On the Mutual Information between disjoint regions in AdS/CFT

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The holographic mutual information $I_{AB}$ between two “small” coplanar and wide separated circular regions $A$ and $B$ in the four dimensional $\mathcal{N} = 4$ SYM gauge theory dual to Type IIB string theory in AdS$_5 \times S^5$, is computed. In our proposal, we first interpret the Ryu-Takayanagi prescription for the holographic computation of the entanglement entropy and the mutual information, in terms of submanifold observables associated with $k$-branes in the AdS/CFT correspondence. Then we provide a long distance expansion for the $I_{AB}$ whose coefficients appear as a byproduct of the operator product expansion for the correlators of these $k$-submanifold observables. It is shown that, while undergoing a phase transition at a critical distance, the holographic mutual information, instead of strictly vanishing, decays with a power law whose leading contributions of order $N^{-1/2}$, originate from the exchange of pairs of the lightest bulk particles between $A$ and $B$. These particles correspond to operators in the boundary field theory with the smallest scaling dimensions.

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INTRODUCTION

Entanglement entropy (EE) and other related information-theoretic quantities such as mutual information (MI) are by now regarded as valuable tools to study different phenomena in quantum field theories and many body systems [1]. These quantities provide a new kind of information that cannot be obtained from more standard observables such as expectation values. Namely, both EE and MI, are sensitive probes able to detect non-local signatures of the theory such as topological order which can not be detected by any local observable. Concretely, the mutual information $I_{AB}$ between two arbitrary regions $A$ and $B$ has certain advantages over the entanglement entropy. First, $I_{AB}$ can be viewed as an entropic correlator between $A$ and $B$ defined by,

$$I_{AB} = S_A + S_B - S_{AUB}, \quad (1)$$

where $S_A, S_B$ is the entanglement entropy of the region $A(B)$ and $S_{AUB}$ is the EE of the two regions. By its definition, $I_{AB}$ is finite and, contrarily to EE, is non UV-cutoff dependent. In addition, the strong subadditivity property of the EE states that when $A$ and $B$ are disconnected, then

$$S_A + S_B \geq S_{AUB}, \quad (2)$$

which immediately leads to realize that $I_{AB} \geq 0$. The strong subadditivity is the most important inequality which entanglement entropy satisfies. A standard approach to compute EE and MI makes use of the replica trick [2]. Unfortunately, these calculations are notoriously difficult to carry out, even in the case of free field theories.

In the context of the AdS/CFT [3], however, Ryu and Takayanagi (RT) have recently proposed a remarkably simple formula [4] to obtain the EE of an arbitrary region $A$ of a $d+1$ dimensional CFT which admits a classical gravity dual given by an asymptotically AdS$_{d+2}$ spacetime. According to the RT formula, the EE is obtained in terms of the area of a certain minimal surface $\gamma_A$ in the dual higher dimensional gravitational geometry; as a result, the entanglement entropy $S_A$ in a CFT$_{d+1}$ is given by the celebrated area law relation

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}, \quad (3)$$

where $d$ is the number of spacetime dimensions of the boundary CFT and $\gamma_A$ is the $d$-dimensional static minimal surface in AdS$_{d+2}$ such that $\partial A = \partial \gamma_A$. The $G_N^{(d+2)}$ is the $d+2$ dimensional Newton constant. The RT formula provides a simple tool to prove the strong subadditivity of EE from the properties of minimal surfaces [5]. Otherwise it has to be laboriously derived from the positive definiteness of the Hilbert space.

This work considers the holographic mutual information between two disconnected regions $A$ and $B$ in the ground state of an strongly coupled quantum field theory with a gravity dual given by the AdS/CFT correspondence. Using (3) in (1), this quantity reads,

$$I_{AB} = \frac{1}{4G_N^{(d+2)}} \left[ \text{Area}(\gamma_A) + \text{Area}(\gamma_B) - \text{Area}(\gamma_{AUB}) \right], \quad (4)$$

