Information dynamics of price and liquidity around the 2017 Bitcoin markets crash

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We study information dynamics between the largest Bitcoin exchange markets during the bubble in 2017-2018. By analysing high-frequency market-microstructure observables with different information theoretic measures for dynamical systems, we find temporal changes in information sharing across markets. In particular, we study time-varying components of predictability, memory, and (a)synchronous coupling, measured by transfer entropy, active information storage, and multi-information. By comparing these empirical findings with several models we argue that some results could relate to intra-market and inter-market regime shifts, and changes in direction of information flow between different market observables.

Bitcoin is a cryptocurrency that was originally designed as a medium of exchange [1], however there is still no strong consensus as to whether it is a currency, a commodity or an asset [2–4]. Cryptocurrencies are usually transmitted and created via distributed peer-to-peer networks with well-defined cryptographic protocols that record the system’s state via public ledger (blockchain). Thus blockchain (or another type of a ledger that maintains knowledge of distributed consensus) is at the heart of digital currencies. The blockchain technology promises benefits such as proving the existence of an asset as well as keeping track of its current and all past ownerships, both in a distributed manner. It also plays a role in crypto-asset price discovery [5].

Electronic coins, such as Bitcoin, are “chains” of digital signatures: an owner transfers the coin by digitally signing a hash of the previous transaction and the public key of the next owner, adding them to the end of the coin. The exchange of cryptocurrencies to fiat money (USD, EUR, GBP, etc.) occurs in cryptocurrency exchange markets which are based on electronic double auctions, operating using limit order books. Here, real-time market data, such as transaction volume, amount of bid and ask orders, and exogenous information, such as news and social media mentions, are sources of information that feed into trading decisions. Within the crypto-market ecosystem, two distinguishable types of information sources exist: intra-market sources and inter-market sources. The first source is related to internal dynamics of a market’s limit order book, e.g., its liquidity and volatility, that may influence trading decisions [6] as well as one another [7]. The second source is related to communication between different markets. Therefore, any two crypto-markets are “connected”: explicitly, when there is a mutually traded currency (or an arbitrage opportunity), or implicitly, when prices of currencies amongst several markets are correlated.

The blockchain ensures that, in the long run, price of a cryptocurrency develops synchronously across exchanges, i.e., the law of one price holds [5]. However, even without the presence of blockchain, one would expect price synchronisation across different venues due to arbitrage [8]. Hence, Bitcoin price is processed collectively, in a distributed manner. Its dynamics is affected both by internal feedback and by exogenous information (e.g., public news), as illustrated in Fig. [1]. It is not clear, however, what type of information is the most relevant for the price formation process. Often, these multiple influences lead to very large fluctuations (high volatility in technical terms) of exchange rates between cryptocurrencies and fiat money, price bubbles, and sudden price crashes. These properties make cryptocurrencies exchange rates a unique laboratory to empirically study the collective dynamics leading to market instabilities, which are well known to be ubiquitous in all financial markets.

Although at low frequencies, the prices of Bitcoin in different markets (and, oftentimes, prices of other cryptocurrencies in relation to Bitcoin) develop in apparent synchrony, the law of one price does not hold at very high frequencies [8, 9]. Furthermore, at sufficiently high frequency, one observes lagged relationships between the prices of the asset(s), i.e., past information about the change in price of one cryptocurrency is informative for predicting the future price change of another cryptocurrency. Such mechanisms are detectable with information transfer measures: linear relationships can be detected via Granger causality [10], while more general nonlinear dependencies can be detected via transfer entropy [11, 12], that is one of constituents of a process’s entropy. Entropy itself relates to computation: at each point in time, computation of the next state of the process. When a system is composed of multiple interacting units, information transfer could be thought of as communication or signalling, and the system as a whole as processing information to determine its collective behaviour at each time step via distributed computation. A study of information dynamics aims to decompose this computation into unique elements, namely, transferred, stored, and modified information, and their changes in space and time [13, 14].

Spatio-temporal patterns of information dynamics (in particular, information transfer and multi-information) within the system of interacting markets was observed to increase in the financial crisis periods, meaning that the system appears to be
In order to study the price and liquidity dynamics in the different venues, we introduce three market microstructure variables.
**II. METHODS**

**A. Information dynamics for stochastic processes**

Consider a system, composed of $N$ stochastic processes $\{X^\alpha_t\}_{\alpha \in \Omega}$, together forming a multivariate process $X^\Omega$. Each stochastic process $X^\alpha_t$ is a collection of random variables $\{X^\alpha_{t_k}\}_{k \in \mathbb{N}^+}$ with $Q$ the total number of observations. For each random variable $X_t$, its realised value is defined as $x_t$. Note that $\Omega$ denotes the finite set of markets in this paper. In the definitions of information dynamics measures, we will use subscripts $X$, $Y$ to refer to information dynamics amongst some two stochastic processes (e.g. $X = X^\alpha$, $Y = X^\beta$). These information dynamics measures will be defined using length-$l$ and length-$k$ collections of random variables, for each time $t$: $Y^{[ij]}_{t-\delta} = \{Y_{t-\delta-t+1}, \ldots, Y_{t-\delta}\}$, $X^{[k]}_{t-1} = \{X_{t-1-k}, \ldots, X_{t-1}\}$, where $\delta$ is a time delay.

**a. Information-theoretic measures for static variables**

The fundamental quantity in this work is Shannon entropy of a random variable $X$, defined as $H(X) = -\sum_{x} p(x) \log p(x)$. Here $x$ is an instance of a random variable, and the sum is over all possible values that $x$ can take. Conditional entropy of $X$ given $Y$ is the average uncertainty that remains about $X$ after learning the values of $Y$: $H(X|Y) = -\sum_{x,y} p(x,y) \log p(x|y)$.

Mutual information between $X$ and $Y$ measures the average amount of information that is communicated in one random variable about another. $I(X,Y) = H(X) - H(X|Y)$. The conditional mutual information between $X$ and $Y$ when $Z$ is known is defined as $I(X,Y|Z) = H(X|Z) - H(X|Y,Z)$.

These information-theoretic measures for static variables can also be adapted to analyse stochastic processes. When temporal information is incorporated, one can quantify: how much information is shared between system’s units at each point in time; how much information about the current state of $X$ is conveyed in the past states of $Y$; how much information about the current state of $X$ is conveyed in the past states of $Y$. These questions can be addressed with information dynamics measures [13, 14] that are summarised below.

**b. Multi-information** Multi-information (MI) is defined as a measure of the deviation from independence of the components in the system [14, 28]:

$$ I_X^\Omega = \left( \sum_{\alpha=1}^N H(X^\alpha_t) \right) - H(X^\Omega_t). $$

Large value of MI is a signature of high synchronous interconnectivity of a system.
c. **Transfer entropy**  Transfer entropy (TE) encapsulates the “distributed nature” of computation. Schreiber [11], and, independently, Paluš et al. [12], defined TE as the amount of information that a source process \( Y \), and, in particular, its past state \( Y_{t-\delta} \) provides about a target’s state \( X_t \) in the context of the target’s immediate past state \( X_{t-1}^{(k)} \) [13]:

\[
T_{Y \rightarrow X}^{(k, l, \delta)} = I(Y_{t-\delta}; X_t | X_{t-1}^{(k)}) = H(X_t | X_{t-1}^{(k)}) - H(X_t | X_{t-1}^{(k)}, Y_{t-\delta}).
\]

This formulation of information transfer quantifies pairwise relationships between variables. When a system is composed of more than two stochastic processes, (3) is known as **apparent transfer entropy**, as it does not account unobserved parent transfer entropy, potentially leading to over-estimation of the total amount of entropy transferred within the system.

