On Subspace-recurrent Operators

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Abstract. In this article, subspace-recurrent operators are presented and it is showed that the set of subspace-transitive operators is a strict subset of the set of subspace-recurrent operators. We demonstrate that despite subspace-transitive operators and subspace-hypercyclic operators, subspace-recurrent operators exist on finite dimensional spaces. We establish that operators that have a dense set of periodic points are subspace-recurrent. Especially, if $T$ is an invertible chaotic or an invertible subspace-chaotic operator, then $T^n$, $T^{-n}$ and $\lambda T$ are subspace-recurrent for any positive integer $n$ and any scalar $\lambda$ with absolute value 1. Also, we state a subspace-recurrence criterion.

1 Introduction and Preliminaries

For a given Banach space $X$ and for a given vector $x \in X$, the orbit of $x$ under $T$ is signified by $\text{orb}(T, x) = \{x, Tx, T^2x, \ldots\}$. If there exists an element $x \in X$ such that $\text{orb}(T, x) = X$ for a bounded and linear operator $T$, then $T$ is called hypercyclic. The notion of hypercyclicity is related to the closed subspace problem and studied by mathematicians for years.

A related topic to hypercyclicity is topological transitivity. Let $U$ and $V$ be two open sets of $X$. A bounded and linear operator $T$ is named topologically transitive if $T^{-n}U \cap V \neq \emptyset$ for some nonnegative integer $n$. It is well known that hypercyclicity and topological transitivity are equivalent on a complete and separable metric space $X$. One can see [6] and [11] for more information. Another central notion in the dynamical system is recurrence. A bounded and linear operator $T : X \to X$ is called recurrent if for any $U \subseteq X$ that is nonempty and open, we have $T^{-n}U \cap U \neq \emptyset$ for some positive integer $n$. That means a recurrent operator, send back to itself, for any open set $U$.

It is not hard to see that transitive operators are recurrent. Also, a vector $x$ is named a recurrent vector for $T$ if we can construct an increasing sequence $\{n_k\}$ that make of positive integers such that $T^{n_k}x \to x$ when $k \to \infty$.

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It is established that any bounded and linear operator on a compact metric space, have some recurrent vectors ([8]).

The notion of recurrence was introduced many years ago in [10] and [7]. It is also considered recently by authors like Glasner [9], Costakis and Parissis [4] and Chen [3]. One can find interesting theorems about the recurrence of composition operators in [17]. Also, Bonilla et al. introduced the concept of frequent recurrence in [2].

Subspace-hypercyclic operators and subspace-transitive operators were presented in [13] by Madore and Martinez-Avendano and by this, they extended the notion of hypercyclic operators and transitive operators. Let $M$ be a closed and nonempty subspace of $X$. A bounded and linear operator $T$ on $X$ is named subspace-hypercyclic with respect to $M$ if there is $x \in X$ so that $\text{orb}(T, x) \cap M = M$. Also, we say $T$ is $M$-transitive if for arbitrary nonempty open subsets $U, V$ of $M$, non-negative integer $n$ can be find such that $T^{-n}U \cap V$ contains a nonempty and relatively open subset of $M$. It is demonstrated in [13] that subspace-transitive operators are subspace-hypercyclic. We can construct subspace-hypercyclic operators that can not be hypercyclic but authors in [1] proved that any hypercyclic operator is subspace-hypercyclic. One can also see [14] and [15].

Now, we interested in knowing that for a closed subspace $M$ of $X$, if for an operator $T$ on $X$ we have $T^{-n}U \cap U$ be nonempty and relatively open for any relatively open and nonempty set $U$ and some positive integer $n$, and call it subspace-recurrence, then what other properties does $T$ has? Clearly, subspace-transitive operators are subspace-recurrent. But is the converse true? Are that mean these operators necessarily subspace-transitive?

Also, authors in [13] showed that subspace-hypercyclic operators and consequently subspace-transitive operators do not exist on finite dimensional spaces. Does it true for subspace-recurrent operators?

