Einstein–Weyl spaces and dispersionless
Kadomtsev–Petviashvili equation from Painlevé I and II.

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Abstract

We present two constructions of new solutions to the dispersionless KP (dKP) equation arising from the first two Painlevé transcendents. The first construction is a hodograph transformation based on Einstein–Weyl geometry, the generalised Nahm’s equation and the isomonodromy problem. The second construction, motivated by the first, is a direct characterisation of solutions to dKP which are constant on a central quadric.

We show how the solutions to the dKP equations can be used to construct some three-dimensional Einstein–Weyl structures, and four-dimensional anti-self-dual null-Kähler metrics.

1 Introduction

Let \( W \) be a three-dimensional complex manifold, with a torsion-free connection \( D \) and a conformal metric \( [h] \). We shall call \( W \) a Weyl space if the null geodesics of \([h]\) are also geodesics for \( D \). This condition is equivalent to \( Dh = \omega \otimes h \) for some one form \( \omega \). Here \( h \) is a representative metric in the conformal class. If we change this representative by \( h \to \phi^2 h \), then \( \omega \to \omega + 2d \ln \phi \). A tensor object \( T \) which transforms as \( T \to \phi^m T \) when \( h \to \phi^2 h \) is said to be conformally invariant of weight \( m \). The pair \(([h], D)\) satisfies the Einstein–Weyl (EW) equations if the symmetrised Ricci tensor of of the Weyl connection is proportional to the conformal metric [2, 8].

In [5] it has been demonstrated that if an Einstein–Weyl space admits a parallel weighted vector, then coordinates can be found in which the metric and the one-form are locally given by

\[
  h = dy^2 - 4dxdt - 4udt^2, \quad \omega = -4uxdt, \quad u = u(x, y, t) \tag{1.1}
\]

and the Einstein–Weyl equations reduce to the dispersionless Kadomtsev–Petviashvili equation

\[
  (u_t - uu_x)_x = uu_y. \tag{1.2}
\]

If \( u(x, y, t) \) is a smooth real function of real variables then (1.1) has signature \((++-)\). One can verify that the vector \( \partial_x \) in the EW space (1.1) is a null vector, covariantly constant in the Weyl connection, and with weight \(-1/2\).

The geometric approach based on the structure (1.1), and the associated twistor theory yielded some new explicit solutions to (1.2)[5]. Other solutions have been obtained in [6, 11].
In this paper we aim to show a nontrivial relation between EW geometry and the dKP equation on one side and the first two Painlevé transcendents on the other. The nontriviality here means that the Painlevé equations do not arise as symmetry reductions of dKP.

Let \( \star : \Lambda^i(W) \to \Lambda^{3-i}(W), i = 1, 2, 3 \) be the Hodge operator corresponding to \([h]\). Using the relations

\[
\begin{align*}
\star dt &= dt \wedge dy, \\
\star dy &= 2dt \wedge dx, \\
\star dx &= dy \wedge dx + 2udy \wedge dt
\end{align*}
\]

we verify that equation (1.2) is equivalent to

\[
d \star du = 0. \tag{1.3}
\]

One needs to consider two classes of solutions:

1. The generic case \( |du|^2 = du \wedge \star du \neq 0 \). The condition (1.3) implies the existence of a foliation \( W = \mathbb{C} \times \Sigma \) of an EW space (1.1) by two-complex-dimensional symplectic manifolds \( \Sigma \) with a holomorphic symplectic two-form \( \omega \) such that

\[
\star du = \omega. \tag{1.4}
\]

We shall consider two cases: \( \Sigma = \mathbb{C}^2, \omega = dp \wedge dq \), and \( \Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1, \omega = (1 + pq)^{-2}dp \wedge dq \). The functions \( p, q : W \to \mathbb{C} \) are local holomorphic coordinates on \( \Sigma \), and in the second case the symplectic two-form is defined on the complement of \( 1 + pq = 0 \).

The idea is to use \((u, p, q)\) as local coordinates, and regard \( x = x(u, p, q), y = y(u, p, q), t = t(u, p, q)\) as dependent variables.

2. The null case \( |du| = 0 \). The existence of smooth \( p \) and \( q \) can not be deduced even locally.

In the next Section we shall see that Case (2) can be integrated completely (Proposition 2.6). In Section 3 we shall show that Case (1) leads to a hodograph transformation between solutions to (1.2) and solutions to the generalised SDiff(\( \Sigma \)) (or so called \( SU(\infty) \)) Nahm equation. The special case of \( SL(2, \mathbb{C}) \) generalised Nahm’s equations turns out to be equivalent to the isomonodromy problem associated with the Painlevé I or Painlevé II equations. The level sets of corresponding solutions to the dKP equations are central quadrics in the \((x, y, t)\) space (Theorem 4.1). This will be established in Section 4. In Section 5 we shall discuss a converse construction: if a solution to dKP is constant on a central quadric in the sense that \( u(x, y, t) \) is determined implicitly by an equation of the form

\[
Q(x, y, t, u) = X^T M(u)X = C, \tag{1.5}
\]

where \( C \) is a constant, \( X^T = (x, y, t) \), and \( M(u) \) is a symmetric matrix whose components depend on \( u \), then it is determined by one of the first two Painlevé equations (Theorem 5.1). Rational solutions to Painlevé II yield explicit new solutions to (1.2). Examples of these are worked out in Section 6. Finally in Section 7 we relate the Painlevé solutions of dKP to Einstein–Weyl and anti-self-dual null-Kähler geometries.
1.1 Users guide

Although the representation (1.4) which underlies the results of this paper has its roots in the EW structure (1.1), the EW geometry is by no means essential to follow the calculations. The reader who is merely interested in solutions to the dKP equation (1.2) could regard the constraint (2.6), the vanishing of differential forms (3.14), or the quadric ansatz (1.5) as starting points.

2 The null case

The condition $du \wedge \ast du = 0$ yields

$$u_x(u_t - uu_x) = u_y^2. \tag{2.6}$$

We aim to find all solutions to the dKP equation (1.2) subject to the above constraint.

