NONLINEAR LANDAU DAMPING FOR THE VLASOV-POISSON SYSTEM IN $\mathbb{R}^3$: THE POISSON EQUILIBRIUM

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Abstract

We prove asymptotic stability of the Poisson homogeneous equilibrium among solutions of the Vlasov-Poisson system in the Euclidean space $\mathbb{R}^3$. More precisely, we show that small, smooth, and localized perturbations of the Poisson equilibrium lead to global solutions of the Vlasov-Poisson system, which scatter to linear solutions at a polynomial rate as $t \to \infty$.

The Euclidean problem we consider here differs significantly from the classical work on Landau damping in the periodic setting, in several ways. Most importantly, the linearized problem cannot satisfy a “Penrose condition”. As a result, our system contains resonances (small divisors) and the electric field is a superposition of an electrostatic component and a larger oscillatory component, both with polynomially decaying rates.

Contents

1. Introduction 1
2. Dynamics of the density and bootstrap setup 7
3. Preliminary estimates 12
4. The contributions of the initial data 21
5. Bounds on the static terms and the type-I reaction term 24
6. Bounds on the type-II reaction term 40
7. Proof of Lemma 5.1 48
8. Proof of Theorem 1.1 54
References 57

1. Introduction

It is believed that the vast majority of ordinary matter in the visible universe takes the form of a plasma, i.e. of an ionised gas, ranging from sparse intergalactic plasma to the interior of stars and neon signs. Both from a theoretical and practical (e.g. nuclear fusion) point of view, the understanding of stability versus instability in plasmas is a formidable yet crucial challenge. We refer to e.g. [10, 7] for physics references in book form.

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1
In this article we consider a hot, unconfined, electrostatic plasma in three dimensions of electrons on a uniform, static, background of ions. Here collisions are neglected and the associated distribution of electrons is modeled by a measure $M(x,v,t)dx dv$ on the phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$, where $M$ satisfies the Vlasov-Poisson equation.

$$m_e(\partial_t + v \cdot \nabla_x)M + qE \cdot \nabla_v M = 0, \quad \text{div}_x E(x,t) = 4\pi \varepsilon_0 \left\{ n_0 + q \int_{\mathbb{R}^3} M dv \right\}. \quad (1.1)$$

In (1.1), $E$ denotes the self-generated electrostatic field, $q = -e < 0$ the charge of an electron, $m_e$ its mass, $\varepsilon_0$ the vacuum resistivity and $n_0 > 0$ the uniform background charge density of the ions.

A particularly simple yet relevant class of solutions to (1.1) are stationary, spatially homogeneous functions $M_0 : \mathbb{R}^3 \to [0,\infty)$ satisfying $q \int_{\mathbb{R}^3} M_0 dv = -n_0$. To understand the role of such equilibria in the overall dynamics of the Vlasov-Poisson equations, an important step is the investigation of their stability. Writing

$$M(x,v,t) = n_0 e^{-1} (M_0(v) + F(x,v,t))$$

the equation for the perturbation $F$ becomes

$$(\partial_t + v \cdot \nabla_x) F + \omega^2_e \nabla \psi \cdot \nabla_v M_0 + \omega^2_e \nabla \psi \cdot \nabla_v F = 0, \quad \Delta_x \psi = \int_{\mathbb{R}^3} F dv,$$

with electrostatic potential $\psi$ and electron plasma frequency $\omega_e$ given by $\omega^2_e := \frac{4\pi \varepsilon_0 m_e e^2}{\varepsilon_0}$. Let

$$f(x,v,t) := F \left( \frac{x}{\omega_e}, v, \frac{t}{\omega_e} \right), \quad \phi(x,t) := \omega^2_e \psi \left( \frac{x}{\omega_e}, \frac{t}{\omega_e} \right),$$

so we obtain the nondimensionalized Vlasov-Poisson system

$$(\partial_t + v \cdot \nabla_x) f + \nabla \phi \cdot \nabla_v f = -\nabla \phi \cdot \nabla_v M_0,$$

$$\Delta_x \phi(x,t) = \rho(x,t) := \int_{\mathbb{R}^3} f(x,v,t) dv,$$  

which will be the focus of the rest of this article.

1.1. Main result. In this paper we investigate the asymptotic stability of a particular homogeneous equilibrium, namely the “Poisson equilibrium”

$$M_0(v) := \frac{C}{(1 + |v|^2)^2}, \quad \hat{M}_0(\xi) = e^{-|\xi|}, \quad (1.3)$$

for a suitable normalization constant $C > 0$. Our main result asserts that smooth localized perturbations of $M_0$ lead to global solutions which scatter to linear solutions, and exhibits two different dynamics in the associated electric field:

**Theorem 1.1.** There exists $\varepsilon > 0$ such that if the initial particle distribution $f_0$ satisfies

$$\sum_{|\alpha| + |\beta| \leq 1} \left\| (v)^{4.5} \partial_x^\alpha \partial_v^\beta f_0(x,v) \right\|_{L^\infty_x L^\infty_v} + \left\| (v)^{4.5} \partial_x^\alpha \partial_v^\beta f_0(x,v) \right\|_{L^1_t L^\infty_x} \leq \varepsilon_0 \leq \varepsilon,$$  

then the Vlasov-Poisson system (1.2) with $M_0$ defined as in (1.3) has a global unique solution $f \in C^1_{x,v,t}(\mathbb{R}^{3+3} \times \mathbb{R}^+)$ that scatters linearly, i.e. there exists $f_\infty \in L^\infty_x$, such that

$$\| f(x,v,t) - f_\infty(x - tv,v) \|_{L^\infty_x} \lesssim \varepsilon_0(t)^{-1/2},$$  

$$\| f(x,v,t) - f_\infty(x - tv,v) \|_{L^\infty_x} \lesssim \varepsilon_0(t)^{-1/2},$$

$$\| f(x,v,t) - f_\infty(x - tv,v) \|_{L^\infty_x} \lesssim \varepsilon_0(t)^{-1/2}.$$
Moreover, the electric field \( E(x,t) = \nabla_x \phi(x,t) \) decomposes into a “static” and an “oscillatory” component with different decay rates
\[
E(t) = E^{\text{stat}}(t) + \Re(e^{-it}E^{\text{osc}}(t)),
\]
\[
\langle t \rangle \|E^{\text{stat}}(t)\|_{L^\infty} + \|E^{\text{osc}}(t)\|_{L^\infty} \lesssim \varepsilon_0(t)^{-2+\delta},
\]
where \( \delta \in (0,1/100] \) is a small parameter.

**Remark 1.2.** (1) Theorem 1.1 appears to be the first nonlinear asymptotic stability result for the Vlasov-Poisson system in \( \mathbb{R}^3 \) in a neighborhood of a smooth non-trivial equilibrium (see [40] for the case of a repulsive point charge).\(^1\) We work with the Poisson homogeneous equilibrium \( M_0 \) mostly for the sake of simplicity, as it leads to explicit formulas such as (1.13)–(1.14). However, we expect that the linear scattering conclusion of the theorem and the underlying analysis extend to more general smooth homogeneous equilibria, at least as long as the resonances of the system (the set where \( 1 + K(\xi,\lambda) \) defined as in (1.9) vanishes) are not too severe. See sections 1.2 and 1.3 for further discussion.

(2) The statement of the theorem, in particular the crucial decay estimates (1.6), depend on the fact that we work in dimension \( d = 3 \). The decay is stronger in higher dimensions \( d \geq 4 \) (so the proof becomes easier), and weaker and insufficient in dimension \( d = 2 \). This is due mainly to the dimension-dependent dispersive estimates Lemma 3.2, and is in sharp contrast with the periodic case \( x \in \mathbb{T}^d \). See section 1.2 for further discussion.

1.2. **Prior works.** The literature of broadly related results is too vast to be surveyed here, so we will focus on a selection of more directly related results. In particular, we restrict our attention to the setting of three spatial dimensions.

1.2.1. *Perturbations of vacuum.* The simplest equilibrium of (1.1) is the vacuum \( n_0 \equiv 0 \) and \( M_0 \equiv 0 \). The resulting equations
\[
(\partial_t + v \cdot \nabla_x) f + \lambda \nabla_x \phi \cdot \nabla_v f = 0, \quad \Delta_x \phi(x,t) = \int f(x,v,t)dv, \quad \lambda \in \{-1, 1\},
\]
are also relevant in astrophysics, where the self-interactions through a gravitational potential are attractive (\( \lambda = -1 \)) rather than repulsive (\( \lambda = +1 \)). In the former case, (1.7) possesses a large variety of localized equilibria, see e.g. [21, 33, 43, 38].

For sufficiently localized initial data, the system (1.7) has been extensively studied (see e.g. [15, 43] for book references). Solutions are global in time under reasonably general regularity assumptions [1, 36, 44, 42]. For small perturbations, the electric field and charge density decay over time [1], but the precise dynamics are only recently clarified (after a series of preliminary works [26, 41, 24, 45, 9]): the *long-range effects* of the electrostatic field lead to asymptotic dynamics that feature a logarithmic correction of linear scattering, a phenomenon known as *modified scattering* [31, 11].

1.2.2. *Nonlinear Landau damping.* As is well-known, the free transport equation \( \partial_t f + v \cdot \nabla_x f = 0 \) exhibits *phase mixing*, which can manifest as time decay in the spatial density \( \rho(t,x) \). It was an observation of Landau [32] (see also [34, Chapter 3, Section 30]) that an interesting mechanism of decay exists also in the linearized Vlasov-Poisson equations near homogeneous equilibria satisfying certain conditions (nowadays called “Penrose criteria”).

\(^1\)After the completion of this work, the authors heard a presentation of Toan T. Nguyen on a related work on asymptotic stability in the relativistic Vlasov-Poisson equation in \( \mathbb{R}^3 \).
The periodic setting \( x \in \mathbb{T}^3 \), is a natural model for a confined plasma and has been studied extensively. After some preliminary works [8, 25], nonlinear Landau damping was proved in the pioneering work [39] (see also [3, 16] for refinements and simplifications). More precisely, for suitable homogeneous equilibria \( M_0 \) that satisfy the Penrose criterion

\[
\inf_{k \in \mathbb{Z}^3 \setminus \{0\}, \lambda \in \mathbb{C}, \Im(\lambda) < 0} \left| 1 + \int_0^\infty e^{-i\lambda s} \hat{M}_0(ks) ds \right| > 0, \tag{1.8}
\]

it was shown that small, highly regular (analytic or Gevrey) perturbations lead to a global solution which scatters linearly, with an electric field that decays exponentially.

The confined nature of the system induces strong nonlinear echoes which, for smooth enough perturbations, are compensated by the fast decay of the linearized equation. Rougher perturbations (e.g. Sobolev) can lead to other stationary solutions [6, 35] or still damp [17]. We note that related mechanisms are at play in 2d fluid mixing, in particular near monotone shear flows [2, 27, 28, 37] or point vortices [29].

In the unconfined setting \( x \in \mathbb{R}^3 \), Theorem 1.1 appears to be the first nonlinear asymptotic stability result of non-trivial, smooth homogeneous equilibria in 3 dimensions. Previous results are mainly concerned with the linearized system, see [13, 14], and the screened case, see [4, 22], in which the low frequency part is screened out.

The key difficulty for the unconfined case is caused by the fact that the Penrose criterion cannot hold. More precisely, for any normalized homogeneous equilibrium \( M_0 \) we have

\[
1 + K(0, \lambda) = 1 - \lambda^{-2} \quad \text{if} \quad \Im(\lambda) < 0 \quad \text{thus} \quad \lim_{\xi \to 0, \lambda \to \pm 1, \Im(\lambda) < 0} |1 + K(\xi, \lambda)| = 0, \tag{1.9}
\]

where \( K(\xi, \lambda) := \int_0^\infty e^{-i\lambda s} \hat{M}_0(\xi s) ds \).

In particular, the key lower bound in the Penrose criterion (1.8) cannot hold. This critical difficulty has been observed in [13, 14, 5, 23] in the study of the linearized system.

In the nonlinear setting the failure of the Penrose condition leads to the presence of small denominators and resonances in the system. As a consequence we have much slower decay and convergence (depending on the dimension of the ambient space), and the global dynamics is completely different compared to the periodic case. In fact our analysis reveals two different types of dynamics for the electric field: an oscillatory part, which decays at almost the critical rate \( \langle t \rangle^{-2} \) and oscillates like \( e^{-it} \) over time, and a static part, which decays faster than the critical rate \( \langle t \rangle^{-2} \). See section 1.3 below for more details.

We note that a suitable analogue of the Penrose condition (1.8) still holds in the screened case investigated in [4, 22]. As a consequence there are no resonances in that case, and one can still prove sufficiently rapid decay of the electric field for sufficiently smooth perturbations.

1.2.3. Comparison with Euler-Poisson models. Finally, we note the analogy with the related case of warm plasmas, when one can use fluid models instead of kinetic ones: the Euler-Poisson equation for electron is classically asymptotically stable (for irrotational data) [18], while the stability of the ion equation was more recently obtained [20]. In this case, the two-fluid models involving both electrons and ions lead to new and interesting phenomena [12, 30, 19].
1.3. Main ideas. As in [22] we use a Lagrangian approach. The left-hand side of (1.2) is a transport equation for $f$, the (backwards) characteristics of which are the curves

\begin{align*}
\partial_s X(x, v, s, t) &= V(x, v, s, t), \\
\partial_s V(x, v, s, t) &= \nabla_x \phi(X(x, v, s, t), s), \\
X(x, v, t) &= x, \\
V(x, v, t) &= v.
\end{align*}

Notice that we can integrate (1.2) to obtain an exact formula for $f(x, v, t)$, namely

\begin{equation}
(1.11)
f(x, v, t) = f_0(X(x, v, 0, t), V(x, v, 0, t)) - \int_0^t \nabla_x \phi(X(x, v, s, t), s) \cdot \nabla_v M_0(V(x, v, s, t)) ds.
\end{equation}

It thus suffices to study the characteristics, which are given through the electric field $\nabla_x \phi$, which in turn arises from the density $\rho$. Through integration in $v$ in (1.11), together with (1.10) we recast (1.2) as a closed system for the density $\rho$ satisfying

\begin{equation}
(1.12)
\rho(x, t) + \int_0^t \int_{\mathbb{R}^3} (t - s) \rho(x - (t - s)v, s) M_0(v) dv ds = \mathcal{N}(x, t),
\end{equation}

where $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ is a sum of initial data contribution $\mathcal{N}_1$ and a nonlinear expression $\mathcal{N}_2$ in $\rho$ (see (2.5)-(2.6) below for precise formulas). This formulation is amenable to a bootstrap approach that we detail further.

1.3.1. Linear analysis and the Penrose condition. The (linear) left-hand side of (1.11) is a Volterra equation, which can be integrated exactly. Using the Laplace-Fourier transform and the explicit formula (1.3), after a few algebraic manipulations it follows from (1.12) that

\begin{equation}
(1.13)
1 + K(\xi, \lambda) = 1 + \int_0^\infty r M_0(r \xi)e^{-i\lambda r} dr = 1 + \frac{1}{(|\xi| + i\lambda)^2} = \frac{(|\xi| + i(\lambda - 1))(|\xi| + i(\lambda + 1))}{(|\xi| + i\lambda)^2}.
\end{equation}

Notice that there is no uniform lower bound for $|1 + K|$ as $\xi \to 0$ and $\lambda \to \pm 1$, i.e. a Penrose condition similar to (1.8) cannot be satisfied. However, we can still obtain an explicit solution of the linear problem,

\begin{equation}
\hat{\rho}(\xi, t) = \tilde{\mathcal{N}}(\xi, t) - \int_0^t \tilde{\mathcal{N}}(\xi, \tau)e^{-(t-\tau)|\xi|} \sin(t - \tau) d\tau.
\end{equation}

This gives a first “solution formula” (in the same spirit as a Duhamel formula) for $\rho$ in terms of the initial data and a nonlinear expression of itself.

We write $\sin(t - \tau) = \sin(t) \cos(\tau) - \cos(t) \sin(\tau)$ and do integration by parts in $\tau$ in (1.14) (similar to a normal form analysis) when the oscillation effect is strong, i.e. in the case $|v||\xi| \ll 1$. As a result, we derive a second formula for the localized density, see (2.12)-(2.14).

The identity (1.14) naturally leads to a decomposition of the density

\begin{equation}
\rho = \Re(e^{-it} \rho^{\text{osc}}) + \rho^{\text{stat}}
\end{equation}

into an oscillatory mode $\Re(e^{-it} \rho^{\text{osc}})$ and a static component $\rho^{\text{stat}}$. The corresponding decomposition at the level of the electric field follows by setting $E^* = \nabla \Delta^{-1} \rho^*$, $* \in \{\text{osc, stat}\}$.
1.3.2. The bootstrap argument and nonlinear analysis. Our main bootstrap argument is at the level of the density function $\rho$. The main idea is to use the decomposition (1.15) and bound the two components $\rho^{\text{osc}}$ and $\rho^{\text{stat}}$ in two different Banach spaces: a stronger space for the static component $\rho^{\text{stat}}$ and a weaker space for the oscillatory component $\rho^{\text{osc}}$. The choice of these two spaces is very important, see (2.22)–(2.23) for the precise definitions. The point is that even though the oscillatory component $\rho^{\text{osc}}$ satisfies weaker bounds, the presence of the factor $e^{-it}$ allows integration by parts in time arguments (normal forms), which lead to improved decay and convergence.

To prove the main bootstrap proposition 2.4 we start from the identity (1.14). The main nonlinear contribution comes from the reaction term

$$\mathcal{N}_2(x, t) = \int_0^t \int_{\mathbb{R}^3} \{ E(x - (t - s)v, s) \cdot \nabla_v M_0(v) - E(X(x, v, s, t), s) \cdot \nabla_v M_0(V(x, v, s, t)) \} \, dv \, ds,$$

(1.16)

where the characteristics $(X, V)(x, v, s, t)$ are defined as in (1.10).

The bootstrap assumptions on $\rho$ and the definitions (1.10) can be used to derive bounds on the deviations $V(x, v, s, t) - v$ and $X(x, v, s, t) - [x - (t - s)v]$ of the linear characteristics. To estimate the reaction term we further localize in $|v| \approx 2^j$ and $x$-frequency $|\xi| \approx 2^k$. We then examine the resulting nonlinear interactions, and analyze them according to the size of the parameters $j$ and $k$. In some cases it is beneficial to integrate by parts in time (normal forms) or in $v$ (improved dispersive estimates), particularly when the oscillatory components $E^{\text{osc}}$ are involved. The analysis of the reaction term (1.16) is the most elaborate part of our proof and covers sections 5, 6, and 7.

Integration by parts in time produces quadratic “main terms” and a large number of “cubic remainders”. Fortunately, in many cases the cubic remainders are of lower order and we can deal with them in systematic fashion using Lemma 5.1.

1.3.3. General equilibria. It is natural to ask if the main conclusions of Theorem 1.1 hold for more general homogeneous equilibria $M_0(v)$. Our choice of the Poisson equilibrium simplifies the analysis, since it leads to explicit formulas such as (1.13)–(1.14), but we expect that the framework we construct here extends to more general situations, provided that the resonances of the system are not too severe.

At the technical level, in the case of more general homogeneous equilibria there is an additional difficulty identified in [5, 23]: even at the linearized level the oscillatory component of the electric field contains an entire family of frequency dependent oscillations instead of just the two discrete modes we have here. This requires a more careful decomposition of the density function and more precise $Z$-norm analysis. We hope to return to these issues soon.

1.4. Organization. The rest of the paper is organized as follows: in section 2 we derive our main formulas for the density function, identify precisely its decomposition (Corollary 2.3), and state our main bootstrap proposition (Proposition 2.4). In section 3 we use the bootstrap assumptions to prove bounds on the modified characteristics (Lemmas 3.5 and 3.6), general dispersive estimates (Lemma 3.2), and bounds on operators defined by Fourier multipliers (Lemma 3.3). In section 4 we prove improved bootstrap estimates on the contributions of the initial data. In sections 5, 6, and 7 we prove improved bootstrap estimates on the contributions of the reaction term. Finally, in section 8 we complete the proof of the main Theorem 1.1.

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2 The choice of these bootstrap spaces is in fact the most important choice in the paper, in the spirit of the “$Z$-norm method” used by the authors in earlier work [30, 19, 31].
2. Dynamics of the density and bootstrap setup

Recall from (1.2) that the perturbation $f$ we study satisfies the equation

$$(\partial_t + v \cdot \nabla_x)f + E \cdot \nabla_v M_0 + E \cdot \nabla_v f = 0,$$

$$\rho(x, t) = \int_{\mathbb{R}^3} f(x, v, t) \, dv, \quad E := \nabla_x \Delta_x^{-1} \rho,$$  \hspace{1cm} (2.1)

where, for a suitable constant $C > 0$,

$$M_0(v) := \frac{C}{(1 + |v|^2)^2}, \quad \hat{M}_0(\xi) = e^{-|\xi|}.$$  \hspace{1cm} (2.2)

To recast this as a problem for the density $\rho$ we introduce the “backwards characteristics” of (2.1): these are the functions $X, V : \mathbb{R}^3 \times \mathbb{R}^3 \times I_T^2 \to \mathbb{R}^3$ obtained by solving the ODE system

$$\partial_s X(x, v, s, t) = V(x, v, s, t), \quad X(x, v, t, t) = x,$$

$$\partial_s V(x, v, s, t) = E(X(x, v, s, t), s), \quad V(x, v, t, t) = v,$$  \hspace{1cm} (2.3)

where $I_T^2 := \{ (s, t) \in [0, T]^2 : s \leq t \}$. The main equation (2.1) gives

$$\frac{d}{ds} f(X(x, v, s, t), V(x, v, s, t), s) = [(\partial_s + v \cdot \nabla_x + E \cdot \nabla_v)f](X(x, v, s, t), V(x, v, s, t), s)$$

$$= -E(X(x, v, s, t), s) \cdot \nabla_v M_0(V(x, v, s, t)).$$

Integrating over $s \in [0, t]$ and letting $M'_0 := \nabla_v M_0$, we have

$$f(x, v, t) = f_0(X(x, v, 0, t), V(x, v, 0, t)) - \int_0^t E(X(x, v, s, t), s) \cdot M'_0(V(x, v, s, t)) \, ds,$$  \hspace{1cm} (2.4)

for any $(x, v, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]$. Adding $\int_0^t \int_{\mathbb{R}^3} E(x - (t - s)v, s) \cdot M'_0(v) \, dvds$ on both sides of (2.4), a further integration in $v$ shows that

$$\rho(x, t) + \int_0^t \int_{\mathbb{R}^3} \rho(x - (t - s)v, s)M_0(v) \, dvds = \mathcal{N}(x, t),$$  \hspace{1cm} (2.5)

for any $(x, t) \in \mathbb{R}^3 \times [0, T]$, where

$$\mathcal{N}(x, t) := \mathcal{N}_1(x, t) + \mathcal{N}_2(x, t),$$

$$\mathcal{N}_1(x, t) := \int_{\mathbb{R}^3} f_0(X(x, v, 0, t), V(x, v, 0, t)) \, dv,$$

$$\mathcal{N}_2(x, t) := \int_0^t \int_{\mathbb{R}^3} \left\{ E(x - (t - s)v, s) \cdot M'_0(v) - E(X(x, v, s, t), s) \cdot M'_0(V(x, v, s, t)) \right\} \, dvds.$$  \hspace{1cm} (2.6)

Since $E = \nabla_x \Delta_x^{-1} \rho$ can be recovered directly from the density $\rho$, together with the characteristic equations (2.3), these equations yield a closed system.

2.1. Solving the density equation. In view of (2.5), the Fourier transform $\hat{\rho}$ of the density satisfies a forced Volterra equation

$$\hat{\rho}(\xi, t) + \int_0^t (t-s)\hat{M}_0((t-s)\xi)\hat{\rho}(\xi, s) \, ds = \hat{H}(\xi, t)$$  \hspace{1cm} (2.7)

where $H$ is a forcing term. Such equations can be solved as follows:
Lemma 2.1. Assume $\rho$ and $H$ satisfy (2.7) and define the convolution kernel $G$

$$G(\xi, \tau) := \delta_0(\tau) - \sin(\tau)e^{-|\tau|}1_+(\tau),$$

where $1_+$ denotes the characteristic function of the interval $[0, \infty)$. Then

$$\hat{\rho}(\xi, t) = \int_{s=0}^{t} G(\xi, s)\tilde{H}(\xi, t-s)ds.$$  \hspace{1cm} (2.8)

Proof. Recall the formulas (2.2), with which we have for any $\lambda \in \mathbb{C}$ with $\Re \lambda \leq 0$

$$K(\xi, \lambda) := \int_0^\infty r\tilde{M}_0(r\xi)e^{-ir\lambda}dr = \int_0^\infty re^{-r(|\xi|+i\lambda)}dr = \frac{1}{(|\xi|+i\lambda)^2}.$$  

Then (2.7) shows that

$$\mathcal{F}_t(\hat{\rho}(\xi, t))(\lambda) + K(\xi, \lambda)\mathcal{F}_t(\hat{\rho}(\xi, t))(\lambda) = \mathcal{F}_t(\hat{H}(\xi, t))(\lambda).$$

The claim follows by computing that

$$\frac{1}{1 + K(\xi, \lambda)} = 1 - \frac{1}{(|\xi| + i\lambda)^2 + 1} = 1 - \frac{1}{2i \left[ \frac{1}{|\xi| + i(\lambda - 1)} - \frac{1}{|\xi| + i(\lambda + 1)} \right]},$$

so

$$\mathcal{F}_t^{-1} \left( \frac{1}{1 + K(\xi, \lambda)} \right)(\tau) = \delta_0(\tau) - e^{-|\tau|}1_+(\tau) =: G(\xi, \tau),$$

as desired. \hspace{1cm} $\square$

Using this we can solve the equations for forcing terms as they appear in (2.5)-(2.6). This introduces the new, real operator

$$D(v, \nabla) := |\nabla| - v \cdot \nabla,$$

and reveals a discretely oscillatory dynamic:

Lemma 2.2. For a given $h : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ let

$$H(x, t) := \int_{\mathbb{R}^3} h(x-tv, v, t)dv.$$  

Then we have two alternative expressions $S^I[h]$ and $S^II[h]$ for the solution of the associated equation (2.7), with

$$S^*\{h\}(x, t) = R^*\{h\}(x, t) + \Re\{e^{-itT^*\{h\}}\}, \hspace{1cm} * \in \{I, II\},$$

where

$$R^I(x, t) = \int_{\mathbb{R}^3} h(x-tv, v, t)dv,$$

$$T^I(x, t) = -i \int_{0}^{t} e^{is}e^{-|s||\nabla|} \int_{\mathbb{R}^3} h(x-sv, v, s)dvds,$$

and

$$R^{II}(x, t) = \int_{\mathbb{R}^3} \frac{D^2}{1 + D^2} h(x-tv, v, t)dv,$$

$$T^{II}(x, t) = e^{-it|\nabla|} \int_{\mathbb{R}^3} \frac{1}{1 - iD} h(x, v, 0)dv$$

$$+ \int_{0}^{t} \int_{\mathbb{R}^3} \frac{e^{is}}{1 - iD} e^{-|s||\nabla|}(\partial_s h)(x - sv, v, s)dvds.$$
Proof of Lemma 2.2. The expressions for \( \mathcal{S}^I[h] \) follow directly from the solution formula (2.8). For \( \mathcal{S}^{II}[h] \), we have
\[
F \{ h(\cdot - vt, v, t) \} = e^{-itv \xi} \tilde{h}(\xi, v, t),
\]
and we observe that
\[
G(\xi, \lambda) e^{i\lambda v \xi} 1_{[0,t]}(\lambda) = \frac{D^2}{1 + D^2} \delta_0(\lambda) + \frac{\cos(t) + D \sin(t)}{1 + D^2} e^{-tD} \delta_0(\lambda - t) + \frac{1}{1 + D^2} \frac{d}{d\lambda} \left\{ (\cos(\lambda) + D \sin(\lambda)) e^{-\lambda D} 1_{[0,t]}(\lambda) \right\},
\]
which gives the claim. \(\square\)

2.2. The main decomposition. Applying Lemma 2.2 to our equations (2.5)-(2.6), the density \( \rho \) naturally decomposes into “static” and “oscillatory” components. We will control these parts using a bootstrap argument and different norms.

To implement this, we further decompose the forcing resp. nonlinear terms of (2.6) in frequency, velocity space, and time. For this we fix an even smooth function \( \varphi : \mathbb{R} \to [0,1] \) supported in \([-8/5,8/5]\) and equal to 1 in \([-5/4,5/4]\), and define
\[
\varphi_k(x) := \varphi(|x|/2^k) - \varphi(|x|/2^{k-1}), \quad \varphi_{\leq k}(x) := \varphi(|x|/2^k), \quad \varphi_{\geq k}(x) := 1 - \varphi(x/2^{k-1}),
\]
for any \( k \in \mathbb{Z} \) and \( x \in \mathbb{R}^d, d \geq 1 \). Let \( P_k, P_{\leq k}, \) and \( P_{\geq k} \) denote the operators on \( \mathbb{R}^3 \) defined by the Fourier multipliers \( \varphi_k, \varphi_{\leq k}, \) and \( \varphi_{\geq k} \) respectively. Moreover, for any \( k \in \mathbb{Z}_+ \), we use \( f_k \) to abbreviate \( P_k f \).

