Averages of alpha-determinants over permutations

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Abstract

We show that certain weighted average of the \( \alpha \)-determinant of a \( kn \) by \( kn \) matrix of the form \( A \otimes 1_{1,k} \), the Kronecker product of a \( kn \) by \( n \) matrix \( A \) and 1 by \( k \) all one matrix \( 1_{1,k} \), over permutations of \( kn \) letters is reduced to the \( k \)-wreath determinant of \( A \) up to constant. The constant is exactly given by the modified content polynomial for the Young diagram \((k^n)\). As a corollary, we give a ‘determinantal’ formula for certain functions on the symmetric groups which are invariant under the left and right translation by a Young subgroup, especially the values of the Kostka numbers for rectangular shapes with arbitrary weight. This corollary gives a generalization of the formula of irreducible characters of the symmetric group for rectangular shapes due to Stanley.

1 Introduction

The \( \alpha \)-determinant of an \( N \) by \( N \) square matrix \( A = (a_{ij}) \) is defined as a parametric deformation of the usual determinant as

\[
\det_\alpha A := \sum_{\sigma \in \mathfrak{S}_N} \alpha^{\nu(\sigma)} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(N)N},
\]

where \( \alpha \) is a complex parameter and \( \nu(\sigma) \) for a permutation \( \sigma \in \mathfrak{S}_N \) is defined to be \( N \) minus the number of disjoint cycles in \( \sigma \). By definition, we see that \( \det_{-1} A = \det A \), \( \det_1 A = \per A \), \( \det_0 A = a_{11} a_{22} \cdots a_{NN} \), where \( \per A \) is the permanent of \( A \). It is Vere-Jones [8] who first introduce such a parametric deformations, which he called the \( \alpha \)-permanent. Here we adopt the modified definition and terminology by Shirai and Takahashi [5]. The \( \alpha \)-determinant is multiplicative only if \( \alpha = -1 \).

Let \( P(\sigma) = (\delta_{i\sigma(j)}) \) be the permutation matrix for a permutation \( \sigma \in \mathfrak{S}_N \). The sum

\[
\sum_{\sigma \in \mathfrak{S}_k} \det_\alpha (AP(\sigma))
\]

is a polynomial in \( \alpha \) which is divisible by \( (1 + \alpha) \cdots (1 + (k - 1)\alpha) \) for a given \( N \) by \( N \) matrix \( A \). Here we regard \( \mathfrak{S}_k \) as a subgroup of \( \mathfrak{S}_N \) consisting of permutations which do not move the \( N - k \) letters \( k + 1, k + 2, \ldots, N \). This fact is used to show that the \( \alpha \)-determinant is weakly alternating when \( \alpha \) is a reciprocal of a negative integer in the sense that \( \det_{-1/k} A \) vanishes whenever more than \( k \) columns or rows in \( A \) are equal (Lemma 3.3). Based on this fact, we define the \( k \)-wreath determinant \( \text{wrdet}_k A \) of a \( kn \times n \) matrix \( A \) by

\[
\text{wrdet}_k A := \det_{-1/k}(\overbrace{a_{11}, \ldots, a_{11}}^{k}, \ldots, \overbrace{a_{n1}, \ldots, a_{n1}}^{k}),
\]

where \( a_{ij} \) is the \( j \)-th column vector of \( A \). This recovers the relative invariance

\[
\text{wrdet}_k (AQ) = \text{wrdet}_k A (\det Q)^k
\]

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with respect to the right translation by any \( n \times n \) matrix \( Q \).

In the extremal case where \( k = N \), we can determine the sum (1.1) explicitly as

\[
\sum_{\sigma \in S_N} \det_\alpha (AP(\sigma)) = \prod_{i=1}^{N-1} (1 + i\alpha) \cdot \text{per} A.
\]

More generally, one can prove

\[
\sum_{\sigma \in S_N} \chi^\lambda(\sigma) \det_\alpha (AP(\sigma)) = f^\lambda f_\lambda(\alpha) \text{Imm}_\lambda A
\]

for any partition \( \lambda \vdash N \). Especially we have

\[
\sum_{\sigma \in S_N} \text{sgn} \sigma \det_\alpha (AP(\sigma)) = \prod_{i=1}^{N-1} (1 - i\alpha) \cdot \det A.
\]

