SOCIAL CHOICE RANDOM UTILITY MODELS OF INTRANSITIVE PAIRWISE COMPARISONS

Rahul Makhijani and Johan Ugander
Management Science and Engineering
Stanford University
rauhmj@stanford.edu, jugander@stanford.edu
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ABSTRACT

There is a growing need for discrete choice models that account for the complex nature of human choices, escaping traditional behavioral assumptions such as the transitivity of pairwise preferences. Recently, several parametric models of intransitive comparisons have been proposed, but in all cases the maximum likelihood problem is non-concave, making inference difficult. In this work we generalize this trend, showing that there cannot exist any parametric model with a concave log-likelihood function that can exhibit intransitive preferences. Given this result, we motivate a new model for analyzing intransitivity in pairwise comparisons, taking inspiration from the Condorcet method (majority vote) in social choice theory. The Majority Vote model we analyze is defined as a voting process over independent Random Utility Models (RUMs). We infer a multidimensional embedding of each object or player, in contrast to the traditional one-dimensional embedding used by models such as the Thurstone or Bradley-Terry-Luce (BTL) models. We show that a three-dimensional majority vote model is capable of modeling arbitrarily strong and long intransitive cycles, and can also represent arbitrary pairwise comparison probabilities on any triplet. We provide experimental results that substantiate our claims regarding the effectiveness of our model in capturing intransitivity for various pairwise choice tasks such as predicting choices in recommendation systems, winners in online video games, and elections.

1 Introduction

Modeling pairwise comparisons is a key challenge at the heart of many ranking problems, with applications to recommendation systems, information retrieval, and predicting the outcomes of diverse pairwise competitions [M. Cattelan and Firth (2013)]. Discrete choice models are applied to a wide range of other domains; the “choice” being modeled is sometimes a choice made by an individual over alternative options, or sometimes a matter of a competition or game “choosing” a player to win that competition. Prominent examples of ranking systems based on discrete choice models are the Elo system for Chess ranking and the TrueSkill system [Herbrich and Graepel (n. d.)] for matching players in online games. Meanwhile pairwise rankings are used in recommendation systems to infer global rankings from pairwise preferences of users [Steffen Rendle (2014), building on Thurstone’s original observation from the 1920’s that comparative judgements are often easier than absolute judgements in many contexts [Thurstone (1927)]. In this work we use these vocabularies of alternatives/choices and games/players interchangeably. Two of the most widely used models for pairwise ranking are the Thurstone [Thurstone (1927)] and the Bradley-Terry-Luce (BTL) [Bradley and Terry (1952)] models of pairwise choice. These two models both belong to a broader class of Random Utility Models (RUMs) [Block (1960); Manski (1977)], which assume that each alternative i is completely described by a random utility $U_i$ with a innate quality $\lambda_i$, also known as the nominal utility, and a zero-mean noise term $\epsilon_i$ such that $U_i = \lambda_i + \epsilon_i$. For a random utility model with n items and utilities $U_1, \ldots, U_n$, it is assumed that the probability that an item is chosen can be written as:

$$P(a \text{ chosen from } S) = Pr(U_a \geq U_c, \forall c \in S),$$

for all possible alternatives $a \in S$ and choice sets $S$. For pairwise choices, this structure reduces to assuming that choices

$$P_{AB} = Pr[A \text{ chosen from } \{A, B\}] = Pr[U_A - U_B > 0],$$

where $P_{AB}$ is the probability that alternative A is chosen over alternative B.
The Thurstone model assumes that each $\epsilon_i$ is i.i.d. Normal while the BTL model assumes that each $\epsilon_i$ is i.i.d. Gumbel (sometimes also called double-exponential). A model is an \textit{independent} RUM if the random variables $\epsilon_i$ are all independent. Thurstone’s model and the BTL model are therefore both independent RUMs. For the Thurstone and BTL models, the difference of two utilities $U_A - U_B$ obeys a Normal distribution and Logistic distribution, respectively. An important general feature of RUMs with i.i.d. noise is that they implicitly assume a specific form of pairwise stochastic transitivity. Informally, if $A$ beats $B$ more often than not and $B$ beats $C$ more often than not, then any such model will assume that $A$ beats $C$ more often than not. In this work we focus on domains where such transitivity does not necessarily hold true. Notice that stochastic transitivity for RUMs with i.i.d. noise follows very specifically from the requirement that the noise be both independent and identically distributed. One of the most famous examples of stochastic intransitivity is the \textit{Efron dice}, which can be formulated as an independent RUM but with noise distributions that are non-identical. The example consists of four “dice” $A, B, C, D$ with the following “sides” that each occur with equal probability:

- $A : 4, 4, 4, 4, 0, 0$
- $B : 3, 3, 3, 3, 3, 3$
- $C : 6, 6, 2, 2, 2, 2$
- $D : 5, 5, 5, 1, 1, 1$

For the above dice it is easy to verify that $P(A > B) = P(B > C) = P(C > D) = P(D > A) = \frac{5}{7}$. But observe that the dice have different means and also very different variances. This example raises an intriguing question, namely whether it is possible to efficiently learn the parameters of an independent RUM that models intransitive pairwise comparisons using distributions defined by a small set of parameters (such the moments of the noise distributions). Several models of intransitive comparisons have recently been proposed, notably the 2-dimensional BTL model \cite{causeur2005} and the Blade-Chest model that will be reviewed in the next section \cite{chen2016}. But these and other models suffer from non-concave log-likelihood functions, meaning that maximum likelihood estimation is challenging. In this work we therefore formally ask: is there some previously overlooked model with a tractable concave log-likelihood that can support intransitive pairwise preferences? Our main result answers this question with a very general “no,” but before we develop it we will review some basic definitions of stochastic transitivity.

\textbf{Stochastic Intransitivity.} There are three basic mathematical definitions of stochastic transitivity. First, \textit{weak stochastic intransitivity} (being the opposite of weak stochastic transitivity) is defined as the existence of three alternatives $A, B, C$ in a set of alternatives such that:

$$P_{AB} > 0.5, P_{BC} > 0.5, P_{CA} > 0.5.$$ 

Second, we distinguish the notion of weak transitivity against the notions of \textit{anti-transitivity}. The notion of anti-transitivity is defined as $P_{AB} > 0.5, P_{BC} > 0.5 \implies P_{CA} > 0.5$, for any three alternatives $A, B$ and $C$. Third, the notion of \textit{strong stochastic intransitivity} is defined as: $P_{CA} > \max\{P_{AB}, P_{BC}\}$, again for any triplet of alternatives $A, B, C$. \textit{Strong stochastic transitivity} therefore requires $P_{AC} > \max\{P_{AB}, P_{BC}\}$ for all triplets. We are specifically focused on weak stochastic transitivity in this work. We show now that any RUM with i.i.d. uncertainties, including the Thurstone and BTL models, exhibits weak stochastic transitivity.

