Subclasses of analytic functions associated with Pascal distribution series

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Abstract

In the present paper we determine necessary and sufficient conditions for the Pascal distribution series to be in the subclasses $S(k, \lambda)$ and $C(k, \lambda)$ of analytic functions. Further, we consider an integral operator related to Pascal distribution series. Some interesting special cases of our main results are also considered.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n. \quad (2)$$

A function $f$ of the form (2) is in $S(k, \lambda)$ if it satisfies the condition

$$\left| \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1 \right| < k, \quad (0 < k \leq 1, \ 0 \leq \lambda < 1, \ z \in U)$$

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and \( f \in \mathcal{C}(k,\lambda) \) if and only if \( zf' \in \mathcal{S}(k,\lambda) \). The class \( \mathcal{S}(k,\lambda) \) was studied by Frasin et al. [7].

We note that \( \mathcal{S}(k,0) = \mathcal{S}(k) \) and \( \mathcal{C}(k,0) = \mathcal{C}(k) \), where the classes \( \mathcal{S}(k) \) and \( \mathcal{C}(k) \) were introduced and studied by Padmanabhan [7] (see also, [11, 16]).

A function \( f \in \mathcal{A} \) is said to be in the class \( \mathcal{R}^\tau(A,B) \) if \( -1 < B < A \leq 1 \), if it satisfies the inequality

\[
\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathbb{U}.
\]

This class was introduced by Dixit and Pal [3].

A variable \( X \) is said to be Pascal distribution if it takes the values \( 0, 1, 2, 3, \ldots \) with probabilities

\[
(1-q)^m, \quad \frac{q^m(1-q)^m}{1!}, \quad \frac{q^m(1-q)^{m+1}(m+1)}{2!}, \quad \frac{q^m(1-q)^{m+1}(m+2)(1-q)^m}{3!}, \ldots,
\]

respectively, where \( q \) and \( m \) are called the parameters, and thus

\[
P(X = r) = \binom{r + m - 1}{m - 1} q^{r}(1-q)^{m}, \quad (m \geq 1, 0 \leq q \leq 1, r = 0, 1, 2, 3, \ldots).
\]

Very recently, El-Deeb et al. [5] (see also, [13, 11]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

\[
\Psi_q^m(z) := z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}(1-q)^m z^n, \quad z \in \mathbb{U},
\]

where \( m \geq 1, 0 \leq q \leq 1 \), and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

\[
\Phi_q^m(z) := 2z - \Psi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}(1-q)^m a_n z^n, \quad z \in \mathbb{U}.
\] (3)

Let consider the linear operator \( \mathcal{I}_q^m : \mathcal{A} \to \mathcal{A} \) defined by the convolution or Hadamard product

\[
\mathcal{I}_q^m f(z) := \Psi_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} q^{n-1}(1-q)^m a_n z^n, \quad z \in \mathbb{U},
\]

where \( m \geq 1 \) and \( 0 \leq q \leq 1 \).

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, by using hypergeometric functions (see for example, [2, 7, 10, 11, 20]) and by using various distributions such as Yule-Simon distribution, Logarithmic distribution, Poisson distribution, Binomial distribution, Beta-Binomial distribution, Zeta distribution, Geometric distribution and Bernoulli distribution (see for example, [4, 6, 8, 9, 12, 13, 18, 15]), in this paper, we determine the necessary and sufficient conditions for \( \Phi_q^m(z) \) to be in our classes \( \mathcal{S}(k,\lambda) \) and \( \mathcal{C}(k,\lambda) \) and connections of these subclasses with \( \mathcal{R}^\tau(A,B) \). Finally, we give conditions for the integral operator \( G_q^m(m,z) = \int_0^z \frac{t^{m-1}}{t^m} \varphi(t) dt \) belonging to the above classes.

2. Preliminary lemmas

To establish our main results, we need the following Lemmas.

