A BOUNDARY VERSION OF CARTAN-HADAMARD AND APPLICATIONS TO RIGIDITY.

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Abstract. In this paper, we prove a version of the classical Cartan-Hadamard theorem for negatively curved manifolds, of dimension $n \neq 5$, with non-empty totally geodesic boundary. More precisely, if $M^n_1$, $M^n_2$ are any two such manifolds, we show that (1) $\partial^\infty \tilde{M^n_1}$ is homeomorphic to $\partial^\infty \tilde{M^n_2}$, and (2) $\tilde{M^n_1}$ is homeomorphic to $\tilde{M^n_2}$. As a sample application, we show that simple, thick, negatively curved P-manifolds of dimension $\geq 6$ are topologically rigid. We include some straightforward consequences of topological rigidity (diagram rigidity, weak co-Hopf property, and Nielsen realization problem).

1. Introduction.

A key aspect in the study of non-positively curved Riemannian manifolds is the large number of rigidity theorems known to hold for these spaces. Two outstanding such theorems are (1) Mostow rigidity [M], stating that in dimension $\geq 3$, homotopy equivalence of irreducible locally symmetric spaces of non-compact type implies isometry of the spaces, and (2) Farrell-Jones topological rigidity [FJ], stating that in dimension $\geq 5$, homotopy equivalence of non-positively curved Riemannian manifolds implies homeomorphism of the spaces.

A natural question is how to extend these theorems to the context of singular spaces satisfying a metric analogue of “non-positive curvature”. In a series of papers [L1, L2, L3], the author introduced the class of hyperbolic P-manifolds, which one can view as some of the simplest non-manifold CAT(-1) spaces, and established Mostow rigidity within this class of spaces. In the present paper, our interest lies in establishing topological rigidity for negatively curved P-manifolds. Let us first recall the definition of a P-manifold:

Definition 1.1. A closed $n$-dimensional piecewise manifold (henceforth abbreviated to P-manifold) is a topological space which has a natural stratification into pieces which are manifolds. More precisely, we define a 1-dimensional P-manifold to be a finite graph. An $n$-dimensional P-manifold ($n \geq 2$) is defined inductively as a closed pair $X_{n-1} \subset X_n$ satisfying the following conditions:

- Each connected component of $X_{n-1}$ is either an $(n-1)$-dimensional P-manifold, or an $(n-1)$-dimensional manifold.
The closure of each connected component of $X_n - X_{n-1}$ is homeomorphic to a compact orientable $n$-manifold with boundary, and the homeomorphism takes the component of $X_n - X_{n-1}$ to the interior of the $n$-manifold with boundary; the closure of such a component will be called a chamber.

Denoting the closures of the connected components of $X_n - X_{n-1}$ by $W_i$, we observe that we have a natural map $\rho : \coprod \partial W_i \to X_{n-1}$ from the disjoint union of the boundary components of the chambers to the subspace $X_{n-1}$. We also require this map to be surjective, and a homeomorphism when restricted to each component. The P-manifold is said to be thick provided that each point in $X_{n-1}$ has at least three pre-images under $\rho$. We will henceforth use a superscript $X^n$ to refer to an $n$-dimensional P-manifold, and will reserve the use of subscripts $X_{n-1}, \ldots, X_1$ to refer to the lower dimensional strata. For a thick $n$-dimensional P-manifold, we will call the $X_{n-1}$ strata the branching locus of the P-manifold.

Intuitively, we can think of P-manifolds as being “built” by gluing manifolds with boundary together along lower dimensional pieces. Examples of P-manifolds include finite graphs and soap bubble clusters. Observe that compact manifolds can also be viewed as (non-thick) P-manifolds. Less trivial examples can be constructed more or less arbitrarily by finding families of manifolds with homeomorphic boundary and gluing them together along the boundary using arbitrary homeomorphisms. We now define the family of metrics we are interested in.

**Definition 1.2.** A Riemannian metric on a 1-dimensional P-manifold (finite graph) is merely a length function on the edge set. A Riemannian metric on an $n$-dimensional P-manifold $X^n$ is obtained by first building a Riemannian metric on the $X_{n-1}$ subspace, then picking, for each $W_i$ a Riemannian metric with non-empty totally geodesic boundary satisfying that the gluing map $\rho$ is an isometry. We say that a Riemannian metric on a P-manifold is negatively curved if at each step, the metric on each $W_i$ is negatively curved.

Observe that, at the cost of scaling the metric of the P-manifold $X$ by a constant, one can assume that the metric on each $W_i$ has sectional curvature bounded above by $-1$. Such a metric on the P-manifold will automatically be locally CAT(-1), and hence the fundamental group of a negatively curved P-manifold is a $\delta$-hyperbolic group. In particular, the universal cover $\tilde{X}$ has a well-defined boundary at infinity, denoted $\partial^\infty \tilde{X}$.

**Definition 1.3.** We say that an $n$-dimensional P-manifold $X^n$ is simple provided its codimension two strata is empty. In other words, the $(n-1)$-dimensional strata $X_{n-1}$ consists of a disjoint union of $(n-1)$-dimensional manifolds. We further assume that, for each chamber $W_i$, the various boundary components of $W_i$ get attached to distinct components of the codimension one strata.

We can now state our main result:
Theorem 1.1 (Topological rigidity). Let $X_1, X_2$ be a pair of simple, thick, negatively curved $P$-manifolds, of dimension $\geq 6$. If $\pi_1(X_1)$ is isomorphic to $\pi_1(X_2)$, then $X_1$ is homeomorphic to $X_2$.

We note that, corresponding to the stratification of a negatively curved $P$-manifold, there is a natural diagram of groups having the property that the direct limit of the diagram is precisely the fundamental group of the $P$-manifold (by the generalized Seifert-Van Kampen theorem). Immediate consequences of the topological rigidity are the following:

Corollary 1.1 (Diagram rigidity). Let $\mathcal{D}_1, \mathcal{D}_2$ be a pair of diagrams of groups, corresponding to a pair of negatively curved, simple, thick $P$-manifolds of dimension $n \geq 6$. Then $\varprojlim \mathcal{D}_1$ is isomorphic to $\varprojlim \mathcal{D}_2$ if and only if the two diagrams are isomorphic.

Corollary 1.2 (weak Co-Hopf property). Let $X$ be a simple, thick, negatively curved $P$-manifold of dimension $n \geq 6$, and assume that at least one of the chambers has a non-zero characteristic number. Then $\Gamma = \pi_1(X)$ is weakly co-Hopfian, i.e. every injection $\Gamma \hookrightarrow \Gamma$ with image of finite index is in fact an isomorphism.

