ON THE ZERO ATTRACTOR OF THE EULER POLYNOMIALS

WILLIAM M.Y. GOH AND ROBERT BOYER

ABSTRACT. We study the limiting behavior of the zeros of the Euler polynomials. When linearly scaled, they approach a definite curve in the complex plane related to the Szegő curve which governs the behavior of the roots of the Taylor polynomials associated to the exponential function. Further, under a conformal transformation, the scaled zeros are uniformly distributed.

1. INTRODUCTION

Eighty years ago Szegő [8] studied the asymptotic behavior of the roots of the Maclurin polynomials associated with the exponential function. He found that if the roots are linearly scaled relative to the degree then the roots approach a curve $S$ (see Figure 2) in the complex plane given by $z \in \mathbb{C}$ such that $\frac{1}{2}e^{\|z\|} = 1$ and $\|z\| > 1$. The behavior of the roots and poles of the Padé approximants and other Taylor polynomials have been analyzed [11].

On the other hand, given any sequence of polynomials $p_n(x)\xi$, where $p_n(x)$ is of degree $n$, asking how are the zeros of $p_n(x)$ distributed in the complex $x$-plane is too general to get a reasonable answer. The best we can hope for is to focus on a special family of polynomials where a definite answer is possible.

In this paper, we initiate the study of the asymptotic behavior of the roots of Euler polynomials $E_n(x)$ which are defined by means of generating functions as

\begin{equation}
\frac{2e^{\xi x}}{e^{\xi} + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{\xi^n}{n!};
\end{equation}

This generating function is listed among the principal generating functions by Louis Comtet [2] for combinatorial applications. Their linearly scaled roots approach a curve related to the Szegő curve $S$ together with an interval on the real axis.

Polynomials of binomial type were introduced by Rota and Mullin [7]. The reason for the name is that if

\begin{equation}
e^\xi D(x) = \sum_{n=0}^{\infty} \frac{\Phi_n(y)}{n!} x^n;
\end{equation}

where $D(x)$ is a polynomial, it follows that

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\[ \phi_n (u + v) = \sum_r (\zeta_r \phi_r (u) \phi_n (v)); \]

which is a reminiscent of the binomial theorem. Herbert Wilf has the same description in his book “generatingfunctionology” [12]. Strictly speaking, \( E_n (x) \) is not a polynomial of binomial type. However, it is close to being binomial type and has the simplest \( D (x) \) function. In this case \( D (\xi) = \xi \). We hope to investigate the zero attractors of this wider class of polynomial families in the future. As evidence, the family of Bernoulli polynomials, for example, are easily handled with the techniques in this paper. A study of the behavior of their real zeros was recently done [10].

Let \( \mathcal{fp}_n (x) \) be a sequence of polynomials. A set \( A \) in the \( x \)-plane is called the zero attractor of zeros of \( \mathcal{fp}_n (x) \) if the following two conditions hold:

a) Let \( A \in \sum_{x \in A} B (x; \varepsilon) \), where \( B (x; \varepsilon) \) is the open disc centered at \( x \) with radius \( \varepsilon \). That is, \( A \in \varepsilon \)-neighborhood of the set \( A \). Then there exists an integer \( n_0 (\varepsilon) \), for all \( n_0 \), all zeros of \( p_n (x) \) are in \( A \in \varepsilon \).

b) For all \( x \in A \) and for all \( \varepsilon > 0 \), there exists an integer \( n_1 (x; \varepsilon) \) and a zero \( r \) of the polynomial \( p_{n_1} (x) \) such that \( r \in B (x; \varepsilon) \).

Condition b) simply says that every point of \( A \) is an accumulation point of zeros of \( \mathcal{fp}_n (x) \).

The Euler polynomials \( E_n (x) \) are defined in (1.1). Since the nearest singularity to the origin of \( \frac{1}{e^x + 1} \) are \( \xi = \pi i \); it is easy to see that for all \( x \in \mathbb{C} \); the power series in (1.1) converges absolutely and uniformly on any compact subset in \( \xi < \pi \). In other words, although the polynomial \( E_n (x) \) is defined for all complex \( x \) but the power series is convergent only for \( \xi \) with \( \xi < \pi \). By the Cauchy residue theorem, we have:

\[ \frac{E_n (x)}{n!} = \frac{2}{2\pi i} \int_{\xi + 1}^{\xi} \frac{e^{\xi n}}{(e^\xi + 1)^{n+1}} d\xi \]

This integral expression is valid for all \( x \in \mathbb{C} \). Let \( x \) be replaced by \( nx \) and we can write the above equation as:

\[ \frac{E_n (nx)}{n!} = \frac{2}{2\pi i} \int_{\xi + 1}^{\xi} \frac{e^{nx \xi}}{\xi (e^{\xi n} + 1)^{n+1}} d\xi \]

The goal of this paper is to study the zero distribution of the polynomial \( E_n (nx) \).

2. A Generalization of the Szegö Approximation

We state two generalizations of the Szegö approximation. Let

\[ S_n (\xi) := \sum_{j=0}^{n} \frac{z^j}{j!} ; \]
**Proposition 1.** Let $S$ be a subset contained in $|z| > 1$ so that the distance between $S$ and the unit circumference $|z| = 1$ is $\delta > 0$ and let $\alpha$ be chosen so $1 < \alpha < 1 = \frac{3}{2}$. Then

$$S_{n} = \frac{1}{e^{nz}} = \frac{(ze^{n}z)^{n}}{2\pi n (z^{n}1)} \left(1 + O(n^{1/3})\right);$$

where the constant in the big $O$ term is uniform for all $z \not\in S$.

**Proof.** By residue theory, we have for $R > 0$

$$S_{n} = \frac{1}{2\pi i} \int_{\gamma_{n} = R} \frac{e^{z} \zeta^{n}}{z^{n} \zeta^{n}} d\zeta;$$

Note that $\zeta = z$ is a removable singularity of the integrand. Therefore, the above expression is valid for all complex $z$. For asymptotics in the region $|z| > \frac{1}{1+\delta}$, we choose the contour to be the circle $|\zeta| = \frac{n}{1}$. Thus

$$S_{n} = \frac{1}{2\pi i} \int_{\gamma_{n} = \eta} \frac{e^{\zeta} \zeta^{n}}{n \zeta^{n}} d\zeta;$$

Since $nz$ is not included inside the contour $|\zeta| = n$, a simple application of Cauchy’s Theorem gives:

$$S_{n} = \frac{(nz)^{n}}{2\pi i} \int_{\gamma_{n} = \eta} \frac{e^{\zeta} \zeta^{n}}{n \zeta^{n}} d\zeta = \frac{e^{n \zeta \ln \zeta}}{2\pi i} \int_{\gamma_{n} = \eta} \frac{e^{\zeta \ln \zeta}}{\zeta \zeta} d\zeta;$$

where $\ln \zeta$ is the principal branch with $\pi < \arg \zeta < \pi$ and $n$ is a positive integer. Now we apply the saddle point method to construct the asymptotics of the integral. Since the critical point is $\zeta = 1$, the neighborhood of $\zeta = 1$ must be carefully analyzed. Let $\eta = n^{\alpha}$ with $1 < \alpha < 1 = \frac{3}{2}$. The contour integral is decomposed into two integrals:

$$1 \int_{\gamma_{n} = \eta} \frac{e^{\zeta \ln \zeta}}{\zeta \zeta} d\zeta = I_{1} + I_{2};$$

where $I_{1}$ is the integral on the circular arc in a small neighborhood of $1: \eta \arg \zeta \eta; \zeta \gamma_{n} = 1$, and the integral $I_{2}$ is along the path in the remaining part of the circle. Set $\zeta = e^{i\theta}$ in $I_{1}$. The Taylor expansion of the integrand in a small neighborhood of $\theta = 0$ is worked out below:

$$e^{n \zeta \ln \zeta} = e^{n e^{i\theta} (\frac{n^{2}}{2} + O(\theta^{3}))} = e^{n e^{- n\theta^{2}/2}} \left(1 + O(n^{1/3})\right).$$
Inserting these estimates in $I_1$ and carrying out some simplifications we get

$$I_1 = \frac{1}{2\pi i} \int Z \eta e^{n \theta^2 z} (1 + O \eta^{1/3 \alpha}) \frac{e^{i\theta} d\theta}{z} = \frac{e^n}{2\pi} \int \frac{Z \eta e^{n \theta^2 z} (1 + O \eta^{1/3 \alpha})}{1 + O \eta^{1 + \alpha}} d\theta$$

Since $1 = 3 < \alpha < 1 = 2$; we have $3\alpha < 1 < \alpha$. So the error term $O \eta^{1/\alpha}$ is absorbed into $O \eta^{1/3 \alpha})$. Hence, for $\delta > 1 + \delta$, we see

$$I_1 = \frac{e^n}{2\pi (1 + \delta)} \int Z \eta e^{n \theta^2 z} (1 + O \eta^{1/3 \alpha})$$

where the big $O$ term holds uniformly for $\delta > 1 + \delta$. If we put $n \theta^2 = u^2$, we get

$$I_1 = \frac{e^n}{2\pi (1 + \delta)} \int Z \eta e^{u^2} \left( 1 + O \eta^{1/3 \alpha} \right)$$

where $\omega = \frac{\eta}{\delta + \frac{2\alpha}{3}}$. Since $\alpha < 1 = 2$, $\omega$ tends to $\infty$ with $n$. But, as $n \to \infty$, $e^{u^2} du = O \psi(\omega^2) = o \psi^{1/3 \alpha}$). We may therefore replace the limits of integration by $\infty$ without altering the error term $1 + O \eta^{1/3 \alpha}$). This gives

$$I_1 = \frac{e^n}{2\pi (1 + \delta)} \int Z \eta e^{u^2} \left( 1 + O \eta^{1/3 \alpha} \right) = \frac{e^n}{2\pi n (1 + \delta)} (1 + O \eta^{1/3 \alpha})$$

after taking $\int e^{u^2} du = \frac{\pi}{2} \theta$ into consideration. To justify that $I_1$ gives the major contribution we obtain an upper estimate for $I_2$:

$$I_2 \leq \frac{1}{2\pi} \int Z \eta e^{n \Re(\zeta)} \frac{e^{\eta \Re(\zeta)}}{\delta} \frac{e^{n \cos \eta \Re(\zeta)}}{\delta}$$

where $C$ is the contour determined by $\eta \gamma \zeta$, $\pi$; and $\gamma = 1$. Obviously, $\Re(\zeta) + \cos \eta$. Note that $\zeta < \pi$ for all $\eta > 0$. Hence

$$I_2 \leq \frac{1}{2\pi} \int e^{n \cos \eta \Re(\zeta)} \frac{e^{n \cos \eta \Re(\zeta)}}{\delta} = \frac{e^{n \cos \eta \Re(\zeta)}}{\delta}$$

Upon using

$$\cos \eta = 1 - \frac{\eta^2}{2} + O(\eta^4)$$

we see that $e^{n \cos \eta} = e^ne^{\frac{1}{2}n^2 \eta^2 (1 + O \eta^{1/3 \alpha})}$. But the factor $e^{\frac{1}{2}n^2 \eta^2} = o(e^{1/\eta})$. Consequently,

$$I_2 = o(\eta^4)$$

This completes the proof of Proposition 1.

