Brane cosmology: an introduction

David Langlois

GReCO, Institut d’Astrophysique de Paris
98 bis, Boulevard Arago, 75014 Paris, France

(Received March 27, 2022)

These notes give an introductory review on brane cosmology. This subject deals with the cosmological behaviour of a brane-universe, i.e., a three-dimensional space, where ordinary matter is confined, embedded in a higher dimensional spacetime. In the tractable case of a five-dimensional bulk spacetime, the brane (modified) Friedmann equation is discussed in detail, and various other aspects are presented, such as cosmological perturbations, bulk scalar fields and systems with several branes.

§1. Introduction

It has been recently suggested that there might exist some extra spatial dimensions, not in the traditional Kaluza-Klein sense where the extra-dimensions are compactified on a small enough radius to evade detection in the form of Kaluza-Klein modes, but in a setting where the extra dimensions could be large, under the assumption that ordinary matter is confined onto a three-dimensional subspace, called brane (more precisely ‘3-brane’, referring to the three spatial dimensions) embedded in a larger space, called bulk.

Although the idea in itself is not completely new, the fact that it might be connected to recent string theory developments has suscitated a renewed interest. In this respect, an inspiring input has been the model suggested by Horava and Witten, sometimes dubbed M-theory, which describes the low energy effective theory corresponding to the strong coupling limit of $E_8 \times E_8$ heterotic string theory. This model is associated with an eleven-dimensional bulk spacetime with 11-dimensional supergravity, the eleventh dimension being compactified via a $Z_2$ orbifold symmetry. The two fixed points of the orbifold symmetry define two 10-dimensional spacetime boundaries, or 9-branes, on which the gauge groups are defined. Starting from this configuration, one can distinguish three types of spatial dimensions: the orbifold dimension, three large dimensions corresponding to the ordinary spatial directions and finally six additional dimensions, which can be compactified in the usual Kaluza-Klein way. It turns out that the orbifold dimension might be larger than the six Kaluza-Klein extra dimensions, resulting in an intermediary picture with a five-dimensional spacetime, two boundary 3-branes, one of which could be our universe, and a “large” extra-dimension. This model provides the motivating framework for many of the brane cosmological models.

This concept of a 3-brane has also been used in a purely phenomenological way by Arkani-Hamed, Dimopoulos and Dvali (ADD) as a possible solution to the
hierarchy problem in particle physics. Their setup is extremely simple since they consider a flat \((4+n)\)-dimensional spacetime, thus with \(n\) compact extra dimensions with, for simplicity, a torus topology and a common size \(R\). From the fundamental (Planck) \((4+n)\)-dimensional mass \(M_*\), which embodies the coupling of gravity to matter from the \((4+n)\)-dimensional point of view, one can deduce the effective four-dimensional Planck mass \(M_P\), either by integrating the Einstein-Hilbert action over the extra dimensions, or by using directly Gauss’ theorem. One then finds

\[
M_P^2 = M_*^{2+n} R^n. \tag{1.1}
\]

On sizes much larger than \(R\), \((4+n)\)-dimensional gravity behaves effectively as our usual four-dimensional gravity, and the two can be distinguished only on scales sufficiently small, of the order of \(R\) or below. The simple but crucial remark of ADD is that a fundamental mass \(M_*\) of the order of the electroweak scale can explain the huge four-dimensional Planck mass \(M_P\) we observe, provided the volume in the extra dimensions is very large. Of course, such a large size for the usual extra-dimensions à la Kaluza-Klein is forbidden by collider constraints, but is allowed when ordinary matter is confined to a three-brane. In contrast, the constraints on the behaviour of gravity are much weaker since the usual Newton’s law has been verified experimentally only down to a fraction of millimeter. This leaves room for extra dimensions as large as this millimeter experimental bound.

The treatment of extra-dimensions can be refined by allowing the spacetime to be curved, by the presence of the brane and possibly by the bulk. In this spirit, Randall and Sundrum have proposed a very interesting model with an Anti de Sitter (AdS) five-dimensional spacetime (i.e. a maximally symmetric spacetime with a negative cosmological constant), and shown that, for an appropriate tension of the brane representing our universe, one recovers effectively four-dimensional gravity even with an infinite extra-dimension. Another model, which will not be discussed in this review, is characterized, in contrast with the previous ones, by a gravity which becomes five-dimensional at large scales and it could have interesting applications for the present cosmological acceleration.

The recent concept of extra-dimensions with branes has been explored in a lot of aspects of particle physics, gravity, astrophysics and cosmology. The purpose of this review is to present a very specific, although very active, facet of this vast domain, dealing with the cosmological behaviour of brane models when the curvature of spacetime along the extra-dimensions, and in particular the brane self-gravity, is explicitly taken into account. For technical reasons, this can be easily studied in the context of codimension one spacetimes and we will therefore restrict this review to the case of five-dimensional spacetimes. Even in this restricted area, the number of works during the last few years is so large that this introductory review is not intended to be comprehensive but will focus on some selected aspects. This also implies that the bibliography is far from exhaustive.
§2. Pre-history of brane cosmology

Since our purpose is to describe (homogeneous) cosmology within the brane, we will model our four-dimensional universe as an infinitesimally thin wall of constant spatial curvature embedded in a five-dimensional spacetime. This means that we need to consider spacetimes with planar (or spherical/hyperboloidal) symmetry along three of their spatial directions, i.e. homogeneous and isotropic along three spatial dimensions. The general metric compatible with these symmetries can be written in appropriate coordinate systems in the form

\[ ds^2 = g_{ab}(z^c)dz^adz^b + a^2(z^c)\gamma_{ij}dx^idx^j, \]  

(2.1)

where \( \gamma_{ij} \) is the maximally symmetric three-dimensional metric, with either negative, vanishing or positive spatial curvature (respectively labelled by \( k = -1, 0 \) or \( 1 \)), and \( g_{ab} \) is a two-dimensional metric (with space-time signature) depending only on the two coordinates \( z^c \), which span time and the extra spatial dimension. We are interested in the history of a spatially homogeneous and isotropic three-brane, which can be simply described as a point trajectory in the two-dimensional \( z \)-spacetime, the ordinary spatial dimensions inside the brane corresponding to the coordinates \( x^i \). Now, once such a trajectory has been defined, it is always possible to introduce a so-called Gaussian Normal (GN) coordinate system so that

\[ g_{ab}dz^adz^b = -n(t,y)^2dt^2 + dy^2. \]  

(2.2)

This coordinate system can be constructed by introducing the proper time on the trajectory and by defining the coordinate \( y \) as the proper distance on (space-like) geodesics normal to the trajectory. The time coordinate defined only on the brane is then propagated off the brane along these normal geodesics. It is convenient to take the origin on the brane so that \( y = 0 \) in this coordinate system represents the brane itself.

Summarizing, it is always possible to write the metric (2.1), in a GN coordinate system where the brane is located at \( y = 0 \), in the form

\[ ds^2 = -n(t,y)^2dt^2 + a(t,y)^2\gamma_{ij}dx^idx^j + dy^2. \]  

(2.3)

This is the most convenient system of coordinates when one wishes to focus on the brane itself since the induced metric in the brane is immediately obtained in its Friedmann-Lemaître-Robertson-Walker (FLRW) form

\[ ds^2_{brane} = -n(t,0)^2dt^2 + a(t,0)^2\gamma_{ij}dx^idx^j. \]  

(2.4)

If \( t \) is the proper time on the brane then \( n(t,0) = 1 \). The GN coordinates can however suffer from coordinate singularities off the brane, and it might be more convenient, as discussed in Section 5, to use more appropriate coordinates in order to describe the bulk spacetime structure.

Having specified the form of the metric, we now turn to the five-dimensional Einstein equations, which can be written in the condensed form

\[ G_{AB} \equiv R_{AB} - \frac{1}{2}Rg_{AB} = -\Lambda g_{AB} + \kappa^2T_{AB}. \]  

(2.5)
where $R_{AB}$ is the five-dimensional Ricci tensor and $R = R^A_A$ its trace. $A$ stands for a possible bulk cosmological constant, whereas $T^A_{AB}$ is the total energy momentum tensor. It includes the energy-momentum tensor of the brane, which is distributional if one assumes the brane infinitely thin (a few works\cite{11,12} have discussed the cosmology for thick branes, which can also be related to the abundant literature on thick domain walls\cite{13}) and thus of the form

$$T^A_{B|\text{brane}} = S^A_B \delta(y) = \text{diag} \left( -\rho_b, p_b, p_b, p_b, 0 \right) \delta(y),$$

where $\rho_b$ is the total brane energy density and $p_b$ is the total brane pressure. The total energy-momentum tensor also includes a possible bulk contribution, which we first assume to vanish (bulk matter will be considered in Section 8).

In the coordinate system (2.3), the components of the Einstein tensor read

$$G_{00} = 3 \frac{\dot{a}^2}{a} - 3n^2 \left( \frac{a''}{a} + \frac{a'^2}{a^2} \right) + 3k \frac{n^2}{a^2},$$

$$G_{ij} = a^2 \gamma_{ij} \left( 2 \frac{a''}{a} + \frac{n''}{n} + \frac{a'^2}{a^2} + 2 \frac{a'n'}{an} \right)$$

$$+ \frac{a^2}{n^2} \gamma_{ij} \left( -2 \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} - 2 \frac{\dot{a}n}{an} \right) - k \gamma_{ij},$$

$$G_{0y} = 3 \left( \frac{n'}{a} - \frac{\dot{a}}{a} \right),$$

$$G_{yy} = 3 \left( \frac{a'^2}{a^2} + \frac{a'n'}{an} \right) - 3 \frac{\dot{a}}{a} \left( \frac{\dot{a}^2}{a^2} - \frac{\dot{a}n}{an} \right) - 3k \frac{1}{a^2},$$

where a dot stands for a derivative with respect to $t$ and a prime a derivative with respect to $y$. One way to solve Einstein’s equations is to insert the full energy-momentum tensor, including the brane energy-momentum tensor with its delta distribution, on the right-hand side of (2.5) and solve the full system of equations. An alternative procedure is to consider Einstein’s equations without the brane energy-momentum tensor, i.e. valid strictly in the bulk, and then impose on the general solutions boundary conditions to take into account the physical presence of the brane. The Einstein tensor is made of the metric up to second derivatives, so formally the Einstein equations with the distributional source are of the form

$$g'' = S \delta(y).$$

If the brane is located at $y = 0$, integrating this equation over $y$ across the brane immediately yields

$$g'(+\epsilon) - g'(-\epsilon) = S.$$  

In other words, the boundary conditions due to the presence of the brane must take the form of a relation between the jump, across the brane, of the first derivative of the metric with respect to $y$ and the brane energy-momentum tensor. In the case of Einstein’s equations, the exact equivalent of the formal relation (2.12) is the so-called
Brane cosmology

5

junction conditions\cite{13}, which can be written in a covariant form as

\[
(K^A_B - K\delta^A_B) = -\kappa^2 S^A_B, \tag{2.13}
\]

where the brackets here denote the jump at the brane, i.e. \( [Q] = Q_{y=0^+} - Q_{y=0^-} \),
and the extrinsic curvature tensor is defined by

\[
K_{AB} = h^C_A \nabla_C n_B, \tag{2.14}
\]

\( n^A \) being the unit vector normal to the brane, and

\[
h_{AB} = g_{AB} - n^A n_B \tag{2.15}
\]

the induced metric on the brane.

