ON THE MONODROMY OF THE INFLECTION POINTS OF PLANE CURVES

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ABSTRACT. We prove that the monodromy group of the inflection points of plane curves of degree \(d\) is the symmetric group \(S_{3d(d-2)}\) for \(d \geq 4\) and in the case \(d = 3\) this group is the group of the projective transformations of \(\mathbb{P}^2\) leaving invariant the nine inflection points of the Fermat curve of degree three.

0. Introduction.

Let \(F(\overline{a}, \overline{z}) = \sum_{k+m+n=d} a_{k,m,n} z_1^k z_2^m z_3^n\) be the homogeneous polynomial of degree \(d\) in variables \(z_1, z_2, z_3\) and of degree one in variables \(a_{k,m,n}, k+m+n=d\). Denote by \(C_d \subset \mathbb{P}^{K_d} \times \mathbb{P}^2\), where \(K_d = \frac{d(d+3)}{2}\), the complete family of plane curves of degree \(d\) given by equation \(F(\overline{a}, \overline{z}) = 0\). Let \(\tilde{h}_d : C_d \rightarrow \mathbb{P}^{K_d}\) and \(h_d : I_d = C_d \cap \mathcal{H}_d \rightarrow \mathbb{P}^{K_d}\) be the restrictions of the projection \(pr_1 : \mathbb{P}^{K_d} \times \mathbb{P}^2 \rightarrow \mathbb{P}^{K_d}\) to \(C_d\) and \(I_d\) respectively, where

\[
\mathcal{H}_d = \{(\overline{a}, \overline{z}) \in \mathbb{P}^{K_d} \times \mathbb{P}^2 | \det \left( \frac{\partial^2 F(\overline{a}, \overline{z})}{\partial z_i \partial z_j} \right) = 0 \}.
\]

It is well-known (see, for example, [1]) that for a generic point \(\overline{a}_0 \in \mathbb{P}^{K_d}\) the intersection of the curve \(C_{\overline{a}_0} = \tilde{h}_d^{-1}(\overline{a})\) and its Hessian curve \(H_{C_{\overline{a}_0}}\) given by \(\frac{\partial^2 F(\overline{a}_0, \overline{z})}{\partial z_i \partial z_j} = 0\) is the set of the inflection points of \(C_{\overline{a}_0}\) containing \(3d(d-2)\) points. Therefore for \(d \geq 3\) we have \(\deg h_d = 3d(d-2)\).

Let \(S_d\) be the subvariety of \(\mathbb{P}^{K_d}\) consisting of the points \(\overline{a}\) such that the curves \(C_{\overline{a}}\) are singular and let \(M_d\) be the closure of subvariety of \(\mathbb{P}^{K_d}\) consisting of the points \(\overline{a}\) such that for \(\overline{a} \in M_d\) the curve \(C_{\overline{a}}\) has a \(r\)-tuple inflection point with \(r \geq 2\), i.e., \(C_{\overline{a}}\) has a smooth point \(p\) such that the tangent line \(L\) to \(C_{\overline{a}}\) at \(p\) and \(C_{\overline{a}}\) have at \(p\) the intersection number \((L, C_{\overline{a}})_p = r + 2 \geq 4\). Let \(B_d = S_d \cup M_d\) (if \(d = 3\) then \(M_3 = \emptyset\)).

Then \(h_d : I_d \setminus h_d^{-1}(B_d) \rightarrow \mathbb{P}^{K_d}\) \(\setminus B_d\) is an unramified covering and therefore it defines a homomorphism \(h_{ds} : \pi_1(\mathbb{P}^{K_d}\setminus B_d, \overline{a}_0) \rightarrow S_{3d(d-2)}\) (here \(S_{3d(d-2)}\) is the symmetric group acting on the set \(I_\overline{a} = C_{\overline{a}_0} \cap I_d\)). The group \(G_d = \text{Im } h_{ds}\) is called the monodromy group of the inflection points of plane curves of degree \(d\).

The main result of the article is the following.
Theorem 1. The group $G_d = S_{3d(d-2)}$ if $d \geq 4$ and $G_3$ is a group of order 216 isomorphic to the group of the projective transformations of $\mathbb{P}^2$ leaving invariant the nine inflection points of the Fermat curve of degree three.

The proof of Theorem 1 is given in Section 1. To prove Theorem 1 some properties of the variety $\mathcal{I}_d$ near $r$-tuple inflection points of curves are investigated in this section. In Section 2 we investigate properties of $\mathcal{I}_d$ near a node of a nodal curve of degree $d$ which will be useful in the further investigations of the variety of the inflection points of plane curves of degree $d$.

1. Proof of Theorem 1

1.1. On the monodromy of dominant morphisms. Let $\mathcal{B} \subset \mathbb{P}^K$ be a reduced hypersurface in the projective space $\mathbb{P}^K$. It is well-known that the fundamental group $\pi_1(\mathbb{P}^K \setminus \mathcal{B}, p)$ is generated by, so called, bypasses $\gamma_{q, L}$ around $\mathcal{B}$, that is, elements presented by loops $\Gamma_{q, L}$ of the following form. Let $L \subset \mathbb{P}^K$ be a germ of a smooth curve intersecting the curve $\mathcal{B}$ at a point $q \in \mathcal{B}$, $L \not\subset \mathcal{B}$, and let $S^1 \subset L$ be a circle of small radius with center at $q$. The right orientation on $\mathbb{P}^K$, defined by complex structure, defines an orientation on $S^1$ and then $\Gamma_{q, L}$ is a loop consisting of a path $l$ lying in $\mathbb{P}^K \setminus \mathcal{B}$ and connecting the point $p$ with some point $q_1 \in S^1$, the loop $S^1$ (with right orientation) starting and ending at $q_1$, and return to the point $p$ along the path $l$.

Let $f : X \to \mathbb{P}^K$ be a dominant morphism, where $X$ is a reduced variety, $\dim X = K$. Then there is a hypersurface $\mathcal{B} \subset \mathbb{P}^K$ (called the discriminant locus of $f$) such that $f : Y = X \setminus f^{-1}(\mathcal{B}) \to \mathbb{P}^K \setminus \mathcal{B}$ is an unramified finite covering. Note that $Y$ is a smooth variety.

Let the degree of the covering $f : Y \to \mathbb{P}^K \setminus \mathcal{B}$ is equal to $n$. Then the covering $f$ defines a homomorphism $f_* : \pi_1(\mathbb{P}^K \setminus \mathcal{B}, p) \to S_n$ (called the monodromy of the covering $f$), where the image $G_f := f_*(\pi_1(\mathbb{P}^K \setminus \mathcal{B}, p))$ (called the monodromy group of $f$) is a subgroup of the symmetric group $S_n$ and it acts on the fibre $f^{-1}(p) = \{p_1, \ldots, p_n\}$ as follows. A loop $\Gamma \subset \mathbb{P}^K \setminus \mathcal{B}$ representing an element $\gamma \in \pi_1(\mathbb{P}^K \setminus \mathcal{B}, p)$ can be lifted to $Y$ and this lift consists of $n$ paths $\Gamma_1, \ldots, \Gamma_n \subset Y$ starting and ending at the points of $f^{-1}(p)$. Therefore this lift defines an action $f_*(\gamma)$ on $f^{-1}(p)$ which sends the start point $p_i \in \Gamma_i$ to the end point of $\Gamma_i$ for each $i = 1, \ldots, n$.

The following Lemma is obvious.

Lemma 1. In notations used above, let

1. $\nu : \tilde{L} \to f^{-1}(L)$ be the normalization of the curve $f^{-1}(L)$, where $L \subset \mathbb{P}^N$;
2. the preimage $\tilde{f}^{-1}(q)$ consist of $k$ points $q_1, \ldots, q_k$, where $\tilde{f} = f \circ \nu$ and $q \in L \cap \mathcal{B}$;
3. $n_i$ be the ramification index of $\tilde{f}$ at $q_i$, $n_1 + \cdots + n_k = n$.

Then $f_*(\gamma_{q, L}) = c_1 \cdots c_k \in S_n$ is the product of $k$ pairwise disjoints cycles $c_i$ of length $n_i$. 

Let \( V \subset \mathbb{P}^K \) be a small neighbourhood (in complex-analytic topology) of a point \( q \in \mathcal{B} \) bi-holomorphic to a polidisk
\[
\Delta^K = \{(z_1, \ldots, z_K) \in \mathbb{C}^K \mid |z_j| < \varepsilon << 1 \text{ for } j = 1, \ldots, K\}
\]
with center at \( q \). The imbedding \( i : V \setminus \mathcal{B} \hookrightarrow \mathbb{P}^K \setminus \mathcal{B} \) induces a homomorphism \( i_* : \pi_1(V \setminus \mathcal{B}) \to \pi_1(\mathbb{P}^N \setminus \mathcal{B}) \) and a homomorphism \( f_{*, loc} = f_* \circ i_* : \pi_1(V \setminus \mathcal{B}) \to G_f \) defined by \( i_* \) and \( f_* \) uniquely up to conjugation in \( G \). If a neighbourhood \( V \) is small enough then the image \( G_{f,q} := f_{*, loc}(\pi_1(V \setminus \mathcal{B})) \) does not depend of \( V \) and it is called the local monodromy group of \( f \) at the point \( q \). The following Claim is well-known.

**Claim 1.** In notations used above, let \( q \) be a smooth point of \( \mathcal{B} \) and a curve \( L \) intersects with \( \mathcal{B} \) transversally at \( q \). Then the local monodromy group \( G_{f,q} \) is cyclic and it is generated by \( f_*(\gamma_{q,L}) \).

