The Double Scaling Limit in Arbitrary Dimensions: A Toy Model

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Colored tensor models generalize matrix models in arbitrary dimensions yielding a statistical theory of random higher dimensional topological spaces. They admit a \(1/N\) expansion dominated by graphs of spherical topology. The simplest tensor model one can consider maps onto a rectangular matrix model with skewed scalings. We analyze this simplest toy model and show that it exhibits a family of multi critical points and a novel double scaling limit. We show in \(D = 3\) dimensions that only graphs representing spheres contribute in the double scaling limit, and argue that similar results hold for any dimension.

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I. INTRODUCTION

Random matrix models [1] provide a statistical theory of random discretized Riemann surfaces. The amplitudes of the ribbon Feynman graphs of their perturbative expansion support a \(1/N\) expansion [2, 3] (where \(N\) is the size of the matrices) indexed by the genus of the surfaces. In the large \(N\) limit the planar graphs corresponding to surfaces of spherical topology dominate [4, 5]. Statistical models of fluctuating geometry can be thought as providing either a regularization or a fundamental description of quantum gravity [6]. The large-\(N\) limit of matrix models yields in two dimensions an analytic description of dynamical triangulations [7, 8], whose link to non-critical string theory in the continuum limit is well-understood [1]. Higher-dimensional models of dynamical triangulations have not been equally successful in providing a sensible continuum limit for quantum gravity [9, 10], although a non-local modification, which goes under the name of causal dynamical triangulations [11] has produced substantial evidence for the emergence of an extended geometry at large scale [12-14].

The family of planar graphs [3-5] dominating the large \(N\) limit of matrix models is summable with finite radius of convergence. When the coupling constant approaches a critical value \(g_c\), the free energy is dominated by graphs with a large number of vertices and exhibits a critical behavior. It is in this regime that the system reaches its continuum limit and the critical exponents can be evaluated.

More general matrix models exhibit a complex critical behavior, specific choices of the potential leading to multi critical points [15]. They correspond to \((m + 1, m)\) conformal matter coupled to Liouville gravity [15], and map on conformal field theories on fixed geometries via the KPZ correspondence [16-19]. The contributions to the free energy (that is the partition function for connected surfaces) at fixed genus exhibit a critical behavior when the coupling \(g\) goes to the same fixed \(g_c\). The corresponding critical exponents are such that the double scaling limit \(N \to \infty, g \to g_c\), with \(N(g - g_c)^{1 + 1/m} = \kappa^{-1}\) fixed combines all genera [20-22] and leads to a well defined continuum theory with fixed Newton’s constant [1].

Random matrices generalize in higher dimensions to random tensors [23-25], whose perturbative expansion performs a sum over random higher dimensional geometries. Although some power counting estimates have been obtained [26-32], and the symmetries of tensor models could be
analyzed [33–35] the key to analytical rather than numerical results, namely the $1/N$ expansion, was missing until recently. That situation has changed with the discovery of such a $1/N$ expansion for colored [39–42] random tensors. The amplitude of their graphs supports a $1/N$ expansion indexed by the degree, a positive integer, which plays in higher dimensions the role the genus played in two dimensions. The leading order graphs, baptized melonic [43], triangulate the $D$-dimensional sphere and form a summable series. When the coupling constant approaches its critical value, the free energy exhibits a critical behavior and, like in matrix models, the colored tensor models reach their continuum limit dominated by triangulations with an infinite number of simplices. The entropy exponent of the melonic series, analogous to the string susceptibility $\gamma_{\text{string}} = -1/2$ of the 1-matrix model for the pure gravity universal class, is $\gamma_{\text{melons}} = 1/2$, [43]. This discovery led to the possibility of new analytical investigations of dynamical triangulations models and their continuum limit in $D \geq 3$ dimensions [43, 44]. Colored random tensors are a promising tractable discretization of quantum gravity in three and more dimensions and the subject is developing fast [45–49]. The understanding of the leading (melonic) order of colored tensor models allows the study of the coupling of statistical systems to random geometries in arbitrary dimension. Thus one can prove that, unlike in two dimensions [50], the Ising model on a random lattice in higher dimensions does not exhibit a phase transition [51], while the dually weighted models (introduced in [52] in two dimensions) counting dynamical triangulations with a non trivial measure factor, do [53]. A detailed introduction and review of colored tensor models is [54].

The Schwinger Dyson equations (SDE) of matrix models translate into constraints (satisfied by the partition function) which obey the Virasoro algebra [55–57]. A similar line of inquiry can be pursued in higher dimensions by integrating all colors but one in a colored tensor model and attributing an independent coupling constant to each effective vertex. The algebra of constraints is, unsurprisingly, much more involved, but still manageable at leading order [44]. The detailed study of this algebra will lead to a precise understanding of the various critical behaviors and a classification of the continuum limits of colored tensor models.

In its full generality this classification is for now out of reach and one must must content with the study of some simplified toy models where a more detailed analysis can be pursued. This is done in the present paper by restricting to a sub sector of the tensor models consisting in a matrix model with skewed scalings. A tensor $T_{\vec{n}_1 \ldots \vec{n}_D}$ with $D$ indices can be seen as a $N \times N^{D-1}$ matrix $T_{n_1 \ldots n_D}$ with $\vec{n} = n^2 \ldots n^D$. General tensor interactions correspond to arbitrary contractions of indices, hence they do not respect this splitting. In the sequel we restrict to the subclass on interactions which do, that is we only take into account interactions of the form $\text{Tr}[(TT^\dagger)^p]$. In order to obtain a sensible theory one needs to adapt appropriately the scalings of the terms with the large parameter $N$. This toy model is natural once one takes into account the full SDE’s of colored tensor model: it consist in restricting to a model whose leading order SDE’s close a Virasoro algebra. The matrix model with skewed scalings can also be interpreted as a model of random surfaces with patches.

Using techniques consecrated for matrix models, but taking into account the effect of the unusual scaling we present in this paper the full solution of the toy model via the all orders SDE’s. We generalize for all $D$ the multi critical points of matrix models [13], and find that their susceptibility exponents are $1 - \frac{1}{m}$, identical with those of multi critical polymers [58]. We the particularize for convenience to $D = 3$ and show that the model admits a double scaling limit $N \to \infty$ $g \to g_c$ with $N(g - g_c)^{1 + \frac{1}{m}} = \kappa^{-1}$ fixed. We then go on to show that the double scaling limit we uncover is not a summation over topologies: indeed only planar graphs with trivial topology contribute to the double scaling limit. Viewed as graphs of a colored tensor model in $D = 3$, these graphs always represent spheres. We argue that similar results hold in all dimensions.

This paper is organized as follows. In section III we present the colored tensor model and its relation to the toy model of a matrix with skewed scalings. In section III we analyze the toy model
and its multi critical points. In section IV we deduce the all orders SDEs and present their iterative solution. In section V we present the double scaling limit of our toy model in $D = 3$, and in section VI we discuss the implications of these results for general tensor models.

