A new upper bound for $|\zeta(1 + it)|$

Timothy Trudgian*
Mathematical Sciences Institute
The Australian National University, ACT 0200, Australia
timothy.trudgian@anu.edu.au

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Abstract

It is known that $|\zeta(1 + it)| \ll (\log t)^{2/3}$. This paper provides a new explicit estimate, viz. $|\zeta(1 + it)| \leq \frac{3}{4} \log t$, for $t \geq 3$. This gives the best upper bound on $|\zeta(1 + it)|$ for $t \leq 10^{2 \cdot 10^5}$.

1 Introduction

Mellin [5] (see also [7, Thm 3.5]) was the first to show that

$$\zeta(1 + it) \ll \log t.$$  \hspace{1cm} (1.1)

This was improved by Littlewood (see, e.g., [7, Thm 5.16]) to

$$\zeta(1 + it) \ll \frac{\log t}{\log \log t}.$$  \hspace{1cm} (1.2)

This was improved by several authors; the best known result (see, e.g. [7, (6.19.2)]) is

$$\zeta(1 + it) \ll (\log t)^{2/3}.$$  \hspace{1cm} (1.3)

Insofar as explicit results are concerned, Backlund [1] made (1.1) explicit by proving that

$$|\zeta(1 + it)| \leq \log t,$$  \hspace{1cm} (1.4)

for $t \geq 50$. Ford [3] has made (1.3) explicit by proving that

$$|\zeta(1 + it)| \leq 72.6(\log t)^{2/3},$$  \hspace{1cm} (1.5)

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1As usual, the Riemann hypothesis gives a stronger result, viz., $\zeta(1 + it) \ll \log \log t$ (see, e.g., [7, §14.18]).
for \( t \geq 3 \). Ford’s result is actually much more general: he obtains excellent bounds for \(|\zeta(\sigma + it)|\) where \( \sigma \) is near 1. Should one be interested in a bound only on \( \sigma = 1 \), one can improve on (1.3) to show that \(|\zeta(1+it)| \leq 62.6(\log t)^{2/3} \). Note that this improves on (1.4) when \( t \geq 10^{10} \). Without a complete overhaul of Ford’s paper it seems unlikely that his methods could furnish a bound superior to (1.4) when \( t \) is at all modest, say \( t \leq 10^{100} \).

To the knowledge of the author there is no explicit bound of the form (1.2). One could follow the arguments of [7, §5.16] to produce such a bound, though this leads to a result that only improves on (1.4) when \( t \) is astronomically large. However one can still use the ideas in [7, §5.16] to reprove (1.1). Indeed if one were lucky, as the author was, one may even be able to supersede (1.4). This fortune is summarised in the following theorem.

**Theorem 1.**

\[ |\zeta(1+it)| \leq \frac{3}{4} \log t, \]

when \( t \geq 3 \).

Good explicit bounds on \(|\zeta(1+it)|\) enable one to bound the zeta-function more effectively throughout the critical strip. Indeed Theorem 1 can be used to improve the estimate on \( S(T) \) given in [8].

## 2 Backlund’s result

To prove (1.4) consider \( \sigma > 1 \) and \( t > 1 \), and write \( \zeta(s) - \sum_{n \leq N} n^{-s} = \sum_{N < n} n^{-s} \). Now invoke the following version of the Euler–Maclaurin summation formula — this can be found in [6, Thm 2.19].

**Lemma 1.** Let \( k \) be a nonnegative integer and \( f(x) \) be \((k+1)\) times differentiable on the interval \([a,b]\). Then

\[
\sum_{a < n \leq b} f(n) = \int_a^b f(t) \, dt + \sum_{r=0}^k \frac{(-1)^{r+1}}{(r+1)!} \left\{ f^{(r)}(b) - f^{(r)}(a) \right\} B_{r+1} + \frac{(-1)^k}{(k+1)!} \int_a^b B_{k+1}(x)f^{(k+1)}(x) \, dx,
\]

where \( B_j(x) \) is the \( j \)th periodic Bernoulli polynomial and \( B_j = B_j(0) \).

