Local Well-posedness of the Coupled KdV-KdV Systems on $\mathbb{R}$

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Abstract

Inspired by the recent successful completion of the study of the well-posedness theory for the Cauchy problem of the Korteweg-de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \quad u|_{t=0} = u_0$$

in the space $H^s(\mathbb{R})$ (or $H^s(\mathbb{T})$), we study the well-posedness of the Cauchy problem for a class of coupled KdV-KdV (cKdV) systems

$$\begin{cases}
u_t + a_1 u_{xxx} = c_{11} uu_x + d_{11} u_x v + d_{12} uv_x, \\
v_t + a_2 v_{xxx} = c_{21} uu_x + d_{21} u_x v + d_{22} uv_x,
\end{cases}$$

$$((u,v)|_{t=0} = (u_0, v_0)$$

in the space $\mathcal{H}^s(\mathbb{R}) := H^s(\mathbb{R}) \times H^s(\mathbb{R})$. Typical examples include the Gear-Grimshaw system, the Hirota-Satsuma system and the Majda-Biello system, to name a few. They usually serve as models to describe the interaction of two long waves with different dispersion relations.

In this paper we look for those values of $s \in \mathbb{R}$ for which the cKdV systems are well-posed in $\mathcal{H}^s(\mathbb{R})$. Our findings enable us to provide a complete classification for the cKdV systems in terms of the analytical well-posedness in $\mathcal{H}^s(\mathbb{R})$ based on its coefficients $a_i$, $c_{ij}$ and $d_{ij}$ for $i, j = 1, 2$. The key ingredients in the proofs are the bilinear estimates in both divergence and non-divergence forms under the Fourier restriction space norms. There are four types of the bilinear estimates that need to be investigated. Sharp results are established for all of them. In contrast to the lone critical index $-\frac{3}{4}$ for the single KdV equation, the critical indexes for the cKdV systems are $-\frac{13}{12}, -\frac{2}{7}, 0$ and $\frac{3}{4}$.

As a result, the cKdV systems are classified into four classes, each of which corresponds to a unique index $s^* \in \{-\frac{13}{12}, -\frac{2}{7}, 0, \frac{3}{4}\}$ such that any system in this class is locally analytically well-posed if $s > s^*$ while the bilinear estimate fails if $s < s^*$.

1 Introduction

1.1 Problem to study

This paper studies the Cauchy problem of a class of coupled KdV-KdV systems posed on the whole line $\mathbb{R}$ of the following general form,

$$\begin{cases}
u_t + A_1 \begin{pmatrix} u_{xxx} \\ v_{xxx} \end{pmatrix} + A_2 \begin{pmatrix} u_x \\ v_x \end{pmatrix} = A_3 \begin{pmatrix} uu_x \\ vv_x \end{pmatrix} + A_4 \begin{pmatrix} uu_x \\ vv_x \end{pmatrix}, \\
\begin{pmatrix} u \\ v \end{pmatrix}|_{t=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},
\end{cases}$$

(1.1)

where $\{A_i\}_{1 \leq i \leq 4}$ are $2 \times 2$ real constant matrices, $u = u(x,t)$, $v = v(x,t)$ are real-valued unknown functions of the two real variables $x$ and $t$, and subscripts adorning $u$ and $v$ connote partial differentiations $\partial_t$ or $\partial_x$.

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It is assumed that there exists an invertible real matrix $M$ such that
\[ A_1 = M \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} M^{-1}, \]
with $a_1a_2 \neq 0$. By regarding $M^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$ as the new unknown functions (still denoted by $u$ and $v$), the system (1.1) can be rewritten in the following form,
\[
\begin{align*}
u_t + a_1u_{xxx} + b_{11}u_x &= -b_{12}v_x + c_{11}uu_x + c_{12}vv_x + d_{11}ux_v + d_{12}uv_x, \\
v_t + a_2v_{xxx} + b_{22}v_x &= -b_{21}u_x + c_{21}uu_x + c_{22}vv_x + d_{21}ux_v + d_{22}uv_x, \\
(u, v)|_{t=0} &= (u_0, v_0).
\end{align*}
\]
This system is called in \textit{divergence form} if $d_{11} = d_{12}$ and $d_{21} = d_{22}$. Otherwise, it is called in \textit{non-divergence form}.

Listed below are a few specializations of (1.1) appeared in the literature.

- **Majda-Biello system:**
  \[
  \begin{align*}
u_t + u_{xxx} &= -vv_x, \\
v_t + a_2v_{xxx} &= -(uv)_x, \\
(u, v)|_{t=0} &= (u_0, v_0),
  \end{align*}
  \]
  where $a_2 \neq 0$. This system was proposed by Majda and Biello in \cite{34} as a reduced asymptotic model to study the nonlinear resonant interactions of long wavelength equatorial Rossby waves and barotropic Rossby waves.

- **Hirota-Satsuma system:**
  \[
  \begin{align*}
u_t + a_1u_{xxx} &= -6a_1uu_x + c_{12}vv_x, \\
v_t + v_{xxx} &= -3uv_x, \\
(u, v)|_{t=0} &= (u_0, v_0),
  \end{align*}
  \]
  where $a_1 \neq 0$. This system was proposed by Hirota-Satsuma in \cite{17} to describe the interaction of two long waves with different dispersion relations.

- **Gear-Grimshaw system:**
  \[
  \begin{align*}
u_t + u_{xxx} + \sigma_3v_{xxx} &= -uu_x + \sigma_1vv_x + \sigma_2(uv)_x, \\
\rho_1v_t + \rho_2\sigma_3u_{xxx} + v_{xxx} + \sigma_4ux &= \rho_2\sigma_3uux - vv_x + \rho_2\sigma_1(uv)_x, \\
(u, v)|_{t=0} &= (u_0, v_0),
  \end{align*}
  \]
  where $\sigma_i \in \mathbb{R} (1 \leq i \leq 4)$ and $\rho_1, \rho_2 > 0$. This system is a special case of (1.1) by setting
  \[
  A_1 = \begin{pmatrix} 1 & \frac{\sigma_3}{\rho_1} \\ \frac{\rho_2\sigma_3}{\rho_1} & \frac{1}{\rho_1} \end{pmatrix},
  \]
  \[ (1.6) \]

Note that $A_1$ in (1.6) is diagonalizable over $\mathbb{R}$ for any $\sigma_3 \in \mathbb{R}$ and $\rho_1, \rho_2 > 0$. Moreover, the eigenvalues of $A_1$ are nonzero unless $\rho_2\sigma_3^2 = 1$. So (1.5) can be reduced to the form (1.2) as long as $\rho_2\sigma_3^2 \neq 1$. This system was derived by Gear-Grimshaw in \cite{15} (also see \cite{5} for the explanation about the physical context) as a model to describe the strong interaction of two-dimensional, weakly nonlinear, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid, where the two waves correspond to different modes.

In this paper we study the well-posedness of the Cauchy problem (1.2) in the space
\[ H^s(\mathbb{R}) \times H^s(\mathbb{R}) \equiv \mathcal{H}^s(\mathbb{R}). \]
The Cauchy problem (1.2) can be viewed as a special example of the following abstract Cauchy problem,

$$\frac{dw}{dt} + Lw = N(w), \quad w(0) = \phi,$$

where $L$ is a linear operator, $N$ is a possibly time-dependent nonlinear operator and the initial datum $\phi$ belongs to a Banach space $X_s$ with index $s \in \mathbb{R}$. The scale of Banach spaces $X_s$ has the property $X_{s_2} \subset X_{s_1}$ if $s_1 \leq s_2$. The well-posedness considered in this paper is understood in the following sense.

**Definition 1.1.** The Cauchy problem (1.7) is said to be well-posed in the space $X_s$ if for any $\delta > 0$ there is a $T = T(\delta) > 0$ such that

(a) for any $\phi \in X_s$ with $\|\phi\|_{X_s} \leq \delta$, (1.7) admits exactly one solution $w$ in the space $C([0,T];X_s)$ satisfying the auxiliary condition

$$w \in Y^T_s$$

where $Y^T_s$ is an auxiliary metric space;

(b) the solution $w$ depends continuously on its initial data $\phi$ in the sense that the mapping $\phi \to u$ is continuous from $\{\phi : \|\phi\|_{X_s} \leq \delta\}$ to $C([0,T];X_s)$.

The well-posedness described by Definition 1.1 is local in character since the time $T$ depends on $\delta$. If $T$ can be specified independently of $\delta$ in Definition 1.1 then (1.7) is said to be globally well-posed in the space $X_s$. On the other hand, the Cauchy problem (1.7) is said to be (locally) uniformly well-posed, $C^k$-well-posed ($k \geq 0$), or analytically well-posed in the space $X_s$ if the corresponding solution map is (locally) uniform continuous, $C^k$ or real analytic.

In this paper, we are looking for an answer to the following problem.

**Problem:** For what values of $s \in \mathbb{R}$ is the Cauchy problem (1.2) well-posed in the space $H^s(\mathbb{R})$?

### 1.2 Literature review

It is beneficial and instructive to the study of the Cauchy problem (1.2) by first reviewing the well-posedness of the Cauchy problem of the single KdV equation

$$u_t + uu_x + u_{xxx} = 0, \quad u(x,0) = u_0(x)$$

posed either on the whole real line $\mathbb{R}$ or on a periodic domain $T$. The study began in the late 1960s with the work of Sjöberg [43,45] and has come to a happy end with the work of Killip and Visan [30]. Looking back, this study, which has lasted more than half a century, can be divided into four stages with four different major approaches developed in the process.

In Stage 1, (1.9) was most studied using traditionally PDE and functional analysis techniques. Sjöberg [43,45] and Temam [49] (see [18,42,50,51] and the references therein for some other works followed) obtained the existence and uniqueness of solutions of (1.9) on $T$ in the space $L^\infty(0,T;H^3(\mathbb{T}))$ (instead of in the space $C([0,T];H^3(\mathbb{T}))$), but without showing the continuity of the associated solution map. The first well-posedness result was due to Bona and Smith [6] who showed that (1.9) is (globally) well-posed in the space $H^k(\mathbb{R})$ or $H^k(\mathbb{T})$ for any integer $k \geq 2$ using a cleverly designed regularization scheme and classical energy estimate method. Then, (1.9) was shown by Bona and Scott [4] to be (globally) well-posed in the space $H^s(\mathbb{R})$ or $H^s(\mathbb{T})$ for any real number $s \geq 2$ using Tartar’s nonlinear interpolation theory [48]. After this, as one of the applications of the semigroup theory, which is a powerful general theory dealing with various quasi-linear evolutionary PDEs, Kato [20,23] showed that (1.9) is locally well-posed in $H^s(\mathbb{R})$ for any $s > \frac{3}{2}$.

In Stage 2, as various smoothing properties of dispersive wave equations were discovered in 1980s (cf. [12,23,25,46], Kenig, Ponce and Vega [24,26,28] were able to exploit the various dispersive smoothing properties of the linear KdV equation to show that (1.9) is locally well-posed in the space $H^s(\mathbb{R})$ for any $s > \frac{3}{2}$ by applying the contraction mapping principle in a carefully constructed Banach space, now known as the Kenig-Ponce-Vega (or KPV) space. As one of the key linear estimate fails when $s < \frac{3}{4}$, one can only show that (1.9) is well-posed in $H^s(\mathbb{R})$ for $s > \frac{3}{4}$ using this approach.
In Stage 3, Bourgain [8] introduced the Fourier restriction spaces $X_{s,b}$ and showed that the Cauchy problem (1.9) is well-posed in both spaces $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ for any $s \geq 0$ by using the contraction mapping principle in $X_{s,b}$. Then Kenig, Ponce and Vega [29] showed that (1.9) is locally well-posed in $H^s(\mathbb{R})$ for any $s > -\frac{3}{4}$ and in $H^s(\mathbb{T})$ for any $s \geq -\frac{1}{2}$. The local well-posedness of (1.9) in the space $H^{-\frac{3}{4}}(\mathbb{R})$ was established by Christ, Colliander and Tao [10]. The thresholds $-\frac{3}{4}$ for $H^s(\mathbb{R})$ and $-\frac{1}{2}$ for $H^s(\mathbb{T})$ are sharp if one requires the solution map to be uniformly continuous, see [10]. Moreover, (1.9) has been shown to be globally well-posed in $H^s(\mathbb{R})$ for $s \geq -\frac{3}{4}$ and in $H^s(\mathbb{T})$ for $s \geq -\frac{1}{2}$ (see [11,16,31]).

In Stage 4, Kappeler and Topalov [19] proved that (1.9) is globally well-posed in the space $H^s(\mathbb{R})$ if one requires the solution map to be uniformly continuous, see [10]. Moreover, (1.9) has been shown by Molinet [35,36] that (1.9) is ill-posed in both $H^s(\mathbb{R})$ and $H^s(\mathbb{T})$ for any $s < -1$, the study of the well-posedness of (1.9) has drawn a satisfactory conclusion.

There is a difference between the well-posedness presented in Stages 1 and 4 and those presented in Stages 2 and 3. For the well-posedness obtained in Stage 1 and 4, the solution of (1.9) depends only continuously on its initial value. By contrast, for the well-posedness established in Stage 2 and 3, one can show the solution of (1.9) depends on its initial value analytically (cf. [52,54]). Thus the Cauchy problem (1.9) is analytically well-posed in $H^s(\mathbb{R})$ for $s \geq -\frac{3}{4}$ and in $H^s(\mathbb{T})$ for $s \geq -\frac{1}{2}$, but is only continuously well-posed in $H^s(\mathbb{R})$ for $-1 \leq s < -\frac{3}{4}$ and in $H^s(\mathbb{T})$ for $-1 \leq s < -\frac{1}{2}$.

Naturally, following the advances of the study of the well-posedness of the Cauchy problem (1.9) for the single KdV equation, there have been many works on the well-posedness of the Cauchy problem (1.2) for the coupled KdV-KdV systems. Here we provide a brief summary of the previous results on $H^s(\mathbb{R})$. As a convenience of the notation, LWP and GWP will stand for local well-posedness and global well-posedness.

- Majda-Biello system (1.3).
  - If $\sigma_2 = 1$, the LWP in $H^s(\mathbb{R})$ for any $s > -\frac{3}{4}$ follows immediately from the single KdV theory. The GWP in $H^s$ for any $s > -\frac{3}{4}$ was justied by Oh [37] via the I-method.
  - If $\sigma_2 \in (0,4) \setminus \{1\}$, Oh [38] proved that (1.3) is locally well-posed in $H^s(\mathbb{R})$ for $s \geq 0$ and ill-posed when $s < 0$ if the solution map is required to be $C^2$. The key ingredient in the proof for the LWP is the bilinear estimate under the Fourier restriction norm. Due to the $L^2$ conservation law of (1.3), its GWP in $H^s(\mathbb{R})$ for $s \geq 0$ automatically holds.

- Hirota-Satsuma system (1.4).
  - Alvarez-Carvajal [1] proved the LWP for (1.4) in $H^s(\mathbb{R})$ for $s > \frac{3}{4}$ via the method in [26].
  - Feng [14] considered a slightly general system:
    \[
    \begin{align*}
    u_t + a_1 u_{xxx} &= -6a_1 uu_x + c_{12}vv_x, \\
    v_t + v_{xxx} &= c_{22}vv_x + d_{22}uv_x, \\
    (u,v)|_{t=0} &= (u_0,v_0).
    \end{align*}
    \]  
    (1.10)

    When $c_{22} = 0$ and $d_{22} = -3$, (1.10) reduces to the original Hirota-Satsuma system (1.4). Feng proved the LWP of (1.10) in $H^s(\mathbb{R})$ for $s \geq 1$ under the assumption that $a_1 \neq 1$ and $c_{12}d_{22} < 0$. The GWP was also shown by the further restriction that $0 < a_1 < 1$.

- Gear-Grimshaw system (1.5).
  - Assume $\sigma_2 = 0$ and $\rho_2 \sigma_3^2 \neq 1$. Bona-Ponce-Saut-Tom [5] proved the LWP of (1.5) in $H^s(\mathbb{R})$ for $s \geq 1$. They also showed the GWP of (1.5) in $H^s(\mathbb{R})$ for $s \geq 1$ under further assumption that $\rho_2 \sigma_3^2 < 1$.
  - Later, further LWP and GWP results were proven by Ash-Cohen-Wang [2], Linares-Panthee [33] and Saut-Tzvetkov [43], where the best LWP result is proven in $H^s(\mathbb{R})$ for $s > -\frac{3}{4}$. However, their argument essentially requires the matrix $A_1$ in (1.6) to be similar to the identity matrix, which means $\sigma_3 = 0$ and $\rho_1 = 1$. Equivalently, if considering the diagonalized system (1.2), their
results are only valid under the assumption that \( a_1 = a_2 \) (see Remark 1.2 in [38] and Remark 3.1 in [1] for more detailed explanations).

- General coupled KdV-KdV systems

  Alvarez-Carvajal [1] considered the diagonalized system (1.2) where \((b_{ij}) = 0, d_{11} = d_{12}\) and \(d_{21} = d_{22}\), i.e.,

  \[
  \begin{align*}
  u_t + a_1 u_{xxx} &= c_{11} u u_x + c_{12} v v_x + d_{11} (uv)_x, \\
  v_t + a_2 v_{xxx} &= c_{21} u u_x + c_{22} v v_x + d_{22} (uv)_x, \\
  (u,v)|_{t=0} &= (u_0, v_0).
  \end{align*}
  \tag{1.11}
  \]

  They proved that (1.11) is locally well-posed in \( H^s(\mathbb{R}) \) for \( s > -\frac{3}{4} \) if \( a_1 = -a_2 \neq 0 \). The key tool in their proof is the bilinear estimate under the Fourier restriction norm. The question whether (1.11) is well-posed in \( H^s(\mathbb{R}) \) when \( |a_1| \neq |a_2| \) is left open in [1]. On the other hand, Alvarez-Carvaja’s result in [1] actually does not apply to the Gear-Grimshaw system (1.5) since \( a_1 = -a_2 > 0 \) implies \( \rho_1 = -1 \) which is against the assumption \( \rho_1 > 0 \).

1.3 Main results on well-posedness

As we have seen from the literature review, the dispersion coefficients \( a_1 \) and \( a_2 \), and other coefficients \((b_{ij}), (c_{ij})\) and \((d_{ij})\), in the systems (1.2) have significant impact on the well-posedness results. The following theorem is the main finding we have obtained so far.

**Theorem 1.2.** Let \( a_1, a_2 \in \mathbb{R} \setminus \{0\} \) and denote \( r = \frac{a_2}{a_1} \). Then (1.2) is locally analytically well-posed in \( H^s(\mathbb{R}) \) for any case in Table 1.

| Case | \( \frac{a_2}{a_1} \) | Coefficients \( b_{ij}, c_{ij} \) and \( d_{ij} \) | \( s \) |
|------|----------------------|------------------------------------------------|------|
| (1)  | \( r < 0 \)          | \( (c_{ij}) = 0, d_{11} = d_{12} \) and \( d_{21} = d_{22} \) | \( s \geq -\frac{13}{12} \) or \( s > -\frac{3}{4} \) |
|      |                      | Otherwise                                           |      |
| (2)  | \( 0 < r < \frac{1}{4} \) | \( c_{12} = d_{21} = d_{22} = 0 \) | \( s > -\frac{3}{4} \) or \( s \geq 0 \) |
|      |                      | Otherwise                                           |      |
| (3)  | \( r = \frac{1}{4} \) | \( c_{21} = d_{11} = d_{12} = 0 \) | \( s \geq 0 \) |
|      |                      | Otherwise                                           |      |
| (4)  | \( \frac{1}{4} < r < 1 \) | arbitrary | \( s \geq 0 \) |
| (5)  | \( r = 1 \)          | \( b_{12} = b_{21} = 0, d_{11} = d_{12} \) and \( d_{21} = d_{22} \) | \( s > -\frac{3}{4} \) |
|      |                      | \( b_{12} = b_{21} = 0, d_{11} \neq d_{12} \) or \( d_{21} \neq d_{22} \) | \( s > 0 \) |
| (6)  | \( 1 < r < 4 \)      | arbitrary | \( s \geq 0 \) |
| (7)  | \( r = 4 \)          | \( c_{12} = d_{21} = d_{22} = 0 \) | \( s \geq 0 \) |
|      |                      | Otherwise                                           |      |
| (8)  | \( r > 4 \)          | \( c_{21} = d_{11} = d_{12} = 0 \) | \( s > -\frac{3}{4} \) |
|      |                      | Otherwise                                           |      |

The well-posedness results presented in Theorem 1.2 are sharp in the sense that the key bilinear estimates used in their proofs are sharp (up to the endpoints), see Theorem 3.3–3.6.
As applications, we apply Theorem 1.2 to a few specializations of (1.2). First, we consider a special class of (1.2) of the following form

\[
\begin{align*}
    u_t + a_1 u_{xxx} &= d_1(uv)_x, \\
    v_t + a_2 v_{xxx} &= d_2(uv)_x, \\
    (u, v)|_{t=0} &= (u_0, v_0).
\end{align*}
\] (1.12)

**Theorem 1.3.** If \(a_1 a_2 < 0\), then the system (1.12) is locally analytically well-posed in \(H^s(\mathbb{R})\) for \(s \geq -\frac{13}{12}\).

