G-Doob-Meyer Decomposition and Its Application in Bid-Ask Pricing for American Contingent Claim Under Knightian Uncertainty

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Abstract The target of this paper is to establish the bid-ask pricing framework for the American contingent claims against risky assets with G-asset price systems (see [5]) on the financial market under Knight uncertainty. First, we prove G-Doob-Meyer decomposition for G-supermartingale. Furthermore, we consider bid-ask pricing American contingent claims under Knight uncertain, by using G-Doob-Meyer decomposition, we construct dynamic superhedge strategies for the optimal stopping problem, and prove that the value functions of the optimal stopping problems are the bid and ask prices of the American contingent claims under Knight uncertain. Finally, we consider a free boundary problem, prove the strong solution existence of the free boundary problem, and derive that the value function of the optimal stopping problem is equivalent to the strong solution to the free boundary problem.

Keywords G-Doob-Meyer decomposition, American contingent claim, optimal stopping problem, free boundary problem, Bid-ask pricing, Knight Uncertainty

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1 Introduction

The earliest, and one of the most penetrating, analysis on the pricing of the American option is by McKean [20]. There the problem of pricing the American option is transformed into a Stefan or free boundary problem. Solving the latter, McKean writes the American option price explicitly up to knowing a certain function – the optimal stopping boundary.

Bensoussan [2] presents a rigorous treatment for American contingent claims, that can be exercised at any time before or at maturity. He adapts the Black and Scholes [1] methodology of duplicating the cash flow from such a claim to this situation by skillfully managing a self-financing portfolio that contains only the basic instruments of the market, i.e., the stocks and the bond, and that entails no arbitrage opportunities before exercise. Bensoussan shows that the pricing of such claims is indeed possible and characterized the exercise time by means of an appropriate optimal stopping problem. In the study of the latter, Bensoussan employs the so-called "penalization method", which forces rather stringent boundedness and regularity conditions on the payoff from the contingent claim.

From the theory of optimal stopping, it is well known that the value process of the optimal stopping problem can be characterized as the smallest supermartingale majorant to the stopping reward. Base on the Doob-Meyer decomposition for the supermartingale, a "martingale" treatment of the optimal stopping problem is used for handling pricing the American option by Karatzas [11], EL Karoui and Karatzas [12], [13].

The Doob decomposition Theorem was proved by and is named for Joseph L. Doob [6]. The analogous theorem in the continuous time case is the Doob-Meyer decomposition theorem proved by Meyer in [18] and [19]. For the pricing American option problem in incomplete Market, Kramkov [15] constructs the optional decomposition of supermartingale with respect to a family of equivalent local martingale measures. He call such a representation optional because, in contrast to the Doob-Meyer decomposition, it generally exists only with an adapted (optional) process C. He apply this decomposition to the problem of hedging European and American style contingent claims in the setting of incomplete security markets. Using the optional decomposition, Frey [8] consider construction of superreplication strategies via optimal stopping which is similar to the optimal stopping problem that arises in the pricing of American-type derivatives on a family of probability space with equivalent local martingale measures.

For the realistic financial market, the asset price in the future is uncertain, the probability distribution of the asset price in the future is unknown – which is called Knight uncertain [14]. The probability distribution of the nature state in the future is unknown, investors have uncertain subjective belief, which makes their consumption and portfolio choice decisions uncertain and leads the uncertain asset price in the future. Pricing contingent claims against such assets under Knight uncertain is open problem. Peng in [22] and [23] constructs G frame work which is a analysis tool for nonlinear system and is applied in pricing European contingent claims under Knight uncertainty [3], [4] and [5].

The target of this paper is to establish the bid-ask pricing frame work for the American contingent claims against risky assets with G-asset price systems (see [5]) on the financial market under Knight uncertain. Firstly, on sublinear expectation space, by using potential theory and sublinear expectation theory we construct G-Doob-Meyer decomposition for G-supermartingale, i.e., a right continuous G-supermartingale could be decomposed as a G-martingale and a right continuous increasing process and the decomposition is unique. Second, we define bid and ask prices of the American contingent claim against the assets with G-asset price systems, and apply the G-Doob-Meyer decomposition to prove that the bid and ask prices of American contingent claims under Knight uncertain could be described by the optimal stopping problems. Finally, we present a free boundary problem, by using the penalization technique (see Friedman [9]) we derive that if there exists strong super-solution to the free boundary problem, then the strong solution to the free bound-
ary problem exists. And by using truncation and regularization technique, we prove that the strong solution to the free boundary problem is the value function of the optimal stopping problem which is corresponding with pricing problem of the American contingent claim under Knight uncertain.

The rest of this paper is organized as follows. In Section 2, we give preliminaries for the sublinear expectation theory. In Section 3 we prove G-Doob-Meyer decomposition for G-supermartingale. In Section 4, using G-Doob-Meyer decomposition, we construct dynamic superhedge strategies for the contingent claim. In section 5, we consider the pricing problem of the American contingent claims under Knight uncertain. In section 6, we consider a free boundary problem, prove the strong solution existence of the free boundary problem, and derive that the solution of the optimal stopping problem is equivalent the strong solution to the free boundary problem.

2 Preliminaries

Let \( \Omega \) be a given set and let \( \mathcal{H} \) be a linear space of real valued functions defined on \( \Omega \) containing constants. The space \( \mathcal{H} \) is also called the space of random variables.

**Definition 2.1** A sublinear expectation \( \hat{E} \) is a functional \( \hat{E} : \mathcal{H} \rightarrow \mathbb{R} \) satisfying

(i) **Monotonicity:**
\[
\hat{E}[X] \geq \hat{E}[Y] \text{ if } X \geq Y.
\]

(ii) **Constant preserving:**
\[
\hat{E}[c] = c \text{ for } c \in \mathbb{R}.
\]

(iii) **Sub-additivity:** For each \( X, Y \in \mathcal{H} \),
\[
\hat{E}[X + Y] \leq \hat{E}[X] + \hat{E}[Y].
\]

(iv) **Positive homogeneity:**
\[
\hat{E}[\lambda X] = \lambda \hat{E}[X] \text{ for } \lambda \geq 0.
\]

The triple \( (\Omega, \mathcal{H}, \hat{E}) \) is called a sublinear expectation space.

In this section, we mainly consider the following type of sublinear expectation spaces \( (\Omega, \mathcal{H}, \hat{E}) \): if \( X_1, X_2, \ldots, X_n \in \mathcal{H} \) then \( \varphi(X_1, X_2, \ldots, X_n) \in \mathcal{H} \) for \( \varphi \in C_{b,Lip}(\mathbb{R}^n) \), where \( C_{b,Lip}(\mathbb{R}^n) \) denotes the linear space of functions \( \varphi \) satisfying
\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y| \text{ for } x, y \in \mathbb{R},
\]
for some \( C > 0, m \in \mathbb{N} \) is depending on \( \varphi \).

For each fixed \( p \geq 1 \), we take \( \mathcal{H}_0^p = \{ X \in \mathcal{H} : \hat{E}[|X|^p] = 0 \} \) as our null space, and denote \( \mathcal{H} / \mathcal{H}_0^p \) as the quotient space. We set \( \|X\|_p := (\hat{E}[|X|^p])^{1/p} \), and extend \( \mathcal{H} / \mathcal{H}_0^p \) to its completion \( \mathcal{H}_p \) under \( \| \cdot \|_p \). Under \( \| \cdot \|_p \) the sublinear expectation \( \hat{E} \) can be continuously extended to the Banach space \( (\mathcal{H}_p, \| \cdot \|_p) \). Without loss generality, we denote the Banach space \( (\mathcal{H}_p, \| \cdot \|_p) \) as \( L_p^p(\Omega, \mathcal{H}, \hat{E}) \). For the G-frame work, we refer to [22] and [23].

In this paper we assume that \( \underline{\mu}, \underline{\pi}, \underline{\sigma} \) and \( \underline{\sigma} \) are nonnegative constants such that \( \underline{\mu} \leq \underline{\pi} \) and \( \underline{\sigma} \leq \underline{\sigma} \).

