The number of eigenvalues of the matrix Schrödinger operator on the half line with general boundary conditions

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Abstract

We prove a bound, of Birman-Schwinger type, on the number of eigenvalues of the matrix Schrödinger operator on the half line, with the most general self adjoint boundary condition at the origin, and with selfadjoint matrix potentials that are integrable and have a finite first moment.

1 Introduction

In this paper we study the matrix Schrödinger operator on the half line

\[ H_{A,B} \psi := -\psi'' + V(x) \psi, \quad x \in (0, \infty), \]

where the prime denotes the derivative with respect to the spatial coordinate \( x \). The wavefunction \( \psi(x) \) will be either a column vector with \( n \) components or an \( n \times n \) matrix-valued function. As it is well known, the most general selfadjoint boundary condition at \( x = 0 \) for the operator (1.1) can be formulated in several equivalent way, see [1]-[10]. However, it was proved in [6]-[10] that, without losing generality, it is useful to state them in terms of constant \( n \times n \) matrices \( A \) and \( B \) as follows,

\[ -B^\dagger \psi(0) + A^\dagger \psi'(0) = 0, \]

\[ -B^\dagger A + A^\dagger B = 0, \]

\[ A^\dagger A + B^\dagger B > 0. \]

Remark that \( A^\dagger B \) is selfadjoint.

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In what follows we suppose that the potential $V$ is a $n \times n$ selfadjoint matrix-valued function

$$V(x) = V(x)^\dagger, \quad x \in \mathbb{R}^+.$$  

(1.5)

By the dagger we designate the matrix adjoint.

Furthermore, we assume that $V(x)$ belongs to the Faddeev class, that is to say, each entry of the matrix $V$ is Lebesgue measurable on $(0, \infty)$ and,

$$\int_0^\infty dx \,(1+x)\|V(x)\| < +\infty.$$  

(1.6)

Here, $\|V(x)\|$ designates the norm of $V(x)$ as an operator on $\mathbb{C}^n$. Of course, (1.6) holds if and only if it holds for each entry of $V$.

As usual, we denote by $L^2$ the standard Hilbert space of measurable functions defined on $(0, \infty)$ with values in $\mathbb{C}^n$.

It is proven in [9] (see also Section 3 below) that the formal differential operator (1.1) has a selfadjoint, bounded below, realization in $L^2$, defined by quadratic forms, with the boundary condition (1.2). We also denote this selfadjoint realization by $H_{A,B}$. For this purpose, we only need that (1.5) holds and that the potential matrix is integrable,

$$\int_0^\infty dx \|V(x)\| < +\infty.$$  

(1.7)

Furthermore [9] $H_{A,B}$ has no singular continuous spectrum and its absolutely continuous spectrum is $[0, \infty)$. Moreover, $H_{A,B}$ has no positive or zero eigenvalues and its negative eigenvalues are of finite multiplicity and can only accumulate at zero. If $V$ is in the Faddeev class (1.6) the number of eigenvalues of $H_{A,B}$ is finite [7], [9].

Note that the matrices $A,B$ in (1.2, 1.4) are not uniquely defined. We can multiply them on the right by an invertible matrix $T$ without affecting (1.2), (1.3) and (1.4), and furthermore,

$$H_{AT,B_T} = H_{A,B}.$$  

(1.8)

Let $U$ be the following matrix

$$U := (B - iA) \,(B + iA)^{-1}.$$  

(1.9)

In proposition 4.2 of [7] it is proven that $B + iA$ is invertible and that $U$ is unitary. Clearly, $U$ is invariant under the transformation $(A, B) \rightarrow (AT, BT)$ for any invertible matrix $T$. Let $\mathcal{M}$ be a unitary matrix that diagonalizes $U$,

$$\mathcal{M}^\dagger U \mathcal{M} = \text{diag} \left\{ e^{2i \cos \theta_1}, e^{2i \cos \theta_2}, \ldots, e^{2i \cos \theta_n} \right\},$$  

(1.10)
where $0 < \cos \theta_j \leq \pi$. In the general case there are $n_N$ values with $\theta_j = \pi/2$ and $n_D$ values with $\theta_j = \pi$, and in consequence there are $n_M$ remaining values, with $n_M := n - n_N - n_D$, such that the corresponding $\theta_j$-values lie in the interval $(0, \pi/2)$ or $(\pi/2, \pi)$. We allow for the special cases where any of $n_N$, $n_D$, and $n_M$ may be zero or $n$. The subscripts N, D, and M refer, respectively, to Neumann, Dirichlet or mixed boundary conditions. In fact, in the representation where the matrices $A, B$ are diagonal $\theta_j = \pi/2$ corresponds to the Neumann boundary condition, $\theta_j = \pi$ to the Dirichlet boundary condition and $\theta$ in $(0, \pi/2) \cup (\pi/2, \pi)$ to mixed boundary conditions. See Section 2 for these issues.

Let us decompose $V$ into its positive and negative parts, i.e.,

$$V(x) = V_+(x) - V_-(x), \text{ with } V_\pm(x) \geq 0.$$  \hspace{1cm} (1.11)

By $V_\pm(x) \geq 0$ we mean that the matrices $V_\pm(x)$ have nonnegative eigenvalues, or equivalently, that they are nonnegative operators on $\mathbf{C}^n$.

We denote by $n_{M,b}$ the number of $\theta_j$ with $0 < \cos \theta_j < \pi/2$. We designate by $\Theta_T$ the following diagonal $n \times n$ matrix,

$$\Theta_T := \text{diag} \left\{ \overline{\tan \theta_1}, \overline{\tan \theta_2}, \cdots, \overline{\tan \theta_n} \right\},$$  \hspace{1cm} (1.12)

where $\overline{\tan \theta_j} = 0$ if $0 < \theta_j \leq \pi/2$, $\overline{\tan \theta_j} = \tan \theta_j$ if $\pi/2 < \theta_j \leq \pi$, for $j = 1, 2, \cdots, n$.

