Interaction of the electro-magnetic precursor from a relativistic shock with the upstream flow. I. Synchrotron absorption of strong electromagnetic waves

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ABSTRACT

This paper is the first in the series of papers aiming to study interaction of the electro-magnetic precursor waves generated at the front of a relativistic shock with the upstream flow. It is motivated by a simple consideration showing that the absorption of such an electro-magnetic precursor could yield an efficient transformation of the kinetic energy of the upstream flow to the energy of accelerated particles. Taking into account that the precursor is a strong wave, in which electrons oscillate with relativistic velocities, the standard plasma-radiation interaction processes should be reconsidered. In this paper, I calculate the synchrotron absorption of strong electro-magnetic waves.

Key words: magnetic fields – radiation mechanisms: non-thermal – shock waves

1 INTRODUCTION

The relativistic wind originating from the rotating, magnetized neutron star (pulsar) terminates at a strong reverse shock, the shocked plasma inflating within the surrounding gas a bubble filled with relativistic particles (mostly electrons and positrons) and magnetic fields. This bubble is called a pulsar wind nebula (PWNe). By now, the overall morphology of PWNe is more or less understood in the scope of MHD models (see, e.g., reviews by (Arons 2007; Kirk et al. 2009; Porth et al. 2017). However, the physical processes giving rise to particle acceleration in PWNe remain obscure; none of the present theories can explain how their spectra formed.

The generic observational feature of PWNe is a flat radio spectrum, $F_{\nu} \propto \nu^{-\alpha}$, with $\alpha$ between 0 and 0.3, extending in some cases out to the infrared. At high frequencies, the spectrum softens, and in the X-ray band, $\alpha > 1$. Such an injection spectrum suggests a very unusual acceleration process. The observed radio spectrum implies a power-law energy distribution of injected electrons, $N(E) \propto E^{-\alpha}$, with a shallow slope $1 < \kappa < 1.6$. Such an energy distribution is remarkable in that most of the particles are found at the low energy end of the distribution, whereas particles at the upper end of the distribution dominate the energy density of the plasma. Specifically in the Crab Nebula, the observed emission spectrum implies that the particles in the energy range from $E_{\text{min}} \sim 100$ MeV to $E_{\text{break}} \sim 1$ TeV are injected into the nebula with a spectral slope $\kappa = 1.6$, so most of the injected energy ($\sim 5 \times 10^{38}$ erg s$^{-1}$) is carried by TeV particles, whereas $\sim 100$ times more particles are found at low energies of less than 100 MeV. This means that the acceleration process somehow transfers most of the total energy of the system to a handful of energetic particles, leaving only a small fraction of the energy for the majority of the particles. This is not what one would normally expect from the conventional first-order Fermi acceleration process, in which the particle flow is randomized at the shock and only a fraction of the upstream kinetic energy is deposited in highly accelerated particles.

It was previously assumed (Lyubarsky 2003) that the unusual particle energy distribution in PWNe may be explained if most of the pulsar spin-down energy is still stored in the striped magnetic field when the flow enters the termination shock. In this case, the alternating magnetic fields annihilate at the shock front, and one can speculate that the radio-to-optical emission of PWNe is generated by pairs accelerated in the course of the reconnection process. Particle in cell (PIC) simulations (Sironi & Spitkovsky 2011) indeed show that the alternating fields easily annihilate at the shock. However, nonthermal particle distributions were found to be generated only if the pair density in the pulsar wind is extremely high, orders of magnitude larger than that compatible with the observed particle density in the nebula. Therefore, an alternative explanation for the unusually flat particle spectrum in PWNe must be sought.

The aim of this series of papers is to investigate the particle acceleration upstream of the shock due to absorption of the electromagnetic precursor wave generated at the shock front. The pulsar wind is magnetized therefore the termination shock is mediated by the Larmor rotation. In this case, the synchrotron maser instability produces strong
low-frequency electromagnetic waves propagating both upstream and downstream of the shock (Langdon et al. 1988, Gallant et al. 1992, Iwamoto et al. 2017) and transferring a few per cent of the upstream energy flow. A strong precursor wave has also been found by Amano & Kirk (2013) and Giacchê & Kirk (2017) who considered interaction of a circularly polarized transverse magnetic shear wave, which models the striped structure of the pulsar wind, with the termination shock. These authors attribute the precursor not to the maser instability but just to wave conversion at a shock discontinuity. In any case, the energy density of the precursor wave exceeds, in the comoving frame of the upstream flow, the plasma energy density therefore when and if the wave is eventually absorbed by the flow, the plasma parameters change significantly even though in the shock frame, the absorbed energy is small as compared with the flow energy.

In order to see why this is the case, consider a body of mass $M$ moving with a high Lorentz factor $\Gamma$ towards a radiation beam. It follows immediately from energy and momentum conservation that after the body absorbs some energy $\varepsilon$, it acquires a Lorentz factor

$$\Gamma_1 \approx \frac{\Gamma}{\sqrt{1 + 4\varepsilon^2 M^2 c^2}}, \quad (1)$$

where $c$ is the speed of light. One sees that the body is decelerated significantly if $\varepsilon > M^2 c^2 / \Gamma^2$. Therefore in the highly relativistic case, the body can decelerate even if the absorbed energy is small. An observer in the lab frame would say that most of the kinetic energy of the body has been transformed into internal energy.

