2-\xi)}, y_2 \sqrt{t}) & \text{if } ld < i, j \le nd
\end{cases}
\end{equation}

As before, for $t \in (0, 1]$ the substitution $A \mapsto A^t$ preserves the constant in the Harnack inequality (Theorem 2.4) and thus we can conclude that (3) holds. \hfill \square

\begin{remark}
 It is not hard to modify the previous proof so that for given $\epsilon' > 0$ there is $\epsilon > 0$ so that

\begin{enumerate}
 \item $B(y_1, \epsilon t^{1/(2-\xi)}) \times B(y_2, \epsilon \sqrt{t}) \subseteq U^t_y$ for every $t \in (0, 1]$ and
 \item $U^t_y \subseteq B(y_1, \epsilon t^{1/(2-\xi)}) \times B(y_2, \epsilon \sqrt{t})$, when $|y_1|^{2-\xi} \le t \le 1$.
 \item $U^t_y \subseteq B(y_1, \epsilon' |y_1|) \times B(y_2, \epsilon' |y_1|^{(2-\xi)/2})$, when $0 < t \le |y_1|^{2-\xi}$.
\end{enumerate}
\end{remark}
We need (2) and (3) in the proof of Theorem 5.12. There we need to find \( \epsilon' > 0 \) so that \( U_t^z \) and \( B(y_1, \epsilon t^{1/(2-\xi)}) \times B(y_2, \sqrt{t}) \) are disjoint whenever \( z \not\in B(y_1, t^{1/(2-\xi)}) \times B(y_2, \sqrt{t}) \) and this is hard to arrange if we don’t have any kind of control over the \( U_t^z \)’s from outside. This required control is provided by (2) and (3) above. The actual choice of \( \epsilon' > 0 \) is done in Lemma 5.10.

Anyway, it is quite easy to make (2) and (3) hold. First of all, it is easy to see that (2) and (3) hold with some \( \epsilon'_0 > 0 \) when \( U_t^y \)’s are defined as in the proof of Proposition 5.2. By letting \( V_t^y := U_t^y/T \) with \( T := (\epsilon'_0/\epsilon')^2 \) we see that \( V_t^y \)’s for \( t \in (0, 1] \) satisfy (1)-(3) above together with the claims of Proposition 5.2. The details are left to the reader.

We will use Proposition 5.2 in this form in the proofs below.

We now have to estimate the tails of the heat kernel. We use a common probabilistic argument for this (killing probabilities). Denote

\[
(5.20) \quad d(x, y)^2 := \max\{|x_1 - y_1|^2 - \xi, |x_2 - y_2|^2\}.
\]

Obviously there is \( C < \infty \) so that

\[
(5.21) \quad C^{-1}d(x, y) \leq \sqrt{|x_1 - y_1|^2 - \xi + |x_2 - y_2|^2} \leq Cd(x, y)
\]

Below, \( P_A^y(\sup_{s \leq t} |X_s - y| \geq \mu) \) denotes the probability of the diffusion \( X \) associated with \( A \) starting from \( y \) at time 0 hitting the set \( \{z : d(y, z) = \mu\} \) before time \( t \).

The following is Proposition 6.5 on page 179 of [1].

**Proposition 5.8.** Suppose \( A \sim^\lambda 1 \) on \( \mathbb{R}^d \). There is \( C < \infty \) depending on \( A \) only through \( \lambda \) such that

\[
(5.22) \quad P_A^y(\sup_{s \leq t} |X_s - y| \geq \mu) \leq C \exp\left\{-\frac{\mu^2}{Ct}\right\}.
\]

**Corollary 5.9.** Suppose \( A \sim^\lambda 1 \) on \( B(0, 2) \subseteq \mathbb{R}^{nd} \). Then there is \( C < \infty \) depending on \( A \) only through \( \lambda \) such that for every \( y \in B(0, 1) \), \( z \in B(y, \frac{1}{2}) \) and \( 0 < t \leq 1 \) we have

\[
(5.23) \quad K_A(t, y, z) \leq Ct^{-\frac{nd}{2}} \exp\left\{-\frac{|y - z|^2}{Ct}\right\}
\]

The proof of this Corollary is quite simple and well-known (folklore) and we shall not prove it here, but the interested reader can reconstruct the argument from the proof of Theorem 5.12 which is a generalization of Corollary 5.9.

Unfortunately we need the following technicality in the proofs of Proposition 5.11 and Theorem 5.12.
Lemma 5.10. Suppose \( \epsilon'' > 0 \) is given. Then there is \( \epsilon' > 0 \) so that if \( d(y,z) \geq \epsilon'' |y_1|^{1-\xi/2} \), we have

\[
\{ z' : d(z,z') \leq \epsilon' |z_1|^{1-\xi/2} \} \subseteq \{ z' : d(z,z') \leq \frac{d(y,z)}{2} \}
\]

and

\[
B(y_1, \epsilon'|y_1|) \times B(y_2, \epsilon'|y_1|^{1-\xi/2}) \cap B(z_1, \epsilon'|z_1|) \times B(z_2, \epsilon'|z_1|^{1-\xi/2}) = \emptyset.
\]

Proof. Let

\[
\alpha := \frac{d(y,z)^{2/(2-\xi)}}{|y_1|},
\]

Then we have

\[
|z_1| \leq |y_1| + |y_1 - z_1| \leq |y_1| + d(y,z)^{2/(2-\xi)} \leq (1 + \alpha)|y_1|.
\]

So to prove (5.24), we just have to find \( \epsilon' > 0 \) so that

\[
\epsilon'((1 + \alpha)|y_1|)^{1-\xi/2} \leq \frac{1}{2} (\alpha |y_1|)^{1-\xi/2},
\]

whenever \( \alpha \geq (\epsilon'')^{2/(2-\xi)} \). By elementary calculus, we see that this is possible.

Using similar reasoning, we see that to prove (5.25) we have to find \( \epsilon' > 0 \) so that

1. \( \epsilon'|y_1| + \epsilon'(1 + \alpha)|y_1| \leq \alpha |y_1| \) and
2. \( \epsilon'|y_1|^{1-\xi/2} + \epsilon'((1 + \alpha)|y_1|)^{1-\xi/2} \leq (\alpha |y_1|)^{1-\xi/2}, \)

when \( \alpha \geq (\epsilon'')^{2/(2-\xi)} \). Again, this is possible. \( \square \)

Proposition 5.11. Suppose \( A \sim^\lambda \sigma(M_{n_1}) \oplus ... \oplus \sigma(M_{n_k}) \oplus \mathbb{1} \) on \( \mathbb{R}^{ld+(n-l)d} \) with \( \sum_{i=1}^{k} (n_i - 1) = l \) and let \( \epsilon'' > 0 \) be given. Then there is \( C < \infty \) such that for \( \mu \geq \epsilon'' |y_1|^{1-\xi/2} \) we have

\[
\mathbb{P}_A^y (\sup_{s \leq t} d(X_s, y) \geq \mu) \leq C \exp\left\{ - \frac{\mu^2}{C t} \right\}.
\]

Proof. Let \( \epsilon' > 0 \) be given by Lemma 5.10. By Corollary 4.24, there is \( C_1 < \infty \) so that if \( d(y,z) \geq \epsilon'|y_1|^{1-\xi/2} \) we have

\[
K_A(t, y, z) \leq C_1 t^{-\frac{ld}{2-\xi} - \frac{(n-l)d}{2}} \exp\left\{ - \frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{C_1 t} \right\}.
\]
Now a direct computation gives
\begin{equation}
\mathbb{P}^y_A(\sup_{s \leq t} d(X_s, y) \geq \mu) \leq \mathbb{P}^y_A(d(X_t, y) \geq \mu/2)
+ \mathbb{P}^y_A(d(X_t, y) \leq \mu/2 \text{ and } \exists s < t : d(X_s, y) = \mu)
\leq \mathbb{P}^y_A(d(X_t, y) \geq \mu/2)
+ \mathbb{P}^y_A(\exists s < t : d(X_s, s) = \mu \text{ and } d(X_s, X_t) \geq \mu/2)
\leq \mathbb{P}^y_A(d(X_t, y) \geq \mu/2) + \sup_{d(y, z) = \mu, s \leq t} \mathbb{P}^z_A(d(X_s, z) \geq \mu/2)
= (\ast).
\end{equation}

By (5.24) of Lemma 5.10, for every \( z \in \mathbb{R}^d \) with \( d(y, z) = \mu \) we have
\begin{equation}
\{ z' : d(z, z') \leq \epsilon|z_1|^{1-\xi/2} \} \subseteq \{ z' : d(z, z') \leq \frac{\mu}{2} \}.
\end{equation}

A fortiori we also have
\begin{equation}
\{ z' : d(y, z') \leq \epsilon|y_1|^{1-\xi/2} \} \subseteq \{ z' : d(y, z') \leq \frac{\mu}{2} \},
\end{equation}
since there are points \( z \in \mathbb{R}^d \) with \( d(y, z) = \mu \) and \( |z_1| \geq |y_1| \).