\textbf{Lemma 1.} For any Random Utility Model with random utilities $U_i = \lambda_i + \epsilon_i$ with i.i.d. $\epsilon_i$ and $\lambda_i \neq \lambda_j, \forall i, j$, weak transitivity holds.

\textit{Proof.} For any three alternatives $A, B,$ and $C$, we have that $\epsilon_A, \epsilon_B,$ and $\epsilon_C$ are i.i.d., implying that $\epsilon_A - \epsilon_B, \epsilon_B - \epsilon_C,$ and $\epsilon_A - \epsilon_C$ are all identically distributed and also symmetric. We see then that $\Pr[U_i - U_j] > 0.5$ iff $\lambda_i > \lambda_j$, and thus the three alternatives are totally ordered by their means.

\textbf{Modelling Intransitivity.} Given the above basic observation, there are three traditional approaches to introducing pairwise intransitivity into choice models: RUMs with possibly dependent distributions, RUMs with non-identical distributions, or departures from the Random Utility Model framework. The study of dependent RUMs has led to an expansive modeling literature, but inference in these settings is generally quite difficult \cite{train2009}. For intransitivity from independent but non-identical RUMs, we refer the reader to the literature on non-transitive dice \cite{savage1960}. None of which has developed tractable methods for inference, and further note that here the ability to introduce intransitivity is bounded, a matter that we elaborate on in Section 5.1. Our work is therefore focused on the last of the three approaches, studying multidimensional parametric representations of alternatives that are not necessarily consistent with i.i.d. RUMs and demonstrate sufficient expressivity for pairwise intransitivity. For our investigation we are inspired by social choice theory, where individual rankings may be transitive but overall the collective social decision procedures are not \cite{johnson2003}. We take inspiration from the Condorcet method of voting, a stronger form of the majority voting procedure, to embrace the intransitivity of this method and apply it to discrete choice modeling. By employing multiple RUMs, each of which are transitive in nature, and running a majority vote procedure, we are able to exhibit quite general forms of intransitivity.
2 Prior work on multidimensional choice models

In this section we provide a brief review of prior works on existing multidimensional models before later introducing our majority vote approach to pairwise intransitivity, which we later evaluate against these other existing models. The use of higher dimensional scores to model pairwise intransitivity is an idea with a notable history (Causseur and Husson 2005; May (1954) and also recent advancements Chen and Joachims (2016a,b); Ragain and Ugander (2016). An intuitive motivation of higher dimensional scores follows from competition, where each dimension can represent the strengths of a player in different skill aspects. For example, in modeling the game of soccer, one might consider three dimensions that represent the offensive, defensive, and endurance skills of a team, and winning a match is some subtle function of all three of these skills.

2.1 Extensions of the BTL model

The standard BTL model with one-dimensional score embeddings consists of \( n \) players with scores \( \lambda_1, ..., \lambda_n \) such that \( \lambda_i \in \mathbb{R}^+ \). Let \( P_{ij} \in (0,1) \) denote the probability that \( i \) beats \( j \). We then have

\[
P_{ij} = S(\lambda_i - \lambda_j),
\]

where \( S(x) = 1/(1 + \exp(-x)) \) is the logistic function. The uniqueness of the scores is asserted by imposing the constraint \( \sum_{i=1}^{n} \lambda_i = 0 \). It can be verified that the likelihood function of the standard BTL model is log-concave (Tsukida and Gupta (2011), meaning that the log-likelihood is concave.

A previous two-dimensional extension of BTL is due to Casseur et al. (Causseur and Husson 2005). In this model, two-dimensional scores are defined by \( \lambda_i \in \mathbb{R}^2 \) and the model is given as

\[
P_{ij} = S(\sigma_{ij} || \lambda_i - \lambda_j ||_2),
\]

where \( || \cdot ||_2 \) denotes the 2-norm, \( S(x) = 1/(1 + \exp(-x)) \), and \( \sigma_{ij} \) is determined from the data: it is assigned the value +1 if player \( i \) wins the majority of the matches between player \( i \) and \( j \) (and \( \sigma_{ji} = -\sigma_{ij} \)). Identifiability restrictions are enforced by imposing the set of constraints:

\[
\sum_{i=1}^{n} \lambda_{i1} = 0, \quad \sum_{i=1}^{n} \lambda_{i2} = 0, \quad \sum_{i=1}^{n} \lambda_{i1} \lambda_{i2} = 0.
\]

The log-likelihood of the score parameters for this two-dimensional BTL model is given by:

\[
\ell(\lambda, a_1, a_2, a_3; \{n_{ij}\}) = \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} n_{ij} \log(P_{ij}) + a_1 \sum_{i}^{n} \lambda_{i1} + a_2 \sum_{i}^{n} \lambda_{i2} + a_3 \sum_{i}^{n} \lambda_{i1} \lambda_{i2},
\]

where \( a_i \) are Lagrange multipliers and \( n_{ij} \) indicate the number of matches between player \( i \) and \( j \). For \( n \) players there are \( 2n + 3 \) free parameters that need to be determined. The parameters can be estimated using the method of gradient descent for minimizing the negative log likelihood (Causseur et al. employ the Newton-Raphson method). However the paper does not address the issue of intransitivity and does not perform any quantitative investigations in this regard. Although the Casseur et al. model can exhibit intransitivity and has a non convex likelihood function; very little has been established regarding the transitive vs. intransitive properties of this model.

The Casseur et al. model can be obviously extended to obtain a higher-dimensional BTL model where the scores \( \lambda_i \) live in more than two dimensions. Note that such extensions require more constraints to impose uniqueness. Specifically, we note that for a 3-dimensional extension the following constraints are sufficient for identifiability:

\[
\sum_{i=1}^{n} \lambda_{ik} = 0, \text{ for } k = 1, 2, 3,
\]

\[
\sum_{i=1}^{n} \lambda_{ik} \lambda_{il} = 0, \text{ for } (k, l) = (1, 2), (2, 3), \text{ and } (1, 3).
\]

The number of parameters for this 3D BTL model are \( 3n \). It is straightforward to verify that the negative log-likelihood function of these extended BTL models are non-convex in general.
2.2 Non-parametric extensions

It is possible to look at non-parametric models where each pairwise probability is its own parameter, i.e. the probability that \( i \) beats \( j \) is simply the probability “parameter” \( p_{ij} \). Such a model would have \( O(n^2) \) parameters that could be estimated using maximum likelihood \([\text{Hunter}(2004)]\). Chatterjee introduced the non-parametric Bradley-Terry model \([\text{Chatterjee}(2015)]\) as such a non-parametric extension of the standard BTL but with the assumption that \( p_{ijk} \geq p_{jik} \) for \( k \neq i, j \) whenever \( i \) beats \( j \). The matrix of pairwise probabilities \( P \) for this model is therefore assumed to satisfy strong stochastic transitivity. Under this assumption, Chatterjee gave a consistent estimator whose expected mean square error (between the actual \( P \) and an estimated matrix \( \hat{P} \)) can be bounded above by \( CN^{-1/3} \), but the analysis only holds under the assumption of strong stochastic transitivity. Without transitivity, no such convergence guarantees exist. Moreover, without transitivity the model has no basis for making inferences about matchups between pairs of alternatives that have not been seen in the data; the non-parametric model’s estimate of \( P_{ij} \) will simply be the empirical fraction of games won by \( i \) over \( j \). This motivates the need for parametric models where the \( p_{ij} \) can be expressed as a function of item-level parameters for \( i \) and \( j \).