Our Birman-Schwinger bound is the following theorem.

**THEOREM 1.1.** Suppose that the boundary conditions are given by (1.2) where the matrices $A, B$ satisfy (1.3), (1.4) and the matrix potential satisfies (1.5, 1.6). Let us denote by $N_{A,B}$ the number of negative eigenvalues of $H_{A,B}$. Then,

$$N_{A,B} \leq n_{M,b} + n_N + \int_0^\infty \text{trace} \left[ V_-(x) (x - \Theta_T) \right] dx.$$  \hspace{1cm} (1.13)

Note that it is necessary to have $n_{M,b}$ and $n_N$ in the right-hand side of (1.13), see Remark 5.1.

There is currently a considerable interest on this problem. Matrix Schrödinger operators on the half line are important in quantum graphs, in quantum wires and in quantum mechanical scattering of particles with internal structure. See, for example, [1]-[5] and [11]-[23], and the references quoted there.

The paper is organized as follows. In Section 2 we introduce notations and definitions and we state preliminary results that we need. In Section 3, following [9], we define the matrix Schrödinger operator as a selfadjoint operator by means of quadratic form techniques and we estimate the difference in the number of bound states of two matrix Schrödinger operators with the same potential and with different boundary conditions. In Section 4 we state results
from [9] on the integral kernel of the resolvent in the case where the potential matrix is identically zero. In Section 5 we prove our Birman-Schwinger bound on the number of bound states.

2 Notations, definitions and preliminary results

We designate by $C^+$ the upper-half complex plane, by $\mathbf{R}$ the real axis, and we let $\mathbb{C}^+ := C^+ \cup \mathbf{R}$. For any $k \in \mathbb{C}^+$ we denote by $k^*$ its complex conjugate. For any matrix $D$ we designate by $D^\dagger$ its adjoint.

By $H_l, l = 1, 2, \cdots, n$, we denote the Sobolev space of order $l$ of all square integrable, complex valued functions defined in $(0, \infty)$ with all distributional derivatives up to order $l$ given by square integrable functions [25]. We designate by $H^1_0((0, \infty))$ the completion of $C^\infty_0((0, \infty))$ in the norm of $H^1$, where $C^\infty_0((0, \infty))$ is the space of all infinitely differentiable complex valued functions with compact support. We use the notation,

$$H_l := \bigoplus_{j=1}^n H_l, \quad l = 1, 2,$$

for the first and second Sobolev spaces of functions with values in $C^n$.

For any trace class operator $G$ we denote by trace($G$) its trace. For any densely defined operator $D$ in a Banach space we denote by $\rho(D)$ its resolvent set, i.e., the open set of all $z \in \mathbb{C}$ such that $D - z$ is invertible and $(D - z)^{-1}$ is bounded. We denote the resolvent of $D$ by $R_D(z) := (D - z)^{-1}$ for $z \in \rho(D)$ [26].

Motivated by the general selfadjoint boundary condition [27, 28, 29] in the scalar case, i.e. when $n = 1$, we consider the case where the matrices $A, B$ are diagonal. We denote this special pair of diagonal matrices by $\tilde{A}$ and $\tilde{B}$, where

$$\tilde{A} := -\text{diag}\{\sin \theta_1, \ldots, \sin \theta_n\}, \quad \tilde{B} := \text{diag}\{\cos \theta_1, \ldots, \cos \theta_n\}.$$  \hspace{1cm} (2.2)

For these matrices the boundary conditions (1.2) are,

$$\cos \theta_j \psi_j(0) + \sin \theta_j \psi_j'(0) = 0, \quad j = 1, 2, \cdots, n.$$  \hspace{1cm} (2.3)

The real parameters $\theta_j$ take values in the interval $(0, \pi]$. The case $\theta_j = \pi/2$ corresponds to the Neumann boundary condition, case $\theta_j = \pi$ corresponds to the Dirichlet boundary condition, and the case where $\theta_j \neq \pi/2, \pi$ corresponds to mixed boundary conditions. We suppose that there are $n_\text{N}$ values with $\theta_j = \pi/2$ and $n_\text{D}$ values with $\theta_j = \pi$, and hence there are $n_\text{M}$ remaining values, with $n_\text{M} := n - n_\text{N} - n_\text{D}$, such that the corresponding $\theta_j$-values lie in the interval $(0, \pi/2)$ or $(\pi/2, \pi)$. The special cases where any of $n_\text{N}, n_\text{D},$ and $n_\text{M}$ may be zero or $n$ are allowed. Observe that $\tilde{A}, \tilde{B}$ satisfy (1.3), (1.4).

In Proposition 4.3 of [7] it is proven that for any pair of matrices $(A, B)$ that satisfy (1.2)-(1.4) there is a pair of
diagonal matrices \((\tilde{A}, \tilde{B})\) as in (2.2), a unitary matrix \(M\) and a two invertible matrices \(T_1, T_2\) such that,
\[
A = M \tilde{A} T_2 M^\dagger T_1, \quad B = M \tilde{B} T_2 M^\dagger T_1. \tag{2.4}
\]
Note that \(T_1, T_2\) in (2.4) correspond, respectively to \(T_1^{-1}, T_2^{-1}\) in Proposition 4.3 of [7]. Furthermore (see the proof of Proposition 4.3 of [7]) the \(\theta_j, j = 1, 2, \cdots, n\) in (2.2) coincide with \(\theta_j, j = 1, 2, \cdots, n\) that appear in the diagonal representation of the matrix \(U\) in (1.10).