**Lemma 2.** Let \( Z \subset \Delta^{K+1} \) be a germ of a reduced complex-analytic variety, \( \dim Z = K \), in the polidisk \( \Delta^{K+1} = \{(z_1, \ldots, z_{K+1}) \in \mathbb{C}^{K+1} \mid |z_j| < \varepsilon \ll 1 \text{ for } j = 1, \ldots, n+1\}, \)
\( o = (0, \ldots, 0) \in Z \), and let the restriction \( f : Z \to \Delta^K \) of the projection \( pr: \Delta^{K+1} \to \Delta^K \), \( pr: (z_1, \ldots, z_{K+1}) \mapsto (z_1, \ldots, z_K) \), has the following properties:

(i) \( f \) is a proper finite holomorphic map, \( \deg f = n \);
(ii) the discriminant locus \( \mathcal{B} \subset \Delta^K \) is a smooth hypersurface;
(iii) the local degree \( \deg_o f = n \);
(iv) there is a germ \( L \subset \Delta^K \), \( \dim L = 1 \), such that \( L \) meets \( \mathcal{B} \) at the point \( o' = f(o) \), \( L \not\subset \mathcal{B} \), and \( E = f^{-1}(L) \) is a smooth curve.

Then \( o \) is a non-singular point of \( Z \).

**Proof.** Without loss of generality, we can assume that \( \mathcal{B} \) is given by \( z_1 = 0 \) and, by Weiershtrass preparation theorem, \( Z \) is given by equation of the form
\[
z^2_{K+1} + \sum_{j=0}^{n-1} \alpha_j(z_1, \ldots, z_K) z^j_{K+1} = 0, \tag{1}
\]
where \( \alpha_j(z_1, \ldots, z_K) \in \mathbb{C}[[z_1, \ldots, z_K]]. \) By Claim \( \square \) \( \alpha_j(0, z_2, \ldots, z_K) = 0 \) for each \( j = 0, \ldots, n - 1 \). Therefore \( z_1 \) is a divisor of each power series \( \alpha_j(z_1, \ldots, z_K), \) \( \alpha_j(z_1, \ldots, z_K) = z_1^{k_j} \beta_j(z_1, \ldots, z_K) \), where \( \beta_j(z_1, \ldots, z_K) \) is a power series coprime with \( z_1 \) and \( k_j \) is a positive integer. Let the germ \( L \) be given by parametrization
\[
z_j = \sum_{l=1}^{\infty} c_{j,l} t^l, \quad j = 1, \ldots, K. \tag{2}
\]
Then the curve \( E = f^{-1}(L) \) is given by \( \square \) and \( \square \). Therefore \( \beta_0(0, \ldots, 0) \neq 0, \) \( c_{1,1} \neq 0, \) and \( k_0 = 1, \) since \( E \) is a smooth curve at \( o \). Now, the smoothness of \( Z \) follows from inequality \( \beta_0(0, \ldots, 0) \neq 0. \) \( \square \)
1.2. On \( r \)-tuple inflection points. In that follows we shall use the following well-known properties of plane curves of degree \( d \geq 3 \) (see, for example [1]). First of all remind that the Hessian curve \( H_C \) of a plane curve \( C \) is independent on the choice of coordinates; \( H_C \) intersects \( C \) at the singular and inflection points of \( C \) if \( C \) does not contain a line as its irreducible component. If a line \( L \) is a component of \( C \), then \( L \) also is a component of \( H_C \). Moreover, if \( z_0 \) is a \( r \)-tuple inflection point of \( C \), then (Theorem 1 in subsection 7.3 in [1]) the intersection number \( (C, H_C)_{z_0} \) at the point \( z_0 \) is equal to \( r \). Therefore we have

Claim 2. Let \( C_{\pi_0} = C \cup (\cup_{j=1}^{k} L_j) \) be the union of a curve \( C \) of degree \( \deg C \geq 3 \) and \( k \) lines \( L_j \) (may be, \( k = 0 \)). Let \( \{z_1, \ldots, z_n\} \) be the set of the inflection points of \( C_{\pi_0} \) which do not lie in \( \cup_{j=1}^{k} L_j \). Then there is a small (in complex analytic topology) neighbourhood \( U \subset \mathbb{P}^{K_4} \) of the point \( \pi_0 \) such that \( H_d^{-1}(U) \) is the disjoint union of \( n + 1 \) open sets \( V_l, l = 1, \ldots, n + 1 \), such that \( (\pi_0, z_l) \in V_l \) for \( l \leq n \). The local degree \( \deg_{\pi_0,z_l} h_d \) of the covering \( h_d \) at a point \( (\pi_0, z_l) \) is equal to \( r_l \) if \( z_l \) is a \( r_l \)-tuple inflection point of \( C_{\pi_0} \). In particular, if \( z_l \) is a simple inflection point (that is, \( r_l = 1 \)) and \( U \) is chosen small enough, then \( h_d|_{V_l} : V_l \to U = h_d(V_l) \) is a bi-holomorphic map.

Proof. The local degree \( \deg_{\pi_0,z_0} h_d \) of the covering \( h_d \) at the point \( (\pi_0, z_0) \) is equal to the intersection number \( (\mathcal{I}, \text{pr}^{-1}_1(\pi_0))_{(\pi_0, z_0)} \) of the variety \( \mathcal{I} \) and the fibre \( \text{pr}^{-1}_1(\pi_0) \) at \( (\pi_0, z_0) \). On the other hand, \( (\mathcal{I}, \text{pr}^{-1}_1(\pi_0))_{(\pi_0, z_0)} \) is equal to the intersection number of \( C_{\pi_0} \) and its Hessian \( H_{C_{\pi_0}} \) at \( z_0 \) in \( \mathbb{P}^2 \).

The following Proposition is well-known, but since I do not know a good reference, a proof will be given.

Proposition 1. The variety \( \mathcal{M}_d \) is an irreducible hypersurface in \( \mathbb{P}^{K_4} \) for each \( d \geq 4 \). There is a non-empty Zariski open neighbourhood \( \mathcal{M}_d \subset \mathcal{M}_d \) such that for each \( \pi \in \mathcal{M}_d \) the curve \( C_{\pi} \) is non-singular and it has \( 3d(d - 2) - 1 \) inflection points (that is, it has the only one multiple \( (r = 2) \) inflection point).

Proof. Denote by \( \mathcal{D}_r \subset C_d, r = 2, 3, \) the subfamilies of curves given by

\[
z_1 S(z - 1, z_2, z_3) + z_2^{r+2} R(z_2, z_3) = 0, \tag{3}
\]

where \( S(z_1, z_2, z_3) \) is the generic homogeneous polynomial of degree \( d - 1 \) in variables \( z_1, z_2, z_3 \) and \( R(z_2, z_3) \) is the generic homogeneous polynomial of degree \( d - (r + 2) \) in variables \( z_2, z_3 \). Denote also by \( D_r = \text{pr}_1(\mathcal{D}_r) \subset \mathbb{P}^{K_4} \) the image of \( \mathcal{D}_r \) under the projection \( \text{pr}_1 \). Obviously, \( D_3 \subset D_2, D_2 \) and \( D_3 \) are irreducible projective varieties, and it is easy to see that \( \dim D_2 = \frac{(d-1)(d+2)}{2} + (d-3) = K_d - 4 \) and \( \dim D_3 = \frac{(d-1)(d+2)}{2} + (d-4) = K_d - 5 \).

\(^1\) In [1], Theorem 1 is proved under the additional assumption that there is not a line among the irreducible components of \( C \). But, it is easy to see that this assumption is not used in the proof of this Theorem.
Similarly, let $D_{2,2} \subset D_2$ be the subfamily of curves given by
\[ z_1z_3S(z_1, z_2, z_3) + z_2^4(R_1(z_1, z_2) + R_2(z_2, z_3)) = 0, \]
where $\deg S(z_1, z_2, z_3) = d - 2$ and $\deg R_1(z_1, z_2) = \deg R_2(z_2, z_3) = d - 4$; $D_{2,2,1} \subset D_2$ the subfamily of curves given by
\[ z_1z_3S(z_1, z_2, z_3) + z_2^4z_3R_1(z_2, z_3) + z_1^4R_2(z_1, z_2)) = 0, \]
where $\deg S(z_1, z_2, z_3) = d - 2$, $\deg R_1(z_1, z_2) = d - 5$, and $\deg R_2(z_2, z_3) = d - 4$; and $D_{2,2,2} \subset D_2$ the subfamily of curves given by
\[ z_1S(z_1, z_2, z_3) + z_2^4z_3R_1(z_2, z_3) = 0, \]
where $\deg S(z_1, z_2, z_3) = d - 1$ and $\deg R_1(z_2, z_3) = d - 8$. As above, denote by $D_{2,2} = \text{pr}_1(D_{2,2})$, $D_{2,2,1} = \text{pr}_1(D_{2,2,1})$, and $D_{2,2,2} = \text{pr}_1(D_{2,2,2})$ the images respectively of $D_{2,2}$, $D_{2,2,1}$, and $D_{2,2,2}$ under the projection $\text{pr}_1$. Obviously, $D_{2,2}$, $D_{2,2,1}$, and $D_{2,2,2}$ are irreducible projective varieties, and it is easy to see that $\dim D_{2,2} = K_d - 7$ and $\dim D_{2,2,1} = \dim D_{2,2,2} = K_d - 8$.

Let $p_1$ be a $r$-tuple point, $r = 2$ or $r \geq 3$, of a curve $C$. We choose homogeneous coordinates $(z_1, z_2, z_3)$ so that the point $p_1$ has coordinates $(0, 0, 1)$ and the line $L_1 = \{z_1 = 0\}$ is the tangent line to $C$ at the point $p_1$. Then an equation of $C$ has the form \[ (3) \].

