MINIMAL POSITIVE STENCILS IN MESHFREE FINITE DIFFERENCE METHODS FOR LINEAR ELLIPTIC EQUATIONS IN NON-DIVERGENCE FORM

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Abstract. We design a monotone meshfree finite difference method for linear elliptic equations in the non-divergence form on point clouds via a nonlocal relaxation method. Nonlocal approximations of linear elliptic equations are first introduced to which a meshfree finite difference method applies. Minimal positive stencils are obtained through a local $l_1$-type optimization procedure that automatically guarantees the stability and, therefore, the convergence of the meshfree discretization for linear elliptic equations. The key to the success of the method relies on the existence of positive stencils for a given point cloud geometry. We provide sufficient conditions for the existence of positive stencils by finding neighbors within an ellipse (2d) or ellipsoid (3d) surrounding each interior point, generalizing the study for Poisson’s equation by Seibold in 2008. It is well-known that wide stencils are in general needed for constructing consistent and monotone finite difference schemes for linear elliptic equations. Our study improves the known theoretical results on the existence of positive stencils for linear elliptic equations when the ellipticity constant becomes small. Numerical algorithms and practical guidance are provided with an eye on the case of small ellipticity constant. We present numerical results in 2d and 3d at the end.

1. Introduction

In this work, we consider numerical approximations to the second-order elliptic equations in non-divergence form

\begin{equation}
-Lu(x) := - \sum_{i,j=1}^{d} a^{ij}(x) \partial_{ij} u(x) = f(x) \quad x \in \Omega
\end{equation}

\begin{equation}
u(x) = g(x) \quad x \in \partial \Omega,
\end{equation}

for a bounded domain $\Omega \subset \mathbb{R}^d$. $A(x) = (a^{ij}(x))_{i,j=1}^{d}$ is a bounded and measurable matrix-valued function and is assumed to be symmetric and positive definite satisfying the uniform ellipticity condition

\begin{equation}
\lambda |\xi|^2 \leq A(x)\xi \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \forall x \in \Omega
\end{equation}

for positive constants $\lambda$ and $\Lambda$. Notice that by dividing both sides of the first equation in (1.1) by $\Lambda$, we can assume without of loss generality that $\Lambda = 1$ and the ratio $\bar{\rho} := \lambda/\Lambda = \lambda \leq 1$.

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Linear elliptic equations in non-divergence form are ubiquitous in science and engineering. They arise in many applications such as stochastic optimal control, materials science, and mathematical finance [18], and they are also recognized as the linearization of fully nonlinear PDEs such as the Monge-Ampère equation [26]. Notice that if the coefficient matrix $A$ is differentiable, then the non-divergence form equation (1.1) can be recast into a divergence form convection-diffusion equation. However, when $A$ is not differentiable, such reformulation no longer exists. PDE theories for non-divergence form elliptic equations are well-established in the literature. Existence, uniqueness and regularity theories are established for different notions of solutions including classical solutions, strong solutions, and viscosity solutions [8, 21, 32, 41].

In terms of numerical methods, non-divergence form elliptic PDEs are much less discussed than the divergence form PDEs because of a lack of variational formulation aforementioned. Discussions on finite element methods can be found in [16, 17, 25, 35, 38, 44] and the references therein. We pursue the direction of monotone finite difference method for elliptic PDEs which has guaranteed convergence under the framework of Barles and Sougnidis [2]. For the linear elliptic equation (1.1), finding monotone finite difference schemes amounts to constructing positive stencils for the approximation of the elliptic operator. Such pursuit dates back to Motzkin and Wasow [34], Kuo and Trudinger [24], and Kocan [23]. What was found in these work was that consistent and positive-type finite difference schemes exist for a given elliptic operator, but the stencil size grows with $\varrho \to 0$, and therefore the so-called wide stencil is a necessary feature of monotone finite difference methods even in the case of linear elliptic equations. Wide-stencil methods are later developed also for fully nonlinear elliptic PDEs [19, 40, 39, 37]. For the linear elliptic PDEs, Kocan [23] gives an estimate of the stencil width for the existence of positive-type finite difference method, and it grows linearly with $\varrho^{-1}$ in 2d and superlinearly with $\varrho^{-1}$ in 3d, which severely impacts the practical use of such scheme for small ratio $\varrho$.

Our work is inspired by the recent development of meshfree methods for nonlocal models [14, 15, 27, 46]. The nonlocal approximation of the linear elliptic operator inspired also by [38], is defined by an integral over an elliptical region where we search for positive stencils once the domain is discretized by a meshfree point cloud. We characterize the size of the elliptical region for guaranteeing the existence of positive stencils and then use an $l_1$-type local optimization method to obtain a minimal positive stencil which in 2d contains only six points and in 3d ten points. Therefore, the resulting linear system is sparse and can be solved efficiently in practice. In addition, our theoretical result states that positive stencils exist within an elliptical searching region where the semi-major axis, denoted by $\delta$, is proportional to $h\varrho^{-1/2}$ (cf. Theorem 3.9), which also significantly improves the theoretical estimate in [23]. In the literature of nonlocal models, such numerical method belongs to the type of asymptotically compatible schemes [45], i.e., for a given problem with $\varrho$ fixed, the size $\delta$ of the nonlocal interaction region can be made proportional to $h$. This is an important feature that differs from many traditional numerical methods that involve nonlocal relaxation/regularization to PDEs. For example, nonlocal relaxation/regularization exists in the vortex method by Beale and Majda [3], the blob method by Craig and Bertozzi [7], and the two-scale method by Nochetto and Zhang [38], and these methods all require the discretization parameter $h$.
decrease faster than the nonlocal regularization parameter, i.e., \( h/\delta \to 0 \), to achieve convergence. We point out that seeking asymptotically compatible schemes is an important idea that could significantly improve the efficiency of numerical methods via nonlocal relaxation. Lastly, we stress that we solve the elliptic equation on meshfree point clouds. Meshfree methods are widely used in computational studies of problems in science and engineering, offering many advantages over traditional mesh- or grid-based numerical methods for problems with complex or moving geometries, discontinuity or other singular behaviors of solutions. Our current study also naturally connects to manifold learning where surface PDEs need to be approximated on point clouds [28].

The rest of the paper is organized as follows. We first present the nonlocal integral approximation to eq. (1.1) in Section 2, as well as the approximation error of the continuum nonlocal model to the elliptic equation. In Section 3, we present the meshfree discretization based on the nonlocal regularized problem. An \( l_1 \)-type local optimization method is proposed in search of positive stencils in an elliptical neighborhood surrounding each node. Theoretical results concerning the neighborhood criteria as well as convergence of numerical solutions are also presented. In Section 4, we discuss point cloud generation and management, the assembling process, and provide complexity analysis and practical guidance for the implementation of the numerical method. We present the 2d and 3d numerical results in Section 5, and make conclusion and further discussions in Section 6.

2. Nonlocal relaxation to elliptic equations

In this section, we discuss the nonlocal integral approximation to eq. (1.1) on which our numerical methods are based.

2.1. Nonlocal elliptic operators in non-divergence form. Nonlocal models have gained much interest in recent years [1, 5, 13]. In [11, 14], nonlocal Laplace operator with a parameter dependence on \( \delta > 0 \) is used as an approximation to the classical Laplace operator \( \Delta \) (when \( A(x) = I \)). The nonlocal Laplace operator is given by

\[
\tilde{L}_\delta u(x) = \int_{B_\delta(0)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|y|}{\delta} \right) (u(x + y) - u(x)) dy
\]

where \( \gamma \) is a nonnegative kernel with

\[
\int_{B_1(0)} |y|^2 \gamma(|y|) dy = 2d.
\]

It can be shown that as \( \delta \to 0 \), \( \tilde{L}_\delta u(x) \to \Delta u(x) \) for a sufficiently smooth function \( u \). Here we consider a more general nonlocal elliptic operator that approximates the classical elliptic operator \( L \) in eq. (1.1) in the \( \delta \to 0 \) limit. Following [38], we define the nonlocal elliptic operator parameterized by \( \delta \) as

\[
L_\delta u(x) = \int_{E_\delta(x)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|M(x)^{-1}y|}{\delta} \right) \det(M(x))^{-1} (u(x + y) - u(x)) dy
\]

where \( M(x) := (A(x))^{1/2} \) is a positive definite matrix and \( E_\delta(z) \) denotes an ellipse with definition

\[
E_\delta(z) := \{ y \in \mathbb{R}^d : M(x)^{-1}(y - z) \in B_\delta(0) \}.
\]

Notice that by our assumption on \( A(x) \), \( E_\delta(z) \) is an elliptical region centered...
at $z$ with semi-axes being $\{\delta \sqrt{\lambda_i(x)}\}_{i=1}^d$ where $\lambda_i(x)$ denotes the $i$-th smallest eigenvalue of $A(x)$. By our assumption, $\rho \leq \lambda_1(x) \leq \cdots \leq \lambda_d(x) \leq 1$. Figure 1 shows a 2d example where the semi-major axis of the ellipse is $\delta \lambda_2^{1/2}(x) \leq \delta$ and the semi-minor axis is $\delta \lambda_1^{1/2}(x) \geq \delta \sqrt{\rho}$. For convenience, we define

$$\rho_{\delta}(x, y) = \frac{1}{\delta^{d+2}} \gamma \left( \frac{|M(x)|^{-1} y}{\delta} \right) \det(M(x))^{-1}$$

and simply write $L_{\delta} u(x) = \int_{\mathcal{E}_\delta^c(0)} \rho_{\delta}(x, y)(u(x + y) - u(x)) dy$.

In the next, we show the consistency between $L_{\delta}$ and $L$ on sufficiently smooth functions. The first result asserts that $L_{\delta} u$ agrees with $Lu$ for all polynomials up to the third order, and the second result gives pointwise truncation error for sufficiently smooth functions. Similar calculations can be found in [38], here we present them for completeness and for the convenience of analyzing our numerical method in Section 3.

**Lemma 2.1.** Let $\mathcal{P}_p(\mathbb{R}^d)$ denote the space of all polynomials up to order $p$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in (\mathbb{Z}^+ \cup \{0\})^d$ with $|\alpha| = \sum_i \alpha_i$. Then for any $x \in \mathbb{R}^d$,

$$\int_{\mathcal{E}_\delta^c(0)} \rho_{\delta}(x, y)y^\alpha dy = 0$$

if $|\alpha|$ is an odd number and

$$L_{\delta} u(x) = Lu(x) \quad \forall u \in \mathcal{P}_3(\mathbb{R}^d).$$

**Proof.** Consider $x \in \mathbb{R}^d$ fixed. If $|\alpha|$ is an odd number we have

$$\int_{\mathcal{E}_\delta^c(0)} \rho_{\delta}(x, y)y^\alpha dy = \int_{\mathcal{E}_\delta^c(0)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|M(x)|^{-1} y}{\delta} \right) \det(M(x))^{-1} y^\alpha dy = \int_{B_\delta(0)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|y|}{\delta} \right) (M(x)y)^\alpha dy = 0.$$ 

This last equality above is due to symmetry of the integration domain and anti-symmetry of the integrand.

