Exponential Convergence of Non-Linear Monotone SPDEs *

Feng-Yu Wang
School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, United Kingdom
wangfy@bnu.edu.cn, F.-Y.Wang@swansea.ac.uk
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Abstract
For a Markov semigroup $P_t$ with invariant probability measure $\mu$, a constant $\lambda > 0$ is called a lower bound of the ultra-exponential convergence rate of $P_t$ to $\mu$, if there exists a constant $C \in (0, \infty)$ such that
\[
\sup_{\mu(f^2) \leq 1} \|P_t f - \mu(f)\|_\infty \leq C e^{-\lambda t}, \quad t \geq 1.
\]

By using the coupling by change of measure in the line of [17], explicit lower bounds of the ultra-exponential convergence rate are derived for a class of non-linear monotone stochastic partial differential equations. The main result is illustrated by the stochastic porous medium equation and the stochastic $p$-Laplace equation respectively. Finally, the $V$-uniformly exponential convergence is investigated for stochastic fast-diffusion equations.

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1 Introduction
It is well known that the solution to the porous medium equation
\[
(1.1) \quad dX_t = \Delta X_t^r \, dt
\]

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decays at the algebraic rate $t^{-\frac{1}{r-1}}$ as $t \to \infty$, where $\Delta$ is the Dirichlet Laplacian on a bounded domain in $\mathbb{R}^d$, $r > 1$ is a constant and $X_t^r := |X_t|^{-1}X_t$, see [1]. This type algebraic convergence has been extended in [3] to stochastic generalized porous media equations. When $r \in (0, 1)$, (1.1) is called the fast-diffusion equation.

Consider, for instance, the stochastic porous medium equation

$$dX_t = \Delta X_t^r dt + dW_t,$$

where $\Delta$ is the Dirichlet Laplacian on $(0, l)$ for some $l > 0$, and $W_t$ is the cylindrical Brownian motion on $L^2(\mathcal{M})$, where $\mathcal{M}$ is the normalized Lebesgue measure on $(0, l)$. By [3, Theorem 1.3], for any $x \in \mathbb{H} := H^{-1}$ (the duality of the Sobolev space w.r.t. $L^2(\mathcal{M})$, see Section 3), the equation has a unique solution starting at $x$, and the associated Markov semigroup $P_t$ has a unique invariant probability measure $\mu$ such that

$$\|P_tf - \mu(f)\|_\infty \leq C\mathcal{L}(f)t^{-\frac{1}{r-1}}, \quad t > 0$$

holds for some constant $C > 0$ and all Lipshitz continuous function $f$, where $\mathcal{L}(f)$ is the Lipschitz constant of $f$, $\|f\|_\infty := \sup_{x \in \mathbb{H}} |f(x)|$ and $\mu(f) := \int_\mathbb{H} f d\mu$.

On the other hand, by using the dimension-free Harnack inequality and a result due to [5], the uniform exponential convergence

$$\|P_tf - \mu(f)\|_\infty \leq Ce^{-\lambda t}\|f\|_\infty, \quad t \geq 0, f \in L^2(\mu)$$

is proved in [8] for some constants $C, \lambda > 0$. Since, according to [17, Theorem 1.2(4)] (see also [8, Theorem 1.4(iv)]) $P_t$ is ultrabounded, i.e. $\|P_t\|_{L^2(\mu) \to L^\infty(\mu)} < \infty$ for $t > 0$, this implies the ultra-exponential convergence

$$\|P_tf - \mu(f)\|_\infty^2 \leq C\{\mu(f^2) - \mu(f)^2\}e^{-\lambda t}, \quad t \geq 1, f \in L^2(\mu)$$

for some constant $C, \lambda > 0$. To see that (1.3) improves (1.2) for large time, we note that

$$\mu(f^2) - \mu(f)^2 = \frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} |f(x) - f(y)|^2 \mu(dx)\mu(dy)$$

$$\leq \mathcal{L}(f)^2 \int_{\mathbb{H} \times \mathbb{H}} |x - y|^{2r} \mu(dx)\mu(dy) =: C'\mathcal{L}(f)^2$$

with constant $C' \in (0, \infty)$ since $\mu(\|\cdot\|_\mathbb{H}) < \infty$, see for instance [3, Theorem 1.3].

However, explicit estimates on the ultra-exponential convergence rate $\lambda$ is not yet available. We note that in [6] an lower bound estimate of exponential convergence rate is presented for a class of semi-linear SPDEs (stochastic partial differential equations). But the main result in [6] does not apply to the present non-linear model, since both [6, Hypothesis 2.4(a)] (i.e. $F$ is a Lipschitz map from $\mathbb{H} \to \mathbb{H}$) and [6, Hypothesis 2.4(b)] (i.e. $\text{Im}(F) \subset L^2(\mathcal{M})$) are not satisfied for the present $F(x) := \Delta x^r$, which is not a well defined map from $\mathbb{H} \to \mathbb{H}$.

In this paper, we aim to present explicit lower bound estimates for the ultra-exponential convergence rate $\lambda$ in (1.3). In the next section, we prove a general result for a class of
non-linear SPDEs considered in [8]. The main tool in the study is the coupling by change of measure constructed in [17] (see also [8]). A general theory on this kind of couplings and applications has been addressed in the recent monograph [18]. The main result is applied to the stochastic porous medium equation and the stochastic \( p \)-Laplace equation in Section 3 and Section 4 respectively. Finally, in Section 5 we investigate the exponential convergence for stochastic fast-diffusion equations.

2 A general result

Let \( V \subset H \subset V^* \) be a Gelfand triple, i.e. \((H, \langle \cdot, \cdot \rangle_H, | \cdot |_H)\) is a separable Hilbert space, \( V \) is a reflexive Banach space continuously and densely embedded into \( H \), and \( V^* \) is the duality of \( V \) with respect to \( H \). Let \( \nu^* \langle \cdot, \cdot \rangle_V \) be the dualization between \( V \) and \( V^* \). We have \( \nu^* \langle u, v \rangle_V = \langle u, v \rangle_H \) for \( u \in H \) and \( v \in V \).

Let \( W = (W_t)_{t \geq 0} \) be a cylindrical Brownian motion on a (possibly different) Hilbert space \((E, \langle \cdot, \cdot \rangle_E, | \cdot |_E)\), i.e. \( W_t := \sum_{i=1}^{\infty} B_i e_i \) for an orthonormal basis \( \{e_i\}_{i \geq 1} \) of \( E \) and a sequence of independent one-dimensional Brownian motions \( \{B_i\}_{i \geq 1} \) on a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). Consider the following stochastic equation:

\[
(2.1) \quad dX_t = b(X_t)dt + QdW_t,
\]

where \( b : V \to V^* \) is measurable and \( Q \in \mathcal{L}_{HS}(E; H) \), the space of all Hilbert-Schmidt linear operators from \( E \) to \( H \), such that the following assumptions hold for some constants \( r > 1 \), and \( C_1, C_2 > 0 \):

(A1) (Hemicontinuity) For any \( v_1, v_2, v \in V \), \( \mathbb{R} \ni s \mapsto \nu^* \langle b(v_1 + sv_2), v \rangle_V \) is continuous.

(A2) (Monotonicity) For any \( v_1, v_2 \in V \), \( \nu^* \langle b(v_1) - b(v_2), v_1 - v_2 \rangle_V \leq C_1 |v_1 - v_2|^2_H \).

