QED spectra in the path integral formalism

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Abstract
Relativistic Hamiltonians, derived from the path integrals, are known to provide a simple and useful formalism for hadrons spectroscopy in QCD. The accuracy of this approach is tested using the QED systems, and the calculated spectrum is shown to reproduce exactly that of the Dirac hydrogen atom, while the Breit-Fermi nonrelativistic expansion is obtained using Foldy-Wouthuizen transformation. Calculated positronium spectrum, including spin-dependent terms, coincides with the standard QED perturbation theory to the considered order \(O(\alpha^s)\).

1 Introduction
The path integral approach to QCD and QED was actively developed since the first formulation in [1, 2] (see [3] for reviews, references, and discussions). The particular line of development is the so-called Fock-Feynman-Schwinger Representation (FFSR) [4, 5, 6], where both relativism and gauge invariance are made explicit. In this latter framework one derives the path-integral relativistic Hamiltonian to be called the Relativistic Hamiltonian (RH), originally exploited in a simple form in [7].

Recently a new integral form of the hadron Green’s function and a rigorous derivation of the RH was done in [8], and we shall use this latter form in what follows.

As it is, the RH formalism is one of the most powerful methods in QCD, which allows to predict spectra and wave functions of hadrons, using a minimal input: current quark masses, string tension, and \(\Lambda_{QCD}\). Therefore it is very important to check its validity for different systems and the accuracy of
results. In the course of derivation some approximations have been done, the significance of which can be made clear by comparison with other relativistic approaches. In the case of the one-particle system in an external field the basic approach is that of the Dirac equation and one can compare results of two approaches – the path integral Hamiltonian and Dirac equation in external fields, e.g. for the Coulomb case in QED. In the case of the linear potential in QCD results can be compared with lattice and experimental data.

It is important that the FFSR is derived for the Green’s functions, and the RH appears in the kernel in the exponent and depends on additional integration variables, which play the role of virtual particle energies. Therefore one encounters the problem of the proper definition of the RH as an operator and its excited states. This topic will also be discussed in comparison with the Dirac formalism, the QED perturbation theory for positronium, and relativistic quark models. As a result, we shall estimate the accuracy of approximations made and shall give the scheme of calculations for the ground and excited states, both in one-particle and two-particle systems. As an additional topic, we compare nonrelativistic expansions for the RH with the known Breit-Fermi expansion.

The plan of the paper is as follows. The short derivation of the RH is done in Section 2. Section 3 is devoted to the Breit-Fermi expansion of the RH. In Section 4 the spectrum of the RH for the hydrogen atom is compared with Dirac and Salpeter equations. The case of positronium and the accuracy of the spectrum of the RH is considered in Section 5. The last Section is devoted to the discussion of results and perspectives.

## 2 Derivation of the relativistic Hamiltonian

We start with the FFSR for the fermion propagator in the external gauge field $A_\mu$ in QED, as well as in QCD in the Euclidean space-time

\[
S = (m + D)^{-1} = (m - \hat{D})(m^2 - \hat{D}^2)^{-1} = \\
= (m - \hat{D}) \int_0^\infty ds e^{-s(m^2 - \hat{D}^2)} = (m - \hat{D}) \int_0^\infty ds (D^4 z) e^{-K W_F}, \quad (1)
\]

\[
(D^4 z)_{xy} = \int \frac{d^4 p}{(2\pi)^4} \prod_k \frac{d^4 \Delta z(k)}{(4\pi \varepsilon)^2} \exp[ip(\sum \Delta z(k) - (x - y))], \quad N\varepsilon = s; \quad (2)
\]
where the kinematic kernel

\[ K = m^2 s + \frac{1}{4} \int_0^s \left( \frac{dz_\mu}{d\tau} \right)^2 d\tau, \quad (3) \]

and the generalized Wilson line is

\[ W_F(x, y) = P_A \exp(i g \int_y^x A_\mu dz_\mu + g \int_0^s \sigma_{\mu\nu} F_{\mu\nu} d\tau), \quad (4) \]

where \( P_A \) is the ordering operator in the case of the nonabelian field \( A_\mu \). Note, that matrices \( \gamma_\mu \) enter \( W_F \) only in the term

\[ \sigma_{\mu\nu} F_{\mu\nu} = \begin{pmatrix} \sigma B & \sigma E_E \\ \sigma E_E & \sigma B \end{pmatrix}, \quad (5) \]

and here \( E_E \) is the Euclidean electric field, which should be replaced by \( E_M \equiv i E \) in the Minkowskian case. Hence all connections of large and small Dirac components are provided by the electric field in \( W_F \) and the factor \((m - \hat{D})\) an (1).