Thus by (5.30) we can conclude that
\begin{equation}
(\ast) \leq C_2 \int_{d(y, z) \geq \mu/2} t^{-\frac{ld}{2-\xi} - \frac{(n-l)d}{2}} \exp\left\{-\frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{C t} \right\} dy
\end{equation}
\begin{align*}
&\leq C_3 \int_{|y_1 - z_1|^{2-\xi} \geq \mu^2} t^{-\frac{ld}{2-\xi}} \exp\left\{-\frac{|y_1 - z_1|^{2-\xi}}{C t} \right\} dy_1
+ C_3 \int_{|y_2 - z_2| \geq \mu} t^{-\frac{(n-l)d}{2}} \exp\left\{-\frac{|y_2 - z_2|^2}{C t} \right\} dy_2
\leq C \exp\left\{-\frac{\mu^2}{C t} \right\}.
\end{align*}

Now we can finish with the local estimates.

**Theorem 5.12.** Suppose that \( A \sim^\lambda \sigma(M_{n_1 + 1}) \oplus \ldots \oplus \sigma(M_{n_k + 1}) \oplus 1 \) on \( B(0, 2) \times B(0, 2) \) with \( l := n_1 + \ldots + n_k < n \) and that \( A \) has a heat kernel. For any \( \epsilon'' \in (0, 1) \) there is \( C < \infty \) so that if \( y \in Q, 0 < t \leq 1 \) and \( \epsilon''|y_1|^{1-\xi/2} \leq d(z, y) \leq \frac{1}{2} \) we have
\begin{equation}
K_A(t, y, z) \leq Ct^{-ld/(2-\xi) - (n-l)d/2} \exp\left\{-\frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{C t} \right\}.
\end{equation}
Moreover, this estimate depends on $A$ only through $\lambda$, $n_1, \ldots, n_k$ and $n$.

**Remark 5.13.** It is not difficult to modify the proof to take into account more general sets. One can replace $B(0, 2) \times B(0, 2)$ with $U := A \times B$ with $A$ and $B$ open, starlike w.r.t. origin, open and satisfying

\[ \bigcup_{y \in Q} \{ z : d(z, y) \leq \frac{1}{2} \} \subset \subset U. \]

Similarly $Q$ can be replaced with $Q' := A \times B$ with $A$ and $B$ closed and starlike w.r.t. origin.

Also $d$ can be replaced with any equivalent metric. (Note in particular that Lemma 5.10 is preserved under replacement by an equivalent metric with possibly a different $\epsilon'$)

**Proof.** If $0 < d(z, y)^2 \leq t$, then there is $C < \infty$ so that

\[ 1 \leq C \exp\left\{-\frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{Ct}\right\}. \]

Thus in view of Corollary 5.3 we only need to prove the claim for $t \leq d(z, y)^2 \leq 1$.

Let $\epsilon' > 0$ be given by Lemma 5.10 and let $\{U_y^t\}$ be a collection of open coverings given by Proposition 5.2 and Remark 5.7 associated with this $\epsilon'$. We may assume $\epsilon' \leq \min\{\frac{\xi}{2}, \frac{1}{2}\}$.

We want to show that $U^t_z$ and $B(y_1, \epsilon' t^{1/(2-\xi)}) \times B(y_2, \epsilon' \sqrt{t})$ are disjoint whenever $z \not\in B(y_1, t^{1/(2-\xi)}) \times B(y_2, \sqrt{t})$. The case $|y_1|^{2-\xi} \leq t \leq 1$ follows easily, since we assumed $\epsilon' \leq \min\{\frac{\xi}{2}, \frac{1}{2}\}$. In case $0 < t \leq |y_1|^{2-\xi}$ we just use Lemma 5.10 to conclude that

\[ B(y_1, \epsilon' t^{1/(2-\xi)}) \times B(y_2, \epsilon' \sqrt{t}) \cap B(z_1, \epsilon' |z_1|) \times B(z_2, \epsilon' |z_1|^{(2-\xi)/2}) = \emptyset, \]

whenever $d(y, z) \geq \epsilon'' |y_1|^{1-\xi/2}$.

By the proof of Corollary 5.3 we have

\[ \frac{\frac{n!}{n^d} \cdot (n-1)!d}{z \in U^t_z} \sup_{z' \in U^t_z} K_{M+1}(t, y, z') \]

\[ \leq C_2 \int_{U^t_z} dy' K_{M+1}(2t, x, y'). \]

By Proposition 5.11 we have

\[ \int_{U^t_z} K_{M+1}(2t, y, z) \leq C_3 \exp\left\{-\frac{|y_1 - z_1|^{2-\xi} + |y_2 - z_2|^2}{C_3 t}\right\}, \]

so we are done. \(\square\)
Remark 5.14. Note that the conclusion of the Theorem depends on \(n_1, \ldots, n_k\) only through \(l\). In particular the estimate obtained above remains the same, when \(\sigma(M_{n_1+1}) \oplus \cdots \oplus \sigma(M_{n_k+1})\) is replaced by \(\sigma(M_2) \oplus l\).

6. Construction of the stationary state

In this section, we shall finally prove Theorem 1.1 modulo some technicalities whose proofs are postponed until Appendix C. To this end, we shall inductively show the following

**Theorem 6.1.** Let \(\chi : \mathbb{R}^d \to \mathbb{R}\) be compactly supported and nonnegative. Then for some \(C_n < \infty\) we have

\[
M_{2n}^{-1}(M_{2n-2}(\cdots(M_2^{-1}\chi \otimes \chi)\cdots) \otimes \chi) \leq C_n \prod_{i=1}^n (1 + |x_{2i-1}|)^{2-\xi-d}.
\]

Obviously Theorem 1.1 follows directly from this.

The following formula is a central tool in this section.

**Proposition 6.2.** Let \(1 \leq l \in \mathbb{N}\). Then

\[
\int_{\mathbb{R}^ld} d^d y |x-y|^{2-\xi-ld} \prod_{i=1}^{l-1} (1 + |y_i|)^{2-\xi-d} \chi(y_i) \leq C \prod_{i=1}^l (1 + |x_i|)^{2-\xi-d}.
\]

The proof of this Proposition can be found in Appendix C.

We want to show that

\[
\int_{\mathbb{R}^{(2n-1)d}} G_{M_{2n}}(x,y) \prod_{i=1}^{n-1} (1 + |y_{2i-1}|)^{2-\xi-d} \chi(y_{2n-1}) dy \leq C \prod_{i=1}^n (1 + |x_i|)^{2-\xi-d}.
\]

We find finitely many sets \(\{A_i\}_{i=1}^k\) so that together with \(\{(x,y) \in \mathbb{R}^{(2n-1)d} \times \mathbb{R}^{(2n-1)d} : |y| \geq \rho|x|\}\) they cover \(\mathbb{R}^{(2n-1)d} \times \mathbb{R}^{(2n-1)d}\). Let 

\(A_i^x := \{x + y : (x,y) \in \mathbb{R}^{nd}\}\).

We shall write the above integral as

\[
\int_{\mathbb{R}^{(2n-1)d}} = \int_{|x-y| \geq \rho |x|} + \sum_{j=0}^{k(x)} \int_{A_j^x \setminus A_{j-1}^x}
\]

and then prove the desired estimate of (6.3) separately for each term of the right-hand side.

We apologise the reader for bouncing around with using \(2n\) and \(n+1\), but for the moment \(n+1\) is more convenient.
We will first reduce everything to investigation of operators \(\sigma(M_2)\oplus t\) using Remark 5.14. What we mean by this is the following: Let

\[
E_C(t, x, y) := \begin{cases} 
C|x|^{-\frac{d}{2}} t^{-\frac{d}{2}} \exp \left\{ -\frac{|x-\xi|^2}{ct} \right\} & \text{if } |y| < \frac{|x|}{2} \\
Ct^{-\frac{d}{2}} \exp \left\{ -\frac{|x-y|^2-\xi}{ct} \right\} & \text{if } |y| \geq \frac{|x|}{2}
\end{cases}
\]

and let

\[
E^n_C(x, y) := \int_0^\infty dt \prod_{i=1}^n E_C(t, x_i, y_i).
\]

We want to find \(C < \infty\) and a finite covering \(\{A_i\}_{i=1}^m\) for \(\mathbb{R}^{nd} \times \mathbb{R}^{nd}\) so that for every \(i \in \{1, ..., m\}\) there is \(L_i \in \mathcal{L}'_{n+1}\) so that

\[
G_{M^L_n}(t, x, x+y) \leq E^n_C(t, x, x+y)
\]

whenever \((x, y) \in LA_i\).