This simple consideration shows that an electromagnetic precursor can have a profound effect on the particle acceleration process, because when this radiation is absorbed in the upstream flow, the kinetic energy of the flow is transformed mostly into internal energy. The particle spectrum is determined by collisionless absorption processes. Therefore the internal energy is not thermalized; on the contrary, one would expect non-thermal particle distributions. As the first step, one has to analyze the decay of a strong EM wave propagating in a pair plasma. A few processes look important: synchrotron absorption, induced scattering, three-wave decay of the pumping wave into an electro-magnetic wave and a magnetosonic wave (stimulated Brillouin scattering), non-linear self-focusing of the wave. Note that the Raman scattering of the electro-magnetic wave into another electro-magnetic wave and the Langmuir wave, does not occur in pair plasmas because the masses of two opposite charges are equal. Understanding which of the many processes dominates in what parameter domain is essential in order to set up the necessary numerical simulations.

An important point is that the wave is strong in the sense that the wave strength parameter,

$$a = \frac{\varepsilon E}{m_e c \omega}, \quad (2)$$

where $\omega$ and $E$ are the wave angular frequency and amplitude, is large. In the field of such a wave, electrons experience oscillations with relativistic velocities (e.g., Landau & Lifshitz 1973). Therefore the standard perturbative approach to plasma-wave interactions could not be used; one has to use methods developed in the field of laser-matter interaction (e.g., Mourou et al. 2000).

There is a vast literature on the interaction of strong waves with plasmas (e.g., reviews by Shukla et al. 1986 and Mourou et al. 2000). However, the parameter range relevant for the case of interest (the wave frequency is much larger than the plasma frequency so that the wave velocity is close to $c$, the electron-positron plasma so that effects relied on the mass difference of the charge carriers, such as Raman scattering, are absent, relativistic mean velocities of particles etc) has attracted little attention. Radiation of relativistic particles oscillating in a strong wave (non-linear Compton scattering) has been thoroughly studied (Gunn & Ostriker 1971, Arons 1972, Blandford 1972, Stewart 1974). In particular, a radiative damping of strong waves has been considered (Asseo et al. 1978, Mochol & Kirk 2013). However, the spontaneous scattering could not play a significant role in the case of interest because this process is unable to take a significant fraction of the flow energy before the flow enters the termination shock (unless a strong radiation source is presented in the system, like in PSR B1259-63). The induced scattering looks more promising (Melrose 1980) derived the kinetic equation for induced scattering of strong waves, however, the process has not been studied thoroughly.

In this paper, I consider synchrotron absorption of strong electromagnetic waves. I address the high frequency case when the wave propagates like in vacuum; then the absorption coefficient may be calculated just by finding the average energy the single electron gains from the vacuum wave. The paper is organized as follows. In the next section, I present equations of motion for an electron in the presence of a strong electro-magnetic wave and a background magnetic field. The exact solutions for the zero background field are reminded and constants of motions of this solution are used as variables in the case of a weak background field. In sect. 3, the motion of the electron guiding centre is found in the case when the wave frequency is large as compared with the Larmor frequency. In sect. 4, small oscillations with respect to the slow Larmor rotation are considered. In sect. 5, the energy exchange between the wave and the electron is found and the absorption coefficient is calculated. The validity of the approximations is analyzed in sect. 6. The obtained results are discussed and qualitatively explained in sect. 7. In Appendix, the Einstein coefficients method is used to derive the classical synchrotron absorption coefficient in the weak wave limit.

2 BASIC EQUATIONS

The absorption coefficient may be found by calculating the work done by the wave on the particles. In the case of a strong wave, one could not consider particle oscillations in the field of the wave as a small perturbation. However, one could exploit the fact that the particle motion in the field of a strong wave may be solved exactly if there is no background magnetic field. This solution may be used in the presence of the background field if the wave frequency significantly exceeds the Larmor frequency; then one could find the particle motion by averaging over the fast wave oscillations.

The wave generated by the maser instability at the shock front is polarized perpendicularly to the magnetic field. When the particles in the upstream flow absorb the wave, they begin to rotate around the magnetic field lines.
The relation between the time, \(\omega\) vector potential \(A\) directed in the wave propagate in the well as the propagation direction, lie in the same plane. Let the ground magnetic field, and the wave polarization vector, as electrons gyrate in the plane perpendicular to the back:

\[v_x = \frac{eE}{\omega \gamma} \cos \eta; \quad \eta = \omega(t - x/c);\]

and the electron equations of motion are written as

\[\frac{mcu_x}{dt} = \frac{eV_x}{c} \left( \frac{\partial A}{\partial x} + B_0 \right);\]

\[\frac{mcu_y}{dt} = \frac{eV_y}{c} \left( \frac{\partial A}{\partial y} + v_x B_0 \right);\]

\[\frac{mc\gamma}{dt} = \frac{eV_y}{c} \frac{\partial A}{\partial t};\]

where \(\gamma = (1 - v^2/c^2)^{-1/2}\) is the electron Lorentz factor, \(u = (v/c)\gamma\) the 4-velocity, \(B_0\) the background magnetic field. The electron is assumed to move in the \(x - y\) plane.

