A STABILIZER FREE WEAK GALERKIN FINITE ELEMENT METHOD ON POLYTOPAL MESH: PART II

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Abstract. A stabilizer free weak Galerkin (WG) finite element method on polytopal mesh has been introduced in Part I of this paper (J. Comput. Appl. Math, 371 (2020) 112699, arXiv:1906.06634.) Removing stabilizers from discontinuous finite element methods simplifies formulations and reduces programming complexity. The purpose of this paper is to introduce a new WG method without stabilizers on polytopal mesh that has convergence rates one order higher than optimal convergence rates. This method is the first WG method that achieves superconvergence on polytopal mesh. Numerical examples in 2D and 3D are presented verifying the theorem.

Key words. weak Galerkin finite element methods, second-order elliptic problems, polytopal meshes

AMS subject classifications. Primary: 65N15, 65N30; Secondary: 35J50

1. Introduction. A stabilizing/penalty term is often used in finite element methods with discontinuous approximations to enforce connection of discontinuous functions across element boundaries. Removing stabilizers from discontinuous finite element method is desirable since it simplifies formulation and reduces programming complexity. A stabilizer free weak Galerkin finite element has been developed in [10] for the following model problem: seek an unknown function $u$ satisfying

\begin{align}
-\Delta u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

where $\Omega$ is a polytopal domain in $\mathbb{R}^d$. The WG method developed in [10] has the following simple formulation without any stabilizers:

\begin{equation}
(\nabla_w u_h, \nabla_w v) = (f, v),
\end{equation}

where $\nabla_w$ is weak gradient. Remove of stabilizing terms from the WG finite element methods is challenging, specially on polytopal mesh. Construction of spaces to approximate $\nabla_w$ is the key of maintaining ultra simple formulation (1.3). The main idea in [10] is to raise the degree of polynomials used to compute weak gradient $\nabla_w$. In [10], gradient is approximated by a polynomial of order $j = k + n - 1$ with $n$ the number of sides of polygonal element. This result has been improved in [11, 12] by reducing the degree of polynomial $j$. In [13, 17], Wachspress coordinates [4] are used to approximate $\nabla_w$, which are usually rational functions, instead of polynomials.

In this paper, we introduce a new stabilizer free WG finite element method on polytopal mesh. This method is the first WG method that achieves superconvergence on polytopal mesh. In this method we use piecewise low order polynomials on a polygonal element to approximate $\nabla_w$ instead of using one piece high order polynomial in [10]. While the stabilizer free WG method in [10] has optimal convergence rates, our new WG method improves the convergence rate from optimality by order one.

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in both an energy norm and the $L^2$ norm. Superconvergence results of the WG methods have been investigated in [8] on simplicial mesh. This method is the first WG method that achieves superconvergence on polytopal mesh, which has been verified theoretically and computationally. Extensive numerical examples are tested for the new WG elements of different degrees in two and three dimensional spaces.

2. Weak Galerkin Finite Element Schemes. Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [9]. Denote by $E_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $E^0_h = E_h \setminus \partial \Omega$ be the set of all interior edges or flat faces. For every element $T \in \mathcal{T}_h$, we denote by $h_T$ its diameter and mesh size $h = \max_{T \in \mathcal{T}_h} h_T$ for $\mathcal{T}_h$.

For a given integer $k \geq 0$, let $V_h$ be the weak Galerkin finite element space associated with $\mathcal{T}_h$ defined as follows

\[(2.1) \quad V_h = \{ v = \{ v_0, v_b \} : v_0|_T \in P_k(T), v_0|_e \in P_k(e), e \subset \partial T, T \in \mathcal{T}_h \}\]

and its subspace $V^0_h$ is defined as

\[(2.2) \quad V^0_h = \{ v : v \in V_h, v_h = 0 \text{ on } \partial \Omega \}.\]

We would like to emphasize that any function $v \in V_h$ has a single value $v_b$ on each edge $e \in E_h$.

For given $T \in \mathcal{T}_h$ and $v = \{ v_0, v_b \} \in V_h$, a weak gradient $\nabla_w v$ is a piecewise polynomial satisfying $\nabla_w v|_T \in \Lambda_k(T)$ and

\[(2.3) \quad (\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T} \quad \forall q \in \Lambda_k(T),\]

where $\Lambda_k(T)$ will be defined in the next section.

For simplicity, we adopt the following notations,

\[
(v, w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (v, w)_T = \sum_{T \in \mathcal{T}_h} \int_T vw \, dx, \\
\langle v, w \rangle_{\partial \mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \langle v, w \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} vw \, ds.
\]

Weak Galerkin Algorithm 1. A numerical approximation for \((1.1)-(1.2)\) can be obtained by seeking $u_h = \{ u_0, u_b \} \in V^0_h$ satisfying the following equation:

\[(2.4) \quad (\nabla_w u_h, \nabla_w v)_h = (f, v_0) \quad \forall v = \{ v_0, v_b \} \in V^0_h.\]

3. Construction of $\Lambda_k(T)$ and Its Properties. The space $H(\div; \Omega)$ is defined as the set of vector-valued functions on $\Omega$ which, together with their divergence, are square integrable; i.e.,

\[H(\div; \Omega) = \{ v \in [L^2(\Omega)]^d : \nabla \cdot v \in L^2(\Omega) \}.\]

We start this section by defining $\Lambda_k(T)$. For any $T \in \mathcal{T}_h$, it can be divided in to a set of disjoint triangles $T_i$ with $T = \bigcup T_i$. Then $\Lambda_k(T)$ can be defined as

\[(3.1) \quad \Lambda_k(T) = \{ v \in H(\div, T) : v|_{T_i} \in RT_k(T_i), \nabla \cdot v \in P_k(T), v \cdot n|_e \in P_k(e), e \subset \partial T \},\]
where $RT_k(T)$ is the usual Raviart-Thomas element \[3\] of order $k$.

