Cooling curves for neutron stars with hadronic matter and quark matter

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The thermal evolution of isothermal neutron stars is studied with matter both in the hadronic phase as well as in the mixed phase of hadronic matter and strange quark matter. In our models, the dominant early-stage cooling process is neutrino emission via the direct Urca process. As a consequence, the cooling curves fall too fast compared to observations. However, when superfluidity is included, the cooling of the neutron stars is significantly slowed down. Furthermore, we find that the cooling curves are not very sensitive to the precise details of the mixing between the hadronic phase and the quark phase and also of the pairing that leads to superfluidity.

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I. INTRODUCTION

Neutron stars are natural laboratories for physics under extreme conditions with their extremely high densities, powerful energy emission, large magnetic fields, and millisecond rotation periods. At the densities near the surface of such a star, atoms break apart into nuclei and electrons. At higher densities, the electrons neutralize with the protons in the nuclei to form neutrons. These stars thus consist of a large fraction of neutrons and are supported from gravitational collapse by the neutron degeneracy pressure, from which the neutron star derives its name.

However, at high densities the existence of more exotic particles is expected. These particles are generated by processes which produce strangeness, such as

\[ n + n \rightarrow n + \Lambda^0 + K^0, \]

where \( n \) is the neutron, \( \Lambda^0 \) the Lambda hyperon, and \( K^0 \) the strange meson. The strange meson can decay via various weak processes and the final products, usually photons and neutrinos, leak out of the star. Therefore, the reverse process is reduced and some net amount of strangeness survives in the dense core of the star [1]. It is generally believed that these strangeness carrying particles, called hyperons, can exist in the center of neutron stars. They coexist with the nucleons as well as the leptons \( e^- \) and \( \mu^- \). The interactions between them are dominated by the complicated nuclear force whose carriers are the mesons. We refer to such a system as the hadronic phase of matter. The system with only nucleons and leptons, i.e., without hyperons, is referred to as the nuclear phase of matter.

At even higher densities, as a consequence of asymptotic freedom, quarks become deconfined from the hadrons. Therefore, strange quark matter, which consists of \( u, d \) and \( s \) quarks, may also exist in the neutron core. There may also be a phase-separated mixture of strange quark matter and hadronic matter in a certain range of densities [2], which we call the mixed phase. At present, there is still a lot unknown about the deconfinement phase transition of quarks. The contemporary knowledge on matter at high densities and temperatures is shown by the schematic QCD phase diagram in Fig. 1. Since the baryon chemical potential \( \mu_B \) is a monotonously increasing function from the surface to the center of the neutron star, the \( \mu_B \) axis can be mapped to the stellar radius and the different phases in the neutron star are explicitly indicated.

Because of the limited knowledge on the state of matter at high densities and the complexity of the interactions between the particles, many effective models for matter inside neutron stars have been constructed. In general, these models supply an equation of state, which determines the particle composition of the neutron star. The observation of neutron stars, in turn, constrains these models. For example, if an equation of state is too soft it is incapable of supporting a very large stellar mass,
such that some models become disfavored whenever the data on the heaviest neutron star is updated. The stellar mass is not the only constraint on the models. The cooling of neutron stars can be studied from their luminosity as a function of time. A proto-neutron star is born with a typical temperature larger than $10^{10}$ K, after which it mainly loses its energy by two processes, namely by neutrino emission everywhere inside the star and by photon radiation at the surface. In the early stages of the thermal evolution of the star, neutrino emission is the dominant cooling effect after which photon radiation ultimately takes over. Since neutrino emission occurs everywhere in the neutron star, it provides a probe for studying the state of matter inside the star.

The most efficient neutrino-emission process is called the direct Urca (DURca) process

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad p + e^- \rightarrow n + \nu_e.$$  \hspace{1cm} (2)

This process is only possible if the proton fraction is more than a certain threshold in order for it to satisfy energy and momentum conservation. Historically, the proton abundance in neutron stars was underestimated. In that scenario, it is reasonable to take into account also the modified Urca (MUrca) process, where a bystander helps satisfy momentum conservation, for example

$$n + n \rightarrow p + n + e^- + \bar{\nu}_e, \quad p + n + e^- \rightarrow n + n + \nu_e,$$
$$n + p \rightarrow p + p + e^- + \bar{\nu}_e, \quad p + p + e^- \rightarrow n + p + \nu_e. \hspace{1cm} (3)$$

Some other processes can have a neutrino emissivity which is smaller or comparable to the modified Urca process, such as neutrino bremsstrahlung and plasmon decay. However, all of them are negligible whenever the direct Urca channel is open. In the hadronic phase, the direct Urca processes are quite rich and can be summarized as

$$b_1 \rightarrow b_2 + l^- + \bar{\nu}_l, \quad b_2 + l^- \rightarrow b_1 + \nu_l, \hspace{1cm} (4)$$

where $b_1$ and $b_2$ denote two different baryons and $l^-$ represents one of the leptons, $e^-$ or $\mu^-$. For strange quark matter, the direct Urca processes are

$$d \rightarrow u + l^- + \bar{\nu}_l, \quad u + l^- \rightarrow d + \nu_l,$$
$$s \rightarrow u + l^- + \bar{\nu}_l, \quad u + l^- \rightarrow s + \nu_l, \hspace{1cm} (5)$$

which are simply the direct Urca processes of the baryons at the quark level. Inside the neutron star, there is no threshold for the direct Urca process in quark matter, since the Fermi momenta of the two quarks and lepton can always satisfy momentum conservation. As mentioned, the neutrino emissivity depends strongly on the type of matter contributing to the process, such that the thermal evolution of a star directly probes its composition. The main goal of this paper is to study the cooling of neutron stars with different types of equations of state and try to constrain them by comparing with observational data.

A large amount of research on the cooling process of neutron stars has been carried out in the last several decades, with a strong focus on the nuclear phase of matter. To the best of our knowledge, a unifying model covering the low-density nuclear phase to the deconfined quark phase, consistent with the QCD phase diagram, has been less thoroughly explored. The strange quark star composed of strange matter, either with or without the nuclear crust, has been considered in some papers. However, the cooling behavior of neutron stars containing also all hyperons and possibly also a mixed phase of strange quark matter and hadronic matter, has not been extensively studied. Two possible reasons for this can be given. First, the hadronic and mixed equations of state are rather soft and are continuously being challenged by new data on heavy neutron stars, such as PSR J1903+0327 which has $M = 1.67 \pm 0.01 \, M_\odot$ [7] and PSR J1614–2230 which even has $M = 1.97 \pm 0.04 \, M_\odot$ [8]. Second, the existence of a mixed phase is being questioned in view of screening and surface effects [2]. With respect to the first concern, we note that most of the observed neutron star masses still lie below the maximum mass a hadronic or mixed equation of state can allow for. According to Ref. [10], most nearby young neutron stars have masses no bigger than 1.4 $M_\odot$. Therefore, at least for the study of the thermal evolution of these stars, the hadronic or mixed phase can still be of great importance. In fact, because of the uncertainties in the interaction between particles at extremely high density, it is hard to exactly determine the equation of state at high density. Although the equations of state used in this paper cannot support neutron stars as massive as those reported above, we can still use them for a discussion on medium-mass neutron stars with more complicated compositions. As for the stability of the mixed phase, the arguments are still indecisive. For example, many details of the surface tension, which strongly influences the stability calculation, are still uncertain. Although it is expected that screening and surface effects diminish the mixed-phase regime, it is far from certain that its existence can be excluded. The mixed phase can in particular have significant effects on the cooling behavior, especially for the heat transport inside the star. However, we show below that its effect will be less important after the star has become isothermal, when the thermal evolution is determined by the heat capacity and neutrino emissivity integrated over the whole volume, and the inner thermal conductivity no longer plays a role.

The temperature of a neutron star is generally much smaller than the typical Fermi energy as a consequence of the very high densities in the star. Therefore, superfluidity may play an important role. According to BCS theory, fermions can form Cooper pairs at low temperatures via an attractive interaction and thereby lower the energy of the system. The resulting pairs, which obey Bose statistics, can form a Bose-Einstein condensate and the system becomes superfluid. Pair formation changes the single-particle dispersion around the Fermi surface.
and consequently the heat capacity and neutrino emissivity will be influenced. We find that, without superfluidity, the cooling of neutron stars is too fast compared with observations. However, by including the effects of superfluidity we obtain a more realistic cooling behavior.

The quark phase is usually referred to as an exotic phase of extremely dense matter, in contrast to the nuclear phase or the hadronic phase. Other exotic phases have also been proposed, among which the pion and kaon condensates have attracted much attention \[14, 15\]. However, the existence of such phases inside the neutron star is still an open question. On the one hand, with such condensates, the equation of state is further softened and thus the corresponding maximum star mass is reduced \[10\], which makes such phases less favored when compared to the observational data of massive stars, as mentioned above. On the other hand, for the cooling process, it was reported that a meson condensate can increase the neutrino emissivity over the typical modified Urca emissivity by several orders of magnitude \[3\], but it is still much less efficient than the direct Urca process. Since in our calculation the direct Urca process is always present, such enhancement from the meson condensate has a negligible effect. Besides, the kaon condensation may even reduce the pressure and cause the star to collapse into a black hole \[15\]. Therefore, considering all the above arguments, we do not include such meson condensates in this paper.

