A class of Randers metrics of scalar flag curvature

Xinyue Cheng, Li Yin, Tingting Li

Abstract

One of the most important problems in Finsler geometry is to classify Finsler metrics of scalar flag curvature. In this paper, we study the classification problem of Randers metrics of scalar flag curvature. Under the condition that $\beta$ is a Killing 1-form, we obtain some important necessary conditions for Randers metrics to be of scalar flag curvature.

Keywords: Randers metric, flag curvature, Killing 1-form, S-curvature, mean Landsberg curvature.

MR(2000) Subject Classification: 53B40, 53C60

1 Introduction

In Finsler geometry, the flag curvature is the natural analogue of sectional curvature in Riemannian geometry and is an important Riemannian geometric quantity. The flag curvature characterizes the shape of Finsler spaces, which was firstly introduced by L. Berwald ([3, 4]). For a Finsler manifold $(M, F)$, the flag curvature $K = K(x, y, P)$ of $F$ is a function of “flag” $P \in T_x M$ and “flagpole” $y \in T_x M$ at $x$ with $y \in P$. A Finsler metric $F$ is said to be of scalar flag curvature if for any non-zero vector $y \in T_x M$, $K = K(x, y)$ is independent of $P$ containing $y \in T_x M$ (that is, the flag curvature is just a scalar function on the slit tangent bundle). In particular, a Finsler metric $F$ is said to be of weakly isotropic flag curvature if its flag curvature is a scalar function on $TM$ in the form $K = \frac{c(y) - c(x)}{f(x, y)} + \sigma(x)$, where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on $M$. A Finsler metric $F$ is of constant flag curvature if $K =$ constant. In 1929, Berwald proved that a projectively flat Finsler metric must be of scalar flag curvature ([5]). However, we can find infinitely many Finsler metrics of scalar flag curvature, which are not projective flat. Therefore, one of the most important problems in Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature. As we known, every two-dimensional Finsler metric is of scalar flag curvature. It is then a natural problem to understand Finsler metrics of scalar flag curvature in dimension $n \geq 3$.

Randers metrics are the simplest non-Riemannian Finsler metrics that can be expressed in the following special form

$$F = \alpha + \beta,$$
where $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a non-zero 1-form on $M$ such that the norm of $\beta$ with respect to $\alpha$ satisfies that $||\beta||_\alpha < 1$. Randers metrics were introduced by physicist G. Randers in 1941 from the standpoint of general relativity [14]. Later on, these metrics were used in the theory of the electron microscope by R. S. Ingarden in 1957, who first named them Randers metrics. Randers metrics form an important class of Finsler metrics with a strong presence in both the theory and applications of Finsler geometry, and studying Randers metrics is an important step to understand general Finsler metrics. It is an important and fundamental problem in Finsler geometry to characterize and classify Randers metrics of scalar flag curvature. Matsumoto and Shimada characterized Randers metrics of constant flag curvature [13]. In 2003, Z. Shen classified locally projectively flat Randers metrics of constant flag curvature [15]. Further, D. Bao, Z. Shen and C. Robles showed that a Randers metric $F$ with navigation data $(h, W)$ is of constant flag curvature if and only if $h$ is a Riemannian metric of constant sectional curvature and $W$ is homothetic with respect to $h$. Based on this, they classified Randers metrics of constant flag curvature via navigation data [1]. Furthermore, the first author, X. Mo and Z. Shen classified projectively flat Randers metrics with isotropic S-curvature [2]. In this case, the metrics must be of weakly isotropic flag curvature $K = \frac{2x \cdot \nabla^\alpha }{F(x, y)} + \sigma(x)$. Later, the first author and Z. Shen classified completely Randers metrics of weakly isotropic flag curvature on the manifolds of dimension $n \geq 3$ [9].

For the current research on the classification problem of Randers metrics of scalar flag curvature, it is very important to find and construct some new examples. B. Chen and L. Zhao constructed a new class of Randers metrics of scalar flag curvature [6]. Further, Z. Shen and Q. Xia proved firstly the following theorem:

**Theorem A** ([17]) Let $\bar{F} = \alpha + \bar{\beta}$ be a Randers metric of weakly isotropic flag curvature on a manifold $M$ with navigation data $(\bar{h}, \bar{W})$ and $V$ be a homothetic vector field on $(M, \bar{F})$ with dilation $c$. Let $F = \alpha + \beta$ be a Randers metric on $M$ such that $\eta := \beta - \bar{\beta}$ is closed. Assume that $\eta$ satisfies the following

$$V^i \eta_j, i + \eta^i V_{ij} = 2 cn_j,$$

where “;” denotes the covariant derivative with respect to the Levi-Civita connection of $\bar{h}$ and $\bar{V}_i := \bar{h}_{ij} \bar{V}^j$, $\eta^i := \bar{h}^{ij} \eta_j$. Then

- $V$ is also homothetic vector field with respect to $F$.
- With this $V$ such that $F(x, -\bar{V}_x) < 1$, define $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ by

$$F \left( x, \frac{y}{F} - V \right) = 1. \quad (1.1)$$

Then $\tilde{F}$ is of scalar flag curvature and its flag curvature is related to that of $F$ by

$$K_{\tilde{F}}(x, y) = K_F(x, y) - c^2,$$

2
where $\tilde{y} := y - \tilde{F}(x, y)V$.

Then, based on Theorem A, they found some new examples of Randers metrics of scalar flag curvature ([17]).

Soon after, Q. Xia studied Randers metrics of Douglas type and proved the following theorem:

**Theorem B** ([18]) Let $F = \alpha + \beta$ be a Randers metric of Douglas type on an $n$-dimensional manifold $M (n \geq 3)$ and $V$ be a conformal vector field on $(M, F)$ with a conformal factor $c(x)$. Let $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric defined from $(F, V)$ given by (1.1). Then each two of the following imply the third one:

1. $F$ is of scalar flag curvature.
2. $\tilde{F}$ is of scalar flag curvature.
3. $V$ is homothetic or $\beta = 0$.

