A compactness theorem of \( n \)-harmonic maps

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Abstract. For \( n \geq 3 \), let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth domain and \( N \subset \mathbb{R}^L \) be a compact smooth Riemannian submanifold without boundary. Suppose that \( \{u_n\} \subset W^{1,n}(\Omega,N) \) are weak solutions to the perturbed \( n \)-harmonic map equation (1.2), satisfying (1.3), and \( u_k \to u \) weakly in \( W^{1,n}(\Omega,N) \). Then \( u \) is an \( n \)-harmonic map. In particular, the space of \( n \)-harmonic maps is sequentially compact for the weak-\( W^{1,n} \) topology.

§1 Introduction

For \( n \geq 2 \), let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth domain, and \( N \subset \mathbb{R}^L \) be a compact smooth Riemannian manifold without boundary, isometrically embedded into the euclidean space \( \mathbb{R}^L \) for some \( L \geq 1 \). For \( 2 \leq p \leq n \), the Sobolev space \( W^{1,p}(\Omega,N) \) is defined by

\[
W^{1,p}(\Omega,N) = \{ u = (u^1, \cdots, u^L) \in W^{1,p}(M, \mathbb{R}^L) \mid u(x) \in N \text{ for a.e. } x \in \Omega \}.
\]

The Dirichlet \( p \)-energy functional \( E_p : W^{1,p}(\Omega,N) \to \mathbb{R} \) is defined by

\[
E_p(u) = \int_{\Omega} |\nabla u|^p \, dx = \int_{\Omega} \left( \sum_{\alpha=1}^{n} \left( \frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\alpha} \right) \right)^p \, dx
\]

where \( \langle \cdot, \cdot \rangle \) is the inner product of \( \mathbb{R}^L \).

Recall that a map \( u \in W^{1,p}(\Omega,N) \) is a \( p \)-harmonic map, if \( u \) is a critical point of \( E_p(\cdot) \) on the space \( W^{1,p}(\Omega,N) \), i.e. \( u \) satisfies the \( p \)-harmonic map equation:

\[
-\text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}A(u)(\nabla u, \nabla u)
\]

(1.1)

in the sense of distributions, where \( \text{div} \) is the divergence operator on \( \mathbb{R}^n \) and \( A(\cdot)(\cdot, \cdot) \) is the second fundamental form of \( N \subset \mathbb{R}^L \).

Since the \( p \)-harmonic map equation (1.1) is an (degenerately) elliptic system with critical nonlinearity in the first order derivatives, the analysis of both the regularity problem for \( p \)-harmonic maps and the limiting behavior of weakly convergent sequences of \( p \)-harmonic maps are very interesting and extremely challenging.

In this paper, we are mainly interested in the compactness problem in the weak topology of the space of \( p \)-harmonic maps. More precisely, we are motivated by the following problem.

**Question A.** For \( n \geq 3 \) and \( 2 \leq p \leq n \), is any weak limit \( u \) in \( W^{1,p}(\Omega,N) \) of a sequence of \( p \)-harmonic maps \( \{u_k\} \subset W^{1,p}(\Omega,N) \) a \( p \)-harmonic map?
For $p = n = 2$, the answer to question A is affirmative, which follows from Hélein’s celebrated regularity theorem [H1]: any 2-harmonic map from a Riemannian surface into any compact Riemannian manifold is smooth.

For $n \geq 3$, the answer to question A remains open in general cases, although many people have made efforts to understand it.

We mention some earlier results in the direction. Schoen-Uhlenbeck [SU] (p = 2), Hardt-Lin [HL] and Luckhaus [L] (p \neq 2) have shown that any weak limit $u \in W^{1,p}$ of a sequence of minimizing $p$-harmonic maps is a strong limit and a minimizing $p$-harmonic map. In particular, question A is true for minimizing $p$-harmonic maps.

Without the minimality assumption, it is known that question A holds for target manifolds $N$ with symmetry, such as $N = S^{L-1}$ is the unit sphere in $\mathbb{R}^L$ (cf. Chen [C], Shatah [S], Evans [E2] §5, and Hélein [H2] §2) or $N = G/H$ is a compact Riemannian homogeneous manifold (cf. Toro-Wang [TW]). Here the symmetry guarantees the existence of Killing tangent vector fields on $N$, under which the nonlinearity of the $p$-harmonic map equation (1.1) is the inner product of a gradient and a divergence free vector field and hence belongs to the Hardy space $H^1(\mathbb{R}^n)$, an improved subspace of $L^1(\mathbb{R}^n)$.

For target manifolds $N$ without symmetry, the idea of the use of Coulomb moving frames, originally due to Hélein [H1] and beautifully explained in his book [H2], has played extremely important roles on the study of regularity of stationary 2-harmonic maps into general target manifolds, through the works by Hélein [H1] ($n = 2$) and Bethuel [B2] ($n \geq 3$) (see also Evans [E1]). Roughly speaking, one can make suitable rotations from a smooth moving frames along $u^*TN$ to obtain a harmonic moving frame $\{e_\alpha\}$ (i.e. a minimizer of $\int |\langle de_\alpha, e_\beta \rangle|^2$).

It turns out that the nonlinearity of 2-harmonic map equation via harmonic moving frames enjoys the Jacobian structure partially. Although this is sufficient for the regularity (and hence convergence) of stationary 2-harmonic maps, it is not good enough for compactness of weakly 2-harmonic maps. On the other hand, in the study on existence of wave maps in $\mathbb{R}^{2+1}$, Freire-Müller-Struwe [FMS1,2] have discovered that for the class of wave maps enjoying the energy monotonicity inequality (e.g. smooth wave maps) in $\mathbb{R}^{2+1}$, the concentration compactness method of Lions [L1,2], in combination with the idea of Coulomb moving frames for wave maps and some end-point analytic estimates, can be used to establish the compactness of wave maps in the class.

When considering $p$-harmonic maps into general target manifolds $N$ for $p \neq 2$, one may encounter the difficulty that is what might be the appropriate construction of Coulomb moving frames (e.g. neither minimizers of $\int |\langle de_\alpha, e_\beta \rangle|^p$ seem to fit the eqn. (1.1) well nor minimizers of $\int |
abla u|^{p-2}|\langle de_\alpha, e_\beta \rangle|^2$ seem to have the $L^p$-estimate). However, we observe that, for $p = n$ case, Uhlenbeck’s construction of Coulomb gauges for Yang-Mills fields [U] can be adopted to obtain Coulomb moving frames along $u^*TN$ under the smallness of $E_n(u)$, see §2 below for the detail and also Wang [W2,3] for its applications to biharmonic maps. With such a Coulomb moving frame along $u^*TN$, we are able to modify the analytic techniques by [FMS2] to show
the compactness of a Palais-Smale sequence (e.g. a sequence of weakly convergent $n$-harmonic maps) of the Dirichlet $n$-energy functional $E_n$ on $W^{1,n}(\Omega, N)$.

