Efficiencies of power plants, quasi-static models and the geometric-mean temperature

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Observed efficiencies of industrial power plants are often approximated by the square-root formula: $1 - \sqrt{T_-/T_+}$, where $T_+(T_-)$ are the highest (lowest) temperature achieved in the plant. This expression can be derived within finite-time thermodynamics, or, by entropy generation minimization, based on finite rates of processes. A closely related quantity is the optimal value of the intermediate temperature for the hot stream, which is given by the geometric-mean value: $\sqrt{T_+T_-}$. It is proposed to model the operation of plants by quasi-static work extraction models, with one reservoir (source/sink) as finite, while the other as practically infinite. No simplifying assumption is made on the nature of the finite system. This description is consistent with two model hypotheses, each yielding a specific value of the intermediate temperature. We show that the expected value of the intermediate temperature, defined as the arithmetic mean, is very closely given by the geometric-mean value. The definition is motivated as the use of inductive inference in the presence of limited information.

Keywords: Power plants; Heat Engines; Thermal efficiency; Inference

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**Introduction:** In recent years, there has been a great interest in extending thermodynamic models to justify the observed performance of industrial power plants [1–4]. The observed efficiencies are usually much less than the Carnot limit

\[ \eta_C = 1 - \frac{T_-}{T_+}, \]  

(1)
evaluated on the basis of the highest \((T_+)\) and the lowest \((T_-)\) temperatures, for the particular plant. Naturally, real machines operate under irreversibilities caused by various factors, like finite rates of heat transfer and fluid flow, internal friction, heat leakage and so on, unlike the idealized quasi-static processes of a reversible cycle. Thus the analysis of irreversible models with finite-rate processes seems a reasonable goal to pursue. One often-studied measure is the efficiency at maximum power of an irreversible model which is then compared with the observed efficiency of these plants.

The earliest known such model is ascribed to Reitlinger [5], which involved a heat exchanger receiving heat from a finite hot stream fed by a combustion process. An analogous model was applied to a steam turbine by Chambadal [6]. The considered heat exchanger in these models is effectively infinite. Novikov [7] considered the heat transfer process between a hot stream and a finite heat exchanger with a given heat conductance. Two, simple but significant, assumptions enter these models: i) constant specific heat of the inlet hot stream and ii) validity of Newton’s law for heat transfer. Further, there appears a floating, temperature variable in between the highest and the lowest values, such as the temperature of the exhaust warm stream, over which the output power can be optimized. This yields an optimal value of the intermediate temperature which is usually found to be \(\sqrt{T_+T_-}\), the geometric mean of \(T_+\) and \(T_-\). Related to this fact, is the conclusion that the efficiency at maximum power is given by an elegant expression:

\[ \eta_{CA} = 1 - \sqrt{\frac{T_-}{T_+}}. \]  

(2)

Due to historical imperative [8], the above expression may be called Reitlinger-Chambadal-Novikov efficiency. However, more recently it was rediscovered by Curzon and Ahlborn (CA) [1]. Thus in the physics literature, it is more popularly addressed as CA-value. This latter model considered finite rates of heat transfer at both the hot and the cold contacts, but also explicitly considered the times allocated to these contacts. The average power per cycle may be optimized over these times [1], or alternately, over the intermediate temperature
variables [9]. This model spawned much activity and the new area borne therefrom was termed Finite-time Thermodynamics [10]. In the engineering literature, the corresponding approach is called Entropy Generation Minimization [2, 11].

A positive indication for the simple thermodynamic approach is that the actually observed efficiencies of industrial plants happen to be quite close to CA-value. Fig.1 shows this comparison as tabulated in Table I. Although the agreement is close, the observed values can be higher, or lower, than CA-value. This apparent discrepancy has stimulated further extensions of models, for instance using the low-dissipation assumption [3], which predict the efficiency at maximum power, to be bounded as:

$$\eta_C \leq \eta \leq \eta_C \left(1 - \frac{1}{1 + \frac{1}{T}}\right).$$

(3)

It is then realized that most of the observed efficiencies fall within these bounds [3, 4].

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Data on the observed efficiencies ($\eta_o$) of power plants, plotted against the CA-value, $\eta_{CA} = 1 - \sqrt{T_-/T_+}$ for respective plants as given in Table I. A point lying on the straight line has the observed efficiency equal to CA-value. So for points above the line, $\eta_o$ is below the CA-value, while the converse is true for the points below this line.}
\end{figure}

Naturally, the question of the actual working constraints and the real optimization targets for each plant, is also relevant. Still, the effectiveness of these simple models, in reproducing the gross features of diverse plants, cannot be denied.