**Proposition 2.1** The EW spaces (1.1) for which $|du| = 0$ (or equivalently (2.6) holds ) are locally given by one of the two forms

$$h = dy^2 + 4Cdxdt + 4(yC' + t + 2tCC')dtdt + C^2dt^2, \tag{2.7}$$

or

$$h = dy^2 + 4(tP - 1)dtds, \tag{2.8}$$

where $C(r)$ and $P(s)$ are arbitrary functions, and $' = d/dr$. The case (2.8) is characterised by the existence of a symmetry $\partial_y$.

**Proof.** We first write equations (2.6), and (1.2) as a system of first order PDEs:

$$w_y = u_t - uu_x, \quad w_x = uy, \quad 0 = u_xw_y - uyw_x.$$ 

Eliminating $w(x, y, t)$ from the first two equations yields the dKP, and the third equation is equivalent to (2.6). The condition $w_x = uy$ implies the existence of $H(x, y, t)$ such that $u = H_x, w = H_y$. The dKP equation becomes

$$H_{yy} - H_{xt} + H_xH_{xx} = 0, \tag{2.9}$$

and (2.6) is equivalent to the Monge equation

$$H_{xx}H_{yy} - H_{xy}^2 = 0. \tag{2.10}$$

To solve (2.10) rewrite it as

$$0 = dH_x \wedge dH_y \wedge dt$$

$$dH = H_xdx + H_ydy + H_tdt$$
and perform the Legendre transform
\[ r = H_x, \quad F(t, y, r) = H(t, y, x(t, y, r)) - r x(t, y, r), \] (2.11)
so that
\[ H_y = F_y, \quad H_t = F_t, \quad x = -F_r. \]
Differentiating these relation with respect to \( r \) yields
\[ H_{xx} = -\frac{1}{F_{rr}}, \quad H_{xy} = \frac{F_{yr}}{F_{rr}}, \quad H_{yy} = \frac{F_{yy}}{F_{rr}} - \frac{F_{ry}^2}{F_{rr}}. \] (2.12)
Equation (2.10) becomes \( F_{yy} = 0 \) with the general solution
\[ F = yA(r, t) + B(r, t). \] (2.13)
Now we shall move on to equation (2.9). We write it as
\[ 0 = dH_x \wedge dx \wedge dy - d\left(\frac{H_x^2}{2}\right) \wedge dt \wedge dy + dH_y \wedge dx \wedge dt \]
and perform the Legendre transform (2.11) to \((t, y, r)\) coordinates. We need to consider two cases:
1. \( dH_y \wedge dx \wedge dt \neq 0 \) (or \( u_y \neq 0 \)). The resulting equation is
\[ F_{rt} + F_{yy}F_{rr} - F_{ry}^2 = r. \]
Substituting the solution (2.13) yields
\[ A = a(r) + c(t), \quad B = \frac{r^2 t}{2} + t \int (a')^2 dr + b(t), \quad \text{where} \ ' = \frac{d}{dr}. \]
To write down the EW structure (1.1) define \( C(r) = a'(r) \), and use (2.12) together with \( u = H_x \). This yields (2.7).
2. \( dH_y \wedge dx \wedge dt = 0 \) (or \( u_y = 0 \)). The resulting equation \( r = F_{rt} \) implies
\[ F = \frac{r^2 t}{2} + m(t) + n(r). \]
Therefore
\[ h = dy^2 + 4(t + n_{rr})drdt, \quad \omega = 4 \frac{1}{t + n_{rr}}. \]
If we now define \( s(r) \) by \( ds = (n_{rr})^{-1}dr \), and put \( P(s) = (n_{rr})^{-1} \) then the EW structure becomes (2.8). This is precisely the EW structure on the \( S^1 \) bundle over a two-dimensional indefinite Weyl manifold constructed in [5] out of solutions to
\( u_t - uu_x = 0. \)
3 The generic case: generalised Nahm’s equations

If \(|du| \neq 0\) then (1.3) implies that there exist holomorphic (or smooth if we seek real solutions) symplectic form \(\omega = f(p, q)dp \wedge dq\) such that

\[
*du = u_x dy \wedge dx + (u_t - 2uu_x)dt \wedge dy + 2u_y dt \wedge dx = \omega.
\]

This enables us to rewrite the dKP equation in terms of differential forms. Define three three-forms in an open set of \(\mathbb{C}^6\) (or \(\mathbb{R}^6\)) by

\[
\begin{align*}
\Omega_1 &= du \wedge dy \wedge dt + \omega \wedge dt, \\
\Omega_2 &= du \wedge dx \wedge dy - d(u^2) \wedge dy \wedge dt + dx \wedge \omega, \\
\Omega_3 &= 2du \wedge dt \wedge dx - \omega \wedge dy.
\end{align*}
\] (3.14)

The dKP equation with \(|du| \neq 0\) is equivalent to

\[
\Omega_1 = 0, \quad \Omega_2 = 0, \quad \Omega_3 = 0.
\]

Selecting a three dimensional surface (an integral manifold) in \(\mathbb{C}^6\) with \((x, y, t)\) as the local coordinates, and eliminating \(p, q\) by cross-differentiating would lead back to (1.2). We are however free to make another choice, and use \((p, q, u)\) as the coordinates. This leads to the following set of first order PDEs:

\[
\begin{align*}
\dot{i} &= \{t, y\} \\
\dot{x} &= \{y, x\} + 2u\{y, t\} \\
\dot{y} &= 2\{t, x\},
\end{align*}
\] (3.15)

where the Poisson structure \(\{,\}\) is induced by \(\omega\), and \(\dot{} = \partial/\partial u\).

