Gravitational Self-force in a Radiation Gauge

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In this, the first of two companion papers, we present a method for finding the gravitational self-force in a modified radiation gauge for a particle moving on a geodesic in a Schwarzschild or Kerr spacetime. An extension of an earlier result by Wald is used to show the spin-weight ±2 perturbed Weyl scalar (ψ0 or ψ4) determines the metric perturbation outside the particle up to a gauge transformation and an infinitesimal change in mass and angular momentum. A Hertz potential is used to construct the part of the retarded metric perturbation that involves no change in mass or angular momentum from ψ0 in a radiation gauge. The metric perturbation is completed by adding changes in the mass and angular momentum of the background spacetime outside the radial coordinate r0 of the particle in any convenient gauge. The resulting metric perturbation is singular only on the trajectory of the particle. A mode-sum method is then used to renormalize the self-force. Gralla shows that the renormalized self-force can be used to find the correction to a geodesic orbit in a gauge for which the leading, O(ρ−1), term in the metric perturbation has spatial components even under a parity transformation orthogonal to the particle trajectory, and we verify that the metric perturbation in a radiation gauge satisfies that condition.

We show that the singular behavior of the metric perturbation and the expression for the bare self-force have the same power-law behavior in \( L = \ell + 1/2 \) as in a Lorenz gauge (with different coefficients). We explicitly compute the singular Weyl scalar and its mode-sum decomposition to subleading order in \( L \) for a particle in circular orbit in a Schwarzschild geometry and obtain the renormalized field. Because the singular field can be defined as this mode sum, the coefficients of each angular harmonic in the sum must agree with the large \( L \) limit of the corresponding coefficients of the retarded field. One may therefore compute the singular field by numerically matching the retarded field to a power series in \( L \). To check the accuracy of the numerical method, we analytically compute leading and subleading terms in the singular expansion of \( \psi_0 \) and compare the numerical and analytic values of the renormalization constants, finding agreement to high precision. Details of the numerical computation of the perturbed metric, the self-force, and the quantity \( h_{\alpha\beta} u^\alpha u^\beta \) (gauge invariant under helically symmetric gauge transformations) are presented for this test case in the companion paper.

I. INTRODUCTION

Among the most important sources for LISA are extreme mass ratio inspirals (EMRIs) of solar mass compact objects into supermassive black holes. LISA could potentially measure hundreds of EMRI events whose wide range of astrophysical and fundamental implications include determination of the Hubble constant, of luminosity distance, mass and spin of galactic black holes; and measurements of the deviation from a Kerr geometry of the central object. Such measurements depend on accurate parameter estimation, for which it is essential to have accurate

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gravitational waveforms available; these require an accurate calculation of the gravitational self-force experienced by the particle.

The gravitational self-force contains both dissipative and conservative parts. The dissipative part is simply the familiar contribution from the half-retarded minus half-advanced Green’s function, smooth at the position of the particle. The more difficult portion of calculating the gravitational self-force is the determination of the conservative part of the field. Estimates of its effect on the phase of the waveform are given, for example in [7–11]). The focus of our work is on this latter problem. A number of authors have worked on this problem in recent years. Poisson’s Living Review [12] is a comprehensive self-contained introduction to the subject; Barack [13] gives an extensive recent review; and Detweiler [14] provides a more concise summary.

A complicating feature of the conservative part of the self-force is a combination of its inherent non-locality in curved space due to scatter off curvature [16–17]) and the singular behavior of its expression near the particle. The calculation can be made tractable through the observation that the field consists of both a regular part that is entirely responsible for the self-force and a singular part that contributes nothing to the self-force.

The other complicating feature of self-force computations is the choice of gauge. The Lorenz gauge is particularly useful for sorting out formal issues but does not lead to separable, decoupled equations in the Kerr spacetime. For this reason we choose to work in a radiation gauge that exploits the separability of the Teukolsky equation [18–19] for the perturbed Weyl scalar $\psi_0$ in the Kerr background. Using the Hertz potential formalism developed for gravity by Kegeles and Cohen [20–21], Chrzanowski [22], Stewart [23] and Wald [24] (which we will call the CCK formalism), it is possible to construct a metric perturbation from source-free solutions to the Teukolsky equation. Explicit forms of the reconstruction are given for perturbations of Schwarzschild and Kerr spacetimes, respectively, by Lousto and Whiting [25] and Ori [26]. With $\psi_0$ written as a sum of angular harmonics (in Kerr, a sum of oblate spheroidal harmonics), the reconstruction yields the part of the perturbed metric that has no change in the mass or angular momentum of the spacetime. There is a radiation-gauge form of a change in mass (an $\ell = 0$ perturbation in the case of a Schwarzschild background) that arises from a Hertz potential in the CCK formalism, but it is singular on a ray through the particle [27–28]. One can find a nonsingular form for the metric of a mass perturbation in a radiation gauge [28], but we know of no advantage to using it. To obtain a gauge in which the perturbed metric is singular only on the particle’s trajectory, we add the change in mass and angular in a gauge for which that part of the metric perturbation is continuous.

An expression for the self-force is then found as a mode-sum in terms of the retarded metric. To renormalize it, one must subtract off a singular part that does not contribute to the self-force. We consider two alternatives, involving either an analytic or numerical determination of the singular part of the expression for the self-force $f^{\alpha}_s$ as a power series in the integer $\ell$ that labels the angular eigenvalues (more precisely, a series involving $L := \ell + 1/2$). Renormalization relies on subtracting from the bare self-force – from the expression for the self-force in terms of the retarded field – a singular part that does not contribute to the self-force. This can be checked by showing that the singular vector field $f^{\alpha}_s$ that one subtracts has vanishing angle-average over a small sphere about the particle. In the case of a circular orbit about a Schwarzschild black hole, the conservative part of the self force is axisymmetric about a radial ray through the particle. We numerically compute the axisymmetric part of $f^{\alpha}_s$ and show that (as in a Lorenz gauge) it coincides with the axisymmetric part of $-m \nabla \psi_0^\perp$, with $\rho$ the geodesic distance to the trajectory. Because the angle-average of $\nabla \psi_0^\perp$ over a sphere of radius $\rho$ about the particle vanishes as $\rho \to 0$, the renormalized self-force gives the first-order correction to geodesic motion in our modified radiation gauge.

The paper is organized as follows: In Sec. II, we introduce the self-force equations and a criterion for their use in a generic gauge; we briefly review features of mode-sum renormalization in a Lorenz gauge; and we review relevant parts of the Newman-Penrose [29–30] formalism and spin-weighted harmonics. In Sec. III, we begin with a list of the steps involved in computing the self-force in a modified radiation gauge. In the subsections that follow, we obtain a simple analytic expression for the singular part of each of the Weyl scalars $\psi_0$ and $\psi_4$; we relate the small-distance behavior of the Weyl scalars to their large $\ell$ behavior, and observe that the singular behavior in $\ell$ of the expression for the self-force has the same behavior (involves the same powers of $\ell$) in a radiation gauge as in a Lorenz gauge; and we show that the perturbed metric obtained from $\psi_0$ is unique up to gauge transformations and the addition of metric perturbations corresponding to infinitesimal changes in mass and spin. We conclude the section by studying the parity of the radiation-gauge part of the perturbed metric; in particular, we show that spatial part of the metric (in the frame of the particle) is even under parity to leading order in the geodesic distance $\rho$ to the trajectory. In Sec. IV, we specialize to a particle in a circular orbit around a Schwarzschild black hole, finding $\psi_0^{\text{ret}}$ and $\psi_0^{\text{ret}}$, the retarded and singular forms of $\psi_0$ and comparing the analytic and numeric methods of renormalizing $\psi_0$. The substantial analytical work involved in the mode-sum expression for the leading and subleading terms of $\psi_0^{\text{ret}}$ is detailed in an appendix. Finally, in Sec. V, we briefly discuss the results.

Conventions

Greek letters early in the alphabet $\alpha, \beta, \ldots$ will be abstract spacetime indices; letters $\mu, \nu, \ldots$ will be concrete spacetime indices, labeling components in Schwarzschild or Boyer-Lindquist coordinates. Bold-face Greek indices
\( \mu, \nu \) will label components along the null Newman-Penrose (NP) tetrad defined in Eq. (18) below. We adopt the + − − − signature of Newman and Penrose and use the standard names \( l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha \) for the null NP tetrad and NP notation for the spin coefficients.

II. REVIEW OF SELF-FORCE AND OF BLACK-HOLE PERTURBATIONS IN AN NP FRAMEWORK.

A. Gravitational self-force

We work in linear perturbation theory, for which the metric perturbation is a solution with point-particle source to the Einstein field equation linearized about a Kerr or Schwarzschild background. With the particle’s mass, trajectory and velocity denoted by \( m, z(\tau) \) and \( u^\alpha \), respectively, the source is given by

\[
T^{\alpha\beta}(x) = mu^\alpha u^\beta \int \delta^4(x, z(\tau))d\tau.
\]

Here \( u^\alpha = u^\alpha(x) \) when \( x \) lies on the particle’s trajectory, and the \( \delta \)-function is normalized by \( \int \delta^4(x, z)\sqrt{|g|}dx = 1 \).

We denote by \( h_{\alpha\beta}^{\text{ret}} \) the retarded solution to the perturbed Einstein equation with this source. As noted by Quinn and Wald and by Detweiler and Whiting \[31\], in the MiSaTaQuWa prescription for finding the self-force \[13, 14\], a particle follows a geodesic of the metric \( g_{\alpha\beta} + h_{\alpha\beta}^{\text{ren}} \), where \( h_{\alpha\beta}^{\text{ren}} \) is given by

\[
h_{\alpha\beta}^{\text{ren}} = h_{\alpha\beta}^{\text{ret}} - h_{\alpha\beta}^s,
\]

with \( h_{\alpha\beta}^s \) a locally defined singular field, chosen to cancel the singular part of \( h_{\alpha\beta}^{\text{ret}} \) and give no contribution to the self-force. The perturbed geodesic equation has the form

\[
a^\alpha := u \cdot \nabla u^\alpha = -(g^{\alpha\delta} - u^\alpha u^\delta) \left( \nabla_\beta h_{\gamma\delta}^{\text{ren}} - \frac{1}{2} \nabla_\delta h_{\gamma\beta}^{\text{ren}} \right) u^\beta u^\gamma.
\]

Here \( a^\alpha \) is the acceleration with respect to the background metric, and the self-force is, by definition, \( f^\alpha = ma^\alpha \).

We will denote by \( a^{\text{ret} \alpha} \) the expression on the right side of Eq. (1), with \( h_{\alpha\beta}^{\text{ren}} \) replaced by \( h_{\alpha\beta}^{\text{ret}} \). Work by Gralla \[32\], following a careful justification of the self-force equations by Gralla and Wald \[33\], gives the following characterization of \( a^\alpha \), based on the vanishing of the angle-averaged singular part of the expression for the self-force: Let \( \rho \) be geodesic distance to the particle trajectory. Let \( h_{\alpha\beta}^{\text{ret}} \) be the retarded metric perturbation in a gauge for which its spatial part near the trajectory is \( O(\rho^{-1}) \) and has even parity to that order. Then the self-force is given in local inertial coordinates about \( P \) by

\[
a^{\text{ren} \mu} = \lim_{\rho \to 0} \int_{S_\rho} a^{\text{ret} \mu} d\Omega
\]

where \( S_\rho \) is a sphere of constant \( \rho \) in a geodesic surface orthogonal to the worldline. That is, the first-order perturbative correction to the geodesic equation is

\[
u^\beta \nabla_\beta u^\alpha = a^{\text{ren} \alpha},
\]

with \( a^{\text{ren} \alpha} \) given by Eq. (4).

One can thus identify the singular part of the self-force with any vector field \( ma^{\alpha} \) near the particle trajectory for which \( a^{\text{ret} \alpha} - a^{\alpha} \) is continuous and

\[
\lim_{\rho \to 0} \int_{S_\rho} a^{\alpha} d\Omega = 0.
\]

Then

\[
a^{\text{ren} \alpha}(P) = \lim_{P' \to P} \{a^{\text{ret} \alpha}(P') - a^\alpha[h^s](P')\}.
\]

A free particle in flat space has no self-force, and the form of its linearized field can be used to obtain \( a^{\alpha} \) in a curved spacetime. Its linearized gravitational field in a Lorenz gauge is the Schwarzschild solution in isotropic coordinates, linearized about flat space

\[
h_{\mu\nu} = \frac{2m}{\rho} \delta_{\mu\nu} = \frac{2m}{\rho} (\eta_{\mu\nu} - 2u_\mu u_\nu).
\]
In a Lorenz gauge for a particle in geodesic motion in a curved background, the singular part \( h^s_{\alpha\beta} \) of the metric perturbation takes this form in local inertial coordinates \( T, X, Y, Z \) centered at any point \( P \) along the trajectory, with \( \partial_T = u \):

\[
h^s_{\mu\nu} = -\frac{2m}{\rho} \delta_{\mu\nu} = \frac{2m}{\rho} \left( g_{\mu\nu} - 2u_{\mu}u_{\nu} \right) |_{\rho = 0}.
\]  

(9)

The corresponding singular part of the self-force is given by

\[
f^s_{\mu\nu} = -m \nabla^\mu \frac{1}{\rho}.
\]  

(10)

As in flat space, the angle-average of \( f^\mu \) over a sphere of constant \( \rho \) vanishes in the small-\( \rho \) limit. In particular, although the leading correction to the flat-space coordinate expression \( x^\alpha / \rho^3 \) can be a term of order \( \rho^0 \), the term has the form \( \epsilon_{ijk} x^i x^j x^k / \rho^3 \); because the term is odd in \( x^i \), we have

\[
\lim_{\rho \to 0} \int_{S^2} \nabla^\mu \frac{1}{\rho} \, d\Omega = 0.
\]  

(11)

Because the self-force involves only the values of \( \nabla^\gamma h^s_{\alpha\beta} \) on the particle’s trajectory, two tensors \( h^s_{\alpha\beta} \) and \( \tilde{h}^s_{\alpha\beta} \) give the same self-force if \( \nabla^\gamma (h^s_{\alpha\beta} - \tilde{h}^s_{\alpha\beta}) \) vanishes on the particle’s trajectory. In particular, Detweiler and Whiting \[31\] show that there is a choice \( h^s_{\alpha\beta} \) of the singular field that is a locally defined solution to the perturbed field equations with the same point-particle source. One can choose local inertial coordinates (THZ, coordinates, for example) for which the Detweiler-Whiting singular solution differs from the form \( \tilde{(9)} \) by terms of order \( \rho^2 \). Following the notation in Detweiler-Whiting, we denote their form of the singular field by an upper-case \( S \).

**B. Mode-sum renormalization**

A review of mode-sum renormalization, is given in Ref. \[13\]. We recall some of its main features in a Lorenz gauge; many of these continue to hold in our (modified) radiation gauge.

In mode-sum renormalization, the retarded field \( h^\text{ret}_{\alpha\beta} \) and the corresponding expression \( a^{\text{ret, } \alpha} \) are written as sums over angular harmonics, labeled by \( \ell, m \). In a Schwarzschild background, these are unambiguously associated with the spherical symmetry of the background. In Kerr, one can use the spherical coordinates of a Boyer-Lindquist chart to define the decomposition. When the retarded field is written as a superposition of angular harmonics, its short-distance singular behavior \( \tilde{(9)} \) is replaced by a large \( L \)-divergence of the mode sum at the position of the particle. (Appendix \[B\] relates the large \( L \) behavior of an function on a sphere to its singular behavior for small \( \rho \).)

For a particle at coordinate radius \( r_0 \), the angular harmonics have finite limits as \( r \to r_0 \) from \( r < r_0 \) or \( r > r_0 \). Denoting by \( h^\text{ret}_{\mu\nu} \) the sum over \( m \) of all harmonics belonging to a given \( \ell \), one has

\[
h^\text{ret, } \pm \nu_{\mu\nu}(P) = \tilde{A}_{\mu\nu} + \tilde{B}_{\mu\nu}/L + O(L^{-3}).
\]  

(12)

Similarly, with \( a^\text{ret, } \alpha_{\mu\nu} \) the contribution to \( a^{\text{ret, } \alpha} \) from \( h^\text{ret}_{\mu\nu} \), the large-\( L \) behavior of \( a^\text{ret, } \alpha_{\mu\nu} \) is given by

\[
a^\text{ret, } \pm \alpha_{\mu\nu}(P) = A^{\pm\alpha} L + B^\alpha + O(L^{-2}).
\]  

(13)

Explicit expressions for the renormalization coefficients \( A^\pm_{\mu\nu} \) and \( B_{\mu\nu} \) have been found for generic orbits in a Kerr background by Barack \[13\], who shows that the first two terms in this expansion reproduce the singular part of the acceleration, \( -\nabla^\nu \frac{1}{\rho} \), up to terms that vanish at the particle:

\[
a^\text{ret, } \pm \alpha_{\mu\nu}(P) = A^{\pm\alpha} L + B^\alpha.
\]  

(14)

The fact that the term of order \( L^{-1} \) vanishes for each component \( a^\pm \) is related to the behavior of a short-distance expansion in which terms of order \( \rho^k \) with \( k \) even occur with an odd number of factors of \( x^i \). The retarded acceleration, expressed as a mode-sum that diverges on the particle trajectory, is regularized by a cutoff \( \ell_{\text{max}} \) in \( \ell \),

\[
a^\text{reg, } \pm \nu_{\mu\nu} = \sum_{\ell=0}^{\ell_{\text{max}}} (a^\text{ret, } \pm \alpha_{\mu\nu}(\ell) - a^\pm_{\mu\nu}(\ell)),
\]  

(15)
and the renormalized acceleration is given by
\[
a_{\text{ren}}^{\mu} = \lim_{t_{\text{max}} \to \infty} \sum_{t=0}^{t_{\text{max}}} (a_{t}^{\text{ren}} - a_{t}^{s\mu}). \tag{16}
\]

### C. Black-hole perturbations in an NP framework

The present method obtains the metric perturbation from components of the Weyl tensor along basis vectors of an NP tetrad
\[
e_{1}^{a} := l^{a}, \quad e_{2}^{a} := n^{a}, \quad e_{3}^{a} := m^{a}, \quad e_{4}^{a} := \bar{m}^{a}, \tag{17}
\]
whose two real null vectors \(l^{a}\) and \(n^{a}\) are along the principle null directions of the Kerr geometry. In particular, the Kinnersley tetrad has in Boyer-Lindquist coordinates the components
\[
(l^{a}) = \left(\frac{r^{2} + a^{2}}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \quad (n^{a}) = \frac{1}{2(r^{2} + a^{2}\cos^{2}\theta)}(r^{2} + a^{2}, -\Delta, 0, a), \quad (m^{a}) = \frac{1}{\sqrt{2}(r + ia\cos\theta)}(ia\sin\theta, 0, 1, i/\sin\theta), \tag{18}
\]
where \(\Delta = r^{2} - 2Mr + a^{2}\). We denote the associated derivative operators by
\[
D = l^{\mu}\partial_{\mu}, \quad \Delta = n^{\mu}\partial_{\mu}, \quad \delta = m^{\mu}\partial_{\mu}, \quad \bar{\delta} = \bar{m}^{\mu}\partial_{\mu}, \tag{19}
\]
where boldface distinguishes these operators from subsequently defined scalars. The nonzero spin coefficients are
\[
\begin{align*}
\beta &= -\frac{\cot\theta}{2\sqrt{2}}, & \pi &= i\frac{a\omega^{2}}{2}\sin\theta, & \tau &= -\frac{i}{\sqrt{2}}a\varrho\sin\theta, \\
\mu &= \frac{1}{2}\omega^{2}\varrho\Delta, & \gamma &= \mu + \frac{1}{2}\varrho\bar{\varrho}(r - M), & \alpha &= \pi - \bar{\beta},
\end{align*}
\tag{20}
\]
where we distinguish \(\varrho\) from \(\rho\) introduced before Eq. (21). The spin-weight \(s = \pm 2\) components, \(\psi_{0}\) and \(\psi_{4}\), of the perturbed Weyl tensor are given by
\[
\begin{align*}
\psi_{0} &= -C_{\alpha\beta\gamma\delta}l^{\alpha}m^{\beta}n^{\gamma}\bar{m}^{\delta}, \\
\psi_{4} &= -C_{\alpha\beta\gamma\delta}n^{\alpha}\bar{m}^{\beta}n^{\gamma}\bar{m}^{\delta},
\end{align*}
\tag{21, 22}
\]
Each satisfies the decoupled Teukolsky equation appropriate to its spin weight:
\[
\mathcal{T}_{s}\psi_{s} := \left\{ \left[ \left( \frac{r^{2} + a^{2}}{\Delta} - a^{2}\sin^{2}\theta \right) \frac{\partial^{2}}{\partial t^{2}} - 2s \left( \frac{M(r^{2} - a^{2})}{\Delta} - r - ia\cos\theta \right) \frac{\partial}{\partial t} + \frac{4Mar}{\Delta} \frac{\partial^{2}}{\partial t\partial\varphi} - \Delta^{-s} \frac{\partial}{\partial r} \left( \Delta^{s+1} \frac{\partial}{\partial r} \right) \right] \psi_{s} - \frac{1}{\sin\theta} \frac{\partial}{\partial \vartheta} \left( \sin\theta \frac{\partial}{\partial \vartheta} \right) - 2s \left[ \frac{\alpha(r - M)}{\Delta} + \frac{i\omega\varrho}{\sin^{2}\theta} \right] \frac{\partial}{\partial \varphi} + \left[ \frac{a^{2}}{\Delta} - \frac{1}{\sin^{2}\theta} \right] \frac{\partial^{2}}{\partial\varphi^{2}} + (s^{2}\omega^{2}\theta - s) \right\} \psi_{s} = 4\pi(r^{2} + a^{2}\cos^{2}\theta)T_{s},
\tag{23}
\]
where
\[
\begin{align*}
\psi_{s=2} &= \psi_{0}, \\
T_{s=2} &= 2(\delta + \bar{\delta} - \bar{\alpha} - 3\beta - 4\tau)(D - 2\epsilon - 2\varrho)T_{13} - (\delta + \bar{\delta} - 2\alpha - 2\beta)T_{11} \\
&\quad + 2(D - 3\epsilon + \bar{\epsilon} - 4\varrho - \bar{\varrho})(\delta + 2\bar{\epsilon} - 2\beta)T_{13} - (D - 2\epsilon + 2\bar{\epsilon} - \bar{\varrho})T_{33}, \\
\psi_{s=-2} &= \varrho^{-4}\psi_{4}, \\
T_{s=-2} &= 2\varrho^{-4}(\Delta + 3\gamma - \bar{\gamma} + 4\mu + \bar{\mu})(\delta - 2\bar{\tau} + 2\alpha)T_{24} - (\Delta + 2\gamma - 2\bar{\gamma} + \bar{\mu})T_{44} \\
&\quad + 2\varrho^{-4}(\delta - \tau + \bar{\beta} + 3\alpha + 4\pi)(\Delta + 2\gamma + 2\mu)T_{24} - (\delta - \tau + 2\beta + 2\alpha)T_{22},
\end{align*}
\tag{24a, 24b}
\]
The scalars $\psi_{0\text{ret}}^r$ and $\vartheta^{-4}\psi_4^r$ can be decomposed into time and angular harmonics,

$$\psi_{0\text{ret}}^r = 2R_{t\text{ret}m}a S_{t\text{ret}m\omega}e^{i(m\phi - \omega t)},$$

(25a)

$$\vartheta^{-4}\psi_4^r = -2R_{t\text{ret}m}a S_{t\text{ret}m\omega}e^{i(m\phi - \omega t)},$$

(25b)

where the $sS_{t\text{ret}m\omega}$ are oblate spheroidal harmonics and where $sR_{t\text{ret}m\omega}$ satisfies the radial equation,

$$\Delta^{-s} \frac{d}{dr} \left( \Delta^{s+1} \frac{dR}{dr} \right) + \left( \frac{K^2 - 2is(r - M)K}{\Delta} + 4is\omega r - \lambda \right) R = -4\pi T_{\delta t\text{ret}m\omega},$$

(26)

with $K = (r^2 + a^2)\omega - ma$. The source, a distribution involving $\delta(r - r_0)$ and its first two derivatives, is obtained from the source term in Eq. (25) using the completeness and orthogonality of the spin-weighted spheroidal harmonics.

