Duality between hydrogen atom and oscillator systems via hidden SO(d,2) symmetry and 2T-physics

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Abstract
The relation between motion in $-1/r$ and $r^2$ potentials, known since Newton, can be demonstrated by the substitution $r \rightarrow r^2$ in the classical/quantum radial equations of the Kepler/Hydrogen problems versus the harmonic oscillator. This suggests a duality-type relationship between these systems. However, when both radial and angular components of these systems are included the possibility of a true duality seems to be remote. Indeed, investigations that explored and generalized Newton’s radial relation, including algebraic approaches based on noncompact groups such as SO(4,2), have never exhibited a full duality consistent with Newton’s. On the other hand, 2T-physics predicts a host of dualities between pairs of a huge set of systems that includes Newton’s two systems. These dualities take the form of rather complicated canonical transformations that relate the full phase spaces of these respective systems in all directions. In this paper we focus on Newton’s case by imposing his radial relation to find an appropriate basis for 2T-physics dualities, and then construct the full duality. Using the techniques of 2T-physics, we discuss the hidden symmetry of the actions (beyond the symmetry of Hamiltonians) for the Hydrogen atom in $D$ dimensions and the harmonic oscillator in $\bar{D}$ dimensions. The symmetries lead us to find the one-to-one relation between the quantum states, including angular degrees of freedom, for specific values of $(D,\bar{D})$, and construct the explicit quantum canonical transformation in those special cases. We find that the canonical transformation has itself a hidden gauge symmetry that is crucial for the respective phase spaces to be dual even when $D \neq \bar{D}$. In this way we
display the surprising beautiful new symmetry of the full duality that generalizes Newton’s radial duality.

Keywords: duality, 2T-physics, radial duality, SO(d,2) symmetry, hydrogen-like atom—harmonic oscillator duality, non-compact symmetry

1. Introduction and brief statement of results

A relation between power-law potentials in the radial Schrödinger equation was encountered when considering the interaction between heavy quarks and antiquarks (‘quarkonium’) [1]. For every potential $V(r) \sim r^{\alpha}$ in the radial Schrödinger equation there exists a related radial Schrödinger equation with a ‘partner potential’ $\bar{V}(\bar{r}) \sim \bar{r}^{\bar{\alpha}}$ obtained from the first by the substitution (see section 2)

$$r = \bar{r}^{\bar{\alpha}/\alpha}, \quad (\alpha + 2)(\bar{\alpha} + 2) = 4. \quad (1)$$

The Kepler or Hydrogen atom (Hatom) problem, with $\alpha = -1$, is thus related to the harmonic oscillator problem (HOsc), with $\bar{\alpha} = 2$. See, e.g., references [2–8]. Moreover, every potential with $-\infty < \alpha < -2$ or $-2 < \alpha < \infty$ has a partner potential with $-\infty < \bar{\alpha} < -2$ or $-2 < \bar{\alpha} < \infty$, respectively, as noted for classical [9–12] and quantum [13–15] systems. However [16], this relation can be traced as far back as Newton [17] and Hooke [18]. Not only did Newton transform the radial equation for the Hatom to the HOsc in order to solve it, but he noted that pairs of potentials related by equation (1) gave congruent orbits with small deviations from circular shape.

The close relationship between the Hatom–HOsc problems in $D = 3$ space dimensions described above is limited to the radial equation. Whether this relationship could be elevated to include the angular degrees of freedom in addition to the radial degrees of freedom remained as an unsolved problem. A pessimism on this issue developed because when the complete set of states of the Hatom, including the orbital angular momentum quantum numbers, are compared to the corresponding complete set of states of the HOsc, one finds that they are different, so the radial duality is not a true full duality between the complete systems.

We mention parallel developments related to the Hatom and HOsc that use spectrum-generating algebra involving noncompact groups such as SO(2,1), SO(4,1), and SO(4,2) [19–33]. Unitary representations of these groups contain an infinity of energy levels of the Hamiltonian, related to each other by group transformations within the same representation of the noncompact group. By suitable identification of generators, one can pick out stepping operators relating eigenstates with different energies. This is an indication that these systems may have some hidden symmetry structure that goes beyond the well known symmetries of the respective Hamiltonians in three spatial dimensions (SO(4) for Hatom, SU(3) for HOsc). However, beyond being dynamical groups, the possible existence of non-compact symmetries remained undetermined within those developments. Furthermore, these efforts did not establish a duality-type relationship between the Hatom and HOsc on the basis of a common non-compact group and its common representations that apply simultaneously to both of these systems.

A full duality between the Hatom–HOsc (and many other systems as well) in every spatial dimension $D$ and 1 time dimension was discovered as a simple prediction of two-time physics (2T-physics) in 1998 [34–36]. A summary of the concepts of 2T-physics appears in appendix A. As briefly explained in the paragraphs containing equation (A15)–(A19) in the appendix, the general duality transformation predicted by 2T-physics between any two 1T-physics shadows (explained in the appendix), that include the Hatom and HOsc shadows, is a non-linear canonical transformation between their phase spaces involving time and
Moreover, for tions beyond equation (1), thus becoming a full duality rather than only a partial radial duality.

The rest of this paper is organized as follows. In section 2 we display generalized radial duality in all dimensions D through radial substitution of the form (1) by relating the radial equations for two different potentials, \( V(r) = \lambda r^2 \) and \( V(r) = \lambda r^\alpha \), including the cases of the HatomD and HOscD. In section 3 we introduce details of separate non-compact groups: SO(D + 1, 2) for the HatomD’s hidden symmetry of its action [35], and Sp(2D, R) for

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4 From here on, subscripts in HatomD or HOscD imply the corresponding system in the indicated number of dimensions, D or D. 

5 When the action has a larger symmetry than the Hamiltonian it is imperative that both the spectrum of the Hamiltonian as well as the dynamics due to interactions are controlled by the symmetry of the action. As an example, consider the
HOsc_D’s dynamical symmetry. We discuss the classification of the respective spectra under these non-compact groups. In section 4 we derive our main result, namely, the duality in terms of canonical transformations. This is done by equating \( L^{MN}(\mathbf{r}, \mathbf{p}) = L^{MN}(\mathbf{r}, \mathbf{p}) \), where \( L^{MN}(\mathbf{r}, \mathbf{p}) \) are the SO\((D + 1, 2)\) generators expressed in terms of the phase space for the Hatom_D, while \( L^{MN}(\mathbf{r}, \mathbf{p}) \) are the subgroup generators for SO\((D + 1, 2) \subset Sp(2D, R)\) expressed in terms of the phase space for the HOsc_D, for dimensions \( D \leq D \). The logic behind this method was introduced in 2T-physics as discussed in the appendix. The method is explicitly applied for the cases \( (D, \bar{D}) = (2, 2) \) and \( (3, 4) \). In section 5 we generalize what was learned in the previous sections and present further examples \( (D, \bar{D}) = (1, 4), (4, 6) \) and \( (5, 8) \). The conclusions are in section 6, where we summarize the information on all the cases we successfully constructed the full duality. For all these cases we established conclusively a one-to-one correspondence between a subset of quantum states of the HOsc_D and all the quantum states of the Hatom_D. Based on this experience we conjecture the full duality satisfies \( D = 2(D - 1) \) for all \( D \geq 2 \), with the same form of canonical transformation and dual quantum states. However, there is room for the formula for \( D \) to be more general as we indicate for \( D \geq 6 \) that we have not analyzed in detail, so this remains open for further investigation.

The appendix summarizes the concepts of 2T-physics on which we have based our methods and shows the deeper spacetime structure hidden in the systems we have discussed in this paper.

2. Radial duality through substitution

The Schrödinger equation with a spherically symmetric potential in \( D \) spatial dimensions, 
\[
\left( -\frac{\hbar^2}{2\mu} \nabla^2 + V(\mathbf{r}) \right) \psi(\mathbf{r}) = E \psi(\mathbf{r}),
\]
is solved in spherical coordinates in a complete angular momentum and energy basis as follows: [35, 38].

\[
\psi(\mathbf{r}) = r^{\frac{D-1}{2}} u(\mathbf{r}) T_{l_1, l_2, \ldots, l_\ell} (\hat{\mathbf{r}}), \quad \text{with} \quad r \equiv |\mathbf{r}|, \quad \hat{\mathbf{r}} \equiv \mathbf{r}/|\mathbf{r}|,
\]
\[
T_{l_1, l_2, \ldots, l_\ell} (\hat{\mathbf{r}}) = \left[ (\hat{\mathbf{r}}_{l_1} \hat{\mathbf{r}}_{l_2} \ldots \hat{\mathbf{r}}_{l_\ell} - \text{trace} + \text{permutations} \right],
\]
\[
- \frac{\hbar^2}{2\mu} u''(r) + \left[ V(r) + \frac{l D (l D + 1) \hbar^2}{2 \mu r^2} - E \right] u(r) = 0,
\]
\[
l_D \equiv l + \frac{D - 3}{2}, \quad \text{with} \quad l = 0, 1, 2, \ldots.
\]

Here, the integer \( l \) parametrizes the eigenvalues of angular momentum in \( D \) dimensions, \( \frac{1}{2} L^2 L_{ij} \to l(l + D - 2) \), while the symbol \( T_{l_1, l_2, \ldots, l_\ell} (\hat{\mathbf{r}}) \) is the angular momentum.

Lorentz symmetry in special relativity, which is a symmetry of the action, but not a symmetry of the Hamiltonian. Recall that the Hamiltonian is the time component of the total momentum that is a Lorentz vector, not a Lorentz scalar. A familiar setting that fits the bill is relativistic field theory. The same is true also in much simpler particle systems, such as the Lorentz-invariant worldline formalism with a gauge symmetry under reparametrizations of proper time \( \tau \). After gauge fixing, such as \( x^i(\tau) = \tau \), the canonical conjugate \( p^\mu \) becomes the Hamiltonian that controls the evolution of the remaining spatial degrees of freedom. The action still has Lorentz symmetry as a hidden non-linear symmetry, but the Hamiltonian \( p^\mu \) is clearly not invariant under the boosts. The hidden symmetry SO\((D + 1, 2)\) of the Hatom’s action, as well as of all the dual shadow’s actions, is easily understood in the worldline formalism as a generalization of the statements above. Then at the quantum level the spectrum of every shadow ends up in the same unitary representation of the non-compact group, thus obeying a full duality. See the appendix to better understand this point.

\(^6\) For alternative approaches in \( D \) dimensions with equivalent conclusions, see also [39–41].

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wavefunction in $D$ dimensions.\footnote{Here are examples for $l = 1, 2, 3, 4,$ taken from [38]:}

\begin{equation}
N_l(D) = \frac{(l + D - 3)!}{(D - 2)! l!} (2l + D - 2).
\end{equation}

For $D = 3$ this reduces to the familiar, $N_l(3) = (2l + 1)l!$, while for $D = 2$ it reduces to $N_0(2) = 1$ or $N_l(0)(2) = 2$, as expected (i.e., angular momentum spin up or down, $\pm l$, in two spatial dimensions). Having taken into account the overall factor $\frac{\alpha + 1}{\alpha}$ in the radial wavefunction, $R(r) = r^{-\frac{\alpha + 1}{2\alpha}} u(r)$, the normalization of the wavefunction in $D$ dimensions, $\int d^D r |\psi(r)|^2 = 1$, reduces to an integral on the half line in one dimension, $\int_0^\infty dr |u(r)|^2 = 1$. Note the effective potential in (2), $V_{\text{eff}} = V(r) + \frac{\lambda}{2} \frac{\alpha}{\alpha + 1} l(l + 1)$, that includes the angular momentum barrier in $D$ dimensions parametrized by $l_0 \equiv l + \frac{\alpha - 3}{2}$.

The relation between the radial Schrödinger equation solutions in a power-law potential $V(r) = \lambda r^\alpha$ and a potential $V(r) = \lambda r^\alpha$ may be derived by substituting $r = \bar{r}^{-\alpha/\alpha}$ in the radial Schrödinger equation in $D$ spatial dimensions given above, taking $u(r) = \bar{u}(|\bar{r}|)$, demanding that $(\alpha + 2)(\bar{\alpha} + 2) = 4$, and with $\beta = -\frac{1}{2} \left(1 + \frac{\alpha}{\alpha + 1}\right) = \frac{\alpha}{4}$ chosen so that no terms with $\bar{u}''(\bar{r})$ occur. Then one recovers a radial Schrödinger equation of the same form for $\bar{u}(\bar{r})$ but with a change of parameters, $(\lambda, \bar{l}_D, \bar{E}) \rightarrow (\bar{\lambda}, l_D, E)$, that are related to each other as follows:\footnote{The version of radial duality in equation (5), involving $l_0$ (see (2)) rather than $l$, is a generalization of the same equation in [1] from $D = 3$ to general $D$ dimensions. Moreover, if the substitution in equation (1) is slightly generalized to $r = \bar{r}^{-\alpha/\alpha}$, with $b > 0$ instead of $b = 1$, then equation (5) is further modified to an uglier form, $E = -\lambda \frac{2\alpha}{4b^2} \bar{\alpha} + \bar{\alpha} - \frac{1}{4b^2} \frac{\lambda}{2} |l| + \frac{1}{4b^2} |l| + \frac{1}{2}$, but this generalization, with $b = \sqrt{2}$ for all $D$, will be needed to fit the canonical transformation derived in equation (47) when $\bar{\alpha} = 2$ and $\alpha = -1$.}

\begin{equation}
- \frac{\hbar^2}{2\mu} \bar{u}''(\bar{r}) + \left[ \lambda r^\alpha + \frac{\hbar^2 \bar{l}_D(\bar{l}_D + 1)}{2\mu \bar{r}^2} - E \right] \bar{u}(\bar{r}) = 0,
\end{equation}

\begin{align}
\bar{E} &= -\lambda \frac{\bar{\alpha}^2}{\alpha^2}, \quad \bar{\lambda} &= -E \frac{\bar{\alpha}^2}{\alpha^2}, \quad \bar{l}_D + \frac{1}{2} = \frac{\bar{\alpha}}{\alpha} |l| + \frac{1}{2},
\end{align}

\begin{align}
T_i &= \bar{r}\bar{f}_i, \quad T_{ij} &= \bar{r}\bar{f}_i \bar{f}_j - \frac{1}{D} \delta_{ij},

T_{i\bar{k}} &= \bar{r}\bar{f}_i \bar{f}_\bar{k} - \frac{1}{D+2} \left( \delta_{i\bar{k}} \bar{f}_j + \delta_{i\bar{j}} \bar{f}_\bar{k} + \delta_{\bar{k}\bar{j}} \bar{f}_i \right),

T_{i\bar{j}\bar{k}} &= \left[ \bar{r}\bar{f}_i \bar{f}_j \bar{f}_\bar{k} - \frac{1}{D+4} \left( \delta_{i\bar{j}} \delta_{\bar{k}\bar{l}} + \delta_{i\bar{l}} \delta_{\bar{k}\bar{j}} + \delta_{\bar{j}i} \delta_{\bar{l}\bar{k}} + \delta_{\bar{l}i} \delta_{\bar{j}\bar{k}} \right) \right].
\end{align}
The last relation in (5) is the solution of the quadratic equation, \( \alpha^2 l D (l D + 1) + \frac{1}{4} \left( \frac{\alpha^2}{\alpha^4} - 1 \right) = \bar{l} D (\bar{l} D + 1) \), that imposes the same form of angular momentum barrier in the effective potential.

This defines the radial duality, under which, coupling constant \( \lambda \) and energy eigenvalue \( E \) trade places up to the factor \( -\frac{1}{(\bar{\alpha}/\alpha)^2} \); furthermore the positive integer angular momenta \( l \), \( \bar{l} \) are related by

\[
\left| \bar{l} + \frac{D - 2}{2} \right| = \left| \frac{\bar{\alpha}}{\alpha} \right| \left| l + \frac{D - 2}{2} \right| .
\]

It is important to emphasize that the dimension \( D \) need not be equal to \( \bar{D} \) but both must be positive integers; hence \( l \) need not be equal to \( \bar{l} \) while satisfying equation (6), but both must be positive integers since they determine the ranks of the angular tensors, \( T_{i_1, i_2, \ldots, i_l} (\hat{r}) \) and \( T_{i_1, i_2, \ldots, i_{\bar{l}}} (\bar{\hat{r}}) \) respectively.

We have not yet given an explicit transformation rule that relates the angular variables \( \hat{r} \) and \( \bar{\hat{r}} \). We would like to keep the possibility of \( D \neq \bar{D} \) open if the claim for duality is only for the radial equation rather than for the complete system. Complete duality requires that the degeneracy of the states should match when \( l, \bar{l} \) are related as in (6), but given that \( N_l(D) \neq N_{\bar{l}}(\bar{D}) \) when \( D \neq \bar{D} \), this non-linear requirement is clearly much too strong. So complete duality guided by the radial equations (2)–(6) seems impossible to satisfy except for special cases of \( D, \bar{D} \) and \( \alpha, \bar{\alpha} \). Later, when we provide an explicit nonlinear relation between the angles \( \hat{r} \) and \( \bar{\hat{r}} \), we will show how for certain dimensions, \( D \leq \bar{D} \), there is a duality between an appropriate subset of the degenerate quantum states by embedding \( \text{SO}(D) \subset \text{SO}(\bar{D}) \).

