The effect of the stress far field on the crack tip behaviour

A. Piva

Received on December 8th, 1978

Abstract

The boundary value problem of an infinite purely elastic sheet with a traction-free crack loaded with a uniform shear and biaxial tension at infinity is solved. It is shown that the singular terms of stress and displacement functions are inadequate to predict the direction of initial crack propagation. The independence of the J vector of the biaxial parameter is proved.
1. INTRODUCTION

The knowledge of stresses and displacements near a fault is one of the main purposes of seismology. However, for a fault model which allows a mathematical description we must conduct ourselves with fairly idealized situations for the present. In this paper we shall start with the two-dimensional model of a crack which has received much attention since the work of Inglis (1913) and has been the main source of progress in fracture mechanics. As far as the crack tip behavior is concerned some other problems can be of geophysical interest as the real effect of the far field on the crack tip stress and displacement distributions.

Recent publications (Carpinteri, 1968; Di Tommaso et al., 1976; Eftis, 1977; Viola, 1978) have shown that the singularities for a plane cracked body subjected to biaxial tension at infinity are insufficient for a correct description of the crack tip behavior.

The arbitrary omission of non-singular terms leading to incorrect predictions of physical quantities such as the maximum tensile stress, the elastic strain energy density and the local elastic strain energy rate. It follows that the crack propagation criteria such as the maximum tensile stress criterion and the minimum strain energy density criterion fail as their predictions (Carpinteri, 1968; Di Tommaso et al., 1976; Eftis, 1977; Viola, 1978) have shown. It is surprising to note that the //-integral which is connected with a crack extension criterion along the direction of the crack, through the expression of a critical value Qc, appears to be independent of the horizontal load applied parallel to the plane of the crack. It seems that the biaxial load affects only the local quantities.
The aims of this paper are to describe the elastic fields of stresses and displacements near the tip of a central crack in an infinite elastic sheet loaded with a uniform shear and biaxial tension at infinity. Some considerations are made on the influence of non-singular terms on crack propagation criteria. The independence of the J vector from the biaxial load parameter is proved.

2. STATEMENT AND SOLUTION OF THE ELASTIC PROBLEM

We consider an infinite purely elastic body with a shear crack loaded at infinity, as shown in Fig. 1. The solution of the elastic problem under these acute or shear stress conditions is reduced to the evolution of the potential function $\Omega(z)$ and
\[ A. \text{ PIVA} \]

Holomorphic functions in a plane parallel to the \( xy \) plane and cut along the trace of the crack. These functions are related to stresses and displacement functions by the well-known relations [7] (Muskelishvili, 1953)

\[ \frac{\partial \sigma_{yy}}{\partial x} = \nu \frac{\partial \sigma_{xx}}{\partial y} = \tau_{xy} \]

where \( \sigma_{xx}, \sigma_{yy} \) are the stress components, \( \tau_{xy} \) is the shear stress, \( \nu \) is the Poisson ratio. We have for the stress distribution the following far field conditions:

\[ \sigma_{xx} = K \tau, \quad \sigma_{yy} = \tau, \quad \tau_{xy} = S; \quad |z| \to \infty \]  

where \( K \) is a real constant which gives a measurement of the biaxiality of the stress field at infinity. Moreover, we have the boundary conditions on the traction-free crack surface:

\[ \sigma_{yy} = 0, \quad \tau_{xy} = 0; \quad \sigma_{xx} = 0 \]  

where \( E \) is a real constant which gives a measurement of the biaxiality of the stress field at infinity. Moreover, we have the boundary conditions on the traction-free crack surfaces:

\[ \sigma_{yy} = 0, \quad \tau_{xy} = 0; \quad \sigma_{xx} = 0 \]
From (2.1), (2.2) and assuming zero rotation at infinity we obtain the following behavior for the derivatives of the potential functions:

\[
\frac{d}{dz} = \frac{\alpha^2 + 3i\beta}{2} + iS
\]

Moreover, after substitution of (2.1) into (2.3) we obtain the Hilbert problem:

\[
\begin{align*}
\frac{d^2}{dz^2} - \left(\alpha^2 + 3i\beta\right)\frac{d}{dz} + \frac{1}{z} &= 0, \quad |z| < r^* \\
\frac{d^2}{dz^2} - \left(\alpha^2 + 3i\beta\right)\frac{d}{dz} + \frac{1}{z} &= 0, \quad |z| > r^* 
\end{align*}
\]

Making use of (2.4) we obtain the solution of (2.5) in the following form:

\[
\begin{align*}
\frac{d}{dz} &= \frac{\alpha^2 + 3i\beta}{2} + iS \\
\frac{d}{dz} &= \frac{\alpha^2 + 3i\beta}{2} + iS
\end{align*}
\]
where \( X(z) = (z^2 - a^2) \) is the Painlevé function of the problem (Muskelishvili, 1953), whose branch is selected in such a way that \( X(z) \rightarrow z \) for large \( z \).