**Definition 2.2** Let \( X_1 \) and \( X_2 \) be two random variables in a sublinear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \), \( X_1 \) and \( X_2 \) are called identically distributed, denoted by \( X_1 \overset{d}{=} X_2 \) if
\[
\hat{E}[\varphi(X_1)] = \hat{E}[\varphi(X_2)] \text{ for } \forall \varphi \in C_{b,Lip}(\mathbb{R}^n).
\]
**Definition 2.3** In a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\), a random variable \(Y\) is said to be independent of another random variable \(X\), if
\[
\hat{E}[\phi(X, Y)] = \hat{E}[\phi(x, Y)]_{x=X}.
\]

**Definition 2.4** (G-normal distribution) A random variable \(X\) on a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E})\) is called G-normal distributed if
\[
aX + b\bar{X} = \sqrt{a^2 + b^2}X \quad \text{for} \ a, b \geq 0,
\]
where \(\bar{X}\) is an independent copy of \(X\).

We denote by \(S(d)\) the collection of all \(d \times d\) symmetric matrices. Let \(X\) be G-normal distributed random vectors on \((\Omega, \mathcal{H}, \hat{E})\), we define the following sublinear function
\[
G(A) := \frac{1}{2}\hat{E}[AX, X], \quad A \in S(d).
\]

**Remark 2.1** For a random variable \(X\) on the sublinear space \((\Omega, \mathcal{H}, \hat{E})\), there are four typical parameters to character \(X\)
\[
\begin{align*}
\mu_X &= \hat{E}X, \\
\mu_X^2 &= -\hat{E}[-X], \\
\sigma_X^2 &= \hat{E}X^2, \\
\sigma_X^2 &= -\hat{E}[-X^2],
\end{align*}
\]
where \([\mu_X, \sigma_X]\) and \([\sigma_X^2, \sigma_X^2]\) describe the uncertainty of the mean and the variance of \(X\), respectively.

It is easy to check that if \(X\) is G-normal distributed, then
\[
\mu_X = \hat{E}X = \mu_X = -\hat{E}[-X] = 0,
\]
and we denote the G-normal distribution as \(N(\{0\}, [\sigma_X^2, \sigma_X^2])\). If \(X\) is maximal distributed, then
\[
\sigma_X^2 = \hat{E}X^2 = \sigma_X^2 = -\hat{E}[-X^2] = 0,
\]
and we denote the maximal distribution (see (2.3)) as \(N(\mu_X, \{0\})\).

Let \(\mathcal{F}\) as a Borel field subsets of \(\Omega\). We are given a family \(\{\mathcal{F}_t\}_{t \in \mathbb{R}}\) of Borel subfields of \(\mathcal{F}\), such that
\[
\mathcal{F}_s \subset \mathcal{F}_t, \quad s < t.
\]

**Definition 2.5** We call \((X_t)_{t \in \mathbb{R}}\) a d-dimensional stochastic process on a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F})\), if for each \(t \in \mathbb{R}\), \(X_t\) is a d-dimensional random vector in \(\mathcal{H}\).

**Definition 2.6** Let \((X_t)_{t \in \mathbb{R}}\) and \((Y_t)_{t \in \mathbb{R}}\) be d-dimensional stochastic processes defined on a sublinear expectation space \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F})\), for each \(t = (t_1, t_2, \ldots, t_n) \in T\),
\[
F^X_t[\phi] := \hat{E}[\phi(X_t)], \quad \forall \phi \in C_{l, \text{Lip}}(\mathbb{R}^{n \times d})
\]
is called the finite dimensional distribution of \(X_t\). \(X\) and \(Y\) are said to be identically distributed, i.e., \(X \overset{d}{=} Y\), if
\[
F^X_t[\phi] = F^Y_t[\phi], \quad \forall t \in T \quad \text{and} \quad \forall \phi \in C_{l, \text{Lip}}(\mathbb{R}^{n \times d})
\]
where \(T := \{t = (t_1, t_2, \ldots, t_n) : \forall n \in N, t_i \in \mathbb{R}, t_i \neq t_j, 0 \leq i, j \leq n, i \neq j\}\).
Definition 2.7 A process $(B_t)_{t \geq 0}$ on the sublinear expectation space $(\Omega, \mathcal{F}, \hat{\mathbb{E}}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+})$ is called a $G$-Brownian motion if the following properties are satisfied:

(i) $B_0(\omega) = 0$;

(ii) For each $t, s > 0$, the increment $B_{t+s} - B_t$ is $G$-normal distributed by $N(\{0\}, [\sigma^2, \sigma])$ and is independent of $(B_1, B_2, \ldots, B_n)$, for each $n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_n \in (0, t]$.

From now on, the stochastic process we will consider in the rest of this paper are all in the sublinear space $(\Omega, \mathcal{F}, \hat{\mathbb{E}}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+})$.

3 G-Doob-Meyer Decomposition for G-supermartingale

Definition 3.1 A G-supermartingale (resp. G-submartingale) is a real valued process $\{X_t\}$, well adapted to the $\mathcal{F}_t$ family, such that

$$(i) \quad \hat{\mathbb{E}}[|X_t|] < \infty \quad \forall t \in \mathbb{R}_+,$$

$$(ii) \quad \hat{\mathbb{E}}[X_{t+s} | \mathcal{F}_s] \leq (\text{resp.} \geq ) X_s \quad \forall t \in \mathbb{R}_+, \text{and} \forall s \in \mathbb{R}_+.$$  \hfill (3.1)

If equality holds in (ii), the process is a G-martingale.

We will consider right continuous G-supermartingales, then if $\{X_t\}$ is right continuous G-supermartingale (ii) in (3.1) holds with $\mathcal{F}_t$ replaced by $\mathcal{F}_{t+}$.

Definition 3.2 Let $A$ be an event in $\mathcal{F}_{t+}$, we define capacity of $A$ as

$$c(A) = \hat{\mathbb{E}}[I_A] \quad \text{(3.2)}$$

where $I_A$ is indicator function of event $A$.

Definition 3.3 Process $X_t$ and $Y_t$ are adapted to the filtration $\mathcal{F}_t$. We call $Y_t$ equivalent to $X_t$, if and only if

$$c(Y_t \neq X_t) = 0.$$  \hfill (3.3)

For a right continuous G-supermartingale $\{X_t\}$ with $\hat{\mathbb{E}}[X_t]$ is right continuous function of $t$, we can find a right continuous G-supermartingale $\{Y_t\}$ equivalent to $\{X_t\}$ by define

$$Y_t(\omega) := X_{t+}(\omega) = \lim_{s \uparrow t} X_s(\omega), \quad \text{for any} \ \omega \in \Omega \ \text{such that the limit exits}$$

$$Y_t(\omega) := 0, \quad \text{otherwise.}$$

Without loss generality, we denote $\mathcal{F}_T = \mathcal{F}_{t+}$.

Definition 3.4 For a positive constant $T$, we define stop time $\tau$ in $[0, T]$ as a positive, random variable $\tau(\omega)$ such that, $\{\tau \leq T\} \in \mathcal{F}_T$.

Let $\{X_t\}$ be a right continuous G-supermartingale, denote $X_\infty$ as the last element of the process $X_t$, then the process $\{X_t\}_{0 \leq t \leq \infty}$ is a G-supermartingale.

Definition 3.5 A right continuous increasing process is a well adapted stochastic process $\{A_t\}$ such that:

(i) $A_0 = 0$ a.s.

(ii) For almost every $\omega$, the function $t \rightarrow A_t(\omega)$ is positive, increasing, and right continuous. Let $A_\infty(\omega) := \lim_{t \rightarrow \infty} A_t(\omega)$, we shall say that the right continuous increasing process is integrable if $\hat{\mathbb{E}}[A_\infty] < \infty$.  

