ENHANCED DISSIPATION AND NONLINEAR ASYMPTOTIC STABILITY OF THE TAYLOR-COUETTE FLOW FOR THE 2D NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper, we study the nonlinear stability of a steady circular flow created between two rotating concentric cylinders. The dynamics of the viscous fluid are described by 2D Navier-Stokes equations. We adopt scaling variables. For the rescaled equations, we prove that the steady flow (Taylor-Couette flow) is asymptotically stable up to a large perturbation of initial data. Back to the original 2D Navier-Stokes equations, this implies an improved transition threshold for the Taylor-Couette flow. The improvement is due to enhanced dissipation and new observations and constructions of weighted $L^2$ norms, which capture a hidden structure between the viscosity constant $\nu$ and (different) rotating speeds and locations of two coaxial cylinders. In particular, we allow the location of the outer cylinder to tend to infinity, which renders the initial fluid kinetic energy not uniformly bounded. Due to enhanced-dissipation effect, we also establish a sharp resolvent estimate, desired space-time bounds and optimal decaying estimates, which lead to the proof of nonlinear asymptotic stability of 2D Taylor-Couette flow.

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1. Introduction

In this paper, we consider the two-dimensional (2D) incompressible Navier-Stokes equations:
\[
\begin{align*}
\partial_t v - \nu \Delta v + v \cdot \nabla v + \nabla p &= 0, \\
\text{div } v &= 0,
\end{align*}
\]
with \(x = (x_1, x_2) \in \Omega = \mathbb{R}^2\) being the space variables and \(t \geq 0\) being the time variable. Here the constant \(\nu > 0\) is called the kinematic viscosity. The unknowns to (1.1) are the velocity field \(v(t, x) = (v_1(t, x), v_2(t, x))\) and the pressure \(p(t, x) \in \mathbb{R}\).

If taking the 2D curl to the above Navier-Stokes equations, then (1.1) is transferred to
\[
\partial_t \omega - \nu \Delta \omega + v \cdot \nabla \omega = 0.
\]
Here \(\omega = \text{curl } v = \partial_1 v_2 - \partial_2 v_1\) is the vorticity field. For 2D case, it is a scalar. And (1.2) is called the vorticity equation formulation. In this paper, we will use (1.2) to understand the asymptotics of solutions to (1.1).

Given \(\omega\) being a solution to (1.2), the velocity field \(v : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}^2\) can be derived via solving the below elliptic system
\[
\text{div } v = 0, \quad \text{curl } v = \omega.
\]
This leads to the 2D Biot-Savart law
\[
v(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)^\perp}{|x - y|^2} \omega(t, y) dy.
\]
For notational simplicity, we write \(v(t, \cdot) = K_{BS} \ast \omega(t, \cdot)\), where \(K_{BS}(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}\) is the Biot-Savart kernel.

1.1. Taylor-Couette flow. In this article, we focus on studying the nonlinear stability and long-time dynamics around the circular flows with the vorticity \(\omega(x) = \omega(|x|)\) being radial. In below, we denote \(\tilde{r}\) to be \(|x|\).

An important physical circular-flow solution to (1.2) is called the Taylor-Couette flow. It describes a steady (circular) flow of viscous fluid bounded between two rotating infinitely long coaxial cylinders and has wide applications ranging from desalination to viscometric analysis. Now we derive its expression.

Denote \(\omega(x) = \omega(\tilde{r})\) and \(\phi(x) = \phi(\tilde{r})\) to be the radial vorticity and the stream function, respectively. Then for 2D case we have
\[
\begin{align*}
\omega(\tilde{r}) &= \Delta \phi = \phi''(\tilde{r}) + \frac{1}{\tilde{r}} \phi'(\tilde{r}), \\
\phi(x) &= \left(\frac{-\partial_2 \phi}{\partial_1 \phi}\right) = \partial_\tilde{r} \phi \partial_{\tilde{r}} = \left(\frac{-x_2}{x_1}\right) \frac{\phi'(\tilde{r})}{\tilde{r}}.
\end{align*}
\]
One can easily check \(\nabla \cdot v = 0\) and via \(\partial_1 = \cos \tilde{\theta} \partial_{\tilde{r}} - \frac{1}{\tilde{r}} \sin \tilde{\theta} \partial_{\tilde{\theta}}, \partial_2 = \sin \tilde{\theta} \partial_{\tilde{r}} + \frac{1}{\tilde{r}} \cos \tilde{\theta} \partial_{\tilde{\theta}}\), one also has
\[
v \cdot \nabla \omega = -\sin \tilde{\theta} \phi' \partial_1 \omega + \cos \tilde{\theta} \phi' \partial_2 \omega = 0.
\]
Hence, a radial vorticity \( \omega(\tilde{r}) \) in (1.4) is a stationary solution to the below 2D incompressible Euler equation

(1.6) \[
\partial_t \omega + v \cdot \nabla \omega = 0.
\]

In particular, if we choose \( \omega = \text{const} \), then we have \( \Delta \omega = 0 \) and \( \omega \) is also a solution to the 2D incompressible Navier-Stokes equation (1.2). For this case, by (1.4) the stream function \( \phi(\tilde{r}) \) reads

(1.7) \[
\phi''(\tilde{r}) + \frac{1}{\tilde{r}} \phi'(\tilde{r}) = 2A_1.
\]

where \( A_1, A_2 \) are constants. This implies \( (\tilde{r}\phi')' = 2A_1\tilde{r} \) and \( \phi' = A_1 + \frac{A_2}{\tilde{r}} \).

Via (1.7) we now derive a solution \( v(x, y) \) to (1.1). And with polar coordinate, we rename \( v(x, y) \) and \( \omega(\tilde{r}) \) to be \( U(\tilde{r}, \tilde{\theta}) \) and \( \Omega(\tilde{r}) \), respectively. Now we have

(1.8) \[
U(\tilde{r}, \tilde{\theta}) = \begin{pmatrix} U^1 \\ U^2 \end{pmatrix} = \begin{pmatrix} -\sin \tilde{\theta} \\ \cos \tilde{\theta} \end{pmatrix} (A_1\tilde{r} + \frac{A_2}{\tilde{r}}) = v_\theta e_\theta, \quad \Omega(\tilde{r}) = 2A_1,
\]

where \( A_1, A_2 \) are constants, \( e_\theta = \begin{pmatrix} -\sin \tilde{\theta} \\ \cos \tilde{\theta} \end{pmatrix} \), and \( v_\theta = A_1\tilde{r} + \frac{A_2}{\tilde{r}} \) is the azimuthal velocity component. Note that \( U(\tilde{r}, \tilde{\theta}) \) in (1.8) is a stationary solution to (1.1) and it is called the Taylor-Couette flow.

1.2. Connection and comparison to the Lamb-Oseen vortex. It is well known that the 2D Navier-Stokes equations (1.2) have a family of self-similar solutions called the Lamb-Oseen vortices, which are of the following form

(1.9) \[
\omega(t, x) = \frac{\alpha}{\sqrt{vt}} G\left(\frac{x}{\sqrt{vt}}\right), \quad v(t, x) = \frac{\alpha}{\sqrt{vt}} v^G\left(\frac{x}{\sqrt{vt}}\right).
\]

Here the vorticity profile and the velocity profile are given by

(1.10) \[
G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4}, \quad v^G(\xi) = \frac{1}{2\pi |\xi|^2} \left(1 - e^{-|\xi|^2/4}\right).
\]

In particular we have \( v^G = K_{BS} \ast G \) with \( K_{BS} \) being the Biot-Savart kernel. And we further denote space

(1.11) \[
Y := L^2(\mathbb{R}^2, G^{-1}d\xi).
\]

An important property of flow obeying (1.2) is that it preserves the mass in \( L^1(\mathbb{R}^2) \), i.e.,

(1.12) \[
\int_{\mathbb{R}^2} \omega(t, x)dx = \int_{\mathbb{R}^2} \omega(0, x)dx = \alpha \quad \text{for any } t > 0.
\]

The parameter \( \alpha \in \mathbb{R} \) is called the circulation Reynolds number. People studying fluid mechanics are especially interested in rapidly rotating vortices, where the circulation \(|\alpha|\) is much larger compared with the kinematic viscosity \( \nu \). This is the regime most relevant with applications to 2D turbulent flows. Recently, there are exciting progress on the linear and nonlinear stability of the Lamb-Oseen vortex. The results in this article are parallel to these developments. And here we review some important results about the Lamb-Oseen vortex.

In 2005, to study the long-time dynamics of the 2D Navier-Stokes equations, Gallay and Wayne [23] introduced the so called scaling variables or similarity variables:

(1.13) \[
\xi = \frac{x}{\sqrt{\nu t}}, \quad \tau = \log t.
\]
And they work under the ansatz
\[ \omega(x,t) = \frac{1}{t}w\left(\frac{x}{\sqrt{vt}}\right) \log t, \quad v(x,t) = \sqrt{\nu \frac{\nu}{t}}u\left(\frac{x}{\sqrt{vt}}\right) \log t. \]

The rescaled vorticity \( w(\xi, \tau) \) now satisfies the evolution equation
\[ \partial_\tau w - (\Delta_\xi + \frac{1}{2} \xi \cdot \nabla_\xi + 1)w + u \cdot \nabla w = 0. \]

When the initial vorticity is integrable, i.e., when \( \alpha \) defined in (1.12) is finite, Gallay and Wayne in [22, 23] proved that the long-time dynamical behaviours of the 2D Navier-Stokes equations can be described and approximated by the Lamb-Oseen vortex. Let \( G \) be defined as in (1.10). They proved

**Proposition 1.1** (Gallay-Wayne [23]). For any \( w_0 \in Y \), the rescaled vorticity equation (1.15) admits a unique global mild solution \( w \in C^0([0, \infty), Y) \) with \( w(0) = w_0 \). This solution satisfies \( \|w(\tau) - \alpha G\|_Y \to 0 \) as \( \tau \to +\infty \), where \( \alpha = \int_{\mathbb{R}^2} w_0(\xi) d\xi \).

Later in 2008, Gallay and Rodrigues in [18, 20] showed that for any finite value of circulation \( |\alpha| \), the equilibrium \( \alpha G \) is asymptotically stable in \( Y \).

**Proposition 1.2** (Gallay-Rodrigues [18, 20]). There exists \( \epsilon > 0 \) such that, for all \( \alpha \in \mathbb{R} \) being finite and for all \( w_0 \in \alpha G + Y_0 \) with \( Y_0 = \{w \in Y | \int_{\mathbb{R}^2} w(\xi) d\xi = 0\} \), if requiring \( \|w_0 - \alpha G\|_Y \leq \epsilon \), then the unique solution to (1.15) with initial data \( w_0 \) satisfies
\[ \|w(\tau) - \alpha G\|_Y \leq \min(1, 2e^{-\frac{\epsilon}{2}})\|w_0 - \alpha G\|_Y \text{ for any } \tau \geq 0. \]

A feature associated with (1.2) that Proposition 1.2 did not address is the enhanced dissipation effect when the circulation Reynolds numbers \( |\alpha| \) is large. Gallay proposed a conjecture on the optimal resolvent estimate for the Oseen vortex. And this conjecture was solved by Li, Wei and Zhang in [29]. Let \( G \) and \( v^G \) be defined as in (1.10). Set \( L \) to be \( \Delta_\xi + \frac{1}{2} \xi \cdot \nabla_\xi + 1 \) and operator \( \Lambda \) to satisfy
\[ \Lambda w = v^G \cdot \nabla w + u \cdot \nabla G \quad \text{with} \quad u = K_{BS} * w. \]

Note that \( \Lambda \) is a nonlocal linear operator. Then the following statement holds

**Proposition 1.3** (Li-Wei-Zhang [29]). Denote \( L_\perp \) and \( \Lambda_\perp \) to be \( L \) and \( \Lambda \)'s restriction to the orthogonal complement of \( \ker \Lambda \) in \( Y \). Define the below pseudospectral bound
\[ \Psi(\alpha) := (\sup_{\lambda \in \mathbb{R}} \left\| (L_\perp - \alpha \Lambda_\perp - i\lambda)^{-1} \right\|_{Y \to Y})^{-1}. \]

Then as \( |\alpha| \to +\infty \), there exists \( C > 0 \) independent of \( \alpha \) satisfying
\[ C^{-1}|\alpha|^{\frac{1}{3}} \leq \Psi(\alpha) \leq C|\alpha|^{\frac{1}{3}}. \]

Based on the linear resolvent estimate in [29], Gallay [19] proved an improved stability result which allows the size of perturbation (size of the basin of attraction) of Oseen’s vortex to be large when \( |\alpha| \gg 1 \). Also, the non-zero frequency of vorticity obeys a faster decaying rate. Rewrite \( w(\xi, \tau) \) as \( w(\theta, r, \tau) \) with polar coordinate and let space \( Y_1 \) be \( \{w \in Y | \int_{\mathbb{R}^2} \xi_i w(\xi) d\xi = 0, i = 1, 2\} \). It holds

**Proposition 1.4** (Gallay [19]). There exist positive constants \( C_1, C_2, \kappa \) such that, for all \( \alpha \in \mathbb{R} \) and all initial data \( w_0 \in \alpha G + Y_1 \) satisfying
\[ \|w_0 - \alpha G\|_Y \leq \frac{C_1 (1 + |\alpha|)^{\frac{\kappa}{6}}}{\log(2 + |\alpha|)} \]
we have that the unique solution to (1.15) in $Y$ (guaranteed by Proposition 1.1) obeys, for any $\tau \geq 0$,
\[
\|w(\cdot, \tau) - \alpha G\|_Y \leq C_2 e^{-\tau} \|w_0 - \alpha G\|_Y,
\]
\[
\| (I - P_r) w(\cdot, \tau) - \alpha G\|_Y \leq C_2 \|w_0 - \alpha G\|_Y \exp\left(-\frac{\kappa (1 + |\alpha|)^{\frac{3}{2}} \tau}{\log(2 + |\alpha|)}\right),
\]
where
\[
P_r w(\theta, r, \tau) = \int_0^{2\pi} w(\theta, r, \tau) d\theta.
\]

1.3. Main results. In this paper, we study fluid dynamics around the Taylor-Couette flow. We are in particular interested in the regime with a high Reynolds number ($\text{Re} = \frac{1}{\nu}$).

The Taylor-Couette flow is the steady flow created between two rotating concentric cylinders. In this paper, we consider the 2D Taylor-Couette flow, and aim to prove its nonlinear asymptotic stability under the 2D incompressible Navier-Stokes equations (1.2). For the Taylor-Couette flow, if the inner cylinder with radius $r_1$ is rotating at constant angular velocity $\omega_1$ and the outer cylinder with radius $r_2$ is rotating at constant angular velocity $\omega_2$, then the azimuthal velocity component is given by
\[
v_\theta = A_1 r + \frac{A_2}{r} = r \omega_{\text{phy}}, \quad A_1 = \omega_1 \frac{\mu - \eta^2}{1 - \eta^2}, \quad A_2 = \omega_1 r_1^2 \frac{1 - \mu}{1 - \eta^2},
\]
where
\[
\mu = \frac{\omega_2}{\omega_1}, \quad \eta = \frac{r_1}{r_2}, \quad \omega_{\text{phy}} := A_1 + \frac{A_2}{r^2}.
\]

One can verify
\[
\omega_1 = \omega_{\text{phy}}(r_1), \quad \omega_2 = \omega_{\text{phy}}(r_2).
\]

Then we can directly compute fluid vorticity from the velocity field as below
\[
\Omega(r) = \partial_1 U^2 - \partial_2 U^1 = \partial_1 (x_1 \omega_{\text{phy}}) - \partial_2 (-x_2 \omega_{\text{phy}}) = 2 \omega_{\text{phy}} + x_1 \partial_1 \omega_{\text{phy}} + x_2 \partial_2 \omega_{\text{phy}} = 2 \omega_{\text{phy}} + r \omega'_{\text{phy}} = 2A_1.
\]

Although $\omega_{\text{phy}}$ is singular at $r = 0$, the above equality indicates that $\Omega(r)$ is regular at $r = 0$.

Before presenting our main theorems, let’s review some related results on the mathematical study of the Taylor-Couette flow or more general circular flows:

- In [47], for 2D incompressible Euler equations, Zillinger established linear inviscid damping with decay rates around the Taylor-Couette flow in an annular region. Note that this restriction on the annular region also avoid the issues of fluid dynamics at the origin and at infinity.

- In [46], for 2D Euler equations, Zelati and Zillinger established linear inviscid damping and proved linear stability for a class of mildly degenerate flows. This class includes $U(r) \sim \frac{1}{r} + r$, which is different from shear flows with strict monotonicity [11].

- Recently, Gallay and Šverák [21] employed variational principles (related to Arnold’s stability criteria) to study orbital stability of steady circular solutions to both Euler’s equations and the Navier-Stokes equation at low viscosity. They proved the stability for the algebraic vortex $\omega(x) = (1 + |x|^2)^{-\kappa}$ with $\kappa > 1$ and for the Lamb-Oseen vortex $\omega(x) = e^{-\frac{|x|^2}{4}}$. Note that the vorticity of the Taylor-Couette flow does not decay when $|x| \to +\infty$. And it has not been addressed in [21].
We proceed to state our main results. In this paper, we work with scaling variables:

\[ \xi = \frac{x}{\sqrt{\nu t}}, \quad \tau = \log t. \]

In [23], Gallay and Wayne rewrite solutions to (1.2) under the ansatz

\[ \omega(x, t) = \frac{1}{t} \nu \left( \frac{x}{\sqrt{\nu t}} \right) \log t, \quad v(x, t) = \sqrt{\nu} \left( \frac{x}{\sqrt{\nu t}} \right) \log t. \]

Then the rescaled vorticity \( w(\xi, \tau) \) satisfies the following evolution equation

\[ \partial_{\tau} w - (\Delta_{\xi} + \frac{1}{2} \xi \cdot \nabla_{\xi} + 1) w + u \cdot \nabla_{\xi} w = 0. \]

The rescaled velocity \( u \) can be expressed via the Biot-Savart law, namely \( u(\cdot, \tau) = K_{BS} \ast w(\cdot, \tau) \). If we write \( \xi \) in polar coordinates

\[ \xi_1 = r \cos \theta, \quad \xi_2 = r \sin \theta \quad \text{with } r \in [0, \infty) \quad \text{and } \theta \in \mathbb{T}. \]

Then we can take Fourier transform with respect to \( \theta \). Denoting \( w_k(\tau, r) = \frac{1}{2\pi} \int_{\mathbb{T}} w(\tau, r, \theta) e^{-ik\theta} d\theta, \) we can write

\[ w(\tau, \xi) = w(\tau, r, \theta) = \sum_{k \in \mathbb{Z}} \omega_k(\tau, r) e^{ik\theta}. \]

Recall that the fluid velocity \( U \) and vorticity \( \Omega \) of the Taylor-Couette flow (1.8) are given by

\[ U(\tilde{r}, \tilde{\theta}) = \begin{pmatrix} U^1 \\ U^2 \end{pmatrix} = \begin{pmatrix} -\sin \tilde{\theta} \\ \cos \tilde{\theta} \end{pmatrix} (A_1 \tilde{r} + A_2) = v_\tilde{\theta} e_\tilde{\theta}, \quad \Omega(r) = 2A_1. \]

Here \( A_1, A_2 \) are constants, \( e_\tilde{\theta} = \begin{pmatrix} -\sin \tilde{\theta} \\ \cos \tilde{\theta} \end{pmatrix} \) and \( v_\tilde{\theta} \) is the azimuthal velocity component.

Via the self-similar transformation, the steady circular solutions (1.8) to the original (1.6) equation are concerted to

\[ (1.17) \quad V = \sqrt{\frac{1}{\nu} U} = \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix} (A_1 e^\tau + \frac{A_2}{\nu |\xi|^2}), \quad W = t\Omega = 2A_1 e^\tau. \]

For 2D Navier-Stokes equations, we work with scaling variables and its equivalent form (1.15). Thus the original problem is transformed into analyzing the stability of (1.17) with equation (1.16). For initial data, via Fourier series we have \( w(0, r, \theta) = w_0(0, r) + \sum_{k \in \mathbb{Z} \setminus \{0\}} w_k(0, r) e^{ik\theta}. \)

For simplicity, we also employ \( w(0), w_0(0), w_k(0) \) to replace \( w(0, r, \theta), w_0(0, r), w_k(0, r) \), respectively.

For future use, we define

\[ (1.18) \quad \|w\|^2_M := \int_0^\infty r e^\frac{r^2}{4} |w(r)|^2 dr, \quad \|w\|_X := \left( \int_0^\infty \frac{|w|^2}{r^2} dr \right)^{\frac{1}{2}}. \]

And we introduce an energy norm \( \mathcal{E}(\tau) \) as below

\[ \mathcal{E}(\tau) = \frac{A_2}{\nu} \left( \frac{1}{r} \right) \omega(\tau) - 2A_1 e^\tau \|w\|_M + \sum_{k \in \mathbb{Z} \setminus \{0\}} (\|\omega_k(\tau)\|_M + |k| \nu \frac{A_2}{\nu} \left( \frac{1}{r} \right) \|\omega_k(\tau)\|_M). \]

For initial data, via Fourier series we have \( w(0, r, \theta) = w_0(0, r) + \sum_{k \in \mathbb{Z} \setminus \{0\}} w_k(0, r) e^{ik\theta}. \) For simplicity, we also adapt \( w(0), w_0(0), w_k(0) \) to denote \( w(0, r, \theta), w_0(0, r), w_k(0, r) \), respectively. In our paper, we are interested in the regime when \( \nu \leq |A_2| \). In particular, we allow \( 0 < \nu \ll 1 \). Our main result of this paper can be stated as follows:
Theorem 1.5 (Main Theorem). For any $|A_2| \geq \nu$, there exist constants $c_0, c, C > 0$ independent of $A_1, A_2, \nu$ such that if the initial data $w(0) = w_0(0) + \sum_{k \in \mathbb{Z} \setminus \{0\}} w_k(0)e^{ik\theta}$ satisfy
\[
\mathcal{E}(0) = \left( \frac{A_2}{\nu} \right)^{\frac{1}{2}} r \frac{\omega_0(0)}{r} \| w_0 \|_M + \sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \frac{\| w_k(0) \|_M}{r} + \| k \frac{1}{\nu} \frac{A_2}{\nu} \frac{\omega_k(0)}{r} \|_M \right) \leq c_0 \left( \frac{A_2}{\nu} \right)^{\frac{1}{2}},
\]
then the solution $w$ of the full nonlinear vorticity formulation (1.15) is global in time and the following stability estimate holds
\[
\left| \frac{A_2}{\nu} \right|^{\frac{1}{2}} \left| \frac{e^{-\gamma} \omega_0(\tau) - 2A_1}{r} \right| \| w \|_M \leq Ce^{-\gamma} \mathcal{E}(0),
\]
and
\[
\sum_{k \in \mathbb{Z} \setminus \{0\}} \left( \frac{\| w_k(\tau) \|_M}{r} + \| k \frac{1}{\nu} \frac{A_2}{\nu} \frac{\omega_k(\tau)}{r} \|_M \right) \leq Ce^{-c \frac{A_2}{\nu}} \mathcal{E}(0).
\]
Hence, it holds
\[
\mathcal{E}(\tau) \leq C \mathcal{E}(0).
\]

In our work, we keep tracking the relation between the enhanced dissipation effect and the coefficients $\nu$ and $A_1, A_2$. We are especially interested in the case with rapidly moving velocity, where the parameter $|A_2|$ is much larger than the kinematic viscosity $\nu$.

Remark 1.6. Compared with Proposition [19] stated above by [19], here we allow larger perturbation. In [19], for the Oseen’s vortex, the size of the basin for attraction is of $|\alpha|^{\frac{1}{2}}$ (where $\alpha \sim \nu^{-1}$), while here for Taylor-Couette flow, $|\frac{A_2}{\nu}|$ is analogous to $|\alpha|$, and the size of perturbation is permitted to be of size $|\frac{A_2}{\nu}|^{\frac{1}{2}}$. Moreover, to control the nonlinear terms in a sharper way, we construct and employ a designed weighted energy space $X$ with $\|w\|_X = \left( \int_0^\infty \|w\|^2 \frac{d\tau}{\tau} \right)^{\frac{1}{2}}$.

This can be fulfilled due to a newly found connection about order of $|\frac{kA_2}{r}|$ between two space-time bounds. Tentative space-time estimates leads us to consider the below two sums
\[
(1.19) \quad \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^\infty L^2} + \frac{kA_2}{\nu} \left| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 L^2} \right| + \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 L^2},
\]
\[
(1.20) \quad \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^\infty X} + \frac{kA_2}{\nu} \left| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 X} \right| + \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 X}.
\]

And we expect (1.19) and (1.20) $|\frac{kA_2}{\nu}|^{\frac{1}{2}}$ are of the same order. Note that in (1.20) the coefficient of $\| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 X} = \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 L^2}$ is 1, and in (1.19) the coefficient in front of $\| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 L^2}$ is $\frac{kA_2}{\nu}^{\frac{1}{2}}$. This observation motivates us to couple $L^2$ and $X$ space together to construct combined energy $E_k(|k| \geq 1)$ of order (1.19) + $|\frac{kA_2}{\nu}|^{\frac{1}{2}}$ (1.20) as below
\[
(1.21) \quad E_k = \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^\infty L^2} + \frac{kA_2}{\nu} \left( \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 L^2} + \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 X} \right) \]
\[
+ |k| \left( \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 L^2} + |k| \left| \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 X} \right| \right) + \frac{kA_2}{\nu} \left| \frac{1}{2} \| e^{e^{\frac{kA_2}{\nu}} \frac{1}{2} \frac{1}{\tau} w_k} \|_{L^2 X} \right|.
\]

It turns out that $E_k$ is the correct energy, which enables us to prove the desired result with improvements. If we only adapt the natural $L^2$ energy (1.19) as in [19], we can prove a stability result allowing perturbation of initial data is less then $|\frac{kA_2}{\nu}|^{\frac{1}{2}}$, while with $X$ space and

\footnote{Recall $\omega = 2A_1e^\gamma$ in (1.14). In scaling variables, we expect $e^{\gamma} \omega_0(\tau) - 2A_1$ term obeys decaying estimate.}
energy, we allow the perturbation to be less than \( \frac{A_2}{r^2} \). By establishing sharp resolvent estimates and employing Gearhart-Prüss type theorem in \cite{39}, we also avoid the loss of \( \log |\alpha| \) in Proposition \ref{1.4}.

**Remark 1.7.** To prove the nonlinear result Theorem \ref{1.5}, we also obtain a sharp pseudospectral bound \( (\Delta \nu)^{\frac{1}{2}} \) for the linearized operator around the Taylor-Couette flow in Section \ref{2}. This sharp bound is consistent with the estimates for the Lamb-Oseen vortices operator in \cite{29} by Li-Wei-Zhang. The obtained pseudospectral bound then implies a sharp enhanced dissipation decaying rate \( e^{-\frac{(\Delta \nu)^{\frac{1}{2}}}{3}} \) for the vorticity equation. Not only we give resolvent estimates from \( L^2 \) space to \( L^2 \) space, we also establish the counterpart from \( H^{-1} \) space and extend the optimal resolvent estimate to the weighted \( L^2 \) space. These estimates together infer desired space-time bounds for nonlinear terms in Section \ref{2}.