Alternatively defining \(s = x + ut\) yields

\[
\begin{align*}
\dot{s} &= \{y, s\} + t \\
\dot{t} &= \{t, y\} \\
\dot{y} &= 2\{t, s\}.
\end{align*}
\] (3.16)

3.1 Generalised Nahm’s equation

Equations (3.16) are contained in a generalisation of the Nahm system proposed in [1]. Consider an \(n\)-dimensional complex Lie group \(G\) with a Lie algebra \(\mathfrak{g}\). Let \([,]_g\) be the Lie bracket in \(\mathfrak{g}\), and let \(\Phi : \mathbb{C} \rightarrow \mathfrak{g} \otimes \mathbb{C}^3\), so that \(\Phi_i = \Phi_{i\alpha}(u)\sigma^\alpha\) where \(u\) is a coordinate on \(\mathbb{C}\), \(\sigma^\alpha, \alpha = 1, \ldots, n\) are generators of \(\mathfrak{g}\), and \(i = 1, 2, 3\). The Nahm equations are

\[
\dot{\Phi}_i = \frac{1}{2} \varepsilon_{ijk} [\Phi_j, \Phi_k]_g.
\] (3.17)

(The summation convention is assumed, except where stated.) To generalise (3.17) introduce a a \(3 \times 3\) trace-free matrix \(B(u)\) such that

\[
\dot{\Phi}_i = \frac{1}{2} \varepsilon_{ijk} [\Phi_j, \Phi_k]_g + B_{ij} \Phi_j.
\] (3.18)
Calderbank [1] claims that equations (3.18) are integrable iff $B(u)$ satisfies the matrix-Riccati equation

$$\dot{B} = 2(B^2)_0,$$

where $(B^2)_0$ is the trace-free part of $B^2$. Consider the system (3.18) with $g = \mathfrak{sl}(2, \mathbb{C})$, and rewrite $\Phi_i$ as a symmetric matrix $\Phi_{AB}$ where $A, B$ are two-dimensional indices, and $\Phi_{AB}$ is symmetric in $A$ and $B$

$$\Phi_i \rightarrow \Phi_{AB}(u) = \frac{1}{\sqrt{2}} \begin{pmatrix} \Phi_1 + \Phi_2 & \Phi_3 \\ \Phi_3 & \Phi_1 - \Phi_2 \end{pmatrix} = \begin{pmatrix} 2S(u) & -Y(u) \\ -Y(u) & 2T(u) \end{pmatrix}.$$ 

Equations (3.18) become

$$\dot{\Phi}_{AB} = \frac{1}{2} \varepsilon^{CD} [\Phi_{AC}, \Phi_{BD}] + \Phi_{CD} \beta^C_A \beta^D_B.$$  

(3.19)

Here $\varepsilon^{CD}$ is a constant symplectic structure on $\mathbb{C}^2$ with $\varepsilon^{01} = 1$, and $\beta : \mathbb{C} \rightarrow \mathfrak{sl}(2, \mathbb{C})$.

To write down a Lax pair for (3.19) we shall work with a homogeneous space for $SL(2, \mathbb{C})$. This is the space $F$ of all pairs $(\pi^A, \eta^B)$, such that $\varepsilon_{AB} \pi^A \eta^B = 1$. We shall restrict ourself to (3.19) for which $\beta$ is nilpotent (and therefore constant as a consequence of the matrix Riccati equation), or $\beta^A_B = -\iota^A \iota^B$, where $\iota_B = (-1, 0)$ and $\iota_B = \varepsilon^{AB} \iota_B = (0, 1)$. Let

$$A = \frac{\Phi_{AB} \pi^A \pi^B}{(\pi C \iota^C)^4} = 2(S + \lambda Y + \lambda^2 T),$$

$$B = -\Phi_{AB} \pi^A \eta^B = Y + 2\lambda T$$

where $\eta^A = \frac{\iota^A}{\pi^A \eta^B}$.

Here $\lambda = \pi_0 / \pi_1$ is the affine coordinate on $\mathbb{CP}^1$, and we put $\pi_1 = 1$.

The Lax pair for (3.19) is in this case given by

$$L_0 = \frac{\partial}{\partial \lambda} - A, \quad L_1 = \frac{\partial}{\partial u} - B.$$  

(3.20)

The integrability $[L_0, L_1] = 0$ of this distribution is equivalent to

$$\dot{S} = [Y, S] + T,$$

$$\dot{T} = [T, Y],$$

$$\dot{Y} = 2[T, S],$$

(3.21)

which is (3.19).

**Remark.** The anonymous referee has pointed out that the generalised Nahm’s equations (3.18) arise as symmetry reductions of anti-self-dual Yang-Mills equations on $\mathbb{C}^4$ by three-dimensional abelian sub-groups of $PGL(4, \mathbb{C})$. When the gauge group is $SL(2, \mathbb{C})$, the corresponding reductions are the six Painlevé equations [12].

## 3.2 $\text{sdiff}(\Sigma)$ generalised Nahms equation and dKP

Now assume that $g$ is the infinite-dimensional Lie algebra $\text{sdiff}(\Sigma)$ of holomorphic symplectomorphisms of a two-dimensional complex symplectic manifold $\Sigma$ with local holomorphic
coordinates \( p, q \) and the holomorphic symplectic structure \( \omega \). Elements of \( \text{sdiff}(\Sigma) \) are represented by the Hamiltonian vector fields \( X_H \) such that

\[
X_H \cdot \omega = dH
\]

where \( H \) is a \( \mathbb{C} \)-valued function on \( \Sigma \). The Poisson algebra of functions which we are going to use is homomorphic to \( \text{sdiff}(\Sigma) \). We shall make the following replacement

\[
[ , ]_g \longrightarrow \{ , \}.
\]

in formulae (3.19,3.21). The components of \( \Phi_{AB} \) are therefore regarded as Hamiltonians generating the symplectomorphisms of \( \Sigma \), and the generalised Nahm equations (3.19) with

\[
\Phi_{AB}(u,p,q) = \begin{pmatrix} 2s(u,p,q) & -y(u,p,q) \\ -y(u,p,q) & 2t(u,p,q) \end{pmatrix}, \quad \beta^A_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

yield (3.16).

