UNIVERSAL PARABOLIC MODULI OVER $\overline{M}_{g,n}$

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ABSTRACT. In this article we will construct a universal moduli space of stable parabolic vector bundles over the moduli space of marked Deligne-Mumford stable curves $\overline{M}_{g,n}$. The objects that appear over the boundary of $\overline{M}_{g,n}$ i.e., over singular curves will remain vector bundles. The total space and the fibers over $\overline{M}_{g,n}$ will have good singularities.

Keywords: Parabolic torsion free sheaves, Gieseker bundle, Deligne-Mumford marked stable curve, universal moduli.

INTRODUCTION

Let $(C, x_1, x_2, \ldots, x^n)$ be a marked semistable curve (1) and $\pi : C \to C'$ be the canonical contraction to the marked stable curve (A) $(C', x_1, x_2, \ldots, x^n)$, where the points $\{x^i : 1 \leq i \leq n\}$ are on the isomorphism locus of $\pi$. Given a vector bundle $E$ on the curve $C$ such that the restriction of $E$ to any rational component($\cong \mathbb{P}^1$) of $C$ is strictly positive (2.2), we define parabolic structures of $E$ at the points $\{x^i : 1 \leq i \leq n\}$ in the sense of Mehta-Seshadri. Following Nagaraj-Seshadri [14], we define such a parabolic bundle $E_*$ to be stable if $\pi_* E_*$ is a $p_2$-stable pure sheaf of dimension 1 (2.5) on the marked stable curve $C'$. We will denote such stable parabolic bundle $E_*$ as pair $(C, E_*)$ (2.3) (2.5) with fixed numerical invariants rank $r$, degree $d$, quasi parabolic structures $r^j$, rational weights $\alpha^j$. Dirk Schütler in [17] has shown that there exist a universal moduli space $\overline{U}_{g,n,r}$ of parabolic pure sheaves over $\overline{M}_{g,n}$. The fiber of the moduli space $\overline{U}_{g,n,r}$ over $[C'] \in \overline{M}_{g,n}$ is the moduli space of $p_2$-stable (1.0.5) parabolic pure sheaves on $C'$ modulo $\text{Aut}(C', x_1, \ldots, x^n)$. Schütler follows the approach of Pandharipande [15]. The novel features in the Gieseker type construction are:

(1) There exist a proper map from the moduli space $\overline{U}_{g,n,r}$ to the moduli space $\overline{U}_{g,n,r}$.

(2) The objects in the degenerate fibers remain locally free.

(3) The resulting moduli space $\overline{U}_{g,n,r}$ has nicer singularities (0.5) (0.6) than $\overline{U}_{g,n,r}$.

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In fact under conditions of semistable=stable (4.4)[2nd paragraph], our moduli stack gives a birational smooth model of the one obtained in [17].

The motivation to study the problem is as follows. Let $G$ be an almost simple, simply connected algebraic group over $\mathbb{C}$ and $\rho : G \to \text{GL}(k, \mathbb{C})$ be a faithful representation of $G$. Let $C$ be a smooth algebraic curve of genus $g \geq 2$ over $\mathbb{C}$. Then the upper half plane $\mathbb{H}$ is the universal cover and $\mathbb{H}/\pi = C$, where $\pi$ is certain Fuchsian group. The $(\pi, G)$ bundles on $\mathbb{H}$ corresponds to parahoric torsors on $C$ [3]. The construction of a parabolic Gieseker moduli space over $\overline{M}_{g,n}$ has an important application. Let $\mathcal{G}$ be a parahoric group scheme over the smooth projective curve $C$ with generic fibre $G$, where the parahoric structure is given at a set of $n$ marked points. Then to construct degenerations of the moduli space $M(\mathcal{G})$ either when the smooth curve degenerates to an irreducible nodal curve or to construct a universal moduli for $\mathcal{G}$-torsors over $\overline{M}_{g,n}$, one can follow the classical construction due to Ramanathan. This can be achieved by viewing the parahoric torsors as “reduction of structure groups” of the parabolic Gieseker bundles. Then the stack of $\mathcal{G}$-torsors gets constructed from the stack of parabolic Gieseker bundles over $\overline{M}_{g,n}$ [16], [2].

There is no torsion free analogue for the degeneration of $G$ bundles when a smooth curve degenerates to a nodal curve except for symplectic and orthogonal case due to Faltings [7]. The only known way to study the degeneration problem of $G$ bundles is by using Gieseker type degeneration [2]. The following is a brief history of Gieseker type degeneration.

The first degeneration of this nature was constructed by Gieseker [9] for the rank 2 and odd degree to prove the Newstead-Ramanan conjecture [9, Theorem 1.1] by degeneration to an irreducible nodal curve. Later Nagaraj-Seshadri defined an equivalent notion of stability more in the spirit of Mumford stability to generalize the construction to arbitrary rank $r$, degree $d$ with $\gcd (r, d) = 1$ [14]; using this notion of stability, Schmitt constructed a universal moduli space of vector bundles over $\overline{M}_g$ [18] when the genus $g$ is greater than 1.

To state the main results of the article we will need the following definitions and notations.

**Definition 0.1.** (1) A marked prestable curve $C$ is a reduced, connected, projective curve whose singularities are nodal singularities with $n$ distinct non-singular points on it.

(2) A marked prestable curve $C$ of genus $g \geq 2$ is called marked semistable if every non singular rational components ($\cong \mathbb{P}^1$) contains at least 2 special points i.e., either marked points or singular points.

**Definition 0.2.** Let $C$ and $D$ be two marked semistable curves and $E_\ast, F_\ast$ be two strictly positive $p_2$-stable parabolic vector bundle on $C$ and $D$ respectively. Then the pair $(C, E_\ast), (D, F_\ast)$ is Aut-equivalent if there exist a marked isomorphism $\phi : C \to D$ such that we have an isomorphism $E_\ast \cong \phi^*F_\ast$ of parabolic bundles [17, definition 4.2.4].
Notation 0.3. Let $V_l$ be a vector space over complex numbers $\mathbb{C}$ and $r^1_{l_i+1} > r^2_{l_i} > \cdots > r^n_{l_i}$ be a decreasing sequence of positive integers for $1 \leq i \leq n$. Let $\text{Flag}(V_l, r^1_{l_i+1}, r^2_{l_i}, \ldots, r^n_{l_i})$ be the flag variety of successive quotient spaces of $V_l$ of the form $V_l \to Q^1_{l_i+1} \to Q^2_l \to \cdots \to Q^n_{l_i}$, where $\dim(Q^j_l) = r^j_l$ for $2 \leq j \leq (l_i + 1)$. Then we denote by $F_l = \prod_{i=1}^n \text{Flag}(V_l, r^1_{l_i+1}, r^2_{l_i}, \ldots, r^n_{l_i})$.

The main result of this article is the following

Theorem 0.4. There exist a projective variety $\overline{M}_{g,n,r}$ over $\mathcal{M}_{g,n}$ such that the fiber over a marked stable curve $C'$ parameterizes aut-equivalence classes of pairs $(C, E_\ast)$ where $C$ is a marked semistable curve whose fixed marked stable model is $C'$ and $E_\ast$ is a stable parabolic Gieseker bundle of fixed numerical type on $C$. Furthermore we have the following commutative diagram

\begin{equation}
\begin{tikzcd}
\overline{M}_{g,n,r} \arrow{r}{\text{projective}} \arrow[bend left=5]{d}[swap]{\kappa_g} & \overline{M}_{g,n,\ast} \arrow{d}{\eta} \\
\overline{M}_{g,n} \arrow{ru}{\pi_\ast}
\end{tikzcd}
\end{equation}

(0.0.1)

The following two theorems describes certain geometric properties of the total space and the fibers of the morphism $\kappa_g$ (0.0.1).

Theorem 0.5. The universal moduli $\overline{M}_{g,n,r}$ is a normal projective variety with finite quotient singularity. The dimension of the moduli space is $3g - 3 + n + r^2(g - 1) + 1 + \dim(F_l(V_l, \mathbb{C}))$.

The variety $\overline{M}_{g,n,r}$ has a distinguished smooth open subvariety consisting of strictly stable bundles.

Theorem 0.6. Let the marked stable curve $C'$ represent an element in $\overline{M}_{g,n}$, such that $\text{Aut}(C', x^1, \ldots, x^n)$ is trivial then $\kappa_g^{-1}([C'])$ has a singularity which is a product of analytic normal crossings.

We will briefly sketch the main steps of the construction. As a first step we will prove that the above mentioned objects of our moduli problem i.e., pairs $(C, E_\ast)$, $C$ is a marked semistable curve and $E_\ast$ is strictly positive $p_2$-stable bundle, form a bounded family. We will rigidify the moduli problem to show that the underlying vector bundles are in an open subvariety $Y_{g,n}$ of a relative Hilbert scheme $\text{Hilb}^{p_2}(X \times \text{Gr}(V_l, r))$. The rigidification will induce a natural action of $\text{SL}(N) \times \text{SL}(V_l)$ on $Y_{g,n}$. We will define a functor which is represented by a generalized flag variety $\mathfrak{sl}_l$ over $Y_{g,n}$. The action of $\text{SL}(N) \times \text{SL}(V_l)$ will lift to an action on $\mathfrak{sl}_l$. We will show that there is a natural morphism from $\mathfrak{sl}_l$ to the flag variety $F_l$ of the torsion free moduli problem. We will show that this morphism is proper. The morphism is defined by the pushforward $\pi_\ast$. The candidate for our universal moduli space is the GIT quotient $\mathfrak{sl}_l / \text{SL}(N) \times \text{SL}(V_l)$. One key element of a moduli space construction using GIT is the choice of a suitable group $G$ (here $G = \text{SL}(N) \times \text{SL}(V_l)$)-linearized ample line bundle on the appropriate Quot scheme (here $\mathfrak{sl}_l$). Here we use in an essential way the ample linearization of the GIT construction in [17]. Indeed, the role that Schützer’s paper plays on our paper is the exact counterpart of the role...
that the moduli space of torsion free sheaves plays on Nagaraj-Seshadri’s paper [14].

The layout of the paper is as follows. In the section I (1) we will give the necessary definitions and review the universal parabolic torsion free moduli construction [17]. In section II (2) we will define the notion of stability of parabolic Gieseker bundle on a marked semistable curve. We will prove that these objects of our moduli problem form a bounded family and we define the parabolic Gieseker functor (2.4) over the Gieseker functor (C.1). In section III (3) we will show that the parabolic Gieseker functor is represented by a flag scheme $\mathfrak{F}_l$ and we will establish a morphism $\eta: \mathfrak{F}_l \rightarrow \mathfrak{F}_l$, where $\mathfrak{F}_l$ is the flag variety (3.1) for the torsion free moduli problem. In section IV (4) we will prove the properness of the morphism $\eta$ and give the proof of the main theorem (0.4). In section V (5) we will prove (0.5) and (0.6).

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1. Preliminaries

In this section first we will give the necessary definitions. Then we will mention key features of [17]. This is essential in our construction since the crucial polarization for our GIT construction will come from the moduli space constructed in [17].

Throughout the section a curve will mean a marked curve with $n$ distinct nonsingular points on it. The standard notation is $C$ unless the markings need specifying. The arithmetic genus of the curve is $g$.

**Definition 1.1.** A pure sheaf $E$ of dimension 1 on a marked prestable curve $C$ is a coherent $\mathcal{O}_C$ module such that every non zero subsheaf $F$ of $E$ has $\dim(\text{supp}(F)) = 1$.

Let $\mathcal{O}_C(1)$ be a polarization on $C$ and $a_i = \deg(\mathcal{O}_C(1)|_{C_i})$, where $C_i$ be the irreducible components of $C$. Without loss of generality we also assume $\sum a_i = 1$. Let $r_i = \text{rank}(E|_{C_i})$. We define total rank of $E$, $\text{totr}(E) := \sum(r_i \cdot a_i)$. The degree of $E$ is defined as, $\deg(E) := \chi(E) - \text{totr}(E)(1 - g)$, where the Euler characteristic $\chi(E) = \dim H^0(E) - \dim H^1(E)$. The Hilbert polynomial with respect to the polarization $\mathcal{O}_C(1)$ denoted by $H$ is $H(t) := \chi(E(t)) = \chi(E) + \text{totr}(E) \cdot t$ by Riemann-Roch theorem.

We now fix the $n$ marked points $\{x^1, x^2, \cdots, x^n\}$ in $C$.

**Definition 1.2.** A quasi parabolic structure (QPS) on $E$ at the points $x^i$ is defined as the following filtration of sheaves

$E = F_1^1E \supseteq F_2^1E \supseteq F_3^1E \supseteq \cdots \supseteq F_{i+1}^1E = E(-x^i)$ (1.0.1)

A parabolic structure is a QPS with added weights

$0 \leq \alpha_1^i < \alpha_2^i < \cdots < \alpha_{i+1}^i < 1$ (1.0.2)
Notation 1.3. We will denote the Hilbert polynomial of $E/F_1F$ by $H^i$ with $\{1 \leq i \leq n\}$ and $\{2 \leq j \leq (l_i + 1)\}$. It is easy to see that $H^i$ are constant integers. Alternatively we will use the notation $r^i_j$ for them.

The above definition (1.0.1) is due to Maruyama-Yokogawa [11]. This is known to be equivalent to the classical notion of parabolic structure in [12] of the following form

$$E_{i+1} = E/E(-x^i) \rightarrow E/F_1E \rightarrow \cdots \rightarrow E/F_nE$$

(1.0.3)

Notation 1.4. Throughout this article a parabolic sheaf will be denoted by $E_\ast$.

The parabolic degree is defined as:

$$\text{par deg}(E_\ast) := \text{deg}(E) + \sum^n_i \sum_j \alpha^i_j \cdot \dim(F_j^iE/F_{j+1}^iE)$$

$$= \text{deg}(E) + \sum_i \sum_i \alpha^i_j \cdot (r^i_j - 1), \ r^i_1 = 0$$

A parabolic sheaf $E_\ast$ is defined to be slope stable (resp. slope semistable) if for any non zero proper saturated subsheaf $F$ of $E$ with the induced parabolic structure on $F$ we have the inequality:

$$\frac{\text{par deg}(F_\ast)}{\sum a_i \cdot s_i} < (\leq) \frac{\text{par deg}(E_\ast)}{\sum a_i \cdot r_i}$$

(1.0.4)

where $s_i$ are the multirank of $F$.

Example 1.5. (A sheaf with different multirank) Let $C$ be a reducible curve with two smooth irreducible components $C_1$ and $C_2$ meeting at a point $p = C_1 \cap C_2$.

Let $E_1, E_2$ be vector bundles of rank $r_1, r_2$ on the curves $C_1, C_2$ respectively. Let $\phi : (E_1)_p \rightarrow (E_2)_p$. We consider the map $\psi : E_1 \oplus E_2 \rightarrow (E_2)_p$ defined by $\psi(s_1, s_2) = \phi(s_1(P)) - s_2(P)$, where $s_1(P), s_2(P)$ are the restriction of the sections $s_1, s_2$ on the fiber $(E_1)_p, (E_2)_p$ respectively. Let $\ker(\psi)$ be $E$. Then $E$ is a pure sheaf on the curve $C$. If we choose $r_1 \neq r_2$ then the multiranks of $E$ are different.