As will be seen, the main difference between the RH and the Dirac equation lies in two points:

1. The RH is a quadratic operator, which stems from the quadratic combination \((m^2 - \hat{D}^2)\), while the Dirac operator is linear in momenta and fields \( A_\mu \). This difference can be seen in the resulting nonrelativistic expansion of both operators and eigenvalues, and is cured by the Foldy-Wouthuizen transformation with the account of the factor \((m - \hat{D})\) in (1).

2. The new element in the relativistic path integral, as compared to its nonrelativistic analog, is the time path in the quantum paths. As shown explicitly in [8], the integration \( ds(D^4z)_{xy} \) in Eq. (11) can be written, using the relations

\[
s = \frac{T}{2\omega}, \quad T \equiv |x_4 - y_4|, \quad d\tau = \frac{dt_E}{2\omega},
\]

so that
\[ \int ds (D^4 z)_{xy} e^{-K} W_F(x, y) = T \int_0^\infty \frac{d\omega}{2\omega^2} \langle D^3 z \rangle_{xy} \exp[-K(\omega)] \langle \Phi_z(x, y) \rangle_{\Delta z_4}, \] (6)

where

\[ K(\omega) = \int_0^T dt_E \left( \frac{\omega}{2} + \frac{m^2}{2\omega} + \frac{\omega}{2} \left( \frac{dz}{dt_E} \right)^2 \right), \] (7)

and one can split the time element in (4),

\[ dz_4 \rightarrow \Delta z_4 = \Delta t_E + \Delta \tilde{z}_4, \]

so that \( \Delta t_E \) is a monotonic Euclidean time interval, while \( \Delta \tilde{z}_4 \) is a stochastic one, with \( \sum_{k=1}^N \Delta \tilde{z}_4(k) = 0 \). Correspondingly, in the integral \( (D z_4)_{x y} = \int \frac{dp_4}{2\pi} \prod_k \frac{d\Delta z_4(k)}{\sqrt{4\omega}} \exp[ip_4(\sum_k \Delta z_4(k) - T)] \) of the Wilson line (4) one can write

\[ \langle \Phi_z(x, y) \rangle_{\Delta z_4} = \int (D z_4)_{x y} W_F(x, y) = W_F(x, y) \sqrt{\frac{\omega}{2\pi T}} \varphi\{A_\mu\}. \] (8)

Here \( \varphi\{A_\mu = 0\} = 1 \) and also \( \varphi = 1 \) for \( A_\mu \) independent of \( z_4 \). As it was argued in [8], the difference \( \varphi\{A\} - 1 \) takes into account the creation of additional particles and hence higher Fock components in the total wave function and higher Fock matrix elements in the Hamiltonian. In terms of one- or two-particle Green’s functions these contributions can be considered as radiative corrections, which are absent in the simplest form of Dirac or Bethe-Salpeter equation. In what follows we shall consider only the minimal Fock component and use the condition \( \varphi\{A\} \equiv 1 \).

Now the monotonic part \( \bar{W}_F \) depends only on 3d trajectories \( z_\mu = \{z(t_E), t_E\} \) and is equal to

\[ \bar{W}_F(x, y) = \exp \left\{ \int_0^T dt_E \left[ ig A_4(t_E) + ig A_i \frac{dz_i}{dt_E} + g \sigma_{\mu\nu} F_{\mu\nu} \right] \right\}. \] (9)

In what follows we shall test the approximation of smooth trajectories with \( \varphi\{A_\mu\} \equiv 1 \) and compare the corresponding results with exact calculations of Dirac equation for the Coulomb potential.

As a result, from (6), (7), (8) one can write the Hamiltonian for a fermion in the electromagnetic field \( \{A(z, t), A_0(z, t)\} \),

\[ H(\omega) = \left( \frac{p - eA}{2\omega} \right)^2 + \frac{m^2 + \omega^2}{2\omega} + eA_0 - \frac{e(\sigma B)}{2\omega} - \frac{i e(\alpha E)}{2\omega}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \] (10)
One can see, that the obtained Hamiltonian contains the parameter $\omega$, which plays the role of the virtual particle energy, to be integrated over in the expression (6) for the Green’s function. There are several ways to proceed and get the final spectra, which are discussed in what follows. In the next Section we compare $H$ with nonrelativistic expansions of the Dirac equation.