The Lax pair for (3.16) is therefore given by

\[
L_0 = \frac{\partial}{\partial \lambda} + X_{H_A}, \quad L_1 = \frac{\partial}{\partial u} + X_{H_B},
\]

(3.23)

where \( X_{H_A} \), and \( X_{H_B} \) are Hamiltonian vector fields with Hamiltonians

\[
H_A = 2(s + \lambda y + \lambda^2 t), \quad H_B = y + 2\lambda t.
\]

To verify this claim we notice that the condition \([L_0, L_1] = 0\) and the standard relation \([X_f, X_g] = -X_{\{f,g\}}\) yield

\[
\partial_u H_A - \partial_\lambda H_B + \{H_A, H_B\} = 0.
\]

Therefore

\[
\dot{s} + \lambda \dot{y} + \lambda^2 \dot{t} = t - \{s, y\} - 2\lambda \{s, t\} - \lambda^2 \{y, t\}
\]

which is (3.16).

### 3.3 \( \text{sl}(2, \mathbb{C}) \) generalised Nahm’s equation and the central cone ansatz.

In this section we shall establish the following result

**Proposition 3.1** The solutions of the \( \text{SL}(2, \mathbb{C}) \) generalised Nahm’s equations (3.21) correspond to solutions of the dKP equation (1.2) which are constant on a central quadric, in the sense that \( u(x,y,t) \) is determined implicitly by an equation of the form

\[
Q(x,y,t,u) = X^T M(u) X = C,
\]

(3.24)

where \( C \) is a constant, \( X^T = (x,y,t) \), and \( M(u) \) is a symmetric matrix whose components depend on \( u \).
Proof. Let
\[
\sigma = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
be a basis of \(\mathfrak{sl}(2, \mathbb{C})\) such that
\[
[\sigma_+, \sigma_-] = 2\sigma, \quad [\sigma, \sigma_\pm] = \pm \sigma_\pm.
\]
Put
\[
S = a_{11}\sigma_+ + a_{12}\sigma + a_{13}\sigma_-, \quad Y = a_{21}\sigma_+ + a_{22}\sigma + a_{23}\sigma_- \\
T = a_{31}\sigma_+ + a_{32}\sigma + a_{33}\sigma_-,
\]
where the coefficients \(a_{ij}\) of the \(3 \times 3\) matrix \(a = a(u)\) are functions of \(u\) (in the next section they will be expressed by solutions to Painlevé I, or Painlevé II). We can construct an explicit embedding of \(\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{sdiff}(\Sigma)\) where by considering the infinitesimal linear or Möbius action of \(SL(2, \mathbb{C})\) on \(\Sigma\). The Hamiltonians corresponding to matrices in \(\mathfrak{sl}(2, \mathbb{C})\) are defined up to the addition of a function of \(u\). Let
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{C}).
\]

- \(\Sigma = \mathbb{C}^2\). The linear action \((p, q) \mapsto (Ap + Bq, Cp + Dq)\) preserves \(\omega = dp \wedge dq\) and is generated by
\[
H_\sigma = \frac{pq}{2}, \quad H_{\sigma_+} = -\frac{q^2}{2}, \quad H_{\sigma_-} = \frac{p^2}{2}.
\]
The Hamiltonians satisfy the algebraic constraint
\[
(H_\sigma)^2 + H_{\sigma_+}H_{\sigma_-} = 0.
\]

- \(\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1\). The group \(SL(2, \mathbb{C})\) acts on one Riemann sphere by Möbius transformation, and on the other by the inverse
\[
(p, q) \mapsto \left(\frac{Ap + B}{Cp + D}, \frac{Dq - C}{Cp + D}, \frac{Bp - A}{Bq + A}\right).
\]
Here \(p\) and \(q\) are independent holomorphic coordinates on both Riemann spheres, and the action preserves \(\omega = (1 + pq)^{-2}dp \wedge dq\). The corresponding Hamiltonians
\[
H_\sigma = \frac{pq - 1}{2pq + 1}, \quad H_{\sigma_+} = \frac{q}{1 + pq}, \quad H_{\sigma_-} = \frac{p}{1 + pq}
\]
satisfy
\[
(H_\sigma)^2 + H_{\sigma_+}H_{\sigma_-} = \frac{1}{4}.
\]
In both case the matrices \((T, Y, S)\) are replaced by functions
\[
t(u, p, q), \quad y(u, p, q), \quad s(u, p, q) = x(u, p, q) + ut(u, p, q).
\]

Regard (3.25) as a system of equations for \((H_{\sigma_x}, H_\sigma, H_{\sigma_1})\), and assume for the moment that \(\det(a(u)) \neq 0\). The algebraic constraints (3.27,3.29) imply
\[
det \begin{pmatrix}
  a_{11} & x + ut & a_{13} \\
  a_{21} & y & a_{23} \\
  a_{31} & t & a_{33}
\end{pmatrix}^2 + det \begin{pmatrix}
  x + ut & a_{12} & a_{13} \\
  y & a_{22} & a_{23} \\
  t & a_{32} & a_{33}
\end{pmatrix} det \begin{pmatrix}
  a_{11} & a_{12} & x + ut \\
  a_{21} & a_{22} & y \\
  a_{31} & a_{32} & t
\end{pmatrix} = C \det(a(u))^2,
\]
(3.30)

(where \(C = \text{const}\)) which is yields to (3.24).

\[
\square
\]

**Remarks.**

- Note that a similarity transformation
  \[
  T \rightarrow g^{-1}Tg, \quad Y \rightarrow g^{-1}Yg, \quad S \rightarrow g^{-1}Sg,
  \]
  \(g = g(u) \in SL(2, \mathbb{C})\) does not change (3.30) or (3.24). This can be checked by an explicit (MAPLE) calculation. It also follows from the fact that the transformation \(H_K \rightarrow H_{\tilde{K}}\) defined by the following diagram
  \[
  \begin{array}{ccc}
  K & \rightarrow & H_K \\
  \downarrow & & \downarrow \\
  g^{-1}Kg & \rightarrow & H_{\tilde{K}},
  \end{array}
  \]
  where
  \[
  K = a\sigma_x + b\sigma + c\sigma_1 \in \text{Map}(\mathbb{C}, \mathfrak{sl}(2, \mathbb{C})),
  \]
  \[
  H_K = aH_{\sigma_x} + bH_\sigma + cH_{\sigma_1} \in \text{Map}(\mathbb{C} \times \Sigma, \mathbb{C})
  \]
  is canonical and preserves the constraints (3.27, 3.29).

