Quantum Transparency of Barriers for Structure Particles

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Penetration of two coupled particles through a repulsive barrier is considered. A simple mechanism of the appearance of barrier resonances is demonstrated that makes the barrier anomalously transparent as compared to the probability of penetration of structureless objects. It is indicated that the probabilities of tunnelling of two interacting particles from a false vacuum can be considerably larger than it was assumed earlier.

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I. INTRODUCTION

Quantum tunnelling through a barrier is one of the most common problems in many trends of physics. Physical processes of tunnelling are usually considered as a penetration of a structureless particle through the barrier; whereas in realistic physics, we are, as a rule, dealing with the problem of penetration of structure particles through the barrier. It is clear that when the spatial size of a barrier is much larger than the typical dimension of an incident complex, we could expect insignificant distinctions of the penetration probability of the complex from that of structureless particles. The situation drastically changes when the size of the complex is larger than the spatial width of the barrier. In this case, there arise mechanisms that increase the barrier transparency (see, for instance, ref. [1] and references therein). The simplest of them arises when only part of the complex interacts with the barrier, i.e. the penetration probability depends on the mass smaller than the complex mass.

In this paper, we consider the mechanism of drastic transparency of a barrier that implies possible formation of a barrier resonance. To this end, it is necessary that at least two particles interact with the barrier. It is easy to imagine the mechanism of formation of such a resonance state. Let through the barrier, only one of particles pass, and forces coupling the pair be sufficient to keep the particles on different sides of the barrier. Then, such a resonance state will live unless one of the particles penetrates through the barrier. The barrier width will determine the lifetime of a resonance of that sort. As will be shown below, the probability of tunnelling through the barrier can reach unity. Physically, this effect is explained by the interference suppression of a reflected wave, since the presence of a barrier resonance is simply described by the effective interaction in the variable of motion of the centre of inertia of the pair, whose spatial form has a local minimum in the barrier centre. Therefore, the suppression of a reflected wave can be explained by the interference phenomenon well known in optics and used in coating lenses – the difference in path between the wave reflected from the first peak and the one reflected from the other peak should equal one-half the wave length.

In this paper, the above effect of transparency is demonstrated analytically and numerically for a pair of identical particles coupled by an oscillator interaction (in what follows, an oscillator) that penetrates through a one-dimensional repulsive barrier of the Gaussian type. This choice of interactions is, on the one hand, due to the system being extremely simple and allowing the reduction of three-dimensional scattering of a three-dimensional pair of particles on a one-dimensional barrier to the scattering of a one-dimensional oscillator on a one-dimensional barrier. On the other hand, just this type of interaction is taken in the literature [2] devoted to the decay probability of false vacuum in the high energy particle collisions (see, for instance [3,4]). It was pointed out that the processes of transition from the false vacuum could be described on the basis of quantum-mechanical tunnelling of a pair of particles through a barrier, but a system was investigated, in which only one particle of the oscillator interacted with the barrier. In this note it is shown that when two particles interact with the barrier, the penetration probability can be essentially higher that in the systems considered earlier.

II. EQUATIONS

Consider the penetration of a pair of identical particles with masses \( m_1 = m_2 = m \) and coordinates \( r_1 \) and \( r_2 \) coupled by an oscillator through the potential barrier \( V_0(x_1) + V_0(x_2) \). The Hamiltonian of this system \( (\hbar = 1) \)

\[
\frac{1}{4m} \Delta_R - \frac{1}{m} \Delta_r + \frac{m \omega^2}{4} r^2 + V_0(R - r/2) + V_0(R + r/2),
\]

written in terms of coordinates of the centre of inertia of the pair \( R = (r_1 + r_2)/2 \) and internal variable of the relative motion \( r = r_1 - r_2 \) describes the three-dimensional motion of a three-dimensional oscillator. Since the potential barrier depends only on one variable, and the oscillator interaction is additive in projections of \( r \), the wave function is factorized, and its nontrivial part describing
scattering depends only on two variables. It is convenient
to represent these variables in the form
\[ x = \sqrt{\frac{m\omega}{2}}(x_1 - x_2), \quad y = \sqrt{\frac{m\omega}{2}}(x_1 + x_2). \]

The Schrödinger equation in these variables is of the form
\[ (-\partial^2_x - \partial^2_y + x^2 + V(x - y) + V(x + y) - E) \Psi = 0, \]
where the energy \( E \) is written in units \( \omega/2 \), and the
potential barrier \( V(x \pm y) = \frac{2}{\sqrt{2\pi}} V_0((x \pm y)/\sqrt{2m\omega}) \) is below written in the convenient form \( V(X) = A \exp(X^2/(2\sigma)) \). Here, the amplitude \( A \) is a parameter
describing the energy height of the barrier, and \( \sigma \) determines its spatial width. Let the process of scattering
go from left to right, and the oscillator initial state
correspond to state \( n \). Then the boundary conditions are
written in the form
\[
\begin{align*}
\lim_{y \to \pm\infty} \Psi &\to \exp(\imath k_n y) \varphi_n(x) = \sum_{j \leq N} S_{nj} \exp(-\imath k_j y) \varphi_j(x), \\
\lim_{y \to -\infty} \Psi &\to \sum_{j \leq N} R_{nj} \exp(\imath k_j y) \varphi_j(x), \\
\lim_{x \to \pm\infty} \Psi &\to 0.
\end{align*}
\]
(2)