Apart from finite-time models, the irreversibilities reducing the efficiency to a lower-than-Carnot value, may also be treated within a quasi-static work extraction models with finite source/sink [13, 16]. Some of these studies also derive the optimal value of an intermediate
\begin{table}
\centering
\begin{tabular}{|l|c|c|c|c|c|c|c|c|}
\hline
Industrial Plant & $T_+$ & $T_-$ & $\eta_0$ & $\eta_{CA}$ & $T_{m}^{(-)}$ & $T_{m}^{(+)}$ & $T_{m}^{(av)}$ & $G(T_+, T_-)$ \\
\hline
Almaraz II (Nuclear, Spain) [2] & 600 & 290 & .34 & .30 & 439.39 & 396.0 & 417.7 & 417.13 \\
Calder Hall (Nuclear, UK) [2] & 583 & 298 & .19 & .29 & 367.9 & 472.23 & 420.07 & 416.81 \\
CANDU (Nuclear, Canada) [1] & 573 & 298 & .30 & .28 & 425.71 & 401.1 & 413.41 & 413.22 \\
Cofrentes (Nuclear, Spain) [2] & 562 & 289 & .34 & .28 & 437.88 & 370.92 & 404.40 & 403.01 \\
Doel 4 (Nuclear, Belgium) [2] & 566 & 283 & .35 & .29 & 435.39 & 367.9 & 401.64 & 400.22 \\
Heysham (Nuclear, UK) [2] & 727 & 288 & .40 & .37 & 480.0 & 436.2 & 458.1 & 457.58 \\
Larderello (Geothermal, Italy) [1] & 523 & 353 & .16 & .18 & 420.24 & 439.32 & 429.78 & 429.67 \\
Sizewell B (Nuclear, UK) [2] & 581 & 288 & .36 & .30 & 450.0 & 371.84 & 410.92 & 409.06 \\
West Thurrock (Coal, UK) [1] & 838 & 298 & .36 & .40 & 465.63 & 536.32 & 500.97 & 499.72 \\
Pressurized water nuclear reactor [12] & 613 & 304 & .33 & .30 & 453.73 & 410.71 & 432.22 & 431.69 \\
Boiling water nuclear reactor [12] & 553 & 304 & .33 & .25 & 453.73 & 370.51 & 412.12 & 410.02 \\
Fast neutron nuclear reactor [12] & 823 & 296 & .40 & .40 & 493.33 & 493.8 & 493.57 & 493.57 \\
(Steam/Mercury, US) [2] & 783 & 298 & .34 & .38 & 451.52 & 516.78 & 484.15 & 483.05 \\
(Steam, UK) [2] & 698 & 298 & .28 & .35 & 413.89 & 502.56 & 458.22 & 456.08 \\
(Gas Turbine, Switzerland) [2] & 963 & 298 & .32 & .44 & 438.24 & 654.84 & 546.54 & 535.7 \\
(Gas Turbine, France) [2] & 953 & 298 & .34 & .44 & 451.52 & 628.98 & 540.25 & 532.91 \\
\hline
\end{tabular}
\caption{Observed efficiencies ($\eta_0$) of power plants working between temperatures $T_+$ and $T_-$, compared with $\eta_{CA} = 1 - \sqrt{T_-/T_+}$. The effective temperatures are defined as: $T_{m}^{(+)} = T_+(1 - \eta_0)$, $T_{m}^{(-)} = T_-/(1 - \eta_0)$, $T_{m}^{(av)} = (T_{m}^{(+)} + T_{m}^{(-)})/2$, and $G(T_+, T_-) = \sqrt{T_+T_-}$.}
\end{table}

The temperature which is the geometric-mean value, and consequently the efficiency at maximum work equals the CA-value. However, here again, the simplifying assumption of a constant heat capacity, say, of the source or the working medium, heavily determines the conclusion.

More recently, it was observed [17] that within linear response theory, the bounds such as Eq. (3), also follow within a quasi-static model of work extraction from finite-sized heat source and sink. Ref. [17] makes no specific assumption about the nature of the heat source/sink. The present study is motivated by this analysis, and aims to study the performance of real plants within a quasi-static approach. More precisely, we address an inverse question. Instead of finding the optimal intermediate temperature and then from it, the efficiency of the process, we use the information on the highest and the lowest tempera-
tures, along with the value of observed efficiency, to infer the intermediate temperature at maximum work. The correspondence between this temperature and efficiency is as follows. If the former is exactly equal to the geometric-mean value, then the efficiency is equal to CA-value. The model of work extraction is based on one system as a finite source, or sink, and the other as an infinite reservoir, which allows for two alternate scenarios: i) when the source is finite and ii) when the sink is finite. The limited information on the actual situation being realized out of these two possibilities, motivates to do an inference analysis. Thus we estimate an expected value of the intermediate temperature scale. Interestingly, this value is found to be very close to the geometric mean of $T_+$ and $T_-$. We also present a quantitative argument for this proximity to the geometric-mean value. Thus our analysis indicates the role of geometric mean from a novel angle which has two distinctive features: first, it is based on maximum work approach, and second, we do not per se assume a specific nature of the finite source or sink.

