The time-dependent Schrödinger equation of dimension $k + 1$: explicit and rational solutions via GBDT and multinodes

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Abstract

A version of the binary Darboux transformation is constructed for a non-stationary Schrödinger equation of dimension $k + 1$, where $k \geq 1$ is the number of space variables. This is an iterated generalized Bäcklund–Darboux transformation version. New families of non-singular and rational potentials and solutions are obtained. Some results are also new for the case that $k = 1$. A certain generalization of a colligation introduced by M S Livšic and a generalization of the $S$-node introduced by L A Sakhnovich have been successfully used in our construction.

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1. Introduction

The time-dependent Schrödinger (TDS), or non-stationary Schrödinger equation is one of the most well-known and important equations in physics. We shall consider the Schrödinger equation, where the potential $q$ is a $p \times p$ matrix function and $\alpha$ is an arbitrary coefficient ($\alpha \in \mathbb{C}, \alpha \neq 0$):

$$Hu = 0, \quad H := \alpha \frac{\partial}{\partial t} + \Delta - q(x, t), \quad u \in \mathbb{C}^p. \quad (1.1)$$

Here $\Delta$ denotes the Laplacian with respect to the spatial variables $x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^k$, $\mathbb{C}$ stands for the complex plane and $\mathbb{R}$ stands for the real axis. Explicit solutions of (1.1) are of essential and permanent interest and various methods have been applied to construct them. For example, rational slowly decaying soliton solutions (lumps) of the Kadomtsev–Petviashvili (KP) and TDS equations were first found in [29]. Since then these, as well as rational solutions of other integrable equations, have been actively studied (see, e.g., [1, 13, 14, 24, 42, 49, 50, 52] and references therein).

Following seminal works of Bäcklund, Darboux and Jacobi, different kinds of Darboux transformations and related commutation and factorization methods have been fruitfully used
to obtain explicit solutions of linear and nonlinear equations (see, e.g., [9–11, 16–19, 22, 23, 30, 32, 33, 41, 51] and numerous references therein). In particular, matrix and operator identities were widely used in these constructions (see [6, 8, 12, 15, 20, 30, 36, 38–40, 42, 46] for various results, discussions and references).

Darboux transformations proved to be especially useful for the construction of explicit solutions of the TDS equation of dimension 1 + 1 (and, correspondingly, for the construction of explicit solutions of the KP equation with two spatial variables). A singular (non-binary) Darboux transformation was used for that purpose in [31] and a binary Darboux transformation for the scalar TDS appeared in a well-known book [32, section 2.4]. Further important results regarding the TDS equation of dimension 1 + 1 can be found in [1, 3, 7, 49] (see also [48] and references therein for generalized TDS equations).

The first discussion on the Darboux transformation for TDS with \( k > 1 \) spatial variables, that we could find, was in the work by Sabatier [34]. In spite of interesting publications (e.g., [2, 35]), the case of linear equations with \( k \) spatial variables \((k > 1)\) is much more difficult (and much less is done, even for the singular Darboux transformation), especially so for \( k > 2 \) (see [35, 47] for some explanations).

In this paper we construct explicit and rational solutions of the TDS equation for all \( k \geq 1 \). We apply the generalized Bäcklund–Darboux transformation (GBDT) approach from [15, 20, 36, 38–40, 42] (see further references and some comparative discussions on this method in [9, 22, 41]). Corresponding results for the case that \( k = 1 \) were announced in [39], and this contains proofs, which are valid for \( k = 1 \) too.

GBDT is partially based on the operator identity (also called \( S \)-node) method [43–45], which, in its turn, takes its roots in the characteristic matrix function and operator colligations introduced by Livšic [25] (see also [26]). In his later works Livšic studied a greatly more complicated case of colligations with several (instead of one) commuting operators [27] (see also [4, 5, 28] and references therein). Correspondingly, for our case of \( k \) spatial variables we need an \( S \)-node with \( k \) matrix identities, which we call \( S \)-multinode.

In section 2 we describe GBDT for the TDS equation. \( S \)-multinodes are introduced in section 3. Using GBDT and multinodes, we construct in section 3 explicit solutions and potentials of the TDS equation and consider examples. Conditions for non-singular and rational solutions and potentials and concrete examples are given in section 4.

We use \( \mathbb{N} \) to denote the set of natural numbers, \( \sigma \) to denote spectrum and the overline means complex conjugation. The notation \( \text{Rank}(A) \) stands for the rank of a matrix \( A \), \( A^* \) is the matrix adjoint to \( A \), \( I_p \) is the \( p \times p \) identity matrix and \( \text{col} \) denotes a column.