The above theorem is surprising since even the Cauchy problem (1.9) of the single KdV equation is ill-posed in \(H^s(\mathbb{R})\) for any \(s < -1\).

**Theorem 1.4.** The Majda-Biello system (1.3), where \(a_2 \neq 0\), is locally (resp. globally) analytically well-posed in \(H^s(\mathbb{R})\) for any case in Table 2 (resp. Table 3).

**Table 2: LWP Results**

| Case | Coefficient \(a_2\) | \(s\) |
|------|----------------------|------|
| (1)  | \(a_2 \in (-\infty, 0) \cup \{1\} \cup \{4, \infty\}\) | \(s > -\frac{3}{4}\) |
| (2)  | \(a_2 \in (0, 1) \cup (1, 4)\) | \(s \geq 0\) |
| (3)  | \(a_2 = 4\) | \(s \geq \frac{3}{4}\) |

**Table 3: GWP Results**

| Case | Coefficient \(a_2\) | \(s\) |
|------|----------------------|------|
| (1)  | \(a_2 = 1\) | \(s > -\frac{3}{4}\) |
| (2)  | \(a_2 \notin \{1, 4\}\) | \(s \geq 0\) |
| (3)  | \(a_2 = 4\) | \(s \geq 1\) |

Remark: in Theorem 1.4, Case (1) and (2) in Table 2 and 3 have been known earlier in Oh [37,38].

**Theorem 1.5.** The Hirota-Satsuma systems (1.4), where \(a_1 \neq 0\), is locally (resp. globally) analytically well-posed in \(H^s(\mathbb{R})\) for any case in Table 4 (resp. Table 5).

**Table 4: LWP Results**

| Case | Coefficients \(a_1\) and \(c_{12}\) | \(s\) |
|------|-----------------------------------|------|
| (1)  | \(a_1 \in (-\infty, 0) \cup (0, \frac{1}{4})\) | \(s > -\frac{3}{4}\) |
| (2)  | \(a_1 \in (\frac{1}{4}, 1) \cup (1, \infty)\) | \(s \geq 0\) |
| (3)  | \(a_1 = 1\) | \(s > 0\) |
| (4)  | \(a_1 = \frac{1}{4}\) | \(s \geq \frac{3}{4}\) |

**Table 5: GWP Results**

| Case | Coefficients \(a_1\) and \(c_{12}\) | \(s\) |
|------|-----------------------------------|------|
| (1)  | \(a_1 \notin \{\frac{1}{4}, 1\}\), \(c_{12} > 0\) | \(s \geq 0\) |
| (2)  | \(a_1 = \frac{1}{4}\), \(c_{12} > 0\) | \(s \geq 1\) |

We finally turn to the Gear-Grimshaw system (1.5) and introduce the condition (1.13) for convenience.

\[
\rho_2 \sigma_2^3 \leq \frac{9}{25} \quad \text{and} \quad \rho_1^2 + \frac{25 \rho_2 \sigma_2^3 - 17}{4} \rho_1 + 1 = 0.
\] (1.13)

**Theorem 1.6.** The Gear-Grimshaw system (1.5), where \(\rho_1, \rho_2 > 0\), is locally (resp. globally) analytically well-posed in \(H^s(\mathbb{R})\) for any case in Table 6 (resp. Table 7).

It should be pointed out that Case (1) in Table 6 is trivial since it directly follows from the proof of the single KdV case.
1.4 Remarks

A few remarks are now in order.

Remark 1.7. While the results presented in Section 1.3 provides a rather thorough description of the analytically well-posedness in $H^s(\mathbb{R})$ for the systems (1.2), the study of the well-posedness of the Cauchy problem of (1.2) in $H^s(\mathbb{R})$ is far from over in comparison to the study of the KdV equation (1.9). We list below a few problems among many to be investigated.

- Question 1.1: For the locally analytically well-posedness results of (1.2) listed in Table 1, it requires $s > \frac{-3}{4}$ in Cases (1), (2), (5) and (8). Can these results be strengthened to include $s = -\frac{3}{4}$?

- Question 1.2: The locally analytically well-posedness results of the systems (1.2) listed in Table 1 are sharp in the sense that the needed bilinear estimates, a key ingredient in the proofs, fail if $s$ is less than the corresponding critical index $s^\ast$. Is the Cauchy problem of (1.2) analytically ill-posed in the space $H^s(\mathbb{R})$ for any $s$ which is less than the corresponding critical index $s^\ast$?

- Question 1.3: Can those locally analytically well-posedness results of the systems (1.2) listed in Table 1 be strengthened to be globally analytically well-posed results?

Remark 1.8. As hinted by the study of the single KdV equation, the answers to both Question 1.1 and Question 1.2 will most likely be positive. For Question 1.1, some more subtly modified Bourgain spaces may need to be constructed, see e.g. [16, 31]. For Question 1.2, some counter examples are needed to show that the solution map fails to be smooth if $s$ is less than the corresponding critical index $s^\ast$. We leave this study to future works since the current paper is already long.

Remark 1.9. For Question 1.3, as long as one can establish a priori global $H^s(\mathbb{R})$ estimates for solutions of the system (1.2), the GWP of (1.2) in $H^s(\mathbb{R})$ follows from the corresponding LWP result. In particular, when there are conserved energy at certain regularity level, the corresponding GWP can be easily verified. For example, we also include some GWP results in Theorem 1.4–1.6. But if the regularity considered in the well-posedness problem is lower than the level provided by the available conserved energy, one may need to apply other methods, such as the I-method [11], to establish the GWP.

Remark 1.10. The single KdV equation has also been intensively studied from control point of views for its controllability and stabilizability (the interested readers are referred to [2,13,22,29,41,52,56] and the references therein for an overview of this subject). Various tools developed in the study of the well-posedness of the single KdV equation have played important roles in studying control theory of the KdV equation. By contrast, there are few studies of the systems (1.2) from control points of view. We expect the results and the tools obtained and developed in the study of the well-posedness of the Cauchy problem of (1.2) will stimulate and play important roles in further studies of the control theory for the coupled KdV-KdV systems.

1.5 Organization

The remaining of the paper is organized as follows. In Section 2, some linear estimates are recalled or proved as a preparation. In Section 3, we present our main results on the bilinear estimates which are the
key ingredients in the proof of the main well-posedness result: Theorem 1.2. The proofs of these bilinear estimates will be postponed to Sections 5 and 6. In Section 4, we prove Theorem 1.2 and its consequences, Theorem 1.3-1.6. Section 5 is devoted to establish the various bilinear estimates, Theorem 3.3 and 3.5, presented in Section 3. Finally, Section 6 is dedicated to justify Theorem 3.4 and 3.6 which exhibit the sharpness of the various bilinear estimates.

2 Preliminaries

Let $\psi \in C^\infty_c(\mathbb{R})$ be a bump function supported on $[-2, 2]$ with $\psi = 1$ on $[-1, 1]$. We will use $C$ and $C_i(i \geq 1)$ to denote the constants. Moreover, $C = C(a, b \ldots)$ means the constant $C$ only depends on $a, b \ldots$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. The notation $A \gtrsim B$ is used similarly. In addition, we will write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Finally, the notation $\langle \cdot \rangle$ means $1 + |\cdot|$.

Consider the Cauchy problem of the following linear KdV equation with $\psi = 1$ on $[-1, 1]$. We will use $C$ and $C_i(i \geq 1)$ to denote the constants. Moreover, $C = C(a, b \ldots)$ means the constant $C$ only depends on $a, b \ldots$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$. The notation $A \gtrsim B$ is used similarly. In addition, we will write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Finally, the notation $\langle \cdot \rangle$ means $1 + |\cdot|$.

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By Holder’s inequality, it suffices to verify
\[
\sup_{\xi, \tau \in \mathbb{R}} \frac{|\xi|}{\langle L_1 \rangle^b \langle L_2 \rangle^{1-b}} \leq C.
\] (2.7)

If $|\xi| \leq 1$, then (2.7) holds for $C = 1$. If $|\xi| > 1$, then it follows from $\frac{1}{2} < b \leq \frac{2}{3}$ that
\[
\langle L_1 \rangle^b \langle L_2 \rangle^{1-b} \geq \langle (L_1) \rangle^{\frac{b}{3}} \geq \langle L_1 - L_2 \rangle^{\frac{2}{3}}.
\]

Since $\alpha_1 \neq \alpha_2$, then it is easy to see that when $|\beta_2 - \beta_1| \leq \epsilon$ for a sufficiently small $\epsilon = \epsilon(\alpha_1, \alpha_2)$, we have
\[
\langle L_1 - L_2 \rangle = |(\alpha_2 - \alpha_1)\xi^3 - (\beta_2 - \beta_1)\xi| + 1 \geq \frac{|\alpha_2 - \alpha_1|}{2} |\xi|^3.
\]

Thus (2.7) also holds when $|\xi| > 1$.

**Proposition 2.3.** If $\alpha_1 = \alpha_2 \neq 0$, then for any $s, b, \beta_1, \beta_2 \in \mathbb{R}$, there does not exist a constant $C = C(\alpha_1, \alpha_2, s, b, \beta_1, \beta_2)$ such that (2.5) holds.

*Proof.* Let $\alpha_1 = \alpha_2 := \alpha$. If there exist $s, b, \beta_1, \beta_2 \in \mathbb{R}$ such that (2.5) holds for some constant $C$, then (2.6) needs to be true for any $f_j \in L^2(\mathbb{R} \times \mathbb{R})$, $j = 1, 2$. We will only prove the statement in the case when $b \geq \frac{3}{2}$ since the situation when $b < \frac{3}{2}$ is similar. When $b \geq \frac{3}{2}$, for any $N \geq 2$, define $f_1(\xi, \tau) = f_2(\xi, \tau) = 1_{E}(\xi, \tau)$ with
\[
E = \{(\xi, \tau) \in \mathbb{R}^2 : N - 1 \leq \xi \leq N, |\tau - \alpha \xi^3 + \beta_1 | \leq 1\},
\]
then for any $(\xi, \tau) \in E$, $|L_1| \leq 1$ and $|L_2| = |L_1 + (\beta_2 - \beta_1)\xi| \leq N$. In addition, the area of $E$ is 2 by direct calculation. As a result, the right hand side of (2.6) equals $2C$ while its left hand side has the following lower bound:
\[
\left| \int \int_{\mathbb{R}} \frac{\xi f_1(\xi, \tau) f_2(\xi, \tau)}{\langle L_1 \rangle^b \langle L_2 \rangle^{1-b}} d\xi d\tau \right| \gtrsim \frac{N}{N^{1-b}} = N^b,
\]
which is impossible when $N \to \infty$.

**3 Main results on bilinear estimates**

Our main well-posedness results in Theorem 1.2 will be proved using the same approach as that developed by Bourgain [7], Kenig-Ponce-Vega [29] in establishing analytical well-posedness of the Cauchy problem of (1.9) in the space $H^s(\mathbb{R})$ for $s > -\frac{4}{3}$. The key ingredient in the approach is the bilinear estimate under the Fourier restriction space (also called Bourgain space). Let us first introduce the definition of this space. For any $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$, denote the polynomial $\phi^{\alpha, \beta}$ as
\[
\phi^{\alpha, \beta}(\xi) = \alpha \xi^3 - \beta \xi.
\] (3.1)

For convenience, $\phi^{\alpha, 0}$ will be denoted as $\phi^\alpha$. Then the Fourier restriction space is defined as follows.

**Definition 3.1.** For any $\alpha, \beta, s, b \in \mathbb{R}$ with $\alpha \neq 0$, the Fourier restriction space $X_{s, b}^{\alpha, \beta}$ is defined to be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm
\[
\|w\|_{X_{s, b}^{\alpha, \beta}} = \|\phi^{\alpha, \beta}(\xi)^{\frac{b}{3}} \tilde{w}(\xi, \tau)\|_{L^2(d\xi d\tau)},
\] (3.2)

where $\langle \cdot \rangle = 1 + |\cdot|$, $\phi^{\alpha, \beta}$ is given by (3.1), and $\tilde{w}$ refers to the space-time Fourier transform of $w$. Moreover, $X_{s, b}^{\alpha, 0}$ is simply denoted as $X_{s, b}^{\alpha}$. On the other hand, for any $T > 0$, $X_{s, b}^{\alpha, \beta}([0, T])$ denotes the restriction of $X_{s, b}^{\alpha, \beta}$ on the domain $\mathbb{R} \times [0, T]$ which is a Banach space when equipped with the usual quotient norm.
The bilinear estimate which was first considered by Bourgain [8] is the following one:

\[ \| \partial_x (w_1 w_2) \|_{X_{s,b}^{1,1}} \leq C \| w_1 \|_{X_{s,b}^{1,1}} \| w_2 \|_{X_{s,b}^{1,1}}, \quad \forall w_1, w_2. \]  \tag{3.3}

Bourgain proved (3.3) for \( s = 0 \) and \( b = \frac{1}{3} \) while the following lemma is due to Kenig, Ponce and Vega.

**Lemma 3.2** (Kenig-Ponce-Vega [29]). The bilinear estimate (3.3) holds for any \( s > -\frac{1}{4} \) and \( b \in \left( \frac{1}{2}, b_0(s) \right) \) with some \( b_0(s) > \frac{1}{2} \), but fails for any \( b \in \mathbb{R} \) if \( s < -\frac{1}{4} \).

In order to deal with the general KdV-KdV systems (1.2), four types of bilinear estimates need to be investigated. In (3.4)–(3.7), \( (D) \) refers to divergence form and \((ND)\) refers to non-divergence form.

(D1):

\[ \| \partial_x (w_1 w_2) \|_{X_{s,b}^{1,1}} \leq C \| w_1 \|_{X_{s,b}^{1,1}} \| w_2 \|_{X_{s,b}^{1,1}}, \quad \forall w_1, w_2. \]  \tag{3.4}

(D2):

\[ \| \partial_x (w_1 w_2) \|_{X_{s,b}^{1,1}} \leq C \| w_1 \|_{X_{s,b}^{1,1}} \| w_2 \|_{X_{s,b}^{1,1}}, \quad \forall w_1, w_2. \]  \tag{3.5}

(ND1):

\[ \| \partial_x (w_1 w_2) \|_{X_{s,b}^{1,1}} \leq C \| w_1 \|_{X_{s,b}^{1,1}} \| w_2 \|_{X_{s,b}^{1,1}}, \quad \forall w_1, w_2. \]  \tag{3.6}

(ND2):

\[ \| w_1 (\partial_x w_2) \|_{X_{s,b}^{1,1}} \leq C \| w_1 \|_{X_{s,b}^{1,1}} \| w_2 \|_{X_{s,b}^{1,1}}, \quad \forall w_1, w_2. \]  \tag{3.7}

Here, \( (\alpha_1, \beta_1) \) (or \( (\alpha_2, \beta_2) \)) stands for \( (\alpha_1, b_{11}) \) or \( (\alpha_2, b_{22}) \). (D1) is used to deal with the square terms \( w u \) and \( v v_x \) in (1.2). (D2) is responsible for the mixed divergence term \( (w v)_x \) when \( d_{11} = d_{12} \) or \( d_{21} = d_{22} \) in (1.2). (ND1) and (ND2) are applied to treat the mixed non-divergence terms \( u x v \) and \( w_x v \) when \( d_{11} \neq d_{12} \) or \( d_{21} \neq d_{22} \). On the other hand, (D1) is different from (D2) since \( w_1 \) and \( w_2 \) live in the same space \( X_{s,b}^{1,1} \) for (D1) but in different spaces for (D2). (ND1) is also slightly different from (ND2). Nevertheless, due to the relation \( (w_1 w_2)_x = (\partial_x w_1) w_2 + w_1 (\partial_x w_2) \), any results for (ND2) can be automatically obtained once the corresponding results are known for (D2) and (ND1). The main challenges of studying the bilinear estimates (3.4)–(3.7) come from either the distinct dispersion coefficients \( \alpha_1 \) and \( \alpha_2 \) or the non-divergence form.

**Theorem 3.3.** Let \( \alpha_1 \alpha_2 < 0 \). Assume \( s \) and \( b \) satisfy one of the following conditions.

(1) \(-\frac{13}{12} \leq s \leq -1 \) and \( \frac{1}{4} - \frac{3}{4} \leq b \leq \frac{4}{3} + \frac{2s}{9}; \)

(2) \(-1 < s < -\frac{3}{4} \) and \( \frac{1}{4} - \frac{3}{8} \leq b \leq 1 + \frac{s}{3}; \)

(3) \( s \geq -\frac{3}{4} \) and \( \frac{1}{2} < b < \frac{7}{4}. \)

Then there exist \( \epsilon = \epsilon(\alpha_1, \alpha_2) \) and \( C = C(\alpha_1, \alpha_2, s, b) \) such that for any \( |\beta_2 - \beta_1| \leq \epsilon, \) (3.3) holds.

![Figure 1: Range of s and b when s < -\frac{3}{4}](image)

For the convenience of the readers, we draw a picture of the range of \( s \) and \( b \) when \( s < -\frac{3}{4} \), see Figure 1. This range is sharp due to Theorem 3.4.
Theorem 3.4. Let $\alpha_1\alpha_2 < 0$ and $\beta_1 = \beta_2 = \beta$. Assume $s$ and $b$ satisfy one of the following conditions.

1. $s < -\frac{13}{12}$ and $b \in \mathbb{R}$;
2. $-\frac{13}{12} \leq s \leq -1$ and $b \notin \left[\frac{1}{4} - \frac{s}{3}, \frac{4}{3} + \frac{2s}{3}\right]$;
3. $-1 < s < -\frac{3}{4}$ and $b \notin \left[\frac{1}{4} - \frac{s}{3}, 1 + \frac{s}{3}\right]$.

Then there does not exist any constant $C = C(\alpha_1, \alpha_2, \beta, s, b)$ such that \((3.3)\) holds.

The results presented in Theorem 3.3 and Theorem 3.4 together are surprising in comparison to the previous results on the bilinear estimate.

- First, in the case of the single KdV equation (1.9), the critical index for the corresponding bilinear estimate (3.3) is $-\frac{3}{4}$. However, when $\alpha_1\alpha_2 < 0$, the critical index of the bilinear estimate (3.5) of type (D2) can be as low as $-\frac{13}{12}$.

- Secondly, for the previous bilinear estimates, $b$ is usually required to be close to $\frac{1}{2}$ as $s$ approaches to the critical threshold. However, for the bilinear estimate (3.5) with $\alpha_1\alpha_2 < 0$ and $-\frac{13}{12} \leq s < -\frac{3}{4}$, $b$ needs to be away from $\frac{1}{2}$. In particular, when $s = -\frac{13}{12}$, $b$ needs to be exactly $\frac{11}{18}$.

