Schmidt rank constraints in quantum information theory

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Abstract
Can vectors with low Schmidt rank form mutually unbiased bases? Can vectors with high Schmidt rank form positive under partial transpose states? In this work, we address these questions by presenting several new results related to Schmidt rank constraints and their compatibility with other properties. We provide an upper bound on the number of mutually unbiased bases of \( \mathbb{C}^m \otimes \mathbb{C}^n \ (m \leq n) \) formed by vectors with low Schmidt rank. In particular, the number of mutually unbiased product bases of \( \mathbb{C}^m \otimes \mathbb{C}^n \) cannot exceed \( m + 1 \), which solves a conjecture proposed by McNulty et al. Then, we show how to create a positive under partial transpose entangled state from any state supported on the antisymmetric space and how their Schmidt numbers are exactly related. Finally, we show that the Schmidt number of operator Schmidt rank 3 states of \( M_m \otimes M_n \ (m \leq n) \) that are invariant under left partial transpose cannot exceed \( m - 2 \).

Keywords Mutually unbiased bases · Schmidt number · Entanglement · Operator Schmidt rank · PPT states

Mathematics Subject Classification 15A69 · 81P40

1 Introduction

The Schmidt rank is a fundamental concept in quantum theory due to its connection to entanglement, so it is natural to wonder how constraints on this rank affect results related to entanglement and other compatible properties. In this work, we investigate three situations, relevant to quantum information theory, where restrictions on this rank are imposed. The first is in the context of mutually unbiased bases.
Consider $s$ orthonormal bases of a $d$-dimensional Hilbert space: $\{ |\Psi_{j1} \rangle, \ldots, |\Psi_{jd} \rangle \}$, $j = 1, \ldots, s$. They are said to be mutually unbiased if

$$| \langle \Psi_{aj}, \Psi_{bi} \rangle | = \frac{1}{\sqrt{d}}$$

for every $\{i, j\} \subset \{1, \ldots, d\}$, $\{a, b\} \subset \{1, \ldots, s\}$ and $a \neq b$.

These bases have been used in state determination, quantum state tomography and cryptography [4, 12, 19, 36, 37]. Determining the maximum number of mutually unbiased bases in an arbitrary dimension $d$ is one of the open problems. It is known that this number cannot exceed $d + 1$. In addition, when $d$ is a prime power, that maximum is exactly $d + 1$ [1, 2, 6, 14, 33, 37].

Here, we address the following problem:

How big is the number of mutually unbiased bases of $\mathbb{C}^m \otimes \mathbb{C}^n$ or $\mathbb{R}^m \otimes \mathbb{R}^n$ formed by vectors with Schmidt rank less or equal to $k$, where $k < m \leq n$?

We show that the number of such bases cannot exceed

$$\frac{k(m^2 - 1)}{m - k} \text{ in } \mathbb{C}^m \otimes \mathbb{C}^n \text{ and } \frac{k(m(m + 1) - 2)}{2(m - k)} \text{ in } \mathbb{R}^m \otimes \mathbb{R}^n.$$  

Note that our upper bound equals $m + 1$, when $k = 1$, in $\mathbb{C}^m \otimes \mathbb{C}^n$. Thus, the number of mutually unbiased product bases of $\mathbb{C}^m \otimes \mathbb{C}^n$ cannot exceed $m + 1$. This solves conjecture 1 proposed by McNulty et al. in [24] (Corollary 2).

The connection of these results with quantum information theory is that they can be interpreted as an upper bound on the number of complementary measurements in bases with little entanglement.

The second situation is in the context of entanglement quantification. At this point, we are interested in the construction of positive under partial transpose states (PPT states) using vectors with high Schmidt rank. This can be accurately done using the notion of the Schmidt number [31, 32].

Given a positive semidefinite Hermitian matrix $\delta = \sum_{i=1}^{n} A_i \otimes B_i \in \mathcal{M}_k \otimes \mathcal{M}_m$, define its Schmidt number by

$$SN(\delta) = \min \left\{ \max_j \{ SR(|w_j|) \}, \, \delta = \sum_{j=1}^{m} |w_j)(w_j| \right\}$$

(This minimum is taken over all decompositions of $\delta$ as $\sum_{j=1}^{m} |w_j)(w_j|$, where $|w_j \rangle \in \mathbb{C}^k \otimes \mathbb{C}^m$ for every $j$ and $SR(|w_j|)$ stands for the Schmidt rank of $|w_j\rangle$).

Recall that $\delta$ is separable if $SN(\delta) = 1$ and entangled if $SN(\delta) > 1$.

A large Schmidt number is associated with an idea of strong entanglement, but entangled PPT states are considered a weaker form of entanglement. Discovering the best possible Schmidt number for PPT states has become an important problem [13, 22, 27, 40].
An example of a PPT state with Schmidt number half of its local dimension has been found recently in [11, Proposition 2]. This state is a mixture of the orthogonal projection on the symmetric space of \( \mathbb{C}^m \otimes \mathbb{C}^m \), which we denote by \( P_{sym}^m \), with a particular pure state. Although it seems delicate, the construction is actually quite robust.

Given any state \( \gamma \) supported on the antisymmetric subspace of \( \mathbb{C}^m \otimes \mathbb{C}^m \), we show that

\[
SN(P_{sym}^m + \epsilon \gamma) = \frac{1}{2} SN(\gamma)
\]

and \( P_{sym}^m + \epsilon \gamma \) is positive under partial transpose for \( \epsilon \in [0, \frac{1}{6}] \) (Theorem 2).

Moreover, if \( \mathbb{C}^m \) contains \( m^2 \) equiangular lines (i.e., a SIC-POVM [3,5,28]), then we can replace \( \frac{1}{6} \) above by \( \frac{m+1}{6m} \) and the result remains the same. The existence of a SIC-POVM in any \( \mathbb{C}^m \) is an open problem. So this little improvement can only be made for some values of \( m \) [26]. But if we know beforehand that \( SN(\gamma) > 2 \), then we can replace \( \frac{1}{6} \) by 1 in the interval above.

These mixtures have been firstly considered in [30] to construct entangled PPT states. It was already noticed in [30] that \( SN(\gamma) > 2 \) would create an entangled mixture. Later in [11,25], it was noticed that \( SN(P_{sym}^m + \epsilon \gamma) \geq \frac{1}{2} SN(\gamma) \), for any \( \epsilon > 0 \), and arbitrary state \( \gamma \) supported on the antisymmetric space. Our new result shows how the Schmidt numbers of this PPT mixture and the original \( \gamma \) are exactly related for sufficiently small \( \epsilon \).

