Approximation Theory of Wavelet Frame Based Image Restoration

Jian-Feng Cai\textsuperscript{a,b,1}, Jae Kyu Choi\textsuperscript{c,2,*}, Jianbin Yang\textsuperscript{d,3}

\textsuperscript{a}Department of Mathematics, The Hong Kong University of Science and Technology, Hong Kong
\textsuperscript{b}HKUST Shenzhen-Hong Kong Collaborative Innovation Research Institute, Futian, Shenzhen, China.
\textsuperscript{c}School of Mathematical Sciences, Tongji University, Shanghai, 200092 China
\textsuperscript{d}Department of Mathematics, Hohai University, Nanjing, 211100 China

Abstract

In this paper, we analyze the error estimate of a wavelet frame based image restoration method from degraded and incomplete measurements. We present the error between the underlying original discrete image and the approximate solution which has the minimal $\ell_1$-norm of the canonical wavelet frame coefficients among all possible solutions. Then we further connect the error estimate for the discrete model to the approximation to the underlying function from which the underlying image comes.

Keywords: Tight wavelet frame, framelet, image restoration, incomplete data recovery, error estimate, $\ell_1$ minimization, uniform law of large numbers, covering number, asymptotic approximation analysis

1. Introduction

In many practical image restoration tasks, observed images or measurements are often incomplete in the sense that features of interest in an image are missing partially and/or corrupted by noise. The corresponding inverse problem is to restore the underlying image $f$ from its incomplete measurements given by

$$g[k] = \begin{cases} (Af)[k] + \eta[k], & k \in \Lambda, \\ \text{unknown}, & k \in \Omega \setminus \Lambda. \end{cases}$$

In (1.1), $\eta$ is a measurement error which is assumed to be uncorrelated with $(Af)|\Lambda$, and $A$ is some linear operator describing the process of imaging, acquisition, and communication. As convention, we consider discrete images as real-valued functions on a two dimensional regular grid $\Omega \subseteq \mathbb{Z}^2$, and $\Lambda$ is a subset of $\Omega$ where $g$ is available or reliable. Nevertheless, the discussions on other dimensions will be almost the same with slight modifications. Note that this paper involves both functions and their discrete counterparts. We shall use regular characters to denote functions and use bold-faced characters to denote their discrete analogies. For example, we use $u$ as an element in a function space, while we use $u$ to denote its corresponding discretized version (the type of discretization will be made clear later).

The measurements could be part of (degraded) images, transmitted data, and sensor data. The restoration of missing data from a given incomplete measurement is an essential part of imaging science area where the final image is utilized for the visual interpretation or for the automatic analysis \cite{51}. In the literature,

\*Corresponding author

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nnumerous methods have been proposed under many different settings, e.g. [9, 11, 26, 42] for image inpainting, [27, 62, 64, 65] for wavelet domain inpainting, [44, 45, 52] for compressed sensing MRI restoration, [28, 36] for sparse view/angle CT restoration, and [19, 25] for miscellaneous applications. That being said, we forgo to give detailed surveys on these various applications and the interested reader should consult the references mentioned above for details. Instead, assuming that we focus on analyzing what happens when \( \rho \) is 0. More precisely, we will show that, under some mild assumptions, the following inequality

\[
\frac{1}{|\Omega|} \left\| u^A - f \right\|_{\ell^2(\Omega)}^2 \leq \frac{2\sigma_{\min}(A)^2}{|\Omega|} \left( \left\| \mathbf{A}u^A - g \right\|_{\ell^2(\Omega)}^2 + \left\| g - \mathbf{A}f \right\|_{\ell^2(\Omega)}^2 \right) \leq 4\sigma_{\min}(A)^2 \eta^2,
\]

which means that if \( \rho = 1 \) and \( \eta = 0 \), the unique solution to (1.3) is the original image \( f \). Hence, we aim to focus on analyzing what happens when \( \rho < 1 \) and \( \eta > 0 \). More precisely, we will show that, under some mild assumptions, the following inequality

\[
\frac{1}{|\Omega|} \left\| u^A - f \right\|_{\ell^2(\Omega)}^2 \leq C_1 \rho^{-1/2} |\Omega|^{-\gamma} (\log_2 |\Omega|)^{3/2} + C_2 \eta^2.
\]
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holds with probability at least $1 - |\Omega|^{-1}$. In (1.4) $\gamma$ is a positive constant related to the regularity of $f$, and $C_1$ and $C_2$ are constants independent of $|\Omega|$, $\rho$, and $\eta$. Briefly speaking, as long as the data set is sufficiently large, one has a pretty good chance to restore $f$ by solving (1.3).

In the literature, there are efficient numerical algorithms developed to solve (1.3) with a guaranteed convergence to minimizer $\|\cdot\|_1$. These numerical algorithms aim to find a sparse approximation of the underlying image using a properly chosen wavelet frame transform. Then a sparsity promoting regularization term (such as the widely used $\ell_1$-norm) is used to regularizing smooth image components while preserving image singularities such as edges and ridges. Hence, one may consider the approximation analysis of the numerical algorithms in the context of compressed sensing (CS) (e.g. $19, 20, 21, 41$). However, we would like to mention that our setting of (1.3) is different from that of CS. First of all, we only assume the decay of wavelet frame coefficients $Wf$ in the sense that $\|\lambda \cdot Wf\|_1$ is bounded rather than the explicit sparsity of $f$ and/or $Wf$, so that we can impose a low regularity assumption on $f$. Second and most importantly, the “sensing matrix” of (1.1) does not satisfy the conditions required for the theoretical analysis of CS. In our setting, the sensing matrix is $R_{\Lambda}A$, where $R_{\Lambda}$ denotes the projection onto $\Lambda$: $R_{\Lambda}f[k] = f[k]$ for $k \in \Lambda$, and $R_{\Lambda}f[k] = 0$ for $k \in \Omega \setminus \Lambda$. Since $\Lambda$ is randomly chosen from all $m$-subsets of $\Omega$ with a given fixed $m$, $R_{\Lambda}$ does not satisfy the concentration inequality in $54$, and $R_{\Lambda}A$ may not satisfy the restricted isometry property condition in general, all of which mean that there may not be information in the measurement $g$ sufficient for the exact restoration $18$. Instead, the main tools are the uniform law of large numbers and an estimation of covering number $29$ of the hypothesis space, which are widely used in classical empirical process $59$ and statistical learning theory $60$. We further mention that our error analysis uses the newly involved estimation for covering number. More precisely, even though we also use the special structure of the set and the max-flow min-cut theorem in graph theory to estimate covering number, we improve the estimate in $18$ Theorem 2.4 by relaxing the constraint on the radius in the covering number.

As a byproduct of the approximation analysis, we can further connect the approximation property of (1.3) to the asymptotic approximation to the underlying function in terms of the shift invariant space generated by a scaled compactly supported basis function. Such an asymptotic approximation is extensively studied in terms of the Strang-Fix condition $31, 40, 49, 63$. Notice that these asymptotic approximations require a certain order of regularity of underlying function in terms of e.g. the Sobolev space $2$, even in the case of $\ell_1$ minimization approach in $40, 63$. Meanwhile, our asymptotic approximation uses the decay condition of wavelet frame coefficients of underlying function and the Bessel property of the shifts of scaled basis function only, as an extension of $18$ for the wavelet frame based image inpainting to the generic image restoration tasks.

The rest of this paper is organized as follows. The explicit formulation of (1.4) is presented in Section 2 More precisely, we begin with introducing basic preliminaries of wavelet frames in Section 2.1. The error estimates for (1.3) in the discrete setting is given in Section 2.2, and in Section 2.3 we further establish the connection to function approximations in terms of the multiresolution approximation. Finally, all technical details are presented in Section 3 and Section 4 concludes this paper with relevant future works.

2. Approximation error of wavelet frame based image restoration

2.1. Preliminaries of wavelet frames

We begin with a brief introduction on wavelet tight frames. For details, one may consult $30, 55$ for theories of frames and wavelet frames, $57$ for a short survey on the theory and applications of frames, and $37, 38$ for more detailed surveys.

A countable set $\{\varphi_n : n \in \mathbb{Z}\} \subseteq L_2(\mathbb{R}^d)$ with $d \in \mathbb{N}$ is called a tight frame of $L_2(\mathbb{R}^d)$ if

$$\|f\|^2_{L_2(\mathbb{R}^d)} = \sum_{n \in \mathbb{Z}} |\langle f, \varphi_n \rangle|^2 \quad \text{for all } f \in L_2(\mathbb{R}^d),$$

(2.1)

where $\langle \cdot, \cdot \rangle$ is the inner product on $L_2(\mathbb{R}^d)$, and $\langle f, \varphi_n \rangle$ is called the canonical coefficient of $f$. 

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For given \( \Psi = \{\psi_1, \cdots, \psi_r\} \subseteq L_2(\mathbb{R}^d) \) and \( J \in \mathbb{N} \), the corresponding quasi-affine system \( \mathcal{X}^J(\Psi) \) generated by \( \Psi \) is defined by the collection of the dilations and the shifts of the elements in \( \Psi \):

\[
\mathcal{X}^J(\Psi) = \{\psi_{\alpha,l,k} : 1 \leq \alpha \leq r, \ l \in \mathbb{Z}, \ k \in \mathbb{Z}^d\}
\]

with \( \psi_{\alpha,l,k} \) being defined as

\[
\psi_{\alpha,l,k} = \begin{cases} 
2d^l \psi_\alpha(2^l \cdot -k) & l \geq J; \\
2^{l-J}d^l \psi_\alpha(2^l \cdot -2^j \cdot k) & l < J.
\end{cases}
\]

When \( \mathcal{X}^J(\Psi) \) forms a tight frame of \( L_2(\mathbb{R}^d) \), each \( \psi_1, \cdots, \psi_r \) is called a (tight) framelet and the entire system \( \mathcal{X}^J(\Psi) \) is called a (tight) wavelet frame. In particular when \( J = 0 \), we simply write \( \mathcal{X}(\Psi) = \mathcal{X}^0(\Psi) \). Note that in the literature, the affine system is widely used, which corresponds to the decimate wavelet (frame) transform. The quasi-affine system, which corresponds to the undecimated wavelet (frame) transformation, was first introduced and analyzed in [55]. Throughout this paper, we only discuss the quasi-affine system (2.3) because it generally performs better in image restoration than the widely used affine system, and the connection to the underlying function in continuum is more natural [15, 16, 39]. The interested reader can find further details on the affine wavelet frame systems and its connections to the quasi-affine wavelet frame systems in [29, 37, 55].

The constructions of framelets \( \Psi \), which are desirably (anti-)symmetric and compactly supported functions, are usually based on a multiresolution analysis (MRA) generated by some refinable function \( \phi \) with a refinement mask \( a_0 \) such that

\[
\phi = 2^d \sum_{k \in \mathbb{Z}^d} a_0[k] \phi(2 \cdot -k).
\]

The idea of an MRA based construction of \( \Psi = \{\psi_1, \cdots, \psi_r\} \subseteq L_2(\mathbb{R}^d) \) is to find finitely supported masks \( a_l \) such that

\[
\psi_\alpha = 2^d \sum_{k \in \mathbb{Z}^d} a_\alpha[k] \phi(2 \cdot -k) \quad \alpha = 1, \cdots, r.
\]

The sequences \( a_1, \cdots, a_r \) are called wavelet frame mask or the high pass filters of the system, and the refinement mask \( a_0 \) is also called the low-pass filter.

The unitary extension principle (UEP) of [55] provides a general theory of the construction of MRA based tight wavelet frames. Briefly speaking, as long as \( \{a_0, a_1, \cdots, a_r\} \) are compactly supported and their Fourier series

\[
\widehat{a}_\alpha(\xi) = \sum_{k \in \mathbb{Z}^d} a_\alpha[k] e^{-i \xi \cdot k}, \quad \alpha = 0, \cdots, r, \ \xi \in \mathbb{R}^d
\]

satisfy

\[
\sum_{\alpha=0}^r |\widehat{a}_\alpha(\xi)|^2 = 1 \quad \text{and} \quad \sum_{\alpha=0}^r \widehat{a}_\alpha(\xi) \overline{\widehat{a}_\alpha(\xi + \nu)} = 0
\]

for all \( \nu \in \{0, \pi\}^d \setminus \{0\} \) and \( \xi \in [-\pi, \pi]^d \), the quasi-affine system \( \mathcal{X}(\Psi) \) with \( \Psi = \{\psi_1, \cdots, \psi_r\} \) defined by (2.5) forms a tight frame of \( L_2(\mathbb{R}^d) \), and the filters \( \{a_0, a_1, \cdots, a_r\} \) form a discrete tight frame on \( l_2(\mathbb{Z}^d) \).

