EXTENSION OF GENERALIZED SOLIDARITY VALUES TO INTERVAL-VALUED COOPERATIVE GAMES

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ABSTRACT. The main purpose of this paper is to extend the concept of generalized solidarity values to interval-valued cooperative games and hereby develop a simplified and fast approach for solving a subclass of interval-valued cooperative games. In this paper, we find some weaker coalition monotonicity-like conditions so that the generalized solidarity values of the $\alpha$-cooperative games associated with interval-valued cooperative games are always monotonic and non-decreasing functions of any parameter $\alpha \in [0,1]$. Thereby the interval-valued generalized solidarity values can be directly and explicitly obtained by computing their lower and upper bounds through only using the lower and upper bounds of the interval-valued coalitions’ values, respectively. The developed method does not use the interval subtraction and hereby can effectively avoid the issues resulted from it. Furthermore, we discuss the effect of the parameter $\xi$ on the interval-valued generalized solidarity values of interval-valued cooperative games and some significant properties of interval-valued generalized solidarity values.

1. Introduction. Cooperative game theory has been extensively studied and successfully applied to some fields such as management, economics, business, insurance, and finance as well as resource allocation. Kirlar et al. [16] achieved the synergy between cryptographic solutions and the cooperative game theory in financial problems of cloud-computing application areas. Crisp cooperative games use real numbers to express the values (or worth) of coalitions. However, in some real situations, coalitions’ values have to be estimated due to uncertainty and complexity. This paper focuses on interval-valued cooperative games which were first introduced by Branzei et al. [6]. Compared with cooperative games, interval-valued cooperative games utilize intervals instead of real numbers to express the coalitions’ values. Obviously, interval-valued cooperative games are a generalization of cooperative games. In recent years, solution concepts and related properties of interval-valued cooperative games can be found in many literatures. Branzei et al. [5] updated the results about interval-valued cooperative games and reviewed

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various existing and potential applications of interval-valued cooperative games in management situations. Palanci et al. [25] investigated the grey Shapley value of a new class of cooperative games where the coalition values are interval grey numbers and characterized the grey Shapley value with the additivity, efficiency, symmetry and dummy player properties. Han et al. [11] proposed the interval-valued core and the interval-valued Shapley-like value of interval-valued cooperative games by defining a new interval order relation and using the Moore’s interval subtraction ([20]). Guan et al. [10] defined a new interval ranking method and hereby investigated the existence of the interval-valued core and discussed the relations between the interval-valued core and the interval-valued dominance core. Based on a partial subtraction operator, Palanci et al. [27] discussed the interval-valued Shapley value and its properties and also introduced the interval-valued Banzhaf value and the interval-valued egalitarian rule. Palanci et al. [26] studied the interval-valued Shapley value and interval-valued core of a transportation interval-valued cooperative game based on the partial subtraction operator. Alparslan Gök [1] used the efficiency, symmetry, and strong monotonicity properties to characterize the interval-valued Shapley value. Alparslan Gök et al. [2] introduced some set-valued solution concepts of interval-valued cooperative games such as the interval-valued core, the interval-valued dominance core, and the interval-valued stable sets. Meng et al. [19] investigated the interval-valued Shapley value of interval-valued fuzzy games based on the extended Hukuhara difference. Hong and Li [12] developed a nonlinear programming method for computing the interval-valued cores of interval-valued cooperative games. However, most of the aforementioned works except from Hong and Li [12] used the partial subtraction operator or the Moore’s interval subtraction ([20]) which is not invertible and usually enlarges uncertainty of the resulted intervals. Therefore, the main goal of this paper is to solve interval-valued cooperative games without using the interval subtraction or the ranking of intervals.

Nowak and Radzik [21] proposed the solidarity value of a cooperative game. Later on, Kamijo and Kongo [14] showed that both the egalitarian value and the solidarity value satisfy the balanced cycle contributions property and presented general axiomatizations of the above two values. Xu et al. [32] introduced an A-potential function and deduced the recursive formula of the Solidarity value. They also characterized the Solidarity value by the quasi-balanced contributions property. Casajus and Huettner [9] proposed the generalized solidarity value of a cooperative game. Calvo and Gutiérrez-López [8] characterized the family of weighted solidarity values by two axiomatizations. Béal et al. [3] introduced a new class of solidarity values combining marginalistic and egalitarian principles. Hu and Li [13] studied the Shapley-solidarity values of cooperative games and gave a new axiomatization of the Shapley-solidarity value.

