Dynamics of planets in retrograde mean motion resonance

Abstract

In a previous paper (Gayon & Bois 2008a), we have shown the general efficiency of retrograde resonances for stabilizing compact planetary systems. Such retrograde resonances can be found when two-planets of a three-body planetary system are both in mean motion resonance and revolve in opposite directions. For a particular two-planet system, we have also obtained a new orbital fit involving such a counter-revolving configuration and consistent with the observational data.

In the present paper, we analytically investigate the three-body problem in this particular case of retrograde resonances. We therefore define a new set of canonical variables allowing to express correctly the resonance angles and obtain the Hamiltonian of a system harboring planets revolving in opposite directions. The acquiring of an analytical “rail” may notably contribute to a deeper understanding of our numerical investigation and provides the major structures related to the stability properties. A comparison between our analytical and numerical results is also carried out.

Keywords Hamiltonian Systems · Planetary Systems · Resonance · Periodic Orbits
1 Introduction

The simplicity and elusiveness of the full three-body problem have captivated mathematicians for centuries. Still today, because of its non-integrability, the three-body problem remains tricky. It has been solved for particular cases, by reducing the number of degrees of freedom. One of the first major results is the determination of families of periodic orbits in the three-body problem under certain assumptions (see for instance Hénon 1976 or Hadjidemetriou 1976).

Since the first observations of extrasolar systems, the interest of the general three-body problem has been boosted, mainly under the coplanar assumption. Moreover, the occurrence of bodies in Mean Motion Resonances (MMR) allowed to study the stability of several two-planet systems (see e.g. Hadjidemetriou 2002, 2008 or Callegari et al. 2004). The secular behavior of two-planet systems has been investigated as well (e.g. Henrard & Libert 2008). Because most presently known extrasolar planets move on eccentric orbits, Beaugé & Michtchenko (2003) investigated the eccentric, coplanar three-body problem to characterize the dynamics of such two-planet systems.

Faced to the detection of multi-planetary systems (one star and 2 planets or more), and to the increasing number of degrees of freedom, numerical methods of global analysis have been performed in order to study their dynamical stability. By combining $N$-body integrators with numerical tools for detection of chaos (notably fast indicators or frequency analyses), the exploration of the $N$-body planetary problem (with $N \geq 3$) is accessible from numerical methods. As a consequence, mechanisms generating stability of multi-planetary systems can be identified and explained. For instance, mechanisms involving a MMR as well as an Apsidal Synchronous Precession$^1$ (ASP hereafter) have been intensively studied (e.g. Lee & Peale 2002, 2003; Bois et al. 2003, Ji et al. 2003). A solution involving both MMR and ASP describes well the stability of eccentric, compact multi-planetary systems, but may not however be unique. We note, for example, that other multi-planetary systems have been found to be mainly controlled by secular dynamics (e.g. Michtchenko et al. 2006 or Ji et al. 2007). In Gayon & Bois (2008a), by using a numerical method of global analysis$^2$ we found a novel stabilizing mechanism involving a retrograde MMR in a two-planet system, which means that the two planets are both in MMR and revolve in opposite direction$^3$ Let us note that the difference between a prograde MMR and a retrograde MMR does not only lie in the retrograde motion of one of the two planets. The mechanisms of stability related to a retrograde MMR and their underlying properties differ from the prograde case. Such stabilities are generally more robust (see Gayon & Bois 2008a, 2008b).

The assumption that two giant planets are in a MMR and revolving in opposite directions around their hosting star is apparently contradicting to the most accepted formation theory of planetary systems, notably to the formation and evolution of the resonant planetary systems (core accretion mechanism combined by

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1 In case of ASP, the apsidal lines on average precess at the same rate.
2 We used the MIPS (Megno Indicator for Planetary Systems) method based on the MEGNO technique (Mean Exponential Growth factor of Nearby Orbits; see Cincotta & Simò 2000).
3 For instance, the inner planet has a prograde direction while the outer planet a retrograde one. In the following, we will also use the terminology of “counter-revolving configuration”.

a planetary migration scenario). However, in Gayon & Bois (2008a), we present two feasible processes leading to planets revolving in opposite directions. The first scenario has been introduced by Nagasawa et al. (2008) and consists in the combination of a planet-planet scattering, a tidal circularization and Kozai mechanisms. Starting from a hierarchical 3-planet system and considering a migration mechanism including a process of planet-planet scattering and tidal circularization, the authors show that close-in planets may be formed. In a few cases, due to the Kozai mechanism, one planet enters a retrograde motion. The second feasible process imagined by Varvoglis (private discussions) and ourselves is related to the capture of free-floating planets. By integrating the trajectories of planet-sized bodies that encounter a coplanar two-body system (a Sun-like star and a Jupiter mass), Varvoglis finds that the probability of capture is significant, and, moreover, that the percentage of free-floating planets captured is higher for retrograde motions than for prograde motions. As a consequence, it would be possible to find one day some planetary systems in counter-revolving configuration.

