Turbulent Magnetohydrodynamic Elasticity: I. Boussinesq-like Approximations for Steady Shear

Peter Todd Williams

Los Alamos National Laboratory, MS P225, P.O. Box 1663, Los Alamos, NM 87545, USA

Abstract

We re-examine the Boussinesq hypothesis of an effective turbulent viscosity within the context of simple closure considerations for models of strong magnetohydrodynamic turbulence. Reynolds-stress and turbulent Maxwell-stress closure models will necessarily introduce a suite of transport coefficients, all of which are to some degree model-dependent. One of the most important coefficients is the relaxation time for the turbulent Maxwell stress, which until recently has been relatively ignored. We discuss this relaxation within the context of magnetohydrodynamic turbulence in steady high Reynolds-number, high magnetic-Reynolds-number shearing flows. The relaxation time for the turbulent Maxwell stress is not limited by the shear time scale, in contrast with Reynolds-stress closure models for purely hydrodynamical turbulence. The anisotropy of the turbulent stress tensor for magnetohydrodynamic turbulence, even for the case of zero mean-field considered here, can therefore not be neglected. This shear-generated anisotropy can be interpreted as being due to an effective turbulent elasticity, in analogy to the Boussinesq turbulent viscosity. We claim that this turbulent elasticity should be important for any astrophysical problem in which the turbulent stress in quasi-steady shear has been treated phenomenologically with an effective viscosity.

Key words:

1 Introduction

The Boussinesq hypothesis of an effective turbulent viscosity is still widely used both in engineering applications and in astrophysics. This is true despite
the passage of over one hundred years since the birth of the concept. There are a very large number of models for turbulent stress that give substantially better results than a simple effective viscosity, and in some situations, such as strongly time-dependent flows, the use of an effective viscosity gives notoriously bad results. Despite this, the concept of an effective turbulent viscosity has proven to be so persistent because it is useful, simple, and because it is based on an appealing analogy between the random motions of turbulent blobs and the kinetic theory of thermal motions of molecules. Thus, effective turbulent viscosity is commonly invoked as a good crude “lowest-order” approximation, for problems in which the Reynolds number is very large.

For problems in which the magnetic Reynolds number is also very large (as is often the case in astrophysics), we ask if a picture of turbulence as colliding blobs of fluid is at all useful even as a crude approximation. A more appropriate analogy for the turbulent component of the field may be not the kinetic theory of point-like molecules, but rather the kinetic theory of long polymers in melt or in solution (Ogilvie, 2001; Williams, 2001). For the crudest possible model, such an analogy would suggest the concept of an effective turbulent elasticity, in addition to the effective turbulent viscosity. Thus, corresponding to the Boussinesq approximation in hydrodynamic turbulence, the corresponding “lowest-order” approximation for the stress in MHD turbulence should include an additional coefficient, in analogy to a simple viscoelastic fluid. Just as the inclusion of elasticity in the stress response of an ordinary laboratory fluid can have dramatic consequences, we suggest that any astrophysical problem in which turbulence has been treated phenomenologically with the Boussinesq viscosity might have dramatically different solutions if an effective elasticity were included as well.

In particular, for the purposes of quasi-steady shear, we claim that the anisotropy of the stress tensor is of much greater significance in the case of MHD turbulence than in the case of purely hydrodynamic turbulence. Thus, while investigations of the idealized problem of homogeneous isotropic hydrodynamic turbulence may lead to a greater understanding of the dynamical importance of turbulence to the large-scale mean flow of a turbulent fluid, we expect that, quite broadly speaking, models for MHD turbulence that are to be of some use in predicting the behavior of the mean flow must address directly the anisotropy of the turbulent Maxwell stresses, irrespective of the presence or absence of any mean field.

Many of the more recent simulations of MHD turbulence in a shearing environment (our main interest in this paper) are simulations (e.g., Hawley et al. (1995)) of the Velikov-Chandrasekhar-Balbus-Hawley magnetorotational instability (MRI), and we will refer to the results of these simulations for guidance. See also Williams (2002) for a comparison of simulations of the MRI with some viscoelastic models.
For our purposes, the dominant feature of simulations of the MRI is that the largest component of the turbulent stress tensors is not the “viscous” $r\theta$ component, but rather the streamwise $\theta \theta$ component. (Note that in simulations of the MRI, $\theta$ is the direction of the shear, and $r$ is the cross-shear direction.) In fact, our original reason for investigating viscoelastic models was as a mechanism to collimate and possibly drive jets using this $\theta \theta$ component of the stress tensor.

Most importantly we wish to point out that the interpretation of the turbulent stress as an effective viscous stress can lead to qualitatively erroneous predictions. In view of the fact pointed out above that the “viscous” component is in fact not even the largest component of the stress tensor, a prescription for the turbulent stress tensor $\Pi_{\text{turb}}$ of the form

$$\Pi_{ij}^{\text{turb}} = \mu_{\text{turb}} \left[ \partial_i v_j + \partial_j v_i - \frac{2}{3} (\partial_k v_k) \delta_{ij} \right] + \mu_{[b]}^{\text{turb}} (\partial_k v_k) \delta_{ij}, \quad (1)$$

where $\mu_{\text{turb}}$ and $\mu_{[b]}^{\text{turb}}$ are effective shear and bulk viscosities respectively, is untenable, even as a gross approximation.

One possible approach to the problem is to generalize the viscosity to a tensor viscosity. This approach ignores the fact, however, that the large azimuthal component of stress in MRI-driven shear turbulence is fundamentally due to the vector advection properties of the induction equation.