Then, by using a special Randers metric $F = |y| + \langle x, y \rangle$ and a conformal vector field $V = xQ$ with $|xQ| < 1$ on $\mathbb{R}^n$, where $Q$ is an antisymmetric matrix, Q. Xia constructed a new Randers metric $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ by solving (1.1) such that $\tilde{F}$ is of scalar flag curvature but $\tilde{F}$ is not locally projectively flat. In this case, $V$ is actually a Killing vector field with respect to $F$ ([18]).

However, although a significant progress has been made in studying Randers metrics of scalar flag curvature as mentioned above, it is still an important open problem in Finsler geometry to characterize and classify Randers metrics of scalar flag curvature.

For a 1-form $\beta = b_i y^i$ on $M$, we say that $\beta$ is a Killing 1-form with respect to a Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ if it satisfies

$$b_{ij} + b_{ji} = 0,$$

where “;” is the covariant derivative with respect to Levi-Civita connection of $\alpha$. In this paper, we study classification problem of Randers metrics of scalar flag curvature. Under the condition that $\beta$ is a Killing 1-form, we obtain some necessary conditions for Randers metrics to be of scalar flag curvature. Firstly, we have the following theorem.

**Theorem 1.1** Let $F = \alpha + \beta$ be a Randers metric on an $n$-dimensional manifold $M$. Suppose that $\beta$ is a Killing 1-form with respect to $\alpha$. If $F$ is of scalar flag curvature $K = K(x, y)$, then $\alpha$ and $\beta$ satisfy the following equations

$$s^m \delta_{nm} = -(n - 1)b^{-2}c(x)\beta, \tag{1.2}$$

$$t_{00} + s_{0,0} = c(x)(\alpha^2 - b^{-2}\beta^2), \tag{1.3}$$

where $t_{ij} = s_{im}s^m_{\ j}$, $t_{00} = t_{i} y^i y^j$ and $b := ||\beta_x||_\alpha$ denotes the norm of $\beta$ with respect to $\alpha$ and $c = c(x)$ is a scalar function on $M$. 
Further, based on Theorem 1.1 and Proposition 4.1 below, we can obtain the following theorem.

**Theorem 1.2** Let $F = \alpha + \beta$ be a Randers metric on an $n$-dimensional manifold $M$. Suppose that $\beta$ is a Killing 1-form with respect to $\alpha$. If $F$ is of scalar flag curvature $K = K(x, y)$, then $\alpha$ and $\beta$ satisfy the following equation

$$
t_0 = \frac{n - 1}{n + 1} (\lambda + cb^{-2}) \beta, \quad (1.4)
$$

where $t_i = b^i t_{ji}$, $t_0 = t_i y^i$, $c = c(x)$ and $\lambda = \lambda(x)$ are scalar functions on $M$.

As an application of Theorem 1.1 and Theorem 1.2, we study Randers metric on $S^3$ constructed by D. Bao and Z. Shen in [2]. Bao-Shen have proved that this Randers metric is of constant flag curvature $K > 1$. We will show that this metric also satisfies (1.2), (1.3) and (1.4) (see Section 5). Besides, Theorem 1.1 and Theorem 1.2 tell us, if a Randers metric $F = \alpha + \beta$ with Killing 1-form $\beta$ does not satisfy one of (1.2), (1.3) and (1.4), then $F$ is not of scalar flag curvature.

### 2 Preliminaries

Let $F$ be a Finsler metric on an $n$-dimensional manifold $M$. The geodesics of $F = F(x, y)$ are characterized by

$$
\frac{d^2 x^i}{dt^2} + 2G^i(x(t), \frac{dx}{dt}(t)) = 0,
$$

where $G^i$ are the geodesic coefficients of $F$.

For any $x \in M$ and $y \in T_x M \setminus \{0\}$, the Riemann curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$
R^i_k = 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m} \frac{\partial G^m}{\partial y^k}.
$$

The trace of the Riemann curvature is Ricci curvature $\text{Ric}$, which is defined by

$$
\text{Ric} = R^m_m. \quad (2.1)
$$

Obviously, the Ricci curvature is a positive homogenous function of degree two in $y$. The Ricci tensor is defined by

$$
\text{Ric}_{ij} := \frac{1}{2} \text{Ric}_{y^i y^j}.
$$

The flag curvature of Finsler manifold $(M, F)$ is the function $K = K(x, y, P)$ of a two-dimensional plane called “flag” $P \subset T_x M$ and a “flagpole” $y \in P \setminus \{0\}$ defined by

$$
K(x, y, P) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - \left[g_y(u, y]\right]^2},
$$

where $g_y$ is the metric induced by $F$.
where $P = \text{Span}\{y, u\}$.

It is known that $F$ is of scalar flag curvature if and only if, in a standard local coordinate system we have (11)

$$R^i_k = K(x, y) \left(F^2 \delta^i_k - FF_{y^k y^i}\right),$$

which means

$$\text{Ric} = (n - 1)K(x, y)F^2.$$ (2.3)

In Finsler geometry, there are some important quantities which all vanish for Riemannian metrics. Hence they are said to be non-Riemannian. Let $\{b_i\}$ be a basis for $T_xM$ and $\{\omega^i\}$ be the basis for $T^*_xM$ dual to $\{b_i\}$. Define the Busemann-Hausdorff volume form by

$$dV_{BH} := \sigma_{BH}(x)\omega^1 \wedge \cdots \wedge \omega^n,$$

where

$$\sigma_{BH}(x) := \frac{\text{Vol}(B^n(1))}{\text{Vol}\{(y^i) \in \mathbb{R}^n|F(x, y^ib_i) < 1\}}.$$  

Here Vol\{\} denotes the Euclidean volume function on subsets in $\mathbb{R}^n$ and $B^n(1)$ denotes the unit ball in $\mathbb{R}^n$.

