ON CUBIC BIRATIONAL MAPS OF $\mathbb{P}^3_C$

JULIE DÉSERTI AND FRÉDÉRIC HAN

ABSTRACT. We study the birational maps of $\mathbb{P}^3_C$. More precisely we describe the irreducible components of the set of birational maps of bidegree $(3,3)$ (resp. $(3,4)$, resp. $(3,5)$).

1. INTRODUCTION

The CREMONA group, denoted Bir($\mathbb{P}^n_C$), is the group of birational maps of $\mathbb{P}^n_C$ into itself. If $n = 2$ a lot of properties have been established (see [2–6] for example). As far as we know the situation is much more different for $n \geq 3$ (see [11,3] for example). If $\psi$ is an element of Bir($\mathbb{P}^2_C$) then $\deg \psi = \deg \psi^{-1}$. It is not the case in higher dimensions where we only have the inequality $\deg \psi^{-1} \leq (\deg \psi)^2$ so one introduces the bidegree of $\psi \in$ Bir($\mathbb{P}^3_C$) as the pair $(\deg \psi, \deg \psi^{-1})$.

For $n = 2$, Bir$_d(\mathbb{P}^2_C)$ is the set of birational maps of the complex projective plane of degree $d$; for $n \geq 3$ denote by Bir$_{d,d'}(\mathbb{P}^n_C)$ the set of elements of Bir($\mathbb{P}^n_C$) of bidegree $(d,d')$, and by Bir$_d(\mathbb{P}^n_C)$ the union $\cup_d$ Bir$_{d,d'}(\mathbb{P}^n_C)$.

In [4] the sets Bir$_2(\mathbb{P}^2_C)$, and Bir$_3(\mathbb{P}^2_C)$ are described: Bir$_2(\mathbb{P}^2_C)$ is smooth, and irreducible in the space of rational maps of the complex projective plane whereas Bir$_3(\mathbb{P}^2_C)$ is irreducible, and rationally connected. In [5] CREMONA studies three types of generic elements of Bir$_2(\mathbb{P}^3_C)$. Then there were some articles on the subject, and finally a precise description of Bir$_2(\mathbb{P}^3_C)$; the left-right conjugacy is the following one

$$\text{PGL}(4; \mathbb{C}) \times \text{Bir}(\mathbb{P}^3_C) \times \text{PGL}(4; \mathbb{C}), \quad (A, \psi, B) \mapsto AB^{-1}.$$

PAN, RONGA and VUST give quadratic birational maps of $\mathbb{P}^n_C$ up to left-right conjugacy ([12 Theorems 3.1.1, 3.2.1, 3.2.2, 3.3.1]). In particular they show that Bir$_2(\mathbb{P}^3_C)$ has three irreducible components of dimension 11, 13, 14; the component of dimension 11 (resp. 13, resp. 14) corresponds to birational maps of bidegree $(2,4)$ (resp. $(2,3)$, resp. $(2,2)$). We will see that the situation is slightly different for Bir$_3(\mathbb{P}^3_C)$; in particular we cannot expect such an explicit list of biclasses because there are infinitely many of them. That’s why the approach is different.

In higher degrees we do not have such a precise description; nevertheless we can find a very fine and classical contribution for Bir$_3(\mathbb{P}^3_C)$ due to HUDSON ([8]); in §A we reproduce Table VI of [8]. HUDSON introduces there some invariants to establish her classification. But it gives rise to many cases, and we also find examples where invariants take values that do not appear in her table. We do not know references explaining how her families fall into irreducible components of Bir$_{3,d}(\mathbb{P}^3_C)$ so we will focus on this natural question.

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**Definition.** An element \( \psi \) of \( \text{Bir}_{3,d}(\mathbb{P}^3_C) \) is **ruled** if the strict transform of a generic plane under \( \psi^{-1} \) is a ruled cubic surface.

Denote by \( \text{rule} \mathcal{D}_{3,d} \) the set of \((3,d)\) ruled maps. Let us remark that there are no ruled birational maps of bidegree \((3,d)\) with \(d \geq 6\). We will detail these families in Lemma 2.1.

We describe the irreducible components of \( \text{Bir}_{3,d}(\mathbb{P}^3_C) \) for \(3 \leq d \leq 5\):

**Theorem A.** Assume that \(2 \leq d \leq 5\). The set \( \text{rule} \mathcal{D}_{3,d} \) is an irreducible component of \( \text{Bir}_{3,d}(\mathbb{P}^3_C) \).

In bidegree \((3,3)\) (resp. \((3,4)\)) there is only another irreducible component; in bidegree \((3,5)\) there are three others.

The closure of the \((3,3)\) ruled birational maps intersects the closure of any irreducible component of \( \text{Bir}_{3,d}(\mathbb{P}^3_C) \).

**Notations 1.1.** Consider a rational map \( \psi \) from \( \mathbb{P}^3_C \) into itself. For a generic line \( \ell \), the preimage of \( \ell \) by \( \psi \) is a complete intersection \( \Gamma_i \); let \( C_2 \) be the union of the irreducible components of \( \Gamma_i \) supported in the base locus of \( \psi \). Define \( C_1 \) by liaison from \( C_2 \) in \( \Gamma_i \). Remark that if \( \psi \) is birational, then \( C_1 = \psi^{-1}(\ell) \). Let us denote by \( p_a(C_i) \) the arithmetic genus of \( C_i \).

It is difficult to find a uniform approach to classify elements of \( \text{Bir}_3(\mathbb{P}^3_C) \). Nevertheless in small genus we succeed to obtain some common detailed results; before stating them, let us introduce some notations.

Let us remark that the inequality \( \deg \psi^{-1} \leq (\deg \psi)^2 \) mentioned previously directly follows from

\[ (\deg \psi)^2 = \deg \psi^{-1} + \deg C_2. \]

**Proposition B.** Let \( \psi \) be a \((3,d)\) birational map.

Assume that \( \psi \) is not ruled, and \( p_a(C_1) = 0 \), i.e. \( C_1 \) is smooth. Then

- \( d \leq 6 \);
- and \( C_2 \) is a curve of degree \(9 - d\), and arithmetic genus \(9 - 2d\).

Suppose \( p_a(C_1) = 1 \), and \( 2 \leq d \leq 6 \). Then

- there exists a singular point \( p \) of \( C_1 \) independent of the choice of \( C_1 \);
- if \( d \leq 4 \), all the cubic surfaces of the linear system \( \Lambda_{p} \) are singular at \( p \);
- the curve \( C_2 \) is of degree \( 9 - d \), of arithmetic genus \(10 - 2d \), and lies on a unique quadric; more precisely \( I_{C_2} = (Q, S_1, \ldots, S_{d-2}) \) where \( \deg Q = 2 \), and \( S_i \) are independent cubics modulo \( Q \).

We denote by \( \text{Bir}_{3,d,p_2}(\mathbb{P}^3_C) \) the subset of non ruled \((3,d)\) birational maps such that \( C_2 \) is of degree \(9 - d\), and arithmetic genus \( p_2 \). One has the following statement:

**Theorem C.** If \( p_2 \in \{3,4\} \), then the sets \( \text{Bir}_{3,3,p_2}(\mathbb{P}^3_C) \) are non empty, and irreducible; \( \text{Bir}_{3,3,p_1}(\mathbb{P}^3_C) \) is empty as soon as \( p_2 \notin \{3,4\} \).

If \( p_2 \in \{1,2\} \), then \( \text{Bir}_{3,4,p_2}(\mathbb{P}^3_C) \) are non empty, and irreducible; \( \text{Bir}_{3,4,p_1}(\mathbb{P}^3_C) \) is empty as soon as \( p_2 \notin \{1,2\} \).

If \( p_2 \in \{-1,1\} \), then \( \text{Bir}_{3,5,p_2}(\mathbb{P}^3_C) \) are non empty, and irreducible but \( \text{Bir}_{3,5,0}(\mathbb{P}^3_C) \) is non empty, and has two irreducible components; one of these components is contained in \( \text{Bir}_{3,5,-1}(\mathbb{P}^3_C) \). The set \( \text{Bir}_{3,5,p_1}(\mathbb{P}^3_C) \) is empty as soon as \( p_2 \notin \{-1,0,1\} \).
Organization of the article. In §2 we explain the particular case of ruled birational maps and set some notations. Then §3 is devoted to liaison theory that plays a big role in the description of the irreducible components of Bir$_{3,3}$ (see §4), Bir$_{3,4}$ (see §5) and Bir$_{3,5}$ (see §6). In the last section we give some illustrations of invariants considered by HUDSON, especially concerning the local study of the preimage of a line. Since HUDSON’s book is very old, let us recall her classification in the first part of the appendix.

2. Definitions and notations

Let $\psi: \mathbb{P}^3_C \dasharrow \mathbb{P}^3_C$ be a rational map given, for some choice of coordinates, by

$$(z_0: z_1: z_2: z_3) \mapsto (\psi_0(z_0, z_1, z_2, z_3): \psi_1(z_0, z_1, z_2, z_3): \psi_2(z_0, z_1, z_2, z_3): \psi_3(z_0, z_1, z_2, z_3))$$

where the $\psi_i$'s are homogeneous polynomials of the same degree $d$, and without common factors. The map $\psi$ is called a CREMONA transformation or a birational map of $\mathbb{P}^3_C$ if it has a rational inverse $\psi^{-1}$. The degree of $\psi$, denoted $\deg \psi$, is $d$. The pair $(\deg \psi, \deg \psi^{-1})$ is the bidegree of $\psi$, we say that $\psi$ is a $(\deg \psi, \deg \psi^{-1})$ birational map. The indeterminacy set of $\psi$ is the set of the common zeros of the $\psi_i$. Denote by $I_\psi$ the ideal generated by the $\psi_i$, and by $\Lambda_\psi \subset H^0(\mathbb{P}^3_C(d))$ the subspace of dimension $4$ generated by the $\psi_i$, and by $\deg I_\psi$ the degree of the scheme defined by the ideal $I_\psi$.

