General quantum Chinos games

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The Chinos game is a non-cooperative game between players who try to guess the total sum of coins drawn collectively. Semicalssical and quantum versions of this game were proposed by F. Guinea and M. A. Martín-Delgado, in J. Phys. A: Math. Gen. 36 L197 (2003), where the coins are replaced by a boson whose number occupancy is the aim of player’s guesses. Here, we propose other versions of the Chinos game using a hard-core boson, one qubit and two qubits. In the latter case, using entangled states, the second player has a stable winning strategy that becomes symmetric for non-entangled states. Finally, we use the IBM Quantum Experience to compute the basic quantities involved in the two-qubit version of the game.

I. INTRODUCTION

Game theory is a field that has fascinated mathematicians since the early 20th century. The idea is to be able to interpret complex problems in many different fields as a game of one or more players who, using logical reasoning, try to optimize their strategies to obtain the highest possible profit. The pioneers who formalized game theory were John von Neumann and Oskar Morgenstern [1], followed soon later by John Nash [2]. A fundamental concept is that of the Nash equilibrium, in which it is assumed that every player knows and chooses their best possible strategy knowing the strategies of the other players. This implies that the Nash equilibrium situation is one in which no player would profit by changing her/his strategy if the other players maintain theirs. Most of multiplayer games tend to reach Nash equilibrium after a certain number of iterations. This area of study appeared with the aim of better understanding the economy, but it quickly extended to biology [3], politics [4], computation and computer science [5] (see review [6]).

In recent decades, physicists have begun to introduce features of the quantum world into game theory in order to gain advantages over classical strategies [7]–[12]. There are several reasons why quantifying games can be interesting. The first is simply because of the number of applications game theory has had in different fields. Moreover, its probabilistic nature makes one want to extend it to quantum probabilities. Another reason is the connection between game theory and quantum information theory. In fact, in the games themselves, players transmit information to other players and, since our world is quantum, it can be interpreted as quantum information [8].

There are several examples of well-known quantum game models. One is the prisoner’s dilemma where, by exploiting the peculiarities of quantum behaviour, both players can escape from the dilemma [8]. Another case is the PQ game (flipping or not flipping a coin a certain number of times each player), where it has been shown that if the first player can use quantum strategies (superposition of both options) she/he will always win no matter what action the second player takes [7]. Not only have quantum games been proposed, but they are already being used to model, for example, human decision-making behaviour [13]–[15].

This paper deals with the well-known (in Spain) Chinos game, which traditionally consists of a group of people who hide a certain number of coins in their hands. The aim of each of them is to guess the total number of coins hidden by all of them. The Chinos game is a variant of the Morra game that is played with fingers instead of coins. It dates back thousands of years to ancient Roman and Greek times [16]. This simple game shows a variety of behavioural patterns that have been used to model financial markets and information transmission [17].

This is a non-cooperative game, in which each player will seek to maximize their chances of victory and minimize those of the other players. For this reason, throughout the work we will always look for this situation in the analysis of the possible strategies. In other words, the Nash equilibrium of the game model will be pursued. We must stress the importance of entanglement in quantum games when looking for the Nash equilibrium [18–20]. It is worth mentioning that some quantum games like the prisoner’s dilemma and the PQ game have been already implemented using IBM quantum computers [21, 22].

The organization of the paper is as follows. In section II we define in an abstract manner the Chinos game. In section III we review a classical Chinos game. In section IV we review the semiclassical model of the Chinos game proposed in [23] and we propose two new semiclassical models. In section V we define a one-qubit game using unitary transformations. In section VI we introduce a two-qubit game that incorporates entanglement. In section VII we implement the latter game using an IBM quantum computer, and finally, in section VIII we state our conclusions and prospects.

II. ABSTRACT CHINOS GAME FOR TWO PLAYERS

Let Alice and Bob have access to a set of objects $\mathcal{C}$. At the start of the game, Alice chooses one object from $\mathcal{C}$, denoted $c_A$, and Bob chooses an object $c_B \in \mathcal{C}$. Both players are unaware of the other player’s choice. Alice and Bob then send $c_A$ and $c_B$ to a device that assigns them an object $g_{AB}$ of a set $\mathcal{G}$. The goal of the players is to guess $g_{AB}$. To do so there is a set $\mathcal{C}$ where Alice and Bob selects the objects $\tilde{c}_A$ and $\tilde{c}_B$ respectively, and send them to the previous device that assigns them the objects $g_A$ and $g_B$ in $\mathcal{G}$. The protocol is shown in Fig. 1.

In order to quantify how good the players’ guesses are, we shall use a distance $d(g,g')$ between objects...
The game is defined by two parameters: $N_p$ that is the number of players and $N_c$ that is the number of coins that each player holds in her/his hand. In each round, each player draws from 0 to $N_c$ coins and guesses the total number of coins that they all draw, with the restriction that the result predicted by the previous players cannot be repeated [23].

