Spin(7)-manifolds with parallel torsion form

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Abstract. Any Spin(7)-manifold admits a metric connection $∇^c$ with totally skew-symmetric torsion $T^c$ preserving the underlying structure. We classify those with $∇^c$-parallel $T^c \neq 0$ and non-Abelian isotropy algebra $\text{iso}(T^c) \leq \text{spin}(7)$. These are isometric to either Riemannian products or homogeneous naturally reductive spaces, each admitting two $∇^c$-parallel spinor fields.

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1. Introduction

In the early '80's physicists tried to incorporate torsion into superstring and supergravity theories in order to get a physically flexible model. Strominger described the mathematics of the underlying superstring theory of type II. It consists of a Riemannian spin manifold $(M^n, g)$ equipped, amongst other things, with a spinor field $\Psi$ and a 3-form $T$ that satisfy a certain set of field equations (see [19]), including

$$∇_X^g Ψ + \frac{1}{4} (X \wedge T) \cdot Ψ = 0 \quad ∀ X ∈ TM^n.$$

We denote by $τ \cdot Ψ$ the Clifford product of a differential form $τ$ with a spinor field $Ψ$. In the theory $T$ is seen as a field strength of sorts, whilst $Ψ$ is the so-called supersymmetry. With the metric connection $∇$ whose torsion is the 3-form $T$,

$$g(∇_X Y, Z) = g(∇^g_X Y, Z) + \frac{1}{2} \cdot T(X, Y, Z) \quad ∀ X, Y, Z ∈ TM^n,$$

the above equation transforms to $∇Ψ = 0$. In other words, the spinor field $Ψ$ is parallel with respect to $∇$, a fact imposing restrictions on the holonomy group $\text{Hol}(∇)$. In the case $T = 0$, i.e. when $∇$ is the Levi-Civita connection of $(M^n, g)$, the holonomy group is one of the following (see [20]):

$$\text{SU}(n), \quad \text{Sp}(n), \quad G_2, \quad \text{Spin}(7).$$

In order to construct models with $T \neq 0$, it is therefore reasonable to study manifolds that admit a metric connection $∇$ with totally skew-symmetric torsion whose $\text{Hol}(∇)$ is

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contained in SU(n), Sp(n), G2 or Spin(7). Surprisingly, the existence of such a connection is unobstructed for Spin(7)-manifolds $M^8$ (see [17]). Furthermore this connection, denoted by $\nabla^c$, is unique, preserves the underlying Spin(7)-structure and makes a non-trivial spinor field parallel. The more general point of view of [2, 12, 18] indicates that structures with parallel torsion form $T^c$,

$$\nabla^c T^c = 0,$$

are of particular interest and provide a starting point to solve the entire system of Strominger’s equations. For example, $\delta T^c = 0$ is automatically satisfied in this setup. Moreover, many geometric properties become algebraically tractable by assuming the parallelism of $T^c$, for this implies, for instance, that the holonomy algebra $\mathfrak{hol}(\nabla^c)$ becomes a subalgebra of the isotropy algebra $\mathfrak{iso}(T^c) \leq \mathfrak{spin}(7)$.

The aim of the paper is the classification of Spin(7)-manifolds with parallel torsion form $T^c \neq 0$ and non-Abelian $\mathfrak{iso}(T^c) \leq \mathfrak{spin}(7)$. We show that the latter fall into eight types. For all of these algebras we describe the admissible torsion forms $T^c$ and Ricci tensors $\text{Ric}^c$ with respect to $\nabla^c$. Finally we discuss the geometry of the space $M^8$ relatively to its holonomy algebra $\mathfrak{hol}(\nabla^c) \leq \mathfrak{iso}(T^c)$ and to the Spin(7)-orbit of the torsion form

$$T^c \in \Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48}.$$ 

The main result is that these spaces are isometric to either a Riemannian product or a homogeneous naturally reductive space; some of them are uniquely determined (see theorem 6.2 and theorem 6.6). Moreover, every structure admits at least two $\nabla^c$-parallel spinor fields. There are examples exhibiting 16 $\nabla^c$-parallel spinor fields and satisfying the additional constraint

$$\text{Ric}^c = \text{Ric}^g_{ij} - \frac{1}{4} T^c_{imn} T^c_{jmn} = 0$$

for the energy-momentum tensor (see examples 6.2 and 6.3).

The paper is structured as follows: In section 2 we state basic facts on metric connections with parallel, totally skew-symmetric torsion. We then specialize to the case of Spin(7)-structures in section 3. Section 4 is devoted to the study of the non-Abelian subalgebras of $\mathfrak{spin}(7)$ used for the algebraic classification (see section 5) in terms of the torsion form. In the last section we discuss the geometry of each of these classes.

## 2. Parallel Torsion

Fix a Riemannian spin manifold $(M^n, g)$, a 3-form $T$, and denote the Levi-Civita connection by $\nabla^g$. The equation

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} \cdot T(X, Y, Z) \quad \forall X, Y, Z \in TM^n,$$

defines a metric connection $\nabla$ with totally skew-symmetric torsion $T$. We will consider the case of parallel torsion, $\nabla T = 0$. Then the 3-form $T$ is coclosed (see [12]), $\delta T = 0$, its differential is given by

$$dT = \sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner T) = 2\sigma^T$$

for a chosen orthonormal frame $(e_1, \ldots, e_n)$, and the curvature tensor $R^\nabla$ of $\nabla$ is a field of symmetric endomorphism of $\Lambda^2$. If there exists a $\nabla$-parallel spinor field $\Psi$ one can
compute the Ricci tensor $\text{Ric}^\nabla$ of $\nabla$ algebraically (see for example [2]),

$$2 \text{Ric}^\nabla(X) \cdot \Psi = (X \lrcorner dT) \cdot \Psi.$$ 

Moreover, the following relation holds (see [1]):

$$4 T^2 \cdot \Psi := 4 T \cdot (T \cdot \Psi) = (2 \text{Scal}^g + \|T\|^2) \cdot \Psi.$$

Consequently, $T^2$ acts as a scalar on the space of $\nabla$-parallel spinor fields, hence it gives an algebraic restriction on $T$.

3. Spin (7)-manifolds

Consider the space $\mathbb{R}^8$, fix an orientation and denote a chosen oriented orthonormal basis by $(e_1, \ldots, e_8)$. The compact simply connected Lie group Spin (7) can be described (see for example [16]) as the isotropy group of the 4-form $\Phi$:

$$\Phi = \phi + *\phi,$$

where $\phi$, $Z$ and $D$ denote the following forms:

$$\phi := (Z \wedge e_7 + D) \wedge e_8, \quad Z := e_{12} + e_{34} + e_{56}, \quad D := e_{246} - e_{235} - e_{145} - e_{136}.$$ 

Here and henceforth we shall not distinguish between vectors and covectors and use the notation $e_{i_1 \ldots i_m}$ for the exterior product $e_{i_1} \wedge \ldots \wedge e_{i_m}$. The so-called fundamental form $\Phi$ is self-dual with respect to the Hodge star operator, $*\Phi = \Phi$, and the 8-form $\Phi \wedge \Phi$ is a non-zero multiple of the volume form of $\mathbb{R}^8$.

A Spin (7)-structure/manifold is a triple $(M^8, g, \Phi)$ consisting of an 8-dimensional Riemannian manifold $(M^8, g)$ and a 4-form $\Phi$ such that there exists an oriented orthonormal adapted frame $(e_1, \ldots, e_8)$ realizing $(\ast)$ at every point. Equivalently, these structures can be defined as a reduction of the structure group of orthonormal frames of the tangent bundle to Spin (7). The space of 3-forms decomposes into two irreducible Spin (7)-modules,

$$\Lambda^3 = \Lambda^3_8 \oplus \Lambda^3_{48},$$

which can be characterized using the fundamental form as

$$\Lambda^3_8 := \{ \ast (\beta \wedge \Phi) : \beta \in \Lambda^1 \}, \quad \Lambda^3_{48} := \{ \gamma \in \Lambda^3 : \gamma \wedge \Phi = 0 \}.$$ 

The subscript specifies the dimension of the respective space. We will denote the projection of a 3-form $T$ onto one of these spaces by $T_8$ or $T_{48}$ respectively.