It is well known that if there is no the background field, \(B_0 = 0\), the above system of equations has two integrals of motion (e.g., Gunn & Ostriker 1971; Landau & Lifshitz 1974). Invariancy with respect to a shift in the \(y\) direction implies conservation of the \(y\) component of the generalized momentum, which means that the quantity

\[w = u_y + a \cos \eta\]

remains constant. One sees that if \(a > 1\), the electron oscillations become relativistic. Invariancy with respect to a transformation \(x \to x + s, t \to t + s\) implies conservation of the quantity

\[g = \gamma - u_x.\]

Making use of the identity \(v_x^2 + u_y^2 + 1 = \gamma^2\), one expresses the velocity components and the electron Lorentz factor via the integrals of motion as

\[v_x = 1 + (w - a \cos \eta)^2 - g^2;\]

\[v_y = 2g(w - a \cos \eta);\]

\[\gamma = 1 + (w - a \cos \eta)^2 + g^2.\]

The relation between the time, \(t\), and the phase, \(\eta\), is found by differentiating eq. 2 with respect to \(t\) and using eq. 3:

\[\frac{d\eta}{dt} = \frac{\omega g}{\gamma}.\]

The electron "sees" the full period of the wave for the time

\[T = \frac{1}{\omega g} \int_0^{2\pi} \gamma d\eta = \frac{\pi}{\omega g^2} \left( 1 + w^2 + \frac{1}{2}a^2 + g^2 \right).\]

The components of the velocity and the Lorentz factor averaged over the wave period are found as

\[\bar{v}_x = \frac{1}{T} \int_0^T v_x dt = \frac{c}{\omega T g} \int_0^{2\pi} u_x d\eta = \frac{1 + w^2 + \frac{1}{2}a^2 - g^2}{1 + w^2 + \frac{1}{2}a^2 + g^2};\]

\[\bar{v}_y = \frac{2\omega g}{1 + w^2 + \frac{1}{2}a^2 + g^2};\]

\[\bar{\gamma} = \frac{(1 + g^2 + w^2 + \frac{1}{2}a^2)^2 + 2a^2 w^2 + \frac{3}{2}a^4}{2g (1 + w^2 + \frac{1}{2}a^2 + g^2)}.\]

The velocity of the electron guiding centre is written as

\[(\bar{\gamma})^2 = (\bar{v}_x)^2 + (\bar{v}_y)^2 = c^2 - \frac{4g^2 (1 + \frac{1}{2}a^2)}{(1 + w^2 + \frac{1}{2}a^2 + g^2)} c^2.\]

In the presence of the background magnetic field, one can find the electron motion if the wave frequency is large as compared with the Larmor frequency,

\[\omega \gg \omega_B \equiv \frac{eB_0}{mc};\]

then the electron motion could be described as rapid oscillations superimposed on a slow Larmor rotation of the guiding centre. In this case, one can conveniently use the "integrals of motion", \(g\) and \(w\), as new unknowns. Differentiating eqs. 3 and 9 in time and making use of eqs. 5 and 6 yields

\[\frac{dg}{dt} = -\omega B v_y;\]

\[\frac{dw}{dt} = -\omega B v_x.\]

Now making use of eqs. 10, 11, 12 and 13, one gets the closed system of equations

\[\frac{dg}{d\eta} = -\frac{\omega B}{\omega} (w - a \cos \eta);\]

\[g \frac{dU}{d\eta} = -\frac{\omega B}{2\omega} [1 + (w - a \cos \eta)^2 - g^2].\]

One sees that one can use \(g^2\) instead of \(g\) as an unknown function. These equations could be solved by separating slow and rapid motions.

### 3 Motion Averaged Over the Rapid Oscillations

Let us present the unknown functions in the form

\[g^2 = G^2 + \psi; \quad w = U + \xi;\]

where \(G\) and \(U\) are a slowly varying quantities defined as

\[G^2 = (2\pi)^{-1} \int_0^{2\pi} g^2 d\eta\]

and

\[U = (2\pi)^{-1} \int_0^{2\pi} u_x d\eta\]

whereas \(\psi\) and \(\xi\) are small rapidly oscillating corrections. Substituting this expansion into eqs. 22 and 23, linearizing in small \(\psi\) and \(\xi\) and averaging in \(\eta\) yields equations describing motion of the guiding centre:

\[\frac{dG^2}{d\eta} = -\frac{2\omega B U}{\omega};\]

\[\frac{dU}{d\eta} = -\frac{\omega B}{2\omega G^2} \left[1 + U^2 + \frac{1}{2} \xi^2 - G^2\right].\]

Dividing the second equation by the first one, one gets a linear equation with respect to \(U^2:\)

\[2G \frac{dU^2}{d\eta} = U^2 + 1 + \frac{1}{2} \xi^2 - G^2.\]
which is solved giving the first integral of the system \(25\) and \(26\):

\[
U^2 + (G - \Gamma)^2 = \Gamma^2 - 1 - \frac{1}{2}v^2,
\]

where \(\Gamma\) is a constant.

According to the above solution, the electron moves along a circle in the \(U-G\) plane. The motion of the guiding centre in the coordinate space is described by eqs. \(15\) and \(16\). Substituting \(w\) and \(g\) by \(U\) and \(G\), correspondingly, and making use of eq. \(28\) yields

\[
\frac{\gamma}{c} = 1 - \frac{G}{\Gamma}; \quad \frac{\gamma}{c} = 1 + \frac{U}{\Gamma}; \quad \gamma = \frac{\sqrt{\Gamma^2 - 1 - \frac{1}{2}a^2}}{\Gamma}.
\]

One sees that the guiding centre of the electron gyrates around the magnetic field with a constant velocity; the Lorentz factor of the averaged motion is found as

\[
\gamma_{\text{bc}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\Gamma}{\sqrt{1 + \frac{1}{2}a^2}}
\]

(30)

The averaged velocity is relativistic if

\[
\Gamma^2 \gg 1 + \frac{1}{2}a^2;
\]

below this condition is assumed to be fulfilled.