We shall consider below the game with only two players, $N_p = 2$, and one coin, $N_c = 1$. The best strategy for Alice is to choose randomly $c_A$ coins and to guess always $g_A = 1$, so as not to reveal information to Bob [23]. This is also based on the fact that with four possible tosses the most likely value of the sum is 1. Bob’s best strategy is to choose $c_B$ at random and make his attempt $g_B$ in an “intelligent” way. This means that, if Bob chooses $c_B=0$, then he must exclude the option $g_B=2$ and if he chooses $c_B=1$, he must exclude $g_B=0$. Table I shows all the possible options assuming that each player follows her/his best strategy.

| $c_A$ | $c_B$ | $g_A$ | $g_B$ | Winner |
|-------|-------|-------|-------|--------|
| 0     | 0     | 1     | 0     | B      |
| 0     | 1     | 1     | 2     | A      |
| 1     | 0     | 1     | 0     | A      |
| 1     | 1     | 2     | 1     | B      |

**TABLE I:** All possible options for the Chinos game with $N_p = 2, N_c = 1$.

It is clear that each player will win half of the time and therefore their winning probabilities are equal, that is

$$P_A = P_B = \frac{1}{2}$$

Moreover, their strategies are stable. In reference [23] it was shown that (2) also holds for two players and a generic number of coins $N_c$. This was called a classical symmetry between the players. These authors also proposed in [23] a semiclassical and quantum version of the Chinos game to test whether the classical symmetry (2) is broken by quantum fluctuations.

From the formal viewpoint introduced in section II, this game involves the following sets:

$$\mathcal{C} = \{0, 1\}, \quad \tilde{\mathcal{C}} = \mathcal{G} = \{0, 1, 2\}.$$  

where $\mathcal{C}$ contains the number of coins, $c_A$ and $c_B$, of Alice and Bob, $\mathcal{G}$ contains their guesses $g_A$ and $g_B$ for the total number of coins $g_{AB}$. The device operates as follows

$$g_A = \tilde{c}_A, \quad g_B = \tilde{c}_B, \quad g_{AB} = c_A + c_B.$$  

The guesses $g_A, g_B$ are identical to the choices $\tilde{c}_A, \tilde{c}_B$, but not in the games we shall consider below. The metric in $\mathcal{G}$ is given by

$$d(g, g') = |g - g'|, \quad g, g' \in \mathcal{G}.$$  

and the minimal distance is set to $d_0 = 1$. The
strategy, i.e. choose randomly among the operators under the choice.

\[ \text{if } g_A = 0 \rightarrow g_B = 1, 2, \quad (6) \]
\[ \text{if } g_A = 1 \rightarrow g_B = 0, 2, \quad \]
\[ \text{if } g_A = 2 \rightarrow g_B = 0, 1, \quad \]

and the intelligent rule to

\[ \text{if } c_{A/B} = 0 \rightarrow \tilde{c}_{A/B} = g_{A/B} = 0, 1, \quad (7) \]
\[ \text{if } c_{A/B} = 1 \rightarrow \tilde{c}_{A/B} = g_{A/B} = 1, 2. \]

IV. SEMICLASSICAL MODELS

A semi-classical version of the Chinos game consists of replacing coins with a quantum degree of freedom and measuring an observable on it. The value taken by that observable is the goal of the players’ guesses [23]. We shall first review the proposal of reference [23] that uses a single boson degree of freedom and later on we shall propose two related games.

Let \( b \) and \( b^† \) be the bosonic creation and annihilation operators satisfying the canonical commutation rule \([b, b^†] = 1\). They generate the bosonic states in the standard way: \( b|0\rangle = 0, \ b^†|n\rangle = \sqrt{n + 1}|n + 1\rangle \) where \(|n\rangle = (b^†)^n|0\rangle/\sqrt{n!}.

We will study the case of two players. Each of them can act on the bosonic state \(|0\rangle\) with one of the following operators

\[ O_1 = I, O_2 = I + b^†/\sqrt{2}, O_3 = I - b^†/\sqrt{2}, O_4 = b^†. \quad (8) \]

This is the quantum analogue of drawing a coin. Let us denote by \( O_i^A \) the operator chosen by Alice and by \( O_i^B \) the operator chosen by Bob. The joint state produced by both players is given by

\[ |\Psi_{i,j}\rangle = N_{i,j}^{-1/2}O_i^A O_j^B |0\rangle = \sum_{n=0}^{2} c_{i,j}(n)|n\rangle, \quad (9) \]

where \( N_{i,j} \) is a normalization constant. The operators \( O_1 \) and \( O_4 \) are equivalent to the classical choices of 0 and 1 coins respectively. In this case the state \(|9\rangle\) is simply \(|0\rangle\), \(|1\rangle\) or \(|2\rangle\), that brings us back to the classical game. Allowing Alice and Bob to use also \( O_2 \) and \( O_3 \) generate linear superpositions in \(|9\rangle\), that leads to probabilistic outcomes. They will be analyzed in terms of the probability of finding the state \(|n\rangle\) in \(|9\rangle\)

\[ p_{i,j}(n) = \langle |n|\Psi_{i,j}\rangle|^2 = c_{i,j}^2(n). \quad (10) \]

Suppose the both players follow the classical strategy, i.e. choose randomly among the operators \( O_i \), \( i = 1, 2, 3, 4 \). The probability of Alice guessing \( n \) under the choice \( O_i^A \) is given by the average over Bob’s choices,

\[ \langle p_i(n) \rangle = \frac{1}{4} \sum_{j=1}^{4} p_{i,j}(n). \quad (11) \]

The results are given in Table II.

| \( O_i^A \) | \( O_i^B \) | \( O_i^A \) | \( O_i^B \) |
|---|---|---|---|
| \(|p(0)|\) | \(1/2\) | \(41/168\) | \(41/168\) | \(0\) |
| \(|p(1)|\) | \(1/2\) | \(59/168\) | \(59/168\) | \(5/12\) |
| \(|p(2)|\) | \(0\) | \(68/168\) | \(68/168\) | \(7/12\) |

Hence if Alice draws \( O_i^A \) her best guess is 0 (or 1) and on drawing \( O_i^B \), \( O_i^A \) and \( O_i^B \), her best guess is 2. Therefore, the total probability of winning for Alice is

\[ P_A = \frac{1}{4} + \frac{1}{2} + \frac{1}{2} + \frac{7}{12} = \frac{53}{112} < \frac{1}{2}. \quad (12) \]

After many rounds Alice realizes that she is losing the game and decides to change her strategy by randomly choosing between the operators \( O_i^A \) and \( O_i^B \).