II. FROM COLORED TENSOR MODELS TO A MATRIX MODEL WITH SKewed SCALINGS

Our starting point is the independent identically distributed colored tensor model with one coupling. We denote $\vec{n}_i$, for $i = 0, \ldots, D$, the $D$-tuple of integers $\vec{n}_i = (n_{ii} - 1, \ldots, n_{i0}, n_{iD}, \ldots, n_{ii} + 1)$, with $n_{ik} = 1, \ldots, N$. This $N$ is the size of the tensors and the large $N$ limit defined in [36–38] represents the limit of infinite size tensors. We set $n_{ij} = n_{ji}$. Let $\bar{\psi}_{i \vec{n}_i}, \psi_{i \vec{n}_i}$, with $i = 0, \ldots, D$, be $D + 1$ couples of complex conjugated tensors with $D$ indices. The independent identically distributed (i.i.d.) colored tensor model in dimension $D$ [54] is defined by the partition function

$$e^{-N^D F_N(\lambda, \bar{\lambda})} = Z_N(\lambda, \bar{\lambda}) = \int \bar{\psi} d\psi e^{-S(\psi, \bar{\psi})},$$

(2.1)

$\sum_n$ denotes the sum over all indices $n_{ij}$ from 1 to $N$. The tensor indices $n_{ij}$ need not be simple integers (they can for instance index the Fourier modes of an arbitrary compact Lie group, or even of a finite group of large order [59]). The partition function of equation (2.1) is evaluated by colored Feynman graphs [39–41]. The tensors have no symmetry properties under permutations of their indices (i.e. all $\bar{\psi}_{i \vec{n}_i}, \psi_{i \vec{n}_i}$ are independent). The colors $i$ of the fields $\psi^i, \bar{\psi}^i$ induce important restrictions on the combinatorics of the graphs. They have two types of vertices, say one of positive (involving $\psi$) and one of negative (involving $\bar{\psi}$). The lines always join a $\psi^i$ to a $\bar{\psi}^i$ and possess a color index, $i$. Any Feynman graph $G$ of this model is an orientable simplicial pseudo manifold [40, 42] and the colored tensor models provide a statistical theory of random triangulations in $D$ dimensions, generalizing random matrix models.

One can picture the topological space associated to a graph in a rather simple manner. The vertices of the graph correspond to the $D$ simplices of the simplicial complex. The halflines of a vertex represent the $D - 1$ simplices bounding a $D$ simplex and have a color. Any lower dimensional subsimplex is colored by the colors of the $D - 1$ simplices sharing it. In figure 1 we sketched the dual complex in $D = 3$ dimensions. The vertices are dual to tetrahedra. A triangle (say 3) is dual to a line (of color 3) and separates two tetrahedra. An edge (say common to the triangles 2 and 3) is dual to a face (and indexed by two colors 2 and 3). A vertex (say common to the triangles 0, 2 and 3) is indexed by the three colors 0, 2 and 3.

![Figure 1. The dual complex in $D = 3$.](image)

The $n$-bubbles of the graph are the maximally connected subgraphs made of lines with $n$ fixed colors. They are associated to the $D - n$ simplices of the pseudo manifold. For instance, the
$D$-bubbles are the maximally connected subgraphs containing all but one of the colors. They are associated to the 0 simplices (vertices) of the pseudo-manifold.

The tensor indices $n_{jk}$ are preserved along the faces of the graph (which are readily identified as bi colored connected subgraphs). The amplitude of a graph with 2$p$ vertices and $F$ faces is

$$A(G) = (\lambda \bar{\lambda})^p N^{-p(D-1)/2 + F}.$$  

(2.2)

We generalize the colored tensor model with one coupling to a model with an infinity of couplings. First we integrate all colors save one, and second we “free” the couplings of the operators in the effective action for the last color. When integrating all colors save one the partition function becomes

$$Z = \int d\psi^0 d\bar{\psi}^0 e^{-S^0(\psi^0, \bar{\psi}^0)}$$

$$S^0(\psi^0, \bar{\psi}^0) = \sum_{B^0} \sum_{\bar{m}} \left( \frac{(\lambda \bar{\lambda})^p}{\text{Sym}(B^0)} \right) \text{Tr}_{B^0}[\bar{\psi}^0, \psi^0] N^{-D(D-1)/2 + F_{B^0}},$$

(2.3)

where the sum over $B^0$ runs over all connected vacuum graphs with colors $1, \ldots, D$ (i.e. over all the possible $D$-bubbles with colors $1, \ldots, D$) and $p$ vertices. The operators Tr$_{B^0}[\bar{\psi}^0, \psi^0]$ in the effective action for the last color are tensor network operators. Every vertex of $B^0$ is decorated by a tensor $\psi^0 \psi^0$ or $\bar{\psi}^0 \bar{\psi}^0$, and the tensor indices are contracted as dictated by the graph $B^0$. We denote $v, \bar{v}$ the positive (resp. negative) vertices of $B^0$, and $l^i_v$ the lines (of color $i$) connecting the positive vertex $v$ with the negative vertex $\bar{v}$. The operators write

$$\text{Tr}_{B^0}[\bar{\psi}^0, \psi^0] = \sum_n \left( \prod_{v, \bar{v} \in B^0} \psi^0_{n_0} \psi^0_{\bar{n}_0} \right) \left( \prod_{i=0}^{D-1} \prod_{l^i_v \in B^0} \delta_{n_i \bar{n}_0}\right),$$

(2.4)

where all indices $n$ are summed. Note that, as all vertices in the bubble belong to an unique line of a given color, all the indices of the tensors are paired. The scaling with $N$ of an operator can be evaluated and the effective action for the last color writes (dropping the index 0)

$$S^D(\psi, \bar{\psi}) = \sum \bar{\psi}^0 \psi^0 + N^{D-1} \sum_{B} \left( \frac{(\lambda \bar{\lambda})^p}{\text{Sym}(B)} \right) N^{-(D-1)/2 - (\frac{2}{D-2})\omega(B)} \text{Tr}_{B}[\bar{\psi}, \psi],$$

(2.5)

with $\omega(B)$ a non negative integer \cite{44, 54}. Attributing to each operator its coupling constant and rescaling the field to $T = \psi N^{D/2}$, we obtain the partition function of colored tensor model with generic potential

$$Z = e^{-N^D F(t_B)} = \int d\bar{T} dT e^{-N^{D-1}S(\bar{T}, T)},$$

$$S(\bar{T}, T) = \sum \bar{T} \bar{v} + \sum_B t_B N^{-\frac{2}{D-2}\omega(B)} \text{Tr}_{B}[\bar{T}, T].$$

(2.6)

Although in the end we deal with an unique tensor $T$, the colors are crucial for the definition of the tensor network operators in the effective action. The initial vertex of the tensor model described a $D$ simplex. The tensor network operators describe (colored) polytopes in $D$ dimensions obtained by gluing simplices along all save one of their boundary $D-1$ simplices around a point (dual to the bubble $B$). This is in strict parallel with matrix models, where higher degree interactions represent polygons obtained by gluing triangles around a vertex. Each index of a tensor $T$ inherits a color.
When evaluating amplitudes of graphs obtained by integrating the last tensor $T$, the tensor network operators act as effective vertices (for instance each comes equipped with its own coupling constant). One can represent a Feynman graph either as graphs with $D + 1$ colors (with the subgraphs with colors $1, \ldots, D$, representing the effective vertices) or directly as a stranded graph for the tensor $T$. The latter is built of stranded lines, and vertices representing the connectivity of the indices of the tensors $T$ in an effective vertex. For example, for tensors with three indices, both

$$T_{a^0 p^1 p^2} T_{b^0 q^1 q^2} T_{c^0 r^1 r^2}, \quad T_{d^0 s^1 s^2} T_{e^0 t^1 t^2} T_{f^0 u^1 u^2} T_{g^0 v^1 v^2} T_{h^0 w^1 w^2},$$

are allowed (leading order $\omega(B) = 0$) vertices. In the representation as stranded graphs one represent the index contractions in the vertex by strands. Furthermore, listing the indices of the tensors $T$ (respectively $\bar{T}$) turning clockwise (respectively anticlockwise) around a vertex, the two vertices above are represented in figure 2.

