ON THE ELLIPTIC VARIATIONAL INEQUALITY FOR A SIMPLIFIED FRICTION PROBLEM

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Abstract

The study investigated variational inequality of the elliptic to the second type of "A Simplified Friction Problem". The operator of the second arrangement partial differential was coupled within elliptic variational inequality. It gives the mathematical and physical demonstration and some characteristics of the solution. The study highlighted the presence, singularity and the second type of EVI solutions characteristic.

Keywords: Variational inequality, EVI first type, EVI second type, Friction Problem, Convex series.

I. Introduction

The variational inequality considers the benefits and essential category problems of no linear resulting from physics, mechanics etcetera. There are two types of Variational Inequality, called EVI first type and EVI second type. A variational inequality referred to variation relationships of functional that all the given variable value be answered, usually related to a convex series.

Friction is the power struggle the proportional movement of fluid layers, strong surfaces, or/and slope of material elements in opposition to each other. The friction divided into different types:

- **Dry friction** aggression proportional side motion in connection of two hard surfaces. Divided dry friction into stable friction between surfaces that not moving, and kinetic friction among connect to the surface.

- **Fluid friction** indicated the layers with sticky liquid friction that are proportional motion of the each one.

- **Lubrication friction** was the friction condition of a solution where a solution split two hard surface.
- **Friction of skin** is a ingredient of traction, power struggle movement of hard body via a liquid.

- **Internal friction** was the powerful movement aggress amide the component build a hard elements, whereas it was suffer from deformity.

During interaction surfaces motion proportional to each other, the two surface friction change kinetic power to heat. This feature might possess dramaticular moments, as explained utilizing of friction generated by dragging firewood parts altogether to flame starting.

Active power is altered to flame at movement occurs, friction resulting, such as moved of viscous fluid. Another considerable moment of several friction types might dress, which might result to mutilation to components or/and degradation action. The tribology science module is friction.

Friction consider not a main power, but occurs due to the electromagnetic power between charges molecules which in contact comprise the surfaces.

Leonardo DaVinci was discussing the sliding friction classic rules at (1452-1519) but his book stays unpublished [V], [II]. At 1699, Guillaume Amontons rediscovered, then, Charles – Augustin de Coulomb developed in 1785. At 1707-1783, Leonhard Euler derived the repose angle weight upon sloped level and distinguished firstly between kinetic and static friction [X]. At 1833, Arthur developed the static friction concept.

Derived the viscous flow equation in 1866 by Osborne Reynolds. This done the friction classic empirical pattern of (fluid, kinetic, and static) mostly today in engineering applications [V].

**i. Definition:** Following [III], the authoritative variational inequality definition is the one of the following:

Assume a Banach space E, a subsect K of E,a functional F: K →E* from K to the dual space E* of the space E, VI problem is the problem resolving respect to x related to K inequality of the below:

\[<F(x),y-x> \geq 0 \quad \forall , y \in k\]

Wherever \(<,>\): E’x: E→R is the dual pairing. Generally, the problem of variational inequality might formulate on any Banach space of infinite-dimensional or finite– dimensional. The study problem footsteps are three as mentioned as follows:

1. Confirmation the solution presence: this footstep revealed the mathematical problem correctness, showed that there is at least a solution.

2. Confirmation the singularity of a particular solution: this step proved the physical problem correctness, showed that possible utilize the solution to physical phenomenon characterize. Principally, It was the important step as variational inequalities modeled many of the issues are physical origin.

3. Solution Find.
I.ii. The Finding of the Minimum Value Problem of an Actual-Function Value of Real Variable:

The is a standard example [II], consider the minimum value of a differentiable function \( f \) on \( I = [a, b] \).

Suppose \( x^* \) be a point in \( I \) where minimally happens. Might occur 3 statuses:

1. if \( a < x^* < b \) then \( f'(x^*) = 0 \),
2. if \( x^* = a \) then \( f'(x^*) \geq 0 \),
3. if \( x^* = b \) then \( f'(x^*) \leq 0 \).

These necessary conditions can be brief as the finding problem of \( x^* \in I \) such that
\[ f'(x^*)(y-x^*) \geq 0 \quad \forall y \in I. \]

The minimum absolute should be between solutions checked (might higher than 1) of another inequality notice that a resolution was a real number. However, consider that the finite dimensional inequality.