2.1. Spectra in \( D, \bar{D} \) dimensions

Let us consider the case of the \( \text{Hatom}_D \) with \( \alpha = -1 \) and \( \text{HOsc}_\bar{D} \) with \( \bar{\alpha} = 2 \). The spectra of the respective Hamiltonians in \( D \) dimensions are well known (using units \( c = 1, \hbar = 1, \mu = 1 \));

| \( \text{Hatom}_D \) | \( \text{HOsc}_\bar{D} \) |
|---|---|
| \( V (\hat{r}), \bar{V} (\bar{\hat{r}}) \) | \( \frac{1}{2} \omega \hat{r}^2 \) |
| \( E_{n, \bar{n}} \) | \( \omega \left( n + \frac{\bar{n}}{2} \right) \), \( n = 1, 2, 3, \ldots \) |
| \( \bar{l}, \bar{\bar{l}} \) | \( \bar{l} \in \{ 0, 2, 4, \ldots \} \), \( \bar{\bar{l}} \in \{ 1, 3, 5, \ldots \} \) |
| \( \bar{\bar{n}} \) | \( \bar{\bar{n}} \in \{ 1, 3, 5, \ldots \} \) |
| \( \bar{\bar{n}} \text{ q.n.} \) | \( \bar{n} = 0, 1, 2, \ldots \) |

The spectra for the \( \text{Hatom}_2 \) and \( \text{HOsc}_2 \) are graphically displayed in equation (8), where \( (n, \bar{n}) \), respectively \( (\bar{n}, \bar{l}) \), label rows and columns. For the \( \text{HOsc}_2 \) case the \( \bar{n}_{\text{even}}, \bar{l}_{\text{even}} \) labels are shown in large bold numbers, \( 0, 2, 4, \ldots \), while the \( \bar{n}_{\text{odd}}, \bar{l}_{\text{odd}} \) labels are shown in smaller numbers, \( 1, 3, 5, \ldots \). The entry at each \( (n, \bar{l}) \) or \( (\bar{n}, \bar{l}) \) pigeon holes is the \( \text{SO}(2) \) angular momentum degeneracy of the state which is in accordance with the dimensions of \( \text{SO}(D) \) representations in (4). The leftmost column of each table lists the total degeneracy for each energy level labelled by \( n \) or \( \bar{n} \). Note that the total degeneracy at level \( n \) for the \( \text{Hatom}_2 \) is \( 2(n - 1) + 1 \) while for the \( \text{HOsc}_2 \) it is \( (2\bar{n} + 1) \). These match the dimensions of representations for \( \text{SO}(3) \) or \( \text{SU}(2) \), namely \( (2J + 1) \), where we identify \( J = (n - 1) \) for \( \text{Hatom}_2 \) and \( J = \frac{\bar{n}}{2} \) for \( \text{HOsc}_2 \).
As shown in [35], for general $D$ or $\bar{D}$ the pigeon holes would be filled with the numbers $N_l(D)$ or $\bar{N}_{\bar{l}}(\bar{D})$ respectively as given in (4). The total degeneracies at the leftmost column for each fixed $n$ or $\bar{n}$ are then

$$
\text{Hatomo}_D : \sum_{l=0}^{n-1} N_l(D) = \frac{(n + D - 3)!}{(D - 1)! (n - 1)!} (2n + D - 3) = N_{n-1}(D + 1),
$$

$$
\text{HOsc}_{\bar{D}} : \sum_{\bar{l}=\text{even or odd}}^n \bar{N}_{\bar{l}}(\bar{D}) = \frac{(\bar{n} + \bar{D} - 1)!}{\bar{n}! (\bar{D} - 1)!}.
$$

For $D = \bar{D} = 2$ these total degeneracies reproduce the results of the previous paragraph and tables in (8), while for $D = \bar{D} = 3$ they match the well known degeneracies, $n^2$ for Hatomo$_3$, and $\frac{1}{2}(\bar{n} + 2)(\bar{n} + 1)$ for HOsc$_3$. For the Hatomo$_n$, the total degeneracy at each $n$ matches the dimension of the SO$(D + 1)$ representation for the completely symmetric traceless tensor, $T_{l_1 l_2 \ldots l_{n-1}}$, of rank $(n - 1)$ in $(D + 1)$ dimensions (single-row Young tableau, $(n - 1)$ boxes, with trace removed). Similarly, for the HOsc$_{\bar{D}}$, the total degeneracy matches the dimension of the completely symmetric SU$(\bar{D})$ tensor with $\bar{n}$ indices (single-row Young tableau, $\bar{n}$ boxes). The underlying reason for these degeneracies is the well known hidden symmetries of the Hamiltonians: SO$(D + 1)$ for the Hatomo$_D$ Hamiltonian and SU$(\bar{D})$ for the HOsc$_{\bar{D}}$ Hamiltonian, as discussed in section 3.

This result is only a small part of the group-theoretical properties of the respective spectra for the Hatomo$_D$ or HOsc$_{\bar{D}}$. As seen in the tables above in (8), for each state in an SO$(D)$ multiplet labelled by a fixed value of $l$ or $\bar{l}$, there exists an infinite tower of states of increasing values of $n$ or $\bar{n}$. It was shown in [35] that these towers form infinite-dimensional irreducible representations of the non-compact groups SO$(1, 2)$ or Sp$(2, \mathbb{R})$ corresponding to the positive discrete series [42, 43] labelled by $|j, m\rangle$ (similar to SU$(2)$ quantum numbers), with

$$
m(j) = j + 1 + n_r, \ n_r = 0, 1, 2, 3, \ldots
$$

where the integer $n_r$ coincides with the usual radial quantum number in equation (7) that emerges when solving the radial equations in (2) or (5). For the towers associated with the SO$(D)$ or SO$(\bar{D})$ multiplets $l$ or $\bar{l}$, the value of $j$ depends on $l$ or $\bar{l}$ as follows [35] (see (25) and (36) for the derivations of $j(l)$ and $\bar{j}(\bar{l})$ respectively):

$$
\text{Hatomo}_D : \begin{cases} 
  j = 0 \text{ if } D = 1 \\
  j(l) = l + \frac{D - 3}{2}, \text{ if } D \geq 2
\end{cases}, \quad \text{HOsc}_{\bar{D}} : \bar{j}(\bar{l}) = \frac{1}{2} \left( \bar{l} + \frac{D - 4}{2} \right). \tag{11}
$$
Hence the overall spectra are direct sums of irreducible representations of direct product groups as follows:

\[
\text{Hatom}_D: \sum_{l=0}^{\infty} \oplus |SO(1, 2)_{l(\bar{l})}, SO(D)_{\bar{l}}\rangle, \quad \text{with} \quad SO(1, 2) \otimes SO(D) \subset SO(D + 1, 2),
\]

\[
\text{HOsc}_D: \begin{cases} 
\sum_{l_{\text{even}}}^{\infty} \oplus |Sp(2, R)_{l(\bar{l})_{\text{even}}}, SO(\bar{D})_{l_{\text{even}}}\rangle, & \text{with} \ Sp(2, R) \otimes SO(\bar{D}) \subset Sp(2\bar{D}, R), \\
\sum_{l_{\text{odd}}}^{\infty} \oplus |Sp(2, R)_{l(\bar{l})_{\text{odd}}}, SO(\bar{D})_{l_{\text{odd}}}\rangle, & 
\end{cases}
\]

(12)

The direct product groups that classify the spectra are themselves subgroups of larger non-compact groups, SO(D + 1, 2) and Sp(2\bar{D}, R) respectively as indicated in (12). These non-compact groups will be discussed in section 3 in more detail. In fact, the full spectrum of the Hatom$_D$ corresponds to a single irreducible representation of SO(D + 1, 2), while the even/odd states of the HOsc$_D$ correspond to two distinct irreducible representations of Sp(2\bar{D}, R) as will be explained in section 3. In both cases these are called singleton representations that have the following Casimir eigenvalues (see equation (32) and (A14)):

\[
\text{SO}(D + 1, 2): C_2 = -\left(\frac{(D + 1)^2}{4} - 1\right), \ C_3, C_4 = \cdots
\]

\[
\text{Sp}(2\bar{D}): C_2^{\text{even}} = C_2^{\text{odd}} = \frac{D}{2} \left(\frac{D}{2} + \frac{1}{4}\right), \ C_3, C_4 = \cdots
\]

(13)

These facts about the spectra of Hatom$_D$ and HOsc$_D$ will be relevant for the full duality we are seeking in this paper, namely a full duality that would be consistent with Newton’s radial duality $r \sim \bar{r}^2$ discussed in the Introduction and details produced in sections 2.2, 4.1, 4.2 and 6.

2.2. Hints of full duality

Armed with the full spectrum, including angles and angular momentum, we now return to the radial duality displayed in equations (2)–(6). Specializing to $D = \bar{D} = 2$, the angular momentum relations (6) become

\[
D = \bar{D} = 2: \ \bar{l} = 2l.
\]

(14)

It is seen graphically in equation (8) that only the $\bar{l} = \text{even}$ (equivalently the $\bar{n} = \text{even}$) HOsc$_2$ states are in one to one correspondence with all the states of the Hatom$_2$, including matching representations of the hidden symmetries SO(3) = SU(2) level by level, at each $(n - 1) = \frac{\bar{n} - \bar{\bar{n}}}{2} = J$, with degeneracy $(2J + 1)$. In particular, the infinite vertical towers for SO(1, 2) = Sp(2, R) also match at each $l_{\text{even}} = 2l$.

Moreover, the $D = 2$ noncompact group SO(3, 2) is the same as the $\bar{D} = 2$ non-compact group Sp(4, R), and according to equation (13) the quadratic Casimir is the same, $C_2 = -\frac{3}{4}$, and so is the only other cubic Casimir, $C_3 = \frac{5}{8}$, according to (A14), (32). Hence the full Hatom$_2$ and even-HOsc$_2$ spectra are in the same irreducible representation of SO(3, 2) = Sp(4, R), just as expected on the basis of 2T-physics dualities as explained in the paragraphs containing
equations (A16)–(A19) in the appendix. These are very encouraging indications of a full duality between the Hatom\(_D\) and the even half of the HOsc\(_2\).

This result raises the question: what is the dual of the odd half of the HOsc\(_2\)? Amazingly, the answer is provided in 2T-physics with spin, that yields a generalization of the Hatom\(_D\) in a particular gauge (see section 5 in [44]). In the current paper we will call this case the dyonic-Hatom\(_D\). The hidden symmetry in this case is again SO(\(D+1,2\)), but the representation is different than the zero spin case, and has a quadratic Casimir given by (see equation (80) in [44] and substitute \(d = D + 1\)) as outlined in the appendix around equation (A20):

\[
\text{Dyonic} - \text{Hatom}_{D^2}, \text{SO} (D + 1, 2), C^\text{spin}_{s=1/2} = - \frac{1}{8} D (D + 3). \tag{15}
\]

Although the physical interpretation of this model was not fully grasped in [44], it was later understood that it corresponds to a hypothetical Hatom whose nucleus is a dyon that has both electric and magnetic charges instead of the usual proton. For \(D = 3\) this matches the model discussed in [45]. The spectrum of the dyonic-Hatom\(_2\) resembles that of the Hatom\(_2\) but instead of \(l\) there appears \(l + \frac{1}{2}\), where the additional \(1/2\) is generated by the dyon. Furthermore, we find that the dyonic-Hatom\(_3\) has Casimir \(C_2 = \frac{3}{2}\), which is the same as the \(C_2\) for Hatom\(_2\) or HOsc\(_2\), and the spectrum matches the spectrum of odd-HOsc\(_2\) since now the generalization of equation (14) is, \(l = 2 (l + \frac{1}{2})\), where \(l\) is odd.

We have found very strong hints that the duals for the even and odd parts of the HOsc\(_2\) are given by Hatom\(_2\), with \(s = 0, \frac{1}{2}\) respectively. In the next section we will display a quantum canonical transformation that establishes the full duality transformation, Hatom\(_2\) ↔ even-HOsc\(_2\), and show that it is compatible with the simple radial substitution, \(r \sim \bar{r}^2\) in equation (1) with \(\alpha = -1, \bar{\alpha} = 2\), that started the current investigation.

After displaying the canonical transformation, we will generalize the method to a few other special values of the pair \(D < D\) that are compatible with the simple radial substitution, \(r \sim \bar{r}^2\). There are also other dualities as non-linear canonical transformations that connect Hatom\(_D\), HOsc\(_D\) (i.e., \(D = D\)) and many other systems in \(D\) spatial and one time dimensions as predicted by 2T-physics [37], but those predicted more general cases, that apply in every dimension \(D\), are at first sight not compatible with Newton’s simple radial substitution, \(r \sim \bar{r}^2\). However, we suspect a further canonical transformation partly related to the one discussed in section 4 must make the general 2T dualities and Newton’s case compatible as well.

3. Non-compact symmetries of the Hatom\(_D\) and HOsc\(_D\)

In this section we discuss the SO(\(D + 1, 2\)) and Sp(\(2D, R\)) generators constructed from the quantum phase space degrees of freedom of the Hatom\(_D\) and HOsc\(_D\). Following the method in [37] (as stated in the paragraphs that contain equations (A17)–(A19) in the appendix) we compare these gauge-invariant generators for different shadows (see appendix) to one another when \(D = D = 2\), and from this we obtain the sought-after canonical transformation that relates the two phase spaces \((r, p) \leftrightarrow (\bar{r}, \bar{p})\), as shown in section 4.1.

3.1. SO(\(D + 1, 2\)) and the Hatom\(_D\)

The subgroup SO(\(D + 1\)) is the well known hidden symmetry for the Hatom\(_D\), which is best explained in the context of 2T-physics because of its extra space dimension (in addition to the extra time dimension) [34]. The hidden symmetry SO(4) in Hatom\(_3\), associated with a conserved Runge–Lenz vector, was recognized already in the 19th century in the study of the Kepler problem in celestial mechanics, and used by Pauli to understand the ‘accidental
degeneracy’ in the spectrum of the Hatom$_3$ [46–48]. Later, in the context of spectrum-generating algebraic techniques, the SO(4) = SU(2) ⊗ SU(2) hidden symmetry was embedded in the non-compact group SO(4, 2) [22–24, 26–32, 49, 50].

Eventually, the underlying reason for the existence of the spectrum-generating algebra was finally understood with the advent of 2T-physics. Namely, SO(D + 1, 2) is far more than an algebraic tool; it is actually a hidden symmetry of the action (not Hamiltonian) for the Hatom$_D$ for any dimension $D$ (see equation (20) in [35]) and for this reason the spectrum of Hatom$_D$ must be described in terms of irreducible representations of SO(D + 1, 2) (see remarks in footnote 4). Part of this symmetry, namely SO(D + 1) ⊃ U(1), is also a symmetry of the Hatom$_D$ Hamiltonian, where SO(D + 1) rotates all spatial dimensions in 2T-physics on an equal footing, and U(1) = SO(2) rotates the two temporal dimensions. In fact, 2T-physics shows that the Hamiltonian is proportional to $(-1)/(L^{00})^2$ where $L^{00}$ is the SO(2) = U(1) generator. In this way the Hatom$_D$ system is a very transparent window to all spatial and temporal dimensions of 2T-physics, and its spectrum displays the action of the remaining $L^M$ or $L^{M1}$ generators that mix spatial and temporal dimensions with each other.

The generators of SO(D + 1, 2) are computed in 2T-physics as $L^{MN} = (X^M P^N - X^N P^M)$. The phase space $(X^M, P^M)$ is not gauge-invariant under the Sp(2, $R$) gauge transformations of the phase space (for each $M$ they transform as a doublet of the local gauge group Sp(2, $R$)), but the combination $L^{MN}$ for the global symmetry SO(D + 1, 2) are gauge-invariant, so the $L^{MN}$ can be evaluated in any gauge (see appendix). It is customary in 2T-physics to label the (D + 1, 2) indices $M$ by $M = (0', 0, 1', 1, 2, \ldots D)$, or by $M = (\mp i', -', 0, 1, 2, \ldots D)$, where

$$X^{i'} = \frac{X^0 + X^i}{2}, \quad X^{-'} = X^0 - X^i, \quad \Leftrightarrow \quad X^0 = X^{i'} + \frac{X^{-'}}{2}, \quad X^i = X^{i'} - \frac{X^{-'}}{2}.$$  

and similarly for $P^M$. Then SO(D + 1, 2)-invariant dot products in flat spacetime take the form

$$X \cdot P = \begin{cases} -X^{0'} P^{0'} - X^0 P^0 + X^i P^i + X \cdot P \quad & \text{if } i = i' \\ -X^{i'} P^{-'} - X^{-'} P^i - X^0 P^0 + X \cdot P \quad & \text{if } i \neq i' \end{cases}.$$  

and similarly for $X \cdot X$ and $P \cdot P$, where the dot product in bold letters is the Euclidean dot product in $D$ spatial dimensions.

In the Hatom shadow (see equations (12)–(21) in [35]), the gauge-fixed version of the D-dimensional Euclidean phase space $(X, P)$ is relabeled as $(\tilde{r}, \tilde{p})$, while the remaining 6 functions of the worldline proper time $\tau$ introduced in the appendix, $(X^{00,1'}, P^{00,1'})$, are gauge-fixed as functions of $(\tilde{r}, \tilde{p}, \tilde{\tau})$. To get there, three gauge parameters of Sp(2, $R$) are used to gauge-fix three functions, and the three constraints $X \cdot X = P \cdot P = X \cdot P = 0$ are explicitly solved to fix 3 more functions. Therefore $(X^{00,1'}, P^{00,1'})$ are all dependent on $(\tilde{r}, \tilde{p}, \tilde{\tau})$ and $\tilde{\tau}(\tau) = \tau$. The gauge-invariant 2T-physics action is then evaluated in this gauge, and it is shown in [35] that it reduces to the 1T-physics action for the Hatom$_D$, \[ \int d\tilde{\tau} \left( \frac{1}{2} \tilde{r} \cdot \tilde{p} - \frac{1}{2} \tilde{r}^2 - \frac{1}{2} \tilde{p}^2 \right). \] Note that the original 2T action has no parameters, so the mass and coupling constants in the Hatom Hamiltonian (and similarly in actions for other shadows) emerge from the gauge-fixing of the phase space $(X^{00,1'}, P^{00,1'})$ in a way similar to the emergence of parameters from ‘moduli’ in M-theory. The gauge-invariant action that one starts with in equation (A3) in the appendix is explicitly invariant under the global symmetry
SO\(D + 1, 2\) that acts linearly on the original phase space \((X^M, p^M)\). Since the global symmetry commutes with the local symmetry, the gauge-fixed action, namely the Hatom\(_D\) action in this paragraph, must have the same non-linearly realized hidden SO\((D + 1, 2)\) symmetry. The generators of this symmetry have to be the gauge-fixed form of the gauge-invariant \(L^{MN}\) which is now a non-linear function of the gauge-fixed \([X^M (\tilde{r}, \tilde{p}, \tilde{t}), P^M (\tilde{r}, \tilde{p}, \tilde{t})]\) in terms of the \(D\)-dimensional Euclidean phase space, \(L^{MN} (r, \mathbf{p})\). Indeed, it was shown [35] that the Hatom\(_D\) action is invariant under the non-linear transformations obtained by applying Poisson brackets between \(\frac{1}{2} \omega_{MN} L^{MN}\) and the phase space \((\tilde{r}, \tilde{p})\), namely

\[
\delta_r \tilde{r} = \frac{1}{2} \omega_{MN} \frac{\partial L^{MN} (\tilde{r}, \tilde{p}, \tilde{t})}{\partial \tilde{p}}, \quad \delta_p \tilde{p} = \frac{1}{2} \omega_{MN} \frac{\partial L^{MN} (\tilde{r}, \tilde{p}, \tilde{t})}{\partial \tilde{r}}.
\]

(18)

where the constants \(\omega_{MN}\) are the global SO\((D + 1, 2)\) parameters. One may reverse this approach by starting from the invariance of the action and use Noether’s theorem to build the \(L^{MN} (\tilde{r}, \tilde{p}, \tilde{t})\). Either way, one finds that \(L^{00} = (X^i p^0 - X^0 p^i)\), evaluated in the Hatom\(_D\) shadow, yields classically (i.e., ignoring quantum ordering)

\[
L^{00} = \frac{Z}{\sqrt{-2H}}, \quad \text{with} \quad H = \left(\frac{1}{2} \mathbf{p}^2 - \frac{Z}{|\mathbf{r}|}\right) = -\frac{Z^2 / 2}{(L^{00})^2}.
\]

(19)

Hence the Hatom\(_D\) Hamiltonian can be written very simply in terms of the gauge-invariant generator \(L^{00}\). Now, this is the generator of a compact SO\((2)\) that rotates the two times into each other, so its eigenvalues must be parametrized by an integer, just like orbital angular momentum, but due to quantum ordering issues the integer may be shifted by a constant that depends on \(D\).