Translated back to the original equations \((1.2)\), initial data at \( \tau = 0 \) is corresponding to data at \( t = 1 \). The above main theorem implies the below improved transition threshold result:

**Theorem 1.8.** *(Transition Threshold)* For any \( |A_2| \geq \nu \), there exist constants \( C_1, C_2 > 0 \) independent of \( A_1, A_2, \nu \) such that if the initial data at \( t = 1 \) satisfies

\[
\|\omega(1) - 2A_1\|_{L^1} \leq C_1|A_2|^{\frac{1}{4}} \nu^{\frac{3}{8}},
\]

then the solution \( \omega(t) \) to the nonlinear vorticity formulation \((1.2)\) is global in time and obeys the following stability estimate

\[
\|\omega(t) - 2A_1\|_{L^1} \leq C_2|A_2|^{\frac{1}{4}} \nu^{\frac{3}{8}}.
\]

**Remark 1.9.** In this theorem, \( |A_2|^{\frac{1}{4}} \nu^{\frac{3}{8}} \) of order \( \nu^{\frac{3}{2}} \) is the transition threshold. Recall on page 6 of \cite{19} Gallay proved (in \( L^1 \) norm) a transition threshold of order \( \nu^{\frac{3}{2}} \) for the flow near the Lamb-Oseen vortices. By scaling consideration, we expect consistent transition thresholds for the Lamb-Oseen vortices and for the Taylor-Couette flow. Our improvement comes from the desired estimates in weighted norms obtained in Theorem \ref{1.5}. In particular, we overcome the difficulty for the cases when \( r_1 \to 0 \) and \( r_2 \to +\infty \).

### 1.4. Derivation of equations.

For notational simplicity, in below we write \( A_1 \) and \( \frac{A_2}{r^2} \) as \( A \) and \( B \) respectively. And we also use polar coordinates

\[
\xi_1 = r \cos \theta, \quad \xi_2 = r \sin \theta \quad \text{with} \quad r \in [0, \infty) \quad \text{and} \quad \theta \in \mathbb{T}.
\]

Define \( \tilde{w} = w - W, \tilde{u} = u - V \) with \( W \) and \( V \) given in \((1.17)\). Via \((1.15)\) and \( \tilde{u} \cdot \nabla_\xi W = 0 \), we have

\[
(1.22) \quad \partial_r \tilde{w} - (\Delta_\xi + \frac{1}{2} \xi \cdot \nabla_\xi + 1)\tilde{w} + (\tilde{u} + V) \cdot \nabla_\xi \tilde{w} = 0.
\]

Since \( \partial_\xi_1 = \cos \theta \partial_r - \frac{1}{r} \sin \theta \partial_\theta \) and \( \partial_\xi_2 = \sin \theta \partial_r + \frac{1}{r} \cos \theta \partial_\theta \), we deduce

\[
-\xi_2 \partial_\xi_1 + \xi_1 \partial_\xi_2 = \partial_\theta, \quad \xi_1 \partial_\xi_1 + \xi_2 \partial_\xi_2 = r \partial_r, \quad \Delta_\xi = \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2.
\]

Thus, equation \((1.22)\) becomes

\[
(1.23) \quad \partial_r \tilde{w} - \left[ (\partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2) + \frac{1}{2} r \partial_r + 1 \right] \tilde{w} + (Ae^\tau + \frac{B}{r^2}) \partial_\theta \tilde{w} + \tilde{u} \cdot \nabla_\xi \tilde{w} = 0.
\]
Recall we have \( \hat{u} = \left( \frac{-\partial_{\xi_2} \varphi}{\partial_{\xi_1} \varphi} \right) \) with \( \varphi \) being the stream function satisfying
\[
\Delta \xi \varphi = \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right) \varphi = \hat{w}.
\]

Then it follows
\[
\hat{u} \cdot \nabla \xi \hat{w} = - \partial_{\xi_2} \varphi \partial_{\xi_1} \hat{w} + \partial_{\xi_1} \varphi \partial_{\xi_2} \hat{w}
= - \left( \sin \theta \partial_r \varphi + \frac{1}{r} \cos \theta \partial_\theta \varphi \right) \left( \cos \theta \partial_r \hat{w} - \frac{1}{r} \sin \theta \partial_\theta \hat{w} \right)
+ \left( \cos \theta \partial_r \varphi - \frac{1}{r} \sin \theta \partial_\theta \varphi \right) \left( \sin \theta \partial_r \hat{w} + \frac{1}{r} \cos \theta \partial_\theta \hat{w} \right)
= \frac{1}{r} (\partial_r \varphi \partial_\theta \hat{w} - \partial_\theta \varphi \partial_r \hat{w}).
\]

Together with
\[
\partial_r \varphi \partial_\theta \hat{w} - \partial_\theta \varphi \partial_r \hat{w} = \partial_r (\varphi \partial_\theta \hat{w}) - \partial_\theta (\varphi \partial_r \hat{w}),
\]
we derive the below (nonlinear) perturbation equation (1.23) for \( \hat{w} \)
\[
(1.24) \quad \partial_r \hat{w} - \left[ \left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \right) + \frac{1}{2} \theta \partial_\theta + 1 \right] \hat{w}_k + (Ae^\tau + \frac{B}{r^2}) \partial_\theta \hat{w}
\]
\[
+ \frac{1}{r} \left[ \left( \partial_r (\varphi \partial_\theta \hat{w}) - \partial_\theta (\varphi \partial_r \hat{w}) \right) \right] = 0.
\]

Take the Fourier transform in \( \theta \) direction. Denote the Fourier coefficient of \( \hat{w} \) and \( \varphi \) to be \( \hat{w}_k \) and \( \hat{\varphi}_k \), respectively. Then equation (1.24) becomes
\[
(1.25) \quad \partial_r \hat{w}_k - \left[ \left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \right) \right] \hat{w}_k + \frac{1}{r} \sum_{l \in \mathbb{Z}} il \partial_r (\hat{w}_l \hat{\varphi}_{k-l}) - ik \sum_{l \in \mathbb{Z}} \hat{\varphi}_{k-l} \partial_r \hat{w}_l = 0
\]
with \( \hat{\varphi}_k \) satisfying \( (\partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2}) \hat{\varphi}_k = \hat{w}_k \).

Denote \( \hat{w}_k := e^{ikAe^\tau} \hat{w}_k \) and \( \hat{\varphi}_k := e^{ikAe^\tau} \hat{\varphi}_k \). We obtain an equivalent form for (1.25)
\[
\partial_r \hat{w}_k - \left[ \left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2} \right) \right] \hat{w}_k + \frac{ikB}{r^2} \hat{w}_k
\]
\[
+ \frac{1}{r} \sum_{l \in \mathbb{Z}} il \partial_r (\hat{w}_l \hat{\varphi}_{k-l}) - ik \sum_{l \in \mathbb{Z}} \hat{\varphi}_{k-l} \partial_r \hat{w}_l = 0
\]
with \( \hat{\varphi}_k \) satisfying \( (\partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2}) \hat{\varphi}_k = \hat{w}_k \).

Let us further define \( w_k := f^{-1} \hat{w}_k \) with \( f = \frac{e^{-2}}{r^2} \). This \( f^{-1} \) weight is designed to make the first derivative \( \frac{1}{r} \partial_r + \frac{1}{2} \theta \partial_\theta \) vanish. With \( w_k \) we deduce
\[
(1.26) \quad \partial_r w_k - \left[ \partial_r^2 - \left( \frac{k^2}{r^2} - \frac{1}{4} \right) + \frac{r^2}{16} - \frac{1}{2} \right] w_k + \frac{ikB}{r^2} w_k
\]
\[
+ \frac{f^{-1}}{r} \sum_{l \in \mathbb{Z}} il \partial_r (w_l \hat{\varphi}_{k-l}) - ik \sum_{l \in \mathbb{Z}} \hat{\varphi}_{k-l} \partial_r \hat{w}_l = 0.
\]
Next we give the explicit forms of nonlinear terms. We first write

\[
\frac{f^{-1}}{r} \left[ \sum_{l \in \mathbb{Z}} i l \partial_r (\tilde{w}_l \tilde{\varphi}_{k-l}) - ik \sum_{l \in \mathbb{Z}} \tilde{\varphi}_{k-l} \partial_r \tilde{w}_l \right]
\]

\[
= \frac{f^{-1}}{r} \left[ \sum_{l \in \mathbb{Z}} i l \partial_r (\tilde{w}_l \tilde{\varphi}_{k-l}) - ik \sum_{l \in \mathbb{Z}} \partial_r (\tilde{\varphi}_{k-l} \tilde{w}_l) + ik \sum_{l \in \mathbb{Z}} \tilde{w}_l \partial_r \tilde{\varphi}_{k-l} \right]
\]

\[
= \frac{f^{-1}}{r} \left[ ik \sum_{l \in \mathbb{Z}} \tilde{w}_l \partial_r \tilde{\varphi}_{k-l} - \sum_{l \in \mathbb{Z}} i(k-l) \partial_r (\tilde{w}_l \tilde{\varphi}_{k-l}) \right]
\]

observing and utilizing below basic equalites

\[
\frac{f^{-1}}{r} \partial_r (\tilde{w}_l \tilde{\varphi}_{k-l}) = \partial_r \left( \frac{f^{-1}}{r} \tilde{w}_l \tilde{\varphi}_{k-l} \right) - \partial_r \left( \frac{f^{-1}}{r} \tilde{w}_l \tilde{\varphi}_{k-l} \right) \tilde{w}_l \tilde{\varphi}_{k-l} \tilde{w}_l \tilde{\varphi}_{k-l}, \quad \partial_r . f = - \left( \frac{1}{2} \frac{1}{r^2} + \frac{1}{4} \frac{1}{r^2} \right) f
\]

we obtain

\[
\frac{f^{-1}}{r} \left[ ik \sum_{l \in \mathbb{Z}} \tilde{w}_l \partial_r \tilde{\varphi}_{k-l} - \sum_{l \in \mathbb{Z}} i(k-l) \partial_r (\tilde{w}_l \tilde{\varphi}_{k-l}) \right]
\]

\[
= \frac{f^{-1}}{r} \left[ ik \sum_{l \in \mathbb{Z}} \tilde{w}_l \partial_r \tilde{\varphi}_{k-l} - \sum_{l \in \mathbb{Z}} i(k-l) \left( \partial_r \left( \frac{w_l \tilde{\varphi}_{k-l}}{r} \right) \right) \tilde{w}_l \tilde{\varphi}_{k-l} \right]
\]

\[
= \left[ ik \sum_{l \in \mathbb{Z}} \tilde{w}_l \partial_r \tilde{\varphi}_{k-l} - \sum_{l \in \mathbb{Z}} i(k-l) \left( \frac{1}{4} - \frac{1}{2} \frac{1}{r^2} \right) w_l \tilde{\varphi}_{k-l} \right]
\]

we denote

\[
f_1 := ik \sum_{l \in \mathbb{Z}} \tilde{w}_l \partial_r \tilde{\varphi}_{k-l} - \sum_{l \in \mathbb{Z}} i(k-l) \left( \frac{1}{4} - \frac{1}{2} \frac{1}{r^2} \right) w_l \tilde{\varphi}_{k-l},
\]

\[
f_2 := \sum_{l \in \mathbb{Z}} i(k-l) \frac{w_l \tilde{\varphi}_{k-l}}{r}.
\]

Then the nonlinear perturbation equation (1.26) can be written as

\[
(\partial_r w_k + L_k w_k + f_1 - \partial_r f_2 = 0,
\]

\[
\left\{ \begin{array}{l}
w_k(0) = w_k|_{r=0}, \quad w_k|_{\partial \Gamma} = 0
\end{array} \right.
\]

with \( \partial \Gamma \) being the boundaries of the viscous fluid. Here

\[
L_k = -[\partial_r^2 - (\frac{k^2 - \frac{1}{2}}{r^2} + \frac{r^2}{16} - \frac{1}{2})] + i \frac{\lambda B}{r^2}.
\]

1.5. Strategy and structure of the paper. In Section 2 we establish the resolvent estimate for the linearized equation

\[-[\partial_r^2 - (\frac{k^2 - \frac{1}{2}}{r^2} + \frac{r^2}{16} - \frac{1}{2})]w + i \beta_k (\frac{1}{r^2} - \lambda)w = F; \quad w|_{r=0} = w|_{r=\infty} = 0
\]

in both \( L^2 \) space and weighted \( L^2 \) space \( X \). Here \( X \) is defined in (1.18). The resolvent estimate is inspired by [29]. A key difference to [29] is that, for here, not only we give the
resolvent estimate from $L^2$ to $L^2$, we also establish the resolvent estimate from $L^2$ to $H^{-1}$. And we further generalize and extend these resolvent estimates to constructed weighted spaces as well. Our estimates in weighted spaces enable us to sharpen previous results and get the desired nonlinear theorem. In later sections to derive a sharp decaying estimate $e^{-c|kB|^r\frac{1}{3}r}$ for the nonlinear problem, we also need to shift the linear operator to the left by $c_2|kB|^\frac{1}{3}$. This means that we establish bounds for $\|F - c_2|\beta_k|^\frac{1}{3}w\|_{L^2}$, $\|F - c_2|\beta_k|^\frac{1}{3}w\|_{H^{-1}}$, $\|F - c_2|\beta_k|^\frac{1}{3}w\|_{L^2}$ and $\|F - c_2|\beta_k|^\frac{1}{3}w\|_{H^{-1}}$. Besides dodging a potential log $|B|$ loss, we also avoid proving the resolvent estimate between $\|w\|_{H^{-1}}$ and $\|F\|_{L^2}$.

In Section 3 we derive space-time estimates for the linearized Navier-Stokes equations. In the aim of applying these estimates to the nonlinear problem, we first study the following inhomogeneous equation (1.27) for $w_k$:

$$\partial_tw_k + Lkw_k + f_1 - \partial_tf_2 = 0, \quad w_k(0) = w_k|_{t=0}. $$

We decompose $w_k$ into two parts with $w_k = w^l_k + w^n_k$. Here $w^l_k$ satisfies the below homogeneous linear equation (3.2) with initial data $w^l_k(0)$

$$\partial_tw^l_k + Lkw^l_k = 0, \quad w^l_k(0) = w_k(0), $$

and $w^n_k$ verifies the inhomogeneous linear equation (3.7) with zero initial data

$$\partial_tw^n_k + Lkw^n_k + f_1 - \partial_tf_2 = 0, \quad w^n_k(0) = 0.$$ 

We obtain the sharp bound for $w^l_k$ by using Gearhart-Prüss type theorem in [39] to avoid log $|B|$, since the linearized operator $L_k$ is accretive in both $L^2$ space and weighted $L^2$ space. Through Fourier transform and applying proofs in Proposition 2.7 and Proposition 2.10 we derive a sharp semigroup bound for $w^n_k$ as well. Putting together, our space-time estimates take the below forms:

$$M(L^2) := \|e^{c|kB|^\frac{1}{3}r}\|_{L^\infty L^2} + |kB|^\frac{1}{3} \|e^{c|kB|^\frac{1}{3}r}w_k\|_{L^2L^2}$$

$$\quad + \|e^{c|kB|^\frac{1}{3}r}\|_{L^2L^2} + \|e^{c|kB|^\frac{1}{3}r}(k + r)w_k\|_{L^2L^2}$$

$$\leq C\left(|kB|^\frac{1}{3}\|e^{c|kB|^\frac{1}{3}r}f_1\|_{L^2L^2} + \|e^{c|kB|^\frac{1}{3}r}f_2\|_{L^2L^2} + \|w_k(0)\|_{L^2}\right)$$

and

$$M(X) := \|e^{c|kB|^\frac{1}{3}r}w_k\|_{L^\infty X} + |kB|^\frac{1}{3} \|e^{c|kB|^\frac{1}{3}r}w_k\|_{L^2X}$$

$$\quad + \|e^{c|kB|^\frac{1}{3}r}\|_{L^2X} + \|e^{c|kB|^\frac{1}{3}r}w_k\|_{L^2X} + \|e^{c|kB|^\frac{1}{3}r}w_k\|_{L^2X}$$

$$\leq C\left(|kB|^\frac{1}{3}\|e^{c|kB|^\frac{1}{3}r}f_1\|_{L^2X} + \|e^{c|kB|^\frac{1}{3}r}f_2\|_{L^2X} + \|w_k(0)\|_{X}\right).$$

The terms singled out in $M(L^2)$ and $M(X)$ will help us to design the correct “energy norm” $E_k$ used in Section 3.

In Section 4 we control the nonlinear terms in a sharpen way and employ a designed weighted energy space. Recall the expression of $M(L^2)$ and $M(X)$. In $M(X)$, the coefficient of $\|e^{c|kB|^\frac{1}{3}r}w_k\|_{L^2X}$ is $|kB|^\frac{1}{3}$. This suggests that a desired energy norm should be of order $M(L^2) + |kB|^\frac{1}{3}M(X)$. 


And it motivates us to couple $L^2$ and $X$ spaces together to construct the combined energy $E_k (k \in \mathbb{Z}/\{0\})$ as below

$$E_k := \| e^{[kB] \frac{1}{3} \tau} w_k \|_{L^\infty L^2} + \| k |B|^\frac{1}{3} e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r} \|_{L^\infty L^2}$$

$$+ \| k |B|^\frac{1}{3} \| e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r^2} \|_{L^2 L^2} + \| k |B|^\frac{1}{3} \| e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r^3} \|_{L^2 L^\infty} \| e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r} \|_{L^2 L^2}.$$ 

With these $E_k$, we prove our main conclusion-Theorem 4.1.

Note that the last term $|kB|^\frac{1}{3} \| e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r} \|_{L^2 L^2}$ in $E_k$ plays a crucial role when estimating nonlinear terms. It is because in $E_k$ we utilize $|kB|^\frac{1}{3} \| e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r} \|_{L^2 L^2}$ rather than $|kB|^\frac{1}{3} \| e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r} \|_{L^2 L^2}$ to control nonlinear terms, that we allow larger $|B|^\frac{1}{3}$ (rather than $|B|^\frac{1}{4}$) perturbation in initial data. Furthermore, our argument is accompanied by including a $|kB|^\frac{1}{3} \| e^{[kB] \frac{1}{3} \tau} \frac{w_k}{r} \|_{L^2 L^\infty}$ term in $E_k$, which helps to control weights in $k$.

1.6. Other related results. Here we list some other related works about enhanced dissipation and nonlinear asymptotic stability in 2D.

For 2D Navier-Stokes, enhanced dissipation effect was proved by Beck-Wayne in [11] for a passive scalar advected by the Kolmogorov flow. The case of $u = (u(y), 0)$ with a finite number of critical points was treated by Bedrossian-Zelati in [7]. The case of the Kolmogorov flow was analyzed by Lin-Xu in [30], Ibrahim-Maekawa-Masmoudi in [24] and Wei-Zhang-Zhao in [43], and they obtained a time-scale of order $O(\nu^{-\frac{1}{2}})$. In [45], Zelati-Delgadino-Elgindi explored its connection to mixing effect. In [10], for the linearized Navier-Stokes system around monotone shear flows with non-slip condition, Chen-Wei-Zhang further developed estimates allowing the couple of the enhanced dissipation and the inviscid-damping effects.

For the 2D Boussinesq equations, in [15] Deng-Wu-Zhang proved enhanced dissipation via studying a non-self-adjoint operator. We also want to refer to a result achieving to establish nonlinear inviscid damping for 2D Euler equations by Ionescu-Jia [25]. There they proved nonlinear asymptotic stability of monotonic shear flows in the channel.

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2. Resolvent estimates in $L^2$ space and in weighted $L^2$ space

In order to establish decays of the linearized equation (1.27), the first step is to study the resolvent equation under the following boundary condition in both $L^2$ space and weighted $L^2$ space. More precisely, we consider

$$- [\partial_r^2 - (\frac{k^2}{r^2} - \frac{1}{2}) + \frac{r^2}{16} - \frac{1}{2}] w + i \beta_k (\frac{1}{r^2} - \lambda) w = F \quad \text{with} \quad w|_{r=0,\infty} = 0,$$

where $\lambda \in \mathbb{R}$. The domain is defined as

$$D_k = \{ w \in H^2_{loc}(\mathbb{R}+, dr) \cap L^2(\mathbb{R}+, dr) : - [\partial_r^2 - (\frac{k^2}{r^2} - \frac{1}{2}) + \frac{r^2}{16} - \frac{1}{2}] w + i \beta_k \frac{w_k}{r^2} \in L^2(\mathbb{R}+, dr) \}.$$ 

Note that for any $|k| \geq 1$, it holds

$$D_k = \{ w \in L^2(\mathbb{R}+, dr) : \partial_r^2 w, \frac{w}{r^2}, r^2 w \in L^2(\mathbb{R}+, dr) \}.$$
We start with deriving the coercive estimates for $\Re\langle F, w \rangle$ and $\Re\langle F, \frac{w}{r^2} \rangle$, which are the real part of $\langle F, w \rangle$ and $\langle F, \frac{w}{r^2} \rangle$, respectively.

2.1. Coercive estimates of the real part.

**Lemma 2.1.** For any $|k| \geq 1$ and $w \in D_k$, it holds

\[ \Re\langle F, w \rangle \gtrsim \|w\|^2_{L^2} + \langle \left( \frac{k^2}{r^2} + r^2 \right) w, w \rangle. \]  

**Proof.** We prove this lemma by considering the following two scenarios:

- **When $|k| = 1$, one can check**

  \[-[\partial_r^2 - \left( \frac{k^2}{r^2} - \frac{1}{4} + \frac{r^2}{16} - \frac{1}{2} \right)]w = -h^{-1}\partial_r\left[ h^2 \partial_r (h^{-1} w) \right] + \frac{1}{2}w,\]

  where $h = r^\frac{3}{2} e^{-\frac{r^2}{8}}$. Via integration by parts we obtain

  \[ \Re\langle F, w \rangle = \Re\left( -[\partial_r^2 - \left( \frac{k^2}{r^2} - \frac{1}{4} + \frac{r^2}{16} - \frac{1}{2} \right)]w, w \right) = \|h\partial_r(h^{-1} w)\|^2_{L^2} + \frac{1}{2}\|w\|^2_{L^2}, \]

  which implies

  \[ \Re\langle F, w \rangle \gtrsim \frac{1}{2}\|w\|^2_{L^2}. \]  

- **When $|k| \geq 2$, observing**

  \[ \frac{k^2}{r^2} - \frac{1}{4} + \frac{r^2}{16} - \frac{1}{2} \geq \frac{2k^2}{3r^2} + \frac{r^2}{16} - \frac{1}{2} \gtrsim \frac{k^2}{r^2} + r^2, \]

  and via employing integration by parts again, we conclude

  \[ \Re\langle F, w \rangle = \|w\|^2_{L^2} - \langle \left( \frac{k^2}{r^2} + r^2 \right) w, w \rangle \gtrsim \|w\|^2_{L^2} + \langle \left( \frac{k^2}{r^2} + r^2 \right) w, w \rangle. \]

This completes the proof of Lemma 2.1. 

With Lemma 2.1, we then can establish
Lemma 2.2. For any \(|k| \geq 1\) and \(w \in D_k\), it holds

\[
|k|^{\frac{3}{2}} \|w\|_{H^1} + |k| \|w\|_{L^2} \lesssim \|F\|_{L^2}, \tag{2.5}
\]

\[
\|w\|_{H^1} + |k|^{\frac{5}{2}} \|w\|_{L^2} \lesssim \|F\|_{H^{-1}}. \tag{2.6}
\]

Proof. By Lemma 2.1, one obtains

\[
\|w'\|_{L^2}^2 + |k| \|w\|_{L^2}^2 \lesssim \Re \langle F, w \rangle,
\]

which implies

\[
|k| \|w\|_{L^2} \lesssim \|F\|_{L^2}.
\]

Therefore, we deduce

\[
\|w\|_{H^1}^2 \lesssim \|w'\|_{L^2}^2 + |k| \|w\|_{L^2}^2 \lesssim \Re \langle F, w \rangle \lesssim |k|^{-1} \|F\|_{L^2}^2.
\]

This completes the proof of (2.5).

To prove (2.6), we apply Lemma 2.1 again

\[
\|w\|_{H^1}^2 \lesssim \|w'\|_{L^2}^2 + |k| \|w\|_{L^2}^2 \lesssim \Re \langle F, w \rangle.
\]

This renders

\[
\|w\|_{H^1} \lesssim \|F\|_{H^{-1}}.
\]

Thus it follows

\[
|k| \|w\|_{L^2}^2 \lesssim \Re \langle F, w \rangle \lesssim \|F\|_{H^{-1}}^2.
\]

This completes the proof of (2.6). \qed

Lemma 2.3. For any \(|k| \geq 1\) and \(w \in D_k\), it holds

\[
\Re \langle F, \frac{w}{r^2} \rangle = \|r^{-1} h \partial_r (h^{-1} w)\|_{L^2}^2 + (k^2 - 1) \|\frac{w}{r^2}\|_{L^2}^2, \tag{2.7}
\]

where \(h = r^\frac{3}{2} e^{-\frac{x^2}{4}}\). Moreover, we have the following estimate

\[
\Re \langle F, \frac{w}{r^2} \rangle \gtrsim \|\frac{w'}{r}\|_{L^2}^2 + k^2 \|\frac{w}{r^2}\|_{L^2}^2 + |k| \|\frac{w}{r}\|_{L^2}^2 + \|w\|_{L^2}^2. \tag{2.8}
\]

Proof. We start with proving (2.7). Denote \(h = r^\frac{3}{2} e^{-\frac{x^2}{4}}\). One can check

\[
-\left[\partial_r^2 - \left(\frac{k^2 - \frac{3}{4} r^2}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right)\right]w = -r^2 h^{-1} \partial_r [r^{-2} h^2 \partial_r (h^{-1} w)] - \frac{2}{r} w' + \frac{k^2 + 2}{r^2} w.
\]

The identities below follow from integration by parts

\[
\Re \langle F, \frac{w}{r^2} \rangle = \Re \langle -\left[\partial_r^2 - \left(\frac{k^2 - \frac{3}{4} r^2}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right)\right]w, \frac{w}{r^2} \rangle
\]

\[
= \|r^{-1} h \partial_r (h^{-1} w)\|_{L^2}^2 - \langle w', \frac{2w}{r^2} \rangle + \left(\frac{k^2 + 2}{r^2} w, \frac{w}{r^2} \right)
\]

\[
= \|r^{-1} h \partial_r (h^{-1} w)\|_{L^2}^2 + (k^2 - 1) \|\frac{w}{r^2}\|_{L^2}^2,
\]

where in the last equality we use

\[
\Re \langle w', \frac{2w}{r^3} \rangle = -\int_0^\infty \frac{1}{r^3} \, d|w|^2 = -\int_0^\infty \frac{3|w|^2}{r^4} \, dr = -3 \|\frac{w}{r^2}\|_{L^2}^2.
\]

\[
\Re \langle F, \frac{w}{r^2} \rangle = \|r^{-1} h \partial_r (h^{-1} w)\|_{L^2}^2 + (k^2 - 1) \|\frac{w}{r^2}\|_{L^2}^2,
\]

where in the last equality we use
Now we continue to prove (2.8). We first observe the following equality through integration by parts
\[ \langle -[\partial_r^2 - \left( \frac{k^2 - \frac{1}{r^2} + \frac{r^2}{16} - \frac{1}{2} \right)]w, w \rangle = \left\| \frac{w'}{r} \right\|^2_{L^2} - \langle w', \frac{2w}{r^2} \rangle + \langle \left( \frac{k^2 - \frac{1}{r^2} + \frac{r^2}{16} - \frac{1}{2} \right)w, w \rangle. \]
Together with (2.9) this gives

(2.10) \[ \Re(F, \frac{w}{r^2}) = \left\| \frac{w'}{r} \right\|^2_{L^2} + \langle \left( \frac{k^2 - \frac{1}{r^2} + \frac{r^2}{16} - \frac{1}{2} \right)w, w \rangle. \]

Then we treat \(|k| \geq 2\) and \(|k| = 1\) separately.