Each angular eigenvalue $\lambda$ is a continuous function of $a$. For a Schwarzschild background, $\lambda$ has the value $(\ell - s)(\ell + s + 1)$, and for Kerr it is labeled by its value of $\ell$ at $a = 0$.

For perturbations of Schwarzschild, tensor components with $s_1$ indices along $m^\alpha$ and $s_2$ indices along $\bar{m}^\alpha$ have angular behavior given by spin-weighted spherical harmonics $sY_{\ell m}(\theta, \phi)$ with spin-weight $s = s_1 - s_2$, where

$$sY_{\ell m} = \left\{ \begin{array}{ll}
[(\ell - s)!/(\ell + s)!]^{1/2} \partial^s Y_{\ell m}, & 0 \leq s \leq \ell, \\
(-1)^s [(\ell + s)!/(\ell - s)!]^{1/2} \partial^{-s} Y_{\ell m}, & -\ell \leq s \leq 0,
\end{array} \right.$$

(27)

with

$$\partial\eta = -(\partial_\theta + i \csc \theta \partial_\phi - s \cot \theta) \eta,$$

$$\bar{\partial}\eta = -(\partial_\theta - i \csc \theta \partial_\phi + s \cot \theta) \eta.$$  

(28)

### D. Reconstruction of the metric perturbation from $\psi_0$ or $\psi_4$

The CCK procedure for obtaining metric perturbations from perturbed Weyl scalars was developed by Chrzanowski and by Cohen and Kegeles [20, 22] (see also Stewart [23]), and a simpler derivation is given in Wald [24]. Discussions in the context of the self-force problem are found in [25, 26] and [28].

The procedure gives the metric perturbation in a radiation gauge, a gauge characterized by the conditions

$$l^\beta h_{\alpha\beta} = g^{\alpha\beta} h_{\alpha\beta} = 0,$$

(29)

or by the corresponding conditions

$$n^\beta h_{\alpha\beta} = g^{\alpha\beta} h_{\alpha\beta} = 0.$$  

(30)

Price, Shankar and Whiting [35, 36] show that a radiation gauge exists locally for vacuum perturbations of any type D vacuum spacetime.

The CCK construction has two parts. Given a solution $\psi_0$ or $\psi_4$ to the source-free $s = \pm 2$ Teukolsky equation, one first finds a Hertz potential, a function $\Psi$ that again satisfies a source-free Teukolsky equation. Then, by taking derivatives of the Hertz potential, one constructs a metric perturbation for which $\psi_0$ and $\psi_4$ are the perturbed Weyl scalars.

There is a striking difference between the asymptotic behavior produced by the CCK procedure and the asymptotic behavior of a metric perturbation in a Lorenz gauge that approximately satisfies Eq. (29) or (30). The difference is related to the terminology used in the literature for the two families of radiation gauges, in which the gauge satisfying $l^\alpha h_{\alpha\beta} = 0$ is called the IRG or ingoing radiation gauge and the gauge satisfying $n^\alpha h_{\alpha\beta} = 0$ is called the ORG or outgoing radiation gauge.

Outgoing solutions in a Lorenz gauge (for example modes for which the metric perturbation has asymptotic behavior $e^{-i\omega u}/r + O(r^2)$), however, satisfy the IRG conditions [29] to leading order: Because $l^\alpha$ is along $\nabla^\alpha u$, we have

$$0 = \nabla^\beta h_{\alpha\beta} = l^\beta h_{\alpha\beta} + O(r^{-2}).$$

(31)

As one might expect, an asymptotically vanishing gauge vector can take one from a metric perturbation in a Lorenz gauge in which the IRG condition is approximately satisfied to a metric perturbation that exactly satisfies the condition: We exhibit in Appendix A an explicit, asymptotically vanishing, gauge vector from a generic outgoing solution.
$h_{\alpha\beta}^{Lor}$ in a Lorenz gauge to an asymptotically flat metric perturbation in an IRG. An analogous gauge transformation leads for incoming radiation to an asymptotically flat metric perturbation satisfying $n^\beta h_{\alpha\beta} = 0$.

Curiously, however, the CCK procedure yields metric perturbations in the two gauges that are asymptotically flat for the opposite cases: In the IRG, with $l^\beta h_{\alpha\beta} = 0$, the CCK procedure yields an asymptotically flat metric for ingoing radiation. For outgoing radiation the CCK procedure yields a metric whose dominant components are asymptotically of order $r$. Similarly, in the ORG gauge, with $n^\beta h_{\alpha\beta} = 0$, the CCK procedure yields an asymptotically flat metric for outgoing radiation. For ingoing radiation the CCK procedure gives a metric whose dominant components are asymptotically of order $r$. This then is the justification for the terms “ingoing” and “outgoing radiation gauge” introduced by Chrzanowski and used in the subsequent literature.

For the gauge (30), a Hertz potential $\Psi$ is related to $\psi_0$ by four angular derivatives or four radial derivatives, and both of these alternatives are listed below. In subsequent sections, we will be concerned only with the specialization of these results to the Schwarzschild spacetime. In this case given a $\Psi$ that satisfies the Teukolsky equation for $\psi_0$, a metric perturbation in the radiation gauge ORG is given by

$$h_{\alpha\beta} = g^{-1} \{ n_\alpha n_\beta (\delta - 3\alpha - \beta + 5\pi)(\delta - 4\alpha + \pi) + \bar{m}_\alpha \bar{m}_\beta (\Delta + 5\mu - 3\gamma + \bar{\gamma})(\Delta + \mu - 4\gamma) - n_\alpha \bar{m}_\beta \} \Psi + c.c.,$$

(32)

which we take as the starting point for the discussion that follows. Note that the sign in this equation is appropriate for a $+ -$ signature and is opposite to that in, for example, Wald [24].

In the ORG, $\Psi$ is related to $\psi_0$ through four angular derivatives according to

$$\psi_0 = \frac{1}{8} [\mathcal{L}^4 \Psi + 12 M \partial t \Psi],$$

(33)

where $\mathcal{L} = \partial - ia \sin \theta \partial_l$. Equivalently, $\mathcal{L}^4 = \mathcal{L}_1 \mathcal{L}_0 \mathcal{L}_{-1} \mathcal{L}_{-2}$, with $\mathcal{L}_\alpha = -[\partial_\alpha - s \cot \theta + ic \csc \theta \partial_\phi] - ia \sin \theta \partial_l$. There is a corresponding equation involving four radial derivatives, namely

$$g^{-4} \psi_4 = \frac{1}{32} \Delta^2 \tilde{D}^4 \Delta^2 \tilde{\Psi},$$

(34)

where $\tilde{D}$ is proportional to $\Delta$, renormalized to make it the radial derivative along the ingoing principal null geodesics (the $t \to -t, \phi \to -\phi$ version of $D$):

$$\tilde{D} := -\frac{2 \Delta}{r^2 + a^2 \cos^2 \theta} \Delta = -\frac{r^2 + a^2}{\Delta} \partial_t + \partial_r - \frac{a}{\Delta} \partial_\phi.$$

(35)

The corresponding equations for the (different) Hertz potential in the IRG are listed in the second line of the summary table below, with $\bar{\mathcal{L}} := \bar{\partial} + ia \sin \bar{\theta} \bar{\partial}_l$.

| Gauge | Gauge conditions | Radial equation | Angular equation |
|-------|------------------|-----------------|-----------------|
| ORG   | $n^\beta h_{\alpha\beta} = 0, \ h = 0$ | $g^{-4} \psi_4 = \frac{1}{32} \Delta^2 \bar{D}^4 \Delta^2 \bar{\Psi}$ | $\psi_0 = \frac{1}{8} [\bar{\mathcal{L}}^4 \bar{\Psi} + 12 M \bar{\partial} t \bar{\Psi}]$ |
| IRG   | $l^\beta h_{\alpha\beta} = 0, \ h = 0$ | $\psi_0 = \frac{1}{8} \bar{D}^4 \bar{\Psi}$ | $g^{-4} \psi_4 = \frac{1}{8} [\bar{\mathcal{L}}^4 \bar{\Psi} - 12 M \bar{\partial} t \bar{\Psi}]$ |

**TABLE I:** Relations between the gauge-invariant Weyl scalars and the Hertz potentials in the two radiation gauges.

The fact that $g^{-4} \psi_4$ and $\psi_0$ satisfy the vacuum Teukolsky equation for spin-weights $\mp 2$ when $\Psi_{\text{ORG}}$ or $\Psi_{\text{IRG}}$ satisfy the spin-weight $\pm 2$ Teukolsky equations follows from the relations in the table together with the commutators

$$\mathcal{T}_2 \bar{D}^4 = \bar{D}^4 \mathcal{T}_{-2}, \quad \mathcal{T}_{-2} \bar{D}^4 \Delta^2 = \Delta^2 \bar{D}^4 \Delta^2 \mathcal{T}_2, \quad \mathcal{T}_2 \bar{L}^4 = \bar{L}^4 \mathcal{T}_{-2}.$$  \hfill (36)

\footnote{\bar{D} is Chandrasekhar’s $\mathcal{D}_2$ and Ori’s $\mathcal{D}_1$. Lousto and Whiting’s Eq. (28) is an incorrect version of Eq. (24), with $\Delta$ (or in their notation) instead of $\bar{D}/2$. This is corrected by Whiting and Price [33], in which $\Delta$ is defined as the GHP prime [37] of $D$. The Ori and Lousto-Whiting papers also have an incorrect factor of two in each of the equations for $\Psi$ that is inherited from an error in Kegeles-Cohen [23].}
These are equivalent to Eqs. (40) and (56) in Chap. 8 of Chandrasekhar [38] and their adjoint relations as defined there.

Explicit solutions to the equations for the Hertz potentials have been presented for a Schwarzschild background by Lousto and Whiting [23] and for Kerr by Ori [26]. Ori shows that the CCK procedure gives a unique Hertz potential in each gauge for each angular harmonic of \( \Psi \). That is, for each harmonic, there is a unique \( \Psi \) that satisfies both the angular equation and the sourcefree Teukolsky equation; there is a unique \( \Psi \) that satisfies both the radial equation and the sourcefree Teukolsky equation; and the two solutions coincide. For \( \psi_0 \) proportional to the harmonic \( 2S_{\ell m \omega} \), \( \Psi_{\text{ORG}} \) is proportional to \( -2S_{\ell -m,-\omega} \).

Note that, although Ori’s metric reconstruction is done mode-by-mode, his statement of uniqueness of solutions does not explicitly restrict it to uniqueness of each angular harmonic. This is, however, a necessary restriction: As shown in Keidl et al. [28], if one requires only that \( \Psi \) satisfy one of the radial or angular equations of Table I together with the appropriate Teukolsky equation, additional freedom remains. This freedom in the Hertz potential corresponds to the addition to the metric perturbation of type D solutions (changes of mass and spin and addition of a perturbed C-metric solution) and with gauge transformations of the perturbed metric.

With \( \psi_0 \) and the ORG \( \Psi \) decomposed in time and angular harmonics, Eq. (38) can be inverted algebraically as follows, at any \( r \) outside the particle’s orbit – for \( r_{\text{min}} < r < r_{\text{max}} \), where \( r_{\text{min}} \) and \( r_{\text{max}} \) are the perihelion and aphelion values of \( r \). The harmonics of \( \Psi \) and \( \psi_0 \) each have the form

\[
\psi_{0 \ell m \omega} = 2R_{\ell m \omega} 2S_{\ell m \omega} e^{i(m\phi - \omega t)}, \quad \Psi_{\ell m \omega} = 2\tilde{R}_{\ell m \omega} 2S_{\ell m \omega} e^{i(m\phi - \omega t)},
\]

where \( R \) and \( S \) are solutions to the radial and angular Teukolsky equations, respectively, and \( \tilde{R} \) is to be determined. Using the identity \( sS_{\ell m \omega} = (-1)^{m+s - s} S_{\ell -m -\omega} \), we can write the harmonic decomposition of \( \Psi \) in the form

\[
\Psi = \sum_{\ell, m, \omega} (-1)^m 2R_{\ell m \omega} 2S_{\ell m \omega} e^{i(m\phi - \omega t)}
\]

\[
= \sum_{\ell, m, \omega} (-1)^m 2\tilde{R}_{\ell -m -\omega} 2S_{\ell m \omega} e^{i(m\phi - \omega t)}.
\]

The Teukolsky-Starobinsky identity (Eqs. (9.59) and (9.61) of Ref. [38]) has the form

\[
\mathcal{L}^4 2S_{\ell m \omega} = D - 2S_{\ell m \omega},
\]

where \( D^2 = \lambda_{CH}^2 (\lambda_{CH} + 2)^2 + 8a\omega(m - a\omega)\lambda_{CH}(5\lambda_{CH} + 6) + 48a^2\omega^2[2\lambda_{CH} + 3(m - a\omega)^2] \), and \( \lambda_{CH} \), the angular eigenvalue used by Chandrasekhar [38], is related to the separation constant \( \lambda \) of Eq. (4.9) of [19] by \( \lambda_{CH} = \lambda + s + 2 \). Because Eq. (38) mixes \( \Psi \) and \( \bar{\Psi} \), its inversion for each angular harmonic involves a linear combination of \( \psi_{0 \ell m \omega} \) and \( \psi_{0 \ell -m -\omega} \). We find that the algebraic inversion gives the ORG Hertz potential in the form

\[
\Psi_{\ell m \omega} = 8(-1)^m D\bar{\psi}_{0 \ell -m -\omega} + 12iM\omega\psi_{0 \ell m \omega} \quad \frac{D^2}{D^2 + 144M^2\omega^2}.
\]

We use this inversion for circular orbits in a Schwarzschild and Kerr background.

For generic orbits, the individual harmonics \( \psi_{0 \ell m \omega} \) do not satisfy the sourcefree Teukolsky equation in the region \( r_{\text{min}} < r < r_{\text{max}} \), where \( r_{\text{min}} \) and \( r_{\text{max}} \) are the values of \( r \) at perihelion and aphelion; and the presence of a source invalidates the algebraic angular inversion. To find \( \Psi \) requires one to integrate one of the radial equations of Table I. With the Kinnersley tetrad, the IRG radial equation for \( \Psi \) has the simplest form: In Kerr coordinates \( u, r, \theta, \phi \), where \( u = t - r^* \), with

\[
\frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta} \quad \text{and} \quad \tilde{\phi} = \phi + a \int_r^\infty \! dr' \frac{1}{\Delta(r')}.
\]

we have \( Df(u, r, \theta, \tilde{\phi}) = \partial_r f(u, r, \theta, \tilde{\phi}) \). The radial equation for \( \Psi_{\text{IRG}} \) is then \( \partial_r^4 \Psi_{\text{IRG}} = 2\psi_0 \), with solution

\[
\tilde{\Psi}_{\text{IRG}} = 2 \int_r^\infty \! dr_4 \int_r^{r_4} \! dr_3 \int_r^{r_3} \! dr_2 \int_r^{r_2} \! dr_1 \psi_0(u, r, \theta, \tilde{\phi}),
\]

satisfying the vacuum Teukolsky equation for ingoing radiation when the outgoing radial null ray does not intersect the particle. To find \( \Psi \) at points on a \( t = \) constant surface, one can use Eq. (42) outside the radial coordinate \( r_o \) of the particle and a corresponding integral from the horizon to \( r \) for \( r < r_o \).
E. Gauge transformations of the self-force

In the form (3), the perturbed geodesic is parameterized so that its tangent is normalized to 1 with respect to the background metric. A gauge transformation of this equation was obtained by Barack and Ori [27], and Appendix A gives an alternative, covariant derivation in terms of an infinitesimal diffeo of the metric and a family of unperturbed geodesics. With the same normalization, changing a background geodesic by a gauge transformation generated by \( \xi^\alpha \) changes its tangent vector by

\[
\delta \xi u^\alpha = (\delta_{(\beta} - u^\alpha u_{\beta}) \xi^\beta;
\]  

and \( u^\alpha + \delta \xi u^\alpha \) satisfies a geodesic equation of the metric perturbed by \( \xi_{\alpha\beta} g_{\alpha\beta} \). With the acceleration defined by

\[
\delta \xi a^\alpha = - (\delta_{(\beta} - u^\alpha u_{\beta}) (u \cdot \nabla) \xi^\beta + R_{\beta\gamma\delta} u^\beta u^\gamma \xi^\delta, \]

and \( a^\alpha u_a = 0 \). Note that the right side vanishes if \( \xi^\alpha \) happens to drag a geodesic of the background spacetime to another geodesic of the background spacetime: This is the equation of geodesic deviation governing the connecting vector joining two neighboring geodesics of \( g_{\alpha\beta} \). For general \( \xi^\alpha \), the right side of Eq. (44) then measures the failure of an infinitesimal diffeo generated by \( \xi^\alpha \) to produce a geodesic of the background metric.

A particle in circular orbit has 4-velocity \( u^\alpha = u^t k^\alpha \), with \( k^\alpha = t^\alpha + \Omega \phi^\alpha \) a Killing vector. The perturbed spacetime with a particle in circular orbit is helically symmetric, symmetric with respect to \( k^\alpha \). We show in Appendix A that, for a gauge transformation that preserves helical symmetry (for \( \xi_k \) = 0), the gauge-transformed self-force is given by

\[
\hat{f}^\alpha = f^\alpha + \frac{1}{2} m (u^i k^j) \xi^\alpha \nabla^\beta \nabla^\gamma (k^\beta k^\gamma). \]

III. METHOD FOR COMPUTING THE SELF-FORCE ON AN ORBITING MASS

A. Overview

In this section we outline the method for computing the self-force in a radiation gauge. Subsequent sections will elaborate on the details of each step of the calculation. The method is a revision of that initially suggested in Refs. [28] and [39]. In broad terms, the steps involved in computing the self-force in a modified radiation gauge are:

A. Compute the retarded Weyl scalar, \( \psi_0^{\text{ret}} \) (or \( \psi_4^{\text{ret}} \)).

B. Use the retarded Weyl scalar to construct a Hertz potential \( \Psi^{\text{ret}} \) as a sum of angular harmonics for \( r \neq r_0 \).

C. Using the CCK formalism described above, reconstruct the retarded metric perturbation in a radiation gauge, and find the perturbation in mass and angular momentum in an arbitrary gauge.

D. Find the expression for the self force as a mode sum involving the retarded field.

E. Find the renormalization coefficients for the singular part \( f^{\alpha\text{s}} \) of the self-force and compute \( f^{\text{ren}\alpha} \).