Then the proof of (6.3) is reduced to the investigation of \(E_C(t, x, y)\) (which is just the natural estimate for \(\sigma(M_n)^{\oplus n}\)).

We’ll first define \(A_i\)'s for symbols \(A \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}) \oplus 1\) on \(B(0, 2) \times B(0, 2)\) by induction on \(l := \sum_{i=1}^k\) and then use these to define \(A_i\)'s for \(\sigma(M_n)\). In this local case we just cover \(\overline{B}(0, 2) \times \overline{B}(0, 2) \times \overline{B}(0, \epsilon)\).

So suppose we have just a uniformly elliptic operator \(A\) on \(B(0, 2)\). Then we just take one set \(A_1 := \{(x, y) : x \in B(0, 1) \text{ and } |x - y| < \frac{1}{2}\}\). Next suppose all the cases \(l' < l\) have been handled. Then by induction hypothesis and compactness of \(\mathbb{S}^{d-1} \times \overline{B}(0, 1)\) there exists a finite set \(\{x_1, ..., x_m\}\) of \(\mathbb{S}^{d-1} \times \overline{B}(0, 1)\) so that there are affine transformations \(K_1, ..., K_m\) so that each \(K_j\) sends \(x_j\) to \(0\) and \(A_{K_j} \sim \bigoplus_{i=1}^k \sigma(M_{n_i+1}) \oplus 1\)

with \(l_j = \sum_{i=1}^k n_i^2 < l\) on \(B(0, 2) \times B(0, 2)\).

Since \(l_j < l\), there are \(\{A_i\}_{i=1}^k\) so that \(\mathbb{S}^{d-1} \times \overline{B}(0, 1) \times \overline{B}(0, \epsilon)\) gets covered by them and each \(A_i\) is just \((L_j^{-1})^\oplus A\) for some associated \(A\) given for \(A_{K_j}\).

Moreover the linear part \(L_j\) of \(K_j\) is of the form

\[
L_j := \begin{pmatrix} M_j & 0 \\
0 & 1 \end{pmatrix},
\]

where \(M_j\) is a \(ld \times ld\)-matrix. So there is a neighbourhood

\[
B_\epsilon := \{(x, y) : x \in \mathbb{S}^{d-1} \times \overline{B}(0, 1) \text{ and } |x - y| < \epsilon\}
\]

so that \(B_\epsilon \subseteq \bigcup_{i=1}^m U_i\) everything is under control.
Let’s define the set $\tilde{A}_i$ as follows:

\[
\tilde{A}_i := \{((rx_1, x_2), (ry_2, r^{1-\xi/2}y_2)) : (x, y) \in A_i, x \in S^{d-1} \times B(0,1) \text{ and } r \in (0,1]\}.
\]

Clearly there is $\epsilon'' > 0$ so that $\{\tilde{A}_i\}_{i=1}^m$ together with (see again Theorem 5.12)

\[
\{\epsilon''|y_1|^{1-\xi/2} \leq d(z, y) \leq \frac{1}{2}\}
\]

cover

\[
\{(x, y) : x \in B(0,1) \times B(0,1) \text{ and } |x - y| < \epsilon\}
\]

for some $\epsilon > 0$.

On this last set $A$ clearly “behaves as” the heat kernel of $\sigma(M_2)^{\oplus l} \oplus 1$, so we have to prove the same for $A_{k\ell}$ on $L_i\tilde{A}_i$. This is a rather easy scaling argument: Pick $\lambda > 0$ so that $A \sim^\lambda \bigoplus_{i=1}^k \sigma(M_{n+1}) \oplus 1$. Let $y \in B(0,1) \times B(0,1)$ and denote $x^y := (|y_1|^{-1}x_1, |y_1|^{\xi/2-1}(y_2-x_2)+x_2)$. Define

\[
B^y_{ij}(z^y) := \begin{cases} 
|y_1|^\xi A_{ij}(z) & \text{ if } 1 \leq i, j \leq ld \\
|y_1|^{\xi/2} A_{ij}(z) & \text{ if } 1 \leq i \leq ld < j \leq nd \text{ or } 1 \leq j \leq ld < i \leq nd \\
A_{ij}(z) & \text{ if } ld < i, j \leq nd.
\end{cases}
\]

A straightforward computation shows that $B^y \sim^\lambda \bigoplus_{i=1}^k \sigma(M_{n+1}) \oplus 1$. By dimensional analysis

\[
G_A(y, z) = |y_1|^{2-\xi-ld-(1-\xi/2)(n-l)d} G_B(y^y, z^y).
\]

Therefore, since the same scaling property holds for $E_C^n$, we can conclude that

\[
G_{A_{k\ell}}(y, z) \leq E_C^n(y, z)
\]

on whole of $L_i\tilde{A}_i$.

Finally for $\sigma(M_n)$ we just cover $S^{nd-1} \times B(0, \rho)$ by the sets described above and conify these. Now if $\sigma(M_{n+1})^L$ “behaves as” $\sigma(M_2)^{\oplus l} \oplus 1$ on $L\Lambda$, then by scaling it “behaves as” $\sigma(M_2)^{\oplus l} \oplus \sigma(M_2)^{\oplus (n-l)} = \sigma(M_2)^{\oplus n}$ on $C\Lambda$, where $C\Lambda$ denotes the conification of $L\Lambda$. 


So we have reduced (6.3) to proving

\[
\int_{\mathbb{R}^{2n-1}d} E_{C}^{2n-1}(x, y) \prod_{i=1}^{n-1} (1 + |Ly_{2i-1}|)^2 - \xi - d \chi(Ly_{2n-1}) \, dy 
\leq C' \prod_{i=1}^{n} (1 + |Lx_{i}|)^{2 - \xi - d}
\]

for arbitrary \( L \in L_{2n} \) and arbitrary \( C > 0 \).

To this end, we split the domain of integration in (6.16) into parts and prove it separately for these parts.

Define the sets \( B_{j}^{x} \) as follows: Assume first that \( \max\{|x_{i}| : 1 \leq i \leq n\} = 1 \). (We then just simply let \( B_{j}^{x} := rB_{j}^{x/r} \) if \( \max\{|x_{i}| : 1 \leq i \leq n\} = r \).

By symmetry we may assume \( |x_{1}| \leq \ldots \leq |x_{n}| = 1 \). Let \( \ell(x) := \#(\{x_{1}, \ldots, x_{n}\} \setminus \{0\}) \) (i.e. the number of distinct strictly positive numbers) and let \( \ell \) be defined by

\[
0 < |x_{\ell_{1}}^{x}| = \ldots = |x_{\ell_{2}}^{x}| < \ldots = |x_{\ell_{\ell \{x\}^{x}}^{x}-1}^{x}| < \ldots < |x_{\ell_{\ell (x)}}^{x}| = \ldots = |x_{n}| = 1
\]

with \( \ell_{k}^{x} \) being the smallest integer so that \( |x_{\ell_{k}}^{x}| > 0 \). Let \( r_{j}^{x} := |x_{\ell_{k}}^{x}| \).

For every \( x \) with \( k(x) > 1 \), we define \( \tilde{x} \) as follows:

1. \( \tilde{x}_{i} = x_{i}/r_{k(x)-1}^{x} \) for \( 1 \leq i < \ell_{k(x)}^{x} \) and
2. \( \tilde{x}_{i} = x_{i} = 1 \) for \( \ell_{k(x)}^{x} \leq i \leq n \).