Corollary 1.3 (Nielsen realization problem). Let $X$ be a simple, thick, negatively curved $P$-manifold of dimension $n \geq 6$, and $\Gamma = \pi_1(X)$. Then the canonical map $\text{Homeo}(X) \to \text{Out}(\Gamma)$ is surjective.

Now recall that a consequence of the classical Cartan-Hadamard theorem is that if $M_1, M_2$ are a pair of closed $n$-dimensional manifolds of non-positive sectional curvature, then the universal covers $\tilde{M}_1$ and $\tilde{M}_2$ are homeomorphic (indeed, are both diffeomorphic to $\mathbb{R}^n$). Another classic result is that for such a manifold $M$, the boundary at infinity of the universal cover $\partial^\infty \tilde{M}$ is always homeomorphic to an $(n - 1)$-sphere $S^{n-1}$. The key to the proof of the previous two theorems is the following analogue of these classic results, in the setting where one allows a non-empty, totally geodesic boundary.

Theorem 1.2 (Cartan-Hadamard). Assume $M_1, M_2$ are a pair of compact, negatively curved Riemannian manifolds of dimension $n \neq 5$, with non-empty, totally geodesic boundary. Then we have:

1. $\partial^\infty \tilde{M}_1$ is homeomorphic to $\partial^\infty \tilde{M}_2$.
2. $\tilde{M}_1$ is homeomorphic to $\tilde{M}_2$.

where $\tilde{M}_i$ is the universal cover of $M_i$.

Note that if $n = 2$, then the boundaries at infinity of the $\tilde{M}_i$ are Cantor sets, and the first statement in the Theorem is just the classical fact that any two Cantor sets are homeomorphic (Brouwer’s characterization theorem).

We now outline the organization of this paper. In Section 2, we will give a proof of Theorem 1.2. The argument relies heavily on a characterization of $n$-dimensional
Sierpinski curves \((n \neq 3)\) due to Cannon [C]. The dimension restriction in Theorem 1.2 arises from the corresponding dimension restriction in Cannon’s work.

In Section 3, we will give outlines of the proofs of Theorems 1.1, as well as proofs of the three corollaries. The arguments for these follow almost verbatim from previous results of the author [L2], [L3]. More precisely, in [L2] the author gave a topological argument allowing, in the case where the simple, thick, negatively curved \(P\)-manifold was actually hyperbolic (i.e. each chamber is isometric to a hyperbolic manifold with non-empty, totally geodesic boundary), for recognizing the fundamental groups of the various chambers and how they are attached together. The argument in [L2] made use of the topology of the boundary at infinity of a hyperbolic manifold with non-empty totally geodesic boundary. Statement (1) of Theorem 1.2, and the fact that the argument in [L2] is purely topological, allows the entire argument to be transferred to the case of simple, thick, negatively curved \(P\)-manifolds, of dimension \(n \neq 5\).

Once one knows how to recognize the fundamental groups of the chambers and how they attach together, we can appeal, in dimension \(n \geq 5\), to the celebrated Topological Rigidity Theorem of Farrell and Jones [FJ]. This allows us to completely determine the topology (up to homeomorphism) of all the chambers, as well as the topology of the codimension one strata. Putting this together yields the desired Theorem 1.1, and translating the topological rigidity result (via the generalized Seifert-Van Kampen) into group theoretic language immediately gives Corollary 1.1. Exploiting the correspondence between subgroups of fundamental groups and coverings of the corresponding space, it is easy to obtain the weak co-Hopf property (Corollary 1.2). The Nielsen realization problem (Corollary 1.3) is immediate from the Theorem 1.1.

**Remark.** (1) We note that topological rigidity fails (trivially) in dimension \(n = 1\). In dimension \(n = 2\), topological rigidity was proved in [L3]. In dimension \(n = 3\), the argument given in the present paper could be extended, provided one had an analogue of Farrell-Jones [FJ] for 3-dimensional manifolds. This analogue is a well-known consequence of the Geometrization Conjecture of Thurston. A proof of the Geometrization Conjecture was announced a few years ago by G. Perelman. The author does not know whether topological rigidity is to be expected in dimensions \(n = 4, 5\) (though the failure of the present proof is due to different reasons in each of these two cases).

(2) It would be interesting to see whether, in statement (2) of Theorem 1.2, one can replace “homeomorphism” by “diffeomorphism”. As the reader will see, the argument in the present paper has no chance of being extended to yield smooth rigidity.

(3) Concerning the hypothesis in Corollary 1.2 on the existence of a non-zero characteristic number for one of the chambers, we point out that the famous Hopf Conjecture on the sign of the Euler characteristic asserts that for a closed, negatively curved, even dimensional manifold \(M^{2n}\), we have the inequality \((-1)^n \chi(M^{2n}) > 0\). It
is easy to see (using a doubling argument) that the Hopf Conjecture, if true, implies that for any compact negatively curved manifold $M$ with *non-empty* totally geodesic boundary, we have $\chi(M) \neq 0$. In particular, the validity of the Hopf Conjecture would yield the desired non-zero characteristic number. We also point out that a much stronger result is known, namely Sela [Se] has shown that a non-elementary $\delta$-hyperbolic group is co-Hopfian if and only if it is freely indecomposable.

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2. Cartan-Hadamard for manifolds with boundary.

We now proceed to prove Theorem 1.2 from the introduction. So let $M_1, M_2$ be a pair of compact, negatively curved manifolds of dimension $n \neq 5$, with non-empty totally geodesic boundary. We will start by establishing property (1), namely that $\partial^\infty \tilde{M}_1$ is homeomorphic to $\partial^\infty \tilde{M}_2$. In order to do this, we will make use of the characterization of Sierpinski curves due to Cannon [C] (generalizing a classic result of Whyburn [W] in dimension $n = 2$). We first start with a definition:

**Definition 2.1.** Let $\{U_i\}$ be a countable collection of pairwise disjoint subsets of $S^n$ satisfying the following four conditions:

1. The collection $\{U_i\}$ forms a null sequence, i.e. $\lim \{\text{diam}(U_i)\} = 0$,
2. $S^n - U_i$ is an $n$-cell for each $i$,
3. $\text{Cl}(U_i) \cap \text{Cl}(U_j) = \emptyset$ for each $i \neq j$ ($\text{Cl}$ denotes closure),
4. $\text{Cl}(\bigcup U_i) = S^n$.

