Two closed geodesics on compact bumpy Finsler manifolds

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Abstract

In this paper, we prove there are at least two closed geodesics on any compact bumpy Finsler n-manifold with \( n \geq 2 \). Thus generically there are at least two closed geodesics on compact Finsler manifolds. Furthermore, there are at least two closed geodesics on any compact Finsler 2-manifold, and this lower bound is achieved by the Katok 2-sphere, cf. [Kat].

Key words: Closed geodesic, Finsler manifold, bumpy.

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1 Introduction and main results

It is well-known that there exist at least two closed geodesics on every compact bumpy Riemannian manifold \( M \) with \( \dim M \geq 2 \), cf. Theorem 4.1.8 of [Kli2]. While its proof depends on the symmetric property for the Riemannian metric, and consequently the proof carries over to the symmetric Finsler case. But for non-symmetric Finsler case, the proof does not work, hence we must develop new methods to handle the problem in this case. This paper is devoted to do this.

Let us recall firstly the definition of the Finsler metrics.

Definition 1.1. (cf. [She]) Let \( M \) be a finite dimensional smooth manifold. A function \( F : TM \to [0, +\infty) \) is a Finsler metric if it satisfies

(F1) \( F \) is \( C^\infty \) on \( TM \setminus \{0\} \),

(F2) \( F(x, \lambda y) = \lambda F(x, y) \) for all \( x \in M \), \( y \in T_x M \) and \( \lambda > 0 \),

(F3) For every \( y \in T_x M \setminus \{0\} \), the quadratic form

\[
g_{x,y}(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \quad \forall u, v \in T_x M,
\]

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is positive definite.

In this case, \((M, F)\) is called a Finsler manifold. \(F\) is symmetric if \(F(x, -y) = F(x, y)\) holds for all \(x \in M\) and \(y \in T_x M\). \(F\) is Riemannian if \(F(x, y)^2 = \frac{1}{2}G(x)y \cdot y\) for some symmetric positive definite matrix function \(G(x) \in GL(T_x M)\) depending on \(x \in M\) smoothly.

A closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [She]). As usual, on any Finsler manifold \((M, F)\), a closed geodesic \(c : S^1 = \mathbb{R}/\mathbb{Z} \to M\) is prime if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the \(m\)-th iteration \(c^m\) of \(c\) is defined by \(c^m(t) = c(mt)\), where \(m \in \mathbb{N}\). The inverse curve \(c^{-1}\) of \(c\) is defined by \(c^{-1}(t) = c(1 - t)\) for \(t \in \mathbb{R}\). Note that unlike Riemannian manifold, the inverse curve \(c^{-1}\) of a closed geodesic \(c\) on a non-symmetric Finsler manifold need not be a geodesic. Two prime closed geodesics \(c\) and \(d\) are distinct if there is no \(\theta \in (0, 1)\) such that \(c(t) = d(t + \theta)\) for all \(t \in \mathbb{R}\). We shall omit the word distinct when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) manifold, two closed geodesics \(c\) and \(d\) are called geometrically distinct if \(c(S^1) \neq d(S^1)\), i.e., their image sets in \(M\) are distinct.

For a closed geodesic \(c\) on \((M, F)\), denote by \(P_c\) the linearized Poincaré map of \(c\). Then \(P_c \in \text{Sp}(2n - 2)\) is symplectic. A closed geodesic \(c\) is called non-degenerate if 1 is not an eigenvalue of \(P_c\). A Finsler manifold \((M, F)\) is called bumpy if all the closed geodesics on it are non-degenerate. Note that bumpy Finsler metrics are generic in the set of Finsler metrics, cf. [Rad4].

The following are the main results in this paper:

**Theorem 1.2.** There exist at least two closed geodesics on every compact bumpy Finsler manifold \((M, F)\) with \(\dim M \geq 2\).

