Quantum version of Wielandt’s Inequality revisited

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September 13, 2018

Abstract

Consider a linear space $L$ of complex $D$-dimensional linear operators, and assume that some power $L^k$ of $L$ is the whole set $\text{End}(\mathbb{C}^D)$. Perez-Garcia, Verstraete, Wolf and Cirac conjectured that the sequence $L^1, L^2, \ldots$ stabilizes after $O(D^2)$ terms; we prove that this happens after $O(D^2 \log D)$ terms, improving the previously known bound of $O(D^4)$.

1 Introduction

The main motivation of this article is a conjecture of Perez-Garcia, Verstraete, Wolf and Cirac [PGVWC07], which can be stated as follows:

Conjecture. Let $L$ be a linear space of $D \times D$ matrices. If $\dim L^k = D^2$ for some $k$, then it also holds for all $k \geq cD^2$ for some constant $c$ not depending on $D$.

Here by $L^k$ we mean the linear space spanned by all products of $k$ matrices in $L$ - for precise definition see Section 2. The state of the art is the main result of [SPGWC10], which instead of the bound $O(D^2)$ provides $O(D^4)$.

There are several motivations to study this conjecture. The original one is the geometry of uniform Matrix Product States. Indeed, as shown in [PGVWC07] the conjecture has direct consequences on the representations of the $W$-state as matrix product states. Let us cite [PGVWC07, Section A]: "The conjectures, if true, can be used to prove a couple of interesting results, one concerning the MPS representation of the $W$-state, and the other concerning the approximation by MPS of ground states of gapped Hamiltonians."

In particular, the conjecture implies that families of tensors described as uniform Matrix Product States may be not closed. Further, as explained in [SPGWC10] it can be regarded as a quantum analogue of Wielandt’s inequality [Wie50]. The result has many other applications including "dichotomy theorems for the zero–error capacity of quantum channels and for the Matrix Product State (MPS) dimension of ground states of frustration-free Hamiltonians" and "new bounds on the required interaction-range of Hamiltonians with unique MPS ground state" [SPGWC10].

Our main new input is to relate this conjecture to another classical open problem in pure algebra. The question is to bound the $k$ under the assumption that $\dim L_L^1 + L_L^2 + \cdots + L_L^k = D^2$. This is an older open problem posed by Paz [Paz84], who conjectured that the correct optimal bound for $k$ in this setting is $2D - 2$. He was able to prove an upper bound of $D^2/3 + 2/3$, which was later improved to $O\left(D^{1.5}\right)$ by Pappacena [Pap97]. The latest best known bound is $O(D \log D)$ by the second author [Shi18]. The $2D - 2$ conjecture is known to hold if $L$ contains a non-derogatory matrix [GLMS18] and for $D \leq 5$ (see [Shi18]). This approach leads to our main theorem:
**Theorem.** Let $L$ be a linear space of $D \times D$ matrices. If $\dim L^k = D^2$ for some $k$, then it also holds for all $k \geq 2D^2 (6 + \log_2(D))$.

In particular, we confirm that the exponent conjectured in [PGVWC07] is indeed equal to two.

After finishing the articles the authors learned that a related topic has been recently studied by Rahaman in [Rah18]. Under additional positivity assumptions the author proves an $O(D^2)$ bound for the index of primitivity. This is a related quantity; however it is not associated to a linear subspace of matrices, but rather a (primitive, positive) operator on the space of matrices. Our results remain independent, apart from the fact that the bound we provide is also a bound for the index of primitivity if the operator admits a Kraus decomposition.

**Acknowledgements**

We would like to thank Khazhgali Kozhasov, Joseph Landsberg, Tim Seynnaeve and Emanuele Ventura for discussions on the topic. MM was supported by Polish National Science Center project 2013/08/A/ST1/00804 affiliated at the University of Warsaw.

2** Notation**

We fix a complex $D$ dimensional vector space $V \simeq \mathbb{C}^D$. Let $L = L^1 \subset \text{End} V$ be a subspace of linear endomorphisms of $V$. We fix a basis $A_1, \ldots, A_{\dim L}$ of $L$ and regard each $A_i$ as a $D \times D$ matrix. Let $L^j$ be the linear subspace of $\text{End}(V)$ spanned by products of (not necessarily distinct) $j$ elements of $L$. In particular, a generator of $L^j$ can be regarded as a word $A_{i_1} \cdots A_{i_j}$ of length $j$.