d. **Conditional transfer entropy**  To discount redundant joint influences of two sources \( Y, Z \) on a target \( X \), (3) is generalised to a measure of transfer entropy from \( Y \) to \( X \) given that \( Z \) can also provide information about \( X \):

\[
T_{Y \rightarrow X|Z}^{(k, l, m, \delta)} = I(Y_{t-\delta}; X_t | X_{t-1}^{(k)}, Z_{t-1}^{(m)}) = H(X_t | X_{t-1}^{(k)}, Z_{t-1}^{(m)}) - H(X_t | X_{t-1}^{(k)}, Y_{t-\delta}, Z_{t-1}^{(m)}).
\]

Finally, the conditional TE can be generalised to account for a multivariate set of processes with the next definition.

e. **Collective transfer entropy**  To find the total amount of information that was collectively transferred to a target \( X \) from all potential sources in the system, we use collective TE, \( T_X \), which accounts for redundancies and synergies among information that sources provide to a target [14]. To compute \( T_X \), we consider a set \( X^\Omega \). The true effect of one source variable \( Y \) to the target variable \( X \) in the universe of \( X^\Omega \) is computed through conditional transfer entropy:

\[ T_{Y \rightarrow X|X^\Omega \setminus \{Y\}} \]

where \( X^\Omega \setminus \{X, Y\} \) = \{\( Z \in X^\Omega \setminus \{X, Y\} \)\}. However, \( T_{Y \rightarrow X|X^\Omega \setminus \{Y\}} \) would not be equal to the total amount of information transferred to \( X \), as only unique information [29] would be accounted for. To calculate the collective TE (see [14] for more details), we sum incrementally collective TE terms. Let us consider an ordered set \( X^\Omega_{X^\Omega} = \{Z \in X^\Omega \setminus \{X, Y\}\} \): \( Z^1, ..., Z^\beta, ..., Z^{N-1} \) and its subset \( X^\Omega_{X^\Omega}^{\alpha, \beta} = \{Z^\alpha \in X^\Omega_{X^\Omega} | \alpha \leq \beta \} \). Collective transfer entropy is then defined as

\[
T_X^{(k)} = \sum_{\beta=1}^{N-1} T_{Y^{\beta} \rightarrow X | X_{X^\Omega}^{\alpha, \beta-1}}.
\]

f. **Active information storage**  Overall, the predictability of the next state of a process \( X \) is characterised by its entropy, whose non-overlapping constituents are the collective transfer entropy \( T_X^{(k)} \), the active information storage (AIS), \( A_X^{(k)} \), defined as memory of the process that is actively in use [14]:

\[
A_X^{(k)} = I(X_{t-1}^{(k)}, X_t) = H(X_t) - H(X_t | X_{t-1}^{(k)}).
\]

AIS is information in the past that contributes to the computation in the next state of \( X \).

g. **Local information dynamics**  The measures defined above can also be considered in a point-wise fashion, as they are expected values of local measures at each observation at time \( t \) as follows [14, 18]:

\[
I_X = (i_X(t))_t = \frac{1}{Q} \sum_{\tau=1}^{Q} \log \frac{p(x_\tau^1, x_\tau^2, ..., x_\tau^N)}{\prod_{\tau=1}^{Q} p(x_\tau^1)},
\]

\[
T_{Y \rightarrow X}^{(k, l, \delta)} = (t_{Y \rightarrow X}^{(k, l, \delta)})_t = \frac{1}{Q} \sum_{\tau=1}^{Q} \log \frac{p(x_{\tau+1}|x_{\tau}^{(k)}), x_{\tau}^{(l)})}{p(x_{\tau+1}|x_{\tau}^{(k)})},
\]

\[
A_X^{(k)} = (a_X^{(k)})_t = \frac{1}{Q} \sum_{\tau=1}^{Q} \log \frac{p(x_{\tau+1}, x_{\tau}^{(k)})}{p(x_{\tau+1})p(x_{\tau}^{(k)})}.
\]

B. **Experimental setup**

a. **Information dynamics**  We use Java Information Dynamics Toolkit – JIDT [30] to compute all information dynamics quantities. For the TE estimator, we choose Kraskov, Stögbauer, Grassberger K-nearest neighbour estimator (KSG) [31], which is minimally parameterised and has been shown to be robust for a wide range of data. To ensure our estimates are reliable, we studied sample-size bias of transfer entropy using the approach of [32] as described in Appendix C We found that \( K = 4 \) for nearest neighbours estimator produces reliable results for our dataset. All values of information dynamics are reported in units of nats.

Throughout the paper, we considered the case where \( l = 1 \). To ensure that we do not over-estimate information transfer for active information storage—the memory of a process in use—of \( X \), we chose \( k = \text{argmax}(A_X^{(k)}) \) for results in Section III A and \( k = \text{argmax}(A_X^{(k)}) \) for results in Section III B.

We then used this \( k \) when computing \( T_{Y \rightarrow X} \) as well as \( T_X \). Lastly, as was demonstrated in [33], when two processes are coupled via non-zero delay \( u \), \( T_{Y \rightarrow X}^{(l)} \) is maximised for \( \delta = u \). Therefore, we consider \( T_{Y \rightarrow X} \) for \( \delta \in [1, 10] \), and select \( \delta \) for which \( T_{Y \rightarrow X}^{(l)} \) is maximal. Such procedure ensures that non-instantaneous coupling across markets is also captured. For the results in Section III B, we do not do this and consider a fixed \( \delta = 1 \).

b. **Rolling windows**  For results discussed in Section III A we considered weekly subsets of the time series, moving each sliding window by three days with respect to the previous one. In such a way we achieve a twofold advantage. First, we reveal temporal patterns in information dynamics. Secondly, we ensure that the data is locally stationary (we expect the time series associated with markets around the crash to be globally non-stationary, however, at sufficiently small time windows intervals we observe stationary time series). In Section III B we considered two non-overlapping time series windows; the data preceding the price crash date (December 17th, 2017) and the remainder.

c. **TE Significance test**  All reported results are significant, where significance is tested against the null hypothesis
that no information transfer exists. For all results, we used 100 surrogate time series to evaluate significance, and, unless otherwise stated, we chose the significance level of 0.05. For the results of Section III A, in each time window a Benjamini–Yekutieli procedure [34] was also performed.

d. Stationarity test For each observable’s time series in all rolling windows, we considered the Augmented Dickey-Fuller (ADF) test [35] whose null hypothesis is a presence of a unit root, and we use significance level of 0.05. We found that we can reject the null hypothesis that there is a unit root with a 5% of significance level for the test statistics for all of the time windows of s, θ time series. Contrary, we found the presence of unit-roots in price time series, so we used price returns (i.e. equal to applying difference operation \( r_t = p_t - p_{t-1} \)), after which all time windows passed the ADF test.

C. Statistics of information dynamics

In the data analysis part, we will consider information dynamics, described in Section II first using one of the observables for each market. In doing so, we assume that the strongest coupling occurs through the same observable (spread in \( \alpha \) has a stronger effect on spread in \( \beta \) than on price returns in \( \beta \)), see Fig. 3 for an illustration. Secondly, we also consider the extent to which market variables are internally coupled via different observables (e.g., quantifying how much spread in \( \alpha \) has an effect on price returns in \( \alpha \)). See Fig. 4 for an illustration. To interpret the results, we aggregate information dynamics measures into several spatio-temporal global and local metrics.

For each information dynamics metric, namely, MI, AIS, and TE, the argument \( w \) denotes the time series window, based on which the reported values are computed. We will also denote one microstructure observable (e.g., \( s \)), with the same capital letter that defines a random variable for a market, e.g., \( T_{X_a \rightarrow Y_b} \) denotes transfer entropy from a market \( \alpha \) to a market \( \beta \) considering the same microstructure observable \( X \), while \( T_{X_a \rightarrow Y_b} \) would indicate that different microstructure observables are considered.

a. Global metrics To quantify the global, system-describing information dynamics, we aggregate results obtained for all markets within a time series window. In particular, in Section III A, we study the extent to which a microstructural variable in one market affects the same microstructural variable in another market. Therefore, we consider the extent of this effect is captured with a sum of transfer entropies (5) between all pairs of markets, namely, the total apparent transfer entropy:

\[
T^\text{app.sys}_X = \sum_{\alpha, \beta | \alpha \neq \beta} T_{X_a \rightarrow Y_b}
\]  

(8)

and the total collective transfer entropy

\[
T^\text{coll.sys}_X = \sum_{\alpha} T_{X_a}
\]  

(9)

These two metrics reflect on the total amount of information transfer within the system.

