FROBENIUS MORPHISMS OVER $\mathbb{Z}/p^2$ AND BOTT VANISHING

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In [1] Bott proved the vanishing theorem

$$H^k(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n} \otimes \mathcal{O}(k)) = 0$$

when $0 < k \leq p$. We will say that a smooth projective variety $X$ has Bott vanishing if for every ample line bundle $L$, $i > 0$ and $j$

$$H^i(X, \Omega^j_X \otimes L) = 0$$

The purpose of this paper is to show that Bott vanishing is a simple consequence of a very specific condition on the Frobenius morphism in prime characteristic $p$. Recall that the absolute Frobenius morphism $F: X \to X$ on $X$, where $X$ is a variety over $\mathbb{Z}/p$ is the identity on point spaces and the $p$-th power map locally on functions. Assume that there is a flat scheme $X^{(2)}$ over $\mathbb{Z}/p^2$, such that $X \cong X^{(2)} \times_{\mathbb{Z}/p^2} \mathbb{Z}/p$. The condition on $F$ is that there should be a morphism $F^{(2)}: X^{(2)} \to X^{(2)}$ which gives $F$ by reduction mod $p$. In this case we will say that the Frobenius morphism lifts to $\mathbb{Z}/p^2$. It is known that a lift of the Frobenius morphism to $\mathbb{Z}/p^2$ leads to a quasi-isomorphism

$$\sigma: \bigoplus_{0 \leq i} \Omega^i_X[-i] \to F_*\Omega^*_X$$

where the complex on the left has zero differentials and $\Omega^*_X$ denotes the de Rham complex of $X$ ([3], Remarques 2.2(ii)). Using duality we prove that $\sigma$ is in fact a split quasi-isomorphism.

In general it is very difficult to decide when Frobenius lifts to $\mathbb{Z}/p^2$. However for varieties which are glued together by monomial automorphisms it is easy. This is the case for toric varieties, where we show that the Frobenius morphism lifts to $\mathbb{Z}/p^2$. This gives natural characteristic $p$ proofs and explanations of the Bott vanishing theorem for (singular and smooth) toric varieties and the degeneration of the Danilov spectral sequence ([2], Theorem 7.5.2, Theorem 12.5).

Paranjape and Srinivas have proved using complex algebraic geometry that if Frobenius for a generalized flag variety $X$ lifts to the $p$-adic numbers $\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^n$, then $X$ is a product of projective spaces [11]. In the last part of this paper we generalize this result by showing that Frobenius for a large class of generalized flag varieties admits no lift to $\mathbb{Z}/p^2$. This is done using a lemma on fibrations linking non-lifting of Frobenius to Bott non-vanishing cohomology groups

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for flag varieties of Hermitian symmetric type over the complex numbers. These cohomology
groups have been studied thoroughly by M.-H. Sato and D. Snow.

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1. Preliminaries

Let $k$ be a perfect field of characteristic $p > 0$ and $X$ a smooth $k$-variety of dimension $n$. By
$\Omega_X$ we denote the sheaf of $k$-differentials on $X$ and $\Omega^j_X = \wedge^j \Omega_X$. The (absolute) Frobenius
morphism $F : X \to X$ is the morphism on $X$, which is the identity on the level of points and
given by $F^#(f) = f^p : \mathcal{O}_X(U) \to F_*\mathcal{O}_X(U)$ on the level of functions. If $\mathcal{F}$ is an $\mathcal{O}_X$-module,
then $F_*\mathcal{F} = \mathcal{F}$ as sheaves of abelian groups, but the $\mathcal{O}_X$-module structure is changed according
to the homomorphism $\mathcal{O}_X \to F_*\mathcal{O}_X$.