In the case of a two-planet system, analytical and numerical methods are complementary. Besides, the acquiring of an analytical “rail” contributes to a deeper understanding of the numerical exploration. As a result, the analytical approach permits to study thoroughly the numerical results. By plotting surfaces of sections, it provides the major structures related to the stability properties. As a consequence, in the present paper, we propose a Hamiltonian expansion based on this novel and feasible stabilizing mechanism introduced in Gayon & Bois (2008a and 2008b). We therefore investigate the three-body problem in the particular case of retrograde resonances, adapting the Hamiltonian approach of Beaugé & Michtchenko (2003) firstly expanded for eccentric, coplanar, and prograde orbits. From this analytical expansion, we can correctly derive the expression of resonance angles in the case of a retrograde MMR. A comparison with the numerical results obtained in Gayon & Bois (2008a) is also carried out.

In the present paper, we firstly introduce the general expressions of the Hamiltonian as well as the canonical variables in the case of two massive planets moving around their host star (Section 2). We note that a dynamical system composed of two satellites orbiting a planet is also an equivalent problem. All the results shown in the present paper may therefore be applied to satellite systems. The consideration of planets revolving in opposite directions involves new expressions for canonical variables presented in Section 3. We adapt the disturbing function of the Beaugé & Michtchenko model to retrograde motions in Section 4. In Section 5, we express the Hamiltonian for retrograde resonances. An orbital fit of the HD 73526 planetary system involving a 2:1 retrograde MMR was found consistent with the observational data in Gayon & Bois (2008a). As a consequence, in Section 6, we apply the analytical model to the HD 73526 planetary system ruled in such conditions and compare our numerical and analytical results. Finally, in Section 7 our results obtained by the analytical expansion are visualized by surfaces of section.
2 General expression of the Hamiltonian

We suppose a star of mass $M_0$ and 2 planets of mass $m_1$ and $m_2$ such as $M_0 \gg m_1, m_2$ and moving around their barycenter. We choose to express the Hamiltonian of the system in the heliocentric frame by using the following Delaunay variables:

\[
M_i \quad L_i = \beta_i \sqrt{\mu_i a_i} \\
\omega_i \quad G_i = L_i \sqrt{1 - e_i^2} \\
\Omega_i \quad H_i = G_i \cos I_i
\]

with $a_i, e_i, I_i, \omega_i$, and $M_i$ the geometrical orbital elements of the planet $i$, and $\mu_i = G \sqrt{M_0 + m_i}$. $G$ is the gravitational constant. $\beta_i = M_0 m_i / (M_0 + m_i)$ is the reduced mass for the planet $i$.

The Hamiltonian of the system (denoted $F$) consists of a Keplerian part ($F_0$) and a disturbing function ($F_1$):

\[
F = F_0 + F_1 \quad \text{with} \quad \begin{cases} 
F_0 = - \sum_{i=1}^{2} \frac{\mu_i^2 \beta_i^3}{2L_i^2} \\
F_1 = -G m_1 m_2 \frac{1}{\Delta} + \frac{m_1 m_2}{M_0} (\dot{x}_1 \dot{x}_2 + \dot{y}_1 \dot{y}_2 + \dot{z}_1 \dot{z}_2)
\end{cases}
\]  

The leading part of the disturbing function, called the direct part, depends on the instantaneous distance between the two planets ($\Delta$). Owing to the choice of the origin of the coordinate systems, the second part of $F_1$ is called the indirect part of the disturbing function. $F_1$ depends on the barycentric velocities $(\dot{x}_i, \dot{y}_i, \dot{z}_i)$ of each planet, expressed in cartesian coordinates.

To study the 3-body problem, different sets of canonical variables can be used, depending on the simplifications carried out for reducing the number of degrees of freedom. For instance, Beaugé & Michtchenko (2003) applied the so-called modified Delaunay variables to solve the coplanar, prograde and eccentric 3-body problem. In the present paper, we define a new set of canonical variables, especially devoted to planets revolving in opposite directions.