Her probability of winning changes to

\[ P_A = \frac{1}{2} + \frac{1}{2} + \frac{7}{12} = \frac{13}{24} > \frac{1}{2}. \quad (13) \]

Bob then notices this imbalance and decides to choose randomly between the operators \( O_i^B \) and \( O_i^A \), which reproduces the classical game where the classical symmetry (2) is restored [23].

The formalization of the game is as follows. Alice and Bob choose \( i, j \in C \) where

\[ C = \{1, 2, 3, 4\}, \quad (14) \]

and send them to the device that constructs the state \(|9\rangle\) belonging to the Hilbert space expanded by the boson states with 0, 1 and 2 occupancies,

\[ G = \text{Span}\{|0\rangle, |1\rangle, |2\rangle\}. \quad (15) \]

Each player’s guess of these occupancies, \( n \), is mapped by device into the basis of \( G \) as

\[ n \in \tilde{C} = \{0, 1, 2\} \rightarrow |n\rangle \in G. \quad (16) \]

To define a distance between two states \(|g\rangle, |g’\rangle \in G\), we use the trace distance between the corresponding density matrices, \( \rho_g = |g\rangle \langle g| \) and \( \rho_{g'} = |g’\rangle \langle g’| \) [24]
\[ d(|g\rangle, |g’\rangle) = \frac{1}{2} \text{tr}[\rho_g - \rho_{g’}] = \sqrt{1 - |\langle g|g’\rangle|^2}. \quad (17) \]

The probability (10) is related to this distance as
\[ d(|n\rangle, |\Psi_{i,j}\rangle) = \sqrt{1 - p_{i,j}(n)}. \quad (18) \]

Hence minimizing the distance of the guess \(|n\rangle\) to the state \(|\Psi_{i,j}\rangle\) is equivalent to maximizing the probability \( p_{i,j}(n) \). The latter quantity is equal to the square of the fidelity between these states.

A. Hard-core boson

We shall next propose a model where the boson operator \( b^† \), used previously, is replaced by a hard-core boson, that is, an operator satisfying the condition:

\[ (b^†)^2 = 0. \quad (19) \]
This forces the elimination of the operator $O_4$ as an option since its use by Alice and Bob would lead to a null move. This also implies that the classical version of the chinos game is not included in the new version.

The study of strategies is based again on the probabilities of obtaining 0 and 1 for each possible move. They are given in Table III.

| $O_1^A$ | $O_2^A$ | $O_3^A$ |
|---------|---------|---------|
| $p(0) = 1$ | $p(0) = 1/2$ | $p(0) = 1/2$ |
| $p(1) = 0$ | $p(1) = 1/2$ | $p(1) = 1/2$ |

TABLE III: Odds of getting 0 or 1 for all possible moves in the game with hard-core bosons.

Following the classical strategy, the players will draw at random among the operators $O_i$, $i = 1, 2, 3$. Averaging over Bob’s choices, the probabilities of Alice getting 0 and 1 are given in Table IV.

| $O_1^A$ | $O_2^A$ | $O_3^A$ |
|---------|---------|---------|
| $\langle p(0) \rangle = 2/3$ | $17/30$ | $17/30$ |
| $\langle p(1) \rangle = 1/3$ | $13/30$ | $13/30$ |

TABLE IV: Average odds of getting 0, 1 or 2 for Alice.

It is clear that Alice’s best guess is 0, regardless her choice of operator, with a winning probability

$$P_A = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{17}{30} = \frac{3}{5} > \frac{1}{2}$$

(20)

To reverse this outcome Bob will choose randomly between the $O_2$ and $O_3$ that seem more favorable from Table IV. However, Alice’s winning probability is still higher than a half,

$$\frac{1}{2} < \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{17}{30} = \frac{3}{5}$$

(21)

Hence, Alice has achieved a winning and stable strategy that breaks the symmetry of the classical game.

To improve the chances of Bob we shall use the following operators

$$O_1 = I, O_2 = cI + sb^1, O_3 = cI - sb^1,$$

(22)

where $c = \cos(\theta), s = \sin(\theta)$ with $\theta \in (0, \pi/2)$. The previous case corresponds to $\theta = \pi/4$. The values $\theta = 0, \pi/2$ are excluded because in these cases $O_2 = \pm O_3$.

The probabilities of obtaining 0 or 1 in the boson occupation are shown in Table V.

| $O_1^B$ | $O_2^B$ | $O_3^B$ |
|---------|---------|---------|
| $p(0) = 1$ | $p(0) = c^2$ | $p(0) = c^2$ |
| $p(1) = 0$ | $p(1) = s^2$ | $p(1) = s^2$ |
| $p(0) = c^2$ | $p(0) = 0$ | $p(0) = 0$ |
| $p(1) = s^2$ | $p(1) = 1$ | $p(1) = 1$ |

TABLE V: Odds of getting 0 or 1 for all possible moves as a function of $\theta$.

