An algebraic approach to representations of the permutation group.

G.Bergdolt

March 29, 2022

1 Abstract

The group algebra of the permutation group is spanned by a set of elements called projectors. The coordinates of permutations expanded in projectors are matrix elements of irreducible representations. The projectors for the permutation group are products of a Young symmetriser and an antisymmetriser. They form non-orthogonal bases of right and left modules. The non-orthogonality is compensated by a constant matrix. It turns out that this reduces the matrix entries to \{-1, 0, +1\}. An algorithm to compute the projectors is given.

2 Introduction

The group of permutations of n objects is called the symmetric group and is denoted $S_n$. A permutation group is a subgroup of $S_n$. Since any symmetric group is a subgroup of a symmetric group with a higher number of symbols we can state: 'A symmetric group is a permutation group'. The converse is not true, since a permutation group is the direct product of symmetric groups relative to disjoint subsets of symbols. In the following the names permutation group and subgroup of a permutation group are used for $S_n$ and its subgroups.

The permutation group is a classical subject, the amount of work devoted to the permutation group can be inferred from the list of references (several hundreds) in the review article of James and Kerber in the collection Encyclopedia of Mathematics\textsuperscript{1).} The origin of the present paper is the chapter entitled 'Symmetric groups' in Littlewood’s book\textsuperscript{2).} It was realized that algebra elements introduced there can be promoted to a basic concept of representation theory. These elements are called projectors below since the relations they satisfy generalise the idempotency of projectors. The projectors for the permutation group, defined as the product of a symmetriser and an antisymmetriser, yield a non-orthogonal matrix representation. The homomorphy of the matrix representation can also be obtained if a constant matrix is intersped in the matrix product. It is known at least since Wigner\textsuperscript{3)} that the representation matrices of a finite group can be taken unitary. Unitarity requires complex numbers for
the matrix entries. In the case of the permutation group, a property called ambivalence (an element and its inverse are in the same conjugation class) entails that the representation matrices can be chosen real orthogonal\(^4\). Orthogonality still involves square roots and hence irrational numbers. Dropping orthogonality, the matrix entries can be integers. When taking the non-orthogonality into account by a constant matrix, the entries are reduced to \([-1, 0, +1]\). This set is the group \(\mathbb{Z}_3\) under addition modulo 3. Whether this fact is of significance or not is an open question.

The projector formalism is described in Section 3. In Section 4 it is shown that the product relations defining projectors are satisfied by the products of a Young symmetriser and a Young antisymmetriser. The proof that matrix entries can be reduced to \([-1, 0, +1]\) is given in Section 6. Units and characters are examined in Section 7. Conclusions are in Section 8. An algorithm for determining the coordinates of projectors is given in Appendix A.

## 3 Projectors

Let \(A\) be the algebra spanned by \(n^2\) elements \(p_{ij}\) called projectors satisfying the relations:

\[
p_{ij}p_{kl} = g_{jk}p_{il}
\]

where \(g_{jk} \in k\) is a scalar, \(k\) denotes a field.

Let \(X, Y \in A\) be defined by coordinates \(X = x^{ij}p_{ij}\) and \(Y = y^{ij}p_{ij}\), where the summation convention is used. The coordinates of \(Z = XY\) are given by:

\[
z^{il} = x^{ij}g_{jk}y^{kl}.
\]

The algebra \(A\) is seen to be a matrix algebra where the product \(\circ\) is defined by a constant matrix \(g\).

\[
x \circ y = xgy.
\]

Note that by multiplying to the right or left by the matrix \(g\) a conventional matrix algebra is obtained. In matrix form \((zg) = (xy)(yg)\).

The direct sum of several such algebras is spanned by projectors \(p_{\lambda ij}\) where \(\lambda\) labels the subalgebras. The defining relations satisfied by the projectors are:

\[
p_{\lambda ij}p_{\mu kl} = \delta_{\lambda \mu}g_{\lambda ij}p_{\lambda \mu l}.
\]

It follows from these relations that

i) the projectors \(p_{\lambda ij}\) with \(\lambda\) and \(i\) fixed span a right ideal,

ii) the projectors with \(\lambda\) and \(j\) fixed span a left ideal,

iii) projectors with \(\lambda\) fixed span a subalgebra.

Assume that \(\lambda\) ranges over all irreducible representations of a finite group: the number of projectors is equal to the number of group elements, i.e. the order of the group. This follows from the theorem: ‘A regular representation contains each irreducible representation with a multiplicity equal to its dimension.’ If
$m_\lambda$ is the dimension of the irreducible representation and $n$ the order of the group, we have
\[ n = \sum_\lambda m_\lambda^2. \]

The right-hand side is the number of projectors. Note that the representation obtained by the direct sum of irreducible representations has dimension
\[ \sum_\lambda m_\lambda < n, \]
and thus is not the regular representation.