Let \( \tilde{\varphi}_0 = \varphi_{\leq 0} \) and \( \tilde{\varphi}_j = \varphi_j \) if \( j \geq 1 \). For any interval \( I \subseteq \mathbb{R} \) let
\[
\varphi_I(x) := \sum_{k \in \mathbb{Z} \cap I} \varphi_k(x), \quad \tilde{\varphi}_I(x) := \sum_{j \in \mathbb{Z}_+ \cap I} \tilde{\varphi}_j(x).
\]
We define the functions \( L_{1,j}, L_{1,j} : \mathbb{R}^3 \times \mathbb{R}^3 \times [0,T] \to \mathbb{R} \) and \( L_{2,j} : \mathbb{R}^3 \times \mathbb{R}^3 \times I_T^2 \to \mathbb{R} \) by
\[
L_{1,j}(x,v,t) := \tilde{\varphi}_j(v) f_0(X(x + tv, v, 0, t), V(x + tv, v, 0, t)),
L_{2,j}(x,v,s,t) := E(x,s) \cdot M_j^I(v) - E(X(x + (t-s)v,v,s,t),s) \cdot M_j^I(V(x + (t-s)v,v,s,t)),
\]
where \( j \in \mathbb{Z}_+ \) and \( M_j^I(v) := \tilde{\varphi}_j(v) \nabla_v M_0(v) \). Then for any \( j \in \mathbb{Z}_+ \) and \( k \in \mathbb{Z} \) we define the functions \( L_{1,j,k} : \mathbb{R}^3 \times \mathbb{R}^3 \times [0,T] \to \mathbb{R} \) and \( L_{2,j,k} : \mathbb{R}^3 \times \mathbb{R}^3 \times I_T^2 \to \mathbb{R} \) by
\[
L_{1,j,k}(x,v,t) := P_k L_{1,j}(x,v,t), \quad L_{2,j,k}(x,v,s,t) := P_k L_{2,j}(x,v,s,t),
\]
where the projections \( P_k \) apply in the \( x \) variable. It follows from (2.6) that
\[
N_1 = \sum_{j \in \mathbb{Z}_+, k \in \mathbb{Z}} N_{1,j,k}, \quad N_{1,j,k}(x,t) := \int_{\mathbb{R}^3} L_{1,j,k}(x - tv, v, t) dv,
N_2 = \sum_{j \in \mathbb{Z}_+, k \in \mathbb{Z}} N_{2,j,k}, \quad N_{2,j,k}(x,t) := \int_0^t \int_{\mathbb{R}^3} L_{2,j,k}(x - (t-s)v, v, s,t) dvds.
\]
Contributions from initial data. For the contributions from \( \mathcal{N}_1 \), using the expressions I of Lemma 2.2 we thus we have the first decomposition

\[
\rho_{1,j,k}(x,t) = R^I_{1,j,k}(x,t) + \Re \{ e^{-itT^I_{1,j,k}(x,t)} \},
\]

\[
\mathcal{F}(R^I_{1,j,k})(\xi,t) := \int_{\mathbb{R}^3} \widehat{L_{1,j,k}}(\xi,v,t)e^{-itv\cdot\xi} dv,
\]

\[
\mathcal{F}(T^I_{1,j,k})(\xi,t) := \int_0^t \int_{\mathbb{R}^3} e^{-(t-s)|\xi|}e^{-isv\cdot\xi}(-i)e^{is} \cdot \widehat{L_{1,j,k}}(\xi,v,s) dv ds.
\]

Alternatively, formulation II of Lemma 2.2 yields a decomposition as

\[
\rho_{1,j,k}(x,t) = R^{II}_{1,j,k}(x,t) + \Re \{ e^{-itT^{II}_{1,j,k}(x,t)} \},
\]

where

\[
\mathcal{F}(R^{II}_{1,j,k})(\xi,t) := \int_{\mathbb{R}^3} \widehat{L_{1,j,k}}(\xi,v,t)e^{-itv\cdot\xi} \frac{D^2}{1+D^2} dv,
\]

\[
\mathcal{F}(T^{II}_{1,j,k})(\xi,t) := e^{-t|\xi|} \int_{\mathbb{R}^3} \frac{1}{1-iD} \widehat{L_{1,j,k}}(\xi,v,0) dv
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} e^{-(t-s)|\xi|}e^{-isv\cdot\xi} \frac{e^{is}}{1-iD} (\partial_s \widehat{L_{1,j,k}})(\xi,v,s) dv ds.
\]

and we have slightly abused notation to denote by \( D \) also its Fourier symbol

\[
D := |\xi| - iv \cdot \xi.
\]

Contributions from nonlinearity: “reaction terms”. Similarly, contributions from \( \mathcal{N}_2 \) can be decomposed with formulation I of Lemma 2.2 to obtain

\[
\rho_{2,j,k}(x,t) = R^I_{2,j,k}(x,t) + \Re \{ e^{-itT^I_{2,j,k}(x,t)} \},
\]

\[
\mathcal{F}(R^I_{2,j,k})(\xi,t) := \int_0^t \int_{\mathbb{R}^3} \widehat{L_{2,j,k}}(\xi,v,\tau,t)e^{-i(t-\tau)v\cdot\xi} dv d\tau,
\]

\[
\mathcal{F}(T^I_{2,j,k})(\xi,t) := \int_0^t \int_{\mathbb{R}^3} 1_+ (s-\tau)e^{-(t-s)|\xi|}(-i)e^{is} \widehat{L_{2,j,k}}(\xi,v,\tau,s)e^{-i(s-\tau)v\cdot\xi} dv d\tau ds.
\]

Alternatively, formulation II gives the expressions

\[
\rho_{2,j,k}(x,t) = R^{II}_{2,j,k}(x,t) + \Re \{ e^{-itT^{II}_{2,j,k}(x,t)} \},
\]

where

\[
\mathcal{F}(R^{II}_{2,j,k})(\xi,t) := \int_0^t \int_{\mathbb{R}^3} \widehat{L_{2,j,k}}(\xi,v,\tau,t)e^{-i(t-\tau)v\cdot\xi} \frac{D^2}{1+D^2} dv d\tau,
\]

\[
\mathcal{F}(T^{II}_{2,j,k})(\xi,t) := \int_0^t \int_{\mathbb{R}^3} 1_+ (s-\tau)e^{-(t-s)|\xi|}e^{-i(s-\tau)v\cdot\xi} \times \frac{e^{is}}{1-iD} (\partial_s \widehat{L_{2,j,k}})(\xi,v,\tau,s) dv d\tau ds.
\]
**Full decomposition.** Depending on the relative sizes of the parameters $k$, $j$, and $t$, we will use either the decompositions (2.11) and (2.16), or the decompositions (2.12) and (2.17). More precisely we define the sets

$$A^I := \{(j,k,m) \in \mathbb{Z}_+ \times \mathbb{Z} \times \mathbb{Z}_+: m < \delta^{-4} \text{ or } j > 19m/20 \text{ or } k + j + \delta m/3 > 0\},$$

$$A^{II} := \{(j,k,m) \in \mathbb{Z}_+ \times \mathbb{Z} \times \mathbb{Z}_+: m \geq \delta^{-4} \text{ and } j \leq 19m/20 \text{ and } k + j + \delta m/3 \leq 0\}.$$

Combining with this splitting of parameters, we have thus established the following full decomposition of the density $\rho$:

**Corollary 2.3.** Assume that $f : \mathbb{R}^3 \times \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ is a regular solution of the system (2.1) and define the function $\rho$ as before. Then we can decompose

$$\rho = \rho^{\text{stat}} + \Re \{e^{-it}\rho^{\text{osc}}\},$$

(2.20)

where, with the definitions (2.11), (2.16), (2.13)–(2.14), and (2.18)–(2.19),

$$\rho^{\text{stat}}(x,t) = \sum_{* \in \{I,II\}} \sum_{(j,k,m) \in A^*} \bar{\varphi}_m(t)[R_{1,j,k}^*(x,t) + R_{2,j,k}^*(x,t)],$$

$$\rho^{\text{osc}}(x,t) = \sum_{* \in \{I,II\}} \sum_{(j,k,m) \in A^*} \bar{\varphi}_m(t)[T_{1,j,k}^*(x,t) + T_{2,j,k}^*(x,t)].$$

(2.21)

The main point of this decomposition is that we will be able to prove stronger control of the low frequencies of the stationary component $\rho^{\text{stat}}$ compared to the time-oscillatory component $\rho^{\text{osc}}$. See Proposition 2.4 below.

### 2.3. Norms and the bootstrap proposition

We are now ready to define our main norms and state the main bootstrap proposition. Let $\mathcal{B}_T$ denote the space of continuous functions on $\mathbb{R}^3 \times [0,T]$ defined by the norm

$$\|f\|_{\mathcal{B}_T} := \sup_{t \in [0,T]} \|f(t)\|_{\mathcal{B}_T^0},$$

$$\|f(t)\|_{\mathcal{B}_T^0} := \sup_{k \in \mathbb{Z}} \{\langle t \rangle^3 \|P_k f(t)\|_{L^\infty} + \|P_k f(t)\|_{L^1}\}.$$  

(2.22)

Assume that $\delta \in (0,1/100]$ is a small parameter and define the norms

$$\|f\|_{\text{Stat}_\delta} := \|\langle t \rangle^{1-2\delta} \langle \nabla_x \rangle f\|_{\mathcal{B}_T},$$

$$\|f\|_{\text{Osc}_\delta} := \|\langle t \rangle^{-\delta} \|f\|_{\mathcal{B}_T} + \|\langle t \rangle^{1-2\delta} \nabla_x f\|_{\mathcal{B}_T}.$$  

(2.23)

We are now ready to state our main bootstrap proposition:

**Proposition 2.4.** There exist $0 < \varepsilon \ll 1$, $\delta \in (0,\frac{1}{100}]$ and $C > 0$ such that the following is true:

Assume that $f : \mathbb{R}^3 \times \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ is a regular solution of the system (2.1) for some $T > 0$ with initial data $f(0) = f_0 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ satisfying the smallness condition

$$\|\langle v \rangle^{4.5} (\partial_x^\alpha \partial_v^\beta f_0)(x,v)\|_{L_x^\infty \cap L_v^\infty} \leq \varepsilon_0 \leq \varepsilon, \quad \alpha, \beta \in \mathbb{N}_0^3, \quad |\alpha| + |\beta| \leq 1,$$

(2.24)

and assume the associated density $\rho$ decomposes as $\rho = \rho^{\text{stat}} + \Re \{e^{-it}\rho^{\text{osc}}\}$ as in Lemma 2.3, which for some $0 < \varepsilon_1 \leq \varepsilon_0^{1/2}$ is bounded as

$$\|\rho^{\text{stat}}\|_{\text{Stat}_\delta} + \|\rho^{\text{osc}}\|_{\text{Osc}_\delta} \leq \varepsilon_1.$$  

(2.25)
Then the functions \( \rho^{\text{stat}} \) and \( \rho^{\text{osc}} \) satisfy the improved bounds
\[
\| \rho^{\text{stat}} \|_{\text{Stat}} + \| \rho^{\text{osc}} \|_{\text{Osc}} \leq \varepsilon_0 + C\varepsilon^2.
\] (2.26)

The proof of Proposition 2.4 will take up sections 4, 5, 6, and 7. We decompose
\[
\rho^{\text{stat}} = \rho^{\text{stat}}_{1,I} + \rho^{\text{stat}}_{1,II} + \rho^{\text{stat}}_{2,I} + \rho^{\text{stat}}_{2,II},
\]
\[
\rho^{\text{osc}} = \rho^{\text{osc}}_{1,I} + \rho^{\text{osc}}_{1,II} + \rho^{\text{osc}}_{2,I} + \rho^{\text{osc}}_{2,II},
\]
where for \( a \in \{1, 2\} \) and \( * \in \{I, II\} \) we define
\[
\rho_{a,*}^{\text{stat}}(x, t) := \sum_{(j, k, m) \in A^*} \tilde{\varphi}_m(t) P_{a,j,k}(x, t),
\] (2.27)
\[
\rho_{a,*}^{\text{osc}}(x, t) := \sum_{(j, k, m) \in A^*} \tilde{\varphi}_m(t) T_{a,j,k}(x, t).
\] (2.28)

Finally, in Section 8 we show how Proposition 2.4 implies our main result Theorem 1.1.

3. Preliminary estimates

In this section we assume that \( f : \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T] \rightarrow \mathbb{R} \) is a regular solution of the system (2.1), and the density function \( \rho \) satisfies the bootstrap assumptions (2.25).

To measure the deviation of the characteristic flow from the free flow we define the functions \( \tilde{Y}, \tilde{W} : \mathbb{R}^3 \times \mathbb{R}^3 \times I^2 \rightarrow \mathbb{R}^3 \) by
\[
\tilde{Y}(x, v, s, t) := X(x + tv, v, s, t) - x - sv,
\]
\[
\tilde{W}(x, v, s, t) := V(x + tv, v, s, t) - v.
\] (3.1)

The definitions (2.3) show that \( \tilde{Y}(x, v, t, t) = 0, \tilde{W}(x, v, t, t) = 0 \) and
\[
\tilde{W}(x, v, s, t) = - \int_s^t E(x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) d\tau,
\]
\[
\tilde{Y}(x, v, s, t) = \int_s^t (\tau - s) E(x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) d\tau.
\] (3.2)

Moreover, for any \( s, t \in [0, T] \) and \( x, v \in \mathbb{R}^3 \) we have
\[
\partial_s \tilde{Y}(x, v, s, t) = \tilde{W}(x, v, s, t), \quad \tilde{Y}(x, v, s, t) = - \int_s^t \tilde{W}(x, v, \tau, t) d\tau.
\] (3.3)

3.1. Bounds on the density function \( \rho \) and electric field \( E \). The decomposition \( \rho = \rho^{\text{stat}} + \Re \{ e^{-it} \rho^{\text{osc}} \} \) in Proposition 2.3 induces a natural decomposition of the electric field
\[
E(t) = E^{\text{stat}}(t) + \Re \{ e^{-it} E^{\text{osc}}(t) \},
\]
\[
E^{\text{stat}}(t) := (\nabla_x \Delta_x^{-1} \rho^{\text{stat}})(t), \quad E^{\text{osc}}(t) := (\nabla_x \Delta_x^{-1} \rho^{\text{osc}})(t).
\] (3.4)

We start with some bounds on the density components \( \rho^{\text{stat}}, \rho^{\text{osc}} \) and on the electric fields \( E^{\text{stat}}, E^{\text{osc}} \).
Lemma 3.1. (i) For any \(k \in \mathbb{Z}\) and \(t \in [0,T]\) we have
\[
\| \langle t \rangle^3 (P_k \rho_{\text{stat}}(t)) \|_{L^\infty} + \| (P_k \rho_{\text{osc}}(t)) \|_{L^1} \lesssim \varepsilon_1 2^{-k} \langle t \rangle^{-1+2\delta},
\]
(3.5)
\[
\| \langle t \rangle^3 (P_k \rho_{\text{osc}}(t)) \|_{L^\infty} + \| (P_k \rho_{\text{osc}}(t)) \|_{L^1} \lesssim \frac{\varepsilon_1}{\langle t \rangle^{-2\delta} 2^k + (2 + t)^{-\delta}},
\]
(3.6)
\[
\| \langle t \rangle^3 (P_k \nabla_x \rho_{\text{osc}}(t)) \|_{L^\infty} + \| (P_k \nabla_x \rho_{\text{osc}}(t)) \|_{L^1} \lesssim \varepsilon_1 \langle t \rangle^{-1+2\delta}.
\]
(3.7)
(ii) Let \(R_j\) denote the Riesz transforms \(R_j = \partial_j |\nabla|^{-1},\) \(j \in \{1,2,3\}\). Then, for any \(t \in [0,T]\)
\[
\| \nabla_x E_{\text{stat}}(t) \|_{L^\infty} + \sum_{j \in \{1,2,3\}} \| R_j E_{\text{stat}}(t) \|_{L^\infty} \lesssim \varepsilon_1 \langle t \rangle^{-3+2\delta},
\]
(3.8)
and
\[
\| \nabla_x E_{\text{osc}}(t) \|_{L^\infty} + \sum_{j \in \{1,2,3\}} \| R_j E_{\text{osc}}(t) \|_{L^\infty} \lesssim \varepsilon_1 \langle t \rangle^{-2+\delta},
\]
(3.9)
\[
\| \nabla_x E_{\text{osc}}(t) \|_{L^\infty} \lesssim \varepsilon_1 \langle t \rangle^{-3+2\delta} \ln(2 + t), \quad \| \partial_t E_{\text{osc}}(t) \|_{L^\infty} \lesssim \varepsilon_1 \langle t \rangle^{-3+2\delta}.
\]
Proof. The bounds (3.5)–(3.7) follow directly from the bootstrap assumptions (2.25) and the definitions (2.22)–(2.23). To prove (3.8)–(3.9) we use (3.5)–(3.7) and the simple estimates \(\| P_k g \|_{L^\infty} \lesssim 2^{3k} \| P_k g \|_{L^1}\). Therefore
\[
\| E_{\text{stat}}(t) \|_{L^\infty} + \sum_{j \in \{1,2,3\}} \| R_j E_{\text{stat}}(t) \|_{L^\infty} \lesssim \sum_{2^k \geq \langle t \rangle^{-1}} 2^{-k} \| (P_k \rho_{\text{stat}}(t)) \|_{L^\infty} + \sum_{2^k \leq \langle t \rangle^{-1}} 2^{2k} \| (P_k \rho_{\text{stat}}(t)) \|_{L^1} \lesssim \varepsilon_1 \langle t \rangle^{-3+2\delta}
\]
and
\[
\| \nabla_x E_{\text{stat}}(t) \|_{L^\infty} \lesssim \sum_{2^k \geq \langle t \rangle^{-1}} \| (P_k \rho_{\text{stat}}(t)) \|_{L^\infty} + \sum_{2^k \leq \langle t \rangle^{-1}} 2^{3k} \| (P_k \rho_{\text{stat}}(t)) \|_{L^1} \lesssim \varepsilon_1 \langle t \rangle^{-4+2\delta} \ln(2 + t).
\]
The bounds (3.8) follow. The bounds (3.9) are similar, using (3.6)–(3.7). \(\square\)

3.2. Linear dispersive estimates. For \(\lambda \in \mathbb{R}\) and multipliers \(m : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}\) we define the operators
\[
\mathcal{I}_m(g; \lambda)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{ix \cdot \xi - i\lambda v \cdot \xi} \tilde{g}(\xi, v) m(\xi, v) d\xi dv.
\]
We have the following lemma:

Lemma 3.2. (i) If \(j \in \{1, \ldots, d\}\) and \(\lambda \neq 0\) then
\[
\mathcal{I}_{\xi_j m}(g; \lambda) = \frac{1}{\lambda} \left\{ \mathcal{I}_{\partial_j m}(g; \lambda) + \mathcal{I}_m(\partial_j g; \lambda) \right\}.
\]
(3.10)
(ii) Let \(K\) denote the inverse Fourier transform in \(\xi\) of the multiplier \(m,\)
\[
K(y, v) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} m(\xi, v) e^{i y \xi} d\xi.
\]
Then
\[ \|\text{I}_m(g; \lambda)\|_{L^1} \lesssim \|g\|_{L^1_L} \|K\|_{L^\infty L^1}, \]
\[ |\lambda|^d \|\text{I}_m(g; \lambda)\|_{L^\infty_L} \lesssim \|g\|_X \|K\|_{L^1_L L^\infty}, \]
\[ \|\text{I}_m(g; \lambda)\|_{L^\infty_L} \lesssim \|g\|_{L^1_L L^\infty_y} \|K\|_{L^\infty L^1_y}, \]
(3.11)
where, by definition,
\[ \|g\|_X := \sup_{p \in \mathbb{R}^d} \int_{\mathbb{R}^3} |g(w, p - w/\lambda)| \, dw. \] (3.12)

(iii) For \( k \in \mathbb{Z} \) let \( m_k(\xi, v) := m(\xi, v)\varphi_k(\xi) \) and \( K_k := \mathcal{F}^{-1}(m_k) \). Then
\[ |\lambda|^d \|\text{I}_{m_k}(g; \lambda)\|_{L^1} \lesssim 2^{-k} \left\{ \|g\|_{L^1_L} \|\nabla_v K_k\|_{L^\infty_L} + \|\nabla_v g\|_{L^1_y} \|K_k\|_{L^\infty L^1_y} \right\}, \]
\[ |\lambda| \|\text{I}_{m_k}(g; \lambda)\|_{L^\infty_L} \lesssim 2^{-k} \left\{ \|g\|_X \|\nabla_v K_k\|_{L^1_L L^\infty_y} + \|\nabla_v g\|_X \|K_k\|_{L^1_y L^\infty_y} \right\}, \]
\[ |\lambda|^d \|\text{I}_{m_k}(g; \lambda)\|_{L^\infty_L} \lesssim 2^{-k} \left\{ \|g\|_{L^1_y L^\infty_y} \|\nabla_v K_k\|_{L^\infty_L L^1_y} + \|\nabla_v g\|_{L^1_y L^\infty_y} \|K_k\|_{L^\infty L^1_y} \right\}. \] (3.13)

Proof. (i) The identities (3.10) follow from the definition and integration by parts in \( v \).
(ii) To prove the bounds (3.11) we rewrite
\[ \text{I}_m(g; \lambda)(x) = \int_{\mathbb{R}^d} K(y, v)g(x - y - \lambda v, v) \, dydv \]
\[ = |\lambda|^{-d} \int_{\mathbb{R}^d} K\left(y, \frac{x - y - w}{\lambda}\right)g\left(w, \frac{x - y - w}{\lambda}\right) \, dydw. \]
The bounds in (3.11) follow from these identities.
(iii) To prove (3.13) we write
\[ m_k(\xi, v) = \xi_a \cdot m_k(\xi, v), \]
and then combine the identity (3.10) and the first two bounds in (3.11). \( \square \)

3.3. **Fourier multipliers.** Some of our main identities, such as those in (2.18)-(2.19), involve the “derivative” operators \( D \). To estimate these contributions efficiently we need to localize in the Fourier space.

For any \( d \geq 1 \) let \( S^\infty = S^\infty(\mathbb{R}^d) \) denote the space of continuous compactly supported multipliers \( m : \mathbb{R}^d \to \mathbb{C} \) defined by the norm
\[ \|m\|_{S^\infty} := \|\mathcal{F}^{-1}m\|_{L^1} < \infty. \]
Notice that if \( m, m' \in S^\infty(\mathbb{R}^d) \) then
\[ \|mm'\|_{S^\infty} \leq \|m\|_{S^\infty} \|m'\|_{S^\infty}. \] (3.14)
Moreover if \( m \in S^\infty(\mathbb{R}^d) \) and \( f \in L^p(\mathbb{R}^d), p \in [1, \infty], \) then
\[ \|\mathcal{F}^{-1}(m\widehat{f})\|_{L^p} \lesssim \|m\|_{S^\infty} \|f\|_{L^p}. \]
Lemma 3.3. (i) For \( j \in \{0, 1, 2\} \), \( k \leq 4 \), and \( v \in \mathbb{R}^3 \) we have
\[
\left\| \frac{D^j}{1 + D^2} \varphi(-4,4)(\xi \cdot v)\varphi_p(1 - (\xi \cdot v)^2)\varphi_k(\xi) \right\|_{S^\infty} \lesssim 2^{-p}, \quad p \geq k,
\]
\[
\left\| \frac{D^j}{1 + D^2} \varphi(-4,4)(\xi \cdot v)\varphi_{\leq k+4}(1 - (\xi \cdot v)^2)\varphi_k(\xi) \right\|_{S^\infty} \lesssim 2^{-k},
\]
\[
\left\| \frac{D^j}{1 + D^2} \varphi(-4,4)(\xi \cdot v)\varphi_k(\xi) \right\|_{S^\infty} \lesssim \min\{1, 2^k(\nu)\}^j,
\]
where \( D = |\xi| - i(\xi, v) \) as in (2.15). Moreover, if \( j \in \{0, 1, 2\} \), \( k \geq 0 \), and \( v \in \mathbb{R}^3 \) then
\[
\left\| \frac{D^j}{1 + D^2} \varphi_k(\xi) \right\|_{S^\infty} \lesssim 2^{(j-2)k}.
\]
(ii) Similarly, for any \( j \in \{0, 1, 2\} \), \( k, p \in \mathbb{Z} \), and \( v \in \mathbb{R}^3 \) we have
\[
\| (\xi \cdot v)^j(1 - (\xi \cdot v)^2)^{1-j}\varphi(-4,4)(\xi \cdot v)\varphi_p(1 - (\xi \cdot v)^2)\varphi_{\leq k}(\xi) \|_{S^\infty} \lesssim 2^{-p},
\]
\[
\| (\xi \cdot v)^j(1 - (\xi \cdot v)^2)^{1-j}\varphi(-2,2)(\xi \cdot v)\varphi_{\leq k}(\xi) \|_{S^\infty} \lesssim \min\{1, 2^{k}(\nu)\}^j,
\]
\[
\| (\xi \cdot v)^j(1 - (\xi \cdot v)^2)^{1-j}\varphi(\xi \cdot v)\varphi_{\leq k}(\xi) \|_{S^\infty} \lesssim 2^{(j-2)q}, \quad q \geq 2.
\]

Proof. (i) By rotation, we may assume that \( v = re_1, \ r \geq 0 \). Using (3.14) it is easy to see that
\[
\left\| D^{j} \varphi_{\leq q'}(\xi \cdot v)\varphi_{\leq k'}(\xi) \right\|_{S^\infty} \lesssim (2^{k'} + 2^{q'})^j
\]
for any \( k', q' \in \mathbb{Z} \) and \( j \in \{0, 1, 2\} \).

We prove first the bounds (3.15). Since \( v = re_1 \) we have
\[
\frac{1}{1 + D^2} = \frac{1}{2} \left( \frac{1}{1 + iD} + \frac{1}{1 - iD} \right) = \frac{1/2}{1 + r\xi_1 + i|\xi|} + \frac{1/2}{1 - r\xi_1 - i|\xi|}.
\]
Notice that \( \varphi(-4,4)(r \xi_1)\varphi_k(\xi) \) vanishes unless \( r \gtrsim 2^{-k} \). Using also (3.19), it is easy to see that
\[
\left\| \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} \left\{ \frac{1}{1 + D^2} \varphi(-4,4)(r \xi_1)\varphi_p(1 - r^2 \xi_1^2)\varphi_k(\xi) \right\} \right\|_{s^\infty} \lesssim 2^{-p} \cdot (2^{-p} r)^{\alpha_1} (2^{-k})^{\alpha_2 + \alpha_3} \mathbf{1}_{\{\xi \in \mathbb{R}^3 : |\xi| \leq 2^k, |\xi_1| \leq 2^p / r\}}(\xi),
\]
if \( p \geq k \), for any integers \( \alpha_1, \alpha_2, \alpha_3 \in [0, 6] \). Using integration by parts it follows that
\[
\left\| \frac{1}{1 + D^2} \varphi(-4,4)(r \xi_1)\varphi_p(1 - r^2 \xi_1^2)\varphi_k(\xi) \right\|_{S^\infty} \lesssim 2^{-p}
\]
if \( p \geq k \). The bounds in the first line of (3.15) follow using also (3.14) and (3.18).

The remaining bounds in (3.15) are easier. Indeed, we can use again (3.19) to prove also the differential inequalities
\[
\left\| \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \partial_{\xi_3}^{\alpha_3} \left\{ \frac{1}{1 + D^2} \varphi(-4,4)(r \xi_1)\varphi_{\leq k+4}(1 - r^2 \xi_1^2)\varphi_k(\xi) \right\} \right\|_{S^\infty} \lesssim 2^{-k} \cdot (2^{-k} r)^{\alpha_1} (2^{-k})^{\alpha_2 + \alpha_3} \mathbf{1}_{\{\xi \in \mathbb{R}^3 : |\xi| \leq 2^k, |\xi_1| \leq 2^k / r\}}(\xi),
\]
for any integers $\alpha_1, \alpha_2, \alpha_3 \in [0, 6]$. The bounds in the second line of (3.15) follow using again integration by parts and the bounds (3.14) and (3.18). Similarly,

$$
\left| \frac{\partial^{\alpha_1}}{\xi_1} \frac{\partial^{\alpha_2}}{\xi_2} \frac{\partial^{\alpha_3}}{\xi_3} \frac{1}{1 + D^2} \varphi_{\leq -2} (r \xi_1) \varphi_k (\xi) \right| \\
\lesssim (2^{-k + r})^{\alpha_1} (2^{-k})^{\alpha_2 + \alpha_3} 1_{\{ \xi \in \mathbb{R}^3 : |\xi| \leq 2^k, |\xi_1| \leq 1/r \}} (\xi).
$$

Moreover, for all $k \in \mathbb{Z}$ and $q \geq \max \{2, k\}$ we write $1 + D^2 = (D + i)(D - i)$ and estimate

$$
\left| \frac{\partial^{\alpha_1}}{\xi_1} \frac{\partial^{\alpha_2}}{\xi_2} \frac{\partial^{\alpha_3}}{\xi_3} \left\{ \frac{D^j}{1 + D^2} \varphi (r \xi_1) \varphi_k (\xi) \right\} \right| \\
\lesssim 2^{j q - 2 q} \cdot (2^{-k} + r 2^{-q})^{\alpha_1} (2^{-k})^{\alpha_2 + \alpha_3} 1_{\{ \xi \in \mathbb{R}^3 : |\xi| \leq 2^k, |\xi_1| \leq 2^{q/r} \}} (\xi).
$$

The desired bounds in the last two lines of (3.15) follow using (3.14) and (3.18) as before. The bounds (3.16) follow in the same way. For $k \geq 0$ we have the differential inequalities

$$
\left| \frac{\partial^{\alpha_1}}{\xi_1} \frac{\partial^{\alpha_2}}{\xi_2} \frac{\partial^{\alpha_3}}{\xi_3} \left\{ \frac{D^j}{1 + D^2} \varphi_{\leq k+4} (r \xi_1) \varphi_k (\xi) \right\} \right| \\
\lesssim 2^{j k - 2 k} \cdot (r + 1) 2^{-k} \alpha_1 (2^{-k})^{\alpha_2 + \alpha_3} 1_{\{ \xi \in \mathbb{R}^3 : |\xi| \leq 2^k, |\xi_1| \leq 2^{k/r} \}} (\xi),
$$

which can be proved by decomposing $1 + D^2 = (D + i)(D - i)$ as before. The bounds (3.16) follow using also the differential inequalities (3.20) to control the contribution of the dyadic regions corresponding to $q \geq k$.

(ii) By rotation, we may assume that $v = r e_1$, $r \geq 0$. We may also assume that $k = 0$, by rescaling $\xi = 2^k \eta$, replacing $v$ by $2^k v$. Using (3.14) we may also replace that factor $\varphi_{\leq 0} (\xi)$ with $\varphi_{\leq 4} (\xi_1) \varphi_{\leq 4} (\xi_2) \varphi_{\leq 4} (\xi_3)$, such that all the multipliers in (3.17) have product structure in the variables $\xi_1, \xi_2, \xi_3$. The desired $S^\infty$ bounds follow easily. \hfill $\square$

3.3.1. The $S^\infty L^\infty$ class. To apply the dispersive bounds in Lemma 3.2 we introduce the space $S^\infty L^\infty (\mathbb{R}^d \times \mathbb{R}^d)$ of multipliers $m = m(\xi, v) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ defined by the norm

$$
\| m \|_{S^\infty L^\infty} := \| F^{-1} m \|_{L^1_x L^\infty_v} = \left\| \sup_{v \in \mathbb{R}^d} \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} m(\xi, v) e^{ix \xi} d\xi \right| \right\|_{L^1_v}.
$$

**Lemma 3.4.** (i) If $m, m' \in S^\infty L^\infty$ then $mm' \in S^\infty L^\infty$ and

$$
\| mm' \|_{S^\infty L^\infty} \leq \| m \|_{S^\infty L^\infty} \| m' \|_{S^\infty L^\infty}.
$$

(ii) If $k \in \mathbb{Z}$, and $j \in \mathbb{Z}_+$ then

$$
\| D^k \varphi (\xi) \|_{S^\infty L^\infty} \lesssim 2^{j + k},
$$

$$
\| (\nabla_v D) \varphi (\xi) \|_{S^\infty L^\infty} \lesssim 2^k.
$$

Moreover, if $j + k \leq -10$ then

$$
\| (1 + D^2)^{-1} \varphi (\xi) \|_{S^\infty L^\infty} \lesssim 1.
$$

**Proof.** The bounds (3.21), (3.22), and (3.23) follow directly from definitions. To prove (3.24) we decompose $(1 + D^2) = (D + i)(D - i)$ and integrate by parts in $\xi$ to show that

$$
\left| \int_{\mathbb{R}^3} \frac{1}{D(\xi, v) \pm i} e^{ix \xi} \varphi (\xi) \varphi_j (v) d\xi \right| \lesssim \varphi_j (v) 2^d (1 + 2^k |x|)^{-d-1}
$$

for any $x, v \in \mathbb{R}^d$. The bounds (3.24) follow. \hfill $\square$
3.4. Pointwise estimates of characteristics. We prove now estimates on the functions \( \tilde{Y} \) and \( \tilde{W} \) defined in (3.1), which measure the nonlinear deviation of the characteristic flow.

