STOCHASTIC INTEGRATION FOR A WIDE CLASS OF GAUSSIAN STATIONARY INCREMENT PROCESSES USING AN EXTENSION OF THE $S$-TRANSFORM

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Abstract. Given a Gaussian stationary increment processes with spectral density, we show that a Wick-Itô integral with respect to this process can be naturally obtained using Hida’s white noise space theory. We use the Bochner-Minlos theorem to associate a probability space to the process, and define the counterpart of the $S$-transform in this space. We then use this transform to define the stochastic integral and prove an associated Itô formula.

1. Introduction

Stochastic integration with respect to general Gaussian processes can be found in [16, Ch.7]. An extension of Itô formula for such integrals has been derived by Nualart and Taqqu [21, 22] and recently in [1] in the setting of white noise analysis.

In this paper we extend the $S$-transform approach to develop stochastic calculus and derive additional results for the family of centered Gaussian processes with covariance function of the form

$$K_m(t, s) = \int_{\mathbb{R}} \frac{e^{i\xi t} - 1}{\xi} \frac{e^{-i\xi s} - 1}{\xi} m(\xi) d\xi$$

where $m$ is a positive measurable even function subject to

$$\int_{\mathbb{R}} \frac{m(u)}{\xi^2 + 1} d\xi < \infty.$$

Note that $K_m(t, s)$ can also be written as

$$K_m(t, s) = r(t) + r(s) - r(t - s),$$

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where
\[ r(t) = \int_{\mathbb{R}} \frac{1 - \cos(t\xi)}{\xi^2} m(\xi) d\xi. \]

This family includes in particular the fractional Brownian motion, which corresponds (up to a multiplicative constant) to
\[ m(\xi) = |\xi|^{1-2H}, \]
where \( H \in (0, 1) \). We note that complex-valued functions of the form
\[ K(t, s) = r(t) + r(s) - r(t-s) - r(0), \]
where \( r \) is a continuous function, have been studied in particular by von Neumann, Schoenberg and Krein. Such a function is positive definite if and only if \( r \) can be written in the form
\[ r(t) = r_0 + i\gamma t + \int_{\mathbb{R}} \left\{ e^{i\xi t} - 1 - \frac{i\xi t}{\xi^2 + 1} \right\} d\sigma(\xi), \]
where \( \sigma \) is an increasing right continuous function subject to \( \int_{\mathbb{R}} d\sigma(\xi) < \infty \). See [20], [18], and see [2] for more information on these kernels.

As in [2], our starting point is the (in general unbounded) operator \( T_m \) on the Lebesgue space of complex-valued functions \( L_2(\mathbb{R}) \) defined by
\[
(1.1) \quad \hat{T}_m f(\xi) = \sqrt{m(\xi)} \hat{f}(\xi),
\]
with domain
\[ D(T_m) = \left\{ f \in L_2(\mathbb{R}) : \int_{\mathbb{R}} m(\xi) |\hat{f}(\xi)|^2 d\xi < \infty \right\}, \]
where \( \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi t} f(t) dt \) denotes the Fourier transform. Clearly, the Schwartz space \( \mathcal{S} \) of smooth rapidly decreasing functions belong to the domain of \( T_m \). The indicator functions
\[ 1_t = \begin{cases} 1_{[0,t]}, & t \geq 0, \\ 1_{[t,0]}, & t \leq 0, \end{cases} \]
also belong to \( D(T_m) \). In [2], and with some restrictions on \( m \), a centered Gaussian process \( B_m \) with covariance function \( K_m(t, s) = \langle T_m 1_t, T_m 1_s \rangle \) was constructed in Hida’s white noise space. In the present paper we chose a different path. We build from \( T_m \) the characteristic functional
\[
(1.2) \quad C_m(s) = e^{-\left\| T_m s \right\|^2_{L_2(\mathbb{R})}}. \]
It has been proved in [3] that \( C_m \) is continuous from \( \mathcal{S} \) into \( L_2(\mathbb{R}) \). Restricting \( C_m \) to real-valued functions and using the Bochner-Minlos theorem, we obtain an analog of the white noise space in which the process \( B_m \) is built in a natural way. Stochastic calculus with respect
STOCHASTIC INTEGRATION AND AN EXTENSION OF S-TRANSFORM

to this process is then developed using an S-transform approach.

We note that when \( m(\xi) = |\xi|^{1-2H} \), and \( H \in \left( \frac{1}{2}, 1 \right) \), the operator \( T_m \) reduces, up to a multiplicative constant, to the operator \( M_H \) defined in [10] and in [5]. The set \( L^2_\phi \) presented in [8, equation (3)], is closely related to the domain of \( T_m \), and the functional \( C_m \) was used with the Bochner-Minlos theorem in [6, (3.5), p. 49]. In view of this, our work generalized the stochastic calculus for fractional Brownian motion presented in these works to the aforementioned family of Gaussian processes.

Note moreover that the function \( \phi \) from the last references defines the kernel associated to the operator \( T_m^*T_m \) via Schwartz’s kernel theorem, with \( m = M_H \). In the general case, the kernel associated to the operator \( T_m^*T_m \) is not a function. This last remark is the source for some of the difficulties arises in extending Wick-Itô integration for fractional Brownian motion such as the distinction between the cases \( H < \frac{1}{2} \), \( H > \frac{1}{2} \) and \( H = \frac{1}{2} \). An advantage of our approach is that such issues do not arise.

There are two main ideas in this paper. The first is the construction of a probability Gaussian space in which a stationary increment process with spectral density \( m \) is naturally defined. This result, being a concrete example of Kolmogorov’s extension theorem on the existence of a Gaussian process with a given spectral density, is interesting in its own right. The second main result deals with developing stochastic integration with respect to the fundamental process in this space. We take an approach based on the analog of the S-transform in our setting, and show that this stochastic integral coincides with the one already defined in [1] but in the framework of Hida’s white noise space.

Possible extensions and applications of the ideas presented here are in the field of stochastic control by considering systems driven by Gaussian noise of an arbitrary spectrum. The case of fractional noise was investigated for example in [15], [17] and [9]. We argue that extensions of some of these works to systems driven by noises of a more general spectrum is straightforward in view of our present work.

The paper consists of five sections besides the introduction. In Section 2 we construct an analog of Hida’s white noise space using the characteristic function \( C_m \). In Section 3 the associated fundamental process
B_m is being defined and studied. The analog of the S-transform is defined and studied in Section 4. In Section 5 we define a Wick-Itô type stochastic integral with respect to B_m, and prove an associated Itô formula. In the last section we relate the present stochastic integral with previous extensions of the Itô integral for non semi-martingales.

2. The m Noise Space

We set \( \mathcal{S}_R \) to be the space of real-valued Schwartz functions, and \( \Omega = \mathcal{S}_R' \). We denote by \( \mathcal{B} \) the associated Borel sigma algebra. Throughout this paper, we denote by \( \langle \cdot, \cdot \rangle \) the duality between \( \mathcal{S}_R' \) and \( \mathcal{S}_R \), and by \( (\cdot, \cdot) \) the inner product in \( L^2(R) \). In case there is no danger of confusion, the \( L^2(R) \) norm will be denoted as \( \| \cdot \| \).