Next, the following proposition states the asymptotics of $S_n(\zeta)$ in the region $\Re(\zeta) < 1$. 
Proposition 2. For \(1=3<\alpha<1=2\), we have

\[
\frac{S_n 1 (nz)}{e^{nz}} = 1 + \frac{(\zeta e^{1/z})^n}{2\pi i (1/z)} + O \left( n^{-1} 3^\alpha \right); \\
\]

where the big O constant holds uniformly for an arbitrary compact set \(K \subset \mathbb{R}(z) < 1\).

Thus the ordinary Szegö approximation is generalized from the open disc \(|z|<1\) to the open half plane \(\mathbb{R}(z) < 1\).

Proof. The proof is actually very similar to that of Proposition 1. Here we use a suitable integral representation for \(S_n 1 (z)\) in the region. We start off with:

\[
S_n 1 (z) = \frac{1}{2\pi i} \int_C \frac{e^{\zeta/z}}{z^n} \frac{dz}{\zeta^n} ; \\
\]

where \(C\) is any closed contour encircling the origin. This integral representation is valid for all complex \(z\). We insert \(nz\) for \(z\) to obtain

\[
S_n 1 (nz) = \frac{1}{2\pi i} \int_{nC} \frac{e^{n\zeta/z}}{nz^n} \frac{dz}{\zeta^n} ; \\
\]

Let \(K\) be an arbitrary compact set in the domain \(\mathbb{R}(z) < 1\). We choose the circular integration contour \(C : \zeta = R + (R + 1)e^{i\theta}; \pi \theta \pi\) with \(R\) so large that \(K\) becomes strictly included inside \(C\). Let \(z\) belong to \(K\) and replace the contour \(C\) by \(nC\). This gives

\[
S_n 1 (nz) = \frac{1}{2\pi i} \int_{nC} \frac{e^{n\zeta/z}}{nz^n} \frac{dz}{\zeta^n} ; \\
\]

After a change of variables \(\zeta = n\zeta\) and the use of the Cauchy Integral Formula, we get

\[
S_n 1 (nz) = e^{nz} \frac{(nz)^n}{2\pi i} \int_C \frac{e^{n\zeta/z}}{z} \frac{dz}{\zeta^n} ; \\
\]

This implies

\[
\frac{S_n 1 (nz)}{e^{nz}} = 1 + \frac{e^{nz} z^n}{2\pi i} \int_C \frac{e^{n\zeta/z}}{\zeta^n} \frac{dz}{z} \ln(\zeta^n) ; \\
\]

where \(C\) is the circle \(\zeta = R + (R + 1)e^{i\theta}; \pi \theta \pi\). The critical point is still \(\zeta = 1\) which is a point in the contour \(C\). It is clear that the saddle point method can be applied to the contour integral. The procedure is very similar to what we did in the previous case. We omit the details. This is how we prove the statement of Proposition 2.

We can rederive the same result by using the integral representation of \(S_n (z)\) in the original derivation of the Szegö approximation. Szegö [9] used

\[
\frac{S_n (nz)}{e^{nz}} = 1 + \frac{n^{n+1} e^{-1} z^n}{n!} \int_0 (ve^{v})^n dv \\
\]
to obtain the result of the proposition in the disc \( \frac{1}{2} \pi j < 1 \). A good elaboration of his approach can indeed offer a proof of Proposition 2. The saddle point method as we presented in the proof is just an easier way to get the result.

**Proposition 3.** *(Jet Wimp) (Uniform Approximation of \( S_n(nt) \) for \( t \to 0 \))*

\[
\frac{S_n(nt)}{e^{nt}} = \delta(\psi) + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{\xi_j}{\xi_j - n\xi_j} \operatorname{Erfc}(\frac{\psi}{n} \xi_j) + O\left(\frac{1}{n}\right)
\]

uniformly for \( t \to 0 \), where \( \xi_j(\psi) = \frac{1}{n} \ln \frac{1}{1 - j^2} \), and

\[
\delta(\psi) = \begin{cases} 1 & \text{for } 0 < t < 1; \\ 0 & \text{for } t = 1; \end{cases}
\]

and for all \( t \);\( \operatorname{Erfc}(\psi) := \int_t^{\infty} e^{-s^2} ds \).

This version of the Szegö approximation was proved by Jet Wimp (personal communication).

3. **A Decomposition for the Euler Polynomial**

The Euler polynomial \( E_n(nt) = n! \) is decomposed as the sum of two polynomials:

**Proposition 4.** Let \( \mu \) be an integer \( 0 \) and let \( F_\mu(\xi) \) be given as

\[
F_\mu(\xi) := \frac{1}{\xi(e^\xi + 1)} + \sum_{k=0}^{\mu} \frac{1}{(2k+1)\pi i} \frac{1}{(2k+1)\pi i} + \frac{1}{(2k+1)\pi i} \frac{1}{(2k+1)\pi i} ;
\]

Then we have

\[
\frac{E_n(nt)}{n!} = M_{n\mu}(x) + K_{n\mu}(x);
\]

where

\[
M_{n\mu}(x) = \frac{1}{\pi i} \int_{\xi = 1} e^{x\xi} \frac{1}{\xi} F_\mu(\xi) d\xi;
\]

\[
K_{n\mu}(x) = 2 \sum_{k=0}^{\mu} \frac{S_{n-1}(nt(2k+1)\pi i)}{(2k+1)\pi i)^{n+1}} + \frac{S_{n-1}(nt(2k+1)\pi i)}{(2k+1)\pi i)^{n+1}} ;
\]

Here \( S_n(\xi) := \sum_{j=0}^{\infty} z^j = j! \), the \( n \)-th partial sum of \( e^\xi \) as usual.

**Proof.** The following integral representation for \( E_n(x) \) is valid for all \( x \in \mathbb{C} \):

\[
\frac{E_n(x)}{n!} = \frac{2}{2\pi i} \int_{\xi = 1} e^{x\xi} \frac{1}{\xi} F_\mu(\xi) d\xi;
\]

Let \( x \) be replaced by \( nx \) to get

\[
\frac{E_n(nt)}{n!} = \frac{2}{2\pi i} \int_{\xi = 1} e^{x\xi} \frac{1}{\xi} F_\mu(\xi) d\xi.
\]
Note that for each integer $k > 0$,

\[
\frac{1}{(\xi k + 1)\pi i \xi (\xi k + 1)\pi i)} \quad \frac{1}{(\xi k + 1)\pi i \xi + (\xi k + 1)\pi i)}
\]

is the sum of singular parts of $\frac{1}{\xi k + 1}$ at $\xi (\xi k + 1)\pi i$ and $(\xi k + 1)\pi i$. Hence $F_\mu(\xi)$ is analytic in the annulus $0 < \xi < (\xi k + 3)\pi$. The bigger $\mu$ is, the bigger the domain of analyticity of $F_\mu(\xi)$ is. Hence

\[
\sum_{k=0}^\mu \frac{1}{(\xi k + 1)\pi i \xi (\xi k + 1)\pi i)} + \frac{1}{(\xi k + 1)\pi i \xi + (\xi k + 1)\pi i)} \quad d\xi
\]

\[(3.1)\]

A typical term of the above sum is:

\[
\frac{2}{2\pi i} \left(\frac{e^{i\xi} \xi}{(\xi k + 1)\pi i \xi (\xi k + 1)\pi i)} \right)^n \frac{1}{(\xi k + 1)\pi i \xi} \quad \frac{1}{(\xi k + 1)\pi i \xi} \quad \frac{1}{(\xi k + 1)\pi i \xi + (\xi k + 1)\pi i)} \quad d\xi
\]

\[
= \frac{2}{2\pi i} \frac{1}{(\xi k + 1)\pi i \xi} \left(\frac{e^{i\xi} \xi}{(\xi k + 1)\pi i \xi} \right)^n \frac{1}{(\xi k + 1)\pi i \xi + (\xi k + 1)\pi i)} \quad d\xi
\]

after using a geometric series expansion. Obviously, the series is uniformly convergent on $\xi = 1$, we carry out the integration term by term. By the Cauchy integral theorem only those terms with $l = n + 1$ survive. Thus we get

\[
\frac{1}{\pi i} \frac{1}{(\xi k + 1)\pi i)}^2 \sum_{l=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]

\[
= \frac{1}{\pi i} \frac{1}{(\xi k + 1)\pi i)}^2 \sum_{l=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]

\[
= \frac{2}{(\xi k + 1)\pi i)}^2 \sum_{l=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]

\[
= \frac{2}{(\xi k + 1)\pi i)}^2 \sum_{j=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]

\[
= \frac{2}{(\xi k + 1)\pi i)}^2 \sum_{j=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]

\[
= \frac{2}{(\xi k + 1)\pi i)}^2 \sum_{j=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]

\[
= \frac{2}{(\xi k + 1)\pi i)}^2 \sum_{j=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]

\[
= \frac{2}{(\xi k + 1)\pi i)}^2 \sum_{j=0}^n \left(\frac{1}{\xi + 1} \right) \frac{1}{(\xi k + 1)\pi i)} \cdot e^{i\xi} \xi \quad d\xi
\]
where \( S_n(z) \) is the \( n \)-th partial sum of \( e^z \); that is, \( S_n(z) = \sum_{j=0}^{n} z^j/j! \). In a similar way, we obtain

\[
\frac{2}{2\pi i} \int_{\mathcal{C}} \frac{e^{xz}}{z} \left( \frac{1}{(2k+1)\pi i} \frac{1}{z + (2k+1)\pi i} \right) d\xi = \frac{2}{(2k+1)\pi i} \sum_{j=0}^{n} e^{(nx)(2k+1)\pi i}
\]

Inserting these back into (3.1), we complete the proof of the proposition.

Proposition 4 is important since it provides asymptotics for the Euler polynomials in various regions. The asymptotics for \( M_{\mu}(x) \) (for any fixed \( \mu \)) can be easily found by the classical saddle point method. It is

\[
M_{\mu}(x) = \frac{1}{\pi} e^{x} \left( \frac{1}{\mu} \right)^n F_{\mu} \left( 1 + \frac{1}{n\pi x} \right) \left( 1 + O \left( \frac{1}{n} \right) \right);
\]

where the big \( O \) term holds uniformly for \( \varepsilon \left( \frac{1}{\pi x} \right) (2\mu + 1)\pi + \varepsilon \). The uniformity can be justified using the fact that \( F_{\mu}(\xi) \) is analytic in \( 0 < \xi < (2\mu + 3)\pi \). The asymptotics for \( K_{\mu}(x) \) can be obtained from the generalized Szegö’s approximation. Using these asymptotic approximations we will prove that the point set \( K \) defined below is the zero attractor of the Euler polynomials and can also determine their density distribution.