Similarly, we also study markets’ synchronisation within one variable via the multi-information: in the definition (2), we consider \( X^{\Omega} \) as a union of random variables that describe a single market observable in \( N \) markets, denoting it as \( T^\text{sys}_X \) in the further discussion.

Lastly, we report the average active information storage, \( A^\text{sys}_X \), defined as

\[
A^\text{sys}_X = \frac{1}{N} \sum_{\alpha=1}^{N} A_{X_\alpha} = \langle A_{X_\alpha} \rangle_{\alpha},
\]  

(10)

i.e., it is the average AIS per market, at each time series window.

b. Market-specific statistics To analyse the market-specific information dynamics measures, we will consider averages of a measure obtained from different time windows \( w \in [1, W] \). The amount of information transfer received by each individual market \( \alpha \), termed an average collective transfer entropy is defined as

\[
T^\text{coll}_X(\alpha) = \frac{1}{W} \sum_{w=1}^{W} T_{X_\alpha}(w) = \langle T_{X_\alpha} \rangle_w,
\]  

(11)

and an individual market’s average AIS is defined as

\[
A_{X_\alpha} = \frac{1}{W} \sum_{w=1}^{W} A_{X\alpha}(w) = \langle A_{X_\alpha} \rangle_w.
\]  

(12)

c. Comparison between inter- and intra- market connectivity In Section III B, we consider the amount of information transfer between microstructure observables within one market. Therefore, we compute the apparent information transfer of the form \( T_{X_a \rightarrow Y_a} \), and contrast the amount of information transfer within the market to the amount of information sharing across markets. To make a fair comparison, we normalise the average values of TE per possible link in a transfer entropy network where nodes are market observables.

For the comparative analysis of inter- and intra- market interactions, consider a directed graph that consists of three nodes that represent our observables. We map information transfer results onto this triadic graph, in which multiple edges are allowed between node pairs. Furthermore, each node is allowed to have multiple self-loops. The directed multi-graph representation depicts information transfer amongst different observables in a market (a somewhat endogenous information flow). In opposition to this type of internal information sharing, we also allow for inter-market connectivity, as discussed in Section III A (this can be thought of as exogenous information flow). Such links are represented as loops in the directed multi-graph, as information is transferred from an observable in one market to the same observable in another. By appropriate edge averaging, we will turn this directed multi-graph into a directed graph that depicts averaged information transfer.

Each loop represents a significant transfer entropy link from one market to another, therefore, at most, there can be \( N(N-1) \) self-loops for each node-observable, since each market can have \( 2(N-1) \) TE links (\( N-1 \) incoming and \( N-1 \) outgoing).
We calculate the averaged TE for an observable \( X \) between all pairs of markets \( \alpha, \beta \) as:

\[
\omega_X^{\text{self}} = \langle T_{X,\alpha \rightarrow X,\beta} \rangle_{\alpha,\beta} = \sum_{\alpha=1}^{N} \sum_{\beta=1 \atop \beta \neq \alpha}^{N} T_{X,\alpha \rightarrow X,\beta}.
\]  (13)

We also compute the average strength of incoming links, \( \omega_X^{\text{in}} \), and outgoing links \( \omega_X^{\text{out}} \):

\[
\omega_X^{\text{in}} = \langle T_{Y,\alpha \rightarrow X,\alpha} \rangle_{\alpha,Y} = \frac{1}{N(N-1)} \sum_{Y \neq X} \sum_{\alpha=1}^{N} T_{Y,\alpha \rightarrow X,\alpha}
\]  (14)

\[
\omega_X^{\text{out}} = \langle T_{X,\alpha \rightarrow Y,\alpha} \rangle_{\alpha,Y} = \frac{1}{N(N-1)} \sum_{Y \neq X} \sum_{\alpha=1}^{N} T_{X,\alpha \rightarrow Y,\alpha}
\]  (15)

Here \( X \) represents one type of observable, e.g., \( s \), and \( Y \) represents a different observable, e.g., \( r \).

D. Models of non-linear information transfer in complex systems

Here we define two case models of stochastic processes coupled via lagged non-linear interaction. By analysing the information-theoretic, spatio-temporal signatures of these models, we will be able to explain in the information dynamics trends observed using real market microstructure data.

i. Vector auto-regressive model of regime shifts

Let us first consider two coupled auto-regressive processes:

\[
X_t = \alpha_1 X_{t-1} + \beta_1 \epsilon_t + K(t) \epsilon_t
\]

\[
Y_t = \alpha_2 Y_{t-1} + \beta_2 \epsilon_{2,t} + K(t) \epsilon_t + C(t) |X_{t-1}|^d,
\]  (16)

where \( \epsilon_t \) and \( \epsilon_{2,t} \) are independent white noise processes, and \( \alpha, \beta, K, C, \) and \( d \) are parameters to be estimated. This model allows for a time-delayed directional coupling \( X \rightarrow Y \), whose time-varying strength is \( C(t) \). In addition to this, let us assume the presence of a common, time-varying hidden driver
whose strength is $K(t)$. $\epsilon$ terms denote independent Gaussian noise. Here $d$ is a constant, that we use to vary the linearity of the causal link from $X$ to $Y$. Note that the hidden driver term $K(t)\epsilon_t$ is present in both variables without any delay. This model is useful to analyse causal interaction amongst system’s sub-units when they are “of the same type”, i.e., representing the same market observable in some pair of markets.

We consider several potential systemic changes, and local information dynamics signatures associated with them. A regime shift itself is defined as a significant change in the strength of a certain type of coupling between two variables, observed at a certain point in time $t_c$. In particular, here we consider a time-varying coupling strength $f$ modelled via logistic function

$$f(t) = \frac{s}{1 + e^{b(t-t_c)}} + C.$$  \hspace{1cm} (17)

Here $s$ relates to the maximum value of the function, and $b$ to the sharpness of the transition. For example, a regime shift of a causal driver may be modelled as a change of a coupling strength term $C(t)$.

As Fig. 5(a) shows, in case of a regime shift in a causal driver, we observe a significant change in the absolute values of local information dynamical measures. This situation is characterised by large TE and large AIS in high coupling regime and no significant change in MI. In the Appendix, we also show that a change in MI is also possible, when $K(t)$’s absolute value is large: its value is high when AIS and TE values are low and vice versa.

Similarly the case of a regime shift of a hidden (common, simultaneous) driver can be modelled via variation in the coupling strength $K(t)$. The result, shown in Fig. 5(b) suggests that the change in a hidden driver’s strength is signified by high MI when coupling is strong, and vice versa when coupling is weak. Note that changes in AIS and TE are possible: the former is possible when $K(t)$ is sufficiently high-valued, while the latter is possible when $C(t)$ is high-valued.

Lastly, we studied a case when $\beta_1 = \beta_2 = \beta(t)$, defined using (17), i.e., the system’s overall uncertainty is a time functional, while $C(t) = K(t) = 0.3$ =const. In this case, we find that all three information dynamic measures can be mutually low-valued the side of transition that models high-uncertainty regime, and mutually low-valued on the other, where the system’s intrinsic uncertainty is low, see Fig. 6(c).