where $\text{Area}(\gamma_{AUB})$ is the area of the minimal surface related to $A \cup B$. Recently, the holographic mutual information [4] has been considered in a quite remarkably amount of different settings [6, 7]. An striking prediction for the holographic MI arises when analyzing the behaviour of the minimal surface $\gamma_{AUB}$. In [6] it is shown how, for certain distances between the two regions, there are minimal surfaces $\gamma^{\text{con}}_{AUB}$ connecting $A$ and $B$. For those regimes, the holographic MI has a nonzero value proportional to the number of degrees of freedom in the gauge theory lying on the boundary of AdS$_{d+2}$. However, when the separation between the two regions is large enough compared to their sizes, then a disconnected surface $\gamma^{\text{dis}}_{AUB}$ with

$$\text{Area}(\gamma^{\text{dis}}_{AUB}) = \text{Area}(\gamma_A) + \text{Area}(\gamma_B), \quad (5)$$

is both topologically allowed and minimal. In this case, (3) yields $S_{AUB} = S_A + S_B$ and a sharp vanishing of $I_{AB}$ then
occurs. This result is quite surprising from a quantum information point of view since, when the MI vanish, the reduced density matrix $\rho_{AB}$ factorizes into $\rho_{AB} = \rho_A \otimes \rho_B$, implying that the two regions are completely decoupled from each other and thus, all the correlations (both classical and quantum) between $A$ and $B$ should be rigorously zero. Indeed, it seems, at least counterintuitive, that all the correlations should strictly vanish at a critical distance, in a field theory in its large $N$ (t’Hooft coupling) limit. This behaviour is a general prediction of the Ryu-Takayanagi formula (3) which is valid for any two regions of any holographic theory [4].

In this paper it is shown that, at least in the cases that have been considered, the HMI [4] between two disjoint regions $A$ and $B$ in the large separation regime, while undergoing a phase transition at a critical distance, instead of strictly vanishing, decays with a law whose leading contributions are given by the exchange of pairs of the lightest bulk particles between $A$ and $B$. These bulk particles correspond to operators in the boundary field theory with small scaling dimensions as stated by the standard AdS/CFT dictionary [3]. In order to achieve this result, first we propose to interpret the HEE and HMI in terms of submanifold observables associated with $k$-branes in the AdS/CFT correspondence [6]. These observables are associated to $k$-dimensional surfaces specified on the boundary of the AdS spacetime and, in the supergravity approximation, they are computed through the minimum area of the $k + 1$-dimensional surfaces ending on them. Then, an operator product expansion (OPE) for the long distance HMI written in terms of these $k$-submanifold observables is provided.

The long distance expansion of the HMI presented here, takes an advantage of the knowledge on the long distance OPE for the correlators of coplanar Wilson loops and it is in accordance with the recent proposal in [9]. There, authors provide an OPE for the long distance HMI between disjoint regions. The proposal departs from an OPE for the mutual information in CFT discussed in [4,10] in which the expected leading contributions come from the exchange of pairs of operators located in $A$ and $B$ each with small scaling dimension $\Delta$, which reads as,

$$I_{AB} \sim \sum C_\Delta \langle O^\Delta(x_A)O^\Delta(x_B) \rangle^2,$$

where $C_\Delta$ comes from squares of OPE coefficients. Thus, when considering a CFT theory with a gravity dual, one must deal with a quantum field theory in a fixed background geometry and the long distance expansion of the MI reduces to an expression similar to (6), where now, one should consider the exchange of the lightest bulk particles.

**k-SUBMANIFOLD OBSERVABLES IN THE AdS/CFT CORRESPONDENCE**

Let us consider, following [6], the Type IIB superstring theory on $X \times S^5$ with $X = \text{AdS}_5$ and a four-manifold $\mathcal{M}$ as a conformal boundary. To compute the expectation value of a Wilson line on a circle $\mathcal{C} \subset \mathcal{M}$, one must consider a path integral with a string whose worldsheet $Y \subset X$ has $\mathcal{C}$ as boundary. In the supergravity approximation, $Y$ obeys the equation of a minimal area surface. Formally, if $\text{Area}(Y)$ is the area of $Y$ and $T$ the string tension, then the expectation value of the Wilson line on $\mathcal{C}$ is

$$\langle W(\mathcal{C}) \rangle = \exp[-T \text{Area}(Y)],$$

where $W(\mathcal{C})$ is the Wilson line operator and $\langle W(\mathcal{C}) \rangle$ is its expectation value.