We consider some \(n \times n\) matrix solutions to the equation
\[
-\psi'' + V(x) \psi = k^2 \psi, \quad x \in (0, \infty), k \in \mathbb{C}^+, \tag{2.5}
\]
assuming that \(V\) satisfies (1.5), (1.7).

The Jost solution (see [24]) to (2.5) is the \(n \times n\) matrix solution satisfying, for \(k \in \mathbb{C}^+\), the asymptotics
\[
f(k, x) = e^{ikx}[I_n + o(1/x)], \quad f'(k, x) = ik e^{ikx}[I_n + o(1/x)], \quad x \to +\infty, \tag{2.6}
\]
where \(I_n\) denotes the \(n \times n\) identity matrix. It is well known [6, 24], that for each fixed \(x\), \(f(k, x)\) and \(f'(k, x)\) are analytic for \(k \in \mathbb{C}^+\) and continuous for \(k \in \overline{\mathbb{C}^+}\). It follows from (2.6) that for each fixed \(k \in \mathbb{C}^+\), each of the \(n\) columns of \(f(k, x)\) decays exponentially to zero as \(x \to +\infty\).

The matrix Schrödinger equation (2.5) also has the \(n \times n\) matrix solution \(g(k, x)\) that satisfies, for each \(k \in \overline{\mathbb{C}^+}\), the following asymptotics [24]
\[
g(k, x) = e^{-ikx}[I_n + o(1/x)], \quad g'(k, x) = -ik e^{-ikx} (I + o(1/x)), \quad x \to \infty. \tag{2.7}
\]

It is proven in [24] that \(g(k, x)\) and \(g'(k, x)\) are analytic in \(k \in \mathbb{C}^+\) and continuous in \(k \in \overline{\mathbb{C}^+}\) for each fixed \(x\). Equation (2.7) implies that each of the \(n\) columns of \(g(k, x)\) grows exponentially as \(x \to +\infty\) for each fixed \(k \in \mathbb{C}^+\).

On page 28 of [24] it is proven that for each \(k \in \overline{\mathbb{C}^+}\), the combined \(2n\) columns of \(f(k, x)\) and of \(g(k, x)\) form a fundamental set of solutions to (2.5). Hence, any column-vector solution \(\omega(k, x)\) to (2.5) can be written as a linear combination of them,
\[
\omega(k, x) = f(k, x) \xi + g(k, x) \eta, \tag{2.8}
\]
for some constant column vectors \(\xi\) and \(\eta\) in \(\mathbb{C}^n\).

Another important \(n \times n\) matrix solution to (2.5) is the regular solutions, \(\varphi_{A,B}(k, x)\) that satisfies the initial conditions,
\[
\varphi_{A,B}(k, 0) = B, \quad \varphi'_{A,B}(k, 0) = A. \tag{2.9}
\]
3 The matrix Schrödinger operator

Here we follow [9] where details and proof are given.

3.1 The case of zero potential

See Subsection 4.1 of [9]. We denote by $H_{0,A,B}$ the self-adjoint realization of $-\frac{d^2}{dx^2}$ with the boundary condition (1.2), namely

$$H_{0,A,B} \psi = -\frac{d^2}{dx^2} \psi, \quad \psi \in D(H_{0,A,B}),$$

(3.1)

where

$$D(H_{0,A,B}) := \{ \psi \in H^2 : -B^\dagger \psi(0) + A^\dagger \psi'(0) = 0 \}.$$  

(3.2)

Note that $H_{0,AT,BT} = H_{0,A,B}$ for all invertible matrices $T$. Recall that in the particular case of the diagonal matrices $\hat{A}, \hat{B}$ (2.2) the boundary conditions (1.2) are given by (2.3). These equations can be written as,

$$\psi'(0) = -\cot \theta_j \psi_j(0), \text{ if } \theta_j \neq \pi, \text{ and } \psi_j(0) = 0, \text{ if } \theta_j = \pi.$$  

(3.3)

Let us construct the quadratic form associated to $H_{0,\hat{A},\hat{B}}$. We denote,

$$H_{1,j} := H_{1,0}, \text{ if } \theta_j = \pi, \text{ and } H_{1,j} := H_1, \text{ if } \theta_j \neq \pi.$$  

(3.4)

We designate,

$$H_{1,\hat{A},\hat{B}} := \oplus_{j=1}^n H_{1,j}.$$  

(3.5)

We define the quadratic form with domain $H_{1,\hat{A},\hat{B}}$,

$$h_{0,\hat{A},\hat{B}}(\varphi, \psi) := (\varphi', \psi') - \sum_{j=1}^n \cot \theta_j \varphi_j(0) \psi_j(0),$$  

(3.6)

where $\cot \theta_j = 0$ if $\theta_j = \pi/2$, or $\theta_j = \pi$, and $\cot \theta_j = \cot \theta_j$ if $\theta_j \neq \pi/2, \pi$.

The symmetric form $h_{0,\hat{A},\hat{B}}$ is closed and bounded below. It follows from Theorems 2.1 and 2.6 in chapter 6 of [26] that $H_{0,\hat{A},\hat{B}}$ is the selfadjoint bounded below operator associated to the quadratic form $h_{0,\hat{A},\hat{B}}$.