Here \( \chi^\lambda \) is the irreducible character of \( S_N \) associated to \( \lambda \), \( f^\lambda \) is the number of standard tableaux with shape \( \lambda \), \( f_\lambda(\alpha) \) is the modified content polynomial for \( \lambda \), and \( \text{Imm}_\lambda A \) is the immanant of \( A \) associated to \( \lambda \). The identity (1.2) is essentially equivalent to the result by Matsumoto and Wakayama [4] on the irreducible decomposition of the \( U(gl_N) \)-cyclic submodule generated by a single polynomial \( \det_\alpha X \). The structure of such cyclic module is the same for almost all values of \( \alpha \), but changes drastically when \( \alpha \) is a reciprocal of a nonzero integer.

The purpose of the paper is to give an analog of (1.3) for the \( k \)-wreath determinant (Theorem 2.2). As corollaries of the main result, we also obtain a formula for certain \( S_\mu \)-biinvariant functions on \( S_N \), where \( S_\mu \) is the Young subgroup of \( S_N \) associated with a partition \( \mu \vdash N \). In particular, we get a formula for Kostka numbers with rectangular shape and arbitrary weight (Corollaries 5.1, 5.4). These corollaries give a generalization of the formula for irreducible characters of the symmetric groups associated to rectangular diagrams which is due to Stanley [6] (Corollary 5.2).

## 2 Weighted averages of alpha-determinants over permutations

Let \( n, k \) be positive integers. We define a linear map \( \varpi_k : M_{kn,n} \to M_{kn} \) by \( \varpi_k(A) := A \otimes 1_{1,k} \), where \( M_{p,q} \) is the set of \( p \times q \) complex matrices, \( M_{p} = M_{p,p} \) is the set of square matrices of size \( p \), \( 1_{p,q} \) is the \( p \times q \) all-one matrix, and \( \otimes \) denotes the Kronecker product of matrices.

\[
A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix} \quad (A = (a_{ij}) \in M_{m,n}).
\]

We note that \( \varpi_k \) commutes with the left translation, that is, \( \varpi_k(PA) = P \varpi_k(A) \) for any \( P \in M_{kn} \) and \( A \in M_{kn,n} \). We also notice that

\[
\varpi_k(A)P(g) = \varpi_k(A)
\]

for any \( A \in M_{kn,n} \) and \( g \in S_k^n = S_{kn} \). For a \( kn \) by \( n \) matrix \( A \in M_{kn,n} \), the \( k \)-wreath determinant of \( A \) is defined by

\[
\text{wrdet}_k A := \det_{-1/k} \varpi_k(A).
\]

The 1-wreath determinant is the ordinary determinant: \( \text{wrdet}_1 A = \det A \). See [1] for basic facts on the wreath determinants.
Example 2.1 \((n = k = 2)\). For \(A = (a_{ij}) \in M_{4,2}\), the 2-wreath determinant of \(A\) is

\[
\text{wrdet}_2 A = \det_{-1/2} A = \det \begin{pmatrix}
a_{11} & a_{11} & a_{12} & a_{12} \\
a_{21} & a_{21} & a_{22} & a_{22} \\
a_{31} & a_{31} & a_{32} & a_{32} \\
a_{41} & a_{41} & a_{42} & a_{42}
\end{pmatrix}
\]

\[
= \frac{1}{4} \left\{ a_{11}a_{21}a_{32}a_{42} + a_{11}a_{22}a_{31}a_{41} \right\} - \frac{1}{8} \left\{ a_{11}a_{22}a_{31}a_{42} + a_{11}a_{22}a_{32}a_{41} + a_{12}a_{21}a_{31}a_{42} + a_{12}a_{21}a_{32}a_{41} \right\}.
\]

We can express \(\text{wrdet}_2 A\) as a sum of products of minor determinants of \(A\) as

\[
\text{wrdet}_2 A = \frac{1}{8} \left| \begin{array}{cccc}
a_{11} & a_{12} & a_{21} & a_{22} \\
a_{31} & a_{32} & a_{41} & a_{42}
\end{array} \right| + \frac{1}{8} \left| \begin{array}{cccc}
a_{11} & a_{12} & a_{21} & a_{22} \\
a_{31} & a_{32} & a_{41} & a_{42}
\end{array} \right|,
\]

which apparently shows the relative invariance \(\text{wrdet}_2 (AQ) = \text{wrdet}_2 A (\det Q)^2\) for \(Q \in M_2\).