One may generalize the method used to compute Wilson loops by considering cases in which the string theory contains $k$-branes. Thus, one associates an observable $W(\Sigma)$ to a $k$-dimensional submanifold $\Sigma \subset \mathcal{M}$. Its expectation value is now computed by a path integral in which a brane is wrapped on a $k + 1$-dimensional submanifold $Y \subset X$ whose boundary is $\Sigma$. In the supergravity approximation, the expectation value of the $k$-surface observable $W(\Sigma)$ is again given formally by (7), with $\text{Area}(Y)$ referring now to the volume of $Y$.

As an example, let us first consider the case of zero-branes, i.e $k = 0$. We denote $\mu$ as the mass of the zerobrane. In the supergravity approximation, one has $\mu \gg 1$. Since $k = 0$, the brane world-volume is a curve $Y \subset X$ which boundary consists of a pair of points $u, v \in \mathcal{M}$. Thus, the observable associated with the endpoints of the brane worldvolume amounts to be the local operator $\Phi(u)\Phi(v)$. To compute its expectation value $\langle \Phi(u)\Phi(v) \rangle$ in the supergravity approximation, one takes $Y$ to be a minimum length geodesic connecting $u$ and $v$ to obtain,

$$\langle \Phi(u)\Phi(v) \rangle = \exp[-\mu \text{Length}(Y)]$$

with $\text{Length}(Y)$ the length of $Y$.

The case with $k = 1$ amounts to the example of Wilson loops in a four-dimensional gauge theory commented above.

**Entanglement Entropy from k-submanifold observables**

Regarding the previous definitions and examples and recalling the RT prescription to compute the holographic EE (4), here we propose to interpret both the HEE and the HMI in terms of the $k$-surface observables defined by,

$$\log \langle W(\Sigma) \rangle = -T \text{Area}(Y),$$

with $\Sigma \equiv \partial \gamma_A$ and $\text{Area}(Y) = \text{Area}(\gamma_A)$. This allows, in principle, to formally write

$$S_A = -\frac{1}{4G_N^{(d+1)}} \frac{1}{T} \log \langle W(\partial \gamma_A) \rangle,$$

as long as the dual string theory contains the $k$-branes through which one might consistently define the $k$-submanifold observables $W(\Sigma)$. 
Let us exemplify this last point with pair of examples. First we consider the Type IIB superstring theory on AdS\(_3\) \(\times S^3\) \(\times T^4\) dual to a two dimensional CFT living in its conformal boundary \(\mathcal{M}_2\). In this theory, 0-branes on AdS\(_3\) arise from 1-branes wrapped on a one-cycle in \(T^4\). We focus on an static 1-dimensional interval \(A \subset \mathcal{M}_2\). According to the RT prescription, one must take the 1-dimensional static minimum length curve \(\gamma_A \subset AdS_3\) whose boundary \(\partial \gamma_A\) is the 0-dimensional surface \(\partial A\), i.e. \(\partial A\) consists of a pair of points \(u, v \in \mathcal{M}_2\). Thus, our proposal establishes that \(S_A\) might be computable from a \(k = 0\)-surface operator \(\Phi(u)\Phi(v)\) with \(\Sigma \in \mathcal{M}_2\) and \(\Sigma = \{u, v\}\).

Let us turn now to the Type IIB superstring theory on AdS\(_3\) \(\times S^5\) dual to the four-dimensional \(\mathcal{N} = 4\) Super Yang-Mills gauge theory living in its conformal boundary \(\mathcal{M}_4\). In this case, we consider an static 2-dimensional region \(A \subset \mathcal{M}_4\). When applying the RT prescription now, one must consider the static 2-dimensional minimum area surface \(\gamma_A \subset AdS_4\) whose boundary \(\partial \gamma_A\) is the 1-dimensional surface \(\partial A\). Regarding (10), \(S_A\) might be computed from a \(k = 1\)-surface operator (Wilson loop) \(W(\Sigma)\) where \(\Sigma \subset \mathcal{M}_4\) is the loop \(C\) given by \(C = \Sigma = \partial \gamma_A = \partial A\). Precisely, this is the case that will be considered in the remainder of this paper.