Define

$$\tau(x, y) := \ln \left[\frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma_{BH}(x)}\right].$$

$\tau = \tau(x, y)$ is well-defined, which is called the distortion of $F$. The distortion $\tau$ characterizes the geometry of tangent space $(T_xM, F_x)$. It is natural to study the rate of change of the distortion along geodesics. For a vector $y \in T_xM \setminus \{0\}$, let $\sigma = \sigma(x)$ be the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$. Put

$$S(x, y) := \frac{d}{dt} \left[\tau(\sigma(t), \dot{\sigma}(t)) \right]_{t=0}.$$  

Equivalently,

$$S(x, y) := \tau\mid_m(x, y)y^m,$$

where “$\mid_m$” denotes the horizontal covariant derivative with respect to Chern connection of $F$. $S = S(x, y)$ is called the $S$-curvature of Finsler metric $F$. The S-curvature $S(x, y)$ measures the rate of change of $(T_xM, F_x)$ in the direction $y \in T_xM$ ([11] [15]). In particular, $S = 0$ for Riemannian metrics.

There are other several important non-Riemannian quantities in Finsler geometry. Let

$$I_i(x, y) := \frac{\partial \tau}{\partial y^i}(x, y) = \frac{1}{2}g^{ik}(x, y)\frac{\partial g_{jk}}{\partial y^i}(x, y).$$

The tensor $I_y := I_i(x, y)dx^i$ is called the mean Cartan torsion of $F$. According to Deicke’s theorem, a Finsler metric $F$ is Riemannian if and only if $I_y = 0$. 


Another important non-Riemannian quantity is the Landsberg curvature $L = L_{ijk}(x, y)dx^i \otimes dy^j \otimes dx^k$ defined by

$$L_{ijk}(x, y) := -\frac{1}{2} FF_{ym} \frac{\partial^3 G^m}{\partial y^i \partial y^j \partial y^k}.$$  

A Finsler metric $F$ is called the Landsberg metric if the Landsberg curvature vanishes, i.e. $L = 0$. Further, let $J_k := g^{ij} L_{ijk}$. Then $J_k dx^k$ is called the mean Landsberg curvature. A Finsler metric $F$ is said to be weakly Landsbergian if $J = 0$.

Now we assume that Finsler metric $F$ is of scalar flag curvature $K = K(x, y)$. Firstly, we have the following identity ([7, 11])

$$S_{ijkl} y^m - S_{ij} = -\frac{n+1}{3} K_i F^2,$$  

where $K_i := K_{yi}$. Further, we have following important identity which connects the mean Cartan torsion $I_i$ and the mean Landsberg curvature $J_i$ with the flag curvature ([11])

$$J_{ijkl} y^m + K F^2 I_i = -\frac{n+1}{3} F^2 K_i.$$  

In the following, let us consider $(\alpha, \beta)$-metrics in the form $F = \alpha \phi(\beta/\alpha)$, where $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i y^i$ is a 1-form on a manifold. Let

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}),$$

$$r^i_j := a^{ik} r_{kj}, \quad s^i_j := a^{ik} s_{kj}, \quad r_j := b^i r_{ij}, \quad s_j := b^i s_{ij},$$

$$q_{ij} := r_{im} s^m_j, \quad t_{ij} := s_{im} s^m_j, \quad q_j := b^i q_{ij}, \quad t_j := b^i t_{ij},$$

where $b^i := a^{ij} b_j$, $(a^{ij})$ is the inverse of $(a_{ij})$ and “;” denotes the covariant derivative with respect to Levi-Civita connection of $\alpha$. We use $(a^{ij})$ and $(a_{ij})$ to raise or lower the indices of a tensor.

For an $(\alpha, \beta)$-metric $F = \alpha \phi(\beta/\alpha)$, $s = \beta/\alpha$, we have the formulas for the mean Landsberg curvature (12) and the mean Cartan torsion (8) of $F$ as follows

$$I_i = -\frac{\Phi(\phi - s \phi')}{2 \Delta \phi \alpha^2} h_i,$$  

$$J_i = -\frac{1}{2 \Delta \alpha^4} \left\{ \frac{2 \alpha^2}{b^2 - s^2} \left[ \frac{\Phi}{\Delta} + (n + 1)(Q + s Q') \right] (s_0 + r_0) h_i \right.$$

$$+ \frac{\alpha}{b^2 - s^2} \left[ \Psi_1 + s \frac{\Phi}{\Delta} \right] (r_0 - 2 \alpha Q s_0) h_i + \alpha \left[ -\alpha Q' s_0 h_i + \alpha Q (\alpha^2 s_i - y_i s_0) \right.$$

$$+ \alpha^2 s_0 + \alpha^2 (r_0 - 2 \alpha Q s_i) - (r_0 - 2 \alpha Q s_0) y_i \frac{\Phi}{\Delta} \right\}. \tag{2.7}$$
Besides, we also have (12)
\[ \bar{J} := J_i b^i = -\frac{1}{2\Delta a^2} \left\{ \Psi_1 (r_{00} - 2\alpha Q s_0) + \alpha \Psi_2 (r_0 + s_0) \right\}, \quad (2.8) \]
where
\[
\begin{align*}
    h_i &= a b_i - s y_i, \\
    Q &= \frac{\phi'}{\phi - s \phi'}, \\
    \Delta &= 1 + s Q + (b^2 - s^2) Q', \\
    \Phi &= -(Q - s Q') \left\{ n \Delta + 1 + s Q \right\} - (b^2 - s^2) (1 + s Q) Q'', \\
    \Psi_1 &= \sqrt{b^2 - s^2} \frac{\Delta}{\Delta^2} \left[ \frac{\sqrt{b^2 - s^2} \Phi}{\Delta^2} \right]' , \\
    \Psi_2 &= 2(n + 1) (Q - s Q') + 3 \frac{\Phi}{\Delta} .
\end{align*}
\]