Then we call the complement $S^n - \bigcup U_i$ an $(n - 1)$-dimensional Sierpinski curve (abbreviated to $\mathcal{S}$-curve).

**Theorem 2.1** (Cannon, 1973). Let $X, Y$ be an arbitrary pair of $(n - 1)$-dimensional $\mathcal{S}$-curves $(n \neq 4)$. Then we have:

- $X$ is homeomorphic to $Y$,
- if $i: X \to S^n$ is an arbitrary embedding, then $i(X) \subset S^n$ is an $(n - 1)$-dimensional $\mathcal{S}$-curve.
- if $h: X \to Y$ is an arbitrary homeomorphism, then $h$ extends to a homeomorphism of the ambient $n$-dimensional spheres.
The scheme of the proof is now clear: considering the double $DM_i$ of the manifold $M_i$ across its boundary, we can view $\tilde{M}_i$ as a totally geodesic subset of $\partial^\infty DM_i$, and hence $\partial^\infty \tilde{M}_i$ as an embedded subset of $\partial^\infty \tilde{DM}_i \cong S^{n-1}$. If we can establish that $\partial^\infty \tilde{M}_i$ is an $(n-2)$-dimensional $S$-curve, Cannon’s theorem will immediately imply that $\partial^\infty \tilde{M}_i$ is homeomorphic to $\partial^\infty \tilde{M}_2$. We now proceed to verify the four conditions of an $(n-2)$-dimensional $S$-curve for $\partial^\infty \tilde{M} \subset \partial^\infty \tilde{DM} \cong S^{n-1}$.

Let us first fix some notation: the collection $\{\partial_i\}$ will be the connected components of $\partial^\infty \tilde{DM} - \partial^\infty \tilde{M}$ inside $\partial^\infty \tilde{DM} \cong S^{n-1}$. We will denote by $\{Y_i\}$ the connected components of $\tilde{DM} - \tilde{M}$. Note that each $Cl(Y_i)$ intersects $\tilde{M}$ along a boundary component, which is a totally geodesic codimension one submanifold of $\tilde{DM}$. We will denote by $Z_i \subset \partial \tilde{M}$ the boundary component corresponding to $Y_i \subset \tilde{DM} - \tilde{M}$. Finally, we observe that each $U_i$ can be identified with a corresponding $\partial^\infty Y_i - \partial^\infty Z_i$, for some suitable component $Y_i$.

**Claim 1:** The collection $\{U_i\}$ forms a null sequence.

**Proof.** At the cost of rescaling the metric on $DM$, we may assume that the sectional curvature is bounded above by $-1$, and hence that $\tilde{DM}$ is a $CAT(-1)$ space. In this situation, Bourdon [E] defined a metric on $\partial^\infty \tilde{DM}$ inducing the standard topology on $\partial^\infty \tilde{DM} \cong S^{n-1}$. The metric is given by:

$$d_\infty(p, q) = e^{-d(*, \gamma_{pq})}$$

where $\gamma_{pq}$ is the unique geodesic joining the points $p, q \in \partial^\infty \tilde{DM}, * \in DM$ a chosen basepoint (and $d$ denotes the distance inside $\tilde{DM}$). Note that different choices of basepoints result in metrics which are Lipschitz equivalent. For convenience, we will pick the basepoint $*$ to lie in the interior of the lift $\tilde{M}$.

Now consider one of the components $U_i$, and let us try to estimate $diam(U_i)$. Note that given any two points $p, q \in Cl(U_i)$, we have that the geodesic $\gamma_{pq} \subset Cl(Y_i)$, where $Y_i$ is the component corresponding to $U_i$. In particular, we see that $d(*, \gamma_{pq}) \geq d(*, Z_i)$, and hence that for any $p, q \in Cl(U_i)$ we have the upper bound:

$$d_\infty(p, q) = e^{-d(*, \gamma_{pq})} \leq e^{-d(*, Z_i)}$$

Since $diam(U_i)$ is the supremum of $d_\infty(p, q)$, where $p, q \in Cl(U_i)$, the above bound yields $diam(U_i) \leq e^{-d(*, Z_i)}$. On the other hand, since $\tilde{M}$ is the universal cover of a compact negatively curved manifold with non-empty boundary, we have that $\lim\{d(*, Z_i)\} = \infty$, where $Z_i$ ranges over the boundary components of $\tilde{M}$. This implies that the collection $\{U_i\}$ forms a null sequence in $\partial^\infty \tilde{DM} \cong S^{n-1}$, as desired.

**Claim 2:** $S^{n-1} - U_i$ is an $(n-1)$-cell for each $i$.

**Proof.** Recall that there exists a homeomorphism $\pi_x : S^{n-1} \cong \partial^\infty \tilde{DM} \rightarrow T_x^1 \tilde{DM} \cong S^{n-1}$, obtained by mapping a point $p \in \partial^\infty \tilde{DM}$ to the unit vector $\dot{\gamma}_{xp}(0)$, where
\( \gamma_{xp} \) is the unit speed geodesic ray originating from \( x \), in the direction \( p \in \partial^\infty \tilde{\partial}M \). Now let \( U_i \) be given, and pick \( x \) to lie on the corresponding \( Z_i \). Note that under the homeomorphism \( \pi_x \), we have that \( \partial^\infty Z_i \) maps homeomorphically to a totally geodesic \( S^{n-2} \subset S^{n-1} \cong T_x^1 \tilde{\partial}M \), while the subset \( U_i \) maps homeomorphically to one of the open hemispheres determined by \( \pi_x (\partial^\infty Z_i) \). In particular, we see that \( \partial^\infty \tilde{\partial}M - U_i \) maps homeomorphically to one of the closed hemispheres determined by \( \pi_x (\partial^\infty Z_i) \), and hence must be an \((n-1)\)-cell, as desired.

**Claim 3:** \( \text{Cl}(U_i) \cap \text{Cl}(U_j) = \emptyset \) for all \( i \neq j \).

**Proof.** Note that by definition we have that \( U_i \cap U_j = \emptyset \), and that \( \text{Cl}(U_i) = U_i \cup \partial^\infty Z_i \), \( \text{Cl}(U_j) = U_j \cup \partial^\infty Z_j \). Hence it is sufficient to show that \( \partial^\infty Z_i \cap \partial^\infty Z_j = \emptyset \) for \( i \neq j \) (since these are codimension one spheres in \( S^{n-1} \cong \partial^\infty \tilde{\partial}M \), with the \( U_i \), \( U_j \) connected components of the respective complements). But a pair of distinct boundary components of \( \tilde{M} \), the universal cover of a compact negatively curved manifold with non-empty totally geodesic boundary, must diverge exponentially (with growth rate bounded below in terms of the upper bound on sectional curvature). In particular, no geodesic ray in \( Z_i \) is within bounded Hausdorff distance of a geodesic ray in \( Z_j \), and hence the boundaries at infinity are pairwise disjoint, as desired.