In order to state our results, we first recall

**Definition.** A sequence of maps $\{u_k\} \subset W^{1,n}(\Omega, N)$ is a Palais-Smale sequence for the Dirichlet $n$-energy functional $E_n$ on $W^{1,n}(\Omega, N)$, if the following two conditions hold: (a) $u_k \to u$ weakly in $W^{1,n}(\Omega, N)$, and (b) $E_n'(u_k) \to 0$ in $(W^{1,n}(\Omega, N))^*$. Here $(W^{1,n}(\Omega, N))^*$ is the dual of $W^{1,n}(\Omega, N)$.

Note that (b) is equivalent to that $u_k$ satisfies the perturbed $n$-harmonic map equation

$$-\text{div}(|\nabla u_k|^{n-2}\nabla u_k) = |\nabla u_k|^{n-2}A(u_k)(\nabla u_k, \nabla u_k) + \Phi_k,$$

in the sense of distributions, and satisfies

$$\lim_{k \to \infty} \|\Phi_k\|_{(W^{1,n}(\Omega, N))^*} = 0. \quad (1.3)$$

The question is whether any weak limit $u$ of a Palais-Smale sequence is an $n$-harmonic map. This is highly nontrivial, since $E_n$ is conformally invariant, i.e. $E_n(u) = E_n(u \circ \Psi)$ for any $C^1$-conformal transformation $\Psi : \Omega \to \Omega$, and the conformal group is non-compact and hence $E_n$ doesn’t satisfy the Palais-Smale condition (cf. [SaU]). Our main result is

**Theorem B.** For $n \geq 3$, assume that $\{u_k\} \subset W^{1,n}(\Omega, N)$ satisfy the equation (1.2), (1.3), and converge weakly to $u$ in $W^{1,n}(\Omega, N)$, then $u \in W^{1,n}(\Omega, N)$ is an $n$-harmonic map.

We would like to remark that for $n = 2$, theorem B has first been proven by Bethuel [B1], later by Freire-Müller-Struwe [FMS2], and also by Wang [W1] with a method different from both [B1] and [FMS2]. For $n \geq 3$, Hungerbühler [H] has obtained the existence of global weak solutions to the $n$-harmonic map flow. Theorem B is applicable to the $n$-harmonic map flow by [H] at infinity time.

As a corollary, we confirm that question A is true for $p = n \geq 3$, i.e.

**Corollary C.** For $n \geq 3$, assume that $\{u_k\} \subset W^{1,n}(\Omega, N)$ are a sequence of $n$-harmonic maps converging weakly to $u$ in $W^{1,n}(\Omega, N)$, then $u$ is an $n$-harmonic map.

The paper is written as follows. In §2, we outline the construction of Coulomb moving frames. In §3, we first recall $H^1(\mathbb{R}^n)$-estimate for functions with Jacobian structure by [CLMS], the duality between $H^1(\mathbb{R}^n)$ and BMO($\mathbb{R}^n$) by [FS], and then give a proof of theorem B.

In this paper, we will use the following notations. For a ball $B = B_r(x) \subset \mathbb{R}^n$, denote $\alpha B = B_{\alpha r}(x)$ for any $\alpha > 0$. For $1 \leq i \leq n$, denote $\wedge^i(\mathbb{R}^n)$ as the $i^{th}$ wedge product of $\mathbb{R}^n$, $C^\infty(\mathbb{R}^n, \wedge^i(\mathbb{R}^n)$ as the space of smooth $i^{th}$ forms on $\mathbb{R}^n$, and $W^{m,p}(\mathbb{R}^n, \wedge^i(\mathbb{R}^n)$ as the space of $i^{th}$ forms on $\mathbb{R}^n$ with $W^{m,p}(\mathbb{R}^n)$ coefficients, for nonnegative integers $m$ and $1 < p < \infty$. Denote by $\mathcal{D}'(\Omega)$ the dual of $C_0^\infty(\Omega)$. Denote $d$ as the exterior differentiation operator on $\mathbb{R}^n$ and $\delta$ as the adjoint operator of $d$. 

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§2 The construction of Coulomb moving frames

This section is devoted to the construction of Coulomb moving frames along \( u^*TN \), under the smallness condition on \( E_n(u) \).

First recall that for any open set \( U \subset \mathbb{R}^n \) and \( u \in W^{1,n}(U, N) \), denote \( u^*TN \) as the pull-back bundle of \( TN \) by \( u \) over \( U \). Denote \( l \) as the dimension of \( N \), we call \( \{ e_\alpha \}_{\alpha=1}^l \) a moving frame along \( u^*TN \), if \( \{ e_\alpha(x) \}_{\alpha=1}^l \) forms an orthonormal base of \( T_{u(x)}N \), the tangent space of \( N \) at the point \( u(x) \), for a.e. \( x \in U \).

Now we have the perturbed \( n \)-harmonic map equation via a moving frame.

**Lemma 2.1.** For \( n \geq 3 \), let \( u \in W^{1,n}(\Omega, N) \) be a weak solution to the perturbed \( n \)-harmonic map equation:

\[
-\text{div}(|\nabla u|^{n-2}\nabla u) = |\nabla u|^{n-2}A(u)(\nabla u, \nabla u) + \Phi, \quad \Phi \in (W^{1,n}(\Omega, N))^*.
\]

(2.1)

Suppose that \( \{ e_\alpha \}_{\alpha=1}^l \) is a moving frame along \( u^*TN \). Then we have, for \( 1 \leq \alpha \leq l \),

\[
-\text{div}(|\nabla u|^{n-2}\nabla u, e_\alpha) = \sum_{\beta=1}^l \langle |\nabla u|^{n-2}\nabla u, e_\beta \rangle \langle \nabla e_\alpha, e_\beta \rangle + \langle \Phi, e_\alpha \rangle
\]

(2.2)

in the sense of distributions.