**Theorem 2.1.** There exists a unique probability measure \( \mu_m \) on \( (\Omega, \mathcal{B}) \) such that

\[
e^{-\frac{4|T_m|}{2}^2} = \int_{\Omega} e^{i\langle \omega, s \rangle} d\mu_m(\omega), \quad s \in \mathcal{S}_R,
\]

**Proof:** The function \( C_m(s) \) is positive definite on \( \mathcal{S}_R \) since

\[
C_m(s_1 - s_2) = \exp \left\{ -\frac{1}{2} \| T_m s_1 \|^2 \right\} \times \exp \left\{ (T_m s_1, T_m s_2) \right\} \times \exp \left\{ -\frac{1}{2} \| T_m s_2 \|^2 \right\}.
\]

Moreover the operator \( T_m \) is continuous from \( \mathcal{S}_R \) (and hence from \( \mathcal{S}_R' \)) into \( L^2(R) \). This was proved in [3], and we repeat the argument for completeness. As in [3] we set \( K = \int_R \frac{m(u)}{1 + u^2} du \) and \( s^* (u) = s(-u) \). We have for \( s \in \mathcal{S}_R \):

\[
\| T_m s \|^2 = \int_R \vert \hat{s}(u) \vert^2 m(u) du
\]

\[
= \int_R \vert (1 + u^2)\hat{s}(u) \vert^2 \frac{m(u)}{1 + u^2} du
\]

\[
\leq K \left( \int_R |s * s^4(\xi)| d\xi + \int_R |s' * (s^5)'(\xi)| d\xi \right)
\]

\[
\leq K \left( \left( \int_R |s(\xi)| d\xi \right)^2 + \left( \int_R |s'(\xi)| d\xi \right)^2 \right),
\]

where we have denoted convolution by \( * \). Therefore \( C_m \) is a continuous map from \( \mathcal{S}_R \) into \( \mathbb{R} \), and the existence of \( \mu_m \) follows from the Bochner-Minlos theorem. \( \Box \)

The triplet \( (\Omega, \mathcal{B}, \mu_m) \) will be used as our probability space.
Proposition 2.2. Let \( s \in \mathcal{S}_\mathbb{R} \). Then:

\[
E[\langle \omega, s \rangle^2] = \| T_m s \|_{L_2(\mathbb{R})}^2.
\]

Proof: We have

\[
e^{-\frac{1}{2}\|T_m s\|^2} = \int_{\Omega} e^{i\langle \omega, s \rangle} d\mu_m(\omega).
\]

Expanding both sides of (2.2) in power series we obtain

\[
E[\langle \omega, s \rangle] = \int_{\mathcal{S}_t} \langle \omega, s \rangle d\mu_m(\omega) = 0.
\]

and

\[
E[\langle \omega, s \rangle^2] = \int_{\mathcal{S}_t} \langle \omega, s \rangle^2 d\mu_m(\omega) = \langle T_m s, T_m s \rangle_{L_2(\mathbb{R})}.
\]

We now want to extend the isometry (2.1) when \( s \) is replaced by \( f \) in the domain of \( T_m \). This involves two separate steps: First, an approximation procedure, and next complexification. The following two propositions deal with the approximation.

Proposition 2.3. The space \( T_m(\mathcal{S}) \) is dense in the range of \( T_m \).

Proof: Let \( f \in \mathcal{D}(T_m) \) be such that \( T_m f \) is orthogonal to \( T_m(\mathcal{S}) \), and let \( h_n, n = 1, 2, \ldots \) denote the Hermite functions. Since these are eigenvectors of the Fourier transform we have:

\[
0 = \langle T_m f, T_m h_n \rangle
= \left( \sqrt{m} \hat{f}, \sqrt{m} \hat{h_n} \right)
= c_n \left( \sqrt{m} \hat{f}, \sqrt{m} h_n \right)
= c_n \left( m \hat{f}, h_n \right),
\]

where \( c_n \in \mathbb{R} \) depends only on \( n \). So \( m \hat{f} \equiv 0 \) since the \( h_n \) form a basis of \( L_2(\mathbb{R}) \). Thus \( \sqrt{m} \hat{f} \equiv 0 \) and so \( T_m f = 0 \). \( \square \)

Theorem 2.4. The isometry (2.1) extends to \( T_m f \) where \( f \) is real-valued and in the domain of \( T_m \).

Proof: We first note that, for \( f \) in the domain of \( T_m \) we have

\[
T_m \overline{f} = \overline{T_m f}.
\]
Indeed, since $m$ is even and real we have
$$\hat{T}_m \bar{f} = \sqrt{m} \hat{f} = (\sqrt{m} \hat{f})^\sharp = (\hat{T}_m f)^\sharp = \hat{T}_m f.$$ 

Let now $f$ be real-valued and in the domain of $T_m$, and let $(s_n)_{n \in \mathbb{N}}$ be a sequence of elements in $\mathcal{F}$ such that
$$\lim_{n \to \infty} \|T_m s_n - T_m f\| = 0. \tag{2.6}$$

In view of (2.5), and since $f$ is real-valued we have
$$\lim_{n \to \infty} \|T_m \overline{s_n} - T_m f\| = 0. \tag{2.7}$$

Together with (2.6) this last equation leads to
$$\lim_{n \to \infty} \|T_m (\text{Re } s_n) - T_m f\| = 0. \tag{2.8}$$

In particular $(T_m (\text{Re } s_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in $L_2(\mathbb{R})$. By (2.1), $(\langle \omega, \text{Re } s_n \rangle)_{n \in \mathbb{N}}$ is a Cauchy sequence in $W_m$. We denote by $\langle \omega, f \rangle$ its limit. It is easily checked that the limit does not depend on the given sequence for which (2.6) holds. \hfill $\square$

We denote by $D_\mathbb{R}(T_m)$ the elements in the domain of $T_m$ which are real-valued.

Let $f, g \in D_\mathbb{R}(T_m)$. The polarization identity applied to
$$\mathbb{E}[\langle \omega, f \rangle^2] = \|T_m f\|^2_{L_2(\mathbb{R})}, \quad f \in D_\mathbb{R}(T_m). \tag{2.9}$$

leads to
$$\mathbb{E}[\langle \omega, f \rangle \langle \omega, g \rangle] = \text{Re} \langle T_m f, T_m g \rangle_{L_2(\mathbb{R})}. \tag{2.10}$$

In view of (2.5), $T_m f$ and $T_m g$ are real and so we have:

**Proposition 2.5.** Let $f, g \in D_\mathbb{R}(T_m)$. It holds that
$$\mathbb{E}[\langle \omega, f \rangle \langle \omega, g \rangle] = \langle T_m f, T_m g \rangle_{L_2(\mathbb{R})}. \tag{2.10}$$

**Proposition 2.6.** $\{\langle \omega, f \rangle, f \in D_\mathbb{R}(T_m)\}$ is a Gaussian process in the sense that, for every linear in the sense that for any $f_1, ..., f_n \in D_\mathbb{R}(T_m)$ and $a_1, ..., a_n \in \mathbb{R}$, the random variable $\sum_{i=1}^n a_i \langle \omega, f_i \rangle$ has a normal distribution.

**Proof.** By (2.2), for $\lambda \in \mathbb{R}$ we have,
$$\mathbb{E}[e^{i\lambda \sum_{i=1}^n a_i \langle \omega, f_i \rangle}] = \int_{\Omega} e^{i\lambda \sum_{i=1}^n a_i \langle \omega, f_i \rangle} d\mu_m(\omega) \tag{2.11}$$
$$= \int_{\Omega} e^{i\omega \lambda \sum_{i=1}^n a_i f_i} d\mu_m(\omega)$$
$$= e^{-\frac{1}{2} \lambda^2 \| \sum_{i=1}^n a_i T_m f_i \|^2}.$$
In particular, we have that for any $\xi_1, ..., \xi_n \in D_R(T_m)$ such that $T_m \xi_1, ..., T_m \xi_n$ are orthonormal in $L_2(\mathbb{R})$ and for any $\phi \in L_2(\mathbb{R}^n)$

\[(2.12)\]

\[
E[\phi(\langle \omega, \xi_1 \rangle, ..., \langle \omega, \xi_1 \rangle)] = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(x_1, ..., x_n) \prod_{i=1}^{n} e^{-\frac{1}{2}x_i^2} dx_1 \cdot ... \cdot dx_n.
\]

**Definition 2.7.** We set $G$ to be the $\sigma$-field generated by \{\langle \omega, f \rangle, f \in D_R(T_m)\}, and denote $W_m \triangleq L_2(\Omega, G, \mu_m)$.

