Riemann $\zeta(3)$-terms in perturbative QED series, conformal symmetry and the analogies with structures of multiloop effects in $N = 4$ supersymmetric Yang-Mills theory

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Abstract

As was discovered recently, the 5-loop perturbative quenched QED approximation to the QED $\beta$-function consist from the rational term and the term proportional to $\zeta(3)$-function. It is stressed, that this feature is also manifesting itself in the conformal invariant pqQED series for the 4-loop approximation to the anomalous mass dimension. The 4-loop pqQED expression for the singlet contribution into the Ellis-Jaffe polarized sum rule is obtained. It coincides with the similar approximation for the non-singlet coefficient functions of the Ellis-Jaffe sum rule and of the Bjorken polarized sum rule. It is stressed that this property is valid in all orders of perturbation theory thanks to the conformal symmetry of pqQED series and to the Crewther relation, which relates non-singlet and singlet coefficient functions of the Ellis-Jaffe sum rule with the coefficient functions of the non-singlet and singlet Adler D-functions. The basic steps of derivation of the Crewther relation in the singlet channel from the AVV triangle diagram are outlined. The similarities between analytical structures of asymptotic series for the coefficient functions in pqQED and for the anomalous dimensions in $N = 4$ conformal invariant supersymmetric Yang-Mills theory are observed. The guess is proposed that the appearance of $\zeta(3)$-terms in the pqQED expressions and the absence of $\zeta(5)$-terms at the same level is the indication of absence of "wrapping" interactions in pqQED.

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1. Introduction

Among consequences of the recent advanced analytical QCD calculations of the 5-loop perturbative corrections to the $e^+e^-$-annihilation Adler $D$-function \cite{1,2,3} and to the Bjorken sum rule of polarized lepton-hadron deep-inelastic scattering \cite{3} is the single-fermion 5-loop contribution to the QED renormalization-group $\beta$-function \cite{1,2,3}.

Single-fermion loop approximation of perturbative QED is sometimes associated with the term "quenched QED" which, rigorously speaking, is commonly accepted in non-perturbative QED studies (for the recent considerations see e.g. \cite{4,5}). In the case of "perturbative quenching" the related Feynman diagrams with internal photon vacuum polarization diagrams are not considered and therefore the coupling constant of QED is renormalized only by the subset of vacuum polarization subgraphs with one external fermion loop and $0 \leq n \leq \infty$ internal photon lines. It can be shown, that within this approximation the QED vacuum polarization function and the related contribution to the QED $\beta$-function do not depend from the choice of renormalization scheme \cite{6}.

It should be stressed that massless "perturbative quenched QED" (pqQED) obey the important property of conformal symmetry. Indeed, within this limit the renormalized QED coupling constant stay fixed and is not running. Conformal symmetry allows to connect multiloop expression for the massless pqQED part of the $e^+e^-$ annihilation Adler $D$-function with the pqQED contribution to the Bjorken polarized sum rule \cite{7}. The details of this statement are explained in Ref. \cite{8}, where the way how to check the appearance of $\zeta(3)$ in the 5-loop result of Ref.\cite{1} by additional 5-loop perturbative QCD calculations was outlined. This way was followed in Ref. \cite{3}, where not only the 5-loop pqQED contribution to the Bjorken polarized sum rule, derived by the back-of-envelope calculations of Ref.\cite{8}, were reproduced, but the explicit 5-loop form of the $\beta$-function factorizable generalization of the Crewther relation in QCD was obtained as well \cite{3}. This relation was discovered previously at the 4-loop level in Ref.\cite{9}. Its all-order validity was proved later in Ref. \cite{10}. Note, that in Ref. \cite{11} new type of the $\beta$-function factorizable QCD generalization of the Crewther relation was proposed. Its more detailed study is in progress.

