COUNTING CLUSTER-TILTED ALGEBRAS OF TYPE $A_n$

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Abstract. The purpose of this paper is to give an explicit formula for the number of non-isomorphic cluster-tilted algebras of type $A_n$, by counting the mutation class of any quiver with underlying graph $A_n$. It will also follow that if $T$ and $T'$ are cluster-tilting objects in a cluster category $C$, then $\text{End}_C(T)$ is isomorphic to $\text{End}_C(T')$ if and only if $T = \tau T'$.

1. Cluster-tilted algebras

The cluster category was introduced independently in [7] for type $A_n$ and in [2] for the general case. Let $D^b(\text{mod } H)$ be the bounded derived category of the finitely generated modules over a finite dimensional hereditary algebra $H$ over a field $K$. In [2] the cluster category was defined as the orbit category $C = D^b(\text{mod } H)/\tau^{-1}$, where $\tau$ is the Auslander-Reiten translation and [1] the suspension functor. The cluster-tilted algebras are the algebras of the form $\Gamma = \text{End}_C(T)^{\text{op}}$, where $T$ is a cluster-tilting object in $C$. See [3].

Let $Q$ be a quiver with no multiple arrows, no loops and no oriented cycles of length two. Mutation of $Q$ at vertex $k$ is a quiver $Q'$ obtained from $Q$ in the following way.

1. Add a vertex $k^*$.
2. If there is a path $i \rightarrow k \rightarrow j$, then if there is an arrow from $j$ to $i$, remove this arrow. If there is no arrow from $j$ to $i$, add an arrow from $i$ to $j$.
3. For any vertex $i$ replace all arrows from $i$ to $k$ with arrows from $k^*$ to $i$, and replace all arrows from $k$ to $i$ with arrows from $i$ to $k^*$.
4. Remove the vertex $k$.

We say that a quiver $Q$ is mutation equivalent to $Q'$, if $Q'$ can be obtained from $Q$ by a finite number of mutations. The mutation class of $Q$ is all quivers mutation equivalent to $Q$. It is known from [11] that the mutation class of a Dynkin quiver $Q$ is finite.

If $\Gamma$ is a cluster-tilted algebra, then we say that $\Gamma$ is of type $A_n$ if it arises from the cluster category of a path algebra of Dynkin type $A_n$.

Let $Q$ be a quiver of a cluster-tilted algebra $\Gamma$. From [3], it is known that if $Q'$ is obtained from $Q$ by a finite number of mutations, then there is a cluster-tilted
algebra $\Gamma'$ with quiver $Q'$. Moreover, $\Gamma$ is of finite representation type if and only if $\Gamma'$ is of finite representation type [3]. We also have that $\Gamma$ is of type $A_n$ if and only if $\Gamma'$ is of type $A_n$. From [3] we know that a cluster-tilted algebra is up to isomorphism uniquely determined by its quiver. See also [8].

It follows from this that to count the number of cluster-tilted algebras of type $A_n$, it is enough to count the mutation class of any quiver with underlying graph $A_n$.

2. Category of diagonals of a regular $n+3$ polygon

We recall some results from [7].

Let $n$ be a positive integer and let $\mathcal{P}_{n+3}$ be a regular polygon with $n+3$ vertices. A diagonal is a straight line between two non-adjacent vertices on the border. A triangulation is a maximal set of diagonals which do not cross. If $\Delta$ is any triangulation of $\mathcal{P}_{n+3}$, we know that $\Delta$ consists of exactly $n$ diagonals.

Let $\alpha$ be a diagonal between vertex $v_1$ and vertex $v_2$ on the border of $\mathcal{P}_{n+3}$. In [7] a pivoting elementary move $P(v_1)$ is an anticlockwise move of $\alpha$ to another diagonal $\alpha'$ about $v_1$. The vertices of $\alpha'$ are $v_1$ and $v'_2$, where $v_2$ and $v'_2$ are vertices of a border edge and rotation is anticlockwise. A pivoting path from $\alpha$ to $\alpha'$ is a sequence of pivoting elementary moves starting at $\alpha$ and ending at $\alpha'$.