3 Nonrelativistic expansions in RH and Dirac equation

We start with the Dirac equation for the hydrogen-like atom, where $A_0 = -\frac{Z\alpha}{r}$, and take into account that the exact form of the fermion Green’s function (for $\varphi\{A\} \equiv 1$) is

$$G(x, y) = \sqrt{\frac{T}{8\pi}} \int_{0}^{\infty} \frac{d\omega}{\omega^{3/2}} (m - \hat{D})_x (x|e^{-H(\omega)T}y).$$

(11)

In (11) the Hamiltonian was defined in Minkowskian space-time, while the final expression is written for the Euclidean time $T$ (see Appendix in [8] for details of derivation).

As the next step we consider the “projection operator” $(m - \hat{D})_x$ in the integral (11), and take into account that the derivative $\frac{\partial}{\partial x_\mu}$ is acting on the Wilson line (4), resulting in the following expression (see Appendix 1 of the second ref. in [7]),

$$(m - \hat{D}) \rightarrow \beta \left( \begin{array}{cc} \omega + m & \sigma \mathbf{p} \\ \sigma \mathbf{p} & \omega - m \end{array} \right).$$

(12)

At this point one needs to diagonalize the whole expression under the integral (11), which allows to give the energy eigenvalues of the Hamiltonian $H$ with the account of the lower components of the wave function. In this way one writes

$$(m - \hat{D}) = \beta U^+ \left( \begin{array}{cc} \omega + \sqrt{\mathbf{p}^2 + m^2} & 0 \\ 0 & \omega - \sqrt{\mathbf{p}^2 + m^2} \end{array} \right) U,$$

(13)

where

$$U = \left( \begin{array}{cc} \cos \vartheta & -\sin \vartheta \\ \sin \vartheta & \cos \vartheta \end{array} \right) = e^{iS}; \quad \tan 2\vartheta = \frac{\sigma \mathbf{p}}{m}, \quad S = \frac{i\gamma \mathbf{p} \theta}{p} = -\beta \alpha_2 \theta.$$

(14)
In a similar way one can write

\[ H(\omega) = U^* \tilde{H}(\omega) U, \quad \tilde{H}(\omega) = e^{iS} H(\omega) e^{-iS}. \quad (15) \]

Our reasoning below and in the next Section follows the arguments from the book [9], and \( \tilde{H}(\omega) \) can be found as a series (see chapter 2 of [9])

\[ \tilde{H}(\omega) = H(\omega) + i[S, H] - \frac{1}{2} [S, [S, H]] - \ldots . \quad (16) \]

As one can see in (14), the series in (16) is in powers of \( \left( \frac{p}{m} \right) \) and gives the higher orders of the nonrelativistic expansion, whereas the first two orders are contained already in \( H(\omega) \). Indeed, keeping for simplicity the first three terms in (10), which we denote as \( H_0(\omega) \),

\[ H_0(\omega) = \frac{(p - eA)^2}{2\omega} + \frac{m^2 + \omega^2}{2\omega} + eA_0, \quad (17) \]

and taking into account that at large \( T \) the integration over \( d\omega \) in (11) can be done using the minimum of \( H_0(\omega) \) in momentum space at some \( \omega = \omega_0 \), one has

\[ \omega_0 = \sqrt{(p - eA)^2 + m^2}, \quad H_0(\omega_0) = \omega_0 + eA_0. \quad (18) \]

Now the nonrelativistic expansion of \( \omega_0 \) and \( H_0(\omega_0) \) in powers of \( \left( \frac{p}{m} \right) \), yields the first terms of the Breit-Fermi expansion, namely, the so-called Pauli Hamiltonian [10], or to be more precise, its positive energy part. Another root of \( \omega_0, \omega_0 = -\sqrt{(p - eA)^2 + m^2} \) is out of the integration region, and in the full Minkowskian integral one would obtain instead

\[ H_{+/\-}(\omega_0) = \beta \sqrt{(p - eA)^2 + m^2} - e(\sigma B) + eA_0. \quad (19) \]

We now turn to the next two terms in (16) and take into account that \( \cos \theta \approx 1 - \frac{p^2}{8m^2}, \quad \sin \theta \approx \frac{\sigma p}{2m}, \) and hence, additional terms from \( eA_0 \) and \( \beta \sigma E \) in (10) yield

\[ U^* H(\omega) U = \cos \theta eA_0 \cos \theta + \ldots = \frac{e\Delta A_0}{8m^2} + \ldots = -\frac{\text{Div} E}{8m^2} + \ldots . \quad (20) \]

In a similar way in (20) one obtains the full \( O(1/m^2) \) form,

\[ \tilde{H} = \left( m + \frac{(p - eA)^2}{2m} - \frac{p^4}{8m^3} \right) + eA_0 - \frac{e}{2m} \sigma B + \ldots . \]

6
\[
+ \left( -\frac{e}{4m^2} \sigma(E \times p) \right) - \frac{e}{8m^2} \text{div} E, \; E = -\nabla A_0. \tag{21}
\]

Note, however, that \( \text{div} E \sim \delta^{(3)}(r) \) and the higher in \( \left( \frac{p}{m} \right) \) terms bring about even higher derivatives of the \( \delta \)-function, which makes the evaluation of this Hamiltonian questionable. Therefore it is more convenient from the beginning to consider the exact solution of the Dirac equation and compare it with the exact eigenvalues of \( H(\omega) \tag{10} \), in this way finding the accuracy of approximations made in the path integral method. This is done in the next Section.