In the third and final situation, we investigate the relationship between the operator Schmidt rank and the Schmidt number of PPT states with some extra conditions.

The operator Schmidt rank (or tensor rank) of \( \delta \in \mathcal{M}_m \otimes \mathcal{M}_n \) is 1, if \( \delta = A_1 \otimes A_2 \neq 0 \). The operator Schmidt rank of an arbitrary \( \gamma \in \mathcal{M}_m \otimes \mathcal{M}_n \setminus \{0\} \) is the minimal number of tensors with operator Schmidt rank 1 that can be added to form \( \gamma \).

We show that the Schmidt number of any state of \( \mathcal{M}_m \otimes \mathcal{M}_n \ (m \leq n) \) invariant under left partial transpose with operator Schmidt rank 3 is at most \( m - 2 \) (Corollary 3). It complements [22, Theorem 5]. This result is also related to the conjecture that says that the Schmidt number of any PPT state of \( \mathcal{M}_m \otimes \mathcal{M}_n \ (m \leq n) \) cannot be \( m \) ([27]).

In particular, this theorem says that every state invariant under left partial transpose with operator Schmidt rank 3 in \( \mathcal{M}_3 \otimes \mathcal{M}_n \) is separable (Theorem 3). This result is a new contribution to an ongoing investigation that relates low operator Schmidt rank to separability.

States with operator Schmidt rank 2 are always separable (see [7, Theorem 58] or [15]). In addition, states of \( \mathcal{M}_2 \otimes \mathcal{M}_m \) with operator Schmidt rank 3 are also separable (see [8, Theorem 19]). However, this is not valid in \( \mathcal{M}_3 \otimes \mathcal{M}_m \ (m \geq 3) \) (see [8, Proposition 25]). The invariance under left partial transpose is a sufficient condition for the separability of states of \( \mathcal{M}_3 \otimes \mathcal{M}_m \) with operator Schmidt rank 3.

This work is organized as follows. In Sect. 2, we obtain an upper bound on the number of mutually unbiased bases formed by vectors with Schmidt rank less or equal to \( k \). In Sect. 3, we constructed entangled PPT states from states supported on the antisymmetric subspace and we show how their Schmidt numbers are exactly related.
In Sect. 4, we prove that the Schmidt number of operator Schmidt rank 3 states of $\mathcal{M}_k \otimes \mathcal{M}_m$ ($k \leq m$) that are invariant under left partial transpose cannot exceed $k - 2$.

## 2 Mutually unbiased bases

In this section, we provide an upper bound on the number of mutually unbiased bases of $\mathbb{C}^m \otimes \mathbb{C}^n$ formed by vectors with Schmidt rank less or equal to $k$ ($k < m \leq n$).

Denote by $\mathcal{M}_k$ the set of complex matrices of order $k$. Identify $\mathcal{M}_k \otimes \mathcal{M}_m \simeq \mathcal{M}_{km}$ and $\mathbb{C}^k \otimes \mathbb{C}^m \simeq \mathbb{C}^{km}$ via Kronecker product. Denote the flip operator by $F_d \in \mathcal{M}_{d} \otimes \mathcal{M}_{d}$ (i.e., $F_d(|a) \otimes |b\rangle = |b\rangle \otimes |a\rangle$, for every $|a\rangle, |b\rangle \in \mathbb{C}^d$).

**Definition 1** Let $P(\rho) = Tr(\rho^2)$, where $\rho$ is a square matrix and $Tr(\rho)$ is its trace. Denote the left and the right partial trace of $\gamma \in \mathcal{M}_m \otimes \mathcal{M}_n$ by $Tr_A(\gamma) \in \mathcal{M}_n$ and $Tr_B(\gamma) \in \mathcal{M}_m$, respectively. Let the Schmidt rank of $|w\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$ be the rank of $Tr_A(|w\rangle\langle w|)$ and denote it by $SR(|w\rangle).

**Remark 1** Let $Y = Tr_A(|w\rangle\langle w|)$. By Cauchy–Schwarz inequality, notice that

$$P(Tr_A(|w\rangle\langle w|)) = Tr(Y^2) \geq \frac{Tr(Y)^2}{rank(Y)}.$$

If $|w\rangle$ is a unit vector, then $Tr(Y) = 1$. Since $rank(Y) = SR(|w\rangle)$,

$$P(Tr_A(|w\rangle\langle w|)) \geq \frac{1}{SR(|w\rangle)}.$$

This last inequality shall be used in Corollary 1.

**Definition 2** Let $\{|e_1\rangle, \ldots, |e_m\rangle\}$ and $\{|f_1\rangle, \ldots |f_n\rangle\}$ be the canonical bases of $\mathbb{C}^m$ and $\mathbb{C}^n$, respectively.

1. Let $|\Phi\rangle = \sum_{j=1}^{n} |f_j\rangle \otimes |f_j\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$.
2. Let $|\Psi\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} |e_i\rangle \otimes |f_j\rangle \otimes |e_i\rangle \otimes |f_j\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^m \otimes \mathbb{C}^n$.
3. Let $Tr_{1,3}(X) \in \mathcal{M}_n \otimes \mathcal{M}_n$ be the partial trace of $X \in \mathcal{M}_m \otimes \mathcal{M}_n \otimes \mathcal{M}_m \otimes \mathcal{M}_n$ (We are tracing out the first and the third sites).
4. Let the functional $f : \mathcal{M}_m \otimes \mathcal{M}_n \otimes \mathcal{M}_m \otimes \mathcal{M}_n \to \mathbb{C}$ be as $f(X) = Tr(Tr_{1,3}(X)|\Phi\rangle\langle\Phi|)$. Note that $f$ is a positive functional, i.e., it sends positive semidefinite Hermitian matrices to nonnegative real numbers.

**Lemma 1** Let $|\omega\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$. Then,

1. $f(|\omega\rangle \langle \omega| \otimes |\overline{\omega}\rangle \langle \overline{\omega}|) = P(Tr_A(|\omega\rangle\langle \omega|))$,
2. $f(|\Psi\rangle\langle \Psi|) = mn^2$,
3. $f(Id_{m \times m} \otimes Id_{n \times n} \otimes Id_{m \times m} \otimes Id_{n \times n}) = m^2 n$. 

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4. $f(F_{mn}) = mn$, 
where $F_{mn} \in M_{mn} \otimes M_{mn}$ is the flip operator. Recall the identification $M_m \otimes M_n \simeq M_{mn}$.