By integrating \([2.6]\) we have also:

\[
\int (K-1)r + 2iS(z) = \int \frac{(K-1)T}{2} X(z) \quad \text{for} \quad \frac{K}{2} = 0
\]

By means of \([2.1]\), \([2.6]\) and \([2.7]\) it is possible to derive the distribution of stresses and displacement functions in the plane.

3. - THE LOCAL CRACK TIP STRESS AND DISPLACEMENT COMPONENTS

Focusing our attention to the crack tip region \( 0 < \theta < \theta = \pi \), where \( \theta = z-a = r e^{i \theta} \) (fig. 2), we can consider the following series of expansions:

\[
X(z) = a^{1/2} \left\{ \left( \frac{1}{4} \right)^n \left[ 1 + \frac{1}{4} \frac{1}{2} \frac{3}{2} \cdot \frac{5}{2} \ldots \right] \right\}
\]

\[
\frac{1}{X} = a^{1/2} \left\{ \left( \frac{1}{4} \right)^n \left[ 1 + \frac{1}{4} \frac{1}{2} \frac{3}{2} \cdot \frac{5}{2} \ldots \right] \right\}
\]
To emphasize the singular behaviour of stresses near the tip we neglect all terms of order \( \frac{1}{\epsilon^1} \) and above after substitution of (3.2) into (2.6), (2.1), and (2.1).

Moreover, in order to maintain the same approximation for the displacement we neglect terms of order \( \frac{1}{\epsilon^2} \) and above after substitution of (3.1) into (2.7) and (2.1).

For the fundamental stress combinations we obtain the following expressions:

\[
\sigma_x + \sigma_y = d \approx \frac{G_{xx}}{V} + \frac{G_{yy}}{V}
\]

\[
\sigma_x - \sigma_y = 2\tau_{xy} = d \approx \frac{\nu G_{xx} + G_{yy}}{V} = \frac{K_1}{V} \frac{K_2}{V}
\]

with

\[
d = \frac{K_1 - K_2}{V}, \quad S = \frac{K_1 + K_2}{V}, \quad \tau_{xy} = \frac{K_1 - K_2}{V}
\]
where $K_1 = rV_{raz}$ and $K_2 = SV_{ITO}$ are the stress intensity factor for the opening and sliding mode respectively. For the complex displacement function we obtain:

$$D = A + C \xi + \frac{1}{A} \left[ \frac{A^2 + 2iE^2}{A^2} \right] $$

where:

$$A = B + c - \frac{1}{2} - \frac{1}{x} \xi$$

We see that, as pointed out also in [3, 4] for the case of simple biaxial tension, the stress and displacement functions (3.3) and (3.4) depend on the far field also through terms which are non singular at the crack tip. The arbitrary omission of these terms can lead to incorrect predictions on stress and displacement distributions and on crack propagation criteria.

4. THE ELASTIC STRAIN ENERGY DENSITY CRITERION

For the plane problem the elastic strain energy density at any point of the plane body is given by:

$$W = \frac{1}{2} \left[ \frac{\sigma_{yz} \partial y}{\partial x} + \frac{\sigma_{zy} \partial x}{\partial y} \right]$$

where $\sigma_{yz}$ and $\sigma_{zy}$ are the shear stresses.
THE EFFECT OF THE STRESS FAR FIELD

and in complex form:

\[ W = \frac{1}{2} \left( (\sigma_x + \sigma_y) \left( \frac{\partial \epsilon}{\partial x} + \frac{\partial \epsilon}{\partial y} \right) + (\sigma_x - \sigma_y + 2\gamma) \left( \frac{\partial \epsilon}{\partial x} - \frac{\partial \epsilon}{\partial y} \right) \right) \]

After substitution of (3.3) and (3.4) into (4.2) we obtain the strain energy density near the crack tip which is:

\[ W = \frac{1}{2} \left( \frac{1}{2} \left( \frac{\partial \epsilon}{\partial x} + \frac{\partial \epsilon}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial \epsilon}{\partial x} - \frac{\partial \epsilon}{\partial y} \right)^2 \right) \]

