Gravity Induced Chiral Condensate Formation and the Cosmological Constant

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It is well known that the covariant coupling of fermionic matter to gravity induces a four-fermion interaction. The presence of this term in a homogeneous and isotropic space-time results in a BCS-like Hamiltonian and the formation of a chiral condensate with a mass gap. We calculate the gap $\Delta$ via a mean-field approximation for minimally coupled fermionic fields in a FRW background and find that it depends on the scale factor. The calculation also yields a correction to the bare cosmological constant $\Lambda_0$ and a non-zero vev for $\langle \psi^\dagger \psi \rangle$ which then behaves as a scalar field. Hence we conjecture that the presence of fermionic matter in gravity provides a natural mechanism for relaxation of the $\Lambda_0$ and explains the existence of a scalar field from (almost) first principles.

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I. INTRODUCTION

Ever since the BCS theory of superconductivity has been discovered, the phenomenon of Cooper pairing has played a seminal role across a wide range of physics, including Pion formation, Technicolor and QCD at high densities. A Cooper pair requires some necessary conditions:

- A Fermi surface.
- Screening resulting in an attractive interaction between fermions.
- A relevant four-fermion interaction.

Another important aspect of the BCS theory is that it signifies that the perturbative vacuum with respect to perturbative phonon or vector boson exchange is unstable; very weak attractive interaction drives the system to a lower energy non-perturbative ground state. In the context of general relativity graviton exchange between fermions is a ripe setting to ask whether or not a BCS condensate can form. This possibility may have consequences, especially for the inflationary paradigm and the cosmological constant problem since the idea that the vacuum is unstable with respect to graviton exchange between fermions can pave a way to solving the cosmological constant problem. In this paper we demonstrate for the first time that gravity naturally incorporates a BCS condensate in a general, $\Lambda$ dominated, FRW space-time. In the context of inflation this condensate can play the role of the inflaton field. We also show that the gap introduces a non-perturbative cancelling correction to the cosmological constant which is consistent with the expectations of the authors as a possible path towards resolving the cosmological constant problem. Other approaches to the dark energy problem as a condensate has been proposed in the past but a concrete microphysical mechanism has been lacking [1,2]. We hope that this work may be useful in providing the correct
microphysics underlying the dark energy problem in terms of condensates.

Recently \cite{3} it was shown that gravity in the presence of a Dirac field induces a non-zero torsion. This torsion turns out to be proportional to the axial current, $J_{\mu 5}$. Inserting the expression for the torsion back into the first-order action we find a new interaction term which is proportional to the square of the axial current and also has a dependence on the Immirzi parameter.\footnote{This four-fermion interaction is not new. As far back as 1922 Cartan proposed that a correct theory of gravity should also contain torsion.} Such a four-fermion interaction is well-known to cause the formation of a chiral condensate. As a consequence $\langle \psi^\dagger \psi \rangle$ develops a non-zero vev and the resulting theory has a mass gap $\Delta$. We also find a negative contribution to the cosmological constant $\Lambda_0$ from the fermionic condensate.

The paper is arranged as follows. In Section 2 we show how the presence of a Dirac term in the first-order action for fermions coupled to gravity, induces the four-fermion interaction. In Section 3 we do the $(3+1)$ decomposition of the resulting Lagrangian and find the Hamiltonian by performing a Legendre transform. This allows us to identify the diffeomorphism, hamiltonian and gauge constraints of the theory. It is the hamiltonian constraint which is responsible for dynamics and we concentrate on it. In Section 4 we write down the hamiltonian constraint for a FRW metric. We then quantize the fermion field, while leaving the background metric classical. In Section 5 we exhibit the Boguliubov transformation on the fermionic ladder operators which is a necessary step in the BCS calculation\footnote{The gap can also be determined via a variational method, however the Boguliubov transformation is simpler and more instructive.}. In Section 6 we diagonalize the matter hamiltonian by applying the Boguliubov transformation and then find the gap equation. We find that the gap has a dependence on the scale factor and acts to negate the cosmological constant term in the hamiltonian constraint. Finally we discuss our results and mention avenues for future research.

II. TORSION AND THE FOUR-FERMI INTERACTION

Our starting point is with the Holst action for General Relativity with a cosmological constant, coupled to fermions. We will calculate the four-fermion interaction induced by Torsion and write the action in Hamiltonian form. The action will be symmetry reduced and after all of the constraints are identified we will show that the fermionic Hamiltonian is a many-body BCS Hamiltonian. Finally we will diagonalize the Hamiltonian and calculate the energy gap.

First, it is convenient to introduce our conventions. Lowercase greek letters $\mu, \nu, ...$ stand for four dimensional spacetime indices $1..4$. Lowercase latin letters denote spatial indices on $\Sigma$. Uppercase latin $I, J, ...$ denote internal indices $1..4$. Lowercase latin letters denote internal indices $1..3$.

The action for gravity coupled with massless fermions is:

$$ S[A, e, \Psi] = S_H + S_D \quad (1) $$
where \( S_H \) is the Holst action and is equivalent to the metric formulation of general relativity:

\[
S_H = \frac{1}{2\kappa} \int d^4x \epsilon^{\mu}_{\nu} \epsilon^{\rho}_{\sigma} F^I_{\mu\nu} - \frac{1}{2\kappa\gamma} \int d^4x \epsilon^\mu e^\nu \epsilon^\nu_\gamma * F^I_{\mu\nu} 
\]

(2)

and \( S_D \) is the action for fermions:

\[
S_D = \frac{i}{2} \int d^4x \epsilon (\bar{\Psi} \gamma^I e^\nu_I D_\mu \Psi - D_\mu \bar{\Psi} \gamma^I e^\nu_I \Psi) 
\]

(3)

where:

\[
D_\mu \Psi = \partial_\mu \Psi - \frac{1}{4} A^{IJ}_\mu \gamma^I \gamma^J \Psi 
\]

(4)

\[
\overline{D}_\mu \bar{\Psi} = \partial_\mu \bar{\Psi} + \frac{1}{4} \bar{\Psi} \gamma^I \gamma^J A^{IJ}_\mu 
\]

(5)

The equation of motion obtained by varying (1) with respect to the four dimensional spin connection \( A^{IJ}_\mu \) yields:

\[
A^{IJ}_\mu = \omega^{IJ}_\mu + C^{IJ}_\mu 
\]

(6)

where \( \omega^{IJ}_\mu \) is the spin connection compatible with the tetrad \( e^\mu_I \) and \( C^{IJ}_\mu \) is the tetrad projection of the contortion tensor:

\[
C^{IJ}_\mu = C^{\nu\delta}_\mu e^\nu_I e^\delta_J 
\]

