Quasibosons composed of two $q$-fermions: realization by deformed oscillators

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Abstract

Composite bosons, here called quasibosons (e.g. mesons, excitons, etc), occur in various physical situations. Quasibosons differ from bosons or fermions as their creation and annihilation operators obey non-standard commutation relations, even for the ‘fermion+fermion’ composites. Our aim is to realize the operator algebra of quasibosons composed of two fermions or two $q$-fermions ($q$-deformed fermions) by the respective operators of deformed oscillators, the widely studied objects. For this, the restrictions on quasiboson creation/annihilation operators and on the deformed oscillator (deformed boson) algebra are obtained. Their resolving proves the uniqueness of the family of deformations and gives explicitly the deformation structure function (DSF) which provides the desired realization. In the case of two fermions as constituents, such realization is achieved when the DSF is a quadratic polynomial in the number operator. In the case of two $q$-fermions, $q \neq 1$, the obtained DSF inherits the parameter $q$ and does not continuously converge when $q \to 1$ to the DSF of the first case.

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1. Introduction

Theoretical treatment of many-particle systems is connected with a number of complications. Some of them can be resolved by introducing the concept of quasiparticle or ‘composite particle’, if this is possible. However, in this way we generally encounter various factors of the internal structure, which cannot be completely encapsulated into internal degrees of freedom of a composite particle. These are the nontrivial commutation relations, or the interaction of the constituents between themselves and with other particles, etc. It is desirable to have an equivalent description of many-(composite-)particle systems, almost as simple as the description of an ideal/point-like particle system, but taking into account the mentioned factors. Deformed bosons or deformed oscillators, see e.g. the review [1], provide possible means for the realization of such an intention. In such a case the basic characteristics of the
factors connected with the internal structure would be encoded in one or more deformation parameters.

A particular realization of the mentioned idea to describe quasibosons [2] (boson-like composites) in terms of deformed Heisenberg algebra was demonstrated by Avancini and Krein in [3], who utilized the quonic [4] version of the deformed boson algebra. Note that if two or more copies (modes) are involved, different modes of quons do not commute [3, 4]. Unlike quons, the deformed oscillators of Arik–Coon (AC) type are independent [5, 6], that is, the operators corresponding to different copies mutually commute.

Regardless of their intrinsic origin and physical motivation, diverse models of deformed oscillators have received much attention during the 1990s and till now. Among the best-known and extensively studied deformed oscillators one encounters the $q$-deformed AC [5] or Biedenharn–Macfarlane (BM) [7, 8] ones, the $q$-deformed Tamm–Dancoff oscillator [9–11], and also the two-parameter $p$, $q$-deformed oscillator [12, 13]. On the other hand, the so-called $\mu$-deformed oscillator is much less studied. Introduced in [14] almost two decades ago, this deformed oscillator essentially differs from the models we have already mentioned and exhibits rather unusual properties [15, 16]. Note that there exists a general approach to the description of deformed oscillators based on the concept of the deformation structure function (DSF) given in [17, 1]. As the extension of the standard quantum harmonic oscillator, deformed oscillators find diverse applications in describing miscellaneous physical systems involving essential nonlinearities, from say quantum optics and the Landau problem to high-energy particle phenomenology and modern quantum field theory, see e.g. [18–27].

Although a great variety of models of deformed oscillators exist as mentioned above, the detailed analysis of possible realizations, on their base, of composite particles along with the interpretation of deformation parameters in terms of the internal structure as far as we know is lacking. To fill this gap, in our preceding paper [28] some steps in that direction were undertaken and first results were obtained. Namely, we carried out the detailed analysis for quasibosons consisting of two ordinary fermions with the ansatz $A_{\alpha}^\dagger = \Phi^{\mu\nu}_{1\alpha} a_\mu^\dagger b_\nu^\dagger$ for the quasiboson creation operator in the $\alpha$th mode, meaning the bilinear combination of the constituents’ creation operators of the general form. The analysis implies the realization of quasibosons by deformed oscillators characterized by the most general DSF $\phi(N)$ which unambiguously determines [17, 1] the deformed algebra within one mode. Our present study further extends the results obtained in [28] by using, instead of the usual fermions, their $q$-deformed analog for the constituents’ operators.

The paper is organized as follows. Section 2, which serves as the base for our analysis, concerns the case of quasibosons whose constituents are ordinary fermions (the particular $q = 1$ case of $q$-fermions). Here, after introducing the creation and annihilation operators for composite quasibosons, we recapitulate main facts and results from [28]. (Note that some of these results, only sketched in [28], here are presented in full detail: in particular, that concerns the extended treatment given in subsection 2.3.) We establish important relations for quasibosons’ operators that include necessary conditions for the representation of quasibosons in terms of deformed bosons to hold. Those conditions are partially solved in subsection 2.1, yielding the DSFs $\phi(N)$ of the effective deformation, and completely solved in subsection 2.3. There we obtain explicitly all possible internal structures for quasibosons with the corresponding matrices $\Phi^{\mu\nu}_{1\alpha}$. In section 3, presenting the further development of the ideas and results of [28], for the constituents’ operators, we take instead of usual fermions their $q$-deformed analogs. The corresponding treatment is performed: the admissible (for the realization under question) structure function $\phi(n)$ and matrices $\Phi^{\mu\nu}_{n\alpha}$ are found as the solution of the necessary conditions for the validity of the realization. Simpler illustrative examples,
along with intermediate proofs, are relegated to appendices. The paper ends with concluding remarks and some outlook.

### 2. System of quasibosons composed of two fermions

The general task of representing the quasibosons consisting of *q*-fermions can be divided into two particular situations: (i) the constituents are pure fermions (*q* = 1) and (ii) the constituents are essentially deformed *q*-fermions (*q* ≠ 1). This section is devoted to the first case: similar to [28] we deal with the system of composite boson-like particles (*quasibosons* [2]) such that each copy (mode) of them is built from two fermions. We study the realization of quasibosons in terms of the set of *independent* identical copies of deformed oscillators of the general form (for some examples of mode-independent systems see [6]).

Let us denote the creation and annihilation operators of the two (mutually anticommuting) sets of usual fermions by $a_\mu^\dagger, b_\nu^\dagger, a_\mu, b_\nu$, respectively, with their standard anticommutation relations, namely

$$\{a_\mu, a_\mu^\dagger\} = a_\mu a_\mu^\dagger + a_\mu^\dagger a_\mu = \delta_{\mu\mu}, \quad \{a_\mu, b_\nu\} = 0,$$

$$\{b_\nu, b_\nu^\dagger\} = b_\nu b_\nu^\dagger + b_\nu^\dagger b_\nu = \delta_{\nu\nu}, \quad \{b_\nu, b_\nu^\dagger\} = 0.$$  \hspace{1cm} (1)

Besides, each of $a_\mu^\dagger, a_\mu$ anticommutes with each of $b_\nu^\dagger, b_\nu$. So, we use these fermions to construct quasibosons. Then, the corresponding quasibosonic creation and annihilation operators $A_\alpha^\dagger$ and $A_\alpha$ (where $\alpha$ labels the particular quasiboson and denotes the whole set of its quantum numbers) are given as

$$A_\alpha^\dagger = \sum_{\mu\nu} \Phi_{\alpha}^{\mu\nu} a_\mu^\dagger b_\nu^\dagger, \quad A_\alpha = \sum_{\mu\nu} \Phi_{\alpha}^{\mu\nu} b_\nu a_\mu^\dagger.$$  \hspace{1cm} (2)

For the matrices $\Phi_{\alpha}$, we assume the following normalization condition:

$$\sum_{\mu\nu} \Phi_{\alpha}^{\mu\nu} \Phi_{\beta}^{\mu\nu} = \text{Tr} \Phi_{\alpha}^\dagger \Phi_{\beta} = \delta_{\alpha\beta}.$$  \hspace{1cm} (3)

One can easily check that

$$[A_\alpha, A_\beta] = [A_\alpha^\dagger, A_\beta^\dagger] = 0.$$  \hspace{1cm} (4)

For the remaining commutator one finds [3]

$$[A_\alpha, A_\beta^\dagger] = \sum_{\mu\nu\mu'\nu'} \Phi_{\alpha}^{\mu\nu} \Phi_{\beta}^{\mu'\nu'} \left( [a_\mu, a_\mu^\dagger] b_\nu b_{\nu'}^\dagger + a_\mu^\dagger, a_\mu [b_\nu, b_{\nu'}^\dagger] \right) = \delta_{\alpha\beta} - \Delta_{\alpha\beta},$$

where

$$\Delta_{\alpha\beta} \equiv \sum_{\mu\nu} \Phi_{\alpha}^{\mu\nu} \Phi_{\beta}^{\mu\nu} a_\mu^\dagger a_\mu + \sum_{\mu\nu} \Phi_{\alpha}^{\mu\nu} \Phi_{\beta}^{\mu\nu} b_\nu b_{\nu}^\dagger.$$  \hspace{1cm} (5)

The entity $\Delta_{\alpha\beta}$ in (5) shows deviation from the pure bosonic canonical relation. Note that if $\Delta_{\alpha\beta} = 0$, then we have $\Phi_{\alpha}^{\mu\nu} = 0$.

Remark that unlike the realization of quasibosonic operators using the quonic version of the deformed oscillator algebra, as was done in [3], in all our analysis we consider (the set of) completely independent copies of deformed oscillators. That is, we assume the validity of (3) and also require $[A_\alpha, A_\beta^\dagger] = 0$ for $\alpha \neq \beta$.

The most simple type of deformed oscillator is the AC *q*-deformation [5]. So it is of interest, first, to try to use this set of *q*-deformed bosons for representing the system of independent quasibosons. However, as was shown in [28], the representation of quasibosons with the *independent* system of *q*-deformed bosons of the AC type leads to inconsistency. For that reason, we set the goal to examine other deformed oscillators in the general form given by their structure function $\phi(N)$. 

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**Necessary conventions.** Our goal is to operate with $A_α$, $A^\dagger_α$ and $N_α$ constructed from $a^α_j$, $a_j$, $b^α_j$, $b_j$ ($N_α$ is some effective number operator for composite particles) as with the elements (operators) of some deformed oscillator algebra, ‘forgetting’ about their internal structure. It means that we are looking for subalgebras of the enveloping algebra $\mathfrak{A}\{A_α, A^\dagger_α, N_α\}$, generated by $A_α$, $A^\dagger_α$, $N_α$, isomorphic to some deformed oscillator algebras $\mathfrak{A}\{A_α, A^\dagger_α, N_α\}$, generated by $A_α$, $A^\dagger_α$, $N_α$:

$$\mathfrak{A}\{A_α, A^\dagger_α, N_α\} \simeq \mathfrak{A}\{A_α, A^\dagger_α, N_α\}.$$  

We will establish necessary and sufficient conditions for the existence of such isomorphism. We also require the isomorphism of representation spaces of the mentioned algebras:

$$L\{a^α_j \cdots b^α_j\} \supset H \simeq H = L\{A^\dagger_α \cdots A^\dagger_n\},$$  

where $L\{\ldots\}$ denotes a linear span. Thus, if the algebra of deformed oscillator operators is given by the relations

$$G_i(A_α, A^\dagger_α, N_α) = 0 \quad \Leftrightarrow \quad G_i(A_α, A^\dagger_α, N_α) A^\dagger_α \cdots A^\dagger_n \{O\} = 0.$$  

then necessary and sufficient conditions for the isomorphism to exist can be written as

$$G_i(A_α, A^\dagger_α, N_α) \equiv 0 \quad \overset{\text{def}}{\iff} \quad G_i(A_α, A^\dagger_α, N_α) A^\dagger_α \cdots A^\dagger_n \{O\} = 0.$$  

Here, the symbol of the weak equality $\equiv$ is introduced which means the equality on all the $n$-(quasi)boson states. Next, we observe that

$$G_i(A_α, A^\dagger_α, N_α) = 0 \quad \Leftrightarrow \quad [G_i, A^\dagger_α]\{O\} = 0$$  

and, by induction,

$$G_i(A^\dagger_α \cdots A^\dagger_n)\{O\} = 0 \quad \Leftrightarrow \quad [\ldots[G_i, A^\dagger_n]\ldots A^\dagger_α]\{O\} = 0.$$  

For a general deformed oscillator defined using the structure function $\phi(N)$, see e.g. [1], relation (7) takes the form

$$[A^\dagger_α, A_α] = \phi(N_α),$$  

$$[A_α, A^\dagger_β] = \phi(N_α + 1) - \phi(N_α),$$  

$$[N_α, A^\dagger_α] = A^\dagger_α, \quad [N_α, A_α] = -A_α.$$  

Here the expressions for $[A_α, A^\dagger_β]$ for any, if any, may be added. Thus, the set of functions $G_i$ applicable in this case reads as follows:

$$G_0(A_α, A^\dagger_α, N_α) = A^\dagger_α A_α - \phi(N_α),$$  

$$G_1(A_α, A^\dagger_α, N_α) = [A_α, A^\dagger_α] - (\phi(N_α + 1) - \phi(N_α)),$$

$$G_2(A^\dagger_α, N_α) = [N_α, A^\dagger_α] - A^\dagger_α.$$  

Such functions $G_i$ are determined by the structure function of deformation $\phi(N_α)$. So relations (7) can be used for deducing the connection between matrices $\Phi^\alpha_\beta$, which determine the operators $A^\dagger_α$, and the DSF $\phi(N_α)$.