Sometimes (in §3 for example) the base locus $\text{Base } \psi$ of $\Lambda_\psi$ is called $F$-locus of $\psi$; its points (resp. curves) are called the fundamental points (resp. fundamental curves) or $F$-points (resp. $F$-curves). We will denote by $\omega_C$ the dualizing sheaf of $C$.

Let us now focus on particular birational maps that cannot be dealt as the others: the ruled birational maps of $\mathbb{P}^3_C$.

Lemma 2.1. Let $\psi$ be a $(3, d)$ ruled birational map, and let $F$ be its base locus. Denote by $F_1$ the maximal 1-dimensional locally COHEN-MACAUlAY subscheme of $F$. Then $F_1$ is the union of a line $\delta$ doubled in $\mathbb{P}^3_C$ with $5 - d$ other lines intersecting $\delta$. Its residual in $F$ is a scheme of length $2d - 4$.

Proof. Let $\psi$ be in rule$3_{3,d}$, and $\delta$ be the double line of all the elements of $\Lambda_\psi$. Of course $\delta \subset F_1$. Let $L$ be a generic subspace of dimension 2 of $\Lambda_\psi$; denote by $\mathcal{C}_L$ the associated $\mathcal{C}_1$. For any point $p$ that lies on $\mathcal{C}_L$ but not on $F_1$ the subset $L$ induces in the plane $\Pi = (\delta, p)$ the pencil of cubics $2\delta \cup \ell$ with $p \in \ell \subset \Pi$. Hence $F_1 \cap \Pi \subset \delta$ and all the components of $\mathcal{C}_L$ are in planes. So $F_1$ is (as a set) the union of $\delta$ and some other lines that intersect $\delta$, that is $F_1 = \delta^2 \cup \ell_1 \cup \ldots \cup \ell_k$ where the $\ell_i$ are some common rules. In particular $k = 5 - \deg \mathcal{C}_1$.

Let us blow up $\delta \subset \mathbb{P}^3_C$:

$$\begin{array}{ccc}
\tilde{\mathbb{P}}^3_C(\delta) & \longrightarrow & \mathbb{P}^3_C \\
\downarrow & & \\
\mathbb{P}^1_C & & \\
\end{array}$$

Denote by $h_i$ the pull back of the hyperplane class of $\mathbb{P}^3_C$. The linear system $\Lambda_\psi$ induces a linear system of $|O_{\tilde{\mathbb{P}}^3_C(\delta)}(2h_1 + h_3)|$. According to [7] Example 9.1.1] one can compute the excess
contribution of a rule:

\[(3(h_3 + 2h_1) - h_3 - h_1) \cdot h_3 \cdot h_1 = (2h_3 + 5h_1) \cdot h_3 \cdot h_1 = 2h_3^2 h_1 = 2 \text{ points.}\]

So \( \psi \) is birational if and only if the length of the residual scheme is

\[(h_3 + 2h_1)^3 - 2k - 1 = h_3^3 + 6h_3^2 h_1 - 2k - 1 = 6 - 2(5 - d) = 2d - 4.\]

\[\square\]

**Lemma 2.2.** The following inclusions hold:

\[\text{ruled}_{3,3} \subset \text{ruled}_{3,4}, \quad \text{ruled}_{3,4} \subset \text{ruled}_{3,5}.\]

**Proof.** Let us start with an element of \( \text{ruled}_{3,5} \) with base curve \( \delta \) and 6 base points \( p_i \) in general position. Then move two of the \( p_i \), for instance \( p_1, p_2 \) until the line \((p_1 p_2)\) intersects \( \delta \). The line \((p_1 p_2)\) is now automatically in the base locus of the linear system \( \Lambda_\psi \), and we obtain like this a generic element of \( \text{ruled}_{3,4} \).

A similar argument allows to prove the first inclusion. \[\square\]

Let us recall the notion of genus of a birational map ([8, Chapter IX]). The *genus* \( g_\psi \) of \( \psi \in \text{Bir}(\mathbb{P}_C^3) \) is the geometric genus of the curve \( h \cap \psi^{-1}(h') \) where \( h \) and \( h' \) are generic hyperplanes of \( \mathbb{P}_C^3 \). The equality \( g_\psi = g_{\psi^{-1}} \) holds.

**Remark 2.3.** If \( \psi \) is a birational map of \( \mathbb{P}_C^3 \) of degree 1 (resp. 2, resp. 3) then \( g_\psi = 0 \) (resp. \( g_\psi = 0 \), resp. \( g_\psi \leq 1 \)).

One can give a characterization of ruled maps of \( \text{Bir}_{3,d}(\mathbb{P}_C^3) \) in terms of the genus.

**Proposition 2.4.** Let \( \psi \) be in \( \text{Bir}_{3,d}(\mathbb{P}_C^3) \), \( 2 \leq d \leq 5 \). The genus of \( \psi \) is zero if and only if \( \psi \) is ruled.

**Proof.** On the one hand the singular locus of the base locus of the base scheme of an element of \( \text{Bir}_{3,d}(\mathbb{P}_C^3) \) has at most isolated singularity if and only if the map is not ruled; on the other hand \( g_\psi = 0 \) if and only if for generic hyperplanes \( h, h' \) of \( \mathbb{P}_C^3 \) the curve \( h \cap \psi^{-1}(h') \) is a singular rational cubic. \( \square\)

3. liaison ([13])

Let us start by this fundamental statement:

**Lemma 3.1** (Liaison sequence). If \( \Gamma_1 \cup \Gamma_2 \) is a complete intersection then

\[
\begin{array}{c}
0 \rightarrow \mathcal{O}_{\Gamma_1} \rightarrow \mathcal{O}_{\Gamma_1 \cup \Gamma_2} \rightarrow \mathcal{O}_{\Gamma_1} \otimes \mathcal{O}_{\Gamma_1 \cup \Gamma_2} \rightarrow 0
\end{array}
\]

Let \( \psi \) be a rational map of \( \mathbb{P}_C^3 \) of degree 3. We have

\[\omega_{\mathcal{O}_{\mathbb{P}_C^3} \cup \mathcal{Q}} = \mathcal{O}_{\mathcal{O}_{\mathbb{P}_C^3} \cup \mathcal{Q}}(2h),\]

where \( h \) an hyperplane of \( \mathbb{P}_C^3 \), and for \( i \in \{1, 2\} \)

\[0 \rightarrow \omega_{\mathcal{Q}_i} \rightarrow \mathcal{O}_{\mathcal{O}_{\mathbb{P}_C^3} \cup \mathcal{Q}_i}(2h) \rightarrow \mathcal{O}_{\mathcal{Q}_i}(2h) \rightarrow 0;\]

this directly implies the following equalities (\( i \in \{1, 2\} \))

\[H^0\omega_{\mathcal{Q}_i}(-h) = H^0 I_{\mathcal{Q}_i}(h), \quad H^0\omega_{\mathcal{Q}_i} = H^0 I_{\mathcal{Q}_i}(2h),\]
\[ h^0 \omega_C(h) + 2 = h^0 I_{C_\omega}(3h), \quad H^0 \omega_C(h) = \frac{H^0 I_{C_\omega}(3h)}{H^0 I_{C_\omega \cup C}(3h)}. \]

Furthermore when \( C_1 \) and \( C_2 \) have no common component, and \( \omega_C \) is locally free, then length \( (C_1 \cap C_2) = \deg \omega_C(2h), \) i.e.
\[ \sum_{p \in C_1 \cap C_2} \text{length}(C_1 \cap C_2)_{[p]} = 2 \deg C_1 - 2p_a(C_1) + 2. \]

In the preimage of a generic point of \( \mathbb{P}^3_C \) by \( \psi \), the number of points that do not lie in the base locus is given by
\[ 3 \deg C_1 - \sum_{p \in C_1 \cap C_2} \text{length}(S \cap C_1)_{[p]} - \sum_{p \in \text{isolated } F\text{-points}} \text{length}(S \cap C_1)_{[p]}, \]
where \( S \in \Lambda_\psi \) is non-zero modulo \( H^0 I_{C_1 \cup C_2}(3h) \), and where ”isolated \( F\)-points” means irreducible components of dimension 0 of the base locus.

**Lemma 3.2.** Let \( \psi \) be a rational map of \( \mathbb{P}^3_C \) of degree 3. The map \( \psi \) is birational if and only if
\[ 1 = 3 \deg C_1 - \sum_{p \in C_1 \cap C_2} \text{length}(S \cap C_1)_{[p]} - \sum_{p \in \text{isolated } F\text{-points}} \text{length}(S \cap C_1)_{[p]}. \]

Remark that the computation of \( \text{length}(S \cap C_1)_{[p]} \) depends on the nature of the singularity of the cubic surface and on the behaviour of \( C_2 \) in that point (see §7).

**Lemma 3.3.** Let \( \psi \) be a \((3, d)\) Cremona map. Assume that \( d \geq 4 \), then \( C_1 \) is not contained in a plane.

**Proof.** Let us suppose for example that \( d = 4 \); then \( C_1 \) is contained in an irreducible cubic surface \( S \). If \( C_1 \) is contained in a plane \( \mathcal{P} \) then all the lines in \( \mathcal{P} \) are quadrisecant to \( S \); contradiction with the irreducibility of \( S \). \( \square \)

**Theorem 3.4.** Let \( \psi \) be a \((3, d)\) birational map, \( 2 \leq d \leq 6 \), that is not ruled. Assume that \( p_\omega(C_1) = 1 \). Then
- there exists a singular point \( p \) of \( C_1 \) independent of the choice of \( C_1 \);
- if \( d \leq 4 \), all the cubic surfaces of the linear system \( \Lambda_\psi \) are singular at \( p \);
- the curve \( C_2 \) is of degree \( 9 - d \), of arithmetic genus \( 10 - 2d \), and lies on a unique quadric;
  more precisely \( I_{C_2} = (Q, S_1, \ldots, S_\ell) \) where \( \deg Q = 2 \), and \( \deg S_i = 3 \).