Clearly for such \( x \), \( k(\tilde{x}) = k(x) - 1 \). We first give the sets \( B_{j}^{x} \) inductively in terms of \( k(x) \) and then explicitly. First of all, for all our \( x \) let

\[
B_{k(x)}^{x} := \{ y \in \mathbb{R}^{nd} : |y_{i} - x_{i}| \leq \frac{1}{2} \text{ for every } i \in \{1, \ldots, n\}\}.
\]

In particular, for \( k(x) = 1 \) everything is done. If \( k(x) > 1 \) and \( B_{j}^{y} \) has been defined for \( y \) with \( k(y) < k(x) \), we just translate \( B_{j}^{x} \) on top of \( x \) and scale it by \( r_{k(x)-1}^{x} \) in the first \( \ell_{k(x)}^{x}-1 \) coordinates and by a factor of \( (r_{k(x)-1}^{x})^{1-\xi/2} \) in the rest of the coordinates. In plain formulae, this is

\[
B_{j}^{x} := \left\{ (ry_{1}, \ldots, ry_{\ell_{k(x)}^{x}-1}^{x}, x_{\ell_{k(x)}^{x}}^{x} + r^{1-\xi/2}(y_{\ell_{k(x)}^{x}}^{x} - x_{\ell_{k(x)}^{x}}^{x}), \ldots, x_{n} + r^{1-\xi/2}(y_{n} - x_{n}) : y \in B_{j}^{y}\right\}.
\]
Thus, explicitly, we have (denoting $\ell_{k(x)+1} = n+1$)

$$B^x_j := \{ y \in \mathbb{R}^{nd} : \forall i < \ell_{j+1} : |x_i - y_i| \leq \frac{1}{2} |x_{\ell_j'}| \text{ and }$$

$$\forall l \in \{ j + 1, \ldots, k(x) \} \forall i \in \{ \ell_l, \ldots, \ell_{l+1} - 1 \} :$$

$$|x_i - y_i| \leq \frac{1}{2} |x_{\ell_l}|^{1-\varepsilon/2} |x_{\ell_{l+1}}^{\ell_l}/2\}. \tag{6.20}$$

For technical reasons related to the fact that the symmetry group of $\sigma(M_{n+1})$ (i.e. $L'_{n+1}$) is rather different from the one of $\sigma(M_2)^{\otimes n}$ we have to modify our covering $\{ B^x_j \}$ a bit, since with our current covering (2) of Lemma 6.5 would not be true. (It is true however for such $L$ for which $\{Ly = 0\} = \{y' = 0\}$ for some other $l'$).

First of all, we can concentrate our investigation to a conical neighbourhood $C$ of the degeneration set, since outside such neighbourhood for $|x - y| \leq \rho |x|$ we have

$$E^n_C(x, y) \leq C|x|^{-\varepsilon} |x - y|^{2-nd} \leq C'|x - y|^{2-\varepsilon-nd}, \tag{6.21}$$

and this is sufficient by the computation in Phase 1 of the proof of Theorem 6.1 below.

Therefore, we pick our conical neighbourhood $C$ and $\rho > 0$ so that the set

$$\bigcup_{x \in C} B(x, 3\rho |x|) \tag{6.22}$$

does not contain any of the degeneration points of $\sigma(M_{n+1})$ that are not degeneration points of $\sigma(M_{n+1})$. Then we just intersect each $B^x_j$ for $x \in C$ with

$$\bigcup_{x \in C} B(x, 2\rho |x|) \tag{6.23}$$

thus forcing (2) of Lemma 6.4 to be true.

After this small diversion, we shall now list some basic properties of $E^n_C$ on these sets needed to establish (6.3). The statements of Lemmata 6.3, 6.4 and 6.5 clearly scale if for general $x$ we define $B^x_j$ to be $rB^x_{j/r}$, where $r := \max\{|x_1|, \ldots, |x_n|\}$. So we can assume that $r = 1$ in the following proofs.
Lemma 6.3. For every $x \in \mathcal{C}$ with $|x_1| \leq \cdots \leq |x_n|$, $j \in \{1, \ldots, k(x)\}$ and $y \in \mathcal{B}_j^x$ there are some positive numbers $a_{j_1}, \ldots, a_n$ so that

$$E_C^n(x, y) \leq C'(\prod_{i=\ell_j^x}^{L} a_i^{-\ell_j^x})(\sum_{i=1}^{\ell_j^x - 1} |x_i - y_i|^{2-\xi} + \sum_{i=\ell_j^x}^{n} a_i^{-\xi} |(x_i - y_i)|^{2-\xi})^{\frac{(n-i)d}{2}}$$

(6.24)

on $\mathcal{B}_j^x \setminus \mathcal{B}_{j-1}^x$.

Proof. This is just a straightforward computation by induction on $k(x)$.

Lemma 6.4. There is $C' < \infty$ such that

$$\int_{\mathcal{B}_j^x \setminus \mathcal{B}_{j-1}^x} E_C^n(x, y) \, dy \leq C'(r_j^x)^{2-\xi}.$$  

(6.25)

Proof. Again by induction on $k(x)$.

Lemma 6.5. Let $L \in \mathcal{L}_{n+1}'$ ($\mathcal{L}_{n+1}'$ was defined in (4.43)). Then there is $C' < \infty$ such that for every $x \in \mathcal{C}$, $j \in \{1, \ldots, k(x)\}$ and $l \in \{1, \ldots, n\}$ the following hold:

1. If $\{y_i = 0 : 1 \leq i \leq \ell_j^x - 1\} \subseteq \{L y_i = 0\}$, then $|L x_l| \leq C' r_j^x$.
2. If $\{y_i = 0 : 1 \leq i \leq \ell_j^x - 1\} \not\subseteq \{L y_i = 0\}$, then $|L x_l| \geq C' r_j^x$.

Proof. As before, without loss of generality we may assume that $\max\{|x_1|, \ldots, |x_n|\} = 1$ and that $|x_1| \leq \cdots \leq |x_n|$.

First we handle (1). Basically it says that if $L y_i$ can be expressed as a linear combination of $y_i$'s with $1 \leq i \leq \ell_j^x - 1$, then $|L x_l|$ is small. Since $\max_{L \in \mathcal{L}_{n+1}'} |L| \leq C$ for some $C < \infty$, it suffices to show that in case of $L = 1$ we have $|x_l| \leq r_j^x$. But this is immediate from the fact that $|x_l| \leq |x_{\ell_j^x}| \leq |x_{\ell_j^x - 1}| = r_j^x$.

Also in the case of (2), we can immediately restrict our attention to the case $L = 1$. This is then immediate using (6.20).

Proof. (of Theorem 6.1) As was said before, it suffices to prove (6.10) for $x \in \mathcal{C}$, since for $x \notin \mathcal{C}$ the whole thing reduces to Phase 1 below using (6.21).

Without loss of generality, we may assume that the support of $\chi$ is so small that if (2) applies to $y_{2n-1}$, then whenever $r_j^x \geq 1$, we have

$$L[\mathbb{R}^{(2n-2)d} \times \text{supp } \chi] \subseteq \{|L y_{2n-1}| < C'/\xi \cdot |L x_{2n-1}|\}.$$  

(6.26)
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Our proof goes as follows. First we split the domain of integration into parts and then we proceed in three phases:

1. Phase 1: Handle the integral \( \int_{|x - y| \geq \rho |x|} \).
2. Phase 2: Handle the integrals \( \int_{B_j \setminus B_{j-1}} \) with \( r_j^x \leq 1 \) and
3. Phase 3: Handle the integrals \( \int_{B_j \setminus B_{j-1}} \) with \( r_j^x \geq 1 \).

First, phase 1: We know that for \( |x - y| \geq \frac{1}{2} |x| \) we have

\[
E_{C}^{2n-1}(x, y) \leq C_1 \left( \sum_{i=1}^{2n-1} |Lx_i - Ly_i| \right)^{2-\xi-2(n-1)d},
\]

so we can conclude that

\[
\int_{|x - y| \geq \rho |x|} d^{(2n-1)d} y E_{C}^{2n-1}(x, y) \prod_{i=1}^{n-1} (1 + |Ly_{2i-1}|)^{2-\xi-d}(|Ly_{2n-1}|)
\]

\[
\leq C_1 \int_{\mathbb{R}^{(2n-1)d}} d^{(2n-1)d} y \left( \sum_{i=1}^{2n-1} |Lx_i - Ly_i| \right)^{2-\xi-(2n-1)d} \prod_{i=1}^{n-1} (1 + |Ly_{2i-1}|)^{2-\xi-d} \chi(|Ly_{2n-1}|) =: (*)
\]

By a change of variables we see that

\[
\int_{\mathbb{R}^{(n-1)d}} \prod_{i=1}^{n-1} d^d y_{2i} \left( \sum_{i=1}^{2n-1} |x_i - y_i| \right)^{2-\xi-(2n-1)d}
\]

\[
= \left( \sum_{i=1}^{n} |x_{2i-1} - y_{2i-1}| \right)^{2-\xi-d} \int_{\mathbb{R}^{(n-1)d}} \prod_{i=1}^{n-1} d^d z_{2i} \cdot (1 + \sum_{i=1}^{n-1} |z_{2i}|)^{2-\xi-(2n-1)d},
\]

so we can conclude that

\[
(*) \leq C_1 \int_{\mathbb{R}^{nd}} \prod_{i=1}^{n} d^d y_{2i-1} \left( \sum_{i=1}^{n} |Lx_{2i-1} - Ly_{2i-1}| \right)^{2-\xi-d} \chi(|Lx_{2n-1}|) =: (\ast^2).
\]
By Proposition 6.2, we have

\[(6.31) \quad (\star^2) \leq C_2 \prod_{i=1}^{n} (1 + |Lx_{2i-1}|)^{2-\xi-d}.\]

Next, some initial preparation for phases 2 and 3: Let \(x \) and \(j \leq k(x)\) be given and let \(U := \{1, 3, \ldots, 2n-1\}\), let \(U_1\) be the set of those \(i \in U\) for which \((2)\) of Lemma 6.5 applies and let \(U_2 := U \setminus U_1\).