![Figure 2. Examples of vertices.](image)

The full model of eq. (2.6) proved for now too complex to be solved analytically. A first step in this direction consists in studying simplified models taking into account only subclasses of tensor network operators. A first subclass is readily identified: one can restrict to vertices such that the tensors share alternatively one and $D - 1$ indices (the vertex on the left in figure 2). Introducing the shorthand notation $T_{\vec{n}}^{n_0}$, where $\vec{n} = (n^2, \ldots, n^D)$, they write

$$T_{\vec{n}}^{n_0} \bar{T}_{p_0}^{\vec{n}} T_{p_0}^{\vec{n}} \cdots = \text{Tr}(TT^{\dagger})^p$$

where $T$ denoted the $N \times N^{D-1}$ matrix with entries $T_{p_0 \vec{n}}$. One can check that all such vertices have degree $\omega(B) = 0$. The restricted colored tensor model is then a model of a random $N \times N^{D-1}$ rectangular matrix. Note that the passage from a stranded graph representation of a Feynman graph to the initial colored representation is trivial: one needs only to collapse all the stranded lines into colored lines of color 0, without collapsing the strands in the vertex. In particular, if a stranded graph is planar, the associated colored graph is also a planar graph. In particular in $D = 3$ a planar colored graph always represents a three sphere. This is expected to generalize in arbitrary dimensions.

### III. A MATRIX MODEL WITH SKEWED SCALINGS.

The most general matrix model with skewed scalings for a $N \times N^{D-1}$ matrix $T$ is defined by the partition function

$$Z = e^{-NDF} = \int dT d\bar{T} e^{-N^{D-1} \text{Tr}(TT^{\dagger})} , \quad V(z) = \sum_{p=1}^{\infty} t_p z^p .$$

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1 There are several ways to see this: as it is planar the graph has trivial homotopy hence it is a sphere, as it is planar it admits a planar jacket, and using it is a sphere, etc.
Rectangular matrix models have already been considered in the literature. However, to the best of our knowledge, this is the first time that a model in which the two indices \( n \) and \( \vec{n} \) have different scaling in \( N \) is considered. This different scalings have profound consequences on the power counting of graphs (the reader can already infer this by comparing the \( D \) dependent scalings in eq. (3.9) with those of usual matrix models) The rest of this paper is dedicated to solving this model.

The Feynman graphs generated by (3.9) are ribbon graphs with two kinds of faces: the “heavy” faces carrying an index \( \vec{n} \) and the “light” faces, carrying an index \( n \). They can therefore be interpreted as random surfaces decorated by patches (corresponding to the light faces). As we will see in the sequel, these light patches completely change the continuum limit of the matrix model. Note that scalings in \( N \) in eq. (3.9) are the unique scalings which lead to a well defined continuum limit. Most of the analysis we perform below relies on classical matrix models techniques. However, at almost every step of the way, the results we obtain are very different from their classical counterparts.

The coupling constants \( t_p \) can be represented as integrals in the complex plane

\[
t_n = \frac{1}{2\pi i} \int_C du \frac{1}{u^{n+1}} V(u), \quad nt_n = \frac{1}{2\pi i} \int_C du \frac{1}{u^n} V'(u),
\]

where the contour \( C \) is the circle at infinity.

**Observables** A set of observables is provided by the loop observables, traces of powers of the \( N \times N \) matrix \( TT^\dagger \). They are computed by deriving w.r.t to the couplings,

\[
\frac{1}{N} \left\langle \frac{\text{Tr}[(TT^\dagger)^q]}{1} \right\rangle = -N^{-D} \frac{1}{Z} \frac{\partial}{\partial t_q} Z = \frac{\partial}{\partial t_q} F_N.
\]

We will denote below the generating function of these observables, also known as the resolvent by \( W(z) \) and its large \( N \) limit by \( W_0(z) \)

\[
W(z) = \frac{1}{z} + \sum_{q=1}^{\infty} \frac{1}{z^{q+1}} \frac{1}{N} \left\langle \frac{\text{Tr}[(TT^\dagger)^q]}{1} \right\rangle = \frac{1}{N} \left\langle \frac{\text{Tr}[(TT^\dagger)^q]}{1} \right\rangle, \quad W_0(z) = \lim_{N \to \infty} W(z).
\]

Remark that the resolvent behaves for \( |z| \to \infty \) as \( \frac{1}{z} \) for any value of \( N \). The loop observables are computed starting from the resolvent via integrations in the complex plane

\[
\frac{1}{N} \left\langle \frac{\text{Tr}[(TT^\dagger)^q]}{1} \right\rangle = \frac{1}{2\pi i} \int_C du u^{q} W(u).
\]

The free energy \( F_N \) is the partition function for connected surfaces. The area of a surface is proportional with the number of vertices, that is

\[
A \sim \frac{1}{F} \sum_{i=2} t_i \partial t_i F.
\]

This area can be computed as a single derivative. Renaming the couplings \( t_1 = \frac{1}{g}, \ t_i = \frac{\alpha_i}{g} \), the potential can be written as \( V(z) = \frac{1}{g} \left( z + \sum_{i=2} \alpha_i z^i \right) \) and the derivative of \( F_N \) w.r.t. \( g \) computes

\[
g \partial g F = -N^{-1} \sum_{i=1} t_i \left\langle \frac{\text{Tr}[(MM^\dagger)^i]}{1} \right\rangle = -\frac{1}{2\pi i} \int_C du \frac{V(u)}{W(u)}.
\]
where we used that fact that \( V(u) \) is an entire function. It follows that the area of the connected surface is \( A \sim g \partial g \ln F \) (in fact one needs to subtract a finite non universal piece from the free energy prior to taking the logarithm). Note that for any analytic function \( f \)

\[
\langle \text{Tr} f(TT^\dagger) \rangle = \frac{1}{2\pi i} \int du \, f(u)W(u),
\]

hence solving the model consists in determining the resolvent \( W(u) \).

### A. Eigenvalue treatment

For random matrices one can change variable in the functional integral and pass to \( N \) eigenvalue integrals. This is well known for square matrices and can be done (see \cite{62} and references therein) also for rectangular ones. In this case one reduces the integral over \( T \) and \( T^\dagger \) to the eigenvalues \( \lambda \) of the square \( N \times N \) hermitian matrices \( TT^\dagger \). Using \cite{62} we can then rewrite the partition function as

\[
e^{-N^D F_N} = Z_N = \int_0^\infty \left( \prod_{i=1}^N d\lambda_i \right) \prod_{1 \leq i < j \leq N} (\lambda_j - \lambda_i)^2 \prod_{i=1}^N \left( \lambda_i^{N^D - 1 - N} e^{-N^D - 1 V(\lambda_i)} \right)
\]

with

\[
G(\lambda_i) = \sum_i V(\lambda_i) - \sum_i \ln \lambda_i + \frac{N}{N^D - 1} \left( \sum_i \ln \lambda_i - \frac{2}{N} \sum_{i<j} \ln |\lambda_j - \lambda_i| \right).
\]

The integral in (3.17) is evaluated by a saddle point whose equations write

\[
\frac{\partial G}{\partial \lambda_i} = V'(\lambda_i) - \frac{1}{\lambda_i} + \frac{N}{N^D - 1} \left( \frac{1}{\lambda_i} - \frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \right).
\]