Theorem 3.5. Let $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ and denote $r = \frac{\alpha_2}{\alpha_1}$. Assume $r$, $s$ and the bilinear estimate type belong to any case in Table 8. Then there exist $b_0 = b_0(s) > \frac{1}{2}$ and $\epsilon = \epsilon(\alpha_1, \alpha_2)$ such that for any $\frac{1}{2} < b \leq b_0$ and for any $|\beta_2 - \beta_1| \leq \epsilon$, the bilinear estimate holds with some constant $C = C(\alpha_1, \alpha_2, s, b)$.

| Type     | $r < 0$ | $0 < r < \frac{1}{4}$ | $r = \frac{1}{4}$ | $r > \frac{1}{4}, r \neq 1$ | $r = 1$ |
|----------|---------|-----------------------|------------------|-----------------------------|--------|
| (D1): 3.4 | $s > -\frac{3}{4}$ | $s > -\frac{3}{4}$ | $s \geq \frac{3}{4}$ | $s \geq 0$ | $s > -\frac{3}{4}$ |
| (D2): 3.5 | $s > -\frac{3}{4}$ | $s \geq \frac{3}{4}$ | $s \geq 0$ | $s > -\frac{3}{4}$ |
| (ND1): 3.6 | $s > -\frac{3}{4}$ | $s \geq \frac{3}{4}$ | $s \geq 0$ | $s > 0$ |
| (ND2): 3.7 | $s > -\frac{3}{4}$ | $s \geq \frac{3}{4}$ | $s \geq 0$ | $s > 0$ |

The indexes in Table 8 are also sharp.

Theorem 3.6. Let $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ and denote $r = \frac{\alpha_2}{\alpha_1}$. Let $\beta_1 = \beta_2 = \beta$. Assume $r$, $s$ and the bilinear estimate type belong to any case in Table 8. Then for any $b \in \mathbb{R}$, there does not exist a constant $C = C(\alpha_1, \alpha_2, \beta, s, b)$ such that the bilinear estimate holds.

| Type     | $r < 0$ | $0 < r < \frac{1}{4}$ | $r = \frac{1}{4}$ | $r > \frac{1}{4}, r \neq 1$ | $r = 1$ |
|----------|---------|-----------------------|------------------|-----------------------------|--------|
| (D1): 3.4 | $s < -\frac{3}{4}$ | $s < -\frac{3}{4}$ | $s < \frac{3}{4}$ | $s < 0$ | $s < -\frac{3}{4}$ |
| (D2): 3.5 | $s < -\frac{3}{4}$ | $s < \frac{3}{4}$ | $s < 0$ | $s < -\frac{3}{4}$ |
| (ND1): 3.6 | $s < -\frac{3}{4}$ | $s < \frac{3}{4}$ | $s < 0$ | $s < 0$ |
| (ND2): 3.7 | $s < -\frac{3}{4}$ | $s < \frac{3}{4}$ | $s < 0$ | $s < 0$ |
There are several things worth mentioning about Theorem 3.5 and 3.6. First, when \( r < 0 \), the critical index for Type (D1) is \(-\frac{3}{2}\) which is much larger than that for Type (D2), see Theorem 3.3. Secondly, when \( r = \frac{1}{4}\), the critical index is \(\frac{3}{2}\) which is much larger than other cases for \( r \neq \frac{1}{4}\). Thirdly, when \( r = 1 \), the critical index is \(-\frac{3}{4}\) for the divergence forms but is 0 for the non-divergence forms.

**Remark 3.7.** Some results in Theorem 3.5 and 3.6 have already been known (or can be proven similarly) in the previous literatures. More specifically, in Table 8 and 9 when \( r = 1 \), Type (D1) and (D2) have been established in [29], when \( r > \frac{1}{4} \) but \( r \neq 1 \), Type (1) and (2) have been justified in [38], and Type (ND1) and (ND2) can be proven similarly. But note that the notations in [38] are slightly different from here. Actually, the roles of \( \alpha_1 \) and \( \alpha_2 \) are interchanged there. In Table 8 with \( r = -1 \), Type (D1) has appeared in [7].

The proofs of Theorem 3.3–3.6 are very technical and tedious, so we postpone them to Section 5 and 6.

### 4 Proofs of the main results on well-posedness

#### 4.1 Proof of Theorem 1.2

The proofs for the local well-posedness results in this paper will use the scaling argument as in [29]. This argument reduces the proofs to the case when the initial data and the elements \( b_{ij} \) in the matrix \( B \) are sufficiently small. Define the functions \( u^\lambda \) and \( v^\lambda \) for \( \lambda \geq 1 \) as follows:

\[
\begin{align*}
\begin{cases}
    u^\lambda(x, t) = \lambda^{-2} u(\lambda^{-1} x, \lambda^{-3} t), & x \in \mathbb{R}, t \in \mathbb{R}, \\
    v^\lambda(x, t) = \lambda^{-2} v(\lambda^{-1} x, \lambda^{-3} t),
\end{cases}
\end{align*}
\]  

(4.1)

Then (1.2) is equivalent to the system below.

\[
\begin{align*}
\begin{cases}
    u^\lambda_t + a_{11} u^\lambda_{xx} + b_{11}^\lambda u^\lambda_x = & -b_{12}^\lambda v^\lambda_x + c_{11} u^\lambda_x u^\lambda_x + c_{12} v^\lambda_x v^\lambda_x + d_{11} u^\lambda v^\lambda_x + d_{12} u^\lambda v^\lambda_x, \\
    v^\lambda_t + a_{22} v^\lambda_{xx} + b_{22}^\lambda v^\lambda_x = & -b_{21}^\lambda u^\lambda_x + c_{21} u^\lambda_x u^\lambda_x + c_{22} v^\lambda_x v^\lambda_x + d_{21} u^\lambda v^\lambda_x + d_{22} u^\lambda v^\lambda_x,
\end{cases}
\end{align*}
\]  

(4.2)

where \( b_{ij}^\lambda = \lambda^{-2} b_{ij} \) and

\[
\begin{align*}
\begin{cases}
    u_0^\lambda(x) = \lambda^{-2} u_0(\lambda^{-1} x), & x \in \mathbb{R}, \\
    v_0^\lambda(x) = \lambda^{-2} v_0(\lambda^{-1} x),
\end{cases}
\end{align*}
\]

Since \( \lambda \geq 1 \) and \( s \geq -\frac{3}{2} \), then

\[
\begin{align*}
\| u_0^\lambda \|_{H^s(\mathbb{R})} \leq \lambda^{-\frac{3}{2}} \| u_0 \|_{H^s(\mathbb{R})}, \\
\| v_0^\lambda \|_{H^s(\mathbb{R})} \leq \lambda^{-\frac{3}{2}} \| v_0 \|_{H^s(\mathbb{R})}.
\end{align*}
\]

Consequently, as \( \lambda \to \infty \),

\[
\begin{align*}
\max_{1 \leq i,j \leq 2} |b_{ij}^\lambda| \to 0 & \quad \text{and} \quad \max_{1 \leq i,j \leq 2} (|c_{ij}| + |d_{ij}|)(\| u_0^\lambda \|_{H^s(\mathbb{R})} + \| v_0^\lambda \|_{H^s(\mathbb{R})} ) \to 0.
\end{align*}
\]  

(4.3)

So in order to prove the local well-posedness of (1.2), it suffices to justify the statement below.

**Proposition 4.1.** Let \( a_1, a_2 \in \mathbb{R} \setminus \{0\} \) and denote \( r = \frac{a_2}{a_1} \). Assume \( r, s \) and the coefficients \( b_{ij}, c_{ij} \) and \( d_{ij} \) belong to any case in Table 4 of Theorem 1.2. Let \( T > 0 \) be given. Then there exists a constant \( \epsilon = \epsilon(a_1, a_2, s, T) \) such that if

\[
\begin{align*}
\max_{1 \leq i,j \leq 2} |b_{ij}| \leq \epsilon & \quad \text{and} \quad \max_{1 \leq i,j \leq 2} (|c_{ij}| + |d_{ij}|)(\| u_0 \|_{H^s(\mathbb{R})} + \| v_0 \|_{H^s(\mathbb{R})} ) \leq \epsilon,
\end{align*}
\]  

(4.4)

then (1.3) admits a unique solution \( (u, v) \in C([0,T]; H^s(\mathbb{R})) \) satisfying the auxiliary condition

\[
\| u \|_{X_{a_1,b}^s([0,T])} + \| v \|_{X_{a_2,b}^s([0,T])} < +\infty.
\]
We only prove Case (1) with Proof of Proposition 4.1. Let $(c_{ij}) = 0, d_{11} = d_{12} := d_1, d_{21} = d_{22} := d_2$ and $s \geq \frac{13}{11}$. Other cases can be proved similarly by using appropriate bilinear estimates presented in Theorem [3.5]. In addition, without loss of generality, we assume $T = 1$. Hence, (1.2) with the assumption (4.4) becomes

\[
\begin{aligned}
\left\{ u_t + a_1 u_{xxx} + b_{11} u_x &= -b_{12} v_x + d_1 (uv)_x, \\
v_t + a_2 v_{xxx} + b_{22} v_x &= -b_{21} u_x + d_2 (uv)_x, \\
(u, v)|_{t=0} &= (u_0, v_0) \in \mathcal{H}^s(\mathbb{R}),
\end{aligned}
\] (4.5)

where $a_1 a_2 < 0$, $s \geq -\frac{13}{11}$ and

\[
\max_{1 \leq i, j \leq 2} |b_{ij}| \leq \epsilon \quad \text{and} \quad (|d_1| + |d_2|)(\|u_0\|_{\mathcal{H}^s(\mathbb{R})} + \|v_0\|_{\mathcal{H}^s(\mathbb{R})}) \leq \epsilon,
\] (4.6)

for some $\epsilon = \epsilon(a_1, a_2, s)$ to be determined.

By virtue of the semigroup operator $S_i = S^{a_i, b_i}$, for $i = 1, 2$, the Cauchy problem (4.5) for $t \in [0, 1]$ can be converted into the integral form

\[
\begin{aligned}
\left\{ u(t) &= \psi(t) \left( S_1(t) u_0 + \int_0^t S_1(t-t') F_1(u(v)(t') dt') \right), \\
v(t) &= \psi(t) \left( S_2(t) v_0 + \int_0^t S_2(t-t') F_2(u(v)(t') dt') \right),
\end{aligned}
\] (4.7)

where $\psi(t)$ is the bump function defined at the beginning of Section 2 and

\[
\begin{aligned}
F_1(u, v) &= -b_{12} v_x + d_1 (uv)_x, \\
F_2(u, v) &= -b_{21} u_x + d_2 (uv)_x.
\end{aligned}
\] (4.8)

This suggests to consider the map $\Phi(u, v) \triangleq (\Phi_1(u, v), \Phi_2(u, v))$, where

\[
\begin{aligned}
\Phi_1(u, v) &= \psi(t) \left( S_1(t) u_0 + \int_0^t S_1(t-t') F_1(u(v)(t') dt') \right), \\
\Phi_2(u, v) &= \psi(t) \left( S_2(t) v_0 + \int_0^t S_2(t-t') F_2(u(v)(t') dt') \right).
\end{aligned}
\] (4.9)

The goal is to show $\Phi$ is a contraction mapping in a ball in an appropriate Banach space, which will imply that the fixed point of $\Phi$ is the desired solution to the Cauchy problem (4.5) for $0 \leq t \leq 1$.

For convenience, let $Y_{s, b} = X^{a_i, b_i}_{s, b}, i = 1, 2, \text{and} \ Y_{s, b} = Y_{s, b}^1 \times Y_{s, b}^2$, equipped with the norm

\[
\| (u, v) \|_{Y_{s, b}} := \| u \|_{Y_{s, b}^1} + \| v \|_{Y_{s, b}^2}.
\]

Define $M_1 = \max_{1 \leq i, j \leq 2} |b_{ij}|$ and $M_2 = \max_{1 \leq i \leq 2} |d_i|$. Then assumption (4.6) becomes

\[
M_1 \leq \epsilon \quad \text{and} \quad M_2 (\| u_0 \|_{\mathcal{H}^s(\mathbb{R})} + \| v_0 \|_{\mathcal{H}^s(\mathbb{R})}) \leq \epsilon.
\] (4.10)

Define

\[
B_{s, b, C}(u_0, v_0) = \{ (u, v) \in Y_{s, b} : \| (u, v) \|_{Y_{s, b}} \leq C(\| u_0 \|_{\mathcal{H}^s(\mathbb{R})} + \| v_0 \|_{\mathcal{H}^s(\mathbb{R})}) \}.
\] (4.11)

In the following, we will choose suitable $\epsilon, b$ and $C$ such that $\Phi$ is a contraction mapping on $B_{s, b, C}(u_0, v_0)$. We will first show that $\Phi$ maps the closed ball $B_{s, b, C}(u_0, v_0)$ into itself. For any $(u, v) \in B_{s, b, C}(u_0, v_0)$, by Lemma 2.1 for any $b > \frac{1}{2}$, there exists a constant $C_1 = C_1(b)$ such that

\[
\begin{aligned}
\| \Phi_1(u, v) \|_{Y_{s, b}^1} &\leq C_1 \| u_0 \|_{\mathcal{H}^s(\mathbb{R})} + C_1 \| F_1(u(v)) \|_{Y_{s, b-1}^1}, \\
\| \Phi_2(u, v) \|_{Y_{s, b}^2} &\leq C_1 \| v_0 \|_{\mathcal{H}^s(\mathbb{R})} + C_1 \| F_2(u(v)) \|_{Y_{s, b-1}^2}.
\end{aligned}
\] (4.12)
Since $F_1(u, v) = -b_1 v_x + d_1(uv)_x$, we will estimate $\|b_1 v_x\|_{Y_{s,b}^{1}}$ and $\|d_1(uv)_x\|_{Y_{s,b}^{1}}$ separately in order to bound $\|F_1(u, v)\|_{Y_{s,b}^{1}}$. Since $a_1 \neq a_2$, it follows from Lemma 2.2 that for any $b \in \left(\frac{1}{2}, \frac{2}{3}\right]$, there exist $\epsilon_1 = \epsilon_1(a_1, a_2)$ and $C_2 = C_2(a_1, a_2)$ such that for any $|b_{22} - b_{11}| \leq \epsilon_1$,

$$\|b_1 v_x\|_{Y_{s,b}^{1}} \leq C_2 |b_{12}| \|v\|_{Y_{s,b}^{2}} \leq C_2 M_1 \|v\|_{Y_{s,b}^{2}}.$$

On the other hand, by Theorem 3.3 there exist $\epsilon = \epsilon(a_1, a_2)$ and $C_3 = C_3(a_1, a_2, s, b^*)$ such that for any $|b_{22} - b_{11}| \leq \epsilon_2$,

$$\|d_1(uv)_x\|_{Y_{s,b}^{1}} \leq C_3 \|d_1\| \|u\|_{Y_{s,b}^{1}} \|v\|_{Y_{s,b}^{2}} \leq C_3 M_2 \|u\|_{Y_{s,b}^{1}} \|v\|_{Y_{s,b}^{2}}.$$

Thus, for this particular $b^*$, taking $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ and $C_4 = \max\{C_1, C_2, C_3\}$, then for any $|b_{11}| + |b_{22}| \leq \epsilon_3$,

$$\|F_1(u, v)\|_{Y_{s,b}^{1}} \leq C_4 \left(M_1 \|v\|_{Y_{s,b}^{2}} + M_2 \|u\|_{Y_{s,b}^{1}} \|v\|_{Y_{s,b}^{2}}\right). \tag{4.13}$$

Analogously, it also holds

$$\|F_2(u, v)\|_{Y_{s,b}^{2}} \leq C_4 \left(M_1 \|u\|_{Y_{s,b}^{1}} + M_2 \|u\|_{Y_{s,b}^{1}} \|v\|_{Y_{s,b}^{2}}\right). \tag{4.14}$$

Adding (4.12), (4.13) and (4.14) together yields that

$$\|\Phi(u, v)\|_{Y_{s,b}^{1}} \leq C_5 \left(\|u\|_{H^s} + \|v\|_{H^s} + M_1 \|u\|_{Y_{s,b}^{1}} + M_2 \|u\|_{Y_{s,b}^{1}} \|v\|_{Y_{s,b}^{2}}\right), \tag{4.15}$$

where the constant $C_5$ only depends on $a_1, a_2, s$ and $b^*$. Actually, since $b^*$ is determined by $s$, $C_5$ only depends on $a_1, a_2$ and $s$. Denote $E_0 = \|u_0\|_{H^s} + \|v_0\|_{H^s}$ and define

$$C^* = 8C_5. \tag{4.16}$$

Then it follows from (4.11) that $\|u, v\|_{Y_{s,b}^{1}} \leq C^* E_0$ for any $(u, v) \in B_{s,b^*}X(u_0, v_0)$. Hence, it follows from (4.15) that

$$\|\Phi(u, v)\|_{Y_{s,b}^{1}} \leq C_5 E_0 + C_5 M_1 C^* E_0 + C_5 M_2 (C^*)^2 E_0^2.$$

Since $C^* = 8C_5$,

$$\|\Phi(u, v)\|_{Y_{s,b}^{1}} \leq C_5 E_0 + 8C_5^2 M_1 E_0 + 64C_5^3 M_2 E_0^2.$$

Now choose

$$\epsilon^* = \min\left\{\frac{\epsilon_3}{2}, \frac{1}{16C_5}, \frac{1}{128C_5^2}\right\}. \tag{4.17}$$

Then for any $(u, v) \in B_{s,b^*}X(u_0, v_0)$, it follows from (4.10) and (4.17) that

$$\|\Phi(u, v)\|_{Y_{s,b}^{1}} \leq 2C_5 E_0 = \frac{C^* E_0}{4},$$

which implies $\Phi(u, v) \in B_{s,b^*}X(u_0, v_0)$. Next for any $(u_j, v_j) \in B_{s,b^*}X(u_0, v_0)$, $j = 1, 2$, the same argument yields

$$\|\Phi(u_1, v_1) - \Phi(u_2, v_2)\|_{Y_{s,b}^{1}} \leq \frac{1}{2}(u_1, v_1) - (u_2, v_2)\|_{Y_{s,b}^{1}}.$$
For the Majda-Biello system (1.3), it is a special case of (1.2) with the coefficients
\[
a_1 = 1; \quad (b_{ij}) = 0; \quad c_{11} = c_{21} = c_{22} = 0, \quad c_{12} = -1; \quad d_{11} = d_{12} = 0, \quad d_{21} = d_{22} = -1.
\] (4.18)

So the LWP results in Theorem 1.4 follow directly from Theorem 1.2. Then according to these LWP results and the conserved energies (4.19), the GWP results in Theorem 1.4 are established (except when \(a_2 = 1\) for which case the GWP was proved for any \(s > -\frac{3}{4}\) by Oh [37] via the I-method).

\[
E_1(u, v) = \int u^2 + v^2 \, dx,
\]
(4.19)

\[
E_2(u, v) = \int u_x^2 + a_2 v_x^2 - uv \, dx.
\]

For the Hirota-Satsuma system (1.4), it is a special case of (1.2) with the coefficients
\[
a_2 = 1; \quad (b_{ij}) = 0; \quad c_{11} = -6a_1, \quad c_{21} = c_{22} = 0; \quad d_{11} = d_{12} = d_{21} = 0, \quad d_{22} = -3.
\] (4.20)

So the LWP results in Theorem 1.5 follow directly from Theorem 1.2. Then according to these LWP results and the conserved energies (4.21), the GWP results in Theorem 1.5 are established.

\[
E_1(u, v) = \int u^2 + \frac{c_{12}}{3} v^2 \, dx,
\]
(4.21)

\[
E_2(u, v) = \int (1 - a_1)u_x^2 + c_{12} v_x^2 - 2(1 - a_1)u + c_{12}uv \, dx.
\]