(A3) (Coercivity) For any \( v \in V \), \( \nu^* \langle b(v), v \rangle_V \leq C_1 - C_2 \|v\|_V^{r+1} \).

(A4) (Growth) For any \( u, v \in V \), \( |\nu^* \langle b(v), u \rangle_V| \leq C_1 \left(1 + \|v\|_V^{r+1} + \|u\|_V^{r+1}\right) \).

Definition 2.1. A continuous \( H \)-valued adapted process \( X \) is called a solution to (2.1), if

\[
\int_0^T \mathbb{E}\|X_t\|_V^{r+1} dt < \infty, \quad T > 0,
\]

and \( \mathbb{P} \)-a.s.

\[
X_t = X(0) + \int_0^t b(X_s) \, ds + \int_0^t QdW_s, \quad t \geq 0,
\]

where the Bochner integral \( \int_0^t b(X_s) \, ds \) is defined on \( V^* \) but indeed takes values in \( H \) for all \( t \geq 0 \).

According to [7] Theorems II.2.1, II.2.2, for any \( x \in H \), the equation (2.1) has a unique solution \( X^x_t \) with initial data \( x \); see also [14] Theorem 2.1 for

\[
K := L^{1+r}([0, T] \times \Omega \to V; dt \times \mathbb{P}) \cap L^2([0, T] \times \Omega \to H; dt \times \mathbb{P}).
\]
Let $P_t$ be the associated Markov semigroup, i.e.

$$P_t f(x) := \mathbb{E} f(X^x_t), \quad f \in \mathcal{B}_b(\mathbb{H}), \; t \geq 0, \; x \in \mathbb{H}.$$ 

For any $u \in \mathbb{H}$, let

$$\|u\|_Q = \inf \{|x|_E : x \in E, Qx = u\},$$

where we set $\inf \emptyset = \infty$ by convention.

The study of (2.1) with the above type assumptions goes back to [12, 13] for non-linear monotone SPDEs. Extensions to stochastic equations with “local conditions” as well as to non-monotone stochastic equations have been made in [9, 11, 15]. As mentioned in the Introduction that in this paper we aim to estimate the ultra-convergence rate of $P_t$. The following is the main result of the paper.

**Theorem 2.1.** Assume that $\text{Ker} Q = \{0\}$. If there exist constants $\theta \in [2, \infty) \cap (r - 1, \infty)$ and $\eta, \delta > 0$ such that

$$2 V \cdot (b(u) - b(v), u - v)_V \leq - \max \{\eta \|u - v\|_Q^\theta |u - v|_H^{r+\theta}, \; \delta |u - v|_H^{r+1}\}$$

holds for all $u, v \in \mathbb{V}$, then $P_t$ has a unique invariant probability measure $\mu$ and (1.3) holds for some constant $C > 0$ and

$$\lambda := \sup_{t > 0} \frac{1}{t} \log \frac{2}{\exp[\alpha t^{\frac{r+1}{r+1}}] - 1} > 0,$$

where

$$\alpha := \left(\frac{\theta r + \theta}{r - 1}\right)^{\frac{r+1}{r+1}} \frac{2 + \theta}{(\theta + 1 - r)^{\frac{2r}{r+1}} \delta^\frac{2(r+1)}{\theta(r+1)} \eta^\frac{2}{\theta}} \in (0, \infty).$$

Moreover,

$$\lambda \geq \frac{(\theta + 1 - r)^{\frac{2r}{r+1}} \delta^\frac{2(r+1)}{\theta(r+1)} \eta^\frac{2}{\theta}}{\theta (2 + \theta)^{\frac{r+1}{r+1}}} \left\{ \log \left(1 + 2e^{-\frac{1+\theta}{2\theta}}\right) \right\}^{\frac{r+1}{r+1}}$$

$$\geq \frac{(\theta + 1 - r)^{\frac{2r}{r+1}} \delta^\frac{2(r+1)}{\theta(r+1)} \eta^\frac{2(r-1)}{\theta}}{e\theta (2 + \theta)^{\frac{r+1}{r+1}}}.$$

**Proof.** By [8, Theorem 1.4] with $\alpha = 1 + r$ (see also [18, Corollary 2.2.4] with $\alpha = r$), $P_t$ has a unique invariant probability measure $\mu$ of full support on $\mathbb{H}$. Moreover, $P_t$ is strong Feller (i.e. $P_t \mathcal{B}_b(\mathbb{H}) \subset C_b(\mathbb{H}), \; t > 0$) and ultra-bounded (i.e. $\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} < \infty, \; t > 0$) with

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \leq \exp \left[c + ct^{\frac{r+1}{r-1}}\right], \; t > 0$$

holding for some constant $c > 0$. So,

$$\|P_t f\|_2^2 = \|P_t f\|_{L^\infty(\mu)}^2 \leq \|P_t\|_{L^2(\mu)\to L^\infty(\mu)}^2 \mu((P_t f)^2), \; t \geq 1.$$
Therefore, it suffices to prove

\[(2.5) \quad \mu((P_t f)^2) \leq C \mu(f^2)e^{-\lambda t}, \quad t \geq 0, \quad f \in L^2(\mu), \quad \mu(f) = 0\]

for some constant \(C \in (0, \infty)\) and the desired constant \(\lambda\), and to verify the claimed lower bounds of \(\lambda\). We shall complete the proof by four steps.

(a) We first construct a coupling by change of measure using the idea of [17]. For fixed \(T > 0\) and \(x, y \in \mathbb{H}\), let \(X_t = X^x_t\) solve (2.1) for \(X_0 = x\), and let \(Y_t\) solve the equation

\[(2.6) \quad dY_t = \left\{ b(Y_t) + \frac{\beta(X_t - Y_t\\H)}{|X_t - Y_t\\H|} \right\} dt + QdW_t, \quad Y_0 = y, \]

where

\[(2.7) \quad \varepsilon := \theta + 1 - r \quad \text{if} \quad 2 + \theta \quad \text{and} \quad \beta := \frac{|x - y\\H|}{\varepsilon T} \geq 0, \quad \text{and} \quad \frac{X_t - Y_t\\H}{|X_t - Y_t\\H|} := 0 \quad \text{if} \quad X_t - Y_t.

As shown in [17, Theorem A.2] (see also [8]) that the equation (2.6) has a unique solution such that \(X_t = Y_t\) for \(t \geq \tau\), where

\[\tau := \inf\{t \geq 0 : X_t = Y_t\}\]

is the coupling time. By (2.2) we have

\[(2.8) \quad d|X_t - Y_t\\H|^2 \leq -\left\{ \eta \||X_t - Y_t\\H|\|^2 |X_t - Y_t\\H|^{r+1-\theta} + 2\beta |X_t - Y_t\\H|^{2-\varepsilon} \right\} dt, \quad t < \tau.

Since \(\eta > 0\), this implies

\[d|X_t - Y_t\\H|^\varepsilon = \frac{\varepsilon}{2} (|X_t - Y_t\\H|^2 \varepsilon - 2) d|X_t - Y_t\\H| \leq -\varepsilon \beta dt, \quad t < \tau.

Thus, if \(T < \tau\) then

\[|X_T - Y_T\\H|^\varepsilon \leq |x - y\\H|^\varepsilon - \varepsilon \beta T = 0,

which is a contradiction since by definition it implies \(\tau \leq T\). Therefore, we have \(\tau \leq T\), so that \(X_T = Y_T\).