**Lemma 3.5.** For any \((s,t) \in T^2_\tau\), we have

\[
\sup_{x,v \in \mathbb{R}^3} |\partial_t \tilde{Y}(x,v,s,t) - (t-s)E(x+tv,t)| \lesssim \varepsilon_1 \langle t-s \rangle^{-2+\delta} \langle s \rangle^{-1+1.1\delta}, \tag{3.25}
\]

\[
\sup_{x,v \in \mathbb{R}^3} |\partial_t \tilde{W}(x,v,s,t) + E(x+tv,t)| \lesssim \varepsilon_1 \langle t-s \rangle^{-2+\delta} \langle s \rangle^{-2+1.1\delta}. \tag{3.26}
\]

**Proof.** It follows from (3.2) and the assumption \( \tilde{Y}(x,v,t,t) = 0 \) that

\[
\partial_t \tilde{Y}(x,v,s,t) = (t-s)E(x+tv,t) + \int_s^t (\tau-s) \nabla_x E(x+\tau v + \tilde{Y}(x,v,\tau,v),\tau) \cdot \partial_t \tilde{Y}(x,v,\tau,t) \, d\tau
\]

and

\[
\partial_t \tilde{W}(x,v,s,t) = -E(x+tv,t) - \int_s^t \nabla_x E(x+\tau v + \tilde{Y}(x,v,\tau,v),\tau) \cdot \partial_t \tilde{Y}(x,v,\tau,t) \, d\tau.
\]

For \( t \in [0,T] \) let

\[
Z_0(t) := \sup_{s \in [0,t]} \sup_{x,v \in \mathbb{R}^3} (t-s)^{-1} |\partial_t \tilde{Y}(x,v,s,t)|.
\]

Using (3.27) and Lemma 3.1 (ii) we have

\[
Z_0(t) \lesssim \varepsilon_1 \langle t \rangle^{-2+\delta} + \frac{\varepsilon_1}{t-s} \int_s^t (\tau-s)(\tau)^{-3+2\delta}(t-\tau)Z_0(t) \, d\tau
\]

\[
\lesssim \varepsilon_1 \langle t \rangle^{-2+\delta} + \varepsilon_1 \int_s^t \langle \tau \rangle^{-2+2\delta} Z_0(t) \, d\tau \lesssim \varepsilon_1 \langle t \rangle^{-2+\delta} + \varepsilon_1 Z_0(t).
\]

Therefore \( Z_0(t) \lesssim \varepsilon_1 \langle t \rangle^{-2+\delta} \). Using again (3.27) and Lemma 3.1 (ii) we have

\[
|\partial_t \tilde{Y}(x,v,s,t) - (t-s)E(x+tv,t)| \lesssim \int_s^t \varepsilon_1 \langle \tau \rangle^{-3+1.1\delta} \langle t-s \rangle \langle t-\tau \rangle^{-2+\delta} \, d\tau
\]

\[
\lesssim \varepsilon_1 \langle t-s \rangle \langle t \rangle^{-2+\delta} \langle s \rangle^{-1+1.1\delta},
\]

which gives the desired estimate (3.25). The estimates (3.26) follow in a similar way. \( \Box \)

**Lemma 3.6.** For any \((s,t) \in T^2_\tau\) and \( x,v \in \mathbb{R}^3 \) we have

\[
|\tilde{Y}(x,v,s,t)| \lesssim \varepsilon_1 \min\{\langle s \rangle^{-1+2\delta} \langle v \rangle, \langle s \rangle^{-1/6}\},
\]

\[
|\tilde{W}(x,v,s,t)| \lesssim \varepsilon_1 \min\{\langle s \rangle^{-2+2\delta} \langle v \rangle, \langle s \rangle^{-7/6}\},
\]

\[
|\nabla_x \tilde{Y}(x,v,s,t)| \lesssim \varepsilon_1 \min\{\langle s \rangle^{-2+2.1\delta} \langle v \rangle, \langle s \rangle^{-7/6}\},
\]

\[
|\nabla_x \tilde{W}(x,v,s,t)| \lesssim \varepsilon_1 \min\{\langle s \rangle^{-3+2.1\delta} \langle v \rangle, \langle s \rangle^{-13/6}\},
\]

\[
|\nabla_y \tilde{Y}(x,v,s,t)| \lesssim \varepsilon_1 \min\{\langle s \rangle^{-1+2.1\delta} \langle v \rangle, \langle s \rangle^{-1/6}\},
\]

\[
|\nabla_y \tilde{W}(x,v,s,t)| \lesssim \varepsilon_1 \min\{\langle s \rangle^{-2+2.1\delta} \langle v \rangle, \langle s \rangle^{-7/6}\}.
\]

(3.28) (3.29) (3.30)
Proof. Step 1. To prove the bounds (3.28) we use the decomposition (3.4) of the electric field, and rewrite the identity in the first line of (3.2) in the form

\[
\tilde{W}(x,v,s,t) = \tilde{W}^{\text{stat}}(x,v,s,t;t) + \tilde{W}^{\text{osc}}(x,v,s,t;t),
\]

\[
\tilde{W}^{\text{stat}}(x,v,s_1,s_2;t) := - \int_{s_1}^{s_2} \left[ E^{\text{stat}} + \Re(e^{-i\tau} P_{\geq 0} E^{\text{osc}}) \right] (x + \tau v + \tilde{Y}(x,v,\tau,t),\tau) d\tau, \tag{3.31}
\]

\[
\tilde{W}^{\text{osc}}(x,v,s_1,s_2;t) := - \int_{s_1}^{s_2} \Re(e^{-i\tau} P_{< 0} E^{\text{osc}})(x + \tau v + \tilde{Y}(x,v,\tau,t),\tau) d\tau.
\]

In view of (3.6) we have \(\|P_{\geq 0} E^{\text{osc}}(\tau)\|_{L^\infty} \lesssim \varepsilon_1 \langle \tau \rangle^{-4+2\delta}\). Thus, using also (3.8), if \(m \geq 0\) and \(s_1 \leq s_2 \in [2^m - 1, 2^m + 1] \cap [0,T]\) then

\[
|\tilde{W}^{\text{stat}}(x,v,s_1,s_2;t)| \lesssim \int_{s_1}^{s_2} \varepsilon_1 \langle \tau \rangle^{-3+2\delta} \, d\tau \lesssim \varepsilon_1 2^{-2m+2\delta}. \tag{3.32}
\]

To bound \(|\tilde{W}^{\text{osc}}|\) we integrate by parts in \(\tau\). Notice that \(\partial_\tau \tilde{Y}(x,v,\tau,t) = -\tilde{W}(x,v,\tau,t)\), see (3.3), and \(|\tilde{W}(x,v,\tau,t)| \lesssim \varepsilon_1 \langle \tau \rangle^{-1+\delta}\) (due to (3.2) and (3.8)–(3.9)). Therefore

\[
|\tilde{W}^{\text{osc}}(x,v,s_1,s_2;t)| \lesssim \int_{s_1}^{s_2} \{ \|(\partial_\tau P_{< 0} E^{\text{osc}})(\tau)\|_{L^\infty} + \|\nabla_x P_{< 0} E^{\text{osc}}(\tau)\|_{L^\infty} \}
\times (|v| + |\tilde{W}(x,v,\tau,t)|) \, d\tau + \sum_{\tau \in \{s_1,s_2\}} \|P_{< 0} E^{\text{osc}}(\tau)\|_{L^\infty} \tag{3.33}
\]

\[
\lesssim \varepsilon_1 2^{-2m+2\delta} \langle v \rangle,
\]

using (3.9) again in the last line.

We prove now that if \(m \geq 0\), \(|v| \geq 2^{4m/5+10}\), and \(s_1 \leq s_2 \in [2^m - 1, 2^m + 1] \cap [0,T]\) then

\[
|\tilde{W}^{\text{osc}}(x,v,s_1,s_2;t)| \lesssim \varepsilon_1 2^{-7m/6}. \tag{3.34}
\]

Indeed, we decompose

\[
\tilde{W}^{\text{osc}}(x,v,s_1,s_2;t) = \sum_{k \leq -1} \Re I_k^{(1)}(x,v,s_1,s_2;t),
\]

\[
I_k^{(1)}(x,v,s_1,s_2;t) := \int_{s_1}^{s_2} e^{-ir} (P_k E^{\text{osc}})(x + \tau v + \tilde{Y}(x,v,\tau,t),\tau) \, d\tau.
\]

Using (3.6) we have \(\|P_k E^{\text{osc}}(\tau)\|_{L^\infty} \lesssim \varepsilon_1 \min(\langle \tau \rangle^{-3}, 2^{3k}) \langle \tau \rangle^{-1+2\delta} 2^{-2k}\) for any \(k \leq 0\), thus

\[
\sum_{k \leq -6m/5+6} |I_k^{(1)}(x,v,s_1,s_2;t)| \lesssim \sum_{k \leq -6m/5+6} \varepsilon_1 2^{2\delta m} 2^k \lesssim \varepsilon_1 2^{-6m/5+2\delta m}, \tag{3.35}
\]

for any \(s_1 \leq s_2 \in [2^m - 1, 2^m + 1] \cap [0,T]\), consistent with the desired bounds in (3.34).

On the other hand, if \(k \in [-6m/5 + 6, 0]\) then we write

\[
e^{-ir} (P_k E^{\text{osc}})(x + \tau v + \tilde{Y}(x,v,\tau,t),\tau) = \frac{1}{(2\pi)^2} \int \varphi_k(\xi) \frac{e^{i\xi}}{e^{i\xi}(x+\tau v+\tilde{Y}(x,v,\tau,t))} \, d\xi.
\]
We insert cutoff functions of the form $\varphi_p(1 - (\xi \cdot v)^2)$ and decompose

$$I^{(1)}_k = J^{(1)}_{k, p_0} + \sum_{p \geq p_0 + 1} J^{(1)}_{k, p},$$

$$J^{(1)}_{k, s}(x, v, s_1, s_2; t) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi_s(1 - (\xi \cdot v)^2)e^{-\imath t(1 - \xi \cdot v)}$$

$$\times \varphi_k(\xi) \hat{E}^{osc}(\xi, \tau)e^{\imath \xi \cdot (x + \bar{Y}(x, v, \tau, t))} d\xi d\tau,$$

where $p_0 := -2m/5$ and $s \in \{p \leq p_0\}$. Since $\|P_k E^{osc}(\tau)\|_{L^\infty} \lesssim \|P_k E^{osc}(\tau)\|_{L^1} \lesssim 2^{-2k}\delta^{-1+2\delta}$ (due to (3.6)) and recalling that $|v| \geq 2^{m/5+10}$, we have

$$|J^{(1)}_{k, p}(x, v, s_1, s_2; t)| \lesssim \int_{s_1}^{s_2} \int_{\mathbb{R}^3} |\varphi_{p_0}(1 - (\xi \cdot v))| \varphi_k(\xi)\|\hat{E}^{osc}(\xi, \tau)\|_{L^\infty} \|\hat{E}^{osc}(\xi, \tau)\|_{L^1} \lesssim \varepsilon_1 2^{-6m/5+2m \delta}. (3.37)$$

On the other hand, for $p \geq p_0 + 1$ we integrate by parts in $\tau$ and estimate

$$|J^{(1)}_{k, p}(x, v, s_1, s_2; t)| \lesssim \sum_{\tau \in \{s_1, s_2\}} \int_{\mathbb{R}^3} \varphi_p(1 - (\xi \cdot v)^2)\frac{1 - \xi \cdot v}{e^{-\imath t(1 - \xi \cdot v)}\varphi_k(\xi)\hat{E}^{osc}(\xi, \tau)e^{\imath \xi \cdot (x + \bar{Y}(x, v, \tau, t))} d\xi |$$

$$+ \int_{s_1}^{s_2} \int_{\mathbb{R}^3} \varphi_p(1 - (\xi \cdot v)^2)\frac{1 - \xi \cdot v}{e^{-\imath t(1 - \xi \cdot v)}\varphi_k(\xi)\frac{d}{d\tau}\{\hat{E}^{osc}(\xi, \tau)e^{\imath \xi \cdot (x + \bar{Y}(x, v, \tau, t))}\} d\xi d\tau.$$}

In view of (3.17), for $F \in \{E^{osc}(\tau), (\partial_\tau E^{osc})(\tau)\}$ we have

$$\left| \int_{\mathbb{R}^3} \varphi_p(1 - (\xi \cdot v)^2)\frac{1 - \xi \cdot v}{e^{-\imath t(1 - \xi \cdot v)}\varphi_k(\xi)\hat{F}(\xi)e^{\imath \xi \cdot (x + \bar{Y}(x, v, \tau, t))} d\xi \right| \lesssim (2^{-p} + 2^{-p/2})\|P_k F\|_{L^\infty}.$$}

Since $\|P_k E^{osc}(\tau)\|_{L^\infty} \lesssim \langle \tau \rangle^{-2+2\delta}$ and $\|P_k(\partial_\tau E^{osc})(\tau)\|_{L^\infty} \lesssim \langle \tau \rangle^{-3+2\delta}$ (due to (3.6)–(3.7)) and $|\partial_\tau \bar{Y}(x, v, \tau, t)| = |\bar{W}(x, v, \tau, t)| \lesssim \varepsilon_1 \langle \tau \rangle^{-1+\delta}$, it follows from the last two estimates that

$$|J^{(1)}_{k, p}(x, v, s_1, s_2; t)| \lesssim \varepsilon_1 (2^{-p} + 2^{-p/2})2^{-2m+4\delta m}$$

for any $s_1 \leq s_2 \in [2^{m} - 1, 2^{m+1}] \cap [0, t]$.

Recalling that $p_0 := -2m/5$ and using (3.36)–(3.37) it follows that $|I^{(1)}_k(x, v, s_1, s_2; t)| \lesssim \varepsilon_1 2^{-6m/5+2m \delta}$. The desired bounds (3.34) follow using also (3.35).

We combine now the bounds (3.32)–(3.34) and divide the interval $[s, t]$ dyadically to conclude that $|\bar{W}(x, v, s, t)| \lesssim \varepsilon_1 \min\{(s)^{-2+2\delta}(v), (s)^{-7/6}\}$ for any $s \leq t \in [0, T]$. This gives the bounds in the second line of (3.28). The bounds in the first line then follow using the identity $\partial_t \bar{Y} = \bar{W}$ and integrating on the interval $[s, t]$.

**Step 2.** To prove the bounds (3.29) we define, for any $t \in [0, T]$,

$$Z_1(t) := \sup_{s \in [0, t]} \sup_{x, v \in \mathbb{R}^3} \left[(v)\frac{1}{(s)^{2-2.1\delta}} + (s)^{7/6}\right]| \nabla_x \bar{Y}(x, v, s, t)| + (s)\nabla_x \bar{W}(x, v, s, t)|.$$}

In view of (3.2) we have

$$\partial_{x_2} \bar{W}(x, v, s, t) = -\int_s^t (\partial_{x_2} E)(x + \tau v + \bar{Y}(x, v, \tau, t), \tau)(\delta_{ab} + \partial_{x_a} \bar{Y}_b(x, v, \tau, t)) d\tau,$$
so we can decompose, as in (3.31)
\[
\partial_{x_{ab}}\tilde{W}(x, v, s, t) = \tilde{W}^{\text{stat}}_{x,1}(x, v, s, t; t) + \tilde{W}^{\text{osc}}_{x,1}(x, v, s, t; t),
\]
\[
\tilde{W}^{\text{stat}}_{x,1}(x, v, s, t; t) := -\int_{s_1}^{s_2} \left[ \partial_{x_{ab}} E^{\text{stat}} + \Re(e^{-i\tau} \partial_{x_{ab}} P_{\geq 0} E^{0}) \right] (x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) \times (\delta_{ab} + \partial_{x_{ab}} \tilde{Y}_b(x, v, \tau, t)) \, d\tau,
\]
\[
\tilde{W}^{\text{osc}}_{x,1}(x, v, s, t; t) := -\int_{s_1}^{s_2} \Re(e^{-i\tau} P_{0} \partial_{x_{ab}} E^{0}) (x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) \partial_{x_{ab}} \tilde{Y}_b(x, v, \tau, t) \, d\tau,
\]
\[
\tilde{W}^{\text{osc}}_{x,2}(x, v, s, t; t) := -\int_{s_1}^{s_2} \Re(e^{-i\tau} P_{0} \partial_{x_{ab}} E^{0}) (x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) \partial_{x_{ab}} \tilde{Y}_b(x, v, \tau, t) \, d\tau.
\]

Assume that \( m \geq 0 \) and \( s_1 \leq s_2 \in [2^{m} - 1, 2^{m+1}] \cap [0, t] \). Since \( |\nabla_v \tilde{Y}(x, v, \tau, t)| \leq Z_1(t)(\tau)^{-7/6} \) it follows from (3.5)–(3.6) that
\[
|\tilde{W}^{\text{stat}}_{x,1}(x, v, s_1, s_2; t)| + |\tilde{W}^{\text{stat}}_{x,2}(x, v, s_1, s_2; t)| \lesssim \varepsilon_1 2^{-3m+2.16m} + \varepsilon_1 Z_1(t)^{-2^{-3m}},
\]
Moreover, as in (3.33), we integrate by parts in \( \tau \) to show that
\[
|\tilde{W}^{\text{osc}}_{x}(x, v, s_1, s_2; t)| \lesssim \varepsilon_1 2^{-3m+2.16m}(\langle v \rangle).
\]
Finally, one can estimate as in the proof of (3.34) to show that if \( |v| \geq 2^{4m/5+10} \) then
\[
|\tilde{W}^{\text{osc}}_{x}(x, v, s_1, s_2; t)| \lesssim \varepsilon_1 2^{-13m/6}.
\]
We combine (3.39)–(3.41) and sum over integers \( m \) satisfying \( 2^m \gtrsim \langle s \rangle \) to conclude that
\[
|\nabla_x \tilde{W}(x, v, s, t)| \lesssim \varepsilon_1 \min(\langle s \rangle^{-3+2.16}(\langle v \rangle), \langle s \rangle^{-13/6}) + \varepsilon_1 Z_1(t)(\langle s \rangle)^{-3}.
\]
Using (3.3) it follows that
\[
\langle s \rangle |\nabla_x \tilde{W}(x, v, s, t)| + |\nabla_v \tilde{Y}(x, v, s, t)| \lesssim \varepsilon_1 \min(\langle s \rangle^{-2+2.16}(\langle v \rangle), \langle s \rangle^{-7/6}) + \varepsilon_1 Z_1(t)(\langle s \rangle)^{-2}.
\]
In particular, using the definition (3.38), \( Z_1(t) \lesssim \varepsilon_1 \), and the desired bounds (3.29) follow.

**Step 3.** The proof of the bounds (3.30) is similar. For \( t \in [0, T] \) we define
\[
Z_2(t) := \sup_{s \in [0, t]} \sup_{v \in \mathbb{R}^3} \left[ (\langle v \rangle)^{-1} (\langle s \rangle^{-1-2.16} + \langle s \rangle^{-1/6}) |\nabla_x \tilde{Y}(x, v, s, t)| + \langle s \rangle |\nabla_v \tilde{W}(x, v, s, t)| \right].
\]
Using the formulas (3.2) and the decomposition (3.4) we write
\[
\partial_{x_{ab}} \tilde{W}(x, v, s, t) = \tilde{W}^{\text{stat}}_{v,1}(x, v, s, t; t) + \tilde{W}^{\text{osc}}_{v,1}(x, v, s, t; t) + \tilde{W}^{\text{osc}}_{v,2}(x, v, s, t; t),
\]
where
\[
\tilde{W}^{\text{stat}}_{v,1}(x, v, s, t; t) := -\int_{s_1}^{s_2} \left[ \partial_{x_{ab}} E^{\text{stat}} + \Re(e^{-i\tau} \partial_{x_{ab}} P_{\geq 0} E^{0}) \right] (x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) \times (\tau \delta_{ab} + \partial_{x_{ab}} \tilde{Y}_b(x, v, \tau, t)) \, d\tau,
\]
\[
\tilde{W}^{\text{osc}}_{v,2}(x, v, s, t; t) := -\int_{s_1}^{s_2} \Re(e^{-i\tau} P_{0} \partial_{x_{ab}} E^{0}) (x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) \partial_{x_{ab}} \tilde{Y}_b(x, v, \tau, t) \, d\tau,
\]
\[
\tilde{W}^{\text{osc}}_{v,2}(x, v, s, t; t) := -\int_{s_1}^{s_2} \tau \Re(e^{-i\tau} P_{0} \partial_{x_{ab}} E^{0}) (x + \tau v + \tilde{Y}(x, v, \tau, t), \tau) \, d\tau.
\]
As in (3.39), it follows from (3.5)–(3.6) that
\[
|\tilde{W}^{\text{stat}}_{v,1}(x, v, s_1, s_2; t)| + |\tilde{W}^{\text{stat}}_{v,2}(x, v, s_1, s_2; t)| \lesssim \varepsilon_1 2^{-2m+2.16m} + \varepsilon_1 Z_2(t)^{-2^{2m}},
\]
provided that $m \geq 0$ and $s_1 \leq s_2 \in [2^m - 1, 2^{m+1}] \cap [0, t]$. Moreover, as in (3.40)–(3.41), we can integrate by parts in $\tau$ to show that
\[
|\tilde{W}_e^{osc}(x, v, s_1, s_2, t)| \lesssim \varepsilon_1 \min(2^{-2m+2.1\delta_m}(v), 2^{-7m/6}).
\]
As before we combine these bounds, sum over integers $m$ satisfying $2^m \gtrsim \langle s \rangle$, and use (3.3) to conclude that
\[
\langle s \rangle|\nabla_{x} \tilde{W}(x, v, s, t)| + |\nabla_{x} \tilde{Y}(x, v, s, t)| \lesssim \varepsilon_1 \min(\langle s \rangle^{-1+2.1\delta}(v), \langle s \rangle^{-1/6}) + \varepsilon_1 Z_2(t) \langle s \rangle^{-1}.
\]
In particular $Z_2(t) \lesssim \varepsilon_1$, using the definition (3.42), and the desired bounds (3.30) follow. □

4. The contributions of the initial data

In this section we bound the contributions of the initial data $f_0$. These contributions are the terms $R^\ast_{1,j,k}$ and $T^\ast_{1,j,k}$, $* \in \{I, II\}$, in the decomposition (2.20)–(2.21), originating from the term $L_1$ defined in (2.9). Our main result in this section is the following:

**Proposition 4.1.** With the notation in (2.27)–(2.28), we have
\[
\|p_{1,I}^{stat}\|_{Stat_1} + \|p_{1,II}^{stat}\|_{Stat_1} + \|\rho_{1,I}^{osc}\|_{Osc_1} + \|\rho_{1,II}^{osc}\|_{Osc_1} \lesssim \varepsilon_0.
\]

We would like to use the defining formulas (2.11) and (2.13)–(2.14) and Lemma 3.2 (with $d = 3$). For this we need several bounds on the functions $L_{1,j,k}$.

**Lemma 4.2.** With $L_{1,j,k} = L_{1,j,k}(x,v,s)$ defined as in (2.9)–(2.10), we have
\[
\sup_{s \in [0,T]} \left\{ \left\| \partial_{x}^{\alpha} \partial_{v}^{\beta} L_{1,j,k}(s) \right\|_{L^\infty_{x} L^1_{v}} + \langle s \rangle^{1-\delta} \left\| (\partial_{x} L_{1,j,k}(s) \right\|_{L^\infty_{x} L^1_{v}} \right\} \lesssim \varepsilon_0 2^{-4.5j},
\]
\[
\sup_{s \in [0,T]} \left\{ \left\| \partial_{x}^{\alpha} \partial_{v}^{\beta} L_{1,j,k}(s) \right\|_{L^\infty_{x} L^\infty_{v}} + \langle s \rangle^{1-\delta} \left\| (\partial_{x} L_{1,j,k}(s) \right\|_{L^\infty_{x} L^\infty_{v}} \right\} \lesssim \varepsilon_0 2^{-4.5j},
\]
\[
\sup_{s \in [1,T]} \left\{ \left\| \partial_{x}^{\alpha} \partial_{v}^{\beta} L_{1,j,k}(s) \right\|_{X_1} + \langle s \rangle^{1-\delta} \left\| (\partial_{x} L_{1,j,k}(s) \right\|_{X_1} \right\} \lesssim \varepsilon_0 2^{-4.5j},
\]
for any $(j, k) \in \mathbb{Z}_+ \times \mathbb{Z}$ and multi-indices $\alpha, \beta$ with $|\alpha| + |\beta| \leq 1$.

**Proof.** We use the initial-data assumptions (2.24). Let
\[
F_0(x, v) := |f_0(x, v)| + |\nabla_{x} f_0(x, v)| + |\nabla_{v} f_0(x, v)|, \quad F_0^\ast(x) := \sup_{v \in \mathbb{R}^3} |F_0(x, v)| \langle v \rangle^{4.5}.
\]

Notice that, for any $(x, v, s) \in \mathbb{R}^3 \times \mathbb{R}^3 \times [0, T]$ we have
\[
L_{1,j,k}(x,v,s) = \varphi_j(v) \int_{\mathbb{R}^3} L_1(x-y,v,s) K_k(y) dy,
\]
where $K_k := F^{-1}(\varphi_k)$. Thus
\[
|\partial_{x}^{\alpha} \partial_{v}^{\beta} L_{1,j,k}(x,v,s)| + \langle s \rangle^{1-\delta} |\partial_{x} L_{1,j,k}(x,v,s)|
\lesssim \varphi_{[j-2,j+2]}(v) \sum_{T \in \{I, \nabla_{x}, \nabla_{v}, \langle s \rangle^{1-\delta} \partial_{x} \}} \int_{\mathbb{R}^3} |TL_1(x-y,v,s)||K_k(y)| dy.
\]

We examine the formula in the first line of (2.9) and use the bounds (3.25)–(3.26), (3.29), and (3.30). Recalling the definition (2.27) and the bounds $|\tilde{W}(x, v, 0, s)| \leq 1$ it follows that
\[
|TL_1(z,v,s)| \lesssim F_0(z + \tilde{Y}(z,v,0,s), v + \tilde{W}(z,v,0,t)) \langle v \rangle \lesssim F_0^\ast(z + \tilde{Y}(z,v,0,s)) \langle v \rangle^{-4.5},
\]
for any $T \in \{ I, \nabla_x, \nabla_v, \langle s \rangle^{1-\delta} \partial_s \}$. Using also (4.2) we have
\[
|\partial^2_\xi \partial^2_\nu L_{1,j,k}(x,v,s)| + \langle s \rangle^{1-\delta} |\partial_s L_{1,j,k}(x,v,s)| \\
\lesssim \tilde{\varphi}_{j-2,j+2}(v)(v)^{-4.5} \int_{\mathbb{R}^3} F_0^*(x-y+\tilde{Y}(x-y,v,0,s))|K_k(y)| dy.
\]

The assumptions (2.24) show that $\|F_0^*\|_{L^\infty_x} + \|F_1^*\|_{L^1_x} \lesssim \varepsilon_0$. The $L^\infty_t L^\infty_x$ bounds in the second line of (4.1) then follow using (4.2). The $L^\infty_t L^1_x$ bounds in the first line also follow, once we notice that the mapping $x \to x-y+\tilde{Y}(x-y,v,0,s)$ is a global change of coordinates on $\mathbb{R}^3$ with Jacobian $\approx 1$ for any $y,v \in \mathbb{R}^3$, due to the bounds (3.29).

Finally, the bounds on the $X_s$ norms in the third line of (4.1) also follow using the definition (3.12) and the observation that the mapping $x \to x-y+\tilde{Y}(x-y,p-x/s,0,s)$ is a global change of coordinates on $\mathbb{R}^3$ with Jacobian $\approx 1$ for any $y,p \in \mathbb{R}^3$, due to the bounds (3.29)–(3.30). This completes the proof of the lemma. \hfill \square

**Proof of Proposition 4.1.**

In view of the definitions, it suffices to prove that for $* \in \{ I, II \}$, $(j,k,m) \in A^*$, $t \in [2^{m-1}, 2^{m+1}]$ (or $t \in [0,2]$ if $m = 0$) we have
\[
2^{m(1-2\delta)}2^{k^+}[\|R^*_{1,j,k}(t)\|_{L^1_t} + 2^{3m}\|T^*_{1,j,k}(t)\|_{L^\infty_t}] \lesssim \varepsilon_0 2^{-\delta j},
\]
(4.3)
\[
(2^{-b} + 2^{m(1-2\delta)}2^{k^+})[\|T^*_{1,j,k}(t)\|_{L^1_t} + 2^{3m}\|T^*_{1,j,k}(t)\|_{L^\infty_t}] \lesssim \varepsilon_0 2^{-\delta j},
\]
(4.4)
\[
2^{m(1-2\delta)}[\|(\partial_t + |\nabla|)T^*_{1,j,k}(t)\|_{L^1_t} + 2^{3m}\|(\partial_t + |\nabla|)T^*_{1,j,k}(t)\|_{L^\infty_t}] \lesssim \varepsilon_0 2^{-\delta j}.
\]
(4.5)

We prove these bounds in several steps.

**Proof of (4.3).** Assume first that $* = I$, so we start from the formula (2.11). We use also Lemma 3.2 with $d = 3$ and $m(\xi,v) = \varphi_{[k-4,k+4]}(\xi)\tilde{\varphi}_{j-4,j+4}(v)$, $\|m\|_{S^{\infty}L^{\infty}} \lesssim 1$. Thus
\[
2^{k^+}\|R^*_{1,j,k}(t)\|_{L^1_t} \lesssim \|L_{1,j,k}(t)\|_{L^1_t} + \|\nabla_x L_{1,j,k}(t)\|_{L^1_t} \lesssim \varepsilon_0 2^{-j},
\]
\[
2^{k^+}\|T^*_{1,j,k}(t)\|_{L^\infty_t} \lesssim \|L_{1,j,k}(t)\|_{L^\infty_t} + \|\nabla_x L_{1,j,k}(t)\|_{L^\infty_t} \lesssim \varepsilon_0 2^{-j},
\]
using the bounds (3.11) and Lemma 4.2. These bounds suffice if $2^m \lesssim 1$.