5. $f(P_{mn}^{\text{sym}2}) = \frac{m^2n + mn}{2}$,
where $P_{mn}^{\text{sym}2} \in M_{mn} \otimes M_{mn}$ is the orthogonal projection on the symmetric subspace of $\mathbb{C}^{mn} \otimes \mathbb{C}^{mn}$. Recall that $P_{sym}^{\text{sym}2} = \frac{1}{2}(I_{dm\times m} \otimes I_{dn\times n} \otimes I_{dm\times m} \otimes I_{dn\times n} + F_{mn})$.

**Proof** The proof of this lemma is straightforward. It is left to the reader. \(\square\)

**Theorem 1** Let $\{|\Psi_{j1}\rangle, \ldots, |\Psi_{j(mn)}\rangle\}$, $j = 1, \ldots, t$, be mutually unbiased bases of a mn-dimensional Hilbert space $\mathcal{H}$. Then,

(a) $\sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(Tr_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \leq (m^2 + t - 1)n$, if $\mathcal{H} = \mathbb{C}^m \otimes \mathbb{C}^n$,

(b) $\sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(Tr_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \leq \left(\frac{m(m+1)}{2} + t - 1\right)n$, if $\mathcal{H} = \mathbb{R}^m \otimes \mathbb{R}^n$.

**Proof** Consider the orthogonal projections $A_1, \ldots, A_t \in M_m \otimes M_n \otimes M_m \otimes M_n$ defined by

$$A_j = \sum_{i=1}^{mn} |\Psi_{ji}\rangle\langle\Psi_{ji}| \otimes |\Psi_{ji}\rangle\langle\Psi_{ji}|.$$ 

By [10, Lemma 34], $A_j A_k = A_k A_j = \frac{1}{mn} |\Psi\rangle\langle\Psi|$, for every $j, k \in \{1, \ldots, t\}$ and $j \neq k$.

Therefore, the matrix

$$B = I_{dm\times m} \otimes I_{dn\times n} \otimes I_{dm\times m} \otimes I_{dn\times n} + \frac{t - 1}{mn} |\Psi\rangle\langle\Psi| - \sum_{j=1}^{t} A_j$$

(1)

is positive semidefinite.

**Case (a)**: $\mathcal{H} = \mathbb{C}^m \otimes \mathbb{C}^n$

By Lemma 1, Eq. (1) and the positivity of $B$, we have

$$f(B) = m^2n + (t - 1)n - \sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(Tr_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \geq 0.$$ 

Finally, $\sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(Tr_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \leq (m^2 + t - 1)n$.

**Case (b)**: $\mathcal{H} = \mathbb{R}^m \otimes \mathbb{R}^n$
In this case, $|\Psi_{ji}\rangle = |\Psi_{ji}\rangle$ for every $j, i$. Therefore, every $A_j$ is supported on the symmetric subspace of $\mathcal{H} \otimes \mathcal{H}$. Hence,

$$P_{sym}^{mn,2} B P_{sym}^{mn,2} = P_{sym}^{mn,2} + \frac{t - 1}{mn} |\Psi\rangle\langle\Psi| - \sum_{j=1}^{t} A_j$$  \hspace{1cm} (2)

is positive semidefinite.

By Lemma 1, Eq. (2) and the positivity of $P_{sym}^{mn,2} B P_{sym}^{mn,2}$, we have

$$f(P_{sym}^{mn,2} B P_{sym}^{mn,2}) = \left(\frac{m(m+1)}{2} + t - 1\right) n - \sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(\text{Tr}_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \geq 0.$$ 

Finally, \(\sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(\text{Tr}_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \leq \left(\frac{m(m+1)}{2} + t - 1\right) n. \hspace{1cm} \square\)

Remark 2 Using the notation of the Proof of Theorem 1, we can describe the equation obtained in [10, Lemma 35] as $B = \text{id}_{m \times m} \otimes \text{id}_{n \times n} \otimes \text{id}_{m \times m} \otimes \text{id}_{n \times n} + |\Psi\rangle\langle\Psi| - \sum_{j=1}^{mn+1} A_j = 0$. Thus,

$$0 = f(B) = mn(m+n) - \sum_{j=1}^{mn+1} \sum_{i=1}^{mn} \mathcal{P}(\text{Tr}_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)).$$

Therefore, we recover the conservation law obtained in [34].

Corollary 1 Let $k < m \leq n$. The number of mutually unbiased bases formed by vectors with Schmidt rank less or equal to $k$ cannot exceed

(a) $\frac{k(m^2 - 1)}{m - k}$ in $\mathbb{C}^m \otimes \mathbb{C}^n$ and
(b) $\frac{k(m(m+1) - 2)}{2(m-k)}$ in $\mathbb{R}^m \otimes \mathbb{R}^n$.

In particular, the number of mutually unbiased product bases cannot exceed $m+1$ in $\mathbb{C}^m \otimes \mathbb{C}^n$.

Proof Let \(\{|\Psi_{j1}\rangle, \ldots, |\Psi_{j(mn)}\rangle\}\), for $j = 1, \ldots, t$, be mutually unbiased bases formed by vectors with Schmidt rank less or equal to $k$.

Since the Schmidt rank of each unit vector $|\Psi_{ji}\rangle$ is less or equal to $k$,

$$\mathcal{P}(\text{Tr}_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \geq \frac{1}{k} \text{ (Remark 1)}.$$ 

By Theorem 1,

(a) $t \frac{mn}{k} \leq \sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(\text{Tr}_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \leq (m^2 + t - 1)n$ in $\mathbb{C}^m \otimes \mathbb{C}^n$ and
(b) $t \frac{mn}{k} \leq \sum_{j=1}^{t} \sum_{i=1}^{mn} \mathcal{P}(\text{Tr}_A(|\Psi_{ji}\rangle\langle\Psi_{ji}|)) \leq \left(\frac{m(m+1)}{2} + t - 1\right) n$ in $\mathbb{R}^m \otimes \mathbb{R}^n$.