**Proof.** Note that for any \( 1 \leq \alpha \leq l \) and a.e. \( x \in \Omega \), we have

\[
\langle e_\alpha(x), A(u(x))(\nabla u(x), \nabla u(x)) \rangle = 0
\]

for \( e_\alpha(x) \in T_{u(x)}N \) and \( A(u(x))(\nabla u(x), \nabla u(x)) \perp T_{u(x)}N \). Then straightforward calculations deduce (2.2) from (2.1).

Now we establish a Coulomb moving frame along \( u^*TN \), with the desired estimates on its connection form. The construction is inspired by an earlier result by the author in the context of biharmonic maps (cf. Wang [W2,3]) and Uhlenbeck’s Coulomb gauge construction for Yang-Mills fields [U].

**Proposition 2.2.** For \( n \geq 3 \) and any ball \( B \subset \mathbb{R}^n \), there exists an \( \epsilon_0 > 0 \) such that if \( u \in W^{1,n}(2B, N) \) satisfies

\[
\|\nabla u\|_{L^n(2B)} \leq \epsilon_0
\]

(2.3)

then there exists a Coulomb moving frame \( \{ e_\alpha \}_{\alpha=1}^l \) along \( u^*TN \) in \( W^{1,n}(B, \mathbb{R}^L) \) such that its connection form \( A = (\langle de_\alpha, e_\beta \rangle) \) satisfies

\[
\delta A = 0 \quad \text{in } B; \quad x \cdot A = 0 \quad \text{on } \partial B
\]

(2.4)

and

\[
\|A\|_{L^n(B)} + \|\nabla A\|_{L^\infty(B)} \leq C\|\nabla u\|_{L^n(2B)}^2.
\]

(2.5)
Proof. Let us first assume $u \in C^\infty(B, N)$. Then $u^*TN|_B$ is a smooth vector bundle over the contractible manifold $B$. Hence $u^*TN|_B$ is trivial and there exists a smooth moving frame $\{\bar{e}_\alpha\}_{\alpha=1}^l$ along $u^*TN$ on $B$. Let $\mathbf{G}$ denote the gauge transformation group of $u^*TN$ consisting of maps $R : B \to SO(l)$. For any $R \in \mathbf{G} \cap W^{1,n}(B, SO(l))$, we rotate $\{\bar{e}_\alpha\}_{\alpha=1}^l$ to get another moving frame $\{e_\alpha = \sum_{\beta=1}^l R^{\alpha\beta} \bar{e}_\beta\}_{\alpha=1}^l$. Then we have

$$de_\alpha = \sum_{\beta=1}^l A^{\alpha\beta} e_\beta, \ 1 \leq \alpha \leq l,$$

where $(A^{\alpha\beta}) = (\langle de_\alpha, e_\beta \rangle)$ is the (matrix-valued) connection form of $u^*TN$.

Let $D$ denote the pull-back covariant derivatives on $u^*TN$. Then the curvature equation of $u^*TN$ is given by: for $1 \leq p, q \leq n$,

$$D_p D_q e_\alpha - D_q D_p e_\alpha = \sum_{\beta=1}^l (D_p (A^{\alpha\beta}_q e_\beta) - D_q (A^{\alpha\beta}_p e_\beta))$$

$$= \sum_{\beta=1}^l \{\partial_p (A^{\alpha\beta}_q) - \partial_q (A^{\alpha\beta}_p) + \sum_{\gamma=1}^l (A^{\alpha\gamma}_q A^{\gamma\beta}_p - A^{\alpha\gamma}_p A^{\gamma\beta}_q) e_\beta\}$$

$$= \sum_{\beta=1}^l F^{\alpha\beta}_{pq} e_\beta$$

(2.6)

or for short,

$$\partial_p A_q - \partial_q A_p + [A_p, A_q] = F_{pq} = R^N (\partial_p u, \partial_q u)$$

(2.7)

where $A_p = (A^{\alpha\beta}_p) = (\langle D \frac{\partial}{\partial x_p} e_\alpha, e_\beta \rangle)$, and $R^N$ is the curvature tensor of $TN$. Here we have used the formula

$$D_p D_q e_\alpha - D_q D_p e_\alpha$$

$$= \sum_{\beta, \delta=1}^l \langle \frac{\partial u}{\partial x_p}, e_\beta \rangle \langle \frac{\partial u}{\partial x_q}, e_\delta \rangle u^* (D^{N^N}_{e_\beta} D^N_{e_\delta} e_\alpha - D^N_{e_\delta} D^{N^N}_{e_\beta} e_\alpha - D^N_{[e_\beta, e_\delta]} e_\alpha)$$

$$= \sum_{\beta, \delta=1}^l \langle \frac{\partial u}{\partial x_p}, e_\beta \rangle \langle \frac{\partial u}{\partial x_q}, e_\delta \rangle u^* (R^N_{e_\beta, e_\delta} (e_\alpha))$$

(2.8)

where $D^N$ is the Levi-Civita connection on $TN$.

For any $R \in \mathbf{G} \cap W^{1,n}(B, SO(l))$, we know that the connection form $\bar{A} = (\langle d\bar{e}_\alpha, \bar{e}_\beta \rangle)$ of $\{\bar{e}_\alpha\}_{\alpha=1}^l$ and the connection form $A = (\langle de_\alpha, e_\beta \rangle)$ of $\{e_\alpha = \sum_{\beta=1}^l R^{\alpha\beta} \bar{e}_\beta\}_{\alpha=1}^l$ is related by

$$A = R^{-1} dR + R^{-1} \bar{A} R.$$  

(2.9)
We also have the curvature \( |F(\tilde{A})|(x) = |F(A)|(x) \) for a.e. \( x \in B \). Therefore the \( L^\infty \)-norm of curvature
\[
\int_B |F(A)|^\infty dx = \int_B |F(\tilde{A})|^\infty dx
\]
is invariant under gauge transformations. Moreover, (2.7) implies that, for a.e. \( x \in B \),
\[
|F(\tilde{A})|(x) = |F(A)|(x) \leq \|\mathcal{R}^N\|_{L^\infty} |\nabla u|^2(x) \leq C_N |\nabla u|^2(x). \tag{2.10}
\]
This implies
\[
\int_B |F(A)|^\infty dx \leq C \int_B |\nabla u|^n dx. \tag{2.11}
\]