Then, phase 2: so suppose \(r^x_j \leq 1\). Then by Lemma 6.4 we have

\[(6.32) \quad \int_{y \in B^x_j \setminus B^x_{j-1}} d^{(n-1)d} y E^{2n-1}_C(x, y) \prod_{i=1}^{n-1} (1 + |Ly_{2i-1}|)^{2-\xi-d} \chi(|Ly_{2n-1}|)
\leq C_3 \sup_{y \in B^x_j \setminus B^x_{j-1}} \prod_{i=1}^{n} (1 + |Ly_{2n-i}|)^{2-\xi-d} =: (\star^3)\]

Since \((1 + |y_i|)^{2-\xi-d} \leq 1\) for any \(i \in U\), we can conclude that

\[(6.33) \quad (\star^3) \leq C_3 \sup_{y \in B^x_j \setminus B^x_{j-1}} \prod_{i=1}^{n} (1 + |Ly_i|)^{2-\xi-d} =: (\star^4)\]

By \((2)\) of Lemma 6.3 we have

\[(6.34) \quad (\star^4) \leq C_4 \prod_{i \in U_2} (1 + |Lx_i|)^{2-\xi-d} =: (\star^5)\]

Since by \((1)\) of Lemma 6.3 we have \(|Lx_i| \leq C'\) for \(i \in U_1\) we can finally conclude that

\[(6.35) \quad (\star^5) \leq C_5 \prod_{i=1}^{n} (1 + |Lx_{2i-1}|)^{2-\xi-d}.\]

Finally, phase 3: If \(r^x_j \geq 1\) and \(2n-1 \in U_2\), then by \(6.26\) we have

\[(6.36) \quad L[\mathbb{R}^{(2n-2)d} \times \text{supp } \chi] \cap B^x_j = \emptyset.\]

by \((2)\) of Lemma 6.5 and thus in this case we have

\[(6.37) \quad \int_{y \in B^x_j} d^{(n-1)d} y E^{2n-1}_C(x, y) \prod_{i=1}^{n-1} (1 + |Ly_{2i-1}|)^{2-\xi-d} \chi(|Ly_{2n-1}|) = 0.\]

So we may assume \(2n-1 \in U_1\). By \((2)\) of Lemma 6.3 we have

\[(6.38) \quad (1 + |Ly_i|)^{2-\xi-d} \leq C_6 (1 + |Lx_i|)^{2-\xi-d}\]
for $i \in U_2$. Therefore

\[(6.39)\]

\[
\int_{y \in B^c_j \setminus B^c_{j-1}} d^{(2n-1)}y E^{2n-1}_C(x, y) \prod_{i=1}^{n} (1 + |Ly_{2i-1}|)^{2-\xi-d}
\]

\[
\leq C_7 \prod_{i \in U_2} (1 + |Lx_i|)^{2-\xi-d} \int_{y \in B^c_j \setminus B^c_{j-1}} d^{(2n-1)}y E^{n}_C(x - y) \cdot \prod_{i \in U_1 \setminus \{2n-1\}} (1 + |Ly_i|)^{2-\xi-d} \chi(Ly_{2n-1}) = (*6).
\]

Writing $l' := 2n - 1 - l$ we get

\[(6.40)\]

\[
\int_{y \in B^c_j \setminus B^c_{j-1}} d^{(2n-1)}y E^{2n-1}_C(x - y) \prod_{i \in U_1 \setminus \{2n-1\}} (1 + |Ly_i|)^{2-\xi-d} \chi(Ly_{2n-1})
\]

\[
\leq C_8 \int_{\mathbb{R}^{(2n-1)d}} d^{(2n-1)}y \left( \prod_{i=l+1}^{n} a_i^{-\frac{ld}{d'}} \right) \left( \sum_{i=1}^{l} |x_i - y_i|^{2-\xi} + \sum_{i=l+1}^{n} a_i^{-\xi} |x_i - y_i|^2 \right)^{1-\frac{ld}{2d'} - \frac{d'}{2}} \cdot \prod_{i \in U_1 \setminus \{2n-1\}} (1 + |Ly_i|)^{2-\xi-d} \chi(Ly_{2n-1}) = (*7),
\]

Note that by the definition of $U_1$ we have that in the expression

\[(6.41)\]

\[
\prod_{i \in U_1 \setminus \{2n-1\}} (1 + |Ly_i|)^{2-\xi-d} \chi(Ly_{2n-1})
\]

depends only on the variables $y_1, \ldots, y_l$.

Using a similar change of variables as in (5.29), we get

\[(*7) \leq C_8 \int_{\mathbb{R}^{ld}} \prod_{i=1}^{l} d^d y_i \left( \sum_{i=1}^{l} |x_i - y_i|^{2-\xi} \right)^{1-\frac{ld}{2d'}}.
\]

\[(6.42)\]

\[
\cdot \prod_{i \in U_1 \setminus \{2n-1\}} (1 + |Ly_i|)^{2-\xi-d} \chi(Ly_{2n-1}) \int_{\mathbb{R}^{ld}} \prod_{i=l+1}^{2n-1} d^d y_i
\]

\[
\cdot \left( \prod_{i=l+1}^{2n-1} a_i^{-\frac{ld}{d'}} \right) \left( 1 + \sum_{i=l+1}^{2n-1} a_i^{-\xi} |x_i - y_i|^2 \right)^{1-\frac{ld}{2d'} - \frac{d'}{2}} = (*8).
\]

By substituting $y'_i = a_i^{-\xi/2} y_i$ we see that the last integral is $\leq C_9$. 
Therefore

\[
\left( **8 \right) \leq C_{10} \int \mathbb{R}^d \prod_{i=1}^l d^d y_i \left( \sum_{i=1}^l |x_i - y_i|^{2-\xi} \right)^{1-\frac{ld}{2-\xi}}.
\]

(6.43)

Noticing that \( \sum_{i=1}^l |x_i - y_i|^{2-\xi} \) is essentially just \( |(x_1 - y_1, ..., x_l - y_l)|^{2-\xi} \) for estimation purposes, using a similar change-of-variables argument as before, we can conclude that

\[
\left( **9 \right) \leq C_{11} \prod_{i \in U_1} (1 + |Lx_i|)^{2-\xi-d},
\]

so

\[
\left( **6 \right) \leq C_{12} \prod_{i=1}^n (1 + |Lx_{2n-1}|)^{2-\xi-d}.
\]

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Appendix A. Poincaré and Sobolev inequalities

The following Theorem was proved in [3].

**Theorem A.1.** Let \( q > 2 \) and let \( w_1 \) and \( w_2 \) be two weights on \( \mathbb{R}^n \) and suppose that \( w_1 \) is \( A_2 \) and that \( w_2 \) is doubling. Suppose also that for all balls \( B' \) and \( B \) with \( B' \subseteq 2B \)

\[
\left( \frac{|B'|}{|B|} \right)^{1/n} \left( \frac{w_2(B')}{w_2(B)} \right)^{1/q} \leq c \left( \frac{w_1(B')}{w_1(B)} \right)^{1/2}
\]

(A.1)

with \( c \) independent of the balls.

Then the Poincaré and Sobolev inequalities hold for \( w_1, w_2 \) with \( q \).

So in order to conclude that the Harnack inequality holds for \( M_n \), it suffices to check the assumptions of above Theorem with \( w_1 = d(x, F)^\xi \) with \( F \) be a finite union of vector subspaces of \( \mathbb{R}^n \) or just \( \{0\} \) and \( w_2 \) either \(|x|^\xi\) or 1.
Lemma A.2. Let $F$ be a finite union of vector subspaces of $\mathbb{R}^n$ or just \{0\}. Suppose $\xi > -n$. Then $w_\xi(x) := d(x, F)^\xi$ satisfies the following: There is a constant $C < \infty$ such that for every $x \in \mathbb{R}^n$ we have

1. If $0 < r < \frac{d(x, F)}{2}$, then $\frac{1}{r}d(x, F)^\xi r^n \leq w_\xi(B(x, r)) \leq Cd(x, F)^\xi r^n$ and
2. If $r \geq \frac{d(x, F)}{2}$, then $\frac{1}{r}r^{n+\xi} \leq w_\xi(B(x, r)) \leq Cr^{n+\xi}$.