For a partition \(\lambda\), \(f_\lambda(\alpha)\) is the modified content polynomial for \(\lambda\)

\[
f_\lambda(x) = \prod_{(i,j) \in \lambda} (1 + (j - i)x),
\]

where we identify \(\lambda\) with its corresponding Young diagram. For instance, we have

\[
f_{(N)}(x) = \prod_{i=1}^{N-1} (1 + ix), \quad f_{(1^N)}(x) = \prod_{i=1}^{N-1} (1 - ix)
\]

for any positive integer \(N\). It is notable that

\[
\det_\alpha 1_N = \sum_{\sigma \in S_N} \alpha^{\nu(\sigma)} = f_{(N)}(\alpha),
\]

where \(1_N = 1_{N,N}\).

Our goal is to prove the

**Theorem 2.2.** For each positive integer \(k\), the equality

\[
\sum_{\sigma \in S_{kn}} \left(-\frac{1}{k}\right)^{\nu(\sigma)} \det_\alpha (\varpi_k(A)P(\sigma)) = f_{(k^N)}(\alpha) \text{wrdet}_k A
\]

holds.

When \(k = 1\), the theorem is reduced to the equality \((1.3)\).

## 3 Proof of the theorem

For later use, we put \(I_{n,k} = I_n \otimes 1_{k,1}\). We postpone the proofs of the lemmas used in this section to \[\text{[3]}\].

### 3.1 Reduction

To prove the theorem, we need the characterization of the \(k\)-wreath determinant.

**Lemma 3.1** (Corollary 5.8 in \[\text{[3]}\]). Suppose that a function \(f: M_{kn,n} \to \mathbb{C}\) satisfies the following conditions.

(W1) \(f\) is multilinear in row vectors.

(W2) \(f(AQ) = f(A)(\det Q)^k\) for any \(Q \in M_n\).
Lemma 3.3 (Lemma 2.3 in [1])

Let \( F \) be a function defined on \( \mathbb{C}^n \). Then

\[
F(P(g)A) = f(A) \quad \text{for any } g \in \mathbb{S}_k^n.
\]

Then \( f \) is equal to the \( k \)-wreath determinant \( \text{wrdet}_k \) up to constant multiple.

To determine the constant factor explicitly in our discussion below, the formula

\[
\text{wrdet}_k \|_{n,k} = \det_{-1/k} \left( I_n \otimes 1_k \right) = \left( \frac{k!}{k} \right)^n
\]
is useful (see Lemma 4.6 in [1]).

Let \( F(\alpha; A) \) be the left-hand side of (2.3). Since the multilinearity of \( F(\alpha; A) \) in row vectors of \( A \) is obvious by its definition, we have only to show the following three equations to obtain the theorem.

\[
\begin{align*}
F(\alpha; AQ) &= F(\alpha; A)(\det Q)^k \quad (Q \in M_n), \\
\text{(A)} \\
F(\alpha; P(g)A) &= F(\alpha; A) \\
\text{(B)} \\
F(\alpha; \|_{n,k}) &= \left( \frac{k!}{k} \right)^n f_{(k^1)}(\alpha). \\
\text{(C)}
\end{align*}
\]

We introduce the two-parameter deformation of the determinant as

\[
det_{\alpha,\beta} A := \sum_{\tau, \sigma \in \mathbb{S}_N} \alpha^{\nu(\tau)} \beta^{\nu(\sigma)} \prod_{i=1}^{N} a_{\tau(i)\sigma(i)}. \quad (3.1)
\]

It is clear that this is symmetric in \( \alpha \) and \( \beta \), i.e. \( \det_{\alpha,\beta} A = \det_{\beta,\alpha} A \). Notice that

\[
F(\alpha; A) = \det_{\alpha,-1/k}(\varpi_k(A)) = \sum_{\sigma \in \mathbb{S}_N} \alpha^{\nu(\sigma)} \det_{-1/k}(\varpi_k(A) P(\sigma)).
\]

Remark 3.2. The equalities (1.1) and (1.3) are readily obtained from the symmetry \( \det_{\alpha,\pm 1} A = \det_{\pm 1,\alpha} A \).

