Transient Cherenkov radiation from an inhomogeneous string excited by an ultrashort laser pulse at superluminal velocity

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(Dated: March 19, 2014)

An optical response of one-dimensional string made of dipoles with a periodically varying density excited by a spot of light moving along the string at the superluminal (sub-luminal) velocity is theoretically studied. The Cherenkov radiation in such system is rather unusual, possessing both transient and resonant character. We show that under certain conditions, in addition to the resonant Cherenkov peak another Doppler-like frequency appears in the radiation spectrum. Both linear (small-signal) and nonlinear regimes as well as different string topologies are considered.

I. INTRODUCTION

The problem of superluminal motion and its existence in nature attracts attention of various researchers for rather long time. At the turn of XIX-XX centuries, O. Heaviside and A. Sommerfeld considered radiation of charged particles moving in vacuum at the velocity greater than the velocity of light in vacuum $c$ (see [1–10] and references therein). However, their works were forgotten for many years because the special theory of relativity “bans” such motions. Further analysis has shown that the “prohibited” are only those motions that involve signal (information) transfer at the superluminal velocity, this strong prohibition being related to the violation of the causality principle [1, 2, 9, 11].

If a charged particle moves faster than light in some medium so called Cherenkov radiation occurs. It is emitted typically into a cone with the angle depending on the ratio of the particle velocity and the speed of light in the media. Similar conical emission can appear also in nonlinear optical parametric processes [12–16]. Not only particles but also spots of light can propagate faster than the phase velocity of light in particular medium [8, 17–21]. Those can be optical pulses and solitons in fibers or filaments [22–29] as well as in other optical systems [30–39]. Cherenkov radiation in various periodically modulated media, with modulations both in space [30, 40, 41] and time [42] was also considered. The intersection point of two wavefronts can also move at the velocity exceeding that of light [8, 43]. Similar situation occurs when a short plane-wave pulse crosses a flat screen (or the plane diffraction grating) [8, 40]. In this case, the intersection of the pulse and the screen moves along the screen at the velocity $V = c / \sin \beta > c$ (here $\beta$ is the angle of wave incidence) [44, 45].

Despite of the great number of various configurations studied in the context of Cherenkov radiation, in all those cases such radiation has the same nature. Namely, it is a result of interference of the secondary waves emitted by the moving “particle”.

The temporal shape of the radiating wave and thus the spectrum of radiation can be significantly different depending on the particular situation. For instance, the spectrum of radiation induced by a charged particle moving faster than the phase velocity of light is rather unstructured [1, 2]. In many other cases, resonances may occur [8, 40]. One important example is the so called Purcell-Smith radiation [40, 46] appearing as a charged particle moves in the vicinity of a periodic structure. Also, a moving and at the same time oscillating dipole emits the Cherenkov radiation characterized with well defined resonance [36], similar situation is realized for optical solitons propagation [47].

In the present article, we consider in details the Cherenkov-type radiation in the case of a one-dimensional (1D) string formed of two-level atoms with a spatially-periodic modulated number density. This system is excited at the superluminal (sub-luminal) velocity at the point of intersection of the string with a moving spot of light. This geometry is imposed by recent advances in optical technologies which allow for the reliable control of matter properties on the spatial level of the order of wavelength of light or even much smaller that allows to create quasi-1D objects (see [48, 49] and references therein). Although our consideration is rather general, we bear in mind the spatial properties nanoantenna or quantum dots arrays [48–55] as well as thin microcapillaries. Similar geometry was recently realized experimentally in [20].

As we show, in the system considered here the Cherenkov emission have somewhat unusual character. It
possesses a narrow-band spectrum, with the central frequency at the resonance of dipoles the string comprises of. In presence of inhomogeneities of the dipoles density a second Doppler-like frequency appears in the spectrum even in a linear regime, that is, when the pump remains weak. We show that this effect does not depend on the string geometry by considering both a straight and a circularly-shaped string. With increasing the pump, when the nonlinear response of two level atoms becomes important, this new frequency may even significantly overcome the resonant one.

The structure of the paper is as following: in Section II the system is described and the possible physical realizations are considered; Section III describes a linear response of the string whereas in Sec. IV the nonlinear dynamics is considered. Concluding remarks are presented in Sec. V.

II. PHYSICAL CONSIDERATIONS

The geometry we would like to consider is illustrated in Fig. 1(a). A short spectrally broadband optical plane wave pulse is emitted by a source 1, passing through lenses 2 and 3 which makes the pulse spatially extended. The source must produce significantly broadband and flat spectrum [56–64], which includes also the resonance frequency \( \omega_0 \) of the dipoles forming the string. This spatially extended short in time and in the axial direction pulse has the form of thin “sheet of light” 4, which illuminates at the angle \( \beta \) the string medium parallel to \( z \)-axis. Similar geometry was recently realized experimentally [20].