**Claim 4:** \( \text{Cl}(\cup U_i) = S^{n-1} \).

**Proof.** Fix a point \( x \in \tilde{M} \), and consider the homeomorphism \( \pi_x : S^{n-1} \cong \partial^\infty \tilde{\partial}M \to T_x^1 \tilde{\partial}M \cong S^{n-1} \). We will show that every point in \( T_x^1 \tilde{\partial}M \cong S^{n-1} \) can be approximated by a sequence of points in \( \pi_x(U_i) \). This will imply that \( T_x^1 \tilde{\partial}M = \text{Cl}(\cup \pi_x(U_i)) \), and since \( \pi_x \) is a homeomorphism, this will immediately imply Claim 4.

Now if \( p \in T_x^1 \tilde{\partial}M \) lies in one of the \( \pi_x(U_i) \), we are done, so let us assume that \( p \in T_x^1 \tilde{\partial}M - \cup \pi_x(U_i) \). Let \( \gamma \) be a unit speed geodesic ray originating from \( x \) with tangent vector \( p \) at the point \( x \). Note that we have that \( \gamma \subset \tilde{M} \subset \partial \tilde{M} \), since we are assuming \( p \in T_x^1 \tilde{\partial}M - \cup \pi_x(U_i) \). Now observe that \( \tilde{M} \) is the universal cover of a compact negatively curved manifold with non-empty totally geodesic boundary, and hence there exists a constant \( K \) with the property that every point in \( \tilde{M} \) is within distance \( K \) of \( \partial \tilde{M} = \bigcup Z_i \) (for instance take \( K = \text{diam}(M) \)).

So for each integer \( k \in \mathbb{N} \), we can find a point \( y_k \in \partial \tilde{M} \) satisfying \( d(\gamma(k), y_k) \leq k \). Now observe that if \( \eta_k \) is the geodesic ray originating from \( x \) and passing through \( y_k \), we have that \( \eta_k(\infty) \in U_{ik} \), where \( Z_{ik} \) is the component of \( \partial \tilde{M} \) containing the point \( y_k \). This implies that \( \tilde{\eta}_k(0) \in T_x^1 \tilde{\partial}M \) lies in the corresponding \( \pi_x(U_{ik}) \), i.e. that the sequence of vectors \( \{ \tilde{\eta}_k(0) \} \subset T_x^1 \tilde{\partial}M \) lies in the set \( \bigcup \pi_x(U_i) \). We are left with establishing that \( \lim \{ \tilde{\eta}_k(0) \} = p \). To see this, we need to estimate the angle between the geodesics \( \eta_k \) and the geodesic \( \gamma \). But this is easy to do: consider the geodesic triangle with vertices \( (x, \gamma(k), y_k) \), and note that \( d(x, \gamma(k)) = k \), while \( d(\gamma(k), y_k) \leq K \).
Applying the Alexandrov-Toponogov triangle comparison theorem, we see that the angle \( \angle(\hat{\eta}_k(0), \hat{\gamma}(0)) \) is bounded above by the angle of a comparison triangle in \( \mathbb{H}^2 \) (recall that we assumed the metrics have been scaled to have upper bound \(-1\) on the sectional curvature). But an easy calculation in hyperbolic geometry shows that if one has a sequence of triangles in \( \mathbb{H}^2 \) of the form \((A_k, B_k, C_k)\) with the property that \( d(A_k, B_k) = k \) and \( d(B_k, C_k) \leq K \), then the angle at the vertex \( A_k \) tends to zero as \( k \) tends to infinity. This implies that \( \lim \{ \angle(\hat{\eta}_k(0), \hat{\gamma}(0)) \} = 0 \), and hence completes the proof of Claim 4.

Appealing to Cannon’s theorem, the four Claims above immediately yield property (1) from Theorem 1.2: if \( M_1, M_2 \) are a pair of compact \( n \)-dimensional negatively curved manifolds with non-empty, totally geodesic boundary, then \( \partial^\infty \tilde{M}_1 \) is homeomorphic to \( \partial^\infty \tilde{M}_2 \). We now proceed to establish property (2): under the hypotheses above, \( \tilde{M}_1 \) is homeomorphic to \( \tilde{M}_2 \). In order to do this, we pick a pair of points \( p_i \) in the interior of the respective \( \tilde{M}_i \), and define subspaces \( \tilde{C}_i \subset \tilde{M}_i \) to be the union of all geodesic rays, emanating from the respective \( p_i \) to points in the corresponding \( \partial^\infty \tilde{M}_i \). Note that each \( \tilde{C}_i \) is homeomorphic to the open cone over \( \partial^\infty \tilde{M}_i \), that is to say the space \( \partial^\infty \tilde{M}_i \times [0, \infty)/\sim \), where the equivalence relation \( \sim \) collapses \( \partial^\infty \tilde{M}_i \times \{0\} \) to a point. From property (1), we conclude that \( \tilde{C}_1 \) is homeomorphic to \( \tilde{C}_2 \). We now proceed to extend the homeomorphism between the subsets \( \tilde{C}_i \subset \tilde{M}_i \) to a homeomorphism between the respective \( \tilde{M}_i \). We will denote by \( C(Y) \) the open cone over any topological space \( Y \).