Let $C$ has two 2-tuple inflection points $p_1$ and $p_2$, and let $L_1$ and $L_2$ be the tangent lines to $C$ respectively at $p_1$ and $p_2$. We have three possibilities depending on the position of the points $p_1$ and $p_2$ with respect to the curve $C$: either $p_1 \notin L_2$ and $p_2 \notin L_1$, or $p_2 \in L_1$, but $L_1 \neq L_2$, or $L_1 = L_2$. In the first case, if we choose homogeneous coordinates $(z_1, z_2, z_3)$ so that the point $p_1$ has coordinates $(0, 0, 1)$ and the line $L_1 = \{z_1 = 0\}$ is the tangent line to $C$ at the point $p_1$, the point $p_2$ has coordinates $(1, 0, 0)$ and the line $L_2 = \{z_3 = 0\}$ is the tangent line to $C$ at the point $p_2$, then an equation of $C$ has the form \[ (1) \]. In the second case, if we choose homogeneous coordinates $(z_1, z_2, z_3)$ so that the point $p_1$ has coordinates $(0, 0, 1)$ and the line $L_1 = \{z_1 = 0\}$ is the tangent line to $C$ at the point $p_1$, the point $p_2$ has coordinates $(0, 1, 0)$ and the line $L_2 = \{z_3 = 0\}$ is the tangent line to $C$ at the point $p_2$, then an equation of $C$ has the form \[ (5) \]. In the third case, if we choose homogeneous coordinates $(z_1, z_2, z_3)$ so that the point $p_1$ has coordinates $(0, 0, 1)$, the point $p_2$ has coordinates $(0, 1, 0)$, and the line $L_1 = \{z_1 = 0\}$ is the tangent line to $C$ at the points $p_1$ and $p_2$, then an equation of $C$ has the form \[ (6) \].

The group $PGL(3, \mathbb{C})$ acts on $\mathbb{P}^{K_d}$ so that $g(\pi) \in M_d$ for each $g \in PGL(3, \mathbb{C})$ and $\pi \in D_2$. Denote by $\Gamma_1 \subset PGL(3, \mathbb{C})$ the subgroup leaving invariant the variety $D_1$, where $\dagger$ is either 2, or 3, or $\{2, 2\}$, or $\{2, 2, 1\}$, or $\{2, 2, 2\}$.

Obviously, the groups $\Gamma_r$, $r = 2$ or $r \geq 3$, contain a subgroup $\Gamma_0$ consisting of the elements of the following form
\[ g = \begin{pmatrix} g_{1,1} & 0 & 0 \\ g_{2,1} & g_{2,2} & 0 \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix} \] (7)
and it is easy to see that $\Gamma_0$ is a subgroup of finite index in $\Gamma_r$, since each non-singular curve $C$ can have only finitely many multiple inflection points. Therefore $\dim \Gamma_r = 5$. Note that if there exists a curve $C_\varpi$, $\varpi \in \mathcal{M}_d$, which has only one multiple inflection point, then $\Gamma_0 = \Gamma_2$.

Similarly, the diagonal group $\Delta$ is a subgroup of $\Gamma_{2,2}$ of finite index; the group consisting of the elements of the form

$$ g = \begin{pmatrix} g_{1,1} & 0 & 0 \\ g_{2,1} & g_{2,2} & 0 \\ 0 & 0 & g_{3,3} \end{pmatrix} \in PGL(3, \mathbb{C}) $$

is a subgroup of $\Gamma_{2,2,1}$ of finite index; and the group consisting of the elements of the form

$$ g = \begin{pmatrix} g_{1,1} & 0 & 0 \\ g_{2,1} & g_{2,2} & 0 \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix} \in PGL(3, \mathbb{C}) $$

is also a subgroup of $\Gamma_{2,2,1}$ of finite index. Therefore $\dim \Gamma_{2,2} = 2$, $\dim \Gamma_{2,2,1} = 3$, and $\dim \Gamma_{2,2,2} = 4$.

Consider the morphism $\nu : PGL(3, \mathbb{C}) \times D_2 \rightarrow \mathbb{P}^{K_d}$ given by $\nu((g, \varpi)) = g(\varpi)$. Obviously, the image $\text{Im} \nu \subset \mathcal{M}_d$ is an everywhere dense subset of $\mathcal{M}_d$. The preimage of a point $\nu((g_0, \varpi_0))$, where $\varpi_0$ is a generic point of $D_2$, is the variety $\{(g, g^{-1}g_0(\varpi_0) \mid g \in \Gamma_r)\}$ of dimension five. Therefore

$$ \dim \nu(PGL(3, \mathbb{C}) \times D_r) = \dim PGL(3, \mathbb{C}) + \dim D_r - \dim \Gamma_r = K_d - r + 1. $$

In particular, $\dim \mathcal{M}_d = \dim \nu(D_2) = K_d - 1$ and therefore $\mathcal{M}_d$ is a hypersurface in $\mathbb{P}^{K_d}$.

Similar calculations give rise $\dim \nu(PGL(3, \mathbb{C}) \times D_{2,2}) = K_d - 2$,

$$ \dim \nu(PGL(3, \mathbb{C}) \times D_{2,2,1}) = K_d - 3, \quad \dim \nu(PGL(3, \mathbb{C}) \times D_{2,2,2}) = K_d - 4. $$

It is well-known that $S_d$ is a divisor in $\mathbb{P}^{K_d}$ and it is easy to see that $\mathcal{M}_d \not\subset S_d$. Therefore $\dim \mathcal{M}_d \cap S_d = K_d - 2$. Now,

$$ \mathcal{M}_d = \mathcal{M}_d \setminus (\nu(PGL(3, \mathbb{C}) \times (D_{2,2} \cup D_{2,2,1} \cup D_{2,2,2}) \cup S_d) $$

is the desired variety, where by $\overline{\mathcal{M}}$ is denoted the closure of a variety $M \subset \mathbb{P}^{K_d}$.

Therefore $\Gamma_2 = \Gamma_0$ and hence $\nu(PGL(3, \mathbb{C}) \times D_2)$ is irreducible, since there is a point $\varpi \in D_2$ such that $C_\varpi$ is non-singular and it has only one multiple inflection point (more precisely, 2-tuple inflection point). \qed
1.3. On a quasi-imbedding of the permutation group \( \mathcal{G}_d \) into \( \mathcal{G}_{d+1} \). Denote by \((G, n)\) a subgroup \( G \) of the symmetric group \( S_n \) acting on a set \( J_n \) of cardinality \( n \) as permutations and call \((G, n)\) a permutation group.

Let \( J_n = J_m \cup J_k \), \( m + k = n \), be a partition of \( J_n \) and \((G, n)\) a permutation group such that the action of \( G \) on \( J_n \) leaves invariant the set \( J_m \). The action of \( G \) on \( J_m \) defines a homomorphism \( \varphi_{n,m} : G \to S_m \), that is, it defines the permutation group \((G_{J_m}, m)\), where \( G_{J_m} = \text{Im } \varphi_{n,m} \). We say that a permutation group \((H, m)\) is quasi-imbedded in a permutation group \((G, n)\) (and denote this quasi-imbedding by \((H, m) \prec (G, n)\)) if

(i) \( n \geq m \) and there is a partition \( J_n = J_m \cup J_{n-m} \) such that \( G \) leaves invariant the set \( J_m \);

(ii) the permutation groups \((G_{J_m}, m)\) and \((H, m)\) are isomorphic as permutation groups.

Remind that the group \( \mathcal{G}_d \) is the image of \( \pi_1(\mathbb{P}^K_d \setminus \mathcal{B}_d, \overline{\sigma}_0) \) under the homomorphism \( h_{d*} : \pi_1(\mathbb{P}^K_d \setminus \mathcal{B}_d, \overline{\sigma}_0) \to S_{3d(d-2)} \), where the symmetric group \( S_{3d(d-2)} \) acts on \( J_{3d(d-2)} := h_{d}^{-1}(\overline{\sigma}_0) \). Therefore in what follows, the group \( \mathcal{G}_d \) will be considered as a permutation group \((\mathcal{G}_d, 3d(d-2))\), but it will be denoted again simply by \( \mathcal{G}_d \).

**Claim 3.** For each \( d \geq 3 \) there is a quasi-imbedding of \( \mathcal{G}_d \) in \( \mathcal{G}_{d+1} \).

**Proof.** For each \( g \in \mathcal{G}_d \) let us choose a continuous loop \( \Gamma : [0, 1] \to \mathbb{P}^K_d \setminus \mathcal{B}_d \) representing an element \( \gamma \in h_{d*}^{-1}(g) \subset \pi_1(\mathbb{P}^K_d \setminus \mathcal{B}_d, \overline{\sigma}_0) \), where \( [a, b] = \{ t \in \mathbb{R} \mid a \leq t \leq b \} \) is a segment in \( \mathbb{R} \). Then the action of \( g \) on \( J_{3d(d-2)} \) is defined by the disjoint union \( h_{d*}^{-1}(\Gamma([0, 1])) = \bigsqcup_{j=1}^{3d(d-2)} l_j([0, 1]) \) of \( 3d(d-2) \) continuous paths \( l_j : [0, 1] \to \mathbb{P}^K_{d+1} \setminus \mathcal{B}_{d+1} \) starting and ending at the points of \( h_{d*}^{-1}(\overline{\sigma}_0) \). Denote \( \overline{\tau}_{\gamma,j} = \text{pr}_2(l_j(\tau)) \).

Since \( h_{d*}^{-1}(\Gamma([0, 1])) \) is one-dimensional as a topological space, we can choose a line \( L \subset \mathbb{P}^2 \) such that \( L \cap \text{pr}_2(\bigsqcup_{j=1}^{3d(d-2)} l_j([0, 1])) = \emptyset \). For each \( \overline{\sigma} \in \mathbb{P}^K_d \) the curve \( C_{\overline{\sigma}} \cup L \) has degree \( d+1 \). Therefore the choice of \( L \) defines an imbedding \( \lambda_d : \mathbb{P}^K_d \hookrightarrow \mathbb{P}^K_{d+1} \) given by \( \lambda_d : \overline{\sigma} \mapsto \overline{b} = \overline{h}_{d+1}(C_{\overline{\sigma}} \cup L) \in \mathbb{P}^K_{d+1} \) for \( \overline{\sigma} \in \mathbb{P}^K_d \).