2 Notation

As is conventionally done in the Reynolds decomposition in hydrodynamic turbulence, it is convenient here to decompose all fluid variables into mean and fluctuating, or “turbulent,” parts. The former will be denoted with an overbar, and the latter with a prime. Thus, for example, $B_i = \bar{B}_i + B'_i$. We employ standard ensemble rules of averaging, so that averaging commutes with time and spatial derivatives. It is also useful to think of this averaging as, say, a time or spatial averaging over some convenient timescale or lengthscale, although strictly speaking this is incorrect, because such averaging introduces additional terms.

Furthermore, for the sake of definiteness, we will make liberal use of Cartesian index notation for vectors and tensors, although we may occasionally use abstract vector notation when there is no chance of ambiguity. To make lengthy equations a bit more palatable, we denote symmetrization and antisymmetrization on indices with curly and square brackets respectively, so
that, e.g.,

\[ A_{ij} \equiv A_{ij} + A_{ji}. \]

The generalization of the given equations to curvilinear (“covariant and contravariant”) form is straightforward.

We use Heaviside-Lorentz units for the magnetic field. The use of Gaussian units and the attendant factors of \(1/\sqrt{4\pi}\) would make lengthy equations even longer, and this would detract from readability. The conversion to Gaussian units is

\[ B_i^{[Gauss]} = \sqrt{4\pi} B_i^{[Heaviside-Lorentz]} \]

The full Maxwell stress tensor, under the approximation that \(|E| \ll |B|\), has components \(B_i B_j - \frac{1}{2} B^2 \delta_{ij}\). The turbulent Maxwell stress is the Maxwell stress due to the turbulent component of the field, \(B'\). We find it more convenient to work with the magnetic cross-correlation tensor \(M_{ij}\) where \(M_{ij} \equiv B'_i B'_j\). Then the turbulent Maxwell stress is

\[ M_{ij} \equiv M_{ij} - M_{kk} \delta_{ij}. \]

3 Invariant Tensor Advection Operators

In perfect flux-freezing the stress tensor \(M_{ij} \equiv B'_i B'_j\) evolves by the action of the (modified) upper-convected invariant derivative \(D_t^{[u]}\) such that \(D_t^{[u]} (M)_{ij} = 0\), where

\[ (D_t^{[u]} M)_{ij} \equiv (\partial_t + \bar{v}_k \partial_k) M_{ij} - (\partial_k \bar{v}_i) M_{kj} - M_{ik} (\partial_k \bar{v}_j) + 2(\partial_k \bar{v}_k) M_{ij}. \]  

(The required modification of the conventional upper-convected invariant derivative is the inclusion of the term proportional to the divergence of the velocity). This relationship is simply a direct result of the vector advection equation for the magnetic field in perfect MHD, and does not rely on breaking the field into mean and fluctuating parts, although we have done so here. This equation shows how, for example, the stress \(M_{ij}\) is distorted by shear (see figure 1).

Generally speaking, tensors formed from a dyad of vectors that obey vector advection equations will often produce a tensor advection equation that is similar to the above:
Consider the scalar advection operator

$$\partial_t + v_k \partial_k.$$ 

This may be generalized to a vector advection operator in more than one way. Note that the vector advection operator acting on either the field $B$ in MHD or the vorticity $\omega$ in simple hydrodynamics is, in the incompressible case (writing the vector as $w$),

$$\partial_t w_i + v_k \partial_k w_i - w_k \partial_k v_i,$$

reflecting that these vectors are stretched and aligned with the shear. In contrast, in the Reynolds decomposition, the advection operator acting on the turbulent velocity field $v'_i$ is

$$\partial_t v'_i + \bar{v}_k \partial_k v'_i + v'_k \partial_k \bar{v}_i + v'_k \partial_k v'_i$$

The last term is the origin of the famous triple-correlation; here we focus on the penultimate term, which is similar to the last term in expression (3) except for a change of sign. Certainly more complicated examples are possible. For instance, the advection of the gradient $g_i \equiv \partial_i \phi$ of a passive scalar $\phi$ is

$$\partial_t g_i + v_k \partial_k g_i + g_k \partial_k v_i - \epsilon_{ijk} g_j \omega_k = 0$$

In all cases above, however, there is a term $w_k \partial_k v_i$ that, depending upon whether it appears with a plus sign or a minus sign, tends to anti-align or align the vector $w_i$ with the shear. Thus, if the stress tensor is formed from vectors that obey such equations, the stress will necessarily be distorted by the Lagrangian deformation of the fluid. Upper- and lower-convected derivatives are two tensor generalizations of the advection operator that arise when the tensors are formed from vectors that are aligned or anti-aligned by the shear.

One finds then that, for example, the Reynolds stress is distorted by the background shear by the advection operator (the lower-convected invariant derivative) which, in the incompressible case, may be written as

$$\partial_t R_{ij} + \bar{v}_k \partial_k R_{ij} + (\partial_k \bar{v}_i) R_{kj} + R_{ik} (\partial_k \bar{v}_j).$$

The second and third terms in this operator definition have long been recognized as being quite important to the theory of hydrodynamical turbulent transport processes. These terms distort the Reynolds stress, and are as well a source for the Reynolds stress. The distortion that shear induces on $R_{ij}$ is opposite to that induced on $M_{ij}$, and in such a manner as to produce a
Fig. 1. The turbulent magnetic field lines are distorted by shear. In equilibrium, the degree of anisotropy is determined by the balance of production, distortion, and dissipation, as described below.

“viscous”-like component to the stress. As the turbulent Maxwell stress and the Reynolds stress appear with different signs in the turbulent stress tensor,

\[
\Pi_{ij}^{\text{turb}} = -R_{ij} + M_{ij},
\]  

(4)

the signs of the viscous component of the Maxwell stress and the Reynolds stress are the same. It is of course just such a distortion of the stress tensor by the deformation of the fluid (including shear) that is the origin of *molecular viscosity*, where in that case the stress tensor is formed from the sums of the contributions \(mv_i v_j\) to the stress tensor from individual particles.