Note that $G$ may be significantly smaller than $\mathcal{B}$, the Borel $\sigma$-field of $\Omega$. For example, if $m \equiv 0$, then $T_m$ is the zero operator and $G = \{\emptyset, \Omega, 0, \Omega \setminus \{0\}\}$.

We will see in the following section that the time derivative, in the sense of distributions, of the fundamental stochastic process $B_m$ in the space $W_m$ has spectral density $m(\xi)$. It is therefore justify to refer $W_m$ as the $m$-noise space.

In the case $m(\xi) \equiv 1$, $T_m$ is the identity over $L_2(\mathbb{R})$ and $\mu_m$ is the white noise measure used for example in [11, (1.4), p. 3]. Moreover, by Theorem 1.9 p. 7 there, $G$ equals the Borel sigma algebra and so the 1-noise space coincides with Hida’s white noise space.

### 3. The Process $B_m$

We now define a process $B_m : \mathbb{R} \rightarrow W_m$ via

\[
B_m(t) \triangleq B_m(t, \omega) \triangleq \langle \omega, 1_t \rangle.
\]

where $1_A$ is the indicator function of the set $A$ and $1_t \triangleq 1_{[0, t]}$. This process plays the role of the Brownian motion for the Ito formula in the space $W_m$. Note that this is the same definition as the Brownian motion in [13], the difference being the probability measure assigned to $(\Omega, \mathcal{B})$.

**Theorem 3.1.** $B_m$ has the following properties:

1. $B_m$ is a centered Gaussian random process.
2. For $t, s \in \mathbb{R}$, the covariance of $B_m(t)$ and $B_m(s)$ is

\[
K_m(t, s) = \int_{\mathbb{R}} \frac{e^{i\xi t} - 1 - e^{-i\xi s} - 1}{\xi} m(\xi) d\xi = (T_m 1_t, T_m 1_s).
\]

3. The process $B_m$ has a continuous version under the condition

\[
\int_{\mathbb{R}} \frac{m(\xi)}{1 + |\xi|^2} d\xi < \infty
\]
Proof: (1) follows from (2.11) and (2.3).
To prove (2), we see that by (2.10) we have
\[ E[B_m(t)B_m(s)] = E[\langle \omega, 1_t \rangle \langle \omega, 1_s \rangle] \]
\[ = \text{Re} \left( T_m 1_t, T_m 1_s \right) \]
\[ = (T_m 1_t, T_m 1_s), \]
since this last expression is real.
The covariance function \( K_m \) is a particular case of the covariance function \( K_\sigma \) presented in [3] and (3) is a consequence of Theorem 10.1 there.

\[ \square \]

We bring here two interesting examples for specific choices of the spectral density \( m \) and the resulting process \( B_m \).

**Example 3.2** (filtered white noise). Consider the spectral density \( m_1(\xi) = 1_{[-\Delta, \Delta]}, \ \Delta \geq 0 \).
The corresponding process \( B_{m_1} \) has the covariance function
\[ K_{m_1}(t, s) = \frac{1}{\sqrt{2\pi}} \int_{-\Delta}^{\Delta} \frac{1 - \cos(t\xi) - \cos(s\xi) + \cos((t-s)\xi)}{\xi^2} d\xi, \quad t, s \in \mathbb{R}. \]
The time derivative of this process also belongs to \( \mathcal{W}_m \), and is a stationary Gaussian process with covariance
\[ \frac{\partial^2}{\partial t \partial s} K_{m_1}(t, s) = \frac{1}{\sqrt{2\pi}} \int_{-\Delta}^{\Delta} e^{i(t-s)\xi} d\xi = \frac{2 \sin(\Delta(t-s))}{t-s}. \]

This process can be obtained in physical models by passing a white noise through a low-pass filter with cut-off frequency \( \Delta \). We can see from the covariance function (3.1) that each time sample \( t_0 \in \mathbb{R} \) is positively correlated with time samples in the interval \( [t_0, t_0 + 2\Delta] \), negatively correlated with time samples in the interval \( (t_0 + \frac{3\pi}{2\Delta}, t_0 + 3\frac{\pi}{2\Delta}) \) and so on with decreasing magnitude of correlation. This behavior may describes well price fluctuation of some financial asset.

**Example 3.3** (band limited fractional Brownian motion). We can combine the spectral density \( m_1 \) from the previous example with the spectral density \( |\xi|^{1-2H} \) of the fractional noise with Hurst parameter \( H \in (0.5, 1) \) to obtain a process with covariance function
\[ K_{m_2}(t, s) = \frac{1}{\sqrt{2\pi}} \int_{-\Delta}^{\Delta} \frac{1 - \cos(t\xi) - \cos(s\xi) + \cos((t-s)\xi)}{\xi^2} |\xi|^{1-2H}, \quad t, s \in \mathbb{R}. \]
This process shares both properties of long range dependency of the fractional Brownian motion with Hurst parameter \( H \) and the ripples
of the filtered noise for its time derivative. As the bandwidth \( \Delta \) approaches infinity, the covariance function \( K_{m_2} \) uniformly converges (up to a multiplicative constant) to the covariance of the fractional Brownian motion.

Our next goal is to develop stochastic calculus based on the process \( B_m \) in the space \( \mathcal{W}_m \). The definition of the Wiener integral with respect to \( B_m \) for \( f \in \mathcal{D}(T_m) \) is straightforward and given by

\[
\int_0^T f(t)dB_m(t) \triangleq \langle \omega, f \rangle.
\]

Note that since

\[
\int_{\mathbb{R}} m(\xi)|\hat{f}(\xi)|^2 d\xi \leq \sup_{\xi \in \mathbb{R}} (1 + \xi^2)|\hat{f}(\xi)|^2 \int_{\mathbb{R}} \frac{m(\xi)}{1 + \xi^2} d\xi < \infty,
\]

a sufficient condition for a function \( f \in L_2(\mathbb{R}) \) to be in the domain of \( T_m \) is

\[
\sup_{\xi \in \mathbb{R}} (1 + \xi^2)|\hat{f}(\xi)|^2 \leq \infty.
\]

This is satisfies in particular if \( f \) is differentiable with derivative in \( L_2(\mathbb{R}) \).

Recall that in the white noise space one defines the Itô-Hitsuda stochastic integral of \( X_t \) on the interval \([a, b]\) as

\[
\int_a^b X_t \diamond \dot{B}_m dt
\]

where \( \dot{B}_m \) denotes the time derivative of the Brownian motion and \( \diamond \) denotes the Wick product. The chaos decomposition of the white noise space is used in order to define the Wick product and appropriate spaces of stochastic distributions where \( \dot{B}_m \) lives.

Chaos decomposition for \( \mathcal{W}_m \) can be obtained by a similar procedure to the one carried in [14, 10, section 3] for the fractional Brownian motion. A space of stochastic distributions that contains \( \dot{B}_m \) and is closed under the Wick product can similarly be defined. However, some inconvenience arises when one tries to obtain a chaos decomposition for \( \mathcal{W}_m \) since any basis to yield it explicitly depends on the spectral density \( m \) through \( T_m \). Moreover, the time derivative of the process \( B_m \) may already exists as an element of \( W_m \) as Example 3.2 teaches us. For those reasons we find that an approach based on the analog of the \( S \) transform in our setting is more general and natural since it uses only the expectation and the Lebesgue integral on the real line.
4. The $S_m$ transform

We now define the analog of the $S$ transform in the space $\mathcal{W}_m$ and study its properties. Some of the results can be carried out immediately from properties of the $S$ transform of the white noise space from [11] for example, but others require more attention.