**HOLOGRAPHIC MUTUAL INFORMATION IN TYPE IIB STRING THEORY ON AdS\(_3\) \(\times S^5\)**

Here we analyze the the HMI (4) between two static circular 2-dimensional regions \(A\) and \(B\) of radius \(a\) separated by a distance \(L \gg a\). Regarding the last paragraph of the previous section, the HEE \(S_A\) and \(S_B\) of the two regions might be written as,

\[ S_{A(B)} = -\frac{1}{4G_N^2} \log \langle W(\gamma_{A(B)}) \rangle \]

with \(C_{A,B} = \partial \gamma_{A,B}\) being two circular and coplanar Wilson loops of radius \(a\). The string tension \(T = \sqrt{\lambda}\) where \(\lambda \gg 1\) is the ’t Hooft coupling of the gauge theory living in the conformal boundary of AdS\(_5\). In addition, the HEE \(S_{A\cup B}\) can be defined through

\[ \langle W(C_A)W(C_B) \rangle = \exp \left[ -\sqrt{\lambda} \text{Area}(\gamma_{A\cup B}) \right], \]

where \(\text{Area}(\gamma_{A\cup B})\) is the minimal area of a 2-dimensional surface \(\gamma_{A\cup B} \subset AdS_5\) connecting \(A\) and \(B\). This surface can be either connected or disconnected depending on the relative position of \(A\) and \(B\) while always obeying that \(\partial \gamma_{A\cup B} = C_A \cup C_B\).

With this at hand, eqs (11) and (12) allow us to write the HMI (4) as,

\[ I_{AB} = \frac{1}{4G_N^2} \log \left( \frac{\langle W(C_A)W(C_B) \rangle}{\langle W(C_A) \rangle \langle W(C_B) \rangle} \right). \]

The strong subadditivity property of the EE for disjoint regions (2) can be used together with eq (11) and (12) to prove the concavity of the two coplanar and disjoint Wilson loops \(C_A\) and \(C_B\) i.e.,

\[ \langle W(C_A) \rangle \langle W(C_B) \rangle \leq \langle W(C_A) W(C_B) \rangle. \]  \hspace{1cm} (14)

Regarding the previous section, this inequality holds provided the Wilson loops, as \(k\)-surface observables in AdS/CFT, are given by exponentials of minimal areas with large coefficients, without any further requirements on the nature of the surfaces or the metric. In terms of minimal surface areas, (14) can be illustrated as follows (11): for certain range of the separation \(L\), the loops are represented in the bulk by a 2-dimensional connected surface \(\gamma_{A,B}^{con}\), with the two loops at its boundary. However, there is a critical distance \(L_c\) for which this minimal surface degenerates into two semispheres \(\gamma_A\) and \(\gamma_B\) which reflects in the saturation of the inequality in (14), i.e \(\langle W(C_A) W(C_B) \rangle = \langle W(C_A) \rangle \langle W(C_B) \rangle\). This is similar to the vanishing of the mutual information commented above while it is equally puzzling. Namely, the pictures for the vanishing of the mutual information and the factorization of Wilson loop correlators presented above result incomplete.

Indeed, our proposal consists in putting forward a suitable OPE for the long distance HMI (13) which may be compatible with the arguments posed in (12), while obeying the general bound (15),

\[ I_{AB} \geq \frac{\langle \langle A \rangle \langle B \rangle \rangle - \langle \langle A \rangle \rangle \langle \langle B \rangle \rangle}{2|\langle A \rangle|^2|\langle B \rangle|^2}. \]  \hspace{1cm} (15)

This subject will be further elaborated below.

**Long distance OPE for the Holographic Mutual Information**

Let us consider now, an OPE for the holographic mutual information (13) between two well separated circular regions \(A\) and \(B\), which accordingly to (4) involves the propagation of the bulk lightest excitations. In (13) it has been computed a supergravity approximation to the long distance expansion of the correlator,

\[ \log \frac{\langle W(C_A)W(C_B) \rangle}{\langle W(C_A) \rangle \langle W(C_B) \rangle}, \]

which integrates the amplitude for the exchange of a supergravity mode in the bulk between two points on the disjoint surfaces \(\gamma_A\) and \(\gamma_B\) and then sums over all modes. Nevertheless, the leading contributions to this long distance expansion only include the exchange of single particle states, as opposed to two particle states (16).