### 2.1. Necessary conditions on $\Phi^\alpha_\beta$ and $\phi(n)$

In the subsequent analysis, we study the independent quasibosons’ systems realized by deformed oscillators without an indication of the particular model of deformation. The aim of this section is to obtain necessary conditions for such realization in terms of the matrices $\Phi^\alpha_\beta$. Note that the results of this section are not sensitive to the form of the definition of $N_α (\cdot)$ as a function of $A_α$, $A^\dagger_α$.  

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Using relations (7)–(10) and taking into account the independence of modes, we arrive at the following weak equalities for the commutators:
\[
\begin{align*}
[A_\alpha, A_\beta^\dagger] &\equiv 0 \quad \text{for} \quad \alpha \neq \beta, \\
[N_\alpha, A_\beta^\dagger] &\equiv A_\beta^\dagger, \quad \quad \quad \quad \quad \quad \quad [N_\alpha, A_\alpha] \equiv -A_\alpha, \\
[A_\alpha, A_\beta] &\equiv \phi(N_\alpha + 1) - \phi(N_\alpha).
\end{align*}
\] (11)

**Treatment of mode independence.** From the first relation in (11) we derive the equivalent requirements of independence in terms of matrices \( \Phi \):
\[
\sum_{\mu \nu} (\Phi_{\beta \mu}^\dagger \Phi_{\alpha \mu} - \Phi_{\beta \nu}^\dagger \Phi_{\alpha \nu} + \Phi_{\alpha \mu}^\dagger \Phi_{\beta \nu}^\dagger) = 0, \quad \alpha \neq \beta,
\] (12)
which can be rewritten in the matrix form
\[
\Phi_{\beta}^\dagger \Phi_{\alpha} + \Phi_{\alpha}^\dagger \Phi_{\beta} = 0, \quad \alpha \neq \beta.
\] (13)

**Conditions on \( \Phi_{\alpha}^\mu \) within one mode \( \alpha \).** Since \( A_\alpha^\dagger A_\alpha \equiv \phi(N_\alpha) \) and \( A_\alpha A_\alpha^\dagger \equiv \phi(N_\alpha + 1) \), we have
\[
[A_\alpha^\dagger A_\alpha, A_\alpha] \equiv 0 \quad \text{and} \quad [\Delta_{\alpha\beta}, N_\alpha] \equiv 0.
\] (14)
The first equality can equivalently be rewritten as
\[
[A_\alpha^\dagger A_\alpha, \Delta_{\alpha\alpha}] = \left[ A_\alpha^\dagger A_\alpha, \sum_{\mu \nu} \Phi_{\alpha \mu}^\dagger \Phi_{\alpha \mu} a_{\mu}^\dagger b_{\nu} + \sum_{\mu \nu} \Phi_{\alpha \mu}^\dagger \Phi_{\alpha \nu}^\dagger b_{\mu}^\dagger b_{\nu} \right] \equiv 0.
\]
The calculation of this commutator gives
\[
[A_\alpha^\dagger A_\alpha, \Delta_{\alpha\alpha}] = 2A_\alpha^\dagger \sum_{\mu \nu} (\Psi_{\alpha \mu}^\dagger)^\mu_{\nu} b_{\nu} a_{\mu} - 2 \sum_{\mu \nu} \Psi_{\alpha \mu}^\dagger a_{\mu}^\dagger b_{\nu}^\dagger A_\alpha \equiv 0,
\] (15)
with \( \Psi_{\alpha} \equiv \Phi_{\alpha} \Phi_{\alpha}^\dagger \Phi_{\alpha} \).

With the account of (2) one can see: the validity of (15) on the one-quasiboson state requires that the commutator with the creation operator on the vacuum should be
\[
\left[ (\Phi_{\alpha \mu}^\dagger \Psi_{\alpha \mu}^\dagger - \Phi_{\alpha \nu}^\dagger \Psi_{\alpha \nu}^\dagger) a_{\mu}^\dagger b_{\mu}^\dagger + \Phi_{\alpha \mu}^\dagger a_{\mu}^\dagger b_{\nu}^\dagger \right] (0) = 0.
\]
and
\[
\Phi_{\alpha \mu}^\dagger a_{\mu}^\dagger b_{\nu}^\dagger - \Phi_{\alpha \mu}^\dagger a_{\nu}^\dagger b_{\mu}^\dagger \Delta_{\alpha\alpha} (0) = 0.
\]
(the summation over repeated indices is meant). From this we obtain the requirement
\[
\Phi_{\alpha \mu}^\dagger a_{\mu}^\dagger b_{\nu}^\dagger = \text{Tr} (\Psi_{\alpha \mu}^\dagger \Phi_{\alpha \mu}^\dagger \Phi_{\alpha \nu}^\dagger \Phi_{\alpha \nu}) \cdot \Phi_{\alpha},
\] (16)
which is also the sufficient one. This requirement guarantees not only the weak equality as in (15) but also the corresponding strong (operator) equality.

Thus, we have two independent requirements (13) and (16) for the matrices \( \Phi_{\alpha} \).

**Relating \( \Phi_{\alpha} \) to the structure function \( \phi(n) \).** Let us derive the relations that involve the DSF \( \phi \). Directly from the system (11) we obtain the initial values for the DSF \( \phi \):
\[
\phi(N_\alpha) \equiv A_\alpha^\dagger A_\alpha \quad \Rightarrow \quad \phi(0) = 0,
\]
\[
\phi(N_\alpha + 1) \equiv A_\alpha A_\alpha^\dagger \quad \Rightarrow \quad \phi(1) = 1.
\]
From (4) and the third relation in (11), we have

$$[A_\alpha, A_\alpha^\dagger] = 1 - \Delta_{\alpha\alpha} \equiv \phi(N_\alpha + 1) - \phi(N_\alpha),$$

or, equivalently,

$$F_{\alpha\alpha} \equiv \Delta_{\alpha\alpha} - 1 + \phi(N_\alpha + 1) - \phi(N_\alpha) \equiv 0.$$

If the conditions (see (11))

$$[N_\alpha, A_\alpha^\dagger] \equiv A_\alpha^\dagger, \quad [N_\alpha, A_\alpha] \equiv -A_\alpha$$

(17)
do hold (it means that for these relations a subsequent verification is needed), then

$$\phi(N_\alpha)A_\alpha^\dagger \equiv A_\alpha^\dagger\phi(N_\alpha + 1) \Rightarrow [\phi(N_\alpha), A_\alpha^\dagger] \equiv A_\alpha^\dagger(\phi(N_\alpha + 1) - \phi(N_\alpha)).$$

As a result, we come to

$$[F_{\alpha\alpha}, A_\alpha^\dagger] \equiv 2(\Phi_\alpha^\dagger\Phi_\alpha^\dagger - A_\alpha^\dagger) + A_\alpha^\dagger(\phi(N_\alpha + 2) - 2\phi(N_\alpha + 1) + \phi(N_\alpha)).$$

(18)

Requiring that this commutator vanishes on the vacuum and taking into account that

$$\phi(0) = 0, \quad \phi(1) = 1,$$

we obtain

$$\Phi_\alpha \Phi_\alpha^\dagger \Phi_\alpha = \left(1 - \frac{1}{2}\phi(2)\right)\Phi_\alpha = \frac{f}{2}\Phi_\alpha,$$

where the deformation parameter $f$ does appear:

$$\frac{f}{2} \equiv 1 - \frac{1}{2}\phi(2) = \text{Tr}(\Phi_\alpha^\dagger\Phi_\alpha^\dagger\Phi_\alpha^\dagger) \quad \text{for all } \alpha.$$

Finding admissible $\phi(n)$ explicitly. Equality (18) can be rewritten as

$$[F_{\alpha\alpha}, A_\alpha^\dagger] \equiv (2 - \phi(2))A_\alpha^\dagger + A_\alpha^\dagger(\phi(N_\alpha + 2) - 2\phi(N_\alpha + 1) + \phi(N_\alpha)).$$

By induction, the equality for the $n$th commutator can be proven:

$$\ldots [F_{\alpha\alpha}, A_\alpha^\dagger] \ldots A_\alpha^\dagger \equiv (A_\alpha^\dagger)^n \sum_{k=0}^{n+1} (-1)^n\phi(N_\alpha + k)$$

(here $C^n_k$ denotes binomial coefficients). The requirement that the $n$th commutator vanishes on the vacuum leads to the recurrence relation

$$\phi(n + 1) = \sum_{k=0}^{n} (-1)^n\phi(N_\alpha + k), \quad n \geq 2.$$ 

(19)

As can be seen, all the values $\phi(n)$ for $n \geq 3$ are determined unambiguously by the two values $\phi(1)$ and $\phi(2)$, which may in general depend on one or more deformation parameters. Taking into account the equality [31]

$$\sum_{k=0}^{n} (-1)^mC^n_m = \begin{cases} 0, & m < n, \\ n!, & m = n, \end{cases}$$

we find: the only independent solutions of (19) are $n$ and $n^2$, as well as their linear combination

$$\phi(n) = \left(1 + \frac{f}{2}\right)n - \frac{f}{2}n^2.$$ 

(20)

This structure function satisfies both the initial conditions and the recurrence relations in (19).

Remark 1. In view of the uniqueness of the solution with fixed initial conditions, formula (20) gives the general solution of (19).
Remark 2. If we take the Hamiltonian in the form $H = \frac{1}{2}(\phi(N) + \phi(N + 1))$, then using the obtained results it is not difficult to derive the three-term recurrence relations for both the DSF and energy eigenspectrum:

$$\phi(n + 1) = \frac{2(n + 1)}{n} \phi(n) - \frac{n + 1}{n - 1} \phi(n - 1),$$

$$E_{n+1} = \frac{4n^2 + 4n - 4}{2n^2 - 1} E_n - \frac{2n^2 + 4n + 1}{2n^2 - 1} E_{n-1}.$$

The latter equality has a typical form of the so-called quasi-Fibonacci [15] relation for the eigenenergies. Note that the general case of deformed oscillators with polynomial structure functions $\phi(N)$ (these are quasi-Fibonacci as well) was studied in [30].