**Remark 3.5.** As soon as \( d = 5 \) the second assertion is not true. Indeed for \( d = 5 \) we obtain two families: one for which all the elements of \( \Lambda_\psi \) are singular, and another one for which it is not the case (§6).

**Proof.** As \( g(C_1) = 0 \) and \( p_\omega(C_1) = 1 \), the singular locus \( \text{Sing} C_1 \) of \( C_1 \) is not empty. Let us show that there is a singular point independent of the choice of \( C_1 \). Let \( S \) be a generic element of \( \Lambda_\psi \). The elements of \( \Lambda_\psi \) give a linear system in \( |O_3(C_1)| \) whose base locus denoted \( \Omega \) is finite. According to BERTINI’s theorem applied on \( S \) one has the inclusion \( \text{Sing} C_1 \subset \Omega \cup \text{Sing} S \). The first assertion thus follows from the fact that \( \Omega \cup \text{Sing} S \) is finite.

Since \( p_\omega(C_1) = 1 \), the curve \( C_2 \) lies on a unique quadric. The arithmetic genus of \( C_2 \) is obtained from \( \deg C_2 - \deg C_1 = p_\omega(C_2) - p_\omega(C_1) \).
The number of cubics containing $C_2$ independent modulo the multiple of $Q_2$ is $d - 2$: the liaison sequence (Lemma 3.1) becomes

$$0 \rightarrow O_{C_1}(h) \rightarrow O_{C_1 \cup C_2}(3h) \rightarrow O_{C_2}(3h) \rightarrow 0$$

one gets that

$$h^0 O_{C_2}(3h) = h^0 O_{C_1 \cup C_2}(3h) - h^0 O_{C_1}(h) = 18 - d.$$  

This implies that

$$h^0 I_{C_2}(3h) = 20 - h^0 O_{C_2}(3h) = d + 2.$$  

If we put away the four multiples of $Q$ one obtains $d + 2 - 4 = d - 2$ cubics.

Assume that $\Lambda_\psi$ contains a smooth element $S$. The cubic surface $S$ can be seen as $\mathbb{P}^2_C$ blown up at 6 points $p_1, \ldots, p_6$; denote by $E_i$ the exceptional divisor associated to $p_i$. On $S$ one has

$$C_2 = 2H - E_1 - \ldots - E_{d-3}$$

where $H = 3h - E_1 - \ldots - E_6$ and

$$\left| \Lambda_\psi \right| \subset \left|[3H - C_2] = [3h - E_{d-2} - \ldots - E_6]\right|.$$

But $h^0 (O_S(3h - E_{d-2} - \ldots - E_6)) = 10 - (6 - (d - 2) + 1) = 1 + d$. If $d \leq 4$, then

$$h^0 (O_S(3h - E_{d-2} - \ldots - E_6)) \leq 5,$$

and that vector space does not contain any subspace of dimension 3 whose generic element is a singular curve. \hfill \Box

**Proposition 3.6.** For $2 \leq d \leq 5$ the set rule$O_{3,d}$ is an irreducible component of $\text{Bir}_{3,d}(\mathbb{P}_C^3)$.  

**Proof.** Let us use the notations introduced in Lemma 2.1. Note that $F_1 \subset C_2$. If $\psi \in \text{Bir}_{3,d}$ is not ruled then at a generic point $p \in F_1$ there exists an element of $\Lambda_\psi$ smooth at $p$. Hence $F_1$ is locally complete intersection at $p$ and $\deg F_1 = \deg C_2$. In particular $\deg I_\psi = 9 - d$.

Consider now an element $\psi$ in rule$O_{3,d}$. There is a line $\ell$ such that $\ell \subset \text{Sing } S$ for any $S \in \Lambda_\psi$; the set $F_1$ has an irreducible component whose ideal is $I_2^\ell$ and $F_1$ is not locally complete intersection. This multiple structure has to be contained in $C_2$ but since $C_2$ is locally complete intersection the inequality $\deg C_2 > \deg F_1$ holds; it can be rewritten $\deg I_\psi < 9 - d$.

The number $\deg I_\psi$ cannot decrease by specialization so we cannot specialize a non-ruled birational map into a ruled one while staying in the same bidegree.

Elements of $\Lambda_\psi$ when $\psi$ is a ruled birational map have no isolated singularities whereas elements of $\Lambda_\psi$ when $\psi$ is a non-ruled birational map have isolated singularities, it is impossible to specialize a ruled birational map into a non-ruled one. \hfill \Box

**Corollary 3.7.** Let $\psi$ be a $(3, \cdot)$ birational map of $\mathbb{P}_C^3$; if the general element of $\Lambda_\psi$ is smooth or if the singularities of a general element of $\Lambda_\psi$ are isolated, then $\deg F_1 = \deg C_2$.

4. $(3,3)$-CREMONA TRANSFORMATIONS

4.1. Some known results.

1. As we will see in Proposition 3.3 this statement is not true if we do not specify "while staying in the same degree".
4.1.1. In the literature one can find different points of view concerning the classification of \((3, 3)\) birational maps. For example HUDSON introduced many invariants related to singularities of families of surfaces and gave four families described in \([8]\) nevertheless we do not understand why the family \(E_{3,3}\) defined below does not appear. PAN chose an other point of view and regrouped \((3, 3)\) birational maps into three families. A \((3, 3)\) birational map \(\psi\) of \(\mathbb{P}^3_3\) is called determinantal if there exists a \(4 \times 3\) matrix \(M\) with linear entries such that \(\psi\) is given by the four \(3 \times 3\) minors of the matrix \(M\); the inverse \(\psi^{-1}\) is also determinantal. Let us denote by \(T^D_{3,3}\) the set of determinantal maps. A \((3, 3)\) CREMONA transformation is a DE JONQUIÈRES one if and only if the strict transform of a general line under \(\psi^{-1}\) is a singular plane rational cubic curve whose singular point is fixed. For such a map there is always a quadric contracted onto a point, the corresponding fixed point for \(\psi^{-1}\) which is also a DE JONQUIÈRES transformation. The DE JONQUIÈRES transformations form the set \(T^D_{3,3}\). PAN established the following ([10] Theorem 1.2)):

\[
\text{Bir}_{3,3}(\mathbb{P}^3_3) = T^D_{3,3} \cup T^I_{3,3} \cup \text{ruled}_{3,3};
\]

in other words an element of \(\text{Bir}_{3,3}(\mathbb{P}^3_3)\) is a determinantal map, or a DE JONQUIÈRES map, or a ruled map.

**Remark 4.1.** One has \(T^D_{3,3} = \text{Bir}_{3,3,3}(\mathbb{P}^3_3)\) and \(T^I_{3,3} = \text{Bir}_{3,3,4}(\mathbb{P}^3_3)\); hence \(\text{Bir}_{3,3,2}(\mathbb{P}^3_3)\) is irreducible for \(p_2 \in \{3, 4\}\).

**Remark 4.2.** The birational involution \((z_0 z_1^2 : z_0^2 z_1 : z_0^2 z_2 : z_1^2 z_3)\) is determinantal, the matrix being

\[
\begin{bmatrix}
  z_0 & z_3 & 0 \\
  -z_1 & 0 & z_2 \\
  0 & 0 & -z_1 \\
  0 & -z_0 & 0
\end{bmatrix},
\]

and also ruled; all the partial derivatives of the components of the map vanish on \(z_0 = z_1 = 0\). The CREMONA transformation \((z_3 : z_0^2 z_1 : z_0^2 z_2 : z_1^2 z_3)\) is a DE JONQUIÈRES and a ruled one.

One has ([9])

\[
T^D_{3,3} \cap T^I_{3,3} = \emptyset, \quad T^D_{3,3} \cap \text{ruled}_{3,3} \neq \emptyset, \quad T^I_{3,3} \cap \text{ruled}_{3,3} \neq \emptyset.
\]

We deal with the usual description of the irreducible components of \(\text{Bir}_{3,3}\) which does not coincide with PAN’s point of view since one of his family is contained in the closure of another one.

4.2. **Irreducible components of the set of \((3, 3)\) birational maps.**

4.2.1. *About HUDSON terminology.* Let us give a few comments about Table VI of [8]. For any subscheme \(X\) of \(\mathbb{P}^3_3\) denote by \(\mathcal{I}_X\) the ideal of \(X\) in \(\mathbb{P}^3_3\). Let \(\psi\) be a \((3, d)\) birational map. A point \(p\) is a double point if all the cubic surfaces of \(\Lambda_p\) are singular at \(p\). A point \(p\) is a binode if all the cubic surfaces of \(\Lambda_p\) are singular at \(p\) with order 2 approximation at \(p\) a quadratic form of rank \(\leq 2\) (but this quadratic form is allowed to vary in \(\Lambda_p\)). In other words \(p\) is binode if there is a degree 1 element \(h\) of \(\mathcal{I}_p\) such that all the cubic belong to \((h \cdot \mathcal{I}_p)\). A point \(p\) is a double point of contact if the general element of \(\Lambda_p\) is singular at \(p\) with order 2 approximation at \(p\) a quadratic form generically constant on \(\Lambda_p\). In other words \(p\) is double point of contact if all the cubics belong to \(\mathcal{I}_p^3 + (Q)\) with \(Q\) of degree 2 and singular at \(p\). A point \(p\) is a point of contact if all the
cubics belong to \( \mathcal{I}_p^2 + \mathcal{I}_S \) where \( S \) is a cubic smooth at \( p \). A point \( p \) is a point of osculation if all the cubics belong to \( \mathcal{I}_p^2 + \mathcal{I}_S \) where \( S \) is a cubic smooth at \( p \).

