On construction of Darboux integrable discrete equations

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Abstract

The problem of discretization of Darboux integrable equations is considered. Given a Darboux integrable continuous equation, one can obtain a Darboux integrable differential-discrete equation, using the integrals of the continuous equation. In the present paper, the discretization of the differential-discrete equations is done using the corresponding characteristic algebras. New examples of integrable discrete equations are obtained.

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1 Introduction

The problem of discretization of the Darboux integrable hyperbolic equations has generated a lot of interest in recent years. A hyperbolic equation

\[ w_{xy} = h(w, w_x, w_y) \]  

(1)
is called Darboux integrable if it admits two non-trivial functions

\[ J(w, w_y, w_{yy}, \ldots) \quad \text{and} \quad G(w, w_x, w_{xx}, \ldots), \]

depending on a finite number of variables, such that for all solutions of equation (1) we have

\[ D_x J = 0 \quad \text{and} \quad D_y G = 0, \]  

(2)

where \( D_x \) is the total \( x \)-derivative operator and \( D_y \) is the total \( y \)-derivative operator. The functions \( J(w, w_y, w_{yy}, \ldots) \) and \( G(w, w_x, w_{xx}, \ldots) \) are called \( x \)- and \( y \)-integrals respectively. For the detailed discussion of the Darboux integrable equations see [1], [17], [18] and references therein.

First, one can look for a differential-discrete equation which is a discretization of equation (1). This differential-discrete equation should also be Darboux integrable. The notion of Darboux integrable differential-discrete equation was introduced in [7]. Let us consider an equation

\[ t_{1x} = g(t, t_1, t_x), \]  

(3)

where \( t(n, x) \) is a function of a continuous variable \( x \), a discrete variable \( n \) and \( t_1 = D_t(n, x) = t(n + 1, x) \) (\( D \) is the shift operator and \( D^k t(n, x) = t(n + k, x) = t_k, k \in \mathbb{Z} \)). Differential-discrete equation (3) is called Darboux integrable if it admits two functions

\[ I(t, t_x, t_{xx}, \ldots) \quad \text{and} \quad F(\ldots, t_{-1}, t, t_1, t_2, t_{-2}, \ldots), \]

depending on a finite number of variables, such that for all solutions of equation (3) we have

\[ D_x F = 0 \quad \text{and} \quad DI = I. \]  

(4)

Such Darboux integrable differential-discrete equations and discrete equations (the definition of Darboux integrable discrete equation is given in the next section) are actively studied nowadays, see [2]-[6].
It was proposed in [9] to use $x$- or $y$- integrals of a continuous equation to obtain its discretization. That is, one looks for a differential-discrete equation that admits a given $x$- or $y$-integral as its $n$-integral. This approach allowed to construct many differential-discrete equations, see [10]-[16]. Moreover, the constructed equations turned out to admit also an $x$-integral, that is the equations are Darboux integrable. Now one can take a constructed differential-discrete equation and consider its further discretization using the corresponding $x$-integral. However, if one tries to find a discrete equation corresponding to a given integral, one obtains a complicated functional equation to be solved, see [9] for some examples.

The Darboux integrability can be also defined in terms of characteristic algebras, which are Lie-Rinehart algebras, see [13], [11] and [12]. We propose to use not only the integrals but also the characteristic algebras for the discretization of the differential-discrete equations.

The paper is organized as follows. In second section we give necessary definitions and description of our approach to discretization. In third section we give examples of discretization for differential-discrete Darboux integrable equations.

2 Preliminaries

In what follows we always consider functions $t(n, x)$ to be dependent on a continuous variable $x$ and a discrete variable $n$ and functions $u(n, m)$ to be dependent on two discrete variables $n$ and $m$. For a function $u(n, m)$ we have the shift operator $D$ such that $Du(n, m) = u(n + 1, m) = u_1$, the shift with respect to the first variable, and the shift operator $D$, $Du(n, m) = u(n, m + 1) = u_1$, the shift with respect to the second variable. Note that $D^k u(n, m) = u(n + k, m) = u_k$ and $D^k u(n, m) = u(n, m + k) = u_k, k \in \mathbb{Z}$.