Hence,
(a) \( t \leq k \left( \frac{m^2 - 1}{m - k} \right) \) in \( \mathbb{C}^m \otimes \mathbb{C}^n \) and
(b) \( t \leq \frac{k(m(m+1) - 2)}{2(m-k)} \) in \( \mathbb{R}^m \otimes \mathbb{R}^n \).

\[ \square \]

**Remark 3** The upper bounds obtained in the last corollary do not depend on \( n \). For instance, in \( \mathbb{C}^m \otimes \mathbb{C}^n \) for \( n \) much larger than \( m \), our upper bound turns out to be much smaller than \( mn + 1 \). In fact, we have

\[
k(m^2 - 1) < mn + 1, \quad \text{if and only if, } k < \frac{mn + 1}{m + n}.
\]

Now, if no restriction on \( n \) is imposed, besides \( n \geq m \), we have \( \frac{k(m^2 - 1)}{m - k} < mn - 1 \) for \( k \leq \frac{m}{2} \). This is interesting because \( mn - 1 \) turns out to be an upper bound on the number of mutually unbiased bases of \( \mathbb{C}^m \otimes \mathbb{C}^n \) formed by vectors with Schmidt coefficients equal to \( \frac{1}{\sqrt{k}} \) and \( n \) is a multiple of \( m \) [29, Theorem 4]. Our result improves this upper bound when \( k \leq \frac{m}{2} \). Nothing can be said about the case \( k > \frac{m}{2} \) with our method. There is an extensive literature on these bases with fixed Schmidt coefficients and their applications to quantum information theory [16,23,35,38,39].

The next corollary solves conjecture 1 in [24].

**Corollary 2** The maximum number of mutually unbiased product bases of \( \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n} \) is less or equal to \( \min_j d_j + 1 \). Note that if \( d_1, \ldots, d_n \) are powers of distinct primes then this maximum number is exactly \( \min_j d_j + 1 \).

**Proof** Assume without loss of generality that \( d_1 + 1 = \min_j d_j + 1 \). Since a product vector in \( \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n} \) is also a product vector in \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \ldots \mathbb{C}^{d_n} \), the maximum number of mutually unbiased product bases of \( \mathbb{C}^{d_1} \otimes \ldots \otimes \mathbb{C}^{d_n} \) is less or equal to the same number in \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \ldots \mathbb{C}^{d_n} \).

By Corollary 1, the maximum number of mutually unbiased product bases of \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \ldots \mathbb{C}^{d_n} \) cannot exceed \( d_1 + 1 \). \( \square \)

### 3 Entangled PPT mixtures

Let us call \( \delta \in \mathcal{M}_k \otimes \mathcal{M}_m \) a state, if \( \delta \) is a positive semidefinite Hermitian matrix with trace 1.

In this section, we show that \( P_{sym}^{k,2} + \epsilon \gamma \) is positive under partial transpose and

\[
SN(P_{sym}^{k,2} + \epsilon \gamma) = \frac{SN(\gamma)}{2}
\]
for sufficiently small \( \epsilon \), where \( \gamma \in \mathcal{M}_k \otimes \mathcal{M}_k \) is any state supported on the antisymmetric subspace of \( \mathbb{C}^k \otimes \mathbb{C}^k \) (Theorem 2). In order to obtain this result, we need the following equation obtained in [20].

If \( \{ |\Psi_{j1}\rangle, \ldots, |\Psi_{jk}\rangle \} \), \( 1 \leq j \leq k + 1 \), are \( k + 1 \) mutually unbiased bases of \( \mathbb{C}^k \), then

\[
2P_{\text{sym}}^{k,2} = \sum_{j=1}^{k+1} \sum_{i=1}^{k} |\Psi_{ji}\rangle\langle \Psi_{ji}| \otimes |\Psi_{ji}\rangle\langle \Psi_{ji}| \in \mathcal{M}_k \otimes \mathcal{M}_k.
\] (3)

**Definition 3** Let the right partial transpose of \( \delta = \sum_{i=1}^{n} A_i \otimes B_i \in \mathcal{M}_k \otimes \mathcal{M}_m \) be \( \delta^\Gamma = \sum_{i=1}^{n} A_i \otimes B_i^\dagger \) (The left partial transpose is defined analogously). Moreover, let us say that \( \delta \) is positive under partial transpose or simply PPT if \( \delta \) and \( \delta^\Gamma \) are positive semidefinite Hermitian matrices.

**Lemma 2** Let \( |a\rangle \in \mathbb{C}^k \) be a unit vector. Then, \( P_{\text{sym}}^{k,2} - \epsilon |a\rangle\langle a| \otimes |a\rangle\langle a| \in \mathcal{M}_k \otimes \mathcal{M}_k \) is separable for \( \epsilon \leq \frac{1}{2} \). In addition, if \( \mathbb{C}^k \) contains a SIC-POVM then the same matrix is separable if \( \epsilon \leq \frac{k+1}{2k} \).

**Proof** Let \( n \) be a prime number greater than \( k \). Let \( \{ |a_{j1}\rangle, \ldots, |a_{jn}\rangle \} \), \( 1 \leq j \leq n + 1 \), be \( n + 1 \) mutually unbiased bases of \( \mathbb{C}^n \) [19]. We can assume without loss of generality that \( |a_{11}\rangle = \left( \begin{array}{c} |a\rangle \\ 0 \end{array} \right) \in \mathbb{C}^k \times \mathbb{C}^{n-k} \).

By Eq. 3,

\[
P_{\text{sym}}^{n,2} = \frac{1}{2} \left( \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ji}\rangle\langle a_{ji}| \otimes |a_{ji}\rangle\langle a_{ji}| \right).
\]

Thus,

\[
B = P_{\text{sym}}^{n,2} - \epsilon |a_{11}\rangle\langle a_{11}| \otimes |a_{11}\rangle\langle a_{11}|
\]

\[
= \left( \frac{1}{2} - \epsilon \right) |a_{11}\rangle\langle a_{11}| \otimes |a_{11}\rangle\langle a_{11}|
\]

\[
+ \frac{1}{2} \left( \sum_{i=2}^{n} |a_{i1}\rangle\langle a_{i1}| \otimes |a_{i1}\rangle\langle a_{i1}| + \sum_{j=2}^{n+1} \sum_{i=1}^{n} |a_{ji}\rangle\langle a_{ji}| \otimes |a_{ji}\rangle\langle a_{ji}| \right).
\]

is separable for \( \epsilon \leq \frac{1}{2} \).