Any Spin (7)-manifold admits (see [17]) a unique metric connection $\nabla^c$ (the characteristic connection) with totally skew-symmetric torsion $T^c$ (the characteristic torsion) preserving the Spin (7)-structure, $\nabla^c \Phi = 0$, and $T^c$ is given by

$$T^c = -\delta \Phi - \frac{7}{6} \ast (\theta \wedge \Phi).$$

Here $\theta \in \Lambda^1$ denotes the so-called Lee form

$$\theta := \frac{1}{7} \ast (\delta \Phi \wedge \Phi) = \frac{6}{7} \ast (\Phi \wedge T^c) = -\frac{1}{7} \ast (*d\Phi \wedge \Phi).$$

The Riemannian scalar curvature $\text{Scal}^g$ and the scalar curvature $\text{Scal}^c$ of $\nabla^c$ are given by (see [17])

$$\text{Scal}^g = \frac{49}{18} \|\theta\|^2 - \frac{1}{2} \|T^c\|^2 + \frac{7}{2} \delta \theta,$$

$$\text{Scal}^c = \text{Scal}^g - \frac{3}{2} \|T^c\|^2.$$
Analyzing the algebraic type of $T^c$ we obtain Cabrera’s description [5] – by differential equations involving the Lee form – of the Fernández classification [10] of Spin (7)-structures. For example, a Spin (7)-structure is of class $\mathcal{W}_1$, i.e. a balanced structure, if and only if the Lee form vanishes. Equivalently, these structures can be characterized by $T^c_8 = 0$. Spin (7)-structures of class $\mathcal{W}_0$ – the so-called parallel structures – are defined by a closed fundamental form, $d\Phi = 0$. These are the structures with vanishing torsion, $T^c = 0$. In [5] Cabrera shows that the Lee form of a Spin (7)-structure of class $\mathcal{W}_2$ (for which $d\Phi = \theta \wedge \Phi$ holds or, equivalently, $T^c_{48} = 0$) is closed, and therefore such a manifold is locally conformally equivalent to a parallel Spin (7)-manifold. These are called locally conformal parallel. Finally, structures of class $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ are characterized by $T^c_8 \neq 0$ and $T^c_{48} \neq 0$. We summarize the previous facts in the following table:

| class                      | characteristic torsion | differential equations |
|----------------------------|------------------------|------------------------|
| $\mathcal{W}_0$ (parallel) | $T^c_8 = 0$, $T^c_{48} = 0$ | $d\Phi = 0$, $\theta = 0$ |
| $\mathcal{W}_2$ (locally conformal parallel) | $T^c_{48} = 0$ | $d\Phi = \theta \wedge \Phi$ |
| $\mathcal{W}_1$ (balanced) | $T^c_8 = 0$ | $\theta = 0$ |
| $\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2$ | $T^c_8 \neq 0$, $T^c_{48} \neq 0$ | — |

We now restrict to parallel characteristic torsion, $\nabla^c T^c = 0$.

**Lemma 3.1.** The following formulae hold in presence of parallel characteristic torsion:

$$
\|\theta\|^2 = \frac{36}{7} \|T^c_8\|^2, \quad \delta \theta = 0.
$$

**Proof.** We prove the second equation,

$$
\delta \theta = - * d * \theta = - \frac{6}{7} * d * (\Phi \wedge T^c) = \frac{6}{7} * d (\Phi \wedge T^c).
$$

The 7-form $\Phi \wedge T^c$ is $\nabla^c$-parallel and the sum $\sum_i (e_i \lrcorner T) \wedge (e_i \lrcorner (\Phi \wedge T))$ vanishes for arbitrary 3-forms $T \in \Lambda^3(\mathbb{R}^8)$.

This Lemma and $(\diamond)$ result in the following proposition:

**Proposition 3.1.** Let $(M^8, g, \Phi)$ be a Spin (7)-manifold with $\nabla^c T^c = 0$. Then the Riemannian scalar curvature $\text{Scal}^g$ and the scalar curvature $\text{Scal}^c$ of $\nabla^c$ are given in terms of the torsion form by

$$
\text{Scal}^g = \frac{27}{2} \|T^c_8\|^2 - \frac{1}{2} \|T^c_{48}\|^2, \quad \text{Scal}^c = 12 \|T^c_8\|^2 - 2 \|T^c_{48}\|^2.
$$

A direct computation shows that for arbitrary 3-forms $T$, vector fields $X$ and spinor fields $\Psi$ the following equation is satisfied:

$$
-4 \left( X \lrcorner \sigma^T \right) \cdot \Psi = (T^2 - 7 \|T^c_8\|^2) \cdot X \cdot \Psi.
$$

The previous proposition together with this equation and the facts of section 2 eventually prove the following:

**Proposition 3.2.** Let $(M^8, g, \Phi)$ be a Spin (7)-manifold with $\nabla^c T^c = 0$. Any $\nabla^c$-parallel spinor field $\Psi$ on $M^8$ satisfies

$$
(T^c)^2 \cdot \Psi = 7 \|T^c_8\|^2 \cdot \Psi, \quad -4 \text{Ric}^c (X) \cdot \Psi = \left( (T^c)^2 - 7 \|T^c_8\|^2 \right) \cdot X \cdot \Psi.
$$

From now on we assume Spin (7)-structures to be non-parallel, $T^c \neq 0$, and to have parallel characteristic torsion, $\nabla^c T^c = 0$. 
4. Subalgebras of spin(7)

It is known that the group Spin(7) \( \subset \text{SO}(8) \) acts on spinors. Let Cliff(R^8) denote the real Clifford algebra of the Euclidean space R^8. We will use the following real representation of this algebra on the space of real spinors \( \Delta_8 := \mathbb{R}^{16} \): 

\[
e_i = \begin{bmatrix} 0 & M_i \\ M_i & 0 \end{bmatrix} \text{ for } i = 1, \ldots, 7, \quad e_8 = \begin{bmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{bmatrix},
\]

\(M_1 := E_{18} + E_{27} - E_{36} - E_{45}, \quad M_2 := -E_{17} + E_{28} + E_{35} - E_{46}, \)
\(M_3 := -E_{16} + E_{25} - E_{38} + E_{47}, \quad M_4 := -E_{15} - E_{26} - E_{37} - E_{48}, \)
\(M_5 := -E_{13} - E_{24} + E_{57} + E_{68}, \quad M_6 := E_{14} - E_{23} - E_{58} + E_{67}, \)
\(M_7 := E_{12} - E_{34} - E_{56} + E_{78}. \)

Here \(E_{ij}\) denotes the standard basis of the Lie algebra \(\mathfrak{so}(8)\). We fix an orthonormal basis \(\Psi_1 := [1, 0, \ldots, 0]^T, \ldots, \Psi_{16} := [0, \ldots, 0, 1]^T\) of real spinors. The 4-form \(\Phi\) corresponds via the Clifford product to the real spinor \(\Psi_0 := \Psi_9 - \Psi_{10} \in \Delta_8\), 

\[\Phi \cdot \Psi_0 = -14 \cdot \Psi_0,\]

and therefore Spin(7) can be seen as the isotropy group of \(\Psi_0\). Its Lie algebra \(\text{spin}(7)\) is the subalgebra of \(\text{spin}(8)\) containing all 2-forms 

\[\omega = \sum_{i<j} \omega_{ij} \cdot e_{ij} \in \Lambda^2(\mathbb{R}^8)\]

such that the Clifford product \(\omega \cdot \Psi_0 = 0\). This is satisfied if and only if 

\[\omega_{18} = -\omega_{27} + \omega_{36} + \omega_{45}, \quad \omega_{28} = \omega_{17} + \omega_{35} - \omega_{46}, \quad \omega_{38} = -\omega_{16} - \omega_{25} - \omega_{47}, \]
\[\omega_{48} = -\omega_{15} + \omega_{26} + \omega_{37}, \quad \omega_{58} = \omega_{14} + \omega_{23} - \omega_{67}, \quad \omega_{68} = \omega_{13} - \omega_{24} + \omega_{57}, \]
\[\omega_{78} = -\omega_{12} - \omega_{34} - \omega_{56}. \]