### 4 Exact Dirac spectrum from RH for the hydrogen-like atoms

Here we study the energy eigenvalues of hydrogen-like atoms. From [10] the RH is

\[
H(\omega) = \frac{p^2}{2\omega} + \frac{m^2 + \omega^2}{2\omega} + eA_0 - \frac{i e(\alpha E)}{2\omega}. \tag{22}
\]

At this point one has two possibilities:

1) to calculate eigenvalues \( M_n(\omega) \) of \( H(\omega) \) and then to find the stationary point \( \omega_0 \) of \( M_n(\omega) \), yielding the actual eigenvalue \( M_n(\omega_0) \). This choice was used in [7] and called “the einbein method”;

2) to define \( \omega = \omega_0 \) from the condition \( \frac{dH(\omega)}{d\omega} \bigg|_{\omega=\omega_1} = 0 \), finding in this way the “stationary value” of the Hamiltonian. This brings us to the (generalized) Salpeter equation [11], extensively studied in the framework of RH, e.g., in [12] and in the relativistic quark model [13] (see [14] for reviews). In our case \( A_0 = -\frac{Z\alpha}{r} \) and \( E = -\nabla A_0 = \frac{Z\alpha}{r} n \).

We start with the simplest (einbein) procedure for the ground state, solving the equation

\[
\left( \frac{p^2}{2\omega} - \frac{Z\alpha}{r} \right) \psi = \varepsilon \psi, \; \varepsilon = -\frac{\omega(Z\alpha)^2}{2n^2}, \; n = 1, 2, \ldots \tag{23}
\]

Inserting \( \varepsilon \) in (22) and neglecting there the last term on the r.h.s., one obtains the expression for the total eigenvalue \( M_n(\omega) \):

\[
H(\omega)\Psi_n = M_n(\omega)\Psi_n, \; M_n(\omega) = \frac{m^2 + \omega^2}{2\omega} - \frac{\omega(Z\alpha)^2}{2n^2}. \tag{24}
\]
As it is prescribed by the $\omega$ integration in (11), the actual energy eigenvalue $M_n(\omega_0)$ should be obtained from $M_n(\omega)$ by the minimization procedure:

$$\left. \frac{\partial M_n(\omega)}{\partial \omega} \right|_{\omega=\omega_0} = 0; \quad \omega_0 = m\sqrt{1 - \left(\frac{Z\alpha}{n}\right)^2} = M_n(\omega_0) \quad (25)$$

This form should be compared with the exact Dirac Hamiltonian eigenvalues $M_n^D$ (see [9]):

$$M_n^D = \frac{m}{\sqrt{1 + \left(\frac{Z\alpha}{n-\delta_j}\right)^2}}, \quad \delta_j = j + \frac{1}{2} - \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2}. \quad (26)$$

It is remarkable that for the ground state with $n = 1, j = \frac{1}{2}$ the einbein approximation gives exactly the same answer, i.e.

$$M_1(\omega_0) = M_1^D \left( j = \frac{1}{2} \right) \quad (27)$$

However, for higher levels the predictions of (25) and (26) differ by $O(m(Z\alpha)^4)$. Moreover, $M_n(\omega_0)$ does not depend on $j$. In general, the einbein method gives a reasonable approximation for not highly excited QCD bound states [12], but in principle does not insure the orthogonality of different wave functions. To overcome this, we turn to the second possibility – the square root or Salpeter equation.

To this end one keeps in $H(\omega)$ (10) the last term for the hydrogen-like atoms, $A = 0, A_0 = -\frac{Z\alpha}{r}$, and (10) has the form,

$$H(\omega) = \frac{p^2 + m^2 + \omega^2 - ie\alpha E}{2\omega} - \frac{Z\alpha}{r}, \quad H\Psi = M(\omega)\Psi. \quad (28)$$

As prescribed in the second (square root or “Salpeter”) method, we define $\omega$ from the minimum of the kinetic part, written in the momentum space,

$$\left. \frac{\partial H(\omega)}{\partial \omega} \right|_{\omega=\omega_0} = 0; \quad \omega_0 = \sqrt{p^2 + m^2 - ie\alpha E}. \quad (29)$$

Hence $H(\omega_0)$ acquires the form

$$\tilde{H}(\omega_0) = \sqrt{p^2 + m^2 - ie\alpha E} - \frac{Z\alpha}{r}, \quad \tilde{H}(\omega_0)\tilde{\Psi}_n = \tilde{M}_n(\omega_0)\tilde{\Psi}_n. \quad (30)$$
Notice that in the chiral representation for \( \gamma \) matrices one can write \(-ie\alpha E \to \mp iz\frac{\sigma n}{r^2}\).