More generally for any linear space $S \subset \text{End}(V)$ we define:

1. $S^j \subset \text{End}(V)$ as the space generated by products of $j$ elements of $S$,
2. $S^{\leq t} := \sum_{j=1}^{t} S^j$.

We say that a matrix $M$ is zero-square if $M^2 = 0$.

3** Quantum version of Wielandt’s Inequality**

Throughout this section we work under the assumption that $\dim L^j = D^2$ for $j$ large enough. We start with a general lemma taken from [Shi18].

**Lemma 3.1** (Claim 13 in [Shi18]). Let $S \subset \mathbb{C}^{n \times n}$, $P \in \mathbb{C}^{p \times n}$, $Q \in \mathbb{C}^{n \times q}$. Let $k$ be the smallest integer such that $P S^k Q \neq 0$. Then, for any $A_1, \ldots, A_k \in S$, we have $\text{rank}(PA_1 \ldots A_k Q) \leq n/k$.

**Lemma 3.2.** Assume that $L^\lambda$ contains a square-zero matrix $H$ of rank $\rho > 0$ with $\lambda \rho \leq D(1 + \log_2 \frac{D}{\rho})$. Then either

1. a square-zero matrix of rank $\rho_1 \in [1, 0.5\rho]$ is contained in $L^{\lambda_1}$ with $\lambda_1 \rho_1 \leq D(1 + \log_2 \frac{D}{\rho_1})$, or
2. a non-nilpotent matrix of rank at most $\rho$ is contained in $L^\Lambda$ with $\Lambda \leq \lambda + 2D/\rho$.

**Proof.** We choose a basis such that

$$H = \begin{pmatrix} O & O & I_\rho \\ O & O & O \\ O & O & O \end{pmatrix}$$

and define $P = (O|O|I_\rho)$ and $Q = (I_\rho|O|O)^\top$. Let $k$ be the smallest integer for which there exist $A_1, \ldots, A_k \in L$ satisfying $PA_1 \ldots A_k Q \neq 0$ (such an integer exists because $L$ generates the whole matrix ring as a C-algebra). Let $A = A_1 \ldots A_k$ and $A' = PAQ$ be the bottom left block of $A$.

**Case 1.** Assume $k \leq 2D/\rho$. If $A'$ is not nilpotent, then $HA$ is a non-nilpotent matrix of rank at most $\rho$, which makes the condition (2) valid. Otherwise, $A'$ is a nilpotent of index $\alpha > 1$, and then $H_1 = (HA)^{\alpha - 1}H$ is a square-zero matrix of non-zero rank $\rho_1 \leq \rho/\alpha$. Note that $H_1$ is spanned by words of length at most:

$$(\alpha - 1)(\lambda + k) + \lambda = \alpha \lambda + (\alpha - 1)k \leq \lambda \rho/\rho_1 + 2D(\frac{\rho}{\rho_1} - 1)/\rho_1 \leq \frac{D}{\rho_1} \left(1 + \log_2 \frac{D}{\rho} + 2(1 - \frac{\rho_1}{\rho})\right).$$
To prove that condition (1) holds it remains to show that:

\[ \log_2 \frac{D}{\rho} + 2(1 - \frac{\rho_1}{\rho}) \leq \log_2 \frac{D}{\rho_1}, \]

which is equivalent to:

\[ 2 + \log_2 \frac{\rho_1}{\rho} \leq 2 \frac{\rho_1}{\rho}. \]

One can easily verify this inequality, as \( 0 \leq \frac{\rho_1}{\rho} \leq \frac{1}{2} \).

**Case 2.** Assume \( k \geq 2D/\rho \). Note that \( HAH \) has \( A' \) at the upper right block and zeros everywhere else. Lemma 3.1 shows that the rank of \( HAH \) is \( \rho_1 \leq D/k \leq 0.5\rho \). Further, \( HAH \) is spanned by words of length at most

\[ 2\lambda + k \leq \lambda \rho/\rho_1 + D/\rho_1 \leq \frac{D}{\rho_1} \left( 2 + \log_2 \frac{D}{\rho} \right) \leq \frac{D}{\rho_1} \left( 1 + \log_2 \frac{D}{\rho_1} \right). \]

Hence, condition (1) holds. \( \square \)

**Lemma 3.3.** There exists \( R > 0 \) and

\[ A \leq \frac{D}{R} \left( 3 + \log_2 \frac{D}{R} \right) \]

such that \( L^A \) contains a non-nilpotent matrix of rank \( R \).