2.3 Blade Chest model

The Blade-Chest model is similarly based on a higher-dimensional embedding of alternatives. In this model, every player \( a \) is represented by two \( d \)-dimensional vectors, a “blade” vector and a “chest” vector, denoted by \( \mathbf{a}_{\text{blade}} \) and \( \mathbf{a}_{\text{chest}} \) respectively. There are two variations of the model, “Blade-Chest distance” and “Blade-Chest inner.” In both models the probability of player \( a \) beating player \( b \) in the match is given by \( S(Q(a, b)) \) where \( S(x) = 1/(1 + \exp(-x)) \) is again the logistic function and \( Q(a, b) \) is a choice of a so-called match-up function between players \( a \) and \( b \). The match-up functions for the distance and inner variations of the Blade-Chest model are:

\[
\begin{align*}
Q_{\text{dist}}(a, b) &= |\mathbf{b}_{\text{blade}} - \mathbf{a}_{\text{chest}}|^2 - |\mathbf{b}_{\text{chest}} - \mathbf{a}_{\text{blade}}|^2 + \gamma_a - \gamma_b, \\
Q_{\text{inner}}(a, b) &= \mathbf{b}_{\text{blade}} \cdot \mathbf{a}_{\text{chest}} - \mathbf{b}_{\text{chest}} \cdot \mathbf{a}_{\text{blade}} + \gamma_a - \gamma_b.
\end{align*}
\]

Here the \( \gamma \) parameters are additional parameters that assure that the BTL model is captured as a clean special case (when all blade and chest vectors are zero vectors). There are \( 2(d + 1)n \) parameters in such Blade-Chest models.

As illustrated in \([\text{Chen and Joachims}(2016a)]\), both models perform well when the dimension \( d \) is large, specifically \( O(n) \), leading to a total of \( O(n^2) \) parameters. Performance improvements over BTL are modest when only a few dimensions are employed. High-dimensional Blade-Chest models are computationally difficult to learn due to the necessary minimization of a non-convex non-separable negative log-likelihood function. Overall, very little has been established regarding the appropriate dimensionality of multidimensional Blade Chest models. These challenges of high dimensionality, combined with the necessary non-convexity of learning, motivates our development of our low-dimensional majority vote model.

3 Non-concavity of intransitiveness

In this section we give our main theoretical result, that intransitiveness necessarily implies non-concavity for the log-likelihood of any parametric choice model. The extended BTL model and both Blade-Chest models are merely different choices of nonlinear pairwise probability functions that fall into this more general analysis. We begin with a useful definition.

**Definition 1.** A pairwise probability function \( f : \mathbb{R}^{2D} \to [0, 1] \) is a function mapping the \( D \)-dimensional parametric representations of two alternatives to a choice probability such that \( f(i, j) + f(j, i) = 1, \forall i, j \).

We will assume that \( f \) satisfies the following two properties:

1. \( f(\mu_1 + x, \mu_2 + x) = f(\mu_1, \mu_2) \) for \( x \in \mathbb{R}^d \)
2. \( f(c \mu_1 + x, c \mu_2) = f(\mu_1, \mu_2) \) for \( c \in \mathbb{R}^+ \)

These two properties correspond to additive and scalar invariance, respectively.

In the BTL model, \( f(a, b) = \Phi(a - b) \) for scalars \( a \) and \( b \), while for the Thurstone model \( f(a, b) = \Phi(a - b) \) again for scalars \( a \) and \( b \), where \( \Phi \) is the cumulative distribution function of the standard Normal distribution.

For the extended BTL model we have that \( f(i, j) = S(\sigma_{ij}||\lambda_i - \lambda_j||_2) \). For the Blade-Chest models we have that \( f_{\text{BC-dist}}(i, j) = S(Q_{\text{dist}}(i, j)) \) and \( f_{\text{BC-inner}}(i, j) = S(Q_{\text{inner}}(i, j)) \). For the Blade-Chest models we note that the dimension \( D \) of the score representation is different from the dimension \( d \) of the embedding: \( D = 2(d + 1)n \).
While these last three models are capable of exhibiting intransitivity, none of their negative log-likelihood functions are convex in the scores parameters of the choice alternatives, making global minimization challenging. This challenge raises a natural question: are there models of intransitive pairwise choice where the log-likelihood function is concave? Notice that for a generic pairwise probability function \( f(i, j) \), the log-likelihood objective of the model given data becomes:

\[
\ell(\lambda; \{n_{ij}\}) = \sum_{i=1}^{n} \sum_{j=1,j \neq i}^{n} n_{ij} \log(f(i, j)).
\]

In order to furnish a concave log-likelihood function, we therefore seek pairwise probability functions \( f(a, b) \) that are log-concave in the score parameters. Can any such \( f \) exhibit intransitivity? We obtain the following key negative result.

**Theorem 1.** For any pairwise probability function \( f(\lambda_1, \lambda_j) \) with \( \lambda_i, \lambda_j \in \mathbb{R}^d \), \( f \) can exhibit an intransitive cycle only if \( f \) is not log-concave.

**Proof.** Let \( A, B \) and \( C \) be three objects with score parameters \( \lambda_i \in \mathbb{R}^d \) and let them form an intransitive cycle, meaning that

\[
P_{AB} = f(\lambda_A, \lambda_B) > 0.5, \\
P_{BC} = f(\lambda_B, \lambda_C) > 0.5, \\
P_{CA} = f(\lambda_C, \lambda_A) > 0.5.
\]

Further assume that \( f \) is log-concave. We will show that this implies a contradiction.

Notice first that the function \( f \) is bounded, \( 0 \leq f(i, j) \leq 1, \forall i, j \). As \( f \) is log-concave we also have the following log-concavity properties for pairs \( f(p, q) \) and \( f(x, y) \) and \( 0 \leq \alpha \leq 1 \):

\[
f(\alpha p + (1 - \alpha)x, \alpha q + (1 - \alpha)y) \geq \min\{f(p, q), f(x, y)\},
\]

\[
f(\alpha p + (1 - \alpha)x, \alpha q + (1 - \alpha)y) \geq f(p, q)^\alpha f(x, y)^{1-\alpha}.
\]

Using this second log-concavity property for \( \alpha = 1/2 \) we have

\[
f((\lambda_A + \lambda_B)/2, (\lambda_B + \lambda_C)/2) \geq \sqrt{f(\lambda_A, \lambda_B)f(\lambda_B, \lambda_C)}.
\]

Further employing \( f(\lambda_A, \lambda_B) > 0.5 \) and \( f(\lambda_B, \lambda_C) > 0.5 \), from Equation (2) and the log-concavity, we have

\[
f((\lambda_A + \lambda_B)/2, (\lambda_B + \lambda_C)/2) > 0.5.
\]

Clearly \( f(-\lambda_A, -\lambda_B) = 0.5 \) because it is a pairwise probability function. Combining this observation with Equation (3), we obtain \( f(\lambda_A, \lambda_B) > 0.5 \) and hence \( f(\lambda_A, \lambda_B) < 0.5 \).