(b) By (2.8), \(\beta \geq 0\) and noting that \(2(\varepsilon - 1) + r + 1 - \theta = -\varepsilon \theta\), we have

\[d|X_t - Y_t\\H|^\varepsilon = \varepsilon |X_t - Y_t\\H|^2 \varepsilon - 1 d|X_t - Y_t\\H| \leq -\varepsilon \eta \frac{||X_t - Y_t\\H|^{\theta}}{|X_t - Y_t\\H|^{\theta}} dt, \quad t < \tau.

Then

\[(2.9) \quad \eta \int_0^\tau \frac{||X_t - Y_t\\H|^{\theta}}{|X_t - Y_t\\H|^r} dt \leq \frac{|x - y\\H|^2}{\varepsilon} = \frac{2 + \theta}{\theta + 1 - r} |x - y\\H|^{2(\theta+1-\rho)/2}\rho}.

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Combining this with $T \leq \tau$ and $\theta \geq 2$, and using the Jensen inequality, we see that

$$\zeta_t := \frac{\beta Q^{-1}(X_t - Y_t)}{|X_t - Y_t|_H^\theta}$$

is well defined in $L^2([0,T] \to E; dt)$. Moreover, since $\theta \geq 2$, by (2.9) and the Hölder inequality we obtain

$$\int_0^T |\zeta_t|_E^2 dt = \int_0^\tau \frac{\beta^2 |X_t - Y_t|_Q^2 dt}{|X_t - Y_t|_H^\theta} \leq \left( \eta \int_0^\tau \frac{|X_t - Y_t|_Q^\theta dt}{|X_t - Y_t|_H^\theta} \right)^\frac{\theta}{\theta - 2} \left( \int_0^\tau \frac{\beta^{2\theta - 2}}{\eta^{2 - \theta}} dt \right)^{\frac{\theta - 2}{\theta}}$$

(2.10)

$$\leq \frac{T^{\frac{\theta - 2}{\theta}} \beta^2}{\eta^\frac{\theta}{\theta - 2}} \left[ (2 + \theta)^{\frac{\theta}{\theta - 2}} |x - y|_H^{\frac{2(\theta + 1 - r)}{\theta}} \right] \frac{(\theta + 1 - r)^{\frac{2(\theta + 1 - r)}{\theta}}}{\eta^\frac{\theta}{\theta - 2}}.$$

Then by the Girsanov theorem,

$$R := \exp \left[ - \int_0^T \langle \zeta_t, dW_t \rangle_E - \frac{1}{2} \int_0^T |\zeta_t|_E^2 dt \right]$$

is a well defined probability density of $\mathbb{P}$, and the process

$$\tilde{W}_t := W_t + \int_0^t \zeta_s ds, \quad t \in [0,T]$$

is a cylindrical Brownian motion on $E$ under the weighted probability measure $d\mathbb{Q} := R d\mathbb{P}$.

Now, rewrite (2.8) by

$$dY_t = b(Y_t) dt + Q d\tilde{W}_t, \quad Y_0 = y.$$

From the weak uniqueness of the solution to (2.1) and $X_T = Y_T$, we conclude that

$$P_T f(y) = \mathbb{E}_\mathbb{Q} f(Y_T) = \mathbb{E}[R f(Y_T)] = \mathbb{E}[R f(X_T)].$$

This together with $P_T f(x) = \mathbb{E}_\mathbb{Q} f(X_T)$ yields that

$$|P_T f(x) - P_T f(y)|^2 = \left| \mathbb{E}[f(X_T)(1 - R)] \right|^2 \leq (P_T f^2(x))(\mathbb{E} R^2 - 1).$$

(2.11)

(c) By (2.10) we have

$$\mathbb{E} R^2 = \mathbb{E} \exp \left[ - \int_0^T \langle \zeta_t, dW_t \rangle_E - \int_0^T |\zeta_t|_E^2 dt \right]$$

(2.12)

$$\leq \exp \left[ \frac{(2 + \theta)^{\frac{2(\theta + 1)}{\theta}} |x - y|_H^{\frac{2(\theta + 1 - r)}{\theta}}}{(\theta + 1 - r)^{\frac{2(\theta + 1 - r)}{\theta}} T^{\frac{\theta + 2}{\theta}} \eta^\frac{\theta}{\theta - 2}} \right] \mathbb{E} e^{-\int_0^T \langle \zeta_t, dW_t \rangle_E - \frac{1}{2} \int_0^T |\zeta_t|_E^2 dt}$$

$$= \exp \left[ \frac{(2 + \theta)^{\frac{2(\theta + 1)}{\theta}} |x - y|_H^{\frac{2(\theta + 1 - r)}{\theta}}}{(\theta + 1 - r)^{\frac{2(\theta + 1 - r)}{\theta}} T^{\frac{\theta + 2}{\theta}} \eta^\frac{\theta}{\theta - 2}} \right].$$
Moreover, (2.2) yields that
\[ d|X_t^x - X_t^y|^2 \leq -\delta |X_t^x - X_t^y|^2 dt, \quad t \geq 0, x, y \in \mathbb{H}, \]
where \( X_t^x \) and \( X_t^y \) solve the equation (2.1) starting at \( x \) and \( y \) respectively. Thus,
\[ |X_t^x - X_t^y|^2 \leq \left( \frac{\delta t(r - 1)}{2} \right)^2, \quad t > 0, x, y \in \mathbb{H}. \]
Substituting this and (2.12) into (2.11) and using the Markov property, we arrive at
\[ |P_{T+s}f(x) - P_{T+s}f(y)|^2 \leq \mathbb{E}[|P_Tf(X_s^x) - P_Tf(X_s^y)|^2] \]
\[ \leq \mathbb{E}\left\{ P_Tf^2(X_s^x) \left( \exp \left[ \frac{(2 + \theta)^{\frac{2(\theta+1)}{\theta}} |X_s^x - X_s^y|^{\frac{2(\theta+1)}{\theta}}}{(\theta + 1 - r)^{\frac{2(\theta+1)}{\theta}} T^{\frac{1}{\theta}}} \right] - 1 \right) \right\} \]
(2.13)
\[ \leq (P_{T+s}f^2(x)) \left( \exp \left[ \frac{C_0}{s^{\frac{2(\theta+1-r)}{\theta(r-1)}} t^{\frac{1}{\theta}}} - 1 \right) \right), \quad T, s > 0, \]
where
\[ C_0 := \eta^{-\frac{2}{\theta}} \left( \frac{2 + \theta}{\theta + 1 - r} \right)^{\frac{2(\theta+1)}{\theta}} \left( \frac{1}{(\theta - 1)} \right)^{\frac{2(\theta+1-r)}{\theta(r-1)}}. \]
For fixed \( t > 0 \), by taking \( s \in (0, t) \) and \( T = t - s \) in (2.13) we obtain
\[ \mu((P_t)^2) = \frac{1}{2} \int_{\mathbb{H} \times \mathbb{H}} |P_t f(x) - P_t f(y)|^2 \mu(dx) \mu(dy) \]
(2.15)
\[ \leq \frac{\mu(f^2)}{2} \inf_{s \in (0, t)} \left\{ \exp \left[ \frac{C_0}{s^{\frac{2(\theta+1-r)}{\theta(r-1)}} (t - s)^{\frac{1}{\theta}}} - 1 \right) \right\}, \quad t > 0, \mu(f) = 0. \]
(d) To calculate the inf in (2.15), let
\[ \alpha_1 = \frac{2(\theta + 1 - r)}{\theta(r-1)}, \quad \alpha_2 = \frac{2 + \theta}{\theta}. \]
We have \( \alpha_1 + \alpha_2 = \frac{r+1}{r-1} \), and by (2.14),
\[ \inf_{s \in (0, t)} s^{\alpha_1} (t - s)^{\alpha_2} = \frac{C_0}{t^{\frac{r+1}{r-1}}} \left( \frac{\theta(r-1)}{2(\theta + 1 - r)} \right)^{\frac{2(\theta+1-r)}{\theta(r-1)}} \left( \frac{\theta}{2 + \theta} \right)^{\frac{2 + \theta}{\theta}} \]
\[ = \frac{\alpha}{t^{\frac{r+1}{r-1}}}, \quad t > 0. \]
Then it follows from (2.15) that
\[ \mu((P_t f)^2) \leq \frac{\mu(f^2)}{2} \left( \exp \left[ \alpha t^{\frac{r+1}{r-1}} \right] - 1 \right), \quad t > 0, \mu(f) = 0. \]
Obviously, there exists $t_0 \in (0, \infty)$ such that