Consider the loop \( \lambda_d(\Gamma([0, 1])) \). By Claim 2 for each \( \tau \in [0, 1] \) there is a small (in complex analytic topology) connected neighbourhood \( U_\tau \subset \mathbb{P}^K_{d+1} \) of the point \( \lambda_d(\Gamma(\tau)) \) such that \( h_{d+1}^{-1}(U_\tau) \) is the disjoint union of \( 3d(d-2) + 1 \) open sets \( V_{\overline{\tau},j} \), \( j = 1, \ldots, 3d(d-2) + 1 \), such that \( (\lambda_d(\Gamma(\tau)), \overline{\tau}_{\gamma,j}) \in V_{\overline{\tau},j} \) and \( h_{d+1} : V_{\overline{\tau},j} \to U_\tau \) is a bi-holomorphic map for \( j \leq 3d(d-2) \), where \( \{\overline{\tau}_{\gamma,1}, \ldots, \overline{\tau}_{\gamma,3d(d-2)}\} \) is the set of the inflection points of the curve \( C_{\Gamma(\tau)} \). Note that we can choose \( U_0 \) and \( U_1 \) such that \( U_0 = U_1 \) and this open set does not depend on \( g \in \mathcal{G}_d \).

Let \( \Delta_\gamma = \{ t \in [0, 1] \mid \tau - \varepsilon_\tau < t < \tau + \varepsilon_\tau \} \) be segments in \([0, 1]\) such that \( \lambda_d(\Gamma(\Delta_\tau)) \subset U_\tau \) for \( 0 < \tau < 1 \) and, similarly, \( \Delta_0 = \{ t \in [0, 1] \mid t < \varepsilon_0 \} \) and \( \Delta_1 = \{ t \in [0, 1] \mid t > 1 - \varepsilon_1 \} \) be segments such that \( \lambda_d(\Gamma(\Delta_0)) \subset U_0 \) and \( \lambda_d(\Gamma(\Delta_1)) \subset U_1 \).

Consider a path \( \Theta : [0, 1] \to \mathbb{P}^K_{d+1} \times [0, 1] \) given by \( \Theta : \tau \mapsto (\lambda(\Gamma(\tau)), \tau) \). Obviously, \( \{U_\tau \times \Delta_\tau\}_{\tau \in [0, 1]} \) is a cover of the path \( \Theta([0, 1]) \). Since \( \Theta([0, 1]) \) is a compact, we can
choose a finite cover
\[ \{ U_0 \times \Delta_0, U_1 \times \Delta_1, \ldots, U_n \times \Delta_n, U_1 \times \Delta_1 \}, \quad 0 = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = 1. \]

It is easy to see that \( U_{\tau_j} \cap U_{\tau_{j+1}} \neq \emptyset \) for each \( j = 0, \ldots, n \). Let us choose a point \( \bar{b}_0 \in U_0 \setminus B_{d+1} \) and points \( \bar{b}_{j+1} \in (U_j \cap U_{\tau_{j+1}}) \setminus B_{d+1} \) for \( j = 0, \ldots, n \). Each variety \( U_{\tau_j} \setminus B_{d+1} \) is connected. Therefore for \( 0 \leq j \leq n+1 \) we can connect the point \( \bar{b}_{j-1,j} \) with \( \bar{b}_{j+1} \) by a continuous path \( \Gamma_j \subset U_j \setminus B_{d+1} \), where \( \bar{b}_{-1,0} = \bar{b}_{n+1,n+2} = \bar{b}_0 \).

Consider the set \( J_{3(d^2-1)} = h_{d+1}^{-1}(\bar{b}_0) = \{ \tilde{q}_1, \ldots, \tilde{q}_3(d^2-1), \tilde{q}_3(d^2-1) \} \), where \( \tilde{q}_j \in V_{0,j} \) for \( j = 1, \ldots, 3d(d-1) \). Denote \( \tilde{J}_{3d(d-2)} = \{ \tilde{q}_1, \ldots, \tilde{q}_3(d^2-1) \} \subset J_{3(d^2-1)} \). The consecutive join of the paths \( \Gamma_j \), \( j = 0, \ldots, n+1 \), is a continuous loop \( \tilde{\Gamma} = \Gamma_0 \cup \ldots \cup \Gamma_{n+1} \subset P^{K_{d+1}} \setminus B_{d+1} \)

starting and ending at \( \bar{b}_0 \). Then the start and end points of the paths \( \tilde{l}_j \), entering in the disjoint union \( h_{d+1}^{-1}(\tilde{\Gamma}) = \bigsqcup_{j=1}^{3(d^2-1)} \tilde{l}_j \) of \( 3(d^2-1) \) continuous paths, are contained in \( J_{3(d^2-1)} \). Let \( \tilde{g} = h_{d+1}^{-1}(\tilde{\Gamma}) \in G_{d+1} \), where \( \tilde{\gamma} \in \pi_1(P^{K_{d+1}} \setminus B_{d+1}, \bar{b}_0) \) is represented by \( \tilde{\Gamma} \). If we number the paths \( \tilde{l}_j \) so that the start point of \( \tilde{l}_j \) is \( \tilde{q}_j \) for \( j \leq 3d(d-2) \), then it easily follows from the construction of \( \tilde{\Gamma} \) that

1. the end point of \( \tilde{l}_j \) lies also in \( \tilde{J}_{3d(d-2)} \) for each \( j \leq 3d(d-2) \);
2. \( \tilde{g} \) leaves invariant the set \( \tilde{J}_{3d(d-2)} \);
3. the restriction of the action of \( \tilde{g} \) to \( \tilde{J}_{3d(d-2)} \) is the same as the action of \( g \) on \( J_{3d(d-2)} \) if we identify \( \tilde{J}_{3d(d-2)} \) with \( J_{3d(d-2)} \).

Denote by \( \tilde{G}_d \) a permutation subgroup of \( G_{d+1} \) generated by the elements \( \tilde{g} \), where \( g \in G_d \). Obviously, the permutation subgroup \( \tilde{G}_d \) defines a quasi-embedding of \( G_d \) in \( G_{d+1} \).

1.4. Behaviour of the covering \( h_d \) near the inflection points of the Fermat curve. Let \( F_d \subset \mathbb{P}^2 \) be the Fermat curve of degree \( d \), i.e., the curve given by equation \( z_1^d + z_2^d + z_3^d = 0 \). It has \( 3d \) the \( (d-2) \)-tuple inflection points \( \bar{x}_{j,l} \), where

\[ \bar{x}_{1,l} = (0, \mu_l, 1), \quad \bar{x}_{2,l} = (\mu_l, 0, 1), \quad \bar{x}_{3,l} = (\mu_l, 1, 0), \quad l = 1, \ldots, d, \quad \mu_l = e^{\pi i (2l-1)/d}. \]

The subgroup \( G_d \) of \( PGL(3, \mathbb{C}) \), generated by

\[ g_1 = \begin{pmatrix} 0, 1, 0 \\ 1, 0, 0 \\ 0, 0, 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1, 0, 0 \\ 0, 0, 1 \\ 0, 1, 0 \end{pmatrix}, \quad g_3 = \begin{pmatrix} e^{2\pi i/d}, 0, 0 \\ 0, 1, 0 \\ 0, 0, 1 \end{pmatrix}, \]

leaves invariant the curve \( F_d \) and acts transitively on the set of its inflection points. As a group, it is isomorphic to \( \mathbb{Z}_d^2 \rtimes S_3 \).

Let \( f_d \in \mathbb{P}^{K_d} \) be the point corresponding to the Fermat curve \( F_d \) and let \( C^{K_d} \subset \mathbb{P}^{K_d} \) be the affine space given by \( a_{0,0,d} \neq 0 \). Denote again the non-homogeneous coordinates in \( C^{K_d} \) by \( a_{k,m,n}, (k, m, n) \neq (0, 0, d) \) (here we assume that \( a_{0,0,d} = 1 \)). Then the point
Consider a neighbourhood
\[ U_\varepsilon = \{ \bar{a} \in \mathbb{C}^{K_d} \mid |a_{k,m,n} - 1| < \varepsilon, \ \text{for} \ (k,m,n) = (d,0,0) \ \text{or} \ (0,d,0) \ \text{and} \]
\[ |a_{k,m,n}| < \varepsilon \ \text{for all} \ (k,m,n) \neq (d,0,0) \ \text{or} \ (0,d,0) \} \]
of the point \( f_d \).

**Claim 4.** For positive \( \varepsilon \ll 1 \) the variety \( U_\varepsilon = h_d^{-1}(U_\varepsilon) \) is the disjoint union of \( 3d \) irreducible varieties \( U_{j,l} \), where for each \( j = 1, 2, 3 \) and \( l = 1, \ldots, d \) the variety \( U_{j,l} \) is defined by the following condition: the \((d-2)\) -tuple inflection point \( \bar{\xi}_{j,l} \) of the curve \( F_d \) lies in \( U_{j,l} \). The restriction \( U_{j,l} \to U_\varepsilon \) of the morphism \( h_d \) to each \( U_{j,l} \) has degree \( d-2 \).

**Proof.** Obviously, Claim 4 is true in the case \( d = 3 \). Therefore we will assume that \( d \geq 4 \).

It is easy to see that if \( \varepsilon \) is small enough, then the variety \( U_\varepsilon = h_d^{-1}(U_\varepsilon) = \bigcup_{j=1}^{3} \bigcup_{l=1}^{d} U_{j,l} \) is the disjoint union of \( 3d \) varieties \( U_{j,l} \) such that \((f_d, \bar{\xi}_{j,l}) \in U_{j,l}\). Therefore to prove Claim 4 it suffices to prove that \( U_{1,1} \) is an irreducible variety, since the induced actions of the group \( G_d \) on \( \mathbb{P}^{K_d} \times \mathbb{P}^2 \) and \( \mathbb{P}^{K_d} \) leave invariant the varieties \( U_\varepsilon \), \( \tilde{h}_d^{-1}(U_\varepsilon) \), and \( U_\varepsilon \), \( g \circ h_d = h_d \circ g \) for each \( g \in G_d \), and this action induces a transitive action on the set of varieties \( U_{j,l} \). Obviously, the restriction of \( h_d \) to each \( U_{j,l} \) has degree \( d-2 \).