4. The Zero Attractor

Let the point set \( K \) be defined by the graph in Figure 1:

The point set \( K \) consists of the curves as indicated and the real interval \([\frac{1}{\pi}, \frac{1}{\pi}][\\text{]}\). \( K \) is symmetric with respect to the imaginary axis in the \( x \)-plane. We will show as the first step that \( K \) contains all accumulation points of the zeros of \( E_n(\eta x) \). In the second step we will carry out the density calculations of the zeros residing in an immediate neighborhood of \( K \). As a consequence, this will establish that every point \( K \) is an accumulation point of zeros of \( E_n(\eta x) \). Hence \( K \) is precisely the zero attractor of \( E_n(\eta x) \). The density calculation gives also the statistical distribution of zeros which includes the information about the fraction of zeros along each segment of \( K \). The work of proving that \( K \) contains all accumulation points is divided into the three parts in the following lemma:

Lemma 1. (a) Let \( \chi \) \( x > \frac{1}{\pi} \); then \( x_0 \) is not an accumulation point of zeros of the Euler polynomials \( E_n(\eta x) \).

(b) There is no accumulation point of zeros in the region \( \chi \] \( \frac{1}{\pi} < \eta \) \( \frac{1}{\pi} \) except for real numbers.

(c) If \( x \) is a non-real accumulation point of zeros in the region \( \frac{1}{\pi} < \chi \) \( \frac{1}{\pi} \); then we must have either

\[
x_\pi i \eta = 1 \quad \text{or} \quad x_\pi i + x_\pi i = 1;
\]
Proof. Let us prove part (a). Now
\[ \frac{E_n(\eta x)}{n!} = \frac{2\pi}{2 \pi i} \left[ \frac{e^{\frac{1}{\xi}} - 1}{\frac{1}{\xi}} \right] \frac{d\xi}{\xi (e^{\xi x} + 1)} \]
where \( f(\xi) = \xi \ln \xi \). We choose the principal branch of \( \ln \xi \) here. To invoke the saddle point method, we need to find the critical points which are roots of \( f'(\xi) = 0 \) or \( 1 \xi = 0 \), that is, \( \xi = 1 \) is the only critical point in question. Observe that
\[ e^{\cos \theta} \text{attains its maximum at } \theta = 0; \]
hence at \( \xi = 1 \). By the saddle point method we have
\[ \frac{E_n(\eta x)}{n!} = \frac{2\pi}{2 \pi i} \left[ \frac{e^{\frac{1}{\xi}} - 1}{\frac{1}{\xi}} \right] \frac{d\xi}{\xi (e^{\xi x} + 1)} \]
where the big \( O \) constant can be made uniform for a given compact set \( K \). From this, we see that for any given set \( K \), there exists \( n_0 \) such that for all \( n > n_0 \) the polynomial \( E_n(\eta x) \) has no zeros in \( K \). Hence the proof of part (a).
Next we prove part (b). Assume that there is a non-real accumulation point of zeros in the region \( \mathcal{R} = \mathbb{C} \setminus \mathbb{R} \), say \( x_0 \). Of course, \( x_0 \neq 0 \), then there exists an integer \( \mu_0 \geq 2 \) and a positive number \( \varepsilon > 0 \) such that

\[
\frac{1}{\mathcal{Q}_0(1)} + \varepsilon < x_0 \frac{1}{\mathcal{Q}_0(1)}
\]

Since \( x_0 \) is an accumulation point of zeros of \( E_n(x) \), there exists an infinite sequence of integers \( n_j \) such that

\[
E_{n_j}(x_n) = 0 \quad \text{and} \quad x_n \to x_0 \quad \text{as} \quad j \to \infty.
\]

We may assume that for all large \( j \), \( x_n \) is in the region

\[
\frac{1}{\mathcal{Q}_0(1)} + \varepsilon < x_0 \frac{1}{\mathcal{Q}_0(1)}
\]

so we apply Proposition 4 with \( \mu \) chosen as \( \mu_0 \) and keep \( \mu_0 \) fixed in the following arguments. Use the asymptotics of \( M_{\mu_0}(x) \), we see that

\[
E_{n_j}(x_n) = 0 \quad \text{implies} \quad (4.1)
\]

\[
e \frac{2}{\pi} \frac{(\alpha_{n_j})^{n_j}}{\mathcal{Q}_0(1)} 
\]

\[
1 + O \left( \frac{1}{n_j} \right) + K_{n_{\mu_0}}(\alpha_{n_j}) = 0,
\]

where the big \( O \) term holds uniformly for all \( x \) in the region

\[
\frac{1}{\mathcal{Q}_0(1)} + \varepsilon < x_0 \frac{1}{\mathcal{Q}_0(1)}
\]

Note that the summation in \( K_{n_{\mu_0}}(\alpha_{n_j}) \) is running from 0 to \( \mu_0 = 1 \) and for all large \( n_j \)

\[
\frac{1}{\mathcal{Q}_0(1)} + \varepsilon < (2k + 1)\pi \quad \text{and} \quad 1 - x_n \frac{1}{\mathcal{Q}_0(1)}
\]

\[
\frac{1}{\mathcal{Q}_0(1)} + \varepsilon < (2k + 1)\pi
\]

A typical term in \( K_{n_{\mu_0}}(\alpha_{n_j}) \) is

\[
S_{n_j}(\alpha_{n_j}) \frac{1}{(2k + 1)\pi i^{n_j+1}}
\]

Introducing \( t = x_n (2k + 1)\pi i \). For \( 0 < k < 2 \), we have \( \varepsilon \). Hence we apply Szegö approximation of Proposition 2. Thus

\[
\frac{S_{n_j}(\alpha_{n_j})}{(2k + 1)\pi i^{n_j+1}} = \frac{e^{\pi j t}}{(2k + 1)\pi i^{n_j+1}}
\]

\[
1 + O \left( \frac{1}{n_j} \right)
\]

Thus

\[
1 + O \left( \frac{1}{n_j} \right)
\]
The big $O$ constant in the above approximation is uniform since $\frac{\mu_0}{2\mu_0} \frac{3}{1} + \varepsilon$.

Introduce the function

$$g(x) := \frac{e^{\pi i t}}{e^{\pi i t}} \text{ or } = (\pi e^{\pi i t})^{1}$$

Thus the above equation when expressed in term of $g(x)$ becomes

$$\frac{S_{n_j \ 1} (\psi_j t)}{(2k+1)\pi i} = \left( e^{\pi i t} \right)^{n_j} \left( \frac{g^{n_j} \left( (2k+1)x_{n_j} \right)}{2k+1}\pi i \right)$$

(4.2)

where $1 < \alpha < 1 = 2$. Since $x_0$ is not real, so we may assume that $x_0$ is in the lower half plane. Thus $x_0 = r_0 e^{-\theta_0}$ with $0 < \theta_0 < \pi$ and $\frac{1}{(2\mu_0 + 1)} \pi < \varepsilon < r_0$, $\frac{1}{(2\mu_0 + 1)} \pi$.

Now,

$$\frac{\pi r_0 \sin \theta_0}{e^{\pi r_0}} ;$$

as a function of $r$ attains its minimum at $r = \frac{1}{\pi \sin \theta_0}$. For each $1 < k < \mu_0$ (remember $\mu_0 = 2$) we have

$$r_0 < (2k + 1)r_0 < (2\mu_0 - 1)r_0 \frac{1}{\pi} \frac{1}{\pi \sin \theta_0} ;$$

Hence

(4.3)

$$\frac{\pi r_0 \sin \theta_0}{e^{\pi r_0}} ;$$

Also, since $x_0$ lies in the interior of the Szegő’s domain

$$z_{0} e^{1} z_{0} 1$$

and $\frac{\pi}{e^{\pi \theta_0}} ;$

Here, $z_0 = \pi x_0 i$. So,

$$\frac{\pi r_0 \sin \theta_0}{e^{\pi r_0}} ;$$

This prepares the behavior of the terms in $K_{n \mu_0} \frac{1}{k \pi i}$ with $0 < k < \mu_0$.

The term corresponding to $k = \mu_0$ \ 1 in the sum $K_{n \mu_0} \frac{1}{k \pi i}$ is

$$S_{n_j \ 1} (\psi_j \mu_0 \frac{1}{(2\mu_0 + 1)} \pi i)$$

which can be estimated using Proposition 3. Let $t_{\mu_0} = \frac{n_j}{n_j} \pi \mu_0 \frac{1}{\pi i}$ and apply the proposition to get

$$S_{n_j \ 1} (\psi_j x_{n_j} \mu_0 \frac{1}{(2\mu_0 + 1)} \pi i) = S_{n_j \ 1} (\psi_j \frac{1}{\pi i}) t_{\mu_0}$$

(4.4)

$$e^{\psi_j \frac{1}{\pi i}} \frac{\sum \xi (t_{\mu_0})}{\pi} t_{\mu_0} \frac{1}{\pi i} \operatorname{Erfc} \frac{\pi}{n_j} \frac{1}{\pi i}$$

Note that $t_{\mu_0} \left( \frac{1}{\mu_0 + 1} \right) + \varepsilon \left( \frac{1}{\mu_0 + 1} \right)$ as $n_j \to \infty$. Recall that

$$g(x) = e^{\pi x_0} = e^{\pi x_0 i} ;$$
where
\[ x_0 = j x_0 e^{i \theta_0}; 0 < \theta_0 < \pi \]
and
\[ \frac{1}{Q \mu_0 + 1} \pi + \epsilon < j x_0 j \]

A detailed study of the Szegő's curve, defined as \( z e^{1/z} = 1 \) and \( j z j = 1 \), shows the following features (see Figure 2):

**Figure 2.**

point B is the intersection point of the curve with the imaginary axis and equals \( i = e^1 \); point C is the intersection point of the curve with the negative real axis and is 0.278 (the unique real root of \( x = e^1 \) point A is the intersection point of curve with the circle \( j z j = 1 \) and \( e^{1/z} = 1 \)). \( \Re \{A\} = 9.861 \times 10^{-2} \) that comes from solving the real root of \( 1 = 0 \), \( x^2 = e^{2x} \), \( x^2 \) for \( x \). Hence, \( z_0 = \pi x_0 i \) falls in the interior of the Szegő’s curve (also to the right of the imaginary axis in the \( z \)-plane). This implies that
\[ j x_0 j = \frac{1}{z_0 e^{1/z_0}} > 1 \]
Recall \( \xi(t_{\mu_0}) = t_{\mu_0} \ln t_{\mu_0} j^{1/2} \). So it is easy to see that
\[ \frac{\xi(t_{\mu_0}) t_{\mu_0}}{t_{\mu_0} j^{1/2}} = O(1) \]
and $\text{Erfc}\left(\frac{\mu_0 - 1}{\sqrt{2} \pi i} \right) = \int_{\sqrt{2} \pi i}^{\infty} e^{-x^2} dx = O(1)$ uniformly. Hence from (4.4) we get

$$S_{nj} \equiv (\mu_0 - 1) \pi i K e^{\gamma^j (1 + \epsilon^2)}$$

for some absolute constant $K$. Now we are ready to see a contradiction from (4.1).

