A GENERAL RELATIVISTIC RAY-TRACING METHOD FOR ESTIMATING THE ENERGY AND MOMENTUM DEPOSITION BY NEUTRINO PAIR ANNIHILATION IN COLLAPSARS

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ABSTRACT

Bearing in mind the application to the collapsar models of gamma-ray bursts (GRBs), we develop a numerical scheme and code for estimating the deposition of energy and momentum due to the neutrino pair annihilation (ν + ν → e⁻ + e⁺) in the vicinity of an accretion tori around a Kerr black hole. Our code is designed to solve the general relativistic (GR) neutrino transfer by a ray-tracing method. To solve the collisional Boltzmann equation in curved spacetime, we numerically integrate the so-called rendering equation along the null geodesics. We employ the Fehlberg (4,5) adaptive integrator in the Runge–Kutta method to perform the numerical integration accurately. For the neutrino opacity, the charged-current (νe, e⁻) and neutrino and antineutrino (νe, e⁺) interactions are taken into account. The numerical accuracy of the developed code is certified by several tests in which we show comparisons with the corresponding analytical solutions. In order to solve the energy-dependent ray-tracing transport, we propose that an adaptive-mesh-refinement approach, which we take for the two radiation angles (θ, φ) and the neutrino energy, is useful in reducing the computational cost significantly. Based on the hydrodynamical data in our collapsar simulation, we estimate the annihilation rates in a post-processing manner. Increasing the Kerr parameter from 0 to 1, it is found that the GR effect can increase the local energy deposition rate by about one order of magnitude, and the net energy deposition rate by several tens of percent. After the accretion disk settles into a stationary state (typically later than ~9 s from the onset of gravitational collapse), we point out that the neutrino-heating timescale in the vicinity of the polar funnel region can be shorter than the dynamical timescale. Our results suggest that the neutrino pair annihilation is potentially as important as the conventional magnetohydrodynamic mechanism for igniting the GRB fireballs.

Key words: accretion, accretion disks – gamma-ray burst: general – magnetohydrodynamics (MHD) – methods: numerical – neutrinos – supernovae: general

Online-only material: color figures

1. INTRODUCTION

Gamma-ray bursts (GRBs) have long attracted the attention of astrophysicists since their accidental discovery in 1970s. Regarding long-duration GRBs, there have been accumulating observations identifying their origin in massive stellar collapse (see, e.g., Woosley & Bloom 2006 for a review). As their central engines, the so-called collapsar has received quite some interest for more than a decade (Woosley 1993; Paczynski 1998; MacFadyen & Woosley 1999).

In the collapsar scenario, central cores with significant angular momenta collapse into a black hole (BH). Neutrinos emitted from the accretion disk heat matter in the polar funnel region to launch the GRB outflows. Paczynski (1990) and Meszaros & Rees (1992) pioneeredly proposed that the energy deposition proceeds predominantly via neutrino and antineutrino annihilation into electron and positron (e.g., ν + ν → e⁻ + e⁺, hereafter “neutrino pair annihilation”). In addition, it is suggested that the strong magnetic fields in the cores of the order of 10¹⁵ G also play an active role both in driving the magneto-driven jets and in extracting a significant amount of energy from the central engine (e.g., Blandford & Znajek 1977; Thompson et al. 2004; Uzdensky & MacFadyen 2007, and see references therein).

However, it is still controversial whether the generation of the relativistic outflows proceeds predominantly via magneto-hydrodynamic (MHD) or neutrino-heating processes. So far, much attention has been paid to the MHD processes (e.g., Proga 2003; Mizuno et al. 2004; Lyutikov 2006; Fujimoto et al. 2006; Nagataki et al. 2007; McKinney & Narayan 2007; Komissarov & Barkov 2007; Barkov & Komissarov 2008; Nagataki 2009; Harikae et al. 2009). A general outcome of these extensive MHD simulations is that the magneto-driven shock waves can blow up massive stars along the rotational axis. Those primary jet-like explosions are first mildly relativistic at most due to too many baryons in the central core (e.g., Takiwaki et al. 2009); however, they could become relativistic as they propagate further out (Nagataki 2009). In such a collapsar environment, explosive nucleosynthesis (e.g., Fujimoto et al. 2006; Nagataki et al. 2007) and neutrino and gravitational-wave signals (e.g., Kawagoe et al. 2009; Hiramatsu et al. 2005) have also been extensively studied.

In contrast to such advances in the MHD studies, there have been only a few studies pursuing the possibility of generating jets by energy deposition via neutrino pair annihilation. This is mainly because neutrino emission from the accretion disk generally becomes highly aspherical, thus demanding us to solve a multidimensional neutrino transfer problem (e.g., Tubbs 1978; Janka & Hillebrandt 1989). This is still computationally very expensive, which is also the case for the neutrino-driven supernova simulations (see references in Janka et al. 2007). MacFadyen & Woosley (1999) were the first to point out the
importance of energy deposition via neutrino pair annihilation in collapsar simulations; however, the energy deposition rates to the polar funnel region were adjusted by hand to produce jets. To the best of our knowledge, the fast and collimated neutrino-heated outflows have not been realized so far in numerical simulations without the artificial energy injection to the polar funnel regions (see, e.g., Aloy et al. 2000; Zhang et al. 2003; Mizuta & Aloy 2009, and references therein).

