The Difference between Optimal Rank-1 Hankel Approximations in the Frobenius Norm and the Spectral Norm

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We present new examples illustrating the fact that optimal Hankel structured rank-1 approximation of matrices is usually different for the Frobenius and spectral norm. Further, we compare our results for optimal rank-1 Hankel approximation to the results of the well-known Cadzow algorithm [3].

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1 Introduction

A Hankel matrix \( H_1 \in \mathbb{C}^{M \times N} \) of rank 1 is either of the form

\[
H_1 = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}
\]

or

\[
H_1 = c z_M z_N^T,
\]

where \( z_N := (1, z, z^2, \ldots, z^{N-1})^T \) and \( c \in \mathbb{C} \setminus \{0\}, z \in \mathbb{C} \).

(1)

Neglecting the first case, the problem of approximating a given matrix \( A \) by a Hankel structured matrix of rank 1 reads

\[
\min_{\|H\|\text{rank }H=1} \|A - H\|^2 = \min_{c, z \in \mathbb{C}} \|A - c z_M z_N^T\|^2 \quad \text{for } A \in \mathbb{C}^{M \times N}.
\]

In [1] we have solved this problem for both the Frobenius and the spectral norm, where for the spectral norm we restricted our considerations to real symmetric matrices \( A \in \mathbb{R}^{N \times N} \) and real parameters \( c \in \mathbb{R} \setminus \{0\} \) and \( z \in \mathbb{R} \).

2 Optimal Rank-1 Hankel Approximation for the Frobenius and the Spectral Norm

Theorem 2.1 ([1], Thm. 3.1) Let \( A = (a_{j,k})_{j,k=0}^{M-1,N-1} \in \mathbb{C}^{M \times N} \) with \( M, N \geq 2 \) and \( |a_{0,0}| \geq |a_{M-1,N-1}| \). Assume that \( \text{rank}(A) \geq 1 \). Then an optimal rank-1 Hankel approximation \( H_1 = \tilde{c} \tilde{z}_M \tilde{z}_N^T \) of \( A \) is determined by

\[
\tilde{z} \in \arg \max_{z \in \mathbb{C}} \frac{z_M^* A z_N}{\|z_M\|_2 \|z_N\|_2}, \quad \tilde{c} := \frac{\tilde{z}_M^* A \tilde{z}_N}{\|z_M\|_2 \|z_N\|_2},
\]

where the vectors \( \tilde{z}_M \) and \( \tilde{z}_N \) are defined by \( \tilde{z} \) via (1) and \( z^* := z^T \).

For a real symmetric matrix \( A \in \mathbb{R}^{N \times N} \), denote by \( \lambda_0, \ldots, \lambda_{N-1} \) the eigenvalues ordered by modulus. Assume that the largest eigenvalue is positive and occurs with multiplicity 1, i.e., \( \|A\|_2 = \lambda_0 > |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{N-1}| \geq 0 \). With \( v_0, \ldots, v_{N-1} \) we denote the corresponding orthonormal eigenvectors. Since we only consider square matrices, we omit the subscript in the definition of the structured vector \( z \) in (1). The first part of Thm. 2.2 generalizes a result from [2].

Theorem 2.2 ([1], Thm. 4.1) Let \( A = (a_{j,k})_{j,k=0}^{N-1,N-1} \in \mathbb{R}^{N \times N} \) be symmetric with \( N \geq 2 \). Assume that \( \text{rank}(A) > 1 \) and \( \lambda_0 = \|A\|_2 > |\lambda_1| \). Let \( H_1 = \tilde{c} \tilde{z} \tilde{z}^T \) be an optimal rank-1 Hankel approximation of \( A \) with regard to the spectral norm.

(i) The optimal error bound \( \|A - H_1\|_2^2 = \|A - \tilde{c} \tilde{z} \tilde{z}^T\|_2^2 = \lambda_1^2 \) is achieved if and only if there exists \( \tilde{z} \in \mathbb{R}^N \) such that the vector \( \tilde{z} \) in (1) satisfies

\[
v_j^T \tilde{z} = 0, \text{ for all } \lambda_j \text{ with } |\lambda_j| = |\lambda_1| \quad \text{and} \quad f(\tilde{z}, \lambda_1) := \frac{1}{\|\tilde{z}\|_2^2} \sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_j^2 - \lambda_1^2} \geq 0,
\]

and if \( \tilde{c} \) is chosen such that

\[
\sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_j + |\lambda_1|} \leq \frac{1}{\tilde{c}} \leq \sum_{j=0}^{N-1} \frac{(v_j^T \tilde{z})^2}{\lambda_j^2 - |\lambda_1|},
\]

where the primed sum indicates that terms of the form \( 0/0 \) might occur for which we use the convention \( 0/0 = 0 \).