4 Reynolds Decomposition

It is helpful to write down in full the Reynolds decomposition for MHD turbulence. Here let us concentrate on the constant-density case, \(\rho = \bar{\rho}\). The continuity equation and “magnetic Gauss’s law” are then

\[
\partial_k \bar{v}_k = \partial_k v'_k = 0
\]

\[
\partial_k \bar{B}_k = \partial_k B'_k = 0
\]

The mean-field velocity transport equation is

\[
\bar{\rho} \partial_t \bar{v}_i + \bar{\rho} \bar{v}_k \partial_k \bar{v}_i = -\partial_i \bar{P} + \bar{B}_k \partial_k \bar{B}_i - \frac{1}{2} \partial_i (\bar{B}^2) - \]

\[
\partial_k R_{ik} + \partial_k M_{ik} - \frac{1}{2} \partial_i M_{kk} + \mu \partial_{kk} \bar{v}_i
\]
The unmodeled transport equation for the Reynolds stress is obtained after some tedious but straightforward algebra:

\[
\partial_t R_{ij} + \bar{v}_k \partial_k R_{ij} + (\partial_k \bar{v}_i) R_{kj} + R_{ik} (\partial_k \bar{v}_j) + \bar{\rho} \partial_k (v'_i v'_j v'_k) = \\
- v'_i (\partial_j (v'_j v'_k)) + (\partial_k \bar{B}'_i) v'_j B'_k + \bar{B}'_k (\partial_k B'_i) v'_j + B'_k (\partial_k B'_i) v'_j - \\
- \frac{1}{2} v'_i (\partial_j (B'_k B'_k)) + \mu \left( \partial_k^2 R_{ij} - 2 (\partial_k v'_i) (\partial_k v'_j) \right)
\]  

(5)

There are three magnetic curvature terms, namely those arising from \( B' \cdot \nabla \bar{B} \), those from \( \bar{B} \cdot \nabla B' \), and those from \( B' \cdot \nabla B' \). Likewise, there are two magnetic pressure terms, namely those arising from the cross-correlation pressure \( \bar{B} B' \) and those from the pure turbulent field pressure \( B' B' \). It should be noted that in all terms in which both the velocity and the field appear, the velocity appears once and the field appears twice.

We write the induction equation as

\[
\partial_t B_i + v_k \partial_k B_i - \bar{B}_k \partial_k v_i - B_i (\partial_k v_k) = \eta \partial_k^2 B_i.
\]

From the induction equation we find that the mean field evolves according to

\[
\partial_t \bar{B}_i + \bar{v}_k \partial_k \bar{B}_i - \bar{B}_k \partial_k \bar{v}_i = \partial_k A_{ik}^{(-)} + \eta \partial_k^2 \bar{B}_i
\]

where we have defined the antisymmetric matrix

\[
A_{ij}^{(-)} \equiv v'_i (\bar{B}'_j) = \bar{v}'_i B'_j - \bar{v}'_j B'_i
\]

(It is also convenient to define the symmetric matrix

\[
A_{ij}^{(+)} \equiv v'_i (\bar{B}'_j) = \bar{v}'_i B'_j + \bar{v}'_j B'_i
\]

although we do not use this here.) From this we can see that the only sources for a mean field \( \bar{B} \) are a stretching component (which also appears in the induction equation itself) and a turbulent source term which is more often written as

\[
\partial_k A_{ik}^{(-)} = (\nabla \times (v' \times B'))_i
\]

As we are mainly interested in the zero-mean-field case, we do not discuss the mean-field induction equation any further.

The turbulent Maxwell-like stress \( M_{ij} \) obeys the transport equation
\[\partial_t M_{ij} + \bar{v}_k \partial_k M_{ij} - M_{ik} (\partial_k \bar{v}_j) - (\partial_k \bar{v}_i) M_{kj} + \partial_k (B'_i B'_j \bar{v}'_k) =
\]
\[-(\partial_k \bar{B}'_i) B'_j \bar{v}'_k + \bar{B}_k (\partial_k \bar{v}'_i) B'_j + (\partial_k \bar{v}'_i) B'_j B'_k +
\]
\[+ \eta \left( \partial^2 M_{ij} - 2(\partial_k B'_i)(\partial_k B'_j) \right) \]  

(6)

We have so far discussed the Reynolds stress and the turbulent Maxwell stress, but a word is in order regarding other approaches to turbulence. In particular we wish to address briefly the dynamics of turbulent vorticity. It is tempting to explore the apparent symmetry in the transport of the vorticity \( \omega \) and the magnetic field \( B \). In high Reynolds-number flow the vorticity tends to concentrate on discrete vortex lines. In the extreme case of a superfluid the vorticity of the fluid is identically zero except on discrete quantized vortex lines, just as in a superconductor the magnetic field is zero except on discrete quantized field lines; there is certainly an interesting symmetry to be explored in the interaction of discrete vortex lines and discrete field lines in the case of turbulence in a superconducting superfluid. One may suspect that this analogy will also hold in the case of high Reynolds-number, high magnetic Reynolds-number turbulence.

However, this apparent symmetry is far from perfect. Neither vector is transported as a passive field. The magnetic field exerts its influence on the bulk flow through the appearance of the divergence of the Maxwell stress as a bulk force in the fluid equations. In contrast, the vorticity does not so much influence the flow as it does determine it, insofar as the velocity is determined directly from the vorticity through the Biot-Savart relation, and conversely the vorticity is simply the curl of the velocity. As a result, a vortex line, if bent, can “wrap itself up” by the self-action of its own vorticity, unlike a field line.