We also assume that the string consists of oscillators (dipoles) with the resonance frequency \( \omega_0 \) and decay constant \( \gamma \), which number density \( N(z) \) varies periodically along the string with the spatial period \( \Lambda_z \).

Moreover, we assume that the string is thin: its thickness is less than the wavelength of light corresponding to the resonance frequency \( \omega_0 \). Such quasi-1D geometry of the system suggests that, if the irradiation and observation angles \( \beta \) and \( \varphi \) are not zero (cf. Fig. 1(a)), the secondary radiation emitted by any dipole will never hit another dipole on its way to the observer.

It should be noted that even for a string which thickness is less than the light wavelength (at resonant frequency) not only linear but also nonlinear response is possible, leading to optical bistability as well as nontrivial dynamical regimes such as pulsations or chaos [65–74].

The electric field created by this excitation observed at the remote position \( Q \) is determined by the solution of the wave equation \( \Box E = \mu_0 \partial_\tau \mathbf{P} \), where \( \Box = \partial_{xx} + \partial_{yy} + \partial_{zz} - 1/c^2 \partial_\tau \) is the d’Alembert operator, \( c \) is the velocity of light in vacuum. In particular, if the source is a single dipole, that is, \( \mathbf{P} \propto \delta(r) \), the observed secondary emission at the point \( r' \) is:

\[
E(r', t) \propto \partial_\tau \mathbf{P}(r, t - |r - r'|/c).
\]  

(1)

FIG. 1. (a) Excitation of a string at superluminal velocity. 1 - a short spectrally broadband laser pulse source, 2,3 -lenses, 4 the plane pulse wave. The intersection of the plane pulse and the medium moves along the string at the velocity \( V = c/\sin \beta > c \). (b) The observation geometry of the string of the length \( Z_m \) emission. The observer is placed far away from the string or in the focus of a lens L which collects the radiation of the string parallel to the \( z \)-axis. (c) The source must produce significantly broadband and flat spectrum [56–64], which includes also the resonance frequency \( \omega_0 \) of the dipoles forming the string.

In the following, we normalize the later relation in such a way that the coefficient of proportionality between \( E \) and \( \omega_0^2 \mathbf{P} \) is one.

Under those circumstances, the response of dipoles to some excitation \( \mathbf{E}_c(t) \) as seen from the point \( Q \) is described by the sum of the responses of separate dipoles over the whole string, taking into account that the response comes to the point \( Q \) delayed in time. If the response of a single dipole to the excitation is described by the function \( g_c(t) \), the resulting expression will be:

\[
E(t, Q) = \int_0^{Z_m} N(z) g_c(\tilde{t}, z) dz,
\]

(2)

where \( \tilde{t} \) is the delayed time depending on the geometry of the system and \( N(z) \) describes the dipole density.

The physical nature of the oscillators forming the string can be very different. In particular, one can use a string of nanoantennas made of the conducting material. Such nanoantennas have indeed the resonance frequencies defined by their plasmonic resonances, which are highly flexible and are determined by the geometry and size of the structures [48–51]. Using such structures the resonance frequency \( \omega_0 \) can be tuned in the wide range from THz up to the visible. The semiconductor quantum dots arrays [52–55] can be also used. One may remark that in the case of semiconductors, strong nonlinearities accomplished by the possibility of pump allow to use such thinner-than-wavelength layers as active elements in quantum well and quantum dot lasers [75–85].

If the exciting pulse is weak and the dipole response is linear, its response to the excitation pulse \( \mathbf{E}_c(t) \) is de-
scribed by the polarization $\mathbf{P}(t)$

$$\ddot{\mathbf{P}} + \gamma \dot{\mathbf{P}} + \omega_0^2 \mathbf{P} = g \mathbf{E}_e(t), \quad (3)$$

where $g$ is the coupling strength to the field.

Considering the small signal (linear) regime we assume also, that the excitation pulse is shorter than the resonant period of oscillators, so that its spectrum not only includes $\omega_0$ but at the same time, is significantly broad and flat [see Fig. 1(c)]. Such pulses can be obtained, for instance, in THz and MIR range using a gaseous ionization-based source pumped by an ultrashort optical pump pulse [56–64].

Under this condition and also assuming that $\gamma \ll \omega_0$, the nonsingular part of the response of the oscillators can be to a good precision described by an excitation function

$$g_e(t) \approx e^{-\gamma t} \cos(\omega_0 t) \Theta(t), \quad (4)$$

where $\Theta(t)$ is the Heaviside step-function.