In order to solve at leading order (3.19), one needs only to note that the last term on the right hand side of eq. (3.19) is suppressed in the large \( N \) limit for \( D \geq 3 \). The saddle point equations therefore **decouple** at leading order and the large \( N \) free energy is

\[
F_\infty = V(z_0) - \ln z_0 , \quad z_0 \text{ the physical solution of } z_0 V'(z_0) = \sum_{n=1}^\infty n t_n z_0^n = 1.
\]

The loop observables and resolvent at leading order are therefore

\[
\lim_{N \to \infty} \frac{1}{N} \langle \text{Tr}[(TT^\dagger)^q] \rangle \langle 1 \rangle = \frac{\partial F_\infty}{\partial t_q} = z_q + \left( V'(z_0) - \frac{1}{z_0} \right) \frac{\partial z_0}{\partial t_q} = z_q,
\]

\[
W_0(z) = \lim_{N \to \infty} \frac{1}{N} \langle \text{Tr} \left( z^{1-DD} \right) \rangle \langle 1 \rangle = \frac{1}{z - z_0}.
\]

In particular \( W_0(z) \) has a **pole** singularity at \( z_0 \), not a cut singularity. This is the first major difference with the usual matrix models, and a direct consequence of the factorization of the saddle point equations at leading order.
Note that $z_0$ has a straightforward interpretation as the leading order connected two point function of the model, and the loop observables factor at leading order into two point functions
\[
\lim_{N \to \infty} \frac{1}{N} \frac{\langle \text{Tr}[(TT^\dagger)^q]\rangle}{\langle 1 \rangle} = \left( \lim_{N \to \infty} \frac{1}{N} \frac{\langle \text{Tr}[TT^\dagger]\rangle}{\langle 1 \rangle} \right)^q.
\]
(3.22)

The large $N$ factorization of the loop observables can readily be understood in terms of graphs. The graphs contributing to the connected correlation $\langle \text{Tr}[(TT^\dagger)^q]\rangle$ are connected vacuum graph with a marked vertex (corresponding to the insertion $\text{Tr}[(TT^\dagger)^q]$). Chose a tree in any graph contributing to this correlation and set its root to be the marked vertex. Going around the tree one finds that there exists exactly one contraction which closes a maximal number of faces with “heavy” index $\vec{n}$. A close inspection reveals that this contraction necessarily connects any two consecutive $TT^\dagger$ on the marked vertex via some connected two point graph, and the factorization follows.

Moreover, as $z_0$ is the two point function, the eq. $z_0V'(z_0) = 1$ is in fact the Schwinger Dyson Equation (SDE)
\[
\frac{1}{Z} \int dTd\bar{T} \frac{\delta}{\delta T_{n1}\bar{n}} e^{-N^{D-1} \text{Tr} V(T\bar{T})} = 0,
\]
combined with the factorization property of the loop observables. Also, $z_0V'(z_0) = 1$ is a generating equation for trees
\[
z_0 = \frac{1}{t_1} - \sum_{p=2} \frac{t_p}{t_1} z_0^p = g - \sum_{p=2} p\alpha_p z_0^p,
\]
(3.24)
and arbitrary weighted trees exhibit multi critical behaviors. The derivative of free energy $F_\infty$ at leading order can be computed either directly from (3.20) or from (3.15) and
\[
g \frac{\partial g}{\partial z_0} = -\frac{1}{2\pi i} \int_C du V(u) \frac{1}{u - z_0} = -V(z_0).
\]
(3.25)

B. Multi Critical Points

Inspired by [15] we choose for potential a polynomial of degree $m$. The saddle point equations
\[
z_0V'(z_0) = 1 \Rightarrow g = z_0 + \sum_{q=2}^m q\alpha_q z_0^q,
\]
(3.26)
prove that the coupling $g$ is a polynomial in $z_0$. The tensor model achieves its continuum limit when
\[
\frac{\partial g}{\partial z_0} = 0,
\]
(3.27)
and a $m$’th multi critical point when
\[
\frac{\partial g}{\partial z_0} = 0 \ldots \frac{\partial^{m-1} g}{(\partial z_0)^{m-1}} = 0, \quad \frac{\partial^m g}{(\partial z_0)^m} \neq 0.
\]
(3.28)
A minimal realization for a $m$’th multi critical model is then obtained if the saddle point equation is
\[
g = g_c - (z_c - z_0)^m, \quad g_c = z_c^m,
\]
(3.29)
corresponding to a choice of potential

\[ V(z) = \frac{1}{g} z^m c \sum_{q=0}^{m-1} \frac{1}{q+1} + \frac{1}{g} \sum_{q=0}^{m-1} \left( \frac{z^m - q - 1}{q + 1} \right) (z_{c} - z)^{q+1}, \quad z_{c} = m - \frac{1}{m-1}, \quad (3.30) \]

where we recall that \( V(0) = 0 \) and the coefficient of \( z \) is \( \frac{1}{g} \). The leading order resolvent is then

\[ W_{0} = \frac{1}{z - z_{0}} = \sum_{k=0}^{r} (z_{c} - z_{0})^{k} = \sum_{k=0}^{r} (g_{c} - g)^{\frac{k}{m}} \]

and the leading non analytic behavior of the derivative of the free energy when \( g \to g_{c} \) computes

\[ g \partial_{g} F_{\infty} \bigg|_{n.a.} \sim (g_{c} - g)^{\frac{1}{m}} \Rightarrow F_{\infty} = (g_{c} - g)^{1 + \frac{1}{m}} \quad (3.32) \]

corresponding to a susceptibility exponent \( \gamma_{m} = 1 - 1/m \), identical with the one of the multi critical polymers of [58]. This is not surprising, taking into account the earlier remark on trees.

IV. SCHWINGER DYSON EQUATIONS AT ALL ORDERS

In this section we go beyond the leading order and derive the SDE’s at all orders. We subsequently present the iterative solution order by order in \( N \). In order to fit the scalings in \( N \) of various terms one needs to take specific choices for the dimension \( D \). We will do this in the next section.

The derivation of SDE’s follows the classical path of [53–57], up to the unusual scalings of various terms with \( N \). We reproduce it below for completeness. For \( q \geq 1 \) we write a SDE

\[
\int dT \frac{\delta}{\delta a_{\alpha \epsilon}} \left[ \langle (T (T^\dagger T)^q) a_{\alpha \epsilon} e^{-N^{D-1} \sum_{p} t_{p} \text{Tr}[(T^\dagger T)^p]} \rangle \right]
\]

\[
= N \langle \text{Tr}[(T^\dagger T)^q] \rangle + \sum_{r=1}^{q-1} \left( \text{Tr}[(T^\dagger T)^{q-r}] \text{Tr}[(T^\dagger T)^r] \right) + N^{D-1} \langle \text{Tr}[(T^\dagger T)^q] \rangle
\]

\[
- N^{D-1} \sum_{p=1}^{r} t_{p} \langle \text{Tr}[(T^\dagger T)^{q+r}] \rangle = 0 , \quad (4.33)
\]

where the double trace term is absent if \( q = 1 \). The case \( q = 0 \) is special and leads to

\[
N^{D} \langle 1 \rangle - N^{D-1} \sum_{p} t_{p} \langle \text{Tr}[(T^\dagger T)^p] \rangle = 0 . \quad (4.34)
\]

