OPTIMAL EXTINCTION RATES FOR THE FAST DIFFUSION EQUATION
WITH STRONG ABSORPTION

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abstract. Optimal extinction rates near the extinction time are derived for non-negative solutions to a fast diffusion equation with strong absorption, the power of the absorption exceeding that of the diffusion.

1. Introduction

Given \( m \in (0, \infty), q \in (0, 1) \), and a non-negative initial condition \( u_0 \in BC(\mathbb{R}^N), u_0 \neq 0 \), it is well-known that the initial value problem

\[
\begin{align*}
    \partial_t u - \Delta u^m + u^q &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \\
    u(0) &= u_0, \quad x \in \mathbb{R}^N,
\end{align*}
\]

has a unique non-negative (weak) solution \( u \) which vanishes identically after a finite time, a phenomenon usually referred to as finite time extinction \[19-21\]. More precisely, introducing the extinction time

\[
T_e := \sup \{ t > 0 : u(t) \neq 0 \} > 0,
\]

then \( T_e \) is finite and satisfies \( T_e \leq \|u_0\|_{\infty}^{(1-q)/(1-q)} \), the latter upper bound being a straightforward consequence of (1.1) and the comparison principle. Moreover, there holds

\[
u(t) \neq 0 \text{ for } t \in [0, T_e) \text{ and } u(t) \equiv 0 \text{ for } t \geq T_e.
\]

When \( q < m \) and \( u_0(x) \to 0 \) as \( |x| \to \infty \), finite time extinction is accompanied by an even more striking phenomenon, the instantaneous shrinking of the support, that is, the positivity set \( \mathcal{P}(t) := \{ x \in \mathbb{R}^N : u(t, x) > 0 \} \) of \( u \) at time \( t \) is a relatively compact subset of \( \mathbb{R}^N \) for all \( t \in (0, T_e) \), even if \( \mathcal{P}(0) = \mathbb{R}^N \) initially \[1,5,7,19\]. Observe that the inequality \( q < m \) is always satisfied when the diffusion is linear \( (m = 1) \) or slow \( (m > 1) \). Additional information on the behaviour of \( \mathcal{P}(t) \) as \( t \to T_e \) is also available when \( m \geq 1 \) and \( N = 1 \) \[6,10,14\].

Once finite time extinction is known to take place, gaining further insight into the underlying mechanism requires to identify the behaviour of \( u(t) \) as \( t \to T_e \), a preliminary step being to determine

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the relevant space and time scales. Simple scaling arguments predict that, for \( r \in [1, \infty] \) and \( u_0 \in L^r(\mathbb{R}^N) \), there is a constant \( \gamma_r > 0 \) (depending on \( N, m, q, u_0, \) and \( r \)) such that

\[
\|u(t)\|_r \sim \gamma_r (T_e - t)^{\alpha - (N\beta/r)},
\]

where

\[
\alpha := \frac{1}{1 - q} > 0, \quad \beta := \frac{q - m}{2(1 - q)} \in \mathbb{R}.
\]

As already observed by several authors \([9, 12, 15]\), a rather simple comparison argument provides a lower bound for the \( L^\infty \)-norm of the form \((1.4)\). Indeed, consider \( t \in (0, T_e) \) and let \( x(t) \in \mathbb{R}^N \) be a point where \( u(t) \) reaches its maximum value, that is, \( u(t, x(t)) = \|u(t)\|_\infty \). Then \( u(t)^m \) also attains its maximum value at this point, so that \( \Delta u^m(t, x(t)) \leq 0 \) and we infer from \((1.1a)\) that (at least formally)

\[
\frac{d}{dt} \|u(t)\|_\infty = \partial_t u(t, x(t)) \leq -u(t, x(t))^q = -\|u(t)\|_\infty^q.
\]

Integrating the above differential inequality over \( (t, T_e) \) gives the expected lower bound

\[
\|u(t)\|_\infty \geq [(1 - q)(T_e - t)]^{1/(1-q)}, \quad t \in [0, T_e).
\]

The derivation of an upper bound of the form \((1.4)\) turns out to be more involved and the results obtained so far are rather sparse: in one space dimension, the upper bound

\[
\|u(t)\|_\infty \leq C_\infty (T_e - t)^{1/(1-q)}, \quad t \in [0, T_e),
\]

is shown in \([17, \text{Proposition 2.2}]\) for \( m = 1 \) and in \([9, \text{Lemma 5.2, Lemma 7.2 & Lemma 9.2}]\) for \( m \in (0, 1) \), the latter being valid only for compactly supported initial data. The proofs are however of a completely different nature: in \([17]\), properties of the linear heat equation are used while the approach in \([9]\) relies on the intersection-comparison technique, which requires in particular the compactness of the support of the initial condition. Still for \( m = 1 \) but in any space dimension, the upper bound \((1.7)\) is derived in \([12, \text{Lemma 2.1}]\) for radially symmetric initial data \( u_0 \) having a non-increasing profile and satisfying \( \Delta u_0 + \mu u_0^q \geq 0 \) in \( \mathbb{R}^N \) for some \( \mu > 0 \). The last case for which \((1.7)\) is proved corresponds to the choice \( m = 2 - q > 1 \) and the proof relies on the derivation of an Aronson-Bénilan estimate, which seems to be only available for this specific choice of the parameters \( m \) and \( q \) \([15]\).