Chaotic operators are an important subset of transitive operators. Remind that we know an operator $T$ on a Banach space $X$ as a chaotic operator, if it is transitive and its periodic points be a dense set in $X$. Also, subspace-chaotic operators are an important subset of subspace-transitive operators. Recall that we say a bounded and linear operator $T$ on $X$ is $M$-chaotic if it is $M$-transitive and the set of its periodic points in $M$ is dense in $M$ ([16]). In this paper, we also want to know the relations between subspace-chaotic and subspace-recurrent operators.

In this paper, $X$ denotes an $F$-space, a complex and complete metrizable topological vector space. Also, $B(X)$ indicates the set of all bounded linear operators on $X$ and we call its elements, by operators. In the article, $M$ shows a nonzero and closed subspace of $X$.

In Section 2, we present some examples of subspace-recurrent operators. We show that there are subspace-recurrent operators that are not subspace-transitive and by this, we conclude that the set of subspace-transitive operators is a strict subset of the set of subspace-recurrent operators. We
also define subspace-recurrent vectors and prove that an operator is $M$-recurrent if and only if it has a dense set of $M$-recurrent vectors.

In Section 3, we prove that if an operator has a dense set of periodic points, then it is subspace-recurrent. Especially, if $T$ is an invertible and chaotic or an invertible subspace-chaotic operator, then $T^n$, $T^{-n}$ and $\lambda T$ are subspace-recurrent for any positive integer $n$ and any scalar $\lambda$ with $|\lambda| = 1$. We show that surprisingly, subspace-recurrent operators exist on finite dimensional spaces.

In Section 4, we give some conditions that under them, the operator becomes subspace-recurrent. Especially, we state a subspace-recurrent criterion.

2 Insight into Subspace-recurrent Operators

At the beginning of this section, we say the main definition.

**Definition 1.** An operator $T$ is said $M$-recurrent or subspace-recurrent with respect to $M$ if for every open and nonempty subset $U$ of $M$, a positive integer $n$ can be find such that $T^{-n}(U) \cap U$ is nonempty and open in $M$.

It is mentioned in Section 1 that subspace-transitive operators are subspace-recurrent. So, we can make an example as follows.

**Example 1.** Assume that $B$ is the backward shift on $l^2$. Let $T = 5B$, and consider

$$M := \{ (a_n)_{n=0}^\infty : a_{2n} = 0 \text{ for all } n \}.$$ 

By [[13], Example 3.7], $T$ is $M$-transitive and accordingly, it is $M$-recurrent.

The next lemma shows that if $T$ is an $M$-recurrent operator, then for open and nonempty set $U \subseteq M$, $T^{-n}(U)$ hit $U$ for infinitely many $n$.

**Lemma 2.1.** Suppose that $T \in B(X)$ is $M$-recurrent. Then,

$$\{ n \in \mathbb{N} : T^{-n}(U) \cap U \text{ is nonempty and open in } M \}$$

is infinite for any open and nonempty subset $U$ of $M$.

**Proof.** Suppose, on the contrary, that the set

$$\{ n \in \mathbb{N} : T^{-n}(U) \cap U \text{ is nonempty and open in } M \}$$

is finite.
is finite for an open set \( U \subseteq M \). Without loose of generality, we can take that
\[
\{ n \in \mathbb{N} : T^{-n}(U) \cap U \text{ is nonempty and open in } M \} = \{ 1, 2, 3, ..., k \}.
\]
So, \( T^{-k}(U) \cap U \) is a nonempty and open subset of \( M \). By definition of an \( M \)-recurrent operator, a natural number \( m \) can be find so that
\[
T^{-m}(T^{-k}(U) \cap U) \cap (T^{-k}(U) \cap U)
\]
is nonempty and open in \( M \). Hence, \( T^{-(m+k)}(U) \cap U \neq \emptyset \) and open in \( M \). But \( m + k \) is greater than \( k \) and this is a contradiction. \( \square \)

Now, we define a subspace-recurrent vector as follows.

**Definition 2.** If an increasing sequence \( \{ n_k \} \) of positive integers exists such that \( T^{n_k}(x) \in M \) and \( T^{n_k}(x) \to x \), where \( x \in M \), we call \( x \) is an \( M \)-recurrent vector. Equivalently, \( x \) is an \( M \)-recurrent vector if for any \( \varepsilon > 0 \) the set \( \{ n \in \mathbb{Z}^+ : T^n(x) \in M \text{ and } d(T^n(x), x) < \varepsilon \} \) is infinite.