Consequently the strategies followed by Alice depend on the value of $\theta$ as follows

- $0 < \theta < \theta_1$: Alice will guess 0 for all $O_1^A$.
- $\theta_1 < \theta < \theta_2$: Alice will guess 0 for $O_1^A$, and 1 for $O_2^A$ or $O_3^A$.
- $\theta_2 < \theta < \pi/2$: Alice will guess 1 for all $O_1^A$.

Averaging the corresponding probabilities yields a value of $P_A$ larger than 1/2 for all $\theta$’s, so reproducing the previous case where $\theta = \pi/4$. An analysis of the strategies that Bob may adopt does not change the situation even at the values of $\theta_1$ and $\theta_2$. We conclude that Alice always has a winning and stable strategy in this game.

The sets $C, \tilde{C}$ and the space $G$ involved in the hard-core boson game are simply the truncation of those used in the boson game, namely

$$C = \{1, 2, 3\}, \tilde{C} = \{0, 1\}, G = \text{span}\{0, 1\}.$$ (24)

while the distance for $G$ is the same as in (18).
V. ONE-QUBIT GAMES

In previous games, the players used the operators $O_i$ that are not unitary, which makes it difficult to carry out experimentally. In this section and the next one we are going to propose two games that solve this difficulty. In the first game we shall replace the hard-core boson by a qubit initialized in the state $|0\rangle$ on which the players act choosing between the following unitary transformations

$$O_1 = I, \quad O_2 = e^{-\frac{i\theta}{2}\sigma_y}, \quad O_3 = e^{\frac{i\theta}{2}\sigma_y}, \quad (25)$$

with $\theta \in (0, \pi/2)$. The operators $O_2$ and $O_3$ are rotations of angles $\theta$ and $-\theta$ around the $y$-axis. The state constructed by the device with the information provided by Alice and Bob is

$$|\Psi_{1,j}\rangle = O_1^j O_2^i |0\rangle$$

This state is already normalized since the operators (25) are unitary.

The probabilities of obtaining the states $|0\rangle$ or $|1\rangle$ are shown in Table VII. Notice the similarities with Table V. Notice the similarities with Table V.

| $O_1^j$ | $O_2^i$ | $O_3^k$ |
|--------|--------|--------|
| $p(0) = 1$ | $p(0) = c^2$ | $p(0) = c^2$ |
| $p(1) = 0$ | $p(1) = s^2$ | $p(1) = s^2$ |

TABLE VII: $c = \cos(\theta/2)$ and $s = \sin(\theta/2)$.

Applying the classical strategies, the probabilities for Alice averaged over Bob’s choices are given in Table VIII and plotted in Fig. 3.

| $O_1^j$ | $O_2^i$ | $O_3^k$ |
|--------|--------|--------|
| $p(0) = 0$ | $p(0) = 0$ | $p(0) = 1$ |
| $p(1) = 1$ | $p(1) = 0$ | $p(1) = 0$ |

TABLE VIII: Average odds for Alice.

As in the previous game there are three regions where Alice’s strategy is winning and stable. However, at their boundaries, Bob is able to balance the game, restoring the classical symmetry. The values of $\theta$ at the boundaries are given by

$$\langle p(0) \rangle_{O_{2,3}}^{\alpha} = \langle p(1) \rangle_{O_{2,3}}^{\alpha} = \frac{1}{2} \Rightarrow \theta_1 = \frac{\pi}{2}, \quad (27)$$

$$\langle p(0) \rangle_{O_{1,2,3}}^{\alpha} = \langle p(1) \rangle_{O_{1,2,3}}^{\alpha} = \frac{1}{2} \Rightarrow \theta_2 = \frac{2\pi}{3}.$$  

The strategies followed by Alice are the following:

- $0 \leq \theta < \theta_1^\prime$: Alice will guess 0 regardless of her choice of operator between $O_1^A$, $O_2^A$ and $O_3^A$.
- $\theta_1^\prime < \theta < \theta_2^\prime$: Alice will guess 0 if she chooses $O_1^A$ and 1 if she chooses $O_2^A$ or $O_3^A$.
- $\theta_2^\prime < \theta < \pi/2$: Alice will guess 1 if she chooses $O_2^A$ and 0 if she chooses $O_1^A$ or $O_3^A$.

At the boundary points we have the following situations:

- $\theta = \theta_1^\prime$: Bob notices that choosing randomly between $O_2^A$ and $O_3^A$ the game becomes symmetric for all Alice’s draws. Table IX shows the corresponding probabilities. Alice has therefore a winning but unstable strategy as Bob can balance it.

| $O_1^j$ | $O_2^i$ | $O_3^k$ |
|--------|--------|--------|
| $p(0) = 1/2$ | $p(0) = 0$ | $p(0) = 1$ |
| $p(1) = 1/2$ | $p(1) = 1$ | $p(1) = 0$ |

TABLE IX: Values given in Table VII for $\theta_1^\prime$ and Bob’s choices $O_2^B$ and $O_3^B$.