In this approach, all fields are written as a sum of time and angular harmonics. The angular harmonics \( \psi_0^{\text{ret}} \) have finite one-sided limits as \( r \to r_0 \):

\[
\psi_0^{\text{ret}\pm} := \lim_{r \to r_0, r_0} \psi_0^{\text{ret}}(t_0, r). \]

For a Schwarzschild background the harmonics are spin-weight 2 spherical harmonics, while for Kerr they are oblate spheroidal harmonics whose form depends on the Kerr parameter \( a \). For a Kerr as well as a Schwarzschild background, however, one can write the singular field in terms of spin-weighted spherical harmonics, where each spin-weighted spherical harmonic is a sum of the form

\[
s S_{\ell m} e^{im\phi} = \sum_{\ell' = \ell_{\text{min}}}^{\infty} b_{\ell'\omega} S_{\ell' m}, \quad \ell_{\text{min}} = \max(|s|, |m|). \]
The computation of $\psi_0^{\text{ret}}$ is straightforward, involving an integration of the radial Teukolsky equation and the computation of spin-weighted spherical harmonics. In computing a Hertz potential $\Psi^{\text{ret}}$ from $\psi_0^{\text{ret}}$, our choice of radiation gauge is dictated by requiring that the CCK-constructed $\Psi^{\text{ret}}$ vanish asymptotically: We use the ORG and find $\Psi^{\text{ret}}$ as the solution of Eq. (41) to the angular equation. The tetrad components of the metric perturbation $h^{\text{ret}}_{\alpha\beta}$ are then obtained from Eq. (32) and are used to compute the expression for the bare self-force in terms of $h^{\text{ret}}_{\alpha\beta}$,

$$f^\alpha[h^{\text{ret}}_{\alpha\beta}]m = a^{\text{ret}}_{\alpha} = -(g^{\alpha\delta} - u^\alpha u^\delta) \left( \nabla_{\beta} h^{\text{ret}}_{\gamma\delta} - \frac{1}{2} \nabla_\delta h^{\text{ret}}_{\gamma\beta} \right) u^\beta u^\gamma.$$

(48)

The remainder of the problem is the mode-sum renormalization of the self-force and the recovery of the part of the metric perturbation that does not arise from $\psi_0$. We discuss them in turn in the next two subsections.

B. Mode-sum renormalization in a radiation gauge

One can carry out a mode sum renormalization by finding the radiation-gauge version of the power series that expresses the singular behavior of $a^{\text{ret}}_{\alpha}$ for $r > r_0$ and $r < r_0$ near the position of the particle. We consider two ways to proceed:

1. One can find an analytic expression for $a^{\text{ret}}_{\alpha}$ as a sum in powers of $L$, starting from an expression we derive below for $\psi_0^{\text{ret}}$, the singular part of $\psi_0^{\text{ret}}$. This analytic way follows the steps just listed in describing the path from $\psi_0^{\text{ret}}$ to $a^{\text{ret}}_{\alpha}$, to successively obtain expressions for $\Psi^{\text{ret}}$, $h^{\text{ret}}_{\alpha\beta}$ and $a^{\text{ret}}_{\alpha}$.

2. The second method is significantly simpler: One simply numerically matches a power series in $L$ to $a^{\text{ret}}_{\ell} \mu(P)$ for successive values of $\ell$.

We begin with a discussion of the analytic method. We find an explicit expression for $\psi_0^{\text{ret}}$ to subleading order in the distance to the particle’s trajectory in terms of components of the tetrad vectors along and orthogonal to the trajectory. We then characterize the powers of $L$ that appear in the power series for $a^{\text{ret}}_{\alpha}$. Finally, we turn to a description of renormalization by numerical matching. In Sec. IV below, we test the numerical method in a relatively simple case by comparing analytic and numerical renormalizations of $\psi_0$ for a particle in circular orbit in a Schwarzschild background.

Note that, in a mode-sum renormalization, the singular parts of the perturbation in the metric and the self-force are determined by the large-$L$ behavior of the retarded fields. They are, in particular, independent of the choice of gauge in which one describes perturbations of mass and angular momentum.

1. Analytic method

The analytic method begins by finding an analytic expression for $\psi_0^{\text{ret}}$ or $\psi_4^{\text{ret}}$. The decomposition of the metric perturbation $h^{\text{ret}}_{\alpha\beta}$ in a Lorentz gauge,

$$h^{\text{ret}}_{\alpha\beta} = h^{\text{ren, Lor}}_{\alpha\beta} + h^{\text{s, Lor}}_{\alpha\beta},$$

(49)

gives a corresponding gauge-invariant decomposition of the perturbed Weyl scalars. From the expression for the perturbed Weyl (or Riemann) tensor in terms of the perturbed metric, we have

$$\psi_0^{\text{ret}} = C_0^{\alpha\beta} h^{\text{ret}}_{\alpha\beta}, \quad \psi_0^{\text{ren}} = C_0^{\alpha\beta} h^{\text{ren}}_{\alpha\beta},$$

$$\psi_4^{\text{ret}} = C_4^{\alpha\beta} h^{\text{ret}}_{\alpha\beta}, \quad \psi_4^{\text{ren}} = C_4^{\alpha\beta} h^{\text{ren}}_{\alpha\beta},$$

(50a)

(50b)

where

$$C_0^{\alpha\beta} = \frac{1}{2} \left( m^\alpha m^\beta l^\gamma l^\delta + l^\alpha l^\beta m^\gamma m^\delta - l^\alpha m^\beta m^\gamma l^\delta - m^\alpha l^\beta l^\gamma m^\delta \right) \nabla_\gamma \nabla_\delta,$$

(51a)

$$C_4^{\alpha\beta} = \frac{1}{2} \left( m^\alpha m^\beta m^\gamma n^\delta + n^\alpha n^\beta m^\gamma m^\delta - n^\alpha m^\beta m^\gamma n^\delta - m^\alpha n^\beta m^\gamma m^\delta \right) \nabla_\gamma \nabla_\delta.$$  

(51b)

Although we will not need the Detweiler-Whiting form of the singular field, it is worth noting that using it would yield a smooth version of $\psi_0^{\text{ren}}$ (or $\psi_4^{\text{ren}}$) that satisfies the sourcefree Teukolsky equation in a neighborhood of the particle’s trajectory. That is, because $h^{\text{ren}}_{\alpha\beta}$ satisfies the perturbed field equation with the same source as $h^{\text{ret}}_{\alpha\beta}$, the
corresponding field $\psi_0^s$ is a local solution to the $s = 2$ Teukolsky equation with the same source as $\psi_0^\text{ret}$, implying that $\psi_0^\text{ret} - \psi_0^s$ satisfies the corresponding homogeneous equation.

Denote by $\epsilon := \sqrt{T^2 + X^2 + Y^2 + Z^2}$ the distance with respect to the Euclidean metric $dT^2 + dX^2 + dY^2 + dZ^2$ of the local inertial coordinates introduced before Eq. (9). We will argue below that the self-force in our modified radiation gauge, like that in a Lorenz gauge, has dominant singular behavior of order $\epsilon^{-2}$ and can be regularized by subtracting leading and subleading terms in $\epsilon$. These arise from the leading and subleading terms in $\epsilon$ in $\psi_0^s$. Instead of directly using Eq. (50a) to compute $\psi_0^s$, it is easier simply to observe that $O_0^{\alpha\beta}$ and $h_0^{\alpha\beta}$ have to subleading order in $\epsilon$ their flat space form. It follows that the value of $\psi_0^s$ has at subleading order in $\epsilon$ its form for a perturbation of flat space. That is, in terms of $T$ and $\rho$, $\psi_0^s$ is the linearized Schwarzschild field of a static particle:

$$C_0^{\alpha\beta} = -\frac{4m}{\rho^3} \left( \delta^{[\alpha}_{[\alpha} \delta^{\beta]}_{\beta]} + 3 \delta^{[\alpha}_{[\alpha} \nabla^\beta]_\beta \rho \nabla_\beta^\rho - 3 \delta^{[\alpha}_{[\alpha} \nabla^\beta]_\beta T \nabla^\beta T - 6 \nabla_{[\alpha} \rho \nabla_{\beta]} T \nabla^{[\alpha} \rho \nabla^{\beta]} T \right).$$  (52)

From this we obtain

$$\psi_0^s = -\frac{6m}{\rho^3} \left( l^T m^T - l^\rho m^T \right)^2,$$  (53)

$$\psi_4 = -\frac{6m}{\rho^3} \left( n^T m^T - n^\rho m^T \right)^2,$$  (54)

where $l^T := l^\alpha \nabla_\alpha T$, $l^T := l^\alpha \nabla_\alpha \rho$, $m^T := m^\alpha \nabla_\alpha T$, $m^\rho := m^\alpha \nabla_\alpha \rho$. One can, of course, also obtain (53) and (54) by applying the operators $O_0^{\alpha\beta}$ and $O_0^{\alpha\beta}$ of Eqs. (51a) and (51b) to the singular metric (9). Eqs. (53) and (54) are valid in general Petrov type D spacetimes.

For each of the quantities $\psi_0^s$, $\Psi$, $h_{\alpha\beta}$, and $a_0^s$, we will use the subscript $\ell$ to denote the sum over $m$ of the contributions from angular harmonics associated with $\ell$, $m$, and we suppress the index $\omega$. The equations that relate $\psi_0^s$ and $\Psi$ do not mix spin-weighted spheroidal harmonics, and $\Psi$ is most simply found as a sum of these harmonics. The equations, (32) and (33), that relate $\Psi$ to $h_{\alpha\beta}$ and $h_{\alpha\beta}$ to $a_0^s$, however, mix spheroidal harmonics on a Kerr background. The large-$L$ expansions of their components can be found in terms of spin-weighted spheroidal harmonics, spin-weighted spherical harmonics, or spherical harmonics. Different choices lead to different definitions of the subleading contributions, because of the mixing of different values of $\ell$ in relating, for example, spin-weighted spherical harmonics to spin-weighted spheroidal harmonics. What matters is only that the same convention is used for the angular harmonics of the retarded and singular fields.

Because $\psi_0^s$ involves two derivatives of $h_{\alpha\beta}$ and can be computed from the Lorenz singular field, its large-$L$ behavior is greater by two powers of $L$ than the large-$L$ behavior at $r = r_0$ of $h_{\alpha\beta}^{\text{Lor}}$ of Eq. (12):

$$\psi_0^{s \pm} := \sum_m \psi_0^{s \pm}(t_0) Y_{\ell m}(\theta_0, \phi_0) = \tilde{A}^{\pm} L^2 + \tilde{B}^{\pm} L + O(L^{-1}).$$  (55)

In Sec. IV and Appendix C we find the large-$L$ expansion of $\psi_0^s$ for a particle in circular orbit in a Schwarzschild background (restricting consideration to the part of $\psi_0^s$ axisymmetric about the position of the particle).

Because the Hertz potential $\Psi^\text{ret}$ involves four integrals of $\psi_0^s$, its singular part has to subleading order in $L$ the form

$$\Psi_\ell^{\pm} = A_\Psi^{\pm} L^2 + B_\Psi^{\pm} L^3.$$  (56)

This behavior also follows from the explicit form (10) of $\Psi_{\ell m\nu\omega}$, together with the fact that, for large $\ell$, the spherical eigenvalues $\lambda$ approach their spherical values, $\lambda/\ell^2 \rightarrow 1$. The leading term in the expansion is then immediate from the leading term in (55), while subleading terms involve the expansion of $\lambda$ (found analytically or numerically) in terms of $L$.

The metric perturbation involves two derivatives of $\Psi$, implying that the singular and retarded fields in a radiation gauge have the same leading power of $L$ as their Lorenz counterparts,

$$h_{\ell \mu\nu}^{\text{RG} \pm} = A_{\mu\nu}^{\pm} + B_{\mu\nu}^{\pm} L + O(L^{-2}).$$  (57)

Finally, the self-force $f_{\ell \mu}^{\text{ren, RG}}$ is computed from $\nabla_{\gamma} h_{\alpha\beta}^{\text{ren, RG}}$, using Eq. (39). As in the Lorenz gauge, the additional derivative gives the behavior

$$a_\ell^{\pm} = A_{\ell \mu}^{\pm} L + B_{\ell \mu}^{\pm} + O(L^{-2}).$$  (58)
The renormalized acceleration at the position of the particle is then given by

$$a_{\text{ren}}^{\text{RG} \mu} = \lim_{\ell_{\text{max}} \to \infty} \sum_{\ell=0}^{\ell_{\text{max}}} \left( a^{\text{ret RG} \mu}_\ell - a^{\text{RG} \mu}_\ell \right).$$

(59)

As noted in the introduction, for a particle in circular orbit in a Schwarzschild background, we find that the large-$L$ expansion of $a^{\text{RG} \alpha}$ agrees through $O(L^0)$ with

$$a^{\text{RG} \alpha} = -\nabla^\alpha \frac{1}{\rho},$$

(60)

differing only by a constant from its form in a Lorenz gauge. This form of $a^{\text{RG} \alpha}$ does not contribute to the self-force, because Eq. (40) is satisfied – the small-$\rho$ angle average of $a^{\text{RG} \alpha}$ vanishes. Thus in this case, we can identify the singular field with its leading and subleading terms as a power series in $L$,

$$a^{\text{RG} \mu}_\ell = A_{\mu \pm} L + B_{\mu \pm}. $$

(61)

For a CCK radiation gauge, there is as yet no general proof that $a^{\text{RG} \alpha}$ is given by its leading and subleading terms. This is true in a Lorenz gauge, and we expect it to hold for a radiation gauge as well, with an argument based on the common property that $a^{\text{RG} \alpha}$ can be expressed (for $r > r_0$ or $r < r_0$) as a power series that begins at $O(\epsilon^{-2})$ and involves positive powers of the coordinate differences $x^\mu - x^\mu_0$ and odd powers of $\rho$.

Without a proof, one must check that the computed $a^{\text{RG} \alpha}$ satisfies Eq. (40).

In the Schwarzschild example below, the coordinate expression for $\psi^{\text{s}}_0$ to subleading order is a sum of more than 25 terms, and to find its large-$L$ expansion we computed the large-$L$ expansion of each term. Finding the corresponding large-$L$ expansion for $a^{\text{RG} \mu}_\ell$ involves finding a large-$L$ expansions of all combinations of three derivatives of each of these terms for each component of $a^{\text{RG} \mu}_\ell$. Without significant insight, this would mean finding the large-$L$ expansion of about 300 terms. Given the simple form that $a^{\alpha \alpha}$ takes for the Schwarzschild circular-orbit there may be a similar form in the generic case and a much simpler way to find it. In its absence, the numerical method is much easier to use.

2. Numerical method

The renormalization coefficients occurring in Eq. (58) for $a^{\text{RG} \mu}_\ell$ are coefficients in the large-$L$ expansion of $a^{\text{ret RG} \mu}_\ell$ evaluated at $r = r_0^\pm$. Consequently, once one has found the numerical values of $a^{\text{RG} \mu}_\ell$, it is not in principle necessary to carry out an additional analytic computation of $a^{\text{RG} \mu}_\ell$. Instead, one can match to the sequence of values $a^{\text{ret RG} \mu}_\ell$ a power series in $L$ of the form

$$a^{\text{ret} \mu}_\ell = A_{\mu \pm} L + B_{\mu \pm} + \sum_{k=2, \text{even}}^{n} \frac{E^\mu_k}{L^k},$$

(62)

finding the $E^\mu_k$ that yield a best fit. A numerical check is that subtraction of the order $L^{-k}$ term reduces the order of the series by one power of $L$. And one should check that the reduction of order holds for values of $L$ larger than those used to obtain the coefficients in the matching. Finally, one checks numerically that $f^{\text{ren} \mu}_{\ell}$ converges to a value $f^{\text{ren} \mu}_{\ell_{\text{max}}}$ as the cutoff $\ell_{\text{max}}$ increases. The disadvantage of the numerical matching method is that, for a given desired accuracy in $a^{\text{ren} \alpha}$, one must compute $a^{\text{ret} \alpha}$ to higher values of $\ell$ than is required when one or more renormalization coefficients are known analytically.

To identify $a^{\text{ret} \alpha}_\ell$ with $A_{\mu \pm} L + B_{\mu \pm}$, one must again show that $a^{\text{ret} \alpha}$ satisfies Eq. (40). This can done if one can find for each component $a^{\alpha \mu}$ expressions in terms of $\rho$ and the coordinate differences $x^\mu - x^\mu_0$ whose large-$L$ expansion is $A_{\mu \pm} L + B_{\mu \pm}$, and if this expression satisfies (40). If the limiting angle-average has a finite value, that finite value must be added back, in accordance with Eq. (41), to find the self-force.

Equivalently, if the first two terms in this $L$-expansion correspond to the terms of order $\epsilon^{-2}$ and $\epsilon^{-1}$ in the position-space expansion, so that the terms involving $E_k$ correspond to terms of order $\epsilon^{-1}$ and higher, then these latter terms sum to zero. We suspect this is the case for a radiation gauge, because the behavior of the singular field as a power series in $L$ implies behavior in $\epsilon$ that is no more singular than in a Lorenz gauge and corresponds for $r > r_0$ and $r < r_0$ to a power series in $\epsilon$. This is consistent with our numerical construction of the singular part of the self-force for a particle in circular orbit in Schwarzschild.
With an analytic knowledge of the first term or terms in the expansion, the numerical method is still useful in finding
subsequent terms, and this has been done to speed convergence in the self-force computations involving mode-sum
renormalization in a Lorenz gauge. In particular, once one knows that \(a_{\ell}^\mu\) can be identified with the leading and
subleading terms \(A^\mu_{\pm} L + B^\mu_{\pm}\), the computation of \(a_{\ell}^\mu\) is given by
\[
a_{\ell}^\mu = \sum_{\ell=0}^{\ell_{\text{max}}} \left( a_{\ell}^\mu - A_{\ell}^\mu L - B_{\ell}^\mu \right) - \sum_{k=2,\text{even}}^{n} \frac{E_k^\mu}{L^k} + \sum_{k=2,\text{even}}^{\infty} E_k^\mu \sum_{\ell=0}^{\ell_{\text{max}}} L^{-k},
\]
with an error of order \(r^{-n+1}\).

To justify the numerical method, we present in Sec. [X] below a comparison of the analytic and numerical de-
termination of renormalization coefficients for the axisymmetric part of \(\psi_0\) and their contribution to the self-force.
The companion paper presents the full numerical computation of the self-force for a particle in circular orbit in a
Schwarzschild background. Checks of the work include a numerical computation of a quantity \(h_{\alpha\beta}u^\alpha u^\beta\) that is invari-
ant under helically symmetric gauge transformations and has previously been computed in Lorenz and Regge-Wheeler
gauges.

C. Mass and Spin: The Remaining Metric

The CCK reconstruction of the metric perturbation from \(\psi_0\) gives a perturbed metric for which there is no change in
the mass and angular momentum. For a Schwarzschild background, this is immediate from the fact that \(\psi_0\)
involve only values of \(\ell\) with \(\ell \geq 2\), because the construction of the perturbed metric preserves the value of \(\ell\). For a
Kerr background, both \(\psi_0\) and \(\Psi\) are a sum of spheroidal harmonics with spin-weight 2, and their expression (77)
in terms of spin-weighted spherical harmonics involves only harmonics with \(\ell \geq 2\). A mass perturbation requires an
\(\ell = 0\) part of the asymptotic perturbed metric at \(O(r^{-1})\). The operator in Eq. (52) mixes different values of \(\ell\), but it
differs from its Schwarzschild form by terms smaller by \(O(r^{-1})\) than its leading terms. That is, both \(\psi_0\) and \(\Psi\)
are \(O(r^{-5})\), the spherically symmetric part of the operator (52) is \(O(r^4)\), and terms that mix different values of \(\ell\) are
\(O(r^3)\), implying \(h_{\alpha\beta}^{\text{RG}}\) has no \(\ell = 0\) part at \(O(r^{-1})\).

A perturbation of angular momentum requires an \(\ell = 1\) contribution at \(O(r^{-2})\) in \(h_{13}\). To see that it also vanishes,
we show that the correction in a Kerr background to the Schwarzschild expression for \(h_{13}^{\text{re}}\) is asymptotically \(O(r^{-3})\).
The operator that relates \(h_{13}^{\text{re}}\) to \(\Psi\) in Eq. (52) is \((\delta - 3\alpha + \beta + 5\pi + \gamma)(\Delta + \mu - 4\gamma) + (\Delta + 5\mu - \bar{\mu} - 3\gamma - \bar{\gamma})(\delta - 4\alpha + \pi))\). Now \(D\) is a derivative along an outgoing null ray. In the Kerr coordinate
\(D(f(u, r, \theta, \phi)) = \partial_r f(u, r, \theta, \phi)\). Because each time harmonic of \(\Phi\) has the form \(\Phi = S(\theta, \phi)e^{-i\omega u}r^{-5}(1 + O(r^{-1}))\), \(D\Phi = S(\theta, \phi)e^{-i\omega u}O(r^{-6})\);
because the spin coefficients are \(O(r^{-1})\) or smaller, \(h_{13}^{\text{re}}\) falls off like \(r^{-2}\), and the correction to its Schwarzschild behavior is at
\(O(r^{-3})\).