2. GBDT for the TDS equation

Let \( Hu = 0 \) (\( H = \alpha \frac{\partial}{\partial t} + \Delta - q \)) be some TDS equation, which we call initial, and let \( \Psi(x, t) \) and \( \Phi(x, t) \) be block rows of \( n \times p \) blocks \( \Psi_r \) and \( \Phi_r \), respectively. Here, \( n \in \mathbb{N} \) is fixed and \( 0 \leq r \leq k \). It is required that \( \Psi \) satisfies equations

\[
H \Psi_r^* = 0, \quad \Psi_r^* = \frac{\partial}{\partial x_r} \Psi_0^* \quad (1 \leq r \leq k),
\]  

where \( H \) is applied to \( \Psi_0^* \) columnwise. We require

\[
\Phi_r = \frac{\partial}{\partial x_r} \Phi_0 \quad (1 \leq r \leq k),
\]

and \( \Phi_0 \) will be discussed a little later.

An \( n \times n \) matrix function \( S \), which we define via \( \Psi \) and \( \Phi \):

\[
\frac{\partial}{\partial x_r} S(x, t) = \Phi_0(x, t) \Psi_0(x, t)^* \quad (1 \leq r \leq k),
\]
is very important in GBDT. Here, \( v_r \) are some \( p \times p \) matrices. It is useful to recall (in the remark below) the case \( k = 1 \).

**Remark 2.1.** For linear equations depending on one variable (and nonlinear equations depending on two variables) the analog of \( \mathcal{S} \) is denoted by \( \mathcal{S} \) and the so-called Darboux matrix is presented as the transfer matrix function of the corresponding depending on two variables) the analog of equation \( \mathcal{S} \). For linear equations depending on one variable (and nonlinear equations)

\[
\frac{\partial}{\partial t} \mathcal{S}(x, t) = \alpha^{-1} \sum_{r=1}^{k} (\Phi_r(x, t) v_r \Psi_0(x, t)^* - \Phi_0(x, t) v_r \Psi_r(x, t)^*),
\]

(2.4)

Remark 2.1. For linear equations depending on one variable (and nonlinear equations depending on two variables) the analog of \( \mathcal{S} \) is denoted by \( \mathcal{S} \) and the so-called Darboux matrix is presented as the transfer matrix function of the corresponding depending on two variables) the analog of equation \( \mathcal{S} \). For linear equations depending on one variable (and nonlinear equations )

\[
\mathcal{H}_d f = \alpha \frac{\partial}{\partial t} f(x, t) - \Delta f(x, t) + f(x, t) q(x, t), \quad f = [f_1, \ldots, f_p].
\]

(2.5)

In view of relations (2.1)–(2.4) and \( \mathcal{H}_d \Phi_0 = 0 \), for the case that \( k = 1 \), we have

\[
S_{x_1} = \alpha^{-1} \left( \left( \frac{\partial}{\partial x_1} \Phi_1 \right) \Psi_0^* - \Phi_0 \left( \frac{\partial}{\partial x_1} \Psi_1^* \right) \right), \quad S_{x_1} := \frac{\partial}{\partial x_1} \frac{\partial}{\partial t} \mathcal{S} ;
\]

\[
S_{x_1 t} = \alpha^{-1} \left( \left( \frac{\partial^2}{\partial x_1^2} \Phi_0 \right) \Psi_0^* - \Phi_0 \left( \frac{\partial^2}{\partial x_1^2} \Psi_0^* \right) \right).
\]

Therefore, the compatibility condition \( S_{x_1 t} = S_{x_1} \) for equations (2.3) and (2.4) is fulfilled.

Though for \( k > 1 \) the situation is more complicated, the equality \( \tilde{\Psi}_0^* = \Psi_0^* \mathcal{S}^{-1} \) for the solution \( \tilde{\Psi}_0^* \) of the transformed TDS \( \tilde{H} f = 0 \) holds also in our case (see theorem 2.3).

**Remark 2.2.** To be more precise, the applicability of Darboux transformation (i.e., GBDT in theorem 2.3) relies on the consistency of equations (2.3) and (2.4) for the matrix function \( \mathcal{S} \). If \( k > 1 \), the consistency conditions are certain rather complicated relations for \( \Psi_0 \) and \( \Phi_0 \), which hardly can be solved in general. However, in section 3 some classes of solutions are described algebraically in terms of multinodes and, further, some particular multinodes are given explicitly in the examples.

Unlike the standard Darboux transformation, we additionally assume the existence of \( \mathcal{S} \) satisfying (2.3) and (2.4) (and do not assume that \( \mathcal{H}_d \Phi_0 = 0 \)) in our theorem below.