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Definition 3.6 An increasing process $A$ is called natural if for every bounded, right continuous $G$-martingale $\{M_t\}_{0 \leq t < \infty}$ we have

$$\hat{E}[\int_{(0,t]} M_s dA_s] = \hat{E}[\int_{(0,t]} M_s dA_s]$$

for every $0 < t < \infty$ (3.3)

Lemma 3.1 If $A$ is an increasing process and $\{M_t\}_{0 \leq t < \infty}$ is bounded, right continuous $G$-martingale, then

$$\hat{E}[M_tA_t] = \hat{E}\left[\int_{(0,t]} M_s dA_s\right].$$

(3.4)

In particular, condition (3.3) in Definition 3.6 is equivalent to

$$\hat{E}[M_tA_t] = \hat{E}[\int_{(0,t]} M_s dA_s].$$

(3.5)

Proof. For a partition $\Pi = \{t_0, t_1, \cdots, t_n\}$ of $[0,t]$, with $0 = t_0 \leq t_1 \leq \cdots \leq t_n = t$, we define

$$M_t^\Pi = \sum_{k=1}^n M_{t_k} I_{[t_{k-1},t_k]}(s).$$

Since $M$ is $G$-martingale

$$\hat{E}\left[\int_{(0,t]} M_t^\Pi dA_s\right] = \hat{E}\left[\sum_{k=1}^n M_{t_k} (A_{t_k} - A_{t_{k-1}})\right]$$

$$= \hat{E}\left[\sum_{k=1}^n M_{t_k} A_{t_k} - \sum_{k=1}^{n-1} M_{t_{k+1}} A_{t_k}\right]$$

$$= \hat{E}[M_tA_t - \sum_{k=1}^{n-1} (M_{t_{k+1}} - M_{t_k}) A_{t_k}]$$

$$= \hat{E}[M_tA_t - \sum_{k=1}^{n-1} (M_{t_{k+1}} - M_{t_k}) A_{t_k}]$$

$$= \hat{E}[M_tA_t],$$

we finish the proof of the Lemma. □

Definition 3.7 A positive right continuous $G$-supermartingale $\{Y_t\}$ with $\lim_{t \to \infty} Y_t(\omega) = 0$ is called a potential.

Definition 3.8 For $a \in [0, \infty)$, a process $\{X_t, t \in [0,a]\}$ is said to be uniformly integrable on $[0,a]$ if

$$\sup_{r \in [0,a]} \hat{E}[|X_r| |X_r| > x] \to 0, \text{ as } x \to 0.$$ 

Definition 3.9 Let $a \in [0, \infty)$, and let $\{X_t\}$ be a right continuous process, we shall say that it belongs to the class (GD) on this interval, if all the random variable $X_T$ are uniformly integrable. The stop time bounded by $a$. If $\{X_t\}$ belongs to the class (GD) on every interval $[0,a], a < \infty$, it will be said to belong locally to the class (GD).

If $\{A_t\}$ is an integrable right continuous, increasing process, then the process $\{-A_t\}$ is a negative $G$-supermartingale, and $\{\hat{E}[A_t | F_t] - A_t\}$ is a potential of the class (GD), which we shall call the potential generated by $\{A_t\}$. 

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Proposition 3.1  (1) Any right continuous G-martingale \{X_t\} belongs locally to the class (GD).

(2) Any right continuous G-supermartingale \{X_t\}, which is bounded from above, belongs locally to the class (GD).

(3) Any right continuous supermartingale \{X_t\}, which belongs locally to the class (GD) and is uniformly integrable, belongs to the class (GD).

Proof. (1) If \(a < \infty\), and \(T\) is a stop time, \(T \leq a\), then G-martingale process \(\{X_t\}\) has \(X_T = \hat{E}[X_a | \mathcal{F}_T]\).

Hence
\[
\hat{E}[|X_T| | X_T > n] \leq \hat{E}[|X_a|] \leq \hat{E}[X_a | X_T > n]
\]
As \(n \cdot c(|X_T| > n) \leq \hat{E}[|X_T|] \leq \hat{E}[|X_a|]\), we have \(c(|X_T| > n) \to 0\) as \(n \to \infty\), then \(\hat{E}[X_a | X_T > n] \leq \hat{E}[X_a | X_T > n]\), from which we prove (1).

(2) If \(a < \infty\), and \(T\) is a stop time, \(T \leq a\), then G-supermartingale process \(\{X_t\}\) has \(X_T \geq \hat{E}[X_a | \mathcal{F}_T]\). Suppose that \(\{X_t\}\) is negative, then
\[
\hat{E}[-X_T | X_T < -n] \leq \hat{E}[-X_a | X_T < -n]
\]
we complete the proof of (2) by using the similar argument in proof (1).

(3) \(\{X_t\}\) is uniformly integrable, we set
\[
X_t = \hat{E}[X_a | \mathcal{F}_t] + (X_t - \hat{E}[X_a | \mathcal{F}_t]).
\]

The first part on the right hand of the above equation \(\hat{E}[X_a | \mathcal{F}_t]\) is a G-martingale, and equivalent to a right continuous process, and from (1) we know that it belongs to the class (GD). We denote the second part in the above equation as \(\{Y_t\}\), it is a potential, i.e., a positive right continuous G-supermartingale, and \(\lim_{t \to \infty} Y_t(\omega) = 0\) a.s.. Next we will prove that \(\{Y_t\}\) belongs to the class (GD).

Since both inf \((T, a)\) and sup \((T, a)\) are stop times
\[
\hat{E}[Y_T | Y_T > a] \leq \hat{E}[Y_T | Y_T > a] + \hat{E}[Y_T | Y_T > a] \leq \hat{E}[Y_T | Y_T > a] + \hat{E}[Y_T].
\]
As \(\lim_{n \to \infty} \hat{E}[Y_n] = 0\) and \(\{Y_t\}\) locally belongs to (GD), i.e., \(\lim_{n \to \infty} \hat{E}[Y_T | Y_T > a] = 0\), which prove that
\[
\lim_{n \to \infty} \hat{E}[Y_T | Y_T > a] = 0.
\]

We complete the proof. \(\Box\)

Lemma 3.2 Let \(\{X_t\}\) be a right continuous G-supermartingale, and \(\{X^n_t\}\) a sequence of decomposed right continuous G-supermartingale:
\[
X^n_t = M^n_t - A^n_t,
\]
where \(\{M^n_t\}\) is G-martingale, and \(\{A^n_t\}\) is right continuous increasing process. Suppose that, for each \(t\), the \(X^n_t\) converge to \(X_t\) in the \(L^1_G(\Omega)\) topology, and the \(A^n_t\) are uniformly integrable in \(n\). Then the decomposition problem is solvable for the G-supermartingale \(\{X_t\}\), more precisely, there is a right continuous increasing process \(\{A_t\}\), and a G-martingale \(\{M_t\}\), such that \(X_t = M_t - A_t\).

Proof. We denote by \(w\) the weak topology \(w(L^1_G(\Omega), L^\infty_G(\Omega))\), a sequence of integrable random variables \(f_n\) converges to a random variable \(f\) in the \(w\)-topology, if and only if \(f\) is integrable, and
\[
\lim_{n \to \infty} \hat{E}[f_n g] = \hat{E}[f g], \quad \forall g \in L^\infty_G(\Omega).
\]
Since the $A_i^n$ are uniformly integrable in $n$, by the properties of the sublinear expectation $\hat{E}[\cdot]$ there exists a $w$-convergent subsequence $A_t^{n_k}$ converge in the $w$-topology to the random variables $A_t$, for all rational values of $t$. To simplify the notations, we shall use $A_t^n$ converge to $A_t$ in the $w$-topology for all rational values of $t$. An integrable random variable $f$ is $\mathcal{F}_t$-measurable if and only if it is orthogonal to all bounded random variables $g$ such that $\hat{E}[g|\mathcal{F}_t]=0$, it follows that $A_t$ is $\mathcal{F}_t$-measurable. For $s < t$, $s$ and $t$ rational,

$$\hat{E}[(A_t^n - A_t^n)_B] \geq 0$$

(3.6)

where $B$ denote any $\mathcal{F}$ set.

As $X^n_t$ converge to $X_t$ in $L^1_c(\Omega)$ topology, which is in a stronger topology than $w$, the $M^n_t$ converge to random variables $M'_t$ for $t$ rational, and the process $\{M'_t\}$ is $G$-martingale; then there is a right continuous $G$-martingale $\{M_t\}$, defined for all values of $t$, such that $c(M_t \neq M'_t) = 0$ for each rational $t$. We define $A_t = X_t + M_t$, $\{A_t\}$ is a right continuous increasing process, or at least becomes so after a modification on a set of measure zero. We complete the proof. □

**Lemma 3.3** Let $\{X_t\}$ be a potential and belong to the class (GD). We consider the measurable, positive and well adapted processes $H = \{H_t\}$ with the property that the right continuous increasing processes

$$A(H) = \{A_t(H, \omega)\} = \{\int_0^t H_t(\omega)ds\}$$

are integrable, and the potentials $Y(H) = \{Y_t(H, \omega)\}$ they generate are majorized by $X_t$. Then, for each $t$, the random variables $A_t(H)$ of all such processes $A(H)$ are uniformly integrable.