Fix a positive integer $n$. Categories of diagonals of regular $(n+3)$-polygons were introduced in [7]. Let $\mathcal{C}_n$ be the category with indecomposable objects all diagonals of the polygon, and we take as objects formal direct sums of these diagonals. Morphisms from $\alpha$ to $\alpha'$ are generated by elementary pivoting moves modulo the mesh relations, which are defined as follows. Let $\alpha$ and $\beta$ be diagonals, with $a$ and $b$ the vertices of $\alpha$ and $c$ and $d$ the vertices of $\beta$. Suppose $P(c)P(a)$ takes $\alpha$ to $\beta$. Then $P(c)P(a) = P(d)P(b)$. Furthermore, if one of the intermediate edges in a pivoting elementary move is a border edge, this move is zero. It is shown in [7] that this category is equivalent to the cluster category defined in Section 1 in the $A_n$ case.

We have the following from [7].

- The irreducible morphisms in $\mathcal{C}_n$ are the direct sums of pivoting elementary moves.
- The Auslander-Reiten translation of a diagonal is given by clockwise rotation of the polygon.
- $\text{Ext}^1_{\mathcal{C}_n}(\alpha, \alpha') = \text{Ext}^1_{\mathcal{C}}(\alpha, \alpha') = 0$ if and only if $\alpha$ and $\alpha'$ do not cross.

It follows that a tilting object in $\mathcal{C}$ corresponds to a triangulation of $\mathcal{P}_{n+3}$.

For any triangulation $\Delta$ of $\mathcal{P}_{n+3}$, it is possible to define a quiver $Q_\Delta$ with $n$ vertices in the following way. The vertices of $Q_\Delta$ are the midpoints of the diagonals.
of $\Delta$. There is an arrow between $i$ and $j$ in $Q_\Delta$ if the corresponding diagonals bound a common triangle. The orientation is $i \to j$ if the diagonal corresponding to $j$ is obtained from the diagonal corresponding to $i$ by rotating anticlockwise about their common vertex. It is known from [7] that all quivers obtained in this way are quivers of cluster-tilted algebras of type $A_n$.

We defined the mutation of a quiver of a cluster-tilted algebra above. We also define mutation of a triangulation at a given diagonal, by replacing this diagonal with another one. This can be done in one and only one way. Let $Q_\Delta$ be a quiver corresponding to a triangulation $\Delta$. Then mutation of $Q_\Delta$ at the vertex $i$ corresponds to mutation of $\Delta$ at the diagonal corresponding to $i$.

It follows that any triangulation gives rise to a quiver of a cluster-tilted algebra, and that a quiver of a cluster-tilted algebra can be associated to at least one triangulation.

Let $\mathcal{M}_n$ be the mutation class of $A_n$, i.e. all quivers obtained by repeated mutation from $A_n$, up to isomorphisms of quivers. Let $\mathcal{T}_n$ be the set of all triangulations of $P_{n+3}$. We can define a function $\gamma : \mathcal{T}_n \to \mathcal{M}_n$, where we set $\gamma(\Delta) = Q_\Delta$ for any triangulation $\Delta$ in $\mathcal{T}_n$. Note that $\gamma$ is surjective.

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If $a$ and $b$ are vertices on the border of a regular polygon, we say that the distance between $a$ and $b$ is the smallest number of border edges between them. Let us say that a diagonal from $a$ to $b$ is close to the border if the distance between $a$ and $b$ is exactly 2. For a quiver $Q_\Delta$ corresponding to a triangulation $\Delta$, let us always write $v_\alpha$ for the vertex of $Q_\Delta$ corresponding to the diagonal $\alpha$.

If $Q$ is a quiver of a cluster-tilted algebra of type $A_n$, we we have the following facts [6, 7, 12].

- All cycles are oriented.
- All cycles are of length 3.
- There does not exist two cycles that share one arrow.

Lemma 3.1. If a diagonal $\alpha$ of a triangulation $\Delta$ is close to the border, then the corresponding vertex $v_\alpha$ in $\gamma(\Delta) = Q_\Delta$ is either a source, a sink or lies on a cycle (oriented of length 3).

Proof. All cycles are oriented and of length 3 in the $A_n$ case. Suppose that $\alpha$ is a diagonal in $\Delta$ which is close to the border. There are only three cases to consider, shown in Figure 1.