3 Set of canonical variables for planets revolving in opposite directions

In this study, we choose to set a prograde direction of revolution for the inner planet (noted 1) and a retrograde one for the outer planet (noted 2). From the previous set of Delaunay variables (1), we therefore define the two following sets of canonical variables for planets 1 and 2 respectively:

\[
\begin{align*}
\lambda_1 & = M_1 + \dot{\omega}_1 & L_1 & = \beta_1 \sqrt{\mu_1 a_1} \\
-\dot{\omega}_1 & = - (\Omega_1 + \omega_1) & L_1 - G_1 & = L_1 (1 - \sqrt{1 - e_1^2}) \\
-\Omega_1 & & G_1 - H_1 & = L_1 \sqrt{1 - e_1^2 (1 - \cos I_1)}
\end{align*}
\]  


The first set of canonical variables (3) corresponds to the well-known modified Delaunay canonical variables while the second one (4) is developed for retrograde planetary motion. \( \lambda_i \) is the mean longitude of the planet \( i \). We note that the longitude of periastron is defined by: \( \tilde{\omega} = \Omega + \omega \), for a coplanar and prograde planetary motion (which is equivalent to consider \( I = 0^\circ \)) and by \( \tilde{\omega} = \Omega - \omega \), for a coplanar and retrograde planetary motion (equivalent to \( I = 180^\circ \)). Restricting the study to a coplanar but counter-revolving problem, the set of canonical variables of the system can be simplified as follows:

\[
\begin{align*}
\lambda_1 &= -M_1 + \ddot{\omega}_1 \\
\lambda_2 &= -L_2 \\
\omega &= -(\Omega_2 - \omega_2) \\
L_1 &= -L_2 - G_2 - H_2 = L_2 \sqrt{1 - e_2^2(1 + \cos I_2)}
\end{align*}
\]

The second set of canonical variables (5) can be simplified as follows:

\[
\begin{align*}
\lambda_1 &= L_1 \\
\lambda_2 &= -L_2 \\
\omega &= L_1 - G_1 \\
G_2 &= L_2
\end{align*}
\]

4 Expansion of the disturbing function

The method used for the expansion of the disturbing function originates from Beaugé & Michtchenko (2003). They expanded an expression for the Hamiltonian that is valid for high eccentricities of one or both planets. Their method provides a good assessment of the disturbing function when planets are close to the collision point. Hence, the solution converges in all points of the phase space except for the singularities (corresponding to planetary collisions). Moreover, the rate of convergence does not depend on the values of eccentricities but rather on the order of magnitude of the distribution function itself. Besides, depending on the studied planetary system, even for very eccentric orbits, the rate of convergence of the disturbing function may be relatively fast.

Classical methods such as Laplace 1799 or Kaula 1962 were expanded for coplanar and quasi-circular orbits and were useful for studying dynamics of the Solar System (or hierarchical systems in general). Nonetheless, due to the detection of eccentric exo-planetary systems, new and specific expansions of the disturbing function were needed. As a consequence, the Beaugé & Michtchenko method seems much more suitable for such eccentric planetary orbits. To apply this method for counter-revolving configurations, we propose to expand again the disturbing function, by considering the prograde motion of the inner planet and the retrograde motion of the outer planet.

4.1 Direct part of the disturbing function

The direct part of the disturbing function that generally encounters problems of convergence may be expressed as a function of the heliocentric radial distances \( r_i \).
of both planets and the $\psi$ angle between both bodies as seen from the star:

$$\frac{1}{A} = (r_1^2 + r_2^2 - 2r_1r_2 \cos \psi)^{-1/2}$$  \hspace{1cm} (6)

We note that for counter-revolving configurations, the angle $\psi$ between both bodies is defined by:

$$\psi = f_1 + f_2 + \Delta \tilde{\omega}$$  \hspace{1cm} (7)

with $f_i$ the true anomaly of the planet $i$, $\Delta \tilde{\omega} = \tilde{\omega}_1 - \tilde{\omega}_2$ and $\tilde{\omega}_{1,2}$ defined in (3) and (4). A concise expression of the previous equation is written in (8) by taking into account the ratio $\rho = r_1/r_2$ such as:

$$\frac{r_2}{\Delta} = (1 + \rho^2 - 2\rho \cos \psi)^{-1/2}$$  \hspace{1cm} (8)