- Solutions to the dKP equations obtained from the ansatz (3.24) with const= 0 (a central cone) are invariant under scaling transformations generated by \(x\partial_x + y\partial_y + t\partial_t\). In fact they form a subclass of all solutions with this symmetry. This subclass is characterised by a ‘general quadric’ ansatz
  \[
  Q(\xi_1, \xi_2, u) = \begin{pmatrix}
  \xi_1 \\
  \xi_2
  \end{pmatrix} \begin{pmatrix}
  m_1 & m_2 \\
  m_3 & m_4
  \end{pmatrix} \begin{pmatrix}
  \xi_1 \\
  \xi_2
  \end{pmatrix} + 2m_5\xi_1 + 2m_6\xi_2 + m_7 = 0,
  \]
  where \(\xi_1 = t/y, \xi_2 = x/y\), and \(m_1, ..., m_7\) depend on \(u\).
4 dKP from the isomonodromy problem associated to PI and PII

In this section we shall relate the generalised Nahm equations (3.21) to the first two Painlevé transcendents, and construct the corresponding solutions to the dKP equation.

Consider the ODE
\[
\frac{d\Psi}{d\lambda} = \hat{A}(\lambda)\Psi,
\]
where \(\hat{A}(\lambda) = 2(\lambda^2\hat{T} + \lambda\hat{Y} + \hat{S})\), (4.31)

anc \(\hat{T}, \hat{Y}, \hat{S}\) are constant elements of \(\mathfrak{sl}(2, \mathbb{C})\). Here \(\Psi(\lambda, z)\) takes values in \(SL(2, \mathbb{C})\). Consider a one-parameter deformation \((\hat{T}(z), \hat{Y}(z), \hat{S}(z))\) of this ODE. It is known [10] that monodromy data around the fourth-order pole at \(\lambda = \infty\) remains constant if \(\Psi(\lambda, z)\) satisfies
\[
\frac{\partial \Psi}{\partial \lambda} = \hat{A}(\lambda)\Psi, \quad \frac{\partial \Psi}{\partial z} = \hat{B}(\lambda)\Psi,
\]
for a certain matrix \(\hat{B}\) whose exact form depends on whether \(\hat{T}\) is diagonalizable or nilpotent. For nilpotent (diagonalizable) \(\hat{T}\) the compatibility conditions \(\Psi_{\lambda z} = \Psi_{z\lambda}\), or
\[
[\partial_\lambda - \hat{A}(\lambda), \partial_z - \hat{B}(\lambda)] = 0
\]
reduce to the first (the second) Painlevé II [9].

- **Diagonalizable \(\hat{T}\).** The explicit parametrisation of \(\hat{T}(z), \hat{Y}(z), \hat{S}(z)\) can be chosen so that
\[
\hat{T} = \sigma, \quad \hat{Y} = \frac{\theta}{2}\sigma_+ - \frac{v}{\theta}\sigma_-, \quad \hat{S} = \left(v + \frac{z}{2}\right)\sigma - \frac{\theta w}{2}\sigma_+ - \frac{wv - \alpha + 1/2}{\theta}\sigma_-.
\]

where \(w, v, \) and \(\theta\) are functions of \(z\), and \(\alpha\) is a parameter. It this case \(\hat{B}(\lambda) = \lambda\hat{T} + \hat{Y}\), and the compatibility conditions for (4.32) are
\[
\frac{dw}{dz} = v + w^2 + \frac{z}{2}, \quad \frac{dv}{dz} = -2wv - \frac{1}{2} + \alpha, \quad \frac{d}{dz}\ln\theta = -w,
\]
and finally
\[
\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha,
\]
which is Painlevé II [9].

- **Nilpotent \(\hat{T}\).** Now
\[
\hat{T} = \frac{1}{2}\sigma_+, \quad \hat{Y} = \frac{w}{2}\sigma_+ + 2\sigma_-, \quad \hat{S} = \frac{2w^2 + z}{4}\sigma_+ - v\sigma - 2w\sigma_-,
\]

where \(w\) and \(v\) depend on \(z\). The matrix \(\hat{B}\) is given by \(\hat{B}(\lambda) = \lambda\hat{T} + (1/2)\hat{Y} + (w/2)\sigma_+\), and the compatibility conditions for (4.32) are
\[
\frac{dw}{dz} = v, \quad \frac{dv}{dz} = 6w^2 + z,
\]
and
\[
\frac{d^2w}{dz^2} = 6w^2 + z,
\]
which is Painlevé I [9].
Replace \( z \) by \( 2u \), and put \( \hat{B} = \hat{B}_1/2 \). The matrices \( \hat{A}, \hat{B}_1 \) are defined up to gauge transformations

\[
\hat{A} \rightarrow A = g^{-1}\hat{A}g - g^{-1}\partial_\lambda g, \quad \hat{B}_1 \rightarrow B = g^{-1}\hat{B}_1 g - g^{-1}\partial_u g,
\]

where \( g = g(u, \lambda) \in SL(2, \mathbb{C}) \).

- **Diagonalizable \( \hat{T} \).** Choose \( g = g(u) \) such that \( g^{-1}\partial_u g = g^{-1}\hat{Y}g \), (which implies \( g = \exp (\hat{Y} du) \)).

- **Nilpotent \( \hat{T} \).** Choose \( g = g(u) \) such that \( g^{-1}\partial_u g = w\sigma_+ \), (so \( g = \exp (-f\sigma_+) \), where \( df/du = -w \)).

After all these transformations

\[
\hat{A} \rightarrow A = 2(\lambda^2 T + \lambda Y + S), \quad \hat{B} \rightarrow B = 2\lambda T + Y,
\]

where \( T = g^{-1}\hat{T}g, Y = g^{-1}\hat{Y}g, S = g^{-1}\hat{S}g \) and the isomonodromy Lax pair (4.32) becomes the Lax pair (3.20) for the generalised Nahm equations (3.19) with nilpotent \( \beta \) (i.e. (3.21)). Moreover making the replacements (3.26,3.28), and using the results of the last subsection we deduce that any solution to Painlevé I, or II yields a solution to the dKP constant on a central quadric.