**Theorem 2.3.** Let matrix functions \( \Psi, \Phi \) and \( \mathcal{S} \) satisfy relations (2.1)–(2.4). Then, in the points of invertibility of \( \mathcal{S} \), the matrix function

\[
\tilde{\Psi}_0^* := \Psi_0^* \mathcal{S}^{-1}
\]

(2.6)

satisfies the transformed TDS equation:

\[
\tilde{H} \tilde{\Psi}_0^* = 0, \quad \tilde{H} := \alpha \frac{\partial}{\partial t} + \Delta - \tilde{q}(x, t),
\]

(2.7)

where

\[
\tilde{q}(x, t) := q(x, t) - 2 \sum_{r=1}^{k} \frac{\partial}{\partial x_r} (\Phi_0(x, t)^* \mathcal{S}(x, t)^{-1} \Phi_0(x, t)) v_r.
\]

(2.8)
Proof. Taking into account (2.1) and definitions of $H$, $\tilde{\Psi}_0$ and $\tilde{H}$ in (1.1), (2.6) and (2.7), respectively, we get

$$\tilde{H}\tilde{\Psi}_0^r = (q - \tilde{q})\tilde{\Psi}_0^r - \alpha\Psi_0^rS^{-1}S_0 \psi + \Psi_0^r\Delta(S^{-1}) - 2\sum_{r=1}^{k} \Psi_0^r S^{-1}S_0 S^{-1}. \quad (2.9)$$

Because of (2.3), we have

$$\Delta(S^{-1}) = \sum_{r=1}^{k} S^{-1}(2S_0 S^{-1}S_0 - \Phi_0 \nu_0 \Psi_0^* - \Phi_0 \nu_0 \Psi_0^*)S^{-1}. \quad (2.10)$$

Finally, using formulas (2.4) and (2.10) and reducing similar terms, we rewrite (2.9) as

$$\tilde{H}\tilde{\Psi}_0^r = (q - \tilde{q})\tilde{\Psi}_0^r - \Psi_0^r S^{-1}r \sum_{r=1}^{k} \Phi_0 \nu_0 \tilde{\Psi}_0^r + 2\Psi_0^r S^{-1}r \sum_{r=1}^{k} \nu_0 \psi_0 S^{-1}r \Phi_0 \nu_0 \tilde{\Psi}_0^r$$

$$- \Psi_0^r S^{-1}r \sum_{r=1}^{k} \Phi_0 \nu_0 \tilde{\Psi}_0^r - 2\sum_{r=1}^{k} \Psi_0^r S^{-1}r \Phi_0 \nu_0 \tilde{\Psi}_0^r. \quad (2.11)$$

Since $\Phi_0 = (\Phi_0)_{\nu_0}$, $\Psi_0 = (\Psi_0)_{\nu_0}$ and (2.3) holds, it follows from (2.11) that for $\tilde{q}$ given by (2.8) the equality $\tilde{H}\tilde{\Psi}_0^r = 0$ is true. \hfill \Box

3. Multinodes and explicit solutions

Definition 3.1. By a matrix $S$-multinode (or, more precisely, by $S_k$-node $\{k, A, B, R, v, C_\Phi, C_\Psi\}$) we call a set of matrices, which consists of $N \times N$ commuting matrices $A_r$ $(1 \leq r \leq k)$, $N \times N$ commuting matrices $B_r$ $(1 \leq r \leq k)$, $p \times p$ matrices $v_r$ $(1 \leq r \leq k)$, and an $N \times N$ matrix $R$, an $N \times p$ matrix $C_\Phi$ and a $p \times N$ matrix $C_\Psi$, such that the matrix identities

$$A_r R - R B_r = C_\Phi v_r C_\Psi, \quad 1 \leq r \leq k \quad (3.1)$$

hold. An operator $S$-multinode is defined in the same way.

The condition that the matrices from $\{A_r\}$ mutually commute and the matrices from $\{B_r\}$ mutually commute,

$$A_r A_{r'} = A_{r'} A_r, \quad B_r B_{r'} = B_{r'} B_r, \quad (3.2)$$

is important in definition 3.1.

For the case that $k = 1$, this definition coincides with the definition of an $S$-node [43–45], and for the case that $R = I_N$ and $B_r = A_r^*$, our definition coincides with the definition of a commutative colligation from [27].

In this section we treat the case $q \equiv 0$, that is,

$$\tilde{q}(x, t) = -2\sum_{r=1}^{k} \frac{\partial}{\partial x_r} (\Psi_0(x, t) S(x, t)^{-1} \Phi_0(x, t)) v_r. \quad (3.3)$$