$$0 < \frac{1}{t_0} \log \frac{2}{\exp \left[ \alpha t_0 \frac{r + 1}{r - 1} \right] - 1} = \sup_{t > 0} \frac{1}{t} \log \frac{2}{\exp \left[ \alpha t \frac{r + 1}{r - 1} \right] - 1} =: \lambda.$$ 

So, (2.16) yields that

$$\mu((P_{t_0} f)^2) \leq \mu(f^2) e^{-\lambda t_0}, \quad \mu(f) = 0.$$ 

Letting $i(t) = \sup\{n \in \mathbb{Z}_+ : n \leq \frac{1}{t_0} \}$ be the integer part of $\frac{1}{t_0}$, combining this with the semigroup property and the $L^2(\mu)$-contraction of $P_t$, we obtain

$$\mu((P_{t} f)^2) \leq \mu((P_{i(t) t_0} f)^2) \leq \mu(f^2) e^{-\lambda t_0 i(t)} \leq \mu(f^2) e^{-\lambda (t - t_0)}, \quad t \geq 0, \mu(f) = 0.$$ 

Thus, (2.5) holds for $C := e^{\lambda t_0}$.

Finally, to derive the desired explicit lower bounds of $\lambda$, we take

$$t = \left( \frac{\alpha}{\log(1 + 2 \exp[-\frac{r + 1}{r - 1}])} \right)^\frac{r + 1}{r - 1}.$$ 

Then

$$\lambda \geq \frac{1}{t} \log \frac{2}{\exp \left[ \alpha t \frac{r + 1}{r - 1} \right] - 1} = (r + 1) \{ \log(1 + 2 \exp \left[ -\frac{r + 1}{r - 1} \right]) \}_{r + 1} \frac{1}{r - 1}$$

$$= \left( \theta + 1 - r \right)^{2 r + 1} \frac{2 \theta^{2 r + 1}}{\eta^{2 r + 1}} \frac{2 (r + 1)}{\eta^{2 r + 1}} \left\{ \log \left( 1 + 2 e^{-\frac{r + 1}{r - 1}} \right) \right\} \frac{r + 1}{r - 1}$$

$$= \left( \theta + 1 - r \right)^{2 r + 1} \frac{2 \theta^{2 r + 1}}{\eta^{2 r + 1}} \frac{2 (r + 1)}{\eta^{2 r + 1}} \left\{ \log \left( 1 + 2 e^{-\frac{r + 1}{r - 1}} \right) \right\} \frac{r + 1}{r - 1} \geq \frac{\left( \theta + 1 - r \right)^{2 r + 1}}{e \theta^2 + 2 \theta + 2 \theta^2} \frac{1}{r - 1},$$

where the last step is due to the fact that

$$\inf_{s \geq 1} \left\{ \log(1 + 2 e^{-s}) \right\} \frac{1}{r} = \lim_{s \to \infty} \left\{ \log(1 + 2 e^{-s}) \right\} \frac{1}{r} = e^{-1}.$$



To conclude this section, we indicate that $P_t$ is ultra-exponential convergent provided

$$2 v^* \langle b(u) - b(v), u - v \rangle_v \leq \gamma \| u - v \|_H^2 - \max \left\{ \eta \| u - v \|_Q^2, \delta \| u - v \|_H^{r + 1 - \theta}, \delta \| u - v \|_H^{1 + r} \right\}$$

holds for some constant $\gamma, \eta > 0$, which is weaker than (2.2). This can be proved as in [8] proof of Theorem 1.5 using the Harnack inequality in [18] Theorem 2.2.1 and the ultraboundedness of $P_t$. When $\gamma > 0$ is small enough, with the coupling constructed in the proof of Theorem 2.2.1 in [18], we may derive explicit lower bounds of the convergence rate $\lambda$ using the argument in the proof of Theorem 2.1. As in this case the resulting estimates will be rather complicated, in Theorem 2.1 we only consider the case that $\gamma = 0$. However, to derive explicit lower bounds of $\lambda$ for any $\gamma > 0$, new techniques are required.
3 Stochastic porous medium equation

Let $\Delta$ be the Dirichlet Laplacian on the interval $(0, l)$ for some $l > 0$, and let $\sigma > 0, r > 1$ be two constants. Let $W_t$ be the cylindrical Brownian motion on $L^2(m)$, where $m(dx) := l^{-1}dx$ is the normalized Lebesgue measure on $(0, l)$. Consider the following stochastic porous medium equation

$$dX_t = \Delta X_t^r dt + \sigma dW_t.$$  

We first verify assumptions $(A1)$-$(A4)$ for an appropriate choice of $(H, V)$. It is well known that the spectrum of $-\Delta$ consists of simple eigenvalues $\{\lambda_k := \frac{\pi^2 k^2}{l^2} \}_{k \geq 1}$. Let $\{e_k\}_{k \geq 1}$ be the corresponding eigenbasis. Then $Q := \sigma I$ is Hilbert-Schmidt from $L^2(m)$ to $H := H^{-1}$, the completion of $L^2(m)$ under the inner product

$$\langle x, y \rangle_H := \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \langle x, e_i \rangle \langle y, e_i \rangle.$$  

Let $V = L^{1+r}(m)$. Then $b(v) := \Delta v^r$ extends to a unique map from $V$ to $V^*$ with

$$V^* \langle b(v), u \rangle_V = -\int_0^l v^r u^r dm, \ u, v \in V.$$  

This implies $(A3)$ and $(A4)$ for $C_1 = C_2 = 1$. Moreover, for any $v_1, v_2, v \in V$,

$$V^* \langle b(v_1 + sv_2), v \rangle_V = -\int_0^l (v_1 + sv_2)^r v^r dm$$

is continuous in $s \in \mathbb{R}$; that is, $(A1)$ holds. Finally, we have (see the proof of Proposition 3.1 below)

$$\langle x, y \rangle_H = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \langle x, e_i \rangle \langle y, e_i \rangle.$$  

Let $\nu = L^{1+r}(m)$. Then $b(v) := \Delta v^r$ extends to a unique map from $\nu$ to $\nu^*$ with

$$\nu^* \langle b(v), u \rangle_{\nu} = -\int_0^l v^r u^r dm, \ u, v \in \nu.$$  

This implies $(A3)$ and $(A4)$ for $C_1 = C_2 = 1$. Moreover, for any $v_1, v_2, v \in \nu$,

$$\nu^* \langle b(v_1 + sv_2), v \rangle_{\nu} = -\int_0^l (v_1 + sv_2)^r v^r dm$$

is continuous in $s \in \mathbb{R}$; that is, $(A1)$ holds. Finally, we have (see the proof of Proposition 3.1 below)

$$(s^r - t^r)(s - t) \geq 2^{1-r}|s - t|^{1+r}, \ s, t \in \mathbb{R}.$$  

Then

$$\nu^* \langle b(v_1) - b(v_2), v_1 - v_2 \rangle_{\nu} \leq -2^{1-r}||v_1 - v_2||_{\nu}^{1+r}, \ v_1, v_2 \in \nu,$$

so that $(A2)$ holds for any positive constant $C_1$. Therefore, for any initial data $x \in \mathbb{H}$ the equation (3.2) has a unique solution starting at $x$. Let $P_t$ be the associated Markov semigroup.