On the other hand, if $m \geq \delta^{-6}$, after using (3.11), (3.13), and Lemma 4.2, we have
\[
(1 + 2^{k^+}2^{3m})\|R^*_{1,j,k}(t)\|_{L^1_t} \lesssim \|L_{1,j,k}(t)\|_{L^1_t} + \|\nabla_v L_{1,j,k}(t)\|_{L^1_t} \lesssim \varepsilon_0 2^{-4j/3},
\]
\[
(1 + 2^{k^+}2^{3m})\|T^*_{1,j,k}(t)\|_{L^\infty_t} \lesssim \|L_{1,j,k}(t)\|_{L^\infty_t} + \|\nabla_v L_{1,j,k}(t)\|_{L^\infty_t} \lesssim \varepsilon_0 2^{-4j/3},
\]

The desired bounds (4.3) follow for $* = I$, since $2^{4j/3}(1 + 2^{k^+}2^{3m}) \gtrsim 2^{\delta j}2^{k^+}2^{m(1-2\delta)}$ for any $(j,k,m) \in A' \cap m \geq \delta^{-6}$.

Assume now that $* = II$, so we start from the formula (2.13). We use also Lemma 3.2 with $d = 3$ and
\[
m'(\xi,v) = \varphi_{[k-4,k+4]}(\xi)\tilde{\varphi}_{j-4,j+4}(v) \frac{D(\xi,v)^2}{1 + D(\xi,v)^2},
\]
which satisfies $\|m'\|_{S^{\infty}L^{\infty}} + \|\nabla_v m'\|_{S^{\infty}L^{\infty}} \lesssim 2^{2j+2k}$ if $j + k \leq -20$ due to Lemma 3.4. Thus, if $(j,k,m) \in A''$, we have
\[
(1 + 2^{k^+}2^{3m})\|R_{1,j,k}^{II}(t)\|_{L^1_t} \lesssim 2^{k^+}2^{j}[\|L_{1,j,k}(t)\|_{L^1_t} + \|\nabla_v L_{1,j,k}(t)\|_{L^1_t}] \lesssim \varepsilon_0 2^{-4j/3}2^{k^+},
\]
\[
(1 + 2^{k^+}2^{3m})\|T_{1,j,k}^{II}(t)\|_{L^\infty_t} \lesssim 2^{k^+}2^{j}[\|L_{1,j,k}(t)\|_{L^\infty_t} + \|\nabla_v L_{1,j,k}(t)\|_{L^\infty_t}] \lesssim \varepsilon_0 2^{-4j/3}2^{k^+}.
\]
Since $2^{j/3}2^{-k}(1 + 2^{k+m}) \gtrsim 2^{2j}2^k2^{m(1-2\delta)}$ for any $(j,k,m) \in A^I$, the desired bounds (4.3) follow if $* = II$.

**Proof of (4.4) when $* = I$.** Using (3.11) and Lemma 4.2 we have

\[
2^k \left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-(t-s)\xi} e^{-iv \cdot \xi} \overline{L_{1,j,k}}(\xi, v, s) d\xi dv \right\|_{L^1_t}
\leq \|L_{1,j,k}(s)\|_{L^1_tL^1_v} + \|\nabla_x L_{1,j,k}(s)\|_{L^1_tL^1_v} \lesssim \varepsilon_0 2^{-j}
\]

and

\[
2^k \left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-(t-s)\xi} e^{-iv \cdot \xi} \overline{L_{2,j,k}}(\xi, v, s) d\xi dv \right\|_{L^\infty_t}
\leq \|L_{1,j,k}(s)\|_{L^1_tL^\infty_v} + \|\nabla_x L_{1,j,k}(s)\|_{L^1_tL^\infty_v} \lesssim \varepsilon_0 2^{-j}
\]

for any $s \in [0,t]$. The bounds (4.4) follow if $* = I$ and $2^m \lesssim 1$, using the identities (2.11).

On the other hand, if $m \geq \delta^{-6}$, $t \in [2^{m-2}, 2^{m+2}]$ and $s \geq t/2$ then we use (3.11), (3.13), and Lemma 4.2 to estimate

\[
(1 + 2^{k+m}) \left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-(t-s)\xi} e^{-iv \cdot \xi} \overline{L_{1,j,k}}(\xi, v, s) d\xi dv \right\|_{L^1_t}
\lesssim (1 + 2^k |t-s|)^{-6} \left[ \|L_{1,j,k}(s)\|_{L^1_tL^1_v} + \|\nabla_v L_{1,j,k}(s)\|_{L^1_tL^1_v} \right]
\lesssim \varepsilon_0 (1 + 2^k |t-s|)^{-6} 2^{-4j/3}
\]

and

\[
(1 + 2^{k+m})2^{3m} \left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-(t-s)\xi} e^{-iv \cdot \xi} \overline{L_{2,j,k}}(\xi, v, s) d\xi dv \right\|_{L^\infty_t}
\lesssim (1 + 2^k |t-s|)^{-6} \left[ \|L_{1,j,k}(s)\|_{X_s} + \|\nabla_v L_{1,j,k}(s)\|_{X_s} \right]
\lesssim \varepsilon_0 (1 + 2^k |t-s|)^{-6} 2^{-4j/3}.
\]

Moreover, if $s \in [0,t/2]$ then we use (3.11) and Lemma 4.2 to estimate

\[
\left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-(t-s)\xi} e^{-iv \cdot \xi} \overline{L_{1,j,k}}(\xi, v, s) d\xi dv \right\|_{L^1_t}
\lesssim (1 + 2^k |t-s|)^{-6} \|L_{1,j,k}(s)\|_{L^1_tL^1_v}
\lesssim \varepsilon_0 (1 + 2^{k+m})^{-6} 2^{-4j/3}.
\]

Since $\|P_l g\|_{L^\infty} \lesssim \|\widehat{P_l g}\|_{L^1} \lesssim 2^{|l|} \|\widehat{P_l g}\|_{L^1} \lesssim 2^{|l|} \|P_l g\|_{L^1}$ for any $l \in \mathbb{Z}$, it follows that

\[
2^{3m} \left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{ix \cdot \xi} e^{-(t-s)\xi} e^{-iv \cdot \xi} \overline{L_{2,j,k}}(\xi, v, s) d\xi dv \right\|_{L^1_t}
\lesssim \varepsilon_0 (1 + 2^{k+m})^{-3} 2^{-4j/3}.
\]

Therefore, using the identities (2.11) and the last four inequalities, we have

\[
(1 + 2^{k+m}) \left[ \|T_{1,j,k}(t)\|_{L^1} + 2^{3m} \|T_{1,j,k}(t)\|_{L^\infty} \right] \lesssim \int_0^t \varepsilon_0 (1 + 2^k |t-s|)^{-2} 2^{-4j/3} ds
\lesssim \varepsilon_0 2^{-4j/3} \min\{2^m, 2^{-k}\}.
\]

Moreover, $2^{-4j/3} \min\{2^m, 2^{-k}\} \lesssim 2^{-\delta j}$ if $(j,k,m) \in A^I$ and $m \geq \delta^{-6}$, and the desired bounds (4.4) follow if $* = I$. 
Proof of (4.4) when \( s = II \). Assume that \((j, k, m) \in A^I\), so \( m \geq \delta^{-4}\), and \( t \in [2^{m-2}, 2^{m+2}]\). If \( s \in [0, t] \) we use (3.11), Lemma 3.4, and Lemma 4.2 to estimate

\[
\left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix \cdot \tilde{\xi}} e^{-(t-s)|\xi|} e^{-i\nu \cdot \tilde{\xi}} \frac{D'}{1 + D^2} (\partial_t \tilde{L}_{1,j,k})(\xi, v, s) \, d\xi \, dv \right\|_{L^1_t} 
\lesssim (1 + 2^k |t - s|)^{-6} \| \partial_s \tilde{L}_{1,j,k}(s) \|_{L^1_t L^1_x} 
\lesssim \varepsilon_0 (1 + 2^k |t - s|)^{-6} 2^{-j} (s)^{-1+\delta},
\]

where \( D' \in \{1, D\} \). If \( s \geq t/2 \) then we use (3.11), Lemma 3.4, and Lemma 4.2 to estimate

\[
2^{3m} \left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix \cdot \tilde{\xi}} e^{-(t-s)|\xi|} e^{-i\nu \cdot \tilde{\xi}} \frac{D'}{1 + D^2} (\partial_t \tilde{L}_{1,j,k})(\xi, v, s) \, d\xi \, dv \right\|_{L^\infty_t} 
\lesssim (1 + 2^k |t - s|)^{-6} \| \partial_s \tilde{L}_{1,j,k}(s) \|_{X_s} 
\lesssim \varepsilon_0 (1 + 2^k |t - s|)^{-6} 2^{-j} 2^{-m+\delta m}.
\]

As before, since \( \| P_{\nu} g \|_{L^\infty} \lesssim 2^{3l} \| P_{\nu} g \|_{L^1} \) for any \( l \in \mathbb{Z} \) we can use (4.6) to estimate

\[
2^{3m} \left\| \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{ix \cdot \tilde{\xi}} e^{-(t-s)|\xi|} e^{-i\nu \cdot \tilde{\xi}} \frac{D'}{1 + D^2} (\partial_t \tilde{L}_{1,j,k})(\xi, v, s) \, d\xi \, dv \right\|_{L^\infty_t} 
\lesssim \varepsilon_0 (1 + 2^k |t - s|)^{-3} 2^{-j} (s)^{-1+\delta}
\]

for \( s \in [0, t/2] \). We use these last three inequalities and the formulas (2.14), and integrate from 0 to \( t \) to conclude that

\[
\| T^{II}_{1,j,k}(t) \|_{L^1_t} + 2^{3m} \| T^{II}_{1,j,k}(t) \|_{L^\infty_t} \lesssim \varepsilon_0 2^{-j} 2^{-m+\delta m} \min(2^m, 2^{-k}).
\]

This suffices to prove the desired bounds (4.4).

Proof of (4.5). The formulas (2.11) show that

\[
(\partial_t + |\nabla|)T^{II}_{1,j,k}(t) = (-i) e^{it} \cdot R^{I}_{1,j,k}(t),
\]

so the bounds (4.5) follow from the bounds (4.3) if \( s = I \). Moreover, using (2.14),

\[
\mathcal{F}\{ (\partial_t + |\nabla|)T^{II}_{1,j,k} \}(t) = \int_\mathbb{R^3} e^{-iv \cdot \tilde{\xi}} \frac{e^{it}}{1 - iD} (\partial_t \tilde{L}_{1,j,k})(\xi, v, t) \, dv.
\]

Using (3.11), Lemma 3.4, and (4.1), if \((j, k, m) \in A^I\), we have

\[
\| (\partial_t + |\nabla|)T^{II}_{1,j,k}(t) \|_{L^1_t} + 2^{3m} \| (\partial_t + |\nabla|)T^{II}_{1,j,k}(t) \|_{L^\infty_t} 
\lesssim \varepsilon_0 2^{-j} 2^{-m+\delta m},
\]

This gives the desired bounds (4.5) if \( s = II \), which completes the proof of Proposition 4.1. \( \square \)

5. Bounds on the static terms and the type-I reaction term

In this section we estimate the \( B_T\)-norm (see (2.22)), of the static terms, which consist of the type-I component \( R^I_{2,j,k} \) defined in (2.16) and the type-II component \( R^{II}_{2,j,k} \) defined in (2.18). Moreover, the estimate of the type-I reaction term \( T^I_{2,j,k} \) will be obtained as a byproduct.

It turns out that the bounds on many error terms appearing in the reduction process not only in this section but also in the next section fit into a general framework. We first define
the trilinear operators and then state some trilinear estimates for the defined trilinear operators, which are better than direct crude estimates because we will exploit the benefit of the decomposition (2.3).

Assume that \( \theta_1, \theta_2 \in [0, 1] \) and \( s \in [0, T] \), and define the trilinear operators

\[
\mathcal{R}_{j,k}(f, g; C)(x, \gamma, \tau, s) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_{j,k}(x - y, v, \tau, s) C(x, y, v, \gamma, \tau, s) \\
\times f(y - (s - \tau)v + \theta_1 \tilde{Y}(y - sv, v, \tau, s)) g(y - (s - \gamma)v + \theta_2 \tilde{Y}(y - sv, v, \gamma, s), \gamma) \, dy \, dv,
\]

where \( \gamma, \tau \in [0, s] \), \((j, k) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \), and the kernel \( K_{j,k} \) satisfies the uniform estimates

\[
|K_{j,k}(y, v, \tau, s)| \lesssim 2^{3k}(1 + 2^k|y|)^{-8} \cdot 2^{-3j} \tilde{\epsilon}_{[j-4, j+4]}(v). \tag{5.2}
\]

Since only the upper bound of kernel will play a role in later argument, we suppress the dependence of operator \( \mathcal{R} \) with respect to the precise formula of kernel and the exact values of \( \theta_1 \) and \( \theta_2 \).

We assume that the coefficient \( C \) is differentiable in \( \gamma \), and define

\[
m(C)(\gamma, \tau, s) := \|C(x, y, v, \gamma, \tau, s)\|_{L^\infty_{x,y,v}},
\]

\[
m'(C)(\gamma, \tau, s) := \|C(x, y, v, \gamma, \tau, s)\|_{L^\infty_{x,y,v}} + \|\partial_\gamma C(x, y, v, \gamma, \tau, s)\|_{L^\infty_{x,y,v}}. \tag{5.3}
\]

Assume that \( m_2 \geq 0 \), \( t_3, t_4 \in [2^{m_2 - 1}, 2^{m_2 + 1}] \cap [0, s] \), and define the integrated trilinear operators

\[
Q_{j,k}(f, g; C)(x, \tau, s) = \int_{t_3}^{t_4} \mathcal{R}(f, g; C)(x, \gamma, \tau, s) \, d\gamma. \tag{5.4}
\]

For the above defined trilinear operators, we have

**Lemma 5.1 (Trilinear estimates).** Let \( s \in [0, T] \), \( \tau \in [0, s] \), \( 2^{m_2 - 1} \leq s \). With the assumptions of the bootstrap proposition 2.4, and the notation and assumptions above, then

\[
\|Q_{j,k}(\nabla E, E; C)(\cdot, \tau, s)\|_{B^0_{2, 1}} \lesssim \varepsilon_2^2 \min\{2^{-1.1m_2}, \langle \tau \rangle^{-1} - 1\} m^*(C)(\tau, s), \tag{5.5}
\]

\[
\|Q_{j,k}(E, \nabla E; C)(\cdot, \tau, s)\|_{B^0_{2, 1}} \lesssim \varepsilon_2^2 \min\{2^{-1.1m_2}, \langle \tau \rangle^{-1} - 1\} m^*(C)(\tau, s), \tag{5.6}
\]

\[
\|Q_{j,k}(E, E; C)(\cdot, \tau, s)\|_{B^0_{2, 1}} \lesssim \varepsilon_2^2 \min\{2^{-0.1m_2}, \langle \tau \rangle^{-0.1}\} m^*(C)(\tau, s), \tag{5.7}
\]

\[
\|Q_{j,k}(\nabla E, \nabla E; C)(\cdot, \tau, s)\|_{B^0_{2, 1}} \lesssim \varepsilon_1^2 \min\{2^{-2.1m_2}, \langle \tau \rangle^{-2.1}\} m^*(C)(\tau, s), \tag{5.8}
\]

\[
\sup_{k_1 \in \mathbb{Z}} \|Q_{j,k}(\nabla^2 E_{k_1}, E; C)(\cdot, \tau, s)\|_{B^0_{2, 1}} \lesssim \varepsilon_1^2 \langle \tau \rangle^{-1} \min\{2^{-1.1m_2}, \langle \tau \rangle^{-1}\} m^*(C)(\tau, s), \tag{5.9}
\]

\[
\sup_{k_1 \in \mathbb{Z}} \|Q_{j,k}(\nabla^2 E_{k_1}, \nabla E; C)(\cdot, \tau, s)\|_{B^0_{2, 1}} \lesssim \varepsilon_1^2 \langle \tau \rangle^{-1} \min\{2^{-2.1m_2}, \langle \tau \rangle^{-2.1}\} m^*(C)(\tau, s), \tag{5.10}
\]

where \( m^*(C)(\tau, s) := \sup_{\gamma \in [t_3, t_4]} m'(C)(\gamma, \tau, s) \).

**Proof.** Postponed to section 7 for better presentation.
5.1. **Large velocity case:** $j \geq 19m/20$. The goal of this section is to prove that the contribution of large velocities is acceptable.

**Proposition 5.2.** Assume that $j \geq 19m/20$

and let $0 \leq t \leq T$, there holds that

$$\| \langle t \rangle \langle \nabla \rangle R_{2,j,k}^I \|_{B_T} \lesssim \varepsilon_1^2 2^{-\delta j},$$

$$\| \langle t \rangle^{-\delta} T_{2,j,k}^I \|_{B_T} + \| \langle t \rangle^{-1-\delta} \nabla_x T_{2,j,k}^I \|_{B_T} \lesssim \varepsilon_1^2 2^{-\delta j}.$$  

**Proof.** The first inequality follows from Lemma 5.3 and Lemma 5.4 below. Recall (2.16). We have

$$(\partial_t + |\nabla|) T_{2,j,k}^I = -ie^{it} R_{2,j,k}^I.$$  

From the above equality, to prove the second estimate in (5.2), it would be sufficient to show

$$\| \langle t \rangle^{-\delta} T_{2,j,k}^I \|_{B_T} + \| \langle t \rangle^{-1-\delta} \nabla_x T_{2,j,k}^I \|_{B_T} \lesssim \varepsilon_1^2 2^{-\delta j}. \quad (5.11)$$

Again, from (5.1), we have

$$T_{2,j,k}^I(x,t) := -i \int_{s=0}^{t} e^{is} e^{-(t-s)} |\nabla| R_{2,j,k}^I(x,s) ds.$$  

Note that,

$$\forall p \in [1, \infty], \quad \| |\nabla| e^{-\lambda |\nabla|} f \|_{L_p} \lesssim \lambda^{-1} \| f \|_{L_p}. \quad (5.12)$$

Using Lemma 5.3, Lemma 5.4, and (5.12), we find that

$$\| e^{is} e^{-(t-s)} |\nabla| R_{2,j,k}^I(x,s) \|_{B^0_t} \lesssim \varepsilon_1^2 (s)^{-1-2^{-\delta j}},$$

$$\| e^{is} e^{-(t-s)} |\nabla| \nabla_x R_{2,j,k}^I(x,s) \|_{B^0_t} \lesssim \varepsilon_1^2 [1 + 2^k(t-s)]^{-1-2^{-\delta j}}.$$  

and upon integration, this gives (5.11).  

It remains to prove Lemma 5.3 and Lemma 5.4. Recall (2.9), (2.10), and (3.1). We have

$$L_{2,j,k}(x,v,s,t) := P_k \left\{ E(x,v) M_j'(v) - E(x + \tilde{Y}_#, s) M_j'(v + \tilde{W}_#) \right\},$$

$$\tilde{Y}_# := \tilde{Y}(x - sv, v, s, t), \quad \tilde{W}_# := \tilde{W}(x - sv, v, s, t).$$

In some cases, for very high frequencies, we will obtain a bound linear in $E$. To ensure that the overall contribution is nonlinearly small, we introduce an integer $K$ such that

$$1 \leq \varepsilon_1 2^K \leq 8. \quad (5.13)$$

In the following lemma, we mainly estimate the contribution from the high frequency case. More precisely, the case $k \geq 0$, in which we have $\langle \nabla \rangle \sim \nabla$.

**Lemma 5.3.** Assume that

$$j \geq 19m/20, \quad (5.14)$$

then there holds that

$$\| \langle t \rangle \nabla_x R_{2,j,k}^I \|_{B_T} \lesssim \varepsilon_1^2 2^{-\delta j}.$$
Proof of Lemma 5.3. We localize the frequency of the electric field and decompose
\[
\partial_{x^p}L_{2,j,k}(x,v,s,t) := P_k[Err_j] + \sum_{k_1 \in \mathbb{Z}} P_{k_1}[\mathcal{L}_{j,k_1}],
\] 
(5.15)
with
\[
\mathcal{L}_{j,k_1} := \left[\partial_{x^p}E_{k_1}(x,s) - \partial_{x^p}E_{k_1}(x + \tilde{Y}_#,s)\right] \cdot M_j'(v),
\]
\[
Err_j := \partial_{x^p}E(x + \tilde{Y}_#,s) \cdot \left[M_j'(v) - M_j'(v + \tilde{W}_#)\right] - \partial_{x^p}\tilde{Y}_#^q\partial_{x^p}E(x + \tilde{Y}_#,s) \cdot M_j'(v + \tilde{W}_#)
- \partial_{x^p}\tilde{W}_# \cdot \nabla_v M_j'(v + \tilde{W}_#)).
\] 
(5.16)

We first estimate the contribution from the error type term $Err_j$. Recall (3.2). After expanding $\tilde{w}$ and $\tilde{Y}$ as in (3.2), we observe that, with notation defined in (5.1), we have
\[
\int_{s=0}^{t} \int_{\mathbb{R}^3} P_k[Err_j](x - (t-s)v,v,s,t)dvds = \int_{\theta=0}^{1} \int_{s=0}^{t} \int_{s}^{t} [R_{j,k}(\nabla E, E, C^1) + R_{j,k}(\nabla E, \nabla E, C^2) + R_{j,k}(E, \nabla E, C^3)](x, \tau, s, t)drdsd\theta,
\]
with
\[
C^1 = 2^{3j} \nabla_v M_j(v + \theta \tilde{W}(x - tv, v, s, t)), \quad C^2 = 2^{3j}(\tau - s)M_j(v + \tilde{W}(x - tv, v, s, t)),
\]
\[
C^3 = -2^{3j} \nabla_v M_j(v + \tilde{W}(x - tv, v, s, t)),
\]
and this leads to acceptable contributions using Lemma 5.1 and the bound on $j$ (5.14).

Now, we move on to the estimate of the main contribution, we split into two cases based on the size of $k_1$ as follows.

**Case 1:** If $k_1 \geq -m/2 + K/2$.

We first consider the $L^1_x$-estimate. Recall $K$ defined in (5.13). Using crude estimates, we find that
\[
\langle t \rangle \left\| \int_{s=0}^{t} \int_{\mathbb{R}^3} \mathcal{L}_{j,k_1}(x - (t-s)v,s)dvds \right\|_{L^1_x} \lesssim \langle t \rangle 2^{-2j} \int_{s=0}^{t} \|\nabla_x E_{k_1}(s)\|_{L^1_x} ds \lesssim \varepsilon_1 \langle t \rangle \int_{s=0}^{t} 2^{-k_1(s)}2^{\delta - 1 - 2j}ds
\]
\[
\lesssim \varepsilon_1 \langle t \rangle^{-1 - 2j} 2^{\delta} \langle t \rangle^{1 + 2\delta}
\]
and the contribution for all $k_1$ with $k_1 \geq -m/2 + K/2$ is acceptable.

Now, we consider the $L^\infty_x$-estimate. If $t/2 \leq s \leq t$, then after using that
\[
\|\partial_{x^p}E_{k_1}(s)\|_{L^\infty_x} \lesssim \varepsilon_1 \langle s \rangle^{2\delta - 4k_1}
\]
and integrating over $v$, we find that
\[
\langle t \rangle 4 \left\| \int_{s=t/2}^{t} \int_{\mathbb{R}^3} \mathcal{L}_{j,k_1}(x - (t-s)v,s)dvds \right\|_{L^\infty_x} \lesssim \varepsilon_1 \langle t \rangle^{2\delta + 1 - k_1 - 2j}
\]
which gives an acceptable contribution given (5.14) and the definition of $K$. 


It remains to consider the case \(0 \leq s \leq t/2\). We can use the characteristics bounds in Lemma 3.6 to verify that the Jacobian of
\[ w \mapsto y := w + \tilde{Y} \left( \frac{t}{t-s}w - sp, p - \frac{w}{t-s} \right), \]
remains uniformly bounded and we deduce that
\[
\|L_{j,k_1}(s)\|_{L^1} \lesssim 2^{-5j} \|P_{k_1} P\|_{L^1} \lesssim \varepsilon_1 2^{-k_1-5j} \langle s \rangle^{2\delta-1}
\]
and using Lemma 3.2, we deduce that
\[
\langle t \rangle^4 \| \int_{s=0}^{t/2} L_{j,k_1}(x-(t-s)v,s)dvds \|_{L^\infty} \lesssim \varepsilon_1 2^{-k_1-5j} \langle t \rangle^{2\delta+1}
\]
which again gives an acceptable contribution.

**Case 2:** If \(k_1 \leq -m/2 + K/2\).

For this case, the derivative in \(E\) is not problematic. Note that, after taking difference between two electric fields, we have
\[
\left[ \partial_x E_{k_1}(x,s) - \partial_v E_{k_1}(x+\tilde{Y}_{\#},s) \right] M_j(v) = \int_{\theta=0}^{1} L_{j,k_1}(x,v,s,t) d\theta,
\]
where
\[
L_{j,k_1}(x,v,s,t) := \tilde{Y}_{\#}^2 \partial_x \partial_v E_{k_1}(x + \theta \tilde{Y}_{\#},s) \cdot M_j(v + \tilde{W}_{\#}).
\]
Recall (5.1). From (3.2), we can develop the characteristics
\[
\tilde{Y}_{\#} = \int_{\tau=s}^{t} (\tau-s)E(x-(\tau-s)v + \tilde{Y}_{1,\#},\tau)d\tau, \quad \tilde{Y}_{1,\#} := \tilde{Y}(x-sv,v,\tau,t).
\]
Therefore, for any fixed \(\theta \in [0,1]\), in terms of the trilinear operator defined in (5.1), we have
\[
\int_{\mathbb{R}^3} P_{k} L_{j,k_1}(x-(t-s)v,v,s,t)dv = \int_{s}^{t} \mathcal{R}_{j,k}(\nabla E_{k_1},E;C^1)(x,\tau,s,t)d\tau
\]
with
\[
C^1 = -2^{k_1}(\tau-s)2^{3j}M_j(v + \tilde{W}_{\#}),
\]
and this leads to an acceptable contribution using Lemma 5.1.

\[\square\]

In the following lemma, we mainly estimate the contribution from the low frequency case. More precisely, the case \(k \leq 0\), in which we have \(\langle \nabla \rangle \sim 1\).

**Lemma 5.4.** Assume that
\[ j \geq 19m/20, \quad k \leq 0, \]
then there holds that
\[
\|R^L_{2,j,k}\|_{B_0^\infty} \lesssim \varepsilon_2^2(t)^{0.9} 2^{-2j}.
\]

**Proof of Lemma 5.4.** We decompose
\[
L_{2,j,k}(x,v,s,t) = L_{a,2,j,k} + L_{b,2,j,k},
\]
(5.20)
where, with \( \tilde{Y}_\# := \tilde{Y}(x - sv, v, s, t) \) and \( \tilde{W}_\# := \tilde{W}(x - sv, v, s, t) \),
\[
L^a_{2,j,k}(x, v, s, t) := P_k \left[ E(x, s) - E(x + \tilde{Y}_\#, s) \right] \cdot M_j'(v),
L^b_{2,j,k}(x, v, s, t) := P_k \left\{ E(x + \tilde{Y}_\#, s) \cdot \left[ M_j'(v) - M_j'(v + \tilde{W}_\#) \right] \right\}.
\]

We can rewrite
\[
E(x, s) - E(x + \tilde{Y}_\#, s) = -\int_{\theta=0}^{1} \tilde{Y}_\# \cdot \nabla_x E(x + \theta \tilde{Y}_\#, s) d\theta,
\]
\[
E(x + \tilde{Y}_\#, s) \cdot \left[ M_j'(v) - M_j'(v + \tilde{W}_\#) \right] = -E(x + \tilde{Y}_\#, s) \cdot \tilde{W}_\# \cdot \int_{\theta=0}^{1} \nabla_v M_j'(v + \theta \tilde{W}_\#) d\theta,
\]
and it suffices to estimate the corresponding integrand uniformly in \( 0 \leq \theta \leq 1 \).

From (3.2), we have (5.18) and the following equality for \( \tilde{W} \),
\[
\tilde{W}_\# = -\int_{\tau=s}^{t} E(x - (s - \tau)v + \tilde{Y}_{1,\#}, \tau) d\tau,
\]
and we arrive at
\[
\forall s \in \{a, b\}, \quad L^s_{2,j,k}(x, v, s, t) = \int_{\theta=0}^{1} \int_{s}^{t} L^{s,\theta}_{2,j,k}(x, v, s, \tau, t) d\tau d\theta,
\]
\[
L^{a,\theta}_{2,j,k}(x, v, s, t) := -P_k \left[ (\tau - s)E^n(x - (s - \tau)v + \tilde{Y}_{1,\#}, \tau) \partial_x E(x + \theta \tilde{Y}_\#, s) \cdot M_j'(v) \right],
L^{b,\theta}_{2,j,k}(x, v, s, t) := P_k \left[ E^n(x - (s - \tau)v + \tilde{Y}_{1,\#}, \tau)E(x + \tilde{Y}_\#, s) \cdot \partial_v M_j'(v + \theta \tilde{W}_\#) \right].
\]

Therefore, for any fixed \( \theta \in [0, 1] \), in terms of the trilinear operator defined in (5.1), we have
\[
\int_{\mathbb{R}^3} L^{a,\theta}_{2,j,k}(x - (t - s)v, v, s, \tau, t) dv = R_{j,k}(\nabla E, E; C^a)(x, \tau, s, t),
\]
\[
\int_{\mathbb{R}^3} L^{b,\theta}_{2,j,k}(x - (t - s)v, v, s, \tau, t) dv = R_{j,k}(E, \nabla E; C^b)(x, \tau, s, t),
\]
where
\[
C^a = (\tau - s)2^{3j} M_j'(v), \quad C^b = 2^{3j} \partial_v M_j'(v + \theta \tilde{w}).
\]

After integrating on \( \max\{s, 2^{m_2}\} \leq \tau \leq 2^{m_2+1} \) and using Lemma 5.1, we obtain the bounds,
\[
\| \int_{\mathbb{R}^3} R_{j,k}(E, \nabla E; C^a)(x, s, \tau, t) d\tau \|_{B^0_t} \lesssim \frac{\varepsilon^2}{\varepsilon^2} 2^{-0.1m_2} 2^{-2j},
\]
\[
\| \int_{\mathbb{R}^3} R_{j,k}(E, E; C^b)(x, s, \tau, t) d\tau \|_{B^0_t} \lesssim \frac{\varepsilon^2}{\varepsilon^2} 2^{-0.1m_2} 2^{-3j},
\]
and, after integrating in \( s \) and using the bound on \( j \), this leads to the bound (5.19). \( \square \)

5.2. **Small velocity case:** \( j \leq 19m/20 \). In this subsection, we mainly estimate \( P_{1,2,j,k}^I \) and \( T_{2,2,j,k}^I \) for the case \( j \leq 19m/20, j + k \geq -\delta m/3 \) and estimate \( R_{2,2,j,k}^{II} \) for the case \( j \leq 19m/20, j + k \leq -\delta m/3 \).
Thanks to the gain of smallness $\mathcal{D}^2$ for $R^I_{2,j,k}$, we can treat $R^I_{2,j,k}$ and $R^{II}_{2,j,k}$ in the same way. For any $* \in \{I, II\}$, we localize the size of $t - s$ and have the following decomposition,

$$R^*_{2,j,k} = \sum_{n \in [0, m + 2] \cap \mathbb{Z}} R^*_{2,j,k,n}, \quad R^*_{2,j,k,n} = \int_{s=0}^{t} \int_{\mathbb{R}^3} \widetilde{\varphi}_n(t - s)L^*_n \varphi_n(x - (t - s)v, v, s, t)dsdv,$$

where

$$L^I_{2,j,k}(x, v, s, t) := L_{2,j,k}(x, v, s, t), \quad L^{II}_{2,j,k}(x, v, s, t) := \int_{\mathbb{R}^3} K_{j,k}(y, v)L_{2,j}(x - y, y, v, s, t)dy,$$

where, for any $j \in [0, 19m/20] \cap \mathbb{Z}$ and $k \in (-\infty, -j - \delta m/2] \cap \mathbb{Z}$, the kernel $K_{j,k}(y, v)$ is defined as follows,

$$K_{j,k}(y, v) := \int_{\mathbb{R}^3} e^{iy \xi} \frac{\mathcal{D}^2}{1 + \mathcal{D}^2} \rho_k(\xi)d\xi,$$

$$|K_{j,k}(y, v)| \lesssim 2^{3k + 2(k+j)}(1 + 2^k |y|)^{-100}.$$  \hfill (5.23)

Therefore, modulo the precise formulas of kernels, $L^I_{2,j,k}(x, v, s, t)$ and $L^{II}_{2,j,k}(x, v, s, t)$ are only different by a factor of $2^{2k+2j}$.