The paper is organized as follows, we first introduce in Sec. II our theoretical framework which includes the relativistic stellar structure, the unified mean-field model describing the state of matter inside the star, and the thermal evolution equations. In Sec. III, we then present the numerical results for the cooling curves of neutron stars with the nuclear, hadronic, and mixed equations of state. Due to the efficiency of the direct Urca process the cooling is seen to be too fast compared to observations. Therefore superfluidity is included in Sec. IV and, as a consequence, the cooling is slowed down and we find a much better agreement with observations.

II. THEORETICAL FRAMEWORK

At the high densities of importance to neutron stars the neutrons, protons, and electrons can be considered to be highly degenerate. In particular, the typical Fermi temperature of the nucleons is about 10^{12} K while the temperature of a proto-neutron star is only 10^{10} K. This provides a great theoretical advantage since it makes it possible to decouple the calculation of the stellar structure, the nuclear model, and the thermal evolution of the star. In other words, the thermal evolution of the star depends on the stellar structure, which in turn is constructed from the nuclear model. The nuclear model will be solved in the zero-temperature limit to give the equation of state. The stellar structure is then obtained from Einstein’s equations using this equation of state.

Although many neutron stars have fast rotation rates and strong magnetic fields, due to the complexity and richness of these phenomena they are beyond our present discussion. In fact, the effect of rotation should be quite small for most of the neutron stars except for millisecond pulsars. Therefore, Einstein’s equations are solved for static stars without magnetic fields. Furthermore, the star interior is approximated by a perfect fluid, i.e., it is hydrostatic, which is valid as long as the energy transported by heat conduction is negligible compared to the total energy. Subsequently, to discuss the thermal evolution, some properties of the stellar matter, such as the heat capacities, conductivities, and emissivities, can be obtained by considering a perturbation of the Fermi surfaces of the various particles. In the following, we deal with these aspects of our theoretical framework separately.

### A. The general relativistic profile of a compact star

The space-time metric for a non-rotating, spherically symmetric star can be written as

\[ ds^2 = c^2[\Phi(r) - r^2 - 2\lambda(r)]dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

(6)

where \( \Phi(r) \) is the metric function related to the gravitational redshift, \( \lambda(r) \) is the metric function related to the gravitational potential, \( m(r) \) is the mass within the sphere of radius \( r \), and \( c \) is the speed of light. As a consequence, the cooling is seen to be too fast compared to observations. Therefore superfluidity is included in Sec. IV and, as a consequence, the cooling is slowed down and we find a much better agreement with observations.

\[ \frac{d\Phi(r)}{dr} = \frac{1}{\rho(r) + p(r)} \frac{dp(r)}{dr}, \]  

(7)

\[ \frac{dp(r)}{dr} = \left[ \rho(r) + p(r) \right] \left[ Gm(r) + 4\pi r^2 \right] \]  

(8)

where \( \rho(r) \) and \( p(r) \) are the pressure and energy density at radius \( r \), respectively. Given an equation of state \( p(\rho) \), the TOV equations can be numerically integrated to provide the structure of the star. The integration starts at \( r = 0 \) with a given pressure \( p(0) = p_c \), until \( p(R) = 0 \) is reached, which defines the radius \( R \) of the star. Outside the star, \( r > R \), the metric reduces to the Schwarzschild form, \( \lambda(r) = 1 - 2GM/c^2r \), where \( G \) is the total gravitational mass of the star. Note that with the help of a local Lorentz transformation we are always able to apply a locally inertial coordinate system \[17\] such that the calculation of the equation of state is performed in a homogeneously flat background. Nevertheless, the TOV equations take into account the effects of general relativity on the structure of the star. However, to calculate this structure we need to supply a realistic equation of state for the matter inside a neutron star.
B. Effective model for matter in neutron stars

To obtain an equation of state for stellar matter, we apply an effective nuclear model which includes all relevant kinds of baryons. The interactions between the baryons are mediated by three types of effective meson fields, namely the scalar meson $\sigma$, the vector meson $\omega$, and the isovector meson $\tilde{\rho}$. The scalar field is coupled to the derivatives of the baryon field as shown in the following Lagrangian density \[ \mathcal{L} = \sum_b \left[ \left( 1 + \frac{g_{b\sigma}}{m_b c^2} \right) \bar{\psi}_b \left( i\hbar c \gamma^\mu \partial_\mu - g_{\omega^0} \gamma^0 \omega^\mu \right) \right. 
\] 
\[ - \frac{1}{2} g_{b\rho} \gamma^\mu \sigma \cdot \rho^\mu \psi_b - m_{\rho^0} \omega^\mu \psi_b \] 
\[ - \frac{m_{\omega}^2 \sigma^2}{2\hbar c} \] 
\[ - \frac{m_{\rho}^2 \rho^\mu \rho^\mu}{2\hbar c} \] 
\[ + \sum_l \left( \bar{l} \left( i\hbar c \gamma^\mu \partial_\mu - m_l c^2 \right) l \right), \] \[ \tag{9} \]

where $c\partial_0 = \partial/\partial t$, the summation over $b$ and $l$ is over all contributing baryon fields $\psi_b$ and lepton fields $\psi_l$, and in the meson kinetic terms $\omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$, and $\tilde{\rho}_{\mu\nu} = \partial_\mu \tilde{\rho}_\nu - \partial_\nu \tilde{\rho}_\mu$ are the antisymmetric field strengths. The derivative coupling of the $\sigma$ field was first introduced by Zimanyi and Moszkowski \[ \text{[18]} \] to remove the problem of a too small or even negative reduced baryon mass (commonly referred to as the effective mass, however, this name is used in this paper only for the effective fermion mass on the Fermi surface) at high density in the standard Walecka model \[ \text{[19]}. \]

This model can be solved within the mean-field approximation. In this approximation all the meson fields are replaced by their ground-state expectation values, which are constants in space-time and their spatial components are zero because the system is assumed to be homogeneous and isotropic. Also the charged components of the isospin vector $\rho^+ \text{ and } \rho^-$, whose sources are the off-diagonal currents of the baryon fields, are zero. Therefore, only the three constant fields $\sigma, \omega^0 \text{ and } \tilde{\rho}^0$ survive in the Euler-Lagrange equations. In the following, we simply denote the last two as $\omega$ and $\rho$. Furthermore, the constant scalar field $\sigma$ can be absorbed into the baryon field and the reduced mass by a rescaling, namely $\Psi_b = \sqrt{1 + \frac{m_{\omega}^2}{m_b c^2}} \psi_b$ and $\bar{m}_b(\sigma) = m_b/(1 + \frac{m_{\omega}^2}{m_b c^2})$. Notice that $\bar{m}_b$ is positive definite and only approaches zero as $\sigma \to \infty$.

In the mean-field approximation the Lagrangian gives rise to the following field equations:

$$\left[ i\hbar c \gamma^\mu \partial_\mu - g_{\omega^0} \gamma^0 \omega - \frac{1}{2} g_{b\rho} \gamma^0 \tau_3 \rho - \bar{m}_b c^2 \right] \Psi_b = 0, \quad \tag{10}$$

$$\sum_b \frac{g_{b\sigma} \langle \bar{\Psi}_b \Psi_b \rangle}{1 + \frac{g_{b\rho}}{m_b c^2}} = \frac{m_b^2 \sigma}{\hbar c} = 0, \quad \tag{11}$$

$$\sum_b g_{\omega b} \langle \bar{\Psi}_b \gamma_5 \Psi_b \rangle - \frac{m_b^2 \omega}{\hbar c} = 0, \quad \tag{12}$$

$$\sum_b g_{\rho b} \left( \frac{1}{2} \bar{\Psi}_b \tau_3 \Psi_b \right) - \frac{m_b^2 \rho}{\hbar c} = 0, \quad \tag{13}$$

where $\tau_3/2$ gives the expectation value of the third component of the baryon isospin $I_b$. The equation of motion of the baryons gives the following energy eigenvalue

$$\mu_b = \epsilon_b = g_{b\omega} \omega + g_{b\rho} \rho + \sqrt{\hbar^2 c^2 k_b^2 + \bar{m}_b c^2}. \quad \tag{14}$$

From the above it is clear that the baryon $b$ only exists when $\mu_b > (g_{b\omega} \omega + g_{b\rho} \rho) > \bar{m}_b c^2$. The equation of motion of the leptons are simply the free Dirac equations and are not listed here. Thus, the leptons obey the simple relations of a free relativistic Fermi gas:

$$n_l = f_l \frac{k_l^3}{6\pi^2}, \quad \mu_l = \epsilon_l = \sqrt{\hbar^2 c^2 k_l^2 + m_l c^4}. \quad \tag{15}$$

For the expectation value of the baryon field we have

$$\langle \bar{\Psi}_b \Psi_b \rangle = \frac{f_b}{2\pi^2} \int_0^{k_b} \frac{\bar{m}_b c^2 k^2 dk}{\sqrt{\hbar^2 c^2 k^2 + \bar{m}_b c^2}}$$

$$= \frac{f_b}{2\pi^2} \frac{\bar{m}_b c}{\hbar k_b} \left[ \sqrt{\hbar^2 k_b^2 + \bar{m}_b c^2} - \bar{m}_b c^2 \log \left( \frac{\hbar k_b}{\bar{m}_b c} + \sqrt{1 + \frac{\hbar^2 k_b^2}{\bar{m}_b c^2}} \right) \right], \quad \tag{16}$$

and

$$\langle \bar{\Psi}_b \gamma_5 \Psi_b \rangle = n_b = f_b \frac{k_b^3}{6\pi^2}. \quad \tag{17}$$

In the above equations $f_i = 2J_i + 1$ is the particle degeneracy, while $k_i$ and $\epsilon_i$ are the Fermi momentum and Fermi energy, respectively.