Resolving the quantum ordering is too complicated in the Hatom\(_D\) shadow using \((\tilde{r}, \tilde{p}, \tilde{t})\). However, since \(L^{00}\) is gauge-invariant, and its commutation rules with all other \(L^{MN}\) are also gauge-invariant, one may choose any convenient gauge to evaluate the \(L^{MN}\), resolve all quantum ordering ambiguities, and then diagonalize \(L^{00}\) to find its gauge-invariant eigenvalues algebraically by using only the commutation rules of the hidden symmetry SO\((D + 1, 2)\) in any shadow. This was done in [35] by choosing the following gauge (evaluated at zero time for that gauge\(^6\))

\[
\begin{pmatrix}
X^M \\
p^M
\end{pmatrix}
= \begin{pmatrix}
+ & -t \\
0 & r \\
1 & p^2 / 2
\end{pmatrix}
\begin{pmatrix}
|\mathbf{r}| & \mathbf{r} \\
|\mathbf{p}| & \mathbf{p}
\end{pmatrix}
\]

(20)

The three numerical entries, \(X^+ = 0, P^+ = 1, P^0 = 0\), are gauge choices, while \((X, \mathbf{P}) = (r, \mathbf{p})\) is just renaming symbols to indicate that we are in another gauge, and the remaining three entries \((r \cdot \mathbf{p}, |\mathbf{r}|, p^2 / 2)\) are computed by solving the tree constraints \(X^2 = P^2 = X \cdot P = 0\) by using the lightcone version (17) of the SO\((D + 1, 2)\) invariant dot product.

Now we evaluate the gauge-invariant \(L^{MN}\) in this gauge and perform the necessary quantum ordering to insure that (i) the \(L^{MN}\) obey the correct SO\((D + 1, 2)\) commutation rules by using only the quantum rules \([r, p] = i \delta_{ij}\), and (ii) the quadratic Casimir eigenvalue for \(C_2 = \frac{1}{2} L^{MN} L_{MN}\) as computed in this gauge, gives the same gauge-invariant result in the

---

\(^6\) Normally, in choosing the gauges there is a non-trivial explicit dependence on time, \(t(\tau) = \tau\), where \(\tau\) is the proper time in the original Lagrangian (A3) in the appendix. Since at this stage we are interested in the equal-time commutation rules of observables, we have chosen \(t(\tau) = \tau = 0\) to simplify as much as possible the gauge-fixed versions of \((X^M, p^M)\) as shown in (20).
appendix, namely \( C_2 = 1 - (D + 1)^2 / 4 \), that was obtained in covariant quantization for physical states, without choosing any gauge. The result that satisfies these physical conditions, with all quantum ordering issues resolved, and insuring hermiticity of the \( \frac{1}{2} (D + 3)(D + 2) \) generators, is as follows [35]:

\[
\begin{align*}
\text{SO} (1, 2) : & \quad L_{mn} = \begin{cases} 
L^{0+} = |r| \\
L^{-+} = \frac{r \cdot p + p \cdot r}{2} \\
L^{0-} = \sqrt{|r|^2} / \sqrt{2} 
\end{cases}, \quad m, n = (+, -, 0) \\
\text{SO} (D) : & \quad L^{ij} = (r^i p^j - r^j p^i), \quad i, j = 1, \ldots, D.
\end{align*}
\]

The subgroup structure, \( \text{SO}(1, 2) \otimes \text{SO}(D) \subset \text{SO}(D + 1, 2) \), that is used to classify the Hatom_0 spectrum as in equations (8)–(12) is indicated above. The coset is in the (vector \( \otimes \) vector) representation of the subgroup. It is also possible to reorganize the generators according to the subgroup, \( \text{SO}(2) \otimes \text{SO}(D + 1) \subset \text{SO}(D + 1, 2) \), where the \( \text{SO}(2) \) generator is \( L^{(0)} \) that rotates the two temporal coordinates \((X^0, X^0)\) into each other and \( \text{SO}(D + 1) \) rotates the spatial coordinates \((X^i, X^j)\) into each other. These subalgebra operators are related to the above according to the lightcone map in (16) as follows:

\[
\begin{align*}
\text{SO} (2) : & \quad L^{00} = \left( L^{0-} + \frac{1}{2} L^{0+} \right) = \sqrt{|r|^2 + 1} / \sqrt{2}, \\
\text{SO} (D + 1) : & \quad L^{ij}, L^{ii} = \left( L^{i-} - \frac{1}{2} L^{i+} \right),
\end{align*}
\]

where \( L^{ii} \) is related (by gauge invariance) to the famous Runge–Lenz vector when \((r, p)\) are rewritten in terms of \((\tilde{r}, \tilde{p})\) (see equation (24) in [35]). Recall that the aim is to find the eigenvalues of \( L^{00} \). This is easily done algebraically [35] by noting that \( L^{00} \) is the compact generator of the \( \text{SO}(1, 2) \) Lie algebra, whose quadratic Casimir is related to \( \text{SO}(D) \) angular momentum as follows:

\[
C_2^{\text{SO}(1,2)} = \frac{1}{2} L_{mn} L_{mn} = L^{0+} L^{0-} + L^{-+} L^{0+} - \left( L^{0+} \right)^2 \\
= \left[ \frac{r^2}{2} + \frac{p^2}{2} - \left( \frac{r \cdot p}{2} \right)^2 + \frac{3}{4} \right] \\
= \left( \frac{D}{2} L_{00} + \frac{(D - 1)(D - 3)}{4} \right). \tag{23}
\]

\[\text{To raise/lower } \text{SO}(1,2) \text{ indices } L_{mn} \text{ use the metric } \eta_{mn} \text{ with nonzero entries: } \eta_{+,+,+} = \eta_{-,+,+} = \eta_{+} = -1.\]
The computation above that yields the numerical contribution \( \frac{1}{2} \) (proportional to \( \hbar^2 \to 1 \)) is performed by watching the orders of operators and using the commutators, \( [\mathbf{r}, \mathbf{p}] = i\hbar \delta \), to change their orders. Since the right hand side shows that \( C_{2}^{\text{SO(1,2)}} \geq -\frac{1}{4} \) for all \( D = 1, 2, 3, \ldots \), only the unitary positive discrete series representation of \( \text{SO}(1, 2) = \text{SL}(2, \mathbb{R}) = \text{SU}(1, 1) \) can occur. Then according to known representation theory of \( \text{SL}(2, \mathbb{R}) \) [48], the eigenvalues of \( C_{2}^{\text{SO(1,2)}} \) and \( L^{00} \) are respectively, \( j(j + 1) \) and \( m(j) = j + 1 + n_{r} \), with \( n_{r} = 0, 1, 2, \ldots \). Simultaneously, \( \text{SO}(D) \) angular momentum is also diagonal, \( \frac{1}{2} L^{ij} L_{ij} \to l(l + D - 2) \). Therefore, we have the following relations among \( (j, l, n_{r}, D) \) [35]

\[
\begin{align*}
    j(j + 1) &= \left[ l(l + D - 2) + \frac{(D - 1)(D - 3)}{4} \right], \Rightarrow j(l) = -\frac{1}{2} \pm \left| \frac{l + D - 2}{2} \right|, \\
    L^{00} \to m(j) &= j(l) + 1 + n_{r}.
\end{align*}
\] (24)

Since this is the positive discrete series we must have \( m(j) > 0 \), which requires \( (j(l) + 1) > 0 \). Accordingly, we can choose the \( \pm \to + \), resolve the absolute value sign, and write:

\[
\begin{align*}
    j(l) &= \begin{cases} 
        0, & \text{if } D = 1 \text{ and } l = 0 \text{ necessarily} \\
        l + \frac{D - 3}{2}, & \text{if } D \geq 2
    \end{cases}, \\
    m(j) &= j(l) + 1 + n_{r} = \begin{cases} 
        (0 + 1 + n_{r}) = n, & \text{for } D = 1 \\
        (l + 1 + n_{r}) + \frac{D - 3}{2} = n + \frac{D - 3}{2}, & \text{for } D \geq 2
    \end{cases}.
\end{align*}
\] (25)

where \( n = (l + 1 + n_{r}) \geq 1 \) since both \( l, n_{r} = 0, 1, 2, \ldots \). This explains the Hatom part in equation (11).

Thus, we have computed algebraically the desired eigenvalue of \( L^{00} \), which then, according to 2T-physics, \(^{11}\) determines the quantum eigenvalue of the Hatom Hamiltonian given in terms of the gauge-invariant \( L^{00} \) in equation (19),

\[
H = -\frac{Z^{2}/2}{\langle L^{00} \rangle} \rightarrow \begin{cases} 
    -\frac{Z^{2}/2}{n^{2}}, & \text{for } D = 1 \\
    -\frac{Z^{2}/2}{(n + \frac{D - 3}{2})^{2}}, & \text{for } D \geq 2
\end{cases}, \quad n = 1, 2, 3, \ldots \] (26)

As anticipated, quantum ordering did produce a quantum shift of the integer \( n \) by the amount \( (D - 3)/2 \). This result agrees with solving the Hatom_{D} radial differential equation, hence the quantum-ordered generators given above are correct since they produce not only the correct spectrum, but also the correct \( \text{SO}(D + 1, 2) \) Casimir eigenvalue, as well as the correct Lie algebra for \( \text{SO}(D + 1, 2) \).

\(^{11}\) Historically, the algebraic computation of the Hatom spectrum followed a different path. It relied on the Runge–Lenz vector that together with orbital angular momentum, complete an \( \text{SO}(D + 1) \) algebra. The quadratic Casimir of this algebra can be shown to be related to the Hatom Hamiltonian, so the spectrum of the Hamiltonian was computed by computing \( C_{2}^{\text{SO}(D+1)} \). The 2T-physics approach showed another way to get there, namely by computing the eigenvalues of \( L^{00} \).
A useful final observation that follows from (21) is to realize that the phase space \((r, p)\) can be written in terms of the gauge-invariants \(L^{r'p'} = |r|\) and \(L^{\beta} = -\frac{1}{2} (|r| p_+ + p'_+ |r|):\)

\[
r' = L^{r'p'}, \quad p' = - \left( L^{r'p'} \right)^{-1/2} L^{\beta} \left( L^{r'p'} \right)^{-1/2}.
\]

The second relation is verified at the quantum level as follows:

\[
- \left( L^{r'p'} \right)^{-1/2} L^{\beta} \left( L^{r'p'} \right)^{-1/2} = \frac{1}{2} \left( \sqrt{|r|} \frac{1}{\sqrt{|r|}} + \frac{1}{\sqrt{|r|}} \frac{p'}{\sqrt{|r|}} \right) = p'.
\]

Equation (27) will be very important to extract the desired canonical transformation.

3.2. \(Sp(2\bar{D}, R)\) and the harmonic oscillator

The HO_{\bar{D}} has a dynamical symmetry \(Sp(2\bar{D}, R)\) that controls its spectrum as described in this section. The harmonic oscillator in \(\bar{D}\) space dimensions, with phase space \((\vec{r}_n, \vec{p}_n), \alpha = 1, 2, \ldots, D\), has dynamics described by the Hamiltonian

\[
H = \left( \frac{\vec{p}^2}{2\mu} + \frac{\mu \omega^2}{2} \right) = \left( a^\dagger a + \frac{\bar{D}}{2} \right), \quad a \equiv \left( \sqrt{\frac{\hbar \omega}{2\mu}} \vec{r} + i \sqrt{\frac{\mu}{2\hbar \omega}} \vec{p} \right).
\]

For convenience (or by a rescaling of the coordinates and momenta) we will take \(\mu = 1, \omega = 1\) and \(\hbar = 1\). The Hamiltonian is invariant under unitary transformations \(a_\alpha \rightarrow U_{\alpha\beta} a_\beta\) where \(U^\dagger U = 1\). These transformations belong to the group \(SU(\bar{D}) \otimes U(1) \subset Sp(2\bar{D}, R)\). Excited levels are of the form \(a_\alpha^\dagger a_\beta^\dagger \ldots a_{\bar{D}}^\dagger |0\rangle\), where \(|0\rangle\) denotes the ground state. The totally symmetric tensor form of the quantum states implies that only totally symmetric representations of \(SU(\bar{D})\) will occur (single-row Young tableaux with \(\bar{n}\) boxes). Thus the spectrum will have degeneracy

\[
\frac{(\bar{D} + \bar{n} - 1)!}{\bar{n}!(\bar{D} - 1)!}, \quad \text{or} \quad \left\{ \frac{\bar{D}}{2}, \frac{\bar{D} + 1}{2}, \frac{\bar{D} + 1}{6}, \ldots \right\},
\]

corresponding to total excitation number, \(\bar{n} = 0, 1, 2, 3, \ldots\), and energies, \(E_\bar{n} = \hbar \omega \left( \bar{n} + \frac{\bar{D}}{2} \right)\).

\(Sp(2\bar{D}, R)\) acts linearly on a \(2\bar{D}\)-dimensional column consisting of \(\bar{D}\) position and \(\bar{D}\) momentum real degrees of freedom in a real basis \(\left( \begin{array}{c} \vec{r} \\ \vec{p} \end{array} \right)\), or equivalently in a pseudo-real basis, \(\left( \begin{array}{c} a \\ a^\dagger \end{array} \right)\). The generators of \(Sp(2\bar{D}, R)\) are formed by the symmetric product of two such columns, so there are \(D(2\bar{D} + 1)\) generators that take the form

\[
\left\{ \frac{\bar{r}_\alpha \bar{r}_\beta + \bar{r}_\beta \bar{r}_\alpha}{2} \frac{\bar{p}_\alpha \bar{p}_\beta + \bar{p}_\beta \bar{p}_\alpha}{2} \right\} \text{ or } \left\{ \frac{a_\alpha^\dagger a_\beta}{2} \frac{a_\beta^\dagger a_\alpha}{2} \frac{a_\alpha a_\beta}{4} \frac{a_\beta a_\alpha}{4} \right\}.
\]

These form the \(Sp(2\bar{D}, R)\) Lie algebra under classical Poisson brackets or quantum commutators. The subset of operators \(\frac{1}{2} \left( a_\alpha^\dagger a_\beta + a_\beta a_\alpha \right)\) or \(\frac{1}{2} \left( \bar{r}_\alpha \bar{p}_\beta + \bar{p}_\beta \bar{r}_\alpha \right)\) that can be decomposed into one symmetric and one antisymmetric tensor form the \(SU(\bar{D}) \otimes U(1)\) subalgebra, where the \(U(1)\) operator, which is the trace of the symmetric tensor, is the Hamiltonian \(H = \frac{1}{2} \left( \vec{p}^2 + \vec{r}^2 \right) = \left( a^\dagger \cdot a + D/2 \right)\). Acting on the harmonic oscillator quantum states in Fock
space, the step-up or step-down generators \( \left( a_\lambda^+, a_\mu^-, a_\alpha a_\beta \right) \) doubly excite or doubly de-excite any given state with total excitation number \( \bar{n} \). For this reason the \( \text{Sp}(2D, R) \) action cannot mix \( \bar{n}_{\text{odd}} \) states with \( \bar{n}_{\text{even}} \) states. Moreover, it is evident that all even (odd) states mix among the even (odd) sets under repeated action of the step up/down operators, so even/odd states form disjoint irreducible representation bases. The Casimir operators \( \bar{C}_k \) that commute with all the generators, in this construction, are all pure numbers (i.e., not operators),

\[
\bar{C}_k = \frac{2}{4^k k!} (2D + 1) \left( 1 - (2D + 1)^{k-1} \right), \quad k = 2, 3, \ldots, D.
\]

For example, the quadratic Casimir is computed explicitly as follows by taking into account

\[
\bar{C}_2 = \left( \frac{2 a_\lambda^+ a_\beta^+ a_\alpha^+ a_\mu^- a_\alpha a_\beta}{4} - \frac{a_\alpha a_\beta a_\lambda^+ a_\mu^- a_\alpha a_\beta}{4} - \frac{a_\lambda^+ a_\mu^- a_\alpha a_\beta}{2} + \frac{a_\lambda^+ a_\mu^- a_\alpha a_\beta}{2} \right) = - \frac{\bar{D}}{2} \left( \frac{\bar{D}}{2} + \frac{1}{4} \right).
\]

So, the Fock space of \( \text{HOsc}_D \) corresponds to two specific fixed representations of \( \text{Sp}(2D, R) \) such that both even and odd states have the same Casimir eigenvalues, \( \bar{C}_k \), even though they are distinct irreducible representations.

Another commuting set of subalgebras of \( \text{Sp}(2D, R) \) that is relevant for our analysis is \( \text{Sp}(2, R) \otimes \text{SO} (\bar{D}) \). The three \( \text{Sp}(2, R) \) generators \( G_{\mu=0,1,2} \) are obtained from the traces of the tensors listed in (31), while the \( \text{SO}(\bar{D}) \) generators \( L_{\alpha\beta} \) correspond to the only antisymmetric tensor constructed from those listed in (31), namely \( L_{\alpha\beta} = (\bar{r}_\alpha p_\beta - \bar{r}_\beta p_\alpha) = \left( a_\lambda^+ a_\beta - a_\lambda a_\beta^+ \right) \).

The remaining coset generators are labelled by representations of \( \text{Sp}(2, R) \otimes \text{SO}(\bar{D}) \), as \( S_{\alpha(\beta)} \), where the pair \( (\alpha\beta) \) corresponds to the irreducible symmetric traceless tensor of \( \text{SO}(\bar{D}) \). So, the \( \text{Sp}(2D, R) \) generators are given as follows (these may be rewritten in terms of \( (a_\lambda^+, a_\lambda) \)), for a total of \( D (2D + 1) \) Hermitian operators:

\[
\text{Sp}(2, R) : G_\mu = \begin{cases} 
G_+ = (G_0 + G_1) = \frac{\bar{r} \cdot \bar{r}}{2} \\
G_0 = \frac{1}{2} (G_0 - G_1) = \frac{\bar{p} \cdot \bar{p}}{4} 
\end{cases}, \\
\text{SO}(\bar{D}) : L_{\alpha\beta} = (\bar{r}_\alpha p_\beta - \bar{r}_\beta p_\alpha), \\
\text{Coset} : S_{\mu(\alpha\beta)} = \begin{cases} 
S_{+(\alpha\beta)} = \frac{\bar{r}_\alpha p_\beta + \bar{r}_\beta p_\alpha}{\bar{D}} - \delta_{\alpha\beta} \frac{\bar{r} \cdot \bar{r}}{\bar{D}} \\
S_{0(\alpha\beta)} = \frac{\bar{r}_\alpha p_\beta + \bar{r}_\beta p_\alpha}{\bar{D}} - \delta_{\alpha\beta} \frac{\bar{p} \cdot \bar{p} + \bar{r} \cdot \bar{r}}{\bar{D}} \\
S_{-(\alpha\beta)} = \frac{\bar{p}_\alpha p_\beta}{\bar{D}} - \delta_{\alpha\beta} \frac{\bar{p} \cdot \bar{p} + \bar{r} \cdot \bar{r}}{\bar{D}} 
\end{cases}.
\]

Note that the \( G_{x,0} \) are Hermitian combinations of the Hermitian \( G_\mu \). The \( \text{Sp}(2, R) \otimes \text{SO}(\bar{D}) \) subalgebras commute with each other, \( [G_\mu, L_{\alpha\beta}] = 0 \), because the \( G_\mu \) are constructed from \( \text{SO}(\bar{D}) \)-invariant dot products. It can be checked that the compact generator \( G_0 \) of \( \text{Sp}(2, R) = \text{SL}(2, R) = \text{SU}(1, 1) \), given by \( G_0 = \frac{1}{2} G_+ + G_- \), is related to half the Hamiltonian \( H \) in (29), and the quadratic Casimir operator, \( \bar{C}_2 (\text{Sp}(2, R)) = (G_0^2 - G_1^2 - G_2^2) = (G_+ G_- + G_- G_+ - G_0^2) \), is related to angular momentum \( \frac{1}{2} L^\mu L_\mu \).
The fact that \( SO(1,2) \) classifies some subsets of the infinite vertical towers of the HOsc\(_D\) suggests that, for a duality to exist, we should identify some subsets of the infinite vertical towers of the HOsc\(_D\) to all the towers of the Hatom\(_D\). We already saw in section 2.2 that when \( D = D = 2 \) this idea actually works. For more general \( 3 \leq D \leq D \), identifying the towers requires at the very least that we require they are in the same representation of \( SO(1,2) = Sp(2,R) \), which means \( j(l) = j(\tilde{l}) \). Using equations (25) and (36), this gives

\[
\begin{align*}
\text{for } D = 1 : & \quad j(l) = 0 = \frac{1}{2} \left( 0 + \frac{D - 4}{2} \right) \quad \text{and } l, \tilde{l} = 0. \\
\text{for } D \geq 2 : & \quad j(l) = l + \frac{D - 3}{2} = \frac{1}{2} \left( \tilde{l} + \frac{D - 4}{2} \right). 
\end{align*}
\]

This condition reproduces equation (6) for \( \frac{D}{n} = -2 \), that was based on the radial duality. This is a very encouraging observation, so we pursue it in this section.