3 Some properties of Randers metrics of scalar flag curvature

Let \( F = \alpha + \beta \) be a Randers metric on an \( n \)-dimensional manifold \( M \) and \( \beta \) be a Killing 1-form (i.e., \( r_{ij} = 0 \)) with respect to \( \alpha \). In this case, \( \phi = 1 + s \) and we have
\[
\begin{align*}
    Q &= 1, \quad \Delta = 1 + s, \quad \Phi = -(n + 1)(1 + s), \\
    \Psi_1 &= \frac{(n + 1)(2s + s^2 + b^2)}{2(1 + s)}, \quad \Psi_2 = -(n + 1), \\
    \Psi_1' &= \frac{(n + 1)(2 + 2s - b^2 + s^2)}{2(1 + s)^2}, \\
    \Psi_1 + s \frac{\Phi}{\Delta} &= \Psi_1 - s(n + 1) = \frac{(n + 1)(b^2 - s^2)}{2(1 + s)}, \\
    2 \Psi_1 - \Psi_2 &= 2 \Psi_1 + (n + 1) = \frac{(n + 1)(1 + 3s + b^2 + s^2)}{1 + s} .
\end{align*}
\]

From these and (2.0)-(2.8), we get
\[
J_i = \frac{n + 1}{2\alpha^2(1 + s)} \left\{ s_0 h_i + \alpha(1 + s) s_0 - \alpha^2 s_i + s_0 y_i \right\}, \quad (3.1)
\]
and
\[
I_i = \frac{n + 1}{2\alpha^2(1 + s)} h_i, \quad \bar{J} = \frac{n + 1}{2\alpha(1 + s)^2} (1 + 3s + b^2 + s^2) s_0 . \quad (3.2)
\]

In the following, we always assume that Randers metric \( F = \alpha + \beta \) is of scalar flag curvature. In this case, by the formulas (4.15) and (5.30) for Ricci curvature of \( F \) in [10], we have
\[
\text{Ric} = 2\alpha s_{\alpha m}^m + (n - 1) \left( \kappa \alpha^2 + t_{00} - \frac{2\alpha^2}{F} t_0 + \frac{3\alpha^2 s_0^2}{F^2} + \frac{\alpha s_0}{F} \right) , \quad (3.3)
\]
where \( \kappa = \kappa(x) \) is a scalar function on \( M \). Obviously, by (2.3) and (3.3), we can get
Lemma 3.1 Let \( F = \alpha + \beta \) be a Randers metric on an \( n \)-dimensional manifold \( M \). If \( F \) is of scalar flag curvature \( K = K(x, y) \) and \( \beta \) is a Killing 1-form with respect to \( \alpha \), then

\[
2\alpha s^m_{0, m} + (n - 1)(\kappa \alpha^2 + t_{00} + \Xi) = (n - 1)KF^2, \tag{3.4}
\]

where

\[
\Xi := -\frac{2\alpha^2t_0}{F} + \frac{3\alpha^2s_0^2}{F^2} + \frac{\alpha s_{0, 0}}{F}. \tag{3.5}
\]

Further, contracting the (3.4) by \( b^i \) yields the following equation

\[
S_{i|m}y^m b^i - S_{i}b^i = -\frac{n + 1}{3}F^2K_{i}b^i. \tag{3.6}
\]

Let \( G^i \) and \( \tilde{G}^i \) denote the geodesic coefficients of \( F \) and \( \alpha \) respectively. The horizontal covariant derivatives \( S_{i|m} \) and \( S_{i:m} \) of \( S_{i} \) with respect to \( F \) and \( \alpha \) respectively are given by

\[
S_{i|m} = \frac{\partial S_{i}}{\partial x^m} - S_{i}\Gamma_{i|m} - \frac{\partial S_{i}}{\partial y^l}N^l_{m}, \tag{3.7}
\]

\[
S_{i:m} = \frac{\partial S_{i}}{\partial y^m} - S_{i}\bar{\Gamma}_{i|m} - \frac{\partial S_{i}}{\partial y^l}\bar{N}^l_{m}, \tag{3.8}
\]

where \( \Gamma_{i|m} = \frac{\partial^2G^i}{\partial y^m} - L_{i|m}, \quad N^l_{m} := \frac{\partial^2\tilde{G}^i}{\partial y^m}, \quad L_{i|m} := g^{il}L_{jim} \) and \( \Gamma_{i|m} := \frac{\partial^2G^i}{\partial y^m} \), \( \bar{N}^l_{m} := \frac{\partial^2\tilde{G}^i}{\partial y^m} \). From (3.7) and (3.8), we have

\[
S_{i|m}y^m = \left\{ S_{i|m} - S_{i}(\Gamma_{i|m} - \tilde{\Gamma}_{i|m}) - \frac{\partial S_{i}}{\partial y^l}(N^l_{m} - \bar{N}^l_{m}) \right\} y^m
\]

\[
= S_{i:m}y^m - S_{i}(N^l_{i} - \bar{N}^l_{i}) - 2\frac{\partial S_{i}}{\partial y^l}(G^l - \tilde{G}^l). \tag{3.9}
\]

Similarly, we have

\[
S_{i} = S_{i} - S_{i}(N^l_{i} - \bar{N}^l_{i}). \tag{3.10}
\]

Substituting (3.9) (3.10) into (3.6) yields

\[
S_{i:m}y^m b^i - 2\frac{\partial S_{i}}{\partial y^l}(G^l - \tilde{G}^l)b^i - S_{i}b^i = -\frac{n + 1}{3}F^2K_{i}b^i. \tag{3.11}
\]

As we known, the relationship between the geodesic coefficients \( G^i \) and \( \tilde{G}^i \) is as follows (10, 11)

\[
G^i = \tilde{G}^i + \alpha s^i_0 - \frac{\alpha s_{0}y^i}{F}. \tag{3.12}
\]

From this, one obtains

\[
N^i_{j} = \tilde{N}^i_{j} + \alpha^{-1}y_j s^i_0 + \alpha s^i_j - \frac{s_0y^i(y_j + \alpha b_j)}{F^2}
\]

\[
= \frac{-\alpha^{-1}y_j s_0y^i + \alpha y_j s^i_j + \alpha s_0\delta^i_j}{F}. \tag{3.13}
\]
Besides, the formula for S-curvature of \( F \) is given by the following

\[
S = -(n+1) \left\{ \frac{\alpha s_0}{F} + \rho_0 \right\}.
\]

(3.14)

where \( \rho := \ln \sqrt{1 - ||\beta||^2}_\alpha \), \( \rho_i = \rho_{x^i} \), and \( \rho_0 = \rho_{y^i} \).