3.2 Proofs of (A) and (B)

Let \( a_1, \ldots, a_n \in \mathbb{C}^k \). We have only to prove (A) when \( Q \) is an elementary matrix. Namely, it is sufficient to verify

\[
\begin{align*}
F(\alpha; (a_1, \ldots, a_j + ca_i, \ldots, a_n)) &= F(\alpha; (a_1, \ldots, a_j, \ldots, a_n)) \quad (i \neq j), \\
\text{(3.2)} \\
F(\alpha; (a_1, \ldots, ca_i, \ldots, a_n)) &= c^k F(\alpha; (a_1, \ldots, a_i, \ldots, a_n)) \quad (c \in \mathbb{C}). \\
\text{(3.3)}
\end{align*}
\]

The equation (3.3) obviously follows from the definition of \( F(\alpha; A) \) and the multilinearity of the \( \alpha \)-determinant in column vectors. The equation (3.2) is guaranteed by the following lemma.

Lemma 3.3 (Lemma 2.3 in [1]). Let \( N \) and \( k \) be positive integers such that \( k < N \). If more than \( k \) column vectors in \( A \in M_N \) are equal, then \( \det_{-1/k} A = 0 \).

The second equality (B) is shown by using (2.1), (A) and the elementary fact

\[
det_{\alpha}(P(\sigma) A) = det_{\alpha}(A P(\sigma)) \quad (A \in M_N, \sigma \in \mathbb{S}_N) \quad (3.4)
\]
as follows: For any \( g \in \mathbb{S}_k^n \), we have

\[
\begin{align*}
F(\alpha; P(g)A) &= \sum_{\sigma \in \mathbb{S}_N} \alpha^{\nu(\sigma)} \det_{-1/k}(\varpi_k(P(g)A) P(\sigma)) \\
&= \sum_{\sigma \in \mathbb{S}_N} \alpha^{\nu(\sigma)} \det_{-1/k}(P(g) \varpi_k(A) P(\sigma)) \\
&= \sum_{\sigma \in \mathbb{S}_N} \alpha^{\nu(g^{-1}\sigma g)} \det_{-1/k}(\varpi_k(A) P(g) P(g^{-1}\sigma g)) = F(\alpha; A).
\end{align*}
\]
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3.3 Proof of (C)

Let $N$ be a positive integer. For each partition $\lambda \vdash N$, $\chi^\lambda$ is the irreducible character of $S_N$ corresponding to $\lambda$ and $f^\lambda$ is the number of standard tableaux with shape $\lambda$. We denote by $K_{\lambda\mu}$ the Kostka number, that is, the number of tableaux with shape $\lambda$ and weight $\mu$. Note that $f^\lambda = K_{\lambda(\mu^N)} = \chi^\lambda(1)$ for each $\lambda \vdash N$. By the hook formula for $f^\lambda$ and the definition of $f_\lambda(x)$, we have

$$f_{(k^n)}(-1/k) = \frac{(kn)!}{k^n f(k^n)}, \quad f_{(k^n)}(1/n) = \frac{(kn)!}{n^k f(k^n)}. \quad (3.5)$$

For each pair $\lambda, \mu$ of partitions of $N$, define

$$\omega^\lambda_\mu(x) := \frac{1}{\mu!} \sum_{\tau \in \Theta_\mu} \chi^\lambda(x\tau) \quad (x \in S_N),$$

where $\Theta_\mu = \Theta_{\mu_1} \times \Theta_{\mu_2} \times \cdots$ is the Young subgroup associated to $\mu$ and $\mu! = \mu_1! \mu_2! \cdots$ is its cardinality. Here we regard the $i$-th component of $\Theta_\mu$ as a subgroup of $S_N$ consisting of permutations of the $\mu_i$ letters $m+1, \ldots, m+\mu_i$ with $m = \sum_{j<i} \mu_j$. It is immediate to see that $\omega^\lambda_\mu$ is $\Theta_\mu$-biinvariant function on $S_N$. It is well known that

$$K_{\lambda\mu} = \frac{1}{\mu!} \sum_{\tau \in \Theta_\mu} \chi^\lambda(\tau) = \omega^\lambda_\mu(1) \quad (3.6)$$

for $\lambda, \mu \vdash N$.