The purpose of this note is to contribute to the validity of \((1.4)\) and derive optimal upper and lower bounds near the extinction time when the parameters \( m \) and \( q \) range in

\[
\frac{(N - 2)_+}{N} < m < q < 1.
\]

Recalling that a lower bound in \( L^\infty \) is already available, see \((1.6)\), we begin with upper bounds.

**Theorem 1.1** (Upper bounds). Assume that \( m \) and \( q \) satisfy \((1.8)\) and consider a non-negative initial condition \( u_0 \in BC(\mathbb{R}^N) \), \( u_0 \not\equiv 0 \), for which there is \( \kappa_0 > 0 \) such that

\[
u_0(x) \leq \kappa_0 |x|^{-2/(q-m)}, \quad x \in \mathbb{R}^N.
\]
Given \( r \in [1, \infty] \), there is \( C_r > 0 \) depending only on \( N, m, q, u_0, \) and \( r \) such that the solution \( u \) to \((1.1a)-(1.1b)\) satisfies
\[
\|u(t)\|_r \leq C_r(T_e - t)^{(N\beta/r)} , \quad t \in (0, T_e) ,
\]
the extinction time \( T_e \) being defined in \((1.2)\).

Theorem 1.1 thus extends the validity of the upper bound \((1.7)\) established in \([9]\) for \( N = 1 \) and \( r = \infty \) to any space dimension \( N \geq 1 \) and \( r \in [1, \infty] \), while relaxing the assumption of compact support required in \([7]\). It is worth mentioning that the validity of \((1.10)\) for \( r \in [1, \infty) \) does not seem to be a simple consequence of \((1.10)\) for \( r = \infty \) since \( u(t) \) is positive everywhere in \( \mathbb{R}^N \) for all \( t \in (0, T_e) \) even if \( u_0 \) is compactly supported, see \([9, \text{Lemma 2.5}]\) and Proposition 1.4 below.

To be able to cope with higher space dimensions and non-compactly supported initial data, the proof of Theorem 1.1 takes a different route from that in \([9]\) and is carried out in two steps: we first show that the algebraic decay at infinity \((1.9)\) enjoyed by \( u_0 \) remains true throughout time evolution and combine it with \((1.1a)\) to prove \((1.10)\) for \( r = 1 \). We next use self-similar variables and Moser’s interation technique to derive \((1.10)\) for all \( r \in (1, \infty) \).

As a consequence of \((1.6)\) and Theorem 1.1 for \( r = \infty \), the correct time scale for the extinction phenomenon is identified. We now supplement the lower bound \((1.6)\) in \( L^\infty \) with another one in \( L^{m+1} \). On the one hand, it allows us to identify the right space scale. On the other hand, its derivation does not rely on the comparison principle but on energy estimates, a technique which is more likely to extend to other problems for which the former might not be available.

**Theorem 1.2** (Lower bound in \( L^{m+1} \)). Assume that \( m \) and \( q \) satisfy \((1.8)\) and consider a non-negative initial condition \( u_0 \in BC(\mathbb{R}^N), u_0 \not\equiv 0 \), such that \( u_0 \in L^{m+1}(\mathbb{R}^N) \). There is \( c_{m+1} > 0 \) depending only on \( N, m, q, \) and \( u_0 \) such that the solution \( u \) to \((1.1a)-(1.1b)\) satisfies
\[
\|u(t)\|_{m+1} \geq c_{m+1}(T_e - t)^{(N\beta/(m+1))} , \quad t \in (0, T_e) ,
\]
the extinction time \( T_e \) being defined in \((1.2)\).

Observing that Theorems 1.1 and 1.2 are shown without using the \( L^\infty \)-lower bound \((1.6)\), the latter may be recovered from these two results by Hölder’s inequality, with a less explicit constant though.

**Corollary 1.3.** Assume that \( m \) and \( q \) satisfy \((1.8)\) and consider a non-negative initial condition \( u_0 \in BC(\mathbb{R}^N), u_0 \not\equiv 0 \), enjoying the decay property \((1.9)\). For \( r \in (m+1, \infty] \), there is \( c_r > 0 \) depending only on \( N, m, q, u_0, \) and \( r \) such that the solution \( u \) to \((1.1a)-(1.1b)\) satisfies
\[
\|u(t)\|_r \geq c_r(T_e - t)^{(N\beta/r)} , \quad t \in (0, T_e) .
\]

Summarizing the outcome of Theorem 1.1, Theorem 1.2, and Corollary 1.3, we have shown that, for all non-negative initial data \( u_0 \in BC(\mathbb{R}^N), u_0 \not\equiv 0 \), enjoying the decay property \((1.9)\), the corresponding solution \( u \) to \((1.1a)-(1.1b)\) is bounded in \( L^r(\mathbb{R}^N), r \in [m+1, \infty] \), from above and from below at time \( t \in (0, T_e) \) by the same power of \( T_e - t \). Such estimates pave the way towards a more precise description of the behaviour of \( u(t) \) as \( t \to T_e \), which is expected to be self-similar. That this
is indeed the case is shown in [8,9] in one space dimension, another building block of the proof being the uniqueness of self-similar solutions [8].