4. Duality as canonical transformations and \( SO(D + 1,2) \)

We are now ready to compare the \( SO(D + 1,2) \) generators \( L^{MN}(r,p) \) in (21) to the \( Sp(2\tilde{D},R) \) generators in (34). These clearly are different Lie algebras; however, they both share the crucial subgroups

\[
\text{Hatom}_D : SO(1,2) \otimes SO(D), \quad \text{and } \text{HOsc}_D : Sp(2,R) \otimes SO(\tilde{D}),
\]

that classify the spectra in the generalized version of equation (8) as described following that equation. The fact that \( SO(1,2) = Sp(2,R) \) suggests that, for a duality to exist, we should identify some subsets of the infinite vertical towers of the HOsc\(_D\) to all the towers of the Hatom\(_D\). We already saw in section 2.2 that when \( D = D = 2 \) this idea actually works. For more general \( 3 \leq D \leq D \), identifying the towers requires at the very least that we require they are in the same representation of \( SO(1,2) = Sp(2,R) \), which means \( j(l) = j(\tilde{l}) \). Using equations (25) and (36), this gives

\[
\begin{align*}
\text{for } D = 1 : & \quad j(l) = 0 = \frac{1}{2} \left( 0 + \frac{D - 4}{2} \right) \quad \text{and } l, \tilde{l} = 0. \\
\text{for } D \geq 2 : & \quad j(l) = l + \frac{D - 3}{2} = \frac{1}{2} \left( \tilde{l} + \frac{D - 4}{2} \right). 
\end{align*}
\]

This condition reproduces equation (6) for \( \frac{D}{n} = -2 \), that was based on the radial duality. This is a very encouraging observation, so we pursue it in this section.
Thus, for sufficiently large \( D \gg D \) we can find an appropriate subgroup \( \text{SO}(D + 1, 2) \subset \text{Sp}(2D, R) \) with generators \( L^{MN}(\mathbf{r}, \mathbf{p}) \) that are some linear combinations of the \( \text{Sp}(2D, R) \) generators in (34). These will look different than the Hatom’s \( L^{MN}(\mathbf{r}, \mathbf{p}) \) in (21) as functions of different phase spaces \((\mathbf{r}, \mathbf{p})_D\) versus \((\mathbf{r}, \mathbf{p})_{\bar{D}}\), but will have the crucial subalgebras in (37), with the imposed condition (38). By the 2T-physics gauge-invariance argument in the appendix, we consider the pair \((L^{MN}(\mathbf{r}, \mathbf{p}), L^{MN}(\mathbf{r}, \mathbf{p}))\) to be the algebraic description of the hidden \( \text{SO}(D + 1, 2) \) symmetry of two shadows, and on this basis we equate them as described in the appendix:

\[
L^{MN}(\mathbf{r}, \mathbf{p}) = L^{MN}(\mathbf{r}, \mathbf{p}) .
\]  

(39)

From this equation we will derive the sought-after canonical transformation that relates the two phase spaces \( (\mathbf{r}, \mathbf{p}) \leftrightarrow (\mathbf{r}, \mathbf{p}) \), as shown below. We will see that this scheme works for certain dimensions \( D \gg D \).

To say that Hatom\(D \) is dual to HOsc\(D \gg D\) via a canonical transformation may seem incomplete since the phase space \((\mathbf{r}, \mathbf{p})_{D, D}\) has certainly more degrees of freedom as compared to \((\mathbf{r}, \mathbf{p})_{D}\). Hence we must expect some constraints on the phase space \((\mathbf{r}, \mathbf{p})_{D, D}\) that reduce the degrees of freedom such that the spectrum of HOsc\(D \gg D\) becomes compatible with Hatom\(D\). This means that the canonical transformation that we will display must have a gauge symmetry that gives rise to the constraints in the form of vanishing gauge generators (thus distinguishing the gauge-invariant subspace). Hence the duality Hatom\(D \leftrightarrow HOscD \gg D\) can be true only on the gauge invariants for which the gauge generators vanish. We will display the gauge symmetry in precisely the form stated in this paragraph. Because of the gauge symmetry, the gauge-invariant sector of HOsc\(D \gg D\) is effectively equivalent to a phase space in \( D \) dimensions.

4.1. Case of \( D = \bar{D} = 2 \) and \( \text{SO}(3, 2) = \text{Sp}(4, R) \)

For \( D = \bar{D} = 2 \) we already have \( \text{SO}(3, 2) = \text{Sp}(4, R) \) so there is no need to search for a subgroup. However the Sp\(4, R\) generators are expressed in the four-dimensional spinor basis as the product of two four-dimensional columns \( \psi_A = \begin{pmatrix} \mathbf{r} \\ \mathbf{p} \end{pmatrix}_A \), whereas the \( \text{SO}(3, 2) \) generators \( L^{MN} \) are expressed in the antisymmetric product of the five-dimensional basis of \( \text{SO}(3, 2) \). All we need to do is convert the symmetric product of the spinor basis \( \frac{1}{2}(\psi_A \psi_B + \psi_B \psi_A) \) discussed in section 3.2 to the antisymmetric product in the vector basis \( L^{MN} \) discussed in section 2.1. This is done by using the antisymmetric product of SO\(3, 2\) gamma matrices, \( (\gamma^{MN})_{AB} = \frac{1}{4}(\gamma^M \gamma^N - \gamma^N \gamma^M)_{AB} \), where the \( 4 \times 4 \) gamma matrices \( \gamma^M_{AB} \) satisfy the Clifford algebra \( \{ \gamma^M, \gamma^N \} = 2i \eta^{MN} \), with the SO\(3, 2\) Minkowski metric \( \eta^{MN} \). Fortunately, for SO\(3, 2\), \( (C^{MN})_{AB} \) where \( C \) is the ‘charge conjugation matrix’, is antisymmetric in the pair \([MN]\) and symmetric in the pair \((AB)\), so we can write

\[
L^{MN} \sim (C^{MN})_{AB} \left( \frac{\psi_A \psi_B + \psi_B \psi_A}{2} \right) ,
\]

(40)

up to a normalization to preserve the correct commutation rules. We can now focus on just a subset of the generators that lead to the desired canonical transformation. These are \( L^{+\gamma}, L^0, L^{\gamma^+} \) for the Hatom\(2\) as given in (21), and \( (S^{+00}, S^{00+}, G_+) \) for the HOsc\(2\) as given in (34). Relating them as in (40) we obtain a simplified version of (40) for this subset:

\[
L^{\gamma^+} = (\gamma^+)_{00} S^{+00}, \quad L^0 = (\gamma^0)_{00} S^{00+}, \quad L^{0\gamma^+} = G_+ ,
\]

(41)
where now $\gamma^i_{\alpha\beta}$ are simply the SO(2) gamma matrices given by $2 \times 2$ Pauli matrices, $\gamma^1 = \sigma_3$ and $\gamma^2 = \sigma_1$ that are correctly normalized. Recalling (27) that says $r' = L^{\alpha\beta} r$ and $p' = -\left(L^{\alpha\beta}\right)^{-1/2} L^\theta \left(L^{\alpha\beta}\right)^{-1/2}$, and using equations (21), (34) and (41), we now can write

$$r' = L^{\alpha\beta} \left(\gamma^i_{\alpha\beta} S^{+\alpha\beta}\right) = \frac{1}{2} \left(r'^2 - r^2\right) \frac{1}{r_1 r_2}.$$  \hspace{1cm} (43)

Explicitly, using $\gamma^i = (\sigma_3, \sigma_1)$ we obtain

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \left(\frac{1}{2} \frac{r_1 r_2}{r_1 r_2} \right) \gamma^i \left(\begin{array}{c} \gamma^i \\ \gamma^i \end{array}\right) - \left(\frac{r_1 p_1 + p_1 r_1 - r_2 p_2 + p_2 r_2}{r_1 p_2 + r_2 p_1}\right) \frac{1}{|r|}. \hspace{1cm} (44)$$

where $|r| \equiv \sqrt{r_1^2 + r_2^2}$. We can verify that this is indeed a canonical transformation by computing the Poisson brackets or quantum commutators as follows:

$$\begin{pmatrix} r' \cdot p' \end{pmatrix} = \left[\frac{1}{2} \frac{1}{r} \left(\gamma^i_{\alpha\beta} p^\alpha \gamma^i_{\lambda\mu} p^\lambda + \frac{1}{2} \frac{1}{|r|} \left(r'^2 \gamma^i \gamma^i - r^2 \gamma^i \gamma^i\right) \frac{1}{|r|}\right) \frac{1}{|r|}\right]$$

$$= \left[\frac{1}{2} \left(\frac{1}{r} \gamma^i \gamma^i + \gamma^i \gamma^i\right) \frac{1}{|r|}\right] - \frac{i}{2} \gamma^i \frac{1}{|r|} \frac{1}{|r|} = i \delta^{ij}. \hspace{1cm} (45)$$

This shows that we obtain the right result, $\left[\frac{1}{2} \frac{1}{r} \left(\gamma^i_{\alpha\beta} p^\alpha \gamma^i_{\lambda\mu} p^\lambda + \frac{1}{2} \frac{1}{|r|} \left(r'^2 \gamma^i \gamma^i - r^2 \gamma^i \gamma^i\right) \frac{1}{|r|}\right) \frac{1}{|r|}\right]$.

This reproduces Newton's radial substitution (1) for the case $\alpha = -1$ and $\tilde{\alpha} = 2$, and adds the transformation of the angles as well, thus completing the radial duality to a full duality in the case of $D = D = 2$. Moreover, we point out the fact that $\theta = 2 \tilde{\theta}$ is consistent with the shapes.
of classical orbits in the Kepler and harmonic oscillator potentials: The perigee is reached once per Kepler orbit, but twice per harmonic oscillator orbit.

Relative to (1) we ended up with an extra factor of $1/2$ in the relation $r = \frac{1}{2}y^2$ in equation (47). This is of no concern: the extra factor of $1/2$ slightly alters only the radial duality rules in (5) as explained in footnote 8.

As a final remark we identify a gauge symmetry that commutes with the canonical transformation. Recall that according to equation (8) only the even $(\hat{n}_{\text{even}}, \hat{\ell}_{\text{even}})$ quantum states of HOsc$_2$ are dual to all the quantum states of the Hatom$_2$. As asserted in the paragraph following equation (39), there must be a gauge symmetry in the canonical transformation that can be used to project out the $(\hat{n}_{\text{odd}}, \hat{\ell}_{\text{odd}})$ quantum states of the HOsc$_2$. Indeed there is such a gauge symmetry in equation (43) or (44) that amounts to the discrete rotation of the HOsc$_2$ phase space by 180 degrees, $(r, p) \rightarrow (-r, -p)$. Only the even quantum states of the HOsc$_2$ are gauge-invariant under this discrete transformation as can be seen from the wavefunction in equation (2) that transforms under the gauge symmetry according to

$$T_{n_1, \ldots, n_l} \left( -\vec{r} \right) = (-1)^D T_{n_1, \ldots, n_l} \left( -\vec{r} \right).$$

(48)

Imposing this gauge symmetry on the quantum states of the HOsc$_2$ eliminates the non-invariant odd states, and gives precisely the duality that is observed in the spectrum for the gauge-invariant even states, as displayed in equation (8).

For the odd states of the HOsc$_2$, we anticipated in section 2.2 that the dyonic-Hatom is the correct dual, based on 2T-physics properties. We will not discuss the details of this structure here in the same manner as above, because we have no space to introduce more technical tools in this paper to deal with the spin degrees of freedom required in the 2T formalism [44].

4.2. Case of $D = 3, \bar{D} = 4$, and SO(4, 2) $\otimes U(1) \subset Sp(8, R)$

The form of the canonical transformation (43) suggests applying it to higher dimensions $D \geq 2$:

$$r' = \frac{1}{2} r^\gamma r, \ p' = \frac{1}{2} \frac{1}{|r|} \left( r^\gamma p + p^\gamma r \right) \frac{1}{|r|}.$$

(49)

So, starting with Hatom$_3$ let us try it for $D = 3$ and some $\bar{D} \geq 3$. A series of questions are: what is $D$, what are the three $D \times \bar{D}$-symmetric matrices, $\gamma_{\alpha\beta}$, and in what SO(3) representation embedded in SO($\bar{D}$), compatible with $\gamma_{\alpha\beta}$, should the components of the SO($\bar{D}$) vector $\vec{r}'$ be classified? As a first try, consider $D = 3$. The relation $r' = \frac{1}{2} r^\gamma r$ indicates that the product of two $r$ vectors is desired to be a vector of SO(3), but $r^\gamma r$ is necessarily a symmetric product of two $r$ vectors which yields SO(3) spin either 0 or 2 but not 1 (vector). Therefore, if (49), which is the only form consistent with equation (1), is adopted as part of the canonical transformation, $\bar{D}$ cannot be 3.$^{12}$

In search for $D$, we observe that the phase space $(\vec{r}', \vec{p}')$ for the Hatom$_3$ is in the vector basis of SO(3), while the spinor basis (which has the smallest dimension possible) is the doublet $Z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right)$ of SU(2) $= SO(3)$. The spinor basis is complex so it contains 4 real numbers. Therefore, for (49) to work for the smallest possible $\bar{D}$, we must take $D = 4$ for the phase space vectors $(\vec{r}, \vec{p})$ of the HOsc$_4$. This suggests to rearrange the 4 real components of $\vec{r}$, which

$^{12}$ There are other forms of canonical transformations between these two systems predicted by 2T-physics that are more general than (1) (see section 4.3). However, in this section we are focusing on dualities consistent with the radial duality in (1).
is a vector of \( \text{SO}(4) \), into two complex numbers of an \( \text{SO}(3) \) spinor \( Z = \left( \frac{z_1}{z_2} \right) \), and similarly for \( \bar{p} \). Then, using the 3 Pauli matrices \( \sigma^i \), we can write an \( \text{SU}(2) = \text{SO}(3) \) covariant relation

\[
\mathbf{r}' = Z \left( \frac{\sigma^i}{2} Z \right)_i, \quad \text{with } z_1 = r_4 + i r_3, \quad z_2 = - r_2 + i r_1.
\]

The second line yields \( |\mathbf{r}'| = \frac{p_2}{2} \), consistent with radial duality (1), including the extra factor of 1/2 that emerged in (47). The first line of (50) can be rewritten in the form (49) to derive three 4 \( \times \) 4 real symmetric \( \gamma^{ij}_{\alpha\beta} \) matrices with \( \alpha, \beta = 1, 2, 3, 4 \). Furthermore, from (50) we obtain the relation between the angular coordinates \( \mathbf{r} \) for the Hatom\(_3\) and the angular coordinates \( \hat{\mathbf{r}} \) for the \( \text{HOsc}_4 \)

\[
\mathbf{r}' = \hat{\mathbf{r}} \cdot \mathbf{r} = Z' \sigma^i Z, \quad \text{with } Z' = \sqrt{\frac{Z}{|\mathbf{r}'|}} = \sqrt{2Z} \left( \frac{\mathbf{r}}{|\mathbf{r}|} \right),
\]

where the doublet \( \hat{\mathbf{r}} \) contains only angular coordinates, thus generalizing the radial duality relation (1) by including the angular relation. This embeds the angular spatial rotations \( \text{SO}(3) \) of the Hatom\(_3\) into the spatial rotations \( \text{SO}(4) \) of the \( \text{HOsc}_4 \). We will give more clarifying details about this embedding below, but first let us complete and verify the canonical transformation.

In addition to the \( Z = \left( \frac{z_1}{z_2} \right) \) doublet we introduce a doublet of canonical conjugates \( \Pi = \left( \begin{array}{c} \pi_1 \\ \pi_2 \end{array} \right) \), to write the second half of the canonical transformation

\[
\mathbf{p}_i = \frac{1}{\sqrt{2Z|Z|}} \left( Z' \sigma_i \Pi + \Pi^\dagger \sigma_i Z \right), \quad \text{with } \pi_1 = \frac{p_4 + i p_3}{2}, \quad \pi_2 = -\frac{p_2 + i p_1}{2}.
\]

Using the \( \text{HOsc}_4 \) commutation rules \( [\hat{\mathbf{r}}^a, \hat{\mathbf{p}}^b] = i \delta^{ab} \), we see that

\[
[Z_a, \Pi_b] = [Z_a', \Pi_b'] = i \delta_{ab}, \quad \text{while } [Z_a, \Pi_b] = [Z_a', \Pi_b'] = 0, \quad \text{for } a, b = 1, 2.
\]

Now we verify that equations (50) and (52), that are equivalent to (49), amount to a canonical transformation, by computing \( [\mathbf{r}', \mathbf{p}'] \) by using only the commutators (53) and obtain \( [\mathbf{r}', \mathbf{p}'] = i \delta^{ij} \) as follows:

\[
[r', p'] = \left( \frac{Z' \sigma^i}{2Z} \right) \frac{1}{\sqrt{2Z|Z|}} \left( Z' \sigma_i \Pi + \Pi^\dagger \sigma_i Z \right) \frac{1}{\sqrt{2Z|Z|}} = i \delta^{ij}.
\]

Now we give a group-theoretical analysis of the duality Hatom\(_3\) \( \leftrightarrow \text{HOsc}_4 \) via the canonical transformation above. Given the reasoning at the beginning of section 4, and using \( (D = 3, D = 4) \), we infer that this duality involves the non-compact groups \( \text{SO}(4, 2) \) and \( \text{Sp}(8, R) \) that classify the corresponding spectra as discussed in sections 2 and 3, provided these obey a subgroup relationship. Indeed in the spinor basis \( \text{SO}(4, 2) = \text{SU}(2, 2) \) is easily recognized as a subgroup of \( \text{Sp}(8, R) \supset \text{SO}(4, 2) \otimes \text{U}(1) \). Then, in the sense of equation (8), the entire spectrum of Hatom\(_3\), in the form of \( \text{SO}(1, 2) \) towers with angular momentum \( l \) for \( \text{SO}(3) \), should correspond to part of the \( \text{HOsc}_4 \) spectrum in the form of \( \text{Sp}(2, R) \) towers with angular momentum \( \tilde{l} \) for \( \text{SO}(4) \). Furthermore, the condition for identical \( \text{SO}(1, 2) = \text{Sp}(2, R) \)
representations for the towers, namely \( j(l) = \widetilde{j(\bar{l})} \) given in (38), must also be satisfied. The last requirement boils down to simply
\[
j(l) = l, \quad j(\bar{l}) = \frac{1}{2} \bar{l}; \quad \Rightarrow \bar{l} = 2l.
\]
(55)

This means the entire \( \text{Hatom}_3 \) spectrum, in the \( SO(4,2) \) singleton representation with Casimir eigenvalues \( C_k(D = 3) \) in equation (A14), must correspond to a subset of the states of the \( \text{HOsc}_4 \) in the even sector with \( (\bar{n}_{\text{even}}, l_{\text{even}}) \), i.e., within the even singleton representation of \( \text{Sp}(8,R) \) with Casimir eigenvalues \( C_k(D = 4) \) in equation (32). This is then consistent with \( SO(4,2) = SU(2,2) \subset \text{Sp}(8,R) \).