To prove that \( U_{1,1} \) is an irreducible variety, consider a family \( C_u \) of curves in \( \mathbb{P}^{K_d} \times \mathbb{P}^2 \) given by
\[
F(u, \bar{\xi}) := z_1^d + z_2^d + z_3^d + u z_1^2 z_3^{d-2} = 0
\]
and its image \( L = \bar{h}(C_u) \simeq \mathbb{C} \). Denote by \( L_\varepsilon = L \cap U_\varepsilon \). The family \( C_u \) lies in \( L \times \mathbb{P}^2 \subset \mathbb{P}^{K_d} \times \mathbb{P}^2 \). It is easy to check that in coordinates \((u; z_1, z_2, z_3)\) the Hessian of the family \( C_u \) is
\[
\mathcal{H}(u, \bar{\xi}) = \det \left( \frac{\partial^2 F(u, \bar{\xi})}{\partial z_i \partial z_j} \right) = d(d-1)z_2^{d-2}z_3^{d-4}H(u, z_1, z_3), \tag{11}
\]
where
\[
H(u, z_1, z_3) = \frac{(d(d-1)z_1^{d-2} + 2uz_3^{d-2})(d(d-1)z_3^{d-2} + (d-2)(d-3)uz_1^2) - 4(d-2)^2u^2 z_1^2 z_3^{d-2}}{(d-4)!} = \frac{d!}{(d-4)!} u z_1^d + \frac{d}{d-1} \left( 2z_1^{d-2}z_3^{d-2} - 2(d-1)(d-2)u z_1^2 z_3^{d-2} \right) + 2d(d-1)u z_3^d.
\]
Therefore the curve \( J = h_d^{-1}(L) \subset \mathbb{C} \times \mathbb{P}^2 \), given by \( F(u, \bar{\xi}) = \mathcal{H}(u, \bar{\xi}) = 0 \), is the union of curves, \( J = J_1 \cup J_2 \cup J_3 \) (if \( d = 4 \) then \( J_3 = \emptyset \)), where \( J_2 \) and \( J_3 \) (if \( d \geq 5 \)) are the intersections of the surface given by (11) and, respectively, two surfaces given
by \( z_2 = 0 \) and \( z_3 = 0 \), and \( J_1 \) is the intersection of the surface given by equation (10) and the surface \( \overline{H}_1 \) given by

\[
\frac{d!}{(d-4)!} u z_1^d + d^2(d-1)^2 z_1^{d-2} z_3^2 - 2(d-1)(d-2) u^2 z_1^2 z_3^{d-2} + 2d(d-1)u z_3^d = 0. \tag{12}
\]

It is easy to see that \((\bigcup_{i=1}^{d} U_{i,1}) \cap (J_2 \cup J_3) = \emptyset \) and \((\bigcup_{i=1}^{d} U_{i,1}) \cap J_1 \subset C \times C^2\), where \( C = \mathbb{P}^2 \setminus \{z_3 = 0\} \). Let \( x = z_1/z_3 \), \( y = z_2/z_3 \) be coordinates in \( C^2 \), then the surface \( H_1 = \overline{H}_1 \cap (C \times C^2) \) is given by equation

\[
\frac{d!}{(d-4)!} u x^d + d^2(d-1)^2 x^{d-2} - 2(d-1)(d-2) u^2 x^2 + 2d(d-1)u = 0. \tag{13}
\]

Since the polynomial in equation (13) depends only on the variables \( x \) and \( u \), the surface \( H_1 \) is isomorphic to the product \( E \times C^1 \), where \( E \) is a curve in \( C^2 \) given by equation (13).

Let us show that the polynomial \( H(u, x) \) in the left side of (13) is irreducible in the ring \( C[u, x] \). Indeed, assume that \( H(u, x) = H_1(u, x)H_2(u, x) \). Then \( H_1(u, x) = A_1(x)u + A_2(x) \) and \( H_2(u, x) = A_3(x)u + A_4(x) \), since \( H(u, x) \) is a polynomial of degree two in variable \( u \) and the polynomials \( 2(d-1)(d-2)x^2 \) and \( \frac{d!}{(d-4)!} x^d + 2d(d-1) \) are coprime. Therefore we have

\[
A_1(x)A_3(x) = -2(d-1)(d-2)x^2, \quad A_2(x)A_4(x) = d^2(d-1)^2 x^{d-2}, \tag{14}
\]

\[
A_1(x)A_4(x) + A_2(x)A_3(x) = \frac{d!}{(d-4)!} x^d + 2d(d-1). \tag{15}
\]

It follows from (14) that \( A_1(x) = b_1 x^{k_1} \), and \( A_3(x) = b_3 x^{2-k_1} \), where \( 0 \leq k_1 \leq 2 \) and \( b_1 b_3 = -2(d-1)(d-2) \in C \). Similarly, \( A_2(x) = b_2 x^{k_2} \) and \( A_4(x) = b_4 x^{d-2-k_2} \), where \( 0 \leq k_2 \leq d-2 \) and \( b_2 b_4 = d^2(d-1)^2 \in C \). Therefore

\[
b_1 b_2 b_3 b_4 = -2d^2(d-1)^3(d-2). \tag{16}
\]

It follows from (13) that

\[
b_1 b_4 x^d + b_2 b_3 x^d = \frac{d!}{(d-4)!} x^d + 2d(d-1)
\]

and therefore

\[
b_1 b_4 = \frac{d!}{(d-4)!} \text{ and } b_2 b_3 = 2d(d-1) \text{ or } b_1 b_4 = 2d(d-1) \text{ and } b_2 b_3 = \frac{d!}{(d-4)!},
\]

but in both cases we have

\[
b_1 b_2 b_3 b_4 = 2 - \frac{d!}{(d-4)!} d(d-1) = 2d^2(d-1)^2(d-2)(d-3). \tag{17}
\]

It follows from (16) and (17) that we have the equality

\[
2d^2(d-1)^2(d-2)(d-3) = 2d^2(d-1)^2(d-2)(d-3),
\]
i.e., \( d = 0, 1 \) or 2, but, by assumption, \( d \geq 3 \) and therefore \( H(u, x) \) is an irreducible polynomial.

Denote by \( S \) the union of the set of critical values of the restriction \( p : E \to \mathbb{C} \cong L \) of the projection \( \text{pr} : (u, x) \mapsto (u) \) to the irreducible curve \( E \) and the set of the images under the projection of the intersection points of \( E \) and the curve given by \( x^d + u x^2 + 1 = 0 \). Note that \( S \) is a finite set. Let \( S = \{0, u_1, \ldots, u_l\} \). Then for \( \varepsilon < \min u_s \), where minimum is taken over all \( u_s \in S \setminus \{0\} \), and for each fixed non-zero value \( u_0 \) of \( u \), \( |u_0| < \varepsilon \), the set \( p^{-1}(0) = \{(u_0, x_1(u_0)), \ldots, (0, x_d(u_0))\} \) consists of \( d \) different points such that two of these points, say \((u_0, x_{d-1}(u_0)) \) and \((0, x_{d}(u_0)) \), lie very far from the point \((0, 0)\) and the other \( d-2 \) points \((u_0, x_1(u_0)), \ldots, (u_0, x_{d-2}(u_0)) \) are very close to the point \((0, 0)\), since the closure of the line \( \{u = 0\} \) meets the closure of the curve \( E \) at infinity with multiplicity two and at the point \((0, 0)\) with multiplicity \( d-2 \).

Therefore
\[
(U_{1,l}^d \cap h_{\varepsilon}^{-1}(u_0)) = \{(u_0, x_s(u_0), y_l(u_0, x_s(u_0)))\}_{1 \leq s \leq d-2, 1 \leq l \leq d},
\]
where \( y_l(u_0, x_s(u_0)) \), \( l = 1, \ldots, d \), are the roots of the equation
\[
x^d(u_0) + y^d + 1 + u_0 x^2(u_0) = 0,
\]
and hence the intersection \( U_{1,1} \cap h_{\varepsilon}^{-1}(u_0) \) consists of \( d-2 \) different points for each \( u_0 \in L_\varepsilon \setminus \{0\} \). Note also that \( U_{1,1} \cap h_{\varepsilon}^{-1}(0) \) is the single point \((0, \xi, 1, 1)\). Therefore to prove that \( U_{1,1} \) is irreducible, it suffices to show that \( U_{1,1} \cap h_{\varepsilon}^{-1}(L_\varepsilon) \) is a smooth curve, since otherwise the curve \( U_{1,1} \cap h_{\varepsilon}^{-1}(L_\varepsilon) \) is the union of several components lying in different irreducible components of \( U_{1,1} \) and meeting at the point \((0, \xi, 1, 1)\) which must be the singular point of \( U_{1,1} \cap h_{\varepsilon}^{-1}(L_\varepsilon) \).

To prove the smoothness of \( U_{1,1} \cap h_{\varepsilon}^{-1}(L_\varepsilon) \) at \((0, \xi, 1, 1)\) note that in non-homogeneous coordinates \((u, x, y_1 = y - \mu)\) the curve \( U_{1,1} \cap h_{\varepsilon}^{-1}(L_\varepsilon) \) is the complete intersection of two surfaces given by equation (13) and the equation \( x^d + (y_1 + \mu_1)^d + 1 + u x^2 = 0 \). It is easy to check that these two surfaces are non-singular at the point \((u, x, y_1) = (0, 0, 0)\) and meet transversally at this point.

**Corollary 1.** Let \( U_{j,l} \subset U_\varepsilon \) be the same as in Claim 4. Then \( U_{j,l} \cap h_{S_d}^{-1}(B_d) \) is a connected smooth variety.

1.5. **Transitivity of the actions of the groups \( G_d \).** In notation used in subsection 1.4, let the base point \( \bar{u}_0 \) of the fundamental group \( \pi_1(\mathbb{P}^{K_d}, B_d, \bar{u}_0) \) lie in the neighbourhood \( U_{\varepsilon}, \varepsilon << 1 \). Then the set \( I_{\bar{u}_0} = h_{d}^{-1}(\bar{u}) \) naturally splits into the union of 3d subsets, \( I_{\bar{u}_0} = \bigsqcup_{j=1}^{3} \bigsqcup_{l=1}^{d} I_{j,l}, \) where \( I_{j,l} = I_{\bar{u}} \cap U_{j,l} = \{p_{j,l,1}, \ldots, p_{j,l,d-2}\} \).

**Claim 5.** The group \( G_d = \text{Im } h_{d*} \subset S_{3d(d-2)} \) acts transitively on the set \( I_{\bar{u}_0} \).

