ALMOST GLOBAL EXISTENCE FOR THE KLEIN-GORDON EQUATION WITH THE KIRCHHOFF-TYPE NONLINEARITY

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Abstract. We prove an almost global existence result for the Klein-Gordon equation with the Kirchhoff-type nonlinearity on $\mathbb{T}^d$ with Cauchy data of small amplitude $\epsilon$. We show a lower bound $\epsilon^{2N-2}$ for the existence time with any natural number $N$. The proof relies on the method of normal forms and induction. The structure of the nonlinearity is good enough that proceeds normal forms up to any order.

1. Introduction. In this paper, we consider the following Cauchy problem of the Kirchhoff-type Klein-Gordon equation

$$\begin{cases}
(\partial_t^2 - \Delta + m^2)v = \langle \nabla v, \nabla v \rangle \Delta v, & (t, x) \in \mathbb{R} \times \Omega, \\
v(0, x) = \epsilon v_0(x), & v_t(0, x) = \epsilon v_1(x),
\end{cases}
$$

(KF)

with periodic boundary conditions $\Omega = \mathbb{T}^d$, where $0 < \epsilon \ll 1, m \geq 0$ and

$$\langle f, g \rangle := \int_{\Omega} f(x)g(x) \, dx \quad \forall f, g \in L^2(\Omega, \mathbb{C}).$$

(Here, we remind readers that our notation $\langle f, g \rangle$ is different from the standard ones, where $\langle f, g \rangle = \int_{\Omega} f(x)g(x) \, dx = \int_{\Omega} f(x)g(x) \, dx$ for any $f, g \in L^2(\Omega, \mathbb{C}).$)

The classical integro-differential equation (KF) ($m = 0$) was introduced in 1876 by Kirchhoff ([8]) as a nonlinear model of the free transversal vibrations of a clamped string. The first mathematical results on the Kirchhoff equation are due to Bernstein ([2]), who considered the Cauchy problem in one space dimension and proved local wellposedness in $H^2 \times H^1$, as well as the global wellposedness for analytic initial data. Later on, the Bernstein’s local existence result was extended to several space dimensions on compact domains or non compact domains.

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There are many known results on the Cauchy problem in the background of Euclidean spaces. The global existence of the solution is obtained in weighted Sobolev spaces. We refer readers to [11] for references concerning those results. For the results on exterior domains, we refer readers to [10, 12] and references therein.

On compact domains, the situation becomes quite different since no dispersion is available for linear equation, due to the fact that the Laplace operator on a compact manifold has pure point spectrum. The local wellposedness in the Sobolev class $H^\frac{s}{2} \times H^\frac{s}{2}$ has been proved by Dickey [4], Medeiros and Miranda [9] and Arosio and Panizzi [1], with existence time of order $\epsilon^{-2}$. Recently, Baldi and Haus [3] proved a lower bound $\epsilon^{-4}$ for the existence time whose proof replies on a normal form transformation. A motivation for this paper arises from the problem whether or not the lifespan of solution of the Cauchy problem is $\epsilon^{-N}$ with uniformly bounded $H^s$ norm, for any $N$ and large enough $s$.

In principle, applied the normal form transformation in [3] by induction, the length of lifespan can be improved better. However, the process is prohibited since one has to deal with the resonant in frequency, where

$$\sum_{i=1}^{p} |n_i| - \sum_{j=p+1}^{q} |n_j| = 0.$$  

In order to overcome the resonant, we introduce a parameter $m$ into the linear operator of the (KF) equation. Actually, it is a Klein-Gordon equation with the Kirchhoff-type nonlinearity, whose solution is a $\mathbb{R}$--valued function.

Before stating our results, we introduce the Sobolev space of zero-mean functions

$$H^0_0(\Omega, \mathbb{R}) := \{ u(x) \in \mathbb{R} | u(x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} u_n e^{inx}, u_n \in \mathbb{C}, \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |n|^{2s}|u_n|^2 < \infty \}. \quad (1.1)$$

Actually, we can split the Cauchy Problem of (KF) into two distinct uncoupled Cauchy Problems. For any initial data $v_0(x)$ and $v_1(x)$, we can see

$$v_0(x) = v_0 + \tilde{v}_0(x), \quad v_1(x) = v_1 + \tilde{v}_1(x), \quad (1.2)$$

where $v_0 = \int_{\Omega} v_0(x) dx$ and $v_1 = \int_{\Omega} v_1(x) dx$. Then the unknown solution $v(t, x)$ is split into the sum of a zero-mean function $\tilde{v}(t, x)$ and the average term $v(t)$.

The average term

$$v(t) = \begin{cases} \epsilon v_0 \cos mt + \frac{\epsilon v_1}{m} \sin mt, & \text{if } m \neq 0; \\ \epsilon v_0 + \epsilon v_1 t, & \text{if } m = 0, \end{cases} \quad (1.3)$$

is the unique solution of

$$\begin{cases} v''(t) + m^2 v = 0, \\ v(0) = \epsilon v_0, v'(0) = \epsilon v_1. \end{cases} \quad (1.4)$$

Thus one has to study the Cauchy Problem for the zero-mean unknown $\tilde{v}(t, x)$ with zero-mean initial data $\epsilon \tilde{v}_0(x)$ and $\epsilon \tilde{v}_1(x)$. For convenience, we still denote $H^0_0(\Omega, \mathbb{R})$ as $H^s(\Omega)$.

The main results of this paper are the following:

**Theorem 1.1.** Let $d \in \mathbb{N}$, there is a zero measure subset $\mathcal{N}$ of $(0, +\infty)$, for any $m \in (0, +\infty) - \mathcal{N}$ and any $N \in \mathbb{N}$, there exists $s_0 > 0$, for any $s \geq s_0$, there are $\epsilon_0 > 0$ and constant $c > 0$, if $(v_0, v_1)$ belongs to the unit ball of $H^{s+1}(\Omega) \times H^s(\Omega)$,
for any $\epsilon \in (0, \epsilon_0)$, the solution $v$ of (KF) exists at least over a time interval of length $c\epsilon^{-2N-2}$ with the uniform bound.

**Theorem 1.2.** Let $m = 0$ and

$$s_0 = \frac{1}{2} \quad \text{if} \quad d = 1, \quad s_0 = 1 \quad \text{if} \quad d \geq 2.$$  

For any $s \geq s_0$, there exist $\epsilon_0 > 0$ and constant $c > 0$, if $(v_0, v_1)$ belongs to the unit ball of $H^{s+1}(\Omega) \times H^s(\Omega)$, then for any $\epsilon \in (0, \epsilon_0)$, the solution $v$ of (KF) exists at least over a time interval of length $c\epsilon^{-4}$ with the uniform bound.

**Remark 1.** Our conclusion mainly shows the lower bound of length of lifespan for $m > 0$. For the case of $m = 0$, we obtain the same results as in [1, 3, 4, 9]. We follow the strategy introduced in [6, 5], of defining the refined energy, and then using normal forms method and induction. We denote $N$ as the number of induction by applying the normal forms method. If without the normal forms ($N = 0$), the lifespan is order $\epsilon^{-2}$ in $H^\frac{1}{2} \times H^\frac{3}{2}$. And if people apply the normal forms once time ($N = 1$), then the lifespan is modified as $\epsilon^{-4}$ in $H^\frac{2}{2} \times H^\frac{4}{2}$, which coincides with [3] for any dimensions (the value of $\nu$ is given by (2.7)). Furthermore, if we apply the method $N$-times repeatedly, then the length of lifespan can be improved better, which is in order $\epsilon^{-2N-2}$ in $H^{s+1} \times H^s$ for $s \geq \frac{1 + N\nu}{2}$ ($\nu$ is given by Proposition 2).

**Remark 2.** For the case of $m = 0$, as far as we know, the normal form can be used only once time. The reason is that the resonant may happen and we can not obtain the lower bound estimate of $\sum_{k \in K} \rho(k)|n_k|$ as similar as (2.8) for $p \geq 3$, where $\rho$ is defined in Definition 2.2 with $K \subset \{1, \ldots, p\}$.

Let us explain the main idea briefly. We want to control the Sobolev energy of the solutions

$$\frac{d}{dt}(\|v(t, \cdot)\|_{H^{s+1}}^2 + \|\partial_t v(t, \cdot)\|_{H^s}^2). \quad (1.5)$$

Using the equation, this quantity may be written as a sum multi-linear expressions in $v, \nabla v$, having a special property coming from the structure of $\int_{\Omega} |\nabla v|^2 \, dx$. For the multi-linear expression homogeneous order of degree $r$ in (1.5), we perturb the Sobolev energy by an expression of homogeneous order of degree $r$ such that

- its time derivatives cancels out the main contribution in (1.5), up to remainders of higher order;
- the perturbation is bounded by powers of $\|v(t, \cdot)\|_{H^{r+1}}$ and $\|\partial_t v(t, \cdot)\|_{H^r}$.