- **When \(|k| \geq 2\),** applying (2.7), we deduce

(2.11) \[ \Re(F, \frac{w}{r^2}) \geq (k^2 - 1)\left\| \frac{w}{r^2} \right\|^2_{L^2} \geq k^2\left\| \frac{w}{r^2} \right\|^2_{L^2}. \]

Combining with (2.11), we hence obtain

\[ \Re(F, \frac{w}{r^2}) \geq \left\| \frac{w'}{r} \right\|^2_{L^2} + \langle \left( \frac{k^2}{r^2} + \frac{r^2}{16} + |k| \right)w, w \rangle. \]

- **When \(|k| = 1\),** we write

\[ -w'' = -g^{-1}\partial_r(r^2g^2\partial_r(r^{-2}g^{-1}w)) - \frac{2}{r}w' - g^{-1}[g'' + \left( \frac{2}{r}g' \right)]w \]

for any function \(g\). Let \(g\) be a real function satisfying

(2.12) \[ \frac{g''}{g} + \frac{2}{r}g' = \frac{2}{r^2}. \]

The existence and explicit form of \(g\) are guaranteed by Lemma A.1 in the Appendix. With (2.9) and (2.12), we deduce

\[ \Re(F, \frac{w}{r^2}) = \left\| rg\partial_r(r^{-2}g^{-1}w) \right\|^2_{L^2} - 3\left\| \frac{w}{r^2} \right\|^2_{L^2} - \langle g^{-1}[g'' + \left( \frac{2}{r}g' \right)]w, \frac{w}{r^2} \rangle \]

\[ + \langle \left( \frac{k^2 - \frac{1}{r^2} + \frac{r^2}{16} - \frac{1}{2} \right)w, w \rangle. \]

\[ = \left\| rg\partial_r(r^{-2}g^{-1}w) \right\|^2_{L^2} - \langle g^{-1}(g'' + \frac{2}{r}g')w, \frac{w}{r^2} \rangle + \langle \left( \frac{k^2 - \frac{5}{r^2} + \frac{r^2}{16} - \frac{1}{2} \right)w, w \rangle \]

\[ = \left\| rg\partial_r(r^{-2}g^{-1}w) \right\|^2_{L^2} + \langle \left( \frac{7}{4r^2} + \frac{r^2}{16} - \frac{1}{2} \right)w, \frac{w}{r^2} \rangle \geq \langle \left( \frac{1}{r^2} + r^2 \right)w, \frac{w}{r^2} \rangle. \]

Together with (2.10), we prove

\[ \Re(F, \frac{w}{r^2}) \geq \left\| \frac{w'}{r} \right\|^2_{L^2} + \langle \left( \frac{1}{r^2} + r^2 \right)w, \frac{w}{r^2} \rangle. \]

This completes the proof of Lemma 2.3.\hfill \Box

In a similar fashion as proving Lemma 2.3, we also obtain

**Lemma 2.4.** For any \(|k| \geq 1\) and \(w \in D_k\), it holds

(2.13) \[ \left\| \frac{w}{r} \right\|_{H^1} + |k| \left\| \frac{w}{r} \right\|_{L^2} \lesssim \left\| \frac{F}{r} \right\|_{L^2}, \]

(2.14) \[ \left\| \frac{w}{r} \right\|_{H^1} + |k| \left\| \frac{w}{r} \right\|_{L^2} \lesssim \left\| \frac{F}{r} \right\|_{H^{-1}}. \]
Proof. Applying Lemma 2.3, one has
\[ \|w'\|^2_{L^2} + |k|\|w\|^2_{L^2} \lesssim |w'(r)\|, \]
which yields
\[ |k|\|w\|^2_{L^2} \lesssim |F(r)|. \]
Therefore, we deduce
\[ \|w\|_{L^2} \lesssim \|F\|_{L^2}. \]
This completes the proof of (2.13).

Using Lemma 2.3 again, we obtain
\[ \|w\|_{H^1} \lesssim \|w'\|_{L^2} + |k|\|w\|_{L^2} \lesssim |w'(r)\|, \]
which gives
\[ \|w\|_{H^1} \lesssim \|F\|_{H^{-1}}. \]
Therefore, we can conclude
\[ \|w\|_{L^2} \lesssim |w'(r)\| \lesssim |F(r)|_{H^{-1}}. \]
This completes the proof of (2.14).

2.2. Resolvent estimates in $L^2$ space. We first derive the resolvent estimates from $L^2$ to $L^2$.

Proposition 2.5. For any $|k| \geq 1$, $\lambda \in \mathbb{R}$ and $w \in D_k$, there exists a constant $C > 0$ independent of $k, \beta_k, \lambda$, such that the following estimate holds
\[ |\beta_k|^\frac{1}{2}|w'|_{L^2} + |\beta_k|^\frac{1}{2}|w|_{L^2} \leq C\|F\|_{L^2}. \]

Proof. Let us first prove $|\beta_k|^\frac{1}{2}|w|_{L^2} \leq C\|F\|_{L^2}$. If $\lambda \leq 0$, with Lemma 2.1 we have
\[ \Re\langle F, w \rangle \gtrsim ((\frac{k^2}{r^2} + r^2)w, w). \]
Together with
\[ |\Im\langle F, w \rangle| \geq |\beta_k|\left(\frac{1}{r^2}w, w\right), \]
this gives
\[ |\langle F, w \rangle| \gtrsim |(r^2 + \frac{|\beta_k|}{r^2})w, w| \geq |\beta_k|^\frac{1}{2}|w|^2_{L^2}. \]
Recall that $\beta_k = kB$ and $|B| \geq 1$. Thus $|\beta_k| = |kB| \geq 1$ for $|k| \geq 1$, and we have
\[ |\langle F, w \rangle| \gtrsim |\beta_k|^\frac{1}{2}|w|^2_{L^2} \geq |\beta_k|^\frac{1}{2}|w|^2_{L^2}. \]
If $\lambda > 0$ and $|\beta_k| \leq k^2$, utilizing Lemma 2.1 again, one obtains
\[ |\langle F, w \rangle| \gtrsim \Re\langle F, w \rangle \gtrsim ((\frac{k^2}{r^2} + r^2)w, w) \geq |k||w|^2_{L^2} \geq |\beta_k|^\frac{1}{2}|w|^2_{L^2} \geq |\beta_k|^\frac{1}{2}|w|^2_{L^2}. \]
It remains to deal with the scenario $\lambda > 0$ and $|\beta_k| \geq k^2$. Denote $r_0 := \frac{1}{\sqrt{|\lambda|}}$. The discussion can be divided into the below three cases:
(1) **Case of** \(|\beta_k| \leq r_0^2\). With Lemma 2.1 we have
\[ \Re(F, w) \geq ((k^2 \frac{1}{r^2} + r^2)w, w). \]

Combining with
\[ |\Im(F, w)| \geq |\beta_k|(\frac{1}{r^2} - \frac{1}{r_0^2})w, w, \]
we deduce
\[ |\langle F, w \rangle| \geq (|\frac{k^2}{r^2} + r^2| + |\beta_k|(\frac{1}{r^2} - \frac{1}{r_0^2}))|w, w| \geq (r^2 + |\beta_k|(\frac{1}{r^2} - \frac{1}{r_0^2}))|w, w|. \]

Since
\[ r^2 + |\beta_k|(\frac{1}{r^2} - \frac{1}{r_0^2}) \geq 2|\beta_k|^\frac{3}{2} - \frac{|\beta_k|}{r_0^2} \geq |\beta_k|^\frac{1}{2}, \]
we then obtain
\[ |\langle F, w \rangle| \geq |\beta_k|^\frac{3}{2}||w||_L^2 \geq |\beta_k|^\frac{1}{2}||w||_L^2. \]

(2) **Case of** \(r_0^2 < |\beta_k| \leq r_0^6\). The condition \(|\beta_k| \geq 1\) implies that \(r_0 \geq 1\). Using Lemma 2.1 we derive the estimate for \(||w||_L^2(\frac{4}{7}, \infty)\)
\[ \Re(F, w) \geq \langle (\frac{k^2}{r^2} + r^2)w, w \rangle \geq ||rw||_L^2 \geq \frac{r_0^2}{4}||w||_L^2(\frac{4}{7}, \infty). \]

The estimate for \(||w||_L^2(0, \frac{4}{7})\) is more subtle. First, we choose \(r_- \in (\frac{4}{7} - \frac{1}{2r_0}, \frac{4}{7})\) such that the following inequality holds
\[ |w'(r_-)|^2 \leq 2r_0 ||w'||_L^2. \]

Notice the imaginary part of \(\langle F, w\chi(0, r_-) \rangle\) obeys
\[ \Im\langle F, w\chi(0, r_-) \rangle = \Im\langle -\partial_r^2 w - (k^2 \frac{1}{r^2} + r^2 - \frac{1}{2})w + i\beta_k(\frac{1}{r^2} - \lambda)w, w\chi(0, r_-) \rangle \]
\[ = \Im\langle -\partial_r^2 w + i\beta_k(\frac{1}{r^2} - \lambda)w, w\chi(0, r_-) \rangle. \]

We hence prove
\[ |\beta_k|(\frac{1}{r^2} - \lambda)|w, w\chi(0, r_-)| \leq ||F||_L^2 ||w||_L^2 + |w'(r_-)w(r_-)|. \]

Using \(\lambda = \frac{1}{r_0^2}\) and \(r_- \in (\frac{4}{7} - \frac{1}{2r_0}, \frac{4}{7})\), we obtain
\[ ||F||_L^2 ||w||_L^2 + |w'(r_-)w(r_-)| \geq |\beta_k|(\frac{1}{r^2} - \frac{1}{r_0^2})w, w\chi(0, r_-) \]
\[ \geq |\beta_k|(\frac{4}{7} - \frac{1}{r_0^2})w, w\chi(0, r_-) \geq \frac{|\beta_k|}{r_0^2} ||w||_L^2(0, r_-). \]

Therefore, \(||w||_L^2\) can be bounded as below
\[ ||w||_L^2 = ||w||_L^2(0, r_-) + ||w||_L^2(r_- \frac{4}{7}) + ||w||_L^2(\frac{4}{7}, \infty) \]
\[ \lesssim \frac{r_0^2}{|\beta_k|}(||F||_L^2 ||w||_L^2 + |w'(r_-)w(r_-)|) + \frac{1}{r_0} ||w||_L^2 + \frac{1}{r_0^2} ||F||_L^2 ||w||_L^2. \]
Together with (2.18), Lemma A.2 and Lemma 2.1, we have
\[
\|w\|_{L^2}^2 \lesssim \frac{\gamma_0^2}{|\beta_k|} (\|F\|_{L^2} \|w\|_{L^2} + r_0^{\frac{1}{2}} \|w'\|_{L^2} \|w\|_{L^\infty}) + \frac{1}{r_0} \|w\|_{L^\infty}^2 + \frac{1}{r_0^2} \|F\|_{L^2} \|w\|_{L^2}
\]
\[
\lesssim \frac{\gamma_0^2}{|\beta_k|} (\|F\|_{L^2} \|w\|_{L^2} + r_0^{\frac{1}{2}} \|w'\|_{L^2} \|w\|_{L^\infty}^2) + \frac{1}{r_0} \|w'\|_{L^2} \|w\|_{L^2} + \frac{1}{r_0^2} \|F\|_{L^2} \|w\|_{L^2}
\]
\[
\lesssim \frac{\gamma_0^2}{|\beta_k|} (\|F\|_{L^2} \|w\|_{L^2} + r_0^{\frac{1}{2}} \|F\|_{L^2} \|w\|_{L^2}^2) + \frac{1}{r_0} \|F\|_{L^2} \|w\|_{L^2}^2 + \frac{1}{r_0^2} \|F\|_{L^2} \|w\|_{L^2}^2.
\]
This yields
\[
\|F\|_{L^2} \|w\|_{L^2} \gtrsim \min\{ \frac{|\beta_k|}{r_0^2}, \left( \frac{|\beta_k|}{r_0^2} \right)^{\frac{3}{2}}, \frac{r_0^2}{|\beta_k|} \} \|w\|_{L^2}^2.
\]
With \( r_0^4 \leq |\beta_k| \leq r_0^6 \), i.e. \( \frac{|\beta_k|}{r_0^4} \geq \frac{|\beta_k|}{r_0^6} \geq r_0^2 \), we conclude
\[
(2.19) \quad \|F\|_{L^2} \|w\|_{L^2} \gtrsim r_0^2 \|w\|_{L^2}^2 \geq |\beta_k| \frac{1}{r_0^4} \|w\|_{L^2}^2.
\]
(3) Case of \( |\beta_k| \geq r_0^6 \). We choose \( r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2}) \) and \( r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta) \) such that the following inequality holds
\[
(2.20) \quad |w'(r_-)|^2 + |w'(r_+)|^2 \leq \frac{4}{\delta} \|w''\|_{L^2}^2.
\]
Here \( \delta \in (0, r_0) \) is a constant which will be determined later.

The following equality is direct
\[
\Im(F, w(\chi_{(0,r_-)} - \chi_{(r_+,\infty)}))
\]
\[
= \Im(-\partial^2_r w - \left( \frac{k^2 - \frac{1}{r}}{r} + \frac{r^2}{16} - \frac{1}{2} \right) w + i\beta_k \left( \frac{1}{r^2} - \lambda \right) w, w(\chi_{(0,r_-)} - \chi_{(r_+,\infty)}))
\]
\[
= \Im(-\partial^2_r w + i\beta_k \left( \frac{1}{r^2} - \lambda \right) w, w(\chi_{(0,r_-)} - \chi_{(r_+,\infty)})).
\]
This implies
\[
|\beta_k| \left( \int_0^{r_-} \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) |w(r)|^2 dr + \int_r^{\infty} \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) |w(r)|^2 dr \right)
\]
\[
\leq \|F\|_{L^2} \|w\|_{L^2} + |w'(r_-)w(r_-)| + |w'(r_+)w(r_+)|.
\]
With \( r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2}) \) and \( r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta) \), it holds
\[
\frac{1}{r^2} - \frac{1}{r_0^2} = \frac{(r_0 - r)(r_0 + r)}{r^2 r_0^2} \geq \frac{r_0 - r}{r^2 r_0} \geq \frac{\delta}{r^2 r_0} \geq \frac{\delta}{r_0^3}
\]
for any \( r \in (0, r_-) \).

In a similar fashion, one obtains
\[
\frac{1}{r^2} - \frac{1}{r_0^2} = \frac{(r - r_0)(r + r_0)}{r^2 r_0^2} \geq \frac{r - r_0}{r^2 r_0} \geq \frac{\delta}{r^2 r_0} \geq \frac{\delta}{r_0^3}
\]
for any \( r \in (r_+, 2r_0) \),
\[
\frac{1}{r^2} - \frac{1}{r_0^2} = \frac{(r - r_0)(r + r_0)}{r^2 r_0^2} \approx \frac{1}{r_0^2} \geq \frac{\delta}{r_0^3}
\]
for any \( r \geq 2r_0 \).

Combining these above inequalities, we get
\[
\|F\|_{L^2} \|w\|_{L^2} + |w'(r_-)w(r_-)| + |w'(r_+)w(r_+)|
\]
\[ \geq |\beta_k| \left( \int_{0}^{r_{-}} \left( \frac{1}{r_{-}^{2}} - \frac{1}{r_{0}^{2}} \right) |w(r)|^2 dr + \int_{r_{+}}^{\infty} \left( \frac{1}{r_{+}^{2}} - \frac{1}{r_{0}^{2}} \right) |w(r)|^2 dr \right) \]
\[ \gtrsim \frac{|\beta_k| \delta}{r_{0}^{3}} \|w\|^2_{L^2((0, r_{-}) \cup (r_{+}, \infty))}. \]

Therefore, \( \|w\|^2_{L^2} \) obeys the estimate
\[ \|w\|^2_{L^2} = \|w\|^2_{L^2((0, r_{-}) \cup (r_{+}, \infty))} + \|w\|^2_{L^2(r_{-}, r_{+})} \]
\[ \gtrsim \frac{r_{0}^{3}}{|\beta_k| \delta} \left( \|F\|_{L^2} \|w\|_{L^2} + |w'(r_-)w(r_-)| + |w'(r_+)w(r_+)| \right) + \delta \|w\|^2_{L^\infty}. \]

Together with (2.20), Lemma A.2 and Lemma 2.1 we deduce
\[ \|w\|^2_{L^2} \lesssim \frac{r_{0}^{3}}{|\beta_k| \delta} \left( \|F\|_{L^2} \|w\|_{L^2} + \|w'\|_{L^2} \|w\|_{L^\infty} + \delta \|w\|^2_{L^\infty} \right) \]
\[ \lesssim \frac{r_{0}^{3}}{|\beta_k| \delta} \left( \|F\|_{L^2} \|w\|_{L^2} + \|w'\|_{L^2} \|w\|_{L^2} \right) + \delta \|w\|^2_{L^2} \]
\[ \lesssim \frac{r_{0}^{3}}{|\beta_k| \delta} \left( \|F\|_{L^2} \|w\|_{L^2} + \|F\|_{L^2} \|w\|_{L^2} \right) + \delta \|w\|^2_{L^2}. \]

It then follows
\[ \|F\|_{L^2} \|w\|_{L^2} \gtrsim \min \left\{ \frac{|\beta_k| \delta}{r_{0}^{3}}, \left( \frac{|\beta_k| \delta^2}{r_{0}^{3}} \right)^{\frac{1}{3}}, \delta^{-2} \right\} \|w\|^2_{L^2}. \]

Take \( \delta = \left( \frac{r_{0}^{3}}{|\beta_k|} \right)^{\frac{1}{3}} \). The condition \( |\beta_k| \geq 1 \) yields
\[ \delta = \left( \frac{r_{0}^{3}}{|\beta_k|} \right)^{\frac{1}{3}} \leq r_{0}. \]

With the basic equality \( \frac{|\beta_k| \delta}{r_{0}^{3}} = \left( \frac{|\beta_k| \delta^2}{r_{0}^{3}} \right)^{\frac{1}{3}} = \delta^{-2} \), we obtain
\[ \|F\|_{L^2} \|w\|_{L^2} \gtrsim \left( \frac{|\beta_k| \delta^2}{r_{0}^{3}} \right)^{\frac{1}{3}} \|w\|^2_{L^2} = \frac{|\beta_k| \delta^2}{r_{0}^{3}} \|w\|^2_{L^2} \geq \frac{|\beta_k| \delta^2}{|\beta_k|^2} \|w\|^2_{L^2} = |\beta_k|^4 \|w\|^2_{L^2}. \]

Combining (2.15), (2.16), (2.17), (2.19) and (2.21), we now establish the following resolvent estimate from \( L^2 \) to \( L^2 \)
\[ |\beta_k|^\frac{1}{2} \|w\|_{L^2} \lesssim \|F\|_{L^2}. \]

By Lemma 2.1 we have
\[ \Re \langle F, w \rangle \gtrsim \|w'\|^2_{L^2}, \]
which along with (2.22) implies the resolvent estimate from \( H^1 \) to \( L^2 \)
\[ \|w'\|^2_{L^2} \lesssim \|F\|_{L^2} \|w\|_{L^2} \lesssim |\beta_k|^{-\frac{3}{4}} \|F\|^2_{L^2}. \]
This completes the proof of Proposition 2.5.

Now we move to establish the resolvent estimates from \( L^2 \) to \( H^{-1} \).
Proposition 2.6. For any $|k| \geq 1$, $\lambda \in \mathbb{R}$ and $w \in D_k$, there exist constants $C, c_2 > 0$ independent of $k, \beta, \lambda$, such that the following estimate holds

$$\|w\|_{H^1} + |\beta_k|^{\frac{1}{2}} \|w\|_{L^2} \leq C \|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}}.$$  

Proof. From Lemma 2.1 we obtain

$$\Re \langle F, w \rangle = \Re \langle F - c_2|\beta_k|^{\frac{1}{2}}w, w \rangle + c_2|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}^2 \gtrsim \|w\|_{L^2}^2 + \langle (\frac{k^2}{r^2} + r^2)w, w \rangle,$$

with $c_2 > 0$ to be determined later. This gives

$$\|w\|_{H^1}^2 \lesssim \|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}} \|w\|_{H^1} + c_2|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}^2.$$  

Thus we have

$$(2.23) \quad \|w\|_{H^1} \lesssim \|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}} + \sqrt{c_2}|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}.$$  

To prove $|\beta_k|^{\frac{1}{2}} \|w\|_{L^2} \leq C \|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}}$, we proceed in the same fashion as for proving Proposition 2.5. If $\lambda \leq 0$, using Lemma 2.1 we have

$$\Re \langle F - c_2|\beta_k|^{\frac{1}{2}} w, w \rangle + c_2|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}^2 \gtrsim \langle (\frac{k^2}{r^2} + r^2)w, w \rangle,$$

Combining with

$$|\Re \langle F - c_2|\beta_k|^{\frac{1}{2}} w, w \rangle| \geq |\beta_k| \langle \frac{1}{r^2} w, w \rangle,$$

and (2.23), we get

$$\|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}}^2 + c_2|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}^2 \gtrsim \langle (r^2 + |\frac{k}{r^2}|)w, w \rangle \geq |\beta_k|^{\frac{3}{2}} \|w\|_{L^2}^2.$$  

It follows that

$$C(\|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}}^2 + c_2|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}^2) \geq |\beta_k|^{\frac{3}{2}} \|w\|_{L^2}^2.$$  

Recall $|\beta_k| = |kB| \geq 1$. Choose $c_2 > 0$ sufficiently small such that $Cc_2 \leq \frac{1}{2}$, we can conclude

$$(2.24) \quad \|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}} \gtrsim |\beta_k|^{\frac{3}{2}} \|w\|_{L^2}^2.$$  

If $\lambda > 0$ and $|\beta_k| \leq k^2$, by Lemma 2.1 and (2.23) we obtain

$$\|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}}^2 + c_2|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}^2 \gtrsim \Re \langle F - c_2|\beta_k|^{\frac{1}{2}} w, w \rangle + c_2|\beta_k|^{\frac{1}{2}} \|w\|_{L^2}^2$$

$$\gtrsim \langle (\frac{k^2}{r^2} + r^2)w, w \rangle \geq |k||w||_{L^2}^2 \geq |\beta_k|^{\frac{3}{2}} \|w\|_{L^2}^2.$$  

Recall $|\beta_k| \geq 1$. Choose $c_2 > 0$ being sufficiently small such that $Cc_2 \leq \frac{1}{2}$, we can conclude

$$(2.25) \quad \|F - c_2|\beta_k|^{\frac{1}{2}} w\|_{H^{-1}} \gtrsim |\beta_k|^{\frac{3}{2}} \|w\|_{L^2}^2.$$  

The remaining scenario is when $\lambda > 0$ and $|\beta_k| \geq k^2$. As the proof for Proposition 2.5, we consider the following three cases. Recall $r_0 = \frac{1}{\sqrt{\lambda}}$. 


(1) **Case of** $|\beta_k| \leq r_0^2$. Utilizing Lemma 2.1 again, it holds

$$ \Re\langle F - c_2|\beta_k|^\frac{3}{2} w, w \rangle + c_2|\beta_k|^\frac{1}{2} \|w\|_{L^2}^2 \gtrsim \langle \left(\frac{k^2}{r^2} + r^2\right) w, w \rangle. $$

Together with

$$ |\Im\langle F - c_2|\beta_k|^\frac{3}{2} w, w \rangle| \geq |\beta_k|^\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right) w, w), $$

this implies

$$ \|F - c_2|\beta_k|^\frac{3}{2} w\|_{H^{-1}} \|w\|_{H^1} + c_2|\beta_k|^\frac{1}{2} \|w\|_{L^2}^2 \gtrsim \langle \left(\frac{k^2}{r^2} + r^2\right) w, w \rangle \gtrsim \langle [r^2 + |\beta_k|(\frac{1}{r^2} - \frac{1}{r_0^2})] w, w \rangle. $$

Since

$$ r^2 + |\beta_k|(\frac{1}{r^2} - \frac{1}{r_0^2}) \gtrsim 2|\beta_k|^\frac{3}{2} - \frac{|\beta_k|}{r_0^2} \geq |\beta_k|^\frac{1}{2}, $$

we then deduce

$$ C(\|F - c_2|\beta_k|^\frac{3}{2} w\|^2_{H^{-1}} + c_2|\beta_k|^\frac{1}{2} \|w\|_{L^2}^2) \gtrsim \langle F, w \rangle \gtrsim \langle g(r) w, w \rangle \gtrsim |\beta_k|^\frac{3}{2} \|w\|_{L^2}^2. $$

Recall $|\beta_k| \geq 1$. Choose $c_2 > 0$ sufficiently small such that $Cc_2 \leq \frac{1}{2}$, we can conclude

(26)

$$ \|F - c_2|\beta_k|^\frac{3}{2} w\|_{H^{-1}} \gtrsim |\beta_k|^\frac{3}{2} \|w\|_{L^2}^2. $$

(2) **Case of** $r_0^2 \leq |\beta_k| \leq r_0^2$. Recall $|\beta_k| \geq 1$, then we have $r_0 \geq 1$. With Lemma 2.1 we obtain the below estimate for $\|w\|_{L^2(\frac{r_0}{2}, \infty)}^2$

$$ \Re\langle F - c_2|\beta_k|^\frac{3}{2} w, w \rangle + c_2|\beta_k|^\frac{1}{2} \|w\|_{L^2}^2 \gtrsim \langle \left(\frac{k^2}{r^2} + r^2\right) w, w \rangle \gtrsim \|rw\|_{L^2}^2 \gtrsim \frac{r_0^2}{4} \|w\|_{L^2(\frac{r_0}{2}, \infty)}^2. $$

As in Proposition 2.3, the estimate for $\|w\|_{L^2(0, \frac{r_0}{2})}$ is more subtle. First, we choose $r_- \in (\frac{r_0}{2} - \frac{1}{r_0}, \frac{r_0}{2})$ so that the following inequality holds

(27)

$$ |w'(r_-)|^2 \leq r_0 \|w'\|_{L^2}^2. $$

Being slightly different from the proof of Proposition 2.3, we require the multiplier for $H^{-1}$ estimate to be of $C^1$ regularity. We introduce a cutoff function $\rho(r)$ as follows:

$$ \rho(r) = \begin{cases} 1, & r \in (0, \frac{r_0}{2} - \frac{2}{r_0}), \\ \sin(\pi \frac{1}{2 r_- - (\frac{r_0}{2} - \frac{2}{r_0})}(r_- - r)), & r \in (\frac{r_0}{2} - \frac{2}{r_0}, r_-), \\ 0, & r \geq r_- \end{cases} $$

Then we consider the following basic equality

$$ \Im\langle F - c_2|\beta_k|^\frac{3}{2} w, w\rho(r) \rangle = \Im\langle -|\beta_k|^\frac{3}{2} w - \left(\frac{k^2}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right) w + i\beta_k(\frac{1}{r^2} - \lambda) w, w\rho(r) \rangle $$

$$ = \Im\langle -\beta_k^2 w + i\beta_k(\frac{1}{r^2} - \lambda) w, w\rho(r) \rangle. $$