Theorem 3.2. Fix some $n \in \mathbb{N}$, let an $n \times N$ matrix $\hat{C}_\Phi$, an $N \times n$ matrix $\hat{C}_\Psi$, an $n \times n$ matrix $S_0$ and a matrix $S_k$-node $\{k, A, B, R, v, C_\Phi, C_\Psi\}$ be given. Then the matrix functions

$$\Phi_0(x, t) = \hat{C}_\Phi e_A(x, t) C_\Phi, \quad e_A(x, t) := \exp \left\{ \sum_{r=1}^{k} x_r A_r \right\} + \alpha^{-1} t \left\{ \sum_{r=1}^{k} A_r^* \right\}, \quad (3.4)$$

$$\Psi_0(x, t)^* = C_\Psi e_B(-x, -t) \hat{C}_\Psi, \quad S = \hat{C}_\Phi e_A(x, t) R e_B(-x, -t) \hat{C}_\Psi + S_0 \quad (3.5)$$

satisfy the conditions of theorem 2.3, where the initial TDS equation is chosen so that $q \equiv 0$ (i.e. the conditions on $\Phi$ and $\Psi$ are satisfied after we add $\Phi_r = (\Phi_0)_{\nu_0}$ and $\Psi_r = (\Psi_0)_{\nu_0}$).
Proof. It is immediate from (3.5) that $\frac{d}{dt} \Psi^*_0 + \Delta \Psi^*_0 = 0$, that is, $H \Psi^*_0 = 0$, and so (2.1) holds. Because of (3.1), (3.4) and (3.5), we have (2.3). It remains to show that (2.4) holds. For that purpose, note that equalities (3.1) imply
\begin{equation}
A_i^2 R - RB_i^2 = A_i (A_i R - RB_i) + (A_i R - RB_i) B_i,
\end{equation}
In view of (3.5) and (3.6) we get
\begin{equation}
\frac{\partial}{\partial t} S = \alpha^{-1} \tilde{C}_\Phi e_A(x, t) \left( \sum_{r=1}^k A_r C_{\Phi} v_j C_{\psi} + C_{\Phi} v_j C_{\psi} B_j \right) e_B(-x, -t) \tilde{C}_\psi.
\end{equation}
Since $\Phi_r = (\Phi_0)_r$ and $\Psi_r = (\Psi_0)_r$, formula (2.4) easily follows from (3.4), (3.5) and (3.7).

Remark 3.3. Clearly, the equality $\alpha \frac{d}{dt} \Phi_0 - \Delta \Phi_0 = 0$ holds for $\Phi_0$ of the form (3.4), that is, we have $H_\beta \Phi_0 = 0$ for the case that $q_d = 0$, and so (2.5) holds. Hence, the transformation, which is addressed in theorem 3.2, is a binary transformation.

Recall that interesting singular (and some stationary binary) vectorial Darboux transformations of a scalar TDS equation into a matrix TDS equation (i.e. into a TDS equation, where potential is a matrix function) were addressed in [35]. It is easy to see that for the case that $\tilde{q} = \{\tilde{q}_{ij}\}_{i, j=1}^p$ and $f = \{f_i\}_{i=1}^p$ are the potential and solution, respectively, of some matrix TDS equation, the functions
\begin{equation}
q^{\omega} = \tilde{q}_{rr} + \left( \sum_{j \neq r} \tilde{q}_{ij} f_j \right) / f_r, \quad f^{\omega} = f_r
\end{equation}
are the potential and solution of a scalar TDS.

Example 3.4. Consider the simplest case $k = 1$ (and $v_1 = I_p$).

If $A_1$ and $B_1$ are diagonal matrices:
\[ A_1 = \text{diag}[a_1, \ldots, a_N], \quad B_1 = \text{diag}[b_1, \ldots, b_N] \quad (a_i \neq b_j), \]
then identity (3.1) leads us to
\[ R = (a_i - b_j)^{-1} (C_{\Phi} C_{\psi})_{ij}, \quad R = [R_{ij}]_{i, j=1}^N, \]
and relations (2.6), (3.3)–(3.5) and (3.9) define the corresponding solution and potential of the transformed matrix TDS equation explicitly (up to the inversion of matrices).

We note that the $N$-soliton solutions to the KP hierarchy can be constructed in this way (see [21, p 3546]).

Example 3.5. Now, consider a simple example for the case that $1 < k \leq p$. We set
\begin{align}
A_1 &= \text{diag}[a_1, \ldots, a_N], \quad A_r = (c_r I_N - A_1)^{-1} \quad (a_i \neq c_r, 1 < r \leq k); \quad \text{(3.10)} \\
B_1 &= \text{diag}[b_1, \ldots, b_N], \quad B_r = (c_r I_N - B_1)^{-1} \quad (b_i \neq c_r, 1 < r \leq k); \quad \text{(3.11)} \\
R &= \{ (a_i - b_j)^{-1} \}_{i, j=1}^N\quad (a_i \neq b_j); \quad v_r = \{ \delta_{r-i} \delta_{r-j} \}_{i, j=1}^p, \quad 1 \leq r \leq k; \quad \text{(3.12)} \\
C_{\Phi} &= [h \quad A_2 h \quad \cdots \quad A_k h \quad \tilde{C}_\Phi], \quad h = \text{col}[1 \quad 1 \quad \cdots \quad 1], \quad \text{(3.13)} \\
C_{\psi} &= \text{col}[h^* \quad h^* B_2 \quad \cdots \quad h^* B_k \quad \tilde{C}_\psi], \quad \text{(3.14)}
\end{align}
Example 3.6. We assume $1 \leq k \leq p, N > 1$ and set