We end up this note with the already mentioned everywhere positivity of solutions to (1.1a)-(1.1b) for positive times prior to the extinction time. As we shall see below, this property holds true for a wider range of the parameters $m$ and $q$, namely $0 < m \leq q < 1$. It is already observed in [9, Lemma 2.5] in one space dimension and we extend it herein to any space dimension. It is worth emphasizing that it includes the case $q = m$ and contrasts markedly with the instantaneous shrinking of the support occurring when $q < m$.

**Proposition 1.4** (Everywhere positivity). Consider $0 < m \leq q < 1$. Let $u_0 \in BC(\mathbb{R}^N)$ be a non-negative initial condition, $u_0 \not\equiv 0$, and denote the corresponding solution to (1.1a)-(1.1b) by $u$ with extinction time $T_e$. For $t \in (0, T_e)$, there holds

$$
P(t) := \{x \in \mathbb{R}^N : u(t, x) > 0\} = \mathbb{R}^N.$$  

(1.13)

Before proving the results stated above, we point out once more that the energy techniques developed herein seem to be rather flexible and are expected to have a wider range of applicability. For instance, a related approach is used in the companion paper [18], where optimal (lower and upper) bounds near the extinction time are established for a different fast diffusion equation (featuring the $p$-Laplacian operator, $p \in (1, 2)$) with a gradient absorption term.

2. Upper bounds near the extinction time

Throughout this section, we assume that $m$ and $q$ satisfy (1.8) and consider a non-negative initial condition $u_0 \in BC(\mathbb{R}^N)$, $u_0 \not\equiv 0$, enjoying the decay property (1.9). Let $u$ be the corresponding solution to (1.1a)-(1.1b).

2.1. $L^1$-estimate. We begin with the propagation throughout time evolution of the algebraic decay (1.9) and set

$$
\kappa_* := \left(\frac{2m(m + q)}{(q - m)^2}\right)^{1/(q - m)}.
$$

(2.1)

**Lemma 2.1.** For $t \in [0, \infty)$ and $x \in \mathbb{R}^N \setminus \{0\}$, there holds

$$
u(t, x) \leq \max\{\kappa_0, \kappa_*\}|x|^{-2/(q - m)}.
$$

Proof. Set $\Sigma_\kappa(x) := \kappa|x|^{-2/(q - m)}$ for $x \in \mathbb{R}^N \setminus \{0\}$, where $\kappa$ is a positive constant yet to be determined. We note that

$$
-\Delta \Sigma^m(x) + \Sigma(x)^q = -\kappa^m \left[\frac{2m(m + q)}{(q - m)^2}|x|^{-2q/(q - m)} - \frac{2m(N - 1)}{(q - m)}|x|^{-2q/(q - m)}\right]
$$

$$
+ \kappa^q |x|^{-2q/(q - m)}
$$

$$
\geq \kappa^m \left(\kappa^{q - m} - \kappa_*^{q - m}\right) |x|^{-2q/(q - m)}
$$

for $x \in \mathbb{R}^N \setminus \{0\}$, so that $\Sigma_\kappa$ is a supersolution to (1.1a) in $\mathbb{R}^N \setminus \{0\}$ for all $\kappa \geq \kappa_*$. We then choose $\kappa = \max\{\kappa_0, \kappa_*\}$ and use the comparison principle to complete the proof of Lemma 2.1. \qed
Proof of Theorem 1.1 \( r = 1 \). Let \( t \in [0, T_e) \). Integrating (1.1a) over \((t, T_e) \times \mathbb{R}^N\) gives
\[
\|u(t)\|_1 = \int_t^{T_e} \int_{\mathbb{R}^N} u(s, x)^q \, dx \, ds .
\] (2.2)
Owing to (1.8), there holds \( 2q/(q - m) > N \) and we infer from Lemma 2.1 and Hölder’s inequality that, for \( s \in (t, T_e) \) and \( R > 0 \),
\[
\int_{\mathbb{R}^N} u(s, x)^q \, dx = \int_{B(0,R)} u(s, x)^q \, dx + \int_{\mathbb{R}^N \setminus B(0,R)} u(s, x)^q \, dx
\leq CR^{N(1-q)}\|u(s)\|^q_1 + \left( \max\{\kappa_0, \kappa_*\} \right)^q \|S^{N-1}\| \int_R^\infty r^{N-1-(2q/(q-m))} \, dr
\leq C \left( R^{N(1-q)}\|u(s)\|^q_1 + R^{(N(q-m)-2q)/(q-m)} \right) .
\]
We next optimize in \( R \) in the previous inequality by setting \( R(s) := \|u(s)\|^{(q-m)/(N(m-q)+2)}_1 \), which satisfies
\[
R(s)^{N(1-q)}\|u(s)\|^q_1 = R(s)^{(N(q-m)-2q)/(q-m)} = \|u(s)\|^{(N(m-q)+2q)/(N(m-q)+2)}_1 .
\]
Consequently, taking \( R = R(s) \) in the previous inequality, we obtain
\[
\int_{\mathbb{R}^N} u(s, x)^q \, dx \leq C \|u(s)\|^{(N(m-q)+2q)/(N(m-q)+2)}_1 ,
\]
which gives, together with (2.2), the positivity of \( N(m - q) + 2q \), and the time monotonicity of \( s \mapsto \|u(s)\|_1 \),
\[
\|u(t)\|_1 \leq C \int_t^{T_e} \|u(s)\|^{(N(m-q)+2q)/(N(m-q)+2)}_1 \, ds
\leq C(T_e - t)\|u(t)\|^{(N(m-q)+2q)/(N(m-q)+2)}_1 ,
\]
from which (1.10) for \( r = 1 \) readily follows.