The key of the Beaugé & Michtchenko (2003) method lies in the expansion in power series in a new variable noted $x$ and corresponding to a measurement of the proximity of the initial condition to the singularity in $1/\Delta$:

$$\frac{r_2}{\Delta} = (1 + x)^{-1/2} \approx \sum_{n=0}^{N} b_n x^n$$  \hspace{1cm} (9)

with $x = \rho^2 - 2\rho \cos \psi$. $r_2/\Delta$ has a singularity at $x = -1$. The determination of the coefficients $b_n$ is performed by the way of a linear regression for $x$ values greater than $-1 + \delta$, $\delta$ being a positive parameter close to zero. A good precision of the direct part of the disturbing function may be reached for a good compromise between the $\delta$ value and the choice of $N$ order in the series expansion. Contrary to classical methods involving Fourier series of the $\psi$ variable or power series in $\rho$, not only the convergence rate of this method is improved but also the expansion of the disturbing function can be applied for eccentric two-planet systems. More details can be found in Beaugé & Michtchenko (2003).

From (9) and by using the explicit expression of $x$, we find:

$$\frac{r_2}{\Delta} \approx \sum_{l=0}^{N} \sum_{k=0}^{l} b_l (-2)^k \binom{l}{k} \rho^{2l-k} \cos^k \psi$$  \hspace{1cm} (10)

Changing from powers of $\cos \psi$ to multiples of $\psi$ and by using the explicit expression of the $\psi$ angle, one obtains:

$$\frac{a_2}{\Delta} \approx \sum_{l=0}^{N} \sum_{u=0}^{N-l} 2 A_{l,u} a_2^{2u+l} \left( \frac{r_1}{a_1} \right)^{2u+l} \left( \frac{r_2}{a_2} \right)^{-2u-l-1} \cos (lf_1 + lf_2 + \Delta \tilde{\omega})$$  \hspace{1cm} (11)

with $A_{l,u} = (-1)^l \sum_{t-u}^{\min(2u,N-l)} b_{l+t+u} \left( \frac{t+l}{t+2l-2u} \right) \gamma_t \gamma_l = \begin{cases} 1/2 \text{ if } l = 0 \\ 1 \text{ if } l > 0 \end{cases}$.

\footnote{For both prograde orbits, the $\psi$ angle is defined by: $\psi = f_1 - f_2 + \Delta \tilde{\omega}$ with $\tilde{\omega}_i = \Omega_i + \omega_i$.}
The direct part of the disturbing function may be expressed in terms of mean anomaly by using the Fourier expansion of the following functions (e.g. Hughes 1981):

\[
\left(\frac{r}{a}\right)^n \cos(lf) = \sum_{m=-\infty}^{\infty} X_{m}^{n,l} \cos(mM) \\
\left(\frac{r}{a}\right)^n \sin(lf) = \sum_{m=-\infty}^{\infty} X_{m}^{n,l} \sin(mM)
\]

(12)

with \(X_{m}^{n,l}\) the Hansen coefficient function of the eccentricity (Kaula 1962):

\[
X_{m}^{n,l} = e^{\text{max}(0, m-\frac{n}{l})} \sum_{j=0}^{\infty} Y_{m}^{n,l} e^{\frac{n}{l} \Delta \omega} \]

(13)

with \(Y_{m}^{n,l}\) the Newcomb operators, \(w_1 = \max(0, m-l)\) and \(w_2 = \max(0, l-m)\).

Let us recall that the Newcomb operators obey to simple recurrence relations (Brouwer & Clemence 1961; Murray & Dermott 1999).

Substituting (13) into (12), we obtain:

\[
\left(\frac{r}{a}\right)^n \cos(lf) = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} B_{n,l,j,m} e^{j \cos(mM)} \\
\left(\frac{r}{a}\right)^n \sin(lf) = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} B_{n,l,j,m} e^{j \sin(mM)}
\]

(14)

The direct part of the disturbing function is therefore expressed as follows:

\[
\frac{d^2}{\Delta} \approx \sum_{j,k=0}^{\infty} \sum_{m,n=-\infty}^{\infty} \sum_{l=0}^{N} \sum_{i=0}^{2N} A_{l,i,j,k,m,n} \alpha^i e^l e^k \cos(mM_1 + nM_2 + l\Delta \omega)
\]

(15)

with \(A_{l,i,j,k,m,n} = 2 B_{i,j,k,m} B_{i-1,j,k,m} - B_{i,j,k,m} + B_{i,j,k,m} B_{i-1,j,k,m}\) and \(B_{i,j,k,m} = Y_{i,j,k,m}^{l,i} e^{l} e^{m} e^{w_1} e^{w_2} e^{\Delta \omega}\).