As far as we know, however, there is no research on interval-valued generalized solidarity values of interval-valued cooperative games. Hence, the primary goal of this paper is to extend the concept of generalized solidarity values to interval-valued cooperative games and develop a simplified method for computing the interval-valued generalized solidarity values of a special class of interval-valued cooperative games. In this method, through adding some conditions such as the size (or coalition) monotonicity-like, we prove that the generalized solidarity values of the cooperative games associated with interval-valued cooperative games are monotonic and non-decreasing functions of any parameter $\alpha \in [0, 1]$. Hereby, the interval-valued generalized solidarity value of the interval-valued cooperative game under
some conditions can be easily attained through determining its lower and upper bounds by utilizing the lower and upper bounds of the interval-valued coalitions’ values, respectively. Moreover, we prove that the derived interval-valued generalized solidarity values of interval-valued cooperative games possess some useful and important properties.

The rest of this paper is organized as follows. Section 2 briefly reviews some notations and concepts of intervals and interval-valued cooperative games. Section 3 introduces the interval-valued generalized solidarity values of interval-valued cooperative games and proposes a fast and simplified method for computing the interval-valued generalized solidarity values for a subclass of interval-valued cooperative games under the defined size monotonicity-like. Some important properties of interval-valued generalized solidarity values are discussed in detail. Section 4 uses a real example to show the applicability and validity of the developed method. Short conclusion ends this paper in Section 5.

2. Notations and concepts of interval and interval-valued cooperative games. To facilitate the sequent discussion, we briefly introduce some notations and concepts of intervals and interval-valued cooperative games. Firstly, an interval is denoted by \( \bar{a} = [a_L, a_R] = \{a \mid a \in \mathbb{R}, a_L \leq a \leq a_R\} \), where \( \mathbb{R} \) is the set of real numbers, \( a_L \in \mathbb{R} \) and \( a_R \in \mathbb{R} \) are the lower bound and the upper bound of the interval \( \bar{a} \), respectively. Let \( \bar{R} \) be the set of intervals on the set \( \mathbb{R} \). Moore [20] gave the following interval arithmetic operations and the order relation between intervals.

**Definition 2.1.** Let \( \bar{a} = [a_L, a_R] \) and \( \bar{b} = [b_L, b_R] \) be two intervals. Then, (1) \( \bar{a} = \bar{b} \) if and only if \( a_L = b_L \) and \( a_R = b_R \); (2) \( \bar{a} + \bar{b} = [a_L + b_L, a_R + b_R] \); (3) \( \gamma \bar{a} = [\gamma a_L, \gamma a_R] \) for any real number \( \gamma \geq 0 \); (4) \( \bar{a} \leq \bar{b} \) if and only if \( a_L \leq b_L \) and \( a_R \leq b_R \).

Secondly, an \( n \)-person interval-valued cooperative game means an ordered-pair \( <N, \bar{v}> \), where \( N = \{1, 2, \cdots, n\} \) is the set of players and \( \bar{v} : 2^N \rightarrow \bar{R} \) is the interval-valued characteristic function of players’ coalitions, with \( \bar{v}(\emptyset) = [0, 0] \). According to the above internal notation, \( \bar{v}(S) \) is the worth (or value) of the coalition \( S(S \subseteq N) \) and is denoted by the internal \( \bar{v}(S) = [v_L(S), v_R(S)] \), where \( v_L(S) \leq v_R(S) \). We usually write \( \bar{v}(S \setminus \{i\}) \), \( \bar{v}(S \cup \{i\}) \), \( \bar{v}(i) \), and \( \bar{v}(i,j) \) instead of \( \bar{v}(S \setminus \{i\}) \), \( \bar{v}(S \cup \{i\}) \), \( \bar{v}(\{i\}) \), and \( \bar{v}((i,j)) \), respectively. In the sequent, an \( n \)-person interval-valued cooperative game \( <N, \bar{v}> \) is often called the interval-valued cooperative game \( \bar{v} \) for short. Let \( \bar{G}^n \) denote the set of \( n \)-person interval-valued cooperative games.

For any interval-valued cooperative games \( \bar{v} \in \bar{G}^n \) and \( \bar{\nu} \in \bar{G}^n \), according to (2) of Definition 2.1, \( \bar{v} + \bar{\nu} \) is defined as an interval-valued cooperative game with the interval-valued characteristic function \( \bar{v} + \bar{\nu} \), where

\[
(\bar{v} + \bar{\nu})(S) = \bar{v}(S) + \bar{\nu}(S) = [v_L(S) + \nu_L(S), v_R(S) + \nu_R(S)]
\]

for any coalition \( S \subseteq N \). Clearly, \( \bar{v} + \bar{\nu} \) is also an interval-valued cooperative game belonging to \( \bar{G}^n \).