One of the most widely used examples is the piecewise linear B-spline [31] for \( L_2(\mathbb{R}) \), which has one
refinable function and two framelets with the associated filters

\[ a_0 = \frac{1}{4} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}, \quad a_1 = \sqrt{\frac{7}{4}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \quad \text{and} \quad a_2 = \frac{1}{4} \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}. \]

Indeed, it can be shown that the above \( \{a_0, a_1, a_2\} \) satisfies (2.6) so that \( \mathcal{X}(\Psi) \) with \( \Psi = \{\psi_1, \psi_2\} \) defined by (2.5) forms a tight frame on \( L_2(\mathbb{R}) \).

One possible way to construct a tight frame on \( L_2(\mathbb{R}) \) is by taking tensor products of univariate tight framelets [15, 16, 30, 37]. Given a set of univariate masks \( \{a_0, a_1, \cdots, a_r\} \), we define two dimensional masks \( a_\alpha[k] \) with \( \alpha = (\alpha_1, \alpha_2) \) and \( k = (k_1, k_2) \) as

\[ a_\alpha[k] = a_{\alpha_1}[k_1]a_{\alpha_2}[k_2], \quad 0 \leq \alpha_1, \alpha_2 \leq r, \quad k = (k_1, k_2) \in \mathbb{Z}^2 \]

so that the corresponding two dimensional refinable function and framelets are defined as

\[ \psi_\alpha(x) = \psi_{\alpha_1}(x_1)\psi_{\alpha_2}(x_2), \quad 0 \leq \alpha_1, \alpha_2 \leq r, \quad x = (x_1, x_2) \in \mathbb{R}^2 \]

with \( \psi_0 = \phi \) for convenience. If the univariate masks \( \{a_l : l = 0, \cdots, r\} \) are constructed from UEP, then it can be verified that \( \{a_\alpha : \alpha \in \{0, \cdots, r\}^2\} \) satisfies (2.6) and thus \( \mathcal{X}(\Psi) \) with

\[ \Psi = \{\psi_\alpha : \alpha \in \{0, \cdots, r\}^2 \setminus \{0\}\} \]

forms a tight frame for \( L_2(\mathbb{R})^2 \).

The advantage of framelet is that we can easily derive the discrete tight frame system by framelet decomposition and reconstruction algorithms of [31]. Throughout this paper, we only consider the MRA forms a tight frame for \( \ell_2(\mathbb{Z}) \). The analysis operator (see, e.g., [37]) with \( \alpha_0 \in Z_2 \), we define two dimensional masks \( a_{\alpha}[k] \) by

\[ a_\alpha[2^{-i}k], \quad k = 2^i\mathbb{Z}^2; \]

\[ 0, \quad k \notin 2^i\mathbb{Z}^2, \quad (2.7) \]

where \( a_\alpha = a_{\alpha_0} \), and * denotes the standard discrete convolution on \( \mathbb{Z}^2 \).

Let \( u \in \ell_2(\mathbb{Z}) \). To define an appropriate discrete framelet transform, or an analysis operator (see, e.g., [37]), we will impose the periodic boundary conditions. Throughout this paper, we identify \( \ell_p(\mathbb{Z}) \) with the space of \( \ell_p \) sequences on \( \mathbb{Z}^2 \) with fundamental period on each variable to be \( N \). We introduce the following periodization operator \( \mathcal{P}_f : \ell_1(\mathbb{Z}^2) \rightarrow \ell_1(\mathbb{Z}) \):

\[ \mathcal{P}_N(u)[k] = \sum_{k' \in \mathbb{Z}^2} u[k + Nk'], \quad k \in \mathbb{Z}. \]

Then for \( a \in \ell_1(\mathbb{Z}^2) \) and \( u \in \ell_2(\mathbb{Z}) \), we define the convolution \( a \odot u \) as

\[ (a \odot u)[k] = \langle \mathcal{P}_N(a)[k-\cdot], u \rangle = \sum_{n \in \Omega} \mathcal{P}_N(a)[k-n]u[n], \quad k \in \Omega. \]

Using the filters defined as (2.7), we define the two dimensional fast (discrete) framelet transform, or the analysis operator (see, e.g., [37]) with \( L \) levels of decomposition as

\[ W u = \{ W_{L, \alpha} u : (l, \alpha) \in \{0, \cdots, L - 1\} \times \mathbb{B} \} \cup \{(L - 1, 0)\}, \]

where \( \mathbb{B} = \{0, \cdots, r\}^2 \setminus \{0\} \) is the framelet band. Then \( W \) is a linear operator with the frame coefficients
Example 2.1. Before we continue, we present some examples related to A1. In particular, if $\lambda W u$ as we shall see later, the parameter $\lambda$ where the discrete convolution $\ast$ is defined as $[2.8]$. The synthesis framelet transform is denoted as $W^T$, the adjoint of $W$. Since we use a tight wavelet frame, we have the following perfect reconstruction formula

$$u = W^T W u$$

(2.10)

for all $u \in \ell_2(\Omega)$.

Finally, using this analysis operator $W$, we define $\|\lambda \cdot W u\|_1$ by

$$\|\lambda \cdot W u\|_1 = \sum_{k \in \Omega} \sum_{l=0}^{L-1} \sum_{\alpha \in \mathbb{B}} \lambda_{l,\alpha}[k] \left| \langle W_{l,\alpha} u \rangle[k] \right| .$$

(2.11)

In particular, if $\lambda_{l,\alpha}[k] = 1$ for all $(l, \alpha, k)$, we simply write it as $\|Wu\|_1$:

$$\|W u\|_1 = \sum_{k \in \Omega} \sum_{l=0}^{L-1} \sum_{\alpha \in \mathbb{B}} \left| \langle W_{l,\alpha} u \rangle[k] \right| .$$

(2.12)

As we shall see later, the parameter $\lambda$ in (2.11) is related to the decay of wavelet frame coefficients, thereby to control the regularity of $u$.

2.2. Error analysis of discrete model $[1.3]$

Throughout the rest of this paper, we denote $\Psi = \{\psi_{\alpha} : \alpha \in \mathbb{B}\}$ as the set of framelets, $\phi$ (or $\psi_0$) as the corresponding refinable function, and $\{a_{l,\alpha} : \alpha \in \mathbb{B} \cup \{0\}\}$ as the associated filters. In this paper, we focus on the tensor product B-spline wavelet frame systems constructed by [55], and $\phi$ is a tensor product B-spline function. We shall refer to $\psi_{\alpha}$ for $\alpha \in \mathbb{B}$ as the tensor product framelets. Note, however, that the analysis can be generalized to the generic MRA based wavelet frame systems whenever the corresponding refinable function is compactly supported with $\int_{\mathbb{R}^2} \phi(x) dx = 1$, Lipschitz continuous, and forms a Riesz basis on the closed shift invariant space $\bigvee \{\phi(\cdot - k) : k \in \mathbb{Z}^2\}$ [13].

Let $u^A$ be a solution to [1.3] with the weighted $\ell_1$-norm of wavelet frame coefficients defined as $[2.11]$ Recall that $\Omega = \{0, 1, \cdots, N - 1\}^2$ and $A$ is a data set uniformly randomly drawn from all $m$-subsets of $\Omega$. Denote by $\rho = m/|\Omega|$ the density of the pixels available. To draw an error between $u^A$ and the underlying image $f$, we assume that the linear operator $A$, and the parameter $\lambda := \{\lambda_{l,\alpha}[k] : l = 0, \cdots, L - 1, \alpha \in \mathbb{B}, k \in \Omega\}$ satisfy the followings:

A1. $\sigma_{\min}(A) > 0$, and both $\sigma_{\min}(A)$ and $\|A\|_\infty$ are independent of $|\Omega|$, i.e. independent of $N$.

A2. $\lambda = \{\lambda_{l,\alpha}[k] : l = 0, \cdots, L - 1, \alpha \in \mathbb{B}, k \in \Omega\}$ satisfies $\lambda_{l,\alpha}[k] = 2^{{\mathcal{Y}(l,\alpha,k)}}$ where $\beta \geq -1$ and $\mathcal{Y} : \Gamma \to \mathbb{N}$ is a positive integer-valued function such that

$$\max \{\mathcal{Y}(l,\alpha,k) : (l,\alpha,k) \in \Gamma\} \leq \frac{1}{2} \log_2 |\Omega|$$

(2.13)

with $\Gamma = \{(l,\alpha,k) : l = 0, \cdots, L - 1, \alpha \in \mathbb{B}, k \in \Omega\}$.

Example 2.1. Before we continue, we present some examples related to A1.

1. Image inpainting: in this case, we have $A = I$, i.e. $\sigma_{\min}(I) = \|I\|_\infty = 1$.

2. Simultaneous Gaussian deblurring and image inpainting: let $Au = a \ast u$ where $a$ is defined as

$$a[k] = C \frac{\exp\left\{\frac{|k|^2}{2\sigma^2}\right\}}{2\pi\sigma^2}, \quad k \in \mathbb{Z}^2,$$
for $\sigma > 0$ and a normalizing constant $C > 0$ is such that $\|a\|_{\ell_1(2^2)} = 1$. Obviously, we have $\|A\|_{\ell_\infty} = \|a\|_{\ell_1(2^2)} = 1$. In addition, by Poisson summation formula (e.g. [53]), we have

$$\tilde{a}(\omega) = C \sum_{\omega' \in \mathbb{Z}^2} \exp \left\{ -\frac{\sigma^2 |\omega - 2\pi \omega'|^2}{2} \right\} \geq C \exp \left\{ -\frac{\sigma^2 \pi^2}{2} \right\}, \quad \omega \in [-\pi, \pi]^2.$$ 

Hence, we can set $\sigma_{\min}(A) = C \exp \left\{ -\frac{\sigma^2 \pi^2}{2} \right\} > 0$, which is independent of $|\Omega|$.

3. Wavelet inpainting: let $A$ be a two dimensional orthonormal wavelet basis transform ($A^T A = AA^T = \mathcal{I}$) with filter banks $\{h_\beta : \beta \in \{0, 1\}^2 \}$ generated by the tensor product of a real-valued univariate conjugate mirror filter $h$:

$$h_\beta[k] = h_{\beta_1}[k_1] h_{\beta_2}[k_2], \quad k \in \mathbb{Z}^2, \quad \beta = (\beta_1, \beta_2) \in \{0, 1\}^2$$

with $h_0 = h$ and $h_1[k] = (-1)^{1-k} h[1-k]$. Then $\|A\|_{\ell_\infty} = \|h\|_{\ell_1(2)}$ and $\sigma_{\min}(A) = \sigma_{\max}(A) = 1$.

**Remark 2.2.** Together with the assumption on $f$ that $\|\lambda \cdot \mathcal{W}f\|_1 < \infty$, $A_2$ is related to the decay of the canonical wavelet frame coefficients $\mathcal{W}f$. Notice that the decay of $\mathcal{W}f$ is further related to the regularity of $f$, which will be clearer in the context of asymptotic function approximation. See Section 2.3 for details.

Under the above assumptions, we note that the underlying true image $f$ satisfies the constraints in (1.3) with $\|\lambda \cdot \mathcal{W}f\|_1 < \infty$. In addition, since $u^A$ is a solution to (1.3), its weighted $\ell_1$-norm $\|\lambda \cdot \mathcal{W}u^A\|_1$ attains the minimum subject to the constraints, and in particular, $\|\lambda \cdot \mathcal{W}u^A\|_1 \leq \|\lambda \cdot \mathcal{W}f\|_1$. This leads us to consider the following set

$$M = \left\{ u \in \ell_\infty(\Omega) : \|\lambda \cdot \mathcal{W}u\|_1 \leq \|\lambda \cdot \mathcal{W}f\|_1, \quad \frac{1}{|A|} \sum_{k \in A} |(Au)[k] - g[k]|^2 \leq \eta^2, \quad u \in [0, M\Omega] \right\} \quad (2.14)$$

as an involved hypothesis space. In (2.14) $M > 0$ is a positive constant related to the boundedness of each pixel value, $\eta > 0$ is a fixed positive constant related to the bound of measurement error, and the weighted $\ell_1$-norm of wavelet frame coefficients $\|\lambda \cdot \mathcal{W}u\|_1$ is defined as (2.11).