**Proof.** In the beginning, let us assume that the group \( G_d \) acts transitively on each subset \( I_{j,l} \). Indeed, by Corollary 1 for each pair \((j, l)\) the variety \( U_{j,l} \cap h_{S_d}^{-1}(B_d) \) is connected. Therefore any two points \( p_{j,l,m_1}, p_{j,l,m_2} \in I_{j,l} \) can be connected by a
smooth path $\gamma \subset U_{j,l} \setminus h_d^{-1}(B_d)$. Then the image $g = h_d(\gamma) \in S_{3d(d-2)}$ of the element $\gamma \in \pi_1(\mathbb{P}^N_d \setminus B_d, \overline{\gamma})$ represented by the loop $h_d(\gamma)$ sends the point $p_{j,l,m_1}$ to $p_{j,l,m_2}$.

Now to complete the proof of Claim 5 it suffices to show that for each pair $(j, l)$ the point $p_{1,1,1} \in I_{1,1}$ can be connected with some point $p_{j,l,m} \in I_{j,l}$ by a smooth path $l \subset I_d \setminus B_d$. For this let us consider an element $g_1 \in G_d \subset PGL(3, \mathbb{C})$ such that $g_1(U_{1,1}) = U_{j,l}$, where the group $G_d$ was introduced in Subsection 1.4. The group $PGL(3, \mathbb{C})$ is connected. Therefore we can find a smooth path $g_t \subset PGL(3, \mathbb{C})$, $t \in [0, 1]$, connecting the elements $g_0 = Id$ and $g_1$ in $PGL(3, \mathbb{C})$. Then the path $g_t(p_{1,1,1})$ lies in $I_d \setminus h_d^{-1}(B_d)$, since for each $t \in [0, 1]$ the curve $g_t(C_{1})$ is smooth and it has not multiple inflection points, and the point $g_t(p_{1,1,1})$ is an inflection point of the curve $g_t(C)$. Note also that the path $g_t(p_{1,1,1})$ connects the point $p_{1,1,1}$ with some point $g_1(p_{1,1,1}) \in U_{j,l}$. As above, by Corollary 1 the point $g_1(p_{1,1,1})$ can be connected with any point lying in $I_{j,l}$ by a path in $U_{j,l} \setminus h_d^{-1}(B_d)$. \hfill $\square$

1.6. Behaviour of the covering $h_d$ near a 2-tuple inflection point. Let $p$ be a 2-tuple inflection point of a curve $C$ of degree $d \geq 4$. Without loss of generality, we can assume that $p = (0, 0, 1)$ and the line $\{z_1 = 0\}$ is the tangent line to $C$ at the point $p$. Then $C$ is given by equation $F(z_1, z_2, z_3) = 0$, where $F(z_1, z_2, z_3)$ is a homogeneous polynomial of the form $z_1^d R(z_2, z_3) + z_1 S(z_1, z_2, z_3)$ and where $R(z_1, z_3)$ is a homogeneous polynomial of degree $d - 4$ such that $R(0, 1) = 1$, and $S(z_1, z_2, z_3)$ is a homogeneous polynomial of degree $d - 1$ such that $S(0, 0, 1) = 1$.

Let $V \subset P^{K_d}$ be a very small neighbourhood of the point $c$ corresponding to the curve $C$ and $V = h_d^{-1}(V) \subset P^{K_d} \times \mathbb{P}^2$. Then $V$ is the disjoint union of two components, $V = V_1 \bigcup V_2$, where $V_1$ contains the point $(c, p) \in P^{K_d} \times \mathbb{P}^2$. Obviously, the restriction of $h_d$ to $V_1$ has degree two, since $p$ is a 2-tuple inflection point of $C$ and therefore under small deformation of $C$ the deformed curves, near the point $p$, have either two different inflection points or one 2-tuple inflection point.

Claim 6. There is a smooth curve $E_1 \subset V_1$ passing through the point $(c, p)$ and such that the restriction $h_d|E_1 : E_1 \rightarrow L_1$ of $h_d$ to $E_1$ is ramified at $(c, p)$ with multiplicity $\deg h_d|E_1 = 2$.

Proof. Consider the family of curves $C_v \subset \Delta \times \mathbb{P}^2 \subset P^{K_d} \times \mathbb{P}^2$ given by $F(z_1, z_2, z_3) + v z_1^2 z_2^4 z_3^{-2} = 0$, where $\Delta = \{|v| < \varepsilon_1\}$ is the disk in $\mathbb{C}$ of small radius $\varepsilon_1$.

Consider the curve $E_1 = V_1 \cap C_v \cap H_v$, were $H_v$ is the Hessian of the family $C_v$. Let $x = z_1/z_3$ and $y = z_2/z_3$. Denote by $R'_j = \frac{\partial R}{\partial z_j}(x, 1)$, $S'_j = \frac{\partial S}{\partial z_j}(x, y, 1)$, $R''_j = \frac{\partial^2 R}{\partial z_j \partial z_l}(x, 1)$, and $R'''_j = \frac{\partial^3 R}{\partial z_j \partial z_l \partial z_m}(x, y, 1)$. In the coordinates $(v, x, y)$ the family $C_v$ is given by
\begin{equation}
y^4 R(y, 1) + vy^2 + x S(x, y, 1) = 0
\end{equation}
and the Hessian $H_v$ is given by
\begin{align}
\begin{bmatrix}
2 S'_1 + x S''_1, \\
S'_2 + x S''_2, \\
S'_3 + x S''_3, \\
12 y^2 R + 8 y S' + y R'' + 2 + x S'''_2, \\
4 y^3 R'_1 + y^2 R'''_2 + 2vy + x S''''_3, \\
y^4 R''_1 + \beta v y^2 + x S''''_3
\end{bmatrix} = 0,
\end{align}
where $\alpha = d - 2$ and $\beta = (d - 2)(d - 3)$.

Let $S'_j(0,0,1) = s_j$ and $S''_j(0,0,1) = s_{j,i}$. By assumption, $S(0,0,1) = 1$. Therefore $S'_3(0,0,1) = d - 1$ and it is easy to see that the differentials at the point $(v, x, y) = (0, 0, 0)$ of the polynomials in the left side of equations (18) and (19) are equal respectively to $dx$ and $(2s_2s_3s_{2,3} - s_2^2s_{3,3})dx - 2(d - 1)\frac{dv}{v}$ (here we denote by $df$ the differential of function $f$ in order not to confuse with degree $d$ of the curve $C$). Therefore the surfaces $C_v$ and $H_v$ are nonsingular at the point $(0, 0, 0)$ and meet transversally at this point, since these differentials are linear independent. It follows from this that $E_1 = V_1 \cap C_v \cap H_v$ is a smooth curve and the differential of $h_{d|E_1} : E_1 \to \Delta$ vanishes at the point $(0, 0, 0)$. Therefore $\deg h_{d|E_1} = 2$ since if a holomorphic surjective map of a smooth curve has degree one, then its differential vanishes nowhere. \hfill $\Box$

**Corollary 2.** Let $C$ be a smooth curve of degree $d \geq 4$ having $3d(d - 2) - 1$ inflection points. Then, in notations used in the proof of Claim 6, $h_d^{-1}(\Delta) \cap \mathcal{V}$ is the disjoint union of $3d(d - 2) - 1$ smooth curves, $h_d^{-1}(\Delta) \cap \mathcal{V} = \bigsqcup_{j=1}^{3d(d-2)-1} E_j$, where $h_{d|E_j} : E_j \to \Delta$ is a bi-holomorphic map for $j = 2, \ldots, 3d(d - 2) - 1$ and $h_{d|E_1}$ is a two-sheeted covering branched at the point $(v = 0) \in L_1$.

Consider a point $(\overline{a}, \overline{z}) \in h_d^{-1}(M_d)$, where $M_d \subset \mathbb{P}K_d$ is the variety defined in Subsection 1.2. Lemma 2 and Claim 6 imply

**Proposition 2.** Let $\overline{a}_0 \in M_d$ and $\overline{z}_0$ be a 2-tuple inflection point of the curve $C_{\overline{a}_0}$. Then $(\overline{a}_0, \overline{z}_0)$ is a smooth point of the variety $\mathcal{L}_d$.

**Corollary 2** and Lemma 1 imply

**Claim 7.** For $d \geq 4$ the group $\mathcal{G}_d \subset S_{3d(d-2)}$ contains a transposition.

1.7. **Case** $d = 3$. Consider the Hesse pencil, that is, the one-dimensional linear system of plane cubic curves given by

$$C_{(t_1, t_2)} : \ t_1z_1^3 + z_2^3 + z_3^3 + t_2z_1z_2z_3 = 0, \quad (t_1, t_2) \in \mathbb{P}^1, \quad (20)$$

We call the surface \( \mathcal{H} \subset L \times \mathbb{P}^2 \subset \mathbb{P}K_3 \times \mathbb{P}^2 \) given in $L \times \mathbb{P}^2$ by equation (20) the body of the Hesse pencil, where $L \simeq \mathbb{P}^1$ and $K_3 = 9$.

It is easy to see that $\mathcal{H}$ is a smooth surface and the restriction $\sigma : \mathcal{H} \to \mathbb{P}^2$ of $pr_2$ to $\mathcal{H}$ is the composition of nine $\sigma$-processes of $\mathbb{P}^2$ with centers at the base points of the Hesse pencil. Let $E_{q_j} = \sigma^{-1}(q_j)$, $j = 1, \ldots, 9$, be the exceptional curve of $\sigma$ over the base point $q_j \in \mathbb{P}^2$ of the pencil. The curves $E_j$ are sections of the projection $pr_1 : L \times \mathbb{P}^2 \to L$.