4.2 Indirect part of the disturbing function

From (2), the indirect part of the disturbing function is given by the function \(T_1\):

\[
T_1 = \frac{m_1 m_2}{M_0} (\dot{x}_1 \dot{x}_2 + \dot{y}_1 \dot{y}_2 + \dot{\dot{z}}_1 \dot{z}_2)
\]

(16)

where \(\dot{x}_i, \dot{y}_i, \text{ and } \dot{\dot{z}}_i\) are the barycentric velocities of the planet \(i\), expressed in cartesian coordinates. In the case of coplanar orbits, the expressions of \(\dot{x}_i\) and \(\dot{y}_i\) take into account the direction of motion of the planet \(i\). We define \(\dot{x}_i\) for a prograde motion and a retrograde one as follows:

\[
\dot{x}_i = \frac{dX_i}{dt} = \frac{\partial x_i}{\partial M_i} dM_i \text{ for a prograde motion} \\
\dot{x}_i = -\frac{\partial x_i}{\partial M_i} dM_i \text{ for a retrograde motion}
\]

(17)

5 A similar equation is obtained for the expression of \(\dot{y}_i\).
Consequently, considering a coplanar problem and planets revolving in opposite directions, the indirect part is equal to:

\[
T_1 = -\frac{Gm_1 m_2}{a_2} \alpha^{-1/2} \left[ \frac{\partial}{\partial M_1} \left( \frac{x_1}{a_1} \right) \frac{\partial}{\partial M_2} \left( \frac{x_2}{a_2} \right) + \frac{\partial}{\partial M_1} \left( \frac{y_1}{a_1} \right) \frac{\partial}{\partial M_2} \left( \frac{y_2}{a_2} \right) \right]
\]  \hspace{1cm} (18)

By using (14) and considering the prograde motion of planet 1 and the retrograde one of planet 2, one obtains:

\[
\begin{align*}
\frac{x_1}{a_1} &= \left( \frac{r_1}{a_1} \right) \cos(\bar{\omega}_1 + f_1) = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} B_{1,j,m} e_1^j \cos(\bar{\omega}_1 + mM_1) \\ \frac{y_1}{a_1} &= \left( \frac{r_1}{a_1} \right) \sin(\bar{\omega}_1 + f_1) = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} B_{1,j,m} e_1^j \sin(\bar{\omega}_1 + mM_1) \\ \frac{x_2}{a_2} &= \left( \frac{r_2}{a_2} \right) \cos(\bar{\omega}_2 - f_2) = \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} B_{1,k,n} e_2^k \cos(\bar{\omega}_2 - nM_2) \\ \frac{y_2}{a_2} &= \left( \frac{r_2}{a_2} \right) \sin(\bar{\omega}_2 - f_2) = \sum_{k=0}^{\infty} \sum_{n=-\infty}^{\infty} B_{1,k,n} e_2^k \sin(\bar{\omega}_2 - nM_2)
\end{align*}
\]  \hspace{1cm} (19)

Substituting these results into (18), we find:

\[
T_1 = -\frac{Gm_1 m_2}{a_2} \sum_{j,l=0}^{\infty} \sum_{m,n=-\infty}^{\infty} \sum_{i=0}^{2N} \bar{A}_i \alpha^i 
\times mn B_{1,1,j,m} e_1^j e_2^k \cos(mM_1 + nM_2 + \Delta\bar{\omega})
\]  \hspace{1cm} (20)

where \( \bar{A}_i \) are constant coefficients such as: \( \alpha^{-1/2} = \sum_{i=0}^{2N} \bar{A}_i \alpha^i \).

By combining (15) and (20), we obtain the complete expression of the disturbing function as follows:

\[
F_1 = -\frac{Gm_1 m_2}{a_2} \sum_{j,k,l=0}^{\infty} \sum_{m,n,-\infty}^{\infty} \sum_{i=0}^{N} \sum_{l=0}^{2N} R_{i,j,k,m,n,l} \alpha^i e_1^j e_2^k \cos(mM_1 + nM_2 + l\Delta\bar{\omega})
\]  \hspace{1cm} (21)

where \( R_{i,j,k,m,n,l} = A_{i,j-k-1} D_{i,j,k,m,n} + \delta_{j,1} \bar{A}_i/m B_{1,1,j,m} B_{1,1,k,n} \) are constant coefficients. \( R_{i,j,k,m,n,l} \) are independent of initial conditions and then require to be determined once. We are now able to express the angles of resonance in the case of counter-revolving configurations as shown in the following section.