Dividing (4.1) by $(\epsilon x_n)^n$, we get

$$\frac{2 F_{j0} \cdot (1 - x_n)}{\pi} 1 + O \left( \frac{1}{n_j} \right) \cdot \left( \epsilon x_n \right)^{n_j} K_n \mu_0 \cdot 1 (\epsilon x_n) = 0$$

By (4.5) the term with summation index $k = \mu_0 - 1$ in $(\epsilon x_n)^{n_j} K_n \mu_0 \cdot 1 (\epsilon x_n)$ is estimated as

$$S_{nj} \equiv (\mu_0 - 1) \pi i (\epsilon x_n)^{n_j} K \frac{e^{1 - \epsilon}}{e x_n \cdot (\mu_0 - 1) \pi}$$

Similarly,

$$S_{nj} \equiv (n_j x_n \cdot \mu_0 \cdot 1) \pi i (\epsilon x_n)^{n_j} K \frac{e^{1 - \epsilon}}{x_n \cdot (\mu_0 - 1) \pi}$$

By (4.2) and (4.3) the dominant term in $(\epsilon x_n)^{n_j} K_n \mu_0 \cdot 1 (\epsilon x_n)$ corresponding to summation indices $0 \cdot k \cdot \mu_0 - 2$ is the term corresponding to $k = 0$; that is, the term

$$\frac{2 g_{nj} (\epsilon x_n)}{\pi i} 1 + \frac{1}{2 \pi n_j 1 + x_n \pi i} g_{nj} (\epsilon x_n) 1 + O \left( \epsilon x_n \right)^{1 - \alpha}$$

The other term with the same index $k = 0$ is

$$\frac{2 g_{nj} (\epsilon x_n)}{\pi i} 1 + \frac{1}{2 \pi n_j 1 + x_n \pi i} g_{nj} (\epsilon x_n) 1 + O \left( \epsilon x_n \right)^{1 - \alpha}$$

which is still dominated by the above term. Note that

$$g (\epsilon x_n) = \frac{\pi x_n i}{e x_n i} = \frac{\pi x_n \sin \theta_n}{e x_n \theta_n} ;$$

where $x_n = x_n e^{i \theta_n}$. Since $x_n = r_0 e^{i \theta_n}$ with $0 < \theta_n < \pi$, this implies $\sin \theta_n = 0$ for all large $n_j$. Hence, $g (\epsilon x_n)$ implies

$$g (\epsilon x_n) \left( \frac{1}{e x_n} \right)^{n_j} \cdot$$
Note also that for $\mu_0 2 \implies \frac{1}{\epsilon^2} > \frac{\epsilon^2}{\epsilon_0 - \frac{1}{\pi}}$. By comparing (4.9) with (4.7) and (4.8) we see the dominant term corresponding to $k = 0$ still dominates the term corresponding to $k = \mu_0 1$. Consequently, we infer from equation (4.6) that the left hand side becomes arbitrarily large as $n_j ! \infty$. Hence the left hand side is non-zero, contradicting to the right side of the equation. This completes the proof of part (b).

We now proceed to prove part (c). We show that if $x_0$ is a non-real accumulation point in the region $\frac{1}{3\pi} < \frac{x}{\pi} < \frac{1}{\pi}$, then either

$$x_0\pi e^{1} x_0\pi i = 1 \text{ or } x_0\pi e^{1} x_0\pi i = 1.$$ 

That $\frac{1}{3\pi}$ and real $x$ such that $\frac{1}{3\pi} < \frac{x}{\pi} < \frac{1}{3\pi}$ are points of the attractor is a consequence of the density calculation. Since by assumption $x_0$ lies in $\frac{1}{3\pi} < \frac{x}{\pi} < \frac{1}{\pi}$, we can certainly choose $\epsilon > 0$ sufficiently small such that $x_0$ is in $\frac{1}{3\pi} + \epsilon \frac{x}{\pi} < \frac{1}{\pi}$. Again as in the previous cases we may assume

$$x_0 = \frac{x}{\pi} \epsilon^{i \theta_0}$$

where $0 < \theta_0 < \pi$

Also the same $\epsilon$ works for an infinite sequence of zeros $x_{n_j}$ in

$$\frac{1}{3\pi} + \epsilon \frac{x}{\pi} \frac{1}{\pi} \epsilon \text{ such that } x_{n_j} ! x_0 \text{ as } n_j ! \infty.$$ 

Now use Proposition with the choice $\mu = 0$. So,

$$\frac{E_n(x)}{n!} = M_{n,0}(x) + K_{n,0}(x);$$

where

$$M_{n,0}(x) = \frac{1}{\pi i} \sum_{\xi \epsilon} \frac{\epsilon^{\pi i}}{\xi^{n+1}} F_0(\xi) d\xi;$$

and

$$K_{n,0}(x) = 2 S_n \frac{(\pi i \pi i)}{(\pi i)^{n+1}} + \frac{S_n}{(1 \pi i)^{n+1}};$$

and

$$F_0(\xi) = \frac{1}{\xi (\xi^2 + 1)} + \frac{1}{\pi i (\xi)} + \frac{1}{(\pi i) (\xi + \pi i)}.$$ 

Now $E_{n_j}(x_J x_{n_j}) = 0$ implies $M_{n_j,0}(x_{n_j}) + K_{n_j,0}(x_{n_j}) = 0$. Since $F_0(\xi)$ is analytic in the region $\epsilon \frac{\xi}{\pi} < 3\pi \epsilon$, the asymptotics for $M_{n_j,0}(x_{n_j})$ is obtained as:

$$M_{n_j,0}(x_{n_j}) = \frac{x}{\pi} \frac{1}{2} (x_{n_j} \epsilon)^{n_j} F_0 \frac{1}{x_{n_j}} \frac{1}{\pi} \frac{1}{n_j x_{n_j}} + O \left( \frac{1}{n_j} \right);$$

where the big $O$ term holds uniformly. Note that since $x_{n_j} ! x_0$ we have $\frac{1}{3\pi} + \epsilon \pi x_{n_j} \pi \epsilon \frac{1}{\pi} (\pi i \epsilon i)$. So $x_{n_j} \pi i$ lies in a compact set in the half plane $\Re x < 1$. Hence,
we can invoke Proposition 2 for the asymptotics for $K_{n,j} (\varphi_{n_j})$. Combining this with (4.11) in (4.10) and expressing the result in terms of the function $g (\nu)$, we obtain

$$
\frac{2}{\pi} F_0 \left( \frac{1}{x_{n_j}} \right) \frac{1}{n_j x_{n_j}} = 1 + O \left( \frac{1}{n_j} \right)
$$

$$
+ 2 \frac{g^{n_j} (\varphi_{n_j})}{\pi i} \frac{1}{n_j x_{n_j} \pi i} = 1 + O \left( \frac{1}{n_j} \pi \right)
$$

$$
+ 2 \frac{g^{n_j} (\varphi_{n_j})}{\pi i} \frac{1}{n_j (1 + x_{n_j}) (\pi i)} = 1 + O \left( \frac{1}{n_j} \pi \right) = 0
$$

Note that there are terms in $\frac{2}{\pi} F_0 \left( 1 = x_{n_j} \right) \frac{1}{n_j x_{n_j}}$ which are to be cancelled in the above equation and the order term $O \left( \frac{1}{n_j} \right)$ is absorbed in $O \left( \frac{1}{n_j} \pi \right)$. This observation leads to a simplification:

$$
(4.12)
\frac{2}{\pi} \frac{1}{e^{1=x_{n_j}} + 1} (1 + O \left( \frac{1}{n_j} \pi \right)) + \frac{2}{\pi i} g^{n_j} (\varphi_{n_j}) \frac{1}{n_j} + \frac{2}{(\pi i)} \frac{g (x_{n_j})}{g (\varphi_{n_j})} = 0
$$

Also note that $\frac{g (x_{n_j})}{g (\varphi_{n_j})} > \frac{g (x_{n_j})}{g (\varphi_{n_j})}$ (strictly greater). This implies

$$
\frac{g (x_{n_j})}{g (\varphi_{n_j})} ! 0 \text{ as } n_j \rightarrow \infty;
$$

Therefore, if $\frac{g (x_{n_j})}{g (\varphi_{n_j})} > 1$, then the left hand side of (4.12) becomes arbitrarily large in modulus for large $n_j$. This is a contradiction. But if $\frac{g (x_{n_j})}{g (\varphi_{n_j})} < 1$, then the left hand side of (4.12) goes to $\frac{2}{\pi} \frac{1}{e^{1=x_{n_j}} + 1}$, as $n_j \rightarrow \infty$, a non-zero number which is still a contradiction. Hence we must have

$$
\frac{g (x_{n_j})}{g (\varphi_{n_j})} = 1
$$

that is,

$$
x_0 \pi i e^{1} x_0 \pi i = 1;
$$

This also shows that $z_0 := x_0 \pi i$ is a point on the Szegő curve:

$$
\frac{1}{z_0} = 1; \frac{1}{z_0} = 1; \pi = 2 < \arg z_0 < \pi = 2;
$$

that is, $\Re z_0 > 0$ or $z_0$ lies in the shaded segment of the curve or $x_0$ is in the rotated Szegő’s curve.

To show there is no real accumulation point $x_0$ satisfying $\frac{1}{\pi} < x_0 < \frac{1}{\pi}$, we can still use the asymptotics that leads to (4.12). In particular, (4.12) still holds. Note that in this case $x_n, \pi i$ lies in the exterior region of the rotated Szegő curve in the $x$-plane. This implies $g (\varphi_{n_j}) < 1$. Hence in the limit as $n_j \rightarrow \infty$ in (4.12) we get

$$
\frac{2}{\pi} \frac{1}{e^{1=x_0} + 1} = 0;
$$

a contradiction. This finishes the proof of part (c).
This establishes the fact that the point set $K$ contains all points of the zero attractor.

5. Determination of the Density of Zeros

The next step is to carry out the density calculation. The Euler polynomials satisfy

$$E_n(1 \cdot x) = (1)^{n+1} E_n(x)$$

We have

$$E_n(\pi x) = (1)^n E_n(1 \cdot nx) = (1)^n E_n(n(x - 1 = n))$$

From this it is easy to see that the zero attractor does have the reflection symmetry with respect to the $y$-axis. The Euler polynomials are polynomials with real coefficients. It obviously has the symmetry with respect to reflection about the $x$-axis. To describe the density we choose the lower half of $K$ for discussion.

![Figure 3.](image)

The image of the points $A, B,$ and $C$ in the $x$-plane under the mapping of $z = \pi i x$ are denoted by $A^0, B^0,$ and $C^0$ respectively. Further, the subsequent images $A^{00}, B^{00},$ and $C^{00}$ are the images of $A^0, B^0,$ and $C^0$ under $\zeta = ze^{1 \cdot z}$ (see Figure 3).

We show:

**Theorem 1.** The image of the zeros of $E_n(\pi x)$ along the arc $BC$ in the $x$-plane are uniformly distributed in the $\zeta$-plane along the corresponding circular arc $B^{00}C^{00}$. As a consequence, the fraction of zeros residing in a neighborhood of the arc $BC$ in the $x$-plane is

$$\frac{\pi}{2\pi} \frac{1-\pi}{2\pi} = \frac{1}{4} \frac{1}{2\pi e}.$$  

**Theorem 2.** The real zeros of $E_n(\pi x)$ falling in the line segment $AC$ are uniformly distributed in the segment $AC$. As a consequence, the fraction of zeros residing in the segment $AC$ is $\frac{2}{\pi e}$.