In the first case $\alpha$ corresponds to a sink. There is no other vertex adjacent to $v_\alpha$ but $v_\beta$, or else the corresponding diagonal of this vertex would cross $\beta$. We have
the same for the second case where $\alpha$ is a source. In the third case $v_\alpha$ lies on a cycle. \qed

Note that if $v_\alpha$ is a sink (or source) then $v_\alpha$ has only one adjacent vertex if and only if $\alpha$ is close to the border.

Lemma 3.2. Let $\Delta$ be a triangulation and let $\gamma(\Delta) = Q_\Delta$ be the corresponding quiver. A quiver $Q'$ obtained from $Q_\Delta$ by factoring out a vertex $v_\alpha$ is connected if and only if the corresponding diagonal $\alpha$ is close to the border.

Proof. Suppose $\alpha$ is close to the border. By Lemma 3.1, $\alpha$ corresponds to a sink, a source or a vertex on a cycle. If $v_\alpha$ is a sink or a source then $v_\alpha$ has only one adjacent vertex, so factoring out $v_\alpha$ does not disconnect the quiver. Suppose $v_\alpha$ lies on a cycle. Then we are in the case shown in the third picture in Figure 1. We see that there can be no other vertex adjacent to $v_\alpha$ except $v_\beta$ and $v_\beta'$, since else the corresponding diagonal would cross $\beta$ or $\beta'$. Hence factoring out $v_\alpha$ does not disconnect the quiver.

Next, suppose that factoring out $v_\alpha$ does not disconnect the quiver. If $v_\alpha$ is a source or a sink with only one adjacent vertex, then $v_\alpha$ is close to the border. If not, first suppose $v_\alpha$ does not lie on a cycle. Then it is clear that factoring out $v_\alpha$ disconnects the quiver, so we may assume that $v_\alpha$ lies on a cycle. Then $\alpha$ is an edge of a triangle consisting of only diagonals (i.e. no border edges), say $\beta$ and $\beta'$. Suppose there is a vertex $v_\delta$ adjacent to $v_\alpha$, with $v_\delta \neq v_\beta$ and $v_\delta \neq v_\beta'$. Then $v_\delta$ can not be adjacent to $v_\beta$ or $v_\beta'$, since then we would have two cycles sharing one arrow. We also see that $v_\delta$ can not be adjacent to any vertex $v_\gamma$ from which there exists a path to $v_\beta$ or $v_\beta'$ not containing $v_\alpha$, or else there would be a cycle of length greater than 3. Therefore factoring out $v_\alpha$ would disconnect the quiver, and this is a contradiction, thus there can be no other vertices adjacent to $v_\alpha$. It follows that
\(\alpha\) can not be adjacent to any other diagonal but \(\beta\) and \(\beta'\), hence \(\alpha\) is close to the border.

\[\square\]

Let \(\Delta\) be a triangulation of \(\mathcal{P}_{n+3}\) and let \(\alpha\) be a diagonal close to the border. The triangulation \(\Delta'\) of \(\mathcal{P}_{n+3-1}\) obtained from \(\Delta\) by factoring out \(\alpha\) is defined as the triangulation of \(\mathcal{P}_{n+3-1}\) by letting \(\alpha\) be a border edge and leaving all the other diagonals unchanged. We write \(\Delta/\alpha\) for the new triangulation obtained. See Figure 2.

\[\text{Figure 2. Factoring out a diagonal close to the border}\]

**Lemma 3.3.** Let \(\Delta\) be a triangulation and \(\gamma(\Delta) = Q_\Delta\). Factoring out a vertex in \(Q_\Delta\) such that the resulting quiver is connected, corresponds to factoring out a diagonal of \(\Delta\) close to the border.

**Proof.** Factoring out a vertex \(v_\alpha\) in \(Q\) such that the resulting quiver is connected, implies that \(\alpha\) is close to the border by Lemma 3.2. Then consider all cases shown in Figure 1. \[\square\]

Note that this means that \(\gamma(\Delta/\alpha) = Q_\Delta/v_\alpha\). We have the following easy fact.

**Proposition 3.4.** Let \(Q\) be a quiver of a cluster-tilted algebra of type \(A_n\), with \(n \geq 3\). Let \(Q'\) be obtained from \(Q\) by factoring out a vertex such that \(Q'\) is connected. Then \(Q'\) is the quiver of some cluster-tilted algebra of type \(A_{n-1}\).