To find the eigenvalues of \( \tilde{H}(\omega_0) \) one can write \( \sqrt{p^2 + m^2 - ie\alpha E} \Psi_n = (\tilde{M}_n + \frac{Z\alpha}{r}) \Psi_n \); multiply it with the Hermitian conjugated equation times \( \beta \),

\[
\Psi_n^* \sqrt{p + m^2 + ie\alpha E} \beta \sqrt{p^2 + m^2 - ie\alpha E} \Psi_n = \Psi_n^* \left( \tilde{M}_n + \frac{Z\alpha}{r} \right) \beta \left( \tilde{M}_n + \frac{Z\alpha}{r} \right) \Psi_n,
\]

obtaining in this way the Hamiltonian

\[
\left\{ p^2 + m^2 \mp iz\frac{\sigma n}{r^2} - \left( \tilde{M}_n + \frac{Z\alpha}{r} \right)^2 \right\} \Psi_n = 0. \tag{32}
\]

Then following the same procedure, as in [9] for the same Hamiltonian (see Appendix for details of derivation), one obtains the exact Dirac spectrum (26). In this way we arrived at the Dirac spectrum starting from the square root form (31), using the quadratic expression (32).

However, direct use of the square root form in the \( x \) space brings about singularities around zero, as can be seen as follows. Indeed, proceeding

\[
\sqrt{p^2 + m^2 - ie\alpha E} \Psi_n = \left( \tilde{M}_n + \frac{Z\alpha}{r} \right) \Psi_n \to \left( p + m^2 - ie\alpha E - (\tilde{M}_n + \frac{Z\alpha}{r})^2 \right) \Psi_n = X \Psi_n \tag{33}
\]

with

\[
X = \left[ \sqrt{p^2 + m^2 - ie\alpha E}, \frac{Z\alpha}{r} \right]. \tag{34}
\]

One can see that \( X \) is a sum of the \( \delta \)-function and its derivatives. These terms can be neglected, if one excludes the small region around the origin. It is interesting that to solve Eq. (33) with \( X = 0 \) one can use (32) with \( \varepsilon_{\lambda,n} = -\frac{M_n(Z\alpha)^2}{2(n - \delta_j)^2} \), and the resulting equation for \( \tilde{M}_n \) is

\[
\tilde{M}_n^2 = m^2 - \frac{M_n^2(Z\alpha)^2}{2(n - \delta_j)^2}, \quad M_n = \frac{m}{\sqrt{1 + (\frac{Z\alpha}{n - \delta_j})^2}}, \tag{35}
\]

with \( \delta_j \) given in (25). Therefore one obtains again the exact spectrum Dirac equation, if in the coordinate space one solves the square root equation, excluding the near-zero region.
Notice, that the case of the Coulomb potential in the square-root (Salpeter-type) equation was studied analytically in [15], and a singularity in the $S$-wave radial wave function $R_0(r) \sim (mr)^{-\nu_0}$, $\nu_0 \approx 0.086583$ was found there, while the spectrum was found in the form ($l = 0$)

$$M_{n0} = \frac{2m}{\sqrt{1 + \alpha^2/4n^2}}. \quad (36)$$

Note, that the term $(-ie\alpha E)$ was not present in [15], and hence $\delta_j$ does not enter in (36).

5 Two-body QED Hamiltonian from the path integral

For two-body systems there is no exact formalism to compare with in QCD, since the Bethe-Salpeter equation is not operative with strong nonperturbative forces. In QED one can use standard perturbation theory and Salpeter equation, which ensure very high accuracy of results. Our aim in this Section is to compare the RH spectrum for two oppositely charged particles (e.g. positronium) with the standard QED calculations. We consider the problem of two charges $e_1$ and $e_2$ and write the two-body Green’s function without stochastic time contributions (radiative corrections) as in [8, 16]

$$G_{e_1e_2}(x,y) = \frac{T}{2\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} \left( D^3z^{(1)}_0 D^3z^{(2)}_0 \right)_{xy} 4tr Y_T(W) \exp(-K_1 - K_2), \quad (37)$$

where

$$Y_T = \frac{1}{4} \Gamma_1(m_1 - i\hat{p}_1)\Gamma_2(m_2 - i\hat{p}_2), \quad (38)$$

$$K_i = \int_0^T dt_E \left( \frac{\omega_i}{2} + \frac{m_i^2}{2\omega_i} + \frac{\omega_i}{2} \left( \frac{dz^{(i)}_0}{dt_E} \right)^2 \right). \quad (39)$$