2.2. Treatment of the quasiboson number operator

The quasiboson number operator $N_\alpha$ can be introduced in different ways. Its definition is dictated by the requirements $G_0 \equiv 0, G_1 \equiv 0$ (recall that $G_0$ and $G_1$ are defined just after (10)) and also by the self-consistency of the realization. A possible definition could be given by the relation $N_\alpha \equiv \phi^{-1}(A_\alpha^\dagger A_\alpha)$, or by $N_\alpha \equiv \phi^{-1}(A_\alpha^\dagger A_\alpha) - 1$. We will not choose some of the two forms of definition, but consider the general possibility:

$$N_\alpha \equiv \chi(A_\alpha^\dagger A_\alpha, \epsilon_\alpha), \quad \text{where} \quad \epsilon_\alpha \equiv 1 - \Delta_\alpha \equiv [A_\alpha, A_\alpha^\dagger].$$

As we have mentioned above, it remains to satisfy relations (17), which allow the function $\chi$ to be defined. Note that the second of them stems by conjugation from the first one,

$$[N_\alpha, A_\alpha^\dagger] \equiv A_\alpha^\dagger.$$

Since we assume the independence of different modes, see (11), we consider the case $\gamma_1 = \gamma_2 = \cdots = \alpha$ in definition (7).

It is useful to denote by $L_\alpha$ the operators

$$L_0 = N, \quad L_{n+1} = [L_n, A_\alpha^\dagger] = \left[ \ldots [N_\alpha, A_\alpha^\dagger] \ldots A_\alpha^\dagger \right], \quad n \geq 0. \quad (22)$$

Taking this into account, condition (21) can be written as

$$L_1|O\rangle = A_\alpha^\dagger|O\rangle, \quad L_n|O\rangle = 0, \quad n > 1. \quad (23)$$

Now consider three useful statements.

Proposition 1. The following relations are true:

$$[\Delta_{\alpha\alpha}, A_\alpha^\dagger] = f A_\alpha^\dagger, \quad [\Delta_{\alpha\alpha}, A_\alpha] = -f A_\alpha, \quad f = 2 \text{Tr}(\Phi_\alpha^\dagger \Phi_\alpha \Phi_\alpha^\dagger \Phi_\alpha).$$

$$[\epsilon_\alpha, A_\alpha^\dagger] = -f A_\alpha^\dagger, \quad [\epsilon_\alpha, A_\alpha] = 0, \quad \Delta_{\alpha\alpha} = \Delta_{\alpha\alpha}^\dagger.$$

This statement is proven straightforwardly.

Proposition 2. For each $n \geq 0$ we have the equalities:

$$[\Delta_{\alpha\alpha} A_\alpha^\dagger]^n, A_\alpha^\dagger] = A_\alpha^\dagger \left[ (A_\alpha^\dagger A_\alpha + \epsilon_\alpha)^n - (A_\alpha^\dagger A_\alpha)^n \right]. \quad (24)$$

$$[\epsilon_\alpha^n, A_\alpha^\dagger] = A_\alpha^\dagger \left[ (-f + \epsilon_\alpha)^n - \epsilon_\alpha^n \right]. \quad (25)$$

Using the propositions 1 and 2 and the exact commuting of $A_\alpha^\dagger A_\alpha$ and $\epsilon_\alpha$ we come to the following:
Proposition 3. For $N_a$ defined as $N_a = \chi (A_a^\dagger A_a, \varepsilon)$, and $n \geq 0$, there is the following equality for the $n$-fold commutator (22):

$$L_n = (A_a^\dagger)^n \chi (A_a^\dagger A_a + n\varepsilon_a - \sigma_n f, \varepsilon_a - n f) - \frac{n}{2} \sum_{k=0}^{n} C_k^n (A_a^\dagger)^{n-k} L_a.$$ 

The proofs of propositions 2 and 3 are given in appendices A and B. Then conditions (23) turn into equalities

$$\begin{align*}
\left\{ A_a^\dagger \chi (A_a^\dagger A_a + \varepsilon_a, \varepsilon_a - f) \right\} (O) &= A_a^\dagger (O), \\
\left\{ (A_a^\dagger)^n \chi (A_a^\dagger A_a + n\varepsilon_a - \sigma_n f, \varepsilon_a - n f) \right\} (O) &= C_n^n (A_a^\dagger)^{n-1} L_a (O) = n (A_a^\dagger)^n (O), \quad n > 1.
\end{align*}$$

To satisfy these, it is necessary that

$$\chi (n - \sigma_n f, 1 - nf) = n, \quad n \geq 1. \quad (26)$$

So, condition (26) guarantees the validity of commutation relations (17), and therefore the consistency of the whole representation of quasibosons by deformed bosons. As one can see, the both definitions $N_a \equiv \phi^{-1} (A_a^\dagger A_a)$ and $N_a \equiv \phi^{-1} (A_a^\dagger A_a) - 1$ satisfy (26). Also, there are other definitions like $N_a \equiv (1 - p)\phi^{-1} (A_a^\dagger A_a) + p (\phi^{-1} (A_a^\dagger A_a) - 1), 0 < p < 1$, which satisfy (26) and lead, as can be checked, to the self-consistent representation of quasibosons.

2.3. General solution for matrices $\Phi_a$

In this subsection, we describe how to find admissible $d_a \times d_b$ matrices $\Phi_a$. These should satisfy the system

$$\begin{align*}
\text{Tr} \left( \Phi_a \Phi_b^\dagger \right) &= \delta_{a\beta}, \\
\Phi_a \Phi_b = f_{\alpha} \Phi_a, \\
\Phi_b \Phi_a^\dagger + \Phi_a \Phi_b^\dagger &= 0. \quad (27)
\end{align*}$$

Consider first the case $f \neq 0$. If the matrix $\Phi_a$ is nondegenerate (that means $d_a = d_b$ and det $\Phi_a \neq 0$) for some $\alpha$, the second relation yields $\Phi_a \Phi_b = f_{\alpha} \Phi_a$. From the third relation at $\gamma = \alpha$, we obtain $\Phi_\beta = 0$, $\forall \beta \neq \alpha$. Then, it follows that only one value of $\alpha$ is possible for which det $\Phi_a \neq 0$. In that case, $\Phi_a$ is an arbitrary unitary matrix. All the rest $\Phi_b = 0$, $\beta \neq \alpha$. That gives the partial nondegenerate solution of the system. Note that other solutions will be degenerate for all $\alpha$.

Let us go over to the analysis of degenerate solutions. At $\gamma = \alpha$ the last equation in (27) reduces to $\Phi_\beta \Phi_\alpha^\dagger + \Phi_\alpha \Phi_\beta^\dagger = 0$; multiplying it by $\Phi_\beta^\dagger$ and utilizing the second relation (note that $f$ is real) we infer

$$K \Phi_\beta \Phi_\alpha^\dagger \equiv \left( \Phi_\beta \Phi_\alpha^\dagger + \frac{f}{2} \right) \Phi_\beta \Phi_\alpha^\dagger = 0, \quad K \equiv \Phi_\alpha \Phi_\alpha^\dagger + \frac{f}{2}. \quad (28)$$

From the second relation of the system (27) we also obtain

$$\forall x \in \text{Im} \Phi_a : \quad \Phi_a \Phi_a^\dagger x = \frac{f}{2} x \quad \Rightarrow \quad \dim \text{Im} \Phi_a \Phi_a^\dagger \geq \dim \text{Im} \Phi_a.$$
Taking into account the latter and the fact that \( \text{Im} \Phi_a \Phi_a^\dagger \subseteq \text{Im} \Phi_a \), we find
\[
\text{Im} \Phi_a \Phi_a^\dagger = \text{Im} \Phi_a. \tag{29}
\]
Applying the Fredholm theorem first to \( \Phi_a \) and then to \( \Phi_a \Phi_a^\dagger \) and using (48), we arrive at the decompositions
\[
\forall \alpha : \quad C^{a_d} = \text{Im} \Phi_a \oplus \text{Ker} \Phi_a^\dagger = \text{Im} \Phi_a \Phi_a^\dagger \oplus \text{Ker} \Phi_a \Phi_a^\dagger.
\]
In each of subspaces \( \text{Im} \Phi_a \) and \( \text{Ker} \Phi_a \Phi_a^\dagger \), which are eigenspaces for \( K \), the operator \( K \) is nondegenerate:
\[
\forall x \in \text{Im} \Phi_a : \quad Kx = fx, \quad \text{and} \quad \forall y \in \text{Ker} \Phi_a \Phi_a^\dagger : \quad Ky = f^\dagger y.
\]
Consequently, the operator \( K \) is nondegenerate on the whole \( C^{a_d} \). Using (28), we find
\[
\forall \alpha \neq \beta : \quad \Phi_\beta \Phi_\alpha^\dagger = 0 \quad \text{or} \quad \Phi_\alpha \Phi_\beta^\dagger = 0.
\]
As a result, we arrive at the system which is equivalent to the initial one (27) and to the respective (for each of the equations) implications (\( \alpha \neq \beta \):
\[
\begin{align*}
\text{Tr}(\Phi_\alpha \Phi_\alpha^\dagger) &= 1, & \Rightarrow \text{dim} \text{Im} \Phi_\alpha \Phi_\alpha^\dagger &= \text{rank} \Phi_\alpha = 2/f \equiv m, \quad \text{for all} \ \alpha, \\
\Phi_\alpha \Phi_\alpha^\dagger \cdot \Phi_\alpha &= (f/2) \cdot \Phi_\alpha, & \Rightarrow \text{Im} \Phi_\alpha &= \text{eigensubspace of} \ \Phi_\alpha \Phi_\alpha^\dagger, \\
\Phi_\alpha \Phi_\beta^\dagger \cdot \Phi_\beta &= 0, & \Rightarrow \forall \beta \neq \alpha \ \text{Im} \Phi_\beta \subset \text{Ker} \Phi_\alpha \Phi_\alpha^\dagger &= \text{Ker} \Phi_\alpha^\dagger, \\
\Phi_\alpha \Phi_\beta^\dagger &= 0. & \Rightarrow \text{Im} \Phi_\beta^\dagger \subset \text{Ker} \Phi_\alpha.
\end{align*}
\]
So, the deformation parameter \( f \) has a discrete range of values determined by \( m \):
\[
f = \frac{2}{m}.
\]

The set of the solutions depends on the relation between \( \sum_a m \) and \( \min(d_a, d_b) \). If \( \sum_a m > \min(d_a, d_b) \), the set of solutions is empty. If \( \sum_a m \leq \min(d_a, d_b) \), then, according to the relations
\[
C^{a_d} = \text{Im} \Phi_a \oplus \text{Ker} \Phi_a^\dagger, \quad \text{Im} \Phi_\beta \subset \text{Ker} \Phi_\alpha^\dagger, \quad \forall \beta \neq \alpha,
\]
the space \( C^{a_d} \) (\( C^{\alpha_d} \)) decomposes into the direct sum of linearly independent subspaces:
\[
C^{\alpha_d} = \left( \bigoplus_a \text{Im} \Phi_a \right) \oplus R, \quad \text{dim} R = n - \sum_a m, \quad \Phi_a R = 0;
\]
\[
C^{\beta_d} = \left( \bigoplus_a \text{Im} \Phi_a^\dagger \right) \oplus \tilde{R}, \quad \text{dim} \tilde{R} = n - \sum_a m, \quad \Phi_a \tilde{R} = 0.
\]
Let \( \{e_1, \ldots, e_m\} \) be the orthonormal basis in the space \( \text{Im} \Phi_a \) and \( U_1(d_a) \) be the corresponding transition matrix to these bases from the initial one of \( C^{\alpha_d} \). Likewise, let \( \{f_1, \ldots, f_m\} \) be the orthonormal basis in the space \( \text{Im} \Phi_\beta^\dagger \) and \( U_2(d_b) \) the corresponding transition matrix from the initial basis in \( C^{\beta_d} \). In the new bases, the transition matrix \( \Phi_a \) is block diagonal:
\[
U_1^\dagger(d_a) \Phi_a U_2(d_b) = \begin{pmatrix}
0 & 0 & 0 \\
0 & \Phi_a & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
The \( m \times m \) matrix \( \Phi_a \) satisfies the equation \( \Phi_a \Phi_a^\dagger = \frac{f}{2} 1_m \). Its general solution can be given through the unitary matrix: \( \Phi_a = \sqrt{f/2} U_\alpha(m) \). Thus, the general solution of the initial system (27) is given in the form
\[
\phi_a = U_1(d_a) \text{diag} \left\{ 0, \sqrt{f/2} U_\alpha(m), 0 \right\} U_2^\dagger(d_b). \tag{30}
\]
In this formula, for every matrix $\Phi_\alpha$, the block $\sqrt{\frac{2}{c}}U_\alpha(m)$ is at its $\alpha$th place, and does not intersect with the corresponding block of any other matrix $\Phi_\beta$ with $\beta \neq \alpha$. To conclude: we have got all possible quasibosonic composite operators, expressed by (2) and (30), which can be realized by the algebra of deformed oscillators.