I.iii. Notations:

- \( V \): real Hilbert space with scalar product \((.,.)\) also related norm \( \| \cdot \| \)
- \( V^* \): The dual space of \( V \).
- \( a(.,.) : V \times V \rightarrow \mathbb{R} \) is a bilinear, continuous, and the \( V \times V \) gives the \( (V) \) - elliptical form.
- A bilinear form \( a(.,.) \) was aforesaid \( V \)-elliptical if there exists a positive constant \( \alpha \) such that \( a(v,v) \geq \alpha \|v\|^2 \quad \forall v \in V \).

Generally, a \((.,.)\) does not assume by us, to be symmetric but in some applications might happen normally
- \( L : V \rightarrow \mathbb{R} \) continuous, linear functional.
- \( K : \) is a closed, convex, non-empty subset of \( V \).
- \( j(.) : V \rightarrow \mathbb{R} = \mathbb{R} \cup \{\infty\} \) is a proper functional, convex and lower semi-continuous (l.s.c).

\( j(.) \) is proper if \( j(v) > -\infty \forall v \in V \) and \( j \neq \infty \)

I.iv. The Second Kind of Elliptical Variational Inequality (EVI)

To get \( u \in V \) where \( u \) is a solution of the problem:
I.v. A Theorem Of Existence And Uniqueness For EVI Of Second Kind [VI], [VII]

The problems $P_1$ have single solution.

Proof:

1. Uniqueness

"Let $u_1$ and $u_2$ be two solutions of $(P_1)$, then we have:

$$a(u_1, v - u_1) + j(v) - j(u_1) \geq L(v - u_1) \quad \forall v \in V, u_1 \in V$$

(1)

$$a(u_2, v - u_2) + j(v) - j(u_2) \geq L(v - u_2) \quad \forall v \in V, u_2 \in V$$

(2)

Since $j(.)$ is an aproper map there exists $v_0 \in V$ such that $-\infty < j(v_0) < \infty$.

Subsequently for $i=1,2$

$$-\infty < j(u_i) \leq j(v_0) - L(v_0 - u_i) + a(u_i, v_0 - u_i)$$

(3)

Which revealed the $j(u_i)$ is finite to $i=1,2$. Subsequently, by substituting and adding, we obtain

$$a\|u_1 - u_2\|^2 \leq a(u_1 - u_2, u_1 - u_2) \leq 0$$

(4)

Hence $u_1 = u_2$.

2. Existence:

To every $u \in V$ and $\rho > 0$ related a problem $\left(\Pi^u_\rho\right)$ of type $(P_1)$ explained as under:

To get $w \in V$, then

$$\left(\Pi^u_\rho\right) \ldots \begin{cases} (w, v - w) + \rho j(v) - \rho j(w) \geq (u, v - w) + \rho L(v - w) - \rho d(u, v - w) \quad \forall v \in V \\ w \in V \end{cases}$$

(5)

The advantage take into consideration this problem on the $(P_1)$ problem was the bilinear form related with $\left(\Pi^u_\rho\right)$ is the inner product of $V$ which is symmetric.
Firstly, let suppose the \( \Pi_{\rho} u \) has a single solution for every \( u \in V \) and \( \rho > 0 \). For every \( \rho \) define the map \( f_{\rho} : V \rightarrow V \) by \( f_{\rho}(u) = w \), where \( w \) is the single solution of \( \Pi_{\rho} u \).

We propose to display that \( f_{\rho} \) is a uniformly rigid constriction charting for appropriately selected \( \rho \).

Let \( u_1, u_2 \in V \) and \( w_i = f_{\rho}(u_i) \), \( i = 1, 2 \). As \( j(.) \) is appropriate, we possess \( j(u_i) \) finite, it able to confirmed as in (3). However, we possess

\[
(w_i, w_2 - w_i) + \rho j(w_i) \geq (u_i, w_2 - w_i) + \rho L(w_2 - w_i) - \rho a(u_i, w_2 - w_i) \quad (6)
\]

\[
(w_2, w_1 - w_2) + \rho j(w_2) \geq (u_2, w_1 - w_2) + \rho L(w_1 - w_2) - \rho a(u_2, w_1 - w_2) \quad (7)
\]

The inequalities add to get

\[
\left\| f_{\rho}(u_1) - f_{\rho}(u_2) \right\|^2 = \left\| w_2 - w_1 \right\|^2 \leq \left( |I - \rho A| \right) \left\| u_2 - u_1 \right\| \left\| w_2 - w_1 \right\| \quad (8)
\]

Hence:

\[
\left\| f_{\rho}(u_1) - f_{\rho}(u_2) \right\| \leq \|I - \rho A\| \left\| u_2 - u_1 \right\|
\]

Which simple to reveal the \( \|I - \rho A\| < 1 \) at \( 0 < \rho < \frac{2\alpha}{\|A\|^2} \).