By a series of direct computations, we have the following

\[
\begin{align*}
F_j &= \alpha^{-1} y_j + b_j, \quad F_{j,k} = \alpha^{-1} a_{jk} - \alpha^{-3} y_j y_k, \quad F_{i} b^i = s + b^2, \quad (3.15) \\
F_{i} &= s_{0i}, \quad F_{i} y^i = 0, \quad F_{i} b^i = -s_0, \quad (3.16) \\
F_{i} s^j_{0} &= s_{0i}, \quad F_{k} s^j_{0} = -s_k. \quad (3.17)
\end{align*}
\]

From (3.12) and (3.14)-(3.17), one obtains

\[
\begin{align*}
S_{,i} b^i &= (n+1) \left\{ - \frac{b^i s_{0,i}}{1+s} - \frac{s_0^2}{\alpha(1+s)^2} - b^i \rho_0; i \right\}, \quad (3.18) \\
S_{i;m} y^m b^i &= (n+1) \left\{ \frac{(b^2 - s^2) s_{0,0}}{\alpha(1+s)^2} + \frac{s_0^2}{\alpha(1+s)^2} - \frac{b^i s_{0,i}}{1+s} - b^i \rho_0; i \right\}, \quad (3.19) \\
\frac{\partial S_{,i}}{\partial y^j} (G^i - \bar{G}^i) b^i &= (n+1) \left\{ - \frac{2s_0^2(s + \bar{b}^2)}{\alpha(1+s)^3} + \frac{t_0(b^2 - s^2)}{(1+s)^2} \right\}. \quad (3.20)
\end{align*}
\]

Plugging (3.18)-(3.20) into (3.11), we obtain the following lemma.

**Lemma 3.2** Let \( F = \alpha + \beta \) be a Randers metric on an \( n \)-dimensional manifold \( M \). If \( F \) is of scalar flag curvature \( K = K(x, y) \) and \( \beta \) is a Killing 1-form with respect to \( \alpha \), then

\[
(n+1) \left\{ \alpha^{-1} \left[ \frac{s_{0,0}(b^2 - s^2)}{(1+s)^2} + \frac{2s_0^2(1 + 3s + 2b^2)}{(1+s)^3} \right] + b^i (\rho_0; i - \rho_{x} ; i) \right\} + 2t_0(s^2 - \bar{b}^2) \left( \frac{b^i(s_{0,i} - s_{x;i})}{1+s} \right) = -\frac{n+1}{3} \alpha^2 (1+s)^2 K_{,i} b^i. \quad (3.21)
\]

On the other hand, contracting (2.5) by \( b^i \) yields the following equation

\[
J_{i;m} y^m b^i + K F^2 I_i b^i = -\frac{n+1}{3} F^2 K_{,i} b^i. \quad (3.22)
\]

Similar to the methods to get Lemma 3.2, we have

\[
J_{i;m} y^m = J_{i;m} y^m - J_i \frac{\partial (G^i - \bar{G}^i)}{\partial y^j} - 2 \frac{\partial J_i}{\partial y^j} (G^i - \bar{G}^i). \quad (3.23)
\]

Substituting (3.23) into (3.22) yields

\[
\begin{align*}
J_{,0} - J_s s^0_{,0} - J_i \frac{\partial (G^i - \bar{G}^i)}{\partial y^j} b^i - 2 \frac{\partial J_i}{\partial y^j} (G^i - \bar{G}^i) + K F^2 I_i b^i \\
= -\frac{n+1}{3} F^2 K_{,i} b^i. \quad (3.24)
\end{align*}
\]
where \( J = J_i b^i \).

It is easy to see that
\[
\begin{align*}
    h_is^i_0 &= \alpha s_0, \\
    h_is^i &= -\alpha s_0, \\
    h_iy^i &= 0, \\
    h_ibi &= \alpha (b^2 - s^2).
\end{align*}
\] (3.25)

From (3.1), (3.2), (3.12) and (3.13), (3.25), by a series of direct computations, one obtains the following
\[
I_i b^i = \frac{(n+1)(b^2 - s^2)}{2\alpha(1+s)},
\] (3.26)
\[
\bar{J};0 = \frac{(n+1)(1+3s+b^2+s^2)s_0+2s_0^2}{2\alpha(1+s)^2},
\] (3.27)
\[
J_is^i_0 = \frac{n+1}{2\alpha(1+s)}\left\{ \frac{s_0^2}{1+s} - (1+s)t_00 - \alpha t_0 \right\},
\] (3.28)
\[
\partial J_i \left( \bar{C}^i - \bar{C}^i \right) = (n+1)\left\{ \frac{1+3s+b^2+s^2}{2(1+s)^2}t_0 + \frac{(1-s-2b^2)s_0^2}{2\alpha(1+s)^3} \right\},
\] (3.29)
\[
J_i \partial \left( \bar{G}^i - \bar{G}^i \right)_{b^i} = \frac{n+1}{2\alpha(1+s)}\left\{ -\alpha(1+2s)t_0 - s(1+s)t_00 - \alpha^2 t_i b^i \right\} - \frac{(2+3s+b^2)s_0^2}{(1+s)^2}.
\] (3.30)

Substituting (3.26)-(3.30) into (3.24), we have the following lemma.