For any interval-valued cooperative game \( \bar{v} \in \bar{G}^n \), it is easy to see that each player should receive an interval-valued payoff from the cooperation due to the fact that each coalition’s value is an interval. Let \( \bar{x}(\bar{v}) = [x_{L}(\bar{v}), x_{R}(\bar{v})] \) be the interval-valued payoff which is allocated to the player \( i \in N \) under the cooperation that the grand coalition is reached. Denote \( \bar{x}(\bar{v}) = (\bar{x}_1(\bar{v}), \bar{x}_2(\bar{v}), \cdots, \bar{x}_n(\bar{v}))^T \), which is the vector of the interval-valued payoffs for all \( n \) players in the grand coalition \( N \). If
Particularly, if the parameter $\xi_\alpha$ solidar

\[ \sum_{i=1}^{n} x_i(\bar{v}) = \bar{v}(N) \text{ and } x_i(\bar{v}) \geq \bar{v}(i) \text{ (} i = 1, 2, \cdots, n, \text{) then the interval-valued payoff vector } \bar{x}(\bar{v}) \text{ satisfies the efficiency and individual rationality.} \]

**Definition 2.2.** Players $i \in N$ and $k \in N$ ($i \neq k$) are said to be symmetric in the interval-valued cooperative game $\bar{v} \in \bar{G}^n$ if $\bar{v}(S \cup i) = \bar{v}(S \cup k)$ for any coalition $S \subseteq N \setminus \{i, k\}$.

Assume that $\sigma$ is any permutation on the player set $N$, we can define a new interval-valued cooperative game $\bar{v}^\sigma \in \bar{G}^n$ of the interval-valued cooperative game $\bar{v} \in \bar{G}^n$, where the interval-valued characteristic function is $\bar{v}^\sigma(S) = \bar{v}(\sigma^{-1}(S))$ for any coalition $S \subseteq N$.

3. A simplified method of computing interval-valued generalized solidarity value. Let us consider any interval-valued cooperative game $\bar{v} \in \bar{G}^n$. For any $\alpha \in [0, 1]$, we can define an $\alpha$-cooperative game $v(\alpha) \in G^n$ associated with the interval-valued cooperative game $\bar{v}$, where the set of players still is $N = \{1, 2, \cdots, n\}$ and the characteristic function $v(\alpha)$ of players’ coalitions is defined as follows:

\[ v(\alpha)(S) = (1 - \alpha)v_L(S) + \alpha v_R(S) \quad (S \subseteq N) \]

and $v(\alpha)(\emptyset) = 0$. The parameter $\alpha \in [0, 1]$ may be interpreted as an attitude factor of players.

In a parallel way to the definition of Casajus and Huettner [9], we easily define the generalized solidarity value of the $\alpha$-cooperative game $v(\alpha)$ as follows:

\[ \rho_{GSV}(v(\alpha)) = (\rho_1^{GSV}(v(\alpha)), \rho_2^{GSV}(v(\alpha)), \cdots, \rho_n^{GSV}(v(\alpha)))^T, \]

whose components are given as follows:

\[ \rho_i^{GSV}(v(\alpha)) = \xi_n \frac{v(\alpha)(N)}{n} + \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [(1 - \xi_{s+1})v(\alpha)(S \cup i) - (1 - \xi_s)v(\alpha)(S)] \quad (i = 1, 2, \cdots, n), \]

where $s = |S|$ and

\[ \xi_s = \frac{s \xi}{(s-1)\xi + 1} \quad (s = 0, 1, 2, \cdots, n). \]

Particularly, if the parameter $\xi \in [0, 1]$, then $\rho^{GSV}(v(\alpha))$ is called the generalized solidarity value of the $\alpha$-cooperative game $v(\alpha)$ associated with the interval-valued cooperative game $\bar{v}$. It is easy to see from Eq.(3) that $\xi_s$ is a monotonic and non-decreasing function of the variable $s$. Namely, $\xi_{s+1} \geq \xi_s$ for $s = 0, 1, 2, \cdots, n$.

In the following, to facilitate finding the condition of the monotonicity of the generalized solidarity value of a cooperative game, we need to give an alternative expression of Eq.(2). Firstly we have

\[ - \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} [(1 - \xi_{s+1})v(\alpha)(N) - (1 - \xi_s)v(\alpha)(N)] \]

\[ = - \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (\xi_s - \xi_{s+1})v(\alpha)(N) \]
Hence, Eq. (2) can be rewritten as follows:
\[
v(\alpha)(N) \sum_{k=0}^{n-1} \frac{k!(n-k-1)!}{n!} (\xi_{k+1} - \xi_k)
\]
\[
v(\alpha)(N) \sum_{k=0}^{n-1} \frac{k!(n-k-1)!}{n!} (\xi_{k+1} - \xi_k) C^k_{n-1}
\]
\[
v(\alpha)(N) \sum_{k=0}^{n-1} \frac{k!(n-k-1)!}{n!} \times \frac{(n-1)!}{k!(n-k)!} (\xi_{k+1} - \xi_k)
\]
\[
v(\alpha)(N) \sum_{k=0}^{n-1} \frac{1}{n} (\xi_{k+1} - \xi_k)
\]
\[
v(\alpha)(N) \frac{1}{n} (\xi_n - \xi_0),
\]
\[\text{i.e.,}\]
\[- \sum_{S \subseteq N \setminus i} \frac{s!}{n!} [(1 - \xi_{s+1})v(\alpha)(N) - (1 - \xi_s)v(\alpha)(N)]
\]
\[= \frac{(\xi_n - \xi_0)}{n} v(\alpha)(N).\]