We offer the symbol \( \overline{\text{Rec}}_M(T) \) to show the set of all \( M \)-recurrent vectors for the operator \( T \).

The next theorem presents an equivalent condition for subspace-recurrence.

**Theorem 2.1.** For an operator \( T \in B(X) \), \( T \) is \( M \)-recurrent if and only if \( \overline{\text{Rec}}_M(T) = M \).

**Proof.** Consider that \( \overline{\text{Rec}}_M(T) = M \). Allow \( U_M \) be an open and nonempty set in \( M \). By hypothesis, \( \overline{\text{Rec}}_M(T) = M \). So, \( U_M \) includes a recurrent vector like \( y \). But \( U_M \) is open. So, there exists \( \varepsilon > 0 \) such that \( B_M = B(y, \varepsilon) \cap M \subseteq U_M \). By Definition 2, \( n \in \mathbb{N} \) can be find such that \( T^n(y) \in M \) and \( ||T^n(y) - y|| < \varepsilon \). Therefore, \( y \in T^{-n}(U_M) \) and hence, \( y \in T^{-n}(U_M) \cap U_M \).

Now, assume that \( T \) is \( M \)-recurrent. Let \( U = B(x_0, \varepsilon_0) \cap M \) for some \( x_0 \in M \) and \( \varepsilon_0 < 1 \). \( T \) is \( M \)-recurrent. So, there exists \( n_1 \in \mathbb{N} \) such that \( U_1 = T^{-n_1}(U) \cap U \) is nonempty and open in \( M \). Therefore, we can find \( x_1 \in M \) and \( \varepsilon_1 < \frac{1}{2} \) such that
\[
U_2 = B(x_1, \varepsilon_1) \cap M \subseteq U_1 = T^{-n_1}(U) \cap U.
\]
Again, \( U_2 = B(x_1, \varepsilon_1) \cap M \) is open in \( M \). So, we can find \( n_2 \) with \( n_2 > n_1 \) so that \( T^{-n_2}(U_2) \cap U_2 \) is nonempty and open in \( M \). Hence, we can find \( x_2 \in M \) and \( \varepsilon_2 < \frac{1}{2^2} \) such that
\[
U_3 = B(x_2, \varepsilon_2) \cap M \subseteq U_2 = B(x_1, \varepsilon_1) \cap M.
\]
By induction, we can make a sequence \( \{ n_k \} \) that is increasing and their elements are positive integers and we can create a sequence \( \{ \varepsilon_k \} \) of real numbers such that \( \varepsilon_k < \frac{1}{2^k} \) and
\[
B(x_k, \varepsilon_k) \cap M \subseteq B(x_{k-1}, \varepsilon_{k-1}) \cap M.
\]

and
\[
T^{m_k}(B(x_k, \varepsilon_k) \cap M) \subseteq T^{n_k}(T^{-n_k}(B(x_{k-1}, \varepsilon_{k-1}) \cap M) \cap (B(x_{k-1}, \varepsilon_{k-1}) \cap M)) \\
\subseteq B(x_{k-1}, \varepsilon_{k-1}) \cap M.
\]

Now, by Cantor theorem, \( \cap_n (B(x_n, \varepsilon_n) \cap M) = \{y\} \) for some \( y \in M \), since \( M \) is complete. So, \( T^{m_k}(y) \to y \) and hence, \( y \) is an \( M \)-recurrent vector.

In the following lemma, we prove that subspace-recurrence of \( T^p \) for a positive integer \( p \) implies subspace-recurrence of \( T \).

**Lemma 2.2.** Suppose that \( T \in B(X) \) and consider \( p > 1 \) is an integer. Then,

(i) If \( T^p \) is an \( M \)-recurrent operator, then \( T \) is an \( M \)-recurrent operator.

(ii) \( \text{Rec}_M(T^p) \subseteq \text{Rec}_M(T) \).