$$A_0 := \frac{i}{2} I_N + \{a_{r,j} \}^N_{r,j=1}, \quad a_s := i \text{ for } s > 0, \quad a_s := 0 \text{ for } s \leq 0; \quad (3.15)$$

$$A_r := (c_r I_N - A_0)^{-1} \quad B_r := (c_r I_N - A_0^*)^{-1} \quad (c_r \neq \pm i/2, 1 \leq r \leq k). \quad (3.16)$$

We see that matrix $A_0$ and therefore matrices $A_r$ and $B_r$ are linear, similar to Jordan cells. The matrix $R$ is a so-called cyclic Toeplitz matrix and is introduced by the equality

$$R := \{T_{r,j} \}^N_{r,j=1} \quad (T_r \in \mathbb{C}, \quad T_0 = 0 \text{ for } s < 0). \quad (3.17)$$

Then, the following matrix identity holds (see, e.g., [37, p 451]):

$$A_0 R - RA_0^* = i g h^*, \quad g = \text{col} \left[ T_0 \quad T_0 + T_1 \quad \cdots \quad \sum_{s=0}^{N-1} T_s \right], \quad (3.18)$$

where $h$ is given in (3.13). Because of (3.16) and (3.18), the identities (3.1) are true, where $v_r$ are given in (3.12) and

$$C_r = i [A_1 g \quad \cdots \quad A_k g \quad \hat{C}_r], \quad C_r = \text{col}[h^* B_1 \quad \cdots \quad h^* B_k \quad \hat{C}_r]. \quad (3.19)$$

It is immediate also that the commutation property (3.2) holds. That is, a multinode, where $k \geq 1$ and matrices $A_r$ and $B_r$ are non-diagonizable, is constructed.

The cases, where matrices $v_r$ had rank 1 and $p \times p$ matrix TDS equations with $p$ spatial variables were included, were addressed in examples 3.5 and 3.6.

Remark 3.7. Clearly, it is quite possible, though somewhat less convenient, to consider $S_{\Delta}$-nodes with matrices $v_r$ of higher ranks in the same way. Recall also an easy transfer (3.8) from a matrix to a scalar TDS.

Definition 3.1 admits an easy generalization for the case of rectangular matrices $R$, whereupon the proof of theorem 3.2 does not require any changes.

Definition 3.8. By the $S_\Delta$-node $\{k, A, B, R, v, C, \hat{C}_r \}$ (with rectangular matrix $R$) we call a set of matrices, which consists of $N_1 \times N_1$ commuting matrices $A_r$ ($1 \leq r \leq k$), $N_2 \times N_2$ commuting matrices $B_r$ ($1 \leq r \leq k$), $p \times p$ matrices $v_r$ ($1 \leq r \leq k$), and of an $N_1 \times N_2$ matrix $R$, an $N_1 \times p$ matrix $C$ and a $p \times N_2$ matrix $\hat{C}_r$, such that the matrix identities (3.1) hold.

Corollary 3.9. Let an $n \times N_1$ matrix $\hat{C}_r$, an $N_2 \times n$ matrix $\hat{C}_r$, an $n \times n$ matrix $\Phi_0$ and a matrix $\Phi_0$ be given. Then the matrix functions $\Phi_0, \Phi_1$ and $\Phi_2$, which are given by formulas (3.4) and (3.5), satisfy equation $H_q \Phi_0 = 0$ and conditions of theorem 2.3, where $q = q_0$. (3.10)-(3.14) determine an $S_\Delta$-node $\{k, A, B, R, \hat{C}_r, \hat{C}_r \}$ and the corresponding explicit expressions for solution and potential of the transformed TDS equation follow.
4. Non-singular, rational, and lump potentials

In this section, we study conditions where the potentials \( \tilde{q} \) and the TDS solutions \( \tilde{\Psi}_0^* \) are non-singular and rational. Our next proposition deals with the construction of rational potentials.

**Proposition 4.1.** Let an \( n \times N_1 \) matrix \( \hat{C}_\Phi \), an \( N_2 \times n \) matrix \( \hat{C}_\Psi \), an \( n \times n \) matrix \( S_0 \) and a matrix \( S \)-node \( \{ k, A, B, R, v, C_\Phi, C_\Psi \} \) be given. Assume additionally that condition (i) below holds:

(i) all the matrices from the set \( \{ A_r \} \cup \{ B_r \} \) are nilpotent.