In the Appendix 5 we also show that the results are robust for a wide range of parameter values $s,b,C$. Although the analysis ensures that the results shown in Fig. 5 are not coincidental, further statistical proof is needed to prove that signatures are persistent in a general spectrum of nonlinearly coupled autoregressive systems. We also note that it is more than possible that a combination of these effects comes into play in real data, and such convolution and its signature is not studied any further in the current work.

b. GARCH model for price returns and spread Next, we analyse a model of coupling between different market microstructure variables, using a variant of generalised autoregressive conditional heteroskedasticity. Our model considers possibly bidirectional coupling between price returns $r$ and spread $s$ when spread is coupled to volatility (variance) of price. Therefore we have coupled time series of the form:

$$r_t = \sigma_t \epsilon_{1,t}, \quad \epsilon_{1,t} \sim \mathcal{N}(0,1)$$

$$\sigma_t^2 = w + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma \epsilon_{t-1}^2,$$

$$s_t^2 = \alpha s_{t-1}^2 + \beta \sigma_{t-1}^2 + c \epsilon_{2,t}, \quad \epsilon_{2,t} \sim \mathcal{N}(0,1).$$

Compared to the standard GARCH(1,1) [32] (where “(1,1)” indicates that there is one variance dependence term, and the second indicates that there is one ARCH term $\epsilon$, $w$ denotes constant base level of volatility, $\alpha, \beta$ determine the influence of past squared returns and past volatility), we have several additional parameters, $a,b,c,\gamma$. The new parameters have the following economic interpretation: $a$ is the (bare, i.e., not volatility driven) persistence of the spread, which is known to be strongly autocorrelated (especially for small tick stocks) [27]. $c$ measures the dispersion of spread and $\epsilon_t$ is the associated innovation noise; $b$ links past volatility to future spread. It might be connected with asymmetric information, since market makers adjust spread according to volatility (see, for example [7]); $\gamma$ links past spread with future volatility. If the spread is large, it is likely that the limit order book is sparse (as empirically shown in [38]). But a sparse, i.e., illiquid, book leads to a more volatile price, since any order can create a large price change [39].

Note that in this formulation, a link from price to spread is indirect, and in fact price returns affect spread at the two-step delay. Therefore in the following figures we will scan through $\delta \in [1,2]$ and use the one for which the observed TE is larger. Depending on parameter values $b, \gamma$ we can have bi- or uni-directional couplings $T_{r \rightarrow s}$ or $T_{s \rightarrow r}$ each data point in the proceeding results is considered at a significance level of 0.01, and results are obtained from 100 independent simulations.

Parameters $\alpha, \beta, a, b, c, w, \gamma$ affect stationarity and damping in the system therefore we need to choose such parameter values that system would be stationary throughout. To find a stationary combination of the parameters, we perform a moment analysis of the two processes (see Appendix E for more details). In Fig. 6 we show that an expected links from spread to returns (top) as well as from returns to spread (bottom) are detectable using KSG estimator for TE, whereas Gaussian estimator cannot detect the non-linearly coupled GARCH variables. We also note that at small time series lengths, spurious bidirectional coupling may be observed, and the observed coupling strength converges to a true one when the time series are sufficiently long.

Since for stationary time series transfer entropy is an expectation value of the logarithm of ratios of conditional probability density functions, defined as per (7), in GARCH model [15], these probability density functions can be estimated using numerical integration and samples of a joint process $\{r_t, s_t\}$. In case of causal link from spread to returns, TE is defined as an expectation value of $\log \frac{p(r_t | r_{t-1}, s_{t-1}, \sigma_{t-1})}{p(r_t | r_{t-1})}$. Here we have an exact probabilistic model, namely

$$p(r_t = x | r_{t-1}, s_{t-1}, \sigma_{t-1}) = \mathcal{N}(x; 0, w + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma \epsilon_{t-1}^2),$$

$$p(r_t | r_{t-1}) = \mathcal{N}(r_t; 0, w).$$
where $N(x; \mu = 0, \sigma^2 = w + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma s_{t-1}^2)$ is a Normal probability density function parameterised with $\mu, \sigma$. Similarly, one can obtain the analytical expression for $p(r_t | r_{t-1})$ via marginalisation. Similarly, transfer entropy from returns to spread is the expectation value of $\log \frac{f(r_t | s_{t-1}, r_{t-1})}{f(r_t | s_{t-1})}$ in Appendix E we show that the denominator, denoted by $\phi(s_t = y)$ can be written as follows

$$\phi(y) = \frac{2y}{\Gamma(1/2) \sqrt{2\pi c^2}} \exp \left(-\frac{y^2 - c^2}{2c^2}\right) 1_{[\sqrt{c^2}]}(y).$$

Here $c_t^2 = a_t^2 + b(w + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma s_{t-1}^2)$ and $1_{[\sqrt{c^2}]}$ stands for an indicator function defined over a subset of real numbers, $\mathcal{A}$. Again, $f(s_t | s_{t-1}, r_{t-2})$ can be obtained by marginalisation.

In the Appendix E we show that the values of information transfer that we obtained empirically and report in Fig. 6 are in agreement with the transfer entropy obtained using numerical integration.

III. EMPIRICAL RESULTS

In this section, we report results of information sharing across markets when a single observable is considered, and when information communication between different observables is taken into account.

A. Inter-market information dynamics for a single observable

Firstly, we consider information dynamics individually for each of the observables: $r, s, \Theta$. Here we look at the following: the total apparent transfer entropy, $T^{app,sys,}\Gamma_{\Theta}^{app,sys}, T^{app,sys,}\Gamma_{\Theta}^{sys}$, multi-information, $I^\Theta_{sys}, I^\Theta_{sys}, I^\Theta_{sys}$, and average active information storage, $A^{sys}_{\Gamma_{sys}}, A^{sys}_{\Theta}, A^{sys}_{\Theta}$. When constructing each of these measures, only significant information dynamics values are considered (the significance is discussed in Section II B). The results obtained for each sliding window are reported in Fig. 6. Top figures show the total apparent transfer entropy as well as multi-information; bottom figures report average active information storage. Overall, we observe that week by week, there were large fluctuations in the amount of information sharing and synchronisation across markets. The patterns of information dynamics for each microstructural variable differ, therefore we discuss them individually.

a. Spread The temporal evolution of information processing in spread suggests that a ‘critical point’ exists nearing price peak and the peak in spread itself. Such point in time is thought to be characterised by a peak in MI amongst agents that constitute the system [40-42]. Authors of [20] claim that stock market crashes exhibit a peak in MI at the point in time when one would expect a significant regime change to take place.
FIG. 7: Top: Multi-information $I^{\text{sys}}$ (denoted with red triangles) and the total apparent transfer entropy $T^{\text{app,sys}}$ (denoted with black circles) across all pairs of markets using the following time series (from left to right): price returns, order imbalance, and spread. The blue dashed vertical lines indicate “critical points”: the date of the peak value obtained by each variable is indicated by green dotted line; the red lines indicate the peak of the bubble, and the blue dashed lines highlight the dates at which multi-information was maximal. Bottom: average active information storage $A^{\text{sys}}$.

In information dynamics amongst spread variables we see a large amount of total apparent transfer entropy, as well as large average AIS, persisting for weeks preceding the price crash. This is in line with a situation simulated in Fig. 5(a), therefore we conclude that spread system variables transit from a strong-coupling regime before the crash to a weak-coupling regime after the crash. Our intuition is further supported by similar results observed by considering $T^{\text{coll,sys}}$, shown in top left of Fig. 8. In the Tab. 8 we report the average collective transfer entropy for the time windows before and after the price crash. As expected, the absolute values of the total collective transfer entropy $T^{\text{coll,sys}}$ are much smaller than the values of apparent transfer entropy. However, we still observe a similar shift from a strong-coupling regime to a weak-coupling regime, centered around the crash date.