**Notations 4.3.** We will denote by \( \mathcal{E}_i \) the \( i \)-th family of Table VI and by \( \mathbb{C}[z_0, \ldots, z_n]_d \) the set of homogeneous polynomials of degree \( d \) in the variables \( z_0, \ldots, z_n \).

4.2.2. General description of \((3,3)\) birational maps. One already describes an irreducible component of \( \text{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C}) \), the one that contains \((3,3)\) ruled birational maps (Proposition 3.6). Hence let us consider the case where the linear system \( \Lambda_p \) associated to \( \psi \in \text{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C}) \) contains a cubic surface without double line.

- If \( C_1 \) is smooth then it is a twisted cubic, we are in the second case of Table VI (see \([4A]\)). In that case \( \psi \) is determinantal; more precisely a \((3,3)\) birational map is determinantal if and only if its base locus scheme is an arithmetically COHEN-MACAULAY curve of degree 6 and (arithmetic) genus 3 (see \([11]\) Proposition 1).

- Otherwise \( \omega_{C_1} = O_{C_1} \) then \( C_2 \) lies on a quadric described by the quadratic form \( Q \). As \( h^0(\omega_{C_1}(h) = 5 \) the curve \( C_2 \) is a \((2,3)\) complete intersection, the ideal of \( C_2 \) is \((Q,S)\), and there exists a point \( p \) such that \( I_p = pQ + (S) \), \( p \in Q \), and \( p \) is a singular point of \( S \) (Theorem \([3,3]\)).

In terms of HUDSON invariants this family is stratified as follows:

- **Description of \( \mathcal{E}_3 \).** The point \( p \) belongs to \( Q \) and the general element of \( \Lambda_p \) has an ordinary quadratic singularity at \( p \) (configuration \((2,0,2,0)\) of Table 1 (see \([7]\)). One can choose \( p = (z_0,z_1,z_2) \), \( Q = z_0L_1 + P_2 \), with \( L_1 \in \mathbb{C}[z_0,z_1,z_2]_1 \), \( P_2 \in \mathbb{C}[z_0,z_1,z_2]_2 \), and \( S \) singular at \( p \).

- **Description of \( \mathcal{E}_{3,5} \).** The point \( p \) lies on \( Q \) (\( p \) is a smooth point or not) and the generic cubic is singular at \( p \) with a quadratic form of rank 2. On other words \( p \) is a binode and this happens when one of the two biplanes is contained in \( T_pQ \), it corresponds to the configuration \((2,0,3,1)\) of Table 1 (see \([7]\)). One can take

\[
p = (z_0,z_1,z_2), \quad Q = z_0L_1 + P_2
\]

and \( S = z_0z_1L_1' + P_3 \) singular at \( p \) with \( P_1 \in \mathbb{C}[z_0,z_1,z_2]_1 \), and \( L_1, L_1' \in \mathbb{C}[z_0,z_1,z_2]_1 \). The generic cubic is singular at \( p \) with a quadratic form of rank 2; this case does not appear in Table VI (see \([4A]\)). Let us denote by \( \mathcal{E}_{3,5} \) the set of the associated \((3,3)\) birational maps. The curve \( C_2 \) has degree 6 and a triple point (in \( Q \)).

- **Description of \( \mathcal{E}_4 \).** One can choose \( p = (z_0,z_1,z_2) \), \( Q \) singular and \( S = (z_0z_1 - z_2^2)L_1 + P_3 \), \( L_1 \in \mathbb{C}[z_0,z_1,z_2,z_3]_1 \), \( P_3 \in \mathbb{C}[z_0,z_1,z_2,z_3]_3 \), and \( S \) singular at \( p \). The point \( p \) is a double point of contact, it corresponds to configuration \((2,0,4,1)\) of Table 1 (see \([7]\)).

**Proposition 4.4.** One has

\[
\dim \mathcal{E}_2 = 39, \quad \dim \mathcal{E}_3 = 38, \quad \dim \mathcal{E}_{3,5} = 35, \quad \dim \mathcal{E}_4 = 35, \quad \dim \mathcal{E}_5 = 31,
\]

and

\[
\overline{\mathcal{E}_2} = T^1_{3,3}, \quad \overline{\mathcal{E}_{3,5}} \subset \overline{\mathcal{E}_3}, \quad \overline{\mathcal{E}_4} \subset \overline{\mathcal{E}_3}, \quad \overline{\mathcal{E}_4} \not\subset \overline{\mathcal{E}_{3,5}}, \quad \overline{\mathcal{E}_{3,5}} \not\subset \overline{\mathcal{E}_4}.
\]

**Proof.** Let us justify the equality \( \dim \mathcal{E}_3 = 38 \). We have to choose a quadric \( Q \) and a point \( p \) on \( Q \), this gives \( 9 + 2 = 11 \). Then we take a cubic surface singular at \( p \) that yields to \( 20 - 4 = 16 \); since
we look at this surface modulo $pQ$ and projectivization one gets $16 - 3 - 1 = 12$ so
\[ \dim E_3 = 11 + 12 + 15 = 38. \]

Let us deal with $\dim E_4$. We take a singular quadric $Q$ this gives $8$. Then we take a cubic singular at $p$, modulo $pQ$ and projectivization this yields to $20 - 4 - 1 - 3 = 12$, and finally one obtains $12 + 8 + 15 = 35$. \hfill \square

4.2.3. Irreducible components.

**Theorem 4.5.** The set $\text{rule} \, \mathcal{D}_{3,3}$ is an irreducible component of $\text{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C})$, and there is only one another irreducible component in $\text{Bir}_{3,3}(\mathbb{P}^3_\mathbb{C})$. More precisely the set of the Jonquières maps $E_3$ is contained in the closure of determinantal ones $E_2$ whereas $\text{rule} \, \mathcal{D}_{3,3} \not\subset E_2$.

**Proof.** Let us consider the matrix $A$ given by
\[
\begin{bmatrix}
0 & 0 & 0 \\
-z_1 & -z_2 & 0 \\
z_0 & 0 & -z_2 \\
0 & z_0 & z_1
\end{bmatrix}
\]
and let $A_i$ denote the matrix $A$ minus the $(i+1)$-th line. If $i > 0$, the $2 \times 2$ minors of $A_i$ are divisible by $z_{i-1}$.

Consider the $3 \times 4$ matrix $B$ given by $[b_{ij}]_{1 \leq i \leq 4, 1 \leq j \leq 3}$ with $b_{ij} \in H^0 O_p(1)$ and, as previously, $B_i$ is the matrix $B$ minus the $(i+1)$-th line. Denote by $\Delta^{j,k}$ the determinant of the matrix $A_0$ minus the $j$-th line and the $k$-th column. The $\Delta^{j,k}$ generate $\mathbb{C}[z_0, z_1, z_2]_2$. One has
\[
\det(A_0 + tB_0) = t \cdot S \quad [t^2]
\]
where
\[
S = (b_{21} + b_{43}) \Delta^{1,1} - (b_{31} - b_{42}) \Delta^{2,1} + (b_{33} - b_{22}) \Delta^{1,2} + b_{23} \Delta^{1,3} + b_{32} \Delta^{2,2} + b_{41} \Delta^{3,1}
\]
is a generic cubic of the ideal $(z_0, z_1, z_2)^2$. For $i > 0$
\[
\det(A_i + tB_i) = \det(A_i + t \cdot (z_{i+1}Q)(-1)^{i+1} = t \cdot (z_{i+1}Q)(-1)^{i+1} \quad [t^2]
\]
where $Q = b_{1,1}z_2 - b_{1,2}z_1 + b_{1,3}z_0$ is the equation of a generic quadric that contains $(0, 0, 0, 1)$. So the map
\[
\left[ \frac{\det(A_0 + tB_0)}{t} : \frac{\det(A_1 + tB_1)}{t} : \frac{\det(A_2 + tB_2)}{t} : \frac{\det(A_3 + tB_3)}{t} \right]
\]
allows to go from $E_2$ to a general element of $E_3$.

Furthermore $E_3$ and $\text{rule} \, \mathcal{D}_{3,3}$ are different components (Proposition 3.6). \hfill \square
5. (3,4)-Cremona Transformations

5.1. General description of (3,4) birational maps. The ruled maps \(ruled_{3,4}\) give rise to an irreducible component (Proposition 3.6). Let us now focus on the case where the linear system \(\Lambda_\psi\) associated to \(\psi \in Bir_{3,4}(\mathbb{P}^4)\) contains a cubic surface without double line.

- First case: \(C_1\) is smooth. From \(h^0\omega_{C_1}(h) = 3\) one gets that \(C_2\) lies on five cubics. Since \(h^0O_C(2h) = 0\) the curve \(C_1\) lies on a quadric, and \(h^0\omega_{C_2} = 1\) thus \(\omega_{C_1} = O_{C_2}\). This configuration corresponds to \(\mathcal{E}_5\) (see §A).