We fix the following basis of \(\text{spin}(7)\):

\[P_1 := e_{35} + e_{46}, \quad P_2 := e_{36} - e_{45}, \quad P_3 := e_{15} + e_{26}, \quad P_4 := e_{16} - e_{25}, \]
\[P_5 := e_{13} + e_{24}, \quad P_6 := e_{14} - e_{23}, \quad P_7 := e_{12} - e_{34}, \quad P_8 := e_{34} - e_{56}, \]
\[Q_1 := 2 \cdot e_{17} - e_{35} + e_{46}, \quad Q_2 := 2 \cdot e_{27} + e_{36} + e_{45}, \quad Q_3 := 2 \cdot e_{37} + e_{15} - e_{26}, \]
\[Q_4 := 2 \cdot e_{47} - e_{16} - e_{25}, \quad Q_5 := 2 \cdot e_{57} - e_{13} + e_{24}, \quad Q_6 := 2 \cdot e_{67} + e_{14} + e_{23}, \]
\[S_1 := e_{18} - e_{27}, \quad S_2 := e_{28} + e_{17}, \quad S_3 := e_{38} - e_{47}, \quad S_4 := e_{48} + e_{37}, \]
\[S_5 := e_{58} - e_{67}, \quad S_6 := e_{68} + e_{57}, \quad S_7 := e_{78} - e_{56}. \]

For a given Lie subalgebra \(g\) of \(\text{spin}(7)\), i.e. \(g \leq \text{spin}(7)\), we denote by \((\Lambda^3(\mathbb{R}^8))_g\) and \((\Delta_8)_g\) the spaces of \(g\)-invariant 3-forms and spinors respectively. We assume \((\Lambda^3(\mathbb{R}^8))_g\) is non-trivial. The action of \(\text{spin}(7)\) coincides on the 8-dimensional vector spaces 

\[\mathbb{R}^8 = \text{span}(e_1, \ldots, e_8), \quad \Lambda^3(\mathbb{R}^8) = \text{span}(\ast(e_1 \wedge \Phi), \ldots, \ast(e_8 \wedge \Phi)).\]
Consequently \( g \) preserves a \( T \in \left( \Lambda^3 \left( \mathbb{R}^8 \right) \right)_g \) with \( T_8 \neq 0 \), if and only if it preserves a vector. A long but elementary computation for the other case \( \left( \Lambda^3 \left( \mathbb{R}^8 \right) \right)_g = \{0\} \) proves that any non-Abelian \( g \) is conjugate to

\[
\mathbb{R} \oplus \text{su} \left( 2 \right) = \text{span} \left( P_7 + 2 P_8 - 4 S_7, P_5, P_6, P_7 \right) < \text{u} \left( 3 \right) < \text{su} \left( 4 \right) < \text{spin} \left( 7 \right).
\]

To conclude, a non-Abelian subalgebra of \( \text{spin} \left( 7 \right) \) that preserves a non-trivial 3-form is either a subalgebra of \( g_2 \) or the algebra \( \mathbb{R} \oplus \text{su} \left( 2 \right) \) above. Dynkin’s results [8, 9] on maximal subalgebrae of exceptional Lie algebrae like \( g_2 \) allow to state the following:

**Theorem 4.1.** Let \( g \) be a non-Abelian subalgebra of \( \text{spin} \left( 7 \right) \). If there exists a non-trivial \( g \)-invariant 3-form \( T \), i.e. \( 0 \neq T \in \left( \Lambda^3 \left( \mathbb{R}^8 \right) \right)_g \), then \( g \) is conjugate to one of the following algebrae:

\[
\begin{align*}
g_2 & = \text{span} \left( P_1, \ldots, P_8, Q_1, \ldots, Q_6 \right) < \text{spin} \left( 7 \right), \\
\text{su} \left( 3 \right) & = \text{span} \left( P_1, \ldots, P_8 \right) < g_2, \\
\text{su} \left( 2 \right) \oplus \text{su}_c \left( 2 \right) & = \text{span} \left( P_5, P_6, P_7, P_7 + 2 P_8, Q_5, Q_6 \right) < g_2, \\
\text{u} \left( 2 \right) & = \text{span} \left( P_7 + 2 P_8, P_5, P_6, P_7 \right) < \text{su} \left( 3 \right), \\
\mathbb{R} \oplus \text{su}_c \left( 2 \right) & = \text{span} \left( P_7, P_7 + 2 P_8, Q_5, Q_6 \right) < \text{su} \left( 2 \right) \oplus \text{su}_c \left( 2 \right), \\
\text{so} \left( 3 \right) & = \text{span} \left( P_1 + P_2, P_2 + P_6, P_7 + P_8 \right) < \text{su} \left( 3 \right), \\
\text{su} \left( 2 \right) & = \text{span} \left( P_5, P_6, P_7 \right) < \text{u} \left( 2 \right), \\
\text{su}_c \left( 2 \right) & = \text{span} \left( P_7 + 2 P_8, Q_5, Q_6 \right) < \mathbb{R} \oplus \text{su}_c \left( 2 \right), \\
\text{so}_{ir} \left( 3 \right) & = \text{span} \left( P_5 - \sqrt{\frac{3}{5}} Q_2, P_6 + \sqrt{\frac{3}{5}} Q_1, P_7 + 3 P_8 \right) < g_2, \\
\mathbb{R} \oplus \text{su} \left( 2 \right) & = \text{span} \left( P_7 + 2 P_8 - 4 S_7, P_5, P_6, P_7 \right) < \text{su} \left( 4 \right) < \text{spin} \left( 7 \right).
\end{align*}
\]

Here \( \text{su}_c \left( 2 \right) \) denotes the centralizer of \( \text{su} \left( 2 \right) \) inside \( g_2 \) which is isomorphic, but not conjugate, to \( \text{su} \left( 2 \right) \). \( \text{so}_{ir} \left( 3 \right) \) denotes the maximal subalgebra of \( g_2 \) generating an irreducible 7-dimensional real representation.

The Lie algebrae \( g_2 \) and \( \mathbb{R} \oplus \text{su} \left( 2 \right) \) are of rank 2. Their maximal tori are given by

\[
t^2 := k \cdot P_7 + l \cdot (P_7 + 2 P_8) < g_2, \quad \tilde{t}^2 := \tilde{k} \cdot P_7 + \tilde{l} \cdot (P_7 + 2 P_8 - 4 S_7) < \mathbb{R} \oplus \text{su} \left( 2 \right).
\]

1-dimensional tori contained in these will be denoted by \( t^1 \) or \( \tilde{t}^1 \) respectively.

### 5. Algebraic classification

Given a non-parallel \( \text{Spin} \left( 7 \right) \)-structure let \( \text{iso} \left( T^c \right) \) be the isotropy algebra of the characteristic torsion \( T^c \) and \( \text{hol} \left( \nabla^c \right) \) the holonomy algebra of the characteristic connection \( \nabla^c \). Obviously, these two are Lie subalgebrae of \( \text{spin} \left( 7 \right) \), and a non-Abelian \( \text{iso} \left( T^c \right) \) is one of the algebrae in theorem 4.1. But not all of those algebrae can occur as the isotropy algebra of a non-trivial 3-form. A direct computation proves the following:

**Proposition 5.1.** If the isotropy algebra \( \text{iso} \left( T \right) < \text{spin} \left( 7 \right) \) of a non-trivial 3-form \( T \) contains \( \text{su}_c \left( 2 \right) \) or \( \text{su} \left( 2 \right) \), then \( \dim \left( \text{iso} \left( T \right) \right) \geq 4 \).

Since we restricted the consideration to parallel characteristic torsion the holonomy algebra \( \text{hol} \left( \nabla^c \right) \) is a subalgebra of \( \text{iso} \left( T^c \right) \),

\[
\text{hol} \left( \nabla^c \right) \leq \text{iso} \left( T^c \right) < \text{spin} \left( 7 \right).
\]
Conversely, fix \( \mathfrak{h} \leq \mathfrak{g} < \mathfrak{spin}(7) \). Suppose there exists a Spin(7)-manifold with \( \mathfrak{hol}(\nabla^c) = \mathfrak{h} \) and \( \nabla^c \)-parallel torsion \( T^c \neq 0 \) satisfying \( \mathfrak{iso}(T^c) = \mathfrak{g} \). Then \( T^c \) is necessarily contained in the space of \( \mathfrak{g} \)-invariant 3-forms \( (\Lambda^3(\mathbb{R}^8))_\mathfrak{g} \) satisfying

\[
(\bullet) \quad (T^c)^2 \cdot \Psi = 7 \|T^c\|^2 \cdot \Psi, \quad -4 \text{Ric}^c(X) \cdot \Psi = \left( (T^c)^2 - 7 \|T^c\|^2 \right) \cdot X \cdot \Psi
\]

for all \( \mathfrak{h} \)-invariant spinors \( \Psi \in (\Delta_8)_\mathfrak{h} \) and all vectors \( X \in \mathbb{R}^8 \) (cf. proposition 3.2). Furthermore, two torsion forms \( 0 \neq T^c_1, T^c_2 \in (\Lambda^3(\mathbb{R}^8))_\mathfrak{g} \) define equivalent geometric structures if they are equivalent under the action of the algebra \( \mathfrak{h} \).