To obtain the contradiction we simply employ log-concavity for the pairs \( \{\lambda_C, \lambda_A\} \) and \( \{0, 0\} \) to obtain \( f(\lambda_A, \lambda_B) > 0.5 \), contradicting the earlier conclusion. \( \square \)

Among the many possible choices of intransitive pairwise probability functions \( f \), the very general family of functions that define a model, we might have suspected that there were some function with favorable inferential properties, i.e. one that yielded an overall log-likelihood function that was concave. The above result tells us that no such function exists. We note that this result covers all multidimensional approaches to non-transitivity, including independent RUMs (non-transitive dice) where one might achieve intransitivity using multiple “score parameters” to separately represent the location, variance, and/or skewness of the random utilities. Such a model does have the capacity to model intransitive choice, but the above result applies even to such models, meaning they can not have a concave log-likelihood function either.

In the absence of a convexity-based optimization argument for one pairwise probability function over another—motivating one choice of model over another—we now turn to a particularly intuitive and low-dimensional pairwise probability function based on the majority vote model from social choice theory.

### 4 Majority Vote Model

In this section we present and investigate a model of pairwise comparisons based on the basic concept of majority vote. In the majority vote model, each object (player) has an attribute representation \( \mu \in \mathbb{R}^k, k \geq 1 \), and the pairwise
probabilities of choosing an object (player) is modeled as a simple dimension-wise majority vote function over the different attribute dimensions.

Consider a contest where there are \( n \) players competing and each player \( i \) has a vector of attributes \( \mu_i \). A game between players \( i \) and \( j \) is won by player \( i \) if \( \mu_i - \mu_j + \xi \) is positive in \( k + 1 \) or more coordinates, and player \( j \) otherwise. Here \( \xi \) is a noise vector defined in terms of a difference distribution, a definition due to Yellott (1977).

**Definition 2.** If \( f \) and \( F \) are the probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of two i.i.d. random variables \( X_1 \) and \( X_2 \), then the difference density and difference distribution functions \( d_f \) and \( D_f \) are the p.d.f. and c.d.f. of \( X_1 - X_2 \) respectively.

The noise vector \( \xi \) in the majority vote model is drawn independently for each game from a multivariate distribution \( G \) with independence between the dimensions such that \( G = \prod_{k=1}^{2n+1} g \), where \( g \) is a distribution function such that the difference distribution function of \( g \) is continuous and strictly increasing over \( \mathbb{R} \). Two useful examples of \( g \) are the Normal distribution and the Gumbel distribution, leading to \( d_g \) being Normal and Logistic, respectively. We refer to the model with Gaussian noise as the Gaussian majority vote model. The Gaussian majority vote model reduces to the Thurstone model if the “voting” is conducted using only one dimension. Similarly, the majority vote model reduces to the standard BTL model if the \( g \) are assumed to be logistic functions and again the “voting” is conducted using only one dimension.

Although this majority vote model can be extended to arbitrarily large dimensions, we focus our analysis on the simple case of \( k = 3 \) dimensions and furnish a complete derivation of the maximum likelihood estimate of the parameter representations in this case. This choice of \( k \) limits the number of parameters to \( 3n \), where \( n \) is the number of alternatives (players). We show in Section 5 of this work that three dimensions is sufficient to (1) model strong and long intransitive cycles and (2) completely express the pairwise choice probabilities of any triplet. Based on these arguments we argue that three dimensions are sufficient for many applications. That said, it is straight-forward to generalize the MLE derivation below to arbitrarily large dimensions.

### 4.1 MLE derivation of the majority vote model for three dimensions

For the three-dimensional model we let \( M \in \mathbb{R}^{3n} \) denote the matrix of attributes \( \mu \) of all the \( n \) players. The \( i^{th} \) row of matrix \( M \) is then \( \mu_i = (\mu_i^1, \mu_i^2, \mu_i^3) \), the attributes of the \( i^{th} \) player. Let \( p_1(i, j), p_2(i, j), \) and \( p_3(i, j) \) be the probabilities that \( \mu_i - \mu_j + \xi \) are positive in the first, second, and third dimensions, respectively. We write \( p_z(i, j) \) as \( p_z \) for notational simplicity. Hence

\[
\begin{align*}
p_1 &= D_g(\mu_i^1 - \mu_j^1), \\
p_2 &= D_g(\mu_i^2 - \mu_j^2), \\
p_3 &= D_g(\mu_i^3 - \mu_j^3).
\end{align*}
\]

For additional notational simplicity we define the win probability \( w_{ij} \) to be the pairwise probability function \( f \) evaluated for two players \( i \) and \( j \),

\[
\begin{align*}
w_{ij} &= f(\mu_i, \mu_j) \\
&= \Pr(\mu_i - \mu_j + \xi \geq 0) \\
&= p_1 p_2 p_3 + (1 - p_1) p_2 p_3 + p_1 (1 - p_2) p_3 + p_1 p_2 (1 - p_3).
\end{align*}
\]

Here the notation \( x \geq y \) indicates that \( x \) is greater than \( y \) in at least two coordinates.

The goal with maximum likelihood estimation is to recover, up to symmetries, the locations of locally optimal \( \mu_i \) of each alternative given observed data about choices/competitions. Here \( N \) denotes the pairwise comparison matrix and \( n_{ij} \) be the number of times \( i \) has been chosen over \( j \). The likelihood function is then

\[
L(M; N) = \prod_{i,j:i\neq j} \binom{n_{ij} + n_{ji}}{n_{ij}} w_{ij}^{n_{ij}} w_{ji}^{n_{ji}}.
\]

If we examine the log-likelihood \( \ell(M; N) \), drop the additive constants, and focus on only pairs \( i < j \) rather than all pairs \( i \neq j \) (as the \( (i, j) \) and \( (j, i) \) terms are identical), we obtain:

\[
\ell(M; N) = \sum_{i,j:i<j} n_{ij} \log(w_{ij}) + n_{ji} \log(w_{ji}).
\]
It is clear that that coordinate-wise gradients are of the form
\[
\nabla_{\mu_i} \ell = \sum_{i<j} n_{ij} \nabla w_{ij} \mu_i + n_{ji} \nabla w_{ji} \mu_i,
\]
where \( \nabla_{\mu_i} \) denotes the derivative of \( \ell \) in the coordinates of the inputs corresponding to \( \mu_i \) only.