To see this more clearly, we display in equation (56) the full spectra of \( \text{Hatom}_3 \) and \( \text{HOsc}_4 \) similar to equation (8) with all their states included. The entry in each pigeon hole labelled by \( (n, l) \) for \( \text{Hatom}_3 \) is the dimension of the \( SO(3) \) \( l \)-representation, i.e., \( (2l + 1) \). Similarly, the entries at fixed values of \( \bar{n} \) or \( \bar{l} \) at the leftmost columns of each table are the dimensions of the hidden symmetries of the Hamiltonians, \( SO(4) \) for the \( \text{Hatom}_3 \) in the \( (n - 1) \)-representation, i.e., \( (n + 1)^2 \). These correspond to the total degeneracies of the energy levels of the respective systems, as discussed in section 3 for general \( D, \bar{D} \).

We still need to indicate the specific subset of \( \text{HOsc}_4 \)-even states that correspond to all the states of \( \text{Hatom}_3 \). We do this by comparing each pair of towers, at \( (\bar{l} = j, \bar{l} = 2j) \) for \( j = 0, 1, 2, \ldots \), since we have already guaranteed that the towers \( (\bar{l} = j, \bar{l} = 2j) \) have the same \( SO(1,2) = \text{Sp}(2,R) \) representation as in (38), and that this is a common subgroup of \( SO(4,2) \subset \text{Sp}(8,R) \). For the pair of towers \( (l = j, \bar{l} = 2j) \) the \( (SO(3)_j, SO(4)_2j) \) dimensions are generally different as seen in the tables in equation (8). For example for \( l = j = 2 \) the tables show that there are 5 \( \text{Hatom}_3 \) \( l = 2 \) towers, versus for the same \( j = 2 \), there are 25 \( \text{HOsc}_4 \) \( \bar{l} = 4 \) towers. Hence for \( j = 2 \), only 5 linear combinations of the 25 \( \text{HOsc}_4 \) towers correspond to the 5 \( \text{Hatom}_3 \) towers.

To figure out methodically the subset of \( \text{HOsc}_4 \) towers that correspond to the \( \text{Hatom}_3 \) towers at each \( \bar{l} = 2l \) we must identify the gauge symmetry inherent in the canonical transformation given above in equations (50), (52) and (53). This gauge symmetry must commute with the \( SO(3) \subset SO(4) \) embedding provided by equation (50), on which we now expand. To do so, note that the spinor \( Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \) is part of a \( 2 \times 2 \) matrix \( M \) that is constructed from the 4-component vector \( \bar{r} \).
\[ M = (i\sigma_1 \bar{r}_1 + i\sigma_2 \bar{r}_2 + i\sigma_3 \bar{r}_3 + \bar{r}_4) = \begin{pmatrix} \bar{r}_4 + i\bar{r}_3 & \bar{r}_2 + i\bar{r}_1 \\ -\bar{r}_2 + i\bar{r}_1 & \bar{r}_4 - i\bar{r}_3 \end{pmatrix} = \begin{pmatrix} z_1 - z_2^* \\ z_2 \\ -z_2^* \\ z_1 \end{pmatrix}. \]

(57)

The SO(4) rotations on \( r^\alpha \) amount now to SU(2)_L \( \otimes \) SU(2)_R = SO(4) transformations on the left and right sides of the matrix \( M \). The SO(3) transformation on the doublet spinor \( Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \) is the SU(2)_L applied on the left side of the matrix \( M \) as seen clearly from (57). The second doublet \( Z_c = \begin{pmatrix} -z_2^* \\ z_1^* \end{pmatrix} \) is the SU(2)_L ‘charge conjugate’ of the first doublet, \( Z_c = (-i\sigma_2 Z^*) \), and transforms also as a doublet under SU(2)_L. The SU(2)_R interchanges the two SU(2)_L doublets \((Z, Z_c)\) and this transformation commutes with SU(2)_R = SO(3). The matrix \( M \) satisfies \( MM^\dagger = M'M = \mathbf{r} \cdot \mathbf{r} \). The canonical transformation (50) for the position coordinates may now be rewritten in terms of \( M \) in the form:

\[ \mathbf{r} \cdot \sigma = M \sigma_3 M^\dagger/2 \text{ or } \mathbf{r}^i = \text{Tr} \left( \sigma^i/2 \sigma_3 M^\dagger M \right) = 1/2 \left( Z^i \sigma^i/2 Z - Z^i \sigma^i/2 Z^* \right) = Z^i \sigma^i/2 Z. \]

(58)

Hence the gauge symmetry that commutes with SU(2)_L = SO(3) must be part of SU(2)_R since SU(2)_R commutes with SU(2)_L. The form of the canonical transformation in equation (58) shows clearly that the SU(2)_R is broken down to U(1) due to the \( \sigma_3 \) insertion on the right side of \( M \). Therefore the gauge symmetry operator is just the third generator of SU(2)_R that we will call \( J_{3R} \). We can build this operator as a Hermitian operator that commutes with the expressions for the canonical transformation in (50) and (52). That is, we can verify the gauge-invariance of the canonical transformation by computing

\[ J_{3R} = 1/2i \left( Z^i \Pi^i - \Pi^i Z \right), \quad [\mathbf{r}, J_{3R}] = [\mathbf{p}, J_{3R}] = 0. \]

(59)

where \( (\mathbf{r}, \mathbf{p}) \) are written in terms of \( (Z, \Pi, Z^i, \Pi^i) \) before performing the commutators. It is easy to see that the finite transformation generated by \( J_{3R} \) amounts to overall phase transformations on the doublets

\[ (Z, \Pi, Z^i, \Pi^i) \rightarrow (e^{i\alpha} Z, e^{i\alpha} \Pi, e^{-i\alpha} Z^i, e^{-i\alpha} \Pi^i), \]

(60)

which is a symmetry of the canonical transformation in (50), (52) and (53). This \( U(1) \) is precisely the same \( U(1) \) that appears in SO(4, 2) \( \otimes \) U(1) \( \subset \) Sp(8, \( R \)), so it commutes with all properties of the hidden SO(4, 2) symmetry of the Hatom3. The vanishing of the operator \( J_{3R} \) is the constraint that must be applied on the quantum states of the HOsc4 in order to project out the gauge-dependent states and keep only the gauge-invariant ‘physical states’. Only the ‘physical states’ of the HOsc4 are dual to all the Hatom3 quantum states.

To see this projection and identification of the ‘physical states’ explicitly, one may decompose the SO(4) traceless symmetric tensors, whose dimensions for the \( l \)-representations appear in (56). We remark that in this case the \( l \)-representation can also be presented as the \( \{ \frac{2}{2}, \frac{1}{2} \} \) representation of SU(2)_L \( \otimes \) SU(2)_R = SO(4), labelled by quantum numbers \( |j_L, m_L; j_R, m_R \rangle \), with \(-j_L \leq m_L \leq j_L \) and \(-j_R \leq m_R \leq j_R \) and where \( j_L = j_R = \frac{l}{2} = l \). So the multiplets \( |l, m_L; l, m_R \rangle \) have dimension, \((2j_L + 1)(2j_R + 1) = (2l + 1)^2 = (l + 1)^2\), that matches the entries in the pigeon holes in the right-side table in equation (56). The gauge-invariant states are only those that have \( J_{3R} \rightarrow m_R = 0 \), hence the physical states are \( |l, m_L, l, 0 \rangle \) while all others with \( m_R \neq 0 \) are to be projected out by demanding gauge-invariant states. Now we see that the
At a fixed bound energy state (planetary-type classical orbits) take $|\ell, m, \ell, 0\rangle$ HOsc$_4$ states are in one-to-one correspondence with the $|\ell, m\rangle$ states of the Hatom$_3$. This proves clearly that we have constructed the duality transformation correctly.

It is also interesting to illustrate the role of the gauge symmetry at the classical level (i.e., not keeping track of operator ordering) as follows. We have already established in (50) that $|\mathbf{r}| = \frac{\sqrt{Z}}{2}$; we now compute also all the SO(3) dot products of the Hatom$_3$ phase space in terms of the HOsc$_4$ phase space as follows:

$$\mathbf{r} \cdot \mathbf{p} = Z^2 \frac{\sigma_3}{2} \frac{Z^2}{Z^2} \mathbf{r}^T \mathbf{r} = \frac{Z^2}{4} = \mathbf{r} \cdot \mathbf{p} = \frac{Z^2}{4}.$$  

$$\mathbf{p}^2 = \frac{Z^2}{2} \mathbf{r}^T \mathbf{r} + \frac{\mathbf{r}^T \mathbf{r} + \frac{Z^2}{2}}{2} = \frac{Z^2}{2} + \frac{(Z^2 - \mathbf{r}^T \mathbf{r})^2}{4} = \frac{Z^2}{4} - \frac{Z^2}{2} - \frac{Z^2}{4} - \frac{(Z^2 - \mathbf{r}^T \mathbf{r})^2}{4}.$$  

(61)

Then compute the classical Hatom$_3$ Hamiltonian in terms of the HOsc$_4$ phase space by using the relations above,

$$H_{\text{Hatom}} = \left( \frac{1}{2} \mathbf{p}^2 - \frac{Z}{r} \right) = \frac{1}{2} \mathbf{p}^2 - \frac{(J_{3\ell})^2}{2r^2} - 2Z.$$

(62)

At a fixed bound energy state (planetary-type classical orbits) take $H_{\text{Hatom}} \rightarrow -|E|$, and rewrite the above relation by multiplying both sides by $r^2$ and re-arranging, to obtain

$$\left( \frac{1}{2} \mathbf{p}^2 + |E| r^2 \right) - \frac{(J_{3\ell})^2}{2r^2} = 2Z.$$  

(63)

For gauge-invariant states of the HOsc$_4$ we must set $J_{3\ell} \rightarrow 0$, resulting in a bound energy state of the HOsc$_4$, $\left( \frac{1}{2} \mathbf{p}^2 + |E| r^2 \right) = 2Z$, that is dual to a bound energy state of the Hatom$_3$. In this way we see again the role of the gauge symmetry, now at the classical level.

Next, we can also keep track of the angular behavior in those orbits. From (57) and (58) we can write the relation between the unit vectors $\mathbf{r}$ and $\mathbf{r}$, in $D = 3$ and $D = 4$ dimensions respectively, as a unitary transformation $U$:

$$\sigma \cdot \mathbf{r} = U\sigma_3 U^\dagger, \quad U = \mathbf{r}_4 + i\sigma_1 \mathbf{r}_1 + i\sigma_2 \mathbf{r}_2 + i\sigma_3 \mathbf{r}_3,$$

(64)

that satisfies $UU^\dagger = U^\dagger U = \mathbf{r} \mathbf{r}^\dagger = 1$. We parametrize the $\mathbf{r}$ unit vector in 3 dimensions in the usual way

$$\mathbf{r}_1 = \sin \theta \cos \phi, \quad \mathbf{r}_2 = \sin \theta \sin \phi, \quad \mathbf{r}_3 = \cos \theta.$$  

(65)

With some hindsight, we parametrize the components of $\mathbf{r}_o$ of the unit vector in 4 dimensions as follows:

$$\mathbf{r}_1 = \sin \hat{\theta} \sin (\hat{\phi} + \hat{\chi}), \quad \mathbf{r}_2 = \sin \hat{\theta} \cos (\hat{\phi} + \hat{\chi}),$$

$$\mathbf{r}_3 = \cos \hat{\theta} \sin (\hat{\chi} - \hat{\phi}), \quad \mathbf{r}_4 = \cos \hat{\theta} \cos (\hat{\chi} - \hat{\phi})$$  

(66)

so that the SU(2) unitary $2 \times 2$ matrix $U$ takes the following form:

$$U = \begin{pmatrix} \cos \hat{\theta} e^{i\hat{\chi} - \hat{\phi}} & -\sin \hat{\theta} e^{-i\hat{\chi} + \hat{\phi}} \\ -\sin \hat{\theta} e^{i\hat{\chi} + \hat{\phi}} & \cos \hat{\theta} e^{-i\hat{\chi} - \hat{\phi}} \end{pmatrix} = e^{-i\hat{\theta}\hat{\sigma}_3} e^{i\hat{\theta}\hat{\sigma}_2} e^{i\hat{\chi}\hat{\sigma}_3}.$$  

(67)
The generators of higher-dimensional system. Because of this holography, the gauge invariants of the unique systems have origins as the Hamiltonians for the 1T systems all have an exact hidden global symmetry $SO(D)$ between the angles of the gauge symmetry generated by $J_{1R}$ (i.e., phase symmetry $U(1) \subset SU(2)_R$). So, $\chi$ is the pure gauge freedom in the expression of the canonical transformation (50). This means an infinite number of HOsc closed trajectories map to the same Hatom trajectories but with correlations that involve the double circling in the Hatom versus the single circling in the HOsc as indicated by equation (68).

Note that $\vec{r}(\theta, \phi)$ is independent of the angle $\bar{\chi}$ that drops out in the expression $U_{\sigma \bar{3}} U^\dagger$ because of the gauge symmetry generated by $J_{1R}$ (i.e., phase symmetry $U(1) \subset SU(2)_R$). So, $\bar{\chi}$ is the pure gauge freedom in the expression of the canonical transformation (50). This means an infinite number of HOsc closed trajectories map to the same Hatom trajectories but with correlations that involve the double circling in the Hatom versus the single circling in the HOsc as indicated by equation (68).

4.3. 2T-physics and more general dualities in any $D$

The appendix summarizes how 2T-physics predicts many shadow 1T systems in the same number of spatial dimensions $D$, and that these are dual to each other for any pair of shadows. Each shadow in $D$ space and 1 time dimensions, being merely a gauge choice of the $Sp(2, R)$ gauge symmetry, holographically captures all the gauge-invariant details of the $(D + 1) + 2$-dimensional structure of the unique $((D + 1) + 2)$-dimensional structure predict dualities among the multitude of 1T-physics systems $\{r, p, t, H\}$. The duality transformations are nothing but $Sp(2, R)$ gauge transformations that take one fixed gauge to another fixed gauge, and in the traditional 1T-physics language, these correspond to canonical transformations between two shadows, $(r, p, t, H) \leftrightarrow (\bar{r}, \bar{p}, \bar{t}, \bar{H})$. We emphasize that the meaning of time and Hamiltonian are different in the two shadows and the canonical transformations involve the time and Hamiltonian, thus explaining how two very different 1T-physics systems (different Hamiltonians), that are traditionally presented as very different ‘physics’, are actually not independent of each other because they are holographic shadows of the same system in $(D + 1) + 2$ dimensions. The usual 1T formalism hides such relationships that are not evident but actually exist as predicted by 2T-physics.

The predictions come in the form of hidden symmetries and dualities that are also measurable features of 1T physics. Establishing the existence of the predicted hidden symmetries and dualities in 1T-physics is tantamount to establishing the existence of the $(D + 1) + 2$-dimensional structure with its $Sp(2, R)$ gauge symmetry in phase space (beyond well studied local gauge symmetry in position space). Among these dualities, the duality for the pair Hatom $\leftrightarrow$ HOsc is only one case among many dualities for a multitude of similar pairs.

When $(D + 1) + 2$ is flat (a very special but very broad case), the corresponding actions (not Hamiltonians) for the 1T systems all have an exact hidden global symmetry $SO(D + 1, 2)$. This has its origins as the global symmetry in flat $(D + 1) + 2$ dimensions for the 2T action. The generators $L^{MN}$ of this global non-compact symmetry are gauge-invariant because they commute with the $SL(2, R)$ generators as seen in (A8). So, each $L^{MN}$ is independent of the shadow, even when it is evaluated in terms of a given shadow labelled by $k$: $\{X^M_{\{k\}}, P^N_{\{k\}}\}$, $k = 1, 2, 3, \ldots$ An infinite set of duality relations between gauge-invariant observables of shadow $k_1$ and shadow $k_2$ are predicted by evaluating any given function of the $L^{MN}$ in those two different shadows as in equation (A19):

$$L^{MN}_{\{k_1\}} = L^{MN}_{\{k_2\}} = L^{MN},$$

Dualities for every function $F: F(L^{MN}_{\{k_1\}}) = F(L^{MN}_{\{k_2\}}).$
These are an infinite set of measurable predictions from 2T-physicsto the dynamics of 1T-physics. From these gauge-invariant predictions we can extract the canonical transformations for the phase spaces \( \{r_{(1)}, p_{(1)}; r_{(2)}, p_{(2)}; H_{(1)}\} \leftrightarrow \{r_{(2)}, p_{(2)}; r_{(1)}, p_{(1)}; H_{(2)}\} \). For examples, see [37].

Turning this relation around, the Hamiltonian \( H_{(1)} \) of each shadow can be expressed as some function \( H_{(2)} \) of the gauge-invariant \( \mathcal{L}_{MN} \) generators of \( \text{SO}(D + 1, 2) \). Examples of such systems that have been explicitly discussed in the past 2T-physicsliterature include Hatom, HOsc, free relativistic massless/massive particles, particles moving in various curved spaces (including some black holes), twistor equivalents for all these, etc. These systems appear to be disjoint in the traditional 1T physics formalism. Actually, they are simply 1T ‘shadows’, with different meanings of 1 time, resulting from various gauge choices of the \( \text{Sp}(2, R) \) symmetry, thus embedding each shadow differently within a given representation of the underlying global symmetry \( \text{SO}(D + 1, 2) \). Furthermore, since the Hamiltonian is a function of the gauge invariant \( \mathcal{L}_{MN} \), the spectrum of energy states in each dual shadow system is captured in the same infinite-dimensional unitary representation of \( \text{SO}(D + 1, 2) \), whose quadratic, cubic, quartic and higher Hermitian Casimir operators are predicted to have the fixed eigenvalues given in equation (A14) that identify the singleton representation. Thus, notably, the Hilbert space of one shadow is mapped to the Hilbert space of another one by a unitary transformation within the fixed singleton representation of \( \text{SO}(D + 1, 2) \). A further notable unification fact, unique to 2T-physics, is that the shadows listed above, and many others, satisfy the same equations in \( (D + 1, 2) \) dimensions, namely simply \( X \cdot X = P \cdot P = X \cdot P = 0 \), that’s all. See (A10) in the appendix for clarification how shadows in \( D + 1 \) dimensions emerge just from these equations.

