Smoother than a circle

or

How Noncommutative Geometry provides the torus with an egocentred metric

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Abstract

This is a non-technical version of [10] written as proceedings of the international conference on "Differential Geometry and its Applications" held in Deva, Romania, on October 2005. Published by Cluj university press.

We give an overview on the metric aspect of noncommutative geometry, especially the metric interpretation of gauge fields via the process of fluctuation of the metric. Connes’ distance formula associates to a gauge field on a bundle $P$ equipped with a connection $H$ a metric. When the holonomy is trivial, this distance coincides with the horizontal distance defined by the connection. When the holonomy is non-trivial, the noncommutative distance has rather surprising properties. Specifically, we exhibit an elementary example on a 2-torus in which the noncommutative metric $d$ is somehow more interesting than the horizontal one since $d$ preserves the $S^1$-structure of the fiber and also guarantees the smoothness of the length function at the cut-locus. In this sense the fiber appears as an object ”smoother than a circle”. As a consequence, from the intrinsic metric point of view developed here, any observer whatever his position on the fiber can equally pretend to be ”the center of the world”.

Let us begin with a toy-cosmology tale. Consider the simplest cosmological model, namely a 1-dimensional universe homeomorphic to a circle $S^1$. The main question for cosmologists is to determine the shape of this universe. Does it look like an Euclidean circle, an egg, or some formless potato? Consider one cosmologist, say $O_1$, located somewhere in this universe and assume his main information about the surrounding world consists in the measurement of the distance that separates him from any other points. For simplicity, and because he believes he is the center of the world, $O_1$ has chosen a parameterization $\phi_1 \in [0, 2\pi]$ of $S^1$ such that $\phi_1 = 0$ corresponds to its own position. Hence all that he knows about the world is encoded within the function

$$d_1(\phi_1) = \text{distance between 0 and } \phi_1.$$  

The same is true for another observer $O_2$ located at $\phi_1 \neq 0$. Also believing he is the center of the world, $O_2$ has chosen a parameterization $\phi_2$ such that zero corresponds to his own position. His knowledge is entirely encoded within a function $d_2(\phi_2)$. Now assume that our observers respectively find

$$d_1(\phi_1) = \min(\phi_1, 2\pi - \phi_1), \quad d_2(\phi_2) = \min(\phi_2, 2\pi - \phi_2).$$
They will agree that the universe is indeed a Euclidean circle. But they won’t agree on who is actually the center of the world. Both of them can equally pretend to be at some very particular point $\phi_i = 0$. Then comes a third cosmologist $O_0$, a theoretician, who explains that the quarrel has no physical meaning and is only a matter of parameterization. Everybody knows, $O_0$ says, that the notion of center has no meaning on a Euclidean circle. All points are on the same footing with respect to each other and the only way to recover a notion of ”center” comes from extra-dimensions, for instance by embedding the 1-dimensional universe into a 2-dimensional Euclidean plane (figure 1). Also, one of the $O_i$’s may argue that the universe they see does not look very smooth since neither of the $d_i$ functions is smooth (see figure 2). In fact, from an intrinsic metric point of view, a Euclidean circle is a collection of discontinuities (any point is a cut-locus for a given observer). As physics is keen on smoothness, our cosmologists may be worried and wonder what metric, if any, could make their $S^1$-universe smoother. It would be something close to the Euclidean metric but that avoids the discontinuity of the derivative at $\phi_i = \pi$ (figure 3). For instance would it be possible to measure

\[ d_i(\phi_i) = \sin \frac{\phi_i}{2} \quad (1) \]
and, in this case, what would be the shape of such a smoother-than-a-circle object?