1.1. The Cartier operator. The universal derivation $d : \mathcal{O}_X \to \Omega_X$ gives rise to a family
of $k$-homomorphisms $d^j : \Omega^j_X \to \Omega_X^{j+1}$ making $\Omega^\bullet_X$ into a complex of $k$-modules which is called
the de Rham complex of $X$. By applying $F_*$ to the de Rham complex, we obtain a complex
$F_*\Omega^\bullet_X$ of $\mathcal{O}_X$-modules. Let $B^i_X \subseteq Z^i_X \subseteq F_*\Omega^i_X$ denote the coboundaries and cocycles in degree
$i$. There is the following very nice description of the cohomology of $F_*\Omega^\bullet_X$ due to Cartier.

**Theorem 1.** There is a uniquely determined graded $\mathcal{O}_X$-algebra isomorphism

$$ C^{-1} : \Omega^\bullet_X \to \mathcal{H}^\bullet(F_*\Omega^\bullet_X) $$

which in degree 1 is given locally as

$$ C^{-1}(da) = a^{p-1}da $$

*Proof.* [7], Theorem 7.2. □

With some abuse of notation, we let $C$ denote the natural homomorphism $Z^i_X \to \Omega^i_X$, which after reduction modulo $B^i_X$ gives the inverse isomorphism to $C^{-1}$. The isomorphism
$\bar{C} : Z^i_X/B^i_X \to \Omega^i_X$ is called the Cartier operator.

2. Liftings of Frobenius to $W_2(k)$

There is a very interesting connection between the Cartier operator and liftings of the Frobenius
morphism to flat schemes of characteristic $p^2$. This beautiful observation was first made
by Mazur in [8]. We go on to explore this next.
2.1. Witt vectors of length two. The Witt vectors $W_2(k)$ ([10], Lecture 26) of length 2 over $k$ can be interpreted as the set $k \times k$, where multiplication and addition for $a = (a_0, a_1)$ and $b = (b_0, b_1)$ in $W_2(k)$ are defined by

$$a \cdot b = (a_0 b_0, a_0 b_1 + b_0^p a_1)$$

and

$$a + b = (a_0 + b_0, a_1 + b_1 + p^{-1} \sum_{j=1}^{p-1} p^{-1} \binom{p}{j} a_0^j b_0^{p-j})$$

In the case $k = \mathbb{Z}/p$, one can prove that $W_2(k) \cong \mathbb{Z}/p^2$. The projection on the first coordinate $W_2(k) \to k$ corresponds to the reduction $W_2(k) \to W_2(k)/p \cong k$ modulo $p$. The ring homomorphism $F^{(2)} : W_2(k) \to W_2(k)$ given by $F^{(2)}(a_0, a_1) = (a_0^p, a_1^p)$ reduces to the Frobenius homomorphism $F$ on $k$ modulo $p$.

2.2. Splittings of the de Rham complex. The previous section shows that there is a canonical morphism $\text{Spec } k \to \text{Spec } W_2(k)$. Assume that there is a flat scheme $X^{(2)}$ over $\text{Spec } W_2(k)$ such that

$$X \cong X^{(2)} \times_{\text{Spec } W_2(k)} \text{Spec } k \quad (1)$$

We shall say that the Frobenius morphism $F$ lifts to $W_2(k)$ if there exists a morphism $F^{(2)} : X^{(2)} \to X^{(2)}$ covering the Frobenius homomorphism $F^{(2)}$ on $W_2(k)$, which reduces to $F$ via the isomorphism (1). When we use the statement that Frobenius lifts to $W_2(k)$ we will always implicitly assume the existence of the flat lift $X^{(2)}$.