So, to the extent that vorticity dynamics may be used – in analogy to polymer dynamics in solution (Chorin, 1988) – as a basis for turbulence models and quasi-viscoelastic behavior in particular, we expect that the corresponding analogy of turbulent field lines to polymers should be much more fruitful. Note also that a stress tensor may be formed directly from the field, whereas the relationship between the turbulent vorticity and the Reynolds stress is not straightforward. For this reason we do not treat the dynamics of turbulent vorticity and we resort entirely to simple Reynolds stress modeling for the stress due to turbulent hydrodynamic motions.
5 Boussinesq-Like Approximations

5.1 Simplest Case: No coupling

An ordinary differential model, such as the viscoelastic Maxwell model, contains in principle all that is needed in order to determine the stress in a fluid, given initial conditions and a rate-of-strain history. In steady, so-called “curvilinear” flows with closed or nearly-closed streamlines (such as the azimuthal Stokes shear flow in the neighborhood of a rotating sphere), it is sometimes useful to rewrite the solutions of these differential equations in closed form.

For example, the upper-convected Maxwell model may be written in closed form as

\[ M = \frac{\eta}{\lambda^2} \int_{-\infty}^{t} \left( C_t^{-1}(\tau) - 1 \right) \exp \left( -(t - \tau)/\lambda \right) d\tau, \]

where \( C_t^{-1} \) is the Finger tensor (see Joseph (1990) p.14). Regardless of whether an integral expression may be found for a given model, another closed-form way of writing the stress for steady, curvilinear flows is to expand the stress as a series solution in the powers of the (symmetrized) spatial derivatives of the velocity. The conventional way of doing this is to make use of powers of the Rivlin-Eriksen tensors \( A^{[n]} \), defined below. The first Rivlin-Eriksen tensor, \( A^{[1]} \), is the symmetrized velocity gradient (defined in terms of the mean velocity \( \bar{v}_i \)), so, with the addition of a “pressure” term proportional to \( \delta_{ij} \), the Boussinesq model may be written as

\[ \Pi_{ij} = -P_{turb}\delta_{ij} + \mu_{turb}A^{[1]}_{ij}, \]

for incompressible flow. (If the flow is compressible, it is conventional to break the rate-of-strain tensor into the divergence and a traceless part as we have done earlier; the coefficient of the former is then the turbulent bulk viscosity.)

A Boussinesq-like approximation may be “derived,” in a very crude sense, from simple turbulence models. A full turbulence model should make consideration of the source of the turbulence, such as an instability such as convection or the MRI. However, we point out that a simple turbulent viscosity works well as a first approximation when solving for the turbulent transport of flow through the lumen of a pipe (i.e. pipe Poiseuille flow), for which there is no known instability, linear or otherwise.

\[ \text{The author is grateful to Michael Steinkamp for pointing this out to him.} \]
The unmodeled Reynolds transport equation for incompressible fluid, under conventional rules of ensemble averaging, and ignoring effects such as rotation and thermal convective driving, is

\[
\partial_t R_{ij} + \bar{v}_k \partial_k R_{ij} + (\partial_k \bar{v}_j)R_{ik} + (\partial_k \bar{v}_i)R_{kj} = -\partial_k (v'_i v'_j v'_k) - \left(v'_i \partial_j P' + v'_j \partial_i P'\right) + \nu \left(\partial^2 R_{ij} - 2(\partial_k v'_i)(\partial_k v'_j)\right).
\]

(7)

(This is simply eqn. (5) rewritten without the terms in which the magnetic field occurs.) The left-hand side of this equation represents the advection and distortion of the Reynolds stress by the mean flow. The right-hand side of the equation is the hard part. The crudest models for the right-hand side of this transport equation will include terms proportional to \(\delta_{ij}\) and terms proportional to \(R_{ij}\). Effectively, we have advection and distortion on the left hand side (this distortion is sometimes referred to as a source, since it changes the total kinetic energy \(\frac{1}{2}R_{kk}\)), and on the right hand side we have a source proportional to \(\delta_{ij}\) and a sink proportional to \(R_{ij}\). It is somewhat abusive to call these terms source and sink terms, but the point is that the one typically has a positive coefficient and the other typically has a negative coefficient. For example, consider the Rotta return-to-isotropy term, which is a model for one of the effects of the correlations of turbulence with pressure fluctuations, i.e. the second of the three terms on the right-hand side of the above equation (note that pressure fluctuations also carry sound away, but this is not included in the Rotta term). This may be written as, e.g.,

\[-C_R \frac{\epsilon}{K} \left(R_{ij} - \frac{1}{3} \delta_{ij} R_{kk}\right),\]

where \(\epsilon\) and \(K\) are the turbulence dissipation rate and the turbulent kinetic energy respectively, and the Rotta coefficient \(C_R\) is a positive constant \(\footnote{Francis Harlow, personal communication}\). Likewise the viscous part of this transport equation — i.e. the last term on the right-hand-side — has a diffusive part, which is usually ignored, and a dissipative part that is most simply modeled as being proportional to the Reynolds stress \(R_{ij}\) itself.