Our error estimate is based on the characterization of the capacity of the hypothesis space [18]. Among such capacities including VC dimension [60], $V_\gamma$-dimension and $P_\gamma$-dimension [3], Rademacher complexities [7, 50], and covering number [29], we choose the covering number because it is the most convenient for metric spaces [18].

**Definition 2.3.** Let $M \subseteq \mathbb{R}^\Omega$ and $r > 0$ be given. The covering number $\mathcal{N}(M, r)$ is defined as

$$\mathcal{N}(M, r) = \inf \left\{ K \in \mathbb{N} : \exists u_1, \ldots, u_K \in M \text{ s.t. } M \subseteq \bigcup_{j=1}^K \left\{ u \in M : \|u - u_j\|_{\ell_\infty(\Omega)} \leq r \right\} \right\},$$

i.e. the minimal number of $\ell_\infty$ balls with radius $r$ in $M$ that covers $M$.

With the above notation, we can present the first relation between the solution $u^A$ to (1.3) and the underlying image $f$. Following the idea of statistical learning theory [60], we first establish the probability of the following event

$$\frac{1}{|\Omega|} \|u^A - f\|_{\ell_2(\Omega)}^2 \leq \epsilon + \frac{16}{3} \sigma_{\min}(A) \eta^2$$

for an arbitrary $\epsilon > 0$ in terms of the covering number with the radius related to $\epsilon$. The proof is postponed to Section 5.1.
Theorem 2.4. Let \( M \) be defined as in (2.14) and \( u^A \) be a solution to (1.3). Then for an arbitrary \( \epsilon > 0 \), the following inequality

\[
P \left\{ \frac{1}{|\Omega|} \left\| u^A - f \right\|_{\ell_2(\Omega)}^2 \leq \epsilon + \frac{16}{3} \sigma^2_{\min}(A) \eta^2 \right\} \geq 1 - \mathcal{N} \left( \mathcal{M}, \frac{\sigma^2_{\min}(A) \epsilon}{12 \|A\|_{\infty}^2 M} \right) \exp \left\{ \frac{-3m\sigma^2_{\min}(A) \epsilon}{256 \|A\|_{\infty}^2 M^2} \right\}
\]

holds for an arbitrary \( m \), where \( m = |\Lambda| \) denotes the number of samples.

Remark 2.5. Notice that the proof of Theorem 2.4 is completed if we estimate the probability of the event

\[
\frac{1}{|\Omega|} \left\| Au^A - Af \right\|_{\ell_2(\Omega)}^2 \leq \sigma^2_{\min}(A) \epsilon + \frac{16}{3} \eta^2.
\]

In this case, we need to use the uniform boundedness constraint \( u[k] \in [0, M] \) for all \( k \in \Omega \), from which we need an assumption on \( \|A\|_{\infty} \). See Section 3.1 for details.

By Theorem 2.4, our error estimate will be completed if we estimate the covering number \( \mathcal{N}(M, r) \). Intuitively, since \( M \) lies in an \( \ell_\infty \) ball \( \{ u \in \ell_\infty(\Omega) : \| u \|_{\ell_\infty(\Omega)} \leq M \} \), it is obvious that

\[
\mathcal{N}(M, r) \leq \left( \frac{2M}{r} \right)^{|\Omega|}
\]

(2.15)
as given in [29]. However, since the above estimation [2.15] is not tight enough to derive an error estimate, we need to find a tighter upper bound for \( \mathcal{N}(M, r) \) to obtain a reasonable error bound. To do this, we adopt the idea given in [15]: using the discrete total variation, we connect the wavelet frame coefficients \( W u \) with the regularity of \( u \). Specifically, we define the discrete gradient of \( u \in \mathbb{R}^\Omega \) as

\[
\nabla u = \{ u[k + e_j] - u[k] : j = 1, 2, k \in \Omega, \& k + e_j \in \Omega \},
\]

(2.16)

where \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \). The discrete total variation \( \| \nabla u \|_1 \) is defined as

\[
\| \nabla u \|_1 := \sum_{j=1}^2 \sum_{k \in \Omega \atop k + e_j \in \Omega} |u[k + e_j] - u[k]|.
\]

(2.17)

Under this setting, we present Lemma 2.6 whose proof is postponed to Section 3.2.

Lemma 2.6. Let \( \mathcal{W} \) be a tensor product B-spline wavelet frame transform defined as (2.9). Then there exists a constant \( C_{\mathcal{W}} \geq 1 \) which is independent of \( |\Omega| \) and \( \ell \) such that

\[
\| \nabla u \|_1 \leq C_{\mathcal{W}} \| \mathcal{W} u \|_1
\]

(2.18)

for all \( u \in \mathbb{R}^\Omega \). In (2.18) \( \| \nabla u \|_1 \) is defined as (2.17) and \( \| \mathcal{W} u \|_1 \) is defined as (2.12).

Lemma 2.6 tells us that, given that \( \lambda = \{ \lambda_{l, \alpha} \} \) satisfies A2, for \( u \in \mathcal{M} \), we have

\[
\| \nabla u \|_1 \leq C_{\mathcal{W}} \| \mathcal{W} u \|_1 = C_{\mathcal{W}} \sum_{k \in \Omega} \sum_{l=0}^{L-1} \sum_{\alpha \in \mathbb{B}} |(\mathcal{W}_{l, \alpha} u)[k]|
\]

\[
\leq C_{\mathcal{W}} \sum_{k \in \Omega} \sum_{l=0}^{L-1} \sum_{\alpha \in \mathbb{B}} |(\mathcal{W}_{l, \alpha} u)[k]| \leq C_{\mathcal{W}} |\Omega| \frac{\max(1-\beta, 0) \max \{ T(l, \alpha, k) : (l, \alpha, k) \in \Gamma \} \sum_{k \in \Omega} \sum_{l=0}^{L-1} 2^{\beta T(l, \alpha, k)} |(\mathcal{W}_{l, \alpha} u)[k]|}{\| \mathcal{W}_f \|_1}
\]

\[
\leq C_{\mathcal{W}} |\Omega| \frac{\max(1-\beta, 0) \max \{ T(l, \alpha, k) : (l, \alpha, k) \in \Gamma \} \sum_{k \in \Omega} \sum_{l=0}^{L-1} 2^{\beta T(l, \alpha, k)} |(\mathcal{W}_{l, \alpha} u)[k]|}{\| \mathcal{W}_f \|_1}.
\]
Proof.\ Theorem 2.7, for the estimation of covering number. Similar to [13, Theorem 2.4], our estimate is also based on the quantized total variation minimization (e.g. [1, 24]) and the max-flow min-cut theorem [32] to introduce a new upper bound of the covering number.

**Theorem 2.7.** Let \( \mathcal{W} \) be a tensor product B-spline wavelet frame transform defined as (2.9). Assume that \( A_2 \) is satisfied with \(-1 < \beta < 1\) and \( \| \lambda \cdot \mathcal{W} f \|_1 \leq C_f |\Omega|^{1/2} \). Let \( M \) be defined as (2.14) for \( r \geq |\Omega|^{-a} \) with \( a \geq 1 \), we have

\[
\ln \mathcal{N}(M, r) \leq \frac{C_a |\Omega|^{\max(1-\beta,1)}}{r} \log_2 |\Omega|,
\]

where \( C_a = 20M(4a + \max\{1 - \beta, 1\})C_{\mathcal{W}}C_f(1 + C_{\mathcal{W}}C_f) \).

**Proof.** See Section 3.3

With the aid of Theorems 2.4 and 2.7, we can establish the explicit formulation of (1.4) in Theorem 2.8. Briefly speaking, for a fixed \( \rho \), as long as the resolution of an image is sufficiently high, the wavelet frame based restoration model (1.3) has a good chance to restore the original image within the measurement error. Further, for a fixed \( |\Omega| \), the error gets smaller as \( \rho \to 1 \), which coincides with our common sense. Finally, Theorem 2.8 takes the results in [18] as a special case because \( \| I \|_\infty = \sigma_{\min}(I) = 1 \).

**Theorem 2.8.** Let \( \mathcal{W} \) be a tensor product B-spline wavelet frame transform. Assume that \( \mathcal{A} \) satisfies A1, \( \lambda \) satisfies A2 with \(-1 < \beta < 1\), and \( f \in [0, M]^|\Omega| \) satisfies \( \| \lambda \cdot \mathcal{W} f \|_{L^2(\Omega)} \leq C_f |\Omega|^{1/2} \). Let \( u^A \) be a solution to (1.3). Then the following inequality

\[
\frac{1}{|\Omega|} \| u^A - f \|_{L^2(\Omega)}^2 \leq \bar{c} \rho^{-a} |\Omega|^{-1} \left[ 4 + 3 \sqrt{5(4a + \max\{1 - \beta, 1\})C_{\mathcal{W}}C_f(1 + C_{\mathcal{W}}C_f)} \right],
\]

holds with probability at least \( 1 - |\Omega|^{-1} \). In (2.20) \( \bar{c} \) is defined as

\[
\bar{c} = \frac{64 \| \mathcal{A} \|_\infty M^2}{3\sigma_{\min}(\mathcal{A})} \left[ 4 + 3 \sqrt{5(4a + \max\{1 - \beta, 1\})C_{\mathcal{W}}C_f(1 + C_{\mathcal{W}}C_f)} \right],
\]

for some fixed \( a > 1 \). In words, \( \bar{c} \) is independent of \( |\Omega|, I, \rho, \) and \( \eta \).

**Proof.** First of all, for any \( \epsilon > 0 \), we can choose \( a \geq 1 \) such that \( \epsilon \geq \frac{12 \| \mathcal{A} \|_\infty M^2}{3\sigma_{\min}(\mathcal{A})} \). With \( r = \frac{\sigma_{\min}(\mathcal{A}) \rho}{2\| \mathcal{A} \|_\infty M} \geq |\Omega|^{-a} \), Theorems 2.4 and 2.7 give that the inequality

\[
\frac{1}{|\Omega|} \| u^A - f \|_{L^2(\Omega)}^2 \leq \epsilon + \frac{16}{3} \sigma_{\min}(\mathcal{A}) \eta^2
\]

holds with probability at least

\[
1 - \mathcal{N} \left( \mathcal{M}, \frac{\sigma_{\min}(\mathcal{A}) \epsilon}{12 \| \mathcal{A} \|_\infty M^2}, \exp \left\{ \frac{-3m \sigma_{\min}(\mathcal{A}) \epsilon}{256 \| \mathcal{A} \|_\infty M^2} \right\} \right)
\geq 1 - \exp \left\{ \frac{240 \| \mathcal{A} \|_\infty M^2(4a + b)C_{\mathcal{W}}C_f(1 + C_{\mathcal{W}}C_f)|\Omega|^{b/2} \log_2 |\Omega| - 3m \sigma_{\min}(\mathcal{A}) \epsilon}{256 \| \mathcal{A} \|_\infty M^2} \right\},
\]
In the above, we use $b = \max\{1 - \beta, 1\}$ for notational simplicity. Hence, if we choose a special $\epsilon^*$ to be the unique positive solution to the following equation

$$
\frac{240 \|A\|^2_{\infty} M^2 (4a + b) C_W C_f (1 + C_W C_f) |\Omega|^{b/2} \log_2 |\Omega|}{\sigma_{\min}^2(A) \epsilon} - \frac{3m \sigma_{\min}^2(A) \epsilon}{256 \|A\|^2_{\infty} M^2} = \ln \frac{1}{|\Omega|},
$$

we have

$$
\frac{1}{|\Omega|} \left\| u^A - f \right\|_{L^2(\Omega)}^2 \leq \epsilon^* + \frac{16}{3} \sigma_{\min}^{-2}(A) \eta^2
$$