**Proof.** It is sufficient to prove that the $A_\infty(H)$ are uniformly integrable.

(1) First we assume that $X_t$ is bounded by some positive constant $C$, then $\hat{E}[A_\infty^2(H)] \leq 2C^2$, and the uniform integrability follows.

We have that

$$A_\infty^2(H, \omega) = 2\int_0^\infty [A_\infty(H, \omega) - A_u(H, \omega)]dA_u(H, \omega)$$

$$= 2\int_0^\infty [A_\infty(H, \omega) - A_u(H, \omega)]H_u(\omega)du.$$

By using the sub-additive property of the sublinear expectation $\hat{E}$

$$\hat{E}[A_\infty^2(H, \omega)] = \hat{E}[\hat{E}A_\infty^2(H, \omega)|\mathcal{F}_t]$$

$$\leq 2\hat{E}[\int_0^\infty H_u\hat{E}[A_\infty(H, \omega) - A_u(H, \omega)|\mathcal{F}_u]du]$$

$$= 2\hat{E}[\int_0^\infty H_uY_u(H)du]$$

$$\leq 2C\hat{E}[\int_0^\infty H_u]$$

$$= 2C\hat{E}[Y_0(H)]$$

$$\leq 2C^2.$$

(2) In order to prove the general case, it will be enough to prove that any $H$ such that $Y(H)$ is majorized by $\{X_t\}$ is equal to a sum $H^c + H_c$, where (i) $A(H^c)$ generates a potential bounded by $c$, and (ii) $\hat{E}[A_\infty[H_c]]$ is smaller than some number $\varepsilon_c$, independent of $H$, such that $\varepsilon_c \rightarrow 0$ as $c \rightarrow 0$. Define

$$H'_c(\omega) = H_c(\omega)I_{\{X_t(\omega) \in [0,c]\}}, \quad H_c = H - H'_c.$$
Set
\[ T^c(\omega) = \inf\{t : \text{such that } X_t(\omega) \geq c\}, \]
as \( c \) goes to infinity \( \lim_{c \to \infty} T^c(\omega) = \infty \), therefore \( X_{T^c} \to 0 \), and the class (GD) property implies that \( \hat{E}[X_{T^c}] \to 0 \). \( T^c \) is a stop time, and \( I_{\{X_0(\omega) \in [0,c]\}} = 1 \) before time \( T^c \). Hence
\[
\hat{E}[A_\infty(H_c)] = \hat{E}\left[ \int_0^\infty H_u(1 - I_{\{X_u(\omega) \in [0,c]\}})du \right]
\leq \hat{E}\left[ \int_0^\infty H_u(1 - I_{\{X_u(\omega) \in [0,c]\}})du \right]
\leq \hat{E}[A_\infty(H) - A_{T^c}(H)]
\leq \hat{E}[\hat{E}[A_\infty(H) - A_{T^c}(H)] | \mathcal{F}_t]
\leq \hat{E}[Y_{T^c}(H)] \leq \hat{E}[X_{T^c}(H)]
\leq \varepsilon_c, \text{ for large enough } c,
\]
from which we prove (ii). We shall prove (i), first we prove that \( Y(H^c) \) is bounded by \( c \):
\[
Y_i(H^c) = \hat{E}[A_{\infty}(H^c) - A_i(H^c)] | \mathcal{F}_t
\leq \hat{E}\left[ \int_0^\infty H_u I_{\{X_u(\omega) \in [0,c]\}}du | \mathcal{F}_t \right]
\leq \hat{E}\left[ \int_0^\infty H_u I_{\{X_u(\omega) \in [0,c]\}}du | \mathcal{F}_t \right]
\leq \hat{E}[Y_{T^c}(H)] \leq \hat{E}[X_{T^c}(H)]
\leq c,
\]
where we set
\[ S^c(\omega) = \inf\{t : \text{such that } X_t(\omega) \leq c\}, \]
and use
\[
\int_t^{S^c(\omega)} H_u I_{\{X_u(\omega) \in [0,c]\}}du = 0.
\]
The inequality \( (3.7) \) holds for each \( t \), therefore for every rational \( t \), and for every \( t \) in consideration of the right continuity, which complete the proof. \( \square \)

**Lemma 3.4** Let \( \{X_t\} \) be a potential and belong to the class (GD), \( k \) is a positive number, define \( Y_i = \hat{E}[X_{t+k} | \mathcal{F}_t] \), then \( \{Y_i\} \) is a G-supermartingale. Denote by \( \{p_tX_t\} \) a right continuous version of \( \{Y_i\} \), then \( \{p_tX_t\} \) is potential.

Use the same notations as in Lemma 3.3. Let \( k \) be a positive number, and \( H_{t,k}(\omega) = (X_t(\omega) - p_tX_t(\omega))/k \). The process \( H_{t,k} = \{H_{t,k}\} \) verify the assumptions of Lemma 3.3 and their potentials increase to \( \{X_t\} \) as \( k \to 0 \).

**Proof.** If \( t < u \)
\[
\hat{E}\left( \int_t^u [X_s - p_tX_s]ds - \int_0^u [X_s - p_tX_s]ds | \mathcal{F}_t \right)
\leq \hat{E}\left( \int_t^u [X_s - p_tX_s]ds | \mathcal{F}_t \right),
\]
For \( s \geq t \), \( \hat{E}[p_tX_s | \mathcal{F}_t] = \hat{E}[\hat{E}[X_{s+k} | \mathcal{F}_s] | \mathcal{F}_t] = \hat{E}[X_{s+k} | \mathcal{F}_t] \). We have that
\[
\hat{E}\left( \int_t^u [X_s - p_tX_s]ds | \mathcal{F}_t \right) \geq \hat{E}\left( \int_t^{u+k} [X_s - p_tX_s]ds | \mathcal{F}_t \right) - \hat{E}\left( \int_t^{u+k} [X_s - p_tX_s]ds | \mathcal{F}_t \right).
\]
by the sub-additive property of the sublinear expectation $\hat{E}$, we derive that

$$
\hat{E}\left[ \frac{1}{k} \int_t^{t+k} X_s ds | F_t \right] - \hat{E}\left[ \frac{1}{k} \int_u^{u+k} X_s ds | F_t \right] = \hat{E}\left[ \frac{1}{k} \int_t^{t+k} X_s ds | F_t \right] - X_t
$$

$$
\geq \hat{E}\left[ \frac{1}{k} \int_t^{t+k} X_s ds | F_t \right] - \frac{1}{k} \int_u^{u+k} \hat{E}[X_s | F_t] ds
$$

$$
\geq -\hat{E}\left[ \frac{1}{k} \int_t^{t+k} (X_s - X_t) ds | F_t \right]
$$

$$
\geq -\frac{1}{k} \int_t^{t+k} \hat{E}[X_s - X_t | F_t] ds
$$

$$
\geq 0.
$$

Hence, we derive that, for any $u, t$ such that $u > t$

$$
\hat{E}\left[ \frac{1}{k} \int_t^{u} [X_s - p_k X_s] ds | F_t \right] \geq 0.
$$

If there exists $s_0 \geq 0$ such that $\frac{1}{k}[X_{s_0} - p_k X_{s_0}] < 0$, the right continuous of $\{X_t\}$ implies that there exists $\delta > 0$ such that $\frac{1}{k}[X_{s} - p_k X_{s}] < 0$ on the interval $[s_0, s_0 + \delta]$. Thus

$$
\hat{E}\left[ \frac{1}{k} \int_{s_0}^{s_0 + \delta} [X_s - p_k X_s] ds | F_{s_0} \right] < 0,
$$

which is contradiction, we prove that $(X_t(\omega) - p_k X_t(\omega))/k$ is a positive, measurable and well adapted process.

Since $\{X_t\}$ is right continuous G-supermartingale

$$
\lim_{s \uparrow t} X_s = X_t
$$

$$
\lim_{k \downarrow 0} Y_t(H_k) = \lim_{k \downarrow 0} \hat{E}\left[ \frac{1}{k} \int_t^{\infty} [X_s - p_k X_s] ds | F_t \right] = \hat{E}\left[ \frac{1}{k} \int_t^{t+k} X_s ds | F_t \right] = \hat{E}\left[ \lim_{k \downarrow 0} \frac{1}{k} \int_t^{t+k} X_s ds | F_t \right] = X_t,
$$

we finish the proof. □

From Lemma 3.2, 3.3, and 3.4 we can prove the following Theorem

**Theorem 3.1** A potential $\{X_t\}$ belongs to the class (GD) if, and only if, it is generated by some integrable right continuous increasing process.