Our starting point is the reversible cycle operating between two infinite heat reservoirs, for which the efficiency is the Carnot value $\eta_C$. As a first step marking a deviation from reversibility, we consider one reservoir as finite, while the other reservoir remains very large compared to the former, or practically infinite. Now, we first assume that system A acts as a finite heat sink at temperature $T_-$, relative to a very large heat source at temperature $T_+$. The two are coupled by an ideal engine which delivers work to a reversible work source, by running in infinitesimal heat cycles, that successively increase the temperature of A, till A comes in equilibrium with the source, see Fig.1 (i). At this point, the extracted work is maximum. Suppose that in this process, the system A moves from an equilibrium state characterized by energy $U_-$ and entropy $S_-$ to an equilibrium state with the corresponding values of $U_+$ and $S_+$. The heat removed from the source is $Q_+ = T_+(S_+ - S_-)$. The heat rejected to the finite sink is $q_+ = U_+ - U_-$. The work extracted is given by $W_+ = Q_+ - q_+$, or

$$W_+ = T_+(S_+ - S_-) - (U_+ - U_-).$$

Then the efficiency at maximum work, $\eta_+ = W_+/Q_+$, is evaluated to be:

$$\eta_+ = 1 - \frac{T_m^{(+)}}{T_+},$$

where we have defined

$$T_m^{(+) = \frac{U_+ - U_-}{S_+ - S_-}}.$$
FIG. 2. Schematic of the engine between a finite system and a heat reservoir, for a given pair of initial temperatures \( (T_+, T_-) \): (i) System A as a finite sink at \( T_- \) and an infinite source at \( T_+ \), coupled with a reversible work source. Work extraction \( W_+ \), Eq. (4), is completed when the temperature of A becomes \( T_+ \). (ii) System B as a finite source at \( T_+ \) and an infinite sink at \( T_- \). Maximum extracted work is \( W_- \), Eq. (8), when the temperature of B becomes \( T_- \).

Further \( T_m^{(+)} \) may be regarded as an effective temperature of an infinite reservoir \([18]\), such that the present situation of an infinite source and a finite sink, at temperatures \( T_+ \) and \( T_- \) respectively, is equivalent to maximum work extraction in a reversible cycle from two infinite reservoirs at temperatures \( T_+ \) and \( T_m^{(+)} \)\( < T_+ \). In the latter case, the extracted work per cycle is: \( W_+ = (T_+ - T_m^{(+)})(S_+ - S_-) \), with (Carnot) efficiency \( 1 - T_m^{(+)} / T_+ \), which is the same as Eqs. (4) and (5).

Now, we assume that the information on the nature of system A is not available, or, in other words, \( T_m^{(+)} \) is not given. But if the quasi-static model with an infinite source and a finite sink, is a good model for the observed performance of an industrial plant, then we may infer the relevant value of the effective temperature \( T_m^{(+)} \), when the theoretical efficiency \( \eta_+ \) for the model is set equal to the observed value \( \eta_0 \). This implies that we write

\[
T_m^{(+)} = T_+ (1 - \eta_0). \tag{7}
\]

Such values for some industrial plants, based on the observed values of their efficiencies, are tabulated in Table I, and also depicted in Fig. 3 in comparison to geometric mean \( G(T_+, T_-) = \sqrt{T_+ T_-} \). The latter value is chosen simply because when \( T_m^{(+)} = G \), the observed
efficiency is equal to the CA-value. Thus the spread of observed efficiency values around
the CA-value in Fig. 1 is translated here, to a spread in the effective temperatures around
the geometric-mean values. More precisely, $T_{m}^{(+)} \gtrsim G$ indicates $\eta_{o} \lesssim \eta_{CA}$, as is evident from
Table I. It is to be noted that inferring the effective temperature from the observed efficiency
does not determine the nature of the system A or the form of fundamental relation $U(S)$.
The latter information is not at our disposal within the assumptions of the model.