Thus far, several methods aiming to implement neutrino pair annihilation into the collapsar simulations have been reported. By estimating the fluxes and spectra of the neutrino emission from the accretion disk via the so-called neutrino leakage scheme, Ruffert et al. (1997) and Ruffert & Janka (1998) proposed to estimate the heating rate by summing up the contributions of the incident neutrino and antineutrino radiation from all directions. Alongside this prescription, Nagataki et al. (2007) have estimated the neutrino heating rates and included them in the hydrodynamical simulation. To reduce the computational time, the assumption of the optical thinness of the accretion disk was added to the prescription of Ruffert & Janka (1998). Even with this potential overestimation of the heating rates, neutrino-driven outflows were not observed in their simulations. More recently, Dessart et al. (2009) have developed a new scheme to estimate the energy deposition rate using the state-of-the-art, multi-angle neutrino-transport solver (Ott et al. 2008a). They discussed the possible formation of neutrino-driven outflow in the postmerger phase of binary neutron star coalescence. Rellying on the neutrino leakage scheme, Harikae et al. (2010) have proposed a special relativistic ray-tracing method to estimate the annihilation rates. Using hydrodynamical data in their collapsar simulation, they pointed out that the neutrino-heated outflow might be formed \( \sim 10 \) s after the initial collapse of the progenitor star.

It should be noted that all of the above schemes neglect the general relativistic (GR) effects for simplicity, which have been reported to significantly enhance the annihilation rates near the accreting BHs (e.g., Jaroszynski 1993, 1996; Salmonson & Wilson 1999; Asano & Fukuyama 2001, 2000; Birkl et al. 2007). Among the GR studies, the numerical method of Birkl et al. (2007) in which a ray-tracing calculation is performed to follow the neutrino trajectories in a Kerr spacetime is one of the most sophisticated. The ray-tracing method has an advantage because it can straightforwardly capture important GR features such as ray bending and redshift. In their scheme, the neutrino number flux emitted from the accretion disk (or from the neutrino spheres) is simply assumed to be conserved along the geodesics. In reality, neutrino emission, absorption, and scattering should occur along the neutrino geodesics, changing their neutrino distribution function simultaneously. Especially in the absence of the charged-current neutrino interactions, the annihilation rates in Birkl et al. (2007) could be overestimated. To improve these, one has to solve the GR neutrino transport equation along each ray; this is what we investigate in this paper.

In this study, we present a numerical code and scheme for calculating the deposition of energy and momentum via neutrino pair annihilation in a Kerr spacetime, in which we solve the GR radiative equation along the null geodesics. The charged-current \( \beta \)-processes, which are dominant in the vicinity of the accretion tori, are taken into account (e.g., Dessart et al. 2009). With these improvements, the newly developed code would provide a more realistic estimation of the annihilation rates than before. We check the numerical accuracy of the developed code by showing several comparisons with analytical solutions, some of which we derive for the first time in this paper. Based on the results of our long-term collapsar simulation (Harikae et al. 2009), we run our new code to estimate the annihilation rate in a post-processing manner and discuss their implications for the dynamics of collapsars.

This paper is organized as follows. In Section 2, we summarize the formulation of the GR ray-tracing method for the collisional Boltzmann equation. Section 3 is devoted to the numerical tests. In Section 4, we estimate the annihilation rates in a post-processing manner using hydrodynamical data in our collapsar simulation. We summarize our results and discuss their implications in Section 5.

2. NEUTRINO PAIR ANNihilation IN GENERAL RELATIVITY

In this section, we summarize the formalism and our strategy to estimate the neutrino-pair-annihilation rates based on the GR radiation transfer. In Section 2.1, we summarize the method to solve the neutrino geodesics in a Kerr spacetime for the collisionless Boltzmann equation. Then in Section 2.2, we describe how to solve the collisional Boltzmann equation along the geodesics.

We assume that the gravitational field, which leads to ray bending and redshift, is given by the central Kerr BH of mass \( M \) and angular momentum parameter \( a \equiv J/M \) (where \( J \) is the angular momentum of the BH and \( 0 \leq a/M \leq 1 \)), whose metric is given in the Boyer–Lindquist coordinates \((t, r, \theta, \phi)\) by

\[
ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -a^2r^2 + \gamma_{ij}(dx^i + \beta^j dt)(dx^j + \beta^i dt).
\]

The lapse function \( \alpha \), the shift vector \( \beta^i \), and the non-vanishing components of the spatial metric \( \gamma_{ij} \) are given as

\[
\alpha = \sqrt{\frac{\Sigma}{A}}, \quad \beta^\theta = -\omega, \quad \gamma_{rr} = \frac{\Sigma}{\Delta},
\]

where \( \Sigma = r^2 + a^2 \cos^2 \theta, \Delta = r^2 - 2Mr + a^2, A = (r^2 + a^2)^2 - a^2\Sigma \sin^2 \theta = \Sigma A + 2Mr(r^2 + a^2), \) and \( \omega = 2aMr/A \) (e.g., Misner et al. 1973). Here we use \( G = c = 1 \) unit and note that the Latin indices \((i, j)\) have the domain of \((r, \theta, \phi)\). For convenience, we also define the dimensionless angular momentum parameter of \( a^* = a/M \).

We first introduce the following three frames: the Boyer–Lindquist frame (BLF), which is given by the center of mass system in curved spacetime; the locally non-rotating frame (LNRF), which is given by the tetrad frame rotating with the central BH to make the dragging effects vanish (i.e., \( e_i = 0 \) with \( e_i \) being the basis of the vierbein); and the rest frame of fluid (RF), which is necessary to define quantities related to radiation such as emissivity and absorptivity. These three frames can be connected to each other by the tetrad and Lorentz transformations. In the following sections, the quantities measured in the LNRF and RF are denoted by the superscripts “L” and “R,” respectively. Variables in the BLF are denoted without any superscripts. A schematic picture between these three frames is illustrated in Figure 1. What we finally need is the annihilation rates measured by the observer in the LNRF (left end of the figure). The neutrino emissivity and absorptivity are naturally...
which can be readily implemented in the GR hydrodynamic simulations via $\nabla_\nu T^{\mu\nu} = Q^\mu$ (e.g., Shibata et al. 2007); this is beyond the scope of this paper.