Now we use the condition (2.3) to approximate \( u \in W^{1,n}(B, N) \) by \( C^\infty(B, N) \) as follows. Let \( \phi : \mathbb{R}^n \to \mathbb{R} \) be a nonnegative, smooth radial mollifying function such that support \( (\phi) \subset B_1 \) and \( \int_{\mathbb{R}^n} \phi = 1 \). For \( 0 < \epsilon < 1 \), let \( \phi^\epsilon(x) = \epsilon^{-n} \phi(\frac{x}{\epsilon}) \) for \( x \in \mathbb{R}^n \), and define
\[
u^\epsilon(x) = \int_{\mathbb{R}^n} \phi^\epsilon(x - y)u(y) dy = \int_{\mathbb{R}^n} \phi(y)u(x - \epsilon y) dy, \quad \forall x \in B.
\]
For any \( \epsilon \in (0, \frac{1}{2}) \) and \( x \in B \), applying the modified Poincaré inequality to \( u_{x,\epsilon}(y) \equiv u(x - \epsilon y) : B \to \mathbb{R}^L \), we have
\[
\int_B |\nu^\epsilon(x) - u_{x,\epsilon}|^n dy \leq C \int_B |\nabla u_{x,\epsilon}|^n dy = C \int_{B_{\epsilon}(x)} |\nabla u|^n dy \leq C \epsilon_0^n. \tag{2.12}
\]
Therefore we have, for any \( \epsilon \in (0, \frac{1}{2}) \),
\[
\max_{x \in B} \text{dist}(u^\epsilon(x), N) \leq C \epsilon_0
\]
so that \( u^\epsilon(B) \subset N_{C \epsilon_0} \). Since the nearest point projection map \( \Pi : N_{C \epsilon_0} \to N \) is smooth for sufficiently small \( \epsilon_0 > 0 \), we have \( u^\epsilon = \Pi \circ u^\epsilon \in C^\infty(B, N) \), \( u^\epsilon \in C^\infty(B, N) \), and \( u^\epsilon \to u \) strongly in \( W^{1,n}(B, N) \) as \( \epsilon \to 0 \), and
\[
\sup_{0 < \epsilon < \frac{1}{2}} \|\nabla u^\epsilon\|_{W^{1,n}(B)} \leq C \|\nabla u\|_{W^{1,n}(2B)} \leq C \epsilon_0. \tag{2.13}
\]
For \( \epsilon \in (0, \frac{1}{2}) \), since \( u^\epsilon T N|_B \) are trivial, there exist smooth moving frames \( \{e_\alpha^\epsilon\}_{\alpha=1}^n \) along \( u^\epsilon T N \) over \( B \). Moreover (2.13) and (2.11) imply that the curvature of the connections \( \tilde{A}_\epsilon = (\langle D e_\alpha^\epsilon, e_\beta^\epsilon \rangle) \) satisfies
\[
\|F(\tilde{A}_\epsilon)\|_{L^\infty(B)} \leq \|\nabla u^\epsilon\|_{W^{1,n}(B)}^2 \leq \|\nabla u\|_{W^{1,n}(2B)}^2 \leq C \epsilon_0^2. \tag{2.14}
\]
Since \( \{e_\alpha^\epsilon\}_{\alpha=1}^l \) are smooth frames and their connections \( \tilde{A}_\epsilon \) satisfy (2.14) with sufficiently small \( \epsilon_0 > 0 \), we can apply Uhlenbeck [U] to conclude that there are \( R_\epsilon : B \to \text{SO}(l) \) satisfying
\[ \nabla R_\epsilon \in L^n(B) \text{ and } \nabla^2 R_\epsilon \in L^2(B) \text{ such that the connection form } A_\epsilon \equiv R_\epsilon^{-1} dR_\epsilon + R_\epsilon^{-1} \bar{A}_\epsilon R_\epsilon \text{ of the moving frame } \{e^\epsilon_\alpha(x) \equiv \sum_{\beta=1}^l R^\epsilon_{\alpha \beta} e^\epsilon_\beta \}_{\alpha=1} \text{ satisfy} \]

\[
\delta A_\epsilon = 0 \text{ in } B, \quad x \cdot A_\epsilon = 0, \text{ on } \partial B, \quad \|A_\epsilon\|_{L^n(B)} + \|\nabla A_\epsilon\|_{L^2(B)} \leq C \|F(\bar{A}_\epsilon)\|_{L^2(B)} \leq C \epsilon_0. \quad (2.15)
\]

We now estimate \( \|\nabla e^\epsilon_\alpha\|_{L^n(B)} \) for \( 1 \leq \alpha \leq l \).

For \( y \in N \), let \( P^\perp(y) = y - \nabla \Pi(y) : \mathbb{R}^L \to (T_N y)^\perp \) denote the orthogonal projection from map \( \mathbb{R}^L \) to the normal space \( (T_N y)^\perp \). Then we have

\[
\nabla e^\epsilon_\alpha = \sum_{\beta=1}^l \langle \nabla e^\epsilon_\alpha, e^\epsilon_\beta \rangle e^\epsilon_\beta + P^\perp(u_\epsilon)(\nabla e^\epsilon_\alpha) = \sum_{\beta=1}^l \langle \nabla e^\epsilon_\alpha, e^\epsilon_\beta \rangle e^\epsilon_\beta - A(u_\epsilon)(e^\epsilon_\alpha, \nabla u_\epsilon) \quad (2.17)
\]

where we have used

\[
P^\perp(u_\epsilon)(\nabla e^\epsilon_\alpha) = -\nabla (P^\perp(u_\epsilon))(e^\epsilon_\alpha) = -A(u_\epsilon)(e^\epsilon_\alpha, \nabla u_\epsilon)
\]

for \( P^\perp(u_\epsilon)(e^\epsilon_\alpha) \). Therefore we have

\[
|\nabla e^\epsilon_\alpha|(x) \leq C(|A_\epsilon| + |\nabla u_\epsilon|(x), \text{ for a.e. } x \in B. \quad (2.18)
\]

This, combined with (2.13) and (2.16), yields

\[
\sum_{\alpha=1}^l \|\nabla e^\epsilon_\alpha\|_{L^n(B)} \leq C \|A_\epsilon\|_{L^n(B)} + \|\nabla u_\epsilon\|_{L^n(B)} \leq C \|\nabla u\|_{L^n(B)}. \quad (2.19)
\]

Therefore, after passing to subsequences, we can assume that \( e^\epsilon_\alpha \to e_\alpha \) weakly in \( W^{1,n}(B) \), strongly in \( L^n(B) \), and a.e. in \( B \). Since \( u_\epsilon \to u \) strongly in \( W^{1,n}(B) \), we have that \( \{e^\epsilon_\alpha\}_{\alpha=1} \subset W^{1,n}(B) \), is a moving frame along \( u^*TN \). Moreover, (2.16) implies that \( A_\epsilon \to A \equiv (\langle d\epsilon_\alpha, e_\beta \rangle) \), the connection form of \( \{e^\epsilon_\alpha\}_{\alpha=1} \), weakly in \( W^{1,n}(B) \). Hence, by taking \( \epsilon \) into zero, (2.15) and (2.16) imply that \( A \) satisfies (2.4) and (2.5). The proof of Proposition 2.2 is complete. \[\Box\]

§3 Proof of theorem B

This section is devoted to the proof of theorem B. First we recall some basic facts on the Hardy space \( \mathcal{H}^1(\mathbb{R}^n) \) and the BMO space \( \text{BMO}(\mathbb{R}^n) \).