The case $f = 0$ in (27). It can be shown that $\Phi_\alpha$ should be zero for such $f$. This is followed by applying the singular value decomposition formula for each of the matrices in the equation $\Phi_\alpha\Phi_\alpha^*\Phi_\alpha = 0$. The fact that $\Phi_\alpha = 0$ means, see (2) and the normalization just after it, that the pure boson being a special $f = 0$ case of the deformed boson with the DSF (20) is unsuitable for the realization of the two-fermion composite quasiboson.

3. Quasibosons with $q$-deformed constituent fermions

Now let us go over to the $q$-generalization of the model considered above. Namely, we adopt nontrivial $q$-deformation for the constituents, the other assumptions being left as above. So, we start from the set of $q$-fermions, see [32], independent in fermionic sense:

\begin{align}
a_{\mu}a^\dagger_{\mu'} + q^{\mu\nu}a^\dagger_{\mu'}a_\mu &= \delta_{\mu\mu'}, \\
b_\nu b^\dagger_\nu + q^{\nu\mu}b^\dagger_\nu b_\mu &= \delta_{\nu\nu'}, \\
a_\mu a_{\nu} + a_\mu a_{\nu} &= 0, & \mu \neq \nu', \\
b_\nu b_{\nu'} + b_\nu b_{\nu'} &= 0, & \nu \neq \nu'.
\end{align}

(31)

The commutation relations (31) within one mode, i.e. for $\mu = \mu'$ and $\nu = \nu'$ completely determine the set of admissible values of the parameter $q$ and the (absence or presence, and the order of) nilpotency of the operators $a^\dagger_{\mu}$ and $b^\dagger_{\nu}$ depending on $q$. More precisely this is reflected in the following statement.

Lemma 1. For the positivity of the norm of $q$-fermion states it is necessary to put $q \in \mathbb{R}$ and $q \leq 1$. If $q = 1$ then $a^\dagger_{\mu}$ and $b^\dagger_{\nu}$ are nilpotent of second order; otherwise, if $q < 1$, the operators $a^\dagger_{\mu}$ and $b^\dagger_{\nu}$ are not nilpotent of any order:

\begin{align}
q = 1 & \implies (a^\dagger_{\mu})^2 = 0, \quad (b^\dagger_{\nu})^2 = 0; \\
q < 1 & \implies (a^\dagger_{\mu})^k \neq 0, \quad (b^\dagger_{\nu})^k \neq 0, \quad k \geq 2.
\end{align}

(33)

Proof. The lemma follows from the expression for the norm of the vector $x = (a^\dagger_{\mu})^n|0\rangle$:

\begin{align}
||x||^2 &= (0|a^\dagger_{\mu}(a^\dagger_{\mu})^n|0) = (0|a^\dagger_{\mu}^{-1}(n^\mu_{\nu} + 1)\mathbb{1}^{-q}(a^\dagger_{\mu})^{k-1}0) = (0|a^\dagger_{\mu}^{-1}(a^\dagger_{\mu})^{k-1}[n^\mu_{\nu} + k]^{-q}0) \\
&= [k]^{-q}(0|a^\dagger_{\mu}^{-1}[a^\dagger_{\mu}]^{k-1}|0) = \cdots = [k]^{-q}[k-1]^{-q} \cdots [1]^{-q},
\end{align}

where the notation $[n]^{-q} \equiv ((-q)^n - 1)/((-q) - 1)$ is nothing but the DSF for the $q$-fermions; $n^\mu_{\nu}$ is the number operator for $q$-fermions of $a$ type. The same considerations apply to the operators $b^\dagger_{\nu}$. This ends the proof. \qed

The $q = 1$ case (i.e. usual fermions with well-known nilpotency of their creation/annihilation operators) was completely analyzed in the preceding section (and also in [28]). Here we restrict ourselves to the case of $q < 1$. Hence (33) holds for any $k$.

The composite quasibosons’ creation and annihilation operators are defined as

\begin{align}
A^\dagger_a &= \sum_{\mu\nu} \Phi^\mu_{\alpha\nu}a^\dagger_{\mu}b^\dagger_{\nu}, \\
A_a &= \sum_{\mu\nu} \Phi^\nu_{\alpha\mu}b_{\nu}a_{\mu},
\end{align}

that is, like in (2). The requirements of the self-consistency of the realization (by deformed bosons) remain intact, see (10) and (11):

\begin{align}
A^\dagger_aA_a &\equiv \phi(N_a), \\
A_aA^\dagger_a &\equiv \phi(N_a + 1),
\end{align}

(34)
\[ [A^\mu_A, A^\nu_B] \cong 0 \Leftrightarrow [A_\alpha, A_\beta] \cong 0, \quad [A_\alpha, A^\mu_\beta] \cong 0, \quad \alpha \neq \beta. \] (35)

\[ [N_\alpha, A^\mu_\beta] \cong A^\mu_\beta, \quad [N_\alpha, A_\beta] \cong -A_\beta. \] (36)

In this case, the requirement of independence \([A^\mu_A, A^\nu_B] \cong 0\), as one can easily check, leads to the following condition on matrices \(\Phi_\alpha\):

\[ \Phi^{\mu \nu}_\alpha \Phi^{\mu \nu}_\beta = \Phi^{\mu \nu}_\alpha \Phi^{\mu \nu}_\beta, \quad \Phi^{\mu \nu}_\alpha \Phi^{\mu \nu}_\beta = \Phi^{\mu \nu}_\alpha \Phi^{\mu \nu}_\beta. \] (37)

The second relation in (34) implies that there should be

\[ A_\alpha (A^\mu_\beta)^n |O\rangle = \Phi (N_\alpha + 1) (A^\mu_\beta)^{n-1} |O\rangle, \quad n = 1, 2, 3, \ldots. \] (38)

Using (36) we obtain

\[ \Phi (N_\alpha + 1) (A^\mu_\beta)^{n-1} |O\rangle = (A^\mu_\beta)^{n-1} \Phi (N_\alpha + n) |O\rangle. \]

As a result, we arrive at

\[ A_\alpha (A^\mu_\beta)^n |O\rangle = \Phi (n) (A^\mu_\beta)^{n-1} |O\rangle, \quad n = 1, 2, 3, \ldots. \] (39)

It can be checked by induction that

\[ A_\alpha (A^\mu_\beta)^n = (-1)^{\sum_1^\beta} \Phi^{\mu \nu}_\alpha \prod_{j=1}^n \Phi^{\mu \nu}_\beta. \]

Then, using equation (39) we arrive at

\[ \Phi (n) \prod_{j=1}^{n-1} \Phi^{\mu \nu}_\alpha \prod_{j=1}^n \Phi^{\mu \nu}_\beta = (-1)^{\sum_1^\beta} \Phi^{\mu \nu}_\alpha \prod_{j=1}^n \Phi^{\mu \nu}_\beta \]

\[ \cdot \left[ \sum_{k=1}^n (-1)^{k-1} \delta_{\mu \nu}, q_1 \sum_{i=1}^{k+1} a^\mu_i + (-1)^k \delta_{\mu \nu}, q_1 \sum_{i=1}^{n-1} a^\mu_i \right] \prod_{r=k}^n \]

\[ \cdot \left[ \sum_{k=1}^n (-1)^{k-1} \delta_{\nu \nu}, q_1 \sum_{i=1}^{k+1} b^\nu_i + (-1)^k \delta_{\nu \nu}, q_1 \sum_{i=1}^{n-1} b^\nu_i \right] \prod_{r=k}^n (O). \] (40)

Note that if (40) holds on the vacuum, the following equality holds on any state:

\[ (-1)^{\sum_1^\beta} \Phi^{\mu \nu}_\alpha \prod_{j=1}^n \Phi^{\mu \nu}_\beta \cdot \left[ \sum_{k=1}^n (-1)^{k-1} \delta_{\mu \nu}, q_1 \sum_{i=1}^{k+1} a^\mu_i \right] \prod_{r=k}^n \]

\[ \cdot \left[ \sum_{k=1}^n (-1)^{k-1} \delta_{\nu \nu}, q_1 \sum_{i=1}^{k+1} b^\nu_i \right] = \Phi (n) \prod_{j=1}^{n-1} \Phi^{\mu \nu}_\alpha a^\mu_i b^\nu_i. \] (41)
As a recursive step, let us consider the following relation valid for \( n + 1 \):

\[
A_\alpha(A^\dagger_n)_{n+1} = (-1)^{n+1}\sum_{l=1}^{n} \prod_{r=p}^{l} \Phi_{\alpha_{l}}^{(\nu_{r+1})} \cdot \left[ \sum_{i=1}^{n+1} (-1)^{i-1} \delta_{\mu_{i}} q_{\nu_{l+1}} \prod_{r=p}^{l} a_{\mu_{i}} \right] \cdot \left[ \sum_{k=1}^{n+1} \prod_{r=p}^{l} b_{\nu_{k}} \right] \cdot (\sum_{i=1}^{n+1} (-1)^{i-1} \delta_{\nu_{i}} q_{\mu_{l+1}} \prod_{r=p}^{l} \alpha_{i}^{\dagger}) + (-1)^{n} \sum_{l=1}^{n} \prod_{r=p}^{l} \Phi_{\alpha_{l}}^{(\nu_{r+1})} \cdot \left[ \sum_{i=1}^{n+1} (-1)^{i-1} \delta_{\mu_{i}} q_{\nu_{l+1}} \prod_{r=p}^{l} a_{\mu_{i}} \right] \cdot \left[ \sum_{k=1}^{n+1} \prod_{r=p}^{l} b_{\nu_{k}} \right]
\]

\[\equiv \phi(n) \prod_{l=1}^{n} \Phi_{\alpha_{l}}^{(\mu_{l+1})} a_{\mu_{l+1}}^{\dagger} b_{\nu_{l+1}}^{\dagger} + (-1)^{n} \sum_{l=1}^{n} \prod_{r=p}^{l} \Phi_{\alpha_{l}}^{(\nu_{r+1})} \cdot \left[ \sum_{i=1}^{n+1} (-1)^{i-1} \delta_{\mu_{i}} q_{\nu_{l+1}} \prod_{r=p}^{l} a_{\mu_{i}} \right] \cdot \left[ \sum_{k=1}^{n+1} \prod_{r=p}^{l} b_{\nu_{k}} \right] \]