Then the solution and potential of the transformed TDS equation, which are given by formulas (2.6) and (3.3), are rational.

**Proof.** In view of corollary 3.9, the matrix functions \( \Phi_0, \Psi_0 \) and \( S \) satisfy the conditions of theorem 2.3. Therefore, using theorem 2.3 we see that the solution and potential of the transformed TDS equation are given by formulas (2.6) and (3.3), respectively.

First, assume that condition (i) holds. It is immediate that all the matrix functions

\[
\exp[x, A_r], \quad \exp[\alpha^{-1} t A_r^2], \quad \exp[-x, B_r], \quad \exp[-\alpha^{-1} t B_r^2] \quad (1 \leq r \leq k)
\]

are matrix polynomials, and thus \( \Phi_0, \Psi_0 \) and \( S \), which are given by (3.4) and (3.5), are matrix polynomials with respect to \( x \) and \( t \). The statement of the proposition follows.

Next, assume that conditions (4.1) hold. Because of (4.1) we have

\[
e_{A}(x, t) = e^{f(x,t)} p_1(x, t), \quad e_{B}(-x, -t) = e^{g(x,t)} p_2(x, t),
\]

where \( p_i \) are the matrix polynomials, whereas \( f \) and \( g \) are the scalar polynomials:

\[
f(x, t) = \sum_{r=1}^{k} \mu_r x_r + \alpha^{-1} \sum_{r=1}^{k} \mu_r^2 t, \quad g(x, t) = -\sum_{r=1}^{k} \lambda_r x_r - \alpha^{-1} \sum_{r=1}^{k} \lambda_r^2 t.
\]

Since \( S_0 = 0 \), we derive from formulas (3.4), (3.5), and (4.2) that \( \Psi_0^* S^{-1} \Phi_0 \) is rational, and so (in view of (3.3)) the potential \( \tilde{q} \) is also rational. \( \square \)

Furthermore, we assume again that \( N_1 = N_2 = N \), that is, \( R \) is a square matrix. The following proposition is immediate from (2.6) and (3.3)–(3.5).

**Proposition 4.2.** Let the conditions of theorem 3.2 hold and let also equalities

\[
\alpha = i, \quad \hat{C}_\Psi = \hat{C}_\Phi^*, \quad S_0 = S_0^*;
\]

\[
C_\Psi = C_\Phi^*, \quad R = R^*; \quad B_r = -A_r^* \quad (1 \leq r \leq k)
\]

be satisfied. Then we have

\[
e_{B}(-x, -t) = e_{A}(x, t)^*, \quad \Phi_0(x, t) = \Psi_0(x, t), \quad S(x, t) = S(x, t)^*.
\]

Furthermore, if the additional relations

\[
R \geq 0, \quad S_0 > 0, \quad or \quad R > 0, \quad \text{Rank}(\hat{C}_\Phi) = n \leq N, \quad S_0 \geq 0
\]

hold, then the inequality \( S(x, t) > 0 \) also holds. Thus, \( S \) is invertible and the solution \( \tilde{\Psi}_0^* \) and potential \( \tilde{q} \) of the transformed TDS are non-singular.
Finally, we consider several concrete examples of non-singular, rational and lump potentials, where \((4.4)\) and \((4.5)\) hold, and

\[
\nu_r = [\delta_{r-j}\delta_{r-j}]^1_{j=1}, \quad 1 \leq r \leq k. \tag{4.8}
\]

Because of \((4.8)\) and the first and third equalities in \((4.5)\), identity \((3.1)\) for \(r = 1\) has the form

\[
A_1R + RA_1^* = \mathbb{Q}, \quad \mathbb{Q} = \mathbb{Q}^*, \quad \text{i.e.}
\]

\[
(zI_N - iA_1)^{-1}R = i(zI_N - iA_1)^{-1}\mathbb{Q}(zI_N + iA_1^*)^{-1} + R(zI_N + iA_1^*)^{-1}. \tag{4.9}
\]

If \(\sigma(iA) \subset \mathbb{C}_+,\) we take residues and derive from \((4.9)\) a well-known representation

\[
R = \frac{1}{2\pi} \int_{-\infty}^{\infty} (zI_N - iA_1)^{-1}\mathbb{Q}(zI_N + iA_1^*)^{-1}dz, \quad R = R^*, \tag{4.10}
\]

that is, the second equality in \((4.5)\) follows now from the first and third equalities.