The \( D = 2 \) case in section 4.1 is a particularly clean example of dualities in flat \((D + 1) + 2\) dimensions, including the spinning case, because there is a clean choice of two-dimensional phase space to describe the two shadows \( \text{Hat}_{(1)} \leftrightarrow \text{HOsc}_{(2)} \) such that the respective time coordinates do not transform \( i = t \). Similarly, the \((D, D) = (3, 4)\) case is the simplest example in which there is a leftover gauge symmetry that makes the respective phase spaces dual to each other. This set of examples, in which \( i = t \), can be generalized to higher dimensions using similar methods involving beautiful group theory as outlined in the next section. For the more general cases in which \( i \neq t \) the canonical transformations are more complicated, dramatic and surprising and were not expected to be possible in 1T-physics, but they do exist. For examples of 5 shadows and related canonical transformations see [37]. Among these there are in particular two versions of dualities that relate to the Hatom \( \leftrightarrow \text{HOsc}_{(0)} \) duality discussed here but the details of the canonical transformation are totally different. These cases include the dualities \( \text{Hat}_{(D)} \leftrightarrow \text{HOsc}_{(D - 1)} \) and \( \text{HOsc}_{(D)} \leftrightarrow \text{HOsc}_{(D)} \) that is implicitly given in [37].

5. Generalizations

We set out searching for a principle underlying the radial duality equation (1) between the hydrogen atom (radial coordinate \( r \)) and the harmonic oscillator (radial coordinate \( r \)) which follows from Newton’s substitution \( r \propto r^2 \). The generalization of radial duality to other power-law potentials and any pair of dimensions \((D, D)\), as given in (2)–(6), showed an important quantum mechanical restriction, \( I + \frac{D - 2}{2} = \frac{|n|}{|n|} [I + \frac{D - 2}{2}] \), where the quantized orbital angular momenta \((l, I)\) and the dimensions \((D, D)\) had to be integers.

---

13 The nonrelativistic \( \text{HOsc}_{(D)} \) is briefly discussed in [37] as a specialized case of the more general ‘shadow-5’. See equation (59) and related discussion in [37]. Thus \( \text{HOsc}_{(D)} \) is dual to all 5 shadows, including the \( \text{Hat}_{(O)} \). The corresponding canonical transformation can be extracted in the same manner as the other cases explicitly discussed in [37]. This is a pretty complicated expression that we may discuss in another paper.
The analysis in section 2 of the quantum spectra for the Hatom$_D$ and HOsc$_D$, with \((-\frac{D}{2}) \approx 2\), showed the disparity between the two systems when angular degrees of freedom are included, so a full duality consistent with Newton’s substitution, \(r \propto r^2\), could not be expected. However, hints did emerge on how a full quantum duality may be possible between a subset of the HOsc$_D$ quantum states and the full Hatom$_D$ quantum states. The non-compact group analysis SO\((D + 1, 2)\) and Sp\((2D, R)\) of the respective spectra made it clear precisely how to proceed to construct the full duality by using 2T-physics as the guiding principle. To implement 2T-physics perspective, it required the embedding of SO\((D + 1, 2)\) into Sp\((2D, R)\) and demanding the identification of the corresponding generators written in terms of the different phase spaces \((\mathbf{r}, \mathbf{p})_D\) and \((\bar{\mathbf{r}}, \bar{\mathbf{p}})_D\) as in equation (38). From this we could extract the canonical transformation that relates the two phase spaces, thus constructing a full duality that includes all directions rather than only the radial direction, while at the same time obtaining a remarkable beautiful symmetry perspective of the full duality.

The explicit canonical transformation between phase spaces (49) in different dimensions, \(D < D\), was clarified by identifying a gauge symmetry group \(G_D\) in the canonical transformation, \((\mathbf{r}, \mathbf{p})_D^{G_D} \rightarrow (\mathbf{r}, \mathbf{p})_D\), such that only the gauge-invariant subsector of the HOsc$_D$ could be dual to the full Hatom$_D$. This gauge symmetry, that we now call \(G_D\), showed precisely which subset of states of HOsc$_D$, that are invariant under the gauge symmetry \(G_D\), are dual to the full Hatom$_D$ spectrum.

This program was carried out explicitly in the previous sections for the pairs, \((D, D)_{G_D} = [(2, 2)_{\text{discr}}, (3, 4)_{\text{h(1)}}]\). Here we sketch how to generalize to the case \(D = 5\) and then to \(D = 1, 4\).

The case \(D = 5\) works exactly the same way as the case \((3, 4)\). The spinor of SO\((5) = \text{USp}(4)\) is a quartet of 4 complex numbers, so this suggests to consider \(D = 8\). Accordingly, we introduce a quartet \(Z\) and let the four complex numbers be constructed from the 8 components of the real vector \(\mathbf{r}^\alpha\) for the HOsc$_8$. Introduce the charge conjugate spinor \(Z_c \equiv CZ\), where \(C\) is the antisymmetric charge-conjugation matrix in spinor space which amounts to be the invariant metric of USp\((4)\). This structure guarantees that the quartet \(Z\) transforms under USp\((4)\) exactly the same way as \(Z\). Now, similarly to the \((3, 4)\) case we construct a \(4 \times 2\) matrix \(M = (Z, Z_c)\) and define an SU\((2)\) transformation that mixes \((Z, Z_c)\) like a doublet. Hence, \(M\) is now classified as \((4, 2)\) under USp\((4) \otimes \text{SU}(2)\). In this way the 8 real numbers \(\mathbf{r}^\alpha\) become a basis for USp\((4) \otimes \text{SU}(2)\) transformations. This is consistent with the fact that SO\((8) \supset \text{SO}(5) \otimes \text{SO}(3)\), not only in the vector basis but also in the SO\((8)\) spinor basis. Accordingly, our starting point is to re-assign \(\mathbf{r}^\alpha\) to the spinor basis of SO\((8)\) rather than the vector basis (recall triality in SO\((8)\)). Thus, USp\((4) = \text{SO}(5)\) will serve to classify the \(D = 5\) vector \(\mathbf{r}^I\) for the Hatom$_5$ and \(G_5 = \text{SU}(2)\) will serve as the gauge symmetry in the relation \(\mathbf{r}^I \sim \mathbf{r}^I \gamma_{\alpha \beta} \mathbf{r}^\alpha\mathbf{r}^\beta\) in the canonical transformation (49), with \(I = 1, 2, 3, 4, 5\) and \(\alpha = 1, 2, \ldots, 8\). This \(G_5 = \text{SU}(2)\) gauge group fits also in the subgroup structure of the hidden symmetry noncompact groups, \(\text{Sp}(16, R) \supset \text{SO}(6, 2) \otimes \text{SU}(2)\), as it should. With this background we are now able to use the \(4 \times 4\) five \(\text{SO}(5) = \text{USp}(4)\) gamma matrices \(\Gamma^I\) to construct the first half of the canonical transformation (49) for the case \((D, D)_{G_D} = (5, 8)_{\text{SU}(2)}\) as follows:

\[
\mathbf{r}' = \frac{1}{4} \left( \mathbf{Z}' \Gamma^I \mathbf{Z} + \mathbf{Z}' \Gamma^I \mathbf{Z}_c \right) = \frac{1}{2} \mathbf{Z}' \Gamma^I \mathbf{Z}.
\]

By taking for example, \(z_1 = r_1 + i\bar{r}_2, z_2 = r_3 + i\bar{r}_4, z_5 = r_5 + i\bar{r}_6, z_4 = r_7 + i\bar{r}_8\), this relation can be rewritten in the form \(\mathbf{r}' \sim \mathbf{r}' \gamma'_{\alpha \beta} \mathbf{r}^\alpha\mathbf{r}^\beta\). The first expression in (70) involving both \((\mathbf{Z}, \mathbf{Z}_c)\) displays the SU\((2)\) gauge symmetry, while the simpler last form is obtained because \(\mathbf{Z}' \Gamma^I \mathbf{Z}_c = \mathbf{Z}' \Gamma^I \mathbf{Z}\), that can be proven by using the properties of the gamma matrices (namely
Now, using the Fierz identity for SO(5) gamma matrices, \( \Gamma'_{\alpha \beta} \Gamma'_{\alpha \beta} = (2\delta_{\alpha,\delta} \delta_{\beta,\delta} - 2C_{\alpha \beta} C_{\delta \delta} - \delta_{\alpha,\delta} \delta_{\beta,\delta}) \), and noting \( ZCZ = ZCZ' = 0 \) due to the antisymmetry of \( C \), we compute \( \mathbf{r} \cdot \mathbf{r} \).

\[
\mathbf{r} \cdot \mathbf{r} = \frac{1}{4} (Z' \Gamma' Z) (Z' \Gamma' Z) = \frac{1}{4} (Z' Z)^2 = \frac{1}{4} (\mathbf{r} \cdot \mathbf{r})^2. \tag{71}
\]

This shows agreement with Newton’s radial substitution \(|\mathbf{r}| = \mathbf{r}^2/2\), while we have included all the angles in both the \( D = 5 \) and \( D = 8 \) systems and satisfied the radial duality condition \( |\hat{l} + \frac{D-2}{2}| = 2 |\hat{l} + \frac{D}{2}| \) with \( \hat{l} = 2l \). The rest of the canonical transformation involving the momenta is constructed steps parallel to the case \((3, 4)\). Further group-theoretical investigation of the Hatom\(_5\) and HOsc\(_8\) spectra, similar to equation \((56)\), confirms that the gauge-invariant subset of the HOsc\(_8\) spectrum exactly matches the full spectrum of the Hatom\(_5\) spectrum.

In the cases \((D, \bar{D})\)\(_G\) = \([(2, 2)]_{\text{discrete}}; (3, 4), (5, 8)_{\text{SU(2)}}\) note the perfect match of the counting of gauge-invariant degrees of freedom for the HOsc\(_D\), namely \( d(D) \equiv (D - \dim(G_D)) = D \), that is identical to the degrees of freedom of Hatom\(_D\). Here \( \dim(G_D) \) is the number of continuous group parameters in the gauge group \( G_D \). In the \( D = 2, 3, 5 \) cases we also note that we find \( D = 2(D - 1) \), so \( l = 2l \) satisfies the crucial radial duality relationships, \( |\hat{l} + \frac{D}{2}| = 2 |\hat{l} + \frac{D-2}{2}| \), given in \((6)\) or \((11)\). How about other dimensions?

Let us start with \( D = 1 \), with the hidden symmetry of the Hatom\(_1\) being SO(2, 2), and knowing \( l = 0 \) since there is no angular momentum, as well as \( j = 0 \) according to \((11)\). Which value of \( D \) for HOsc\(_D\) is dual to this system? According to equation \((11)\), since \( j = 0 \) for the Hatom\(_1\) tower, we must have the \( \hat{l} = 0 \) tower of \( j(0) = \frac{D-4}{4} = 0 \). So we must have \( D = 4 \), with the 4 components of \( \mathbf{r} \) arranged into a complex doublet \( Z = \left( \begin{array}{c} z_1^1 \\ z_2^1 \end{array} \right) \) of the gauge group \( G_1 \equiv \text{SU}(2)_L \subset \text{SO}(4) \). This leads to the SO(4) invariant canonical transformation \((49)\), i.e., \( r = \frac{1}{2}Z'Z = \frac{1}{2} (z_1^1 z_2^1 + z_2^1 z_1^1) = \frac{1}{2} \mathbf{r}^2 \), which is consistent with Newton’s substitution \((1)\). Note that angular degrees of freedom have been included in the canonical transformation, although trivially due to the gauge symmetry \( G_1 \equiv \text{SU}(2)_L \). The \((D, \bar{D}) = (1, 4)\) version of the radial duality condition \( |\hat{l} + \frac{D-2}{4}| = 2 |\hat{l} + \frac{D-6}{4}| \) is also satisfied since \( \hat{l} = 0 \) due to the SO(4) gauge symmetry while \( l = 0 \) due to \( D = 1 \). Furthermore, the effective number of gauge invariant degrees of freedom for HOsc\(_4\), \( d(D) = D - \dim(G_D) = 4 - 3 = 1 \), matches the number of degrees of freedom \( D = 1 \) for Hatom\(_1\).

Next consider \( D = 4 \) and analyze all the requirements of the duality we have discussed up to now to find \( D \). The radial duality condition \((6)\) with \( D = 4 \) and assuming \( D > 2 \), becomes \( l = 2l + \frac{D-2}{2} \). The hidden symmetry of the Hatom\(_4\) is SO(5, 2) and we must embed this into Sp\((2D) \supset \text{SO}(5, 2) \otimes G_4\), where \( G_4 \) is the gauge symmetry of the canonical transformation \((49)\), so \( G_4 \) must also satisfy \( \text{SO(\bar{D})} \supset \text{SO}(4) \otimes G_4\). These are severe restrictions on \( D \). If we consider the hint, that \( l = 2l \) that worked so far in the cases \( D = 1, 2, 3, 5 \) might also work for \( D = 4 \), then we should choose \( D = 6 \). Then \( G_4 \) should satisfy \( \text{SO(6)} \supset \text{SO}(4) \otimes G_4\), where \( \text{SO}(4) \) applies to the vector \( \mathbf{r} \) of the Hatom\(_4\), while \( \text{SO}(6) \) applies to the vector \( \mathbf{r}' \) of the HOsc\(_6\). This pins down \( G_4 \equiv U(1) \otimes U(1) \) that fits the counting of gauge-invariant degrees of freedom in HOsc\(_6\), namely \( d(D) = D - \dim(G_D) = 6 - 2 = 4 \), matching \( D = 4 \) the number of degrees of freedom in Hatom\(_4\). This \( G_4 = U(1) \otimes U(1) \) is consistent also with the subgroup structure of the relevant hidden symmetry non-compact groups, \( \text{Sp}(12, R) \supset \text{SO}(5, 2) \otimes (U(1) \otimes U(1)) \).

With this information the relation \( \mathbf{r}' \sim r' \alpha \gamma_{\alpha} \mathbf{r}^\dagger \) can now be constructed by re-arranging the 6 real numbers of \( \mathbf{r}' \) into a complex quartet \( Z \), but anticipating that this complex quartet contains
only 6 real numbers rather than the natural 8 real numbers. We begin with the quartet spinor \( W \) of \( \text{SO}(6) = \text{SU}(4) \) that contains four complex numbers \( W_a = (w_1, w_2, w_3, w_4) \). This is also a spinor of \( \text{USp}(4) = \text{SO}(5) \) and can be used as in equation (70) to construct an \( \text{SO}(5) \) vector out of two \( \text{SO}(5) \) spinors, \( \mathbf{r}^i = \frac{1}{4} W^i \Gamma^i W \) where \( i = 1, 2, 3, 4, 5 \). We now focus on the \( \text{SO}(4) = \text{SU}(2) \otimes \text{SU}(2) \) subgroup of \( \text{SO}(5) = \text{USp}(4) \) and identity \( I \to i = 1, 2, 3, 4 \) with the vector \( \mathbf{r}^i \) of the \( \text{Hatom}_4 \). Also for \( I \to 5 \) we impose the following constraint on the quartet \( W \)

\[
r^5 = \frac{1}{2} W^i \Gamma^5 W = 0.
\]  

The solution of this constraint is the quartet is \( Z(\mathbf{r}) \equiv W(\text{solution}) \) which is parametrized by only 6 real numbers that can be related to the six \( r^a \) of the \( \text{HOsc}_6 \). Hence we have

\[
r^{i=1,2,3,4} = \frac{1}{2} Z^i(\mathbf{r}) \Gamma^{i=1,2,3,4} Z(\mathbf{r}), \text{ and } Z^i(\mathbf{r}) \Gamma^5 Z(\mathbf{r}) = 0. \tag{73}
\]

Clearly, this relation is covariant under the rotation group \( \text{SO}(4) \) as a subgroup of \( \text{SO}(5) \). Moreover, we compute \( r \cdot r = \frac{1}{4} \left( Z^i(\mathbf{r}) Z^i(\mathbf{r}) \right)^2 = \frac{1}{4} (r \cdot r)^2 \tag{74} \)

which agrees with Newton’s radial substitution \( r = r^2/2 \). Of course, our transformation (73) includes all the angular variables for both unit vectors in 4-dimensions (3 angles) and 6-dimensions (5 angles). To display how \( Z(\mathbf{r}) \) depends on the 5 angles of \( \mathbf{r} \) we work in a basis in which \( \Gamma^5 \) is diagonal, \( \Gamma^5 = \text{diag}(1, 1, -1, -1) \), and write the quartet \( Z(\mathbf{r}) \) that satisfies the required properties \( Z^i(\mathbf{r}) \Gamma^5 Z(\mathbf{r}) = 0 \) as follows

\[
Z(\mathbf{r}) = r \left( e^{i \phi} \begin{pmatrix} e^{i \chi_+} \cos \theta_+ \\ \bar{e}^{-i \chi_+} \sin \theta_+ \\ e^{i \chi_-} \cos \theta_- \\ \bar{e}^{-i \chi_-} \sin \theta_- \end{pmatrix} \right), \quad Z^i(\mathbf{r}) Z(\mathbf{r}) = 0, \quad Z^i(\mathbf{r}) Z(\mathbf{r}) = r \cdot r. \tag{75}
\]

The relation \( r^{i=1,2,3,4} = \frac{1}{4} Z^i \Gamma^{i=1,2,3,4} Z \) has a \( G_d \equiv \text{U}(1) \otimes \text{U}(1) \) gauge symmetry. The first \( \text{U}(1) \) amounts to an overall phase transformation on \( Z(\mathbf{r}) \); this can be used to gauge-fix \( \phi \to 0 \). The second \( \text{U}(1) \) amounts to a translation of \( \theta_\pm \) in opposite directions, \( \theta_\pm \to \theta_\pm + \alpha \), so that the sum \( \left( \theta_+ + \theta_- \right) \) remains invariant; this can be used to gauge-fix \( \theta_\pm \to \hat{\theta}_\pm \). Once gauge fixed, \( Z(\mathbf{r}) \) has only 4 parameters, \( (r, \hat{\theta}, \chi_+, \chi_-) \), which is the expected number of gauge invariants according to \( d(D) = D - \dim(G_d) = 6 - 2 = 4 \), that matches \( D = 4 \) for \( \text{Hatom}_4 \). Further study of the spectra of \( \text{Hatom}_4 \) versus \( \text{HOsc}_6 \) reveals the perfect duality between the respective spectra, once the \( \text{U}(1) \otimes \text{U}(1) \) gauge-invariant subset of states of \( \text{HOsc}_6 \) are identified.