(7)

On solving for \( C^{IJ}_\mu \) in terms of the fermionic field and inserting the resulting expression for \( A^{IJ}_\mu \) in (1) one obtains the following:

\[
S[e, \Psi] = \frac{1}{16\pi G} \int d^4x \epsilon^\mu I e^\nu J F^{IJ}_{\mu\nu}[\omega(e)] + \frac{i}{2} \int d^4x \epsilon (\bar{\Psi} \gamma^I e^\nu_I D_\mu [\omega(e)] \Psi - \overline{D}_\mu [\omega(e)] \bar{\Psi} \gamma^I e^\nu_I \Psi) + S_{int}[e, \Psi] + S_b 
\]

(8)

where \( S_{int} \) is the four fermion interaction:\n
\[
S_{int} = -\frac{3}{2} \pi G \frac{\gamma^2}{\gamma^2 + 1} \int d^4x e (\bar{\Psi} \gamma_5 \gamma^I \Psi)(\bar{\Psi} \gamma_5 \gamma^I \Psi) = \frac{3}{2} \pi G - \frac{\gamma^2}{\gamma^2 + 1} \int d^4x e (\gamma^I)^2 
\]

(9)

\( ^3 \) A detailed derivation is included in the Appendix
and $S_b$ is a boundary term, given by:

$$ S_b = -\frac{3}{4\kappa\gamma} \oint_{\partial\Sigma} d^3x \Gamma_{\mu\nu\rho}^a j^a_{\mu} $$

(10)

Before we proceed to the $(3 + 1)$ decomposition of the above action, we write the Dirac action in terms of Weyl spinors. This will make the decomposition simpler and will also illustrate an important property of the left and right handed spinors.

We expand the second term in (4)

$$ A_{\mu}^{IJ} \gamma_I \gamma_J = A_{\mu}^{i0} \gamma_i \gamma_0 + A_{\mu}^{i0} \gamma_0 \gamma_i + A_{\mu}^{ij} \gamma_i \gamma_j $$

$$ = 2A_{\mu}^{0i} \gamma_0 \gamma_i + A_{\mu}^{ij} \gamma_i \gamma_j $$

$$ = 2A_{\mu}^{0i} \left( -\sigma_i \begin{array}{c} 0 \\ 0 \end{array} \right) + iA_{\mu}^{ijk} \begin{array}{c} \sigma_i \\ 0 \end{array} \right) $$

$$ = 2i \begin{pmatrix} A_{\mu}^{i+} \sigma_i & 0 \\ 0 & A_{\mu}^{i-} \sigma_i \end{pmatrix} $$

(11)

In the second line we have used the fact that $A_{\mu}^{IJ}$ is antisymmetric in the internal indices and that the gamma matrices anticommute. In the third we have used the expressions for the gamma matrices given in the appendix to expand out the matrix products. In the fourth we have used the definition of the self and anti-self dual parts of the connection:

$$ A_{\mu}^{i+} = \frac{1}{2} \epsilon^{ijk} A_{\mu}^{jk} + iA_{\mu}^{0i} $$

$$ A_{\mu}^{i-} = \frac{1}{2} \epsilon^{ijk} A_{\mu}^{jk} - iA_{\mu}^{0i} $$

(12)

Now writing the Dirac spinor $\Psi$ in terms of the Weyl spinors $\psi, \eta$, we see that (11) becomes:

$$ D_{\mu} \Psi = \begin{pmatrix} D_{\mu}^{\psi} \\ D_{\mu}^{\eta} \end{pmatrix} $$

(13)

where $D_{\mu}^{\psi} = \partial_{\mu} \psi - \frac{i}{2} A_{\mu}^{i+} \sigma_i \psi$ and $D_{\mu}^{\eta} = \partial_{\mu} \eta - \frac{i}{2} A_{\mu}^{i-} \sigma_i \eta$. Thus the left(right) handed spinors couple to the self(anti-self) dual parts of the connection.

We now proceed with the $(3+1)$ decomposition of (5).

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4 In the following we essentially follow the Appendix of [4], filling in some of the steps. We have included this derivation to make the paper self-contained.
Consider a spacelike slice $\Sigma$ of the spacetime manifold $\mathcal{M}$ with unit normal $n^\mu$. Then the Dirac action is:

$$2S_D = i \int d^3x \, dt \sqrt{q} (\bar{\Psi} \gamma_\mu D_\mu \Psi - \text{c.c.}) (q^{\mu\nu} - n^\mu n^\nu)$$

$$= i \int d^3x \, dt \sqrt{q} (\bar{\Psi} \gamma^a D_a \Psi + \bar{\Psi} \gamma^0 n^\nu D_\nu \Psi - \text{c.c.})$$

$$= i \int d^3x \, dt \sqrt{q} (\psi^\dagger \sigma^a D^+_a \psi - \eta^\dagger \sigma^a D^-_a \eta - \text{c.c.}) + \sqrt{q} (t^\mu - N^\mu) (\psi^\dagger D^+_\mu \psi + \eta^\dagger D^-_\mu \eta - \text{c.c.})$$

$$= i \int d^3x \, dt \sqrt{q} (\psi^\dagger \sigma^a D^+_a \psi - \eta^\dagger \sigma^a D^-_a \eta - \text{c.c.})$$

$$+ \sqrt{q} (\psi^\dagger \dot{\psi} + \eta^\dagger \dot{\eta} - \frac{i}{2} A^{IC}_i \psi \sigma_i \eta - \frac{i}{2} \bar{A}^{IC}_i \eta \sigma_i \psi - c.c.)$$

$$- \sqrt{q} N^\alpha (\psi^\dagger D^+_a \psi + \eta^\dagger D^-_a \eta - c.c.)$$

(14)

In the first line we have used the decomposition of the metric $g_{\mu\nu}$ on $\mathcal{M}$ in terms on the metric $g_{ab}$ on $\Sigma$ and the unit normal $n^\mu$ to $\Sigma$. $q^{\mu\nu}$ projects tensors and derivatives on $\mathcal{M}$ to tensors and derivatives on $\Sigma$. $D_\mu$ and $D_a$ denote the covariant derivative on $\mathcal{M}$ and its restriction to $\Sigma$ respectively. In the second line we have used the freedom to fix the gauge in the internal space such that the contraction of $\gamma_\mu$ and $n^\mu$ gives us $-\gamma^0$. In the third the decomposition of $n^\mu$ in terms of the timelike vector field $t^\mu$, the lapse $N$ and the shift $N^\mu$, and the expression of the covariant derivative in terms of the self and anti-self dual parts of the connection is used. In the last line we have noted that the restriction of $A^a_{\mu+}$ to $\Sigma$ is the Ashtekar connection $\Gamma^a_\mu + iK^a_\mu$. The time component of $A^a_{\mu+}$ is written as $A'^a_{\mu}$ in the fifth line.