Now it is easy to see that $L_{\delta} u(x) = Lu(x) = 0$ when $u$ is a constant or $u(z) = (z - x)^\alpha$ with $|\alpha|$ being an odd number. On the other hand, notice that

$$\int_{B_\delta(0)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|y|}{\delta} \right) y_i y_j dy = 0 \quad i \neq j$$

and

$$\int_{B_\delta(0)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|y|}{\delta} \right) y_i^2 dy = \frac{1}{d} \int_{B_\delta(0)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|y|}{\delta} \right) |y|^2 dy = 2 \quad \forall i.$$
We have

\[
\int_{\mathcal{E}_\delta(0)} \rho_\delta(x, y)(y \otimes y) dy = \int_{\mathcal{E}_\delta(0)} \frac{1}{\delta^{d+2}} \left( \frac{|M(x)^{-1} y|}{\delta} \right) \det(M(x))^{-1} (y \otimes y) dy
\]

\[
= \int_{B_\delta(0)} \frac{1}{\delta^{d+2}} \left( \frac{|y|}{\delta} \right) (M(x)y \otimes M(x)y) dy
\]

\[
= M(x) \int_{B_\delta(0)} \frac{1}{\delta^{d+2}} \left( \frac{|y|}{\delta} \right) (y \otimes y) dy M(x)
\]

\[
= 2(M(x))^2 = 2A(x),
\]

and thus \( \mathcal{L}_\delta u(x) = Lu(x) \) when \( u(z) = (z - x)^\alpha \) with \( |\alpha| = 2 \). Since \( \mathcal{L}_\delta(x) \) agrees
with \( Lu(x) \) for \( u(z) = (z - x)^\alpha \) with \( |\alpha| \leq 3 \), eq. (2.6) is true. \( \square \)

We now consider an open bounded domain \( \Omega \subset \mathbb{R}^d \). For \( x \in \Omega \) near the boundary
of \( \Omega \), the definition in eq. (2.2) requires the values of \( u \) outside \( \Omega \). Therefore nonlocal
equations on bounded domains are usually accompanied by volumetric constraints (\cite{14}) imposed over a boundary layer surrounding \( \Omega \). In our case, we need to define
the boundary interaction layer as

\[
\Omega_{\delta_0} = \{ x \in \mathbb{R}^d \setminus \Omega : \text{dist}(x, \partial \Omega) < \delta_0 \}.
\]

We then denote the extended domain \( \Omega_\delta := \Omega \cup \Omega_{\delta_0} \). The consistency of \( \mathcal{L}_\delta \) to \( L \)
indicated in the following lemma.

**Lemma 2.2.** Let \( \mathcal{L}_\delta u \) be defined by (2.2) and \( C > 0 \) being a generic constant. Let \( \delta_0 > 0 \) be a fixed number.

1. If \( u \in C^2(\overline{\Omega_{\delta_0}}) \), then \( |\mathcal{L}_\delta u(x) - Lu(x)| \to 0 \) as \( \delta \to 0 \) for all \( x \in \Omega \).
2. If \( u \in C^{k,\alpha}(\overline{\Omega_{\delta_0}}) \) for \( k = 2 \) or \( 3 \) and \( \alpha \in (0, 1] \), then

\[
|\mathcal{L}_\delta u(x) - Lu(x)| \leq C|u|_{C^{k,\alpha}(\overline{\Omega_{\delta_0}})} \delta^{k-2+\alpha}
\]

for all \( x \in \Omega \) and \( \delta \leq \delta_0 \).

**Proof.** Notice that

\[
u(x + y) - u(x) = \int_0^1 \frac{du}{dt} u(x + ty) dt = \int_0^1 \nabla u(x + ty) \cdot y dt.
\]

Therefore

\[
\mathcal{L}_\delta u(x) = \int_{\mathcal{E}_\delta(0)} \rho_\delta(x, y)y^T \int_0^1 \nabla u(x + ty) dt dy
\]

\[
= \int_{\mathcal{E}_\delta(0)} \rho_\delta(x, y)y^T \cdot \int_0^1 (\nabla u(x + ty) - \nabla u(x)) dt dy
\]

where we have used \( \int_{\mathcal{E}_\delta(0)} \rho_\delta(x, y) y dy = 0 \) for the second line above. Then use

\[
\nabla u(x + ty) - \nabla u(x) = \int_0^1 D^2 u(x + sty) \cdot ty ds,
\]

we have

\[
\mathcal{L}_\delta u(x) = \int_{\mathcal{E}_\delta(0)} \rho_\delta(x, y)(y \otimes y) : \int_0^1 t \int_0^1 D^2 u(x + sty) ds dt dy
\]
for all \( x \in \Omega \). On the other hand, from calculations in Lemma 2.1, one can show for \( x \in \Omega \),

\[
Lu(x) = \int_{\mathcal{E}_t^a}(y \otimes y) \frac{(y \otimes y)}{2} : D^2u(x)dy
\]

\[
= \int_{\mathcal{E}_t^a} \rho_\delta(x,y)(y \otimes y) \int_0^1 t \int_0^1 (D^2u(x)stdy)dsdtdy.
\]

For \( u \in C^2(\bar{\Omega}) \), we have \( D^2u(x + sty) \rightarrow D^2u(x) \) as \( \delta \rightarrow 0 \) since \( y \in \mathcal{E}_t^a(0) \). Therefore case (1) holds. For \( u \in C^{2,\alpha}(\bar{\Omega}) \), we have

\[
|D^2u(x + sty) - D^2u(x)|_\infty \leq |u|_{C^{2,\alpha}(\bar{\Omega})} |sty|^{\alpha}.
\]

So

\[
|L_\delta u(x) - Lu(x)| \leq C|u|_{C^{2,\alpha}(\bar{\Omega})} \int_{\mathcal{E}_t^a}(x,y) |y|^{2+\alpha}dy
\]

\[
\leq C|u|_{C^{2,\alpha}(\bar{\Omega})} \int_{\mathcal{E}_t^a}(x,y) |y|^{2}dy
\]

\[
= C|u|_{C^{2,\alpha}(\bar{\Omega})} \int_{\mathcal{E}_t^a}(x,y) \frac{1}{\delta^{d+2\gamma\delta}} \left( \frac{|y|}{\delta} \right)^{d+2\gamma\delta} |y^T A(x)y|dy
\]

\[
\leq C|u|_{C^{2,\alpha}(\bar{\Omega})} \int_{\mathcal{E}_t^a}(x,y) |y|^{\alpha}dy
\]

\[
= 2dC|u|_{C^{2,\alpha}(\bar{\Omega})} \delta^{\alpha}.
\]

Finally, if \( u \in C^{3,\alpha}(\bar{\Omega}) \), then we can write

\[
L_\delta u(x) - Lu(x)
\]

\[
= \int_{\mathcal{E}_t^a}(x,y) (y \otimes y) : \int_0^1 t \int_0^1 (D^2u(x + sty) - D^2u(x))dsdtdy
\]

\[
= \int_{\mathcal{E}_t^a}(x,y) (y \otimes y \otimes y) : \int_0^1 t^2 \int_0^1 s \int_0^1 (D^3u(x + rsty)drdsdtdy
\]

\[
= \int_{\mathcal{E}_t^a}(x,y) (y \otimes y \otimes y) : \int_0^1 r^2 \int_0^1 s \int_0^1 (D^3u(x + rsty) - D^3u(x))drdsdtdy.
\]

Therefore, by the same reasoning as before, we have

\[
|L_\delta u(x) - Lu(x)| \leq C|u|_{C^{3,\alpha}(\bar{\Omega})} \int_{\mathcal{E}_t^a}(x,y) |y|^{3+\alpha}dy \leq 2dC|u|_{C^{3,\alpha}(\bar{\Omega})} \delta^{1+\alpha}.
\]

\[
\square
\]

3. Meshfree discretization

Meshfree methods have been widely used in simulations, see [6, 30] and the references cited therein. Moving least squares (MLS) and radial basis functions (RBF) have emerged among the most popular methods for function approximations on scattered datasets [50]. Generalized moving least squares (GMLS) was later developed for the direct approximation of operators such as derivatives of functions on scattered datasets [33]. The key idea in MLS and GMLS is a local fitting of data using least squares approximation. They can also be written as a weighted \( l_2 \)-type
local optimization under certain reproducing conditions. In [42, 10], weighted $l_1$-type optimization was discussed for the sparsity of stencils. It was shown that using $l_1$-type optimization, the number of nonzero weights generated is at most the number of constraints in the reproducing condition. Such property is important to keep the linear system sparse, especially when the elliptic problem is nearly degenerate ($\rho \ll 1$), in the case of which the searching region becomes large (cf. Theorem 3.9).

For the rest of this section, we discuss the meshfree method for solving eq. (1.1) based on the nonlocal relaxation and the convergence of the numerical method.

### 3.1. Optimization based meshfree discretization.

Our numerical method is inspired by the meshfree finite difference method presented in [42] for solving the classical Poisson equation. The focus here is on the generation of positive stencils which lead to monotone schemes. The desirability of positive stencils was observed in other meshless methods, see e.g., [12, 29], although there was no guarantee of positive stencils in these works. We now present a reformulation of the meshfree method in [42] as a nonlocal relaxation method on which a generalization to elliptic equations is based. Given a point cloud $X = \{x_i\} \subset \mathbb{R}^d$ with $h$ being its associated fill distance to be defined later, it is proposed in [42] to discretize the Laplace operator by

$$\Delta u(x_i) \approx \Delta_h u(x_i) = \sum_{x_j \in B_h(x_i)} \beta_{j,i} (u(x_j) - u(x_i)), \tag{3.1}$$

where the weights $\{\beta_{j,i}\}$ are determined by the following linear minimization problem in order to achieve the so-called minimal positive stencils:

$$\{\beta_{j,i}\} = \arg\min_j \frac{\beta_{j,i}}{W(|x_j - x_i|)} \quad \text{s.t. } \beta_{j,i} \geq 0 \text{ and } \Delta u(x_i) = \Delta_h u(x_i) \forall u \in \mathcal{P}_p(\mathbb{R}^d) \tag{3.2}$$

In [42], the polynomial space $\mathcal{P}_p(\mathbb{R}^d)$ is taken to be $\mathcal{P}_2(\mathbb{R}^d)$ with $p = 2$. The parameter $\delta$ in eq. (3.1) is determined in relation to $h$ such that the feasible set of the minimization problem is non-empty. The weight function $W(r)$ is suggested in [42] as $W(r) = r^{-\alpha}$ for $\alpha > 0$. It is not hard to see that when we choose the nonlocal kernel function $\gamma(r) = C r^{-\alpha} \chi(|r| < 1)$ that satisfies eq. (2.1), then by letting $\beta_{j,i} = \frac{1}{\delta^{d+2}} \gamma \left( \frac{|x_j - x_i|}{\delta} \right) \omega_{j,i}$, the minimization problem (3.2) is equivalent to

$$\{\omega_{j,i}\} = \arg\min_j \sum \omega_{j,i} \quad \text{s.t. } \omega_{j,i} \geq 0 \text{ and } \mathcal{L}_\delta u(x_i) = \mathcal{L}_\delta^h u(x_i) \forall u \in \mathcal{P}_p(\mathbb{R}^d). \tag{3.3}$$

where $\mathcal{L}_\delta^h$ is the nonlocal approximate operator defined by

$$\mathcal{L}_\delta^h u(x_i) = \sum_{x_j \in B_h(x_i)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|x_j - x_i|}{\delta} \right) \omega_{j,i} (u(x_j) - u(x_i)). \tag{3.4}$$