The variable \(U\), which is the averaged vertical component of the electron 4-velocity, varies from \(U = -\sqrt{\Gamma^2 - 1 - \frac{1}{2}a^2} \approx -\Gamma\) to \(U = \sqrt{\Gamma^2 - 1 - \frac{1}{2}a^2} \approx \Gamma\) and vanishes twice during the rotation period, at the upper and the lower points of the electron orbit. The electron moves in the direction of the wave in the upper part of the orbit and towards the wave in the lower point. At these points, \(G\) reaches minimum and maximum, correspondingly:

\[
G_{\text{min, max}} = \Gamma \pm \sqrt{\Gamma^2 - 1 - \frac{1}{2}a^2} \approx \left\{ \frac{1 + \frac{1}{2}a^2}{2\Gamma} \right\}.
\]

The averaged Lorentz factor, eq. \(17\), varies along the Larmor orbit as

\[
\Xi = 1 + \left( \frac{a}{\Gamma} \right)^2 \frac{U^2 + \frac{3}{2}a^2}{2G^2}.
\]

(33)

One sees that in the case of weak waves, \(a \ll 1\), the constant \(\Xi\) is just the Lorentz factor of the electron. Inspection of eqs. \(33\) and \(28\) shows that for strong waves at the condition \(31\), \(\Xi\) remains close to \(\Gamma\) in the most of the Larmor orbit and only in the upper part increases reaching

\[
\Xi_{\text{max}} = \left( 1 + \frac{3a^4}{2(2 + a^2)^2} \right) \Gamma
\]

in the upper point. Beyond the upper point, \(\Xi\) decreases and goes to \(\Gamma\) again therefore when considering only motion averaged over the rapid oscillations, one could not find the net energy gain due to the absorption of the wave. One has to find the corrections \(\psi\) and \(\xi\), which will be done in the next section.

In order to find the dependence of the variables on time, one can use eq. \(12\). Substituting \(\gamma\) from eq. \(12\) and integrating, one gets the relation between the phase and the time; for \(\eta \gg 1\) it looks like

\[
t = \int \frac{1 + (w - a \cos \eta)^2 + g^2}{2g^2} d\eta.
\]

(35)

Neglecting oscillating parts of \(w\) and \(g\), one can substitute them by \(U\) and \(G\), correspondingly. Substituting the other oscillating terms by their averaged values, one gets

\[
t = \int \frac{1 + w^2 + \frac{3}{2}a^2 + g^2}{2g^2} d\eta = -\frac{1}{\omega_B} \int \frac{1 + U^2 + \frac{3}{2}a^2 + G^2}{UG} dG.
\]

(36)

Here in the last equality, the integration variable has been substituted by \(G\) with the aid of eq. \(25\). The integral is performed after expressing \(U\) via \(G\) with the aid of eq. \(28\), then one finds

\[
G = \Gamma - \left( \Gamma^2 - 1 - \frac{1}{2}a^2 \right)^{1/2} \cos \Omega(t - t_0);
\]

(37)

\[
\Omega = \frac{\omega_B}{\Gamma}.
\]

(38)

One sees that in the presence of a strong wave, the guiding centre of the electron experiences Larmor rotation around the background magnetic field. Taking into account that the constant \(\Gamma\) is equal to the average Lorentz factor in the most of the orbit, the Larmor period is not affected by the wave.

### 4 Oscillations with Respect to the Averaged Motion

In order to find oscillations with respect to the average motion of the guiding centre, one linearizes eqs. \(22\) and \(23\) in small \(\psi\) and \(\xi\) and eliminates the zeroth order terms by setting

\[
\frac{d\psi}{d\eta} = \frac{2\omega_B}{\omega} \left( a \cos \eta - \xi \right); \quad \frac{d\xi}{d\eta} = \frac{\omega_B}{\omega} \left( \frac{U_\xi - aU \cos \eta + a^2}{4 \cos 2\eta - 1} \psi \right).
\]

(39)

(40)

Eliminating \(\xi\) and making use of eq. \(26\), one gets a single equation for \(\psi\)

\[
d^2\psi \frac{d\psi}{d\eta^2} + \frac{\omega_B}{\omega} \frac{U}{G^2} \frac{d\psi}{d\eta} + \frac{\omega_B^2}{\omega^2 G^4} \left( 1 + \frac{3}{2}a^2 + U^2 \right) \psi - \frac{2\omega_B a}{\omega} \sin \eta + \frac{\omega_B^2 a^2}{2\omega^2 G^2} \cos 2\eta.
\]

(41)

In this equation, \(U\) and \(G\) are related by eq. \(28\); the dependence of these functions on \(\eta\) may be found by substituting eq. \(28\) into eq. \(25\).