where at the last stage we have used (41). Substituting the last expression for \( A_\alpha(A^\dagger_n)_{n+1} \) into (39) rewritten for \( n \to n + 1 \), we deduce the following relation that involves the linear combination:

\[
\sum_{\mu_{1}, \ldots, \mu_{n+1}} B_{\mu_{1}, \ldots, \mu_{n+1}, \nu_{1}, \ldots, \nu_{n+1}} (\Phi_{\alpha_{1}}^{(\nu_{1})}, \ldots, \Phi_{\alpha_{n+1}}^{(\nu_{n+1})}) \cdot c_{\mu_{1}, \ldots, \mu_{n+1}, \nu_{1}, \ldots, \nu_{n+1}} = 0,
\]

where the coefficients are
and regardless of any permutations within each set. So let us extract in (42) the terms with

\[ B_{\mu_1...\mu_n,v_1...v_n} = \sum_{q=1}^{n-1} \sum_{\delta_{\mu_1\mu_2}} \sum_{\delta_{v_1v_2}} (-1)^{\sum_{i=1}^{n} b_{\mu_i} + b_{v_i}} \prod_{i=1}^{n} \Phi_{\mu_1...\mu_n,v_1...v_n}^{\mu_1\mu_2} \Phi_{\alpha}^{\mu_1\mu_2} \Phi_{\alpha}^{v_1,v_2} \prod_{i=1}^{n} \Phi_{\mu_1...\mu_n,v_1...v_n}^{\mu_1\mu_2} \prod_{i=1}^{n} \Phi_{\alpha}^{\mu_1\mu_2} \prod_{i=1}^{n} \Phi_{\alpha}^{v_1,v_2} \]

and the basis elements are

\[ e_{\mu_1...\mu_n,v_1...v_n} = a^{\mu_1}_{\mu_1} b^{v_1}_{v_1} ... a^{\mu_n}_{\mu_n} b^{v_n}_{v_n} |O\rangle. \]

These basis elements are independent for differing sets of indices \( \mu_1, \ldots, \mu_n \) and \( v_1, \ldots, v_n \) regardless of any permutations within each set. So let us extract in (42) the terms with

\[ B_{\mu_1...\mu_n,v_1...v_n}(\Phi_{\alpha}, q) = 0, \]

that can be rewritten in the following form:

\[ \sum_{i=1}^{n} (-1)^{\mu_1\mu_2-1} + 2(\delta_{\mu_1\mu_2} + \delta_{v_1v_2})(q^n - q^{i-1} - 2) + (\delta_{\mu_1\mu_2} + \delta_{v_1v_2})(q^n - q^{i-1} + 1) + \delta_{\mu_1\mu_2} \delta_{v_1v_2}(q^n - q^{i-1})^2 \]

Performing the summation over \( i, \mu, v \) on the left-hand side, we find

\[ ((-1)^n - 1)(\Phi_{\alpha} \Phi_{\alpha}^\dagger \Phi_{\alpha}^\dagger)(\Phi_{\alpha}^\dagger)^{\mu_1\mu_2}(\Phi_{\alpha}^\dagger)^{v_1v_2}(\Phi_{\alpha}^\dagger)^{n-1} + \frac{1}{2} (-q)^n + \frac{q - 1}{q + 1} q^n - \frac{q}{q + 1} (1 - 1)^n \]

\[ \times \left[ (\Phi_{\alpha} \Phi_{\alpha}^\dagger \Phi_{\alpha}^\dagger)^{\mu_1\mu_2} + (\Phi_{\alpha} \Phi_{\alpha}^\dagger \Phi_{\alpha}^\dagger)^{v_1v_2} \right]^n\]

\[ + \frac{q - 1}{q + 1} (q^n - 1)^2 |\Phi_{\alpha}^\dagger|^2 (\Phi_{\alpha}^\dagger)^{\mu_1\mu_2}(\Phi_{\alpha}^\dagger)^{v_1v_2}(\Phi_{\alpha}^\dagger)^{n-1} = [\phi(n + 1) - \phi(n)](\Phi_{\alpha}^\dagger)^{\mu_1\mu_2} \]

For all the indices \((\mu_1, v_1)\) for which \(\Phi_{\alpha}^{\mu_1\mu_2} \neq 0\), the last equation can be divided by \((\Phi_{\alpha}^{\mu_1\mu_2})^n\).

Summing (43) over \( n \) from \( n = 1 \) to \( n = s \) and then replacing in the resulting equality \( s \to n \to 1 \), we obtain

\[ \left( \frac{1}{2} - \frac{1}{2} (-1)^n \right) \left( \frac{\Phi_{\alpha} \Phi_{\alpha}^\dagger \Phi_{\alpha}^\dagger}{\Phi_{\alpha}^\dagger} \right)^{\mu_1\mu_2} + \left[ \frac{n}{2} - \frac{1}{2} (-1)^n \right]^2 |\Phi_{\alpha}^\dagger|^2 \]

\[ + \frac{1 - (-1)^n}{2} \left[ [n]_{-q} - \frac{1}{2} (-1)^n \right]^2 (\Phi_{\alpha}^\dagger)^{\mu_1\mu_2}(\Phi_{\alpha}^\dagger)^{v_1v_2}(\Phi_{\alpha}^\dagger)^{n-1} = \phi(n) - n, n \geq 2. \]

Note that the functions \((\frac{1}{2} - \frac{1}{2} (-1)^n)\), \([n]_{-q} - \frac{1}{2} (-1)^n \) and \(\frac{1}{2} (-1)^n([n]_{-q} - 1)\) as functions of \( n \) are independent for the admissible values of \( q \). Hence \((\Phi_{\alpha} \Phi_{\alpha}^\dagger \Phi_{\alpha}^\dagger)^{\mu_1\mu_2}/(\Phi_{\alpha}^\dagger)^{\mu_1\mu_2}\) and \((\Phi_{\alpha} \Phi_{\alpha}^\dagger)^{v_1v_2}/\Phi_{\alpha}^\dagger(\Phi_{\alpha}^\dagger)^{v_1v_2}\) do not depend on \((\mu_1, v_1)\) if \(\Phi_{\alpha}^{\mu_1\mu_2} \neq 0\):

\[ (\Phi_{\alpha} \Phi_{\alpha}^\dagger \Phi_{\alpha}^\dagger)^{\mu_1\mu_2}/(\Phi_{\alpha}^\dagger)^{\mu_1\mu_2} = p_1, \]

\[ |\Phi_{\alpha}^\dagger|^2 = p_2, \]

\[ (\Phi_{\alpha} \Phi_{\alpha}^\dagger + (\Phi_{\alpha} \Phi_{\alpha}^\dagger)^{\mu_1\mu_2} = p_3, \]

where \( p_1, p_2 \) and \( p_3 \) are some numerical parameters. Thus, we obtain

\[ \phi(n) = n - \left( n - \frac{1}{2} (-1)^n \right) p_1 + \left( [n]_{-q} - \frac{1}{2} (-1)^n \right)^2 p_2 + \frac{1}{2} (-1)^n ([n]_{-q} - 1)p_3. \]

(44)
Let us now consider the terms in equation (42) with \( n \) equated indices \( \mu_1 = \cdots = \mu_n \), and with \( n - 1 \) equated indices in the set \( (v_1, \ldots, v_n) \), the remaining one being different. Denote the \( n - 1 \) equal indices by \( v_1 \), and the differing one (suppose it occupies the \( k \)th position) by \( v_2 \).

Due to the linear independence of the mentioned terms from the others we obtain the equation

\[
\sum_{k=1}^{n} B^{\mu_1 \cdots (\mu_k v_1 \cdots v_k) \cdots v_2} e_{\mu_1 \cdots \mu_k v_1 \cdots v_k \cdots v_2} |v_1 \cdots v_2| = 0, \quad \text{i.e.} \quad \sum_{k=1}^{n} (-1)^k B^{\mu_1 \cdots (\mu_k v_1 \cdots v_k) \cdots v_2} |v_1 \cdots v_2| = 0.
\]

Introducing auxiliary notations

\[
\begin{aligned}
X &= \Phi_{\alpha}^{(v_1 \cdots v_2)} \Phi_{\alpha}^{(v_2)}, \\
Y &= \Phi_{\alpha}^{(v_1 \cdots v_2)} (\Phi_{\alpha} \Phi_{\alpha})^{(v_1 \cdots v_2)}, \\
Z &= (\Phi_{\alpha}^{(v_1 \cdots v_2)})^2 (\Phi_{\alpha}^{(v_2)}),
\end{aligned}
\]

after performing all the summations in (45) we obtain

\[
\begin{aligned}
[X p_2] &(-1)^n q^{2n} + [(-q^n + 2q^2 - 3q + 4)p_2 X] q^{2n} \\
+ &[(q^2 - 2 - q)p_2 + 2p_3]X + (-2 - q)Y - nq^n \\
+ &[((-3q^3 + 17q^2 + q^4 - 26 - 5q)p_2 + (-4q + 2q^2 + 2)p_3)X \\
+ &((6 + 5q - q^3 - 2q^2)Y + (4q - 10q^2 + 6)Z)q^n \\
+ &[((-3q^3 + q^2 - 2)p_2 + (2 - 2q)p_3)X \\
+ &((q^2 + 3q - 2)Y + (2 - 2q)Z)q^n \\
+ &[(q^2 + 3 - 4q)p_2 + (-4q^2 + 2q + 2)p_3)X \\
+ &((4q^2 - q - 5)Y + (3q^2 + 1)Z)(-1)^n \\
+ &[(2p_1 + (-3q^3 + 2p_2 - 2p_3)X + Y + (3q - 3)Z)n \\
+ &[(8 - 8q^2)p_1 + (23 - 3q - 19q^2 + 7q^3)p_2 + (8q - 4q^3 + 2q^2 - 6)p_3)X \\
+ &((4q^3 - 5 - 12q + 5q^2)Y + (-3q^3 - 11 + 3q + 11q^2)Z) = 0.
\end{aligned}
\]

Extracting the coefficients of this system at the linearly independent functions \((-1)^n q^{2n}, q^{2n}, nq^n, q^n, (-q)^n, (-1)^n, n, 1\) (considered as the elements of the vector space of functions of \( n \)), we arrive at the following system:

\[
\begin{aligned}
X p_2 &= 0, \\
[ - q^3 + 2q^2 - 3q + 4]p_2 X &= 0, \\
[(q^2 - 2 - q)p_2 + 2p_3]X + [-2 - q]Y &= 0, \\
[(-3q^3 + 17q^2 + q^4 - 26 - 5q)p_2 + (-4q + 2q^2 + 2)p_3]X + [6 + 5q - q^3 - 2q^2]Y \\
+ &[4q - 10q^2 + 6]Z = 0, \\
[(-q^3 + q^2 - 2)p_2 + (2 - 2q)p_3]X + [q^2 + 3q - 2]Y + [-2 - 2q]Z = 0, \\
[(q^2 + 3 - 4q)p_2 + (-4q^2 + 2q + 2)p_3]X + [4q^2 - q - 5]Y + [3q^2 + 1]Z = 0, \\
[2p_1 + (-3q^3 + 5)p_2 - 2p_3]X + Y + [3q - 3]Z = 0, \\
[(8 - 8q^2)p_1 + (23 - 3q - 19q^2 + 7q^3)p_2 + (8q - 4q^3 + 2q^2 - 6)p_3]X \\
+ [4q^3 - 5 - 12q + 5q^2]Y + [-3q^3 - 11 + 3q + 11q^2]Z = 0.
\end{aligned}
\]

The solution of this system is \((q \neq 1)\):

\[
\begin{aligned}
X &= \Phi_{\alpha}^{(v_1 \cdots v_2)} \Phi_{\alpha}^{(v_2)} = 0, \\
Y &= \Phi_{\alpha}^{(v_1 \cdots v_2)} (\Phi_{\alpha} \Phi_{\alpha})^{(v_1 \cdots v_2)} = 0, \\
Z &= (\Phi_{\alpha}^{(v_1 \cdots v_2)})^2 (\Phi_{\alpha}^{(v_2)})^{v_2} = 0.
\end{aligned}
\]
This set of conditions is equivalent to
\[ \Phi_{\mu_1}^{\nu_1} \Phi_{\mu_2}^{\nu_2} = 0, \]  
which means that the matrix \( \Phi_{\mu} \) cannot contain two nonzero elements in any one row.