Before enumerating the zeros, we mention some well-known facts about the Szegő curve: The mapping $\zeta = ze^{1 \cdot z}$ is conformal in $\mathbb{D}$ in $|z| < 1$. The Szegő curve is defined by:
The mapping \( \zeta = ze^{1/z} \) is in a neighborhood of \( z = 1 \). It is 2-to-1 in a neighborhood of \( z = 1 \) since \( \frac{d\zeta}{dz} \big|_{z=1} = 0 \) and \( \frac{d^2\zeta}{dz^2} \big|_{z=1} \neq 0 \). Therefore, to enumerate the number of zero images inside a contour we will face a difficulty of inverting the function in a neighborhood of 1 in the \( \zeta \)-plane. For this reason we will choose a contour that does not enclose 1. But how many zero images are left out in a neighborhood of 1? We will show that it is of order \( o(n) \). To prove this we use Jensen’s inequality. However, the use of Jensen’s inequality requires a knowledge of the function \( E_n(nx) \) in a neighborhood of \( i=\pi \). Fortunately, this knowledge is sufficiently provided by Proposition 1 and Proposition 2.

**Theorem 3.** (Jensen’s Inequality) Let \( h(z) \) be analytic in the disc \( |z| < R \) and let 0 < \( r < R \) and \( m \) be the number of zeros of \( h(z) \) in the disc \( |z| < r \). Then we have the following inequality:

\[
\frac{R}{r} m \leq \max_{|z|=r} |h(z)| \leq \frac{1}{\ln 2} (\ln (1 + O(\varepsilon))) + \ln \frac{K}{\varepsilon}
\]

A proof can be found in most books of complex analysis. Let \( \gamma^{(\varepsilon)}_n \) denote the number of zeros of \( E_n(nx) \) which lie in the disc \( x < \frac{1}{\pi \varepsilon} \) in the \( x \)-plane.

**Proposition 5.** For all sufficiently small \( \varepsilon > 0 \), there exists \( n_0(\varepsilon) \) such that for all \( n > n_0 \) we have

\[
\gamma^{(\varepsilon)}_n \leq \frac{1}{\ln 2} (\ln (1 + O(\varepsilon))) + \ln \frac{K}{\varepsilon}
\]

where the big \( O \) constant and \( K \) are absolute.

**Proof.** Since \( E_n(nx) \) does not vanish in \( x < \frac{1}{\pi \varepsilon} \), the function

\( h_n(x) := E_n(nx) \frac{\Gamma(n)}{n! (nx)^n} \)

is well-defined and has the same number of zeros as \( E_n(nx) \) in \( x < \frac{1}{\pi \varepsilon} \). We apply Theorem 3 to \( h_n(x) \) on the disc \( x < \frac{1}{\pi \varepsilon} \). Thus

\[
2\gamma^{(\varepsilon)}_n = \max_{\frac{1}{\pi \varepsilon} < x < \frac{1}{\pi \varepsilon} + 2\varepsilon} |h_n(x)|
\]

We need asymptotic estimates for \( h_n\left(\frac{1}{\pi \varepsilon}\right) \) and \( \max_{\frac{1}{\pi \varepsilon} < x < \frac{1}{\pi \varepsilon} + 2\varepsilon} |h_n(x)| \). First, for \( h_n\left(\frac{1}{\pi \varepsilon}\right) \) we apply Proposition 4 with \( \mu = 0 \). Thus

\[
\frac{E_n(nx)}{n!} = M_n(\chi) + K_n(\chi);
\]
where
\[ M_{n,0}(\xi) = \frac{2}{2\pi i} \sum_{\xi \neq 0} e^{\xi \xi^{-1} n} F_0(\xi) d\xi; \]
and
\[ K_{n,0}(\xi) = 2 \frac{S_n}{(\pi i)^{n+1}} \left( \frac{x}{n\xi} \right)^n + \frac{S_n}{(\pi i)^{n+1}} : \]
The asymptotics of \( M_{n,0}(\frac{1}{\pi i}) \) is known:
\[ (5.2) \quad M_{n,0}(\frac{1}{\pi i}) = e^{\frac{1}{2} \pi i} 1 + O(\frac{1}{n}) : \]
Recall
\[ K_{n,0}(\frac{1}{\pi i}) = 2 \frac{S_n}{(\pi i)^{n+1}} + \frac{S_n}{(\pi i)^{n+1}} : \]
We will show that \( K_{n,0}(\frac{1}{\pi i}) \) is the dominant term for \( E_{n,1}(\frac{1}{\pi i}) \). Use Proposition 3 to get (with \( t = n = \xi = \frac{1}{\pi i} \) and \( n \gg 1 \)):
\[ (5.3) \quad S_n(\xi) = e^{\frac{1}{2} \pi i} \frac{n}{\xi^n} \text{Erfc}(\frac{n}{\pi i} \xi) + O(\frac{1}{n}) : \]
where \( \xi = \frac{1}{n} \ln \frac{n}{\sqrt{2\pi}} = 1 + O(\frac{1}{n}) \). So
\[ \frac{n}{\sqrt{2\pi}} \ln \frac{n}{\sqrt{2\pi}} = 1 + O(\frac{1}{n}) : \]
Recall that
\[ \text{Erfc}(x) = \int_x^\infty e^{-s^2} ds = \int_0^\pi e^{-\frac{1}{2} \pi \xi^2} d\xi = \frac{1}{\sqrt{2\pi}} + O(\frac{1}{n}) : \]
as \( x \to 0 \). Hence
\[ \text{Erfc}(\frac{n}{\pi i} \xi) = \text{Erfc}(\frac{1}{2} \frac{n}{\pi i} \xi) = \frac{1}{\sqrt{2\pi}} + O(\frac{1}{n}) : \]
Inserting these estimates back into (5.3) we have
\[ S_n(\xi) = e^{\frac{1}{2} \pi i} + O(\frac{1}{n}) : \]
Similarly using Proposition 3 we have
\[ S_n(\frac{1}{n}) = O(\frac{1}{n}) : \]
These estimates give

\[ K_{n0} \left( \frac{1}{\pi i} \right) = \left( \frac{1}{\pi i} \right) \left( e^{\pi i} \right)^n 1 + O \left( \frac{1}{n} \right) : \]

Comparing (5.2) with the above we see that the order of \( M_{n0} \left( \frac{1}{\pi i} \right) \) is small than that of \( K_{n0} \left( \frac{1}{\pi i} \right) \) by a factor of \( \frac{\mathcal{P}}{n} \). Hence

\[ \frac{E_n (\pi i)}{n!} = \left( \frac{1}{\pi i} \right) \left( e^{\pi i} \right)^n 1 + O \left( \frac{1}{n} \right) ; \]

and

\[ h_n \left( \frac{1}{\pi i} \right) := \frac{E_n (\pi i)}{n!} \frac{\mathcal{P}}{n} = \frac{\mathcal{P}}{n} 1 + O \left( \frac{1}{n} \right) ; \]

We now estimate \( \max_{\frac{1}{\pi i} \leq \frac{1}{2e}} \phi_n (x) \). To this end we use Proposition 4 with \( \mu = 0 \):

\[ \frac{E_n (\pi i)}{n!} = M_{n0} (x) + K_{n0} (x) ; \]

where

\[ M_{n0} (x) = \left( \frac{1}{\pi i} \right) \int_{\phi}^{1} e^{\xi i} \xi F_0 (\xi) d\xi ; \]

\[ K_{n0} (x) = 2 \left( \frac{S_{n-1} (\pi i)}{\pi} \right)^{n+1} + \left( \frac{S_{n} (\pi i)}{\pi} \right)^{n+1} ; \]

The asymptotics of \( M_{n0} (x) \) is:

\[ M_{n0} (x) = \frac{2}{\pi} (\pi e)^n F_0 \left( \frac{1}{x} \right) \frac{\mathcal{P}}{n} 1 + O \left( \frac{1}{n} \right) ; \]

Care must be exercise to handle the asymptotics of \( S_{n-1} (\pi i) \). Recall that \( x \) is on the circumference \( x \frac{1}{\pi i} = 2e \).

Choose two points \( p \) and \( q \) (see Figure 4) such that \( p \) is the mid-point of the circular arc between the circle \( \hat{x} \hat{j} = \frac{1}{\xi} \) and the horizontal tangent line to the circle \( \hat{x} \hat{j} = \frac{1}{\xi} \) at the point \( x = \frac{1}{\pi i} \). The point \( q \) is similarly selected. Note that as \( \varepsilon > 0^+ \), the distance between \( p \) and the circle \( \hat{x} \hat{j} = \frac{1}{\xi} \) is of order \( O (\psi^2) \). This is so because the horizontal line \( y = \frac{1}{\xi} \) is tangent to the circle \( \hat{x} \hat{j} = \frac{1}{\xi} \). The image of the arc \( q \hat{t} \) under the map \( z = x \pi i \) is a circular arc in the \( z \)-plane where Proposition 1 is applicable. Thus for all \( x \) on the arc \( q \hat{t} \) we have

\[ S_{n-1} (\pi i) = \left( \frac{2}{\pi i} \right)^n \frac{\mathcal{P}}{n} \frac{1}{(\pi i)^{n+1}} 1 + O \left( \frac{1}{n} \right) ; \]

(5.4)
where $O_\epsilon$ stands for the $\epsilon$-dependence for the big $O$ constant. Taking (5.4) into consideration we get

$$h_n(x) = \frac{x}{\pi} F_0 \left( \frac{1}{x} \right) \frac{1}{x} \left( 1 + O \left( \frac{1}{n} \right) \right)$$

$$+ \frac{2 g^n(x)}{\pi} g^n \left( \frac{x}{\pi i} \right) \frac{1}{x} \left( 1 + O \left( \frac{1}{n} \right) \right)$$

$$+ \frac{2 \rho}{\pi} \bar{g}^n \left( \frac{x}{\pi i} \right) \frac{1}{x} \left( 1 + O \left( \frac{1}{n} \right) \right)$$

where the third term is obtained from an analogous application of Proposition 2 to $S_{n-1}$ ($nx\pi i$). We now estimate $h_n(x)$ as follows. The first term

$$\frac{x}{\pi} F_0 \left( \frac{1}{x} \right) \frac{1}{x} \left( 1 + O \left( \frac{1}{n} \right) \right)$$

is obviously $K$, an absolute constant. The second term

$$\frac{2 g^n(x)}{\pi i} g^n \left( \frac{x}{\pi i} \right) \frac{1}{x} \left( 1 + O \left( \frac{1}{n} \right) \right)$$

is $O_\epsilon$, and the third term

$$\frac{2 \rho}{\pi} \bar{g}^n \left( \frac{x}{\pi i} \right) \frac{1}{x} \left( 1 + O \left( \frac{1}{n} \right) \right)$$

is $O_\epsilon$. Therefore, the $O_\epsilon$ term is negligible.