The relativistic electron exchanges energy with the wave in the upper part of the orbit, where it moves in the direction of the wave thus remaining for a long time in phase with the wave. If it follows from eqs. \(31\) and \(32\) that in the upper part of the orbit, \(G \ll \Gamma\); then eq. \(28\) is reduced to

\[
G = \frac{1 + \frac{3}{2}a^2 + U^2}{2\Gamma^2}.
\]

(42)

Substituting this relation into eq. \(25\) and integrating, one gets

\[
\eta = \phi - \frac{\omega}{2\omega_B \Gamma^2} \left[ \left( 1 + \frac{1}{2}a^2 \right) U + \frac{U^3}{3} \right],
\]

(43)

where \(\phi\) is the phase of the wave when the electron passes the upper point of the orbit. Eqs. \(12\) and \(13\) describe, in parametric form, motion of the electron guiding centre in the upper part of the orbit. Recall that \(U\) is the averaged
over rapid oscillations vertical component of the electron 4-velocity; it passes zero at the upper point of the orbit.

Let us now solve eq. (41) in the upper part of the orbit. Instead of substituting directly eqs. (42) and (43) into the equation, one can conveniently introduce a new independent variable

$$ z = - \frac{U}{\sqrt{1 + \frac{1}{2} a^2}} $$

Then

$$ \eta = \phi + S \left[ z + \frac{3}{2} \right] $$

$$ S = \frac{\omega}{2 \omega_B} \left( 1 + \frac{1}{2} a^2 \right)^{3/2} $$

$$ G = \left( 1 + \frac{1}{2} a^2 \right) \frac{1 + z^2}{2 \Gamma} $$

Now eq. (41) takes the form

$$ (1 + z^2) \frac{d^2 \psi}{dz^2} - 4z \frac{d\psi}{dz} + 4\psi = - \frac{aS}{2(1 + a^2)} \left\{ (1 + z^2)^3 \sin \left[ \phi + S \left( z + \frac{3}{2} \right) \right] \right\}. $$

One can check easily that the corresponding homogeneous equation is satisfied by $\psi = z$ and $\psi = 1 - 2z^2 - \frac{z^2}{3}$. Then variation of constants yields the solution of eq. (47) in the form

$$ \psi = \frac{aS}{2(1 + a^2)} \int_{-\infty}^{\infty} \left\{ (1 + z^2)^3 \sin \left[ \phi + S \left( z + \frac{3}{2} \right) \right] \right\} dz $$

$$ \times \left\{ \sin \left[ \phi + S \left( z' + \frac{3}{2} \right) \right] \right\} + \frac{a}{2S} \left( 1 + \frac{1}{2} a^2 \right)^{1/2} \left( 1 + z^2 \right) \cos 2 \left[ \phi + S \left( z' + \frac{3}{2} \right) \right] \right\} dz'. $$

5 THE ENERGY EXCHANGE BETWEEN THE ELECTRON AND THE WAVE

Variation of the particle energy could be found by differentiating eq. (42) for the particle Lorentz factor and making use of eqs. (22) and (23):

$$ \frac{d\gamma}{d\eta} = - \frac{\omega a}{\omega_B} \frac{d\omega}{d\eta} \sin \eta. $$

The energy gain after passing the upper part of the orbit is found by integrating eq. (50). In all practical cases, the absorption by an ensemble of homogeneously distributed electrons is of interest; therefore the result should be averaged in phases, $\langle \cdots \rangle = (2\pi)^{-1} \int_0^{2\pi} \cdots \, d\phi$. Performing integration by parts, one gets

$$ \langle \Delta \gamma \rangle = - \frac{\omega a}{\omega_B} \left\{ \int_{-\infty}^{\infty} \frac{d\eta}{d\eta} \sin \eta \, d\eta \right\} = \frac{\omega a}{\omega_B} \left\{ \int_{-\infty}^{\infty} g \cos \eta \, d\eta \right\}. $$

In order to get a non-zero result after averaging in phases, one has to take into account oscillations of the electron with respect to the guiding centre.

It follows from the expansion (24) that

$$ g = \sqrt{G^2 + \psi} = G + \frac{\psi}{2G} $$

Then the particle energy gain is written as

$$ \langle \Delta \gamma \rangle = \frac{\omega a^2}{2\omega_B} \left\{ \int_{-\infty}^{\infty} \frac{\psi}{G} \, d\eta \right\} $$

$$ = \frac{a \omega^2}{2\omega_B^2} \left( 1 + \frac{1}{2} a^2 \right)^{1/2} \int_{-\infty}^{\infty} \left\{ \psi \cos \left[ \phi + S \left( z + \frac{3}{2} \right) \right] \right\} dz $$

where in the last equality, eqs. (45), (46) and (47) were used.

When substituting the solution (49) into eq. (53), the term with $\cos 2 \left( \phi + S \left( z' + \frac{3}{2} \right) \right)$ vanishes after the averaging in phases. The term with $\sin \left[ \phi + S \left( z' + \frac{3}{2} \right) \right]$ is transformed as

$$ \langle \sin \eta' \cos \eta \rangle = \frac{1}{2} \sin (\eta' - \eta) = \frac{1}{2} \sin \left[ S \left( z' - z + \frac{3}{2} \right) \right]. $$

Then one finds

$$ \langle \Delta \gamma \rangle = \frac{a \omega^2 \alpha}{4\omega_B^2 \Gamma} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} \left( z' - z \right) \left[ 1 + 2z' \right] $$

$$ + \frac{8 \pi^2 a^2 S^{2/3} \Gamma}{3 (1 + \frac{1}{2} a^2)} \text{Ai}' \left( S^{2/3} \right) \text{Ai}' \left( S^{2/3} \right) - 4 S^{4/3} \Gamma \text{Ai} \left( S^{2/3} \right); $$

where

$$ \text{Ai} (t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos \left( tx + \frac{1}{3} t^3 \right) \, dx $$

is the Airy function. One sees that the range of $z$ satisfying the condition $S \left( z + \frac{3}{2} \right) \sim 1$, which corresponds, according to eq. (45), to $\eta \sim 1$, contributes to the integrals. This means that the electron gains energy at $\eta \sim 1$, i.e. when it moves in phase with the wave at the upper part of the orbit.