Based on this observation, we propose the discretization of nonlocal elliptic operator (2.2) as

$$\mathcal{L}_\delta u(x_i) = \sum_{x_j \in \mathcal{L}_\delta^h(x_i)} \rho_\delta(x_i, x_j - x_i) \omega_{j,i} (u(x_j) - u(x_i)). \tag{3.5}$$
where \( \rho(\mathbf{x}_i, \mathbf{x}_j - \mathbf{x}_i) = \frac{1}{\delta^{d+2}} \gamma \left( \frac{|M(\mathbf{x}_i)^{-1}(\mathbf{x}_j - \mathbf{x}_i)|}{\delta} \right) \det(M(\mathbf{x}_i))^{-1} \). The weights \( \{\omega_{j,i}\} \) in eq. (3.5) are solved by the minimization problem

\[
\{\omega_{j,i}\} = \arg\min \sum_j \omega_{j,i} \\
\text{s.t. } \omega_{j,i} \geq 0 \quad \text{and} \quad \mathcal{L}_h^\delta u(\mathbf{x}_i) = \mathcal{L}_h^\delta u(\mathbf{x}_i) \quad \forall u \in \mathcal{P}_p(\mathbb{R}^d). 
\]

The well-posedness (3.6) is guaranteed only if the feasible set is non-empty. We will discuss in Section 3.4 the neighborhood criteria for non-emptiness of the feasible set

\[
S_{\delta,h,p}(\mathbf{x}_i) := \{ \{\omega_{j,i}\} : \omega_{j,i} \geq 0 \quad \text{and} \quad \mathcal{L}_h^\delta u(\mathbf{x}_i) = \mathcal{L}_h^\delta u(\mathbf{x}_i) \quad \forall u \in \mathcal{P}_p(\mathbb{R}^d) \}. 
\]

3.2. Boundary treatment. For an open and bounded domain \( \Omega \subset \mathbb{R}^d \), we take a point cloud \( X = \{\mathbf{x}_i\}_{i=1}^M \subset \Omega_{\delta} \) and define its associated fill distance

\[
h := \sup_{\mathbf{x} \in \Omega_{\delta}} \min_{1 \leq i \leq M} |\mathbf{x} - \mathbf{x}_i| 
\]

following the convention in [50]. Assume that \( \{\mathbf{x}_i\}_{i=1}^N \subset \Omega \). For \( \mathbf{x}_j \in \Omega \) near the boundary of \( \partial \Omega \), the definition in eq. (3.5) may require the value of \( u(\mathbf{x}_j) \) for \( \mathbf{x}_j \in \Omega_{I_\delta} \). Therefore, extensions of the boundary values from \( \partial \Omega \) to \( \Omega_{I_\delta} \) are needed. However, it is usually hard to find an easy way to do the extension, especially in higher dimensions, to guarantee a second-order convergence rate for nonlocal solutions. We propose an alternative way for the boundary treatment.

\[
\mathbf{x}_j \quad \text{associated with} \quad \mathbf{x}_{i_1} \\
\mathbf{x}_{j} \\
\mathbf{x}_{i_1} \\
\partial \Omega \\
\mathbf{x}_j \quad \text{associated with} \quad \mathbf{x}_{i_2} \\
\mathbf{x}_{j_1} \\
\mathbf{x}_{j_2} \\
\text{Figure 2. Illustration of the projection}
\]

For \( \mathbf{x}_i \in \Omega \) and \( \mathbf{x}_j \in E^\delta_{\mathbf{x}_i}(\mathbf{x}_i) \), we define \( \overline{\mathbf{x}}_j = \mathbf{x}_j \) if \( \mathbf{x}_j \in \overline{\Omega} \), otherwise \( \overline{\mathbf{x}}_j \in \partial \Omega \) is defined as the projection of \( \mathbf{x}_j \) onto \( \partial \Omega \) such that the line from \( \mathbf{x}_i \) to \( \overline{\mathbf{x}}_j \) is contained in \( \overline{\Omega} \). Notice that the projected point \( \overline{\mathbf{x}}_j \) depends on both \( \mathbf{x}_j \) and \( \mathbf{x}_i \). Here for notational convenience, we have omitted the dependence on \( i \) and simply denoted the projected point as \( \overline{\mathbf{x}}_j \). See Figure 2 as an illustration of the projection. We then define the approximate operator associated with \( \delta, h \) and \( \Omega \) as

\[
\mathcal{L}_h^\delta u(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in E^\delta_{\mathbf{x}_i}(\mathbf{x}_i)} \rho(\mathbf{x}_i, \overline{\mathbf{x}}_j - \mathbf{x}_i) \omega_{j,i}(u(\overline{\mathbf{x}}_j) - u(\mathbf{x}_i)) 
\]
where \( \{ \omega_{j,i} \} \) is solved from
\[
\{ \omega_{j,i} \} = \underset{\{ \omega_{j,i} \} \in \mathcal{S}_{\delta,h,p}(x_i)}{\text{argmin}} \sum_j \omega_{j,i}
\]
where \( \mathcal{S}_{\delta,h,p}(x_i) \) is the feasible set defined as
\[
\mathcal{S}_{\delta,h,p}(x_i) := \{ (\omega_{j,i}) : \omega_{j,i} \geq 0 \text{ and } \mathcal{L}_{\delta,\Omega}^h u(x_i) = \mathcal{L}_\delta u(x_i) \forall u \in \mathcal{P}_p(\mathbb{R}^d) \}.
\]
We will address in Section 3.4 the feasibility of the minimization problem (3.10).

With the definition of the discrete operator \( \mathcal{L}_{\delta,\Omega}^h \), we define the discrete problem as to find a function \( u_h^\delta : \{ x_i \}_{i=1}^N \cup \partial \Omega \to \mathbb{R} \) such that
\[
\left\{
\begin{array}{ll}
-\mathcal{L}_{\delta,\Omega}^h u_h^\delta(x_i) = f(x_i) & x_i \in \Omega \\
u_h^\delta(x) = g(x) & x \in \partial \Omega
\end{array}
\right.
\]
3.3. Convergence analysis. We first provide a truncation error analysis for the discrete operator \( \mathcal{L}_{\delta,\Omega}^h \).

**Lemma 3.1.** Take a point cloud \( X = \{ x_i \}_{i=1}^M \subset \Omega_3 \) with \( \{ x_i \}_{i=1}^N \subset \Omega \). Assume also that \( \mathcal{S}_{\delta,h,p}(x_i) \) is not empty and \( C > 0 \) is a generic constant.

(1) If \( p \geq 2 \) and \( u \in C^2(\bar{\Omega}) \), then \( |\mathcal{L}_{\delta,\Omega}^h u(x_i) - Lu(x_i)| \to 0 \) as \( \delta \to 0 \) for all \( x_i \in \Omega \).

(2) If \( p \geq 2 \) and \( u \in C^{2,\alpha}(\bar{\Omega}) \) for \( \alpha \in (0,1] \), then \( |\mathcal{L}_{\delta,\Omega}^h u(x_i) - Lu(x_i)| \leq C|u|_{C^{2,\alpha}(\bar{\Omega})}\delta^\alpha \) for all \( x_i \in \Omega \).

(3) If \( p \geq 3 \) and \( u \in C^{3,\alpha}(\bar{\Omega}) \) for \( \alpha \in (0,1] \), then \( |\mathcal{L}_{\delta,\Omega}^h u(x_i) - Lu(x_i)| \leq C|u|_{C^{3,\alpha}(\bar{\Omega})}\delta^{1+\alpha} \) for all \( x_i \in \Omega \).

**Proof.** Consider a fixed \( x_i \in \Omega \). The proof follows closely from the proof of Lemma 2.2 by noticing that \( \{ \omega_{j,i} \} \in \mathcal{S}_{\delta,h,p}(x_i) \) implies that
\[
\int_{E^\alpha(i)(0)} \rho_\delta(x_i,y)y^\alpha dy = \sum_{x_j \in E^\alpha(i)(x_i)} \rho_\delta(x_i,x_j)(x_j - x_i)^\alpha \omega_{j,i}
\]
for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in (\mathbb{Z}^+ \cup \{0\})^d \) with \( |\alpha| = \sum_i \alpha_i \leq p \).

**Remark 3.2.** In practice, we often observe superconvergence for \( p = 2 \), which is likely due to symmetry. When \( p = 2 \) and \( u \in C^{3,\alpha}(\bar{\Omega}) \), a more precise error estimate for \( x_i \in \Omega \) is given by
\[
|\mathcal{L}_{\delta,\Omega}^h u(x_i) - Lu(x_i)| \leq C \left( |u|_{C^3(\bar{\Omega})}T_3(x_i) + |u|_{C^{3,\alpha}(\bar{\Omega})}\delta^{1+\alpha} \right)
\]
where \( T_3(x_i) = \max_{|\alpha|=3} \sum_{x_j \in E^\alpha(i)(x_i)} \rho_\delta(x_i,x_j)(x_j - x_i)^\alpha \omega_{j,i} \).

**Lemma 3.3** (Discrete maximum principle). Let \( \Omega \subset \mathbb{R}^d \) be an open, bounded and simply connected domain. Take a point cloud \( X = \{ x_i \}_{i=1}^M \subset \Omega_3 \) with \( \{ x_i \}_{i=1}^N \subset \Omega \). Assume that there exists \( x_i \in \Omega \) such that \( E^\alpha(i)(x_i) \cap \Omega^c \neq \emptyset \). If \( u \in C\left( \{ x_i \}_{i=1}^N \cup \partial \Omega \right) \) and \( \mathcal{L}_{\delta,\Omega}^h u(x_i) \geq 0 \) for all \( x_i \in \Omega \), then
\[
\max_{x_i \in \Omega} u(x_i) \leq \max_{x \in \partial \Omega} u(x).
\]
Proof. First notice that \( \max_{x \in \partial \Omega} u(x) \) is well-defined since \( \partial \Omega \) is a closed set and \( u \) is continuous on \( \partial \Omega \). Assume that \( \max_{x_i \in \Omega} u(x_i) > \max_{x \in \partial \Omega} u(x) \), then there exists \( x_k \in \Omega \) such that

\[
u(x_k) = \max_{x_i \in \Omega} u(x_i) \geq u(x) \quad \forall x \in \{x_i\}_{i=1}^{N} \cup \partial \Omega.
\]

Therefore

\[
L_{0,\Omega}^{h} u(x_k) \leq 0.
\]

By the assumption, we must have \( L_{\delta,\Omega} u(x_k) = 0 \) and \( u(x_j) = u(x_k) \) for \( x_j \in \mathcal{E}_{\delta}^{\omega}(x_k) \cap \Omega \). Continue this process we can shown that \( u \) is constant on \( \{ x_i \}_{i=1}^{N} \subset \Omega \). Choose \( x_i \in \Omega \) such that there exists \( x_j \in \mathcal{E}_{\delta}^{\omega}(x_i) \cap \Omega \), so \( x_j \in \partial \Omega \). However, since we can argue that \( L_{\delta,\Omega}^{h} u(x_i) = 0 \), it implies \( u(x_i) = u(x_j) \), which is contradiction to \( \max_{x_i \in \Omega} u(x_i) > \max_{x \in \partial \Omega} u(x) \). \( \square \)

\noindent \textbf{Theorem 3.4.} Take a point cloud \( X = \{ x_i \}_{i=1}^{M} \subset \Omega_\delta \) and assume that \( \mathcal{X}_{\delta,h,p}(x_i) \) is not empty. Let \( u \) and \( u_{h}^{\delta} \) be the solutions to eqs. (1.1) and (3.12) respectively.