For the Gear-Grimshaw system (1.5), we first write it into the vector form:
\[
\begin{pmatrix}
\dot{u}_t \\
\dot{v}_t
\end{pmatrix} + A_1 \begin{pmatrix}
\frac{u_{xxx}}{v_{xxx}} \\
v_{xxx}
\end{pmatrix} + A_2 \begin{pmatrix}
u_x \\
v_x
\end{pmatrix} = A_3 \begin{pmatrix}
u_{ux} \\
u_{vx}
\end{pmatrix} + A_4 \begin{pmatrix}
u_{ux} v \\
u_{vx}
\end{pmatrix},
\]
(4.22)

\[
(u, v)|_{t=0} = (u_0, v_0),
\]

where
\[
A_1 = \begin{pmatrix}
\frac{1}{\rho_1} & \frac{\sigma_3}{\rho_1} \\
0 & \frac{1}{\rho_1}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
0 & 0 \\
0 & \frac{1}{\rho_1}
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
-1 & \frac{\sigma_1}{\rho_1} \\
\frac{\rho_2 \sigma_2}{\rho_1^2} & -\frac{1}{\rho_1}
\end{pmatrix}, \quad A_4 = \begin{pmatrix}
\frac{\sigma_2}{\rho_1^2} & \frac{\sigma_2}{\rho_1} \\
\frac{\rho_2 \sigma_1}{\rho_1^2} & \frac{\rho_2 \sigma_1}{\rho_1}
\end{pmatrix}.
\]

When \(\rho_2 \sigma_2^2 \neq 1\), \(A_1\) has two nonzero eigenvalues \(\lambda_1\) and \(\lambda_2\):

\[
\lambda_1 = \frac{\rho_1 + 1}{2\rho_1} + \sqrt{\frac{(\rho_1 - 1)^2 + 4\rho_1 \rho_2 \sigma_2^2}{2\rho_1}}, \quad \lambda_2 = \frac{\rho_1 + 1}{2\rho_1} - \sqrt{\frac{(\rho_1 - 1)^2 + 4\rho_1 \rho_2 \sigma_2^2}{2\rho_1}}.
\] (4.23)

So there exists an invertible real-valued matrix \(M\) such that \(A_1 = M \begin{pmatrix}
\lambda_1 \\
\lambda_2
\end{pmatrix} M^{-1}\). By regarding \(M^{-1} \begin{pmatrix} u \\ v \end{pmatrix}\) as the new unknown functions (still denoted by \(u\) and \(v\)), (4.22) can be rewritten as

\[
\begin{pmatrix}
\dot{u}_t \\
\dot{v}_t
\end{pmatrix} + \lambda_1 \begin{pmatrix}
u_{xxx} \\
v_{xxx}
\end{pmatrix} + B \begin{pmatrix}
u_x \\
v_x
\end{pmatrix} = C \begin{pmatrix}
u_{ux} \\
u_{vx}
\end{pmatrix} + D \begin{pmatrix}
u_{ux} v \\
u_{vx}
\end{pmatrix},
\]
(4.24)

\[
(u, v)|_{t=0} = (u_0, v_0),
\]
where \( d_{11} = d_{12}, \ d_{21} = d_{22} \). In addition, \( B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) if \( \sigma_4 = 0 \). Define
\[
  r = \frac{\lambda_2}{\lambda_1}. \tag{4.25}
\]

Then it follows from (4.23) that \( r < 1 \). Moreover, since both \( \rho_1 \) and \( \rho_2 \) are positive numbers, we have \( r < 0 \iff \rho_2 \sigma_3^2 > 1 \). Moreover, \( r = \frac{1}{4} \) if and only if (1.13) holds, that is
\[
  \rho_2 \sigma_3^2 \leq \frac{9}{25} \quad \text{and} \quad \rho_1^3 + \frac{25 \rho_2 \sigma_3^2 - 17}{4} \rho_1 + 1 = 0.
\]

Based on the above observations, the LWP results in Theorem 1.6 follow from Theorem 1.2. Then according to these LWP results and the conserved energies (4.26), the GWP results in Theorem 1.6 are established.

\[
  E_1(u, v) = \int \rho_2 u^2 + \rho_1 v^2 \, dx,
\]
\[
  E_2(u, v) = \int \rho_2 u_x^2 + v_x^2 + 2 \rho_2 \sigma_3 u_x v_x - \frac{\rho_2}{3} v_3^3 + \rho_2 \sigma_2 u^2 v + \rho_2 \sigma_1 u v^2 - \frac{1}{3} v_3^3 - \sigma_4 v^2. \tag{4.26}
\]

## 5 Proofs of the bilinear estimates

The goal of this section is to prove Theorem 3.3 and Theorem 3.5.

### 5.1 Idea of the proofs

The main idea of treating the bilinear estimates of different types are similar, and is thus explained only for the following divergence form with \( \beta_i = 0 \) (\( i = 1, 2, 3 \)).

\[
  \| \partial_x (w_1 w_2) \|_{X_{\sigma, \beta}^s} \lesssim \| w_1 \|_{X_{\sigma, \beta}^s} \| w_2 \|_{X_{\sigma, \beta}^s}, \quad \forall \ w_1, w_2. \tag{5.1}
\]

By duality and Plancherel identity, (5.1) is equivalent to (see e.g. [47])

\[
  \left| \int \int \sum_{i=1}^3 f_i(\xi, \tau_i) \frac{\xi_3 \langle \xi_3 \rangle^s}{\langle \xi_1 \rangle^s (\xi_2)^s} \prod_{i=1}^3 \langle L_i \rangle^b (\xi_3)^{1-b} \right| \leq C \prod_{i=1}^3 \| f_i \|_{L^2(\xi)}, \quad \forall \left\{ f_i \right\}_{1 \leq i \leq 3}, \tag{5.2}
\]

where
\[
  L_i = \tau_i - \phi_{\alpha_i}^\prime (\xi_i) = \tau_i - \alpha_i \xi_i^3, \quad 1 \leq i \leq 3.
\]

In (5.2), the loss of the spatial derivative in the bilinear estimate (5.1) is reflected in the term \( \frac{\xi_3 \langle \xi_3 \rangle^s}{\langle \xi_1 \rangle^s (\xi_2)^s} \) and the gain of the time derivative is reflected in the term \( \langle L_1 \rangle^b (\xi_2)^{1-b} \). Then how to compensate the loss of the spatial derivative from the gain of the time derivative is the key issue. Denote
\[
  K_1 = \frac{\xi_3 \langle \xi_3 \rangle^s}{\langle \xi_1 \rangle^s (\xi_2)^s} \quad \text{and} \quad K_2 = \langle L_1 \rangle^b (\xi_2)^{1-b}.
\]

Then the main idea is to control \( K_1 \) by taking advantage of \( K_2 \). Since \( \sum_{i=1}^3 \xi_i = 0 \), then \( \langle \xi_3 \rangle \leq \langle \xi_1 \rangle \langle \xi_2 \rangle \). As a result, \( K_1 \) is a decreasing function in \( s \), which means the smaller \( s \) is, the more likely the bilinear estimate will fail. So the question is how to find the smallest \( s \) such that the bilinear estimate holds. Noticing that \( L_i \) contains the time variable \( \tau_i \), so a single \( L_i \) can barely have any contributions. Since \( \sum_{i=1}^3 \tau_i = 0 \), then
\[ \sum_{i=1}^{3} L_i = - \sum_{i=1}^{3} \alpha_i \xi_i^3 \] is a function only in \( \xi_i, 1 \leq i \leq 3 \). Define

\[ H(\xi_1, \xi_2, \xi_3) := \sum_{i=1}^{3} \alpha_i \xi_i^3. \]

Then it is obvious that \( K_2 \gtrsim |H|^{\min\{b, 1-b\}} \), which may be used to control \( K_1 \). Thus, \( H(\xi_1, \xi_2, \xi_3) \) plays a fundamental role. In addition, \( H \) measures to what extent the spatial frequencies \( \xi_1, \xi_2 \) and \( \xi_3 \) can resonate with each other. Because of this, \( H \) is called the resonance function (see Page 856 in [47]). Unfortunately, \( |H| \) is not always large, the situation may become complicated near the region where \( H \) vanishes. We shall call the zero set of \( H \) to be the resonance set. Usually, the worst situation occurs near the resonance set and this trouble is called resonant interactions (see Page 856 in [47]).

In the following, we will investigate the resonance function and the resonance set in three typical situations (again \( \{\beta_i\}_{i=1}^{3} \) are assumed to be zero for simplicity).

- In the classical case when \( \alpha_1 = \alpha_2 = \alpha_3 \), the resonance function \( H_0 \) is in a very simple form:

\[ H_0(\xi_1, \xi_2, \xi_3) = 3\alpha_1 \xi_1 \xi_2 \xi_3. \]

The resonance set consists of three hyperplanes: \( \{\xi_i = 0\}, i = 1, 2, 3 \).

- For the bilinear estimate of Type (D1), the resonance function \( H_1 \) is

\[ H_1(\xi_1, \xi_2, \xi_3) = \alpha_1 \xi_1^3 + \alpha_1 \xi_2^3 + \alpha_2 \xi_3^3. \]

By writing \( \xi_2 = -(\xi_1 + \xi_3) \),

\[ H_1(\xi_1, \xi_2, \xi_3) = \xi_3 \left[ (\alpha_2 - \alpha_1) \xi_2^2 - 3\alpha_1 \xi_1 \xi_3 - 3\alpha_1 \xi_2^2 \right]. \]

So \( \{\xi_3 = 0\} \) belongs to the resonance set. If \( \xi_3 \neq 0 \), then \( H_1 \) can be rewritten as

\[ H_1(\xi_1, \xi_2, \xi_3) = -3\alpha_1 \xi_1^3 h_r \left( \frac{\xi_1}{\xi_3} \right), \]

where \( r = \frac{\alpha_2}{\alpha_1} \) and

\[ h_r(x) := x^2 + x + \frac{1 - r}{3}. \]

So the resonance set is determined by the roots of \( h_r \).

- Similarly, for the bilinear estimate of Type (D2), the resonance function \( H_2 \) is

\[ H_2(\xi_1, \xi_2, \xi_3) = \alpha_1 \xi_1^3 + \alpha_2 \xi_2^3 + \alpha_1 \xi_3^3. \]

By writing \( \xi_3 = -(\xi_1 + \xi_2) \),

\[ H_2(\xi_1, \xi_2, \xi_3) = \xi_2 \left[ (\alpha_2 - \alpha_1) \xi_2^2 - 3\alpha_1 \xi_1 \xi_3 - 3\alpha_1 \xi_2^2 \right]. \]

So \( \{\xi_2 = 0\} \) belongs to the resonance set. If \( \xi_2 \neq 0 \), then \( H_2 \) can be rewritten as

\[ H_2(\xi_1, \xi_2, \xi_3) = -3\alpha_1 \xi_2^3 h_r \left( \frac{\xi_1}{\xi_2} \right). \]

Again the resonance set is determined by the roots of \( h_r \).

Due to the above observations, the function \( h_r \) is crucial in determining the resonance set. The roots of \( h_r \) have three possibilities.
(1) If \( r < \frac{1}{4} \), then \( h_r \) does not have any real roots.

(2) If \( r = \frac{1}{4} \), then \( h_r \) has one real root \(-\frac{1}{2}\) of multiplicity 2.

(3) If \( r > \frac{1}{4} \), then \( h_r \) has two distinct real roots.

As we have seen, the structure of \( H_1 \) is analogous to that of \( H_2 \). In addition, \( H_0 \) is just a special case of \( H_2 \) when \( r = 1 \). So in the following, we will just focus on \( H_2 \) to discuss the effect of the resonance set on the threshold of \( s \).

(i) \( r = 1 \). This agrees with the classical case and \( h_r \) has two roots \( x_{1r} = -1 \) and \( x_{2r} = 0 \). As we have seen that the resonance set of \( H_0 \) consists of three hyperplanes \( \{\xi_i = 0\}, 1 \leq i \leq 3 \). When \( s < 0 \), by writing \( \rho = -s \), then \( \rho > 0 \) and

\[
K_1 \sim (\xi_1)^{\rho} (\xi_2)^{\rho} |\xi_3|^{1-\rho}.
\]

As a result, \( K_1 \) is also small near the resonance set \( \{\xi_i = 0\}, 1 \leq i \leq 3 \), which means the resonant interactions do not cause too much trouble. This is why the sharp index for the bilinear estimate of the divergence form can be as low as \(-\frac{3}{4}\) as shown in Lemma 3.2.

(ii) \( r < \frac{1}{4} \). In this case, there exists a positive constant \( \delta_r \) such that \( h_r(x) \geq \delta_r \) for any \( x \in \mathbb{R} \). Consequently, the resonance set is only a single hyperplane \( \{\xi_2 = 0\} \). Moreover, \(|\xi_2| < 1 \) and \(|\xi_1| \sim |\xi_3|\) near this hyperplane. As a result, \( |K_1| \sim |\xi_3| \) does not depend on \( s \) at all, which means the resonant interactions have no effect on \( s \) in this case. So there is hope to obtain an even smaller threshold for \( s \). Actually, for Type (D2) with \( r < 0 \), \( s \) can be as small as \(-\frac{13}{12}\).

(iii) \( r > \frac{1}{4} \) and \( r \neq 1 \). In this case, \( h_r \) has two distinct nonzero roots \( x_{1r} \) and \( x_{2r} \). Therefore,

\[
H_2(\xi_1, \xi_2, \xi_3) = -3\alpha_1\xi_2^2 \left( \frac{\xi_1}{\xi_2} - x_{1r} \right) \left( \frac{\xi_1}{\xi_2} - x_{2r} \right).
\]

The resonance set consists of three different hyperplanes: \( \{\xi_2 = 0\}, \{\xi_1 = x_{1r}, \xi_2\} \) and \( \{\xi_1 = x_{2r}, \xi_2\} \). If \( s < 0 \), then near the hyperplane \( \{\xi_1 = x_{1r}, \xi_2\} \) or \( \{\xi_1 = x_{2r}, \xi_2\} \) with large \( \xi_2 \), the resonance function \( H_2 \) is small while \( K_1 \) is large. Thus, the bilinear estimate is likely to fail. Actually, the threshold for \( s \) in this case is \( s \geq 0 \). This has already been pointed out by Oh [38].

(iv) \( r = \frac{1}{4} \). In this case,

\[
H_2(\xi_1, \xi_2, \xi_3) = -3\alpha_1\xi_3^3 \left( \frac{\xi_1}{\xi_2} + \frac{1}{2} \right)^2.
\]

The resonance set consists of two hyperplanes \( \{\xi_2 = 0\} \) and \( \{\xi_1 = -\frac{1}{2}\xi_2\} \). But the resonance interaction is significant near the hyperplane \( \{\xi_1 = -\frac{1}{2}\xi_2\} \) due to the square power. Consequently, the situation is expected to be worse. Actually, the critical indices for \( s \) are different in these two cases.

- If \( r < 0 \), then \( \alpha_1\xi_1^2 \) will not match \( \alpha_2\xi_2^2 \) regardless of the values of \( \xi_1 \) and \( \xi_2 \). So the coherent interactions do not occur in this case and the sharp index for \( s \) is \(-\frac{13}{12}\).

- If \( r > 0 \), then \( \alpha_1\xi_1^2 = \alpha_2\xi_2^2 \) when \( \xi_1 = \pm\sqrt{r}\xi_2 \). So the coherent interactions occur along the hyperplanes \( \{\xi_1 = \pm\sqrt{r}\xi_2\} \). It turns out that the critical index for this case is \(-\frac{3}{4}\).

The above arguments revealed the difficulties for the bilinear estimate of the divergence form. These difficulties play the similar role in the nondivergence case. But the nondivergence form can bring additional trouble. Let us compare (D2) and (ND1) with \( r = 1 \) and \( \beta_i = 0 (i = 1, 2) \). In this case, the resonance
functions for (D2) and (ND1) are the same, both of them are equal to $H_0$. However, the terms $K_1$ and $\tilde{K}_1$ coming from the loss of the spatial derivative for (D1) and (ND1) are different. More precisely,

$$K_1 = \frac{\xi_3(\xi_1)^s}{(\xi_1)^s(\xi_2)^s} \quad \text{and} \quad \tilde{K}_1 = \frac{\xi_1(\xi_3)^s}{(\xi_1)^s(\xi_2)^s}.$$ 

Consider $s = -\frac{3}{4}$, then

$$|K_1| \sim \left| \frac{\langle \xi_3 \rangle^{\frac{3}{4}} - \langle \xi_2 \rangle^{\frac{3}{4}}}{\langle \xi_3 \rangle^{\frac{3}{4}}} \right| \quad \text{and} \quad |\tilde{K}_1| \sim \left| \frac{\langle \xi_1 \rangle^{\frac{3}{4}} - \langle \xi_2 \rangle^{\frac{3}{4}}}{\langle \xi_3 \rangle^{\frac{3}{4}}} \right|.$$ 

Previously, the worst region for (D2) is when $|\xi_1| \sim |\xi_2| \gg |\xi_3|$ and this forces $s$ to be greater than $-3/4$. But in this region, it is easily seen that $\tilde{K}_1$ is even much larger than $K_1$. So there is no hope to control $\tilde{K}_1$ as well when $s$ is near $-3/4$. Actually, it will be shown that the critical index for (ND1) is 0.

In summary, there are three main troubles in establishing the bilinear estimates (3.4)-(3.7).

(T1) : resonant interactions;

(T2) : coherent interaction;

(T3) : the nondivergence form in the region $|\xi_1| \sim |\xi_2| \gg |\xi_3|$.

Generally speaking, (T1) is the most significant trouble and (T2) and (T3) are of the same level of influence. In most cases, these troubles do not occur at the same place, then the strategy is simply to divide the region suitably and deal with one trouble in each region. However, if more than one trouble occur at the same place, then the situation is expected to be worse. In the following, we provide Table 10 to present the main trouble and the critical indexes for $s$ in each case for the bilinear estimates (3.4)-(3.7). The sign “+” indicates the situation when two troubles occur at the same place.

|        | $r < 0$ | $0 < r < \frac{1}{4}$ | $r = \frac{1}{4}$ | $r > \frac{1}{4}$, $r \neq 1$ | $r = 1$ |
|--------|---------|----------------------|------------------|-------------------------------|---------|
| (D1): (3.4) | (T2) | (T2) | (T1)+(T2) | (T1) | (T2) |
| (D2): (3.5) | None | (T2) | (T1)+(T2) | (T1) | (T2) |
| (ND1): (3.6) | (T3) | (T2) or (T3) | (T1)+(T2) | (T1) | (T2)+(T3) |
| (ND2): (3.7) | (T3) | (T2) or (T3) | (T1)+(T2) | (T1) | (T2)+(T3) |

5.2 Auxiliary lemmas

Lemma 5.1. Let $\rho_1 > 1$ and $0 \leq \rho_2 \leq \rho_1$ be given. There exists a constant $C = C(\rho_1, \rho_2)$ such that for any $\alpha, \beta \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{dx}{(x-\alpha)^{\rho_1}(-x-\beta)^{\rho_2}} \leq \frac{C}{(\alpha + \beta)^{\rho_2}}. \quad (5.3)$$

The proof for this lemma is standard and therefore omitted, we just want to remark that $\langle \alpha + \beta \rangle = \langle (x-\alpha)+(-x-\beta) \rangle$, this observation will be used in the estimate (5.3).
Lemma 5.2. If \( \rho > \frac{1}{2} \), then there exists \( C = C(\rho) \) such that for any \( \sigma_i \in \mathbb{R} \), \( 0 \leq i \leq 2 \), with \( \sigma_2 \neq 0 \),

\[
\int_{-\infty}^{\infty} \frac{dx}{(\sigma_2 x^2 + \sigma_1 x + \sigma_0)^\rho} \leq C |\sigma_2|^{1/2}.
\]

(5.4)

Similarly, if \( \rho > \frac{1}{3} \), then there exists \( C = C(\rho) \) such that for any \( \sigma_i \in \mathbb{R} \), \( 0 \leq i \leq 3 \), with \( \sigma_3 \neq 0 \),

\[
\int_{-\infty}^{\infty} \frac{dx}{(\sigma_3 x^3 + \sigma_2 x^2 + \sigma_1 x + \sigma_0)^\rho} \leq C |\sigma_3|^{1/3}.
\]

(5.5)

Proof. We refer the reader to the proof of Lemma 2.5 in [3] where (5.5) was proved. The similar argument can also be applied to obtain (5.4).