- Second case: \(C_1\) is a singular curve of degree 4 not contained in a plane (see Lemma 3.3) so \(\omega_{C_1} = O_{C_1}\). The curve \(C_1\) lies on two quadrics and \(C_2\) on six cubics (\(h^0\omega_{C_1}(h) = 4\)). Let \(p\) be the singular point of \(C_1\); all elements of \(\Lambda_\psi\) are singular at \(p\) (Theorem 3.4). Let us first prove that \(p\) belongs to \(C_2\): having a singular point outside \(C_2\) requires at least three linear conditions and since there are only six cubics that contain \(C_2\) we do not have enough cubics to get a birational map. The curve \(C_2\) is linked to a line \(\ell\) in a \((2,3)\) complete intersection \(Q_2 \cap S_1\) (with \(\deg Q_2 = 2\) and \(\deg S_1 = 3\)) hence \(C_2 = (Q_2, S_1, S_2)\) with \(\deg S_2 = 3\) and \(p^2 \cap (Q_2, S_1, S_2) = (Q_2p, S_1, S_2)\) and \(I_p \subset (Q_2p, S_1, S_2)\). More precisely, in general, the image of \(\mathbb{P}^3\) by \((Q_2p, S_1, S_2)\) is a quadric in \(\mathbb{P}^4\) so we need a base point to get a birational map. Indeed, in general, \(C_1\) is a nodal cubic at \(p\) and so does \(C_2\), it is the configuration \((2,0,2,0)\) of Table 1 (see §B). By restriction of liaison exact sequence (Lemma 3.1) note that length \((C_1 \cap C_2) = \deg \omega_{C_1}^\vee(2h) = 4.2 = 8\); but length \((C_1 \cap C_2)_{(p)} = 2\) so there is 6 base points. Hence

\[
3\deg C_1 - \text{length}(C_1 \cap S)_{(p)} - \sum_{i=1}^{6} \text{length}(C_1 \cap C)_{(p_i)} = 12 - 4 - 6 = 2.
\]

In other words \(I_p = (Q_2p, S_1, S_2) \cap (p_1)\) where \(p_1\) denotes an ordinary point in general position.

Let us give some explicit examples, the generic one and the degeneracies considered by Hudson:

- Description of \(\mathcal{E}_7\). The quadric \(Q_2\) is smooth at \(p\), and \(\text{rk} Q_2\) is maximal. Hence the point \(p = (z_0, z_1, z_2)\) is an ordinary quadratic singularity of the generic element of \(\Lambda_\psi\), we are in the configuration \((2,0,2,0)\) of Table 1 (see §D). One can choose

\[
\ell = (z_1, z_2), \quad Q_2 = z_0z_3 - z_1z_2, \quad S_1 = z_3Q_1 + z_1Q_4, \quad S_2 = z_2Q_3 + z_0Q_4
\]

with \(Q_3, Q_4 \in \mathbb{C}[z_0, z_1, z_2, 2], \text{rk} Q_3 \geq 2\).

There are two ways to obtain a binode:

1. \(Q_2\) is smooth at \(p\) and one of the two biplanes is contained in \(T_pQ_2\) (family \(\mathcal{E}_{7.5}\));
2. \(Q_2\) is an irreducible cone with vertex \(p\) (family \(\mathcal{E}_8\)).

- Description of \(\mathcal{E}_{7.5}\). In that case, \(p = (z_0, z_1, z_2)\) is a binode, we are in the configuration \((2,0,3,1)\) of Table 1 (see §D), and one can take without loss of generality:

\[
\ell = (z_2, z_3), \quad Q_2 = z_0z_3 - z_1z_2, \quad S_1 = z_3Q_1 + z_2Q_4, \quad S_2 = z_1Q_3 + z_0Q_4
\]

with \(Q_3 = z_0(\alpha_0z_0 + \alpha_1z_1 + \alpha_2z_2)\) and \(Q_4 \in \mathbb{C}[z_0, z_1, z_2, 2]\). The curve \(C_2\) is reducible: it contains \((z_0, z_2)\). The set of such maps is denoted \(\mathcal{E}_{7.5}\), this case does not appear in Table VI but should appear.
• Description of $\mathcal{E}_8$. We can take $\ell = (z_0, z_1)$, $Q_2 = z_0^2 - z_1z_2$.

$$S_1 = z_0(z_3L + Q_3) + z_1Q_4, \quad S_2 = z_2(z_3L + Q_3) + z_0Q_4.$$ 

with $L \in \mathbb{C}[z_0, z_1, z_2]$, $Q_i \in \mathbb{C}[z_0, z_1, z_2]$. This corresponds to the configuration $(2, 0, 3, 0)$ of Table 1 (see §7).

• Description of $\mathcal{E}_9$. The rank of $Q_2$ is 2, the point $p = (z_0, z_1, z_2)$ is a double point of contact, we are in the configuration $(2, 0, 4, 1)$ of Table 1 (see §7), and one can take

$$\ell = (z_1, z_3), \quad Q_2 = z_0z_1, \quad S_1 = z_3Q_3 + z_1Q_4, \quad S_2 = z_0Q_3$$

with $Q_3, Q_4$ generic elements of $\mathbb{C}[z_0, z_1, z_2]$. Let us see that $C_2$ is the union of a cubic and two lines:

$$(Q_2, S_1) = (z_0, z_3Q_3(0, z_1, z_2) + z_1Q_4(0, z_1, z_2)) \cup (z_1, z_3Q_3(z_0, 0, z_2))$$

hence

$$(Q_2, S_1) \setminus \ell = (z_0, z_3Q_3(0, z_1, z_2) + z_1Q_4(0, z_1, z_2)) \cup (z_1, Q_3(z_0, 0, z_2)).$$

• Description of $\mathcal{E}_{10}$. One can take

$$\ell = (z_1, z_3), \quad Q_2 = z_0z_1, \quad Q_3 = z_0^2 + z_1z_2, \quad S_1 = z_3Q_3 + z_1Q_4, \quad S_2 = z_0Q_3$$

where $Q_3$ denotes an element of $\mathbb{C}[z_0, z_1, z_2]$. Let us see that $C_2$ is the union of a triple line and a conic:

$$(Q_2, S_1) = (z_0, z_1(z_2z_3 + Q_4(0, z_1, z_2))) \cup (z_1, z_3(z_0^2 + z_1z_2))$$

thus

$$(Q_2, S_1) \setminus \ell = (z_0, z_1) \cup (z_0, z_2z_3 + Q_4(0, z_1, z_2)) \cup (z_1, z_0^2).$$

The general element of $\mathbb{A}_\psi$ has a binode and a double point of contact (configuration $(1, 0, 4, 2)$ and $(2, 0, 4, 1)$ of Table 1, see §7). Note that HUDSON details this case carefully ([8, Chap. XV]).

**Proposition 5.1.** One has the following properties:

$$\dim \mathcal{E}_6 = 38, \quad \mathcal{E}_{7.5} \cup \mathcal{E}_8 \subset \overline{\mathcal{E}_7}$$

and

- a generic element of $\mathcal{E}_{7.5}$ is not a specialization of a generic element of $\mathcal{E}_8$;
- a generic element of $\mathcal{E}_8$ is not a specialization of a generic element of $\mathcal{E}_{7.5}$;
- a generic element of $\mathcal{E}_9$ is a specialization of a generic element of $\mathcal{E}_8$.

**Proof.** The arguments to establish $\dim \mathcal{E}_6 = 38$ are similar to those of the proof of Proposition 4.4.

Let us justify that a generic element of $\mathcal{E}_{7.5}$ is not a specialization of a generic element of $\mathcal{E}_8$ (we take the notations of §5.1): as we see when $\psi \in \mathcal{E}_8$ the quadric $Q_2$ is always singular and it is not the case when $\psi \in \mathcal{E}_{7.5}$. Conversely if $\psi$ belongs to $\mathcal{E}_{7.5}$ then $C_2$ is reducible but if $\psi$ belongs to $\mathcal{E}_8$ the curve $C_2$ can be irreducible and reduced; hence a generic element of $\mathcal{E}_8$ is not a specialization of a generic element of $\mathcal{E}_{7.5}$. \qed
5.1.1. Irreducible components.

**Theorem 5.2.** The set \( \text{rule}d_{3,4} \) is an irreducible component of \( \text{Bir}_{3,4}(\mathbb{P}^3_{\mathbb{C}}) \) and there is only one another irreducible component in \( \text{Bir}_{3,4}(\mathbb{P}^3_{\mathbb{C}}) \).

**Proof.** According to Proposition 3.3, the set \( \text{rule}d_{3,4} \) is an irreducible component of \( \text{Bir}_{3,4}(\mathbb{P}^3_{\mathbb{C}}) \).

Any element \( \psi \) of \( E_7 \cup E_7.5 \cup E_8 \cup E_9 \cup E_{10} \) satisfies the following property:

\[
I_{\psi} = (Q_2p, S_1, S_2) \cap p_1
\]

where \( p \in Q_2, p_1 \) a general point,

\[
Q_2 = \det \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}, \quad S_1 = L_1Q_3 + L_2Q_4, \quad S_2 = L_3Q_3 + L_4Q_4
\]

with \( L_i \in \mathbb{C}[z_0, z_1, z_2, z_3] \), \( Q_i \in \mathbb{C}[z_0, z_1, z_2] \). So \( E_7, E_7.5, E_8, E_9 \) and \( E_{10} \) belong to the same irreducible component.

It remains to show that this component is \( E_6 \); let us consider

\[
J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -z_2 & z_3 & L_0 \\ z_2 & 0 & L_1 & L_2 \\ -z_3 & -L_1 & 0 & L_3 \\ -L_0 & -L_2 & -L_3 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} z_2 \\ z_1 \\ z_0 \\ tz_3 \end{bmatrix}
\]

with \( L_i \) linear forms and

\[
M_t = \begin{bmatrix} Jv \\ Nv \end{bmatrix} = \begin{bmatrix} tz_3 & z_0 & -z_1 & -z_2 \\ tz_3L_0 + Q_2 & q_1 & q_2 & q_3 \end{bmatrix}
\]

with \( Q_2 = z_0z_3 - z_1z_2, \quad q_1 = z_2^2 + z_0L_1 + tz_3L_2, \quad q_2 = -z_2z_3 - z_1L_1 + tz_3L_3, \quad q_3 = -z_2L_0 - z_1L_2 - z_0L_3. \)

For generic \( L_i \)'s and \( t \neq 0 \) the \( 2 \times 2 \) minors of \( M_t \) generate the ideal of a generic elliptic quintic curve as in \( E_6 \). For \( M_0 \) the \( 2 \times 2 \) minors become \( Q_2z_0, Q_2z_1, Q_2z_2, S_1, S_2, S_3 \) with

\[
S_1 = -z_2Q_2, \quad S_2 = -z_1q_3 + z_2q_2, \quad S_3 = z_0q_3 + z_2q_1.
\]

Therefore the ideal \( M_2 \) generated by these minors is

\[
(Q_2z_0, Q_2z_1, Q_2z_2, S_2, S_3).
\]

Denote by \( \ell \) the line defined by \( L_t = (z_1, z_3) \). According to

\[
z_3S_3 = -z_2S_2 + Q_2(q_3 + L_1z_2) \quad \& \quad z_4S_3 = -z_0S_2 - z_2^2Q_2
\]

\( M_2 \) is the ideal of the residual of \( L \) in the complete intersection of ideals \( (Q_2, S_2) \).