Since a finite group can be reconstructed from the set of its irreducible representations, the equality of the number of projectors and group elements entails that the projectors form a basis of the group algebra.

Elements of the group algebra $kS_n$ satisfying these relations are defined in the next section.

## 4 Projectors for the permutation group

The representation theory of the permutation group can be sketched as follows: Irreducible representations of $S_n$ are characterised by partitions of $n$. A partition defines a Young frame. Young frames filled with $n$ symbols are Young tableaux. A lexical order of the symbols is adopted. A Young frame with the symbols in rows and columns in lexical order is a standard Young tableau. The number of standard Young tableaux for a given partition is the dimension of the irreducible representation. A Young tableau is defined by a partition and the sequence of symbols obtained by reading the tableau from left to right, top to bottom. The sequences are ordered by the first differing symbol in the sequences. This yields an ordering of Young tableaux.

Let $i$ label Young tableaux and let $P_i$ be the row symmetriser of the tableau $T_i$, i.e. the sum of all permutations permuting symbols in rows of $T_i$. Let $N_i$ be the column antisymmetriser i.e. the sum of signed permutations permuting the symbols in columns of $T_i$ with sign $+$ for even, $-$ for odd permutations. Let $\sigma_{ij}$ be the permutation which permutes the sequence of tableau $T_i$ into the sequence of tableau $T_j$. We have $N_i\sigma_{ij} = \sigma_{ij}N_j$ and $P_i\sigma_{ij} = \sigma_{ij}P_j$, i.e. $\sigma_{ij}$ is an intertwiner.

Define $p_{ij} \in kS_n$ by:
\[ p_{ij} = P_i\sigma_{ij}N_j. \]

Proposition: The elements $p_{ij}$ satisfy the projector relations (1).

Proof: The set of permutations permuting the symbols in columns of a Young tableau form a subgroup of $S_n$ as do the permutations permuting the symbols in rows. Denote these subgroups also by $N_j$ and $P_j$ for tableau $T_j$. For a permutation $v_j \in N_j$ we have $N_j v_j = \pi(v_j)N_j$ where $\pi(v_j)$ is the parity of $v_j$. For a permutation $h_k \in P_k$ we have $h_kP_k = P_k$.

If an odd permutation is contained in $N_j$ and in $P_k$ we have $N_jP_k = -N_jP_k$. 

3
and the product is null. This is the case if two symbols are in the same column in $T_j$ and the same row in $T_k$. The transposition of the two symbols is an odd permutation contained in $N_j$ and $P_k$. If no permutation except the identity is contained in $N_j$ and $P_k$ a permutation $v_j \in N_j$ can be found which puts a symbol of $T_j$ in the same row as in $T_k$. A permutation $h_k \in P_k$ then exists which puts the symbols in the same position as in $T_k$, hence $\sigma_{jk} = v_j h_k$.

It follows that:

$$p_{ijkl} = P_i \sigma_{ij} N_j P_k \sigma_{kl} N_l = \pi(v_j)(P_i N_i)^2 \sigma_{il}.$$  

From Littlewood 2) we have the result $(P_i N_i)^2 = (n!/m) P_i N_i$ where $m$ is the number of standard Young tableaux. Relation (1) follows with

$$g_{jk} = \frac{n!}{m} \pi(v_j).$$

If the Young tableaux are ordered according to the ordering defined above then $N_i P_j = 0$ if $T_i < T_j$ and the $g$ matrix is lower triangular. The proof of the proposition 1 If $T_i < T_j$ the first differing symbols are in the same column in $T_i$ and the same row in $T_j$ is given by Littlewood 2). As shown above this entails that $N_i P_j = 0$.

Let $\lambda, \mu$ label partitions and let $N^\lambda_i$, $P^\mu_i$ denote the corresponding anti-symmetriser and symmetriser. According to another result of Littlewood 2) $(P^\lambda_i N^\lambda_i)(P^\mu_j N^\mu_j) = 0$ if $\lambda \neq \mu$. Hence the general relation (2) is satisfied by the projectors.

Remark: Another set of projectors is provided by $p^*_{ij} = N_i \sigma_{ij} P_j$. The sets $\{p_{ij}\}$ and $\{p^*_{ij}\}$ are related by the involution $s \rightarrow s^{-1}$ of $S_n$.