Study of the origin of the appearance of the "puzzling $\zeta(3)$-term" in the 5-loop correction of the pqQED $\beta$-function is an interesting theoretical problem. Contributions of Riemann $\zeta(3)$-functions were already detected at the intermediate stages of the diagram-by-diagram calculations during the 3-loop calculations, which were performed in the works of Refs. \cite{12,13,14}. But after summing up all corresponding 3-loop graphs the terms, proportional to $\zeta(3)$, cancelled in the ultimate result. The details of these cancellations were followed up in Ref. \cite{13} and in Ref.\cite{15} later on. In the work of Ref. \cite{13} the explicit diagrammatic analysis was made, while the analysis of Ref.\cite{15} was performed within the knot-theory formalism. In Ref. \cite{16} the guess was formulated that the rationality of the 3-loop level expression is related to really existing property of the conformal symmetry of the pqQED series. Using this guess and neglecting integrals, which generate $\zeta(3)$-terms at the 3-loop level, the authors of Ref.\cite{16} reproduced the original result of Ref.\cite{12}.

Analytical calculations of the 4-loop QED $\beta$-function in the class of modified minimal subtractions renormalization scheme (and of the $\overline{\text{MS}}$-scheme among them) were finally completed in Ref.\cite{17}. In these calculations in addition to $\zeta(3)$-term, the terms proportional to $\zeta(5)$ appeared as well, but both types of transcendental functions disappeared in the expression for the 4-loop correction to the $\beta$-function in the pqQED approximation. The absence of the $\zeta$-functions at the 4-loop level were confirmed in Ref.\cite{18} using the Schwinger-Dyson equations, where the rational 4-loop expression of $O(A^4)$ contribution to the pqQED $\beta$-function, obtained in Ref.\cite{17}, was reproduced as well. The background field method
calculations of Ref. [19] clarified the origin of rationality of this order $A^4$ term.

In view of the 3 and 4-loop cancellations of the transcendental Riemann functions at the
3- and 4-loop levels, the explicit manifestation of the $\zeta(3)$-function at the 5-loop level was
considered as a puzzle. In this paper we will show that this is not the puzzle at all, but
the regular feature, which is consistent with the structure of perturbative series in another
conformal invariant model, namely $N = 4$ supersymmetric (SUSY) Yang-Mills theory.

2. Manifestations of $\zeta(3)$ functions in perturbative quenched QED series and
its cancellation in the 4-loop expression for Ellis-Jaffe sum rule.

It is important to have a look, how in perturbative quenched QED series
$\zeta(3)$-terms are manifesting themselves. Consider first the original result of Re fs.[1],[3] for the 5-loop
expression of the pqQED $\beta$-function, namely

$$
\beta_{\text{QED}}^{[1]}(A) = \frac{4}{3} A + 4A^2 - 2A^3 - 46A^4 + \left(\frac{4157}{6} + 128\zeta(3)\right)A^5 + O(A^6) \\
= \frac{4}{3} A \times C_D^{\text{ns}}(A) .
$$

The coefficient function is defined from the QCD expression for the
non-singlet contribution to the $e^+ e^-\text{-annihilation}$ Adler D-function

$$
D^{\text{ns}}(A_s) = 3 \sum_F Q_F^2 C_D^{\text{ns}}(A_s) .
$$

The QED and QCD perturbation-theory expansion parameters are normalized as $A = \alpha/(4\pi)$ and $A_s = \alpha_s/(4\pi)$ with $\alpha$ and $\alpha_s$ being the renormalized QED and QCD coupling
constants.

It is interesting to have a look whether in perturbative quenched QED there are any
other renormalization group function for the gauge-invariant operators, which contain $\zeta(3)$-
function in high order corrections.

Consider first perturbative series for the anomalous mass dimension in pqQED. Its
expression differs from the anomalous dimension of the operator $\overline{\Psi} \Psi$ by overall sign only,
and therefore, for the reason of rigour it is better not to introduce mass term in the QED
lagrangian, and consider massless conformal invariant limit of the QED series for the
anomalous dimension function $\gamma_{\overline{\Psi} \Psi}(A) = -\gamma_m(A)$. Its expression can be obtained from the 4-loop
QCD calculations of the mass anomalous dimension function $\gamma_m(\alpha_s)$, performed in Ref.[20]
and in Ref.[21] independently. It is more convenient to use the results of [20], since this work
contains the explicit dependence of the 4-loop expression for $\gamma_m(\alpha_s)$ from Casimir operators
$C_F, C_A$, normalization factor $T_F$ and the number of quarks flavours $N_F$. The choice $C_F = 1$,
$C_A = 0$, $T_F = 1$ and $N_F = 0$ corresponds to the case of pqQED approximation. The pqQED
expression for the anomalous dimension of the gauge-invariant operator $\overline{\Psi} \Psi$ has the following
form

$$
\gamma_{\overline{\Psi} \Psi}^{\text{pqQED}}(A) = -3A - \frac{3}{2} A^2 - \frac{129}{2} A^3 + \left(\frac{1261}{8} + 336\zeta(3)\right)A^4 + O(A^5)
$$