According to Sih's theory (Sih, 1973) on the prediction of the direction of initial crack extension, crack growth will start along the radial direction for which the local elastic strain energy density attains a minimum. In view of this hypothesis the growing direction will be a solution of the following two conditions:

\[ \frac{dW}{dr} = 0 \quad \text{and} \quad \frac{d^2W}{dr^2} > 0 \]

where the radial distance \( r \) is very small, but unspecified.
Recently, an estimation of the range of $\alpha$ was given in the case of pure tension applied at infinity [2, 3]. In contrast to [4, 5], the case of a biaxially loaded crack is valid as a function of $\alpha$, as confirmed by experimental data. The same feature can be shown by substitution of [4.3] in [4.4]. Assuming that $\alpha = \alpha_0 \sigma$ where $\alpha$ is a real positive number we can write the strain energy $\alpha_0$ as:

$$8n \frac{S}{r_0} W(\alpha_0) = Cn \left( 1 + n \alpha_0 \right) \frac{S}{r_0} W(\alpha_0) + \frac{S}{r_0} W(\alpha_0) + \left( 1 + n \alpha_0 \right) \frac{S}{r_0} W(\alpha_0) + \left( 1 + n \alpha_0 \right) \frac{S}{r_0} W(\alpha_0)$$

Choosing $\alpha = 10^3$ is apparent with the condition on the distance $r$, from the crack border, the results of plotting the expression [4.5] to $\alpha$ in the case of crack extension corresponding to the minimum of the strain energy density occurs as $\alpha = 0$ for $\alpha = 0$ and in general...
for $K < -1$ for all values of the Poisson ratio. For $K = 0$ the angle occurs at $\theta = 90^\circ$ or $\theta = 0^\circ$ depending on the values of the Poisson ratio and for $K = 2$ at $\theta = 0^\circ$. The graphs show that for $K < -1$ the angle of initiation of crack extension occurs at $\theta = 0^\circ$ for $v = 0.3$ and an increase of the Poisson ratio shifts this angle to $\theta = 90^\circ$. For $v = 1$ the angle of initiation crack extension is predicted to be greater than $90^\circ$ for any value of $K$ and the crack should bend back towards its centre. Since this kind of behaviour has never been reported this portion of the curves is shown by dashed lines. It seems that in the above circumstance the Sih criterion is inadequate.
5. THE MAXIMUM NORMAL STRESS CRITERION

The maximum normal stress criterion proposed firstly by Yoffe (1951) and reviewed by Erdogan and Sih (1963) is concerned with the direction of crack extension too and maintains that a crack will begin to extend radially along the plane on which the stress normal to this plane attains a positive maximum value. If \( \theta_p \) is the angle of the tangent to this plane, the criterion requires that, near the tip of the crack, we should have:

\[
\left[ \frac{\sigma_{np}}{\sigma_p} \right] > 0 , \quad \left( \frac{\partial \sigma_{np}}{\partial \theta_p} \right)_{\theta_p} = 0 , \quad \left( \frac{\partial^2 \sigma_{np}}{\partial \theta_p^2} \right)_{\theta_p} < 0
\]

(5.1)
The effect of the stress far field 

\[ \text{Diagram} \]
Remembering that:

\[ J_x + O_y = J_y + 2 \Omega \]

we obtain from [3.3] the following expression:

\[ \mathbf{a} = (p \mathbf{e} - \mathbf{p} \mathbf{e})^2 (1 - \text{Re} \mathbf{S} + \text{Re} \mathbf{S}^2) \]

\[ \text{Vs} \mathbf{e} \mathbf{e} \]

\[ \text{[5.2]} \]

As obtained from [3.1] the following expression:

\[ \mathbf{c} = \mathbf{c} \left( -\mathbf{c} \right) - \mathbf{c} \left( \mathbf{c} \mathbf{c} \mathbf{c} \right) \]

\[ \text{[5.3]} \]
The variation of the angle \( \theta_0 \) of initial crack extension is shown in Figs. 7 and 8 where the function \( \frac{\sigma_{yy}(0)}{\sigma_{xx}} \) is plotted for different values of \( K = \frac{a}{r_0} \). This last condition satisfies (5.1) and the requirements \( a = 0.07 \). The best condition is the same for \( n = 1 \). It can be seen in Fig. 7 that for \( n = 0 \) and \( K = 2 \) the maximum occurs at \( \theta_0 = 0^\circ \). As far as \( K \) increases, \( \theta_0 \) increases up to \( 80^\circ \) for \( K = 6 \). In Fig. 8 it can be observed that for \( n = 1 \) the maximum of the normal stress occurs from \( \theta_0 = 30^\circ \) to \( \theta_0 = 70^\circ \) depending on the value of \( K \).