A criteria for existence of $n$- and $x$- integrals of equation (3) can be formulated in terms of the so-called characteristic algebras. In what follows we always assume that $t, t_\pm, t_\pm^2, \ldots$ and $t_x, t_{xx}, t_{xxx} \ldots$ are independent dynamical variables. Let us introduce the criteria for existence of $x$-integral first. Define an operator

$$Z = t_x \frac{\partial}{\partial t} + t_{1x} \frac{\partial}{\partial t_1} + t_{-1x} \frac{\partial}{\partial t_{-1}} + \ldots,$$

which corresponds to the total derivative operator $D_x, D_x F = Z F$, and an
operator

$$W = \frac{\partial}{\partial t_x}. \quad (6)$$

We have that $ZF = 0$, $WF = 0$. Clearly the function $F$ is also annulled by all possible commutators of these operators. Thus we define the characteristic $x$–algebra, denoted by $L_x$, as a Lie-Rinehart algebra generated by the operators $Z$ and $W$. The algebra $L_x$ is considered over the ring of functions depending on a finite number of dynamical variables. In general, all algebras and linear spaces of operators introduced in this paper are considered over the ring of functions depending on finite number of dynamical variables.

**Theorem 1.** [7] Equation (3) admits a non-trivial $x$–integral if and only if the corresponding characteristic $x$-algebra $L_x$ is finite dimensional.

Now we introduce the criteria for the existence of the $n$–integral of equation (3). Following [7] we define an operator

$$Y_0 = \frac{\partial}{\partial t_1} \quad (7)$$

and operators

$$Y_k = D^{-k} \frac{\partial}{\partial t_1} D^k, \quad k = 1, 2, 3, \ldots; \quad (8)$$

$$X_k = \frac{\partial}{\partial t_{-k}}, \quad k = 1, 2, \ldots. \quad (9)$$

**Theorem 2.** [7] Equation (3) admits a non-trivial $n$–integral if and only if the following conditions are satisfied.

1. The linear space generated by the operators $\{Y_k\}_{k=0}^{\infty}$ has a finite dimension. Let us denote the dimension by $N$.
2. The Lie-Rinehart algebra generated by the operators $\{Y_k\}_{k=0}^{N}$ and $\{X_k\}_{k=1}^{N}$ has a finite dimension.

The characteristic $n$-algebra, denoted by $L_n$, is a Lie-Rinehart algebra generated by the operators $\{Y_k\}_{k=1}^{N}$ and $\{X_k\}_{k=1}^{N}$ from the above theorem.

Let us consider a discrete equation

$$u_1 = f(u, u_1, u_i). \quad (10)$$

In what follows we always assume that $u, u_1, u_2, \ldots$ and $u_1, u_2, \ldots$ are independent dynamical variables. A sequence of functions $\{J_k(u_{-j}, \ldots, u_r)\}_{-\infty}^{\infty}$,
depending on finite number of dynamical variables $u_{-j}, \ldots, u_r$, is called an $m$-integral for equation (10) if $\bar{D}J_i = J_{i+1}$, $i \in \mathbb{Z}$.

Note that a shift of an $m-$integral $\left\{D^pJ_k(u_{-j}, \ldots, u_r)\right\}_{k=1}^\infty$, where $p$ is fixed, is also an $m-$integral.

The notion of an $n$-integral for equation (10) is defined in a similar way. Existence of $m-$ and $n-$integrals determines Darboux integrability of a discrete equation. Equation (10) is called Darboux integrable if it admits non-trivial $m$- and $n$-integrals.