Define the space \( \mathcal{K}(\mathfrak{h}) \) of algebraic curvature tensors with values in \( \mathfrak{h} \) by

\[
\mathcal{K}(\mathfrak{h}) := \{ R \in \Lambda^2(\mathbb{R}^8) \otimes \mathfrak{h} : \sigma_{X,Y,Z} \{ R(X,Y,Z,V) \} = 0 \ \forall X,Y,Z,V \in \mathbb{R}^8 \}.
\]

Here \( \sigma_{X,Y,Z} \) denotes the cyclic sum over \( X,Y,Z \). If the space \( \mathcal{K}(\mathfrak{hol}(\nabla^c)) \) is trivial for a Spin(7)-manifold with parallel torsion, the curvature operator \( R^c : \Lambda^2(\mathbb{R}^8) \to \mathfrak{hol}(\nabla^c) \) of the characteristic connection is \( \nabla^c \)-parallel (see [7]) and thus \( \mathfrak{hol}(\nabla^c) \)-invariant. A case-by-case study proves the following:

**Proposition 5.2.** Let \( \mathfrak{h} \leq \mathfrak{g} < \mathfrak{spin}(7) \) with \( \mathfrak{g} \) non-Abelian and suppose there exists a non-trivial \( \mathfrak{g} \)-invariant 3-form. Then \( \mathcal{K}(\mathfrak{h}) \) is non-trivial if and only if \( \mathfrak{su}(2) \leq \mathfrak{h} \).

The recipe to obtain necessary conditions on \( T^c \) and \( \text{Ric}^c \) goes as follows:

1. Fix \( \mathfrak{h} = \mathfrak{hol}(\nabla^c) \leq \mathfrak{iso}(T^c) = \mathfrak{g} \) with \( \mathfrak{g} < \mathfrak{spin}(7) \) non-Abelian.
2. Determine the spaces \( (\Lambda^3(\mathbb{R}^8))_\mathfrak{g} \) and \( (\Delta_8)_\mathfrak{h} \).
3. Solve (\( \bullet \)).
4. Quotient out the action of \( \text{inv}(\Lambda^3(\mathbb{R}^8))_\mathfrak{g} \) on \( T^c \neq 0 \).
5. If \( \mathfrak{su}(2) \not\leq \mathfrak{h} \), analyze the \( \mathfrak{h} \)-invariance and the symmetry of \( R^c \).

Applying this, we determine \( T^c \) and \( \text{Ric}^c \) for all admissible combinations of \( \mathfrak{hol}(\nabla^c) \) and non-Abelian \( \mathfrak{iso}(T^c) \),

\[
\mathfrak{iso}(T^c) = \mathfrak{g}_2, \ \mathfrak{su}(3), \ \mathfrak{su}(2) \oplus \mathfrak{su}_c(2), \ \mathfrak{u}(2), \ \mathbb{R} \oplus \mathfrak{su}_c(2), \ \mathfrak{so}(3), \ \mathfrak{so}_\text{ir}(3), \ \mathbb{R} \oplus \mathfrak{su}(2).
\]

The condition of (5), \( \mathfrak{su}(2) \not\leq \mathfrak{hol}(\nabla^c) \), is satisfied for \( \mathfrak{hol}(\nabla^c) \leq \mathbb{R} \oplus \mathfrak{su}_c(2), \ \mathfrak{so}(3), \ \mathfrak{so}_\text{ir}(3) \) or \( \mathfrak{t}^2 \). For clarity we define the following forms:

\[
\begin{align*}
Z_1 & := e_{12} + e_{34}, & Z_2 & := e_{56}, & Z_3 & := e_{12} - e_{34}, & D_1 & := e_{246} - e_{145}, \\
D_2 & := -e_{235} - e_{136}, & D_3 & := -e_{135} + e_{245}, & D_4 & := e_{146} + e_{236}, & D_5 & := e_{123} - e_{356},
\end{align*}
\]

so that we have

\[
Z = Z_1 + Z_2, \quad D = D_1 + D_2, \quad \tilde{D} := D_3 + D_4.
\]

5.1. The cases \( \mathfrak{iso}(T^c) = \mathfrak{g}_2, \ \mathfrak{su}(2) \oplus \mathfrak{su}_c(2), \ \mathbb{R} \oplus \mathfrak{su}_c(2), \ \mathfrak{so}_\text{ir}(3) \). The characteristic torsion is an element of the family

\[
T^c = a_1 \cdot (Z \wedge e_7 + D) + b_1 \cdot ((Z_1 - 6 Z_2) \wedge e_7 + D) + b_2 \cdot (Z_3 \wedge e_8),
\]

where \( a_1, b_1, b_2 \in \mathbb{R} \). The constraints on the torsion parameters relative to the considered isotropy algebras are arranged in the following table:

| \mathfrak{iso}(T^c) | \mathfrak{g}_2, \mathfrak{so}_\text{ir}(3) | \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) | \mathbb{R} \oplus \mathfrak{su}_c(2) |
|---|---|---|---|
| constraints | \( b_1 = b_2 = 0 \) | \( b_1 \neq 0, b_2 = 0 \) | \( b_2 \neq 0 \) |
The corresponding Spin (7)-structure is of type \( \mathcal{W}_1 \) or \( \mathcal{W}_2 \) if and only if \( a_1 = 0 \) or \( b_1 = b_2 = 0 \) respectively. The characteristic Ricci tensor \( \text{Ric}^c \) has the shape
\[
\text{Ric}^c = \text{diag} (\lambda, \lambda, \lambda, \lambda, \kappa, \kappa, 0)
\]
depending on the parameters of the torsion form,
\[
\lambda = 3 \ (a_1 + b_1) \ (4 \ a_1 - 3 \ b_1) - b_2^2, \quad \kappa = 4 \ (a_1 + b_1) \ (3 \ a_1 - 4 \ b_1).
\]

We proceed with the holonomy classification. System \((\bullet)\) becomes inconsistent for \( \mathfrak{su} (2) \leq \mathfrak{h} \leq \mathfrak{g} = \mathfrak{g}_2 \). Moreover, we deduce \( \kappa = 0 \) in the case of \( \mathfrak{hol} (\nabla^c) = \mathfrak{u} (2) \) or \( \mathfrak{su} (2) \) with \( \mathfrak{iso} (T^c) = \mathfrak{su} (2) \oplus \mathfrak{su}_c (2) \).

5.1.1. The subcases \( \mathfrak{hol} (\nabla^c) \leq \mathbb{R} \oplus \mathfrak{su}_c (2), \mathfrak{so} (3) \). Applying step (5) we are able to compute the characteristic curvature tensor
\[
R^c = r_1 \cdot (P_7 \otimes P_7) + r_2 \cdot ((P_7 + 2 \ P_8) \otimes (P_7 + 2 \ P_8) + Q_5 \otimes Q_5 + Q_6 \otimes Q_6),
\]
which depends on the torsion parameters in the following way:
\[
r_1 = \frac{3}{8} \ k - \lambda = -\frac{3}{2} \ (a_1 + b_1) \ (5 \ a_1 - 2 \ b_1) + b_2^2, \quad r_2 = -\frac{1}{8} \ k.
\]

We arranged the necessary conditions on the parameters \( r_1 \) and \( r_2 \) for each holonomy algebra \( \mathfrak{hol} (\nabla^c) \leq \mathbb{R} \oplus \mathfrak{su}_c (2) \) or \( \mathfrak{so} (3) \) in the following table:

| \( \mathfrak{hol} (\nabla^c) \) | \( \mathfrak{su}_c (2) \) | \( t^2, t^1 \ [l = 0] \) | \( \mathfrak{so} (3), t^1 \ [l \neq 0], 0 \) |
|---|---|---|---|
| constraints | \( r_1 = 0 \) | \( r_2 = 0 \) | \( r_1 = r_2 = 0 \) |

Here 0 denotes the zero algebra.