In a similar way, we can derive the condition
\[ \Phi_{\alpha}^{\mu_1} \Phi_{\alpha}^{\nu_2} = 0, \]  
implying that the matrix \( \Phi_{\alpha} \) cannot contain two nonzero elements in any one column.

Next, the same analysis as in the previous two paragraphs is performed for those terms in (42) for which: in the set \((\mu_1, \ldots, \mu_n)\) there is only one index (denoted by \(\mu_2\)) different from the other \((n - 1)\) equal ones (all denoted by \(\mu_1\)), and likewise for \(\nu\)-indices—in the set \((\nu_1, \ldots, \nu_n)\) there is only one index (denoted by \(\nu_2\)) different from the other, equal ones (all denoted by \(\nu_1\)). As a result, we derive
\[ \Phi_{\mu}^{\nu_1} \Phi_{\alpha}^{\nu_2} = 0. \]  
That is, the matrix \( \Phi_{\mu} \) cannot have two nonzero elements in differing rows and columns.

And using the previous conditions (46) and (47) we obtain that the matrix \( \Phi_{\alpha} \) cannot contain two nonzero elements. As a consequence, we obtain the following values for the parameters \(p_1, p_2\) and \(p_3\):
\[ p_1 = p_2 = 1, \quad p_3 = 2. \]

Then, the following expression for the DSF results from (44):
\[ \phi(n) = ([n]_{-q})^2. \]  
(49)

The mode independence conditions contained in (37) and equalities (46), (47) and (48) enable us to determine the solution for \( \Phi_{\alpha} \): the only nonzero elements in matrices \( \Phi_{\mu} \) and \( \Phi_{\nu} \) are situated at the intersection of different rows and different columns:
\[ \Phi_{\mu}^{\alpha} = \Phi_{\mu}^{\alpha} \delta_{\mu,\mu_0(\alpha)} \delta_{\nu,\nu_0(\alpha)}, \quad |\Phi_{\mu}^{\alpha} \delta_{\nu,\nu_0(\alpha)}| = 1. \]  
(50)

For the illustrative purpose, a more detailed treatment of two particular examples including also the omitted steps of the derivation above is provided in appendix C. The first example concerns the case with only one possible value of \(\mu, \nu = 1\) for the constituent \(q\)-fermion modes. The second example concerns the case of two-mode constituents, i.e. of two possible values of \(\mu, \nu = 1, 2\).

It remains to satisfy the commutation relations (36) by means of the correct definition of \(N_\alpha\). Let \(N_\alpha\) be defined as \(N_\alpha \equiv \chi (A_{\alpha}^\dagger A_\alpha A_\alpha^\dagger)\), and the matrices \(\Phi_\alpha\) are those already found in (50). Taking into account the latter we have
\[ A_\alpha A_\alpha^\dagger \cdot (A_\alpha^\dagger)^n |O\rangle = [n + 1]_{-q} (A_\alpha^\dagger)^n |O\rangle, \quad A_\alpha^\dagger A_\alpha \cdot (A_\alpha)^n |O\rangle = [n]_{-q} (A_\alpha)^n |O\rangle. \]

Then (36) is equivalent to
\[ \chi (A_{\alpha}^\dagger A_\alpha, A_\alpha A_\alpha^\dagger) (A_{\alpha}^\dagger)^n |O\rangle - A_{\alpha}^\dagger \chi (A_{\alpha} A_\alpha, A_\alpha A_\alpha^\dagger) (A_{\alpha}^\dagger)^n |O\rangle = A_{\alpha}^\dagger (A_{\alpha}^\dagger)^n |O\rangle \]
\[ \Leftrightarrow \chi (A_{\alpha}^\dagger A_\alpha, [n + 2]_{-q}) (A_{\alpha}^\dagger)^{n+1} |O\rangle - A_{\alpha}^\dagger \chi (A_\alpha A_\alpha^\dagger, [n + 1]_{-q}) (A_{\alpha}^\dagger)^n |O\rangle = (A_{\alpha}^\dagger)^{n+1} |O\rangle \]
\[ \Leftrightarrow \chi ([n + 1]_{-q}, [n + 2]_{-q}) (A_{\alpha}^\dagger)^{n+1} |O\rangle - \chi ([n]_{-q}, [n + 1]_{-q}) (A_{\alpha}^\dagger)^n |O\rangle = (A_{\alpha}^\dagger)^{n+1} |O\rangle \]
\[ \Leftrightarrow \chi ([n + 1]_{-q}, [n + 2]_{-q}) - \chi ([n]_{-q}, [n + 1]_{-q}) = 1, \quad n \geq 0. \]

Thus, the condition \( \chi ([n]_{-q}, [n + 1]_{-q})^{n+1} \equiv \chi ([n + 1]_{-q}, [n + 2]_{-q}) - \chi ([n]_{-q}, [n + 1]_{-q}) = 1, \quad n = 0, 1, \ldots \) is necessary and sufficient for (36) to hold.
Remark 3. Expression (49) for the structure function is valid only when \(q \neq 1\), i.e. when \(a_{1}^\dagger\) and \(a_{1}^\dagger\) are not nilpotent of any order. If \(q = 1\), it is the DSF (20) which provides the realization. Thus, the unifying formula for the DSF (of those deformed oscillators that give realization) for quasibosons composed of two \(q\)-fermions can be written as

\[
\phi(n) = \begin{cases} 
(n-q)^2 & q < 1; \\
\left(1 + \frac{1}{n}\right)n - \frac{1}{2}n^2, & q = 1, \quad m \in \mathbb{N}.
\end{cases}
\]

The absence of a continuous limit from (49) to (20) when \(q \to 1\) or in other words the discontinuity of (51) at the \(q = 1\) point is explained as follows. If \(q \neq 1\), then there is an infinite number of basis elements \(\{(a_{1}^\dagger)^{k_1}\cdots(a_{d_{1}}^\dagger)^{k_{d_{1}}} \cdots (b_{1}^\dagger)^{k_{d_{1}}}\cdots(b_{d_{1}}^\dagger)^{k_{d_{1}}}|O\rangle | k_i, l_i \geq 0, \sum_{i=1}^{d_{1}}k_i = \sum_{j=1}^{d_{1}}l_j = n, n = 0, 1, 2, \ldots\}\) of the subspace of composite bosons’ states. The latter results in an infinite number of requirements (42) thus imposing a considerable restriction on \(\Phi_\alpha^\mu\). On the other hand, if \(q = 1\), then there is only a finite number, equal to \(\sum_{k=1}^{\min(d_{1},d_{2})}C_{d_{1}}^{k}C_{d_{2}}^{k} = C_{d_{1}+d_{2}}^{\min(d_{1},d_{2})}\) of the basis elements: \(|O\rangle, a_{1}^\dagger b_{1}^\dagger|O\rangle, \ldots, a_{1}^\dagger \ldots a_{d_{1}}^\dagger b_{d_{1}}^\dagger \ldots b_{d_{1}}^\dagger|O\rangle\), that leads to a finite number of requirements (42). Moreover, in this case only a few requirements among them are independent, see (27).

4. Conclusions and outlook

As shown in our preceding paper [28] and in section 2 above, the problem of realization of ‘fermion+fermion’ quasi-bosons by means of deformed oscillators has nontrivial solutions. In the case of pure fermions as constituents, the structure function \(\phi\) of the relevant deformation is found in the form (20) quadratic in the number operator \(N\), with a discrete valued deformation parameter \(f = 2/m\). This is the only DSF for which the realization (isomorphism) is possible. In addition, necessary and sufficient conditions on the matrices \(\Phi_{\alpha}\) in the construction (2) of quasibosons, for such representation to be self-consistent, are obtained and expression (30) gives their general solution.

In this paper, the novel generalization was carried out, as presented in section 3. This is the case of quasibosons made up of two constituents which are \(q\)-deformed fermions (31)–(32). For this generalization, again, we have derived the relations for the defining matrices \(\Phi_{\alpha}\) and solved them. Detailed analysis led us at \(q \neq 1\) to the resulting structure function (49) of the deformed oscillator which provides the exact realization of the quasibosons made up of two \(q\)-fermions. The principal distinction of the situation treated herein from the case considered in section 2 (following [28]) is such that, while the pure fermions are nilpotent, the \(q\)-deformed fermions for \(q \neq 1\) are not nilpotent of any order, see (33). Since the second-order nilpotency of usual fermions (as the no-deformation limit of \(q\)-fermions) abruptly appears at \(q = 1\) according to lemma 1, there is no direct transition from the DSF (49) to DSF (20), as a result of the continuous \(q \to 1\) limit. See also remark 3 including (51) on this issue.

The general strategy of the developed approach is to explore deformed bosons as tools for the realization of quasibosons, which should give considerable simplification (in the algebraic sense) in subsequent applications, achieved when the algebra representing the initial system of composite particles is treated as the algebra corresponding to some deformed oscillator. The obtained results and the developed approach have a potential application to: various problems in (sub)nuclear physics (with such composite particles as hadrons, nucleon complexes) like the study of pairing in nuclei [33]; bipartite entangled composites [34] in the quantum information theory (where the role of quasibosons can be played e.g. by biphotons [35]); Bose–Einstein condensation of composite bosons [24] and other thermodynamic questions including the equation of state for many composite bosons systems. Also, the developed formalism can
be applied to modeling physical particles or quasiparticles such as excitons, biphonons and cooperons in the corresponding directions of condensed matter physics. Concerning excitons, there already exists [36] the description of interacting excitons using infinite series in their creation operators. Besides, excitons were modeled [37, 38] by $q$-deformed version of bosons, however, without taking into account their compositeness.

As the next steps we intend to study more complicated situations, say the case of quasibosons composed of two (deformed) bosons, or from two generally deformed fermions. Also, in our nearest plan there is the analysis of composite (quasi)fermions. Yet another path of extension is to treat quasi-independent quasibosons, i.e. those with noncommuting different modes.

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Appendix A. Proof of proposition 2

As our treatment below concerns only one mode $\alpha$, we will omit the index $\alpha$. Let us first prove the equality (24). For $n = 0$ this is trivial. Put $n = 1$: \[
[A^\dagger A, A^\dagger] = A^\dagger [A, A^\dagger] = A^\dagger (1 - \Delta_{\alpha\alpha}) = A^\dagger = A^\dagger [(A^\dagger A + \varepsilon)^1 - (A^\dagger A)^1].
\]

Then we proceed by induction. Assuming that the equality holds for $n$, let us prove that it is valid for $n + 1$: \[
(A^\dagger A)^{n+1}, A^\dagger = [A^\dagger A (A^\dagger A)^n, A^\dagger] = [A^\dagger A, A^\dagger] (A^\dagger A)^n + A^\dagger A [(A^\dagger A)^n, A^\dagger] = A^\dagger \varepsilon (A^\dagger A)^n + A^\dagger AA^\dagger [(A^\dagger A + \varepsilon)^n - (A^\dagger A)^n] = A^\dagger \varepsilon (A^\dagger A)^n + A^\dagger (A^\dagger A + \varepsilon)^n - A^\dagger (A^\dagger A)^n - A^\dagger \varepsilon (A^\dagger A)^n = A^\dagger [(A^\dagger A + \varepsilon)^{n+1} - (A^\dagger A)^{n+1}].
\]

Consider the second equation. When $n = 0$ it is also trivial. For $n = 1$, we have \[
[\varepsilon, A^\dagger] = -fA^\dagger = A^\dagger [(-f + \varepsilon) - \varepsilon].
\]

The step of induction is \[
[\varepsilon^{n+1}, A^\dagger] = [\varepsilon^n, A^\dagger] = -fA^\dagger \varepsilon^n + \varepsilon A^\dagger [(-f + \varepsilon)^n - \varepsilon^n] = -fA^\dagger \varepsilon^n + (fA^\dagger + A^\dagger \varepsilon) [(-f + \varepsilon)^n - \varepsilon^n] = A^\dagger [(-f + \varepsilon)^{n+1} - \varepsilon^{n+1}].
\]

Thus, the proposition is proven.