In Tab 8 we analyse AIS and total collective TE for individual markets in the two market regimes (leading up to and following the price crash). The averages are calculated from either the observed AIS values in the windows that precede the crash or follow it. Markets with small $A_s$ are, on average, “information absorbers” (e.g., Gemini, Coinbase): their order book spreads are responsive to either exogenous information sources or inter-market dynamics, or are generally intrinsically unpredictable. Other markets, such as BTC-e use a lot of information from their memory to compute the next market spread. Lastly, Fig. 8 shows that those markets that were shown to have large $T_s$ also are those with the largest $A_s$. On the other hand, spread in markets such as Gemini, Coinbase, Bitfinex, are driven by exogenous information as both $T^{\text{coll}}$ and $A_s$ is small. Therefore, given information available from market microstructure observables, they are the least predictable. Tab. 8 shows that Bitfinex, Gemini, and Coinbase are exchanges that experienced the smallest influx of information from other exchanges prior to crash. We also observed that these markets have small average AIS, suggesting that they are “independent” from other exchanges and could primarily be driven by exogenous information sources. On the contrary, Binance, Poloniex absorb the largest amount of information from other markets. They were also found to have large average AIS. Such markets could be labelled as “information absorbers”.

b. Price returns and order imbalance Fig. 7 shows that information dynamics patterns for $\theta, r$ are different from those of $s$. Firstly, for both observables, temporal evolution
of MI is similar: somewhat symmetric around the crash date, minimal at the crash date, and fluctuating from high MI to low MI with a period of around 2-3 weeks. These observations suggest that similar processes were driving the system at either side of the price bubble. The peak of the bubble is characterised by a clear, although not dramatic, reduction in both TE and MI, and to a lesser extent, reduction in AIS. We observe that for $\alpha$ (and to a smaller extent, for $r$), all three measures—AIS, TE, MI—follow the same pattern, that is, when TE is large, AIS, MI are also large. We were able to simulate this scenario in Fig. 5(c) by changing the autoregressive parameters responsible for the amount of uncertainty in the signals. When the amount of noise, unique for each system’s constituents, is varied, the three information dynamics measures either increase or decrease in magnitude simultaneously as a response, being high-valued when the uncertainty is low. The minimal values of information dynamical quantities at the peak of the bubble suggest that the system was unpredictable, chaotic and not synchronised.

B. Information flow between market microstructure observables

Here we consider information sharing between different market microstructure observables and compare it to intermarket information sharing. We study two non-overlapping time series windows: the time series preceding the price crash date (2017-11-01 to 2017-12-17), and the remainder (2017-12-17 to 2018-01-31).

We report the results in Tab. 9. Altogether, they suggest that the system’s connectivity as a whole moved from a high-coupling regime to a low-coupling regime with one significant exception for price returns. Namely, for price returns, where we see $\omega_{\text{self}}$ larger than $\omega_{\text{in}}, \omega_{\text{out}}$ prior to crash date is smaller than after the crash. Lastly, we also observe a larger average AIS for returns prior to the crash. Furthermore, $\omega_{\text{in}}, \omega_{\text{out}}$ is of the same order as $\omega_{\text{self}}$, suggesting that internal market dynamics played a significant role in price movement in the period.

For the other two observables, we observe that both inter- and intra- market connectivity becomes smaller after the crash. For spread, we observe that a relatively large inter-market connectivity drops by an order of magnitude after the crash.
In this paper we studied patterns of information dynamics in the Bitcoin system, describing it with market microstructure observables: price returns, order imbalance, and spread—three measures that quantify the state of market makers and market takers. Our analysis contrasted micro-level information processing within important Bitcoin trading venues during two distinguishable states: while price was, on average, ascending, and during the period in which price was, on average, dropping. We found persisting intra-market connectivity at the minute level frequency. Furthermore, we found that at high frequency, markets are interconnected via order book spread, and the interconnectivity is enhanced prior to price crash, suggesting a potential regime shift. On the contrary, the inter-market connectivity via price returns and order imbalance appeared symmetric around the price crash. We also observed the dip in all information dynamic measures at the time of the crash, suggesting that the system was asynchronous and unpredictable. We also observed information flow between different market observables. By contrasting the two market regimes, we found that the system shifted from a strongly interconnected state to a sparsely connected one.

To supplement these findings, we analysed several simulated econometric models. With the model of auto-regressive non-linearly coupled variables that have time-varying drivers, we found several simplified mechanism that could explain the empirical observations. With the generalised auto-regressive conditional heteroskedasticity model (GARCH) coupled with spread dynamics, we studied a link between returns and spread, as well as potential of detecting it using means of information transfer. Our findings suggest that different types of regime changes can be distinguished when a collection of information dynamics measures is used together.

All in all, our findings suggest that prior to a price crash, Bitcoin system was in a strongly coupled state. In particular, this is clear when liquidity marker—spread—is considered. The drop in all forms of information at the point at which the price is maximal suggests a reduction in predictability of the system. It may also indicate that such a system is susceptible to various types of perturbations, however we leave such hypothesis for further research. We also make a note that the results of Fig. (right column) for \( s \) are reminiscent of a second order phase transition \([40][41]\) and could be a predictor of a financial crash \([42]\). Of course other explanations are possible for this empirical observation and future studies could further clarify its microscopic origin.

Our findings, although interesting, have limited explanatory capacity due to the granularity of order-book data, presence of noise and effects due to unobserved (exogenous) system drivers. We assumed that the minute-level snapshots of the limit order book data are sufficiently high-frequency to reveal true interactions across and within markets. Of course, this data is an approximation of an, in reality, practically continuous-time dynamics of the markets. Finally, since we limited our analysis to Bitcoin traded against USD(T), analysis of larger market microstructure datasets, consisting of a bigger variety of currency pairs and forming larger informa-

**FIG. 9 & TABLE II:** Top: Directed multi-graphs of information flow between the three market observables before the price crash (left) and after the price crash (right). Saturation of edges indicates the strength of information transfer. Self-loops represent transfer entropy from one market to another when considering the same observable. Bottom: Amount of apparent information transfer between market microstructure observables per possible link. The last column shows an average active information storage. The first part of the table reports results at 5-min level frequency of re-sampling, whereas the second one is for a minute level frequency of re-sampling.
tion transfer networks would undoubtedly provide richer insights as to why we observe increased market co-integration at the time of a price bubble. Lastly, it would be important to repeat our analysis using other similar datasets, concerning market microstructure during price bubbles. Observing similar results would suggest that information dynamics framework may be a generally applicable tool for early warning signals of financial crashes.

AUTHOR CONTRIBUTIONS

All authors designed the research. V.V. conducted experiments, analysed the data, and derived mathematical models. F.L. and N.A.F. derived mathematical models and supervised the work. All authors interpreted the results and wrote the manuscript.

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DATA AVAILABILITY STATEMENT

The data that support the analysis of the Section III A and Appendix D of this study have been synthetically generated by the authors and the results can be reproduced using the equations and parameters described in the article.

Appendix A: Data alignment

As stated in the main text, the limit order books are snapshots sampled at approximately minute level frequency. However, the exact time of a snapshot is not the zeroth second of a minute, the intervals between consecutive snapshots in one market are not identical, due to different technological reasons. Furthermore, for a given minute, the times at which the snapshots are taken in different markets need not be exactly the same. The trading data on the other hand is stored in continuous time, as each trading event is being registered with its exact execution time. Derivatives obtained from each type of data therefore have to be carefully aligned for the consequent analysis to always respect the order of time.

To illustrate the data alignment problem, consider some two consecutive LOB snapshots in BTC-e market: the first limit order book snapshot is taken on 2017-11-01 at $t_1$ =00:00:03.021 and the second one is taken approximately one minute later, at $t_2$ =00:01:02.798. Further consider Binance market, where the first limit order book snapshots are taken at $t_1' =00:00:01.267$ and $t_2' =0:01:01.274$. One can already see that $t_1 \neq t_1'$ and that $t_2 - t_1 \neq 60s$.

Let us first consider the data alignment for analysis in Section III A where we consider information transfer between markets given one observable, e.g. spread. Firstly, for the observable based on trading data, we simply re-sample the trades at the minute level so that the trades that happened between 00:00:00.000 and 00:00:59.999 would represent the order imbalance at time $\tau =00:01:00.000$. For observables based on limit order books, we assume that the snapshot is representative of a limit order book that would be observed at the zeroth second of the minute, namely, at precisely HH:mm:00.000 for every hour HH and minute mm. In Fig. 10 we show the histogram of the exact snapshot seconds within a minute of each limit order book. It is clear that the majority of the limit order books were captured at first halves of a minute. Since we chose to use information dynamics that assumes discrete time series, the limit order book snapshot times are first floored to the closest minutes, after which spread and price returns observables are extracted.