Let us now forget our cosmology tale for a while and ask a more serious - and apparently unrelated - question. In noncommutative geometry the metric information is encoded within the Dirac operator. Specifically, via Connes’ distance formula (see eq.4 below) one is able to recover from purely algebraic data the geodesic distance on a riemannian compact smooth spin manifold $M$,

$$-i\gamma^\mu \partial_\mu \iff \text{Riemannian geodesic distance.}$$

Physics not only deals with spin manifolds but also with gauge theories, that is to say bundles $P$ equipped with a connection $H$ (and an associated 1-form $A_\mu$). Therefore it is quite natural to wonder what distance $d$ is encoded within the covariant Dirac operator,

$$-i\gamma^\mu (\partial_\mu + A_\mu) \iff ?$$

(2)

It turns out that for a very simple example of covariant Dirac operator, the distance $d$ is precisely the one expected by our cosmologists in (1). This example is treated in detail in [10]. We give here a non-technical account of this result.

Before entering into the details let us recall that gauge fields already have a well known metric interpretation in terms of horizontal distance $d_H$. This distance, also called Carnot-Carathéodory or sub-Riemannian distance [12], is by definition the length of the shortest path whose tangent vector is everywhere horizontal with respect to the connection $H$ (see figure 3),

$$d_H(p, q) = \inf_{\dot{c}(t) \in H_{c(t)}P} \int_0^1 \|\dot{c}(t)\| \, dt$$

(3)

where $c(0) = p$, $c(1) = q$ is a smooth curve in $P$. In [4] Connes has pointed out the link between $d_H$ and $d$. In [10] we have examined this link in detail, showing the importance of the holonomy of the connection on this matter. We also worked out an example in which the two distances are not equal. This is this example, related to the toy-cosmology metric (1), that we study in detail below.

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A connection $H$ is the splitting of the tangent space $TP$ into a horizontal subspace (the kernel of the connection 1-form) and a vertical subspace,

$$T_pP = V_pP \oplus H_pP.$$
Figure 4: With shortest horizontal path a helix of radius 1, \( d_H(p_0, p_2) = 4\pi \).

1 The distance formula in noncommutative geometry

Noncommutative geometry aims at understanding the geometry of a space whose algebra of functions (defined on it) is noncommutative. Such objects are well described in terms of spectral triples \((\mathcal{A}, \mathcal{H}, D)\), where \(\mathcal{A}\) is an associative \(*\)-algebra (commutative or not) represented by \(\pi\) on a Hilbert space \(\mathcal{H}\) and \(D\) is an operator on \(\mathcal{H}\). Together with a chirality \(\Gamma\) and a real structure \(J\) also acting on \(\mathcal{H}\), these three elements satisfy a set of 7 conditions providing the necessary and sufficient conditions for

1) an axiomatic definition of Riemannian spin geometry in terms of commutative algebra,

2) its natural extension to the noncommutative framework.

Explicitly, given a spin manifold \(M\) one checks that

\[
(C^\infty(M), L_2(M, S), -i\gamma^\mu \partial_\mu)
\]

is a spectral triple, with \(L_2(M, S)\) the space of square integrable spinors on \(M\). Conversely starting from a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) with \(\mathcal{A}\) the algebra of smooth functions over a compact, Riemannian manifold \(N\), one obtains that \(N\) is indeed a spin manifold with corresponding Dirac operator \(D\) (modulo a torsion term). Moreover the geodesic distance corresponding to the Riemannian structure of \(N\) is given by

\[
d_{geo}(x, y) = \sup_{f \in C^\infty(N)} \{|f(x) - f(y)| / \|[D, f]\| \leq 1\}.
\]  \hspace{1cm} (4)

Extension to the noncommutative framework is obtained by dropping the commutativity of \(\mathcal{A}\). "Points" are recovered as pure states \(\mathcal{P}(\mathcal{A})\) of \(\mathcal{A}\), in analogy with the commutative case where,

\(^b\)To get familiar with this formula, one can study the example \(N = \mathbb{R}, \mathcal{D} = \frac{d}{dx}, \mathcal{H} = L_2(\mathbb{R})\). Then \(\|[D, f]\| = \|f'\| = \sup_{z \in \mathbb{R}} |f'(z)|\) so that \(\|f\| \leq |x - y| = d_{geo}(x, y)\). This upper bound is attained by the function \(z \mapsto z\).