Now, let \( U_{k \times n} = (I_{k \times k} 0_{k \times n-k}) \) and note that

\[
(U \otimes U)B(U^* \otimes U^*) = P_{\text{sym}}^{k,2} - \epsilon |a\rangle\langle a| \otimes |a\rangle\langle a|.
\]

So \( P_{\text{sym}}^{k,2} - \epsilon |a\rangle\langle a| \otimes |a\rangle\langle a| \in \mathcal{M}_k \otimes \mathcal{M}_k \) is separable too for \( \epsilon \leq \frac{1}{2} \).
Next, if $\mathbb{C}^k$ contains a SIC-POVM, then we can write

$$P_{\text{sym}}^{k,2} = \sum_{i=1}^{k^2} \frac{k + 1}{2k} |v_i\rangle\langle v_i| \otimes |v_i\rangle\langle v_i|$$

where $|v_1\rangle = |a\rangle$ ([26, Definition 2.1]). Thus, $P_{\text{sym}}^{k,2} - \epsilon |a\rangle\langle a| \otimes |a\rangle\langle a| = (\frac{k + 1}{2k} - \epsilon) |v_1\rangle\langle v_1| \otimes |v_1\rangle\langle v_1| + \sum_{i=2}^{k^2} \frac{k + 1}{2k} |v_i\rangle\langle v_i| \otimes |v_i\rangle\langle v_i|.$

In this case, $P_{\text{sym}}^{k,2} - \epsilon |a\rangle\langle a| \otimes |a\rangle\langle a|$ is separable for $\epsilon \leq \frac{k+1}{2k}$. \hfill $\square$

**Lemma 3** Let $|a_1\rangle, |a_2\rangle$ be orthonormal vectors of $\mathbb{C}^k$ and $|s\rangle = |a_1\rangle \otimes |a_2\rangle + |a_2\rangle \otimes |a_1\rangle$. Consider $B = P_{\text{sym}}^{k,2} - \epsilon |s\rangle\langle s| \in M_k \otimes M_k$. Then,

(a) $SN(B) \leq 2$ for $\epsilon \leq \frac{1}{2}$ and arbitrary $k$,
(b) $SN(B) = 1$ for $\epsilon \in \left[0, \frac{1}{12}\right]$ and arbitrary $k$,
(c) $SN(B) = 1$ for $\epsilon \in \left[0, \frac{1}{12k}\right]$, if $\mathbb{C}^k$ contains a SIC-POVM.

**Proof** Part (a): Let $|a_1\rangle, \ldots, |a_k\rangle$ be an orthonormal basis of $\mathbb{C}^k$. Since $P_{\text{sym}}^{k,2}$ is the projection on the symmetric subspace of $\mathbb{C}^k \otimes \mathbb{C}^k$,

$$P_{\text{sym}}^{k,2} = \sum_{i=1}^{k} |a_i\rangle\langle a_i| \otimes |a_i\rangle\langle a_i|$$

$$+ \sum_{1 \leq i < j \leq k} \frac{1}{2} (|a_i\rangle \otimes |a_j\rangle + |a_j\rangle \otimes |a_i\rangle)(|a_i\rangle \otimes \langle a_j| + \langle a_j| \otimes |a_i\rangle).$$

Hence,

$$B = P_{\text{sym}}^{k,2} - \epsilon |s\rangle\langle s| = \sum_{i=1}^{k} |a_i\rangle\langle a_i| \otimes |a_i\rangle\langle a_i| + \left(\frac{1}{2} - \epsilon\right) |s\rangle\langle s|$$

$$+ \sum_{1 \leq i < j \leq k} \frac{1}{2} (|a_i\rangle \otimes |a_j\rangle + |a_j\rangle \otimes |a_i\rangle)(|a_i\rangle \otimes \langle a_j| + \langle a_j| \otimes |a_i\rangle).$$

Thus, $SN(B) \leq 2$ for $\epsilon \leq \frac{1}{2}$.

Parts (b) and (c): Let $\{|e_1\rangle, |e_2\rangle\}$, $\{|v_1\rangle, |v_2\rangle\}$ and $\{|w_1\rangle, |w_2\rangle\}$ be 3 mutually unbiased bases of $\mathbb{C}^2$, where $\{|e_1\rangle, |e_2\rangle\}$ is the canonical basis [19].

By Eq. 3,

$$2P_{\text{sym}}^{2,2} = \sum_{i=1}^{2} |e_i\rangle\langle e_i| \otimes |e_i\rangle\langle e_i| + \sum_{i=1}^{2} |v_i\rangle\langle v_i| \otimes |v_i\rangle\langle v_i| + \sum_{i=1}^{2} |w_i\rangle\langle w_i| \otimes |w_i\rangle\langle w_i|.$$
Moreover, since $P_2^{2,2}$ is the projection on the symmetric subspace of $\mathbb{C}^2 \otimes \mathbb{C}^2$,

$$2P_2^{2,2} = \sum_{i=1}^{2} 2(|e_i\rangle \langle e_i| \otimes |e_i\rangle \langle e_i|) + |v\rangle \langle v|,$$

where $|v\rangle = |e_1\rangle \otimes |e_2\rangle + |e_2\rangle \otimes |e_1\rangle$.

Next, define the isometry $U_{k \times 2}$ as $U|e_1\rangle = |a_1\rangle$ and $U|e_2\rangle = |a_2\rangle$.

Note that

$$(U \otimes U)(2P_2^{2,2})(U^* \otimes U^*)$$

$$= \sum_{j=1}^{2} 2|a_j\rangle \langle a_j| \otimes |a_j\rangle \langle a_j| + |s\rangle \langle s| = \sum_{i=1}^{6} |b_i\rangle \langle b_i| \otimes |b_i\rangle \langle b_i|,$$

where $|b_1\rangle = |a_1\rangle$, $|b_2\rangle = |a_2\rangle$, $|b_3\rangle = U|v_1\rangle$, $|b_4\rangle = U|v_2\rangle$, $|b_5\rangle = U|w_1\rangle$, $|b_6\rangle = U|w_2\rangle$.

In addition, $|b_1\rangle, \ldots, |b_6\rangle$ are unit vectors, since $U$ is an isometry.

Thus,

$$2P_{sym}^{k,2} - \epsilon \left( \sum_{j=1}^{2} 2|a_j\rangle \langle a_j| \otimes |a_j\rangle \langle a_j| + |s\rangle \langle s| \right)$$

$$= 2P_{sym}^{k,2} - \epsilon \left( \sum_{i=1}^{6} |b_i\rangle \langle b_i| \otimes |b_i\rangle \langle b_i| \right)$$

$$= \sum_{i=1}^{6} \frac{1}{6} \left( 2P_{sym}^{k,2} - 6\epsilon |b_i\rangle \langle b_i| \otimes |b_i\rangle \langle b_i| \right)$$

$$= \sum_{i=1}^{6} \frac{1}{3} \left( P_{sym}^{k,2} - 3\epsilon |b_i\rangle \langle b_i| \otimes |b_i\rangle \langle b_i| \right),$$

is separable for $\epsilon \in \left[0, \frac{1}{6}\right]$, when $k$ is arbitrary, or for $\epsilon \in \left[0, \frac{k+1}{6k}\right]$, when $\mathbb{C}^k$ contains a SIC-POVM by Lemma 2.