The oscillator wave functions \( \varphi_j(x) \) obey the
Schrödinger equation
\[ (-\partial^2_x + x^2 - \varepsilon_j) \varphi_j = 0, \]
with energy \( \varepsilon_j = 2j + 1 \) \((j = 0, 1, 2, \ldots)\), momenta
\( k_j = \sqrt{E - \varepsilon_j} \), and with \( N \) being the number of the
last open channel \((E - \varepsilon_{N+1} < 0)\). Below, we consider
an oscillator composed of bosons, whose spectrum
is conveniently numbered from 1. Thus, in what follows,
\( \varepsilon_j = 4j - 3 \) \((j = 1, 2, \ldots)\).

We define the probabilities of penetration \( W_{ij} \) and
reflection \( D_{ik} \) as the ratio of the density of a transmitted
or reflected flux to that of incident particles, i.e.,
\[ W_{ij} = |R_{ij}|^2 \frac{k_j}{k_i}, \quad D_{ij} = |S_{ij}|^2 \frac{k_j}{k_i}. \]
It is clear that \( \sum_{j \leq N} W_{ij} + D_{ij} = 1 \).

This problem of determination of penetration (reflection)
probabilities requires solution of a two-dimensional
differential equation. The aim of this paper is to demonstrate
the quantum transparency of a barrier. Therefore,
we take advantage of the well-known adiabatic approximation
successfully applied in various three-body problems
(see, for instance, review \[3\]). To this end, we introduce basis functions \( \Phi_i \) obeying the equation
\[ (-\partial^2_x + x^2 + V(x - y) + V(x + y) - \varepsilon_i(y)) \Phi_i(x; y) = 0, \]
(3)

and use them for the expansion \( \Psi(x, y) = \sum_i f_i(y) \Phi_i(x; y) \). Inserting this expansion into Eq.(3)
and projecting onto the basis, we arrive at the system of equations
\[ ((-\partial^2_y + \varepsilon_i - E) \delta_{ij} - Q_{ij} \partial_y - \partial_y Q_{ij} + P_{ij}) f_j = 0, \]
(4)

where the effective interaction in channel \( i \): \( E_i = \varepsilon_i + P_i \)
corresponds to the diagonal part of the interaction, and
functions derived in projecting \( Q_{ij} = \langle \Phi_i, \partial_y \Phi_j \rangle \) and
\( P_{ij} = \langle \partial_y \Phi_i, \partial_y \Phi_j \rangle \) correspond to the coupling of channels. Brackets mean integration over the whole region
of \( x \). By definition, the functions \( Q_{ij} \) is antisymmetric,
and \( P_{ij} \) is positive. As a rule, the coupling of channels
is small, and the processes of scattering can be described
by a limited number of equations. As in our case the
spectrum of Eq.(3) is discrete, a good description of the scattering processes is achieved with the use of all the
channels open in energy \( \varepsilon \). At large \( |y| \), the effective
energy \( E_i \to \varepsilon_i \), and \( \Phi_i(x; y) \to \varphi_i(x) \), which allows us to easily rewrite the boundary conditions \( \Psi \) in the channel form.

Considering the boson case, we show for one channel
that the effective interaction \( E_i \) \((i = 1, 2)\) possesses a clear minimum, resulting in the resonance mechanism of transparency, and that the inclusion of the second channel does not change this picture.

III. ONE-CHANNEL APPROXIMATION

Within the chosen approach, the effect of quantum transparency is observed even in the one-channel approximation, i.e., in the Born-Oppenheimer approximation. In Fig.[II] the dependencies \( E_1(y) \) are drawn, determined by numerical solution of Eq.(3) at \( \sigma = 0.01 \) and three values of the amplitude \( A \) denoted by letters A, B, and
C, respectively. These values of parameters were taken
to demonstrate the formation of a potential well that provides the resonance peculiarities of scattering. For comparison, shown in Fig.[III] are the initial potentials of barriers at \( x = 0 \), i.e., \( 2V(y) \), that describe the scattering of structureless (or extremely bound) particles. For convenience, they are shifted by the binding energy of a pair.
FIG. 1. Effective energies of interaction: 1–2V and 2–E1. For explanation, see the text.

FIG. 2. The probabilities of penetration through barriers: 1 for 2V and 2 for E1. For explanation, see the text.