Since the whole star is considered to be in equilibrium, the various processes, such as the direct Urca process in Eq. \[ \text{[2]} \] and those involving the hyperons in Eq. \[ \text{[1]} \], impose relations between the chemical potentials of the particles. During these processes the baryon number and electric charge are conserved, which gives rise to the two chemical potentials $\mu_B$ and $\mu_Q$ for baryon number and electric charge, respectively. The chemical potential for an arbitrary particle with baryon number $B_i$ and charge $Q_i$, such as a baryon, lepton or quark, can be written as $\mu_i = B_i \mu_B + Q_i \mu_Q$. The particle data for all the fermions used in this paper are listed in Table.\[ \text{[4]} \]
TABLE I: Particle Data [21]. Here, $M$ is the particle mass, $B$ is the baryon number, $J$ is the spin, $I_3$ is the 3-component of the isospin, and $Q$ is the charge.

| particle | $M$/MeV | $B$ | $J$ | $I_3$ | $Q/e$ |
|----------|---------|-----|-----|-------|-------|
| p        | 938.27  | 1   | 1/2 | 1/2   | +1    |
| n        | 939.57  | 1   | 1/2 | -1/2  | 0     |
| $\Lambda^0$ | 1115.7  | 1   | 1/2 | 0     | 0     |
| $\Sigma^+$ | 1189.4  | 1   | 1/2 | 1     | +1    |
| $\Sigma^0$ | 1192.6  | 1   | 1/2 | 0     | 0     |
| $\Sigma^-$ | 1197.4  | 1   | 1/2 | -1    | -1    |
| $\Delta^{++}$ | 1232.1  | 1   | 3/2 | +3/2  | +2    |
| $\Delta^+$ | 1232.1  | 3/2 | +1/2| +1    |       |
| $\Delta^0$ | 1232.1  | 3/2 | -1/2| 0     |       |
| $\Delta^-$ | 1232.1  | 3/2 | -3/2| -1    |       |
| $\Xi^0$ | 1315.1  | 1   | 1/2 | +1/2  | 0     |
| $\Xi^-$ | 1322.1  | 1   | 1/2 | -1/2  | -1    |
| $\Omega^-$ | 1672.1  | 1   | 3/2 | 0     | -1    |

| Lepton | $\mu^-$ | 105.7 | 1/2 | 0 | -1 |
| Quark  | u        | 2.4   | 1/3 | 1/2 | +1/2 | +2/3 |
|        | d        | 4.9   | 1/3 | 1/2 | -1/2 | -1/3 |
|        | s        | 10.5  | 1/3 | 1/2 | 0    | -1/3 |

The coupling constants $g_{\sigma b}$, $g_{\omega b}$ and $g_{\rho b}$ for nucleons can be fitted with data of the interactions at densities near the nuclear saturation point. However, for hyperons the values are not known. Here the coupling constants are simply chosen to be equal among the different baryons [2]. Their values are $(g_{\sigma}/m_\sigma)^2 = 7.487$ fm$^2$, $(g_{\omega}/m_\omega)^2 = 2.615$ fm$^2$ and $(g_{\rho}/m_\rho)^2 = 4.774$ fm$^2$. Notice that the ratio of the coupling constant and the corresponding meson mass is enough to solve this model, because we can always rescale the meson fields with their masses, so the meson mass will not appear in Eqs. (10–12).

Given the chemical potentials $\mu_B$ and $\mu_Q$ as the input, together with the neutrality condition $\sum_i n_i Q_i = 0$, Eqs. (10–12) form a set of self-consistent nonlinear equations which can be solved numerically. With the Fermi momenta $k_i$ of the particles as the output, we can calculate the energy density and the pressure of the hadronic matter as:

$$\rho_H = \sum_b \frac{f_b c}{16\pi^2 \hbar^3} \left[ h k_b \sqrt{h^2 k_b^2 + \tilde{m}_b^2 c^2} (2h^2 k_b^2 + \tilde{m}_b^2 c^2) - \tilde{m}_b^4 c^4 \log \left( \frac{h k_b}{\tilde{m}_b c} + \sqrt{1 + \frac{h^2 k_b^2}{\tilde{m}_b^4 c^2}} \right) \right]$$

$$p_H = \sum_b \frac{f_b c}{48\pi^2 \hbar^3} \left[ h k_b \sqrt{h^2 k_b^2 + \tilde{m}_b^2 c^2} (2h^2 k_b^2 - 3\tilde{m}_b^2 c^2) + 3\tilde{m}_b^4 c^4 \log \left( \frac{h k_b}{\tilde{m}_b c} + \sqrt{1 + \frac{h^2 k_b^2}{\tilde{m}_b^4 c^2}} \right) \right]$$

The relation between the pressure $p$ and the energy density $\rho$ gives the equation of state, which is shown in Fig. 2 and is subsequently used in the TOV equations to get the stellar structure. The particle densities as a function of baryon density are shown in Fig. 3. Fig. 4 gives the particle composition inside the maximum mass star with the hadronic equation of state $(M = 1.523$ $M_\odot, R = 9.72$ km). Remember that the model is used to calculate the equation of state for the whole range of the density, however, this description may not be appropriate near the star surface, i.e., the crust. Nevertheless, the outer crust only affects low mass stars significantly, while we are mainly concerned with maximum mass stars. Furthermore, in view of the cooling behavior, the outer crust has negligible neutrino emissivity compared to the direct Urca process in the core.

Compared to the hadronic phase, the quark phase is more easily described. Quark matter is treated as a free Fermi gas, whereby we assume that asymptotic freedom has taken effect at the very high densities in the center of the star. Thus Eq. (15) is also valid for quarks, where the lepton chemical potential $\mu_l$ is substituted with the quark chemical potential $\mu_q$ and the degeneracy becomes $f_L = 3 \times (2I_3 + 1) = 6$ because of the extra color degrees of freedom. The fact that the quark phase has a different vacuum than the hadronic phase, which has a nonzero expectation value of the gluon field, can be taken into account by the so-called bag model. This model adds a
constant shift called the bag constant \( B_c \) to the pressure and energy density of the quarks, such that

\[
\rho_q = \sum_q \frac{f_q}{16\pi^2 a^3} \left[ \hbar k_q \sqrt{\hbar^2 k_q^2 + m_q^2 c^2 (2\hbar^2 k_q^2 + m_q^2 c^2)} \right] - m_q^4 c^4 \log \left( \frac{\hbar k_q}{m_q c} + \sqrt{1 + \frac{\hbar^2 k_q^2}{m_q^2 c^2}} \right) + \sum_l \frac{f_l}{16\pi^2 a^3} \left[ \hbar k_l \sqrt{\hbar^2 k_l^2 + m_l^2 c^2 (2\hbar^2 k_l^2 + m_l^2 c^2)} \right] - m_l^4 c^4 \log \left( \frac{\hbar k_l}{m_l c} + \sqrt{1 + \frac{\hbar^2 k_l^2}{m_l^2 c^2}} \right) + B_c, \tag{20}
\]

\[
p_q = \sum_q \frac{f_q}{48\pi^2 a^3} \left[ \hbar k_q \sqrt{\hbar^2 k_q^2 + m_q^2 c^2 (2\hbar^2 k_q^2 + m_q^2 c^2)} \right] + 3m_q^4 c^4 \log \left( \frac{\hbar k_q}{m_q c} + \sqrt{1 + \frac{\hbar^2 k_q^2}{m_q^2 c^2}} \right) + \sum_l \frac{f_l}{48\pi^2 a^3} \left[ \hbar k_l \sqrt{\hbar^2 k_l^2 + m_l^2 c^2 (2\hbar^2 k_l^2 + m_l^2 c^2)} \right] + 3m_l^4 c^4 \log \left( \frac{\hbar k_l}{m_l c} + \sqrt{1 + \frac{\hbar^2 k_l^2}{m_l^2 c^2}} \right) - B_c, \tag{21}
\]

where the leptons still contribute because the unequal masses of the \( u, d \) and \( s \) quarks result in unequal densities of these flavors, such that even in the pure quark phase charge neutrality cannot be satisfied without leptons.