It only remains to prove that one can obtain the generic element of \( E_7 \) with a good choice of the \( L_i \)'s, in other words it remains to prove that \( S_2 \) is generic among the cubics singular at \( p \) that contain \( \ell \). Modulo \( Q_2 \) one can assume that \( q_3 = -z_3a + b \), with \( a \) (resp. \( b \)) an element of \( \mathbb{C}[z_1, z_2]_1 \) (resp. \( \mathbb{C}[z_0, z_1, z_2]_2 \)). Then

\[
S_2 = -z_3(z_1a + z_3^2) + z_1(b - z_2L_1);
\]

in conclusion \( S_2 = z_3A + z_2B \) for generic \( A \) and \( B \) in \( \mathbb{C}[z_0, z_1, z_2]_2 \). \( \square \)
5.2. Relations between $\text{Bir}_{3,3}(\mathbb{P}_C^3)$ and $\text{Bir}_{3,4}(\mathbb{P}_C^3)$. One can now state the following result:

**Proposition 5.3.** The closure $\text{rule}_{3,3}$ intersects the closure of any irreducible component of $\text{Bir}_{3,4}(\mathbb{P}_C^3)$.

**Proof.** According to Lemma 2.2 it sufficient to prove that $\text{rule}_{3,3}$ intersects the closure of $(3,4)$ birational maps that are non ruled.

Let us consider an element $\psi$ of $\text{Bir}_{3,4}(\mathbb{P}_C^3)$ whose $C_2$ is the union of the following lines

$$\delta = (z_0, z_1^2), \quad (z_0 - \varepsilon z_2, z_1), \quad \ell_1 = (z_0, z_3), \quad \ell_2 = (z_1, z_2).$$

Denote by $J_\varepsilon = (z_0, z_1^2) \cap (z_0 - \varepsilon z_2, z_1) \cap (z_0, z_3) \cap (z_1, z_2)$. One can check that

$$J_\varepsilon = (z_0 z_1, z_0^2 z_2 + \varepsilon z_0 z_3^2, z_1^2 z_3).$$

Set $I_{\varepsilon} = z_0 z_1 (z_0, z_1, z_2) + (z_0^2 z_2 + \varepsilon z_0 z_3^2, z_1^2 z_3)$. For a general $p_2$ the map $\psi_{p_2}$ defined by $I_{\varepsilon} \cap p_2$ is birational; for $\varepsilon$ non zero $\psi_{p_2} \in \text{Bir}_{3,4} \setminus \text{rule}_{3,4}$, and $\psi_{p_2}$ belongs to $\text{rule}_{3,3}$. \hfill $\Box$

As in the case of $(3,3)$ birational maps one has the following statement:

**Theorem 5.4.** If $p_2 \in \{1, 2\}$, then $\text{Bir}_{3,4,p_2}(\mathbb{P}_C^3)$ is non empty and irreducible.

6. $(3,5)$-Cremona Transformations

6.1. General description of $(3,5)$ birational maps. We already know an irreducible component of the set of $(3,5)$ birational maps: $\text{rule}_{3,5}$ (Proposition 3.6). Let us now consider a $(3,5)$-Cremona transformation $\psi$ such that $\Lambda_{\psi}$ contains a cubic surface without double line.

- First case. Assume that $C_1$ is smooth, then $C_2$ has genus $-1$ and does not lie on a quadric; $C_2$ is the disjoint union of a twisted cubic and a line, that is $\psi$ belongs to $\mathcal{E}_{12}$. Indeed suppose that $\psi \notin \mathcal{E}_{12}$, then $C_2$ is the union of two smooth conics $\Gamma_1$ and $\Gamma_2$ that do not intersect. Any $\Gamma_1$ is contained in a plane $\mathcal{P}$. Denote by $\ell$ the intersection $\mathcal{P}_1 \cap \mathcal{P}_2$. As $\#(\Gamma_1 \cup \Gamma_2) = 4$, all the cubic surfaces that contain $\Gamma_1 \cup \Gamma_2$ contain $\ell$. So $\ell \subset C_2$: contradiction.

- Second case. Suppose now that $C_1$ is not smooth so $p_\psi(C_1) \geq 1$, and

$$p_\psi(C_2) = \deg C_2 - \deg C_1 + p_\psi(C_1) \geq 0.$$

One has the following alternative:

- either $p_\psi(C_1) = 1$, then $C_1$ is singular at $p$. The curve $C_2$ lies on one quadric $Q_2$ and $p_\psi(C_2) = 0$. Since $h^0\omega_{C_2} = h^0I_{C_1}(2h)$, the curve $C_1$ does not lie on a quadric. According to Theorem 3.4 one has $I_{C_1} = (Q_2, S_1, \ldots, S_{d-2})$.

a) Assume first that all the elements of $\Lambda_{\psi}$ are singular at $p$. Then we can show that $\psi$ has two base-points: on the one hand length $(C_1 \cap C_2) = \deg \omega_{C_1}^p(2h) = 10$; in general $C_1$ and $C_2$ are nodal at $p$ and $C_1 \cap C_2$ has 8 other points of simple intersection. On the other hand

$$\text{length}(C_1 \cap S)_{(p)} = 4,$$

and one conclude with Lemma 3.2. Note that as $p_\psi(C_2) = 0$ and $C_2$ is singular, $C_2$ is reducible; hence $C_2$ is the union of a line and a degree 3 curve of arithmetic genus 0, or the union of two conics. So if $\psi$ is not in $\mathcal{E}_{44}$ then $C_2$ is the union of two smooth conics $\Gamma_1$ and $\Gamma_2$ that intersect at a point $p$. Any $\Gamma_1$ lies in a plane $\mathcal{P}_1$; let us
set \( \ell = P_1 \cap P_2 \). Any element of \( \Lambda_p \), singular at \( p \) and containing \( \Gamma_1 \cup \Gamma_2 \), contains \( \ell \) because \( \ell \) contains two distinct points of \( \Gamma_1 \cup \Gamma_2 \) different from \( p \); therefore \( \ell \subset \mathcal{C}_2 \); contradiction. Hence \( \psi \) belongs to \( \mathcal{E}_{14} \).

\( b) \) Suppose now that \( \Lambda_p \) contains a smooth element at \( p \). Then \( p \) is a point of contact, all the cubic surfaces are tangent. Since \( C_2 \subset Q_2 \) is linked to a curve of degree 2 and genus \(-2\), in general \( Q_2 \) is smooth, and \( \mathcal{C}_2 \) is a smooth rational curve on \( Q_2 \). Set \( Q_2 = z_0 z_3 - z_1 z_2, \ell_1 = (z_0, z_1), \) and \( \ell_2 = (z_2, z_3) \); one has

\[ \mathcal{J} = \mathcal{J}_{1,1} (z_0 z_2 z_0, z_1 z_2, z_1 z_3). \]

Let \( S_0 \) be the element of \( \mathcal{J} \) given by \( a z_0 z_2 + b z_0 z_3 + c z_1 z_3 \) where \( a, b, c \) belongs to \( \mathbb{C}[z_0, z_1, z_2, z_3] \); one has \( I_{C_2} = (S_0, Q_2) : \mathcal{J} = (Q_2, S_0, S_1, S_2) \) with

\[ S_1 = z_0^2 a + z_0 z_2 b + z_1^2 c, \quad S_2 = z_2^2 a + z_2 z_3 b + z_3^2 c. \]

The dimension of \( H^0 I_{C_2} (3h) \) is 7; indeed one has the following seven cubics:

\[ Q_2 : (z_0, z_1, z_2, z_3), S_0, S_1, S_2. \]

With a similar argument as in \( a) \) we get that \( \psi \) has no base point. The map \( \psi \) belongs to \( \mathcal{E}_{23} \).

• or \( p_u(C_1) = 2 \). By computing linear conditions one can check that we can not have two points of contact; using \( \mathcal{E}_{27} \) we get that \( p \) is either a D.p's, or a binode, or a double point of contact. In the generic case \( p \) is a D.p's. Set \( p = (z_0, z_1, z_2), p = (z_0, z_1, z_2) \). In general \( C_2 \) belongs to \( \mathcal{C}_2 \) (configuration \((3, 0, 1, 0)\) of \( \mathcal{E}_{27} \) hence \( C \) lies on two quadrics given by \( Q_1, Q_2 : (z_0, z_1, z_2, z_3) \), \( p_u(C_2) = 1 \), and in general \( C_2 \) is smooth and irreducible. The point \( p \) lies on \( C_2 \) otherwise the contribution to \( p_u(C_1) \) is too big. With a similar argument as previously we get that \( \psi \) has, in this case, two base points: indeed

\[ \text{length}(C_1 \cap C_2) = 2 \deg \omega_C (2h) = 2 \deg C_2 = 8 \quad \text{length}(C_1 \cap S)(p) = 6 \]

and Lemma 3.2 allows us to conclude. As \( h^0 I_{C_1} (2h) = h^0 \omega_C = 1 \) the curve \( C_1 \) lies on a quadric. Then

\[ I \psi = (p(Q_1, Q_2)) \cap (p_1) \cap (p_2) \]

and \( \psi \in \mathcal{E}_{13} \).