For the example snapshots we consider here $t_1$ and $t_2$ would be rounded to the closes smaller minutes: $\tau_1 = [t_1] =00:00:00.000$, $\tau_2 = [t_2] =00:01:00.000$ from where we obtain $s_{t_1}, s_{t_2}$ and $r_{t_2}$.

We note that the problems would occur only when there is a systematic delay between the exact snapshot times for a given minute in a pair markets. If for some pair of $\alpha, \beta$, $t^\alpha < t^\beta$ in the majority of minutes, one may observe a spurious link $\beta \rightarrow \alpha$. To test whether this could be the case, we computed the differences between the exact snapshot times at each minute between pairs of markets. We report several observed distributions and note that the distributions do not appear skewed and the mean of the distributions is close to 0, see Fig. 11

We could also observe spurious information transfer if mapping to the closest minute is not done systematically. For instance, information transfer from BTC-e (whose first snapshots are taken at $t_1$, $t_2$ and are later rounded to $\tau_1$, $\tau_2$) to Binance (whose first snapshots are taken at $t_1' < t_1$, $t_2' < t_2$ and are

![Graph showing histogram of exact capture seconds](attachment:image.png)

FIG. 10: Histogram of the exact capture second of each minute snapshot of the limit order books used in this paper.
Therefore, issues would arise only if we mapped, e.g.,
time \tau to \tau' over and above the information about \alpha at
time \tau_1 that is already available from its own past at time \tau_1.
Therefore our results are limited to measuring
information dynamics at the inter-minute granularity.

For the second part of the analysis, presented in Section III B we consider information transfer between
different observables. Therefore we need to ensure that we do not
create acausal links. To compare the information flow from
spread at \tau_2 to order imbalance within that minute, we must
only consider the trades that occurred in the interval \{\tau_1, \tau_2\},
((0:00:03.021,00:01:02.798) in the example) when calculating \Theta. Similarly, the price returns at \tau_2 must be the price
difference between \tau_2 and \tau_1. Therefore we consider the time
index of order books, where the frequency is approximately
1min, and aggregate trading data based on that. For results,
shown in Fig. 9 where we consider lower frequency data, we
take the first data point within each interval.

Appendix B: Daily order imbalance

Table III reports the average of the daily sum of realised
order imbalance, see Eq. (1), expressed in either quote or base
currency. The table shows that before the crash date December
17th, 2017, the order imbalance was positive and signifi-
cantly larger than after the crash date, when it was negative in
all markets, except for Coinbase and Binance.

Appendix C: Choice of K in K-nearest neighbours

At the heart of transfer entropy is estimation of differential
conditional mutual information, which in turn relies on the
quality of probability distributions of our variables. Mutual
information involves averages of logarithms of \(P\), the under-
lying probability distribution. Since, for small \(P\), \(\log P \to \infty\),
the ranges of values for our variables \(X, Y, Z\) where \(P\) is small
and hence cannot be sampled and estimated reliably from data
contribute disproportionately to the value of information [32].

Information estimators that use continuity of real valued
data to overcome issues related to undersampling, have proved
to be the most successful for a wide range of datasets and ap-
lications. Amongst those, one of the most successful such
estimators is the Kraskov, Stögbauer, and Grassberger [31],
which we will refer to as KSG. The KSG estimates informa-
tion transfer based on the statistics of distances between
neighbouring data points. To implement this, KSG uses the
max(\Delta_x, \Delta_y, \Delta_z) metric to define the distance between two
points that are (\Delta_x, \Delta_y, \Delta_z) away from each other in the joint
space \{(x, y, z)\}. For each given test point, \(n_x\) is the neighbour
count strictly within \(\epsilon\) in the \(z\) marginal space, and \(n_{xz}\) and \(n_{yz}\)
are the neighbour counts strictly within \(\epsilon\) in the joint \{(x, z)\} and
\{(y, z)\} spaces, respectively. Conditional mutual information
is then defined in terms of these variables [13]:

\[ I(X, Y|Z) = \psi(K) - E [\psi(n_{xz}) - \psi(n_{yz}) + \psi(n_{x})], \]  

(C1)

where \(\psi\) is a digamma function and averaging is over the sam-
ple.

As pointed out in [32], for any information estimator, it is
essential to ensure that such estimators (there a KSG estima-
tor for MI rather than conditional MI was considered) is min-
imally biased towards the sample size, and the choice of \(K\) is
optimal: that is, for independent sub-samples of the dataset
the variance of obtained result is minimal.

The details of this approach for the choice of \(K\) are thor-
oughly described in [32]. Here we simply note that we tested

![FIG. 11: Differences between capture seconds at the same minute for a pair of markets. The title of each subfigure indicates \(\alpha - \beta\) for the histogram that shows \(t^\alpha - t^\beta\). The x-axis is in units of 1min.](image)
several choices of nearest neighbours for a pair of sample time series for which we identified significant transfer entropy with estimator using $K = 4$ nearest neighbours. As Fig. 12 shows, this choice of $K$ has a small bias for the sample size: even if we slice the time series in 4-5 parts, the value of TE does not increase significantly (specify). Furthermore, the variance for values for individual subsamples is also minimal. Note for $K = 1$ we have a strong sample-size dependent bias as well as large variance.

**FIG. 12:** Bias and variance of transfer entropy for subsamples of size $n$ for spread time series from Bitstamp to HitBTC where we identified a significant information transfer of 0.0052 nats using $K = 4$ nearest neighbours.

**Appendix D: Information transfer metrics around regime shifts in auto-regressive time series**

Here we perform a parameter sensitivity analysis for the results, shown in Fig. 5. Note that we do not consider the parameters $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ as variables and set them to $\alpha_1 = \alpha_2 = 0.2$ and $\beta_1 = \beta_2 = 1$. In all figures, we also chose $d = 0.5$. We note that for a definite conclusion regarding these information dynamical signatures, a thorough analysis of the relation to these parameters should also be conducted.

In Fig. 13 we show that a signature of high TE and AIS with no change in MI indicates strong causal coupling, and this statement is robust for a wide range of parameters. It is worth pointing out, however, that at high levels of $K$, a significant amount of MI is also detectable in a weak coupling regime.

The case of a shift in the hidden driver (illustrated in Fig. 5(b)) is signed with a high MI value in a strong-coupling regime and vice versa in a weak coupling regime. We find that for a wide range of parameter values that characterise the strength and steepness of the regime transition function, as well as the strength of causal coupling, MI is significantly larger in the strong coupling regime. Although for some parameter values AIS is anti-correlated with MI, it is not a robust signifier based on the sensitivity analysis. We also note that, not surprisingly, the strength of the coupling is directly related to the magnitude of MI.

**FIG. 13:** Expected values of information dynamical variables when $C(t)$ is a sigmoid functional of time, (17), parameterised by $b$, $s$, in the presence of a constant hidden driver of strength $c$. Information dynamic measures are significantly different in the two regimes, as defined by the values of the sigmoid function.

Lastly, Fig. 15 shows that a change in overall system’s uncertainty (the amount of uncorrelated random noise) is detectable with a signature of having low TE, MI and AIS in a high uncertainty regime, and vice versa in a low uncertainty regime.

**Appendix E: Modelling the non-linear relationship between price and spread**

Consider VAR toy example

$$X_t = \alpha_1 X_{t-1} + \beta_1 \varepsilon_{1,t} + K(t) \varepsilon_t$$

$$Y_t = \alpha_2 Y_{t-1} + \beta_2 \varepsilon_{2,t} + K(t) \varepsilon_t + C(t) (X_{t-1})^d,$$

with $d = 1$ and $d = 2$. For simplicity, we set $\alpha_1, \alpha_2 = 0$, $\beta_1, \beta_2 = 0.1$, $C(t) = 1$ and $K(t) = 0$. In Fig. 16 we show that
when the non-linear relation between the variables is present, KSG estimator can detect a large amount of information transfer. In case of linear relation we show a convergence between two measures in the large N limit, while a non-linear relationship is only detectable with a KSG estimated TE.