The SDEs can be written as

\[
L_{m} Z = 0 ,
\]

\[
L_{q} = -\left( N^{-D+2} + 1 \right) \frac{\partial}{\partial t_{q}} + N^{-2D+2} \sum_{r=1}^{q-1} \frac{\partial}{\partial t_{q-r} \partial t_{r}} + \sum_{p=1}^{t_{p}} \frac{\partial}{\partial t_{q+p}} ,
\]

\[
L_{1} = -\left( N^{-D+2} + 1 \right) \frac{\partial}{\partial t_{1}} + \sum_{p=1}^{t_{p}} \frac{\partial}{\partial t_{1} + p} ,
\]

\[
L_{0} = N^{D} \sum_{p=1}^{t_{p}} \frac{\partial}{\partial t_{q+p}} , \quad (4.35)
\]
and, like for the usual matrix models, the $L_m$'s respect the Virasoro algebra. Introducing the derivative of the resolvent

$$W(z, z) = - \sum_{p=1} \frac{1}{z^p} \frac{d}{dp} W(z) ,$$

(4.36)

a straightforward computations expresses the double trace observable as

$$\sum_{t=2} \frac{1}{z^{t+1}} \sum_{r=1}^{t-1} \left\langle \frac{1}{1} \right| \text{Tr}\left[(TT^\dagger)^{t-r} \right| \text{Tr}\left[(TT^\dagger)^r \right] \right\rangle = N^{-D+2}W(z, z) + N^2 \left(zW(z)^2 - 2W(z) + \frac{1}{z}\right) .$$

(4.37)

Adding the equations (4.33) and (4.34) with well chosen coefficients leads to

$$\left[ \frac{1}{z} - \frac{1}{N} \sum_{p=1} \frac{1}{z^p} \frac{1}{1 \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^p \right] \right\rangle} \right] + (N^{-1} + N^{-D+1}) \frac{1}{z^2} \left\langle 1 \right| \frac{1}{1 \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^1 \right] \right\rangle} - \frac{1}{N} \sum_{p=1} \frac{1}{z^p} \frac{1}{1 \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^{1+p} \right] \right\rangle}$$

$$+ \sum_{q=2} \left[ (N^{-1} + N^{-D+1}) \frac{1}{z^{q+1}} \left\langle 1 \right| \frac{1}{1 \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^q \right] \right\rangle} + N^{-D} \sum_{r=1}^{q-1} \frac{1}{z^{q+1}} \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^{q-r} \right| \text{Tr}\left[(T^\dagger T)^r \right] \right\rangle \right\rangle$$

$$- \frac{1}{N} \sum_{p=1} \frac{1}{z^{q+1}} \frac{1}{1 \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^{q+p} \right] \right\rangle} = 0 .$$

(4.38)

Recalling that

$$pt_p = \frac{1}{2\pi i} \int_C du \frac{1}{u^p} V'(u) ,$$

(4.39)

we obtain

$$\frac{1}{z} + \frac{1}{N} \sum_{q=1} \frac{1}{z^{q+1}} \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^q \right] \right\rangle + N^{-D+1} \sum_{q=1} \frac{1}{z^{q+1}} \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^q \right] \right\rangle$$

$$+ N^{-D} \sum_{q=2} \frac{1}{z^{q+1}} \sum_{r=1}^{q-1} \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^{q-r} \right| \text{Tr}\left[(T^\dagger T)^r \right] \right\rangle$$

$$- \frac{1}{N} \sum_{p=1} \sum_{q=0}^{1} \frac{1}{2\pi i} \int_C du \frac{1}{u^p} V'(u) \frac{1}{z^{q+1}} \left\langle 1 \right| \text{Tr}\left[(TT^\dagger)^{q+p} \right] \right\rangle = 0 ,$$

(4.40)

and a straightforward computations finally yields the loop insertion equation

$$\frac{1}{2\pi i} \int_C du \frac{uV'(u)}{z-u} W(u)$$

$$= \left( 1 - zV'(z) \right) W(z) + N^{-D+2} \left(zW(z)^2 - W(z)\right) + N^{-2D+2}W(z, z) .$$

(4.41)
Note that for $D = 2$ we obtain the usual loop insertion equation for a non hermitian matrix (see for instance [53]). The loop insertion equation simplifies further. Indeed, at leading order $W_0(z)$ depends on $t_i$ only implicitly trough $z_0$. This generalizes order by order to $W(z)$. Noting that

$$z_0 V'(z_0) - 1 = 0 \Rightarrow \left[ zV'(z) \right]'_{z=z_0} = \sum_{n=1} \frac{n z_0^n}{t} = 0 \Rightarrow \frac{\partial z_0}{\partial t_q} = - \frac{q z_0^q}{\sum_{n=1} z_0^n}$$

where we supposed that $\left[ zV'(z) \right]'_{z=z_0} \neq 0$ (which holds for instance for multi critical models), we can evaluate

$$W(z, z) = - \sum q = 1 \frac{z^q}{q} \left( \frac{z}{z_0} \frac{\partial z_0}{\partial t_q} \right)^{W(z)} = \frac{1}{z_0} \left[ \frac{\partial z_0}{\partial t_q} \right]^{W(z)}_{z_0}$$

and for $z > z_0$ we obtain

$$W(z, z) = \frac{z z_0 \partial z_0}{(z - z_0)^2} \left[ \frac{\partial z_0}{\partial t_q} \right]^{W(z)}_{z_0}$$

and the equation (4.41) becomes

$$\frac{1}{2\pi i} \int_C du \frac{uV'(u)}{z - u} W(u) = \left(1 - zV'(z)\right) W(z) + N^{D+2} \left( zW(z)^2 - W(z) \right)$$

$$+ N^{-D+2} \frac{z z_0 \partial z_0}{(z - z_0)^2} \left[ \frac{\partial z_0}{\partial t_q} \right]^{W(z)}_{z_0}.$$  

A. Iterative Solution

The loop insertion equation (4.45) can be solved order by order in $N$. We start by expanding the resolvent in powers of $N$,

$$W(z) = W_0(z) + \sum_n N^{-n} W_n(z).$$

From eq. (4.45) we conclude that the first non trivial correction is $W_{D-2}$, and it respects the equation

$$\frac{1}{2\pi i} \int_C du \frac{uV'(u)}{z - u} W_{D-2}(u) - \left(1 - zV'(z)\right) W_{D-2}(z) = z W_0^2 - W_0 = \frac{z_0}{(z - z_0)^2}.$$  

Note that

$$\frac{1}{2\pi i} \int_C du \frac{uV'(u)}{z - u} \left( \frac{1}{(z - z_0)^{n+1}} - \left(1 - zV'(z)\right) \frac{1}{(z - z_0)^{n+1}} \right)$$

$$= \sum_{p=0}^n \frac{1}{(z - z_0)^{p+1}} \frac{1}{(n - p)!} \left[ uV'(u) \right]^{(n-p)}_{z_0} - \frac{1}{(z - z_0)^{n+1}}$$

$$= \sum_{p=0}^{n-1} \frac{1}{(z - z_0)^{p+1}} \frac{1}{(n - p)!} \left[ uV'(u) \right]^{(n-p)}_{z_0}.$$
where we took into account $z_0 V'(z_0) = 1$. Proposing the ansatz

$$W_{D-2} = \frac{a}{(z - z_0)^3} + \frac{b}{(z - z_0)^2}, \quad (4.49)$$

The equation writes

$$a\left(\frac{1}{(z - z_0)^2} [uV'(u)]'_{z_0} + \frac{1}{z - z_0} \frac{1}{2} [uV'(u)]''_{z_0}\right) + b\left(\frac{1}{z - z_0} [uV'(u)]'_{z_0}\right) = \frac{z_0}{(z - z_0)^2}, \quad (4.50)$$

with solution

$$a = \frac{z_0}{[uV'(u)]_{z_0}} \quad b = -\frac{z_0 [uV'(u)]''_{z_0}}{2 \left\{ [uV'(u)]'_{z_0}\right\}^2}. \quad (4.51)$$

The next non-trivial term is $W_{2D-4}$. Subsequent terms are $W_{3D-6}$ and $W_{2D-2}$ (generated by the last line in eq. (4.45)). We obtain non-trivial corrections $W_n$ for $n = p(D - 2) + q(2D - 2)$. In order to derive the full solution we will in the sequel chose $D = 3$.