Keeping in mind the 1st remark after Proposition 3.1 we shall work out the details of this construction and find the matrix \( M(u) \) in (3.24).

- For PI we verify that \( \det (a) = v/2 \neq 0 \), and so the formula (3.30) can be used to read off the matrix \( M(u) \)

\[
x^2 + w^2 y^2 - w\left(\frac{w^2}{4} - 4w^3\right)t^2 - 4xtw^2 + 2wxy + \left(\frac{w^2}{4} - 4w^3\right)yt = \hat{C}w^2, \quad (4.35)
\]

where \( w = w(u) \) satisfies the rescaled PI equation

\[
\ddot{w}/4 = 6w^2 + 2u, \quad (4.36)
\]

and the constant \( \hat{C} \) can always be set to 0 or 1.

- For PII we find that \( \det (a) = vw - (\alpha - 1/2)/2 \). Imposing the constraint \( \det (a) = 0 \) on PII leads to \( v = 0 \) and \( \alpha = 1/2 \). We shall examine this case later, and now we shall concentrate on the generic case \( v \neq 0 \). The quadric becomes

\[
x^2v - y^2w(vw - (\alpha - 1/2)) + \frac{1}{2}t^2 \left((\alpha - 1/2)^2 + 4wv(vw - (\alpha - 1/2)) + 2v^3\right) + xy(\alpha - 1/2) - ytv(\alpha - 1/2) - 2txv^2 = \hat{C}(2wv - (\alpha - 1/2))^2, \quad (4.37)
\]

where

\[
v = \frac{1}{2}\dot{w}(u) - w(u)^2 - u, \quad (4.38)
\]

\[
\frac{1}{4}\dot{w} = 2w^3 + 2wu + \alpha, \quad (4.39)
\]

and \( \hat{C} = 0, \) or 1.

We still have to consider \( \det (a) = 0 \) (or \( v = 0, \alpha = 1/2 \)). Putting \( w(u) = -\dot{f}/(2f) \) reduces (4.39) to the Airy equation

\[
\ddot{f} + 4uf = 0.
\]
We shall recover \( u(x, y, t) \) explicitly from the matrices \((\hat{T}, \hat{Y}, \hat{S})\). Applying the formula
\[
e^{g}Ke^{-g} = K + [g, K] + \frac{1}{2!}[g, [g, K]] + ...
\]
which holds for any square matrices \( g \) and \( K \) we find
\[
T = \sigma + \frac{1}{2} [\int f du] \sigma_{+}, \quad Y = \frac{1}{2} f \sigma_{+}, \quad S = u \sigma + \left( \frac{1}{4} \mathbf{j} + \frac{u}{2} \int f du \right) \sigma_{+},
\]
and the equations (3.15) are satisfied with
\[
x = -\frac{f q^2}{8}, \quad y = -\frac{f q^2}{4}, \quad t = \frac{pq}{2} - \frac{[\int f du] q^2}{4}
\]
Differentiating the constraint \( 2xf = y \mathbf{j} \) we find that \( u_{t} = xu_{x} + yu_{y} = 0 \), and \( u(\xi) \) (where \( \xi = x/y \)) satisfies
\[
2\xi u_{\xi} + uu_{\xi} + 1 = 0.
\]
This ODE is reducible to a special kind of Abel’s equation.

We can summarise the results of this section in the following theorem

**Theorem 4.1** The \( SL(2, \mathbb{C}) \) generalised Nahm equations (3.21) are equivalent to Painlevé I if \( T \) is nilpotent and to Painlevé II if \( T \) is diagonal. The corresponding solutions to the \( dKP \) equations are constant on a central quadrics \( Q(x, y, t, u) = X^{T}M(u)X = C \) where \( X^{T} = (x, y, t) \), and \( M(u) \) can be read-off form (4.35) and (4.37).

5 The quadric ansatz

The solutions to \( dKP \) found in Proposition 3.1 have a form which is reminiscent of some solutions of the \( SU(\infty) \) Toda field equations constant on central quadrics found in [14].

We shall now present a converse to Proposition 3.1, and show that the existence of Painlevé solutions to \( dKP \) follows from what we may call the quadric ansatz, which may have a wider utility. This ansatz can be made whenever we have a non-linear PDE of the form

\[
\frac{\partial}{\partial x^{i}} \left( b^{ij}(u) \frac{\partial u}{\partial x^{j}} \right) = 0,
\]
(5.40)

where \( u \) is a function of coordinates \( x^{i}, i = 1, ..., n \). The ansatz is to seek solutions constant on central quadrics or equivalently to seek a matrix \( M(u) = (M_{ij}(u)) \) so that a solution of equation (5.40) is determined implicitly as in (1.5) by

\[
Q(x^{i}, u) = M_{ij}(u)x^{i}x^{j} = C.
\]
(5.41)

(This ansatz is motivated by the work of Darboux [3] orthogonal curvilinear coordinates.) We differentiate (5.41) implicitly to find

\[
\frac{\partial u}{\partial x^{i}} = -\frac{2}{Q} M_{ij}x^{j}, \quad \text{where} \quad \dot{Q} = \frac{\partial Q}{\partial u}.
\]
(5.42)

Now we substitute this into (5.40) and integrate once with respect to \( u \). Introducing \( g(u) \) by

\[
\dot{g} = \frac{1}{2} b^{ij} M_{ij} = \frac{1}{2} \text{trace} (bM)
\]
(5.43)
we obtain

\[(g\dot{M}_{ij} - M_{ik} b^k M_{mj})x^i x^j = 0,\]

so that as a matrix ODE

\[g\dot{M} = MbM.\]  \hspace{1cm} (5.44)

This equation simplifies if written in terms of another matrix \(N(u)\) where

\[N = -M^{-1}\]  \hspace{1cm} (5.45)

for then

\[g\dot{N} = b,\]  \hspace{1cm} (5.46)

and \(g\) can be given in terms of \(\triangle = \text{det} (N)\) by

\[g^2 \triangle = \zeta = \text{constant}.\]  \hspace{1cm} (5.47)

Restricting to three dimensions with \((x^1, x^2, x^3) = (x, y, t)\), the \(SU(\infty)\) Toda field equation

\[u_{xx} + u_{yy} + (e^u)_{tt} = 0\]

is given by (5.40) with

\[b(u) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^u \end{pmatrix}\]

and as was shown in [14], in this case (5.46) can be reduced to Painlevé III.