In (37) the function $W$ is the vacuum averaged contour integral over paths of charges $e_1$ and $e_2$ in the e.m. field $A_\mu$

$$W = \langle \exp \left( \sum_{k=1,2} \left( e_k^i \int A_\mu(z^k_0)dz^{k}_0 + e_k \int_0^T \frac{dt_E}{2\omega_k} (\sigma_{\mu\nu} F_{\mu\nu}) \right) \right) \rangle \equiv e^{-\nu_T}. \quad (40)$$
In the case $e_1 = -e_2$, $W$ is the gauge invariant QED analogue of the Wilson loop, and below we shall consider this case for simplicity. To get rid of the c.m. motion one integrates over $d^3(x - y)$ and obtains

$$\int d^3(x - y) G_{e_1 e_2}(x, y) = \frac{T}{2\pi} \int_0^\infty \frac{d\omega_1}{\omega_1^{3/2}} \int_0^\infty \frac{d\omega_2}{\omega_2^{3/2}} Y_T d^3(x - y) e^{iP(x - y)} \times$$

$$\times \langle x \mid e^{-H(\omega_1, \omega_2, p_1, p_2)T} \mid y \rangle, \quad (41)$$

$$H(\omega_1, \omega_2, p_1, p_2) = \sum_i \frac{p_i^2 + m_i^2}{2\omega_i} + \frac{P^2}{2(\omega_1 + \omega_2)}, \quad \omega = \frac{\omega_1\omega_2}{\omega_1 + \omega_2}. \quad (42)$$

Since the last term on the r.h.s. in (42) vanishes, one is left with the c.m. Hamiltonian,

$$H(\omega_1, \omega_2, p) = \sum_{i=1,2} \frac{m_i^2 + \omega_i^2}{2\omega_i} + \frac{P^2}{2\tilde{\omega}} + \hat{V}, \quad (43)$$

where the potential $\hat{V}$ is to be found from the cluster expansion of the Wilson loop. Keeping only the $O(e^2)$ terms (bilocal correlators), one has (see [16, 17] for details)

$$\hat{V} = V_C(r) + \frac{(\sigma_1 \sigma_2 V_4(r) + S_{12} V_3)}{12\omega_1 \omega_2} + \left(\frac{\sigma_1 L}{4\omega_1^2} + \frac{\sigma_2 L}{4\omega_2^2}\right) \frac{1}{r} V_0'(r) + \left(\frac{\sigma_1 + \sigma_2}{2\omega_1 \omega_2}\right)^2 \frac{V_2(r)}{r}, \quad (44)$$

where

$$V_C(r) = \int_0^r \lambda d\lambda \int_0^\infty d\nu D^{(2)}(\lambda, \nu), \quad (45)$$

$$V_4(r) = \int_0^\infty d\nu \left(3D^{(2)}(r, \nu) + 2r^2 \frac{D^{(2)}(r, \nu)}{\partial \nu^2}\right), \quad (46)$$

$$V_3(r) = -r^2 \frac{\partial}{\partial r^2} \int_0^\infty d\nu D^{(2)}(r, \nu), \quad (47)$$

$$V_0'(r) = r \int_0^\infty d\nu D^{(2)}(r, \nu), \quad V_2'(r) = r \int_0^\infty d\nu D^{(2)}(r, \nu), \quad (48)$$

and $D^{(2)}(\lambda, \nu)$ is the quadratic correlator

$$e^2 \langle F_{\mu\nu}(x) F_{\lambda\rho}(y) \rangle = \frac{1}{2} \left[ \frac{\partial}{\partial u_\mu} (u_\lambda \delta_{\nu\rho} - u_\rho \delta_{\lambda\nu}) + \left( \begin{array}{c} \mu \leftrightarrow \nu \end{array} \right) \right] D^{(2)}(u). \quad (49)$$
with \( u = x - y \). To the lowest order \( D^{(2)}(u) \) is \((e_1 = -e_2 = e)\)

\[
D^{(2)}(u) = \frac{4\alpha}{\pi u^4}, \quad \alpha = \frac{e^2}{4\pi}. \tag{50}
\]