Definition 1.6. Following D. Schütte [17, definition 4.3.11] we call a parabolic sheaf $E_\ast$ to be $p_2$-stable (resp. $p_2$- semistable) if for any non zero proper saturated subsheaf $F$ of $E$ with the induced parabolic structure on $F$, multirank $s_i$, and Hilbert polynomial $H(F/F_j^iF) = s^i_j$, we have the inequality:

$$H(E, y) \cdot \left( H(F, x) - \frac{1}{n} \left( \sum_{i=1}^{n} l_{i+1} \sum_{j=1} c^i_{j} \cdot s^i_{j} \right) \right) < (\leq) H(E, y) \cdot \left( H(F, x) - \frac{1}{n} \left( \sum_{i=1}^{n} l_{i+1} \sum_{j=2}^{c^i_{j} \cdot r^i_{j}} \right) \right)$$

(1.0.5)

where $c^i_{j} := \alpha^i_j - \alpha^i_{j-1}, 2 \leq j \leq (l_i + 1)$ and $\alpha^i_{l_i+1} = 1$. The above inequality of polynomials in two variables $x, y$ is according to the lexicographic ordering in $\mathbb{Q}[x, y]$ (1.9).

Remark 1.7. The notion of $p_2$- stability that has been used to construct the universal moduli space in [17] is a modification of the notion of slope stability (1.0.4).
the definition [17, definition 4.3.11] the Hilbert polynomials \( H_1(E, x) \) and \( H_1(E, y) \) will become constants for curves. The constant is \( \dim(E/F_2E) = r_j \). Also for a sub-sheaf \( F \) of \( E \), the Hilbert polynomial \( p(F/F_2F, y) \) is the constant \( s_j \). Therefore putting all these quantities together the definition [17, definition 4.3.11] will take the form of \((1.0.5)\).

**Remark 1.8.** In the above definition of \( p_2 \)-stability, the product of polynomials that appears on both side of \((1.0.5)\) are of the form \((a_1y + b_1)(a_2x + b_2)\). In the next paragraph we will define lexicographic ordering for this set of polynomials.

**Definition 1.9.** Given two polynomials \( f = a_1xy + b_1y + c_1x + d_1 \) and \( g = a_2xy + b_2y + c_2x + d_2 \) in \( \mathbb{Q}[x, y] \), we will define \( f \leq g \) if \( a_1 < a_2 \) or \( a_1 = a_2 \) and \( b_1 < b_2 \) or \( a_1 = a_2, b_1 = b_2 \) and \( c_1 < c_2 \) or \( a_1 = a_2, b_1 = b_2, c_1 = c_2 \) and \( d_1 \leq d_2 \).

For few more details see Appendix B.

### 1.1. Universal moduli of parabolic sheaves

We will briefly give an outline of the universal moduli of semi stable parabolic pure sheaves as in [17].

Objects of the moduli space are \( p_2 \)-stable(semistable) \((1.0.5)\) pure sheaves \( E \) of uniform rank \( r \), i.e., \( r_i = r \) for all \( i \), on marked stable curves \( C \) \((A.1)\). We will fix once and for all rank \( r \), \( \text{deg} \ d \), quasi parabolic structures \( s_j \) and weights \( \alpha_j \in \mathbb{Q} \). This data is called numerical type of this moduli problem.

By (semi)stable locus, (semi)stable points of a variety \( X \) under the action of a reductive group \( G \) we mean it in the sense of GIT.

**Notation 1.10.** Let \( \mu : X \to \mathbb{S}_{g,n} \) \((A.1.6)\) be the local universal family of marked stable curves with natural sections \( \sigma_i : \mathbb{S}_{g,n} \to X, 1 \leq i \leq n \) corresponding to the marked points. Let \( D^i \) be their associated divisors. The base \( \mathbb{S}_{g,n} \) is the Quot scheme for the moduli problem of \( \mathbb{M}_{g,n} \) \((A.1.6)\). Let \( \mathcal{O}_X(1) \) be the relative ample line bundle \( \left( \omega_{X/\mathbb{S}_{g,n}}(\sum \sigma_i) \right)^{\otimes \mu} \).

**Remark 1.11.** In several places for notational reasons we will denote \( \mathbb{S}_{g,n} \) by \( \mathbb{S} \).

By the boundedness of the family of \( p_2 \)-stable sheaves on marked stable curves of fixed numerical type \((B.4)\) there exist \( l_0 \) such that \( \forall \ l \geq l_0 \) and \( \forall \ s \in \mathbb{S}_{g,n} \) we have:

1. \( H^i(X_s, \mathcal{E}(l)) = 0 \)
2. \( \text{H}^0(X_s, \mathcal{E}(l)) \otimes \mathcal{O}_{X_s} \to \mathcal{E}(l) \) is surjective.

where \( \mathcal{E} \) is a \( p_2 \)-stable sheaf on \( X_s \).

Since degree of \( \mathcal{E}(l) \) is independent of \( s \in \mathbb{S}_{g,n} \), from the cohomology vanishing \( H^i(X_s, \mathcal{E}(l)) = 0 \) and the Riemann-Roch theorem, it follows \( \text{dim} \ H^0(X_s, \mathcal{E}(l)) \) is independent of \( s \in \mathbb{S}_{g,n} \). We will rigidify the moduli problem by fixing an isomorphism \( V_l \cong H^0(\mathcal{E}(l)) \). We consider the relative Quot scheme over \( \mathbb{S}_{g,n} \):

\[
\text{Quot}_{X/\mathbb{S}_{g,n}}^{\mathcal{O}_X(1)}(V_l \otimes \mathcal{O}_{X_s}, \mathcal{H}) \tag{1.1.1}
\]

where \( \mathcal{H} \) is the Hilbert polynomial \( \mathcal{H}(\mathcal{E}(l), t) = \chi(\mathcal{E}(l)) + r \cdot t = d + r \cdot l + r \cdot (1 - g) + r \cdot t \). We will denote the Quot scheme \((1.1.1)\) by \( \text{Q}_q(\mu, V_l, \mathcal{H}) \) where \( \mu : X \to \mathbb{S}_{g,n} \) is the structure map. The points of the Quot scheme \( \text{Q}_q(\mu, V_l, \mathcal{H}) \) are coherent \( \mathcal{O}_X \) modules \( \mathcal{F} \) which are flat over \( \mathbb{S}_{g,n} \) with a surjection \( V_l \otimes \mathcal{O}_X \to \mathcal{F} \to 0 \) and for \( s \in \mathbb{S}_{g,n} \) the
fiber $\mathcal{F}_s$ over $X_s$ has Hilbert polynomial $H$ with respect to the ample line bundle $\mathcal{O}_{X_s}(1)$.

**Remark 1.12.** Note that from the above (2), the $p_2$-stable sheaves $\mathcal{E}(l)$ are points of $Q_g(\mu, V_l, H)$.

The rigidification will induce an action of $\text{SL}(V_l)$ on $Q_g(\mu, V_l, H)$. There is a natural group action of $\text{SL}(N) \times \text{SL}(V_l)$ on $Q_g(\mu, V_l, H)$ where $\text{SL}(\mathbb{N})$ action comes from it’s action on $\mathcal{X}$ (A). The universal quotient $\mathcal{U}$ fits in the universal short exact sequence

$$0 \to \mathcal{K} \to V_l \otimes \mathcal{O}_{X \times \mathbb{G}_m} \to \mathcal{U} \to 0$$ (1.1.2)

where the kernel $\mathcal{K}$ is called the universal subsheaf.

Let $F_l$ be the flag variety over $Q_g(\mu, V_l, H)$ which parameterizes parabolic sheaves $\mathcal{E}_*$ of fixed numerical type $(\mathbb{H}, r^*_j, \alpha^*_j)$ such that $\mathcal{E} \in Q_g(\mu, V_l, H)$. More precisely for $s \in S_{r, n}$ a closed point of the fiber $(F_l)_s$ is given by a quotient $V_l \otimes \mathcal{O}_{X_s} \to \mathcal{E}$ in the fiber $(Q_g(\mu, V_l, H))_s$, along with a sequence of quotients $\mathcal{E} \to Q^1_{l+1} \to Q^1_l \to \cdots \to Q^1_2$, where $Q^1_i$ are skyscraper sheaves supported at the parabolic point $x^l_i$ for $i = 1, 2, \ldots, n$ and $\dim(Q^1_j) = r^*_j$ for $j = 2, 3, \ldots, l_{i+1}$. This has been constructed inductively by constructing successive Quot schemes using the universal quotient sheaf $\mathcal{U}$ in [17, 4.6]. The flag variety $F_l$ represents the following functor:

$$F_l : \text{Sch}/S_{r, n} \to \text{Sets}$$ (1.1.3)

for a $S_{r, n}$ scheme $T$, we define $F_l(T)$ to be the set

$\{\text{parabolic filtrations of } T\text{-flat sheaves } \mathcal{E}_T \text{ of the form (B.0.7)} \text{ which has a quotient representation } V_l \otimes \mathcal{O}_{X_T} \to \mathcal{E}_T \text{ such that for all } t \in T, \mathcal{E}_t \text{ has Hilbert polynomial } \mathbb{H} \text{ and } (\mathcal{E}/F^t_j \mathcal{E})_t \text{ has dimension } r^*_j \}$

The natural action of $\text{SL}(N) \times \text{SL}(V_l)$ on $Q_g(\mu, V_l, H)$ will lift to an action on $F_l$. The GIT quotient $F_l // \text{SL}(V_l) \times \text{SL}(N)$ is the candidate for the universal moduli space. The moduli functor associates to every scheme $T$ the set

$\{\text{equivalence classes of flat family of } p_2\text{-stable}^1 \text{ parabolic sheaves } \mathcal{E}_* \text{ (B.5)} \text{ on the family of marked stable curves } \mu_T : C_T \to T \text{ such that } \mathcal{E}_* \text{ has uniform rank } r, \text{ degree } d \text{ and } (\mathcal{E}/F^t_j \mathcal{E})_t \text{ has dimension } r^*_j \}$

Two flat families of $p_2$-stable sheaves $\mathcal{E}_*$ and $\mathcal{E}'_*$ on $C_T/T$ and $C'_T/T$ are equivalent if there exist a $T$ isomorphism $\psi : C_T \to C'_T$ and a line bundle $L$ on $T$ such that we have a parabolic isomorphism $\mathcal{E}_* \cong \psi^* \mathcal{E}'_* \otimes \mu^*_T L$. Let

$$Q_g^r(\mu, V_l, H) \to Q_g(\mu, V_l, H)$$ (1.1.4)

be the closed subscheme of uniform rank $r$ [15, Lemma 8.1.1]. The uniform rank Quot scheme $Q_g^r(\mu, V_l, H)$ is invariant under the action of $\text{SL}(N) \times \text{SL}(V_l)$. By an abuse of notation $F_l$ continues to denote the base change of $F_l$ under the morphism (1.1.4).

The flag variety $F_l$ can be embedded as a closed subscheme inside the product of relative Grassmannians over $S_{r, n}$ [17, 4.41]

$$F_l \hookrightarrow \text{Gr} \times_S (\text{Gr}^1_{l+1} \times_S \cdots \times_S \text{Gr}^1_2) \times_S \cdots \times_S (\text{Gr}^n_{l+1} \times_S \cdots \times_S \text{Gr}^n_2)$$ (1.1.5)

\(^1\text{The moduli space in [17] has been constructed for } p_2\text{-semistable sheaves. A notion of } S\text{-equivalence has been defined for } p_2\text{ semistable sheaves.}\)
where \( \text{Gr} := \text{Gr}_S(V_l \otimes \mu_* \mathcal{O}_X(k), H(k)) \) and \( \text{Gr}^i_j := \text{Gr}_S(V_l \otimes \mu_* \mathcal{O}_X(k), r^i_j(k)) \).

Here the pushforward of the relative ample line bundle \( \mathcal{O}_X(k) \) has been defined with respect to the structure map \( \mu : \mathcal{X} \rightarrow S_{g,a} \). The embedding is \( \text{SL}(V_l) \) equivariant.

We will get the following \( \mathbb{Q} \)-divisor on the product of Grassmannians on RHS of (1.1.5)

\[
L_{\beta, \beta_j^i} := \mathcal{O}_{Gr}(\beta) \otimes_{\mathbb{Q}} \mathcal{O}_{Gr}(\beta_j^i)
\]

where \( \beta, \beta_j^i \) are rational numbers. For \( a \gg 0 \), \( L_{\beta, \beta_j^i} \mid_{F_l} \) is a relative very ample linearization for the \( \text{SL}(V_l) \) action on \( F_l \).

**Remark 1.13.** It is enough to study the GIT problem of \( \text{SL}(V_l) \) action on a fiber \((F_l)_s \) for \( s \in S_{g,a} \) by using [20, Lemma 1.13].

**Notation 1.14.** A point in the image of the map in (1.1.5) over a point \( s \in S_{g,a} \) will correspond to a parabolic sheaf \( \mathcal{E}_s \) together with a morphism \( \gamma : V_l \rightarrow H^0(\mathcal{E}(l)) \) which is induced by the quotient map \( V_l \otimes \mathcal{O}_X \rightarrow \mathcal{E}(l) \).

For a suitable choice of linearization weights \( \beta, \beta_j^i \) [17, 4.49] we will get the following lemma

**Lemma 1.15.** There exist \( l_0 \in \mathbb{N} \) such that for all \( l \geq l_0 \) there exist \( K(l) \) with the property that for all \( k \geq K(l) \), a point \((\mathcal{E}_s, \gamma)\) in the flag variety \((F_l)_s \) is \( \text{SL}(V_l) \) stable with respect to the linearization \( L_{\beta, \beta_j^i} \mid_{F_l} \) if and only if \( \mathcal{E}_s \) is a \( p_2 \)-stable sheaf of uniform rank \( r \) and \( \gamma : V_l \rightarrow H^0(\mathcal{E}(l)) \) is an isomorphism.

**Proof.** For a proof see [17, Theorem 4.8.1]. \( \square \)

**Remark 1.16.** Notice that the linearization \( L_{\beta, \beta_j^i} \mid_{F_l} \) depends on the integer \( k \) since it is induced from the embedding (1.1.5) which depends on \( k \).

We will denote the product on the R.H.S (1.1.5) by \( \text{Gr}_S(l,k) \). Let \( \text{Gr}(l,k) \) be the fiber of \( \text{Gr}_S(l,k) \) at a closed point \( s \in S_{g,a} \). Then

\[
\text{Gr}(l,k) = \text{Gr}(V_l \otimes H^0(\mathcal{O}_X(k), H(k)) \mid \mathbb{C} \prod_{i=1}^a \prod_{j=2}^{1+m} \text{Gr}(V_l \otimes H^0(\mathcal{O}_X(k)), r^i_j(k)).
\]

The flag variety \( F_l \) can be embedded as a locally closed subscheme [17, p. 139]

\[
F_l \hookrightarrow S_{g,a} \times \mathbb{C} \text{Gr}(l,k) \hookrightarrow S_{g,a} \times \mathbb{C} \text{Gr}(l,k)
\]

There exist a natural action of \( \text{SL}(N) \) on \( S_{g,a} \) and a linearization \( L_{a,b} \) for the \( \text{SL}(N) \) action on \( S_{g,a} \) with nice properties [17, p. 47], [5, pp. 23-24], (A.5). We will write \( \mathcal{M} = L_{a,b} \). We have mentioned the linearization \( L_{\beta, \beta_j^i} \) for the \( \text{SL}(V_l) \) action on \( \text{Gr}(l,k) \). The embedding (1.1.7) is an \( \text{SL}(N) \times \text{SL}(V_l) \)-equivariant embedding. The following \( \mathbb{Q} \)-divisor

\[
L_{\beta, \beta_j^i} \otimes \mathcal{M} \otimes b
\]

on \( S_{g,a} \times \mathbb{C} \text{Gr}(l,k) \) gives a linearization for the action of \( \text{SL}(N) \times \text{SL}(V_l) \) on \( \text{Gr}(l,k) \) and therefore on \( F_l \) by pullback. We will denote this linearization by \( L \).