We define the diagonal matrix

$$\Theta := \text{Diag}\{\hat{\cot} \theta_1, \hat{\cot} \theta_2, \cdots \hat{\cot} \theta_n\}.$$  

(3.7)

The quadratic form associated to $H_{0,A,B}$ is given by,
\[ h_{0,A,B}(\varphi,\psi) := (\varphi',\psi') - \sum_{j=1}^{n} \langle M\Theta M^\dagger \varphi(0),\psi(0) \rangle, \quad (3.8) \]

where by \( \langle \cdot,\cdot \rangle \) we denote the scalar product in \( \mathbb{C}^n \), and the domain of \( h_{0,A,B} \) is given by

\[ D(h_{0,A,B}) = \mathcal{H}_{1,A,B} \quad \text{where} \quad \mathcal{H}_{1,A,B} := M\mathcal{H}_{1,\tilde{A},\tilde{B}} \subset \mathcal{H}_1. \quad (3.9) \]

### 3.2 The case of integrable potential

See Subsection 4.2 of [9]. Suppose that \( V \) satisfies (1.5), (1.7). Let us define the following quadratic form,

\[ h_{A,B}(\varphi,\psi) := h_{0,A,B}(\varphi,\psi) + (V\varphi,\psi), \quad D(h_{A,B}) = \mathcal{H}_{1,A,B}. \quad (3.10) \]

The symmetric form \( h_{A,B} \) is closed, and bounded below. Let us denote by \( H_{A,B} \) the associated bounded below selfadjoint operator (see theorems 2.1 and 2.6 in chapter 6 of [26]). Note that

\[ D(H_{A,B}) = \{ \psi \in \mathcal{H}_{1,A,B} : -B^\dagger \psi(0) + A^\dagger \psi'(0) = 0, -\psi'' + V\psi \in L^2 \}. \]

It proven in [9] that under the transformation (2.4)

\[ H_{V,A,B} = MH_{M^\dagger VM,\tilde{A},\tilde{B}} M^\dagger, \quad (3.11) \]

where we made explicit the dependence in \( V \) of the matrix Schrödinger operator. Actually, this follows from the fact that \( H_{A,B} \) is the selfadjoint operator associated to the quadratic form \( h_{A,B} \) and from the uniqueness of the selfadjoint operator associated to a quadratic form (Theorem 2.1 and Theorem 2.6 of [26]).

We wish to estimate the difference in the number of bound states of two matrix Schrödinger operators with the same potential and with different boundary conditions. For simplicity we assume that \( V \) satisfies (1.5), (1.7), that it is bounded and that \( V(x) = 0, 0 \leq 0 \leq \delta \) for some \( \delta > 0 \). In this case \( H_{A,B} \) is just the operator sum of \( H_{0,A,B} \) and \( V \) and \( D(H_{A,B}) = D(H_{0,A,B}) \). Since the number of bound states is invariant under unitary transformations by (3.11) it is enough to study the case where the boundary matrices are diagonal. So, let us consider two sets of diagonal matrices by \( (\tilde{A}_1,\tilde{B}_1) \) and \( (\tilde{A}_2,\tilde{B}_2) \) where

\[ \tilde{A}_j := -\text{diag}\{\sin \theta_{j,1},\ldots,\sin \theta_{j,n}\}, \quad \tilde{B}_j := \text{diag}\{\cos \theta_{j,1},\ldots,\cos \theta_{j,n}\}, \quad j = 1,2. \quad (3.12) \]

Let \( L := \{l_1, l_2, \cdots, l_q\} \) where \( 1 \leq q \leq n \) be a subset of \( \{1,2,\cdots,n\} \), and assume that \( \theta_{1,l_m} = \theta_{2,l_m}, m = 1,2,\cdots,q \). We denote,
\[ D_L := \{ \varphi \in C^\infty([0, \infty), \mathbb{C}^n) : \text{For } m = 1, 2, \cdots, q, \varphi_m \text{ has compact support in } [0, \infty), \text{and} \]
\[ \cos \theta_{j,m} \varphi_m(0) + \sin \theta_{j,m} \varphi'_m(0) = 0. \text{ For } m \in \{1, 2, \cdots, n\} \setminus \{l_1, l_2, \cdots, l_q\}, \]
\[ (3.13) \]
\[ \varphi_m \text{ has compact support in } (0, \infty) \} . \]

Let us denote by \( h_0 \) the symmetric operator that acts as \(-\Delta + V\) on the domain \( D(h_0) = C_0^\infty(0, \infty)\), and let us designate by \( h_L \) the symmetric operator that acts as \(-\Delta + V\) on the domain \( D(h_L) = D_L \). Note that \( H_{A_j, B_j}, j = 1, 2 \) are selfadjoint extensions of \( h_L \).

Let us compute the deficiency indices of \( h_L \). We denote by \( h \) the adjoint of \( h_0 \). By Theorem 3.6 of [30] and since \( V \) is bounded, \( D(h) = \{ \varphi \in L^2 : \varphi \text{ and } \varphi' \text{ are absolutely continuous in } (0, \infty) \text{ and } \varphi'' \in L^2 \} \).