To evaluate the annihilation rates, we have yet to determine $f^{\nu}_{\nu}(\vec{v})$ in Equation (3). It is noted that the distribution function is invariant under the tetrad transformation as

$$f^{L}_{\nu}(\bar{p}^L_{\nu}, r) = f^{\nu}_{\nu}(\bar{p}^\nu_{\nu}, r), \quad (9)$$

where $f^{\nu}_{\nu}(\bar{v})$ is the distribution function in the BLF. $f^{\nu}_{\nu}(\bar{v})$ is determined by the GR Boltzmann transport equation (Misner & Sharp 1964) as

$$\frac{df^{\nu}_{\nu}}{d\lambda} = p^\nu \frac{df^{\nu}_{\nu}}{dx^\nu} = \left( \frac{df^{\nu}_{\nu}}{dx^\nu} \right)_{\text{coll}}, \quad (10)$$

$$D \frac{Dx^\nu}{d\lambda} = \frac{\partial}{\partial x^\nu} - \Gamma^\nu_{\alpha\beta} p^\nu \frac{\partial}{\partial p^\alpha}, \quad (11)$$

where $(df^{\nu}_{\nu}/dx^\nu)_{\text{coll}}$ represents the collision term. In the context of photon propagation from the accretion disk, there have been extensive studies to determine the geodesics (e.g., Bardeen et al. 1973; Cunningham & Bardeen 1973; Cunningham 1975; Misner, Thorne & Wheeler 1973; Shibata et al. 2007); this is

$$\nabla_\nu T^{\mu\nu} = Q^\mu$$

In the Boyer–Lindquist coordinates, the Lagrangian $\mathcal{L}$ for describing the geodesics of massless particles in Kerr geometry (e.g., Misner et al. 1973) is given as

$$2\mathcal{L} = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$$

$$= \left( \frac{1}{r^2} \frac{1}{2} (1 - \frac{2M}{r}) \right) \frac{1}{\Sigma} \left( \frac{1}{r^2} + \frac{a^2}{\Sigma} \right) + \frac{\Sigma}{r^2} \sin^2 \theta \phi^2 + \frac{\Sigma}{r^2} \cos^2 \theta \phi^2,$$

where overdots denote differentiation with respect to an affine parameter $\lambda$. With three constants of motion,

$$E \equiv -p_t,$$

$$L_\phi \equiv p_\phi,$$

$$C \equiv \left[ L_z^2 \cos^2 \theta - a L_r^2 \right] \cos^2 \theta + p_r^2,$$

one obtains the equations governing the orbital trajectory (e.g., Carter 1968; Bardeen et al. 1972),

$$p_r = \frac{E - \omega L_\phi}{\alpha^2},$$

$$p_\phi = \frac{E - \omega L_z}{a^2},$$

$$p_t = \frac{1}{r^2} + \frac{a^2}{\Sigma} \left( \frac{1}{r^2} \frac{1}{2} (1 - \frac{2M}{r}) \right) \frac{1}{\Sigma} \left( \frac{1}{r^2} + \frac{a^2}{\Sigma} \right) + \frac{\Sigma}{r^2} \sin^2 \theta \phi^2 + \frac{\Sigma}{r^2} \cos^2 \theta \phi^2.$$
\[ n(x) \] is the proper number density of the external medium with which the neutrinos interact, and thus measured in its own local rest frame. \( Q(x, p) \) is the emission rate per particle of the medium \( Q_e \) plus a further increase due to scattering \( Q_s \), which can be therefore written as

\[ Q(x, p) = Q_e(x, e^R) + Q_s(x, p), \]

\[ Q_s(x, e^R) = \frac{j(x, e^R)}{4\pi(e^R)^2}, \]

\[ Q_s(x, p) = \int e^R d\Omega(x, p') \xi(x; p' \rightarrow p) f(x, p'), \]

where \( j \) is the emissivity and \( \xi(x; p' \rightarrow p) \) is the so-called invariant phase function, describing the momentum transfer due to scattering. \( \kappa \) in Equation (23) is the invariant absorption coefficient. \( d\Omega(x, p) \) is the solid angle in the momentum space of \( p \) at position \( x \). \( e^R \) is the neutrino energy measured in the local proper frame that is related to the quantities in the BLF as

\[ e^R = -p^R_0 = -u^a p_a. \]

The formal solution of Equation (23) can be given as

\[ f(\epsilon, \Omega) = \int_{\lambda_0}^{\lambda_s} n(\lambda^*) Q(\lambda, f) e^{-\int_{\lambda_0}^{\lambda_s} n(\lambda^*) \epsilon(\lambda^*) d\lambda^*} d\lambda^*. \]

which is referred to as the rendering equation of the radiation transport problem (e.g., Zink 2008). Note that the integration with respect to \( \lambda \) starts from a given target point \( (\lambda_0, \lambda_s) \), where the neutrino pair annihilation occurs, propagated backward to the neutrino sources along the geodesics. This backward ray tracing terminates when it hits the outermost boundary of our computational domain or when the optical depth for each neutrino energy exceeds unity, indicating the surface of the neutrino spheres, both of which are represented by \( \lambda_s \) in Equation (28). Note that we set the inner boundary of the target region as the surface of the ergosphere because we consider an idealized situation where the energy released inside the ergosphere will terminate in the BH, playing no important role in energizing a GRB.

In solving the rendering equation, we neglect the scattering terms \( Q_s \) (Equation (26)), which are not only difficult to treat by the ray-tracing technique but also a major undertaking in the radiative transport problem in general. The integration in the rendering Equation (28) is done explicitly along the geodesics. In doing so, we determine each integration step by restricting the maximum change of neutrino opacity for all the neutrino energy bins to be less than 10%. By this choice, our code can safely pass some test problems (see Section 3).

