ON NON-ADMISSIBLE IRREDUCIBLE MODULO $p$ REPRESENTATIONS OF $GL_2(\mathbb{Q}_{p^2})$

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Abstract. We use a Diamond diagram attached to a 2-dimensional reducible split mod $p$ Galois representation of $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_{p^2})$ to construct a non-admissible smooth irreducible mod $p$ representation of $GL_2(\mathbb{Q}_{p^2})$ following the approach of Daniel Le.

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1. Introduction

Let $p$ be a prime number, $\mathbb{Q}_p$ be the field of $p$-adic numbers, and $\overline{\mathbb{F}}_p$ be the algebraic closure of the finite field $\mathbb{F}_p$ of cardinality $p$. The study of the admissibility of smooth irreducible representations of connected reductive $p$-adic groups goes back to Harish-Chandra ([HC70]). Building upon his work, Jacquet proved that every such representation over the field of complex numbers is admissible ([Jac75], see also Bernstein [Ber74]). This result was extended by Vignéras to smooth irreducible representations over any algebraically closed field of characteristic not equal to $p$ ([Vig96]). It is also known that every smooth irreducible representation of $GL_2(\mathbb{Q}_p)$ over $\mathbb{F}_p$ is admissible (see Berger [Ber12]). Let $\mathbb{Q}_{p^2}$ be the unramified extension of $\mathbb{Q}_p$ of degree 2. In this paper, we establish the existence of a non-admissible smooth irreducible $\overline{\mathbb{F}}_p$-linear representation of $GL_2(\mathbb{Q}_{p^2})$, for $p > 2$, following the approach of Daniel Le ([Le19]). Our result supports the viewpoint of Breuil and Paškūnas that the mod $p$ (and $p$-adic) representation theory of $GL_2(\mathbb{Q}_{p^2})$ is more complicated than that of $GL_2(\mathbb{Q}_p)$ ([BP12], see also Schraen [Sch15]).

Let $G = GL_2(\mathbb{Q}_{p^2})$, $K = GL_2(\mathbb{Z}_{p^2})$, and $\Gamma = GL_2(\mathbb{F}_{p^2})$, where $\mathbb{Z}_{p^2}$ is the ring of integers of $\mathbb{Q}_{p^2}$ with residue field $\mathbb{F}_{p^2}$. Fix an embedding $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$. Let $I$ and $I_1$ denote the Iwahori and the pro-$p$ Iwahori subgroups of $K$ respectively, and $K_1$ denote the first principal congruence subgroup of $K$. Write $N$ for the normalizer of $I$ (and of $I_1$) in $G$. As a group, $N$ is generated by $I$, the center $Z$ of $G$, and by the element $\Pi = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. All representations

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considered in this paper from now on are over $\overline{\mathbb{F}}_p$-vector spaces. For a character $\chi$ of $I$, $\chi^*$ denotes its $\Pi$-conjugate sending $g$ in $I$ to $\chi(\Pi g \Pi^{-1})$.

A weight is a smooth irreducible representation of $K$. The $K$-action on such a representation factors through $\Gamma$ and thus any weight is described by a 2-tuple $(r_0, r_1) \otimes \det^m := \text{Sym}^r_0 \mathbb{F}_p^2 \otimes (\text{Sym}^r_1 \mathbb{F}_p^2)_{\text{Prob}} \otimes \det^m$ of integers with $0 \leq r_0, r_1 \leq p - 1$ together with a determinant twist for some $0 \leq m < p^2 - 1$ ([Bre07], Lemma 2.16 and Proposition 2.17). Given a weight $\sigma$, its subspace $\sigma^{I_1}$ of $I_1$-invariants has dimension 1. If $\chi_\sigma$ denotes the corresponding smooth character of $I$, then there exists a unique weight $\sigma^*$ such that $\chi_{\sigma^*} = \chi_\sigma^*$ ([Pas04], Theorem 3.1.1).