**Figure 4.**
The third term
\[ 2^\Omega \frac{ng^n(x)}{(\pi i)} 1 \times g^n(x) \frac{1}{2\pi n} \frac{1 + O_\epsilon \phi_1^{3\alpha}}{1 + x\pi i} K; \]

because \( x\pi i \) lies outside of the Szegö curve and \( j g(x) \) \( j < 1 \). Hence for \( x \) on the arc \( ptq \) on \( x = \frac{1}{\pi i} = 2\epsilon \) we have
\[ j h_n(x) j K 1 + O_\epsilon \phi_1^{3\alpha}; \]

where \( K \) is an absolute constant. When \( x \) in the arc \( psq \) on the circumference \( x = \frac{1}{\pi i} = 2\epsilon \), we invoke Proposition 2. Thus
\[ h_n(x) = \frac{1}{\pi} F_0 \left( \frac{1}{x} \right) \frac{1}{x} 1 + O \left( \frac{1}{n} \right) + \frac{2^\Omega \frac{ng^n(x)}{(\pi i)}}{\pi i} \]

The magnitude of \( g(x) \) for \( x \) on \( psq \) is estimated as follows. Recall \( x\pi i = 1 + 2\epsilon \pi ie^{i\theta} \).

This implies
\[ x\pi i e^{i\theta} \]

This yields
\[ j g(x) j = \frac{1}{x\pi i e^{i\theta}} \frac{e^{2\epsilon\pi}}{1} e^{2\epsilon\pi}; \]

Therefore, we infer from (5.5) that
\[ j h_n(x) j K \frac{e^{2\epsilon\pi}}{1} \frac{n^\Omega}{2\pi n} 1 + O_\epsilon \phi_1^{3\alpha}. \]

Note that in deriving the above equation the fact that \( j g(x) j < 1 \) was still used. Combine these estimates to get
\[ \max j h_n(x) j K \frac{e^{2\epsilon\pi}}{1} \frac{n^\Omega}{2\pi n} 1 + O_\epsilon \phi_1^{3\alpha}; \]

Recall that
\[ 2^\Omega \left( \max j h_n(x) j \right) \frac{1}{\pi} \frac{1}{x\pi i} \frac{e^{2\epsilon\pi}}{1} \frac{n^\Omega}{2\pi n} 1 + O_\epsilon \phi_1^{3\alpha}; \]

\[ K \left[ \frac{e^{2\epsilon\pi}}{1} \frac{n^\Omega}{2\pi n} 1 + O_\epsilon \phi_1^{3\alpha} \right] \]
This implies that
\[ \gamma_e^{(n)} \ln K \frac{e^{2\pi n}}{2\pi} + O_e(n^{-2}) = \ln 2. \]

This finishes the proof of Proposition 5.

We now come to determine the density of zeros. First of all, we refer to the mapping relation in Figure 3. In general, we let \( N_n(\alpha;\beta) \) be the number of image points of zeros of \( E_n(nx) \) in the \( \zeta \)-plane that fall in the angular sector \( \alpha \leq \arg \zeta \leq \beta \). Now let an arbitrary \( \theta \) be given in the interval \( 0 < \theta < \frac{\pi}{2} \).

Our goal is to establish the following proposition:

**Proposition 6.** \( \lim_{n! \to \infty} \frac{1}{n!} N(0;\theta) = \frac{\theta}{2\pi} \).

**Proof.** Recall that \( N_n(0;\theta) \) is the number of image points in the \( \zeta \)-plane that fall in the angular sector \( \theta \leq \arg \zeta \leq \theta \). We alleviate the problem that the straight edges of the above sector may contain some image points by perturbation. The reason for requiring that no image points of zeros fall on the straight edges is to guarantee an application of the argument principle. Since the Euler polynomials are polynomials of rational coefficients, the roots are algebraic numbers in the \( x \)-plane. This implies that the totality of image points in the \( \zeta \)-plane is countable. Hence we can choose an arbitrary small number \( \varepsilon > 0 \) so that the straight edges of \( 3\varepsilon = \arg \zeta \) avoids all images points of zeros. The following arguments assume a fixed \( \varepsilon > 0 \) and \( n \) will be eventually. It is now obvious that we have the following inequality:

\[ N_n(0;\theta) \sim N_n(3\varepsilon;\theta + \varepsilon_n^0) = O^{\gamma_e^{(n)}} \]

where a null sequence \( \varepsilon_n^0 > 0 \) is chosen so that no image points are on the straight edge \( \arg \zeta = \theta + \varepsilon_n^0 \). Define

\[ f_n(\zeta) := \frac{E_n(\zeta \phi)}{n! (\phi \zeta)^n} \]

Note that: (1) the image points are zeros of \( f_n(\zeta) \) and (2) \( f_n(\zeta) = h_n(x(\zeta)) \), where \( h_n(x) \) was defined in (5.1). Here the function \( x(\zeta) \) is the inverse map of the map from the \( x \)-plane to the \( \zeta \)-plane. \( x(\zeta) \) is 1-1 except in a small neighborhood of \( \zeta = 1 \).

We introduce the contour \( \Gamma_{3\varepsilon;\theta + \varepsilon_n^0} \) defined as (see Figure 5): \( \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \):

\[ \begin{align*}
\Gamma_1 & : re^{i3\varepsilon}; \quad 0 \leq r \leq e^{\varepsilon_n^0} + 1 + \varepsilon \geq 2; \\
\Gamma_2 & : (1 + \varepsilon = 2) e^{i\theta}; \quad 3\varepsilon \phi \leq \theta + \varepsilon_n^0; \\
\Gamma_3 & : re^{i\theta + \varepsilon_n^0}; \quad 0 \leq r \leq e^{\varepsilon_n^0} + 1 + \varepsilon \geq 2; \\
\Gamma_4 & : (1 + \varepsilon = 2) e^{i\theta}; \quad 3\varepsilon \phi \leq \theta + \varepsilon_n^0.
\end{align*} \]
We apply the argument principle to enumerate the number of image points. Thus,

\[
N_n (\beta \varepsilon; \theta + \varepsilon_n) = \frac{1}{2\pi i} \int_{\Gamma_{3\varepsilon;\theta+\varepsilon_n}} \frac{d}{d\zeta} f_n (\zeta) d\zeta
\]

where \(I_i = \mathbb{R}, i \in \{1, 2, 3, 4\}\). We shall handle \(I_2\) first. If we pull the contour \(\Gamma_2\) back in the \(z\)-plane, it falls in a compact set in the half plane \(\Re(z) < 1\) so that we can use Proposition 2 for the asymptotics of \(S_n (1/z)\). Thus with an application of Proposition 4 (with \(\mu = 0\)) we have

\[
E_n (\psi x)^{\frac{p}{n}} \frac{e^x}{n!} = \frac{x}{n^2} F_0 (\psi) \frac{1}{x^2} + O \left( \frac{1}{n} \right)
\]

Recall

\[
F_0 (\xi) = \frac{1}{\xi (\xi^2 + 1)} + \frac{1}{\pi i (\xi^2 + 1)} + \frac{1}{(\pi i) (\xi^2 + 1)};
\]
so

\[ F_0 \left( \frac{1}{x} \right) = \frac{x}{e^{1-x} + 1} + \frac{1}{\pi i (1 + x\pi i)} + \frac{1}{(\pi i) (1 + x\pi i)} : \]

Canceling some common terms to simplify the expression we get

\[ E_n \frac{\varphi(x)}{n! (\varphi(x))^n} = \frac{x}{2 \pi e^{1-x} + 1} \]

(F.8) \[ 1 + O \left( \frac{1}{n} \right) \]

Furthermore since the pulled back \( x \)'s lie outside the zero attractor so \( \varphi \left( x, \varphi(x) \right) < 1 \). In this way we see that

\[ \lim_{n \to \infty} E_n \frac{\varphi(x)}{n! (\varphi(x))^n} = \frac{2\pi}{e^{1-x} + 1} \]

or

\[ h_n \left( \varphi(x) \right) ! = \frac{2\pi}{e^{1-x} + 1} ; \text{ uniformly} \]

or equivalently

\[ h_n \left( \varphi(\zeta) \right) ! = \frac{2\pi}{e^{1-x} \varphi(\zeta) + 1} ; \text{ uniformly for } \zeta \in \Gamma_2 \]

By a theorem of uniform convergence of analytic functions we get

\[ \frac{d}{\zeta} h_n \left( \varphi(\zeta) \right) ! = \frac{e^{1-x} \varphi(\zeta) dx d\zeta}{e^{1-x} \varphi(\zeta) + 1} \]

uniformly for \( \zeta \in \Gamma_2 \). Recall \( f_n (\zeta) = h_n \left( \varphi(\zeta) \right) \). Hence by integrating the above along \( \Gamma_2 \) we get

\[ \frac{1}{2\pi} \int_{\Gamma_2} \int_{\zeta} \frac{d}{\zeta} f_n (\zeta) ! = \frac{1}{2\pi} \int_{\Gamma_2} \int_{\zeta} e^{1-x} \varphi(\zeta) dx d\zeta \]

This implies

\[ \frac{1}{2\pi} \int_{\Gamma_2} \int_{\zeta} \frac{1}{2\pi} ! = O \left( \frac{1}{n} \right) ; \]

Let us handle \( I_4 \) now. The asymptotics in (5.8) still hold good in this case. But since \( x \) now is inside the zero attractor so

\[ \varphi \left( x, \varphi(x) \right) > 1 \text{ and still } \varphi \left( x, \varphi(x) \right) > \varphi \left( x \right) \]
Rewrite (5.8) in the form
\[
\frac{f_n(\zeta)}{2\pi n} \zeta^n = \frac{\zeta^n}{2\pi n} e^{i\pi n + 1} \left( 1 + O\left( \frac{1}{n}\right) + O \epsilon^{1/3\alpha_1} \right) + \frac{p}{2\pi} \frac{\zeta}{\pi i} + \frac{p}{2\pi} \frac{\zeta}{\pi i} \left( \zeta g(x) \right)^n:
\]
(5.9)