Recall that \( f \in L^1(\mathbb{R}^n) \) belongs to the Hardy space \( \mathcal{H}^1(\mathbb{R}^n) \) if

\[
f_* := \sup_{\epsilon > 0} |\phi_\epsilon * f| \in L^1(\mathbb{R}^n)
\]
where \( \phi_\varepsilon(x) := \varepsilon^{-n} \phi(\frac{x}{\varepsilon}) \) for a fixed nonnegative \( \phi \in C_0^\infty(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} \phi \, dy = 1 \). Note that \( \mathcal{H}^1(\mathbb{R}^n) \) is a Banach space with the norm

\[
\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} := \|f\|_{L^1(\mathbb{R}^n)} + \|f\|_{L^1(\mathbb{R}^n)}.
\]

An important property of \( f \in \mathcal{H}^1(\mathbb{R}^n) \) is the cancellation identity \( \int_{\mathbb{R}^n} f \, dy = 0 \) (cf. Fefferman-Stein [FS]).

Recall also that \( f \in L^1_{loc}(\mathbb{R}^n) \) belongs to the BMO space \( \text{BMO}(\mathbb{R}^n) \) (cf. John-Nirenberg [JN]), if

\[
\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup\left\{ \frac{1}{|B|} \int_B |f - f_B| \, dy : \text{any ball } B \subset \mathbb{R}^n \right\} < \infty
\]

where \( f_B = \frac{1}{|B|} \int_B f \, dy \) is the average of \( f \) over \( B \). By the Sobolev inequality we have \( W^{1,n}(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n) \) and

\[
\|f\|_{\text{BMO}(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^n(\mathbb{R}^n)}. \tag{3.1}
\]

The celebrated theorem of Fefferman-Stein [FS] says that the dual of \( \mathcal{H}^1(\mathbb{R}^n) \) is \( \text{BMO}(\mathbb{R}^n) \). Moreover

\[
|\int_{\mathbb{R}^n} fg \, dy| \leq C \|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \|g\|_{\text{BMO}(\mathbb{R}^n)}. \tag{3.2}
\]

Now we recall an important result of Coifman-Lions-Meyer-Semmes [CLMS], see also [E1].

**Proposition 3.1** ([CLMS]). For any \( 1 < p < \infty \), denote \( p' = \frac{p}{p-1} \). Let \( f \in W^{1,p}(\mathbb{R}^n) \), \( g \in W^{1,p'}(\mathbb{R}^n, \Lambda^1(\mathbb{R}^n)) \), and \( h \in W^{1,n}(\mathbb{R}^n) \). Then \( df \cdot \delta g \in \mathcal{H}^1(\mathbb{R}^n) \) and

\[
\|df \cdot \delta g\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)} \|\nabla g\|_{L^{p'}(\mathbb{R}^n)}. \tag{3.3}
\]

In particular, we have

\[
|\int_{\mathbb{R}^n} (df \cdot \delta g, h) \, dy| \leq C \|\nabla f\|_{L^p(\mathbb{R}^n)} \|\nabla g\|_{L^{p'}(\mathbb{R}^n)} \|\nabla h\|_{L^n(\mathbb{R}^n)}. \tag{3.4}
\]

We also recall the following pointwise convergence result, which is essentially due to Hardt-Lin-Mou [HLM] (see also [F]).

**Lemma 3.2** ([HLM]). Suppose that \( \{u_k\} \subset W^{1,n}(\Omega, \mathbb{R}^L) \) are weak solutions to

\[
-\text{div}(|\nabla u_k|^{n-2}\nabla u_k) = f_k + \Phi_k, \tag{3.5}
\]

where \( f_k \to 0 \) in \( L^1(\Omega, \mathbb{R}^L) \), and \( \Phi_k \to 0 \) in \( (W^{1,n}(\Omega, \mathbb{R}^L))^* \). Assume that \( u_k \rightharpoonup u \) weakly in \( W^{1,n}(\Omega, \mathbb{R}^L) \). Then, after taking possible subsequences, we have \( \nabla u_k \to \nabla u \) a.e. in \( \Omega \). In particular, \( \nabla u_k \to \nabla u \) strongly in \( L^q(\Omega, \mathbb{R}^L) \) for any \( 1 \leq q < n \).
After these preparations, we are ready to give a proof of theorem B. It turns out the crucial step is to show the following weak compactness under the smallness condition on $E_n$.

Lemma 3.3 ($\epsilon$-weak compactness). For any $n \geq 3$, there exists an $\epsilon_1 > 0$ such that if \{u_k\} $\subset W^{1,n}(2B, N)$ satisfy both the equation (1.2) and the condition (1.3) with $\Omega$ replaced by $2B$, $u_k \to u$ weakly in $W^{1,n}(2B, N)$, and satisfy
\[
\int_{2B} |\nabla u_k|^n \, dx \leq \epsilon_1^n, \quad \forall k \geq 1.
\] (3.6)

Then $u \in W^{1,n}(B, N)$ is an $n$-harmonic map.

Proof. For the convenience, we will write both equation (1.1) and (1.2) by using $d$ and $\delta$ from now on.