**Proof.** Proof of (i) is clear by definition. For proving (ii), let \( x \in \text{Rec}_M(T^p) \). Hence, one can find an increasing sequence \( \{n_k\} \) of positive integers with \( (T^p)^{n_k}(x) \in M \) and \( (T^p)^{n_k}(x) \to x \). So, \( T^{pn_k}(x) \to x \) and \( pn_k \) is increasing. Therefore, \( x \in \text{Rec}_M(T) \).

\( \square \)

3 Periodic Points and Subspace-recurrent Operators

In this section, we discuss operators that have a dense set of periodic points. We consider relations between them and subspace-recurrent operators. We begin by saying a lemma. The proof of the lemma is not hard by using the definition of the subspace-recurrent vector.

**Lemma 3.1.** Consider that \( x \in M \) is a periodic point for \( T \). Then \( x \) is an \( M \)-recurrent vector for \( T \).

By Lemma 3.1, we can present the subsequent example.

**Example 2.** Assume that \( T = 2B \), where \( B \) is the backward shift on \( l^2 \). It is shown in [16] that \( T \oplus I \) is subspace-chaotic with respect to \( M := l^2 \oplus \{0\} \). So, the set of periodic points for \( T \) in \( M \) is dense in \( M \). But any periodic point is an \( M \)-recurrent vector by Lemma 3.1. So, \( T \) has a dense set of \( M \)-recurrent vectors. Hence, \( T \) is an \( M \)-recurrent operator by Theorem 2.1.

Moreover, we can extend our statements as it is shown in the next theorem.
**Theorem 3.1.** Let $T \in B(X)$. Suppose that the set of periodic points of $T$ is dense in $X$. Then $T$ is subspace-recurrent with respect to a closed and nontrivial subspace $M$ of $X$.

**Proof.** Assume that $Per(T)$ is the set of periodic points for $T$. According to the hypothesis, $Per(T) = X$. So, by [11, Theorem 2.1], one can find a closed and nontrivial subspace $M$ of $X$ such that $Per(T) \cap M = M$. Hence, the set of periodic points of $T$ in $M$ is dense in $M$ and so, $T$ has a dense set of $M$-recurrent vectors in $M$. Consequently, $T$ is an $M$-recurrent operator by Theorem 2.1.

It is established in [13] that subspace-hypercyclic operators and consequently subspace-transitive operators do not exist on finite dimensional spaces. But in the next example, we make examples of subspace-recurrent operators on finite dimensional spaces.

**Example 3.** (a) The tent map $T : [0, 1] \to [0, 1]$ is determined by,

$$T(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2}; \\ 2 - 2x, & \frac{1}{2} < x \leq 1. \end{cases}$$

The tent map has a periodic point in any interval $[\frac{m}{2^n}, \frac{m+1}{2^n}]$ [[11], Example 1.24]. Hence, it's periodic points make a dense subset of $[0, 1]$.

(b) Recall that a rotation $T : \mathbb{T} \to \mathbb{T}$ is defined by $z \to e^{i\alpha}z$ where $\alpha \in [0, 2\pi)$. If $T$ is a rational rotation, we can detect $N \geq 1$ such that $T^N = I$ [[11], Example 1.24]. So, every point is a periodic point for a rational rotation.

By Theorem 3.1, these operators are subspace-recurrent.

By Example 3, and this fact that subspace-hypercyclic operators and subspace-transitive operators don't exist on finite dimensional spaces, we can say the following corollaries.

**Corollary 3.2.** There are finite dimensional spaces that support subspace-recurrent operators.

**Corollary 3.3.** There exist subspace-recurrent operators, where they are not subspace-hypercyclic nor subspace-transitive.

In fact, we can deduce that the set of subspace-transitive operators is a proper subset of subspace-recurrent operators.

The next theorem shows that if the set of periodic points of $T$ is dense in $M$, then $T^n$, $\lambda T$ with $|\lambda| = 1$ and $T^{-n}$, when $T$ is invertible, are all $M$-recurrent.

**Theorem 3.4.** Consider that $T \in B(X)$. If the set of periodic points of $T$ is dense in $M$, then:
(i) $T^n$ is an $M$-recurrent operator, for every $n \in \mathbb{N}$.