\( \square \)

If the power \( \rho \) in Lemma 5.2 is greater than 1, then stronger conclusions hold.

Lemma 5.3. Let \( \rho > 1 \) be given. There exists a constant \( C = C(\rho) \) such that for any \( \sigma_i \in \mathbb{R} \), \( 0 \leq i \leq 2 \), with \( \sigma_2 \neq 0 \),

\[
\int_{-\infty}^{\infty} \frac{dx}{(\sigma_2 x^2 + \sigma_1 x + \sigma_0)^\rho} \leq C |\sigma_2|^{-\frac{1}{2}} \left( \sigma_0 - \frac{\sigma_1^2}{4\sigma_2} \right)^{-\frac{1}{2}}.
\]

(5.6)

Proof. It suffices to consider the case when \( \sigma_2 > 0 \). By rewriting

\[
\sigma_2 x^2 + \sigma_1 x + \sigma_0 = \sigma_2 \left( x + \frac{\sigma_1}{2\sigma_2} \right)^2 + \sigma_0 - \frac{\sigma_1^2}{4\sigma_2}
\]

and doing a change of variable \( y = \sqrt{\sigma_2} \left( x + \frac{\sigma_1}{2\sigma_2} \right) \), it reduces to show for any \( \alpha \in \mathbb{R} \),

\[
\int_{-\infty}^{\infty} \frac{dy}{(y^2 + \alpha)^\rho} \leq C(\alpha)^{-\frac{1}{2}},
\]

for which, the verification is straightforward and left to the readers.

\( \square \)

Lemma 5.4. Let \( \rho > 1 \) be given. There exists a constant \( C = C(\rho) \) such that for any \( \sigma_i \in \mathbb{R} \), \( 0 \leq i \leq 2 \),

\[
\int_{-\infty}^{\infty} \frac{dx}{(x^3 + \sigma_2 x^2 + \sigma_1 x + \sigma_0)^\rho} \leq C (3\sigma_1 - \sigma_2^2)^{-\frac{1}{2}}.
\]

(5.7)

Proof. By the change of variable \( y = x + \frac{\sigma_2}{3} \),

\[
\int_{-\infty}^{\infty} \frac{dx}{(x^3 + \sigma_2 x^2 + \sigma_1 x + \sigma_0)^\rho} = \int_{-\infty}^{\infty} \frac{dy}{(y^3 + b_1 y + b_0)^\rho},
\]

where

\[
b_1 = \sigma_1 - \frac{1}{3} \sigma_2^2, \quad b_0 = \frac{2}{27} \sigma_2^3 - \frac{1}{3} \sigma_1 \sigma_2 + \sigma_0.
\]

Thus, (5.7) reduces to justify

\[
\int_{-\infty}^{\infty} \frac{dy}{(y^3 + b_1 y + b_0)^\rho} \leq C(b_1)^{-\frac{1}{2}}
\]

(5.8)

for some constant \( C \) which only depends on \( \rho \). If \( |b_1| \leq 1 \), then (5.8) follows from (5.5) in Lemma 5.2. If \( |b_1| > 1 \), we define \( g(y) = y^3 + b_1 y + b_0 \) and find \( g'(y) = 3y^2 + b_1 \). If \( |g'(y)| \geq |b_1|^{1/4} \), then

\[
\int \frac{dy}{(g(y))^\rho} \leq \int \frac{1}{|b_1|^{1/4} (g(y))^\rho} dy \leq C(b_1)^{-\frac{1}{4}}.
\]

If \( |g'(y)| \leq |b_1|^{1/4} \), then the measure of the set of these \( y \) values is at most \( O(|b_1|^{-\frac{1}{4}}) \), so the integral of \( (g(y))^{-\rho} \) on this set is also bounded by \( C(b_1)^{-1/4} \).
For the proof of the bilinear estimate, it is usually beneficial to transfer it to an estimate of some 
weighted convolution of $L^2$ functions as pointed out in [11, 47]. The next lemma is one of such an 
example for the general bilinear estimate whose proof is standard by using duality and Plancherel theorem. For the 
convenience of notation, we denote $\xi = (\xi_1, \xi_2, \xi_3)$ and $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ to be the vectors in $\mathbb{R}^3$ and define 

$$A := \left\{ (\vec{\xi}, \vec{\tau}) \in \mathbb{R}^6 : \sum_{i=1}^{3} \xi_i = \sum_{i=1}^{3} \tau_i = 0 \right\}. \quad (5.9)$$

**Lemma 5.5.** Given $s$, $b$ and $(\alpha_i, \beta_i)1 \leq i \leq 3$, the bilinear estimate 

$$\|\partial_s (w_1 w_2)\|_{X^{s,b-1}_r} \leq C \|w_1\|_{X^{s,b}_r} \|w_2\|_{X^{s,b}_r}, \quad \forall \{w_i\}_{i=1,2},$$

is equivalent to 

$$\int_A \xi_3 (\xi_3) \prod_{i=1}^{3} f_i (\xi_i, \tau_i) \leq C \prod_{i=1}^{3} \|f_i\|_{L^1_{\xi}}, \quad \forall \{f_i\}_{1 \leq i \leq 3}. \quad (5.10)$$

where 

$$L_i = \tau_i - \phi^{\alpha_i, \beta_i}(\xi_i), \quad i = 1, 2, 3. \quad (5.11)$$

### 5.3 Resonance functions and the characteristic quadratic function

Based on the discussion in Section 5.1, the resonance function plays an essential role in establishing bilinear 
estimates. Now we follow [47] to give a formal definition to this function in the most general form.

**Definition 5.6 (47).** Let $((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ be a triple in $(\mathbb{R}^* \times \mathbb{R})^3$. Define the resonance function $H$ associated to this triple by 

$$H(\xi_1, \xi_2, \xi_3) = \sum_{i=1}^{3} \phi^{\alpha_i, \beta_i}(\xi_i), \quad \forall \sum_{i=1}^{3} \xi_i = 0. \quad (5.12)$$

The resonance set of $H$ is defined to be the zero set of $H$, that is 

$$\left\{ (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} \xi_i = 0, H(\xi_1, \xi_2, \xi_3) = 0 \right\}. \quad (5.13)$$

In particular, we introduce the notations of $H_0$, $H_1$ and $H_2$.

1. The resonance function associated to the triple $((\alpha_1, \beta_1), (\alpha_1, \beta_1), (\alpha_1, \beta_1))$ is denoted as $H_0$:

$$H_0(\xi_1, \xi_2, \xi_3) = \sum_{i=1}^{3} \phi^{\alpha_1, \beta_1}(\xi_i), \quad \forall \sum_{i=1}^{3} \xi_i = 0. \quad (5.14)$$

This applies to the classical case [3.3] or the case $r = 1$ in Table 8. By direct calculation,

$$H_0(\xi_1, \xi_2, \xi_3) = 3\alpha_1 \xi_1 \xi_2 \xi_3.$$

2. The resonance function associated to the triple $((\alpha_1, \beta_1), (\alpha_1, \beta_1), (\alpha_2, \beta_2))$ is denoted as $H_1$:

$$H_1(\xi_1, \xi_2, \xi_3) = \phi^{\alpha_1, \beta_1}(\xi_1) + \phi^{\alpha_1, \beta_1}(\xi_2) + \phi^{\alpha_2, \beta_2}(\xi_3), \quad \forall \sum_{i=1}^{3} \xi_i = 0. \quad (5.15)$$
This applies to the bilinear estimate of Type (D1). By direct calculation and writing \( \xi_2 = -(\xi_1 + \xi_3) \),

\[
H_1(\xi_1, \xi_2, \xi_3) = \xi_3 \left[ (\alpha_2 - \alpha_1)\xi_2^2 - 3\alpha_1\xi_1\xi_3 - 3\alpha_1\xi_1^2 \right] + (\beta_1 - \beta_2)\xi_3.
\] (5.16)

If \( \xi_3 = 0 \), then \( H_1 = 0 \). If \( \xi_3 \neq 0 \), then \( H_1 \) can be rewritten as

\[
H_1(\xi_1, \xi_2, \xi_3) = -3\alpha_1\xi_3^2 h_r \left( \frac{\xi_1}{\xi_3} \right) + (\beta_1 - \beta_2)\xi_3,
\] (5.17)

where \( r = \frac{\alpha_2}{\alpha_1} \) and

\[
h_r(x) := x^2 + x + \frac{1 - r}{3}.
\] (5.18)

(3) The resonance function \( H_2 \) associated to the triple \((\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_1, \beta_1)\) is denoted as \( H_2 \):

\[
H_2(\xi_1, \xi_2, \xi_3) = \phi^{\alpha_1, \beta_1}(\xi_1) + \phi^{\alpha_2, \beta_2}(\xi_2) + \phi^{\alpha_1, \beta_1}(\xi_3), \quad \forall \sum_{i=1}^{3} \xi_i = 0.
\] (5.19)

This applies to the bilinear estimates of Type (D2), (ND1) and (ND2). By direct calculation and writing \( \xi_3 = -(\xi_1 + \xi_2) \),

\[
H_2(\xi_1, \xi_2, \xi_3) = \xi_2 \left[ (\alpha_2 - \alpha_1)\xi_2^2 - 3\alpha_1\xi_1\xi_2 - 3\alpha_1\xi_1^2 \right] + (\beta_1 - \beta_2)\xi_2.
\] (5.20)

If \( \xi_2 = 0 \), then \( H_2 = 0 \). If \( \xi_2 \neq 0 \), then \( H_2 \) can be rewritten as

\[
H_2(\xi_1, \xi_2, \xi_3) = -3\alpha_1\xi_3^2 h_r \left( \frac{\xi_1}{\xi_2} \right) + (\beta_1 - \beta_2)\xi_2,
\] (5.21)

where \( r = \frac{\alpha_2}{\alpha_1} \) and \( h_r \) is as defined in (5.18).

According to the above computation, the quadratic function \( h_r \) is essential to determine the behavior of \( H_1 \) and \( H_2 \), thus, it is a characterization of the coupled KdV-KdV systems.

**Definition 5.7.** The quadratic function \( h_r \) in (5.18) is called the characteristic quadratic function associated to the coupled KdV-KdV systems (1.2).

### 5.4 Proof of Theorem 3.3

For the convenience of the proof, we introduce some notations below. For any \( r \in \mathbb{R} \), we define \( h_r \) as in (5.18) and define \( p_r : \mathbb{R} \to \mathbb{R} \) by

\[
p_r(x) = x^2 + 2x + 1 - r.
\] (5.22)

For fixed \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and for any \( \xi, \tau \in \mathbb{R} \), define \( P_{\xi,\tau} \) and \( Q_{\xi,\tau} \) from \( \mathbb{R} \) to \( \mathbb{R} \) as

\[
P_{\xi,\tau}(x) = (\alpha_1 - \alpha_2)x^3 + 3\alpha_1\xi x^2 + (3\alpha_1\xi^2 + \beta_2 - \beta_1) x + \phi^{\alpha_1, \beta_1}(\xi) - \tau,
\] (5.23)

\[
Q_{\xi,\tau}(x) = 3\alpha_1\xi^2 x^2 + 3\alpha_1\xi^2 x + \phi^{\alpha_1, \beta_1}(\xi) - \tau,
\] (5.24)

where \( \phi^{\alpha_1, \beta_1}(\xi) = \alpha_1\xi^3 - \beta_1\xi \). In the case when \( (\alpha_1, \beta_1) = (\alpha_2, \beta_2) \), \( P_{\xi,\tau} \) reduces to \( Q_{\xi,\tau} \).

**Proof of Theorem 3.3.** We will provide details for Case (1) and then briefly mention Case (2) and Case (3).

**Proof of Case (1).**

For \(-\frac{13}{12} \leq s \leq -1 \) and \( \frac{1}{4} - \frac{s}{3} \leq b \leq \frac{4}{3} + \frac{2s}{3} \), let \( \rho = -s \). Then

\[
1 \leq \rho \leq \frac{13}{12}, \quad \frac{1}{4} + \frac{\rho}{3} \leq b \leq \frac{4}{3} - \frac{2\rho}{3}.
\] (5.25)
According to Lemma \(5.5\) it suffices to prove
\[
\int_A \frac{|\xi_3|\langle\xi_1\rangle^p\langle\xi_2\rangle^p \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle\xi_3\rangle^p\langle L_1\rangle^b\langle L_2\rangle^b\langle L_3\rangle^{1-b}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2_{\psi}}, \quad \forall \{f_i\}_{1 \leq i \leq 3},
\]
where \(A\) is as defined in \((5.9)\) and
\[
L_1 = \tau_1 - \phi^{\alpha_1, \beta_1}(\xi_1), \quad L_2 = \tau_2 - \phi^{\alpha_2, \beta_2}(\xi_2), \quad L_3 = \tau_3 - \phi^{\alpha_3, \beta_3}(\xi_3).
\]
The resonance function \(H_2\) is as defined in \((5.19)\).

Since \(r := \frac{\alpha_3}{\alpha_1} < 0\), the function \(h_r\), as defined in \((5.18)\), has no real roots, so there exists \(\delta_1 = \delta_1(\alpha_1, \alpha_2)\) such that
\[
h_r(x) \geq \delta_1(1 + x^2), \quad \forall x \in \mathbb{R}.
\]
Then according to \((5.21)\), there exists \(\delta_2 = \delta_2(\alpha_1, \alpha_2)\) such that
\[
|H_2(\xi_1, \xi_2, \xi_3)| \geq \delta_2|\xi_2|(|\xi_1^2 + \xi_2^2| - |\beta_1 - \beta_2||\xi_2|).
\]
Now if \(|\beta_2 - \beta_1| \leq \epsilon_1\) with sufficiently small \(\epsilon_1\) depending only on \(\delta_2\),
\[
\langle H_2(\xi_1, \xi_2, \xi_3) \rangle \geq \frac{\delta_2}{2} |\xi_2|(|\xi_1^2 + \xi_2^2|).
\]
Since \(\sum_{i=1}^{3} \xi_i = 0\), the above estimate implies that
\[
\langle H_2(\xi_1, \xi_2, \xi_3) \rangle \gtrsim |\xi_2|^3 \sum_{i=1}^{3} \xi_i^2.
\]
Define \(\text{MAX} = \max\{\langle L_1 \rangle, \langle L_2 \rangle, \langle L_3 \rangle\}\). Then it follows from \(H_2 = -\sum_{i=1}^{3} L_i\) that \(\text{MAX} \gtrsim \frac{1}{3}(H_2)\). Therefore,
\[
\text{MAX} \gtrsim |\xi_2|^3 \sum_{i=1}^{3} \xi_i^2.
\]
Decompose the region \(A\) as \(\bigcup_{i=0}^{3} A_i\), where
\[
A_0 = \{(\xi, \tau) \in A : |\xi_1| \leq 1 \text{ or } |\xi_2| \leq 1\},
A_i = \{(\xi, \tau) \in A : |\xi_1| > 1, |\xi_2| > 1 \text{ and } \langle L_i \rangle = \text{MAX}\}, \quad 1 \leq i \leq 3.
\]

**Contribution on \(A_0\):**
Since \(\langle \xi_1 \rangle \langle \xi_2 \rangle \lesssim \langle \xi_3 \rangle\) when \(|\xi_1| \leq 1 \text{ or } |\xi_2| \leq 1\),
\[
\int_A \frac{|\xi_3|\langle\xi_1\rangle^p\langle\xi_2\rangle^p \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle\xi_3\rangle^p\langle L_1\rangle^b\langle L_2\rangle^b\langle L_3\rangle^{1-b}} \lesssim \int_A \frac{|\xi_3| \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{\langle L_1\rangle^b\langle L_2\rangle^b\langle L_3\rangle^{1-b}} = \int \int \frac{|\xi_3| |f_3|}{\langle L_3\rangle^{1-b}} \left( \int \int \frac{|f_1 f_2|}{\langle L_1\rangle^b\langle L_2\rangle^b} d\tau_2 d\xi_2 \right) d\tau_3 d\xi_3.
\]

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In order to bound the above integral by \( C \prod_{i=1}^{3} \| f_i \|_{L^2_{\xi}} \), it suffices to show

\[
\sup_{\xi_3, \tau_3} \frac{\| \xi_3 \|}{(L_3)^{1-b}} \left( \int \int \frac{d\tau_2 d\xi_2}{(L_1)^{2b}(L_2)^{2b}} \right)^{1/2} \leq C
\]  

due to the same argument as in [29] via the Cauchy-Schwartz inequality. Next, for any fixed \( \xi_3 \) and \( \tau_3 \), we will estimate

\[
\int \int \frac{d\tau_2 d\xi_2}{(L_1)^{2b}(L_2)^{2b}}.
\]

Since \( \tau_1 = -\tau_2 - \tau_3 \) and \( \xi_1 = -\xi_2 - \xi_3 \), \( L_1 \) can be written as

\[
L_1 = -\tau_2 - \tau_3 - \phi^{\alpha_1,\beta_1}(-\xi_2 - \xi_3).
\]

Meanwhile, recalling \( L_2 = \tau_2 - \phi^{\alpha_2,\beta_2}(\xi_2) \), it then follows from Lemma 5.1 that

\[
\int \frac{d\tau_2}{(L_1)^{2b}(L_2)^{2b}} \lesssim \frac{1}{(L_1 + L_2)^{2b}}.
\]

So (5.33) is reduced to

\[
\sup_{\xi_3, \tau_3} \frac{\| \xi_3 \|}{(L_3)^{1-b}} \left( \int \frac{d\xi_2}{(L_1 + L_2)^{2b}} \right)^{1/2} \leq C,
\]

or equivalently,

\[
\sup_{\xi_3, \tau_3} \frac{\| \xi_3 \|^2}{(L_3)^{2(1-b)}} \int \frac{d\xi_2}{(L_1 + L_2)^{2b}} \leq C.
\]

By direct calculation, we find

\[
L_1 + L_2 = P_{\xi_3, \tau_3}(\xi_2),
\]

where \( P_{\xi_3, \tau_3} \) is as defined in (5.23) with \((\xi, \tau)\) being replaced by \((\xi_3, \tau_3)\). Hence, (5.35) is further reduced to

\[
\sup_{\xi_3, \tau_3} \frac{\| \xi_3 \|^2}{(L_3)^{2(1-b)}} \int \frac{d\xi_2}{(P_{\xi_3, \tau_3}(\xi_2))^{2b}} \leq C.
\]

There are two situations.

- \( |\xi_3| \leq 1 \). In this situation, it suffices to prove \( \int \frac{d\xi_2}{(P_{\xi_3, \tau_3}(\xi_2))^{2b}} \) is bounded. Since \( P_{\xi_3, \tau_3}(\xi_2) \) is a cubic function in \( \xi_2 \), the boundedness of this integral follows from Lemma 5.2

- \( |\xi_3| \geq 1 \). In this situation,

\[
P'_{\xi_3, \tau_3}(\xi_2) = 3(\alpha_1 - \alpha_2)\xi_2^3 + 6\alpha_1\xi_3\xi_2 + 3\alpha_1\xi_3^2 + \beta_2 - \beta_1.
\]

When \( \xi_2 \neq 0 \),

\[
P'_{\xi_3, \tau_3}(\xi_2) = 3\alpha_1\xi_2^3p_r \left( \frac{\xi_3}{\xi_2} \right) + \beta_2 - \beta_1,
\]

where \( p_r \) is as defined in (5.22). Since \( r < 0 \), \( p_r \) does not have any real roots. Therefore, there exists \( \delta_3 = \delta_3(\alpha_1, \alpha_2) \) such that

\[
p_r(x) \geq \delta_3(1 + x^2), \quad \forall x \in \mathbb{R}.
\]

As a result, there exists \( \delta_4 = \delta_4(\alpha_1, \alpha_2) \) such that

\[
|P'_{\xi_3, \tau_3}(\xi_2)| \geq \delta_4(\xi_2^3 + \xi_3^2) - |\beta_2 - \beta_1|.
\]

Since \( |\xi_3| \geq 1 \), when \( |\beta_2 - \beta_1| \) is sufficiently small,

\[
|P'_{\xi_3, \tau_3}(\xi_2)| \gtrsim \xi_2^3 + \xi_3^2.
\]
Hence,
\[ \int \frac{d\xi_2}{(P_{\xi_3,\tau_3}(\xi_2))^2b} \lesssim \frac{1}{|\xi_3|^2} \int_{\mathbb{R}} \frac{|P'_{\xi_3,\tau_3}(\xi_2)|}{(P_{\xi_3,\tau_3}(\xi_2))^2b} d\xi_2 \lesssim \frac{1}{|\xi_3|^2}, \]
which also justifies (5.37).