**Theorem 5.1** Solutions to the dKP equation (1.2) constant on the central quadric (1.5) are implicitly given by solutions to Painlevé I or II:

- If \((M^{-1})_{33} \neq 0\) then
  \[x^2 v - y^2 w(wv - (\alpha - 1/2)) + \frac{1}{2} t^2 \left( (\alpha - 1/2)^2 + 4wv(wv - (\alpha - 1/2)) + 2v^3 \right) + xy(\alpha - 1/2) - ytv(\alpha - 1/2) - 2txv^2 = \hat{C}(2wv - (\alpha - 1/2))^2,\]  \hspace{1cm} (5.48)

  where \(\alpha\) is a constant, \(v\) is given by (4.38), and \(w\) is a solution to the rescaled Painlevé II (4.39).

- If \((M^{-1})_{33} = 0\) and \((M^{-1})_{23} \neq 0\) then
  \[x^2 + w^2 y^2 - w\left(\frac{u^2}{4} - 4w^3\right) t^2 - 4xtw^2 + 2wxy + \left(\frac{u^2}{4} - 4w^3\right)yt = \hat{C}w^2,\]  \hspace{1cm} (5.49)

  where \(w(u)\) satisfies the rescaled Painlevé I (4.36).

- If \((M^{-1})_{33} = (M^{-1})_{23} = 0\) then
  \[\frac{y^2}{4} + \left(\sin(u)^3 \cos(u) - u \sin(u)^2 + \gamma^2 \cos(u)^4\right) t^2 - \sin(u)^2 tx - \gamma \cos(u)^2 ty = \hat{C} \tan(u)^2\]  \hspace{1cm} (5.50)

  where \(\gamma\) is a constant.

The constant \(\hat{C}\) can always be set to 0 or 1.
Proof. For the dKP equation we have (5.40) with

\[ b(u) = \begin{pmatrix} -u & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}. \]  

(5.51)

(Note that \(-b(u)\) is the inverse of \(h\) in (1.1)). Equation (5.46) implies that \(N(u)\) can be written as

\[ N(u) = \begin{pmatrix} Y & \beta & Z \\ \beta & X & \epsilon \\ Z & \epsilon & \phi \end{pmatrix} \]  

(5.52)

where \(\beta, \epsilon\) and \(\phi\) are constants, while

\[ \dot{Y} = -g^{-1}u, \quad \dot{X} = -g^{-1}, \quad \dot{Z} = \frac{1}{2}g^{-1}. \]  

(5.53)

Now

\[ \eta = X + 2Z = \text{constant} \]  

(5.54)

and we can eliminate \(Z\) in favour of \(X\) and another constant. To eliminate \(Y\) in favour of \(X, \dot{X}\) and another constant we use (5.47) and (5.54):

\[ \Delta = (\phi X - \epsilon^2)Y - \frac{1}{4}X^3 + \frac{1}{2}\eta X^2 - X(\frac{1}{4}\eta^2 + \epsilon\beta) - \beta^2 \phi + \epsilon\beta\eta = \zeta \dot{X}^2 \]  

(5.55)

so that, provided \(\phi\) and \(\epsilon\) are not both zero

\[ Y = (\phi X - \epsilon^2)^{-1}(\zeta \dot{X}^2 + \frac{1}{4}X^3 - \frac{1}{2}\eta X^2 + X(\frac{1}{4}\eta^2 + \epsilon\beta) + \beta^2 \phi - \epsilon\beta\eta). \]  

(5.56)

If \(\phi = \epsilon = 0\) then (5.55) is a first order ODE for \(X\)

\[ \zeta \dot{X}^2 = -\frac{1}{4}X(\eta - X)^2 \]

which is readily solved yielding (5.50). Otherwise we obtain \(\dot{Y}\) from (5.56) and substitute it into the equation

\[ \dot{Y} - u \dot{X} = 0 \]  

(5.57)

obtained form (5.53). This is the desired second order ODE for \(X\) which we shall investigate below.

First we use some freedom of redefinition to absorb some of the constants. Note that the linear change of coordinates

\[ \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} 1 & c_1 & c_2 \\ 0 & 1 & 2c_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}, \quad \dot{u} = u - c_1^2 - \frac{1}{2}c_1 - \frac{1}{2}c_2 \]  

(5.58)

is a symmetry of the dKP equation [5]. This transformation preserves the quadric ansatz but changes the constants as follows

\[ \begin{aligned}
\hat{\phi} &= \phi \\
\hat{\epsilon} &= \epsilon + 2c_1\phi \\
\hat{\beta} &= \beta + (c_2 + 2c_1^2)\epsilon + 2c_1c_2\phi + c_1\eta \\
\hat{\eta} &= \eta + 6c_1\epsilon + (4c_1^2 + 2c_2)\phi
\end{aligned} \]  

(5.59)
If $\phi \neq 0$, we can use this transformation to set $\epsilon = \eta = 0$. Now introduce functions $v(u)$ and $w(u)$ and a constant $\alpha$ by

$$X = -2\phi v, \quad Y = 2\phi (vw - (\alpha - \frac{1}{2})) + \phi v^2, \quad \alpha = \frac{1}{2} + \frac{\beta}{\phi}$$

(5.60)

to find that the equations (5.55) and (5.57) with $\zeta = -\phi/16$ are equivalent to (4.39) with $v$ given by (4.38). This case reduces to Painlevé II. The quadric (1.5) can be written as (5.48), which is identical to the previously obtained (4.37).