The semistable locus of \( F_l \) is closed in the semistable locus of \( S_{g,a} \times \mathbb{C} \text{Gr}(l,k) \) for the \( \text{SL}(N) \times \text{SL}(V_l) \) action with respect the linearization \( L \) for \( a \gg 0 \) [15, 8.2, 30],

\[
L \]
Therefore the GIT quotient $F \sslash \SL(V_l)$ exists as a projective variety which is denoted by $\mathcal{M}_{g,n}$. The following proposition together with (1.15) gives the moduli theoretic interpretation of $\SL(N) \times \SL(V_l)$ stable points of $F_l$ with respect to the linearization $L$ for $b \gg 0$.

**Proposition 1.17.** A point of $F_l$ is stable for the $\SL(V_l)$ action with respect to the linearization $L_{\beta,\beta}^{\otimes b} |_{F_l}$ if and only if the point is GIT stable for the $\SL(N) \times \SL(V_l)$ action under the linearization $L|_{F_l}$.

**Proof.** For a proof we refer to [17, Proposition 5.1.2]. \hfill \Box

2. **Moduli Problem**

The aim of this section is to state the moduli problem and then define a representable functor which is represented by a flag variety that parameterizes all parabolic structures of fixed numerical type on semi-stable curves. This functor is built together with a morphism to the Gieseker functor (C.1).

2.1. **Parabolic Gieseker bundles.**

**Definition 2.1.** Let $C$ be a marked semistable curve of genus $g \geq 2$ (1) such that $\pi : C \to C'$ is the collapsing morphism and $C'$ is the fixed marked stable model. Then $C$ is called a marked Gieseker curve if $\pi_\ast \omega_{C'} \cong \omega_{C}$, where $\omega_{C'}$ and $\omega_{C}$ are the dualizing sheaves of $C'$ and $C$ respectively.

Let $z_1, z_2, \ldots, z_c$ are the nodes of $C'$ such that $\pi^{-1}(z^j) = R^j$ is a chain of projective lines i.e., $R^j = \cup R^j_\iota$ such that $R^j_\iota \cong \mathbb{P}^1$, $R^j_\iota \cap R^j_{\iota'}$ is a singleton set if $|\iota - \iota'| = 1$, otherwise empty and $R^j$ meets other components of $C$ at exactly two points $p^j_1, p^j_2$.

We fix the notation $E$ to denote a vector bundle on $C$, where $C$ will denote a marked Gieseker curve. The rank and degree of $E$ are denoted by $r$ and $d$ respectively. Then $E|_{R^j_\iota} \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a^j_{\iota i})$.

**Definition 2.2.** The bundle $E$ is called positive if $a^j_{\iota i} \geq 0$ for all $j, \iota, i$. The bundle $E$ is called strictly positive if $E$ is positive and for all $j, \iota$ there exist $i$ such that $a^j_{\iota i} > 0$. The bundle $E$ is strictly standard if it is strictly positive and $0 \leq a^j_{\iota i} \leq 1$.

**Definition 2.3.** A vector bundle $E$ on $C$ is called a Gieseker vector bundle if $E$ is strictly positive and $\pi_\ast E$ is a pure sheaf of dimension 1 on $C'$. 
Let \( \{x^1, x^2, \ldots, x^n\} \) are the marked points on \( C \). The marked points \( \{x^1, x^2, \ldots, x^n\} \) are not on the chain of projective lines \( R^j \) which are contracted to get a canonical marked stable curve.

A parabolic structure on \( E \) at the points \( x^i \) is a quasi parabolic structure (QPS) which is a decreasing filtration of sheaves

\[
E = F_1^i E \supseteq F_2^i E \supseteq F_3^i E \supseteq \cdots \supseteq F_{l_i+1}^i E = E(-x^i)
\]

(2.1.1)

together with an increasing sequence of weights

\[
0 \leq \alpha_1^i < \alpha_2^i < \cdots < \alpha_{l_i}^i < 1
\]

(2.1.2)

The map \( \pi \) gives a canonical isomorphism \( C - \cup R^j \cong C' - \{z^j : 1 \leq j \leq c\} \). Therefore it is natural to use the same notation \( x^i \) for the points \( \pi(x^i) \) on \( C' \). We will fix the line bundle \( (\omega_C(\sum x^i)^{\otimes \rho}) \) on \( C \) which is isomorphic with the pullback \( \pi^* (\omega_{C'}(\sum x^i))^{\otimes \rho} \). We will denote the line bundle \( (\omega_C(\sum x^i))^{\otimes \rho} \) by \( \mathcal{O}_C(1) \).

Once and for all we will fix numerical type (a set of invariants) for our moduli problem — rank(\( E \)) = \( r \), degree of \( E \) = \( d \), Hilbert polynomials of \( E/F_j^i E \) with respect to \( \mathcal{O}_C(1) \) is \( r_j^i \) which is the constant \( \dim_C E/F_j^i E \) and weights \( \alpha_j^i \in \mathbb{Q} \).

The parabolic Gieseker bundle will be denoted by \( (E_\ast) \). Since \( \pi_\ast \) is a left exact functor we have a filtration of sheaves on \( C' \):

\[
\pi_\ast E = \pi_\ast F_1^i E \supseteq \pi_\ast F_2^i E \supseteq \cdots \supseteq \pi_\ast F_{l_i+1}^i E = \pi_\ast (E(-x^i)) \cong (\pi_\ast E)(-x^i)
\]

(2.1.3)

**Lemma 2.4.** The parabolic sheaves \( E_\ast \) on \( C \) and \((\pi_\ast E)_\ast \) on \( C' \) have the same numerical type \((r, d, r_j^i, \alpha_j^i)\).

**Proof.** It is clear that ranks will remain same under pushforward \( \pi_\ast \). Since genus(\( C \)) = genus(\( C' \)) and \( H^i(C, E) \cong H^i(C', \pi_\ast E) \) for \( i \geq 0 \) [14, Proposition 3], by using Riemann-Roch theorem we will get \( \deg(E) = \deg(\pi_\ast E) = d \).

Let \( U^i \) be a neighbourhood at \( x^i \) such that restriction of \( \pi \) induces an isomorphism \( \pi : V^i = \pi^{-1}(U^i) \cong U^i \). Since both \( E/F_j^i E \) and \( \pi_\ast E/\pi_\ast F_j^i E \) are supported at points \( x^i \) we will get

\[
E/F_j^i E \cong E|_{V^i}/F_j^i E|_{V^i} \cong \pi_\ast (E|_{V^i})/\pi_\ast (F_j^i E|_{V^i}) \cong (\pi_\ast E)|_{U^i}/(\pi_\ast F_j^i E)|_{U^i}
\]

(2.1.4)

The last isomorphism in (2.1.4) is due to [10, Proposition 9.3] and we have \((\pi_\ast E)|_{U^i}/(\pi_\ast F_j^i E)|_{U^i} \) is isomorphic with \( \pi_\ast E/\pi_\ast F_j^i E \). Therefore the QPS \( r_j^i \) remains same under \( \pi_\ast \). Since we will assign the same weights \( \alpha_j^i \) for the filtration (2.1.3) the numerical type will remain the same. \( \square \)

**Definition 2.5.** The parabolic Gieseker bundle \( E_\ast \) is called stable if \( \pi_\ast E \) with respect to the parabolic structure (2.1.3) is \( p_2 \)-stable (1.0.5).

**Remark 2.6.** Similarly if we define semistability of \( E_\ast \) to be the \( p_2 \)-semistability of \( \pi_\ast E \), then it will not be a GIT semistability notion (4.4). One needs to impose additional condition along with \( \pi_\ast E \) to be \( p_2 \)-semistable. A. Schmitt in [18, Definition 2.2.10] has worked out a notion of semistability for Gieseker vector bundles. But in this article we will restrict our attention to stable parabolic Gieseker vector bundles.
Let $\mathcal{S}$ be the family

\[ \mathcal{S} = \{ \text{"Aut- equivalence" classes of pairs } (C, E) \text{ where } C \text{ is a marked Gieseker curve and } E \text{ is a stable parabolic Gieseker bundle on } C \text{ of fixed numerical type : } \]

\[ C \to C' \text{ is the contraction morphism to the marked stable curve } [C'] \in \overline{M}_{g,n}, \]

where $[C']$ denotes the isomorphism class of the curve $C'$.

Two pairs $(C, E)$ and $(D, F)$ are called Aut- equivalent if there exist a marked isomorphism $\psi : C \cong D$ such that we have a parabolic isomorphism $E \cong \psi^* F$.

Our aim is to give a scheme structure on the set $\mathcal{S}$.

### 2.2. Boundedness

In this subsection we will prove that the objects in the set $\mathcal{S}$ form a bounded family. In particular we will have the following result

**Lemma 2.7.** There exist $l_0 \gg 0$ such that for all $l \geq l_0$ we have the cohomology vanishing $H^1(C, E(l)) = 0$ and the natural map $H^0(C, E(l)) \otimes \mathcal{O}_C \to E(l)$ is surjective for all $(C, E) \in \mathcal{S}$.

Furthermore the natural morphism $C \to Gr(H^0(E(l)), r)$ is a closed embedding.

**Proof.** We will denote the partial normalization $\tilde{C} = C \cup \cup_{R_j}$ of $C'$ by $\tilde{C}$ and $\nu : \tilde{C} \to C'$ is the normalization morphism. Then the inclusion $i : \tilde{C} \hookrightarrow C$ is a closed immersion.

\[ \begin{align*}
\tilde{C} & \xrightarrow{\nu} C \\
\downarrow & \\
C' & \xrightarrow{i} C \\
\downarrow & \\
\pi & \\
\end{align*} \] (2.2.1)

The family of sheaves $\{ \pi_* E : E \in \mathcal{S} \}$ is contained inside the family

\[ \{ p_2 \text{ semistable sheaves } E_s \text{ on } X_s \text{ with fixed numerical type } (r, d, r_j^i, \alpha_j^i) : s \in S_{e_n} \} \]

Hence by (B.4) there exist $l_0 \gg 0$ such that for all $l \geq l_0$ we will have the following

1. The cohomology vanishing $H^1(C', \pi_* E(l)) = 0$.
2. The natural map $H^0(C', \pi_* E(l)) \otimes \mathcal{O}_{C'} \to \pi_* E(l)$ is surjective i.e., $\pi_* E(l)$ is globally generated.

for all $(C, E) \in \mathcal{S}$

Since $H^i(C, E(l)) \cong H^i(C', \pi_* E(l))$, we have $H^1(C, E(l)) = 0$ for $l \geq l_0$.

Our aim is to show that $\{ E(l)|_{\tilde{C}} : E \in \mathcal{S} \}$ is a bounded family. Then the lemma will follow from (C.1).

Since the diagram (2.2.1) commutes, we have $\nu = \pi \circ i$ and so $\nu^* \pi_* E(l) = i^* \pi_* E(l)$. From the adjoint property we get a natural map $\pi^* \pi_* E \to E$. Applying $i^*$ will give us the following short exact sequence

\[ 0 \to \nu^* \pi_* E(l)/\operatorname{Tor}(\nu^* \pi_* E(l)) \to E(l)|_{\tilde{C}} \to \mathcal{Q} \to 0 \] (2.2.2)
since $\nu$ is an isomorphism except for finitely many points $Q$ is a torsion sheaf. We will denote $\nu^* \pi_* E(l) / \text{Tor}(\nu^* \pi_* E(l))$ by $\tilde{E}$. So it is enough to show $\tilde{E}$ is globally generated and $H^1(\tilde{C}', \tilde{E}) = 0$.

By definition of $\tilde{E}$ we have the quotient map $\nu^* \pi_* E(l) \to \tilde{E}$. Applying $\nu_*$ and using the natural map $\pi_* E(l) \to \nu_* \nu^* \pi_* E(l)$, we will get the following short exact sequence on $C'$

$$0 \to \pi_* E(l) \to \nu_* \tilde{E} \to Q' \to 0$$

(2.2.3)

where $Q'$ is a torsion sheaf. Since $\pi_* E(l)$ is globally generated, by the five lemma $\nu_* \tilde{E}$ is globally generated. We also get $H^1(C', \nu_* \tilde{E}) = 0$ and $H^0(C', \tilde{E}) \cong H^0(C', \nu_* \tilde{E})$. The sheaf $\tilde{E}$ is locally free at $p^i_1, p^i_2$. We can check the equality $(\nu_* \tilde{E})_{2i} = (\tilde{E})_{p^i_1} \oplus (\tilde{E})_{p^i_2}$. Hence the global generation of $\nu_* \tilde{E}$ will imply that $\tilde{E}$ is globally generated. □

2.3. Relative divisors on families of semi-stable curves. We fix a natural number $l \geq l_0$ such that proposition (2.7) holds. It follows from proposition (2.7) that $\dim H^1(C, E(l))$ is independent of the pair $(C, E) \in \mathcal{S}$ since $r, d$ are fixed. We will fix a vector space $V_i$ over $C$ such that $\dim_C H^0(C, E(l)) = \dim_{\nu} V_i$. We will rigidify the moduli problem by adding a new data, namely an isomorphism $H^0(C, E(l)) \cong V_i$. Let $\Gr(V_i, r)$ be the Grassmannian of $r$ dimensional quotients of the $C$ vector space $V_i$. Let $\mathcal{G} = \mathcal{G}(r, d)$ be the functor (C.1). Since the map $i : C \hookrightarrow \Gr(V_i, r)$ is a closed embedding (2.7), it follows that the map $\pi \times i : C \hookrightarrow C' \times \Gr(V_i, r)$ is also a closed embedding. The vector bundle $E(l)$ on $C$ is the pullback of the tautological quotient bundle on $\Gr(V_i, r)$. This shows the functor (C.1) is natural to consider in this context. The family $\mathcal{X}/\mathcal{S}_{x, s}$ has natural sections $\sigma_i : \mathcal{S}_{x, s} \to \mathcal{X}$ which will give us divisors on the semi-stable curves $C$.

Let $D^i$ be the associated divisors corresponding to sections $\sigma_i$, $1 \leq i \leq n$. So $D^i \to \mathcal{S}_{x, s}$ is flat. Let $T$ be a $\mathcal{S}_{x, s}$ scheme. By the base change $T \to \mathcal{S}_{x, s}$ we get the flat morphism $D^i \times_T T \to T$. We will denote $D^i \times_T T$ by $D^i_T$. So $D^i_T$ are relative divisors$^2$ of $\mathcal{X} \times_T T$.