In Fig.2, we present the probabilities of penetration of a pair through barriers determined by numerical solution of Eq.(4) and corresponding to the potentials drawn in Fig.1. It is clearly seen that for A = 1, the scattering of an oscillator and a structureless particle with a doubled mass are slightly different. At A = 5, the resonance component of scattering appears, and at A = 10, a clear resonance with W11 = 1 at the peak is observed for energy Er ≈ 8.12. It is just this behaviour that is defined by the term "quantum transparency of barriers". For comparison, the probability of penetration through a barrier 2V(y) is as few as ≈ 0.012.

Complete transparency of a barrier may happen to be somewhat surprising. Simple analogies with optical phenomena were presented in the Introduction. Below, we write simple expressions valid in the case of rectangular barriers and in the quasi-classical approximation which testify to the possibility for a barrier being completely transparent. To this end, we take the potential of form C given in Fig.1 with two clear peaks. Since the one-dimensional problem of penetration through a barrier can be found in many textbooks (see, e.g., [6]), here we present only the scheme of solution of the problem of penetration through a two-peak barrier. Denoting 3 regions of classically allowed motion from left to right by numbers 1,2,3 and introducing upper indices for the amplitudes and probabilities of penetration from the region marked by the left index to the region marked by the right index, we easily obtain

\[ R^{(13)} = \frac{R^{(12)}R^{(23)}}{1 - |S^{(21)}S^{(23)}|} \]

For simplicity, the lower index of channel 1 is omitted. Then the probability of penetration through a two-peak channel is expressed through the probabilities of penetration through each peak as follows:

\[ W^{(13)} = \frac{W^{(12)}W^{(23)}}{1 + |S^{(21)}|^2|S^{(23)}|^2 - 2|S^{(21)}||S^{(23)}|\cos(\theta)} \]

where \(\theta\) is a doubled difference of phases (or action in quasi-classics) of motion between the left and right peaks. Time-reversal invariance leads to the principle of detailed balance (see, e.g. [6]) that in our case results in the equality \(|S^{(21)}| = |S^{(12)}|\).

For a symmetric potential \(W^{(12)} = W^{(23)}\), the penetration probability \(W^{(13)}\) reaches a maximum at \(\theta = 2\pi n\) (n=1,2,...). Note that it is just the condition that in a quasi-classical approach determines the spectrum of bound states for infinitely broad peaks. Provided that \(|S^{(ij)}|^2 = 1 - W^{(ij)}\), it is not difficult to verify that at these energies, \(W^{(13)} = 1\), i.e. a complete transparency occurs.

Parameters of the barrier potential V were chosen so that the resonance energy Er would be higher than the energy of the second channel \(E_2 = 5\). This is necessary for proving that the inclusion of inelastic processes does not change the resonance picture of transparency.

It is necessary to note, that the parameters of a potential V differ from parameters of a potential of Ref. [2], where power height of a barrier is comparable to energy of elementary excitation \((V(0) = 1, \omega = 1/2)\). The parameters of potentials will become close if magnitude \(\omega\) from Ref. [2] will be in 50 times less.

IV. TWO-CHANNEL APPROXIMATION

In Fig.3, we show the results of numerical solution of Eq.(5) for the second channel. It is seen that the coupling functions of channels Q12 and P12 are about 2 ordered as small as diagonal values E2. The effective energy E2 is more complicated in form than E1 and can also generate extra resonances, whose correct consideration requires inclusion of the third channel (energies above 9). This goes
beyond the scope of our problem of demonstrating the transparency of a barrier, and here we only mention the presence of peaks of $E_1$ in $E_2$.

![FIG. 3. Components of the second channel. For explanation, see the text.](image)

The second peak at energy $E_r \approx 9.6$ shown in Fig. 4 cannot be considered reliable since at these energies, account is to be made of the third channel. This peak demonstrates as interesting peculiarity – all the probabilities of both channel 1 and channel 2 amount to 1/4. Moreover, the behaviour of probabilities of transition from state 2 ($W_{22}, W_{21}, D_{21}$) in the energy region of the second peak (not shown in Fig. 4) is visually indistinguishable from the behaviour of inelastic components from state 1. Owing to the principle of detailed balance $[6]$, the equalities $W_{21} = W_{12}$ and $D_{21} = D_{12}$ demonstrate only the accuracy of calculation. Of surprise is the behaviour of the probability $W_{22}$ whose energy dependence with 5% accuracy reproduces the behaviour of inelastic components around this peak.

### V. CONCLUSION

The considered mechanism of transparency of barriers for a coupled pair of particles is clearly observed for narrow and high barriers, as compared to characteristic sizes and energy of an oscillator. These conditions do not eliminate spatially asymmetric barrier potentials since the symmetry of effective interaction $E_i(y)$ is determined only by particles being identical. Therefore, the effects of quantum transparency can occur in various fields of physics. Here note is to be made that when it is allowable to describe the processes of the false vacuum disintegration in high energy collisions by means of quantum tunnelling of a pair of particles coupled by an oscillator interaction, there arise mechanisms of the resonance transparency of a barrier, much increasing the decay probabilities of the false vacuum.

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