Using the chain rule we see that
\[
\nabla_{\mu_i} w_{ij} = \left( \frac{dw_{ij}}{d\mu_1} \frac{dp_1}{dp_2} \frac{dp_2}{d\mu_2} \frac{dp_3}{d\mu_3} \right)'.
\]
We note that we can rearrange \( w_{ij} \) for convenience:
\[
\begin{align*}
w_{ij} &= p_2 p_3 + p_1 (1 - p_2) p_3 + p_1 p_2 (1 - p_3) \\
&= p_2 p_3 + p_1 (p_2 + p_3 - 2 p_2 p_3).
\end{align*}
\]
An advantage of this majority vote model is the ease with which the gradients can be calculated for gradient-based optimization methods such as stochastic gradient descent [Bottou 2008]. For the Gaussian majority vote model we note that \( \frac{\partial \mu_i}{\partial \mu_j} = \phi(\mu_i - \mu_j) \) where \( \phi \) is the standard normal PDF, and obtain:
\[
\nabla w_{ij} = \left( \phi(\mu_1 - \mu_3)(p_2 + p_3 - 2 p_2 p_3), \\
\phi(\mu_2 - \mu_3)(p_1 + p_3 - 2 p_1 p_3), \\
\phi(\mu_3 - \mu_1)(p_1 + p_2 - 2 p_1 p_2) \right)'.
\]
Also using that \( w_{ji} = 1 - w_{ij} \) and \( \nabla w_{ji} = -\nabla w_{ij} \), we obtain the following overall gradient:
\[
\begin{align*}
\frac{\partial \ell}{\partial \mu_i} &= \sum_{i<j} \left( \frac{n_{ij}}{w_{ij}} - \frac{n_{ji}}{1 - w_{ij}} \right) \nabla w_{ij} \\
&= \sum_{i<j} \left[ \left( \frac{n_{ij}}{w_{ij}} - \frac{n_{ji}}{1 - w_{ij}} \right) \phi(\mu_i - \mu_j)(p_2 + p_3 - 2 p_2 p_3) \right].
\end{align*}
\]
The partial derivatives with respect to \( \mu_{i2} \) and \( \mu_{3} \) parameters can be calculated similarly. In general it is clear (directly, but also from the capacity for intransitivity and our theoretical result in the earlier section) that the maximization problem is non-concave with the potential for multiple distinct local maxima.

5 Characteristics of the Majority Vote Model

A possibly desirable property of a model capturing intransitivity is the ability to represent all pairwise win probabilities between players, but the ability to represent any \( n \times n \) matrix of pairwise probabilities would clearly require \( \binom{n}{2} \) parameters. In this work we are interested in models with few parameters, at most \( O(n) \) and specifically \( 3n \) for our 3D majority vote model, and seek focused results that establish the expressiveness of the majority vote model despite only having \( 3n \) parameters. Specifically, we show that it is capable of representing arbitrarily long intransitive cycles and that it is capable of representing any submatrix of probabilities between a triplet of players (though this is not the same as simultaneously being able to represent any submatrix on all triplets).

5.1 Cycle expressivity of the majority vote model

The random utility model (RUM) framework is an important framework for the characterization of various ranking models [Block 1960; Manski 1977]. Random Utility Models in general do not necessarily assume stochastic transitivity, not even for independent RUMs: consider the example of non-transitive Efron dice [Savage 1960] discussed in the introduction. There are, however, limits on how “strong” the intransitivity for independent RUMs can be: Trybula provides conditions for existence of independent random variables \( X_1, X_2, \ldots, X_n \) such that \( \min \{ P(X_1 > X_2, X_2 > X_3) \} \)
We apply the inequality with
where \( l \in \{1, 2, 3\} \). The sequence \( c_n \) is increasing and converges to 3/4. It is easy to determine that \( c_3 = (\sqrt{5} - 1)/2 \) and \( c_4 = 2/3 \). In this subsection we show that majority vote models exhibit no such bound on the strength of intransitivity.

Our majority vote model can be understood as an aggregate process over a set of independent RUM models. This independence means that there is no statistical dependence between the random variables used to represent players, yet we will show that the 3D majority vote model exceeds Trybula’s bounds. Specifically, we will prove that the intransitivity of the majority vote model is not bounded away from 1 by any constant. As a result we are able to conclude that Majority Vote choice model, despite not relying on any dependence between individuals or heterogeneous noise distributions, is qualitatively different from any univariate independent RUM.

**Theorem 2.** Given \( \epsilon > 0 \) and \( n \) i.i.d. random variables with distribution \( g \) whose pairwise difference follow the difference distribution function \( D_g \) which is continuous, strictly increasing on \( \mathbb{R} \) with finite variance \( c \), there exists an attribute matrix \( M \) for \( n \) players such that

\[
\min\{P(X_1 > X_2), P(X_2 > X_3), ..., P(X_n > X_1)\} \geq 1 - \epsilon.
\]

**Proof.** The proof is by construction. Let \( X_i \) and \( X_{i+1} \) denote cyclically consecutive players, where \( X_n \) is cyclically adjacent to \( X_1 \). For all \( i \) we have that:

\[
\Pr(X_i > X_{i+1}) = p_2p_3 + p_1p_3 + p_1p_2 - 2p_1p_2p_3.
\]

As before, we write \( p_2(i, j) \) as \( p_2 \) when the intended \( (i, j) \) pair is unambiguous. As defined before \( p_1 = D_g(\mu_1^l - \mu_1^{l+1}) \) where \( l \in \{1, 2, 3\} \). For our construction an appropriate \( \mu \) is selected for the first three players, where we choose \( \mu_1, \mu_2, \) and \( \mu_3 \) to be

\[
\mu_1 = (a, 3a, -a), \quad \mu_2 = (-a, a, 3a), \quad \mu_3 = (3a, -a, a),
\]

where \( a \in \mathbb{R}^+ \) is a constant to be determined. As \( D_g \) is a continuous cumulative distribution function there exists a constant \( a_g \in \mathbb{R}^+ \) s.t. \( \forall a \geq a_g \) for the random variables \( X_1 \) and \( X_2 \), \( p_1 \geq 1 - \frac{a}{2}, p_2 \geq 1 - \frac{a}{2} \). As \( p_3 \geq 0 \) and \( p_1 + p_2 - 2p_1p_2 \geq 0 \), we observe that

\[
\Pr(X_1 > X_2) \geq p_1p_2,
\]

and hence \( \Pr(X_1 > X_2) \geq 1 - \epsilon \). A similar argument holds for the pair \( \{X_2, X_3\} \).