For $s \in \mathcal{S}_\mathbb{R}$ we define the analog of the Wick exponential in the space $\mathcal{W}_m$:

$$e^{(\omega,s)} : \triangleq e^{(\omega,s) - \frac{1}{2} \|T_m s\|^2}$$

**Definition 4.1.** The $S_m$ transform of $\Phi \in \mathcal{W}_m$ is defined by

$$(S_m \Phi)(s) \triangleq \int_\Omega e^{(\omega,s)} : \Phi(\omega) d\mu_m(\omega) = \mathbb{E} \left[ : e^{(\omega,s)} : \Phi(\omega) \right], \quad s \in \mathcal{S}_\mathbb{R}.$$ 

The following result is similar to [1, Theorem 2.2].

**Theorem 4.2.** Let $\Phi, \Psi \in \mathcal{W}_m$. If $(S_m \Phi)(s) = (S_m \Psi)(s)$ for all $s \in \mathcal{S}_\mathbb{R}$, then $\Phi = \Psi$.

**Proof:** By linearity of the $S_m$ transform, it is enough to prove

$$\forall s \in \mathcal{S}, \quad (S_m \Phi)(s) = 0 \Rightarrow \Phi = 0.$$

Let $\{\xi_n\}_{n \in \mathbb{N}} \subset \mathcal{S}_\mathbb{R}$ be a countable dense set in $L^2(\mathbb{R})$ and denote by $\mathcal{G}_n$ the $\sigma$-field generated by $\{\langle \omega, \xi_1 \rangle, \ldots, \langle \omega, \xi_n \rangle\}$. We may choose $\{\xi_n\}_{n \in \mathbb{N}}$ such that $\{T_m \xi_n\}_{n \in \mathbb{N}}$ are orthonormal. For every $n \in \mathbb{N}$, $E[\Phi | \mathcal{G}_n] = \phi_n (\langle \omega, \xi_1 \rangle, \ldots, \langle \omega, \xi_n \rangle)$ for some measurable function $\phi_n : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mathbb{E} \Phi = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \phi_n(x) e^{-\frac{1}{2} x' x} dx < \infty,$$

where $x'$ denotes the transpose of $x$; see for instance [7, Proposition 2.7, p. 7]. Thus, for $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, using (2.12) we obtain

$$0 = \int_\Omega e^{(\omega, \sum_{k=1}^n t_k \xi_k)} : \Phi(\omega) d\mu_m = \int_\Omega e^{(\omega, \sum_{k=1}^n t_k \xi_k)} : \mathbb{E} [\Phi | \mathcal{G}_n] d\mu_m(\omega)$$

$$= e^{-\frac{1}{2} \sum_{k=1}^n t_k \|T_m \xi_k\|^2} \int_\Omega e^{\sum_{k=1}^n t_k (\omega, \xi_k)} \phi_n (\langle \omega, \xi_1 \rangle, \ldots, \langle \omega, \xi_n \rangle) d\mu_m(\omega)$$

$$= e^{-\frac{1}{2} \sum_{k=1}^n t_k \|T_m \xi_k\|^2} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} e^{\sum_{k=1}^n t_k x_k} \phi_n (x_1, \ldots, x_n) e^{-\frac{1}{2} \sum_{k=1}^n x_k^2} dx_1 \cdots dx_n$$

$$= \int_{\mathbb{R}^n} \phi_n(x) e^{-\frac{1}{2} (x-t)'(x-t)} dx.$$ 

By properties of the Fourier transform, we get that $\phi_n = 0$ for all $n \in \mathbb{N}$. Since $\bigcup_{n \in \mathbb{N}} \mathcal{G}_n = \mathcal{G}$ we have $\Phi = 0$. \qed
Definition 4.3. A stochastic exponential is a random variable of the form
\[ e^{\langle \omega, f \rangle}, \quad f \in \mathcal{D}_R(T_m). \]
We denote by \( \mathcal{E} \) the family of linear combinations of stochastic exponentials.

Since \( e^{\langle \omega, f \rangle} := e^{-\frac{1}{2} \| T_m f \|^2} e^{\langle \omega, f \rangle} \), the following claim is a direct consequence of Theorem 4.2.

Proposition 4.4. \( \mathcal{E} \) is dense in \( \mathcal{W}_m \).

Definition 4.5. A stochastic polynomial is a random variable of the form
\[ p (\langle \omega, f_1 \rangle, ..., \langle \omega, f_n \rangle), \quad f_1, ..., f_n \in \mathcal{D}_R(T_m). \]
for some polynomial \( p \) in \( n \) variables. We denote the set of stochastic polynomials by \( \mathcal{P} \).

Corollary 4.6. The set of stochastic polynomials is dense in \( \mathcal{W}_m \).

Proof. We first note that the stochastic polynomials indeed belong to \( \mathcal{W}_m \) because the random variables \( \langle \omega, f \rangle \) are Gaussian and hence have moments of any order.

Let \( \Phi \in \mathcal{W}_m \) such that \( \mathbb{E} [\Phi p] = 0 \) for each \( p \in \mathcal{P} \). Then any \( f \in \mathcal{D}_R(T_m) \),
\[
\mathbb{E} [e^{\langle \omega, f \rangle} \Phi(\omega)] = \mathbb{E} \left[ \sum_{n=0}^{\infty} \frac{(\langle \omega, f \rangle)^n}{n!} \Phi(\omega) \right] = \sum_{n=0}^{\infty} \frac{\mathbb{E} [\langle \omega, f \rangle]^n \Phi(\omega)]}{n!} = 0,
\]
where interchanging of summation is justified by Fubini’s theorem since
\[
\sum_{n=0}^{\infty} \mathbb{E} \left[ \left| \frac{(\langle \omega, f \rangle)^n}{n!} \Phi(\omega) \right| \right] \leq \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \mathbb{E} [\langle \omega, f \rangle^{2n}] \mathbb{E} [\Phi(\omega)^2] 
\]
\[
\leq \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(n!)^2} \| T_m f \|_{L^2(\mathbb{R})}^n \sqrt{\mathbb{E} [\Phi(\omega)^2]} 
\]
\[
\leq \sum_{n=0}^{\infty} \frac{2^n}{n!} \| T_m f \|_{L^2(\mathbb{R})}^n \sqrt{\mathbb{E} [\Phi(\omega)^2]} 
\]
\[
= e^{2\| T_m f \|_{L^2(\mathbb{R})}^2} \cdot \sqrt{\mathbb{E} [\Phi(\omega)^2]} < \infty.
\]
(We have used the Cauchy-Schwarz inequality and the moments of a Gaussian distribution).

We have showed that \( \mathbb{E} [e^{\langle \omega, f \rangle} \Phi(\omega)] = 0 \) for any \( f \in \mathcal{D}_R(T_m) \) so by Theorem 4.2 we obtain \( \Phi = 0 \) in \( \mathcal{W}_m \). \( \square \)
Lemma 4.7. Let \( f, g \in \mathcal{D}_\mathbb{R}(T_m) \). Then
\[
E[\langle \omega, f \rangle : \langle \omega, g \rangle :] = e^{(T_m f, T_m g)_{L_2(\mathbb{R})}}.
\]

Proof:
\[
E[\langle \omega, f \rangle :] = e^{-\frac{1}{2}\|T_m f\|^2} E[e^{\langle \omega, f \rangle}] = 1,
\]
since \( E[e^{\langle \omega, f \rangle}] \) is the moment generating function of the Gaussian random variable \( \langle \omega, f \rangle \) with variance \( \|T_m f\|^2 \) valued at 1.