- $\theta = \theta_2^\prime$: At this point all the curves in Fig. 3 intersect at the value of 1/2. Bob does not even have to change his strategy to make the game symmetrical again. Moreover, Alice can do nothing to prevent this. This is observed by evaluating the probabilities shown in Table VII at $\theta_2^\prime$.

We conclude that in the one-qubit Chinos game with unitary operators, Alice will have a winning and stable strategy for any angle $\theta \in (0, \pi)$ except at the values $\theta_1^\prime$ and $\theta_2^\prime$. In the former, Bob can choose a strategy that balances the game, and in the latter, the game is symmetrized without the need for Bob to change his strategy. This result is essentially the same as the one obtained using a hard-core boson, where Alice has always a winning stable strategy.

In this one-qubit game, the sets $\mathcal{C}$, $\tilde{\mathcal{C}}$ and the space $\mathcal{G}$ are the same as those of the hard-core boson given in eq.(24).
VI. TWO-QUBITS GAMES

In the previous quantum games the players’ guesses where mapped by the device onto the states belonging to the orthonormal basis of the Hilbert space $\mathcal{G}$. Along the lines of reference [23], we shall next propose games where this condition is not imposed. This is implemented as follows. Alice and Bob will guess the states $|g^A\rangle$ and $|g^B\rangle$ respectively, with the condition that Bob’s state should be orthogonal to Alice’s state, that is

$$
\langle g^A | g^B \rangle = 0.
$$

(28)

In view of eq.(17), this implies that the distance between the players’ guess is exactly 1, i.e. $d_0 = 1$. This condition is the quantum version of the classical rule where players cannot repeat the guesses of the previous ones. Alice and Bob payoffs evaluate how successful each player was in predicting the state generated jointly $|\Psi_{a,b}\rangle$,

$$
f^A = |\langle g^A | \Psi_{a,b} \rangle|^2, \quad f^B = |\langle g^B | \Psi_{a,b} \rangle|^2.
$$

(29)

Eq.(17) relates these quantities to the distance of the players’ guess to the joint state. The winner of the game is the one with the highest payoff.

We shall consider a two qubit system on the computational basis $|i_0, i_1\rangle$ with $i_0, i_1 = 0, 1$. The operators that generates the joint state $|\Psi_{a,b}\rangle$ will be those that produce the Bell states acting on $|0,0\rangle$. The Bell states form an orthonormal basis of maximally entangled states of a two-qubit system given by

$$
|\phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \quad |\psi^\pm\rangle = \frac{|10\rangle \pm |11\rangle}{\sqrt{2}}.
$$

(30)

They can be constructed as

$$
|\phi^+\rangle = Bell(0,0)|00\rangle, \quad |\phi^-\rangle = Bell(0,1)|00\rangle, \quad |\psi^+\rangle = Bell(1,0)|00\rangle, \quad |\psi^-\rangle = Bell(1,1)|00\rangle.
$$

(31)

with

$$
Bell(i_0,i_0) = CNOT(id \otimes H)(X^{i_0} \otimes X^{i_1}),
$$

(32)

and where $H$ is the Hadamard gate, $X$ the NOT gate and $CNOT$ the gate

$$
CNOT|i_0,i_0\rangle = |i_1 \oplus i_0, i_0\rangle,
$$

(33)

where $i_0$ is the control qubit and $i_1$ is the target qubit.

We shall use in what follows the notation

$$
O_0 = Bell(0,0), \quad O_1 = Bell(0,1), \quad O_2 = Bell(1,0), \quad O_3 = Bell(1,1).
$$

(34)

Using these operators Alice and Bob will generate the joint state

$$
|\psi_{a_0,b_0}\rangle = O^A_{a_0}O^B_{b_0}|00\rangle,
$$

(35)

that is normalized because the operators (34) are unitary. An interesting property of all the joint states (35) is that they are maximally entangled. This can be proved computing the reduced density matrix of the qubit 0

$$
\rho_1 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}.
$$

(36)

Alice’s and Bob’s guesses are chosen as

$$
|g^A_{a_1,a_2}\rangle = O^A_aO^A_{a_2}|00\rangle; \quad a_1, a_2 = 0, 1, 2, 3,
$$

(37)

$$
|g^B_{b_1,b_2}\rangle = O^B_bO^B_{b_2}|00\rangle; \quad b_1, b_2 = 0, 1, 2, 3,
$$

(38)

that should be orthogonal to satisfy (28),

$$
\langle g^A_{a_1,a_2}|g^B_{b_1,b_2}\rangle = 0.
$$

(39)

The payoffs of the players are given by eq.(29) with $a, b$ replaced by $a_0, b_0$.

It is convenient to define a metric in the space expanded by the operators $O_i O_j$ acting on $|00\rangle$ [23],

$$
G_{(i_1,j_1),(i_2,j_2)} = \langle 00|O^T_{j_1}O_{i_1}O_{i_2}O_{j_2}|00\rangle,
$$

(40)

where $i_1,j_1,j_2,i_2 = 0, 1, 2, 3$. In terms of this metric the condition eq. (38) becomes

$$
G_{(a_1,a_2),(b_1,b_2)} = 0,
$$

(41)

and the payoffs of each player,

$$
f^A_{(a_1,a_2),(a_0,b_0)} = |G_{(a_1,a_2),(a_0,b_0)}|^2,
$$

$$
f^B_{(b_1,b_2),(a_0,b_0)} = |G_{(b_1,b_2),(a_0,b_0)}|^2.
$$

(42)

TABLE X: Metric $G_{(i_1,j_1),(i_2,j_2)}$ defined in eq. (39) using the operators of eq. (34) organized in the sets (42).