We claim that

**Proposition 5.5.** Assuming the bootstrap assumptions (2.25), for any fixed $j \in [0, 19m/20] \cap \mathbb{Z}, n \in [0, m + 2] \cap \mathbb{Z}$, there holds that

$$\forall k \in [-j - \delta m/3, +\infty) \in \mathbb{Z}, \quad \| \langle t \rangle R^I_{2,j,k,n} \|_{B^0_T} \lesssim \varepsilon^2 (1 + 2^{-(k+j)}2^{\delta m/10}),$$

$$\forall k \in (-\infty, -j - \delta m/3) \in \mathbb{Z}, \quad \| \langle t \rangle R^{II}_{2,j,k,n} \|_{B^0_T} \lesssim \varepsilon^2 2^{\delta m/10}$$

and in particular

$$\| \langle t \rangle^{1-\delta} \langle \nabla \rangle R^I_{2,j} \|_{B_T} \lesssim \varepsilon^2,$$

$$\| \langle t \rangle^{-\delta} T^I_{2,j} \|_{B_T} + \| \langle t \rangle^{1-\delta} \nabla x,t T^I_{2,j} \|_{B_T} \lesssim \varepsilon^2.$$  \hfill (5.24)

**Proof of Proposition 5.5.** The basic bound follows from Lemma 5.6 and Lemma 5.7 for the case of bounded $n$, and from Lemma 5.8 and Lemma 5.9 below. Here we show that it implies (5.24).

The first bound in (5.24) follows by summation with respect to $k$ for fixed $j, n$ and the fact that there are at most $m^2$ number of $j \in [0, 19m/20] \cap \mathbb{Z}$ and $n \in [0, m + 2] \cap \mathbb{Z}$. For the second bound, we use that

$$T^I_{2,j,k}(x, t) = -i \int_{s=0}^{t} e^{is} e^{-\langle t-s \rangle |x|} R^I_{2,j,k}(x, s)ds$$

and using that, for $0 \leq s \leq t$,

$$\| e^{is} e^{-\langle t-s \rangle |x|} R^I_{2,j,k} \|_{B^0_T} \lesssim \| R^I_{2,j,k} \|_{B^0_T},$$

$$(t-s)\| e^{is} \nabla |x| e^{-\langle t-s \rangle |x|} R^I_{2,j,k} \|_{B^0_T} \lesssim \| R^I_{2,j,k} \|_{B^0_T},$$

and that the kernel $K_{j,k}(y, v)$ is well-localized.

\hfill (5.24)
a crude integration gives
\[ \|T_{2,j,k}^I\|_{B_1^0} \lesssim \varepsilon_1^2 2^{5m/3} \int_{s=0}^t \langle s \rangle^{-1} ds, \]
\[ \|\nabla T_{2,j,k}^I\|_{B_1^0} \lesssim \int_{s=0}^{t-1} \|R_{2,j,k}^I(s)\|_{B_1^0} \frac{ds}{t-s} + 2^k \int_{s=t-1}^t \|R_{2,j,k}^I(s)\|_{B_1^0} ds \]
and since \((\partial_t + |\nabla|) T_{2,j,k}^I = -ie^{it} R_{2,j,k}^I\), direct summation gives the second line of (5.24).

\[ \square \]

In the next two Lemmas, we first consider the case when \(t-s\) is relatively small, i.e. the case \(n \in [0,10] \cap \mathbb{Z}\).

**Lemma 5.6.** Assuming the bootstrap assumptions (2.25), for any fixed \(j \in [0,19m/20] \cap \mathbb{Z}, n \in [0,10] \cap \mathbb{Z}\), there holds that
\[ \sup_{k \in [-j-\delta m/3, +\infty]} \|\langle t \rangle \nabla_x R_{2,j,k,n}^I\|_{B_1^0} \lesssim \varepsilon_1^2, \]
\[ \sup_{k \in (-\infty, -j-\delta m/3]} \|\langle t \rangle \nabla_x R_{1,j,k,n}^I\|_{B_1^0} \lesssim \varepsilon_1^2. \]

**Proof of Lemma 5.6.** We use again the decomposition (5.15). For late time, the integral over \(s\) does not incur a loss and therefore the contribution of the error term in (5.16) is acceptable.

For \(k_1 \geq m/4 + K/2\), with \(K\) defined in (5.13), we find that
\[ \| \int_{\mathbb{R}^3} \nabla_x E_{k_1}(x + (t-s)v + \theta \bar{Y}(x-tv,v,s,t),s) \cdot M_j^*(v) dv \|_{B_1^0} \lesssim \varepsilon_1 2^{-k_1-2j} \langle s \rangle^{2\delta-1} \]
and each time in \(L_{2,j,k}\) gives an acceptable contribution separately.

For \(k_1 \leq m/4 + K/2\), as in (5.17), we take the difference between two electric fields. As \(t \geq s \geq t - 2^{10}\), from rough bilinear estimates, we have
\[ \| \int_{\mathbb{R}^3} L_{j,k_1}^0(x - (t-s)v,v,s,t) dv \|_{B_1^0} \lesssim \varepsilon_1^2 2^{-2j} 2^{k_1-k} \langle t \rangle^{-2+2\delta}, \]
and therefore, after summing over \(k_1\) and integrating over \(s\), we have
\[ \sum_{k_1 \in (-\infty, m/4+K/2] \cap \mathbb{Z}} \langle t \rangle^{3/2} || \int_0^t \int_{\mathbb{R}^3} \bar{\varphi}_n(t-s) P_k[L_{j,k_1}^0](x - (t-s)v,v,s,t) dv ds \|_{B_1^0} \lesssim \varepsilon_1^{3/2}. \]

Hence finishing the proof. \[ \square \]

**Lemma 5.7.** Assuming the bootstrap assumptions (2.25), for any fixed \(j \in [0,19m/20] \cap \mathbb{Z}, n \in [0,10] \cap \mathbb{Z}\), we have
\[ \forall k \in [-j-\delta m/3, +\infty) \cap \mathbb{Z}, \|\langle t \rangle R_{2,j,k,n}^I\|_{B_1^0} \lesssim \varepsilon_1, \]
\[ \forall k \in (-\infty, -j-\delta m/3] \cap \mathbb{Z}, \|\langle t \rangle R_{1,j,k,n}^I\|_{B_1^0} \lesssim \varepsilon_1. \]

**Proof.** For this case, we use the decompositions in (5.20) and (5.21) and equalities in (5.22). Again, as \(t \geq s \geq t - 2^{10}\), using Lemma 5.1, we have
\[ \| \int_0^t \int_{\mathbb{R}^3} \bar{\varphi}_n(t-s) L_{2,j,k}(x - (t-s)v,v,s,t) dv ds \|_{B_1^0} \]
\[ \lesssim \int_0^t \bar{\varphi}_n(t-s) \left( \| \int_{s}^t R_{j,k}(\nabla E, E; C^a)(x,\tau, s,t) d\tau \|_{B_1^0} + \| \int_s^t R_{j,k}(E, \nabla E; C^b)(x,\tau, s,t) d\tau \|_{B_1^0} \right) ds \]
\[ \leq \varepsilon^{2-\frac{1}{2}m}2^{-2j}. \]

Hence finishing the proof. \( \square \)

From now on, we assume that \( n \geq 10 \). We first isolate the main contribution.

**Lemma 5.8.** For any \( \star \in \{I, II\} \), there holds that
\[
\int_{s=0}^{t} \varphi_n(t-s) \int_{\mathbb{R}^3} L_{2,j,k}^\star(x-(t-s)v, v, s, t) ds dv = I_{n,j,k}^\star + Rem_{n,j,k}^\star
\]
where \( Rem_{n,j,k}^\star \) denotes acceptable terms:
\[
\forall k \in [-j - \delta m/3, +\infty) \cap \mathbb{Z}, \quad \|\langle t \rangle Rem_{n,j,k}^I \|_{B_T} \leq \varepsilon 2^{-(k+j)}2^{\delta m/10},
\]
\[
\forall k \in (-\infty, -j - \delta m/3) \cap \mathbb{Z}, \quad \|\langle t \rangle Rem_{n,j,k}^II \|_{B_T} \leq \varepsilon 2^{\delta m/10},
\]
and the main terms \( I_{n,j,k}^\star, \star \in \{I, II\} \), can be written as a linear superposition of terms of the form
\[
I_{n,j,k}^\star = \sum_{-3m \leq k_{\min} \leq k_{\max} \leq 3m} 2^{-5j-k-n} \int_{\{0 \leq s \leq t \}} \varphi_n(t-s) \int_{\mathbb{R}^3} \mathcal{J}_{kk_{1}k_{2}}^I [P_{k_{1}}P_{k_{2}}] dvdsd\tau,
\]
\[
\mathcal{J}_{kk_{1}k_{2}}^I[f,g](x,v,\tau,s,t) = \mathcal{J}_{kk_{1}k_{2}}[f,g](x,v,\tau,s,t),
\]
\[
\mathcal{J}_{kk_{1}k_{2}}^II[f,g](x,v,\tau,s,t) = \int_{\mathbb{R}^3} K_{j,k}(y,v) \mathcal{J}_{kk_{1}k_{2}}[f,g](x-y,v,\tau,s,t) dy
\]
\[
\times P_{k_{2}-2,k+2}^s \mathcal{R}^1 [(\mathcal{R}^2 f)(x-(t-s)v,v) \cdot (\mathcal{R}^3 g)(x-(t-\tau)v,\tau)],
\]
(5.25)

where \( \mathcal{R}^j \) denotes normalized Calderon-Zygmund operators,
\[
\max\{k_1, k_2\} \geq k-2, \quad \omega_{k_1}(\tau,s) \in \{\tau-s, 2^{-k_1}\}, \quad \omega_{k_2}(\tau,s) \in \{\tau-s, 2^{-k_2}\}, \quad \|b\|_{L_{s,v}^{\infty}} + 2^{n}\|\partial_v b\|_{L_{s,v}^{\infty}} \leq 1.
\]

**Proof of Lemma 5.8.** Recall (5.2). Since the only difference between \( L_{2,j,k}^I(x,v,s,t) \) and \( L_{2,j,k}^{II}(x,v,s,t) \) lies in the kernels, which play a minor role, it would be sufficient to consider \( L_{2,j,k}^I(x,v,s,t) \) in detail here. We observe that for any function \( F \),
\[
\int_{\mathbb{R}^3} P_k F(x-(t-s)v,v,s,t) dv = (2^k(t-s))^{-1} \int_{\mathbb{R}^3} [(t-s)\partial_{x^r} + \partial_{v^r}] P_k^r F(x-(t-s)v,v,s,t) dv
\]
\[
= (2^k(t-s))^{-1} \int_{\mathbb{R}^3} P_k^r (\partial_{v^r} F)(x-(t-s)v,v,s,t) dv,
\]
where \( P_k^r \) has the same properties as \( P_k \). In particular, we see that \( k \geq 0 \) is only advantageous, and we compute that
\[
\partial_{v^r} L_{2,j,k}(x,v,s,t) = P_k \left\{ E(x,s) - E(x + \bar{Y}_#,s) \right\} \partial_{v^r} M_j^r(v)
\]
\[
+ P_k \left\{ E(x + \bar{Y}_#,s) \left[ \partial_{v^r} M_j^r(v) - \partial_{v^r} M_j^r(v + \bar{W}_#) \right] \right\}
\]
\[
+ P_k \left\{ \bar{Z}_{r,#}^r \partial_{x#} E(x + \bar{Y}_#,s) M_j^r(v + \bar{W}_#) \right\}
\]
\[
- P_k \left\{ E(x + \bar{Y}_#,s) \nu_{r,#}^q \cdot \partial_{v^r} M_j^r(v + \bar{W}_#) \right\},
\]
where
\[ \tilde{Z}^q_{r,\#} : = \left[ (s \partial_{x^r} - \partial_{v^r}) \bar{Y}^q \right] (x - sv, v, s, t), \quad \nu^q_{r,\#} : = \left[ (s \partial_{x^r} - \partial_{v^r}) \bar{W}^q \right] (x - sv, v, s, t). \]

Recall (3.2). As a result of direct computations, we have
\[
\tilde{Z}^q_{r,\#}(x, v, s, t) = - \int_s^t (\tau - s)^2 \partial_{x^r} E^q_{k_1}(x + \theta \bar{Y}_{\#}, s) E^q_{k_2}(x - (s - \tau)v + \bar{Y}_{1,\#}, \tau) \partial_{v^r} M_j^1(v) d\theta
\]
\[+ \int_s^t (\tau - s)^2 \partial_{x^r} E^q_{k_1}(x + \bar{Y}_{\#}, s) E^q_{k_2}(x - (s - \tau)v + \bar{Y}_{1,\#}, \tau) \partial_{v^r} M_j^1(v + \theta \bar{W}) d\theta
\]
\[- (\tau - s)(\partial_{v^r} - s \partial_{x^r}) \bar{Y}^q (x - sv, v, s, t) \partial_{x^r} E^q_{k_1}(x + \bar{Y}_{\#}, s) \partial_{v^r} E^q_{k_2}(x - (s - \tau)v + \bar{Y}_{1,\#}, \tau) M_j^1(v + \bar{W})
\]
\[- (\tau - s) E^q_{k_1}(x + \bar{Y}_{\#}, s) \cdot \partial_{x^r} E^q_{k_2}(x - (s - \tau)v + \bar{Y}_{1,\#}, \tau) \partial_{v^r} M_j^1(v + \bar{W})
\]
\[- (\partial_{v^r} - s \partial_{x^r}) \bar{Y}^q (x - sv, v, s, t) E^q_{k_1}(x + \bar{Y}_{\#}, s) \partial_{x^r} E^q_{k_2}(x - (s - \tau)v + \bar{Y}_{1,\#}, \tau) \partial_{v^r} M_j^1(v + \bar{W}), \]
and \( \bar{Y}_{1,\#} \) is defined in (5.18).

For the case \( k_{\text{min}} : = \min\{k_1, k_2\} \leq -4m \) or the case \( k_{\text{max}} : = \max\{k_1, k_2\} \geq 4m \), we use crude estimates directly for \( \mathcal{M}_{k_1, k_2} \), and obtain that
\[
\| \int_{s=0}^{t} \varphi_n(t-s) \int_{\mathbb{R}^3} P_k \left[ \mathcal{M}_{k_1, k_2} \right] (x - (t - s)v, v, s, t) dv d\tau ds \|_{L^q_0} \lesssim \varepsilon_2^{2m + 4\delta m + 2k_{\text{min}} - k_{\text{max}}^2}.
\]

From the above estimate it is clear that we can treat the case \( \min\{k_1, k_2\} \leq -4m \) and the case \( \max\{k_1, k_2\} \geq 4m \) as error type terms.

For the main case \( -4m \leq \min\{k_1, k_2\} \leq \max\{k_1, k_2\} \leq 4m \). Recall (5.18) and (5.1). Extracting the main term, we can approximate
\[
\mathcal{M}_{k_1, k_2} (x, v, s, t) = \left[ \mathcal{M}^0_{k_1, k_2} + \mathcal{M}^{\text{err}}_{k_1, k_2} \right] (x, v, s, t),
\]
where
\[
\mathcal{M}^0_{k_1, k_2} := E^q_{k_1}(x, s) E^q_{k_2}(x - (s - \tau)v, \tau) \partial_{v^r} M_j^1(v)
\]
\[- (\tau - s) \left[ E^q_{k_1}(x, s) \partial_{x^r} E^q_{k_2}(x - (s - \tau)v, \tau) \partial_{v^r} M_j^1(v) + \partial_{x^r} E^q_{k_1}(x, s) E^q_{k_2}(x - (s - \tau)v, \tau) \partial_{v^r} M_j^1(v) \right]
\]
\[- (\tau - s)^2 \partial_{x^r} E^q_{k_1}(x, s) \partial_{x^r} E^q_{k_2}(x - (s - \tau)v, \tau) M_j^1(v), \]
\[
\mathcal{M}_{k_1,k_2}^{err} = \sum_{i=1,2,3} \mathcal{M}_{k_1,k_2}^{err;i}
\]  
(5.26)

with
\[
\mathcal{M}_{k_1,k_2}^{err;1} = -\int_0^1 (\tau - s)^2 \overset{\text{v.p.}}{\overline{Y}}_{\#,1} \partial_{x^s} E_{k_1}(x + \overline{Y}_\#, s) \partial_{x^r} \partial_{x^r} E_{k_2}^q(x - (s - \tau)v + \theta \overline{Y}_\#, \tau) M_j'(v + \overline{W}_#) d\theta
\]
\[
+ \int_0^1 (\tau - s)^2 \overset{\text{v.p.}}{\overline{Y}}_{\#,1} \partial_{x^s} E_{k_1}(x + \overline{Y}_\#, s) \cdot \partial_{x^p} \partial_{x^r} E_{k_2}^q(x - (s - \tau)v + \theta \overline{Y}_\#, \tau) \partial_{v^q} M_j'(v + \overline{W}_#) d\theta
\]
\[
\mathcal{M}_{k_1,k_2}^{err;2} = \int_0^1 \int_0^1 \theta \overline{W}_{#}^p E_{k_1}(x,s) E_{k_2}^q(x - (s - \tau)v, \tau) \partial_{v^p} \partial_{v^q} \partial_{v^r} M_j'(v + \theta \overline{W}_#) d\theta d\tau
\]
\[
- \int_0^1 (\tau - s)^2 \overline{W}_{#}^p \partial_{x^q} E_{k_1}(x, s) \partial_{x^r} E_{k_2}^q(x - (s - \tau)v, \tau) \partial_{v^r} M_j'(v + \overline{W}_#) d\theta
\]
\[
- \int_0^1 (\tau - s)^2 \overline{W}_{#}^p E_{k_1}(x, s) \cdot \partial_{x^r} \partial_{x^r} E_{k_2}^q(x - (s - \tau)v, \tau) \partial_{v^r} M_j'(v + \overline{W}_#) d\theta
\]
\[
\mathcal{M}_{k_1,k_2}^{err;3} = -\int_0^1 \int_0^1 (\tau - s) \theta \overline{Y}_{\#,1} \partial_{x^p} \partial_{x^q} E_{k_1}(x + \theta \overline{Y}_\#, s) E_{k_2}^q(x - (s - \tau)v, \tau) \partial_{v^r} M_j'(v) d\theta d\tau
\]
\[
- \int_0^1 (\tau - s)^2 \theta \overline{Y}_{\#,1} \partial_{x^p} \partial_{x^q} E_{k_1}(x + \theta \overline{Y}_\#, s) \partial_{x^r} E_{k_2}^q(x - (s - \tau)v, \tau) M_j'(v + \overline{W}_#) d\theta.
\]

Direct inspection shows that all terms in \( \mathcal{M}_{k_1,k_2}^0 \) lead to terms of the form (5.25) and we can apply Lemma 5.10 to conclude. \( \square \)

We focus on the terms in (5.25) and have the following Lemma.

**Lemma 5.9.** With the hypothesis of Lemma 5.8 and the notations inside its proof, we have that
\[
\forall k \in [-j - \delta m/3, +\infty) \cap \mathbb{Z}, \quad \| (t)^j \langle \nabla \rangle_{r,j,k} I_{n,j,k}^I \|_{BT} \lesssim 2^{k_1 - 2(k + j) + \delta m/10} M_1^2,
\]
\[
\forall k \in (-\infty, -j - \delta m/3] \cap \mathbb{Z}, \quad \| (t)^j \langle \nabla \rangle_{r,j,k} I_{n,j,k}^{II} \|_{BT} \lesssim 2^{\delta m/10} M_1^2.
\]
Proof. We focus on the estimate of $I^I_{n,j,k}$. With minor modifications, the estimate of $I^I_{n,j,k}$ can be obtained as a byproduct. Recall (5.25). For simplicity, when $\{g_1, g_2\} = \{\rho\}$, we will omit the input and simply write $\bar{3}_{kk_1k_2} = \bar{3}_{kk_1k_2}^{T,\rho}$.

Without loss of generality, we assume that $k_1 \leq k_2$ and apply the decomposition (2.20) for the first input. Otherwise, for the case $k_2 \leq k_1$ we apply the decomposition in (2.20) for the second input, use the same strategy with the only difference that we do integration by parts in $\tau$ instead of $s$ for the oscillatory part, which gives us better estimates as we have $\tau \geq s$.

Note that we have $k \leq k_2 + 10$. Moreover, we use the following partition to localize the size of $\tau$,

$$
1_{[0,\bar{t}]}(\tau) = \sum_{\bar{m} \in [0, m+2\ell] \cap \mathbb{Z}} \bar{\varphi}_m(\tau) 1_{[0,\bar{t}]}(\tau).
$$

**Case 1:** The $L^1$-estimate.

We first estimate the static part. A crude estimate gives that

\[
\begin{aligned}
\int_0^t \int_s^t 2^{-5j-k-n} \varphi_n(t-s) \varphi_m(\tau) \otimes \bar{3}_{kk_1k_2}^{R, \rho} \|_{L^1} d\tau ds & \lesssim \int_0^t \int_s^t 2^{-2j-k-n} \varphi_n(t-s) \\
\times \varphi_m(\tau) 2^{2\text{max}\{-k_1, \bar{m}\}} \min\{|P_{k_1} R\|_{L^\infty}, \|P_{k_2} \rho\|_{L^1}, \|P_{k_1} R\|_{L^1}, \|P_{k_2} \rho\|_{L^\infty}\} d\tau ds & \lesssim \int_0^t \int_s^t 2^{-2j-2k-n} \varphi_n(t-s) \varphi_m(\tau) 2^{2\text{max}\{-k_1, \bar{m}\}} 2^{3\text{min}(k_1, -\bar{m})} (s)^{2\delta - 1} (\tau)^{2\delta - 1} ds d\tau \\
& \lesssim 2^{-2j-2k-m-\delta \bar{m}} \varepsilon_1^2.
\end{aligned}
\]

We now consider the contribution of the oscillatory field. Let $p_0 := -100\delta \bar{m}$ and

$$
T_{\leq p_0} := \mathcal{C}_{\leq p_0} T, \quad \mathcal{C}_{\leq p_0} f := \mathcal{F}^{-1}\{\varphi_{\leq p_0}(2^{-p}(1 - v \cdot \xi)) \hat{f}(\xi)\}.
$$

Now, we estimate the contribution of the $T_{\leq p_0}$-part. Due to the cutoff function $\varphi_{\leq p_0}(2^{-p}(1 - v \cdot \xi))$ on the Fourier side for the $T$-part, we know that $j + k_1 \geq -\delta \bar{m}$. From the volume of support of $\xi$ of the cutoff function $\varphi_{\leq p_0}(2^{-p}(1 - v \cdot \xi))$ and the $S^\infty$-estimate of the symbol in (3.18), we have

\[
\begin{aligned}
\int_0^t \int_s^t 2^{-5j-k-n} \varphi_n(t-s) \varphi_m(\tau) \otimes \bar{3}_{kk_1k_2}^{T_{\leq p_0}, \rho} \|_{L^1} d\tau ds & \lesssim \int_0^t \int_s^t 2^{-2j-k-n} \varphi_n(t-s) \\
\times \varphi_m(\tau) 2^{2\text{max}\{-k_1, \bar{m}\}} \min\{|P_{k_1} T_{\leq p_0}\|_{L^\infty}, \|P_{k_2} \rho\|_{L^1}, \|P_{k_1} T_{\leq p_0}\|_{L^1}, \|P_{k_2} \rho\|_{L^\infty}\} d\tau ds & \lesssim \int_0^t \int_s^t 2^{-2j-2k-n} \varphi_n(t-s) \varphi_m(\tau) 2^{-1}(s)^{2\delta - 1} (\tau)^{2\delta - 1} ds d\tau \\
& \lesssim 2^{-2j-2k-m-\delta \bar{m}} \varepsilon_1^2.
\end{aligned}
\]