1. If \( p \geq 2 \) and \( u \in C^{2,\alpha}(\Omega) \) for \( \alpha \in (0,1] \), then

\[
\max_{x_i \in \Omega} |u(x_i) - u_{h}^{\delta}(x_i)| \leq C |u|_{C^{2,\alpha}(\Omega)} \delta^\alpha.
\]

2. If \( p \geq 3 \) and \( u \in C^{3,\alpha}(\Omega) \) for \( \alpha \in (0,1] \), then

\[
\max_{x_i \in \Omega} |u(x_i) - u_{h}^{\delta}(x_i)| \leq C |u|_{C^{3,\alpha}(\Omega)} \delta^{1+\alpha}.
\]

\noindent \textbf{Remark 3.5.} Using the truncation error estimate in Remark 3.2, one can show that if \( p = 2 \) and \( u \in C^{3,\alpha}(\Omega) \), then

\[
\max_{x_i \in \Omega} |u(x_i) - u_{h}^{\delta}(x_i)| \leq C (|u|_{C^{3}(\Omega)} \tau + |u|_{C^{3,\alpha}(\Omega)} \delta^{1+\alpha}).
\]

where

\[
\tau = \max_{x_i \in \Omega} T_{3}(x_i)
\]

In practice, \( \tau \) might be a very small number depending on the point cloud. Therefore, superconvergence may be observed.

\noindent \textbf{Proof of Theorem 3.4.} We only show the proof for the first case and the second case can be similarly shown. Denote \( e_{i}^{h}(x) = u(x) - u_{h}^{\delta}(x) \) for \( x \in \{ x_i \}_{i=1}^{N} \cup \partial \Omega \) and \( T_{3}^{h}(x_i) = L_{\delta,\Omega}^{h} u(x_i) - L u(x_i) \) for \( x_i \in \Omega \). Notice that \( L_{\delta,\Omega}^{h} e_{i}^{h}(x_i) = T_{3}^{h}(x_i) \) for \( x_i \in \Omega \). By Lemma 3.1, we have

\[
K := \max_{x_{i} \in \Omega} |T_{3}^{h}(x_i)| \leq C |u|_{C^{2,\alpha}(\Omega)} \delta^\alpha.
\]

Take \( x^{*} \in \Omega \) such that \( \Omega \subset B_{R}(x^{*}) \) for some \( R > 0 \). Then define \( \Phi(x) = (x - x^{*})T(x - x^{*})/(2d) \), we have

\[
L_{\delta,\Omega}^{h} \Phi(x) = A(x) : D^{2} \Phi(x) = \frac{1}{d} \sum_{i=1}^{d} a_{ii}(x) = \frac{1}{d} \sum_{i=1}^{d} e_{i}^{T} A(x) e_{i} \geq \lambda,
\]

where \( e_{i} \in \mathbb{R}^{d} \) is the unit vector with the \( i \)-th component equal 1. Therefore we have

\[
L_{\delta,\Omega}^{h} \left( \frac{K}{\lambda} \Phi + e_{i}^{h} \right) (x_i) \geq 0 \quad \forall x_i \in \Omega,
\]
and by Lemma 3.3, we have
\[
\max_{x_i \in \Omega} e^h_\delta(x_i) \leq \max_{x_i \in \Omega} \left( \frac{K}{\lambda} \Phi(x_i) + e^h_\delta(x_i) \right) \leq \max_{x \in \partial \Omega} \left( \frac{K}{\lambda} \Phi(x) + e^h_\delta(x) \right)
\]
\[
= \frac{K}{\lambda} \max_{x \in \partial \Omega} \Phi(x) \leq \frac{K}{\lambda} \frac{R^2}{2d} \leq \frac{C R^2}{2d} |u|_{C^{2,\alpha}(\Omega)} \delta^n.
\]
Similar estimates can be done for \(-e^h_\delta(x_i)\) and therefore the proof is complete. □

3.4. Neighborhood criteria. In this subsection, we will discuss the neighborhood criteria that guarantee positive stencils. We only discuss the case \(p = 2\) in this subsection. The case \(p = 3\) is much harder to characterize which will be left for future work.

First of all, there is a sufficient criterion for positive stencils for solving the Laplace equation, and it is presented as a cone condition in [42] for \(d = 2\) or \(d = 3\). For any \(x_i \in \Omega\) and unit vector \(v \in \mathbb{R}^d\), we define an associated cone \(C^\delta_v(x_i)\) in \(B^\delta(x_i)\) by
\[
C^\delta_v(x_i) := \left\{ x \in B^\delta(x_i) : x^Tv \geq \frac{1}{\sqrt{1 + \sigma_d}} |x|^2 \right\}
\]
where \(\sigma_d = \sqrt{2} - 1\) (a cone with total opening angle 45°) for \(d = 2\) and \(\sigma_d = \sqrt{(3 - \sqrt{6})/6}\) (a cone with total opening angle 33.7°) for \(d = 3\). With a rephrasing of words, we quote the result in [42, Theorems 9 and 10] in the following lemma.

**Lemma 3.6** (Theorems 9 and 10 in [42]). Take a point cloud \(X = \{x_i\}_{i=1}^M \subset \Omega^\delta \subset \mathbb{R}^d\) and let \(x_i \in \Omega\) be fixed. If for any unit vector \(v \in \mathbb{R}^d\), \(C^\delta_v(x_i) \cap X \setminus \{x_i\} \neq \emptyset\), then the feasible set to problem (3.3) with \(p = 2\) is not empty.

To discuss the neighborhood criteria for our problem, we first notice that for \(x_i \in \Omega\), one can define a one-to-one mapping between \(B^\delta(x_i)\) and \(E^\delta_i(x_i)\) by
\[
T_i x = x_i + M(x_i)(x - x_i) \quad x \in B^\delta(x_i).
\]
The inverse of \(T_i\) is then given by
\[
T_i^{-1} x = x_i + M(x_i)^{-1}(x - x_i) \quad x \in E^\delta_i(x_i).
\]
We also denote \(T_i(D) = \{y = T_i x : x \in D\}\) and \(T_i^{-1}(D) = \{y = T_i^{-1} x : x \in D\}\) for any set \(D \subset \mathbb{R}^d\).

![Figure 3. Illustration of \(T_i\) and \(T_i^{-1}\)](image-url)
Lemma 3.7. Take a point cloud $X = \{x_i\}_{i=1}^M \subset \Omega_2 \subset \mathbb{R}^d$ and let $x_i \in \Omega$ be fixed. For any $x_j \in \mathcal{E}_2^\delta(x_i) \cap X \setminus \{x_i\}$, we write $\tilde{x}_j = T_i^{-1}x_j$. Let $L_\delta^0 u(x_i)$ be defined by eq. (3.5) and

$$L_\delta^0 u(x_i) := \sum_{\tilde{x}_j \in B_\delta(x_i)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|\tilde{x}_j - x_i|}{\delta} \right) \tilde{\omega}_{j,i}(u(\tilde{x}_j) - u(x_i)).$$

The following statements are equivalent.

1. There exists $\{\tilde{\omega}_{j,i} \geq 0\}$ such that $L_\delta^0 u(x_i) = \tilde{L}_\delta^0 u(x_i) \forall u \in \mathcal{P}_2(\mathbb{R}^d)$. 
2. There exists $\{\omega_{j,i} \geq 0\}$ such that $L_\delta^0 u(x_i) = L_\delta u(x_i) \forall u \in \mathcal{P}_2(\mathbb{R}^d)$. 

Proof. By definition, we see that for any $x_j \in \mathcal{E}_2^\delta(x_i) \iff \tilde{x}_j \in B_\delta(x_i)$. We show that (1) and (2) are equivalent by letting $\tilde{\omega}_{j,i} := \det(M(x_i))^{-1}\omega_{j,i}$. Assume that (2) is true, then since $L_\delta u = 0$ if $u \in \mathcal{P}_1(\mathbb{R}^d)$, we have

$$0 = \sum_{x_j \in \mathcal{E}_2^\delta(x_i)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|M(x_i)x_j - x_i|}{\delta} \right) \det(M(x_i))^{-1}\omega_{j,i}M(x_i)^{-1}(x_j - x_i)$$

$$= \sum_{\tilde{x}_j \in B_\delta(x_i)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|\tilde{x}_j - x_i|}{\delta} \right) \det(M(x_i))^{-1}\omega_{j,i}(\tilde{x}_j - x_i).$$

By letting $\tilde{\omega}_{j,i} := \det(M(x_i))^{-1}\omega_{j,i}$, we see that $L_\delta^0 u(x_i) = 0 = \tilde{L}_\delta^0 u(x_i) \forall u \in \mathcal{P}_1(\mathbb{R}^d)$. Next by using $L_\delta^0 u(x_i) = L_\delta u(x_i) = A(x_i) : D^2u(x_i)$ for $u \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$2 = M(x_i)^{-1}(2A(x_i))M(x_i)^{-1}$$
$$= M(x_i)^{-1} \sum_{x_j \in \mathcal{E}_2^\delta(x_i)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|M(x_i)x_j - x_i|}{\delta} \right) \det(M(x_i))^{-1}\omega_{j,i}((x_j - x_i) \odot (x_j - x_i))M(x_i)^{-1}$$
$$= \sum_{\tilde{x}_j \in B_\delta(x_i)} \frac{1}{\delta^{d+2}} \gamma \left( \frac{|\tilde{x}_j - x_i|}{\delta} \right) \tilde{\omega}_{j,i}(\tilde{x}_j - x_i) \odot (\tilde{x}_j - x_i).$$

This implies that $L_\delta^0 u(x_i) = \tilde{L}_\delta^0 u(x_i) = \Delta u(x_i)$ when $u \in \mathcal{P}_2(\mathbb{R}^d)$. Therefore (2) implies (1). Similarly, we can show (1) also implies (2).

The following result is an implication of Lemma 3.6 and Lemma 3.7.

Corollary 3.8. Take a point cloud $X = \{x_i\}_{i=1}^M \subset \Omega_2 \subset \mathbb{R}^d$ and let $x_i \in \Omega$ be fixed. If for any unit vector $v \in \mathbb{R}^d$, $T_i(C_v^\delta(x_i)) \cap X \setminus \{x_i\} \neq \emptyset$, then $S_{\delta,h,2}(x_i)$ and $\overline{S}_{\delta,h,2}(x_i)$ are not empty.

Proof. First of all, from Lemma 3.6 and Lemma 3.7, it is easy to see that $T_i(C_v^\delta(x_i)) \cap X \setminus \{x_i\} \neq \emptyset$ for all unit vector $v \in \mathbb{R}^d$ implies $S_{\delta,h,2}(x_i)$ (as defined in eq. (3.7)) is not empty. Now if we define a new point cloud $\overline{X}$ by replacing all $x_j \in X \cap \mathcal{E}_2^\delta(x_i) \setminus \{x_i\}$ with $\overline{x}_j$ in $X$. Then since $\overline{x}_j$ lies on the line connecting $x_i$ and $x_j$, we see that

$$T_i(C_v^\delta(x_i)) \cap X \setminus \{x_i\} \neq \emptyset \implies T_i(C_v^\delta(x_i)) \cap \overline{X} \setminus \{x_i\} \neq \emptyset,$$

and the latter implies $S_{\delta,h,2}(x_i)$ is not empty by the same reasoning. See an illustration of the sets $T_i(C_v^\delta(x_i)) \cap X \setminus \{x_i\}$ and $T_i(C_v^\delta(x_i)) \cap \overline{X} \setminus \{x_i\}$ in Figure 4.
Although Corollary 3.8 is a complete characterization of a sufficient condition for the well-posedness of eq. (3.10). It is hard to use in practice. In the following, we proceed to give a sufficient condition that is easy to use in the case \( d = 2 \). We leave the proof of the following theorem in the Appendix A.