For players \( i \geq 4 \) the \( \mu \) values are generated with the following recursive equation:

\[
\mu_i = (\mu_{i-1} + \mu_1)/2, \quad \text{for } i = 4, \ldots, n.
\]

Now we proceed to show the inequality \( \Pr(X_i > X_{i+1}) \geq 1 - \epsilon \) for \( i = 3 \) and onward. As \( p_2 \geq 0 \) and \( p_1 + p_2 - 2p_1p_2 \geq 0 \), we observe that

\[
\Pr(X_i > X_{i+1}) \geq p_1p_3.
\]

By the given construction of the \( \mu \) parameters, \( \mu_i - \mu_{i+1} \) is decreasing in dimensions one and three for \( i = 3, \ldots, n \). Hence the product \( p_1p_3 \) is decreasing for players \( X_i \) and \( X_{i+1} \) as \( i \) is increasing since \( g \) is a strictly increasing function.

Define \( h(i, j) = p_1(i, j)p_3(i, j) \) for players \( i \) and \( j \). By construction we automatically have \( h(3, 4) > h(4, 5) > \ldots > h(n - 1, n) = h(n, 1) \). Hence we only need to prove the inequality \( p_1(i, j)p_3(i, j) \geq 1 - \epsilon \) for players \( i = n, j = 1 \).

For players \( X_n \) and \( X_1 \) we then have:

\[
\Pr(X_n > X_1) \geq p_1p_3, \quad \text{where } p_1 = p_3 = D_g\left(\frac{2a}{2n-3}\right).
\]

Now recall Chebyshev’s inequality,

\[
P(Z \geq z) \leq \frac{\text{Var}(Z)}{z^2}.
\]

We apply the inequality with \( Z \) set to the first and the third dimensions of \( X_n - X_1 \). As the variance is bounded, \( \text{Var}(Z) \leq c \) where \( c \) is a positive constant and \( \Pr(Z \geq z) = 1 - \phi(z) \), we have that \( D_g(Z) \geq 1 - \frac{c}{z^2} \). Using this
property with \( z = \frac{2n}{\epsilon^2} \), we have

\[
\Pr(X_n > X_1) \geq p_1 p_3 \geq \left(1 - \frac{\epsilon}{\epsilon^2}\right)^2 \geq \left(1 - \frac{22n - 8\epsilon}{a^2}\right)^2 \geq 1 - \frac{2^{2n-7}\epsilon}{a^2},
\]

where for the last part we employ the fact that \((1 - x)^2 \geq 1 - 2x\) when \( x > 0 \). We conclude that \( 1 - \frac{2^{2n-7}\epsilon}{a^2} \geq 1 - \epsilon \)
when \( \epsilon \geq \frac{2^{2n-7}\epsilon}{a^2} \). Combining this analysis with the earlier basic analysis for the initial \( i = 1, 2, 3 \), we can choose \( a \)
such that \( a \geq \max\{a_g, \frac{2^{2n-7}\epsilon}{\sqrt{\epsilon}}\} \) to complete the proof.

The above theorem has two important implications. First, large intransitive cycles can be created using the majority vote model in merely three dimensions. Second, we can conclude that the majority vote model for a broad class of
difference densities \( d_g \) is not equivalent to any independent RUM model.

### 5.2 Triplet expressivity

The lemma below shows how the three-dimensional majority vote model can represent all the pairwise win probability
relations for a set of any three players.

**Lemma 2.** Given \( \epsilon > 0 \) and a matrix \( W \) of pairwise win probabilities for any three players, there exists an attribute
matrix \( M \) which can be used to construct a matrix \( \tilde{W} \) of pairwise win probabilities from a Majority Vote model such that \( \max_{i,j} |W_{ij} - \tilde{W}_{ij}| \leq \epsilon \).

**Proof.** The proof is by construction. We are given a pairwise probability matrix \( W \) with elements \( w_{ij} \). Let \( c_1 = \tilde{D}_g^{-1}(w_{23}) \). Define \( c_2 \) and \( c_3 \) similarly. Let \( \alpha_1 = \frac{1-w_{12}}{w_{23}} \). Define \( \alpha_2 \) and \( \alpha_3 \) similarly in terms of \( w_{23} \) and \( w_{31} \)
respectively. For the three players we can construct \( M \) as defined above to be

\[
\begin{bmatrix}
c_3 & 0 & 0 \\
0 & -C + c_1 & C \\
C & -C & c_2
\end{bmatrix},
\]

where \( \tilde{D}_g \) is continuous and strictly increasing over \( \mathbb{R} \) and \( C \) is a positive constant such that

- \( \tilde{D}_g(-C) \leq \frac{\epsilon}{2} \)
- \( \tilde{D}_g(C - c_i) \geq 1 - \alpha_i \epsilon \) for \( i = 1, 2, 3 \).

The existence of \( C \) is guaranteed due to the continuity of \( \tilde{D}_g \).

We need to prove \( |w_{i,j} - \tilde{w}_{i,j}| \leq \epsilon \) for \( 1 \leq i \neq j \leq 3 \). Consider \( i = 1 \) and \( j = 2 \). We have \( \tilde{w}_{12} = p_1 p_2 + p_2 p_3 + p_3 p_1 - 2p_1 p_2 p_3 \). With \( p_1 = w_{12}, p_3 \leq \frac{\epsilon}{2}, \) and \( 1 - \frac{\alpha_1 \epsilon}{2} \leq p_2 \leq 1 \) we have

\[
\tilde{w}_{12} \leq w_{12} + \left(\epsilon/2\right) + (\epsilon/2)w_{12} \leq w_{12} + \epsilon.
\]

Next we need to prove the bound in the other direction. Since

\[
\tilde{w}_{12} \geq p_1 p_2 - 2p_1 p_2 p_3,
\]

we have \( \tilde{w}_{12} \geq w_{12} - \epsilon \).

An identical argument can be applied to the other two pairs within the triplet, yielding the requisite bound on the
maximum over pairs \((i, j)\), as desired.

It is not hard to verify that a three-dimensional Majority Vote model is not fully expressive for larger subsets of players
(in particular, subsets of five players or more), but we consider the two expressivity results we contribute here, for
cycles and for triplets, to be sufficiently flexible to model realistic competition data. Most significantly, we note that
neither of these results can be achieved by any independent RUM model.
6 Experimental Results

We now study the predictive performance of the different parametric models of intransitive pairwise choice we have discussed: higher dimensional BTL models, the Blade Chest model, and our Gaussian Majority Vote model. We evaluate each model on nine different datasets where each model is furnished with a comparable number of free parameters \((3n)\) for comparison. Let \(N\) represent an \(n \times n\) count matrix of the number of pairwise encounters between the \(n\) players and where \(n_{ij}\) indicates how many games have been won by \(i\) in all the pairwise encounters between \(i\) and \(j\). Each pairwise comparison dataset was randomly split into training and test sets using 5-fold cross-validation, and the results were averaged across the five random splits. The training count matrix \(N_{\text{train}}\) was used to estimate the parameters in the models which are then used to find the prediction probabilities for pairwise comparisons on the test dataset \(N_{\text{test}}\).