Thus we get
\[
E[\langle \omega, f \rangle : \langle \omega, g \rangle :] = e^{(T_m f, T_m g)_{L_2(\mathbb{R})}} E[\langle \omega, f + g \rangle :] = e^{(T_m f, T_m g)_{L_2(\mathbb{R})}}.
\]

□

The following formula is useful in calculating the \( S_m \) transform of a multiplication of two random variables, and can be easily proved using Lemma 4.7.

\[
S_m \left( \langle \omega, f \rangle : \langle \omega, g \rangle : \right) = e^{(T_m s, T_m f)_{L_2(\mathbb{R})}} e^{(T_m s, T_m g)} e^{(T_m s, T_m g)}, \quad f, g \in \mathcal{D}_\mathbb{R}(T_m).
\]

Proposition 4.8. Let \( \{\Phi_n\} \) be a sequence in \( \mathcal{W}_m \) that converges in \( \mathcal{W}_m \) to \( \Phi \). Then for any \( s \in \mathcal{S}_\mathbb{R} \) the sequence of real numbers \( \{S_m(\Phi_n)(s)\} \) converges to \( S_m(\Phi)(s) \).

Proof. For any \( s \in \mathcal{S}_\mathbb{R} \),
\[
|S_m\Phi_n(s) - S_m\Phi(s)| = |E \left[ \langle \omega, s \rangle : (\Phi_n - \Phi) \right]| \leq \sqrt{E \left[ \langle \omega, s \rangle^2 \right]} \cdot \sqrt{E \left[ (\Phi_n - \Phi)^2 \right]}.
\]
By direct calculation \( E \left[ \langle \omega, s \rangle^2 \right] = e^{\|T_m s\|^2} \) and since \( E \left[ (\Phi_n - \Phi)^2 \right] \rightarrow 0 \), the claim follows. □

In the statement of Theorem 4.9 recall that \( T_m \) is a bounded operator from \( \mathcal{S}_\mathbb{R} \) into \( L_2(\mathbb{R}) \) and so its adjoint is a bounded operator from \( L_2(\mathbb{R}) \) into \( \mathcal{S}_\mathbb{R}^\prime \).

Theorem 4.9. For \( \Phi \in \mathcal{W}_m \) and \( s \in \mathcal{S}_\mathbb{R} \),
\[
S_m\Phi(s) = \int_\Omega \Phi(\omega + T_m^* T_m s) d\mu_m(\omega).
\]
STOCHASTIC INTEGRATION AND AN EXTENSION OF S-TRANSFORM

Proof. Assume first that \( \Phi (\omega) =: e^{(\omega, s_1)} \), where \( s_1 \in \mathcal{S}_R \). We have by Lemma 4.7 that

\[
\int_{\Omega} \Phi (\omega + T_m s) d\mu_m (\omega) = \int_{\Omega} e^{(\omega, s_1)} e^{(T_m s, s_1)} d\mu_m (\omega)
\]

\[
= e^{(T_m s, s_1)} \int_{\Omega} e^{(\omega, s_1)} d\mu_m (\omega)
\]

\[
= e^{(T_m s, T_m s_1)} 1
\]

\[
= S_m \Phi (s).
\]

The result may be extended by linearity to any \( \Phi \in \mathcal{E} \) which is a dense subset of \( \mathcal{W}_m \) by Propositions 4.4 and 4.8.

We can find the \( S_m \) transform of powers of \( \langle \omega, f \rangle \) for \( f \in \mathcal{D}_R (T_m) \) by the formula for Hermite polynomials.

**Corollary 4.10.** For \( f \in \mathcal{D}_R (T_m) \) and \( s \in \mathcal{S}_R \), we have that

\[
(T_m s, T_m f)^n = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{2} \right)^m \left( S_m (\omega, f)^{n-2m} \right)(s) \|T_m f\|^{2m} / m!(n-2m)!
\]

in particular

\[
(S_m (\omega, f))(s) = (T_m f, T_m s)
\]

and

\[
(S_m (\omega, f)^2)(s) = (T_m f, T_m s)^2 + \|T_m s\|^2
\]

**Proof.** From Lemma 4.7 we get that

\[
(S_m : e^{(\omega, f) :}))(s) = e^{(T_m s, T_m f)},
\]

then,

\[
e^{-\frac{1}{2}\|T_m f\|^2} S_m \left( \sum_{k=0}^{\infty} \frac{\langle \omega, f \rangle^k}{k!} \right)(s) = \sum_{k=0}^{\infty} \frac{(T_m s, T_m f)^k}{k!}
\]

By the linearity of the \( S_m \) transform and Fubini’s theorem, and replacing \( f \) by \( tf \) with \( t \in \mathbb{R} \) we compare powers of \( t \) at both sides to get (4.4). □

This last corollary can also be formulated in terms of the Hermite polynomials. Recall that the \( n_{th} \) Hermite polynomial with parameter \( t \in \mathbb{R} \) is defined by

\[
h_n^{[t]}(x) \triangleq n! \sum_{m=0}^{\lfloor n/2 \rfloor} \left( -\frac{1}{2} \right)^m x^{n-2m} \cdot t^{2m} / m!(n-2m)!
\]
(see for instance [19, p. 33]). For \( f \in \mathcal{D}(T_m) \) we define
\[
\tilde{h}_n (\langle \omega, f \rangle) \triangleq h_n^{[\|T_m f\|]} (\langle \omega, f \rangle) = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m \langle \omega, f \rangle^{n-2m} \cdot \|T_m f\|^{2m}}{m!(n-2m)!},
\]
and we also set \( \tilde{h}_0 = 1 \).

So by (4.4) we have that
\[
(4.10) \quad \left( S_m \tilde{h}_n (\langle \omega, f \rangle) \right) (s) = (T_m s, T_m f)^n.
\]

Using Equation 4.4 and Lemma 4.7 one can easily verify the following result:

**Proposition 4.11.** Let \( f \in \mathcal{D}_t(T_m) \). It holds that:
\[
(4.11) \quad e^{\langle \omega, f \rangle} := \sum_{k=0}^{\infty} \frac{\tilde{h}_k (\langle \omega, f \rangle)}{k!}
\]

It is possible to define a Wick product in \( \mathcal{W}_m \) using the \( S_m \) transform.

**Definition 4.12.** Let \( \Phi, \Psi \in \mathcal{W}_m \). The Wick product of \( \Phi \) and \( \Psi \) is the element \( \Phi \diamond \Psi \in \mathcal{W}_m \) that satisfies
\[
S_T (\Phi \diamond \Psi) (s) = (S_T \Phi) (s) (S_T \Psi) (s), \quad \forall s \in \mathcal{S}_T,
\]
if it exists.

As this definition suggests, in general the Wick product is not stable in \( \mathcal{W}_m \).

From (4.10), the Wick product of Hermite polynomials satisfies
\[
\tilde{h}_n (\langle \omega, f \rangle) \diamond \tilde{h}_k (\langle \omega, f \rangle) = \tilde{h}_{n+k} (\langle \omega, f \rangle), \quad n, k \in \mathbb{N}, \quad f \in \mathcal{D}_t(T_m).
\]

### 5. The stochastic integral

We now use the \( S_m \)-transform to define a Wick-Itô stochastic integral and prove an Itô formula for this integral. In the next section we also show that for particular choice of \( m \), our definition of the stochastic integral coincide with previously defined Wick-Itô stochastic integrals for fractional Brownian motion; see [8, 6]. We set
\[
\mathcal{B}_s (t) = S_m (B_m(t)) (s).
\]

By (4.5) we see that
\[
(5.1) \quad \mathcal{B}_s (t) = (T_m s, T_m 1_t) = \int_{\mathbb{R}} m(\xi) \hat{s}(\xi) \frac{e^{i \xi t} - 1}{\xi} d\xi.
\]
This function is absolutely continuous with respect to Lebesgue measure and its derivative is

\[(5.2) \quad (B_s(t))' = \int_{\mathbb{R}} m(\xi)\hat{s}(\xi)e^{i\xi t}d\xi.\]

We note that when \(T_m\) is a bounded operator from \(L_2(\mathbb{R})\) into itself we have by a result of Lebesgue (see \([23, \text{p. 410}]\)), \((B_s(t))' = (T_m^* T_m s)(t)\) (a.e.).