This gives

$$ |\beta_k|(\frac{1}{r^2} - \lambda) w, w\rho(r) \rangle \lesssim \|F - c_2|\beta_k|^\frac{3}{2} w\|_{H^{-1}} \|\rho w\|_{H^1} + r_0 \|w\|_{L^2} \|w\|_{L^2}. $$
Thus, with $\lambda = \frac{1}{r_0}$ and $r_- \in (\frac{r_0}{2}, \frac{r_0}{4})$ we obtain
\[
\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \| pw \|_{H^1} + r_0 \| w \|_{L^2} \| w' \|_{L^2} \\
\geq |\beta_k| (\frac{4}{r_0^2} - \frac{1}{r_0^2}) \| w, w \rho(r) \| \geq |\beta_k| \| w \|_{L^2(0, \frac{r_0}{2}, \frac{r_0}{4})}^2.
\]
Since
\[
\| pw \|_{H^1} \lesssim \| w \|_{H^1} + r_0 \| w \|_{L^2},
\]
then $\| w \|_{L^2}$ can be bounded by
\[
\| w \|_{L^2}^2 = \| w \|_{L^2(0, \frac{r_0}{2}, \frac{r_0}{4})}^2 + \| w \|_{L^2(\frac{r_0}{2}, \frac{r_0}{4}, \frac{r_0}{4})}^2 + \| w \|_{L^2(\frac{r_0}{4}, \infty)}^2 \\
\leq \frac{r_0^2}{|\beta_k|} (\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \| w \|_{H^1} + r_0 \| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \| w \|_{L^2}) + \frac{1}{r_0} \| w \|_{L^\infty}^2 \\
+ \frac{1}{r_0^2} (\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \| w \|_{H^1} + r_0 \| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \| w \|_{L^2}) \\
+ \frac{1}{r_0} \| w' \|_{L^2} \| w \|_{L^2} + \frac{1}{r_0^2} (\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \| w \|_{L^2}) \\
\leq C\left( \frac{r_0^2}{|\beta_k|} (\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} + c_2|\beta_k|^\frac{1}{2} \| w \|_{L^2} + r_0 \| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \| w \|_{L^2}) \\
+ \frac{1}{r_0} (\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} + \sqrt{c_2}|\beta_k|^\frac{1}{2} \| w \|_{L^2}) \| w \|_{L^2} \\
+ \frac{1}{r_0} (\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} + c_2|\beta_k|^\frac{1}{2} \| w \|_{L^2}) \right)\).
\]
Thus we obtain
\[
C(c_2, \frac{r_0^2}{|\beta_k|}, \sqrt{c_2}, \frac{|\beta_k|^\frac{1}{2}}{r_0}) \leq C(c_2|\beta_k|^{-\frac{1}{2}} + \sqrt{c_2} + c_2),
\]
Recall $|\beta_k| \geq 1$. Choose $c_2 > 0$ satisfying
\[
C(c_2|\beta_k|^{-\frac{1}{2}} + \sqrt{c_2} + c_2) \leq \frac{1}{2},
\]
we can conclude
\[
\| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \geq \min\{\frac{|\beta_k|}{r_0^2}, (\frac{|\beta_k|^2}{r_0^3})^2, r_0^2\} \| w \|_{L^2}^2.
\]
Noting that $r_0^4 \leq |\beta_k| \leq r_0^6$, i.e. $\frac{|\beta_k|^4}{r_0^2} \geq \frac{|\beta_k|}{r_0^2} \geq r_0^2$, we hence arrive at
\[
(2.28) \quad \| F - c_2|\beta_k|^\frac{1}{2} w \|_{H^{-0}} \geq r_0^2 \| w \|_{L^2} \geq |\beta_k|^\frac{1}{2} \| w \|_{L^2}^2.
\]
(3) **Case of** \( |\beta| \geq r_0^6 \). Similarly, as in the previous case, we choose \( r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2}) \) and \( r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta) \) such that the following inequality holds

\[
|u'(r_-)|^2 + |u'(r_+)|^2 \leq \frac{4}{9} \|w'\|_{L^2}^2, \tag{2.29}
\]

Here \( \delta > 0 \) is a constant which will be determined later. As in Proposition 2.5, we define a \( C^1 \) cutoff function \( \rho \) with domain \((0, \infty)\) as follows

\[
\rho(r) = \begin{cases} 
1, & r \in (0, r_0 - 2\delta), \\
\sin\left(\frac{\pi}{2} \frac{1}{r_0 - (r_0 - 2\delta)} (r_0 - 2\delta - r)\right), & r \in (r_0 - 2\delta, r_-), \\
0, & r \in (r_-, r_+), \\
\sin\left(\frac{\pi}{2} \frac{1}{r_0 + 2\delta - r_+} (r_0 + 2\delta - r_+)\right), & r \in (r_+, r_0 + 2\delta), \\
-1, & r \geq r_0 + 2\delta.
\end{cases}
\]

Observing

\[
\Im\langle F - c_2 |\beta|^\frac{1}{3} w, w\rho(r) \rangle = \Im\langle -|\partial_r^2 w - \left(\frac{k^2 - \frac{3}{4}}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right)w\rangle + i \beta_0 \left(\frac{1}{r^2} - \frac{1}{2}\right) w, w\rho(r) \rangle
\]

we obtain

\[
|\beta| \left( \int_{r_0 - 2\delta}^{r_0} \left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)|w(r)|^2 dr + \int_{r_0 + 2\delta}^{\infty} \left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)|w(r)|^2 dr \right) \lesssim \|F - c_2 |\beta|^\frac{1}{3} w\|_{H^{-1}} \|\rho w\|_{H^1} + \delta^{-1} \|w\|_{L^2} \|w'|_{L^2} \|
\]

With \( r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2}) \) and \( r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta) \), it holds

\[
\frac{1}{r_0^2} - \frac{1}{r^2} = \frac{r_0 - r}{r^2 r_0} \geq \frac{r_0 - r}{r^2 r_0} \geq \frac{\delta}{r_0^3} \geq \frac{\delta}{r_0^3} \text{ for any } r \in (0, r_-).
\]

Likewise, we have

\[
\frac{1}{r_0^2} - \frac{1}{r^2} = \frac{r - r_0}{r^2 r_0} \geq \frac{r - r_0}{r^2 r_0} \geq \frac{\delta}{r_0^3} \geq \frac{\delta}{r_0^3}, \text{ for any } r \in (r_+, 2r_0),
\]

\[
\frac{1}{r_0^2} - \frac{1}{r^2} = \frac{r - r_0}{r^2 r_0} \approx \frac{1}{r_0^2} \geq \frac{\delta}{r_0^3}, \text{ for any } r \geq 2r_0.
\]

Plugging in all these above inequalities, we deduce

\[
\begin{align*}
&\|F - c_2 |\beta|^\frac{1}{3} w\|_{H^{-1}} \|\rho w\|_{H^1} + \delta^{-1} \|w\|_{L^2} \|w'|_{L^2} \\
\geq &|\beta| \left( \int_{0}^{r_0 - 2\delta} \left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)|w(r)|^2 dr + \int_{r_0 + 2\delta}^{\infty} \left(\frac{1}{r^2} - \frac{1}{r_0^2}\right)|w(r)|^2 dr \right) \\
\geq & \frac{1}{r_0^3} \|w\|_{L^2}^2 \left( 0, r_0 - 2\delta, r_0, r_0 + 2\delta, \infty \right)
\end{align*}
\]

Since

\[
\|\rho w\|_{H^1} \lesssim \|w\|_{H^1} + \delta^{-1} \|w\|_{L^2},
\]
then \( \|w\|_{L^2}^2 \) can be bounded as follows
\[
\|w\|_{L^2}^2 = \|w\|_{L^2}^2(0, r_0 - 2\delta, (r_0 - 2\delta, r_0 + 2\delta)) + \|w\|_{L^2}^2(r_0 - r_0 + 2\delta)
\]
\[
\leq \frac{r_0^3}{|\beta_k|^2} \left( \|F - c_2|\beta_k|^\frac{1}{2} w\|_{H}^2 + c_2|\beta_k|^\frac{1}{2} \|w\|_{L^2}^2 + \delta^{-1}\|F - c_2|\beta_k|^\frac{1}{2} w\|_{H^{-1}} \|w\|_{L^2}^2 \right)
\]
\[
+ \delta^{-1}\|w\|_{L^2}^2(2\delta, r_0 + 2\delta) + \delta\|w\|_{L^2}^2.
\]
Employing (2.29), Lemma A.2 and (2.23) we obtain
\[
\|w\|_{L^2}^2 \leq \frac{r_0^3}{|\beta_k|^3} \left( \|F - c_2|\beta_k|^\frac{1}{2} w\|_{H}^2 + c_2|\beta_k|^\frac{1}{2} \|w\|_{L^2}^2 + \delta^{-1}\|F - c_2|\beta_k|^\frac{1}{2} w\|_{H^{-1}} \|w\|_{L^2}^2 \right)
\]
\[
+ \delta^{-1}\|w\|_{L^2}^2(2\delta, r_0 + 2\delta) + \delta\|w\|_{L^2}^2.
\]
Take \( \delta = (\frac{r_0^3}{|\beta_k|^3})^2 \), then the condition \( |\beta_k| \geq \max\{r_0^6, 1\} \) yields
\[
\delta = (\frac{r_0^3}{|\beta_k|^3})^2 \leq r_0.
\]
Notice that \( |\beta_k| \geq r_0^6 \), we deduce
\[
C(c_2) \frac{r_0^3}{|\beta_k|^3} + \sqrt{c_2} \frac{r_0^3}{|\beta_k|^3} + \sqrt{c_2} \delta |\beta_k|^\frac{1}{2} = C(c_2) \frac{r_0^3}{|\beta_k|^3} + 2\sqrt{c_2} \frac{r_0}{|\beta_k|^3} \leq C(c_2) + 2\sqrt{c_2},
\]
Choose \( c_2 > 0 \) such that
\[
C(c_2) + 2\sqrt{c_2} \leq \frac{1}{2},
\]
we have
\[
\|F - c_2|\beta_k|^\frac{1}{2} w\|_{H^{-1}} \geq \min\left\{ \frac{|\beta_k|\delta}{r_0^3}, (\frac{|\beta_k|\delta^2}{r_0^3})^2, \delta^{-2}\right\} \|w\|_{L^2}^2.
\]
With the basic equality \( \frac{|\beta_k|\delta^2}{r_0^3} = (\frac{|\beta_k|\delta^2}{r_0^3})^2 = \delta^{-2} \), we conclude
\[
(2.30) \quad \|F - c_2|\beta_k|^\frac{1}{2} w\|_{H^{-1}} \geq (\frac{r_0}{|\beta_k|^\frac{1}{3}})^2 \|w\|_{L^2}^2 = |\beta_k|^\frac{1}{3} \|w\|_{L^2}^2 \geq |\beta_k|^\frac{1}{3} \|w\|_{L^2}^2 = |\beta_k|^\frac{1}{3} \|w\|_{L^2}^2.
\]
Combining (2.24), (2.25), (2.26), (2.28) and (2.30), we therefore establish the following resolvent estimate from \( L^2 \) to \( H^{-1} \)
\[
|\beta_k|^\frac{1}{3} \|w\|_{L^2}^2 \leq \|F - c_2|\beta_k|^\frac{1}{2} w\|_{H^{-1}}.
\]
Applying (2.23), we have
\[ |w|_{H^1} \lesssim \| F - c_2 |\beta_k|^\frac{1}{4} w \|_{H^{-1}} + \sqrt{c_2} |\beta_k|^\frac{1}{4} \| w \|_{L^2} \lesssim \| F - c_2 |\beta_k|^\frac{1}{4} w \|_{H^{-1}}.\]

This completes the proof of Proposition 2.6. \(\square\)

The following proposition can be derived directly from Proposition 2.5 and Proposition 2.6 if we choose \(c_2 > 0\) being small enough.

**Proposition 2.7.** For any \(|k| \geq 1, \lambda \in \mathbb{R}\) and \(w \in D_k\), there exist constants \(C, c_2 > 0\) independent of \(k, \beta, \lambda\), such that the following estimates hold
\[ |\beta_k|^{\frac{1}{4}} \| w' \|_{L^2} + |\beta_k|^{\frac{1}{4}} \| w \|_{L^2} \leq C \| F - c_2 |\beta_k|^\frac{1}{4} w \|_{L^2},\]
and
\[ |w|_{H^1} + |\beta_k|^{\frac{1}{4}} \| w \|_{L^2} \leq C \| F - c_2 |\beta_k|^\frac{1}{4} w \|_{H^{-1}}.\]

### 2.3. Resolvent estimates in the weighted \(L^2\) space \(X\).

To prove sharp decaying estimates for the nonlinear problem, we design and employ a new energy space. Here we introduce a weighted space \(L^2\) space as follows
\[ \| w \|_X = (\int_0^\infty \| w \|_{r^2}^2 \, dr)^{\frac{1}{2}}.\]

As we proceed in the last subsection, we start with deriving the resolvent estimates from \(L^2(\frac{1}{r^2})\) to \(L^2(\frac{1}{r^2})\).

**Proposition 2.8.** For any \(|k| \geq 1, \lambda \in \mathbb{R}\) and \(w \in D_k\), there exists a constant \(C > 0\) independent of \(k, \beta, \lambda\), such that the following inequality holds
\[ |\beta_k|^{\frac{1}{4}} \| w' \|_{L^2} + |\beta_k|^{\frac{1}{4}} \| w \|_{L^2} \leq C \| F - c_2 |\beta_k|^\frac{1}{4} w \|_{L^2}.\]

**Proof.** We first prove \(|\beta_k|^{\frac{1}{4}} \| w' \|_{L^2} \leq C \| F \|_{L^2}.\) The discussion is analogous to proofs of Proposition 2.5 and Proposition 2.6. If \(\lambda \leq 0\), using Lemma 2.3 we have
\[ \Re \langle F, \frac{w}{r^2} \rangle \geq \langle (\frac{k^2}{r^2} + r^2)w, \frac{w}{r^2} \rangle.\]

This together with
\[ |\Re \langle F, \frac{w}{r^2} \rangle | \geq |\beta_k| \langle \frac{1}{r^2} w, \frac{w}{r^2} \rangle,\]

yields
\[ \| F \|_{L^2} \| \frac{w}{r^2} \|_{L^2} \geq \langle (r^2 + |\beta_k|^\frac{1}{4})w, \frac{w}{r^2} \rangle \geq |\beta_k|^{\frac{1}{4}} \| w \|_{L^2}^2.\]

If \(\lambda > 0\) and \(|\beta_k| \leq k^2\). With Lemma 2.3 we have
\[ \| F \|_{L^2} \| \frac{w}{r^2} \|_{L^2} \geq |\beta_k|^{\frac{1}{4}} \| w \|_{L^2}.\]

If \(\lambda > 0\) and \(|\beta_k| \geq k^2\), we consider the following three cases. Recall \(r_0 = \frac{1}{\sqrt{\lambda}}\).

1. **Case of \(|\beta_k| \leq r_0^2\).** Via integration by parts we get
\[ \Re \langle F, \frac{w}{r^2} \rangle = -2 \Re \langle \frac{w'}{r^2}, \frac{w}{r^2} \rangle + \beta_k \langle (\frac{1}{r^2} - \frac{1}{r_0^2})w, \frac{w}{r^2} \rangle,\]
Together with Lemma 2.23 this gives
\[ |\beta_k| \langle \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) w, \frac{w}{r^2} \rangle \leq \| \mathcal{M}(F, \frac{w}{r^2}) \| + 2 \| \frac{w'}{r} \|_{L^2} \| \frac{w}{r^2} \|_{L^2} \lesssim \frac{F}{r} \| L^2 \| \frac{w}{r} \| L^2 \].

Employing Lemma 2.23 again, we have
\[ \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim \langle r^2 + |\beta_k| \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) w, \frac{w}{r^2} \rangle. \]

Since
\[ r^2 + |\beta_k| \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) \geq 2 |\beta_k|^2 - \frac{|\beta_k|^2}{r_0^2} \geq |\beta_k|^2, \]
we conclude (2.33)
\[ \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim |\beta_k|^2 \| \frac{w}{r} \|_{L^2} \gtrsim |\beta_k|^2 \| \frac{w}{r} \|_{L^2}^2. \]

(2) Case of \( r_0^2 \leq |\beta_k| \leq r_0^6 \). Since \( |\beta_k| \geq 1 \), it renders \( r_0 \geq 1 \). Utilizing Lemma 2.23 we obtain the estimate of \( \| \frac{w}{r} \|_{L^2(\frac{r_0}{4}, \infty)} \) as follows
\[ \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim \| \frac{w}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim \frac{r_0^2}{4} \| \frac{w}{r} \|_{L^2(\frac{r_0}{4}, \infty)}^2. \]

We continue to estimate \( \| \frac{w}{r} \|_{L^2(0, \frac{r_0}{4})} \). Choose \( r_- \in \left( \frac{r_0}{2} - \frac{1}{r_0}, \frac{r_0}{4} \right) \) such that the following inequality holds
\[ \frac{|w(r_-)|^2}{r_-^2} \leq 2r_0 \| \frac{w}{r} \|_{L^2}^2. \]

Observing the basic equality
\[ \mathcal{M}(F, \frac{w\chi(0,r_-)}{r^2}) = \mathcal{M}(\frac{\partial_x w}{2} - (\frac{k^2 - \frac{1}{2}}{r^2} + \frac{r^2}{16} - \frac{1}{2})w + i\beta_k(1 - \lambda)w, \frac{w\chi(0,r_-)}{r^2}) \]
\[ = \mathcal{M}(\frac{\partial_x w}{2} + i\beta_k(1 - \lambda)w, \frac{w\chi(0,r_-)}{r^2}), \]
we deduce
\[ |\beta_k| \langle \left( \frac{1}{r^2} - \lambda \right) w, \frac{w\chi(0,r_-)}{r^2} \rangle \leq \frac{F}{r} \| L^2 \| \frac{w}{r} \| L^2 + \frac{|w(r_-)w(r_-)|}{|r_-|^2}. \]

It follows from the fact that \( \lambda = \frac{1}{r_0} \) and \( r_- \in \left( \frac{r_0}{2} - \frac{1}{r_0}, \frac{2r_0}{2} \right) \)
\[ \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} + \frac{|w(r_-)w(r_-)|}{|r_-|^2} \gtrsim |\beta_k| \langle \left( \frac{4}{r_0^2} - \frac{1}{r_0^2} \right) w, \frac{w\chi(0,r_-)}{r^2} \rangle \gtrsim |\beta_k| \| \frac{w}{r} \|_{L^2(0,r_-)}^2. \]

Therefore, \( \| \frac{w}{r} \|_{L^2} \) can be bounded by
\[ \| \frac{w}{r} \|_{L^2}^2 = \| \frac{w}{r} \|_{L^2(0,r_-)}^2 + \| \frac{w}{r} \|_{L^2(r_0,r_-)}^2 + \| \frac{w}{r} \|_{L^2(r_0, \infty)}^2 \]
\[ \lesssim \frac{r_0^2}{|\beta_k|} \left( \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} + \frac{|w(r_-)w(r_-)|}{|r_-|^2} \right) + \frac{1}{r_0} \| \frac{w}{r} \|_{L^\infty} + \frac{1}{r_0^2} \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2}. \]

Together with (2.34), Lemma A.2, this yields
\[ \| \frac{w}{r} \|_{L^2}^2 \lesssim \frac{r_0^2}{|\beta_k|} \left( \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} + r_0^2 \| \frac{w}{r} \|_{L^2} \| \frac{w}{r} \|_{L^\infty} \right) + \frac{1}{r_0} \| \frac{w}{r} \|_{L^\infty} + \frac{1}{r_0^2} \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2}. \]
By Lemma 2.3 we obtain
\[ \| \frac{w'}{r} \|_{L^2}^2 + \| (\frac{w}{r})' \|_{L^2}^2 \lesssim \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2}, \]
which implies
\[
\| \frac{w}{r} \|_{L^2}^2 \lesssim \frac{r_0^2}{|\beta_k|} \left( \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} + \frac{r_0^2}{|\beta_k|} \| (\frac{w}{r})' \|_{L^2}^2 \right) + \frac{1}{r_0^2} \| (\frac{w}{r})' \|_{L^2} \| \frac{w}{r} \|_{L^2} \]
\[ + \frac{1}{r_0^2} \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2}. \]

Then we conclude
\[ \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim \min \left\{ \frac{|\beta_k|}{r_0^2} \left( \frac{r_0^2}{|\beta_k|} \right)^{\frac{4}{3}}, \frac{r_0^2}{r_0^2} \right\} \| \frac{w}{r} \|_{L^2}^2. \]

With \( r_0^4 \leq |\beta_k| \leq r_0^6 \Rightarrow \frac{|\beta_k|}{r_0^2} \geq \frac{r_0^2}{r_0^2} \), we deduce
\[ (2.35) \]
\[ \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim \frac{r_0^2}{|\beta_k|} \| \frac{w}{r} \|_{L^2}^2 \geq \| \frac{\lambda}{r} \|_{L^2}. \]

(3) **Case of** \( |\beta_k| \geq r_0^6 \). Choose \( r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2}) \) and \( r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta) \) such that it holds
\[ (2.36) \]
\[ \frac{|w'(r_-)|^2}{r_-^2} + \frac{|w'(r_+)|^2}{r_+^2} \leq \frac{4}{\delta} \| \frac{w'}{r} \|_{L^2}^2. \]

Here \( \delta \in (0, r_0) \) is a constant to be determined later. Noting
\[ \Im \langle F, \frac{w(\chi(0,r_-)) - \chi(r_+,\infty)}{r_2} \rangle \]
\[ = \Im \langle -[\partial_{\hat{r}} w - (\frac{k^2}{r_2} + \frac{1}{16} - \frac{1}{2}) w] + i\beta_k \frac{1}{r_2} - \lambda \rangle \| \frac{w(\chi(0,r_-)) - \chi(r_+,\infty)}{r_2} \rangle \]
\[ = \Im \langle -[\partial_{\hat{r}} w + i\beta_k \frac{1}{r_2} - \lambda \rangle w, \frac{w(\chi(0,r_-)) - \chi(r_+,\infty)}{r_2} \rangle \],
we obtain
\[ |\beta_k| \left( \int_0^{r_-} \frac{1}{r_2} \| \frac{w(r)}{r} \|_{L^2}^2 \, dr + \int_{r_+}^{\infty} \frac{1}{r_2} \| \frac{w(r)}{r} \|_{L^2}^2 \, dr \right) \]
\[ \leq \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \left( \frac{|w'(r_-)|w(r_-)|}{r_-^2} + \frac{|w'(r_+)|w(r_+)|}{r_+^2} \right). \]
With \( r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2}) \) and \( r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta) \), it holds
\[
\frac{1}{r^2} - \frac{1}{r_0^2} = \frac{(r_0 - r)(r_0 + r)}{r^2 r_0^2} \geq \frac{r_0 - r}{r^2 r_0} \gtrsim \frac{\delta}{r^2 r_0} \gtrsim \frac{\delta}{r_0^3}, \quad \text{for any } r \in (0, r_-).
\]

Similarly, it can be verified that
\[
\frac{1}{r_0^2} - \frac{1}{r^2} = \frac{(r - r_0)(r + r_0)}{r^2 r_0^2} \geq \frac{r - r_0}{r^2 r_0} \gtrsim \frac{\delta}{r^2 r_0} \gtrsim \frac{\delta}{r_0^3}, \quad \text{for any } r \in (r_+, 2r_0),
\]
\[
\frac{1}{r_0^2} - \frac{1}{r^2} = \frac{(r - r_0)(r + r_0)}{r^2 r_0^2} \approx \frac{1}{r_0^2} \gtrsim \frac{\delta}{r_0^3}, \quad \text{for any } r \geq 2r_0.
\]

Combining all inequalities above, we deduce
\[
\| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} + \left| \frac{|w'(r_-)w(r_-)|}{r_-^2} \right| + \left| \frac{|w'(r_+)w(r_+)|}{r_+^2} \right| \\
\gtrsim \frac{\beta_k \delta}{r_0^3} \left( \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} + \left| \frac{|w'(r_-)w(r_-)|}{r_-^2} \right| + \left| \frac{|w'(r_+)w(r_+)|}{r_+^2} \right| + \| \frac{w}{r} \|_{L^\infty}^2 \right).
\]

Hence \( \| \frac{w}{r} \|_{L^2} \) can be bounded by
\[
\| \frac{w}{r} \|_{L^2}^2 = \| \frac{w}{r} \|_{L^2((0, r_-) \cup (r_+, \infty))}^2 + \| \frac{w}{r} \|_{L^2(r_-, r_+)}^2 \leq \frac{r_0^3}{|\beta_k| \delta} \left( \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} + \delta^{-\frac{1}{2}} \| \frac{w}{r} \|_{L^2} \| \frac{w}{r} \|_{L^\infty} \right) + \delta \| \frac{w}{r} \|_{L^\infty}^2.
\]

Together with (2.36), Lemma A.2 and Lemma 2.3, this gives
\[
\frac{\| \frac{w}{r} \|_{L^2}^2}{\| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2}} \lesssim \frac{r_0^3}{|\beta_k| \delta} \left( \frac{\| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \| \frac{w}{r} \|_{L^\infty}}{\delta^{-\frac{1}{2}} \| \frac{w}{r} \|_{L^2}^2} \right) + \delta \| \frac{w}{r} \|_{L^\infty}^2.
\]

Therefore, we obtain
\[
\| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim \min \left\{ \frac{|\beta_k| \delta}{r_0^3}, \frac{\| \frac{w}{r} \|_{L^2}}{\delta^{\frac{2}{3}}}, \delta^{-2} \right\} \| \frac{w}{r} \|_{L^2}^2.
\]

Taking \( \delta = \left( \frac{r_0^3}{|\beta_k|} \right)^{\frac{2}{3}} \), and noting \( |\beta_k| \geq 1 \), we have
\[
\delta = \left( \frac{r_0^3}{|\beta_k|} \right)^{\frac{2}{3}} \leq r_0.
\]

With the basic equality \( \frac{|\beta_k| \delta}{r_0^3} = \left( \frac{|\beta_k| \delta^{\frac{2}{3}}}{r_0^3} \right)^{\frac{2}{3}} = \delta^{-2} \), we arrive at
\[
(2.37) \quad \| \frac{F}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \gtrsim |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2.
\]

Plugging in (2.31), (2.32), (2.33), (2.35) and (2.37), we therefore establish the following resolvent estimate from \( L^2 \left( \frac{1}{r^2} \right) \) to \( L^2 \left( \frac{1}{r^2} \right) \)
\[
(2.38) \quad |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2} \lesssim \| \frac{F}{r} \|_{L^2}.
\]
In addition, by Lemma 2.3 and (2.38) we deduce
\[ \| u' \|_{L^2}^2 \lesssim \| F_r \|_{L^2} \| w \|_{L^2} \lesssim | \beta_k | \frac{1}{r} \| F_r \|_{L^2}^2. \]
This completes the proof of Proposition 2.8.

Finally, we are ready to derive the resolvent estimates from \( L^2(\frac{1}{r^2}) \) to \( H^{-1}(\frac{1}{r^2}) \).

**Proposition 2.9.** For any \( |k| \geq 1 \), \( \lambda \in \mathbb{R} \) and \( w \in D_k \), there exist constants \( C, c_2 > 0 \) independent with \( k, \beta_k, \lambda \), such that the following inequality holds
\[ \| \frac{w}{r} \|_{H^1} + | \beta_k |^{\frac{1}{2}} \| \frac{w}{r} \|_{L^2} \leq C \| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{H^{-1}}. \]

**Proof.** By Lemma 2.3 we obtain
\[ \Re \langle F, \frac{w}{r^2} \rangle = \Re \langle F - c_2 | \beta_k | \frac{1}{r} w, \frac{w}{r^2} \rangle + c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}^2 \geq \| \frac{w'}{r} \|_{L^2}^2 + \langle (\frac{k^2}{r^2} + r^2) w, \frac{w}{r^2} \rangle \geq \| \frac{w}{r} \|_{H^1}^2, \]
with \( c_2 > 0 \) to be determined later. It follows directly that
\[ \| \frac{w}{r} \|_{H^1}^2 \lesssim \| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \frac{w}{r} \|_{H^{-1}} + c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}. \]
Thus we have
\[ (2.39) \quad \| \frac{w}{r} \|_{H^1} \lesssim \| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \frac{w}{r} \|_{H^{-1}} + \sqrt{c_2} | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}. \]
We proceed in the similar fashion as in Proposition 2.6 to show \( | \beta_k |^{\frac{1}{2}} \| \frac{w}{r} \|_{L^2} \leq C \| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \frac{w}{r} \|_{H^{-1}}. \) Likewise, we divide the proof into different cases. If \( \lambda \leq 0 \), using Lemma 2.3 we obtain
\[ \Re \langle F - c_2 | \beta_k | \frac{1}{r} w, \frac{w}{r^2} \rangle + c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}^2 \geq \langle (\frac{k^2}{r^2} + r^2) w, \frac{w}{r^2} \rangle. \]
This together with
\[ | \Im \langle F - c_2 | \beta_k | \frac{1}{r} w, \frac{w}{r^2} \rangle | \geq | \beta_k | \langle \frac{1}{r^2} w, \frac{w}{r^2} \rangle, \]
and (2.39) gives
\[ \| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \frac{w}{r} \|_{H^{-1}} + c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}^2 \geq \langle (r^2 + \frac{1}{r^2}) w, \frac{w}{r^2} \rangle \geq | \beta_k |^{\frac{1}{2}} \| \frac{w}{r} \|_{L^2}^2. \]
Then we can deduce
\[ C(\| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \frac{w}{r} \|_{H^{-1}} + c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}^2) \geq | \beta_k |^{\frac{1}{2}} \| \frac{w}{r} \|_{L^2}^2. \]
Notice \( | \beta_k | \geq 1 \). Choose \( c_2 > 0 \) sufficiently small satisfying \( Cc_2 \leq \frac{1}{2} \), we conclude
\[ (2.40) \quad \| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \frac{w}{r} \|_{H^{-1}} \geq | \beta_k |^{\frac{1}{2}} \| \frac{w}{r} \|_{L^2}^2. \]
If \( \lambda > 0 \) and \( | \beta_k | \leq k^2 \), it follows from Lemma 2.3 and (2.39)
\[ \| \frac{F_r}{r} - c_2 | \beta_k | \frac{1}{r} \frac{w}{r} \|_{H^{-1}} + c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}^2 \geq \Re \langle F - c_2 | \beta_k | \frac{1}{r} w, \frac{w}{r^2} \rangle + c_2 | \beta_k | \frac{1}{r} \| \frac{w}{r} \|_{L^2}^2 \]
\[ \geq \langle (\frac{k^2}{r^2} + r^2) w, \frac{w}{r^2} \rangle \geq | k | \| \frac{w}{r} \|_{L^2}^2 \geq | \beta_k |^{\frac{1}{2}} \| \frac{w}{r} \|_{L^2}^2. \]
Note $|\beta_k| \geq 1$. Choose $c_2 > 0$ sufficiently small satisfying $Cc_2 \leq \frac{1}{2}$, we conclude

$$
(2.41) \quad \| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} \geq \| \beta_k \|_{L^2}^2.
$$

If $\lambda > 0$ and $|\beta_k| \geq k^2$, the discussion can be separated into three cases. Recall $r_0 = \frac{1}{\sqrt{\lambda}}$.