Apply this to \( f(n) = n^{-s} \), with \( k = 1 \), \( a = N \) and with \( b \) dispatched to infinity. Thus

\[
\zeta(s) - \sum_{n \leq N} n^{-s} = \frac{N^{1-s}}{s-1} + \frac{1}{2N^s} - \frac{s(s+1)}{12N^{s+1}} - \frac{s(s+1)}{2} \int_N^\infty \frac{\{x\}^2 - \{x\} + \frac{1}{6}}{x^{s+2}} \, dx,
\]

(2.1)

\(^2\)The integral inequality on [3, p. 622], originally verified for \( y \geq 0 \), can now be evaluated at \( y = 0 \) only.
where, since the right-side converges for $\Re(s) > -1$, the equation remains valid when $s = 1 + it$. Hence one can estimate the sum in (2.1) using
\[
\sum_{n \leq N} \frac{1}{n} \leq \log N + \gamma + \frac{1}{N},
\]
which follows from partial summation, and in which $\gamma$ denotes Euler’s constant.

Now if $N = [t/m]$, where $m$ is a positive integer to be chosen later, (2.1) and (2.2) combine to show that
\[
|\zeta(1 + it)| - \log t \leq -\log m + \gamma + \frac{1}{t} + \frac{m}{2(t - m)} + \frac{m^2(1 + t)(4 + t)}{24(t - m)^2}.
\]

The aim is to choose $m$ and $t_0$ such that $t \geq t_0$ guarantees the right-side of (2.3) to be negative. It is easy to verify that when $m = 3$, choosing $t = 49.385 \ldots$ suffices. Thus (1.4) is true for all $t \geq 50$; a quick computation shows that (1.4) remains true for $t \geq 2001 \ldots$

It seems impossible to improve upon (1.4) without a closer analysis of sums of the form $\sum_{a<n \leq 2a} n^{-it}$. Taking further terms in the Euler–Maclaurin expansion in (2.1) does not achieve an overall saving; choosing $N = [t^\alpha]$ for some $\alpha < 1$ in (2.2) means that the integral in (2.1) is no longer bounded.

The next section aims at securing a good bound for $\sum_{a<n \leq 2a} n^{-it}$ for ‘large’ values of $a$. For ‘small’ values of $a$ one may estimate the sum trivially. The inherent optimism is that, when combined, these two estimates give an improvement on (1.4).

### 3 Exponential sums: beyond Backlund

The following is an explicit version of Theorem 5.9 in [7].

**Lemma 2** (Cheng and Graham). Assume that $f(x)$ is a real-valued function with two continuous derivatives when $x \in (a, 2a]$. If there exist two real numbers $V < W$ with $W > 1$ such that
\[
\frac{1}{W} \leq |f''(x)| \leq \frac{1}{V}
\]
for $x \in [a + 1, 2a]$, then
\[
\left| \sum_{a<n \leq 2a} e^{2\pi f(n)} \right| \leq \frac{1}{5} \left( \frac{a}{V} + 1 \right) (8W^{1/2} + 15).
\]

**Proof.** See Lemma 3 in [2].

Applying Lemma 2 to $f(x) = -(2\pi)^{-1}t \log x$ gives
\[
\left| \sum_{a<n \leq 2a} n^{-it} \right| \leq t^{1/2} \left\{ \frac{8}{5} \frac{\sqrt{\pi}}{2\pi} + \frac{16\sqrt{2\pi a}}{5t} + \frac{3t^{1/2}}{2\pi a} + 3t^{-1/2} \right\},
\]
(3.1)