The difficulty is to find conditions so that

- the existence of perturbed Sobolev energy should be guaranteed;
- the remainders of higher order after the perturbation have the same property as before.

To ensure the first item, where the key point is to avoid causing the resonances, we introduce an extra parameter $m$ by which to settle the invertibility of a small divisor (2.6). In fact, if we let $m = 0$, then the inverse of $|n_1| - |n_2|$ can be obtained if $|n_1| \neq |n_2|$ (case of $p = 2$), which is used in [3] for a normal form transformation. But for the further progress of induction, for example the case of $p = 3$, the resonances may occur that $|n_1| + |n_2| - |n_3| = 0$ even if $|n_1| \neq |n_2| \neq |n_3|$. Fortunately, Proposition 2 shows that with the exception of a zero measure set, almost all $m$ guarantee the invertibility of small divisor (2.6), which was firstly proved by Delort.
and Szeftel in [6, 5] to solve the long-time existence for Klein-Gordon equation with the power type nonlinearity.

The second item allows us to perform such normal form process up to any order, leading to almost global existence of solution, where the proper structure of nonlinearity is essential. Zhang and us introduced a null condition for the power-type nonlinearity in [7]. But for our equation, the Kirchhoff-type nonlinearity is good enough to keep the process. Since by Plancherel’s theorem
\[ \langle \nabla v, \nabla v \rangle = \sum_n |n|^2 |\hat{v}_n|^2, \]
it guarantees a symmetric structure of energy estimate in frequency space. The symmetric structure confirms that the frequency does not fall on the characteristic manifold \( Z_p^p \) for any dimension. The details can be found in Lemma 4.1.

The rest of this paper is organized as follows. In Section 2, we introduce the notation and the key Propositions for the proof. In Section 3, we introduce two elementary linear transformations to equation (KF). The equation (KF) is transformed into a quasilinear system (QLS), whose solutions are \( C^- \)-valued. Section 4 is devoted to long-time existence.

2. Notation and preliminary results. Throughout this paper, all the function spaces we consider are made up of complex-valued functions. For a complex-valued function \( u \), we decompose it into Fourier series
\[ u(t, x) := \sum_{n \in \mathbb{Z}^d} u_n(t, x), \]
where \( u_n(t, x) = e^{inx} \int_\Omega u(t, y)e^{-iny} \, dy \). A simple computation can show that
\[ \Lambda_m^s u_n = (n_m^s u_n, \nabla_x u_n = i nu_n, \partial_t u_n = (\partial_t u)_n, \]
\[ \Lambda_m^s \partial_t u_n = (\nabla_x u_n = -i u_n, \partial_t u_n = (\partial_t u)_n, \]
where \( \Lambda_m^s = (\sqrt{-\Delta_x + m^2})^s \) and \( (n_m^s) = (\sqrt{|n|^2 + m^2})^s \).

For the numbers \( a_1, \cdots, a_p, p \in \mathbb{N} \), we define the second largest element of this family by
\[ \max_2(a_1, \cdots, a_p) = \max \left( \{a_1, \cdots, a_p\} - \{a_{i_0}\} \right), \]
where \( i_0 \) is the index such that
\[ a_{i_0} = \max(a_1, \cdots, a_p). \]

It is convenient to introduce the notation
\[ M(a_1, \cdots, a_p) = \max \{a_1, \cdots, a_p\}, \]
\[ \mu(a_1, \cdots, a_p) = \begin{cases} 1, & \text{if } p = 1, \\ \max_2(a_1, \cdots, a_p) + 1, & \text{if } p \geq 2. \end{cases} \]

Definition 2.1. For \( p \in \mathbb{N} \), let \( K \) be a nonempty set of \( \{1, \cdots, p\} \), which is denoted as
\[ K = \{k_1, \cdots, k_{|K|}\} \text{ satisfying } k_1 < k_2 < \cdots < k_{|K|}. \]

We define a projection mapping \( P_K \) that
\[ P_K : \mathbb{Z}^{d \times p} \to \mathbb{Z}^{d \times |K|} \]
with
\[ P_K(n_1, \cdots, n_p) = (n_{k_1}, \cdots, n_{k_{|K|}}). \]
We now introduce the sets $H_p$ and $Z^p_\rho$, by which we deal with a small divisor by normal forms (see Section 4).

**Definition 2.2.** For $p \in \mathbb{N}$, let $K$ be a nonempty subset of $\{1, \cdots , p\}$. We denote by $H_{zK}$ the set of all mapping

$$\rho : K \to \{+,-\},$$

such that for any $j \in K$,

$$(u_n)^{\rho(j)} = \begin{cases} u_n, & \text{if } \rho(j) = +, \\ u_n^{-1}, & \text{if } \rho(j) = -. \end{cases}$$

Moreover, we set $\overline{H_{zK}}$ be the subset of $H_{zK}$ defined as follows:

- If $zK$ is odd, we set $\overline{H_{zK}} = \emptyset$,
- If $zK$ is even, we set $\overline{H_{zK}} = \{ \rho \in H_{zK} : z\rho^{-1}(+) = z\rho^{-1}(-) \}$.

We define a subspace $Z^{zK}_p$ of $Z^{d\times zK}$ as

- If $zK$ is odd or $zK$ is even and $\rho \notin \overline{H_{zK}}$, we set $Z^{zK}_p = \emptyset$,
- If $zK$ is even and $\rho \in \overline{H_{zK}}$, we set $Z^{zK}_p = \{ \vec{n} \in Z^{d\times zK} : \text{there is } \sigma \in \mathcal{S}_K \text{ with } \sigma^2 = Id, \rho \circ \sigma = -\rho \text{ such that for any } j \in K, |n_j| = |\sigma(n_j)| \}$.

Especially, when $K = \{1, \cdots , p\}$, we denote $H_{zK}, \overline{H_{zK}}$ and $Z^{zK}_p$ as $H_p, \overline{H_p}$ and $Z^p_\rho$, respectively.

Away from the manifold $Z^{zK}_p$, a small divisor function, which variables are $|n_k|$ for all $k \in K$, is denoted as

$$F^\rho_{m,K} := \sum_{k \in K} \rho(k) \sqrt{m^2 + |n_k|^2}. \quad (2.6)$$

When $K = \{1, \cdots , p\}$, we denote $F_{m,K}^\rho$ as $F^\rho_m(|n_1|, \cdots , |n_p|)$.

The next Propositions are the key ingredients in our proof of almost global existence.

**Proposition 1.** Let $\rho \in H_2$. For any $m \geq 0$, there exists $c > 0$ and

$$\nu = 0 \quad \text{if} \quad d = 1, \quad \nu = 1 \quad \text{if} \quad d \geq 2, \quad (2.7)$$

such that for each $(n_1, n_2) \in (\mathbb{Z}^2_\rho)^c$, we have

$$|F^\rho_m(|n_1|, |n_2|)| \geq c\mu(|n_1|, |n_2|)^{-\nu}. \quad (2.8)$$

**Proof.** $(2.8)$ is obviously true for $\rho \notin H_2$ with $c = 1$ and $\nu = 0$. In the case of $\rho \in H_2$ and $|n_1| \neq |n_2|$, suppose $\mu(|n_1|, |n_2|) = |n_2| + 1$,

$$|F^\rho_m(|n_1|, |n_2|)| = \sqrt{m^2 + |n_1|^2} - \sqrt{m^2 + |n_2|^2} \geq \frac{1}{\sqrt{m^2 + 1}} (|n_1| - |n_2|). \quad (2.9)$$

In dimension $d = 1$, one has $|n_1| - |n_2| \geq 1$. Then $(2.8)$ hold with $c = \frac{1}{\sqrt{m^2 + 1}}$ and $\nu = 0$. And in dimension $d \geq 2$, if $|n_1| - |n_2| \geq 1$, then $|n_1| - |n_2| \geq \frac{1}{|n_2|}$. Let $|n_1| - |n_2| < 1$, then $|n_2| < |n_1| < |n_2| + 1$. So we can observe that

$$|n_1| - |n_2| = \frac{|n_1|^2 - |n_2|^2}{|n_1| + |n_2|} \geq \frac{1}{|n_1| + |n_2|} \geq \frac{1}{3|n_2|} \geq \frac{1}{3} \mu(|n_1|, |n_2|)^{-1}. \quad (2.10)$$
Then (2.8) holds with \( c = \frac{1}{3\sqrt{m+1}} \) and \( \nu = 1 \).