In the case when the signal has large enough amplitude, the response of the dipoles becomes nonlinear. The time relaxation of the polarization, $\Delta \omega t$ where $\Theta(\omega_0 t)$

$$\Phi(t, z) = \frac{d_{12}}{\hbar} \int_{-\infty}^{t} E(t', z) dt',$$

is the pulse area [86, 87]. The change of $N$ and $P$ can be represented as the rotation of a unit vector in the $(x, y)$ plane in such a way that the $x$ component of the vector corresponds to $v$, and the $y$ component to $w$. Then the function $\Phi$ is the angle of rotation of this vector: $\Phi = \pi$ corresponds to a complete transition of the particle to the upper level ($\pi$-pulse), and $\Phi = 2\pi$ corresponds to a complete return to the ground level (2$\pi$-pulse).

III. A LINEAR RESPONSE

A. Straight string: general considerations

In this section we consider the case when our string has the form of a straight line of the length $Z_m$ [Fig. 1(b)]. The observer is located at the very large distance from the string or in the focal point $Q$ of the lens $L$. The string consists of identical dipole oscillators having the resonance frequency $\omega_0$ ($\lambda_0$ is the corresponding wavelength) and the decay rate $\gamma$. The number density of oscillators along the Oz axis varies periodically with the period $\Lambda_z$:

$$N(z) = \frac{1}{2} \left( 1 + a \cos \frac{2\pi}{\Lambda_z} z \right), \quad (11)$$

where $a \leq 1$ is the amplitude of density oscillations. This equation describes a sort of 1D diffraction grating formed of particles possessing proper resonance frequency. In the following, for simplicity, we take $a = 1$.

At the initial time moment $t = 0$ the excitation point crosses the point $z = 0$ and starts to propagate at the velocity $V$ along the string towards its other end. We suppose that the exciting pulse is linearly polarized and the polarization direction is perpendicular to the plane of Fig. 1(b). The oscillators start to emit electromagnetic radiation according to the response law Eqs. (4)-(1). We now consider this secondary radiation propagating at the angle $\varphi$ to the string, which reaches the amplitude detector at the point Q (Fig. 1(b)). Under those circumstances we can restrict ourself by a single linear polarization, thus obtaining a scalar problem.

The electric field emitted by the oscillator located in $z$ (as being observed in the same point) is proportional, according to Eq. (4), to:

$$E(t, z) = \exp \left[ -\gamma \left( t - \frac{z}{V} \right) \right] \cos \left[ \omega_0 \left( t - \frac{z}{V} \right) \right] \Theta \left[ t - \frac{z}{V} \right], \quad (12)$$

where the delayed argument describes the fact that the excitation appears in the point $z$ at the moment delayed by $z/V$. Instead of point $Q$ one may consider the electric field at the plane S orthogonal to the direction $\varphi$ passing
through the point $Z_m$. Propagation time from this reference plane to $Q$ is constant and will be omitted in the following analysis.

The light propagation time from the point $z$ to the reference plane is given by $\frac{Z_m - z}{c} \cos \varphi$. Thus, the electric field emitted at the point $z$ will have at the reference plane the value:

$$E_{\text{ref}}(t, z) = \exp \left[ -\frac{\gamma}{2} f(t, z) \right] \cos [\omega_0 f(t, z)] \Theta[f(t, z)],$$

where $f(t, z) = t - \frac{z}{c} - \frac{Z_m - z}{c} \cos \varphi$.

The total field observed in $Q$ is obtained by the integration of Eq. (13) over the whole string:

$$E(t, \varphi) = \int_0^Z N(z) \exp \left[ -\frac{\gamma}{2} f(t, z) \right] \cos [\omega_0 f(t, z)] \times$$

$$\times \Theta[f(t, z)] dz.$$  

(14)

The analytical solution of Eq. (14) in the case of $\gamma = 0$ is given in the Appendix. As one can see from analytical calculation in Appendix [see Eq. (A6)] the response contains the resonance frequency of oscillators $\omega_0$ together with a new component given, by the expression:

$$\Omega_1 = 2\pi \left( \frac{V/\Lambda_z}{1 - \frac{1}{c} \cos \varphi} \right).$$

(15)

The inverse numerator of Eq. (15) is the time interval which the excitation spot needs to cross the single oscillation period of $N(z)$. When this time is equal to the period of the oscillations ($V/c = \Lambda_z/\omega_0$) formula (15) leads to:

$$\Omega_{1D} = \frac{\omega_0}{1 - \frac{1}{c} \cos \varphi}.$$  

(16)

This relation formally coincides with that one for the Doppler frequency shift [44, 45], so we will call it the Doppler frequency but will keep in mind that its physical origin differs from that of the Doppler effect.