When $p_0 < p \leq 0, p \in \mathbb{Z}$, we let

$$
T_p := \mathcal{C}_p T, \quad \mathcal{C}_p f := \mathcal{F}^{-1}\{\varphi(2^{-p}(1 - v \cdot \xi)) \hat{f}(\xi)\}, \quad \mathcal{C}_0 f := f - \sum_{p \in (p_0, 0) \cap \mathbb{Z}} \mathcal{C}_p f - \mathcal{C}_{p_0} f, \quad \bar{\mathcal{C}}_p f := \mathcal{F}^{-1}\{\varphi(2^{-p}(1 - v \cdot \xi)) \hat{f}(\xi)\},
$$
and we do a normal form using the following integration by parts formula for any \( t_1, t_2 \in [0, t] \),
\[
\int_{t_1}^{t_2} \omega(\tau, s)e^{-is\hat{T}_p(\xi, s)e^{-i(t-s)v\cdot\xi}}ds
= ie^{-it_2\omega(\tau, t_2)}\frac{\hat{T}_p(\xi, t_2)}{1-v\cdot\xi}e^{-i(t-t_2)v\cdot\xi} - ie^{-it_1\omega(\tau, t_1)}\frac{\hat{T}_p(\xi, t_1)}{1-v\cdot\xi}e^{-i(t-t_1)v\cdot\xi}
\]
\[
- i \int_{t_1}^{t_2} e^{-i(s-t)\omega(\tau, s)} \frac{1}{1-v\cdot\xi} \left[ \partial_s \omega(\tau, s)\hat{T}_p(\xi, s) + \omega(\tau, s)\partial_s \hat{T}_p(\xi, s) \right] ds
\]
with
\[
\omega(\tau, s) := (t-s)^{-1}\omega_{k_1}(\tau, s)\omega_{k_2}(\tau, s)\varphi_n(t-s),
\]
\[
|\omega(\tau, s)| \leq (t-s)^{-1}\left[2^{-k_1} + \tau\right] \cdot \left[2^{-k_2} + \tau\right] \varphi_n(t-s),
\]
\[
|\partial_s \omega(\tau, s)| \leq [(\tau + 2^{-k_1})(t-s)^{-1} + 2^{-2k_1}(t-s)^{-2}] \varphi_{n-10,n+10}(t-s).
\]
Using that \( T(0) = 0 \), this gives
\[
\Re \int_{s=0}^{\tau} \omega(\tau, s)e^{-is\hat{T}_p(x-(t-s)v, s)}ds
= 2^{-p}\Re (ie^{-i\tau\omega(\tau, \tau)C_p}T(x-(t-\tau)v, \tau)
- 2^{-p}\Re (i \int_{s=0}^{\tau} e^{-is\hat{T}_p(\partial_s \omega(\tau, s)T_p(x-(t-s)v, s) + \omega(\tau, s)\partial_s T_p(x-(t-s)v, s))}ds)
\]
and, letting
\[
M(s, \tau) := \min\{\|P_{k_1}T_p(s)\|_{L^\infty}, \|P_{k_1}T_p(\tau)\|_{L^1}, \|P_{k_1}T_p(s)\|_{L^1}\|\rho_{k_2}(\tau)\|_{L^\infty}\}
\]
\[
\lesssim \varepsilon_1^2 2^{-k_2} \langle s \rangle^{2\delta-1} \min\{2^{3k_1}, \langle \tau \rangle^{-3}\},
\]
\[
M'(s, \tau) := \min\{\|P_{k_1}\partial_s T_p(s)\|_{L^\infty}, \|P_{k_1}\partial_s T_p(\tau)\|_{L^1}, \|P_{k_1}\partial_s T_p(s)\|_{L^1}\|\rho_{k_2}(\tau)\|_{L^\infty}\}
\]
\[
\lesssim \varepsilon_1^2 2^{-k_2} \langle s \rangle^{2\delta-1} \langle \tau \rangle^{2\delta-1} \min\{2^{3k_1}, \langle \tau \rangle^{-3}\},
\]
a crude estimate then gives that
\[
\int_0^t \int_s^t 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_{\hat{m}}(\tau) \|\mathfrak{J}_{kk_1k_2}[\rho, \rho]\|_{L^1} d\tau ds
\]
\[
\lesssim \int_0^t 2^{-2j-k-p} \tilde{\varphi}_{\hat{m}}(\tau) \left[(t-s)^{-1}2^{\max\{-k_1, \hat{m}\}}\|P_{k_1}T_p(\tau)\|_{L^\infty}\|\rho_{k_2}(\tau)\|_{L^1}\right] d\tau
+ \int_0^t (t-s)^{-1}2^{\max\{-k_1, \hat{m}\}} M'(s, \tau) + ((t-s)^{-1}(2^{-k_1} + \tau) + (t-s)^{-2}2^{-2k_1}) M(s, \tau) ds d\tau
\]
\[
\lesssim \int_0^t 2^{-2j-2k-p} \tilde{\varphi}_{\hat{m}}(\tau) (t-\tau)^{-1} 2^{\max\{-k_1, \hat{m}\}} \min\{2^{3k_1}, 2^{-3\hat{m}}\} \langle \tau \rangle^{4\delta-1} d\tau
\]
\[
\lesssim 2^{-2j-2k-m-\delta \hat{m}} \varepsilon_1^2.
\]
To sum up, after combining the obtained estimates (5.2), (5.2), and (5.2), we have
\[
\int_0^t \int_s^t 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_{\hat{m}}(\tau) \|\mathfrak{J}_{kk_1k_2}[\rho, \rho]\|_{L^1} d\tau ds \lesssim 2^{-2j-2k-m-\delta \hat{m}} \varepsilon_1^2.
\]
\textbf{Case 2:} The \( L^\infty \)-estimate.
We first handle the contribution from the static part. If $t/4 \leq s \leq t$, we find that
\[
\int_{t/4}^{t} \int_{s}^{t} 2^{-5j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \|3_{kk_1k_2}[\rho^{stat}, \rho]\|_{L^\infty} d\tau ds \\
\lesssim \int_{t/4}^{t} \int_{s}^{t} 2^{-2j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \left[2^{-k_1} + \tau\right] \left[2^{-k_2} + \tau\right] \|P_{k_1} \rho^{stat}\|_{L^\infty} \|P_{k_2} \rho\|_{L^\infty} d\tau ds \\
\lesssim 2^{-2(k+j)-5m+10\delta m} \varepsilon_1^2.
\]
Let $\bar{\tau} := \max\{s, \min\{2^{-k_1}, t/2\}\}$. If $0 \leq s \leq t/4$ and $\min\{2^{-k_1}, t/2\} \leq \tau \leq t$, we use the dispersion inequality (3.13) for $\rho^{stat}$ and the bootstrap assumption for $\rho$ and we get
\[
\int_{0}^{t/4} \int_{s}^{t} 2^{-5j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \|3_{kk_1k_2}[\rho^{stat}, \rho]\|_{L^\infty} d\tau ds \\
\lesssim \int_{0}^{t/4} \int_{s}^{t} 2^{-5j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \left[2^{-k_1} + \tau\right] \left[2^{-k_2} + \tau\right] \|\mathcal{I}_{\varphi_j}(\rho^{stat}_k(s); t-s)\|_{L^\infty} d\tau ds \\
\times \|P_{k_2} \rho(\tau)\|_{L^\infty} d\tau ds \\
\lesssim \varepsilon_1^2 2^{-2(j+k)-4m-\tilde{m}+10\delta m}.
\]
If $0 \leq s \leq t/4$ and $s \leq \tau < \bar{\tau}$, in which case we have $s \geq \min\{2^{-k_1}, t/2\}$, we use the dispersion (3.13) on $\rho$ or use the dispersion inequality (3.13) for $\rho^{stat}$. As a result, we have
\[
\int_{0}^{t/4} \int_{s}^{t} \bar{\tau} 2^{-5j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \|3_{kk_1k_2}[\rho^{stat}, \rho]\|_{L^\infty} d\tau ds \\
\lesssim \int_{0}^{t/4} \int_{s}^{t} \min\{2^{-k_1}, t/2\} 2^{-5j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \left[2^{-k_1} + \tau\right] \left[2^{-k_2} + \tau\right] \\
\times \min\{\|P_{k_1} R(s)\|_{L^\infty}, \|\mathcal{I}_{\varphi_j}(\rho^{stat}_k(s); t-s)\|_{L^\infty}\} d\tau ds \\
\lesssim \varepsilon_1^2 2^{-2(j+k)-4m-\tilde{m}+10\delta m}.
\]
After combining the obtained estimates (5.2–5.2), we have
\[
\int_{0}^{t} \int_{s}^{t} 2^{-5j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \|3_{kk_1k_2}[\rho^{stat}, \rho]\|_{L^\infty} d\tau ds \\
\lesssim \varepsilon_1^2 2^{-2(j+k)-4m-\tilde{m}+10\delta m}.
\]
Now, we move on to the estimate of the contribution from the oscillatory part. As in the $L^1$-estimate we let $p_0 = -10\delta \tilde{m}$, and also use the definitions introduced in (5.2) and (5.2). We first estimate the contribution of the $T_{\leq p_0}$-part, in which case we have $j + k_1 \geq -\delta \tilde{m}$. If $t/4 \leq s \leq t$, we find that
\[
\int_{t/4}^{t} \int_{s}^{t} 2^{-5j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \|3_{kk_1k_2}[T_{\leq p_0}, \rho]\|_{L^\infty} d\tau ds \\
\lesssim \int_{t/4}^{t} \int_{s}^{t} 2^{-2j-k-n} \varphi_n(t-s) \bar{\varphi}_m(\tau) \left[2^{-k_1} + \tau\right]^2 \|P_{k_1} T_{\leq p_0}\|_{L^\infty} \|P_{k_2} \rho\|_{L^\infty} d\tau ds \\
\lesssim \int_{t/4}^{t} \int_{s}^{t} 2^{-2j-k-n-4m+2\delta \tilde{m}} \varphi_n(t-s) \bar{\varphi}_m(\tau) \left[2^{-k_1} + \tau\right]^2 \min\{2^{2k_1-j+p_0}, 2^{-3m}\} d\tau ds \\
\lesssim \int_{t/4}^{t} \int_{s}^{t} 2^{-2j-k-n-4m+2\delta \tilde{m}} \varphi_n(t-s) \bar{\varphi}_m(\tau) \left[2^{-k_1} + \tau\right]^2 \min\{2^{3k_1+\delta \tilde{m}+p_0}, 2^{-3m}\} d\tau ds
\]
If $0 \leq s \leq t/4$ and $t/2 \leq \tau \leq t$, we use the dispersion inequality (3.13) for $T_{\leq p_0}$,

$$
\int_0^{t/4} \int_{s/2}^t 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \|J_{kk_1k_2}[T_{\leq p_0}, \rho]\| L^\infty d\tau ds
\lesssim \int_0^{t/4} \int_{s/2}^t 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \left[2^{-k_1 + \tau}\right]^2 \|I(P_{k_1}T_{\leq p_0}(s); t-s)\| L^\infty \|P_{k_2}\rho\| L^\infty d\tau ds
\lesssim \int_0^{t/4} \int_{s/2}^t 2^{-2j-2k-n-4m+2\delta m} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \left[2^{-k_1 + \tau}\right]^2 \min\{2^{3k_1+\delta m+p_0}, 2^{-3m}\} d\tau ds
\lesssim 2^{-2(j+k)-4m-10\delta m} \varepsilon_1^2.
$$

Let $s' := \min\{\max\{s, 2^{-k_1-p_0/4}\}, t/2\}$. If $0 \leq s \leq t/4$ and $s' \leq \tau \leq t/2$, again, we use the dispersion inequality (3.13) for $T_{\leq p_0}$ and the bootstrap assumption for $\rho$ and we get

$$
\int_0^{t/4} \int_{s'}^{t/2} 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \|J_{kk_1k_2}[T_{\leq p_0}, \rho]\| L^\infty d\tau ds
\lesssim \int_0^{t/4} \int_{s'}^{t/2} 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \left[2^{-k_1 + \tau}\right]^2 \|I(P_{k_1}T_{\leq p_0}(s); t-s)\| L^\infty \|P_{k_2}\rho\| L^\infty d\tau ds
\lesssim \int_0^{t/4} \int_{s'}^{t/2} \varepsilon_1^2 2^{-2(j+k)-4m} \min\{1, (2^{k_1}(s))^{-1}\}(\tau)^{4\delta - 2} d\tau ds
\lesssim \varepsilon_1^2 2^{-2(j+k)-4m+p_0/4+4\delta m} \lesssim \varepsilon_1^2 2^{-2(j+k)-4m-10\delta m}.
$$

If $0 \leq s \leq t/4$ and $s \leq \tau \leq s'$, we use the dispersion inequality (3.13) for $T_{\leq p_0}$ or use the dispersion inequality (3.13) for $\rho$. As a result, we have

$$
\int_0^{t/4} \int_{s'}^s 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \|J_{kk_1k_2}[T_{\leq p_0}, \rho]\| L^\infty d\tau ds
\lesssim \int_0^{t/4} \int_{s'}^s 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \left[2^{-k_1 + \tau}\right]^2 \|P_{k_1}T_{\leq p_0}(s)\| L^\infty \|I(\rho_{k_2}(\tau); t-\tau)\| L^\infty d\tau ds
\lesssim \int_0^{t/4} \int_{s'}^s \tilde{\varphi}_m(\tau) 2^{-2(k+j)-4m}(\tau)^{4\delta - 1} 2^{-2k_1-p_0/2} \min\{2^{3k_1+p_0+\delta m}, (s)^{-3}\} d\tau ds
\lesssim \varepsilon_1^2 2^{-2(j+k)-4m+p_0/6+4\delta m} \lesssim \varepsilon_1^2 2^{-2(j+k)-4m-10\delta m}.
$$

Lastly, we consider the contribution from the oscillatory part of $T_p$, $p_0 < p \leq 0, p \in \mathbb{Z}$. As in the $L^1$-estimate, we integrate by parts in $s$ once, see (5.2).

As a result, as in the estimate of the contribution from $T_{\leq p_0}$, after using the same strategies for different sizes of $s$ and $\tau$, we have

$$
\int_0^t \int_s^t 2^{-5j-k-n} \varphi_n(t-s) \tilde{\varphi}_m(\tau) \|J_{kk_1k_2}[T_p, \rho]\| L^\infty d\tau ds \lesssim \int_0^t 2^{-5j-k-n-p} \varphi_n(t-s)
\times \left[2^{2\max\{-k_1, m\}} \|I_{\varphi_j(v)}(P_{k_1}T_{p}(s); t-s)\| L^\infty \|\rho_{k_2}(s)\| L^\infty \tilde{\varphi}_m(s) + \int_s^t \tilde{\varphi}_m(\tau) \left[2^{2\max\{-k_1, m\}} \times \min\{1, (2^{k_1}(s))^{-1}\}(\tau)^{4\delta} \left[2^{3k_1+p_0+\delta m}, (s)^{-3}\right] \right] d\tau ds
\times \{\|I_{\varphi_j(v)}(P_{k_1}T_{p}(s); t-s)\| L^\infty \|\rho_{k_2}(\tau)\| L^\infty, \|P_{k_1}\partial_s T_{p}(s)\| L^\infty \|I_{\varphi_j(v)}(P_{k_2}\rho(\tau); t-\tau)\| L^\infty \} + (2^{\max\{-k_1, m\}} + 2^{-n-2k_1}) \min\{1, (2^{k_1}(s))^{-1}\}(\tau)^{4\delta} \left[2^{3k_1+p_0+\delta m}, (s)^{-3}\right] \right] d\tau ds
\times \{\|I_{\varphi_j(v)}(P_{k_1}T_{p}(s); t-s)\| L^\infty \|\rho_{k_2}(\tau)\| L^\infty, \|P_{k_1}T_{p}(s)\| L^\infty \}.
$$
\[
\times \| I_{\varphi_j(v)}(P_{k_2} \rho(\tau); t - \tau) \|_{L^\infty} \right) d\tau \right] ds \\
\lesssim \varepsilon_1^2 2^{-2(k+j) - p - 4m - \bar{m} + 10\bar{m}} \lesssim \varepsilon_1^2 2^{-2(j+k) - 4m - 10\bar{m}}.
\]

Recall (5.2). After combining the obtained estimates (5.2) and (5.2–5.2) and summing with respect to \( \bar{m} \), our desired estimate (5.9) holds.

For the error type terms \( M_{err}^{err,1} \) appearing in (5.26) we have:

**Lemma 5.10.** With the hypothesis of Lemma 5.8 and the notations inside its proof, for any fixed \( k_1, k_2 \in [-3m, 3m] \cap \mathbb{Z} \), we have that

\[
\forall k \in [-j - \delta m/3, +\infty) \cap \mathbb{Z}, \quad \langle t \rangle \left\| \int_{0 \leq s \leq \tau \leq t} 2^{-k-n} \varphi_n(t - s) \times \int_{\mathbb{R}^3} P_k M_{err,1}^{err,1}(x - (t - s)v, v, s, \tau, t) dv ds d\tau \right\|_{B^0_1} \lesssim \varepsilon_1^2 2^{-(k+j)}.
\]

Moreover, for the kernel defined in (5.23), we have

\[
\forall k \in (-\infty, -j - \delta m/3] \cap \mathbb{Z}, \quad \langle t \rangle \left\| \int_{0 \leq s \leq \tau \leq t} 2^{-k-n} \varphi_n(t - s) \times \int_{\mathbb{R}^3} P_k M_{err,1}^{err,1}(x - y - (t - s)v, v, s, \tau, t) dy dv ds d\tau \right\|_{B^0_1} \lesssim \varepsilon_1^2.
\]

**Proof.** Recall (5.26). We estimate three components one by one as follows.

- The estimate of \( M_{err,1}^{err,1} \).

Recall (5.2) and the definition of \( \tilde{Y}_{1,\#} \) in (5.18). We first consider the case \( s \in [t/2, t] \). From the rough \( L^1 - L^\infty \)-type estimate for the \( L^1 \)-estimate and the \( L^\infty - L^\infty \)-type estimate for the \( L^\infty \)-estimate, and the estimate (3.28) in Lemma 3.6, we have

\[
\left\| \int_{t/2}^t \int_s^t 2^{-k-n} \varphi_n(t - s) \varphi_n(t - s) \int_{\mathbb{R}^3} P_k M_{err,1}^{err,1}(x - (t - s)v, v, s, \tau, t) dv ds d\tau \right\|_{B^0_1} \lesssim 2^{-k-2j} \sqrt{m/6 + 18m} \varepsilon_1^3.
\]

Now, we focus on the case \( s \in [0, t/2] \), in which we have \( 2^n \sim t - s \sim t \). For the \( L^2 \)-estimate, we use \( L^1 - L^\infty \)-type bilinear estimate. For the \( L^\infty - L^\infty \)-estimate, we use the dispersion estimate (3.11) for the input evaluated at time \( s \) and the \( L^\infty - L^\infty \) type bilinear estimate. As a result, from the estimate (3.28) in Lemma 3.6, we have

\[
\left\| \int_0^{t/2} \int_s^t 2^{-k-n} \varphi_n(t - s) \varphi_n(t - s) \int_{\mathbb{R}^3} P_k M_{err,1}^{err,1}(x - (t - s)v, v, s, \tau, t) dv ds d\tau \right\|_{B^0_1} \lesssim 2^{-k-2j-m} \int_0^{t/2} \int_s^t (\tau)^{-7/6 + 2\delta} (s)^{-1} \varepsilon_1^2 d\tau ds \lesssim 2^{-k-2j-m} \varepsilon_1^2.
\]

- The estimate of \( M_{err,2}^{err,2} \).

Recall (5.2). In terms of the trilinear operator defined in (5.1), we have

\[
P_k M_{err,2}^{err,2}(\tau, s, t) = \int_0^1 \int_0^1 \mathcal{R}_{j,k}(E_{k_1}, E_{k_2}; C_0)(\tau, s, t) + \mathcal{R}_{j,k}(\nabla_x E_{k_1}, E_{k_2}; C_1)(\tau, s, t) + \mathcal{R}_{j,k}(\nabla_x E_{k_1}, \nabla_x E_{k_2}; C_2)(\tau, s, t)
+ \mathcal{R}_{j,k}(\nabla_x E_{k_1}, \nabla_x E_{k_2}; C_3)(\tau, s, t) + \mathcal{R}_{j,k}(\nabla_x E_{k_1}, \nabla_x E_{k_2}; C_4)(\tau, s, t)d\theta d\theta_1,
\]

where...
where trilinear forms are implicitly depending on $\theta, \theta_1$. Moreover, coefficients $C_i, i \in \{1, 2, 3\}$, are given as follows,

$$C_0 = 2^{3j} \theta \bar{W}_p^p \partial_{v_0} \partial_{v_0} \partial_{v_0} M'_j (v + \theta \bar{W}_p)$$

$$C_1 := 2^{3j} \bar{Y}_p^p \partial_{v_0} \partial_{v_0} M'_j (v), \quad C_2 := 2^{3j} (\tau - s) \bar{W}_p^p \partial_{v_0} \partial_{v_0} M'_j (v + \theta \bar{W}_p)$$

$$C_3 = (\tau - s) \bar{Y}_p^p \partial_{v_0} M'_j (v + \bar{W}_p), \quad C_4 = (\tau - s)^2 \bar{W}_p^p \partial_{v_0} M'_j (v + \theta \bar{W}_p).$$

Therefore, from the estimates $(5.5), (5.6),$ and $(5.8)$ in Lemma 5.1, the estimates of $\bar{Y}$ and $\bar{W}$ in $(3.28)$ in Lemma 3.6, we have

$$\| \int_0^t \int_s^t 2^{-k-n} \varphi_n (t - s) \varphi_n (t - s) \int_{\mathbb{R}^3} P_k M_{k_1, k_2}^p (x - (t - s)v, v, s, \tau, t) dv d\tau ds \|_{B^0_k} \lesssim 2^{-k-2j-m} \varepsilon_1^2 \int_0^t \langle s \rangle^{-1} ds \lesssim 2^{-k-2j-m} \varepsilon_1^2.$$  

• The estimate of $M_{k_1, k_2}^p$.

Recall $(5.2)$. In terms of the trilinear operator defined in $(5.1)$, we have

$$P_k M_{k_1, k_2}^p (\tau, s, t) = \int_0^1 \int_0^1 \mathcal{R}_{j,k}(\nabla_x^2 E_{k_1}, E_{k_2}; \bar{C}_1) (\tau, s, t) + \mathcal{R}_{j,k}(\nabla_x^2 E_{k_1}, \nabla_x E_{k_2}; \bar{C}_2) (\tau, s, t) d\theta d\theta_1,$$

where

$$\bar{C}_1 := 2^{3j} (\tau - s) \bar{Y}_p^p \partial_{v_0} M'_j (v), \quad \bar{C}_2 := 2^{3j} (\tau - s)^2 M'_j (v + \bar{W}_p).$$

From the estimates $(5.9)$ and $(5.10)$ in Lemma 5.1, and the estimate $(3.28)$ in Lemma 3.6, we have

$$\| \int_0^t \int_s^t 2^{-k-n} \varphi_n (t - s) \varphi_n (t - s) \int_{\mathbb{R}^3} P_k M_{k_1, k_2}^p (x - (t - s)v, v, s, \tau, t) dv d\tau ds \|_{B^0_k} \lesssim 2^{-k-2j-m} \varepsilon_1^2 \int_0^t \langle s \rangle^{-0.1} \langle s \rangle^{-1+\delta} ds \lesssim 2^{-k-2j-m} \varepsilon_1^2.$$  

To sum up, our desired estimates $(5.10)$ and $(5.10)$ hold from the obtained estimates $(5.2-5.2)$ and the estimate of the kernel $K_{k,j}(y, v)$ in $(5.23)$.

6. Bounds on the type-II reaction term

In this section we prove the following:

**Proposition 6.1.** With the notation in $(2.27)-(2.28)$, we have

$$\| \rho_{2,2}^{osc} \|_{osc_{ij}} \lesssim \varepsilon_0. \quad (6.1)$$

The rest of this section is concerned with the proof of Proposition 6.1. We start from the definition $(2.9)$ and calculate $(\partial_t L_{2,j})(x, v, s, t)$. For this we notice that

$$X(x + (t - s)v, v, s, t) = \bar{Y}(x - sv, v, s, t) + x,$$

$$V(x + (t - s)v, v, s, t) = \bar{W}(x - sv, v, s, t) + v,$$
see (3.1). Therefore

\[(\partial_t L_{2,j})(x,v,s,t)\]

\[= \left[ - (\partial_i E^i)(x + \tilde{Y}(x-sv,v,s,t),s)(\partial_t \tilde{Y}^i)(x-sv,v,s,t)\partial_t M_0(v + \tilde{W}(x-sv,v,s,t)) \right.\]

\[- E_i(x + \tilde{Y}(x-sv,v,s,t),s)(\partial_t \tilde{W}^i)(x-sv,v,s,t)\partial_t \partial_t M_0(v + \tilde{W}(x-sv,v,s,t)) \right] \varphi_j(v).\]

Using the formulas (3.2) we have

\[(\partial_t \tilde{Y})(x-sv,v,s,t) = (t-s)E(x+(t-s)v,t) + Err^1(x+(t-s)v,v,s,t),\]

\[Err^1(x,v,s,t):= \int_s^t (\tau-s)(\nabla E)\nabla(x-tv+\tau v + \tilde{Y}(x-tv,v,\tau,\tau),\tau) \cdot \partial_t \tilde{Y}(x-tv,v,\tau,\tau)d\tau,\]

\[(6.2)\]

\[\partial_t \tilde{W}(x-sv,v,s,t) = -E(x+(t-s)v,t) - Err^2(x+(t-s)v,v,s,t),\]

\[Err^2(x,v,s,t):= \int_s^t (\nabla E)\nabla(x-tv+\tau v + \tilde{Y}(x-tv,v,\tau,\tau),\tau) \cdot \partial_t \tilde{Y}(x-tv,v,\tau,\tau)d\tau,\]

\[(6.3)\]

Therefore

\[(\partial_t L_{2,j})(x,v,s,t) = [N_1(x+(t-s)v,v,s,t) + N_2(x+(t-s)v,v,s,t)] \varphi_j(v),\]

\[(6.4)\]

where

\[N_1(y,v,s,t) = -(\partial_i E^i)(y-(t-s)v+\tilde{Y}(y-tv,v,s,t),s) \cdot (t-s)E^i(y,t)\partial_t M_0(v + \tilde{W}(y-tv,v,s,t)) + E^i(y-(t-s)v+\tilde{Y}(y-tv,v,s,t),s) \cdot E^i(y,t)\partial_t \partial_t M_0(v + \tilde{W}(y-tv,v,s,t)),\]

\[(6.5)\]

\[N_2(y,v,s,t) = N_1(y,v,s,t) + N_2(y,v,s,t),\]

\[N_1(y,v,s,t) := -(\partial_i E^i)(y-(t-s)v+\tilde{Y}(y-tv,v,s,t),s) \cdot Err^1(y,v,s,t) \times \partial_t M_0(v + \tilde{W}(y-tv,v,s,t)),\]

\[(6.6)\]

\[N_2(y,v,s,t) := E^i(y-(t-s)v+\tilde{Y}(y-tv,v,s,t),s) \cdot Err^2(y,v,s,t) \times \partial_t \partial_t M_0(v + \tilde{W}(y-tv,v,s,t)).\]

In view of (2.28), for (6.1) it suffices to prove that, for any \((j,k,m) \in A^{II}\),

\[\|\varphi_{m}(t)T_{2,j,k}^{II}\|_{B^0} \lesssim \varepsilon_1^{2j}.\]

Recall (2.19). We use the formulas (6.4)–(6.6) to decompose \(T_{2,j,k}^{II} = X_{j,k}^1 + X_{j,k}^2\), where

\[X_{j,k}^a(\xi,t) := \int_0^t \int_0^t \int_{\mathbb{R}^3} 1_+ (s-\tau)e^{-(t-s)\xi} \frac{e^{is}}{1 - iD} \varphi_k(\xi) \varphi_j(v) \hat{N}_a(\xi,\tau,\tau)dv d\tau ds,\]

\[(6.7)\]

for \(a \in \{1,2\}\).

We estimate first the remainder terms components \(X_{j,k}^2\).

**Lemma 6.2.** For any \((j,k,m) \in A^{II}\) we have

\[\|\varphi_{m}(t)X_{j,k}^2\|_{Osc\delta} \lesssim \varepsilon_1^{2j}.\]
Proof. With $D = |\xi| - iv \cdot \xi$ as before let
\[ L_{j,k}(z, v, s, t) := \tilde{\varphi}_j(v) \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{iz \cdot \xi} e^{-(t-s)|\xi|} e^{is |\xi|} e^{itD} \varphi_k(\xi) d\xi. \] (6.8)

Since $(j, k, m) \in A^I$ we have $k + j \leq -10$, so we can integrate by parts several times in $\xi$ in the definition above to see that
\[ |L_{j,k}(z, v, s, t)| \lesssim \tilde{\varphi}_{j-4,j+4}(v)(1 + |t-s|2^k)^{-8} 2^k (1 + 2^k |z|)^{-10}. \]

Using (6.6) and recalling that $|\tilde{W}(y - tv, v, \tau, s)| \lesssim \varepsilon_1$ (see (3.28)) we decompose
\[ X_{j,k}^2 = X_{j,k}^{2,1} + X_{j,k}^{2,2}, \]
\[ X_{j,k}^{2,l}(x, t) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbf{1}_l(s - \tau) L_{j,k}(x - y, v, s, t) N^1_2(y, v, \tau, s) dy dv d\tau ds. \]

Step 1. We consider first the case $l = 1$, and examine the formula for $N^1_2$ in (6.6) and the formula for the functions $\tilde{E}r^1$ in (6.2). We have
\[ X_{j,k}^{2,1}(x, t) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_{j,k}(x - y, v, s, t)(-\partial_i E^i)(y - (s - \tau)v + \tilde{Y}(y - sv, v, \tau, s), \tau) \times (\gamma(\tau)(\partial_a E^a)(y - (s - \gamma)v + \tilde{Y}(y - sv, v, \gamma, s), \gamma) \times (\partial_i \tilde{Y}^a)(y - sv, v, \gamma, s)\partial_l M_0(v + \tilde{W}(y - sv, v, \tau, s)) dy dv d\gamma d\tau ds. \]

Notice that, with the notation in (5.1), for a suitable kernel $K_{j,k}$ satisfying (5.2) we have
\[ X_{j,k}^{2,1}(x, t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{R}_{j,k}(\partial_i E^i, \partial_a E^a; C^1_{al})(x, \gamma, \tau, s; t) d\gamma d\tau ds, \]
\[ C^1_{al}(y, v, \gamma, \tau, s; t) := -(\gamma(\tau)(1 + |t-s|2^k)^{-8}(\partial_i \tilde{Y}^a)(y - sv, v, \gamma, s)) \times 2^{j+8} \partial_l M_0(v + \tilde{W}(y - sv, v, \tau, s)). \] (6.9)

Clearly, using (3.25)–(3.26) and Lemma 3.1 (ii) we have
\[ |\langle (\partial_a \tilde{Y}^a)(y - sv, v, \gamma, s) \rangle| \leq \varepsilon_1(s)^{-1+\delta}. \]

Since $\|\partial_l M_0(v)\varphi_{j-4,j+4}(v)\|_{L^\infty} \lesssim 2^{-5j}$, with the notation in (5.3) we have
\[ m'(C^1_{al})(\gamma, \tau, s) \lesssim \langle (1 + |t-s|2^k)^{-8}s^{-1+\delta}2^{-2j}. \] (6.10)

We decompose $X_{j,k} = X_{j,k}^{2,1} + X_{j,k}^{2,2}$ where
\[ X_{j,k}^{2,1}(x, t) = \int_{t/2}^{t} \int_{0}^{s} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{R}_{j,k}(\partial_i E^i, \partial_a E^a; C_{al})(x, \gamma, \tau, s; t) d\gamma d\tau ds, \]
\[ X_{j,k}^{2,2}(x, t) = \int_{t/2}^{t} \int_{0}^{s} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{R}_{j,k}(\partial_i E^i, \partial_a E^a; C_{al})(x, \gamma, \tau, s; t) d\gamma d\tau ds. \]

Using now (5.8) and recalling that $k \leq 0$ it follows that
\[ \|X_{j,k}^{2,1}(., t)\|_{L^1_x} \lesssim \int_{0}^{t/2} \int_{0}^{s} (1 + |t-s|2^k)^{-8}s^{-1+\delta}2^{\frac{2}{\varepsilon_1}2^{-2j}}(1 + 2^k(t))^{-8}. \] (6.11)
Therefore, using Sobolev embedding,
\[ \langle t \rangle^3 \| X_{j,k}^{2,1,1}(., t) \|_{L_\infty} \lesssim \varepsilon_1^2 2^{-2j} 2^{3k} \langle t \rangle^3 \| (1 + 2^k(t))^{-8} \|_{L_\infty} \lesssim \varepsilon_1^2 2^{-2j} \langle t \rangle^3 \| \|
\]

Moreover, using again (5.8), we can also estimate
\[
\| X_{j,k}^{2,1,2}(., t) \|_{L_1^2} + \langle t \rangle^3 \| X_{j,k}^{2,1,2}(., t) \|_{L_\infty} \\
\lesssim \int_0^t \int_0^\infty (1 + |t - s| 2^k)^{-8} \langle s \rangle^{-1 + \delta} \varepsilon_1^2 \langle \tau \rangle^{-1} 2^{-j} d\tau ds \\
\lesssim \varepsilon_1^2 2^{-2j} \langle t \rangle^{-1 + \delta} \min\{2^{-k}, \langle t \rangle\}. 
\]

These three bounds show that
\[ \| \tilde{\varphi}_m(t) \|_{L_\infty}^{2,1} X_{j,k}^{2,1} \|_{B^0_1} + \| \tilde{\varphi}_m(t) \|_{L_\infty}^{1-2\delta} \| \nabla_x X_{j,k}^{2,1} \|_{B^0_1} \lesssim \varepsilon_1^2 2^{-2j}. \]

We show now that
\[ \| \tilde{\varphi}_m(t) \|_{L_\infty}^{1-2\delta} (\partial t + |\nabla_x|) X_{j,k}^{2,1} \|_{B^0_1} \lesssim \varepsilon_1^2 2^{-2j}. \]

Indeed, the identities (6.9) show that
\[
(\partial_t + |\nabla_x|) X_{j,k}^{2,1}(x, t) = \int_0^t \int_\tau^t \mathcal{R}_{j,k}(x, \gamma, \tau, t) \|_{B^0_1} \|_{B^0_1} d\gamma d\tau.
\]

It follows from (5.8) and (6.10) that
\[
\| (\partial_t + |\nabla_x|) X_{j,k}^{2,1}(., t) \|_{B^0_1} \lesssim \int_0^t \varepsilon_1^2 (\tau)^{1 - 1 + \delta} 2^{-2j} 2^{-1} d\tau \lesssim \varepsilon_1^2 (\tau)^{1 - 1 + \delta} 2^{-2j}.
\]

The desired bounds (6.14) follow, which completes the analysis of the term $X_{j,k}^{2,1}$.