**Proposition 2.** Let \( p \in \mathbb{N} \) and \( \rho \in H_p \), then there exists a zero measure subset \( \mathcal{N} \) of \((0, +\infty)\) such that for every \( m \in (0, +\infty) - \mathcal{N} \), there are constants \( c > 0 \) and \( \nu \in \mathbb{N} \) such that for each \( (n_1, \ldots, n_p) \in (\mathbb{Z}_p^m)^c \), we have

\[
|F_m^p((n_1), \ldots, |n_p|)| \geq c\mu(|n_1|, \ldots, |n_p|)^{-\nu}. \tag{2.11}
\]

**Proof.** Estimate (2.11) is immediate for \( p = 1 \). For \( p \geq 2 \), Proposition 2.1.5 in [5] implied that: there exists a zero measure subset \( \mathcal{N} \) of \((0, +\infty)\), for any \( m \in (0, +\infty) - \mathcal{N} \), there are \( c_0 > 0, N_0 > 0 \) such that for each \( (n_1, \ldots, n_p) \in (\mathbb{Z}_p^m)^c \),

\[
|F_m^p((n_1), \ldots, |n_p|)| \geq c_0(1 + M(|n_1|, \ldots, |n_p|))^{-N_0}. \tag{2.12}
\]

We consider \( \mu(|n_1|, \ldots, |n_p|) < \delta(1 + M(|n_1|, \ldots, |n_p|)) \) with \( \delta \) small enough satisfying \( 16\rho^2(1 + m^2)\delta^2 < 1 \). Since \( \mu(|n_1|, \ldots, |n_p|) \geq 1 \), then \( \delta M(|n_1|, \ldots, |n_p|) \geq 1/2 \) and then

\[
M^2 > 16\rho^2m^2\delta^2M^2 + 16\rho^2\delta^2M^2 > 4\rho^2m^2 + 16\rho^2\delta^2M^2 = 4\rho^2(m^2 + 4\delta^2M^2).
\]

Thus, there exists a constant \( c > 0 \) such that

\[
|F_m^p((n_1), \ldots, |n_p|)| \geq \sqrt{m^2 + M^2} - \sum_{\substack{k=1 \atop |n_k| \neq M}}^p \sqrt{m^2 + |n_k|^2} - \sqrt{m^2 + M^2} - (p - 1)\sqrt{m^2 + (\mu - 1)^2} \geq \sqrt{m^2 + M^2} - p\sqrt{m^2 + 4\delta^2M^2} \geq \frac{M}{2} \geq c\mu^{-1}. \tag{2.13}
\]

If \( \mu(|n_1|, \ldots, |n_p|) \geq \delta(1 + M(|n_1|, \ldots, |n_p|)) \), then it follows from (2.12) that

\[
|F_m^p((n_1), \ldots, |n_p|)| \geq c_0\delta^{N_0}\mu^{-N_0}. \tag{2.14}
\]

Hence, we have proved the estimate (2.11).

**Remark 3.** For any nonempty subset \( K \subseteq \{1, \ldots, p\} \), Proposition 2 is still true for \( F_{m,K}^p \). Indeed, if we denote

\[
K := \{k_1, \ldots, k_{|K|}\} \text{ with } k_1 < k_2 < \cdots < k_{|K|},
\]

then

\[
F_{m,K}^p = \sum_{i=1}^{k_{|K|}} \rho(k_i) \sqrt{m^2 + |n_{k_i}|^2}.
\]

We apply Proposition 2 with \( p = \sharp K \), then the same conclusion holds for \( F_{m,K}^p \).

### 3. The transformations of equation (KF).

In this section, as P. Baldi and E. Haus did in the article [3], we introduce two standard, classical linear transformations that transform the original equation (KF) into a quasilinear system.

We denote

\[
u = (-i\partial_t + \Lambda_m)v, \tag{3.1}
\]

so that

\[
\dot{v} = \Lambda_m^{-1}\frac{u + \bar{u}}{2}, \quad \partial_tv = i\frac{u - \bar{u}}{2}, \tag{3.2}
\]

where \( \bar{u} \) is the conjugate of \( u \).

If we use the Fourier multiplier

\[
|\nabla| : H^s \rightarrow H^{s-1}, \quad e^{ij\cdot x} \rightarrow |j|e^{ij\cdot x}
\]
and set

\[ Q(u, \bar{u}) = \frac{1}{8} \langle |\nabla|^{-1} A_m^{-1}(u + \bar{u}), |\nabla|^{-1} A_m^{-1}(u + \bar{u}) \rangle, \]

(3.3)

then the equation \((KF)\) can be rewritten as

\[
\partial_t \left( \begin{array}{c} u \\ \bar{u} \end{array} \right) = i \left( \begin{array}{c} 1 + Q(u, \bar{u}) \\ -Q(u, \bar{u}) \end{array} \right) A_m \left( \begin{array}{c} u \\ \bar{u} \end{array} \right) \\
- im^2 \left( \begin{array}{cc} Q(u, \bar{u}) & Q(u, \bar{u}) \\ -Q(u, \bar{u}) & -Q(u, \bar{u}) \end{array} \right) \Lambda^{-1} \left( \begin{array}{c} u \\ \bar{u} \end{array} \right) \right) \quad \text{(QLS-u)}
\]

with the initial data

\[
\left( \begin{array}{c} u(0) \\ \bar{u}(0) \end{array} \right) = \epsilon \left( \begin{array}{c} -iv_1(x) + \Lambda_m v_0(x) \\ iv_1(x) + \Lambda_m v_0(x) \end{array} \right).
\]

(3.4)

By diagonalization, for any \(x \geq 0\) one has

\[
\begin{pmatrix} 1 & \theta(x) \\ \theta(x) & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 + x & x \\ -x & 1 - x \end{pmatrix} \begin{pmatrix} 1 & \theta(x) \\ \theta(x) & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{1 + 2x} & 0 \\ 0 & -\sqrt{1 + 2x} \end{pmatrix}
\]

(3.5)

where

\[
\theta(x) := \frac{1 - \sqrt{1 + 2x}}{1 + \sqrt{1 + 2x}}
\]

(3.6)

with the properties

\[-1 < \theta(x) \leq 0 \quad \text{for } x \geq 0, \]

(3.7)

and

\[
\frac{1 - \theta(x)}{1 + \theta(x)} = \sqrt{1 + 2x}.
\]

(3.8)

We define a linear transformation that

\[
\begin{pmatrix} u \\ \bar{u} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},
\]

(3.9)

where \(\mathcal{M} = \mathcal{M}(\theta) := \frac{1}{\sqrt{1 - \theta^2}} \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}\) and \(\theta\) is given by (3.6).

By a simple computation, we can see \(\eta_1 = \eta_2\), and then for convenience, we denote

\[
\begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = \mathcal{M}^{-1} \begin{pmatrix} u \\ \bar{u} \end{pmatrix}.
\]

(3.10)

By the definition of \((\eta, \bar{\eta})\), we have

\[
\eta + \bar{\eta} = \sqrt{\frac{1 - \theta}{1 + \theta}} (u + \bar{u}), \quad \eta - \bar{\eta} = \sqrt{\frac{1 + \theta}{1 - \theta}} (u - \bar{u}).
\]

(3.11)

Then it follows from (3.8) with \(x = Q(u, \bar{u})\) and (3.11) that

\[
Q(\eta, \bar{\eta}) = \sqrt{1 + 2Q(u, \bar{u})} \cdot Q(u, \bar{u}).
\]

(3.12)

The function \(x \mapsto x\sqrt{1 + 2x}\) is invertible, and we denote by \(\varphi\) its inverse,

\[
y = x\sqrt{1 + 2x} \Leftrightarrow x = \varphi(y).
\]

(3.13)

Therefore, one can express \(Q(u, \bar{u})\) in terms of \((\eta, \bar{\eta})\) as

\[
Q(u, \bar{u}) = \varphi(Q(\eta, \bar{\eta})) := P(\eta, \bar{\eta}).
\]