with probability at least $1 - |\Omega|^{-1}$. Indeed, solving (2.21) yields (2.20)

$$
\epsilon^* = \frac{128 \|A\|^2_{\infty} M^2}{3m \sigma_{\min}^2(A) \sqrt{m}} \left[ \ln |\Omega| + \sqrt{\ln^2 |\Omega| + \frac{45}{4} m (4a + b) C_W C_f (1 + C_W C_f) |\Omega|^{b/2} \log_2 |\Omega|} \right]
$$

$$
\leq \frac{64 \|A\|^2_{\infty} M^2}{3m \sigma_{\min}^2(A) \sqrt{m}} \left[ 4 \ln |\Omega| + 3 \sqrt{5 (4a + b) C_W C_f (1 + C_W C_f) (1 + C_W C_f) |\Omega|^{b/2} \log_2 |\Omega|} \right]
$$

$$
\leq \frac{64 \|A\|^2_{\infty} M^2}{3m \sigma_{\min}^2(A) \sqrt{m}} \left[ 4 + 3 \sqrt{5 (4a + b) C_W C_f (1 + C_W C_f)} \right] \rho^{1/2} |\Omega|^{b/2} \log_2 |\Omega|^{3/2},
$$

as $b - 2 = - \min\{1 + \beta, 1\}$. In addition, from the definition of $\epsilon^*$, we have

$$
\epsilon^* \geq \frac{128 \|A\|^2_{\infty} M^2}{3m \sigma_{\min}^2(A) \sqrt{m}} \sqrt{C_W C_f (1 + C_W C_f) |\Omega|^{b/2} \log_2 |\Omega|}
$$

$$
\geq \frac{16 \|A\|^2_{\infty} M}{\sigma_{\min}^2(A) \sqrt{m}} \sqrt{C_W C_f (1 + C_W C_f)} \rho^{1/2} \sqrt{\log_2 |\Omega|} |\Omega|^{b/2} \log_2 |\Omega|^{1/2} \geq \frac{12 \|A\|^2_{\infty} M}{\sigma_{\min}^2(A) \sqrt{m}} |\Omega|^{-a}.
$$

In the final inequality, we use the fact that for $\beta \in (-1, 1)$, we have $b - 2 = - \min\{1 + \beta, 1\} \in (-1, 0)$ while $a \geq 1$. This completes the proof. \(\square\)

**Remark 2.9.** From the proof of Theorem 2.8, we can see that the constant $a \geq 1$ is to apply Theorem 2.7 to Theorem 2.4. Notice that the lower bound of the probability of

$$
\frac{1}{|\Omega|} \left\| u^A - f \right\|_{L^2(\Omega)}^2 \leq \epsilon + \frac{16}{3} \sigma_{\min}^{-2}(A) \eta^2
$$

decreases as $\epsilon > 0$ decreases. Meanwhile, we estimate the critical $\epsilon^* > 0$ (i.e. the positive solution to (2.21)) by fixing the probability to be no smaller than $1 - |\Omega|^{-1}$, which in turn increases the upper bound (or the confidence interval) of $\epsilon^* > 0$ as reflected by the constant $a \geq 1$. Nevertheless, we forgo the effect of $a \geq 1$ in (2.20) to the error estimate in order not to dilute the focus of this paper. In fact, $a \geq 1$ becomes fixed once we fix the admissible lower bound of $\epsilon > 0$ in Theorem 2.4 to apply Theorem 2.7.

To conclude, we present some numerical results to demonstrate the theoretical error bound in (2.20) and the empirical restoration error under different settings of $|\Omega|$ and $\rho$. Specifically, we fix $N = 2^J$ and $\eta = 0$, and we consider the following noise-free case

$$
\min_{u \in [0, M]^n} \| \lambda \cdot W u \|_1 \quad \text{subject to} \quad (Au)[k] = g[k], \ k \in \Lambda,
$$

so that (2.20) takes the form of

$$
\frac{1}{22^J} \left\| u^A - f \right\|_{L^2(\Omega)}^2 \leq \bar{c} \rho^{-1/2} J^{3/2} 2^{-J \min\{\frac{1+\beta}{2}, \frac{1}{2}\}}
$$
Approximation Theory of Wavelet Frame Based Image Restoration

Figure 1: Simulation results for the image inpainting. Fig. 1a describes the simulation results when $N = 512$ is fixed, and Fig. 1b depicts the simulation results when $\rho = 0.5$ is fixed.

Figure 2: Simulation results for the simultaneous deblurring and inpainting. Fig. 2a describes the simulation results when $N = 512$ is fixed, and Fig. 2b depicts the simulation results when $\rho = 0.5$ is fixed.

with probability at least $1 - 2^{-2J}$.

In this simulation, two types of $\mathcal{A}$ are used; one is the identity operator $\mathcal{I}$ (i.e. the image inpainting) and the other is a convolution operator with a low-pass filter generated by the Matlab built-in function “special(’gaussian’,9,1)” (i.e. the simultaneous deblurring and inpainting). More precisely, we generate a piecewise linear phantom as an underlying true image $f$, and we implement the simulation for each $\mathcal{A}$ as follows. To see the behavior of error with respect to the sample density $\rho$, we fix $N = 512$ (i.e. $J = 9$), and for each $\rho \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$, we test (2.22) with 100 realizations of $\Lambda$. To see the behavior of error with respect to $|\Omega|$, we fix $\rho = 0.5$, and for each $J \in \{5, 6, 7, 8, 9, 10\}$ (or the resolution $J$), we again test (2.22) with 100 realizations of $\Lambda$. In any case, we choose $\mathcal{W}$ to be the tensor product piecewise linear B-spline wavelet frame with 1 level of decomposition, and (2.22) is solved by the split Bregman algorithm [17]. Finally, we choose the largest empirical error to compare with the smallest theoretical error in (2.23) by setting $\beta = 0$ in (2.23). In particular, it is in general difficult to determine the constants explicitly, we calculate the above error by assuming that the equality holds in the worst case, i.e. in the lowest sample density, or the lowest resolution case. The results are presented in Figs. 1 and 2 respectively. Specifically, Figs. 1a and 2a show the results when $N = 512$ is fixed and $\rho$ is varying, and Figs. 1b and 2b demonstrate when $\rho = 0.5$ is fixed and $N$ is varying. We can easily see that, in any case, the empirical restoration error does not exceed the theoretical error in (2.23) which empirically demonstrates that Theorem 2.8 provides a reasonable upper bound for the restoration error with high probability.
2.3. Asymptotic function approximation via MRA

In this section, we apply Theorem 2.8 to further connect with the function approximation given that the underlying true image is obtained by the samples of a function. All functions we consider are defined on \( \Omega = [0, 1)^2 \), and we consider the \( 2^J \times 2^J \) Cartesian grid \( \Omega \) defined as
\[
\Omega = \{0, 1, \cdots, 2^J - 1\}^2 \simeq 2^{-J} \mathbb{Z}^2 \cap \Omega.
\]
In other words, we implicitly identify the \( 2^J \times 2^J \) grid \( \Omega \) with the \( 2^J \times 2^J \) mesh of \( \Omega \). For the consistency with the discrete image cases, we identify \( L_p(\Omega) \) with the space of \( L_p \) functions on \( \mathbb{R}^2 \) with fundamental period on each variable to be 1. For the wavelet frame decomposition (e.g. \([31, 37]\)) of \( f \in L_2(\Omega) \), we write
\[
f = \sum_{l \in \mathbb{Z}} \sum_{k \in \Omega} \sum_{\alpha \in \mathbb{B}} \langle f, \psi_{\alpha,l,k} \rangle \psi_{\alpha,l,k}
\]
by identifying \( \psi_{\alpha,l,k} \) with its periodized version
\[
\psi_{\alpha,l,k}^\text{per} = \sum_{k' \in \mathbb{Z}^2} \psi_{\alpha,l,k}(\cdot + 2^\max\{l,J\} k')
\]
with a slight abuse of notation.

In the literature of wavelet frame, the discrete image \( f \) is interpreted as discrete sampling of an underlying function \( f \in L_2(\Omega) \) via the inner product with the corresponding refinable function \( \phi \). More precisely, let \( \phi_{j,k} = 2^J \phi(2^J \cdot -k) \). Then we can write \( f \) as
\[
f[k] = 2^J \langle f, \phi_{j,k} \rangle. \tag{2.24}
\]
With this \( f \), we implicitly use the following interpolated function
\[
f_j = \sum_{k \in \Omega} f[k] \phi(2^J \cdot -k) = \sum_{k \in \Omega} \langle f, \phi_{j,k} \rangle \phi_{j,k} \tag{2.25}
\]
to approximate \( f \).

There are extensive studies on the approximation order of \( f_j \) in \([2.25]\) to the original \( f \), and most of them are based on certain properties of the refinable function \( \phi \). Relevant works include the wavelet frame based scattered data restoration \([63]\) and the wavelet frame based image denoising/inpainting \([40]\). Briefly speaking, these works commonly assume the Strang-Fix condition on \( \phi \) (e.g. \([31, 49]\)) to approximate the underlying function in the Sobolev space of a certain order. Meanwhile, the underlying function that we concern in this paper does not satisfy certain order of regularity. Instead, we only impose some decay condition of the wavelet frame coefficients for the analysis of the approximation order. More precisely, we assume that there is \( \beta \geq -1 \) such that
\[
C_f := \sum_{l=-L}^{\infty} 2^{3l} \sum_{k \in \Omega} \sum_{\alpha \in \mathbb{B}} |\langle f, \psi_{\alpha,l,k} \rangle| < \infty. \tag{2.26}
\]
Note that the above decay condition links to the regularity of the underlying function \( f \) when the wavelet frame system satisfies some mild conditions \([10, 18, 46]\).

Under the decay condition given in \([2.26]\) we present the following approximation of underlying function in Theorem 2.10. Briefly speaking, as long as the image resolution is sufficiently large, the approximated function constructed from the restored image gives a good approximation of the underlying function where the underlying true image comes from.

**Theorem 2.10.** Let \( \mathcal{W} \) be a tensor product B-spline wavelet frame transform. Assume that A1 is satisfied,
Remark 2.11. Before proving Theorem 2.10, we would like to mention that, whenever
the underlying function $f \in L_2(\Omega)$ satisfies (2.26) with $-1 < \beta < 1$. Let $u^\Lambda$ be a solution to (1.3) with
$g$ in (1.1) generated by $f$ in (2.24). If $f(x) \in [0,M] \ a.e. \ x \in \Omega$, we have the followings:
1. The inequality
   \[
   \frac{1}{2^J} \left\| u^\Lambda - f \right\|_{L_2(\Omega)}^2 \leq \tilde{c} \rho^{-\frac{\beta}{2}} J \min \left\{ \frac{1}{4}, \frac{1}{4} \right\} + \frac{16}{3} \sigma_{\min}(A) \eta^2
   \]
   holds with probability at least $1 - 2^{-2J}$, where $\tilde{c}$ is a constant independent of $J$ (i.e. independent of $|\Omega|$), $L$, $\rho$, and $\eta$.
2. Let $u^\Lambda_J$ be defined as
   \[
   u^\Lambda_J = \sum_{k \in \Omega} u^\Lambda[k] \phi(2^J \cdot - k).
   \]
   Then
   \[
   \left\| u^\Lambda_J - f \right\|_{L_2(\Omega)}^2 \leq C_1 \rho^{-\frac{\beta}{2}} J \min \left\{ \frac{1}{4}, \frac{1}{4} \right\} + C_2 \eta^2 + C_3 2^{-(\beta + 1)J}
   \]
   holds with probability at least $1 - 2^{-2J}$, where $C_1$, $C_2$, and $C_3$ are three positive constants independent of $J$, $L$, $\rho$, and $\eta$.