Defining $A^a_i := \Re(A^{IC}_i)$ and evaluating the complex conjugate terms explicitly we get:

$$S_D = \frac{i}{2} \int d^3x \, dt \sqrt{q} (\psi^\dagger \dot{\psi} + \eta^\dagger \dot{\eta} - c.c.) - i \sqrt{q} A^a_i (\psi^\dagger \sigma_i \psi + \eta^\dagger \sigma_i \eta)$$

$$- \sqrt{q} N^\alpha (\psi^\dagger D_a \psi + \eta^\dagger D_a \eta - c.c.)$$

$$+ N \left[ E^a_i (\psi^\dagger \sigma^a D^a_a \psi - \eta^\dagger \sigma^a D^-_a \eta - c.c.) + i[K_a, E^a]_k (\psi^\dagger \sigma_k \psi + \eta^\dagger \sigma_k \eta) \right]$$

(15)

Here $D_a \psi = \partial_a \psi - \frac{i}{2} \Gamma^a_\mu \sigma_i \psi$. We can easily see that contributions of the Dirac action to the gauss, scalar and diffeomorphism constraints are the coefficients of $A'^a_\mu$, $N$ and $N^a$ respectively. The decomposition of $S_{int}$ is easily done and we obtain the following form:

$$S_{int} = -\frac{3}{2} \pi G \gamma^2 \gamma^2 + 1 \int d^3x \, dt \sqrt{q} N \left[ (\psi^\dagger \sigma^a \psi + \eta^\dagger \sigma^a \eta)^2 - (-\psi^\dagger \psi + \eta^\dagger \eta)^2 \right]$$

(16)

From (15) we see that Lagrange multiplier of the matter contribution to the gravitational gauss
constraint is $A^i_j$. In order to get this Lagrange multiplier one must first start with the 3+1 decomposition of the self-dual gravitational action and then take its real part. The self-dual gravitational action is:

$$S_{SD} = \frac{1}{\kappa} \int d^4x \, e^a_I \epsilon^i_{ij} + F_{ab}^j \, J$$

(17)

$+F_{ab}^j \, J$ is the curvature of the self-dual connection and $e^a_I$ is the usual tetrad. Doing the 3+1 decomposition in the usual manner yields:

$$S_{SD} = \frac{1}{\kappa} \int d^3x \, dt \left[ -i \tilde{E}_b^i \dot{A}^i_b - iA^i_c D_b(E_b^i) - iN^a \text{tr}[F_{ab} \tilde{E}_b^a] + \frac{N}{2\sqrt{q}} \text{tr}(F_{ab} [\tilde{E}^a, \tilde{E}^b]) \right]$$

(18)

where $\tilde{E}_b^i$ is the densitized triad, $F_{ab}^i$ is the curvature of the restriction $A^i_b$ to $\Sigma$ of the complex self-dual connection, and the trace and commutators are taken in the Lie-algebra of $su(2)$.

Taking the real part of the above action and using the fact that $A^i_a = \Gamma^i_a + iK^i_a$ we get:

$$S_{real} = \frac{1}{\kappa} \int d^3x \, dt \left\{ \tilde{E}_b^i K^i_b + A^i_o [K_b, \tilde{E}_b^i] + 2N^a D_a K^i_b \tilde{E}_b^i + \frac{N}{2\sqrt{q}} (R^i_{ab} - [K_a, K_b]^i) [\tilde{E}^a, \tilde{E}^b]_i \right\}$$

(19)

From (15) we see that the momenta conjugate to $\psi$ and $\psi^\dagger$ are $\frac{i}{2} \phi$ and $\frac{-i}{2} \phi$ respectively. Then doing the Legendre transform on $S_{real} + S_D + S_{int}$ we get the following Hamiltonian:

$$H_{G+D+int} = \int d^3x \, A^i \left\{ \frac{1}{\kappa} [K_b, \tilde{E}^b]_i + j^i \right\}$$

$$+ N \left\{ \frac{1}{2\sqrt{q}} (R^i_{ab} - [K_a, K_b]^i) [\tilde{E}^a, \tilde{E}^b]_i + \frac{i}{2\sqrt{q}} \tilde{E}_i^a (\xi^i \sigma^j D_a \xi - \rho^j \sigma^i D_a \rho - \text{c.c.}) \right\}$$

$$+ \frac{1}{2} [K_a, \tilde{E}^a]^j_k j_k - \frac{3}{2} \gamma^2 G \frac{\gamma^2}{\gamma^2 + 1} [J^2 - (-\xi^i \xi^j + \rho^i \rho^j)^2] + \sqrt{q} \Lambda_0$$

$$+ N^a \left\{ \frac{2}{\kappa} D_a \tilde{E}_i^b + \frac{i}{2} (\xi^i D_a \xi + \rho^i D_a \rho - \text{c.c.}) \right\}$$

(20)

where $\xi = \frac{q}{2} \psi; \rho = \frac{q}{2} \eta$ and $j^i = (\eta^i \sigma^j + \rho^i \sigma^j)/2$ is the axial current. We must change variables to make the matter fields half-densities, because otherwise the connection would become complex [4]. The hamiltonian is manifestly a sum of constraints and the form of each constraint is easy to read off from (20). It is important to note the gravitational Gauss constraint now has a matter contribution. In the third line we have also added a term coming from the bare cosmological constant.
IV. SYMMETRY REDUCTION AND QUANTIZATION

We make the ansatz that the background metric is FRW with scale factor $a$. The basic gravitational variables are:

$$E_i^a = a^2 \delta_i^a, \quad K_i = a^2 \dot{a} \delta_i^a, \quad R_{ab} = 0$$  \quad (21)

We assume, for the moment, that the axial current is zero and hence the Gauss constraint is satisfied. We also assume that the matter contribution to the diffeomorphism constraint is zero. Later we shall find that these statements are true when we quantize the fermionic field. We are left with the Hamiltonian constraint and this reduces to:

$$H = H_G + H_D + H_{int} = -\frac{3}{\kappa} a^3 H^2 + a^3 \Lambda_0 + \frac{i}{a} \left( \xi^\dagger \sigma^a \partial_a \xi - \rho^\dagger \sigma^a \partial_a \rho \right) + \frac{3 \kappa}{32 a^3 \gamma^2 + 1} \left[ \xi^\dagger \xi - \rho^\dagger \rho \right] = 0$$  \quad (22)

where $H = (\dot{a}/a)$ is the Hubble parameter.