To complete the metric reconstruction one needs to add the contributions from a change in the mass and angular
momentum of the spacetime outside \(r = r_0\). Satisfying the perturbed field equation also requires a discontinuous
gauge transformation associated with a change in the system’s center of mass. That nothing further is needed follows
from an minor extension of a theorem by Wald that implies that a perturbed vacuum metric is determined up to
gauge transformations and the addition of a Petrov type D perturbation of the black-hole geometry [33]. There are
four kinds: (i) an infinitesimal change in the black hole’s mass and (ii) in its spin; (iii) the perturbative version of the
C-metric and (iv) the perturbative version of the Kerr-NUT solution. The type D perturbations are all stationary
and axisymmetric, and only the mass and angular momentum perturbations are smooth in the region exterior to the
black hole: The C and Kerr-NUT perturbations are each singular on their axis of symmetry, coinciding for a Kerr
background, with the axis of symmetry of the Kerr geometry. If we were dealing with a source-free perturbation,
regularity at the horizon and at infinity would rule out the addition of Kerr-NUT and C-metric perturbations, and,
after a choice of gauge, we would be left with changes in mass and angular momentum (and gauge transformations).
To extend the argument to \(h_{\alpha\beta}^{\text{re}}\) in our case, we note that smoothness of each time-harmonic of \(\psi_0^{\text{re}}\) for \(r \neq r_0\),
together with the explicit form (10), implies that \(\Psi^{\text{re}}\) is smooth for \(r \neq r_0\). That is, smoothness of \(\psi_0^{\text{re}}\) implies
that the coefficients of each angular harmonic in its decomposition fall off faster than any power of \(L\). Eq. (10) implies
that the coefficients of the angular harmonics of \(\Psi\) fall off still faster in \(L\). Thus each time harmonic of \(\Psi\) is smooth
and has no contribution from a C or Kerr-NUT perturbation.

The contributions to the retarded field of mass and spin were examined by Detweiler and Poisson [44] and by
Price [45]. With the Hertz potential restricted to \(\ell \geq 2\) (for a Schwarzschild or Kerr background), there is no local
contribution to mass and angular momentum. For \(r < r_0\), this is the appropriate solution for the retarded field.
For \(r > r_0\), one has to determine in any gauge four parameters corresponding to changes in mass and spin (two
correspond to a change in the spin direction and are gauge transformations); and three parameters associated with gauge transformations for \( r > r_0 \) that eliminate an asymptotic dipole. The change in mass and in the magnitude of angular momentum along its direction in the background Kerr spacetime can be found from the integrals

\[
\delta M = \int_S (2T^\alpha_{\beta} - \delta^\alpha_{\beta} T) t^\beta dS, \\
\delta J = -\int S T^\alpha_{\beta} \phi^\beta dS
\]

over any hypersurface intersecting the trajectory. (For a Schwarzschild background, all components of \( \delta J \) are determined in this way.) The remaining parameters are determined by jump conditions in the field equations across \( r = r_0 \).

Although the metric perturbations corresponding to changes in mass and spin can be written in a radiation gauge, we do not see a good reason to do so. In particular, the radiation gauge form of a mass perturbation that arises from a Hertz potential has a singularity on the radial ray through the particle. (There is an alternative radiation-gauge form of a mass perturbation that is nonsingular on the axis of symmetry \( [33] \), but it has no obvious advantage over a mass perturbation in another gauge.) Instead we use for \( h_{\alpha\beta}^{\text{ref}} \) a radiation gauge for \( \ell \geq 2 \), together with an arbitrary convenient gauge for \( \ell < 2 \).

### D. Leading order parity of \( \psi_0, \Psi \) and \( h_{\alpha\beta} \)

We now consider the parity of the perturbed metric in a radiation gauge. Gralla’s criterion is that the projection of the perturbed metric to a surface orthogonal to the particle’s 4-velocity is even under parity to leading order in \( \rho \). Parity here means the locally defined diffeo that exchanges points at the same geodesic distance on opposite sides of each geodesic orthogonal to the particle trajectory; and the associated hypersurface spanned by geodesics orthogonal to a point \( P \) of the trajectory is the surface onto which \( h_{\alpha\beta} \) is projected. Because the invariance under parity is only to leading order, we can define parity in terms of local inertial coordinates \( T, X, Y, Z \) as the diffeo \( \mathcal{P} : (T, X, Y, Z) \mapsto (T, -X, -Y, -Z) \). The Weyl tensor has to leading order in \( \rho \) its flat-space form, implying that it is even (invariant) under both parity and time-reversal, where time reversal, \( T \), exchanges points with coordinates \( \pm T, X, Y, Z \). (Its behavior under both parity and time-reversal will be needed in the argument below.)

We first show that these symmetries are retained by the singular form of \( \psi_0 \), given by Eq. \( [34] \). Because the tetravector fields \( l^\alpha \) and \( m^\alpha \) (and \( n^\alpha \)) are smooth, they are constant near a point \( P \) on the trajectory to leading order in \( \rho \) (as long as the particle is not on the \( \theta = 0, \pi \) axis, where \( m^\alpha \) is not defined). The corresponding scalars \( l^\alpha \nabla_\alpha T \) and \( m^\alpha \nabla_\alpha T \) are then even to leading order under parity. Because the cartesian components of \( \nabla_\alpha \rho \) have opposite signs at diametrically opposite points \( (T, X, Y, Z) \) and \( (T, -X, -Y, -Z) \), the scalars \( l^\alpha \nabla_\alpha \rho \) and \( m^\alpha \nabla_\alpha \rho \) are odd under parity. Then \( (l^T m^\rho - l^\rho m^T)^2 \) is even and hence \( \psi_0 \) is even under \( \mathcal{P} \) to leading order in \( \rho \).

Similarly, the scalars \( l^\alpha \nabla_\alpha T \) and \( m^\alpha \nabla_\alpha T \) are odd to leading order under time-reversal; and \( l^\alpha \nabla_\alpha \rho \) and \( m^\alpha \nabla_\alpha \rho \) are even, implying that \( (l^T m^\rho - l^\rho m^T)^2 \) and \( \psi_0 \) are even to leading order under time reversal.

To show that \( \Psi \) is even under parity to leading order in \( \rho \) requires additional steps. First, because \( \Psi \) is obtained from \( \psi_0 \) as a sum of angular harmonics, we use the fact that the leading order parts of \( \psi_0 \) and \( \Psi \) in \( \rho \) for small \( \rho \) are associated by the transform to angular harmonics with the leading order in \( L \) part of its angular harmonics, for large \( L \), as described in Appendix \( [32] \). In particular the leading terms in \( \rho \) of \( \psi_0 \) and \( \Psi \) are, respectively, \( O(\rho^{-3}) \) and \( O(\rho) \), and they correspond to the large-\( L \) terms of order \( O(L^2) \) and \( O(L^{-2}) \) in the angular harmonics. Second, Eq. \( [30] \), expressing \( \Psi \) in terms of \( \psi_0 \), involves angular harmonics of \( \psi_0 \) on a surface of constant \( t \), a surface that is not perpendicular to the particle trajectory; because of this, the plane tangent to the surface is not invariant under parity. It is, however, invariant under \( \mathcal{PT} \), implying that the restriction of \( \psi_0 \) to a constant \( t \) surface is invariant to leading order under \( \mathcal{PT} \) (Restricted to the constant \( t \) surface, \( \mathcal{PT} \) is a parity transformation about \( P \).) It follows that \( \Psi \) and \( h_{\alpha\beta} \) are also invariant to leading order under \( \mathcal{PT} \). We give the argument in terms of the the angular equation \( [32] \) for \( \Psi \). The right side of this equation is dominated for large \( L \) by its first term, having to leading order the form

\[
\psi_0 = \frac{1}{8} \mathcal{L}^4 \bar{\Psi},
\]

that is, because \( \mathcal{L}^4 \) is, for large \( L \), quartic in \( \lambda \) and \( \omega \), it dominates the second term. (One cannot look at the large \( L \) limit with \( \omega \) fixed, because \( \omega \) is not independent of \( L \): For a circular orbit, for example, \( \omega = m \Omega \).) For the particle at \( \theta_0 \neq 0, \pi \), \( \mathcal{L} \) has to leading order in \( \rho \) the form \( \mathcal{L} = \mathcal{L}_0 + i \csc \theta_0 \partial_\phi - ia \sin \theta_0 \partial_i \). In Boyer-Lindquist coordinates, \( \mathcal{PT} \) is given to leading order in \( \rho \) by \( (t, r, \theta, \phi) \rightarrow (2t_0 - t, 2r_0 - r, 2\theta_0 - \theta, 2\phi_0 - \phi) \), implying that each term in this
leading form of $\mathcal{L}$ is odd under $\mathcal{P}\mathcal{T}$. Then $\mathcal{L}^4$ is even to leading order:

$$\mathcal{L}^4 \mathcal{P}\mathcal{T} = \mathcal{P}\mathcal{T} \mathcal{L}^4.$$  \hspace{1cm} (66)

The parts of $\Psi$ that are even and odd under $\mathcal{P}\mathcal{T}$ then have sources that differ by one order in $L$:

$$\frac{1}{2}(1 + \mathcal{P}\mathcal{T})[\mathcal{L}^4(\Psi^{\text{odd}} + \Psi^{\text{even}}) + 12M\partial_i\Psi] = \mathcal{L}^4\Psi^{\text{even}}[1 + O(L^{-1})] = 8\psi_0^{\text{even}},$$  \hspace{1cm} (67a)

$$\frac{1}{2}(1 - \mathcal{P}\mathcal{T})[\mathcal{L}^4(\Psi^{\text{odd}} + \Psi^{\text{even}}) + 12M\partial_i\Psi] = \mathcal{L}^4\Psi^{\text{odd}} + O(\Psi^{\text{even}} \times L^3) = 8\psi_0^{\text{odd}}.$$  \hspace{1cm} (67b)

With a source smaller by one order in $L$, the algebraic inversion \hspace{1cm} (10) then gives $\Psi^{\text{odd}}$ smaller than $\Psi^{\text{even}}$ by one power of $L$.

Next note that, because $\rho$ is independent of $T$ and the tetrad vectors are smooth (and hence constant to lowest order in $\rho$), $\psi_0$ is independent of $T$ to lowest order in $\rho$. That is, translating $\psi_0$ by $\Delta T$ changes it only by a term of order $\psi_0\Delta T$, from the $O(\Delta T)$ change in the tetrad vectors. Now time-translating $\Psi$ from a point on the $t = 0$ surface to a point on the relatively boosted $T = 0$ surface through the same point $P$ of the trajectory involves a translation by a time $\Delta T$ proportional to $\rho$ and hence changes $\Psi$ only to subleading order in $\rho$. Thus $\Psi$ is invariant under $\mathcal{P}$ to leading order in $\rho$.

Finally, a tensor $T_{\alpha\beta}$ is invariant under $\mathcal{P}$ if $T_{\alpha\beta} = \mathcal{P}^* T_{\alpha\beta}$, where the pullback $\mathcal{P}^* T_{\alpha\beta}$ has components in coordinates $\{x^\mu\}$ given by $(\mathcal{P}^* T)_{\mu\nu}(Q) := \partial\mu \mathcal{P}\sigma \partial\nu \mathcal{P}^* T_{\sigma\tau}[\mathcal{P}(Q)]$. (We have used $\mathcal{P}^{-1} = \mathcal{P}$.) In terms of the coordinates $(T, X, Y, Z)$, the requirement that the spatial projection of $h_{\alpha\beta}$ is invariant under parity is equivalent to the condition that each spatial component $h_{ij}$ is even under parity

$$h_{ij}(T, X, Y, Z) = (\mathcal{P}h)_{ij}(T, X, Y, Z) = h_{ij}(T, -X, -Y, -Z).$$  \hspace{1cm} (68)

That this condition is satisfied follows from Eq. (65) for $h_{\alpha\beta}$ in terms of $\Psi$ and the fact that the leading, $O(\rho^{-1})$, part of $h_{\alpha\beta}$ comes entirely from terms quadratic in the derivative operators, in $\Delta$, $\delta$ and $\delta^i$. Because $\Psi$ is independent of $T$ to leading order in $\rho$, each derivative operator along a tetrad vector involves only the spatial $(X^i)$ components of the vector, implying that the quadratic derivatives $\Delta^2 \Psi, \ldots, \delta^2 \Psi$ all have even parity to leading order. Finally, the lowest-order constancy of the tetrad vectors implies that the products of their components $n_i n_j, \ldots, \bar{m}_i \bar{m}_j$ in local inertial coordinates are even to leading order in $\rho$ under parity. We conclude that the projection of $h_{ij}^{\text{ret, RG}}$ orthogonal to the 4-velocity is even under parity at leading order in $\rho$: to $O(\rho^{-1})$.

**IV. PARTICLE IN CIRCULAR ORBIT IN A SCHWARZSCHILD GEOMETRY**

As a simplest explicit example of the method, we consider a particle of mass $m$ in circular orbit at radial coordinate $r_0$ about a Schwarzschild black hole of mass $M$. In this section we compare the numerical and analytic renormalization methods by looking at the mode-sum renormalization of the axisymmetric part of $\psi_0$. We first compute the retarded field; we then find an analytic expression for the singular field to subleading order; finally, we obtain the renormalization coefficients of the singular field numerically, finding agreement to high accuracy with their analytic values. The numerical computation of the self-force is described in the companion paper.

We work in Schwarzschild coordinates and adopt the notation

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (69)

where

$$f(r) := 1 - \frac{2M}{r}.$$  \hspace{1cm} (70)

The Kinnersley tetrad vectors have components

$$(l^\mu) = (1/f(r), 1, 0, 0), \quad (n^\mu) = \frac{1}{2}(1, -f(r), 0, 0), \quad (m^\mu) = \frac{1}{\sqrt{2r}}(0, 0, 1, i/\sin \theta).$$  \hspace{1cm} (71)

\hspace{1cm} 2 In fact, the argument shows that each component $h_{\mu\nu}$, regarded as a scalar, has even parity at leading order in $\rho$. 

With this choice of tetrad the nonzero spin coefficients are
\[ \theta = \frac{1}{r}, \quad \beta = -\alpha = \cot \theta \frac{2}{2\sqrt{2}r}, \quad \gamma = \frac{M}{2r^2}, \quad \mu = -\frac{1}{2r} \left( 1 - \frac{2M}{r} \right), \] (72)

with corresponding Christoffel symbols
\[ \Gamma^1_{12} = -\Gamma^2_{22} = 2\gamma \] (73a)
\[ \Gamma^1_{43} = \Gamma^1_{34} = \Gamma^4_{24} = \Gamma^4_{34} = \Gamma^3_{14} = -\vartheta \] (73b)
\[ \Gamma^3_{33} = \Gamma^4_{44} = -\Gamma^4_{34} = -\Gamma^3_{34} = 2\beta. \] (73d)

The only nonzero component of the background Weyl tensor is
\[ \Psi_2 = -\frac{M}{r^3}. \] (74)

The particle's 4-velocity is
\[ u^\alpha = u^t(t^\alpha + \Omega \phi^\alpha), \] (75)

with \( t^\alpha \) and \( \phi^\alpha \) timelike and rotational Killing vectors and with \( u^t = \sqrt{1 - 3M/r_0} \). Its energy and angular momentum per unit mass, \( E := -u_\alpha t^\alpha \) and \( J := u_\alpha \phi^\alpha \), are given by
\[ E = -\frac{r_0 - 2M}{\sqrt{r_0^2 - 3Mr_0}}, \quad J^2 = \frac{r_0^2 M}{r_0 - 3M}. \] (76)

From Eq. (32), the nonzero components of the metric perturbation are
\[ h_{11} = -\frac{r^2}{2} (\bar{\psi}^2 + \bar{\vartheta}^2 \hat{\Psi}), \] (77)
\[ h_{33} = -r^4 \left[ \frac{1}{4} (\partial_\beta^2 - 2f \partial_\beta r + f^2 \partial_\gamma^2) - \frac{3(r - M)}{2r^2} \partial_t + 3r^2 - 2M \right] \bar{\Psi}, \] (78)
\[ h_{13} = \frac{r^3}{2\sqrt{2}} \left( \partial_t - f \partial_r - \frac{2}{r} \right) \bar{\vartheta}. \] (79)

To compute the self-acceleration (3) in terms of these tetrad components, we use the relations
\[ t^\alpha = \frac{1}{2} f \ell^\alpha + n^\alpha, \quad \phi^\alpha = -\frac{ir}{\sqrt{2}} (m^\alpha - \bar{m}^\alpha), \quad \nabla^\alpha r = -\frac{1}{2} f l^\alpha + n^\alpha, \] (80)

to write
\[ a^r = (u^t)^2 \left[ \frac{1}{2} f_0 l^\alpha - n^\alpha \right] \left( \frac{1}{2} f_0 l^\beta + n^\beta - i \frac{\Omega r_0}{\sqrt{2}} (m^\beta - \bar{m}^\beta) \right) \left( \frac{1}{2} f_0 t^\gamma + n^\gamma - i \frac{\Omega r_0}{\sqrt{2}} (m^\gamma - \bar{m}^\gamma) \right) (\nabla_\beta h_{\alpha \beta} - \frac{1}{2} \nabla_{\alpha h_{\beta \gamma}}). \] (81)

Then, expanding the covariant derivatives and using Eqs. (72) and (76), we find
\[ a^r = (u^t)^2 \left\{ f_0 \left[ \frac{1}{16} f_0 D + \frac{3}{8} \Delta + \frac{i}{8} \Omega (\bar{\delta} - \bar{\vartheta}) - \frac{M}{2 \sqrt{2} r_0^2} \right] h_{11} + f_0 \left[ \frac{1}{8} r_0 D - \frac{1}{4} \frac{1}{r_0 f_0} \Delta + \frac{1}{2} \frac{M}{r_0^2} \right] h_{33} + \left( - \frac{i}{\sqrt{2}} \Omega r_0 \Delta - \frac{1}{4 \sqrt{2} r_0^3} (\bar{\delta} - \bar{\vartheta}) + \frac{i}{2 \sqrt{2}} \Omega \right) h_{13} + c.c. \right\}. \] (82)

A. The retarded field

The retarded fields \( \psi^\text{ret}_4 \) and \( \psi^\text{ret}_0 \) are simplest to compute in coordinates for which the unperturbed orbit lies in the \( \theta = \pi/2 \) plane. The particle’s trajectory is then given by \( \phi = \Omega t \), where \( \Omega = \sqrt{M/r_0^3} \). From Eq. (7), its stress-energy tensor is
\[ T^{\alpha \beta} = \frac{m}{u^t r_0^3} u^\alpha u^\beta \delta(r - r_0) \delta(\cos(\theta)) \delta(\phi - \Omega t) \]
\[ = \sum_{\ell,m} \frac{m}{u^t r_0^3} u^\alpha u^\beta \delta(r - r_0) Y_{\ell m}(\theta, \phi) Y_{\ell m}^{\dagger} \left( \frac{\pi}{2}, \Omega t \right). \] (83)
Because the source and the background metric are both helically symmetric, Lie derived by \( t^\alpha + \Omega \phi^\alpha \), the retarded fields – metric and Weyl tensor – will also be helically symmetric. Because the tetrad vectors \( \hat{\mathbf{e}} \) are also helically symmetric, the symmetry is shared by the scalars \( \psi^{\text{ret}} \) and \( \psi^{\text{ret}}_4 \), which therefore only involve \( \phi \) and \( t \) in the combination \( \phi - \Omega t \). In the harmonic decomposition, \( \phi \) and \( t \) then occur only in the combination \( e^{im(\phi - \Omega t)} \), and the frequency associated with each value of \( m \) is \( \omega = m\Omega \).