**Proof.** If \( L \) contains a non-nilpotent matrix, then we are done. Otherwise, there is a matrix \( A \in L \) of nilpotency index \( \lambda_0 + 1 > 1 \). The matrix \( A^{\lambda_0} \) is square-zero, belongs to \( L^{\lambda_0} \), and has rank \( \rho_0 \in [1, D/(\lambda_0 + 1)] \). Now we repeatedly apply Lemma 3.2 until we end up under the condition (2) of it; we obtain a sequence \( (\lambda_0, \rho_0), \ldots, (\lambda_\tau, \rho_\tau) \). We write \( R = \rho_\tau \) and assume that we fall into case (2) of Lemma 3.2 after applying it to \( (\lambda_{\tau-1}, \rho_{\tau-1}) \). As \( \lambda_{\tau-1}\rho_{\tau-1} \leq D(1 + \log_2 \frac{D}{\rho_{\tau-1}}) \) we get:

\[ \lambda_\tau \leq \lambda_{\tau-1} + \frac{2D}{\rho_{\tau-1}} \leq \frac{D(3 + \log_2 \frac{D}{\rho_{\tau-1}})}{\rho_{\tau-1}} \leq \frac{D}{R} \left( 3 + \log_2 \frac{D}{R} \right). \]

\( \square \)

Our aim is to bound from above the smallest \( j \) for which \( \dim L^j = D^2 \). From now on we set:

\[ \mathcal{I} := \min \{ j : \dim L^j = D^2 \}. \]

The following Lemma is based on the techniques presented in [SPGWC10, Section 3]. We include a complete proof for the sake of completeness.

**Lemma 3.4.** Suppose we have a non-nilpotent matrix \( B \in L^A \) of rank \( R \). Then \( \mathcal{I} \leq A(R+1)D \).

**Proof.** **Step 0:** After rescaling \( B \), we may assume there exists an eigenvector \( v \) with \( Bv = v \). By passing from the sequence \( L^j \) to the subsequence \( L^{j-A} \) we may assume that \( B \in L^1 \) and we want to prove that \( \dim L^{(R+1)D} = D^2 \).

**Step 1:** Consider the sequence of vector subspaces of \( \mathbb{C}^D \) defined by:

\[ M_j := (L^1 + \cdots + L^j)v. \]

Clearly, \( M_1 \subset M_2 \subset \ldots \). Further, if \( M_j = M_{j+1} \), then \( M_j = M_{j+k} \) for any \( k \). Indeed, the former equality is equivalent to \( L_{j+1}v \subset (L^1 + \cdots + L^j)v \). In such a case, by induction on \( k \) we have:

\[ L^{j+1+k}v = \bigoplus_{A \in L^k} AL^{j+1}v \subset \bigoplus_{A \in L^{k+1}, 1 \leq i \leq j} AL^i v \subset M_{j+k} \subset M_j. \]

Hence, \( \dim M_j < \dim M_{j+1} \) unless \( M_j = M_{j+1} = \ldots \). As the sequence \( L^j \) is spanning, we must have \( M_j = \mathbb{C}^D \) for large \( j \). We conclude that \( M_k = \mathbb{C}^D \) for \( k \geq D \), by dimension count. It follows that for any \( w \in \mathbb{C}^D \) there exist such elements \( W_i \in L^i \) that:

\[ w = \sum_{i=1}^{D} W_i v. \]
However, then we also have \( w = \sum_{i=1}^{D} W_i B^{D-i} v \) and \( W_i B^{D-i} \in L^D \). We have proved that \( L^D v = \mathbb{C}^D \).