Equation Eq. (15) is valid for arbitrary $V$ and has the same form as the one appearing in the case of Purcell-Smith radiation. The appearance of this frequency and other related questions will be studied in detail in the next section.

### B. Straight string: the linear response dynamics

Now we explore the temporal and spectral shape of the linear string response defined by Eq. (14) and its dependence on the system parameters.

We start from the numerical simulations of Eq. (14) for some “typical” parameter values. Namely, we choose the normalized parameters as: $\frac{V}{c} = 2$, $\frac{\omega_0}{\omega_0} = 9.55$, $\frac{\Lambda_z}{\Lambda_0} = 5$, $\frac{Z_m}{\gamma} = 22.22$. The real-world values of the parameters corresponding to this set depend on the resonance frequency of the oscillators in the string. For instance, assuming $\omega_0 = 2\pi \times 10$ ps$^{-1}$ (frequency for which the $\delta$-function assumption from Fig. 1(c) is especially easy fulfilled), we will have $\Lambda_z = 150 \mu m$, $Z_m = 1.4$ mm, and $\gamma = 2.8$ ps$^{-1}$. Another example is the pump at optical frequencies $\omega_0 = 2\pi \times 375$ ps$^{-1}$, $\Lambda_z = 4 \mu m$, $Z_m = 40 \mu m$, $\gamma = 106.04$ ps$^{-1}$.

The numerical solution of integral Eq. (14) and its spectrum are shown in Fig. 2 (in normalized units) for the above mentioned parameters, and assuming $\varphi = 0$ (observation point is on the same line as the string) in Fig. 2(a,b) and the Cherenkov angle $\varphi = 60^\circ$ in Fig. 2(c,d).

As one can see, the resonant response at $\omega = \omega_0$ dominates in both cases. Nevertheless, for $\varphi = 0$ additional frequency arises. As it is seen in Fig. 2(a), the new frequency appears in the transient process for the time interval from approx. $t_1/T_0 = 20$ to approx. $t_2/T_0 = 47$, at the moment $t_1$ the excitation spot reaches the end of the string. During the period $t_1$ to $t_2$, the radiation from the points $z = Z_m$ to $z = 0$ arrives to the observation plane. As the result of the interference of the incoming waves, a transition process occurs. It lasts until the moment $t_2$. Only the decaying emission with the frequency $\omega_0$ remains at the later time.

For the superluminal velocity of the excitation the denominator of Eq. (15) approaches zero if:

$$\cos \varphi_0 = c/V,$$  

(17)

which coincides with the condition for Cherenkov radiation. Fig. 2(c) corresponds to the Cherenkov emission

![Fig. 2. Time dependence of the field $E(t)$ according to Eq. (14) (a,c) and its spectral intensity $I(\omega)$ (b,d) normalized to their maximal values vs. normalized time $t/T_0$ and frequency $\omega/\omega_0$, for $\frac{V}{c} = 2$, $\frac{\omega_0}{\omega_0} = 9.55$, $\frac{\Lambda_z}{\Lambda_0} = 5$, $\frac{Z_m}{\gamma} = 22.22$ for the observation angle $\varphi = 0$ (a,b) and $\varphi = 60^\circ$ (c,d), the later corresponds to the Cherenkov emission angle.](image)
assuming $\varphi = 1$ condition Eq. (17) is fulfilled, we have $\Omega_1 = \infty$, and the radiation from all points of the grating (the resonance medium) comes to the reference plane simultaneously, thus no transient process occurs.

Analogously, +1st- and -1st diffraction orders maxima are defined by the relation:

$$\cos \varphi_{\pm 1} = \frac{\pm \lambda_0}{\Lambda_z} + \frac{c}{V},$$

which for the parameters of Fig. 3(a) gives the angles $\varphi_{+1} = 45.57$ and $\varphi_{-1} = 72.54$ degree correspondingly. For those angles, we have $\omega_0 = \Omega_1$ as well. For all values of $\varphi$ different from the one given by Eq. (18), the Doppler frequency $\Omega_1$ is not equal to $\omega_0$. It should be noted however that in this case the radiation intensity is smaller than for the Cherenkov angle.

Dependence of the Eq. (14) solution spectrum on the system parameters is presented in Fig. 3. In particular, the dependence on the observation angle $\varphi$ is presented in Fig. 3(a), on the excitation speed $V$ in Fig. 3(b) and on the grating period $\Lambda_z$ (cf. Eq. (11)) in Fig. 3(c).

Dependence of the string response spectrum on $V/c$ assuming $\varphi = 60$ degree is presented in Fig. 3(b). The other parameters are the same as in Fig. 2(c). One can clearly see the frequency branch corresponding to the resonance $\omega = \omega_0$, as well as the another one corresponding to the frequency shift given by Eq. (15).