The evaluation metric used is root mean square error (RMSE) between the predicted probabilities \(\tilde{p}\) from the model and the empirical probabilities \(\hat{p}\) from the test data. The RMSE is scaled by the number of players:

\[
\text{RMSE} = \frac{1}{n} \sqrt{\sum_{i,j \in n, j \neq i} (\hat{p}(i, j) - \tilde{p}(i, j))^2}.
\]

The source code and the datasets used for testing in the following section will be made available on our website.

6.1 Synthetic data: univariate intransitive dice

Taking inspiration from intransitive dice, in this section we first study how well our model can capture the behavior of intransitive sets of 3 and 4 dice. For the set of 3 dice, we use the following construction:

\[
\begin{align*}
A & : 2, 2, 4, 9, 9 \\
B & : 1, 1, 6, 8, 8 \\
C & : 3, 3, 5, 7, 7
\end{align*}
\]

where \(\Pr(A > B) = \Pr(B > C) = \Pr(C > A) = \frac{5}{9}\). For the set of 4 dice we use the construction due to Efron from the introduction, where \(\Pr(A > B) = \Pr(B > C) = \Pr(C > D) = \Pr(D > A) = \frac{2}{3}\). We generate a synthetic count matrix \(N\) from comparisons of these independent random random variables. The data is simulated in two realms: 1,000 games representing limited data, and 10,000 games representing close to asymptotic data.

Table 1. RMSE for Intransitive dice, 1,000 games.

| Dataset | 3D BTL | Blade Chest (best) | Majority Vote (MV) |
|---------|--------|--------------------|--------------------|
| 3 dice  | 0.6 ± 0.03 | 0.03 ± 0.01 | 0.03 ± 0.01 |
| 4 dice  | 0.3 ± 0.3  | 0.05 ± 0.03 | 0.05 ± 0.03 |

Table 2. RMSE for Intransitive dice, 10,000 games.

| Dataset | 3D BTL | Blade Chest (best) | Majority Vote (MV) |
|---------|--------|--------------------|--------------------|
| 3 dice  | 0.01 ± 0.004 | 0.01 ± 0.004 | 0.01 ± 0.004 |
| 4 dice  | 0.02 ± 0.01  | 0.02 ± 0.01 | 0.03 ± 0.02 |

In Tables 1 and 2 we show the RMSE for the three different intransitive models trained on and predicting games played between intransitive dice. The 3D BTL model can model the set of 3 intransitive dice but not the set of 4. Meanwhile the Blade Chest and Majority Vote model can model both the set of 3 and set of 4. The results for Blade Chest represent the performance of the best of the “inner” and “distance” variations of the model. For the three dice all models have 9 parameters; for the four dice all models have 12 parameters.

6.2 Real data: elections, games, and jokes

We now test the models of intransitive comparison discussed in this work on three datasets: a commonly studied set of election datasets (A5, A9, A17, A48, County Meath and Dublin West) \cite{Tideman2006}, the Jester joke rating dataset (a dataset underlying a recommendation system) \cite{Goldberg1999}, and the Super Street Fighter IV dataset (pairwise match-up data from competitive play between different video game characters) \cite{Chen2016a}. The election and Jester datasets are ranking datasets which were converted to pairwise comparisons datasets.
In general it is empirically rare to see large amount of intransitivity in pairwise competitions (Chen and Joachims, 2016a). This observation is consistent with the intuition that a sports competitor will tend to be roughly as good at offense as she or he is at defense. Large imbalances would imply that a competitor could greatly improve their performance by developing one skill or another, which is why skilled competitors tend to focus on addressing their weaknesses. Overall, the varied skills of a single player tend to be correlated. In Table 3 we count the number of intransitive triplets in the empirical datasets, meaning the number of triplets (out of all possible triplets) where the empirical win percentages are not transitive. For most datasets these counts are every small.

**Table 3.** Details of datasets and the number of intransitive triplets in each dataset.

| Dataset     | Players, n | Games    | Intransitive Triplets |
|-------------|------------|----------|------------------------|
| Election A5 | 16         | 44298    | 5                      |
| Election A9 | 12         | 95888    | 3                      |
| Election A17| 13         | 21037    | 18                     |
| Election A48| 10         | 25848    | 0                      |
| Election A81| 11         | 20803    | 3                      |
| Election CM | 10         | 356038   | 0                      |
| Election DW | 13         | 726178   | 0                      |
| Jester      | 100        | 333956   | 90                     |
| Street Fighter | 35     | 25000    | 476                    |
We see that across the election datasets there is relatively little difference between the different

comparisons in the 3-dimensional model following the same ordering as the Thurstone model. There are, however, local
deviations where certain characters pairwise matchups have predicted win probabilities that deviate significantly from

The non-concave likelihoods of all models were maximized using the Nelder–Mead simplex algorithm. The optimization

was repeated five times to ensure that the algorithm did not converge to poor local optima. For each of the three

intransitive models, except for with the Jester dataset. The figure also counts the average number of intransitive triplets

(triplets exhibiting weak intransitivity) for the predicted probability matrix (obtained from the MLE estimates of the

parameters) for each for the three models. For all models we see that these counts roughly replicate the structure of the
datasets.

For the Jester dataset we observe a large difference in the RMSE performance between our 3D Gaussian Majority V ote
model and the other intransitive models, with our model reporting an RMSE that is 32% lower than the best Blade-Chest
model. For the Jester dataset the 3D BTL, Blade-Chest, and 3D Gaussian Majority V ote models all have the same
number of parameters.

In Figure 2 and 3 we show the matrices of win probabilities for the Street Fighter dataset under the 1-dimensional
Thurstone and a 3-dimensional Gaussian Majority V ote model. The matrices are both arranged according to the order of
the Thurstone skill parameters. We observe that the two models exhibit largely the same win probabilities, with most
comparisons in the 3-dimensional model following the same ordering as the Thurstone model. There are, however, local
deviations where certain characters pairwise matchups have predicted win probabilities that deviate significantly from
the the total ordering. The most intransitive triplet of the learned model (measured by the maximum minimum pairwise
probability) was 0.59, which is less than the bound for intransitive dice (0.75) discussed in Section 5.1. We also see
many deviations from the 1-dimensional Thurstone predictions that are not themselves violations of transitivity.

The non-concave likelihoods of all models were maximized using the Nelder–Mead simplex algorithm. The optimization
was repeated five times to ensure that the algorithm did not converge to poor local optima. For each of the three
intransitive models, the varied local optima we obtained for a given model had the same objective value within a small
margin of error. The RMSE results are provided in Table 4. Figure 1 provides a visual comparison for some of the
RMSE values indicated in Table 4. For the Blade Chest models we again use the best results between the “inner” and
“distance” variations. We see that across the election datasets there is relatively little difference between the different
intransitive models, except for with the Jester dataset. The figure also counts the average number of intransitive triplets
(triplets exhibiting weak intransitivity) for the predicted probability matrix (obtained from the MLE estimates of the
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the the total ordering. The most intransitive triplet of the learned model (measured by the maximum minimum pairwise
probability) was 0.59, which is less than the bound for intransitive dice (0.75) discussed in Section 5.1. We also see
many deviations from the 1-dimensional Thurstone predictions that are not themselves violations of transitivity.

