Convexity of Second-Order Cone Program in the Right Hand Side Parameter

Shourya Bose

Abstract
In this note we prove that the optimum value of a second-order cone program (SOCP) is convex in the right hand side (RHS) parameter.

1 Introduction
Consider the following SOCP.
\[
J(b, d_1, \cdots, d_N) \triangleq \min_x c^\top x
\]
subject to:
\[
Ax \leq Gb + h \tag{1a}
\]
\[
\|C_i x\|_2 \leq d_i, \quad i = 1, \ldots, N \tag{1b}
\]
In the above problem, \(x, c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, G \in \mathbb{R}^{m \times p}, b \in \mathbb{R}^p, h \in \mathbb{R}^m, \) and for \(i = 1, \cdots, M, C_i \in \mathbb{R}^{m_i \times n} \) and \(d_i \in \mathbb{R}_{\geq 0}. \) For notational convenience, we define \(l \triangleq [b^\top, d_1, \cdots, d_N]^\top, \) and correspondingly \(J(l)\) represents \(J(b, d_1, \cdots, d_N). \) It is well-known that when problem \((1)\) is a linear program, i.e. the constraits \((1b)\) are absent, then \(J\) is convex in \(b [1, \text{ Theorem 5.1}]. \) In this note, we establish that a general SOCP problem of the form \((1)\), i.e. \(J(l)\) is also convex in \(l.\)

2 Main Result
Proposition 1 (Convexity of \(J(l)\) in \(l)\). \(J(l)\) is convex in \(l\) for all \(l \in \Gamma, \) where \(\Gamma\) denotes the set of all \(l\) for which there exists at least one feasible \(x\) such that constraits \((1a)\) and \((1b)\) hold strictly.

Before presenting a proof of Proposition 1, we present a helper lemma which is a standard result.

*Department of Electrical and Computer Engineering, University of California, Santa Cruz
Lemma 1 (Dual problem of (1), [2, Problem 5.43]). The dual problem of SOCP (1) is given as

\[
\mathcal{D}(l) \overset{\Delta}{=} \max_{u \geq 0, w_i \geq 0, v_i} -(Gb + h)^\top u - \sum_{i=1}^{N} d_i v_i
\]

subject to:

\[
c + A^\top u - \sum_{i=1}^{N} C_i^\top w_i = 0
\]  

(2a)

\[
\|w_i\|_2 \leq v_i, \quad i = 1, \ldots, N
\]  

(2b)

Proof of Lemma 1. Consider adding \( N \) dummy variables \( y_i \overset{\Delta}{=} C_i x \) to problem (1). The resulting problem, and the dual variables corresponding to the constraints in parentheses is given as follows.

\[
\min_{x,y} \quad c^\top x
\]

subject to:

\[
Ax \leq Gb + h \quad (u)
\]

\[
\|y_i\|_2 \leq d_i, \quad i = 1, \ldots, N \quad (v_1, \ldots, v_N)
\]

\[
y_i = C_i x, \quad i = 1, \ldots, N \quad (w_1, \ldots, w_N)
\]

The Lagrangian of problem (3) is given as

\[
\mathcal{L} = c^\top x + u^\top (Ax - Gb - h) + \sum_{i=1}^{N} \left[ v_i (\|y_i\|_2 - d_i) + w_i^\top (y_i - C_i x) \right]
\]

\[
= \left( c + A^\top u - \sum_{i=1}^{N} C_i^\top w_i \right)^\top x - (Gb + h)^\top u - \sum_{i=1}^{N} v_i d_i + \sum_{i=1}^{N} \left[ v_i \|y_i\|_2 + w_i^\top y_i \right]
\]

Note that \( \mathcal{L} \) is unbounded below in \( x \) unless \( c + A^\top u - \sum_{i=1}^{N} C_i^\top w_i = 0 \), which recovers constrain (2a). Furthermore, note that the dual objective function,

\[-b^\top u - \sum_{i=1}^{N} v_i d_i\]

is also a part of \( \mathcal{L} \). It only remains to compute \( \inf_y \sum_{i=1}^{N} (v_i \|y_i\|_2 + w_i^\top y_i) \) in order to recover the full structure of the dual problem, for which the following identity is helpful.

\[
\inf_y (v \|y\|_2 + w^\top y) = \begin{cases} 0, & \text{if } \|w\|_2 \leq v, \\ -\infty, & \text{otherwise.} \end{cases}
\]

An explanation of the above equation is as follows, thanks to [3]. We will use the fact that \( v \geq 0 \) due to dual feasibility constrain.

- **Case 1:** \( \|w\|_2 \leq v \): It follows from Cauchy-Schwartz inequality that

\[-w^\top y \leq \|w\|_2 \|y\|_2 \leq v \|y\|_2,
\]

and therefore \( w^\top y + v \|y\|_2 \geq 0 \), and correspondingly \( \inf_y (v \|y\|_2 + w^\top y) = 0 \).
• **Case 2**: \( \|w\|_2 > v \): Choose \( y = -sw \) for some \( s > 0 \). We have,

\[
v \|y\|_2 + w^\top y = sv \|w\|_2 - s \|w\|_2^2 = s \|w\|_2 (v - \|w\|_2) < 0.
\]

Since the choice of \( s \) is not predetermined, the above term can be made arbitrarily negative, and therefore \( \inf_y (v \|y\|_2 + w^\top y) = -\infty \).

From the above, we conclude that constrains (2b), viz. \( \|w_i\|_2 \leq v_i \) must be added in order to prevent \( \mathcal{L} \) from becoming unbounded below.

We now present the proof of Proposition 1.

**Proof of Proposition 1.** Due to assumption of strict feasibility, strong duality holds [2, Section 5.2.3], and therefore \( \mathcal{J}(l) = \mathcal{D}(l) \) for all \( l \in \Gamma \). Define the set

\[
\mathcal{A} \doteq \{ (u, \{v_i\}_{i=1}^N, \{w_i\}_{i=1}^N) : \text{constrains (2a) and (2b) are satisfied} \}.
\]

The dual problem \( \mathcal{D}(l) \) can be written as

\[
\mathcal{D}(l) = \max_{u,v} \left( -(Gb + h)^\top u - \sum_{i=1}^N d_i v_i \right)
\]

using Lemma 1. Recall that \( l = [b^\top, d_1, \ldots, d_N]^\top \). Since \( -(Gb + h)^\top u - \sum_{i=1}^N d_i v_i \) is an affine (and therefore convex) function of \( l \), \( \mathcal{D}(l) \) is the pointwise supremum of a family of convex functions parameterized by set \( \mathcal{A} \), and is therefore convex [2, Section 3.2.3].

**References**

[1] D. Bertsimas and J. N. Tsitsiklis, *Introduction to linear optimization*. Athena Scientific Belmont, MA, 1997, vol. 6.

[2] S. P. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.

[3] littleO (https://math.stackexchange.com/users/40119/littleo), “Dual of a second order cone program (socp),” Mathematics Stack Exchange, uRL:https://math.stackexchange.com/q/2738200 (version: 2018-04-15). [Online]. Available: https://math.stackexchange.com/q/2738200