Furthermore, if \(\dim M = 2\), the bumpy condition is not needed, and we have the following:

**Theorem 1.3.** There exist at least two closed geodesics on every compact Finsler manifold \((M, F)\) with \(\dim M = 2\).

**Remark 1.4.** In 1973, Katok in [Kat] found some non-reversible Finsler metrics on CROSSs (compact rank one symmetric spaces) with only finitely many prime closed geodesics and all closed geodesics are non-degenerate. The number of closed geodesics on \(S^n\) that one obtains in these examples is \(2[\frac{a+1}{2}]\), where \([a] = \max\{k \in \mathbb{Z} | k \leq a\}\) for \(a \in \mathbb{R}\), cf. [Zil].

We are aware of a number of results concerning closed geodesics on Finsler manifolds. According to the classical theorem of Lyusternik-Fet [LyF] from 1951, there exists at least one closed geodesic on every compact Riemannian manifold. The proof of this theorem is variational and carries over to the Finsler case. In [BaL], V. Bangert and Y. Long proved that on any Finsler 2-sphere \((S^2, F)\), there exist at least two closed geodesics. In [Rad3], H.-B. Rademacher studied the existence and stability of closed geodesics on positively curved Finsler manifolds. In [DuL1] of Duan and Long and in [Rad4] of Rademacher, they proved there exist at least two closed geodesics on any bumpy Finsler \(n\)-sphere independently. In [Rad5], Rademacher proved there exist at least two closed
geodesics on any bumpy Finsler $\mathbb{CP}^2$. In [DLW], Duan, Long and Wang proved there exist at least two closed geodesics on any compact simply-connected bumpy Finsler manifold. In [LiX], Liu and Xiao proved there exist at least two non-contractible closed geodesics on any bumpy Finsler $\mathbb{RP}^n$. In [LLX], Liu, Long and Xiao proved there exist at least two non-contractible closed geodesics on any bumpy Finsler $S^n/\Gamma$, where $\Gamma$ is a finite group acts on $S^n$ freely and isometrically.

In this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbb{Q}$-coefficients. For terminologies in algebraic topology we refer to [GrH]. For $k \in \mathbb{N}$, we denote by $\mathbb{Q}^k$ the direct sum $\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}$ of $k$ copies of $\mathbb{Q}$ and $\mathbb{Q}^0 = 0$. For an $S^1$-space $X$, we denote by $X$ the quotient space $X/S^1$.

2 Critical point theory for closed geodesics

Let $M = (M, F)$ be a compact Finsler manifold, the space $\Lambda = \Lambda_M$ of $H^1$-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries, c.f. [Kli1]-[Kli3]. This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt. \quad (2.1)$$

It is $C^{1,1}$ and invariant under the $S^1$-action, cf. [Mer]. The critical points of $E$ of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional $E$ is well defined along any closed geodesic $c$ on $M$, which we denote by $E''(c)$. As usual we define the index $i(c)$ of $c$ as the maximal dimension of subspaces of $T_c \Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of $c$ so that $\nu(c) + 1$ is the dimension of the null space of $E''(c)$, cf. Definition 2.5.4 of [Kli3]. In the following, we denote by

$$\Lambda^\kappa = \{ d \in \Lambda \mid E(d) \leq \kappa \}, \quad \Lambda^- = \{ d \in \Lambda \mid E(d) < \kappa \}, \quad \forall \kappa \geq 0. \quad (2.2)$$

For a closed geodesic $c$ we set $\Lambda(c) = \{ \gamma \in \Lambda \mid E(\gamma) < E(c) \}$.

For $m \in \mathbb{N}$ we denote the $m$-fold iteration map $\phi_m : \Lambda \to \Lambda$ by $\phi_m(\gamma)(t) = \gamma(mt)$, for all $\gamma \in \Lambda, t \in S^1$, as well as $\gamma^m = \phi_m(\gamma)$. If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of $\gamma$ is the order of the isotropy group $\{ s \in S^1 \mid s \cdot \gamma = \gamma \}$. For a closed geodesic $c$, the mean index $\hat{i}(c)$ is defined as usual by $\hat{i}(c) = \lim_{m \to \infty} i(c^m)/m$.