The entries of the metric (39) are given in Table X and has a block diagonal structure in terms of the following subsets,

$$
Set_1 = \{00, 22, 12, 30\}, \quad Set_2 = \{13, 31, 01, 23\},
$$

$$
Set_3 = \{02, 20, 10, 32\}, \quad Set_4 = \{11, 33, 03, 21\}.
$$

(43)
Since these sets are mutually orthogonal, if Alice chooses a state belonging to one set, then Bob has to choose a state belonging to a different set to satisfy condition (40).

With this metric, \( f^A \) and \( f^B \) can be either 0 or 1, that is Alice and Bob can either win or lose. In particular, for any of Alice’s choices, she can win with a certain Bob’s choice of \( b_0 \), i.e.

\[
\forall a_0, a_1, a_2, \exists b_0 / f^A(a_1, a_2, (a_0, b_0)) = 1. \tag{43}
\]

The situation of Bob is different. His choice of \( b_0 \) reduces the possible winning guesses to the following sets

\[
\text{if } b_0 = 0 \text{ or } 2 \implies (b_1, b_2) \in \text{Set}_1 \text{ or } \text{Set}_3, \tag{44}
\]

\[
\text{if } b_0 = 1 \text{ or } 3 \implies (b_1, b_2) \in \text{Set}_2 \text{ or } \text{Set}_4
\]

as can be verified in Table X.

Let us consider an example. Suppose that Alice chooses at random among the four sets of eq. (42) and that Bob chooses \( b_0 = 0 \). We shall consider the following cases:

- Alice selects \((a_1, a_2) \in \text{Set}_1\) then Bob, applying eq.(44) and the orthogonality rule (40), will choose \((b_1, b_2) \in \text{Set}_3\). The payoffs (41) for \( a_0 = 0, 1, 2, 3 \) and \( b_0 = 0 \) are given by

\[
\begin{array}{cccc}
\text{Set}_1 & 00 & 10 & 20 & 30 \\
f^A & 1 & 0 & 1 & 0 \\
\text{Set}_3 & 0 & 1 & 1 & 0
\end{array}
\]

- Alice selects \((a_1, a_2) \in \text{Set}_2\) then Bob choosing \((b_1, b_2) \in \text{Set}_1 \cup \text{Set}_3\). Their payoffs are

\[
\begin{array}{cccc}
\text{Set}_2 & 00 & 10 & 20 & 30 \\
f^A & 0 & 0 & 0 & 0 \\
\text{Set}_3 & 1 & 0 & 0 & 1 \\
\end{array}
\]

- Alice selects \((a_1, a_2) \in \text{Set}_3\). This case is similar to the choice of \text{Set}_1.

- Alice selects \((a_1, a_2) \in \text{Set}_4\). This case is similar to the choice of \text{Set}_2.

The average payoffs of Alice and Bob are given by

\[
\langle f^A \rangle = \frac{1}{2} \left( \frac{2}{4} + 0 \right) = \frac{1}{4}, \tag{45}
\]

\[
\langle f^B \rangle = \frac{1}{2} \left( \frac{2}{4} + \frac{2}{4} \right) = \frac{1}{2}, \tag{46}
\]

that gives the normalized probabilities

\[
P_A = \frac{1}{3}, \quad P_B = \frac{2}{3}.
\]

In view of this result, Alice will try to reverse the situation, but it will not be possible since, according to eq. (43), given \( a_0 \), any set can make her win depending on the value of \( b_0 \). We conclude that in this Chinos game Bob has a winning and stable strategy. This is in contrast with with game using bosons [23] where Alice has a winning and stable strategy.

### A. States with different entanglement

We shall next extend the previous model replacing the CNOT gate with a CU gate given by

\[
CU(|i_1 \otimes |i_0\rangle) = (U(\theta b_0)|i_1\rangle \otimes |i_0\rangle), \tag{47}
\]

where

\[
U(\theta) = e^{-i\frac{\theta}{2}X}, \tag{48}
\]

is a rotation around the x-axis of angle \( \theta \). The rotated Bell operators will be defined as

\[
RBell(i_1, i_0) = CU(id \otimes H)(X^{i_1} \otimes X^{i_0}). \tag{49}
\]

Obviously, for \( \theta = 0 \) we recover the states and operators considered previously.

The operators (49) will be denoted as

\[
O'_0 = RBell(0, 0); \quad O'_1 = RBell(0, 1);
\]

\[
O'_2 = RBell(1, 0); \quad O'_3 = RBell(1, 1). \tag{50}
\]

and using them Alice and Bob will generate the states

\[
|\psi_{a_0, b_0}'\rangle = O'_{a_0}O'_{b_0}|00\rangle. \tag{51}
\]

TABLE XI: Metric \( G'_{(i_1,i_2),(j_1,j_2)} \) as a function of the parameter \( \theta \) where \( c = \cos \frac{\theta}{2} \) and \( s = i \sin \frac{\theta}{2} \).