**Theorem 2.** If the Frobenius morphism on $X$ lifts to $W_2(k)$ then there is a split quasi-isomorphism

$$0 \to \bigoplus_{0 \leq i} \Omega^i_X[-i] \xrightarrow{\varphi} F_* \Omega^\bullet_X.$$

**Proof.** For an affine open subset $\text{Spec } A^{(2)} \subseteq X^{(2)}$ there is a ring homomorphism $F^{(2)} : A^{(2)} \to A^{(2)}$ such that

$$F^{(2)}(b) = b^p + p \cdot \varphi(b)$$

where $\varphi : A^{(2)} \to A = A^{(2)}/pA^{(2)}$ is some function and $p \cdot : A \to A^{(2)}$ is the $A^{(2)}$-homomorphism derived from tensoring the short exact sequence of $W_2(k)$-modules

$$0 \longrightarrow p W_2(k) \longrightarrow W_2(k) \longrightarrow p W_2(k) \longrightarrow 0$$

with the flat $W_2(k)$ module $A^{(2)}$ identifying $A \cong A^{(2)}/pA^{(2)}$ with $p A^{(2)}$. We get the following properties of $\varphi$:

$$\varphi(a + b) = \varphi(a) + \varphi(b) - \sum_{j=1}^{p-1} p^{-1} \binom{p}{j} \bar{a}^j \bar{b}^{p-j}$$

and

$$\varphi(ab) = \bar{a}^p \varphi(b) + \bar{b}^p \varphi(a)$$

where $\bar{\cdot}$ means reduction mod $p$. Now it follows that

$$a \mapsto a^{p-1} da + d\varphi(\bar{a})$$
where $\tilde{a}$ is any lift of $a$, is a well defined derivation $\delta : A \to Z^1_{\text{Spec } A} \subseteq F_*\Omega^1_{\text{Spec } A}$, which gives a homomorphism $\varphi : \Omega^1_{\text{Spec } A} \to Z^1_{\text{Spec } A} \subseteq F_*\Omega^1_{\text{Spec } A}$. This homomorphism can be extended via the algebra structure to give an $A$-algebra homomorphism $\sigma : \oplus_i \Omega^i_{\text{Spec } A} \to Z^i_{\text{Spec } A} \subseteq F_*\Omega^i_{\text{Spec } A}$, which composed with the canonical homomorphism $Z^i_{\text{Spec } A} \to H^i(F_*\Omega^i_{\text{Spec } A})$ gives the inverse Cartier operator. Since an affine open covering $\{\text{Spec } A^{(2)}\}$ of $X^{(2)}$ gives rise to an affine open covering $\{\text{Spec } A^{(2)}/pA^{(2)}\}$ of $X$, we have proved that $\sigma$ is a quasi-isomorphism of complexes inducing the inverse Cartier operator on cohomology.

Now we give a splitting homomorphism of $\sigma_i : \Omega^i_X \to F_*\Omega^i_X$. Notice that $\sigma_0 : \mathcal{O}_X \to F_*\mathcal{O}_X$ is the Frobenius homomorphism and that $\sigma_i$ ($i > 0$) splits $C$ in the exact sequence

$$0 \longrightarrow B^i_X \longrightarrow Z^i_X \overset{C}{\longrightarrow} \Omega^i_X \longrightarrow 0$$

The natural perfect pairing $\Omega^i_X \otimes \Omega^{n-i}_X \to \Omega^i_X$ gives an isomorphism between $\mathcal{H}om_X(\Omega^{n-i}_X, \Omega^i_X)$ and $\Omega^i_X$. It is easy to check that the homomorphism

$$F_*\Omega^i_X \to \mathcal{H}om_X(\Omega^{n-i}_X, \Omega^i_X) \cong \Omega^i_X$$

given by $\omega \mapsto \varphi(\omega)$, where $\varphi(\omega)(z) = C(\sigma_{n-i}(z) \wedge \omega)$, splits $\sigma_i$. 

**2.3. Bott vanishing.** Let $X$ be a normal variety and let $j$ denote the inclusion of the smooth locus $U \subseteq X$. If the Frobenius morphism lifts to $W_2(k)$ on $X$, then the Frobenius morphism on $U$ also lifts to $W_2(k)$. Define the Zariski sheaf $\tilde{\Omega}^i_X$ of $i$-forms on $X$ as $j^*\Omega^i_U$. Since $\text{codim}(X-U) \geq 2$ it follows ([5], Proposition 5.10) that $\tilde{\Omega}^i_X$ is a coherent sheaf on $X$.