We thus write

\[
\partial_t R_{ij} + \bar{v}_k \partial_k R_{ij} + (\partial_k \bar{v}_j)R_{ik} + (\partial_k \bar{v}_i)R_{kj} = a_R \delta_{ij} - s_{-1} R_{ij}
\]

where \(a_R\) and \(s_{-1}\) are numerical coefficients (which may depend upon certain characteristics of the flow, such as the shear, the buoyant convective driving, and so forth). Note that \(s_{-1}\) is a relaxation time. We may then solve for \(R_{ij}\), in
the case of steady shear (so that the advective operator \( \partial_t + \bar{v}_k \partial_k \) gives zero),
by plugging the above equation back into itself to obtain an infinite series,

\[
\mathbf{R}_{ij} = s_R a_R \delta_{ij} - s_R^2 a_R A_{ij}^{[1]} + s_R^3 a_R A_{ij}^{[2]} - s_R^4 a_R A_{ij}^{[3]} + \ldots
\]

This may be seen from the following relation for \( A^{[n]} \):

\[
A_{ij}^{[n+1]} = \partial_t A_{ij}^{[n]} + \bar{v}_k \partial_k A_{ij}^{[n]} + (\partial_k \bar{v}_i) A_{kj}^{[n]} + (\partial_k \bar{v}_j) A_{ik}^{[n]}. 
\]

In fact, it is not at all necessary to assume that the “source” term is proportional to \( \delta_{ij} \). If we instead write

\[
\partial_t \mathbf{R}_{ij} + \bar{v}_k \partial_k \mathbf{R}_{ij} + (\partial_k \bar{v}_i) \mathbf{R}_{kj} + (\partial_k \bar{v}_j) \mathbf{R}_{ik} = S_{ij}^{[R]} - s^{-1}_R \mathbf{R}_{ij}
\]

where \( S_{ij}^{[R]} \) is some unspecified tensor, we obtain

\[
\mathbf{R}_{ij} = s_R S_{ij}^{[R]} - s_R^2 \mathbb{L}(S_{ij}^{[R]}) + s_R^3 \mathbb{L}^2(S_{ij}^{[R]}) - s_R^4 \mathbb{L}^3(S_{ij}^{[R]}) + \ldots
\]

where \( \mathbb{L} \) is the linear operator

\[
\mathbb{L}(S)_{ij} = (\partial_k \bar{v}_i) S_{kj} + S_{ik}^{[R]} (\partial_k \bar{v}_j),
\]

so we may formally write

\[
\mathbf{R}_{ij} = e^{-s_R^2} (s_R S_{ij}^{[R]})_{ij}.
\]

If the relaxation time \( s_R \) is much shorter than the shear time scale (which we denote \( 1/\gamma_s \)), then one expects the series to converge rapidly.

Despite our formal solution above, let us concentrate on the original case, \( S_{ij}^{[R]} = a_R \delta_{ij} \). To first order in the shear one obtains then

\[
\mathbf{R}_{ij} = s_R a_R \delta_{ij} - s_R^2 a_R (\partial_i \bar{v}_j + \partial_j \bar{v}_i) = P_{\text{turb}} - \nu_{\text{turb}} (\partial_i \bar{v}_j + \partial_j \bar{v}_i).
\]

The exact form of the Reynolds stress in this case, where \( s_R \gamma_s \ll 1 \), is actually quite sensitive to the source term, which we have here modeled as being proportional to \( \delta_{ij} \). Nevertheless, the approximation that one obtains — an effective turbulent pressure and an effective turbulent viscosity — has proven quite useful over the past century.

The product \( s_R \gamma_s \), which one might call a “turbulent hydrodynamic Weissenburg-Deborah number,” is in fact generally less than or of the order of one in pure
hydrodynamic turbulence. One can make the argument in very rough fashion using eddy-viscosity type arguments. Colliding turbulent blobs lose their identity on the order of a coherence timescale, which is on the order of an “eddy turnover” timescale, which is of the order of the turbulent lengthscale $\ell_{\text{turb}}$ divided by the turbulent velocity scale $v_{\text{turb}}$. The magnitude of the difference in the corresponding background mean flow from one side of the eddy to the other is $\ell_{\text{turb}} \gamma_s$. If this velocity is greater than the turbulent velocity, the background shear will rapidly disrupt the eddy. Thus $\ell_{\text{turb}} \gamma_s < v_{\text{turb}}$. But this implies that $s_R \gamma_s < 1$, if $s_R \sim \ell_{\text{turb}} / v_{\text{turb}}$. Likewise, in the case of the continuum approximation in hydrodynamics, the molecular collision frequency must exceed the shear rate for the concept of a Stokes molecular viscosity to be terribly useful.

The admittedly very crude argument given above does not hold at all in the case of a high magnetic-Reynolds-number turbulent magnetic field, however. In perfect flux-freezing, a magnetic field line maintains its identity, or “coherence,” indefinitely.

In reality, dissipative effects will become important at some point. This is true even in ideal MHD, we expect, if we extend the word “dissipative” to include the effects of turbulent cascades and inverse cascades, which will take energy on intermediate scales and transport that energy either to very small scales or to larger scales. (Whether energy that is transported to small scales is ultimately dissipated in the molecular sense is not necessarily relevant.) It is not at all clear, however, that these quasi-dissipative effects become important on a shear time scale.

Consider the transport equation for the turbulent magnetic cross-correlation tensor $M_{ij}$ as written in eqn. (6). If we write

$$\partial_t M_{ij} + \bar{v}_k \partial_k M_{ij} - (\partial_k \bar{v}_i) M_{kj} - (\partial_k \bar{v}_j) M_{ki} = S_{ij}^{[M]} - (1/s_M) M_{ij}$$

we obtain, in a similar matter to the above,

$$M_{ij} = e^{+s_M L(s_M S_{ij}^{[M]})}.$$  

Again, taking the simplest case, $S_{ij}^{[M]} = a_M \delta_{ij}$, we obtain

$$M_{ij} = s_M a_M \delta_{ij} + s_M^2 a_M (\partial_i \bar{v}_j + \partial_j \bar{v}_i) + s_M^3 a_M A_{ij}^{[2]} + \ldots$$

Note that the signs of the coefficients here do not alternate, so that if the relaxation times were equal, $s_R = s_M$, then the series solution for $M$ might not be expected to converge as rapidly as that for $R$. In reality, however, we
note that for simple shear flow (or “curvilinear” flow), both series converge exactly with only three terms, as $A^{[n]} = 0$ for $n > 2$.