As discussed in the introduction, there could also be a mixed phase between the hadronic phase and the quark phase. These two phases are taken to be in equilibrium, i.e., the two phases have the same temperature (both set to zero here), pressure, and chemical potentials. Charge neutrality then determines the volume fraction of these two phases. For the three possible phases (hadronic, quark, and mixed) the system will be in the one with the highest pressure, or equivalently, the lowest grand potential, which determines the phase transition behavior. According to Eq. \([21]\), the bag constant determines the scale at which deconfinement sets in. In other words, the larger the bag constant, the later the quark phase sets in with increasing density. Throughout this paper the bag constant is taken to be \( B_c = 230 \text{ MeV/fm}^3 \) unless indicated specifically. In Fig. 5 it is shown how the phase transition takes place between the two phases. Similar to Fig. 5 for the hadronic phase, the various particle densities for the mixed phase are shown in Fig. 6 as a function of baryon density. Fig. 7 shows the particle composition inside the maximum mass star with the mixed equation of state \((M = 1.479 \text{ M}_\odot; R = 9.81 \text{ km})\).

C. Thermal evolution equations

The equations governing the thermal evolution of a spherically symmetric star, given by the metric in Eq.
expressed in terms of their baryon densities instead of thei r number densities.

\[ \frac{n}{n_B} \text{ versus baryon density } n_B \text{ for the mixed phase case.} \]

FIG. 6: The particle densities \( n_i/n_B \) versus baryon density \( n_B \) for the mixed phase case. The densities of the quarks are expressed in terms of their baryon densities instead of their number densities.

\[ c_V \cdot \frac{\partial (T e^{\Phi})}{\partial t} = -\frac{e^{-\lambda}}{4\pi r^2} \frac{\partial (Le^{2\Phi})}{\partial r} - q_\nu e^{2\Phi} - q_\gamma e^{2\Phi}, \tag{22} \]

\[ \kappa \cdot \frac{\partial (T e^{\Phi})}{\partial r} = -\frac{Le^{\lambda + \Phi}}{4\pi r^2}, \tag{23} \]

where \( c_V \) is the specific heat capacity at constant volume, \( \kappa \) is the thermal conductivity, \( q_\nu \) and \( q_\gamma \) are the neutrino and photon emissivity, respectively. The photon emissivity \( q_\gamma \) is only nonzero on the surface \( r = R \). It is convenient to define the redshifted temperature \( \tilde{T} = T e^\Phi \) inside the star. Similarly, \( Le^{2\Phi} \) represents the redshifted luminosity corresponding to the heat current. In order to compare with observations, the photon luminosity is of great importance. Here it is assumed to obey the black body radiation law \( L_\gamma = 4\pi R^2 q_\gamma = 4\pi R^2 T_\gamma^4 \), where \( \sigma = 5.67 \times 10^{-5} \text{ erg/cm}^2\text{K}^4\) is the Stefan-Boltzmann constant and \( T_\gamma \) is the surface temperature. According to general relativity, an observer at infinity will measure the gravitationally red-shifted temperature and luminosity. The observational quantities are thus \( T_\infty = T_\gamma e^{\Phi(R)} \) and \( L_\infty = L_\gamma e^{2\Phi(R)} \). As is generally accepted, a very thin layer of the outer crust of the neutron star will act as a thermal insulator causing the temperature at the surface to be much lower than inside the star. The relation between the surface temperature and the inner temperature depends on the chemical composition of this envelop. We will simply locate such a layer at the surface, neglecting its thickness, and adopt the \( T_\infty - T \) relation given by Potekhin et al. \[22\], as shown in Fig. 8 and set \( T \) to be the inside temperature at \( r = R \). Note that the surface temperature \( T_\infty \) introduced above is the outside temperature at \( r = R \).

The parameters \( c_V, \kappa, q_\nu, \) and \( q_\gamma \) need to be determined to solve the thermal evolution equations, Eqs. \[22\] and \[23\]. The specific heat \( c_V \) can simply be obtained by summing up all the contributions of the different fermions in the model. In the low-temperature approximation, Fermi-liquid theory gives

\[ c_{V_i} = \frac{m_i^* k_i}{3\hbar^2} k_B T, \tag{24} \]

where \( k_B \) is Boltzmann’s constant, \( m_i^* = \hbar v_i/k_i \) is the effective mass on the Fermi surface, and \( v_i = d\epsilon_i/\left|d\hbar k\right|_{k=k_i} \) is the Fermi velocity of quasi-particle \( i \). For leptons the effective mass is easily obtained since they are described as a free Fermi gas, i.e., \( m_l^* = \sqrt{k^2 m_i^2 + m_i^2} = \mu_l/c^2 \). Similarly for quarks, \( m_q^* = \mu_q/c^2 \). For baryons the effective mass follows from the
the lepton state, where $m_n$ is the mass of the light elements in the envelop and $g_{14}$ is the surface gravity in unit of $10^9$ cm/s$^2$. The different curves correspond to $\log \eta = -\infty$, $-16$, $-14$, $-12$, $-10$, $-8$, $-6$ and $0$ from bottom to top, respectively. These results are for the maximum mass neutron star with the hadronic equation of state, where $g_{14} = 2.918$.

The dispersion relation in Eq. (14)

$$v_b = \frac{d\epsilon_b(k)}{dk} \bigg|_{k=k_b} = \frac{\hbar k_b}{\sqrt{\hbar^2 k_b^2/c^2 + m_b^2}},$$

such that $m_b^* = \sqrt{\hbar^2 k_b^2/c^2 + m_b^2}$. The effect of interactions is, in the mean-field approximation, only present in the reduced mass $m_b$ of the baryons.

The general expression for the neutrino emissivity has the form $q_vi = C_i T^s$, where $C_i$ and $s$ are different constants for each kind of process. For the direct Urca process we have $s = 6$ and for the modified Urca process $s = 8$. The difference in the exponents shows the inefficiency of the modified Urca process since the temperature $T$ of a neutron star is much smaller than the typical Fermi temperature $T_F$. The neutrino emissivities for the various processes are summarized in Ref. [23]. For the general baryon direct Urca process, cf. Eq. (1), the emissivity is

$$q_{\nu12l} \approx 1.207 \times 10^{25} \frac{\mu m_{\nu12l}^2}{(1 \text{ MeV}) m_\Lambda^6} R_{12} T_0^6 \Theta_{12l} \text{ erg/cm}^3 \text{s},$$

where $T_0$ is the temperature in units of $10^9$ K, $\Theta_{12l}$ is 1 only if the Fermi momenta of the two baryons $b_1$, $b_2$ and the lepton $l$ can form a triangle, otherwise $\Theta_{12l} = 0$. The coefficients $R_{12}$ vary depending on the baryons involved in the process. In our mean-field model the maximum mass star contains massive hyperons up to $\Xi$, such that all possible direct Urca processes given in Ref. [23] need to be included, as listed in Table [11]. For the quark direct

| $\Lambda^0p$ | $\Sigma^-$ | $\Sigma^-$ | $\Sigma^-$ | $\Xi^-$ | $\Xi^-$ | $\Xi^-$ | $\Xi^-$ | $\Xi^-$ |
|------------|-----------|-----------|-----------|--------|--------|--------|--------|--------|
| 1          | 0.0394    | 0.0125    | 0.2055    | 0.6052 | 0.0175 | 0.0282 | 0.0564 | 0.2218 |

Urca emissivities, we refer to Ref. [23]:

$$q_{\nu\text{udl}} \approx 4.773 \times 10^{18} \frac{\hbar^2 c^2[(k_u + k_l)^2 - k_d^2]\mu_d T_0^6}{1 \text{ MeV}^3} \text{ erg/cm}^3 \text{s},$$

$$q_{\nu\text{usl}} \approx 2.552 \times 10^{17} \frac{\hbar^2 c^2[(k_u + k_l)^2 - k_s^2]\mu_s T_0^6}{1 \text{ MeV}^3} \text{ erg/cm}^3 \text{s},$$

where the efficiency of the strange quark channel is reduced because it changes strangeness. Since the direct Urca channels are open, it is not necessary to consider other neutrino-emission processes.

In the present model, the thermal conductivity $\kappa$ cannot be easily obtained since it strongly depends on the interactions between the particles in the complicated hadronic phase. For the mixed case, the interface between the phases may introduce even more uncertainty. Therefore, the thermal relaxation period inside the star is not considered here and the calculation simply starts when the star has become isothermal, as indicated in the introduction. Because of this simplification the interesting early stage behavior of the cooling cannot be discussed. This disadvantage may also be one of the reasons why such hadronic models are not extensively studied. However, the typical thermal relaxation time is usually less than a century and most of the available observational data is of the later stages, i.e., after the star has become isothermal, such that the results obtained after the relaxation period has ended are still useful. Consequently, the star will be treated as an isothermal object with a constant temperature $T$ and the equations are rather simplified. Instead of two coupled partial differential equations in Eqs. (22) and (23), we have now only one ordinary differential equation:

$$C_V \frac{dT}{dt} = -Q_\nu - L_{\infty},$$

where the capital letters represent the integrated parameters over the volume of the star, namely

$$C_V = 4\pi R \sum_i \int_0^R \tilde{c}_V(r) e^{\lambda(r)} r^2 e^{\Phi(r)} dr,$$

$$Q_\nu = 4\pi \tilde{T} r \sum_i \int_0^R \tilde{q}_\nu(r) e^{\lambda(r)} r^2 e^{(s-2)\Phi(r)} dr,$$

where the $T$-dependent parts are taken outside of the radial integral and the temperature-independent prefactors of $\tilde{c}_V$ and $\tilde{q}_\nu$ are denoted with a bar. The factors of $e^{\lambda(r)}$ in the denominators are a consequence of the construction of the isothermal temperature $T$. Note that
the isothermal temperature does not take into account the temperature decrease near the surface of the star, which occurs in a very thin layer close to the surface.