**Theorem 3.** Let $X$ be a projective normal variety such that $F$ lifts to $W_2(k)$. Then

$$H^s(X, \tilde{\Omega}^r_X \otimes L) = 0$$

for $s > 0$ and $L$ an ample line bundle.

**Proof.** Let $U$ be the smooth locus of $X$ and let $j$ denote the inclusion of $U$ into $X$. On $U$ we have by Theorem 2 a split sequence

$$0 \to \Omega^r_U \to F_*\Omega^r_U$$

which pushes down to the split sequence ($F$ commutes with $j$)

$$0 \to \tilde{\Omega}^r_X \to F_*\tilde{\Omega}^r_X$$

Now tensoring with $L$ and using the projection formula we get injections for $s > 0$

$$H^s(X, \tilde{\Omega}^r_X \otimes L) \hookrightarrow H^s(X, \tilde{\Omega}^r_X \otimes L^p)$$

Iterating these injections and using that the Zariski sheaves are coherent one gets the desired vanishing theorem by Serre’s theorem. 

$\square$
2.4. Degeneration of the Hodge to de Rham spectral sequence. Let $X$ be a projective normal variety with smooth locus $U$. Associated with the complex $\tilde{\Omega}_X^\bullet$ there is a spectral sequence

$$E_1^{pq} = H^q(X, \tilde{\Omega}_X^p) \implies H^{p+q}(X, \tilde{\Omega}_X^\bullet)$$

where $H^\bullet(X, \tilde{\Omega}_X^\bullet)$ denotes the hypercohomology of the complex $\tilde{\Omega}_X^\bullet$. This is the Hodge to de Rham spectral sequence for Zariski sheaves.

**Theorem 4.** If the Frobenius morphism on $X$ lifts to $W_2(k)$, then the spectral sequence degenerates at the $E_1$-term.

**Proof.** As complexes of sheaves of abelian groups $\tilde{\Omega}_X^\bullet$ and $F_*\tilde{\Omega}_X^\bullet$ are the same so their hypercohomology agree. Applying hypercohomology to the split injection (Theorem 2)

$$\sigma : \bigoplus_{0 \leq i} \tilde{\Omega}^i_{X/k}[-i] \to F_*\tilde{\Omega}_X^\bullet$$

we get

$$\sum_{p+q=n} \dim_k E_1^{pq} = \dim_k H^n(X, \tilde{\Omega}_X^\bullet) = \dim_k H^n(X, F_*\tilde{\Omega}_X^\bullet) \geq \sum_{p+q=n} \dim_k H^q(X, \tilde{\Omega}_X^p) = \sum_{p+q=n} \dim_k E_1^{pq}$$

Since $E_1^{pq}$ is a subquotient of $E_1^{pq}$, it follows that $E_1^{pq} \cong E_1^{pq}$ so that the spectral sequence degenerates at $E_1$. \qed

3. Toric varieties

In this section we briefly sketch the definition of toric varieties following Fulton [4] and demonstrate how the results of Section 2 may be applied.

3.1. Convex geometry. Let $N$ be a lattice, $M = \text{Hom}_\mathbb{Z}(N, \mathbb{Z})$ the dual lattice, and let $V$ be the real vector space $V = N \otimes \mathbb{R}$. It is natural to identify the dual space of $V$ with $M \otimes \mathbb{R}$, and we think of $N \subset V$ and $M \subset V^*$ as the subsets of integer points.

By a cone in $N$ we will mean a subset $\sigma \subset V$ taking the form $\sigma = \{r_1v_1 + \cdots + r_sv_s \mid r_i \geq 0\}$ for some $v_i \in N$. The vectors $v_1, \ldots, v_s$ are called generators of $\sigma$. We define the dual cone to be $\sigma^\vee = \{u \in V^* \mid \forall v \in \sigma : \langle u, v \rangle \geq 0\}$. One may show that $\sigma^\vee$ is a cone in $M$. A face of $\sigma$ is any set $\sigma \cap u^\perp$ for some $u \in \sigma^\vee$. Any face of $\sigma$ is clearly a cone in $N$, generated by the $v_i$ for which $\langle u, v_i \rangle = 0$.