Remark 2.11. Before proving Theorem 2.10, we would like to mention that, whenever $J \in \mathbb{N}$ satisfies
\[
J \geq \frac{-2 \log_2 \eta}{\beta + 1},
\]
(2.29) can be further written as
\[
\left\| u^\Lambda_J - f \right\|_{L_2(\Omega)}^2 \leq C_1 \rho^{-\frac{\beta}{2}} J \min \left\{ \frac{1}{4}, \frac{1}{4} \right\} + (C_2 + C_3) \eta^2.
\]
In other words, whenever the mesh is sufficiently dense (i.e. $J$ is sufficiently large), the $L_2$ distance between
the interpolation of the restored image and the original unknown function is bounded by the restoration
error of the discrete image restoration problem (1.3) only.

The rest is devoted to the proof of Theorem 2.10. To begin with, we introduce the following lemma on
the Bessel property of \{ $\phi(2^J \cdot - k) : k \in \Omega$ \}.

Lemma 2.12. Let $\phi$ be a tensor product B-spline of order $n$. For $J \in \mathbb{N}$, we have the followings.
1. For $u \in L_2(\Omega)$, we have
   \[
   \sum_{k \in \Omega} \left| \langle u, \phi(2^J \cdot - k) \rangle \right|^2 \leq \frac{n^2}{2^J} \left\| u \right\|_{L_2(\Omega)}^2.
   \]
2. For $u \in l_2(\Omega)$, we have
   \[
   \left\| \sum_{k \in \Omega} u[k] \phi(2^J \cdot - k) \right\|_{L_2(\Omega)}^2 \leq \frac{n^2}{2^J} \left\| u \right\|_{l_2(\Omega)}^2.
   \]
Proof. Notice that $0 \leq \phi \leq 1$, and the direct computation shows that $\|\phi\|_{L^2(\mathbb{R}^2)}^2 \leq 1$, and $\|\phi(2^J \cdot k)\|^2_{L^2(\mathbb{R}^2)} \leq 2^{-2J}$. Then by the Schwarz inequality, we have

$$|\langle u, \phi(2^J \cdot k) \rangle| \leq 2^{-J} \left( \int_{\Omega} |u(x)|^2 \sum_{k \in \Omega} \mathbf{1}_{S_k}(x) \, dx \right)^{1/2}$$

where $S_k = \text{supp} \left( \phi(2^J \cdot k) \right)$. Since $\sum \mathbf{1}_{S_k} = n^2$, we have

$$\sum_{k \in \Omega} |\langle u, \phi(2^J \cdot k) \rangle|^2 \leq 2^{-2J} \int_{\Omega} |u(x)|^2 \left( \sum_{k \in \Omega} \mathbf{1}_{S_k}(x) \right) \, dx = \frac{n^2}{2^J} \|u\|_{L^2(\Omega)}^2,$$

which proves (2.30). For (2.31), let $v \in L^2(\Omega)$. We have

$$\left( \sum_{k \in \Omega} u[k] \phi(2^J \cdot k), v \right) = \sum_{k \in \Omega} u[k] \langle \phi(2^J \cdot k), v \rangle.$$

By the Schwarz inequality, we have

$$\sum_{k \in \Omega} |u[k]| \left| \langle \phi(2^J \cdot k), v \rangle \right| \leq \left( \sum_{k \in \Omega} |u[k]|^2 \right)^{1/2} \left( \sum_{k \in \Omega} \left| \langle \phi(2^J \cdot k), v \rangle \right|^2 \right)^{1/2}.$$

By (2.30), we further have

$$\left| \left( \sum_{k \in \Omega} u[k] \phi(2^J \cdot k), v \right) \right| \leq \left( \sum_{k \in \Omega} |u[k]|^2 \right)^{1/2} \frac{n}{2^J} \|v\|_{L^2(\Omega)}.$$

Since $v \in L^2(\Omega)$ is arbitrary, we have (2.31) by the converse of Hölder’s inequality with $p = q = 2$ (e.g. [43]). This completes the proof. □

Proof of Theorem 2.10 Notice that in this case, we have $|\Omega| = 2^{2J}$. In addition, for $\lambda$ defined as (2.27), we have

$$\Upsilon(l, \alpha, k) = J - l - 1 < J := \frac{1}{2} \log_2 |\Omega|, \quad l = 0, \ldots, L - 1, \quad \alpha \in \mathbb{B}, \quad k \in \Omega,$$

which shows that our choice of $\lambda$ satisfies A2. When the wavelet frame transform is applied to $f$ in (2.24) the refinable equations (2.4) and (2.5) and the definitions of $\mathcal{X}^j(\Psi)$ (2.3) and (2.3) lead to

$$(\mathcal{W}_{l, \alpha} f)[k] = 2^j (f_\alpha, \psi_{\alpha, J-l, k}), \quad \alpha \in \mathbb{B} \cup \{0\}, \quad 0 \leq l \leq L - 1$$

with $\psi_{0, l, k} = \phi_{l, k}$ for simplicity. Then together with (2.26), we have

$$\|\lambda \cdot \mathcal{W} f\|_1 \leq 2^J \sum_{l = J - L}^{\infty} 2^{2l} \sum_{k \in \Omega} \sum_{\alpha \in \mathbb{B}} |\langle f, \psi_{\alpha, l, k} \rangle| = C_J 2^J,$$

which shows that $\|\lambda \cdot \mathcal{W} f\|_1 \leq C_J 2^J$ with $C_J = C_f$. Hence, since we have $\|f\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \leq M$ by Hölder’s inequality (e.g. [13], (2.28)) follows from Theorem 2.8 by setting

$$\tilde{c} = \frac{64M^2 \|\mathcal{A}\|_{L^2}^2}{3 \sigma_{\min}(\mathcal{A})^2} \left[ 4 + 3 \sqrt{5(4a + \max\{1 - \beta, 1\})C_J (1 + C_J)} \right] \sqrt{2},$$

where $a = \min\{0, 1/2, 1 - \beta\}$. \hfill $\square$
for some \( a \geq 1 \), which is independent of \( J, L, \rho, \) and \( \eta \).

For (2.29) note that we have
\[
\|u^\Lambda - f\|_{L^2(\Omega)} \leq \|u^\Lambda - f\|_{L^2(\Omega)} + \|f - f\|_{L^2(\Omega)} \leq \frac{n}{2J} \|u^\Lambda - f\|_{L^2(\Omega)} + \|f - f\|_{L^2(\Omega)},
\]
where the last inequality comes from (2.31) in Lemma 2.12 and \( n \) denotes the order of \( \phi \) (the tensor product B-spline used). Hence, we have
\[
\|u^\Lambda - f\|_{L^2(\Omega)} \leq 2 \left( \frac{n^2}{2J} \|u^\Lambda - f\|_{L^2(\Omega)} + \|f - f\|_{L^2(\Omega)} \right),
\]
and by (2.28) the proof is completed when we estimate \( \|f - f\|_{L^2(\Omega)} \). The standard wavelet frame decomposition gives
\[
f = \sum_{k \in \Omega} \langle f, \phi_{J,k} \rangle \phi_{J,k} + \sum_{l=J}^{\infty} \sum_{k \in \Omega} \sum_{\alpha \in B} \langle f, \psi_{\alpha,l,k} \rangle \psi_{\alpha,l,k}.
\]
Then the Bessel property of \( X^J(\Psi) \) gives
\[
\|f - f\|_{L^2(\Omega)}^2 \leq \sum_{l=J}^{\infty} \sum_{k \in \Omega} \sum_{\alpha \in B} |\langle f, \psi_{\alpha,l,k} \rangle|^2.
\]
Note that by Hölder’s inequality, we have
\[
|\langle f, \psi_{\alpha,l,k} \rangle| \leq \|f\|_{L^\infty(\Omega)} \|\psi_{\alpha,l,k}\|_{L^1(\mathbb{R}^2)} = 2^{-l} \|f\|_{L^\infty(\Omega)} \|\psi_{\alpha}\|_{L^1(\mathbb{R}^2)} \leq 2^{-l} M \|\psi_{\alpha}\|_{L^1(\mathbb{R}^2)},
\]
for all \( l \geq J \), and this leads to
\[
\|f - f\|_{L^2(\Omega)}^2 \leq M \left( \max_{\alpha \in B} \|\psi_{\alpha}\|_{L^1(\mathbb{R}^2)} \right) \sum_{l=J}^{\infty} 2^{l} \sum_{k \in \Omega} \sum_{\alpha \in B} |\langle f, \psi_{\alpha,l,k} \rangle|.
\]
Therefore, by letting
\[
C_1 = 2n^2 \bar{c}, \quad C_2 = \frac{32}{3} \sigma_\min(\mathcal{A}) n^2, \quad \text{and} \quad C_3 = 2C_f M \left( \max_{\alpha \in B} \|\psi_{\alpha}\|_{L^1(\mathbb{R}^2)} \right),
\]
all of which are independent of \( J, L, \rho, \) and \( \eta \), we obtain (2.29) and this concludes Theorem 2.10

\[ \square \]

### 3. Technical proofs

This section is devoted to the technical details left in Section 2. More precisely, we present the proofs of Theorem 2.4, Lemma 2.6, and Theorem 2.7.
Lemma 3.2. Then we have the following lemma.

Lemma 3.1. Assume that a random variable ξ on a probability space X satisfies μ := Eξ ≥ 0 and |ξ − μ| ≤ B almost surely. If Eξ^2 ≤ cEξ for some c > 0, the following ratio probability inequality, which can be derived from Bernstein inequality [29, 60], holds for arbitrary ϵ > 0 and 0 < γ ≤ 1.

\[
P\left\{ z \in X^m : \frac{\mu - \frac{1}{m} \sum_{i=1}^m \xi(z_i)}{\sqrt{\mu + \epsilon}} > \gamma \sqrt{\epsilon} \right\} \leq \exp\left\{ \frac{3\gamma^2 m \epsilon}{6c + 2B} \right\}
\]

Let ξ be a random variable which is drawn from the uniform distribution on Ω. For u ∈ M, we introduce ξ = |(Au)| - |(Af)|. Then ξ is a random variable which is drawn from the uniform distribution on Ω, and satisfies 0 ≤ ξ ≤ |A|^2 M^2. Define

\[
\mathcal{E}(u) = E\xi = E\left( |(Au)| - |(Af)| \right) = \frac{1}{|Ω|} \sum_{k \in Ω} |(Au)[k] - |(Af)[k]|. \tag{3.1}
\]

We also define \( \mathcal{E}_Λ(u) \) as a conditional expectation of ξ given Λ:

\[
\mathcal{E}_Λ(u) = E\left( |\xi| \right| Λ) = \frac{1}{|Λ|} \sum_{k \in Λ} |(Au)[k] - |(Af)[k]|. \tag{3.2}
\]

Then we have the following lemma.

Lemma 3.2. Let M be defined as \[2.14\]. For u₁, u₂ ∈ M, we have

\[
|\mathcal{E}(u_1) - \mathcal{E}(u_2)| \leq 2 \| A \|_∞ M \| u_1 - u_2 \|_{l_∞(Ω)} \tag{3.3}
\]

\[
|\mathcal{E}_Λ(u_1) - \mathcal{E}_Λ(u_2)| \leq 2 \| A \|_∞ M \| u_1 - u_2 \|_{l_∞(Ω)}. \tag{3.4}
\]

Proof. We have

\[
|\mathcal{E}(u_1) - \mathcal{E}(u_2)| = \frac{1}{|Ω|} \left| \sum_{k \in Ω} |(Au_1)[k] - |(Af)[k]| - \sum_{k \in Ω} |(Au_2)[k] - |(Af)[k]| \right|
\]

\[
\leq \frac{1}{|Ω|} \left| \sum_{k \in Ω} |(Au_1)[k] - (Au_2)[k]| \right|
\]

\[
\leq \frac{1}{|Ω|} \left| \sum_{k \in Ω} |(Au_1)[k] + (Au_2)[k] - 2(Af)[k]| \right|
\]

\[
\leq \| A(u_1 + u_2) - 2f \|_{l_∞(Ω)} \| A(u_1 - u_2) \|_{l_1(Ω)}
\]

\[
\leq 2 \| A \|_∞ M \| u_1 - u_2 \|_{l_∞(Ω)},
\]

where we have used the fact that \( u_1, u_2, f \in [0, M] Ω \) in the last inequality; that is, \( \| u_1 + u_2 - 2f \|_{l_∞(Ω)} \leq 2M \). \( \Box \)

With the aid of Lemmas 3.1 and 3.2, we can present the following ratio probability inequality involving the space M defined as \[2.14\].
Approximation Theory of Wavelet Frame Based Image Restoration

**Proposition 3.3.** Let $\mathcal{M}$, $\mathcal{E}(u)$ and $\mathcal{E}_\Lambda(u)$ be respectively defined as (3.2), (3.1) and (3.5) Then for any $\epsilon > 0$ and $0 < \gamma < 1$, we have

$$
\mathbb{P} \left\{ \sup_{u \in \mathcal{M}} \frac{\mathcal{E}(u) - \mathcal{E}_\Lambda(u)}{\sqrt{\mathcal{E}(u)} + \epsilon} > 4\gamma \sqrt{\epsilon} \right\} \leq \mathcal{N} \left( \frac{\gamma \epsilon}{2 \| A \|_\infty^2 M} \right) \exp \left\{ - \frac{3\gamma^2 me}{8 \| A \|_\infty^2 M^2} \right\}
$$

where $\mathcal{N}(\mathcal{M}, r)$ is the covering number of $\mathcal{M}$ with respect to $\| \cdot \|_{\ell_\infty(\Omega)}$.