Let $*$ be the convolution product defined by

$$(\phi_1 * \phi_2)(x) = \sum_{\sigma \in S_N} \phi_1(x\sigma)\phi_2(\sigma^{-1})$$

for $\phi_1, \phi_2 : S_N \to \mathbb{C}$. Recall that the irreducible characters satisfy

$$\chi^\lambda \chi^\rho = \delta^\lambda_{\lambda} \frac{N!}{\lambda!} \chi^\lambda \quad (\lambda, \rho \vdash N). \quad (3.7)$$

We need the following Fourier expansion formula.

**Lemma 3.4** (Fourier expansion of $\alpha^{(\cdot)}$).

$$\alpha^{(\sigma)} = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\sigma) \chi^\lambda(\sigma) \quad (\sigma \in S_N). \quad (3.8)$$

By (3.7) and (3.8), we have

$$\alpha^{(\cdot)} * \beta^{(\cdot)} = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) f_\lambda(\beta) \chi^\lambda.$$

Hence it follows that

$$\det_{\alpha, \beta} X = \frac{1}{N!} \sum_{\lambda \vdash N} f^\lambda f_\lambda(\alpha) f_\lambda(\beta) \text{Imm}_\lambda X,$$

where $\text{Imm}_\lambda X$ is the immanant associated to $\lambda$ defined by

$$\text{Imm}_\lambda X = \sum_{\sigma \in S_N} \chi^\lambda(\sigma) x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(N)} = \sum_{\sigma \in S_N} \chi^\lambda(\sigma) \det_0(P(\sigma^{-1})X). \quad (3.9)$$

For a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \vdash N$, define

$$1_\mu := \begin{pmatrix} 1_{\mu_1} & 1_{\mu_2} & \cdots & 1_{\mu_l} \end{pmatrix}.$$
For example, we have $1_{(k^n)} = I_n \otimes 1_k$. We have

$$\omega^\lambda_{\mu}(g) = \frac{1}{\mu!} \text{Im}_\lambda(P(g)1_\mu)$$

since

$$\det_0(P(g)1_\mu) = \begin{cases} 1 & g \in \mathfrak{S}_\mu, \\ 0 & g \notin \mathfrak{S}_\mu. \end{cases}$$

Thus it follows that

$$\det_{\alpha, \beta}(P(g)1_\mu) = \frac{\mu!}{N!} \sum_{\lambda \vdash n} f^\lambda f^\lambda(\alpha)f^\lambda(\beta)\omega^\lambda_{\mu}(g) \quad (g \in \mathcal{S}_N).$$

As a particular case, we have

$$\det_{\alpha, \beta}(I_n \otimes 1_k) = \frac{(k!)^n}{(kn)!} \sum_{\lambda \vdash n} f^\lambda f^\lambda(\alpha)f^\lambda(\beta).$$

by putting $N = kn, \mu = (k^n)$ and $g = 1$ because of (3.10).

Now we further assume that $\beta = -1/k$ in (3.11). By definition, we have $f^\lambda(-1/k) = 0$ unless $\lambda_1 \leq k$. On the other hand, $K_{\lambda, (k^n)} = 0$ unless $l(\lambda) \leq n$. Therefore, the summand in (3.11) vanishes unless $\lambda = (k^n)$. Thus it follows that

$$F(\alpha; \mathbb{I}_{n, k}) = \det_{\alpha, -1/k}(I_n \otimes 1_k) = \frac{(k!)^n}{(kn)!} f^{(k^n)}(\alpha)f^{(k^n)}(-1/k) = \frac{(k!)^n}{k^{kn}} f^{(k^n)}(\alpha),$$

where we use (3.5) in the last equality. This completes the proof of the theorem.