5 The resonant average Hamiltonian

Considering planets close to a MMR and revolving in opposite directions, we define the MMR ratio by \( p + q/p \) with \( p \neq 0 \) and \( p < 0 \). For instance, when two planets revolve in opposite direction and have a period ratio of 2, the \( q \) order of
resonance is equal to 3 and \( p = -1 \). We set \( s = p/q \) and define the following set of canonical variables in the case of a retrograde MMR:

\[
\begin{align*}
\lambda_1 & : J_1 = L_1 + s(I_1 + I_2) \\
\lambda_2 & : J_2 = -L_2 - (1 + s)(I_1 + I_2) \\
\sigma_1 & = (1 + s)
\lambda_2 - s\lambda_1 - \tilde{\omega}_1 \\
\sigma_2 & = (1 + s)
\lambda_2 - s\lambda_1 - \tilde{\omega}_2
\end{align*}
\]

(22)

where \( \sigma_1 \) and \( \sigma_2 \) are the resonant angles while \( I_1 \) and \( I_2 \) are the conjugate momenta depending on the eccentricities \( e_1 \) and \( e_2 \).

Let \( \theta \) be the following angle of the disturbing function such as : \( \theta = mM_1 + nM_2 + l\Delta\tilde{\omega} \). Considering the new set of variables, the expression of \( \theta \) becomes:

\[
\theta = m\sigma_1 - n\sigma_2 + l(\sigma_2 - \sigma_1) + [m(p + q) - np]Q
\]

(23)

where \( Q = (\lambda_1 - \lambda_2) \) is the synodic angle. Hence, the disturbing function depends on three angular variables \( (\sigma_1, \sigma_2, Q) \). Considering the new set of canonical variables \( (\sigma_1, \sigma_2, Q, \lambda_2; I_1, I_2, qJ_1, J_1 + J_2) \), we find the following constant of motion:

\[
J_1 + J_2 = \text{constant}
\]

(24)

As a consequence, our system has therefore two integrals of motion: the \( F \) Hamiltonian and \( J_{\text{tot}} = J_1 + J_2 \). It is well known that the frequency of the angle \( Q \) is much higher than that of \( \sigma_i \). As a consequence, we consider only long period perturbations in the Hamiltonian. The system is then averaged with respect to the synodic angle as follows:

\[
F_1 = \frac{1}{2\pi} \int_0^{2\pi} F_1 dQ
\]

(25)

The averaging over the \( Q \) variable written in (25) implies a third constant of motion, namely \( J_1 \). As a consequence, the system consists of four degrees of freedom and three constants of motion. From now on, the system can be reduced to four independent variables, namely \( \sigma_1, \sigma_2, I_1 \) and \( I_2 \). Taking into account the condition of MMR \( (n = m(p + q)/p) \), we obtain the final expression of the disturbing function (26) and the average Hamiltonian (27) respectively, for a retrograde resonance:

\[
F_1 = -\frac{Gm_1m_2}{a_2} \sum_{j=0}^{l_{\text{max}}} \sum_{k=0}^{k_{\text{max}}} \sum_{m=-m_{\text{max}}}^{m_{\text{max}}} \sum_{l=0}^{l_{\text{max}}} \sum_{i=0}^{2N} R_{i,j,k,m,l} \times \alpha_i e_1^j e_2^k \cos((m - l)\sigma_1 + (l - n)\sigma_2)
\]

(26)

\[
F = -\sum_{i=1}^{2} \frac{\mu_1^2\Omega_i^2}{2L_i^2} - \frac{Gm_1m_2}{a_2} \sum_{i,j,k,m,l} R_{i,j,k,m,l} \alpha_i e_1^j e_2^k \cos((m - l)\sigma_1 + (l - n)\sigma_2)
\]

(27)

Due to D’Alembert’s properties of the disturbing function, some coefficients are null if one (or more) of the following conditions is reached: 1) \( j < |m - l| \), 2) \( k < |l - n| \), 3) \( m - l \) even (odd) number and \( j \) odd (even) number, 4) \( l - n \) even (odd) number and \( k \) odd (even) number.
6 Comparison with numerical methods