Since \( h_0 \subset h_L \) we have that, \( h_L^* \subset h \). As by Theorem 3.2 of [30] if \( \psi \in D(h), \psi \) and \( \psi' \) are continuous a \( x = 0 \), it follows integrating by parts that,
\[ (h_L \varphi, \psi) - (\varphi, h_L^* \psi) = \varphi'(0) \psi(0) - \varphi(0) \psi'(0), \forall \varphi \in D(h_L), \forall \psi \in D(h_L^*). \]

It follows that \( \cos \theta_{j,m} \psi_{l_m}(0) + \sin \theta_{j,m} \psi'_{l_m}(0) = 0, m = 1, 2, \cdots, q \) for all \( \psi \in D(h_L^*). \)

To compute the deficiency indices of \( h_L \) we have to calculate the number of linearly independent solutions of,
\[ (-\Delta + V) \varphi^\pm = \pm i \varphi^\pm, \text{ with } \varphi^\pm \in D(h) \]
(3.14)
such that
\[ \cos \theta_{j,m} \varphi^\pm_m(0) + \sin \theta_{j,m} \varphi'^\pm_m(0) = 0, m = 1, 2, \cdots, q. \]
(3.15)
By (2.8) and since \( \varphi^\pm \in L^2 \),
\[ \varphi^\pm = f(\sqrt{\pm i}, x) v^\pm, \text{ for some } v^\pm \in \mathbb{C}^n. \]
But, since \( V(x) = 0, 0 \leq x \leq \delta, \) for some \( \delta > 0, \)
\[ f(\sqrt{\pm i}, x)_{l, j} = f_l(\sqrt{\pm i}, \delta)_{l, j}, l, j = 1, \cdots, n, 0 \leq x \leq \delta, \]
where \( f_l(\sqrt{\pm i}, x) = \left( a_{\pm l} e^{i \sqrt{\pm i} x} + b_{\pm l} e^{-i \sqrt{\pm i} x} \right) \) for some complex numbers \( a_{\pm l}, b_{\pm l}, l = 1, 2, \cdots, n \). Then,
\[ \cos \theta_{j,m} \varphi^\pm_{l_m}(0) + \sin \theta_{j,m} \varphi'^\pm_{l_m}(0) = \left( \cos \theta_{j,m} f_{l_m}(\sqrt{\pm i}, 0) + \sin \theta_{j,m} f'_{l_m}(\sqrt{\pm i}, 0) \right) v^\pm_{l_m}. \]
We prove that \( \cos \theta_{j,m} f_{l_m}(\sqrt{\pm i}, 0) + \sin \theta_{j,m} f'_{l_m}(\sqrt{\pm i}, 0) \neq 0 \) for \( m = 1, 2, \cdots, q \). Suppose, on the contrary, that some \( \cos \theta_{j,m} f_{l_m}(\sqrt{\pm i}, 0) + \sin \theta_{j,m} f'_{l_m}(\sqrt{\pm i}, 0) = 0 \). Take \( \xi^\pm = (\xi^\pm_1, \xi^\pm_2, \cdots, \xi^\pm_n) \in \mathbb{C}^n \) with \( \xi^\pm_{l_m} = 1 \) and \( \xi^\pm_l = 0, l \neq l_m. \)
Then, \( f(\sqrt{-i}, x)\xi^\pm \in L^2 \) will satisfy the boundary condition \(-\hat{\mathcal{B}}^\dagger f(\sqrt{-i}, 0)\xi^\pm + \hat{\mathcal{A}}^\dagger f'(\sqrt{-i}, 0)\xi^\pm = 0\) and this is not possible since as \( \hat{\mathcal{H}}_{\hat{\mathcal{A}},\hat{\mathcal{B}}} \) is selfadjoint it can not have complex eigenvalues.

Then, (3.15) implies that \( v_m^\pm = 0, m = 1, 2, \cdots q \). Hence, as it follows from (2.6) that the columns of \( f(k, x) \) are linearly independent solutions of (2.5) the number of linearly independent solutions \( \varphi^\pm \) of (3.14) that satisfy (3.15) is \( n - q \), and it follows that both deficiency indices of \( h_L \) are equal to \( n - q \).

Since \( \hat{\mathcal{H}}_{\hat{\mathcal{A}},\hat{\mathcal{B}}}, j = 1, 2 \) are selfadjoint extensions of \( h_L \) it follows from Lemma 2 and Theorem 3 in Section 3 of Chapter 9 of [31] that,

\[
N_{\hat{\mathcal{A}}_1,\hat{\mathcal{B}}_1} - q \leq N_{\hat{\mathcal{A}}_2,\hat{\mathcal{B}}_2} \leq N_{\hat{\mathcal{A}}_1,\hat{\mathcal{B}}_1} + q.  
\tag{3.16}
\]

### 4 The resolvent with zero potential

See Subsection 5.1 of [9] for the results below. We first consider the case of the diagonal matrices \( \hat{\mathcal{A}}, \hat{\mathcal{B}} \) given in (2.2). Let us denote by \( R_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(z) \) the resolvent of \( H_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}} \).

Let, \( R_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(z)(x,y) \) be the integral kernel of \( R_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(z) \). Then, we have that,

\[
R_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(z)(x,y) = \begin{cases} 
\varphi_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(x,k)e^{iky} [J_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(k)]^{-1}, & x \leq y, \\
e^{ikx} \varphi_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(y,k) [J_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(k)]^{-1}, & x \geq y,
\end{cases} \tag{4.1}
\]

where, \( k := \sqrt{z}, \text{Im} k \geq 0, \varphi_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(k,x) \) is the regular solution (2.9) with zero potential,

\[
\varphi_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(k,x) = \begin{cases} 
-\frac{1}{k} \sin kx, & \text{if } \theta_j = \pi, \\
-\cos kx, & \text{if } \theta_j = \pi/2, \\
\frac{1}{k} \cos \theta_j \sin kx - \sin \theta_j \cos kx, & \text{if } \theta_j \neq \pi, \pi/2,
\end{cases} \tag{4.2}
\]

and and \( J_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}} \) is the Jost matrix,

\[
J_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(k) = \text{diag} \{\cos \theta_1 + ik \sin \theta_1, \ldots, \cos \theta_{nm} + ik \sin \theta_{nm}, -I_{n_0}, ik I_{n_0}\}. \tag{4.3}
\]