We switch to comoving co-ordinates in order to take care of the factor of $1/a$ in $H_D$. $H_D$ reduces to $i(\xi^\dagger \sigma^a \partial_a \xi - \rho^\dagger \sigma^a \partial_a \rho)$. We can then expand $\xi$ and $\rho$ in terms of Fourier modes:

$$\xi(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ \xi^{\dagger}_k e^{-ikx} + \xi_k e^{ikx} \right\}$$  \quad (23a)

$$\rho(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ \xi_{\dagger k} e^{-ikx} + \xi_k e^{ikx} \right\}$$  \quad (23b)

where $\xi_k$ ($\xi^{\dagger}_k$) is a spinor\(^5\) of density weight $1/2$\(^6\) along the direction $\hat{k}$ in momentum space and with helicity $1/2$ and $-1/2$ respectively. Thus we can write the quantized field in the usual manner in terms of anticommuting annihilation and creation operators:

$$\hat{\xi}(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ a_k \xi_k + b^\dagger_{-k} \xi_k \right\} e^{-ikx}$$  \quad (24a)

$$\hat{\rho}(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ \bar{b}_k \xi_k + \bar{a}_{-k} \xi_k \right\} e^{-ikx}$$  \quad (24b)

\(^5\) The expressions for these spinors are given in the Appendix
\(^6\) Because as mentioned earlier the fermionic fields must be half-densities
\( \rho \) and \( \xi \) are independent fields, therefore we have used \( ^{\dagger} \) to distinguish their operators. These fields satisfy the anticommutation relations:

\[
\{ \hat{\xi}_{\alpha}^{\dagger}(x), \hat{\xi}_{\beta}(y) \} = \{ \hat{\rho}_{\alpha}^{\dagger}(x), \hat{\rho}_{\beta}(y) \} = (2\pi)^{3}\delta_{\alpha\beta}\delta^{3}(x, y)\sqrt{q} \quad (25a)
\]

\[
\{ \hat{\xi}_{\alpha}(x), \hat{\xi}_{\beta}^{\dagger}(y) \} = \{ \hat{\rho}_{\alpha}(x), \hat{\rho}_{\beta}^{\dagger}(y) \} = 0 \quad (25b)
\]

The above expressions for the quantized field can be used to easily verify that the spatial current and the matter contribution to the diffeomorphism constraint are zero, as stated previously. Using the orthogonality of spinors of opposite helicity, the quantized form of the free Dirac Hamiltonian is easily found to be:

\[
\hat{H}_{D} = \hat{H}_{\xi} + \hat{H}_{\rho} = \int \frac{d^{3}k}{(2\pi)^{3}} |k| (a_{k}^{\dagger}a_{k} + b_{-k}^{\dagger}b_{-k} + \bar{a}_{-k}\bar{a}_{-k} + \bar{b}_{k}\bar{b}_{k}) \quad (26)
\]

\textbf{V. BOGULIUBOV TRANSFORMATION}

The four-fermi interaction is identical to the one which describes the formation of a condensate in BCS theory\(^7\). Due to this interaction the true vacuum is not the one corresponding to the Dirac equation but one in which particles and antiparticles of opposite momenta and helicity are paired\(^7\). The interacting part is non-diagonal in the present variables. In order to diagonalize the full matter Hamiltonian we have to perform a Boguliubov transformation, which is a linear canonical transformation to new annihilation and creation operators. We get a new ground state corresponding to these operators. This BCS ground state is a condensate of Cooper pairs. Excitations of this "vacuum" are produced by the action of the new operators whose physical effect is to break up Cooper pairs and produce free fermions and antifermions.

\[
\alpha_{k} = u_{k}a_{k} - v_{k}b_{-k}^{\dagger} \quad (27a)
\]

\[
\beta_{-k} = u_{k}b_{-k} + v_{k}a_{k}^{\dagger} \quad (27b)
\]

Then the new variables \( \alpha_{k} \) and \( \beta_{-k} \) satisfy anticommutation relations if \( u_{k}^{2} + v_{k}^{2} = 1 \). In terms of the new variables, the old ones are:

\[
a_{k} = u_{k}\alpha_{k} + v_{k}\beta_{-k}^{\dagger} \quad (28a)
\]

\(^7\) In BCS theory the pairing happens between particles of opposite momenta. However, here we have left and right handed fermions therefore the pairing must include the helicity
\[ b_{-k} = u_k \beta_{-k} - v_k \alpha_k^\dagger \]  

(28b)

In the new variables \( \hat{H}_\xi \) becomes:

\[
\hat{H}_\xi = \int \frac{d^3k}{(2\pi)^3} |k| (2v_k^2 + (u_k^2 - v_k^2)(m_k + n_{-k}) + 2u_kv_k\Sigma_k) 
\]

(29)

where \( m_k = \alpha_k^\dagger \alpha_k \), \( n_{-k} = \beta_{-k}^\dagger \beta_{-k} \) are the new number operators and \( \Sigma_k = \alpha_k^\dagger \beta_{-k}^\dagger + \beta_{-k} \alpha_k \) is the off-diagonal part.

VI. FOUR-FERMION TERM

The interaction hamiltonian is an attractive four-fermion term which causes the formation of the fermion condensate. In the this section we use the mode expansion for the fermion field to expand this term and then apply the Boguliubov transformation to it.

The four-fermion term is:

\[
\hat{H}_{\text{int}} = \frac{3\kappa}{32 \alpha^3 \gamma^2 + 1} \int d^3x \left( \hat{\xi}^\dagger \rho \hat{\rho} \right)^2 
\]

\[
= \frac{\alpha}{a^3} \int d^3x \left( \hat{\xi}^\dagger \hat{\xi} - \rho^\dagger \rho \right) 
\]

\[
= \hat{H}_1 + \hat{H}_2 + \hat{H}_{\rho \xi} + \hat{H}_{\xi \rho} 
\]

(30)

where \( \alpha = \frac{3\kappa}{32 \gamma^2 + 1} \). Now we can write \( \rho \) as:

\[
\hat{\rho}(x) = \int \frac{d^3k}{(2\pi)^3} \left\{ \bar{b}_{-k} \hat{\xi}_k \sigma_k + a_k^\dagger \hat{\xi}_k^\dagger \right\} e^{ikx} 
\]

(31)

by doing changing variables from \( k \) to \( -k \) in the integration. Then by comparing (31) and (24a) we see that one can switch from \( \xi \) to \( \rho \) (or vice versa) by changing \( a_k \leftrightarrow \bar{b}_{-k} \).