5 Matrix representations of $S_n$

Denote by $s_a, a = 1, \ldots, n!$ the permutations of $S_n$. Expanding $s_a$ in the projector basis we have

$$s_a = x^\lambda_{a ij} P_{\lambda ij}.$$  

(4)

where $x^\lambda_{a ij}$ are coordinates and the summation is over $\lambda ij$. From the product relations (2) it follows that the coordinates with $\lambda$ fixed provide an irreducible matrix representation of $S_n$. If $s_c = s_a s_b$ then

$$x^\lambda_{c il} = x^\lambda_{a ij} g_{\lambda jk} x^\lambda_{b kl}$$

where the summation is over $j$ and $k$.

The relation (3) defines the projectors as linear combinations of permutations. Hence:

$$p_{\lambda ij} = y_{\lambda ij}^a s_a$$  

(5)

where the summation is over $a$.

Note that the coordinates $y_{\lambda ij}^a$ have values $\{-1, 0, +1\}$. 


Proof: The subgroups \( N_i \) and \( P_j \) have no permutation in common except the Identity \( e \). If \( c', c'' \in N_i \) and \( r', r'' \in P_j \) are such that \( c'r' = c''r'' \) then since \( r''^{-1}r' \in N_i \) and \( c''c'^{-1} \in P_j \) we have \( r' = r'' \) and \( c' = c'' \).

The linear system (4) is the inverse of (5); it follows that the matrices of coordinates are inverses \( (x) = (y)^{-1} \). The matrix of coordinates \( (y) \) is given by the definition of projectors. The problem of determining the irreducible representations of \( S_n \) is then solved in principle.

Example: The group \( S_3 \)

Define the permutations by the sequence obtained applying the permutation to \([123]\):

\[ s_1 = [123], s_2 = [132], s_3 = [213], s_4 = [231], s_5 = [312], s_6 = [321] \]

The partitions are labeled: \( 1= (3) \), \( 2= (2,1) \), \( 3= (1,1,1) \).

The definition of projectors yields the linear system:

\[
\begin{pmatrix}
 p_{111} \\
p_{211} \\
p_{212} \\
p_{221} \\
p_{222} \\
p_{111}
\end{pmatrix}
= \begin{pmatrix}
 1 & 1 & 1 & 1 & 1 & 1 \\
 1 & 0 & 1 & 0 & -1 & -1 \\
 0 & 1 & -1 & 1 & -1 & 0 \\
 0 & 1 & 0 & 1 & -1 & -1 \\
 1 & 0 & -1 & 1 & 0 & 1 \\
 1 & -1 & -1 & 1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
 s_1 \\
s_2 \\
s_3 \\
s_4 \\
s_5 \\
s_6
\end{pmatrix}
\]

The projector relations (2) are satisfied with a \( g \) matrix \( g = \text{diag}(6,3,3,3,3,6) \).

The linear system can be inverted; permutations are related to projectors by:

\[
\begin{pmatrix}
 s_1 \\
s_2 \\
s_3 \\
s_4 \\
s_5 \\
s_6
\end{pmatrix}
= \begin{pmatrix}
 1 & 1 & 0 & 0 & 1 & 1 \\
 1 & 0 & 1 & 1 & 0 & -1 \\
 1 & -1 & 1 & -1 & 0 & 1 \\
 1 & 1 & -1 & 0 & -1 & -1 \\
 1 & 0 & -1 & 1 & -1 & 1 \\
 1 & -1 & 0 & -1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
 p_{111} / 6 \\
p_{211} / 3 \\
p_{212} / 3 \\
p_{221} / 3 \\
p_{222} / 3 \\
p_{131} / 6
\end{pmatrix}
\]

As pointed out the entries of the representation matrices are the coordinates of the permutations. Moreover the normalisation factors of the projectors and the entries of the \( g \) matrix cancel so that the two dimensional representation can be read off the matrix above:

\[
 s_1 = \begin{pmatrix}
 1 & 0 \\
 0 & 1
\end{pmatrix},
 s_2 = \begin{pmatrix}
 0 & 1 \\
 1 & 0
\end{pmatrix},
 s_3 = \begin{pmatrix}
 1 & 0 \\
 -1 & -1
\end{pmatrix},
 s_4 = \begin{pmatrix}
 -1 & -1 \\
 1 & 0
\end{pmatrix},
 s_5 = \begin{pmatrix}
 0 & 1 \\
 -1 & -1
\end{pmatrix},
 s_6 = \begin{pmatrix}
 -1 & -1 \\
 0 & 1
\end{pmatrix}.
\]

6 Reduced entries

An algorithm to compute the coordinates of projectors expanded in permutations is described in Appendix A.