Now let $\sigma$ be a strongly convex cone in $N$, this means that $\{0\}$ is a face of $\sigma$ or equivalently that no nontrivial subspace of $V$ is contained in $\sigma$. We define $S_\sigma$ to be the semi group $\sigma^\vee \cap M$. Since $\sigma^\vee$ is a cone in $M$, $S_\sigma$ is finitely generated.

3.2. Affine toric varieties. If $k$ is any commutative ring the semigroup ring $k[S_\sigma]$ is a finitely generated commutative $k$-algebra, and $U_\sigma = \text{Spec} k[S_\sigma]$ is an affine scheme of finite type over $k$. Schemes of this form are called affine toric schemes.
3.3. Glueing affine toric varieties. Let \( \tau = \sigma \cap u^\perp \) be a face of \( \sigma \). One may assume that \( u \in S_\sigma \). Then it follows that \( S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u) \), so that \( k[S_\tau] = k[S_\sigma]_u \). In this way \( U_\tau \) becomes a principal open subscheme of \( U_\sigma \). This may be used to glue affine toric schemes together. We define a fan in \( N \) to be a nonempty set \( \Delta \) of strongly convex cones in \( N \) satisfying that the faces of any cone in \( \Delta \) are also in \( \Delta \) and the intersection of two cones in \( \Delta \) is a face of each. The affine varieties arising from cones in \( \Delta \) may be glued together to form a scheme \( X_k(\Delta) \) as follows. If \( \sigma, \tau \in \Delta \), then \( \sigma \cap \tau \in \Delta \) is a face of both \( \tau \) and \( \sigma \), so \( U_{\sigma \cap \tau} \) is isomorphic to open subsets \( U_{\sigma \tau} \) in \( U_\sigma \) and \( U_{\tau \sigma} \) in \( U_\tau \). Take the transition morphism \( \phi_{\sigma \tau} : U_{\sigma \tau} \to U_{\tau \sigma} \) to be the one going through \( U_{\sigma \cap \tau} \). A scheme \( X_k(\Delta) \) arising from a fan \( \Delta \) in some lattice is called a toric scheme.

3.4. Liftings of the Frobenius morphism on toric varieties. Let \( X = X_k(\Delta) \) be a toric scheme over the commutative ring \( k \) of characteristic \( p > 0 \). We are going to construct explicitly a lifting of the absolute Frobenius morphism on \( X \) to \( W = W_2(k) \). Define \( X^{(2)} \) to be \( X_W(\Delta) \). Since all the rings \( W[S_\sigma] \) are free \( W \)-modules, this is clearly a flat scheme over \( W_2(k) \). Moreover, the identities \( W[S_\sigma] \otimes_W k \cong k[S_\sigma] \) immediately give an isomorphism \( X^{(2)} \times_{\text{Spec } W} \text{Spec } k \cong X \).

For \( \sigma \in \Delta \), let \( F^{(2)}_\sigma : W[S_\sigma] \to W[S_\sigma] \) be the ring homomorphism extending \( F^{(2)} : W \to W \) and mapping \( u \in S_\sigma \) to \( u^p \). It is easy to see that these maps are compatible with the transition morphisms, so we may take \( F^{(2)} : X^{(2)} \to X^{(2)} \) to be the morphism which is defined by \( F^{(2)}_\sigma \) locally on \( \text{Spec } W[S_\sigma] \). This gives the lift of \( F \) to \( W_2(k) \) and completes the construction.