(1) **Case of** $|\beta_k| \leq r_0^3$. Recall that Lemma 2.3 yields

$$
(2.42) \quad \Re \langle F - c_2 |\beta_k|^{\frac{1}{3}} w, \frac{w}{r^2} \rangle + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2 \gtrsim \| \frac{k^2}{r^2} + r^2 \| w, \frac{w}{r^2} \rangle.
$$

Since

$$
\Im \langle F - c_2 |\beta_k|^{\frac{1}{3}} w, \frac{w}{r^2} \rangle = \Im \langle -w', \frac{w}{r^2} \rangle + \beta_k \langle \frac{k^2}{r^2} - \frac{1}{r^2} \rangle w, \frac{w}{r^2} \rangle
$$

$$
= \Re \langle 2w', \frac{w}{r^2} \rangle + \beta_k \langle \frac{k^2}{r^2} - \frac{1}{r^2} \rangle w, \frac{w}{r^2} \rangle,
$$

and note (2.39), it holds

$$
|\beta_k| \langle \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) w, \frac{w}{r^2} \rangle \leq \left\| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} \| \frac{w}{r} \|_{H^1} + 2 \| \frac{w'}{r} \|_{L^2} \| \frac{w}{r} \|_{L^2} \right\|
$$

$$
\leq \left\| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} \| \frac{w}{r} \|_{H^1} + 2 \| \frac{w}{r} \|_{H^1} \| \frac{w}{r} \|_{H^1} \right\|
$$

$$
\lesssim \left\| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2 \right\|.
$$

Combining with (2.42), we obtain

$$
\| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2 \gtrsim \| \frac{k^2}{r^2} + r^2 \| w, \frac{w}{r^2} \rangle.
$$

Observing

$$
r^2 + |\beta_k| \langle \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) \rangle \geq 2 |\beta_k|^{\frac{1}{3}} - \frac{|\beta_k|}{r_0^2} \geq |\beta_k|^{\frac{1}{2}},
$$

we deduce

$$
\| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2 \gtrsim \| \beta_k \|_{L^2}^2.
$$

Recall $|\beta_k| \geq 1$. Choose $c_2 > 0$ sufficiently small so that $Cc_2 \leq \frac{1}{2}$, we conclude

$$
(2.43) \quad \| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} \gtrsim \| \beta_k \|_{L^2}^2.
$$

(2) **Case of** $r_0^3 \leq |\beta_k| \leq r_0^6$. Then $r_0 \geq 1$. Utilizing Lemma 2.3 and (2.39), we obtain

$$
\| \frac{F - c_2 |\beta_k|^{\frac{1}{3}} w}{r} \|_{H^{-1}} + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2 \gtrsim \Re \langle F - c_2 |\beta_k|^{\frac{1}{3}} w, \frac{w}{r^2} \rangle + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2
$$

$$
\gtrsim \langle \frac{k^2}{r^2} + r^2 \rangle w, \frac{w}{r^2} \rangle \geq \| \frac{k^2}{r^2} + \| \frac{w}{r^2} \|_{L^2}^2 \geq \| \frac{r_0^2}{4} \| \frac{w}{r^2} \|_{L^2}^2(\mathcal{M}, \infty).
$$
We move to estimate $\|\frac{w'}{r}\|_{L^2(0, \frac{r_0}{r})}^2$. Pick $r_- \in \left(\frac{r_0}{2} - \frac{1}{r_0}, \frac{r_0}{2}\right)$ such that the following inequality holds

$$\left|\frac{w'(r_-)}{r_-}\right|^2 \leq r_0 \left|\frac{w'}{r}\right|^2_{L^2}.$$ 

We construct a cutoff function

$$\rho(r) = \begin{cases} 1, & r \in \left(0, \frac{r_0}{2} - \frac{2}{r_0}\right), \\ \sin \left(\frac{r_0}{2} - \frac{1}{r_0} - \frac{2}{r_0}\right) (r_- - r), & r \in \left(\frac{r_0}{2} - \frac{2}{r_0}, r_-\right), \\ 0, & r \geq r_- . \end{cases}$$ 

Starting from with the following basic equality

$$\Im \langle F - c_2|\beta_k|^\frac{3}{2} w, \frac{w p(r)}{r^2} \rangle = \Im \langle -[\partial^2_r w - \left(\frac{k^2 - \frac{3}{2}}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right) w] + i\beta_k(\frac{1}{r^2} - \lambda) w, \frac{w p(r)}{r^2} \rangle$$

we obtain

$$|\beta_k|((\frac{1}{r^2} - \lambda) w, \frac{w p(r)}{r^2}) \lesssim \|F - c_2|\beta_k|^\frac{3}{2} w, \frac{w p(r)}{r} \|_{H^{-1}} \|w p(r)\|_{H^1} + r_0 \|\frac{w'}{r}\|_{L^2} \|w\|_{L^2} + \|\frac{w'}{r}\|_{L^2} \|\frac{w}{r^2}\|_{L^2}.$$ 

Thus with $\lambda = \frac{1}{r_0^2}$ and $r_- \in \left(\frac{r_0}{2} - \frac{1}{r_0}, \frac{r_0}{2}\right)$ we have

$$\|F - c_2|\beta_k|^\frac{3}{2} w, \frac{w p(r)}{r} \|_{H^{-1}} \|w p(r)\|_{H^1} + r_0 \|\frac{w'}{r}\|_{L^2} \|w\|_{L^2} + \|\frac{w'}{r}\|_{L^2} \|\frac{w}{r^2}\|_{L^2} \lesssim |\beta_k|((\frac{4}{r_0^2} - \frac{1}{r_0}) w, \frac{w p(r)}{r^2}) \geq \frac{|\beta_k|}{r_0^2} \|w\|^2_{L^2(0, \frac{r_0}{2} - \frac{2}{r_0})}.$$

Note

$$\|\frac{p w}{r} \|_{H^1} \lesssim \|w\|_{H^1} + r_0 \|\frac{w}{r}\|_{L^2}.$$ 

Combining with (2.39), we deduce

$$\|F - c_2|\beta_k|^\frac{3}{2} w, \frac{w p(r)}{r} \|_{H^{-1}} \|w p(r)\|_{H^1} + r_0 \|\frac{w'}{r}\|_{L^2} \|w\|_{L^2} + \|\frac{w'}{r}\|_{L^2} \|\frac{w}{r^2}\|_{L^2} \lesssim \|F - c_2|\beta_k|^\frac{3}{2} w, \frac{w p(r)}{r} \|_{H^{-1}} \|F - c_2|\beta_k|^\frac{3}{2} w \|_{H^{-1}} + r_0 \|\frac{w}{r}\|_{L^2} + \|\frac{w}{r^2}\|_{L^2}+ r_0 \left(\|\frac{F - c_2|\beta_k|^\frac{3}{2} w}{r} \|_{H^{-1}} + \sqrt{c_2}|\beta_k|^\frac{1}{2} \|\frac{w}{r}\|_{L^2}\right) \|\frac{w}{r^2}\|_{L^2}$$

$$+ \|\frac{F - c_2|\beta_k|^\frac{3}{2} w}{r^2}\|_{H^{-1}} + \sqrt{c_2}|\beta_k|^\frac{1}{2} \|\frac{w}{r}\|_{L^2}\|^2_{L^2}.$$
Therefore, with Lemma A.2, we now can bound \( \| \frac{w}{r} \|_{L^2} \) by

\[
\| \frac{w}{r} \|_{L^2}^2 = \| \frac{w}{r} \|_{L^2(0, \frac{r}{2} - \frac{r_0}{2})}^2 + \| \frac{w}{r} \|_{L^2(\frac{r}{2} - \frac{r_0}{2}, \frac{r}{2})}^2 + \| \frac{w}{r} \|_{L^2(\frac{r}{2}, \infty)}^2 
\]

\[
\lesssim \frac{r_0^3}{|\beta_k|} \left[ \left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1}} \left( \left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1} + r_0 \| \frac{w}{r} \|_{L^2}} \right) + \sqrt{c_2} r_0 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2} + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2} \right] + \frac{1}{r_0} \| \frac{w}{r} \|_{L^2}^2 
\]

\[
\lesssim \frac{r_0^3}{|\beta_k|} \left[ \left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1}} \left( \left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1} + r_0 \| \frac{w}{r} \|_{L^2}} \right) + \sqrt{c_2} r_0 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2} + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2} \right] + \frac{1}{r_0} \| \frac{w}{r} \|_{L^2}^2 
\]

\[
\leq C \left\{ \frac{r_0^3}{|\beta_k|} \left[ \left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1}} \left( \left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1} + r_0 \| \frac{w}{r} \|_{L^2}} \right) + \sqrt{c_2} r_0 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2} + c_2 |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2} \right] \right\}. 
\]

The assumption \( r_0^3 \leq |\beta_k| \leq r_0^3 \) implies

\[
C(c_2 \frac{r_0^3}{|\beta_k|^{\frac{1}{3}}} + \sqrt{c_2} \frac{r_0^3}{|\beta_k|^{\frac{1}{3}}} + \sqrt{c_2} \frac{|\beta_k|^{\frac{1}{3}}}{r_0} + c_2 \frac{|\beta_k|^{\frac{1}{3}}}{r_0}) 
\]

\[
\leq C(c_2 |\beta_k|^{- \frac{1}{3}} + \sqrt{c_2} |\beta_k|^{- \frac{1}{3}} + \sqrt{c_2} + c_2). 
\]

Recalling \( |\beta_k| \geq 1 \), we can choose \( c_2 > 0 \) small enough to obtain

\[
C(c_2 |\beta_k|^{- \frac{1}{3}} + \sqrt{c_2} |\beta_k|^{- \frac{1}{3}} + \sqrt{c_2} + c_2) \leq \frac{1}{2}. 
\]

Therefore, we deduce

\[
\left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1}} \geq \min \left\{ \frac{|\beta_k|^{\frac{1}{3}}}{r_0^3}, \left( \frac{|\beta_k|}{r_0^3} \right)^2, r_0^2 \right\} \| \frac{w}{r} \|_{L^2}^2. 
\]

By \( r_0^3 \leq |\beta_k| \leq r_0^3 \), i.e., \( |\beta_k|^2 \leq \frac{|\beta_k|^{\frac{1}{3}}}{r_0^3} \geq r_0^2 \), we arrive at

\[
(2.44) \quad \left\| \frac{F - c_2 |\beta_k|^{ \frac{1}{3}} w}{r} \right\|_{H^{-1}} \geq r_0^2 \| \frac{w}{r} \|_{L^2}^2 \geq |\beta_k|^{\frac{1}{3}} \| \frac{w}{r} \|_{L^2}^2. 
\]
(3) **Case of** $|\beta| \geq r_0^\delta$. We pick up $r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2})$ and $r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta)$ such that the following inequality holds

$$\frac{|w'(r_-)|^2}{r_-^2} + \frac{|w'(r_+)|^2}{r_+^2} \leq \frac{4}{\delta} \frac{|w'|}{r} \|w\|_{L^2},$$

Here $\delta \in (0, r_0)$ is a constant to be determined later. As in Proposition 2.6 the $H^{-1}$ estimate requires the cutoff function $\rho(r)$ is of $C^1$ regularity. Let us introduce

$$\rho(r) = \begin{cases} 1, & r \in (0, r_0 - 2\delta), \\ \sin \left( \frac{\pi}{2} \frac{r_0 - (r_0 - 2\delta)}{(r_- - r)} \right), & r \in (r_0 - 2\delta, r_-), \\ 0, & r \in (r_-, r_+), \\ \sin \left( \frac{\pi}{2} \frac{r_0 + 2\delta - r_+}{(r_+ - r)} \right), & r \in (r_+, r_0 + 2\delta), \\ -1, & r \geq r_0 + 2\delta. \end{cases}$$

Write

$$\Im(\mathcal{F} - c_2 |\beta|^\frac{3}{2} w, \frac{w\rho(r)}{r^2}) = \Im(-|\beta|^2 w - (\frac{k^2}{r^2} - \frac{1}{2}) w + i\beta_k \frac{1}{r^2} - \lambda) w, \frac{w\rho(r)}{r^2}).$$

This gives

$$|\beta_k| (\int_0^{r_0 - 2\delta} \frac{1}{r^2} - \frac{1}{r_0^2} \frac{|w(r)|^2}{r^2} dr + \int_{r_0 + 2\delta}^\infty \frac{1}{r_0^2} - \frac{1}{r^2} |w(r)|^2 dr)$$

$$\lesssim \frac{\mathcal{F} - c_2 |\beta|^\frac{3}{2} w}{r} \|w\|_{H^{-1}} + \|\rho w\|_{H^1} + \delta^{-1} \|w'\|_{L^2} \|w\|_{L^2} + \|\frac{w'}{r}\|_{L^2} \|\frac{w}{r}\|_{L^2}.$$

With $r_- \in (r_0 - \delta, r_0 - \frac{\delta}{2})$ and $r_+ \in (r_0 + \frac{\delta}{2}, r_0 + \delta)$, we have

$$\frac{1}{r_0^2} - \frac{1}{r_0^2} = \frac{(r_0 - r)(r_0 + r)}{r_0^2 r_0} \geq \frac{r_0 - r}{r_0^2 r_0} \geq \frac{\delta}{r_0^3}, \text{ for any } r \in (0, r_-),$$

Analogously, we obtain

$$\frac{1}{r_0^2} - \frac{1}{r_0^2} = \frac{(r - r_0)(r + r_0)}{r_0^2 r_0} \geq \frac{r - r_0}{r_0^2 r_0} \geq \frac{\delta}{r_0^3}, \text{ for any } r \in (r_+, 2r_0),$$

$$\frac{1}{r_0^2} - \frac{1}{r_0^2} = \frac{(r - r_0)(r + r_0)}{r_0^2 r_0} \approx \frac{1}{r_0^2} \geq \frac{\delta}{r_0^3}, \text{ for any } r \geq 2r_0.$$
together with (2.39), we obtain

\[
\frac{F - c_2\beta_k \frac{1}{3} w}{r} \|H^{-1}\|H^1 + \delta^{-1} \|w\|L^2 + \|w\|L^2 + \|w\|L^2 + \|w\|L^2
\]

\[
\lesssim \|F - c_2\beta_k \frac{1}{3} w\|H^{-1}(\|w\|H^1 + \delta^{-1} \|w\|L^2) + \delta^{-1} \|w\|H^1 \|w\|L^2 + \|w\|^2_3
\]

\[
\lesssim \|F - c_2\beta_k \frac{1}{3} w\|H^{-1}(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \sqrt{c_2} \beta_k \frac{1}{3} \|w\|L^2 + \delta^{-1} \|w\|L^2)
\]

\[
+ \delta^{-1}(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \sqrt{c_2} \beta_k \frac{1}{3} \|w\|L^2) \|w\|L^2
\]

\[
+ \|F - c_2\beta_k \frac{1}{3} w\|^2_2 \|H^{-1} + c_2\beta_k \frac{1}{3} \|w\|^2_2 \|L^2 \| + \delta \|w\|^2_\infty
\]

\[
\lesssim \|F - c_2\beta_k \frac{1}{3} w\|H^{-1}(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \sqrt{c_2} \beta_k \frac{1}{3} \|w\|L^2 + \delta^{-1} \|w\|L^2)
\]

\[
+ \delta^{-1}(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \sqrt{c_2} \beta_k \frac{1}{3} \|w\|L^2) \|w\|L^2
\]

\[
+ \|F - c_2\beta_k \frac{1}{3} w\|^2_2 \|H^{-1} + c_2\beta_k \frac{1}{3} \|w\|^2_2 \|L^2 \| + \delta \|w\|^2_\infty \|w\|L^2
\]

\[
\lesssim \|F - c_2\beta_k \frac{1}{3} w\|H^{-1}(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \delta^{-1} \|w\|L^2) + \sqrt{c_2} \beta_0 \|w\|L^2
\]

\[
+ c_2 |\beta_k| \frac{1}{3} \|w\|^2_2 \|L^2\| + \delta(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \sqrt{c_2} \beta_k \frac{1}{3} \|w\|^2_2 \|L^2\|)
\]

\[
\leq C\left\{ \frac{r_0^3}{|\beta_k|^4} \|F - c_2\beta_k \frac{1}{3} w\|H^{-1}(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \delta^{-1} \|w\|L^2) + \sqrt{c_2} \beta_0 \|w\|^2_2 \|L^2\| + c_2 |\beta_k| \frac{1}{3} \|w\|^2_2 \|L^2\| + \delta(\|F - c_2\beta_k \frac{1}{3} w\|H^{-1} + \sqrt{c_2} \beta_k \frac{1}{3} \|w\|^2_2 \|L^2\|) \right\}
\]

Take \( \delta = \left( \frac{r_0^3}{|\beta_k|^4} \right)^{\frac{1}{3}} \), then the condition \(|\beta_k| \geq \max\{r_0^3, 1\} \) yields

\[
\delta = \left( \frac{r_0^3}{|\beta_k|^4} \right)^{\frac{1}{3}} \leq r_0.
\]

Recall \(|\beta_k| \geq 1\), we have

\[
C(c_2 \frac{r_0^3}{|\beta_k|^4} + \sqrt{c_2} \frac{r_0^3}{|\beta_k|^4} + \sqrt{c_2} \beta_0 \frac{1}{3}) = C(c_2 \frac{r_0^2}{|\beta_k|^2} + 2\sqrt{c_2} \frac{r_0}{|\beta_k|^2}) \leq C(c_2 + 2\sqrt{c_2}).
\]
Lemma 2.11. There exist
Proof.

\begin{align*}
|F - c_2| \beta_k |^{\frac{1}{3}} w | r & \leq \min \{ |\beta_k|^{\delta} (\frac{|\beta_k|^{\delta^2}}{r_0^{2}}), \delta^{-2} \} \|w\|_{L^2}^2. \\
\end{align*}

With the basic equality $|\beta_k|^{\delta} (\frac{|\beta_k|^{\delta^2}}{r_0^{2}}) = \delta^{-2}$, we arrive at
\begin{equation}
(2.45) \|F - c_2| \beta_k |^{\frac{1}{3}} w | r \|_{H^{-1}}^2 \geq |\beta_k|^{\frac{1}{3}} \|w\|_{L^2}^2.
\end{equation}

Combining (2.30), (2.41), (2.43), (2.44) and (2.45), we therefore establish the following resolvent estimate from $L^2(\frac{1}{r^2})$ to $H^{-1}(\frac{1}{r^2})$
\begin{align*}
|\beta_k|^{\frac{1}{3}} \|w\|_{L^2} \lesssim \|F - c_2| \beta_k |^{\frac{1}{3}} w | r \|_{H^{-1}}.
\end{align*}

Applying (2.39), we obtain
\begin{align*}
\|w\|_{H^1} \lesssim \| F - c_2| \beta_k |^{\frac{1}{3}} w | r \|_{H^{-1}} + \sqrt{c_2} |\beta_k|^{\frac{1}{3}} \|w\|_{L^2} \lesssim \| F - c_2| \beta_k |^{\frac{1}{3}} w | r \|_{H^{-1}}.
\end{align*}

This completes the proof of Proposition 2.9.

The following proposition follows directly from Proposition 2.8 and Proposition 2.9 if we choose $c_2 > 0$ small enough.

Proposition 2.10. For any $|k| \geq 1$, $\lambda \in \mathbb{R}$ and $w \in D_k$, there exist constants $C, c_2 > 0$ which independent with $k, \beta_k, \lambda$, such that
\begin{align*}
|\beta_k|^{\frac{1}{3}} \|w\|_{L^2} + |\beta_k|^{\frac{1}{3}} \|w\|_{L^2} \leq C \| F - c_2| \beta_k |^{\frac{1}{3}} w | r \|_{L^2},
\end{align*}
and
\begin{align*}
\|w\|_{H^1} + |\beta_k|^{\frac{1}{3}} \|w\|_{L^2} \leq C \| F - c_2| \beta_k |^{\frac{1}{3}} w | r \|_{H^{-1}}.
\end{align*}

2.4. Sharpness of the resolvent bound $|\beta_k|^{\frac{1}{3}}$. In the last subsection, we prove that the resolvent bound $|\beta_k|^{\frac{1}{3}}$ in Proposition 2.9 is sharp. We have

Lemma 2.11. There exist $\lambda \in \mathbb{R}$ and non-zero $w \in C^0_{\infty}(\mathbb{R}, R^3)$ such that it holds
\begin{align*}
\|F\|_{L^2} \leq C|\beta_1|^{\frac{1}{3}} \|w\|_{L^2}.
\end{align*}

Proof. Choose $\lambda = \frac{1}{r_0^6}$ and $\beta_1 = r_0^6$ for some $r_0 \geq 1$. We pick up a function $\eta(r)$ as below
\begin{align*}
\eta(r) = \begin{cases} 
1(r - 1), & 0 \leq r \leq 1, \\
0, & r \geq 1.
\end{cases}
\end{align*}

We construct
\begin{align*}
w(r) = \begin{cases} 
(r - r_0)(r_0 + 1 \frac{1}{r_0} - r), & r_0 \leq r \leq r_0 + \frac{1}{r_0}, \\
0, & r \in (0, r_0) \cup (r_0 + \frac{1}{r_0}, \infty).
\end{cases}
\end{align*}
One can verify
\[ \|w\|_{L^2} = r_0^{-\frac{3}{4}} \|\eta\|_{L^2}, \quad w'' = -2. \]

Then it holds
\[ \| -[\partial_r^2 - (3 \frac{1}{4} r^2 + \frac{r^2}{16} - \frac{1}{2})]w\|_{L^2} \lesssim (1 + r_0^2)\|w\|_{L^2} \lesssim r_0^2\|w\|_{L^2}. \]

In addition, we have
\[ \|\beta_1\| (\frac{1}{r_0^2} - \frac{1}{r^2})w\|_{L^2} = |\beta_1| \| (r - r_0)(r + r_0) \|_{L^2} \leq |\beta_1| \| r + r_0 \|_{L^2} \lesssim |\beta_1| \|w\|_{L^2}. \]

Thus we deduce
\[ \| -[\partial_r^2 - (3 \frac{1}{4} r^2 + \frac{r^2}{16} - \frac{1}{2})]w + i\beta_1 (\frac{1}{r^2} - \frac{1}{r_0^2})\|_{L^2} \lesssim (r_0^2 + |\beta_1|)\|w\|_{L^2}. \]

Together with \(|\beta_1| = r_0^6\) this implies
\[ \| -[\partial_r^2 - (3 \frac{1}{4} r^2 + \frac{r^2}{16} - \frac{1}{2})]w + i\beta_1 (\frac{1}{r^2} - \frac{1}{r_0^2})\|_{L^2} \lesssim r_0^2\|w\|_{L^2} = |\beta_1|\|w\|_{L^2}. \]

\[ \square \]

3. Space-time estimate for the linearized Navier-Stokes equations

In this section, for the linearized 2D Navier-Stokes equation in the vorticity formulation \([1,27]\), we establish the space-time estimate. We consider
\[
\begin{cases}
\partial_r w_k + L_k w_k + f_1 - \partial_r f_2 = 0, \\
w_k(0) = w_k|_{r=0}, \quad w_k|_{r=\infty} = 0.
\end{cases}
\]

Here \(L_k = -[\partial_r^2 - (\frac{k^2}{r^2} - \frac{1}{r^2} + \frac{r^2}{16} - \frac{1}{2})] + ikB\) and \(f_1, f_2\) are nonlinear terms in Section 1.4.

We also introduce the space-time norm
\[ \|g\|_{L^p L^2} = \|\|g\|_{L^2}\|_{L^p_t(0,\infty)}, \quad \|g\|_{L^p X} = \|\|g\|_X\|_{L^p_t(0,\infty)}, \]

where
\[ \|g\|_{L^2_r} = \left( \int_0^\infty |g(r)|^2 dr \right)^{\frac{1}{2}}, \quad \|g\|_X = \left( \int_0^\infty \frac{|g(r)|^2}{r^2} dr \right)^{\frac{1}{2}}. \]

3.1. Pseudospectral bound and semigroup bound.

3.1.1. Pseudospectral bound. Recall in [31] an operator \(L\) in a Hilbert space \(H\) is called accretive if
\[ \Re\langle Lf, f\rangle \geq 0, \quad \text{for any } f \in D(L). \]

The operator \(L\) is called \(m\)-accretive if in addition all \(\lambda > 0\) belong to the resolvent set of \(L\) (see [27] for more details). The pseudospectral bound of \(L\) is defined to be
\[ \Psi(L) = \inf\{\| (L - i\lambda)f \| : f \in D(L), \lambda \in \mathbb{R}, \|f\| = 1\}. \]

Recall the operator \(L_k\) in \([1,27]\) is given by
\[ L_k w = -[\partial_r^2 - (\frac{k^2}{r^2} - \frac{1}{r^2} + \frac{r^2}{16} - \frac{1}{2})]w + ikB \frac{1}{r^2} w. \]

And we will use the weighted-\(L^2\) space \(X\). Its norm is expressed as \(\|w\|_X = (\int_0^\infty \frac{|w|^2}{r^2} dr)^{\frac{1}{2}}\).

Note that \(-\partial_r^2\) is an operator with the compact resolvent. Since \(L_k\) is a relatively compact
perturbation for $-\partial_x^2$ in the domain $D_k$, it is hence clear that the operators $L_k$ has the compact resolvent and only point spectrum.

Employing Lemma 2.1 and Lemma 2.3 we obtain

$$\Re((L_k - i\lambda)w, w)_{L^2} \geq |\lambda||w||_X^2, \quad \Re((L_k - i\lambda)w, w)_X \geq |\lambda||w||_X^2,$$

for any $\lambda \in \mathbb{R}$.

This indicates that $L_k$ is accretive, and is also m-accretive. Recall in Proposition 2.8 and Proposition 3.2 we establish the resolvent estimates

$$\|(L_k - i\lambda)w\|_{L^2} \geq C|kB|^\frac{1}{2}||w||_{L^2}, \quad \|(L_k - i\lambda)w\|_X \geq C|kB|^\frac{1}{2}||w||_X.$$

This implies the following lemma for the pseudospectral bounds of $L_k$:

**Lemma 3.1.** Let $\Psi$ be defined as in (3.3), it holds $\Psi(L_k(L^2 \to L^2)) \geq C|kB|^\frac{1}{2}, \Psi(L_k(X \to X)) \geq C|kB|^\frac{1}{2}$.