V. THE DOUBLE SCALING LIMIT IN $D = 3$

In $D = 3$ we obtain a nonzero correction $W_n$ for all $n$

$$W(z) = W_0(z) + \sum_{i=1}^{\infty} N^{-n} W_n(z). \quad (5.52)$$

Note that for any even $D$ (both $D = 4$ and in $D = 2$ for instance) non-trivial corrections $W_n$ arise only for even $n$. The equation (4.45) translates for $W_n$ into

$$\frac{1}{2\pi i} \int_{C} du \frac{uV'(u)}{z - u} W_n(u) - \left(1 - z V'(z)\right) W_n(z) = \sum_{p=0}^{n-1} W_p(z) W_{n-1-p}(z) - W_{n-1}(z) + \frac{zz_0}{(z - z_0)^2 [uV'(u)]'_{z_0}} \partial_{z_0} W_{n-4}(z), \quad (5.53)$$

where by convention $W_q = 0$ for $q < 0$. To solve the equation for $W_n$, we propose the ansatz

$$W_n(z) = \sum_{k=1}^{2n} \frac{f_k^p(z_0)}{(z - z_0)^{k+1}} \quad n \geq 1, \quad W_0 = \frac{1}{z - z_0}, \quad (5.54)$$

and substituting we obtain for $n \geq 1$

$$\sum_{p=1}^{2n} f_p^p(z_0) \sum_{k=0}^{p-1} \frac{1}{(z - z_0)^{k+1}} \frac{1}{(p - k)!} \left[uV'(u)\right]^{(p-k)}_{z_0}$$

$$= (z - z_0) \sum_{p=0}^{n-1} W_p(z) W_{n-1-p}(z) + \sum_{p=0}^{n-1} W_p(z) W_{n-1-p}(z) - W_{n-1}(z)$$

$$+ \frac{z_0}{(z - z_0)^2 [uV'(u)]'_{z_0}} \partial_{z_0} W_{n-4}(z) + \frac{z_0^2}{(z - z_0)^2 [uV'(u)]'_{z_0}} \partial_{z_0} W_{n-4}(z). \quad (5.55)$$
Exchanging the sums on the left hand side we obtain

\[
\sum_{k=0}^{2n-1} \frac{1}{(z - z_0)^{k+1}} \sum_{p=k+1}^{2n} f_n^p(z_0) \frac{1}{(p-k)!} \left[uV'(u)\right]_{z_0}^{(p-k)} \cdot \sum_{p=0}^{n-1} \sum_{p=0}^{n-1} W_p(z)W_{n-1-p}(z) + (z - z_0) \sum_{p=0}^{n-1} W_p(z)W_{n-1-p}(z) - W_{n-1}(z)
\]

\[
= z_0 \sum_{p=0}^{n-1} W_p(z)W_{n-1-p}(z) + (z - z_0) \sum_{p=0}^{n-1} W_p(z)W_{n-1-p}(z) - W_{n-1}(z)
\]

\[
+ \frac{z_0}{(z - z_0)^2} \frac{\partial z_0}{uV'(u)} \left[ uV'(u) \right]_{z_0} \sum_{p=0}^{n-1} \sum_{p=0}^{n-1} W_p(z)W_{n-1-p}(z).
\]

The equation (5.56) allows to determine iteratively the coefficients \( f_n^p(z_0) \). In order to prove that a multi critical model admits a double scaling limit we are interested only in the behavior of these coefficients when \( z_0 \) approaches \( z_c \). Using the solution at first order we see that at leading order in \( z_c - z_0 \),

\[
f_1^2(z_0) = \frac{d_1^2}{(z_c - z_0)^{m-1}}, \quad f_1^1(z_0) = \frac{d_1^1}{(z_c - z_0)^{m}},
\]

with \( d_1^2 \) and \( d_1^1 \) some numerical coefficients. We will show by induction that the generic scaling formula

\[
f_n^k(z_0) = \frac{d_n^k}{(z_c - z_0)^{n(m+1)-k}},
\]

holds. The scaling at order \( n \) is fixed from (5.56). We analyze one by one the scalings of the various terms on the right hand side for \( n \geq 2 \).

The first term we analyze is

\[
\sum_{p=0}^{n-1} W_p(z)W_{n-1-p}(z) = 2W_0W_{n-1} + \sum_{p=0}^{n-2} W_p(z)W_{n-1-p}(z),
\]

where the second sum is absent when \( n = 2 \). It writes

\[
2W_0W_{n-1} + \sum_{p=0}^{n-2} \sum_{k_1=1}^{2p} 2^{n-2-2p} \sum_{k_2=1}^{2n-2} \frac{f_{k_1}^p(z_0)}{(z - z_0)^{k_1+1}} \frac{f_{k_2}^{k_1-p}(z_0)}{(z - z_0)^{k_2+1}}
\]

\[
= \sum_{t=1}^{2n-2} \frac{2f_{n-1}^t(z_0)}{(z - z_0)^{t+2}} + \sum_{p=0}^{n-2} \sum_{k_1=1}^{2p} 2^{n-2} \frac{1}{(z - z_0)^{t+2}} \frac{f_{k_1}^p(z_0)}{(z - z_0)^{k_1+1}} \frac{f_{n-1-p}^{k_1}(z_0)}{(z - z_0)^{k_1+1}}
\]

\[
\sim \sum_{k=1}^{2n-2} \frac{1}{(z - z_0)^{k+1}} \frac{1}{(z_c - z_0)^{n-1}(m+1)-k+1},
\]

hence the first line of the right hand side of (5.56) is dominated when \( z_0 \to z_c \) by the first term. The derivatives of \( W_{n-4} \) behave for \( z_0 \to z_c \) like

\[
\partial_{z_0} W_{n-4} \sim \sum_{k} \frac{1}{(z - z_0)^{k+1}} \frac{1}{(z_c - z_0)^{n-4}(m+1)-k+1},
\]

hence the second line on the right hand side of (5.56) scales at most like

\[
\sum_{k} \frac{1}{(z - z_0)^{k+1}} \frac{1}{(z_c - z_0)^{n-3}(m+1)-k+3},
\]