**Theorem 3.9.** Let \( d = 2 \) or \( d = 3 \) and \( h \) be the fill distance defined in eq. (3.8). Let \( \lambda_1 = \lambda_1(\mathbf{x}_i) \) denote the smallest eigenvalue of \( \mathbf{A}(\mathbf{x}_i) \). Then there exists a constant \( C > 0 \) such that if

\[
h \leq C \delta \sqrt{\lambda_1},
\]

then \( S_{\delta,h,2}(\mathbf{x}_i) \) and \( S_{\delta,h,2}(\mathbf{x}_i) \) are not empty. This implies if \( h \leq C \delta \sqrt{\varrho} \), then positive stencil exists.

**Remark 3.10.** Notice that for a given point \( \mathbf{x} \), the elliptical searching region surrounding \( \mathbf{x} \) has semi-axes \( \{ \delta \sqrt{\lambda_i(\mathbf{x})} \}_{i=1}^d \) where \( \lambda_i(\mathbf{x}) \) denotes the \( i \)-th smallest eigenvalue of \( \mathbf{A}(\mathbf{x}_i) \). By Theorem 3.9, we can choose an elliptical neighborhood of \( \mathbf{x} \) whose volume is proportional to \( h^d \left( \prod_{i=1}^d \varrho_i \right)^{-1/2} \), where \( \varrho_i = \varrho_i(\mathbf{x}) := \lambda_1(\mathbf{x})/\lambda_i(\mathbf{x}) \geq \varrho \). This implies that the number of points within the searching neighborhood of \( \mathbf{x} \) is proportional to \( \varrho_i^{-1/2} \leq \varrho^{-1/2} \) in 2d and \( (\varrho_2 \varrho_3)^{-1/2} \leq \varrho^{-1} \) in 3d. However, we do not have an explicit estimate of the constant \( C \) in Theorem 3.9. In practice, we estimate this constant numerically which is described in detail in Section 4.3.

**Remark 3.11.** Combining Theorem 3.9 with Theorem 3.4, one can take \( h = C \delta \sqrt{\varrho} \) and then the convergence rate is given by

\[
(3.16) \quad \max_{\mathbf{x}_i \in \Omega} |u(\mathbf{x}_i) - u_\delta^h(\mathbf{x}_i)| \leq C |u|_{C^{2+k,0}(\overline{\Omega})} \varrho^{-(k+\alpha)/2} h^{k+\alpha}.
\]

for \( k = 0 \) or \( k = 1 \) and \( \alpha \in (0,1] \).

4. Algorithm Design & Complexity Analysis

In this section, we explain our algorithms in detail, mainly focusing on point cloud generation and matrix assembly.
4.1. Point cloud generation. In order to perform numerical experiments, some criteria need to be given on the point cloud geometry. For this, we first need to define two geometric quantities with respect to point clouds in addition to the fill distance defined in eq. (3.8). For a point cloud $X = \{x_i\}_{i=1}^M \subset \Omega_\delta$, we define the separation distance $\zeta$ as

$$\zeta := \frac{1}{2} \min_{1 \leq i < j \leq M} |x_i - x_j|.$$  

(4.1)

In addition, for the points $\{x_i\}_{i=1}^N$ inside $\Omega$, we denote $\kappa$ the minimum distance to the boundary, i.e.,

$$\kappa := \min_{1 \leq i \leq N} \text{dist}(x_i, \partial \Omega).$$  

(4.2)

With these geometric quantities, we now define proper point clouds to be used in numerical experiments.

**Definition 4.1.** Let $X = \{x_i\}_{i=1}^M \subset \Omega_\delta$ be a point cloud with its geometric quantities $h$, $\zeta$ and $\kappa$ defined in eqs. (3.8), (4.1) and (4.2), respectively. Given a set of positive constants $\{c_h, c_\zeta, c_\kappa\}$, we say $X$ is a proper point cloud (with respect to the constants $\{c_h, c_\zeta, c_\kappa\}$) if it satisfies the following conditions:

(i) $h \leq c_h \left( \frac{|\Omega_\delta|}{|X|} \right)^{1/d}$

(ii) $\zeta \geq c_\zeta h$

(iii) $\kappa \geq c_\kappa h$

Notice that $|\Omega_\delta|$ denotes the $d$-dimensional Lebesgue measure of $\Omega_\delta$ and $|X| = M$.

**Remark 4.2.** Condition (ii) in Definition 4.1 essentially requires the point cloud to be quasi-uniform ([50]), and condition (iii) prevents interior points to be too close to the boundary which is important for the linear programming algorithm to work well for solving eq. (3.10). Notice that if (ii) is satisfied, then there exists $C = C(c_\zeta, d) > 0$ such that $h \leq C \left( \frac{|\Omega_\delta|}{|X|} \right)^{1/d}$. In practice, we impose condition (i) with a chosen constant $c_h > 0$ to have explicit control over the fill distance. In our numerical experiments in Section 5, we take $c_h = 1$, $c_\zeta = 0.175$, and $c_\kappa = 0.25$.

**Remark 4.3.** Since the domain $\Omega_\delta$ may be irregular, in practice, we always generate point clouds on a larger bounding box of $\Omega_\delta$, as indicated by figs. 5a to 5d. The formula for the fill distance in eq. (3.8) and condition (i) Definition 4.1 are then modified accordingly.

To generate a proper point cloud, we first initialize a random point cloud using the Quasi-Monte Carlo method [36] (see fig. 5a), then adjust this point cloud to make it proper. Adjustment contains three steps in each loop:

*Step 1.* add points until $h$ satisfies condition (i) (see fig. 5b);

*Step 2.* map points until $\kappa$ satisfies condition (iii) (see fig. 5c);

*Step 3.* merge points until $\zeta$ satisfies condition (ii) (see fig. 5d).

Adjustment stops when the conditions in Definition 4.1 are satisfied. In practice, it usually takes a few loops to make the point cloud proper.

**Remark 4.4.** One can also use the Quasi-Monte Carlo method for generating the initial point cloud and perform only step 2 without the adding and merging steps. Our adjustment algorithm provides explicit control over the fill distance and the separation distance and it leads to smaller fill distances for the same number of
**Figure 5.** The process of proper point cloud generation. The grey circular domain is $\Omega$. The square domain is a bounding box of $\Omega$. (A): initialize a point cloud by the Quasi-Monte Carlo method. (B): use the Voronoi diagram [48, 49] for the calculation of the fill distance, and then add the green points to the point cloud so that $h$ satisfies condition (i). (C): map points near the boundary of $\Omega$ to the interior so that $\kappa$ satisfies condition (iii). (D): merge points whose distances are less than $2c_\zeta h$ so that $\zeta$ satisfies condition (ii). Notice that after merging of points, the fill distance may increase, as a result, the adjustment loop may be needed again.

interior points compared with point clouds without adjustment. This is a trade-off situation, namely, one can save memory by using extra time adjusting the initialized point cloud, or vice versa.

4.2. Matrix assembly. The major effort in matrix assembly is the generation of the weights $\{\omega_{j,i}\}$ defined in eq. (3.10). Notice that with respect to each point cloud and coefficient matrix $A(x)$, we need to solve $N$ number of linear minimization problems to get the weights where $N$ denotes the number of interior points. For each $x_i \in \Omega$, we first need to find all the points inside the searching area, i.e., the domain $\mathcal{E}_\delta(x_i)$ for a given $\delta > 0$, then solve the linear minimization problem to get the stencil.
We now describe the process of finding points inside an elliptical searching area. Notice that the proper point cloud is quasi-uniform, an easy way to accelerate this procedure is dividing the domain into same-size axis-aligned blocks \([42, 50]\). We call these blocks voxels. Alternatively, point clouds can also be managed by k-d trees \([4, 50]\). To search neighbors in the given elliptical area \(E^\delta(x_i)\), we first compute which voxel contains the current point \(x_i\), then search all the neighboring voxels that intersect non-trivially with \(E^\delta(x_i)\). If a voxel that intersects non-trivially with the searching area is contained in that area, then we add all the points in the voxel to the result set; otherwise, points in the voxel need to be checked one by one. The intersection algorithm of voxels with ellipses/ellipsoids is crucial and we now describe it below.

Let \(H\) denote a \(d\)-dimensional (hyper)rectangle and \(E\) a \(d\)-dimensional ellipsoid. \(H\) and \(E\) are both open sets. We present an intersection detection algorithm in Algorithm 1 that distinguishes the following cases:

Case 1. \(H\) does not intersect with \(E\);
Case 2. \(H\) is contained in \(E\);
Case 3. \(H\) intersects with \(E\) but is not contained in \(E\).

**Algorithm 1:** Intersection detection of hyperrectangles with ellipsoids.

```python
function intersection(H, E)
    if the center of H is inside E then
        if all the vertices of H are inside E then
            return Case 2;
        else
            return Case 3;
    else if the center of E is inside H then
        return Case 3;
    else
        if any one of the faces of H intersects with E then
            return Case 3;
        else
            return Case 1;
```

Notice that in Algorithm 1, the most time-consuming part is the intersection detection of faces of \(H\) with \(E\). In 2d, the faces of a rectangle are line segments. Intersection detection of a line segment with an ellipse is relatively easy to carry out. One can first find the intersection (if exists) of the underlying line with the ellipse, which is a line segment (see fig. 6a as an illustration), by solving a quadratic equation. Then the intersection of two line segments can be easily checked. In 3d, to check whether a face of a 3d rectangle intersects with an ellipsoid, we first find the intersection area (if exists) of the underlying plane with the ellipse. Then since the intersection area (see fig. 6b as an illustration) is an ellipse, the problem is then reduced to the intersection detection of two-dimensional rectangles with ellipses. This can be further extended to higher dimensions, and a \(d\)-dimensional intersection problem can be reduced to a \(d-1\)-dimensional problem by this reasoning. Let \(Q_1(d)\) denote the complexity of the intersection algorithm in \(d\) dimensions. Notice that
a \(d\)-dimensional hyperrectangle has \(2d\) faces, we can then deduce the recurrence relation
\[
Q_I(d) \leq cdQ_I(d - 1) \quad \text{with} \quad Q_I(1) = \mathcal{O}(1).
\]
for some constant \(c > 0\) independent of \(d\). Finally, the recurrence relation leads to
\[
Q_I(d) = \mathcal{O}(c^d d!).
\]

**Remark 4.5.** The recursive algorithm for face-ellipsoid intersection detection gives a complexity (eq. (4.3)) that grows quickly with dimension. In this work, we only consider \(d = 2\) or \(d = 3\) so that \(Q_I(2)\) or \(Q_I(3)\) can be treated as constants. It will be of future interest to explore better intersection detection algorithms in higher dimensions.

**Remark 4.6.** The complexity for testing whether a hyperrectangle is contained in an ellipsoid is less than \(Q_I(d)\). In fact, since a hyperrectangle has \(2^d\) vertices, this gives the complexity \(O(2^d d^2)\). Notice that when the ellipsoid is not aligned with the axes, we need to do the mapping with complexity \(O(d^2)\) first, and then check with complexity \(O(d)\) for each vertex.