According to Eq. (15) and Fig. 2(c), $\Omega_1$ decreases with the increasing of $V/c$ for $V/c > 2$ and increases for $V/c < 2$. From Eq. (15) it also follows that $\Omega_1 \to \infty$ for $V \to 2c$ (when $\varphi = 60$ degree). This also coincides with the typical behavior of the Doppler frequency shift.

The dependence of the string response spectrum on the modulation period $\Lambda_z/\lambda_0$ is presented in Fig. 3(c) for $V/c = 3, \varphi = 60$ degree. As it can be seen, $\Omega_1$ decreases with increasing of the $\Lambda_z/\lambda_0$.

Up to now we have considered the case when the string is excited by a spot of light moving at the superluminal velocity. Another interesting case if the exciting spot moves at the sub-luminal velocity.

Such situation can be realized not only using the scheme in Fig. 1, but also using an electron beam moving with some velocity $u$ at an angle $\psi$ to the boundary of the string. In this case, the velocity of the intersection of the incident beam with the boundary of the medium is $V = u/\sin \psi$ [45].

The example of numerical solution of the integral Eq. (14) assuming $V/c = 0.7$ and $\varphi = 0$ are shown in Fig. 4. Other parameters are the same as in the Fig. 2. One can see that the additional frequency component arises during the transient process from approx. $t_1/T_0 = 47$ to approx. $t_2/T_0 = 70$. At the time moment $t_1$ the radiation from the point $z = 0$ reaches the end of the medium. At the time moment $t_2$ the radiation from the point $z = Z_m$ appears at the observation point. Later on, only decaying oscillations at the frequency $\omega_0$ remain.

Dependence of the string response spectrum on the observation angle $\varphi$ as well as on the grating period $\Lambda_z$ is presented in Fig. 5. As one can see, the situation in the case $V < c$ is in many respects similar to the case of the superluminal velocity. In particular, $\Omega_1$ decreases with the increase of $\varphi$ as well as with the increase of $\Lambda_z$. On the other hand, the Cherenkov angle at which $\Omega_1 = \infty$ is never achieved.

$$\cos \varphi_{\pm 1} = \frac{\pm \lambda_0}{\Lambda_z} + \frac{c}{V},$$

FIG. 3. Dependence of the spectral intensity $I(\omega)$ of the string response according to Eq. (14) on the observation angle $\varphi$ (a), the excitation velocity $V$ (b) and on the string density modulation period $\Lambda_z$ (c). The other parameters coincide with those ones given in Fig. 2(c,d). The spectral intensity is presented in the logarithmic scale.

FIG. 4. Time dependence (a) of the string response field $E(t)$ according to Eq. (14), and (b) its spectral intensity $I(\omega)$ normalized to the maximum values vs. normalized time $t/T_0$ and frequency $\omega/\omega_0$, for $\frac{V}{c} = 0.7, \frac{\lambda_0}{\Lambda_z} = 9.55, \frac{\lambda_0}{\chi_2} = 5, \frac{\lambda_0}{\gamma} = 22.22$ and observation angle $\varphi = 0$. 

FIG. 5. Dependence of the string response spectrum on the observation angle $\varphi$ as well as on the grating period $\Lambda_z$.
In this section we consider completely different topol-
gogy of the string depicted in Fig. 6. Namely, the string
parameters coincide with ones in Fig. 4. Note the logarithmi
c scale in the plot.

C. Circular string: general considerations

In this section we consider completely different topol-
gy of the string depicted in Fig. 6. Namely, the string
made of the dipoles owing the same resonance frequency
\( \omega_0 \) as before is arranged along the circle of radius \( R \). The
dipole density is modulated along the string in a period-
ically with the angular period \( \Lambda_\phi \) as:

\[
N(\phi) = \frac{1}{2} \left( 1 + a \cos(\frac{2\pi}{\Lambda_\phi} \phi) \right),
\]

(19)

As in Eq. (11), we assume the modulation amplitude
\( a = 1 \). In the center of the circle a source of a short
spectrally broad optical pulse (see Fig. 1(c)) is located,
which quickly rotates, so that the cross-section point (yel-
low point in Fig. 6) moves at the velocity \( V \) along the
circle.