Therefore, we are led to consider an alternative OPE for the HMI which still may take an advantage of the knowledge on the long distance expansion for correlators of coplanar Wilson loops while regarding the exchange of two particle states. To this end, let us first consider the long distance correlator of the Wilson loop \(C_A\) with each operator \(O_B^{\Delta}\) that is expected to appear in the other loop,

\[ \log \frac{\langle W(C_A,L)O_B^{\Delta}(0) \rangle}{\langle W(C_A,L) \rangle}. \]  \hspace{1cm} (17)
FIG. 1: Two static circular 2-dimensional regions $A$ and $B$ (shaded grey) of radius $a$ separated by a long distance $L \gg a$ whose boundaries $\partial A$ and $\partial B$ define the circular loops $C_A$ and $C_B$ respectively. $z$ represents the radial coordinate in AdS. Top Left: The emission of a supergravity particle (dotted line) from the Wilson loop $C_A$ onto a point (X) on the boundary of AdS at a distance $L$ where the operator $O_B$ contained in $C_B$ is inserted. Bottom Left: The emission of a particle from the Wilson loop $C_B$ onto a point (X) on the boundary of AdS at a distance $L$ where the operator $O_A$ contained in $C_A$ is inserted. Right: A leading contribution to the long distance OPE for $I_{AB}$ is given by the exchange of a pair of the lightest supergravity particles between the loops $C_A$ and $C_B$.

In (17), $\Delta_k$ is the scaling dimension of the operator $O_B^{\Delta_k}$ and it also has been explicitly shown the distance between the operators. As stated above, in the supergravity approximation, the Wilson loop is related with a 1-brane (string) 2-dimensional world-volume $\subset$ AdS$_5$ ending on the boundary of the spacetime. The correlator (17) is calculated by treating the string worldsheet as an external source for a number of propagating bulk fields in AdS and then computing the string effective action for the emission of the supergravity state associated to $\Delta_k$ onto the point on the boundary where the operator $O_B$ is inserted (13) (see Figure 1 top left). In (13) it is argued that the leading contribution to (17) is given by

$$\log \frac{\langle W(C_A, L) O_B^{\Delta_k}(0) \rangle}{\langle W(C_A, L) \rangle} \sim c_{\Delta_k} \left( \frac{a}{L^2} \right)^{\Delta_k}.$$  (18)

Similarly, one should also consider the reciprocal situation, that is, the long distance correlator of the Wilson loop $C_B$ with each operator $O_A^{\Delta_m}$ that is expected to appear in $C_A$ (see Figure 1 bottom left) whose main contribution is again given by

$$\log \frac{\langle W(C_B, 0) O_A^{\Delta_m}(L) \rangle}{\langle W(C_B, 0) \rangle} \sim c_{\Delta_m} \left( \frac{a}{L^2} \right)^{\Delta_m}.$$  (19)

Once provided these expansions, here we propose an ansatz for the long distance OPE of the HMI (13) which jointly considers the mutual exchange of bulk particles between $C_A$ and $C_B$ given by (18) and (19).

$$I_{AB} \sim \frac{\lambda^{-1/2}}{4 G_N} \sum_{k,m} \log \frac{\langle W(C_A) O_B^{\Delta_k} \rangle}{\langle W(C_A) \rangle} \log \frac{\langle W(C_B) O_A^{\Delta_m} \rangle}{\langle W(C_B) \rangle} \sim \sum_k (c_{\Delta_k})^2 \left( \frac{a}{L^2} \right)^{2\Delta_k} + \sum_{k \neq m} c_{\Delta_k} c_{\Delta_m} \left( \frac{a}{L^2} \right)^{\Delta_k + \Delta_m}.$$  (20)

As a result, the leading contributions to this OPE are given by the exchange of pairs of the lightest supergravity particles (see Figure 1 right) and its coefficients arise as a byproduct of the OPE coefficients for the Wilson loop correlators appearing in (18) and (19).