5.1.2. The subcase \( \mathfrak{hol} (\nabla^c) = \mathfrak{so}_{ir} (3) \). There exists only one \( \mathfrak{so}_{ir} (3) \)-invariant curvature tensor \( R^c : \Lambda^2 (\mathbb{R}^8) \rightarrow \mathfrak{so}_{ir} (3) \), namely the projection onto the algebra \( \mathfrak{so}_{ir} (3) \),
\[
R^c = -a_1^2 \cdot (U_1 \otimes U_1 + U_2 \otimes U_2 + U_3 \otimes U_3).
\]

Here \( (U_1, U_2, U_3) \) denotes the following basis of \( \mathfrak{so}_{ir} (3) \):
\[
U_1 := \sqrt{5/2} \ P_5 - \sqrt{3/2} \ Q_2, \quad U_2 := \sqrt{5/2} \ P_6 + \sqrt{3/2} \ Q_1, \quad U_3 := P_7 + 3 \ P_8.
\]

5.2. The cases \( \mathfrak{iso} (T^c) = \mathfrak{so} (3), \mathfrak{su} (3) \). There are two admissible families of characteristic torsions. The first one depends on a single positive parameter,
\[
T^c_I = a_1 \cdot Z \wedge e_7, \quad a_1 \in \mathbb{R}, \ a_1 > 0,
\]
whilst the second is a 3-parameter family
\[
T^c_{II} = a_1 \cdot \bar{D} + a_2 \cdot (2 \ D_1 + 5 \ D_2 + 3 \ D_3) + b_1 \cdot (D_1 - D_2 - 2 \ D_3)
\]
with \( a_1, a_2, b_1 \in \mathbb{R}, \ b_1 > 0 \). The isotropy algebra of type I is the algebra \( \mathfrak{su} (3) \), i.e. \( \mathfrak{iso} (T^c_I) = \mathfrak{su} (3) \). The condition \( \mathfrak{iso} (T^c_{II}) = \mathfrak{su} (3) \) holds if and only if \( a_1 = 0, \ b_1 = \frac{3}{2} \ a_2, \)
\[
T^c_{II} \sim D = D_1 + D_2.
\]

The Spin (7)-structures of this subsection are not of type \( \mathcal{W}_2 \). They are of type \( \mathcal{W}_1 \) if only if \( T^c \) is of type \( II \) and \( a_1 = a_2 = 0 \). The characteristic Ricci tensor is
\[
\text{Ric}^c = \text{diag} (\lambda, \lambda, \lambda, \lambda, \lambda, \lambda, 0, 0)
\]
and depends on the torsion type
\[
\lambda_I = 2 \ a_1^2, \quad \lambda_{II} = 4 \ a_1^2 + 4 \ (2 \ a_2 + b_1) \ (5 \ a_2 - b_1)
\]
The basis \((V_1, V_2, V_3)\) of \(\mathfrak{so}(3)\) is
\[
V_1 := \sqrt{1/2} \left( P_1 + P_5 \right), \quad V_2 := \sqrt{1/2} \left( P_2 + P_6 \right), \quad V_3 := P_7 + P_8.
\]
For \(\mathfrak{hol}(\nabla^c) = t^1 < \mathfrak{so}(3)\) and in the case of trivial holonomy, \(\mathfrak{hol}(\nabla^c) = \mathfrak{o}\), the parameter \(\lambda\) has to vanish necessarily.

5.2.2. The subcase \(\mathfrak{hol}(\nabla^c) = t^2\). The characteristic curvature \(R^c : \Lambda^2 (\mathbb{R}^8) \rightarrow \mathfrak{so}(3)\) is
\[
R^c = -\frac{\lambda}{4} \cdot (3 (P_7 \otimes P_7) + (P_7 + 2 P_8) \otimes (P_7 + 2 P_8))
\]
for the parameter \(\lambda\) above.

5.3. The case \(\mathfrak{iso} (T^c) = \mathfrak{u}(2)\). The characteristic torsion is an element of one of the following two 3-parameter families:
\[
T^c_I = a_1 \cdot (Z_1 + 5 Z_2) \wedge e_8 + a_2 \cdot (Z_1 + 5 Z_2) \wedge e_7 + b_1 \cdot (Z_1 - 2 Z_2) \wedge e_7,
\]
\[
T^c_H = a_1 \left( (Z_1 - 2 Z_2) \wedge e_8 + \frac{7}{4} D \right) + a_2 \left( (Z_1 - 2 Z_2) \wedge e_7 + \frac{7}{4} D \right) + b_1 \cdot (Z_1 - 2 Z_2) \wedge e_7.
\]
Here \(a_1, a_2, b_1 \in \mathbb{R}, \ b_1 > 0\). Not all parameter configurations are admissible: the isotropy algebra \(\mathfrak{iso} (T^c_I)\) of type \(I\) contains the algebra \(\mathfrak{su} (2) \oplus \mathfrak{su}_c (2)\) if \(a_1 = 0\) and \(b_1 = -a_2\), and the condition \(\mathfrak{su} (2) \oplus \mathfrak{su}_c (2) \leq \mathfrak{iso} (T^c_H)\) holds if \(a_1 = 0\) and \(b_1 = \frac{3}{4} a_2\). The isotropy algebra is \(\mathfrak{iso} (T^c) = \mathfrak{su} (3)\) if and only if \(a_1 = 0\) and \(b_1 = \frac{3}{4} a_2\) (for type \(I\)) or \(a_1 = 0\) and \(b_1 = -a_2\) (for type \(II\)) respectively. These four cases have to be excluded.

The considered Spin(7)-structures are not of type \(\mathcal{W}_2\). Those of type \(\mathcal{W}_1\) satisfy \(a_1 = a_2 = 0\). In this particular case both torsion families coincide, i.e. \(T^c_I = T^c_H\). The Ricci tensor of \(\nabla^c\) is
\[
Ric^c = \text{diag} (\lambda, \lambda, \lambda, \lambda, \kappa, \kappa, 0, 0).
\]
The constants \(\lambda\) and \(\kappa\) depend on the torsion type,
\[
\lambda_I = 6 a_1^2 + (a_2 + b_1) \cdot (6 a_2 - b_1), \quad \kappa_I = 10 a_1^2 + 2 (a_2 + b_1) \cdot (5 a_2 - 2 b_1),
\]
\[
\lambda_{II} = \frac{45}{4} \left( a_1^2 + a_2^2 \right) - 2 a_2 b_1 - b_1^2, \quad \kappa_{II} = \frac{33}{4} \left( a_1^2 + a_2^2 \right) - 8 a_2 b_1 - 4 b_1^2.
\]

5.3.1. The subcase \(\mathfrak{hol}(\nabla^c) \leq t^2\). Proposition 5.2 allows to compute the curvature tensor of the characteristic connection,
\[
R^c = r_1 \cdot (P_7 \otimes P_7) + r_2 \cdot (P_7 + 2 P_8) \otimes (P_7 + 2 P_8),
\]
where \(r_1\) and \(r_2\) are given in terms of the parameters \(\lambda\) and \(\kappa\) above,
\[
r_1 = \frac{1}{4} \kappa - \lambda, \quad r_2 = -\frac{1}{4} \kappa.
\]
Conditions on these parameters, given a specific holonomy algebra \(\mathfrak{hol}(\nabla^c) \leq t^2\), are the following:
\[
\begin{array}{c|c|c|c|c}
\text{iso}(\nabla^c) & t^1[k = 0] & t^1[l = 0] & t^1[k, l \neq 0], o \\
\text{constraints} & r_1 = 0 & r_2 = 0 & r_1 = r_2 = 0 \\
\end{array}
\]

The condition \(r_1 = r_2 = 0\) can only be realized for \(T^c_1\) with \(a_1 = 0\) and \(b_1 = -a_2\), one of the excluded possibilities. Consequently, there exists no Spin(7)-structure with parallel characteristic torsion, \(\text{iso}(T^c) = u(2)\) and \(\text{hol}(\nabla^c) = t^1[k, l \neq 0]\) or \(o\).