3.5. Bott vanishing and the Danilov spectral sequence. Since toric varieties are normal we get the following corollary of Section 2:

**Theorem 5.** Let \( X \) be a projective toric variety over \( k \). Then

\[
H^q(X, \hat{\Omega}_X^p \otimes L) = 0
\]

where \( q > 0 \) and \( L \) is an ample line bundle. Furthermore the Danilov spectral sequence

\[
E_1^{pq} = H^q(X, \hat{\Omega}_X^p) \Longrightarrow H^{p+q}(X, \hat{\Omega}_X^\cdot)
\]

degenerates at the \( E_1 \)-term.

**Remark 1.** One may use the above to prove similar results in characteristic zero. The key issue is that we have proved that Bott vanishing and degeneration of the Danilov spectral sequence holds in any prime characteristic.

4. Flag varieties

In this section we generalize Paranjape and Srinivas result on non-lifting of Frobenius on flag varieties not isomorphic to \( \mathbb{P}^n \). The key issue is that one can reduce to flag varieties with rank 1 Picard group. In many of these cases one can exhibit ample line bundles with Bott non-vanishing.

We now set up notation. Let \( G \) be a semisimple algebraic group over \( k \) and fix a Borel subgroup \( B \) in \( G \). Recall that (reduced) parabolic subgroups \( P \supset B \) are given by subsets of the simple root subgroups of \( B \). These correspond bijectively to subsets of nodes in the Dynkin diagram associated with \( G \). A parabolic subgroup \( Q \) is contained in \( P \) if and only if the
simple root subgroups in $Q$ is a subset of the simple root subgroups in $P$. A maximal parabolic subgroup is the maximal parabolic subgroup not containing a specific simple root subgroup.

We shall need the following result from the appendix to [9]

**Proposition 1.** If the sequence

$$0 \rightarrow B^1_X \rightarrow Z^1_X \overset{\gamma}{\rightarrow} \Omega^1_X \rightarrow 0$$

splits, then the Frobenius morphism on $X$ lifts to $W_2(k)$.

We also need the following fact derived from ([6], Proposition II.8.12 and Exercise II.5.16(d))

**Proposition 2.** Let $f : X \rightarrow Y$ be a smooth morphism between smooth varieties $X$ and $Y$. Then for every $n \in \mathbb{N}$ there is a filtration $F^0 \supseteq F^1 \supseteq \ldots$ of $\Omega^n_X$ such that

$$F^i/F^{i+1} \cong f^*\Omega^i_Y \otimes \Omega^{n-i}_{X/Y}.$$ 

**Lemma 1.** Let $f : X \rightarrow Y$ be a surjective, smooth and projective morphism between smooth varieties $X$ and $Y$ such that the fibers have no non-zero global $n$-forms, where $n > 0$. Then there is a canonical isomorphism

$$\Omega^\bullet_Y \rightarrow f_\ast \Omega^\bullet_X$$

and a splitting $\sigma : \Omega^1_Y \rightarrow Z^1_Y$ of the Cartier operator $C : Z^1_X \rightarrow \Omega^1_X$ induces a splitting $f_\ast \sigma : \Omega^1_Y \rightarrow Z^1_Y$ of $C : Z^1_Y \rightarrow \Omega^1_Y$.

**Proof.** Notice first that $\mathcal{O}_Y \rightarrow f_\ast \mathcal{O}_X$ is an isomorphism of rings as $f$ is projective and smooth. The assumption on the fibers translates into $f_\ast \Omega^n_{X/Y} \otimes k(y) \cong H^0(X_y, \Omega^n_{X_y}) = 0$ for geometric points $y \in Y$, when $n > 0$. So we get $f_\ast \Omega^n_{X/Y} = 0$ for $n > 0$. By Proposition 2 this means that all of the natural homomorphisms $\Omega^\bullet_Y \rightarrow f_\ast \Omega^\bullet_X$ induced by $\mathcal{O}_Y \rightarrow f_\ast \mathcal{O}_X \rightarrow f_\ast \Omega^1_X$ are isomorphisms giving an isomorphism of complexes

$$0 \rightarrow \Omega_Y \rightarrow \Omega^1_Y \rightarrow \Omega^2_Y \rightarrow \cdots$$

and

$$0 \rightarrow f_\ast \mathcal{O}_X \rightarrow f_\ast \Omega^1_X \rightarrow f_\ast \Omega^2_X \rightarrow \cdots$$