## 4 Proofs of the lemmas

Here we prove Lemmas used in the previous section. The proof of Lemma 3.1 below is different from the one given in [1], and is rather elementary. The proof of Lemma 3.3 is just a revision of the one given in [1]. Lemma 3.4 is proved by using Okounkov-Vershik theory [7] on representations of symmetric groups.

### 4.1 Proof of Lemma 3.1

Let $f: M_{kn,n} \to \mathbb{C}$ be a function satisfying the conditions (W1)–(W3). We put

$$\phi(\sigma) = f(P(\sigma)1_{nk})$$

for $\sigma \in \mathfrak{S}_{kn}$. Notice that

$$\phi(\tau \sigma \tau') = \phi(\sigma) \quad (\tau, \tau' \in \mathfrak{S}^k, \sigma \in \mathfrak{S}_{kn})$$

by (W3) and the invariance $P(\tau')1_{nk} = 1_{nk}$. Let $I$ and $J$ be fixed complete systems of representatives of the coset $\mathfrak{S}_{kn}/\mathfrak{S}^k$ and the double coset $\mathfrak{S}^k \backslash \mathfrak{S}_{kn}/\mathfrak{S}^k$ respectively, and define $I(\sigma)$ for each $\sigma \in J$ to be the subset of $I$ such that $\bigcup_{\tau \in I(\sigma)} \tau \mathfrak{S}^k = \mathfrak{S}^k \sigma \mathfrak{S}^k$. Notice that $\phi(\tau) = \phi(\sigma)$ for each $\tau \in I(\sigma)$.

By (W1), we have

$$f(A) = \sum_{1 \leq j_1, \ldots, j_k \leq n} a_{1j_1} \ldots a_{kn,j_k} f((e_{j_1} \ldots e_{j_k}))$$

for $A = (a_{ij}) \in M_{kn,n}$, where $e_1, e_2, \ldots, e_n$ are the standard basis vectors of $\mathbb{C}^n$. By (W2), $f((e_{j_1} \ldots e_{j_k}))$ vanishes unless the matrix rank of $(e_{j_1} \ldots e_{j_k})$ equals $n$, or $(j_1, \ldots, j_k)$ is a permutation of $(1, \ldots, 1, \ldots, n, \ldots, n)$. Hence it follows that

$$f(A) = \sum_{\tau \in I} \phi(\tau) a_{\tau(1)1} \ldots a_{\tau(kn)n} = \sum_{\sigma \in J} \phi(\sigma) \sum_{\tau \in I(\sigma)} a_{\tau(1)1} \ldots a_{\tau(kn)n}.$$
If we take $A = I_{n,k}X$, $X = (x_{ij}) \in M_n$, then we have
\[ f(I_{n,k}X) = \sum_\sigma \phi(\sigma) \sum_{\tau \in I(\sigma)} x_\tau^{M(\tau)} = \sum_\sigma \phi(\sigma) \# I(\sigma)x_\tau^{M(\sigma)}, \]
where
\[ x_\tau^{M(\sigma)} = \prod_{i,j} m_{ij}(\sigma), \quad m_{ij}(\sigma) = \# \{ s \mid (i-1)k < s \leq ik, (j-1)k < \sigma(s) \leq jk \}. \]

Notice that $M(\sigma)$ depends only on the double coset $\mathfrak{S}_k^\alpha \mathfrak{S}_k^\beta$. On the other hand, by (W2), we have $f(I_{n,k}X) = f(I_{n,k})(\det X)^k$. Hence we get
\[ \phi(\sigma) = \frac{f(I_{n,k})}{\# I(\sigma)} \times \text{coefficient of } x_\tau^{M(\sigma)} \text{ in } (\det X)^k. \quad (4.1) \]
As a result, we have
\[ f(A) = \frac{f(I_{n,k})}{\wr \det A} \wr \det A \]
as desired.