**Definition 5.1.** Let \(M \in \mathbb{R}\) be a Borel set and \(X : M \to \mathcal{W}_m\) a stochastic process. The process \(X\) will be called integrable over \(M\) if for any \(s \in \mathcal{S}_R\), \((S_m X_t)(s)B_s(t)'\) is integrable on \(M\), and if there is a \(\Phi \in \mathcal{W}_m\) such that

\[S_m \Phi(s) = \int_M (S_m X_t)(s)B_s(t)' dt.\]

for any \(s \in \mathcal{S}_R\). If \(X\) is integrable, \(\Phi\) is uniquely determined by Theorem 4.2 and we denote it by \(\int_M X_t dB_m(t)\).

If \(T = \text{Id}_{L_2(\mathbb{R})}\), this definition coincides with the Hitsuda-Skorohod integral \([11, \text{Chapter 8}]\). See also Section 5.

Note that since

\[|B_s(t)'| \leq \int_{\mathbb{R}} m(\xi)|\hat{s}(\xi)|d\xi \leq \sup_\xi |(1 + \xi^2)\hat{s}(\xi)| \int_{\mathbb{R}} \frac{m(\xi)}{1 + \xi^2} d\xi < \infty,
\]

for any \(s \in \mathcal{S}_R\) there exist a constant \(K_s\) such that

\[|\int_M S_m X_t(s)B_s(t)' dt| \leq K_s \int_M |\mathbb{E} [X_t : e^{i(\omega,s)}]| dt \leq K_s \mathbb{E} [\mathbb{E} [e^{i(\omega,s)}]] \int_M |E[X_t]| dt.
\]

Thus a sufficient condition for the integrability of \(S_m X_t(s)B_s(t)\)' is \(\int_M E|X_t|dt < \infty\).

We note that general conditions for the integrability of a stochastic process on \(\mathcal{W}_m\) cannot be easily obtained. Inconvenient conditions for integrability is the price we pay for not relying on stochastic distributions.

**Theorem 5.2.** Any non-random \(f \in \mathcal{D}_R(T_m)\) is integrable and we have,

\[(5.3) \quad \int_0^r f(t) dB_m(t) = \langle \omega, 1_{[0,r]} f \rangle.\]
Proof. In virtue of (4.5) and the definition of the stochastic integral, we need to show that
\[
\int_0^\tau f(t)B_s(t)dt = (T_m s, T_m 1_{[0,\tau]} f).
\]

Using formula (5.2) and Fubini’s theorem, we have:
\[
\int_0^\tau f(t)B_s(t)dt = \int_0^\tau f(t) \left( \int_{\mathbb{R}} m(\xi)\hat{s}(\xi)e^{it\xi}d\xi \right) dt \\
= \int_{\mathbb{R}} m(\xi)\hat{s}(\xi) \left( \int_0^\tau f(t)e^{it\xi}dt \right) d\xi \\
= \int_{\mathbb{R}} m(\xi)\hat{s}(\xi) \left( \hat{f}(1_{[0,\tau]}(\xi)) \right) d\xi \\
= (T_m s, T_m 1_{[0,\tau]} f).
\]

\[\square\]

Proposition 5.3. The stochastic integral has the following properties:

(1) For \(0 \leq a < b \in \mathbb{R}\),
\[
B_m (b) - B_m (a) = \int_a^b dB_m (t)
\]

(2) Let \(X : M \rightarrow \mathcal{W}_m\) an integrable process. Then
\[
\int_M X_t dB_m (t) = \int_{\mathbb{R}} 1_M X_t dB_m (t).
\]

(3) Let \(X : M \rightarrow \mathcal{W}_m\) an integrable process. Then
\[
\mathbb{E} \left[ \int_M X_t dB_m (t) \right] = S_m \left( \int_M X_t dB_m (t) \right) (0) = 0
\]

(4) The Wick product and the stochastic integral can be interchanged: Let \(X : \mathbb{R} \rightarrow \mathcal{W}_m\) an integrable process and assume that for \(Y \in \mathcal{W}_m\), \(Y \odot X_t\) is integrable. Then,
\[
Y \odot \int_{\mathbb{R}} X_t dB_T (t) = \int_{\mathbb{R}} Y \odot X_t dB_T (t)
\]
Proof. The proof of the first three items is easy and we omit it. The last item is proved in the following way:

\[
S_m(Y \diamond \int \mathbb{R} X_t dB_m(t))(s) = (S_m Y)(s) \int_M (S_m X_t)(s) dB_T
\]

\[
= \int_M (S_m Y)(s)(S_m X_t)(s) dB_m
\]

\[
= S_m(\int \mathbb{R} Y \diamond X_t dB_m(t))(s).
\]

Example 5.4. For \( \tau \geq 0 \), we have by equation (4.6),

\[
\int_0^\tau (T_m s, T_m 1_t) \frac{d}{dt} (T_m s, T_m 1_t) dt = \frac{1}{2} (T_m s, T_m 1_\tau)^2
\]

\[
= \frac{1}{2} S_m (\langle \omega, 1_t \rangle \quad \| T_m 1_t \|^2) (s).
\]

Then \( B_m \) is integrable on the interval \([0, \tau]\), and we have

\[
\int_0^\tau B_m(t) dB_m(t) = \frac{1}{2} B_m(\tau)^2 - \frac{1}{2} \| T_m 1_\tau \|^2.
\]

This reduces to the well known result if \( m(\xi) \equiv 1 \) and \( T_m \) is then the identity operator.

Example 5.5. Let \( \tilde{h}_n \) be defined by (4.9). A similar argument to the one in (5.2) will show that any \( f \) such that for any \( f_1 t \in \mathcal{D}(T_m) \) we have that the function \( t \mapsto (T_m s, T_m f_1 t) \) is differentiable with time derivative

\[
\frac{d}{dt} (T_m s, T_m f_1 t) = f(t) \int \mathbb{R} m(\xi) \tilde{s}(\xi) e^{-i \xi t} d\xi = f(t) B_s(t)'.
\]

By a similar argument to Theorem 5.2 we have

\[
\frac{1}{n+1} S_m \left( \tilde{h}_{n+1} (\langle \omega, 1_\tau f \rangle) (t) \right) (s) = \frac{1}{n+1} (T_m s, T_m 1_\tau f)
\]

\[
= \int_0^\tau (T_m s, T_m f_1 t)^n f(t) B_s(t)' dt
\]

\[
= S_m \left( \int_0^\tau f(t) \tilde{h}_n (\langle \omega, 1_t f \rangle) dB_m(t) \right) (s).
\]

Thus,\n
\[
(5.4) \quad \int_0^\tau f(t) \tilde{h}_n (\langle \omega, 1_t f \rangle) dB_m(t) = \frac{1}{n+1} \tilde{h}_{n+1} (\langle \omega, 1_\tau f \rangle).
\]
It follows from (5.4) that for any polynomial \( p \) and \( f \) with \( 1_t f \in \mathcal{D}(T_m) \) the process \( p(\langle \omega, 1_t f \rangle) \) is integrable. This result can be easily extended to the process 

\[
t \mapsto e^{\langle \omega, 1_t f \rangle}, \quad 1_t f \in \mathcal{D}(T_m),
\]

and we also obtain

**Corollary 5.6.**

\[
\int_0^\tau f(t) : e^{\langle \omega, 1_t f \rangle} : dB_m(t) =: e^{\langle \omega, 1_t f \rangle} : -1.
\]