A frequent assumption in the brane cosmology literature has been, for simplicity, to keep the orbifold nature of the extra dimension in the Horava-Witten model and thus impose a mirror symmetry across the brane, although some works have relaxed this assumption\cite{14,15}. This enables us to replace the jump in the extrinsic curvature by twice the value of the extrinsic curvature at the location of the brane. The junction conditions (2.13) then imply

\[
K_{AB} = -\frac{\kappa^2}{2} \left( S_{AB} - \frac{1}{3} S g_{AB} \right), \tag{2.16}
\]

where \( S \equiv g^{AB} S_{AB} \) is the trace of \( S_{AB} \). If one uses the GN coordinate system with the metric (2.3) and substitutes the explicit form of the brane energy-momentum tensor (2.6), the junction conditions reduce to the two relations

\[
\left( \frac{n'}{n} \right)_{0^+} = \frac{\kappa^2}{6} \left( 3p_b + 2\rho_b \right), \quad \left( \frac{a'}{a} \right)_{0^+} = -\frac{\kappa^2}{6} \rho_b. \tag{2.17}
\]

Going back to the bulk Einstein equations, one can solve the component \( G_{0y} = 0 \) (see 2.9) to get

\[
\dot{a}(t, y) = \nu(t) n(t, y), \tag{2.18}
\]

and the integration of the component \((0 - 0)\) with respect to \( y \) and of the component \((y - y)\) with respect to time yields the first integral

\[
(aa')^2 - \nu^2 a^2 - ka^2 + \frac{A}{6} a^4 + C = 0, \tag{2.19}
\]

where \( C \) is an arbitrary integration constant. When one evaluates this first integral at \( y = 0 \), i.e. in our brane-universe, substituting the junction conditions given above in (2.16), one ends up with the following equation\cite{13,18}

\[
H^2 \equiv \frac{a_0^2}{a^2} = \frac{\kappa^4}{36} \rho_b^2 + \frac{A}{6} - \frac{k}{a^2} + \frac{C}{a^4}, \tag{2.20}
\]

where the subscript ‘0’ means evaluation at \( y = 0 \). This equation relates the Hubble parameter to the energy density but it is different from the usual Friedmann equation.
$H^2 = (8\pi G/3)\rho$. The most remarkable feature of (2.20) is that the energy density of the brane enters quadratically on the right hand side in contrast with the standard four-dimensional Friedmann equation where the energy density enters linearly. As for the energy conservation equation, it is unchanged in this five-dimensional setup and still reads

$$\dot{\rho}_b + 3H(\rho_b + p_b) = 0, \quad (2.21)$$

as a consequence of the component $(0 - y)$ of Einstein’s equations combined with the junction conditions (2.17).

In the simplest case where $\Lambda = 0$ and $C = 0$, one can easily solve the cosmological equations (2.20)-(2.21) for a perfect fluid with equation of state $p_b = w\rho_b$ ($w$ constant). One finds that the evolution of the scale factor is given by

$$a_0(t) \propto t^{1/(1+w)}.$$  \hfill (2.22)

In the most interesting cases for cosmology, radiation and pressureless matter, the evolution of the scale factor is thus given by, respectively, $a \sim t^{1/4}$ (instead of the usual $a \sim t^{1/2}$) and $a \sim t^{1/3}$ (instead of $a \sim t^{2/3}$). Such behaviour is problematic because it cannot be reconciled with nucleosynthesis. Indeed, the standard nucleosynthesis scenario depends crucially on the balance between the microphysical reaction rates and the expansion rate of the universe. Any drastic change in the evolution of the scale factor between nucleosynthesis and now would dramatically modify the predictions for the light element abundances. After discussing some particular solutions in the next section, the subsequent section will present a brane cosmological model with much nicer features.

§3. Cosmological solutions for ‘domain walls’

It is instructive to consider the brane cosmological solutions for the simplest equation of state, $p_b = -\rho_b$, which also characterizes the so-called domain walls. With this particular equation of state, the cosmological equations can be explicitly integrated \cite{31,9,13}. Indeed, the energy density is necessarily constant, as implied by (2.21), and the Friedmann equation (2.20) is of the form

$$\frac{\dot{a}^2}{a^2} = \alpha - \frac{k}{a^2} + \frac{C}{a^4}, \quad (3.1)$$

where

$$\alpha \equiv \frac{\kappa^4}{36}\rho_b^2 + \frac{\Lambda}{6} \quad (3.2)$$

is a constant. The case $\alpha = 0$ is referred to as a ‘critical’ brane (or domain wall), while $\alpha < 0$ and $\alpha > 0$ correspond to subcritical and supercritical branes respectively. Defining $X \equiv a^2$, the Friedmann equation (3.1) can be rewritten as

$$\frac{\dot{X}^2}{4} = \alpha X^2 - kX + C, \quad (3.3)$$

which is analogous to the equation for a point particle with kinetic energy on the left hand side and (minus) the potential energy on the right hand side. For a critical
brane \((\alpha = 0)\), one immediately finds the following three solutions, depending on the spatial curvature of the brane:

\[
\begin{align*}
    a &= \sqrt{2} C^{1/4} t^{1/2}, \quad (k = 0), \\
    a &= \left(t^2 + 2\sqrt{C} t\right)^{1/2}, \quad (k = -1), \\
    a &= \sqrt{C - t^2}, \quad (k = +1),
\end{align*}
\]

which in fact correspond to the usual FLRW solutions with radiation, the term proportional to \(C\) playing the effective rôle of radiation.

For a non critical brane, integration of \((3.3)\) yields the following solutions,

\[
\begin{align*}
    a^2 &= \sqrt{\beta} \alpha \sinh(2\sqrt{\alpha} t) + \frac{k}{2\alpha} \quad (\alpha > 0, \ \beta > 0) \quad (3.7) \\
    a^2 &= \pm \sqrt{-\beta} \alpha \cosh(2\sqrt{\alpha} t) + \frac{k}{2\alpha} \quad (\alpha > 0, \ \beta < 0) \quad (3.8) \\
    a^2 &= \sqrt{-\beta} \alpha \sin(2\sqrt{\alpha} t) + \frac{k}{2\alpha} \quad (\alpha < 0, \ \beta > 0), \quad (3.9)
\end{align*}
\]

with

\[
\beta = C - \frac{k^2}{4\alpha}. \quad (3.10)
\]

The particular solutions for \(\beta = 0\) (and \(\alpha > 0\)) are

\[
a^2 = \frac{k}{2\alpha} \pm e^{2\sqrt{\alpha} t}. \quad (3.11)
\]

In all the above equations the (additive) integration constant defining the origin of times is not written explicitly and one can always use this time shift to rewrite any solution in a more adequate form (for instance so that \(a = 0\) at \(t = 0\)). Moreover, all solutions have their time reversed counterpart obtained by changing \(t\) into \(-t\).

The analytical expressions \((3.8)\) or \((3.9)\) cover very different cosmological behaviours depending on the value of their parameters. Using the analogy with the point particle mentioned before, one can see that the global cosmological behaviour depends on the number of positive roots of the quadratic ‘potential’ on the right hand side of \((3.3)\). For \(\alpha > 0\), \(k = 1\) and \(0 < C < 1/(4\alpha)\), the potential has two positive roots and this leads to two solutions, one expanding from the singularity \(a = 0\) and recontracting to \(a = 0\), the other contracting from infinity to a minimum scale factor and then expanding. These two types of solutions correspond to the two branches of \((3.8)\). For \(\alpha > 0\) and \(C > 0\) with \(k = 0, -1\), or \(C > 1/(4\alpha)\) with \(k = 1\), there is no positive root and the corresponding cosmology starts from the singularity \(a = 0\) and then expands indefinitely. This behaviour is described by the solution \((3.7)\) and the solution \((3.8)\) for \(k = -1\). For \(\alpha < 0\) and \(C > 0\), there is a single positive root, and the cosmological solution starts expanding from \(a = 0\) and then recollapses back to \(a = 0\). This corresponds to the solution \((3.9)\). For \(\alpha < 0, k = -1\)
and $1/(4\alpha) < C < 0$, the solution is confined between the two positive roots, which gives an oscillating cosmology between a minimum scale factor and a maximum scale factor. This is also described by the solution (3.9). Finally, for $\alpha > 0$ and $C < 0$, there is a single positive root and the solution corresponds to a contraction from infinity followed, after a bounce at a minimum scale factor, by an expansion back to infinity. This is described by the solution (3.8).

§4. Simplest realistic brane cosmology

As explained earlier, the simplest model of self-gravitating brane cosmology, that of a brane embedded in an empty bulk with vanishing cosmological constant (in fact a Minkowski bulk because of the symmetries), does not appear compatible with the standard landmarks of modern cosmology. It is thus necessary to consider more sophisticated models in order to get a viable scenario, at least as far as homogeneous cosmology is concerned.

An instructive exercise is to look for non trivial (i.e. with a non empty brane) static solutions. In the simplest case $C = 0$, one immediately sees from (2.20) that a static solution, corresponding to $H = 0$, can be obtained with a negative cosmological constant and an energy density $\rho_b$ satisfying

$$\kappa^2 \rho_b = \pm \sqrt{-6\Lambda}. \quad (4.1)$$

One can check that this is compatible with the other equations, in particular the conservation equation in the brane (2.21), which imposes that the matter equation of state is that of a cosmological constant, i.e. $p_b = -\rho_b$. It turns out that this configuration is exactly the starting point of the two models due to Randall and Sundrum (RS)\(^{14,15}\). From the point of view of brane cosmology, embodied in the unconventional Friedmann equation (2.20), the RS models thus appear as the simplest non trivial static brane configurations. The case of the single brane model\(^{16}\), with a positive tension, is particularly interesting, because ordinary four-dimensional gravity, at least at linear order, is effectively recovered on large enough lengthscales\(^{22}\), with

$$8\pi G \equiv M_p^{-2} = \kappa^2 \ell^{-1}, \quad (4.2)$$

where $\ell$ is the Anti de Sitter (AdS) lengthscale defined by the (negative) cosmological constant,

$$\Lambda = -\frac{6}{\ell^2}. \quad (4.3)$$

Since usual gravity is recovered, the generalization of this model to cosmology seems a priori a good candidate for a viable brane cosmology.

Let us therefore consider a brane with the total energy density

$$\rho_b = \sigma + \rho, \quad (4.4)$$

where $\sigma$ is a tension, constant in time, and $\rho$ the energy density of ordinary cosmological matter. Substituting this decomposition into (2.20), one obtains\(^{18,19,24}\)

$$H^2 = \left(\frac{\kappa^4}{36} \sigma^2 - \mu^2\right) + \frac{\kappa^4}{18} \sigma \rho + \frac{\kappa^4}{36} \sigma^2 - \frac{k}{a^2} + \frac{C}{a^4} \quad (4.5)$$
where $\mu \equiv \ell^{-1}$ is the AdS mass scale. If one fine-tunes the brane tension and the bulk cosmological constant like in (4.1) so that

$$\frac{\kappa^2}{6} \sigma = \mu,$$

(4.6)

the first term on the right hand side of (4.5) vanishes and, because of (4.2), the tension is proportional to Newton's constant,

$$\kappa^2 \equiv 8\pi G = \frac{\kappa^4}{6} \sigma = \kappa^2 \mu.$$

(4.7)

The second term in (4.5) then becomes the dominant term if $\rho$ is small enough and is exactly the linear term of the usual Friedmann equation, with the same coefficient of proportionality.

The third term on the right hand side of (4.5), quadratic in the energy density, provides a high-energy correction to the Friedmann equation which becomes significant when the value of the energy density approaches the value of the tension $\sigma$ and dominates at still higher energy densities. In the very high energy regime, $\rho \gg \sigma$, one recovers the unconventional behaviour (2.22), not surprisingly since the bulk cosmological constant is then negligible.