Note that \( \zeta = ze^1 \) and \( \zeta_j = 1 \) \( \epsilon = 2 \) since \( \zeta \geq \gamma_4 \). Also, \( \zeta g(x) = \gamma \) \( \epsilon = 2 \). Hence we see that
\[
\lim_{n \to \infty} \frac{f_n(\zeta)}{2\pi n} \zeta^n
\]
uniformly and we get similarly
\[
\frac{d}{d\zeta} \frac{f_n(\zeta)}{f_n(\zeta)} + n \frac{\zeta}{\zeta !} 0
\]
uniformly for \( \zeta \geq \gamma_4 \). Integration along \( \gamma_4 \) gives
\[
\Im(\vartheta_4) = \Im \left( \int \left( \frac{\zeta}{\zeta} \right) d\zeta \right) = n \Im \left( \theta + \epsilon_n^0 \ 3\epsilon \right) + o(1):
\]
Note that the orientation of \( \gamma_4 \) gives the correction of the “−” sign. Hence,
\[
\Im(\vartheta_4) = n \Im \left( \theta + \epsilon_n^0 \ 3\epsilon \right) + o(1):
\]
This implies that
\[
\frac{1}{n} \frac{1}{2\pi} \Im(\vartheta_4) = \frac{1}{2\pi} \left( \theta + \epsilon_n^0 \ 3\epsilon \right) + o(1):
\]
Caution must be exercised when we handle \( I_1 \) (\( I_3 \) can be analogously handled). Although (5.9) still holds good for \( I_1 \), but \( \zeta_j \) varies from 1 \( \epsilon = 2 \) to 1 \( \epsilon = 2 \). It is not clear how (5.9) becomes useful to us. We now employ a number-theoretic argument and Theorem 3 to obtain a useful estimate for \( I_1 \). The argument we follow is inspired by that of K. Chandrasekharan [11]. Let \( \gamma \) be the number of points on \( \Gamma_1 \) so that \( \Re(f_n(\zeta)) = 0 \). We insert these points into \( \Gamma_1 \) and decompose \( I_1 \) as the sum of integrals whose end points are two consecutive points where \( \Re(f_n(\zeta)) = 0 \) except possibly the beginning integral and the final integral. A typical integral to consider is of the form
\[
\Im \left( \int \left( \frac{\zeta}{\zeta} \right) d\zeta \right), \text{ where } a \text{ and } b \text{ are two consecutive roots of } \Re(f_n(\zeta)) = 0 \text{ along } \Gamma_1.
\]
By a change of variable, we have
\[
\Im \left( \int \left( \frac{\zeta}{\zeta} \right) d\zeta \right) = \Im \left( \int \frac{d\zeta}{C_n} \right);
\]
where the contour \( C_n \) is the image of the segment \( ab \) under the map \( \zeta ! \ f_n(\zeta) \). We comment that \( C_n \) intersects the imaginary axis at its two endpoints and at no other
interior point of $C_n$. No matter what, we always have by Cauchy’s Theorem

$$
\int_{C_n} \frac{d\zeta}{\zeta} = \int_{K_{ab}} \frac{d\zeta}{\zeta},
$$

where $K_{ab}$ denotes the semi-circle with segment $\frac{f_n}{f_n} (a)f_n (\Phi)$ as the base so that $K_{ab}$ lies in the same half plane as $C_n$. Note that $K_{ab} d\zeta = \zeta$ is either $0, i\pi$, or $i\pi$. Hence

$$
\int_a^b \frac{f_n (\zeta)}{f_n (\zeta)} d\zeta = \int_{K_{ab}} \frac{d\zeta}{\zeta} \pi.
$$

It follows that

(5.10) \hspace{1cm} \int \phi_1) \mathcal{J} \phi + 1) \pi:

Here we have $(\phi + 1)\pi$ as an upper estimate because of the possible inclusion of the beginning and final integrals. To estimate $l$, we shall use the Jensen’s inequality: Define

$$
\Phi_n (\zeta) := h_n (\phi (\zeta e^{i3\epsilon})) + h_n (x (\zeta e^{i3\epsilon}) \frac{2}{2};
$$

where $h_n (\phi)$ was defined in (5.1). Let us note the following two properties of $\Phi_n (\zeta)$:

1. $\Phi_n (\zeta)$ is an analytic function of $\zeta$.
2. When $\zeta$ is real and $1 \epsilon = 2 \zeta 1 + \epsilon = 2$, then $\zeta e^{i3\epsilon} \Gamma_1$ and $x (\zeta e^{i\epsilon})$ and further $x (\zeta e^{-i\epsilon})$ are complex conjugates in the $x$-plane.

Hence

$$
\Phi_n (\zeta) = \Re h_n (\phi (\zeta e^{i3\epsilon}));
$$

The reason we choose $3\epsilon$ instead of $\epsilon$ is for convenience as the sequel will show. Now we regard $\Phi_n (\zeta)$ as an analytic function defined on the disc $\zeta : 1 + \epsilon = 2 \Phi$ in the $\zeta$-plane. (This is so because the circle $\zeta e^{i3\epsilon} : \zeta (1 + \epsilon = 2)$ does not include $\zeta = 1$ in its interior, for otherwise, $\phi (\zeta e^{i3\epsilon})$ would not be well-defined in the $\zeta e^{i3\epsilon} : \zeta (1 + \epsilon = 2)$.) Note that the disc $\zeta e^{i3\epsilon} : \zeta (1 + \epsilon = 2)$ does not contain the real interval $1 \epsilon = 2, 1 + \epsilon = 2$. Each root of $\Phi_n (\zeta)$ along $\Gamma_1$ is a real root of $\Phi_n (\zeta)$ in $\Phi : \zeta (1 + \epsilon = 2)$ (possitive complex roots are counted in $\Phi$), then obviously

(5.11) \hspace{1cm} l \Phi:

Apply Theorem 3 in the disc $\zeta e^{i3\epsilon} : \zeta (1 + \epsilon = 2)$ to get

(5.12) \hspace{1cm} \frac{2\epsilon}{\epsilon} \Phi = 2\Phi \max_{\Phi_n (1 + \epsilon = 2)} \Phi_n (\zeta) ;

Recall the definition of $\Phi_n (1 + \epsilon = 2)$:

$$
\Phi_n (1 + \epsilon = 2) = \frac{1}{2} h_n (\phi ((1 + \epsilon = 2) e^{i3\epsilon})) + h_n (x ((1 + \epsilon = 2) e^{i3\epsilon}))
$$
The point $\zeta = (1 + \varepsilon=2)e^{3i\varepsilon}$ lies in a region in the $\zeta$-plane so that the pull back $z ((1 + \varepsilon=2)e^{3i\varepsilon})$ lies in the half plane $\Re (\zeta) < 1$. Therefore, we can use (5.5) to determine the asymptotics of $h_n (x ((1 + \varepsilon=2)e^{3i\varepsilon}))$. Similarly for $h_n (x ((1 + \varepsilon=2)e^{3i\varepsilon}))$. In this case we have

$$g (x ((1 + \varepsilon=2)e^{3i\varepsilon})) < g (x ((1 + \varepsilon=2)e^{3i\varepsilon})) < 1;$$

so

$$(5.13) \lim_{n \to \infty} \frac{\pi e^{h_n (1 + \varepsilon=2)}}{\pi e^{1+x_{\varepsilon} + 1}};$$

where

$$x_{\varepsilon} = x ((1 + \varepsilon=2)e^{3i\varepsilon});$$

A study of the curves defined by

$$\Re \left( \frac{1}{e^{\varepsilon+1}} \right) = 0,$$

that is, $e^{1+x} + 1 = si$; $2 \Re$ leads to

$$x = \frac{1}{\ln (1 + s^2 + i \arg (1 + si)};$$

These curves represent points in the $x$-plane where $e^{1+x} + 1 = purely imaginary numbers. We have branches of the curve which is a consequence of the multi-valueness of $\arg (1 + si)$. These curves cluster at $x = 0$. The point $x_{\varepsilon} = x ((1 + \varepsilon=2)e^{3i\varepsilon})$ is in a small vicinity of $x = 1 = \pi$. Note that

$$\frac{dx}{ds} = \frac{i}{\pi};$$

This means that the curve has a vertical tangent at $x = 1 = \pi$. In the $\zeta$-plane

$$\frac{d}{d\varepsilon} (1 + \varepsilon=2)e^{3i\varepsilon} = \frac{1}{2} + 3i;$$

It is easy to see that the pull back of $x_{\varepsilon}$ does not lie on any of the branches of the curve

$$\Re \left( \frac{1}{e^{1+x} + 1} \right) = 0;$$

Moreover, since $e^{\pi i} + 1 = 0$, it is not hard to see that

$$\Re \left( \frac{1}{e^{1+\varepsilon} + 1} \right) \frac{K}{\varepsilon};$$

for some positive constant $K$. This implies from (5.13),

$$\frac{K}{\varepsilon};$$

Next we estimate

$$\max_{\zeta, (1 + \varepsilon=2); \neq 2e} \Phi_n (\zeta) = \max_{\zeta, (1 + \varepsilon=2); \neq 2e} \frac{E_n (x (\zeta)) \frac{1}{\varepsilon}}{n! (1 + \varepsilon=2)^{2e} (n \varepsilon (x (\zeta))^2) };$$
we can still use (5.3) for the purpose. This is so because the pullback $z(\zeta)$ is still in $\Re(z) < 1$. Proceeding similarly and using (5.5), we get

$$\max_{\zeta \in (1 + \epsilon z; 2 \epsilon)} \frac{1}{\ln n} (1 + O(\epsilon))^{n \epsilon} n + O_\epsilon(\epsilon^{1 - 3\alpha})$$

where the big oh constant in $O(\epsilon)$ is an absolute constant. Further, combining (5.11) and (5.12), we get

$$\Re(I_1) \frac{1}{\ln n} (1 + O(\epsilon))^{n \epsilon} n + O_\epsilon(\epsilon^{1 - 3\alpha})$$

Similarly we obtain

$$\Re(I_3) \frac{1}{\ln n} (1 + O(\epsilon))^{n \epsilon} n + O_\epsilon(\epsilon^{1 - 3\alpha})$$

Now from (5.6)

$$\frac{1}{n} N_n(\theta; \theta) = \frac{1}{n} N_n(3\epsilon; \theta + \epsilon_0) = \frac{1}{n} O(\epsilon^{\frac{3n}{2}})$$

Inserting the estimate from $\Re(I_4)$, we have

$$\limsup_{n \to \infty} \frac{N_n(\theta; \theta)}{n} = \frac{\theta}{2\pi} + O(\epsilon)$$

An analogous lower bound can be obtained, that is,

$$\liminf_{n \to \infty} \frac{N_n(\theta; \theta)}{n} = \frac{\theta}{2\pi} - O(\epsilon)$$

Since $\epsilon$ can be made arbitrarily small, we conclude

$$\lim_{n \to \infty} \frac{1}{n} N_n(\theta; \theta) = \frac{\theta}{2\pi}$$

This ends the proof of Theorem 1.

We next calculate the density of zeros in the interval $[0; \frac{1}{\epsilon}]$. The strategy will be the same to that of Theorem 1. However, the technical details are slightly different. We outline the steps below.

**Lemma 2.** (a) For every $\epsilon > 0$, let $N_\epsilon$ be the number of zeros of $E_n(\epsilon x)$ in the disc $\epsilon x : \epsilon x(1 + O(\epsilon))^n$.

(b) For all $0 < b < \frac{1}{\epsilon}$, we construct the rectangular contour $C_1 \subseteq C_2 \subseteq C_3 \subseteq C_4 = C$ as shown in Figure 6.

Let $N_n(\epsilon; b)$ be the number of zeros of $E_n(\epsilon x)$ enclosed inside the contour $C$. In general, we introduce the notation: let $N_n(\epsilon; b)$ be the number of zeros of $E_n(\epsilon x)$...
enclosed in the rectangular contour with vertices $a + i\delta; a + i\delta; b + i\delta; b + i\delta$. Then for all $\varepsilon > 0$ and for all sufficiently small $\delta > 0$ we have

$$\frac{1}{n} N_n(\varepsilon; b) = b + O(\delta) + O(\varepsilon) + O(\tan \frac{\delta}{\varepsilon}) + O(\max_j \frac{1}{\ln 2}) \ln (K_\varepsilon (1 + O(\varepsilon))^\varepsilon_n):$$

A consequence of the above two lemmas is: $\lim_{n \to \infty} \frac{1}{n} N_n(0; b) = b$. Thus the real roots of $E_n(\varepsilon x)$ are uniformly distributed in $[1/\pi e, 1/\pi e]$. 