6.2. Irreducible components.

**Proposition 6.1.** One has the inclusion: \( \mathcal{E}_{14} \subset \mathcal{E}_{12} \).

The set \( \text{Bir}_{3,5}(\mathbb{P}^3) \) has four irreducible components: \( \mathcal{E}_{12}, \mathcal{E}_{13}, \mathcal{E}_{23}, \) and \( \mathcal{E}_{27} = \text{ruled}_{3,5} \).

**Proof.** Let us first prove that \( \mathcal{E}_{14} \subset \mathcal{E}_{12} \). If \( \psi \) belongs to \( \mathcal{E}_{12} \), or to \( \mathcal{E}_{14} \) the curve \( C_2 \) is the union of a line \( \ell \) and a twisted cubic \( \Gamma \) such that length \( (\ell \cap \Gamma) \leq 1 \). Let \( I_\ell \) (resp. \( I_\Gamma \)) be the ideal of \( \ell \) (resp. \( \Gamma \)). In both cases the ideal of \( C_2 \) is \( I_\ell \cdot I_\Gamma \). We can thus specialize an element of \( \mathcal{E}_{12} \) to get an element of \( \mathcal{E}_{14} \).

Note that \( \mathcal{E}_{12} \not\subset \mathcal{E}_{13} \) (resp. \( \mathcal{E}_{12} \not\subset \mathcal{E}_{23} \)): if \( \psi \) is in \( \mathcal{E}_{12} \) then the associated \( C_2 \) does not lie on a quadric whereas if \( \psi \) belongs to \( \mathcal{E}_{13} \) (resp. \( \mathcal{E}_{23} \)) then the associated \( C_2 \) lies on two quadrics (resp. one quadric). Conversely \( \mathcal{E}_{13} \not\subset \mathcal{E}_{12} \) (resp. \( \mathcal{E}_{23} \not\subset \mathcal{E}_{12} \)): if \( \psi \) is an element of \( \mathcal{E}_{13} \) (resp. \( \mathcal{E}_{23} \)), then the associated \( C_2 \) is smooth and irreducible whereas the associated \( C_2 \) of a general element of \( \mathcal{E}_{12} \) is the disjoint union of a twisted cubic and a line.
Let us now justify that \( E_{23} \not\subset E_{13} \): the linear system of an element of \( E_{23} \) has a smooth surface whereas the linear system of an element of \( E_{13} \) does not. Conversely \( E_{13} \not\subset E_{23} \); indeed \( h^0 I_{C_1}(3h) = 6 \) for a birational map of \( E_{13} \) and \( h^0 I_{C_2}(3h) = 7 \) for a birational map of \( E_{23} \).

**Remark 6.2.** Nevertheless it is possible to specialize an element of \( E_{23} \) to obtain a map that belongs to \( E_{13} \) (in particular when we look at the case \( Q_2 \) singular), i.e. \( E_{13} \cap E_{23} \neq \emptyset \).

In bidegree \((3,5)\) the description of \( \text{Bir}_{3,5,p_2}(\mathbb{P}_C^3) \) is very different from those of smaller bidegrees.

**Theorem 6.3.** If \( p_2 \in \{-1, 1\} \), then \( \text{Bir}_{3,5,p_2}(\mathbb{P}_C^3) \) is non empty and irreducible. Nevertheless \( \text{Bir}_{3,5,0}(\mathbb{P}_C^3) \) is non empty and has two irreducible components: one formed by the birational maps of \( E_{44} \) and the other one by the elements of \( E_{23} \).

### 7. Relations with Hudson’s invariants

To prove the birationality of a linear system of cubics, the local properties of \( C_1 \) and \( C_2 \) are required. For instance to apply Lemma 3.2 one needs to understand the support of \( C_1 \cup C_2 \) and the local intersection of \( C_1 \) with a general element of \( \Lambda_\psi \) at any point of \( C_1 \cup C_2 \). So in the following table we make a schematic picture of the tangent cone to \( C_i \) at \( p \) in the different cases considered by Hudson (see §4.2.1 for Hudson’s terminology).

**Convention.** If the point is black (resp. white) then \( C_2 \) does not pass (resp. passes through) through the point. We precise \((\tilde{d}_1, \tilde{p}_1, \tilde{d}_2, \tilde{p}_2)\) where \( \tilde{d}_i \) (resp. \( \tilde{p}_i \)) is the degree (resp. the arithmetic genus) of the tangent cone to \( C_i \) at \( p \).
| Binode | D.p. of Contact |
|--------|----------------|
| (6, 4, 0, -) | (5, 2, 1, 0) |
| (3, 0, 3, 0) | (2, 0, 4, 1) |
| (1, 0, 5, 2) | (0, , 6, 4) |

| Binode | D.p.'s |
|--------|--------|
| (5, 3, 0, -) | (4, 2, 1, 0) |
| (4, 1, 1, 0) | (3, 1, 2, 0) |
| (3, 0, 2, 0) | (2, 0, 3, 0) |
| (2, 0, 3, 1) | (1, 0, 4, 1) |
| (1, 0, 4, 2) | (0, , 5, 3) |

| Binode | Pt of Osculation |
|--------|------------------|
| (4, 1, 0, -) | (3, 0, 1, 0) |
| (2, 0, 2, 0) | (1, 0, 3, 0) |
| (0, , 4, 1) | (0, , 5, 3) |

| Binode | Pt of Contact |
|--------|--------------|
| (3, 1, 0, -) | (2, 0, 1, 0) |
| (1, 0, 2, 0) | (0, , 3, 1) |

| Binode | |
|--------||
| (2, 0, 0, -) | (1, 0, 1, 0) |
| (0, , 2, 0) | (0, , 3, 1) |

Table 1
APPENDIX A. HUDSON’S TABLE

In this appendix we give a reproduction of what HUDSON called “Cubic Space Transformations”. The first (resp. second, resp. third, resp. fourth) table concerns birational maps of bidegrees $(3,2)$, $(3,3)$ and $(3,4)$ (resp. $(3,5)$, resp. $(3,6)$, resp. $(3,7)$, $(3,8)$ and $(3,9)$).
| number | degrees | D.p. of contact | binode | D. p.'s | pt of osculation | pt of contact | ordinary pts | F-curves | Remarks |
|--------|---------|----------------|--------|---------|----------------|--------------|--------------|----------|---------|
| 1      | 3–2     | · · · · ·       | · · · · | · ·     | · · · · ·      | ·            | ·            | $l^2, l_1, l_2, l_3$ | 3 generators meet double line |
| 2      | 3–3     | · · · · ·       | · · · · | · ·     | · ·           | ·            | ·            | $\omega_6$ (genus 3)  | |
| 3      | 1       | · · · · ·       | · · · · | · ·     | · ·           | ·            | ·            | $\omega_6 \equiv O^2$ (genus 3) | |
| 4      | 1       | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | 2            | $l^2, l_1, l_2$ | 2 generators meet double line |
| 5      | 1       | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | ·            | $\omega_6 \equiv O^4$ (rational) | |
| 6      | 3–4     | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | 1            | $\omega_5$ (genus 1) | |
| 7      | 1       | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | 1            | $\omega_5 \equiv O^2_1$ (genus 1) | |
| 8      | 1       | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | 1            | $\omega_5 \equiv O^2_1(2)$ (rational) | |
| 9      | 1       | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | 1            | $\omega_3 \equiv O^2_1, l_1 \equiv O_1, l_2 \equiv O_1$ | |
| 10     | 1       | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | 1            | $\omega_2 \equiv O_1 O_2, l \equiv O_1 O_2$ (osculation) | |
| 11     | 1       | · · · · ·       | · · · · | · ·     | · · · · ·     | ·            | 4            | $l^2, l_1$ | $(\phi)$ touch plane along $l$ generator meets double line |
| number | D.p. of contact | binode | D. p.'s | pt of osculation | pt of contact | ordinary pts | F-curves | Remarks |
|--------|----------------|--------|---------|------------------|---------------|--------------|----------|---------|
| 12     | ·               | ·      | ·       | ·                | ·             | 2            | $\omega_3$ (rational), $l$ |
| 13     | ·               | ·      | 1       | ·                | ·             | 2            | $\omega_4 \equiv O_1$ (genus 1) |
| 14     | ·               | ·      | 1       | ·                | ·             | 2            | $\omega_3 \equiv O_1$ (rational), $l \equiv O_1$ |
| 15     | ·               | 1      | ·       | ·                | ·             | 2            | $\omega_4 \equiv O_1^2(2)$ |
| 16     | ·               | 1      | ·       | ·                | ·             | 2            | $\omega_2 \equiv O_1(1), l_1 \equiv O_1(1), l_2 \equiv O_1$ |
| 17     | 1               | ·      | ·       | ·                | ·             | 2            | $\omega_3 \equiv O_1(2), l_1 \equiv O_1$ |
| 18     | 1               | ·      | ·       | ·                | ·             | 2            | $l \equiv O_1$ (contact), $l_1 \equiv O_1, l_2 \equiv O_1$ (φ) touch quadric |
| 19     | ·               | ·      | 2       | ·                | ·             | 2            | $\omega_3 \equiv O_1O_2$ (rational), $l \equiv O_1O_2$ |
| 20     | ·               | 1      | 1       | ·                | ·             | 2            | $\omega_3 \equiv O_1^2, l_1 \equiv O_1$ |
| 21     | 1               | ·      | 1       | ·                | ·             | 2            | $l \equiv O_1O_2$ (contact), $l_1 \equiv O_1, l_2 \equiv O_1$ (φ) touch plane |
| 22     | 1               | 1      | ·       | ·                | ·             | 2            | $l \equiv O_1O_2(1)$ (osculation), $l_1 \equiv O_1$ (φ) touch plane |
| 23     | ·               | ·      | ·       | 1                | ·             | $\omega_4$ (rational) |
| 24     | ·               | ·      | 1       | ·                | ·             | $\omega_4 \equiv O_1^2$ |
| 25     | ·               | 1      | ·       | ·                | 1             | $\omega_3 \equiv O_1^2(1), l \equiv O_1(1)$ |
| 26     | 1               | ·      | ·       | ·                | 1             | $l_1 \equiv O_1, l_2 \equiv O_1, l_3 \equiv O_1, l_4 \equiv O_1$ |
| 27     | ·               | ·      | ·       | ·                | ·             | 6            | $l^2$ |