Proof. Since for $y \in B(x, r) \subseteq B(x, \frac{d(x, F)}{2})$ we have $(\frac{d(x, F)}{2})^\xi \leq w_\xi(y) \leq (\frac{3d(x, F)}{2})^\xi$, the first estimate follows.

For the second estimate, since $w_\xi(x, r) = |x|^{n+\xi}w_\xi(\hat{x}, \frac{r}{|x|})$ we see that by scaling it suffices to prove the inequality for $x \in \mathbb{S}^{n-1}$. To conclude the proof, it suffices to prove that

\[ 0 < \lim_{r \to \infty} \sup_{x \in \mathbb{S}^{n-1}} \frac{1}{r^{n+\xi}}w_\xi(B(x, r)) = \lim_{r \to \infty} \inf_{x \in \mathbb{S}^{n-1}} \frac{1}{r^{n+\xi}}w_\xi(B(x, r)) < \infty. \]

The computation is omitted. \qed

Naturally, the choice of the borderline at $\frac{d(x, F)}{2}$ was arbitrary. We can and will put the borderline at $\epsilon d(x, F)$ with $\epsilon \in (0, 1)$ depending on the situation.

Proposition A.3. Let $0 < \xi < 2$ and let $F$ be a finite union of vector subspaces of $\mathbb{R}^n$ of codimension $\geq 2$ or just \{0\}. Let $w_1 = C_1d(x, F)^\xi$ and $w_2 = C_2|x|^\xi$ with $0 < C_1, C_2 < \infty$. Then the Poincaré and Sobolev inequalities hold for $w_1$, $w_2$ with $q := \frac{2n}{n+\xi-2}$ and for $w_1$, 1 with $q$.

Proof. By Lemma A.2 both $w_1$ and $w_2$ are $A_2$, so it suffices to prove the scaling assumption in Theorem A.1 with $q$. Now Lemma A.2 implies that there is a constant $C < \infty$ such that for every $x \in \mathbb{R}^n$ and $r > 0$ and every ball $B' := B(x', r') \subseteq B(x, 2r)$ we have

1. If $0 < r < \frac{d(x, F)}{4}$, then $C^{-1}|x|^{n}r^m \leq w(B') \leq C|x|^\xi r^m$.
2. If $\frac{d(x, F)}{4} < r$, then $C^{-1}r^{m+\xi} \leq w(B') \leq Cr^{\xi}r^m$.

Here $w$ stood for either $w_1$ or $w_2$. Thus we have for some $C < \infty$ the following:

1. If $0 < r < \frac{d(x, F)}{4}$, then
   \[ C^{-1}\left(\frac{r^m}{r^n}\right) \leq \left(\frac{w(B')}{w(B)}\right) \leq C\left(\frac{r^m}{r^n}\right). \]
2. If $\frac{d(x, F)}{4} < r$, then
   \[ C^{-1}\left(\frac{r^{m+\xi}}{r^{n+\xi}}\right) \leq \left(\frac{w(B')}{w(B)}\right) \leq C\left(\frac{r^m}{r^n}\right). \]
Therefore, the claim reduces to finding $C < \infty$ such that for every $r$ and $r'$ with $r' \leq 2r$ we have:

1. If $0 < r < \frac{d(x,F)}{4}$, then

\[
\left( \frac{r'}{r} \right) \left( \frac{r'^n}{r^n} \right)^{\frac{n-2+\varepsilon}{2n}} \leq C \left( \frac{r^n}{r'^n} \right)^{1/2}
\]

and

\[
\left( \frac{r'}{r} \right)^{\frac{n+\varepsilon}{2}} \leq C \left( \frac{r'}{r} \right)^{\frac{n+\varepsilon}{2}}
\]

2. If $\frac{d(x,F)}{4} < r$, then

\[
\left( \frac{r'}{r} \right) \left( \frac{r'^n}{r^n} \right)^{\frac{n-2+\varepsilon}{2n}} \leq C \left( \frac{r^n}{r'^n} \right)^{1/2}
\]

and

\[
\left( \frac{r'}{r} \right)^{\frac{n+\varepsilon}{2}} \leq C \left( \frac{r'}{r} \right)^{\frac{n+\varepsilon}{2}}
\]

Obviously, such a $C$ exists, so our claim has been proved. 

---

**Appendix B. Proofs for \([4.3]\)**

**Proof.** (of Proposition \([4.10]\)) Let $C_1 := \inf \{ \langle v, \sigma(M_{n+1})(x)v \rangle : |x| = |v| = 1 \text{ and } x \in A \}$ and $C_2 := \sup \{ \langle v, \sigma(M_{n+1})(x)v \rangle : |x| = |v| = 1 \text{ and } x \in A \}$. Since $A$ is conical with $A \cap S^{d-1}$ compact and disjoint from the degeneration set, we have $C_1 > 0$.

For $x \in A$ we have

\[
C_1 \sum_{i=1}^{n} |x_i|^\xi |v_i|^2 \leq C_1 \sum_{i=1}^{n} |x|^\xi |v_i|^2
\]

\[
= C_1 |x|^\xi |v|^2 
\]

\[
\leq \sigma(M_n) 
\]

\[
\leq C_2 |x|^\xi |v|^2
\]

\[
= C_2 \sum_{i=1}^{n} |x|^\xi |v_i|^2
\]

\[
\leq C_2 \left( \frac{\sqrt{n}}{\varepsilon} \sum_{i=1}^{n} |x_i|^\xi |v_i|^2 \right),
\]

(B.1)
where the last inequality follows from the fact that

\[ |x|^\xi = \left( \sum_{i=1}^{n} |x_i|^2 \right)^{\xi/2} \]

\[ \leq n^{\xi/2} \max\{|x_i|^\xi : 1 \leq i \leq n\} \]

\[ \leq \left( \frac{\sqrt{n}}{\xi} \right)^{\xi} \min\{|x_i|^\xi : 1 \leq i \leq n\} \]

\[ \leq \left( \frac{\sqrt{n}}{\xi} \right)^{\xi} |x_i|^\xi. \]

(B.2)

Proof. (of Lemma 4.11) We write \[ |\langle v_i, (d(x_i + x_{i+1}) - d(x_i))v_{i+1} \rangle| \leq |\langle v_i, (d(x_i + x_{i+1}))v_{i+1} \rangle| + |\langle v_i, d(x_i)v_{i+1} \rangle| \] and estimate the two terms separately.

Since \( d \) is differentiable in the ball \( B(x_{i+1}, \frac{1}{2}|x_{i+1}|) \) a simple application of the mean value theorem of elementary calculus gives

\[ |\langle v_i, (d(x_i + x_{i+1}))v_{i+1} \rangle| \leq \sup_{0 \leq r \leq 1} \langle v_i, (x_i \cdot \nabla) d(x_i + r x_{i+1})v_{i+1} \rangle \]

\[ \leq \sup_{\frac{1}{2} \leq |y| \leq \frac{3}{2}} |\langle \hat{v}_i, (\hat{x}_i \cdot \nabla) d(y)\hat{v}_{i+1} \rangle||x_i||x_{i+1}|^{\xi-1}|v_i||v_{i+1}| \]

\[ := C|x_i||x_{i+1}|^{\xi-1}|v_i||v_{i+1}| \]

\[ = C \left( \frac{|x_i|}{|x_{i+1}|} \right)^{1-\xi/2} |x_i|^{\xi/2}|x_{i+1}|^{\xi/2}|v_i||v_{i+1}| \]

\[ \leq C \left( \frac{|x_i|}{|x_{i+1}|} \right)^{1-\xi/2} (|x_i|^\xi|v_i|^2 + |x_{i+1}|^\xi|v_{i+1}|^2). \]

(B.3)

Similarly,

\[ |\langle v_i, d(x_i)v_{i+1} \rangle| \leq (1 + \frac{\xi}{d-1})|x_i|^\xi|v_i||v_{i+1}| \]

\[ \leq (1 + \frac{\xi}{d-1}) \left( \frac{|x_i|}{|x_{i+1}|} \right)^{\xi/2} |x_i|^{\xi/2}|x_{i+1}|^{\xi/2}|v_i||v_{i+1}| \]

\[ \leq \left( \frac{1}{2} + \frac{\xi}{2d-2} \right) \left( \frac{|x_i|}{|x_{i+1}|} \right)^{\xi/2} (|x_i|^\xi|v_i|^2 + |x_{i+1}|^\xi|v_{i+1}|^2). \]

(B.4)

Therefore, by setting \( E := \max\{ \frac{C}{2}, \frac{1}{2} + \frac{\xi}{2d-2} \} \), we can conclude our claim. \( \square \)
Proof. (of Lemma 4.12) We just estimate $|\langle v_i, (d(v_{i,j}) - d(v_{i+1,j}))v_j \rangle|$ and the other part is estimated similarly.