\[\square\]

2.2. Scaling variables and \( L^r \)-estimates, \( r \in (1, \infty) \). The next step is to take advantage of the just derived \( L^1 \)-upper bound to derive the corresponding ones in \( L^r \) for \( r \in (1, \infty) \). To this end, we introduce the scaling variables
\[
s := \ln \left( \frac{T_e}{T_e - t} \right) , \quad y := x(T_e - t)^\beta , \quad (t, x) \in [0, T_e) \times \mathbb{R}^N ,
\] (2.3)
and the new unknown function \( v \) defined by
\[
u(t, x) = (T_e - t)^\alpha v \left( \ln(T_e) - \ln(T_e - t), x(T_e - t)^\beta \right) , \quad (t, x) \in [0, T_e) \times \mathbb{R}^N ,
\] (2.4)
or, equivalently,
\[
v(s, y) = T_e^{-\alpha} e^{\alpha s} u \left( T_e(1 - e^{-s}), yT_e^{-\beta} e^{\beta s} \right) , \quad (s, y) \in [0, \infty) \times \mathbb{R}^N .
\] (2.5)
It readily follows from (1.1a)-(1.1b) that \( v \) solves
\[
\partial_t v(s, y) = \alpha v(s, y) + \beta y \cdot \nabla v(s, y) + \Delta v^m(s, y) - v(s, y)^q, \quad (s, y) \in (0, \infty) \times \mathbb{R}^N, \quad (2.6)
\]
\[
v(0, y) = v_0(y) := T_e^{-\alpha} u_0 \left( yT_e^{-\beta} \right), \quad y \in \mathbb{R}^N. \quad (2.7)
\]
Since
\[
\|u(t)\|_r = (T_e - t)^{\alpha - (N\beta/r)} \|v(s)\|_r, \quad t \in (0, T_e),
\]
for all \( r \in [1, \infty] \), we realize that an upper bound such as (1.10) on \( \|u(t)\|_r \) for \( t \in (0, T_e) \) obviously follows from a uniform upper bound on \( \|v(s)\|_r \) for \( s \geq 0 \), the converse being true as well. In particular, it follows from (2.8) and Theorem 1.1 for \( r = 1 \) that
\[
\|v(s)\|_1 \leq C_1, \quad s \geq 0, \quad (2.9)
\]
and we may assume without loss of generality that \( C_1 \geq 1 \).

We now aim at using a bootstrap argument to deduce from (2.6) and (2.9) that \( v \) belongs to \( L^\infty(0, \infty; L^r(\mathbb{R}^N)) \) for all \( r \in (1, \infty] \). To this end, Moser’s iteration technique is a suitable tool and the way we apply it is inspired from [2, Theorem 3.1]. But since [2, Theorem 3.1] is devoted to the slow diffusion case \( m > 1 \), some technical aspects of its proof do not seem to apply directly here and we borrow additional arguments from the proof of [3, Proposition 2].

**Lemma 2.2.** Let \( r \in (0, \infty) \). There is \( C_{r+1} > 0 \) depending only on \( N, m, q, u_0 \), and \( r \) such that
\[
\|v(s)\|_{r+1} \leq C_{r+1}, \quad s \geq 0.
\]

**Proof.** Let \( r \in [2 - m, \infty) \). Multiplying (2.6) by \( v^r \), integrating over \( \mathbb{R}^N \), and using integration by parts, we obtain
\[
\frac{1}{r+1} \frac{d}{ds} \|v\|_{r+1}^r + rm \int_{\mathbb{R}^N} v^{r+m-2} |\nabla v|^2 dy + \int_{\mathbb{R}^N} v^{r+q} dy = \left( \alpha - \frac{N\beta}{r+1} \right) \|v\|_{r+1}^r.
\]
\[
\frac{d}{ds} \|v\|_{r+1}^r + 4mr(r+1) \|v\|_{r+1}^r \|\nabla v^{(m+r)/2}\|_2^2 \leq \alpha(r+1) \|v\|_{r+1}^r.
\]

Since \( 4mr(r+1) \geq 2m(m+r)^2 \), we end up with
\[
\frac{d}{ds} \|v\|_{r+1}^r + 2m \|\nabla v^{(m+r)/2}\|_2^2 \leq \alpha(r+1) \|v\|_{r+1}^r. \quad (2.10)
\]

We next fix \( \zeta \in (2/m, 2^* \) where \( 2^* := 2N/(N-2) \) (with \( 2^* = \infty \) for \( N = 1, 2 \)). On the one hand, it follows from the Gagliardo-Nirenberg inequality that
\[
\|v^{(m+r)/2}\|_{\zeta} \leq C \|\nabla v^{(m+r)/2}\|_2^\theta \|v^{(m+r)/2}\|_1^{1-\theta}, \quad (2.11)
\]
with
\[
\theta := \frac{2N(\zeta - 1)}{(N+2)\zeta}.
\]