3.1.2. Semigroup bounds. To prove decay estimates, we appeal to semigroup bounds. To obtain them from pseudospectral bounds, we employ the following Gearhart-Prüss type lemma established by Wei in [39].

**Lemma 3.2** (Wei [39]). Let $L$ be a m-accretive operator in a Hilbert space $H$. Then we have

$$||e^{-tL}||_H \leq e^{-\tau q_1 + \frac{3}{2}}, \quad \text{for any } t \geq 0.$$

We are now ready to study the homogeneous linear equation

$$\partial_{\tau}w'_k + L_kw'_k = 0, \quad w'_k(0) = w_k(0).$$

Utilizing the language of semigroup theory, we can write

$$w'_k(\tau) = e^{-\tau L_k}w_k(0).$$

And we obtain

**Proposition 3.3.** Let $w'_k$ be a solution of (3.2) with $w_k(0) \in L^2$ and $w_k(0) \in X$. Then for any $|k| \geq 1$ ($k \in \mathbb{Z}$), there exist constants $C, c_3 > 0$ being independent of $B,k$, such that the following inequalities hold

$$||w'_k(\tau)||_{L^2} \leq Ce^{-c_3|kB|^\frac{1}{2}\tau}||w_k(0)||_{L^2}, \quad \text{for any } \tau \geq 0,$$

(3.4)

$$||w'_k(\tau)||_X \leq Ce^{-c_3|kB|^\frac{1}{2}\tau}||w_k(0)||_X, \quad \text{for any } \tau \geq 0.$$

Moreover, for any $|k| \geq 1$ ($k \in \mathbb{Z}$) and any $c' \in (0, c_3)$, it holds

$$||kB|^\frac{1}{2}||e^{-c'|kB|^\frac{1}{2}\tau}w'_k(\tau)||_{L^2L^2} \leq C||w_k(0)||^2_{L^2},$$

(3.5)

$$||kB|^\frac{1}{2}||e^{-c'|kB|^\frac{1}{2}\tau}w'_k(\tau)||_X^2 \leq C||w_k(0)||^2_{X}.$$

**Proof.** To derive the semigroup bounds (3.3) and (3.4), we can directly apply Lemma 3.1 and Lemma 3.2. We proceed to use (3.3). For any $c' \in (0, c_3)$ we deduce

$$2c'||kB|^\frac{1}{2}||e^{c'|kB|^\frac{1}{2}\tau}w'_k(\tau)||_{L^2}^2 \leq 2Cc'||kB|^\frac{1}{2}e^{-2(c_3 - c')|kB|^\frac{1}{2}\tau}||w_k(0)||^2_{L^2}.$$ Integrating with respect to $\tau$ on both sides yields

$$2c'||kB|^\frac{1}{2}||e^{c'|kB|^\frac{1}{2}\tau}w'_k(\tau)||_{L^2}^2 \leq \int_0^\infty 2Cc'||kB|^\frac{1}{2}e^{-2(c_3 - c')|kB|^\frac{1}{2}\tau}d\tau||w_k(0)||^2_{L^2} \lesssim ||w_k(0)||^2_{L^2}.$$ In a similar fashion, by (3.3), for any $c' \in (0, c_3)$ we have

$$2c'||kB|^\frac{1}{2}||e^{c'|kB|^\frac{1}{2}\tau}w'_k(\tau)||_{X}^2 \leq 2Cc'||kB|^\frac{1}{2}e^{-2(c_3 - c')|kB|^\frac{1}{2}\tau}||w_k(0)||^2_{X},$$
which implies
\[ 2c'|kB|^{\frac{1}{3}}\|e^{c'|kB|^{\frac{1}{3}}\tau}w_k^l\|_{L^2}^2 \leq \int_0^\infty 2CC'|kB|^{\frac{1}{3}}e^{-2(c_2-c')}|kB|^{\frac{4}{3}}\tau\|w_k(0)\|^2_X \lesssim \|w_k(0)\|^2_X. \]

This completes the proof of Proposition 3.3. □

3.2. **Space-time estimates for non-zero frequency.** Recall the full nonlinear equation (1.27) as follows
\[
\begin{aligned}
\partial_\tau w_k + L_k w_k + f_1 - \partial_\tau f_2 &= 0, \\
w_k(0) &= w_k|_{\tau=0}, \quad w_k|_{\tau=0,\infty} = 0,
\end{aligned}
\]
where \(L_k = -[\partial_\tau^2 - \left(\frac{k^2-\frac{1}{4}}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right)] + ik\frac{r}{r^2}\) and
\[
f_1 = ik \sum_{l \in \mathbb{Z}} \frac{\partial_\tau \hat{w}_k}{r} + \sum_{l \in \mathbb{Z}} i(k-l)\left(\frac{1}{4} - \frac{1}{2r^2}\right)w_{l} \hat{w}_{k-l}, \quad f_2 = \sum_{l \in \mathbb{Z}} i(k-l)\frac{w_{l} \hat{w}_{k-l}}{r},
\]
with \((\partial_\tau^2 + \frac{r}{r^2}\partial_\tau - \frac{k^2}{r^2})\hat{w}_k = f w_k, \quad f = \frac{\hat{\omega}^2}{r^2}.\)

We decompose the solution \(w_k\) into two parts: let \(w_k = w_k^l + w_k^n\) so that \(w_k^l\) satisfies the below homogeneous linear equation with initial data \(w_k(0)\)
\[ \partial_\tau w_k^l + L_k w_k^l = 0, \quad w_k^l(0) = w_k(0), \]
and \(w_k^n\) satisfies the following inhomogeneous linear equation with zero initial data
\[ \partial_\tau w_k^n + L_k w_k^n + f_1 - \partial_\tau f_2 = 0, \quad w_k^n(0) = 0. \]

We first establish the following space-time estimate for the linear part \(w_k^l\) in \(L^2\) space.

**Lemma 3.4.** Let \(w_k^l\) be a solution to (3.2) with initial data \(w_k(0) \in L^2\), \(c_3\) be the same as in Proposition 3.3 and \(c' \in (0, c_3)\). For any \(|k| \geq 1(k \in \mathbb{Z})\), it holds
\[
(3.8) \quad \|e^{c'|kB|^{\frac{1}{3}}\tau}w_k^l\|_{L^\infty L^2} + |kB|^{\frac{1}{3}}\|e^{c'|kB|^{\frac{1}{3}}\tau}w_k^l\|_{L^2}^2
+ \|e^{c'|kB|^{\frac{1}{3}}\tau}\partial_\tau w_k^l\|_{L^2}^2 + \|e^{c'|kB|^{\frac{1}{3}}\tau}(\frac{|k|}{r} + r)w_k^l\|_{L^2}^2 \lesssim \|w_k(0)\|_{L^2}^2.
\]

**Proof.** Employing integration by parts we obtain
\[
\begin{aligned}
\Re\langle \partial_\tau w_k^l - [\partial_\tau^2 - \left(\frac{k^2-\frac{1}{4}}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right)]w_k^l + ik\frac{r}{r^2}w_k^l, w_k^l \rangle
= \frac{1}{2}\partial_\tau \|w_k^l\|_{L^2}^2 + (-[\partial_\tau^2 - \left(\frac{k^2-\frac{1}{4}}{r^2} + \frac{r^2}{16} - \frac{1}{2}\right)]w_k^l, w_k^l) = 0.
\end{aligned}
\]
Together with Lemma 2.1, this gives that there exists a constant \(c_0 > 0\) such that
\[ \partial_\tau \|w_k^l\|_{L^2}^2 + c_0(\|\partial_\tau w_k^l\|_{L^2}^2 + \|\frac{|k|}{r} + r\|w_k^l\|_{L^2}^2) \leq 0. \]

Multiplying \(e^{2c'|kB|^{\frac{1}{3}}\tau}\) on both sides, we deduce
\[ \partial_\tau \|e^{c'|kB|^{\frac{1}{3}}\tau}w_k^l\|_{L^2}^2 + c_0(\|e^{c'|kB|^{\frac{1}{3}}\tau}\partial_\tau w_k^l\|_{L^2}^2 + \|e^{c'|kB|^{\frac{1}{3}}\tau}(\frac{|k|}{r} + r)w_k^l\|_{L^2}^2)
\leq 2c'|kB|^{\frac{1}{3}}\|e^{c'|kB|^{\frac{1}{3}}\tau}w_k^l\|_{L^2}^2. \]
Thus we obtain the space-time estimate of $w_k^l$ as follows
\[
\| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^\infty_t L^2} + c_0 \left( \| e^{c'|kB|^{\frac{2}{3}} r} \partial_r w_k^l \|_{L^2_t L^2} + \| e^{c'|kB|^{\frac{2}{3}} r} \left( \frac{|k|}{r} + r \right) w_k^l \|_{L^2_t L^2} \right) \\
\leq 2c'|kB|^{\frac{2}{3}} \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^\infty_t L^2} + \| w_k(0) \|_{L^2}.
\]
Combining with Proposition 3.3, we arrive at
\[
\| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^\infty_t L^2} + |kB|^{\frac{2}{3}} \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^2_t L^2} \\
+ \| e^{c'|kB|^{\frac{2}{3}} r} \partial_r w_k^l \|_{L^2_t L^2} + \| e^{c'|kB|^{\frac{2}{3}} r} \left( \frac{|k|}{r} + r \right) w_k^l \|_{L^2_t L^2} \lesssim \| w_k(0) \|_{L^2}.
\]
This completes the proof of Lemma 3.4. □

Analogously, we derive the space-time estimate for the linear part $w_k^l$ in weighted $L^2$ space.

**Lemma 3.5.** Let $w_k^l$ be a solution to (3.2) with initial data $w_k(0) \in X$, $c_3$ be the same as in Proposition 3.3 and $c' \in (0, c_3)$. For any $|k| \geq 1 (k \in \mathbb{Z})$, it holds
\[
(3.9) \quad \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^\infty_t L^2} + |kB|^{\frac{2}{3}} \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^2_t L^2} \\
+ \| e^{c'|kB|^{\frac{2}{3}} r} \partial_r w_k^l \|_{L^2_t L^2} + \| e^{c'|kB|^{\frac{2}{3}} r} \left( \frac{|k|}{r} + r \right) w_k^l \|_{L^2_t L^2} \lesssim \| w_k(0) \|_{L^2}.
\]

**Proof.** Via integration by parts we have
\[
\Re \langle \partial_r w_k^l - \left( \frac{k^2}{r^2} - \frac{1}{4} \right) \partial_r w_k^l + \frac{ikB}{r^2} w_k^l, u_k^l \rangle \\
= \frac{1}{2} \partial_r \| w_k^l \|_{L^2}^2 + \left( \frac{k^2}{r^2} - \frac{1}{4} \right) \partial_r \| w_k^l \|_{L^2}^2 = 0.
\]
Together with Lemma 2.3, this yields that there exists a constant $c_0 > 0$ such that
\[
\partial_r \| w_k^l \|_{L^2}^2 + c_0 \left( \| \partial_r w_k^l \|_{L^2}^2 + \| (\frac{|k|}{r} + r) w_k^l \|_{L^2}^2 \right) \leq 0.
\]
Multiplying $e^{c'|kB|^{\frac{2}{3}} r}$ on both sides, the inequality above becomes
\[
\partial_r \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^2}^2 + c_0 \left( \| e^{c'|kB|^{\frac{2}{3}} r} \partial_r w_k^l \|_{L^2}^2 + \| e^{c'|kB|^{\frac{2}{3}} r} \left( \frac{|k|}{r} + r \right) w_k^l \|_{L^2}^2 \right) \\
\leq 2c'|kB|^{\frac{2}{3}} \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^2}^2.
\]
This implies the space-time estimate of $w_k^l$ as below
\[
\| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^\infty_t L^2} + c_0 \left( \| e^{c'|kB|^{\frac{2}{3}} r} \partial_r w_k^l \|_{L^2_t L^2} + \| e^{c'|kB|^{\frac{2}{3}} r} \left( \frac{|k|}{r} + r \right) w_k^l \|_{L^2_t L^2} \right) \\
\leq 2c'|kB|^{\frac{2}{3}} \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^\infty_t L^2} + \| w_k(0) \|_{L^2}.
\]
Plugging in Proposition 3.3, we conclude
\[
\| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^\infty_t L^2} + |kB|^{\frac{2}{3}} \| e^{c'|kB|^{\frac{2}{3}} r} w_k^l \|_{L^2_t L^2} \\
+ \| e^{c'|kB|^{\frac{2}{3}} r} \partial_r w_k^l \|_{L^2_t L^2} + \| e^{c'|kB|^{\frac{2}{3}} r} \left( \frac{|k|}{r} + r \right) w_k^l \|_{L^2_t L^2} \lesssim \| w_k(0) \|_{L^2}.
\]
This completes the proof of Lemma 3.5. □
Next we establish the space-time estimates for \( w^n_k \), which satisfies the following inhomogeneous equation (3.7) with zero initial data

\[
\begin{align*}
\partial_t w^n_k + L_k w^n_k + f_1 - \partial_r f_2 &= 0, \quad w^n_k(0) = 0.
\end{align*}
\]

Multiply both sides by \( e^{c_2|kB|^\frac{3}{4} \tau} \), with \( c_2 \) being the same constant as in Proposition 2.7 and in Proposition 2.10. The above inhomogeneous equation then becomes

\[
\begin{align*}
\partial_r (e^{c_2|kB|^\frac{3}{4} \tau} w^n_k) + \left( L_k - c_2|kB|^\frac{1}{2} \right) (e^{c_2|kB|^\frac{3}{4} \tau} w^n_k) + e^{c_2|kB|^\frac{3}{4} \tau} (f_1 - \partial_r f_2) &= 0, \quad w^n_k(0) = 0.
\end{align*}
\]

Define \( \tilde{w}^n_k = e^{c_2|kB|^\frac{3}{4} \tau} w^n_k \) and \( \tilde{f}_j = e^{c_2|kB|^\frac{3}{4} \tau} f_j \) for \( j = 1, 2 \). We then derive the following space-time estimate for the nonlinear part \( w^n_k \) in \( L^2 \) space.

**Lemma 3.6.** Let \( w^n_k \) be a solution to (3.7) with zero initial data. For any \( |k| \geq 1 (k \in \mathbb{Z}) \), it holds

\[
\begin{align*}
&\|e^{c_2|kB|^\frac{3}{4} \tau} w^n_k\|_{L^\infty_t L^2_x} + |kB|^\frac{1}{2} \|e^{c_2|kB|^\frac{3}{4} \tau} w^n_k\|_{L^2_t L^2_x} + \|e^{c_2|kB|^\frac{3}{4} \tau} \partial_r w^n_k\|_{L^2_t L^2_x} \\
&+ \|e^{c_2|kB|^\frac{3}{4} \tau} (|k|/r + r) \tilde{w}^n_k\|_{L^2_t L^2_x} \lesssim |kB|^{-\frac{1}{4}} \|e^{c_2|kB|^\frac{3}{4} \tau} f_1\|_{L^2_t L^2_x} + \|e^{c_2|kB|^\frac{3}{4} \tau} f_2\|_{L^2_t L^2_x}.
\end{align*}
\]

**Proof.** We first take the Fourier transform in \( \tau \) and define

\[
\tilde{w}^n_k(\lambda, r) = \int_0^\infty \tilde{w}^n_k(\tau, r)e^{-i\tau \lambda} d\tau,
\]

\[
F_j(\lambda, r, k) = \int_0^\infty \tilde{f}_j(\tau, r, k)e^{-i\tau \lambda} d\tau \quad \text{for } j = 1, 2.
\]

The inhomogeneous equation (3.7) is transferred into the following resolvent equation

\[
(i\lambda + L_k - c_2|kB|^\frac{1}{2}) \tilde{w}^n_k(\lambda, r) = -F_1 + \partial_r F_2.
\]

We further decompose \( \tilde{w}^n_k \) as \( \tilde{w}^n_k = \tilde{w}^{n,1}_k + \tilde{w}^{n,2}_k \), where \( \tilde{w}^{n,1}_k \) and \( \tilde{w}^{n,2}_k \) satisfy

\[
(i\lambda + L_k - c_2|kB|^\frac{1}{2}) \tilde{w}^{n,1}_k = -F_1 \quad \text{and} \quad (i\lambda + L_k - c_2|kB|^\frac{1}{2}) \tilde{w}^{n,2}_k = \partial_r F_2.
\]

respectively.

Applying Proposition 2.7, we obtain

\[
|kB|^{-\frac{1}{4}} \|\partial_r \tilde{w}^{n,1}_k\|_{L^2_x} + |kB|^{\frac{1}{4}} \|\tilde{w}^{n,1}_k\|_{L^2_x} \leq C \|F_1\|_{L^2_x},
\]

and

\[
\|\partial_r \tilde{w}^{n,2}_k\|_{L^2_x} + |kB|^{\frac{1}{8}} \|\tilde{w}^{n,2}_k\|_{L^2_x} \leq C \|\partial_r F_2\|_{H^{-1}} \leq C \|F_2\|_{L^2_x}.
\]

Combining the above inequalities, we deduce

\[
\begin{align*}
&|kB|^{-\frac{1}{4}} \|\partial_r \tilde{w}^n_k\|_{L^2_x} + \|\tilde{w}^n_k\|_{L^2_x} \lesssim |kB|^{-\frac{1}{4}} \|F_1\|_{L^2_x} + |kB|^{-\frac{1}{4}} \|F_2\|_{L^2_x}.
\end{align*}
\]

Therefore, using Plancherel’s theorem, it holds

\[
\begin{align*}
&|kB|^{-\frac{1}{4}} \|\partial_r \tilde{w}^n_k\|_{L^2_t L^2_x} + \|\tilde{w}^n_k\|_{L^2_t L^2_x} \\
&\lesssim |kB|^{-\frac{1}{4}} \|\partial_r F_1\|_{L^2_t L^2_x} + |kB|^{-\frac{1}{4}} \|F_2\|_{L^2_t L^2_x}.
\end{align*}
\]

\[
(3.11)
\]

\[\text{Note that the zero initial condition in (3.10) ensures that the Fourier transform is well-defined for } \lambda \in (0, \infty).\]
Via integration by parts, we also have
\[
\Re(\partial_r w^n_k - (\frac{k^2}{r^2} - \frac{1}{8}) + \frac{r^2}{16} - \frac{1}{2})w^n_k + \frac{\partial r}{\partial r} w^n_k + f_1 - \partial_r f_2, e^{2\beta_2 |k| B^\frac{1}{2} L^\tau w^n_k})
\]
\[
= \frac{1}{2} \partial_r \| \tilde{w}_k^n \|^2_{L^2} + \langle \partial_r^2 - (\frac{k^2 - \frac{1}{8}}{r^2} + \frac{r^2}{16} - \frac{1}{2}) \| \tilde{w}_k^n - c_2 |k| B^\frac{1}{2} \tilde{w}_k^n + \tilde{f}_1 - \partial_r \tilde{f}_2, w^n_k \rangle = 0.
\]
Together with Lemma 2.1, this yields that there exists a constant $c_0 > 0$ such that
\[
c_2 |k| B^\frac{1}{2} \| \tilde{w}_k^n \|^2_{L^2}
\]
\[
\geq \frac{1}{2} \partial_r \| \tilde{w}_k^n \|^2_{L^2} + c_0 (\| \partial_r \tilde{w}_k^n \|^2_{L^2} + \| (\frac{|k|}{r} + r) \tilde{w}_k^n \|^2_{L^2}) + \Re \langle \tilde{f}_1 - \partial_r \tilde{f}_2, \tilde{w}_k^n \rangle
\]
\[
= \frac{1}{2} \partial_r \| \tilde{w}_k^n \|^2_{L^2} + c_0 (\| \partial_r \tilde{w}_k^n \|^2_{L^2} + \| (\frac{|k|}{r} + r) \tilde{w}_k^n \|^2_{L^2}) + \Re \langle \tilde{f}_1, \tilde{w}_k^n \rangle + \Re \langle \tilde{f}_2, \partial_r \tilde{w}_k^n \rangle.
\]
By Cauchy-Schwarz inequality, we obtain
\[
\partial_r \| \tilde{w}_k^n \|^2_{L^2} + 2c_0 (\| \partial_r \tilde{w}_k^n \|^2_{L^2} + \| (\frac{|k|}{r} + r) \tilde{w}_k^n \|^2_{L^2})
\]
\[
\lesssim (\| \tilde{f}_1 \|^2_{L^2} + |k| B^\frac{1}{2} \| \tilde{f}_2 \|^2_{L^2}) (\| \tilde{w}_k^n \|^2_{L^2} + \| \partial_r \tilde{w}_k^n \|^2_{L^2}) + |k| B^\frac{1}{2} \| \tilde{w}_k^n \|^2_{L^2}.
\]
Combining with (3.11), we arrive at
\[
\| \tilde{w}_k^n \|^2_{L^2} + \| \partial_r \tilde{w}_k^n \|^2_{L^2} + \| (\frac{|k|}{r} + r) \tilde{w}_k^n \|^2_{L^2} + |k| B^\frac{1}{2} \| \tilde{w}_k^n \|^2_{L^2}
\]
\[
\lesssim (\| \tilde{f}_1 \|^2_{L^2} + |k| B^\frac{1}{2} \| \tilde{f}_2 \|^2_{L^2}) (\| \tilde{w}_k^n \|^2_{L^2} + \| \partial_r \tilde{w}_k^n \|^2_{L^2}) + |k| B^\frac{1}{2} \| \tilde{w}_k^n \|^2_{L^2}
\]
\[
\lesssim |k| B^\frac{1}{2} (\| \tilde{f}_1 \|^2_{L^2} + |k| B^\frac{1}{2} \| \tilde{f}_2 \|^2_{L^2}) \lesssim |k| B^\frac{1}{2} \| \tilde{f}_1 \|^2_{L^2} + \| \tilde{f}_2 \|^2_{L^2}.
\]
This completes the proof of Lemma 3.6. \qed

We then proceed in the same fashion and derive the space-time estimate for the nonlinear part $w^n_k$ in weighted $L^2$ space.

**Lemma 3.7.** Let $w^n_k$ be a solution to (3.7) with zero initial data. For any $|k| \geq 1 (k \in \mathbb{Z})$, it holds
\[
\| e^{2\beta_2 |k| B^\frac{1}{2} L^\tau \tilde{w}_k^n} \|^2_{L^\infty X} + |k| B^\frac{1}{2} \| e^{2\beta_2 |k| B^\frac{1}{2} L^\tau \tilde{w}_k^n} \|^2_{L^2 X}
\]
\[
+ \| e^{2\beta_2 |k| B^\frac{1}{2} \partial_r \tilde{w}_k^n} \|^2_{L^2 X} + \| e^{2\beta_2 |k| B^\frac{1}{2} L^\tau (\frac{|k|}{r} + r) \tilde{w}_k^n} \|^2_{L^2 X}
\]
\[
\lesssim |k| B^\frac{1}{2} (\| e^{2\beta_2 |k| B^\frac{1}{2} \partial_r \tilde{f}_1} \|^2_{L^2 X} + \| e^{2\beta_2 |k| B^\frac{1}{2} L^\tau \tilde{f}_2} \|^2_{L^2 X}) + \| e^{2\beta_2 |k| B^\frac{1}{2} L^\tau \tilde{f}_2} \|^2_{L^2 X},
\]
where $w^n_k$ satisfies the equation (3.7).

**Proof.** We still take the Fourier transform with respect to $t$ and get
\[
\hat{w}_k^n (\lambda, r) = \int_0^\infty \tilde{w}_k^n (\tau, r) e^{-ir\lambda} d\tau, \quad F_j (\lambda, r, k) = \int_0^\infty \hat{f}_j (\tau, r, k) e^{-ir\lambda} d\tau \quad \text{with } j = 1, 2.
\]
The inhomogeneous equation (3.7) then becomes
\[
(i\lambda + L_k - c_2 |k| B^\frac{1}{2}) \hat{w}_k^n (\lambda, r) = -F_1 + \partial_r F_2 = -F_1 + r \partial_r \left( \frac{F_2}{r} \right) + \frac{F_2}{r}.
\]
As in Lemma 3.6 we decompose $\hat{w}_k^n = \hat{w}_k^{n1} + \hat{w}_k^{n2}$, where $\hat{w}_k^{n1}$ and $\hat{w}_k^{n2}$ satisfy
\[
(i\lambda + L_k - c_2 |k| B^\frac{1}{2}) \hat{w}_k^{n1} = -F_1 + \frac{F_2}{r} \quad \text{and} \quad (i\lambda + L_k - c_2 |k| B^\frac{1}{2}) \hat{w}_k^{n2} = r \partial_r \left( \frac{F_2}{r} \right),
\]
respectively. It follows from Proposition 2.10 that
\[ |kB|^\frac{1}{r} \| \partial_r \tilde{w}_k^n \|_X + |kB|^\frac{1}{r} \| \tilde{w}_k^n \|_X \leq C \| F_1 - \frac{F_2}{r} \|_X, \]
and
\[ \| \partial_r \tilde{w}_k^n,2 \|_X + |kB|^\frac{1}{r} \| \tilde{w}_k^n,2 \|_X \leq C \| \partial_r (\frac{F_2}{r}) \|_{H^{-1}} \leq C \| \frac{F_2}{r} \|_{L^2} \leq C \| F_2 \|_X. \]
Hence we obtain
\[ |kB|^{-\frac{1}{r}} \| \partial_r \tilde{w}_k^n,2 \|_X + |kB|^{-\frac{1}{r}} \| \tilde{w}_k^n,2 \|_X \lesssim |kB|^{-\frac{1}{r}} \| F_1 - \frac{F_2}{r} \|_X + |kB|^{-\frac{1}{r}} \| F_2 \|_X. \]
Therefore, utilizing Plancherel's theorem we deduce
\[
|kB|^{-\frac{1}{r}} \| \partial_r \tilde{w}_k^n \|_{L^2_X} + \| \tilde{w}_k^n \|_{L^2_X} 
\approx |kB|^{-\frac{1}{r}} \| \partial_r \tilde{w}_k^n \|_{L^2(R)} + \| \tilde{w}_k^n \|_{L^2(R)} 
\lesssim |kB|^{-\frac{1}{r}} \| F_1 - \frac{F_2}{r} \|_{L^2(R)} + |kB|^{-\frac{1}{r}} \| F_2 \|_{L^2(R)} 
\approx |kB|^{-\frac{1}{r}} \| \tilde{f}_1 - \tilde{f}_2 \|_{L^2_X} + |kB|^{-\frac{1}{r}} \| \tilde{f}_2 \|_{L^2_X}. \]
(3.12)