Again an application of mean value theorem gives us

\begin{equation}
(B.5) \quad |\langle v_i, (d(x_{i,j}) - d(x_{i+1,j}))v_j \rangle| \\
\leq \sup_{0 \leq r \leq 1} \langle v_i, (x_i \cdot \nabla) d(x_{i+1,j} + rx_i)v_j \rangle \\
\leq \sup_{\frac{1}{2} \leq |y| \leq \frac{3}{2}} |\langle \dot{v}_i, (\dot{x}_i \cdot \nabla) d(y)\rangle| |x_i||x_{i+1,j}|^{\xi-1}|v_i||v_j| \\
:= C|x_i||x_{i+1,j}|^{\xi-1}|v_i||v_j| \\
\leq C\left(\frac{|x_i|}{|x_{i+1,j}|}\right)^{1-\xi/2}\left(\frac{|x_{i+1,j}|}{|x_j|}\right)^{\xi/2}|x_i|^{\xi/2}|x_j|^{\xi/2}|v_i||v_j| \\
\leq \frac{C}{2}\left(\frac{|x_i|}{|x_{i+1,j}|}\right)^{1-\xi/2}\left(\frac{|x_{i+1,j}|}{|x_j|}\right)^{\xi/2}\left(|x_i|^\xi |v_i|^2 + |x_{i+1,j}|^{\xi}|v_{i+1,j}|^2\right).
\end{equation}

\proofsymbol

Proof. (of Lemma 4.13) First we make a split:

\begin{equation}
(B.6) \quad |\langle v_i, (d(x_{i,j}) - d(x_{i+1,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j \rangle| \\
\leq |\langle v_i, (d(x_{i,j}) - d(x_{i+1,j})v_j \rangle| + |\langle v_i, d(x_{i,j-1})v_j \rangle| + |\langle v_i, d(x_{i+1,j-1})v_j \rangle|.
\end{equation}

Now the first term is estimated exactly as in the previous Lemma and the latter as follows. (Actually we only estimate the second one, the third one is handled identically).

\begin{equation}
(B.7) \quad |\langle v_i, d(x_{i,j-1})v_j \rangle| \\
\leq (1 + \frac{\xi}{d-1})|x_{i,j-1}|^{\xi}|v_i||v_j| \\
\leq (1 + \frac{\xi}{d-1})\left(\frac{|x_{i,j-1}|}{|x_i|}\right)^{\xi}\left(\frac{|x_i|}{|x_j|}\right)^{\xi/2}|x_i|^{\xi/2}|x_j|^{\xi/2}|v_i||v_j| \\
\leq \left(\frac{1}{2} + \frac{\xi}{2d-2}\right)\left(\frac{|x_i|}{|x_j|}\right)^{\xi}\left(\frac{|x_i|}{|x_j|}\right)^{\xi/2}\left(|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2\right) \\
\leq \left(\frac{1}{2} + \frac{\xi}{2d-2}\right)3^{\xi}\left(\frac{|x_i|}{|x_j|}\right)^{\xi/2}\left(|x_i|^{\xi}|v_i|^2 + |x_j|^{\xi}|v_j|^2\right).
\end{equation}
\proofsymbol
Proof. (of Lemma 4.14) Two applications of the mean value theorem give us

\[ |\langle v_i, (d(x_{i,j}) - d(x_{i,j-1}) + d(x_{i+1,j-1}))v_j \rangle| \]
\[ \leq \sup_{0 \leq r \leq 1} \langle v_i, ((x_i \cdot \nabla)d(x_{i+1,j} + rx_i) - (x_i \cdot \nabla)d(x_{i+1,j-1} + rx_i)v_j \rangle \]
\[ \leq \sup_{0 \leq r, r' \leq 1} |\langle v_i, (\dot{x}_i \cdot \nabla)(\dot{x}_j \cdot \nabla)d(y)\dot{v}_j \rangle| |x_i||x_j||x_{i+1,j-1}|^{\xi-2}|v_i||v_j| \]
\[ := 2E|x_i||x_j||x_{i+1,j-1}|^{\xi-2}|v_i||v_j| \]
\[ \leq E\left(\frac{|x_i|}{|x_{i+1,j-1}|}\right)^{1-\xi/2}\left(\frac{|x_j|}{|x_{i+1,j-1}|}\right)^{1-\xi/2}(|x_i|^\xi|v_i|^2 + |x_j|^\xi|v_j|^2) \]

\[ \Box \]

Proof. (of Lemma 4.15) We estimate the terms individually. The mean value theorem gives us

\[ |\langle v_i, (d(x_{i,j}) - d(x_A))v_j \rangle| \]
\[ \leq 2^{\xi/2}C\left( \sum_{k \in [i,j]\setminus A} |x_k||x_A|^{\xi-1}|v_i||v_j| \right) \]
\[ \leq 2^{\xi/2}C\left( \sum_{k \in [i,j]\setminus A} \frac{|x_k|}{|x_A|} \right)^{1-\xi/2}\left( \frac{|x_A|}{|x_j|} \right)^{\xi/2}. \]

Since we assumed that \( \sum_{k \in [i,j]\setminus A} |x_k| \leq \frac{1}{2} \min\{ |x_{k,l}| : k, l \in A, k \leq l \} \), we have

\[ 2^{\xi/2}C\left( \sum_{k \in [i,j]\setminus A} \frac{|x_k|}{|x_A|} \right)^{1-\xi/2}\left( \frac{|x_A|}{|x_j|} \right)^{\xi/2} \]
\[ \leq 2^{\xi/2-1}C\left( \frac{\sum_{k \in [i,j]\setminus A} |x_k|}{\min\{ |x_{k,l}| : k, l \in A, k \leq l \}} - \sum_{k \in [i,j]\setminus A} |x_k| \right)^{1-\xi/2}\left( \frac{|x_A|}{|x_j|} \right)^{\xi/2} \]
\[ \cdot (|x_i|^\xi|v_i|^2 + |x_j|^\xi|v_j|^2) \]
\[ \leq C\left( \frac{\sum_{k \in [i,j]\setminus A} |x_k|}{\min\{ |x_{k,l}| : k, l \in A, k \leq l \}} \right)^{1-\xi/2}\left( \frac{\sum_{k \in A} |x_k|}{|x_j|} \right)^{\xi/2} \]
\[ \cdot (|x_i|^\xi|v_i|^2 + |x_j|^\xi|v_j|^2) \]
Similar estimates hold for the other terms, except when \( A = \{i, j\} \), which causes modifications to the last pair of terms. Then

\[
|\langle v_i, d(x_{i+1,j-1})v_j \rangle| \\
\leq (1 + \frac{\xi}{d-1})|x_{i+1,j-1}|^\xi|v_i||v_j| \\
\leq \left( \frac{1}{2} + \frac{\xi}{2d-2} \right) \left( \sum_{k \in [i,j] \setminus A} \frac{|x_k|}{|x_j|} \right)^{\xi/2} (|x_i|^\xi|v_i|^2 + |x_j|^\xi|v_j|^2).
\]

(\text{B.11})

\[\square\]

\textbf{Appendix C. Proof of Proposition 6.2}

In order to prove Proposition 6.2 we first need a Lemma.

\textbf{Lemma C.1.} Let \( 2 \leq l \in \mathbb{N} \). Then there is \( C < \infty \) such that

\[
(\text{C.1}) \int_{\mathbb{R}^d} d^d y \ (k + |x - y|)^{2-\xi-ld} (1 + |y|)^{2-\xi-d} \leq C k^{2-\xi-(l-1)d} (1 + |x|)^{2-\xi-d}.
\]

\textit{Proof.} We split the domain of integration into three parts and estimate these separately:

\[
(\text{C.2}) \quad \int_{\mathbb{R}^d} d^d y \ (k + |x - y|)^{2-\xi-ld} (1 + |y|)^{2-\xi-d} \\
= \int_{|x-y| \leq |x|/2} + \int_{|y| \leq |x|/2} + \int_{|y|,|x-y| \geq |x|/2} =: (\ast^1) + (\ast^2) + (\ast^3)
\]

To estimate \((\ast^1)\) we note that in \(|x - y| \leq |x|/2\) we have \(|x|/2 \leq |y|\) which implies that in \(|x - y| \leq |x|/2\) we have

\[
(1 + |y|)^{2-\xi-d} \leq (1 + |x|/2)^{2-\xi-d} \leq C_1 (1 + |x|)^{2-\xi-d}.
\]