### 4.2 Proof of Lemma 3.3

It is enough to prove that the sum $\sum_\theta \phi(\theta)$ is divisible by $(1 + \alpha) \ldots (1 + (k-1)\alpha)$. We see that $\sum_\theta \phi(\theta)$ is equal to
\[ \sum_{\tau \in \mathfrak{S}_N} \left( \sum_{\sigma \in \mathfrak{S}_k} \alpha^{\nu(\tau_0)} \right) \prod_{i=1}^N a_{\tau(i)i}. \]
For each $\tau \in \mathfrak{S}_N$, there uniquely exists $\tau_0 \in \mathfrak{S}_k$ such that $\nu(\tau_0) = \nu(\tau_0^{-1}) + \nu(\tau_0\sigma)$ for any $\tau \in \mathfrak{S}_k$. In fact, if we define $g_i$ and $\tau_i$ for $i = n, n-1, \ldots, 1$ recursively by
\[ \tau_n = \tau; \quad g_i = (i \; \tau_i(i)), \quad \tau_{i-1} = g_i \tau_i, \]
then we have $\tau_0 = g_k g_{k-1} \ldots g_1$. It follows that
\[ \sum_{\sigma \in \mathfrak{S}_k} \alpha^{\nu(\tau_0)} = \alpha^{\nu(\tau_0^{-1})} \det_\alpha 1_k = \alpha^{\nu(\tau_0^{-1})}(1 + \alpha) \ldots (1 + (k-1)\alpha). \]

Hence the sum $\sum_\theta \phi(\theta)$ is divisible by $(1 + \alpha) \ldots (1 + (k-1)\alpha)$.

### 4.3 Proof of Lemma 3.4

Let $X_1, \ldots, X_N$ be the Jucys-Murphy elements of the group algebra $\mathbb{C}\mathfrak{S}_N$:
\[ X_k = (1 \; k) + (2 \; k) + \cdots + (k-1 \; k) \quad (1 \leq k \leq N). \]
It is elementary to see that
\[ \phi := \sum_{\sigma \in \mathfrak{S}_N} \alpha^{\nu(\sigma)} \sigma = (1 + \alpha X_1)(1 + \alpha X_2) \ldots (1 + \alpha X_N), \]
which is central since $\nu$ is a class function. So it is a linear combination of the projections
\[ P_\lambda := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \chi^\lambda(\sigma) \sigma \quad (\lambda \vdash N). \]

For each partition $\lambda \vdash N$, let $\{v_T\}_{T \in \text{Stab}(\lambda)}$ be the Gelfand-Tsetlin basis (or Young basis) of the irreducible representation $\mathcal{S}_\lambda$ of $\mathfrak{S}_N$ associated to $\lambda$, where $\text{Stab}(\lambda)$ is the set of standard tableaux with shape $\lambda$. It is known that if the number written in the $(i,j)$-position of $T$ is $k$, then
\[ X_k v_T = (j-i) v_T. \]
Hence it follows that
\[ \phi_{\lambda T} = \prod_{(i,j) \in \lambda} (1 + (j-i)\alpha) \nu_T = f_\lambda(\alpha) \nu_T \]
for any \( T \in \text{STab}(\lambda) \). Thus we have
\[ \text{Tr} \phi \big|_{\mathcal{S}_\lambda} = f_\lambda(\alpha), \]
so that we get
\[ \phi = \sum_{\lambda \vdash N} \text{Tr} \phi \big|_{\mathcal{S}_\lambda} \rho_\lambda = \sum_{\sigma \in \mathcal{S}_N} \left( \frac{1}{N!} \sum_{\lambda \vdash N} f_\lambda(\alpha) \chi^\lambda(\sigma) \right) \sigma \]
as desired.

## 5 Corollaries of the discussion

We obtain the following “determinantal” formula of the values of \( \omega_{\mu}^{(k^n)} \) and the Kostka numbers \( K_{\lambda \mu} \) for rectangular-shaped Young diagrams \( \lambda \) as a byproduct of the discussion above.