A criteria for existence of $m$- and $n$-integrals can be formulated in terms of the so-called characteristic algebras. We consider the existence criteria for the $m$-integral (for the $n$-integral the corresponding criteria is formulated in a similar way). Following [18] we define operators

$$\tilde{Y}_0 = \frac{\partial}{\partial u_1}$$

(11) and

$$\tilde{Y}_k = \bar{D}^{-k} \frac{\partial}{\partial u_1} \bar{D}^k, \quad k = 1, 2, \ldots;$$

(12)

$$\tilde{X}_k = \frac{\partial}{\partial u_{-k}}, \quad k = 1, 2, \ldots.$$ (13)

**Theorem 3.** [18] Equation (10) admits a non-trivial $n-$integral if and only if the following conditions are satisfied

1. The linear space generated by the operators $\{\tilde{Y}_k\}_{k=0}^\infty$ has a finite dimension. Let us denote the dimension by $\tilde{N}$.
2. The Lie-Rinehart algebra generated by the operators $\{\tilde{Y}_k\}_{k=0}^N$ and $\{\tilde{X}_k\}_{k=1}^{\tilde{N}}$ has a finite dimension.

The Lie-Rinehart algebra generated by the operators $\{\tilde{Y}_k\}_{k=0}^N$ and $\{\tilde{X}_k\}_{k=1}^{\tilde{N}}$ from the above theorem is called the characteristic $m$-algebra, denoted by $L_m$.

Given a Darboux integrable differential-discrete equation (3) we have an $x-$integral $F(t, t_1, \ldots, t_j)$. We would like to find a discrete equation (10) that admits same function $F$ as its $m-$integral. For simplicity we assume that $\bar{J}_k = F$ for all $k$, that is the equality

$$\bar{D}F = F$$

(14) holds. The equality (14) in general gives a complicated equation for the function $f$ from equation (10), namely

$$F(u, u_1, \ldots, u_j) = I(u_1, u_{j1}, \ldots, u_{j1}) = I(u_1, f, f(u_1, u_2, f), \ldots),$$

(15)
that is not easy to solve.

To find the discrete equation we propose to use characteristic algebras. Operators of the characteristic algebra $L_x$ of a given differential-discrete equation and operators of characteristic algebra $\tilde{L}_m$ of the corresponding discrete equation annul the same function $F$. The operator $\tilde{Y}_1 \in \tilde{L}_m$ has the form (see [18])

$$\tilde{Y}_1 = \frac{\partial}{\partial u} + \alpha \frac{\partial}{\partial u_1} + \frac{1}{\alpha - 1} \frac{\partial}{\partial u_{-1}} + \ldots,$$

(16)

where

$$\alpha = \tilde{D}^{-1} \frac{\partial}{\partial u_1} f.$$  

(17)

We assume that the operator $\tilde{Y}_1$ can be identified with the operator

$$[X, Z] = \frac{\partial}{\partial t} + \frac{\partial t_{1x}}{\partial t} \frac{\partial}{\partial t_1} + \frac{\partial t_{-1x}}{\partial t} \frac{\partial}{\partial t_{-1}} + \ldots,$$

(18)

$[X, Z] \in L_x$ (note that $\frac{\partial t_{1x}}{\partial t} = \frac{1}{\alpha - 1}$). Thus, the coefficient $\alpha = \tilde{D}^{-1} \frac{\partial}{\partial u_1} f$ is identified with the coefficient $\frac{\partial}{\partial t} g$. So, if we take function $\frac{\partial}{\partial t} g(t, t_1, t_x)$ and replace the variables as follows $t = u_1$, $t = u_{11}$ and $t_x = A(u, u_1)$ (the function $A$ to be found later) we obtain an equation for $\frac{\partial}{\partial u_1} f$

$$\frac{\partial f}{\partial u_1} = \frac{\partial}{\partial t} g(t, t_1, t_x)|_{t=u_1, t=u_{11}, t_x=A(u,u_1)}.$$  

(19)

The above equation determines the function $f$ up to some unknown functions of $u, u_1$. The unknown functions can be found using equality (14).