**Theorem 3.2** (G-Doob-Meyer’s Decomposition)

1. $\{X_t\}$ is a right continuous G-supermartingale if and only if it belongs to the class (GD) on every finite interval. More precisely, $\{X_t\}$ is then equal to the difference of a G-martingal $M_t$ and a right continuous increasing process $A_t$,

$$
X_t = M_t - A_t.
$$

(3.8)

2. If the right continuous increasing process $A$ is natural, the decomposition is unique.
Theorem 3.3 following G-Doob-Meyer’s Theorem there exist the following decomposition
\[ X_t' = M_t' - A_t', \]
where \( M_t' \) is a G-martingal, and \( A_t' \) is a right continuous increasing process.

Let \( a \to \infty \), as in Lemma 3.4 the expression of the \( Y_i (H_k) \) that the \( A_t' \) depend only on the values of \( \{X_t'\} \) on intervals \([0, t + \varepsilon]\), with \( \varepsilon \) small enough. As \( a \to \infty \), they don’t vary any more once \( a \) has reached values greater than \( t \), us again Lemma 3.2 we finish the proof of the Theorem.

(2) Assume that \( X \) admits both decompositions
\[ X_t = M_t' - A_t' = M_t'' - A_t'', \]
where \( M_t' \) and \( M_t'' \) are G-martingale and \( A_t' \), \( A_t'' \) are natural increasing process. We define
\[ \{C_t := A_t' - A_t'' = M_t' - M_t''\}. \]

Then \( \{C_t\} \) is a G-martingale, and for every bounded and right continuous G-martingale \( \{\xi_t\} \), from Lemma 3.1 we have
\[ \hat{E} \{\xi_t (A_t' - A_t'')\} = \hat{E} \left( \int_{\{0, a\}} \xi_t - dC_t = \lim_{n \to \infty} \sum_{j=1}^{m} \xi_{t_j}^n (C_{t_{j-1}}^{(n)} - C_{t_j}^{(n)}) \right], \]
where \( \Pi_n = \{j_1, \cdots, j_{m_n}\}, n \geq 1 \) is a sequence of partitions of \([0, t]\) with \( \max 1 \leq j \leq m_n (t_j - t_{j-1}) \) converging to zero as \( n \to \infty \). Since \( \xi \) and \( C \) are both G-martingale, we have
\[ \hat{E} \{\xi_t^{(n)} (C_{t_{j-1}}^{(n)} - C_{t_j}^{(n)}) \} = 0, \]
and thus \( \hat{E} \{\xi_t (A_t' - A_t'')\} = 0. \)

For an arbitrary bounded random variable \( \xi \), we can select \( \{\xi_t\} \) to be a right-continuous equivalent process of \( \{\hat{E}[\xi_t|\mathcal{F}_t]\} \), we obtain that \( \hat{E}[\xi_t (A_t' - A_t'')] \) = 0. We set \( \xi = I_{A_t' \neq A_t''} \) therefore \( c(A_t' \neq A_t'') = 0. \)

By Theorem 3.2 and G-martingale decomposition Theorem in \([23]\) and \([25]\), we have the following G-Doob-Meyer’s Theorem

**Theorem 3.3** \( \{X_t\} \) is a right continuous G-supermartingale, there exists a right continuous increasing process \( A_t \) and adapted process \( \eta_t \), such that
\[ X_t = \int_0^t \eta_s d B_s - A_t, \]
where \( B_t \) is G-Brownian motion.

## 4 Superheding strategies and optimal stopping

### 4.1 Financial model and G-asset price system

We consider a financial market with a nonrisky asset (bond) and a risky asset (stock) continuously trading in market. The price \( P(t) \) of the bond is given by
\[ dP(t) = rP(t) dt, P(0) = 1, \]
\[ \text{(4.1)} \]
where \( r \) is the short interest rate, we assume a constant nonnegative short interest rate. We assume the risk asset with the G-asset price system \(((S_u)_{u\geq t}, \hat{E}) \) (see [5]) on sublinear expectation space \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F}_t)\) under Knightian uncertainty, for given \( t \in [0, T] \) and \( x \in \mathbb{R}^d \)

\[
dS_u^t, x = S_u^t dB_t = S_u^t (dt + d\hat{B}_t),
\]

\[
S_t^x = x
\]

(4.2)

where \( B_t \) is the generalized G-Brownian motion. The uncertain volatility is described by the G-Brownian motion \( \hat{B}_t \). The uncertain drift \( b_t \) can be rewritten as

\[
b_t = \int_0^t \mu_u dt
\]

where \( \mu_t \) is the asset return rate ([3]). Then the uncertain risk premium of the G-asset price system

\[
\theta_t = \mu_t - r,
\]

(4.3)

is uncertain and distributed by \( N([\mu - r, \mu - r], \{0\}) \) ([3]), where \( r \) is the interest rate of the bond.

Define

\[
\tilde{B}_t := B_t - rt = b_t + \hat{B}_t - rt,
\]

(4.4)

we have the following G-Girsanov Theorem (presented in [4], [5] and [10])

**Theorem 4.1 (G-Girsanov Theorem)** Assume that \((B_t)_{t\geq 0}\) is generalized G-Brownian motion on \((\Omega, \mathcal{H}, \hat{E}, \mathcal{F}_t)\), and \( \tilde{B}_t \) is defined by (4.4), there exists G-expectation space \((\hat{\Omega}, \hat{\mathcal{H}}, \hat{E}, \hat{\mathcal{F}}_t)\) such that \( \tilde{B}_t \) is G-Brownian motion under the G- expectation \( \hat{E} \), and

\[
\hat{E}[\tilde{B}_t^2] = E^G[\tilde{B}_t^2], \quad -\hat{E}[-\tilde{B}_t^2] = -E^G[-\tilde{B}_t^2].
\]

(4.5)

By the G-Girsanov Theorem, the G-asset price system (4.2) of the risky asset can be rewritten on \((\Omega, \mathcal{H}, E^G, \mathcal{F}_t)\) as follows

\[
dS_u^t, x = S_u^t (rdt + d\hat{B}_t),
\]

\[
S_t^x = x,
\]

(4.6)

then by G-Ito formula we have

\[
S_u^x = x \exp\left(r(u-t) + \tilde{B}_{u-t} - \frac{1}{2}(<\tilde{B}_u> - <\tilde{B}_t>), u > t\right)
\]

(4.7)

### 4.2 Construction of superreplication strategies via optimal stopping

We consider the following class of contingent claims:

**Definition 4.1** We define a class of contingent claims with the nonnegative payoff \( \xi \in L^2_\mathcal{G}(\Omega_T) \) has the following form

\[
\xi = f(S_T^x)
\]

(4.8)

for some function \( f : \Omega \rightarrow \mathbb{R} \) such that the process

\[
f_u := f(S_u^x)
\]

(4.9)

is bounded below and càdlàg.
We consider a contingent claim $\xi$ with payoff defined in Definition 4.1 written on the stock prices $S_t$ with maturity $T$. We give definitions of superhedging (resp. subhedging) strategy and ask (resp. bid) price of the claim $\xi$.

**Definition 4.2**

1. A self-financing superstrategy (resp. substrategy) is a vector process $(Y, \pi, C)$ (resp. $(-Y, \pi, C)$), where $Y$ is the wealth process, $\pi$ is the portfolio process, and $C$ is the cumulative consumption process, such that

$$
dY_t = rY_t dt + \pi_t dB_t - dC_t, \quad (4.10)$$

$$
(\text{resp.} -dY_t = -rY_t dt + \pi_t dB_t - dC_t) \quad (4.11)
$$

where $C$ is an increasing, right-continuous process with $C_0 = 0$. The superstrategy (resp. substrategy) is called feasible if the constraint of nonnegative wealth holds

$$Y_t \geq 0, \quad t \in [0,T].$$

2. A superhedging (resp. subhedging) strategy against the European contingent claim $\xi$ is a feasible self-financing superstrategy $(Y, \pi, C)$ (resp. substrategy $(-Y, \pi, C)$) such that $Y_T = \xi$ (resp. $-Y_T = -\xi$). We denote by $\mathcal{H}(\xi)$ (resp. $\mathcal{H}'(-\xi)$) the class of superhedging (resp. subhedging) strategies against $\xi$, and if $\mathcal{H}(\xi)$ (resp. $\mathcal{H}'(-\xi)$) is nonempty, $\xi$ is called superhedgeable (resp. subhedgeable).