**Proof.** It is already known from [4] that \(Q'\) is the quiver of a cluster-tilted algebra. Suppose \(\Delta\) is a triangulation of \(\mathcal{P}_{n+3}\) such that \(\gamma(\Delta) = Q\). Such a \(\Delta\) exists since \(\gamma\) is surjective. It is enough, by Lemma 3.2, to consider vertices corresponding to a diagonal close to the border. By Lemma 3.3, factoring out a vertex corresponding to a diagonal \(\alpha\) close to the border, corresponds to factoring out \(\alpha\). Then the resulting triangulation of \(\mathcal{P}_{(n-1)+3}\) corresponds to a quiver of a cluster-tilted algebra of type \(A_{n-1}\), since it is a triangulation. \[\square\]

Now we want to do the opposite of factoring out a vertex close to the border. If \(\Delta\) is a triangulation of \(\mathcal{P}_{n+3}\), we want to add a diagonal \(\alpha\) such that \(\alpha\) is a diagonal
close to the border and such that $\Delta \cup \alpha$ is a triangulation of $P_{(n+1)+3}$. Consider any border edge $m$ on $P_{n+3}$. Then we have one of the cases shown in Figure 3.

![Figure 3.](image)

We can extend the polygon at $m$ for each case in Figure 3 and add a diagonal $\alpha$ to the extension. See Figure 4 for the corresponding extensions at $m$.

![Figure 4.](image)

It follows that for a given diagonal $\beta$, there are at most three ways to extend the polygon with a diagonal $\alpha$ such that $\alpha$ is adjacent to $\beta$, and it is easy to see that these extensions give non-isomorphic quivers.

For a triangulation $\Delta$ of $P_{n+3}$, let us denote by $\Delta(i)$ the triangulation obtained from $\Delta$ by rotating $\Delta$ $i$ steps in the clockwise direction. We define an equivalence relation on $T_n$, where we let $\Delta \sim \Delta(i)$ for all $i$. We define a new function $\tilde{\gamma} : (T_n/\sim) \to M_n$ induced from $\gamma$. This is well defined, for if $\Delta = \Delta'(i)$ for an $i$, then obviously $Q_\Delta = Q_{\Delta'}$ in $M_n$. And hence since $\gamma$ is a surjection, we also have that $\tilde{\gamma}$ is a surjection. We actually have the following.

**Theorem 3.5.** The function $\tilde{\gamma} : (T_n/\sim) \to M_n$ is bijective for all $n \geq 2$.

**Proof.** We already know that $\tilde{\gamma}$ is surjective.

Suppose $\tilde{\gamma}(\Delta) = \tilde{\gamma}(\Delta')$ in $M_n$. We want to show that $\Delta = \Delta'$ in $(T_n/\sim)$ using induction.
It is easy to check that \((T_3/\sim) \to M_3\) is injective. Suppose \((T_{n-1}/\sim) \to M_{n-1}\) is injective. Let \(\alpha\) be a diagonal close to the border in \(\Delta\), with image \(v_\alpha\) in \(Q\), where \(Q\) is a representative for \(\tilde{\gamma}(\Delta)\). Then the diagonal \(\alpha'\) in \(\Delta'\) corresponding to \(v_\alpha\) in \(Q\) is also close to the border. We have \(\tilde{\gamma}(\Delta/\alpha) = \tilde{\gamma}(\Delta'/\alpha') = Q/v_\alpha\) by Lemma 3.3 and hence, by hypothesis, \(\Delta/\alpha = \Delta'/\alpha'\) in \((T_n/\sim)\).

We can obtain \(\Delta\) and \(\Delta'\) from \(\Delta/\alpha = \Delta'/\alpha'\) by extending the polygon at some border edge. Fix a diagonal \(\beta\) in \(\Delta\) such that \(v_\alpha\) and \(v_\beta\) are adjacent. This can be done since \(Q\) is connected. Let \(\beta'\) be the diagonal in \(\Delta'\) corresponding to \(v_\beta\). By the above there are at most three ways to extend \(\Delta/\alpha\) such that the new diagonal is adjacent to \(\beta\). It is clear that these extensions will be mapped by \(\tilde{\gamma}\) to non-isomorphic quivers. Also there are at most three ways to extend \(\Delta'/\alpha'\) such that the new diagonal is adjacent to \(\beta'\), and all these extensions are mapped to non-isomorphic quivers, thus \(\Delta = \Delta'\) in \((T_n/\sim)\).