The total heat capacity and the neutrino emissivities for both the direct and modified Urca processes as a function of the radial coordinate are shown in Figs. 9 and 10 for the case of a maximum mass neutron star with the hadronic and mixed equations of state, respectively. Here “total” means summed over all particles which contribute, but not integrated over the star volume. The quantities were obtained at $T = 10^9$ K, but the values at different temperatures can be obtained directly from the temperature dependence of $c_V$ and $q_\nu$. The discontinuities in the emissivity are due to the step functions in the expressions for the emissivities. These step functions are a consequence of the low-temperature approximation, which does not take into account any further momentum fluctuations around the Fermi surfaces. Also $q_{M\text{Urca}}$ is calculated only for the nucleon processes shown in Eq. 3, with the following emissivity

$$q_{\nu Mn} \approx 1.882 \times 10^{19} \frac{m_n^3 m_p^3 h^2 2k_p k_l}{(1 \text{ MeV})\mu_m m_n m_p} T_9^8 \text{ erg/cm}^3 \text{s},$$

$$q_{\nu Mp} \approx \left(\frac{m_p^3}{m_n^3}\right)^2 \frac{(3k_p + k_l - k_n)^2}{8k_p k_l} \Theta_{Mp} q_{\nu Mn},$$

(31)

where the subscript $Mn$ or $Mp$ means the process is modified by a bystander neutron $n$ or proton $p$, and $\Theta_{Mp}$ is 1 only if $3k_p + k_l > k_n$ as required by momentum conservation in the $Mp$ process. The modified Urca processes are only calculated when the nucleon direct Urca process is forbidden, since the formulae are only valid in this case. The small region of overlap where both $q_{D\text{Urca}}$ and $q_{M\text{Urca}}$ are nonzero is due to the different thresholds for the direct Urca processes with electrons $e^-$ and muons $\mu^-$.

III. NUMERICAL RESULT

In this section the three different equations of state for the hadronic matter, the mixed phase of hadronic and quark matter, and nuclear matter are compared. Since the maximum masses are not very large for the last two cases, we concentrate on the star with the maximum allowable mass for each equation of state. The initial temperature is always set as $T = 10^9$ K at $t = 0$, which corresponds to the time at which the isothermal condition is reached. As mentioned previously, the thermal relaxation period is not included in our calculation, which is about several decades according to calculations with nuclear matter [3]. The results are plotted in a logarithmic time scale, such that our results can be shown directly with most of the observational data even though the data include the thermal relaxation era of the neutron stars, that is negligible compared to the age of the stars. However, this makes it hard to draw a comparison with the recent observation of the cooling of the Cassiopeia A neutron star [23], which provides direct evidence for the cooling process. This neutron star is too young and is believed to have become isothermal quite recently, such that the thermal relaxation period is not negligible.

In order to avoid confusion, our results always show cooling curves starting at $t = 1$ yr, when the result has become not very sensitive to the initial conditions. In fact, the effect of the initial temperature is limited to the very beginning. The higher the initial temperature is, the faster the dependence dies away, which can be seen clearly from the asymptotic temperature dependence of the neutrino-emission dominated era

$$\tilde{T} = \left\{\frac{2}{C_0} Q_0 t + \tilde{T}(0)\right\}^{\frac{1}{2}} ,$$

(32)

where $C_0$ and $Q_0$ are the constant coefficients of the $\tilde{T}$-
dependent factors in Eqs. (29) and (30). This asymptotic solution can be easily obtained by dropping the photon radiation term in Eq. (28) and keeping only the leading neutrino-emission process with the smallest exponent $s$. Note that the exponent $s$ is never smaller than 6 for any of the neutrino-emission processes. It is quite clear that the late-time behavior of the neutrino-emission dominated era is thus completely determined by the coefficients of the heat capacity and the neutrino emissivity and not by the initial condition $\tilde{T}(0)$.

For the hadronic equation of state, Fig. 11 shows the evolution of the temperatures $\tilde{T}$ and $T_\infty$, where it can be seen that the cooling process can be divided into two stages, namely the neutrino-emission dominated and photo-radiation dominated era. The two eras can be clearly seen from the energy loss due to the different processes as shown in Fig. 12 where the switch from neutrino emission to photon radiation occurs at $t \approx 3 \times 10^4$ yr. In Fig. 12 the energy emission due to the modified Urca processes is also shown demonstratively by considering only the processes involving the nucleons in Eq. (3). This, of course, underestimates the energy loss due to the modified Urca processes, however, the magnitude is expected to be of the same order. The energy loss by these processes is seen to be less than $10^{-9}$ of the total energy loss, such that the modified Urca processes are always negligible in the present calculation. In Figs. 11 and 12 the $T_\infty$-$T$ relation is used with $\eta = 10^{-10}$. It is found that different values of $\eta$ only slightly change the cooling process, as shown in Fig. 13 where two groups of cooling curves with extremely large and almost vanishing $\eta$ are compared. The difference is that, for the neutrino-emission dominated era, $T_\infty$ can differ by about a factor of 2 but the inner temperature is almost not influenced. With less massive elements in the envelop, i.e., smaller $\eta$, the turning point into a photon-radiation dominated era is a little sharper and the final temperatures are a little lower. In any case, these effects are not very significant and can hardly be distinguished with the present accuracy of observational data. Without loss of generality, the value $\eta = 10^{-10}$ will be used in this section. Here we should also point out that the calculation cannot be carried out after $\tilde{T}$ is smaller than $10^4$ K, because then the $T_\infty$-$T$ relation given in Ref. [22] is no longer valid.

The thermal evolution of the neutron star with the mixed equation of state is shown in Fig. 14 which is similar to the purely hadronic case. The straight lines represent the asymptotic behavior, which are solved using Eq. (28) by neglecting $Q_\nu$ or $Q_\gamma$ for the neutrino-emission dominated or photon-radiation dominated era, respectively. For example, in the neutrino-emission dominated era, for the hadronic equation of state the temperatures scale with time as $\tilde{T} \sim 1.6 \times 10^9 (t/\text{yr})^{-1/4}$K, $T_\infty \sim 1.1 \times 10^7 (t/\text{yr})^{-0.602/4}$K, where the power 0.602 comes from the asymptotic approximation for the $T_\infty$-$T$ relation at low temperature, $T_\infty \propto T^{0.602}$. For the mixed phase, the temperature scalings are just slightly higher: $\tilde{T} \sim 1.77 \times 10^9 (t/\text{yr})^{-1/4}$K, $T_\infty \sim 1.13 \times 10^7 (t/\text{yr})^{-0.602/4}$K. This is due to the fact that the thermal parameters, after being integrated over the whole star, are not quite different for the two cases even though the stars have different structure and mass. For example, the hadronic case we have $C_V = 3.331 \times 10^{39} T_9$ erg/K and $Q_\nu = 1.227 \times 10^{47} (T_9)^6$ erg/s, while for the mixed phase $C_V = 2.839 \times 10^{39} T_9$ erg/K and $Q_\nu = 7.280 \times 10^{46} (T_9)^6$ erg/s. To demonstrate the effect of the mixed phase, the same curves are shown in Fig. 15 with a smaller bag constant of $B_c = 170$ MeV/fm$^3$, where we see that it causes a slightly higher temperature. As pointed out earlier, a smaller bag constant causes the deconfined phase of quarks to appear at lower densities, such that the volume of the mixed phase is increased. In this case we find $C_V = 2.156 \times 10^{39} T_9$ erg/K and
Fig. 13: The temperature evolution of the neutron star with the hadronic equation of state with different $T_\infty$-$T$ relation. The solid curves correspond to $\eta = 1$ while the dashed curves correspond to $\eta = 10^{-20}$. The upper group is $\tilde{T}$ and the lower group is $T_\infty$.

Fig. 14: The same as in Fig. 13 but with the mixed equation of state.

Fig. 15: The same as in Fig. 13 but with $B_c = 170$ MeV/fm$^3$.

Fig. 16: The same as in Fig. 13 but with the nuclear equation of state.

$Q_\nu = 2.590 \times 10^{46} (\tilde{T}_9)^6$ erg/s. Nevertheless, the total effect on the cooling process is not drastically changed. According to these results, we expect that even if the mixed phase is reduced because of screening and surface effects, the cooling behavior is not changed considerably.