If $\phi = 0$, we can use the transformation (5.59) to set $\beta = \eta = 0$. Now introduce $w$ by

$$X = 2\epsilon w, \quad Y = \frac{\epsilon}{2} \left( \frac{\dot{w}^2}{4} - 4w^3 \right)$$

(5.61)

to find that (5.55) and (5.57) with $\zeta = -\epsilon/32$ are equivalent to the rescaled Painlevé I (4.36). The quadric (1.5) is this time (5.49) (which is (4.35)).

If $\dot{C} \neq 0$, then we can set $\dot{C} = 1$ by (possibly complex) rescaling of $(x, y, t)$.

\[\square\]

6 Examples

All solutions to PI are transcendental [13]. On the other hand for certain values of $\alpha$ PII admits particular solutions expressible in terms of ‘known’ functions. For $\alpha = n \in \mathbb{Z}$ (4.39) possesses rational solutions, and for $\alpha = n + 1/2$ there exists a class of solutions expressible by Airy functions. For example if $\alpha = 1$, then (4.39) is satisfied by $w = -1/(2u)$. Now $v = -u$, and (4.37) becomes cubic in $u$

$$8t^2 u^3 + 16xu^2 + (8x^2 - 4yt)u - (t^2 + 4xy) = c, \quad c = \text{const.}$$

The three roots of this cubic give three solutions to dKP. A root which yields a real solution is

$$u(x, y, t) = \frac{3\sqrt{A + 12t\sqrt{B}}}{12t} + \frac{6yt + 4x^2}{3t\sqrt{A + 12t\sqrt{B}}} - \frac{2x}{3t},$$

(6.62)

$$A = 144xyt + 64x^3 + 108t^3 - 108tc,$n

$$B = -96y^3t - 48y^2x^2 + 216xyt^2 + 96x^3t + 81t^4 - 96\frac{cx^3}{t} - 216cxy - 162c^2t^2 + 81c^2.$$n

The three roots of this cubic give three solutions to dKP. A root which yields a real solution is

$$w_\alpha = -w_{(\alpha-1)} - \frac{2\alpha + 1}{w_{(\alpha-1)} + 2w_{(\alpha-1)}^2 + 2u}.$$n

Here $w_\alpha = w_\alpha(u)$ is an arbitrary solution of PII with a parameter $\alpha$. We can therefore formulate the following

**Corollary 6.1** If the quadric (1.5) is rational in $u$ then it is necessarily of the form (5.48) and it can be found explicitly.
For example if $\alpha = 2$ and $\hat{C} = 0$ then $w = 1/(2u) - 3u^2/(1 + 2u^3), v = -(1 + 2u^3)/(2u^2),$ and (4.37) is
\[
\begin{align*}
(8xt - 2y^2) + 8t^2u - (6yt - 4x^2)u^2 - (-4y^2 - 16xt)u^3 - (12yx - 11t^2)u^4 \\
- (-8x^2 + 12yt)u^5 + 16xtu^6 + 8t^2u^7 = 0.
\end{align*}
\]

7 Final comments on associated geometries

We can now look back at our initial motivation and use the solutions to the dKP equation we have obtained to construct the Einstein–Weyl structures. To write down the EW structures in a way which explicitly depends on solutions to PI or PII we shall change coordinates in (1.1) to $(t, y, u),$ and regard $x$ as a function. Both cones (5.49) and (5.48) can be written as
\[
x^2 + 2ax + b = 0,
\]
where $a = a(t, y, u), b = b(t, y, u).$ For example
\[
a = wy - 2tw^2, \quad b = w^2y^2 + (\dot{w}^2/4 - 4w^3)(yt - wt^2) - \hat{C}\dot{w}^2.
\]
in the case of PI. Now we choose one root of (7.63), and differentiate (7.63) with respect to $x$ to find an expression for $u_x.$ The resulting EW structure is
\[
h = dy^2 - 4(udt - d(a + \sqrt{a^2 - b^2}))dt, \quad \omega = \frac{8\sqrt{a^2 - b^2}}{2(a + \sqrt{a^2 - b^2})a_u - b_u}dt.
\]

There is a class of four-dimensional anti-self-dual geometries associated to (7.64).

**Definition 7.1** A null-Kähler structure on a four-manifold $M$ consists of an inner product $g$ of signature $(+++)$ and a real rank-two endomorphism $N : TM \to TM$ parallel with respect to this inner product such that
\[
N^2 = 0, \quad \text{and} \quad g(NX, Y) + g(X, NY) = 0
\]
for all $X, Y \in TM.$ A null-Kähler structure is anti-self-dual (ASD) if the self-dual part of the Weyl tensor vanishes.

In [4] it has been shown that all ASD null Kähler structures with a symmetry which preserves $N$ are locally given by solutions to the dKP equation (1.2) and its linearisation.
\[
V_{yy} - V_{xt} + (uV)_{xx} = 0.
\]

Given $u$ and $V$ the associated ASD null Kähler metric with a symmetry $\partial_z$ is given by
\[
g = V(dy^2 - 4dxdt - 4udt^2) - V^{-1}(dz + \beta)^2,
\]
where the one-form $\beta$ is a solution to the monopole equation
\[
*(dV + (1/2)\omega V) = d\beta,
\]
and the null-Kähler two-form is $\Omega := g(N, \cdot) = dz \wedge dt.$

Now assume that $u(x, y, t)$ is given in terms of PI or PII, so that $(h, \omega)$ are of the form (7.64). A general $V$ will then lead to metrics (7.66) with just one symmetry. If
$V = \text{const.}u_x$ then (7.65) reduces to (1.2) and $g$ is (pseudo) hyper-Kähler. Another particular solution of the monopole equation is picked out by the quadric ansatz: If $u$ is a solution of dKP given by the quadric ansatz (1.5) then $V = \partial u/\partial C$ satisfies (7.65). However by implicit differentiation of (1.5) this is just

$$V = \frac{1}{X^7MX}.$$

There is a class of solutions to (7.65) such that the resulting ASD null-Kähler metrics are homogeneous of Bianchi type $VIII$. These Bianchi geometries will be characterised elsewhere.

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