Neglecting the energy and momentum transfer via neutrino scattering, the neutrino Boltzmann equation for \( v_\nu \) and \( \vec{v}_\nu \) now reads

\[ \frac{df}{d\lambda} = n(Q_e (1 - f) - \kappa f) = n(Q_e - \kappa^* f), \]

where the Pauli blocking term \( (1 - f) \) is now taken into account. Note that the rendering equation is also valid in this case by replacing \( \kappa \) in Equation (28) with \( \kappa^* \equiv (Q_e + \kappa) \). As for the opacity sources of neutrinos \( (\kappa^*) \), electron capture in proton and nuclei, positron capture in neutrons, and neutrino scattering with nucleon and nuclei are included (Fuller et al. 1985; Takahashi et al. 1978; Bruenn 1985). Here, \( \kappa^* \) is estimated as \( \kappa^* = \sum[n_{\text{target}} \cdot \sigma(\epsilon^R)] \) with \( n_{\text{target}} \) and \( \sigma(\epsilon^R) \) being the target number density of each reaction and the corresponding cross section, respectively. The neutrino emission illuminated from the accretion disk mainly comes from the optically thick region, where the charged current \( \beta \)-equilibrium should be nearly satisfied. Hence, we estimate the neutrino emissivity as \( Q_e = \kappa^* f_{\text{TD}} \), where \( f_{\text{TD}} \equiv (1/(e^{\epsilon^R/T} + 1)) \) is the Fermi–Dirac neutrino distribution function with a vanishing chemical potential.

An AMR approach that we propose in this paper is another important tool for saving the computational cost of the ray-tracing calculation. For example, a number of rays are required for estimating the annihilation rates correctly in the vicinity of the accretion disk, in which the neutrino-heated outflow is expected to be produced. It is therefore of primary importance...
to perform AMR with respect to the angular direction of the rays. Second, the energy bin of neutrinos is better treated by AMR because the neutrino distribution function can be more accurately determined if the finer energy bins are cast for the relevant energy scales. The actual implementation procedure is given as follows. Given a point \( x \), we search the maximum intensity \( I(\epsilon, \theta, \phi) \) among the neighboring points for all the directions and for all the energy bins and call it \( I_{\text{max}} \). Then we focus on the energy bins and angular directions, which satisfy \( I_{\text{rel}}(\epsilon, \theta, \phi) \geq K I_{\text{max}} \), where we set \( K = 0.01 \). Only for the domain of \( (\epsilon, \theta, \phi) \) satisfying the condition do we cast finer mesh points. In actual implementation, we perform this selection procedure for every three-dimensional space, which not only saves the computational costs but also maintains good accuracy in estimating the annihilation rates.

3. NUMERICAL TESTS

Before applying the newly developed code to collapsars, we check the accuracy of our code. In Sections 3.1 and 3.2, we show a comparison of the neutrino trajectory between the numerical and analytical solutions, by which we check the numerical accuracy of the collisionless Boltzmann solution. In Section 3.3, we demonstrate the capability of our code to capture the images around the accreting BHs, the so-called BH shadow problem. In the case of the collisional Boltzmann equation, we perform the numerical tests to reproduce the radiation fields shedding from a spherical light bulb, which will be presented in Section 3.4.

3.1. Geodesics in the \((r-\theta)\) Plane

By a straightforward, albeit tedious, calculation, one can obtain the well-known analytical form of the null geodesics in the \((r-\theta)\) plane around a Kerr BH (e.g., Carter 1968; Bardeen et al. 1972; Cadez et al. 1998; Li et al. 2005). Figure 2 shows the geodesics near the BH in the case of \( a^* = 0 \) (left) or \( a^* = 0.999 \) (right), obtained either numerically (points) or analytically (lines). In both cases, neutrinos are initially injected from the right edge of the figure with different impact parameters (for different \( Z \) in the figure). They are shown to be dragged by the gravity of the BH, whose surface is indicated by the black line in the center. Note that the reflection at \( X = 0 \) or \( Z = 0 \) is just for visualization. For example, at \( X = 0 \), the rays keep on propagating to the left (\( X < 0 \)) in reality. For the numerical solutions, we vary the two different parameters \( (\delta = 10^{-3}, 10^{-4}) \) that regulate the numerical convergence in the adaptive integrator (see Section 2.1). We find that the regulation parameter of \( \delta = 10^{-4} \) is sufficient to trace the trajectory in good agreement with the analytical solution, which we take in the following calculations.

3.2. Geodesics in the \((r-\phi)\) Plane

Now we move on to show the geodesics in the \((r-\phi)\) plane. Since the analytical solution becomes very complicated in this case, we consider the special case, \( L = aE \) (Chandrasekhar 1983). In this case, the evolution equations (Equations (16)–(19)) are greatly simplified as

\[
\dot{r} = \pm E, \quad (30)
\]

\[
\dot{\phi} = \frac{aE}{\Delta}. \quad (31)
\]

Combining these equations, the geodesics in the \((r-\phi)\) plane becomes

\[
\frac{d\phi}{dr} = \pm \frac{a}{\Delta}, \quad (32)
\]

\[
\pm \phi = \frac{a}{r_+ - r_-} \log \left( \frac{r}{r_+} - 1 \right) - \frac{a}{r_+ - r_-} \log \left( \frac{r}{r_-} - 1 \right), \quad (33)
\]

where \( r_\pm \) is the position of the event horizon.

Figure 3 is the same as Figure 2, but for the geodesics in the \((r-\phi)\) plane around an extremely rapidly rotating BH of \( a^* = 0.999 \) (note again that \( a^* = a/M \) is the dimensionless Kerr parameter). In the following, we call the case of \( a^* = 0.999 \) extreme Kerr for simplicity. As shown, our numerical integration can reproduce the analytical solution without visible errors. These results show that our code can correctly trace the null geodesics in the Kerr geometry.

3.3. BH Shadow for Neutrinos

In this section, we demonstrate the capability of our code to capture the images around the accreting BHs, which is often referred to as the BH shadow problem.