**Corollary 5.1.** For any \( g \in \mathfrak{S}_{kn} \) and \( \mu \vdash kn \), it holds that
\[ \omega_{\mu}^{(k^n)}(g) = \frac{f^{(k^n)} \det_{-1/k,1/n}(P(g)1_\mu)}{\mu! \det_{-1/kn}1_{kn}}. \]  
(5.1)
In particular, it holds that
\[ K_{(k^n)\mu} = \frac{f^{(k^n)} \det_{-1/k,1/n}1_\mu}{\mu! \det_{-1/kn}1_{kn}}. \]  
(5.2)

**Proof.** We first notice that \( \text{det}_{-1/kn}1_{kn} = (kn)!/(kn)^{kn} \) by (2.3). Putting \( N = kn, \alpha = -1/k \) and \( \beta = 1/n \) in (5.1), we have
\[ \det_{-1/k,1/n}(P(g)1_\mu) = \frac{\mu!}{(kn)!} \sum_{\lambda \vdash kn} f_\lambda(-1/k) f_\lambda(1/n) \omega_{\mu}^{(k^n)}(g). \]
Since \( f_\lambda(-1/k) = 0 \) unless \( \lambda \leq k \) and \( f_\lambda(1/n) = 0 \) unless \( l(\lambda) = n \), the summand in the right-hand side of the equation above vanishes unless \( \lambda = (k^n) \). Therefore we have
\[ \det_{-1/k,1/n}(P(g)1_\mu) = \frac{\mu!}{f^{(k^n)}(kn)!} \omega_{\mu}^{(k^n)}(g) = \frac{\mu!}{f^{(k^n)}(kn)^{kn} \omega_{\mu}^{(k^n)}(g)} \]
which implies (5.1). The equation (5.2) is readily obtained by putting \( g = 1 \) in (5.1). \( \square \)

By putting \( \mu = (1^{kn}) \) in Corollary 5.1, we have a formula of irreducible characters for rectangular diagrams.

**Corollary 5.2.** It holds that
\[ \chi^{(k^n)}(g) = -\frac{\det_{-1/k,1/n}(P(g)1_{kn})}{\det_{-1/kn}1_{kn}}. \]
(5.3)

**Remark 5.3.** Assume that \( m \leq N = kn \). Let \( i : \mathfrak{S}_m \to \mathfrak{S}_N \) be the natural inclusion. Then we have
\[ \frac{\det_{\alpha,\beta}(P(i(w)))}{\det_{\alpha,\beta}1_N} = \sum_{\sigma \in \mathfrak{S}_m} \alpha^{\nu(\omega(\sigma))} \beta^{\nu(\sigma^{-1})}. \]
Hence, for \( w \in \mathfrak{S}_m \), the formula (5.2) gives Stanley’s formula [6]
\[ \frac{N!}{(N-m)!} \chi^{(k^n)}(i(w)) = (-1)^m \sum_{\sigma \in \mathfrak{S}_m} (-k)^{\chi^{(k^n)}(1)} \kappa(\sigma^{-1}), \]  
(5.4)
where \( \kappa(\sigma) \) denotes the number of disjoint cycles in \( \sigma \).
We look at another particular case where $\mu = (k^n)$. For each $\lambda \vdash N$, we put $\omega^\lambda := \omega^\lambda_\mu$.

**Corollary 5.4.** Let $n, k$ be positive integers. For any $g \in \mathfrak{S}_{kn}$,

$$\omega^{(k^n)}(g) = \frac{\wrdet_k \left( P(g)I_{n,k} \right)}{\wrdet_k I_{n,k}}$$

holds.

**Proof.** As we see in the proof of Corollary 5.1, we have

$$\det_{-1/k,1/n}(P(g)1_{(k^n)}) = \frac{\left( \frac{k!}{(kn)!} f_{(k^n)}(1/n) \right) \omega^{(k^n)}(g)}{\omega^{(k^n)}(g) f_{(k^n)}(1/n)} \wrdet_k I_{n,k}.$$ 

On the other hand, by Theorem 2.2, we have

$$\det_{-1/k,1/n}(P(g)1_{(k^n)}) = \det_{-1/k,1/n} \left( \omega_k \left( P(g)I_{n,k} \right) \right) = f_{(k^n)}(1/n) \wrdet_k \left( P(g)I_{n,k} \right).$$

Combining these two, we obtain the desired conclusion. $\square$

**Remark 5.5.** By Corollary 5.4 and 4.1, we have

$$\omega^{(k^n)}(\sigma) = \text{coefficient of } x^{M(\sigma)} \text{ in } (\det X)^k$$

for $\sigma \in \mathfrak{S}_{kn}$.

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