We call a closed geodesic satisfying the isolation condition, if the following holds:

(Iso) For all $m \in \mathbb{N}$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of $E$.

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).
Using singular homology with rational coefficients we consider the following critical $\mathbb{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\overline{C}_s(E, c) = H_* \left( (\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1 \right).$$

\textbf{Proposition 2.1.} (cf. Satz 6.11 of [Rad2] or Proposition 3.12 of [BaL]) Let $c$ be a prime closed geodesic on a Finsler manifold $(M, F)$ satisfying (Iso). Then we have

$$\overline{C}_q(E, c^m) \equiv H_q \left( (\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1 \right)$$

$$= \left( H_i(c^m)(U_{c^m} \cup \{c^m\}, U_{c^m}^-) \otimes H_{q-i}(c^m)(N_{c^m} \cup \{c^m\}, N_{c^m}^-) \right)^+ \mathbb{Z}^m$$

(i) When $\nu(c^m) = 0$, there holds

$$\overline{C}_q(E, c^m) = \begin{cases} \mathbb{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbb{Z}, \text{ and } q = i(c^m), \\ 0, & \text{otherwise}. \end{cases}$$

(ii) When $\nu(c^m) > 0$, there holds

$$\overline{C}_q(E, c^m) = H_{q-i}(c^m)(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)^{-i(c^m) - i(c)} \mathbb{Z}^m,$$

where $N_{c^m}$ is a local characteristic manifold at $c^m$ and $N_{c^m}^- = N_{c^m} \cap \Lambda(c^m)$, $U_{c^m}$ is a local negative disk at $c^m$ and $U_{c^m}^- = U_{c^m} \cap \Lambda(c^m)$, $H_*(X, A)^\mathbb{Z} = \{ \xi \} \in H_*(X, A) \mid T_* \xi = \pm \xi \}$ where $T$ is a generator of the $\mathbb{Z}_m$-action.

\textbf{Definition 2.2.} The Euler characteristic $\chi(c^m)$ of $c^m$ is defined by

$$\chi(c^m) \equiv \chi \left( (\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1 \right),$$

$$\equiv \sum_{q=0}^{\infty} (-1)^q \dim \overline{C}_q(E, c^m).$$

Here $\chi(A, B)$ denotes the usual Euler characteristic of the space pair $(A, B)$.

The average Euler characteristic $\hat{\chi}(c)$ of $c$ is defined by

$$\hat{\chi}(c) = \lim_{N \to \infty} \frac{1}{N} \sum_{1 \leq m \leq N} \chi(c^m).$$

By Remark 5.4 of [Wan], $\hat{\chi}(c)$ is well-defined and is a rational number. In particular, if $c^m$ are non-degenerate for $\forall m \in \mathbb{N}$, then

$$\hat{\chi}(c) = \begin{cases} (-1)^i(c), & \text{if } i(c^2) - i(c) \in 2\mathbb{Z}, \\ \frac{(-1)^i(c)}{2}, & \text{otherwise}. \end{cases}$$

Set $\overline{T}^0 = \overline{T}^0 M = \{ \text{constant point curves in } M \} \equiv M$. Let $(X, Y)$ be a space pair such that the Betti numbers $b_i = b_i(X, Y) = \dim H_i(X, Y; \mathbb{Q})$ are finite for all $i \in \mathbb{Z}$. As usual the Poincaré series of $(X, Y)$ is defined by the formal power series $P(X, Y) = \sum_{i=0}^{\infty} b_i t^i$. We need the following results on Betti numbers.
For a compact and simply-connected Finsler manifold $M$ with $H^*(M; \mathbb{Q}) \cong T_{d,h+1}(x)$ with the generator $x$ of degree $d$ and height $h + 1$, if $d$ is odd, then $x^2 = 0$ and $h = 1$ in $T_{d,h+1}(x)$, thus $M$ is rationally homotopy equivalent to $S^d$ (cf. [Rad1] or [Hin]).