Finally, the last term in (4.5) behaves like radiation and arises from the integration constant $C$. This constant $C$ is analogous to the Schwarzschild mass, as will be shown in the next section, and it is related to the bulk Weyl tensor, which vanishes when $C = 0$. In a cosmological context, this term is constrained to be small enough at the time of nucleosynthesis in order to satisfy the constraints on the number of extra light degrees of freedom (this will be discussed quantitatively just below). In the matter era, this term then redshifts quickly and would be in principle negligible today.

To summarize the above results, the brane Friedmann equation (2.20), for a RS type brane (i.e. satisfying (4.4) and (4.6)), reduces to

$$H^2 = \frac{8\pi G}{3} \rho \left( 1 + \frac{\rho}{2\sigma} \right) + \frac{C}{a^4},$$

(4.8)

which shows that, at low energies, i.e. at late times, one recovers the usual Friedmann equation. Going backwards in time, the $\rho^2$ term becomes significant and makes brane cosmology deviate from the usual FLRW behaviour.

It is also useful, especially as a preparation to the equations for the cosmological perturbations, (see Section 7), to notice that the generalized Friedmann equation (4.8) can be seen as a particular case of the more general effective four-dimensional Einstein’s equations for the brane metric $g_{\mu\nu}$, obtained by projection on the brane. Using the Gauss equation and the junction conditions, and decomposing the total energy-momentum tensor of the brane into a pure tension part and an ordinary matter part so that

$$S_{\mu\nu} = -\sigma g_{\mu\nu} + \tau_{\mu\nu},$$

(4.9)

one arrives to effective four-dimensional Einstein equations, which read

$$G_{\mu\nu} + \Lambda_4 g_{\mu\nu} = \kappa^2 \tau_{\mu\nu} + \kappa^2 \Pi_{\mu\nu} - E_{\mu\nu},$$

(4.10)
with
\[ A_4 = \frac{A}{2} + \frac{\kappa^4}{12} \sigma^2, \]
\[ \Pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\alpha} \tau_{\nu}^{\alpha} + \frac{1}{12} \tau_{\mu\nu}^{\alpha} + \frac{1}{24} \left( 3 \tau_{\alpha\beta} \tau^{\alpha\beta} - \tau^2 \right) g_{\mu\nu}, \] (4.11)
and where
\[ E_{\mu\nu} = C^\mu_{\mu\nu\nu} \] (4.12)
is the projection on the brane of the five-dimensional Weyl tensor. In this new form, the effective Einstein’s equations are analogous to the standard four-dimensional equations with the replacement of the usual matter energy-momentum tensor \( \tau_{\mu\nu} \) by the sum of an effective matter energy-momentum tensor,
\[ T_{\mu\nu}^{\text{eff}} = \tau_{\mu\nu} + \frac{\kappa^4}{\kappa_4^4} \Pi_{\mu\nu}, \] (4.13)
which is constructed only with \( \tau_{\mu\nu} \), and of an additional part that depends on the bulk, which defines what we will call the Weyl energy-momentum tensor
\[ T_{\mu\nu}^{\text{Weyl}} = -\kappa_4^{-2} E_{\mu\nu}. \] (4.14)
Although this formulation might appear very simple, the reader should be warned that this equation is in fact directly useful only in the cosmological case, where \( E_{\mu\nu} \) reduces to the arbitrary constant \( C \). In general, \( E_{\mu\nu} \) will hide a dependence on the brane content and only a detailed study of the bulk can in practice provide the true behaviour of gravity in the brane.

Let us now work out a few explicit cosmological solutions. For an equation of state \( p = w \rho \), with \( w \) constant, one can integrate explicitly the conservation equation (2.21) to obtain as usual
\[ \rho = \rho_0 a^{-q}, \quad q \equiv 3(w + 1). \] (4.15)
Substituting in the Friedmann equation (4.3), one gets
\[ \frac{a^2}{a^2} = \alpha + \frac{\kappa^4}{18} \sigma \rho_0 a^{-q} + \frac{\kappa^4}{36} \rho_0^2 a^{-2q} + \frac{C}{a^4} - \frac{k}{a^2}, \] (4.16)
with
\[ \alpha = \frac{\kappa^4}{36} \sigma^2 - \mu^2. \] (4.17)
In the case \( C = 0 \) and \( k = 0 \), defining \( X \equiv a^q \), the Friedmann equation (4.16) reduces to the form
\[ \frac{X^2}{q^2} = \alpha X^2 + \beta X + \xi, \quad \beta = \frac{\kappa^4}{18} \sigma \rho_0, \quad \xi = \frac{\kappa^4}{36} \rho_0^2, \] (4.18)
which is similar to (3.3). For a critical brane, i.e. \( \alpha = 0 \), the corresponding cosmological evolution is given by
\[ a^q = \frac{q^2}{4} \beta t^2 + q \sqrt{\xi} t. \] (4.19)
It is clear from this analytical expression that there is a transition, at a typical time of the order of $\mu^{-1}$, i.e. of the AdS lengthscale $\ell$, between an early high energy regime characterized by the behaviour $a \propto t^{1/q}$ and a late low energy regime characterized by the standard evolution $a \propto t^{2/q}$.

For a non critical brane, the cosmological solutions are similar to the expressions (3.8) and (3.9) [the expression (3.7) does not apply here because of the relation between the coefficients $\alpha$, $\beta$ and $\xi$]. After choosing an appropriate origin of time (so that $a(0) = 0$), they can be rewritten in the form

$$a^q = \sqrt{\frac{\xi}{\alpha}} \sinh (q\sqrt{\alpha}t) + \frac{\beta}{2\alpha} \left[ \cosh (q\sqrt{\alpha}t) - 1 \right] \quad (\alpha > 0) \quad (4.20)$$

and

$$a^q = \sqrt{\frac{\xi}{-\alpha}} \sin (q\sqrt{\alpha}t) + \frac{\beta}{2\alpha} \left[ \cos (q\sqrt{\alpha}t) - 1 \right] \quad (\alpha < 0). \quad (4.21)$$

The solution (4.20) describes a brane cosmology with a positive cosmological constant. This implies that, by fine-tuning adequately the brane tension, one can obtain a cosmology with, like in (4.19), an unconventional early phase followed by a conventional phase, itself followed by a period dominated by a cosmological constant.

Fig. 1. Evolution of the scale factor for $\alpha = 0$ (continuous line), $\alpha > 0$ (dotted line) and $\alpha < 0$ (dashed line)

In the present section, we have so far considered only the metric in the brane. But the metric outside the brane can be also determined explicitly by solving the full system of Einstein’s equations. The metric coefficients defined in (2.3) are given by

$$a(t, y) = \frac{a_0(t)}{\sqrt{2}} \left\{ (1 - \eta^2) - \frac{C}{\mu^2 a_0^4} + \left[ (1 + \eta^2) + \frac{C}{\mu^2 a_0^4} \right] \cosh(2\mu y) - 2\eta \sinh(2\mu |y|) \right\}^{1/2} \quad (4.22)$$
and
\[ \frac{\dot{n}(t, y)}{a_0(t)} = \frac{\dot{a}(t, y)}{a_0(t)} \]  
(4.23)

which, in the special case \( C = 0 \), reduce to the much simpler expressions
\[ a(t, y) = a_0(t) (\cosh \mu y - \eta \sinh \mu |y|) \]  
(4.24)
\[ n(t, y) = \cosh \mu y - \tilde{\eta} \sinh \mu |y|, \]  
(4.25)

with
\[ \eta = 1 + \frac{\rho}{\sigma}, \quad \tilde{\eta} = \eta + \frac{\dot{\eta}}{H_0}. \]  
(4.26)

In the limit \( \rho = 0 \), i.e. \( \rho_b = \sigma \), which implies \( \eta = \tilde{\eta} = 1 \), one recovers the RS metric \( a(y) = n(y) = \exp(-\mu |y|) \).

In summary, we have obtained a cosmological model, based on a braneworld scenario, which appears to be compatible with current observations, because it converges to the standard model at low enough energies. Let us now quantify the constraints on the parameters of this model. As mentioned above an essential constraint comes from nucleosynthesis: the evolution of the universe since nucleosynthesis must be approximately the same as in usual cosmology. This is the case if the energy scale associated with the tension is higher than the nucleosynthesis energy scale, i.e.
\[ M_c \equiv \sigma^{1/4} > 1 \text{ MeV}. \]  
(4.27)

Combining this with (4.7) this implies for the fundamental mass scale (defined by \( \kappa^2 \equiv M^{-3} \))
\[ M > 10^4 \text{ GeV}. \]  
(4.28)

However, one must not forget another constraint, not of cosmological nature: the requirement to recover ordinary gravity down to scales of the submillimeter order. This requires
\[ \ell < 10^{-1} \text{ mm}, \]  
(4.29)

which yields in turn the constraint
\[ M > 10^8 \text{ GeV}. \]  
(4.30)

Therefore the most stringent constraint comes, not from cosmology, but from gravity experiments in this particular model.

There are also constraints on the Weyl parameter \( C \). The ratio
\[ \epsilon_W = \frac{\rho_{Weyl}}{\rho_{rad}} = \frac{C \sigma}{2 \mu^2 \rho_{rad} a^4} \]  
(4.31)

is constrained by the number of additional relativistic degrees of freedom allowed during nucleosynthesis, which is usually expressed as the number of additional light neutrino species \( \Delta N_\nu \). A typical bound \( \Delta N_\nu < 1 \) implies \( \epsilon_W < 8 \times 10^{-2} \) at the time of nucleosynthesis. It is also important to stress that if \( C \) has been considered here as an arbitrary constant, a more refined, and more realistic, treatment must take
into account the fact that cosmology is only approximately homogeneous: the small inhomogeneities in the brane can produce bulk gravitons and the energy outflow carried by these gravitational waves can feed the asymptotic ‘Schwarzschild mass’, i.e. the Weyl parameter $C$ which would not be any longer a constant (in agreement with the fact that the bulk is then no longer vacuum). This process is at present under investigation and the first estimates give a Weyl parameters of the order of the nucleosynthesis limit $^{28}$, $^{29}$.

So far, we have thus been able to build a model, which reproduces all qualitative and quantitative features of ordinary cosmology in the domains that have been tested by observations. The obvious next question is whether this will still hold for a more realistic cosmology that includes perturbations from homogeneity, and more interestingly, whether brane cosmology is capable of providing predictions that deviate from usual cosmology and which might tested in the future. This question will be addressed, but unfortunately not answered, in the section following the next one which presents the brane cosmological solutions in a totally different way.

§5. A different point of view

In the previous sections, we have from the start restricted ourselves to a particular system of coordinates, namely a GN coordinate system. This choice entails no physical restriction (at least locally) and is very useful for a ‘brane-based’ point of view, but there also exists an alternative approach for deriving the brane cosmological solutions given earlier, which relies on a ‘bulk-based’ point of view $^{30}$, $^{31}$ and corresponds to a more appropriate choice of coordinates for the bulk. The link between the two points of view can be understood from a general analysis $^{21}$ which consists in solving the five-dimensional Einstein equations for a generic metric of the form (2.1). It turns out that simple coordinates emerge, expressing in a manifest way the underlying symmetries of the solutions. This approach leads to the following simple form for the solutions (with the required ‘cosmological symmetries’) of Einstein’s equations in the bulk (2.3) [with $T_{AB} = 0$]:

$$ ds^2 = -f(R) \, dt^2 + \frac{dR^2}{f(R)} + R^2 \gamma_{ij} \, dx^i \, dx^j, $$

(5.1)

where

$$ f(R) \equiv k - \frac{4}{6} R^2 - \frac{C}{R^2}. $$

(5.2)

The above metric is known as the five-dimensional Schwarzschild-Anti de Sitter (Sch-AdS) metric (AdS for $\Lambda < 0$, which is the case we are interested in; for $\Lambda > 0$, this the Schwarzschild-de Sitter metric). It is clear from (5.2) that $C$ is indeed the five-dimensional analog of the Schwarzschild mass, as said before (the $R^{-2}$ dependence instead of the usual $R^{-1}$ is simply due to the different dimension of spacetime).