**Lemma 3.1.** Let $\Pi_h : H(\text{div}, \Omega) \to H(\text{div}, \Omega) \cap \otimes \Lambda_k(T)$ be defined in (3.8) below. For any $v \in H(\text{div}, \Omega)$ and for all $T \in T_h$, we have,

\[
\begin{align*}
(3.2) & \quad (\Pi_h v, \ w)_T = (v, w)_T \quad \forall w \in [P_{k-1}(T)]^d, \\
(3.3) & \quad (\Pi_h v \cdot n, q)_e = (v \cdot n, q)_e \quad \forall q \in P_k(e), e \subset \partial T, \\
(3.4) & \quad (\nabla \cdot v, q)_T = (\nabla \cdot \Pi_h v, q)_T \quad \forall q \in P_k(T), \\
(3.5) & \quad -\langle \nabla \cdot v, v_0 \rangle_{T_h} = (\Pi_h v, \nabla w)_T \forall v \in \{v_0, v_h\} \subset V_h^0, \\
(3.6) & \quad ||\Pi_h v - v|| \leq Ch^{k+1} ||v||_{k+1}.
\end{align*}
\]

**Proof.** A similar interpolation operator $\Pi_h$ is studied in [3] which does not require one piece polynomial $\Pi_h v$ on one face of $T$, i.e., (3.8g) is omitted below in the definition of $\Pi_h$. The proof here is very similar to the one in [5].

We assume no additional inner vertex(edges) is introduced in subdividing a polygon/polyhedron $T$ in to $n$ triangles/tetrahedrons ($T_1$). That is, we have precisely $n - 1$ internal edges/triangles which separate $T$ into $n$ parts. For simple notation, only one face $e_1$ of $T$ is subdivided in to $m$ edges/triangles, $e_{1,1}, \ldots, e_{1,m}$. Note that in 3D a non-triangular polygonal face has to be subdivided in to several triangles. We limit the proof to 3D. We need only omit the fourth equation (3.8d) in (3.8) to get a 2D proof.

On $n$ tetrahedrons, a function of $\Lambda_k$ can be expressed as

\[
(3.7) \quad v_h|_{T_{i_0}} = \sum_{i+j+l \leq k} \left[ \begin{array}{c} 1,ij \ 2,ij \ 3,ij \end{array} \right] x^i y^j z^l + \sum_{i+j+l=k} \left[ \begin{array}{c} 4,ij \ 1,ij \ 2,ij \ 3,ij \end{array} \right] x^i y^j z^l, \quad i_0 = 1, \ldots n.
\]

$v_h|_{e_{i_0}}$ is determined by

\[
\frac{n(k+1)(k+2)(k+3)}{2} + \frac{n(k+1)(k+2)}{2} = \frac{n(k+1)(k+2)(k+4)}{2}
\]

coefficients. For any $v \in H(\text{div}; T)$, $\Pi_h v \in \Lambda_k(T)$ is defined by

\[
\begin{align*}
(3.8a) & \quad \int_{e_{i_0} \subset \partial T} (\Pi_h v - v) \cdot n_{ij} p_k dS = 0 \quad \forall p_k \in P_k(e_{ij}), e_{ij} \neq e_{1,\ell}, \ell \geq 2, \\
(3.8b) & \quad \int_{T} (\Pi_h v - v) \cdot n_{1} p_{k-1} d\mathbf{x} = 0 \quad \forall p_{k-1} \in P_{k-1}(T), \\
(3.8c) & \quad \int_{T_{i}} (\Pi_h v - v) \cdot n_{2} p_{k-1} d\mathbf{x} = 0 \quad \forall p_{k-1} \in P_{k-1}(T_{i}), \ i = 1, \ldots n, \\
(3.8d) & \quad \int_{T_{i}} (\Pi_h v - v) \cdot n_{3} p_{k-1} d\mathbf{x} = 0 \quad \forall p_{k-1} \in P_{k-1}(T_{i}), \ i = 1, \ldots n, \\
(3.8e) & \quad \int_{e_{i_0} \subset \partial T} P_{\Pi_h v} \cdot n_{ij} p_k dS = 0 \quad \forall p_k \in P_k(e_{ij}), \\
(3.8f) & \quad \int_{T_{i}} \nabla \cdot (\Pi_h v|_{T_{i}} - \Pi_h v|_{T_{i}}) p_k d\mathbf{x} = 0 \quad \forall p_k \in P_k(T_{i}), \ i = 2, \ldots, n, \\
(3.8g) & \quad \int_{e_{1,1}} (\Pi_h v|_{e_{1,1}} - \Pi_h v|_{e_{1,1}}) \cdot n_{p_k} dS = 0 \quad \forall p_k \in P_k(e_{1,1}), \ i = 2, \ldots, m.
\end{align*}
\]
where \(e_{ij}\) is the \(j\)-th face triangle of \(T_i\) with a fixed normal vector \(n_{ij}\), \(n_1\) is a unit vector not parallel to any internal face normal \(n_{ij}\), \((n_1, n_2, n_3)\) forms a right-hand orthonormal system, \([\cdot]\) denotes the jump on a face triangle, \(\Pi_h v|_{T_i}\) is understood as a polynomial vector which can be used on another tetrahedron \(T_1, e_{12} \subset e_1 \subset \partial T\) is a face triangle of \(T_i, n\) is a normal vector on \(e_1\), and \(\Pi_h v_{|e_{12}}\) is extended to the whole \(e_1\) as one polynomial. The linear system (3.8) of equations has the following number of equations,

\[
(n + 3 - m) \frac{(k + 1)(k + 2)}{2} + (2n + 1) \frac{k(k + 1)(k + 2)}{6} \\
+ (n - 1) \frac{(k + 1)(k + 2)}{2} + (n - 1) \frac{(k + 1)(k + 2)(k + 3)}{6} \\
+ (m - 1) \frac{(k + 1)(k + 2)}{2} \\
= \frac{n(k + 1)(k + 2)(k + 4)}{2},
\]

which is exactly the number of coefficients for a \(v_h\) function in (3.7). Thus we have a square linear system. The system has a unique solution if and only if the kernel is \(\{0\}\).