(3.14)
It follows from (3.14) and (3.3) that
\[
\frac{d}{dt} P(\eta, \bar{\eta}) = \frac{d}{dt} Q(u, \bar{u}) = \frac{1}{4} (|\nabla| \Lambda_m^{-1} (u_t + \eta_t), |\nabla| \Lambda_m^{-1} (u + \bar{u})).
\]
And then by the system (QLS-u) and (3.11), we have
\[
\frac{d}{dt} P(\eta, \bar{\eta}) = \frac{i}{4} (|\nabla| (u - \bar{u}), |\nabla| \Lambda_m^{-1} (u + \bar{u})) = \frac{i}{4} \left( (|\nabla| \Lambda_m^{-\frac{1}{2}} \eta, |\nabla| \Lambda_m^{-\frac{1}{2}} \bar{\eta}) - (|\nabla| \Lambda_m^{-\frac{1}{2}} \bar{\eta}, |\nabla| \Lambda_m^{-\frac{1}{2}} \eta) \right).
\]
(3.15)
Subsequently, we express the system (QLS-u) in the terms of \((\eta, \bar{\eta})\). By the linear transformation (3.10),
\[
\partial_t (u, \bar{u}) = \partial_t \{ M(\eta, \bar{\eta}) [\eta, \bar{\eta}] \} = M(\eta, \bar{\eta}) \{ \partial_t \eta, \partial_t \bar{\eta} \} + \frac{d}{dt} \{ M(\eta, \bar{\eta}) \} [\eta, \bar{\eta}],
\]
(3.16)
where \( M(\eta, \bar{\eta}) = M(\theta(P(\eta, \bar{\eta})) \).
It follows from the definition of \( M \) and \( \theta \) that
\[
\frac{d}{dt} \{ M(\eta, \bar{\eta}) \} = \frac{1}{(1 - \theta^2)^\frac{1}{2}} \begin{pmatrix} \theta & 1 \\ 1 & \theta \end{pmatrix} \frac{d}{dt} \theta,
\]
(3.17)
\[
\frac{d}{dt} \theta = \theta'(P) \frac{d}{dt} P = \frac{-i}{2(1 + \sqrt{1 + 2P})} \left( |\nabla| \Lambda_m^{-\frac{1}{2}} \eta, |\nabla| \Lambda_m^{-\frac{1}{2}} \bar{\eta} \right)
\]
(3.18)
\[
\frac{8(1 + 2P)}{\sqrt{1 - \theta^2}} \begin{pmatrix} \theta & 1 \\ 1 & \theta \end{pmatrix}.
\]
(3.19)
According to (3.5), we observe that
\[
\begin{pmatrix} 1 + Q(u, \bar{u}) \\ -Q(u, \bar{u}) \end{pmatrix} \begin{pmatrix} \Lambda_m u \\ \Lambda_m \bar{u} \end{pmatrix} = M(\eta, \bar{\eta}) \sqrt{1 + 2P(\eta, \bar{\eta})} \Lambda_m \begin{pmatrix} \eta \\ -\bar{\eta} \end{pmatrix},
\]
(3.20)
and
\[
-i m^2 \begin{pmatrix} Q(u, \bar{u}) \\ Q(u, \bar{u}) \end{pmatrix} \begin{pmatrix} \Lambda_m^{-1} u \\ \Lambda_m^{-1} \bar{u} \end{pmatrix} = -M(\eta, \bar{\eta}) i m^2 \frac{P}{\sqrt{1 + 2P}} \Lambda_m \begin{pmatrix} \eta + \bar{\eta} \\ -\eta - \bar{\eta} \end{pmatrix}.
\]
(3.21)
Then, the system (QLS-u) can be transformed that
\[
\begin{pmatrix} \partial_t \eta \\ \partial_t \bar{\eta} \end{pmatrix} + M^{-1} \left( \frac{d}{dt} M \right) \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = i \sqrt{1 + 2P} \begin{pmatrix} \Lambda_m \eta \\ -\Lambda_m \bar{\eta} \end{pmatrix} - i \frac{m^2 P}{\sqrt{1 + 2P}} \Lambda_m \begin{pmatrix} \eta + \bar{\eta} \\ -\eta - \bar{\eta} \end{pmatrix}.
\]
(3.22)
The form of \( M^{-1} \) and (3.19) imply the following form:
\[
M^{-1} \left( \frac{d}{dt} M \right) \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix} = \frac{-i}{8(1 + 2P)} \left( |\nabla| \Lambda_m^{-\frac{1}{2}} \eta, |\nabla| \Lambda_m^{-\frac{1}{2}} \bar{\eta} \right) \begin{pmatrix} \eta \\ \bar{\eta} \end{pmatrix}.
\]
(3.23)
Since \( P(\eta, \bar{\eta}) = Q(u, \bar{u}) \), (3.8) and (3.11) imply that
\[
P(\eta, \bar{\eta}) = \frac{1}{8(1 + 2P)^\frac{3}{2}} \left( |\nabla| \Lambda_m^{-1} (\eta + \bar{\eta}), |\nabla| \Lambda_m^{-1} (\eta + \bar{\eta}) \right).
\]
(3.24)
Therefore, if we denote the operators \( |\nabla|_m^\alpha := |\nabla| \Lambda_m^{-\alpha} (\alpha \in \mathbb{R}) \) and use the notations of Section 2, we can have the system on variables \((\eta, \bar{\eta})\) from (3.22)-(3.24)
that
\[
\partial_t \left( \eta \right) = i \sqrt{1 + 2P} \Lambda_m \left( \eta \right) + \frac{i}{8(1 + 2P)} \left( \sum_{\rho \in H_1} \rho(1) \langle |\nabla|^{\frac{1}{2m}} \eta, |\nabla|^{\frac{1}{2m}} \theta \rangle \langle \eta, \Lambda^2 \eta \rangle \right) \left( \eta \right)
\]
\[
- \frac{im^2}{8(1 + 2P)} \left( \sum_{\rho \in H_3} \langle |\nabla|^{\frac{1}{2m}} \eta, |\nabla|^{\frac{1}{2m}} \theta \rangle \langle \eta, \Lambda^2 \eta \rangle \right) \left( \begin{array}{c} 1 \\ -1 \end{array} \right).
\]  
(QLS)

At the end of section, we express the initial data of (QLS). The linear transformation (3.10) shows that
\[
\left( \eta \eta \right) = \frac{1}{\sqrt{1 - \theta^2}} \left( \begin{array}{c} 1 \\ -1/\theta \\ 1 \end{array} \right) \left( u \bar{u} \right),
\]
then
\[
\eta(0) = \frac{1}{\sqrt{1 - \theta_0^2}} (u(0) - \theta_0 \bar{u}(0)),
\]  
(3.25)
where \( \theta_0 = \theta(Q(u(0), \bar{u}(0))) \) and \( Q(u(0), \bar{u}(0)) = \frac{\epsilon}{\epsilon^2} \|v_0\|_{H^1}^2 \leq \frac{\epsilon^2}{2} \).

Since \( \|u(0)\|_{H^s} \leq \|v_0\|_{H^s} + \|v_1\|_{H^s} \leq 2\epsilon \), then (3.7)-(3.8) deduce that, if \( \epsilon < 1 \),
\[
\|\eta(0)\|_{H^s} \leq \frac{1 - \theta_0}{\sqrt{1 - \theta_0^2}} \|u(0)\|_{H^s} = \sqrt{\frac{1 - \theta_0}{1 + \theta_0}} \|u(0)\|_{H^s}
\]
\[
= (1 + 2Q(u(0), \bar{u}(0))) \|u(0)\|_{H^s} \leq \epsilon m \|u(0)\|_{H^s} \leq 4 \epsilon.
\]  
(3.26)