(ii) $\lambda T$ is $M$-recurrent, where $\lambda \in \mathbb{C}$ and $|\lambda| = 1$.

(iii) $T^{-n}$ is $M$-recurrent for every $n \in \mathbb{N}$, when $T$ is invertible.

**Proof.** For proving (i), let $n$ be a positive integer greater than 1. It is sufficient to show that $T^n$ has a dense set of periodic points in $M$. Let $x \in M$ be a periodic point for $T$. So, we can detect $p \in \mathbb{N}$ so that $T^p(x) = x$. In fact,

$$(T^n)^p(x) = T^{np}(x) = \underbrace{T^n \cdots T^n}_p(x) = x.$$ 

This completes the proof.

For proving (ii), note this point that if $x$ is any periodic point for $T$, then for $\lambda$ with $|\lambda| = 1$, $\frac{x}{\lambda}$ is a periodic point for $\lambda T$. For this, suppose that $T^p(x) = x$. Note that,

$$\left(\lambda T\right)^p\left(\frac{x}{\lambda}\right) = \lambda^p T^p\left(\frac{x}{\lambda}\right) = \lambda^p \frac{1}{\lambda^p} T^p(x) = x.$$ 

Also, we know that:

$$\left\{\frac{x}{\lambda} : x \in Per(T)\right\} = \frac{1}{\lambda}\left\{x : x \in Per(T)\right\} = \left\{x : x \in Per(T)\right\} = M.$$ 

So, $\lambda T$ has a dense set of periodic points in $M$ and hence, it is an $M$-recurrent operator.

For proving (iii), let $x \in M$ be a periodic point for $T$. Consequently, we can discover $p \in \mathbb{N}$ such that $T^p(x) = x$. So, $T^{-p}(T^p(x)) = T^{-p}(x)$. But $T$ is invertible. Therefore, $T^{-p}(T^p(x)) = x$ and hence, $T^{-p}(x) = x$. So, $x$ is a periodic point for $T^{-1}$. Then $Per(T^{-1}) \cap M$ is dense in $M$. Similar to (i), we have $T^{-n}$ is $M$-recurrent.

\[\square\]

Costakis, Manoussos and Parissis proved in [5] that if $T$ be an invertible operator, then the recurrence of $T$ and $T^{-1}$ are equivalent. By Theorem 3.4, we can conclude that if $T$ is invertible and its periodic points in $M$ are dense in $M$, then $T$ is $M$-recurrent if and only if $T^{-1}$ is $M$-recurrent.

Now, the following question arises:

**Question:** Assume that $T$ is an invertible operator. Can we infer that $M$-recurrence of $T$ and $T^{-1}$ are equivalent?

Also, we have the following corollaries from Theorem 3.4.

**Corollary 3.5.** Consider that $T \in B(X)$ is an invertible operator. If $T$ is an $M$-chaotic operator, then:
(i) $T^n$ is an $M$-recurrent operator, for any $n \in \mathbb{N}$.

(ii) $\lambda T$ is an $M$-recurrent operator, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

(iii) If $T$ is invertible, then $T^{-n}$ is an $M$-recurrent operator for any $n \in \mathbb{N}$.

Proof. If $T$ is an $M$-chaotic operator, by definition of a subspace-chaotic operator, $T$ has a dense set of periodic points in $M$. So, Theorem 3.4 completes the proof.

Corollary 3.6. Consider that $T \in B(X)$ is an invertible operator. If $T$ is a chaotic operator, then:

(i) $T^n$ is subspace-recurrent, for any $n \in \mathbb{N}$.

(ii) $\lambda T$ is subspace-recurrent operator, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

(iii) If $T$ is invertible, then $T^{-n}$ is subspace-recurrent for any $n \in \mathbb{N}$.

Proof. According to the hypothesis, $T$ is chaotic. So, by definition of a chaotic operator, $T$ has a dense set of periodic points in $X$. Like to proof of Theorem 3.4, $T^n$, has a dense set of periodic points in $X$. By Theorem 3.1, there is a closed and nontrivial subspace $M_n$ of $X$ such that $T^n$ is $M_n$-recurrent. Similarly, $\lambda T$, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $T^{-n}$ are subspace-recurrent.