\(^c\)State: positive linear mapping \(\tau\) from \(\mathcal{A}\) to \(\mathbb{C}\). Pure state \(\omega\): state that does not decompose as a convex combination of other states, \(\omega \neq \lambda \tau_1 + (1 - \lambda)\tau_2\).
by Gelfand duality, $\mathcal{P}(C^\infty(M)) \simeq M$ with explicit homeomorphism
\[ x \mapsto \omega_x \text{ such that } \omega_x(f) = f(x) \] (5)
for any $f \in C^\infty(M)$. Formula (4) rewritten as
\[ d(\omega_1, \omega_2) = \sup_{a \in A} \{|\omega_1(a) - \omega_2(a)| / \| [D, a] \| \leq 1 \} \] (6)
defines a distance $d$ between states which
- makes sense whether $A$ is commutative or not,
- is consistent with the classical case, $d = d_{\text{geo}}$, when $A = C^\infty(M)$,
- does not involve notions ill-defined in a quantum context such as paths between points.

In [8] we computed $d$ in an $n$-point space ($A = \mathbb{C}^n$) as well as for other finite dimensional algebras. For instance $A = M_2(\mathbb{C})$ yields a metric on the 2-sphere$^d$. In fact finite dimensional spectral triples are particularly interesting in products of geometries. Namely given a spin manifold $M$ and a spectral triple $(A_I, \mathcal{H}_I, D_I)$, one defines
\[ A = C^\infty(M) \otimes A_I, \mathcal{H} = L^2(M, S) \otimes \mathcal{H}_I, D = -i\gamma^\mu \partial_\mu \otimes 1_I + \gamma^5 \otimes D_I \] (7)
which again is a spectral triple. $\mathcal{P}(A)$ is the set of couples
\[ (\omega_x \in \mathcal{P}(C^\infty(M)), \omega \in \mathcal{P}(A_I)), \]
so that for a finite dimensional $A_I$ the spectral triple describes a geometry which is the product of a discrete space $\mathcal{P}(A_I)$ by a continuous one $\mathcal{P}(M)$. For instance $A_I = \mathbb{C}^2$ yields a two-sheeted model, two copies of $M$ indexed by the pure states of $\mathbb{C}^2$. On each copy the distance coincides with the geodesic distance of $M$ while it remains finite between the sheets, although there is no "path" between them. Note that some metric aspects of sums, rather than products, of spectral triples have been studied very recently$^2$.

2 Fluctuation of the metric

Inspired by the commutative case, $Diff(M) = Aut(C^\infty(M))$, one describes the symmetries of a noncommutative geometry in terms of automorphisms of $A$. The group $Aut(A)$ naturally splits into inner automorphisms,
\[ In(A) : \alpha_u(a) = uu^*a \]
given by unitary elements $(uu^* = 1)$ of $A$, and outer automorphisms
\[ Out(A) = Aut(A)/In(A). \]
The latter have a nice interpretation in quantum field theory as flow of time$^5$. The former are characteristic of the noncommutative case (otherwise $In(A)$ is trivial). Remarkably, the action of $In(A)$ on a geometry $(A, \mathcal{H}, D)$ via the replacement of the representation $\pi$ by $\pi \circ \alpha_u$ is equivalent to replacing $D$ by
\[ D_A \doteq D + A + JA^{-1} \] (8)
where $A \doteq uu^*[D, u^*]$. This appears as a particular instance of the so-called fluctuations of the metric, defined in a more general manner by taking
\[ A = \Sigma a_i [D, b_i] \quad a_i, b_i \in A. \] (9)$^d\mathcal{P}(M_2(\mathbb{C})) = \mathbb{C}P^1$ is in one to one correspondence with $S^2$; see eq. (11) below.
In the case of the product geometry (7), explicit computations yield

\[ A = H - i\gamma^\mu A_\mu \]