4. The proof of Theorems 1.1 and 1.2. In this section, we give the proof of Theorems 1.1 and 1.2. We have to study for some large \( s \) the equivalent Sobolev energy
\[
E^0_s(\eta) = (\Lambda^s \eta, \Lambda^s \bar{\eta}).
\]  
(4.1)
By the local theory, the solution to equation (QLS) does not blow up as long as the energy is finite. Thus we compute its derivation with respect to \( t \) according to the system (QLS)
\[
\frac{d}{dt} E^0_s(\eta) = 2R(\partial_t \eta, \Lambda^2 \eta) = 2R(i \sqrt{1 + 2P}(\Lambda \eta, \Lambda^2 \eta))
\]
\[
+ 2R \left( \frac{i}{8(1 + 2P)} \left( \sum_{\rho \in H_1} \rho(1) \langle |\nabla|^{\frac{1}{2m}} \eta, |\nabla|^{\frac{1}{2m}} \theta \rangle \langle \eta, \Lambda^2 \eta \rangle \right) \right)
\]
\[
+ 2R \left( \frac{im^2}{8(1 + 2P)} \left( \sum_{\rho \in H_3} \langle |\nabla|^{\frac{1}{2m}} \eta, |\nabla|^{\frac{1}{2m}} \theta \rangle \langle \eta, \Lambda^2 \eta \rangle \right) \right)
\]
\[
= - \frac{1}{4(1 + 2P)} \sum_{\rho \in H_1} \rho(1) \langle |\nabla|^{\frac{1}{2m}} \eta, |\nabla|^{\frac{1}{2m}} \theta \rangle \langle \eta, \Lambda^2 \eta \rangle
\]
\[
+ \frac{m^2}{4(1 + 2P)} \sum_{\rho \in H_3} \langle |\nabla|^{\frac{1}{2m}} \eta, |\nabla|^{\frac{1}{2m}} \theta \rangle \langle \eta, \Lambda^2 \eta \rangle.
\]  
(4.2)
Taking the Fourier series into account with the notation \( |n|^{2\beta} = |n|^2 (|n|^{2\beta - 2} \), the derivation of energy has the following forms
\[
\frac{d}{dt} E^0_s(\eta) = - \frac{3}{4(1 + 2P)} \sum_{\rho \in H_2, n_1, n_2} \rho(2) |n_2|_m (n_1^{2\beta - 2} \prod_{k=1} b_{n_k} \eta_{n_k} \eta_{n_k}).
\]
\[
+ \frac{m^2 \Im}{4(1 + 2P)} \sum_{\rho \in H_\nu, n_1, n_2} |n_2|^2_{m_1}(n_1)_{m_2}^{-1} \prod_{k=1}^2 (\eta_{n_k} \eta_{-n_k})^{\rho(k)} \\
+ \frac{m^2 \Im}{2(1 + 2P)} \sum_{n_1, n_2} |n_2|^2_{m_1}(n_1)_{m_2}^{-1} |\eta_{n_2}|^2 (\eta_{n_1} \eta_{-n_1}). \quad (4.3)
\]

Then, we consider the terms, possessing the similar constructions in (4.3), with the frequencies being restricted on the characteristic manifold \(Z^\rho_p\).

**Lemma 4.1.** Let \(p \in \mathbb{N}\) and \(\rho \in H_p\). Assume \(f(r_1, \cdots, r_p)\) is an arbitrary real valued function on \([0, \infty)^p\). For any \((n_1, \cdots, n_p) \in Z^\rho_p\),

\[
\Im \sum_{(n_1, \cdots, n_p) \in Z^\rho_p} f(|n_1|, \cdots, |n_p|) \prod_{k=1}^p (\eta_{n_k} \eta_{-n_k})^{\rho(k)} = 0. \quad (4.4)
\]

**Proof.** If \(p\) is odd or \(p\) is even and \(\rho \notin H_p\), (4.4) is immediate since \(Z^\rho_p = \emptyset\). If \(p\) is even and \(\rho \in H_p\), let \(l = \frac{p}{2}\) and suppose

\[
\rho(k) = \begin{cases} +, & \text{if } k = 1, \cdots, l; \\ - , & \text{if } k = l+1, \cdots, p. \end{cases}
\]

and \(|n_k| := r_k \text{ for } k = 1, \cdots, l\).

We define

\[
\mathcal{S}_p[\rho] := \{ \sigma \in \mathcal{S}_p : \sigma^2 = Id, \rho \circ \sigma = -\rho \} \text{ for } p \text{ even and } \rho \in H_p,
\]

and for any \(\sigma \in \mathcal{S}_p[\rho]\), let

\[
Z^\rho_p[\sigma] := \{(n_1, \cdots, n_p) \in \mathbb{Z}^d \times \mathbb{Z}^p : |n_k| = |n_{\sigma(k)}| = r_k, 1 \leq k \leq l\},
\]

so that, by the definition of \(Z^\rho_p\),

\[
Z^\rho_p = \bigcup_{\sigma \in \mathcal{S}_p[\rho]} Z^\rho_p[\sigma].
\]

For any \(\sigma \in \mathcal{S}_p[\rho]\),

\[
\Im \sum_{(n_1, \cdots, n_p) \in Z^\rho_p[\sigma]} f(|n_1|, \cdots, |n_p|) \prod_{k=1}^p (\eta_{n_k} \eta_{-n_k})^{\rho(k)}
\]

\[
= \Im \sum_{r_1, \cdots, r_l} \tilde{f}(r_1, \cdots, r_l) \prod_{k=1}^l \sum_{|n_k| = |n_{\sigma(k)}| = r_k} (\eta_{n_k} \eta_{n_k} \eta_{-n_k} \eta_{-n_k})\cdot \quad (4.5)
\]

For each \(k \in \{1, \cdots, l\}\), if \((\alpha, \beta) \in \{(n_k, n_{\sigma(k)}) | |n_k| = |n_{\sigma(k)}| = r_k\}\), which corresponds to the term

\[
\eta_{\alpha} \eta_{\alpha} (\eta_{\beta} \eta_{-\beta}),
\]

then \((\beta, \alpha) \in \{(n_k, n_{\sigma(k)}) | |n_k| = |n_{\sigma(k)}| = r_k\}\), which corresponds to the term

\[
\eta_{\beta} \eta_{-\beta} (\eta_{\alpha} \eta_{\alpha}).
\]

Obviously,

\[
\eta_{\alpha} \eta_{\alpha} (\eta_{\beta} \eta_{-\beta}) + \eta_{\beta} \eta_{-\beta} (\eta_{\alpha} \eta_{\alpha}) \in \mathbb{R}.
\]

So by the symmetry of summation, we have, for each \(\sigma \in \mathcal{S}_p[\rho]\),

\[
\Im \sum_{(n_1, \cdots, n_p) \in Z^\rho_p[\sigma]} f(|n_1|, \cdots, |n_p|) \prod_{k=1}^p (\eta_{n_k} \eta_{-n_k})^{\rho(k)} = 0. \quad (4.6)
\]
Since the set \( \{ Z^p_\rho(\sigma) \}_{\sigma \in \Sigma^p[\rho]} \) are not disjoint, in order to show (4.4), we need to verify that the sum also vanishes when frequencies are restricted to the intersections of \( Z^p_\rho(\sigma) \).

Consider \( \bigcap_{q=1}^Q Z^p_\rho(\sigma_q) \), where \( 2 \leq Q \leq \# \Sigma_p = l! \). For any \( (n_1, \ldots, n_p) \in \bigcap_{q=1}^Q Z^p_\rho(\sigma_q) \), we let \( L = \{ r_1, \ldots, r_l \} \) and denote
\[
\{ r_1, \ldots, r_l \} := \{ r_1, \ldots, r_l \}
\]
and
\[
A_s := \{ k \in \{ 1, \ldots, l \} | |n_k| = \tilde{r}_s \} \quad (1 \leq s \leq L).
\]

For any \( \sigma_1 \) and \( \sigma_2 \) \((1 \leq q_1 \neq q_2 \leq Q)\), since for any \( 1 \leq s \leq L \), by the definition of \( Z^p_\rho(\sigma_q) \)(i = 1, 2),
\[
\{ n_{\sigma_1}(k) | k \in A_s \} = \{ n_{\sigma_2}(k) | k \in A_s \}. \tag{4.7}
\]

In fact, if there exists \( k \in A_s \) such that \( n_{\sigma_1}(k) \in \{ n_{\sigma_1}(k) | k \in A_s \} \) and \( n_{\sigma_1}(k) \notin \{ n_{\sigma_2}(k) | k \in A_s \} \), then for any \( t \in A_s \), \( \sigma_1(k) \neq \sigma_2(t) \), and there exist \( k' \in \{ 1, \ldots, l \} \) but \( k' \notin A_s \) such that \( \sigma_1(k) = \sigma_2(k') \). By the definition of \( Z^p_\rho(\sigma_1) \) and \( Z^p_\rho(\sigma_2) \), we know
\[
|n_k| = |n_{\sigma_1}(k)| = \tilde{r}_s = |n_{\sigma_2}(k')| = |n_{k'}|.
\]

By the definition of \( A_s \), it means \( k' \in A_s \), which is a contradiction.