It is manifestly static (since the metric coefficients are time-independent), which means that the solutions of Einstein’s equations have more symmetries than assumed a priori. In fact, this is quite analogous to what happens in four-dimensional general relativity when one looks for vacuum solutions with only spherical symmetry:
one ends up with the Schwarschild geometry, which is static. The above result for empty five-dimensional spacetimes can thus be seen as a generalization of Birkhoff’s theorem, as it is known in the four-dimensional case.

Since the solutions of the bulk Einstein equations, with the required symmetries, are necessarily Sch-AdS, it is easy to infer that the solutions (4.22) or (4.25), obtained in the particular GN coordinate system correspond to the same geometry as (5.1) but written in a more complicated coordinate system, as can be checked by finding the explicit coordinate transformation going from (5.1) to (2.3).

If the coordinates in (5.1) are much simpler to describe the bulk spacetime, it is not so for the brane itself. Indeed, whereas the brane is ‘at rest’ (at $y = 0$) in the GN coordinates, its position $R$ in the new coordinate system, will be in general time-dependent. This means that the brane is moving in the manifestly static reference frame (5.1). The trajectory of the brane can be defined by its coordinates $T(\tau)$ and $R(\tau)$ given as functions of a parameter $\tau$. Choosing $\tau$ to be the proper time imposes the condition

$$g_{ab}u^a u^b = -f \dot{T}^2 + \frac{\dot{R}^2}{f} = -1,$$

where $u^a = (\dot{T}, \dot{R})$ is the brane velocity and a dot stands for a derivative with respect to the parameter $\tau$. The normalization condition (5.3) yields $\dot{T} = \sqrt{f + \dot{R}^2/f}$ and the components of the unit normal vector (defined such that $n_a u^a = 0$ and $n_a n^a = 1$) are, up to a sign ambiguity,

$$n_a = \left(\dot{R}, -\sqrt{f + \dot{R}^2/f}\right).$$

The four-dimensional metric induced in the brane worldsheet is then directly given by

$$ds^2 = -d\tau^2 + R(\tau)^2 d\Omega_4^2,$$

and it is clear that the scale factor of the brane, denoted $a$ previously, can be identified with the radial coordinate of the brane $R(\tau)$.

The dynamics of the brane is then obtained by writing the junction conditions for the brane in the new coordinate system (2.16). The ‘orthogonal’ components of the extrinsic curvature tensor are given by

$$K_{ij} = \frac{\sqrt{f + \dot{R}^2}}{R} g_{ij},$$

which, after insertion in the junction conditions (2.16), implies for a mirror symmetric brane (see Section 9.2 for an asymmetric brane)

$$\frac{\sqrt{f + \dot{R}^2}}{R} = \frac{\kappa^2}{6} \rho_b.$$  

After taking the square of this expression, substituting (5.2) and rearranging, we get

$$\frac{R^2}{R^2} = \frac{\kappa^4}{36 \rho_b^2} + \frac{A}{6} + \frac{C}{R^4} - \frac{k}{R^2},$$
which, upon the identification between $a$ and $R$, is exactly the Friedmann equation (2.20) obtained before. There is additional information in the ‘longitudinal’ part of the junction conditions. The ‘longitudinal’ component of the extrinsic curvature tensor is given by

$$K_{\tau\tau} \equiv K_{AB} u^A u^B = u^A u^B \nabla_A n_B = -n_B a^B,$$

(5.9)

where $a^B$ is the acceleration, i.e. $a^B = u^A \nabla_A u^B$. Since $u_B a^B = 0$, the acceleration vector is necessarily of the form $a^B = a n^B$, and $K_{\tau\tau} = -a$. Finally, using the fact that $\partial / \partial \tau$ is a Killing vector, one can easily show that $a = n^{-1} (du_T / d\tau)$. Substituting in the junction conditions, one finds

$$\frac{1}{R} \frac{d}{d\tau} \left( \sqrt{f + \dot{R}^2} \right) = -\frac{\kappa^2}{6} (2\rho_b + 3p_b)$$

(5.10)

Combining with the first relation (5.7), it is immediate to rewrite the above expression as the traditional cosmological conservation equation

$$\dot{\rho}_b + 3\frac{\dot{R}}{R} (\rho_b + p_b) = 0.$$  

(5.11)

One has thus established the complete equivalence between the two pictures: in the first, the brane sits at a fixed position in a GN coordinate system and while it evolves cosmologically the metric components evolve with time accordingly; in the second, one has a manifestly static bulk spacetime in which the brane is moving, its cosmological evolution being simply a consequence of its displacement in the bulk (phenomenon sometimes called ‘mirage cosmology’). Let us add that the metric (5.1) describes in principle only one side of the brane. In the case of a mirror symmetric brane, the complete spacetime is obtained by gluing along the brane worldsheet, two copies of a portion of Sch-AdS spacetime. For an asymmetric brane, one can glue two (compatible) portions from different Sch-AdS spacetimes, as illustrated in Section 9.2.

§6. Inflation in the brane

The archetypical scenario of nowadays early universe cosmology is inflation. Since the infancy of brane cosmology, brane inflation has attracted some attention. The simplest way to get inflation in the brane is to detune the brane tension from its Randall-Sundrum value (4.1), and to take it bigger so that the net effective four-dimensional cosmological constant is positive. This leads to an exponential expansion in the brane, as illustrated before in (3.11) (we take here $k = 0$ and $\mathcal{C} = 0$ for simplicity). In the GN coordinate system, the metric corresponding to this situation is given by the specialization of the bulk metric (4.25) to the case $\eta = \tilde{\eta}$. The metric components $a(t, y)$ and $n(t, y)$ are then separable, i.e. they can be written as

$$a(t, y) = a_0(t) \mathcal{A}(y), \quad n = \mathcal{A}(y),$$

(6.1)

with

$$\mathcal{A}(y) = \cosh \mu y - \left( 1 + \frac{\rho}{\sigma} \right) \sinh \mu |y|.$$  

(6.2)
Of course, this model is too naive for realistic cosmology since brane inflation would never end. Like in standard cosmology, one can replace the cosmological constant, or here the deviation of the brane tension from its RS value, by a scalar field $\phi$ whose potential $V(\phi)$ behaves, during slow-roll motion, like an effective tension. The simplest scenario is to consider a four-dimensional scalar field confined to the brane. The brane cosmology formulas established above then apply, provided one substitutes for the energy density $\rho$ and pressure $p$ the appropriate expressions for a scalar field, namely

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi).$$

Like in the analysis for ordinary matter, the brane inflationary scenarios will be divided into two categories, according to the typical value of the energy density during inflation:

- high energy brane inflation if $\rho_\phi > \sigma$
- low energy brane inflation if $\rho_\phi \ll \sigma$, in which case the scenario is exactly similar to four-dimensional inflation.

New features appear for the high energy scenarios: for example, it is easier to get inflation because the Hubble parameter is bigger than the standard one, producing a higher friction on the scalar field. It enables inflation to take place with potentials usually too steep to sustain it. Because of the modified Friedmann equation, the slow-roll conditions are also changed.

For high energy inflation, the predicted spectra for scalar and tensor perturbations are also modified. It has been argued that, since the scalar field is intrinsically four-dimensional, the modification of the scalar spectrum is due simply to the change of the background equations of motion. However, the gravitational wave spectrum requires more care because the gravitational waves are five-dimensional objects. This question will be treated in detail in the next section. Let us also mention another type of inflation in brane models, based by a bulk five-dimensional scalar field that induces inflation within the brane.

One of the main reasons to invoke inflation in ordinary four-dimensional cosmology is the horizon problem of the standard big bang model, i.e. how to get a quasi-homogeneous CMB sky when the Hubble radius size at the time of last scattering corresponds to an angular scale of one degree. Rather than adapt inflation in brane cosmology, one may wonder whether it is possible to solve the horizon problem altogether without using inflation. A promising feature of brane cosmology in this respect is the possibility for a signal to propagate more rapidly in the bulk than along the brane, thus modifying ordinary causality. This property can also be found in non cosmological brane models, usually called Lorentz violation. Unfortunately, in the context of the Sch-AdS brane models presented here, a quantitative analysis (for $C = 0$) has shown that the difference between the gravitational wave horizon (i.e. for signals propagating in the bulk) and the photon horizon (i.e. for signals confined to the brane) is too small to be useful as an alternative solution to the horizon problem. Note also that the homogeneity problem of standard cosmology, i.e. why the universe was so close to homogeneity in its infancy, might
appear more severe in the braneworld context, where one must explain both the homogeneity along the ordinary spatial dimensions and the inhomogeneity along the fifth dimension.

§ 7. Cosmological perturbations

With the homogeneous scenario presented in section 4 as a starting point, one would like to explore the much richer, and much more difficult, question of cosmological perturbations and, in particular, investigate whether brane cosmology leads to new effects that could be tested in the forthcoming cosmological observations, in particular of the anisotropies of the Cosmic Microwave Background (CMB). Brane cosmological perturbations is a difficult subject and although there are now many published works with various formalisms on this question, no observational signature has yet been predicted. Rather than entering into the technicalities of the subject, for which the reader is invited to consult the original references, this section will try to summarize a few results concerning two different, but illustrative, aspects of perturbations: on one hand, the evolution of scalar type perturbations on the brane; on the other hand, the production of gravitational waves from quantum fluctuations during an inflationary phase in the brane.

7.1. Scalar cosmological perturbations

In a metric-based approach, there are various choices for the gauge in which the metric perturbations are defined. We will choose here a GN gauge, which has the advantage that the perturbed brane is still positioned at $y = 0$ and that only the four-dimensional part of the metric is perturbed. One can then immediately identify the value at $y = 0$ of the metric perturbations with the usual cosmological metric perturbations defined by a brane observer. The most general metric with linear scalar perturbations about a FLRW brane is

$$g_{\mu\nu} = \begin{pmatrix} -(1 + 2A) & aB_{ij} \\ aB_{ij} & a^2 \left\{ (1 + 2\mathcal{R})\gamma_{ij} + 2E_{ij} \right\} \end{pmatrix},$$

(7.1)

where $a(t)$ is the scale factor and a vertical bar denotes the covariant derivative of the three-dimensional metric $\gamma_{ij}$.