### 5.4. The case \(\text{iso}(T^c) = \mathbb{R} \oplus su(2)\).

Here the characteristic torsion form \(T^c\) is an element of the 1-parameter family

\[
T^c = b_1 \cdot (D_3 - D_4), \quad b_1 \in \mathbb{R}, \quad b_1 > 0,
\]

and Ric\(^c\) is given by

\[
\text{Ric}^c = \text{diag}\left(0, 0, 0, 0, -4b_1^2, -4b_1^2, 0, 0\right).
\]

The corresponding Spin(7)-structure is of type \(\mathcal{W}_1\). The Ricci tensor of a \(\tilde{T}^2\)-invariant and symmetric curvature operator is an element of the 3-parameter family

\[
\text{diag}\left(\alpha + \beta + \gamma, \alpha + \beta + \gamma, \alpha + \beta - \gamma, \alpha + \beta - \gamma, 4\beta, 4\beta, 16\beta, 16\beta\right), \quad \alpha, \beta, \gamma \in \mathbb{R}.
\]

Thus \(\mathbb{R} \oplus su(2)\) is the only admissible (i.e. \(T^c \neq 0\)) characteristic holonomy algebra for \(\text{iso}(T^c) = \mathbb{R} \oplus su(2)\).

### 5.5. The admissible isotropy and holonomy algebras.

Summarizing the previous subsections, the following table provides an overview of the isotropy and holonomy algebras which comply with the requirements of steps (1) to (5) and lead to non-vanishing characteristic torsion:

| iso\((T^c)\) | \(\text{hol}(\nabla^c)\) |
|----------------|------------------|
| \(g_2\) | \(g_2, su(2) \oplus su_c(2)\) |
| \(su(3)\) | \(su(3), u(2)\) |
| \(su(2) \oplus su_c(2)\) | \(R \oplus su_c(2), su_c(2), so_{ir}(3)\) |
| \(u(2)\) | \(su(2)\) |
| \(R \oplus su_c(2)\) | \(R \oplus su_c(2), su_c(2), t^2, t^1, o\) |
| \(so(3)\) | \(so(3), t^2, t^1\) |
| \(so_{ir}(3)\) | \(so_{ir}(3)\) |

### 6. Geometric results

In this section we discuss the geometries related to the algebraic cases of section 5.

The most important tool in these considerations is the splitting theorem of de Rham generalized to geometric structures with totally skew-symmetric torsion (see [6]):

**Theorem 6.1.** Let \((M^n, g, T)\) be a complete, simply connected Riemannian manifold with 3-form \(T\). Suppose the tangent bundle

\[
TM^n = TM_+ \oplus TM_-
\]

splits under the action of the holonomy group of \(\nabla_X Y = \nabla^\n_X Y + \frac{1}{2} \cdot T(X, Y, \cdot)\) so that

\[(*) \quad T(X_+, X_-, \cdot) = 0, \quad T(X_+, Y_+, \cdot) \in TM_+, \quad T(X_-, Y_-, \cdot) \in TM_-\]
for all $X_+, Y_+ \in TM_+$ and $X_-, Y_- \in TM_-$. Let $T = T_+ + T_-$ denote the corresponding decomposition of the 3-form $T$. Then $(M, g, T)$ is isometric to a Riemannian product

$$(M_+, g_+, T_+) \times (M_-, g_-, T_-).$$

The condition $\nabla T = 0$ results in $\nabla^+ T_+ = 0$, $\nabla^- T_- = 0$ for

$$\nabla^+_X Y := \nabla^+_X g + \frac{1}{2} \cdot T_+ (X, Y, \cdot), \quad \nabla^-_X Y := \nabla^-_X g + \frac{1}{2} \cdot T_- (X, Y, \cdot).$$

We split the consideration in the same manner as in section 5 and start with the most obvious cases.

6.1. The case $\frak{iso}(T^c) = \frak{so}_{ir}(3)$. The characteristic holonomy $\frak{hol}(\nabla^c)$ is equal to $\frak{so}_{ir}(3)$, the $\nabla^c$-parallel torsion form $T^c$ is proportional to $(Z \wedge e_7 + D)$ and the tangent bundle of $(M^8, g, \Phi)$ splits into the following $\frak{so}_{ir}(3)$-invariant components:

$$TM^8 = E \oplus \mathbb{R} \cdot e_8.$$ 

There exist two spinor fields which are parallel with respect to $\nabla^c$, namely $\Psi_1 - \Psi_2$ and $\Psi_9 - \Psi_{10}$ (cf. section 4). The curvature tensor $R^c$ is uniquely determined, $\frak{so}_{ir}(3)$-invariant, $\nabla^c$-parallel and $R^c = R^c|_E \oplus 0|_{\mathbb{R} \cdot e_8}$ (see section 5). Since the torsion form $T^c$ does not depend on $e_8$ and $\nabla^c e_8 = 0$, we conclude that $e_8$ is $\nabla^c$-parallel. Consequently, a complete and simply connected $M^8$ is the Riemannian product of a 7-dimensional manifold $Y^7$ with $\mathbb{R}$. We furthermore conclude that the space $Y^7$ is isometric to a naturally reductive, nearly parallel $G_2$-structure with fundamental form $(Z \wedge e_7 + D)$ and characteristic holonomy algebra $\frak{so}_{ir}(3)$ (see [12]). Consider the embedding of $SO(3)$ into $SO(5)$ given by the 5-dimensional irreducible $SO(3)$-representation. This gives rise to the homogeneous naturally reductive space $SO(5)/SO_{ir}(3)$. With [11,14] we obtain that $Y^7$ is isometric to $SO(5)/SO_{ir}(3)$.

Theorem 6.2. A complete, simply connected Spin (7)-manifold with parallel characteristic torsion, $\nabla^c T^c = 0$, and $\frak{iso}(T^c) = \frak{so}_{ir}(3)$ is isometric to the Riemannian product $SO(5)/SO_{ir}(3)$ with $\mathbb{R}$.

6.2. The cases $\frak{iso}(T^c) = \frak{so}(3), \frak{su}(3)$. The Spin (7)-structure $(M^8, g, \Phi)$ admits two $\nabla^c$-parallel vector fields $e_7$, $e_8$ and four $\nabla^c$-parallel spinor fields $\Psi_1, \Psi_2, \Psi_9, \Psi_{10}$. Moreover, the differential forms $Z$ and $D$ are parallel with respect to $\nabla^c$, and we can reconstruct a Spin (7)-structure by using $(\ast)$.

There are two types of characteristic torsion. First we discuss type I: $T^c \sim Z \wedge e_7$ and $\frak{iso}(T^c) = \frak{su}(3)$. The torsion form vanishes along $e_8$, i.e. $e_8 \perp T^c = 0$, and the tangent bundle splits into two $\frak{hol}(\nabla^c)$-invariant components,

$$TM^8 = E \oplus \mathbb{R} \cdot e_8.$$ 

Consequently, a complete, simply connected $M^8$ is isometric to the Riemannian product of a 7-dimensional manifold $Y^7$ with $\mathbb{R}$. The torsion 3-form $T^c$ is contained in the space $\Lambda^3_2(\mathbb{R}^7) \oplus \Lambda^3_2(\mathbb{R}^7)$ of the decomposition of $\Lambda^3(\mathbb{R}^7)$ into irreducible $G_2$-components. Up to isometry the space $Y^7$ admits a cocalibrated $G_2$-structure with fundamental form $(Z \wedge e_7 + D)$ and a characteristic connection with totally skew-symmetric torsion equal to $T^c$. The holonomy algebra of this connection coincides with $\frak{hol}(\nabla^c)$. Consequently, $Y^7$ is homothetic to an $\eta$-Einstein Sasakian 7-manifold with contact vector field $e_7$ and fundamental form $Z$ (see [11]).
**Theorem 6.3.** Let $(M^8, g, \Phi)$ be a complete, simply connected Spin $(7)$-manifold with parallel characteristic torsion $T^c$ and $\mathfrak{iso}(T^c) = \mathfrak{su}(3)$. Suppose that the torsion form is of type I, i.e. $T^c \sim Z \wedge e_7$. Then $M^8$ is isometric to the Riemannian product of a 7-dimensional, simply connected, $\eta$-Einstein

$$\text{Ric}^\theta = 10 \cdot \tilde{g} - 4 \cdot e_7 \otimes e_7$$

Sasakian manifold $(Y^7, \tilde{g}, e_7, Z)$ with $\mathbb{R}$. Conversely, such a product admits a Spin $(7)$-structure with parallel characteristic torsion and $\mathfrak{hol}(\nabla^c)$ contained in $\mathfrak{su}(3)$.