Then again we find evidence of the influence of non-singular terms on the predictions of the angle of initial crack extension.
6. THE LOCAL ELASTIC STRAIN ENERGY RATE AND THE J-FUNCTION

The elastic strain energy $W$ due to the presence of the crack is independent of the horizontal stress applied at infinity. In fact, we have:

$$W = \frac{\pi T^2 (\mathbf{P} + 1)}{8k}. \tag{6.1}$$

Moreover, the local elastic strain energy $W_c$ obtained by integrating of [6.1] over a circular region centered at the crack tip with radius $R < 2.5 a$ has the following expression:

$$W_c = \frac{\pi T^2}{8k} \left[ \frac{3(3 + K^2) + (2 + 3K)a^2}{\sqrt{1 - 2K}} \right]. \tag{6.2}$$

From [6.1] and [6.2] we obtain the global and local rates of change of the elastic strain energy with the crack size:

$$\frac{dW}{da} = \frac{\pi T^2 (\mathbf{P} + 1)}{4k} \left( \frac{1}{2} \right). \tag{6.3}$$

$$\frac{dW_c}{da} = \frac{\pi T^2}{8k} \left[ \frac{3(3 + K^2) + (2 + 3K)a^2}{\sqrt{1 - 2K}} \right]. \tag{6.4}$$
We see that the global strain energy rate is independent of the lateral stress at infinity which otherwise has a significant influence on the local elastic strain energy rate. This behaviour was pointed out also in [4, 5] for the case of simple biaxial tension at infinity.

Rice (1968) showed that the energy release rate for a two-dimensional crack extending in its plane in a homogeneous material was equal to the path-independent integral $J_1$ formulated by Eshelby (1956) in the theory of lattice defects, and applied to crack problems by Sanders (1960) and Cherepanov (1969). Knowles and Stenberg (1972) generalized $J_1$ to a vector $J$, corresponding to the energy release rate for a movement in any direction of the crack edge. Rice and Rice (1973) showed that the energy release rate $J$ for a movement along a particular direction can be computed by using the density of functions of a complex variable. Recently some authors (Bui, 1976; Carpinteri, 1979) showed that the energy release due to the extension of the crack along some direction is equal to the component of the $J$ vector along the direction of extension.

Here we formulate an expression for the $J_1$ integral in terms of the fundamental combinations of stresses and the derivatives of the complex displacement. The two components of the $J$ vector are:

\[
J_1 = \frac{dW}{d\Gamma} = \left( \frac{\partial W}{\partial \sigma_{xx}} + \frac{\partial W}{\partial \sigma_{xy}} \right) \frac{\partial u}{\partial x} + \frac{\partial W}{\partial \sigma_{yy}} \frac{\partial u}{\partial y} + \frac{\partial W}{\partial \sigma_{xy}} \frac{\partial v}{\partial x} + \frac{\partial W}{\partial \sigma_{yx}} \frac{\partial v}{\partial y},
\]

and

\[
J_2 = \frac{dW}{d\Gamma} = \left( \frac{\partial W}{\partial \sigma_{xx}} + \frac{\partial W}{\partial \sigma_{xy}} \right) \frac{\partial v}{\partial x} + \frac{\partial W}{\partial \sigma_{yy}} \frac{\partial v}{\partial y} + \frac{\partial W}{\partial \sigma_{xy}} \frac{\partial u}{\partial x} + \frac{\partial W}{\partial \sigma_{yx}} \frac{\partial u}{\partial y}.
\]
were $W$ is the strain energy density [4.3] and $F$ is a small circle surrounding the crack tip. Introducing the complex force acting on the element $ds$ of $F$:

$$Tds = (Tx + iTy)ds = (axvdy - axydx) + i(awxdy - awydx)$$

we can write from (7.1) and (7.2):

$$\frac{1}{\pi} = \frac{1}{\pi} \int Tds + T_5ds$$

This expression was obtained also by Hellen and Blackburn (1975). Moreover, we can write:

$$J = \int Tds + T_5ds = \frac{1}{\pi} \frac{1}{\sqrt{\lambda}}$$

together with:

$$T = \frac{1}{2} \left( \frac{1}{\sqrt{\lambda}} - a_{11} - a_{22} \right)$$
Finally, with the help of relations (4.2) and (7.4) the line element \[ \text{(7.5)} \] acquires the following form:

\[
\begin{align*}
7 & = \int_2 \left[ \varphi_s - \varphi_z \right] \left( \begin{bmatrix} \delta_x & \delta_y \end{bmatrix} - \begin{bmatrix} \delta_z \end{bmatrix} \right) \cdot \left( \begin{bmatrix} \delta_x & \delta_y \end{bmatrix} - \begin{bmatrix} \delta_z \end{bmatrix} \right) \text{d}s,
\end{align*}
\]

From (3.3) and (3.4) the integrals (7.7) split into three classes of integrals. The terms of the first class vanish identically; those of the second class, which are integrals containing the bi-axial load parameter \( K \) and the radius of the loaded zone, cancel one another out. The integrals of the third class are \( K \)-path-independent.

The resulting integral is:

\[
J = \int_2 \left( \begin{bmatrix} \delta_x & \delta_y \end{bmatrix} - \begin{bmatrix} \delta_z \end{bmatrix} \right) \cdot \left( \begin{bmatrix} \delta_x & \delta_y \end{bmatrix} - \begin{bmatrix} \delta_z \end{bmatrix} \right) \text{d}s,
\]

from which we obtain the well-known result:

\[
J = \frac{1}{4\pi} \int_2 \left( \begin{bmatrix} \delta_x & \delta_y \end{bmatrix} - \begin{bmatrix} \delta_z \end{bmatrix} \right) \cdot \left( \begin{bmatrix} \delta_x & \delta_y \end{bmatrix} - \begin{bmatrix} \delta_z \end{bmatrix} \right) \text{d}s.
\]

Thus, the value of the \( J \)-integral is independent of the zero in polar second appearing in the expressions of stresses and displace-
7. CONCLUSION

Assuming a two-dimensional crack as a model of a fault some results are obtained on the basis of recent progress in fracture mechanics which could be of interest in Geophysics.

As far as the crack tip behaviour is concerned the influence of the stress far field through terms which are non-singular is pointed out. The arbitrary omission of these terms should cause an incorrect prediction on stress and displacement distributions and on local crack propagation criteria. It appears that the biaxiality of the stress far field affects only the local strain energy release rate and the independence of the /-vector from the biaxial load parameter is shown.
REFERENCES

BUDIANSKY B, RICE J.R., 1973. - « J. Applied Mech. », 40, 201-205.

Bui H.D., 1976. - Arch. Mech. Stosowanij, 28, p. 649-659.

CARPINTERI A., 1978. - « Tech. Note. 1st. di Scienza delle Costruzioni », Univ. di Bologna.

CARPINTERI A., DI TOMMASO A., VIOLA E., 1978. - Proceedings 4° Congresso Naz. AIMETA, Firenze.

CHEREPANOV G.L., 1967. - P.M.M., 25, p. 476-488.

Di TOMMASO A., NOBILE L., VIOLA E., 1976. - Proceedings 3° Congresso Naz. AIMETA, Firenze, 1976.

EFTIS J., SUBRAMONIAN N., LIEBOWITZ H., 1977. - Eng. Fract. Mech., p. 182-210.

ERDOGAN F., SIH G.C., 1963. - J. Bas. Eng., 85D, p. 519-525.

ESHELBY J.R., 1956. - « Solid State Physics », 3.

HELLEN T.K., BLACKBURN W.S., 1975. - Int. J. of Fract., 11, p. 605-617.

INGLIS C.E., 1913. - « Trans. Inst. Naval Arch. », 55, Part 1, p. 219-230.

KNOWLES J.K., STENBERG E., 1972. - Arch. Rat. Mech., 44, p. 187-211.

MUSKELISHVILI N.I., 1953. - « Some Basic Problems of the Mathematical Theory of Elasticity. » Groniningen, Noordhoff.

RICE J.R., 1968. - « J. Applied Mech. », 35, p. 379-386.

SANDERS J.L., 1960. - J. Applied Mech., 27, p. 352-353.

SIH G.C., 1973. - « Methods of analysis and solution of crack problems », G.C. Sih, Noordhoff Int.

VIOLA E., 1978. - Tech. Note, 1st. di Scienza delle Costruzioni, Univ. di Bologna.

YOFFE E.H., 1951. - Phyl. Magazine, 32, p. 739-750.