Let $\Delta \in \mathcal{G}(T)$. We consider the induced morphism $\pi_T : \Delta \to \mathcal{X} \times_T T$ which is the collapsing morphism of the family of curves $\{\Delta_t : t \in T\}$. The components of each fiber, where the restriction of the relative dualizing sheaf $\omega_{\Delta_t/T}$ is trivial, are getting contracted. So the morphism $\pi_T$ is a birational morphism.

$$\begin{array}{ccc}
D^i_T \times (\mathcal{X} \times_T T) & \xrightarrow{\text{closed}} & \Delta \\
\downarrow & \text{closed} & \downarrow \\
D^i_T & \xrightarrow{\text{closed}} & \mathcal{X} \times_T T \\
\end{array}$$

(2.3.1)

Since for $t \in T$, the divisor $D^i_T$ is supported on the nonsingular locus of the curves $\mathcal{X}_t$, we see that $D^i_T$ sits inside the isomorphism locus of the birational map

---

$^2$The word “relative” in relative divisors means the map $D^i_T \to T$ is flat
π_\text{T}. Thus the morphism \( D_\text{T}^i \times_{(X \times S T)} \Delta \cong D_\text{T}^i \) is an isomorphism.

We will use the same notation \( D_\text{T}^i \) to denote the divisor \( D_\text{T}^i \times_{(X \times S T)} \Delta \) on \( \Delta \). The use of this notation should be clear from the context.

**Remark 2.8.** The family \((\Delta, D_\text{T}^i)\) over \( T \) is a family of marked Gieseker curves \((2.1)\) (which are marked semi stable curves) with respect the morphism \(\pi_\text{T} \). For any closed point \( x \in X \times S T \), \( \pi_\text{T}^{-1}(x) \) is either a singleton set or a connected chain of projective lines \( R_i \). This map has the the following property:

\[
(\pi_\text{T})_* \mathcal{O}_\Delta \cong \mathcal{O}_{X \times S T} \quad (2.3.2)
\]

For a proof of this we refer to [1, Proposition 6.7].

2.4. **Parabolic Gieseker functor.** For any closed subscheme \( \Delta \in \mathcal{G}(T) \) \((C.1)\), let \( E \) be be the pullback of the tautological quotient bundle on \( \text{Gr}(V_\ell, r) \) to the closed subscheme \( \Delta \). We define the following functor \( \mathcal{G}(r, d, r_j^i) \)

\[
\mathcal{G}(r, d, r_j^i) : \text{Sch}/S_{g,n} \to \text{Sets} \quad (2.4.1)
\]

where a \( S_{g,n} \)-scheme \( T \) is sent to a filtration of the bundle \( E \) of the form

\[
E = F_1^i E \supset F_2^i E \supset F_3^i E \supset \cdots \supset F_{l+1}^i E = E(-D_\text{T}^i) \quad (2.4.2)
\]

such that the filtration has the following properties:

1. The quotients \( E/F_j^i E \) are flat over \( T \) \( \forall \ i, j \).

2. For all \( t \in T \) the restriction of the filtration \((2.4.2)\) induces a filtration of the bundle \( E_t \) on \( \Delta_t \) \((1.0.1)\) with respect to the divisors \( D_t^i \).

3. The quotients \((E/F_j^i E)_t\) are supported on \( D_t^i \) and are of dimension \( r_j^i \).

There is a forgetful morphism of functors \( F : \mathcal{G}(r, d, r_j^i) \to \mathcal{G} \) that sends the parabolic vector bundles to the underlying vector bundles.

3. **Some technical lemmas**

In this section we will construct a flag variety \( \mathfrak{F}_t \) which represents the functor \((2.4)\). Then we will establish a relationship between \( \mathfrak{F}_t \) and the flag variety \( \mathfrak{F}_t \) for the torsion free parabolic moduli which represents the functor \((1.1.3)\).

We see that for all pair \((C, E_s) \in \mathfrak{G}_t \), the embedding of the curve \( C \) in \( X \times \text{Gr}(V_\ell, r) \) has same Hilbert polynomial. We will denote the polynomial by \( p(t) \). Let \( \text{Hilb}^{p(t)}(X \times \text{Gr}(V_\ell, r)) \) be the relative Hilbert scheme over \( S_{g,n} \). Let \( T \) be a scheme over \( S_{g,n} \) and \( \Delta \in \mathcal{G}(T) \) \((C.1)\). Then by definition \((C.1)\) we have the induced morphism \( \pi_\text{T} : \Delta \to X \times S T \). Then the second condition of the definition is equivalent to \( \Delta_t \) being a prestable curve of genus \( g \) and \( \omega_\Delta, \cong \pi_t^* \omega_{X_t} \), where \( t \) maps to \( s \) and \( \pi_t : \Delta_t \to X_t \) is the restriction of the morphism \( \pi_\text{T} \). Both of these two conditions are open conditions i.e., \( \{ t \in T : \Delta_t \text{ is a prestable curve of genus } g \text{ and } \omega_\Delta, \cong \pi_t^* \omega_{X_t} \} \) is an open subset of \( T \) \([9, p. 179] \).
Let $0 \to K \to V_l \otimes O_\Delta \to E \to 0$ be the pullback of the tautological short exact sequence on the Grassmannian $Gr(V_l, r)$. Then the third condition in (C.1) is equivalent to the conditions $\dim H^0(K_l) \leq 0$, $\dim H^0(E_l) \leq \dim(V_l)$ and $\dim H^1(E_l) \leq 0$. By the upper semicontinuity of cohomology, these three cohomological conditions are open conditions. Therefore there exist an open subvariety $Y_{g,n}$ of $\text{Hilb}^{p(t)}(\mathcal{X} \times Gr(V_l, r))$ over the base $\mathcal{S}_{g,n}$ which represents the Gieseker functor (C.1).

3.1. Flag variety. Let the closed subscheme $\Delta \hookrightarrow Y_{g,n} \times_S (\mathcal{X} \times Gr(V_l, r))$ be the universal curve. Let $V_l \otimes O_\Delta \to Q$ be the universal quotient on $\Delta$. By the construction (2.3) there exist relative divisors $D^i$ on the universal curve $\Delta$ over $Y_{g,n}$

$$
\begin{array}{c}
D^i \\
\downarrow \text{flat} \\
\Delta \\
\downarrow \leftarrow \leftarrow \leftarrow Y_{g,n}
\end{array}
$$

(3.1.1)

Let $Q_{|D^i}$ be the restriction of the rank $r$ universal bundle $Q$ on $D^i$. We will consider the relative flag variety over $Y_{g,n}$ of locally free quotients of the vector bundle $Q_{|D^i}$ on $D^i$ of rank in decreasing order $(r_{l,1}, r_{l,2}, \cdots, r_{l,2})$. We will denote this relative flag variety by $\mathfrak{F}_l$. Let $\Delta^i$ be the fiber product $\Delta \times_Y \mathfrak{F}_l$. Then there exist a universal family which is a filtration of sheaves on $\Delta^i$ of the form

$$
P^i_1 Q = F^i_1 Q \supset F^i_2 Q \supset \cdots \supset F^i_{l+1} Q = P^i_1 Q(-D^i)$$

(3.1.2)

where $P_1 : \Delta \to \Delta$ is the projection map.

We have a natural closed embedding of $\mathfrak{F}_l$ inside a product of Grassmannians

$$
\mathfrak{F}_l \hookrightarrow Y_{g,n} \times (\text{Gr}(V_l, r_{l,1+1}) \times \cdots \times \text{Gr}(V_l, r_{l,2}))
$$

(3.1.3)

Let $\mathfrak{F}_l$ be the fiber product $\mathfrak{F}_l \times_Y \mathfrak{F}_l \times_Y \cdots \times_Y \mathfrak{F}_l$. There will be $n$ universal filtrations on $\Delta \times_Y \mathfrak{F}_l$ which are the pullbacks of the filtrations (3.1.2) on $\Delta^i$ corresponding to the $n$ divisors $D^i$. The pullbacks are taken for the projections $\Delta \times_Y \mathfrak{F}_l \to \Delta^i$ and will remain filtrations since the relative flag varieties $\mathfrak{F}_l$ are flat over $Y_{g,n}$. The flag variety $\mathfrak{F}_l$ along with these $n$ universal filtrations will represent the functor (2.4).

We have the following closed embedding of the relative flag variety $\mathfrak{F}_l$

$$
\mathfrak{F}_l \hookrightarrow Y_{g,n} \times (\text{Gr}(V_l, r_{l,1+1}) \times \cdots \times \text{Gr}(V_l, r_{l,2})) \cdots \times (\text{Gr}(V_l, r_{l,n+1}) \times \cdots \times \text{Gr}(V_l, r_{l,2}))
$$

(3.1.4)

3.2. Group action. The group $\text{SL}(N)$ and $\text{SL}(V_l)$ acts on $\mathcal{X}$ and $\text{Gr}(V_l, r)$ respectively. Thus the product $\text{SL}(N) \times \text{SL}(V_l)$ induces a natural action on $\mathcal{X} \times \text{Gr}(V_l, r)$ and therefore on $\text{Hilb}^{p(t)}(\mathcal{X} \times \text{Gr}(V_l, r))$. Note that the open subvariety $Y_{g,n}$ is invariant under the action of $\text{SL}(N) \times \text{SL}(V_l)$. Thus $\text{SL}(N) \times \text{SL}(V_l)$ will induce a natural action on the flag variety $\mathfrak{F}_l$. In particular, let $g = (g_1, g_2)$ be an element in $\text{SL}(N) \times \text{SL}(V_l)$ and $([C], E_s)$ be in $\mathfrak{F}_l$ where $[C] \in Y_{g,n}$. Let $(g \cdot [C]) = [D]$ i.e., $D$ is the image of the curve $C$ under the automorphism $g : \mathcal{X} \times \text{Gr}(V_l, r) \to \mathcal{X} \times \text{Gr}(V_l, r)$. Then the action of $g$ on $\mathfrak{F}_l$ is $g \cdot ([C], E_s) = ([D], g_1 E_s)$. The closed embedding (3.1.4) is $\text{SL}(N) \times \text{SL}(V_l)$ equivariant.
3.3. Relation between flag varieties corresponding to the parabolic pure sheaves and parabolic Gieseker bundles. The universal curve $\Delta \rightarrow Y_{g,n} \times S$ $(X \times \text{Gr}(V_i, r))$ induces the proper birational morphism (canonical contraction) $\pi : \Delta \rightarrow Y_{g,n} \times S X$ which has the property $\pi_* \mathcal{O}_\Delta \cong \mathcal{O}_{X \times S Y}(2.8)$. We have the universal quotient $V_i \otimes \mathcal{O}_\Delta \rightarrow Q$ flat over $Y_{g,n}$. Thus applying $\pi_*$ we will get the morphism of sheaves on $Y_{g,n} \times S X$
\[ V_i \otimes \mathcal{O}_{X \times S Y} \rightarrow \pi_* Q \] (3.3.1)

The above morphism can be shown to be surjective using the isomorphism $(\pi_* Q)_t \cong (\pi_t)_* Q_t$ for $t$ in $Y_{g,n}$ due to (C.2). Also we have $\pi_* Q$ is flat over $Y_{g,n}$. Therefore by the definition of the Quot scheme functor the quotient (3.3.1) will induce a morphism $\theta : Y_{g,n} \rightarrow Q^r_{g}(\mu, V_i, H)$.

We will now state and prove a result which is crucial to prove the properness of certain morphism (4.3).

**Proposition 3.1.** The following is a fiber product diagram

\[ \tilde{\mathcal{G}}_I \quad \xymatrix{ & Y_{g,n} \ar[d]^\eta \ar[dl]_\theta \ar@{=}[d] \ar[r] & Q^r_{g}(\mu, V_i, H) \ar[d]^\theta \ar[l]_\eta \ar[dl]_\phi } \]

(3.3.2)

The natural morphism $\eta : \tilde{\mathcal{G}}_I \rightarrow F_I$ is $SL(N) \times SL(V_i)$ equivariant.

**Proof.** Let $Q$ be the usual Quot scheme functor which is represented by $Q^r_{g}(\mu, V_i, H)$ (1.1.4) and recall that $F_I$ be the functor (1.1.3) defining the flag variety $F_I$. Thus to prove $\tilde{\mathcal{G}}_I \cong Y_{g,n} \times Q^r_{g}(\mu, V_i, H) F_I$, it is enough to prove the following isomorphism between functors:

\[ \mathcal{G}(r, d, r^i_j) \xymatrix{ \ar[r]^{T_1} & \mathcal{G} \times Q F_I \ar[l]_{T_2} } \]

such that $T_1 \circ T_2 = id$ and $T_2 \circ T_1 = id$.

Given a $S$ scheme $T$, let $(\Delta, E_\ast)$ be an element in $\mathcal{G}(r, d, r^i_j)(T)$ with $\Delta \in \mathcal{G}(T)$. We define the morphism $T_1$ as

\[ T_1(\Delta, E_\ast)) = (\Delta, (\pi_T)_* E_\ast) \]

(3.3.4)

We need to prove that $(\pi_T)_* E_\ast \in F_I(T)$ to make the definition well defined. For that we have to check $(\pi_T)_* E_\ast$ satisfies the conditions in the definition (B.5).

1. We will show that the quotients $(\pi_T)_*(E/\pi_T)_j E_f^j E$ are flat over $T$. By definition the quotients $E/F^j_i E$ are flat over $T$ and is supported on $D^j_i$. Since $\pi_T : \Delta \rightarrow X_T$ restricted to $D^j_i$ is an isomorphism, $(\pi_T)_*(E/F^j_i E)$ is flat over $T$. Therefore it is enough to show that $(\pi_T)_* (E/F^j_i E) \cong (\pi_T)_* (E/\pi_T)_j E_f^j E$.

From the filtration of $E^*_\ast$ we get the short exact sequence

\[ 0 \rightarrow F^j_i E \rightarrow E \rightarrow E/F^j_i E \rightarrow 0 \]

(3.3.5)

Applying the pushforward $(\pi_T)_*$ we get

\[ 0 \rightarrow (\pi_T)_* F^j_i E \rightarrow (\pi_T)_* E \rightarrow (\pi_T)_* (E/F^j_i E) \rightarrow R^1(\pi_T)_* F^j_i E \]

(3.3.6)
We want to show \( R^1(\pi_T)_* F^i_1 E = 0 \). The coherent sheaf \( R^1(\pi_t)_* (F^i_1 E)_t \) is the sheaf associated to the graded module \( \oplus_{n \geq 0} H^1(\Delta_t, (F^i_1 E)_t(n)) \) where tensor product is taken with respect the line bundle \( \mathcal{O}_{\Delta_t}(1) = \pi^* \mathcal{O}_{\Delta}(1) \). Since by definition \( E_* \) induces a filtration \((E_t)_* \) when restricted to \( \Delta_t \), we get the short exact sequence

\[
0 \to (F^i_1 E)_t \to E_t \to Q \to 0 \tag{3.3.10}
\]

where the quotients of the filtration \((E_t)_* \) is supported at the divisors \( x^i \) and we have \( Q \subseteq E_t|_{x^i} \). Since \( E_t \) is globally generated and \( H^1(E_t) = 0 \), the cohomology long exact sequence corresponding to the short exact sequence (3.3.7) will give \( H^1(\Delta_t, (F^i_1 E)_t) = 0 \). Using similar arguments we will have \( H^1(\Delta_t, (F^i_1 E)_t(n)) = 0 \) for all \( n \geq 1 \). Therefore the sheaf \( R^1(\pi_t)_* (F^i_1 E)_t = 0 \) for all \( t \in T \). By the lemma (C.2) we have the isomorphism \((R^1(\pi_T)_* F^i_1 E)_t \cong R^1(\pi_t)_* (F^i_1 E)_t = 0 \). Since each fiber over \( t \in T \) vanishes, the sheaf \( R^1(\pi_T)_* F^i_1 E \) is the 0-sheaf.