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subject to $8\pi a^2 > t$. Now take $A_1 t^{1/2} < a \leq \lfloor t/m \rfloor$ for some constant $A_1$ and positive integer $m$ to be determined later. If $t \geq t_0$, then (3.1) shows that

$$\left| \sum_{a<n \leq 2a} n^{-it} \right| \leq A_2 t^{1/2},$$

and hence, by partial summation,

$$\left| \sum_{a<n \leq 2a} n^{-1-it} \right| \leq A_2 a^{-1} t^{1/2} \leq \frac{A_2}{A_1},$$

where

$$A_2 = \frac{8}{5} \sqrt{\frac{2}{\pi}} + \frac{16\sqrt{2\pi}}{5m} + \frac{3}{2\pi A_1} + 3t_0^{-1/2}.$$  

One may now apply (3.2) to each of the sums on the right-side of

$$\left| \sum_{A_1 t^{1/2} < n \leq \lfloor t/m \rfloor} \frac{1}{n^{1+it}} \right| = \sum_{\frac{1}{2}(t/m) < n \leq \lfloor t/m \rfloor} + \sum_{\frac{1}{4}(t/m) < n \leq \frac{1}{2}(t/m)} + \cdots.$$ 

There are at most

$$\frac{1}{2} \log t - \log(mA_1) + \log 2$$

such sums. This gives an upper bound for $\sum n^{-1-it}$ when $n > A_1 t^{1/2}$. When $n \leq A_1 t^{1/2}$ one may use (2.2) to estimate the sum trivially.

4 Proof of Theorem 1

In $\zeta(s) - \sum_{n \leq N} n^{-s} = \sum_{N<n} n^{-s}$ use Euler–Maclaurin summation (Lemma 1) to $k$ terms. Choosing $N - 1 = \lfloor t/m \rfloor$, recalling (3.2) and (3.3), and estimating all complex terms trivially gives

$$\left| \zeta(1 + it) \right| \leq \log \{ \frac{1}{2} + \frac{A_2}{2A_1 \log 2} \} + \frac{A_2 \log 2 - \log(mA_1)}{A_1 \log 2} + \log A_1 + \gamma$$

$$+ \frac{1}{A_1 t_0^{1/2}} + \frac{m}{2t} + \frac{1}{t} + \sum_{r=1}^k \frac{|B_{r+1}|}{(r+1)!} (1 + t) \cdots (r + t) \left( \frac{m}{t} \right)^{r+1}$$

$$+ \frac{(1 + t) \cdots (k + 1 + t)}{(k + 1) \cdot (k + 1)!} \max |B_{k+1}(x)| \left( \frac{m}{t} \right)^{k+1}.$$  

Note that each term in the $r$-sum in (4.1) is $O_{m,k}(t^{-1})$. This is cheap relative to the last term which is $O_{m,k}(1)$. Thus one can take $k$ somewhat large to reduce

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3This is to ensure that, in Lemma 2, $W > 1$ — see (4.2).
4To ensure that this is a non-empty interval see (4.2).
the burden of the final term. For a given $t_0$, when $t \geq t_0$ one can optimise (4.1) over $k$, $m$ and $A_1$ subject to

$$A_1 > \frac{1}{\sqrt{8\pi}}, \quad mA_1 \leq t_0^{1/2}. \tag{4.2}$$

One finds that, when $k = 14$, $m = 6$, $A_1 = 23$ then $|\zeta(1 + it)| \leq 0.749818\ldots$ for all $t \geq 10^8$. A numerical check on Mathematica suffices to extend the result to all $t \geq 2.391\ldots$, whence Theorem 1 follows.

4.1 Improvements

Lemma 2 is unable to furnish a value less than $\frac{1}{12}$ in Theorem 1. On the other hand, by verifying that $|\zeta(1 + it)| < \frac{1}{12} \log t$ for $t$ larger than $10^8$ one will improve slightly on Theorem 1.

One could also take an analogue of Lemma 2 that incorporates higher derivatives. Such a result, giving explicit bounds on exponential sums of a function involving $k$ derivatives, is given in [4, Prop. 8.2]. It is unclear how much could be gained from pursuing this idea.

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