6. Conclusion and outlook

We hope the outline given above is sufficient for the cases \( D = 1, 4, 5 \). We have not included the group-theoretical details for \( D = 1, 4, 5 \) in the current paper because it would take too much space, but if interest persists, we may do it in a future publication.
We compile a list of the cases we have discussed up to now:

| $D$ | $\tilde{D}$ | Spinors $\text{SO}(D)$ | $\text{Sp} \left( 2\tilde{D}, R \right) \supset \text{SO} \left( D + 1, 2 \right) \otimes G$ |
|-----|-------------|------------------------|---------------------------------------------------------------------|
| 1   | 1           | $\text{1}_\text{real}$ (four copies) | $\text{Sp} \left( 8, R \right) \supset \text{SO} \left( 2, 2 \right) \otimes \text{SU} \left( 2 \right)$ |
| 2   | 2           | $\text{1}_\text{complex}$ | $\text{Sp} \left( 4, R \right) \supset \text{SO} \left( 3, 2 \right) \otimes \text{discrete}$ |
| 3   | 4           | $\text{2}_\text{complex}$ plus one constraint | $\text{Sp} \left( 8, R \right) \supset \text{SO} \left( 4, 2 \right) \otimes \text{U} \left( 1 \right)$ |
| 4   | 6           | $\text{4}_\text{complex}$ | $\text{Sp} \left( 12, R \right) \supset \text{SO} \left( 5, 2 \right) \otimes \text{U} \left( 1 \right) \otimes \text{U} \left( 1 \right)$ |
| 5   | 8           | $\text{4}_\text{complex}$ | $\text{Sp} \left( 16, R \right) \supset \text{SO} \left( 6, 2 \right) \otimes \text{SU} \left( 2 \right)$ |

The third column gives information on the re-classification of the $\text{SO}(D)$ vector $r$ as representations under $\text{SO}(D) \times G_D$. In every case a spinor representation of $\text{SO}(D)$ is included but in the case $D = 4$ we started with a complex quartet of $\text{SO}(D = 6)$ and imposed constraints to reduce it effectively to two less real parameters. Furthermore, in the case $D = 1$ there is a repetition of representations (although trivial in this case). These should be taken as clues for how to proceed for larger dimensions $D \geq 6$.

As a result of our experience with $D = 1$ to 5, we witness that $\tilde{l} = 2l$ applies to all the cases so far. Combined with the radial duality requirement, $\left| l + \frac{D-2}{2} \right| = 2 \left| l + \frac{D-2}{2} \right|$, we see that $D = 2(D - 1)$, for $2 \leq D \leq 5$, except for the case of $D = 1$ for which $D = 4$. The pattern $\tilde{l} = 2l$, along with $D = 2(D - 1)$, may be taken as a conjecture for further investigations to determine $\tilde{D}$ once $D \geq 6$ is given, but it is not necessary that this conjecture, based on $\tilde{l} = 2l$, should hold at larger $D$. In any case, we have enough circumstantial evidence to expect that there is a duality $\text{Hat}_{D,D+1} \leftrightarrow \text{HO}_{sc_D}$ for every dimension $D$. This is in harmony with the general prediction from 2T-physics.

We note that so far only the even states of the $\text{HO}_{sc_D}$ participate in the duality. As outlined in section 2.2, a subset of the odd states of the $\text{HO}_{sc_D}$ are supposed to be dual to the dyonic-Hatom according to 2T-physics. The detailed analysis of this prediction is left to future research.

We generalized Newton’s radial duality to a full duality in the form of a canonical transformation consistent with 2T-physics and its hidden symmetry $\text{SO}(D + 1, 2)$ that applies to all shadows, beyond the two shadows, Hatom and HOsc, we discussed here. As emphasized in section 4.3 2T-physics offers other forms of canonical transformations that are more complicated and a priori do not seem compatible with Newton’s radial duality. The basis of the phase spaces in those cases are different (for example $D = D$ for all $D$) but must ultimately be related to the bases discussed in this paper by some canonical transformations, especially after solving explicitly all the gauge constraints for the gauge groups $G_D$ introduced in the current paper (since then the $\text{HO}_{sc_D}$ system in the gauge-invariant sector is reduced to a smaller phase space effectively in $D$ dimensions). We leave the resolution of these and related questions to future investigations.

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Appendix A. Concepts of 2T-physics

In this appendix we briefly outline the essential features of 2T-physics that are relevant for the reader to better understand the deeper spacetime meaning of the material in the text. The features of 2T-physics are what prompted Peter Freund to suggest to J Rosner that the dualities encountered in the radial Schrödinger equation, that was originally discovered by Newton [17], and later encountered while analyzing quark–antiquark bound states [1], could be related to the more general dualities of the shadows predicted by 2T-physics.

2T-physics is based on having an additional Sp(2, R) gauge symmetry acting on phase space [34, 44] beyond the gauge symmetries that occur in traditional classical or quantum mechanics, field theory [51–53] or string theory (for reviews, see [36, 54, 55]). This gauge symmetry adds gauge degrees of freedom to conventional 1T-physics such that all physical systems in D space and 1 time dimensions are elevated to a covariant description in (D + 1) space and 2 time dimensions with one extra time and one extra space dimension in flat or curved spacetime.14 The higher-dimensional perspective is very powerful and reveals correlations that actually exist in nature but remain well hidden and are impossible or very hard to capture in the traditional 1T-physics formalism. However, the predictions can be verified within 1T-physics either through theoretical computations or experiments. The dualities of the shadows that Peter Freund alluded to are part of general predictions by 2T-physics.

Shadows are 1T-physics systems obtained from 2T-physics by making 1 + 1 gauge choices (in phase space, not just position space) and solving two constraints, thus eliminating 1 + 1 position and momentum degrees of freedom. Any shadow captures holographically all the gauge-invariant physical phenomena that occur in (D + 1) + 2 dimensions. For several examples of shadows, see [37] and references therein to previous similar 2T gauge choices. The remaining D spatial and 1 temporal dynamical phase space degrees of freedom, including time and its canonical conjugate Hamiltonian, are embedded in (D + 1) + 2-dimensional phase space in an infinite number of geometrical or algebraic configurations that create very different perspectives for D + 1-dimensional observers (of gauge-invariants) to view the phenomena that occur in (D + 1) + 2 spacetime. For this reason, the concepts of time and Hamiltonian are different for the observers in different shadows. The duality of the shadows is expressed in the language of 1T-physics via canonical transformations that include time and Hamiltonian, thus establishing unexpected connections to one another among many 1T-physics systems with diverse Hamiltonians. All the connected shadows are actually gauge-fixed forms of the same unique higher-dimensional system, and these dualities are just Sp(2, R) gauge transformations.

14 The reason for the extra 1 + 1 dimensions is closely related to the number and signature of the gauge parameters in Sp(2, R) = SO(1, 2), that has 1 spacelike gauge parameter and two timelike gauge parameters. By contrast, 1T-physics has only 1 timelike gauge parameter as recognized in the familiar τ-reparametrization gauge symmetry in the world-line formalism. The difference then is 1 + 1 additional gauge degrees of freedom in 2T-physics versus 1T-physics. Even though the number of physical spatial (hence ghost free) degrees of freedom, D, is the same in both cases, the higher-dimensional perspective turns out to be much richer in predictions of hidden symmetries and dualities as compared to 1T-physics. Since we mentioned τ-reparametrization, it may be interesting to note that τ-reparametrization gauge symmetry amounts to general relativity based on local (i.e., τ-dependent) translation invariance in one temporal dimension. By contrast Sp(2, R) = SO(1, 2) gauge symmetry amounts to conformal gravity based on local conformal transformations on the worldline, noting that SO(1, 2) is the conformal group in the space of a single temporal dimension [34]. This is another way to understand the extra 1 + 1 gauge parameters on the worldline and hence the extra 1 + 1 gauge degrees of freedom as part of target phase space, and the associated extra 1 + 1 constraints.
that take one fixed gauge to another fixed gauge. All connected shadows in 1T-physics are the same unique system in 2T-physics and obey the same equations in \((D + 1) + 2\) dimensions. Thus, 2T-physics in \((4 + 2)\) dimensions provides an unprecedented unification that is manifested in our experience in \(3 + 1\) dimensions in the form of dualities and hidden symmetries. The widely recognized conformal symmetry, SO(4, 2) in relativistic 1T-physics in \(3 + 1\) dimensions, emerges from one of the simplest shadows in 2T-physics directly because of the one extra space and one extra time dimension. This is the shadow for the free relativistic massless particle; by now we call this case the ‘conformal shadow’. Experimentally observing or theoretically verifying the predicted dualities, hidden symmetries and their consequences, in particular those related to conformal symmetry, is one form of experimental evidence for the underlying \((D + 1) + 2\) dimensions.

In this paper the simple cases of the shadows for Hydrogen atom, Harmonic oscillator and a third shadow, closely resembling the conformal shadow (but with \(r\) and \(p\) interchanged (see equation (20) and footnote 1 in [35])), are discussed in section 3 with the purpose of trying to understand if Newton’s radial duality is part of the larger unifying features of 2T-physics.

In the simplest context of 2T-physics, the worldline formalism for a single spin \(= 0\) particle, the Sp(2,\(R\)) gauge symmetry acts on the phase space degrees of freedom of the particle \(\{X^M(\tau), P_M(\tau)\}\), making position and momentum in \((D + 1) + 2\) dimensions interchangeable and on equal footing in the formulation of all classical and quantum physics for each particle. The signature with two temporal dimensions in target space, no less and no more, and any number of spatial dimensions, is not an input, but rather it is an output of the Sp(2,\(R\)) gauge symmetry as explained below. The gauge-invariant subspace of the phase space \(\{X^M, P_M\}\) is unitary, causal, and is physically sensible just like 1T-physics, but with more predictions than the traditional formulation of 1T-physics.

The Sp(2,\(R\)) gauge symmetry has three gauge parameters, \(\varepsilon^a(\tau), a = 0, 1, 2\), local on the worldline, and three corresponding generators, \(Q_a(X, P)\), that are functions of the phase space. Here \(Q_0\) is the compact generator (as in SU(2)) while \(Q_1, Q_2\) are non-compact. The equal-\(\tau\) Poisson brackets, \(\{X^M, P_N\} = \delta^M_N\), are invariant under the infinitesimal canonical transformations generated by each \(Q_a\), namely

\[
\delta \equiv \left\{ X^M, P_N \right\} = 0, \quad \delta_{\epsilon} \left\{ X^M, P_N \right\} = \left\{ X^M, Q_a \right\} \equiv \epsilon^a(\tau) \frac{\partial Q_a}{\partial P_N},
\]

\[
\delta_{\epsilon} \left\{ P_M, Q_a \right\} \equiv \epsilon^a(\tau) \frac{\partial Q_a}{\partial X^M}.
\]

(A1)

Note that no spacetime metric \(g_{MN}\) is involved in any of these expressions since \(X^M\) is defined with a contravariant index and \(P_M\) is defined with a covariant index. Therefore, this formalism applies in any curved spacetime.

Generally the \(Q_a(X, P)\) are non-linear functions of phase space [37, 55, 56] when the particle on the worldline moves in the presence of any set of background fields in \((D + 1) + 2\) dimensions, such as gravity, electromagnetism, high spin fields, etc. The \(Q_a\) are required to form the Lie algebra of Sp(2,\(R\)) under classical Poisson brackets,

\[
\left\{ Q_0, Q_1 \right\} = Q_2, \left\{ Q_2, Q_0 \right\} = Q_1, \left\{ Q_1, Q_2 \right\} = -Q_0.
\]

(A2)

15 By definition, the Poisson bracket between any two functions of phase space is \(\{F, G\} \equiv \frac{\partial F}{\partial X^M} \frac{\partial G}{\partial P_N} - \frac{\partial F}{\partial P_M} \frac{\partial G}{\partial X^M}\).
and similarly under quantum commutators (after quantum ordering the expressions for the \( Q_a (X, P) \)). The minus sign on the right hand side of the last commutator is the difference between SU(2) and Sp(2, R) = SL(2, R) = SU(1, 1) = SO(1, 2). The requirement of closure as a Lie algebra turns into a restriction on the background fields such that they must obey certain subsidiary conditions, covariant in \((D + 1) + 2\) dimensions, that follow from (A2). Then the physical content of the background fields amount to the same content of fields (of every spin) in one less time and one less space dimension, leaving no room for Kaluza–Klein-type additional degrees of freedom in those background fields [37, 56].

The general worldline Lagrangian for the dynamics of a single particle moving in any background field and subject to the Sp(2, R) gauge symmetry is given by [34, 37, 55]:

\[
L(\tau) = \frac{dX^M(\tau)}{d\tau} P_M(\tau) - A^a(\tau) Q_a(X(\tau), P(\tau)) - H(X(\tau), P(\tau)), \quad (A3)
\]

where \( A^a (\tau) \) is the gauge field for Sp(2, R). The infinitesimal gauge transformations with local gauge parameters \( \varepsilon^a (\tau) \) are based on the canonical transformations in (A1) and the Yang–Mills-type transformation of the gauge field

\[
\delta X^M = \varepsilon^a \frac{\partial Q_a}{\partial P_M}, \quad \delta P_M = -\varepsilon^a \frac{\partial Q_a}{\partial X^M}, \quad \delta \eta = \frac{d\varepsilon^a}{d\tau} + \eta^{ab} \varepsilon_{bc} A^c \varepsilon^d. \quad (A4)
\]

where \( \eta^{ab} \) is the Killing metric of Sp(2, R), \( \varepsilon_{abc} \) is the Levi-Civita symbol and the combinations \( \eta^{ab} \varepsilon_{bcd} \) amount to the structure constants of the Lie algebra in equation (A2). Under these gauge transformations the Lagrangian transforms to a total \( \tau \)-derivative [55]

\[
\delta \tau L(\tau) = \frac{d}{d\tau} \left( \varepsilon^a(\tau) \frac{\partial L}{\partial P_M(\tau)} P_M(\tau) - \varepsilon^a(\tau) Q_a(X(\tau), P(\tau)) \right), \quad (A5)
\]

provided the Hamiltonian \( H \) is gauge-invariant, which means it commutes with the generators under Poisson brackets, \( \{H, Q_a\} = 0 \). Even when the Hamiltonian is zero the theory based on equation (A3) is extremely rich in physical content. Therefore, in almost all discussions of 2T-physics in the literature so far, including in the present paper, the Hamiltonian is chosen to be zero, \( H = 0 \).

When the \((D + 1) + 2\) -dimensional spacetime is flat and there are no background fields, the three Sp(2, R) real (or Hermitian) generators are rearranged to simple expressions,

\[
Q_a \rightarrow ((Q_0 - Q_1), (Q_0 + Q_1), Q_2) = \left( \frac{X \cdot X}{2}, -\frac{P \cdot P}{2}, \frac{X \cdot P}{2} \right), \quad (A6)
\]

where a flat spacetime metric \( \eta_{MN} \) with signature \((D + 1), 2\) is used for the dot products. This is how SO(D + 1, 2) becomes relevant as a global symmetry of the 2T-physics action (A3). Note that under Poisson brackets these quadratic expressions of phase space close to the Lie algebra of Sp(2, R) as required in (A2). The canonical transformations in equation (A1) and (A4) reduce to a linear transformation on the phase space that treats \((X^M, P^M)\) as a collection of Sp(2, R) doublets, one for every spacetime direction \( M \). The finite (as opposed to infinitesimal) Sp(2, R) gauge transformation is then linear

\[
\begin{pmatrix} X^M(\tau) \\ P_M(\tau) \end{pmatrix} \rightarrow \exp \left( \begin{pmatrix} \varepsilon^2(\tau) & \varepsilon^1(\tau) + \varepsilon^0(\tau) \\ \varepsilon^1(\tau) - \varepsilon^0(\tau) & -\varepsilon^2(\tau) \end{pmatrix} \right) \begin{pmatrix} X^M(\tau) \\ P_M(\tau) \end{pmatrix}. \quad (A7)
\]

So in this special case of linear Sp(2, R) transformations, Sp(2, R) is the same as a \( 2 \times 2 \) SL(2, R) matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with unit determinant. Note that the \( \varepsilon_0 \) transformation is compact and as a
subgroup it is a local SO(2) transformation. To treat \((X^M, P^M)\) as a Sp(2, R) doublet as well as an SO\((D + 1, 2)\) vector, the \(M\) index is raised for the momentum, \(P^M = \eta^{MN} P_N\), by using the inverse metric \(\eta^{MN}\) that is introduced in this flat background. In this form it is evident that the Sp(2, R) gauge transformation commutes with the global SO\((D + 1, 2)\) Lorentz-type target spacetime transformations.

In the flat background, the Sp(2, R) generators in (A6), as well as the corresponding Lagrangian (A3), are symmetric under linear SO\((D + 1, 2)\) Lorentz transformations of the elementary degrees of freedom \((X^M, P^M)\). Using Noether’s theorem, the generators of the conserved SO\((D + 1, 2)\) global symmetry are constructed and verified that they commute with the \(Q_a\):

\[
Q_a \equiv (X^2, P^2, X \cdot P), \quad L^{MN} = (X^M p^N - X^N p^M), \quad [Q_a, L^{MN}] = 0. \tag{A8}
\]

Since the Sp(2, R) generators are SO\((D + 1, 2)\)-invariant dot products, they had to commute with the \(L^{MN}\) (classically using Poisson brackets, and quantum mechanically using quantum commutators based on the fundamental commutator, \([X^M, P^N] = i\eta^{MN}\)). This statement also means that the \(L^{MN}\) are Sp(2, R) gauge invariants since they commute with the \(Q_a\). The gauge invariant sector of phase space is identified as the observables \(F\) (functions of phase space) that commute with the Sp(2, R) generators, \([Q_a, F(X, P)] = 0\). All gauge invariants are all possible functions of the \(L^{MN}\). Hence, for the flat background case, these are all the Sp(2, R) gauge-invariant physical observables:

All physical observables in flat background: \(F(L^{MN})\). \tag{A9}

These functions \(F(L^{MN})\) need not be SO\((D + 1, 2)\)-invariant. Since we have identified the gauge invariants, if one wishes, one may add to the Lagrangian (A3) any Hamiltonian, \(H(X, P) = H(L^{MN})\), as long as it is any function of the \(L^{MN}\). Such a Hamiltonian may break the global SO\((D + 1, 2)\) symmetry without destroying the essential Sp(2, R) gauge symmetry. Even with a broken global SO\((D + 1, 2)\) there still remains an underlying SO\((D + 1, 2)\) group structure that can be used to keep track of the hidden higher-dimensional nature of all related physics. In most of the 2T-physics literature the discussion has concentrated on the case of a zero Hamiltonian and unbroken SO\((D + 1, 2)\).