The full turbulent stress tensor is

$$\Pi_{ij} = -R_{ij} + M_{ij} - \frac{1}{2} M_{kk}\delta_{ij},$$

recalling that the Maxwell stress tensor $\tau^{[\text{Maxwell}]}$ (if $|E| \ll |B|$) is

$$\tau_{ij}^{[\text{Maxwell}]} = B_i B_j - \frac{1}{2} B^2 \delta_{ij},$$

so that the turbulent Maxwell stress tensor $M_{ij}$ is

$$M_{ij} = M_{ij} - \frac{1}{2} M_{kk}\delta_{ij}.$$ 

We may then write, again for the simplest case where $S_{ij}^{[R]} = a_R\delta_{ij}$ and $S_{ij}^{[M]} = a_M\delta_{ij}$, that

$$\Pi_{ij} = -\left(s_R a_R + \frac{1}{2} s_M a_M + (\gamma s_M)^2 s_M a_M\right)\delta_{ij} +$$

$$+ \left(s_R^2 a_R + s_M^2 a_M\right)A_{ij}^{[1]} + \left(-s_R^3 a_R + s_M^3 a_M\right)A_{ij}^{[2]}.$$

The identification of the term proportional to $\delta_{ij}$ above as a turbulent pressure is not unique, as the rest of the terms are not all traceless. This ambiguity naturally arises from the identification of $B^2/2$ as the magnetic pressure. Consider the case $M_{ij} = m\delta_{ij}$. Then the turbulent Maxwell stress is

$$M_{ij} = M_{ij} - \frac{1}{2} M_{kk}\delta_{ij} = m\delta_{ij} - \frac{3}{2}m\delta_{ij} = -\frac{1}{2}m\delta_{ij}$$

so that the Maxwell stress is proportional to $\delta_{ij}$, which looks like a pressure, except that the nominal magnetic pressure contribution to the stress is actually $-(3/2)m\delta_{ij}$. With this caveat in mind, then, we identify the term proportional to $\delta_{ij}$ as a pressure. This enables us to write

$$\Pi_{ij} = -P_{\text{turb}}\delta_{ij} + \mu_{\text{turb}} A_{ij}^{[1]} + \zeta_{\text{turb}} A_{ij}^{[2]},$$

where $\zeta_{\text{turb}}$ is the effective turbulent elasticity. (One alternative choice for defining the turbulent pressure is to assume that the pressure is proportional to the trace of the full stress tensor, so that

$$\Pi_{ij} = -P_{\text{turb}}\delta_{ij} + \mu_{\text{turb}} A_{ij}^{[1]} + \zeta_{\text{turb}} \left(A_{ij}^{[2]} - \frac{1}{3} A_{kk}^{[2]}\delta_{ij}\right);$$
it is not immediately clear if this is a preferable definition or not.) We may then define an effective turbulent relaxation rate as

\[ s_{\text{eff}} = \frac{\zeta_{\text{turb}}}{\mu_{\text{turb}}} \]

and an effective turbulent Weissenburg number

\[ \text{We}_{\text{turb}} = \gamma_s s_{\text{eff}} \]

The elastic term might also be thought of as a higher-order viscous term, for this restricted case of steady curvilineal flow.

5.2 Inclusion of coupling between Reynolds and turbulent Maxwell stresses

In fact, we also obtain an expression of the form (8) if we take the more general case that

\[
\partial_t R_{ij} + \bar{v}_k \partial_k R_{ij} + (\partial_k \bar{v}_j) R_{ik} + (\partial_k \bar{v}_i) R_{kj} = a_R \delta_{ij} + b_{RM} M_{ij} - s^{-1}_R R_{ij} \\
\partial_t M_{ij} + \bar{v}_k \partial_k M_{ij} - (\partial_k \bar{v}_i) M_{kj} - (\partial_k \bar{v}_j) M_{ik} = S_{ij}^{[M]} + b_{MR} R_{ij} - (1/s_M) M_{ij}
\]

That is to say, in this case, we make use of the available tensors \( \delta_{ij}, R_{ij}, \) and \( M_{ij} \). The new terms (with coefficients \( b_{RM} \) and \( b_{MR} \)) represent exchange terms. As such, the coefficients must have opposite signs. Such a form may be motivated on the general grounds that energy should be exchanged between the Reynolds stress and the turbulent Maxwell stress; further motivation is found through a more careful analysis of the transport equations as discussed below.

Let us rewrite the \( M_{ij} \) transport equation, highlighting a few terms, and replacing the resistive term with a simple dissipative term. We have

\[
\partial_t M_{ij} + \bar{v}_k \partial_k M_{ij} - M_{ik}(\partial_k \bar{v}_j) - (\partial_k \bar{v}_j) M_{kj} = - (\partial_k \bar{B}'_{(i)} \bar{B}'_{(j)} \bar{v}'_k v'_k + \bar{B}'_k(\partial_k \bar{v}'_{(i)} \bar{B}'_{(j)}) - \partial_k(\bar{B}'_{(i)} \bar{B}'_{(j)} \bar{v}'_k) + (\partial_k \bar{v}'_{(i)} \bar{B}'_{(j)} \bar{B}'_k) - s^{-1}_M M_{ij} (9)
\]

Terms written in light grey are those in which the mean field \( \bar{B}_i \) appears. The terms in green exchange energy between \( R_{ij} \) and \( M_{ij} \) through the respective terms \( v'_k \partial_k B'_i \) and \( \bar{B}'_k \partial_k v'_i \) that appear in the induction equation. (It might seem that we should discuss energy exchange between the Reynolds stress \( R \)
and the Maxwell stress \( \mathbf{M} \) rather than between \( \mathbf{R} \) and \( M \). However, note that the turbulent energy density \( U_{\text{turb}} \) is

\[
U_{\text{turb}} = \frac{1}{2} R_{kk} - \frac{1}{2} M_{kk} = \frac{1}{2} R_{kk} + \frac{1}{2} M_{kk},
\]

so that we recover that the turbulent field energy density is \( (B_k' B_k')/2 \), as it should be.