3 Examples

In this section we derive several examples of discretization of differential-discrete Darboux integrable equations.

Example 4. Consider a differential-discrete equation

$$t_{1x} = (1 + Re^{t+t_1})t_x + \sqrt{R^2e^{2(t+t_1)} + 2Re^{t+t_1}} \sqrt{t_x^2 - 4}$$

(20)
with the x-integral

\[ F_1 = \sqrt{Re^{2t_1} + 2e^{t_1-t}} + \sqrt{Re^{2t_1} + 2e^{t_1-t}} \]  

(equation (3*b) in [9]). The corresponding discrete equation is given by

\[
e^{-u_1} = e^{-u_1} \left( e^{u} \sqrt{e^{-u} - u_1} + e^{u} \sqrt{R} \right)^2 + \sqrt{2R} \left( e^{u} \sqrt{e^{-u} - u_1} + e^{u} \sqrt{R} \right)
\]  

(22)

Equation (22) admits the m-integral

\[ \tilde{F}_1 = \sqrt{Re^{2u_1} + 2e^{u_1-u} + \sqrt{Re^{2u_1} + 2e^{u_1-u}}} \]  

(23)

One can also find an n-integral for the given equation

\[ \tilde{G}_1 = \frac{e^{-u} + e^{-u_1}}{e^{-u_1} + e^{-u_2}}. \]  

(24)

So equation (22) is Darboux integrable. The derivation of this example is given below

**Example 5.** Let us consider the equations of the form

\[ t_{1x} = K(t, t_1) t_x \]  

(25)

given in [9] (equations (2*b), (5*), (6*)) . Such equations admit an x-integral \( F(t, t_1) \), where the function \( F \) satisfies the equation

\[ F_t + K(t, t_1) F_{t_1} = 0. \]  

(26)

For such equations our approach leads to an obvious discrete equation

\[ F(u, u_1) = F(u_1, u_{11}) \]  

(27)

with m-integral \( F(u, u_1) \). The existence of n-integrals for such equations requires further investigation. Some results on a similar classification problems can be found in [19].
Remark 6. For the autonomous equations of the form

\[ t_{1x} = t_x + A(t, t_1) \]  

(28)
given in [9] (equations (2\*a), (3\*a)). Our approach leads to a simple discrete equations

\[ u_{11} = u_1 + u - u_1, \]  

(29)
in case of equation (2\*a), and

\[ u_{11} = u_1 + u_1 - u, \]  

(30)
in case of equation (3\*a).

Now let us derive the discrete equation given in the Example 4. We are looking for an equation (10) which is a discretization of equation (20). Since

\[ \frac{\partial}{\partial t_x} \left[ (1 + Re^{t+t_1})t_x + \sqrt{R^2e^{2(t+t_1)} + 2Re^{t+t_1}} \sqrt{t_x^2 - 4} \right] = (1 + Re^{t+t_1}) + \sqrt{R^2e^{2(t+t_1)} + 2Re^{t+t_1}} t_x \sqrt{t_x^2 - 4} \]  

(31)

we assume that

\[ \frac{\partial f}{\partial u_1} = 1 + Re^{u_1} + T(u, u_1) \sqrt{R^2e^{2(u_1)+f} + 2Re^{u_1}+f} \]  

(32)

where \( T \) is a function of \( u, u_1 \). By solving equation (32) we get

\[ \sqrt{e^{-f-u_1}} + \frac{R}{2} = e^{-u_1} E(u, u_1) + C(u, u_1). \]  

(33)

Since \( u_{11} = f \) we have

\[ \sqrt{e^{-u_{11}-u_1}} + \frac{R}{2} = e^{-u_1} E(u, u_1) + C(u, u_1). \]  

(34)
or

\[ e^{-u_{11}} = e^{-u_1}E^2(u, u_1) + e^{u_1} \left( C^2(u, u_1) - \frac{R}{2} \right) + 2E(u, u_1)C(u, u_1). \]  