Let \(v = 0\) in (3.8). Though \(\Pi_h v\) is a \(P_{k+1}\) polynomial, \(\Pi_h v \cdot n_{ij}\) is a \(P_k\) polynomial when restricted on \(e_{ij}\). This can be seen by the normal format of plane equation for triangle \(e_{ij}\). By the first equation (3.8a), \(\Pi_h v \cdot n_{ij} = 0\) on \(e_{ij}, e_{ij} \neq e_{12}, \ell \geq 2\). By the seventh equation (3.8g), \(\Pi_h v_{|e_{12}} \cdot n = \Pi_h v_{|e_{11}} \cdot n = 0, \ell \geq 2\). In other words, (3.8g) ensures \(\Pi_h v \cdot n\) is a one-piece polynomial on \(e_1\) and (3.8a) enforces it to zero. By the sixth equation (3.8f), \(\nabla \cdot \Pi_h v\) is a one-piece polynomial on the whole \(T\). Because \(\nabla \cdot \Pi_h v\) is continuous on inner interface triangles and is a \(P_k(e_{ij})\) polynomial on all the outer face triangles, by the first five equations in (3.8), we have

\[
\int_T (\nabla \cdot \Pi_h v)^2 dx = \sum_{i=1}^{n} \left( \int_{T_i} -\Pi_h v \cdot \nabla (\nabla \cdot \Pi_h v) dx + \int_{\partial T_i} \Pi_h v \cdot n (\nabla \cdot \Pi_h v) dS \right) \\
= \sum_{i=1}^{n} \sum_{j=1}^{3} \int_{T_i} -(\Pi_h v \cdot n_j) (n_j \cdot \nabla (\nabla \cdot \Pi_h v)) dx \\
= 0.
\]

That is,

\[
(3.9) \quad \nabla \cdot \Pi_h v = 0 \text{ on } T.
\]

Starting from a corner tetrahedron \(T_1\), we have its three face triangles, \(e_{11}, e_{12}\) and \(e_{13}\), on the boundary of \(T\). The forth face triangle \(e_{14}\) of \(T_1\) is shared by \(T_2\). By the selection of \(n_1\), the normal vector \(n_{14} = c_1 n_1 + c_2 n_2 + c_3 n_3\) of \(e_{14}\) has a non zero \(c_1 \neq 0\). a 2D polynomial \(p_k \in P_k(e_{14})\) can be expressed as \(p_k(x_2, x_3)\), where we use \((x_1, x_2, x_3)\) as the coordinate variables under the system \((n_1, n_2, n_3)\). Viewing this polynomial as a 3D polynomial, i.e. extending it constantly in \(x_1\)-direction, we have

\[
p_k(x_1, x_2, x_3) = p_k(x_2, x_3), \quad (x_1, x_2, x_3) \in T_1.
\]
By \((3.9)\) and the third and fourth equations of \((3.8)\), it follows that

\[
0 = \int_{T_1} (\nabla \cdot \Pi_h \mathbf{v}) p_k \, dx \\
= - \int_{T_1} [(\Pi_h \mathbf{v} \cdot \mathbf{n}_1) \partial x_1 p_k + (\Pi_h \mathbf{v} \cdot \mathbf{n}_2) \partial x_2 p_k + (\Pi_h \mathbf{v} \cdot \mathbf{n}_3) \partial x_3 p_k] \, dx \\
+ \int_{c_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS \\
= - \int_{T_1} (\Pi_h \mathbf{v} \cdot \mathbf{n}_1) \cdot 0 \, dx + 0 + \int_{c_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS \\
\tag{3.10}
= \int_{c_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS \quad \forall p_k \in P_k(e_{14}).
\]

Next, for any \(p_{k-1} \in P_{k-1}(T_1)\), we let \(p_k \in P_k(T_1)\) be one of its anti-\(x_1\)-derivative, i.e., \(\partial x_1 p_k = p_{k-1}\). Thus, by \((3.9)\), \((3.8c)\), \((3.8d)\) and \((3.10)\), we get

\[
0 = \int_{T_1} \nabla \cdot \Pi_h \mathbf{v} p_k \, dx \\
= - \int_{T_1} [(\Pi_h \mathbf{v} \cdot \mathbf{n}_1) \partial x_1 p_k + 0 + 0] \, dx + \int_{c_{14}} (\Pi_h \mathbf{v}) \cdot \mathbf{n}_{14} p_k \, dS \\
\tag{3.11}
= - \int_{T_1} (\Pi_h \mathbf{v} \cdot \mathbf{n}_1) p_{k-1} \, dx \quad \forall p_{k-1} \in P_{k-1}(T_1).
\]

Continuing work on \(T_1\), by \(\nabla \cdot \Pi_h \mathbf{v} = 0\), all \(a_{ij,kl} = 0\) in \((3.7)\), since the divergence of each such term is non-zero and independent of the divergence of other terms. Thus \(\Pi_h \mathbf{v}|_{T_1}\) is in \([P_k(T_1)]^d\), instead of \(RT_k(T_1)\). It can be linearly expanded by the three projections on three linearly independent directions. In particular, on a corner tetrahedron \(T_1\) we have three outer triangles \(e_{1j}\) on \(\partial T\). On \(T_1\),

\[
\Pi_h \mathbf{v} = A \begin{pmatrix} \Pi_h \mathbf{v} \cdot \mathbf{n}_{11} \\ \Pi_h \mathbf{v} \cdot \mathbf{n}_{12} \\ \Pi_h \mathbf{v} \cdot \mathbf{n}_{13} \end{pmatrix} = A \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix},
\]

where \(p_1, p_2\) and \(p_3\) are scalar \(P_k\) polynomials, and \(A\) is a \(3 \times 3\) scalar matrix.