**II.** Corresponding to the following base change diagram

\[
\begin{array}{ccc}
\Delta_t & \to & \Delta \\
\downarrow \pi_t & & \downarrow \pi_T \\
\tilde{X}_t & \to & \tilde{X}_T
\end{array}
\]  

we will have the commutative diagram:

\[
\begin{array}{cccc}
(\pi_t)_* E_t & \leftarrow & (\pi_t)_* (F^i_1 E)_t & \leftarrow \cdots & \leftarrow & (\pi_t)_* E_t \leftarrow (\pi_t)_* E_t \leftarrow (\pi_t)_* E_t \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
(\pi_* E)_t & \leftarrow & (\pi_* F^i_1 E)_t & \leftarrow \cdots & \leftarrow & (\pi_* F^i_1 E)_t & \leftarrow (\pi_* E)_t \leftarrow (\pi_* E)_t \\
\end{array}
\]

(3.3.9)

By definition of \( E_* \) the upper row is a filtration. By our argument in the above paragraph the coherent sheaf \( R^1(\pi_t)_* (F^i_1 E)_t = 0 \). Therefore by lemma (C.2) the vertical arrows are isomorphisms. By the commutativity of the diagram the lower row is also a filtration.

Now we are going to define the morphism \( T_2 (3.3.3) \). Let \((\Delta, E_*) \in \mathcal{G}_Q F_1(T) \). Let \( E \) be the natural vector bundle on \( \Delta \) such that \((\pi_T)_* E = E \) which also implies \((\pi_T)_* E(-D^i_T) = E(-D^i_T) \). The filtration of sheaves \( E_* \) is equivalent to the sequence of quotients:

\[
E/E(-D^i_T) = E|_{D^i_T} \to E/(F^i_1 E) \to \cdots \to E/(F^i_2 E) \tag{3.3.10}
\]

Since \( R^1(\pi_T)_* E(-D^i_T) = 0 \), from the short exact sequence

\[
0 \to E(-D^i_T) \to E \to E|_{D^i_T} \to 0 \tag{3.3.11}
\]

it follows \((\pi_T)_* E|_{D^i_T} = E|_{D^i_T} \). Since \( \pi_T \) induces canonical isomorphism on \( D^i_T \), we will construct the following sequence of quotients on \( \Delta \) from (3.3.10)

\[
E|_{D^i_T} = (\pi_T^{-1})_* E|_{D^i_T} \to (\pi_T^{-1})_* (E/F^i_1 E) \to \cdots \to (\pi_T^{-1})_* (E/F^i_2 E) \tag{3.3.12}
\]

which is equivalent to a filtration of sheaves \( E_* \). We define

\[
T_2(\Delta, E_*) = (E_*) \tag{3.3.13}
\]
(I) It is clear from construction of the filtration $E_*$ that quotients are flat sheaves over $T$ since they are isomorphic with $E/F_i E$.

(II) To prove that the restriction of $E_*$ on $\Delta_r$ induces filtration, we can use similar arguments that has been used to prove property (II) for the morphism $T_1$.

From the definition of $T_1$ and $T_2$ it is clear $T_1 \circ T_2 = \text{id}$ and $T_2 \circ T_1 = \text{id}$. □

4. MODULI CONSTRUCTION

4.1. Properness. The image of the morphism $\theta$ lands inside the open subscheme $R$ of $Q^r_\mu(V, H)$ consisting of points $q \in Q^r_\mu(V, H)$ where $q$ maps to $s \in S_g$, such that the quotient map $V \otimes O_{X_s} \to U_q$ induces an isomorphism $V \cong H^0(X_s, U_q)$. We denote by $R_f$ the open subscheme of $R$ such that the sheaf $U_q$ is torsion free on $X_s$. Let $Y_f := \theta^{-1}(R_f)$.

Let $C$ be a smooth curve and $\zeta : C \to S_{g,n}$ be a morphism. We denote the base change $X \times S$ by $X'_C$. Further we assume that $\mu^{-1}(p) = X_p$ for $p \in C$ is the only singular fiber of the family $\mu : X'_C \to C$.

Lemma 4.1. [4, Proposition 4.2] Let $E_C$ be a flat family of torsion free sheaves on $X'_C$. Then there exist a family of marked Gieseker curves $X'_C$ with the canonical contraction $\pi : X'_C \to X_C$ such that $E_C = (\pi^* E_C / \text{Tor})$ obtained by going modulo torsion is a family of Gieseker vector bundles and furthermore we will also have an isomorphism $\pi_* (E_C) \cong E_C$.

Proof. All of the isolated singular points of the surface $X'_C$ will lie inside the only singular fiber of the family $\mu : X'_C \to C$.

Proposition 4.2. The morphism $\theta : Y_f \to R_f$ is proper.

Proof. Similar properness results has been proved in [14, Proposition 10], [18, Theorem 2.1.2]. Since the map $\theta$ is quasi projective in a relative set up (over $S_{g,n}$), this allows us to use a particular form of valuative criterion, called horizontal properness [4, Definition 4.4].

Since the morphism $\theta$ is quasi projective, there exist a projective morphism $\bar{\theta} : Z \to R_f$ with the following diagram

$$
\begin{array}{ccc}
Y_f & \xrightarrow{open} & Z \\
\downarrow \theta & & \downarrow \bar{\theta} \\
R_f & \xrightarrow{\text{projective}} & Z
\end{array}
$$
Let $S^0$ be the open subvariety of $S_{g,n}$ representing the locus of marked nonsingular curves. Let $\pi : D \to D'$ be a morphism of prestable curves with equal genus such that $\pi^*\omega_{C'} \cong \omega_C$ and $D'$ be a nonsingular curve. This will imply that $\pi$ is an isomorphism. Therefore by the definition of the functor $G$ (2.4.1), $\theta|_{S_{g,n}} : Y^f|_{S_{g,n}} \cong R^f|_{S_{g,n}}$ is an isomorphism. Thus in particular it is a proper morphism. We have $R^f|_{S_{g,n}}$ is an open subvariety of $R$. We will take the closure $\overline{Y^f|_{S_{g,n}}}$ inside $Z$. Without loss of generality we can assume $Z = \overline{Y^f|_{S_{g,n}}}$. Thus to prove properness of $\theta$ it is enough to show $Y^f = \overline{Y^f|_{S_{g,n}}}$ (4.1.2).

Let $x$ be a point in $\overline{Y^f|_{S_{g,n}}} \setminus Y^f|_{S_{g,n}}$, then we can assume there exist a smooth curve $C$ with a morphism $\kappa : C \to \overline{Y^f|_{S_{g,n}}}$ such that $\kappa(C \setminus p) \subseteq Y^f|_{S_{g,n}}$ and $\kappa(p) = x$ for some point $p \in C$. This will induce the map $\tilde{\theta} \circ \kappa : C \to R^f$. We will show that there exist a morphism $\kappa' : C \to Y^f$ with the following commutative diagram

$$
\begin{array}{ccc}
C \setminus p & \xrightarrow{\kappa} & Y^f \\
\downarrow & & \downarrow \theta \\
C & \xrightarrow{\tilde{\theta} \circ \kappa} & R^f \\
\end{array}
$$

The map $\tilde{\theta} \circ \kappa$ will be induced by the flat family of torsion free sheaves $E_C$ on family of marked stable curves $X_C (= X \times_{S} C)$ over $C$ along with a quotient representation $V_l \otimes O_{X_C} \to E_C$. Hence by (4.1) we will get the following quotient which is flat over $C$

$$
V_l \otimes O_{X_C} \to E_C
$$

where $X'_C$ is a family of marked Gieseker curves and $E_C$ is a family of Gieseker bundles over $C$.

The locally free quotient in (4.1.4) will define a morphism to the Grassmannian

$$
X'_C \to C \times Gr(V_l, r)
$$

Let $\pi : X'_C \to X_C$ is the contraction morphism. We also have $\pi_*E_C \cong E_C$ (4.1). Hence the family of bundles $E_C$ on $X'_C$ satisfies the conditions of (2.7). Therefore the morphism (4.1.5) is a closed embedding. It is clear that the family $X'_C$ satisfies the conditions of the Gieseker functor (2.4.1). Thus the family will define a morphism $\kappa' : C \to Y^f|_{g,n}$ which will agree with the morphism $\kappa$ on $C \setminus p$. Since $\overline{Y^f|_{S_{g,n}}}$ is a separated scheme we will have $\kappa' = \kappa$. Therefore we have $Y^f|_{g,n} = \overline{Y^f|_{S_{g,n}}}$.

Let $F_l^f$ is the base change $F_l \times_{Q_g(\mu, V_l, H)} R^f$ and $\tilde{F}_l^f$ is the base change $\tilde{F}_l \times_{Y_{g,n}} Y^f|_{g,n}$. By Proposition (3.1) we have $\tilde{F}_l \cong F_l \times_{Q_g(\mu, V_l, H)} Y_{g,n}$. Therefore we have $\tilde{F}_l^f = \tilde{F}_l \times_{Y_{g,n}} Y^f|_{g,n} \cong \left( F_l \times_{Q_g(\mu, V_l, H)} Y_{g,n} \right) \times_{Y_{g,n}} Y^f|_{g,n} \cong F_l \times_{Q_g(\mu, V_l, H)} Y^f|_{g,n} \cong \tilde{F}_l^f$. 

□
\[ F'_l \times_{I^l} Y^f_{g,n} \text{ i.e., we have a modified cartesian product diagram} \]

\[
\begin{array}{ccc}
\mathfrak{N}_l^f & \longrightarrow & Y^f_{g,n} \\
\downarrow \eta & & \downarrow \theta \\
F'_l & \longrightarrow & R^f
\end{array}
\quad (4.1.6)
\]

**Corollary 4.3.** The morphism \( \eta : \mathfrak{N}_l^f \to F'_l \) is proper (by lemma (4.2)). Let \( F''_l \) be the open subscheme of \( p_2 \)-stable sheaves in \( F'_l \). Let \( \mathfrak{N}''_l = \eta^{-1}(F''_l) \) be the parabolic vector bundles on marked semistable curves whose pushforward is \( p_2 \)-stable. Then (again by base change we see) the morphism \( \eta : \mathfrak{N}''_l \to F'_l \) is proper.

### 4.2. Proof of main theorem (0.4).

The proof essentially follows the methods used in [14, p. 180]. But here we have to deal with few more complexities. For the sake of completeness we will give the argument.

The first step is to embed \( \mathfrak{N}_l^f \) as a locally closed subscheme in a projective variety. We have the following locally closed embedding (3.1.4)

\[
\mathfrak{N}_l^f \hookrightarrow \mathfrak{N} = Y_{g,n} \times \left( \prod Gr(V_i, r_i^{j+1}) \times \cdots \times Gr(V_i, r_i^j) \right) \times \cdots \times Gr(V_i, r_i^{n+1}) \times \cdots \times \left( \prod Gr(V_i, r_i^s) \right)
\]

(4.2.1)

where the first embedding is an open embedding and the second one is a closed embedding. We will denote the product \( \prod_{i=1}^n \prod_{j=2}^{r_i^{j+1}} Gr(V_i, r_i^j) \) by \( Gr(F) \). Since the family \( X \) has the closed embedding \( X \hookrightarrow S_{g,n} \times \mathbb{P}^N \) (A.1.6) we get the closed embedding

\[
\text{Hilb}^p(t)(X \times Gr(V_i, r)) \hookrightarrow \text{Hilb}^p(t)(S_{g,n} \times \mathbb{P}^N \times Gr(V_i, r))
\]

(4.2.2)

By the universal property of Hilbert scheme the relative Hilbert scheme \( \text{Hilb}^p(t)(S_{g,n} \times \mathbb{P}^N \times Gr(V_i, r)) \) over \( S_{g,n} \) is isomorphic with \( S_{g,n} \times_{\mathbb{P}^N} \text{Hilb}^p(t)(\mathbb{P}^N \times Gr(V_i, r)) \). The inclusion of \( S_{g,n} \) in its closure \( \overline{S_{g,n}} \) is an open immersion and \( Y \) is an open subscheme of \( \text{Hilb}^p(t)(X \times Gr(V_i, r)) \). Hence as a composition of finitely many locally closed immersion we have the locally closed immersion of \( \mathfrak{N}_l^f \) in a projective variety

\[
\mathfrak{N}_l^f \hookrightarrow \overline{S_{g,n}} \times_{\mathbb{P}^N} \text{Hilb}^p(t)(\mathbb{P}^N \times Gr(V_i, r)) \times_{\mathbb{P}^N} \text{Gr}(\text{F})
\]

(4.2.3)

The embedding (4.2.1) is SL(\(N\)) \times SL(\(V_i\)) equivariant and the closed embedding \( X \hookrightarrow S_{g,n} \times \mathbb{P}^N \) is SL(\(N\)) equivariant. Therefore the embedding of \( \mathfrak{N}_l^f \) (4.2.3) is SL(\(N\)) \times SL(\(V_i\)) equivariant embedding.

Since (4.2.3) is a locally closed embedding, \( \mathfrak{N}_l^f \) is open in \( \overline{\mathfrak{N}_l^f} \) with the closure being taken inside the projective variety at the R.H.S of (4.2.3). We denote the closure \( \overline{\mathfrak{N}_l^f} \) by \( Z_1 \).

Let \( \mathcal{O}_{Z_1}(1) \) be an ample linearization of the SL(\(N\)) \times SL(\(V_i\)) action on \( Z_1 \). Recall the embedding (1.1.7) of the flag variety \( F_l \). We get the following equivariant

---

3 Since \( S_{g,n} \) is a locally closed subscheme of \( I \) (A.1.1), \( S_{g,n} \) is open inside \( \overline{S_{g,n}} \) with the closure being taken in \( I \).
commutative diagram:
\[
\begin{array}{ccc}
\mathfrak{F}^f & \longrightarrow & Z_1 \\
\mathfrak{F}^f & \downarrow\eta & \downarrow \mathfrak{F}^f \\
S^g, n \times \text{Gr}(l, k) & & \text{Gr}(l, k)
\end{array}
\] (4.2.4)

Let \( Z \) be the graph closure of the vertical rational morphism. Then \( Z \hookrightarrow Z_1 \times S^g, n \times \text{Gr}(l, k) \) is a projective variety. Recall the linearization \( L = L_{\beta, \beta^i} \otimes M^\otimes b \) (1.1.8) of the \( \text{SL}(N) \times \text{SL}(V_l) \) action on \( S^g, n \times \text{Gr}(l, k) \). We will choose the linearization \( L \otimes a \otimes O_{Z_1}(1) \) on \( Z \). We have the following equivariant commutative diagram
\[
\begin{array}{ccc}
\mathfrak{F}^f & \longrightarrow & Z \\
\mathfrak{F}^f & \downarrow\eta & \downarrow \mathfrak{F}^f \\
S^g, n \times \text{Gr}(l, k) & & \text{Gr}(l, k)
\end{array}
\] (4.2.5)

where \( \lambda \) is the 2\(^{nd} \) projection.

The open locus of marked nonsingular curves \( S^\phi_{g, a} \) is irreducible. We consider the restriction \( Y^f|_{S^g_{g, a}} \). The family \( Y^f|_{S^g_{g, a}} \to S^\phi_{g, a} \) is flat and the fibers are irreducible. Therefore the total space \( Y^f|_{S^g_{g, a}} \) is irreducible. From the proof of properness (4.2) we have \( Y^f|_{S^g_{g, a}} = Y^f \). This implies \( Y^f \) is irreducible. The fibers of the flat morphism \( \mathfrak{F}^f_i \to Y^f \) are flag varieties. Thus \( \mathfrak{F}^f_i \) is irreducible. So \( Z \) is an irreducible projective variety.