**Lemma 3.3** Let \( F = \alpha + \beta \) be a Randers metric on an \( n \)-dimensional manifold \( M \). If \( F \) is of scalar flag curvature \( K = K(x,y) \) and \( \beta \) is a Killing 1-form with respect to \( \alpha \), then
\[
\left\{ \alpha^{-1} \left[ \frac{s_0(1+3s+b^2+s^2)}{2(1+s)^2} + \frac{t_0(1+s)}{2} + \frac{s_0^2(1+6s+5b^2)}{2(1+s)^3} \right] \\
- \frac{t_0(s+b^2)}{(1+s)^2} + \frac{\alpha t_i b^i}{2(1+s)} + \frac{\alpha(1+s)(b^2 - s^2)K}{2} \right\} (n+1)
= -\frac{n+1}{3} \alpha^2 (1+s)^2 K_i b^i.
\] (3.31)

Lemmas 3.1, 3.2 and 3.3 are fundamental for the proofs of our main theorems.

**4 Proofs of The Theorems**

In this section, we will prove Theorem 1.1 and 1.2. Firstly, we will prove the following proposition which plays an important role in our proofs.

**Proposition 4.1** Let \( \alpha = \sqrt{\alpha_{ij}y^i y^j} \) be a Riemann metric and \( \beta = b_i y^i \) be a 1-form on a manifold. Then
\[
s^i_{0;i} = ^\alpha \text{Ric}_{0i} + r^i_{i;0} - r^i_{0;i},
\] (4.1)
where “;” denotes the covariant derivative with respect to Levi-Civita connection of $\alpha$. $\alpha_{\text{Ric}}$ is the Ricci curvature of $\alpha$ and $\alpha_{\text{Ric}}^0 := (\frac{1}{2}\alpha_{\text{Ric}})_{y^iy^j}y^i y^j$.

**Proof**: By the definition of $r_{ij}$, we have

$$b_{i;k} + b_{k;i} = 2r_{ik;j}, \quad (4.2)$$
$$- b_{k;i} - b_{j;k} = -2r_{kj;i}. \quad (4.3)$$

On the other hand, by Ricci identity, we get

$$b_{i;jk} - b_{k;i} = \alpha R_{imjk} b^m, \quad (4.4)$$
$$b_{k;i} - b_{j;k} = -\alpha R_{kmij} b^m, \quad (4.5)$$
$$b_{j;k} - b_{j;i} = \alpha R_{jmki} b^m. \quad (4.6)$$

Here, $\alpha_{\text{Ric}}$ denotes the Riemann curvature tensor of $\alpha$. Adding all the equations from (4.2) to (4.6), we can get

$$s_{ij;k} = -\alpha R_{kmij} b^m + r_{ik;j} - r_{kj;i}. \quad (4.7)$$

Contracting (4.7) with $\alpha^p$ yields

$$s^p_{j;k} = \alpha R^p_{jk} b^m + r^p_{kj} - r^p_{kj;i}, \quad (4.8)$$

where $\alpha R^p_{jk} := \alpha R_{jk} a^p$ and $\alpha R^p_{jk} = \alpha R_{jk} b^p$. Contracting (4.8) with $y^i$ yields

$$s^p_{0;k} = \alpha R^p_{0j} b^m + r^p_{k0} - r^p_{k0;i}. \quad (4.9)$$

Letting $p = k$ in (4.9), we obtain (4.1). Q.E.D.

Now we are in the position to prove our main theorems.

**Proof of Theorem 1.1**: Let $F = \alpha + \beta$ be a Randers metric. Assume that $F$ is of scalar flag curvature $K = K(x, y)$ and $\beta$ is a Killing 1-form. Then (3.4), (3.21) and (3.31) all hold by Lemma 3.1, Lemma 3.2 and Lemma 3.3.

By $\frac{b^2 - s^2}{\alpha} \leq 0$ and plugging $s = \frac{\beta}{\alpha}$ into the resulting equation, we can obtain

$$\frac{1}{2\alpha(\alpha + \beta)} (\Xi_3 \alpha^3 + \Xi_2 \alpha^2 + \Xi_1 \alpha + \Xi_0) = 0,$$

which is equivalent to the following

$$\Xi_3 \alpha^3 + \Xi_2 \alpha^2 + \Xi_1 \alpha + \Xi_0 = 0, \quad (4.10)$$

where

$$\Xi_3 = \kappa b^2 + t_i b^i, \quad (4.11)$$
$$\Xi_2 = 2b^i (\rho_i - \rho_0) + s_i s_0 - s_{0;i} + \frac{2b^2 s_m}{n-1} \quad (4.12)$$
$$\Xi_1 = -\kappa \beta \beta^2 + [2b^i (\rho_i - \rho_0) - 2t_0] \beta^2 + t_{00} - 3s_0^2 + t_0 \alpha_0 + s_0 \alpha_0, \quad (4.13)$$
$$\Xi_0 = 2\beta \left( s_{0;0} + t_{00} - \frac{\beta s_m}{n-1} \right). \quad (4.14)$$
Further, by the definition of $\rho$ and because of $r_{ij} = 0$, we have $\rho_i = \rho_{x^i} = -\frac{s_i}{1-\beta^2}$. Then, we can get

$$\rho_{i;0} - \rho_{0;i} = \frac{(s_{0;i} - s_{i;0})}{1 - \beta^2}.$$ 

On the other hand, it is clear that $(b^2)_{i;j} = (b^2)_{j;i}$. By a direct computation, we get $s_{i;j} = s_{j;i}$, that is, $s_{i;0} = s_{0;i}$. Hence, (4.12), (4.13) can be rewritten as

$$\Xi_2 = \frac{2b^2 s_m_{0;m}}{n - 1},$$  
(4.15)  
$$\Xi_1 = -\kappa \beta^2 - 2t_0 \beta + t_00(1 + b^2) - 3s_0^2 + s_{0;0}.$$  
(4.16)