By the first equation in \((3.8)\), \(p_1\) vanishes on \(e_{11}\) and

\[
p_1 = \lambda_1 q_{k-1} \quad \text{on} \quad T_1,
\]

where \(\lambda_1\) is a barycentric coordinate of \(T_1\) (which is a linear function assuming 0 on \(e_{11}\)), and \(q_{k-1}\) is a \(P_{k-1}(T)\) polynomial. Let \(p_k \in P_k(T)\) be an anti-\(x\)-derivative of \((\mathbf{n}_{11}) q_{k-1}\), i.e., \(\nabla p_k|_{e_{11}} = (\mathbf{n}_{11}) q_{k-1}\). Note that \(\nabla p_k|_{e_{12}}\) and \(\nabla p_k|_{e_{13}}\) can be anything (of \(y\) and \(z\) functions) which result in zero integrals below. By \((3.11)\), \((3.8c)\) and \((3.8d)\), since \(\nabla \cdot \Pi_h \mathbf{v} = 0\), we get

\[
\int_{T_1} \lambda_1 q_{k-1}^2 \, dx = \int_{T_1} \Pi_h \mathbf{v} \cdot (\mathbf{n}_{11} q_{k-1}) \, dx = 0.
\]

Since \(\lambda_1 > 0\) in \(T_1\), we conclude with \(q_{k-1} = 0\) and \(p_1 = 0\). Repeating the analysis, as \(p_2 = 0\) on \(e_{12}\) and \(p_3 = 0\) on \(e_{13}\), we get \(p_2 = p_3 = 0\) and \(\Pi_h \mathbf{v} = 0\) on \(T_1\).
Adding the equations (3.10) and (3.11) to (3.8). \( T_2 \) would be a new corner tetrahedron with three no-flux boundary triangles. Repeating the estimates on \( T_1 \), it would lead \( \Pi_h \mathbf{v} = 0 \) on \( T_2 \). Sequentially, we obtain \( \Pi_h \mathbf{v} = 0 \) on all \( T_i \), i.e., on the whole \( T \). (3.2) follows (3.8b), (3.8c) and (3.8d). (3.8a) and (3.8g) imply (3.3). For a \( \mathbf{v} \in H(\text{div}; \Omega) \) and a \( v \in P_k(T) \), we have, by (3.8), (3.10) and (3.11),

\[
(\nabla \cdot (\mathbf{v} - \Pi_h \mathbf{v}), v)_T = \sum_{i=1}^n \left( \int_{T_i} (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \nabla v d\mathbf{x} + \int_{\partial T_i} (\mathbf{v} - \Pi_h \mathbf{v}) \cdot \mathbf{n} v dS \right) = 0.
\]

That is, (3.4) holds.

It follows from (3.4) and (2.3) that for \( v = \{v_0, v_b\} \in V_h^0 \)

\[
-(\nabla \cdot \mathbf{v}, v_0)_{T_h} = -(\nabla \cdot \Pi_h \mathbf{v}, v_0)_{T_h} = -(\nabla \cdot \Pi_h \mathbf{v}, v_0)_{T_h} + (v_h, \Pi_h \mathbf{v} \cdot \mathbf{n})_{\partial T_h} = (\Pi_h \mathbf{v}, \nabla w v)_{T_h},
\]

which proves (3.5).

Since \( [P_k(T)]^3 \subset \Lambda_k \) and \( \Pi_h \mathbf{v} = \mathbf{v} \) for all \( \mathbf{v} \in [P_k(T)]^3 \). On one size 1 \( T \), by the finite dimensional norm-equivalence and the shape-regularity assumption on sub-triangles, the interpolation is stable in \( L^2(T) \), i.e.,

(3.12) \[ ||\Pi_h \mathbf{v}||_T \leq C ||\mathbf{v}||_T. \]

After a scaling, the constant \( C \) in (3.12) remains same for all \( h > 0 \). It follows that

\[
||\Pi_h \mathbf{v} - \mathbf{v}||^2 \leq C \sum_{T \in T_h} (||\Pi_h (\mathbf{v} - p_k \mathbf{v})||^2_T + ||p_k \mathbf{v} - \mathbf{v}||^2_T)
\]

\[
\leq C \sum_{T \in T_h} ||\mathbf{v} - p_k \mathbf{v}||^2_T + ||p_k \mathbf{v} - \mathbf{v}||^2_T
\]

\[
\leq C \sum_{T \in T_h} h^{2k+2} ||\mathbf{v}||^2_{k+1,T}
\]

\[
= C h^{2k+2} ||\mathbf{v}||^2_{k+1,T}
\]

where \( p_k \mathbf{v} \) is the \( k \)-th Taylor polynomial of \( \mathbf{v} \) on \( T \). \( \Box \)

Let \( Q_0 \) and \( Q_b \) be the two element-wise defined \( L^2 \) projections onto \( P_k(T) \) and \( P_k(c) \) with \( c \subset \partial T \) on \( T \) respectively. Define \( Q_h u = \{Q_0 u, Q_b u\} \in V_h \). Let \( Q_h \) be the element-wise defined \( L^2 \) projection onto \( \Lambda_k(T) \) on each element \( T \).