**Remark 6.1.** Simply connected Sasakian manifolds $(M^7, g, \xi, \varphi)$ which admit the Ricci tensor of the last theorem can be constructed via the Tanno deformation of a 7-dimensional Einstein-Sasakian structure $(\tilde{M}^7, \tilde{g}, \tilde{\xi}, \tilde{\varphi})$. This deformation is

$$\varphi := \tilde{\varphi}, \quad \xi := a^2 \cdot \tilde{\xi}, \quad g := a^{-2} \cdot \tilde{g} + (a^{-4} - a^{-2}) \cdot \tilde{\eta} \otimes \tilde{\eta}$$

with the deformation parameter $a^2 = \frac{3}{4}$ (see [15]). We recommend the article [4] for further constructions of Sasakian structures of $\eta$-Einstein type.

**Example 6.1.** The algebraic classification in section 5 proves that for $\mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$ or $\mathfrak{t}^2$ the corresponding Spin $(7)$-structure $M^8$ is a homogeneous naturally reductive space. Since the curvature tensor of the characteristic connection does not depend on $e_8$, we can conclude the same for the Sasakian manifold $Y^7$ and denote $Y^7 = G/H$. In [11] $Y^7$ was identified for characteristic holonomy $\mathfrak{h} = \mathfrak{hol}(\nabla^c) = \mathfrak{so}(3)$. Here the corresponding naturally reductive space is isometric to the Stiefel manifold $Y^7 = \text{SO}(5)/\text{SO}(3)$. We finally discuss the case $\mathfrak{h} = \mathfrak{hol}(\nabla^c) = \mathfrak{t}^2$. The Lie algebra $\mathfrak{g}$ of the 9-dimensional automorphism group $G$ is given by $\mathfrak{g} = \mathfrak{t}^2 \oplus \mathbb{R}^7$ with the bracket

$$[A + X, B + Y] = ([A, B] - R^c(X, Y)) + (A \cdot Y - B \cdot X - T^c(X, Y)).$$

It turns out that the corresponding Killing form is non-degenerate, and thus $\mathfrak{g}$ is semisimple. Consequently, $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(2) \oplus c(\mathfrak{su}(2))$, where $c(\mathfrak{su}(2))$ denotes the centralizer of $\mathfrak{su}(2)$ inside $\mathfrak{spin}(7)$.

We proceed with torsion type II. The torsion form does not depend on $e_7$ and $e_8$, and the tangent bundle splits into the following $\mathfrak{hol}(\nabla^c)$-invariant components:

$$TM^8 = E_1 \oplus E_2.$$ 

Here $E_2$ is spanned by $\{e_7, e_8\}$. The torsion form $T^c$ belongs to the $A_2^6(\mathbb{R}^6) \oplus A_1^7(\mathbb{R}^6)$-component of the decomposition of $A^3(\mathbb{R}^6)$ under the action of $U(3)$ (see [3]). Consequently, if $M^8$ is simply connected and complete, then it is isometric to the Riemannian product of $\mathbb{R}^2$ with an almost Hermitian manifold $X^6$ of Gray-Hervella type $W_1 \oplus W_3$, with Kähler form $Z$ and characteristic holonomy contained in $\mathfrak{iso}(T^c)$. The latter structures have been exhaustively studied in [3, 21].

**Theorem 6.4.** A complete, simply connected Spin $(7)$-manifold with parallel characteristic torsion of type II in the class $\mathfrak{iso}(T^c) = \mathfrak{so}(3)$ or $\mathfrak{su}(3)$ is isometric to the Riemannian product $X^6 \times \mathbb{R}^2$, where $X^6$ is an almost Hermitian manifold of Gray-Hervella type $W_1 \oplus W_3$ with characteristic holonomy contained in $\mathfrak{iso}(T^c)$.

**Remark 6.2.** Consider the special case $\mathfrak{hol}(\nabla^c) = \mathfrak{su}(3) = \mathfrak{iso}(T^c)$ and torsion type II. Here $T^c$ is proportional to $D$ and $X^6$ is isometric to a strictly (i.e. non-Kähler) nearly Kähler manifold. Conversely, any Riemannian product $X^6 \times \mathbb{R}^2$ with $X^6$ a
strictly nearly Kähler manifold admits a Spin(7)-structure with parallel characteristic torsion and characteristic holonomy contained in \( \mathfrak{su}(3) \).

6.3. The cases \( \mathfrak{iso}(T^c) = \mathfrak{g}_2, \mathfrak{su}(2) \oplus \mathfrak{su}_c(2), \mathbb{R} \oplus \mathfrak{su}_c(2) \). The vector field \( e_8 \), the spinor fields \( \Psi_1 - \Psi_2, \Psi_9 - \Psi_{10} \) and the globally defined 3-form \( (Z \wedge e_7 + D) \) are parallel with respect to \( \nabla^c \). The tangent bundle \( TM^8 \) of \( (M^8, g, \Phi) \) splits into two components preserved by \( \nabla^c \),

\[
TM^8 = E \oplus \mathbb{R} \cdot e_8.
\]

The torsion form \( T^c \) satisfies

\[
e_8 \mathcal{J} T^c = b_2 \left( e_{12} - e_{34} \right)
\]

and the real parameter \( b_2 \) vanishes if and only if \( \mathfrak{iso}(T^c) = \mathfrak{g}_2 \) or \( \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \) (see section 5). Consequently, we split the discussion into \( \mathfrak{iso}(T^c) = \mathfrak{g}_2 \) or \( \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \) and \( \mathfrak{iso}(T^c) = \mathbb{R} \oplus \mathfrak{su}_c(2) \).

Suppose \( \mathfrak{iso}(T^c) = \mathfrak{g}_2 \) or \( \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \). Here the torsion form does not depend on \( e_8 \), and \( \nabla^c e_8 = 0 \). Therefore a complete and simply connected \( M^8 \) is isometric to the Riemannian product of a 7-manifold \( Y^7 \) with \( \mathbb{R} \). The 3-form \( T^c \) is contained in the component \( \Lambda^3 \left( \mathbb{R}^7 \right) \oplus \Lambda^3 \left( \mathbb{R}^7 \right) \) of the decomposition of 3-forms on \( \mathbb{R}^7 \) into \( G_2 \)-irreducible components (see [12]). Consequently, the space \( Y^7 \) is isometric to a cocalibrated \( G_2 \)-manifold with fundamental form \( (Z \wedge e_7 + D) \) and parallel characteristic torsion \( T^c \).

The corresponding characteristic holonomy of \( Y^7 \) is the subalgebra \( \mathfrak{hol}(\nabla^c) \) of \( \mathfrak{iso}(T^c) \). Finally, we can reconstruct the considered Spin(7)-structure using equation \((\ast)\).

**Theorem 6.5.** Let \( (M^8, g, \Phi) \) be a complete, simply connected Spin(7)-manifold with parallel characteristic torsion \( T^c \) and \( \mathfrak{iso}(T^c) = \mathfrak{g}_2 \) or \( \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \). Then \( M^8 \) is isometric to the Riemannian product of \( \mathbb{R} \) with a cocalibrated \( G_2 \)-manifold with parallel characteristic torsion and characteristic holonomy contained in \( \mathfrak{iso}(T^c) \).

Those \( Y^7 \) with non-Abelian characteristic holonomy were extensively studied in [11]. We provide an example for \( Y^7 \) with Abelian characteristic holonomy. This restricts necessarily to \( \mathfrak{iso}(T^c) = \mathfrak{su}(2) \oplus \mathfrak{su}_c(2) \).