Combining with \(\xi_0 = 0\) which is directly derived from Eq.(3), we easily obtain
\[- \sum_{S \subseteq N \setminus i} \frac{s!}{n!} [(1 - \xi_{s+1})v(\alpha)(N) - (1 - \xi_s)v(\alpha)(N)] = \xi_n \frac{v(\alpha)(N)}{n}
\]
Hence, Eq.(2) can be rewritten as follows:
\[
\rho_i^{\text{GSV}}(v(\alpha)) = \sum_{S \subseteq N \setminus i} \frac{s!}{n!} [(1 - \xi_{s+1})(v(\alpha)(S \cup i) - v(\alpha)(N)) - (1 - \xi_s)(v(\alpha)(S) - v(\alpha)(N))]\quad (i = 1, 2, \ldots, n).
\]

Clearly, the generalized solidarity value \(\rho_i^{\text{GSV}}(v(\alpha))\) \((i = 1, 2, \ldots, n)\) of the \(\alpha\)-cooperative game \(v(\alpha)\) is a continuous function of the parameter \(\alpha \in [0, 1]\).

**Lemma 3.1.** For a given parameter \(\xi \in [0, 1]\), if any interval-valued cooperative game \(\bar{v} \in \hat{G}^n\) satisfies the following system of inequalities
\[
(1 - \xi_{s+1})(v_R(S \cup i) - v_L(S \cup i)) + (\xi_{s+1} - \xi_s)(v_R(N) - v_L(N)) \geq (1 - \xi_s)(v_R(S) - v_L(S)) \quad (S \subseteq N \setminus i; i = 1, 2, \ldots, n),
\]
then the generalized solidarity value \(\rho_i^{\text{GSV}}(v(\alpha))\) \((i = 1, 2, \ldots, n)\) of the \(\alpha\)-cooperative game \(v(\alpha) \in G^n\) is a monotonic and non-decreasing function of the parameter \(\alpha \in [0, 1]\).

**Proof of Lemma 3.1.** For any \(\alpha \in [0, 1]\) and \(\alpha' \in [0, 1]\), it is derived from Eqs.(1) and (4) that
\[
\rho_i^{\text{GSV}}(v(\alpha)) - \rho_i^{\text{GSV}}(v(\alpha')) = (\alpha - \alpha') \sum_{S \subseteq N \setminus i} \frac{s!}{n!} \left\{ (1 - \xi_{s+1})[(v_R(S \cup i) - v_L(S \cup i)) - (v_R(N) - v_L(N))] - (1 - \xi_s)[(v_R(S) - v_L(S)) - (v_R(N) - v_L(N))] \right\},
\]
where \(i = 1, 2, \cdots, n\). If \(\alpha \geq \alpha'\), then it easily follows from Eq. (5) that

\[
\rho_i^{GSV}(v(\alpha)) - \rho_i^ {GSV}(v(\alpha')) \geq 0 \quad (i = 1, 2, \cdots, n),
\]

which directly infers that

\[
\rho_i^ {GSV}(v(\alpha)) \geq \rho_i^ {GSV}(v(\alpha')) \quad (i = 1, 2, \cdots, n).
\]

Accordingly, the generalized solidarity value \(\rho_i^{GSV}(v(\alpha)) \quad (i = 1, 2, \cdots, n)\) is a monotonic and non-decreasing function of the parameter \(\alpha \in [0, 1]\). Lemma 3.1 has been proven. \(\square\)

Thus, according to Lemma 3.1, if any interval-valued cooperative game \(\bar{v} \in \bar{G}^n\) satisfies Eq. (5), then the lower and upper bounds of the interval-valued generalized solidarity value \(\bar{\rho}^{GSV}(\bar{v}) = (\rho_1^{GSV}(\bar{v}), \rho_2^{GSV}(\bar{v}), \cdots, \rho_n^{GSV}(\bar{v}))^T\) can be attained at the lower and upper bounds of the interval \([0, 1]\), respectively, i.e., \(\rho_i^{GSV}(\bar{v}) = \left[\rho_i^{GSV}(v(0)), \rho_i^{GSV}(v(1))\right]\). Thus, according to Eq. (2) or Eq. (4), we can directly and explicitly define the interval-valued generalized solidarity value \(\bar{\rho}^{GSV}(\bar{v})\), whose components are given as follows:

\[
\begin{align*}
\rho_i^{GSV}(\bar{v}) = & \left[\xi_n \frac{v_L(N)}{n} + \sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}[(1 - \xi_{s+1})v_L(S \cup i) - (1 - \xi_s)v_L(S)],
\end{align*}
\]

\[
\begin{align*}
\xi_n \frac{v_R(N)}{n} + \sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}[(1 - \xi_{s+1})v_R(S \cup i) - (1 - \xi_s)v_R(S)]
\end{align*}
\]