It is well-known (see, for example, [11]) that the base points of the Hesse pencil are the inflection points of each smooth member of the pencil. The Hesse pencil has four degenerate members, $C_{(0, 1)}$, $C_{(1, -3)}$, $C_{(1, -3e^{2\pi i/3})}$, and $C_{(1, -3e^{4\pi i/3})}$. Each of the degenerate members is the union of three lines. Therefore

$$\mathcal{L}_3 \cap \mathcal{H} = C_{(0, 1)} \cup C_{(1, -3)} \cup C_{(1, -3e^{2\pi i/3})} \cup C_{(1, -3e^{4\pi i/3})} \cup (\bigcup_{j=1}^{9} E_j).$$
The group $Hes \subset PGL(3, \mathbb{C})$ of the projective transformations leaving invariant the set of the inflection points of the Fermat curve $F_3 = C(1,0)$ is well investigated (see, for example, [I]). The order of $Hes$ is equal to 216 and the action of $Hes$ on the 9 inflection points of $F_3$ defines an imbedding $Hes \subset S_9$ such that $Hes$ is a 2-transitive subgroup of $S_9$. The orbit of the Fermat curve $F_3$ under the action of $Hes$ consists of four members of the Hesse pencil: $F = C(1,0), C(1,6e^{2\pi i/3}), C(1,6e^{4\pi i/3})$. We choose three continuous paths $l_j, j = 0, 1, 2,$ in $L \setminus \{(0,1), (1,-3), (1,-3e^{2\pi i/3}), (1, -3e^{4\pi i/3})\}$ connecting the points $(1, 6e^{2j\pi i/3})$ with the point $(1,0)$.

It is well-known ([3], [4]) that the set of nine inflection points of a plane cubic curve is a projectively rigid set, that is, for each two smooth plane cubic curves $C_1$ and $C_2$ there is a projective transformation of the plane sending the set of the inflection points of $C_1$ onto the set of the inflection points of $C_2$. Therefore there is an imbedding $\varphi: PGL(3, \mathbb{C}) \to (\mathbb{P}^2)^9$ given for $\tau \in PGL(3, \mathbb{C})$ by

$$\varphi: \tau \mapsto (\tau(q_1), \ldots, \tau(q_9)) \in (\mathbb{P}^2)^9,$$

where $\{q_1, \ldots, q_9\} \subset \mathbb{P}^2$ is the set of the inflection points of the Fermat curve $F_3$. (Note that $\varphi$ depends on the numbering of the inflection points of the Fermat curve $F_3$, that is, there are $9!$ such imbeddings.) Denote by $P = \varphi(PGL(3, \mathbb{C})) \subset (\mathbb{P}^2)^9$.

**Claim 8.** We have $G_3 = Hes \subset S_9$.

**Proof.** Consider the homomorphism $h_{3*}: \pi_1(\mathbb{P}^K_3 \setminus S_3, f) \to S_9$, where $f = (1,0) \in L \subset \mathbb{P}^K_3$ is the point corresponding to the Fermat curve $F_3$. Then $G_3 = h_{3*}(\pi_1(\mathbb{P}^K_3 \setminus S_3, f))$ acts on the set $h_{3*}^{-1}(f) = \{q_1, \ldots, q_9\}$.

Let us show that $Hes \subset G_3$. For this, consider an element $g_1 \in Hes \subset PGL(3, \mathbb{C})$. Since $PGL(3, \mathbb{C})$ is a connected variety, we can choose a continuous path $g_t \subset PGL(3, \mathbb{C}), 0 \leq t \leq 1$, connecting $g_0 = Id$ with $g_1$. The path $g_t$ defines a continuous path $l(t) \subset \mathbb{P}^K_3 \setminus S_3$ such that $C_{l(t)} = g_t(F_3)$. We have $C_{l(1)}$ is a member of the Hesse pencil, since $g_1 \in Hes$. Denote by $\Gamma \subset \mathbb{P}^K_3 \setminus S_3$ the path $l(t)$ if $C_{l(1)} = F_3$ and $l(t) \cup l_j$ if $C_{l(1)} = C(1,6e^{2\pi i/3})$. Then the loop $\Gamma$ represents an element $\gamma \in \pi_1(\mathbb{P}^K_3 \setminus S_3, f)$ and it is easy to see that the action of $h_{3*}(\gamma)$ on $h_{3*}^{-1}(f) = \{q_1, \ldots, q_9\}$ is the same as the action of $g_1$.

Let us show that $G_3 \subset Hes$. For this, consider an element $g \in G_3$ and a continuous loop $\Gamma(t) \subset \mathbb{P}^K_3 \setminus S_3$ starting and ending at $f$ and representing an element $\gamma \in \pi_1(\mathbb{P}^K_3 \setminus S_3, f)$ such that $h_{3*}(\gamma) = g$. We lift the loop $\Gamma(t)$ to $\mathcal{I}_3$ and this lift consists of 9 continuous paths $\Gamma_1(t), \ldots, \Gamma_9(t) \subset \mathcal{I}_3$ starting and ending at the points of $h_{3*}^{-1}(f)$. So, we obtain 9 continuous paths $pr_2(\Gamma_1(t)), \ldots, pr_2(\Gamma_9(t)) \subset \mathbb{P}^2$. If we number the paths $\Gamma_j(t)$ so that $\Gamma_j(0) = q_j$, then $(pr_2(\Gamma_1(t)), \ldots, pr_2(\Gamma_9(t)))$ is a continuous path in $P$, since $\{pr_2(\Gamma_1(t)), \ldots, pr_2(\Gamma_9(t))\}$ is the set of the inflection points of smooth plane cubic curves. Therefore there is an element $\tau \in PGL(3, \mathbb{C})$ such that $\tau(q_j) = pr_2(\Gamma_j(1)) \in h_3^{-1}(f)$ for $j = 1, \ldots, 9$, that is, $g = \tau \in Hes$. \qed
1.8. **Case** $d = 4$. Consider the Klein curve $Kl \subset \mathbb{P}^2$ given by
\[ z_1^3z_2 + z_1^3z_3 + z_1z_2^3 + z_2z_3^3 + z_1z_3^3 + z_2z_3^3 = 0. \]

It is well-known (see, for example, [2]) that the automorphism group $Aut(Kl)$ of $Kl$ have the following properties. The order of $Aut(Kl)$ is equal to 168 and $Aut(Kl) \simeq PSL(2, \mathbb{Z}/7)$, the group $Aut(Kl)$ can be represented as a subgroup of $PGL(3, \mathbb{C})$ leaving invariant the curve $Kl$ and the set of inflection points is an orbit under the action $Aut(Kl)$, the order of the stabilizer of each inflection point is equal to 7. In particular, the action of $Aut(Kl)$ on the set of the inflection points of $Kl$ is transitive.

**Claim 9.** There is an imbedding $Aut(Kl) \subset G_4 \subset S_{24}$ such that $Aut(Kl)$ is a transitive subgroup of $S_{24}$.

**Proof.** Let $\pi_0 \in \mathbb{P}^{Kl}$ be the point corresponding to the curve $Kl$ and $g_1$ an element of $Aut(Kl) \subset PGL(3, \mathbb{C})$. Since $PGL(3, \mathbb{C})$ is connected, there is a continuous path $g_t \subset PGL(3, \mathbb{C})$, $t \in [0, 1]$, connecting $g_0 = Id$ and $g_1$. Then the loop $\Gamma \subset \mathbb{P}^{Kl} \setminus B_4$ given by $g_t(\pi_0)$, $t \in [0, 1]$, defines an element $\gamma \in \pi_1(\mathbb{P}^{Kl} \setminus B_4, k)$ such that the action of $h_4(\gamma) \in S_{24}$ on the set $h_4^{-1}(\pi_0)$ of the inflection points of $Kl$ coincides with the action of $g_1 \in Aut(Kl)$. \hfill \Box

By Claim 3 and 9, the group $G_4 \subset S_{24}$ has the following properties:

1. $G_4$ contains a subgroup $Aut(Kl)$ which acts transitively on the set $I_{24} = \{1, 2, \ldots, 23, 24\}$;
2. there are a partition $I_{24} = J_1 \cup J_2$, $|J_1| = 9$, $|J_2| = 15$, and a quasi-imbedding $G_3 \prec G_4$ such that $J_1$ is invariant under the action of $\tilde{G}_3 \subset G_4$ (see subsection 1.3) and the action of $\tilde{G}_3$ on $J_1$ is 2-transitive;
3. the group $G_4$ contains a transposition.

**Claim 10.** Properties (1) – (3) imply $G_4 = S_{24}$.

**Proof.** We say that a subset $J \subset I_{24}$ is 2-transitive (with respect to the action of $G_4$) if for each two pairs $\{j_1, j_2\} \subset J$ and $\{j_3, j_4\} \subset J$ there is an element $g \in G_4$ such that $g(\{j_1, j_2\}) = \{j_3, j_4\}$.

Denote by $\tilde{J} \subset I_{24}$ a 2-transitive subset of maximal cardinality. Obviously, if $J \subset I_{24}$ is a 2-transitive subset then for each $g \in G_4$ the set $g(J)$ is also 2-transitive, and if $J_1$ and $J_2$ are 2-transitive subsets such that the cardinality $|J_1 \cap J_2| \geq 2$, then $J_1 \cup J_2$ is also 2-transitive. Therefore it is easy to see that there are two possibilities: either $\tilde{J} = I_{24}$, or $|\tilde{J}| = 12$, since, by property (2), the cardinality $|\tilde{J}| \geq 9$ and, by property (1), $G_4$ acts transitively on $I_{24}$.

Let us show that the second case is impossible. Indeed, in this case it follows from transitivity of the action of $G_4$ that for each $g \in G_4$ either $g(\tilde{J}) = \tilde{J}$, or $g(\tilde{J}) = I_{24} \setminus \tilde{J}$. Therefore the action of $G_4$ on $I_{24}$ induces an action on the set $\{\tilde{J}, I_{24} \setminus \tilde{J}\}$ of cardinality two, that is, there is an epimorphism $\varphi : G_4 \to \mathbb{Z}_2$. But, in this case the restriction $\varphi|_{Aut(Kl)} : Aut(Kl) \to \mathbb{Z}_2$ is also an epimorphism, since $Aut(Kl)$ acts transitively
on the set $I_{24}$. On the other hand, $\text{Aut}(Kl) \simeq PSL(2, \mathbb{Z}_7)$ is a simple group and therefore $\varphi|_{\text{Aut}(Kl)}$ must be trivial homomorphism.