Cubic Space Transformations of bidegree $(3, 5)$
| number | D.p. of contact | binode | D. p.'s | pt of osculation | pt of contact | ordinary pts | F-curves | Remarks |
|--------|----------------|--------|---------|-----------------|---------------|-------------|----------|---------|
| 28     | .              | .      | .       | .               | .             | 3           | $l$ (contact), $l_1$ |         |
| 29     | .              | .      | 1       | .               | .             | 3           | $O_3$ (plane, genus 1) |         |
| 30     | .              | .      | 1       | .               | .             | 3           | $O_2$, $l \equiv O_1$ |         |
| 31     | .              | .      | 1       | .               | .             | 3           | $l \equiv O_1$ (contact), $l_1$ |         |
| 32     | .              | .      | 1       | .               | .             | 3           | $l \equiv O_1$ (osculation) |         |
| 33     | .              | 1      | .       | .               | .             | 3           | $O_2 \equiv O_1(1)$, $l \equiv O_1$ |         |
| 34     | 1              | .      | .       | .               | .             | 3           | $O_3 \equiv O_1^2$ |         |
| 35     | 1              | .      | .       | .               | .             | 3           | $l \equiv O_1$ (contact), $l_1 \equiv O_1$ | (φ) touch quadric |
| 36     | .              | .      | 2       | .               | .             | 3           | $O_2 \equiv O_1$, $l \equiv O_1O_2$ |         |
| 37     | .              | 1      | 1       | .               | .             | 3           | $O_2 \equiv O_1(1)O_2$, $l \equiv O_1O_2$ |         |
| 38     | .              | 1      | 1       | .               | .             | 3           | $l \equiv O_1O_2$, $l_1 \equiv O_1(1)$, $l_2 \equiv O_1(1)$ |         |
| 39     | 1              | .      | 1       | .               | .             | 3           | $l \equiv O_1O_2$ (contact), $l_1 \equiv O_1$ | (φ) touch plane |
| 40     | 1              | 1      | .       | .               | .             | 3           | $l \equiv O_1O_2(1)$ osculation | (φ) touch plane |
| 41     | .              | .      | .       | 1               | 1             | $l_1, l_2, l_3$ |         |
| 42     | .              | .      | 1       | .               | 1             | $O_3 \equiv O_1$ (rational) |         |
| 43     | .              | .      | 1       | .               | 1             | $l_1 \equiv O_1$, $l_2 \equiv O_1$, $l_3$ |         |
| 44     | .              | 1      | .       | .               | 1             | $O_3 \equiv O_1^2(1)$ |         |
| 45     | .              | 1      | .       | .               | 1             | $O_2 \equiv O_1(1)$, $l \equiv O_1(1)$ |         |
| 46     | .              | 1      | .       | .               | 1             | $l \equiv O_1(1)$ (contact), $l_1 \equiv O_1$ | (φ) touch quadric |
| 47     | 1              | .      | .       | .               | 1             | $l_1 \equiv O_1$, $l_2 \equiv O_1$, $l_3 \equiv O_1$ |         |
| 48     | .              | .      | 2       | .               | 1             | $l \equiv O_1O_2$, $l_1 \equiv O_1$, $l_2 \equiv O_2$ |         |
| 49     | .              | 1      | 1       | .               | 1             | $l \equiv O_1(1)O_2$ (contact), $l_2 \equiv O_1$ | (φ) touch plane |
| 50     | .              | .      | 3       | .               | 1             | $l_1 \equiv O_2O_3$, $l_2 \equiv O_3O_1$, $l_3 \equiv O_1O_2$ | O_2 on fixed plane at O_1 |
| 51     | .              | .      | .       | 1               | .             | $O_3$ (rational) |         |
| 52     | .              | .      | 1       | 1               | .             | $O_3 \equiv O_1^2$ |         |

Cubic Space Transformations of bidegree $(3, 6)$
| number | degrees | D.p. of contact | binode | D. p.'s | point of osculation | point of contact | ordinary points | F-curves | Remarks |
|--------|---------|-----------------|--------|---------|---------------------|-----------------|----------------|----------|---------|
| 53     | 3–7     | 1               | -      | -       | -                   | -               | 4              | $l \equiv O_1$ (contact) | (φ) touch quadric |
| 54     |         |                 | 2      | -       | -                   | -               | 4              | $l \equiv O_1O_2, l_1$ |                     |
| 55     |         |                 | 1      | 1       | -                   | -               | 4              | $l \equiv O_1O_2, l_1 \equiv O_1(1)$ |                     |
| 56     |         |                 | 1      | 1       | -                   | -               | 4              | $l \equiv O_1O_2$ (contact) | (φ) touch plane |
| 57     |         |                 | 1      | 1       | -                   | -               | 1              | $\omega_2$ |                     |
| 58     |         |                 | 1      | -       | -                   | 1               | 2              | $\omega_2 \equiv O_1(1)$ |                     |
| 59     |         |                 | 1      | -       | -                   | -               | 2              | $l_1 \equiv O_1, l_2 \equiv O_1$ |                     |
| 60     |         |                 | -      | 1       | -                   | -               | 2              | $l \equiv O_1, l_1$ |                     |
| 61     |         |                 | -      | 1       | -                   | -               | 2              | $l_1 \equiv O_1(1), l_2 \equiv O_1$ |                     |
| 62     |         |                 | -      | 1       | -                   | -               | 2              | $l \equiv O_1(1)$ (contact) | (φ) touch quadric |
| 63     |         |                 | -      | 2       | -                   | -               | 2              | $l \equiv O_1O_2, l_1 \equiv O_1$ |                     |
| 64     |         |                 | -      | 1       | 1                   | -               | 2              | $l_1 \equiv O_1(1)O_2$ (contact) | (φ) touch plane |
| 65     |         |                 | -      | 1       | 1                   | -               | 1              | $\omega_2 \equiv O_1$ | $O_2$ on fixed plane at $O_1$ |
| 66     |         |                 | -      | 1       | 1                   | -               | 1              | $l_1 \equiv O_1(1), l_2 \equiv O_1(1)$ |                     |
| 67     | 3–8     | -               | 1      | 1       | -                   | -               | 5              | $l \equiv O_1O_2$ |                     |
| 68     |         |                 | 1      | -       | -                   | 1               | 3              | $l \equiv O_1$ |                     |
| 69     |         |                 | 1      | -       | 2                   | 1               | 1              | $l \equiv O_1$ |                     |
| 70     |         |                 | -      | 1       | 1                   | -               | 2              | $l$ |                     |
| 71     |         |                 | -      | 1       | 1                   | -               | 2              | $l \equiv O_1(1)$ |                     |
| 72     | 3–9     | 1               | -      | -       | -                   | 1               | 4              |                     | 4-point contact at $O_2$ |
| 73     |         |                 | -      | 1       | -                   | -               | 3              |                     |                     |
| 74     |         |                 | -      | 1       | -                   | 3               | 1              |                     |                     |
| 75     |         |                 | -      | 1       | 1                   | -               | 2              |                     |                     |
REFERENCES

[1] D. Avritzer, G. Gonzalez-Sprinberg, and I. Pan. On singular quadratic complexes, quintic curves and Cremona transformations. Rend. Circ. Mat. Palermo (2), 61(2):201–240, 2012.

[2] S. Cantat. The Cremona group in two variables. available at http://perso.univ-rennes1.fr/serge.cantat/publications.html.

[3] S. Cantat. Morphisms between Cremona groups and a characterization of rational varieties. available at http://perso.univ-rennes1.fr/serge.cantat/publications.html.

[4] D. Cerveau and J. Déserti. Transformations birationnelles de petit degré. Cours Spécialisés [Specialized Courses]. Société Mathématique de France, Paris, to appear.

[5] L. Cremona. Sulle trasformazioni razionali nello spazio. Annali di Mat., V:131–162, 1871-1873.

[6] J. Déserti. Some properties of the Cremona group, volume 21 of Ensaios Matemáticos [Mathematical Surveys]. Sociedade Brasileira de Matemática, Rio de Janeiro, 2012.

[7] W. Fulton. Intersection theory, volume 2 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, second edition, 1998.

[8] H. P. Hudson. Cremona Transformations in Plane and Space. Cambridge University Press. 1927.

[9] I. Pan. Sur les transformations de Cremona de bidegré (3,3). PhD thesis, Université de Genève, 1996.

[10] I. Pan. Sur les transformations de Cremona de bidegré (3,3). Enseign. Math. (2), 43(3-4):285–297, 1997.

[11] I. Pan. Une remarque sur la génération du groupe de Cremona. Bol. Soc. Brasil. Mat. (N.S.), 30(1):95–98, 1999.

[12] I. Pan, F. Ronga, and T. Vust. Transformations birationnelles quadratiques de l’espace projectif complexe à trois dimensions. Ann. Inst. Fourier (Grenoble), 51(5):1153–1187, 2001.

[13] C. Peskine and L. Szpiro. Liaison des variétés algébriques. I. Invent. Math., 26:271–302, 1974.

E-mail address: deserti@math.jussieu.fr

E-mail address: han@math.jussieu.fr

INSTITUT DE MATHEMATIQUES DE JUSSIEU, UMR 7586, UNIVERSITE PARIS 7, BATIMENT SOPHIE GERMAIN, CASE 7012, 75205 PARIS CEDEX 13, FRANCE.