\((i = 1, 2, \cdots, n), \quad (6)\)

or

\[
\begin{align*}
\rho_i^{GSV}(\bar{v}) = & \left[\sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}[(1 - \xi_{s+1})(v_R(S \cup i) - v_R(N)) - (1 - \xi_s)(v_R(S) - v_R(N))],
\end{align*}
\]

\[
\sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}[(1 - \xi_{s+1})(v_L(S \cup i) - v_L(N)) - (1 - \xi_s)(v_L(S) - v_L(N))]
\]

\((i = 1, 2, \cdots, n). \quad (7)\)

Obviously, if \(\xi = 0\), then the interval-valued generalized solidarity value \(\rho_i^{GSV}(\bar{v})\) is reduced to the interval-valued Shapley value \(\bar{\phi}_i^{SH}(\bar{v})\), i.e.,

\[
\bar{\rho}_i^{GSV, 0}(\bar{v}) = \left[\sum_{S \subseteq N \backslash i} \frac{s!(n-s-1)!}{n!}(v_L(S \cup i) - v_L(S)),
\right]
\]
\[
\sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} (v_R(S \cup i) - v_R(S)) = \tilde{\sigma}^{SH}(\bar{v}).
\]

Similarly, if \( \xi = 1 \), then the interval-valued generalized solidarity value \( \tilde{\rho}_i^{GSV \xi}(\bar{v}) \) is reduced to the interval-valued equal division value \( \tilde{\rho}_i^{ED}(\bar{v}) \), i.e.,

\[
\tilde{\rho}_i^{GSV,1}(\bar{v}) = \left[ \sum_{S \subseteq N : i \in S} \frac{(s-1)!}{n!s} \sum_{j \in S} (v_L(S) - v_L(S \setminus j)), \right. \\
\left. \sum_{S \subseteq N : i \in S} \frac{(s-1)!}{n!s} \sum_{j \in S} (v_R(S) - v_R(S \setminus j)) \right] = \tilde{\rho}_i^{SV}(\bar{v}).
\]

If \( \xi = 1/2 \), then the interval-valued generalized solidarity value \( \tilde{\rho}_i^{GSV \xi}(\bar{v}) \) is reduced to the interval-valued solidarity value \( \tilde{\rho}_i^{SV}(\bar{v}) \), i.e.,

\[
\tilde{\rho}_i^{GSV,0.5}(\bar{v}) = \left[ \sum_{S \subseteq N : i \in S} \frac{(s-1)!}{n!s} \sum_{j \in S} (v_L(S) - v_L(S \setminus j)), \right. \\
\left. \sum_{S \subseteq N : i \in S} \frac{(s-1)!}{n!s} \sum_{j \in S} (v_R(S) - v_R(S \setminus j)) \right] = \tilde{\rho}_i^{SV}(\bar{v}).
\]

Eq.(5) is one of the main contributions in this paper, which can be called as the size (or coalition) monotonicity-like condition due to the fact that the condition is related to the lengths of interval-valued values of players’ coalitions.

Furthermore, for a given parameter \( \xi \in [0, 1] \), if any interval-valued cooperative game \( \bar{v} \in \bar{G}^n \) satisfies the size monotonicity-like, i.e., Eq.(5), then its interval-valued generalized solidarity value \( \tilde{\rho}^{GSV \xi}(\bar{v}) \) possesses some useful and important properties, which are summarized as in the following Theorem 3.2.

**Theorem 3.2.** For a given parameter \( \xi \in [0, 1] \), if any interval-valued cooperative game \( \bar{v} \in \bar{G}^n \) satisfies Eq.(5), then \( \bar{v} \) always has a unique interval-valued generalized solidarity value \( \tilde{\rho}^{GSV \xi}(\bar{v}) \), which possesses the efficiency, additivity, symmetry, and anonymity.

**Proof of Theorem 3.2.** (1) Existence and uniqueness. It is obvious from the above discussion that \( \tilde{\rho}^{GSV \xi}(\bar{v}) \) given by Eq.(6) (or Eq.(7)) is the interval-valued generalized solidarity value of the interval-valued cooperative game \( \bar{v} \in \bar{G}^n \) which satisfies Eq.(5). Moreover, the interval-valued generalized solidarity value \( \tilde{\rho}^{GSV \xi}(\bar{v}) \) is unique due to the size monotonicity-like and continuous. Therefore, any interval-valued cooperative game \( \bar{v} \in \bar{G}^n \) which satisfies Eq.(5) always has a unique interval-valued generalized solidarity value \( \tilde{\rho}^{GSV \xi}(\bar{v}) \), which is given by Eq.(6) (or Eq.(7)).