3. The ask-price $X(t)$ at time $t$ of the superhedgeable claim $\xi$ is defined as

$$X(t) = \inf \{x \geq 0 : \exists (Y_t, \pi_t, C_t) \in \mathcal{H}(\xi) \text{ such that } Y_t = x \},$$

and bid-price $X'(t)$ at time $t$ of the subhedgeable claim $\xi$ is defined as

$$X'(t) = \sup \{x \geq 0 : \exists (-Y_t, \pi_t, C_t) \in \mathcal{H}'(-\xi) \text{ such that } -Y_t = -x \}.$$

Under uncertainty, the market is incomplete and the superhedging (resp. subhedging) strategy of the claim is not unique. The definition of the ask-price $X(t)$ implies that the ask-price $X(t)$ is the minimum amount of risk for the buyer to superhedging the claim, then it is coherent measure of risk of all superstrategies against the claim for the buyer. The coherent risk measure of all superstrategies against the claim can be regard as the sublinear expectation of the claim, we have the following representation of bid-ask price of the claim via optimal stopping (Theorem 4.2).

Let $(\mathcal{G}_t)$ be a filtration on $G$-expectation space $(\Omega, \mathcal{G}, E^G, (\mathcal{F}_t)_{t \geq 0})$, and $\tau_1$ and $\tau_2$ be $(\mathcal{G}_t)$-stopping times such that $\tau_1 \leq \tau_2$ a.s. We denote by $\mathcal{G}_{\tau_1, \tau_2}$ the set of all finite $(\mathcal{G}_t)$-stopping times $\tau$ with $\tau_1 \leq \tau \leq \tau_2$.

For given $t \in [0,T]$ and $x \in R^d_+$, we define the function $V^{Am} : [0,T] \times \Omega \rightarrow R$ as the value function of the following optimal-stopping problem

$$V^{Am}(t, S_t) := \sup_{v \in \mathcal{G}_{\tau_1, \tau_2}} E_t^G[f_v] \quad (4.12)$$

$$= \sup_{v \in \mathcal{G}_{\tau_1, \tau_2}} E_t^G[f(S_v)] \quad (4.13)$$

**Proposition 4.1** Consider two stopping times $\underline{\tau} \leq \check{\tau}$ on filtration $\mathcal{F}$. Let $(f_t)_{t \geq 0}$ denote some adapted and RCLL-stochastic process, which is bounded below. Then we have for two points $s, t \in [0, \check{\tau}]$ and $s < t$

$$\text{ess sup}_{\tau \in \mathcal{G}_{\check{\tau}}} \{E^G_t[f_\tau]\} = E^G_t[\text{ess sup}_{\tau \in \mathcal{G}_{\check{\tau}}} \{E^G_\tau[f_\tau]\}] \quad (4.14)$$

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Proof. By the consistent property of the conditional G-expectation, for \( \tau \in \mathcal{F}_T \), \( s, t \in [0, \tau] \) and \( s < t \)

\[
E_t^G[f_t] = E_t^G[E_t^G[f_t]] \\
\leq E_t^G[\text{ess sup}_{\tau \in \mathcal{G}} \{E_t^G[f_t]\}],
\]

thus we have

\[
\text{ess sup}_{\tau \in \mathcal{G}} \{E_t^G[f_t]\} \leq E_t^G[\text{ess sup}_{\tau \in \mathcal{G}} \{E_t^G[f_t]\}].
\]

There exists a sequences \( \{\tau_n\} \rightarrow \tau^t \in [\tau, \infty) \) as \( n \rightarrow \infty \), such that

\[
\lim_{n \rightarrow \infty} E_t^G[f_{\tau_n}] = E_t^G[f_{\tau^t}] = \text{ess sup}_{\tau \in \mathcal{G}} \{E_t^G[f_t]\}, \tag{4.15}
\]

notice that

\[
E_t^G[\text{ess sup}_{\tau \in \mathcal{G}} \{E_t^G[f_t]\}] \\
= E_t^G[E_t^G[f_{\tau^t}]] \\
= E_t^G[f_{\tau^t}] \\
\leq \text{ess sup}_{\tau \in \mathcal{G}} \{E_t^G[f_t]\},
\]

we prove the Proposition. \( \square \)

Proposition 4.2 The process \( V^{\text{Am}}(t, S_t)_{0 \leq t \leq T} \) is a G-supermartingale in \( (\Omega, \mathcal{F}, E^G, \mathcal{G}, \mathcal{F}) \).

Proof. By Proposition 4.1 for \( 0 \leq s \leq t \leq T \)

\[
E_t^G\left[ \sup_{\tau \in \mathcal{F}_{s,t}} E_t^G[f(S_{\tau})] \right] \\
= \sup_{\tau \in \mathcal{F}_{s,t}} E_t^G[f(S_{\tau})].
\]

Since \( \mathcal{F}_{s,t} \subseteq \mathcal{F}_{s,T} \), we have

\[
\sup_{\tau \in \mathcal{F}_{s,t}} E_t^G[f(S_{\tau})] \\
\leq \sup_{\tau \in \mathcal{F}_{s,T}} E_t^G[f(S_{\tau})].
\]

Thus, we derive that

\[
E_t^G\left[ \sup_{\tau \in \mathcal{F}_{s,t}} E_t^G[f(S_{\tau})] \right] \\
\leq \sup_{\tau \in \mathcal{F}_{s,T}} E_t^G[f(S_{\tau})].
\]

We prove the Proposition. \( \square \)

Theorem 4.2 Assume that the uncertain financial market consists of the bond which has the price process satisfying 4.1 and \( d \) risky assets with the price processes as the G-asset price systems 4.2 and can trade freely, the contingent claim \( \xi \) which is written on the \( d \) assets with the maturity \( T > 0 \) has the class of the payoff defined in Definition 4.1 and the function \( V^{\text{Am}}(t, S_t) \) is defined in 4.12. Then there exists a superhedging (resp. subhedging) strategy for \( \xi \), such that, the process \( V = (V_t)_{0 \leq t \leq T} \) defined by

\[
V_t := e^{-r(T-t)}V^{\text{Am}}(t, S_t), \quad \text{(resp. } e^{-r(T-t)}\text{ess sup}_{\tau \in \mathcal{F}_{s,T}} E_t^G[-f_\tau]) \tag{4.16}
\]

is the ask (resp. bid) price process against \( \xi \).
Proof. The value function for the optimal stop time $V^{Am}(t, S_t)$ is a $G$-supermartingale, it is easily to check that $e^{-rt}V_t$ is $G$-supermartingale. By G-Doob-Meyer decomposition Theorem [3.2]

$$e^{-rt}V_t = M_t - \bar{C}_t$$

(4.17)

where $M_t$ is a $G$-martingale and $\bar{C}_t$ is an increasing process with $\bar{C}_0 = 0$. By G-martingale representation Theorem ([23] and [25])

$$M_t = E_G[M_T] + \int_0^t \eta_s d\tilde{B}_t - K_t$$

(4.18)

where $\eta_s \in H^1_G(0, T)$, $-K_t$ is a $G$-martingale, and $K_t$ is an increasing process with $K_0 = 0$. From the above equation, we have

$$e^{-rt}V_t = E_G[M_T] + \int_0^t \eta_s d\tilde{B}_t - (K_t + \bar{C}_t),$$

(4.19)

hence $(V_t, e^{rt} \eta_t, \int_0^t e^{rs} d(\bar{C}_s + K_s) ds)$ is a superhedging strategy.