In addition, integration by parts yields
\[
\Re \langle \partial_r w_k^n - (\frac{\partial^2}{r^2} - \frac{1}{r^2} + \frac{r^2}{16} + \frac{1}{2}) w_k^n + \frac{ib_k}{r} w_k^n, f_1 - \partial_r f_2 - \frac{c_{2|kB|^\frac{1}{r} w_k^n}}{r^2} \rangle 
= \frac{1}{2} \partial_r \| \tilde{w}_k^n \|_{L^2}^2 + \Re \langle k \frac{\tilde{f}_1 - \tilde{f}_2 + \partial_r \tilde{w}_k^n}{r^2} \rangle = 0. \]
Together with Lemma 2.3, this implies that there exists a constant \( c_0 > 0 \) such that
\[
c_2|kB|^\frac{1}{r} \| \tilde{w}_k^n \|_{X}^2 
\geq \frac{1}{2} \| \partial_r \tilde{w}_k^n \|_{X}^2 + c_0(\| \partial_r \tilde{w}_k^n \|_{X}^2 + \| (\frac{|k|}{r} + r) \tilde{w}_k^n \|_{X}^2) + \Re \langle (\tilde{f}_1 - \partial_r \tilde{f}_2, \tilde{w}_k^n) \rangle 
\geq \frac{1}{2} \| \partial_r \tilde{w}_k^n \|_{X}^2 + c_0(\| \partial_r \tilde{w}_k^n \|_{X}^2 + \| (\frac{|k|}{r} + r) \tilde{w}_k^n \|_{X}^2) + \Re \langle (\tilde{f}_1, \tilde{w}_k^n) \rangle - \Re \langle (\tilde{f}_2, \partial_r \tilde{w}_k^n - 2 \tilde{w}_k^n) \rangle. \]
Thus we obtain
\[
\| \partial_r \tilde{w}_k^n \|_{X}^2 + 2c_0(\| \partial_r \tilde{w}_k^n \|_{X}^2 + \| (\frac{|k|}{r} + r) \tilde{w}_k^n \|_{X}^2) \lesssim \| \tilde{f}_1 \|_X \| \tilde{w}_k^n \|_X + \| \tilde{f}_2 \|_X (\| \partial_r \tilde{w}_k^n \|_X + \| \frac{|k|}{r} \tilde{w}_k^n \|_X) + |kB|^\frac{1}{r} \| \tilde{w}_k^n \|_{X}^2. \]
With (3.12), we conclude
\[
|kB|^\frac{1}{r} \| \tilde{w}_k^n \|_{L^\infty_X} + |kB|^\frac{1}{r} \| \tilde{w}_k^n \|_{L^2_X} + \| \partial_r \tilde{w}_k^n \|_{L^2_X} + |kB|^\frac{1}{r} \| \tilde{w}_k^n \|_{L^2_X}^2 \lesssim |kB|^{-\frac{1}{r}} (\| \tilde{f}_1 \|_{L^2_X} + \| \frac{|f|}{r} \|_{L^2_X})^2 + \| \tilde{f}_2 \|_{L^2_X}^2. \]
This completes the proof of Lemma 3.11 \( \square \)

Now we can state the two key conclusions of this section. We call them the space-time estimates for equation (1.27).
Proposition 3.8. Assume $w_k$ is a solution to (1.27) with $w_k(0) \in L^2$ and $e^{c'|kB|^12\tau}f_1$, $e^{c'|kB|^12\tau}f_2 \in L^2L^2$ for some $c' > 0$, then there exist constants $C, c > 0$ independent of $B, k$ so that

\begin{equation}
\|e^{c'|kB|^12\tau}w_k\|_{L^\infty L^2} + |kB|^12\|e^{c'|kB|^12\tau}w_k\|_{L^2L^2} \\
+ \|e^{c'|kB|^12\tau}\partial_r w_k\|_{L^2L^2} + \|e^{c'|kB|^12\tau}(\frac{|k|}{r} + r)w_k\|_{L^2L^2} \\
\leq C\left[|kB|^{-12}(\|e^{c'|kB|^12\tau}f_1\|_{L^2L^2} + \|e^{c'|kB|^12\tau}f_2\|_{L^2L^2}) + \|w_k(0)\|_X\right].
\end{equation}

Proof. This result follows directly from our prepared Lemma 3.4 and Lemma 3.6 by taking $c = \min\{c', c_2\}$. \Box

Proposition 3.9. Assume $w_k$ is a solution to (1.27) with $w_k(0) \in X$ and $e^{c'|kB|^12\tau}f_1, e^{c'|kB|^12\tau}f_2$, $e^{c'|kB|^12\tau}\frac{f_k}{r} \in L^2X$ for some $c' > 0$, then there exist constants $C, c > 0$ independent of $B, k$ so that

\begin{equation}
\|e^{c'|kB|^12\tau}w_k\|_{L^\infty X} + |kB|^12\|e^{c'|kB|^12\tau}w_k\|_{L^2X} \\
+ \|e^{c'|kB|^12\tau}\partial_r w_k\|_{L^2X} + \|e^{c'|kB|^12\tau}(\frac{|k|}{r} + r)w_k\|_{L^2X} \\
\leq C\left[|kB|^{-12}(\|e^{c'|kB|^12\tau}f_1\|_{L^2X} + \|e^{c'|kB|^12\tau}f_2\|_{L^2X}) + \|w_k(0)\|_X\right].
\end{equation}

Proof. This result follows directly from Lemma 3.5 and Lemma 3.7 by taking $c = \min\{c', c_2\}$. \Box

3.3. Space-time estimates for zero frequency. Recall from (1.27) that $w_0$ satisfies

$$\partial_tw_0 - (\partial_r^2 + \frac{1}{4r^2} - \frac{r^2}{16} + \frac{1}{2})w_0 - \sum_{l \in \mathbb{Z}/\{0\}} il\left(\frac{1}{4} - \frac{1}{2r^2}\right)w_l\hat{\varphi}_{-l} + \sum_{l \in \mathbb{Z}/\{0\}} il\partial_r\left(\frac{w_l\hat{\varphi}_{-l}}{r}\right) = 0,$$

where $\hat{\varphi}_k$ verifies $(\partial_r^2 + \frac{1}{4}\partial_r - \frac{k^2}{r^2})\hat{\varphi}_k = f w_k$ with $f = \frac{r^2}{r^2}$. For simplicity, we denote the nonlinear terms as

$$N = - \sum_{l \in \mathbb{Z}/\{0\}} il\left(\frac{1}{4} - \frac{1}{2r^2}\right)w_l\hat{\varphi}_{-l} + \sum_{l \in \mathbb{Z}/\{0\}} il\partial_r\left(\frac{w_l\hat{\varphi}_{-l}}{r}\right).$$

And we obtain

Proposition 3.10. Let $w_0$ be the solution to (1.27) with initial data $\frac{w_0(0)}{r} \in L^2$, then $w_0(\tau)$ obeys

$$\frac{\|w_0\|_{L^\infty L^2}}{r} + \frac{\|\partial_r w_0\|_{L^2L^2}}{r} + \frac{\|w_0\|_{L^2L^2}}{r^2} + \|w_0\|_{L^2L^2}$$

$$\leq \sum_{l \in \mathbb{Z}/\{0\}} |l|\|w_l\hat{\varphi}_{-l}\|_{L^2L^2} + \sum_{l \in \mathbb{Z}/\{0\}} |l|\|w_l\hat{\varphi}_{-l}\|_{L^2L^2} + \frac{\|w_0(0)\|_X}{r^2}.$$

Proof. For any $C^2$ function $g$, one can check

$$-w_0'' = -g^{-1}\partial_r[r^2g^2\partial_r(r^{-2}g^{-1}w_0)] - \frac{2}{r}w_0' - g^{-1}[g'' + \left(\frac{2}{r}g\right)']w_0.$$
We will need a real-valued function \( g \) satisfying
\[
\frac{g''}{g} + \frac{2g'}{rg} = -\frac{3}{r^2}.
\]

The existence of \( g \) is guaranteed by Lemma A.1 in the Appendix (with \( A = 2, B = -3 \)).

Employing (2.9) and (3.15), we deduce
\[
0 = \langle \partial_r w_0 - [\partial_r^2 + \frac{1}{4r^2} - \frac{r^2}{16} + \frac{1}{2}]] w_0 + N, \frac{w_0}{r^2} \rangle
\]
\[
= \frac{1}{2} \partial_r \| \frac{w_0}{r} \|^2_{L^2} + \| rg \partial_r (r^{-2} g^{-1} w_0) \|^2_{L^2} - 3 \| \frac{w_0}{r^2} \|^2_{L^2} - \langle g^{-1} [g'' + (\frac{2}{r} g')] w_0, \frac{w_0}{r^2} \rangle
\]
\[
+ \langle \langle -\frac{r^2}{4r^2} + \frac{r^2}{16} - \frac{1}{2} \rangle \rangle w_0, \frac{w_0}{r^2} \rangle + \Re \langle N, \frac{w_0}{r^2} \rangle
\]
\[
= \frac{1}{2} \partial_r \| \frac{w_0}{r} \|^2_{L^2} + \| rg \partial_r (r^{-2} g^{-1} w_0) \|^2_{L^2} - \langle \langle \frac{r^2}{4r^2} + \frac{r^2}{16} - \frac{1}{2} \rangle \rangle w_0, \frac{w_0}{r^2} \rangle
\]
\[
+ \langle \langle \langle -\frac{r^2}{4r^2} + \frac{r^2}{16} - \frac{1}{2} \rangle \rangle w_0, \frac{w_0}{r^2} \rangle + \Re \langle N, \frac{w_0}{r^2} \rangle
\]
\[
= \frac{1}{2} \partial_r \| \frac{w_0}{r} \|^2_{L^2} + \| rg \partial_r (r^{-2} g^{-1} w_0) \|^2_{L^2} + \langle \langle \frac{r^2}{4r^2} + \frac{r^2}{16} - \frac{1}{2} \rangle \rangle w_0, \frac{w_0}{r^2} \rangle
\]
\[
- \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{1}{4} \langle 1 - \frac{1}{2r^2} \rangle w_l \tilde{\varphi}_{-l}, \frac{w_0}{r^2} \rangle - \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{w_l \tilde{\varphi}_{-l}}{r}, \partial_r \frac{w_0}{r^2} \rangle - \frac{2}{r^3} \frac{w_0}{r^2} \rangle.
\]

Note that there exists \( c_0 > 0 \) such that \( \frac{7}{4r^2} + \frac{r^2}{16} - \frac{1}{2} \geq c_0 \langle \frac{r^2}{r^2} + r^2 \rangle \), we hence obtain
\[
0 \geq \frac{1}{2} \partial_r \| \frac{w_0}{r} \|^2_{L^2} + \| rg \partial_r (r^{-2} g^{-1} w_0) \|^2_{L^2} + \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{w_l \tilde{\varphi}_{-l}}{r}, \partial_r \frac{w_0}{r^2} \rangle - \frac{2}{r^3} \frac{w_0}{r^2} \rangle.
\]

In addition, we also find
\[
0 = \langle \partial_r w_0 - [\partial_r^2 + \frac{1}{4r^2} - \frac{r^2}{16} + \frac{1}{2}] \rangle w_0, \frac{w_0}{r^2} \rangle
\]
\[
- \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{1}{4} \langle 1 - \frac{1}{2r^2} \rangle w_l \tilde{\varphi}_{-l}, \frac{w_0}{r^2} \rangle - \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{w_l \tilde{\varphi}_{-l}}{r}, \partial_r \frac{w_0}{r^2} \rangle - \frac{2}{r^3} \frac{w_0}{r^2} \rangle
\]
\[
= \frac{1}{2} \partial_r \| \frac{w_0}{r} \|^2_{L^2} + \| \partial_r \frac{w_0}{r} \|^2_{L^2} - \frac{13}{4} \| \frac{w_0}{r^2} \|^2_{L^2} + \frac{1}{16} \| w_0 \|^2_{L^2} - \frac{1}{2} \| w_0 \|^2_{L^2}
\]
\[
- \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{1}{4} \langle 1 - \frac{1}{2r^2} \rangle w_l \tilde{\varphi}_{-l}, \frac{w_0}{r^2} \rangle - \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{w_l \tilde{\varphi}_{-l}}{r}, \partial_r \frac{w_0}{r^2} \rangle - \frac{2}{r^3} \frac{w_0}{r^2} \rangle.
\]

For any \( C_0 > 0 \), adding \( C_0 \times (3.16) \) and (3.17), one obtains
\[
\frac{1}{2} C_0 \partial_r \| \frac{w_0}{r} \|^2_{L^2} + C_0 \| rg \partial_r (r^{-2} g^{-1} w_0) \|^2_{L^2} \leq \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{1}{4} \langle 1 - \frac{1}{2r^2} \rangle w_l \tilde{\varphi}_{-l}, \frac{w_0}{r^2} \rangle + \Re \langle \sum_{l \in \mathbb{Z}/\{0\}} \imath l \langle \frac{w_l \tilde{\varphi}_{-l}}{r}, \partial_r \frac{w_0}{r^2} \rangle - \frac{2}{r^3} \frac{w_0}{r^2} \rangle \rangle.
\]
Choose $C_0$ being large enough such that
\[ \frac{C_0 c_0}{2} \left( \frac{1}{r^2} + r^2 \right) w_0, \frac{w_0}{r^2} \geq \frac{13}{4} \left\| \frac{w_0}{r^2} \right\|_{L^2}^2 + \frac{1}{2} \left\| \frac{w_0}{r^2} \right\|_{L^2}^2. \]

Then we deduce
\[ \partial_r \left\| \frac{w_0}{r} \right\|_{L^2}^2 + \left\| \partial_r \frac{w_0}{r} \right\|_{L^2}^2 + \left( \frac{1}{r^2} + r^2 \right) w_0, \frac{w_0}{r^2} \right\|_{L^2}^2 + \left\| \sum_{l \in \mathbb{Z}/\{0\}} \frac{i l \| w_l \|_{L^2}^2}{r} \right\|_{L^2}^2. \]

This yields
\[ \partial_r \left\| \frac{w_0}{r} \right\|_{L^2}^2 + \left\| \partial_r \frac{w_0}{r} \right\|_{L^2}^2 + \left( \frac{1}{r^2} + r^2 \right) w_0, \frac{w_0}{r^2} \right\|_{L^2}^2 + \left\| \sum_{l \in \mathbb{Z}/\{0\}} \frac{i l \| w_l \|_{L^2}^2}{r} \right\|_{L^2}^2. \]

Hence we obtain
\[ \left\| \frac{w_0}{r} \right\|_{L^2 L^2}^2 + \left\| \partial_r \frac{w_0}{r} \right\|_{L^2 L^2}^2 + \left\| \frac{w_0}{r^2} \right\|_{L^2 L^2}^2 + \left\| \sum_{l \in \mathbb{Z}/\{0\}} \frac{i l \| w_l \|_{L^2}^2}{r} \right\|_{L^2 L^2}^2. \]

And this completes the proof of Proposition 3.10.

4. Nonlinear stability

This section is devoted to the proof of the main Theorem 4.1. For the 2D Navier-Stokes equation, it is well-known that smooth initial data could lead to smooth global solutions. Here we are interested in solutions’ asymptotically behaviors, especially for initial data close to the Taylor-Couette flow. We first recall the energy functional $E_k$ (1.21):
\[ E_k = \| e^{i k B} \hat{\tau} w_k \|_{L^\infty L^2} + k B \| \left( e^{i k B} \hat{\tau} w_k, \frac{w_k}{r^2} \right) \|_{L^2 L^2} + \| e^{i k B} \hat{\tau} \frac{w_k}{r^2} \|_{L^2 L^2} \]
\[ + \| k B \| \| e^{i k B} \hat{\tau} \frac{w_k}{r^2} \|_{L^2 L^\infty} + \| k B \| \| e^{i k B} \hat{\tau} \frac{w_k}{r^2} \|_{L^2 L^2}, \quad |k| \geq 1, \]
\[ E_0 = \| \frac{w_0}{r} \|_{L^\infty L^2} + \| \frac{w_0}{r^2} \|_{L^2 L^\infty} + \| \frac{w_0}{r^2} \|_{L^2 L^2} + \| w_0 \|_{L^2 L^2}, \quad k = 0, \]
where $w_k$ is a solution of the nonlinear vorticity formulation (1.27).

Before we provide the proof, let us recall the nonlinear vorticity formulation (1.27) from Section 1.3:
\[ \begin{cases} \partial_r w_k + L_k w_k + f_1 - \partial_r f_2 = 0, \\ w_k(0) = w_k|_{\tau=0}, \quad w_k|_{\tau=0, \infty} = 0, \end{cases} \]
where $L_k = -\left[ \partial_r^2 - \left( \frac{k^2 - \frac{1}{16}}{r^2} + \frac{r^2 - \frac{1}{2}}{r^2} \right) \right] + i \frac{\beta_k}{r^2}$ with $\beta_k = kB$ and
\[ f_1 = \sum_{l \in \mathbb{Z}} \frac{w_l \hat{\varphi}_{k-l}}{r} + \sum_{l \in \mathbb{Z}} i (k - l) \left( \frac{1}{4} - \frac{1}{2r^2} \right) w_l \hat{\varphi}_{k-l}, \quad f_2 = \sum_{l \in \mathbb{Z}} i (k - l) \frac{w_l \hat{\varphi}_{k-l}}{r}, \]
with \(\varphi_k\) and \(\varphi_k\) satisfy \((\partial_r^2 + \frac{1}{r} \partial_r - \frac{k^2}{r^2})\varphi_k = e^{-\frac{r^2}{2r^2}}w_k\) and \(\varphi_k := r^{\frac{1}{2}}\varphi_k\). For \(\varphi_k\), we further deduce

\[r^{-\frac{1}{2}}(\partial_r^2 - \frac{k^2}{r^2})\varphi_k = r^{-\frac{1}{2}}e^{-\frac{r^2}{2r^2}}w_k,\]

which yields

\[(\partial_r^2 - \frac{k^2}{r^2})\varphi_k = e^{-\frac{r^2}{2r^2}}w_k.\]

Employing (4.1), the nonlinear terms \(f_1, f_2\) can be further expressed as

\[f_1 = ik \sum_{l \in \mathbb{Z}} w_l \frac{\partial_r (r^{-\frac{1}{2}}\varphi_{k-l})}{r} + \sum_{l \in \mathbb{Z}} i(k-l)(\frac{1}{4} - \frac{1}{2r^2}) \frac{w_l \varphi_{k-l}}{r^2}, \quad f_2 = \sum_{l \in \mathbb{Z}} i(k-l) \frac{w_l \varphi_{k-l}}{r^2}.\]

The following is the main result of this paper:

**Theorem 4.1.** For any \(|B| \geq 1\), there exist constants \(c_0, C > 0\) independent of \(B\) so that if the initial data satisfies

\[
\sum_{k \in \mathbb{Z}/\{0\}} \|w_k(0)\|_{L^2}^2 + \sum_{k \in \mathbb{Z}/\{0\}} \|kB\|^{\frac{1}{2}} \|w_k(0)\|_r \|B\|^{\frac{1}{2}} \|w_0(0)\|_r \leq c_0 |B|^{\frac{3}{4}},
\]

then the solution to the nonlinear vorticity formulation (1.27) exists globally in time, and there exists a constant \(C > 0\) independent of \(B\) such that

\[
\sum_{k \in \mathbb{Z}} E_k \leq C \left( \sum_{k \in \mathbb{Z}/\{0\}} \|w_k(0)\|_{L^2}^2 + \sum_{k \in \mathbb{Z}/\{0\}} \|kB\|^{\frac{1}{2}} \|w_k(0)\|_r \|B\|^{\frac{1}{2}} \|w_0(0)\|_r \right) \leq C_0 |B|^{\frac{3}{4}}.
\]

**4.1. Interactions between zero frequency and non-zero frequency.** Now we establish the estimate involving the interaction between zero frequency and other frequency. Let’s start with the terms including \(w_0 \varphi_k\).

**Lemma 4.2.** For any \(k \in \mathbb{Z}\setminus\{0\}\), it holds that

\[
|k| \left( \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} + \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} \right)
\]

\[
+ \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} + \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} \right) \leq |B|^{-\frac{1}{4}} |k|^{-\frac{1}{4}} \tilde{E}_0 E_k.
\]

**Proof.** Applying Hölder’s inequalities for the nonlinear terms in \(L^2 L^2\) norm, we obtain

\[
|k| \left( \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} + \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} \right)
\]

\[
+ \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} + \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} \right) \leq |k| \left( \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} + \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{w_0 \varphi_k}{r^\frac{1}{2}}\|_{L^2 L^2} \right).
\]

Combining above inequalities with (4.1) and the calculation Lemma 4.5 proved in appendix (with \(\beta = -2, -1, 0, 1\)), we derive

\[
|k| \left( \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{1}{r^\frac{1}{2}} \varphi_k\|_{L^2 L^2} + \|e^{\frac{1}{2}k |B|^\beta \tau} \frac{1}{r^\frac{1}{2}} \varphi_k\|_{L^2 L^2} \right)
\]
Therefore, we deduce Lemma 4.3. 

For any \( k \) we have \( k \), 

\[ \text{This together with (4.1) and Lemma A.5 yields for any } k \in \mathbb{Z} \backslash \{0\}, \text{ we have} \]

\[ |k| \left( \left\| e^{c|kB|^\frac{1}{3} \tau} \varphi_k \right\|_{L^\infty} + \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{w_0 \varphi_k'}{r^\frac{3}{2}} \right\|_{L^2} \right) \lesssim |k| |\tau|^\frac{1}{6} E_0 \]

**Proof.** Employing Hölder’s inequalities, we can prove

\[ |k| \left( \left\| e^{c|kB|^\frac{1}{3} \tau} \varphi_k \right\|_{L^\infty} + \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{w_0 \varphi_k'}{r^\frac{3}{2}} \right\|_{L^2} \right) \]

\[ \lesssim |k| \left\| \frac{w_0}{r^\frac{3}{2}} \right\|_{L^\infty} \left( \left\| e^{c|kB|^\frac{1}{3} \tau} \varphi_k \right\|_{L^\infty} + \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{\varphi_k'}{r^\frac{3}{2}} \right\|_{L^2} \right). \]

This together with (4.1) and Lemma A.5 yields

\[ |k| \left( \left\| e^{c|kB|^\frac{1}{3} \tau} \varphi_k \right\|_{L^\infty} + \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{w_0 \varphi_k'}{r^\frac{3}{2}} \right\|_{L^2} \right) \lesssim |k| \left\| \frac{w_0}{r^\frac{3}{2}} \right\|_{L^\infty} \left( \left\| e^{c|kB|^\frac{1}{3} \tau} \varphi_k \right\|_{L^\infty} + \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{\varphi_k'}{r^\frac{3}{2}} \right\|_{L^2} \right). \]

Hence we deduce

\[ |k| \left( \left\| e^{c|kB|^\frac{1}{3} \tau} \varphi_k \right\|_{L^\infty} + \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{w_0 \varphi_k'}{r^\frac{3}{2}} \right\|_{L^2} \right) \]

\[ \lesssim \left\| \frac{w_0}{r^\frac{3}{2}} \right\|_{L^\infty} \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{w_0 \varphi_k'}{r^\frac{3}{2}} \right\|_{L^2} \lesssim |B|^{-\frac{1}{6}} E_0 |kB|^{-\frac{1}{6}} E_k = |k|^{-\frac{1}{6}} |B|^{-\frac{1}{6}} E_0 E_k. \]

For quadratic terms containing \( w_k \partial_r (r^{-\frac{1}{2}} \varphi_0) \), we have

**Lemma 4.4.** For any \( k \in \mathbb{Z} \backslash \{0\} \), the following inequality holds

\[ |k| \left( \left\| e^{c|kB|^\frac{1}{3} \tau} w_k \partial_r (r^{-\frac{1}{2}} \varphi_0) \right\|_{L^2} + \left\| e^{c|kB|^\frac{1}{3} \tau} \frac{w_k \partial_r (r^{-\frac{1}{2}} \varphi_0)}{r^\frac{3}{2}} \right\|_{L^2} \right) \lesssim |k|^{-\frac{1}{6}} |B|^{-\frac{1}{6}} E_0 E_k. \]
\textbf{Proof.} Recall that $\varphi_0$ and $w_0$ satisfy
\[
(\partial_r^2 + \frac{1}{r}\partial_r)(r^{-\frac{1}{2}}\varphi_0) = \frac{e^{-\frac{r^2}{2}}}{r^{\frac{3}{2}}} w_0.
\]

Construct $G := \partial_r(r^{-\frac{1}{2}}\varphi_0)$, one can check $\partial_r(r^{-\frac{1}{2}}\varphi_0) = G$. Observing the inequality as below,
\[
\|G\|_{L^\infty} + \frac{G}{r}\|\|_{L^\infty} = \int_0^r s^2 e^{-\frac{s^2}{2}} w_0 ds + \int_0^r s^2 e^{-\frac{s^2}{2}} w_0 ds dL
\]
\[
\leq \left(\int_0^r \frac{s^2 e^{-\frac{s^2}{2}}}{r^2} ds\right)^\frac{1}{2} \|w_0\|_{L^2} + \left(\int_0^r \frac{s^2 e^{-\frac{s^2}{2}}}{r^2} ds\right)^\frac{1}{2} \|w_0\|_{L^2} \leq \|w_0\|_{L^2}.
\]
Together with Hölder’s inequality, we conclude
\[
|k| \left(\|e^{i[kB]^\frac{1}{2}r_w}w_k\partial_r(r^{-\frac{1}{2}}\varphi_0)\|_{L^2 L^2} + \|e^{i[kB]^\frac{1}{2}r_w}w_k\partial_r(r^{-\frac{1}{2}}\varphi_0)\|_{L^2 L^2} \right)
\]
\[
\leq |k| \|e^{i[kB]^\frac{1}{2}r_w}w_k\|_{L^2 L^2} \|G\|_{L^\infty} \|L^\infty L^\infty\| \|w_0\|_{L^\infty L^\infty} \leq |k| \|e^{i[kB]^\frac{1}{2}r_w}w_k\|_{L^2 L^2} \|w_0\|_{L^\infty L^\infty}
\]
\[
\leq |kB|^{-\frac{1}{2}} |B|^{-\frac{1}{2}} E_k E_0 = |k|^{-\frac{1}{2}} |B|^{-\frac{1}{2}} E_k E_0.
\]
\[
\square
\]

\textbf{4.2. Interactions between non-zero frequency spaces.} We now move to control non-linear terms containing only non-zero frequency. Firstly, observing the following inequality
\[
(4.2) \quad |kB|^\frac{1}{2} \leq |lB|^\frac{1}{2} + |(k - l)B|^\frac{1}{2} \quad \text{for any } k, l \in \mathbb{Z},
\]
we obtain

\textbf{Lemma 4.5.} For any $k \in \mathbb{Z}\setminus\{0\}$, we have
\[
\|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2} + \|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2}
\]
\[
+ \|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2} + \|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2}
\]
\[
\leq \|B|^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} |l|^{-\frac{1}{2}} |k - l|^{-\frac{1}{2}} E_k E_{k-l}.
\]

\textbf{Proof.} Utilizing (4.2) we obtain
\[
|k - l| \left(\|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2} + \|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2}
\]
\[
+ \|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2} + \|e^{i[kB]^\frac{1}{2}r_w} \sum_{l \in \mathbb{Z}\setminus\{0,k\}} i(k - l) \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \|_{L^2 L^2}
\]
\[
\leq |k - l| \left(\|e^{i[(k - l)B]^\frac{1}{2}r_w} \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \varphi_k \|_{L^\infty L^\infty} + \|e^{i[(k - l)B]^\frac{1}{2}r_w} \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \varphi_k \|_{L^\infty L^\infty}
\]
\[
+ \|e^{i[(k - l)B]^\frac{1}{2}r_w} \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \varphi_k \|_{L^\infty L^\infty} + \|e^{i[(k - l)B]^\frac{1}{2}r_w} \frac{w_k \varphi_{k-l}}{r^\frac{3}{2}} \varphi_k \|_{L^\infty L^\infty}
\right).
\]
Combining (4.1) and the calculation Lemma [A.3] proved in appendix (with $\beta = -4, -2, 0, 2$), it follows

$$
|k - l| \left( \left\| e^{c[(k-l)B]^\frac{1}{2} r} \varphi_{k-l} \right\|_{L^\infty L^\infty} + \left\| e^{c[(k-l)B]^\frac{1}{2} r^2} \varphi_{k-l} r^2 \right\|_{L^\infty L^\infty} \right)
$$

+ \left( \left\| e^{c[(k-l)B]^\frac{1}{2} r} \varphi_{k-l} \right\|_{L^\infty L^\infty} + \left\| e^{c[(k-l)B]^\frac{1}{2} r^2} \varphi_{k-l} r^2 \right\|_{L^\infty L^\infty} \right)

\lesssim |k - l| \left( |lB|^{-\frac{1}{2}} E_l |k - l|^{-\frac{1}{2}} (k - l)^{-\frac{1}{2}} E_k \right).