At a small or a large $S$, one finds simple relations

$$ \langle \Delta \gamma \rangle = \frac{8 \pi^2 a^2 \Gamma}{3 (1 + \frac{1}{2} a^2)} \left\{ \frac{n}{S^{2/3} \Gamma^{(1/3)}} S^{2/3}; \quad S \ll 1; \right\} $$

$$ \left\{ \frac{S e^{-3/2}}{S^{2/3} \Gamma^{(1/3)}}; \quad S \gg 1; \right\} $$

where $\Gamma(x)$ is the gamma-function. One sees that the particle energy gain in one Larmor period is maximal at $S \sim 1$. In a weak wave, $a \ll 1$, the energy gain is always small, $\langle \Delta \gamma \rangle \ll 1$. In a strong wave, $a \gg 1$, it becomes significant, $\langle \Delta \gamma \rangle \sim \Gamma$, at $S \sim 1$.

The absorption cross-section, $\sigma$, is defined such that the energy absorbed by an electron per unit time is $\sigma$ times the Poynting flux in the wave. The electron absorbs on the average the energy $mc^2 \langle \Delta \gamma \rangle$ per rotation period, $T_B = 2\pi \Gamma / \omega_B$,  

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therefore one can write
\[ mc^2 \langle \Delta \gamma \rangle \frac{\omega_B}{2\pi} = \sigma \frac{E^2}{8\pi}. \]
Substituting eq. (57) yields finally
\[ \sigma = 2^{13/3} \pi r_e c \omega_B^{1/3} \left[ \frac{\omega}{\omega_0} \right]^{2/3} \times \left\{ \frac{\omega}{\omega_0} \right\}^{2/3} - 4 \left[ \frac{\omega}{\omega_0} \right]^{1/3} \frac{\omega}{\omega_0} \left[ \frac{\omega}{\omega_0} \right]^{2/3} \right\} \]
\[ = \frac{2^{7/3}}{3} \pi r_e c \omega_B^{1/3} \omega_0^{2/3} \psi \left( \frac{3}{2} \right) e^{-2 \omega_0} \left( \frac{\omega}{\omega_0} \right)^{2/3}, \] where
\[ \omega_0 = \frac{2 \omega_B B^2}{(1 + \frac{3}{2} a^2)^{3/2}}, \]
\[ r_e = \frac{e^2}{mc^2} \] is the classical electron radius. At a small \( a \), this expression reduces to the classical expression for the synchrotron absorption (see Appendix).

6 VALIDITY OF THE PERTURBATIVE SOLUTION

The above results are based on the perturbative solution, which is valid if \( \psi \ll G^2 \). In the upper part of the electron trajectory, where the electron exchanges energy with the wave, \( G \) is small therefore this condition could be violated. In order to check validity of the obtained solution, let us estimate \( \psi \) directly from eq. (11), which is simpler than finding estimates from the exact solution (12). Substituting eq. (17) and (14), one can write this equation as
\[ \frac{d^2 \psi}{d\eta^2} - \frac{2}{S(1 + z)^2} \frac{d\psi}{d\eta} + \frac{4}{S^2(1 + z)^2} \psi = 0. \]
Taking into account that \( \frac{d\eta}{d\psi} \sim \psi \), one can estimate \( \psi \) just balancing terms in the equation.

At \( S \gg 1 \), the lhs of the equation is dominated by the first term and the rhs is dominated by the first term. Therefore \( \psi \sim \omega_B a/\omega \), which implies
\[ \psi \sim \frac{a}{\omega} \frac{1}{S(1 + z)^2} \ll 1. \]
One sees that at large \( S \), i.e. at large frequencies, \( \psi \) remains small as compared with \( G^2 \) at any \( a \). It is no surprise that the approximate solution is valid in this case, because it follows from eq. (15) that at \( S \gg 1 \), the phase of the wave, \( \eta \), rapidly varies when \( z \), and therefore \( U \) and \( G \), vary slowly, which was an initial assumption of our perturbation method.