**Remark 4.3.** For the case that \(\mathbb{Q} \geq 0,\) representation \((4.10)\) implies \(R \geq 0.\)

**Example 4.4.** Let \(k = p = 3, N = 2\) and let \(A_1\) be a \(2 \times 2\) Jordan cell:

\[
A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \mu_0I_2 + A_0, \quad \kappa := \mu_0 + \overline{\mu_0} \neq 0; \tag{4.11}
\]

\[
A_r = (A_1 - c_rI_2)^{-1}, \quad c_r = -\overline{c_r} \quad (r = 2, 3). \tag{4.12}
\]

It is immediate from \((4.11)\) and \((4.12)\) that

\[
A_r = (\mu_0 - c_r)^{-1}I_2 = (\mu_0 - c_r)^{-1}A_0, \tag{4.13}
\]

and (similar to \((3.13)\) but with different choices of \(A_r\) and \(h)\) we put

\[
C_\psi = C_\phi^*, \quad C_\phi = \left[h \quad A_2h \quad A_3h \right] = \begin{bmatrix} 0 & -(\mu_0 - c_2)^{-2} & -(\mu_0 - c_3)^{-2} \\ 1 & (\mu_0 - c_2)^{-1} & (\mu_0 - c_3)^{-1} \end{bmatrix}, \tag{4.14}
\]

where \(h = \text{col}[0 \quad 1].\) We recover \(R\) from the identity

\[
A_1R + RA_1^* = hh^*, \tag{4.15}
\]

which is equivalent (in view of \(B_1 = -A_1^*, \quad (4.8)\) and \((4.14)\)) to relation \((3.1)\) for \(r = 1.\) That is, we rewrite \((4.15)\) in the form

\[
\kappa R + \begin{bmatrix} R_{21} + R_{12} \\ R_{22} \end{bmatrix} = hh^*, \tag{4.16}
\]

where \(R_{ij}\) are the entries of \(R.\) Using \((4.16)\) (and starting from the recovery of \(R_{22}\)) we easily get a unique \(R\) satisfying \((4.15):\

\[
R = \begin{bmatrix} 2\kappa^{-3} & -\kappa^{-2} \\ \kappa^{-2} & \kappa^{-3} \end{bmatrix}, \quad \det R = \kappa^{-4}. \tag{4.17}
\]

Identities \((3.1)\) for \(r > 1\) easily follow from \((4.15),\) and so we obtain an \(S_3\)-node \([3, A, B, R, v, C_\phi, C_\psi]\). Moreover, remark 4.3 and the second equality in \((4.17)\) yield

\[
R > 0 \quad \text{for} \quad \kappa = \mu_0 + \overline{\mu_0} > 0. \tag{4.18}
\]

Therefore, the conditions of theorem \(3.2\) and proposition \(4.2\) are fulfilled.

Since \(A_0^0 = 0,\) formula \((4.13)\) and second relations in \((3.4)\) and \((4.11)\) imply

\[
e_a(x, t) = e^{\Omega_1(x, t)}(I_2 + \Omega_1(x, t)A_0) = e^{\Omega_1(x, t)} \begin{bmatrix} 1 & \Omega_1(x, t) \\ 0 & 1 \end{bmatrix}, \tag{4.19}
\]
\[ \Omega_0(x, t) := \mu_0 x_1 + (\mu_0 - c_2)^{-1} x_2 + (\mu_0 - c_3)^{-1} x_3 - i(\mu_0^2 + (\mu_0 - c_2)^{-2} + (\mu_0 - c_3)^{-2})t, \]
\[ (4.20) \]
\[ \Omega_1(x, t) := x_1 - (\mu_0 - c_2)^{-2} x_2 - (\mu_0 - c_3)^{-2} x_3 - 2i(\mu_0 - (\mu_0 - c_2)^{-3} - (\mu_0 - c_3)^{-3})t. \]
\[ (4.21) \]

**Corollary 4.5.** Let \( k = p = 3, N = 2 \) and \( \kappa = \mu_0 + \eta_0 > 0 \). Define matrices \( C_\Psi, C_\Phi \) and \( R \) via (4.14) and (4.17). Choose \( C_\Psi = \tilde{C}_\Psi \) and \( \delta_0 > 0 \). Then relations (2.6) and (3.3)–(3.5), where \( e_A \) is given by (4.19)–(4.21) and \( e_B (-x, -t) = e_A(x, t)^* \), explicitly define non-singular solutions and potentials of TDS.

The cases \( N > 2 \) can be addressed in the same way.