Note, that the accurate derivation of the spin-dependent terms, valid both for QCD and QED, taking into account the proper positions of \((m_i - \hat{D}_i)\) terms, is done in \([16]\). In the QED case substituting \(D^{(2)}\) from (50) one obtains the familiar results \((S_{12} = \frac{1}{4}(3\sigma_1 \sigma_2 n - \sigma_1 \sigma_2))\)

\[
V_C(r) = -\frac{\alpha}{r}, \quad \frac{1}{r}V'_0 = \frac{1}{r}V'_2 = \frac{\alpha}{r^3}; \quad V_3 = \frac{3\alpha}{r^3}, \quad V_4 = 8\pi\alpha\delta^{(3)}(r). \tag{51}
\]

These expressions coincide with the corresponding nonrelativistic spin-dependent potentials, when \(\omega_i = m_i\), but in our case \((44), (51)\) are applicable in the relativistic case to the order \(O(\alpha^5)\). Note, that in the case of positronium the additional term in \(\hat{V}\) appears due to the annihilation diagram, which in the nonrelativistic limit is

\[
V_5 = \frac{\pi\alpha}{2m^2}(\sigma_1 \sigma_2 + 3) \tag{52}
\]

One can now proceed as in (28), (29), but treating all terms in \(\hat{V}\) (44) as a perturbation, except for \(V_C(r)\), and for \(m_1 = m_2 = m\), \(\Delta \hat{V} \equiv \hat{V} - V_C(r)\) one obtains

\[
\hat{H}_{e,-e} = 2\sqrt{p^2 + m^2} - \frac{\alpha}{r} + \Delta \hat{V} \equiv H_0 + \Delta \hat{V}. \tag{53}
\]

Again, as in (32), for \(\Psi_n^{(0)}, H_0\Psi_n^{(0)} = M_n^{(0)}\Psi_n^{(0)}, \) one has

\[
\{4(p^2 + m^2) - (M_n^{(0)} + \frac{\alpha}{r})^2\}\Psi_n^{(0)} = 0 \tag{54}
\]

and the analog of the angular operator \(\hat{N}^2\) (see Appendix) is now diagonal with eigenvalues \(\lambda(\lambda + 1) = L(L + 1) - \frac{\alpha^2}{4}\), yielding the eigenvalues \(\varepsilon_n = -\frac{M_n^{(0)}\alpha^2}{8\bar{n}^2}, \quad \bar{n} = n - \delta_L\), with

\[
\delta_L = L - \sqrt{(L + \frac{1}{2})^2 - \frac{\alpha^2}{4} + \frac{1}{2}}. \tag{55}
\]

Finally one obtains for \(M_n^{(0)}\),

\[
M_n^{(0)} = \frac{2m}{\sqrt{1 + \frac{\alpha^2}{4\pi}}} \tag{56}
\]
The expansion in $\alpha^2$ produces the expected result,

$$M_n^{(0)} = 2m - \frac{\alpha^2 m}{4n^2} + \ldots \approx 2m - \frac{\alpha^2 m}{4n^2} + O(\alpha^4). \tag{57}$$

At this point we can compare the accuracy of our expressions (56) with the account of the potentials $V_4, V_5$ in (50), (51) to the results of QED perturbation theory for the orthopositronium ($1^3S_1 - 2^3S_1$) interval $\Delta E$ (see reviews [18, 19] for results and discussions). From [18], Table 5, one obtains in perturbation theory

$$\Delta E_{PT} = \Delta E_{PT}(\alpha^2) + \Delta E_{PT}(\alpha^4) + \Delta E_{PT}(\alpha^n, n \geq 5),$$

where

$$\Delta E_{PT}(\alpha^2) = \frac{12336907351}{10^9} \cdot MHz, \tag{58}$$

$$\Delta E_{PT}(\alpha^4) = -82.0056 \cdot 10^3 MHz, \tag{59}$$

$$\Delta E_{PT}(\alpha^5) = -1.5014 \cdot 10^3 MHz. \tag{60}$$

At the same time our Eq. (56) contributes the same amount in the order $O(\alpha^2)$, $\Delta E_{RH}(\alpha^2) = \Delta E_{PT}(\alpha^2)$, while in $O(\alpha^4)$ its contribution from $M_n^{(0)}$ is $\Delta E'_{RH}(\alpha^4) = 23.9515582 \cdot 10^3 MHz$, and from the potentials $V_4, V_5$ one obtains $\Delta E''_{RH}(\alpha^4) = -102.1933153 \cdot 3 MHz$, so that the total contribution in the order $O(\alpha^4)$ is

$$\Delta E_{RH}(\alpha^4) \equiv \Delta E'_{RH}(\alpha^4) + \Delta E''_{RH}(\alpha^4) = -78.2417571 \cdot 10^3 MHz, \tag{61}$$

which should be compared to $\Delta E_{PT}(\alpha^4)$, Eq. (59). One can see, that the difference between these numbers is of the order $O(10^{-6})$ of the total result for $\Delta E$, and is in the realm of the $O(\alpha^5)$ corrections. Note also, that the relativistic $O(\alpha^4)$ corrections, coming from the square root expression (56), are of the vital importance for the resulting accuracy. In this way we have proved, that the square root of the two-body Hamiltonian (53) is able to provide the high accuracy for the positronium spectrum.