This completes the proof of the lemma. \qed

We also have

**Lemma 4.6.** For any $k \in \mathbb{Z} \setminus \{0, k\}$, it holds

$$
|k| \left( \left\| e^{c[kB]^\frac{1}{2} r} \right\|_{L^2 L^2} + \left\| e^{c[kB]^\frac{1}{2} r^2} \right\|_{L^2 L^2} \right)
$$

+ \left( \left\| e^{c[kB]^\frac{1}{2} r} \right\|_{L^2 L^2} + \left\| e^{c[kB]^\frac{1}{2} r^2} \right\|_{L^2 L^2} \right)

\lesssim |lB|^{-\frac{1}{2}} \left( \frac{1}{2} k - l \right)^{-\frac{1}{2}} E_l E_{k-l} + \frac{1}{2} |k - l|^{-\frac{1}{2}} E_l E_{k-l}.

**Proof.** Applying (4.2) we have

$$
|k - l| \left( \left\| e^{c[(k-l)B]^\frac{1}{2} r} \varphi_{k-l} \right\|_{L^\infty L^\infty} + \left\| e^{c[(k-l)B]^\frac{1}{2} r^2} \varphi_{k-l} r^2 \right\|_{L^\infty L^\infty} \right)
$$

+ \left( \left\| e^{c[(k-l)B]^\frac{1}{2} r^2} \varphi_{k-l} r^2 \right\|_{L^\infty L^\infty} + \left\| e^{c[(k-l)B]^\frac{1}{2} r^4} \varphi_{k-l} r^4 \right\|_{L^\infty L^\infty} \right)

\lesssim |k - l| \left( \left\| e^{c[(k-l)B]^\frac{1}{2} r} \varphi_{k-l} \right\|_{L^\infty L^\infty} + \left\| e^{c[(k-l)B]^\frac{1}{2} r^2} \varphi_{k-l} r^2 \right\|_{L^\infty L^\infty} \right)
\[ + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{\varphi_{k-l}}{r} \|_{L^\infty L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} (r \frac{2}{8}) \|_{L^\infty L^2} \).\]

Together with (4.41), the calculation Lemma A.5 proved in appendix (with \( \beta \) chosen to be \(-2 \) and \( 0 \)) this gives

\[ |k - l| \left( \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} \right) \]

\[ \leq \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} \left( \| e^{c(k-l)B^{\frac{1}{2}}} \frac{\varphi_{k-l}}{r} \|_{L^\infty L^\infty} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{\varphi_{k-l}}{r} \|_{L^\infty L^\infty} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{\varphi_{k-l}}{r} \|_{L^\infty L^\infty} \right). \]

Employing (4.41) and calculation Lemma A.5 proved in appendix (with \( \beta = -2, 0 \)), we obtain

\[ |k| \left( \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} \right) \]

\[ \leq |k| \left( \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} \right) \]

\[ \leq |l| \left( \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} \right) \]

\[ \leq |l| \left( \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \frac{w_k \varphi_{k-l}}{r} \|_{L^2 L^2} \right) \]

Therefore, with applying the basic inequality \(|k| \lesssim |k - l||l| \) for \( l \neq 0, k \), we conclude

\[ |k| \left( \| e^{c(k-l)B^{\frac{1}{2}}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \frac{w_k \varphi_{k-l}}{r^2} \|_{L^2 L^2} + \| e^{c(k-l)B^{\frac{1}{2}}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \frac{w_k \varphi_{k-l}}{r^2} \|_{L^2 L^2} \right) \]

\[ \leq |k| \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \left( |l| \| e^{c(k-l)B^{\frac{1}{2}}} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \frac{w_k \varphi_{k-l}}{r^2} \|_{L^2 L^2} \right). \]
This completes the proof of this lemma.

Finally, we prove the following bound for terms involving $w_l \phi_{-l}$.

**Lemma 4.7.** For nonlinear terms involving $w_l \phi_{-l}$ ($l \neq 0$), we have the bounds as below

$$\sum_{l \in \mathbb{Z}/\{0\}} |l||w_l \phi_{-l}|_{L^2 L^2} + \sum_{l \in \mathbb{Z}/\{0\}} |l||w_l \phi_{-l}|_{L^2 L^2} \lesssim |B|^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}/\{0\}} |l|^{-1} E_l E_{-l}.$$  

**Proof.** Recall (4.1) and for $\phi_k, \phi_{\breve{k}}$ we have $\breve{\phi}_k = \frac{w_k}{r^2}$, and

$$(\partial_r^2 - \frac{k^2}{r^2} - \frac{1}{r^2}) \phi_k = e^{\frac{r^2}{8}} w_k.$$  

Employing Hölder’s inequalities, one has

$$\sum_{l \in \mathbb{Z}/\{0\}} |l||w_l \phi_{-l}|_{L^2 L^2} + \sum_{l \in \mathbb{Z}/\{0\}} |l||w_l \phi_{-l}|_{L^2 L^2} \lesssim \sum_{l \in \mathbb{Z}/\{0\}} |l||w_l|_{L^2 L^2} \left(\left|\phi_{-l}\right|_{L^\infty L^\infty} + \left|\phi_{-l}\right|_{L^\infty L^\infty}\right),$$  

Combining with (4.1) and Lemma A.3, we derive

$$|l|\left(\left|\phi_{-l}\right|_{L^\infty L^\infty} + \left|\phi_{-l}\right|_{L^\infty L^\infty}\right) \lesssim |l|^{-\frac{1}{2}} \left(\left|w_{-l}\right|_{L^\infty L^2} + \left|w_{-l}\right|_{L^\infty L^2}\right) \lesssim |l|^{-\frac{1}{2}} \left|w_{-l}\right|_{L^\infty L^2}.$$  

Hence we conclude

$$\sum_{l \in \mathbb{Z}/\{0\}} il w_l \phi_{-l} ||L^2 L^2 || + \sum_{l \in \mathbb{Z}/\{0\}} il \frac{w_l \phi_{-l}}{r^2} ||L^2 L^2 || \lesssim \sum_{l \in \mathbb{Z}/\{0\}} \left|w_l\right|_{L^2 L^2} |l|^{-\frac{1}{2}} \left|w_{-l}\right|_{L^\infty L^2} \lesssim \sum_{l \in \mathbb{Z}/\{0\}} |l|^{-\frac{1}{2}} |lB|^{-\frac{1}{2}} E_l |lB|^{-\frac{1}{2}} E_{-l} \lesssim |B|^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}/\{0\}} |l|^{-1} E_l E_{-l}.$$  

4.3. **Proof of Theorem 4.1.** With above estimates at hand, we are now ready to prove the main theorem of this paper.

**Proof of Theorem 4.1.** Recall that in Proposition 3.8 and Proposition 3.9 we have derived the following space-time estimates

$$\|e^{i|k|\frac{r}{r^2}} w_k\|_{L^\infty L^2} + \|kB\|\|e^{i|k|\frac{r}{r^2}} w_k\|_{L^2 L^2} + \|e^{i|k|\frac{r}{r^2}} \partial_r w_k\|_{L^2 L^2} + \|e^{i|k|\frac{r}{r^2}} (\frac{|k|}{r} + r) w_k\|_{L^2 L^2} \lesssim |k|^{-\frac{1}{2}} \|e^{i|k|\frac{r}{r^2}} f_1\|_{L^2 L^2} + \|e^{i|k|\frac{r}{r^2}} f_2\|_{L^2 L^2} + \|w_k(0)\|_{L^2}.$$  


and
\[
|kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} w_k||_{L^\infty} + |kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} w_k||_{L^2} + |kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} w_k||_{L^2} + |kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} w_k||_{L^2} \\
\lesssim (||e^{[kB]\frac{1}{2}\tau} f_1||_{L^2} + ||e^{[kB]\frac{1}{2}\tau} f_2||_{L^2}) + |kB|^\frac{1}{2} ||w_k(0)||_X,
\]
where \(X = L^2(\frac{1}{r^2})\) is the weighted \(L^2\) space we construct, and \(f_1, f_2\) are nonlinear terms given by
\[
f_1 = ik \sum_{l \in \mathbb{Z}} \frac{\partial_l (r^{-\frac{1}{2}} \varphi_{k-l})}{r} + \sum_{l \in \mathbb{Z}} i(k - l)(\frac{1}{4} - \frac{1}{2r^2}) \frac{w_l \varphi_{k-l}}{r^2}, \quad f_2 = \sum_{l \in \mathbb{Z}} i(k - l) \frac{w_l \varphi_{k-l}}{r^2}.
\]
Using the calculation Lemma [A.3] proved in appendix (with \(\alpha = \frac{3}{2}\)), we obtain
\[
|k| ||\frac{w_k}{r^2}||_{L^2} \lesssim |k| ||\frac{w_k}{r^2}||_{L^2} + |k| ||\frac{w_k}{r^2}||_{L^2} + k^2 ||\frac{w_k}{r^2}||_{L^2},
\]
This implies
\[
|k| ||e^{[kB]\frac{1}{2}\tau} \frac{w_k}{r^2}||_{L^2} \lesssim ||e^{[kB]\frac{1}{2}\tau} \frac{w_k}{r^2}||_{L^2} + k^2 ||e^{[kB]\frac{1}{2}\tau} \frac{w_k}{r^2}||_{L^2},
\]
Therefore, following Proposition 3.8, Proposition 3.9 and Proposition 3.10, we deduce
\[
E_k = ||e^{[kB]\frac{1}{2}\tau} w_k||_{L^\infty} + |kB|^\frac{1}{2} \left(||e^{[kB]\frac{1}{2}\tau} w_k||_{L^2} + ||e^{[kB]\frac{1}{2}\tau} w_k||_{L^2} \right) + |kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} w_k||_{L^2} \\
\lesssim |kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} f_1||_{L^2} + |kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} f_2||_{L^2} + |kB|^\frac{1}{2} ||w_k(0)||_{L^2} + |kB|^\frac{1}{2} ||w_k(0)||_X, \quad |k| \geq 1,
\]
\[
E_0 = |B|^\frac{1}{2} \left(||\frac{w_0}{r^2}||_{L^\infty} + ||\frac{w_0}{r^2}||_{L^2} + ||\frac{w_0}{r^2}||_{L^2} + ||\frac{w_0}{r^2}||_{L^2} \right), \quad k = 0.
\]
To bound the terms on the right hand side, we utilize the following inequality
\[
(4.3) \quad |k| \lesssim |k - l||l| \quad \text{if } k \neq 0, l.
\]
With (4.3), Lemma 4.2, Lemma 4.3, Lemma 4.4, Lemma 4.5 and Lemma 4.6, we have that the first term \(|kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} f_1||_{L^2}\) can be controlled via
\[
|kB|^\frac{1}{2} ||e^{[kB]\frac{1}{2}\tau} f_1||_{L^2} \]
Lemma 4.4, Lemma 4.5 and Lemma 4.6, we obtain

\[ \| e^{c[kB]^{1/4} \tau} \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \frac{w_{l} \partial_{r} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} + \| e^{c[kB]^{1/4} \tau} \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

+ \| e^{c[kB]^{1/4} \tau} \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} + \| e^{c[kB]^{1/4} \tau} \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

\[ \| e^{c[kB]^{1/4} \tau} \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

\[ \| e^{c[kB]^{1/4} \tau} \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

\[ \| e^{c[kB]^{1/4} \tau} \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

\[ \| E_k \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

Finally, we arrive at

\[ E_k \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

\[ \| e^{c[kB]^{1/4} \tau} f_1 \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]

\[ \| e^{c[kB]^{1/4} \tau} f_2 \|_{L^2} \sum_{l \in \mathbb{Z} \setminus \{0, k\}} i(k - l) \frac{w_{l} \varphi_{k-l}}{r^{2}} \| L^2 \|_{L^2} \]
In addition, using Lemma 4.4, we also find
\[ E_0 \lesssim |B|^{-\frac{1}{4}} \sum_{l \in \mathbb{Z}/\{0\}} E_l E_{-l} + |B|^{\frac{1}{4}} \frac{w_0(0)}{r} \|_{L^2}. \]

With the bootstrap assumption
\[ \sum_{k \in \mathbb{Z}} E_k \leq c_0 B^\frac{1}{2}, \]
we obtain
\[ \sum_{k \in \mathbb{Z}} E_k \]
\[ \leq C \left[ |B|^{-\frac{1}{4}} \left( \sum_{k \in \mathbb{Z}} E_k \right)^2 + \sum_{k \in \mathbb{Z}/\{0\}} \|w_k(0)\|_{L^2} + \sum_{k \in \mathbb{Z}/\{0\}} |kB|^{\frac{1}{4}} \|w_k(0)\|_X + |B|^{\frac{1}{4}} \frac{w_0(0)}{r} \|_{L^2} \right] \]
\[ \leq C \left[ c_0 \sum_{k \in \mathbb{Z}} E_k + \sum_{k \in \mathbb{Z}/\{0\}} \|w_k(0)\|_{L^2} + \sum_{k \in \mathbb{Z}/\{0\}} |kB|^{\frac{1}{4}} \frac{w_k(0)}{r} \|_{L^2} + |B|^{\frac{1}{4}} \frac{w_0(0)}{r} \|_{L^2} \right]. \]

Choose \( c_0 \leq \frac{1}{4} C^{-1} \), we hence prove
\[ \sum_{k \in \mathbb{Z}} E_k \lesssim \sum_{k \in \mathbb{Z}/\{0\}} \|w_k(0)\|_{L^2} + \sum_{k \in \mathbb{Z}/\{0\}} |kB|^{\frac{1}{4}} \frac{w_k(0)}{r} \|_{L^2} + |B|^{\frac{1}{4}} \frac{w_0(0)}{r} \|_{L^2}. \]

The above completes the proof of Theorem 1.1.

4.4. Proof of Theorem 1.8

As a corollary of the above main result, we also obtain an improved transition threshold for the original equations (1.2). Let \( \tilde{\omega} := \frac{w - 2A_1}{t} \). We have
\[ \int_{\mathbb{R}^2} |\tilde{\omega}(t, x)| dx = \nu \int_{\mathbb{R}^2} \left| \frac{w - 2A_1 t}{t} \right| |d\xi = \nu \int_{\mathbb{R}^2} |w(\tau, \xi) - 2A_1 e^\tau| d\xi \]
\[ = \nu \int_0^{2\pi} \int_0^\infty \int_0^\infty |w(\tau, r, \theta) - 2A_1 e^\tau| dr d\theta. \]

Note that \( w(\tau, \xi) = w(\tau, r, \theta) = \sum_{k \in \mathbb{Z}} \omega_k(\tau, r)e^{ik\theta} \). And recall the following estimate for \( \tau \geq 0 \) in Theorem 1.5
\[ \mathcal{E}(\tau) = \|A_2\|_{L^2} \left( \frac{\|w_0(\tau) - 2A_1 e^\tau\|}{r} \right) \|_{L^2} + \sum_{k \in \mathbb{Z}/\{0\}} (\|\omega_k(\tau)\|_{L^2} + |k|^{\frac{1}{4}} \frac{A_2}{\nu} \|\omega_k(\tau)\|_{L^2}) \|_{L^2} \leq C_{c_0} \|A_2\|_{L^2}. \]

We thus obtain
\[ \int_{\mathbb{R}^2} |\tilde{\omega}(t, x)| dx = \nu \int_0^{2\pi} \int_0^\infty \int_0^\infty r |w(\tau, r, \theta) - 2A_1 e^\tau| dr d\theta \]
\[ \leq 2\pi \nu \int_0^\infty \int_0^\infty r |\omega_0(\tau) - 2A_1 e^\tau| + \sum_{k \in \mathbb{Z}/\{0\}} |\omega_k| dr \]
\[ \leq 2\pi \nu \int_0^\infty r^3 e^{-\frac{r^2}{4\tau}} dr \left( \int_0^\infty r^{-1} e^{-\frac{r^2}{4\tau}} |\omega_0(\tau) - 2A_1 e^\tau|^2 dr \right)^{\frac{1}{2}} \]
\[ + 2\pi \nu \sum_{k \in \mathbb{Z}/\{0\}} \left( \int_0^\infty r e^{-\frac{r^2}{4\tau}} dr \right)^{\frac{1}{2}} \left( \int_0^\infty e^{-\frac{r^2}{4\tau}} |\omega_k(\tau)|^2 dr \right)^{\frac{1}{2}} \]
$$\begin{align*}
&= 2\pi \nu \left( \int_0^\infty r^3 e^{-\frac{r^2}{4}} dr \right)^{\frac{1}{2}} \left\| \frac{\omega_0(\tau) - 2A_e}{r} \right\|_{\mathcal{M}} \\
&+ 2\pi \nu \left( \int_0^\infty r e^{-\frac{r^2}{4}} dr \right)^{\frac{1}{2}} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\| \omega_k(\tau) \right\|_{\mathcal{M}} \\
&\leq C \nu \mathcal{E}(\tau) \leq C c_0 \left| \frac{A_2}{\nu} \right| \frac{\nu}{4} = C c_0 |A_2| \frac{\nu}{4} \frac{\nu}{2}.
\end{align*}$$

Hence, for initial data at $t = 1$, if the initial perturbation satisfies

$$\int_{\mathbb{R}^2} |\tilde{\omega}(1, x)| dx \lesssim \nu \mathcal{E}(0) \leq C c_0 |A_2| \frac{\nu}{4} \frac{\nu}{2},$$

then we proof

$$\int_{\mathbb{R}^2} |\tilde{\omega}(t, x)| dx \lesssim \nu \mathcal{E}(\tau) \lesssim \nu \mathcal{E}(0) \leq C c_0 |A_2| \frac{\nu}{4} \frac{\nu}{2}, \quad \text{for all } t \geq 1.$$

This is corresponding to the improved transition threshold stated in the Theorem 1.8.

**Appendix A. Calculation lemmas**

**Lemma A.1.** For any $A, B \in \mathbb{R}$, the below ODE admits a global solution (with variable $r \in (0, \infty)$)

(A.1) \quad \frac{g''}{g} + \frac{A}{r} g' = \frac{B}{r^2}.

**Proof.** Let $G := \frac{g'}{g}$. By noting that

$$G' = \frac{g'' g - g'^2}{g^2} = \frac{g''}{g} - G^2,$$

we can transfer the above equation (A.1) to

$$G' + G^2 + \frac{A}{r} G = \frac{B}{r^2}.$$ 

Denote $F = G + \frac{A}{2r}$, we deduce $F' + F^2 = \frac{C}{r^2}$ with $C = \frac{A^2}{4} + \frac{A}{2} + B$. Dividing both sides by $F^2$ and via introducing $H = \frac{1}{F}$, one obtains

$$-H' + 1 = C \left( \frac{H}{r} \right)^2,$$

Let $K = \frac{H}{r}$, we then have

$$-(rK' + K) + 1 = CK^2,$$

or equivalently

$$K' = \frac{-CK^2 - K + 1}{r}.$$

The above separable equation can be solved via

$$\int \frac{dK}{-CK^2 - K + 1} = \log r.$$

Therefore, we deduce

$$g(r, A, B) = C_0 \exp \left( \int_0^r G(s) ds \right) = C_0 \exp \left( \int_0^r \frac{1}{sK(s)} - \frac{A}{2s} ds \right),$$

□
Lemma A.2. For any $w \in L^2(\Gamma), w|_{r \in \partial \Gamma} = 0$, it holds
\[ \|w\|_{L^\infty} \leq 2\|w\|_{L^2} \|w\|_{L^2}. \]

Proof. Since $w|_{r \in \partial \Gamma} = 0$, we have
\[ |w(r)|^2 = \int_0^r \partial_s(|w(s)|^2)ds = \int_0^r (w'(s)\overline{w}(s) + w(s)\overline{w}'(s))ds \leq 2\|w\|_{L^2} \|w\|_{L^2}, \]
\[ \square \]

Lemma A.3. For any $w \in L^2(\Gamma)$ and $\alpha \in \mathbb{R}$, it holds
\[ \| \frac{w}{r^\alpha} \|_{L^\infty} \lesssim \| \frac{w'}{r^{\alpha-1}} \|_{L^2} + \frac{w}{r^{\alpha+1}} \|_{L^2} \]

Proof. Choose $r_1 \in (0, 1)$ such that \[ \frac{|w(r)|^2}{r_1^{2\alpha}} \leq \| \frac{w}{r^{\alpha-1}} \|_{L^2(0, 1)} \]
\[ \leq \| \frac{w}{r^{\alpha+1}} \|_{L^2(0, 1)} \]
This completes the proof of this lemma. \[ \square \]

Lemma A.4. Let $w = \varphi'' - \frac{k^2}{r^2} \varphi \in L^2(\Gamma)$ with $w|_{r \in \partial \Gamma} = 0, \varphi|_{r \in \partial \Gamma} = 0$. Then for any $|k| \geq 1$ and $\beta \in \mathbb{R}$ it holds
\[ \Re(-w, r^\beta \varphi) \geq \| r^{\frac{\alpha}{2}} \varphi' \|_{L^2}^2 + k^2 \| r^{\frac{\beta}{2}} - 1 \varphi \|_{L^2}^2. \]

Proof. For any $C^2$ real function $g$ with variable $r$, we have the following identity:
\[ -\varphi'' = -g^{-1}\partial_r(r^\beta g^2 \partial_r(r^{-\beta} g^{-1} \varphi)) + \frac{\beta}{r} \varphi' - g^{-1}[g'' - (\frac{\beta}{r} g') \varphi]. \]

Via integration by parts we obtain
\[ \langle -w, r^\beta \varphi \rangle = \langle -g^{-1}\partial_r(r^\beta g^2 \partial_r(r^{-\beta} g^{-1} \varphi)) + \frac{\beta}{r} \varphi' - g^{-1}[g'' - (\frac{\beta}{r} g') \varphi + k^2 - \frac{1}{4} \varphi, \frac{\varphi}{r^\beta} \rangle \]
\[ = \| r^{-\frac{\alpha}{2}} g \partial_r(r^\beta g^{-1} \varphi) \|_{L^2}^2 + \beta \langle \varphi', r^\beta - 1 \varphi \rangle - \langle g^{-1}[g'' - (\frac{\beta}{r} g') \varphi, r^\beta \varphi \rangle \]
\[ + (k^2 - \frac{1}{4}) \| r^{\beta - 1} \varphi \|_{L^2}^2. \]
This together with
\[ \Re(\varphi', r^{\beta - 1} \varphi) = -\frac{1}{2} \int_\Gamma r^{\beta - 1} d|\varphi|^2 = -\frac{\beta - 1}{2} \int_\Gamma r^{\beta - 2} |\varphi|^2 dr, \]
gives
\[ \Re(-w, r^\beta \varphi) = \| r^{-\frac{\alpha}{2}} g \partial_r(r^\beta g^{-1} \varphi) \|_{L^2}^2 - \langle g^{-1}[g'' - (\frac{\beta}{r} g') \varphi, r^\beta \varphi \rangle \]
\[ + (k^2 - \frac{1}{4} - \frac{\beta(\beta + 1)}{2}) \| r^{\beta - 1} \varphi \|_{L^2}^2. \]
Choosing \( g = g(r, A, B) \) as in Lemma A.1 with \( A = -\beta, B = \frac{1}{4} + \frac{\beta(\beta+1)}{2} \), then \( \frac{d''}{g} - \frac{\beta g'}{r g} = \frac{1}{r} \). Hence we deduce
\[
\Re \langle -w, r^\beta \varphi \rangle = \| r^{-\beta} g \partial_r (r^\beta g^{-1} \varphi) \|_{L^2}^2 + k^2 \| r^\beta -1 \varphi \|_{L^2}^2 \geq k^2 \| r^\beta -1 \varphi \|_{L^2}^2.
\]
Employing integration by parts and (A.2), we also have
\[
\Re \langle -w, r^\beta \varphi \rangle = \langle -\varphi'' + \frac{k^2 - \frac{1}{4}}{r^2} \varphi, r^\beta \varphi \rangle
\]
(A.4)
\[
= \| r^\beta \varphi' \|_{L^2}^2 + \beta \Re \langle \varphi', r^\beta \varphi \rangle + (k^2 - \frac{1}{4}) \| r^\beta -1 \varphi \|_{L^2}^2
\]
\[
= \| r^\beta \varphi' \|_{L^2}^2 + (k^2 - \frac{1}{4} - \frac{\beta(\beta - 1)}{2}) \| r^\beta -1 \varphi \|_{L^2}^2.
\]
Combining (A.4) and (A.3), we now conclude
\[
\Re \langle -w, r^\beta \varphi \rangle \geq \| r^\beta \varphi' \|_{L^2}^2 + k^2 \| r^\beta -1 \varphi \|_{L^2}^2.
\]
\[
\text{ Lemma A.5. Under the same conditions of Lemma A.4, we have }
\]
\[
\| r^{-\beta+1} \varphi'' \|_{L^2} + |k| \| r^{-\beta+1} \varphi' \|_{L^\infty} + |k| \| r^{-\beta+1} \varphi' \|_{L^2} + |k| \| r^{-\beta+1} \varphi \|_{L^2} + k^2 \| r^{-\beta+1} \varphi \|_{L^2} \lesssim \| r^{-\beta+1} w \|_{L^2}.
\]
\[
\text{ Proof. Recall in Lemma A.4 we have the following inequality }
\]
\[
\Re \langle -w, r^\beta \varphi \rangle \geq \| r^\beta \varphi' \|_{L^2}^2 + k^2 \| r^\beta -1 \varphi \|_{L^2}^2.
\]
Combining with
\[
\Re \langle -w, r^\beta \varphi \rangle \leq \| r^{-\beta+1} w \|_{L^2} \| r^{-\beta+1} \varphi \|_{L^2} \leq |k|^{-1} \| r^{-\beta+1} w \|_{L^2} (\| r^{-\beta+1} \varphi \|_{L^2} + |k| \| r^{-\beta+1} \varphi \|_{L^2})
\]
and
\[
\| r^\beta \varphi' \|_{L^2}^2 + k^2 \| r^\beta -1 \varphi \|_{L^2}^2 \lesssim (\| r^\beta \varphi' \|_{L^2} + |k| \| r^\beta -1 \varphi \|_{L^2})^2,
\]
we obtain
(A.6)
\[
|k| \| r^\beta \varphi' \|_{L^2} + k^2 \| r^\beta -1 \varphi \|_{L^2} \lesssim \| r^{-\beta+1} w \|_{L^2}.
\]
Employing Lemma A.3 (with \( \alpha = -\frac{\beta}{2} + 1 \)) and (A.6) we get
\[
\| r^{-\beta+1} \varphi \|_{L^2} \lesssim \| r^\beta \varphi' \|_{L^2} \| r^\beta -1 \varphi \|_{L^2} + \| r^\beta -1 \varphi \|_{L^2}^2 \lesssim |k|^{-3} \| r^{-\beta+1} w \|_{L^2}^2.
\]
To derive the bound for \( \| r^{-\beta+1} \varphi'' \|_{L^2} \), we apply the following equality
\[
\Re \langle w, r^{\beta+2} \varphi'' \rangle = \| r^{\beta+1} \varphi'' \|_{L^2}^2 - (k^2 - \frac{1}{4}) \Re \langle \varphi, r^{\beta+2} \varphi'' \rangle.
\]
By Hölder’s inequality we have
\[
\| r^{\beta+1} \varphi'' \|_{L^2} \leq \| r^{\beta+1} \varphi'' \|_{L^2} \| r^{\beta+1} w \|_{L^2} + k^2 \| r^{\beta+1} \varphi'' \|_{L^2} \| r^{\beta+1} -1 \varphi \|_{L^2},
\]
which yields
(A.7)
\[
\| r^{\beta+1} \varphi'' \|_{L^2} \leq \| r^{\beta+1} w \|_{L^2} + k^2 \| r^{\beta+1} -1 \varphi \|_{L^2} \lesssim \| r^{\beta+1} w \|_{L^2}.
\]
Here in the second inequality we appeal to (A.6).
Finally, we utilize Lemma A.3 again with $\alpha = -\frac{\beta + 1}{2}$. Together with (A.6) and (A.7), we deduce

$$
\| r^{\frac{\beta + 1}{2}} \varphi' \|_{L^\infty} \lesssim \| r^{\frac{\beta + 1}{2}} \varphi'' \|_{L^2} + \| r^{\frac{\beta}{2}} \varphi' \|_{L^2} \lesssim |k|^{-2} \| r^{\frac{\beta + 1}{2}} w \|_{L^2}^2.
$$

\[ \square \]

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