Now using the anticommutation relations for the fermionic fields we can write \( \hat{H}_1 \) as:

\[
\hat{H}_1 = \alpha \int d^3x \xi^\dagger \hat{\xi} + \frac{\alpha}{a^3} \int d^3x \xi^\dagger \xi \sigma_k \xi^\dagger \sigma_k \xi^\dagger \xi = \hat{N}_\xi + \hat{H}_{\xi \xi} 
\]

(32)
Using (24a) and (27) \( \hat{N}_\xi \) becomes:

\[
\hat{N}_\xi = \alpha \int \frac{d^3k}{(2\pi)^3} \left[ a_k^\dagger a_k - b_{-k}^\dagger b_{-k} \right] = \alpha \int \frac{d^3k}{(2\pi)^3} \left[ m_k - n_{-k} \right]
\]

Likewise for \( \hat{\rho} \) we have:

\[
\hat{H}_2 = \alpha \int d^3x \hat{\rho}^\dagger \hat{\rho} + \frac{\alpha^2}{a^3} \int d^3x \hat{\rho}_{\alpha \beta}^\dagger \hat{\rho}_{\beta \alpha}^\dagger = \hat{N}_\rho + \hat{H}_{\rho\rho}
\]

and \( \hat{N}_\rho \) is:

\[
\hat{N}_\rho = \alpha \int \frac{d^3k}{(2\pi)^3} \left[ b_{-k}^\dagger b_k - a_{-k}^\dagger a_{-k} \right] = \alpha \int \frac{d^3k}{(2\pi)^3} \left[ \bar{n}_{-k} - \bar{m}_{-k} \right]
\]

To explicitly evaluate \( \hat{H}_{\xi \xi} \) and \( \hat{H}_{\rho \xi} \) we use the mode expansion (24) and the anticommutation relations of the fermionic operators. Then \( \hat{H}_{\rho \rho} \) and \( \hat{H}_{\rho \xi} \) are obtained by simply using the substitution \( a_k \leftrightarrow b_{-k} \). After some algebra we obtain the following expression:

\[
\hat{H}_\xi + \hat{H}_{\xi \xi} + \hat{N}_\xi = \int \frac{d^3k}{(2\pi)^3} \left[ \alpha \int \frac{d^3k d^3p_\rho d^3p_{\rho'}}{(2\pi)^6} \left\{ \left( \xi_{k\uparrow} \xi_{p\uparrow} \right) \left( \xi_{k\uparrow} \xi_{p\uparrow} \right) + \delta^3(k' - p' - p) a_k^\dagger a_k b_{-k}^\dagger b_{-k} - a_{-k}^\dagger a_{-k} b_k^\dagger b_k \right\} + \right. \\
- \alpha \int \frac{d^3k d^3p d^3k' d^3p'}{(2\pi)^6} \left\{ \left( \xi_{k\uparrow} \xi_{p\uparrow} \right) \left( \xi_{k'\uparrow} \xi_{p'\uparrow} \right) + \delta^3(k - k' + p - p') a_k^\dagger a_k b_{-k} b_{-k} - a_{-k}^\dagger a_{-k} b_k b_k \right\} + \right. \\
+ \left. \delta^3(k - k' - p - p') a_k^\dagger a_k b_{-k}^\dagger b_{-k} - a_{-k}^\dagger a_{-k} b_k^\dagger b_k \right\} + \left. \int \frac{d^3k}{(2\pi)^3} \left[ a_k^\dagger a_k - b_{-k}^\dagger b_{-k} \right] \right]
\]

In the last line the As denote the operator products and the Vs denote the spinor products. Also in the above expression and henceforth we only use dedensitized spinors. There is a factor of \( a^3 \) in front of the whole expression which we set to 1 for now. The factor is re-introduced later when appropriate.

Now using momentum conservation we can simplify \( A_1 \) as follows.

\[
A_1 = a_k^\dagger a_k^\dagger a_{k'} a_{k'} = a_k^\dagger a_{k'-q} a_{k'+q}
\]
Using Wick’s theorem and the operator identities in the Appendix the above expression can be written as:

\[ A_1 = N(A_1) + \left\{ -N(a_k^\dagger a_{k-q}) a_k^{\dagger} d_{k'}^{\dagger} q - N(a_k^\dagger a_{k'+q}) a_k^{\dagger} d_{k-q}^{\dagger} q + N(a_k^\dagger a_{k'+q}) a_k^{\dagger} d_{k-q}^{\dagger} q \\
+ N(a_k^\dagger a_{k-q}) d_{k'}^{\dagger} a_{k'+q} - d_{k}^{\dagger} a_{k-q} d_{k'}^{\dagger} a_{k'+q} + d_{k}^{\dagger} a_{k'+q} a_{k-q} \right\} \]

\[ = N(A_1) + \left\{ -N(a_k^\dagger a_{k}) v_k^2 \delta_q,0 - N(a_k^\dagger a_{k'}) v_k^2 \delta_q,0 + N(a_k^\dagger a_{k}) v_k^2 \delta_{k',k-q} + N(a_k^\dagger a_{k'}) v_k^2 \delta_{k',k-q} - v_k^2 v_{k'}^2 \delta_q,0 + v_k^2 v_{k'}^2 \delta_{k',k-q} \right\} \]  

(38)

Inserting the above expression for \( A_1 \) into \( \boxed{36} \) and integrating first over the delta function in \( \boxed{36} \) and then over the delta functions in \( \boxed{38} \) we obtain after relabelling some indices and some algebraic manipulations we have:

\[ -\alpha \int \frac{d^3kd^3k'd^3q}{(2\pi)^9} A_1 V_1(k,k',q) = \]

\[ -N(V_1) + \alpha \int \frac{d^3kd^3k'd^3q}{(2\pi)^9} \left[ N(a_k^\dagger a_{k}) v_k^2 + N(a_k^\dagger a_{k'}) v_k^2 \right] \left[ 1 - \left( \xi_{k'}^{\dagger} \xi_{k'} \right) \left( \xi_{k'}^{\dagger} \xi_{k'} \right) \right] \]  

(39)

where \( N(V_1) \) is quartic in the creation and annihilation operators.