**Proof.** Let $K = \mathcal{N} \left( \mathcal{M}, \frac{\gamma \epsilon}{2 \| A \|_\infty^2 M} \right)$ and we choose $\{u_1, \ldots, u_K\} \subseteq \mathcal{M}$ such that

$$
\mathcal{M} \subseteq \bigcup_{j=1}^{K} \left\{ u : \| u - u_j \|_{\ell_\infty(\Omega)} \leq \frac{\gamma \epsilon}{2 \| A \|_\infty^2 M} \right\}.
$$

For each $j = 1, \ldots, K$, we consider the random variable $\xi = |(A u_j)[\xi] - (A f)[\xi]|^2$ where $\xi$ is a random variable drawn from the uniform distribution on $\Omega$. Note that $\xi$ is a random variable on $\Omega$, and

$$
\mu = \mathbb{E} \xi = \frac{1}{|\Omega|} \sum_{k \in \Omega} |(A u_j)[k] - (A f)[k]|^2 = \mathcal{E}(u_j).
$$

Then since $u_j, f \in \mathcal{M}$, we have $0 \leq \xi \leq \| A \|_\infty^2 M^2$, $|\xi - \mathbb{E} \xi| \leq \| A \|_\infty^2 M^2$, and

$$
\mathbb{E} \xi^2 = \mathbb{E} \left( |(A u_j)[\xi] - (A f)[\xi]|^4 \right) \leq \| A \|_\infty^2 M^2 \mathbb{E} \left( |(A u_j)[\xi] - (A f)[\xi]|^2 \right) = \| A \|_\infty^2 M^2 \mathbb{E} \xi.
$$

In addition,

$$
\frac{1}{m} \sum_{k \in \Lambda} \xi(k) = \frac{1}{|A|} \sum_{k \in \Lambda} |(A u_j)[k] - (A f)[k]|^2 = \mathcal{E}_\Lambda(u_j).
$$

Applying Lemma 3.1 to $\xi$ with $B = c = \| A \|_\infty^2 M^2$, we have

$$
\mathbb{P} \left\{ \frac{\mathcal{E}(u_j) - \mathcal{E}_\Lambda(u_j)}{\sqrt{\mathcal{E}(u_j)} + \epsilon} > \gamma \sqrt{\epsilon} \right\} \leq \exp \left\{ - \frac{3\gamma^2 me}{8 \| A \|_\infty^2 M^2} \right\}.
$$

For each $u \in \mathcal{M}$, we can find $u_j$ such that $\| u - u_j \|_{\ell_\infty(\Omega)} \leq \frac{\gamma \epsilon}{2 \| A \|_\infty^2 M}$. Then by Lemma 3.2, we have

$$
|\mathcal{E}_\Lambda(u) - \mathcal{E}_\Lambda(u_j)| \leq 2 \| A \|_\infty^2 M \| u - u_j \|_{\ell_\infty(\Omega)} \leq \gamma \epsilon,
$$

$$
|\mathcal{E}(u) - \mathcal{E}(u_j)| \leq 2 \| A \|_\infty^2 M \| u - u_j \|_{\ell_\infty(\Omega)} \leq \gamma \epsilon.
$$

In other words,

$$
\frac{|\mathcal{E}_\Lambda(u) - \mathcal{E}_\Lambda(u_j)|}{\sqrt{\mathcal{E}(u)} + \epsilon} \leq \gamma \sqrt{\epsilon} \quad \text{and} \quad \frac{|\mathcal{E}(u) - \mathcal{E}(u_j)|}{\sqrt{\mathcal{E}(u)} + \epsilon} \leq \gamma \sqrt{\epsilon}.
$$

In particular, the second inequality implies that

$$
\mathcal{E}(u) + \epsilon = \mathcal{E}(u_j) - \mathcal{E}_\Lambda(u) + \mathcal{E}(u) + \epsilon \leq \gamma \sqrt{\epsilon} \sqrt{\mathcal{E}(u)} + \epsilon + \mathcal{E}(u) + \epsilon
$$

$$
\leq \sqrt{\epsilon} \sqrt{\mathcal{E}(u)} + \epsilon \leq \frac{\mathcal{E}(u) + 2\epsilon}{2} + \mathcal{E}(u) + \epsilon \leq 2(\mathcal{E}(u) + \epsilon).
$$

Hence, for an arbitrary $u \in \mathcal{M}$ with $\| u - u_j \|_{\ell_\infty(\Omega)} \leq \frac{\gamma \epsilon}{2 \| A \|_\infty^2 M}$, we have $\sqrt{\mathcal{E}(u_j)} + \epsilon \leq 2 \sqrt{\mathcal{E}(u)} + \epsilon$. 
Given that $\frac{E(u) - E_A(u)}{\sqrt{E(u)} + \epsilon} > 4\gamma \sqrt{\epsilon}$, we have
\[
\frac{E(u_j) - E_A(u_j)}{2\sqrt{E(u)} + \epsilon} > \frac{E(u_j) - E(u)}{2\sqrt{E(u)} + \epsilon} + \frac{E(u) - E_A(u)}{2\sqrt{E(u)} + \epsilon} + \frac{E_A(u) - E_A(u_j)}{2\sqrt{E(u)} + \epsilon} > -\frac{\gamma \sqrt{\epsilon}}{2} + 2\gamma \sqrt{\epsilon} = \gamma \sqrt{\epsilon}.
\]
Then since we have $\sqrt{E(u) + \epsilon} \leq 2\sqrt{E(u) + \epsilon}$ for any $u \in M$ with $\|u - u_j\|_\infty \leq \frac{\gamma \epsilon}{2\|A\|_\infty M}$, it follows that if $\frac{E(u) - E_A(u)}{\sqrt{E(u) + \epsilon}} > 4\gamma \sqrt{\epsilon}$ holds, then the following inequality
\[
\frac{E(u_j) - E_A(u_j)}{\sqrt{E(u_j) + \epsilon}} \geq \frac{E(u_j) - E_A(u)}{2\sqrt{E(u)} + \epsilon} > \gamma \sqrt{\epsilon}
\]
holds. Hence, for each fixed $j = 1, \cdots, K$,
\[
P\left\{ \sup_{u \in \{u: \|u - u_j\|_\infty(a) \leq \frac{\gamma \epsilon}{2\|A\|_\infty M}\}} \frac{E(u) - E_A(u)}{\sqrt{E(u)} + \epsilon} > 4\gamma \sqrt{\epsilon}\right\} \leq P\left\{ \frac{E(u_j) - E_A(u_j)}{\sqrt{E(u_j)} + \epsilon} > \gamma \sqrt{\epsilon}\right\}.
\]
Then since $M \subseteq \bigcup_{j=1}^K \left\{ u : \|u - u_j\|_\infty(a) \leq \frac{\gamma \epsilon}{2\|A\|_\infty M}\right\}$, together with (3.5) we have
\[
P\left\{ \sup_{u \in M} \frac{E(u) - E_A(u)}{\sqrt{E(u)} + \epsilon} > 4\gamma \sqrt{\epsilon}\right\} \leq \sum_{j=1}^K P\left\{ \sup_{u \in \{u: \|u - u_j\|_\infty(a) \leq \frac{\gamma \epsilon}{2\|A\|_\infty M}\}} \frac{E(u) - E_A(u)}{\sqrt{E(u)} + \epsilon} > 4\gamma \sqrt{\epsilon}\right\} \leq \sum_{j=1}^K \left\{ \frac{E(u_j) - E_A(u_j)}{\sqrt{E(u_j)} + \epsilon} > \gamma \sqrt{\epsilon}\right\} \leq \mathcal{N}\left(M, \frac{\gamma \epsilon}{2\|A\|_\infty M}\right) \exp\left\{ -\frac{3\gamma^2 m \epsilon}{8\|A\|_\infty^2 M^2}\right\}.
\]
This concludes Proposition 3.3.

Now, we can provide the proof of Theorem 2.4.

**PROOF OF THEOREM 2.4** By Proposition 3.3 for arbitrary $\epsilon > 0$ and $0 < \gamma < 1$, the inequality
\[
\sup_{u \in M} \frac{E(u) - E_A(u)}{\sqrt{E(u)} + \epsilon} \leq 4\gamma \sqrt{\epsilon}
\]
holds with probability at least
\[
1 - \mathcal{N}\left(M, \frac{\gamma \epsilon}{2\|A\|_\infty M}\right) \exp\left\{ -\frac{3\gamma^2 m \epsilon}{8\|A\|_\infty^2 M^2}\right\}.
\]
Therefore, for all $u \in M$, the inequality
\[
E(u) - E_A(u) \leq 4\gamma \sqrt{\epsilon} \sqrt{E(u) + \epsilon}
\]
(3.6)
holds with the same probability. Then by taking \( \gamma = \sqrt{2}/8 \) and \( u = u^A \in M \) in (3.6) it follows that
\[
\mathcal{E}(u^A) - \mathcal{E}_\Lambda(u^A) \leq \frac{1}{2} \sqrt{2\epsilon (\mathcal{E}(u^A) + \epsilon)}
\]
holds with probability at least \( 1 - N\left(M, \frac{\epsilon}{\sqrt{2\|A\|_\infty^2 M}}\right) \exp \left\{ - \frac{3m\epsilon}{256\|A\|_\infty^2 M^2} \right\} \). Besides, since
\[
\mathcal{E}_\Lambda(u^A) \leq \frac{2}{|A|} \sum_{k \in A} |(Au^A)[k] - g[k]|^2 + \frac{2}{|A|} \sum_{k \in A} |g[k] - (Af)[k]|^2 \leq 4\eta^2,
\]
combining (3.7) and (3.8) yields
\[
\mathcal{E}(u^A) \leq \frac{1}{2} \sqrt{2\epsilon (\mathcal{E}(u^A) + \epsilon)} + 4\eta^2.
\]
Noting that \( N\left(M, \frac{\epsilon}{\sqrt{2\|A\|_\infty^2 M}}\right) \leq N\left(M, \frac{\epsilon}{12\|A\|_\infty^2 M}\right) \), we have
\[
\mathbb{P}\left\{ \mathcal{E}(u^A) \leq \epsilon + \frac{16}{3}\eta^2 \right\} \geq 1 - N\left(M, \frac{\epsilon}{12\|A\|_\infty^2 M} \right) \exp \left\{ - \frac{3m\epsilon}{256\|A\|_\infty^2 M^2} \right\}.
\]
Finally, since
\[
\left\| u^A - f \right\|_{\ell_2(\Omega)}^2 \leq \sigma_{\min}(A)^2 \sum_{k \in \Omega} |(Au^A)[k] - (Af)[k]|^2 = \sigma_{\min}(A)^2 \mathcal{E}(u^A),
\]
we therefore have
\[
\mathbb{P}\left\{ \left\| u^A - f \right\|_{\ell_2(\Omega)}^2 \leq \epsilon + \frac{16}{3}\sigma_{\min}(A)^2\eta^2 \right\} \geq \mathbb{P}\left\{ \mathcal{E}(u^A) \leq \sigma_{\min}(A)^2\epsilon + \frac{16}{3}\eta^2 \right\} \geq 1 - N\left(M, \frac{\sigma_{\min}(A)^2\epsilon}{12\|A\|_\infty^2 M} \right) \exp \left\{ - \frac{3m\epsilon}{256\|A\|_\infty^2 M^2} \right\}.
\]
This concludes Theorem 2.4. \( \square \)

3.2. Proof of Lemma 2.6

Before we prove Lemma 2.6, we first mention that the authors in [18] have presented the similar results for the generic MRA based wavelet frame systems, using the convergence rate of a stationary subdivision algorithm [22, 48, 61]. However, since our definition of \( \ell_1 \)-norm of wavelet frame coefficients does not include the low-pass filter coefficients, our proof is based on the technique different from [18]. In fact, since we restrict ourselves to the tensor product B-spline wavelet frame systems, our proof follows the line described in [63], based on the definition of B-spline wavelet frame system in [55].