The scalars \( \psi^{\text{ret}}_4 \) and \( \psi^{\text{ret}}_0 \) satisfy the Bardeen-Press equation \( \Box \), the \( a = 0 \) form of the Teukolsky equation, namely

\[
T_s \psi := \left[ \frac{r^4}{\Delta} \partial_r^2 - 2s \left( \frac{My^2}{\Delta} - r \right) \partial_t - \Delta^{-s} \frac{\partial}{\partial r} \left( \Delta^{s+1} \frac{\partial}{\partial r} \right) - \partial \partial \right] \psi = 4\pi r^2 T_s.
\]

We will work with \( \psi^{\text{ret}}_0 \), whose source, from Eq. \( (24a) \), has the form

\[
T_{s=2} = -2(\delta - 2\beta) \delta T_{11} + 4(D - 4\varphi)(\delta - 2\beta) T_{13} - 2(D - 5\varphi)(D - \varphi) T_{33}
\]

\[
= T^{(0)} + T^{(1)} + T^{(2)},
\]

where the superscripts indicate the maximum number of radial derivatives in each term. From Eq. \( (83) \), these terms have the explicit forms

\[
T^{(0)} = - \sum_{\ell,m} \frac{m\Omega}{r_0^5} \delta(r - r_0)[(\ell - 1)\ell(\ell + 1)(\ell + 2)]^{1/2} 2Y_{\ell m}(\theta, \phi) \tilde{Y}_{\ell m} \left( \frac{\pi}{2}, \Omega t \right),
\]

\[
T^{(1)} = 2 \sum_{\ell,m} \frac{m\Omega u}{r_0^5} \left[ i\delta'(r - r_0) + \left( \frac{m\Omega}{r_0^3} + \frac{4i}{r_0^3} \right) \delta(r - r_0) \right] [(\ell - 1)(\ell + 2)]^{1/2} 2Y_{\ell m}(\theta, \phi) \tilde{Y}_{\ell m} \left( \frac{\pi}{2}, \Omega t \right),
\]

\[
T^{(2)} = \sum_{\ell,m} \frac{m^2 \Omega^2 u^2}{r_0^5} \left[ \delta''(r - r_0) + \left( \frac{6}{r_0^3} - 2i\Omega f_0 \right) \delta'(r - r_0)
\right.
\]

\[
\left. - \left( \frac{2\Omega^2}{r_0^2 f_0^2} + \frac{6\Omega u}{r_0^2 f_0} - \frac{2i\Omega M f_0}{r_0^2 f_0^2} - \frac{4}{r_0^3} \right) \delta(r - r_0) \right] 2Y_{\ell m}(\theta, \phi) \tilde{Y}_{\ell m} \left( \frac{\pi}{2}, \Omega t \right).
\]

Each mode of \( \psi_0 \) or \( \psi_4 \),

\[
\psi = e^{-i\omega t} R(r) Y_{\ell m}(\theta, \phi),
\]

has radial eigenfunction \( R_0 \) or \( R_4 \) satisfying the radial equation corresponding to its spin-weight:

\[
\Delta R''_0 + 6(r - M) R'_0 + \left[ \frac{\omega^2 r^4}{\Delta} + \frac{4i\omega r^2 (r - 3M)}{\Delta} - (\ell - 2)(\ell + 3) \right] R_0 = 0,
\]

\[
\Delta R''_4 - 2(r - M) R'_4 + \left[ \frac{\omega^2 r^4}{\Delta} - \frac{4i\omega r^2 (r - 3M)}{\Delta} - (\ell - 1)(\ell + 2) \right] R_4 = 0,
\]

where the prime denotes a derivative with respect to the radial coordinate \( r \). Solutions to these equations are related by

\[
R_0 = \frac{R_4}{r^3 f_0^2}.
\]

To compute \( \psi^{\text{ret}}_0 \), it is helpful to define a Green’s function \( \tilde{G}(r, r') \) as the solution to

\[
\Delta \tilde{G}'' + 6(r - M) \tilde{G}' + \left( \frac{\omega^2 r^4}{\Delta} + \frac{4i\omega r^2 (r - 3M)}{\Delta} - (\ell - 2)(\ell + 3) \right) \tilde{G} = \frac{\delta (r - r') \Delta^{1/2}}{r^3}.
\]

namely

\[
\tilde{G}(r, r') = - \sum_{\ell,m} A_{\ell m} \left[ \frac{[\Delta (r')]^{5/2}}{r'^3} \right] R_H(r_<) R_\infty(r_>).
\]

where \( R_H \) and \( R_\infty \) are solutions to the radial equation for \( \psi_0 \) that are regular at the horizon and at infinity, respectively, and the quantity

\[
A_{\ell m} := \frac{1}{\Delta^3 (R_H R'_\infty - R_\infty R'_H)}
\]
is a constant, independent of \( r \). The full spatial Green’s function \( G(x, x') \equiv G(r, \theta, \phi; r', \theta', \phi') \) is then given by

\[
G(x, x') = -\sum_{\ell, m} A_{\ell m} \frac{[\Delta_r(r')]^{5/2}}{r'^3} R_H(r_<) R_\infty(r_>) \cdot 2 Y_{\ell m}(\theta, \phi) 2 \bar{Y}_{\ell m}(\theta', \phi').
\]  

(96)

The Weyl scalar \( \psi_0 \) is defined and smooth everywhere except on the trajectory of the particle. It is given in terms of the source and the Green’s function \( G(x, x') \) by

\[
\psi_0 = 4\pi \int T(t, x') G(x, x') r'^2 dV'
= 4\pi \int \left( T_0^{(0)} + T_0^{(1)} + T_0^{(2)} \right) G(x, x') r'^2 dV'
=: \psi_0^{(0)} + \psi_0^{(1)} + \psi_0^{(2)},
\]  

(97)

where the superscripts on \( \psi_0 \) correspond to the three terms in Teukolsky source function defined in Eq. (85). The three terms in Eq. (97) have outside the particle trajectory\(^3\) the form

\[
\psi_0^{(0)} = 4\pi u t' \frac{\Delta_r^2}{r_0^2} \sum_{\ell m} A_{\ell m} [(\ell - 1)(\ell + 1)(\ell + 2)]^{1/2} R_H(r_<) R_\infty(r_>) 2 Y_{\ell m}(\theta, \phi) \bar{Y}_{\ell m} \left( \frac{\pi}{2}, \Omega t \right),
\]  

(98)

\[
\psi_0^{(1)} = 8im\Omega u t' \Delta_0 \sum_{\ell m} A_{\ell m} [(\ell - 1)(\ell + 2)]^{1/2} 2 Y_{\ell m}(\theta, \phi) \bar{Y}_{\ell m} \left( \frac{\pi}{2}, \Omega t \right) \times
\left\{ [im\Omega r_0^2 + 2r_0] R_H(r_<) R_\infty(r_>) + \Delta_0 [R_H(r_0) R_\infty(r) \theta(r - r_0) + R_H(r) R'_\infty(r_0) \theta(r_0 - r)] \right\},
\]  

(99)

\[
\psi_0^{(2)} = -4\pi m^2 u t' \sum_{\ell m} A_{\ell m} Y_{\ell m}(\theta, \phi) \bar{Y}_{\ell m} \left( \frac{\pi}{2}, \Omega t \right) \times
\left\{ [30r_0^4 - 80Mr_0^3 + 48M^2r_0^2 - m^2\Omega^2 r_0^2 - 2\Delta_0^2 - 24\Delta_0 r_0(r_0 - M) + 6im\Omega r_0^4(r_0 - M)] R_H(r_<) R_\infty(r_>) + 2(6\Delta_0^2 - 20Mr_0^3 + 16M^2r_0^2 - 3\Delta_0 r_0^2 + im\Omega\Delta_0 r_0^4) [R_H(r_0) R_\infty(r) \theta(r - r_0) + R'_\infty(r_0) R_H(r) \theta(r_0 - r)] + r_0^2 \Delta_0^2 [R_H(r_0) R_\infty(r) \theta(r - r_0) + R'_\infty(r_0) R_H(r) \theta(r_0 - r)] \right\}.
\]  

(100)

B. The singular field

Because the conservative part of the self-force is radial, it is axisymmetric about a radial ray through the particle. We will compare the analytic to the numerical determination of the singular part of a Weyl scalar by looking at the axisymmetric part of \( \psi_0 \) and (in Appendix D) its contribution to the self-force. We outline the calculation of the leading and subleading terms in the axisymmetric part of the singular field \( \psi_0^s \) as a sum of angular harmonics \( 2Y_0(\Theta) \) whose coefficients are polynomials in \( \ell \), with angular coordinates \( \Theta \) and \( \Phi \) chosen so that the \( \Theta = 0 \) line (at fixed \( t \)) is the radial line through the particle. Details of the conversion from a small distance expansion to a large \( L \) expansion are left to Appendix C.

The analytic expression for the resulting renormalization coefficients is then compared to a numerical determination by matching the retarded field to a power series in \( L \). Remarkably, although the subleading part of \( \psi_0^s \) is a lengthy expression – Eq. (99) – we will see that its axisymmetric part, written as a sum over angular harmonics, vanishes. Because the angular harmonics \( 2Y_{\ell m} \) are complete in \( L_2(S^2) \), this means that, as a distribution, \( \psi_0^s \) has support at \( \Theta = 0 \), where \( 2Y_{00}(\Theta) = 0 \).

The expression for the retarded field is simplest for coordinates in which the orbit is in the \( \theta = \pi/2 \) plane. Expressing the singular field \( \psi_0^s \) of Eq. (99) as a sum of angular harmonics is simplest if angular coordinates \( \Theta \) and \( \Phi \) are chosen

\(^3\) Note that the formal integral of the Green’s function also gives a \( \delta \)-function contribution with support on the trajectory, namely

\[-4\pi m^2 u t' f_0^{-1} \delta(r - r_0) \delta(\cos \theta) \delta(\phi - \Omega t).\]
with the particle at $\Theta = 0$, as we have just described. To compute the difference $\psi_0^\text{ren} = \psi_0^\text{ret} - \psi_0^\ast$, one must rotate $\psi_0^\text{ret}$ to the coordinate position of or $\psi_0^\ast$ or vice-versa. Following the conventions of Detweiler et al. [47] (henceforth DMW), $\Theta$ and $\Phi$ are related to $\theta$ and $\phi$ by a rotation of the form

$$
\begin{align*}
\sin \theta \cos \phi &= \cos \Theta, \\
\sin \theta \sin \phi &= \sin \Theta \cos \Phi, \\
\cos \theta &= \sin \Theta \sin \Phi.
\end{align*}
$$

With the usual association of Cartesian coordinates $x, y, z$ to $r, \theta, \phi$ and of $\hat{x}, \hat{y}, \hat{z}$ to $r, \Theta, \Phi$, the map is $x = \hat{z}, y = \hat{x}, z = \hat{y}$.

Eq. (53) for $\psi_0^\ast$ involves the components $l^T = l^a \nabla_a T$ and $l^\rho = l^a \nabla_a \rho$. We obtain these to subleading order in terms of Schwarzschild coordinates: That is, with $\epsilon$ the distance from the particle’s position $P$ with respect to the positive-definite metic $g_{a\beta} + 2u_a u_\beta$, the leading and subleading terms in $T$ and $\rho$ are $O(\epsilon)$ and $O(\epsilon^2)$, respectively. The corresponding leading and subleading terms of $l^T$ and $l^\rho$ are then $O(1)$ and $O(\epsilon)$.

Expansions of $\rho$ and of the local inertial coordinates $T, X, Y, Z$ about a point $P$ in terms of Schwarzschild coordinates are given, for example, in Ref. [47]. To subleading order, $T$ has the form

$$
T = (E(t-t_0) - J \sin \Theta \cos \Phi) + \left(\frac{EM}{r_0 f_0}(t-t_0)(r-r_0) - \frac{J}{r_0} (r-r_0) \sin \Theta \cos \Phi \right).
$$

It is convenient to work with $\rho^2$ instead of $\rho$; to subleading order, we have $\rho^2 = \rho^{(2)} + \rho^{(3)}$, where the order $\epsilon^2$ and $\epsilon^3$ contributions are, respectively,

$$
\rho^{(2)} = \frac{(r-r_0)^2}{f_0} + (r_0^2 + J^2 \cos^2 \Phi) \sin^2 \Theta - 2EJ \sin \Theta \cos \Phi (t-t_0) + \frac{J^2 f_0}{r_0^2} (t-t_0)^2,
$$

$$
\rho^{(3)} = \frac{M}{r_0^2} \left( 1 + \frac{2J^2}{r_0^2} \right) (t-t_0)^2 (r-r_0) + \frac{2JE(M-r_0)}{f_0 r_0^2} (t-t_0)(r-r_0) \sin \Theta \cos \Phi - \frac{M}{r_0^2 f_0} (r-r_0)^3 \\
+ r_0 \sin^2 \Theta (r-r_0) + 2r_0 \sin^2 \Theta \sin^2 \Phi (r-r_0) + \frac{2E^2 r_0}{f_0} \sin^2 \Theta \cos^2 \Phi (r-r_0).
$$

We use Eq. (53). This expression omits $\delta$-functions with support at the position of the particle. These do not contribute to the renormalization of the retarded field if the renormalization is done by subtracting the singular field from the retarded field in a neighborhood of the particle, averaging over a sphere surrounding the particle, and then taking the limit as the radius of the sphere shrinks to zero (the Quinn-Wald prescription [16]). In a mode-sum regularization, we discard $\delta$-functions with support at the particle in both the singular and the retarded field.

The background Kinnersley tetrad written in terms of $\Theta$ and $\Phi$ is

$$
l^a = \left(\frac{1}{f}, 1, 0, 0\right), \quad m^a = \frac{1}{\sqrt{2}r} \left(0, 0, 1, \frac{i}{\sin \Theta}\right), \quad \text{where again } f = \left(1 - \frac{2M}{r}\right)
$$

We now expand the needed tetrad components to subleading (quadratic) order in Schwarzschild coordinates, about their values at $P$, using superscripts $(0)$ and $(1)$ as above to denote orders in $\epsilon$ and writing $l^a = l^{(0)a} + l^{(1)a}$,
\[ m^\alpha = m^{(0)} + m^{(1)} \alpha. \] Using Eqs. (102), (103), (104), and (105), we have

\begin{align*}
\tau(0)^T &= \frac{E}{f_0}, \\
\tau(1)^T &= \frac{EM \rho}{r_0 f_0}(t - r_0) + \frac{EM \rho}{r_0 f_0}(t - t_0) - \frac{J}{r_0} \sin \Theta \cos \Phi, \\
m(0)^T &= -\frac{J}{\sqrt{2} r_0} e^{-i \Phi}, \\
m(1)^T &= 0, \\
\tau(1)^\rho &= \frac{J^2}{r_0^2 f_0}(t - t_0) - \frac{JE}{f_0 \rho} \sin \Theta \cos \Phi + \frac{1}{f_0 \rho} (r - r_0), \\
\tau(2)^\rho &= \frac{M}{r_0^2 f_0 \rho} \left[ 1 + \frac{2J^2}{r_0^2} \right] (t - t_0) (r - r_0) + \frac{JE(M - r_0)}{f_0^2 r_0^2 \rho} (r - r_0) \sin \Theta \cos \Phi + \frac{M(r_0^2 + 2J^2)}{2 r_0^2 \rho} (t - t_0)^2 \\
&+ \frac{JE(M - r_0)}{f_0^2 r_0^2 \rho} (t - t_0) \sin \Theta \cos \Phi - \frac{3M}{2 r_0^2 \rho^2} r_0^2 (r - r_0)^2 - \frac{r_0}{2 \rho} \sin^2 \Theta + \frac{r_0}{\rho} \sin^2 \Theta \sin^2 \Phi \\
&+ \frac{E^2 r_0}{f_0 \rho} \sin^2 \Theta \cos^2 \Phi - \frac{2MJ^2}{r_0^2 f_0 \rho} (r - r_0) (t - t_0) - \frac{MJE}{f_0^2 r_0^2 \rho} (r - r_0) \sin \Theta \cos \Phi, \\
m(1)^\rho &= \frac{\sqrt{2}}{2 r_0 \rho} \left( \frac{r_0^2 + J^2 \cos \Phi e^{-i \Phi}}{f_0} \right) \sin \Theta - \frac{\sqrt{2} \rho E}{2 r_0 \rho} (t - t_0) e^{-i \Phi}, \\
m(2)^\rho &= \frac{\sqrt{2} J^2 e^{-i \Phi} \cos \Phi}{2 r_0^2 f_0 \rho} (r - r_0) \sin \Theta - \frac{\sqrt{2} J E M e^{-i \Phi}}{2 r_0^3 f_0 \rho} (t - t_0) (r - r_0).
\end{align*}

The terms in parentheses in Eq. (53) are then given to subleading order by

\begin{equation}
(\tau^T m^\rho - m^T \tau^\rho)^2 = \left( \tau(0)^T m^{(1)} \rho - m^{(0)} T^T \tau(1)^\rho \right)^2 \\
+ 2 \left( \tau(0)^T m^{(1)} \rho - m^{(0)} T^T \tau(1)^\rho \right) \left( \tau(0)^T m^{(2)} \rho - m^{(0)} T^T \tau(2)^\rho + \tau(1)^T m^{(1)} \rho - m^{(1)} T^T \tau(1)^\rho \right); \tag{107}
\end{equation}

and \( \rho^{-5} \) is given by

\begin{equation}
\frac{1}{\rho^5} = \frac{1}{\rho^{(2)^5}} - \frac{5}{2 \rho^{(3)^5}}. \tag{108}
\end{equation}

implying

\begin{align*}
\psi_0^\rho &= -\frac{3}{2} \frac{m}{\rho^{(2)^5}} \left( \tau(0)^T m^{(1)} \rho - m^{(0)} T^T \tau(1)^\rho \right)^2 + \frac{15}{4} \frac{m}{\rho^{(2)^5}} \rho^{(3)} \left( \tau(0)^T m^{(1)} \rho - m^{(0)} T^T \tau(1)^\rho \right)^2 \\
&- 3 \frac{m}{\rho^{(2)^5}} \left( \tau(0)^T m^{(1)} \rho - m^{(0)} T^T \tau(1)^\rho \right) \left( \tau(0)^T m^{(2)} \rho - m^{(0)} T^T \tau(2)^\rho + \tau(1)^T m^{(1)} \rho - m^{(1)} T^T \tau(1)^\rho \right) + O(\epsilon^{-1}). \tag{109}
\end{align*}

The expression for \( \psi_0^\rho \) as a mode sum is obtained from the value of this expression at \( t = t_0 \). The full expression, including terms involving \( t - t_0 \) is given in Appendix (C). We denote by \( \psi_0^{\rho - L} \) and \( \psi_0^{\rho - SL} \) the leading and subleading parts of the singular field, respectively. Using Eqs. (102), (104) and (106), we obtain for \( \psi_0^\rho \) to subleading order the
Coefficients of $\psi_0$ and $\psi_{sL}$ are given by

$$
\psi_0^{sL} = -\frac{3mE^2r_0^2\sin^2\Theta}{f_0^2} + \frac{3mJ^2e^{-2i\Phi}(r-r_0)^2}{\hat{\rho}^5} - \frac{3mJEe^{-i\Phi}(r-r_0)\sin\Theta}{f_0^2},
$$

$$
\psi_0^{s}\psi_{sL} = -\frac{15mJ^2Me^{-2i\Phi}(r-r_0)^5}{2f_0^4r_0^4} + \frac{15mJEe^{-i\Phi}\sin\Theta(r-r_0)^4}{f_0^4r_0^4} + \frac{15me^{-2i\Phi}J^2\left((-i\sin\Phi)e^{2\Phi} + \frac{f_0^2}{r_0^2}\cos^2\Phi\right)(r-r_0)^3\sin^2\Theta}{\hat{\rho}^7} + \frac{15mJEe^{-i\Phi}(r_0^2 + 2J^2\cos^2\Phi)(r-r_0)^2\sin^3\Theta}{f_0^2} + \frac{15m^2J^2E^2(J^2 + r_0^2 + J^2\cos 2\Phi)(r-r_0)^4\sin\Theta}{2f_0^7} + \frac{9m(e^{-2i\Phi}J^2M)(r-r_0)^3}{f_0^4r_0^4} + \frac{3me^{-i\Phi}Jr_0E\sin^3\Theta}{f_0^2} + \frac{15me^{-i\Phi}MJE(r-r_0)^2\sin\Theta}{f_0^2r_0^4} + \frac{9mJ^2(r_0^2 + J^2\cos^2\Phi)(r-r_0)^3}{r_0f_0^4r_0^4}
$$

The axisymmetric part of each of these terms is proportional to an expression of the form

$$
\frac{\delta^{k_1}\sin^{k_2}\Theta}{\delta^2 + 1 - \cos\Theta)^{k+1/2}},
$$

where $k_1$, $k_2$ and $k$ are positive integers, and $\delta$, given by Eq. [110], is proportional to $r-r_0$. In Appendix [C], following DMW, we use the generating function for Legendre polynomials,

$$
\frac{1}{(e^T + e^{-T} - 2u)^{1/2}} = \sum_{\ell} e^{-(\ell+1/2)|T|} P_\ell(u), \quad T \neq 0,
$$

and its derivatives to express each term as a sum of Legendre polynomials and their derivatives. We then use a relation between the spin-weighted harmonics $\nu Y_{\ell m}$ and Legendre polynomials to write the series in terms of the harmonics $2Y_{\ell 0}$. The leading order part of $\psi_0$ then has the form

$$
\langle \psi_{sL}^0 \rangle_{r_0}(\Theta) = \frac{-m(r_0 - 3M)^{3/2}}{r_0^2} \left\langle \frac{1}{\chi^{5/2}} \right\rangle \sum_{\ell=2}^\infty \sqrt{\frac{4\pi(\ell + 2)!}{(\ell - 2)!(2\ell + 1)}} 2Y_{\ell 0}(\Theta, 0),
$$

where $\langle \psi_{sL}^0 \rangle$ is the axisymmetric part of $\psi_0$.

Finally, each subleading term is proportional as a distribution to the sum $\sum_{l=0}^\infty (l + 1/2) P_\ell(\cos\theta)$. That sum is a $\delta$-function with support at $\Theta = 0$, and its projection along $2Y_{\ell 0}$ therefore vanishes for all $\ell$. Eq. [114] thus gives $\langle \psi_{sL}^0 \rangle$ to subleading order in $L$.

**C. Comparison with numerical determination of $\psi_0$**

We complete this section with a comparison of the analytic form of $\langle \psi_{sL}^0 \rangle$ with its numerical value obtained by matching $\psi_{sL}^{\text{ret}}$ to a power series in $L$. A comparison of the numerically and analytically determined contributions to the self-force from the axisymmetric part of $\psi_0$ is given in Appendix [C].