The analytical structure of this series was already investigated in Ref.[18] using the
Shwinger-Dyson approach. In view of the appearance of $\zeta(3)$-term in the pqQED part
for the QED $\beta$-function (see Eq.(1)), it is worth to attract more attention to the appearance
of $\zeta(3)$-term in the 4-loop correction in Eq.(4). Moreover, the 4-loop manifestation of $\zeta(3)$-
term in the conformal invariant expression of Eq.(4) indicate that the similar feature may
manifest itself in other pqQED series as well. The anticipating its manifestation cancellations of $\zeta(3)$-terms at the intermediate stages of lower order calculation should also hold in the series of Eq. (11). This statement is the consequence of the experience gained in the process of evaluation of 3-loop counter-terms in QCD during the 4-loop calculations, which result in the publications of the works of Refs. [17], [22], [23].

Note, that the expression for Eq. (11) follows from the calculations of the renormalization group function of "vertex operator". In the case of calculations of renormalization-group quantities, related to two-point functions, $\zeta(3)$-term is appearing one loop later, namely at the 5-loop order (see Eq. (1)). It enters the expressions for the non-singlet coefficient functions of the 5-loop $O(A^4)$-corrections to the $e^+e^-$-annihilation Adler $D$-function and the Bjorken polarized deep-inelastic scattering sum rule, defined in QCD as

$$ Bj_{jp}(Q^2) = \int_0^1 \left[ g_1^{lp}(x, Q^2) - g_1^{ln}(x, Q^2) \right] dx = \frac{1}{6} g_\mu C_{B_{jp}}^{ns}(A_s(Q^2)) \ . $$

Indeed, comparing Eq. (1) with Eq. (2), one can get:

$$ C_{ns}^{D} = 1 + 3A - \frac{3}{2} A^2 - \frac{69}{2} A^3 + \left( \frac{4157}{8} + 96\zeta(3) \right) A^4 + O(A^5) \ . $$

The similar 5-loop expression for the coefficient function of the Bjorken sum rule, given in Ref. [8] and confirmed by diagram-by-diagram calculations in Ref. [3], reads:

$$ C_{B_{jp}}^{ns} = 1 - 3A + \frac{21}{2} A^2 - \frac{3}{2} A^3 - \left( \frac{4823}{8} + 96\zeta(3) \right) A^4 + O(A^5) \ . $$

These quantities do not contain anomalous dimension terms.

The logic of the discussions presented above leads to the conclusion that in the pqQED series for the quantities, which are related with the non-zero anomalous renormalization constant, $\zeta(3)$-should cancel down 1-loop prior their manifestation in Eq. (6) and Eq. (7), namely on the level of $O(A^3)$-corrections.

To verify this statement consider now the Ellis-Jaffe sum rule of the deep-inelastic scattering of polarized leptons on protons. In QCD it is defined as

$$ E_{Jp}(Q^2) = \int_0^1 g_1^{lp}(x, Q^2) dx = C_{B_{jp}}^{ns}(A_s(Q^2)) \left( \frac{1}{12} a_3 + \frac{1}{36} a_8 \right) + C_{E_{Jp}}^{s}(Q^2) \frac{1}{9} \Delta\Sigma(Q^2) \ \ (8) $$

where $a_3 = \Delta u - \Delta d$, $a_8 = \Delta u + \Delta d - 2\Delta s$, $\Delta u$, $\Delta d$ and $\Delta s$ are the polarized distributions and $\Delta\Sigma$ depends from the scheme choice. In the $\overline{MS}$-scheme it is defined as $\Delta\Sigma = \Delta u + \Delta d + \Delta s$, while in the Adler-Bardeen scheme it contains the additional additive contribution from polarized gluon distribution $\Delta G$.