**Lemma 3.2.** Let \( \phi \in H^1(\Omega) \), then on any \( T \in T_h \),

(3.13) \[ \nabla_w Q_h \phi = Q_h \nabla \phi. \]

**Proof.** Using (2.3), the definition of \( \Lambda_k(T) \) and integration by parts, we have that for any \( \mathbf{q} \in \Lambda_k(T) \)

\[
(\nabla_w Q_h \phi, \mathbf{q})_T = -(Q_0 \phi, \nabla \cdot \mathbf{q})_T + (Q_h \phi, \mathbf{q} \cdot \mathbf{n})_{\partial T} = -\phi, \nabla \cdot \mathbf{q} + (\phi, \mathbf{q} \cdot \mathbf{n})_{\partial T} = (\nabla \phi, \mathbf{q})_T = (Q_h \nabla \phi, \mathbf{q})_T.
\]
which implies the desired identity (3.13). We have proved the lemma. □

For any $v \in V_h$, let

\[(3.14) \quad ||v||^2 = (\nabla w v, \nabla w v)_{T_h} \, .\]

We introduce a discrete $H^1$ semi-norm as follows:

\[(3.15) \quad \|v\|_{1,h} = \left( \sum_{T \in \mathcal{T}_h} \left( \|\nabla v_0\|_T^2 + h^{-1} T \|v_0 - v_b\|_{\partial T}^2 \right) \right)^{\frac{1}{2}} .\]

It is easy to see that $\|v\|_{1,h}$ define a norm in $V^0_h$. The following lemma indicates that $\|\cdot\|$ is equivalent to the $||\cdot||$ in (3.14).

**Lemma 3.3.** There exist two positive constants $C_1$ and $C_2$ such that for any $v = \{v_0, v_b\} \in V_h$, we have

\[(3.16) \quad C_1 \|v\|_{1,h} \leq ||v|| \leq C_2 \|v\|_{1,h} .\]

**Proof.** For any $v = \{v_0, v_b\} \in V_h$, it follows from the definition of weak gradient \[2.3\] and integration by parts that

\[(3.17) \quad (\nabla v, q)_T = (\nabla v_0, q)_T + (v_b - v_0, q \cdot n)_{\partial T}, \quad \forall q \in \Lambda_k(T).\]

By letting $q = \nabla w v$ in (3.17) we arrive at

\[(\nabla w v, \nabla w v)_T = (\nabla v_0, \nabla w v)_T + (v_b - v_0, \nabla w v \cdot n)_{\partial T}.\]

From the trace inequality (5.1) and the inverse inequality we have

\[
\|\nabla w v\|_T^2 \leq \|\nabla v_0\|_T \|\nabla w v\|_T + \|v_b - v_0\|_{\partial T} \|\nabla w v\|_{\partial T},
\]

\[
\leq \|\nabla v_0\|_T \|\nabla w v\|_T + C h^{-1/2} \|v_0 - v_b\|_{\partial T} \|\nabla w v\|_{\partial T},
\]

which implies

\[
\|\nabla w v\|_T \leq C \left( \|\nabla v_0\|_T + h^{-1/2} \|v_0 - v_b\|_{\partial T} \right),
\]

and consequently

\[
\|v\| \leq C_2 \|v\|_{1,h} .
\]

Next we will prove $C_1 \|v\|_{1,h} \leq \|v\|$. The construction of $\Lambda_k(T)$ implies there exists $q_0 \in \Lambda_k(T)$ such that

\[(3.18) \quad (\nabla v_0, q_0)_T = 0, \quad (v_b - v_0, q_0 \cdot n)_{\partial T} = \|v_0 - v_b\|_{\partial T}^2 .\]

and

\[(3.19) \quad \|q_0\|_T \leq C h^{1/2} \|v_b - v_0\|_{\partial T} .\]

Letting $q = q_0$ in (3.17), we get

\[(3.20) \quad (\nabla v, q_0)_T = \|v_b - v_0\|_T^2 .\]
It follows from Cauchy-Schwarz inequality and (3.19) that
\[ \|v_b - v_0\|_e^2 \leq C \|\nabla v\|_T \|q_0\|_T \leq Ch_T^{1/2} \|\nabla v\|_T \|v_0 - v_b\|_e, \]
which implies
\[ h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \leq C \|\nabla v\|_T. \]

It follows from the trace inequality, the inverse inequality and (3.21),
\[ \|\nabla v_0\|^2_T \leq \|\nabla w\|_T \|\nabla v_0\|_T + Ch_T^{-1/2} \|v_0 - v_b\|_{\partial T} \|\nabla v_0\|_T \leq C \|\nabla w\|_T \|\nabla v_0\|_T. \]
Combining the above estimate and (3.21), by the definition (3.15), we prove the lower bound of (3.16) and complete the proof of the lemma.

**Lemma 3.4.** The weak Galerkin finite element scheme (2.4) has a unique solution.

**Proof.** If \(u_h^{(1)}\) and \(u_h^{(2)}\) are two solutions of (2.4), then \(\epsilon_h = u_h^{(1)} - u_h^{(2)}\) \(\in V_h^0\) would satisfy the following equation
\[ (\nabla w \epsilon_h, \nabla w v) = 0, \quad \forall v \in V_h^0. \]
Then by letting \(v = \epsilon_h\) in the above equation we arrive at
\[ \|\epsilon_h\|^2 = (\nabla w \epsilon_h, \nabla w \epsilon_h) = 0. \]
It follows from (3.16) that \(\|\epsilon_h\|_{1,h} = 0\). Since \(\|\cdot\|_{1,h}\) is a norm in \(V_h^0\), one has \(\epsilon_h = 0\). This completes the proof of the lemma.

**4. Error Equations.** Let \(\epsilon_h = Q_h u - u_h\). Next we derive two error equations that \(\epsilon_h\) satisfies. One will be used in energy norm error analysis and another one for \(L^2\) error estimate.