Let $\epsilon_1 > 0$ be the same constant as in Proposition 2.2. Then we have that for any $k \geq 1$ there is a Coulomb moving frame $\{e_{\alpha}^k\}_{\alpha=1}^l$ along $u_k^*TN$ such that the connection form $A_k = (\langle de_{\alpha}^k, e_{\beta}^k \rangle)$ satisfies
\[
\delta A_k = 0 \text{ in } B; \quad x \cdot A_k = 0 \text{ on } \partial B
\] (3.7)
and
\[
\|A_k\|_{L^n(B)} + \|\nabla A_k\|_{L^\frac{n}{n-1}(B)} \leq C\|\nabla u_k\|_{L^n(B)}^2.
\] (3.8)

Moreover, similar to (2.19), we have
\[
\max_{\alpha=1}^l \|\nabla e_{\alpha}^k\|_{L^n(B)} \leq C\|\nabla u_k\|_{L^n(B)} \leq C\epsilon_1, \quad \forall k \geq 1.
\] (3.9)

Therefore we may assume, after passing to subsequences, that $e_{\alpha}^k \to e_{\alpha}$ weakly in $W^{1,n}(B, \mathbb{R}^L)$ and strongly in $L^n(B, \mathbb{R}^L)$, $A_k \to A$ weakly in $W^{1,\frac{n}{n-1}}(B)$ and strongly in $L^\frac{n}{n-1}(B)$. It is easy to see that $\{e_{\alpha}^k\}_{\alpha=1}^l$ is a moving frame along $u^*TN$, and $A = (\langle de_{\alpha}, e_{\beta} \rangle)$ satisfies
\[
\delta A = 0 \text{ in } B; \quad x \cdot A = 0 \text{ on } \partial B,
\] (3.10)
and
\[
\|A\|_{L^n(B)} + \|\nabla A\|_{L^\frac{n}{n-1}(B)} \leq C\liminf_k \|\nabla u_k\|_{L^n(B)}^2 \leq C\epsilon_1^2.
\] (3.11)

Using these moving frames, Lemma 2.1 yields that for any $1 \leq \alpha \leq l$
\[
-\delta(|du_k|^{n-2}du_k, e_{\alpha}^k) = \sum_{\beta=1}^l \langle |du_k|^{n-2}du_k, e_{\alpha}^k \rangle \cdot (de_{\alpha}^k, e_{\beta}^k) + \langle \Phi_k, e_{\alpha}^k \rangle.
\] (3.12)

It follows from Lemma 3.2 that we can assume that $\nabla u_k \to \nabla u$ strongly in $L^q(\Omega)$ for any $1 \leq q < n$. Therefore we have
\[
|du_k|^{n-2}du_k \to |du|^{n-2}du, \quad \text{weakly in } L^{\frac{n}{n-1}}(2B).
\] (3.13)
This implies
\[-\delta(\|du_k\|^2 du_k, e^k_\alpha) \rightarrow -\delta(\|du\|^2 du, e_\alpha), \text{ in } D'(B) \quad (3.14)\]
as \(k \rightarrow \infty\), for all \(1 \leq \alpha \leq l\).

It is readily seen that for any \(\phi \in C^\infty_0(B)\) we have
\[|\langle \Phi_k, e^k_\alpha \rangle_{(W^{1,n} \phi, W^{1,n})}| \leq \|\Phi_k\|_{(W^{1,n}(B, N))} \|e^k_\alpha \phi\|_{W^{1,n}(B)} \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (3.15)\]

In order to prove that \(u\) is an \(n\)-harmonic map, it suffices to prove that for any \(1 \leq \alpha, \beta \leq l\)
\[\langle |du_k|^{-2} du_k, e^k_\alpha \rangle \cdot \langle de^k_\alpha, e^k_\beta \rangle \rightarrow \langle |du|^{-2} du, e_\beta \rangle \langle de_\alpha, e_\beta \rangle, \text{ in } D'(B). \quad (3.16)\]

To prove (3.16), we first let \(\bar{u}_k \in W^{1,n}(R^n, R^L)\) and \(\bar{e}^k_\alpha \in W^{1,n}(R^n, R^L)\) be the extensions of \(u_k\) and \(e^k_\alpha\) from \(B\) respectively such that
\[\|\nabla \bar{u}_k\|_{L^n(R^n)} \leq C\|\nabla u_k\|_{L^n(B)}, \quad \|\nabla \bar{e}^k_\alpha\|_{L^n(R^n)} \leq C\|\nabla e^k_\alpha\|_{L^n(B)}. \quad (3.17)\]

For \(\langle |d\bar{u}_k|^{-2} d\bar{u}_k, e^k_\beta \rangle \in L^{\frac{n}{n-1}}(R^n, \Lambda^1(R^n))\), the Hodge decomposition theorem (cf. Iwaniec-Martin [IM]) implies that there are \(f^k_\beta \in W^{1,\frac{n}{n-1}}(R^n)\) and \(g^k_\beta \in W^{1,\frac{n}{n-1}}(R^n, \Lambda^2(R^n))\) such that \(dg^k_\beta = 0\),
\[\langle |d\bar{u}_k|^{-2} d\bar{u}_k, e^k_\beta \rangle = df^k_\beta + \delta g^k_\beta, \quad (3.18)\]
and
\[\|\nabla f^k_\beta\|_{L^{\frac{n}{n-1}}(R^n)} + \|\nabla g^k_\beta\|_{L^{\frac{n}{n-1}}(R^n)} \leq C\|\nabla u_k\|_{L^n(B)}^{\frac{n}{n-1}}. \quad (3.19)\]

It follows from (3.19) that we may assume \(f^k_\beta \rightarrow f_\beta, g^k_\beta \rightarrow g_\beta\) weakly in \(W^{1,\frac{n}{n-1}}(R^n)\). Therefore, by taking \(k\) to infinity, (3.18) implies
\[\langle |du|^{-2} du, e_\beta \rangle = df_\beta + \delta g_\beta; \quad dg_\beta = 0, \text{ in } B. \quad (3.20)\]
Moreover, (3.18) gives
\[\langle |du_k|^{-2} du_k, e^k_\beta \rangle \cdot \langle de^k_\alpha, e^k_\beta \rangle = df^k_\beta \cdot \langle de^k_\alpha, e^k_\beta \rangle + \delta g^k_\beta \cdot \langle de^k_\alpha, e^k_\beta \rangle, \text{ in } B. \quad (3.21)\]
Since \(df^k_\beta \rightarrow df_\beta\) weakly in \(L^{\frac{n}{n-1}}(B)\), \(\langle de^k_\alpha, e^k_\beta \rangle \rightarrow \langle de_\alpha, e_\beta \rangle\) weakly in \(L^n(B)\), and \(\delta \langle de^k_\alpha, e^k_\beta \rangle = 0\) in \(B\), we can apply the Div-Curl Lemma (cf. Evans [E2] page 53) to conclude
\[df^k_\beta \cdot \langle de^k_\alpha, e^k_\beta \rangle \rightarrow df_\beta \cdot \langle de_\alpha, e_\beta \rangle, \text{ in } D'(B). \quad (3.22)\]
In fact, (3.22) follows directly from the integrations by parts: for any \(\phi \in C^\infty_0(B)\),
\[\int_{R^n} df^k_\beta \cdot \langle de^k_\alpha, e^k_\beta \rangle \phi dx = -\int_{R^n} f^k_\beta \langle de^k_\alpha, e^k_\beta \rangle \cdot d\phi dx \rightarrow -\int_{R^n} f_\beta \langle de_\alpha, e_\beta \rangle \cdot d\phi dx = \int_{R^n} df_\beta \cdot \langle de_\alpha, e_\beta \rangle \phi\]
as \( k \to \infty \). Here we have used both (3.7) and (3.10), i.e. \( \delta \langle de^k_\alpha, e^k_\beta \rangle = \delta \langle de_\alpha, e_\beta \rangle = 0 \), in \( B \).