Finally,

$$P_{sym}^{k,2} - \epsilon |s\rangle \langle s| = \frac{1}{2} (2P_{sym}^{k,2} - 2\epsilon |s\rangle \langle s|)$$

$$= \frac{1}{2} \left[ 2P_{sym}^{k,2} - 2\epsilon \left( \sum_{j=1}^{2} 2|a_j\rangle \langle a_j| \otimes |a_j\rangle \langle a_j| + |s\rangle \langle s| \right) \right]$$

$$+ \epsilon \left( \sum_{j=1}^{2} 2|a_j\rangle \langle a_j| \otimes |a_j\rangle \langle a_j| \right).$$
We have just noticed that the first summand above is separable for \( 2\epsilon \in [0, \frac{1}{6}] \), when \( k \) is arbitrary, or for \( 2\epsilon \in \left[0, \frac{k+1}{6k}\right] \), when \( \mathbb{C}^k \) contains a SIC-POVM.

Therefore, \( P_{\text{sym}}^{k,2} - \epsilon|s\rangle\langle s| \) is separable for \( \epsilon \in \left[0, \frac{1}{12}\right] \), when \( k \) is arbitrary, or for \( \epsilon \in \left[0, \frac{k+1}{12k}\right] \), when \( \mathbb{C}^k \) contains a SIC-POVM. \( \square \)

**Lemma 4** Let \( |v\rangle \) be a unit antisymmetric vector of \( \mathbb{C}^k \otimes \mathbb{C}^k \). Consider \( B = P_{\text{sym}}^{k,2} + \epsilon|v\rangle\langle v| \in M_k \otimes M_k \). Then, \( B \) is PPT and

(a) \( SN(B) \leq \max \left\{ \frac{SR(|v\rangle)}{2}, 2 \right\} \) for \( \epsilon \in [0, 1] \) and arbitrary \( k \),

(b) \( SN(B) \leq \frac{SR(|v\rangle)}{2} \) for \( \epsilon \in \left[0, \frac{1}{8}\right] \) and arbitrary \( k \),

(c) \( SN(B) \leq \frac{SR(|v\rangle)}{2} \) for \( \epsilon \in \left[0, \frac{k+1}{6k}\right] \), if \( \mathbb{C}^k \) contains a SIC-POVM.

**Proof** Let \( SR(|v\rangle) = 2n \). By [17, Corollary 4.4.19.], there are positive numbers \( \lambda_1, \ldots, \lambda_n \) and orthonormal vectors \( |v_1\rangle, \ldots, |v_n\rangle, |w_1\rangle, \ldots, |w_n\rangle \) of \( \mathbb{C}^k \) and such that

\[
|v\rangle = \sum_{i=1}^{n} \lambda_i (|v_i\rangle \otimes |w_i\rangle - |w_i\rangle \otimes |v_i\rangle) \quad \text{and} \quad 2 \left( \sum_{i=1}^{n} \lambda_i^2 \right) = 1.
\]

Define \( |m_i\rangle = \lambda_i (|v_i\rangle \otimes |w_i\rangle + |w_i\rangle \otimes |v_i\rangle) \) for \( i = 1, \ldots, n \). By induction on \( n \), we can easily show that

\[
P_{\text{sym}}^{k,2} + \epsilon|v\rangle\langle v| = P_{\text{sym}}^{k,2} - \epsilon |m_1\rangle\langle m_1| - \cdots - \epsilon |m_n\rangle\langle m_n|
\]

\[
+ \sum_{i_1,\ldots,i_n=1}^{2} \frac{\epsilon}{2^n} |v_{i_1\ldots i_n}\rangle\langle v_{i_1\ldots i_n}|,
\]

where \( |v_{i_1\ldots i_n}\rangle = |m_n\rangle + (-1)^{i_1} |m_{n-1}\rangle + \cdots + (-1)^{i_{n-1}} |m_1\rangle + (-1)^{i_n} |v\rangle \).

Hence,

\[
B = P_{\text{sym}}^{k,2} + \epsilon|v\rangle\langle v| = \sum_{i=1}^{n} 2\lambda_i^2 \left( P_{\text{sym}}^{k,2} - \frac{\epsilon}{2} \frac{|m_i\rangle\langle m_i|}{\lambda_i^2} \right) + \sum_{i_1,\ldots,i_n=1}^{2} \frac{\epsilon}{2^n} |v_{i_1\ldots i_n}\rangle\langle v_{i_1\ldots i_n}|.
\]

(4)

Next, by Lemma 3,

(a) \( SN \left( P_{\text{sym}}^{k,2} - \frac{\epsilon}{2} \frac{|m_i\rangle\langle m_i|}{\lambda_i^2} \right) \leq 2 \), when \( k \) is arbitrary and \( \frac{\epsilon}{2} \in [0, \frac{1}{2}] \),

(b) \( SN \left( I_d + F - \frac{\epsilon}{2} \frac{|m_i\rangle\langle m_i|}{\lambda_i^2} \right) = 1 \), when \( k \) is arbitrary and \( \frac{\epsilon}{2} \in [0, \frac{1}{12}] \),

(c) \( SN \left( I_d + F - \frac{\epsilon}{2} \frac{|m_i\rangle\langle m_i|}{\lambda_i^2} \right) = 1 \), when \( \mathbb{C}^k \) contains a SIC-POVM and \( \frac{\epsilon}{2} \in [0, \frac{k+1}{12k}] \).
In addition, notice that \( SR(|v_{i_1, \ldots, i_n}\rangle) = n = \frac{SR(|v\rangle)}{2} \).

So Eq. 4 provides a way to write \( B \) using only vectors with Schmidt rank less or equal to

(a) \( \max \left\{ \frac{SR(|v\rangle)}{2}, 2 \right\} \), when \( k \) is arbitrary and \( \epsilon \in [0, 1] \),

(b) \( \frac{2(\sum_{i=1}^{n} \lambda_i^2)}{2} \), when \( k \) is arbitrary and \( \epsilon \in \left[ 0, \frac{1}{6} \right] \),

(c) \( \frac{SR(|v\rangle)}{2} \), when \( \mathbb{C}^k \) contains a SIC-POVM and \( \epsilon \in \left[ 0, \frac{k+1}{6k} \right] \).