For \(x_i \in \Omega\), let \(q(x_i)\) denote the number of points in the searching area \(E^x_{\delta}(x_i)\), then the corresponding searching process needs \(O(q(x_i))\) intersection detections. For point clouds managed by k-d trees, it can be shown that \(O(q(x_i) \log M)\) intersection detections are needed for such range query [50]. Once we find all the points inside \(E^x_{\delta}(x_i)\), we proceed to solve the linear minimization problem eq. (3.10). We adopt the simplex method [9], which in average has a linear complexity in \(q(x_i)\) when the dimension is fixed [43].

Combining the above discussions, when the dimension \(d\) is fixed, the total average complexity of finding a stencil for a given interior point \(x_i\) is \(O(q(x_i))\). Note that by the quasi-uniform assumption (condition (2) in Definition 4.1), we have
\[
q(x_i) \leq C|E^x_{\delta}(x_i)| h^d
\]
for some \(C > 0\). The volume \(|E^x_{\delta}(x_i)|\) depends on \(\delta\) and the coefficient matrix \(A(x_i)\). In practice, we take \(h\) to be proportional to \(\delta\), and therefore \(|E^x_{\delta}(x_i)| = \mathcal{O}(1)\) considering \(A(x)\) to be fixed. As a result, the total complexity of searching for neighbors near a given point can be considered as a constant. In the near degenerate case, i.e., \(q \ll 1\), \(q(x_i)\) may grow with the decrease of \(q\) as mentioned in Remark 3.10.

**Figure 6.** Intersection illustration

(A) \(d = 2\), ellipse intersects with line  
(B) \(d = 3\), ellipsoid intersects with plane
Traversing all \( N \) number of interior points, we can get all the weights \( \{ \omega_{j,i} \} \) to complete the matrix assembly process. Therefore, the whole complexity of assembling a matrix is given by \( O(N) \) for a fixed problem. In addition, notice that the weights generation process is embarrassingly parallelizable, the actual computational time can be further reduced by parallelization.

**Remark 4.7.** From the \( l_1 \) type minimization, we get a minimal positive stencil, and therefore the assembled matrix is sparse. It is also recommended that a reindexing process be applied to the point cloud to reduce the bandwidth of the assembled matrix. The simplest way to do this is to sort all the interior points by coordinates so that the index distance between two close points is not too large.

**Remark 4.8.** One may encounter memory issues using exact solvers when the linear system gets large. Iterative methods can be used in the case of large and sparse linear systems. We use the biconjugate gradient stabilized method (BiCGSTAB) \([47]\) to approximately solve the sparse linear system when \( N \) is large.

### 4.3 Searching area estimate

Theorem 3.9 does not specify the constant \( C > 0 \) which determines the searching neighborhoods. Here we discuss how to determine the searching neighborhoods in practice. For a given fill distance \( h \), we let \( \delta = h/(C\sqrt{\overline{\rho}}) \), where the determination of \( C > 0 \) is described below. Then the searching neighborhood of a point \( x \) is the domain \( E_{x,\delta} \).

We now discuss the choice of \( C > 0 \) in practice. We first discuss the 2d case, and then use the 2d result to approximately estimate the searching area in 3d. Without loss of generality, we fix \( x_i \in \Omega \) and assume that

\[
A(x_i) = \begin{pmatrix} \varrho & 0 \\ 0 & 1 \end{pmatrix}.
\]

According to Lemma 6.1, we need to find the smallest radius of the inscribed circles of the domains \( \{ T_i(C^\rho_j(x_i)) \} \) \( \forall \overline{\rho} \in \mathbb{R}^2 \). The problem is a rescaling of the case \( \delta = 1 \), as illustrated by fig. 7. Therefore we only need to consider the case \( \delta = 1 \) and find \( r(\varrho) := \min_{\overline{\rho} \in \mathbb{R}^2} T_i(C^\rho_j(x_i)) \). The detailed procedure for finding \( r(\varrho) \) numerically is provided in Appendix B. One may choose \( C = \min_{\overline{\rho} \in [0,1]} r(\varrho)/\sqrt{\overline{\rho}} \) and then by letting \( \delta = h/(C\sqrt{\overline{\rho}}) \) we have the desired relation \( h \leq \delta r \). In practice, we find that the \( r(\varrho)/\sqrt{\overline{\rho}} \) is a bit larger with smaller \( \varrho > 0 \) using the estimate.
of \( r(\varrho) \). Therefore, for different values of \( \varrho \in (0, 1) \), we suggest taking \( C \) by the following formula in 2d

\[
C = C_{2d}(\varrho) := 0.352\chi_{(0,0.01)}(\varrho) + 0.344\chi_{(0.01,0.1)}(\varrho) + 0.276\chi_{(0.1,1)}(\varrho).
\]

In 3d, it is difficult to estimate the radius of the inscribed ball in \( T_i(C^v_i(x_i)) \). Therefore, we only take the intersecting ellipses of a given ellipsoid with the three planes that go through its principal axes, and perform the 2d estimate described above to obtain an estimate of the constant \( C > 0 \) in 3d. The result is given as follows.

\[
C = C_{3d}(\varrho) := 0.276\chi_{(0,0.01)}(\varrho) + 0.264\chi_{(0.01,0.1)}(\varrho) + 0.224\chi_{(0.1,1)}(\varrho).
\]

In practice, we find that solutions often exist for even smaller searching areas and this means that one may take even larger values of \( C > 0 \) to further decrease the computational cost. We suggest taking \( C = \sqrt{3}C_{2d}(\varrho) \approx 1.7C_{2d}(\varrho) \) in 2d and \( C = \sqrt{18}C_{3d} \approx 2.6C_{3d}(\varrho) \) in 3d first, and if no solution exists resetting \( C = C_{2d}(\varrho) \) in 2d and \( C = C_{3d}(\varrho) \) in 3d. This procedure could reduce the number of points in a searching neighborhood by a large factor.

5. Numerical Results

In this section, we report the results of numerical experiments for the study of the numerical accuracy of our method. We present 2d numerical results in Section 5.1 and 3d numerical results in Section 5.2.

5.1. 2d numerical tests. We test our numerical algorithm in 2d using two domains. The first domain is a unit disk given by \( \{x_1^2 + x_2^2 < 1\} \), and the second domain is an L-shaped domain given by \((-1,1)^2 \setminus [0,1]^2\). The nonlocal kernel function is chosen as \( \gamma(r) = C r^{-3} \chi_{(r<1)} \) so that eq. (5.1) is satisfied. We implement the numerical algorithm with \( p = 2 \). Smooth manufactured solutions are used in our tests with the right-hand side of eq. (1.1) computed based on them.

5.1.1. Tests for continuous coefficient matrices. We first test our algorithm for continuous coefficient matrices. Our baseline is \( A_0(x) = I \) with \( \varrho = 1 \). A list of coefficient matrices used in numerical experiments is given below.

| # | \( A(x) \)                                                                 | \( \varrho \) |
|---|----------------------------------------------------------------------------|-------------|
| 1 | \( \begin{pmatrix} 1 - 0.5|x_1| & 0 \\ 0 & 0.25 + 0.25|x_2| \end{pmatrix} \) | 0.2500     |
| 2 | \( \frac{1}{2.21} \begin{pmatrix} 2 - |x_1| & 0.5 \\ 0.5 & 0.5 + 0.5|x_2| \end{pmatrix} \) | 0.0864     |
| 3 | \( \begin{pmatrix} 1 - 0.5|x_1| & 0 \\ 0 & 0.025 + 0.025|x_2| \end{pmatrix} \) | 0.0250     |
| 4 | \( \begin{pmatrix} 1 - 0.5|x_1| & 0 \\ 0 & 0.0025 + 0.0025|x_2| \end{pmatrix} \) | 0.0025     |
| 5 | \( \frac{1}{2.001} \begin{pmatrix} 2 - |x_1| \left(0.5 - x_2\right) & 0.025 \\ 0.025 & 0.01 \left(0.0025 x_1 \exp(x_2)\right) \end{pmatrix} \) | 0.0014     |

Notice that the value \( \varrho \) is computed approximately in the domain \([-1,1]^2\), which contains both the unit disk and the L-shaped domain as subsets. Numerical results are presented in figs. 8 to 10. We observe second-order convergence in \( h \) for all cases, which is better than the theoretical analysis in Theorem 3.4 for \( p = 2 \). This
superconvergence phenomenon may likely be due to the cancellation of terms as mentioned in Remarks 3.2 and 3.5. In figs. 8 and 9, we test our method on two manufactured solutions \( u_1^{(2d)}(x_1, x_2) = x_1 x_2 + \cos(x_1) \exp(x_2) \) and \( u_2^{(2d)}(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1 (x_1 + 2x_2)) \). The numerical errors in these graphs grow as \( \rho \) becomes smaller as predicted by theory. In some very special cases, the numerical errors may behave differently as \( \rho \to 0 \), and one example is with \( u_3^{(2d)}(x_1, x_2) = x_1^2 + \sin(x_2) \exp(x_2^2 - 1) \) illustrated by fig. 10. The reason for this abnormal behavior is because the elliptic operator degenerates to \( \partial_x^2 \) as \( \rho \to 0 \) by our choices of \( A(x) \) and the exact solution in this case is a second order polynomial in \( x_1 \) which can be exactly reproduced by our method.

![Figure 8](image1.png)

**Figure 8.** 2d tests on the unit disk domain with continuous coefficient matrices

![Figure 9](image2.png)

**Figure 9.** 2d tests on the L-shaped domain with continuous coefficient matrices

5.1.2. Tests for discontinuous coefficient matrices. We now show numerical results for discontinuous coefficient matrices. Notice that when \( A(x) \) is discontinuous, the elliptic equation in the non-divergence form cannot be recast into a variational form. Therefore, non-variational methods are especially important in this case. We divide the computational domains into smaller blocks and define piecewise constant coefficient matrices with respect to the blocks. More specifically, for \( n \in \mathbb{N} \), we
divide the domain $[-1,1]^2$ into $(2n+1)^2$ blocks, and define the corresponding piecewise constant coefficient matrix

$$A_\psi(x,n) := (B_\psi(x,n) + B_\psi^T(x,n) + 4I)/8,$$

where $B_\psi(x,n)$ is generated by mt19937 [31] (a pseudorandom number generator) with seed $\psi(x,n) := \text{round}(x_1 * n) + 2 + \text{round}(x_2 * n) * 3 \mod 2^{32}$. Here round($x$) maps $x$ to the closest integer. A list of coefficient matrices used in our experiments is given below.

| #  | $A(x)$          | $\rho$ | Discription          |
|----|-----------------|--------|----------------------|
| 6  | $A_\psi(x,10^{10})$ | 0.2500 | dense blocks         |
| 7  | $A_\psi(x,10^4)$  | 0.2500 | medium blocks        |
| 8  | $A_\psi(x,10^9)$  | 0.2500 | loose blocks         |

In addition, we have the last example matrix:

$$A_9(x) = \begin{cases} A_2(x), & x_1 < 0 \\ A_3(x), & \text{otherwise} \end{cases} \quad \text{with} \quad \rho = 0.0250.$$

Numerical results are presented in figs. 11 and 12. We observe similar second-order convergence in $h$ for all cases.