where \( H \) is a scalar field on \( M \) with value in \( A_I \) (the Higgs field in the standard model) and \( A_\mu \) is a 1-form field with value in \( \text{Lie}(U(A_I)) \), that is to say a gauge field. Therefore via the fluctuations of the metric both the Higgs field and gauge fields acquire a metric interpretation. In [11] we focused on the Higgs field only, assuming \( A_\mu = 0 \). In [4] Connes considers the example \( H = 0 \) with an internal geometry \( A_I = M_n(\mathbb{C}) \). The vanishing of \( H \) is obtained by taking \( D_I = 0 \) so that the fluctuated Dirac operator

\[ D_A = -i\gamma^\mu(\partial_\mu + A_\mu) \]

is nothing but the usual covariant Dirac operator on the \( U(n) \) trivial bundle

\[ P = M \times \mathbb{C}P^{n-1} \]

with connection 1-form \( A_\mu \). The latest defines both a Carnot-Carathéodory distance \( d_H \) via (8) and a noncommutative distance \( d \) via (6) (with \( D_A \) instead of \( D \) ). It is not difficult to show that whatever \( M \) and \( n \), \( d \leq d_H \). Also \( d = d_H \) as soon as the holonomy of the connection is trivial. However when the holonomy is non trivial \( d \) does not necessarily equals \( d_H \). We examine below a simple example, the 2-torus \( S^1 \times S^1 \), in which the two distances have strongly different characteristics: while \( d_H \) forgets about the bundle structure of \( P \), \( d \) deforms the Euclidean metric on the fiber \( S^1 \) in a rather intriguing way, providing it with the toy-cosmology metric (1).

### 3 An egalitarian and vanity-preserving metric

Take \( M = S^1 \) and \( A_I = M_2(\mathbb{C}) \), that is to say

\[ \mathcal{A} = C^\infty(M) \otimes M_2(\mathbb{C}) \approx C^\infty(M, M_2(\mathbb{C})). \]

\( \mathcal{P}(\mathcal{A}) \) is the trivial bundle on the circle with fiber \( \mathbb{C}P^1 \), mapped to the 2-sphere via the Hopf map

\[ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{C}P^1 \mapsto \begin{pmatrix} x_\xi = 2\text{Re}(\xi_1\xi_2) \\ y_\xi = 2\text{Im}(\xi_1\xi_2) \\ z_\xi = |\xi_1|^2 - |\xi_2|^2 \end{pmatrix} \in S^2. \]

Let us fix a trivialization on \( P \) and write \( \xi_x \) for the point in the fiber over \( x \) corresponding to \( \xi \in \mathbb{C}P^1 \). Take

\[ A_\mu = \begin{pmatrix} 0 & 0 \\ 0 & \theta \end{pmatrix} \]

with \( \theta \in]0,1[ \). Then for any \( \xi_x, \zeta_y \in P \) one easily computes that

\[ d(\xi_x, \zeta_y) = +\infty \]

if and only if \( z_\xi \neq z_\zeta \). Hence the set of pure states at finite noncommutative distance from \( \xi_x \) is a two-torus

\[ T_\xi = M \times S^1. \]

*Note the slight abuse of notation: we canceled out the \( JAJ^{-1} \) term in (8) since it commutes with the representation and so does not play any role in the computation of the distance.*
Let us parameterize the $S^1$ fiber by $\phi \in [0, 2\pi]$ such that
\begin{equation}
0 = \xi_x = \xi_x^0 \tag{12}
\end{equation}
and define $\xi_x^k = 2k\theta\pi$ as the end point of the horizontal lift starting at $\xi_x^{k-1}$ of the basis $S^1$ (see figure 5). Assuming the basis has radius 1, the horizontal distance is
\[ d_H(\xi_x, \xi_x^k) = 2k\pi. \]
In case $\theta$ is irrational, any neighborhood of 0 contains some $\phi_k \doteq 2k\theta\pi \mod [2\pi]$ with $k$ arbitrarily large. Hence, as plotted in figure 6, $d_H$ ”destroys” the $S^1$ structure of the fiber. On the contrary a rather long calculation carried out in [10] shows that
\[ d(0, \phi) = C \sin \frac{\phi}{2} \tag{13} \]
for any $\phi \in [0, 2\pi]$, with
\[ C = \frac{4\pi|\xi_1\xi_2|}{|\sin \theta\pi|} \]
a constant. This is nothing but the metric expected in [1], which is smooth at the cut locus $\phi = \pi$. Hence from the metric point of view the fiber over $x$ equipped with the noncommutative distance $d$ is smoother than a circle.
There still remains the question asked by our cosmologists: what does $S^1$ equipped with $(13)$ look like? In polar coordinates $d(0, \phi)$ is the Euclidean length on the cardioid $\frac{C}{2}(1 + \cos \phi)$