\( A_2 \) can be dealt with in a similar manner and after some computations we find:

\[ -\alpha \int \frac{d^3kd^3k'd^3q}{(2\pi)^9} (A_1 V_1 + A_2 V_2) = \]

\[ -N(V_1 + V_2) - \alpha \int \frac{d^3kd^3k'd^3q}{(2\pi)^9} 2 \left[ N(a_k^\dagger a_{k}) + N(b_{-k}^\dagger b_{-k}) + v_k^2 \right] v_{k'}^2 \left[ \left( \xi_{k'}^{\dagger} \xi_{k'} \right) \left( \xi_{k'}^{\dagger} \xi_{k'} \right) - 1 \right] \]  

(40)

The term with \( A_3 \) yields:

\[ -\alpha \int \frac{d^3kd^3k'd^3q}{(2\pi)^9} A_3 V_3 = -N(V_3) - \left\{ \left[ N(a_k^\dagger a_{k}) + N(b_{-k}^\dagger b_{-k}) + v_k^2 \right] v_{k'}^2 \left[ \left( \xi_{k'}^{\dagger} \xi_{k'} \right) \left( \xi_{k'}^{\dagger} \xi_{k'} \right) + 1 \right] \right\\
+ \left[ N(a_k^\dagger b_{-k}^\dagger) + N(b_{-k} a_{k}) + u_k v_k \right] u_{k'} v_{k'} R \left[ \left( \xi_{k'}^{\dagger} \xi_{k'} \right) \left( \xi_{k'}^{\dagger} \xi_{k'} \right) \right] \right\} \]  

(41)

Above we have dealt with the terms of \( \hat{H}_{\xi\xi} \). Doing similar manipulations with \( \hat{H}_{\xi\rho} \) we find:

\[ \hat{H}_{\xi\rho} = \hat{H}_{\rho\xi} = -\alpha \int \frac{d^3kd^3k'd^3q}{(2\pi)^9} \left( a_k^\dagger a_{k} - b_{-k}^\dagger b_{-k} + b_{-k}^\dagger b_{-k} + a_k^\dagger a_{k} \right) \]  

(42)
In the above equation we have a seemingly divergent integral over the momenta $k'$. This is dealt with by imposing a momentum cutoff. We get:

\[ \int \frac{d^3k}{2\pi^3} = \frac{1}{2\pi^2} \int k^2 dk = \frac{1}{2\pi^2} \int_0^{\hbar \omega_D} E^2 dE = \frac{(\hbar \omega_D)^3}{6\pi^2} = C_1 \]

The sum of (29), (39), (40), (41) and the first half of (42) gives us the matter hamiltonian corresponding only to the field $\xi$. The other half corresponding to $\rho$ can be is identical except for the substitution $a_k \rightarrow \bar{b}_{-k}$.

\[
\hat{H}(\xi) = \int \frac{d^3k}{(2\pi)^3} \left\{ \alpha^+_k a_k + b^+_k \bar{b}_{-k} \right\} (|k| - C_1 \alpha) - \alpha \int \frac{d^3k}{(2\pi)^3} \left\{ N(a^+_k b^+_k) + N(b_{-k} a_k) + u_k v_k \right\} u_k v_k V_1' + \\
+ \left\{ N(a^+_k a_k) + N(b^+_k b_{-k}) + v_k^2 \right\} v_k^2 V_2' \\
= \int \frac{d^3k}{(2\pi)^3} \left\{ (u_k^2 - v_k^2) (m_k + n_k) + 2u_k v_k \Sigma_k + 2v_k^2 \right\} (|k| - C_1 \alpha) - \alpha \int \frac{d^3kd^3k'}{(2\pi)^6} \left\{ ((u_k^2 - v_k^2)\Sigma_k + 2u_k v_k \Sigma_k + v_k^2) v_k^2 V_2' \right\} (43)
\]

where:

\[
\Sigma_k = \alpha_k^+ \beta_{-k}^+ - \beta_{-k} \alpha_k
\]

and,

\[
V_1'(k, k') = R \left\{ (\xi^\dagger_k \xi_{k'}) (\xi^\dagger_k \xi_{k'}) \right\} \quad (44a)
\]

\[
V_2'(k, k') = (\xi^\dagger_k \xi_{k'}) (\xi^\dagger_k \xi_{k'}) + (\xi^\dagger_k \xi_{k'}) (\xi^\dagger_k \xi_{k'}) = 1 \quad (44b)
\]

Where in the second line we have used the expressions for spinors given in the Appendix. Now we can easily apply the Bogoliubov transformation to the above hamiltonian and then collect terms according to their operator coefficients. This process yields:

\[
\hat{H}(\xi) = -\hat{N}(V) + \int \frac{d^3k}{(2\pi)^3} \left\{ (m_k + n_{-k}) [u_k^2 - v_k^2] E_k + 2u_k v_k \Delta_k \right\} + \Sigma_k \left[ 2u_k v_k E_k - (u_k^2 - v_k^2) \Delta_k \right] + \\
+ [2v_k^2 E_k + E' v_k^2 - u_k v_k \Delta_k] = -\hat{N}(V) + \hat{K}_1 + \hat{K}_2 + \hat{U} \quad (45)
\]

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where:

\[
E' = \alpha \int \frac{d^3k'}{(2\pi)^3} \frac{v^2_{k'}}{2} V''_{\pm}(k, k')
\]  

(46a)

\[
E_k = |k| - C_1 - E'
\]  

(46b)

\[
\Delta_k = \alpha \int \frac{d^3k'}{(2\pi)^3} V'_1(k, k') u_{k'} v_{k'}
\]  

(46c)

In order to make the full matter Hamiltonian diagonal we set the coefficient of \(\Sigma\) in (45) to zero. This allows us to solve for \(u_k\) and \(v_k\) in terms of \(\Delta_k\) and \(E_k\).

Since \(u^2_k + v^2_k = 1\), it is natural to use trigonometric variables. We set \(u_k = \cos \theta\) and \(v_k = \sin \theta\). Then we have:

\[
2u_k v_k E_k - (u^2_k - v^2_k) \Delta_k = 0
\]  

(47)

\[
\Rightarrow \sin(2\theta) E_k = \cos(2\theta) \Delta_k
\]

\[
\Rightarrow \tan(2\theta) = \frac{\Delta_k}{E_k}, \quad \sin(2\theta) = \frac{\Delta_k}{\sqrt{\Delta^2_k + E^2_k}} = \frac{\Delta_k}{\epsilon_k}, \quad \cos(2\theta) = \frac{E_k}{\epsilon_k}
\]  

(48)

Using the above the various terms in (45) become:

\[
\hat{U} = \int \frac{d^3k}{(2\pi)^3} \{ \left(1 - \frac{E_k}{\epsilon_k}\right) \left(E_k + \frac{E'_k}{2}\right) - \frac{\Delta^2_k}{2\epsilon_k} \}
\]  

(49a)

\[
\hat{K}_1 = \int \frac{d^3k}{(2\pi)^3} \epsilon_k (m_k + n_{-k})
\]  

(49b)

From (49b) it is clear that the spectrum is now bounded from below by \(\Delta_k\) which therefore is the mass gap.