**Contribution on A_3:**
Since \( \langle \xi_i \rangle \sim |\xi_i| \) when \( |\xi_i| > 1 \) for \( i = 1, 2, \)
\[ \int \frac{|\xi_i| \langle \xi_i \rangle^\rho \langle \xi_2 \rangle^\rho \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_i \rangle^\rho (L_1)^b(L_2)^b(L_3)^{1-b}} \lesssim \int \int \frac{|\xi_2||f_2|}{\langle \xi_3 \rangle^\rho (L_3)^{1-b}} \left( \int \int \frac{|\xi_1| \langle \xi_2 \rangle^\rho |f_1 f_2|}{\langle L_1 \rangle^b(L_2)^b} d\tau_2 d\xi_2 \right) d\tau_3 d\xi_3. \tag{5.41} \]
In order to bound the above integral by \( C \prod_{i=1}^3 \|f_i\|_{L^2_\xi} \), similar to the derivation from (5.32) to (5.37), it suffices to show
\[ \sup_{\xi_3, \tau_3} \frac{|\xi_3|^2}{\langle \xi_3 \rangle^{2\rho} (L_3)^{2(1-b)}} \int \frac{|\xi_1 \xi_2|^{2\rho}}{(P_{\xi_3,\tau_3}(\xi_2))^2b} d\xi_2 \leq C, \tag{5.42} \]
where \( P_{\xi_3,\tau_3}(\xi_2) \) is the same as (5.36). Then by analogous derivation from (5.38) to (5.40), it also holds
\[ |P'_{\xi_3,\tau_3}(\xi_2)| \gtrsim |\xi_2|^2 + |\xi_3|^2. \]
Moreover, since \( \sum_{i=1}^3 \xi_i = 0 \),
\[ |P'_{\xi_3,\tau_3}(\xi_2)| \gtrsim \sum_{i=1}^3 \xi_i^2 \gtrsim |\xi_1 \xi_2|. \tag{5.43} \]
As a result, (5.42) is reduced to
\[ \sup_{\xi_3, \tau_3} \int \frac{|\xi_3|^2|\xi_1 \xi_2|^{2\rho-1}}{\langle \xi_3 \rangle^{2\rho} (L_3)^{2(1-b)}} \frac{|P'_{\xi_3,\tau_3}(\xi_2)|}{(P_{\xi_3,\tau_3}(\xi_2))^2b} d\xi_2 \leq C. \]

Since \( \langle L_3 \rangle = \text{MAX on } A_3 \) and
\[ \int \frac{|P'_{\xi_3,\tau_3}(\xi_2)|}{(P_{\xi_3,\tau_3}(\xi_2))^2b} d\xi_2 < \infty, \]
it suffices to show
\[ \frac{|\xi_3|^2|\xi_1 \xi_2|^{2\rho-1}}{\langle \xi_3 \rangle^{2\rho} (\text{MAX})^{2(1-b)}} \leq C. \tag{5.44} \]
To this end, note that \( |\xi_3| \leq \langle \xi_3 \rangle^\rho \) as \( \rho \geq 1 \). Moreover, it follows from (5.30) that \( \langle \text{MAX} \rangle \gtrsim |\xi_1 \xi_2|^3 \). Consequently,
\[ \frac{|\xi_3|^2|\xi_1 \xi_2|^{2\rho-1}}{\langle \xi_3 \rangle^{2\rho} (\text{MAX})^{2(1-b)}} \lesssim \frac{|\xi_1 \xi_2|^{2\rho-1}}{|\xi_1 \xi_2|^{3(1-b)}} = |\xi_1 \xi_2|^{3b+2\rho-4}. \]
Noticing the restriction (5.25) implies \( 3b + 2\rho - 4 \leq 0 \), so \( |\xi_1 \xi_2|^{3b+2\rho-4} \leq 1 \).

**Contribution on A_1:**
Since \( \langle L_1 \rangle = \text{MAX on } A_1 \),
\[ \frac{1}{(L_1)^b(L_3)^{1-b}} \leq \frac{1}{(L_1)^{1-b}(L_3)^b}. \]
Therefore,

\[
\int \frac{|\xi_3|^2 |\xi_3|^2 (\xi_2) |^p \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_3 \rangle^p (L_1)^{1-b} (L_2)^{1-b} (L_3)^{1-b}} \lesssim \int \frac{|\xi_3|^2 |\xi_1\xi_2|^p \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_3 \rangle^p (L_1)^{1-b} (L_2)^{1-b} (L_3)^{1-b}} = \int \frac{|f_2| |\xi_1|^p}{\langle L_1 \rangle^{1-b}} \left( \int \frac{|\xi_3|^2 |\xi_3|^2 |f_2 f_3|}{\langle \xi_3 \rangle^p (L_2)^{1-b} (L_3)^{1-b}} d\tau_2 d\xi_2 \right) d\tau_1 d\xi_3.
\]

Then similar to the derivation from (5.32) to (5.35), it suffices to show

\[
\sup_{\xi_1, \tau_1} \frac{|\xi_1|^2}{(\langle L_1 \rangle)^{2(1-b)}} \int \frac{|\xi_2|^2 |\xi_3|^2}{\langle \xi_3 \rangle^p (L_2 + L_3)^{2p}} d\xi_2 \leq C. \tag{5.45}
\]

For any fixed \((\xi_1, \tau_1)\), writing \(\tau_3 = -\tau_2 - \tau_1\) and \(\xi_3 = -\xi_2 - \xi_1\), then by direct calculation, we find

\[
L_2 + L_3 = P_{\xi_1, \tau_1}(\xi_2),
\]

where \(P_{\xi_1, \tau_1}\) is as defined in (5.23) with \((\xi, \tau)\) being replaced by \((\xi_1, \tau_1)\). Hence, (5.45) is further reduced to

\[
\sup_{\xi_1, \tau_1} \frac{|\xi_1|^2}{(\langle L_1 \rangle)^{2(1-b)}} \int \frac{|\xi_2|^2 |\xi_3|^2}{\langle \xi_3 \rangle^p (P_{\xi_1, \tau_1}(\xi_2))^{2p}} d\xi_2 \leq C. \tag{5.47}
\]

Then by analogous derivation from (5.38) to (5.40), for sufficiently small \(|\beta_2 - \beta_1|\), we have

\[
|P'_{\xi_1, \tau_1}(\xi_2)| \gtrsim \xi_1^2 + |\xi_2| \geq |\xi_1\xi_2|.
\]

Based on this estimate, the rest argument is similar to that for the region \(A_3\) after (5.43).

**Contribution on \(A_2\):** First, we decompose \(A_2\) into three parts: \(A_2 = \bigcup_{i=1}^3 A_{2i}\) with

\[
\begin{align*}
A_{21} &= \{ (\xi, \tau) \in A_2 : |\xi_1| < \frac{1}{2} |\xi_2| \}, \\
A_{22} &= \{ (\xi, \tau) \in A_2 : \frac{1}{2} |\xi_2| \leq |\xi_1| \leq \frac{3}{4} |\xi_2| \}, \\
A_{23} &= \{ (\xi, \tau) \in A_2 : |\xi_1| > \frac{3}{4} |\xi_2| \}.
\end{align*} \tag{5.48}
\]

- On \(A_{21}\) or \(A_{23}\), since \(\langle L_2 \rangle = \text{MAX},\)

\[
\frac{1}{\langle L_2 \rangle^{1-b} (L_3)^{1-b}} < \frac{1}{\langle L_2 \rangle^{1-b} (L_3)^{1-b}}.
\]

Thus,

\[
\int \frac{|\xi_3|^2 |\xi_1|^p (\xi_2) |^p \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_3 \rangle^p (L_1)^{1-b} (L_2)^{1-b} (L_3)^{1-b}} \lesssim \int \frac{|\xi_3|^2 |\xi_1\xi_2|^p \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_3 \rangle^p (L_1)^{1-b} (L_2)^{1-b} (L_3)^{1-b}} = \int \frac{|f_2| |\xi_1|^p}{\langle L_1 \rangle^{1-b}} \left( \int \frac{|\xi_3|^2 |\xi_3|^2 |f_2 f_3|}{\langle \xi_3 \rangle^p (L_2)^{1-b} (L_3)^{1-b}} d\tau_2 d\xi_2 \right) d\tau_1 d\xi_3.
\]

Then similar to the derivation from (5.32) to (5.35), it suffices to show

\[
\sup_{\xi_2, \tau_2} \frac{|\xi_2|^2}{(\langle L_2 \rangle)^{2(1-b)}} \int \frac{|\xi_1|^2 |\xi_3|^2}{\langle \xi_3 \rangle^p (L_1 + L_3)^{2p}} d\xi_1 \leq C. \tag{5.49}
\]
For any fixed \((\xi_2, \tau_2)\), writing \(\tau_3 = -\tau_1 - \tau_2\) and \(\xi_3 = -\xi_1 - \xi_2\), then by direct calculation, we find

\[
L_1 + L_3 = Q_{\xi_2, \tau_2}(\xi_1),
\]

(5.50)

where \(Q_{\xi_2, \tau_2}\) is as defined in (5.24) with \((\xi, \tau)\) being replaced by \((\xi_2, \tau_2)\). Hence, (5.49) is further reduced to

\[
\sup_{\xi_2, \tau_2} \left| \frac{\xi_2}{(L_2)^{2(1-b)}} \right| \int \frac{|\xi_1|^{2p}|\xi_3|^2}{(\xi_3)^{2p}(Q_{\xi_2, \tau_2}(\xi_1))^{2b}} d\xi_1 \leq C.
\]

(5.51)

Again by direct calculation,

\[
Q'_{\xi_2, \tau_2}(\xi_1) = 6\alpha_1\xi_2\xi_1 + 3\alpha_1\xi_2^2 = 3\alpha_1\xi_2(2\xi_1 + \xi_2).
\]

(5.52)

According to the definition of \(A_{21}\) and \(A_{23}\) in (5.48), either \(|\xi_1| < \frac{1}{2}|\xi_2|\) or \(|\xi_1| > \frac{3}{2}|\xi_2|\), so it follows from (5.52) that

\[
|Q'_{\xi_2, \tau_2}(\xi_1)| \gtrsim |\xi_1\xi_2|.
\]

Based on this estimate, the rest argument is similar to that for the region \(A_3\) after (5.43).

- On \(A_{22}\), we have \(|\xi_1| \sim |\xi_2| \sim |\xi_3|\), so

\[
\int \frac{|\xi_1|(|\xi_1|)^{p}(\xi_2)^{p} \prod_{i=1}^{3} |f_i(\xi_1, \tau_i)|}{(\xi_3)^{p}(L_1)^{b}(L_2)^{b}(L_3)^{1-b}} \lesssim \int \frac{|\xi_2|^{1+p} \prod_{i=1}^{3} |f_i(\xi_1, \tau_i)|}{(L_1)^{b}(L_2)^{b}(L_3)^{1-b}} = \int \frac{|f_2|\|\xi_2|^{1+p}}{(L_2)^{b}} \left( \int \frac{|f_1f_3|}{(L_1)^{b}(L_3)^{1-b}} d\tau_1 d\xi_1 \right) d\tau_2 d\xi_2.
\]

Then similar to the derivation from (5.32) to (5.35), it suffices to show

\[
\sup_{\xi_2, \tau_2} \frac{|\xi_2|^{2(1+p)}}{(L_2)^{2b}} \int \frac{d\xi_1}{(L_1 + L_3)^{2(1-b)}} \leq C.
\]

That is to prove

\[
\sup_{\xi_2, \tau_2} \frac{|\xi_2|^{2(1+p)}}{(L_2)^{2b}} \int \frac{d\xi_1}{(Q_{\xi_2, \tau_2}(\xi_1))^{2(1-b)}} \leq C,
\]

(5.53)

where \(Q_{\xi_2, \tau_2}(\xi_1) = L_1 + L_3\) is as defined in (5.50). By (5.50) and (5.24),

\[
Q_{\xi_2, \tau_2}(\xi_1) = 3\alpha_1\xi_2\xi_1^2 + 3\alpha_1\xi_2^2\xi_1 + \alpha_{\xi_1, \beta_1}(\xi_2) - \tau_2.
\]

In other words, \(Q_{\xi_2, \tau_2}(\xi_1)\) is a quadratic function in \(\xi_1\) with the leading coefficient \(3\alpha_1\xi_2\). Since (5.25) implies that \(2(1 - b) > \frac{1}{2}\), then it follows from Lemma 5.2 that

\[
\int \frac{d\xi_1}{(Q_{\xi_2, \tau_2}(\xi_1))^{2(1-b)}} \lesssim |\xi_2|^{-\frac{b}{2}}.
\]

Therefore, (5.53) reduces to

\[
\sup_{\xi_2, \tau_2} \frac{|\xi_2|^{\frac{3}{2}+2p}}{(L_2)^{2b}} \leq C.
\]

(5.54)

Since \(\langle L_2 \rangle = \text{MAX on } A_{22}\), it follows from (5.30) that \(\langle L_2 \rangle \gtrsim |\xi_2|^3\). Hence,

\[
\frac{|\xi_2|^{\frac{3}{2}+2p}}{(L_2)^{2b}} \lesssim |\xi_2|^{\frac{3}{2}+2p-6b}.
\]

Finally, due to the restriction \(b \geq \frac{1}{4} + \frac{\rho}{3}\) in (5.25), we have \(\frac{3}{2} + 2p - 6b \leq 0\) and \(|\xi_2|^{\frac{3}{2}+2p-6b} \leq 1\).
Proof of Case (2).

Let \( \rho = -s \). By the assumption in this case,
\[
\frac{3}{4} < \rho < 1, \quad \frac{1}{4} + \frac{\rho}{3} \leq b \leq 1 - \frac{\rho}{3}.
\] (5.55)

As in the proof for Case (1), we first decompose \( A = \bigcup_{i=0}^{3} A_i \) as in (5.31).

- On \( A_0 \), the proof is the same as that for Case (1).
- On \( A_3 \), it again reduces to prove (5.44). Since \( \rho < 1 \), it suffices to show
  \[
  \frac{|\xi_3|^{2-2\rho}|\xi_1\xi_2|^{2\rho-1}}{(\text{MAX})^{2(1-b)}} \leq C.
  \] (5.56)

By (5.36),
\[
\text{MAX} \gtrsim \max\{|\xi_2\xi_3^2|, |\xi_1^3\xi_2|, |\xi_2|^{3}\} \geq 1.
\]

Then it follows from \( \frac{3}{4} < \rho < 1 \) that
\[
|\xi_3|^{2-2\rho}|\xi_1\xi_2|^{2\rho-1} = |\xi_2\xi_3^2|^{1-\rho}|\xi_1^3\xi_2|^{\rho-3}\xi_3^{4\rho-3}\xi_2^{4\rho-3} \lesssim (\text{MAX})^{2\rho/3}.
\]

Finally, (5.56) holds since (5.55) implies \( \frac{2\rho}{3} \leq 2(1-b) \).

- On \( A_1 \), similarly, it reduces to prove (5.56) which can be justified exactly the same as above.
- On \( A_2 \), we also decompose \( A_2 \) as (5.48). The arguments on \( A_{21} \) and \( A_{23} \) are again reduced to prove (5.56). The argument on \( A_{22} \) is the same as that for Case (1) thanks to the condition \( b \geq \frac{1}{4} + \frac{\rho}{3} \) in (5.55).

Proof of Case (3).

Since \( \langle \xi_1 \rangle \langle \xi_2 \rangle \geq \langle \xi_3 \rangle \), the left hand side of (5.26) is an increasing function in \( \rho \). So it suffices to consider the case when \( s = -\frac{3}{4} \). Then it can be justified in the same way as that for Case (2).

\[ \square \]

5.5 Proof of Theorem 3.5

First, we want to point out several cases in Table 8 which have been known or can be proved similarly.

- When \( r = 1 \), Type (D1) and (D2) with \( s > -\frac{3}{4} \) were established in [29].

- When \( r > \frac{1}{4} \) but \( r \neq 1 \), Type (D1) and (D2) with \( s \geq 0 \) have been justified in [38]. The situations for Type (ND1) and (ND2) can be treated similarly.

- When \( r = -1 \), Type (D1) was proved in [1].

In all of the above results, it is assumed that \( \beta_1 = \beta_2 = 0 \). But as we have seen from the proof of Theorem 3.3, even if \( \beta_1 \) or \( \beta_2 \) is not equal to 0, they will not affect the conclusion as long as \( |\beta_2 - \beta_1| \) is small.

For the rest cases in Table 8 we will only provide proofs for the following typical ones.

1. Among the cases when \( r < 0 \) or \( 0 < r < \frac{1}{4} \), we will only prove Type (ND1) with \( 0 < r < \frac{1}{4} \), see Section 5.5.1. There are two reasons. Firstly, the cases when \( 0 < r < \frac{1}{4} \) is generally more difficult than the cases when \( r < 0 \). Secondly, the non-divergence cases is more challenging than the divergence cases.

2. When \( r = \frac{1}{4} \), the justifications for all four types are analogous, so we will still only focus on Type (ND1), see Section 5.5.2.

3. When \( r = 1 \), Type (D1) and (D2) have been known and Type (ND1) and (ND2) are similar, so we will again only deal with Type (ND1), see Section 5.5.3.
As discussed above, only Type (ND1) will be investigated, so we list some common notations which will be used in Sections 5.5.1–5.5.3. First, we define the set $A$ as \( A := \{ (\tilde{\xi}, \tilde{\tau}) \in \mathbb{R}^6 : \sum_{i=1}^{3} \xi_i = \sum_{i=1}^{3} \tau_i = 0 \} \).

Then for any \((\xi, \tau) \in A\), we denote
\[
L_1 = \tau_1 - \phi^{\alpha_1, \beta_1}(\xi_1), \quad L_2 = \tau_2 - \phi^{\alpha_2, \beta_2}(\xi_2), \quad L_3 = \tau_3 - \phi^{\alpha_3, \beta_3}(\xi_3).
\]

The resonance function is \( H_2 \) as defined in (5.19). That is
\[
H_2(\xi_1, \xi_2, \xi_3) = \phi^{\alpha_1, \beta_1}(\xi_1) + \phi^{\alpha_2, \beta_2}(\xi_2) + \phi^{\alpha_3, \beta_3}(\xi_3) = - \sum_{i=1}^{3} L_i.
\]

In addition, we write \( \text{MAX} = \max\{L_1, L_2, L_3\} \). It is obvious that \( \text{MAX} \geq |H_2(\xi_1, \xi_2, \xi_3)| \). Finally, we denote the functions \( h_r, p_r, P_{\xi, \tau} \) and \( Q_{\xi, \tau} \) as in (5.18), (5.22), (5.23) and (5.24) respectively.