The dipoles of the string response to the excitation
emitting the secondary waves. Here we will concentrate
on the behavior of the string response field \( E(t) \) observed
in the center of circle. The electric field formed in the
center of the circle by the element \( dE_\phi \) located at the
point which angular coordinate is \( \phi \) is given by:

\[
dE_\phi(t) = N(\phi) \exp \left( -\frac{\gamma}{2} f_\phi(t, \phi) \right) \cos[\omega_0 f_\phi(t, \phi)] \times
\]

\[
\times \Theta[f_\phi(t, \phi)] d\phi, \tag{20}
\]

where \( f_\phi(t, \phi) = t - \frac{R_\phi - R}{c} \). For one round pass of the
excitation, the total electric field is obtained by integra-
tion (20) over \( \phi ):

\[
E(t, \phi) = \int_0^{2\pi} N(\phi) \exp \left[ -\frac{\gamma}{2} f_\phi(t, \phi) \right] \cos[\omega_0 f_\phi(t, \phi)] \times
\]

\[
\times \Theta[f_\phi(t, \phi)] d\phi. \tag{21}
\]

The analytical solution of Eq. (21) in the case of \( \gamma = 0 \)
is given in the Appendix. As one can see from analytical
calculation in Appendix [see Eq. (A8)] the response con-
tains the resonance frequency of oscillators \( \omega_0 \) together
with a new component, given by the expression:

\[
\Omega_2 = 2\pi \frac{V}{\Lambda_\phi R}. \tag{22}
\]

Eq. (22) also has a simple physical meaning, namely this
is the frequency at which the intersection point crosses
the inhomogeneity oscillations. Under the condition

\[
\frac{V}{c} = \frac{\Lambda_\phi R}{\lambda_0}, \tag{23}
\]

the new frequency is equal to the resonance one.

Note also that Eq. (22) is valid if the observer is located
anywhere on the axis passing through the center of the
circle perpendicularly to its plane.

D. Circular string: the linear response dynamics

We start from a typical situation in the spectrum when
the frequency \( \Omega_2 \) is clearly visible. Namely, we take the
following parameters: \( V/c = 3.75 \), \( \frac{\Lambda_\phi R}{\lambda_0} = 2 \), \( \omega_0/\gamma = 22.2 \), \( \omega_0/\Omega_2 = 0.53 \). Assuming the same \( \omega_0 = 2\pi \times 10 \)
ps\(^{-1} \) as in the Sec. III B and \( R = 3 \) cm, we obtain
\( \Lambda_\phi = 0.002 \) rad\(^{-1} \), \( \gamma = 2.8 \) ps\(^{-1} \). The transient process
for these parameters calculated using Eq. (21) is shown in
Fig. 7. For these particular parameters, the frequency of
oscillators practically doubles that of the transient emis-
sion, it results in the high-amplitude beatings clearly seen
in Fig. 7. Once the transition process is finished, the ob-
server at O records the ordinary decaying oscillations.
This conclusion is also valid in the case when the excita-
tion pulse moves at the sub-luminal velocity or precisely

FIG. 5. Dependence of the spectral intensity \( I(\omega) \) of the string
response according to Eq. (14) on the observation angle \( \varphi \) (a),
and the string density modulation period \( \Lambda_\phi \) (b). The other
parameters coincide with ones in Fig. 4. Note the logarithmi
c scale in the plot.

FIG. 6. Circular geometry of the string. The source of a
short pulse with a broad spectrum (c) is located in the center
of the circle and quickly rotates. The cross-section of the
pulse an medium (yellow dot) moves at the velocity \( v \) along
the string (black circle). As in the previous case, the string
is made of dipoles characterized with resonance frequency \( \omega_0 \)
and the dipoles number density is modulated along the string
periodically with the angular period \( \Lambda_\phi \).

FIG. 7. For these particular parameters, the frequency of
oscillators practically doubles that of the transient emis-
sion, it results in the high-amplitude beatings clearly seen
in Fig. 7. Once the transition process is finished, the ob-
server at O records the ordinary decaying oscillations.
This conclusion is also valid in the case when the excita-
tion pulse moves at the sub-luminal velocity or precisely
at the velocity of light. In all these cases, the radiation spectrum at the center of circle will possess a new frequency, with only exception of the resonance Eq. (23).

In order to illustrate the dependence $\Omega_2$ on the parameters of system, we present the radiation spectrum in dependence on $V$ (Fig. 8(a)) and $R$ (Fig. 8(b)) whereas the other parameters are as in the Fig. 7. As it can be easily seen from Eq. (22), the new frequency increases with the increase of $V$ and decreases with $R$.

In the presented circular case, the role of the angle (if the excitation velocity is fixed) plays the radius of the circle. The Cherenkov resonance corresponds then to the $R$ value defined by Eq. (23).

**IV. STRONG PUMP AND NONLINEAR DYNAMICS**

In the present section we investigate a response of a straight string in the regime when the pump is strong enough, so that the response of the string is significantly nonlinear.