The entanglement of the states (51) depends on the value of \( \theta \). The density matrix obtained by tracing over the qubit 0 reads

\[
\rho' = \frac{1}{4} \left( \begin{array}{cc}
1 & k \\
k^* & 1
\end{array} \right), \quad k = \pm i \sin(\theta/2), \tag{52}
\]

and has eigenvalues \( \lambda_{\pm} = \frac{1}{2}(1 \pm \sin(\theta/2)) \) that are invariant under the replacement \( \theta \to 2\pi - \theta \). This symmetry allows us to restrict ourselves to the interval \( \theta \in [0, \pi] \). The maximal entanglement corresponds to \( \theta = 0 \), and the minimum to \( \theta = \pi \) where the states (51) become separable.
The metric $G'_{(i_1,j_1),(i_2,j_2)}$ is defined by

$$G'_{(i_1,j_1),(i_2,j_2)} = \langle 00|O_{i_1}^t O_{j_1}^t O_{i_2}^t O_{j_2}^t |00\rangle,$$

and its entries are given in Table XI, which coincide with those in Table X for $\theta = 0$. When $\theta \neq 0$ these entries $G'$ can be organized into the following pairs

$$P_1 = \{00, 22\}, \quad P_2 = \{12, 30\},$$
$$P_3 = \{13, 31\}, \quad P_4 = \{01, 23\},$$
$$P_5 = \{02, 20\}, \quad P_6 = \{10, 32\},$$
$$P_7 = \{11, 33\}, \quad P_8 = \{03, 21\},$$

that are related to the sets (42) as

$$Set_1 = P_1 \cup P_2, \quad Set_2 = P_3 \cup P_4,$$
$$Set_3 = P_5 \cup P_6, \quad Set_4 = P_7 \cup P_8.$$  

We shall next analyze the different strategies of Alice and Bob. Suppose that Alice chooses at random among the pairs given in eq. (54) and that Bob chooses $b_0 = 0$. This leads to the following cases:

- Alice selects $(a_1, a_2) \in P_1$ then Bob, applying the orthogonality rule (40), will choose $(b_1, b_2) \in P_{4,5,6,7,8}$. The payoffs (41) for $a_0 = 0, 1, 2, 3$ and $b_0 = 0$ are given by

$$f^B = |s|^2$$

- Alice selects $(a_1, a_2) \in P_3$ then Bob, applying the orthogonality rule (40), will choose $(b_1, b_2) \in P_{2,5,6,7,8}$. The payoffs (41) are given by

$$f^A = 0$$

- The cases where Alice selects $P_{2,4,5,6,7,8}$ are evenly distributed among the previous ones.

Taking into account that $1 + c^2 \geq |s|^2$, the best strategy for Bob is to select the pairs $P_5$ and $P_6$ with equal probability. In this case, the average payoffs of Alice and Bob are given by

$$\langle f^A \rangle = \frac{1}{2} \left( \frac{1 + c^2}{4} + \frac{|s|^2}{4} \right) = \frac{1}{4},$$
$$\langle f^B \rangle = \frac{1}{2} \left( \frac{1 + c^2}{4} + \frac{1 + c^2}{4} \right) = 1 + c^2,$$

that yields the normalized probabilities

$$P_A = \frac{1}{2 + c^2}, \quad P_B = \frac{1 + c^2}{2 + c^2}.$$  

For $\theta = 0$, one recovers eq. (46), while for $\theta = \pi$, one gets $P_A = P_B = 1/2$. Fig. 4 plots the values of (57), that shows. The highest the entanglement of the states (51) the highest probability has Bob for winning over Alice. Indeed the ratio of their probabilities can be related to the purity of the states, $	ext{tr}(\rho_1^2)$,

$$\frac{P_B}{P_A} = 3 - 2\text{tr}(\rho_1^2).$$  

\[\text{FIG. 4: Normalized probabilities (57) as a function of } \theta \in [0, \pi]\]

B. The order matters

In the quantum games based on the boson, hard-core boson and one-qubit, the operators $O^A$ and $O^B$, that generate the joint state, commute. However, in the two-qubit game, the operators (34) or (50) do not. This implies that the order of their action on the initial state may lead to different outcomes. We shall next analyze the case $\theta = 0$. The joint state is now created by Alice acting first and followed by Bob

$$\tilde{\psi}_{a_0, b_0} = O^B_{b_0} O^A_{a_0}\langle 00\rangle.$$  

The definition of the guess states (37) remains the same. This implies that the overlaps between the guess states and the joint state yield a matrix $\tilde{G}$ that is the partial transposed of the matrix $G$, that is

$$\tilde{G}_{(i,j),(a_0,b_0)} = G_{(i,j),(b_0,a_0)}.$$  

Hence, the payoffs on the Alice-first game, are given by those of the Bob-first game as

$$\tilde{f}^A_{(a_1,a_2),(a_0,b_0)} = f^A_{(a_1,a_2),(b_0,a_0)},$$
$$\tilde{f}^B_{(b_1,b_2),(a_0,b_0)} = f^B_{(b_1,b_2),(b_0,a_0)}.$$  

In this new game, one can expect that Alice will have some advantage because her choice of $a_0$ will restrict possible winning guesses. Nevertheless, this does not happen because Alice’s guess is done before Bob’s one, and then she will reveal information about the value of $a_0$ to Bob. This leads in turn to a symmetric game. Let us illustrate this result with an example. Suppose
that Alice chooses \(a_0 = 0\). Replacing \(b_0\) in eq.(44) by \(a_0\), one finds that the winning sets are \(S_{el1}\) or \(S_{el3}\). So, she will choose at random among both. However, Bob, knowing this information, will notice that \(a_0\) is equal to 0 or 2. Then, he will choose the set which is not chosen by Alice among the pair of sets.