In order to extend the homeomorphism, let us view \( \tilde{M} \) as a subset in \( \widetilde{DM} \). Since each \( \partial^\infty Z_i \subset \partial^\infty \widetilde{DM} \cong S^{n-1} \) separates, the subset \( C(\partial^\infty Z_i) \subset X \) separates \( \widetilde{DM} \). Let us denote by \( H_i \) the unique connected component of \( \partial^\infty \widetilde{DM} - (Z_i \cup C(\partial^\infty Z_i)) \) that contains both \( Z_i \) and \( C(\partial^\infty Z_i) \) in its closure. Observe that \( H_i \subset \tilde{M} \), and that we have a decomposition of \( \tilde{M} \) as \( X \cup_i H_i \), where each \( H_i \) attaches to \( X \) along the boundary component \( C(\partial^\infty Z_i) \cong \mathbb{R}^{n-1} \). Property (2) will now follow from the following:

**Lemma 2.1.** Each \( H_i \) is homeomorphic to \([0, 1] \times \mathbb{R}^{n-1}\).

**Proof.** Consider the point \( x \in \widetilde{DM} \) from which we cone to obtain \( X = C(\partial^\infty \tilde{M}) \), and observe that in \( T_x^1 \widetilde{DM} \cong S^{n-1} \), we have that the set of directions to points in \( \partial^\infty Z_i \) form an embedded codimension one sphere \( S^{n-2} \) inside \( T_x^1 \widetilde{DM} \). Denoting by \( S \subset T_x^1 \widetilde{DM} \) this embedded codimension one sphere, we further observe that the geodesics joining \( x \) to any point in \( H_i - C(\partial^\infty Z_i) \) have the property that they all lie in a common component \( D \) of \( T_x^1 \widetilde{DM} - S \) (and \( D \) is homeomorphic to an open (\( n-1 \))-dimensional cell).

Next we note that given any direction \( v \in D \), the unit speed geodesic ray \( \gamma_v(t) \) has the property that its distance from the subset \( C(\partial^\infty Z_i) \) tends to infinity as \( t \to \infty \). Since the subset \( H_i \) lies within finite Hausdorff distance of \( C(\partial^\infty Z_i) \), this implies that
the subset $R_v := \{ t \mid \gamma_v(t) \in H_i \}$ is bounded. Continuity ensures that the subset $R_v$ is closed inside $[0, \infty)$, and it is clear that it is open at any $t \in R_v$ with the property that $\gamma_v(t) \in H_i - Z_i$. We now claim that the set $R_v \subset [0, \infty)$ has only one boundary point. Indeed, if it had two such points $t_1 < t_2$, then from the comments above, we must have that $\gamma_v(t_1), \gamma_v(t_2) \in Z_i$ which implies, since $Z_i$ is totally geodesic, that $\gamma_v \subset Z_i$. But we know that $\gamma_v(0) = x \notin Z_i$, yielding a contradiction.

So we see that for each $v \in D$, the subset $R_v \subset [0, \infty)$ is a compact subset containing precisely one boundary point. This implies that it is a subinterval of $[0, \infty)$ of the type $[0, \phi(v)]$, where $\phi(v)$ is a real number depending on the chosen direction $v \in D$. Now note that the function $\phi : D \to [0, \infty)$ is a continuous function, tending to infinity as we approach $\partial D = S$. Furthermore, for each point $y \in H_i - C(\partial^\infty Z_i)$, there is unique $v(y) \in D$ and a unique $t_y \in [0, \phi(v)]$ with the property that $\gamma_{v(y)}(t_y) = y$.

Now fix a homeomorphism $\rho$ from $D$ to the upper hemisphere in the standard $(n - 1)$-dimensional sphere $S^{(n-1)} \subset \mathbb{R}^n$. Construct a map $\bar{\rho} : H_i - C(\partial^\infty Z_i) \to \mathbb{R}^n$ by setting $\bar{\rho}(y) := \rho(y) \cdot \Phi_{\phi(v)}(t_y)$, where the functions $\Phi_s : [0, s] \to [0, 1]$ are homeomorphisms varying continuously with respect to $s$, and having the property that the map $\Phi_\infty : [0, \infty) \to [0, 1]$ defined by $\Phi_\infty(t) := \lim_{s \to \infty} \Phi_s(t)$ is a homeomorphism. Observe that the map $\bar{\rho}$ is a homeomorphism from $H_i - C(\partial^\infty Z_i)$ to the subset

$$\{ \vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_n > 0, ||\vec{v}|| \leq 1 \}$$

The homeomorphism $\bar{\rho}$ aligns (using $\rho$) the directions $D$ pointing from $x$ into the subspace $H_i - C(\partial^\infty Z_i)$ with directions at the origin in $\mathbb{R}^n$ pointing into the upper hemisphere, and then scales (using the functions $\Phi_s$) the intervals so that each geodesic segment $\gamma_v \cap [H_i - C(\partial^\infty Z_i)]$ maps to the unit length radial geodesic segment in the direction $\rho(v)$. Now observe that the choice of the scaling functions $\Phi_s$ implies that this homeomorphism $\bar{\rho}$ extends to a homeomorphism from $H_i$ to the subset:

$$\{ \vec{v} \in \mathbb{R}^n \mid v_n > 0, ||\vec{v}|| \leq 1 \} \cup \{ \vec{v} \in \mathbb{R}^n \mid v_n = 0, ||\vec{v}|| < 1 \}$$

But the subset of $\mathbb{R}^n$ described above is clearly homeomorphic to $[0, 1] \times \mathbb{R}^{n-1}$, concluding the proof of Lemma 2.1.

To conclude the proof of Property (2), we take the homeomorphism from $X_1$ to $X_2$. Note that each connected component of $\tilde{M}_1 - X_1$ is given by some $H_i \cong [0, 1] \times \mathbb{R}^{n-1}$, attached to a corresponding $C(\partial^\infty Z_i) \subset X_1$, and furthermore, the homeomorphism $\partial^\infty \tilde{M}_1 \to \partial^\infty \tilde{M}_2$ takes each $\partial^\infty Z_i \subset \partial^\infty \tilde{M}_1$ homeomorphically to a corresponding $\partial^\infty Z_i' \subset \tilde{M}_2$. This yields homeomorphisms between each $C(\partial^\infty Z_i) \subset X_1$ and the corresponding $C(\partial^\infty Z_i') \subset X_2$. On the level of the corresponding $H_i \subset \tilde{M}_1$ and $H_i' \subset \tilde{M}_2$, this gives a homeomorphism between the subsets $\{1\} \times \mathbb{R}^{n-1}$ in the respective topological splittings $H_i \cong [0, 1] \times \mathbb{R}^{n-1} \cong H_i'$. Extending in the obvious manner, we obtain compatible homeomorphisms between the various $H_i \subset \tilde{M}_1$ and the (bijectively associated) $H_i' \subset \tilde{M}_2$. Compatibility ensures that when we glue the $H_i$ to $X_1$, we still
obtain a homeomorphism onto the space obtained by gluing the $H_i'$ to $X_2$. But the resulting spaces are $\tilde{M}_1$ and $\tilde{M}_2$ respectively, completing the proof of Property (2), and hence of Theorem 1.2.