For the neutrino sources, we assume a thin accretion disk with a Keplerian rotation profile. We set the mass of the BH to be \( 2 M_\odot \) surrounded by the accretion disk, whose inner and
outer radii are set to be the last stable orbit of the BH (r_{\text{los}}) and 15GM/c^2 with the disk thickness of \pi/10 (rad), respectively. The accretion disk is set to have a uniform density, temperature, and electron fraction of 10^{13} \text{ g cm}^{-3}, 5 \times 10^{11} \text{ K}, and 0.3, respectively. We focus only on the electron-type neutrino in this test problem.

Figure 4 shows one example of the neutrino images around the accreting BHs seen from the viewing angle of \theta_{\text{view}} = 72^\circ from the spin axis of the accretion disk. No visible differences are seen between the two panels, in which the left and right panels are obtained from either the analytical or numerical integration of the geodesics. This supports the validity of our numerical integration of the rendering equation (Equation (28)).

Figure 5 shows a variety of images seen from various viewing angles. For example, when we see the accretion disk from the equatorial plane (bottom right), we can observe neutrinos not only from the disk from the front side, but also from the opposite side because of the bending of the trajectory.

Figure 6 shows the images for different neutrino energies while the viewing angle is kept fixed (\theta_{\text{view}} = 72^\circ). For lower energy neutrinos (such as for 5 MeV (left panel)), the disk luminosity is shown to be almost north–south symmetric, while it becomes highly asymmetric for higher energy neutrinos (such as for 40 MeV (right panel)). As the neutrino energy becomes lower, the position of the neutrino sphere is formed deeper inside the accretion disk, by which we can see the regions closer to the BH (Figure 6). Since the angular velocity of the Keplerian disk is larger for regions at a farther distance from the center, the deformation of the images due to the Doppler effects can be more remarkably seen for the high-energy neutrinos. It is interesting to note that in the case of maximally rotating BHs (bottom panels), the BH shadows become asymmetric even for the low-energy neutrinos due to the frame-dragging effects (bottom two left panels). Such features of the photon shadow in the vicinity of massive BHs in our Galactic center have been considered to give important information that reveals the mass and spin of the BHs (e.g., Takahashi & Watarai 2007; Nagakura & Takahashi 2010). Although this may not be the case for GRBs due to their cosmological distances, the bending of neutrinos may have impact on the gravitational radiation generated by anisotropic neutrino emission (e.g., Epstein 1978; Kotake et al. 2009a). This is one possible extension of this study.

3.4. Neutrino Pair Annihilation from a Spherical Neutrino Sphere

For the collisional Boltzmann equation in GR, it is commonly not trivial to derive analytical solutions for a radiative transport problem. In the following, we derive the analytical solution for radiation fields shedding from a spherical light bulb into a
uniform medium outside. We hope that the analytical solution may be useful to check newly developed codes for the radiative transport in curved space.

In the following numerical tests, the neutrino sphere with a radius of 50 km is assumed to have a surface temperature of \( T = 5 \text{ MeV} \) on which the neutrino distribution function takes a Fermi–Dirac shape with a vanishing chemical potential. The numerical domain \([50 \text{ km}:300 \text{ km}]\) is covered with \( n_r = 100 \) radial mesh points. The fiducial values of the energy and angular bins for the ray-tracing calculation are set to be \((n_\theta, n_\phi, n_\rho) = (16, 32, 16)\), which we will change to see the numerical convergence.

To find the analytical solution in Equation (3), we first take the simplest case of \( u^\tau = 0, u^\nu = u^\rho = 0 \). In this case, the momenta \( p^L_\alpha \) in the LNRF can be expressed in the BLF as

\[
p^L_\alpha = \tilde{\omega}^\rho_\alpha p_\rho,
\]

where

\[
\tilde{\omega}^\rho_\alpha = \left( \begin{array}{cccc} \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & \frac{\bar{\rho}}{\sqrt{\gamma_\rho}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\gamma_\rho}} & 0 \\ 0 & \frac{\bar{\rho}}{a} & 0 & 0 \end{array} \right).
\]

In this way, the solid angle between the two frames can be readily shown to be the same \( (d\Omega^L = d\Omega) \). Similarly, the volume element of the phase space and the neutrino energy in the local rest frame can be expressed by the variables in the BLF as follows:

\[
d^3 p^L = -\left( p^L_\rho \right)^3 dp_\rho d\Omega^L = -\left( \tilde{\omega}^\rho_\alpha \right)^3 (p_\rho)^3 dp_\rho d\Omega.
\]

and

\[
\epsilon^R = -u^\tau p_\tau = -u^0 p_0.
\]

where

\[
u^0 = \frac{1}{\sqrt{-g_{\mu\nu}u^\mu u^\nu}} = \frac{1}{\sqrt{-g_{00} - g_{33}(v^3)^2}},
\]

where we define \( \nu^\nu = u^\nu/\nu^0 \).

Inserting these results into Equation (3), we obtain the following analytical forms of the energy and momentum deposition rate respectively as

\[
Q^L_\tau(r) = 2cKG_{L}^{\bar{\epsilon}} \xi^R_\tau(r) E^R_\nu(r) N^R_\nu(r) F(r),
\]

\[
Q^L_\nu(r) = 2cKG_{L}^{\bar{\epsilon}} \xi^R_\nu(r) E^R_\nu(r) N^R_\nu(r) G(r).
\]

Here \( \xi_\nu \) reflects the GR correction to the neutrino energy as

\[
\xi_\nu(r) \equiv \frac{\epsilon^L}{\epsilon^R} = \sqrt{-g_{00}(R) - g_{33}(v^3)^2} = \frac{1 - 2M/R - (\hat{r}^3)^2}{1 - 2M/R}.
\]

Figure 7. Comparison of the energy deposition rate from the spherical light-bulb test (see the text for detail) for a given angular resolution of ray-tracing calculation \( (n_\theta = 32) \) with or without the AMR treatment. Note that in this test we assume Minkowskian geometry.