The 4-loop QCD corrections to the coefficient function of the singlet part of Ellis-Jaffe sum rule were calculated in Ref. [24] using the method of the dimensional regularization. In the framework of the dimensional regularization the final expression for the singlet coefficient function can be presented as the ratio of two functions [24]:

$$ C_{E_{Jp}}^{s} = C_{E_{Jp}}^{s}/Z_5^{s} \ . $$

Here $Z_5^{s}$ is the finite singlet renormalization constant of the operator $\Psi\gamma_{\mu}\gamma_{5}\Psi$, which should be calculated within dimensional regularization and the $\overline{MS}$-scheme. This finite constant is similar to the finite constant $Z_5^{\alpha s}$ in the definition of the non-singlet axial operator $\Psi\gamma_{\mu}\gamma_{5}(\lambda^a/2)\Psi$ within dimensional regularization. It enters in the procedure of calculations
of high order QCD corrections to the Bjorken polarized sum rule at the 3-loop \cite{25,26} and 4-loop \cite{27} levels. In view of the property, that the expression for $Z_5$ differs from $Z_5^{ns}$ by the corrections, which come from the light-by-light-type scattering graphs \cite{28}, in the pqQED limit these constants coincide. Therefore, the 4-loop corrections in Eq.(9) are determined by the ratio of the following pqQED expressions for the coefficient function

$$C_{EJ}^s = 1 - 7 A + \frac{89}{2} A^2 - \left(\frac{1397}{6} - 96 \zeta(3)\right) A^3 + O(A^4) \ .$$

(10)

and for the finite renormalization constants, namely

$$Z_s^5|_{pqQED} = Z_s^5|_{pqQED} = 1 - 4 A + 22 A^2 + \left(-\frac{370}{3} + 96 \zeta(3)\right) A^3 + O(A^4) \ .$$

(11)

The expressions of Eq.(10) and of Eq.(11) are extracted from the results of calculations of Ref. \cite{24} and Ref. \cite{27} correspondingly. Notice the appearances of $\zeta(3)$-terms in the coefficients of the $O(A^3)$-corrections to Eq.(10) and Eq.(11). However, these terms cancel each other in our new ultimate 4-loop pqQED result for the coefficient of order $A^3$ approximation to the singlet coefficient function:

$$C_{EJp}^s(A) = 1 - 3 A + \frac{21}{2} A^2 - \frac{3}{2} A^3 + O(A^4)$$

(12)

and coincide with the similar expression for the pqQED series of Eq.(7).

At the possible next step of analytical calculations of Eq.(10) $\zeta(5)$ must manifest itself. Indeed, not presented yet next term in the result of Eq.(11) for the renormalization constant $Z_5|_{pqQED}$, evaluated during the calculations of 5-loop perturbative corrections to the Bjorken polarized sum rule \cite{3}, must contain $\zeta(5)$ function, while corresponding $\zeta(7)$-terms should cancel in the expressions for its $O(A^3)$-corrections. However, $\zeta(3)$ should remain in the expression of the coefficient of the $O(A^4)$-correction to Eq.(12), since the following identity

$$C_{EJp}^s(A) = C_{Bjp}^{ns}(A)$$

(13)

holds in pqQED in all orders of perturbation theory and is the consequence of the of the axial variant of Crewther relation.

Let me outline the basic steps of the proof of this statement in the momentum space. These steps were first discussed in Ref. \cite{29} together with more detailed proof of the original non-singlet Crewther relation in the momentum space \cite{30}.

The proof is based on the application of the operator product expansion approach to the 3-point function with the axial singlet current :

$$T_{\mu\alpha\beta}(p, q) = i \int <0|TA_{\mu}(y)V_{\alpha}(x)V_{\beta}(0)|0> e^{ipx+iqy}dxdy$$

(14)

where $A_{\mu} = \bar{\psi}\gamma_{\mu}\gamma_5\psi$. Keeping the singlet structure in the operator-product expansion of the two non-singlet vector currents, one can get

$$i \int TV_{\alpha}V_{\beta} e^{ipx} dx|_{p^2 \to \infty} \approx C_{\alpha\beta\rho}^{SI} A_{\rho}(0) + \text{other structures} \ .$$

(15)

\footnote{The Crewther relation \cite{7} was originally derived in the coordinate space.}
where
\[ C^{SI,ab}_{\mu\nu\alpha} \sim i\delta^{ab}\epsilon_{\mu\nu\alpha\beta} \frac{q^\beta}{q^2} C^s_{EJp}(a_s) \]  
(16)