Using the final expression of the Hamiltonian, the $J_1$ and $J_2$ constants of motion as well as the following set of canonical variables,
\[ \sigma_1 = (1 + s)\lambda_2 - s\lambda_1 - \bar{\omega}_1 \quad I_1 = L_1(1 - \sqrt{1 - e_1^2}) \]
\[ \sigma_2 = (1 + s)\lambda_2 - s\lambda_1 - \bar{\omega}_2 \quad I_2 = -L_2(1 - \sqrt{1 - e_2^2}) \]

we can firstly integrate a two-planet system both in counter-revolving configuration and in MMR and secondly, compare the analytical results with numerical ones. Using a Bulirsch-Stoer method, we integrate numerically the following differential equations:
\[ \dot{I}_i = -\frac{\partial \bar{F}}{\partial \sigma_i} = -\frac{\partial \bar{F}_1}{\partial \sigma_i} \]
\[ \dot{\sigma}_i = \frac{\partial \bar{F}}{\partial I_i} = \frac{\partial \bar{F}_1}{\partial I_i} + \frac{\partial F_0}{\partial I_i} \]

For the integration of (29), we use our initial conditions for the HD 73526 system, located very close to the 2:1 retrograde MMR (i.e. at the edge of a V-shape structure in a stability map in $[a, e]$ orbital elements; see Gayon & Bois, 2008a):
\[ M_0 = 1.08 M_\odot \]
\[ M_1 = 2.9 M_{\oplus} \quad M_2 = 2.5 M_{\oplus} \]
\[ a_1 = 0.66 \text{ AU} \quad a_2 = 1.05 \text{ AU} \]
\[ e_1 = 0.19 \quad e_2 = 0.14 \]
\[ \sigma_1 = 94 \text{ (deg)} \quad \sigma_2 = 94 \text{ (deg)} \]

Such initial conditions (30) are sufficiently close to the retrograde MMR to apply our analytical expansion. These initial conditions (more particularly the semi-major axes) are averaged for the analytical expansion.

Fig. 1 shows the time evolution of eccentricity of both planets. Dots represent the numerical solution while black curves the analytical one. Both solutions express the same behavior, with a relative error of 1.2% and 10%, on average, over the eccentricity of the inner orbit and the outer orbit, respectively (absolute errors being 0.002 for $e_1$ and 0.005 for $e_2$ on average). Due to the averaging of the Hamiltonian over short periods, numerical dots are scattered on both sides of the analytical solution.

Fig. 2 shows the time variation of the variable $\Delta \sigma$. Because the $e_2$ eccentricity periodically reaches the zero value, the numerical method does not always permit to determine the value of the $\Delta \sigma$ angle. This phenomenon is expressed by the vertical scattering of dots from 0 to 360 degrees. As a consequence, surfaces of

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6 We forecast to study various initial conditions in a forthcoming paper, notably for a “full” retrograde resonance.

7 More details on the used numerical method and corresponding results may be found in Gayon & Bois (2008a).
section obtained numerically comprise this numerical bias. The analytical method is therefore more reliable when the eccentricity of a planet reaches values close to zero. Besides, the analytical approach ensures the properties of stability found for each planetary system.

**Fig. 1** Time variation of the eccentricites $e_1$ and $e_2$ for the HD 73526 planetary system in 2:-1 MMR. Red dots represent the numerical solution while black curves the analytical one. Used initial conditions are given by: $M_0 = 1.08 M_\odot$; $M_1 = 2.9 M_{Jup}$; $M_2 = 2.5 M_{Jup}$; $a_1 = 0.66$; $a_2 = 1.05$; $e_1 = 0.19$; $e_2 = 0.14$; $\sigma_1 = 94$; $\sigma_2 = 94$.

**Fig. 2** Time variation of the $\Delta \sigma$ variable (where $\Delta \sigma = \sigma_1 - \sigma_2$) for the HD 73526 planetary system in 2:-1 MMR. Red dots represent the numerical solution while black curves the analytical one. Used initial conditions are the same as in Fig. 1.
7 Surfaces of section

Nature of planetary systems can be directly deduced from the behavior of the resonance variables. As a consequence, from the previous Hamiltonian expansion (27) of the three-body problem, we can plot surfaces of section and study the dynamics of two planets in retrograde MMR.

Fig. 3 shows surfaces of section plotted for the HD 73526 planetary system previously defined. The energy level corresponding to the initial conditions (30) was numerically evaluated as $-0.13835250$ (units of AU, solar mass, and year). With our analytical expansion, we obtain the Hamiltonian value of $-0.13835197$, which is in good agreement with our numerical results. Panel (a) of Fig. 3 corresponds to the inner planet and is plotted in the $(e_1 \cos \sigma_1, e_1 \sin \sigma_1)$ parameter space. The section plane for the inner planet is chosen such as $\sigma_2 = 0$. Similarly, panel (b) is plotted for the outer planet in the $(e_2 \cos \sigma_2, e_2 \sin \sigma_2)$ parameter space and using the plane $\sigma_1 = 0$. The initial conditions corresponding to (30) are plotted in red. All the solutions found for $\bar{F} = -0.138352$ are quasi-periodic.