The retarded Weyl scalar $\psi_{sL}^{\text{ret}}$ is computed by integrating the radial Teukolsky equation for each value of $\ell$. The coefficients of $2Y_{\ell 0}$ are matched to a power-series in $L$ of the form

$$
\psi_{sL}^{\text{ret}} = \frac{\sqrt{4\pi(\ell + 2)!}}{(2\ell + 1)!(\ell - 2)!} \left( A + \frac{B}{L} + \frac{C}{L^2} + \cdots \right).
$$

Shown below is a table of the fractional error in $A$ and $B$ when found numerically and compared to the analytic form given by Eq. [114]. Details of the numerical methods and checks of numerical accuracy are given in the companion paper.
V. BRIEF DISCUSSION

The methods discussed in this paper for finding the self-force in a radiation gauge have been used to find the self-force on a particle in circular orbit in a Schwarzschild spacetime, and work on orbits in a Kerr background is now underway. The advantage of a radiation gauge is the ease with which the retarded field can be computed. A disadvantage is the difficulty in analytically computing the singular field \( h_{\alpha\beta} \) from \( \psi_0 \). We have avoided this difficulty by using a numerical matching procedure to find the singular field, and the companion paper shows that the numerical matching reproduces the renormalization coefficients for gauge-invariant quantities to machine accuracy, for the Schwarzschild example.

We have shown that the perturbed metric in a radiation gauge generically has even parity to leading order in geodesic distance \( \rho \) to the particle trajectory. Using the renormalized field to compute the perturbed geodesic then relies on showing that the singular field gives no contribution to the self-force. The companion paper checks this for a particle in circular orbit in a Schwarzschild spacetime, in which the self-force is symmetric about a radial line through the particle. We numerically compute the axisymmetric part of the singular field and find that to subleading order it coincides with the axisymmetric part of \(-m \nabla^\alpha \rho^{-1}\); it thus gives no contribution to the self-force at order \( \rho^0 \).

This regular behavior of the singular part of the self-force may seem remarkable, given the line singularity in a radiation gauge that arises when one includes the perturbed mass in a radiation-gauge metric obtained from a Hertz potential. It is less surprising, however, if one considers a particle at rest in flat space. When principal null directions coincide with the axisymmetric part of \( \ell \), the self-force of course vanishes, with the contribution from the singular field coinciding with that from the retarded field; but the contribution from each is nonzero and gauge invariant under time-independent gauge transformations. This implies that the singular part of the expression for the self-force in a radiation gauge has its Lorenz form \(-m \nabla^\alpha \rho^{-1}\). For a circular orbit, the result is implied by the fact that the gauge transformation of the self-force can be written in a form, Eq. (37), that involves no derivatives of the gauge vector \( \xi^\alpha \), together with the fact that \( \xi^\alpha \) is \( O(\rho^0) \). For generic orbits, the gauge transformation involves \((\mathbf{u} \cdot \nabla)^2 \xi^\alpha\), and it remains to be seen whether the singular part of the self force can again be identified with \(-m \nabla^\alpha \rho^{-1}\).

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| \( r_0/M \) | \( A_{\text{analytic}} \) | \( A_{\text{numerical}} \) | \( |\Delta A/A| \) | \( |\Delta B| \) |
|---|---|---|---|---|
| 8 | -0.002507548110466834 | -0.002507548108782573 | 6.717 \times 10^{-10} | 8.496 \times 10^{-10} |
| 10 | -0.001214016915072354 | -0.001214016915092580 | 1.666 \times 10^{-11} | 2.125 \times 10^{-11} |
| 15 | -0.000335610432965837 | -0.000335610432973954 | 2.419 \times 10^{-12} | 6.003 \times 10^{-13} |
| 20 | -0.0001370231924969076 | -0.0001370231924969057 | 1.365 \times 10^{-14} | 3.853 \times 10^{-14} |
| 25 | -0.0000688282866736571 | -0.00006882828667362483 | 5.706 \times 10^{-13} | 7.462 \times 10^{-13} |
| 30 | -0.00003933557520091981 | -0.00003933557520088896 | 7.843 \times 10^{-13} | 1.067 \times 10^{-14} |
| 35 | -0.00002455304484569332 | -0.00002455304484594233 | 8.549 \times 10^{-13} | 8.341 \times 10^{-15} |
| 40 | -0.00001634080095821354 | -0.00001634080095821053 | 7.960 \times 10^{-13} | 5.325 \times 10^{-15} |
| 45 | -0.00001141847437793787 | -0.00001141847437792919 | 7.605 \times 10^{-13} | 3.673 \times 10^{-15} |
| 50 | -0.000008290448479296679 | -0.000008290448479290278 | 7.722 \times 10^{-13} | 2.807 \times 10^{-15} |
| 55 | -0.000006208226467936966 | -0.000006208226467932644 | 6.961 \times 10^{-13} | 1.912 \times 10^{-15} |
| 60 | -0.000004768831841073202 | -0.000004768831841069988 | 6.739 \times 10^{-13} | 1.433 \times 10^{-15} |
| 70 | -0.000002990259098529844 | -0.000002990259098527988 | 6.209 \times 10^{-13} | 8.488 \times 10^{-16} |
| 80 | -0.000001996831417921701 | -0.000001996831417920439 | 6.316 \times 10^{-13} | 5.893 \times 10^{-16} |

TABLE II: The fractional error in the renormalization coefficient \( A \) and the error in \( B \) for \( \langle \psi_0 \rangle \) is given here for a particle in circular orbit in a Schwarzschild background at radius \( r_0 \). \( \Delta A \) and \( \Delta B \) are the differences between the coefficients obtained numerically and by using the analytic expression (37). The analytic value of \( B \) is zero.
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Appendix A: Gauge transformations

1. Gauge transformation from Lorenz to radiation gauge

We consider here the perturbed radiation-gauge metrics that describe asymptotically flat vacuum metric perturbations involving no linear change in mass or angular momentum. For harmonic time dependence, outgoing perturbations of this kind have, in a Lorenz (transverse-tracefree) gauge, the asymptotic behavior \( \hat{h}_{\alpha \beta} = \hat{h}_{\alpha \beta} e^{-i\omega u} \), with

\[
\hat{h}_{11} = O(r^{-3}), \quad \hat{h}_{12} = O(r^{-2}), \quad \hat{h}_{13} = O(r^{-2}),
\]

and they therefore satisfy the IRG condition to \( O(r^{-2}) \):

\[
l^3 \hat{h}_{\alpha \beta} = O(r^{-2}), \quad \hat{h} = 0.
\]

We first find a corresponding asymptotically flat radiation-gauge metric perturbation by exhibiting a gauge transformation from the given Lorenz-gauge metric to an asymptotically flat metric satisfying the exact IRG conditions.

We then show that there is an asymptotically vanishing gauge transformation to an asymptotically flat ORG metric perturbation.

IRG metric for outgoing radiation.

We obtain as follows the gauge transformation from a Lorenz gauge to an IRG metric perturbation,

\[
h_{\alpha \beta}^{ret, RG} = h_{\alpha \beta}^{ret} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha.
\]

We begin with the transformation for a Schwarzschild background and then generalize it to Kerr. The transformation is described most simply in coordinates \( u, r, \theta, \phi \), with \( u \) the outgoing null coordinate. The harmonics of \( \xi^\alpha \) and \( h_{\alpha \beta}^{ret} \) are then given by

\[
\xi_1 = \xi_1 Y_{\ell m} e^{-i\omega u}, \quad \xi_2 = \xi_2 Y_{\ell m} e^{-i\omega u}, \quad \xi_3 = \xi_3 Y_{\ell m} e^{-i\omega u},
\]

with the corresponding spin-weighted harmonics for the tetrad components of the metric perturbation,

\[
h_{\mu \nu}^{ret} = \hat{h}_{\mu \nu} Y_{\ell m} e^{-i\omega u}.
\]

The radiation gauge condition \( l^3 h_{\alpha \beta} = 0 \) has components

\[
\nabla_1 \xi_1 = -\frac{1}{2} h_{11}^{ret}
\]
\[
\nabla_1 \xi_2 + \nabla_2 \xi_1 = -h_{12}^{ret}
\]
\[
\nabla_1 \xi_3 + \nabla_3 \xi_1 = -h_{13}^{ret}.
\]

These equations have the explicit form,

\[
\partial_r \hat{\xi}_1 = -\frac{1}{2} \hat{h}_{11},
\]

\[
\partial_r \hat{\xi}_2 + \left( i\omega - \frac{1}{2} \partial_r - \frac{M}{r^2} \right) \hat{\xi}_1 = -\hat{h}_{12},
\]

\[
(\partial_r - \frac{1}{r}) \hat{\xi}_3 - \frac{1}{r} [\ell(\ell + 1)/2]^{1/2} \hat{\xi}_1 = -\hat{h}_{13}.
\]

with solution

\[
\hat{\xi}_1 = \frac{1}{2} \int_r^\infty dr' \hat{h}_{11}(r')
\]
\[
\hat{\xi}_2 = \int_r^\infty dr' \left[ \hat{h}_{12}(r') + \frac{1}{4} f(r') \hat{h}_{11}(r') - \left( i\omega + \frac{M}{r'^2} \right) \xi_1(r') \right]
\]
\[
\hat{\xi}_3 = r \int_r^\infty dr' \left[ \frac{1}{r'} \hat{h}_{13} - \frac{1}{r'^2} [\ell(\ell + 1)/2]^{1/2} \hat{\xi}_1 \right].
\]
A result of Price et al. [36] shows, for a vacuum perturbation satisfying the condition \( l^2 h_{\alpha \beta} = 0 \), that the remaining gauge condition, \( h = 0 \), is also satisfied. That is, when \( h_{\mu \nu} = 0 \), we have \( h = -2h_{34} \); and from Eq. (16) of that paper, the perturbed Einstein equation \( \delta (G_{11} - 8\pi T_{11}) = 0 \) implies

\[
h_{34} = a^0 \left( \frac{\hat{\varrho}}{\varrho} + \frac{\varrho}{\hat{\varrho}} \right) + b^0 (\varrho + \hat{\varrho}),
\]

(A9)

where \( a^0 \) and \( b^0 \) are functions of \( u, \theta, \phi \). Now \( \hat{h}_{34} = \nabla_3 \xi_4 + \nabla_4 \xi_3 \), and Eq. (A8) implies \( \xi_3 = O(r^{-1}) \), whence \( \hat{h}_{34} = O(r^{-2}) \). Since the right side of Eq. (A9) is \( O(r^{-2}) \) only if \( a^0 = b^0 = 0 \), we have \( h_{34} = 0 \).

For each harmonic, it is not difficult to show that Eqs. (A8) give the unique solution to Eqs. (A7) for which \( k_{12} \) vanish asymptotically for \( \omega \neq 0 \) and vanish faster than \( r^{-1} \) for \( \omega = 0 \): Any other gauge transformation differs from the solution (A8) by a solution to the homogeneous equations, to Eqs. (A7) with \( h_{\mu \nu} = 0 \). Their general solution is

\[
\hat{\xi}_1 = k_1,
\]

(A10a)

\[
\hat{\xi}_2 = -(i\omega + M/r)k_1 + k_2,
\]

(A10b)

\[
\hat{\xi}_3 = -[i(\ell + 1)/2]^{1/2}k_1 + k_3 r.
\]

(A10c)

For \( \omega \) nonzero, \( h_{22} \) vanishes asymptotically only if \( k_1 = 0 \) and \( k_2 = 0 \); and \( h_{33} \) vanishes asymptotically only if \( k_3 = 0 \). Similarly, for \( \omega = 0 \), \( h_{22} = o(r^{-1}) \) only if \( k_1 = k_2 = 0 \); and \( h_{33} = o(r^{-1}) \) only if \( k_3 = 0 \).

For a Kerr background, we similarly find the harmonics of the gauge vector in Kerr coordinates, \( u, r, \theta, \phi \) of Eq. (H1). Harmonics of the gauge vector have the form

\[
\xi_1 = \hat{\xi}_1 (r, \theta) e^{i(m\phi - \omega u)}, \quad \xi_2 = \hat{\xi}_2 (r, \theta) e^{i(m\phi - \omega u)}, \quad \xi_3 = \hat{\xi}_3 (r, \theta) e^{i(m\phi - \omega u)}.
\]

(A11)

The corresponding harmonics for the metric perturbation are

\[
h^\text{ret}_{\mu \nu} = h_{\mu \nu} (r, \theta) e^{i(m\phi - \omega u)}.
\]

(A12)

The gauge transformation for a Kerr background is governed by the equations

\[
D\xi_1 = -\frac{1}{2} h^\text{ret}_{11},
\]

(A13a)

\[
D\xi_2 + (\Delta - \gamma - \bar{\gamma})\xi_1 + (\bar{\tau} - \pi)\xi_3 + (\tau - \bar{\pi})\xi_4 = -h^\text{ret}_{12},
\]

(A13b)

\[
(\delta - 2\bar{\pi})\xi_1 + (D + \bar{\varrho})\xi_3 = -h^\text{ret}_{13},
\]

(A13c)

The components \( \xi_{\mu} \) are given successively by

\[
\hat{\xi}_1 = \frac{1}{2} \int_r^\infty dr' \hat{h}_{11},
\]

(A14a)

\[
\hat{\xi}_3 = -\frac{1}{\varrho} \int_r^\infty dr' \bar{\varrho} \left[ \hat{h}_{13} - (\delta - 2\bar{\pi})\hat{\xi}_1 \right],
\]

(A14b)

\[
\hat{\xi}_2 = \int_r^\infty dr' \left[ \hat{h}_{12} + (\Delta - \gamma - \bar{\gamma})\hat{\xi}_1 + (\pi - \bar{\tau})\hat{\xi}_3 + (\bar{\tau} - \tau)\hat{\xi}_4 \right].
\]

(A14c)

Asymptotic regularity follows from the asymptotic behavior (A1) of components along \( l^a \) of outgoing waves in a Lorenz gauge. And the Price et al. result again implies \( h = 0 \).

**IRG perturbed metric for ingoing radiation**

The Hertz potential construction yields an asymptotically flat IRG form for each ingoing asymptotically flat metric perturbation. We find the gauge transformation from transformation from a Lorenz gauge to this asymptotically flat IRG. For simplicity, we restrict consideration to a Schwarzschild background.
In Eqs. (A4) and (A5) the outgoing null coordinate $u$ is replaced by the ingoing null coordinate $v = t + r^*$. In Eqs. (A7) $\partial_r$ is the replaced by $e^{-2\omega r^*} \partial_r e^{2\omega r^*}$, and the solution has the form

$$\xi_1 = \frac{1}{2} e^{-2\omega r^*} \int_r^\infty \frac{d\rho}{\rho} e^{2\omega \rho} h_{11}(\rho)$$  \hspace{1cm} (A15a)$$

$$\xi_2 = e^{-2\omega r^*} \int_r^\infty \frac{d\rho}{\rho} e^{2\omega \rho} \left[ \hat{h}_{12}(\rho) + \frac{1}{4} f(\rho) \hat{h}_{11}(\rho) - \left( \frac{1}{2} + \frac{M}{\rho^2} \right) \xi_1 \right]$$  \hspace{1cm} (A15b)$$

$$\xi_3 = \int_r^\infty \frac{d\rho}{\rho} e^{2\omega \rho} \left[ \frac{1}{\rho^2} (\hat{h}_{13} - \frac{1}{\rho^2} (\ell + 1)/2)^{1/2} \xi_1 \right].$$  \hspace{1cm} (A15c)

Now, however, because the radiation is ingoing, $\hat{h}_{11} \sim e^{i\omega r^*}/r$, when $\omega \neq 0$. Asymptotic flatness follows from the relation

$$\int_r^\infty \frac{d\rho}{\rho} e^{ikr^*} = \frac{1}{ik} + O(r^{-1}).$$

When $\omega = 0$, asymptotic flatness follows from the asymptotic conditions (A1). Again, because $\xi_3 = O(r^{-1})$, we have $\nabla_3 \xi_4 = O(r^{-2})$, and the Price et al. relation then implies $h = -2h_{34} = 0$.

2. Gauge transformations of the self force

A gauge transformation of the self-force was obtained by Barack and Ori [27]. We give an alternate, covariant derivation, mention a second kind of gauge freedom, and obtain a simpler form of a gauge transformation of the self-force for a particle in circular orbit.

A gauge transformation is an infinitesimal diffeomorphism that drags an unperturbed geodesic of the background metric to a neighboring curve that is a geodesic of the dragged-along metric. This can be stated precisely in terms of a congruence of timelike geodesics through a neighborhood of a point. This can be stated precisely in terms of the original metric by $\xi_u g_{\alpha\beta}$, and the perturbed geodesic through $P$ has 4-velocity altered by $\delta u^\alpha = \xi u^\alpha$. The perturbed geodesic equation associated with a perturbation that is pure gauge has the form

$$\tilde{\delta} (u^\beta \nabla_\beta u^\alpha) = \xi (u^\beta \nabla_\beta u^\alpha) = 0.$$  \hspace{1cm} (A16)

Writing

$$\xi (u^\beta \nabla_\beta u^\alpha) = (\xi u^\beta) \nabla_\beta u^\alpha + u^\beta \nabla_\beta \xi u^\alpha + u^\beta [\xi, \nabla_\beta] u^\alpha$$  \hspace{1cm} (A17)

and

$$[\xi, \nabla_\beta] u^\alpha = u^\gamma \nabla_\beta \nabla_\gamma u^\alpha - R^\alpha_{\gamma\beta\delta} u^\gamma \xi^\delta,$$  \hspace{1cm} (A18)

with the perturbed acceleration defined by $\tilde{\delta} a^\alpha := \tilde{\delta} u^\beta \nabla_\beta u^\alpha + u^\beta \nabla_\beta \tilde{\delta} u^\alpha$, we have

$$\tilde{\delta} a^\alpha = -(u \cdot \nabla)^2 \xi^\alpha + R^\alpha_{\beta\gamma\delta} u^\beta u^\gamma \xi^\delta.$$  \hspace{1cm} (A19)

In this form, $u^\alpha + \tilde{\delta} u^\alpha$ is normalized to $-1$ with respect to the perturbed metric $g_{\alpha\beta} + h_{\alpha\beta}$, and the geodesic equation is affinely parameterized with respect to the perturbed metric. If the geodesic is parameterized so that its tangent is normalized to 1 with respect to the background metric, and we denote by $\delta u^\alpha$ the change in $u^\alpha$ with that normalization, then

$$\delta u^\alpha = (\delta^\alpha_\beta - u^\alpha u_\beta) \tilde{\delta} u^\beta = (\delta^\alpha_\beta - u^\alpha u_\beta) \xi u^\beta.$$  \hspace{1cm} (A20)

With $\delta a^\alpha := \delta u^\beta \nabla_\beta u^\alpha - u^\beta \nabla_\beta \delta u^\alpha$, Eq. (A19) implies

$$\delta a^\alpha = -(\delta^\alpha_\beta - u^\alpha u_\beta)(u \cdot \nabla)^2 \xi^\beta + R^\alpha_{\beta\gamma\delta} u^\beta u^\gamma \xi^\delta,$$  \hspace{1cm} (A21)

and $a^\alpha u_\alpha = 0$. Note that the right side vanishes if $\xi^\alpha$ happens to drag a geodesic of the background spacetime to another geodesic of the background spacetime: This is the equation of geodesic deviation governing the connecting
vector joining two neighboring geodesics of \( g_{\alpha\beta} \). For general \( \xi^\alpha \), the right side of Eq. \( (A21) \) then measures the failure of \( \xi^\alpha \) to produce a geodesic of the background metric.

The effect of a gauge transformation on the perturbed geodesic equation \( (B1) \) is to replace \( \delta u^\alpha \) by \( \delta u^\alpha + \delta \xi^\alpha \) and \( a^\alpha \) by \( a^\alpha + \delta \xi^\alpha \).

There is a second kind of gauge freedom, an infinitesimal change in the background geodesic to which one compares a geodesic in the perturbed spacetime. The perturbed geodesic at an initial point of the trajectory can then be changed from \( u^\alpha \) to \( u^\alpha + \delta u^\alpha \), with \( a^\alpha = 0 \). This allows one to regard the right side of Eq. \( (A21) \) as the change in the perturbed geodesic equation for a geodesic through an initial point \( P \) with the same initial tangent vector as that in the original gauge.

For a particle in circular orbit in a Schwarzschild background, Eq. \( (A23) \) takes the form \( (49) \)

\[ \delta \xi u^r = \frac{3\Omega^2}{(1 - 3M/r)} \xi^r. \]  

**(Appendix B: singularity and large \( \ell \) behavior)**

Mode-sum renormalization involves relations, like the formal harmonic decomposition

\[ (1 - \cos \theta)^{-1/2} = \sum_{\ell=0}^{\infty} \sqrt{\frac{4\pi}{\ell + 1/2}} Y_{\ell0}(\theta), \]  

(B1)

between the short-distance (ultraviolet) singular behavior of a function and the large-\( \ell \) behavior of its harmonics. The angle \( \theta \) is geodesic distance on the unit 2-sphere \( S \) to the origin \( \theta = 0 \), and in this example, a function that behaves like \( \theta^{-1} \) has angular harmonics that behave like \( L^{-1/2} \), where, as in the body of the paper, \( L = \ell + 1/2 \). In this appendix, we review how one characterizes the large \( \ell \) behavior of functions whose explicit angular harmonics are not known; and we give a precise meaning, in terms of distributions on the 2-sphere, to formal expressions like the divergent right side of Eq. \( (B1) \) and to the angular harmonics of functions like \( f = \theta^{-n} \) for which the integral \( \int d\Omega f Y_{\ell m} \) diverges.

We initially restrict the discussion to ordinary spherical harmonics and then generalize it to spin-weighted harmonics.