**Lemma 4.1.** For any \(v \in V_h^0\), the following error equation holds true
\[ (\nabla w \epsilon_h, \nabla w v)_{\Omega_h} = \ell(u,v), \quad (4.1) \]
where
\[ \ell(u,v) = (Q_h \nabla u - \Pi_h \nabla u, \nabla w v)_{\Omega_h} \]

**Proof.** For \(v = \{v_0, v_b\} \in V_h^0\), testing (1.1) by \(v_0\) and using (3.5), we arrive at
\[ (f, v_0) = -(\nabla \cdot \nabla u, v_0)_{\Omega_h} = (\Pi_h \nabla u, \nabla w v)_{\Omega_h}. \]
It follows from (3.13) and (4.2)
\[ (\nabla w Q_h u, \nabla w v)_{\Omega_h} = (f, v_0) + \ell(u,v). \]
The error equation follows from subtracting (2.4) from (4.3),
\[ (\nabla w \epsilon_h, \nabla w v)_{\Omega_h} = \ell(u,v) \quad \forall v \in V_h^0. \]
This completes the proof of the lemma.

**Lemma 4.2.** For any \(v \in V_h^0\), the following error equation holds true
\[ (\nabla w \epsilon_h, \nabla w v)_{\Omega_h} = \ell_1(u,v), \quad (4.4) \]
where
\[ \ell_1(u, v) = \langle (\nabla u - Q_h \nabla u) \cdot n, v_0 - v_b \rangle_{\partial \Omega_h}. \]

**Proof.** For \( v = \{v_0, v_b\} \in V_h^0 \), testing (1.1) by \( v_0 \) and using integration by parts and the fact that \( \sum_{T \in T_h} \langle \nabla u \cdot n, v_b \rangle_{\partial T} = 0 \), we arrive at
\[ (\nabla u, \nabla v_0)_{\Omega_h} - \langle \nabla u \cdot n, v_0 - v_b \rangle_{\partial \Omega_h} = (f, v_0). \]

It follows from integration by parts, (2.3) and (3.13) that
\[ (\nabla u, \nabla v_0)_{\Omega_h} = (Q_h \nabla u, \nabla v_0)_{\Omega_h} = -\langle v_0, \nabla \cdot (Q_h \nabla u) \rangle_{\partial \Omega_h} + \langle v_0 - v_b, Q_h \nabla u \cdot n \rangle_{\partial \Omega_h} \]
\[ = (\nabla w Q_h u, \nabla w v_0)_{\Omega_h} + \langle v_0 - v_b, Q_h \nabla u \cdot n \rangle_{\partial \Omega_h}. \]
Combining (4.5) and (4.6) gives
\[ (\nabla w Q_h u, \nabla w v_0)_{\Omega_h} = (f, v_0) + \ell_1(u, v). \]

The error equation follows from subtracting (2.4) from (4.7),
\[ (\nabla w e_h, \nabla w v_0)_{\Omega_h} = \ell_1(u, v) \quad \forall v \in V_h^0. \]
This completes the proof of the lemma. \(\square\)

**5. Error Estimates.** For any function \( \varphi \in H^1(T) \), the following trace inequality holds true (see [9] for details):
\[ \|\varphi\|_{\partial T}^2 \leq C \left( h_T^{-1} \|\varphi\|_T^2 + h_T \|\nabla \varphi\|_T^2 \right). \]

**Theorem 5.1.** Let \( u_h \in V_h \) be the weak Galerkin finite element solution of (2.4). Assume the exact solution \( u \in H^{k+2}(\Omega) \). Then, there exists a constant \( C \) such that
\[ \|Q_h u - u_h\| \leq Ch^{k+1} |u|_{k+2}. \]

**Proof.** Letting \( v = e_h \) in (4.1) gives
\[ \|e_h\|^2 = \ell(u, e_h). \]
The definitions of \( Q_h \) and \( \Pi_h \) imply
\[ |\ell(u, e_h)| = |(Q_h \nabla u - \Pi_h \nabla u, \nabla w e_h)_{\Omega_h}| \]
\[ \leq \left( \sum_T \|Q_h \nabla u - \Pi_h \nabla u\|_{T}^2 \right)^{1/2} \|\epsilon_h\| \]
\[ \leq C h^{k+1} |u|_{k+2} \|\epsilon_h\|. \]
Combining (5.3) and (5.4), we arrive
\[ \|e_h\| \leq Ch^{k+1} |u|_{k+2}. \]
which completes the proof of the theorem. 

The standard duality argument is used to obtain $L^2$ error estimate. Recall $\epsilon_h = \{\epsilon_0, \epsilon_b\} = Q_h u - u_h$. The considered dual problem seeks $\Phi \in H^1(\Omega)$ satisfying

$$
-\Delta \Phi = \epsilon_0 \quad \text{in } \Omega.
$$

Assume that the following $H^2$ regularity holds

$$
\|\Phi\|_2 \leq C|\epsilon_0|.
$$

**Theorem 5.2.** Let $u_h \in V_h$ be the weak Galerkin finite element solution of (2.4). Assume that the exact solution $u \in H^{k+2}(\Omega)$ and (5.6) holds true. Then, there exists a constant $C$ such that for $k \geq 1$

$$
|Q_0 u - u_0| \leq Ch^{k+2}\|u\|_{k+2}.
$$

**Proof.** Testing (5.5) by $\epsilon_0$ and using the fact that $\sum_{T \in T_h} (\nabla \Phi \cdot n, \epsilon_b)_{\partial T} = 0$ give

$$
\|\epsilon_0\|^2 = - (\Delta \Phi, \epsilon_0)
$$

(5.8)