**Proposition 2.3.** (cf. Theorem 2.4, Remark 2.5 of [Rad1] and Lemma 2.5, 2.6 of [DuL2]) Let $M$ be a compact simply-connected manifold with $H^*(M; \mathbb{Q}) \cong T_{d,h+1}(x)$. Then the Betti numbers of the free loop space of $M$ defined by $b_q = \text{rank}H_q(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Q})$ for $q \in \mathbb{Z}$ are given by

(i) If $h = 1$ and $d \in 2N + 1$, then we have

$$b_q = \begin{cases} 
2, & \text{if } q \in K \equiv \{k(d-1) | 2 \leq k \in \mathbb{N}\}, \\
1, & \text{if } q \in \{d - 1 + 2k | k \in \mathbb{N}_0\} \setminus K, \\
0 & \text{otherwise}.
\end{cases} \quad (2.7)$$

(ii) If $h = 1$ and $d \in 2N$, then we have

$$b_q = \begin{cases} 
2, & \text{if } q \in K \equiv \{k(d-1) | 3 \leq k \in (2N + 1)\}, \\
1, & \text{if } q \in \{d - 1 + 2k | k \in \mathbb{N}_0\} \setminus K, \\
0 & \text{otherwise}.
\end{cases} \quad (2.8)$$

(iii) If $h \geq 2$ and $d \in 2N$. Let $D = d(h + 1) - 2$ and

$$\Omega(d, h) = \{k \in 2N - 1 | iD \leq k - (d - 1) = iD + jd \leq iD + (h - 1)d \}
\text{ for some } i \in \mathbb{N} \text{ and } j \in [1, h - 1]. \quad (2.9)$$

Then we have

$$b_q = \begin{cases} 
0, & \text{if } q \text{ is even or } q \leq d - 2, \\
\frac{2^{-(d-1)}}{q} + 1, & \text{if } q \in 2N - 1 \text{ and } d - 1 \leq q < d - 1 + (h - 1)d, \\
h + 1, & \text{if } q \in \Omega(d, h), \\
h, & \text{otherwise}.
\end{cases} \quad (2.10)$$

By a similar proof of Theorem 5.5 of [Wan], we have the following mean index identity:

**Proposition 2.4.** (cf. Theorem 3.1 of [Rad1] and Satz 7.9 of [Rad2]) Let $(M, F)$ be a compact simply-connected Finsler manifold with $H^*(M, \mathbb{Q}) = T_{d,h+1}(x)$ and possess finitely many prime closed geodesics. Denote prime closed geodesics on $(M, F)$ with positive mean indices by $\{c_j\}_{1 \leq j \leq q}$ for some $q \in \mathbb{N}$. Then the following identity holds

$$\sum_{j=1}^{q} \hat{\chi}(c_j) = B(d, h) = \begin{cases} 
\frac{h(h+1)d}{2d(h+1) - 4}, & \text{if } d \text{ is even,} \\
\frac{d+1}{2d-2}, & \text{if } d \text{ is odd (then } h = 1),
\end{cases} \quad (2.11)$$

where $\text{dim } M = hd$.