The perturbed energy-momentum tensor for matter on the brane, with background energy density $\rho$ and pressure $P$, can be given as

$$\tau^\mu_\nu = \begin{pmatrix} -(\rho + \delta\rho) & a(\rho + P)(v + B)_{ij} \\ -a^{-1}(\rho + P)v^i & (P + \delta P)\delta_{ij} + \delta\pi^i_j \end{pmatrix},$$

(7.2)

where $\delta\pi^i_j = \delta\pi^k_{ij} - \frac{1}{3}\delta^i_j\delta\pi^k|_k$ is the tracefree anisotropic stress perturbation. The
perturbed quadratic energy-momentum tensor, defined in (4.11), is
\[\Pi^\mu_\nu = \frac{\rho}{12} \begin{pmatrix} -\rho + 2\delta \rho & 2a(\rho + P)(v + B)_{ij} \\ -2a^{-1}(\rho + P)v^{|i} & \{2P + \rho + 2(1 + P/\rho)\delta \rho + 2\delta P\} \delta^i_j - (1 + 3P/\rho)\delta \pi^i_j \end{pmatrix}.\]

The last term on the right hand side of the effective four-dimensional Einstein equations (4.10) is the projected Weyl tensor \(E^\mu_\nu\). Although it is due to the effect of bulk metric perturbations not defined on the brane, one can parametrize it as an effective energy-momentum tensor (4.14)
\[T^\mu_\nu_{Weyl} = \begin{pmatrix} -(\rho_W + \delta \rho_W) & a\delta q_W^{|i} \\ -a^{-1}\delta q_W^{|i} + a^{-1}(\rho_W + P_W)B^{|i} & (P_W + \delta P_W)\delta^i_j + \delta \pi_W^i_j \end{pmatrix}.
\]

With these definitions, one can write explicitly the perturbed effective Einstein equations on the brane, which will look exactly as the four-dimensional ones for the geometrical part but with extra terms due to \(\Pi^\mu_\nu\) and \(T^\mu_\nu_{Weyl}\) for the matter part. In addition to the effective four-dimensional Einstein equations, the five-dimensional Einstein equations also provide three other equations. Two of them are equivalent to the conservation of the matter energy and momentum on the brane, i.e. of the tensor \(\tau^\mu_\nu\). The final one yields an equation of state for the Weyl fluid, which in the 4-dimensional equations follows from the symmetry properties of the projected Weyl tensor, requiring \(P_W = \frac{1}{3}\rho_W\) in the background and \(\delta P_W = \frac{1}{3}\delta \rho_W\) at first order. The equations of motion for the effective energy and momentum of the projected Weyl tensor are provided by the 4-dimensional contracted Bianchi identities, which are \textit{intrinsically four-dimensional}, only being defined on the brane and not part of the five-dimensional field equations. The contracted Bianchi identities (\(\nabla_\mu G^\mu_\nu = 0\)) and energy-momentum conservation for matter on the brane (\(\nabla_\mu \tau^\mu_\nu = 0\)) ensure, using Eq. (4.10), that
\[\nabla_\mu E^\mu_\nu = \kappa^4 \nabla_\mu \Pi^\mu_\nu.
\]

In the background, this tells us that \(\rho_W\) behaves like radiation, as we knew already, and for the first-order perturbations we have
\[\delta \rho_W + 4H\delta \rho_W + 4\rho_W \dot{\mathcal{R}} + a^{-1} \left[\nabla^2 \delta q_W + \frac{4}{3}\rho_W \nabla^2 \left(\frac{1}{4}\dot{E} - B\right)\right] = 0.\]

This means that the effective energy of the projected Weyl tensor is conserved independently of the quadratic energy-momentum tensor. The only interaction is a momentum transfer \(\bar{\rho}\mathcal{E}\), as shown by the perturbed momentum conservation equation
\[\delta q_W + 4H\delta q_W + a^{-1} \left[\frac{4}{3}\rho_W A + \frac{1}{3}\delta \rho_W + \frac{2}{3}(\nabla^2 + 3k)\delta \pi_W \right] = \frac{\rho + P}{a\sigma} \left[\delta \rho - 3Ha\delta q - (\nabla^2 + 3k)\delta \pi\right].\]
where the right hand side represents the momentum transfer from the quadratic energy-momentum tensor.

It is also possible to construct gauge-invariant variables corresponding to the curvature perturbation on hypersurfaces of uniform density, both for the brane matter energy density and for the total effective energy density (including the quadratic terms and the Weyl component). These quantities are extremely useful because their evolution on scales larger than the Hubble radius can be solved easily. However, their connection to the large-angle CMB anisotropies involves the knowledge of anisotropic stresses due to the bulk metric perturbations. This means that for a quantitative prediction of the CMB anisotropies, even at large scales, one needs to determine the evolution of the bulk perturbations.

In summary, we have obtained a set of equations for the brane linear perturbations, where one recognizes the ordinary cosmological equations but modified by two types of corrections:

- modification of the homogeneous background coefficients due to the additional $\rho^2$ terms in the Friedmann equation. These corrections are negligible in the low energy regime $\rho \ll \sigma$.
- presence of source terms in the equations. These terms come from the bulk perturbations and cannot be determined solely from the evolution inside the brane. To determine them, one must solve the full problem in the bulk (which also means to specify some initial conditions in the bulk). In the effective four-dimensional perturbation equations, these terms from the fifth dimension appear like external source terms, in a way somewhat similar to the case of “active seeds” due to topological defects.

7.2. Inflationary production of gravitational waves

Let us turn now to another facet of the brane cosmological perturbations: their production during a brane inflationary phase. We concentrate on the tensor perturbations, which are subtler than the scalar perturbations, because gravitational waves have an extension in the fifth dimension. The brane gravitational waves can be defined by a perturbed metric of the form

$$ds^2 = -n^2 dt^2 + a^2 \left[ \delta_{ij} + E_{ij}^{TT} \right] dx^i dx^j + dy^2,$$

(7.8)

where the ‘TT’ stands for transverse traceless. The linearized Einstein equations for the metric perturbations give a wave equation, which reads in Fourier space ($E_{ij} = E e^{i\vec{k} \cdot \vec{x}} e_{ij}$)

$$\ddot{E} + 3H_0 \dot{E} + \frac{k^2}{a_0^2} E = \mathcal{A}^2 E'' + 4 \mathcal{A} \dot{\mathcal{A}} E'. $$

(7.9)

where $\mathcal{A}$ is defined in (6.2). This equation being separable, one looks for solutions $E = \psi_m(t) e_m(y)$, where the time-dependent part obeys an ordinary FLRW Klein-Gordon equation while the $y$-dependent part must satisfy the Schrödinger type equation

$$\frac{d^2 \psi_m}{dz^2} - V(z) \psi_m = -m^2 \psi_m, $$

(7.10)
after introducing the new variable \( z - z_b = \int_0^y d\tilde{y}/A(\tilde{y}) \) (with \( z_b = H_0^{-1} \sinh^{-1}(H_0/\mu) \)) and the new function \( \Psi_m = A^{3/2} \xi_m \). The potential is given by

\[
V(z) = \frac{15H_0^2}{4 \sinh^2(H_0 z)} + \frac{3}{2}H_0^2 - 3\mu \left[ 1 + \frac{\rho}{\sigma} \right] \delta(z - z_b) .
\]  

(7.11)

The non-zero value of the Hubble parameter signals the presence of a gap\(^{[3]}\) between

\[
\text{Fig. 2. Potential for the graviton modes in a de Sitter brane}
\]

the zero mode \((m = 0)\) and the continuum of Kaluza-Klein modes \((m > 0)\). The zero mode corresponds to \( E_0 = \text{const} \) and the constant is determined by the normalization

\[
2 \int_{z_b}^{\infty} |\Psi_0|^2 dz = 1 .
\]

One finds

\[
E_0 = \sqrt{\mu} F(H_0/\mu) , \quad F(x) = \left\{ \sqrt{1 + x^2} - x^2 \ln \left[ \frac{1}{x} + \sqrt{1 + \frac{1}{x^2}} \right] \right\}^{-1/2} .
\]  

(7.12)

Asymptotically, \( F \simeq 1 \) at low energies, i.e. for \( H_0 \ll \mu \), and \( F \simeq \sqrt{3H_0/(2\mu)} \) at very high energies, i.e. for \( H_0 \gg \mu \). One can then evaluate the vacuum quantum fluctuations of the zero mode by using the standard canonical quantization. To do this explicitly, one writes the five-dimensional action for gravity at second order in the perturbations. Keeping only the zero mode and integrating over the fifth dimension, one obtains

\[
S_g = \frac{1}{8\kappa^2} \sum_{+,x} \int d\eta d^3 k a_0^2 \left[ \left( \frac{d\varphi_0}{d\eta} \right)^2 + k^2 \varphi_0^2 \right] .
\]  

(7.13)

This has the standard form of a massless graviton in four-dimensional cosmology, apart from the overall factor \( 1/8\kappa^2 \) instead of \( 1/8\kappa^4 \). It follows that quantum fluctuations in each polarization, \( \varphi_0 \), have an amplitude of \( 2\kappa(H_0/2\pi) \) on super-horizon scales. Quantum fluctuations on the brane at \( y = 0 \), where \( E_0 = E_0 \varphi_0 \), thus have the typical amplitude\(^{[4]}\)

\[
\frac{1}{2\kappa_4} \delta E_{\text{brane}} = \left( \frac{H_0}{2\pi} \right) F(H_0/\mu) .
\]  

(7.14)
The same result can be obtained in a bulk-based approach\textsuperscript{[3]}. At low energies, $F = 1$ and one recovers exactly the usual four-dimensional result but at higher energies the multiplicative factor $F$ provides an enhancement of the gravitational wave spectrum amplitude with respect to the four-dimensional result. However, comparing this with the amplitude for the scalar spectrum\textsuperscript{[38]}, one finds that, at high energies ($\rho \gg \sigma$), the tensor over scalar ratio is in fact suppressed with respect to the four-dimensional ratio. An open question is how the gravitational waves will evolve during the subsequent cosmological phases, the radiation and matter eras.

\section*{§ 8. Non-empty bulk}

Out of simplicity, a majority of works in brane cosmology have focused on a brane-universe embedded in a five-dimensional empty bulk, i.e. with only gravity propagating in the bulk. But many works have also considered extra fields in the bulk, either motivated by M/string theory\textsuperscript{[64, 65, 66]}, or simply in a purely phenomenomogical approach\textsuperscript{[67, 68]}, in some cases with the objective to solve some standing problems such as the cosmological constant problem\textsuperscript{[29]} or more specific problems like the question of the radion stabilization\textsuperscript{[70, 71]}. The most common extra field considered in the literature is not surprisingly the scalar field although one can find other generalizations such as including gauge fields\textsuperscript{[72, 73]}. We will consider here only models with a bulk scalar field and start from the action

$$S = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} (\nabla A_\phi)(\nabla A_\phi - V(\phi)) \right] + \int_{\text{brane}} d^4x L_m[\varphi_m, \tilde{h}_{\mu\nu}],$$

(8.1)

where it is assumed that the four-dimensional metric $\tilde{h}_{\mu\nu}$, which is minimally coupled to the four-dimensional matter fields $\varphi_m$ in the brane, is conformally related to the induced metric $h_{\mu\nu}$, i.e.

$$\tilde{h}_{\mu\nu} = e^{2\xi(\phi)} h_{\mu\nu}. \quad (8.2)$$

In the terminology of scalar-tensor theories, one would say that $h_{\mu\nu}$ corresponds to the Einstein frame while $\tilde{h}_{\mu\nu}$ corresponds to the Jordan frame. One can also define two brane matter energy-momentum tensors, $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$, respectively with respect to $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu},$

$$T_{\mu\nu} = \frac{2}{\sqrt{-h}} \frac{\delta L_m}{\delta h_{\mu\nu}}, \quad \tilde{T}_{\mu\nu} = \frac{2}{\sqrt{-\tilde{h}}} \frac{\delta L_m}{\delta \tilde{h}_{\mu\nu}},$$

(8.3)

and they are related by $\tilde{T}_\nu^\mu = e^{-4\xi} T_\nu^\mu$. The corresponding energy densities, $\rho$ and $\tilde{\rho}$, and pressures, $p$ and $\tilde{p}$, will be related by the same factor $e^{-4\xi}$. Variation of the action (8.1) with respect to the metric $g_{AB}$ yields the five-dimensional Einstein equations (2.5), where, in addition to the (distributional) brane energy-momentum, there is now the scalar field energy-momentum tensor

$$T^\phi = \partial_A \phi \partial_B \phi - g_{AB} \left[ \frac{1}{2} (\nabla C \phi)(\nabla C \phi) + V(\phi) \right].$$

(8.4)
Variation of (8.1) with respect to $\phi$ yields the equation of motion for the scalar field, which is of the Klein-Gordon type, with a distributional source term since the scalar field is coupled to the brane via $\tilde{h}_{\mu\nu}$. This implies that there is an additional junction condition, now involving the scalar field at the brane location and which is of the form

$$\left[n^A \partial_A \phi\right] = -\xi' e^{\xi T} = -\xi'T,$$

(8.5)

where $T = -\rho + 3P$ is the trace of the energy-momentum tensor. In summary, we now have a much more complicated system than for the empty bulk, since, in addition to the brane dynamics, one must solve for the dynamics of the scalar field. Although a full treatment would require a numerical analysis, a few interesting analytical solutions have been derived in the literature\cite{74,75,76,77,78,79}, out of which we give below two examples.