**Step 2.** We consider now the case $i = 2$, and examine the formula for $N_{2,j}^2$ in (6.6) and the formula for the functions $E_{\gamma} r^2$ in (6.3). As before, with the notation in (5.1), for a suitable kernel $\mathcal{K}_{j,k}$ satisfying (5.2) we have
\[
X_{j,k}^{2,1} = X_{j,k}^{2,2,1} + X_{j,k}^{2,2,2},
\]

\[
X_{j,k}^{2,2,1}(x, t) := \int_0^{t/2} \int_0^s \int_\tau^s \mathcal{R}_{j,k}(x, \gamma, \tau, t) \|_{B^0_1} \|_{B^0_1} d\gamma d\tau ds,
\]

\[
X_{j,k}^{2,2,2}(x, t) := \int_0^{t/2} \int_0^s \int_\tau^s \mathcal{R}_{j,k}(x, \gamma, \tau, t) \|_{B^0_1} \|_{B^0_1} d\gamma d\tau ds,
\]

\[
C_{alj}(v, \gamma, \tau, t) := (1 + |t - s| 2^k)^{-8} (\partial_\gamma \tilde{\varphi}_m(y - sv, v, \gamma, s)) \times 2^{3j} (\partial_t \partial_t M_0)(v + \tilde{W}(y - sv, v, \gamma, s)) \tilde{\varphi}_{j-4,j+4}(v).
\]

As in (6.10) we have
\[
m'(C_{alj}^2(\gamma, \tau, s)) \lesssim (1 + |t - s| 2^k)^{-8} \langle s \rangle^{-1 + \delta} 2^{-2j}.
\]

Then we estimate, using (5.6) and Sobolev embedding
\[
\| X_{j,k}^{2,1,2}(., t) \|_{L_1^2} + \langle t \rangle^3 \| X_{j,k}^{2,1,2}(., t) \|_{L_\infty} \lesssim \varepsilon_1^2 2^{-2j} \langle t \rangle^3 \| (1 + 2^k(t))^{-5} \|
\]

and
\[
\| X_{j,k}^{2,2,2}(., t) \|_{L_1^2} + \langle t \rangle^3 \| X_{j,k}^{2,2,2}(., t) \|_{L_\infty} \lesssim \varepsilon_1^2 2^{-2j} \langle t \rangle^{-1 + \delta} \min\{2^{-k}, \langle t \rangle\}.
\]
as in (6.11)–(6.12). Then we notice that
\[(\partial_t + |\nabla|)X_{j,k}^{2,2}(x, t) = \int_0^t \int_\gamma R(E^l, \partial_\alpha E^l, C_{adi}^2)(x, \gamma, t; t) \, d\gamma \, dt,
\]
so we can estimate using (5.8)
\[\|\partial_t + |\nabla|\|X_{j,k}^{2,2}(\cdot, t)\|_{B^p} \lesssim \int_0^t \frac{\varepsilon_1^2}{\varepsilon_1^3(t) - 1 + 2^{j-1}, \tau^{-1}} \, dt \lesssim \frac{\varepsilon_1^2}{\varepsilon_1^3(t) - 1 + 2^{j-1}}.\]
These bounds show that
\[\|
\|\bar{\varphi}_m(t)\|_{B_T}^2 + \|
\|\bar{\varphi}_m(t)\|_{B_T}^2 + \|
\|\bar{\varphi}_m(t)\|_{B_T}^2 \lesssim \varepsilon_1^2 2^{-j},
\]
and the lemma follows using also (6.13)–(6.14).

We estimate now the main terms $X_{1,1}^{1,j}$ defined in (6.7). Using the identities (6.5), we rewrite
\[N_1 = N_1^1 + N_1^2 + N_1^3,
\]
\[N_1^1(y, v, \tau, s) := \frac{d}{dv_i} \left\{ E^l(y - (s - \tau)v + \bar{Y}(y - sv, v, \tau, s), \tau)E^l(y, s) \right\}
\times \partial_t M_0(v + \bar{W}(y - sv, v, \tau, s)) \right\},
\]
\[N_1^2(y, v, \tau, s) := - (\partial_a E^l(y - (s - \tau)v + \bar{Y}(y - sv, v, \tau, s), \tau)Y_{i,a}^l(y, v, \tau, s)
\times E^l(y, s) \partial_t M_0(v + \bar{W}(y - sv, v, \tau, s)),
\]
\[N_1^3(y, v, \tau, s) := - E^l(y - (s - \tau)v + \bar{Y}(y - sv, v, \tau, s), \tau)E^l(y, s)
\times \bar{W}_{i,a}^l(y, v, \tau, s)(\partial_a \partial_t M_0)(v + \bar{W}(y - sv, v, \tau, s)),
\]
where
\[\bar{Y}_{i,a}^l(y, v, \tau, s) := [(\partial_{v_i} - s \partial_{v_a})\bar{Y}^l](y - sv, v, \tau, s),
\]
\[\bar{W}_{i,a}^l(y, v, \tau, s) := [(\partial_{v_i} - s \partial_{v_a})\bar{W}^l](y - sv, v, \tau, s).
\]

From the above decomposition, we have
\[X_{j,k}^1(x, t) = \sum_{l=1,2,3} X_{j,k}^{1,l}(x, t),
\]
\[X_{j,k}^{1,l}(x, t) := \int_0^t \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} 1(\tau - s) \partial_j \mathcal{L}_{j,k}(x - y, v, s, t)N_1^l(y, v, \tau, s) \, dy \, dv \, d\tau \, ds.
\]

We estimate first the main term:

**Lemma 6.3.** For any fixed $k \in \mathbb{Z}$, we have
\[\mathcal{\bar{\varphi}_m(t)X_{j,k}^{1,1}\|_{Osc} \lesssim \varepsilon_1^2\].

**Proof.** Recall (6.16) and (6.15). After integrating by parts in $v$, we have
\[X_{j,k}^{1,l}(x, t) = - \int_0^t \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\partial_{v_i} \mathcal{L}_{j,k})(x - y, v, s, t)E^l(y, s)
\times E^l(y - (s - \tau)v + \bar{Y}(y - sv, v, \tau, s), \tau)\partial_t M_0(v + \bar{W}(y - sv, v, \tau, s)) \, dy \, dv \, d\tau \, ds.
\]
Recall the definition of the kernel $L_{j,k}$ in (6.8). Firstly, we sum the cutoff functions $\tilde{\varphi}_j(v)$. Secondly, we take the $\partial_v$ derivative. Lastly, we apply an inhomogenous dyadic decomposition for $v$ again. As a result, we have

$$
\sum_{j \in [0,19m/20] \cap [0,-k-\delta m/3] \cap \mathbb{Z}} \partial_v L_{j,k}(y, v, s, t) = \sum_{j \in [0,19m/20+10] \cap \mathbb{Z}} \tilde{L}_{j,k}^1(y, v, s, t) + \tilde{L}_{j,k}^2(y, v, s, t),
$$

where

$$
\tilde{L}_{j,k}^1(y, v, s, t) = \frac{\tilde{\varphi}_j(v)}{(2\pi)^3} \int_{\mathbb{R}^3} e^{iz\xi} e^{-(t-s)|\xi|} \partial_v \left[ \frac{e^{is}}{1 - iD} \psi \right] \varphi_k(\xi) d\xi,
$$

$$
\tilde{L}_{j,k}^2(y, v, s, t) = \frac{\tilde{\varphi}_j(v)}{(2\pi)^3} \int_{\mathbb{R}^3} e^{iz\xi} e^{-(t-s)|\xi|} \frac{e^{is}}{1 - iD} \partial_v \left[ \psi \varphi_k(\xi) \right] d\xi.
$$

By doing iterated integration by parts in $\xi$, we have

$$
|\tilde{L}_{j,k}^1(y, v, s, t)| \lesssim (1 + |t - s|2^k)^{-8} 2^{3k} (1 + 2^k|y|)^{-10} \varphi_{j-4,j+4}(v),
$$

$$
|\tilde{L}_{j,k}^2(y, v, s, t)| \lesssim (1 + |t - s|2^k)^{-8} 2^{3k} 2^{-\min\{19m/20, -k-\delta m/3\}} (1 + 2^k|y|)^{-10}
$$

$$
\times \varphi_j(v) 1_{j \in \min\{19m/20, -k-\delta m/3\} + [-5, 5] \cap \mathbb{Z}},
$$

where, for any $a, b, c \in \mathbb{R}$, we use $a + [b, c]$ to denote $[a + b, a + c]$.

From the decomposition (6) and the above estimates of kernels, in terms of the trilinear operator defined in (5.1), we have

$$
| \sum_{j \in [0,19m/20] \cap [0,-k-\delta m/3] \cap \mathbb{Z}} X_{j,k}^{1,1} (x, t) |
$$

$$
\lesssim \sum_{j \in [0,19m/20+10] \cap \mathbb{Z}, s \in \{1, 2\}} \int_0^t (1 + |t - s|2^k)^{-8} \int_0^s R_{j,k}(E, E; C_*) (x, \tau, s, s) d\tau |ds|
$$

where

$$
C_1 = 2^{k+3j} \partial_t M_0 (v + \tilde{W}(y - sv, v, \tau, s)) \varphi_{j-4,j+4}(v),
$$

$$
C_2 := 2^{3j-\min\{19m/20, -k-\delta m/3\}}
\times \partial_t M_0 (v + \tilde{W}(y - sv, v, \tau, s)) \varphi_j(v) 1_{j \in \min\{19m/20, -k-\delta m/3\} + [-5, 5] \cap \mathbb{Z}}.
$$

Moreover, if we distribute one derivative of $\nabla_y$ when $s \in [t/2, t]$, we have

$$
2^k | \sum_{j \in [0,19m/20] \cap [0,-k-\delta m/3] \cap \mathbb{Z}} \sum_{s \in \{1, 2\}} \int_0^{t/2} 2^k (1 + |t - s|2^k)^{-8} \int_0^s R_{j,k}(E, E; C_*) (x, \tau, s, s) d\tau |ds|
$$

$$
+ \int_{t/2}^t (1 + |t - s|2^k)^{-8} \left[ \int_0^s R_{j,k}(\nabla E, E; \tilde{C}_1) (x, \tau, s, s) d\tau \right] + \left[ \int_0^s R_{j,k}(E, \nabla E; C_*) (x, \tau, s, s) d\tau \right] |ds|
$$

$$
+ \int_0^s R_{j,k}(E, E; \tilde{C}_2) (x, \tau, s, s) d\tau |ds|
$$
where
\[ \tilde{C}_1^* = C_*(1 + \nabla_y \tilde{Y}(y - sv, v, \tau, s)), \]
\[ \tilde{C}_2^* = C_* \frac{\partial_y \partial_t M_0(v + \tilde{W}(y - sv, v, \tau, s))}{\partial_t M_0(v + \tilde{W}(y - sv, v, \tau, s))} \partial_y \tilde{W}(y - sv, v, \tau, s). \]

To sum up, from the above estimates and the trilinear estimates (5.5–5.7) in Lemma 5.1, for any fixed \( j \in [0, 19m/20 + 10] \cap \mathbb{Z} \), we have

\[ (1 + 2^{k+m}) \| \sum_{j \in [0,19m/20]\cap[0,-\delta m/3]\cap\mathbb{Z}} X_{j,k}^{1,1}(x,t) \|_{B_0^2} \leq \int_0^t \varepsilon_2^2 2^{-3j+4\delta m} 1_{j \in 19m/20 + [5,5]\cap\mathbb{Z}} + \varepsilon_2^2 (1 + |t-s|2^k) - 8 2^{k+4\delta m} \langle s \rangle^{-0.1} ds \leq \varepsilon_2^2. \]

Lastly, we estimate \((\partial_t + |\nabla|)X_{j,k}^{1,1}(x,t)\). Recall (6.16) and (6.15). As a result of direct computations, after integrating by parts in \( v \), we have

\[ (\partial_t + |\nabla|)X_{j,k}^{1,1}(x,t)(x,t) = - \int_0^t \int_{\mathbb{R}^3} \left( \partial_{y_l} \mathcal{L}_{j,k}(x - y, v, t, t) \right) E^i(y, t) \times E^i(y - (t - \tau)v + \tilde{Y}(y - tv, v, \tau, t)) \partial_t M_0(v + \tilde{W}(y - tv, v, \tau, t)) dydvdt. \]

Recall (6). Similar to what we did in (6), for the kernel determined by \( \tilde{L}_{j,k}^1(y, v, t, t) \) and \( \tilde{L}_{j,k}^2(y, v, t, t) \) when \( k \geq -19m/20 - \delta m/3 \), we distribute one derivative. Moreover, since for the kernel \( \tilde{L}_{j,k}^2(y, v, t, t) \) when \( k < -19m/20 - \delta m/3 \), we use \( j \in [19m/20 + [-5,5] \cap \mathbb{Z} \), we use the smallness of \( 2^{-2j} \) from the derivative of the Poisson equilibrium. As a result, we have

\[ \| \sum_{j \in [0,19m/20]\cap[0,-\delta m/3]\cap\mathbb{Z}} (\partial_t + |\nabla|)X_{j,k}^{1,1}(x,t)(x,t) \|_{B_0^2} \leq \varepsilon_2^2 2^{-m}. \]

To sum up, our desired estimate (6.3) follows from the above estimate and the obtained estimate (6).

Finally we estimate the remainders arising in the decomposition (6.15).

**Lemma 6.4.** With \( X_{j,k}^{1,l} \) defined as in (6.16), for any \((j, k, m) \in A^{1l} \) and \( l \in \{2,3\} \) we have

\[ \| \tilde{\varphi}_m(t) X_{j,k}^{1,l} \|_{\text{Osc}_d} \lesssim \varepsilon_2^2 2^{-\delta j}. \]

**Proof.** Recall (6). As a result of direct computations, we have

\[ \tilde{Y}_{i,a}(y, v, \tau, s) = - \int_{\tau}^s (\gamma - \tau)(s - \gamma) \partial_{x_i} E^a(x - (s - \gamma)v + \tilde{Y}(x - sv, v, \gamma, s), \gamma) \]

\[-(\gamma - \tau)(\partial_{v_i} - s \partial_{x_i}) \tilde{Y}'(x - sv, v, \gamma, s) \partial_{x_i} E^a(x - (s - \gamma)v + \tilde{Y}(x - sv, v, \gamma, s), \gamma) d\tau, \]

\[ \tilde{W}_{i,a}(y, v, \tau, s) = - \int_{\tau}^s (s - \gamma) \partial_{x_i} E^a(x - (s - \gamma)v + \tilde{Y}(x - sv, v, \gamma, s), \gamma) \]

\[-(\partial_{v_i} - s \partial_{x_i}) \tilde{W}'(x - sv, v, \gamma, s) \partial_{x_i} E^a(x - (s - \gamma)v + \tilde{Y}(x - sv, v, \gamma, s), \gamma) d\tau. \]

Recall (6.15) and (6.16). From the above results of computations, the following decomposition holds:

\[ \forall l \in \{2,3\}, \quad X_{j,k}^{1,l} = X_{j,k}^{1,l;1} + X_{j,k}^{1,l;2}, \quad (6.17) \]
where
\[ X_{j,k}^{1,2} := \int_0^t \int_0^s \int_0^s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\gamma - \tau) (s - \gamma) \mathcal{L}_{j,k}(x - y, v, s, t) E^t(y, s) \partial_t M_0(v + \tilde{W}(y - sv, v, \tau, s)) \]
\[ \times \partial_x E^t(x - (s - \gamma)v + \tilde{Y}(x - sv, v, \tau, s), \gamma) (\partial_a E^t)(y - (s - \tau)v + \tilde{Y}(y - sv, v, \tau, s), \tau) d\gamma d\tau ds, \]
\[ X_{j,k}^{1,2} := \int_0^t \int_0^s \int_0^s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\gamma - \tau) \mathcal{L}_{j,k}(x - y, v, s, t) E^t(y, s) \partial_t M_0(v + \tilde{W}(y - sv, v, \tau, s)) \]
\[ \times (\partial_{v_n} - s \partial_{x_n}) \tilde{Y}^d(x - sv, v, \gamma, s) \partial_x E^t(x - (s - \gamma)v + \tilde{Y}(x - sv, v, \gamma, s), \gamma) \]
\[ \times (\partial_a E^t)(y - (s - \tau)v + \tilde{Y}(y - sv, v, \tau, s), \tau) d\gamma d\tau ds. \]

Firstly, we estimate the main terms \( X_{j,k}^{1,l}, l \in \{2,3\} \). From the bilinear estimates (5.5) and (5.8) in Lemma 5.1, we have
\[ (1 + 2^{k+m}) \left( \| X_{j,k}^{1,2} \|_{B^0_t} + \| X_{j,k}^{1,3} \|_{B^1_t} \right) \]
\[ \lesssim \int_0^t \int_0^s \varepsilon_1^2 (1 + 2^{k+m}) 2^{-2j} (1 + 2^k |t - s|)^{-8} (s)^{-1+\delta} (\tau)^{-1} d\tau ds \lesssim \varepsilon_1^2 2^{\delta m - 2j}. \quad (6.18) \]

Now, we estimate the above error type terms \( X_{j,k}^{1,l}, l \in \{2,3\} \). From the estimates (3.29) and (3.30) in Lemma 3.6, the dispersive estimate (3.11) in Lemma 3.2, the rough \( L^\infty - L^1 \)-type bilinear estimates for the \( L^1 \)-estimate, the rough \( L^\infty - L^1 \)-type bilinear estimates for the \( L^\infty \)-estimate, we have
\[ (1 + 2^{k+m}) \left( \| X_{j,k}^{1,2} \|_{B^0_t} + \| X_{j,k}^{1,3} \|_{B^1_t} \right) \]
\[ \lesssim \int_0^t \int_0^s \varepsilon_1^2 (1 + 2^{k+m}) 2^{-2j} (1 + 2^k |t - s|)^{-8} (s)^{-1+\delta} (\tau)^{-7/6} d\tau ds \lesssim \varepsilon_1^2 2^{\delta m - 2j}. \quad (6.19) \]

Hence, to sum up, from the decomposition (6.17) and the obtained estimates (6.18) and (6.19), we have
\[ \sum_{l \in \{2,3\}} (1 + 2^{k+m}) \| X_{j,k}^{1,l} \|_{B^0_t} \lesssim \varepsilon_1^2 2^{\delta m - 2j}. \]

Lastly, it remains to estimate \( (\partial_t + |\nabla|) X_{j,k}^{1,l}, l \in \{2,3\} \). As a result of direct computations, we have
\[ \forall l \in \{2,3\}, \quad (\partial_t + |\nabla|) X_{j,k}^{1,l} = Y_{j,k}^{1,l,1} + Y_{j,k}^{1,l,2}, \]
where
\[ Y_{j,k}^{1,l,1} := \int_0^t \int_\tau \int_{\mathbb{R}^3} (\gamma - \tau) (t - \gamma) \mathcal{L}_{j,k}(x - y, v, t, t) E^t(y, t) \partial_t M_0(v + \tilde{W}(y - tv, v, \tau, t)) \]
Proof of Lemma 7. We bound first the contributions of the static part. In view of (5.5), we have
\[ \sum_{l \in \{2, 3\}, i \in \{1, 2\}} (1 + 2^{k+m}) \| Y_{j, k}^{1, l, i} \|_{B^j_\infty} \lesssim c_2 2^{-m_0-2j}. \]
To sum up, our desired estimate holds (6.4) from the above estimate and the obtained estimate (6).

7. Proof of Lemma 5.1

The rest of the section is concerned with the proof of Lemma 5.1. The main idea is to decompose the electric field corresponding to the variable \( \gamma \) into its static and oscillatory parts, and then integrate by parts in \( \gamma \) in certain cases.

7.1. Proof of (5.5). We may assume that \( C \) and \( K_{j,k} \) are real-valued, decompose \( E(t) = E^{\text{stat}}(t) + \Re \{ e^{-it} E^{\text{osc}}(t) \} \) as in (3.4), and then decompose dyadically in frequency, so
\[ Q(\nabla E, E; C) = \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{s \in \{\text{stat}, \text{osc}\}} \Re \{ Q(\nabla E, c_s(\gamma) P_{k_2} E^*; C) \}, \]
where \( c_{\text{stat}} = 1 \) and \( c_{\text{osc}} = e^{-i\gamma} \).

**Step 1.** We bound first the contributions of the static part. In view of (3.29)
\[ \int_{\mathbb{R}^3} |h(y - (s - \mu)v + \theta \tilde{Y}(y - sv, v, \mu, s))| \, dy \lesssim \|h\|_{L^1}, \]
for any \( h \in L^1(\mathbb{R}^3), \theta \in [0, 1], v \in \mathbb{R}^3, \) and \( \mu \leq s \leq t \). Moreover
\[ \int_{\mathbb{R}^3} |h(y - (s - \mu)v + \theta \tilde{Y}(y - sv, v, \mu, s))| \, dv \lesssim \langle s \rangle^{-3} \|h\|_{L^1}, \]
provided that \( h \in L^1(\mathbb{R}^3), \theta \in [0, 1], y \in \mathbb{R}^3, \) and \( \mu \leq s \leq t \) satisfy \( s - \mu \geq \langle s \rangle/8 \).
We use Lemma 3.1 (i) to see that
\[
\|R_{j,k}(P_k(\nabla E),P_k E^{stat};C)(\cdot,\gamma,\tau,s)\|_{L_1^1} \lesssim \|K_{j,k}(\cdot,\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^1} m(C)(\gamma,\tau,s)
\times 2^{-k_2} \min\{\|P_k\rho(\tau)\|_{L^1},\|P_k \rho^{\text{stat}}(\gamma)\|_{L^\infty},\|P_k\rho(\tau)\|_{L^\infty}\}
\lesssim m(C)(\gamma,\tau,s) \cdot \frac{\varepsilon_1^2}{(\tau)^{1-2\delta}2k_1 + (\gamma)^{-\delta}2^{-k_2-k_2^2}(\gamma)^{-1+2\delta}} \min\{2^{3k_1},(\gamma)^{-3},2^{3k_2},(\gamma)^{-3}\},
\]  
(7.3)
where the operator \(R_{j,k}\) is defined as in (5.1). Similarly, using the (7.1)–(7.2) and the observation that \(\|K(\cdot,\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^1} \lesssim 1\), it follows that if \(s-\gamma \geq (s)/8\) and \(s-\tau \geq (s)/8\) then
\[
\|R_{j,k}(P_k(\nabla E),P_k E^{stat};C)(\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^\infty} \lesssim 2^{-k_2} m(C)(\gamma,\tau,s)
\times \min\{\|P_k\rho(\tau)\|_{L^\infty},\|P_k \rho^{\text{stat}}(\gamma)\|_{L^\infty},\|P_k\rho(\tau)\|_{L^\infty}\}
\lesssim m(C)(\gamma,\tau,s) \cdot \frac{\varepsilon_1^2}{(\gamma)^{1-2\delta}2k_1 + (\tau)^{-\delta}2^{-k_2-k_2^2}(\gamma)^{-1+2\delta}} \min\{2^{3k_1},(\tau)^{-3},2^{3k_2},(\gamma)^{-3}\}.
\]  
(7.4)
Similar bounds hold if \(s-\gamma \leq (s)/8\) or if \(s-\tau \leq (s)/8\). In all cases, using Lemma 3.1 (i),
\[
\|R_{j,k}(P_k(\nabla E),P_k E^{stat};C)(\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^\infty} \lesssim m(C)(\gamma,\tau,s) \cdot \frac{\varepsilon_1^2}{(\gamma)^{1-2\delta}2k_1 + (\tau)^{-\delta}2^{-k_2-k_2^2}(\gamma)^{-1+2\delta}} \min\{2^{3k_1},(\gamma)^{-3},2^{3k_2},(\gamma)^{-3}\}.
\]
(7.4)
Therefore, using (5.4),(7.3), and (7.4), we have
\[
\sum_{k_1,k_2 \in \mathbb{Z}} \left\{ \|Q_{j,k}(P_k(\nabla E),P_k E^{stat};C)(\cdot,\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^1} + \langle s \rangle^3 \|Q_{j,k}(P_k(\nabla E),P_k E^{stat};C)(\cdot,\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^\infty} \right\}
\lesssim \int_{s/3}^{t_4} m(C)(\gamma,\tau,s) \cdot \varepsilon_1^2 \langle \gamma \rangle^\delta \langle \gamma \rangle^{-1+2\delta} \min\{\langle \gamma \rangle^{-2},\langle \tau \rangle^{-2}\} d\gamma
\lesssim \varepsilon_1^2 \gamma^{2\delta m_2} \langle \tau \rangle^{-\delta} \min\{2^{-2m_2},\langle \tau \rangle^{-2}\} m^*(C)(\tau,s).
\]
\textbf{Step 2.} We bound now the contributions of the oscillatory parts, and we will show that
\[
\sum_{k_1,k_2 \in \mathbb{Z}} 2^{-k_2} \|Q(P_k R_1^\rho, e^{-i\gamma} P_k R_2^\rho \omega;C)(\cdot,\cdot,\gamma,\tau,s)\|_{B_{1,\infty}^\infty}
\lesssim \varepsilon_1^2 \min\{2^{-1.1m_2},\langle \tau \rangle^{-1.1}\} m^*(C)(\tau,s),
\]  
(7.5)
where \(R_1^\rho, R_2^\rho\) are operators defined by Hörmander-Michlin multipliers satisfying the inequalities
\[
\sup_{|\alpha| \leq 8} \sup_{\xi \in \mathbb{R}^3} |\xi|^{|\alpha|} |D_\xi^\alpha R_1^\rho(\xi)| \leq 1, \quad l \in \{1,2\}.
\]  
(7.6)
Indeed, we estimate first as in (7.3)–(7.4) to see that for any \(k_1,k_2 \in \mathbb{Z}\)
\[
\|R(P_k R_1^\rho, e^{-i\gamma} P_k R_2^\rho \omega;C)(\cdot,\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^1} + \langle s \rangle^3 \|R(P_k R_1^\rho, e^{-i\gamma} P_k R_2^\rho \omega;C)(\cdot,\cdot,\gamma,\tau,s)\|_{L_{1,\infty}^\infty}
\lesssim m(C)(\gamma,\tau,s) \cdot \frac{\varepsilon_1^2}{(\tau)^{1-2\delta}2k_1 + (\gamma)^{-\delta}2^{-k_2-k_2^2}(\gamma)^{-1+2\delta}} \min\{2^{3k_1},\langle \gamma \rangle^{-3},2^{3k_2},\langle \gamma \rangle^{-3}\},
\]  
(7.7)
where we use (3.6) instead of (3.5). The bounds (7.5) follow if \( m_2 \in [0, 400] \). On the other hand, if \( m_2 \geq 400 \) then we have

\[
\sum_{\{k_1, k_2 \in \mathbb{Z} : |k_2 + m_2| \geq (1 + 10\delta)m_2/9 \text{ or } k_1 \leq -9m_2/8 \}} 2^{-k_2} \| Q_{j, k}(P_{k_1} R'_1 \rho, e^{-i\gamma} P_{k_2} R'_2 \rho^{osc}; C)(\cdot, \tau, s) \|_{B^0_{\infty}} \\
\leq \varepsilon_2^2 \min \{ 2^{-1.1m_2}, (\tau)^{-1.1} \} m^* (C)(\tau, s).
\]

It remains to bound the contributions coming from indices \( k_2 \in [-10(1 + 10\delta)m_2/9, -(8 - 10\delta)m_2/9] \) and \( k_1 \geq -9m_2/8 \) when \( m_2 \geq 400 \). For this we integrate by parts in \( \gamma \). We start by writing

\[
e^{-i\gamma} P_{k_2} R'_2 \rho^{osc}(y - (s - \gamma)v + \theta_2 \bar{Y}(y - sv, v, \gamma, s), \gamma) \\
= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-i\gamma} \varphi_{k_2}(\xi) R'_2(\xi) \rho^{osc}(\xi, \gamma) e^{i\xi \cdot (y - (s - \gamma)v + \theta_2 \bar{Y}(y - sv, v, \gamma, s))} d\xi.
\]

Therefore, using the definitions

\[
Q_{j, k}(P_{k_1} R'_1 \rho, e^{-i\gamma} P_{k_2} R'_2 \rho^{osc}; C)(x, \tau, s) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{t_3}^{t_1} K_{j, k}(x - y, v, \tau, s) \\
\times C(x, y, v, \gamma, \tau, s)(P_{k_1} R'_1 \rho)(y - (s - \tau)v + \theta_1 \bar{Y}(y - sv, v, \tau, s), \tau) \\
\times \rho^{osc}(\xi, \gamma) e^{i\gamma(-1 + \xi \cdot v)} \varphi_{k_2}(\xi) R'_2(\xi) e^{i\xi \cdot (y - sv + \theta_2 \bar{Y}(y - sv, v, \gamma, s))} d\gamma d\tau dy dv d\xi.
\]

We insert cutoff functions of the form \( \varphi_p(1 - (\xi \cdot v)^2) \) and use the integration by parts formula

\[
\int_{t_3}^{t_1} e^{i\gamma(-1 + \xi \cdot v)} C(x, y, v, \gamma, \tau, s) \rho^{osc}(\xi, \gamma) \varphi_p(1 - (\xi \cdot v)^2) e^{i\xi \cdot (y - sv + \theta_2 \bar{Y}(y - sv, v, \gamma, s))} d\gamma \\
= \int_{t_3}^{t_1} (\partial_\gamma C)(x, y, v, \gamma, \tau, s) I_p^1(\xi, y, v, \gamma, s) d\gamma + \int_{t_3}^{t_1} C(x, y, v, \gamma, \tau, s) I_p^2(\xi, y, v, \gamma, s) d\gamma \\
+ \sum_{j \in \{3, 4\}} (-1)^{j+1} C(x, y, v, t_j, \gamma, \tau, s) I_p^1(\xi, y, v, t_j, s),
\]

where

\[
I_p^1(\xi, y, v, \gamma, s) := -\frac{e^{i\gamma(-1 + \xi \cdot v)}}{i(-1 + \xi \cdot v)} \rho^{osc}(\xi, \gamma) \varphi_p(1 - (\xi \cdot v)^2) e^{i\xi \cdot (y - sv + \theta_2 \bar{Y}(y - sv, v, \gamma, s))},
\]

\[
I_p^2(\xi, y, v, \gamma, s) := -\frac{e^{i\gamma(-1 + \xi \cdot v)}}{i(-1 + \xi \cdot v)} \varphi_p(1 - (\xi \cdot v)^2) e^{i\xi \cdot (y - sv + \theta_2 \bar{Y}(y - sv, v, \gamma, s))} \{ (\partial_\gamma \rho^{osc})(\xi, \gamma) \\
+ \rho^{osc}(\xi, \gamma)(i\theta_2 \xi \cdot (\partial_\gamma \bar{Y})(y - sv, v, \gamma, s)) \}.
\]

For \( p \in \mathbb{Z} \) and \( \ell \in \{1, 2\} \) let

\[
H_{p, k_2}^\ell(y, v, \gamma, s) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} I_p^\ell(\xi, y, v, \gamma, s) \varphi_{k_2}(\xi) R'_2(\xi) d\xi,
\]

(7.10)

For \( p \in \mathbb{Z} \) let also

\[
G_{p, k_2}(y, v, \gamma, s) := \frac{e^{-i\gamma}}{(2\pi)^3} \int_{\mathbb{R}^3} \varphi_{p \leq p}(1 - (\xi \cdot v)^2) \rho^{osc}(\xi, \gamma) \varphi_{k_2}(\xi) \\
\times R'_2(\xi) e^{i\xi \cdot (y - (s - \gamma)v + \theta_2 \bar{Y}(y - sv, v, \gamma, s))} d\xi.
\]
In view of these identities, for any $p_0 \in \mathbb{Z}$ we can thus decompose

$$Q_{j,k}(P_{k_1}R'_1\rho, e^{-i\gamma} P_{k_2} R'_2\rho^{osc}; C) = X_{\leq p_0, k_1, k_2} + \sum_{p \geq p_0 + 1} (Y^1_{p,k_1,k_2} + Y^2_{p,k_1,k_2})$$

(7.11)

where, with $C' := (\partial_{\gamma} C) \cdot 1_{[t_3, t_4]}(\gamma) + C \cdot (\delta_0(\gamma - t_3) - \delta_0(\gamma - t_4))$,

$$X_{\leq p_0, k_1, k_2}(x, \tau, s) := \int_{t_3}^{t_4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_{j,k}(x - y, v, \tau, s) C(x, y, v, \gamma, \tau, s) \times (P_{k_1}R'_1\rho)(y - (s - \tau)v + \theta_1 \bar{Y}(y - sv, v, \tau, s), \tau) G_{\leq p_0, k_2}(y, v, \gamma, s) dy dv d\gamma,$$

$$Y^1_{p,k_1,k_2}(x, \tau, s) := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_{j,k}(x - y, v, \tau, s) C'(x, y, v, \gamma, \tau, s) \times (P_{k_1}R'_1\rho)(y - (s - \tau)v + \theta_1 \bar{Y}(y - sv, v, \tau, s), \tau) H^1_{p,k_2}(y, v, \gamma, s) dy dv d\gamma,$$

(7.12)

$$Y^2_{p,k_1,k_2}(x, \tau, s) := \int_{t_3}^{t_4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K_{j,k}(x - y, v, \tau, s) C(x, y, v, \gamma, \tau, s) \times (P_{k_1}R'_1\rho)(y - (s - \tau)v + \theta_1 \bar{Y}(y - sv, v, \tau, s), \tau) H^2_{p,k_2}(y, v, \gamma, s) dy dv d\gamma.$$  

(7.13)

**Step 3.** We will now prove bounds on the functions $H^1_{p,k_2}$ and $G_{\leq p_0, k_2}$. We claim that

$$\|H^1_{p,k_2}(\cdot, v, \gamma, s)\|_{L^1_y} + \langle \gamma \rangle^3 \|H^1_{p,k_2}(\cdot, v, \gamma, s)\|_{L^\infty_y} \lesssim \varepsilon_1 2^{-p - p/4 + 2\varepsilon_1},$$

$$\|H^2_{p,k_2}(\cdot, v, \gamma, s)\|_{L^1_y} + \langle \gamma \rangle^3 \|H^2_{p,k_2}(\cdot, v, \gamma, s)\|_{L^\infty_y} \lesssim \varepsilon_1 2^{-p - p/4 + 2\varepsilon_1}$$

(7.14)

and, if $p_0, k_2 \leq -10$,

$$\|G_{\leq p_0, k_2}(\cdot, v, \gamma, s)\|_{L^\infty_y} \lesssim \varepsilon_1 2^{p_0} (v)^{-1} \langle \gamma \rangle^{-1 - 2\varepsilon_1},$$

(7.15)

for any $v \in \mathbb{R}^3$. Indeed, we examine the definitions (7.9)–(7.10) and rewrite

$$H^1_{p,k_2}(y, v, \gamma, s) = -e^{-i\gamma} \tilde{H}^1_{p,k_2}(y - (s - \gamma)v + \theta_2 \bar{Y}(y - sv, v, \gamma, s), v, \gamma),$$

$$\tilde{H}^1_{p,k_2}(x, v, \gamma) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \varphi_{k_2}(\xi) R'_1(\xi) \frac{\varphi_p(1 - (\xi \cdot v)^2)}{i(-1 + \xi \cdot v)} \rho^{osc}(\xi, \gamma) e^{i\xi \cdot x} d\xi.$$  

It follows from Lemma 3.3 (ii) that

$$\|\tilde{H}^1_{p,k_2}(\cdot, v, \gamma)\|_{L^2} \lesssim 2^{-p - p/4} \|\rho^{osc}(\cdot, \gamma)\|_{L^q}, \quad q \in [1, \infty],$$

and the desired bounds in the first line of (7.14) follow using (3.6) and (7.1). The bounds on $H^2_{p,k_2}$ in the second line of (7.14) follow by a similar argument.