Then by (4.7), it is easy to see that
\[
\bigcap_{q=1}^Q Z^p_\rho(\sigma_q) = \left\{ (n_1, \ldots, n_p) : \begin{array}{ll}
|n_k| = |n_{\sigma_1}(k)|, & \text{for } 1 \leq k \leq l; \\
|n_k| = |n_{k'}|, & \text{if } k, k' \in A_s \text{ for some } s.
\end{array} \right\}.
\]

Therefore, we have
\[
\exists \sum_{(n_1, \ldots, n_p) \in \bigcap_{q=1}^Q Z^p_\rho(\sigma_q)} f(|n_1|, \ldots, |n_p|) \prod_{k=1}^p (\eta_{n_k} \eta_{-n_k})^{\rho(k)} \tag{4.8}
\]
\[
= \exists \sum_{r_1, \ldots, r_l, r_k \neq r_k} f(r_1, \ldots, r_l) \prod_{k=1}^l \sum_{|n_k| = |n_{\sigma_1}(k)| = r_k} (\eta_{n_k} \eta_{-n_k}) (\eta_{n_{\sigma_1}(k)} \eta_{-n_{\sigma_1}(k)}).
\]

Using the same argument for proving (4.6), we have
\[
\exists \sum_{(n_1, \ldots, n_p) \in \bigcap_{q=1}^Q Z^p_\rho(\sigma_q)} f(|n_1|, \ldots, |n_p|) \prod_{k=1}^p (\eta_{n_k} \eta_{-n_k})^{\rho(k)} = 0. \tag{4.9}
\]

By (4.6) and (4.9), we show (4.4) to hold. \( \square \)

It follows from Lemma 4.1 with \( p = 2 \) that
\[
\frac{d}{dt} P^0_s(\eta) = -\frac{3}{4(1 + 2P)} \sum_{\rho \in H_2, (n_1, n_2) \notin \Sigma^2_\rho} \rho(2)|n_2|_m \langle n_1 \rangle^{2s}_m \prod_{k=1}^2 (\eta_{n_k} \eta_{-n_k})^{\rho(k)}
\]
\[
+ \frac{m^2 3}{4(1 + 2P)} \sum_{\rho \in H_2, (n_1, n_2) \notin \Sigma^2_\rho} |n_2|_m^0 \langle n_1 \rangle^{2s-1}_m \prod_{k=1}^2 (\eta_{n_k} \eta_{-n_k})^{\rho(k)}
\]
Lemma 4.2. Let 

\[ \tilde{E}^0_\mu(\eta) = - \frac{3}{4(1 + 2P)} \sum_{\rho \in H_2, \ (n_1, n_2) \notin \mathbb{Z}^2} \sum_{\rho(1) = -1} \frac{\rho(2) |n_2| m(n_1) |n_1|^{2s-1}}{i2\sqrt{1 + 2P|n_1|, |n_2|}} \prod_{k=1}^{2} (\eta_{n_k} \eta_{-n_k})^{\rho(k)} \]

\[ + \frac{m^2 \overline{\rho}}{4(1 + 2P)} \sum_{n_1, n_2} \rho(\frac{1}{2}) |n_2| m(n_1) |n_1|^{2s-1} \left\langle \eta_{n_1} \eta_{-n_1} \right\rangle \]

\[ + \frac{m^2 \overline{\rho}}{2(1 + 2P)} \sum_{n_1, n_2} \rho(\frac{1}{2}) |n_2| m(n_1) |n_1|^{2s-1} \left\langle \eta_{n_1} \eta_{-n_1} \right\rangle \]

We define a perturbation of energy:

\[ \tilde{E}^0_\mu(\eta) = - \frac{3}{4(1 + 2P)} \sum_{\rho \in H_2, \ (n_1, n_2) \notin \mathbb{Z}^2} \sum_{\rho(1) = -1} \frac{\rho(2) |n_2| m(n_1) |n_1|^{2s-1}}{i2\sqrt{1 + 2P|n_1|, |n_2|}} \prod_{k=1}^{2} (\eta_{n_k} \eta_{-n_k})^{\rho(k)} \]

\[ + \frac{m^2 \overline{\rho}}{4(1 + 2P)} \sum_{n_1, n_2} \rho(\frac{1}{2}) |n_2| m(n_1) |n_1|^{2s-1} \left\langle \eta_{n_1} \eta_{-n_1} \right\rangle \]

\[ + \frac{m^2 \overline{\rho}}{2(1 + 2P)} \sum_{n_1, n_2} \rho(\frac{1}{2}) |n_2| m(n_1) |n_1|^{2s-1} \left\langle \eta_{n_1} \eta_{-n_1} \right\rangle \]

It follows from Proposition 1 and \( \mu([n_1], [n_2]) \leq |n_2| + 1 \) that, if \( s \geq \frac{1 + \nu}{2} \),

\[ |\tilde{E}^0_\mu(\eta)| \lesssim \sum_{n_1, n_2} \frac{m^2 \overline{\rho}}{2(1 + 2P)} \left\langle \eta_{n_1} \eta_{-n_1} \right\rangle \lesssim_m \| \eta \|^2 \|

In order to express the derivation of \( \tilde{E}^0_\mu(\eta) \), we need the following Lemma:

**Lemma 4.2.** Let \( (\eta, \tilde{\eta}) \) be the \( \mathbb{C} \)-valued solution of system (QLS), then \( (\eta, \tilde{\eta}) \) satisfies the following properties:

\[ \frac{d}{dt} P(\eta, \tilde{\eta}) = i \sum_{\rho \in H_1} \sum_{n} \overline{\rho}(1) |\tilde{n}| m(\eta \tilde{\eta}) \tilde{\rho}(1) ; \]

\[ \partial_t \eta_n = i \sqrt{1 + 2P} \langle n \rangle_m \eta_n \]

\[ + \frac{i}{8(1 + 2P)} \sum_{\rho \in H_1} \sum_{n} \overline{\rho}(1) |\tilde{n}| m(\eta \tilde{\eta}) \tilde{\rho}(1) \tilde{\eta}_n \]

\[ - \frac{m^2}{8(1 + 2P)} \sum_{\rho \in H_2} \sum_{n} |\tilde{n}|^2 \langle n \rangle_m^{-1} (\eta \tilde{\eta}) \tilde{\rho}(1) \tilde{\rho}(2) \]

\[ - \frac{m^2}{4(1 + 2P)} \sum_{\rho \in H_1} \sum_{n} |\tilde{n}|^2 \langle n \rangle_m^{-1} |\tilde{\eta}|^2 \tilde{\rho}(1) \]

\[ \frac{d}{dt} (\eta \eta_n - n) = 2i \sqrt{1 + 2P} \langle n \rangle_m (\eta \eta_n - n) \]

\[ - \frac{im^2}{4(1 + 2P)} \sum_{\rho \in H_1} \sum_{n} |\tilde{n}|^2 \langle n \rangle_m^{-1} (\eta \tilde{\eta}) \tilde{\rho}(1) \tilde{\eta}_n \]

\[ - \frac{im^2}{2(1 + 2P)} \sum_{\rho \in H_1} \sum_{n} |\tilde{n}|^2 \langle n \rangle_m^{-1} |\tilde{\eta}|^2 \tilde{\eta}_n \]

\[ + \frac{i}{8(1 + 2P)} \sum_{\rho \in H_1} \sum_{n} \overline{\rho}(1) |\tilde{n}| m(\eta \tilde{\eta}) \tilde{\rho}(1) \tilde{\eta}_n^2 + |\eta_n|^2 \]

\[ - \frac{im^2}{8(1 + 2P)} \sum_{\rho \in H_1} \sum_{n} |\tilde{n}|^2 \langle n \rangle_m^{-1} (\eta \tilde{\eta}) \tilde{\rho}(1) \tilde{\eta}_n^2 + |\eta_n|^2 \]

\[ - \frac{im^2}{4(1 + 2P)} \sum_{\rho \in H_1} \sum_{n} |\tilde{n}|^2 \langle n \rangle_m^{-1} |\tilde{\eta}|^2 (|\eta_n|^2 + |\eta_n|-n)^2) ; \]
\[
\frac{d}{dt} \| \eta_t \|^2 = \frac{i}{8(1+2P)} \sum_{\bar{n}} \sum_{\bar{\rho} \in H_2} \bar{\rho}(1) |\bar{n}|_m (\eta_{\bar{n}} \eta_{-\bar{n}}) \bar{\rho}(1) (\eta_{\bar{n}} \eta_{-\bar{n}}) \bar{\rho}(2) \\
+ \frac{im^2}{8(1+2P)} \sum_{\bar{n}} \sum_{\bar{\rho} \in H_2} \bar{\rho}(2) |\bar{n}|_m (\eta_{\bar{n}} \eta_{-\bar{n}}) \bar{\rho}(1) (\eta_{\bar{n}} \eta_{-\bar{n}}) \bar{\rho}(2) \\
+ \frac{im^2}{4(1+2P)} \sum_{\bar{n}} \sum_{\bar{\rho} \in H_1} \bar{\rho}(1) |\bar{n}|_m (\eta_{\bar{n}} \eta_{-\bar{n}}) \bar{\rho}(1) (\eta_{\bar{n}} \eta_{-\bar{n}}) \bar{\rho}(2). \tag{4.15}
\]

**Proof.** (4.12) follows from (3.15) by taking the Fourier series. And conclusions (4.13), (4.14) and (4.15) are deduced by the system (QLS).