A basic $0$-diagram is a triplet $(D_0, D_1, r)$ consisting of a smooth $KZ$-representation $D_0$, a smooth $N$-representation $D_1$ and an $IZ$-equivariant isomorphism $r : D_1 \xrightarrow{\sim} D_0^1$ with the trivial action of $p$ on $D_0$ and $D_1$. Given such a diagram such that $D_0^{K_1}$ has finite dimension, the smooth injective $K$-envelope $\text{inj}_KD_0$ admits a non-canonical $N$-action which glues together with the $K$-action to give a smooth $G$-action on $\text{inj}_KD_0$ ([BP12], Theorem 9.8). The $G$-subrepresentation of $\text{inj}_KD_0$ generated by $D_0$ is smooth admissible and its $K$-socle equals the $K$-socle $\text{soc}_KD_0$ of $D_0$.

From now on, assume that $p$ is odd. Let $\rho : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^2) \to GL_2(\overline{\mathbb{F}}_p)$ be a continuous generic Galois representation such that $p$ acts trivially on its determinant and $D(\rho)$ be the set of weights, called Diamond weights, associated to $\rho$ as described in [BP12], Section 11. Breuil and Paškūnas attach a family of basic $0$-diagrams $(D_0(\rho), D_1(\rho), r)$, called Diamond diagrams, to $\rho$ such that $\text{soc}_KD_0(\rho) = \bigoplus_{\sigma \in D(\rho)} \sigma$ ([BP12], Theorem 13.8).

For a finite unramified extension $F$ of $\mathbb{Q}_p$ of degree at least 3, Le uses a Diamond diagram attached to an irreducible $\rho : \text{Gal}(\overline{\mathbb{Q}}_p/F) \to GL_2(\overline{\mathbb{F}}_p)$ to construct an infinite dimensional smooth irreducible representation of $GL_2(F)$ ([Le19]). His strategy does not work for a Diamond diagram attached to an irreducible Galois representation of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^2)$ because such a diagram does not have suitable $\Pi$-action dynamics. However, for $F = \mathbb{Q}_p^2$, we observe that a Diamond diagram attached to a reducible split $\rho$ has an indecomposable subdiagram with suitable $\Pi$-action dynamics so that Le’s method can be used to obtain a non-admissible irreducible representation of $G = GL_2(\mathbb{Q}_p^2)$.

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2. Reducible Diamond diagram

Let $\omega_2$ be Serre’s fundamental character of level 2 for the fixed embedding $\mathbb{F}_p^2 \hookrightarrow \overline{\mathbb{F}}_p$, and let $\rho : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^2) \to GL_2(\overline{\mathbb{F}}_p)$ be a continuous reducible split generic Galois representation. The restriction of $\rho$ to the inertia subgroup is, up to a twist by some character, isomorphic to

\[
\begin{pmatrix}
\omega_2^{ro_1+1+(r_1+1)p} & 0 \\
0 & 1
\end{pmatrix}
\]
for some $0 \leq r_0, r_1 \leq p - 3$, not both equal to 0 or equal to $p - 3$ ([Bre07], Corollary 2.9 (i) and [BP12], Definition 11.7 (i)). Define the weight

$$\sigma := (r_0 + 1, p - 2 - r_1) \otimes \det^{p - 1 + r_1 p}.$$  

Then the set of Diamond weights for $\rho$ is given by

$$\mathcal{D}(\rho) = \{(r_0, r_1), \sigma, \sigma^s, (p - 3 - r_0, p - 3 - r_1) \otimes \det^{r_0 + 1 + (r_1 + 1)p}\}$$

([BP12], Lemma 11.2 or Section 16, Example (ii)). Fix a Diamond diagram $(D_0(\rho), D_1(\rho), r)$ attached to $\rho$, and identify $D_1(\rho)$ with $D_0(\rho)^{\Pi_1}$ as IZ-representations via $r$. There is a direct sum decomposition $D_0(\rho) = \bigoplus_{\nu \in D(\rho)} D_{0, \nu}(\rho)$ of $K$-representations with $\soc K D_{0, \nu}(\rho) = \nu$ ([BP12], Proposition 13.4).