**5.5.1 Type (ND1) with \( 0 < r < \frac{1}{4} \) and \( s > -\frac{3}{4} \)**

Let \( \rho = -s \). Then \( \rho < \frac{3}{4} \). Similar to the argument as in the proof of Lemma 5.5, one only needs to show
\[
\int_{A} \frac{|\xi_1|^{\rho} (\xi_2)^{\rho} \prod_{i=1}^{3} |f_i(\xi_i, \tau_i)|}{(\xi_3)^{\rho} (L_1)^{b} (L_2)^{b} (L_3)^{1-b}} \leq C \prod_{i=1}^{3} \|f_i\|_{L^2_{\xi}} \quad \forall \{f_i\}_{1 \leq i \leq 3}.
\]  

(5.57)

Since \( \frac{\langle \xi_1 \rangle}{\langle \xi_3 \rangle} \geq 1 \), it suffices to consider the case when \( \frac{9}{16} \leq \rho < \frac{3}{4} \). Assume
\[
\frac{1}{2} < b \leq \frac{3}{4} - \frac{\rho}{3} := b_0.
\]  

(5.58)

Since \( 0 < r < \frac{1}{4} \), the function \( h_r \) has no real roots. Then by the similar argument from (5.28) to (5.30) in Section 5.4, there exist \( \epsilon \) and \( \delta \), which only depend on \( \alpha_1 \) and \( \alpha_2 \), such that whenever \( |\beta_2 - \beta_1| \leq \epsilon \), it holds
\[
\text{MAX} \geq \delta |\xi_2|^{2} \sum_{i=1}^{3} \xi_i^2.
\]  

(5.59)

Decompose the region \( A = \bigcup_{i=0}^{3} A_i \) as in (5.31), that is
\[
A_0 = \{(\xi, \tau) \in A : |\xi_1| \leq 1 \text{ or } |\xi_2| \leq 1\}, \quad A_i = \{(\xi, \tau) \in A : |\xi_1| > 1, |\xi_2| > 1 \text{ and } (L_i) = \text{MAX}\}, \quad 1 \leq i \leq 3.
\]  

(5.60)

Among the above regions \( \{A_i\}_{i=0}^{3} \), the most challenging region is \( A_3 \), so we will only show how we estimate on this region next.

**Contribution on \( A_3 \)**:

Similar to the derivation for (5.42), it suffices to show
\[
\sup_{\xi_3, \tau_3} \frac{1}{(\xi_3)^{2\rho} (L_3)^{2(1-b)}} \int_{\xi_3, \tau_3} \frac{|\xi_1|^{2(1+\rho)} |\xi_2|^{2\rho}}{(P_{\xi, \tau}(\xi_2))^{2b}} d\xi_2 \leq C.
\]  

(5.61)
So there exists a positive constant $\sigma$ satisfying $p_r$ has two roots $x_{1r} = -1 - \sqrt{r}$ and $x_{2r} = -1 + \sqrt{r}$ which satisfy

$$-2 < x_{1r} < -1 < x_{2r} < 0.$$ 

So there exists a positive constant $\sigma$, depending only on $r$, such that

$$[x_{1r} - 2\sigma_r, x_{1r} + 2\sigma_r] \subset (-2, -1) \text{ and } [x_{2r} - 2\sigma_r, x_{2r} + 2\sigma_r] \subset (-1, 0).$$

The region $A_3$ is accordingly decomposed further as $A_3 = \bigcup_{i=1}^{3} A_{3i}$, where

$$A_{31} = \{(\vec{\xi}, \vec{\tau}) \in A_3 : |\frac{\xi_3}{\xi_2} - x_{1r}| \geq \sigma_r \text{ and } |\frac{\xi_2}{\xi_3} - x_{2r}| \geq \sigma_r\},$$

$$A_{32} = \{(\vec{\xi}, \vec{\tau}) \in A_3 : |\frac{\xi_3}{\xi_2} - x_{1r}| < \sigma_r\},$$

$$A_{33} = \{(\vec{\xi}, \vec{\tau}) \in A_3 : |\frac{\xi_2}{\xi_3} - x_{2r}| < \sigma_r\}.$$

- On $A_{31}$, since $\frac{\xi_3}{\xi_2}$ is away from the roots of $p_r$, there exists $\delta$, depending only on $r$, such that

$$p_r \left(\frac{\xi_3}{\xi_2}\right) \geq \delta \left[1 + \left(\frac{\xi_1}{\xi_2}\right)^2\right].$$

Hence,

$$|P'_{\xi_3, \tau_3} (\xi_2)| \geq 3|\alpha_1|\delta(\xi_2^2 + \xi_3^2) - |\beta_2 - \beta_1|.$$ 

When $|\beta_2 - \beta_1|$ is sufficiently small,

$$|P'_{\xi_3, \tau_3} (\xi_2)| \gtrsim \xi_2^2 + \xi_3^2 \gtrsim \xi_1^2.$$ 

Then

$$\text{LHS of (5.61)} \lesssim \sup_{\xi_3, \tau_3} \int_{\mathbb{R}} \frac{|\xi_1\xi_2|^{2\rho}}{(|\xi_3|^{2\rho} (L_3)^{2(1-b)})^2} \frac{|P'_{\xi_3, \tau_3} (\xi_2)|}{(P_{\xi_3, \tau_3} (\xi_2))^{2\beta}} \, d\xi_2.$$ 

In order to prove the boundedness of the above integral, it suffices to show

$$\frac{|\xi_1\xi_2|^{2\rho}}{(L_3)^{2(1-b)}} \lesssim C.$$ 

Since $\langle L_3 \rangle = \text{MAX}$, it follows from (5.59) that $\langle L_3 \rangle \gtrsim |\xi_2|(|\xi_1^2 + \xi_3^2| \gtrsim |\xi_1\xi_2|^2$. Finally, due to the restriction $\rho < \frac{3}{4}$ and the choice (5.58) for $b$, we have $3(1-b) \geq 2\rho$. Therefore,

$$\langle L_3 \rangle^{2(1-b)} \gtrsim |\xi_1\xi_2|^{3(1-b)} \gtrsim |\xi_1\xi_2|^{2\rho},$$

which implies (5.63).

- On $A_{32}$, it is easily seen that $|\xi_1| \sim |\xi_2| \sim |\xi_3|$. Then $\langle L_3 \rangle = \text{MAX} \gtrsim |\xi_2| \sum_{i=1}^{3} \xi_i^2 \gtrsim |\xi_3|^3$. Therefore,

$$\text{LHS of (5.61)} \sim \sup_{\xi_3, \tau_3} \int_{\mathbb{R}} \frac{|\xi_4|^4\rho^2}{(|\xi_3|^{2\rho} (L_3)^{2(1-b)})^2} \, d\xi_2 \lesssim \sup_{\xi_3, \tau_3} \frac{|\xi_4|^4\rho^2}{(|\xi_3|^{2\rho} (L_3)^{2(1-b)})^2} \int_{\mathbb{R}} \frac{d\xi_2}{(P_{\xi_3, \tau_3} (\xi_2))^{2\beta}}.$$
Since
\[ P_{\xi_3, \tau_3}(\xi_2) = (\alpha_1 - \alpha_2)\xi_3^2 + 3\alpha_1\xi_3\xi_2^2 + (3\alpha_1\xi_3^2 + \beta_2 - \beta_1)\xi_2 + \phi^{\alpha_1, \beta_1}(\xi_3) - \tau_3, \]
then by dividing the leading coefficient \(\alpha_1 - \alpha_2\), we have
\[ \langle P_{\xi_3, \tau_3}(\xi_2) \rangle \sim \langle \xi_3^3 + \sigma_2\xi_2^2 + \sigma_1\xi_2 + \sigma_0 \rangle, \]
where
\[ \sigma_2 = \frac{3\alpha_1\xi_3}{\alpha_1 - \alpha_2}, \quad \sigma_1 = \frac{3\alpha_1\xi_3^2 + \beta_2 - \beta_1}{\alpha_1 - \alpha_2}, \quad \sigma_0 = \frac{\phi^{\alpha_1, \beta_1}(\xi_3) - \tau_3}{\alpha_1 - \alpha_2}. \]
Consequently, it follows from Lemma 5.4 and direct calculation that
\[ \int \frac{d\xi_2}{\langle P_{\xi_3, \tau_3}(\xi_2) \rangle^{2\rho}} \lesssim \langle 3\sigma_1 - \sigma_2^2 \rangle^{-\frac{\rho}{2}} \]
\[ \sim \langle -9\alpha_1\alpha_2\xi_3^2 + 3(\alpha_1 - \alpha_2)(\beta_2 - \beta_1) \rangle^{-\frac{\rho}{2}}. \quad (5.65) \]
Since \(|\xi_3| \sim |\xi_1| \gtrsim 1\), when \(|\beta_2 - \beta_1|\) is sufficiently small, it follows from (5.65) that
\[ \int \frac{d\xi_2}{\langle P_{\xi_3, \tau_3}(\xi_2) \rangle^{2\rho}} \lesssim \langle \xi_3^2 \rangle^{-\frac{\rho}{2}} \sim |\xi_3|^{-\frac{\rho}{2}}. \]
Hence, it follows from (5.64) that
\[ \text{LHS of (5.61)} \lesssim \sup_{\xi_3 \geq 1} \frac{|\xi_3|^{4\rho + 2}}{\langle \xi_3 \rangle^{2\rho}|\xi_3|^{6b + 2\rho - 3}} \lesssim \sup_{\xi_3 \geq 1} |\xi_3|^{6b + 2\rho - 3} \leq C, \]
where the last inequality is due to \(6b + 2\rho - \frac{3}{2} \leq 0\) (see (5.58)).

- On \(A_{33}\), the argument is similar to that for \(A_{32}\).

5.5.2 Type (ND1) with \(r = \frac{1}{2}\) and \(s \geq \frac{3}{4}\)

Similar to the argument as in the proof of Lemma 5.5, it suffices to prove
\[ \int_A \frac{|\xi_1|\langle \xi_3 \rangle^2 \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_1 \rangle^{2\rho}|\langle \xi_2 \rangle|^{1-b}} \leq C \prod_{i=1}^3 \|f_i\|_{L^2}^{\rho}, \quad \forall \{f_i\}_{1 \leq i \leq 3}. \]
As \(\frac{\langle \xi_3 \rangle}{\langle \xi_1 \rangle}\langle \xi_2 \rangle \leq 1\), we only need consider the case of \(s = \frac{3}{4}\), i.e.,
\[ \int_A \frac{|\xi_1|\langle \xi_3 \rangle^2 \prod_{i=1}^3 |f_i(\xi_i, \tau_i)|}{\langle \xi_1 \rangle^{\frac{1}{4}}\langle \xi_2 \rangle^{\frac{1}{4}}\langle \xi_3 \rangle^b} \leq C \prod_{i=1}^3 \|f_i\|_{L^2}^{\frac{1}{4}}, \quad \forall \{f_i\}_{1 \leq i \leq 3}. \quad (5.66) \]
Assume \(b \in (\frac{1}{4}, b_0]\) with \(b_0 = 1\). Similar as before, it reduces to show
\[ \sup_{\xi_3, \tau_3} \frac{\langle \xi_3 \rangle^2}{\langle \xi_3 \rangle^{2(1-b)}} \int \frac{|\xi_1|^2}{\langle \xi_1 \rangle^2\langle \xi_2 \rangle^2\langle \xi_3 \rangle^{2b}} d\xi_2 \leq C, \quad (5.67) \]
where \(P_{\xi_3, \tau_3}(\xi_2) = L_1 + L_2\) is the same as (5.36) but with \(r = \frac{1}{4}\). More precisely,
\[ P_{\xi_3, \tau_3}(\xi_2) = \frac{3}{4}\alpha_1\xi_2^3 + 3\alpha_1\xi_3\xi_2^2 + (3\alpha_1\xi_3^2 + \beta_2 - \beta_1)\xi_2 + \phi^{\alpha_1, \beta_1}(\xi_3) - \tau_3. \quad (5.68) \]
Taking derivative with respect to $\xi_2$, then

$$
P'_{\xi_3, \gamma_3}(\xi_2) = \frac{9}{4} \alpha_1 \xi_2^2 + 6\alpha_1 \xi_3 \xi_2 + 3\alpha_1 \xi_2^3 + \beta_2 - \beta_1.
$$

When $\xi_2 \neq 0$, it can be rewritten as

$$
P'_{\xi_3, \gamma_3}(\xi_2) = 3\alpha_1 \xi_2 p\left(\frac{\xi_3}{\xi_2}\right) + \beta_2 - \beta_1, \tag{5.69}
$$

where the function $p$ is just the function $p_r$, as defined in (5.22), with $r = \frac{1}{4}$. That is

$$
p(x) = x^2 + 2x + \frac{3}{4}
$$

Since the function $p$ has two roots $-\frac{3}{2}$ and $-\frac{1}{2}$, we further decompose the domain $A$ as $A = \bigcup_{i=0}^{3} B_i$ with

$$
B_0 = \{(\xi, \tau) \in A : |\xi| \leq 1\},
$$

$$
B_1 = \{(\xi, \tau) \in A : |\xi| > 1, \frac{\xi_3}{\xi_2} + \frac{3}{2} \geq \frac{1}{10} \text{ and } \frac{|\xi_3|}{|\xi_2|} + \frac{1}{2} \geq \frac{1}{10}\},
$$

$$
B_2 = \{(\xi, \tau) \in A : |\xi| > 1, \frac{\xi_3}{\xi_2} + \frac{3}{2} < \frac{1}{10}\},
$$

$$
B_3 = \{(\xi, \tau) \in A : |\xi| > 1, \frac{\xi_3}{\xi_2} + \frac{1}{2} < \frac{1}{10}\}.
$$

Among these regions $\{B_i\}_{i=0}^{3}$, the most difficult analysis occurs on $B_2$ (or equivalently on $B_3$), so next we will just focus on $B_2$. It is easily seen that $|\xi_1| \sim |\xi_2| \sim |\xi_3|$ on $B_2$, so

$$
\text{LHS of } (5.67) \lesssim \sup_{\xi_3, \gamma_3} \frac{|\xi_1|^{\frac{3}{2}}}{(L_3)^2(1-b)} \int \frac{1}{(P_{\xi_3, \gamma_3}(\xi_2))^{2b}} d\xi_2. \tag{5.70}
$$

By dividing the leading coefficient $\frac{3}{4}\alpha_1$ in (5.68), we get

$$
(P_{\xi_3, \gamma_3}(\xi_2)) \sim (\xi_3^3 + \sigma_2 \xi_2^2 + \sigma_1 \xi_2 + \sigma_0),
$$

where

$$
\sigma_2 = 4\xi_3, \quad \sigma_1 = 4\xi_3^2 + \frac{4(\beta_2 - \beta_1)}{3\alpha_1}, \quad \sigma_0 = \frac{4}{3\alpha_1} \left(\phi^{\alpha_1, \beta_1}(\xi_3) - 1\right).
$$

Then it follows from Lemma 5.4 that

$$
\int \frac{1}{(P_{\xi_3, \gamma_3}(\xi_2))^{2b}} d\xi_2 \lesssim (3\sigma_1 - \sigma_2)^{-\frac{b}{2}} \lesssim (3\sigma_1 - \sigma_2)^{-\frac{b}{2}} = \left(-4\xi_3^2 + \frac{4(\beta_2 - \beta_1)}{\alpha_1}\right)^{-\frac{b}{2}}.
$$

Since $|\xi_3| \sim |\xi_1| \geq 1$, when $|\beta_2 - \beta_1|$ is sufficiently small, $\left(-4\xi_3^2 + \frac{4(\beta_2 - \beta_1)}{\alpha_1}\right) \sim \xi_3^2$. Consequently,

$$
\int \frac{1}{(P_{\xi_3, \gamma_3}(\xi_2))^{2b}} d\xi_2 \lesssim |\xi_3|^{-\frac{b}{2}},
$$

which implies the boundedness of the right hand side of (5.70).
Moreover, since $\xi$ (5.73), we have $\xi$ Taking derivative respect to $Q$, where $Q$

Decompose the region $A$ as $A = \bigcup_{i=0}^{2} B_i$, where

$B_0 = \{(\vec{\xi}, \vec{\tau}) \in A : |\xi_1| \leq 1\}$,
$B_1 = \{(\vec{\xi}, \vec{\tau}) \in A : |\xi_1| > 1, |\xi_3| \geq \frac{1}{2}|\xi_1|\}$,
$B_2 = \{(\vec{\xi}, \vec{\tau}) \in A : |\xi_1| > 1, |\xi_3| < \frac{1}{2}|\xi_1|\}$.

Among the above regions, the most difficult analysis occurs on $B_2$, so next we will just focus on this part.

**Contribution on $B_2$:**

Since $|\xi_3| < \frac{1}{2}|\xi_1|$, then

$$\frac{4}{5} |\xi_2| \leq |\xi_1| \leq \frac{4}{3} |\xi_2|$$  \hspace{1cm} (5.73)

and

$$\int \frac{|\xi_1| |\xi_3| |f_i(\xi, \tau_i)|}{(\xi_1)^{s} (\xi_2)^{s} (L_1)^{b} (L_2)^{b} (L_3)^{1-b}} \lesssim \iint \frac{|\xi_2|^{1-s} |f_2|}{(L_2)^{b}} \left( \iint \frac{|f_1 f_3| d\tau_1 d\xi_1}{(L_1)^{b} (L_3)^{1-b}} \right) d\tau_2 d\xi_2.$$  \hspace{1cm} (5.74)

Thus, similar to the derivation for (5.59) in Section 5.4, it suffices to prove

$$\sup_{\xi_2, \tau_2} \frac{|\xi_2|^{2(1-s)}}{(L_2)^{2b}} \iint \frac{d\xi_1}{(Q_{\xi_2, \tau_2}(\xi_1))^{2(1-b)}} \leq C,$$

where $Q_{\xi_2, \tau_2}(\xi_1) = L_1 + L_3$ is as defined in (5.50). More specifically,

$$Q_{\xi_2, \tau_2}(\xi_1) = 3\alpha_1 \xi_2 \xi_1^2 + 3\alpha_1 \xi_2 \xi_1^2 + \phi^{\alpha_1, \beta_1}(\xi_2) - \tau_2.$$  \hspace{1cm} (5.75)

Taking derivative respect to $\xi_1$,

$$Q'_{\xi_2, \tau_2}(\xi_1) = 3\alpha_1 \xi_2 (2\xi_1 + \xi_2).$$  \hspace{1cm} (5.76)

Due to (5.73), we have

$$|Q'_{\xi_2, \tau_2}(\xi_1)| \gtrsim |\xi_2|^2.$$  \hspace{1cm} (5.77)

Moreover, since $r = 1$, then $\alpha_1 = \alpha_2$ and

$$|\phi^{\alpha_1, \beta_1}(\xi_2) - \tau_2| = |\phi^{\alpha_2, \beta_2}(\xi_2) - \tau_2 + (\beta_2 - \beta_1)\xi_2| = | - L_2 + (\beta_2 - \beta_1)\xi_2|.$$  \hspace{1cm}

When $|\beta_2 - \beta_1|$ is sufficiently small,

$$| - L_2 + (\beta_2 - \beta_1)\xi_2| \leq |L_2| + |\xi_2|^3.$$  \hspace{1cm}
As a result, it follows from (5.75) and (5.73) that
\[
\left| Q_{\xi_2, \tau_2}(\xi_1) \right| \leq C|\xi_2|^3 + |L_2|.
\] (5.78)

Then by (5.78) and (5.77), we obtain
\[
\int \frac{d\xi_1}{(Q_{\xi_2, \tau_2}(\xi_1))^{2(1-b)}} \leq \int \frac{1}{|Q'_{\xi_2, \tau_2}(\xi_1)|} \frac{|Q'_{\xi_2, \tau_2}(\xi_1)|}{|Q_{\xi_2, \tau_2}(\xi_1)|} d\xi_1
\]
\[
\lesssim \frac{1}{|\xi_2|^2} \int_0^C |\xi_2|^3 + |L_2| \frac{dy}{\langle y \rangle^{2(1-b)}}.
\]
Hence,
\[
\text{LHS of (5.74)} \lesssim \sup_{\xi_2, \tau_2} \frac{|\xi_2|^{3(2b-1)} + \langle L_2 \rangle^{2b-1}}{|\xi_2|^{2}}.
\] (5.79)

Since \(|\xi_2| \gtrsim 1\) and (5.72) implies \(6b - 2s - 1 \leq 2\), the boundedness of (5.79) is justified.