We define

\[
\tilde{E}_0^0 := \{ (\rho, n_1, n_2, n_3) \in H_3 \times \mathbb{Z}^{d \times 3} | \rho(1) = -\rho(2), |n_1| = |n_2| \}.
\]

By the construction of \( \tilde{E}_0^0(\eta) \), \( \frac{d}{dt} \tilde{E}_0^0(\eta) \) contains not only the terms of (4.10), but also the terms of the form

\[
\exists \sum_{(\rho, n_1, n_2, n_3) \notin \tilde{E}_0^0} C(m, P) H(\rho) \prod_{\tau=1}^3 G_{\tau}(|n_{\tau}|) \left( \frac{1}{\rho(1)|n_1|_m + \rho(2)|n_2|_m} + \frac{1}{\rho(1)|n_1|_m} \right) \\
\times \left( \sum_{K \subset \{1,2,3\}} \prod_{k \in K} (\eta_{n_k} | \eta_{-n_k}) \prod_{l \neq k} |\eta_{n_l}|^2 \right), \tag{4.16}
\]

where \( C(m, P) \) is a real-valued smooth function on \( m \) and \( P \), satisfying

\[
C(m, P) \sim_m (1 + 2P)^{-5/2}, \tag{4.17}
\]

and \( G_{\tau}(|n_{\tau}|) \) (1 \( \leq \tau \leq 3 \) are real-valued functions on variables \( |n_{\tau}| \) such that

\[
\left| \prod_{\tau=1}^3 G_{\tau}(|n_{\tau}|) \right| \lesssim (|n_1|_m^2 |n_2| |n_3|). \tag{4.18}
\]

If we define

\[
\mathcal{E}_0^1(\eta) := \frac{d}{dt} (E_0^0(\eta) - E_0(\eta)),
\]

it is easy to see from (4.16)-(4.18) and Proposition 1 that

\[
\left| \mathcal{E}_0^1(\eta) \right| \lesssim_m \sum_{n_1, n_2, n_3} \langle n_1 \rangle_{m}^{2s} |n_2| |n_3| \mu(|n_1|, |n_2|) \nu \\
\times \left( \sum_{K \subset \{1,2,3\}} \prod_{k \in K} |\eta_{n_k}| |\eta_{-n_k}| \prod_{l \neq k} |\eta_{n_l}|^2 \right). \tag{4.19}
\]

Since \( \mu(|n_1|, |n_2|) \leq |n_2| + 1 \), then if \( s \geq \frac{1+\nu}{2} \),

\[
\left| \mathcal{E}_0^1(\eta) \right| \lesssim_m \| \eta \|_{H^{s}}^6.
\]

Furthermore, we claim that the resonant terms in (4.16) vanish. In fact, the resonant terms only appear for \( \sharp K = 2 \), where we can see the terms of the form

\[
\exists \sum_{(\rho, n_1, n_2, n_3) \notin \tilde{E}_0^0} C(m, P) H(\rho) \prod_{\tau=1}^3 G_{\tau}(|n_{\tau}|) \left( \frac{1}{\rho(1)|n_1|_m + \rho(2)|n_2|_m} + \frac{1}{\rho(1)|n_1|_m} \right)
\]
vanish in the second term \((s)
\). Repeat the above argument, one can see by Propositions 1 and 2 that, if
\(E_1 = 0\), then
\[\|E_2\|_2^2 = 0\] and
\[\|E_3\|_2^2 = 0\]. Furthermore, if we denote the set
\[\Omega^1 = \{(\rho, \eta) \in H_3 \times \mathbb{Z}^d \times 3 \mid \exists K \subset \{1, 2, 3\} \text{ such that } \rho|_K \text{ is even and } \rho|_{1\neq K}, P_K(\eta) \in \mathbb{Z}_\rho^K\},\]
then \(E_2^1(\eta)\) consists of the terms possessing the form
\[\sum_{(\rho, n_1, n_2, \eta) \in \Omega^1} C(m, P) \prod_{\tau = 1}^3 G_\tau(|n_\tau|) F(\rho, |n_1|, |n_2|) \times \left( \sum_{K \subset \{1, 2, 3\}, K \neq 0} \prod_{K \neq 0} (\eta_{n_k} n_{-n_k})^{\rho(k)} \prod_{l=1}^3 \|\eta_n\|_2^2 \right).\] (4.21)

Here, \(C(m, P)\) and \(G_\tau\) are defined as (4.17) and (4.18), and \(F\) is a real-valued function on \(\rho, |n_1|\) and \(|n_2|\) such that
\[|F| \lesssim \mu(|n_1|, |n_2|)^\nu,\]
where \(\nu\) is given by Proposition 1.

The next step is to define \(H_2(\eta)\), which consists of the terms according to (4.21) that
\[\sum_{(\rho, n_1, n_2, \eta) \in \Omega^1} C(m, P) \prod_{\tau = 1}^3 G_\tau(|n_\tau|) F(\rho, |n_1|, |n_2|) \times \left( \sum_{K \subset \{1, 2, 3\}, K \neq 0} \frac{1}{2i\sqrt{1 + 2PF_{m,K}}} \prod_{K \neq 0} (\eta_{n_k} n_{-n_k})^{\rho(k)} \prod_{l=1}^3 \|\eta_n\|_2^2 \right).\] (4.22)

Repeat the above argument, one can see by Propositions 1 and 2 that, if \(s \geq \frac{1+2\nu}{2}\),
\[|\tilde{E}_2^2(\eta)| \lesssim_m \sum_{n_1, n_2, n_3} (n_1)^{2s} |n_2||n_3| \mu(|n_1|, |n_2|, |n_3|)^{2\nu} \times \left( \sum_{K \subset \{1, 2, 3\}, K \neq 0} \prod_{K \neq 0} \|\eta_n\|_2^2 \prod_{l=1}^3 \|\eta_n\|_2^2 \right) \lesssim_m \|\eta\|_{H^s}.\] (4.23)

Furthermore, if we denote
\[E_2^s(\eta) := \frac{d}{dt} E_0(\eta) - \frac{d}{dt} E_1(\eta) - \frac{d}{dt} E_2(\eta),\]
and the set
\[
\Omega^2 := \{(\rho, \vec{n}) \in H_4 \times \mathbb{Z}^{d+1} \mid \text{there exists } K \subset \{1, 2, 3, 4\} \text{ such that } \\
\sharp K \text{ is even and } \rho|_K \in \overline{\Pi}_{\rho|_K}, P_K(\vec{n}) \in \mathbb{Z}^{\sharp K}_{\rho|_K}\},
\]

then \(\mathcal{E}_s^2(\eta)\) consists of the terms as the form
\[
\exists \sum_{(\rho, \vec{n}) \notin \Omega^2} C(m, P) \prod_{\tau=1}^4 G_{\tau}(|n\tau|) F(\rho, |n_1|, |n_2|, |n_3|)
\times \left( \sum_{K \subset \{1, 2, 3, 4\}} \prod_{k \in K} (\eta_{mk} \eta_{mk})^{\rho(k)} \prod_{l=1}^4 |\eta_{ml}|^2 \right),
\] (4.24)

where \(C(m, P)\) is a real-valued smooth function on \(m\) and \(P\), satisfying
\[
C(m, P) \sim_m (1 + 2P)^{-7/2},
\] (4.25)
and \(G_{\tau}(|n\tau|)\) (\(1 \leq \tau \leq 4\)) are real-valued functions on variables \(|n\tau|\) such that
\[
\left| \prod_{\tau=1}^4 G_{\tau}(|n\tau|) \right| \lesssim \langle n \rangle_{\tau=1}^{2\nu} |n_2||n_3||n_4|,
\] (4.26)
and \(F\) is a real-valued function on \(\rho, |n_1|, |n_2|\) and \(|n_3|\) such that
\[
|F| \lesssim \mu(|n_1|, |n_2|, |n_3|)^{2\nu},
\]
where \(\nu\) is given by Proposition 2.