For the nuclear equation of state, the maximum mass is $1.719 \, M_\odot$ with $R = 10.04$ km and its thermal evolution is shown in Fig. 16. To summarize, the luminosity for the three types of equations of state are plotted in Fig. 17 where a comparison with observational data is also made. In all three cases the cooling is too fast compared with the data. The star with the nuclear equation of state cools a little faster, since the direct Urca processes with hyperons in the hadronic phase and the mixed phase are not as efficient as the nucleon direct Urca process [23]. However, the difference is quite small such that it might be hard to distinguish these different equations of state simply by their thermal evolution after the isothermal condition is reached.

IV. EFFECT OF SUPERFLUIDITY

From the above results, we can see that our neutron stars cool too fast because the direct Urca process is open for all the three types of equations of state. An explanation could be that the neutrino emissivity is overestimated by neglecting the possibility of superfluidity. As pointed out in the introduction, it is generally believed that superfluidity appears inside neutron stars due to the attractive part of the nuclear force, for both the nucleons and the hyperons. Due to the presence of an energy gap $\Delta$, associated with the binding energy of the Cooper pairs, the number of excitations near the Fermi surface is suppressed by a factor of $\exp (-\Delta/k_B T)$ when the temperature is smaller than a certain critical temperature $T_c$. Therefore, pairing reduces the heat capacity and the neutrino emissivity significantly when the star cools down to temperatures below $T_c$, thus the cooling
behavior is drastically changed. In this section we will explore the effect of superfluidity to the cooling process of neutron stars.

A. Superfluidity inside neutron stars

In our present model for neutron stars, there are different fermions, namely the baryons, the leptons, and the quarks. The typical \( T_c \) of baryons is about \( 0.1 - 1 \) MeV, which is close to the initial temperature of neutron stars, such that the superfluidity of baryons is very important to the cooling process. As for the leptons, such as electrons, they could become superfluid due to interactions via phonons, however, the typical \( T_c \) is very small, namely in the order of several kelvin. Hence lepton superfluidity is expected to be unimportant to the neutron star cooling process. For quarks in the mixed phase, the gaps can be about \( 50 - 100 \) MeV \[12\], which is much higher than the typical temperatures of the stars we are concerned with. The quark contribution will thus be suppressed more strongly than the baryonic contribution. Without superfluidity the quark contributions to the heat capacity and the neutrino emissivity were shown to be of the same order as the baryonic ones, such that with superfluidity they are completely negligible at the same stellar temperature. Throughout this section we thus only consider the contributions to \( C_V \) and \( q_\nu \) of non-superfluid leptons and superfluid baryons but neglect those of the superfluid quarks.

Superfluidity not only reduces the neutrino emissivity but also opens a different channel of neutrino emission based on the breaking and formation of Cooper pairs. At temperatures not far below \( T_c \) thermal fluctuations can cause pair break-up into single-particle excitations, which subsequently reform into pairs. Neutrino emission due to such processes is quite efficient at temperatures slightly below \( T_c \) and can even surpass the direct Urca process in some cases \[27\]. However, its contribution also decreases dramatically when \( T \) becomes small due to the exponential reduction factor mentioned previously. According to a similar argument as above, we only need to consider such neutrino-emission processes involving baryon pairs.

By taking into account superfluidity, the dominant neutrino-emission process is no longer simply determined by the density, which is crucial for the threshold of the direct Urca process, but also depends on the reduced temperature \( \tau = T/T_c \). However, it is still true that the modified Urca process can be neglected as long as the direct Urca process is open, since the modified Urca process involves more particles and is more strongly suppressed due to the energy gaps of all superfluid participants. Thus we again do not consider the modified Urca process since the direct Urca process is always open in our study. According to Ref. \[27\] electron-electron bremsstrahlung becomes a dominant process when all baryons are strongly superfluid at low temperature. So for a correct estimate of the neutrino emission at low temperatures we also include lepton-bremsstrahlung processes. Based on the discussion above, in order to include the effect of superfluidity we should recalculate \( C_V \) and \( q_\nu \) for all baryons, neglect those of the quarks in the mixed phase, keep the lepton contributions unchanged, and also include the newly introduced neutrino emission due to baryon pair break-up and lepton bremsstrahlung. We next discuss these effects separately.

To describe the superfluidity of the particles, it is necessary to specify the type of pairing. At low density, the singlet-state nuclear interaction is attractive, while at higher densities it becomes repulsive. It is believed, however, that the triplet-state interaction still provides an attractive channel at high densities, such that triplet-state pairing is expected in the core of the neutron star. This transition happens around the nuclear saturation density \( n_0 = 0.16 \) fm\(^{-3}\). The proton pairing is usually taken to be in the singlet channel, even in the stellar core, due to their low concentration. However, in our model, as seen from Figs. 4 and 7 the densities of neutrons and protons in the core are quite close to each other, namely \( n_p \sim 0.2 - 0.3 \) fm\(^{-3}\) and \( n_n \sim 0.4 \) fm\(^{-3}\). It thus seems that both the protons and the neutrons should form triplet-state pairs. The hyperons, which can also become superfluid, are usually taken to pair in the singlet channel. But in our model there are some hyperons, e.g. \( \Lambda^0 \), which can have densities comparable to the neutrons and protons. As was discussed earlier, the details of the interactions between the hyperons are not well established and thus there are some ambiguities in dealing with hyperons with largely varying densities. For the singlet-state pairing of neutrons, protons \[24\] and \( \Lambda^0 \)'s \[28\] the results of many models largely agree on the order

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig17.pdf}
\caption{The luminosity of neutron stars with different equations of state, with solid, dashed, and dotted curves representing the hadronic, the mixed, and the nuclear equation of state, respectively. The observational data are from Ref. \[27\], with the numbers indicating the corresponding stars as: 1 - RX J0822-4247, 2 - 1E 1207.4-5209, 3 - RX J0002+6246, 4 - PSR 0833-45 (Vela), 5 - PSR 1706-44, 6 - PSR 0538+2817, 7 - PSR 0656+14, 8 - PSR 0633+1748 (Geminga), 9 - PSR 1055-52, 10 - RX J1856.5-3754, and 11 - RX J0720.4-3125.}
\end{figure}
of magnitude of the pairing gaps and their density dependence. As discussed, triplet-state pairing is of great interest to our model, however, there is little known about this kind of pairing except for the case of neutrons for which it is still highly model dependent. In fact, the existence of triplet-state pairing inside neutron stars is still uncertain. According to observations of the cooling neutron stars, it seems necessary to have all baryons in the superfluid state. For example, the effect of superfluidity on the thermal evolution of neutron stars with hyperons in the core has been studied by Schaab et al. [29], which only included the singlet-state pairing of $\Lambda^0$ at low densities but did not include the triplet-state pairing of $\Lambda^0$ at high densities and ignored the pairing of other hyperons. The cooling was found to be too fast for heavy stars since not all the direct Urca processes were suppressed.

To estimate the effect of superfluidity, we make the simplifying assumption that, regardless of the density, the neutrons, protons, and $\Lambda^0$'s pair in the triplet channel, the remaining hyperons pair in the singlet channel, and the critical temperatures of all the baryons are taken to be equal. Equating all critical temperatures of the various baryons is consistent with our model, since all baryons couple equally to the meson fields. Although our pairing mechanism seems quite crude, we expect that, at least qualitatively, the effect of pairing on the thermal evolution of the star will be taken into account. To discuss our simplifications, we first need to go into the details of how the specific heat and neutrino emissivity are reduced by superfluidity.

1. Reduction factor

The new expressions for $c_V$ or $q_v$ are easily obtained by multiplying the original expressions by a reduction factor $R_c$ or $R_q$. These reduction factors are functions of the reduced temperature $\tau$ and also depend on the type of superfluidity considered. A rather systematic calculation of these reduction factors has already been carried out (see review [27] and references therein). In general, the different types of triplet-state pairing can be represented by the projections $m_J = 0, \pm 1, \pm 2$ of the total angular momentum. We only present the most studied $m_J = 0$ case, which is denoted as type-B pairing. Singlet-state pairing is referred to as type A. By introducing the dimensionless variables $\tau = T/T_c$ and $v(\tau) = \Delta(T)/k_B T_c$, the properties of superfluidity can be described independent of $T_c$. Numerical fits for the energy gap are given by

$$v_A = \sqrt{1 - \tau} \left( 1.456 - \frac{0.157}{\sqrt{\tau}} + \frac{1.764}{\tau} \right),$$

$$v_B = \sqrt{1 - \tau} \left( 0.7893 + \frac{1.188}{\tau} \right),$$

and $v \equiv 0$ at $\tau > 1$. The reduction factors for $c_V$ are

$$R_{cA} = \left( 0.4186 + \sqrt{1.014 + 0.2510v_A^2} \right)^{2.5} \times e^{1.456 - \sqrt{2.120} + v_A^2},$$

$$R_{cB} = \left( 0.6893 + \sqrt{0.6241 + 0.07975v_B^2} \right)^2 \times e^{1.934 - \sqrt{3.740} + v_B^2}.$$  

(35)