**Proposition 3.1.** For the equation (3.1), $P_t$ has a unique invariant probability measure $\mu$ such that (1.3) holds for some constant $C > 0$ and $\lambda$ defined in (2.3) for

$$\alpha := \frac{l^{-1-r}(3 + r)(r + 1)^{2(r+1)}}{(2\pi)^{1-r}\sigma^2(r - 1)^{2(r+1)}}.$$  

Moreover,

$$\lambda \geq \frac{(2\pi)^{\frac{4}{r+1}} \sigma^{\frac{2(r-1)}{r+1}} \{\log(1 + 2 \exp[-\frac{r+1}{r}])\}^\frac{r-1}{r+1}}{(r + 1)l^{\frac{4}{r+1}}(3 + r)^{\frac{r-1}{r+1}}} \geq \frac{(2\pi)^{\frac{4}{r+1}} \sigma^{\frac{2(r-1)}{r+1}}}{e(r + 1)l^{\frac{4}{r+1}}(3 + r)^{\frac{r-1}{r+1}}}.$$

Proof. We first prove (3.2). Obviously, we may assume that $s > t$ by considering the following two situations respectively.

(i) $s > t \geq 0$. Since $0 \leq s \mapsto s^r$ is convex, we have

$$\frac{d}{ds} (s^r - t^r) = \frac{rt}{(s - t)^{r+1}} (t^{r-1} - s^{r-1}) \leq 0.$$

So,

$$\inf_{s > t} \frac{s^r - t^r}{(s - t)^r} = \lim_{s \to \infty} \frac{s^r - t^r}{(s - t)^r} = 1, \quad t \geq 0.$$

Then (3.2) holds since $2^{1-r} \leq 1$.

(ii) $s \geq 0 > t$. By the Jensen inequality we have

$$s^r - t^r = 2 \left( \frac{s^r}{2} + \frac{|t|^r}{2} \right) \geq 2 \left( s + \frac{|t|}{2} \right)^r = 2^{1-r} (s + |t|)^r = 2^{1-r} (s - t)^r.$$

Thus, (3.2) holds.

Now, let $b(x) = \Delta x^r, x \in V := L^{r+1}(m)$. Since $Q = \sigma I$, we have $\| \cdot \|_Q = \frac{1}{\sigma} \| \cdot \|_2$. Combining this with $\| \cdot \|_{r+1} \geq \| \cdot \|_2, \lambda_1 = \frac{\pi^2}{r-1}$ and the definition of $| \cdot |_H$, we obtain

$$\|x\|_{r+1} \geq \|x\|_2 = \max \left\{ \sqrt{\lambda_1} |x|_H, \sigma \|x\|_Q \right\} = \max \left\{ \frac{\pi}{r} |x|_H, \sigma \|x\|_Q \right\}.$$

Then, due to (3.2), for any $\theta \in (r-1, r+1] \cap [2, r+1]$,

$$2 \nu_v \langle b(x) - b(y), x - y \rangle_v = -2 \int_0^1 (x^r - y^r)(x - y)dm \leq -2 \nu^{r+1} \|x - y\|_{r+1} \leq - \max \left\{ \eta \|x - y\|_{Q}^{\theta} |x - y|_{H}^{r+1-\theta}, \delta |x - y|_{H}^{r+1}\right\}, \quad x, y \in V := L^{r+1}(m)$$

holds for

$$\eta := 2^{2-r} \sigma^\theta \left( \frac{\pi}{r} \right)^{r+1-\theta}, \quad \delta := 2^{2-r} \left( \frac{\pi}{r} \right)^{r+1}.$$

Therefore, by Theorem 2.1 (1.3) holds for some constant $C \in (0, \infty)$ and

$$\lambda := \sup_{t > 0, \theta \in (r-1, r+1] \cap [2, r+1]} \frac{1}{t} \log \frac{2}{\exp \left[ \alpha \phi(t^{\frac{r+1}{r}}) \right] - 1},$$
where

\[ \alpha_{\theta} := \frac{4^{\frac{r+3}{r-1}}}{\sigma^2(\theta + 1 - r)^{\frac{2r}{r-1}}} \left( \frac{1}{\pi} \right)^{\frac{1}{r-1}} \left( \frac{\theta r + \theta}{r - 1} \right)^{\frac{r+1}{r-1}} \]

\[ = \frac{4^{\frac{r+3}{r-1}}}{\sigma^2} \left( \frac{1}{\pi} \right)^{\frac{1}{r-1}} \left( \frac{r + 1}{r - 1} \right)^{\frac{r+1}{r-1}} \left( \frac{\theta}{\theta + 1 - r} \right)^{\frac{r+1}{r-1}} \frac{2 + \theta}{\theta + 1 - r}. \]

Noting that \( r \geq 1 \) implies \( \theta + 1 - r \leq \theta \), so that \( \alpha_{\theta} \) is decreasing in \( \theta \), we obtain

\[ \inf_{\theta \in (r-1, r+1] \cap [2, r+1]} \alpha_{\theta} = \alpha_{r+1} = \frac{l^{\frac{r-1}{r-1}}(3 + r)(r + 1)^{\frac{2(r+1)}{r-1}}}{(2\pi)^{\frac{1}{r-1}} \sigma^2(r - 1)^{\frac{r+1}{r-1}}} =: \alpha. \]

So, (1.3) holds for some \( C \in (0, \infty) \) and the desired \( \lambda \). Moreover, as in the proof of Theorem 2.1 that the desired lower bound estimates follows by taking in (2.3)

\[ t = \left( \frac{\alpha}{\log(1 + 2 \exp[-\frac{r+1}{r-1}] \right)^{\frac{r-1}{r+1}}}. \]

To conclude this section, let us recall a corresponding result in the linear case, i.e. \( r = 1 \). Let \( R = \sigma I \) and \( T_t = e^{\epsilon \Delta} \). In this case, for any \( p > 2 \) there exist constants \( C_p, t_p \in (0, \infty) \) such that

(3.3) \[ \| P_t - \mu \|_{L^2(\mu) \rightarrow L^2(\mu)} \leq C_p e^{-\lambda_1 t}, \quad t \geq t_p. \]

To see this, we observe that \( \sigma W_t \) is a Wiener process on \( \mathbb{H} \) with variance operator \( Q e_i := \sigma^2, i \geq 1 \). Taking \( M = 0, R = Q \) and \( T_t = e^{\epsilon \Delta} \), we see that assumptions in \([16, Corollary 1.4]\) hold for \( h_1(t) = e^{-\lambda_1 t/2} \) and \( h_2(t) = 0 \), so that

(3.4) \[ \| P_t - \mu \|_{L^2(\mu) \rightarrow L^2(\mu)} \leq e^{-\lambda_1 t}, \quad t \geq 0. \]

Moreover, according to \([2, Theorem 4 c]\), \( P_t \) is hypercontractive, i.e. for any \( p > 2 \) there exists a constant \( t_p > 0 \) such that \( \| P_t \|_{L^2(\mu) \rightarrow L^p(\mu)} = 1 \) holds for \( t \geq t_p \). Combining this with (3.4) we prove (3.3). Note that in this linear case \( P_t \) is not ultra-bounded, so that we do not have the ultra-exponential convergence as in (1.3).