Setting $u = \Phi$ and $v = \epsilon_h$ in (4.1) yields

$$
(\nabla \Phi, \nabla \epsilon_0)_{T_h} = (\nabla w Q_h \Phi, \nabla \epsilon_h)_{T_h} + (Q_h \nabla \Phi \cdot n, \epsilon_0 - \epsilon_b)_{\partial T_h}.
$$

(5.9)

Substituting (5.9) into (5.8) and using (4.4) yield

$$
\|\epsilon_0\|^2 = (\nabla w \epsilon_h, \nabla w Q_h \Phi)_{T_h} - (\nabla \Phi - Q_h \nabla \Phi \cdot n, \epsilon_0 - \epsilon_b)_{\partial T_h}
$$

(5.10)

Next we estimate the two terms on the right hand side of (5.10). Using the Cauchy-Schwarz inequality, the trace inequality (5.1) and the definitions of $Q_h$ and $H_h$ we obtain

$$
|\ell_1(u, Q_h \Phi)| \leq |(\nabla u - Q_h \nabla u \cdot n, Q_0 \Phi - Q_h \Phi)_{\partial T_h}|
$$

$$
\leq \left( \sum_{T \in T_h} \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} \|Q_0 \Phi - Q_h \Phi\|_{\partial T}^2 \right)^{1/2}
$$

$$
\leq C \left( \sum_{T \in T_h} h \|\nabla u - Q_h \nabla u\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h^{-1} \|Q_0 \Phi - Q_h \Phi\|_{\partial T}^2 \right)^{1/2}
$$

$$
\leq C h^{k+2}\|u\|_{k+2}\|\Phi\|_2.
$$

Using the Cauchy-Schwarz inequality, the trace inequality (5.1), (3.16) and (5.2), we have

$$
|\ell_1(\Phi, \epsilon_h)| = \left| \sum_{T \in T_h} \langle (\nabla \Phi - Q_h \nabla \Phi) \cdot n, \epsilon_0 - \epsilon_b \rangle_{\partial T} \right|
$$

$$
\leq C \sum_{T \in T_h} \|\nabla \Phi - Q_h \nabla \Phi\|_{\partial T} \|\epsilon_0 - \epsilon_b\|_{\partial T}
$$

$$
\leq C \left( \sum_{T \in T_h} h_T \|\nabla \Phi - Q_h \nabla \Phi\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} \|\epsilon_0 - \epsilon_b\|_{\partial T}^2 \right)^{1/2}
$$

$$
\leq C h^{k+2}\|u\|_{k+2}\|\Phi\|_2.
$$
Combining the two estimates above with (5.10) yields
\[ \| \epsilon_0 \|^2 \leq Ch^{k+2} |u|_{k+2} \| \Phi \|_2. \]

It follows from the above inequality and the regularity assumption (5.6),
\[ \| \epsilon_0 \| \leq Ch^{k+2} |u|_{k+2}. \]

We have completed the proof. \( \Box \)

6. Numerical Experiments. We solve the Poisson problem (1.1)-(1.2) on the unit square domain with the exact solution
\[ u = \sin(\pi x) \sin(\pi y). \]

We first use the uniform square grids shown in Figure 6.1. We then compute the problem with several We list the computational results in Table 6.1. As proved, we have one order of super-convergence for both \( L^2 \) errors and \( H^1 \)-like errors.

![Fig. 6.1. The first three levels of wedge grids used in Table 6.4](image)

We compute the solution (6.1) again on a type of quadrilateral grids, shown in Figure 6.2. Here to avoid convergence to parallelograms under the nest refinement of quadrilaterals, we fix the shape of quadrilaterals on all levels of grids. We list the computation in Table 6.2. Again, the data confirm the theoretic convergence rates.

![Fig. 6.2. The first three levels of grids, for Table 6.2](image)

Next we solve the same problem (6.1) on a type of grids with quadrilaterals and hexagons, shown in Figure 6.3. We list the result of computation in Table 6.3 where we obtain one order of superconvergence in all cases. Lastly, we solve a 3D problem (1.1)-(1.2) on the unit cube domain \( \Omega = (0, 1)^3 \) with the exact solution
\[ u = \sin(\pi x) \sin(\pi y) \sin(\pi z). \]
Table 6.1
Error profiles and convergence rates on square grids shown in Figure 6.1 for (6.1).

| level | $\|Q_h u - u_h\|_0$ | rate | $\|Q_h u - u_h\|$ | rate |
|-------|----------------------|------|----------------------|------|
|       | by the $P_0-P_0(\Lambda_0)$ WG element |       | by the $P_1-P_1(\Lambda_1)$ WG element |       |
| 6     | 0.1101E-02 1.99       | 0.1988E+00 0.99 |
| 7     | 0.2756E-03 2.00       | 0.9951E-01 1.00 |
| 8     | 0.6892E-04 2.00       | 0.4977E-01 1.00 |
|       | by the $P_2-P_2(\Lambda_2)$ WG element |       | by the $P_2-P_2(\Lambda_2)$ WG element |       |
| 6     | 0.2722E-04 2.99       | 0.6952E-02 2.00 |
| 7     | 0.3407E-05 3.00       | 0.1739E-02 2.00 |
| 8     | 0.4261E-06 3.00       | 0.4347E-03 2.00 |
|       | by the $P_3-P_3(\Lambda_3)$ WG element |       | by the $P_3-P_3(\Lambda_3)$ WG element |       |
| 5     | 0.8248E-06 4.00       | 0.3106E-03 3.00 |
| 6     | 0.5156E-07 4.00       | 0.3884E-04 3.00 |
| 7     | 0.3313E-08 3.96       | 0.4855E-05 3.00 |
|       | by the $P_4-P_4(\Lambda_4)$ WG element |       | by the $P_4-P_4(\Lambda_4)$ WG element |       |
| 3     | 0.2491E-03 2.00       | 0.9033E-01 1.00 |
| 4     | 0.6231E-04 2.00       | 0.4518E-01 1.00 |
| 5     | 0.1558E-04 2.00       | 0.2259E-01 1.00 |