Now let us consider the case \( S \ll 1 \). In this case, we have to consider a few ranges of \( z \) separately. If \( z \gg S^{-1/3} \), both the lhs side and the rhs of the equation are dominated by their first terms therefore the estimate (55) remains valid in this case too. In the small range \( S^{-1/3} < z < S^{-1/3} \), the lhs is dominated by the last term whereas the rhs is still dominated by the first term therefore one finds
\[ \psi \sim \frac{a}{\omega} S^2 z^5, \]
which yields
\[ \psi \sim \frac{a}{G^2} \frac{1}{(1 + \frac{3}{2} a^2)^{1/2}} S z^5 \ll 1. \]
In the case \( z < S^{-1/4} \), both the lhs and the rhs of the equation are dominated by their last terms; then
\[ \psi \sim \frac{a^2}{\omega} \frac{1}{(1 + \frac{3}{2} a^2)^{1/2}} S (1 + z^2), \]
and
\[ \psi \sim \frac{a}{G^2} \frac{1}{(1 + \frac{3}{2} a^2)^{1/2}} (1 + z^2). \]
One sees that in the case of weak wave, \( a \ll 1 \), the condition \( \psi \ll G^2 \) is fulfilled at any \( z \) therefore our approximation is valid everywhere. In the case of strong waves, \( a \gg 1 \), it is valid in the most of the Larmor orbit with the exception of a region \( z \sim 1 \), where \( \psi \ll G^2 \). Note that the energy exchange between the wave and the electron occurs at \( \eta \sim 1 \). For \( S \ll 1 \), this corresponds to \( z \sim S^{-1/3} \gg 1 \); it is this range of \( z \) that contributes to integrals in eq. (56). Taking into account that our approximation is valid at \( z \gg 1 \) and is marginally fulfilled \( z \sim 1 \), one concludes that the expressions for the particle energy gain and for the absorption coefficients, eqs. (56)-(61), are valid at \( S \ll 1 \).

Now let us consider the case \( S \sim 1 \). Then \( \psi \sim a \omega_B/\omega \) and
\[ \psi \sim \frac{a}{G^2} \frac{1}{(1 + \frac{3}{2} a^2)^{1/2}} (1 + z^2). \]
One sees that for weak waves, \( a \ll 1 \), our approximation is valid at any \( z \) whereas for strong waves, it becomes marginally correct at \( z \sim 1 \). At \( S \sim 1 \), the electrons gain energy at \( z \sim 1 \) therefore one finally concludes that the expression (61) for the synchrotron absorption cross-section is always correct for weak waves whereas for strong waves, it is correct in the high and low frequency limits, \( S \gg 1 \) and \( S \ll 1 \), and could be used as an estimate for \( S \sim 1 \).

7 DISCUSSION

It is well known that the synchrotron emission and absorption occur in a wide frequency range at high harmonics of the rotation period. According to the standard theory (e.g., Landau & Lifshitz 1975, Melrose 1986), the characteristic frequency is \( \omega_0 \sim \omega_B \gamma^3 \), which corresponds to \( \gamma^3 \) harmonics. It was shown in this paper that for strong waves, the synchrotron absorption occurs at high harmonics too but the characteristic frequency significantly decreases, see eq. (55), therefore the frequency range for an efficient absorption also decreases significantly. Let us discuss the physical origin of the phenomenon.

The rotation frequency in the presence of a high-frequency electromagnetic wave is given by eq. (55), where \( \Gamma \) is the constant of motion, which is equal to the particle Lorentz factor if the wave is weak; in a strong wave, it is equal to the average particle Lorentz factor in the most of the orbit. Therefore the rotation period is not affected by the wave. On the other hand, the rotational motion averaged over the rapid oscillations occurs with the Lorentz
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The phase of the wave seen by the electron guiding centre, eq. 1, is

$$\eta = \omega \left( t - \frac{v}{\Omega} \sin \Omega t \right). \quad (71)$$

The electron absorbs radiation at a small fraction of the trajectory, where it moves together with the wave thus remaining relatively long time in phase with the average. Then the frequency “seen” by the electron, \( \frac{d\eta}{dt} \), is minimum; in our case, this occurs near the upper point of the orbit, \( t = 0 \). Expanding around this point yields

$$\eta = \omega \left( \frac{t}{2 \gamma^2 c} + \frac{1}{6} \Omega^2 t^3 \right) = \frac{\omega}{2 \Omega^2 \gamma^2 c} \left[ \gamma \Omega t + \frac{1}{3} (\gamma \Omega t)^3 \right]. \quad (72)$$

One now sees that the characteristic synchrotron frequency is \( \omega_0 \sim \gamma \Omega^2 c \), which reproduces formula (63). At \( \omega \ll \omega_0 \), the electron exchanges energy with the wave during the time interval \( \Omega \sim (\omega_0/\Omega)^{1/3} \) corresponding to \( \Delta \eta \sim 1 \); beyond this interval, the oscillation frequency rapidly grows so that the energy exchange does not occur on the average. At \( \omega \gg \omega_0 \), the electron experiences a few oscillations while \( \gamma \Omega t < 1 \), when the wave frequency \( \frac{d\eta}{dt} \) remains constant; then the average energy exchange is small.

Other absorption mechanisms, as well as the application of the obtained results to the termination shocks in PWNe, will be discussed in the next papers of the series.