On the other hand, since \( (m+r)/2 \in [1, (m+r)/2] \) for \( r \geq 2-m \), we infer from Hölder’s inequality that
\[
\|v^{(m+r)/2}\|_{(m+r)/2} \leq \|v\|_{(m+r)/2}^{\zeta(m+r)(m+r-2)/(\zeta(m+r)-2)} \|v\|_1^{(\zeta-1)(m+r)/(\zeta(m+r)-2)}. \quad (2.12)
\]
We deduce from (2.11) and (2.12) that
\[
\| v \|_{\zeta(m+r)/2}^{(m+r)/2} = \| v^{(m+r)/2} \|_{\zeta} \leq C \| \nabla v^{(m+r)/2} \|^\theta_2 \left( \| v \|_{\zeta}^{(m+r)/2} \right)^{1-\theta} \\
\leq C \| \nabla v^{(m+r)/2} \|^\theta_2 \left[ \| v \|_{\zeta(m+r)/2}^{(m+r-2)/2(\zeta(m+r)-2)} \| v \|_1^{(\zeta-1)(m+r)/(\zeta(m+r)-2)} \right]^{1-\theta},
\]
hence
\[
\| v \|_{\zeta(m+r)/2}^{(m+r)[N(m+r)+2-N]/N[\zeta(m+r)-2]} \leq C \| \nabla v^{(m+r)/2} \|^2 \| v \|^{{2N-(N-2)\zeta(m+r)/N[\zeta(m+r)-2]}}. \tag{2.13}
\]
Moreover, since \( \zeta > 2/m \) and \( m < 1 \), we have \( 2r \leq m[\zeta(m+r) - 2] \), hence
\[
\frac{2N - (N - 2)\zeta}{N[\zeta(m+r) - 2]} \leq \frac{2N - (N - 2)\zeta}{N} \frac{m(m+r)}{2r} \leq \frac{m[2N - (N - 2)\zeta]}{N},
\]
so that
\[
\| v \|_1^{2N-(N-2)\zeta(m+r)/N[\zeta(m+r)-2]} \leq C_1^{2N-(N-2)\zeta(m+r)/N[\zeta(m+r)-2]} \leq C_1^m[2N-(N-2)\zeta]/N. \tag{2.14}
\]
Also,
\[
1 - \frac{N[\zeta(m+r) - 2]}{\zeta[N(m+r) + 2 - N]} = \frac{2N - \zeta(N-2)}{\zeta[N(m+r) + 2 - N]} > 0 ,
\]
and we infer from (2.13), (2.14), and Young’s inequality that
\[
\| v \|_{\zeta(m+r)/2}^{m+r} \leq \frac{N[\zeta(m+r) - 2]}{\zeta[N(m+r) + 2 - N]} \| v \|_{\zeta(m+r)/2}^{(m+r)[N(m+r)+2-N]/N[\zeta(m+r)-2]} \]
\[
+ \frac{2N - \zeta(N-2)}{\zeta[N(m+r) + 2 - N]} \| v \|_{\zeta(m+r)/2} \]
\[
\leq C \| \nabla v^{(m+r)/2} \|^2 + 1.
\]
Therefore, there is \( \nu \in (0, 1) \) depending only on \( N, m, q \), and \( u_0 \) such that
\[
\nu \left( \| v \|_{\zeta(m+r)/2}^{m+r} - 1 \right) \leq \| \nabla v^{(m+r)/2} \|^2. \tag{2.15}
\]
Moreover, since \( r + 1 \in [1, \zeta (m + r) / 2] \), it follows from (2.9) and H"older’s and Young’s inequalities that
\[
\| v \|_{r+1}^{r+1} \leq \| v \|_{\zeta (m+r)/2}^{|r+1|} \| v \|_1^{(r+1) \zeta (m+r)/2} \| v \|_1^{(r+1)(r+m)/(r+m)}
\]
\[
\leq C_1 \| v \|_{\zeta (m+r)/2}^{r(r+m)/(r+m)} \| v \|_1^{(r+1)(r+m)/(r+m)}
\]
\[
\leq C_1 \| v \|_{\zeta (m+r)/2}^{r(r+m)/(r+m)} + \delta \| v \|_{\zeta (m+r)/2}^{(r+1)(r+m)/(r+m)}
\]
(2.16)

Next, let \( \sigma > 1 \) to be chosen appropriately later on and set
\[
I_r(s) := \int_{\mathbb{R}^N} v(s, y) \zeta_{[r+\sigma(1-m)]/\sigma(\zeta-2)+2} dy , \quad s \geq 0 .
\]

Since \( \sigma(\zeta-2)+2 \in [\zeta, \sigma \zeta] \) and
\[
\| v \|_{r+1}^{r+1} \leq \| v \|_{\zeta (m+r)/2}^{\sigma(1-m)} \| v \|_{\zeta (m+r)/2}^{(r+1)\delta} + \frac{1}{\delta \sigma^{\sigma(\zeta-2)+2}/\zeta}
\]
(2.17)