We have seen that the matrix of coordinates of permutations expanded in projectors can be obtained by inversion of the matrix of projector coordinates.
There is however a shortcut which avoids the inversion of an \( n \!) \text{ dimensional matrix. The coordinates } (x) \text{ and } (y) \text{ are related by:}

\[
y_{\lambda ij} = \frac{n!}{m} \sum_{rs} g_{\lambda j r} x_b^{\lambda r} g_{\lambda s r}.
\]  

(6)

Here \( b \) and \( b^{-1} \) are a permutation and its inverse. A proof of relation (6) is given in Appendix B. We now prove the claim made in the Introduction: Representation matrices exist with entries restricted to \( \{-1, 0, +1\} \).

In relation (6) \( \lambda \) is fixed. Set \( f = \frac{n!}{m} \). Writing the quantities with indices \( i,j,r,s \) as matrices (6) reads:

\[
\tilde{y}(b^{-1}) = f^{-1} g x(b) g.
\]

The integer factor \( f \) was incorporated in the \( g \) matrix in Section (4) hence \( g' = f^{-1} g \) is a matrix with entries \( \{-1, 0, +1\} \). If projectors are renormalised as \( p'_{ij} = f^{-1} p_{ij} \) the relations \( p'_{ij} p'_{kl} = g'_{ij} g'_{kl} \) are satisfied and the coordinates \( x'^{ij} = f x^{ij} \) are the matrix elements of a representation. Relation (6) is now:

\[
\tilde{y}(b^{-1}) = g' x'(b) g'.
\]

If \( x'(b) \) defines a representation with matrix \( g' \) then \( g' x'(b) g' \) defines a representation with matrix \( g'^{-1} \).

Proof: \((g' x'(b) g') g'^{-1} (g' x'(c) g') = g'(x(b) g' x'(c)) g' = (g' x'(bc) g')\). Note that the transposed matrix \( \tilde{y}(b^{-1}) \) on the left-hand side of (6) is a matrix with entries \( \{-1, 0, +1\} \) so that the claim is proved. The matrix \( g'^{-1} \) is the inverse of a lower triangular matrix with reduced entries. It follows that the entries of the matrix \( g'^{-1} \) are integers but not that they are reduced to \( \{-1, 0, +1\} \).

7 Units and conjugation classes

Let \( C_{\rho} \) be the sum of the permutations in a conjugation class. Recall that the number of conjugation classes of a finite dimensional group is equal to the number of irreducible representations. \( C_{\rho} \) commutes with all elements of the group algebra.

Define:

\[
U_{\lambda} = \sum_{ij} g_{\lambda ij}^{-1} p_{\lambda ij}.
\]

From the product relations (3) it follows that \( U_{\lambda} p_{\mu kl} = \delta_{\lambda \mu} p_{\lambda kl} \) and \( p_{\mu kl} U_{\lambda} = \delta_{\mu \lambda} p_{\lambda kl} \). Further (\( U_{\lambda} \))^2 = \( U_{\lambda} \). \( U_{\lambda} \) is idempotent, leaves elements of the subalgebra \( \lambda \) invariant and annihilates all others, i.e. \( U_{\lambda} \) is a unit of the subalgebra \( \lambda \). \( U_{\lambda} \) commutes with all elements of the algebra, hence \( U_{\lambda} \) is a sum of entire conjugation classes:

\[
U_{\lambda} = \sum_{\rho} \chi_{\lambda}^{\rho} C_{\rho}.
\]  

(7)
The coefficients are group characters of $S_n$. A permutation cannot be in several classes and the elements of $g_{\lambda}^{-1}$ are integers. It follows that the characters defined above are integers.

Example: The group $S_3$.

Permutations in cycle notation are denoted by ( ) and in sequence notation by [ ]. The conjugation classes are:

\[ C_1 = (1)(2)(3), \]
\[ C_2 = (12) + (23) + (13), \]
\[ C_3 = (123) + (132). \]

The units are given by:

\[ U_1 = [123] + [132] + [312] + [213] + [231] + [321], \]
\[ U_2 = 2[123] - [312] - [231], \]
\[ U_3 = [123] - [132] + [312] - [213] + [231] - [321]. \]

The linear system (7) is then:

\[
\begin{pmatrix}
U_1 \\
U_2 \\
U_3
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 \\
2 & 0 & -1 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
C_1 \\
C_2 \\
C_3
\end{pmatrix}.
\]