Similarly, for the Reynolds stress, we write

\[
\begin{align*}
\partial_t R_{ij} + \bar{v}_k \partial_k R_{ij} + (\partial_k \bar{v}_i) R_{kj} + R_{ik} (\partial_k \bar{v}_j) + \bar{\rho} \partial_k (\bar{v}_j \bar{v}_j) = \\
-\bar{v}_i B_k' \partial_i \partial_k B_k' + (\partial_k \bar{B}_i') v_j' B_k' + \bar{B}_k' (\partial_k B_i') v_j' + B_k' (\partial_k B_i') v_j' - \\
-\bar{v}_i' \left( \partial_j (B_k' B_k') \right) - \frac{1}{2} \bar{v}_j' \left( \partial_i (B_k' B_k') \right) - s_R^{-1} R_{ij}
\end{align*}
\]

Again, \( \mathbf{R} - \mathbf{M} \) exchange terms are highlighted in green.

One would expect the exchange terms to be proportional to the mean of \( v' \cdot (B' \cdot \nabla B' - \frac{1}{2} \nabla (B')^2) \); this is easily confirmed. The trace of the (zero mean-field) exchange terms in the Reynolds stress transport equation is

\[
\text{Tr(terms)} = 2 \bar{v}_i' \bar{B}_k' \partial_i \partial_k B_k' - \bar{v}_i' \partial_i (B_k' B_k')
\]

whereas the trace of the exchange terms in the Maxwell-like (i.e. \( M \)) stress transport equation is

\[
\text{Tr(terms)} = 2 \bar{v}_i' \bar{B}_k' \partial_i \partial_k B_k' - \bar{v}_i' \partial_i (B_k' B_k') + 2 \partial_k \left[ (\bar{v}_i' B_k') B_k' \right]
\]

this last term being in conservative form and so identically zero in the homogeneous case.

In a more sophisticated model, the exchange between \( \mathbf{R}_{ij} \) and \( M_{ij} \) might be modeled through a transport equation for the triple-correlation exchange terms described above. Here, however, we simply model this exchange in the transport equations for \( \mathbf{R}_{ij} \) and \( M_{ij} \) by assuming that the exchange is proportional to these tensors themselves.

With this assumption, \( \mathbf{R}_{ij} \) and \( M_{ij} \) have solutions of the form

\[
\begin{align*}
\mathbf{R}_{ij} &= r_0 \delta_{ij} + r_1 A_{[1]}^{ij} + r_2 A_{[2]}^{ij} \\
\mathbf{M}_{ij} &= m_0 \delta_{ij} + m_1 A_{[1]}^{ij} + m_2 A_{[2]}^{ij}
\end{align*}
\]
So, the expression (8) is again obtained. Specifically, the solution is

\[
\begin{align*}
    r_0(1 - s_R b_{RM} s_M b_{MR}) &= s_R a_R + s_R b_{RM} s_M a_M \\
    m_0(1 - s_R b_{RM} s_M b_{MR}) &= s_M a_M + s_M b_{MR} s_R a_R \\
    r_{i+1}(1 - s_R b_{RM} s_M b_{MR}) &= s_R b_{RM} m_i - r_i \\
    m_{i+1}(1 - s_R b_{RM} s_M b_{MR}) &= -s_M b_{MR} r_i + m_i
\end{align*}
\]

5.3 Effects of Rotation and Stratification

It might be expected that rotation is not something to be put into a turbulence transport model; it ought to exist already in the transport model, so that rotation is just a matter of doing the proper coordinate transformation. This philosophy is also in keeping with the principle of material frame indifference, according to which a fluid should not know (at least as far as its constitutive relations are concerned) about the overall rotation of the system. Actually the issue is a bit more subtle than this, since the form of the Reynolds and Maxwell stress transport equations described above (or, typically, in their unmodeled state) will be invariant with respect to transformation to a rotating reference frame, but this means that the action of the Coriolis force on the turbulent blobs in the rotating reference frame is ignored (see, e.g., the discussions in \textit{Pope} (2000) and in \textit{R"udiger} (1989)). Inclusion of the Coriolis force may appear as the introduction of new terms with symmetrized products of the rotation tensor \( \Omega_{ij} \equiv \epsilon_{ijk} \Omega_k \), as described below. Such considerations are important for a model of the MRI, as well as for the more simple case of Taylor instability when the Rayleigh stability criterion is not satisfied. The inclusion of the effects of rotation is an ongoing project that we intend for a future paper. These considerations must be addressed in order to reproduce the result from simulations that the \( rr \) component of the stress is larger than the \( zz \) component of the stress, not to mention the presence of the MRI itself.

It would also be nice to address the effects of stratification on the turbulence, such as is a necessary ingredient for the treatment of convective MHD turbulence. Consideration of turbulent elasticity should be important for the turbulence in the solar convective zone (SCZ), for example, even though the turbulence in the SCZ is not driven (primarily, at least) by MHD instabilities but rather purely hydrodynamical ones. For incompressible but variable-density stratified fluids in the laboratory, one finds an additional driving term in the Reynolds stress transport equation that is proportional to the product of the pressure gradient and the density gradient. In cases of astrophysical interest, where compressibility becomes important, the important consideration is the direction of the entropy gradient (or, the gradient in the potential
temperature). This will necessarily introduce anisotropies into the turbulent stress tensor, just as a consideration of the sources of the MRI will produce anisotropies that are not adequately addressed by the considerations outlined in this paper, as mentioned above. Inclusion of these effects may appear as the introduction of new terms with symmetrized products of the vectors $g_i$ and $\partial_i P$ and $\partial_i \bar{\rho}$, or other thermodynamic derivatives, depending upon the choice made by the modeler for the independent thermodynamic quantities. Again, a proper inclusion of these effects must follow from a model that is built from the transport equations in which — unlike what we have presented here — the assumption of incompressibility is relaxed. This is also an ongoing project that we intend for a future paper.