**Example 5.7.** Let \( f \in \mathcal{D}_\mathbb{R}(T_m) \). Using (4.3) we can obtain

\[
S_m \left( e^{\langle \omega, f \rangle} : e^{\langle \omega, 1_t f \rangle} : dB_m(t) \right)(s) = S_m \left( e^{\langle \omega, f \rangle} : e^{\langle \omega, 1_t \rangle} : - e^{\langle \omega, f \rangle} \right)(s) = e^{(T_m s, T_m f)} \left( e^{(T_m s, T_m 1_t)} e^{(T_m f, T_m 1_t)} - 1 \right).
\]

On the other hand,

\[
S_m \left( \int_0^\tau e^{\langle \omega, f \rangle} : e^{\langle \omega, 1_t \rangle} : dB_m(t) \right)(s) = e^{(T_m s, T_m f)} \int_0^\tau e^{(T_m s, T_m 1_t)} e^{(T_m f, T_m 1_t)} \frac{d}{dt} (T_m s, T_m 1_t) \, dt
\]

\[
= e^{(T_m s, T_m f)} \left( e^{(T_m s, T_m 1_t)} e^{(T_m f, T_m 1_t)} - 1 \right)
\]

\[
- \int_0^\tau e^{(T_m s, T_m 1_t)} e^{(T_m f, T_m 1_t)} \frac{d}{dt} (T_m f, T_m 1_t) \, dt.
\]

So in general for an integrable stochastic process \( X \) and a random variable \( Y \),

\[
Y \int_0^\tau X_t dB_m(t) \neq \int_0^\tau Y X_t dB_m(t).
\]

6. Itô’s formula

In this section we prove an Itô’s formula. We begin by proving an extension of the classical Girsanov theorem to our setting.

**Theorem 6.1.** Let \( f \in \mathcal{D}(T_m) \), and let \( \mu \) be the measure defined by \( \mu(A) = E[ e^{\langle \omega, T_m f \rangle} : 1_A] \). The process

\[
\tilde{B}_m(t) \triangleq B_m(t) - (T_m f, T_m 1_t),
\]

is Gaussian and satisfies

\[
\mathbb{E}_\mu[\tilde{B}_m(t) \tilde{B}_m(s)] = (T_m 1_t, T_m 1_s).
\]

**Proof:** We will first prove that for all \( t \geq 0 \), \( \tilde{B}_m(t) \) is a Gaussian random variable relative to the measure \( \mu \) by considering its moment generating function \( \mathbb{E}_\mu \left[ e^{\lambda \tilde{B}_m(t)} \right], \lambda \in \mathbb{R} \),
STOCHASTIC INTEGRATION AND AN EXTENSION OF $S_m$-TRANSFORM 19

$E_\mu \left[ e^{\lambda \tilde{B}_m(t)} \right] = E \left[ e^{\langle \omega, T_m f \rangle - \frac{1}{2} \|T_m f\|^2} e^{\lambda \langle \omega, 1_t \rangle - \lambda (T_m 1_t, 1_t)} \right]$

$= e^{-\lambda (T_m 1_t, 1_t)} e^{-\frac{1}{2} \|T_m f\|^2} E \left[ e^{\langle \omega, T_m f + \lambda 1_t \rangle} \right] .$

Since $\langle \omega, \phi + \lambda 1_t \rangle$ is a zero mean Gaussian random variable with variance

$\|T_m (f + \lambda 1_t)\|^2 = \|T_m f\|^2 + \lambda^2 \|T_m 1_t\|^2 + 2 \lambda (T_m \phi, T_m 1_t),$

its moment generating function evaluated at 1 is given by

$E \left[ e^{\langle \omega, T_m f + \lambda 1_t \rangle} \right] = e^{\frac{1}{2} \|T_m f\|^2} e^{\frac{1}{2} \lambda^2 \|T_m 1_t\|^2} e^{\lambda (T_m f, T_m 1_t)},$

and we conclude from (6.1) that

$E_\mu \left[ e^{\lambda \tilde{B}_m(t)} \right] = e^{\frac{1}{2} \lambda^2 \|T_m 1_t\|^2}.$

Thus for all $t \geq 0$, $\tilde{B}_m(t)$ is a zero mean Gaussian random variable on $(\Omega, \mathcal{G}, \mu_m)$. Similar arguments will show that any linear combination of time samples is a Gaussian variable, and thus $\tilde{B}_m(t), t \geq 0$ is a Gaussian process. Finally, by the polarization formula,

$E_\mu [ \tilde{B}_m(t) \tilde{B}_m(s) ] = (T_m 1_{[0,t]}, T_m 1_{[0,s]}).$

We now interpret integrals of the type $\int_0^\tau \Phi(t) dt$, where for every $t \in [0, \tau], \Phi(t) \in W_m$, as Pettis integrals, that is as

$E \left[ \left( \int_0^\tau \Phi(t) dt \right) \Psi \right] = \int_0^\tau E[\Phi(t) \Psi] dt, \quad \forall \Psi \in W_m,$

under the hypothesis that the function $t \mapsto E[\Phi(t) \Psi]$ belongs to $L_1([0, \tau], dt)$ for every $\Psi \in W_m$. See [12] pp. 77-78. We note that if $X$ is moreover pathwise integrable and such that the pathwise integral belongs to $W_m$, then

$\int_0^\tau E[|X_t|] dt < \infty,$

and we can apply Fubini’s theorem to show that both integrals coincide. It is also clear from the definition of the Pettis integral that it commutes with the $S_m$ transform.
We define the conditions
\[
E \left[ |F(t, X_t)| : e^{\langle \omega, s \rangle} \right] < \infty \tag{6.4}
\]
\[
E \left[ \left| \frac{\partial F}{\partial t}(t, X_t) \right| : e^{\langle \omega, s \rangle} \right] < \infty \tag{6.5}
\]
\[
E \left[ \left| \frac{\partial F}{\partial x}(t, X_t) \right| : e^{\langle \omega, s \rangle} \right] < \infty \tag{6.6}
\]
for \( F \in C^{1,2}([0, \infty), \mathbb{R}) \).

We shall now develop an Itô formula for a class of stochastic processes of the form,
\[
X_t(\omega) = \int_0^t f(t) dB_m(t) = \langle \omega, 1_{[0,\tau]} f \rangle, \quad \tau \geq 0, \quad I_{[0,\tau]} f \in \mathcal{D}(T_m).
\]

**Theorem 6.2.** Let \( F \in C^{1,2}([0, \infty), \mathbb{R}) \) satisfying (6.4)–(6.6), and assume that the function \( \|T_m 1_t f\|^2 \) is absolutely continuous with respect to the Lebesgue measure as a function of \( t \). Then we have,
\[
F(\tau, X_\tau) - F(0, 0) = \int_0^\tau \frac{\partial}{\partial t} F(t, X_t) dt + \int_0^\tau f(t) \frac{\partial}{\partial x} F(t, X_t) dB_m(t) + \frac{1}{2} \int_0^\tau \frac{d}{dt} \| T_m 1_t f \|^2 \frac{\partial^2}{\partial x^2} F(t, X_t) dt
\]
\[\text{in } \mathcal{W}_m.\]

This proof is based on [19, Section 13.5].