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APPENDIX. SYNCHROTRON ABSORPTION OF WEAK WAVES

As a consistency check, let us find the absorption coefficient of weak waves for the configuration used in this paper. This could be conveniently done by making use of the Einstein coefficient method. The evolution of the photon occupation number, \( n_k \), is governed by the kinetic equation, which is written with account of the detailed balance principle in the form

$$\frac{d n_k}{dt} = \int W(p, k) \{ f_p (1 + n_k) - n_k f_{p-k} \} \frac{d^3 p}{(2\pi)^3}, \quad (73)$$

where \( f_p \) is the electron distribution function, \( W(p, k) \) the probability for spontaneous emission of a photon with the wave vector \( k \) by an electron with the momentum \( p \). We are interested in synchrotron emission/absorption of highly relativistic electrons rotating perpendicularly to the magnetic field therefore the element of the phase volume may be conveniently written in the cylindrical coordinates as \( d^3 p = c^{-2} d\varepsilon d\phi dp_z \), where \( \varepsilon = cp \) is the electron energy, whereas the electron distribution function may be presented as

$$f_p = \left( 2\pi c \right)^2 \frac{N(\varepsilon)}{\varepsilon} \delta(p_z), \quad (74)$$

where \( N(\varepsilon) \) is the number density of electrons per unit energy range.

We consider radiation in the plane \( k_z = 0 \); in this case, the emission probability depends on the electron energy, \( \varepsilon \), the photon frequency, \( \omega \), and the angle \( \theta \) between \( p \) and \( k \). A highly relativistic electron radiates in the direction of motion therefore one can write \( W(p, k) = 2\pi Y(\varepsilon, \omega) \delta(\varphi - \varphi') \), where the angle \( \varphi' \) shows the direction of the photon in the \( x-y \) plane, \( \varphi' = k_y/k_x \). Then the kinetic equation is written as

$$\frac{dn_k}{dt} = \int_0^\infty Y(\varepsilon, \omega) \left\{ N(\varepsilon) (1 + n_k) - n_k \frac{\varepsilon}{\varepsilon - \hbar \omega} N(\varepsilon - \hbar \omega) \right\} d\varepsilon \quad (75)$$

Instead of the photon occupation number, \( n_k \), one can conveniently use the radiation intensity,

$$I = \frac{\hbar \omega^3}{(2\pi c)^3} n_k. \quad (76)$$

Substituting \( n_k \) by \( I \) and expanding in small \( \hbar \omega \ll \varepsilon \), one reduces the kinetic equation to the standard form of the radiation transfer equation

$$\frac{dI}{dt} = -n_0 \int_0^\infty Y(\varepsilon, \omega) N(\varepsilon) d\varepsilon \quad (77)$$

where

$$n_0 = -\hbar \omega \int_0^\infty Y(\varepsilon, \omega) \frac{d}{d\varepsilon} \left( \frac{N(\varepsilon)}{\varepsilon} \right) d\varepsilon = \hbar \omega \int_0^\infty \frac{N(\varepsilon)}{\varepsilon} \frac{d}{d\varepsilon} (\varepsilon Y(\varepsilon, \omega)) d\varepsilon \quad (79)$$

the absorption coefficient. Now the absorption cross-section may be presented as

$$\sigma = \frac{\hbar \omega}{\varepsilon} \frac{d}{d\varepsilon} \varepsilon Y(\varepsilon, \omega), \quad (80)$$

i.e. the absorption is related to the spontaneous emission power.

The radiation of an electron gyrating perpendicularly to the magnetic field is calculated, e.g., in Landau & Lifshitz (1974). The electron radiates in harmonics of the rotation frequency, \( \omega = n\omega_B/\gamma \). The emission power in the rotation plane at the \( n \)-th harmonic is found as

$$dI_n = \frac{n^2 c^2 \omega_B^2 v^2}{2\pi c^2 \gamma^2 c^2} J_n^2 \left( \frac{nv}{c} \right) d\Omega, \quad (81)$$

where \( J_n(x) \) is the derivative of the Bessel function of \( n \)-th order. For high harmonics, \( n \gg 1 \), the emission power in a frequency interval \( d\omega = (\omega_B/\gamma) d\Omega \) is presented as

$$dP = dI_n \frac{\omega}{\omega_B} d\omega. \quad (82)$$

On the other hand, it follows from eq. (28) that the single electron emission power is presented as

$$dP = \frac{\hbar \omega^3}{(2\pi c)^3} Y d\omega d\Omega. \quad (83)$$

Comparing these two expressions, one finds

$$Y = \frac{4\pi^2 c^2 \gamma v^2}{\hbar \omega_B} J_n^2 \left( \frac{nv}{c} \right). \quad (84)$$
In the case of interest, \( v \approx c, n \gg 1 \), one can use the asymptotic relation
\[
J_n(n\xi) = \left( \frac{2}{n} \right)^{1/3} \text{Ai} \left( 2^{1/3} n^{2/3} (1 - \xi) \right); \quad n \gg 1; \quad \xi \approx 1;
\]
\[\text{(85)}\]
to yield
\[
Y = \frac{10^{13/3} \pi^2 e^2 c^2 \omega_B^{1/3}}{h \omega^{1/3} \gamma^{1/3}} \text{Ai}' \left( R^{2/3} \right); \quad R = \frac{\omega}{2\omega_B \gamma^2}. \quad \text{(86)}
\]
Substituting this expression into eq. (80), one gets the synchrotron absorption cross-section in the form
\[
\sigma = \frac{2^{1/3} \pi^2 e^2 c^2 \omega_B^{1/3}}{3 \omega^{1/3} \gamma^{1/3}} \text{Ai}' \left( R^{2/3} \right) \left[ \text{Ai}' \left( R^{2/3} \right) - 4R^{4/3} \text{Ai} \left( R^{2/3} \right) \right]. \quad \text{(87)}
\]
One sees that this expression coincides with eq. (61) at \( a \ll 1 \), when \( R = S \).

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