8 Conclusion

The approach is restricted to finite groups. With respect to the usual representation theory the approach described above could be called natural. Orthonormalisation requires irrational numbers as matrix entries. In the $g$ matrix scheme entries are integers restricted to $\{-1, 0, +1\}$. For example the set of matrices with reduced entries given in section 5 is closed under matrix multiplication. The $g$ matrix defines a vector space isomorphism between right and left modules and denotes an intrinsic property of the $kS_n$ algebra. The algorithm described in Appendix B has been implemented in a computer program. For $n < 4$ the $g$ matrices are diagonal. The first non-diagonal $g$ matrix occurs for $n = 5$ and partition $(3,2)$. The first matrix $g^{-1}$ with entries not in $\{-1, 0, +1\}$ occurs for $n = 7$ and partition $(3, 2, 2)$. The representations matrices are always matrices with reduced entries.

The reduced entries feature has not found applications to physics up to now. The theory of coherent states in quantum mechanics shows that orthonormalisation is not an unavoidable feature.

References

[1] G.James and A.Kerber, The Representation Theory of the Symmetric Group, Addison Wesley Reading Mas. 1981
Appendix A According to definition (3) $p_{ij}$ is a sum of terms of the form $\pi(v_i)h_j\sigma_{ji}v_i$ where $h_j$ is a row permutation of $T_j$, $v_i$ a column permutation of $T_i$ and $\pi(v_i) = +1$ for even, $-1$ for odd permutations. If a permutation $s$ can be factorised as $s = h_j\sigma_{ji}v_i$ the coordinate of $p_{ji}$ along $s$ is given by $y_s^{ij} = \pi(v_i)$. Define the action of a permutation on a Young tableau as a right action: $T_i \rightarrow T_is = T_s$ The conjugates of the permutations $v_i$ by $s$ : $sv_is^{-1} = v_s$ form a subgroup ; the subgroup of column permutations of $T_s$. Note that the parity of $v_i$ and $v_s$ are the same. We have $v_i^{-1}s = \sigma_{ij}h_j = vs^{-1}$.

If this permutation acts on $T_i$ we obtain $T_s v_s^{-1} = T_jh_j$. A column permutation of $T_s$ exists which puts the symbols in the same rows as in $T_j$. This yields the following algorithm to compute the coordinates of a projector.

Determine the non-standart Young tableau $T_s = T_is$. Set to 0 a parameter $k$. Scan the columns of the non standard Young tableau $T_s$ and take the sequence of symbols in a column. Locate the same symbols in $T_j$ ordered according to the rows they occupy. This is not possible if two such symbols are in the same row and in that case the algorithm ends with a zero coordinate. Otherwise the two sequences of symbols define a permutation. Determine the cycle structure of the permutation. For a cycle of length $l$ add $l - 1$ to the parameter $k$. The coordinate is given by: $y_{ij}^s = 1 - 2(k \text{ mod } 2)$

Note that the matrix elements of $g$ are obtained by setting $s = e$.

Appendix B Since the $(x)$ and $(y)$ matrices are inverses we have

$$\sum_b g_{\lambda ij}^b x_{b}^{ikl} = \delta^a_\lambda \delta_k^i \delta^l_j,$$

$$\sum_{\lambda ij} x_{a}^{\lambda ij} g_{\lambda ij}^b = \delta_{a}^b.$$

where $a, b$ are permutations. Recall the representation relation

$$x_{c}^{\lambda i} = \sum_{jk} x_{a}^{\lambda ij} g_{\lambda jk} x_{b}^{\lambda kl},$$

where $c = ab$. Now

i) multiply by $g_{\lambda ij}^a$ and sum over $a,$
ii) multiply by $y_{\mu ks}^c$ and sum over $\mu ks$, the result is:

$$y_{\lambda ij}^c = \sum_{rs} x_b^{\lambda rs} g_{\lambda jr} y_{\lambda ks}^c.$$  \hspace{1cm} (8)

If $c = b$ the relation is:

$$y_{\lambda ij}^c = \sum_{rs} x_b^{\lambda rs} g_{\lambda jr} y_{\lambda ks}^b.$$ 

Summing over $b$ gives:

$$n! y_{\lambda ij}^c = m_{\lambda} g_{\lambda ji}.$$

If in (8) $c = e$ we have:

$$y_{\lambda ij}^{e-1} = \sum_{rs} g_{\lambda jr} x_b^{\lambda rs} y_{\lambda ks}^e.$$ 

From the last two relations, relation (6) follows.