We have the following version of the Morse inequality.
Theorem 2.5. (Theorem 6.1 of [Rad2]) Suppose that there exist only finitely many prime closed geodesics \( \{c_j\}_{1 \leq j \leq p} \) on \((M, F)\), and \(0 \leq a < b \leq \infty\) are regular values of the energy functional \(E\). Define for each \(q \in \mathbb{Z}\),

\[
M_q(\Lambda^0, \Lambda^a) = \sum_{1 \leq j \leq p, a < E(c_j^m) < b} \text{rank} \mathcal{U}_q(E, c_j^m),
\]

\[
b_q(\Lambda^0, \Lambda^a) = \text{rank} H_q(\Lambda^0, \Lambda^a).
\]

Then there holds

\[
M_q(\Lambda^0, \Lambda^a) - M_{q-1}(\Lambda^0, \Lambda^a) + \cdots + (-1)^q M_0(\Lambda^0, \Lambda^a) \geq b_q(\Lambda^0, \Lambda^a) - b_{q-1}(\Lambda^0, \Lambda^a) + \cdots + (-1)^q b_0(\Lambda^0, \Lambda^a),
\]

(2.12)

\[
M_q(\Lambda^0, \Lambda^a) \geq b_q(\Lambda^0, \Lambda^a).
\]

(2.13)

3 Proof of main theorems

In this section, we give the proofs of the main theorems. We prove by contradiction, by [LyF] we suppose the following holds:

(C) There is only one prime closed geodesic \(c\) on \((M, F)\).

Proof of Theorem 1.2. We have the following two cases:

Case 1. \(\pi_1(M)\) is a infinite group.

Sub-Case 1.1. \(\pi_1(M) = \mathbb{Z}\).

In this case, there must be infinitely many closed geodesics on \((M, F)\) by the same proof of [BaH] since their proof works well in the Finsler case. This proves Theorem 1.2 in this case.

Sub-Case 1.2. \(\pi_1(M) \neq \mathbb{Z}\).

In this case, for any \(\alpha \in \pi_1(M)\), Let \(c_\alpha \in \Lambda\) such that \([c_\alpha] = \alpha\) and \(E(c_\alpha) = \inf \{E(d) | d \in \Lambda, [d] = \alpha\}\). Then \(c_\alpha\) is a closed geodesic, and we have \(c_\alpha = c^{m_\alpha}\) for some \(m_\alpha \in \mathbb{N}\) by the assumption (C). Therefore \(\pi_1(M) = \{[c^m]\}_{m \in \mathbb{Z}}\), this contradicts to \(\pi_1(M)\) is infinite and \(\pi_1(M) \neq \mathbb{Z}\). This proves Theorem 1.2 in this case.

Case 2. \(\pi_1(M)\) is a finite group.

In this case, let \(p : \tilde{M} \rightarrow M\) be the universal covering of \(M\) and \(\tilde{F} = p^*(F)\). Then it is clear that \((\tilde{M}, \tilde{F})\) is a compact Finsler manifold and it is locally isometric to \((M, F)\). Moreover, if \(\tilde{d}\) is a closed geodesic on \((\tilde{M}, \tilde{F})\), then \(d = p(\tilde{d})\) is a closed geodesic on \((M, F)\) and \(P_d = P_{\tilde{d}}\). Since \((M, F)\) is bumpy, \(d\) is non-degenerate, i.e. \(1 \notin \sigma(P_d)\), therefore \(\tilde{d}\) is non-degenerate, hence \((\tilde{M}, \tilde{F})\) is bumpy. Now we have the following two sub-cases:

Sub-Case 2.1. \(H^*(\tilde{M}; \mathbb{Q}) \neq T_{d, h+1}(x)\).

In this case, there must be infinitely many closed geodesics on \((\tilde{M}, \tilde{F})\) by [ViS] of M. Vigué-Poirrier and D. Sullivan. In fact, the Betti numbers of the free loop space \(\Lambda \tilde{M}\) are unbounded and
the theorem in [GrM] of Gromoll-Meyer can be applied to the Finsler manifolds as well. Hence there
must be infinitely many closed geodesics on \((M, F)\) since \(\pi_1(M)\) is finite. This proves Theorem 1.2
in this case.

**Sub-Case 2.2.** \(H^*(\tilde{M} ; Q) = T_{d,h+1}(x)\).

In this case, if \(\pi_1(M) = 0\), then Theorem 1.2 holds by [DLW], thus it remains to consider the
case that \(\pi_1(M) \neq 0\) and there are finitely many prime closed geodesics on \((\tilde{M}, \tilde{F})\).