We denote respectively by \( p = [1 \ 1] \) one dimensional 1st order average filter and by \( q = [1 \ -1] \) one dimensional 1st order difference filter. Then we note that for \( u \in \mathbb{R}^\Omega \), the direct computation gives
\[
\left\| \nabla u \right\|_1 \leq \left\| q_{e_1}[-] \ast u \right\|_{\ell_1(\Omega)} + \left\| q_{e_2}[-] \ast u \right\|_{\ell_1(\Omega)}
\]
where \( \left\| \nabla u \right\|_1 \) is defined as (2.17) and the convolution \( \ast \) is defined as (2.8). In (3.9) \( q_{e_1} \) and \( q_{e_2} \) with the standard basis \( \{e_1, e_2\} \) for \( \mathbb{R}^2 \) is defined as
\[
q_{e_1}[k] = q[k_1]\delta[k_2] \quad \text{and} \quad q_{e_2}[k] = \delta[k_1]q[k_2]
\]
where \( \delta \) is the one dimensional discrete Dirac delta; \( \delta[k] = 1 \) for \( k = 0 \), and \( \delta[k] = 0 \) for \( k \neq 0 \).
Now, we note that for $L \geq 1$, we have

$$
\sum_{k \in \Omega} \sum_{\alpha \in \mathbb{B}} |(W_{0,\alpha}u)[k]| \leq \sum_{l=0}^{L-1} \sum_{\Omega} \sum_{\alpha \in \mathbb{B}} |(W_{l,\alpha}u)[k]|.
$$

Hence, (3.9) tells us that, if we find $C > 0$ independent of $J$ such that

$$
\|q_{e_{j}}[- \cdot] \odot u\|_{\ell_{1}(\Omega)} + \|q_{e_{j}}[- \cdot] \odot u\|_{\ell_{1}(\Omega)} \leq C \sum_{k \in \Omega} \sum_{\alpha \in \mathbb{B}} |(W_{0,\alpha}u)[k]|,
$$

we obtain (2.18) by simply setting $C_{W} = \max \{C, 1\}$.

Recall that the first equality of UEP (2.6) is rewritten as

$$
\sum_{\alpha \in \mathbb{B} \cup \{0\}} (a_{\alpha} * a_{\alpha}[- \cdot])[k] = \delta[k] = \begin{cases} 
1 & \text{if } k = 0; \\
0 & \text{if } k \neq 0,
\end{cases}
$$

where $*$ is the standard discrete convolution on $\mathbb{Z}^{2}$. Then for $j = 1, 2$, we have

$$
q_{e_{j}}[- \cdot] \odot u = \sum_{\alpha \in \mathbb{B} \cup \{0\}} (a_{\alpha} * a_{\alpha}[- \cdot]) * q_{e_{j}}[- \cdot] \odot u,
$$

and by the discrete Young’s inequality (e.g. [8]), we have

$$
\|q_{e_{j}}[- \cdot] \odot u\|_{\ell_{1}(\Omega)} \leq \sum_{\alpha \in \mathbb{B} \cup \{0\}} \|a_{\alpha}\|_{\ell_{1}(\mathbb{Z}^{2})} \|a_{\alpha}[- \cdot] * q_{e_{j}}[- \cdot] \odot u\|_{\ell_{1}(\Omega)}.
$$

(3.12)

From the definition of B-spline wavelet frame system, the Fourier series of one dimensional filters $a_{\alpha}$ is

$$
\hat{a}_{0}(\xi) = e^{-ij_{r}^{\frac{\alpha}{2}} \cos \left(\frac{\xi_{1}}{2}\right)},
$$

(3.13)

$$
\hat{a}_{\alpha}(\xi) = -i^\alpha e^{-ij_{r}^{\frac{\alpha}{2}}} \left(\frac{r}{\alpha}\right) \cos^{-\alpha} \left(\frac{\xi_{1}}{2}\right) \sin^{\alpha} \left(\frac{\xi_{1}}{2}\right), \quad \alpha = 1, \cdots, r
$$

(3.14)

where $j_{r} = 1$ if $r$ is odd; $j_{r} = 0$ if $r$ is even. Noting that the Fourier series of $p$ and $q$ are

$$
\hat{p}(\xi) = 2e^{-j_{r}^{\frac{\alpha}{2}} \cos \left(\frac{\xi_{1}}{2}\right)}, \quad \text{and} \quad \hat{q}(\xi) = -2ie^{j_{r}^{\frac{\alpha}{2}} \sin \left(\frac{\xi_{1}}{2}\right)},
$$

we define $p^{\alpha}$ and $q^{\alpha}$ with $\alpha \in \mathbb{N}$ as

$$
p^{\alpha} = p * p * \cdots * p \quad \text{and} \quad q^{\alpha} = q * q * \cdots * q,
$$

and we set $p^{0} = q^{0} = \delta$, the one dimensional discrete Dirac delta. Then by (3.13) and (3.14) we have

$$
a_{0}[k] = 2^{-r}p^{r}[k - j_{r}]
$$

(3.15)

$$
a_{\alpha}[k] = (-1)^{\alpha+1}2^{-r} \left(\frac{r}{\alpha}\right) p^{\alpha * q^{\alpha}[k - j_{r}]}, \quad \alpha = 1, \cdots, r.
$$

(3.16)
Using (3.15) and (3.16) and the discrete Young’s inequality, we have
\[
\|a_0\|_{l_1(\mathbb{Z}^2)} = 1 \quad \text{and} \quad \|a_{\alpha}\|_{l_1(\mathbb{Z}^2)} \leq \sqrt{\frac{r}{\alpha}} \quad \alpha \in \mathbb{R},
\]
where \(r = (r, r)\), \((\alpha) = (\alpha_1, \alpha_2)\) for multi-indices \(\alpha = (\alpha_1, \alpha_2)\), \(\beta = (\beta_1, \beta_2)\), and the first equality comes from the fact that \(\hat{a}_0(0) = \sum a_0[k] = 1\). From (3.12), (3.13) becomes
\[
\left\|q_{e_1}[-\cdot] \circledast u\right\|_{l_1(\Omega)} \leq \sum_{\alpha \in \mathbb{B}(\mathbb{N})} \sqrt{\frac{r}{\alpha}} \left\|a_{\alpha}[-\cdot] \circledast q_{e_1}[-\cdot] \circledast u\right\|_{l_1(\Omega)}.
\]
It is obvious that, for \(\alpha \in \mathbb{B}\), we have
\[
\left\|a_{\alpha}[-\cdot] \circledast q_{e_1}[-\cdot] \circledast u\right\|_{l_1(\Omega)} \leq \left\|q_{e_1}[-\cdot]\right\|_{l_1(\mathbb{Z}^2)} \left\|a_{\alpha}[-\cdot] \circledast u\right\|_{l_1(\Omega)} = 2 \left\|\mathcal{W}_{0,\alpha} u\right\|_{l_1(\Omega)}
\]
by the discrete Young’s inequality. Hence, we need to bound \(\left\|a_0[-\cdot] \circledast q_{e_1}[-\cdot] \circledast u\right\|_{l_1(\Omega)}\) by using \(\left\|\mathcal{W}_{0,\alpha} u\right\|_{l_1(\Omega)}\) for some suitable \(\alpha \in \mathbb{B}\). To do this, we first note that
\[
a_0 \ast q[k] = r^{-\frac{2}{p}} p \ast a_1[k - j_r].
\]
Then by the discrete Young’s inequality, we have
\[
\left\|a_0[-\cdot] \circledast q_{e_1}[-\cdot] \circledast u\right\|_{l_1(\Omega)} \leq \frac{2}{\sqrt{r}} \left\|a_{e_1}[-\cdot] \circledast u\right\|_{l_1(\Omega)} = \frac{2}{\sqrt{r}} \left\|\mathcal{W}_{0,e_1} u\right\|_{l_1(\Omega)}.
\]
Combining these altogether, we have
\[
\left\|q_{e_1}[-\cdot] \circledast u\right\|_{l_1(\Omega)} + \left\|q_{e_2}[-\cdot] \circledast u\right\|_{l_1(\Omega)} \leq 4 \sum_{\alpha \in \mathbb{B}} \sqrt{\frac{r}{\alpha}} \left\|\mathcal{W}_{0,\alpha} u\right\|_{l_1(\Omega)} + \frac{2}{\sqrt{r}} \left\|\mathcal{W}_{0,e_1} u\right\|_{l_1(\Omega)} + \frac{2}{\sqrt{r}} \left\|\mathcal{W}_{0,e_2} u\right\|_{l_1(\Omega)} \\
\leq 6 \max_{\alpha \in \mathbb{B}} \sqrt{\frac{r}{\alpha}} \left\|\mathcal{W}_{0,\alpha} u\right\|_{l_1(\Omega)}
\]
where the last inequality follows from the fact that \(r \geq 1\) and \((\alpha) \geq 1\). Therefore, letting
\[
C = 6 \max_{\alpha \in \mathbb{B}} \sqrt{\frac{r}{\alpha}},
\]
we obtain (3.11) and this concludes Lemma 2.6.

3.3. Proof of Theorem 2.7

Theorem 2.7 is to estimate the covering number. The proof follows the line similar to [18, Theorem 2.4]. However, since we improve [18, Theorem 2.4] by relaxing the constraint of the radius \(r\), we include the detailed proof for the sake of completeness. It is obvious that if \(u \in [0, M] \mathbb{N}\), then \(\|u\|_{l_\infty(\Omega)} \leq M\). In addition, if \(A2\) is satisfied, together with the definition of \(M\) in (2.14) and Lemma 2.6, we have
\[
\|\nabla u\|_1 \leq C_{\mathcal{W}} \|\mathcal{W} u\|_1 \leq C_{\mathcal{W}} 2^{\max\{-\beta, 0\}} \max\{\mathbb{N}(\lambda, \alpha, k) : (\lambda, \alpha, k) \in \Gamma\} \|\lambda \cdot \mathcal{W} u\|_1.
\]
Since $\Upsilon : \Gamma \to \mathbb{N}$ satisfies (2.13) for $u \in \mathcal{M}$, we have
\[
\|\nabla u\|_1 \leq C_{\mathcal{W}|\Omega}^{\max(\ell,\tilde{\omega})} \|\lambda \cdot \mathcal{W}f\|_1.
\]
In other words, $\mathcal{M} \subseteq \tilde{\mathcal{M}}$ and $\mathcal{N}(\mathcal{M}, r) \leq \mathcal{N}(\tilde{\mathcal{M}}, r)$ where $\tilde{\mathcal{M}}$ is defined as (2.19). Hence, it suffices to bound the covering number $\mathcal{N}(\tilde{\mathcal{M}}, r)$. In addition, it is easy to see that if there exists a finite set $F \subseteq \tilde{\mathcal{M}}$ such that
\[
\tilde{\mathcal{M}} \subseteq \bigcup_{q \in F} \left\{ u : \|u - q\|_{\ell_\infty(\Omega)} \leq r \right\},
\]
we have $\mathcal{N}(\tilde{\mathcal{M}}, r) \leq |F|$. What we need now is to construct an appropriate set $F$ by exploiting the specific structure of $\tilde{\mathcal{M}}$, so that $|F|$ has an appropriate upper bound.