The second ingredient in the singlet version of the Crewther relation appears after consideration of vacuum expectation value of the product of two axial currents
\[ i \int <0|TA_\mu(x)A_\nu(0)|0>e^{iqx}dx = \Pi^{SI}_{\mu\nu}(q^2) \]  
(17)

In this channel one can also define Adler function and its coefficient function \( C^s_D(A_s) \) as well. Taking now the conformal symmetry limit, it is possible to get the singlet variant of the Crewther relation [29], namely
\[ C^s_{EJp}(A) \times C^s_D(A)|_{\text{conf-sym}} = 1 \]  
(18)

This expression should be compared with the similar expression for the non-singlet Crewther relation [30], which reads
\[ C^{ns}_{Bjp}(A) \times C^s_D(A)|_{\text{conf-sym}} = 1 \]  
(19)

In the massless pqQED approximation the following identity takes place
\[ C^s_D(A) = C^{ns}_D(A) \]  
(20)

Comparing now Eq.(18) with Eq. (19) and taking into account Eq.(20), I find that the expression of Eq.(13) is indeed valid in all orders of perturbation theory.

Note once more that pqQED is the conformal invariant version of massless perturbative QED. Therefore, in order to understand deeper the status and nature of the manifestation of odd \( \zeta \)-functions, it is important to have a look to the structure of perturbative series in some other conformal invariant theory and \( N=4 \) SUSY Yang-Mills theory in particular.

3. Analytic structure for the anomalous dimension of the Konishi operator in \( N=4 \) SUSY Yang-Mills theory.

To demonstrate that the explicit manifestation of transcendental \( \zeta(3) \)-terms in high order perturbation theory corrections to renormalization group quantities does not contradict conformal symmetry let us turn to the behaviour of perturbative series for the anomalous dimensions in the massless \( N=4 \) SUSY Yang-Mills theory. Since its renormalization group \( \beta \)-function is identically equal to zero, this theory possess the property of explicit conformal symmetry. The validity of this property at the 3-loop level was discovered in Ref. [31] by perturbative methods. Soon afterwards the absence of renormalization of the coupling constant in this theory was proved within the light-cone quantization approach [32].

The absence of the coupling constant renormalization does not mean that there are no ultraviolet divergencies in the massless \( N=4 \) Yang-Mills theory. Indeed, calculations of anomalous dimensions of various operators in this quantum field theory give non-zero results (see e.g. Refs. [33]- [45]).

Among the most interesting are the ones, related to analytical evaluation of the anomalous dimension of the Konishi operator in high levels of perturbation theory. The operator is defined as
\[ O_K = tr \Phi^i \Phi^i \]  
(21)
where \( \Phi^i \) is the complex adjoint scalar field. The expression for the anomalous dimension of this operator obey the interesting property, namely the transcendental functions \( \zeta(3) \) and
\( \zeta(5) \) are manifesting themselves starting from the 4-loop perturbative corrections. Indeed, the direct quantum field theory perturbative calculation, performed in terms of Feynman diagrammatic \[38\], gave the following result

\[
\gamma_K(\lambda) = 12\lambda - 48\lambda^2 + 336\lambda^3 - \lambda^4 \left( 2496 - 576\zeta(3) + 1440\zeta(5) \right) + O(\lambda^5) \tag{22}
\]

where \( \lambda = g^2N_c/(4\pi)^2 \) and \( N_c \) is the "number of colours" of \( SU(N_c) \) gauge group. Note, that in \( N = 4 \) SUSY Yang Mills gauge theory the values of Casimir operators are fixed as \( C_F = C_A = T_F N_F \). Another interesting feature of \( N = 4 \) SUSY Yang-Mills theory is that the property of AdS/CFT correspondence \[46\]-[48] links \( N = 4 \) SUSY Yang-Mills with the theory of superstrings in \( AdS_5 \times S^5 \).

This property opens the second way for the calculations of anomalous dimensions in \( N = 4 \) SUSY Yang-Mills theory using quantum field theory of the superstring in \( AdS_5 \times S^5 \) and taking into account its integrability property. This was done in Ref.\[37\], where the coefficients of the series in Eq.(22) were calculated prior the work of Ref. \[38\].