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(ii) If there is no $\tilde{z}$ satisfying (2), then the optimal rank-1 Hankel approximation of $A$ possesses the error $\tilde{\lambda} := \|A - \tilde{c}\tilde{z}\tilde{z}^T\|_2 = \|A - \tilde{c}\tilde{z}\tilde{z}^T\|_F$, where $\tilde{\lambda}$ is the minimal number in $\{|\lambda_1|, \lambda_0\}$ satisfying $\max_{z \in \mathbb{R}} f(z, \tilde{\lambda}^2) = 0$. Further, we have

$$\tilde{z} \in \arg \max_{z \in \mathbb{R}} f(z, \tilde{\lambda}^2) \quad \text{and} \quad \tilde{c} := \left( \sum_{k=0}^{N-1} \frac{\langle v_k^T \tilde{z} \rangle^2}{\lambda_k - \tilde{\lambda}} \right)^{-1} > 0.$$

### 3 Examples and Comparisons

In the beginning of this section, we present two examples and the respective optimal rank-1 Hankel approximations. Then, by the means of these examples, we emphasize the difference between the optimal approximation w.r.t. the Frobenius norm and the one w.r.t. the spectral norm. The resulting approximation errors (rounded to four digits) are presented in Table 1.

**Example 3.1** Consider the matrix $A = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix}$ with eigenvectors $v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ corresponding to the ordered eigenvalues $\lambda_0 = 12, \lambda_1 = 11, \lambda_2 = 1$.

For the best rank-1 Hankel approximation in the Frobenius norm, according to Thm. 2.1, we compute

$$\tilde{z} = \arg \max_{z \in \mathbb{R}} \frac{z^T A z}{|z|^2} = \arg \max_{z \in \mathbb{R}} \frac{12 + 3z^2 + 8z^3 + 9z^4}{1 + z^2 + z^4} = 0 \quad \text{and} \quad \tilde{c} = \frac{\tilde{z}^T A \tilde{z}}{|\tilde{z}|^2} = 12.$$

Applying Thm. 2.2(i) we obtain several rank-1 Hankel matrices that attain the minimal spectral norm error: For both $z = 0$ and $z = -1/2$ the conditions (2) are satisfied. From (3) we obtain $1 \leq \tilde{c} \leq 23$ for $z = 0$ and $32/31 \leq \tilde{c} \leq 4416/307$ for $\tilde{z} = -1/2$.

In this example, the eigenvector corresponding to the largest eigenvalue is of the structure of $z$ from (1) for $z = 0$. Therefore Cadzow’s algorithm (see [1, 3]) terminates after one iteration and leads to the optimal result also acquired by Thm. 2.1.

**Example 3.2** Consider the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{pmatrix}$ with eigenvectors $v_0 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ corresponding to the eigenvalues $\lambda_0 = 11, \lambda_1 = 3, \lambda_2 = 1$.

By Thm. 2.1 we obtain the optimal parameters $\tilde{z} = 2.5410$ and $\tilde{c} \approx 0.2189$ (rounded to four digits) for optimal rank-1 Hankel approximation w.r.t. the Frobenius norm.

Regarding the optimal approximation w.r.t. the spectral norm, the conditions (2) cannot be satisfied in this example, since $\langle v_1^T z \rangle = 1$ for all $z \in \mathbb{R}$. Thus, we apply Thm. 2.2(ii) in form of a bisection iteration and obtain $\tilde{z} \approx 4.1113$ and $\tilde{c} \approx 0.0145$.

| Alg spectral | Alg Frobenius | Cadzow Alg |
|--------------|---------------|-------------|
| Example 3.1  | 11            | 11          | 11          |
| $\|A - \tilde{c}\tilde{z}\tilde{z}^T\|_2$ | 11.0454 - 16.3774 | 11.0454 | 11.0454 |
| Example 3.2  | 3.1748        | 3.5801      | 3.5294      |
| $\|A - \tilde{c}\tilde{z}\tilde{z}^T\|_2$ | 5.0727        | 3.9001      | 3.9122      |

Table 1: Comparison of the approximation errors in the Frobenius and the spectral norm.

Example 3.1 confirms a theoretical result from [1]: The optimal rank-1 Hankel approximations w.r.t. the Frobenius and spectral norm coincide with the optimal unstructured rank-1 approximation if the eigenvector corresponding to the largest eigenvalue already is of the structure indicated in (1). At the same time we observe that this condition is not necessary for the approximation in the spectral norm to achieve the error bound of the unstructured approximation since (2) is a weaker condition. Furthermore, the approximation in the spectral norm is not unique and it might be a good idea to choose $\tilde{c}$ in (3) such that also the Frobenius norm is minimized.

In Example 3.2 it is evident that the results for the Frobenius norm and the spectral norm differ significantly as predicted in [1]. Further, this example backs the long-standing suspicion that Cadzow’s algorithm in general does not lead to optimal results since for both the spectral norm and the Frobenius norm we find better approximations.

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