(A color version of this figure is available in the online journal.)

where \( \hat{r}^3 = r \sin \theta r^3 \) and \( R \) is the radius of the neutrino sphere. The following two quantities are the energy-weighted integration of the neutrino distribution function on the neutrino sphere (namely \( f_\nu(r, p^R_\nu) \)),

\[
E^R_\nu(r) = \int (\epsilon^R_\nu)^4 f_\nu(r, p^R_\nu) d\epsilon^R_\nu = \frac{(kT(r_\nu))^5}{(hc)^3} F_3(0),
\]

\[
N^R_\nu(r) = \int (\epsilon^R_\nu)^3 f_\nu(r, p^R_\nu) d\epsilon^R_\nu = \frac{(kT(r_\nu))^4}{(hc)^3} F_3(0),
\]

where \( T(r_\nu) \) is set to be 5 MeV. Finally, geometrical factors of \( F(r) \) and \( G(r) \) are given as

\[
F(r) \equiv \int \left[ 1 - \sin \theta \sin \phi \cos (\phi_\nu - \phi_\nu) - \cos \theta \cos \phi_\nu \right]^2 \times d\Omega_\nu d\Omega;
\]

\[
x = \frac{2\pi^2}{3} (1 - x^4)(5 + 4x + x^2),
\]

\[
G(r) \equiv \frac{\pi^2}{6} (1 - x^4)(1 + x)(8 + 9x + 3x^2),
\]

To emphasize the importance of AMR for our ray-tracing calculation (e.g., Section 2.2), we show in Figure 7 our comparison of the energy deposition rates calculated with or without AMR treatment. A good agreement with the analytical solution can be obtained by utilizing the AMR technique. We take \( n_\theta = 32 \) with AMR to be the fiducial value in the following test calculations. A visualization of AMR is also given in Figure 8.

In Figure 9, we compare the analytical solutions (line) with the corresponding numerical solutions in the following four cases: the Minkowskian case with or without rotation (circle, cross)
and the Schwarzschild case with or without rotation (square, triangle). For models with rotation, we set \( \sqrt{1 - \left(\frac{v}{c}\right)^2} = 2 \). For models with the Schwarzschild geometry, we put a point mass of \( M = 3 M_\odot \) inside the neutrino sphere.

Here we present the test calculations to check our implementation of the two GR factors in Equation (41): the gravitational redshift \(-g_{00}(R)\) and the tetrad transformation \(-g_{00}(r)\). We purposely neglect each factor one by one, and compare it to the analytical solution. Figure 10 depicts the numerical solutions including both (cross), without the gravitational redshift (circle), and without the tetrad transformation (triangle). The analytical solutions (lines) are shown to be reproduced only when both of them are appropriately included.

4. APPLICATION TO THE COLLAPSRAR MODEL

Having checked the accuracy of our code in previous sections, we are now in a position to show an application of our code in the collapsar’s environment. As in Harikae et al. (2010), we estimate the annihilation rates in a post-processing manner using the hydrodynamic data obtained in our long-term collapsar simulations. By comparing the neutrino-heating timescale to the advection timescale of material in the polar funnel regions (see Harikae et al. 2010 for details), we discuss the possibility of generating neutrino-driven outflows there. Paying particular attention to the GR effects on the annihilation rates, we discuss their possible impacts on the collapsar dynamics.

As for the hydrodynamic data (such as density, electron fraction, and entropy), we take the ones at 9.1 s after the onset of gravitational collapse for model J0.8 (Figure 11), which show a clear accretion disk and a BH system with the Schwarzschild geometry. As in Harikae et al. (2010), we discuss the possibility of generating neutrino-driven outflows there. Paying particular attention to the GR effects on the annihilation rates, we discuss their possible impacts on the collapsar dynamics.

Figure 8. One example describing the casting of rays with (green) or without AMR technique (red). In this case, the rays are cast to estimate the annihilation rate at a given point (seen as a convergent point of the rays) outside the ergosphere \( r_{\text{ergo}} \). The concentration of the rays is seen (green), which is helpful to correctly estimate the heating rates with reduced computational cost. Note that in this figure only selected rays are chosen for illustrative purpose. This plot is selected from the BH shadow problems (Section 3.3) for the visualization of the AMR.

Figure 9. Same as Figure 7 but for the Minkowskian case with or without rotation (circle, cross) and the Schwarzschild case with or without rotation (square, triangle). See the text for details. For models with rotation, we set \( \sqrt{1 - \left(\frac{v}{c}\right)^2} = 2 \). For models with the Schwarzschild geometry, we put a point mass of \( M = 3 M_\odot \) inside the neutrino sphere.

(A color version of this figure is available in the online journal.)

Figure 10. Same as Figure 7 but for the numerical solution without the gravitational redshift (circle), without the correction due to the tetrad transformation (triangle), and the one including both (cross). Agreement with the analytical solution (line) can be seen when the two ingredients are included (See the text for details.)

(A color version of this figure is available in the online journal.)

Figure 11. Hydrodynamic configuration employed in the ray-tracing calculation. This is the snapshot at 9.1 s after the onset of gravitational collapse for model J0.8 when the accretion disk is in a stationary state (see Harikae et al. 2010 for more details). The logarithmic density (in g cm\(^{-3}\), left-half) and temperature (in K, right-half) are shown. The white solid line denotes the area where the density is equal to \( 10^{11} \) g cm\(^{-3}\), representing the surface of the accretion disk. The central black circle \((\approx 4 M_\odot)\) represents the inner boundary of our computations.

(A color version of this figure is available in the online journal.)
tracing calculation. The position of the inner boundary of the computational domain is set to be $4\,M_{\odot}$, which mimics the event horizon of the BH. We set the Kerr parameter by hand as $a^* = 0$ for the Schwarzschild geometry and $a^* = 0.999$ for the extreme Kerr geometry.