By Corollary 3.6, we can make examples as follows.

Example 4. Recall that the Birkhoff’s operator on $H(\mathbb{C})$, space of entire functions, is defined by

$$T_a f(z) = f(z + a), \quad a \neq 0.$$  

Birkhoff’s operators are chaotic[11], Example 2.35. Therefore, for any $n \in \mathbb{N}$ and any scalar $\lambda$ with $|\lambda| = 1$, $T_a^n$, $T_{-a}^n$ and $\lambda T_a$ are subspace-recurrent.

4 Some Sufficient Conditions for Subspace-recurrence

In this section, we give conditions for an operator to be subspace-recurrent. We state a subspace-recurrence criterion and we construct an example by this criterion.

Lemma 4.1. Assume that $T \in B(X)$ and assume that $Z$ is a dense subset of $M$. If we can build an increasing sequence $\{n_k\}$ of positive integers so that:

(i) $T^{n_k} x \to x$ for any $x \in Z$,
(ii) \( T^{n_k}(M) \subseteq M \),
then \( T \) is \( M \)-recurrent.

Proof. Consider \( U \subseteq M \) is a relatively open and nonempty set. By hypothesis, \( Z \) is dense in \( M \). So, there is \( x \in U \cap Z \). By condition (i), we can detect a positive integer \( n_k \) such that \( T^{n_k}x \in U \). Consequently, \( x \in T^{-n_k}(U) \cap U \). But \( T^{n_k}(M) \subseteq M \) and hence, \( T^{n_k}|_M : M \to M \) is continuous. Therefore \( T^{-n_k}(U) \) is open in \( M \). Hence, \( T^{-n_k}(U) \cap U \) is nonempty and open in \( M \).

\[ \Box \]

**Theorem 4.1.** (Subspace-recurrence Criterion) Assume that \( T \in B(X) \) and assume that \( M \) is a closed and nonzero subspace of \( X \). Suppose that a dense set \( Z \) of \( M \) and an increasing sequence \( \{ n_k \} \) of positive integers are existed so that:

(i) \( T^{n_k}x \to 0 \) for any \( x \in Z \),

(ii) For every \( x \in Z \), a sequence \( \{ x_k \} \) can be determined such that \( x_k \in M \) and \( x_k \to 0 \) and \( T^{n_k}x_k \to x \),

(iii) \( T^{n_k}(M) \subseteq M \).

Then \( T \) is \( M \)-recurrent.

Proof. Let \( U \subseteq M \) be a relatively open set. By hypothesis, \( Z \) is dense in \( M \). So, \( x \in U \cap Z \) and \( \varepsilon > 0 \) can be find such that \( B(x, \varepsilon) \cap M \subseteq U \).

By (i), \( T^{n_k}x \to 0 \) and by (ii), there exists a sequence \( \{ x_k \} \) in \( M \) such that \( x_k \to 0 \) and \( T^{n_k}x_k \to x \). Hence, we can find a positive integer \( k \) such that:

\[ ||T^{n_k}(x)|| < \frac{\varepsilon}{2}, ||x_k|| < \varepsilon \text{ and } ||T^{n_k}(x_k) - x|| < \frac{\varepsilon}{2}. \]

Now,

\[ ||(x + x_k) - x|| = ||x_k|| < \varepsilon. \]

Therefore, \( x + x_k \in U \). Also, we have:

\[ ||T^{n_k}(x + x_k) - x|| = ||T^{n_k}(x) + T^{n_k}(x_k) - x|| \leq ||T^{n_k}(x)|| + ||T^{n_k}(x_k) - x|| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Hence, \( T^{n_k}(x + x_k) \in U \) and then \( x + x_k \in T^{-n_k}(U) \cap U \). But \( T^{-n_k}(U) \) is open in \( M \), by the continuity of since \( T^{n_k}|_M \). Therefore, \( T^{-n_k}(U) \cap U \) is nonempty and open in \( M \).

\[ \Box \]
Example 5. Let $T = \lambda B$, where $\lambda$ is a scalar with $|\lambda| > 1$. Let

$$M := \{\{a_n\} \in l^2 : a_{3k} = 0 \text{ for all } k\}.$$ 

Consider $Z$ be the set consists of every finite sequence in $l^2$. So, $Z$ is dense in $l^2$. Let $x \in Z$. Then, we can detect a natural number $m$ such that for any $k > m$, we have $x_k = 0$. Hence, $T^{m_k}x \to 0$. But $x$ is an arbitrary element of $Z$. Hence, condition (i) of Theorem 4.1 holds.