However, the scenario of work extraction from a finite system coupled to an infinite
reservoir, via a reversible process, is consistent with an alternate picture too. This involves
that, for the same initial temperatures $T_+$ and $T_-$, we consider a finite source (B) at $T_+$, coupled to an infinite sink at $T_-$, see Fig. 2 (ii). Here again, we can extract maximal work by utilizing the temperature gradient between B and the reservoir, till B comes to be at temperature $T_- [19]$. Assuming that the system goes from some equilibrium state $(U'_+, S'_+)$ to another one $(U'_-, S'_-)$, the heat removed from the source is $Q_- = U'_+ - U'_-$ and the amount rejected to sink is $q_- = T_- (S'_+ - S'_-)$. So the extracted work is $W_- = Q'_- - q'_-$, or

$$W_- = (U'_+ - U'_-) - T_- (S'_+ - S'_-).$$ (8)

This is called exergy in the engineering literature [20]. The efficiency $\eta_- = W_-/Q_-$ is given by

$$\eta_- = 1 - \frac{T_-}{T_m^{(-)}},$$ (9)

where

$$T_m^{(-)} = \frac{U'_+ - U'_-}{S'_+ - S'_-}.$$ (10)

It is clear from the expressions for $W_-$ and $\eta_-$, that an equivalence exists between the above model and that of work extraction in a reversible cycle from two infinite reservoirs at $T_-$ and $T_m^{(-)} (> T_-)$.

Now again, to apply the above case to model an industrial plant, we can equate the observed efficiency to the theoretical efficiency, as $\eta_o = \eta_-$ and infer the corresponding effective temperature $T_m^{(-)}$ from Eq. (9), as

$$T_m^{(-)} = \frac{T_-}{1 - \eta_o}.$$ (11)

It is clear that $T_+ > T_m^{(-)} > T_-$. The calculated values of $T_m^{(-)}$, based on the observed efficiencies of the plants, are also tabulated in Table I, and shown graphically in Fig. 4 in comparison to $G(T_+, T_-)$.

The following remark seems appropriate here. An analogy may be made between the above model and the earlier irreversible models such as proposed by Chambadal [2, 6], in which an intermediate temperature $T_w$ (of the warm exit stream) enters the analysis and the theoretical efficiency of the model is $1 - T_-/T_w$ (compare with Eq. (9)). For an optimal value $T_w = \sqrt{T_+ T_-}$, the power output becomes optimal with the corresponding efficiency equal to CA-value. The crucial differences, from our model, are twofold: a) we study quasi-static maximum work extraction process b) the nature of system B, the finite source, is not specified.
Now as observed in Figs. 3 and 4, the values of effective temperatures seem to be distributed, apparently in a random fashion, about the geometric-mean value. However, it is remarkable to note that for a given plant, the calculated values of $T_m^{(+)}$ and $T_m^{(-)}$ are such that, to a high accuracy, they are equidistant from the geometric mean $G$. More precisely, if we define an average value of temperature $T_m^{(av)}$, as $T_m^{(+)} - T_m^{(-)} = T_m^{(av)} - T_m^{(-)}$, then this value $T_m^{(av)}$ is very close to the $G$ value for that situation. In other words, we define an average scale of temperature as the arithmetic mean of the two inferred temperatures $T_m^{(±)}$, and so given by

$$T_m^{(av)} = \frac{1}{2} \left[ T_+ (1 - \eta_o) + \frac{T_-}{1 - \eta_o} \right].$$

(12)

Then the above average value is found to be closely approximated by $G(T_+, T_-)$ for most observed cases of industrial plants, see Fig. 5, as well as Table I.

Now, we turn to a more quantitative characterization of the above observation. It is easy to see that $T_m^{(av)}$ takes a minimum value of $\sqrt{T_+ T_-}$, w.r.t to $\eta_o$ ($dT_m^{(av)}/d\eta_o = 0$, and $d^2T_m^{(av)}/d\eta_o^2 > 0$), at $\eta_o = 1 - \sqrt{T_-/T_+}$. This implies that any possible value of $T_m^{(av)}$ is equal to or greater than $\sqrt{T_+ T_-}$, and so will lie on or below the straight line plotted in Fig. 5.

