On the mass term of the Dirac equation

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Abstract

We consider the generalization of the Dirac equation where the mass term is an arbitrary matrix $M$. A general form of $M$, consistent with the mass shell constraint, is derived and proven to be equivalent to the original Dirac equation.

1 Introduction

The original way [1] in which Dirac obtained relativistic equation for fermions seems to leave certain ambiguities related to the choice of the mass term. This led some authors [2] to discuss the possibility of generalizing the term by considering certain mass matrices $M$ instead of the usual matrix $m \mathbf{1}$. We would like to point out that consistency conditions actually imply that $M$ must be given by $M = me^{(i\alpha-\beta)\gamma^5}$ with $\alpha \in [0, 2\pi]$ and $\beta \in \mathbb{R}$, of which the cases $\beta = 0$ and $\alpha = 0$ were discussed in [2]. The mass term $M$ can be obtained from the Dirac

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equation by an appropriate change of the phases and the norms of the Weyl spinors.

2 General mass term

Consider a non hermitian $x^\mu$ dependent matrix $M$ and assume that the corresponding Dirac equations $D_M \psi = 0$, $D_M = -i\gamma^\mu \partial_\mu + M$ holds. For an arbitrary operator $D$ the consistency conditions $DD_M \psi = 0$ have to be satisfied. Due to the mass shell constraint $p_\mu p^\mu = m^2$, $p_\mu = -i\partial_\mu$, useful conditions will come from operators $D$ which involve the $i\gamma^\mu \partial_\mu$ operator. Let us consider

$$0 = D_M D_M \psi = (m^2 - M^2 - i\gamma^\mu \partial_\mu M) \psi - i[\gamma^\mu, M] \partial_\mu \psi \quad (1)$$

(we use the conventions $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1$, $\gamma^5 = +i\gamma^0\gamma^1\gamma^2\gamma^3$). One can also consider other equations e.g.

$$0 = D_M D_M \psi = D_M D_M \psi = D_M D_M \psi = D_0 D_M \psi \quad (2)$$

however as it turns out they do not give new constraints.

If $M$ is equal to $m1$, equation (1) is trivially satisfied (equations in (2) are either trivial or give the Dirac equation $D_m \psi = 0$). For general $M$ we obtain some nontrivial, first order, differential equations for $\psi$. These equations must reduce to the Dirac equation $D_M \psi = 0$ - otherwise we would obtain an independent equation for fermions. Concentrating on Eqn. (1) we conclude that

$$[\gamma^\mu, M] = A\gamma^\mu, \quad (3)$$

$$m^2 - M^2 - i\gamma^\mu \partial_\mu M = AM \quad (4)$$

for some matrix $A$.

In order to solve (3) and (4) it is useful to multiply Eqn. (3) from the r.h.s. by $\gamma^\mu$ (no sum) which in particular implies the following equations

$$\gamma^i M \gamma^i - \gamma^j M \gamma^j = 0, \quad 1 \leq i < j \leq 3.$$

Using explicit representation for gamma matrices we find that the general solution of the letter is

$$M = a(x) + b(x)\gamma^5, \quad a(x), b(x) \in \mathbb{C} \quad (5)$$
which using (3) gives $A = -2b(x)\gamma^5$ hence (4) gives

$$m^2 = a(x)^2 - b(x)^2 + \gamma^\mu \partial_\mu \left(a(x) + b(x)\gamma^5\right).$$

(6)

The r.h.s. in (6) should be proportional to the unit matrix hence $\partial_\mu a = \partial_\mu b = 0$. Therefore the general solution of (3) and (4) and hence of the constraint (1) is given by

$$M = a + b\gamma^5, \quad a, b \in \mathbb{C},$$

$$m^2 = a^2 - b^2.$$  

(7)

It turns out that (7) also solves other constraint (2). Let us consider the first equation in (2)

$$0 = D_M^\dagger D_M \psi = (m^2 - M^\dagger M - i\gamma^\mu \partial_\mu M)\psi - i(\gamma^\mu M - M^\dagger \gamma^\mu)\partial_\mu \psi.$$  

(8)

The equations following from (8) are

$$\gamma^\mu M - M^\dagger \gamma^\mu = B\gamma^\mu,$$

$$m^2 - M^\dagger M = BM$$

(9)

(10)

for some matrix $B$. Substituting (7) to (9) we find that

$$B = 2i|a|\sin \alpha - 2|b|\cos \beta \gamma^5, \quad \alpha := \text{Arg}(a), \quad \beta := \text{Arg}(b)$$

which substituted to (10) gives two equations

$$m^2 - |a|^2 - |b|^2 = 2|a||b|\sin \alpha - 2|b|\cos \beta,$$

$$-2|a||b|\cos(\alpha - \beta) = 2|a||b|\sin \alpha - 2|a||b|\cos \beta.$$  

(11)

(12)

Equation (11) is actually equivalent to the second equation in (7) while (12) is an identity hence (8) gives no new constraints on $a$ and $b$. The same conclusion holds for the remaining equations in (2).

Using the parametrization for the complex circle in (7)

$$a = m(\cos \alpha \cosh \beta - i \sin \alpha \sinh \beta),$$

$$b = mi(\sin \alpha \cosh \beta + i \cos \alpha \sinh \beta)$$

with $\alpha \in [0, 2\pi]$ and $\beta \in \mathbb{R}$ we can write $M$ in the compact form

$$M = me^{i(\alpha - \beta)\gamma^5}.$$  

(13)
Finally let us observe that this form of $M$ can be obtained from the Dirac equation with $M = m \mathbf{1}$. Noting that in the Weyl representation we have

\[ i\sigma^\mu \partial_\mu \psi_L = me^{-i\alpha + \beta} \psi_R, \]
\[ i\bar{\sigma}^\mu \partial_\mu \psi_R = me^{i\alpha - \beta} \psi_L \]

where $\psi_R, \psi_L$ are Weyl spinors and choosing $\tilde{\psi}_L = e^{i\alpha - \beta} \psi_L, \tilde{\psi}_R = e^{-i\alpha - \beta} \psi_R$ (which could be interpreted as the chiral transformation with the complex angle) the Dirac equation for Weyl spinors $\tilde{\psi}_R, \tilde{\psi}_L$ transforms into the standard form with $M = m$.

It is interesting to note that choosing $M$ not belonging to (13) breaks in general the explicit relativistic invariance of equations. As an example let us consider $M = m \gamma_0$. The condition (1) implies that $i\gamma^j \partial_j \psi = 0$ and hence $(i\partial_0 - m)\psi = 0$ which clearly is not Lorentz invariant. This example is particularly interesting (among other choices e.g. $M = \gamma^i$) as there are no negative energy solutions for this choice i.e. the plane-wave ansatz $\psi = ue^{-ikx}$ for positive energy solutions and $\psi = ve^{ikx}$ for negative energy solutions, implies $k_i = 0, k_0 = m$ for four basis spinors $[u_s]_t = \delta_{st}, s, t = 1, 2, 3, 4$ and no solutions for the $v$ spinor.

3 Conclusions

In this paper we considered generalizations of the Dirac equation where the mass term is replaced by an arbitrary matrix $M$. It follows that a simple consistency condition (1) implies that $M$ must be of the form (13) which in turn can be obtained from the original Dirac equation by a suitable redefinition of the wavefunction. Therefore the choice $M = m \mathbf{1}$ is already general.

References

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