8.1. Moving brane in a static bulk

Taking example on the empty bulk case, where the cosmology in the brane is induced by the motion of the brane in a static bulk, it seems natural to look for brane cosmological solutions corresponding to a moving brane in a static bulk geometry with a static scalar field. In the case of an exponential potential,

$$V(\phi) = V_0 \exp\left(-\frac{2}{\sqrt{3}} \lambda \kappa \phi\right),$$

(8.6)

it turns out that there exists a simple class of static solutions\cite{80} (the full set of static solutions is larger\cite{78}), which correspond to a metric of the form

$$ds^2 = -h(R)dT^2 + \frac{R^{2\lambda^2}}{h(R)}dR^2 + R^2 d\vec{x}^2,$$

(8.7)

with

$$h(R) = -\frac{\kappa^2 V_0/6}{1 - (\lambda^2/4)} R^2 - CR^{4\lambda^2-2},$$

(8.8)

where $C$ is an arbitrary constant, and a scalar field given by

$$\frac{\kappa}{\sqrt{3}} \phi = \lambda \ln(R).$$

(8.9)

The next step is then to insert a moving (mirror-symmetric) brane in the above bulk configuration\cite{83,84}. This is possible if the three junction conditions, two for the metric and one for the scalar field, are satisfied. The two metric junction conditions are easily derived by repeating the procedure given in section 5 for the metric (8.7) and will be generalizations of (5.7) and (5.10). The condition on the scalar field comes from (8.5).

These three junction conditions can be rearranged to give the following three relations:

- a generalized Friedmann equation,

$$H^2 = \frac{\kappa^2}{36} \rho^2 - \frac{h(R)}{R^{2+2\lambda^2}} = \frac{\kappa^2}{36} \rho^2 + \frac{\kappa^2 V_0/6}{1 - (\lambda^2/4)} R^{-2\lambda^2} + CR^{-4-\lambda^2},$$

(8.10)
• a (non-) conservation equation for the energy density,
\[ \dot{\rho} + 3H(\rho + p) = (1 - 3w)\xi'\rho\dot{\phi}, \quad (8.11) \]
• a constraint on the brane matter equation of state, which must be related to the conformal coupling according to the expression
\[ 3w - 1 = \frac{\kappa}{\sqrt{3}} \frac{\lambda}{\xi'}. \quad (8.12) \]
The latter constraint can accommodate a constant \( w \), provided the conformal coupling is of the form \( \xi(\phi) = \xi_1\phi \), but it is not very sensible for realistic cosmological scenarios. If there is no coupling between the brane and the scalar field, i.e. \( \xi' = 0 \), then \( \lambda \) must vanish and one recovers the familiar brane solutions with an empty bulk where \( w \) is unconstrained. It is also worth noticing that, when going to the Jordan frame, i.e. using the scale factor \( \tilde{a} = e^{\xi}a \) and the time \( \tilde{t} \), such that \( d\tilde{t} = e^{\xi}dt \), the conservation equation for the energy density transforms into its familiar form
\[ \frac{d\tilde{\rho}}{d\tilde{t}} + 3\tilde{H}(\tilde{\rho} + \tilde{p}) = 0. \quad (8.13) \]

8.2. Non-static bulk solutions

In contrast with the empty bulk case, where the required symmetries impose the bulk to be static, there now exist cosmological solutions with non static bulk configurations because the generalized Birkhoff’s theorem no longer applies. The example below provides an explicit illustration of this fact. Still for an exponential potential
\[ V = V_0 \exp \left( \alpha \frac{\kappa}{\sqrt{3}} \phi \right), \quad (8.14) \]
a solution of the Einstein/Klein-Gordon bulk equations is given by the metric
\[ ds^2 = \frac{18}{\kappa^2 V_0} e^{\alpha^2 T} e^{-\alpha \sqrt{\alpha^2 - 4R}} \left(-dT^2 + dR^2\right) + e^{4T} dx^2, \quad (8.15) \]
and the scalar field configuration
\[ \phi = \alpha^2 T - \alpha \sqrt{\alpha^2 - 4R}. \quad (8.16) \]
It is possible to embed a (mirror) symmetric brane in this spacetime, with an equation of state \( p_0 = w\rho_0 \) ( \( w \) constant). This leads, for a brane observer in the Einstein frame, to a cosmology with the power-law expansion
\[ a(t) \sim t^p, \quad p = \frac{1}{3(1 + w)} \left[ 1 + \frac{4(2 + 3w)}{\alpha^2} \right], \quad (8.17) \]
with \( p \) between the values \( p = 1 \) and \( p = 1/(3(1 + w)) \), since the solution is defined only for \( \alpha^2 \geq 4 \); the scalar field in the brane is given by
\[ \frac{\kappa}{\sqrt{3}} \phi_0 = -\frac{6\alpha(2 + 3w)}{\alpha^2 + 4(2 + 3w)} \ln a \quad (8.18) \]
and the energy density is proportional to Hubble parameter,

$$\frac{\kappa^2}{6} \rho_b = \frac{\alpha \sqrt{\alpha^2 - 4}}{2(\alpha^2 + 4(2 + 3w))} H.$$  \hfill (8.19)

It is then straightforward to go to the Jordan frame. As in the previous example, these solutions are very contrived and, moreover, they can be of interest only for the very early universe because of their Brans-Dicke nature.

§9. Multi-brane cosmology

The last part of this review will be devoted to scenarios where several branes are involved. So far, we have concentrated our attention on a single brane, supposed to correspond to our accessible universe. Nothing forbids however the presence of other branes which, because of the required cosmological symmetries, would be ‘parallel’ or ‘concentric’ with respect to our brane-universe. A very important property in the case of an empty bulk spacetime (apart the branes themselves of course), which is not always well appreciated, is that the cosmology in one brane is completely independent of the evolution of any other brane. The only influence of another brane on, say, our brane-universe is contained in the single constant $C$, which appears in the Friedmann equation (2.20). But $C$ is just determined at any given time by a kind of compatibility condition between any two branes and the empty spacetime separating them. This compatibility condition is then automatically satisfied at subsequent times whatever the evolution of each brane. Below, we first study the cosmology of two parallel (mirror-symmetric) branes. We then consider the general problem of collisions in a multi-brane system.

9.1. Two-brane system

Sometimes motivated by the old prejudice that extra dimensions must be compact (although this is no longer necessary as illustrated by the Randall-Sundrum model), a lot of attention has been paid to two-brane systems. This is in particular the case with the first Randall-Sundrum model, which was supposed to solve the hierarchy problem, although this model is incompatible with our gravity and should be completed for instance with a stabilization mechanism in order to be phenomenologically valid. This subsection will present a few results concerning systems with two parallel (planar) branes separated by vacuum (with a negative cosmological constant if one wishes to recover standard cosmology in our brane). The two branes will be assumed to be mirror symmetric.

We will adopt the brane point of view and use a GN coordinate system based on our brane-universe. As already shown, the metric then reads

$$ds^2 = -n(t, y)^2 dt^2 + a(t, y)^2 d\Sigma^2 + dy^2,$$  \hfill (9.1)

where the explicit expressions for $a(t, y)$ and $n(t, y)$ are given in (4.22) and (4.23). While $y = 0$ is the position of our brane-universe, the position of the second brane, at any time $t$, will be expressed in terms of its coordinate $y = R(t)$. It represents the relative distance between the two branes, as measured by an observer at rest.
Brane cosmology

with respect to our brane-universe, and will be called the *cosmological radion*. Using the definition of the extrinsic curvature tensor (2.14), one can compute the junction conditions for the second brane $82)$. The ‘orthogonal’ components give

$$\frac{a'}{a} + \frac{\dot{a}}{a} \frac{\dot{\mathcal{R}}}{n^2} = \frac{\kappa^2}{6} \rho_1 \left(1 - \frac{\mathcal{R}^2}{n^2}\right)^{1/2},$$

whereas the ‘longitudinal’ component yields

$$\frac{\ddot{\mathcal{R}}}{n^2} + \frac{n'}{n} \left(1 - 2 \frac{\dot{\mathcal{R}}^2}{n^2}\right) = -\frac{\kappa^2}{6} (2 \rho_1 + 3 P_1) \left(1 - \frac{\mathcal{R}^2}{n^2}\right)^{3/2}.$$

The quantities $\rho_1$ and $P_1$ refer to the total energy density and pressure of the second brane. All the $y$-dependent quantities in the above equations are of course evaluated at $y = \mathcal{R}(t)$.

If the second brane is at rest, i.e. $\dot{\mathcal{R}} = 0$, then the junction conditions are, with the important exception of the sign, the same relations as written before in (2.17). The equilibrium condition thus consists of two relations relating the energy densities and pressures in the two branes, involving as well the value of the radion $\mathcal{R}_{eq}$. They can be rewritten in the very simple form

$$\theta_0 + \theta_1 = \tilde{\theta}_0 + \tilde{\theta}_1 = \mu \mathcal{R}_{eq},$$

where $\eta_i = \tanh \theta_i$ and $\tilde{\eta}_i = \tanh \tilde{\theta}_i$ ($\eta$ and $\tilde{\eta}$ have been defined in (4.26)). Note that in the Randall-Sundrum two-brane configuration, where the two branes are at rest relative to each other, $\theta_0 = \tilde{\theta}_0 = +\infty$ and $\theta_1 = \tilde{\theta}_1 = -\infty$, and $\mathcal{R}_{eq}$ can take any value.

If the second brane is not at rest, then the above relations (9.2) and (9.3) represent generalized junction conditions when the brane moves with respect to the coordinate frame. It is also instructive to consider linearized radion fluctuations about an equilibrium value, i.e. $\delta \mathcal{R} = \mathcal{R} - \mathcal{R}_{eq}$. Combining the two linearized junction conditions, one can obtain

$$\frac{\ddot{\mathcal{R}}}{n^2} + \frac{3 \dot{a}}{a} \frac{\dot{\mathcal{R}}}{n^2} + m_{eff}^2 \delta \mathcal{R} = -\frac{\kappa^2}{6} \delta T_1,$$

where the effective square mass is given by

$$m_{eff}^2 = \mu^2 \left(4 - 3 \eta_1^2 - \tilde{\eta}_1^2\right).$$

This equation for the radion allows one to make the connection with the traditional view that the radion should be seen, from the four-dimensional point of view, as a scalar field coupled to the trace of energy-momentum tensor. Note that a RS brane corresponds to a vanishing radion mass, whereas the radion is unstable for a de Sitter brane and stable for an AdS brane.