A feature in the linear case is that the exponential convergence rate \( \lambda_1 \) is independent of \( \sigma \). Note that for \( r > 1 \) the lower bound estimates of \( \lambda \) presented in Proposition 3.1 are increasing to \( \infty \) as \( \sigma \uparrow \infty \). But if we let \( r \downarrow 1 \) in these estimates, the lower bounds of \( \lambda \) tend to \( \frac{2\lambda_1}{\sigma} \) (of course, the other constant \( C \) will tend to \( \infty \) since \( P_t \) is not ultracontractive for \( r = 1 \)), which is also independent of \( \sigma \). This indicates that the power of \( \sigma \) included in the lower bound estimates of \( \lambda \) presented in Proposition 3.1 is suitable when \( r \) goes down to 1.
4 Stochastic $p$-Laplace equation

Again let $D = (0, l)$ for some $l > 0$ and $m$ be the normalized volume measure. For $p > 2$, let $\mathbb{H}_0^p$ be the closure of $C_0^\infty(D)$ with respect to the norm

$$\|f\|_{1,p} := \|f\|_p + \|\nabla f\|_p,$$

where, since $D$ is one-dimensional, $\nabla f := f'$. The $p$-Laplacian on $D$ is defined by

$$\Delta_p f = \nabla (|\nabla f|^{p-2} \nabla f), \quad f \in C^2(D).$$

Consider the SPDE

(4.1) \[ dX_t = \Delta_p X_t dt + QdW_t, \]

where $W_t$ is a cylindrical Brownian motion on $L^2(m)$, and $Q \in \mathcal{L}(\mathbb{H})$ is such that

(4.2) \[ Qe_i = q_i e_i, \quad q_i^2 \geq \frac{\sigma^2}{l^2}, \quad \sum_{i=1}^{\infty} q_i^2 < \infty \]

holds for some constants $\sigma > 0$ and $\{q_i\}_{i \geq 1} \subset \mathbb{R}$, recall that $\{e_i\}_{i \geq 1}$ is the eigenbasis of $-\Delta$ with respect to the eigenvalues $\lambda_i := \left(\frac{i \pi}{l}\right)^2, i \geq 1$.

To apply Theorem 2.1 let $E = \mathbb{H} = L^2(m), V = \mathbb{H}_0^p$ and $r = p - 1 > 1$. Then $Q \in \mathcal{L}_{HS}(E, \mathbb{H})$ by (4.2), and $b := \Delta_p$ extends to a unique operator from $V$ to $V^*$ with

(4.3) \[ \nu \cdot \langle b(v), u \rangle_V := -\int_0^l |\nabla v|^{p-2} \langle \nabla v, \nabla u \rangle dm, \quad u, v \in V. \]

Thus, (A1) and (A4) with $C_1 = 1$ hold. Next, since for any $f \in C_0^\infty(D)$ we have

$$\int_0^l |f|^p dm = \int_0^l \left( \int_0^x f'(s) ds \right)^p m(dx) \leq \int_0^l x^{p-1} m(dx) \int_0^l |f'(s)|^p ds \leq \left( \int_0^l |f'|^p dm \right) \int_0^l x^{p-1} dx = \frac{l^p}{p} \int_0^l |f'|^p dm,$$

it follows that

$$\|f\|_{1,p} \leq (1 + lp^{-\frac{1}{p}}) \|\nabla f\|_p, \quad f \in V.$$

From this and (4.3), it is easy to see that (A3) holds for $C_1 = 0$ and some $C_2 > 0$. Moreover, according to [8, Example] we have

(4.4) \[ 2\nu \cdot \langle b(u) - b(v), u - v \rangle_V \leq -2^{p-1} \|\nabla (u - v)\|_2^p, \quad u, v \in V. \]

Then (A2) holds for $C_1 = 0$. Therefore, for any $x \in \mathbb{H}$ the equation (4.1) has a unique solution starting at $x$. Let $P_t$ be the Markov semigroup associated to (4.1).
Proposition 4.1. For the equation (4.1), \( P_t \) has a unique invariant probability measure \( \mu \) such that (1.3) holds for some constant \( C > 0 \) and \( \lambda \) defined in (2.3) for \( \alpha := \left( \frac{p^2 l^2}{\pi^2 (p - 2)} \right)^{\frac{p}{p - 2}} \frac{2 + p}{\sigma^2 2^{\frac{4(p - 1)}{p - 2}}} \).

Moreover,
\[
\lambda \geq \frac{\pi^2 2^{\frac{4(p - 1)}{p}} \sigma^2}{pl^2 (2 + p)^{\frac{2(p - 2)}{p}}} \left\{ \log \left( 1 + 2e^{-\frac{p}{\pi^2}} \right) \right\}^{\frac{p}{p - 2}} \geq \frac{\pi^2 2^{\frac{4(p - 1)}{p}} \sigma^2}{epl^2 (2 + p)^{\frac{2(p - 2)}{p}}}
\]

Proof. By the Poincaré inequality we have
\[
\mathbf{m}(\|\nabla (u - v)\|^2) \geq \frac{\pi^2}{l^2} \|u - v\|_2^2 = \frac{\pi^2}{l^2} \|u - v\|_H^2.
\]

Next, by (4.2)
\[
\|\nabla (u - v)\|_2^2 = \frac{\pi^2}{l^2} \sum_{i=1}^{\infty} i^2 \langle u - v, e_i \rangle_H^2 \geq \frac{\pi^2 \sigma^2}{l^2} \sum_{i=1}^{\infty} 1 \langle u - v, e_i \rangle_H^2 = \frac{\pi^2 \sigma^2}{l^2} \|u - v\|_Q^2.
\]

Combining this with (4.4) and (4.5), we arrive at
\[
2 \nu \langle b(u) - b(v), u - v \rangle_Y \leq -2^{p-1} \left( \frac{\pi}{l} \right)^p \max \left\{ \sigma^p \|u - v\|_Q^p, \|u - v\|_H^p \right\}.
\]

This implies (2.2) for
\[
r = p - 1, \quad \theta = p, \quad \eta = 2^{p-1} \left( \frac{\pi \sigma}{l} \right)^p, \quad \delta = 2^{p-1} \left( \frac{\pi}{l} \right)^p.
\]

Therefore, the proof is finished by Theorem 2.1.