Note that this also means that \(\Delta = \Delta'(i)\) for an \(i\) if and only if \(Q_\Delta \simeq Q_{\Delta'}\) as quivers.

Now, let \(T\) be a cluster-tilting object of the cluster category \(C\). This object corresponds to a triangulation \(\Delta\) of \(P_{n+3}\), and all tilting objects obtained from rotation of \(\Delta\) gives the same cluster-tilted algebra. No other triangulation gives rise to the same cluster-tilted algebra.

The Catalan number \(C(i)\) can be defined as the number of triangulations of an \(i\)-polygon with \(i - 3\) diagonals. The number is given by the following formula.

\[
C(i) = \frac{(2i)!}{(i+1)!i!}
\]

We now have the following.

**Corollary 3.6.** The number \(a(n)\) of non-isomorphic basic cluster-tilted algebras of type \(A_n\) is the number of triangulations of the disk with \(n\) diagonals, i.e.

\[
a(n) = C(n+1)/(n+3) + C((n+1)/2)/2 + (2/3)C(n/3),
\]

where \(C(i)\) is the \(i\)'th Catalan number and the second term is omitted if \((n+1)/2\) is not an integer and the third term is omitted if \(n/3\) is not an integer.

These numbers appeared in a paper by W. G. Brown in 1964 [1]. See Table 1 for some values of \(a(n)\).

We have that if \(T\) is a cluster-tilting object in \(C\), then the cluster-tilted algebras \(\text{End}_C(T)\) and \(\text{End}_C(\tau T)\) are isomorphic. In the \(A_n\) case we also have the following.

**Theorem 3.7.** Let \(T\) and \(T'\) be tilting objects in \(C\), then the cluster-tilted algebras \(\text{End}_C(T)\) and \(\text{End}_C(T')\) are isomorphic if and only if \(T' = \tau^iT\) for an \(i \in \mathbb{Z}\).
Proof. Let $\Delta$ be the triangulation of $\mathcal{P}_{n+3}$ corresponding to $T$ and let $\Delta'$ be the triangulation corresponding to $T'$. If $T' \not\simeq \tau^iT$ for any $i$, then $\Delta'$ is not obtained from $\Delta$ by a rotation, and hence $\text{End}_\mathcal{C}(T)$ is not isomorphic to $\text{End}_\mathcal{C}(T')$ by Theorem 3.5. □

Proposition 3.8. Let $\Gamma$ be a cluster-tilted algebra of type $A_n$. The number of non-isomorphic cluster-tilting objects $T$ such that $\Gamma \simeq \text{End}_\mathcal{C}(T)$ has to divide $n + 3$.

Proof. Let $T$ be a tilting object in $\mathcal{C}$ corresponding to the triangulation $\Delta$. Denote by $\Delta(i)$ the rotation of $\Delta$ $i$ steps in the clockwise direction. Let $0 < s \leq n$ be the smallest number of rotations needed to obtain the same triangulation $\Delta$, i.e. the smallest $s$ such that $\Delta = \Delta(s)$. It is clear from the above that $T \not\simeq T'$, where $T'$ corresponds to $\Delta(t)$ with $0 < t < s$, hence $s$ is the number of non-isomorphic tilting objects giving the same cluster-tilted algebra. Now we only need to show that $s$ divides $n + 3$, but this is clear. □

The proof of the following is easy and is left to the reader. First recall from [10, Proposition 3.8] that there are exactly $C(n)$ non-isomorphic tilting objects in the cluster category for type $A_n$, where $C(n)$ denotes the $n$th Catalan number.

Proposition 3.9. Consider the $A_n$ case.

- There are always at least 2 non-isomorphic cluster-tilting objects giving the same cluster-tilted algebra.
- There are at most $n + 3$ non-isomorphic cluster-tilting objects giving the same cluster-tilted algebra.
- Let $\Gamma$ be a cluster-tilted algebra of type $A_n$. If $n + 3$ is prime, there are exactly $n + 3$ non-isomorphic cluster-tilting objects giving $\Gamma$. In this case there are $C(n)/n + 3$ non-isomorphic cluster-tilted algebras, where $C(n)$ denotes the $n$th Catalan number.

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