**Proof.** Let \( s \in \mathcal{S} \) and \( f \in \mathcal{D}(T_m) \). It follows from Theorem 6.1 that for every \( t \in [0, \tau] \), \( X_t(\omega) = \langle \omega, 1_t f \rangle \) is normally distributed under the measure
\[
\mu_s(A) \triangleq E \left[ 1_A \exp \left\{ \langle \omega, s \rangle - \frac{1}{2} \| T_m s \|^2 \right\} \right] = E \left[ 1_A : e^{\langle \omega, s \rangle} \right],
\]
with mean
\[
(T_m s, T_m 1_{[0,t]} f)
\]
and variance
\[
\| T_m 1_{[0,t]} f \|^2.
\]
Thus,

\[(6.9)\]

\[
(S_m F(t, X_t))(s) = \mathbb{E} \left[ e^{(\omega, s)} : F(t, X_t) \right] = \int_{\mathbb{R}} F(t, u + (T_m 1_{[0,t]} f, T_m s)) \rho \left( \|T_m 1_{[0,t]} f\|^2, u \right) du,
\]

where \(\rho(w, u) = \frac{1}{\sqrt{2\pi w}} e^{-\frac{u^2}{2w}}\) and satisfies,

\[(6.10)\]

\[
\frac{\partial}{\partial w} \rho = \frac{1}{2} \frac{\partial^2}{\partial u^2} \rho.
\]

Integrating by part we obtain:

\[(6.11)\]

\[
\int_{\mathbb{R}} F(t, u) \frac{\partial^2}{\partial u^2} \rho(w, u) du = \int_{\mathbb{R}} \frac{\partial^2}{\partial u^2} F(t, u) \rho(w, u) du.
\]

In view of (6.4), (6.6) we may differentiate under the integral sign by (6.9), (6.10) and (6.11) and obtain for \(0 \leq t \leq \tau\),

\[
\frac{d}{dt} S_m (F(t, X_t))(s) = \int_{\mathbb{R}} \frac{\partial}{\partial t} F(t, u + (T_m 1_{[0,t]} f, T_m s)) \rho \left( \|T_m 1_{[0,t]} f\|^2, u \right) du
\]

\[
+ \int_{\mathbb{R}} \frac{\partial}{\partial x} F(t, u + (T_m 1_{[0,t]} f, T_m s)) \frac{d}{dt} (T_m 1_{[0,t]} f, T_m s) \rho \left( \|T_m 1_{[0,t]} f\|^2, u \right) du
\]

\[
+ \int_{\mathbb{R}} F(t, u + (T_m 1_{[0,t]} f, T_m s)) \frac{d}{dt} \|T_m 1_{[0,t]} f\|^2 \frac{\partial}{\partial t} \rho \left( \|T_m 1_{[0,t]} f\|^2 \right) du
\]

\[
= S_m \left( \frac{\partial}{\partial t} F(t, X_t) \right)(s) + \frac{d}{dt} (T_m s, T_m 1_{[0,t]} f) S_m \left( \frac{\partial}{\partial x} F(t, X_t) \right)(s)
\]

\[
+ \frac{1}{2} \frac{d}{dt} \|T_m 1_{[0,t]} f\|^2 \cdot S_m \left( \frac{\partial^2}{\partial x^2} F(t, X_t) \right)(s).
\]

Hence,

\[(6.12)\]

\[
S_m (F(\tau, X_\tau) - F(0, 0))(s) = \int_0^\tau S_m \left( \frac{\partial}{\partial t} F(t, X_t) \right)(s) dt
\]

\[
+ \int_0^\tau \frac{d}{dt} (T_m s, T_m 1_{[0,t]} f) S_m \left( \frac{\partial}{\partial x} F(t, X_t) \right)(s) dt
\]

\[
+ \frac{1}{2} \int_0^\tau \frac{d}{dt} \|T_m 1_{[0,t]} f\|^2 \cdot S_m \left( \frac{\partial^2}{\partial x^2} F(t, X_t) \right)(s) dt.
\]
Note that,
\[ S_m \left( \int_0^\tau f(t) \frac{\partial}{\partial x} F(t, X_t) dB_m(t) \right) (s) = \int_0^\tau \frac{d}{dt} (T_m s, T_m 1_{[0,t]} f) S_m \left( \frac{\partial}{\partial x} F(t, X_t) \right) (s) dt. \]

Thus we may use Fubini’s theorem to interchange the \( S_m \)-transform and the pathwise integral, and obtain that the \( S_m \)-transform of the right hand side of (6.8) is exactly the right hand side of (6.12) and the theorem is proved. \( \square \)

7. Relation with other white-noise extensions of Wick-Itô integral

Recall that the white noise space correspond to \( m(\xi) \equiv 1 \), so denoting it \( W_1 \) is consistent with our notation, and \( S_1 \) is the classical \( S \)-transform of the white noise space. We define a map \( \tilde{T}_m : W_m \rightarrow W_1 \) by describing its action on the dense set of stochastic polynomials in \( W_m \):
\[ \tilde{T}_m \langle \omega, f \rangle^n = \langle \omega, T_m f \rangle^n, \quad f \in D(T_m). \]
Note that since \( D(T_m) \subset L_2(\mathbb{R}) \) this map is well defined. It easy to see that \( \tilde{T}_m \) is an isometry of Hilbert spaces. By continuity we obtain that
\[ \tilde{T}_m e^{\langle \omega, f \rangle} = e^{\langle \omega, T_m f \rangle}, \quad f \in D(T_m), \]

hence
\[ \left( S_1 \tilde{T}_m e^{\langle \omega, f \rangle} \right) (T_m s) = e^{\langle T_m s, T_m f \rangle} = \left( S_m e^{\langle \omega, f \rangle} \right) (s). \]

So this relations between the \( S_1 \) and the \( S_m \) transform is extended such that for any \( \Phi \in W_m \),
\[ \left( S_1 \tilde{T}_m \Phi \right) (T_m s) = (S_m \Phi) (s). \]

Let \( X : [0, \tau] \rightarrow W_m \) be a stochastic process. We have defined its Itô integral as the unique element \( \Phi \in W_m \) (if exists) having \( S_m \)-transform
\[ (S_m \Phi) (s) = \int_0^\tau (X_t) (s) \frac{d}{dt} (T_m s, T_m 1_t) (s) dt. \]

This suggests that if we define in the white noise the process \( \tilde{B}_m \) as \( \langle \omega, T_m 1_t \rangle \) and stochastic integral with respect to \( \tilde{B}_m \) as the unique element \( \Phi \in W_1 \) (if exists) having \( S_1 \)-transform
\[ (S_1 \Phi) (s) = \int_0^\tau (X_t) (s) \frac{d}{dt} (s, T_m 1_t)) (s) dt, \]

(7.1)
both definitions coincides in the sense that

\begin{equation}
\widetilde{T}_m \int_0^\tau X_t dB_m(t) = \int_0^\tau \widetilde{T}_m X_t dB_m(t).
\end{equation}

Recall that the fractional brownian motion can be obtained in our setting by taking \( m(\xi) = \frac{1}{2} |\xi|^{1-2H} \), \( H \in (0, 1) \), which results in \( T_m = M_H \), where \( M_H \) defined in [10]. In the white noise, the fractional Brownian motion can be defined by the continuous version of the process \( \{\langle \omega, M_{Ht}\rangle\}_{t \geq 0} \).

An approach based on the definition described in (7.1) for the fractional Brownian motion was given in [4]. Due to Theorem 3.4 there, under appropriate conditions our definition of the Itô integral in the case of \( T_m = M_H \) coincide in the sense of (7.2) with the Hitsuda-Skorohod integral. Stochastic integration in the white noise setting for the family of stochastic processes considered in this paper can be found in [1], and its equivalence to the integral described here can be obtained by a similar argument to that of Theorem 3.4 in [4].

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