Appendix B. Proof of proposition 3

When $n = 0$, the equality reduces to the definition of $N$. Let us prove it for $n = 1$. Present $\chi$ as the formal series: \[
\chi(x, y) = \sum_{n,m=1}^\infty b_{nm} x^m y^n, \quad [x, y] = 0.
\]
Then,

\[ L_1 = [\chi (A^\dagger, \epsilon), A^\dagger] = \sum_{n,m=1}^\infty b_{nm}[ (A^\dagger A)\chi, A^\dagger] + \sum_{n,m=1}^\infty b_{nm}(A^\dagger A)^\dagger [\epsilon, A^\dagger] \]

\[ = \sum_{n,m=1}^\infty b_{nm}[ (A^\dagger A + \epsilon)\chi - (A^\dagger A)^\dagger \chi, A^\dagger] + \sum_{n,m=1}^\infty b_{nm}(A^\dagger A)^\dagger [(-f + \epsilon)\chi, A^\dagger] \]

\[ = A^\dagger [\chi (A^\dagger A + \epsilon, \epsilon - f) - \chi (A^\dagger A, \epsilon)] = A^\dagger \chi (A^\dagger A + \epsilon, \epsilon - f) - A^\dagger N. \]

Next, proceed by induction. The induction step is

\[ L_{n+1} = [L_n, A^\dagger] = (A^\dagger)^n[\chi (A^\dagger A + n\epsilon - \sigma_n f, \epsilon - nf), A^\dagger] - \sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} [L_k, A^\dagger] \]

\[ = (A^\dagger)^n[\chi (A^\dagger A + n\epsilon - \sigma_n f, \epsilon - nf), A^\dagger] - \sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} L_{k+1}. \quad (B.1) \]

Let us transform the commutator in the last expression,

\[ [\chi (A^\dagger A + n\epsilon - \sigma_n f, \epsilon - nf), A^\dagger] = \sum_{n,m=1}^\infty b_{nm}[ (A^\dagger A + n\epsilon - \sigma_n f)^\dagger, A^\dagger] (\epsilon - nf)^m \]

\[ + \sum_{n,m=1}^\infty b_{nm}(A^\dagger A + n\epsilon - \sigma_n f)^\dagger [(\epsilon - nf)^m, A^\dagger] \]

\[ = A^\dagger [\chi (A^\dagger A + (n+1)\epsilon - \sigma_{n+1} f, \epsilon - (n+1) f) - A^\dagger \chi (A^\dagger A + n\epsilon - \sigma_n f, \epsilon - nf)], \]

where we have used that \( \sigma_{n+1} = \sigma_n + n \). Consequently,

\[ (A^\dagger)^n[\chi (A^\dagger A + n\epsilon - \sigma_n f, \epsilon - nf), A^\dagger] \]

\[ = (A^\dagger)^{n+1} \chi (A^\dagger A + (n+1)\epsilon - \sigma_{n+1} f, \epsilon - (n+1) f) \]

\[ - A^\dagger (A^\dagger)^n \chi (A^\dagger A + n\epsilon - \sigma_n f, \epsilon - nf). \quad (B.2) \]

Taking into account the induction assumption, the last term takes the form

\[ A^\dagger \cdot (A^\dagger)^n \chi (A^\dagger A + n\epsilon - \sigma_n f, \epsilon - nf) = A^\dagger L_n + A^\dagger \sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} L_k. \]

Substituting this expression into (B.2) and then the resulting expression into (B.1), we obtain

\[ L_{n+1} = (A^\dagger)^{n+1} \chi (A^\dagger A + (n+1)\epsilon - \sigma_{n+1} f, \epsilon - (n+1) f) \]

\[ - A^\dagger L_n - A^\dagger \sum_{k=0}^{n-1} C_n^k (A^\dagger)^{n-k} L_{k+1} \]

\[ = (A^\dagger)^{n+1} \chi (A^\dagger A + (n+1)\epsilon - \sigma_{n+1} f, \epsilon - (n+1) f) - \sum_{k=0}^{n} C_{n+1}^k (A^\dagger)^{n+1-k} L_k. \]

Thus, the proposition is proven.

Appendix C. Particular examples: \( \mu, \nu = 1 \) and \( \mu, \nu = 1, 2 \)

For both the examples we assume \( q \neq 1 \).
Example 1. Let us consider first the simplest case when only one mode is possible for a composite boson’s constituents: $\mu, v = 1$. The number of the modes $\alpha$ is not significant here as further treatment concerns only one fixed arbitrary mode $\alpha$. Then, the creation and annihilation operators $A^\alpha_u$ and $A_\alpha$ of the composite boson according to (2) reduce to (the fixed index $\alpha$ is omitted)

$$A^- = \Phi^{11^i}\bar{b}^j_1, \quad A = \Phi^{11^j}b_1.$$

We require the composite bosons to be algebraically represented by deformed bosons, i.e. that defining equality $A\bar{A}^\dagger = \phi (N + 1)$ for deformed bosons holds on any $n$-composite bosons state $(A^\dagger)^n|O\rangle$

$$AA^\dagger \cdot (A^\dagger)^n|O\rangle = \phi (N + 1)(A^\dagger)^n|O\rangle
\Leftrightarrow \Phi^{11^i}[n_l^i + 1]_q[n_l^i + 1]_q(A^\dagger)^n|O\rangle = \phi (n + 1)(A^\dagger)^n|O\rangle,$$

where we have introduced the number operators $n_l^i$ and $n_l^i$ for $q$-deformed constituent fermions. Taking into account the normalization condition $|\Phi^{11^i}|^2 = 1$ and the equality $n_l^i(A^\dagger)^n|O\rangle = n(A^\dagger)^n|O\rangle$, $i = a, b$, relation (C.2) is rewritten as

$$(n + 1)_q[n + 1]_q(A^\dagger)^n|O\rangle = \phi (n + 1)(A^\dagger)^n|O\rangle.$$

The latter implies $\phi (n) = ([n]_q)_2^2$.

Example 2. Next, let us consider the case of two modes $\mu = 1, 2$ and $v = 1, 2$. For the creation and annihilation operators $A^\alpha_u$ and $A_\alpha$, we obtain (the fixed $\alpha$ is omitted as before)

$$A^\dagger = \sum_{\mu, v=1}^2 \Phi^{\mu\nu}a^\dagger_\mu b^\nu_1 = \Phi^{11^i}a^\dagger_1 b^1_1 + \Phi^{12^i}a^\dagger_1 b^2_1 + \Phi^{21^i}a^\dagger_2 b^1_1 + \Phi^{22^i}a^\dagger_2 b^2_1,$$

$$A = \sum_{\mu, v=1}^2 \Phi^{\mu\nu}b_\mu a_\nu = \Phi^{11^i}b_1 a_1 + \Phi^{12^i}b_1 a_2 + \Phi^{21^i}b_2 a_1 + \Phi^{22^i}b_2 a_2.$$
with the initial conditions

\[ P_n^0(0) = P_{n+1}^0(0) = P_n^0(0) = P_{n+1}^0(n) = 1, \]
\[ P_n^j(j) = 0 \quad \text{if} \quad j > \min(k, l) \quad \text{or} \quad j < \max(0, k + l - n). \]

In what follows we need only a few coefficients \( P_n^j(j) \) for which we give their explicit expressions, as the solutions of the above recurrence relations:

\[ P_n^0(0) = P_{n+1}^0(0) = \frac{1 - (-1)^n}{2}, \quad (C.8) \]
\[ P_n^1(0) = -n + \frac{1 - (-1)^n}{2}, \quad P_n^1(1) = n, \quad (C.9) \]
\[ P_n^2(0) = P_{n+1}^2(0) = \frac{1 + (-1)^n}{2} + \frac{3}{4}n^2 - \frac{3}{4}, \quad (C.10) \]
\[ P_{n+1}^1(1) = -n + \frac{1 - (-1)^n}{2}, \quad (C.12) \]
\[ P_{n+1}^2(1) = (n - 1)\frac{1 + (-1)^n}{2} - n^2 + 1, \quad (C.14) \]
\[ P_{n+1}^2(2) = n(n + 1)/2. \quad (C.15) \]

Now, rewrite the lhs and rhs of (C.5) respectively as

\[ AA^{n+1}|O\rangle = \sum_{k,l=0}^{n} (-1)^{n(n-1)/2} [k + 1]_{-q}[l + 1]_{-q}\Phi^{kl}\Phi^{n+1,k/l+1}(\Phi) \]
\[ + (-1)^{j}k + 1 \quad \text{if} \quad n + 1 - l], -q\Phi^{k,l+1}(\Phi) \]
\[ + (-1)^{j}n + 1 - k], -q[l + 1], -q\Phi^{k+1,l+1}(\Phi) \]
\[ + (-1)^{j+k}[n + 1 - k], -q[n + 1 - l], -q\Phi^{kl+1}(\Phi) \]
\[ (a_2^3)^{k} (a_1^3)^{n-k} (b_1^3)^{l} (b_0^3)^{n-l}|O\rangle, \]
\[ \phi(N + 1)(A^+)^n|O\rangle = \phi(n + 1)\sum_{k,l=0}^{n} (-1)^{n(n-1)/2}C_{kl}^n(\Phi) (a_2^3)^{k} (a_1^3)^{n-k} (b_1^3)^{l} (b_0^3)^{n-l}|O\rangle. \]

Since the vectors \((a_2^3)^{k} (a_1^3)^{n-k} (b_1^3)^{l} (b_0^3)^{n-l}|O\rangle, k, l = 0, \ldots, n, n = 0, 1, 2, \ldots, \) form a basis in the Hilbert space, we may equate the corresponding summands, and thus the requirement (C.5) for \( n \geq 1 \) is equivalent to the following system of equations:

\[ [k + 1]_{-q}[l + 1]_{-q}\Phi^{kl}\Phi^{n+1,k/l+1}(\Phi) + (-1)^{j}k + 1 \quad \text{if} \quad n + 1 - l], -q\Phi^{k+1,l+1}(\Phi) \]
\[ + (-1)^{j}n + 1 - k], -q[l + 1], -q\Phi^{kl+1}(\Phi) \]
\[ + (-1)^{j+k}[n + 1 - k], -q[n + 1 - l], -q\Phi^{kl+1}(\Phi) \]
\[ -\phi(n + 1)C_{kl}^n(\Phi) = 0, \quad k, l = 0, \ldots, n, n = 1, 2, \ldots \quad (C.16) \]
Taking here $k = l = 0$, and using (C.7) and expressions (C.8)–(C.9), we arrive at the equations

\[
\begin{align*}
(-n + \frac{1 + (-1)^n}{2} - 1)\Phi^{22}\Phi^{21}\Phi^{21}(\Phi^{11})^{n-1} + (n + 1)\Phi^{22}\Phi^{21}(\Phi^{11})^{n} & \\
& + \frac{1 + (-1)^n}{2}[n + 1] - q\Phi^{21}\Phi^{21}(\Phi^{11})^{n} + \frac{1 + (-1)^n}{2}[n + 1] - q\Phi^{21}\Phi^{12}(\Phi^{11})^{n} & \\
& + [n + 1]^{-q}\Phi^{12}(\Phi^{11})^{n} = \phi(n + 1)(\Phi^{11})^{n}, \quad n = 1, 2, \ldots.
\end{align*}
\]