(2) Efficiency. It follows from Eq.(6) that

\[
\sum_{i \in N} \tilde{\rho}_i^{GSV \xi}(\bar{v}) = \sum_{i \in N} \left\{ \frac{v_L(N)}{n} + \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} [(1 - \xi_{s+1})v_L(S \cup i) - (1 - \xi_s)v_L(S)] \right\}
\]

\[
= \sum_{i \in N} \xi_n \frac{v_L(N)}{n} + \sum_{i \in N} \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} [(1 - \xi_{s+1})v_L(S \cup i) - (1 - \xi_s)v_L(S)]
\]
Similarly, we can prove that \( \sum_{s \subseteq N : s \neq \emptyset} \rho_{s} \) satisfies Eq.(5) always possesses the additivity. Hence, according to Definition 2.1, we obtain \( \rho_{\bar{G}S} \). Namely, the interval-valued generalized solidarity value \( \rho_{\bar{G}S} \) of any interval-valued cooperative game \( \bar{v} \in \bar{G}^{n} \) which satisfies Eq.(5) always possesses the efficiency.

(3) Additivity. If \( \bar{v} \in G^{n} \) and \( \bar{v} \in G^{n} \) satisfy Eq.(5), then it follows from Eq.(6) that

\[
\rho_{\bar{G}S}(\bar{v} + \bar{v}) = \xi_{n}(v_{L} + v_{L})(N) + \sum_{s \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!}[(1 - \xi_{s+1})(v_{L} + v_{L})(S \cup i) - (1 - \xi_{s})(v_{L} + v_{L})] \]

\[
= \xi_{n}(v_{L} + v_{L})(N) + \sum_{s \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!}[(1 - \xi_{s+1})v_{L}(S \cup i) - (1 - \xi_{s})v_{L}(S)] \]

\[
= \rho_{\bar{G}S}(\bar{v}) + \rho_{\bar{G}S}(\bar{v}) \quad (i = 1, 2, \ldots, n).
\]

Similarly, we can prove that \( \rho_{\bar{G}S}(\bar{v} + \bar{v}) = \rho_{\bar{G}S}(\bar{v}) + \rho_{\bar{G}S}(\bar{v}) \) (\( i = 1, 2, \ldots, n \)). Hence, we have \( \rho_{i} \bar{G}S(\bar{v} + \bar{v}) = \rho_{i} \bar{G}S(\bar{v}) + \rho_{i} \bar{G}S(\bar{v}) \) (\( i = 1, 2, \ldots, n \)). Thus, we obtain

\[
\rho_{\bar{G}S}(\bar{v} + \bar{v}) = \rho_{\bar{G}S}(\bar{v}) + \rho_{\bar{G}S}(\bar{v}),
\]

i.e., the interval-valued generalized solidarity value \( \rho_{\bar{G}S} \) of any interval-valued cooperative game \( \bar{v} \in \bar{G}^{n} \) which satisfies Eq.(5) always possesses the additivity.

(4) Symmetry. If the players \( i \in N \) and \( k \in N (i \neq k) \) are symmetric in the interval-valued cooperative game \( \bar{v} \) which satisfies Eq.(5), then according to Definition 2.2, we have \( \bar{v}(S \cup i) = \bar{v}(S \cup k) \), i.e., \( v_{L}(S \cup i) = v_{L}(S \cup k) \) and \( v_{R}(S \cup i) = v_{R}(S \cup k) \). According to Eq.(6), we have
4. Analysis of a specific example. There are three companies, numbered by 1, 2, and 3, respectively. They all have the ability to produce separately, while they can make more profits if they work together. Due to the incomplete and uncertain information, their profits are usually estimated by using intervals. Hence, the problem of profits allocation for the companies can be regarded as a three-person interval-valued cooperative game \( \bar{\upsilon} \), where the companies 1, 2, and 3 are regarded as the players 1, 2, and 3, respectively, the grand coalition is \( N = \{1, 2, 3\} \), and the interval-valued characteristic functions \( \bar{\upsilon} \) are defined as follows: \( \bar{\upsilon}'(1) = [1, 1.7], \bar{\upsilon}'(2) = [1.5, 2.3], \bar{\upsilon}'(3) = [2, 2.5], \bar{\upsilon}'(1, 2) = [4, 4.8], \bar{\upsilon}'(1, 3) = [4.3, 5.5], \bar{\upsilon}'(2, 3) = [5, 6], \) and \( \bar{\upsilon}'(1, 2, 3) = [8, 9.6] \). Let us compute some interval-valued generalized solidarity values \( \bar{\rho}^{GSV}_i (\bar{\upsilon}) \) of the interval-valued cooperative game \( \bar{\upsilon} \).
Obviously, the interval-valued cooperative game \( \bar{G}_S \) satisfies Eq.(5). According to Eq.(6), we have \( \rho_{L_1}^{G_{S,V,0}}(\bar{v}') \approx 2.14, \rho_{R_1}^{G_{S,V,0}}(\bar{v}') \approx 2.69, \rho_{L_2}^{G_{S,V,0}}(\bar{v}') \approx 2.73, \rho_{R_2}^{G_{S,V,0}}(\bar{v}') \approx 3.23, \rho_{L_3}^{G_{S,V,0}}(\bar{v}') \approx 3.13, \) and \( \rho_{R_3}^{G_{S,V,0}}(\bar{v}') \approx 3.68. \) Namely, when the grand coalition is reached, the profit intervals \([2.14, 2.69], [2.73, 3.23], [3.13, 3.68]\) are distributed to the companies (i.e., players) 1, 2, and 3, respectively.