The angular harmonics of the retarded fields are limits as \( \ell \to \infty \) of \( \ell_m \) of angular harmonics that behave like \( \theta^{-\ell} \). For general \( \ell \), the angular harmonics of \( f(\theta,\phi) \) on \( S \) is square integrable if and only if \( \ell^n f_{\ell m} \) is square summable: With \( D_n \) the covariant derivative operator of the metric \( ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) on \( S \), the angular
Laplacian is \( D^2 := D_a D^a \), and we have
\[
\int d\Omega D_{a_1} \cdots D_{a_n} f D^{a_1} \cdots D^{a_n} f = \int d\Omega f (-D^2)^n f < \infty
\]
\[
\iff \sum_{\ell>0,m} [\ell(\ell+1)]^n |f_{\ell m}|^2 < \infty.
\]

One extends this relation to functions like \((1 - \cos \theta)^{-3/2}\) that are not square integrable by regarding them as distributions obtained by taking derivatives of functions like \((1 - \cos \theta)^{1/2}\) that are square integrable. If \( f \) is any distribution on \( S \), \( f_{\ell m} = \int d\Omega f \bar{Y}_{\ell m} \) exists, because \( Y_{\ell m} \) is smooth. Thus, for example, writing
\[
(2 \alpha^2 D_a D^a + \alpha + 2) (1 - \cos \theta) \frac{\partial}{\partial r} = (1 - \cos \theta) \frac{\partial}{\partial r} - 1
\]
gives \( f = (1 - \cos \theta)^{-3/2} \) as a distribution with
\[
f_{\ell 0} = \int d\Omega (1 - \cos \theta)^{-1/2} (-2 D_a D^a + 1/2) Y_{\ell 0} \quad \text{(already well defined)},
\]
or
\[
f_{\ell 0} = \int d\Omega (1 - \cos \theta)^{1/2} (2 D_a D^a + 3/2) (-2 D_a D^a + 1/2) Y_{\ell 0}.
\]

For any distribution \( f \), the right side of Eq. (B4) is finite for some negative \( n \), with increasingly singular distributions corresponding to increasingly negative values of \( n \). To obtain a theorem that relates in this way the short-distance and large-\( \ell \) behavior of distributions, we define the standard spaces of functions whose first \( n \) derivatives are square integrable, the Sobolev spaces \( H_n \), and then extend the definition to spaces \( H_r \) of distributions, where \( r \) can be negative (and need not be an integer).

Recall that the Hilbert space \( L_2(S) \) is the completion of smooth \((C^\infty)\) functions in the norm \( \| f \| \) defined by \( \| f \|^2 = \int d\Omega |f|^2 \). The operator \(-D^2 + \frac{1}{4}\) is positive definite with eigenvalues \( L^2 \).

**Definition.** For positive integers \( n \), the Sobolev space \( H_n(S) \) is the completion of smooth functions in the norm \( \| f \|_n \) defined by
\[
\| f \|^2_n = \int d\Omega \left| f \left(-D^2 + \frac{1}{4}\right)^n f \right|.
\]

Because the right-hand side is a sum of terms of the form
\[
C \int d\Omega f D^{2k} f, \quad k = 0, \ldots, n,
\]
a function has finite norm if and only if the function and its first \( n \) derivatives are square integrable. In particular, \( H_0(S) = L_2(S) \).

Because the operator \(-D^2 + \frac{1}{4}\) is positive definite, it has a well-defined square root, and we can write the definition of the norm in the more concise form
\[
\| f \|_n = \| D^n f \|, \quad \text{with} \quad D := \left(-D^2 + \frac{1}{4}\right)^{1/2}.
\]

The action of the operator \( D \) on a distribution \( f \) is given by
\[
D f = \sum_{\ell m} L f_{\ell m} Y_{\ell m}.
\]
Equivalently, \( D f \) is defined by its action on smooth functions \( g \),
\[
D f(g) = f(D g), \quad \text{or} \quad \int d\Omega \bar{g} D f = \int d\Omega \langle D \bar{g} \rangle f = \sum_{\ell m} L \bar{g}_{\ell m} f_{\ell m}.
\]
We can now define $H_r(S)$ for all real $r$:

**Definition.** The Sobolev space $H_r(S)$ is the completion of smooth functions in the norm $\| \cdot \|_r$ defined by

$$\| f \|_r = |D^r f|.$$  

(B11)

With inner product defined by

$$\langle g, f \rangle_r = \int d\Omega D^r g D^r f,$$

(B12)

the space $H_r$ is a Hilbert space.

The relation

$$\| f \|_r^2 = \sum_{\ell m} L^{2r} |f_{\ell m}|^2,$$

(B13)

implies that a distribution $f$ is in $H_r$ if and only if the sequence $(f_{\ell m})$ of its angular harmonics has finite norm

$$\|(f_{\ell m})\|_r^2 := \sum_{\ell m} L^{2r} |f_{\ell m}|^2.$$

(B14)

Formally, for each $r$, one can turn the set of sequences $(f_{\ell m})$ of complex numbers into a Hilbert space\(^4\) with norm $\|(f_{\ell m})\|_r$.

A function $f$ is smooth if and only if it is in $H_r$ for all $r$, implying that the elements of a sequence $(f_{\ell m})$ are the angular harmonics of a smooth function if and only $f_{\ell m}$ falls off faster than any power of $\ell$: $\lim_{\ell \to \infty} \ell^n |f_{\ell m}| = 0$. A second immediate consequence of the correspondence is the fact that $D$ maps $H_r$ to $H_{r+1}$ and that the Laplacian maps $H_r$ to $H_{r+2}$. (In fact, these maps are isomorphisms.)

The function $\theta^{-1}$ is nearly square-integrable, with the integral $\int d\Omega \theta^{-2}$ diverging only logarithmically and $\int d\Omega \theta^{-2+\epsilon}$ finite for all $\epsilon > 0$. This suggests that $\theta^{-1} \in H_{-\epsilon}$ for all $\epsilon > 0$, and that is in fact the case: From Eq. (B1), the function $f = (1 - \cos \theta)^{-1/2}$ satisfies

$$\|(f_{\ell m})\|_{-\epsilon} = \sum_{\ell m} |f_{\ell m}|^2 L^{-2\epsilon} = 4\pi \sum_{\ell} L^{-1-2\epsilon} < \infty, \text{ all } \epsilon > 0.$$  

(B15)

(Again, for $\epsilon = 0$, the sum diverges logarithmically.) Thus $(1 - \cos \theta)^{-1/2} \in H_{-\epsilon}$. Because $[2(1 - \cos \theta)]^{-1/2}$ differs from $\theta^{-1}$ by $O(\theta^2)$, the function $\theta^{-1}$ and any other function with the same singular behavior belong to $H_{-\epsilon}$. From Eq. (B1), successive applications of $D^2$ imply $\theta^{-n} \in H_{-n-\epsilon}$. If a function has singular behavior $\theta^{-n}$ for integer $n$ and if $f_{\ell m}$ has singular behavior $L^s$, for some $s$, it follows that $s = n - \frac{1}{2}$.

Finally, a formal sum $f = \sum_{\ell m} f_{\ell m} Y_{\ell m}(\theta, \phi)$, like the right side of Eq. (B1), has the meaning $f(g) = \sum g_{\ell m} f_{\ell m}$, for smooth $g$.

Finally we must relate this formalism to angular harmonics of functions $f(r, \theta, \phi)$ that are singular at $r = r_0$ and smooth for $r \neq r_0$, when those harmonics are found as

$$\lim_{r \to r_0} \int d\Omega f(r, \theta, \phi) Y_{\ell m}.$$  

(B16)

We suppose that $f$ can be written in the form $D^r F$, where $F(r, \theta, \phi)$ is continuous everywhere and smooth for $r \neq r_0$ and where $D$ is an operator for which $D$ and $D^4$ have domains that include $C^\infty(S)$. Then, for $g$ smooth,

$$\int d\Omega D^n F g = \int d\Omega F D^4 g \quad \Rightarrow \quad \lim_{r \to r_0} \int d\Omega D^n F g = \lim_{r \to r_0} \int d\Omega F D^4 g = \int d\Omega \lim_{r \to r_0} F D^4 g = \int d\Omega F(r_0) D^4 g(r_0).$$

---

\(^4\) The inner product is $\sum L^{2r} g_{\ell m} f_{\ell m}$, and the Hilbert space is the completion in the norm $\| \cdot \|_r$ of sequences for which $\lim_{\ell \to \infty} |f_{\ell m}|^{2r} = 0$ (these are the sequences corresponding to smooth functions).
Because this last expression is, by definition, the action of the distribution $f(r_0) = D^n F(r_0)$ on $g$, we have the claimed equivalence

$$\lim_{r \to r_0} \int d\Omega f g = \int d\Omega f(r_0)g(r_0). \quad (B17)$$

**Appendix C: Mode-sum representation of $\langle \psi_0^2 \rangle$**

We present here details of the computation $\psi_0^2$ as a mode sum, outlined in Sect. \(\text{IV B}\). We first give the expression for $\psi_0^2$ to subleading order in Schwarzschild coordinates, including terms involving $t - t_0$. As in Sect. \(\text{IV B}\) we denote by $\psi_0^\text{SL}$ and $\psi_0^\text{L}$, respectively.

Eqs. \(103, 104\) and \(106\) yield for $\psi_0^2$ to subleading order the explicit form

$$\psi_0^\text{SL} = \frac{15m}{4\rho^3} \left[ -\frac{2J^2M e^{-2\Phi}}{f_0^3 r_0^6} (r - r_0)^5 - \frac{4JME e^{-2\Phi}}{f_0^3 r_0^4} \sin\Theta(r - r_0)^4 + \frac{4J^2M e^{-2\Phi}}{f_0^3 r_0^6} (t - t_0)(r - r_0)^4 ight. \\
- \frac{2ME^2}{f_0^3} (r - r_0)^3 \sin^2\Theta + \frac{2J^2(2J^2 \cos^2\Phi + r_0^2) e^{-2\Phi}}{f_0^3 r_0^5} (r - r_0)^3 \sin^2\Theta + \frac{4e^{-2\Phi} J^4 M}{f_0^2 r_0^6} (r - r_0)^3 (t - t_0)^2 \\
+ \frac{4JE e^{-2\Phi} (e^{2\Phi} M r_0^2 + J^2 (M - r_0) \cos\Phi)}{f_0^3 r_0^4} (r - r_0)^3 (t - t_0) \sin\Theta + \frac{4JE e^{-2\Phi} (r_0^2 + 2J^2 \cos^2\Phi)}{f_0^2 r_0} (r - r_0)^2 \sin^2\Theta \\
+ \frac{4J^2 e^{-2\Phi} [r_0 (t - t_0) + f_0 r_0 + e^{2\Phi} (r_0 - M) E^2 + J^2 f_0^3 \cos 2\Phi]}{f_0^3 r_0^4} (t - t_0)^2 (r - r_0)^2 \sin^2\Theta \\
+ \frac{4e^{-3\Phi} (1 + 2e^{2\Phi}) E (M - r_0) J^3}{f_0^6 r_0^5} (t - t_0)^2 (r - r_0)^2 \sin\Theta - \frac{4e^{-2\Phi} J^2 M (2J^2 + r_0^2)}{f_0^5 r_0^6} (r - r_0)^2 (t - t_0)^3 \\
+ \frac{2r_0 E^2 (J^2 + r_0^2 + J^2 \cos 2\Phi)}{f_0^2} (r - r_0)^4 \sin^4\Theta + \frac{2e^{-\Phi} J^3 E \left((5 - e^{-2\Phi}) - (3 + e^{2\Phi}) \frac{r_0}{M}\right)}{r_0 f_0} (t - t_0)(r - r_0)^3 \sin^3\Theta \\
+ \frac{1}{f_0^2 r_0^3} 2e^{-2\Phi} (r_0 (J^2 (-2M + (2 + f_0) r_0) + e^{2\Phi} r_0 (2J^2 + Mr_0)) E^2 + f_0^2 J^4 \cos 2\Phi) (r - r_0)(t - t_0)^2 \sin^2\Theta \\
- \frac{4e^{-2\Phi} (r_0 - M) J^3 E (3e^{2\Phi} + e^{-\Phi})}{2r_0^3 f_0} (t - t_0)^3 (r - r_0) \sin\Theta + \frac{2e^{-2\Phi} J^4 (r_0 - M)}{r_0^3 r_0^6} (t - t_0)^4 (r - r_0) \\
- \frac{3m}{\rho^5} \left[ -\frac{3e^{-2\Phi} J r_0 E}{f_0} (r - r_0)^3 \sin^3\Theta + \frac{3e^{-2\Phi} J^2 E}{r_0^3} (t - t_0)^3 - \frac{5 e^{-\Phi} M J E}{r_0^3 f_0^3} (r - r_0)^2 \sin\Theta \\
- \frac{5 e^{-2\Phi} J^2 M}{r_0^3 f_0^6} (r - r_0)^2 (t - t_0) - \frac{3J^2 f_0}{r_0^4 f_0} (r - r_0)^2 \sin^2\Theta + \frac{6 e^{-\Phi} M J E}{r_0^3 f_0^2} (r - r_0)(t - t_0) \sin\Theta \\
- \frac{3 e^{-2\Phi} M J^2}{r_0^3 f_0} (r - r_0)(t - t_0)^2 + \frac{3J^2}{r_0} (t - t_0) \sin^2\Theta \right] + O(\epsilon^{-1}). \quad (C1)$$
We repeat the form at \( t = t_0 \) given in the text, in order to label each term with a subscript for later reference.

\[
\psi^{\text{e-L}}_0 = \left[ \frac{3mE^2 r_0^2 \sin^2 \Theta}{f_0^2} \right]_{1} + \left[ -\frac{3mJ^2 e^{-2\Phi} (r - r_0)^2}{\rho^5} \right]_{2} + \left[ -\frac{3mEJ e^{-\Phi} (r - r_0) \sin \Theta}{f_0^2} \right]_{3},
\]

\[
\psi^{\text{e-SL}}_0 = \left[ \frac{15mJ^2 r_0 M e^{-2\Phi} (r - r_0)^3}{2f_0^4 r_0^4} \right]_{1} + \left[ \frac{15mJ M e^{-\Phi} \sin \Theta (r - r_0)^4}{f_0^4 r_0^4} \right]_{2}
+ \left[ \frac{15m e^{-2\Phi} J^2 (\sin \Phi + f^2) (r - r_0)^2 \sin^2 \Theta}{r_0 f_0^2} \right]_{3}
+ \left[ \frac{15m e^{-\Phi} J r_0 E (r - r_0)^2 \sin \Theta}{f_0^2} \right]_{4}
+ \left[ \frac{15m e^{-\Phi} M J E (r - r_0)^2 \sin \Theta}{r_0^2 f_0^3} \right]_{5}
+ \left[ \frac{9 m (e^{-2\Phi} J^2 M) (r - r_0)^2}{f_0^2 r_0 \rho^5} \right]_{6}
+ \left[ \frac{3 m e^{i \Phi} J r_0 E \sin^3 \Theta}{f_0} \right]_{7}
+ \left[ \frac{15 m e^{-i \Phi} M J E (r - r_0)^2 \sin \Theta}{r_0^2 f_0^3} \right]_{8}
+ \left[ \frac{9 m J^2 (r - r_0) \sin^2 \Theta}{r_0 f_0 \rho^5} \right]_{9}.
\]
(C3)

The axisymmetric part of each of these terms is to be written as a sum over \( 2Y_0(\Theta, 0) \) at \( r = r_0 \). Each term in the singular field involves the leading part of \( \rho^2 \), namely

\[
\tilde{\rho}^2 := A(r - r_0)^2 + B(1 - \cos \Theta) = \rho^{(2)}_{t=t_0} + O(\Theta^4),
\]
(C4)

where

\[
A = \frac{r_0}{r_0 - 2M}, \quad B = 2r_0^2 - 2M \chi(\Phi), \quad \chi(\Phi) = 1 - \frac{M \sin^2 \Phi}{r_0 - 2M}.
\]
(C5)

We follow the notation of DMW, writing

\[
\tilde{\rho}^2 := B(\delta^2 + 1 - \cos \Theta), \quad \delta^2 := A(r - r_0)^2 / B.
\]
(C6)

Then the leading term in \( \psi_0^{\text{e}} \) is given by

\[
\psi_0^{\text{e-L}} = \frac{-3mE^2 r_0^2}{f_0^2 B^{5/2}} \sin^2 \frac{\Theta}{\delta^2 + 1 - \cos \Theta}^{5/2} - \frac{3mJ^2 e^{-2\Phi}}{f_0^2 r_0^4 AB^{3/2}} \frac{\delta^2}{(\delta^2 + 1 - \cos \Theta)^{5/2}} - \frac{3mEJ e^{-\Phi}}{f_0^2 B^2 \sqrt{A}} \frac{\delta \sin \Theta}{(\delta^2 + 1 - \cos \Theta)^{5/2}}.
\]
(C7)

The axisymmetric part of the above expression is achieved by angle averaging over \( \Phi \). Substituting the values of \( \lambda, B, E, J \) and \( J \) in the above expression, we obtain

\[
\langle \psi^{\text{e-L}}_0 \rangle_{t_0} (\Theta) = \frac{1}{2^{5/2}} \int_0^{2\pi} f(\Phi) d\Phi.
\]
(C8)

where \( \langle f(\Phi) \rangle = (2\pi)^{-1} \int_0^{2\pi} f(\Phi) d\Phi \).

We start with a form of generating function of the Legendre polynomials given by

\[
\frac{1}{(e^T + e^{-T} - 2u)^{1/2}} = \sum_{\ell} e^{-(\ell + 1/2)|T|} P_\ell(u), \quad T \neq 0,
\]
(C9)

and set \( u = \cos \Theta, |T| = \sqrt{2}\delta \). In the limit \( T \rightarrow 0^{\pm} \), the sum does not converge, but it is well-defined as a distribution:

\[
\lim_{T \rightarrow 0^{\pm}} \frac{1}{(e^T + e^{-T} - 2u)^{1/2}} = \sum_{\ell} P_\ell(u),
\]
(C10)
where the symbol \( \overset{=}{\text{means equality of both sides as distributions on the sphere. In particular, the regularization proceeds by imposing a cutoff } \ell_{\text{max}} \text{ on the singular field and on the retarded field; the projection } \mathcal{P} \text{ of the distribution } (C9) \text{ onto the subspace } \ell \leq \ell_{\text{max}} \text{ is the smooth function}
\[
\lim_{T \to 0} \mathcal{P} \frac{1}{(e^T + e^{-T} - 2u)^{1/2}} = \sum_{\ell=0}^{\ell_{\text{max}}} P_\ell(u), \tag{C11}
\]
and the subsequent functions obtained by taking derivatives can be regarded as projections of the corresponding derivatives of the distribution \( (C10) \). In particular, taking successive derivatives with respect to \( T \) gives the relation (Eq. (D11) of DMW)
\[
\lim_{T \to 0} \frac{1}{(e^T + e^{-T} - 2u)^{k+1/2}} \equiv \sum_{\ell} \frac{(2\ell + 1)}{2(2k - 1)T^{2k-1}} P_\ell(u). \tag{C12}
\]
Consider now the expression \( (C7) \) for the leading term in \( \psi_0^2 \). The first term is proportional to \( \frac{\sin^2 \Theta}{(\ell+1-\cos \Theta)^{1/2}} \), where \( \delta \) is proportional to \( T \). We express this term as a sum of Legendre polynomials by differentiating \( (C9) \) twice with respect to \( u \) to obtain
\[
\frac{1}{(e^T + e^{-T} - 2u)^{5/2}} = \frac{1}{3} \sum_{l=0}^{\infty} e^{-(l+1/2)T} P''_l(u). \tag{C13}
\]

The second term is proportional to \( \frac{\delta^2}{(e^T + e^{-T} - 2u)^{1/2}} \). The fact that it is \( O(e^{-3}) \) suggests that it can be written as a linear combination of derivatives of \( \frac{T^2}{(e^T + e^{-T} - 2u)^{5/2}} = \alpha \partial_u \frac{1}{(e^T + e^{-T} - 2u)^{1/2}} + \beta \partial_u^2 \frac{1}{(e^T + e^{-T} - 2u)^{1/2}}, \tag{C14}
\]
and solving for \( \alpha \) and \( \beta \) gives \( \alpha = \beta = \frac{1}{2+u} \). Then
\[
\frac{T^2}{(e^T + e^{-T} - 2u)^{5/2}} = \frac{1}{2+u} \sum_{\ell} e^{-(l+1/2)T} \left[ P'_\ell(u) + \left( \ell + \frac{1}{2} \right)^2 P_\ell(u) \right], \tag{C15}
\]
or
\[
\lim_{\delta \to 0} \frac{\delta^2}{(\delta^2 + 1 - \cos \Theta)^{5/2}} \overset{=}{\text{sum}} \frac{2^{3/2}}{3} \sum_{\ell} \left[ P'_\ell + \left( \ell + \frac{1}{2} \right)^2 P_\ell \right]. \tag{C16}
\]
The axisymmetric part of the third term (its angle average over \( \Phi \)) vanishes.