The equation of motion derived from the Lagrangian, by minimizing with respect to the gauge field, \(\partial L / \partial A^\tau = 0\), demands the constraints \(Q_a = 0\). This defines the physical space as being the gauge invariants for which the gauge generators must vanish. In the flat background (A6) this restricts the classical phase space to only the solutions of the simple constraints

If flat background: \(X^2 = 0, P^2 = 0, X \cdot P = 0.\) \tag{A10}

The vanishing of the generators on shell is the simple statement that the physical subspace of the phase space that obeys these equations is gauge invariant. Here is where the reader can see why two times are required for non-trivial physical content in the solution of these constraints. If the flat background metric \(\eta_{MN}\) were purely Euclidean (no timelike dimensions in target space, but there still is the evolution parameter \(\tau\)), the only solution is \(X^M(\tau) = P^M(\tau) = 0\), which is zero physics content. If the flat background metric \(\eta_{MN}\) were Minkowski with only one timelike direction in target space, then the only solution would be that \(X^M(\tau)\) should be parallel to \(P^M(\tau)\) and both lightlike; but this has zero angular momentum \(L^{MN} = 0\), which implies there are no non-trivial gauge-invariant \(F(L^{MN})\); so again no physical content. If the flat background metric \(\eta_{MN}\) contained three or more timelike dimensions, then the gauge-invariant sector of the theory would violate causality and also have ghosts (negative norm states in the quantum
Invariantsubset of quantum states, \( \text{Sp}(2, R) \) is insufficient gauge symmetry to remove them. So less than two times and more than two times are eliminated as unphysical theories. Therefore the \( \text{Sp}(2, R) \) gauge symmetry is the fundamental underlying principle that requires two timelike directions in target spacetime,\(^{16}\) no less and no more, in order to have unitarily and causally sensible non-trivial physical content.\(^{17}\)

When the metric has just two timelike dimensions, no less and no more, the gauge-invariant sector of the theory is causal and has no ghosts. When the background is not flat one can similarly construct a parallel argument. A more general reasoning confirms this conclusion: the signature of the gauge parameters \( \varepsilon^\alpha \) is such that the non-compact parameters \( \varepsilon^1,2 \) are timelike and the compact parameter \( \varepsilon^0 \) is spacelike. This means that \( \varepsilon^\alpha (\tau) \), together with the constraints \( Q_a = 0 \), can remove from the \((D + 1)\)-dimensional phase space \( \{X^\mu, P_\mu\} \) precisely one space and two timelike degrees of freedom, leaving behind only \( D \) independent spacelike physical degrees of freedom. This fully gauge-fixed spacelike phase space configuration has no ghosts and is causal (evolving with \( \tau \)), just as in ordinary non-relativistic or relativistic 1T-physics. This is why two times in target space, no less and no more, amounting to one extra space and one extra time dimension (compared to a 1T-worldline theory with \( \tau \)-reparametrization gauge symmetry) is predicted by the larger gauge symmetry \( \text{Sp}(2, R) \).

In a covariant quantization formalism (without making gauge choices to solve the constraints \((A10)\)) the gauge transformations already vanish for the observables identified above as \( F \{L^\alpha\} \) since \( [Q_a, L^\alpha] = 0 \), hence these observables are gauge-invariant. To implement the vanishing of the \( Q_a \) in the covariantly quantized theory, one requires gauge-invariant states, namely \( Q_a |\text{gauge invariant}\rangle = 0 \). To identify these physical states, one begins by classifying all the quantum states (gauge-invariant and non-invariant) by the commuting symmetries \( \text{Sp}(2, R) \otimes \text{SO}(d, 2) \) of the action \((A3)\), where \( d = D + 1 \) refers to the spatial dimensions,

\[
|\text{all states}\rangle = |\text{Sp}(2, R) \times \text{SO}(d, 2)\rangle. \tag{A11}
\]

All possible unitary representations of both noncompact symmetries may appear. The gauge-invariant subset of quantum states, \( Q_a |\text{Sp}(2, R) \times \text{SO}(d, 2)\rangle = 0 \), can only be the unique singlet of \( \text{Sp}(2, R) \), so this nails down the physical states as being singlets under \( \text{Sp}(2, R) \) and unitary representations under \( \text{SO}(d, 2) \). The question still remains: Which unitary representations of \( \text{SO}(D + 1, 2) \)? This depends on the background fields from which the \( Q_a (X, P) \) are constructed. In the flat background there is a definite prediction.

In the flat background the physical state condition takes the form

\[
\text{If flat background: } X \cdot X |\psi_{\text{phys}}\rangle = 0, \quad P \cdot P |\psi_{\text{phys}}\rangle = 0, \quad (X \cdot P + P \cdot X) |\psi_{\text{phys}}\rangle = 0. \tag{A12}
\]

This is a conceptually fundamental viewpoint because it attributes the signature and dimension of spacetime to emanate from the properties of the \( \text{Sp}(2, R) \) gauge symmetry. It may be compared to the statement that the \( \text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \) gauge symmetry is the underlying fundamental principle for the existence and nature of the electroweak and QCD forces, because it is the gauge symmetry that requires the introduction of the Yang–Mills gauge fields as the carriers of forces with the patterns of interactions in the Standard Model of particles and fields. Similar statements also apply to the gauge symmetries in general relativity and string theory. Extending the \( \text{Sp}(2, R) \) gauge symmetry concept to field theory has in fact demanded that all known physics be (and in fact is) formulated with an additional space and an additional time dimension.

The question arises of whether a higher gauge symmetry could allow more timelike dimensions. This has been tried many times over the past 20 years but it has never worked out in the sense that either the physical content is empty (too much gauge symmetry, too many constraints) or the structure of the noncompact gauge group (signature of the gauge parameters \( \varepsilon^\alpha \)) is incompatible with the signature structure of spacetime to remove all the ghosts. It seems very likely that there is a no-go theorem, but such a theorem has not been conclusively proven.
These quantum states must now automatically fall into irreducible representations of the global symmetry \( SO(D + 1, 2) \). Therefore all gauge-invariant quantum physics derived from the Lagrangian (A3), both quantum observables \( F(L^{MN}) \) as well quantum states \( |\psi_{\text{phys}}\rangle \), are automatically predicted to have an underlying \( SO(D + 1, 2) \) structure that reveals the underlying \( (D + 1) + 2 \) dimensions, even if the \( SO(D + 1, 2) \) global symmetry may be broken by adding some \( SO(D + 1, 2) \) non-invariant Hamiltonian in (A3). This provides the inescapable prediction of 2T-physics regarding the presence of the underlying \( (D + 1) + 2 \) dimensions. This aspect may remain hidden in the usual formalism of 1T-physics, but specific predictions made by 2T-physics become the practical tool for uncovering the hidden \( (D + 1) + 2 \) structure in 1T-physics.

What are the physical unitary representations of \( SO(D + 1, 2) \) that are predicted for the system (A3)? For this we should compute the predicted Casimir eigenvalues. At the classical level (ignoring quantum ordering) the quadratic Casimir is

\[
C_{2} = \frac{1}{2} L^{MN} L_{MN} = (X^2 p^2 - (X \cdot P)^2) = 0,
\]

where the vanishing occurs only in the physical sector that satisfies the constraints (A10). Similarly, all Casimir operators,

\[
C_k = \frac{(D-k)^2}{k!} (L_{M_1 M_2}^{M_1 M_2} L_{M_2 M_3}^{M_2 M_3} \cdots L_{M_{n-1} M_n}^{M_{n-1} M_n}),
\]

vanish in the physical sector of the phase space, in the classical theory. However, in the quantum theory, by implementing the quantum constraints (A12) while respecting the ordering of the quantum operators \( (X, P) \) as they appear in \( C_k \), one finds that the Casimir operators are diagonal on the physical states, \( C_k |\psi_{\text{phys}}\rangle = \lambda_k |\psi_{\text{phys}}\rangle \), where the \( \lambda_k \) are definite non-trivial eigenvalues given by\(^{18}\) (see equation (9) in [35] and equation (2.9) in [57])

\[
\begin{align*}
C_2 &= \left( 1 - \frac{(D+1)^2}{4} \right), \\
C_3 &= \frac{D+1}{3!} \left( 1 - \frac{(D+1)^2}{4} \right), \\
C_4 &= \frac{1}{4!} \left( 1 - \frac{(D+1)^2}{4} \right) \left( 1 + \frac{3(D+1)^2}{4} \right) \cdots.
\end{align*}
\]

This is just a single infinite-dimensional unitary representation which is identified as the ‘singleton’ representation of \( SO(D + 1, 2) \). So all gauge-invariant physical quantum states of the system must be assembled into the unique singleton representation with the specific \( C_k \) quantum numbers given above.

This is the result of ‘covariant quantization’ (without choosing any gauges) of the 2T-physics particle given by the Lagrangian (A3) with \( H = 0 \) and a flat background. The full set of quantum states corresponds to the states of the infinite-dimensional unitary singleton representation of \( SO(D + 1, 2) \) taken in any basis. That is, along with the Casimir operators, a simultaneously diagonalizable subset of operators constructed from \( L^{MN} \) that defines the basis is not specified, so any such basis will do. There clearly are an infinite set of combinations \( F (L^{MN}) \) of simultaneously diagonalizable observables, so there are an infinite set of quantum bases. Adding in (A3) a nontrivial gauge-invariant Hamiltonian \( H (L^{MN}) \neq 0 \), or choosing a gauge for the \( (X^M, P^M) \) that favors some orientation within \( SO(D + 1, 2) \), would influence the choice of basis, but this would not change the singleton representation that is already fixed by the \( C_k \).

The same gauge theory may also be treated by working in specific gauge choices and solving the constraints \( Q_n = 0 \) explicitly. Then one finds an infinite number of solutions of the

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\(^{18}\) The reader is alerted that the definition of \( C_k \) given here may differ from previous 2T-physics papers by inessential overall normalization factors for the cases \( n \geq 3 \). Moreover, in the broader literature, the Casimirs for \( n \geq 3 \) may in some definitions amount to a linear combination of our \( C_k \).
constraints in equation (A10) some of which are discussed explicitly in several papers [35, 37, 54]. These solutions are called ‘shadows’. The shadows are gauge-fixed versions of the phase space \((X^M, P^M)\) that solve the constraints, \(X^2_{\text{fixed}} = P^2_{\text{fixed}} = X_{\text{fixed}} \cdot P_{\text{fixed}} = 0\). When only two out of three gauge choices are made, and two out of three constraints are solved explicitly, each solution, \((X^M_{(k)}, P^M_{(k)})\) labelled by \(k = 1, 2, 3, \ldots\), is parametrized in terms of a 1T-sub-phase-space, \((r_{(k)}, p_{(k)}, t_{(k)}, h_{(k)})\), in one less space and one less time dimension,

\[
\text{Shadows: } X^M_{(k)} (r_{(k)}, p_{(k)}, t_{(k)}, h_{(k)}) , \quad P^M_{(k)} (r_{(k)}, p_{(k)}, t_{(k)}, h_{(k)}) . \quad (A15)
\]

Here \((t_{(k)} (\tau), h_{(k)} (\tau))\) is a temporal canonical pair at the same level as the \(D\) spatial canonical pairs \((r_{(k)} (\tau), p_{(k)} (\tau))\). An infinite set of shadows exist due to the fact that \(D\) spatial plus 1 temporal phase space can be embedded in \((D + 1) + 2\)-dimensional phase space in an infinite number of non-linear geometric or algebraic ways. At this stage the gauge-fixed Lagrangian (A3) takes the form of a particle on the worldline in 1T-physics with a remaining gauge symmetry and a remaining constraint

\[
L^2_{\text{fixed}} = \left[ x^\mu_{(k)} p_{\mu(k)} - A (\tau) Q (x_{(k)}, p_{(k)}) \right] , \quad x^\mu_{(k)} \equiv (t_{(k)}, r_{(k)}), \quad p_{\mu(k)} \equiv (h_{(k)}, p_{(k)}) . \quad (A16)
\]

A total \(\tau\)-derivative \(\frac{\partial}{\partial \tau} (x^\mu, p_\mu, \tau)\) is dropped from \(L^2_{\text{fixed}}\) since it does not affect the physics (see [37] for a nontrivial role of this total derivative for building canonical transformations among shadows). When the remaining third gauge choice is made by taking \(t_{(k)} (\tau) = \tau\), the solution of the third constraint, \(Q (x, p) = 0\), yields an expression for an \(h_{(k)}\) that depends on the remaining dynamical spatial degrees of freedom, \(h_{(k)} = H_k (r_{(k)}, p_{(k)}, t_{(k)})\). Then the gauge-fixed form of the original action (A3) or of (A16) for the \(k\)th shadow takes the standard form in 1T-physics:

\[
L^2_{\text{fixed}} = r_{(k)} \cdot p_{(k)} - H_k (r_{(k)}, p_{(k)}, t_{(k)}) . \quad (A17)
\]

Here the emerging Hamiltonians \(H_k (r_{(k)}, p_{(k)}, t_{(k)})\) for the shadows are different for each distinct solution labelled by \(k\). Examples of shadows that emerge from the Lagrangian (A3) with a flat background and \(H = 0\) include: the free massless relativistic particle, free massive non-relativistic particle, Hatom, HOsc, particle moving in various curved backgrounds including the expanding Universe, twistor equivalent of all these, and many others. All of these systems (an infinite set, but only a few studied) are united by the fact that they obey the same equations in the higher dimensions, namely \(X^2 = P^2 = X \cdot P = 0\), that’s all! In the lower \(D\) dimensions these shadows are all duals to each other; since they are gauge equivalent, each shadow in \(D\)-space and 1-time dimension holographically captures all the gauge-invariant physics content available in the \((D + 1)\)-space and 2-time dimensions, as described below. The physics interpretation for 1T observers, like us humans, is different for each shadow, because the gauge choice of time and Hamiltonian, \((t_{(k)} = \tau, h_{(k)} = H_k = (r_{(k)}, p_{(k)}, \tau))\), as embedded in \((D + 1) + 2\) dimensions creates many different 1T observational perspectives of the same phenomena that occur in \((D + 1) + 2\) dimensions.

It is now evident that the shadow Lagrangian \(L_{(k)}\) in (A17) has a hidden \(\text{SO}(D + 1, 2)\) symmetry since it is merely a gauge-fixed form of the original action (A3) that has the global \(\text{SO}(D + 1, 2)\) symmetry that commutes with the gauge symmetry. The transformation laws for the hidden symmetry are given by computing equal-\(\tau\) Poisson brackets of \((r_{(k)}, p_{(k)})\) with the \(L_{MN}\) evaluated for that shadow,
Then the shadow Lagrangian transforms to a total derivative, so the action, \( \int d\tau L_{\text{mixed}} (\tau) \), is SO\( (D + 1, 2) \) invariant. For examples, see [37, 58]. Applying Noether’s theorem by starting from the transformation rules above (without being informed that they are assembled into equation (A14) (for examples of such properties of the shadows see [35, 58]).

After quantization, quantum ordering for each shadow must be performed such that the \( L_{MN}^{(i)} (r_{(i)}, p_{(i)}, \tau) \) close correctly, under equal \( \tau \) quantum commutators based on \( [r_{(i)}, p_{(i)}] = i\delta^{ij} \), to form the SO\( (D + 1, 2) \) Lie algebra, and yield the same Casimir eigenvalues given in equation (A14) (for examples of such properties of the shadows see [35, 58]).

The \( L_{MN} \) are gauge-invariant because they commute with the \( \text{SL}(2, R) \) generators as seen in (A8). So, each \( L_{MN} \) is independent of the shadow, even when it is evaluated in terms of a given shadow \( (X_{(i)}, P_{(i)}) \). The physical gauge-invariant observables \( F (L_{MN}) \) in (A9) are then identified as functions of a smaller \( D \)-dimensional Euclidean phase space for each shadow, \( F (L_{MN}^{(i)}) \). An infinite set of duality relations between gauge-invariant observables of shadow \( k_1 \) and shadow \( k_2 \) are predicted by evaluating any given function of the \( L_{MN} \) in those two different shadows.

Dualities for every function: \( F (L_{MN}^{(i)}) = F (L_{MN}^{(j)}) = F (L_{MN}) \). (A19)

These are an infinite set of measurable predictions from 2T-physics for the dynamics of 1T-physics. From these gauge-invariant predictions we can extract the canonical transformations for the phase spaces \( (r_{(i)}, p_{(i)}, t_{(i)}, H_{(i)}) \leftrightarrow (r_{(i)}, p_{(i)}, t_{(i)}, H_{(i)}) \). For examples, see [37].

For spinning particles of spin \( s \), fermions \( \psi_i^M \) with \( i = 1, 2, \ldots, 2s \), are added to the bosonic phase space \( (\psi_i^M, X^M, P_M) \), and then the gauge group is OSp\( (2s|2) \) that contains the previous \( \text{Sp}(2, R) \) [44]. Fermions can also be added via spacetime supersymmetry ([59] and section 3.1 in [60]). Repeating the same reasoning that leads to the shadows, now we obtain 1T-physics that includes spin, and the associated predictions for the corresponding shadows. In particular one bit of information relevant for the current paper is the SO\( (D + 1, 2) \) Casimir eigenvalues analogous to (A14) that unite all the shadows in the same unitary representation. For spin \( s \) it is given by [51]

\[
C_{2s} = \frac{s - 1}{4} ((D + 1) + 2s)(D + 1) + 2s - 2), C_{3s}, \ldots \quad (A20)
\]

The resulting dualities are far richer than just Hatom \( \leftrightarrow \text{HOsc} \) and includes many shadows with spin that fit in the same spinning representation of SO\( (D + 1, 2) \) with the \( C_{2s} \) given above. The spinning 1T-physics shadows include the massless spin 1/2 particle (Dirac equation when quantized [44]) and the dyonic-Hatom whose properties are outlined in section 2.2. Moreover, instead of flat backgrounds, the generalized version of 2T-physics on the worldline includes background fields on which the phase space \( (\psi_i^M, X^M, P_M) \) propagates [56]. Although there are an infinite set of shadows, only some of the shadows, including backgrounds that provide interactions, have been explicitly explored in the language of canonical transformations in the 1T-physics framework [37]. Little use has been made of the physics predictions provided by these classical or quantum dualities. These can be useful both for experimental predictions to verify aspects of the hidden dimensions as well as for trying to solve difficult problems that may be more tractable in some dual shadow version [37].
In addition, these concepts have been generalized to field theory \cite{51, 54, 55} including the Standard Model in \((4 + 2)\) dimensions \cite{52}, gravity \cite{53}, their coupling to each other, and their supersymmetric and higher-dimensional generalizations. So, 2T-physics in \((4 + 2)\) dimensions presents all the physics we know that actually works as a shadow in \((3 + 1)\) dimensions. The emerging theory in the ‘conformal shadow’ is closely related to the familiar \((3 + 1)\)-dimensional Standard Model coupled to gravity but yields an improved standard theory \cite{61} with a predicted local scale invariance (Weyl symmetry). The familiar form of Weyl transformation in \((3 + 1)\) is shown to come from local general coordinate transformations in the extra \(1 + 1\) dimensions and is present because of the local \(\text{Sp}(2, \mathbb{R})\) that acts on the phase space \((X^M, P^M)\). These are manifestations of the underlying \(4 + 2\) dimensions in forms that are recognizable in \(3 + 1\)-dimensional theories of fundamental nature. Due to the Weyl symmetry the improved theory \cite{61} is geodesically complete at cosmological and black hole-type singularities. It also shows how all dimensionful parameters in the Standard Model of particle physics (Newton constant, Higgs vacuum, cosmological constant) come from a single source that spontaneously breaks the Weyl symmetry \cite{61, 62}, thus opening a window to the extra dimensions. These consequences of the improved standard theory are consistent with all we know while providing new insight not available before for the source of mass and its relation to the symmetries of the underlying \(4 + 2\) dimensions.

2T-physics also potentially predicts the existence of many field theories that are dual to the Standard Model and general relativity which may be put to use as computational tools that take advantage of dualities in field theory \cite{63}. A small part of this duality was used to explore the geodesic completeness of the improved standard theory \cite{61} based on its Weyl symmetry and explore the new antigavity region behind the cosmological singularity \cite{62, 64, 65} as well as beyond the black hole singularity \cite{66}. Little effort has been made to expand on such predictions of 2T-physics and the more general dualities in field theory. Hence a lot more research needs to be dedicated to further exploration in these directions.

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