The second aspect is related to the observation made earlier that the deviations in values of effective temperatures $T_m^{(±)}$ from the corresponding $G$-values, reflect the fact how the observed efficiency deviates from the CA value. However, the deviations from $G$-values, are
suppressed considerably in case of $T_{m}^{(av)}$. This may be argued by considering small deviations ($\epsilon$) in efficiency from the CA value, and expanding $T_{m}^{(av)}$ in powers of $\epsilon$. Let $\eta_{o} = \eta_{CA} + \epsilon$. Then we see that up to second order:

$$T_{m}^{(av)} = \sqrt{T_{+}T_{-}} + \frac{1}{2} \sqrt{\frac{T_{+}^{3}}{T_{-}}} \epsilon^{2} + O(\epsilon^{3}).$$

Thus, the first non-zero correction from the geometric-mean value is of second order in $\epsilon$, while it is of the first order for $\eta_{o}$, or $T_{m}^{(\pm)}$. Clearly, the magnitude of fluctuations is suppressed in the case of $T_{m}^{(av)}$.

Finally, we address the meaning of the average effective temperature. If we again consider the two extreme situations envisaged in Fig. [2], then they are mutually exclusive, or one may say, they are counterfactual. The average temperature is not necessarily seen in an actual realization, except for the special case $T_{m}^{(+)} = T_{m}^{(-)} = \sqrt{T_{+}T_{-}}$. In this sense, the meaning one can attach to the definition of $T_{m}^{(av)}$, can be given best in the language of inductive inference [21, 22]. In latter terms, the average temperature represents an expected scale of the effective temperature, in view of our inability to choose between the two alternatives (i) and (ii), where each represent our hypothesis for the model of work extraction applicable to the plant. In inference, when one of, say, two mutually exclusive hypotheses cannot be given a preference over the other, then we must assign equal weights to the inferences derived from each [22, 23]. The effective temperatures $T_{m}^{(\pm)}$ are our inferred values from the given data on the observed efficiency, and, so $T_{m}^{(av)}$, defined with equal weights assigned to both the inferences, represents the expected value of the effective temperature, commensurate with the information at our disposal. Any deviation from equal weights would imply that we have some extra piece of information about the process, and thus would be inconsistent [24].

Concluding, we have proposed to model the observed efficiencies, using the quasi-static models of work extraction where one reservoir (hot or cold) is finite while the other is practically infinite. This brings an extra scale of intermediate temperature into the analysis. We note two consistent models depending on which reservoir is taken as finite, and consequently, we get two possible values of the intermediate temperature. The limited information on the working conditions does not allow to prefer one model hypothesis over the other and so an equally-weighted average represents the expected scale of intermediate temperature. For most of the power plants, this inferred value is found to be quite close to the geometric mean of the highest and the lowest temperatures. Thus we rediscover the significance of
the geometric-mean temperature, which was emphasized in earlier irreversible models of plants [5–7, 25], but there it was often based on simplifying assumptions such as constant heat capacity for the hot stream, and Newton’s law for heat transfer. Such assumptions are not important in our analysis and we do not make use of a particular nature of the finite reservoir. In the present approach, if the deviations in the observed efficiency from the CA-value are small, then the geometric-mean temperature appears as a rational estimate for the intermediate temperature.

An inference based approach has been used earlier to study models of thermodynamic machines with limited information. The emergence of CA efficiency from inference has been noted in quantum heat engine [26] and mesoscopic models like Feynman’s ratchet [27]. For classical models of work extraction from two finite reservoirs, the results for efficiency at maximum work are reproduced through inference, beyond linear response [28]. Further, reversible models with limited information have been related to irreversible models through inference based reasoning [29]. In the present context, it is remarkable how simple, but general considerations can lead to estimates close to the geometric-mean value for the intermediate temperature. It is indeed curious to ask, how the inference approach leads to the same conclusions as the simple irreversible models with constant heat capacity. Does it indicate a connection between the simplicity of objective modelling and assuming minimal information in inference? More research is needed for a deeper understanding of the connection between the use of limited information and thermodynamic modelling.

The purpose of inductive inference is not just to provide an academic and philosophical point of view which may sometimes rival the more objective modelling we usually resort to in science, but also seek how methods of inference can give insight into the actual state of affairs, while incorporating the prior information, normally not considered in thermodynamic models. In this context, we note that although most of the data on plants yield an expected value of the intermediate temperature close to geometric-mean value, still there are a couple of not-so-insignificant deviations in Fig. 5 near the top right corner. A valid question is, why do these examples differ from the rest of the cases? Does it indicate other measures of optimization being used in the actual operation? As far as the available data is concerned, we note these plants operate under higher temperature gradients than most other plants. Thus, one may argue that our model and the assumptions may not be good approximations for large temperature differences. In any case, once we have noticed deviations from the
expected behavior in these plants, further studies on the actual working conditions may be undertaken, which may then yield information to generalize the inference analysis, or may help to improve the performance, at par with other plants.

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