(2) If \( \xi = 0.25 \), then according to Eq.(3), we have \( \xi_0 = 0, \xi_1 = 0.25, \xi_2 = 0.4, \) and \( \xi_3 = 0.5. \) Obviously, the interval-valued cooperative game \( \bar{v}' \) satisfies Eq.(5). Thus, according to Eq.(6), we get

\[
\rho_{L_1}^{G_{S,V,0.25}}(\bar{v}') = \xi_3 \frac{v'_1(N)}{3} + \sum_{s \subseteq (2,3)} \frac{1}{3!} (1 - \xi_2) v'_1(S \cup 1) - (1 - \xi_3) v'_1(S) \approx 2.31
\]

and

\[
\rho_{R_1}^{G_{S,V,0.25}}(\bar{v}') = \xi_3 \frac{v'_2(N)}{3} + \sum_{s \subseteq (2,3)} \frac{1}{3!} (1 - \xi_2) v'_2(S \cup 1) - (1 - \xi_3) v'_2(S) \approx 2.85.
\]

Analogously, we have

\[
\rho_{L_1}^{G_{S,V,0.25}}(\bar{v}') \approx 2.71, \rho_{R_2}^{G_{S,V,0.25}}(\bar{v}') \approx 3.23, \rho_{L_3}^{G_{S,V,0.25}}(\bar{v}') \approx 2.98, \rho_{R_3}^{G_{S,V,0.25}}(\bar{v}') \approx 3.52.
\]

Namely, when the grand coalition is reached, the companies 1, 2, and 3 obtain the profit intervals \([2.31, 2.85], [2.71, 3.23], [2.98, 3.52]\), respectively.

(3) If \( \xi = 0.5 \), then according to Eq.(3), we obtain \( \xi_0 = 0, \xi_1 = 0.5, \xi_2 = 2/3 \) and \( \xi_3 = 0.75. \) Obviously, the interval-valued cooperative game \( \bar{v}' \) satisfies Eq.(5). Thus, according to Eq.(6), we obtain \( \rho_{L_1}^{G_{S,V,0.5}}(\bar{v}') \approx 2.45, \rho_{R_1}^{G_{S,V,0.5}}(\bar{v}') \approx 2.99, \rho_{L_2}^{G_{S,V,0.5}}(\bar{v}') \approx 2.69, \rho_{R_2}^{G_{S,V,0.5}}(\bar{v}') \approx 3.22, \rho_{L_3}^{G_{S,V,0.5}}(\bar{v}') \approx 2.86, \) and \( \rho_{R_3}^{G_{S,V,0.5}}(\bar{v}') \approx 3.53. \)
3.39. Namely, when the grand coalition is reached, the profit intervals [2.45, 2.99], [2.69, 3.22], and [2.86, 3.39] are distributed to the companies 1, 2, and 3, respectively.

(4) If $\xi = 0.75$, then according to Eq.(3), we have $\xi_0 = 0$, $\xi_1 = 0.75$, $\xi_2 = 6/7$, and $\xi_3 = 0.9$. Clearly, the interval-valued cooperative game $\bar{v}'$ satisfies Eq.(5). Thus, according to Eq.(6), we obtain $\rho_1^{\text{GSV},0.75}(\bar{v}') \approx 2.56$, $\rho_2^{\text{GSV},0.75}(\bar{v}') \approx 3.10$, $\rho_3^{\text{GSV},0.75}(\bar{v}') \approx 2.68$, $\rho_1^{\text{GSV},0.75}(\bar{v}') \approx 3.21$, $\rho_2^{\text{GSV},0.75}(\bar{v}') \approx 2.76$, and $\rho_3^{\text{GSV},0.75}(\bar{v}') \approx 3.29$. Namely, when the grand coalition is reached, the companies 1, 2, and 3 obtain the profit intervals [2.56, 3.10], [2.68, 3.21], and [2.76, 3.29], respectively.