5 Exponential convergence for stochastic fast-diffusion equations

In this section we consider the equation (2.1) in Section 2 for \( r \in (0, 1) \), i.e. the stochastic fast-diffusion equation. In this case, we do not have the ultra-exponential convergence, but we are able to derive a weaker version of exponential convergence by combining the Harnack inequality with a result of [5], see [8] for the study of the equation for \( r \geq 1 \). To see the difference between the case of \( r \geq 1 \) and that of \( r \in (0, 1) \), we come back to the specific equation (3.1). When \( r \geq 1 \) all assumptions in [5] Theorem 2.5] can be easily verified (see the proof of Theorem 1.5 in [8]), but when \( r \in (0, 1) \) one needs additional conditions (see (5.7) below) which exclude the equation (3.1) where \( Q := \sigma I \) for some \( \sigma > 0 \).
From now on, we let $r \in (0, 1)$ and consider the equation (2.1) such that assumptions (A1)-(A4) hold. Let $P_t$ be the associated Markov semigroup. We aim to investigate the $V$-uniformly exponential convergence

$$\|P_t - \mu\|_V := \sup_{|f| \leq V} \|\frac{|P_t f - \mu(f)|}{V}\|_\infty \leq C e^{-\lambda t}, \quad t \geq 0$$

for some constants $C, \lambda > 0$, where $\mu$ is the invariant probability measure of $P_t$ and $V \geq 1$ is a continuous function on $\mathbb{H}$. Obviously, (5.1) is equivalent to

$$\sup_{|f| \leq V} |P_t f(x) - \mu(f)| \leq CV(x)e^{-\lambda t}, \quad t \geq 0, \quad x \in \mathbb{H}$$

used in [3, Definition 2.3].

**Theorem 5.1.** If there exists a non-negative measurable function $h$ on $\mathbb{V}$ such that $\{h \leq R\}$ is relatively compact in $\mathbb{H}$ for any $R > 0$, and

$$\nu\langle b(u), u \rangle_V \leq \alpha - \eta \{h(u) \vee \|u\|_\mathbb{H}\}^{1+r}, \quad u, u \in \mathbb{H},$$

$$\nu\langle b(u) - b(v), u - v \rangle_V \leq -\frac{\eta \|u - v\|_Q^g}{|u - v|_{\mathbb{H}}^{\theta - 2} \{h(u) \vee h(v)\}^{1-r}}, \quad u, v \in \mathbb{V}$$

hold for some constants $\alpha, \eta > 0$ and $\theta \geq \frac{4}{1+r}$. Then $P_t$ has a unique invariant probability measure $\mu$, and for any $\gamma > 0$, there exist two constants $C, \lambda > 0$ such that (5.1) holds for $V := \exp[\gamma(1 + \|\cdot\|_\mathbb{H}^{1+r})].$

**Proof.** By (5.2) and the Itô formula, we see that

$$\frac{1}{n} \int_0^n \mathbb{E}h(X_t^0)^{1+r} dt \leq \frac{2\alpha + \|Q\|_{HS}^2}{2\eta} < \infty, \quad n \geq 0.$$ 

Since $h$ has relatively compact level sets in $\mathbb{H}$, this implies that the sequence $\{\frac{1}{n} \int_0^n \delta_0 P_t dt\}_{n \geq 1}$ is tight and each of its weak limit point gives rise to an invariant probability measure of $P_t$. Now, according to the proof of [3, Theorem 2.5(1)], it suffices to verify

(i) (Assumption 2.1 in [3]): $P_t$ is strong Feller (i.e. $P_t \mathcal{B}_b(\mathbb{H}) \subset C_b(\mathbb{H}), t > 0$) and $P_1 U(x) > 0$ holds for any $t > 0, x \in \mathbb{H}$ and non-empty open set $U \subset \mathbb{H}$. 

(ii) (Assumption 2.2 in [3]): For any $r > 0$ there exists $t_0 > 0$ and a compact subset $K$ of $\mathbb{H}$ such that $\inf_{|x|_{\mathbb{H}} \leq r} \mathbb{E}1_K(X_{t_0}^x) > 0$.

(iii) (In place of (2.4) in [3]): There exist constants $\beta, k, c > 0$ such that $\mathbb{E}V(X_t^x) \leq kV(x)e^{-\beta t} + c, \quad t \geq 0, \quad x \in \mathbb{H}$.
Firstly, according to [18, Theorem 2.3.1] (see also [10, Theorem 1.1] under a more specific framework), for any \( p > 1 \) there exists a continuous function \( \Phi_p \) on \( \mathbb{H} \times \mathbb{H} \times (0, \infty) \) with \( H(x, x, t) = 0 \) such that the Harnack inequality

\[
|P_t f(x)|^p \leq (P_t |f|^p)(y) e^{\Phi_p(x, y; t)}, \quad x, y \in \mathbb{H}, \ t > 0, \ f \in \mathcal{B}_0(\mathbb{H})
\]

holds. By [18, Theorem 1.4.1] (see also [19, Proposition 3.1]), this implies that the invariant probability measure of \( P_t \) is unique with full support on \( \mathbb{H} \), and \( P_t \) is strong Feller and has a strictly positive density with respect to the unique invariant probability measure \( \mu \). Therefore, (i) holds.

Next, since \( h \) has relatively compact level sets in \( \mathbb{H} \), it follows from (5.4) that for any \( t_0 > 0 \) and compact set \( K \) in \( \mathbb{H} \). Indeed, (5.4) implies \( c_0 := \mathbb{E} h(X_{t_0}) < \infty \) for some \( t_0 > 0 \), so that we may take \( K \) being the closure of \( \{ h \leq c_0 + 1 \} \). Then it follows from (5.5) that for any \( r > 0 \),

\[
\inf_{|x| \leq r} P_{t_0} 1_K(x) \geq (P_{t_0} 1_K(0))^p \inf_{|x| \leq r} e^{-\Phi_p(0, x, t_0)} > 0.
\]

Thus, (ii) holds.

Finally, by (5.2) and the Itô formula, we have

\[
d|X_t|_{\mathbb{H}}^2 \leq (2\alpha + \|Q\|_{HS}^2 - 2\eta|X_t|_{\mathbb{H}}^{1+r})dt + 2\langle X_t, QdW_t \rangle_{\mathbb{H}}.
\]

Then for any \( \gamma > 0 \),

\[
de^{\gamma(1+|X_t|_{\mathbb{H}}^{2})^{(1-r)/2}} \leq \frac{\gamma(1-r)}{2} e^{\gamma(1+|X_t|_{\mathbb{H}}^{2})^{(1-r)/2}} (1 + |X_t|_{\mathbb{H}}^2)^{-\frac{1+r}{2}} d|X_t|_{\mathbb{H}}^2
\]

\[
+ 2\gamma^2 (1 + |X_t|_{\mathbb{H}}^2)^{-1+(1+r)} \|Q\|_{E \to \mathbb{H}}^2 |X_t|_{\mathbb{H}}^2 e^{\gamma(1+|X_t|_{\mathbb{H}}^{2})^{(1-r)/2}} dt
\]

\[
\leq \gamma e^{\gamma(1+|X_t|_{\mathbb{H}}^{2})^{(1-r)/2}} \left\{ (1-r)(2\alpha + \|Q\|_{HS}^2) + 2\|Q\|_{E \to \mathbb{H}}^2 (1 + |X_t|_{\mathbb{H}}^2)^{-\frac{1+r}{2}} \right\} dt + dM_t
\]

\[
\leq \{ C_1 - C_2 e^{\gamma(1+|X_t|_{\mathbb{H}}^{2})^{(1-r)/2}} \} dt + dM_t
\]

for some constants \( C_1, C_2 > 0 \) and some local martingale \( M_t \). This implies

\[
de^{C_2 t + \gamma(1+|X_t|_{\mathbb{H}}^{2})^{(1-r)/2}} \leq C_1 e^{C_2 t} dt + e^{C_2 t} dM_t.
\]