Notice that $R_{cA} = 2.426$ and $R_{cB} = 2.188$ are greater than 1 at $\tau = 1$, such that the specific heat is discontinuous as the temperature falls below $T_c$. The neutrino emissivity for the direct Urca process is different if both or only one of the two involved baryons is superfluid. For the case with only one superfluid baryon

$$R_{qA} = \left( 0.2312 + \sqrt{0.5911 + 0.02068v_A^2} \right)^{2.5} \times e^{3.427 - \sqrt{11.74} + v_A^2},$$

$$R_{qB} = \left( 0.2546 + \sqrt{0.5556 + 0.01649v_B^2} \right)^5 \times e^{2.701 - \sqrt{7.295} + v_B^2}.$$  

(38)

FIG. 18: The reduction factors $R_c$ (solid curves) and $R_q$ (dashed curves) for different types of superfluidity, where type-A (thick curves) is always lower than type-B (thin curves) pairing. Notice the jump of $R_c$ at $\tau = 1$. The dotted curve represents $R_{cB}^1$ used for the direct Urca process where both baryons are superfluid and have the same $T_c$. Also $R_{q_A}$ (dot-dashed curve) and $R_{q_A}^2$ (thick dot-dashed curve) are shown for comparison.
distinguished by the type of pairing (A or B) of the neutrons and protons, labeled as AA, BA or BB, where the first letter signifies the type of pairing for the neutrons and the second the pairing type for the protons. We do not present their complicated expressions here. The most important factor we need is for the BB case, which as far as we know has not been explicitly calculated yet. Therefore, we use \( R_{BB}(\tau_1)R_{BB}(\tau_2) \) instead of \( R_{BB}(\tau_1, \tau_2) \) for those direct Urca processes with both baryons paired in the triplet-state, which we refer to as the standard setting. Furthermore, we even use this expression for all the direct Urca processes with two superfluid baryons, independent of the pairing type. Therefore, the effect of superfluidity on the emissivity for the BB-type process is expected to be overestimated \([27]\). On the other hand, the cooling process is the order of magnitude of each expected to be overestimated \([31]\). On the other hand, the lepton-bremsstrahlung processes are expected to be dominant, such that in this regime any ambiguity due to the type of pairing is expected to be unimportant.

2. New neutrino emission processes due to superfluidity

The neutrino emissivity associated with Cooper pair break-up is given by \([31]\)

\[
q_{\nu b}^C = 1.17 \times 10^{21} N_\nu \frac{m_n^4 \hbar k_b}{m_n^2 c^2} a_b F(v) T_9^7 \text{erg/cm}^3\text{s},
\]  

where \( N_\nu = 3 \) is the number of neutrino flavors, \( a_b \) is a numerical factor from the electroweak neutral currents and depends on the quark composition of the baryon and the pairing type. As far as we know, no calculation has been carried out which included the triplet-state pairing of \( \Lambda^0 \). We simply use \( a_{\Lambda} = a_n \), considering that the quark composition of \( \Lambda^0 \) is similar to the neutron with one \( d \) quark replaced by one \( s \) quark, while their contributions to the neutral current are the same. This treatment is different from the approximation in Ref. \([32]\), where the contribution from quarks other than \( u \) and \( d \) is neglected. This may cause some uncertainties, but what matters in the cooling process is the order of magnitude of each process. Since \( a_b \) appears as a multiplier rather than an exponent in Eq. \([33]\), such an inaccuracy will not be magnified during the calculation. In fact, we will see that due to the assumption of a uniform \( T_c \), the neutrino emission associated with Cooper pair break-up is always negligible compared to the direct Urca process even if the suppression due to superfluidity is included. The factor \( a_b \) for the various baryons used in the calculation is shown in Table \([\text{III}]\).

The function \( F(v) \) plays the same role as the reduction factor and is fitted in Ref. \([31]\) to:

\[
F_A = (0.602 v_A^2 + 0.5942 v_A^2 + 0.288 v_A^2) \left( 0.5547 + \frac{0.1983 + 0.0113 v_A^2}{\sqrt{c^2 - 0.54 + 4v_A^2}} \right)^{1/2},
\]

\[
F_B = \frac{1.204 v_B^2 + 3.733 v_B^2 + 0.3191 v_B^2}{1 + 0.3511 v_B^2} \left( 0.7591 + \frac{0.05803 + 0.3145 v_B^2}{1 + 0.2131 + 4v_B^2} \right)^2 e^{-0.4616 - \frac{0.2131 + 4v_B^2}{v_B^2}}.
\]

Furthermore, the neutrino emissivity due to the lepton-bremsstrahlung process \( ee \) is given by \([33]\)

\[
q_{\nu}^{ee} = 2.089 \times 10^{11} \frac{\hbar c k_e}{1\text{MeV} y_s} T_9^8 \text{erg/cm}^3\text{s},
\]

where \( y_s \) is a dimensionless parameter representing the effect of screening in the plasma, \( y_s = k_{sc}/2k_e \), with \( k_{sc} \) the screening wavenumber. This wavenumber is obtained for the case of static screening in the limit of zero temperature \([34]\) by the Thomas-Fermi expression:

\[
k_{sc}^2 = \frac{4\pi c^2}{\hbar^2} \left( \sum_{\nu} \frac{m_{\nu}}{k_\nu^2} + \sum_{b} \frac{m_b^* k_b Z_b}{k_b} \right),
\]

where the summation over \( b \) is over all charged baryons, and \( Z_b \) represents the effect of superfluidity on the baryons. Since lepton bremsstrahlung only becomes important if all the baryons are highly superfluid, for which \( Z_b \) becomes negligibly small compared to the lepton terms, we can simply omit the terms related to the baryons. Subsequently, \( y_s \) is simplified to

\[
y_s = \sqrt{\frac{\alpha \mu \hbar Q \sum_{\nu} \frac{k_\nu}{\hbar c k_e^2}}{\pi e k_e}},
\]

where \( \alpha \) is the fine-structure constant. Other processes involving non-relativistic muons are calculated in Ref. \([35]\). In our model the muons are relativistic, so we adopt the similarity criterion \([3]\) and get

\[
q_{\mu}^{\nu} = \frac{k_\nu}{k_e} q_{\nu}^{ee}, \quad q_{\nu}^{\mu} = \frac{k_\mu}{k_e} q_{\mu}^{ee}.
\]
relevant parameter for the thermal evolution is
\[ \tau \]
equal agrees qualitatively with the result in Ref. [27] for the magnitude smaller than for the direct Urca process. This with Cooper pair break-up is always about 5 orders of from the maximum mass star for each type of equation of state. Again, we only present the results for very high densities. Therefore, we can still use the effect of the energy gap at the Fermi surface is very small all states in momentum space and it is expected that the change very little. In fact, we can always neglect the contribution to the neutrino emissivity associated with Cooper pair break-up.

Finally, it should be pointed out that superfluidity can also affect the equation of state. However, the equation of state used in the TOV equations is an integral over all states in momentum space and it is expected that the effect of the energy gap at the Fermi surface is very small for very high densities. Therefore, we can still use the stellar structure obtained from the previous unaffected equation of state. Again, we only present the results from the maximum mass star for each type of equation of state.

B. Numerical results

To summarize, in Fig. 19 we present the neutrino emissivities of the discussed processes as a function of reduced temperature \( \tau \) after the inclusion of superfluidity. Due to our assumption of equal \( T_c \), the emissivity associated with Cooper pair break-up is always about 5 orders of magnitude smaller than for the direct Urca process. This agrees qualitatively with the result in Ref. [27] for the equal \( T_c \) case of nuclear matter. For the neutron star with the mixed equation of state, the situation is similar since the leading contribution of triplet-state paired baryons changes very little. In fact, we can always neglect the contribution to the neutrino emissivity associated with Cooper pair break-up.

To simulate the thermal evolution of the star is similar to the case without superfluidity, but now the coefficients \( C_V \) and \( Q_\nu \) in Eqs. (29) and (30) include the temperature-dependent reduction factors. Note that the relevant parameter for the thermal evolution is \( \tilde{T} \) rather than \( T \), such that the reduction factors are functions of \( \tau(r) = \tilde{T} e^{-\Phi(r)}/T_c \), which is not constant throughout the star if we use a constant \( T_c \). This radial dependence should be included in the integration of Eqs. (29) and (30). Since the \( \tau \) dependence of the reduction factor is nonlinear, the variables \( \tilde{T} \) and \( r \) cannot be separated, which means that the numerical integration must be carried out at each instance to solve the differential equation, cf. Eq. (28). To circumvent this we introduce an additional approximation by taking \( T_c e^{\Phi(r)} \) a constant, such that the reduced temperature \( \tau \) is constant inside the star. This decouples \( \tau \) and \( r \) in Eqs. (29) and (30), which means we do the radial integration only once. Then Eq. (28) is independent of the radial coordinate and the reduction effect is represented by several extra \( \tau \)-dependent terms in the coefficients. As is well known, the critical temperature \( T_c \) is a function of the density, however, the dependence is often uncertain. Because of the discrepancies between the pairing models, it is even unclear whether \( T_c \) for each type of baryon increases or decreases with baryon density in the range of interest. In fact, setting \( T_c = \text{const} \) throughout the star also implies some kind of density dependence because the baryon density changes with the radius. In other words, as an ansatz, \( T_c e^{\Phi(r)} = \text{const} \) is as reasonable as \( T_c = \text{const} \), even though the former seems unnatural because of its dependence on the macroscopic stellar structure. We use the former ansatz first because of its simplicity. Later we make a comparison between these two ansätze. In any case, \( e^{-\Phi(r)} \) only varies smoothly and monotonically around 2, for example, from about 2.2 at \( r = 0 \) to about 1.4 at \( r = R \) for the maximum mass neutron star with the hadronic equation of state, so we expect the difference between these two ansätze will not be of several orders in magnitude.