(35)
To find the functions $E$, $C$ we use equality $\bar{DF}_1 = F_1$. We have
\[
e^{u_{1\bar{1}}} \sqrt{e^{-u_{1\bar{1}}-u_1} + \frac{R}{2} + e^{u_{1\bar{1}}} \sqrt{e^{-u_{1\bar{1}}-u_{2\bar{1}}} + \frac{R}{2}}} = e^{u_{1\bar{1}}} \sqrt{e^{-u_{1\bar{1}}-u_1} + \frac{R}{2} + e^{u_{1\bar{1}}} \sqrt{e^{-u_{1\bar{1}}-u_{2\bar{1}}} + \frac{R}{2}}}. \tag{36}
\]

Using equation (34) we can write
\[
\sqrt{e^{-u_{2\bar{1}}-u_{1\bar{1}}} + \frac{R}{2}} = e^{-u_{1\bar{1}}} E(u_1, u_2) + C(u_1, u_2) \tag{37}
\]
and rewrite the equality (36) as
\[
e^{u_{1\bar{1}}} (e^{-u_{1\bar{1}}} E(u, u_1) + C(u, u_1)) + e^{u_{1\bar{1}}} (e^{-u_{1\bar{1}}} E(u_1, u_2) + C(u_1, u_2)) = e^{u_{1\bar{1}}} \sqrt{e^{-u_{1\bar{1}}-u_1} + \frac{R}{2} + e^{u_{1\bar{1}}} \sqrt{e^{-u_{1\bar{1}}-u_{2\bar{1}}} + \frac{R}{2}}} \tag{38}
\]
By differentiating the equality (38) with respect to $u_{1\bar{1}}$ we get
\[
\frac{\partial u_{1\bar{1}}}{\partial u_1} = \frac{e^{-u_{1\bar{1}}} E(u, u_1)}{e^{-u_{1\bar{1}}} E(u, u_1) + C(u, u_1) + C(u_1, u_2)}, \tag{39}
\]
which implies that
\[
u_{1\bar{1}} = -\ln(e^{-u_{1\bar{1}}} E(u, u_1) + C(u, u_1) + C(u_1, u_2)) + \tilde{K}(u, u_1) \tag{40}
\]
or
\[
e^{-u_{1\bar{1}}} = K(u, u_1)(e^{-u_{1\bar{1}}} E(u, u_1) + C(u, u_1) + C(u_1, u_2)). \tag{41}
\]
Now comparing the equality (41) with (35) we get $C(u, u_1) = \pm \sqrt{\frac{R}{2}}$ and $K(u, u_1) = E(u, u_1)$. Thus,
\[
e^{-u_{1\bar{1}}} = E^2(u, u_1)e^{-u_{1\bar{1}}} \pm \sqrt{2RE}(u, u_1). \tag{42}
\]
By substituting the above expression for $e^{-u_{1\bar{1}}}$ into the equality (38) and differentiating the equality (38) with respect to $u_{2\bar{1}}$ we get
\[
\frac{\partial E(u_1, u_2)}{\partial u_2} = \frac{\partial}{\partial u_2} \left( e^{u_{1\bar{1}}} \sqrt{e^{-u_{1\bar{1}}-u_2} + \frac{R}{2}} \right), \tag{43}
\]
that is
\[ E(u_1, u_2) = e^{u_1} \sqrt{e^{-u_1 - u_2} + \frac{R}{2} + M(u_1)} \] (44)

So, we rewrite the equality (42) as
\[ e^{-u_1} = e^{-u_1} (e^u \sqrt{e^{-u_1} + \frac{R}{2} + M(u)})^2 \pm \sqrt{2Re^u \sqrt{e^{-u_1} + \frac{R}{2} + M(u)}}. \] (45)

We use equality (36) and have \( M(u) = e^u \sqrt{\frac{R}{2}} \). Thus, we obtain equation (22).

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