Table 6.2
Error profiles and convergence rates on quadrilateral grids shown in Figure 6.2 for (6.1).

| level | $\|Q_h u - u_h\|_0$ | rate | $\|Q_h u - u_h\|$ | rate |
|-------|----------------------|------|----------------------|------|
|       | by the $P_0-P_0(\Lambda_0)$ WG element |       | by the $P_1-P_1(\Lambda_1)$ WG element |       |
| 6     | 0.2491E-03 2.00       | 0.9033E-01 1.00 |
| 7     | 0.6231E-04 2.00       | 0.4518E-01 1.00 |
| 8     | 0.1558E-04 2.00       | 0.2259E-01 1.00 |
|       | by the $P_2-P_2(\Lambda_2)$ WG element |       | by the $P_2-P_2(\Lambda_2)$ WG element |       |
| 6     | 0.2722E-04 2.99       | 0.6952E-02 2.00 |
| 7     | 0.3407E-05 3.00       | 0.1739E-02 2.00 |
| 8     | 0.4261E-06 3.00       | 0.4347E-03 2.00 |
|       | by the $P_3-P_3(\Lambda_3)$ WG element |       | by the $P_3-P_3(\Lambda_3)$ WG element |       |
| 5     | 0.9093E-06 4.00       | 0.3256E-03 3.00 |
| 6     | 0.5686E-07 4.00       | 0.4071E-04 3.00 |
| 7     | 0.3554E-08 4.00       | 0.5090E-05 3.00 |
|       | by the $P_4-P_4(\Lambda_4)$ WG element |       | by the $P_4-P_4(\Lambda_4)$ WG element |       |
| 2     | 0.7967E-03 5.22       | 0.4984E-01 4.34 |
| 3     | 0.2629E-04 4.92       | 0.3181E-02 3.97 |
| 4     | 0.8342E-06 4.98       | 0.1998E-03 3.99 |
| 5     | 0.2618E-07 4.99       | 0.1251E-04 4.00 |
Table 6.3
Error profiles and convergence rates on polygonal grids shown in Figure 6.3 for (6.1).

| level | $\|Q_h u - u_h\|_0$  | rate | $\|Q_h u - u_h\|$  | rate |
|-------|-----------------|------|-----------------|------|
|       | by the $P_0 - P_0(\Lambda_0)$ WG element                       |
| 6     | 0.1892E-03      | 2.00 | 0.8731E-01      | 1.00 |
| 7     | 0.4731E-04      | 2.00 | 0.4367E-01      | 1.00 |
| 8     | 0.1183E-04      | 2.00 | 0.2184E-01      | 1.00 |
|       | by the $P_1 - P_1(\Lambda_1)$ WG element                       |
| 6     | 0.3602E-05      | 3.00 | 0.1791E-02      | 2.00 |
| 7     | 0.4504E-06      | 3.00 | 0.4477E-03      | 2.00 |
| 8     | 0.5631E-07      | 3.00 | 0.1119E-03      | 2.00 |
|       | by the $P_2 - P_2(\Lambda_2)$ WG element                       |
| 6     | 0.1850E-07      | 4.00 | 0.1655E-04      | 3.00 |
| 7     | 0.1156E-08      | 4.00 | 0.2068E-05      | 3.00 |
| 8     | 0.7299E-10      | 3.99 | 0.2586E-06      | 3.00 |
|       | by the $P_3 - P_3(\Lambda_3)$ WG element                       |
| 5     | 0.7478E-08      | 5.00 | 0.4103E-05      | 4.00 |
| 6     | 0.2339E-09      | 5.00 | 0.2565E-06      | 4.00 |
| 7     | 0.7941E-11      | 4.88 | 0.1603E-07      | 4.00 |

Fig. 6.4. The first three levels of wedge grids used in Table 6.4

Here we use a uniform wedge-type (polyhedron with 2 triangle faces and 3 rectangle faces) grids, shown in Figure 6.4. Here each wedge is subdivided in to three tetrahedrons with three rectangular faces being cut in to two triangles, when defining
piecewise $RT_k$ weak gradient space $\Lambda_k$. The results are listed in Table 6.4, confirming the one order superconvergence in the two norms for all polynomial-degree $k$ elements.

Table 6.4

| level | $\|Q_h u - u_h\|_0$ | rate | $\|Q_h u - u_h\|$ | rate |
|-------|-----------------|------|-----------------|------|
| by the $P_0-P_0(\Lambda_0)$ WG element |
| 5     | 0.0010162       | 2.0  | 0.1241692       | 1.0  |
| 6     | 0.0002548       | 2.0  | 0.0621798       | 1.0  |
| 7     | 0.0000637       | 2.0  | 0.0311019       | 1.0  |
| by the $P_1-P_1(\Lambda_0)$ WG element |
| 4     | 0.0011585       | 2.9  | 0.1242568       | 2.0  |
| 5     | 0.0001470       | 3.0  | 0.0312093       | 2.0  |
| 6     | 0.0000184       | 3.0  | 0.0078116       | 2.0  |
| by the $P_2-P_2(\Lambda_0)$ WG element |
| 4     | 0.0001439       | 4.0  | 0.0284085       | 3.0  |
| 5     | 0.0000090       | 4.0  | 0.0035641       | 3.0  |
| 6     | 0.0000006       | 4.0  | 0.0004459       | 3.0  |
| by the $P_3-P_3(\Lambda_0)$ WG element |
| 3     | 0.0004607       | 4.9  | 0.0770284       | 3.9  |
| 4     | 0.0000148       | 5.0  | 0.0048815       | 4.0  |
| 5     | 0.0000005       | 5.0  | 0.0003062       | 4.0  |

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