Recall that combining (1.15) and (1.17) will imply that a point \((C', \mathcal{E}', \gamma, V_l \otimes \mathcal{O}_{C'}, \mathcal{E})\) in \( F_l \) is \( p_2 \)-stable and \( \gamma : V_l \cong H^0(\mathcal{E}(l)) \) is an isomorphism (i.e., the point belongs in the open locus of \( p_2 \)-stable sheaves \( F^s_l \subseteq F^f_l \)) if and only if it’s image in \( S^g_{g, a} \times \text{Gr}(l, k) \) is GIT stable with respect to the linearization \( L \) of the \( \text{SL}(N) \times \text{SL}(V_l) \) action.

With respect to the linearization \( \mathcal{L}^\otimes b \otimes \mathcal{O}_{Z_1}(1) \) on \( Z \) we apply the GIT lemma (B.6) for \( a \gg 0 \). We get \( \lambda^{-1}(F^s_l) \hookrightarrow Z^s \) and the following diagram
\[
\begin{array}{ccc}
\mathfrak{F}^s_l & \longrightarrow & \lambda^{-1}(F^s_l) \\
\mathfrak{F}^s_l & \downarrow\eta & \downarrow \mathfrak{F}^s_l \\
F^s_l & & \text{Gr}(l, k)
\end{array}
\] (4.2.6)

Since the morphism \( \lambda \) is the base change of the projective morphism \( \lambda \) in (4.2.5) it is proper. The morphism \( \eta \) is proper (4.3). This implies \( i \) is proper. Also \( i \) is an open immersion. Irreducibility of \( \lambda^{-1}(F^s_l) \to Z^s \) means \( i \) is an isomorphism. This means we have a GIT quotient \( \mathfrak{F}^s_l / (\text{SL}(N) \times \text{SL}(V_l)) \) as a quasi projective variety, denoted by \( \mathcal{U}_{y, n, r} \). We also have a proper birational morphism \( \mathcal{U}_{y, n, r} \to F^s_l / (\text{SL}(N) \times \text{SL}(V_l)) = \mathcal{U}_{y, n, r} \).
Remark 4.4. Note that by the GIT lemma (B.6) we have
\[ \lambda^{-1}(F_i^s) \subseteq Z^s \subseteq Z^{ss} \subseteq \lambda^{-1}(F_i^{ss}) \]
(4.2.7)
We will have a morphism \( Z^{ss} \to F_i^{ss} \). The GIT quotient \( Z^{ss} \sslash (\text{SL}(N) \times \text{SL}(V_i)) \) which is a projective variety, can be called universal moduli space of semistable parabolic Gieseker vector bundles. In [18, Definition 2.2.10] a moduli theoretic interpretation of semistable locus is given in the context of vector bundles.

One way to give modular interpretation of points in \( Z^{ss} \) is to make \( F_i^s = F_i^{ss} \) which can be done by choosing weights \( \alpha^s_j \) and quasi parabolic structures \( r^s_j \) in such a way that \( p_2\)-stability=\( p_2\)-semistability. This can be done by choosing \( r, d, r_i^s \) such that \( g.c.d (r, d, r_i^s) = 1 \) and choosing suitable weights \( \alpha^s_j \). In fact for generic weights \( \alpha^s_j \) this will be true.

5. PROPERTIES OF THE MODULI SPACE \( \overline{\mathcal{M}}_{g,n,r} \)

In this section we will prove (0.5), (0.6). For the proofs we will need to study the deformation of a marked semistable curve.

Let \( C \) be a marked semistable curve and \( \pi : C \to C' \) be the canonical contraction to it’s stable model. There exist a formal universal deformation of \( (C', x^1, \cdots, x^n) \)
\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\sigma_i} & \mathcal{M} = \mathbb{C}[[t_1, t_2, \cdots, t_8]] \\
\mu & & \\
\end{array}
\] (5.0.1)
where \( \sigma_i \) for \( 1 \leq i \leq n \) are sections of the morphism \( \mu \) and \( \mathbb{M} = \dim(\text{Ext}^1(\mathcal{O}_C, \mathcal{O}_C(-D))) \). The divisor \( D = x^1 + \cdots + x^n \) is the divisor associated to the marked points [6, pp. 79-80]. Let the nodes of the curve \( C' \) be \( p_1, p_2, \cdots, p_8 \). Using Schlessinger’s theory it can be proved \( \mathcal{O}_{\mathcal{C}, p_i} \cong \mathbb{C}[[u_i, v_i, t_1, \cdots, t_8]] \) where \( u_i, v_i \) are the local coordinates at the node \( p_i \) of the curve \( C' \) [6, p. 82].

Recall that \( R^1, R^2, \cdots, R^m \) are the chains of \( \mathbb{P}^1 \)’s in \( C \) which are contracted by \( \pi \) to the nodes \( p_1, p_2, \cdots, p_8 \) respectively where \( R^i = \cup_{j=1}^{t_i} R_j^i \). Let \( N = \text{Spec} \mathbb{C}[[t_{1j}, t_{2j}, \cdots, t_{8j}]] \). Let \( \mathcal{C}_N \) be the restriction of the family \( \mathcal{C} \) to the closed subscheme \( N \) of \( \mathcal{M} \). By the local universal property (A.4) there exist a morphism \( \mathcal{N} \to S_{g,s} \) such that \( \mathcal{C}_N \cong \mathcal{N} \times_{S_{g,s}} \mathcal{N} \).

Let \( W = \text{Spec}(\mathbb{C}[[t_{ij} : 1 \leq i \leq m, \forall i, 1 \leq j \leq t_i]]) \) and the morphism \( W \to N \) be defined by \( t_i \to t_{i1} \cdot t_{i2} \cdot \cdots \cdot t_{in} \).

Lemma 5.1. [9, Lemma 4.2] [18, 3.3.1] There exist deformation \( Z_W \) over \( W \) of the curve \( C \) with a morphism \( \psi : Z_W \to X \times_{S_{g,s}} W \) which restrict to the contraction morphism \( \pi : C \to C' \) over the unique closed point of \( W \). The morphism \( \psi \) has the property \( \psi^* \omega_{X \times_{S_{g,s}} W / W} \cong \omega_{Z_W / W} \) relative to \( W \). The closed subscheme of \( W \), where the fibers of \( \psi \) are singular, is defined by \( \prod_{i=1}^m t_{ij} = 0, i = 1, 2, \cdots, m \).

We will denote this subvariety of \( W \) by \( Q \). The family \( Z_W \) over \( W \) has the following versal property:
Proposition 5.2. [9, Proposition 4.5] Let $A$ be an Artin local ring over $C$ such that we have a morphism $T = \text{Spec}(A) \to S_{g,n}$. Let $Z'$ over $T$ be a deformation of the marked semistable curve $C$ with a morphism $\psi' : Z' \to X \times_{S_{g,n}} T$ over $T$ which restricts to the canonical contraction $\pi : C \to C'$ over the unique closed point of $T$. Then there exist a morphism $T \to \mathcal{W}$ such that the deformation $Z'$ and the morphism $\psi'$ is isomorphic with $Z''$ and $\psi''$ where $Z''$ and $\psi''$ are the base change of $\mathcal{Z}_W$ and $\psi$.

Proposition 5.3. [14, Appendix I, III] The quasi projective variety $Y$ is regular.

Proof. The variety $Y$ represents the functor (C.1) and the universal family of curves is $\Delta \to Y \times_{S_{g,n}} (X \times \text{Gr}(V_i, r))$. Let $y \in Y$ be a point which is represented by the curve $C$ i.e., $\Delta_y = C$. We can define a local Gieseker functor $G_{\text{loc}} : \text{Artin local ring/S}_{g,n} \to \text{Sets}$ which sends a $S_{g,n}$ scheme $\text{Spec}(A)$ to $\Delta$ in $G(\text{Spec}(A))$ such that the fiber of $\Delta$ over the closed point in $\text{Spec}(A)$ is the curve $C$. Let $G_C : \text{Artin local ring/S}_{g,n} \to \text{Sets}$ be the functor which sends $\text{Spec}(A)$ to flat family of marked semistable curves $\Delta$ over $\text{Spec}(A)$ with a morphism $\psi : \Delta \to X \times_{S_{g,n}} \text{Spec}(A)$ such that the closed fiber is $C$ and $\psi^* \omega_{X,A} \cong \omega_{\Delta/A}$. We have a natural forget morphism $F : G_{\text{loc}} \to G_C$ which can be proved to be formally smooth [14, Appendix I]. On the other hand using the versal property of the deformation $\mathcal{Z}_W$ over $\mathcal{W}$ it can be checked that $(\mathcal{Z}_W, \mathcal{W})$ is a versal family for the functor $G_C$. Hence $Y$ is regular at $y$. \qed

Proof of Theorem (0.5). The morphism $\mathfrak{g}_l \to Y$ is flat, projective and the fibers of the morphism are flag varieties of equal dimension. Therefore the morphism is smooth. Since the quasi projective variety $Y$ is regular (5.3), we have $\mathfrak{g}_l$ is regular. Being an open subset of $\mathfrak{g}_l$, the stable locus $\mathfrak{g}_l^s$ is also regular. Thus $\mathfrak{g}_l^s / \text{SL}(N) \times \text{SL}(V_i)$ is a normal quasi projective variety. We have $Z^{ss}$ as an open subvariety of $\mathfrak{g}_l^s$ (4.4). Hence $Z^{ss}$ is regular. Therefore the semistable moduli $Z^{ss} / \text{SL}(N) \times \text{SL}(V_i)$ is a normal projective variety.

The map $\eta : \mathfrak{g}_l^s \to F_l^s$ is $\text{SL}(N) \times \text{SL}(V_i)$ equivariant. The stabilizer of a point in $F_l^s$ for the $\text{SL}(V_i)$ action is the finite cyclic group of order $\dim(V_i)$. Therefore the stabilizer of a point in $\mathfrak{g}_l^s$ for the $\text{SL}(V_i)$ action is also the same finite group. Hence the $\text{SL}(V_i)$ action will factor through the $\text{PGL}(V_i)$ action which will act freely. The stabilizer of a point $(C, E_s)$ for $\text{SL}(N)$ action is the finite automorphism group $\text{Aut}(C', x_1, x_2, \ldots, x^n)$, where $C'$ is the marked stable model of the marked Gieseker curve $C$. Since the total space $\mathfrak{g}_l^s$ is regular, it follows from the étale slice theorem that the universal moduli space $\mathfrak{g}_l^s / \text{SL}(N) \times \text{SL}(V_i)$ has finite quotient singularity.

A stable parabolic Gieseker bundle $E_s$ on a marked semistable curve $C$ is called strictly stable if for any automorphism $\phi : (C, x_1, \ldots, x^n) \to (C, x_1, \ldots, x^n)$ such that there exist a parabolic isomorphism $\phi^* E_s \cong E_s$ then $\phi = \text{id}$. The strictly stable locus $\mathfrak{g}_l^s$ is an open subvariety of $\mathfrak{g}_l^s$. By definition the action of $\text{SL}(N) \times \text{SL}(V_i)$ on $\mathfrak{g}_l^s$ is free. Therefore $\mathfrak{g}_l^s \to (\mathfrak{g}_l^s / \text{SL}(N) \times \text{SL}(V_i))$ is a principal bundle. Thus $\mathfrak{g}_l^s / \text{SL}(N) \times \text{SL}(V_i)$ is a regular subvariety. \qed


Proof of Theorem (0.6). We have the morphism \( \kappa_g : \mathcal{H}_{g,n,r} \to \mathcal{M}_{g,n} \). Let \([C_0]\) be a point in \( \mathcal{M}_{g,n} \) represented by a point \( s \in \mathcal{S}_{g,n} \) and we are writing \( C_0 \) for \( \mathcal{X}_s \). Then the variety \( Y_s \) is a subvariety of \( \text{Hilb}^{p(l)}(\mathcal{C}_0 \times \text{Gr}(V_l, r)) \). We will have the flag variety \((\mathcal{F}_s)_s \to Y_s \). Then the fiber of the morphism \( \kappa_g \) at \([C_0]\) is \((\mathcal{F}_s)_s \sslash (\text{Aut}(C_0, x^1, \cdots, x^m) \times \text{SL}(V_l)) \).

We want to prove that \((\mathcal{F}_s)_s \) has the singularity which is a product of analytic normal crossings. Since the forget morphism \((\mathcal{F}_s)_s \to Y_s \) is smooth, it is enough to show that \( Y_s \) has the same type of singularity. Let \( h \in Y_s \) be a point represented by a marked semistable curve \( C \). Let \( R = \text{Spec}(\mathcal{O}_{h,Y_s}) \) be a formal neighbourhood at \( h \). Let \( \Delta_R \to R \times (\mathcal{C}_0 \times \text{Gr}(V_l, r)) \) be the restriction of the universal curve to \( R \). We have the induced morphism \( \psi_R : \Delta_R \to R \times C_0 \). Then by the versal property (5.2) there exists a morphism \( R \to W \) which will factor through the subvariety \( \mathcal{Q} \). Now it can be proved using methods similar to (5.3) that \( R \to \mathcal{Q} \) is formally smooth.

Let \((\mathcal{F}_s)_s \to (\mathcal{F}_s)_s \) be the open subvariety of \( p_2 \)-stable sheaves. The action of \( \text{SL}(V_l) \) on \((\mathcal{F}_s)_s \) is free since the action on \((F_1)_s \) is free (5.5) and the morphism \( \mathcal{F}_s \to F_1 \) is \( \text{SL}(V_l) \) equivariant. Thus \((\mathcal{F}_s)_s \) is a principal \( \text{SL}(V_l) \) bundle over the geometric quotient \((\mathcal{F}_s)_s / \text{SL}(V_l) \). Therefore the fiber of \( \kappa_g \) at \([C_0]\) has the singularity which is a product of analytic normal crossing modulo the action of a finite group. In particular when the curve has no nontrivial automorphisms it is of the type of product of analytic normal crossing singularities.

Recall that the open subvariety \( F^*_l \) of \( F_l \) is the locus of \( p_2 \)-stable sheaves which has the natural induced isomorphism \( V_l \cong H^0(E) \). A parabolic sheaf \( E \) is \( p_2 \)-stable if and only if for any non zero proper saturated subsheaf \( F \) of \( E \) with the induced parabolic structure on \( F \) we have either \( \text{par}_\mu(F) < \text{par}_\mu(E) \) or \( \text{par}_\mu(F) = \text{par}_\mu(E) \) and \( \mu(F) > \mu(E) \).