Combining (2.10), (2.15), and (2.17) leads us to
\[
\frac{d}{ds} \| v \|_{r+1}^{r+1} + 2m \nu (\| v \|_{\zeta (m+r)/2}^{r+1} - 1) \leq \frac{d}{ds} \| v \|_{r+1}^{r+1} + 2m \| \nabla v \|_{\zeta (m+r)/2}^{r+1}
\]
\[
\leq \alpha(r+1) \| v \|_{r+1}^{r+1} + \frac{\alpha(r+1)}{\delta \sigma^{\sigma(\zeta-2)+2}/\zeta}
\]
We then choose \( \delta = m \nu / \alpha(r+1) \) in the above inequality to obtain
\[
\frac{d}{ds} \| v \|_{r+1}^{r+1} + m \nu \| v \|_{\zeta (m+r)/2}^{r+1} \leq 2m \nu + \frac{\alpha(r+1)}{\delta \sigma^{\sigma(\zeta-2)+2}/\zeta}
\]
We finally use (2.16) to estimate from below the second term of the left-hand side of the previous inequality and end up with
\[
\frac{d}{ds} \| v \|_{r+1}^{r+1} + m \nu \| v \|_{r+1}^{r+1} \leq 2m \nu + m \nu C_1^{\zeta (\zeta-2)/\zeta (m-2)} + \frac{\alpha(r+1)}{\delta \sigma^{\sigma(\zeta-2)+2}/\zeta}
\]
(2.18)
We first choose
\[ \sigma = \frac{\zeta(m + r) - 2}{\zeta m - 2} > 1 \]
in (2.18) and observe that this choice guarantees that
\[ \zeta[m + r + \sigma(1 - m)] = \sigma(\zeta - 2) + 2. \]
Consequently, \( I_r = \|v\|_1 \) and we deduce from (2.9) and (2.18) that there is \( C(r) > 0 \) depending on \( N, m, q, u_0, \) and \( r \) such that
\[ \frac{d}{ds}\|v\|_{r+1} + m\nu\|v\|_{r+1} \leq C(r). \]
Integrating the previous differential inequality entails that
\[ \sup_{s \geq 0} \|v(s)\|_{r+1} < \infty. \] (2.19)
The validity of (2.19) extends to all \( r \in (0, 2 - m) \) by (2.9) and Hölder’s inequality.

To complete the proof of Lemma 2.2, we are left to check the boundedness of \( v \) in \( L^\infty(\mathbb{R}^N) \). To this end, we take \( \sigma = \sigma_0 := 2(\zeta - 1)/\zeta(\zeta - 2) > 1 \) in (2.18) and obtain, after integration with respect to time,
\[ \|v(s)\|_{r+1} \leq \|v_0\|_{r+1} e^{-m\nu s} + 2 + C_1 \frac{\zeta(m+r) - 2}{(\zeta m - 2)} \]
\[ + \left( \frac{\alpha(r + 1)}{m\nu} \right)^{\sigma_0} \sup_{s \in [0,s]} \mathcal{I}_r(s), \]
\[ \leq \|v_0\|_1 \|v_0\|_\infty + 2 + C_1 \frac{\zeta(m+r) - 2}{(\zeta m - 2)} \]
\[ + \left( \frac{\alpha(r + 1)}{m\nu} \right)^{\sigma_0} \sup_{s \in [0,s]} \mathcal{I}_r(s) \],
and
\[ \mathcal{I}_r = \|v\|_{\lfloor (r+m) + \sigma_0(1-m)/2 \rfloor} \cdot \]
Therefore, there are \( K_0 > 0 \) and \( K_1 > 0 \) depending only on \( N, m, q, \) and \( u_0 \) such that
\[ \sup_{s \geq 0} \{\|v(s)\|_{r+1}^m\} \leq K_0 \left( K_1^{r+1} + (1 + r)^{\sigma_0} \sup_{s \geq 0} \left\{\|v(s)\|_{\lfloor (r+m) + \sigma_0(1-m)/2 \rfloor} \right\} \right). \] (2.20)
We now define the sequence \((r_j)_{j \geq 0}\) by
\[ 1 + r_{j+1} = 2(1 + r_j) - (1 - m)(\sigma_0 - 1), \quad j \geq 0, \quad r_0 := 2 - m, \]
and set
\[ V_j := \sup_{s \geq 0} \left\{\|v(s)\|_{r_j+1}^m\}, \quad j \geq 0. \]
For \( j \geq 0 \), we take \( r = r_{j+1} \) in (2.20) and realize that
\[
V_{j+1} \leq K_0 \left( K_1^{1+r_{j+1}} + (1 + r_{j+1})^{\sigma_0} V_j^2 \right)
\]
\[
\leq K_0 (1 + r_{j+1})^{\sigma_0} \max \left\{ K_1^{1+r_{j+1}}, V_j^2 \right\}, \quad j \geq 0.
\]
Since \( \sigma_0 - 1 < 1/(1 - m) \) thanks to the constraint \( \zeta > 2/m \), one has \( 1 + r_0 - (1 - m)(\sigma_0 - 1) > 0 \) and we are in a position to apply [22, Lemma A.1], which we recall in Lemma 2.3 below for completeness, to conclude that there is \( K_2 > 0 \) depending only on \( m, \zeta, K_0, \) and \( K_1 \) such that
\[
V_j^{1/(1+r_j)} \leq K_2, \quad j \geq 0.
\]
Equivalently,
\[
\sup_{s \geq 0} \{ \|v(s)\|_{1+r_j} \} \leq K_2, \quad j \geq 0,
\]
and letting \( j \to \infty \) entails that \( \|v(s)\|_{\infty} \leq K_2 \) for all \( s \geq 0 \), thereby completing the proof of Lemma 2.2.