In the numerical calculation with the first ansatz, we set \( T_c e^{\Phi(r)} = \tilde{T}(0) = 10^9 \text{ K} \) such that the baryons are superfluid from the start. This implies we have \( T_c \approx 2.2 \times 10^9 \text{ K} \) at the central density while \( T_c \approx 1.4 \times 10^9 \text{ K} \) near the surface. If, however, \( \tilde{T}(0) \) were to be set higher, the reduction factors would at first not take effect and the direct Urca process would thus quickly reduce \( \tilde{T} \) down to \( T_c \) after which it would be suppressed due to the onset of superfluidity and the cooling slowed down. This process takes only several seconds. On the other hand, if \( \tilde{T}(0) \) would have been slightly lower, for example, above 0.1\( T_c \), then in the beginning the superfluid suppression would not be very strong and the direct Urca process could still cool the star to low temperature, i.e., the highly superfluid regime, within one year. Since the isothermal assumption we used here is virtually a later stage behavior, given a reasonable \( T_c \) and the well-accepted range of \( \tilde{T}(0) \) of neutron stars, the result we presented in this section depends little on the accurate values of the temperature. As before, we take the neutron star with the hadronic equation of state as the basic example, then we study the stars with the mixed and nuclear equation of state.

![Graph showing neutrino emissivities of various processes](image)
We find several general properties of the cooling behavior of stars containing superfluid matter from the numerical results. One obvious difference with the non-superfluid case is the shape of the cooling curve. Since the neutrino-emission processes, which dominate the early stage cooling, are strongly suppressed with decreasing temperature, the temperature stays high for a longer time and the shift to the photon radiation era takes place at higher temperatures. In contrast to the large slope in the non-superfluid case, we find a rather slowly decreasing plateau around $T \sim 10^8$ K and $T_\infty \sim 10^9$ K, followed by a much sharper turn to the photon radiation stage as can be seen in the following figures. Another difference can be seen in that the parameter $\eta$ in the $T_\infty - T$ relation plays a more significant role. Because the neutrino-emission processes are suppressed due to superfluidity, photon radiation becomes more important at higher temperatures and the dependence on how the inner temperature is screened by the surface layer becomes more visible. We present the luminosity curves with different $\eta$ in Fig. 20 where we see that the observational data favor moderate values of $\eta$. In the following, we usually use $\eta = 10^{-13}$.

Here some other general properties are summarized. First, we find that the heat capacity $c_V$ of the baryons is quickly reduced as a consequence of superfluidity. Starting from $t \approx 1$ yr, the heat capacity is completely dominated by the lepton contribution, such that the pairing type of the baryons is seen to be of no importance to the heat capacity. Second, as was already shown in Fig. 19 neutrino emission via Cooper pair break-up is always negligible compared to the direct Urca process.

We also find that the role of lepton bremsstrahlung in the cooling process is negligible since in the later stages when it dominates the neutrino emission, photon radiation has already become the dominant cooling effect. Therefore, the cooling process is determined only by the direct Urca process and photon radiation. The various energy loss rates are shown in Fig. 21. Third, the direct Urca process depends on the type of pairing. However, this dependence is quite moderate. We show the different cooling curves for the various pairing types in Fig. 22 for singlet-state pairing, we see that neutrino emission is more strongly suppressed and the resulting temperature is slightly higher than the other types of pairing. This result confirms that the cooling behavior is not strongly dependent on the type of pairing of the baryons.
By comparing Fig. 20 and Fig. 17, we see that the luminosity curves in the superfluid case are much closer to observations than in the non-superfluid case. However, the curves do not agree with all the observational data except for the younger stars, namely number 1-5. This mismatch may lie in some important mechanisms which are neglected in the present calculation. For example, the magnetic field of the neutron star can affect the photon radiation at the surface such that it no longer obeys the standard black-body radiation law and shift the photon-radiation era to a later time. This reduction of photon radiation also decreases the luminosity, which is unfavorable according to the cooling curves shown in Fig. 20. However, there are some known heating mechanisms which are not considered here, for example, due to magnetic field energy and rotational energy (for more details, see Ref. [12] and references therein), such that the temperature and the luminosity can remain higher. Besides, note that sources such as number 10 and 11 may be old magnetars. As a consequence, the cooling history may be anomalous. Both are a result of the decay of the strong magnetic field: age estimates assume magnetic braking of rotation with a constant magnetic field, whereas the decay of the magnetic field results in heating of the neutron star. We discuss these issues in future work.

Now we discuss the second ansatz, $T_e = \text{const}$. Since $e^{-\Phi(r)} > 1$, if we set $T_e = \tilde{T}(0) = 10^9$ K we will have a lower $T_e$ than in the previous ansatz and the baryons will not be superfluid everywhere inside the star at $t = 0$. To make a proper comparison between the two, we set $T_e = 1.9 \times 10^9$ K, where the extra factor 1.9 is the middle value of $e^{-\Phi(r)}$ inside the star (the outer value is taken at $r = 7$ km rather than at the surface since the direct Urca process starts around here, see Figs. 9 and 10). With this value of $T_e$, at $t = 0$, the baryons near the surface are superfluid while those in the core are not, since $\tau = \tilde{T}e^{-\Phi(r)}/T_e$ is bigger in the core. Nevertheless we expect the integrated results will be comparable to those of the previous ansatz. As a comparison, in Fig. 23, $C_V$ and $Q_\nu$ are shown for these two ansätze. We see that the difference between the two increases with decreasing temperature and only becomes significant when their values are very small. Therefore, it is not surprising to find that their cooling curves are similar, as is shown in Fig. 24. The two curves are quite close except that the second ansatz has a little lower luminosity, because with $T_e = \text{const}$ the higher temperature in the core delays the suppression of the direct Urca process, such that the total cooling rate is larger in the beginning. Furthermore, the result of the ansatz $T_e = 10^9$ K mentioned above is also shown, where we can see that the high cooling rate due to the direct Urca process remains for a longer time and the luminosity is even more reduced after the neutron star has become completely superfluid. Here we see again that different initial conditions have little influence on the later-stage behavior.

Finally, we briefly present the results for the neutron stars with the mixed and nuclear equation of state. For the nuclear case, we always take the neutrons and protons to pair in the triplet channel. The general arguments still apply, i.e., we only need to consider the direct Urca process and the photon radiation as the dominant cooling mechanisms. Without too much difference, the simple ansatz $T_e e^{\Phi(r)} = \text{const}$ will be used. The cooling behavior is mainly determined by the heat capacity, the direct Urca neutrino emissivity, and the photon radiation. These three parameters are not quite different among the various equations of state, even including the
than baryons, but their contribution to the heat capacity is relatively stronger. This is because, without superfluidity, the luminosity of the nuclear case in Fig. 25 is quite helpful in order to remove the uncertainties concerning baryon superfluidity at high density. Note that the cooling process is almost completely determined by the direct Urca process and photon radiation even after including the effects due to superfluidity. The cooling curves of the neutron stars with the three equations of state are still quite similar when superfluidity is included. However, the behavior in the early stages before the stars become isothermal could be significantly different, but such a study requires the knowledge of the thermal conductivity of these complex systems. The geometrical structure of the mixed phase is also expected to play an important role, although no decisive conclusion has yet been drawn.

As we have seen, the fast cooling in the non-superfluid case did not agree with observations. In order to remedy this discrepancy, superfluidity was introduced, which significantly reduces the efficiency of the direct Urca process as well as the heat capacity. The resulting cooling curve is much closer to the observational data. We also found that the particular pairing type of the superfluid baryons is not very important to the thermal evolution of the star. The thermal evolution after the star becomes isothermal is not strongly dependent on the initial temperature of the star or on the critical temperature related to superfluidity, as long as they are within reasonable ranges. The robustness of the results with superfluidity is quite helpful in order to remove the uncertainties concerning baryon superfluidity at high density. Note that the cooling process is almost completely determined by the direct Urca process and photon radiation even after including the effects due to superfluidity. The cooling curves of the neutron stars with the three equations of state are still quite similar when superfluidity is included. We expect that a calculation including the magnetic field and rotation gives an even better agreement with the observational data. Besides, since our nuclear model is geared originally towards nuclear matter near the saturation point, a further improvement of it may play an important role in getting a better agreement with the observations. By incorporating the properties at higher densities, e.g., the coupling constants for hyperons, it is possible to get cooling curves covering most of the observational data, as in Ref. [12].

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