(5) If $\xi = 1$, then according to Eq.(3), we have $\xi_0 = 0$ and $\xi_1 = \xi_2 = \xi_3 = 1$. Obviously, the interval-valued cooperative game $\bar{v}'$ satisfies Eq.(5). According to Eq.(6), we have

$$\rho_i^{\text{GSV},1}(\bar{v}') = \frac{\nu'_L(N)}{3}, \frac{\nu'_R(N)}{3} \quad (i = 1, 2, 3).$$

Thus, we obtain $\rho_1^{\text{GSV},1}(\bar{v}') = \rho_2^{\text{GSV},1}(\bar{v}') = \rho_3^{\text{GSV},1}(\bar{v}') = [8/3, 3.2]$. Namely, when the grand coalition is reached, the profit intervals [8/3, 3.2], [8/3, 3.2], and [8/3, 3.2] are distributed to the companies 1, 2, and 3, respectively.

Therefore, for some special values of the parameter $\xi$, we obtain the interval-valued generalized solidarity values $\hat{\rho}^{\text{GSVC}}(\bar{v}')$ of the interval-valued cooperative game $\bar{v}'$, depicted as in Table 1.

| $\rho_i^{\text{GSV}}(\bar{v}')$ | $\xi = 0$ | $\xi = 0.25$ | $\xi = 0.5$ | $\xi = 0.75$ | $\xi = 1$ |
|------------------------|----------|-------------|-------------|-------------|----------|
| $\rho_1^{\text{GSV}}(\bar{v}')$ | [2.14, 2.69] | [2.31, 2.85] | [2.45, 2.99] | [2.56, 3.10] | [8/3, 3.2] |
| $\rho_2^{\text{GSV}}(\bar{v}')$ | [2.73, 3.23] | [2.71, 3.23] | [2.69, 3.22] | [2.68, 3.21] | [8/3, 3.2] |
| $\rho_3^{\text{GSV}}(\bar{v}')$ | [3.13, 3.68] | [2.98, 3.52] | [2.86, 3.39] | [2.76, 3.29] | [8/3, 3.2] |

Table 1 shows that the interval-valued generalized solidarity values change as the parameter $\xi$ changes. It is easy to see from Table 1 that with the increasing of the parameter $\xi$, the interval-valued generalized solidarity values of the company 1 are increasing, i.e.,

$$\rho_1^{\text{GSV},0}(\bar{v}') \leq \rho_1^{\text{GSV},0.25}(\bar{v}') \leq \rho_1^{\text{GSV},0.5}(\bar{v}') \leq \rho_1^{\text{GSV},0.75}(\bar{v}') \leq \rho_1^{\text{GSV},1}(\bar{v}')$$

whereas the interval-valued generalized solidarity values of the companies 2 and 3 are decreasing, i.e.,

$$\rho_i^{\text{GSV},0}(\bar{v}') \geq \rho_i^{\text{GSV},0.25}(\bar{v}') \geq \rho_i^{\text{GSV},0.5}(\bar{v}') \geq \rho_i^{\text{GSV},0.75}(\bar{v}') \geq \rho_i^{\text{GSV},1}(\bar{v}') \quad (i = 2, 3).$$

Therefore, the parameter $\xi$ greatly affects the interval-valued generalized solidarity values of interval-valued cooperative games.

5. Conclusions. We introduce the concept of interval-valued generalized solidarity values of interval-valued cooperative games and find the weaker condition of the size monotonicity-like so that the interval-valued generalized solidarity values can be directly and explicitly obtained by computing their lower and upper bounds through only using the lower and upper bounds of the interval-valued coalitions’ values, respectively. The developed method does not use the interval subtraction and hereby can effectively avoid the issues resulted from it. This is remarkably
different from the existing methods which directly used the Moore’s interval subtraction or interval partial operator. At the same time, some significant properties of interval-valued generalized solidarity values are given.

Recently, there are some investigations on fuzzy matrix games ([4], [17], [18]), rough matrix games ([29]), and stochastic differential games ([22]). In the future we will extend the generalized solidity value to the cooperative games under the above uncertain environments, such as cooperative games with coalitions’ values represented by fuzzy sets, rough sets or stochastic variables. There are also some literatures about real-life applications in different fields such as transportation problem ([30]), inventory model ([28]), supply chain network design problem ([7]), stochastic optimal control problem ([31]), and robust optimization problem ([15], [23], [24]). The generalized solidity value may be applied to solve some problems in the above fields, such as cost allocation in inventory management and transportation problems.

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