Hence, \( SN(B) \leq \max \left\{ \frac{SR(|v\rangle)}{2}, 2 \right\} \) in case (a) and \( SN(B) \leq \frac{SR(|v\rangle)}{2} \) in cases (b) and (c).

It remains to prove that \( B \) is PPT for \( \epsilon \in [0, 1] \).

It is not difficult to check that \( \| |v\rangle\langle v| \Gamma \|_{\infty} = \max\{\lambda_1^2, \ldots, \lambda_n^2\} \). Since \( 2 \left( \sum_{i=1}^{n} \lambda_i^2 \right) = 1 \), we obtain

\[
\| |v\rangle\langle v| \Gamma \|_{\infty} \leq \frac{1}{2}.
\]

Finally, \( (P_{sym}^{k,2})^{\Gamma} \) is positive definite and its minimum eigenvalue is \( \frac{1}{2} \). So

\[
B^{\Gamma} = (P_{sym}^{k,2})^{\Gamma} + \epsilon |v\rangle\langle v|^{\Gamma}
\]

is positive semidefinite for \( \epsilon \in [0, 1] \). \( \square \)

**Theorem 2** Let \( \gamma \) be a state supported on the antisymmetric subspace of \( \mathbb{C}^k \otimes \mathbb{C}^k \).

Consider \( B = P_{sym}^{k,2} + \epsilon \gamma \in M_k \otimes M_k \). Then, \( B \) is PPT and

(a) \( SN(B) = \frac{SN(\gamma)}{2} \), if \( SN(\gamma) > 2 \) or \( SN(\gamma) = 2 \) for \( \epsilon \in \left[ 0, 1 \right] \) and arbitrary \( k \),

(b) \( SN(B) = \frac{SN(\gamma)}{2} \) for \( \epsilon \in \left[ 0, \frac{1}{6} \right] \) and arbitrary \( k \),

(c) \( SN(B) = \frac{SN(\gamma)}{2} \) for \( \epsilon \in \left[ 0, \frac{k+1}{6k} \right] \), if \( \mathbb{C}^k \) contains a SIC-POVM.

**Proof** First, by [11, Proposition 1], \( SN(B) \geq \frac{SN(\gamma)}{2} \) for every positive \( \epsilon \).

Next, let \( \gamma = \sum_{i=1}^{l} \beta_i |v_i\rangle\langle v_i| \), where \( \sum_{i=1}^{l} \beta_i = 1 \), \( \beta_i > 0 \) and \( |v_i\rangle \) is a unit antisymmetric vector of \( \mathbb{C}^k \otimes \mathbb{C}^k \) such that \( SR(|v_i\rangle) \leq SN(\gamma) \), for every \( i \).

Finally, since

\[
B = P_{sym}^{k,2} + \epsilon \gamma = \sum_{i=1}^{l} \beta_i (P_{sym}^{k,2} + \epsilon |v_i\rangle\langle v_i|),
\]

\[
SN(B) \leq \max \left\{ SN(P_{sym}^{k,2} + \epsilon |v_1\rangle\langle v_1|), \ldots, SN(P_{sym}^{k,2} + \epsilon |v_l\rangle\langle v_l|) \right\}.
\]

So the result follows by Lemma 4. \( \square \)
4 Low operator Schmidt rank and separability

States of $\mathcal{M}_3 \otimes \mathcal{M}_m$ with operator Schmidt rank 3 are in general not separable [8, Proposition 25]. Here, we prove that invariance under left partial transpose is a sufficient condition for separability of such states. This is a new result relating low operator Schmidt rank to separability (see [7, Theorem 58] and [8, Theorem 19]).

As a corollary, we show that the Schmidt number of any state of $\mathcal{M}_k \otimes \mathcal{M}_m$ ($k \leq m$) invariant under left partial transpose with operator Schmidt rank 3 cannot be greater than $k - 2$. This result complements [22, Theorem 5].

In this section, let $\text{Im}(\delta)$ denote the image of $\delta \in \mathcal{M}_k \otimes \mathcal{M}_m$ within $\mathbb{C}^k \otimes \mathbb{C}^m$.

The next lemma is well known (e.g., [9, Lemma 3.42]).

**Lemma 5** Any state $A \in \mathcal{M}_k \otimes \mathcal{M}_m$ with operator Schmidt rank $n$ can be written as $A = \sum_{i=1}^{n} \gamma_i \otimes \delta_i$, where $\gamma_i \in \mathcal{M}_k$, $\delta_i \in \mathcal{M}_m$ are Hermitian matrices such that $\text{Im}(\gamma_i) \subset \text{Im}(\gamma_1)$ and $\text{Im}(\delta_i) \subset \text{Im}(\delta_1)$, for every $i$, and $\gamma_1, \delta_1$ are positive semidefinite.

**Theorem 3** Let $A \in \mathcal{M}_3 \otimes \mathcal{M}_k$ be a state which is invariant under left partial transpose. If its operator Schmidt rank is less or equal to 3, then $A$ is separable.

**Proof** We can assume that the operator Schmidt rank of $A$ is 3, since every state with operator Schmidt rank less than 3 is separable by [7, Theorem 58].

First, let us assume that $A$ is positive definite. Let $A = \sum_{i=1}^{3} \gamma_i \otimes \delta_i$ be the decomposition described in Lemma 5.

Note that $\gamma_1, \gamma_2, \gamma_3$ are real symmetric matrices, since $A$ is invariant under left partial transpose. Moreover, $\gamma_1 \in \mathcal{M}_3$ must be positive definite, otherwise $A$ would not be positive definite (since $\text{Im}(\gamma_i) \subset \text{Im}(\gamma_1)$ for every $i$).

Let $\gamma_1 = R^2$, where $R \in \mathcal{M}_3$ is real, symmetric and invertible.

Now, let

$$B = (R^{-1} \otimes I)A(R^{-1} \otimes I) = I d_{3 \times 3} \otimes \delta_1 + R^{-1} \gamma_2 R^{-1} \otimes \delta_2 + R^{-1} \gamma_3 R^{-1} \otimes \delta_3.$$ 

Since $R^{-1} \gamma_2 R^{-1}$ is real symmetric, there is an orthogonal matrix $O \in \mathcal{M}_3$ and a real diagonal matrix $D \in \mathcal{M}_3$ such that $O R^{-1} \gamma_2 R^{-1} O^t = D$.

Next, let

$$C = (O \otimes I)B(O^t \otimes I) = I d_{3 \times 3} \otimes \delta_1 + D \otimes \delta_2 + M \otimes \delta_3,$$

where $M \in \mathcal{M}_3$ is real symmetric.