Now define

$$D_0 := D_{0, \sigma}(\rho) \oplus D_{0, \sigma^s}(\rho) \text{ and } D_1 := D_0^{\Pi_1}.$$  

It follows from [BP12], Theorem 15.4 (ii) that $(D_0, D_1, r)$ is an indecomposable subdiagram of $(D_0(\rho), D_1(\rho), r)$. Set

$$\tau := (r_0 + 2, r_1) \otimes \det^{p - 2 + (p - 1)p} \text{ and } \tau' := (p - 1 - r_0, p - 3 - r_1) \otimes \det^{r_0 + (r_1 + 1)p}.$$  

The graded pieces of the socle filtrations of $D_{0, \sigma}(\rho)$ and $D_{0, \sigma^s}(\rho)$ are as follows ([BP12], Theorem 14.8 or Section 16, Example (ii)):

$$D_{0, \sigma}(\rho) : \quad \sigma \quad \tau \oplus \tau^s \quad (p - 4 - r_0, r_1 - 1) \otimes \det^{r_0 + 2}$$

$$D_{0, \sigma^s}(\rho) : \quad \sigma^s \quad \tau' \oplus \tau'^s \quad (r_0 - 1, p - 4 - r_1) \otimes \det^{(r_1 + 2)p}.$$  

We have from [BP12], Corollary 14.10 that

$$D_1 = \chi_\sigma \oplus \chi_\tau \oplus \chi_\sigma^s \oplus \chi_\tau^s \oplus \chi_\sigma^s \oplus \chi_\tau^s.$$  

For an IZ-representation $V$ and an IZ-character $\chi$, we write $V^\chi$ for the $\chi$-isotypic part of $V$.

3. THE INFINITE DIMENSIONAL DIAGRAM AND THE CONSTRUCTION

Let $D_0(\infty) := \bigoplus_{i \in \mathbb{Z}} D_0(i)$ be the smooth $KZ$-representation with component-wise $KZ$-action, where there is a fixed isomorphism $D_0(i) \cong D_0$ of $KZ$-representations for every $i \in \mathbb{Z}$. Following [Le19], we denote the natural inclusion $D_0 \hookrightarrow D_0(i) \hookrightarrow D_0(\infty)$ by $\iota_i$, and write $v_i := \iota_i(v)$ for $v \in D_0$ for every $i \in \mathbb{Z}$. Let $D_1(\infty) := D_0(\infty)^{\Pi_1}$. We define the $\Pi$-action on $D_1(\infty)$ as follows. Let $\lambda = (\lambda_i) \in \prod_{i \in \mathbb{Z}} \mathbb{F}_p^\times$. For all integers $i \in \mathbb{Z}$, define

$$\Pi v_i := \begin{cases} (\Pi v)_i & \text{if } v \in D_1^{\chi^s}, \\ (\Pi v)_{i+1} & \text{if } v \in D_1^{\chi^s}, \\ \lambda_i (\Pi v)_i & \text{if } v \in D_1^{\chi^s}. \end{cases}$$  

This uniquely determines a smooth $N$-action on $D_1(\infty)$ such that $p = \Pi^2$ acts trivially on it. Thus we get a basic 0-diagram $D(\lambda) := (D_0(\infty), D_1(\infty), \can)$ with the above actions where $\can$ is the canonical inclusion $D_1(\infty) \hookrightarrow D_0(\infty)$. 
Theorem 3.1. There exists a smooth representation \( \pi \) of \( G \) such that

1. \( (\pi|_{KZ}, \pi|_N, \text{id}) \) contains \( D(\lambda) \),
2. \( \pi \) is generated by \( D_0(\infty) \) as a \( G \)-representation, and
3. \( \text{soc}_K \pi = \text{soc}_K D_0(\infty) \).