**Remark.** (1) As mentioned in the introduction, there is no hope of the present argument giving a smooth classification of the universal cover, as the homeomorphism is “built” from a homeomorphism between the (non-manifold) boundary at infinity of the universal cover.

(2) The hypothesis of *strict* negative curvature, rather than non-positive curvature, is used in two places. First of all, in the proof of Claim 3, to ensure that distinct connected lifts of boundary components yield subsets of the boundary at infinity that are disjoint. In the case of non-positive curvature, there is the possibility of two such distinct lifts of boundary components containing geodesic rays that are asymptotic. This can only occur if there exists a (semi-infinite) flat strip isometric to $[0, \infty) \times [0, r]$ (for some positive real number $r$) with $[0, \infty) \times \{0\}$ mapping to one boundary component, and $[0, \infty) \times \{r\}$ mapping to the other boundary component. Hence to obtain Claim 3, one can weaken the curvature hypothesis somewhat, by allowing non-positive curvature, but requiring the fact that there do not exist any such flat strips. More problematic is the use of strict negative curvature in the proof of Claim 1: in the presence of zero curvature, we cannot use the Bourdon metric on the boundary at infinity. We can alternatively use the homeomorphic projection to the unit tangent space at a point $x$ in the interior of the manifold, but we still have a problem: in the zero curvature setting, we can have a sequence of boundary components $Z_i$ with $d(x, Z_i) \to \infty$, but with the $Z_i$ projecting to subsets with diameter uniformly bounded away from zero. It is not clear what hypothesis would be needed to avoid this difficulty.

(3) Note that compactness of the manifold $M^n$ is used superficially in the proofs of Claims 1, 3 and 4. Indeed the same argument classifies topologically any simply connected Riemannian $n$-dimensional ($n \neq 5$) manifold $X$ having the following properties:

- $X$ has each boundary component totally geodesic and complete, used to show Claim (2),
- $X$ is semi-geodesically complete, i.e. every geodesic segment with endpoints not lying on distinct boundary components is extendible,
- for any two distinct boundary components $Z_i, Z_j$, there exists a constant $\epsilon_{ij} > 0$ such that for all $p_i \in Z_i, p_j \in Z_j$, we have the lower bound $d(p_i, p_j) \geq \epsilon_{ij}$, used in Claim (3),
- sectional curvature bounded above by $-a^2$ and bounded below by $-b^2$, used in Claims (1) and (3),
- the topological dimension of the boundary at infinity satisfies $\dim(\partial^{\infty}X) < n - 1$, used in Claim (4).
The first two bullets are used to construct the complete simply connected Riemannian manifold $\bar{X} \supset X$ by repeated reflections in the boundary components (which serves as a substitute for $\tilde{DM}$), and hence an inclusion $\partial^\infty X \subset \partial^\infty \bar{X} \cong S^{n-1}$. We leave it to the interested reader to verify that, with the conditions mentioned above, the proofs of Claims (1)-(4) go through with minimal changes (the author includes in the list above the number of the Claims whose proofs require the specified bullet).

3. Topological rigidity and applications.

In this section, we explain the proofs of Theorem 1.1 and 1.2 from the introduction. We first start with a definition:

**Definition 3.1.** Define the 1-tripod $T$ to be the topological space obtained by taking the join of a one point set with a three point set. Denote by $*$ the point in $T$ corresponding to the one point set. We define the $n$-tripod $(n \geq 2)$ to be the space $T \times \mathbb{D}^{n-1}$, and call the subset $* \times \mathbb{D}^{n-1}$ the spine of the tripod $T \times \mathbb{D}^{n-1}$. The subset $* \times \mathbb{D}^{n-1}$ separates $T \times \mathbb{D}^{n-1}$ into three open sets, which we call the open leaves of the tripod. The union of an open leaf with the spine will be called a closed leaf of the tripod. We say that a point $p$ in a topological space $X$ is $n$-branching provided there is a topological embedding $f : T \times \mathbb{D}^{n-1} \rightarrow X$ such that $p \in f(* \times \mathbb{D}^{n-1})$.

It is clear that the property of being $n$-branching is invariant under homeomorphisms. Note that, in a simple, thick P-manifold of dimension $n$, points in the codimension one strata are automatically $n$-branching. One can ask whether this property can be detected at the level of the boundary at infinity. This is the content of the following:

**Proposition 3.1 (Characterization of branching points).** Let $X$ be an $n$-dimensional, simple, thick, negatively curved P-manifold, and $p \in \partial^\infty \bar{X}$. Then $p$ is $(n-1)$-branching if and only if there exists a geodesic ray $\gamma$, entirely contained in the lift of the branching locus, and satisfying $\gamma(\infty) = p$.

**Proof.** First observe that if $p \in \partial^\infty \bar{X}$ coincides with $\gamma(\infty)$, for some $\gamma$ entirely contained in a connected component $\mathcal{B}$ of the lift of the branching locus, then from the thickness hypothesis, there exist $\geq 3$ lifts of chambers that contain $\gamma$ in their common intersection $\mathcal{B}$. Focusing on three such lifts of chambers, call them $Y_i, Y'_i, Y''_i$, we can successively extend each of these in the following manner: form subspaces $Y_{i+1}, Y'_{i+1}, Y''_{i+1}$ from the subspaces $Y_i, Y'_i, Y''_i$ by choosing, for each boundary component of $Y_i, Y'_i, Y''_i$ distinct from $\mathcal{B}$, an incident lift of a chamber (note that each boundary component is a connected component of the lift of the branching locus). Finally, set $Y_\infty := \cup_i Y_i$, and similarly for $Y'_\infty, Y''_\infty$. Now observe that, by construction, the three subsets $Y_\infty, Y'_\infty, Y''_\infty$ have the following properties:

- they are totally geodesic subsets of $\bar{X}$,
• their pairwise intersection is precisely $B$, their (common, totally geodesic) boundary component,
• doubling them across their boundary $B$ results in a simply connected, negatively curved, complete Riemannian manifold.

The first property ensures that the boundary at infinity of the space $Y_\infty \cup Y'_\infty \cup Y''_\infty$ embeds in $\partial^\infty \bar{X}$. The third property ensures that $\partial^\infty Y_\infty \cong \partial^\infty Y'_\infty \cong \partial^\infty Y''_\infty \cong \mathbb{D}^{n-1}$. The second property ensures that $S^{n-2} \cong \partial^\infty B \subset \partial^\infty \bar{X}$ coincides with the boundary of the three embedded $\mathbb{D}^{n-1}$. Since $p \in \partial^\infty B$, this immediately implies that $p$ is $(n-1)$-branching, yielding one of the two desired implications.