Note that $C$ is positive definite and has the following format:

$$C = \begin{bmatrix} F_1 & m_{21} \delta_3 & m_{31} \delta_3 \\ m_{21} \delta_3 & F_2 & m_{32} \delta_3 \\ m_{31} \delta_3 & m_{32} \delta_3 & F_3 \end{bmatrix},$$

where $m_{ij}$ is the $ij$ entry of the real symmetric matrix $M$ and $\delta_3, F_1, F_2, F_3$ are Hermitian matrices. Since $C$ is positive definite, $F_1 \in \mathcal{M}_k$ is also positive definite.
Assume that $m_{21}, m_{31} \neq 0$ (If one or both are equal to 0 then the proof is simpler). Note that

$$
\left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & m_{31} & -m_{21}
\end{array} \right) \otimes \text{Id}_{k \times k} \right) C \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & m_{31} \\
0 & 0 & -m_{21}
\end{array} \right) \otimes \text{Id}_{k \times k} = \begin{bmatrix}
F_1 & m_{21} \delta_3 & 0 \\
m_{21} \delta_3 & H_2 & H_3 \\
0 & H_3 & H_4
\end{bmatrix}.
$$

Next, let $F_1 = U U^*$ for an invertible $U$. Thus,

$$(\text{Id}_{3 \times 3} \otimes U^{-1}) \begin{bmatrix}
F_1 & m_{21} \delta_3 & 0 \\
m_{21} \delta_3 & H_2 & H_3 \\
0 & H_3 & H_4
\end{bmatrix} (\text{Id}_{3 \times 3} \otimes U^{-1})^* = \begin{bmatrix}
\text{Id}_{k \times k} & L & 0 \\
L & O_2 & O_3 \\
0 & O_3 & O_4
\end{bmatrix}.$$

Note that $L$ is Hermitian, since $L = U^{-1}(m_{21} \delta_3)(U^{-1})^*$.

Now,

$$
\begin{bmatrix}
\text{Id}_{k \times k} & L & 0 \\
L & O_2 & O_3 \\
0 & O_3 & O_4
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & O_2 - L^2 & O_3 \\
0 & O_3 & O_4
\end{bmatrix} + \begin{bmatrix}
\text{Id}_{k \times k} & L & 0 \\
L & L^2 & 0 \\
0 & 0 & 0
\end{bmatrix}.
$$

The second summand above is a well-known separable matrix, since $L$ is Hermitian (see [21, Theorem 1] and [18, Lemma 3]).

In addition, the first summand can be embedded in $\mathcal{M}_2 \otimes \mathcal{M}_k$. Since there are only three sub-blocks forming this matrix ($O_2 - L^2$, $O_3$ and $O_4$), its operator Schmidt rank is less or equal to 3. Moreover, it is positive semidefinite, since

$$
\begin{bmatrix}
0 & 0 & 0 \\
-L & \text{Id} & 0 \\
0 & 0 & \text{Id}
\end{bmatrix} \begin{bmatrix}
\text{Id} & L & 0 \\
L & O_2 & O_3 \\
0 & O_3 & O_4
\end{bmatrix} \begin{bmatrix}
0 & -L & 0 \\
0 & \text{Id} & 0 \\
0 & 0 & \text{Id}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 \\
0 & O_2 - L^2 & O_3 \\
0 & O_3 & O_4
\end{bmatrix}.
$$

Therefore, the first summand of Eq. 5 is also separable by [8, Theorem 19]. Hence, the sum is separable. Since all the local operations used are reversible and preserve separability, $A$ is separable.

Now, for the positive semidefinite case: Given $\epsilon > 0$, define $A(\epsilon) = (\gamma_1 + \epsilon \text{Id}) \otimes (\delta_1 + \epsilon \text{Id}) + \gamma_2 \otimes \delta_2 + \gamma_3 \otimes \delta_3$.

Note that $A(\epsilon)$ has operator Schmidt rank less or equal to 3, is invariant under partial transpose ($\epsilon \text{Id} + \gamma_1, \gamma_2, \gamma_3$ are symmetric) and is positive definite ($A(\epsilon) = A + \epsilon \text{Id} \otimes \delta_1 + \gamma_1 \otimes \epsilon \text{Id} + \epsilon^2 \text{Id} \otimes \text{Id}$). By the first case, $A(\epsilon)$ is separable and so is $\lim_{\epsilon \to 0^+} A(\epsilon) = A$.

**Corollary 3** Let $A \in \mathcal{M}_k \otimes \mathcal{M}_m$ ($k \leq m$) be a positive semidefinite Hermitian matrix which is invariant under left partial transpose. If its operator Schmidt rank is equal to 3, then $SN(A) \leq k - 2$.

**Proof** Let us show that $SN(A)$ cannot be $k - 1$, since $SN(A) < k$ was already proved in [22, Theorem 5].
If $SN(A) = k - 1$, then $A$ contains an entangled sub-block, $B \in M_3 \otimes M_m$, which is invariant under left partial transpose (see [22, Theorem 4] and [22, Theorem 5] for details).

By the construction of $B$, its operator Schmidt rank is less or equal to the operator Schmidt rank of $A$, which is 3. Therefore, $B$ is separable by Theorem 3. Absurd! ☐

**Summary and conclusion**

In this work, we obtained results on the number of mutually unbiased bases and the Schmidt number of states under certain constraints. The connection between these different results is the type of restrictions imposed. These restrictions were made on the Schmidt rank of the tensors used in the results.

We obtained an upper bound on the number of mutually unbiased bases of $\mathbb{C}^m \otimes \mathbb{C}^n$ formed by vectors with Schmidt rank less or equal to $k$ ($k < m \leq n$). It solved a conjecture on mutually unbiased product bases in a straightforward way.

We found an interval for the values of $\epsilon$ such that the Schmidt number of $p_{sym}^{k,2} + \epsilon \gamma$ equals half of the Schmidt number of $\gamma$ for all states $\gamma$ supported on the antisymmetric subspace of $\mathbb{C}^k \otimes \mathbb{C}^k$. This common interval provided a flexible method to create positive under partial transpose states with high Schmidt numbers.

Finally, we proved that invariance under left partial transpose is a sufficient condition for the separability of operator Schmidt rank 3 states of $M_3 \otimes M_m$. As a corollary, we proved that the Schmidt number of operator Schmidt rank 3 states of $M_k \otimes M_m$ ($k \leq m$) that are invariant under left partial transpose cannot exceed $k - 2$.

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**Declarations**

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