Assume that $(Y_t, \pi_t, C_t)$ is a superhedging strategy against $\xi$, then

$$e^{-rt}V_t = e^{-rT}\xi - \int_0^T \pi_t d\tilde{B}_t + C_t.$$  

(4.20)

Taking conditional G-expectation on the both sides of the equation (4.20) and notice that the process $C_t$ is an increasing process with $C_0 = 0$, we derive

$$e^{-rt}Y_t \geq E_G^t[e^{-rT}\xi]$$

(4.21)

which implies that

$$Y_t \geq E_G^t[e^{-r(T-t)}\xi] \geq e^{-r(T-t)}\xi \sup_{f \in F_T} E_G^t[f]\sup_{\nu \in \mathcal{F}_T} E_G^t[f]\sup_{\nu \in \mathcal{F}_T} E_G^t[f]$$

from which, we prove that $V_t = e^{-r(T-t)}V^{Am}(t, S_t)$ is the ask price against the claim $\xi$ at time $t$. Similarly we can prove that $-e^{-r(T-t)}\sup_{f \in F_T} E_G^t[-f]$ is the bid price against the claim $\xi$ at time $t$. □

5 Free Boundary and Optimal Stopping Problems

For given $t \in [0, T]$, $x \in \mathbb{R}^d$, the G-asset price system (4.2) of the risky asset can be rewritten as follows

$$\begin{cases} 
    dS_t^x = S_t^x(\nu dt + d\tilde{B}_t) \\
    S_t^x = x
\end{cases}$$

We define the following deterministic function

$$u^d(t, x) := e^{-r(T-t)}V^{Am}(t, S_t^x),$$

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where

\[ V^{Am}(t,S_t^x) = \text{ess sup}_{v \in F_t,T} E^G_t[f(S_v^x)]. \]

From Theorem 2, the price of an American option with expiry date \( T \) and payoff function \( f \), is the value function of the optimal stopping problem

\[ u^a(t,x) := e^{-r(T-t)} \text{ess sup}_{v \in F_t,T} E^G_t[f(S_v)] - x. \]  \hfill (5.22)

We define operator \( L \) as follows:

\[ Lu = G(D^2 u) + rDu + \partial_t u, \]

where \( G(\cdot) \) is the sublinear function defined by equation (2.1). We consider the free boundary problem

\[
\begin{cases}
  Lu := \max \{ Lu - ru, f - u \} = 0, & \text{in } [0,T] \times \mathbb{R}^d, \\
  u(T,\cdot) = f(T,\cdot), & \text{in } \mathbb{R}^d.
\end{cases}
\]  \hfill (5.23)

Denote

\[ S_T := [0,T] \times \mathbb{R}^d, \]

for \( p \geq 1 \)

\[ S^p(S_T) := \{ u \in L^p(S_T) : D^2 u, Du, \partial_t u \in L^p(S_T) \}. \]

and for any compact subset \( D \) of \( S_T \), we denote \( S^p_{\text{loc}}(D) \) as the space of functions \( u \in S^p(D) \).

**Definition 5.1** A function \( u \in S^1_{\text{loc}}(S_T) \cap C(\mathbb{R}^d \times [0,T]) \) is a strong solution of problem (5.23) if \( L u = 0 \) almost everywhere in \( S_T \) and it attains the final datum pointwisely. A function \( u \in S^1_{\text{loc}}(S_T) \cap C(\mathbb{R}^d \times [0,T]) \) is a strong super-solution of problem (5.23) if \( L u \leq 0 \).

We will prove the following existence results

**Theorem 5.1** If there exists a strong super-solution \( \bar{u} \) of problem (5.23) then there also exists a strong solution \( u \) of (5.23) such that \( u \leq \bar{u} \) in \( S_T \). Moreover \( u \in S^p_{\text{loc}}(S_T) \) for any \( p \geq 1 \) and consequently, by the embedding theorem we have \( u \in C^1_{B,\text{loc}}(S_T) \) for any \( \alpha \in [0,1] \).

**Theorem 5.2** Let \( u \) be a strong solution to the free boundary problem (5.23) such that

\[ |u(t,x)| \leq C(\lambda) x^2, \quad (t,x) \in S_T, \]  \hfill (5.24)

form some constants \( C,\lambda \) with \( \lambda \) sufficiently small so that

\[ E^G[\exp(\lambda \sup_{t \leq \tau \leq T} |S_{\tau}^x|)] < \infty. \]

holds. Then we have

\[ u(t,x) = e^{-r(T-t)} \text{ess sup}_{v \in F_t,T} E^G_t[f(S_v)], \]  \hfill (5.25)

i.e., the solution of the free boundary problem is the value function of the optimal stopping problem. In particular such a solution is unique.
5.1 Proof of Theorem 5.2

We employ a truncation and regularization technique to exploit the weak interior regularity properties of $u$, for $R > 0$ we set for $R > 0$, $B_R := \{ x \in \mathbb{R}^d | |x| < R \}$, and for $x \in B_R$ denote by $\tau_R$ the first exit time of $S^x_u$ from $B_R$, it is easy check that $E^G[\tau_R]$ is finite. As a first step we prove the following result: for every $(t, x) \in [0, T) \times B_R$ and $\tau \in \mathcal{F}_{t,T}$ such that $\tau \in [t, \tau_R]$, it holds

$$u(t, x) = E^G[u(\tau, S^x_t)] - E^G\left[\int_t^\tau Lu(s, S^x_s)ds\right]. \quad (5.26)$$

For fixed, positive and small enough $\varepsilon$, we consider a function $\varepsilon^R$ on $\mathbb{R}^{d+1}$ with compact support and such that $\varepsilon^R = u$ on $[t, T - \varepsilon] \times B_R$. Moreover we denote by $(\varepsilon^R_{n,n})_{n \in \mathbb{N}}$ a regularizing sequence obtained by convolution of $\varepsilon^R$ with the usual mollifiers, then for any $p \geq 1$ we have $\varepsilon^R_{n,n} \in \mathcal{S}(\mathbb{R}^{d+1})$ and

$$\lim_{n \to \infty} ||\varepsilon^R_{n,n} - \varepsilon^R||_{L^p([t, T - \varepsilon] \times B_R)} = 0. \quad (5.27)$$

By G-Lu formula we have

$$\varepsilon^R_{n,n}(\tau, S^x_t) = \varepsilon^R_{n,n}(t, x) + \frac{1}{2} \int_t^\tau D^2 \varepsilon^R_{n,n}ds < B + \int_t^\tau rD\varepsilon^R_{n,n}ds + \int_t^\tau \partial_t \varepsilon^R_{n,n}ds + \int_t^\tau D\varepsilon^R_{n,n}dB_s, \quad (5.28)$$

which implies that

$$E^G[u(\varepsilon^R_{n,n}(\tau, S^x_t))] = u(\varepsilon^R_{n,n}(t, x)) + \int_t^\tau L\varepsilon^R_{n,n}ds. \quad (5.29)$$

We have

$$\lim_{n \to \infty} \varepsilon^R_{n,n}(t, x) = \varepsilon^R(t, x)$$

and, by dominated convergence

$$\lim_{n \to \infty} E^G[u(\varepsilon^R_{n,n}(\tau, S^x_t))] = E^G[u(\varepsilon^R(\tau, S^x_t))].$$

We have

$$|E^G[\int_t^\tau L\varepsilon^R_{n,n}(s, S^x_s)ds] - E^G[\int_t^\tau L\varepsilon^R(s, S^x_s)ds]| \leq E^G[|\int_t^\tau L\varepsilon^R_{n,n}(s, S^x_s) - L\varepsilon^R(s, S^x_s)|ds],$$

by sublinear expectation representation Theorem (see [23]) there exists a family of probability space $Q$, such that

$$E^G[|\int_t^\tau L\varepsilon^R_{n,n}(s, S^x_s) - L\varepsilon^R(s, S^x_s)|ds] = \text{ess sup}_{P \in Q} E_P[|\int_t^\tau L\varepsilon^R_{n,n}(s, S^x_s) - L\varepsilon^R(s, S^x_s)|ds].$$