Contributions from the Lightest bulk fields

For 10-dimensional supergravity compactified on $X_5 \times S^5$, the ten-dimensional fields may be written as,

$$\Psi = \sum_{k,l} \phi_{k,l} Y_{k,l}.$$  (21)
where $\phi_k$ is a five dimensional field and $Y_{k,l}$ are the spherical harmonics on $S^5$ with total angular momentum $k$. The full spectrum of 10D-supergravity compactified on $S^5$ was obtained in [14] but, in what follows, we will focus only in the lightest scalars $s_k$, whose exchange will dominate the long distance behaviour of $\langle W(\gamma) \rangle$. These light scalar states correspond to operators of the lowest dimension in the OPE for the Wilson loop and HMI. These fields have the following five dimensional masses $m_{s_k}$,

$$m_{s_k}^2 = k(k-4) \quad k \geq 2 \quad (22)$$

Note that the field $s_k$ has a negative $m^2$ for $k = 2, 3$. However, these modes are not tachyonic, since they propagate on a space of negative curvature.

In [13] it has been shown in full detail how to obtain that,

$$\log \frac{\langle W(C) O^{\Delta_k} \rangle}{\langle W(C) \rangle} \sim 2^{k/2} \sqrt{\pi k} \sqrt{\frac{g_s}{N}} Y_{k,l} \left( \frac{a}{L^2} \right)^k, \quad (23)$$

where $g_s$ is the string interaction strength. From this expression and [18] one may identify $\Delta_k = k$ while both determine the OPE coefficients as

$$c_{\Delta_k} = 2^{k/2} \sqrt{\pi k} \sqrt{\frac{g_s}{N}} Y_{k,l} = 2^{\Delta_k/2-1} \sqrt{\Delta_k} \sqrt{\frac{N}{\lambda}}, \quad (24)$$

and compute the leading contributions from the lightest scalar ($k = 2$) to the HMI [20],

$$I_{AB} \sim \frac{\lambda^{-1/2}}{4G_N^5} (c_{\Delta_2})^2 \left( \frac{a}{L^2} \right)^{2\Delta_2} = \frac{\sqrt{\lambda}}{\pi} \left( \frac{a}{L^2} \right)^4. \quad (25)$$

In the last equality we have used (see appendix)

$$c_{\Delta_2} = \frac{2\sqrt{2}}{N}, \quad (26)$$

$$G_N^5 = \frac{\pi}{2N^2}. \quad (27)$$

Thus, the $N$ dependence for the long distance holographic $I_{AB}$ between two wide separated circular regions of the four dimensional $\mathcal{N} = 4$ SYM theory provided by the OPE [20], is of order $\sqrt{N}$. This $N$ dependence is subleading with respect to the $N^2$ dependence which is expected to hold when a fully connected minimal surface $\gamma_{\text{conn}}$ between $A$ and $B$ is allowed, contrarily to the regimes which have been considered here. In this sense, one might claim that the HMI experiences a phase transition marked by this change in the $N$ dependence of the leading contributions to $I_{AB}$. Furthermore, the non-zero quantum corrections to the long distance $I_{AB}$ given by [20], enable us to say that, the HMI does not suffer a sharp vanishing due to large $N$ effects but instead, it smoothly decays following a power law given by [25] while parametrically saturates the bound [15].

**CONCLUSIONS**

In this paper, we have computed the holographic mutual information $I_{AB}$ between two “small” coplanar and wide separated circular regions $A$ and $B$ in the four dimensional $\mathcal{N} = 4$ SYM gauge theory dual to Type IIB string theory in $\text{AdS}_5 \times S^5$. We have interpreted the Ryu-Takayanagi prescription for the holographic computation of the entanglement entropy and the mutual information in terms of submanifold observables associated with $k$-branes in the $\text{AdS}/\text{CFT}$ correspondence. We have also provided a long distance expansion for the $I_{AB}$ whose coefficients appear as a byproduct of the operator product expansion for the correlators of these $k$-submanifold observables. The results show that in the regimes under consideration, the mutual information undergoes a phase transition at a critical distance marked by a change in the $N$ dependence of its leading contributions. Furthermore, the $I_{AB}$, instead of strictly vanishing, smoothly decays with a power law shaped by the exchange of pairs of the lightest bulk particles between $A$ and $B$.

We feel that our results may help to elucidate the behaviour of the mutual information in AdS/CFT despite the limited applicability of the method. Namely, our procedure to compute the holographic entanglement entropy and mutual information only allows to study cases in which the $k$-submanifold observables $\langle W(\partial \gamma) \rangle$ may be consistently defined through the brane content of the dual string theory under consideration. In this sense, in addition to those cases analyzed or commented here, it is worth to note that the method would be also applicable to the case of small and wide separated 3-dimensional regions lying on the 6-dimensional conformal boundary of $\text{AdS}_7$ through the OPE for the correlators of Wilson surface operators [13, 15].