The proof of Theorem 1.1 for \( r \in (1, \infty] \) is now a straightforward consequence of (2.8) and Lemma 2.2.

**Lemma 2.3.** Let \( a > 1, b \geq 0, c \in \mathbb{R}, C_0 \geq 1, C_1 \geq 1, \) and \( p_0 > 0 \) be given such that \( p_0(a-1)+c > 0 \). We define the sequence \( (p_k)_{k \geq 0} \) of positive real numbers by \( p_{k+1} = ap_k + c \) for \( k \geq 0 \) and assume that \( (Q_k)_{k \geq 0} \) is a sequence of positive real numbers satisfying
\[
Q_0 \leq C_1^{p_0}, \quad Q_{k+1} \leq C_0 p_k^b \max \left\{ C_1^{p_k+1}, Q_k^a \right\}, \quad k \geq 0.
\]
Then the sequence \( \left( Q_k^{1/p_k} \right)_{k \geq 0} \) is bounded.

### 3. Lower bound near the extinction time

We now turn to the lower bound near the extinction time in \( L^{m+1}(\mathbb{R}^N) \).

**Proof of Theorem 1.2.** For \( t \in [0, T_e] \), we define
\[
X(t) := \|u(t)\|_{m+1}^{m+1} \quad \text{and} \quad Y(t) := \int_{\mathbb{R}^N} u(t,x)^{m+q} \, dx.
\]
Let \( t \in (0, T_e) \). It follows from (1.1a) that
\[
\frac{1}{m+1} \frac{dX}{dt}(t) + \|\nabla u_m(t)\|_2^2 + Y(t) = 0.
\]
Since \( 1 < \frac{m+q}{m} < \frac{m+1}{m} < 2^* := \frac{2N}{(N-2)_+} \)
by (1.8) we infer from the Gagliardo-Nirenberg inequality that
\[ X(t)^{m/(m+1)} = \|u(t)^m\|_{(m+1)/m} \leq C \|\nabla u^m(t)\|^\theta_2 \|u(t)^m\|^{1-\theta}_{(m+q)/m} \]
\[ \leq CY(t)^{m(1-\theta)/(m+q)} \|\nabla u^m(t)\|^\theta_2, \]
where
\[ \theta := \frac{2Nm(1-q)}{(m+1)\left[m(N+2) - q(N-2)\right]}. \]
Consequently, since \( u(t) \neq 0 \) as \( t \in (0, T_e) \),
\[ \|\nabla u^m(t)\|_2^2 \geq CX(t)^{2m/\theta(m+1)}Y(t)^{-2m(1-\theta)/\theta(m+q)}, \]
which gives, together with (3.1),
\[ \frac{dX}{dt}(t) + CX(t)^{2m/\theta(m+1)}Y(t)^{-2m(1-\theta)/\theta(m+q)} + (m+1)Y(t) \leq 0. \tag{3.2} \]
Setting
\[ \xi := 1 + \frac{2m(1-\theta)}{\theta(m+q)} > 1 \quad \text{and} \quad \gamma := \frac{2m}{\theta \xi(m+1)}, \]
it follows from Young’s inequality that
\[ X(t)^\gamma = X(t)^\gamma Y(t)^{-\xi(1-\gamma)/\xi}Y(t)^{(1-\gamma)/\xi} \leq \frac{1}{\xi}X(t)^\xi Y(t)^{1-\xi} + \frac{\xi-1}{\xi}Y(t) \]
\[ \leq X(t)^{2m/\theta(m+1)}Y(t)^{-2m(1-\theta)/\theta(m+q)} + Y(t) \]
Combining this inequality with (3.2) leads us to the differential inequality
\[ \frac{dX}{dt}(t) + CX(t)^\gamma \leq 0, \quad t \in (0, T_e). \tag{3.3} \]
Now,
\[ \gamma = \frac{2m}{\theta \xi(m+1)} = \frac{2m(m+q)}{(2m+\theta(q-m))(m+1)} \]
\[ = \frac{2m(m+q)[m(N+2) - q(N-2)]}{2m\left[(m+1)[m(N+2) - q(N-2)] + N(q-m)(1-q)\right]} \]
\[ = \frac{m(N+2) - q(N-2)}{m(N+2) - qN + 2} \in (0, 1), \]
and we integrate (3.3) over \((t, T_e)\) to obtain
\[ -X(t)^{1-\gamma} + (1-\gamma)C(T_e - t) \leq 0, \quad t \in (0, T_e). \]
Noticing that
\[ (m+1)\alpha - N\beta = \frac{m+1}{1-q} - \frac{N(q-m)}{2(1-q)} = \frac{m(N+2) - qN + 2}{2(1-q)} = \frac{1}{1-\gamma}, \]
the lower bound (1.11) readily follows from the previous inequality. \qed
Proof of Corollary 1.3. We first note that, owing to (1.8), there holds $2/(q - m) > N$ and (1.9) entails that $u_0 \in L^1(\mathbb{R}^N)$. Since $u_0$ also belongs to $L^\infty(\mathbb{R}^N)$, we conclude that $u_0 \in L^{m+1}(\mathbb{R}^N)$.