It is sometimes useful to derive from a model with extra-dimensions an effective four-dimensional theory by integrating out the extra degrees of freedom. The
braneworld models are however very particular, in contrast with the traditional Kaluza-Klein type models, because matter is confined on branes. Let us consider a cosmological two-brane system with two mirror symmetric branes. As stressed earlier, an important property at the homogeneous level is that the second brane does not influence our brane-universe (and reciprocally) with the exception of the constant $C$, which can be reinterpreted in terms of the second brane, but which does not change with time. This non-influence can seem at odds with the usual intuition built on an effective four-dimensional description, where the Friedmann equation is expected to involve the energy densities of both branes. We will give below a brief idea of how the two viewpoints can be reconciled.

In order to construct a four-dimensional effective action describing the two-brane system, one can start from the full five-dimensional action, assuming for simplicity $k = 0$ and $C = 0$, and that the matter in each brane is a pure tension, in which case both branes undergo inflation\(^8\). One substitutes the $y$-dependence of the bulk fields given in this case (with $k = 0$ and $C = 0$ for simplicity) by (6.1-6.2) and $n = N(t)A(y)$. The integration over the fifth dimension yields the following (homogeneous) four-dimensional action\(^9\)

$$S = \frac{1}{\kappa^2 \mu} \int d^4 x N a_0^3 \left[ (4) R \psi_1(\mathcal{R}) + 12 \mu^2 \psi_2(\mathcal{R}) + 6 \mu A_1^2 \frac{\dot{a}_0}{N a_0} \frac{\dot{\mathcal{R}}}{N} + 3 \mu A_1^3 \left( \frac{\dot{a}_0}{N a_0} + \mu A_1' \frac{\dot{\mathcal{R}}}{A_1 N} \right) \ln \left( \frac{N A_1 - \mathcal{R}}{N A_1 + \mathcal{R}} \right) + \kappa^2 \mu \sigma_0 - \kappa^2 \mu \sigma_1 A_1^4 \sqrt{1 - \frac{\mathcal{R}^2}{N^2 A_1^4}} \right],$$

(9.7)

with $A_1 \equiv A(\mu \mathcal{R})$ and $A_1' \equiv A'(\mu \mathcal{R})$, while the $\psi$'s are dimensionless functions of $\mu \mathcal{R}$ defined by

$$\psi_1(\mathcal{R}) = \int_0^{\mu \mathcal{R}} d\xi A (\xi)^2, \quad \psi_2(\mathcal{R}) = \int_0^{\mu \mathcal{R}} d\xi A (\xi)^2 \left( A_1' (\xi)^2 + A (\xi)^2 \right),$$

(9.8)

and $(4) R$ is the (homogeneous) four-dimensional Ricci scalar

$$(4) R = \frac{6}{N^2} \left( \frac{\ddot{a}_0^2}{a_0^2} + \frac{\ddot{a}_0}{a_0} - \frac{\dot{a}_0}{N} \frac{\dot{N}}{N} \right).$$

(9.9)

The variation with respect to $N$ of this action, yields a Friedmann-like equation which reads

$$H_0^2 \psi_1 + 2\mu^2 \psi_2 + \mu \left( H_0 + \mu \frac{A_1'}{A_1} \frac{\dot{\mathcal{R}}}{1 - (\mathcal{R}/A_1)^2} \right) \frac{A_1^2 \dot{\mathcal{R}}}{1 - (\mathcal{R}/A_1)^2} = \kappa^2 \mu \frac{\sigma_0}{6} + \kappa^2 \mu \frac{\sigma_1}{6} \sqrt{1 - (\mathcal{R}/A_1)^2} A_1^4.$$  

(9.10)

setting $N = 1$ after variation. What is worth noticing is that this equation involves the energy densities in a linear way, and not in a quadratic way as is characteristic of brane cosmology. The solution to this apparent paradox lies in the fact that
the equation also involves the radion. As seen above, the junction condition (9.2) for the second brane provides a relation between the energy densities of the two branes and the radion. It turns out that, using this junction condition, one can simultaneously get rid of the radion and of the energy density $\sigma_1$ and recover the ‘usual’ unconventional Friedmann equation quadratic in $\sigma_0$. But, if one ignores the junction condition, one will in general lose some information about the fifth dimensional aspect of the problem and find solutions which are not physical. This is only in the low energy limit, i.e. near the RS configuration, where the effective potential for the radion is quasi-flat, and for small radion velocities, that the four-dimensional action yields the correct solutions. One must be therefore very cautious when trying to infer the cosmological behaviour of an intrinsically five-dimensional model from its effective four-dimensional description.

9.2. Collision of branes

As soon as the spacetime contains several branes and that these branes move with respect to each other, they might collide. A fascinating possibility, which has been actively explored recently [4], [7], [8], is that the Big-Bang is such a brane collision. Rather than entering into the details of these various models, let us point out here a simple and general analysis of the collision of (parallel or concentric) branes separated by vacuum, i.e. branes separated by patches of Sch-AdS spacetimes (allowing for different Schwarzschild-type mass and cosmological constant in each region) with the metric [7]. Although we are interested here by 3-branes embedded in a five-dimensional spacetime, this analysis is immediately applicable to the case of $n$-branes moving in a $(n+2)$-dimensional spacetime, with the analogous symmetries.

To analyze the collision, it is convenient to introduce an angle $\alpha$, which characterizes the motion of the brane with respect to the coordinate system (5.1), defined by

$$\alpha = \sinh^{-1}(\epsilon \dot{R}/\sqrt{f}),$$

(9.11)

where $\epsilon = +1$ if $R$ decreases from “left” to “right”, $\epsilon = -1$ otherwise. Considering a collision involving a total number of $N$ branes, both ingoing and outgoing, thus separated by $N$ spacetime regions one can label alternately branes and regions by integers, starting from the leftmost ingoing brane and going anticlockwise around the point of collision (see Figure). The branes will thus be denoted by odd integers, $2k - 1$ ($1 \leq k \leq N$), and the regions by even integers, $2k$ ($1 \leq k \leq N$). Let us introduce, as before, the angle $\alpha_{2k-1|2k}$ which characterizes the motion of the brane $B_{2k-1}$ with respect to the region $R_{2k}$, and which is defined by

$$\sinh \alpha_{2k-1|2k} = \frac{\epsilon_{2k} \dot{R}_{2k-1}}{\sqrt{f_{2k}}}.$$  

(9.12)

Conversely, the motion of the region $R_{2k}$ with respect to the brane by the Lorentz angle $\alpha_{2k|2k-1} = -\alpha_{2k-1|2k}$. It can be shown that the junction conditions for the branes can be written in the form

$$\tilde{\rho}_{2k-1} \equiv \pm \frac{\kappa^2}{3} \rho_{2k-1} R = \epsilon_{2k} \sqrt{f_{2k}} \exp(\pm \alpha_{2k-1|2k}) - \epsilon_{2k-2} \sqrt{f_{2k-2}} \exp(\mp \alpha_{2k-2|2k-1}),$$

(9.13)
with the plus sign for ingoing branes \((1 \leq k \leq N_{\text{in}})\), the minus sign for outgoing branes \((N_{\text{in}} + 1 \leq k \leq N)\). An outgoing positive energy density brane thus has the same sign as an ingoing negative energy density brane.

The advantage of this formalism becomes obvious when one writes the geometrical consistency relation that expresses the matching of all branes and spacetime regions around the collision point. In terms of the angles defined above, it reads simply

\[
2N \sum_{i=1}^{2N} \alpha_{i|i+1} = 0. \tag{9.14}
\]

Moreover, introducing the generalized angles \(\alpha_{jj'} = \sum_{i=j}^{j'-1} \alpha_{i|i+1}\), if \(j < j'\), and \(\alpha_{jj'} = -\alpha_{jj'}\), the sum rule for angles \((9.14)\) combined with the junction conditions \((9.13)\) leads to the laws of energy conservation and momentum conservation. The energy conservation law reads

\[
\sum_{k=1}^{N} \tilde{\rho}_{2k-1} \gamma_{jj'} |2k-1| = 0, \tag{9.15}
\]

where \(\gamma_{jj'} \equiv \cosh \alpha_{jj'}\) corresponds to the Lorentz factor between the brane/region \(j\) and the brane/region \(j'\) and can be obtained, if \(j\) and \(j'\) are not adjacent, by combining all intermediary Lorentz factors (this is simply using the velocity addition rule of special relativity). The index \(j\) corresponds to the reference frame with respect
to which the conservation rule is written. Similarly, the momentum conservation law in the $j$-th reference frame can be expressed in the form

$$\sum_{k=1}^{N} \tilde{\rho}_{2k-1} \gamma_{2k-1;jj} \beta_{2k-1;jj} = 0,$$

with $\gamma_{jj'} \beta_{jj'} \equiv \sinh \alpha_{jj'}$. One thus obtains, just from geometrical considerations, conservation laws relating the brane energies densities and velocities before and after the collision point. These results apply to any collision of branes in vacuum, with the appropriate symmetries of homogeneity and isotropy. An interesting development would be to extend the analysis to branes with small perturbations and investigate whether one can find scenarios which can produce quasi-scale invariant adiabatic spectra, as seems required by current observations.

Acknowledgements

I would like to thank the organizers of the brane cosmology workshop at the Yukawa Institute for inviting me to a very stimulating meeting (and also for their financial support). Let me also thank Geneviève and Thierry Pichevin for their warm hospitality in Pospoder, where the first part of this review was written. I am also grateful to my new-born son Nathan for his approving silence during the last stage of this work.