Letting \( \tau_n \uparrow \infty \) be a sequence of stopping times such that \( (M_{t \wedge \tau_n})_{t \geq 0} \) is a martingale for every \( n \geq 1 \), we obtain

\[
e^{C_2 t} \mathbb{E} V(X_t) = e^{C_2 t} \mathbb{E} e^{\gamma(1+|X_t|_{\mathbb{H}}^{2})^{(1-r)/2}} = \mathbb{E} \lim_{n \to \infty} e^{C_2 t \wedge \tau_n + \gamma(1+|X_{t \wedge \tau_n}|_{\mathbb{H}}^{2})^{(1-r)/2}}
\]

\[
\leq \lim_{n \to \infty} \mathbb{E} e^{C_2 t \wedge \tau_n + \gamma(1+|X_{t \wedge \tau_n}|_{\mathbb{H}}^{2})^{(1-r)/2}} \leq e^{\gamma(1+|X_0|^{2})^{(1-r)/2}} + \lim_{n \to \infty} \mathbb{E} \int_0^{t \wedge \tau_n} C_1 e^{C_2 s} ds
\]

\[
= e^{\gamma(1+|X_0|^{2})^{(1-r)/2}} + \frac{C_1(e^{C_2 t} - 1)}{C_2} \leq V(X_0) + \frac{C_1}{C_2} e^{C_2 t}, \quad X_0 \in \mathbb{H}.
\]

this implies (iii) for some \( \beta = C_2, k = 1 \) and \( c = \frac{C_1}{C_2} \).
To illustrate Theorem 5.1, we let $\Delta, m, H := H^{-1}, \{\lambda_i, e_i\}_{i \geq 1}, E := L^2(m)$ and $W_t$ be in Section 3, and consider the equation

\begin{equation}
(5.6) \quad dX_t = \Delta X_t^{1+r} dt + Q dW_t
\end{equation}

for some $r \in (0, 1)$, and $Qe_i := q_i e_i (i \geq 1)$ with $\{q_i\}_{i \geq 1} \subset \mathbb{R}$ satisfying

\begin{equation}
(5.7) \quad \|Q\|_{HS}^2 := \sum_{i=1}^{\infty} \frac{q_i^2}{\lambda_i} < \infty, \quad \inf_{i \geq 1} |q_i| \frac{r+1}{r} > 0
\end{equation}

for some constants $\theta \geq \frac{4}{r+1}$ and $\varepsilon \in (\frac{1-r}{2(1+r)}, 1)$. Since $\lambda_i = \frac{\pi^2 i^2}{r^2}$, if $r \in (\frac{1}{2}, 1)$ then for any $\kappa \in (\frac{1}{2}, \frac{1+3r}{8}) \neq 0$, $q_i := \lambda_i^{\frac{1}{2} - \kappa} (i \geq 1)$ satisfies $(5.7)$ for $\theta = \frac{4}{r+1}$ and $\varepsilon = 1 - \frac{4\kappa}{1+r} \in (\frac{1-r}{2(1+r)}, 1)$.

**Corollary 5.2.** Let $P_t$ be the Markov semigroup associated to $(5.6)$, such that $(5.7)$ holds for some constants $\theta \geq \frac{4}{r+1}$ and $\varepsilon \in (\frac{1-r}{2(1+r)}, 1)$. Then the assertion of Theorem 5.1 holds.

**Proof.** By $(5.7)$ we have $Q \in \mathcal{L}_{HS}(E, H)$. Then it is easy to see that assumptions $(A1)$-$(A4)$ hold for $b(u) := \Delta u^r$ for $u \in \mathbb{V} := L^{1+r}(m)$, provided $\mathbb{V}$ is continuously embedded into $H$. In fact, according to the proof of Corollary 3.2 in [10], since $d := 1 \in (0, \frac{2(c+1)}{1-r})$ due to $(5.7)$, the classical Nash inequality

\begin{equation}
(5.8) \quad \|f\|_2^{2+r} \leq C m(|\nabla f|^2), \quad f \in C_0^1((0, l)), \quad m(|f|) = 1
\end{equation}

for some constant $C > 0$ implies that

\begin{equation}
(5.9) \quad \|x\|_{r+1}^2 \geq c \sum_{i=1}^{\infty} \frac{m(xe_i)^2}{\lambda_i^\varepsilon}, \quad x \in \mathbb{V}
\end{equation}

holds for some constant $c > 0$. Since $\varepsilon < 1$, this implies that $\mathbb{V}$ is compactly (hence, also continuously) embedded into $H$. So, it remains to verify conditions $(5.2)$ and $(5.3)$ in Theorem 5.1 for $h(u) := \|u\|_{r+1}$ and some constants $\alpha, \eta > 0$.

Since by $(5.8)$ we have $h(u)^2 := \|u\|_{r+1}^2 \geq c \lambda_1^{1-c} \|u\|_1^2$, $(5.2)$ with some $\eta > 0$ and any $\alpha > 0$ follows from the fact that $\mathbb{V}, (b(u), u)_{\mathbb{V}} = -h(u)^{1+r}$. Next, by $(5.7)$ we have $|q_i| \geq c_1 \lambda_i^{\frac{1}{r} - \frac{r}{2}}$ for some constant $c_1 > 0$ and all $i \geq 1$. Combining this with $(5.8)$ we obtain

\begin{equation}
(5.9) \quad \|x\|_Q^2 \left( \sum_{i \geq 1} \frac{m(xe_i)^2}{\lambda_i^\varepsilon} \right)^{\theta - 2} \leq \left( \sum_{i \geq 1} \frac{\lambda_i^{\theta - 2} m(xe_i)^2}{|q_i|^\theta} \right)^{\theta - 2}
\end{equation}

for some constants $c_2, c_3 > 0$. Moreover, by the Hölder inequality and noting that

\begin{equation}
\mathbb{m}((|u| \vee |v|)^{1+r}) \leq h(u)^{1+r} + h(v)^{1+r} \leq 2(h(u) \vee h(v))^{1+r},
\end{equation}

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we obtain
\[
\|u - v\|_{r+1}^{1+r} := m(|u - v|^{1+r}) \leq m\left(|u - v|^2(|u| \lor |v|)^{r-1}\right)^{\frac{1+r}{2}} m\left((|u| \lor |v|)^{1+r}\right)^{\frac{1-r}{2}} \leq 2^{\frac{1+r}{2}} m\left(|u - v|^2(|u| \lor |v|)^{r-1}\right)^{\frac{1+r}{2}} \{h(u) \lor h(v)\}^{\frac{1-r}{2}}.
\]

Combining this with (5.9) we arrive at
\[

\bar{v}. \langle b(u) - b(v), u - v \rangle_v = -m\left((u^r - v^r)(u - v)\right) \leq -r m\left(|u - v|^2(|u| \lor |v|)^{r-1}\right) \leq -\eta \|u - v\|^2 \|Q\|_{Q}\|u - v\|^2 \{h(u) \lor h(v)\}^{1-r} \leq -\eta \|u - v\|^2 \{h(u) \lor h(v)\}^{1-r}
\]

for some constant $\eta > 0$ and all $u, v \in \mathbb{V}$. Thus, (5.3) holds.

\[
\square
\]

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