4.1. Effect of General Relativity on Energy and Momentum Deposition

To clarify the GR effects, we compare the annihilation rates in the Minkowskian ($M = 0$), Schwarzschild, and extreme Kerr geometries ($a^* = 0.999$). Figure 12 shows the energy deposition rate $Q_\nu$ (contour; Equation (3)) and the normalized momentum transfer rate $Q_\nu/Q_\ell$ (vector; e.g., Equation (3)) for the Minkowskian geometry (top left/top right: without/with special relativistic corrections), the Schwarzschild geometry (bottom left), and the extreme Kerr geometry ($a^* = 0.999$; bottom right). The spatial vector is visualized by showing the spatial velocity vector $v \equiv Q_\ell/Q_\nu$, which is normalized by the speed of light ($c = 1$) being represented by the arrow (top right in each panel). The central black circle ($\approx 4\,M_{\odot}$) represents the inner boundary of the computational domain. Note that the triangular regions colored by black closely coincide with the surface of the accretion disk (e.g., Figure 11).

(A color version of this figure is available in the online journal.)

From Figure 12, it can be seen in the Minkowskian geometry (top two panels) that the direction of momentum transfer is generally radially outward, while in the Schwarzschild and Kerr geometries (bottom two panels), the direction, especially in the vicinity of the BH, tends to be directed toward the center as a result of GR bending. This bending effect, acting to suppress the outward momentum transfer, does harm to the launch of the neutrino-driven outflow. On the other hand, it does well to the energy deposition, because it enhances the head-on collision especially in the polar funnel regions.

In fact, it can be seen that the deposition rate for the Schwarzschild and Kerr geometries becomes larger than for the Minkowskian geometry (Figure 13). In the bluish region which corresponds to the polar funnel region, the energy deposition rate for the extreme Kerr geometry is enhanced by several factors compared to the Minkowskian geometry. Interestingly, it is mentioned that the heating rate is enhanced by about one order of magnitude near the equatorial plane in the vicinity of the BH. This is because the neutrino rays are concentrated there, reflecting the conical shape of the accretion disk (triangular blackish regions at the sides). This concentration near the equatorial plane is found to be suppressed for the maximally rotating BH mainly due to the frame-dragging effect.

To see the GR effects on the net energy deposition, we calculate the total energy deposition rate

$$Q_{\nu}^{\text{tot}} = \int \sqrt{-g} \, Q_\ell \, dV,$$

and the one with the outgoing momentum as (Jaroszynski 1993; Birkl et al. 2007)

$$Q_{\nu}^{\text{out}} = \int \sqrt{-g} \, Q_\ell \, dV \bigg|_{Q_\ell > 0},$$

where the contributions with the outgoing radial component of the momentum vector ($Q_\ell$) are counted. As shown in Table 1, the net deposition rate and efficiency for the extreme Kerr geometry increase up to 16% (18% for $Q_{\nu}^{\text{out}}$) compared to the Minkowskian geometry. Our results support previous studies showing that GR can enhance the heating rate, and thus is good for the formation of neutrino-driven outflow (e.g., Birkl et al. 2007). From Table 1, the deposition rate and efficiency are barely influenced by the spin of the BH. However, it should be noted that we have included the spin effects only in the radiative transport. As pointed out by Asano & Fukuyama (2001) and Birkl et al. (2007), the spin effects, such as on the structure of the spacetime (i.e., the inner-most stable circular orbit becomes smaller for the rapidly rotating BH) and also on the accretion disk, should be more important in affecting the heating rate. To clarify this point, we need hydrodynamic data based on the GR simulations of collapsars, which we need to investigate as a sequel to this study.

4.2. Condition for Outflow Formation

Based on the annihilation rates in the last section, we compare the two timescales in this section, the neutrino-heating timescale

\[ T_{\nu} = \frac{\int \sqrt{-g} \, Q_\nu \, dV}{\sum \sqrt{-g} \, Q_\ell \, dV} \]

and the collision timescale

\[ T_{\text{col}} = \frac{1}{\sqrt{-g} \, \sum \sqrt{-g} \, Q_\ell \, dV} \]

...
and the dynamical timescale. We then anticipate if the neutrino-heating outflows could or could not be produced in the polar funnel regions.

To trigger the neutrino-heating explosion, the neutrino-heating timescale should be smaller than the advection timescale, which is characterized by the free-fall timescale in the polar funnel regions. This condition is akin to the condition of the successful neutrino-driven explosion in the case of core-collapse supernovae (e.g., Bethe 1990 and see collective references in Janka et al. 2007). The heating timescale is the timescale for a fluid to absorb the energy by neutrino heating, comparable to the gravitational binding energy for making the fluid gravitationally unbound, and which may be defined as $\tau_{\text{heat}} \equiv \rho \Phi / Q_\nu$. Here, $\Phi$, the local gravitational potential, is taken to be the sum of the pseudo-Newtonian potential and self-gravity in the flat spacetime (Harikae et al. 2010) and $\rho$ is the local matter density. Then the dynamical timescale is defined as $\tau_{\text{dyn}} \equiv \sqrt{3\pi/16G\bar{\rho}}$, where $\bar{\rho}$ is the average density at a certain radius and we take $\bar{\rho}(r) \equiv 3M(r)/4\pi r^3$.