Conversely, assume that $p \in \bar{X}$ is not of the form $\gamma(\infty)$, where $\gamma$ is contained entirely in a connected component of the lift of the branching locus. Consider a geodesic ray $\gamma$ satisfying $\gamma(\infty) = p$, and note that there are two possibilities:

• there exists a connected lift $W$ of a chamber with the property that $\gamma$ eventually lies in the interior of $W$, and is not asymptotic to any boundary component of $W$, or
• $\gamma$ intersects infinitely many connected lifts of chambers.

In both these cases, we would like to argue that $p$ cannot be $(n-1)$-branching.

Let us consider the first of these two cases, and assume that there exists an embedding $f : T \times \mathbb{D}^{n-2} \to \partial^\infty \bar{X}$ satisfying $p \in f(\{\ast\} \times \mathbb{D}_0^{n-2})$. Picking a point $x$ in the interior of $W$, one can consider the composition $\pi_x \circ f : T \times \mathbb{D}^{n-2} \to lk_x \cong S^{n-1}$, where $lk_x$ denotes a small enough $\epsilon$-ball centered at the point $x$, and the map $\pi_x$ is induced by geodesic retraction. Note that the map $\pi_x$ is not injective: the points in $lk_x$ where $\pi_x$ is injective coincides with $\pi_x(\partial^\infty W)$ (i.e. for every $q \in \partial^\infty W$, we have $\pi_x^{-1}(\pi_x(q)) = \{q\}$, and the latter are the only points in $\partial^\infty \bar{X}$ with this property). Note that, from Theorem 1.2, along with part (2) of Cannon’s theorem, this subset of injective points $I \subset lk_x$ is an $(n-2)$-dimensional Sierpinski curve. Furthermore, the hypothesis on the point $p$ ensures that $\pi_x p$ does not lie on one of the boundary spheres of the $(n-2)$-dimensional Sierpinski curve $I$. But now in [L2] the following result was established:

**Theorem:** Let $F : T \times \mathbb{D}^{n-2} \to S^{n-1}$ be a continuous map, and assume that the sphere $S^{n-1}$ contains an $(n-2)$-dimensional Sierpinski curve $I$. Let $\{U_i\}$ be the collection of embedded open $(n-1)$-cells whose complement yield $I$, and let $Inj(F) \subset S^{n-1}$ denote the subset of points in the target where the map $F$ is injective. Then $F(\{\ast\} \times \mathbb{D}_0^{n-2}) \cap [I - \cup_i(\partial U_i)] \neq \emptyset$, implies that $[\cup_i(\partial U_i)] - Inj(F) \neq \emptyset$. In other words, this forces the existence of a point in some $\partial U_i$ which has at least two pre-images under $F$.

Actually, in [L2] this Theorem was proved using purely topological arguments under some further hypotheses on the open cells $U_i$. But parts (1) and (3) of Cannon’s Theorem allows the exact same proof to apply in the more general setting, just by
composing with a homeomorphism taking the arbitrary Sierpinski curve to the one used in the proof in [L2].

To conclude, we apply the Theorem above to the composite map \( F := \pi_x \circ f : T \times \mathbb{D}^{n-2} \to \text{lk}_x \). The point \( f^{-1}(p) \in \{ \ast \} \times \mathbb{D}^{n-2}_0 \) has image lying in \( I - \bigcup_i (\partial U_i) \), which tells us that \( F(\{ \ast \} \times \mathbb{D}^{n-2}_0) \cap [I - \bigcup_i (\partial U_i)] \neq \emptyset \). The Theorem implies that there exists a point \( q \) in some \( \partial U_i \subset I \) which has at least two pre-images under the composite map \( F = \pi_x \circ f \). Since the map \( \pi_x \) is actually injective on the set \( I \), this implies that the map \( f \) had to have two pre-images at the point \( \pi^{-1}_x(q) \in \partial^\infty \tilde{X} \), contradicting the fact that \( f \) was an embedding. This resolves the first of the two possible cases.

For the second of the two cases (where the geodesic ray \( \gamma \) passes through infinitely many lifts of chambers), a simple separation argument (see Sections 3.2, 3.3 in [L2]) shows that if there exists a branching point of the second type, there must also exist a branching point of the first type. But we saw above that there cannot exist any branching points of the first type. This concludes the proof of Proposition 3.1.

Now given the characterization of branching points, let us see how to show Theorem 1.1. So assume that we are given a pair \( X_1, X_2 \) of simple, thick, negatively curved P-manifolds of dimension \( n \geq 6 \), and that we are told that \( \pi_1(X_1) \cong \pi_1(X_2) \). This immediately implies that \( \tilde{X}_1 \) is quasi-isometric to \( \tilde{X}_2 \), and hence that \( \partial^\infty \tilde{X}_1 \) is homeomorphic to \( \partial^\infty \tilde{X}_2 \). Let \( B_i \) denote the union, in each respective \( \partial^\infty \tilde{X}_i \), of the boundaries at infinity of the individual connected components of the lift of the branching locus. Note that each \( B_i \) is a union of countably many, pairwise disjoint, embedded \( S^{n-2} \) inside \( \partial^\infty \tilde{X}_i \) (each \( S^{n-2} \) arising as the boundary at infinity of a single connected component of the lift of the branching locus). Now the characterization of branching points in Proposition 3.1 implies that, under the homeomorphism between \( \partial^\infty \tilde{X}_1 \) and \( \partial^\infty \tilde{X}_2 \), we have that \( B_1 \) must map homeomorphically to \( B_2 \).

In particular, connected components of \( B_1 \) must map homeomorphically to connected components of \( B_2 \). A result of Sierpinski [S1] implies that the connected components in each case are precisely the individual \( S^{n-2} \) in the countable union. This yields a bijection between connected components of the lift of the branching locus in the respective \( \tilde{X}_i \). Furthermore, the homeomorphism must restrict to a homeomorphism between the complements of the \( B_i \) in the respective \( \partial^\infty \tilde{X}_i \). The connected components of this complement are either:

- isolated points, corresponding to \( \gamma(\infty) \), where \( \gamma \) is a geodesic ray passing through infinitely many connected lifts of chambers, and
- components with \( \geq 2 \) points, which are in bijective correspondence with connected lifts of chambers in the respective \( \tilde{X}_i \) (see [L2] Section 3.2).
This yields a bijective correspondence between lifts of chambers in \(\tilde{X}_1\) and lifts of chambers in \(\tilde{X}_2\). Furthermore, the closure of the components containing \(\geq 2\) correspond canonically with \(\partial^\infty W_i\), where \(W_i\) is the bijectively associated connected lift of a chamber.