Obviously, by the definition of \(\mathcal{E}_s^2(\eta)\), if \(s \geq \frac{1+2\nu}{2}\), we have
\[
|\mathcal{E}_s^2(\eta)| \lesssim \|\eta\|^s_H.
\]

The following step is the classical induction process. For any \(r \in \mathbb{N}\), we define the sets
\[
\Omega^r := \{(\rho, \vec{n}) \in H_{r+2} \times \mathbb{Z}^{d+1(r+2)} \mid \text{there exists } K \subset \{1, \cdots, r+2\} \text{ such that } \\
\sharp K \text{ is even and } \rho|_K \in \overline{\Pi}_{\rho|_K}, P_K(\vec{n}) \in \mathbb{Z}^{\sharp K}_{\rho|_K}\}.
\]

By induction, we denote \(\widetilde{\mathcal{E}}_s^r(\eta)\) consisting of the terms as the form
\[
\sum_{(\rho, \vec{n}) \notin \Omega^r} C_E(m, P) F_E(\rho, |n_1|, \cdots, |n_{r+1}|) \prod_{\tau=1}^{r+2} G_{\tau}(|n\tau|)
\times \left( \sum_{K \subset \{1, \cdots, r+2\}} \frac{1}{2t\sqrt{1 + 2PF_{m,K}}} \prod_{k \in K} (\eta_{mk} \eta_{mk})^{\rho(k)} \prod_{l=1}^{r+2} |\eta_{ml}|^2 \right),
\] (4.27)

and \(\mathcal{E}_{s}^{r+1}(\eta)\) consisting of the terms as the form
\[
\sum_{(\rho, \vec{n}) \notin \Omega^{r+1}} C_E(m, P) F_E(\rho, |n_1|, \cdots, |n_{r+2}|) \prod_{\tau=1}^{r+3} G_{\tau}(|n\tau|)
\times \left( \sum_{K \subset \{1, \cdots, r+3\}} \prod_{k \in K} (\eta_{mk} \eta_{mk})^{\rho(k)} \prod_{l=1}^{r+3} |\eta_{ml}|^2 \right),
\] (4.28)
where $C_E(m, P)$ and $C_F(m, P)$ are real-valued smooth functions on $m$ and $P$, satisfying

$$C_E(m, P) \sim_m (1 + 2P)^{-3/2 - r} \quad \text{and} \quad C_F(m, P) \sim_m (1 + 2P)^{-5/2 - r},$$

and $G_r([n_r])$ ($1 \leq r \leq r + 3$) are real-valued functions on variables $|n_r|$ such that

$$\left| \prod_{r=1}^{\kappa} G_r([n_r]) \right| \lesssim (n_1)_{2s} \prod_{r=2}^{\kappa} |n_r|, \quad \kappa = r + 2 \text{ or } \kappa = r + 3,$$

and $F_E$ and $F_F$ are real-valued functions on $\rho$ and $|n_1|, \ldots, |n_{r+1}|$ or $|n_1|, \ldots, |n_{r+2}|$ respectively, such that

$$|F_E| \lesssim \mu((|n_1|, \ldots, |n_{r+1}|)^{r\nu} \quad \text{and} \quad |F_F| \lesssim \mu((|n_1|, \ldots, |n_{r+2}|)^{(r+1)\nu},$$

where $\nu$ is given by Propositions 1 and 2.

Since $\mu(|n_1|, \ldots, |n_{s}|) \leq \max(|n_2|, \ldots, |n_{s}|) + 1$ ($\kappa = r + 1$ or $r + 2$), it follows from (4.27)-(4.31) that, if $s \geq \frac{1 + (r+1)\nu}{2}$,

$$|\widetilde{E}_s^0(\eta)| \lesssim_m \|\eta\|_{H^s}^{2r+4} \quad \text{and} \quad |\widetilde{E}_s^{r+1}(\eta)| \lesssim_m \|\eta\|_{H^{s+6}}^{2r+6}.$$

Hence, it follows from (4.32) that for any $N \in \mathbb{N}$, let $s \geq \frac{1 + N\nu}{2}$

$$\frac{d}{dt} E_s^0(\eta) - \sum_{r=0}^{N-1} \frac{d}{dt} \widetilde{E}_s^r(\eta) = E_s^N(\eta) = O(\|\eta\|_{H^{s+4}}^{2N+4}).$$

By definition, we have

$$E_s^0(\eta)(t) \sim_m \|\eta(t)\|_{H^s}^2.$$  

It follows from (3.26), (4.32)-(4.33) and Newton-Leibniz’s formula that, there exists a constant $C_m > 1$ depending on $m$, if $C_m \epsilon^2 < 1/2$, then

$$E_s^0(\eta)(T)$$

$$\leq E_s^0(\eta)(0) + \sum_{r=0}^{N-1} \widetilde{E}_s^r(\eta)(0) + \sum_{r=0}^{N-1} \widetilde{E}_s^r(\eta)(T) + \int_0^T O(\|\eta(t)\|_{H^{s+4}}^{2N+4}) \, dt$$

$$\leq E_s^0(\eta)(0) + \sum_{r=0}^{N-1} \left( E_s^0(\eta)(0) \right)^{r+2} + \sum_{r=0}^{N-1} \left( E_s^0(\eta)(T) \right)^{r+2} + C_m \int_0^T \left( E_s^0(\eta)(t) \right)^{N+2} \, dt$$

$$\leq 2C_m \epsilon^2 + \sum_{r=0}^{N-1} \left( E_s^0(\eta)(T) \right)^{r+2} + C_m \int_0^T \left( E_s^0(\eta)(t) \right)^{N+2} \, dt.$$

We claim: For any fixed $m$ and $N \in \mathbb{N}$, there exists $\epsilon_0 > 0$ satisfying $(4C_m)^{N+2} \epsilon_0^2 < 1$ and $20C_m \epsilon_0^2 < 1$, then for any $\epsilon \in (0, \epsilon_0)$, we have

$$E_s^0(\eta)(t) < 4C_m \epsilon^2, \quad \forall t \in [-\epsilon \zeta^{-N-2}, \epsilon \zeta^{-N-2}]$$

for some constant $\zeta = C_N, m$.

If it is not true, by continuity of $E_s^0(\eta)$ on $t$, there exists a positive time $T_0 < \epsilon \zeta^{-N-2} \zeta = 1$, which is the first point such that

$$E_s^0(\eta)(T_0) = 4C_m \epsilon^2.$$

Then, (4.35) gives that

$$4C_m \epsilon^2 \leq 2C_m \epsilon^2 + \sum_{r=0}^{N-1} (4C_m \epsilon^2)^{r+2} + C_m \int_0^{T_0} (4C_m \epsilon^2)^{N+2} \, dt$$
\begin{equation}
\leq 2C_m\epsilon^2 + \frac{(4C_m\epsilon^2)^2}{1 - 4C_m\epsilon^2} + C_m(4C_m\epsilon^2)^N + 2T_0 \\
< 3C_m\epsilon^2 + 4^{N+2}C_m^{N+3}\epsilon^{2N+4}C\epsilon^{-2N-2} < 4C_m\epsilon^2,
\end{equation}

which is a contradiction.

Therefore, our claim is concluded and the solution \((\eta, \bar{\eta})\) of (QLS) exists at least over a time interval of length of order \(\alpha\epsilon^{-2N-2}\) with uniformly bounded Sobolev norms on that interval if \(s \geq \frac{1+N\nu}{2}\).

Back to the original question, the linear transformations (3.2), (3.10) and properties of \(\theta\) (3.7)-(3.8) deduce that

\[
v = \frac{1 + \theta}{2\sqrt{1 - \theta^2}} \Lambda^{-1}_m(\eta + \bar{\eta}) = \frac{1}{2} \sqrt{\frac{1 + \theta}{1 - \theta}} \Lambda^{-1}_m(\eta + \bar{\eta}) = \frac{1}{2(1 + \|v\|_{\dot{H}^1})^{\frac{1}{2}}} \Lambda^{-1}_m(\eta + \bar{\eta})
\]

and

\[
\partial_t v = \frac{i}{2} \sqrt{\frac{1 - \theta}{1 + \theta}} (\eta - \bar{\eta}) = \frac{i}{2} \sqrt{\frac{1 - \theta}{1 + \theta}} (\eta - \bar{\eta}) = \frac{i}{2}(1 + \|v\|_{\dot{H}^1})^{\frac{1}{2}} (\eta - \bar{\eta}).
\]

Firstly, we estimate \(v\) in \(\dot{H}^1\) by (4.38) that

\[
\|v\|_{\dot{H}^1} \lesssim \frac{1}{(1 + \|v\|_{\dot{H}^1})^{\frac{1}{2}}} \|\eta\|_{L^2},
\]

which deduces

\[
\|v\|_{\dot{H}^1} \lesssim \|\eta\|_{L^2}^{\frac{2}{3}}
\]

if \(\eta\) is bounded in \(L^2\).

And then, it follows from (4.38) and (4.39) that

\[
\|v\|_{H^{s+1}} \lesssim \|\eta\|_{H^s} \quad \text{and} \quad \|\partial_t v\|_{H^s} \lesssim \|\eta\|_{H^s}.
\]

Therefore, \(v\) and \(\partial_t v\) have bounded Sobolev norms as long as \((\eta, \bar{\eta})\) has, and the proof of Theorems is completed.

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