Figure 7: A cardioid.

(see figure 7). So one could be tempted to believe that the fiber is indeed a cardioid. One has to be careful with this interpretation. The noncommutative distance $d$ is invariant under translation on the fiber: the identification of $\xi_x$ with $\phi = 0$ in $(12)$ is arbitrary; identifying $0$ with $\zeta_x$ with $\zeta \neq \xi$ and $z_\xi = z_\zeta$ would lead to a similar result $d(0, \phi) = C \sin \frac{\phi}{2}$. On the contrary the Euclidean distance on the cardioid is not invariant under translation. Therefore, assuming that our observers $O_1, O_2$ are respectively located at $\xi_x$ and $\zeta_x$, each of them measures $d_i(\phi_i) = d(0, \phi_i)$, $i = 1, 2$. Both agree that the fiber they are lying on is a cardioid and both pretend to be localized at this particular point opposite to the ”cusp” of the cardioid (see figure 8). Both are equally right and the quarrel is not just a matter of parameterization as on the Euclidean circle. On a cardioid all points are not on the same footing, the cusp and the point opposite to the cusp in particular have unique properties (they are their own image under the axial symmetry $\sigma$ that leaves the cardioid globally invariant). The very particular position of $O_1$ according to its own point of view (he is his own image by $\sigma$) seems in contradiction with $O_2$’s point of view (for whom $O_1$ is not $\sigma$-invariant). Moreover, extra-dimensions are of no help: measuring $(11)$ is not compatible with some $O_0$’s ”outside” riemannian point of view. In fact the contradiction only comes from the implicit assumption that the fiber is a riemannian manifold. What $(13)$ shows is that the fiber equipped with $d$ is not a riemannian manifold. It is a geometry in which everyone can on the same footing pretend to be the center of the world and in which, at the same time, the notion of center (i.e. of points with particular symmetry properties) still has a meaning. In brief, whereas the price to pay for equality on the Euclidean circle is either to renounce the notion of center or to recover it from an extra-dimension (both options are hard for $O_i$’s own vanity), the noncommutative distance on the fiber of $T_{\xi}$ is both egalitarian and vanity-preserving.

4 Conclusion

As a conclusion let us mention several groups of questions. First, there is still a lot to do to clarify the physical meaning of this ”smoothness from an intrinsic metric point of view”. In particular it would be interesting to check whether the same properties can be observed with a fiber of higher dimension, like the 2 or the 3- sphere. Similarly one should deal with other base-manifolds than $S^1$. As explained in $(10)$, the link between the holonomy of the connection and
the possibility for the noncommutative distance to equal the Carnot-Carathéodory one yields a nice question for sub-Riemannian geometers.\[12\] given a minimal horizontal curve, is it possible to deform it keeping its length fixed and reducing the number of times its projection on $M$ self-intersects? The answer is obviously "no" for $M$ of dimension 1 but seems unknown for dimension greater than 3.

Another open question is to compute the metric when both the scalar part $H$ of the fluctuation and the gauge part $A_\mu$ are different from 0. Finally let us underline that \[10\] was intended to be a preliminary step towards the study of the metric aspect of the noncommutative torus.\[13\] The situation there should be quite similar, except that the pure state space is then a twisted bundle.

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