Now recall that the homeomorphisms between \(\partial^\infty \tilde{X}_1\) and \(\partial^\infty \tilde{X}_2\) has the additional property that it is equivariant with respect to the respective \(\pi_1(X_i)\) actions on the \(\partial^\infty \tilde{X}_i\). We also have the following Lemma relating the dynamics on \(\partial^\infty \tilde{X}\) with the action on \(\tilde{X}\) (the argument is identical to that given in [L1, pg. 212]) :

**Lemma 3.1.** Let \(B_i\) be a connected component of the lift of the branching locus in \(\tilde{X}\), and let \(W_i\) be a connected lift of a chamber in \(\tilde{X}\). Then we have:

- \(\text{Stab}_{\pi_1(X)}(B_i) = \text{Stab}_{\pi_1(X)}(\partial^\infty B_i)\), and
- \(\text{Stab}_{\pi_1(X)}(W_i) = \text{Stab}_{\pi_1(X)}(\partial^\infty W_i)\).

where the action on the left hand side is the obvious action of \(\pi_1(X)\) on \(\tilde{X}\) by deck transformations, and the action on the right hand side is the induced action of \(\pi_1(X)\) on \(\partial^\infty \tilde{X}\).

Observe that equivariance of the homeomorphism implies that the bijective correspondence between connected lifts of chambers descends to a bijective correspondence between the chambers in \(X_1\) and the chambers in \(X_2\) (since two connected lifts of chambers cover the same chamber in \(X_i\) if and only if the two lifts have stabilizers which are conjugate in \(\pi_1(X_i)\)). Similarly, the bijective correspondence between connected components of the lifts of the branching loci descends to a bijective correspondence between the connected components of the branching loci in \(X_1\) with those in \(X_2\). Furthermore, by equivariance of the homeomorphism, we have that chambers (or connected components of the branching loci) that are bijectively identified have isomorphic fundamental groups. Separation arguments identical to the ones in [L1, Lemmas 2.1-2.4] ensures that the bijective correspondence also preserves the incidence relation between chambers and components of the codimension one strata (and that the isomorphisms between the various fundamental groups respect the incidence structure).

To conclude, we apply the celebrated Farrell-Jones topological rigidity theorem for non-positively curved manifolds [FJ]. This implies that, corresponding to the bijections between chambers (and components of the branching loci), one has homeomorphisms between the corresponding chambers that induce the isomorphisms on the level of the fundamental groups. Note that, a priori, the various homeomorphisms between chambers might not be compatible with the gluing maps. But by construction, the attaching maps all induce the same maps on the fundamental group \(\pi_1(B_i)\) of each individual component \(B_i\) of the branching locus. By Farrell-Jones, this implies that the restriction to \(B_i\) of the maps induced by the various homeomorphisms of incident chambers are all pairwise isotopic. Hence at the cost of deforming the
homeomorphism in a collared neighborhood of the boundary of each chamber, we may assume that the homeomorphisms respect the gluing maps. But attaching together these individual homeomorphisms on chambers now induces a \emph{globally defined} homeomorphism from $X_1$ to $X_2$. This concludes the sketch of Theorem 1.1.

To obtain Corollary 1.1, we merely note that the generalized Seifert-Van Kampen theorem implies that both $\pi_1(X_i)$ can be expressed as the direct limit of a diagram of groups, with vertex groups given by the fundamental groups of the chambers (and of the components of the branching locus), and edge morphisms induced by the inclusion of the components of the branching locus into the incident chambers. Now an abstract isomorphism between the direct limits corresponds to an isomorphism from $\pi_1(X_1)$ to $\pi_1(X_2)$. From Theorem 1.1, this isomorphism is induced by a homeomorphism from $X_1$ to $X_2$, and hence must take chambers to chambers and components of the branching locus to components of the branching locus. This implies the existence of isomorphism between the groups attached to the vertices in the diagram for $\pi_1(X_1)$ to the groups attached to the corresponding vertices in the diagram for $\pi_1(X_2)$. Furthermore, these isomorphisms commute (up to inner automorphisms, due to choice of base points) with the corresponding edge morphisms. But this is precisely the definition of diagram rigidity. This concludes the sketch of Corollary 1.1.

Next let us explain the argument for Corollary 1.2. Since the space $X$ is a $K(\Gamma, 1)$, any injection $i : \Gamma \hookrightarrow \Gamma$ with image of finite index yields a finite cover $\hat{i} : \hat{X} \to X$ with $\pi_1(\hat{X}) \cong \Gamma$, and $\hat{i} \circ \pi_1(X) = i(\Gamma)$. Now Theorem 1.2 implies that $\hat{X}$ is homeomorphic to $X$, so this yields a covering map $\hat{i} : \hat{X} \to X$, whose degree coincides with the index of the group $i(\Gamma)$ in $\Gamma$. Hence it is sufficient to show that this covering has degree one. But we know that $X$ contains a chamber with a non-zero characteristic number. Since there are finitely many chambers, consider a chamber $W$ for which this characteristic number has the largest possible magnitude $|r| \neq 0$. Then we know that under a covering of degree $d$, characteristic numbers scale by the degree, so we conclude that the pre-image chamber $\hat{i}^{-1}(W)$ has characteristic number of magnitude $d \cdot |r|$. By maximality of the characteristic number of $W$, we conclude that $d = 1$, as desired. Note that in this argument, it is crucial that the image $i(\Gamma)$ has finite index in $\Gamma$. If this is not the case, then the covering space $\hat{X}$ is non-compact. Since compactness was an essential ingredient in the proof of Theorem 1.1, one cannot conclude in this situation that $\hat{X}$ is homeomorphic to $X$.

Finally, for Corollary 1.3, take any element $\alpha \in Out(\Gamma)$. Then there exists an element $\hat{\alpha} \in Aut(\Gamma)$ which projects to $\alpha$ under the canonical map $Aut(\Gamma) \to Out(\Gamma)$. From Theorem 1.1, we have a self-homeomorphism $\phi \in Homeo(X)$ with the property that $\phi_* = \alpha$, concluding the proof of Corollary 1.3.

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