**Lemma 5.4.** Every \( p_2 \)-stable sheaf is simple.

**Proof.** We will prove that for a \( p_2 \)-stable sheaf \( \text{Par-End}(E) \cong \mathbb{C} \). Let \( \phi : E \to E \) be a parabolic endomorphism. Let \( p \) be a point of \( C \). Then \( \phi_p : E_p \to E_p \) is an automorphism of vector spaces over \( \mathbb{C} \). Let \( \lambda \) be an eigenvalue. We will get the parabolic endomorphism \( \phi - \lambda \cdot I : E \to E \) which will be denoted by \( \psi \). Let \( F \) be the kernel of \( \psi \) which is a non zero subsheaf. If \( F = E \) then we are done, so we will assume \( F \subsetneq E \). The morphism \( \psi \) will induce an isomorphism \( E/F \cong \psi(E) \) where \( \psi(E) \) is a subsheaf of \( E \). Since \( F \) is a saturated subsheaf, we will give the induced parabolic structure on \( F \) and \( \psi(E) \). Since \( E \) is \( p_2 \)-stable, by the above definition, let \( \text{par}_\mu(F) < \text{par}_\mu(E) \), then \( \text{par}_\mu(\psi(E)) > \text{par}_\mu(E) \) which contradicts \( p_2 \)-stability of \( E \). So let us assume \( \text{par}_\mu(F) = \text{par}_\mu(E) \) and \( \mu(F) > \mu(E) \), which will again give a contradiction. Thus we have \( F = E \) which means \( \phi = \lambda \cdot I \) for some \( \lambda \in \mathbb{C} \). \( \square \)

**Lemma 5.5.** The action of \( \text{SL}(V_l) \) on \( F^*_l \) is free.

**Proof.** Note that the action of \( \text{SL}(V_l) \) is fiber preserving. Let \((F_1)_s \) be the fiber of \( F^*_l \) over a point \( s \in \mathcal{S}_{g,n} \) which is represented by the marked stable curve \( \mathcal{X}_s = C \). The fiber \((F_s)_s \) is over the uniform rank \( \text{Quot} \) scheme \( \text{Quot}^r(V_l \otimes \mathcal{O}_C, H) \) which is a subscheme of the \( \text{Quot} \) scheme \( \text{Quot}(V_l \otimes \mathcal{O}_C, H) \). We will prove that the stabilizer
Stab_{SL(V)}(\mathcal{E}_*) = \text{Par-Aut}(\mathcal{E}_*)$, where $\mathcal{E}_*$ is an element of $F_l$. This combined with (5.4) will give us the result.

Let $g$ be an element of $\text{SL}(V)$ and $\mathcal{E}_* \in F_l$. We will have the following diagram which depicts the action of $g$ on $\mathcal{E}_*$.

If $g$ is a stabilizer of $\mathcal{E}_*$, then we will have $\ker(k) = \ker(k')$ and $\ker(k_j) = \ker(k'_j)$ for $2 \leq j \leq (l_i + 1)$. Then we will have the following commutative diagram

which will induce an isomorphism $\phi : \mathcal{E} \to \mathcal{E}$. Using $\ker(k_j) = \ker(k'_j)$ in the same way we will get an isomorphism $\phi_j : \mathcal{E}/F_1^j \mathcal{E} \to \mathcal{E}/F_2^j \mathcal{E}$ which is compatible with $\phi$. Hence we will get a parabolic automorphism $\phi : \mathcal{E}_* \to \mathcal{E}_*$ in a natural way.

Conversely, let $\phi : \mathcal{E}_* \to \mathcal{E}_*$ be a parabolic automorphism. We will have the quotient representation $k : V \otimes O_C \to \mathcal{E}$ along with the induced isomorphism $H^0(k) : V \to H^0(\mathcal{E})$. We will define the unique automorphism $g : V \to V$ so that the following diagram is commutative

We will prove that $g$ is a stabilizer of $\mathcal{E}_*$ in $F_l$. We will show that $\ker(k) = \ker(k')$ for the action of $g$ (5.0.2), similarly it will follow $\ker(k_j) = \ker(k'_j)$. From the definition of $g$ we will get the following commutative diagram
Since $\phi$ is an automorphism and $k' = \phi \circ k$, we have $\ker(k) = \ker(k')$.

Both of these two morphisms are natural in the sense that there is no choice involved. From the definitions it follows that they are inverses of each other. \hfill \Box

APPENDIX A. MODULI SPACE OF MARKED STABLE CURVES

**Definition A.1.** (1) A marked stable curve is a marked prestable curve (1)
which has finite automorphism (preserving marked points) group.

(2) (combinatorial definition) A marked prestable curve is called a marked stable curve if every nonsingular rational component has at least 3 special points (either marked points or singular points) and every nonsingular elliptic component has at least 1 special points.

**Definition A.2.** (1) A family of marked stable curves is a flat morphism $\mu : Y \to T$ with sections $\sigma_i : T \to Y$ such that $(\mu^{-1}(t), \sigma_1(t), \ldots, \sigma_n(t))$ is a marked stable curve for all $t \in T$. The family will be denoted by $(\mu : Y \to T, \sigma_1, \ldots, \sigma_n)$

(2) Two families $(\mu : Y \to T, \sigma_1, \ldots, \sigma_n)$ and $(\mu' : Y' \to T, \sigma'_1, \ldots, \sigma'_n)$ are called equivalent if there exist an isomorphism $\phi : Y \cong Y'$ over $T$ which is compatible with sections i.e., $\phi \circ \sigma_i = \sigma'_i$. The equivalence will be denoted by $(\mu : Y \to T, \sigma_1, \ldots, \sigma_n) \equiv (\mu' : Y' \to T, \sigma'_1, \ldots, \sigma'_n)$

There exist a moduli space of isomorphism classes of marked stable curves of genus $g$ and $n$ marked points which is denoted by $\overline{M}_{g,n}$. This moduli space is the Deligne-Mumford compactification of the moduli space $M_{g,n}$ of isomorphism classes of smooth curves of genus $g$ and $n$ marked points. We will briefly mention the method of GIT construction of $\overline{M}_{g,n}$ [5]. We will assume genus $g \geq 2$ for the rest of the section.

**A.1. Brief construction of $\overline{M}_{g,n}$.** Let $(C, x_1, x_2, \ldots, x^n)$ be a marked prestable curve. The dualizing sheaf $\omega_C$ is the sheaf of logarithmic 1 form $f$ on the normalization $\overline{C}$ which are regular except simple poles at $\{p_1^j, p_2^j : 1 \leq j \leq c\}$ such that $\text{Res}_{p_1^j}(f) + \text{Res}_{p_2^j}(f) = 0$ where $p_1^i, p_2^i$ are inverse image of the node $z^j$. Let $\mathcal{L}$ be the twisted line bundle $\omega_C(x^1 + x^2 + \ldots + x^n)$.

**Lemma A.3.** If $(C, x^1, x^2, \ldots, x^n)$ is a marked stable curve then $\mathcal{L}^{\otimes \rho}$ is very ample if $\rho \geq 3$. We also note that $H^1(C, \mathcal{L}^{\otimes \rho}) = 0$ for $\rho \geq 2$.

**Proof.** This can be proved using arguments similar to [6, Theorem 1.2] \hfill \Box

We will consider the linear system corresponding to the very ample line bundle $\mathcal{L}^{\otimes \rho}$ with $\rho$ large enough e.g., $\rho \geq 5$ will be enough to embedd the curve $C$ in a projective space.

Let the dimension of the linear system $\dim(H^0(C, \mathcal{L}^{\otimes \rho}))$ be $N$. The moduli problem is rigidified with the choice of an isomorphism $H^0(C, \mathcal{L}^{\otimes \rho}) \cong \mathbb{C}^N$ and hence it induces an action of $\text{SL}(N)$. Let $p(t)$ be the Hilbert polynomial of the curve $C$ with respect to the line bundle $\mathcal{L}^{\otimes \rho}$ i.e., $p(n) = \chi(\mathcal{L}^{\otimes \rho}n)$. Let $\text{Hilb}^{p(t)}(\mathbb{P}^{N-1})$ be the Hilbert scheme of curves in $\mathbb{P}^{N-1}$ of Hilbert polynomial $p(t)$. Let the closed
subscheme $\mathcal{C} \hookrightarrow \text{Hilb}^{(t)}(\mathbb{P}^{N-1}) \times \mathbb{P}^{N-1}$ be the universal curve. We will consider the closed subscheme of incidence

$$I \hookrightarrow \text{Hilb}^{(t)}(\mathbb{P}^{N-1}) \times \underbrace{\mathbb{P}^{N-1} \times \cdots \times \mathbb{P}^{N-1}}_{n \text{ times}}$$

(A.1.1)

consisting of $(n+1)$ tuple $\{([C], x^1, x^2, \ldots, x^n) : [C] \in \text{Hilb}^{(t)}(\mathbb{P}^{N-1})$ and $x^i \in C \}$. We will have a natural forgetful map

$$\text{Hilb}^{(t)}(\mathbb{P}^{N-1}) \times \underbrace{\mathbb{P}^{N-1} \times \cdots \times \mathbb{P}^{N-1}}_{n \text{ times}} \twoheadrightarrow \text{Hilb}^{(t)}(\mathbb{P}^{N-1})$$

(A.1.2)

Taking the product with the universal curve $\mathcal{C}$ we get the following closed subscheme

$$\mathcal{C} \times \underbrace{\mathbb{P}^{N-1} \times \cdots \times \mathbb{P}^{N-1}}_{n \text{ times}} \hookrightarrow \text{Hilb}^{(t)}(\mathbb{P}^{N-1}) \times \underbrace{\mathbb{P}^{N-1} \times \cdots \times \mathbb{P}^{N-1}}_{n \text{ times}}$$

(A.1.3)

We will take the following base change

$$\begin{array}{ccc}
\mathcal{C} \times (\mathbb{P}^{N-1})^n & \hookrightarrow & \mathcal{C} \times (\mathbb{P}^{N-1})^n \\
\mathbb{P}^{N-1} \times (\mathbb{P}^{N-1})^n & \twoheadrightarrow & \text{Hilb}^{(t)}(\mathbb{P}^{N-1}) \times (\mathbb{P}^{N-1})^n
\end{array}$$

(A.1.4)

The incidence subscheme $I$ is called the Hilbert scheme of marked curves and the closed subscheme $\mathcal{C}_I$ is called the marked universal curve. We consider the locally closed subscheme $J \hookrightarrow I$ whose points has the following property:

1. The point $([C], x^1, x^2, \ldots, x^n)$ has to be a marked prestable curve (1).

2. The embedding of the curve $C$ in $\mathbb{P}^{N-1}$ has to be non degenerate i.e., the image does not lie inside a hyperplane.

3. There exist an isomorphism $\omega_C(x^1 + x^2 + \cdots + x^n)^{\otimes \rho} \cong O_{\mathbb{P}^{N-1}}(1)|_C$.

Prestability and non degeneracy are open conditions. That the third property is a closed condition is verified in [8, p. 58]. Therefore we will get a locally closed subscheme $J$ of $I$. The elements of $J$ parameterizes the marked stable curves. Restricting the marked universal curve we get the closed subscheme $\mathcal{C}_J \hookrightarrow J \times \mathbb{P}^{N-1}$. The family $\mathcal{C}_J \to J$ has natural sections $\sigma_i : J \to \mathcal{C}_J$ which assigns the marked points $x^i$ to each point $([C], x^1, \ldots, x^n) \in J$. This family has the local universal property for the moduli problem of $\overline{M}_{g,n}$ in the following sense:

**Proposition A.4.** Let $\{\mu : Y \to T, \sigma_1, \ldots, \sigma_n\}$ be a family of marked stable curves. Then for all $t$ in $T$ there exist a neighbourhood $U$ of $t$ and a morphism $U \to J$ such that $Y|_U \equiv \mathcal{C}_J \times J U$.

**Proof.** For a proof we refer to [5, Proposition 3.4].

From the rigification we have an action of $SL(N)$ on $\mathbb{P}^{N-1}$. Thus $SL(N)$ will induce a natural action on $\text{Hilb}^{(t)}(\mathbb{P}^{N-1})$ and so $SL(N)$ will act diagonally on $I$. It is clear that $J$ and $\mathcal{J}$ is invariant under the action of $SL(N)$ with the closure of $J$ being taken in the projective variety $I$. The main GIT problem in [5] is the study
of the \( \text{SL}(N) \) action on \( I \).

Let the vector space \( W = H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)) \) and \( W_n = \wedge^p W \). Then we will have a closed embedding \( \text{Hilb}^p(\mathbb{P}^{n-1}) \hookrightarrow \mathbb{P}(W) \). This embedding is the composition of the Grothendieck embedding with the Plücker embedding. We choose the following ample line bundle

\[
L_{m,m'} = \left( \mathcal{O}_{\mathbb{P}(W)}(1) \boxtimes \left( \bigotimes_{i=1}^n \mathcal{O}_{\mathbb{P}^{n-1}}(m') \right) \right)|_I
\]

which gives a linearization for the \( \text{SL}(N) \) action on \( I \) [5, pp. 17–18]. Now we will state the main result of [5].

**Theorem A.5** ([5], Theorem 6.3, Propositions 6.8, 6.9). There exist natural numbers \( m, m' \) (depends only on \( g, n \)) such that

\[
\text{ss}(L_m, m') = \text{ss}(L_m, m') = \text{ss}. 
\]

The GIT quotient \( J/\text{SL}(N) \) is a projective variety. In fact the quotient is an orbit space which is denoted by \( \mathcal{M}_{g,n} \). The moduli space \( \mathcal{M}_{g,n} \) is the coarse moduli space of the following functor.

**Definition A.6.** We define the moduli functor

\[
\mathcal{M}_{g,n} : \text{Sch}/\mathbb{C} \to \text{Sets}
\]

which assigns a scheme \( T \) to the set of equivalence classes of families of marked stable curves \( (\mu : Y \to T, \sigma_1, \cdots, \sigma_n) \).

For the purpose of the paper we will identify the flat and projective family

\[
\begin{align*}
\mathcal{E}_3 & \\
\sigma_1 & \downarrow \sigma_3 \downarrow \\
J & \quad J
\end{align*}
\]

as \( \mathcal{X} \to S_{g,n} \). We will have the relative dualizing sheaf (a line bundle) \( \omega_{\mathcal{X}/S_{g,n}} \) and the relative very ample line bundle \( \omega_{\mathcal{X}/S_{g,n}}(\sigma_1 + \sigma_2 + \cdots + \sigma_n) \).

**Appendix B. Universal moduli space of parabolic torsion free sheaves**

**Lemma B.1.** A parabolic sheaf \( \mathcal{E}_* \) is \( p_2 \)-stable(\( p_2 \)-semistable) if and only if for any non zero proper saturated sub sheaf \( \mathcal{F} \) of \( \mathcal{E} \) with the induced parabolic structure on \( \mathcal{F} \) we have either \( \text{par} \mu(\mathcal{F}_*) \leq \text{par} \mu(\mathcal{E}_*) \) or \( \text{par} \mu(\mathcal{F}_*) = \text{par} \mu(\mathcal{E}_*) \) and \( \mu(\mathcal{F}) > (\geq ) \mu(\mathcal{E}). \)

**Proof.** This easily follows from the definitions. For a proof we refer to [17, Remark 4.3.13].

**Remark B.2.** We will have the following relations:

\[
\{ \text{par} \mu \text{ stable sheaves} \} \subseteq \{ \text{par} p_2 \text{ stable sheaves} \} \subseteq \{ \text{par} p_2 \text{ semistable sheaves} \}
\]

The inclusions are in general strict.

\footnote{This is a standard notation used for the expression in the LHS of (1.0.4)}
Remark B.3. As mentioned in the above remark (B.2) the notion of slope stability (1.0.4) and $p_2$ stability (1.0.5) are not equivalent even in the case of curves. We will provide an example to show that the R.H.S inclusion is strict.