This means that the middle arrow in the commutative diagram

$$0 \rightarrow B^1_Y \rightarrow Z^1_Y \overset{\gamma}{\rightarrow} \Omega_Y \rightarrow 0$$

is an isomorphism and the result follows. $\square$

**Corollary 1.** Let $Q \subseteq P$ be two parabolic subgroups of $G$. If the Frobenius morphism on $G/Q$ lifts to $W_2(k)$, then the Frobenius morphism on $G/P$ lifts to $W_2(k)$. 
Using Künneth it is easy to deduce that parabolic subgroups $P$

4.3. Bott non-vanishing for flag varieties. In this section we search for specific maximal

In specific cases one can prove using the “standard” exact sequences that certain flag varieties
do not have Bott vanishing. We go on to do this next.

Let $Y$ be a smooth divisor in a smooth variety $X$. Suppose that $Y$ is defined by the sheaf
of ideals $I \subseteq \mathcal{O}_X$. Then ([6], Proposition II.8.17(2) and Exercise II.5.16(d)) gives for $n \in \mathbb{N}$ an exact sequence of $\mathcal{O}_Y$-modules

$$0 \to \Omega_Y^{n-1} \otimes I/I^2 \to \Omega_Y^n \otimes \mathcal{O}_Y \to \Omega_Y^n \to 0$$

From this exact sequence and induction on $n$ it follows that $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^j \otimes \mathcal{O}(m)) = 0$, when $m \leq j$ and $j > 0$.

4.1. Quadric hypersurfaces in $\mathbb{P}^n$. Let $Y$ be a smooth quadric hypersurface in $\mathbb{P}^n$, where $n \geq 4$. There is an exact sequence

$$0 \to \mathcal{O}_Y(1 - n) \to \mathcal{O}_{\mathbb{P}^n}^1 \otimes \mathcal{O}(3 - n) \otimes \mathcal{O}_Y \to \Omega_Y^1 \otimes \mathcal{O}_Y(3 - n) \to 0$$

From this it is easy to deduce that

$$H^{n-2}(Y, \Omega_Y^1 \otimes \mathcal{O}_Y(3 - n)) \cong H^1(Y, \Omega_Y^{n-2} \otimes \mathcal{O}_Y(n - 3)) \cong k$$

using that $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^j \otimes \mathcal{O}(m)) = 0$, when $m \leq j$ and $j > 0$.

4.2. The incidence variety in $\mathbb{P}^n \times \mathbb{P}^n$. Let $X$ be the incidence variety of lines and hyperplanes in $\mathbb{P}^n \times \mathbb{P}^n$, where $n \geq 2$. Recall that $X$ is the zero set of $x_0 y_0 + \cdots + x_n y_n$, so that there is an exact sequence

$$0 \to \mathcal{O}(1) \times \mathcal{O}(1) \to \mathcal{O}_{\mathbb{P}^n} \times \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

Using Künneth it is easy to deduce that

$$H^{2n-2}(X, \mathcal{O}_X^1 \otimes \mathcal{O}(1 - n) \times \mathcal{O}(1 - n)) \cong H^1(X, \mathcal{O}_X^{2n-2} \otimes \mathcal{O}(n - 1) \times \mathcal{O}(n - 1)) \cong k$$

4.3. Bott non-vanishing for flag varieties. In this section we search for specific maximal

where $i > 0$. These are instances of Bott non-vanishing. This will be used in Section 4.4 to prove non-lifting of Frobenius for a large class of flag varieties.