(5.62)
and when \(z_0 \to z_c\) is dominated by the terms in the first line. Identifying the coefficients of \(\frac{1}{(z-z_0)^{m+1}}\) in eq. (5.56) we conclude that

\[
\sum_{p=k+1}^{2n} f_n^p(z_0)[uV]^n(p-k) \sim \frac{1}{(z_c-z_0)^{n-m+1}(m+1)} ,
\]

and taking into account that \([uV]^n(p-k) \sim (z_c-z_0)^{m-p+k}\), we get

\[
f_n^p(z_0) \sim \frac{1}{(z_c-z_0)^{nm+n-p}} ,
\]

reproducing (5.58). It follows that the leading contributions to the resolvent write

\[
W(z) = \frac{1}{z-z_0} + \sum_{n=1}^N \frac{\sum_{k=1}^d (z_c-z_0)^k (z-z_0)^{k+1}}{n} \frac{d_k}{N^n} \frac{1}{(z_c-z_0)^{n(m+1)}} ,
\]

Taking into account that \(z_c - z_0 = (g_c - g)^{1/m}\), the resolvent writes when \(g \to g_c\) as

\[
W(z) = \frac{1}{z-z_c} + \sum_{k} \frac{1}{(z-z_c)^{k+1}} (g_c-g)^{\frac{1}{m}} F_k(\kappa)
\]

with \(\kappa^{-1} = N(g_c - g_0)^{1+\frac{1}{m}}\). The graphs contributing to the double scaling limit can be easily understood in terms of stranded graphs starting from the SDE in equation (4.41). The term on the left hand side represents the connection of two effective vertices by a tree line. The terms on the left hand side represent a marked vertex decorated by a loop line. The loop line divides the marked vertex into two smaller effective vertices, corresponding to \(\frac{1}{1} \langle \text{Tr}[(TT^\dagger)^{q-r}] \text{Tr}[(T^\dagger T)^r] \rangle\). Either these two smaller vertices do not reconnect again (corresponding to the terms involving \(W(z)\) and \(W(z)^2\), i.e. the correlation splits into two connected correlations \(\langle \text{Tr}[(TT^\dagger)^{q-r}] \rangle \langle \text{Tr}[(T^\dagger T)^r] \rangle\) in which case the total graph is planar. Or the two smaller vertices reconnect with at least a line (the terms generated by \(W(z, z)\)), in which case the correlation does not split \(\langle \text{Tr}[(TT^\dagger)^{q-r}] \text{Tr}[(T^\dagger T)^r] \rangle\), and the initial loop line and the new line cross yielding a non planar graph. All the contributions of \(W(z, z)\) are suppressed in the double scaling limit with respect to those of \(W^2(z)\). It ensues that only planar graphs corresponding to spheres in \(D = 3\) dimension contribute in the double scaling limit.

VI. CONCLUSION

We have shown that a \(N \times N^{D-1}\) matrix model admits multi critical points and a double scaling limit. The model constitutes a toy model for higher dimensional tensor models. In \(D = 3\) the double scaling limit only graphs having the topology of a sphere contribute. This is expected to hold for all \(D\). Indeed, form eq. (4.41) we see that the term \(W(z, z)\) is increasingly suppressed with respect to \(W^2(z)\), hence only planar graphs should dominate for all \(D\). Such graphs are homotopically trivial in all dimensions, and are expected to represent spheres.

Although the model we analyze is very simple, and the (prospective) double scaling limit of colored tensor models is expected to be much more involved one can draw several conclusions about the major features of such an expansion. First, one should not expect all terms in the \(1/N\)
expansion to contribute: unlike in usual matrix models, even in the double scaling many terms are suppressed. In fact, somewhat surprisingly, one should not expect the double scaling limit to be a summation over topologies: it seems possible that only spherical graphs contribute. Third, the exact interpretation of this double scaling limit is yet unclear. In two dimensions one notices that the free energy in the double scaling limit is a series in $\kappa^\chi$, where $\chi$ is the Euler character of a graph. As the latter is just the evaluation of the Einstein Hilbert action, one concludes that the double scaling limit corresponds to a continuum limit with a finite (renormalized) Newton’s constant. In higher dimension the precise interpretation of the parameter $\kappa$ is far from clear.