**Example 6.2.** If \( \mathfrak{hol}(\nabla^c) = 0 \) the torsion parameters satisfy \( b_2 = 0 \) and \( b_1 = -a_1 \). Moreover, the spinor fields \( \Psi_1, \ldots, \Psi_{16} \) are \( \nabla^c \)-parallel, the characteristic curvature tensor vanishes \( R^c = 0 \) and the torsion form \( T^c \) is proportional to \( Z_2 \wedge e_7 = e_{567} \). In particular, the latter implies that \( T^c \) does not depend on the \( \nabla^c \)-parallel vector fields \( e_1, e_2, e_3, e_4 \) and \( e_8 \). Computing the Lie bracket \([\cdot, \cdot]\) of the Lie algebra corresponding to \( M^8 \) via \( T^c(X, Y, Z) = -g([X, Y], Z) \) results in the conclusion that a complete and simply connected \( M^8 \) is isometric to the Riemannian product \( \mathbb{R}^5 \times SU(2) \).

We proceed with \( \mathfrak{iso}(T^c) = \mathbb{R} \oplus \mathfrak{su}_c(2) \). With the algebraic considerations of section 5 we immediately obtain the following result:

**Proposition 6.1.** Any Spin(7)-manifold with parallel characteristic torsion \( T^c \) and \( \mathfrak{iso}(T^c) = \mathbb{R} \oplus \mathfrak{su}_c(2) \) is isometric to a homogeneous naturally reductive space.

**Example 6.3.** Let \( \mathfrak{hol}(\nabla^c) = 0 \). Then the spinor fields \( \Psi_1, \ldots, \Psi_{16} \) are \( \nabla^c \)-parallel, \( R^c = 0 \) and the characteristic torsion is proportional to one of the following two 3-forms:

\[
\alpha_\pm = (Z_1 - 2 Z_2) \wedge e_7 + D \pm \sqrt{3} \cdot (Z_3 \wedge e_8).
\]
Each of these 3-forms can be reconstructed with 
\[ \alpha_\pm = -g([X,Y], Z) \]
using the bracket \([\cdot, \cdot]\) of the respective Lie algebra
\[ su(3) = \text{span}(P_4, P_3, P_1, P_2, -P_5, -P_6, P_7, \pm \sqrt{1/3} (P_7 + 2P_8)) . \]
We conclude that the considered Spin(7)-manifold \(M^8\) is isometric to SU(3).

6.4. The case \( iso(T^c) = \mathbb{R} \oplus su(2) \). The characteristic holonomy \( hol(\nabla^c) \) is necessarily equal to \( \mathbb{R} \oplus su(2) \). A Spin(7)-structure \((M^8, g, \Phi)\) with non-trivial parallel characteristic torsion and \( hol(\nabla^c) = \mathbb{R} \oplus su(2) = iso(T^c) \) admits two \( \nabla^c\)-parallel 2-forms \( Z_1 = e_{12} + e_{34} \) and \( Z_2 = e_{56} \), two \( \nabla^c\)-parallel spinor fields \( \Psi_9 \) and \( \Psi_{10} \), and the tangent bundle \( TM^8 \) splits into the sum of two \( \mathbb{R} \oplus su(2)\)-invariant subbundles,
\[ TM^8 = E_1 \oplus E_2 . \]

Here \( E_2 \) is spanned by \( \{e_7, e_8\} \). The torsion form \( T^c \) does not depend on \( e_7 \) and \( e_8 \), and therefore a complete, simply connected \( M^8 \) is isometric to the Riemannian product of a 2-dimensional manifold with a 6-dimensional manifold \( X^6 \). The space \( X^6 \) is isometric to an almost Hermitian manifold with Kähler form \( Z = Z_1 + Z_2 \) and non-trivial parallel characteristic torsion \( T^c \). This torsion form is contained in the \( \Lambda^3_1(\mathbb{R}^6)\)-component in the decomposition of \( \Lambda^3(\mathbb{R}^6) \) under the action of \( U(3) \) (see [3]). Analyzing the representation of \( Hol(\nabla^c) \subset U(1) \times U(2) \subset U(3) \) on \( \mathbb{R}^6 \cong E_1 \), we conclude with [14] that \( X^6 \) carries the structure of a twistor space and is homothetic to either \( \mathbb{CP}^3 \) or \( \mathbb{P}(1,2) \). The representation of \( Hol(\nabla^c) \) on \( \mathbb{R}^2 \cong E_2 \) defines a non-trivial rotation. The 2-dimensional component is consequently isometric to \( S^2 \). We can reconstruct the considered Spin(7)-structure from the Hermitian structure of its components,
\[ \Phi = \frac{1}{2} \cdot \omega \wedge \omega + \text{Re}(F) , \]
where
\[ \omega := Z + e_7 \wedge e_8 , \quad F := (e_1 + i e_2) \wedge (e_3 + i e_4) \wedge (e_5 + i e_6) \wedge (e_7 + i e_8) . \]

Finally we obtain the following result.

**Theorem 6.6.** A complete, simply connected Spin(7)-manifold with parallel characteristic torsion \( T^c \) and \( iso(T^c) = \mathbb{R} \oplus su(2) \) is isometric to the Riemannian product of a 2-sphere with either the projective space \( \mathbb{CP}^3 \) or the flag manifold \( \mathbb{F}(1,2) \), both equipped with their standard nearly Kähler structure from the twistor construction.

6.5. The case \( iso(T^c) = u(2) \). The differential forms \( Z_1, Z_2, D \), the vector fields \( e_7, e_8 \) and the spinor fields \( \Psi_1, \Psi_2, \Psi_9, \Psi_{10} \) are parallel with respect to \( \nabla^c \) and the tangent bundle of \((M^8, g, \Phi)\) splits into the following \( u(2)\)-invariant components:
\[ TM^8 = E_1 \oplus E_2 \oplus \mathbb{R} \cdot e_7 \oplus \mathbb{R} \cdot e_8 . \]

The subbundle \( E_2 \) is spanned by \( e_5 \) and \( e_6 \).

There are two types of characteristic torsion. We start to discuss the case of torsion type \( I \). Setting
\[ TM_+ = E_1 \oplus ((a_2 + b_1) \cdot e_7 + a_1 \cdot e_8) , \quad TM_- = E_2 \oplus \left( \left( a_2 - \frac{2}{3} b_1 \right) \cdot e_7 + a_1 \cdot e_8 \right) \]
and $T = T_+ + T_- = T^c$ satisfies system (*) and

$$T_+ = Z_1 \wedge ((a_2 + b_1) \cdot e_7 + a_1 \cdot e_8), \quad T_- = 5 \cdot Z_2 \wedge \left( \left( a_2 - \frac{2}{5} b_1 \right) \cdot e_7 + a_1 \cdot e_8 \right).$$

The equation $e_7 \wedge T^c = 0$ or $e_8 \wedge T^c = 0$ holds, if $a_2 = b_1 = 0$ or $a_1 = 0$ respectively. Consequently, a simply connected and complete $M^8$ is isometric to the Riemannian product of a 5-manifold with a 3-manifold each carrying a Sasakian structure. The respective fundamental forms are $Z_1$ and $Z_2$.

**Theorem 6.7.** Let $(M^8, g, \Phi)$ be a complete, simply connected Spin (7)-manifold with parallel characteristic torsion $T^c$ and $\text{iso}(T^c) = u(2)$. Suppose that the torsion form is of type I. Then $M^8$ is isometric to the Riemannian product of a Sasakian 3-manifold with a 5-dimensional Sasakian structure.

We proceed with the discussion of torsion type II. Solving the equation $X \wedge T^c = 0$ leads to $X = a_1 \cdot e_7 - (a_2 + b_1) \cdot e_8 \neq 0$ (see section 5). Consequently, a complete, simply connected $M^8$ is isometric to the Riemannian product of a 7-dimensional integrable $G_2$-manifold $Y^7$ with $\mathbb{R}$ (see [13]).

**Theorem 6.8.** Let $(M^8, g, \Phi)$ be a complete, simply connected Spin (7)-manifold with parallel characteristic torsion $T^c$ and $\text{iso}(T^c) = u(2)$. Suppose that the torsion form is of type II. Then $M^8$ is isometric to the Riemannian product of $\mathbb{R}$ with an integrable $G_2$-manifold with parallel characteristic torsion and characteristic holonomy contained in $u(2)$.

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