Proof. Let \( \Omega \) be the smooth injective \( K \)-envelope of \( D_0 \) equipped with the \( KZ \)-action such that \( p \) acts trivially. The smooth injective \( I \)-envelope \( \text{inj}_I D_1 \) of \( D_1 \) appears as an \( I \)-direct summand of \( \Omega \). Let \( e \) denote the projection of \( \Omega \) onto \( \text{inj}_I D_1 \). There is a unique \( N \)-action on \( \text{inj}_I D_1 \) compatible with that of \( I \) and compatible with the action of \( N \) on \( D_1 \). By [BPT12], Lemma 9.6, there is a non-canonical \( N \)-action on \((1 - e)(\Omega) \) extending the given \( I \)-action. This gives an \( N \)-action on \( \Omega \) whose restriction to \( IZ \) is compatible with the action coming from \( KZ \) on \( \Omega \).

Now let \( \Omega(\infty) := \bigoplus_{i \in \mathbb{Z}} \Omega(i) \) with component-wise \( KZ \)-action where there is a fixed isomorphism \( \Omega(i) \cong \Omega \) of \( KZ \)-representations for every \( i \in \mathbb{Z} \). We wish to define a compatible \( N \)-action on \( \Omega(\infty) \). As before, denote the natural inclusion \( \Omega \hookrightarrow \Omega(i) \hookrightarrow \Omega(\infty) \) by \( \iota_i \), and write \( v_i := \iota_i(v) \) for \( v \in \Omega \). Let \( \Omega_\chi \) denote the smooth injective \( I \)-envelope of an \( I \)-character \( \chi \). Thus, from [2], we have \( e(\Omega) = \text{inj}_I D_1 = \Omega_{\chi_0} \oplus \Omega_{\chi_1} \equiv \Omega_{\chi_0} \oplus \Omega_{\chi_1} \oplus \Omega_{\chi_2} \oplus \Omega_{\chi_3} \oplus \Omega_{\chi_3^*} \). If \( v \in (1 - e)(\Omega) \), we define \( \Pi v_i := (\Pi v)_i \) for all integers \( i \). Otherwise, we define \( \Pi v_i := (\Pi v)_i \) if \( v \in \Omega_{\chi_0} \); \( \Pi v_i := (\Pi v)_{i+1} \) if \( v \in \Omega_{\chi_1} \), and \( \Pi v_i := \lambda_i(\Pi v) \) if \( v \in \Omega_{\chi_2} \). By demanding that \( \Pi^2 \) acts trivially, this defines a smooth \( N \)-action on \( \Omega(\infty) \) which is compatible with the \( N \)-action on \( D_1(\infty) \), and whose restriction to \( IZ \) is compatible with the action coming from \( KZ \) on \( \Omega(\infty) \). By [Pas04], Corollary 5.5.5, we have a smooth \( G \)-action on \( \Omega(\infty) \). We then take \( \pi \) to be the \( G \)-representation generated by \( D_0(\infty) \) inside \( \Omega(\infty) \). It follows easily from the construction that \( \pi \) satisfies the properties (1), (2) and (3).

Theorem 3.2. If \( \lambda_i \neq \lambda_0 \) for all \( i \neq 0 \), then any smooth representation \( \pi \) of \( G \) satisfying the properties (1), (2), and (3) of Theorem 3.1 is irreducible and non-admissible.

Proof. Let \( \pi' \subseteq \pi \) be a non-zero subrepresentation of \( G \). By property (3), we have either \( \text{Hom}_K(\sigma, \pi') \neq 0 \) or \( \text{Hom}_K(\sigma^*, \pi') \neq 0 \). We consider the case \( \text{Hom}_K(\sigma, \pi') \neq 0 \); the other case is treated analogously. There exists a non-zero \( (c_i) \in \bigoplus_{i \in \mathbb{Z}} \mathbb{F}_p \) such that

\[
\left( \sum_i c_i \iota_i \right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0.
\]

We claim that

\[
(3.3) \quad \left( \sum_i c_i \iota_{i+j} \right)(D_0) \subset \pi' \quad \text{for all} \quad j \in \mathbb{Z}.
\]