Now, to complete the proof of Claim 10, it suffices to apply property (3), since $G_4$ acts 2-transitively on $I_{24}$ and therefore all transpositions of $S_{24}$ are contained in $G_4$. \hfill \Box

1.9. The end of the proof of Theorem 1. To complete the proof of Theorem 1 we need in the following

**Lemma 3.** Let $G$ be a subgroup of the symmetric group $S_m$ acting on a finite set $M$ of cardinality $m$. Assume that

- (i) $G$ acts transitively on $M$;
- (ii) there is a subgroup $G_1$ of $G$ and a subset $M_1$ of $M$ such that
  - (ii)$_1$ $M_1$ is invariant under the action of the group $G_1$,
  - (ii)$_2$ $2m_1 \geq m + 2$, where $m_1$ is the cardinality of the set $M_1$,
  - (ii)$_3$ the action of the group $G_1$ on $M_1$ is 2-transitive;
- (iii) the group $G$ contains a transposition.

Then $G = S_m$.

**Proof.** By (ii)$_3$, for each element $g \in G$ the subgroup $gG_1g^{-1}$ of $G$ acts 2-transitively on $g(M_1)$ and by (ii)$_2$, the group $G$ acts 2-transitively on $M_1 \cup g(M)$, since there are at least two elements in the intersection of $M_1 \cap g(M_1)$. It follows from (i) that for each element $x \in M$ there is an element $g \in G$ such that $x \in g(M_1)$. Therefore $G$ acts 2-transitively on $M$ and hence applying (iii) the group $G \subset S_m$ contains all transpositions. \hfill \Box

Now, applying induction on $d$, Theorem 1 follows from Claims 3, 5, 7, 8, 10 and Lemma 3 since

$$2[3d(d-2)] \geq [3(d+1)(d-1)] + 2$$

for $d \geq 4$.

2. Behaviour of the covering $h_d$ near a node of a nodal curve

2.1. On the subset of $S_d$ consisting of the points corresponding to the nodal curves. Denote by $N_d$ a subvariety of $S_d$ consisting of the points $\overline{\omega} \in S_d$ such that the set of singular points of the curves $C_{\pi}$ consist of the only one ordinary node. The following Proposition is well-known.

**Proposition 3.** The variety $S_d$ is an irreducible hypersurface in $\mathbb{P}^{K_d}$ for each $d \geq 3$. The variety $N_d$ is a non-empty Zariski open subset of $S_d$.

**Proof** of this proposition is similar to the proof of Proposition 1 and therefore it will be omitted. \hfill \Box

**Claim 11.** The variety $N_d \subset \mathbb{P}^{K_d}$ is smooth.
Proposition 4. Let \( \overline{a}_0 \in \mathcal{N}_d \) and \( \overline{z}_0 \) be the singular point of \( C_{\overline{a}_0} \). Then the local monodromy group \( \mathcal{G}_{d,\overline{a}_0} \subset S_{3d(d-2)} \) at the point \( \overline{a}_0 \) is a cyclic group of order 3 and it is generated by the product of two disjoint cycles of length 3.
Proof. Without loss of generality we can assume that \( C_{\mathfrak{a}_0} \) is given by equation \( F(\mathfrak{a}_0, z) = 0 \), where

\[
F(\mathfrak{a}_0, z) = z_1 z_2 z_3^{d-2} + z_3^{d-3} \sum_{j+k=3} \alpha_{j,k,d-3} z_1^j z_2^k + R(z_1, z_2, z_3)
\]

and where \( R \) is a polynomial of degree \( \leq d - 4 \) in the variable \( z_3 \).

Note that \( a_{3,0,d-3} \neq 0 \) and \( a_{0,3,d-3} \neq 0 \) if \( \mathfrak{a}_0 \in \mathfrak{N}_d \). Indeed, if, for example, \( a_{3,0,d-3} = 0 \) then it is easy to see that the line \( L \) given by \( t \mathfrak{u} + \mathfrak{a}_0 \), where \( t \in \mathbb{C} \) and in \( \mathfrak{u} \) all coordinates except the coordinate \( u_{0,1,d-1} \) are equal to zero and \( u_{0,1,d-1} = 1 \), lies in \( \mathcal{M}_d \).

Consider a one-parametric family \( C_{\mathfrak{a}_i} \) given by equation

\[
F(\mathfrak{a}_i, z) + tz_3^d = 0
\]

and its projection \( \text{pr}_1(C_{\mathfrak{a}_i}) = L = \{ \mathfrak{a}_t = \mathfrak{a}_0 + t \mathfrak{u} \} \subset \mathbb{P}^{K_d} \), where in \( \mathfrak{u} \) all coordinates except the coordinate \( v_{0,0,d} \) are equal to zero and \( v_{0,0,d} = 1 \). By Claim \( \square \) \( L \) meets \( S_d \) transversally at \( \mathfrak{a}_0 \).

In non-homogeneous coordinates \( x = \tilde{x}_z, y = \tilde{y}_z \) we have \( \mathfrak{a}_0 = (0, 0) \), the family \( C_{\mathfrak{a}_i} \) in \( L \times \mathbb{C}^2 \subset \mathbb{P}^{K_d} \times \mathbb{P}^2 \) is given by equation

\[
t + xy + \sum_{i+j=3} a_{i,j,d-3} x^i y^j + \text{terms of higher degree} = 0,
\]

and everybody can easily check that its Hessian \( H_{C_{\mathfrak{a}_i}} \) is given by equation

\[
2(d - 2)^2(xy - 3a_{3,0,d-3} x^3 + a_{2,1,d-3} x^2 y + a_{1,2,d-3} xy^2 - 3a_{0,3,d-3} y^3) + (d(d-1)) (1 + 4a_{2,1,d-3} x + 4a_{1,2,d-3} y) t + r_1(x, y) + tr_2(x, y) = 0,
\]

were \( r_1(x, y) = \sum_{i+j \geq 4} b_{i,j} x^i y^j \) and \( r_2(x, y) = \sum_{i+j \geq 2} c_{i,j} x^i y^j \) are some polynomials.

Consider the curve \( Z = h_d^{-1}(L) = C_{\mathfrak{a}_0} \cap H_{C_{\mathfrak{a}_0}} \subset X \), where \( X \) is a surface in \( \mathbb{C}^3 \) given by equation \( (24) \). It is easy to see that \( X \) is isomorphic to \( \mathbb{C}^2 = \text{Spec} \mathbb{C}[x, y] \) and \( Z \) in \( X \) is given by equation

\[
(d^2 - 7d + 8)xy - 6(d - 2)(a_{3,0,d-3} x^3 + a_{0,3,d-3} y^3) - 2(d^2 + 2d - 4)(a_{2,1,d-3} x^2 y + a_{1,2,d-3} xy^2) + \text{terms of higher degree} = 0,
\]

since

\[
t = -(xy + \sum_{i+j=3} a_{i,j,d-3} x^i y^j + \text{terms of higher degree}).
\]

It follows from \( (26) \) that \( Z \) has a node at the point \( p = (\mathfrak{a}_0, z_0) \). To resolve this point, consider the \( \sigma \)-process \( \sigma : \tilde{X} \rightarrow X \) with center at \( p \). The surface \( \tilde{X} \) is covered by two open neighbourhoods isomorphic to \( \mathbb{C}^2 \), \( \tilde{X} = U_1 \cup U_2 \). The coordinates in \( U_j \), \( j = 1, 2 \), are \( x_j, y_j \) and \( \sigma_{|U_j} \) is given by \( x = x_1 \) and \( y = x_1 y_1 \), and \( \sigma_{|U_2} \) is given by \( x = x_2 y_2 \) and \( y = y_2 \). Therefore the proper inverse image \( \sigma^{-1}(Z) \cap U_1 \) is given by equation

\[
(d^2 - 7d + 8)y_1 - 6(d - 2)^2(a_{3,0,d-3} x_1 + a_{0,3,d-3} x_1^3) - 2(d^2 + 2d - 4)(a_{2,1,d-3} x_1 y_1 + a_{1,2,d-3} x_1^2 y_1^2) + \text{terms of higher degree} = 0,
\]
and therefore the curve $\tilde{Z} = \sigma^{-1}(Z)$ is non-singular at the point $p_1 = \tilde{Z} \cap U_1 \cap E$, where $E$ is the exceptional divisor of $\sigma$.

Since $a_{3,0,d-3} \neq 0$, the coordinate $x_1$ is a local parameter in $\tilde{L}$ at the point $p_1$ and
\[ y_1 = \frac{6(d-2)^2a_{3,0,d-3}}{d^2 - 7d + 8}x_1 + \sum_{j=2}^{\infty} b_j x_1^j. \]

It follows from (27) that
\[ t = \frac{x_1^2 y_1 + a_{3,0,d-3}x_1^3}{(\frac{6(d-2)^2}{d^2 - 7d + 8} + 1)a_{3,0,d-3}x_1^3} + \text{terms of higher degree}. \]

Therefore the covering $h_d \circ \sigma : \tilde{Z} \to L$ is ramified at $p_1$ with multiplicity three.

Similar calculations (which will be omitted) show that the covering $h_d \circ \sigma : \tilde{Z} \to L$ also is ramified at $p_2 = \tilde{Z} \cap U_2 \cap E$ with multiplicity three, since $a_{0,3,d-3} \neq 0$. \qed

Let $\nu_d : I_d \to I_d$ be the normalization of the variety $I_d$ and $\nu_{d-1} = h_d \circ \nu_d : I_d \to \mathbb{P}^{K_d}$. The following Proposition is an easy corollary of Lemma 1, Claim 1, and Proposition 4.

**Proposition 5.** Let $a_0 \in \mathfrak{m}_d$ and $z_0$ is the singular point of the curve $C_{a_0}$. Then

(i) the variety $I_d$ is smooth at the point $p = (a_0, z_0)$;
(ii) $\nu_d^{-1}(p) = \{q_1, q_2\}$ consists of two points;
(iii) $h_d$ is ramified along $\nu_d^{-1}(I_d)$ and the local degree $\deg_{q_j} h_d = 3$ for $j = 1, 2$.

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