Denote by \(\{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k\}\) the prime closed geodesics on \((\tilde{M}, \tilde{F})\), then we have \(k \geq 2\) by [DLW].
Thus for \(1 \leq i \leq k\), we have \(p(\tilde{d}_i) = c^{m_i}\) for some \(m_i \in \mathbb{N}\) by the assumption \((C)\). By translating
the parameters if necessary, we may assume \(p(\tilde{d}_j(0)) = c^{m_j}(0) = c(0)\) for \(1 \leq j \leq k\).

Note that for each \(i \in \{2, \ldots, k\}\), there exists a covering transformation \(f_i : (\tilde{M}, \tilde{F}) \to (\tilde{M}, \tilde{F})\)
such that \(f_i(\tilde{d}_1(0)) = \tilde{d}_1(0)\). By the definition of the Finsler metric \(\tilde{F}\) on \(\tilde{M}\), the map \(f_i\) is an
isometry on \((\tilde{M}, \tilde{F})\). Therefore \(f_i(\tilde{d}_i)\) is a closed geodesic started at \(f_i(\tilde{d}_i(0)) = \tilde{d}_1(0)\). By the
property of covering transformation, we have
\[
p(f_i(\tilde{d}_i(t))) = p(\tilde{d}_i(t)) = c^{m_i}(t) = c(m_i t), \quad \forall t \in \mathbb{R}.
\]
On the other hand,
\[
p(\tilde{d}_1(t)) = c^{m_1}(t) = c(m_1 t), \quad \forall t \in \mathbb{R}.
\]
Hence we have \(f_i(\tilde{d}_i(t)) = \tilde{d}_1(\frac{m_i}{m_1} t)\) for \(\forall t \in \mathbb{R}\). Since \(f_i(\tilde{d}_i)\) is a closed geodesic and \(\tilde{d}_1\) is a prime
closed geodesic, we have \(m_1 \mid m_i\). Exchanging \(\tilde{d}_1\) and \(\tilde{d}_i\), we obtain \(m_i \mid m_1\), and then \(m_i = m_1\).
This yields \(f_i(\tilde{d}_i) = \tilde{d}_1\). Since \(f_i\) is an isometry on \((\tilde{M}, \tilde{F})\), it preserves the energy functional, i.e.,
\(E(\gamma) = E(f_i(\gamma))\) for any \(\gamma \in \Lambda(M)\). This implies \(i(\tilde{d}_i^m) = i(\tilde{d}_1^m)\) for any \(m \in \mathbb{N}\). By Proposition
2.1, we have
\[
M_q \equiv M_q(\tilde{\Lambda}, \tilde{\Lambda}^0) = \sum_{1 \leq j \leq k, \ m \in \mathbb{N}} \text{rank} \mathcal{C}_q(E, \tilde{d}_j^m)
= \sum_{1 \leq j \leq k} \#\{m \mid i(\tilde{d}_j^m) \in 2\mathbb{Z} \text{ and } q = i(\tilde{d}_j^m)\}
= k \#\{m \mid i(\tilde{d}_1^m) \in 2\mathbb{Z} \text{ and } q = i(\tilde{d}_1^m)\}
\]
By Bott formula, c.f. [Bot], \(i(\tilde{d}_1^m) \geq i(\tilde{d}_1)\) for \(m \in \mathbb{N}\), then we have \(M_q = 0\) for \(q < i(\tilde{d}_1)\) by (3.3).
By Proposition 2.1 and (3.3), we have \(M_{i(\tilde{d}_1)} = k \# \{m \mid i(\tilde{d}_1^m) = i(\tilde{d}_1)\}\) and \(M_{i(\tilde{d}_1)+1} = 0\). We claim
that \(i(\tilde{d}_1) = d - 1\). In fact, by Proposition 2.3 and Theorem 2.5, \(M_{d-1} \geq b_{d-1} = 1\), this implies
\(i(\tilde{d}_1) \leq d - 1\). By (3.3), there exists \(m \in \mathbb{N}\) such that \(d - 1 = i(\tilde{d}_1^m)\) and \(i(\tilde{d}_1^m) - i(\tilde{d}_1) \in 2\mathbb{Z}\), this
implies \(i(\tilde{d}_1) - (d - 1) \in 2\mathbb{Z}\). Thus if \(i(\tilde{d}_1) < d - 1\), we must have \(i(\tilde{d}_1) < d - 2\) holds. Hence by
Theorem 2.5 and Proposition 2.3,
\[
-k \#\{m \mid i(\tilde{d}_1^m) = i(\tilde{d}_1)\}
\]
This contradiction proves that $i(\tilde{d}_1) = d - 1$. By Theorem 2.5 and Proposition 2.3 again,