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**Appendix**

We show some terms of the $\mathcal{N} = 4$ SYM/AdS$_5$ duality dictionary and the calculations giving the $N$ dependence of $I_{AB}$. First, we introduce some basic identities related with the Kaluza Klein compactification of a 10-dimensional supergravity theory into a 5-dimensional one and other identities related with the dictionary of the duality,

$$G_N^{10} = 8\pi^6 \alpha'^4 g_s^2, \quad G_N^5 = \frac{G_N^{10}}{\pi^4 R^5} \quad (27)$$

$$R^4 = 4\pi g_s \alpha'^2 N$$

$$g_s = 2\pi g_Y^2 M, \quad \lambda = g_Y^2 N$$
where $\alpha'$ and $g_s$ are the string tension and string interaction strength respectively, $R$ is the radius of AdS$_5$ (that will be settled to 1 after our calculations), $N$ is the number of colors of the boundary gauge theory with a $g_{YM}$ coupling both defining the t’Hooft coupling $\lambda$.

From these expressions it is straightforward to obtain that,

$$G^5_{\mathcal{N}} = \frac{\pi}{2N^2},$$

(28)

after settling $R = 1$ in the calculations.

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[1] I. R. Klebanov, D. Kutasov and A. Murugan, Nucl.Phys B796, 274-293, (2008); J. L.F. Barbón and C. A. Fuertes, JHEP 0804 096,(2008).

[2] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. B424, 443-467, (1994); P. Calabrese and J. Cardy, J. Stat. Mech. P06002 (2004); P. Calabrese , J. Cardy and E. Tonni, J. Stat. Mech P11001, (2009).

[3] O. Aharony, S. S. Gubser, J.M Maldacena, H. Ooguri and Y. Oz, Phys. Rept. 323 183-386 (2000); J. M. Maldecena, Adv. Theor. Math. Phys. 2 231-252 (1998); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B428 105-114, (1998).

[4] S. Ryu and T. Takayanagi, Phys. Rev. Lett. 96, 181602 (2006); JHEP 0608 045 (2006); T. Nishioka, S. Ryu, and T. Takayanagi, J.Phys.A 42 504008 (2009); T. Takayanagi, Class. Quantum Grav, 29, 153001 (2012).

[5] M. Headrick and T. Takayanagi, Phys. Rev. D76 106013 (2007)

[6] M Headrick, Phys.Rev D82 126010 (2010).

[7] J. Molina-Vilaplana and P. Sodano, JHEP 1110, 011 (2011); W. Fischler, A. Kundu and S. Kundu, [arXiv:1212.4764]; P. Hayden, Matthew Headrick and A. Maloney, [arXiv:1107.2940]; A. Allais and E. Tonni, JHEP 1201:102,(2012); V. Balasubramanian, A. Bernamonti, N. Copland, B. Craps, F. Galli, Phys.Rev. D84 105017, (2011).

[8] C. R. Graham and E. Witten, Nucl.Phys. B546 52-64 (1999).

[9] T. Faulkner, A. Lewkowycz and J. Maldacena, [arXiv:1307.2892]

[10] H. Casini and M. Huerta, JHEP 0903, 048 (2009); P. Calabrese, J. Cardy and E. Tonni, J. Stat. Mech. 1101, P01021 (2011); J. Cardy, [arXiv:1304.7985][hep-th].

[11] D. J. Gross and H. Ooguri, Phys.Rev.D58 106002 (1998).

[12] M.M. Wolf, F. Verstraete, M.B. Hastings, J.I. Cirac, Phys. Rev. Lett. 100, 070502 (2008).

[13] D. Berenstein, R. Corrado, W. Fischler and J. Maldacena, Phys.Rev D59 105023 (1999).

[14] HJ Kim, LJ Romans and P. van Nieuwenhuizen, Phys.Rev D32 389-399 (1985).

[15] R. Corrado, B. Florea, R. McNees, Phys.Rev D60 085011 (1999)

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