This calculation is based on the application of the Bethe Anzatz quantization. Note, that using this ansatz it is possible to separate pure weak-coupling contribution from the one, which interpolates between strong and weak coupling \[49\] and is responsible for the contribution of the Lücher corrections \[50\].

In other words, its application allowed to demonstrate that at the level of order \( \lambda^4 \) extra contributions, which describe "wrapping effects" of Lücher corrections \[50\], are manifesting themselves. These effects are detectable both at strong coupling constant regime (see e.g. Ref. \[51\]) and weak coupling constant regime \[36\].

Perturbation-theory oriented clarification of these words is encoded in the results of Ref. \[37\]. Indeed, the 4-loop expression for \( \gamma_K \) can be decomposed into two terms, namely

\[
\gamma_K = \gamma_{\text{asymp}}(\lambda) + \gamma_{\text{wr}}(\lambda) \tag{23}
\]

where

\[
\gamma_{\text{asymp}} = 12\lambda - 48\lambda^2 + 336\lambda^3 - \left( 2820 + 288\zeta(3) \right)\lambda^4 + O(\lambda^5) \tag{24}
\]

\[
\gamma_{\text{wr}}(\lambda) = \left( 324 + 864\zeta(3) - 1440\zeta(5) \right)\lambda^4 + O(\lambda^5) \tag{25}
\]

The result of Eq.(24) was first obtained in Ref. \[35\]. The analytical calculations of overall order \( \lambda^4 \)-contribution and of its two parts are in agreement with the calculations performed with superspace diagrammatic formalism \[36\]. In its turn, the total expression for the order \( \lambda^4 \)-approximation of Eq.\[23\], obtained in Ref. \[37\] and Ref. \[36\], coincide with the result of Eq.(22), obtained in Ref. \[38\] from direct Feynman diagrams calculations.

This independent calculation gave real confidence in the correctness of final analytical expression and in the fact that the asymptotic part of 4-loop result for \( \gamma_K \) (see Eq.(24)) does not contain \( \zeta(5) \)-contribution, which, together with additional pure rational and \( \zeta(3) \)-contributions, enter into 4-loop "wrapping" effects (for the diagrammatic explanation of the appearance of \( \zeta(5) \) in Eq.(24) see Ref. \[39\]).

The results of Eq.(24) should be compared with the pqQED ones, given in Sec.2. Compared with each other they indicate, that Riemann \( \zeta(3) \)-puzzle is not the puzzle, but the regular feature of the asymptotic series in the conformal-invariant theories. Following this conclusion, one should expect manifestation of \( \zeta(3) \) and \( \zeta(5) \) terms in the next-to-presented above coefficients of the corresponding asymptotic perturbative series in the conformal
invariant theories. This feature is realized in the results of calculations of 5-loop corrections to the anomalous dimension of Eq. (24) in $N = 4$ SUSY Yang-Mills theory \cite{41,42}. Note that $\zeta(7)$-terms are appearing in the 5-loop "wrapping contributions" only (see e.g. \cite{43}, \cite{44}). Moreover, $\zeta$-functions counting rule, namely the appearance of extra $\zeta$-functions in high order wrapping contributions, is supported by the results of six loop calculations (see Ref. \cite{40} and Ref. \cite{45}), which demonstrate the appearance of $\zeta(9)$-terms.

**Conclusions.**

In this work we introduce the way of explaining the structure of analytical expression for high order corrections in asymptotic perturbative series for the anomalous dimensions and coefficient functions of gauge-invariant operators in $p\!q$QED. The arguments, presented in this work, are useful for realizing that the appearance of $\zeta(3)$-terms in the $p\!q$QED series is rather regular feature. This feature is supported by the property of conformal symmetry. Indeed, the $\zeta$-functions counting rules are also satisfied for dealing with coefficients of the asymptotic perturbative series for the anomalous dimensions of operators in super-conformal $N = 4$ SUSY Yang-Mills gauge theory in the case when "wrapping interactions" are not taken into account. These interactions are responsible for the interpolation into the regime of large values of coupling constant.

At present I do not know whether it is possible to find the signals of the existence of these interactions in the strong-coupling phase of quenched QED. In the case if these interactions do exist, they may signal about themselves through the explicit manifestations of higher transcendentailities, and $\zeta(5)$ in particular.

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