**Step 2:** We fix a basis, starting from \( v \), in which \( B \) is in Jordan normal form. We assume that first \( s \) eigenvalues of \( B \) are nonzero. Clearly \( s \leq R \). Let \( P \) be the projector onto the vector space \( V' \subset V \) spanned by first \( s \) basis vectors. We consider the following spaces of matrices \( M_j := PL^j \). We claim that \( \dim \tilde{M}_j < \dim \tilde{M}_{j+1} \), unless \( \dim \tilde{M}_j = sD \), i.e. it is maximal possible. Indeed, let \( W_1, \ldots, W_k \) be a basis of \( M_j \). These are linearly independent operators from \( V \) to \( V' \). We set \( \tilde{W}_i := BW_i \). As \( PB = BP \) we have \( \tilde{W}_i \in M_{j+1} \). Further, as the restriction of \( B \) to \( V' \) is invertible, we see that \( \tilde{W}_i \) are linearly independent. We see that \( \dim \tilde{M}_j \leq \dim \tilde{M}_{j+1} \). If equality holds, then \( \tilde{W}_i \) span \( \tilde{M}_{j+1} \). In this situation we have \( \tilde{M}_{j+k+1} = \tilde{M}_{j+1} L^k = B \tilde{M}_j L^k = B \tilde{M}_{j+k} \). In particular, \( \dim \tilde{M}_{j+k} \) is constant for \( k \geq 0 \). As the sequence \( L_i \) is spanning this can happen only if \( \dim \tilde{M}_j = sD \). By dimension count, it follows that \( \dim \tilde{M}_j = sD \).

We have \( B^R = \tilde{M}_s D = B^R = \tilde{M}_s D \). As \( B^R \) restricted to \( V' \) is an isomorphism we see that \( B^R \subset L^D \cup L_{D+D} - R - s \) contains all linear maps from \( V \) to \( V' \). In particular, for any \( w \in V \) there exists such \( M \in L_{RD} \) of rank one that \( M w = v \).

**Step 3:** We prove that \( \dim L^{(R+1)D} = D^2 \), by showing that all rank one matrices belong to \( L^{(R+1)D} \). Fix arbitrary two vectors \( v_1, v_2 \in V \). We construct a rank one matrix in \( L^{(R+1)D} \) that sends \( v_1 \) to \( v_2 \). By Step 1 there exists such a matrix \( M_1 \in L^D \) that \( M_1 v = v_2 \). By Step 2 there exists such a matrix \( M_2 \in L_{RD} \) of rank one that \( M_2 v_1 = v_2 \). Clearly \( M_1 M_2 \in L^{(R+1)D} \) is of rank one and \( M_1 M_2 v_1 = v_2 \), which finishes the proof of the Lemma.

**Theorem 3.5.** We have \( I \leq 2D^2 \left( 6 + \log_2 D \right) \), i.e. \( \dim L^k = D^2 \) for some \( k \) if and only if \( \dim L^{\left( 2D^2 \left( 6 + \log_2 D \right) \right)} = D^2 \).

**Proof.** By Lemma 3.3 we know there exists a rank \( R \) non-nilpotent matrix \( A \in L^A \) with

\[
A \leq \frac{D}{R} \left( 3 + \log_2 \frac{D}{R} \right).
\]

Applying this to Lemma 3.4 we obtain:

\[
I \leq \left( \frac{D}{R} \left( 3 + \log_2 \frac{D}{R} \right) \right) (R + 1)D.
\]

As \( 1 \leq R \leq D \) the above value is maximized for \( R = 1 \) which gives the result.

**Remark 3.6.** One could consider a ‘dual’ question:

Suppose \( L^j = 0 \) for some \( j \), what are the bounds on \( j \)?

This is much easier, as in fact \( L^j = 0 \) if and only if \( L^D = 0 \). Indeed, if \( L^D = 0 \) for some \( j \) we know that \( \sum_{j=1}^{\infty} L^j \) is an algebra of nilpotent matrices. In particular, it is a Lie algebra consisting of nilpotent matrices. Thus by Engel’s theorem, all matrices in the algebra can be simultaneously brought into upper-diagonal form. Hence, \( L^D = 0 \).

Clearly, \( D \) is optimal, as demonstrated by an example when \( L^1 \) consists of all (nilpotent) strictly upper-diagonal matrices.

**Remark 3.7.** As one can see, the most problematic case is when \( L^1 \) contains only nilpotent matrices. Of course, still it is possible that \( L^j = \text{End}(V) \) for some \( V \) - examples can be found e.g. in [MOR91].

**Remark 3.8.** We point out that even if \( L^j = \text{End}(V) \) for some \( j \) it is not true that the sequence \( \dim L^j \) has to be weakly monotonic. An example can be found in [Šid64].

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