Let $\mathcal{O}(1)$ be the ample generator of $\text{Pic} Y$. By flat base change one may produce examples of Bott non-vanishing for $Y$ for fields of arbitrary prime characteristic by restricting to the field of the complex numbers. This has been done in the setting of Hermitian symmetric spaces, where the cohomology groups $H^p(Y, \mathcal{O}^n \otimes \mathcal{O}(n))$ have been thoroughly investigated by Sato [12] and Snow [13][14]. We now show that these examples exist. In each of the following subsections $Y$ will denote $G/P$, where $P$ is the maximal parabolic subgroup not containing the root subgroup corresponding to the marked simple root in the Dynkin diagram. These flag manifolds are the irreducible Hermitian symmetric spaces.
4.3.1. Type A.

If \( Y \) is a Grassmann variety not isomorphic to projective space (\( Y = G/P \), where \( P \) corresponds to leaving out a simple root which is not the left or right most one), one may prove ([13], Theorem 3.3) that
\[
\text{H}^1(Y, \Omega_Y^3 \otimes \mathcal{O}(2)) \neq 0
\]

4.3.2. Type B.

Here \( Y \) is a smooth quadric hypersurface in \( \mathbb{P}^n \), where \( n \geq 4 \) and Bott non-vanishing follows from Section 4.1.

4.3.3. Type C.

By ([14], Theorem 2.2) it follows that
\[
\text{H}^1(Y, \Omega_Y^2 \otimes \mathcal{O}(1)) \neq 0
\]

4.3.4. Type D.

For the maximal parabolic \( P \) corresponding to the leftmost marked simple root, \( Y=G/P \) is a smooth quadric hypersurface in \( \mathbb{P}^n \), where \( n \geq 4 \) and Bott non-vanishing follows from Section 4.1. For the maximal parabolic subgroup corresponding to one of the two rightmost marked simple roots we get by ([14], Theorem 3.2) that
\[
\text{H}^2(Y, \Omega_Y^4 \otimes \mathcal{O}(2)) \neq 0
\]
4.3.5. Type $E_6$. 

By ([14], Table 4.4) it follows that
\[ H^3(Y, \Omega^5 \otimes \mathcal{O}(2)) \neq 0 \]

4.3.6. Type $E_7$. 

By ([14], Table 4.5) it follows that
\[ H^4(Y, \Omega^6 \otimes \mathcal{O}(2)) \neq 0 \]

4.3.7. Type $G_2$. 

Here $Y$ is a smooth quadric hypersurface in $\mathbb{P}^6$ and Bott non-vanishing follows from Section 4.1.

4.4. Non-lifting of Frobenius for flag varieties. We now get the following

**Theorem 6.** Let $Q$ be a parabolic subgroup contained in a maximal parabolic subgroup $P$ in the list 4.3.1 - 4.3.7. Then the Frobenius morphism on $G/Q$ does not lift to $W_2(k)$. Furthermore if $G$ is of type $A$, then the Frobenius morphism on any flag variety $G/Q \ncong \mathbb{P}^m$ does not lift to $W_2(k)$.

**Proof.** If $P$ is a maximal parabolic subgroup in the list 4.3.1-4.3.7, then the Frobenius morphism on $G/P$ does not lift to $W_2(k)$. By Corollary 1 we get that the Frobenius morphism on $G/Q$ does not lift to $W_2(k)$. In type $A$ the only flag variety not admitting a fibration to a Grassmann variety $\ncong \mathbb{P}^m$ is the incidence variety. Non-lifting of Frobenius in this case follows from Section 4.2. □

**Remark 2.** The above case by case proof can be generalized to include projective homogeneous $G$-spaces with non-reduced stabilizers. It would be nice to prove in general that the only flag variety enjoying the Bott vanishing property is $\mathbb{P}^m$. We know of no other visible obstruction to lifting Frobenius to $W_2(k)$ for flag varieties than the non-vanishing Bott cohomology groups.
FROBENIUS MORPHISMS OVER $\mathbb{Z}/p^2$ AND BOTT VANISHING

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