6.3 Spectral analysis of dimensions

In order to understand the role of higher dimensions in modeling skill, we perform an analysis of the singular values of
the attribute matrix $\hat{M}$ learned from data for our 3D Gaussian Majority V ote model. In general if the dimensions of $M$
are uncorrelated then this indicates the presence of multiple uncorrelated skills. If the variance of $M$ is concentrated
on just one dimension, the competition data can be viewed as mostly transitive. We use the explained variance of the
vectors of the singular value decomposition as measures of how correlated the skill dimensions of the learned
embeddings are. In Table 5 we observe that most, but not all, of the variance is explained by the first singular vector
of $\hat{M}$ in all the datasets we examine. The “least one-dimensional” model is learned for the Street Fighter data, in the
sense that the leading singular vector explains the last percentage of the variance. If of interest, the leading dimension
across which the maximum variance is spread could be used as a method for producing ranked orderings of all the
alternatives/players.

To complement the empirical analysis, we also simulate data which is inherently one, two, and three dimensional,
respectively. This is done by simulating a 3D Majority V ote model with a $\mu$ vector that is drawn from a uniform
distribution for each player which is 1, 2, or 3-dimensional. For the 3-dimensional $\mu$, the Gaussian Majority V ote model
is then simulated. For the 1-dimensional and 2-dimensional $\mu$, a version of the Gaussian Majority V ote model is then
simulated where the excess dimensions are unbiased coin-flips. We use this process to produce a pairwise comparison
matrices for 50 players with 10,000 games for each pair. We train the majority vote model based on the the simulated

| Dataset     | 3D BTL | 3D BTL IT | BC  | BC IT  | 3D Gaussian MV | MV IT |
|-------------|--------|-----------|-----|--------|----------------|-------|
| Election A5 | 0.033 ± 0.009 | 10        | 0.029 ± 0.004 | 1    | 0.026 ± 0.001 | 1     |
| Election A9 | 0.018 ± 0.002 | 3         | 0.017 ± 0.002 | 2    | 0.018 ± 0.002 | 1     |
| Election A17| 0.046 ± 0.022 | 12        | 0.031 ± 0.002 | 5    | 0.032 ± 0.003 | 3     |
| Election A48| 0.027 ± 0.002 | 1         | 0.027 ± 0.003 | 0    | 0.025 ± 0.002 | 1     |
| Election A81| 0.035 ± 0.003 | 4         | 0.032 ± 0.004 | 1    | 0.031 ± 0.004 | 1     |
| Election CM | 0.014 ± 0.002 | 0         | 0.014 ± 0.002 | 0    | 0.015 ± 0.005 | 0     |
| Election DW | 0.017 ± 0.002 | 0         | 0.014 ± 0.002 | 0    | 0.017 ± 0.003 | 0     |
| Jester      | 0.031 ± 0.005 | 80        | 0.031 ± 0.002 | 142  | 0.021 ± 0.001 | 49    |
| Street Fighter | 0.064 ± 0.004 | 410       | 0.06 ± 0.002 | 187  | 0.06 ± 0.003 | 167   |
Figure 2. A heat map for the predicted pairwise probabilities matrix estimated using a Thurstone model. The matrix entries are sorted according to ranks suggested by the Thurstone model.

Table 5. Fraction of the variance explained by the singular vectors of the attribute matrix of the Gaussian Majority Vote model for different datasets.

| Dataset      | $\sigma_1$  | $\sigma_2$  | $\sigma_3$  |
|--------------|-------------|-------------|-------------|
| Election A5  | 0.62 ± 0.08 | 0.29 ± 0.04 | 0.09 ± 0.04 |
| Election A9  | 0.61 ± 0.05 | 0.30 ± 0.03 | 0.08 ± 0.05 |
| Election A17 | 0.65 ± 0.10 | 0.31 ± 0.10 | 0.03 ± 0.01 |
| Election A48 | 0.66 ± 0.05 | 0.24 ± 0.03 | 0.09 ± 0.04 |
| Election A81 | 0.68 ± 0.05 | 0.25 ± 0.03 | 0.07 ± 0.02 |
| Election CM  | 0.64 ± 0.09 | 0.24 ± 0.05 | 0.11 ± 0.05 |
| Election DW  | 0.83 ± 0.03 | 0.11 ± 0.03 | 0.05 ± 0.02 |
| Jester       | 0.72 ± 0.01 | 0.18 ± 0.01 | 0.08 ± 0.01 |
| Street Fighter| 0.53 ± 0.02 | 0.28 ± 0.02 | 0.18 ± 0.03 |

data, repeating the training five times, and calculate the fraction of variance explained by each singular vector of the learned $\mu$.

We see that even when the data is truly three-dimensional, there is still a “dominant” vector that explains most of the variation. When the data is two or one dimensional, the majority of the variation can be explained by fewer singular vectors, as expected.

7 Conclusions

In this work we show that intransitivity implies non-convexity in the log-likelihood function for parametric choice models. We further introduce a Majority Vote model for learning preference relations from pairwise comparison data.
Figure 3. A heat map for the predicted pairwise probabilities matrix estimate using the 3-dimensional Gaussian Majority Vote model. The matrix entries are sorted according to ranks suggested by the Thurstone model, as in Figure 2. We see several cycles in the probability matrix which differ from the predictions of the Thurstone model.

Table 6. Fraction of the variance explained by the singular vectors of the attribute matrix of the Gaussian Majority Vote model for synthetic data.

| Dataset               | $\sigma_1$       | $\sigma_2$       | $\sigma_3$       |
|-----------------------|-------------------|-------------------|-------------------|
| One Dimensional Data  | 0.81 ± 0.08       | 0.13 ± 0.05       | 0.06 ± 0.05       |
| Two Dimensional Data  | 0.58 ± 0.09       | 0.26 ± 0.05       | 0.16 ± 0.05       |
| Three Dimensional Data| 0.51 ± 0.04       | 0.29 ± 0.05       | 0.20 ± 0.03       |

This model can represent intransitive relations and is rich enough to incorporate other independent RUM models. The model also showed results that were comparable or slightly better than earlier multidimensional models when predicting outcomes of elections, online video games, or pairwise joke ratings, with the added benefit of simplicity and interpretability. A significant benefit of the majority vote model is the ability to interpret the set of three- (or more) dimensional representations as a point cloud in an embedded space, and we give a straight-forward spectral approach to measuring the “dimensionality” of intransitive data using singular values. Where this work has focused on pairwise comparisons, it would be interesting to attempt extensions of both the analysis and the model from pairwise choices to modeling choices or competitions from larger sets.
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