6 Sharpness of bilinear estimates

In this section we prove Theorem 3.4 and 3.6 which establish the sharpness of all the bilinear estimates in Theorem 3.3 and 3.5. We first fix some notations. First, we define \(A\) as in (5.9), that is
\[
A = \left\{ (\vec{\xi}, \vec{\tau}) \in \mathbb{R}^6 : \sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \tau_i = 0 \right\}.
\]
Secondly, for any set \(E \in \mathbb{R}^2\), we denote its Lebesgue measure by \(|E|\). The following is a simple result which will be used frequently in this section.

Lemma 6.1. Let \(E_i \subset \mathbb{R}^2 (1 \leq i \leq 3)\) be bounded regions such that \(E_1 + E_2 \subset -E_3\), i.e.,
\[
-(\xi_1 + \xi_2, \tau_1 + \tau_2) \in E_3, \quad \forall (\xi_i, \tau_i) \in E_i, i = 1, 2.
\] (6.1)

Then
\[
\int_A \prod_{i=1}^3 \mathbb{1}_{E_i}(\xi_i, \tau_i) = |E_1||E_2|.
\]

The proof of this lemma follows from (6.1) by rewriting the left hand side of the above equation as
\[
\iint_{E_1} \left( \iint_{E_2} \mathbb{1}_{E_3}(-(\xi_1 + \xi_2), -(\tau_1 + \tau_2)) d\xi_2 d\tau_2 \right) d\xi_1 d\tau_1.
\]

6.1 Proof of Theorem 3.4

Proof of Case (1).

Fix \(\alpha_1, \alpha_2, \beta \in \mathbb{R}\) with \(\alpha_1 \alpha_2 < 0\). Suppose there exist \(s < -\frac{13}{12}\), \(b \in \mathbb{R}\) and \(C = C(\alpha_1, \alpha_2, \beta, s, b)\) such
that the bilinear estimate (3.5) holds. Then it follows from Lemma 5.5 that
\[
\int_A \frac{\xi_3(\xi_3)^{\alpha} \prod_{i=1}^3 f_i(\xi_i, \tau_i)}{(\xi_1)^{\alpha}(\xi_2)^{\beta}(L_1)^b(L_2)^b(L_3)^{1-b}} \leq C \prod_{i=1}^3 \|f_i\|_{L^2_\xi}, \quad \forall \{f_i\}_{1 \leq i \leq 3}, \tag{6.2}
\]
where
\[
L_1 = \tau_1 - \phi^{\alpha_1,\beta}(\xi_1), \quad L_2 = \tau_2 - \phi^{\alpha_2,\beta}(\xi_2), \quad L_3 = \tau_3 - \phi^{\alpha_1,\beta}(\xi_3). \tag{6.3}
\]
Let \(r = \frac{\alpha_2}{\alpha_1} \). Then \(r < 0 \). The resonance function is \(H_2\) as calculated in (5.20) with \(\beta_1 = \beta_2 = \beta\), that is
\[
H_2(\xi_1, \xi_2, \xi_3) = \xi_2 \left[(\alpha_2 - \alpha_1)\xi_2^2 - 3\alpha_1\xi_1\xi_2 - 3\alpha_1\xi_1^2\right] = -3\alpha_1\xi_2 \left(\frac{1-r}{3}\xi_2^2 + \xi_1\xi_2 + \xi_1^2\right). \tag{6.4}
\]
So \(|H_2| \sim |\xi_2|(|\xi_1^2 + \xi_2^2|)\) due to the fact that \(r < 0\).

• **Claim A:** If (6.2) holds, then
\[
b \leq \frac{4 + 2s}{3}. \tag{6.5}
\]
For any large number \(N > 0\), let
\[
B_1 = \{(\xi_1, \tau_1) : N - 1 \leq \xi_1 \leq N, \quad |\tau_1 - \phi^{\alpha_1,\beta}(\xi_1)| \leq 1\},
\[
B_2 = \{(\xi_2, \tau_2) : -N - 2 \leq \xi_2 \leq -N - 1, \quad |\tau_2 - \phi^{\alpha_2,\beta}(\xi_2)| \leq 1\}.
\]
For any \((\xi_1, \tau_1) \in B_1\) and \((\xi_2, \tau_2) \in B_2\), \((\xi_1, \tau_1) = -(\xi_2 + \tau_1 + \tau_2)\) satisfies \(1 \leq \xi_3 \leq 3\). Since \(|\xi_1 - N| \leq 1\) and \(|\xi_2 + N| \leq 2\),
\[
\phi^{\alpha_1,\beta}(\xi_1) + \phi^{\alpha_2,\beta}(\xi_2) = \alpha_1(\xi_1 - N + N)^3 + \alpha_2(\xi_2 + N - N)^3 - \beta(\xi_1 + \xi_2) = (\alpha_1 - \alpha_2)N^3 + O(N^2). \tag{6.6}
\]
Moreover, since \(|\tau_1 - \phi^{\alpha_1,\beta}(\xi_1)| \leq 1\) and \(|\tau_2 - \phi^{\alpha_2,\beta}(\xi_2)| \leq 1\), it follows from (6.6) that
\[
|\tau_3 + (\alpha_1 - \alpha_2)N^3| = O(N^2).
\]
Thus, for a suitably large constant \(C_1\), the set
\[
B_3 := \{(\xi_3, \tau_3) : 1 \leq \xi_3 \leq 3, \quad |\tau_3 + (\alpha_1 - \alpha_2)N^3| \leq C_1N^2\}
\]
satisfies \(B_1 + B_2 \subseteq -B_3\). In addition, \(|B_1| = |B_2| = 2\) and \(|B_3| \sim N^2\). Choosing \(f_i = \mathbb{I}_{B_i} (1 \leq i \leq 3)\) in (6.2) yields
\[
C \prod_{i=1}^3 |B_i|^{\frac{2}{3}} \geq \int_A \frac{\xi_3(\xi_3)^{\alpha} \prod_{i=1}^3 f_i(\xi_i, \tau_i)}{(\xi_1)^{\alpha}(\xi_2)^{\beta}(L_1)^b(L_2)^b(L_3)^{1-b}}. \tag{6.7}
\]
For any \((\xi_i, \tau_i) \in B_i, 1 \leq i \leq 3\), it holds that
\[
|L_1| \leq 1, \quad |L_2| \leq 1, \quad |H_2(\xi_1, \xi_2, \xi_3)| \sim N^3.
\]
So \(|L_3| = |H_2 + L_1 + L_2| \sim N^3\). It then follows from (6.7) and Lemma 6.1 that
\[
N \geq \frac{1}{N^{2s}N^{3(1-b)}} \int_A \prod_{i=1}^3 \mathbb{I}_{B_i}(\xi_i, \tau_i) = \frac{|B_1||B_2|}{N^{2s}N^{3(1-b)}} \sim \frac{1}{N^{2s}N^{3(1-b)}}.
\]
which implies (6.5).

**Claim B:** If (6.2) holds, then

\[ b \geq \frac{1}{4} - \frac{s}{3}. \]  \hspace{1cm} (6.8)

Similarly, for large number \( N \), let

\[
B_1 := \{ (\xi_1, \tau_1) : N - N^{-\frac{4}{3}} \leq \xi_1 \leq N, \quad |\tau_1 - \phi^{\alpha_1, \beta}(\xi_1)| \leq 1 \},
\]

\[
B_3 := \{ (\xi_3, \tau_3) : N - N^{-\frac{4}{3}} \leq \xi_3 \leq N, \quad |\tau_3 - \phi^{\alpha_1, \beta}(\xi_3)| \leq 1 \}.
\]

For any \((\xi_1, \tau_1) \in B_1 \) and \((\xi_3, \tau_3) \in B_3 \), \((\xi_2, \tau_2) = -(\xi_1 + \xi_3, \tau_1 + \tau_3) \) satisfies

\[ -2N \leq \xi_2 \leq -2N + 2N^{-\frac{1}{2}}. \]

As

\[
\phi^{\alpha_1, \beta}(\xi_1) + \phi^{\alpha_1, \beta}(\xi_3) = \alpha_1 \xi_1^3 + \alpha_1 \xi_3^3 - \beta(\xi_1 + \xi_3)
\]

\[
= \alpha_1 (\xi_1 + \xi_3) \left[ \frac{(\xi_1 + \xi_3)^2}{4} + \frac{3(\xi_1 - \xi_3)^2}{4} \right] + \beta \xi_2
\]

\[
= -\frac{\alpha_1 \xi_3^3}{4} + O(1) + \beta \xi_2,
\]

it follows from \(|\tau_1 - \phi^{\alpha_1, \beta}(\xi_1)| \leq 1 \) and \(|\tau_3 - \phi^{\alpha_1, \beta}(\xi_3)| \leq 1 \) that

\[ |\tau_2 - \frac{\alpha_1}{4} \xi_3^2 + \beta \xi_2| = O(1). \]

Thus, for a suitably large constant \( C_2 \), the set

\[ B_2 := \{ (\xi_2, \tau_2) : -2N \leq \xi_2 \leq -2N + 2N^{-\frac{1}{2}}, \quad |\tau_2 - \frac{\alpha_1}{4} \xi_3^2 + \beta \xi_2| \leq C_2 \} \]

satisfies \( B_1 + B_3 \subseteq -B_2 \). Moreover, \(|B_1| \sim |B_3| \sim |B_2| \sim N^{-\frac{1}{2}} \). Choosing \( f_i = \mathbb{1}_{B_i} \) \( (1 \leq i \leq 3) \) in (6.2) yields

\[
C \prod_{i=1}^{3} |B_i|^{\frac{1}{2}} \geq \int_A \frac{\xi_3^{(\xi_3)^s} \prod_{i=1}^{3} \mathbb{1}_{B_i}(\xi_i, \tau_i)}{\langle \xi_1 \rangle^{s} \langle \xi_2 \rangle^{s} \langle L_1 \rangle^b \langle L_2 \rangle^b \langle L_3 \rangle^{1-b}}. \]  \hspace{1cm} (6.9)

As for any \((\xi_i, \tau_i) \in B_i, 1 \leq i \leq 3, \)

\[ |L_1| \leq 1, \quad |L_3| \leq 1, \quad |H_2(\xi_1, \xi_2, \xi_3)| \sim N^3, \]

we have \(|L_2| = |H_2 + L_1 + L_3| \sim N^3 \). It then follows from (6.9) and Lemma 6.1 that

\[ N^{-\frac{3}{2}} \geq \frac{N^{1+s}}{N^{2s} N^{3s}} \int_A \prod_{i=1}^{3} \mathbb{1}_{B_i}(\xi_i, \tau_i) = \frac{N^{1+s} |B_1||B_3|}{N^{2s} N^{3s}} \sim \frac{1}{N^s N^{3s}}, \]

which implies (6.8).

Combining (6.5) and (6.8) together yields \( s \geq \frac{-13}{12} \), which contradicts to the assumption \( s < \frac{-13}{12} \).

**Proof of Case (2).**

If \( \frac{-13}{12} \leq s \leq -1 \), the same arguments as in the proofs of Claim A and Claim B show that (6.5) and (6.8) are necessary conditions on \( b \) if the bilinear estimate (3.5) holds.

**Proof of Case (3).**
Let \(-1 < s < -\frac{3}{4}\). The same argument as in the proof of Claim B shows \(b \geq \frac{1}{4} - \frac{s}{3}\). To obtain the desired upper bound for \(b\), let
\[
B_1 := \{(\xi_1, \tau_1) : N - 1 \leq \xi_1 \leq N, \quad \frac{1}{2}N \leq |\tau_1 - \phi^{\alpha_1, \beta}(\xi_1)| \leq N\},
\]
\[
B_2 := \{(\xi_2, \tau_2) : N - 2 \leq \xi_2 \leq -N - 1, \quad \frac{1}{2}N \leq |\tau_2 - \phi^{\alpha_2, \beta}(\xi_2)| \leq N\}.
\]
Then similar to the procedure in the proof of Claim A in Case (1), there exists a suitably large constant \(C_3\) such that the set
\[
B_3 := \{(\xi_3, \tau_3) : -2N \leq \xi_3 \leq -2N + 2, \quad |\tau_3 + (\alpha_1 + \alpha_2)N^3| \leq C_3N^2\}
\]
has the property \(B_1 + B_2 \subset -B_3\). In addition, \(|B_1| = |B_2| = 2\) and \(|B_3| \sim N^2\). Choosing \(f_i = -1_{B_i}\) \((1 \leq i \leq 3)\) in (6.2) yields
\[
C \prod_{i=1}^{3} |B_i|^{\frac{3}{2}} \geq \int_{A} \left| -\xi_3(\xi_3)^{\alpha_3} \prod_{i=1}^{3} 1_{B_i}(\xi_i, \tau_i) \right| \frac{\xi_3(\xi_3)^{\alpha_3}}{\langle \xi_1 \rangle^{r} \langle \xi_2 \rangle^{s} \langle L_1 \rangle^{b} \langle L_2 \rangle^{b} \langle L_3 \rangle^{1-b}}. \quad (6.10)
\]
For any \((\xi_i, \tau_i) \in B_i, 1 \leq i \leq 3,\)
\[
|L_1| \leq 1, \quad |L_2| \leq 1, \quad |H_2(\xi_1, \xi_2, \xi_3)| \sim N^3, \quad |L_3| \sim N^3.
\]
It then follows from (6.10) and Lemma 6.1 that \(b \leq 1 + \frac{s}{3}\).

### 6.2 Proof of Theorem 3.6

First, we want to point out several cases in Table 9 which have been known or can be proved similarly. When \(r = 1\), the bilinear estimates of Type (D1) and (D2) have been known to fail if \(s < -\frac{3}{4}\), see [29]. When \(r > \frac{1}{4}\) but \(r \neq 1\), the bilinear estimates of Type (D1) and (D2) do not hold for \(s < 0\), see [38]. The situations for Type (ND1) and (ND2) can be treated similarly.

For the rest cases in Table 9, we will only prove the failure of the bilinear estimates in the following five cases since other cases are similar. **Case (1):** Type (D1) with \(r < 0\) and \(s < -\frac{3}{4}\). **Case (2):** Type (D2) with \(0 < r < \frac{1}{4}\) and \(s < -\frac{3}{4}\). **Case (3):** Type (ND1) with \(r < 0\) and \(s < -\frac{3}{4}\). **Case (4):** Type (ND1) with \(r = \frac{1}{4}\) and \(s < -\frac{3}{4}\). **Case (5):** Type (ND1) with \(r = 1\) and \(s < 0\).

Moreover, the general strategy for all the above cases is very similar to that in the proof of Theorem 3.4 as shown above. The key ingredient is to construct suitable \(\{B_i\}_{i=1}^{3}\). So in the following, we will only write out the sets \(\{B_i\}_{i=1}^{3}\) that works for the argument, but omit the detailed computations which can be easily carried out.

**Proof of Case (1).** For any large number \(N > 0\), define
\[
B_1 := \{(\xi_1, \tau_1) : N - 1 \leq \xi_1 \leq N, \quad \frac{1}{2}N \leq |\tau_1 - \phi^{\alpha_1, \beta}(\xi_1)| \leq N\},
\]
\[
B_2 := \{(\xi_2, \tau_2) : -N - 2 \leq \xi_2 \leq -N - 1, \quad \frac{1}{2}N \leq |\tau_2 - \phi^{\alpha_2, \beta}(\xi_2)| \leq N\}.
\]
Then we choose a suitably large constant \(C_1\) such that the set
\[
B_3 := \{(\xi_3, \tau_3) : 1 \leq \xi_3 \leq 3, \quad |\tau_3 - 3\alpha_1N^2\xi_3| \leq C_1N\}
\]
satisfies \(B_1 + B_2 \subset -B_3\).

**Proof of Case (2).** For large number \(N > 0\), define
\[
B_1 := \{(\xi_1, \tau_1) : N - N^{-\frac{1}{2}} \leq \xi_1 \leq N, \quad |\tau_1 - \phi^{\alpha_1, \beta}(\xi_1)| \leq 1\},
\]
\[
B_2 := \{(\xi_2, \tau_2) : r^{-\frac{1}{2}}N - N^{-\frac{1}{2}} \leq \xi_2 \leq r^{-\frac{1}{2}}N, \quad |\tau_2 - \phi^{\alpha_2, \beta}(\xi_2)| \leq 1\}.
\]
Then we choose a suitably large constant \( C \) such that the set
\[
B_3 := \left\{ (\xi_3, \tau_3) : -(1+r^{-\frac{3}{2}})N \leq \xi_3 \leq -(1+r^{-\frac{1}{2}})N + 2N^{-\frac{3}{2}}, |\tau_3 - 2\alpha_1(1+r^{-\frac{1}{2}})N^3 + (\beta - 3\alpha_1N^2)\xi_3| \leq C \right\}
\]
satisfies \( B_1 + B_2 \subseteq -B_3 \).

**Proof of Case (3).** For large number \( N > 0 \), define
\[
B_1 := \left\{ (\xi_1, \tau_1) : N - N^{-\frac{3}{2}} \leq \xi_1 \leq N, \quad \left| \tau_1 - \phi^{\alpha_1,\beta}(\xi_1) \right| \leq N^{\frac{3}{2}} \right\},
\]
\[
B_2 := \left\{ (\xi_2, \tau_2) : -N - N^{-\frac{1}{2}} \leq \xi_2 \leq -N, \quad \left| \tau_2 - \phi^{\alpha_2,\beta}(\xi_2) \right| \leq N^{\frac{1}{2}} \right\}.
\]
Then we choose a suitably large constant \( C \) such that the set
\[
B_3 := \left\{ (\xi_3, \tau_3) : 0 \leq \xi_3 \leq 2N^{-\frac{1}{2}}, \quad |\tau_3 + (\alpha_1 - \alpha_2)N^3| \leq C N^{\frac{3}{2}} \right\}
\]
satisfies \( B_1 + B_2 \subseteq -B_3 \).

**Proof of Case (4).** For large \( N > 0 \), define
\[
B_1 = \left\{ (\xi_1, \tau_1) : N - N^{-\frac{1}{2}} \leq \xi_1 \leq N, \quad \left| \tau_1 - \phi^{\alpha_1,\beta}(\xi_1) \right| \leq 1 \right\},
\]
\[
B_2 = \left\{ (\xi_2, \tau_2) : -2N - N^{-\frac{1}{2}} \leq \xi_2 \leq -2N, \quad \left| \tau_2 - \phi^{\alpha_2,\beta}(\xi_2) \right| \leq 1 \right\}.
\]
Then we choose a suitably large constant \( C \) such that the set
\[
B_3 := \left\{ (\xi_3, \tau_3) : N \leq \xi_3 \leq N + 2N^{-\frac{1}{2}}, \quad |\tau_3 + 2\alpha_1N^3 + (\beta - 3\alpha_1N^2)\xi_3| \leq C \right\}
\]
satisfies \( B_1 + B_2 \subseteq -B_3 \).

**Proof of Case (5).** For large number \( N > 0 \), define
\[
B_1 := \left\{ (\xi_1, \tau_1) : N - N^{-2} \leq \xi_1 \leq N, \quad \left| \tau_1 - \phi^{\alpha,\beta}(\xi_1) \right| \leq 1 \right\},
\]
\[
B_2 := \left\{ (\xi_2, \tau_2) : -N - 2N^{-2} \leq \xi_2 \leq -N, \quad \left| \tau_2 - \phi^{\alpha,\beta}(\xi_2) \right| \leq 1 \right\}.
\]
Then we choose a suitably large constant \( C \) such that the set
\[
B_3 = \left\{ (\xi_3, \tau_3) : N^{-2} \leq \xi_3 \leq 3N^{-2}, \quad |\tau_3 + \beta\xi_3| \leq C \right\}
\]
has the property \( B_1 + B_2 \subseteq -B_3 \).

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