We can, however, comment briefly upon the combined effects of rotation and stratification as they manifest themselves in stellar convective zones through the so-called “Λ-effect” (Rüdiger, 1989). This introduces additional terms that, in the SCZ, are to lowest order proportional to the dyad product $(\vec{g} \times \vec{\Omega})\vec{g}$. (Note that inclusion of the rotation $\Omega_i$ violates one of the assumptions that forms the basis of the expansion in terms of Rivlin-Eriksen tensors, namely material frame indifference, but again, this is to be expected for a turbulence model that takes account of Coriolis forces, as mentioned above.)

For the Λ-effect as described by Rüdiger (1989) (in particular, section 4.6), the additional terms appear in the prescription for the equilibrium Reynolds stress (as opposed to the Reynolds stress transport equation), and for the lowest-order Λ-effect the symmetrized product mentioned above,

$$R_{ij}^{\Lambda} = \bar{\rho} \Lambda (g_i g_k \Omega_{kj} + \Omega_{ik} g_k g_j) + \ldots,$$

produces additional components $R_{r\phi}^{\Lambda}$ and, to higher order, $R_{\theta\phi}^{\Lambda}$. These decouple from the elastic terms in the stress, so that the total turbulent stress tensor may be written as

$$\Pi_{ij} = -P_{ij}^{\text{turb}} + \Pi_{ij}^{\text{visc}} + \Pi_{ij}^{\text{elastic}} + \Pi_{ij}^{\Lambda}.$$

The form of the turbulent stress tensor may then be written as an expression of the form (compare Rekowski and Rüdiger (1998) eqn. 17)

$$\Pi_{ij} = -P \delta_{ij} + \bar{\rho} N_{ij \ell k}(\partial_k \bar{v}_\ell) + \bar{\rho} Z_{ijklmn}(\partial_k \bar{v}_\ell)(\partial_m \bar{v}_n) - \bar{\rho} \Lambda_{ijk} \Omega_k$$

so that the various effects are simply additive.
6 Conclusions

We have discussed some modifications to the basic Boussinesq ansatz that are motivated by an analogy between the turbulent component of the magnetic field in MHD turbulence and the dynamics of polymers in solution. This is part of an ongoing project begun by us to study MHD turbulence beginning with the Reynolds decomposition of the field, with particular focus on the small-scale tangled component in preference to the large-scale mean-field component. It is our belief that undue focus on the latter, apparently for the sake of understanding dynamo behavior, has obscured the important dynamical effects that the former may have in practical MHD turbulent flow problems, and that furthermore the Boussinesq effective viscosity had been adopted for such flows without sufficient consideration as to whether the initial motivation for an effective turbulent viscosity carried over from hydrodynamics to MHD. While a viscoelastic picture of MHD turbulence, built from — or at least inspired by — a polymer analogy, may not be “correct” in some absolute sense (as pointed out by a previous anonymous referee), we feel that it is at least useful, which is all that a model can ask to be. Of course, there are many effects that naturally arise from viscoelastic models, as described at length in any textbook on the matter, which we hope to discuss in future work. For example, the turbulent medium supports Alfvén waves that can be treated easily within a viscoelastic picture; these turbulent Alfvén waves have a complex dispersion relation, reflecting the dissipative nature of turbulence. These waves do not appear at all in a purely viscous model. We take this as a further indication that a viscoelastic picture of MHD turbulence may be useful.

Here we have focused on one aspect common to any viscoelastic model of MHD turbulence, which is to say any model that takes into account the distortion of the turbulent Maxwell stress by the inclusion of the proper tensor derivatives and a relaxation time, namely the appearance of significant anisotropy of the turbulent stress tensor in the direction of the shear. This effect should be important in any shearing environment in MHD turbulence, including but not limited to the MRI (and in particular the inner regions of accretion disks and the jet-launching process) and to the turbulence in stellar convection zones.
References

Chorin, A. J., 1988. Spectrum, dimension, and polymer analogies in fluid turbulence. Phys. Rev. Lett. 60 (19), 1947–1949.
Hawley, J. F., Gammie, C. F., Balbus, S. A., 1995. Local Three-dimensional Magnetohydrodynamic Simulations of Accretion Disks. ApJ 440, 742–763.
Joseph, D. D., 1990. Fluid Dynamics of Viscoelastic Fluids. Springer-Verlag, New York.
Ogilvie, G. I., 2001. Non-linear fluid dynamics of eccentric disks. Mon. Not. Royal Astron. Soc. 325, 231–248, astro-ph/0102245.
Pope, S. B., 2000. Turbulent flows. Cambridge University Press, Cambridge.
Rekowski, B. V., Rüdiger, G., 1998. Differential rotation and meridional flow in the solar convection zone with AKA-effect. A&A 335, 679–684.
Rüdiger, G., 1989. Differential Rotation and Stellar Convection. Akademie-Verlag, Berlin.
Williams, P. T., 2001. Viscoelastic analogy for the acceleration and collimation of astrophysical jets. xxx.lanl.gov astro-ph/0111603.
Williams, P. T., 2002. Protostellar jets driven by a disorganized magnetic field. IAOC Workshop, “Galactic Star Formation Across the Mass Spectrum”, La Serena, Chile, March 2002, to be published in ASP conference series, ed. J.M.de Buizer astro-ph/0206230.