Now, let $S$ be the forward shift on $l^2$ and let $x_k = \frac{1}{\lambda^{3k}}S^{3k}x$, where $x$ is an arbitrary and fix element of $Z$. It is not hard to see that $x_k \in M$. Also, since $|\lambda| > 1$, we have

$$||x_k|| = \frac{1}{|\lambda^{3k}|}||x|| \to 0.$$ 

On the other hand:

$$T^{3k}(x_k) = T^{3k}(\frac{1}{\lambda^{3k}}S^{3k}x) = (\lambda B)^{3k}(\frac{1}{\lambda^{3k}}S^{3k}x) = x.$$ 

So, condition (ii) of Theorem 4.1 holds.

Also, for any $x \in M$,

$$T^{3k}(x_0, x_1, x_2, x_3, \ldots, x_6, \ldots, x_{3n}, \ldots) = (x_{3k}, x_{3k+1}, x_{3k+2}, x_{3k+3}, \ldots).$$ 

and $x_{3n} = 0$ for any $n$. Consequently $T^{3k}(M) \subseteq M$. Hence, condition (iii) of Theorem 4.1 holds and therefore $T$ is an $M$-recurrent operator.

References

[1] N. Bamerni, V. Kadets and A. Kilicman, Hypercyclic operators are subspace-hypercyclic, J. Math. Anal. Appl., 435 (2016), 1812-1815.

[2] A. Bonilla, K. G. Grosse-Erdmann, A. Lopez-Martinez and A. Peris, Frequently recurrent operators, arXiv:2006.11428v1.

[3] C. C. Chen, Recurrence of cosine operator functions on groups, Canad. Math. Bull., 59 (2016), 693–704.

[4] G. Costakis and I. Parissis, Szemeredi’s theorem, frequent hypercyclicity and multiple recurrence, Math. Scand., 110 (2012), 251–272.

[5] G. Costakis, A. Manoussos and I. Parissis, Recurrent linear operators, Complex. Anal. Oper. Th., 8 (2014), 1601–1643.
[6] C. T. J. Dodson, A review of some recent work on hypercyclicity, Balk. J. Geo. App., 19 (2014), 22–41.

[7] H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory, Princeton University Press, 1981.

[8] H. Furstenberg, Poincare recurrence and number theory, B. Am. Math. Soc., 5 (1981), 211–234.

[9] E. Glasner, Classifying dynamical systems by their recurrence properties, Journal of the Juliusz Schauder Center, 24 (2004), 21–40.

[10] W. H. Gottschalk and G. H. Hedlund, Topological dynamics, American Mathematical Society, 1994.

[11] K. G. Grosse-Erdmann and A. Peris Manguillot, Linear chaos, Springer, 2011.

[12] C. M. Le, On subspace-hypercyclic operators, Proc. Amer. Math. Soc., 139 (2011), 2847–2852.

[13] B. F. Madore and R. A. Martinez-Avendano, Subspace hypercyclicity, J. Math. Anal. Appl., 373 (2011), 502–511.

[14] R. A. Martinez-Avendano and O. Zatarain-Vera, Subspace-hypercyclicity for Toeplitz operators, J. Math. Anal. Appl., 422 (2015), 60–68.

[15] M. Moosapoor, Common subspace-hypercyclic vectors, Int. J. Pure Appl. Math., 118 (2018), 865–870.

[16] S. Talebi and M. Moosapoor, Subspace-chaotic operators and subspace-weakly mixing operators, Int. J. Pure Appl. Math., 78 (2012), 879–885.

[17] Z. Yin, Chaotic dynamics of composition operators on the space of continuous functions, Int. J. Bifurcat. Chaos, 27 (2017), 1–12.

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