**Figure 10.** 2d tests for a separable function

**Figure 11.** 2d tests on the unit disk domain with discontinuous coefficient matrices
5.2. 3d numerical tests. 3d numerical tests are performed over the unit sphere given by \( \{ x_1^2 + x_2^2 + x_3^2 < 1 \} \) and a 3d L-shaped domain given by \((-1, 1)^3 \times [-1, 1] \times [0, 1] \). The nonlocal kernel function is chosen to be the same one as in the 2d case. We again test our algorithm for smooth manufactured solutions and \( p = 2 \).

5.2.1. Tests for continuous coefficient matrices. For the test on continuous coefficient matrices, we use the following list of coefficient matrices. Notice again that our baseline case is \( A_0(x) = I \) with \( q = 1 \).

\[
\begin{align*}
\# & & A(x) & & q \\
1 & & \begin{pmatrix} 1 - 0.5|x_1| & 0 & 0 \\ 0 & 0.5 - 0.25|x_2| & 0 \\ 0 & 0 & 0.25 + 0.25|x_3| \end{pmatrix} & & 0.2500 \\
2 & & \frac{1}{2.21} \begin{pmatrix} 2 - |x_1| & 0 & 0 \\ 0 & 0.5 + 0.5|x_2| & 0 \\ 0 & 0 & 1 - 0.5|x_3| \end{pmatrix} & & 0.0864 \\
3 & & \begin{pmatrix} 1 - 0.5|x_1| & 0 & 0 \\ 0 & 0.05 - 0.025|x_2| & 0 \\ 0 & 0 & 0.025 + 0.025|x_3| \end{pmatrix} & & 0.0250 \\
4 & & \begin{pmatrix} 1 - 0.5|x_1| & 0 & 0 \\ 0 & 0.005 - 0.0025|x_2| & 0 \\ 0 & 0 & 0.0025 + 0.0025|x_3| \end{pmatrix} & & 0.0025 \\
5 & & \frac{1}{2.001} \begin{pmatrix} 2 - |x_1(0.5 - x_2)| & -0.02 & 0.005 \\ -0.02 & 0.005 + 0.005|x_1 + x_3| & -0.001 \\ 0.005 & -0.001 & 0.01 + 0.0025x_2 \exp(x_3) \end{pmatrix} & & 0.0014
\end{align*}
\]

Here the value \( q \) is computed approximately in the domain \([-1, 1]^3\). Numerical results are presented in figs. 13 and 14 for the two manufactured solutions \( u_1^{(3d)}(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3 + \cos(x_1) \exp(x_2 + x_3) \) and \( u_2^{(3d)}(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^4 \cos(x_1(x_1 + 2x_2 + 2x_3)) \). We observe similar second-order convergence in \( h \) for all cases and numerical errors grow as \( q \) becomes smaller.

5.2.2. Tests for discontinuous coefficient matrices. We now show numerical results for discontinuous coefficient matrices. Again, we divide the computational domains into smaller blocks and define piecewise constant coefficient matrices with respect
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Figure 13. 3d tests on the unit sphere domain with continuous coefficient matrices

Figure 14. 3d tests on the 3d L-shaped domain with continuous coefficient matrices

to the blocks. More specifically, for \( n \in \mathbb{N} \), we divide the domain \([-1, 1]^3\) into \((2n + 1)^3\) blocks, and define the corresponding piecewise constant coefficient matrix

\[
A_\psi(x, n) := \left( B_\psi(x, n) + B_\psi^T(x, n) + 4I \right) / 10,
\]

where \( B_\psi(x, n) \) is generated by \texttt{mt19937} with seed \( \psi(x, n) := \text{round}(x_1 * n) * 2 + \text{round}(x_2 * n) * 3 + \text{round}(x_3 * n) * 5 \mod 2^{32} \). A list of coefficient matrices used in our experiments is given below.

| # | \( A_\psi(x, n) \) | \( \rho \) | discription  |
|---|-----------------|-------|--------------|
| 6 | \( A_\psi(x, 10^{10}) \) | 0.1847 | dense blocks |
| 7 | \( A_\psi(x, 10^4) \) | 0.1847 | medium blocks |
| 8 | \( A_\psi(x, 10^6) \) | 0.1847 | loose blocks |

In addition, we have the last example matrix:

\[
A_9(x) = \begin{cases} 
A_2(x), & x_1 < 0 \\
A_3(x), & \text{otherwise}
\end{cases} \quad \text{with} \quad \rho = 0.0250.
\]

Numerical results are presented in figs. 15 and 16. We observe similar second-order convergence in \( h \) for all cases.
6. Conclusion

In this paper, we have presented a monotone meshfree finite difference method for linear elliptic equations in non-divergence form via integral relaxation. Minimal positive stencils are found through an $l_1$-type minimization problem within a local elliptical searching neighborhood of each point in a meshfree point cloud. For the treatment of Dirichlet boundary conditions, a mapping strategy near the boundary is incorporated into the numerical scheme. Convergence is guaranteed by the consistency and monotonicity of the scheme and efficient solvers can be designed by the sparsity of the resulting linear system. It is essential to characterize the shape and size of the elliptical searching neighborhood for the guarantee of positive stencils. Our theoretical result improves the previously known result for the stencil sizes when $\varrho$, the ratio between the smallest and the largest eigenvalues of the coefficient matrix, is a very small number. More precisely, our theory predicts that within an elliptical region with semi-major axis proportional to $\varrho^{-1/2} h$, we are able to find a positive stencil. The searching region determines the size of the $l_1$-type minimization problem, and therefore the efficiency of our algorithm. Our theory predicts that the number of points within the searching neighborhood grows with $1/\varrho$ with a rate not worse than $O(\varrho^{-1/2})$ in 2d and $O(\varrho^{-1})$ in 3d.
We present algorithms for point cloud management and matrix assembly. We also give practical guidance for finding the elliptical searching neighborhood and present numerical experiments. Numerical tests are presented in both 2d and 3d for several different domains and coefficient matrices, including the near degenerate cases when \( \theta \ll 1 \). While theoretical convergence in \( h \) for the numerical method (when the polynomial order \( p = 2 \)) is only first order, we observe second-order convergence in all cases for manufactured smooth solutions. The super-convergence is likely due to the cancellation of odd order terms for the stencils obtained from the \( l_1 \)-type minimization problem. A rigorous explanation for this phenomenon is still an open question.

Our current study focuses on the case of \( p = 2 \) and \( d \in \{2, 3\} \) with Dirichlet boundary conditions. Future work includes higher order methods, problems in higher dimensions, and Neumann boundary value problems. Extending the study to surface PDEs is also a natural direction for future research. While we only test our algorithm for smooth manufactured solutions, adaptive methods will be useful when solutions display singularity. For adaptive point cloud management, some data structures that support fast insertion and deletion may be needed, for instance, R-tree [22] and scapegoat k-d tree [20]. The topic of monotone schemes for solving PDEs has a long history in numerical analysis. While our new ideas, inspired by the recent development of nonlocal modeling and meshfree methods, are presented for the linear elliptic equations, extending them to other types of PDEs is also possible for future research.

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APPENDIX A

We show the proof of Theorem 3.9 in Appendix A. We begin with some useful lemmas before proving the theorem.

Lemma 6.1. Let \( r(v, x_i) \) denote the radius of the inscribed ball in \( T_i(C_v^\delta(x_i)) \) and \( h \) be the fill distance associated with \( X = \{x_i\}_{i=1}^M \subset \Omega_\delta \). If

\[
h < \min_{v \in \mathbb{R}^d, |v| = 1} r(v, x_i),
\]

then \( S_{\delta,h,2}(x_i) \) and \( \overline{S}_{\delta,h,2}(x_i) \) are not empty.

Proof. Notice that by the definition of the fill distance in eq. (3.8), there are no holes with a radius larger than \( h \). Suppose \( S_{\delta,h,2}(x_i) \) or \( \overline{S}_{\delta,h,2}(x_i) \) is empty, then by Corollary 3.8, there exists \( v \) such that \( T_i(C_{v}^\delta(x_i)) \) contains no point in \( X \setminus \{x_i\} \). Therefore the inscribed ball in \( T_i(C_{v}^\delta(x_i)) \) is a hole with radius larger than \( h \) by the assumption, which gives a contradiction. \( \square \)

From the lemma above, our goal is then to get a lower bound for

\[
\min_{v \in \mathbb{R}^d, |v| = 1} r(v, x_i)
\]
for each \( \mathbf{x}_i \in \Omega \). We first present a result in 2d which will also be useful for the 3d estimates. In 2d, we assume that \( C^v_\delta(\mathbf{x}_i) \) is a cone with total opening angle \( 2\phi \). In addition, without loss of generality, we fix \( \mathbf{x}_i \in \Omega \) and assume that

\[
(6.1) \quad A(\mathbf{x}_i) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.
\]

From symmetry, we only need to consider \( \mathbf{v}(\theta) = (\cos(\theta), \sin(\theta)) \) for \( \theta \in [0, \frac{\pi}{2}] \).

**Lemma 6.2.** Consider \( d = 2 \) and \( A(\mathbf{x}_i) \) given by eq. (6.1). Assume that \( C^v_\delta(\mathbf{x}_i) \) is a cone with total opening angle \( 2\phi \) for \( \phi \in (0, \frac{\pi}{2}] \), and \( r(\mathbf{v}, \mathbf{x}_i) \) denote the radius of the inscribed circle in \( T_i(C^v_\delta(\mathbf{x}_i)) \). In addition, let \( \mathbf{v}(\theta) = (\cos(\theta), \sin(\theta)) \) for \( \theta \in [0, \frac{\pi}{2}] \). Then, there exists a constant \( c = c(\phi) > 0 \) such that

\[
\min_{\theta \in [0, \frac{\pi}{2}]} r(\mathbf{v}(\theta), \mathbf{x}_i) \geq c\delta \sqrt{q}.
\]

**Proof.** We try to fit a cone in \( T_i(C^v_\delta(\mathbf{x}_i)) \) and then find the inscribed circle in the cone. First, notice that for a cone with a radius \( R \) and total opening angle \( \alpha \in (0, \pi) \), the radius of the inscribed circle is given by the formula

\[
\frac{\sin(\alpha/2)}{1 + \sin(\alpha/2)} R \geq \frac{1}{2} \sin(\alpha/2) R.
\]

Notice that \( \frac{1}{2} \sin(\alpha/2) R \) increases with \( \alpha \in (0, \pi) \) and \( R \). Now for \( \theta \in [0, \frac{\pi}{2}] \), we let \( \Gamma(\theta) \) denote the opening angle of \( T_i(C^v_\delta(\mathbf{x}_i)) \) and define

\[
R(\theta) := \min_{\varphi \in [\theta-\phi, \theta+\phi]} \delta \sqrt{\varphi \cos^2(\varphi) + \sin^2(\varphi)} = \min_{\varphi \in [\theta-\phi, \theta+\phi]} \delta \sqrt{\varphi + (1 - q) \sin^2(\varphi)},
\]

then it is easy to see that a cone with radius \( R(\theta) \) and total opening angle \( \Gamma(\theta) \) is contained in \( T_i(C^v_\delta(\mathbf{x}_i)) \). Therefore we have

\[
\min_{\theta \in [0, \frac{\pi}{2}]} r(\mathbf{v}(\theta), \mathbf{x}_i) \geq \min_{\theta \in [0, \frac{\pi}{2}]} \frac{1}{2} \sin(\Gamma(\theta)/2) R(\theta).
\]