Therefore

\[
(\ast^1) \leq C_1 \int_{|x-y| \leq |x|/2} d^d y \ (k + |x - y|)^{2-\xi-ld} (1 + |x|)^{2-\xi-d} \\
= C_1 k^{2-\xi-(l-1)d} (1 + |x|)^{2-\xi-d} \int_{|x-y| \leq |x|/(2k)} d^d y \ (1 + |x - y|)^{2-\xi-ld} \\
\leq C_1 k^{2-\xi-(l-1)d} (1 + |x|)^{2-\xi-d} \int_{\mathbb{R}^d} d^d y \ (1 + |x - y|)^{2-\xi-ld} \\
\leq C_2 k^{2-\xi-(l-1)d} (1 + |x|)^{2-\xi-d}.
\]
We make in a similar estimate in (\(\star^2\)): in \(|y| \leq |x|/2\) we have \(|x|/2 \leq |x-y|\) which implies that in \(|y| \leq |x|/2\) we have

\[
(k + |x-y|)^{2-\xi-ld} \leq (k + |x|/2)^{2-\xi-ld} \leq C_3(k + |x|)^{2-\xi-ld}.
\]

Now we can compute:

\[
(\star^2) \leq C_3 \int_{|y| \leq |x|/2} d^d y \, (k + |x|)^{2-\xi-ld}(1 + |y|)^{2-\xi-d} \leq C_4(k + |x|)^{2-\xi-ld} \leq C_4 k^{2-\xi-(l-1)d} |x|^{-d} |x|^d \leq C_5 k^{2-\xi-(l-1)d} (1 + |x|)^{2-\xi-d}.
\]

To treat the case \(|x| \leq 1\), we compute:

\[
C_4(k + |x|)^{2-\xi-ld}|x|^d \leq C_4 k^{2-\xi-(l-1)d} |x|^{-d} |x|^d \leq C_5 k^{2-\xi-(l-1)d} (1 + |x|)^{2-\xi-d}.
\]

If \(|x| \geq 1\) we have

\[
C_4(k + |x|)^{2-\xi-ld}|x|^{2-\xi} \leq C_4 k^{2-\xi-(l-1)d} |x|^{-d} |x|^{2-\xi} = C_4 k^{2-\xi-(l-1)d} |x|^{2-\xi-d}.
\]

Finally, we handle (\(\star^3\)). When \(|y|, |x-y| \geq |x|/2\), we have \(|y|/3 \leq |x-y|\). Since this might not be obvious, we compute: Since \(B(x, |x|/2) \subseteq B(0, 3|x|/2)\), we have

\[
|y|/3 = |x|/2 + \frac{1}{3} d(y, B(0, 3|x|/2)) \leq |x|/2 + d(y, B(0, 3|x|/2)) \leq |x|/2 + d(y, B(x, |x|/2)) = |x-y|.
\]

Therefore, when \(|y|, |x-y| \geq |x|/2\), we have

\[
(k + |x-y|)^{2-\xi-ld}(1 + |y|)^{2-\xi-d} \leq C_6(k + |y|)^{2-\xi-ld}(1 + |y|)^{2-\xi-d}
\]

and thus

\[
(\star^3) \leq C_6 \int_{|y| \geq |x|/2} d^d y \, (k + |y|)^{2-\xi-ld}(1 + |y|)^{2-\xi-d} =: (\star^4)
\]

We split the analysis of (\(\star^4\)) into two subcases: \(|x| \geq 2\) and \(|x| \leq 2\).
If $|x| \geq 2$, then we have

$$\begin{align*}
(*^4) &\leq C_6 \int_{|y| \geq |x|/2} (k + |y|)^{2-\xi - ld} |y|^{2-\xi - d}
\end{align*}$$

(C.12)

$$= C_7 k^{2(2-\xi) - ld} \int_{|y| \geq |x|/(2k)} d^d y (1 + |y|)^{2-\xi - ld} |y|^{2-\xi - d}$$

$$= C_8 k^{2(2-\xi) - ld} (1 + |x|/k)^{2(2-\xi) - ld} =: (*^5)$$

If $|x| \leq k$, then

$$(*^5) = C_9 k^{2(2-\xi) - ld} \leq C_9 k^{2-\xi - (l-1)d} (1 + |x|)^{2-\xi - d}.$$  

On the other hand, if $k \leq |x|$, then

$$(*^5) = C_8 |x|^{2(2-\xi) - ld} \leq C_{10} k^{2-\xi - (l-1)d} (1 + |x|)^{2-\xi - d}.$$

If instead of $|x| \geq 2$ we have $|x| \leq 2$ in $(*)^4$, we compute

(C.15)

$$(*^4) \leq C_6 \int_{\mathbb{R}^d} d^d y (k + |y|)^{2-\xi - ld} (1 + |y|)^{2-\xi - d}$$

$$\leq C_{11} \int_{|y| \leq 1} d^d y (k + |y|)^{2-\xi - ld} + C_{11} \int_{|y| \geq 1} (k + |y|)^{2-\xi - ld} |y|^{2-\xi - d}$$

$$\leq C_{12} k^{2-\xi - (l-1)d} \int_{\mathbb{R}^d} d^d y (1 + |y|)^{2-\xi - ld} + C_{12} k^{2-\xi - (l-1)d}$$

$$\leq C_{13} k^{2-\xi - (l-1)d} (1 + |x|)^{2-\xi - d}.$$  

\[\Box\]

Proof. (of Proposition 6.2.) Without loss of generality, we may assume that $\chi$ is the characteristic function of the unit ball. First we integrate $y_l$ out:

Write $k := \sum_{i=1}^{l-1} |x_i - y_i|$. Now we have

$$\int_{y_l \in B(0,1)} d^d y_l (k + |x_l - y_l|)^{2-\xi - ld}$$

(C.16)

$$\leq C_1 \begin{cases} 
(k + |x_l|)^{2-\xi - ld} & \text{if } |x_l| \geq 2, \\
k^{2-\xi - (l-1)d} & \text{if } |x_l| \leq 2 \text{ and } k \leq 1 \text{ and } \\
k^{2-\xi - ld} & \text{if } |x_l| \leq 2 \text{ and } k \geq 1.
\end{cases}$$
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The first case, i.e. $|x_l| \geq 2$, is computed by a repeated application of Lemma \text{C.1}:

\[
\int_{\mathbb{R}^{(l-1)d}} \prod_{i=1}^{l-1} d^d y_i \left( |x_l| + \sum_{i=1}^{l-1} |x_i - y_i| \right)^{2-\xi-d} \prod_{i=1}^{l-1} (1 + |y_i|)^{2-\xi-d}
\]

\[
\leq C_2 (1 + |x_{l-1}|)^{2-\xi-d} \int_{\mathbb{R}^{(l-1)d}} \prod_{i=1}^{l-1} d^d y_i.
\]

\[
\cdot \left( |x_l| + \sum_{i=1}^{l-2} |x_i - y_i| \right)^{2-\xi-(l-1)d} \prod_{i=1}^{l-2} (1 + |y_i|)^{2-\xi-d}
\]

\[
(C.17)
\]

In the second case, i.e. $|x_l| \leq 2$ and $k \leq 1$, we get

\[
\int_{k \leq \mathbb{R}^d} \prod_{i=1}^{l-1} d^d y_i k^{2-\xi-(l-1)d} \prod_{i=1}^{l-1} (1 + |y_i|)^{2-\xi-d}
\]

\[
\leq \sup_{k \leq \mathbb{R}^d} \prod_{i=1}^{l-1} (1 + |y_i|)^{2-\xi-d} \int_{k \leq \mathbb{R}^d} \prod_{i=1}^{l-1} d^d y_i k^{2-\xi-(l-1)d}
\]

\[
\leq C' \prod_{i=1}^{l-1} (1 + |x_i|)^{2-\xi-d}.
\]

\[
(C.18)
\]

The third case, i.e. $|x_l| \leq 2$ and $k \geq 1$, uses the following trick:

\[
\int_{k \geq \mathbb{R}^d} \prod_{i=1}^{l-1} d^d y_i k^{2-\xi-d} \prod_{i=1}^{l-1} (1 + |y_i|)^{2-\xi-d}
\]

\[
\leq C_2' \int_{\mathbb{R}^{(l-1)d}} \prod_{i=1}^{l-1} d^d y_i (1 + k)^{2-\xi-d} \prod_{i=1}^{l-1} (1 + |y_i|)^{2-\xi-d} = (*)
\]

\[
(C.19)
\]

Now repeating the computation of the first case, we get:

\[
(*) \leq C_3' \prod_{i=1}^{l-1} (1 + |x_i|)^{2-\xi-d} \leq C_4' \prod_{i=1}^{l} (1 + |x_i|)^{2-\xi-d}.
\]

\[
(C.20)
\]
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