**Lemma 2.1.** [7] A function \( f \) of the form [2] is in \( \mathcal{S}(k,\lambda) \) if and only if it satisfies

\[
\sum_{n=2}^{\infty} |a_n((1-\lambda) + k(1+\lambda)) - (1-\lambda)(1-k)] |a_n| \leq 2k
\] (4)
where $0 < k \leq 1$ and $0 \leq \lambda < 1$. The result is sharp.

**Lemma 2.2.** A function $f$ of the form (2) is in $\mathcal{C}(k,\lambda)$ if and only if it satisfies

$$
\sum_{n=2}^{\infty} n[n((1 - \lambda) + k(1 + \lambda)) - (1 - \lambda)(1 - k)] |a_n| \leq 2k
$$

where $0 < k \leq 1$ and $0 \leq \lambda < 1$. The result is sharp.

**Lemma 2.3.** If $f \in \mathcal{R}^+(A,B)$ is of the form (1), then

$$
|a_n| \leq (A - B)\frac{|\tau|}{n}, \quad n \in \mathbb{N} - \{1\}.
$$

The result is sharp for the function

$$
f(z) = \int_0^z (1 + (A - B)\frac{\tau t^{n-1}}{1 + Bt^{n-1}})dt, \quad (z \in \mathbb{U}; n \in \mathbb{N} - \{1\}).
$$

### 3. Necessary and sufficient conditions

For convenience throughout in the sequel, we use the following identities that hold at least for $m \geq 2$ and $0 \leq q < 1$:

$$
\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n = \frac{1}{(1-q)^m}, \quad \sum_{n=0}^{\infty} \binom{n+m-2}{m-2} q^n = \frac{1}{(1-q)^{m-1}},
$$

$$
\sum_{n=0}^{\infty} \binom{n+m}{m} q^n = \frac{1}{(1-q)^{m+1}}, \quad \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n = \frac{1}{(1-q)^{m+2}}.
$$

By simple calculations we derive the following relations:

$$
\sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n - 1 = \frac{1}{(1-q)^{m-1}},
$$

$$
\sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1} = qm \sum_{n=0}^{\infty} \binom{n+m}{m} q^n = \frac{qm}{(1-q)^{m+1}},
$$

and

$$
\sum_{n=3}^{\infty} (n-1)(n-2) \binom{n+m-2}{m-1} q^{n-1} = q^2m(m+1) \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n = \frac{q^2m(m+1)}{(1-q)^{m+2}}.
$$

Unless otherwise mentioned, we shall assume in this paper that $0 < k \leq 1$, $0 \leq \lambda < 1$, while $m \geq 1$ and $0 \leq q < 1$.

Firstly, we obtain the necessary and sufficient conditions for $\Phi^m_q$ to be in the class $\mathcal{S}(k,\lambda)$.

**Theorem 3.1.** We have $\Phi^m_q \in \mathcal{S}(k,\lambda)$, if and only if

$$
((1 - \lambda) + k(1 + \lambda)) \frac{q^m}{(1-q)^{m+1}} \leq 2k.
$$
Proof. Since
\[ \Phi_q^m(z) = z - \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m z^n \] (7)
in view of Lemma 2.1 it suffices to show that
\[ \sum_{n=2}^{\infty} [n((1 - \lambda) + k(1 + \lambda)) - (1 - \lambda)(1 - k)] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \leq 2k. \] (8)
Writing
\[ n = (n - 1) + 1 \]
in (8) we have
\[ \sum_{n=2}^{\infty} [n((1 - \lambda) + k(1 + \lambda)) - (1 - \lambda)(1 - k)] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \]
\[ \sum_{n=2}^{\infty} [(n - 1)((1 - \lambda) + k(1 + \lambda)) + 2k] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \]
\[ = [(1 - \lambda) + k(1 + \lambda)] \sum_{n=2}^{\infty} (n - 1) \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \]
\[ + 2k \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \]
\[ = ((1 - \lambda) + k(1 + \lambda)) \frac{q m}{1 - q} + 2k (1 - (1 - q)^m). \]
But this last expression is bounded above by 2k if and only if (6) holds.