\[
\begin{array}{cccc}
\hat{f}^A & 00 & 01 & 02 & 03 \\
S_{el1} & 1 & 0 & 0 & 1 \\
\hat{f}^B & 00 & 01 & 02 & 03 \\
S_{el3} & 0 & 1 & 1 & 0 \\
\end{array}
\]

This scenario ends with each one winning half of the times on average. The same probabilities will appear for every value of \(a_0\), so the normalized probabilities are

\[P_A = P_B = \frac{1}{2}.
\]

The sets \(C, \tilde{C}\) and the space \(G\) involved in the two-qubit game are given by

\[
\begin{align*}
C &= \{0, 1\}, \quad \tilde{C} = \{(0,0),(0,1),(1,0),(1,1)\}, \quad (63)
G &= \text{Span}\{\{00\}, \{01\}, \{10\}, \{11\}\},
\end{align*}
\]

where the distance in \(G\) is defined by (17).

**VII. SIMULATION ON AN IBM QUANTUM COMPUTER**

The previous games are completely characterized by the metric \(G\) given in eq.(39), and its extended version \(G'\) given in (53). Their values are given in Tables X and XI respectively. In this section, we shall present the results obtained using an IBM quantum computer to find the matrix \(G\) that corresponds to the case with \(\theta = 0\). We have also studied the case \(\theta = \pi\), but it will not be presented here.

To compute the matrix \(G\) we apply two \(O_i\)-operators and two \(O_j\)-operators on the state \(|00\rangle\). The probability of measuring the state \(|00\rangle\) provides \(|G|^2\). Some entries of \(G\) have phases but they are not relevant to the strategies of the game. Nevertheless, we have included them in the results.

The circuit to compute \(G_{(2,2),(3,0)}\) is given by

We used the qubits \(q0\) and \(q1\) of the quantum computer ibm-q-Manila whose topology is

The number of shots was set to its maximum, 8192, obtaining the results plotted in Fig. 5. The experimental value of 0.964 is very close to the theoretical value of 1 given in Table X. This is of course due to the noise and decoherence in this computer. We have determined experimentally all the entries of \(|G|\) and collect them in Table XII, including the signs. The theoretical values equal to 1 have an averaged error of 2-3%, while the null values have an average error of 10-24%. This implies that the orthogonality condition (38) has to a more relaxed, say to \(|\langle g_A | g_B \rangle | \leq 0.25\), if the game is to be close to the theoretical one.

![Fig. 5: Histogram of the probabilities obtained to measure |G_(2,2),(3,0)|^2 after 8192 shots](image)

**TABLE XII: Experimental values of G_(43,31),(52,23) measured obtained on ibmq-Manila.**

| \(G_{(2,2),(3,0)}\) | 0.90 | 0.92 | 0.94 | 0.96 | 0.98 | 1.00 |
|-----------------|------|------|------|------|------|------|
| 0.10 | 0.10 | 0.10 | 0.10 | 0.10 | 0.10 |
| 0.20 | 0.20 | 0.20 | 0.20 | 0.20 | 0.20 |
| 0.30 | 0.30 | 0.30 | 0.30 | 0.30 | 0.30 |
| 0.40 | 0.40 | 0.40 | 0.40 | 0.40 | 0.40 |

**VIII. CONCLUSIONS AND PROSPECTS**

We have presented in this work a general formalism of the Chinos game between two players that includes the quantum games introduced in reference [23] using a boson and extend them to a hard-core boson, a one-qubit system, and a two-qubits system.

A general Chinos game is defined by: i) two sets, \(C, \tilde{C}\), containing the players’ choices and guesses, ii) a device that maps the previous data to a metric space \(G\) whose distance is used to compute the players’ payoffs, iii) a restriction rule on the second player, and iv) an intelligence rule to optimize the players’ guesses. In the games we have considered, the sets \(C, \tilde{C}\) are a discrete and finite collection of integers. In the classical Chinos game the space \(G\) is also discrete and finite, but in the quantum Chinos games \(G\) is a Hilbert space. We have employed the trace norm of density matrices to define the payoffs and the restriction rule, which for pure states is in direct correspondence with the fidelity. However, one can consider mixed states in which case...
the two criteria, i.e. distance or fidelity, can lead to different results.

The game based on two-qubits shows an interesting interplay between entanglement and the success probabilities of the players. Moreover, the game is sensitive to the way the joint state is created. In the one and two qubit games, the device can be implemented on a quantum computer, which has allowed us to simulate the basic matrix involved in the game using an IBM quantum computer.

The rules of the Chinos game make it similar to those considered in the quantum decision theory whose mathematics is based on the theory of quantum measurement [13]-[15]. These classes of games can also be categorized as abstract economics because the player’s strategies are not independent of each other [12]. Finally, we would like to notice that the formalism presented here can be extended in a natural way to more than two players. It will be interesting to explore its application to quantum communication protocols.

Acknowledgements.-

GS acknowledges financial support through the Spanish MINECO grant PGC2018- 095862-B-C21, the Comunidad de Madrid grant No. S2018/TCS-4432, the Centro de Excelencia Severo Ochoa Program SEV-2016-0597 and the CSIC Research Platform on Quantum Technologies PTI-001.

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