From (4.10) we obtain the following fundamental equations

$$\Xi_2 \alpha^2 + \Xi_0 = 0,$$  
(4.17)  
$$\Xi_4 \alpha^2 + \Xi_1 = 0.$$  
(4.18)

Rewrite (4.17) as

$$\Xi_2 \alpha^2 = -\Xi_0,$$  
(4.19)

Since $\alpha^2$ is an irreducible polynomial in $y$, from (4.19) and (4.14), (4.15), we have

$$s_{0;0} + t_{00} - \frac{\beta s_m_{0;m}}{n - 1} = c(x)\alpha^2,$$  
(4.20)

where $c = c(x)$ is a scalar function on $M$. Substituting (4.20) into (4.19) yields

$$\frac{b^2 s_m}{n - 1} = -c(x)\beta,$$  
(4.21)

which means that (1.2) holds. Plugging (1.21) into (4.20), we get

$$s_{0;0} + t_{00} = c(x)(\alpha^2 - b^{-2}\beta^2),$$

which is just (1.3). This completes the proof of Theorem 1.1. Q.E.D.

Finally, let us give the proof of Theorem 1.2.

**Proof of Theorem 1.2**: Let $F = \alpha + \beta$ be a Randers metric. Assume that $F$ is of scalar flag curvature $K = K(x, y)$. By (5.30) in [10], we have the following

$$\alpha \text{Ric} = (n - 1)\lambda(x)\alpha^2 + (n + 1)t_{00},$$

where $\lambda := \lambda(x)$ is a scalar function on $M$. Further, by a direct computation, we have

$$\alpha \text{Ric}_{ij} = \left(\frac{1}{2}\alpha \text{Ric}\right)_{y^i'y^j} = (n - 1)\lambda(x)a_{ij} + (n + 1)t_{ij}.$$  
(4.22)
From Proposition 4.1 and by (4.22) and the assumption that $\beta$ is a Killing 1-form, we get

$$s^m_{0,m} = (n-1)\lambda(x)\beta + (n+1)t_0. \quad (4.23)$$

On the other hand, by Theorem 1.1, we have

$$s^m_{0,m} = -(n-1)cb^{-2}\beta. \quad (4.24)$$

Comparing (4.23) with (4.24), we obtain (1.4). Q.E.D.

5 Example: Bao-Shen metric on $S^3$

In this section, we will prove that, for Bao-Shen metric on $S^3$ constructed in [2], $\alpha$ and $\beta$ satisfy (1.2), (1.3) and (1.4).

We view $S^3$ as a Lie group and let $\eta^1, \eta^2$ and $\eta^3$ be the standard right invariant 1-form on $S^3$ such that $d\eta^1 = 2\eta^2 \wedge \eta^3$, $d\eta^2 = 2\eta^3 \wedge \eta^1$, $d\eta^3 = 2\eta^1 \wedge \eta^2$.

For a constant $K > 1$, write $\varepsilon := \sqrt{K}$, $\delta := \pm \sqrt{K-1}$. Let $\omega^1 := \varepsilon \eta^1$, $\omega^2 := \eta^2$, $\omega^3 := \eta^3$. Then

$$dw^1 = 2\varepsilon \omega^2 \wedge \omega^3,$$  
$$dw^2 = \frac{2}{\varepsilon} \omega^3 \wedge \omega^1,$$  
$$dw^3 = \frac{2}{\varepsilon} \omega^1 \wedge \omega^2.$$  

Consider the following Riemannian metric on $S^3$:

$$\alpha := \sqrt{\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3}. \quad (5.1)$$

The Levi-Civita connection forms $(\omega^i_j)$ of $\alpha$ are given by $d\omega^i = \omega^j \wedge \omega^i_j$ and $\omega_{ij} = -\omega_{ji}$, where $\omega_{ij} := \omega^k_{ij} \delta_{kj}$. Concretely, we have

$$\begin{pmatrix}
\omega^1_1 & \omega^1_2 & \omega^1_3 \\
\omega^2_1 & \omega^2_2 & \omega^2_3 \\
\omega^3_1 & \omega^3_2 & \omega^3_3 \\
\end{pmatrix} = \begin{pmatrix}
0 & -\varepsilon \omega^3 & \varepsilon \omega^2 \\
\varepsilon \omega^3 & 0 & (\varepsilon - \frac{2}{\varepsilon}) \omega^1 \\
-\varepsilon \omega^2 & (\frac{2}{\varepsilon} - \varepsilon) \omega^1 & 0 \\
\end{pmatrix}$$

Further, let

$$\beta := \frac{\delta}{\varepsilon} \omega^1. \quad (5.2)$$

Then we have

$$b_1 = \frac{\delta}{\varepsilon}, \quad b_2 = 0, \quad b_3 = 0.$$  

Obviously, this $\beta$ has constant length with respect to Riemannian metric $\alpha$:

$$b := \|\beta\|_\alpha = \frac{\|\delta\|}{\varepsilon}.$$
By the Riemannian metric $\alpha$ defined by (5.1) and the $1$-form $\beta$ defined by (5.2), D. Bao and Z. Shen constructed the following family of Randers metrics on $S^3$:

$$F(x, y) = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \pm \sqrt{\frac{K-1}{K}} y^1, \quad \delta = \pm \sqrt{\frac{K-1}{K}}.$$  (5.3)

Further, they have proved that this family of Randers metrics has constant flag curvature $K$ but nonzero Douglas tensor (see [2]). This family of Randers metrics is the first example of Finsler metrics of constant flag curvature which are not projectively flat.