Note that the \( K \)-socle of \( \left( \sum_i c_i \iota_i \right)(D_{0,\sigma}(\rho)) \) is \( \left( \sum_i c_i \iota_i \right)(\sigma) \) which is irreducible. Hence, \( \left( \sum_i c_i \iota_i \right)(D_{0,\sigma}(\rho)) \cap \pi' \neq 0 \) implies that \( \left( \sum_i c_i \iota_i \right)(\sigma) \subset \pi' \). The map \( \delta \) defined in [BPT12], Section 15 takes \( \sigma \) to \( \sigma^* \) and vice versa. Therefore, the arguments in the proof of [BPT12], Theorem 19.10 (i) and Lemma 19.7 imply that

\[
(3.4) \quad \left( \sum_i c_i \iota_i \right)(D_{0,\delta(\sigma)}(\rho)) = \left( \sum_i c_i \iota_i \right)(D_{0,\sigma^*}(\rho)) \subset \pi'.
\]
Repeating the argument now for $σ^*$, we see that $(\sum_i c_it_i)(D_{0,σ}(ρ)) \subset π'$. Thus,

$$(\sum_i c_it_i)(D_0) \subset π'.$$

Therefore,

$$(\sum_i c_it_i)(D_1^{Xσ}) \subset π' \quad \text{and} \quad (\sum_i c_it_i)(D_1^{Xσ'}) \subset π'.$$

Since $π'$ is stable under the $Π$-action, we have

$$(\sum c_{i+1})(D_0,σ(ρ)) \subset π' \quad \text{and} \quad (\sum c_{i-1})(D_0,σ(ρ)) \cap π' \neq 0.$$

In particular,

$$(\sum_i c_{i+1})(D_0,σ(ρ)) \subset π' \quad \text{and} \quad (\sum_i c_{i-1})(D_0) \subset π'.$$

By the same arguments as above, we find that

$$(\sum_i c_{i+1})(D_0) \subset π' \quad \text{and} \quad (\sum_i c_{i-1})(D_0) \subset π'.$$

The claim is now proved by repeatedly using the $Π$-action.

For $(d_i) ∈ \bigoplus_{i∈Z} F_p$, let $#(d_i)$ denote the number of non-zero $d_i$'s. Among all the non-zero elements $(c_i)$ of $\bigoplus_{i∈Z} F_p$ for which $(\sum_i c_it_i)(D_0) \subset π'$, we pick one with $#(c_i)$ minimal. We may also assume that $c_0 \neq 0$ using (3.3). We now show that $(c_i) = 1$. Assume to the contrary that $(c_i) > 1$. Since $(\sum_i c_it_i)(D_1^{Xσ}) \subset π'$ and $π'$ is stable under the $Π$-action, we have

$$(\sum_i \lambda_i c_it_i)(D_1^{Xσ}) \subset π'.$$

Since $(\sum_i \lambda_0 c_it_i)(D_1^{Xσ})$ is also clearly in $π'$, subtracting it from the above, we get

$$(\sum_i (λ_i - λ_0)c_it_i)(D_1^{Xσ}) \subset π'.$$

Writing $(c'_i) := ((λ_i - λ_0)c_i)$, we see that

$$(\sum_i c'_it_i)(D_0,σ(ρ)) \cap π' \neq 0.$$

Following the same arguments as in the previous paragraph, we get that $(\sum_i c'_it_i)(D_0) \subset π'$. However, the hypothesis $λ_i ≠ λ_0$ for all $i ≠ 0$, and the assumption $#(c_i) > 1$ imply that $(c'_i)$ is non-zero and $#(c'_i) = #(c_i) - 1$ contradicting the minimality of $#(c_i)$. Therefore, we have $c_0t_0(D_0) \subset π'$. So $uo(D_0) \subset π'$. Using (3.3) again, we get that $\bigoplus_{j∈Z} t_j(D_0) = D_0(∞) \subset π'$. By property (2), we have $π' = π$.

Non-admissibility of $π$ is clear because $π^{K_1} ⊃ soc_K π$ and $soc_K π$ is not finite dimensional by property (3). □
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