Now we need the compensated compactness result (cf. Lions [L1,2]), which was developed by Freire-Müller-Struwe [FMS1,2] in the context of wave maps on \( \mathbb{R}^{2+1} \).

**Lemma 3.4.** Under the same notations. After taking possible subsequences, we have

\[
d \delta g^k_\beta \cdot \langle de^k_\alpha, e^k_\beta \rangle \to \delta g_\beta \cdot \langle de_\alpha, e_\beta \rangle + \nu, \quad \text{in } B
\]

(3.23)

where \( \nu \) is a signed Radon measure given by

\[
\nu = \sum_{j \in J} a_j \delta_{x_j}
\]

(3.24)

where \( J \) is at most countable, \( a_j \in \mathbb{R} \), \( x_j \in B \), and \( \sum_{j \in J} |a_j| < +\infty \).

**Proof.** For the simplicity, we only outline a proof based on suitable modifications of [FMS2].

First we observe that

\[
\begin{align*}
\delta g^k_\beta \cdot \langle de^k_\alpha, e^k_\beta \rangle &- \delta g_\beta \cdot \langle de_\alpha, e_\beta \rangle \\
&= \delta (g^k_\beta - g_\beta) \cdot \langle d(e^k_\alpha - e_\alpha), e^k_\beta \rangle + \delta g_\beta \cdot \langle d(e^k_\alpha - e_\alpha), e_\beta \rangle \\
&\quad + (\delta g^k_\beta \cdot \langle de_\alpha, e^k_\beta \rangle - \delta g_\beta \cdot \langle de_\alpha, e_\beta \rangle) \\
&= \delta (g^k_\beta - g_\beta) \cdot \langle d(e^k_\alpha - e_\alpha), e^k_\beta \rangle + I_k + II_k.
\end{align*}
\]

The dominated convergence theorem implies

\[
I_k, II_k \to 0, \quad \text{in } L^1(B), \quad \text{as } k \to \infty.
\]

Therefore (3.23) and (3.24) is equivalent to

\[
\delta (g^k_\beta - g_\beta) \cdot \langle d(e^k_\alpha - e_\alpha), e^k_\beta \rangle \to \nu
\]

(3.25)

where \( \nu \) is the Radon measure given by (3.24).

Since \( |\nabla (e^k_\alpha - e_\alpha)|^n \), \( |\nabla (g^k_\beta - g_\beta)|^{n-\tau} \) are bounded in \( L^1(B) \), we may assume, after taking subsequences, that there is a nonnegative Radon measure \( \mu \) on \( B \) such that

\[
\left( \sum_{\alpha=1}^l |\nabla (e^k_\alpha - e_\alpha)|^n + \sum_{\beta=1}^l |\nabla (g^k_\beta - g_\beta)|^{n-\tau} \right) dx \to \mu
\]

as convergence of Radon measures on \( B \).

Let \( S = \{ x \in B : \mu(\{ x \}) \equiv \lim_{r \to 0} \mu(B_r(x)) > 0 \} \). Then it follows from \( \mu(B) < +\infty \) that \( S \) is at most a countable set. Now we want to show

\[
\text{supp}(\nu) \subset S.
\]

(3.26)
It is easy to see that (3.26) yields (3.24) and hence the conclusion of Lemma 3.4.

To see (3.26), we proceed as follows. For \( \phi \in C_0^\infty(B) \), we have

\[
\langle \nu, \phi^3 \rangle = \lim_{k \to \infty} \int_{\mathbb{R}^n} \phi \delta (g^k_{\beta} - g_{\beta}) \cdot (\phi d(e^k_{\alpha} - e_{\alpha}) , \phi e^k_{\beta}) \, dx
\]

\[
= \lim_{k \to \infty} \int_{\mathbb{R}^n} [\delta(\phi (g^k_{\beta} - g_{\beta})) - d\phi \cdot (g^k_{\beta} - g_{\beta})] \cdot [d(\phi (e^k_{\alpha} - e_{\alpha})) - (e^k_{\alpha} - e_{\alpha}) \, d\phi] , \phi e^k_{\beta}) \, dx
\]

\[
= \lim_{k \to \infty} \int_{\mathbb{R}^n} \delta(\phi (g^k_{\beta} - g_{\beta})) \cdot d(\phi (e^k_{\alpha} - e_{\alpha})) , \phi e^k_{\beta}) \, dx
\]

(3.27)

where we have used

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} [(g^k_{\beta} - g_{\beta}) \, d\phi \cdot (\phi d(e^k_{\alpha} - e_{\alpha}) , \phi e^k_{\beta}) - \delta(\phi (g^k_{\beta} - g_{\beta})) \cdot (e^k_{\alpha} - e_{\alpha}) \, d\phi , \phi e^k_{\beta}) \, dx = 0.
\]

Note that Proposition 3.1 implies \( H_k \equiv \delta(\phi (g^k_{\beta} - g_{\beta})) \cdot d(\phi (e^k_{\alpha} - e_{\alpha})) \) is bounded in \( \mathcal{H}^1(\mathbb{R}^n) \), and (3.22) implies \( H_k \to 0 \) in \( \mathcal{D}'(\mathbb{R}^n) \). Therefore we have that \( H_k \to 0 \) weak* in \( \mathcal{H}^1(\mathbb{R}^n) \). On the other hand, since \( \phi e_{\beta} \in W^{1,n}(\mathbb{R}^n) \), we have \( \phi e_{\beta} \in \text{VMO}(\mathbb{R}^n) \), where \( \text{VMO}(\mathbb{R}^n) \subset \text{BMO}(\mathbb{R}^n) \) is the closure of \( C_0^\infty(\mathbb{R}^n) \) in the BMO norm. It is well-known [FS] that the dual of \( \text{VMO}(\mathbb{R}^n) \) is \( \mathcal{H}^1(\mathbb{R}^n) \). Hence we have