Figure 14 depicts the ratio of the dynamical $\tau_{\text{dyn}}$ to the heating timescales $\tau_{\text{heat}}$ for the Schwarzschild (left) and extreme Kerr geometries (right), showing in both cases that the ratio becomes greater than unity in the polar funnel regions (compare Figure 11). Figure 15 shows the energy deposition rate along the polar axis of Figure 12 for the Minkowskian geometry without or with the special relativistic correction (indicated by “Newtonian” and “SR”), and for the Schwarzschild and extreme Kerr geometries (indicated by “GR ($a^* = 0$)” and “GR ($a^* = 0.999$)”). Note that energy deposition drops sharply from the Newtonian to the SR case (left panel). This is the outcome of the special relativistic beaming effects. Since the rotational velocity of the accretion disk is perpendicular to the polar direction, the special relativistic beaming effect suppresses the neutrino emission toward the polar region (see Harikae et al. 2010 for more details). When the GR bending effects are taken into account, the deposition rate becomes larger again (see “GR ($a^* = 0$)” and “GR ($a^* = 0.999$)”). Reflecting this situation, $\tau_{\text{dyn}}/\tau_{\text{heat}}$ becomes smallest for the case with SR and largest for the Newtonian case (the right panel of Figure 15). It is important that the ratio in the case of the Schwarzschild and extreme Kerr geometries, which reflect the nature of the collapsar’s environment, becomes larger than unity inside a 100 km radius in the vicinity of the rotational axis (Figure 15, right). This indicates the possible formation of neutrino-driven outflows there, if coupled with the collapsar’s hydrodynamics.

### 5. SUMMARY AND DISCUSSION

In light of collapsar models of GRBs, we developed a numerical scheme and code for estimating the deposition of energy and momentum due to neutrino pair annihilation ($\nu + \bar{\nu} \rightarrow e^- + e^+$) in the vicinity of an accretion tori around a Kerr BH. We designed our code to calculate the GR neutrino transfer by a ray-tracing method. To solve the collisional Boltzmann equation in Kerr geometry, we numerically integrated the so-called rendering equation along the null geodesics. For the neutrino opacity, the charged-current $\beta$-processes, which are dominant in the vicinity of the accretion tori, are taken into account. We employed the Fehlberg (4,5) adaptive integrator in the Runge-Kutta method in order to perform the numerical integration accurately. We checked the numerical accuracy of the developed code with several tests in which we showed comparisons with the corresponding analytical solutions. In order to solve the energy-dependent ray-tracing transport, we proposed that an AMR approach, which we took for the two radiation angles ($\theta, \phi$) and the neutrino energy, is efficient in reducing the computational cost. Based on the hydrodynamical data in our collapsar simulation, we estimated the annihilation rates in a post-processing manner. It is found that the GR effect can increase the local energy deposition rate by about one order of magnitude, and the net energy deposition rate by several tens of percents. After the accretion disk settles into a stationary state (typically later than $\sim 9$ s from the onset of gravitational collapse), we pointed out that the neutrino-heating timescale can be smaller than the dynamical timescale inside a 100 km radius in the vicinity of the rotational axis. Our results suggest that the neutrino-driven outflows can possibly be launched there.

For further investigation, we need to include several important ingredients ignored in this study. We plan to develop a GRMHD code for collapsars, which is indispensable to the outcome of this paper. By changing the precollapse magnetic fields and rotation systematically, we hope to clearly understand how the outflow formation in collapsars could change from a neutrino-driven mechanism to an MHD-driven one. The neutrino oscillation from the Mikheyev–Smirnov–Wolfenstein (MSW) effect (see references in both Kotake et al. 2006 and Kawagoe et al. 2009) could be important, albeit in a much later phase than we considered in this paper. When the density in the polar funnel regions drops to as low as $\rho \lesssim 10^5$ g cm$^{-3}$ later, the neutrino oscillation could operate for neutrinos traveling from the accretion disk to the polar funnel. If this is the case, the incoming neutrino spectra to the polar funnel regions and the pair annihilation rates there could be affected significantly. It is also noted that the effects of neutrino self-interaction remain

| Geometry         | $Q_{\nu}^{\text{tot}}$ (erg s$^{-1}$) | $Q_{\nu}^{\text{heat}}$ (erg s$^{-1}$) | Efficiency (%) |
|------------------|--------------------------------------|---------------------------------------|----------------|
| Minkowski        | $6.18 \times 10^{50}$                | $5.71 \times 10^{50}$                | 0.510          |
| Schwarzschild    | $7.15 \times 10^{50}$                | $6.09 \times 10^{50}$                | 0.590          |
| Extreme Kerr     | $7.08 \times 10^{50}$                | $6.13 \times 10^{50}$                | 0.585          |

Note. Efficiency is evaluated as $Q_{\nu}^{\text{tot}}/L_\nu$, where $L_\nu$ is the total neutrino luminosity.
to be studied; these have been attracting great attention in the theory of core-collapse supernovae (e.g., Duan et al. 2006). As in the case of core-collapse supernovae (e.g., Kotake et al. 2009b; Ott et al. 2008b), studies of gravitational-wave emissions from collapsars might provide us with a new window with which to probe into the central engine (e.g., Hiramatsu et al. 2005; Suwa & Murase 2009). As a sequel to this work, we plan to implement the ray-tracing calculation in GRMHD simulation and to clarify issues one by one. We hope that this study is a very first step toward the meeting of GR with neutrino transport, which is indispensable for understanding collapsar engines.

S.H. is grateful to T. Kajino for helpful exchanges. T.T. and K.K. express thanks to K. Sato, S. Yamada, and S. Nagataki for continuing encouragements. Numerical computations were in part carried out on XT4 and the general common use computer system at the Center for Computational Astrophysics, CfCA, and the National Astronomical Observatory of Japan. This study was supported in part by the Grants-in-Aid for the Scientific Research from the Ministry of Education, Science and Culture of Japan (Nos. S19104006, 19540309, and 20740150).

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Figure 15. Comparison of energy deposition rates (left) and $\tau_{\text{heat}}$ along the rotational axis (right) between the Minkowskian geometry without or with the special relativistic correction (indicated by “Newtonian” and “SR”), and for the Schwarzschild and extreme Kerr geometries (indicated by “GR ($a^* = 0$)” and “GR ($a^* = 0.999$)”).

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