We designate by \( R_{0,A,B}(z) \) the resolvent of \( H_{0,A,B} \). Then, by (2.4), (3.11),

\[
R_{0,A,B}(z) = M R_{0,\hat{\mathcal{A}},\hat{\mathcal{B}}}(z) M^1, \quad z \in \rho(H_{0,A,B}). \tag{4.4}
\]

Its integral kernel is given by,

\[
R_{0,A,B}(z)(x,y) = \begin{cases} 
\varphi_{0,A,B}(x,k)e^{iky} [J_{0,A,B}(k)]^{-1}, & x \leq y, \\
e^{ikx} \varphi_{0,A,B}(y,k) [J_{0,A,B}(k)]^{-1}, & x \geq y.
\end{cases} \tag{4.5}
\]

where \( J_{0,A,B} = B - ikA \) is the Jost matrix. The following estimate holds [9],

\[
|R_{0,A,B}(z)(x,y)| \leq CD(k) e^{-\text{Im}k|x-y|}, D(k) := \max \left[ \frac{1}{|k|}, \frac{1}{|\cos \theta_1 + ik \sin \theta_1|}, \ldots, \frac{1}{|\cos \theta_{nm} + ik \sin \theta_{nm}|} \right], \tag{4.6}
\]

where \( k = \sqrt{z} \).
5 The number of eigenvalues

Let us first consider the case of zero potential. It is convenient to go to the diagonal representation where the boundary matrices are given by (2.2) and the boundary conditions by (2.3) and with \( n_N, n_D, \) and \( n_M \) defined below (2.3). It follows from a simple calculation that the \( n_N \) Neumann boundary conditions and the \( n_D \) Dirichlet boundary conditions give rise to no eigenvalues. Also the mixed boundary conditions with \( \tan \theta_j < 0 \) produce no eigenvalues. However, the mixed boundary conditions with \( \tan \theta_j > 0 \) produce the eigenvalue \(- (\cot \theta_j)^2\). To deal with this issue we will take advantage of the fact that changing \( q \) boundary conditions can only add \( q \) eigenvalues, to turn the mixed boundary conditions with \( \tan \theta_j > 0 \) into boundary conditions with \( \tan \theta_j < 0 \). We will also find convenient to turn the Neumann boundary conditions into mixed boundary conditions with \( \tan \theta_j < 0 \), to avoid a singularity at zero energy that appears in the integral kernel of the resolvent with zero potential (4.1) when there are Neumann boundary conditions present.

For any bounded below selfadjoint operator \( H \) let us denote

\[
\mu_n(H) := \sup_{\varphi_1, \varphi_2, \ldots, \varphi_{n-1}} \inf_{\varphi \in Q(H), \|\varphi\|=1, \varphi \text{ orthogonal to } \varphi_1, \varphi_2, \ldots, \varphi_{n-1}} (H\varphi, \varphi),
\]

(5.7)

where \( Q(H) \) is the quadratic form domain of \( H \). Then, by the min-max principle (Theorem XIII.1 of [32]), either there are \( n \) eigenvalues, repeated according to multiplicity, below the bottom of the essential spectrum and \( \mu_n(H) \) is the \( n \)th eigenvalue repeated according to multiplicity or \( \mu_n \) is the bottom of the essential spectrum and \( \mu_n = \mu_{n+1} = \mu_{n+2} = \cdots \).

Recall that under (1.5), (1.6) the operator \( H_{A,B} \) has a finite number of negative eigenvalues, it has no zero or positive eigenvalues and the essential spectrum is given by \([0, \infty)\) and it is absolutely continuous. For these results see [9].

For any \( E < 0 \) we denote by \( N_{A,B}(E) \) the number of eigenvalues, repeated according to multiplicity, of \( H_{A,B} \) that are smaller than \( E \).

Let \( V_\pm \) be defined as in (1.11). Let us denote by \( H_{A,B,V_-} \) the selfadjoint operator associated to the quadratic form (3.10), but with \( V_- \) instead of \( V \). By \( N_{A,B,V_-}(E) \) we denote the number of eigenvalues, repeated according to multiplicity, of \( H_{A,B,V_-} \) that are smaller than \( E \). By the minimax principle,

\[
N_{A,B}(E) \leq N_{A,B,V_-}.
\]

It follows that it is enough to prove Theorem 1.1 in the case \( V \leq 0 \).

We have proven in [9] that,

\[
R_{A,B}(z) = R_{0,A,B}(z) - R_{0,A,B}(z) V_2 (I + V_1 R_{0,A,B}(z) V_2)^{-1} V_1 R_{0,A,B}(z), \quad z \in \rho(H_{0,A,B}) \cap \rho(H_{A,B}).
\]

(5.8)

where, since \( V \leq 0 \), \( V_1 := \sqrt{|V|}, V_2 := -\sqrt{|V|} \).
We say that $V \in C_0^\infty(0, \infty)$ if each of the entries of $V$ is a function in $C_0^\infty(0, \infty)$. Let $Q_j(x) \in C_0^\infty((0, \infty)$ be a sequence of matrix potentials such that, $Q_j(x) \geq 0$ and

$$\lim_{j \to \infty} \int_0^\infty \| \sqrt{|V|(x)} - Q_j(x) \|^2 = 0.$$ 

Denote

$$V_j = V_{1,j} V_{2,j},$$

where $V_{1,j} := Q_j$, $V_{2,j} := -Q_j$.

Let us denote by $H_{A,B,j}$ the selfadjoint operator associated to the quadratic form (3.10) with $V_j$ instead of $V$. Note that since $V_j \in C_0^\infty(0, \infty)$ the operator $H_{A,B,j}$ is just the operator sum $H_{0,A,B} + V_j$.