That confirms the \( f_{\rho} \) was uniform a constrict conclude mapping and subsequently possess a single point \( u \). The \( u \) turning out as the \((P_1)\) solution as \( f_{\rho}(u) = u \) involves.

\[
(u, v - u) + \rho j(v) - \rho j(u) \geq (u, v - u) + \rho L(v - u) - \rho a(u, v - u) \quad \forall v \in V.
\]

However

\[
a(u, v - u) + j(v) - j(u) \geq L(v - u) \quad \forall v \in V \quad (9)
\]

Hence single solution possess by \((P_1)\)’.”
II. An Example of the Second Second of EVI
"A Simplified Friction Problem"

II.i. Notations
* \( \Omega \): a bounded domain in \( \mathbb{R}^2 \).
* \( \Gamma = \partial \Omega \).
* \( x = \{ x_1, x_2 \} \) a generic point of \( \Omega \).

\[
\nabla = \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\}
\]

* \( C^m(\overline{\Omega}) \): M-times Space continuously differentiable real valued functions so that all the derivatiation up to be m are continuous in \( \overline{\Omega} \).
* \( C_0^m(\Omega) = \{ v \in C^m(\overline{\Omega}) : \text{supp} (v) \text{ is a compact subset of } \Omega \} \)

\[
\| v \|_{m,p,\Omega} = \sum_{|\alpha| \leq m} \left\| D^\alpha \right\|_{L(\Omega)}^p 
\text{for } v \in C^m(\overline{\Omega}) \text{ where } \alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 
\text{non-negative integers, } |\alpha| = \alpha_1 + \alpha_2 \text{ and } D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}};
\]

* \( w^{m,p}(\Omega) \): completion of \( C^m(\overline{\Omega}) \) in the above norm defined.
* \( w_0^{m,p}(\Omega) \): completion of \( C_0^m(\Omega) \) in the above norm.
* \( H^m(\Omega) = w^{m,2}(\Omega) \).
* \( H_0^m(\Omega) = w_0^{m,2}(\Omega) \).

II.ii. The Problem Mathematical Interpretation
Let \( \Omega \) be a bounded domain of \( \mathbb{R}^2 \) with a smooth boundary \( \Gamma = \partial \Omega \), we define

\[
V = H^1(\Omega)
\]

\[
a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx
\]

\[
L(v) = \langle f, v \rangle, \quad f \in V^*
\]

\[
j(v) = g \int_\Gamma |v| \, d\Gamma
\]

Where \( g > 0 \), \( \gamma v \) denotes the trace of \( v \) on \( \Gamma \).
III. "Friction Problem. Uniqueness and "Existence Results

Theorem: The VI

\[
\begin{align*}
    & \{a(u, v - u) + j(v) - j(u) \geq L(v - u) \quad \forall v \in V \\
    \quad & u \in V
\end{align*}
\]

Possess a single solution.

Proof:

Since a (.,.) bilinear is the frequent \(H^1(\Omega)\) scalar product, and continuous of \(L\), according (1.5) theorem, we get that (14) has a unique solution.

It is enough to confirm the \(j(.)\) was l.s.c, appropriate and convex.

Indeed \(j(.)\) is a semi norm on \(V\). However, utilizing Schwartz I in \(L^2(\Gamma)\) also the truth that

\[
\gamma \in L(H^1(\Omega), L^2(\Gamma)) \text{ we possess}
\]

\[
|j(u) - j(v)| \leq |j(u - v)| \leq g\Gamma^{1/2} \gamma(u - v)_{L^2(\Gamma)} \leq c\|u - v\|_V
\]

To some what of \(C\) constant.

Hence \(j(.)\) is Lipschitz \(V\) continuous, therefore \(j(.)\) is l.s.c., is \(j(.)\) shown properly and convex. Hence confirmed the theorem.

**Remark 3.1:** Subsequently, \(a(.,.)\) is symmetric, the solution \(u\) of (14) is identified like a single solution of the minimization problem

\[
\begin{align*}
    & \{J(u) \leq J(v) \quad \forall v \in V \\
    \quad & u \in V
\end{align*}
\]

where \(J(v) = \frac{1}{2}a(v,v) + j(v) - L(v)\)

IV. Result

In this paper, we get the elliptic variational inequality for a simplified friction problem.

V. Conclusion

In this paper, we can conclude the variational inequality approach gives the most appropriate formulation.
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