We next use the same techniques to express the subleading terms in \( \psi_0^2 \) as power series in \( P_\ell \) with coefficients polynomial in \( \ell \). The axisymmetric parts of the second, fourth, seventh and eighth terms vanish. In each of the remaining terms, Eq. \( (C12) \) is used to expand an expression involving a power of \( \delta^2 + 1 - \cos \Theta \), and in each case we find that the term is proportional as a distribution to the sum \( \sum_{l=0}^{\infty} (l+1/2)P_l(\cos \theta) \); that sum is a \( \delta \)-function with support at \( \Theta = 0 \) \( (48) \):
\[
\sum_{l=0}^{\infty} (l+1/2)P_l(\cos \theta) = \delta(1 - \cos \theta). \tag{C17}
\]

Because \( 2Y_{l,0} \) vanishes at \( \Theta = 0 \), the expansion of the singular field in terms of \( 2Y_{l,0} \) has no subleading contribution.

We verify the claimed form of each of the subleading terms in Eq. \( (C3) \) as follows: The first term is proportional to \( \delta^5/(\delta^2 + 1 - \cos \Theta)^{7/2} \), and Eq. \( (C12) \) gives
\[
\lim_{\delta \to 0^+} \frac{\delta^5}{(\delta^2 + 1 - \cos \Theta)^{7/2}} = \frac{2}{5} \sum_{\ell} \left( \ell + \frac{1}{2} \right) P_\ell(\cos \Theta). \tag{C18}
\]
The third term is proportional to
\[
\frac{\delta^3 \sin^2 \Theta}{(\delta^2 + 1 - \cos \Theta)^{7/2}} = \frac{2\delta^3}{(\delta^2 + 1 - \cos \Theta)^{7/2}} - \frac{2\delta^5}{(\delta^2 + 1 - \cos \Theta)^{7/2}} + O(\epsilon^{-1}),
\]
and Eq. (C12) gives
\[
\lim_{\delta \to 0} \frac{\delta^3 \sin^2 \Theta}{(\delta^2 + 1 - \cos \Theta)^{7/2}} = \frac{8}{15} \sum_\ell \left( \ell + \frac{1}{2} \right) P_\ell(\cos \Theta).
\]

The fifth term is proportional to
\[
\frac{\delta \sin^4 \Theta}{(\delta^2 + 1 - \cos \Theta)^{5/2}} = \frac{4\delta}{(\delta^2 + 1 - \cos \Theta)^{5/2}} - \frac{4\delta^5}{(\delta^2 + 1 - \cos \Theta)^{5/2}} - \frac{4\delta^3 \sin^2 \Theta}{(\delta^2 + 1 - \cos \Theta)^{5/2}},
\]
and Eq. (C12) gives
\[
\lim_{\delta \to 0} \frac{\delta \sin^4 \Theta}{(\delta^2 + 1 - \cos \Theta)^{5/2}} = \frac{64}{15} \sum_\ell \left( \ell + \frac{1}{2} \right) P_\ell(\cos \Theta).
\]

The sixth term is proportional to
\[
\lim_{\delta \to 0} \frac{\delta^3}{(\delta^2 + 1 - \cos \Theta)^{5/2}} = \frac{2}{3} \sum_\ell \left( \ell + \frac{1}{2} \right) P_\ell(\cos \Theta).
\]

The ninth term is proportional to
\[
\frac{\delta^2 \sin^2 \Theta}{(\delta^2 + 1 - \cos \Theta)^{5/2}} [1 - \cos \Theta + O(\Theta^4)] = \frac{2\delta}{(\delta^2 + 1 - \cos \Theta)^{5/2}} - \frac{2\delta^3}{(\delta^2 + 1 - \cos \Theta)^{5/2}} + O(\epsilon^{-1});
\]
again using Eq. (C12) we have
\[
\lim_{\delta \to 0} \frac{\delta^2 \sin^2 \Theta}{(\delta^2 + 1 - \cos \Theta)^{5/2}} = \frac{8}{3} \sum_\ell \left( \ell + \frac{1}{2} \right) P_\ell(\cos \Theta).
\]

The axisymmetric parts of the second, fourth, seventh and the eighth terms vanish.

We have obtained the leading terms as series of Legendre polynomials, and we now convert them to series involving \(2Y_{\ell 0}\). We begin with \(P^{(2)}_\ell\). We use the relation between \(Y_{\ell m}\) and \(D^\ell_{s m}(\Theta, \Phi, 0)\), the representation matrix for the rotation group [44] to write \(2Y_{\ell 0} = Y_{\ell 2}\). We next convert the series in terms of Legendre polynomials to series involving \(2Y_{\ell 0}\). Then, using \(P^{(m)}_\ell = \left[ \frac{4\pi}{(2\ell + 1)(\ell - m)!} \right]^{1/2} Y_{\ell m}\), we have
\[
P^{(2)}_\ell(\cos \Theta) = \sum_\ell^\ell C_{\ell \ell'} 2Y_{\ell'0}(\Theta, 0),
\]
where
\[
C_{\ell \ell'} = \left[ \frac{4\pi(\ell - 1)!(\ell + 1)(\ell + 2)}{(2\ell + 1)} \right]^{1/2} \delta_{\ell \ell'}.
\]

Regarding \(P_\ell(\cos \Theta)\) and \(P'_\ell(\cos \Theta)\) as elements of \(L_2(S^2)\), we find
\[
P_\ell(\cos \Theta) = \sum_{n=2}^\infty A_{\ell n} P_n^{(2)}(\cos \Theta),
\]
\[
P'_\ell(\cos \Theta) = \sum_{n=2}^\infty B_{\ell n} P_n^{(2)}(\cos \Theta),
\]

\[\text{5 That is, the coefficients } A_{\ell n}, \text{ and } B_{\ell n} \text{ are obtained from the inner products of } P_n^{(2)} \text{ with } P_\ell(\cos \Theta) \text{ and } P'_\ell(\cos \Theta), \text{ and the relations are implied by } L_2 \text{ completeness of } 2Y_{\ell m}.\]
where

\[ A_{\ell n} := \frac{(2n + 1)(n - 2)!}{2(n + 2)!} \left[ -\frac{2\ell(\ell - 1)}{(2\ell + 1)} \delta_{\ell, m} + 4d_{\ell, n} \right], \quad \text{(C30)} \]

\[ B_{\ell n} := \frac{(2n + 1)(n - 2)!}{2(n + 2)!} 2\ell(\ell + 1)d_{\ell-1, n}, \quad \text{(C31)} \]

with

\[ d_{\ell, n} = \begin{cases} 1, & \text{if } n - \ell \text{ is a positive even integer}, \\ 0, & \text{otherwise}. \end{cases} \]

The form of these coefficients was found by numerical experiment for \( \ell, n < 10 \) and then checked as rational numbers for larger values.

The second term in the expression for the leading part of the singular field is proportional to the right side of Eq. (C16); and we now show for each \( \ell \) that the bracketed expression vanishes as an element of \( L_2 \). We have

\[ P'_\ell + \left( \ell + \frac{1}{2} \right)^2 P_\ell = \sum_{\ell'=0}^{\infty} \left[ B_{\ell \ell'} + \left( \ell + \frac{1}{2} \right)^2 A_{\ell \ell'} \right] C_{\ell \ell'} 2 Y_{\ell 0}. \quad \text{(C32)} \]

For \( \ell' \) even, the sum over \( \ell \) in Eq. (C16) is proportional to

\[ \frac{2(\ell' + 2)!}{(2\ell' + 1)(\ell' - 2)!} \sum_{\ell=0}^{\infty} \left[ A_{\ell \ell'} + \left( \ell + \frac{1}{2} \right)^2 B_{\ell \ell'} \right] = \sum_{\ell=0}^{\infty} -\left( \ell + \frac{1}{2} \right)\ell(\ell - 1)\delta_{\ell, \ell'} + \sum_{\ell=0, \text{even}}^{\ell'-2} 4 \left( \ell + \frac{1}{2} \right)^2 + \sum_{\ell=1, \text{odd}}^{\ell'-1} 2\ell(\ell + 1) \quad \text{(C33)} \]

The second and third sums are given by

\[ \sum_{\ell=0, \text{even}}^{\ell'-2} (2\ell + 1)^2 = \sum_{p=0}^{\ell'/2-1} (4p + 1)^2 = \frac{1}{6} \ell'(4\ell'^2 - 6\ell' - 1), \quad \text{(C34)} \]

and

\[ \sum_{\ell=1, \text{odd}}^{\ell'-1} 2\ell(\ell + 1) = \sum_{p=0}^{\ell'/2-1} [2(2p + 1)^2 + 2(2p + 1)] = \frac{1}{6} \ell'(\ell' + 2)(2\ell' - 1), \quad \text{(C35)} \]

and the identity

\[ \sum_{\ell=0}^{\infty} \left[ A_{\ell \ell'} + \left( \ell + \frac{1}{2} \right)^2 B_{\ell \ell'} \right] = 0, \quad \text{for } \ell' \text{ even}, \quad \text{(C36)} \]

follows. A similar manipulation yields the same identity for \( \ell' \) odd. We conclude that the projection of the distribution

\[ \sum_{\ell=0}^{\ell_{\max}} \left[ P'_\ell + \left( \ell + \frac{1}{2} \right)^2 P_\ell \right] \]

along \( 2 Y_{\ell 0} \) vanishes.

Then, of the three terms in Eq. (C7) for \( \langle \psi_0^s \rangle \), only the term proportional to \( \sin^2 \Theta / \tilde{\rho}^{5/2} \) is nonzero when written as a sum over \( 2 Y_{\ell 0} \); and the axisymmetric part of \( \psi_0^s \) has, to subleading order, the form

\[ \langle \psi_0^s \rangle_{\rho_0} (\Theta) = -\frac{m(r_0 - 3M)^{3/2}}{r_0^3(r_0 - 2M)^{5/2}} \left( \frac{1}{\chi^{5/2}} \right) \sum_{\ell=2}^{\infty} \sqrt{\frac{4\pi(\ell + 2)!}{(\ell - 2)!}} \frac{1}{(2\ell + 1)} 2 Y_{\ell 0}(\Theta, 0). \quad \text{(C37)} \]
Appendix D: Comparison of analytic and numerical computation of self-force

To show how accurately one can recover the leading and the subleading terms in $L$ in the mode sum expression for $a^r$ by numerically matching a power series in $L$ to the values of $a^r_{ret}$, we will present an example where we know $A$ and $B$ analytically: the contribution to the self-force from the part of $h_{11}$ that is axisymmetric about a radial line through a particle in circular orbit in a Schwarzschild background. The contribution to the self-acceleration from $h_{11}$ is

$$\langle a[h_{11}] \rangle = \frac{(1 - 2M/r)^2}{8(1 - 3M/r)} \left[ 2\partial_t - \left(1 - \frac{2M}{r}\right) \partial_r - \frac{6M}{r^2} \right] \langle h_{11} \rangle \quad (D1)$$

To compute the leading and subleading terms analytically requires us to find the leading and subleading terms of the radial and time derivatives of $\psi^S_0$. From Eq. (D1), we have

$$\langle \partial_t \psi^S_0(t = t_0) \rangle = \left\langle \frac{6\mu J^2(r - r_0)e^{-2i\Phi}}{r_0^2 f_0 \rho^5} + \frac{6\mu EJ \sin^3 \Theta e^{-i\Phi}}{f_0 \rho^5} \right. - \frac{15\mu JE \sin \Theta \cos \Phi}{f_0 \rho^7} \left( E^2 r_0^2 \sin^2 \Theta + \frac{J^2}{r_0^2} (r - r_0)^2 e^{-2i\Phi} + 2JE \sin \Theta (r - r_0) e^{-i\Phi} \right) \right\rangle$$

$$= \left\langle \frac{6\mu J^2(r - r_0)e^{-2i\Phi}}{r_0^2 f_0 \rho^5} - \frac{30\mu J^2 E^2 (r - r_0) \sin^2 \Theta \cos \Phi e^{-i\Phi}}{f_0^2 \rho^7} \right\rangle,$$  

(D2)

with only two terms surviving the angle-average over $\Phi$. The first term is proportional to $\frac{\delta}{(\delta^2 + 1 - u)^{5/2}}$ where $\delta \propto (r - r_o)$ and $u = \cos \Theta$. Taking $\partial_a \partial_r$ of Eq. (D2), we have

$$\lim_{\delta \to 0^+} \frac{\delta}{(\delta^2 + 1 - u)^{5/2}} \cong \frac{4}{3} \sum_{\ell} (\ell + 1/2) P^{(2)}_{\ell}(u) = \frac{1}{6} \sum_{\ell} P^{(2)}_{\ell}(u).$$  

(D3)

The second term is proportional to $\frac{\delta \sin^2 \Theta}{(\delta^2 + 1 - \cos \Theta)^{7/2}}$. A straightforward calculation gives us

$$\lim_{\delta \to 0^+} \frac{\delta \sin^2 \Theta}{(\delta^2 + 1 - \cos \Theta)^{7/2}} \cong \frac{8}{15} \sum_{\ell} (\ell + 1/2) P^{(2)}_{\ell}(\cos \Theta)$$  

(D4)

Therefore, we get

$$\langle \partial_r \psi^S_{0-L} \rangle_{t_0, r \to r_0} = \left( \frac{\mu M (r_0 - 3M)}{4} \langle \frac{e^{-2i\Phi}}{r_0^2} \rangle + \frac{\mu M (r_0 - 3M)}{r_0^{7/2}} \langle \frac{\sin^2 \Theta}{r_0^{5/2}} \rangle \right)$$

$$\times \sum_{\ell} \sqrt{\frac{4\pi (2\ell + 1)(\ell + 2)!}{(\ell - 2)!}} Y_{\ell, 0}(\Theta)$$

(D5)

A similar calculation for the radial derivative of the leading singular field gives us

$$\langle \partial_r \psi^S_{0-L} \rangle_{t_0, r \to r_0} = \left( \frac{\mu M (r_0 - 3M)^2}{8} \langle \frac{e^{-2i\Phi}}{r_0^{5/2}} \rangle \right)$$

$$\times \sum_{\ell} \sqrt{\frac{4\pi (2\ell + 1)(\ell + 2)!}{(\ell - 2)!}} Y_{\ell, 0}(\Theta)$$

(D6)

Here the superscript S-L refers to the singular leading. The dot and prime represent time and radial derivatives, respectively.

To find the $t$ and $r$ derivatives of the subleading terms, we need to assume the following result, directly verified only for small $n$:

$$\lim_{\delta \to 0^+} \frac{\delta^{2n}}{(\delta^2 + 1 - \cos \Theta)^{n+3/2}} = 0, \quad n \geq 1.$$  

(D7)
Using Eqs. (D2) and (D3), and (D7), we have

\[ \langle \partial_t \psi_{0}^{\text{S-SL}} \rangle_{t_0, r \to r_0} = \left( \frac{5 \mu M (r_0 - 3M)^{3/2}}{r_0^4 (r_0 - 2M)^{5/2}} \left( \frac{\cos^2 \Phi}{\chi^{7/2}} \right) - \frac{3 \mu M (r_0 - 3M)^{3/2}}{r_0^4 (r_0 - 2M)^{5/2}} \left( \frac{1}{\chi^{5/2}} \right) \right) \]

\times \sum_{\ell} \sqrt{\frac{4 \pi (\ell + 2)!}{(2\ell + 1)(\ell - 2)!}} Y_{\ell,0}(\Theta)

\[ \langle \partial_t \psi_{0}^{\text{S-SL}} \rangle_{t_0, r \to r_0} = \left( \frac{15 \mu (r_0 - 3M)^{5/2}}{29/2 r_0^7 (r_0 - 2M)^{7/2}} \left( \left( \frac{\sin^2 \Phi}{\chi^{7/2}} \right) + \frac{r_0 - M}{r_0 - 3M} \left( \frac{\cos^2 \Phi}{\chi^{7/2}} \right) \right) + \frac{3 \mu M (r_0 - 3M)^{3/2}}{r_0^4 (r_0 - 2M)^{5/2}} \left( \frac{1}{\chi^{5/2}} \right) \right) \]

\times \sum_{\ell} \sqrt{\frac{4 \pi (\ell + 2)!}{(2\ell + 1)(\ell - 2)!}} Y_{\ell,0}(\Theta)

Here the subscript S-SL refers to singular subleading.

From Eqs. (40), (32) and (D1), we find

\[ \langle a[h_{11}] \rangle = AL + B + O(L^{-2}), \]

with

\[ A = \frac{-5c_1 M + 4(4c_2 M + c_5 (r_0 - 3M))}{8r_0^{5/2} (r_0 - 2M)^{1/2}} \]

\[ B = \frac{(r_0 - 3M)^{1/2}(-20c_3 M + 2c_4 (7M - 2r_0) + 5c_6 (r_0 - 3M))}{2r_0^3 (r_0 - 2M)^{1/2}} \]

where

\[ c_1 = \left( \frac{\cos \Phi}{\chi^2} \right), \quad c_2 = \left( \frac{\cos^2 \Phi}{\chi^2} \right), \quad c_3 = \left( \frac{\sin^2 \Phi}{\chi^2} \right), \quad c_4 = \left( \frac{1}{\chi^2} \right), \quad c_5 = \left( \frac{\sin^2 \Phi}{\chi^{7/2}} \right), \quad c_6 = \left( \frac{\cos^2 \Phi}{\chi^{7/2}} \right) \]

In using Eq. (40) to calculate \( A \) and \( B \), we ignore the term involving \( \partial_t \Psi \propto m \Omega \Psi \), because it smaller than the leading term by three powers of \( \ell \).

We numerically calculate \( a^r_{\text{ret}}[h_{11}] \) by matching it to a series in \( L \) of the form

\[ a^r_{\text{ret}}[h_{11}] = AL + B + \frac{D}{L^2} + \frac{E}{L^4} + \cdots. \]

Shown below is a table of the fractional error in \( A \) and \( B \) when found numerically for the self-force's contribution from the axisymmetric part of \( h_{11} \).

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| $\ell$ | $A_{\text{analytic}}$ | $A_{\text{numerical}}$ | $|\Delta A/A|$ | $B_{\text{analytic}}$ | $B_{\text{numerical}}$ | $|\Delta B/B|$ |
|---|---|---|---|---|---|---|
| 8 | 0.0102526970132717 | 0.010252697099652308 | $1.634 \times 10^{-10}$ | 0.005653763844987715 | 0.005653764469505427 | $1.105 \times 10^{-7}$ |
| 10 | 0.00610420354814939 | 0.00610420354275933 | $8.330 \times 10^{-11}$ | 0.003815279709214152 | 0.003815279708107976 | $5.213 \times 10^{-8}$ |
| 15 | 0.002550224308053105 | 0.002550224308053105 | $2.860 \times 10^{-11}$ | 0.001847266656645128 | 0.001847266656645128 | $1.403 \times 10^{-8}$ |
| 20 | 0.00136176969970571 | 0.00136176969970571 | $1.332 \times 10^{-11}$ | 0.001087734862490035 | 0.001087734862490035 | $5.892 \times 10^{-9}$ |
| 25 | 0.000551286914190482 | 0.000551286914125439 | $7.606 \times 10^{-12}$ | 0.0007157902702909011 | 0.0007157902702909011 | $3.134 \times 10^{-12}$ |
| 30 | 0.000586691431647096 | 0.000586691431647096 | $3.127 \times 10^{-12}$ | 0.000506413382867952 | 0.000506413382867952 | $1.361 \times 10^{-12}$ |
| 35 | 0.000427439057333508 | 0.000427439057333508 | $1.996 \times 10^{-12}$ | 0.000377039757432567 | 0.000377039757432567 | $8.384 \times 10^{-10}$ |
| 40 | 0.000325251349350586 | 0.000325251349350586 | $1.350 \times 10^{-12}$ | 0.000291563478173744 | 0.000291563478173744 | $5.631 \times 10^{-10}$ |
| 45 | 0.000255782677933768 | 0.000255782677933768 | $9.438 \times 10^{-13}$ | 0.0002321631293434098 | 0.0002321631293434098 | $3.891 \times 10^{-10}$ |
| 50 | 0.000206159202491534 | 0.000206159202491534 | $7.033 \times 10^{-13}$ | 0.000189205218212492 | 0.000189205218212492 | $2.854 \times 10^{-10}$ |
| 55 | 0.000170795176380504 | 0.000170795176380504 | $5.015 \times 10^{-13}$ | 0.0001571744717361175 | 0.0001571744717361175 | $2.045 \times 10^{-10}$ |
| 60 | 0.00014255994920264520 | 0.00014255994920264520 | $3.730 \times 10^{-13}$ | 0.0001326282836118128 | 0.0001326282836118128 | $1.530 \times 10^{-10}$ |
| 65 | 0.000104333547903106 | 0.000104333547903106 | $2.276 \times 10^{-13}$ | 0.0000980883612137512 | 0.0000980883612137512 | $9.291 \times 10^{-11}$ |
| 70 | 0.0000796517646622090 | 0.0000796517646622090 | $1.770 \times 10^{-13}$ | 0.0000754722513438541 | 0.0000754722513438541 | $6.928 \times 10^{-11}$ |

TABLE III: The table compares values of regularization parameters calculated analytically to values obtained numerically by matching the retarded field to a series in $(\ell + 1/2)$; the quantity that is regularized is $(a^{\text{ret}}[h_{11}])$, as described in the text. The first column lists orbital radius in units of Schwarzschild mass; the second and the fifth columns list the analytically computed leading and the subleading regularization parameters A and B; the third and the sixth columns list the numerically obtained values of A and B, and the fourth and the seventh columns list fractional differences between the analytic and numerical values.