Now we do a rough calculation to estimate the value of \(\Delta_k\). First let:

\[
u_k = (\frac{1}{2} + x_k)^{\frac{1}{2}}, \quad v_k = (\frac{1}{2} - x_k)^{\frac{1}{2}}
\]  

(50)

Then (47) becomes:
\[ 2E_k \left( \frac{1}{4} - x_k^2 \right)^{\frac{1}{2}} - 2x_k \Delta_k = 0 \]
\[ \Rightarrow x_k = \pm \frac{E_k}{2 \sqrt{E_k^4 + \Delta_k^2}} \]  

(51)

Inserting the solution for \( x_k \) into the expression (46c) for \( \Delta_k \), we get the gap equation:

\[ \Delta_k = \alpha \int \frac{d^3 k'}{(2\pi)^3} \frac{V'_1(k,k') \Delta_{k'}}{2 \sqrt{E_{k'}^2 + \Delta_{k'}^2}} \]  

(52)

In the above expression the potential \( V'_1 \sim O(1) \). We use a mean-field approximation to set the value of this potential to a constant \( V_a \).

Then (52) becomes:

\[ \Delta \approx \alpha V_a D(0) \int_{-\hbar \omega_D}^{\hbar \omega_D} \frac{dE_k}{2 \sqrt{E_k^4 + \Delta^2}} \]
\[ \Rightarrow 1 \approx \frac{\alpha V_a D(0)}{2} \ln \frac{\sqrt{(\hbar \omega_D)^2 + \Delta^2 + \hbar \omega_D}}{\sqrt{(\hbar \omega_D)^2 + \Delta^2 - \hbar \omega_D}} \]
\[ \Rightarrow \Delta \approx \frac{2\hbar \omega_D \exp^\nu}{\exp^\nu - 1} \left( \nu = \frac{2}{\alpha V_a D(0)} \right) \]  

(53)

where in the first line we have used the fact that \( \frac{d^3 k}{2\pi} \approx D(0) dE_k \). \( D(0) \) is the density of states at the fermi surface. We note that the gap depends on the Immirzi parameter which is contained in \( \alpha \). Now the density of states, \( D(E) \), for a field in a 3-dimensional box of volume \( V \) is:

\[ D(E) = \frac{dN}{dE} = \frac{dN}{dK} \frac{dK}{dE} = \frac{VE^2}{\pi^2} \]  

(54)

The volume of our co-moving box, and hence \( D(E) \), scales as \( a^3 \). Therefore the gap is an increasing function with respect to \( t \). This behavior is shown in Fig [1]

The gap has different behavior in the strong \((V_a D(0) >> 1)\) and weak \((V_a D(0) << 1)\)coupling limits, corresponding to \( a >> 1 \) and \( a << 1 \).

\[ \Delta \sim 2\hbar \omega_D \exp^\nu \quad \text{Weak coupling} \]
\[ \Delta \sim 2\hbar \omega_D \alpha V_a D(0) \quad \text{Strong coupling} \]  

(55)
For small $a$ the gap is exponentially suppressed and for large $a$ it grows as $a^3$. In particular, in an inflating background the gap grows as $\exp^{3Ht}$.

One has to keep in mind that this is a semiclassical calculation and breaks down for small $a$ as we enter a non-perturbative regime where quantum gravitational fluctuations of the metric must be taken into account.

VII. DISCUSSION

Eqn. (49a) is the expression for the potential energy of the fermi gas. The gap equation (52) has two solutions. The trivial solution is zero and corresponds to the free fermi gas. In this case (49a) reduces to the Hartree-Fock potential energy for the free fermi gas $[5]$. When the condensate forms the potential is reduced by the amount given by the last term in (52)\(^8\). The full Hamiltonian constraint (22) now becomes:

$$\frac{1}{V} \int d^3 x \ a^3 \mathcal{H} = -\frac{3}{\kappa} a^3 H^2 + a^3 (\Lambda_0 - \Lambda_{corr}) + a^3 \int \frac{d^3 k}{(2\pi)^3} \sqrt{E_k^2 + \Delta_k^2} (m_k + \bar{m}_k + n_{-k} + \bar{n}_k)$$

(56)

where $V$ is the volume of integration over the three-manifold. The correction to the cosmological constant is given by two times the last term of (49a)\(^9\):

$$\Lambda_{corr} = 2 \int \frac{d^3 k}{(2\pi)^3} \frac{\Delta_k^2}{2 \sqrt{E_k^2 + \Delta_k^2}} \approx \frac{2 \Delta^2}{\alpha}$$

(57)

where the third expression is obtained by using the approximation discussed at the end of the previous section. In $[7, 9]$ a perturbative one-loop calculation done for fermions coupled to gravity

\(^8\) The other terms in (52) are also affected when he have a condensate. However, this is perturbation is negligible compared to the that due to the gap term.

\(^9\) We have a contribution from the left and right handed spinors.
via a quartic potential showed that the cosmological constant must be proportional to $\Delta^2$. Here we have done a non-perturbative calculation to demonstrate that this expectation is indeed borne out albeit it is the correction $\Lambda_{\text{corr}}$, and not $\Lambda_0$, which is proportional to $\Delta^2$.

**VIII. CONCLUSION**

In this work we have demonstrated that when a covariant coupling to fermions in General relativity induces a four fermion coupling, the Hamiltonian reduces to a BCS theory. The gravitational field also induces a chemical potential which creates a Fermi-surface. By employing the appropriate Boguliobov transformation we were able to diagonalize this Hamiltonian and evaluate the energy gap. This gap played the role of negating the cosmological constant. In a time dependent background the gap is also time dependent. However, further analysis is needed to make this expectation concrete and we leave this up to a future work. By extending our mechanism to the full theory and incorporating fermionic spin-networks (open holonomies) we hope to show that the BCS mechanism persists in the semiclassical approximation.

There are questions this work raises that need to be explored further:

- All fermionic species, regardless of whether they couple to Yang-Mills or electromagnetic fields, also couple to gravity. Therefore a four-fermi interaction is induced for all fermions and does not distinguish between different species. In the real world we have fermions that condense (quarks) and those that are free (electrons and neutrinos). It remains to be understood how in this mechanism can one include interactions which would distinguish between different species, allowing some to condense and others to remain free and we will pursue this in a forthcoming paper.