In panels (a) and (b) of Fig. 3 we represent the circulation state of the $\sigma_1$ and $\sigma_2$ variables in black and red colors. The libration state about $0^\circ$ is plotted in blue; such initial conditions are located inside the 2 : 1 retrograde MMR. Nevertheless, for all the sets of initial conditions required for plotting Fig. 3 the time variation of $\sigma_1$ and $\sigma_2$ expresses a circulation (in time) of both variables (not shown here). This time circulation seems contrary to the possible behavior in libration of $\sigma_1$ and $\sigma_2$ shown on our surfaces of section. Such initial condition sets inside the 2 : 1 retrograde MMR and their behaviors could therefore point out a particular characteristic of retrograde MMR. Such a characteristic would deserve a specific study.

![Fig. 3 Surfaces of section for the inner (a) and the outer (b) planets of the HD 73526 system found close to a 2:-1 MMR. The value of $\bar{F}$ is $-0.138352$. Initial conditions (30) are represented by the red curve. Black and blue curves respectively correspond to circulation and libration of the $\sigma_1$ variables in the plane $\sigma_2 = 0$.](image)
8 Conclusion

In the present paper, we investigated the three-body problem in the particular case of retrograde resonances. Our study is derived from the Hamiltonian approach of Beaugé & Michtchenko (2003), which was expanded for eccentric, coplanar and prograde orbits. To apply this method to the retrograde MMR case, we expanded again the disturbing function, when considering a prograde motion of the inner planet and a retrograde motion of the outer planet. Hence, we defined a new set of canonical variables, which allow us to express correctly the angles of resonance in the case of counter-revolving configurations.

Although the exploration of the N-body problem is accessible from numerical methods, the acquiring of an analytical “rail” notably contributes to a deeper understanding of the numerical investigation. As shown in Section 6, the analytical method also permits a better determination of the resonance variables. Moreover, from the behavior of the resonance angles displayed in surfaces of section, we can directly infer the local behavior of a 3-body system, that is to say its stability or its chaoticity.

Until now, no planetary system has been truly detected in counter-revolving configuration. However, given the efficiency for stability of retrograde MMR (see Gayon & Bois 2008b), such a detection might occur. The work presented in this paper is firstly based on a dynamical study of theoretical two-planet systems. Nevertheless, a dynamical system composed of two satellites orbiting a planet is an equivalent problem. Since some satellites of Saturn are found in counter-revolving motions, our expansion of the three-body problem solved in the case of retrograde motions could then be applied for satellites of the Solar System.

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A Appendix

This appendix shows the analytical expansion when the body moving on a retrograde orbit is the inner planet. The new set of canonical variables for the resonant averaged Hamiltonian is written as follows (equivalent to 22):

\[
\begin{align*}
\lambda_1 &= -M_1 + \tilde{\omega}_1, \\
\lambda_2 &= -M_2 + \tilde{\omega}_2, \\
\sigma_1 &= (1 + s)\lambda_2 - s\lambda_1 - \tilde{\omega}_1, \\
\sigma_2 &= (1 + s)\lambda_2 - s\lambda_1 - \tilde{\omega}_2, \\
J_1 &= -L_1 + s(I_1 + I_2), \\
J_2 &= -L_2 - (1 + s)(I_1 + I_2)
\end{align*}
\]  
(31)

with \( \lambda_1 = -M_1 + \tilde{\omega}_1, \lambda_2 = -M_2 + \tilde{\omega}_2, \tilde{\omega}_1 = \Omega_1 - \omega_1 \) and \( \tilde{\omega}_2 = \Omega_2 + \omega_2 \).

Although the set of canonical variables changes, the expression of the resonant averaged Hamiltonian remains the same:

\[
F = -\sum_{j=1}^{2} \frac{\mu_j^{3/2}}{2l_j^2} - \frac{Gm_1m_2}{a_2^2} \sum_{i,j,k,m,l} \tilde{R}_{j,k,m,l} a_1 e_1^l e_2^k \cos((m - l)\sigma_1 + (l - n)\sigma_2)
\]  
(32)
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