Let $r \in (m + 1, \infty]$ and $t \in (0, T_e)$. We infer from Theorem 1.1 Theorem 1.2 and Hölder’s inequality that

$$c_{m+1}^{m+1}(T_e - t)^{(m+1)\alpha - N\beta} \leq \|u(t)\|_{m+1}^{m+1} \leq \|u(t)\|_r^{r m/(r-1)} \|u(t)\|_1^{(r-1-m)/(r-1)}$$

$$\leq C_1(T_e - t)^{(\alpha - N\beta)(r-1-m)/(r-1)} \|u(t)\|_r^{r m/(r-1)},$$

from which (1.12) readily follows. \hfill \Box

4. EVERYWHERE POSITIVITY

In this section, we assume that $0 < m \leq q < 1$ and consider a non-negative initial condition $u_0 \in BC(\mathbb{R}^N)$, $u_0 \neq 0$. We denote the corresponding solution to (1.1a)-(1.1b) by $u$ and define its extinction time by (1.2). As in [9], the proof relies on an upper bound for $\partial_t u$ which we establish now.

Lemma 4.1. For $t > 0$ there holds

$$\partial_t u(t) \leq \frac{u(t)}{(1-m)t} \quad \text{in} \quad \mathbb{R}^N.$$  

When $q = m$, Lemma 4.1 is a consequence of [4, Theorem 2], the proof relying on an homogeneity argument. Though the operator $-\Delta u^m + u^q$ is not homogeneous, we may still adapt the proof of [4, Theorem 2] when $q \geq m$.

Proof. Given a non-negative initial condition $u_0 \in BC(\mathbb{R}^N)$, we denote the corresponding solution to (1.1a)-(1.1b) at time $t \geq 0$ by $S(t)u_0$. Recall that, if $u_0$ and $v_0$ are two non-negative functions in $BC(\mathbb{R}^N)$ satisfying $u_0 \geq v_0$, then the comparison principle entails $S(t)u_0 \geq S(t)v_0$ for all $t \geq 0$.

Step 1. We first claim that, for $\lambda \geq 1$,

$$S(\lambda t)u_0 \leq \lambda^{1/(1-m)} S(t) \left( \lambda^{1/(m-1)} u_0 \right), \quad t \geq 0.$$  

Indeed, setting $u(t) := S(t)u_0$ for $t \geq 0$, the function $v$ defined by $v(t) := \lambda^{1/(m-1)} S(\lambda t)u_0$ satisfies

$$\partial_t v(t, x) - \Delta v^m(t, x) + v(t, x)^q = \lambda^{m/(m-1)} \partial_t u(\lambda t, x) - \lambda^{m/(m-1)} \Delta u^m(\lambda t, x) + \lambda^{q/(m-1)} u(\lambda t, x)^q$$

$$= (\lambda^{q/(m-1)} - \lambda^{m/(m-1)}) u(\lambda t, x)^q \leq 0.$$  

Since $v(0) = \lambda^{1/(m-1)} u_0 \leq u_0$, we infer from the comparison principle that (4.1) holds true.
Step 2. Now, fix $t > 0$ and consider $h > 0$. Since $\lambda = (1 + h/t) > 1$ and $m \in (0, 1)$, we infer from (4.1) and the comparison principle that
\[ S(t + h)u_0 - S(t)u_0 = S(\lambda t)u_0 - S(t)u_0 \leq \lambda^{1/(1-m)}S(t) \left( \lambda^{1/(m-1)}u_0 \right) - S(t)u_0 \leq \left[ \left( 1 + \frac{h}{t} \right)^{1/(1-m)} - 1 \right] S(t)u_0. \]
Dividing the above inequality by $h$ and passing to the limit as $h \to 0$ complete the proof.

We now argue as in the proof of [9, Lemma 2.5] to complete the proof of Proposition 1.4.

Proof of Proposition 1.4. Fix $t \in (0, T_e)$ and assume for contradiction that $u(t, x_0) = 0$ for some $x_0 \in \mathbb{R}^N$. By (1.1a) and Lemma 4.1, there holds
\[ -\Delta u^m(t) + u(t)^q + u(t)^{(1-m)}t \geq 0 \quad \text{in} \quad \mathbb{R}^N, \]
so that $u(t)^m$ is a supersolution to
\[ -\Delta w + dw = 0 \quad \text{in} \quad \mathbb{R}^N, \]
with $d(x) := u(t, x)^{q-m} + u(t, x)^{1-m}/((1-m)t)$ for $x \in \mathbb{R}^N$. Since $t > 0$ and $m \leq q < 1$, the function $d$ is non-negative and bounded and we infer from the strong maximum principle [16, Theorem 8.19] that $u(t)^m \equiv 0$ in $\mathbb{R}^N$, contradicting $t < T_e$. Consequently, $u(t)^m$ is positive everywhere in $\mathbb{R}^N$ and the proof of Proposition 1.4 is complete.

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