To prove the bounds (7.15) we notice that if $p_0, k_2 \leq -10$ then the functions $G_{\leq p_0, k_2}$ are non-trivial only if $|v| \geq 1$. The support of the integral defining $G_{\leq p_0, k_2}$ has volume $\lesssim 2^{k_2} 2^{p_0} (v)^{-1}$, so, using also (3.6),

$$|G_{\leq p_0, k_2}(y, v, \gamma, s)| \lesssim 2^{k_2} 2^{p_0} (v)^{-1} \cdot \|P_{k_2} \rho^{osc}(\cdot, \gamma)\|_{L^1} \lesssim \varepsilon_1 2^{p_0} (v)^{-1} \langle \gamma \rangle^{-1 + 2\varepsilon_1} 2^{k_2},$$

for any $y, v \in \mathbb{R}^3$, as claimed in (7.15).
Step 4. We return now to the proof of the main bounds (7.5). In view of the discussion in Step 2, it suffices to show that if $m_2 \geq 400$ then
\[ \sum_{\{k_1, k_2 \in \mathbb{Z}: |k_2 + m_2| \leq (1+100\delta)m_2/9 \text{ and } k_1 \geq -9m_2/2\}} 2^{-k_2} \| \mathcal{Q}_{j,k}(P_{k_1} R'_1 \rho, e^{-i\gamma} P_{k_2} R'_2 \rho^{osc}; C)(\cdot, \tau, s) \|_{B^0_2} \]
\[ \lesssim \varepsilon_1^2 \min\{2^{-3m_2}, (\tau)^{-1.1}\} m^*(C)(\tau, s). \tag{7.16} \]

Recall that $\| \mathcal{K}(\cdot, \tau, s) \|_{L^1_{\gamma,e}} + \| \mathcal{K}(\cdot, \tau, s) \|_{L^1_{\gamma}L^\infty} \lesssim 1$ (see (5.2)). We use the decomposition (7.11) with $p_0 := -3m_2/5$. The function $X_{\leq p_0, k_1,k_2}$ is nontrivial only if $k_2 + j \geq -20$. From the estimate (7.1), we have
\[ \| X_{\leq p_0, k_1,k_2}(\cdot, \tau, s) \|_{L^1_{\gamma}} \lesssim \int_{t_3}^{t_4} m(C)(\gamma, \tau, s) \times \min\{\| P_{k_1} \rho(\tau) \|_{L^1}, \| G_{\leq p_0, k_2}(\cdot, \gamma, s) \|_{L^\infty_{\gamma,e}}, \| P_{k_1} \rho(\tau) \|_{L^\infty_{\gamma}} \| G_{\leq p_0, k_2}(\cdot, \gamma, s) \|_{L^\infty_{\gamma}L^1_{\gamma}} \} d\gamma \]
\[ \lesssim \varepsilon_1^2 \sum_{\{k_1, k_2 \in \mathbb{Z}: |k_2 + m_2| \leq (1+100\delta)m_2/9 \text{ and } k_1 \geq -9m_2/2\}} \min\{2^{-4m_2}, 2^{m_2} \langle \tau \rangle^{-3}, 2^{m_2} m_2 2^{3k_1}\}. \tag{7.17} \]

Now, we proceed to the $L^\infty_{\gamma}$-estimate. By using Lemma 3.1 (i) and (7.15), moreover, using the trivial estimate if $s - \tau \leq \langle s \rangle/8$ and using also (7.2), if $s - \tau \geq \langle s \rangle/8$, we have
\[ \| X_{\leq p_0, k_1,k_2}(\cdot, \tau, s) \|_{L^\infty_{\gamma}} \lesssim \int_{t_3}^{t_4} m(C)(\gamma, \tau, s) \| G_{\leq p_0, k_2}(\cdot, \gamma, s) \|_{L^\infty_{\gamma}} \times \min\{\| P_{k_1} \rho(\tau) \|_{L^\infty_{\gamma}}, \langle s \rangle^{-3}\| P_{k_1} \rho(\tau) \|_{L^1_{\gamma}} \} d\gamma. \]

Similarly, if we use the same strategy for $G_{\leq p_0, k_2}(\cdot, \cdot, s)$ based on the size of $s - \gamma$, we have
\[ \| X_{\leq p_0, k_1,k_2}(\cdot, \tau, s) \|_{L^\infty_{\gamma}} \lesssim \int_{t_3}^{t_4} m(C)(\gamma, \tau, s) \langle s \rangle^{-3} \| G_{\leq p_0, k_2}(\cdot, \gamma, s) \|_{L^\infty_{\gamma}L^1_{\gamma}} \| P_{k_1} \rho(\tau) \|_{L^\infty_{\gamma}} d\gamma. \]

In both cases we see easily that
\[ \langle s \rangle^{-3} \| X_{\leq p_0, k_1,k_2}(\cdot, \tau, s) \|_{L^\infty_{\gamma}} \lesssim \varepsilon_1^2 \sum_{\{k_1, k_2 \in \mathbb{Z}: |k_2 + m_2| \leq (1+100\delta)m_2/9 \text{ and } k_1 \geq -9m_2/2\}} \min\{2^{-4m_2}, 2^{m_2} \langle \tau \rangle^{-3}, 2^{m_2} m_2 2^{3k_1}\}. \tag{7.18} \]

Therefore, recalling that $-j \leq 20 + k_2$, we have
\[ \sum_{\{k_1, k_2 \in \mathbb{Z}: |k_2 + m_2| \leq (1+100\delta)m_2/9 \text{ and } k_1 \geq -9m_2/2\}} 2^{-k_2} \| X_{\leq p_0, k_1,k_2}(\cdot, \tau, s) \|_{B^0_2} \lesssim \varepsilon_1^2 \min\{2^{-1.1m_2}, (\tau)^{-1.1}\} m^*(C)(\tau, s). \tag{7.19} \]
as desired.

Now, we proceed to estimate $Y^1_{p_0, k_1,k_2}(x, \tau, s)$ in (7.12). Similarly, we use (7.14) to estimate, for any $p \geq p_0 = -3m_2/5$,
\[ \| Y^1_{p_0, k_1,k_2}(\cdot, \tau, s) \|_{L^1_{\gamma}} \lesssim \sup_{\gamma \in [t_3, t_4]} m(C)(\gamma, \tau, s) \{\| P_{k_1} \rho(\tau) \|_{L^1}, \| H_{p_0, k_2}(\cdot, \cdot, \gamma, s) \|_{L^\infty_{\gamma,e}}, \| P_{k_1} \rho(\tau) \|_{L^\infty_{\gamma}} \times \| H_{p_0, k_2}(\cdot, \cdot, \gamma, s) \|_{L^\infty_{\gamma}L^1_{\gamma}} \}
\]
\[ \lesssim \varepsilon_1^2 \min\{2^{-3m_2}, (\tau)^{-3}, 2^{3k_1}\}. \tag{7.20} \]
Moreover, using also (7.2), if \( s - \tau \geq \langle s \rangle / 8 \) then
\[
\| Y_{p,k_1,k_2}(\cdot, \tau, s) \|_{L^\infty_x} \lesssim \sup_{\gamma \in \{\tau, 3\}} m(C)(\gamma, \tau, s) \| H_{p,k_2}(\cdot, \gamma, s) \|_{L^\infty_x}
\times \min\{\| P_{k_1} \rho(\tau) \|_{L^\infty}, \langle s \rangle^{-3} \| P_{k_1} \rho(\tau) \|_{L^1} \} d\gamma,
\]
with a simpler bound in the case \( s - \tau \leq \langle s \rangle / 8 \). Similarly, if we use the same strategy for \( H_{p,k_2}(\cdot, \gamma, s) \) based on the size of \( s - \gamma \), we have
\[
\| Y_{p,k_1,k_2}(\cdot, \tau, s) \|_{L^\infty_x} \lesssim \sup_{\gamma \in \{\tau, 3\}} m(C)(\gamma, \tau, s) \| H_{p,k_2}(\cdot, \gamma, s) \|_{L^\infty_x} L^1_y \| P_{k_1} \rho(\tau) \|_{L^\infty}.
\]
In both cases we obtain
\[
\langle s \rangle^3 \| Y_{p,k_1,k_2}(\cdot, \tau, s) \|_{L^\infty_x} \lesssim \varepsilon^2 \min\{\| P_{k_1} \rho(\tau) \|_{L^\infty}, \langle s \rangle^{-3} \| P_{k_1} \rho(\tau) \|_{L^1} \} m(C)(\tau, s).
\]
Therefore, after combining the obtained estimates (7.20) and (7.21), we have
\[
\sum_{\{k_1, k_2 \in \mathbb{Z} : |k_2 + m_2| \leq (1 + 100 \delta) m_2 / 9 \text{ and } k_1 \geq -9 m_2 / 8 \}} \sum_{p \geq -3 m_2 / 5} 2^{-k_2} \| Y_{p,k_1,k_2}(\cdot, \tau, s) \|_{L^p_y} \lesssim \varepsilon^2 \min\{\| P_{k_1} \rho(\tau) \|_{L^\infty}, \langle s \rangle^{-3} \| P_{k_1} \rho(\tau) \|_{L^1} \} m(C)(\tau, s).
\]
Lastly, the contributions of the functions \( Y_{p,k_1,k_2}(x, \tau, s) \) can be estimated in a similar way, and the bounds (7.16) follow using also (7.19). This completes the proof of the main bounds (5.5).

7.2. Proof of (5.6). Step 1. As in subsection 7.1 we decompose
\[
Q_{j,k}(E, \nabla E; C) = \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{s \in \{\text{stat, osc}\}} Q_{j,k}(P_{k_1} E, c_s(\gamma) P_{k_2} (\nabla E^s); C),
\]
where \( c_{\text{stat}} = 1 \) and \( c_{\text{osc}} = e^{-i\gamma} \) as before. As in (7.3), (7.4), and (7.7), we have
\[
\| R_{j,k}(P_{k_1} E, P_{k_2} (\nabla E^{\text{stat}}); C)(\cdot, \gamma, \tau, s) \|_{L^1_x} + \langle s \rangle^3 \| R_{j,k}(P_{k_1} E, P_{k_2} (\nabla E^{\text{stat}}); C)(\cdot, \gamma, \tau, s) \|_{L^\infty_x} \lesssim m(C)(\gamma, \tau, s) \cdot \frac{\varepsilon^2 2^{-k_1}}{\langle \tau \rangle^{1-28 2^{k_1}}} (2^{-k_2} (\gamma)^{-1+2\delta}) \min\{2^{3k_1}, (\gamma)^{-3}, 2^{3k_2}, (\gamma)^{-3}\}
\]
and
\[
\| R_{j,k}(P_{k_1} E, e^{-i\gamma} P_{k_2} (\nabla E^{\text{osc}}); C)(\cdot, \gamma, \tau, s) \|_{L^1_x} + \langle s \rangle^3 \| R_{j,k}(P_{k_1} E, e^{-i\gamma} P_{k_2} (\nabla E^{\text{osc}}); C)(\cdot, \gamma, \tau, s) \|_{L^\infty_x} \lesssim m(C)(\gamma, \tau, s) \cdot \frac{\varepsilon^2 2^{-k_1}}{\langle \tau \rangle^{1-28 2^{k_1}}} (2^{-k_2} (\gamma)^{-1+2\delta}) \min\{2^{3k_1}, (\gamma)^{-3}, 2^{3k_2}, (\gamma)^{-3}\}.
\]
These bounds are sufficient to estimate the contributions of the static component \( E^{\text{stat}} \) and of the oscillatory component \( E^{\text{osc}} \) corresponding to frequencies \( \{k_1, k_2 \in \mathbb{Z} : |k_2 + m_2| \geq (1 + 100 \delta) m_2 / 9 \text{ or } k_1 \leq -9 m_2 / 8 \} \) (or when \( 2^{m_2} \lesssim 1 \)).
Step 2. As before, it remains to prove that if \( m_2 \geq 400 \) then
\[
\sum_{\{k_1,k_2 \in \mathbb{Z} : |k_2+m_2| \leq (1+100)m_2/9 \text{ and } k_1 \geq -9m_2/8 \}} 2^{-k_1} \| \mathcal{Q}_{j,k}(P_{k_1} R'_1 \rho, e^{-i\gamma} P_{k_2} R'_2 \rho^{osc}; C)(.,\tau,s) \|_{B^0_2}
\lesssim \varepsilon_1^2 \min \{2^{-1.1m_2}, \langle \tau \rangle^{-1.1} \} m^*(C)(\tau, s).
\]
(7.24)

where \( R'_1, R'_2 \) are operators defined by multipliers satisfying the differential inequalities (7.6).

We integrate by parts, using (7.8)–(7.9), and decompose \( \mathcal{Q}(P_{k_1} R'_1 \rho, e^{-i\gamma} P_{k_2} R'_2 \rho^{osc}; C) \) as in (7.10)–(7.13) with \( p_0 = -3m_2/5 \). Then we use the bounds (7.17)–(7.18) and recall that the functions \( X_{\leq p_0,k_1,k_2} \) are nontrivial only if \(-j \leq 20 + k_2\), so
\[
\sum_{|k_2+m_2| \leq (1+100)m_2/9, k_1 \in [-9m_2/8, \infty) \cap \mathbb{Z}} 2^{-k_1} \| X_{\leq p_0,k_1,k_2}(.,\tau,s) \|_{B^0_2}
\lesssim \sum_{|k_2+m_2| \leq (1+100)m_2/9, k_1 \in [-9m_2/8, \infty) \cap \mathbb{Z}} 2^{-k_1} \varepsilon_1^2 2^2 \delta m_2 \langle \tau \rangle^{2\delta} \min \{2^{p_0 - j + 2k}, 2^{m_2} \langle \tau \rangle^{-3}, 2^{m_2} \langle \tau \rangle^{-1} \} m^*(C)(\tau, s)
\lesssim \varepsilon_1^2 \min \{2^{-1.1m_2}, \langle \tau \rangle^{-1.1} \} m^*(C)(\tau, s).
\]

Similarly, using (7.20)–(7.21), for \( l \in \{1,2\} \) we estimate
\[
\sum_{|k_2+m_2| \leq (1+100)m_2/9, k_1 \in [-9m_2/8, \infty) \cap \mathbb{Z}, p \geq -3m_2/5} 2^{-k_1} \| Y^l_{p,k_1,k_2}(.,\tau,s) \|_{B^0_2}
\lesssim \sum_{|k_2+m_2| \leq (1+100)m_2/9, k_1 \in [-9m_2/8, \infty) \cap \mathbb{Z}, p \geq -3m_2/5} \varepsilon_1^2 2^{3-p/2} - p/2 \langle \tau \rangle^{2\delta} 2^{-k_1 - k_2 - m_2 (1 - 2\delta)} \times \min \{2^{-3m_2}, \langle \tau \rangle^{-3}, 2^{3k_1} \} m^*(C)(\tau, s)
\lesssim \varepsilon_1^2 \min \{2^{-1.1m_2}, \langle \tau \rangle^{-1.1} \} m^*(C)(\tau, s).
\]

The desired bounds (7.24) follow.

7.3. Proofs of (5.7)–(5.10). These estimates can be proved in the same way. We first decompose the electric field in the second position into its static and oscillatory components, and decompose dyadically in frequency. Then we estimate the contributions of the static components using bounds similar to (7.22) (with different factors of \( 2^{-k_1} \) or \( 2^{-k_2} \)) and the contributions of most of the oscillatory components using bounds similar to (7.23).

After these reductions we are left with the contributions of the oscillatory components coming from frequencies \( \{k_1,k_2 \in \mathbb{Z} : |k_2+m_2| \leq m_2/8 \text{ and } k_1 \geq -9m_2/8 \} \). Here we integrate by parts in \( \gamma \), as before, and use the bounds (7.17)–(7.18) and (7.20)–(7.21) to control the corresponding contributions.

8. Proof of Theorem 1.1

In this section we show how the bootstrap of Proposition 2.4 implies the main result, Theorem 1.1.

Proof of global existence and uniqueness. Let \( \varepsilon > 0 \) and \( \delta > 0 \) as in Proposition 2.4 be given, and let \( f_0 \) satisfying the assumptions (1.4) (resp. (2.4)) be given. Then by standard local existence theory we can find \( T > 0 \) and a unique solution \( f \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,T]) \) to (1.2),
Moreover, by Corollary 2.3 we can decompose the associated density \( \rho(x, t) \) as in (2.20), and by continuity (note that \( \rho^{\text{osc}}(x, 0) = 0 \) we can assume that \( T > 0 \) is chosen such that the smallness assumption (2.25) of Proposition 2.4 is met.

Noting that \( f \) can be recovered exactly by integrating along the characteristics, i.e. that by (2.4) we have
\[
f(x, v, t) = f_0(X(x, v, 0, t), V(x, v, 0, t)) + [M_0(V(x, v, 0, t)) - M_0(v)],
\]
and the characteristics in turn are determined by \( \rho \) and its derivative through the system of ODEs (2.3), we can apply the conclusion (2.26) of Proposition 2.4 to extend the solution \( f \) to a larger time interval \([0, T']\) with \( T' > T \), on which the splitting of the density according to Corollary 2.3 and the corresponding smallness condition (2.25) still hold. Hence we obtain a unique global solution \( f \in C^1(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+) \) with associated density \( \rho = \rho^{\text{stat}} + \mathcal{R}\{e^{-it} \rho^{\text{osc}}\} \) satisfying for all \( t > 0 \) that
\[
\|\rho^{\text{stat}}\|_{\text{Stat}} + \|\rho^{\text{osc}}\|_{\text{Osc}} \lesssim \varepsilon_0.
\]
The corresponding splitting (1.6) of the electric field follows directly by (3.4). \( \square \)

We prove next the scattering statement (1.5) of Theorem 1.1:

**Proposition 8.1.** Assume \( f \) is a global solution as constructed above, so that in particular the associated density satisfies (8.2). Then there exists \( f_\infty \in L^\infty_{x,v} \) such that
\[
\|f(x, v, t) - f_\infty(x - tv, v)\|_{L^\infty_{x,v}} \lesssim \varepsilon_1(t)^{-\frac{1}{2}}.
\]

**Remark 8.2.** Using formula (8.1) and with additional simple work on the characteristics, one could obtain the formula
\[
f_\infty(x, v) := (f_0 \circ \Phi)(x, v) + \left[M_0(\Phi^2(x, v)) - M_0(v)\right]
\]
where \( \Phi = (\Phi^1, \Phi^2) \) is a \( C^1 \) symplectic diffeomorphism.

Proposition 8.1 essentially follows from the following lemma:

**Lemma 8.3.** Under the assumptions of Proposition 8.1, there exist \((\tilde{Y}_\infty, \tilde{W}_\infty) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3 \) such that for each fixed \( s \), we have that
\[
\langle t \rangle^\frac{1}{2}(v)^{-1} \left[|\tilde{Y}(x, v, s, t) - \tilde{Y}_\infty(x, v, s)| + |\tilde{W}(x, v, s, t) - \tilde{W}_\infty(x, v, s)|\right] \lesssim \varepsilon_1,
\]
uniformly in \( 0 \leq s \leq t < \infty \).

We first show how Proposition 8.1 follows from Lemma 8.3.

**Proof of Proposition 8.1 assuming Lemma 8.3.** Write for simplicity \( \tilde{Y}_\infty(x, v) := \tilde{Y}_\infty(x, v; 0) \) and \( \tilde{W}_\infty(x, v) := \tilde{W}_\infty(x, v; 0) \) and let
\[
G(y, u) := f_0(y + \tilde{Y}_\infty(y, u), u + \tilde{W}_\infty(y, u)) + \left[M_0(u + \tilde{W}_\infty(y, u)) - M_0(u)\right]
\]
be the (putative) scattering data from Lemma 8.3. By (8.1) we can rewrite
\[
f(x, v, t) - G(x - tv, v)
\]
\[
= f_0(x - tv + \tilde{Y}(x - tv, v, 0, t), v + \tilde{W}(x - tv, v, 0, t)) - f_0(x - tv + \tilde{Y}_\infty(x - tv, v), v + \tilde{W}_\infty(x - tv, v))
\]
\[
+ M_0(v + \tilde{W}(x - tv, v, 0, t)) - M_0(v + \tilde{W}_\infty(x - tv, v))
\]
\[
= \delta Y(x, v, t) \cdot A_x(x, v, t) + \delta W(x, v, t) \cdot A_v(x, v, t) + \delta W(x, v, t) \cdot B(x, v, t),
\]
where

\[
\delta Y := \tilde{Y}(x - tv, v, 0, t) - \tilde{Y}_\infty(x - tv, v), \quad Y_0 := \tilde{Y}(x - tv, v, 0, t), \quad Y_\infty := \tilde{Y}_\infty(x - tv, v)
\]

\[
\delta W := \tilde{W}(x - tv, v, 0, t) - \tilde{W}_\infty(x - tv, v), \quad W_0 := \tilde{W}(x - tv, v, 0, t), \quad W_\infty := \tilde{W}_\infty(x - tv, v)
\]

and

\[
A_x := \int_{\theta=0}^{1} \nabla_x f_0(x - tv + \theta Y_0 + (1 - \theta)Y_\infty, v + \theta W_0 + (1 - \theta)W_\infty) d\theta,
\]

\[
A_v := \int_{\theta=0}^{1} \nabla_v f_0(x - tv + \theta Y_0 + (1 - \theta)Y_\infty, v + \theta W_0 + (1 - \theta)W_\infty) d\theta,
\]

\[
B := \int_{\theta=0}^{1} \nabla_v M_0(v + \theta W_0 + (1 - \theta)W_\infty) d\theta.
\]

By our choice of background \( M_0 \) in (1.3) and the assumptions on the initial data (2.24), together with the estimate (3.28) in Lemma 3.6, which ensures that \( \langle v \rangle \sim \langle v + \tilde{W} \rangle \), we directly see that

\[
\|\langle v \rangle A_x\|_{L^2_{\infty,y}} + \|\langle v \rangle A_v\|_{L^2_{\infty,y}} \lesssim \varepsilon_0, \quad \|\langle v \rangle B\|_{L^2_{\infty,y}} \lesssim 1.
\]

Letting \( f_\infty := G \), together with the bound (8.3) for \( \delta Y \) and \( \delta W \) the claim follows. \( \Box \)

It remains to prove Lemma 8.3.

Proof of Lemma 8.3. Given \( 0 \leq s \leq t \leq t' \), we can write

\[
\Delta_{s,t',t'}(x,v) := \tilde{Y}(x,v,s,t') - \tilde{Y}(x,v,s,t)
\]

\[
= \int_{\tau=t}^{t'} (\tau - s) E(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) d\tau
\]

\[
+ \int_{\tau=s}^{t} (\tau - s) \left[ E(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) - E(x + \tau v + \tilde{Y}(x,v,\tau,t), \tau) \right] d\tau
\]

\[
= e_{s,t,t'}(x,v) + r_{s,t,t'}(x,v).
\]

A crude estimate gives

\[
|r_{s,t,t'}(x,v)| \lesssim \int_{\tau=s}^{t} \tau \|\nabla E(\tau)\|_{L^\infty_x} \cdot |\Delta_{\tau,t,t'}(x,v)| d\tau
\]

and observing that

\[
g(\tau) := \tau \|\nabla E(\tau)\|_{L^\infty_x} \lesssim \varepsilon_1(\tau)^{2\delta - 2}
\]

belongs to \( L^1_\tau \) and using Grönwall’s Lemma, we see that it suffices to bound \( e_{s,t,t'}(x,v) \) by \( \varepsilon_1 \langle t \rangle^{-\frac{1}{2}} \) to prove the claim for \( \tilde{Y} - \tilde{Y}_\infty \). To obtain this, we can decompose

\[
e_{s,t,t'}(x,v) = e_{s,t,t'}^{stat}(x,v) + \Re (e_{s,t,t'}^{osc}(x,v)),
\]

\[
e_{s,t,t'}^{(s)}(x,v) = \int_{\tau=t}^{t'} (\tau - s) E^s(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) d\tau.
\]

On the one hand, we see that

\[
|e_{s,t,t'}^{stat}(x,v)| \lesssim \int_{\tau=t}^{t'} \tau \|E^{stat}(\tau)\|_{L^\infty_x} d\tau \lesssim \varepsilon_1 \int_{\tau=t}^{t'} (\tau)^{2\delta - 2} d\tau \lesssim \varepsilon_1 \langle t \rangle^{2\delta - 1}.
\]
We now consider
\[ \varepsilon_{s,t,t'}^{osc}(x,v) = \int_{\tau=t}^{\tau=t'} (\tau - s) e^{-i\tau} E_{osc}^\tau(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) d\tau \]
and do a direct normal form transformation to get
\[ \varepsilon_{s,t,t'}^{osc}(x,v) = \left[ i(\tau - s) e^{-i\tau} E_{osc}^\tau(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) \right]_{\tau=t}^{\tau=t'} \]
\[ - i \int_{\tau=t}^{\tau=t'} e^{-i\tau} E_{osc}^\tau(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) d\tau \]
\[ - i \int_{\tau=t}^{\tau=t'} e^{-i\tau(\tau - s)} (\partial_\tau + |\nabla|) E_{osc}^\tau(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) d\tau \]
\[ - i \int_{\tau=t}^{\tau=t'} e^{-i\tau(\tau - s)} \left( \varepsilon_j^\tau \cdot \partial_{\varepsilon_j} + |\nabla| \right) \nabla \Delta^{-1} \rho_{osc}^\tau(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) d\tau. \]

Another crude integration gives that
\[ |\varepsilon_{s,t,t'}^{osc}(x,v)| \lesssim \varepsilon_1 \langle t \rangle^{2\delta - 1}. \]

We can control the \( \tilde{W} \) variation using a similar decomposition:
\[ |\tilde{W}(x,v,s,t') - \tilde{W}(x,v,s,t)| \leq \int_{\tau=t}^{\tau=t'} \left| E(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) \right| d\tau \]
\[ + \int_{\tau=s}^{\tau=t} \left| E(x + \tau v + \tilde{Y}(x,v,\tau,t'), \tau) - E(x + \tau v + \tilde{Y}(x,v,\tau,t), \tau) \right| d\tau \]
\[ \lesssim \varepsilon_1 \int_{\tau=t}^{\tau=t'} \langle \tau \rangle^{2\delta - 2} d\tau + \int_{\tau=s}^{\tau=t} \| \nabla E(\tau) \|_{L^\infty} \cdot |\Delta_{t,t',t}(x,v)| d\tau \]
\[ \lesssim \varepsilon_1 \langle t \rangle^{2\delta - 1} + \varepsilon_1^2 \langle t \rangle^{2\delta - 1} \int_{\tau=s}^{\tau=t} \langle \tau \rangle^{2\delta - 3} d\tau. \]

\[ \square \]

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