A more realistic example considers possibly bidirectional coupling between price returns $r$ and spread $s$ when spread is coupled to volatility (variance) of price. Therefore we have coupled time series of the form \[\text{(18)}.\]

Note that in this formulation, a link from price to spread is indirect, and in fact price returns affect spread at the two-step delay. Therefore in the following figures we will scan through $\delta \in [1, 2]$ and choose the delay for which the observed TE is larger. Depending on parameter values $b, \gamma$ we can have bi- or uni-directional couplings $T_{s \rightarrow r}$ or $T_{r \rightarrow s}$. Each datapoint in the proceeding results is considered at a significance level of 0.01, and results are obtained from 100 independent simulations.

Parameters $\alpha, \beta, a, b, c, w, \gamma$ affect stationarity and damping in the system therefore we need to choose such parameter values that system would be stationary throughout. To find a stationary combination of the parameters, we need to consider the moments of the two processes.

First, to compute the unconditional mean values $\sigma^2$ and $s^2$ of volatility and spread, let us consider

\[
\sigma^2 = \text{Var}[r_i] = E[r_i^2] = E[E[r_i^2|\mathcal{F}_{t-1}]] = E[w + \alpha r_{t-1}^2 + \beta \sigma_{t-1}^2 + \gamma s_{t-1}^2] = w + (\alpha + \beta) \sigma^2 + \gamma s^2
\]
and similarly
\[ s^2 = E[s_t^2] = E[as_t^2 + b\sigma_t^2 + c\epsilon_t^2] = as^2 + b\sigma^2 + c \]

Solving the system we obtain
\[
\sigma^2 = \frac{w(1-a) + yc}{(1-\alpha - \beta)(1-a) - \gamma b} \tag{E2}
\]
\[
s^2 = \frac{b\sigma^2 + c}{1-a} \tag{E3}
\]
From the first equation we see that unconditional volatility increases with \(\gamma, c\), and \(b\), i.e., there is a contribution of illiquidity to volatility. From the second equation, for large volatility \((\sigma^2 \gg b/c)\) we obtain the well known proportionality \(s \propto \sigma\) between liquidity and volatility. The stationary condition is obtained by imposing that these two quantities are positive, hence, since \(a < 1\), it must be
\[
(1 - \alpha - \beta)(1-a) - \gamma b > 0. \tag{E4}
\]
Volatility can be eliminated from the equations (18) and by recursive substitution we get
\[
\sigma_t^2 = \frac{w}{1-\beta} + \frac{\alpha}{\beta} \sum_{i=1}^\infty \beta^i \sigma_{t-i} + \gamma \sum_{i=1}^\infty \beta^i s_{t-i}
\]
\[
s_t^2 = \frac{bw}{1-\beta} + \frac{as_{t-1} + b\gamma}{\beta^2} \sum_{i=2}^\infty \beta^i s_{t-i} + \frac{ba}{\beta^2} \sum_{i=2}^\infty \beta^i r_{i-1} + c\epsilon_t^2
\]
i.e., the volatility is the sum of an Exponentially Weighted Moving Average (EWMA) of past square returns and of an EWMA of past square spreads and the same holds true for squared spread. Squaring the last two equations in (18) and taking expectations one gets
\[
E[\sigma^4] = w^2 + 3\alpha^2 E[\sigma^4] + \beta^2 E[\sigma^4] + \gamma^2 E[s^4]
+ 2w\alpha \sigma^2 + 2w\beta \sigma^2 + 2w\gamma s^2
+ 2\alpha \beta E[\sigma^4] + 2\alpha \gamma E[\sigma^2 s^2] + 2\beta \gamma E[\sigma^2 s^2]
\]
and
\[
E[s^4] = \sigma^4 E[s^2] + b^2 E[\sigma^4] + 3c^2
+ 2ab E[\sigma^2 s^2] + 2ac s^2 + 2bc \sigma^2
\]
where we have used the fact that \(E[\epsilon_t^4] = E[\epsilon_t^4] = 3\) (because they are Gaussian) and we introduced in the equations the unconditional means \(\sigma^2\) and \(s^2\) in Eq. (E2).

In order to close the system we need an expression for \(E[\sigma^2 s^2]\). This can be obtained by taking the product of the last two equations in (18) and taking the expectation, obtaining
\[
E[\sigma^2 s^2] = was^2 + wb\sigma^2 + wc + \alpha aE[\sigma^2 s^2]
+ abE[\sigma^4] + ac \sigma^2 + a\beta E[\sigma^2 s^2] + \beta b E[\sigma^4]
+ \beta c \sigma^2 + \gamma aE[\sigma^4] + \gamma bE[\sigma^2 s^2] + \gamma c s^2
\]

We now have a system of three equations and three unknowns \((E[\sigma^4], E[s^4], E[\sigma^2 s^2])\) whose solution gives the second moments. As usual, these moments will be finite in the parameter region where they are positive.

The solutions for fourth moments are cumbersome to write in full form. However, we checked that for the parameter combinations that we will study next the expectation of \(\sigma^2, s^2, \sigma^4, s^4\) are all well-defined. In particular, we considered the three following cases:

- Information transfer from spread to volatility, and, therefore to price returns; no information flow from price returns to spread: \(\alpha = 0.1, \beta = 0.4, a = 0.8, b = 0.0, c = 0.1, \gamma = 0.9, \omega = 0.1\)

- Information transfer from volatility, and, therefore, price returns, to spread; no information flow from spread to price returns: \(\alpha = 0.1, \beta = 0.1, a = 0.1, b = 0.9, c = 0.1, \gamma = 0.0, \omega = 0.1\)

- Bi-directional coupling from price returns to spread and vice versa: \(\alpha = 0.1, \beta = 0.5, a = 0.1, b = 0.5, c = 0.1, \gamma = 0.5, \omega = 0.1\).

First set of parameter values gives \(\sigma^2 \approx 0.13, s^2 = 0.24, \sigma^4 \approx 0.01 + 0.04\sigma^2, s^4 \approx 0.04 + 0.02s^2 + 0.24\sigma^2\). All values are positive. For the second set of parameters, we get \(\sigma^2 \approx 1.1, s^2 \approx 0.5, \sigma^4 \approx 0.2 + 1.4s^2 + 0.2\sigma^2, s^4 \approx 0.08 + 0.44s^2\). Lastly, for the third set of parameters, we find \(\sigma^2 \approx 1.27, s^2 \approx 0.82, \sigma^4 \approx 0.2 + 1.4s^2 + 0.2\sigma^2, s^4 \approx 0.08 + 0.44s^2\). Therefore all parameter values are suitable.

\[\text{Expected link } r \leftrightarrow s\]

FIG. 17: TE from spread to returns (left) and reverse (right), \(\alpha = 0.1, \beta = 0.5, a = 0.1, b = 0.5, c = 0.1, \gamma = 0.5\).

1. Simulation results

The three cases of coupling between spread and price returns are shown in Fig. 16 and Fig. 17. The first case, shown
in Fig. 6(top) considers $\gamma = 0.9$ and therefore coupling from spread to returns is strong. The figure shows that Gaussian estimate of TE is always close to zero, whereas KSG estimate of TE is clearly larger in the direction from spread to price returns (left figure) as opposed to the reverse (right figure). Note that in some simulations significant non-zero coupling is observed, when Gaussian estimate is used. However, on average, the magnitude of information transfer estimated with Gaussian estimate is smaller in contrast with KSG estimated TE.

The second case, shown in Fig. 6(bottom) considers a situation where $\gamma = 0.0$, therefore the link from spread to returns should not be observed, whereas parameter $b$, that indicates the strength of a reciprocal coupling is set to a large value $b = 0.9$. Here transfer entropy detects a significant flow only from price returns to spread. Similarly to the previous case, Gaussian estimate is also found significant and different from zero in some of the simulations, however the magnitude of TE is smaller than that obtained with KSG estimate.