By calculation we have

\[
\Gamma(\theta) = \begin{cases} 
\arctan(\sqrt{q}^{-1} \tan(\theta + \phi)) - \arctan(\sqrt{q}^{-1} \tan(\theta - \phi)), & \theta \in [0, \pi/2 - \phi), \\
\pi + \arctan(\sqrt{q}^{-1} \tan(\theta + \phi)) - \arctan(\sqrt{q}^{-1} \tan(\theta - \phi)), & \theta \in (\pi/2 - \phi, \pi/2],
\end{cases}
\]

and

\[
R(\theta) = \begin{cases} 
\delta \sqrt{q}, & \theta \in [0, \phi], \\
\delta \sqrt{q + (1 - q) \sin^2(\theta - \phi)}, & \theta \in [\phi, \pi/2].
\end{cases}
\]

For \( \theta \in [0, \phi] \), \( \Gamma(\theta) \) decreases and \( R(\theta) = \delta \sqrt{q} \), so

\[
\min_{\theta \in [0, \phi]} \frac{1}{2} \sin(\Gamma(\theta)/2) R(\theta) = \frac{1}{2} \sin(\Gamma(\phi)/2) \delta \sqrt{q} \geq \frac{1}{2} \sin(\phi/2) \delta \sqrt{q}
\]

where we have used \( 2\phi \leq \Gamma(\phi) \leq \pi/2 \).
For $\theta \in [\pi/2 - 2\phi, \pi/2]$, $\Gamma(\theta)$ decreases and $R(\theta)$ increases, so

$$\min_{\theta \in [\pi/2 - 2\phi, \pi/2]} \frac{1}{2} \sin(\Gamma(\theta)/2) R(\theta)$$

$$\ge \frac{1}{2} \sin(\pi/2)/2 R(\pi/2 - 2\phi) \ge \frac{\delta}{2} \sin(\pi/2)/2 \sin(\pi/8)$$

$$= \frac{\sin(\pi/8)\delta}{2} \sqrt{\frac{\theta}{\sqrt{\theta + \tan^2(\pi/2 - \phi)}}} \ge \frac{\sin(\pi/8)}{2\sqrt{1 + \tan^2(3\pi/8)}} \delta \sqrt{\rho},$$

where we have used $\phi \le \pi/8$ and $\Gamma(\pi/2) = 2\arctan(\sqrt{\rho}\cot(\pi/2 - \phi))$.

Now for $\theta \in [\phi, \pi/2 - 2\phi]$, we use the formulas for $\Gamma(\theta)$ and $R(\theta)$ to compute $\frac{1}{2} \sin(\Gamma(\theta)/2) R(\theta)$. Denote

$$\alpha = \arctan(\sqrt{g^{-1}}\tan(\theta + \phi)) \quad \text{and} \quad \beta = \arctan(\sqrt{g^{-1}}\tan(\theta - \phi)).$$

Use the formula

$$\sin \left( \frac{\alpha - \beta}{2} \right) = \sqrt{\frac{1 - \cos(\alpha - \beta)}{2}} = \sqrt{\frac{1 - \cos(\alpha) \cos(\beta) - \sin(\alpha) \sin(\beta)}{2}},$$

and the fact that

$$\sin(\alpha) = \frac{\sqrt{g^{-1}}\tan(\theta + \phi)}{\sqrt{1 + g^{-1}\tan^2(\theta + \phi)}}, \quad \cos(\alpha) = \frac{1}{\sqrt{1 + g^{-1}\tan^2(\theta + \phi)}},$$

$$\sin(\beta) = \frac{\sqrt{g^{-1}}\tan(\theta - \phi)}{\sqrt{1 + g^{-1}\tan^2(\theta - \phi)}}, \quad \cos(\beta) = \frac{1}{\sqrt{1 + g^{-1}\tan^2(\theta - \phi)}},$$

we can obtain the formula for $\frac{1}{2} \sin(\Gamma(\theta)/2) R(\theta)$ where $\theta \in [\phi, \pi/2 - 2\phi]$. In particular, denote $g_1(\theta) = \tan^2(\theta - \phi)$, $g_2(\theta) = \tan^2(\theta + \phi)$, and $g_3(\theta) = \sin^2(\theta - \phi)$, we find

$$\frac{1}{2} \sin(\Gamma(\theta)/2) R(\theta)$$

$$= \frac{\delta}{2\sqrt{2}} \sqrt{\frac{\sqrt{(\theta + g_1(\theta))(\theta + g_2(\theta))} - \theta - \sqrt{g_1(\theta)g_2(\theta)}}{(\theta + g_1(\theta))(\theta + g_2(\theta))} (\theta + (1 - \theta)g_3(\theta))}$$

$$= \frac{\delta}{2} \sqrt{\frac{(\sqrt{g_1(\theta)})^{1/2} - (\sqrt{g_2(\theta)})^{1/2}}{(\theta + g_1(\theta))(\theta + g_2(\theta))} (\theta + (1 - \theta)g_3(\theta))}$$

$$= \frac{\delta}{2} \sqrt{\frac{(\theta + g_1(\theta))(\theta + g_2(\theta))}{(\theta + g_1(\theta))(\theta + g_2(\theta)) + \theta + \sqrt{g_1(\theta)g_2(\theta)}}}$$

$$= \frac{\delta}{2} G(\rho, \theta).$$

Notice that $G(\rho, \theta)$ defined in the above is a continuous function on $(\rho, \theta) \in [0, 1] \times [\phi, \pi/2 - 2\phi]$. Therefore it attains a minimum value at some $(\rho^*, \theta^*) \in [0, 1] \times [\phi, \pi/2 - 2\phi]$. Next we show that we must have $G(\rho^*, \theta^*) > 0$. Indeed, if $\rho^* > 0$, then it is easy to see that $G(\rho^*, \theta^*) > 0$. Now if $\rho^* = 0$, then

$$G(0, \theta) = \sqrt{\frac{(\sqrt{g_1(\theta)})^{1/2} - (\sqrt{g_2(\theta)})^{1/2}}{2g_1(\theta)g_2(\theta)}} g_3(\theta)$$

$$= (\tan(\theta + \phi) - \tan(\theta - \phi)) \frac{\sin(\theta - \phi)}{2\tan(\theta + \phi) \tan(\theta + \phi)}$$

$$= (\tan(\theta + \phi) - \tan(\theta - \phi)) \frac{\cos(\theta - \phi)}{2\tan(\theta + \phi)} > 0.$$
for any \( \theta \in [\phi, \pi/2 - 2\phi] \). Therefore, we can take
\[
c = \min \left\{ \frac{\sin(\phi)}{2}, \frac{\sin(\pi/8)}{2\sqrt{1 + \tan^2(3\pi/8)}}, G(q^*, \theta^*)/2 \right\} > 0
\]
for the claim to be true. \( \square \)

**Proof of Theorem 3.9.** Let \( \lambda_j = \lambda_j(x_i) \) denotes the \( j \)-th smallest eigenvalue of \( A(x_i) \). For \( d = 2 \), we apply Lemma 6.2 with \( \phi = \pi/8 \) on a rescaled ellipse of \( T_i(B_3(x_i)) \), we get
\[
\min_{\theta \in [0, \pi/2]} r(v(\theta), x_i) \geq c\delta \sqrt{\lambda_2} \frac{\sqrt{\lambda_1}}{\lambda_2} = c\delta \sqrt{\lambda_1}.
\]

Now consider \( d = 3 \). First of all, we can assume without loss of generality that \( \lambda_3 = 1 \) by the method of rescaling. By the discussions at the beginning of Section 3.4, \( C^w_\theta(x_i) \) is a 3d cone with total opening angle \( 33.7^\circ \) for a given unit vector \( v \in \mathbb{R}^3 \). Let \( P_v \subset \mathbb{R}^3 \) be a 2d plane that contains the vector \( v \), then we see that \( P_v \cap B_5(x_i) \) is a circular domain and \( P_v \cap C^w_\theta(x_i) \) is a 2d cone with total opening angle \( 33.7^\circ \). With the transform \( T_i \), we see that \( T_i(P_v \cap B_3(x_i)) \) is a 2d ellipse and \( T_i(P_v \cap C^w_\theta(x_i)) \) is a section of the ellipse. Therefore, the 2d calculations in Lemma 6.2 can be applied. Notice that for each \( P_v \), there exists \( \rho_1 \) and \( \rho_2 \) with \( \lambda_1 \leq \rho_1 \leq \rho_2 \leq 1 \) such that the lengths of the semi-axes of the ellipse \( T_i(P_v \cap B_3(x_i)) \) are given by \( \sqrt{\rho_1} \) and \( \sqrt{\rho_2} \). We can then rescale the ellipse \( T_i(P_v \cap B_3(x_i)) \) and use Lemma 6.2 with \( \phi = 33.7^\circ/2 \) we see that the radius of the inscribed circle in \( T_i(P_v \cap C^w_\theta(x_i)) \) has a lower bound
\[
\tilde{c}(\delta \sqrt{\rho_2}) \sqrt{\frac{\rho_1}{\rho_2}} \geq \tilde{c}\delta \sqrt{\lambda_1},
\]
for some \( \tilde{c} = \tilde{c}(\phi) > 0 \). Notice that \( P_v \) is an arbitrary plane that contains \( v \), and in addition, the average length of the line segments that connect \( T_i(x_i) \) and the edge of \( T_i(P_v \cap C^w_\theta(x_i)) \) is at the same scale for different plane \( P_v \). Therefore, it can be shown that there exists \( c > 0 \) such that
\[
\min_{\theta \in [0, \pi/2]} r(v(\theta), x_i) \geq c\delta \sqrt{\lambda_1}.
\]

At last, by Lemma 6.1, whenever \( h \leq c\delta \sqrt{\lambda_1(x_i)} \), \( S_{s,h,2}(x_i) \) and \( \overline{S_{s,h,2}}(x_i) \) are not empty. By our assumption, \( \varrho \leq \lambda_1(x_i) \) for all \( x_i \), and therefore \( h \leq c\delta \sqrt{\varrho} \) implies the existence of positive stencils. \( \square \)

**Appendix B**

We show the details of finding the radius of the inscribed circle in \( T_i(C^w_\theta(x_i)) \) that contained in an ellipse given by \( x^2/\varrho + y^2 = 1 \). Let \((x_0, y_0)\) be the center of the inscribed circle with radius \( r \), then it can only sit on the angle bisection of \( T_i(C^w_\theta(x_i)) \). Let \((x_1, y_1)\) and \((x_2, y_2)\) represent the two corner points of \( T_i(C^w_\theta(x_i)) \). Then by some elementary calculations, there exists some \( t > 0 \) such that
\[
(x_0, y_0) = t \left( \sqrt{x_1^2 + y_1^2} x_1 + \sqrt{x_1^2 + y_1^2} x_2, \sqrt{x_1^2 + y_1^2} y_1 + \sqrt{x_1^2 + y_1^2} y_2 \right)
\]
and
\[
r = t|x_1 y_2 - y_1 x_2|.
\]
To determine $t > 0$, we find the closest point to the circle center on the ellipse and choose $t > 0$ such that the point is also on the circle. The closest point to the circle center on the ellipse can be found by the minimization problem

$$
\min_\theta \| (\sqrt{2} \cos(\theta), \sin(\theta)) - (x_0, y_0) \|_2^2,
$$

which can be solved by, e.g., Newton’s method. At last, one may use a numerical method, e.g., the bisection method, to determine $t > 0$ such that the point is also on the circle.

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