**Example 4.6.** Let \( k = p = 3 \) and \( N = 3 \). Set
\[ A_1 = \mu_0 I_3 + A_0, \quad A_r = (A_1 - c I_3)^{-1} = -\sum_{i=0}^{2} (c_r - \mu_0)^{-i-1} A_0^i, \]
\[ (4.22) \]
\[ c_r = -\overline{\kappa} \quad (r = 2, 3); \quad \kappa := \mu_0 + \eta_0 \neq 0, \]
\[ (4.23) \]
\[ A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_\Phi = C_\Psi^* = [h \ A_2 h \ A_3 h]. \]
\[ (4.24) \]

In a way, which is quite similar to considerations from example 4.4, we can show that an \( S_3 \)-node \( \{3, A, B, R, v, C_\Phi, C_\Psi \} \) appears, if we put
\[ B_r = -A_r^* \quad (1 \leq r \leq 3), \quad R = \begin{bmatrix} 6\kappa^{-5} & -3\kappa^{-4} & \kappa^{-3} \\ -3\kappa^{-4} & 2\kappa^{-3} & -\kappa^{-2} \\ \kappa^{-3} & -\kappa^{-2} & \kappa^{-1} \end{bmatrix}. \]
\[ (4.25) \]

Moreover, we have \( \det R = \kappa^{-9} \neq 0 \), and so (4.18) is valid for \( R \) of the form (4.25) too. Finally, we note that since \( A_0^3 = 0 \), formulas (3.4) and (4.22) imply
\[ e_A(x, t) = e^{\Omega_0(x, t)} (I_3 + \Omega_1(x, t) A_0 + \Omega_2(x, t) A_0^2) \]
\[ = e^{\Omega_0(x, t)} \begin{bmatrix} 1 & \Omega_1(x, t) & \Omega_2(x, t) \\ 0 & 1 & \Omega_1(x, t) \\ 0 & 0 & 1 \end{bmatrix}. \]
\[ (4.26) \]
\[ \Omega_2(x, t) := \frac{1}{2} \Omega_1(x, t)^2 + (\mu_0 - c_2)^{-3} x_2 + (\mu_0 - c_3)^{-3} x_3 - i(1 + (\mu_0 - c_2)^{-4} + (\mu_0 - c_3)^{-4})t. \]
\[ (4.27) \]

**Corollary 4.7.** Let \( k = p = 3, N = 3 \) and \( \kappa = \mu_0 + \eta_0 > 0 \). Define matrices \( C_\Psi, C_\Phi \) and \( R \) via (4.24) and (4.25). Choose \( C_\Psi = \tilde{C}_\Psi \) and \( \delta_0 > 0 \). Then relations (2.6) and (3.3)–(3.5), where \( e_B (-x, -t) = e_A(x, t)^* \) and \( e_A \) is given by (4.26) (using functions \( \Omega_0 \), which are introduced in (4.20), (4.21) and (4.27)), explicitly define non-singular solutions and potentials of TDS.

If we take the \( S_3 \)-node from example 4.6 and set \( \delta_0 = 0 \), then relations (4.1) hold. Therefore, taking into account proposition 4.1, we see that the potential \( \tilde{q} \) is rational. Usually, in the study of lumps it is required that the corresponding potentials (or solutions) are not only rational but also non-singular. To choose non-singular \( \tilde{q} \), recall that (4.18) is valid for \( R \) of the form (4.25). Hence, in view of proposition 4.2, conditions
\[ \kappa = \mu_0 + \eta_0 > 0, \quad \text{Rank}(\tilde{C}_\Psi) = n \leq 3 \]
\[ (4.28) \]

imply that \( \tilde{q} \) is non-singular, and our next corollary follows.
Corollary 4.8. Let \( k = p = 3, N = 3, \) and \( \alpha = i. \) Define matrices \( C_\Psi, \) \( C_\Phi, \) and \( R \) via (4.24) and (4.25). Let numbers \( \kappa, \mu_0, n \) and matrix \( \hat{C}_{\Phi} = \hat{C}_{\Psi}^* \) satisfy (4.28), and set \( S_0 = 0. \) Then relations (2.6) and (3.3)–(3.5), where \( e_B(x, -t) = e_A(x, t) \) and \( e_A \) is given by (4.26), explicitly define non-singular solutions and potentials of TDS. Moreover, the potentials \( \tilde{q} \) are not only non-singular but also rational.

5. Conclusion

Thus, a version of the binary Darboux transformation is constructed for the TDS equation, where \( k \geq 1, \) in theorem 2.3. This is an iterated GBDT version. (No binary Darboux transformations for the TDS equation, where \( k > 1, \) were known before.)

Interesting classes of solutions and potentials are described algebraically in terms of multinodes in theorem 3.2. Namely, a certain generalization of a commutative colligation introduced by M S Livšč and a generalization of the \( S \)-node introduced by L A Sakhnovich, which we call \( S \)-multinode (see definition 3.1 of the \( S \)-multinode), are used in our construction, and could be useful also in constructions of explicit solutions for other multidimensional systems. Another interesting possibility is the application to generalized multidimensional nonlinear Schrödinger equations from [34].

New families of non-singular and rational potentials and solutions are obtained using propositions 4.1 and 4.2. Some results are also new for the case that \( k = 1. \)

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