References

[1] V.A. Rubakov, M.E. Shaposhnikov, Phys. Lett. B 125, 139 (1983); M. Visser, Phys. Lett. B 159, 22 (1985); E.J. Squires, Phys. Lett. B 167, 286 (1986); G.W. Gibbons, D.L. Wiltshire, Nucl. Phys. B 287, 717 (1987); K. Akama, Lect. Notes Phys. 176, 267 (1982) [arXiv:hep-th/0001113].
[2] P. Horava and E. Witten, Nucl. Phys. B 460, 506 (1996) [arXiv:hep-th/9510209]; Nucl. Phys. B 475, 94 (1996) [arXiv:hep-th/9603142].
[3] N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys.Lett. B429, (1998) 263; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B436, 257 (1998); N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. D59 (1999) 086004.
[4] C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner and H. E. Swanson, Phys. Rev. Lett. 86, 1418 (2001) [arXiv:hep-ph/0011011].
[5] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 3370 [arXiv:hep-ph/9905221].
[6] L. Randall and R. Sundrum, Phys. Rev. Lett. 83 (1999) 4690 [arXiv:hep-th/9906064].
[7] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485, 208 (2000) [arXiv:hep-th/0005016].
[8] C. Deffayet, Phys. Lett. B 502, 199 (2001) [arXiv:hep-th/0010186]; C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D 65, 044023 (2002) [arXiv:astro-ph/0105068].
[9] R. Maartens, arXiv:gr-qc/0101050.
[10] V. A. Rubakov, Phys. Usp. 44, 871 (2001) [Usp. Fiz. Nauk 171, 913 (2001)] [arXiv:hep-th/0104152].
[11] P. Kanti, I. I. Kogan, K. A. Olive and M. Pospelov, Phys. Rev. D 61, 106004 (2000) [arXiv:hep-ph/9912206].
[12] P. Mounaix and D. Langlois, Phys. Rev. D 65, 103523 (2002) [arXiv:hep-th/0206081].
[13] M. Cvetic and H. H. Soleng, Phys. Rept. 282, 159 (1997) [arXiv:hep-th/9604001].
[14] P. Binétruy, C. Deffayet and D. Langlois, Nucl. Phys. B 655 (2000) 269 [arXiv:hep-th/9905012].
[15] W. Israel, Nuovo Cim. B 44S10, 1 (1966) [Erratum-ibid. B 48, 463 (1966)].
[16] H. Stoica, S. H. Tye and I. Wasserman, Phys. Lett. B 482, 205 (2000) [arXiv:hep-th/0004120].
[17] R. A. Battye, B. Carter, A. Meninno and J. P. Uzan, Phys. Rev. D 64, 124007 (2001) [arXiv:hep-th/0100500].
[18] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B 477 (2000) 285 [arXiv:hep-th/9910219].
[19] N. Kaloper, Phys. Rev. D 60, 123506 (1999) [arXiv:hep-th/9905210].
[20] T. Nihei, Phys. Lett. B 465, 81 (1999) [arXiv:hep-th/090487].
[21] P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. 17, 4745 (2000) [arXiv:hep-th/0007177].
[22] J. Garriga and T. Tanaka, Phys. Rev. Lett. 84 (2000) 2778 [arXiv:hep-th/9911055].
[23] C. Csaki, M. Graesser, C. F. Kolda and J. Terning, Phys. Lett. B 462, 34 (1999) [arXiv:hep-ph/9906513]; J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. 83, 4245 (1999) [arXiv:hep-ph/9906528].
[24] E. E. Flanagan, S. H. Tye and I. Wasserman, Phys. Rev. D 62, 044039 (2000) [arXiv:hep-th/000498].
[25] T. Shiromizu, K. i. Maeda and M. Sasaki, Phys. Rev. D 62, 024012 (2000) [arXiv:gr-qc/0010076].
[26] K. Ichiki, M. Yahiro, T. Kajino, M. Orito and G. J. Matthews, Phys. Rev. D 66, 043521 (2002) [arXiv:astro-ph/0203272].
[27] S. Mukohyama, T. Shiromizu and K. i. Maeda, Phys. Rev. D 62, 024028 (2000) [Erratum-ibid. D 63, 029901 (2000)] [arXiv:hep-th/9912287].
[28] H. A. Chamblin and H. S. Reall, Nucl. Phys. B 562, 133 (1999) [arXiv:hep-th/9903225].
[29] A. Kehagias and E. Kiritsis, JHEP 9911, 022 (1999) [arXiv:hep-th/9910174].
[30] D. J. Chung and J. March-Russell, Nucl. Phys. B 604, 312 (2001) [arXiv:hep-th/0012143].
[31] D. Langlois, L. Sorbo and M. Rodriguez-Martinez, arXiv:hep-th/0206146, to appear in Phys. Rev. Lett.
[32] P. Kraus, JHEP 9912 (1999) 011 [arXiv:hep-th/9910149].
[33] D. Ida, JHEP 0009, 014 (2000) [arXiv:gr-qc/9912002].
[34] S. Mukohyama, T. Shiromizu and K. i. Maeda, Phys. Rev. D 62, 024028 (2000) [Erratum-ibid. D 63, 029901 (2000)] [arXiv:hep-th/9912287].
Brane cosmology

[50] C. van de Bruck, M. Dorca, R. H. Brandenberger and A. Lukas, Phys. Rev. D 62, 123515 (2000) [arXiv:hep-th/0005032].

[51] K. Koyama and J. Soda, Phys. Rev. D 62, 123502 (2000) [arXiv:hep-th/0005239]; Phys. Rev. D 65, 023514 (2002) [arXiv:hep-th/0108001].

[52] C. van de Bruck, M. Dorca, C. J. Martins and M. Parry, Phys. Lett. B 495, 183 (2000) [arXiv:hep-th/0009056].

[53] D. Langlois, Phys. Rev. Lett. 86, 2212 (2001) [arXiv:hep-th/0010063].

[54] H. A. Bridgman, K. A. Malik and D. Wands, Phys. Rev. D 63, 084012 (2001) [arXiv:hep-th/0010133].

[55] N. Deruelle, T. Dolezel and J. Katz, Phys. Rev. D 63, 083513 (2001) [arXiv:hep-th/0010215].

[56] C. van de Bruck, M. Dorca, C. J. Martins and M. Parry, Phys. Lett. B 495, 183 (2000) [arXiv:hep-th/0009056].

[57] D. Langlois, R. Maartens, M. Sasaki and D. Wands, Phys. Rev. D 63, 084009 (2001) [arXiv:hep-th/0012044].

[58] H. A. Bridgman, K. A. Malik and D. Wands, Phys. Rev. D 65, 043502 (2002) [arXiv:astro-ph/0107245].

[59] D. S. Gorbunov, V. A. Rubakov and S. M. Sibiryakov, JHEP 0110, 015 (2001) [arXiv:hep-th/0108017].

[60] B. Leong, P. Dunsby, A. Challinor and A. Lasenby, Phys. Rev. D 65, 104012 (2002) [arXiv:gr-qc/0111033].

[61] C. Deffayet, arXiv:hep-th/0205084.

[62] A. Riazuelo, F. Vernizzi, D. Steer and R. Durrer, arXiv:hep-th/0205220.

[63] J. Garriga and M. Sasaki, Phys. Rev. D 62, 043523 (2000) [arXiv:hep-th/9912118].

[64] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram Phys. Rev. D 59, 086001 (1999) [arXiv:hep-th/9803239]; Nucl. Phys. B 552, 246 (1999) [arXiv:hep-th/9806051]; J. A. Lukas, B. A. Ovrut and D. Waldram, Phys. Rev. D 60, 086001 (1999) [arXiv:hep-th/9806024].

[65] H. S. Reall, Phys. Rev. D 59, 103506 (1999) [arXiv:hep-th/9809193].

[66] J. E. Lidsey, Phys. Rev. D 64, 063507 (2001) [arXiv:hep-th/0106081].

[67] K. i. Maeda and D. Wands, Phys. Rev. D 62, 124009 (2000) [arXiv:hep-th/0008188].

[68] A. Mennim and R. A. Battye, Class. Quant. Grav. 18, 2171 (2001) [arXiv:hep-th/0008192].

[69] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and R. Sundrum, Phys. Lett. B 480, 193 (2000) [arXiv:hep-th/0001197]; S. Kachru, M. B. Schulz and E. Silverstein, Phys. Rev. D 62, 045021 (2000) [arXiv:hep-th/0001206]; C. Csaki, J. Erlich, C. Grojean and T. J. Hollowood, Nucl. Phys. B 584, 359 (2000) [arXiv:hep-th/0004133]; C. Csaki, J. Erlich and C. Grojean, Nucl. Phys. B 604, 312 (2001) [arXiv:hep-th/0012143]; D. Youm, Nucl. Phys. B 589, 315 (2000) [arXiv:hep-th/9904103]; S. Forste, Z. Lalak, S. Lavignac and H. P. Nilles, Phys. Lett. B 481, 360 (2000) [arXiv:hep-th/0002164]; S. Forste, Z. Lalak, S. Lavignac and H. P. Nilles, JHEP 0009, 034 (2000) [arXiv:hep-th/0006139]; J. M. Cline and H. Firouzjahi, Phys. Lett. B 495, 271 (2001) [arXiv:hep-th/0008187].

[70] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. 83 (1999) 4922 [arXiv:hep-ph/9907447].

[71] P. Kanti, J. L. Kogan, K. A. Olive and M. Pospelov, Phys. Rev. D 61, 106004 (2000) [arXiv:hep-ph/9912269]; C. Csaki, M. Graesser, L. Randall and J. Terning, Phys. Rev. D 62, 045015 (2000) [arXiv:hep-ph/9911407]; J. Lesgourgues, S. Pastor, M. Peloso and L. Sorbo, Phys. Lett. B 489, 411 (2000) [arXiv:hep-ph/0004082].

[72] B. Carter and J. P. Uzan, Nucl. Phys. B 606, 45 (2001) [arXiv:gr-qc/0101010].

[73] C. Grojean, F. Quevedo, G. Tasinato and I. Zavala C., JHEP 0108, 005 (2001) [arXiv:hep-th/0106120].

[74] P. Brax and A. C. Davis, Phys. Lett. B 497, 289 (2001) [arXiv:hep-th/0011043]; P. Brax, C. van de Bruck and A. C. Davis, JHEP 0110, 026 (2001) [arXiv:hep-th/0108215].

[75] A. Feinstein, K. E. Kunze and M. A. Vazquez-Mozo, Phys. Rev. D 64, 084015 (2001) [arXiv:hep-th/0105183]; K. E. Kunze and M. A. Vazquez-Mozo, Phys. Rev. D 65, 044002 (2002) [arXiv:hep-th/0109083].

[76] D. Langlois and M. Rodríguez-Martínez, Phys. Rev. D 64, 123507 (2001) [arXiv:hep-th/0106245].

[77] S. C. Davis, JHEP 0203, 054 (2002) [arXiv:hep-th/0106271]; JHEP 0203, 058 (2002) [arXiv:hep-th/0203058].
[78] C. Charmousis, Class. Quant. Grav. 19, 83 (2002) [arXiv:hep-th/0107126].
[79] E. E. Flanagan, S. H. Tye and I. Wasserman, Phys. Lett. B 522, 155 (2001) [arXiv:hep-th/0100074].
[80] R. G. Cat, J. Y. Ji and K. S. Soh, Phys. Rev. D 57, 6547 (1998) [arXiv:gr-qc/9708063].
[81] T. Tanaka and X. Montes, Nucl. Phys. B 582, 259 (2000) [arXiv:hep-th/0001092].
[82] P. Binétruy, C. Deffayet and D. Langlois, Nucl. Phys. B 615 (2001) 219 [arXiv:hep-th/0101234].
[83] C. Charmousis, R. Gregory and V. A. Rubakov, Phys. Rev. D 62 (2000) 067505 [arXiv:hep-th/9912160]; Z. Chacko and P. J. Fox, Phys. Rev. D 64 (2001) 024015 [arXiv:hep-th/0010234]; U. Gen and M. Sasaki, arXiv:gr-qc/0201031.
[84] H. B. Kim and H. D. Kim, Phys. Rev. D 61, 064003 (2000) [arXiv:hep-th/9909053].
[85] D. Langlois and L. Sorbo, Phys. Lett. B 543, 155 (2002) [arXiv:hep-th/0203034].
[86] J. Khoury, B. A. Ovrut, P. J. Steinhardt and N. Turok, Phys. Rev. D 64, 123522 (2001) [arXiv:hep-th/0103239]; R. Kallosh, L. Kofman and A. D. Linde, Phys. Rev. D 64, 123523 (2001) [arXiv:hep-th/0104073]; P. J. Steinhardt and N. Turok, arXiv:hep-th/0111030.
[87] M. Bucher, Phys. Lett. B 530 (2002) 1. [arXiv:hep-th/0107148].
[88] U. Gen, A. Ishibashi and T. Tanaka, Phys. Rev. D 66, 023519 (2002) [arXiv:hep-th/0110286].
[89] D. Langlois, K. i. Maeda and D. Wands, Phys. Rev. Lett. 88, 181301 (2002) [arXiv:gr-qc/0111013].