For this purpose, let $\kappa = \lfloor 2M/r \rfloor$, and we define
\[
\mathcal{R} = \{-\kappa r/2, (-\kappa + 1)r/2, \ldots, \kappa r/2\}. \tag{3.17}
\]
By [18] Lemma 4.4, for each $u \in \tilde{\mathcal{M}}$, there exists $Q(u) \in \mathcal{R}|\Omega$ such that
\[
\|u - Q(u)\|_{\ell_\infty(\Omega)} \leq r/2 \text{ and } \|\nabla (Q(u))\|_1 \leq \|\nabla u\|_1.
\]
Let
\[
\tilde{F} = \left\{ q \in \ell_\infty(\Omega) : q = Q(u) \text{ for some } u \in \tilde{\mathcal{M}} \right\} \subseteq \mathcal{R}|\Omega.
\]
Notice that for each $q \in \tilde{F}$, there may be more than one $u \in \tilde{\mathcal{M}}$ such that $q = Q(u)$.

For each $q \in \tilde{F}$, choose $u_q \in \tilde{\mathcal{M}}$ such that $\|u_q - q\|_{\ell_\infty(\Omega)} \leq r/2$, and define $F = \left\{ u_q : q \in \tilde{F} \right\}$. For an arbitrary $u \in \tilde{\mathcal{M}}$, there exists $q \in \tilde{F}$ such that $\|u - q\|_{\ell_\infty(\Omega)} \leq r/2$. This implies
\[
\|u - u_q\|_{\ell_\infty(\Omega)} \leq \|u - q\|_{\ell_\infty(\Omega)} + \|q - u_q\|_{\ell_\infty(\Omega)} \leq r,
\]
by the definition of $u_q$. Therefore,
\[
\tilde{\mathcal{M}} \subseteq \bigcup_{u_q \in F} \left\{ u : \|u - u_q\|_{\ell_\infty(\Omega)} \leq r \right\} \text{ and } \mathcal{N}(\tilde{\mathcal{M}}, r) \leq |F| \leq |\tilde{F}|.
\]
Thus, the covering number $\mathcal{N}(\tilde{\mathcal{M}}, r)$ is bounded by any upper bound of $|\tilde{F}|$. Notice that each $q \in \tilde{F}$ is uniquely determined by $\nabla q$ and $q[1, \ldots, 1]$. Since $\tilde{F}$ is a subset of $\mathcal{R}|\Omega$, there are $2\kappa + 1$ choices for $q[1, \ldots, 1]$. It remains to count the number of choices in $\nabla q$. Define
\[
\nabla \tilde{F} = \left\{ \nabla q : q \in \tilde{F} \right\}.
\]
Then we need to bound $|\nabla \tilde{F}|$.

To do this, we first consider the uniform upper bound of $\|\nabla q\|_1$ for $q \in \tilde{F}$. By the definition of $\tilde{F}$ and [18] Lemma 4.4, for each $q \in \tilde{F}$, there exists $u \in \mathcal{M}$ such that $\|\nabla q\|_1 \leq \|\nabla u\|_1$. Since $\|\lambda \cdot \mathcal{W}f\|_1 \leq C_f |\Omega|^{1/2}$ by assumption, we further have
\[
\|\nabla q\|_1 \leq \|\nabla u\|_1 \leq C_{\mathcal{W}}C_f |\Omega|^{1/2}
\]
Approximation Theory of Wavelet Frame Based Image Restoration

with \( b = \max\{1 - \beta, 1\} \) for notational simplicity. Hence, for all \( q \in \tilde{F} \), we have \( \|\nabla q\|_1 \leq Kr/2 \), where

\[
K = \left\lfloor \frac{2C_{\mathcal{W}C_f} |\Omega|^{b/2}}{r} \right\rfloor.
\]

In addition, since \( q \in \mathbb{R}^\Omega \), each element of \( \nabla q \) has to be a multiple of \( r/2 \), which means that the range of \( \nabla q \) is a subset of \( \{-Kr/2, -(K - 1)r/2, \ldots, Kr/2\}^2 \). Recall that there are \( R = 2 \left( |\Omega| - |\Omega|^{1/2} \right) \) elements in \( \nabla q \). Hence, \( \|\nabla \tilde{F}\| \) can be bounded by the number of all possible integer solutions of the following inequality

\[
|x_1| + |x_2| + \cdots + |x_{R-1}| + |x_R| \leq K.
\]

That is (e.g. [56]),

\[
\|\nabla \tilde{F}\| \leq 1 + 2 \sum_{k=1}^{K} \sum_{s=1}^{\min\{k,R\}} 2^s \binom{R}{R - s} \binom{k - 1}{s - 1} \leq 2^{K+1} \binom{R + K - 1}{K} \leq 2 |2 (R + K - 1)|^K.
\]

Hence, we have

\[
\|\nabla \tilde{F}\| \leq (4\kappa + 2) |2 (R + K - 1)|^K.
\]

In other words, using \( r \geq |\Omega|^{-a} \) with \( a \geq 1 \) and \( K - 1 \leq 2C_{\mathcal{W}C_f} |\Omega|^{a+b/2} \), we have

\[
\ln \mathcal{N}(M, r) \leq \frac{2C_{\mathcal{W}C_f} |\Omega|^{b/2}}{r} \ln (2 (R + K - 1)) + \ln \left( \frac{8M}{r} + 2 \right) \leq \frac{2C_{\mathcal{W}C_f} |\Omega|^{b/2}}{r} \ln \left( \frac{4 |\Omega| + 4C_{\mathcal{W}C_f} |\Omega|^{b/2}}{r} \right) + \ln \frac{10M}{r} \leq \frac{2C_{\mathcal{W}C_f} |\Omega|^{b/2}}{r} \ln \left( 1 + C_{\mathcal{W}C_f} \right) 40M |\Omega|^{2a+b/2}
\]

where we use the fact that \( a + b/2 \geq 1 \) from the choice of \( a \) and \( b \) in the final inequality. Therefore, we have

\[
\ln \mathcal{N}(M, r) \leq \frac{20M(4a + b)C_{\mathcal{W}C_f}(1 + C_{\mathcal{W}C_f}) |\Omega|^{b/2}}{r} \log_2 |\Omega|,
\]

and this completes the proof. \( \square \)

4. Conclusion

In this paper, we present an approximation analysis of wavelet frame based restoration from degraded and incomplete measurements. By the combination of the uniform law of large numbers and the estimation for its involved covering number of a hypothesis space of the solution, we establish unified approximation properties of the wavelet frame based image restoration model, together with the improved estimate of covering number over the previous one in [18]. In addition, thanks to the underlying multiresolution analysis...
structure, we further connect the error analysis in the discrete setting to the approximation of underlying function where the discrete data comes from. For the future works, we plan to study an approximation property of wavelet frame based image restoration from partial Fourier samples, where $A = \mathcal{F}$ (the unitary discrete Fourier transform) satisfies $\|F\|_\infty = \|\Omega\|^{1/2}$ and $\sigma_{\text{min}}(F) = \sigma_{\text{max}}(F) = 1$ in our framework. We may also plan to establish an approximation of tight frame based missing data restoration on the graph \[33\], and the multiresolution approximation of functions on the manifold \[35\].

References

[1] A. Chambolle, A., Pock, T., [2021] ©2021. Approximating the total variation with finite differences or finite elements, in: Geometric partial differential equations. Part II. Elsevier/North-Holland, Amsterdam. volume 22 of Handb. Numer. Anal., pp. 383–417. doi:10.1016/bs.hna.2020.2010.005

[2] Adams, R.A., 1975. Sobolev Spaces. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London. Pure and Applied Mathematics, Vol. 65.

[3] Alon, N., Ben-David, S., Cesa-Bianchi, N., Haussler, D., 1997. Scale-sensitive dimensions, uniform convergence, and learnability. J. ACM 44, 615–31. doi:10.1145/263867.263927

[4] Aubert, G., Kornprobst, P., 2006. Mathematical Problems in Image Processing. Partial Differential Equations and the Calculus of Variations. Foreword by Olivier Faugeras. volume 147 of Appl. Math. Sci. 2nd ed., Springer, New York.

[5] Bar, L., Brook, A., Sochen, N., Kiryati, N., 2006. Image deblurring in the presence of impulse plus Gaussian noise. Inverse Probl. Imaging. 10, 1101–11. doi:10.1109/TIP.2005.858979

[6] Bartlett, P.L., Jordan, M.I., McAuliffe, J.D., 2006. Convexity, classification, and risk bounds. J. Amer. Statist. Assoc. 101, 138–56. doi:10.1198/016214506000009097

[7] Beckner, W., 1975. Inequalities in Fourier analysis. Ann. of Math. (2) 102, 159–82. doi:10.1215/S0012-7094-75-04503-1

[8] Beckner, W., 1975. Inequalities in Fourier analysis. Ann. of Math. (2) 102, 159–82. doi:10.1215/S0012-7094-75-04503-1

[9] Bertalmio, M., Vese, L., Sapiro, G., Osher, S., 2003. Simultaneous cartoon and texture inpainting. Inverse Probl. Imaging 4, 379–95. doi:10.1007/s10851-009-0169-7

[10] Cai, J.F., Chan, R.H., Nikolova, M., 2010a. Fast two-phase image deblurring under impulse noise. J. Math. Imaging Vision 36, 46–53. doi:10.1007/s10851-009-0169-7

[11] Cai, J.F., Chan, R.H., Nikolova, M., 2010a. Fast two-phase image deblurring under impulse noise. J. Math. Imaging Vision 36, 46–53. doi:10.1007/s10851-009-0169-7

[12] Cai, J.F., Chan, R.H., Nikolova, M., 2008. Two-phase approach for deblurring images corrupted by impulse plus Gaussian noise. Inverse Probl. Imaging 2, 187–204. doi:10.3934/ipi.2008.2.187

[13] Cai, J.F., Chan, R.H., Nikolova, M., 2010b. Simultaneous cartoon and texture inpainting. Adv. Comput. Harmon. Anal. 41, 94–138. doi:10.1007/s10687-010-0169-7

[14] Cai, J.F., Chan, R.H., Shen, Z., 2007. Deconvolution: a wavelet frame approach. Numer. Math. 106, 529–87. doi:10.1007/s00211-005-0075-0

[15] Candeas, E.J., Romberg, J., Tao, T., 2006. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. IEEE Trans. Inform. Theory 52, 489–509. doi:10.1109/TIT.2005.862083

[16] Candeas, E.J., Tao, T., 2005. Decoding by linear programming. IEEE Trans. Inform. Theory 51, 4203–15. doi:10.1109/TIT.2005.858979

[17] Candeas, E.J., Tao, T., 2006. Near-optimal signal recovery from random projections: universal encoding strategies? IEEE Trans. Inform. Theory. 52, 5406–25. doi:10.1109/TIT.2006.885507

[18] Cavaretta, A.S., Dahmen, W., Micchelli, C.A., 1991. Stationary subdivision. Mem. Amer. Math. Soc. 93, vi+186.

[19] Chambolle, A., Darbon, J., 2009. On total variation minimization and surface evolution using parametric maximum flows. Int. J. of Comput Vis. 84, 288–99. doi:10.1007/s10851-009-0169-7

[20] Chan, T.F., Kang, S.H., Shen, J., 2002. Euler’s elastica and curvature-based inpainting. SIAM J. Appl. Math. 63, 564–92. doi:10.1137/s0036139901390088
[62] Wen, Y.W., Chan, R.H., Yip, A.M., 2012. A primal-dual method for total-variation-based wavelet domain inpainting. IEEE Trans. Image Process. 21, 106–14. doi:10.1109/TIP.2011.2159983

[63] Yang, J., Stahl, D., Shen, Z., 2017. An analysis of wavelet frame based scattered data reconstruction. Appl. Comput. Harmon. Anal. 42, 480–507. doi:10.1016/j.acha.2015.09.008

[64] Ye, X., Zhou, H., 2013. Fast total variation wavelet inpainting via approximated primal-dual hybrid gradient algorithm. Inverse Probl. Imaging 7, 1031–50. doi:10.3934/1pi.2013.7.1031

[65] Zhang, X., Chan, T.F., 2010. Wavelet inpainting by nonlocal toral variation. Inverse Probl. Imaging 4, 191–210. doi:10.3934/1pi.2010.4.191