$$-k \# \{ m \mid i(\tilde{d}_1^m) = i(\tilde{d}_1) = d - 1 \} = M_d - M_{d-1} + \cdots + (-1)^d M_0 \geq b_d - b_{d-1} + \cdots + (-1)^d b_0 = -1.$$  

(3.5)

This contradicts to $k \geq 2$ and proves Theorem 1.2.

**Proof of Theorem 1.3.** We have the following two cases:

**Case 1.** $\pi_1(M)$ is a infinite group.

Note that the same proof of Theorem 1.2 yields two closed geodesics and proves Theorem 1.3 in this case.

**Case 2.** $\pi_1(M)$ is a finite group.

As in Thereom 1.2, let $p: \tilde{M} \to M$ be the universal covering of $M$ and $\tilde{F} = p^*(F)$. Then $(\tilde{M}, \tilde{F})$ is a compact simply-connected Finsler manifold of dimension 2. Due to the classification of surfaces, cf. Chapter 9 of [Hir], $\tilde{M} = S^2$.

Clearly it is sufficient to consider the case that $(S^2, \tilde{F})$ possess finitely many closed geodesics. Denote by $\{ \tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k \}$ the prime closed geodesics on $(S^2, \tilde{F})$. Thus for $1 \leq i \leq k$, we have $p(\tilde{d}_i) = e^{m_i}$ for some $m_i \in N$ by the assumption (C). As in Theorem 1.2, we have $m_i = m_1$ and $i(\tilde{d}_i^m) = i(\tilde{d}_1^m)$ and $k_l(\tilde{d}_i^m)^{\pm 1} = k_l(\tilde{d}_1^m)^{\pm 1}$ for any $m \in N$, $l \in Z$. Therefore we obtain $\hat{\chi}(\tilde{d}_i) = \hat{\chi}(\tilde{d}_1)$. By Lemma 4.3 and Theorem 4.4 of [LoW], we have $\hat{i}(\tilde{d}_1) > 0$. By Proposition 2.4 or Theorem 4.4 of [LoW], we have

$$k \frac{\hat{\chi}(\tilde{d}_1)}{\hat{i}(\tilde{d}_1)} = \sum_{j=1}^{k} \frac{\hat{\chi}(\tilde{d}_j)}{\hat{i}(\tilde{d}_j)} = B(2, 1) = 1.$$  

(3.6)

Since $\hat{\chi}(\tilde{d}_1) \in Q$, this implies $\hat{i}(\tilde{d}_1) \in Q$ also. Thus there is no irrationally elliptic closed geodesic on $(S^2, \tilde{F})$ since such a closed geodesic $d$ must satisfy $i(d) \notin Q$, cf. Section 3 of [LoW]. This contradicts to [LoW] which claims there exist at least two irrationally elliptic prime closed geodesics on every Finsler 2-sphere possessing only finitely many prime closed geodesics. This proves Theorem 1.3.

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