Since $\tau \leq \tau_R$

$$\text{ess sup}_{P \in Q} E_P[|\int_t^\tau L\varepsilon^R_{n,n}(s, S^x_s) - L\varepsilon^R(s, S^x_s)|ds] \leq \text{ess sup}_{P \in Q}[|\int_t^{T-\varepsilon} L\varepsilon^R_{n,n}(s, y) - L\varepsilon^R(s, y)|ds + \int_{B_R} |\int_s^{T-\varepsilon} L\varepsilon^R_{n,n}(s, y) - L\varepsilon^R(s, y)|dF(t, x; s, y)ds].$$
where \( \Gamma_p(t, x; \cdot) \in L^q([t, T] \times B_R) \), for some \( \bar{q} > 1 \), is the transition density of the solution of
\[
dX_s^t = X_s^t (rds + \sigma_s \, dW_s)
\]
where \( W_p \) is Wiener process in probability space \((\Omega, P, F, F_t^p)\), and \( \sigma_s \) is adapted process such that \( \sigma_s \in [S, T] \). By Hölder inequality, we have \( (1/\bar{p} + 1/\bar{q} = 1) \)
\[
\int_t^{T-\varepsilon} \int_{B_R} |Lu^{E,R,n}(s, y) - Lu^{E,R}(s, y)||\Gamma_p(t, x; s, y)dyds \leq \|Lu^{E,R,n}(s, y) - Lu^{E,R}(s, y)\|_{L^q([t, T] \times B_R)} \|\Gamma_p(t, x; s, y)\|_{L^p([t, T] \times B_R)}
\]
then, we obtain that
\[
\lim_{n \to \infty} E^G[\int_t^T Lu^{E,R,n}(s, S_s^x)] = E^G[\int_t^T Lu^{E,R}(s, S_s^x)].
\]
This concludes the proof of (5.26), since \( u^{E,R} = u \) on \([t, T - \varepsilon] \times B_R \) and \( \varepsilon > 0 \) is arbitrary.
Since \( Lu \leq 0 \), we have for any \( \tau \in \mathcal{F}_t,T \)
\[
E^G \int_t^\tau Lu(s, S_s^x) ds \leq 0.
\]
we infer from (5.26) that
\[
u(t, x) \geq E^G [\nu(\tau \wedge R, S_{\tau \wedge R}^x)].
\]
Next we pass to the limit as \( R \to +\infty \); we have
\[
\lim_{R \to +\infty} \tau \wedge R = \tau,
\]
and by the growth assumption (5.24)
\[
|\nu(\tau \wedge R, S_{\tau \wedge R}^x)| \leq C \exp(\lambda \sup_{t \leq s \leq T} |S_s^x|^2).
\]
As \( R \to +\infty \)
\[
\nu(t, x) \geq E^G [\nu(\tau, S_{\tau}^x)] \geq E^G [f(\tau, S_{\tau}^x)].
\]
This shows that
\[
\nu(t, x) \geq \sup_{\tau \in \mathcal{F}_t,T} E^G [f(\tau, S_{\tau}^x)]
\]
We conclude the proof by putting
\[
\tau_0 = \inf \{ s \in [t, T] ; u(s, S_s^x) = f(s, S_s^x) \}
\]
Since \( Lu = 0 \) a.e. where \( u > \phi \), it holds
\[
E^G \int_{t}^{\tau_0 \wedge \tau R} Lu(s, S_s^x) ds = 0
\]
and from (5.26) we derive that
\[
\nu(t, x) = E^G [\nu(\tau_0 \wedge \tau R, S_{\tau_0 \wedge \tau R}^x)]
\]
Repeating the previous argument to pass to the limit in \( R \), we obtain
\[
\nu(t, x) = E^G [\nu(\tau_0, S_{\tau_0}^x)] = E^G [f(\tau_0, S_{\tau_0}^x)].
\]
Therefore, we finish the proof. \( \square \)
5.2 Free boundary problem

Here we consider the free boundary problem on a bounded cylinder. We denote the bounded cylin-
ders as the form 
\[ \{0, T\} \times H_n, \]
where \((H_n)\) is an increasing covering of \(\mathbb{R}^d\). We will prove the existence
of a strong solution to problem
\[
\begin{aligned}
\max \{Lu, f - u\} &= 0, & \text{in } H(T) := [0, T] \times H, \\
u|_{\partial_pH(T)} &= f, 
\end{aligned}
\tag{5.30}
\]
where \(H\) is a bounded domain of \(\mathbb{R}^d\) and
\[
\partial_pH(T) := \partial H(T) \setminus \{T\} \times H
\]
is the parabolic boundary of \(H(T)\).

We assume the following condition on the payoff function

**Assumption 5.1** The payoff function \(\xi = f(S_t, x, u)\) have the following assumption expressed by the
sublinear function
\[
-G(-D^2f) \geq c \quad \text{in } H,
\tag{5.31}
\]
where \(G(\cdot)\) is the sublinear function defined by equation \((2.1)\).

**Theorem 5.3** We assume the Assumption ?? holds. Problem \((5.30)\) has a strong solution \(u \in S^1_{\text{loc}}(H(T)) \cap C(H(T))\). Moreover \(u \in S^p_{\text{loc}}(H(T))\) for any \(p > 1\).

**Proof.** The proof is based on a standard penalization technique (see Friedman [9]). We consider a
family \((\beta_\varepsilon)_{\varepsilon \in [0, 1]}\) of smooth functions such that, for any \(\varepsilon\), the function
\(\beta_\varepsilon\) is increasing, bounded on \(\mathbb{R}\) and has bounded first order derivative, such that
\[
\beta_\varepsilon(s) \leq \varepsilon, \quad s > 0, \quad \lim_{\varepsilon \to 0^+} \beta_\varepsilon(s) = -\infty, \quad s < 0.
\]
We denote by \(f^\delta\) as the regularization of \(f\), and consider the following penalized and regularized
problem and denote the solution as \(u_{\varepsilon, \delta}\)
\[
\begin{aligned}
Lu = \beta_\varepsilon(u - f^\delta), & \quad \text{in } H(T), \\
u|_{\partial_pH(T)} &= f^\delta, 
\end{aligned}
\tag{5.32}
\]
Lions [17], Krylov [16] and Nisio [21] prove that problem \((5.32)\) has a unique viscosity solution
\(u_{(\varepsilon, \delta)} \in C^{2,\alpha}(H(T)) \cap C(H(T))\) with \(\alpha \in [0, 1]\).

Next, we firstly prove the uniform boundedness of the penalization term:
\[
|\beta_\varepsilon(u_{\varepsilon, \delta} - f^\delta)| \leq c, \quad \text{in } H(T),
\tag{5.33}
\]
with \(c\) independent of \(\varepsilon\) and \(\delta\).

By construction \(\beta_\varepsilon \leq \varepsilon\), it suffices to prove the lower bound in \((5.33)\). By continuity, \(\beta_\varepsilon(u_{\varepsilon, \delta} - f^\delta)\) has a minimum \(\zeta\) in \(H(T)\) and we may suppose
\[
\beta_\varepsilon(u_{\varepsilon, \delta}(\zeta) - f^\delta(\zeta)) \leq 0,
\]
otherwise we prove the lower bound. If \(\zeta \in \partial_pH(T)\) then
\[
\beta_\varepsilon(u_{\varepsilon, \delta}(\zeta) - f^\delta(\zeta)) = \beta_\varepsilon(0) = 0.
\]
On the other hand, if \( \zeta \in H(T) \), then we recall that \( \beta_\varepsilon \) is increasing and consequently \( u(\varepsilon, \delta) - f(\delta) \) also has a (negative) minimum in \( \zeta \). Thus, we have

\[
Lu(\varepsilon, \delta)(\zeta) - Lf(\delta)(\zeta) \geq u(\varepsilon, \delta)(\zeta) - f(\delta)(\zeta).
\]

By the Assumption 5.1 on \( f \), we have that \( Lf(\delta)(\zeta) \) is bounded uniformly in \( \delta \). Therefore, by (5.34), we deduce

\[
\beta_\varepsilon(u(\varepsilon, \delta)(\zeta) - f(\delta)(\zeta)) = Lu(\varepsilon, \delta)(\zeta) \geq Lf(\delta)(\zeta) \geq c,
\]

where \( c \) is a constant independent on \( \varepsilon, \delta \) and this proves (5.33).

Secondly, we use the \( S^p \) interior estimate combined with (5.33), to infer that, for every compact subset \( D \) in \( H(T) \) and \( p \geq 1 \), the norm \( \| u_{\varepsilon, \delta} \|_{S^p(D)} \) is bounded uniformly in \( \varepsilon \) and \( \delta \). It follows that \( u_{\varepsilon, \delta} \) converges as \( \varepsilon, \delta \rightarrow 0 \), weakly in \( S^p \) on compact subsets of \( H(T) \) to a function \( u \). Moreover

\[
\limsup_{\varepsilon, \delta} \beta_\varepsilon(u_{\varepsilon, \delta} - f(\delta)) \leq 0,
\]

so that \( Lu \leq f \) a.e. in \( H(T) \). On the other hand, \( Lu = f \) a.e. in the set \( \{ u > f \} \).

Finally, it is straightforward to verify that \( u \in C(H(T)) \) and assumes the initial-boundary conditions, by using standard arguments based on the maximum principle and barrier functions. □

**Proof of Theorem 5.1** The proof of Theorem 5.1 about the existence theorem for the free boundary problem on unbounded domains is similar in [7] by using Theorem 5.3 about the existence theorem for the free boundary problem on the regular bounded cylindrical domain. □

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