The last figure, Fig. 17, considers a case when both $\gamma \neq 0$ and $b \neq 0$. Although the link from spread to returns is much weaker than the reverse, we nevertheless observe significant flow in both directions.

In all cases considered, TE estimates show no apparent size-dependent value drift, and therefore we assume there is no sample size-related bias, as long as $N \geq 1000$.

2. Theoretical values for transfer entropy between price returns and spread

The mathematical simplicity of our model (18) allows us to estimate the TE using numerical integration techniques. By comparing the simulation results, obtained using a data-dependent estimator of TE, with the results obtained analytically, we can attest the usability of such estimator for the type of data that could arise from the model considered.

a. Information transfer from spread to price returns To proceed, note that transfer entropy from spread to price returns is the difference of two conditional entropy ($H(\cdot|\cdot)$) terms, namely:

$$T_{s \rightarrow r} = H(r_{t-1}|r_{t-1}) - H(r_{t-1}|s_{t-1}).$$

It can also be rewritten in terms of expectation of $f(r_{t}|r_{t-1}, s_{t-1})$, where we sample points $\{r_t, r_{t-1}, s_{t-1}\}$, $\{r_t, s_t\}$ obtained from (18). Using a natural logarithm log yields a result units of nats, and $f(\cdot)$ denotes probability density function (p.d.f).

First, let us consider the case when $b = 0$ (Fig. 6) we derive the theoretical distribution densities for model-based transfer entropy from spread to returns. To obtain the p.d.f. $f(r_{t}|r_{t-1}, s_{t-1})$, one can sample values of $f(r_{t}|r_{t-1}, s_{t-1}, \sigma)$ and integrate over the hidden volatility variable:

$$f(r_{t}|r_{t-1}, s_{t-1}) = \int f(\sigma)f(r_{t}|r_{t-1}, s_{t-1}, \sigma)d\sigma.$$

Here we assume independence of $\sigma$. Further marginalising we obtain the conditional p.d.f. of price returns:

$$f(r_{t}|r_{t-1}) = \int \int f(\sigma)f(s)f(r_{t}|r_{t-1}, s, \sigma)d\sigma ds.$$

Finally, in case of our model defined in (18), we know the exact probabilistic model:

$$f(r_t = x|\cdot) = E_{f(\cdot)}[f(r_t = x|\cdot)] = \mathcal{N}(\mu, \sigma^2).$$

where $\mathcal{N}(\mu, \sigma^2)$ is the normal probability density function parameterised with $\mu, \sigma$.

Now, compute the quantity $T_{s \rightarrow r}$, we perform Monte Carlo sampling:

$$f(r_{t} = x|\cdot) = E_{f(\cdot)}[f(r_{t} = x|\cdot)] = \mathcal{N}(\mu, \sigma^2).$$

On the left panel of Fig. 18 and right panel of Fig. 19 we show agreement between the transfer entropy obtained from such analytical integration, and the result we found in Fig. 6 using a K-nearest neighbour data-driven estimator.

b. Information transfer from price returns to spread Similarly to the case before, transfer entropy from returns to spread is the difference of two conditional entropies:

$$T_{r \rightarrow s} = H(s_{t}|s_{t-1}, s_{t-2}) - H(s_{t}|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2}).$$

that can also be rewritten in terms of expectation value

$$\log\frac{f(s_{t}|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2})}{f(s_{t}|s_{t-1}, s_{t-2})}.$$
Here we need to estimate a conditional p.d.f. of spread given last two spread values. We do this again via marginalisation of hidden variables:

\[
    f(s_t|s_{t-1}, s_{t-2}) = \int \int f(s_t, r_{t-1}, r_{t-2}|s_{t-1}, s_{t-2})dr_{t-1}dr_{t-2}.
\]

The conditional p.d.f. of spread given last two spread and price return values, \( f(s_t|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2}) \), is obtained by marginalising over a hidden volatility variable:

\[
    f(s_t|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2}) = \int f(s_t, \sigma_t|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2})d\sigma_t.
\]  

(E5)

For simplicity, let us denote the conditional density of spread with \( \Phi(x) \)

\[
    f(s_t = y|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2}) = \Phi(y).
\]

Then the value of spread at time \( t \) is connected to the conditional variables as

\[
    s_t^2 = a s_{t-1}^2 + b(w + \alpha r_{t-2}^2 + \beta \sigma_{t-2}^2 + \gamma r_{t-2}^2) + c e^2,
\]

where \( e^2 \) is chi-square distribution. Note that spread can be rewritten as

\[
    s_t = \sqrt{c_t^* + c e^2},
\]

where \( \sqrt{c e^2} \) is Nakagami distribution. Then we have

\[
    \Phi(y) = \frac{2\sqrt{0.5}}{\Gamma(0.5)\sqrt{c}} \exp\left(-\frac{x^2}{2c}\right),
\]

and \( c_t^* + c e^2 \) is shifted Gamma distribution. Under the monotone transformation of random variable \( y = g(x) = \sqrt{c_t^* + x} \), we can compute the p.d.f. of \( s_t \) as:

\[
    \Phi(y) = f_s(g^{-1}(y)) \left| \frac{d}{dy}(g^{-1}(y)) \right|,
\]

where,

\[
    f_s(x) = \frac{1}{\Gamma(1/2)\sqrt{2c}} x^{-1/2} \exp\left(-\frac{x}{2c}\right),
\]

and

\[
    g^{-1}(y) = y^2 - c_t^*
\]

Finally we get conditional probability density for spread \( f(s_t = y|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2}) = \Phi(y) \)

\[
    \Phi(y) = \frac{2y}{\Gamma(1/2)\sqrt{2c}} \exp\left(-\frac{y^2 - c_t^*}{2c}\right) I_{\sqrt{c_t^*}}(y),
\]

where \( c_t^* = a s_{t-1}^2 + b(w + \alpha r_{t-2}^2 + \beta \sigma_{t-2}^2 + \gamma r_{t-2}^2) \) and \( I_{\sqrt{c_t^*}} \) stands for an indicator function defined over a subset of real numbers, \( \sqrt{c_t^*} \). Now we can get a numerical Monte Carlo estimate of the model-based transfer entropy as

\[
    E_{\sigma_t \sim f(s_t, \sigma_t)}[f(s_t|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2}, \sigma_t)] = \frac{1}{N} \sum_{i=1}^{N} \Phi(y; c, c_t^*) = e_{t-1}^2 + b(w + \alpha r_{t-2}^2 + \beta \sigma_{t-2}^2 + \gamma r_{t-2}^2).
\]

and

\[
    f(s_t = y|s_{t-1}, s_{t-2}) = E_{r_{t-1, r_{t-2}} \sim f(r_{t-1}, r_{t-2})}[f(s_t = y|s_{t-1}, s_{t-2}, r_{t-1}, r_{t-2})].
\]

In Fig. [18] and Fig. [19] we show agreements of transfer entropy obtained from such analytical integration, in comparison to transfer entropy values from returns to spread with KSG estimates shown in corresponding panels of Fig. [6]. For the case shown in Fig. [17], where we have bidirectional flow, namely, \( b > 0 \) and \( Y > 0 \), deriving theoretical density distributions requires special attention since we need to account for dependence between spread and volatility in marginalisation step, \( E[5] \), that falls out of the scope of current work.

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[36] Note that the results reported in Fig. 5 are local information dynamics values, and each local value is estimated for each individual run. For non-stationary time series, an appropriate treatment would be to obtain each local value from an ensemble of time series available [43]. However, in financial time series, usually only one trajectory is available. What we aimed to show with our controlled vector auto-regressive experiments of regime shifts in Fig. 5 was that certain regime shifts are detectable even from a single time series realisation, and the information theoretic signature is visible for a wide range of such non-stationary time series (see Appendix).
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