By (4.6), the operators $V_1 R_{0,A,B}(z) V_2$ and $V_{1,j} R_{0,A,B}(z) V_{2,j}$ are Hilbert-Schmidt for $z \in \mathbb{C}_\pm$ and

$$\lim_{j \to \infty} V_{1,j} R_{0,A,B}(z) V_{2,j} = V_1 R_{0,A,B}(z) V_2,$$

where the limit is on the Hilbert-Schmidt norm. We designate by $R_{A,B,j} := (H_{A,B,j} - z)^{-1}$. Then, by (5.8),

$$\lim_{j \to \infty} R_{A,B,j}(z) = R_{A,B}(z), \quad z \in \mathbb{C}_\pm,$$

in operator norm. Hence, $H_{A,B,j}$ converges to $H_{A,B}$ in norm resolvent sense, and by Theorem VIII.23 in [33], for any $E < 0$ that is not an eigenvalue of $H_{A,B}$

$$\lim_{j \to \infty} P_{(-\infty,E)}(H_{A,B,j}) = P_{(-\infty,E)}(H_{A,B}),$$

in operator norm, where $P_{(-\infty,E)}(H_{A,B,j})$, respectively, $P_{(-\infty,E)}(H_{A,B})$ denote the spectral projector for $(-\infty, E)$ of $H_{A,B,j}$ and $H_{A,B}$. Note that

$$N_{A,B}(E) = \dim P_{(-\infty,E)}(H_{A,B}) L^2, \quad N_{A,B,j}(E) = \dim P_{(-\infty,E)}(H_{A,B,j}) L^2.$$

Then, by Theorem 6.32 in chapter one of [26],

$$\lim_{j \to \infty} N_{A,B,j}(E) = N_{A,B}(E), \quad (5.9)$$

and actually the equality in (5.9) is obtained for a large enough finite $j$. In conclusion, it is enough to prove (1.13) for $V \in C_0^\infty(0, \infty)$, with $V \leq 0$ and this what we proceed to do now.

It is convenient to go to the diagonal representation where the boundary matrices are given by (2.2) and the boundary conditions by (2.3) and with $n_N$, $n_D$, and $n_M$ defined below (2.3)

Let us consider the operator,

$$H_{A,B}(\lambda) := H_{0,A,B} + \lambda V, \quad (5.10)$$

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for $0 \leq \lambda \leq 1$. By perturbation theory [26] the eigenvalues of $H_{\tilde{A},\tilde{B}}(\lambda)$ are continuous functions of $\lambda$ (they are branches of holomorphic functions) and are non increasing functions of $\lambda$ because $V \leq 0$. Then, by the mini max principle the $\mu_n(H_{\tilde{A},\tilde{B}}(\lambda))$ are continuous non increasing functions of $\lambda$. Suppose, moreover, that there are no Neumann boundary conditions and no mixed boundary conditions with $0 < \theta_j < \pi/2$. Then, since $H_{\tilde{A},\tilde{B},0}$ has no bound states, $\mu_n(H_{\tilde{A},\tilde{B}}(0)) = 0$.

The crunch of the Birman-Schwinger [34], [35] method is the following. By the min-max principle,

$$N(H_{\tilde{A},\tilde{B}})(E) = \# \left\{ n : \mu_n(H_{\tilde{A},\tilde{B}}(1)) = E \right\}. \quad (5.11)$$

However, as $\mu_n(H_{\tilde{A},\tilde{B}}(0)) = 0$, since $H_{\tilde{A},\tilde{B}}$ has no eigenvalues, and $\mu_n(H_{\tilde{A},\tilde{B}}(\lambda))$ are non increasing functions,

$$N(H_{\tilde{A},\tilde{B}})(E) = \# \left\{ n : \mu_n(H_{\tilde{A},\tilde{B}}(\lambda)) = E, \text{ for some } 0 < \lambda < 1 \right\} \leq \sum_{\lambda: 0 < \lambda < 1, \mu_n(\lambda) = E, n = 1, 2, \cdots} \frac{1}{\lambda}. \quad (5.12)$$

Suppose that $E < 0$ is an eigenvalue of $H_{\tilde{A},\tilde{B}}(\lambda)$ with eigenvector $\varphi$, i.e.,

$$\left( H_{\tilde{A},\tilde{B}} + \lambda V - E \right) \varphi = 0. \quad (5.13)$$

Equation (5.13) holds if and only if,

$$\lambda V \varphi = V_1 \left( R_{0,\tilde{A},\tilde{B}}(E) V_1 \right) V_1 \varphi. \quad (5.14)$$

But (5.14) is equivalent to

$$\mathcal{B} \psi = \frac{1}{\lambda} \psi \quad \text{for some non zero } \psi \in L^2, \quad (5.15)$$

where $\mathcal{B}$ is the operator

$$\mathcal{B} := V_1 R_{0,\tilde{A},\tilde{B}}(E) V_1. \quad (5.16)$$

Note that $\mathcal{B}$ is a selfadjoint and non negative operator. Then $E$ is an eigenvalue of $H_{\tilde{A},\tilde{B}}(\lambda)$, if and only if $1/\lambda$ is an eigenvalue of $\mathcal{B}$ and the multiplicity is the same. Equivalently $\mu_n(\tilde{A},\tilde{B},\lambda) = E$, for some $n = 1, 2, \cdots$ if and only $1/\lambda$ is an eigenvalue $\mathcal{B}$ and the number of $n$’s such that $\mu_n(\tilde{A},\tilde{B},\lambda) = E$ is equal to the multiplicity of the eigenvalue $1/\lambda$ of $\mathcal{B}$. Denote by $\rho_1, \rho_2, \cdots$ the eigenvalues of $\mathcal{B}$ repeated according to multiplicity. Hence, by (5.12)

$$N(H_{\tilde{A},\tilde{B}})(E) \leq \sum_{\{\rho_n: \rho_n > 1\}} \rho_n \leq \text{trace } \mathcal{B}. \quad (5.17)$$