- How would the generic inflationary scenario be modified due to the presence of the gap? $\langle \bar{\Psi}\Psi \rangle$ develops a non-zero vev and is a scalar. Can this composite scalar then play the role of the scalar field in cosmology?

- We have not studied the effects of gravitational perturbations on the condensate. In particular if $\Gamma_i^a$ is non-zero then the number operator would be modified by a term proportional to $e_i^a \Gamma_i^a$. This would increase the chemical potential thereby decreasing $\Lambda_{\text{corr}}$. The effect of these perturbations and the other questions mentioned above will be studied in a future work.

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APPENDIX

1. Weyl Representation

For the internal space we use the metric with signature \((-+++)\). For this signature, the gamma matrices in the Weyl representation are:

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \sigma^a & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix};
\] (A.1)

2. Spinors

The expressions for the dedensitized spinors are:

\[
\xi_{k\uparrow} = \begin{pmatrix} \cos \theta \frac{2}{\pi} \\ \sin \theta \frac{2}{\pi} \exp^{i\phi} \end{pmatrix} \quad \xi_{k\downarrow} = \begin{pmatrix} -\sin \theta \frac{2}{\pi} \\ \cos \theta \frac{2}{\pi} \exp^{i\phi} \end{pmatrix}
\] (A.2)

It is manifest the above spinors are orthogonal and by changing \(k\) to \(-k\) one can check the useful identity \(\xi_{k\uparrow} = -\xi_{-k\downarrow}\). A factor \(q^\pm\) is required to convert the above spinors into half-densities.

3. Certain operator contractions

Following are operator identities used in the main text:

\[
m_k = \alpha^+_k \alpha_k; \quad n_{-k} = \beta^+_{-k} \beta_{-k}; \quad \Sigma_k = \alpha^+_k \beta^+_{-k} - \beta_{-k} \alpha_k \]
\[
N(a^+_k a_k) = u^2_k m_k - v^2_k n_{-k} + u_k v_k \Sigma_k
\]
\[
N(b^+_k b_{-k}) = u^2_k n_{-k} - v^2_k m_k + u_k v_k \Sigma_k
\]
\[
N(b_{-k} a_k) = u^2_k \beta_{-k} \alpha_k - v^2_k \alpha^+_k \beta^+_{-k} - u_k v_k (m_k + n_{-k})
\]
\[
N(a^+_k b^+_k) = u^2_k \alpha^+_k \beta^+_{-k} - v^2_k \beta_{-k} \alpha_k - u_k v_k (m_k + n_{-k})
\]
\[
\Lambda^+_k a_{k'} = b^+_k b_{-k'} = v^2_k \delta_{k,k'}; \quad b^+_k a_{k'} = a^+_k b^+_k = u_k v_k \delta_{k,k'}
\] (A.3) (A.4) (A.5) (A.6) (A.7) (A.8)
4. Derivation of the torsion term

The following is essentially the content of [3]. We include this derivation here in order to keep this paper self-contained. The Holst action can be written as:

\[ S_H = \frac{1}{2\kappa} \int d^4x \ e I^\mu e J^\nu P^{IJ}_{KL} F^{KL}_{\mu\nu} \] (A.10)

where:

\[ P^{IJ}_{KL} = \delta^I_{[K} \delta^J_{L]} - \frac{1}{2\gamma} \epsilon^{IJ}_{KL} \] (A.11)

whose inverse is:

\[ P^{-1}^{IJ}_{KL} = \frac{\gamma^2}{\gamma^2 + 1} \left( \delta^I_{[K} \delta^J_{L]} - \frac{1}{2\gamma} \epsilon^{IJ}_{KL} \right) \] (A.12)

Variation of the Holst action w.r.t the connection yields:

\[ \frac{\delta S_H}{\delta A^K_{\mu}} = -\frac{1}{\kappa} D_\mu \left( e I^\mu e J^\nu \right) P^{IJ}_{KL} \] (A.13)

Likewise variation of the Dirac action w.r.t yields:

\[ \frac{\delta S_D}{\delta A^K_{\mu}} = -\frac{\bar{\psi}}{8} \gamma^{[K} e^\gamma_{\mu] e J^\nu e I^\nu} \psi = \frac{\bar{\psi}}{4} \epsilon_{KLM} \gamma^M \gamma^5 \] (A.14)

In the second line we have used the identity: \( \{\gamma^{[K} \gamma^\nu e J^\mu] e I^\nu e I^\nu \} = 2i \epsilon_{KLM} \gamma^M \gamma^5 \). Therefore the variation of the action \( S_H + S_D \) w.r.t to the connection yields:

\[ D_\mu \left( e I^\mu e J^\nu \right) P^{IJ}_{KL} = \frac{\kappa e}{4} \epsilon_{KLM} \gamma^M j^a M \] (A.15)

where \( j^a M \) is the axial current given by \( \bar{\psi} \gamma^M \gamma^5 \psi \).

Writing the connection as \( A^K_{\mu} = \omega^K_{\mu} + C^K_{\mu} \) where \( \omega \) is the connection compatible with the tetrad, and using \( P^{-1}^{IJ}_{KL} \) (A.15) becomes:

\[ C_{\mu[P} e^\nu_{Q]} + C_{[PQ]} e^\nu M = \frac{\kappa}{4} \frac{\gamma^2}{\gamma^2 + 1} e^\nu M j^a M \left( e^{MI}_{PQ} + \frac{1}{\gamma} \delta^I_{[PQ]} \right) \] (A.16)
Tracing over $\nu$ and $P$ we obtain:

$$C_{\mu Q}^{\nu} = \frac{3}{8} j_a Q \frac{\gamma}{\gamma^2 + 1} \tag{A.17}$$

From the above two equation we obtain:

$$C_{[PQ]R} = \kappa \frac{\gamma^2}{4 \gamma^2 + 1} j_a \left\{ \epsilon_{MPQR} - \frac{1}{2\gamma} \delta_{M[P} \delta_{QR]} \right\} \tag{A.18}$$

where we raise and lower indices using the tetrad. Then we have:

$$C_{PQR} = C_{[PQ]R} + C_{[RP]Q} + C_{[QR]P} \tag{A.19}$$

and finally:

$$C^{IJ}_{\mu} = \kappa \frac{\gamma^2}{4 \gamma^2 + 1} j_a \left\{ \epsilon_{MK}^{IJ} \epsilon^{K}_{\mu} - \frac{1}{2\gamma} \delta^{[IJ}_{M} \epsilon^{K}_{\mu]} \right\} \tag{A.20}$$

Inserting the above expression into the first-order gravity+matter action yields [8]. The contribution comes only from $S_D$. The Holst action yields the boundary term [10].

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