**Proof.** Let us show part (a) first. Since $\frac{E_n(x)}{n!}$ is analytic on the disc $\{x : |x| < 2\pi e\}$, we apply Theorem 3 to get

$$\frac{2\varepsilon}{n} E_n(\varepsilon) = 2^n \max_{|y| < 2\pi e} \frac{E_n(\varepsilon x)}{n!} :$$

$\frac{E_n(0)}{n!}$ can be obtained from the integral representation:

$$\frac{E_n(0)}{n!} = \frac{1}{\pi i} \int_{|\xi| = 1} \frac{1}{(e^\xi + 1)^{n+1}} d\xi.$$ 

Here caution must be exercised because for even $n$ except $n = 0$, $E_n(0) = 0$. This means that $E_n(x)$ has a zero at $x = 0$ for all even $n > 2$. Therefore, we must modify the function if we still want to use Jensen’s inequality. We consider $\frac{1}{n!} E_n(x)$ when $n$
is even. In this case, an equivalent Jensen’s inequality is:

\[
2^{N_e} \frac{\max_j x_j + 2e}{x_n!} \frac{E_n(\eta x)}{n!} \leq \lim_{x \to 0} \frac{\frac{1}{x} E_n(\eta x)}{n!}
\]

We will show the former inequality has the main features in the proof, while the latter can be treated similarly. From integral representation \((n\) is odd now\) we apply the Darboux method \([5]\). In this case the nearest singularities are \(\pi i\). One shows that

\[
\frac{E_n(0)}{n!} = \frac{1}{\pi i} \int_{\xi = 1}^{\xi = 1} \frac{1}{\xi + \pi i} + \frac{1}{\xi + \pi i} d\xi + o\left(\frac{1}{\pi n}\right)
\]

\[(5.15)\]

The corresponding quantity when \(n\) is even is

\[
\lim_{x \to 0} \frac{\frac{1}{x} E_n(\eta x)}{n!} = \frac{1}{n!} \lim_{x \to 0} \frac{e^{nx\xi}}{2\pi i} \int_{\xi = 1}^{\xi = 1} \frac{1}{e^{\xi} + 1} \xi^{n+1} d\xi
\]

Now we know that

\[
\frac{H_j}{\xi + 1} \int_{\xi = 1}^{\xi = 1} \frac{1}{e^{\xi} + 1} \xi^{n+1} d\xi = 0 \quad (n \text{ is even here})
\]

We use the above to rewrite \(\lim_{x \to 0} \frac{1}{x} E_n(\eta x)\). Thus

\[
\lim_{x \to 0} \frac{E_n(\eta x)}{n!} = \frac{1}{2\pi i} \int_{\xi = 1}^{\xi = 1} \frac{e^{nx\xi}}{e^{\xi} + 1} \xi^{n+1} d\xi
\]

which is still of the same feature as in \((5.15)\). In this way we see it does not matter whether \(n\) is odd or even, they can be handled in a similar way. Now we consider

\[
\max_{\xi = 2e} \frac{E_n(\eta x)}{n!}
\]

We use Proposition \([4]\) with \(\mu_1\) chosen sufficiently large so that \(\frac{1}{2(\mu_1 + 1)\pi} < 2e < \frac{1}{2(\mu_1 + 1)\pi}\). Here the additional assumption that the arbitrarily small number \(2e\) is not of the form \(\frac{1}{2(\mu_1 + 1)\pi} m\quad 0\), will not hurt the arguments we present here. Then with \(\mu = \mu_1 \quad 1\) in Proposition \([4]\) we get

\[
\frac{E_n(\eta x)}{n!} = M_{n,\mu_1} (x) + K_{n,\mu_1} (x)
\]
The asymptotics for $M_{n; \mu_1} (x)$ is

$$M_{n; \mu_1} (x) = \frac{2}{\pi} (xe)^n F_{\mu_1} \left( \frac{1}{x} \right) \frac{1}{nx} 1 + O \left( \frac{1}{n} \right)$$

and the asymptotics for $K_{n; \mu_1} (x)$ comes from applying Proposition 2. This is so because $\gamma (2k + 1)\pi i j \neq j \pi i 1$ for all $0 < k < 1$. When $\gamma j = 2\epsilon$, we find $\gamma (2k + 1)\pi i j < 2\epsilon Q_{\mu_1} 1 < 1$. This means the Proposition 2 is applicable. Furthermore, the largest term comes from $k = 0$. Thus

$$E_n (\theta x) = \frac{2}{\pi} (xe)^n F_{\mu_1} \left( \frac{1}{x} \right) \frac{1}{nx} 1 + O \left( \frac{1}{n} \right)$$

$$+ \frac{2g^n (x)}{\pi i} \left( 1 + o (1) \right) \frac{2g^n (x)}{(\pi i)} \left( 1 + o (1) \right)$$

(5.16)

Note that on $\gamma j = 2\epsilon$, $x$ is in the interior of the rotated Szegő curve $x \pi i e^{-\pi i} = 1$. Hence $\gamma (x) j > 1$, $\gamma (x) j > 1$. On the upper semicircle of $\gamma j = 2\epsilon$, $\gamma (x) j > \gamma (x) j$ while on the lower semicircle of $\gamma j = 2\epsilon$, $\gamma (x) j > \gamma (x) j$ and on the real axis $\gamma (x) j = \gamma (x) j$ No matter what we always have

$$\gamma (x) j \frac{e^{\pi i}}{e^{\gamma j} \gamma} = 1 + O (\epsilon) \gamma j = 2\epsilon$$

Similarly $\gamma (x) j \frac{1 + O (\epsilon)}{e^{\gamma j} \gamma}$. Hence from (5.16) we conclude

$$\frac{E_n (\theta x)}{n!} = \frac{K_\epsilon (1 + O (\epsilon))^n}{\pi^n}$$

for some absolute constant $K_\epsilon$ depending only on $\epsilon$. Using (5.14) and (5.15), we get

$$2N_\epsilon = \max \gamma j \geq 2\epsilon \frac{E_n (\theta x)}{n!} \frac{E_n (\theta x)}{n!} \frac{K_\epsilon (1 + O (\epsilon))^{n}}{\pi^n} \frac{(\frac{A}{\pi})}{(\frac{A}{\pi})} = K_\epsilon (1 + O (\epsilon))^{n}$$

Taking logarithms we get

$$N_\epsilon = \frac{1}{\ln 2} \ln (K_\epsilon (1 + O (\epsilon))^{n}$$

This ends the proof of part (a).

We now prove part (b).

$$N (\epsilon; b) = \frac{1}{2\pi} \Re \int \frac{h_0 (x)}{c h_n (x)} dx$$
where $h_n(x) = \frac{E_n(\eta x)}{n! (\xi x)^n}$. Decomposing $C$ into $C_1; C_2; C_3,$ and $C_4$ as shown in Figure 6, we get

\begin{equation}
N; \eta; \frac{1}{2\pi} \mathcal{J}(U_1) + \frac{1}{2\pi} \mathcal{J}(U_2) + \frac{1}{2\pi} \mathcal{J}(U_3) + \frac{1}{2\pi} \mathcal{J}(U_4);
\end{equation}

where

\begin{equation}
J_i = \frac{h_n^0(x)}{C_i} \int_{C_i} \frac{h_n^0(x)}{dx}; \quad 1 \leq i \leq 4.
\end{equation}

Let us focus on $J_1$ first. We infer that (5.16) still works for all $x$ on $C_1$ provided $\Im(x) = \delta$ is sufficiently small. This is so, for $\Im(x) = \eta(x)\frac{e^{x\xi}}{e^\delta}; 0 < \theta < \pi$. This observation leads to

\begin{equation}
\frac{E_n(\eta x)}{n! (\xi x)^n} = \frac{\mathcal{E}_n}{\mathcal{E}_n} 1 \times 1 \times 1 + O(1) + 2g^n(x, \pi i (1 + O(1)));
\end{equation}

that is, $g^n(x)$ is the dominant term among the terms in $K_n\mathcal{H}_1^1(x)$. Hence $h_n(x)$ $g^n(x)$ is uniformly. This implies

\begin{equation}
\frac{h_n^0(x)}{h_n(x)} \sim 1 \times 1 \times 0; \quad \text{uniformly}
\end{equation}

Integrating the above along $C_1$ we get

\begin{equation}
\frac{1}{2\pi n} \Im(U_1) \sim \frac{1}{2\pi} \int_{C_1} \frac{1}{x} \int_x dx
\end{equation}

that is,

\begin{equation}
\lim_{n \to \infty} \frac{1}{2\pi n} \Im(U_1) = \frac{b}{2} + O(\delta) + O(\xi) + O(\tan \frac{1}{\epsilon})
\end{equation}

Similarly, we obtain

\begin{equation}
\lim_{n \to \infty} \frac{1}{2\pi n} \Im(U_2) = \frac{b}{2} + O(\delta) + O(\xi) + O(\tan \frac{1}{\epsilon})
\end{equation}

(In this case, $g^n(x)$ is becomes dominant, rather than $g^n(x)$.) An estimate for $\frac{1}{2\pi} \Im(U_2)$ comes from the observation that the change of arguments for $g(x)$ on the vertical segment $b \leq i \delta; b$ is of order $O(\delta)$ (5.18). Hence,

\begin{equation}
\frac{1}{2\pi} \Im(U_2) = O(\delta).
\end{equation}
Similar to (5.14) we can likewise prove that
\[
\frac{1}{2\pi n} \Im(J_4) = \frac{1}{n \ln 2} \ln (K_ε (1 + O(ε))^n) :
\]
Inserting all these estimates into (5.17) we get
\[
\frac{1}{n} N_n (ε;b) = b + O(δ) + O(ε) + O(\tan(\frac{δ}{ε})) + \frac{1}{n \ln 2} \ln (K_ε (1 + O(ε))^n) :
\]
Again taking (5.14) into account we get
\[
\frac{1}{n} N_n (0;b) = b + O(δ) + O(ε) + O(\tan(\frac{δ}{ε})) + \frac{2}{n \ln 2} \ln (K_ε (1 + O(ε))^n) :
\]
This implies
\[
\limsup_{n \to \infty} \frac{1}{n} N_n (0;b) = b + O(δ) + O(ε) + O(\tan(\frac{δ}{ε})) + \ln (1 + O(ε)) :
\]
We get a similar lower bound for
\[
\liminf_{n \to \infty} \frac{1}{n} N_n (0;b) = b + O(δ) + O(ε) + O(\tan(\frac{δ}{ε})) + \ln (1 + O(ε)) :
\]
But the above is true for all ε > 0 and all δ > 0, hence
\[
\lim_{n \to \infty} \frac{1}{n} N_n (0;b) = b :
\]
In this way we have established all claims we have made in this paper.

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DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PA
E-mail address: wgoh@math.drexel.edu

DEPARTMENT OF MATHEMATICS, DREXEL UNIVERSITY, PHILADELPHIA, PA
E-mail address: rboyer@math.drexel.edu