By (4.1) $\mathcal{B}$ is an integral operator with continuous kernel, $\mathcal{B}(x, y)$,

$$\mathcal{B}(x, y) := V_1(x) R_{0,\tilde{A},\tilde{B}}(x, y) V_1(y). \quad (5.18)$$

Then (see [36])

$$N(H_{\tilde{A},\tilde{B}})(E) \leq - \int_0^\infty \sum_{j=1}^n V_{j,j}(x) R_{0,\tilde{A},\tilde{B}}(E)(x, x) \, dx. \quad (5.19)$$

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Taking the limit as $E \to 0$ (note that since there are only a finite number of eigenvalues and zero is not an eigenvalue $N(H_{\tilde{A},\tilde{B}})(E)$ is constant for $-E$ small enough) and using (4.1), (4.2) and (4.3)

$$N_{\tilde{A},\tilde{B}} := N(H_{\tilde{A},\tilde{B}})(0) \leq - \int_{0}^{\infty} \sum_{j=1}^{n} V_{j,j}(x) (x - \tan \theta_{j}) \, dx.$$  \hspace{1cm} (5.19)

Let us now go back to the general case where we allow for Neumann boundary conditions and for mixed conditions with $0 < \theta_{j} < \pi/2$. We define

$$\theta_{j,+} := \begin{cases} \theta_{j}, & \text{if } \pi/2 < \theta_{j} \leq \pi, \\ \pi, & \text{if } 0 < \theta_{j} \leq \pi/2, \end{cases} \quad j = 1, 2, \ldots, n,$$

and let us define $\tilde{A}+, \tilde{B}+$ as in (2.2) with $\theta_{j,+}$ instead of $\theta_{j}$ and,

$$H_{\tilde{A},\tilde{B},+} := H_{0,\tilde{A},\tilde{B}} + V.$$

Then, by (3.16) and (5.19)

$$N_{\tilde{A},\tilde{B}} \leq n_{M,b} + n_{N} - \int_{0}^{\infty} \sum_{j=1}^{n} V_{j,j}(x) (x - \tan \theta_{j,+}) \, dx,$$  \hspace{1cm} (5.20)

where we denote by $n_{M,b}$ the number of $\theta_{j}$ with $0 < \theta_{j} < \pi/2$. Recall that $n_{N}$ is the number of Neumann boundary conditions on the diagonal representation where the boundary matrices are given by (2.2).

Let us consider the general case of matrices $A, B$ that satisfy Equation (1.3), (1.4). Then, equation (5.20) holds with the with the matrices $\hat{A}, \hat{B}$ in (2.4) and with the potential $\tilde{V} := M^{\dagger}VM$. Finally, by (3.11), since the number of eigenvalues is invariant under a unitary transformation, and by the cyclicity of the trace, we have that equation (1.13) holds.

**REMARK 5.1.** The numbers $n_{M,b}$ and $n_{D}$ are necessary in (1.13). Suppose that $\tilde{V}$ is a diagonal matrix.

$$\tilde{V}(x) = \text{diag} \left\{ \tilde{V}_{1}(x), \tilde{V}_{2}(x), \ldots, \tilde{V}_{n}(x) \right\}.$$  \hspace{1cm}

Then, as mentioned above, for each $0 < \theta_{j} < \pi/2$ produces the eigenvalue $-(\cot \theta_{j})^{2}$ if the corresponding $\tilde{V}_{j}$ is identically zero. Then, there are $n_{M,b}$ eigenvalues even if $\tilde{V}$ is identically zero. Moreover, each $\theta_{j} = \pi/2$ produces an eigenvalue if $\tilde{V}_{j} = \lambda Q(x)$, if $Q(x) \leq 0$ it is not identically zero, and for any $\lambda > 0$. By the proof of Theorem 1.1 it is enough to prove that for any $\lambda > 0$, there is an $E < 0$ such that the following operator in $L^{2}(0, \infty)$,

$$B_{N} := \sqrt{Q}(x) (-\Delta_{N} + E)^{-1} \sqrt{Q}(x),$$  \hspace{1cm} (5.21)

has an eigenvalue larger than $1/\lambda$, where $-\Delta_{N}$ is the selfadjoint realization of $-\Delta$ in $L^{2}(0, \infty)$ with Neumann boundary condition at $x = 0$. But, as $B_{N}$ is Hilbert-Schmidt it is enough to prove that,

$$\lim_{E \to 0} \|B_{N}\| = \infty,$$
and this will follow if we find a $\varphi$ so that,

$$
\lim_{E \to 0} \|B_N \varphi\| = \infty.
$$

(5.22)

The operator $-\Delta_N$ is diagonalized by the cosine transform,

$$
\hat{\varphi}(k) := \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx) \varphi(x) \, dx.
$$

Then,

$$
\lim_{E \to 0} \|B_N \varphi\|^2 = \lim_{E \to 0} \int_0^\infty \frac{1}{(k^2 - E)^2} |\hat{\varphi}(k)|^2 \, dk = \infty,
$$

if we take $\varphi \in C_0^\infty(0, \infty), \varphi \geq 0$, so that $\hat{\varphi}(0) > 0$. Then, there is always a diagonal potential $\hat{V}(x) = \lambda Q(x)$ such that there are $n_N$ negative eigenvalues for any $\lambda > 0$, what shows that the integer $n_N$ is necessary in (1.13).

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