6 Discussion of results

We have calculated the spectrum of the hydrogen-like atoms in QED, using our RH, derived in the framework of the path integral. This spectrum exactly coincides with the spectrum of the Dirac equation.

It was shown above, that in the first approach (the einbein approximation), where the eigenvalues are functions of virtual energy $\omega$, one obtains a
reasonable result for the relativistic ground state energy, however for higher eigenvalues corrections are of the order of $(Z\alpha)^4$.

At the same time the second approach, where the virtual energy is defined on the operator level, provides the square-root-type Hamiltonian, which yields the exact Dirac spectrum. In this way our results support the so-called Salpeter approach in the relativistic quark models, which was so successful in predicting hadronic states [12, 13, 17, 20]. However, in QCD the string correction needs to be taken into account, to provide orbital and radial Regge trajectories [12] in good agreement with experiment.

We have also shown, how the Breit-Fermi nonrelativistic expansion is obtained from our RH, when Foldy-Wouthuizen transformation is applied.

Finally, the case of two oppositely charged particles was considered and all interaction terms, including spin-dependent ones, were derived and included in the resulting Hamiltonian. The latter contains both kinematic relativistic effects and lowest order dynamic effects, and our formalism allows to distinguish between two contributions. A short comparison to the standard QED perturbation results is done for the $(2^3S_1 - 1^3S_1)$ energy interval of positronium showing a good accuracy of the RH for the positronium spectrum.

Summarizing these results, one can consider RH as a reliable tool for the studies both in QED and of hadronic properties in QCD with the proper comparison with lattice and experimental results.

Another important line of development is the theory of QED systems in strong magnetic field, where the RH approach was formulated in [8, 16], and a new phenomenon of the magnetic focusing is found in [21].

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Explicit solution of Eq. (33)

Following [9] we write Eq. (32) in the form

\[ \{-\Delta_r + \frac{\hat{N}^2}{r^2} - \frac{2Z\alpha}{r} M_n - (M_n^2 - m^2)\} \psi_n = 0, \quad (A. 1) \]

where \( \hat{N}^2 \) in the chiral representation for the matrices \( \alpha_i \) in the term \(-i\alpha E\) is written in the diagonal form as \( \mp iZ\alpha (\sigma^i) \). For the total angular momentum \( J = L + \frac{\sigma}{2} \) with eigenvalues \( j = \frac{1}{2}, \frac{3}{2}, ... \) one can define \( \hat{N}^2 \) as the matrix in the states \( l_{\pm} = j \pm \frac{1}{2} \), which has the form

\[ \hat{N}^2 = \begin{pmatrix} l_+ (l_+ + 1) - (Z\alpha)^2 & \mp iZ\alpha \\ \mp iZ\alpha & l_- (l_- + 1) - (Z\alpha)^2 \end{pmatrix}. \quad (A. 2) \]

The eigenvalues of \( \hat{N}^2 \) are found from (A. 2) to be

\[ \hat{N}^2 = \lambda(\lambda + 1), \quad \lambda = \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2 - 1}, \quad \sqrt{\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2}, \quad (A. 3) \]

and writing \( \lambda = \left(j + \frac{1}{2}\right) - \delta_j \), one can define the radial quantum number \( n_r \), pertinent to \( \Delta_r, n_r = 0, 1, 2, ... \) and the solution of the reduced Coulomb problem (the first three terms in (A. 1)) is

\[ \varepsilon_n = -\frac{(Z\alpha)^2 M_n}{2\tilde{n}^2}, \quad (A. 4) \]

where \( \tilde{n} = n_r + \lambda + 1 = n_r + j + \frac{1}{2} + 1 - \delta_j = n - \delta_j \), \( n = 1, 2, ... \)

Finally, from (A. 1) one finds that \( M_n^2 - m^2 = 2M_n\varepsilon_n \), or

\[ M_n = \frac{m}{\sqrt{1 + \frac{(Z\alpha)^2}{(n - \delta_j)^2}}}. \quad (A. 5) \]

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