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \delta(\phi (g^k_{\beta} - g_{\beta})) \cdot d(\phi (e^k_{\alpha} - e_{\alpha})) , \phi e_{\beta}) \, dx = 0.
\]

(3.28)

Putting (3.28) together with (3.27) and applying (3.4), we have

\[
\| \langle \nu, \phi^3 \rangle \|
\leq C \lim_{k \to \infty} \| \nabla (\phi (e^k_{\beta} - e_{\beta})) \|_{L^{n}(\mathbb{R}^n)} \| \nabla (\phi (e^k_{\alpha} - e_{\alpha})) \|_{L^{n}(\mathbb{R}^n)} \| \nabla (\phi (g^k_{\beta} - g_{\beta})) \|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}
\]

\[
\leq C \lim_{k \to \infty} \{ \| \phi \nabla (e^k_{\beta} - e_{\beta}) \|_{L^{n}(\mathbb{R}^n)} + \| \nabla \phi \|_{L^{\infty}} \| e^k_{\beta} - e_{\beta} \|_{L^{n}(B)}
\]

\[
\cdot \| \phi \nabla (e^k_{\alpha} - e_{\alpha}) \|_{L^{n}(\mathbb{R}^n)} + \| \nabla \phi \|_{L^{\infty}} \| e^k_{\alpha} - e_{\alpha} \|_{L^{n}(B)}
\]

\[
\cdot \| \phi \nabla (g^k_{\beta} - g_{\beta}) \|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} + \| \nabla \phi \|_{L^{\infty}} \| g^k_{\beta} - g_{\beta} \|_{L^{\frac{n}{n-1}}(B)}\}
\]

\[
\leq C((\mu, \phi^m)) \frac{1}{n} ((\mu, \phi^m)) \frac{1}{n} (\mu, \phi^{n+1}) \frac{1}{n} (\mu, \phi^{n+1}) \frac{1}{n} (\mu, \phi^{n+1}) \frac{1}{n} \quad (3.29)
\]

where we have used

\[
\lim_{k \to \infty} (\| e^k_{\alpha} - e_{\alpha} \|_{L^{n}(B)} + \| g^k_{\beta} - g_{\beta} \|_{L^{\frac{n}{n-1}}(B)}) = 0.
\]

By choosing \( \phi_i \in C_0^\infty(B) \) such that \( \phi_i \to \lambda_B, (y) \), the characteristic function of a ball \( B_r(y) \), we then have

\[
\nu(B_r(y)) \leq C \mu(B_r(y)) \frac{n+1}{n}.
\]

(3.30)
Therefore $\nu$ is absolutely continuous with respect to $\mu$. Moreover, for any $y \not\in S$, the Radon-Nikodym derivative

$$\frac{d\nu}{d\mu}(y) = \lim_{r \to 0} \frac{\nu(B_r(y))}{\mu(B_r(y))} \leq C \lim_{r \to 0} \mu(B_r(y))^{\frac{1}{n}} = 0.$$
Then $\Sigma$ is a finite subset and
\[ |\Sigma| \leq C \epsilon_1^n, \quad C \equiv \limsup_{k \to \infty} \int_\Omega |\nabla u_k|^n \, dx < +\infty. \]

For any $x_0 \in \Omega \setminus \Sigma$, there exists an $r_0 > 0$ such that $\mu(B_{4r_0}(x_0)) < \epsilon_1^n$. Since
\[ \limsup_{k \to \infty} \int_{B_{2r_0}(x_0)} |\nabla u_k|^n \, dx \leq \mu(B_{4r_0}(x_0)), \]
we can assume that there exists $k_0 \geq 1$ such that
\[ \int_{B_{2r_0}(x_0)} |\nabla u_k|^2 \, dx \leq \epsilon_1^n, \quad \forall k \geq k_0. \]
Therefore Lemma 3.3 implies that $u$ is an $n$-harmonic map in $B_{r_0}(x_0)$. Since $x_0 \in \Omega \setminus \Sigma$ is arbitrary, we conclude that $u$ is an $n$-harmonic map in $\Omega \setminus \Sigma$.

To show $u$ is an $n$-harmonic map in $\Omega$, observe that $\text{Cap}_n(\Sigma) = 0$ (cf. [EG]). Therefore there are a sequence $\{\eta_i\}$ of functions on $\mathbb{R}^n$ such that $0 \leq \eta_i \leq 1$, $\Sigma \subset \text{int} \{\eta_i = 1\}$, $\int_{\mathbb{R}^n} |\nabla \eta_i|^n \, dx \to 0$, $\eta_i \to 0$ boundedly a.e. (3.33)

Then, for any $\phi \in C_0^\infty(\Omega, \mathbb{R}^L)$, we have
\[ \int_\Omega |du|^{n-2} du \cdot d\phi \, dx = \int_\Omega |du|^{n-2} du \cdot d((1 - \eta_i)\phi) \, dx + \int_\Omega |du|^{n-2} du \cdot (\eta_i d\phi + d\eta_i \phi) \, dx. \]

From (3.33) we conclude that $(1 - \eta_i)\phi$ is compactly supported in $\Omega \setminus \Sigma$ and therefore,
\[ \int_\Omega |du|^{n-2} du \cdot d((1 - \eta_i)\phi) \, dx = \int_\Omega |du|^{n-2} A(u)(du, du)(1 - \eta_i)\phi \, dx. \]

By the dominated convergence theorem and (3.33), we have
\[ \lim_{i \to \infty} \int_\Omega |du|^{n-2} A(u)(du, du) \eta_i \phi \, dx = 0, \quad \lim_{i \to \infty} \int_\Omega |du|^{n-2} du \cdot \eta_i d\phi \, dx = 0. \]

Applying (3.34) we have
\[ |\int_\Omega |du|^{n-2} du \cdot d\eta_i \phi \, dx| \leq \|\phi\|_{L_\infty(\Omega)} \left( \int_\Omega |Du|^n \, dx \right)^{\frac{n-1}{n}} \left( \int_{\mathbb{R}^n} |D\eta_i|^n \, dx \right)^{\frac{1}{n}} \to 0 \]
as $i \to \infty$. Therefore we have
\[ \int_\Omega |du|^{n-2} du \cdot d\phi \, dx = \int_\Omega |du|^{n-2} A(u)(du, du) \phi \, dx, \quad \forall \phi \in C_0^\infty(\Omega, \mathbb{R}^L). \]
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