Due to the normalization condition the matrix $\Phi$ has at least one nonzero element. Let this be $\Phi^{11} \neq 0$ (for any other nonzero element the subsequent treatment is analogous). Then, from the last equation after dividing by $(\Phi^{11})^{n}$ and replacing $n + 1 \rightarrow n$ we obtain the following expression for the structure function:

\[
\phi(n) = \left(\frac{1 + (-1)^n}{2} - n\right)\Phi^{22}\Phi^{21}\Phi^{12}(\Phi^{11})^{n-1} + n\Phi^{22}\Phi^{21} + \frac{1 + (-1)^n}{2}[n - q]\Phi^{21}\Phi^{21}
\]

\[
+ \frac{1 + (-1)^n}{2}[n - q]\Phi^{12}\Phi^{12} + [n]^{-q}\Phi^{12}\Phi^{11}.
\]

(C.17)

Next, let us take $k = 1, l = 0$ in (C.16). Then, using (C.7) together with (C.8)–(C.12), after respective transformations we rewrite (C.16) in the following form:

\[
f_1(n)\Phi^{22}\Phi^{12}(\Phi^{11})^2 + f_2(n)\Phi^{22}\Phi^{22}\Phi^{21}\Phi^{11} + f_3(n)\Phi^{22}\Phi^{21}\Phi^{11} + f_4(n)\Phi^{22}\Phi^{22}(\Phi^{11})^2
\]

\[
+ f_5(n)\Phi^{21}(\Phi^{11})^2\Phi^{11} + f_6(n)\Phi^{12}(\Phi^{11})^2\Phi^{11} = 0,
\]

(C.18)

where

\[
f_1(n) = [2] - qP^{21}_{n+1}(0) - P^{10}_{n}(0)P^{11}_{n+1}(0) = [2] - q\frac{3n + (-1)^n}{2} + (n + 1)\frac{1 + (-1)^n}{2},
\]

\[
f_2(n) = [2] - qP^{21}_{n+1}(1) - P^{10}_{n}(0)P^{11}_{n+1}(1) = [2] - q\frac{1 + (-1)^n}{2} - (n + 1)\frac{1 + (-1)^n}{2},
\]

\[
f_3(n) = [-n] - qP^{11}_{n+1}(0) - [n + 1] - qP^{10}_{n}(0)P^{11}_{n+1}(0) = [n] - q\left(n + \frac{1 + (-1)^n}{2}\right),
\]

\[
f_4(n) = [-n] - qP^{11}_{n+1}(1) = [n] - q(n + 1),
\]

\[
f_5(n) = [2] - qP^{11}_{n+1}(0) - [n + 1] - qP^{10}_{n}(0)P^{11}_{n+1}(0) = [2] - q[n + 1] - q\frac{n + 1}{4},
\]

\[
f_6(n) = [-n] - qP^{11}_{n+1}(0) - [n + 1] - qP^{10}_{n}(0)P^{11}_{n+1}(0) = -[n + 1] - q[n] - q\frac{1 + (-1)^n}{2} - [n + 1]^2 - q\frac{1 + (-1)^n}{2}.
\]

Using the linear independence of $f_6(n)$ from $f_1(n), \ldots, f_5(n)$ and that $\Phi^{11} \neq 0$, from (C.18) we obtain

\[
\Phi^{12}(\Phi^{11})^2f_6(n) = 0, \quad \Rightarrow \quad \Phi^{12} = 0.
\]

(C.19)

Likewise, considering the case of $k = 0, l = 1$ in (C.16) we deduce

\[
\Phi^{12} = 0.
\]

(C.20)

Now examine the case $k = l = 1$ in (C.16). Substituting (C.7) together with (C.9)–(C.15) into (C.16), after dividing by $(\Phi^{22})^{n-3}$, we obtain

\[
g_1(n)\Phi^{22}(\Phi^{12})^2(\Phi^{12})^2 + g_2(n)\Phi^{22}\Phi^{22}\Phi^{21}\Phi^{12}\Phi^{11} + g_3(n)\Phi^{22}(\Phi^{22})^2(\Phi^{11})^2
\]

\[
+ g_4(n)\Phi^{21}(\Phi^{21})^2\Phi^{12}\Phi^{11} + g_5(n)\Phi^{22}\Phi^{21}(\Phi^{11})^2
\]

\[
+ g_6(n)\Phi^{21}(\Phi^{12})^2\Phi^{11} + g_7(n)\Phi^{12}\Phi^{22}(\Phi^{12})^2
\]

\[
+ g_8(n)\Phi^{21}\Phi^{22}(\Phi^{11})^2 + g_9(n)\Phi^{12}\Phi^{22}(\Phi^{11})^3 = 0,
\]

(C.21)
where
\[
g_1(n) = [2]^{2}_{-\frac{1}{q}} p^{22}_{n+1}(0) - p^{11}_{n+1}(0)p^{11}_{n}(0),
\]
\[
g_2(n) = [2]^{2}_{-\frac{1}{q}} p^{22}_{n+1}(0) - p^{11}_{n+1}(0)p^{11}_{n}(1) - p^{11}_{n+1}(1)p^{11}_{n}(0),
\]
\[
g_3(n) = [2]^{2}_{-\frac{1}{q}} p^{22}_{n+1}(2) - p^{11}_{n+1}(1)p^{11}_{n}(1) = ([2]^{2}_{-\frac{1}{q}} - 2) n(n + 1),
\]
\[
g_4(n) = g_6(n) = -[2]_{-\frac{1}{q}} [n]_{-\frac{1}{q}} p^{22}_{n+1}(0) - [n + 1]_{-\frac{1}{q}} p^{10}_{n+1}(0)p^{11}_{n}(0),
\]
\[
g_5(n) = g_7(n) = -[2]_{-\frac{1}{q}} [n]_{-\frac{1}{q}} p^{22}_{n+1}(1) - [n + 1]_{-\frac{1}{q}} p^{10}_{n+1}(0)p^{11}_{n}(1),
\]
\[
g_8(n) = [n]^{2}_{-\frac{1}{q}} p^{11}_{n+1}(0) - [n + 1]^{2}_{-\frac{1}{q}} p^{11}_{n}(0),
\]
\[
g_9(n) = [n]^{2}_{-\frac{1}{q}} p^{11}_{n+1}(1) - [n + 1]^{2}_{-\frac{1}{q}} p^{11}_{n}(1) = [n]^{2}_{-\frac{1}{q}} (n + 1) - [n + 1]^{2}_{-\frac{1}{q}} n.
\]

Taking into account (C.19) and (C.20) relation (C.21) reduces to
\[
g_3(n)\Phi^{22}(\Phi^{11})^{2} + g_9(n)\Phi^{11}\Phi^{22}(\Phi^{11})^{3} = 0.
\]

Using again the linear independence of the functions \(g_6(n)\) and \(g_9(n)\) and recalling that \(\Phi^{11} \neq 0\) from (C.22) we obtain
\[
\Phi^{22}(\Phi^{11})^{3} = 0 \implies \Phi^{22} = 0.
\]

Substitution of this element along with two previous ones (C.19) and (C.20) in (C.17), and the use of the normalization condition gives the resulting expression for the structure function in the case under consideration: \(\phi(n) = [n]^{2}_{-\frac{1}{q}}\).

Remark that if \(q = 0\), the uncertainty "00" in \(\phi(n)\) at \(n = 0\) is resolved using the condition \(\phi(0) = 0\) that results in \(\phi(n) = \theta(n)\).

References

[1] Bonatsos D and Daskaloyannis C 1999 Prog. Part. Nucl. Phys. 43 537
[2] Perkins W R 2002 Int. J. Theor. Phys. 41 823
[3] Avancini S and Krein G 1995 J. Phys. A: Math. Gen. 28 685
[4] Greenberg O W 1991 Phys. Rev. D 43 4111
[5] Arik M and Coon D D 1976 J. Math. Phys. 17 524
[6] Jagannathan R et al 1992 J. Phys. A: Math. Gen. 25 6429
[7] Biedenharn L C 1989 J. Phys. A: Math. Gen. 22 L873
[8] Macfarlane A J 1989 J. Phys. A: Math. Gen. 22 4581
[9] Odaka K, Kishi T and Kamufuchi S 1991 J. Phys. A: Math. Gen. 24 L591
[10] Chaturvedi S, Srinivasan V and Jagannathan R 1993 Mod. Phys. Lett. A 8 3727
[11] Gavrilik A M and Rebesh A P 2007 Mod. Phys. Lett. A 22 949
[12] Chakrabarti A and Jagannathan R 1991 J. Phys. A: Math. Gen. 24 L711
[13] Arik M et al 1992 Z. Phys. C 55 89–95
[14] Jannussis A 1993 J. Phys. A: Math. Gen. 26 L233
[15] Gavrilik A M, Kachurik I I and Rebesh A P 2010 J. Phys. A: Math. Theor. 43 245204
[16] Gavrilik A M and Rebesh A P 2011 Eur. Phys. J. A 47 55
[17] Meljanac S, Mileković M and Pallua S 1994 Phys. Lett. B 328 55
[18] Man’ko V I et al 1997 Phys. Scr. 55 528
[19] Rego-Monteiro M, Rodrigues L M C S and Wulck S 1998 Phys. Rev. Lett. 76 1098
[20] Rego-Monteiro M, Rodrigues L M C S and Wulck S 1998 Physica A 259 245
[21] Anchishkin D, Gavrilik A and Iorgov N 2000 Eur. Phys. J. A 7 229
[22] Gavrilik A M 2001 Nucl. Phys. B 102 298
[23] Anchishkin D, Gavrilik A and Iorgov N 2000 Mod. Phys. Lett. A 15 1637
[24] Avancini S S, Marinelli I R and Krein G 2003 J. Phys. A: Math. Gen. 36 9045
[25] Adamska L V and Gavrilik A M 2004 J. Phys. A: Math. Gen. 37 4787
[26] Gavrilik A M 2006 SIGMA 2 074
[27] Ribeiro-Silva C I, Curado E M F and Rego-Monteiro M A 2008 J. Phys. A: Math. Theor. 41 145404
[28] Gavrilik A M, Kachurik I I and Mishchenko Yu A 2011 Ukr. J. Phys. 56 948 (available at http://ujp.bitp.kiev.ua/files/file/papers/569/560911p.pdf)
[29] Gavrilik A M and Rebesh A P 2008 Mod. Phys. Lett. A 23 921
[30] Gavrilik A M and Rebesh A P 2010 J. Phys. A: Math. Theor. 43 095203
[31] Korn G A and Korn T M 1967 Mathematical Handbook for Scientists and Engineers (New York: McGraw-Hill)
[32] Viswanathan K S et al 1992 J. Phys. A: Math. Gen. 25 L335
[33] Sviratcheva K D et al 2004 Phys. Rev. Lett. 93 152501
[34] Gavrilik A M and Mishchenko Yu A 2011 arXiv:1108.0936
[35] Shih Y 2003 Rep. Prog. Phys. 66 1009
[36] Combescot M and Betbeder-Matibet O 2008 Phys. Rev. B 78 125206
[37] Harouni Bagheri M, Roknizadeh R and Naderi M H 2009 J. Phys. B: At. Mol. Opt. Phys. 42 095501
[38] Liu Y-X et al 2001 Phys. Rev. A 63 023802