In the following, we carry out our calculations in the moving coframe $\{\omega^1, \omega^2, \omega^3\}$ and the dual moving frame $\{e_1, e_2, e_3\}$. The covariant differentiation formulas of $\beta$ with respect to $\alpha$ are given by

$$b_{i;j} = (db_i - b_s \omega^s_i)(e_j),$$
$$b_{i;j;k} = (db_{i;j} - b_{s;i} \omega^s_i - b_{i;s} \omega^s_j)(e_k).$$

Straightforward calculations give:

$$\begin{pmatrix}
  b_{1:1} & b_{1:2} & b_{1:3} \\
  b_{2:1} & b_{2:2} & b_{2:3} \\
  b_{3:1} & b_{3:2} & b_{3:3}
\end{pmatrix} = \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & -\delta \\
  0 & \delta & 0
\end{pmatrix}.$$  (5.4)

Then, we have $r_{ij} = 0$, that is, $\beta$ is a Killing $1$-form with respect to $\alpha$. Further, we have

$$s_{11} = 0, \quad s_{12} = 0, \quad s_{13} = 0,$$
$$s_{21} = 0, \quad s_{22} = 0, \quad s_{23} = -\delta,$$
$$s_{31} = 0, \quad s_{32} = \delta, \quad s_{33} = 0.$$  

It is obvious that $s_1 = s_2 = s_3 = 0$. Hence, we have

$$s_0 = 0, \quad s_{0:0} = 0.$$  (5.4)

By the definition, $t_{ij} = s_{im}s^{m}_{j}$, we can get

$$t_{11} = 0, \quad t_{12} = 0, \quad t_{13} = 0,$$
$$t_{21} = 0, \quad t_{22} = -\delta^2, \quad t_{23} = 0,$$
$$t_{31} = 0, \quad t_{32} = 0, \quad t_{33} = -\delta^2.$$  

Then we have

$$t_{00} = -\delta^2 ((y^2)^2 + (y^3)^2)$$  (5.5)

and $t_1 = t_2 = t_3 = 0$, which means

$$t_0 = 0.$$  (5.6)

Further, by $s_{ij;k} = (ds_{ij} - s_{pj}\omega^p_i - s_{ip}\omega^p_j)(e_k)$, we have the following

$$s_{11;1} = 0, \quad s_{12;1} = 0, \quad s_{13;1} = 0,$$
$$s_{21;2} = \delta \epsilon, \quad s_{22;2} = 0, \quad s_{23;2} = 0,$$
$$s_{31;3} = \delta \epsilon, \quad s_{32;3} = 0, \quad s_{33;3} = 0.$$  

14
It is easy to see that
\[ s_{0,m}^0 = 2\delta v_1. \]  
(5.7)

Then, from (5.1), (5.2) and (5.4), (5.5) and (5.7), we find that \((1.2)\) and \((1.3)\) hold for
\[ c(x) = -(K - 1). \]  
(5.8)

At the same time, by (5.6), it is easy to check that \((1.4)\) holds for \(\lambda = K\) and \(c(x) = -(K - 1)\).

References

[1] D. Bao, C. Robles and Z. Shen, Zermelo navigation on Riemannian manifolds, J. Differ. Geom., 66(2004), 391-449.

[2] D. Bao and Z. Shen, Finsler metrics of constant positive curvature on the Lie group \(S^3\), J. London Math. Soc., 66(2)(2002), 453-467.

[3] L. Berwald, Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus, Math. Z., 25(1926), 40-73.

[4] L. Berwald, Parallelübertragung in allgemeinen Räumen, Atti Congr. Intern. Mat. Bologna., 4(1928), 263-270.

[5] L. Berwald, Über die n-dimensionalen Geometrien konstanter Krümmung, in denen die Geraden die kürzesten sind, Math. Z., 30(1929), 449-469.

[6] B. Chen and L. Zhao, A note on Randers metrics of scalar flag curvature, Canad. Math. Bull., 55 (2012), 474-486.

[7] X. Cheng, X. Mo and Z. Shen, On the flag curvature of Finsler metrics of scalar curvature, J. of the London Math. Soc., 68(2)(2003), 762-780.

[8] X. Cheng, H. Wang and M. Wang, \((\alpha, \beta)\)-metrics with relatively isotropic mean Landsberg curvature, Publ. Math. Debrecen., 72(3-4)(2008), 475-485.

[9] X. Cheng and Z. Shen, Randers metrics of scalar flag curvature, Journal of the Australian Mathematical Society, 87(3)(2009), 359-370.

[10] X. Cheng and Z. Shen, Finsler Geometry–An approach via Randers spaces, Springer and Science Press, 2012.

[11] S. S. Chern and Z. Shen, Riemann-Finsler Geometry, Nankai Tracts in Mathematics, Vol. 6, World Scientific, 2005.

[12] B. Li and Z. Shen, On a class of weak Landsberg metrics, Sci. China. A., 50(1)(2007), 75-85.
[13] M. Matsumoto and H. Shimada, *The corrected fundamental theorem on Randers spaces of constant curvature*, Tensor, N. S., 63(2002), 43-47.

[14] G. Randers, *On an asymmetric metric in the four-space of general relativity*, Physical Review, 59(1941), 195-199.

[15] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances in Mathematics, 128(2)(1997), 306-328.

[16] Z. Shen, *Projectively flat Randers metrics with constant flag curvature*, Math. Ann., 325(2003), 19-30.

[17] Z. Shen and Q. Xia, *On a class of Randers metrics of scalar flag curvature*, Internat J. Math., 24(7)(2013): 1350055.

[18] Q. Xia, *On the flag curvature of a class of Randers metric generated from the navigation problem*, J. Math. Anal. Appl., 397(2013), 415-427.

Xinyue Cheng
School of Mathematical Sciences
Chongqing Normal University
Chongqing 401331, P. R. of China
E-mail: chengxy@cqut.edu.cn & chengxy@cqnu.edu.cn

Li Yin
School of Sciences
Chongqing University of Technology
Chongqing 400054, P. R. China
E-mail: yinli@2015.cqut.edu.cn

Tingting Li
School of Sciences
Chongqing University of Technology
Chongqing 400054, P. R. China
E-mail: litt@2015.cqut.edu.cn