Theorem 3.2. We have \( \Phi_q^m \in C(k, \lambda) \) if and only if
\[ [(1 - \lambda) + k(1 + \lambda)] \frac{q^2 m(m + 1)}{(1 - q)^{m+2}} \]
\[ + [\lambda(1 - \lambda)(2 + k) + 3k(1 + \lambda)] \frac{q m}{(1 - q)^{m+1}} \leq 2k. \] (9)
Proof. In view of Lemma 2.2 we must show that
\[ \sum_{n=2}^{\infty} n[n((1 - \lambda) + k(1 + \lambda)) - (1 - \lambda)(1 - k)] \left( \frac{n + m - 2}{m - 1} \right) q^{n-1}(1 - q)^m \leq 2k. \] (10)
Writing
\[ n^2 = (n - 1)(n - 2) + 3(n - 1) + 1 \] and \( n = (n - 1) + 1 \),
we get
\[ \sum_{n=2}^{\infty} n[ (1-\lambda) + k(1+\lambda) ] - (1-\lambda)(1-k)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \]

\[ \frac{[(1-\lambda) + k(1+\lambda)]}{m} \sum_{n=3}^{\infty} (n-1)(n-2) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \]

\[ + [(1-\lambda)(2+k) + 3k(1+\lambda)] \sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \]

\[ + 2k \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \]

\[ = \frac{[ (1-\lambda) + k(1+\lambda)] q^2 m(m+1)}{(1-q)^2} \]

\[ + [(1-\lambda)(2+k) + 3k(1+\lambda)] \frac{q m}{(1-q)^2} + 2k (1 - (1-q)^m) . \]

Therefore, we see that the last expression is bounded above by 2k if (9) is satisfied.

\[ \square \]

4. Inclusion Properties

Making use of Lemma 2.3, we will study the action of the Pascal distribution series on the classes \( \mathcal{S}(k,\lambda) \) and \( \mathbb{C}(k,\lambda) \).

**Theorem 4.1.** Let \( m > 1 \). If \( f \in \mathcal{R}^r(A,B) \), then \( T_q^m \in \mathcal{S}(k,\lambda) \) if

\[ (A - B)|\tau| [[(1-\lambda) + k(1+\lambda)] (1 - (1-q)^m)] \]

\[ - \frac{(1-\lambda)(1-k)}{q(m-1)} [(1-q) - (1-q)^m - q(m-1)(1-q)^m] \]

\[ \leq 2k. \] (11)

**Proof.** In view of Lemma 2.1, it suffices to show that

\[ \sum_{n=2}^{\infty} n[ (1-\lambda) + k(1+\lambda) ] - (1-\lambda)(1-k)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m |a_n| \leq 2k. \]

Since \( f \in \mathcal{R}^r(A,B) \), then by Lemma 2.3 we have

\[ |a_n| \leq \frac{(A - B)|\tau|}{n}. \] (12)

Thus, we have

\[ \sum_{n=2}^{\infty} n[ (1-\lambda) + k(1+\lambda) ] - (1-\lambda)(1-k)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m |a_n| \]

\[ \leq (A - B)|\tau| \sum_{n=2}^{\infty} [ (1-\lambda) + k(1+\lambda)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \]

\[ - \sum_{n=2}^{\infty} \frac{1}{n} [(1-\lambda)(1-k)] \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \]

\[ = (A - B)|\tau| [(1-\lambda) + k(1+\lambda)] (1 - (1-q)^m) \]

\[ - \frac{(1-\lambda)(1-k)}{q(m-1)} [(1-q) - (1-q)^m - q(m-1)(1-q)^m] \]

But this last expression is bounded by 2k, if (11) holds, which completes the proof of Theorem 4.1. \[ \square \]
Applying Lemma 2.2 and using the same technique as in the proof of Theorem 4.1 we have the following result:

**Theorem 4.2.** Let \( m \geq 1 \). If \( f \in \mathcal{R}^\tau(A,B) \), then \( I_m^q \in \mathcal{C}(k,\lambda) \) if

\[
(A - B)|\tau| \left[(1 - \lambda) + k(1 + \lambda)\frac{q^m}{1 - q} + 2k(1 - q)^m\right] \leq 2k. \tag{13}
\]

5. **An integral operator**

**Theorem 5.1.** If \( m \geq 1 \), then the integral operator

\[
G_q^m(m,z) = \int_0^z \frac{\Phi_q^m(t)}{t} dt \tag{14}
\]

is in \( \mathcal{C}(k,\lambda) \) if and only if inequality (6) is satisfied.

**Proof.** According to (14) it follows that

\[
G_q^m(m,z) = z - \sum_{n=2}^{\infty} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m \frac{z^n}{n}
\]

then by Lemma 2.2 we need only to show that

\[
\sum_{n=2}^{\infty} n[(1 - \lambda) + k(1 + \lambda)] - (1 - \lambda)(1 - k)] \times \frac{1}{n} \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m \leq 2k,
\]

or, equivalently

\[
\sum_{n=2}^{\infty} n[(1 - \lambda) + k(1 + \lambda)] - (1 - \lambda)(1 - k)] \left(\frac{n + m - 2}{m - 1}\right) q^{n-1}(1 - q)^m \leq 2k. \tag{15}
\]

The remaining part of the proof of Theorem 5.1 is similar to that of Theorem 3.1 and so we omit the details. \( \square \)

**Theorem 5.2.** If \( m > 1 \), then the integral operator \( G_q^m(m,z) \) given by (14) is in \( \mathcal{S}(k,\lambda) \) if and only if

\[
[(1 - \lambda) + k(1 + \lambda)](1 - (1 - q)^m)
- \frac{(1 - \lambda)(1 - k)}{q(m - 1)}[(1 - q) - (1 - q)^m - q(m - 1)(1 - q)^m]
\]

\[
\leq 2k.
\]

The proof of Theorem 5.2 is lines similar to the proof of Theorem 5.1, so we omitted the proof of this theorem.

6. **Corollaries and consequences**

By specializing the parameter \( \lambda = 0 \) in the above theorems we obtain the following corollaries.

**Corollary 6.1.** We have \( \Phi_q^m \in \mathcal{S}(k) \), if and only if

\[
\frac{q^m(1 + k)}{(1 - q)^{m+1}} \leq 2k. \tag{16}
\]
Corollary 6.2. We have $\Phi_q^m \in \mathcal{C}(k)$ if and only if
\[
q^2 \frac{m(m+1)(1+k)}{(1-q)^{m+2}} + 2q \frac{m(1+2k)}{(1-q)^{m+1}} \leq 2k.
\] (17)

Corollary 6.3. Let $m > 1$. If $f \in \mathcal{R}^r(A,B)$, then $I_q^m \in \mathcal{S}(k)$ if
\[
(A - B)|r| \left[ \frac{(1+k)(1-(1-q)^m)}{q(m-1)} \right] \leq 2k.
\] (18)

Corollary 6.4. Let $m \geq 1$. If $f \in \mathcal{R}^r(A,B)$, then $I_q^m f \in \mathcal{C}(k)$ if
\[
(A - B)|r| \left[ (1+k) \frac{m}{1-q} + 2k (1-(1-q)^m) \right] \leq 2k.
\] (19)

Corollary 6.5. If $m \geq 1$, then the integral operator $G_q^m(m,z)$ given by (14) is in $\mathcal{C}(k)$ if and only if inequality (16) is satisfied.

If $m > 1$, then the integral operator $G_q^m(m,z)$ given by (14) is in $\mathcal{S}(k)$ if and only if
\[
(1+k) \frac{(1-(1-q)^m)}{q(m-1)} \leq 2k.
\]

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