The response of the string, assuming the scalar approximation and also taking into account that the secondary radiation never comes back to the string and thus no nonlinear propagation takes place (unless the observation angle $\varphi = 0$) is described by Eq. (2). Taking into account Eq. (9) we obtain:

$$E(t, \varphi) = \int_{0}^{z_{m}} N(z) P[f(t, z)] \sin[\omega_{0} f(t, z)] \, dz. \quad (24)$$

In this section, as it was already mentioned, we consider pulses with relatively narrow spectrum, to be consistent with the approximations for which Eq. (9) were derived. The pulses we use are nevertheless still short enough to clearly observe the frequency $\Omega_1$. The result of numerical solution of the integral Eq. (24) assuming the parameters of Fig. 2 and the observation angle $\varphi = 71$ degree, total pulse area $S = \pi/2$, $\Omega R = 0.07 \omega_{0}$ and $\tau_{p} = 2T_{0}$ is presented in the Fig. 9.

As one can see from the Fig. 9(a), analogously to the linear case, the system demonstrates a short pulse in a transient regime. Its duration is equal to few periods of optical oscillations $T_{0}$. After a transient process decaying emission at the frequency $\omega_{0}$ is observed. As in the linear case, the two frequencies are observed as shown in Fig. 9(b). One may note that the peak corresponding to the frequency $\Omega_1$ is more pronounced than in the linear case.

Fig. 10 illustrates the dependence of the secondary emission spectrum on the total pulse area $\Phi$ (cf. Eq. (10)). Pulse area was changed via modification of the Rabi frequency (pulse amplitude), keeping the pulse duration constant. From Fig. 10 one can observe two branches corresponding to the resonance frequency $\omega_{0}$ and to the Doppler-like one $\Omega_1$. Analysis of Fig. 10 shows that in strongly nonlinear regime when the pulse area is large the radiation on the resonance frequency $\omega_{0}$ has even smaller intensity than the one on the frequency $\Omega_1$. The periodic structure revealed in Fig. 10 in dependence on $\Phi$ is explained by phase relations between the periodic term $\sin \Phi$ entering $P(t, z)$ [cf. Eq. (24), Eq. (9) and Eq. (10)] and the period of spatial inhomogeneity of dipole density $N(z)$. 

**FIG. 7.** (a) Time dependence of the electric field $E(t)$ excited by the string and (b) the corresponding intensity spectrum $I(\omega)$ in the center of the circle for the circular scheme depicted in Fig. 6 and the parameters $V/c = 3.75$, $\Delta_{\omega} = 2$, $\omega_{0}/\gamma = 22.2$, $\omega_{0}/\Omega = 0.53$.

**FIG. 8.** (a) Dependence of the radiation spectrum on the normalized propagation speed $V$ of the excitation (a) and the radius of the circle $R$ (b). Other parameters are as in the Fig. 7.
in the case of the straight string and Ω₂ in the circular case) appears in the transient regime, when some of the secondary waves excited by the exciting pulse have not yet reached the observation plane. The dynamics of the radiation after this moment is trivial and contains only decaying oscillations on the resonant frequency ω₀. In the strong-signal regime, when the nonlinearity in the string response becomes significant, the new frequency may even significantly overcome the resonant one.

The behavior described there can find its application, for instance, to shape the broad spectra and short pulses in desired way using rather compact setup.

**ACKNOWLEDGMENTS**

R.M. Arkhipov would like to acknowledge the support of EU FP7 ITN PROPHET, Grant No. 264687. I. B. is thankful to German Research Foundation (DFG) for the financial support in the framework of the Collaborative Research Center SFB 910 and project BA 41561-1.

**Appendix A: Analytical solutions of Eq. (14) and Eq. (21).**

In this appendix we provide an analytical solutions of Eq. (14) and Eq. (21) when γ = 0. To obtain such expression we first rearrange the argument of Θ - function in Eq. (14) as \( f(t,z) = t - \frac{z}{c} - \frac{Z_0}{c} \cos φ = t - z/W - Z_0/c \cos φ \), where \( W \) is the effective velocity defined as:

\[
\frac{1}{W} = \frac{1}{V} - \frac{1}{c/\cos φ}. \tag{A1}
\]

One can see that \( W \) can be interpreted as the velocity of the projection of the cross-section point to the axis parallel to the observation plane in Fig. 1(b). Using this parameter we can rewrite the integral for the pulse response in the form:

\[
E(t) = \int N(z)h_0 \left( t' - \frac{z}{W} \right) dz, \tag{A2}
\]

where \( t' = t - \frac{Z_0}{c} \cos φ \), and the function \( h_0(t) \) denotes the response of a dipole located at \( z = 0 \) and excited with the excitation in the form of delta-function \( δ(t) \):

\[
h_0(t) = \cos(ω₀t)\Theta(t). \tag{A3}
\]

The integral in Eq. (A2) has different form depending on the sign of \( W \):

\[
E(t) = \int_{0}^{Wt'} N(z)h_0 \left( t' - \frac{z}{W} \right) dz \text{ for } W > 0, \tag{A3}
\]

\[
E(t) = \int_{Z_m}^{Wt'} N(z)h_0 \left( t' - \frac{z}{W} \right) dz \text{ for } W < 0. \tag{A4}
\]

If \( W > 0 \) the emitting element of the string moves in positive direction starting from zero as being seen by the
observer. In the opposite situation, when \( W < 0 \), it is seen as moving in the negative direction from \( Z_m \) to 0.