Let us consider the rank 2 vector bundle $E = \mathcal{O} \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$. Let $P$ and $Q$ be the parabolic points on $\mathbb{P}^1$. At both $P$ and $Q$ we will define the following same parabolic structure with same weights on $E$

$$E \supseteq \mathcal{O} \oplus \mathcal{O} \supseteq \mathcal{O}(-1) \oplus \mathcal{O}$$  \hspace{1cm} (B.0.1)

with attached weights

$$0 < \frac{1}{4} < \frac{3}{4} < 1$$  \hspace{1cm} (B.0.2)

Then $\text{par } \mu(E) = \frac{\deg(E) + \frac{1}{2} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4}}{2} = \frac{3}{2}$.

Let $L$ be a rank 1 saturated subbundle of $E$. We consider the induced maps $L \to \mathcal{O}$ and $L \to \mathcal{O}(1)$. If either of them is zero then we will have either $L \cong \mathcal{O}$ or $L \cong \mathcal{O}(1)$, since $L$ is saturated. If both of them is nonzero, then either $L \cong \mathcal{O}$ or $\deg(L) \leq -1$.

If $\deg(L) \leq -1$ then we can see that $\text{par } \mu(L) < \text{par } \mu(E)$. In both cases when $L \cong \mathcal{O}$ we will have the following induced parabolic structure

$$\mathcal{O} \supseteq \mathcal{O} \supseteq \mathcal{O}(-1)$$  \hspace{1cm} (B.0.3)

with attached weights

$$0 < \frac{1}{4} < \frac{3}{4} < 1$$  \hspace{1cm} (B.0.4)

Then $\text{par } \mu(\mathcal{O}) = 0 + \frac{3}{4} + \frac{3}{4} = \frac{6}{4} = \text{par } \mu(E)$ and $\mu(\mathcal{O}) < \mu(E)$. Then by (B.1) $E$ can not be $p_2$-semistable.

If $L \cong \mathcal{O}(1)$, then we will have the following induced parabolic structure

$$\mathcal{O}(1) \supseteq \mathcal{O} \supseteq \mathcal{O}$$  \hspace{1cm} (B.0.5)

with attached weights

$$0 < \frac{1}{4} < \frac{3}{4} < 1$$  \hspace{1cm} (B.0.6)

Then we will get $\text{par } \mu(\mathcal{O}(1)) = 1 + \frac{1}{4} + \frac{1}{4} = \frac{6}{4} = \text{par } \mu(E)$. Hence $E$ is $\mu$ semistable.

Lemma B.4 (boundedness). Consider the local universal family $\mu : \mathcal{X} \to \mathbb{S}_{s,n}$ (A.1.6). Then the family of objects \{p$_2$ semistable sheaves with fixed numerical type $(r, d, r'_i, \alpha'_i)$ on $\mathcal{X}_s : s \in \mathbb{S}_{s,n}$ \} is bounded.

Proof. For the proof we refer to [17, Theorem 4.5.2]. \hfill \square

We have the natural closed embedding $\mathcal{X} \hookrightarrow \mathbb{S}_{s,n} \times \mathbb{P}^{N-1}$. Thus for any $s \in \mathbb{S}_{s,n}$ we have $\mathcal{X}_s \hookrightarrow \mathbb{P}^{N-1}$. Therefore the sheaves in (B.4) are all p$_2$ semistable sheaves of dimension 1 in $\mathbb{P}^{N-1}$. So the notion of boundedness makes sense.
Definition B.5 (flat family of parabolic sheaves). Let $T$ be a $\mathcal{S}_{g,n}$ scheme. A flat family of parabolic sheaves $\mathcal{E}_*$ means a coherent sheaf $\mathcal{E}$ on $\mathcal{X}_T$ flat over $T$ with

I. A Parabolic filtration:

$$\mathcal{E} = F^0\mathcal{E} \supset F^1\mathcal{E} \supset F^2\mathcal{E} \supset \cdots \supset F^l\mathcal{E} = \mathcal{E}(-D_T)$$

such that $\mathcal{E}/F^j\mathcal{E}$ are flat over $T \forall i, j$.

II. Along with attached weights:

$$0 \leq a_i^1 < a_i^2 < \cdots < a_i^l < 1$$

such that for $t \in T$ the restriction of $\mathcal{E}_*$ is a parabolic sheaf on $\mathcal{X}_t$ (1.0.1) with respect to the divisors $D^i_t$.

Let $G$ be a reductive group acting on two projective variety $X$ and $Y$. Let $\mathcal{L}, \mathcal{M}$ be two linearization of the $G$ action on $X$ and $Y$ respectively. Then $G$ will have a natural action on $X \times Y$. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the two projections.

Lemma B.6. There exist an integer $b \gg 0$ such that the following holds:

$$p_2^{-1}Y^s(\mathcal{M}) \subset (X \times Y)^s(p_1^*\mathcal{L} \otimes b \mathcal{L}) \subset (X \times Y)^{ss}(p_1^*\mathcal{M} \otimes p_2^*\mathcal{L}) \subset p_2^{-1}Y^{ss}(\mathcal{M})$$

Proof. This result has been mentioned in several places. This goes back to [19, Proposition 5.1]. See also [13, Proposition 2.18, Proposition 3.3.1], [17, Proposition 5.1.1].

Appendix C. Some lemmas from Nagaraj-Seshadri

C.1. Gieseker functor. The Gieseker functor has been defined in [9], [14, Definition 7]. The basic notations has been defined in the introduction of the subsection (2.3). We recall $\mathcal{X}$ over $\mathcal{S}_{g,n}$ is the local universal family for the moduli problem of $\overline{\mathcal{M}}_{g,n}$ (A.1.6). Let $\mathcal{G} = \mathcal{G}(r; d)$ be the functor

$$\mathcal{G} : \mathfrak{Sch}/\mathcal{S}_{g,n} \to \mathfrak{Sets}$$

$$\mathcal{G}(T) = \text{Set of closed subschemes } \Delta \hookrightarrow \mathcal{X} \times_S T \times_S \text{Gr}(V_i, r)$$

such that it satisfies the following property:

1. The induced projection morphism $\Delta \to T \times_S \text{Gr}(V_i, r)$ is also a closed immersion.

2. The family $\Delta \to T$ is a flat family of curves. For a given $t \in T$ maps to $s \in \mathcal{S}_{g,n}$ such that $\Delta_s$ is a marked semi-stable curve (2.1) and the induced morphism $\Delta_s \to \mathcal{X}_s$ is the collapsing map to its marked stable model. In addition the closed subscheme $\Delta_t \to \mathcal{X}_s \times_S \text{Gr}(V_i, r)$ has Hilbert polynomial $p(t)$. The Hilbert polynomial is defined with respect to the line bundle $\text{det}(E_t)$ where $E_t$ is pullback of the tautological quotient bundle of $\text{Gr}(V_i, r)$.

3. The vector bundle $E_t$ on $\Delta_t$ has rank $r$, degree $d$ such that $\text{dim } V_i = d + r(1 - q)$. The bundle $E_t$ has a natural quotient representation $V_i \otimes \mathcal{O}_{\Delta_t} \to E_t \to 0$ which is the pullback of the universal short exact sequence on $\text{Gr}(V_i, r)$. The induced map in cohomology has to be an isomorphism $V_i \cong H^0(E_t)$. This implies $H^1(E_t) = 0$. 

Proposition C.1. Let $C$ be a marked semistable curve and $\tilde{C}'$ be the closure $C \setminus R'$. Recall that $C'$ is the partial normalization $C \setminus \cup R'$ of the marked stable curve $C'$. Let $E$ be a strictly positive vector bundle $(2.1)$ on the curve $C$ which satisfies the following conditions

1. We have the cohomology vanishing $H^1(\tilde{C}', I_{p_1p_2}^*E|_{\tilde{C}'}) = 0$. In particular this implies that $H^0(\tilde{C}', E|_{\tilde{C}'}) \to E_{p_1} \oplus E_{p_2}$ is surjective.
2. The canonical map $H^0(\tilde{C}', I_{p_1p_2}^*E|_{\tilde{C}'}) \to I_{p_1p_2}^*E|_{\tilde{C}'}/I_{p_1p_2}^2E|_{\tilde{C}'}$ is surjective.
3. The canonical map $H^0(\tilde{C}', I_2^*E|_{\tilde{C}'}) \to E|_{\tilde{C}'}/I_2^2E|_{\tilde{C}'}$ is surjective for $x \in \tilde{C}' \setminus Z$, where $Z = \cup \{p_1, p_2\}$.
4. The canonical map $H^0(\tilde{C}', I_2^*E|_{\tilde{C}'}) \to E_{s_1} \oplus E_{s_2}$ is surjective for all pair of points $s_1, s_2 \in \tilde{C}' \setminus Z$ such that $s_1 \neq s_2$.

Then we will have $H^1(C, E) = 0$. The global sections $H^0(E)$ generates $E$ and the natural morphism $C \to \text{Gr}(H^0(C, E), r)$ is a closed immersion.

Proof. This can be proved using similar methods of [14, Proposition 4].

Lemma C.2. Suppose we have the following commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{\pi} & W \\
\downarrow{p} & & \downarrow{q} \\
T & \xleftarrow{q} & W
\end{array}
$$

(C.1.1)

such that $p$ and $q$ are projective morphisms (which implies $\pi$ is proper), $\pi_*O_Z = O_W$ and $p$ is flat. Let $E$ be a vector bundle on $Z$ such that $R^i(\pi_1)_*E_i = 0$ for $i \geq 1$, then the higher direct images of $E$ behaves well with respect to restriction i.e., $(R^i\pi_*E)_i \cong R^i(\pi_1)_*E_i$ for $i \geq 0$.

Proof. For the proof of $i = 0$ case we refer to [14, Lemma 4]. Using similar argument with little modification one can prove that the sheaf $R^i\pi_*E = 0$ for $i > 0$. Here we will briefly mention the arguments for the case of $i > 0$.

Let $O_W(1)$ be a relative ample line bundle. Let $O_Z[1]$ be the pullback $\pi^*O_W(1)$. Let $E[n]$ be the tensor product $E \otimes O_Z[n]$. Without loss of generality we can assume $T = \text{spec}(A)$. Since $q$ is projective by Serre correspondence $R^i\pi_*E$ is the sheaf associated to the graded module $\oplus_{n \geq 0}H^0(q_*R^i\pi_*E(n)))$.

We will use the Grothendieck spectral sequence for composite of two functors $E_2^{i,j} = R^iq_*(R^j\pi_*(E(n))) \implies R^{i+j}p_*(E[n])$. From the spectral sequence it follows that $R^i\pi_*(E[n]) \cong q_*R^i\pi_*(E(n)))$ for all $i \geq 0$ and for sufficiently large $n$. Therefore we have

$$
\oplus_{n \geq 0}H^0(q_*(R^i\pi_*E(n))) \cong \oplus_{n \geq 0}H^0(R^i\pi_*(E[n]))
$$

(C.1.2)

Using the projection formula We will have

$$
R^i(\pi_1)_*(E_i[n]) \cong (R^i\pi_1)_*(E_i(n)) = 0
$$

(C.1.3)

for $i > 0$ and for $n \geq 0$. This implies $H^i(Z_1, E_i[n]) \cong H^i(W_1, (\pi_1)_*E_i(n))$ for all $i \geq 0$. Since $O_W(n)$ is relatively ample, there exist $n_0$ such that for $i > 0$, $H^i(W_1, (\pi_1)_*E_i(n)) = 0$ for $n \geq n_0$ and for all $t$. This in turn implies that
$H^i(Z_t, E_t[n]) = 0$ for $n \geq n_0$ and $i > 0$. From the semicontinuity theorem of Grauert it follows $R^i\pi_*(E[n]) = 0$ for $i > 0$ and $n \geq n_0$. From (C.1.2) it follows that graded module associated to the sheaf $R^i\pi_*E$ vanishes after finitely many terms of the grading. Therefore the associated sheaf $R^i\pi_*E = 0$. 

\[ \square \]

References

[1] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. Geometry of algebraic curves. Vol. I, volume 267 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, New York, 1985. doi:10.1007/978-1-4755-5323-3.

[2] V Balaji. Torsors on semistable curves and degenerations. Proceedings-Mathematical Sciences, 132(1):1–63, 2022.

[3] V. Balaji and C. S. Seshadri. Moduli of parahoric $G$-torsors on a compact Riemann surface. J. Algebraic Geom., 24(1):1–49, 2015. doi:10.1090/s0273-0979-14-02364-x.

[4] Vikraman Balaji, Pabitra Barik, and Donihakkalu S. Nagaraj. A degeneration of moduli of Hitchin pairs. Int. Math. Res. Not. IMRN, (21):6581–6625, 2016. doi:10.1093/imrn/rnv356.

[5] Elizabeth Baldwin and David Swinarski. A geometric invariant theory construction of moduli spaces of curves. Int. Math. Res. Pap. IMRP, (1):Art. ID rp. 004, 104, 2008.

[6] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. Inst. Hautes Études Sci. Publ. Math., (36):75–109, 1969. URL: http://www.numdam.org/item?id=PMIHES_1969__36__75_0.

[7] Gerd Faltings. Moduli-stacks for bundles on semistable curves. Math. Ann., 304(3):489–515, 1996. doi:10.1007/BF01446303.

[8] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In Algebraic geometry—Santa Cruz 1995, volume 62 of Proc. Sympos. Pure Math., pages 45–96. Amer. Math. Soc., Providence, RI, 1997. doi:10.1090/pspum/062.2/1492534.

[9] D. Gieseker. A degeneration of the moduli space of stable bundles. J. Differential Geom., 19(1):173–206, 1984. URL: http://projecteuclid.org/euclid.jdg/1214438427.

[10] Robin Hartshorne. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[11] M. Maruyama and K. Yokogawa. Moduli of parabolic stable sheaves. Math. Ann., 293(1):77–99, 1992. doi:10.1007/BF02100913.

[12] D. Mumford, J. Fogarty, and F. Kirwan. Geometric invariant theory, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]. Springer-Verlag, Berlin, third edition, 1994. doi:10.1007/978-3-642-57916-5.

[13] D. S. Nagaraj and C. S. Seshadri. Degenerations of the moduli spaces of vector bundles on curves. II. Generalized Gieseker moduli spaces. Proc. Indian Acad. Sci. Math. Sci., 109(2):165–201, 1999. doi:10.1007/BF02841533.

[14] Rahul Pandharipande. A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles. J. Amer. Math. Soc., 9(2):425–471, 1996. doi:10.1090/S0894-0347-96-00173-7.

[15] A. Ramanathan. Moduli for principal bundles over algebraic curves. I. Proc. Indian Acad. Sci. Math. Sci., 106(3):301–328, 1996. doi:10.1007/BF02867438.

[16] Dirk Schütz. Universal moduli of parabolic sheaves on stable marked curves. PhD thesis, Oxford University, UK, 2011. URL: https://ora.ox.ac.uk/objects/uuid:b0260f8e-6654-4bec-b670-5e925fd403dd.

[17] Alexander Schmitt. The Hilbert compactification of the universal moduli space of stable marked curves. J. Differential Geom., 66(2):169–209, 2004. URL: http://projecteuclid.org/euclid.jdg/1102538609.

[18] C. S. Seshadri. Quotient spaces modulo reductive algebraic groups. Ann. of Math. (2), 95:511–556; errata, ibid. (2) 96 (1972), 599, 1972. doi:10.2307/1970870.

[19] Carlos T. Simpson. Moduli of representations of the fundamental group of a smooth projective variety. I. Inst. Hautes Études Sci. Publ. Math., (79):47–129, 1994. URL: http://www.numdam.org/item?id=PMIHES_1994__79__47_0.