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[1] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” Phys. Rept. 254, 1 (1995) arXiv:hep-th/9306153.
[2] G. ’t Hooft, “A PLANAR DIAGRAM THEORY FOR STRONG INTERACTIONS,” Nucl. Phys. B 72, 461 (1974).
[3] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” Commun. Math. Phys. 59, 35 (1978).
[4] V. A. Kazakov, “Bilocal Regularization of Models of Random Surfaces,” Phys. Lett. B 150, 282 (1985).
[5] F. David, “A Model Of Random Surfaces With Nontrivial Critical Behavior,” Nucl. Phys. B 257, 543 (1985).
[6] J. Ambjorn, B. Durhuus and T. Jonsson, “Quantum geometry. A statistical field theory approach,” Cambridge, UK: Univ. Pr., 1997. (Cambridge Monographs in Mathematical Physics). 363 p
[7] F. David, “Simplicial quantum gravity and random lattices,” Les Houches Sum. Sch. 1992:0679-750. arXiv:hep-th/9303127
[8] J. Ambjorn, “Quantization of geometry,” Fluctuating Geometries in Statistical Mechanics and Field Theory: Proceedings. Edited by F. David, P. Ginsparg and J. Zinn-Justin. North-Holland, 1996. pp. 77-195. (ISBN 0-444-82294-1). arXiv:hep-th/9411179
[9] R. Loll, “Discrete approaches to quantum gravity in four dimensions,” Living Rev. Rel. 1 (1998) 13 arXiv:gr-qc/9805049.
[10] J. Ambjorn and J. Jurkiewicz, “Scaling in four-dimensional quantum gravity,” Nucl. Phys. B 451 (1995) 643 arXiv:hep-th/9503006.
[11] J. Ambjorn, A. Gorlich, J. Jurkiewicz and R. Loll, “CDT—an Entropic Theory of Quantum Gravity,” arXiv:1007.2560 [hep-th].
[12] J. Ambjorn, J. Jurkiewicz and R. Loll, “Reconstructing the universe,” Phys. Rev. D 72, 064014 (2005) arXiv:hep-th/0505154.
[13] J. Ambjorn, A. Gorlich, J. Jurkiewicz, R. Loll, J. Gizbert-Studnicki and T. Trzesniewski, “The Semi-classical Limit of Causal Dynamical Triangulations,” Nucl. Phys. B 849 (2011) 144 arXiv:1102.3929 [hep-th].
[14] J. Ambjorn, S. Jordan, J. Jurkiewicz and R. Loll, “A second-order phase transition in CDT,” arXiv:1108.3932 [hep-th].
[15] V. A. Kazakov, “The Appearance of Matter Fields from Quantum Fluctuations of 2D Gravity,” Mod. Phys. Lett. A 4, 2125 (1989).
[16] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, “Fractal Structure of 2D Quantum Gravity,” Mod. Phys. Lett. A 3, 819 (1988).
[17] F. David, “Conformal Field Theories Coupled to 2D Gravity in the Conformal Gauge,” Mod. Phys. Lett. A 3, 1651 (1988).
[18] J. Distler, H. Kawai, “Conformal Field Theory and 2D Quantum Gravity Or Who’s Afraid of Joseph Liouville?,” Nucl. Phys. B321, 509 (1989).
[19] B. Duplantier “Conformal Random Geometry” Les Houches, Session LXXXIII, 2005, Mathematical Statistical Physics, A. Bovier, F. Dunlop, F. den Hollander, A. van Enter and J. Dalibard, eds., pp. 101-217, Elsevier B. V. (2006): arXiv:math-ph/0608053.
[20] E. Brezin and V. A. Kazakov, “EXACTLY SOLVABLE FIELD THEORIES OF CLOSED STRINGS,” Phys. Lett. B 236, 144 (1990).
[21] M. R. Douglas, S. H. Shenker, “Strings in Less Than One-Dimension,” Nucl. Phys. B335, 635 (1990).
[22] D. J. Gross, A. A. Migdal, “Nonperturbative Two-Dimensional Quantum Gravity,” Phys. Rev. Lett. 64, 127 (1990).
[23] J. Ambjorn, B. Durhuus and T. Jonsson, “Three-Dimensional Simplicial Quantum Gravity And Generalized Matrix Models,” Mod. Phys. Lett. A 6, 1133 (1991).
[24] M. Gross, “Tensor models and simplicial quantum gravity in > 2-D,” Nucl. Phys. Proc. Suppl. 25A, 144 (1992).
[25] N. Sasakura, “Tensor model for gravity and orientability of manifold,” Mod. Phys. Lett. A 6, 2613 (1991).
[26] L. Freidel, R. Gurau and D. Oriti, “Group field theory renormalization - the 3d case: Power counting of divergences,” Phys. Rev. D 80, 044007 (2009) arXiv:0905.3772 [hep-th].
[27] J. Magnen, K. Noui, V. Rivasseau and M. Smerlak, “Scaling behaviour of three-dimensional group field theory,” Class. Quant. Grav. 26, 185012 (2009) arXiv:0906.5477 [hep-th].
[28] J. Ben Geloun, J. Magnen and V. Rivasseau, “Bosonic Colored Group Field Theory,” Eur. Phys. J. C 70, 1119 (2010) arXiv:0911.1719 [hep-th].
[29] J. Ben Geloun, T. Krajewski, J. Magnen and V. Rivasseau, “Linearized Group Field Theory and Power Counting Theorems,” Class. Quant. Grav. 27, 155012 (2010) arXiv:1002.3592 [hep-th].
[30] V. Bonzom and M. Smerlak, “Bubble divergences: sorting out topology from cell structure,” arXiv:1103.3961 [gr-qc].
[31] V. Bonzom and M. Smerlak, “Bubble divergences from cellular cohomology,” Lett. Math. Phys. 93, 295 (2010) arXiv:1004.5196 [gr-qc].
[32] S. Carrozza and D. Oriti, “Bounding bubbles: the vertex representation of 3d Group Field Theory and the suppression of pseudo-manifolds.” arXiv:1104.5158 [hep-th].
[33] N. Sasakura, “Tensor models and 3-ary algebras,” arXiv:1104.1463 [hep-th].
[34] N. Sasakura, “Tensor models and hierarchy of n-ary algebras,” Int. J. Mod. Phys. A26, 3249-3258 (2011) arXiv:1104.5312 [hep-th].
[35] N. Sasakura, “Super tensor models, super fuzzy spaces and super n-ary transformations,” Int. J. Mod. Phys. A26, 4203-4216 (2011). arXiv:1106.0379 [hep-th].
[36] R. Gurau, “The 1/N expansion of colored tensor models,” Annales Henri Poincare 12, 829 (2011) arXiv:1011.2728 [gr-qc].
[37] R. Gurau, V. Rivasseau, “The 1/N expansion of colored tensor models in arbitrary dimension,” Europhys. Lett. 95, 50004 (2011). arXiv:1101.4182 [gr-qc].
[38] R. Gurau, “The complete 1/N expansion of colored tensor models in arbitrary dimension,” arXiv:1102.5759 [gr-qc].
[39] R. Gurau, “Colored Group Field Theory,” Commun. Math. Phys. 304, 69 (2011) arXiv:0907.2582 [hep-th].
[40] R. Gurau, “Lost in Translation: Topological Singularities in Group Field Theory,” Class. Quant. Grav. 27, 235023 (2010) arXiv:1006.0714 [hep-th].
[41] R. Gurau, “Topological Graph Polynomials in Colored Group Field Theory,” Annales Henri Poincare 11, 565 (2010) arXiv:0911.1945 [hep-th].
[42] F. Caravelli, “A simple proof of orientability in the colored Boulatov model,” arXiv:1012.4087 [math-ph].
[43] V. Bonzom, R. Gurau, A. Riello, V. Rivasseau, “Critical behavior of colored tensor models in the large N limit,” Nucl. Phys. B853, 174-195 (2011). arXiv:1105.3122 [hep-th].
[44] R. Gurau, “A generalization of the Virasoro algebra to arbitrary dimensions,” Nucl. Phys. B 852, 592 (2011) arXiv:1105.6072 [hep-th].
[45] J. B. Geloun, “Ward-Takahashi identities for the colored Boulatov model,” arXiv:1106.1847 [hep-th].
[46] J. B. Geloun, “Classical Group Field Theory,” arXiv:1107.3122 [hep-th].
[47] A. Baratin, F. Girelli and D. Oriti, “Diffeomorphisms in group field theories,” Phys. Rev. D 83, 104051 (2011) [arXiv:1101.0590 [hep-th]].

[48] D. Oriti, L. Sindoni, “Towards classical geometrodynamics from Group Field Theory hydrodynamics,” New J. Phys. 13, 025006 (2011). [arXiv:1010.5149 [gr-qc]].

[49] J. B. Geloun and V. Bonzom, “Radiative corrections in the Boulatov-Ooguri tensor model: The 2-point function,” Int. J. Theor. Phys. 50, 2819 (2011) [arXiv:1101.4294 [hep-th]].

[50] V. A. Kazakov, “Ising model on a dynamical planar random lattice: Exact solution,” Phys. Lett. A 119, 140 (1986).

[51] V. Bonzom, R. Gurau, V. Rivaussetau, “The Ising Model on Random Lattices in Arbitrary Dimensions,” [arXiv:1108.6269 [hep-th]].

[52] V. A. Kazakov, M. Staudacher and T. Wynter, “Characters expansion methods for matrix models of dually weighted graphs”, Commun. Math. Phys. 177 (1996) 451–468 [arXiv:hep-th/9502132].

[53] D. Benedetti, R. Gurau, “Phase Transition in Dually Weighted Colored Tensor Models,” [arXiv:1108.5389 [hep-th]].

[54] R. Gurau, J. P. Ryan, “Colored Tensor Models - a review,” [arXiv:1109.4812 [hep-th]].

[55] J. Ambjorn, J. Jurkiewicz and Yu. M. Makeenko, “Multiloop correlators for two-dimensional quantum gravity,” Phys. Lett. B 251, 517 (1990).

[56] M. Fukuma, H. Kawai and R. Nakayama, “Continuum Schwinger-Dyson Equations and universal structures in two-dimensional quantum gravity,” Int. J. Mod. Phys. A 6, 1385 (1991).

[57] Yu. Makeenko, “Loop equations and Virasoro constraints in matrix models,” [arXiv:hep-th/9112058].

[58] J. Ambjorn, B. Durhuus, T. Jonsson, “SUMMING OVER ALL GENERA FOR d > 1: A TOY MODEL,” Phys. Lett. B244, 403-412 (1990).

[59] B. Bahr, B. Dittrich, J. P. Ryan, “Spin foam models with finite groups,” [arXiv:1103.6264 [gr-qc]].

[60] J. P. Ryan, “Tensor models and embedded Riemann surfaces,” [arXiv:1104.5471 [gr-qc]].

[61] J. P. Ryan, in preparation.

[62] P. Di Francesco, “Rectangular matrix models and combinatorics of colored graphs,” Nucl. Phys. B648, 461-496 (2003). [cond-mat/0208037].