Eqs. (A3)-(A4) are valid for \( 0 < t < Z_m/W \) (transient regime) assuming \( \nu_z = \frac{2\pi}{\lambda_z} \). In particular for \( W > 0 \) one can obtain:

\[
E(t) = \frac{W}{\omega_0} \sin (\omega_0 t') + \frac{W}{W^2 \nu_z^2 - \omega_0^2} [\nu_z W \sin (\nu_z W t') - \omega_0 \sin (\omega_0 t')] . \quad (A5)
\]

On the other hand, for \( W < 0 \) we have:

\[
E(t) = \frac{W}{\omega_0} \sin \left( \omega_0 (t' - \frac{Z_m}{W}) \right) + \frac{W^2 \nu_z \sin (\nu_z W t') - \nu_z W^2 \sin (\nu_z Z_m) \cos \left[ \omega_0 (t' - \frac{Z_m}{W}) \right] - \omega_0 W \cos (\nu_z Z_m) \sin \left[ \omega_0 (t' - \frac{Z_m}{W}) \right]}{\nu_z^2 W^2 - \omega_0^2} . \quad (A6)
\]

The last equations contain the oscillating terms with the frequencies \( \omega_0 \) and \( \Omega_1 = \nu_z W \) which coincides with the Eq. (15).

For \( t \geq Z_m/W \), that is when the excitation pulse comes out of the string, we have:

\[
E(t) = \int_{0}^{Z_m} N(z) \cos \left[ \omega_0 \left( t' - \frac{z}{W} \right) \right] dz = \frac{W}{\omega_0} \left[ \sin (\omega_0 t') - \sin \left( \omega_0 (t' - \frac{Z_m}{W}) \right) \right] + \frac{\nu_z W^2 \sin (\nu_z Z_m) \cos \left[ \omega_0 (t' - \frac{Z_m}{W}) \right]}{\nu_z^2 W^2 - \omega_0^2} + \frac{\omega_0 W \cos (\nu_z Z_m) \sin \left[ \omega_0 (t' - \frac{Z_m}{W}) \right]}{\nu_z^2 W^2 - \omega_0^2} . \quad (A7)
\]

This term describes the oscillations with the frequency \( \omega_0 \) after transition process stops.

In the case of circular geometry for transient process (\( R < t < \frac{2\pi R}{V} + \frac{R}{c} \) if \( V > c \)) one can obtain:

\[
E(t) = \int_{0}^{2\pi} N(\phi) \cos \left[ \omega_0 \left( t'' - \frac{R \phi}{V} \right) \right] d\phi = \frac{V}{R \omega_0} \sin \left( \omega_0 (t'' - \frac{2\pi R}{V}) \right) + \frac{R \omega_0 V \cos (2\pi \nu_\phi) \sin \left[ \omega_0 (t'' - \frac{2\pi R}{V}) \right]}{\nu_\phi^2 V^2 - R^2 \omega_0^2} . \quad (A8)
\]

Here \( t'' = t - \frac{R}{c} \), \( \nu_\phi = \frac{2\pi}{\lambda_\phi} \). The last expression contains terms oscillating on the frequencies \( \Omega_2 \) and \( \omega_0 \). After transition process ends (\( V > c \), \( t > \frac{2\pi R}{V} + \frac{R}{c} \)), we have:

\[
E(t) = \int_{0}^{2\pi} N(\phi) \cos \left[ \omega_0 \left( t'' - \frac{R \phi}{V} \right) \right] d\phi = \frac{V}{R \omega_0} \left[ \sin (\omega_0 t'') - \sin \left( \omega_0 (t'' - \frac{2\pi R}{V}) \right) \right] + \frac{R \omega_0 V \cos (2\pi \nu_\phi) \sin \left[ \omega_0 (t'' - \frac{2\pi R}{V}) \right]}{\nu_\phi^2 V^2 - R^2 \omega_0^2} - \sin (\omega_0 t'') . \quad (A9)
\]

which contains only terms oscillating with the frequency \( \omega_0 \).

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