Pulsations and stability of stars with phase transition

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The general characteristics of stars with first-order phase transition (PT1) are reviewed at short. The model of two-incompressible-phase star with PT1 is considered in some detail.

I. INTRODUCTION

First-order phase transition (PT1) is characterized by density jump from \( \rho_1 \) to \( \rho_2 \) with \( q = \rho_2/\rho_1 > 1 \), at some pressure \( P_0 \). PT1 leads to density discontinuity inside the star while pressure is continuous across boundary between phases. Also continuous are gravity- and pressure-induced forces inside the star. Here we review some results of studying such stars get mainly (but not only) by author.

II. THREE METHODS

There are three main methods of star’s equilibrium and stability analysis:

- Static Criterion (Mass - Central Pressure dependence)
- Dynamical Principle (pulsation frequency)
- Variational (Energetic) Principle (variation of total energy).

Stability loss/restoration (critical equilibrium states) according to these methods are defined as follows:

- Mass extremum, \( dM/dP_c = 0 \)
- Zero frequency of pulsation (of the lowest mode), \( \omega^2 = 0 \)
- Equilibrium energy extremum, \( \delta^2 E = 0 \).
In fig.1 two first principles are shown qualitatively. Unstable equilibrium states are shown by broken line.

FIG. 1. Static (lower part) and dynamical (upper part) principles in the equilibrium and stability of stars
III. NEWTON THEORY OF GRAVITATION

Here are some general results in NTG for stars with PT1:

- At \( q > 1.5 \), stability loss occurs at \( P_c = P_0 \) for any EoS, while recover of stability for larger \( P_c \) depends on EoS.

- At \( q_{\text{min}}(\gamma) < q < 1.5 \), stability loss occurs only at \( P_c > P_0 \); \( \gamma = 1 + 1/n \), \( \gamma \) and \( n \) are adiabatic and polytropic indices. E.g., \( q_{\text{min}} = 1.46, 1.33, 1.20, \) and \( 1.09 \) for \( n = 1, 1.5, 2 \) and \( 2.5 \).

The critical value of (relative) mass of new-phase core is larger for smaller \( q \)'s and for smaller \( \gamma \)'s (softer EoS's).

E.g., critical value of new-phase core \( x_{\text{crit}} = 8/9 \) \((q - 3/2)\) for \( \gamma = 1 \) \((n = 1)\).

- At slow rotation with angular velocity \( \Omega \), in spherical approximation, \( q_{\text{crit}} = 3/2 - \Omega^2 / 4 \pi G \rho_1 \), for stability loss at \( P_c = P_0 \) for any EoS.

- For PT1, starting at finite radius ("neutral core"),
  the larger size of neutral core, the larger value of \( q_{\text{crit}} \).

E.g., \( q_{\text{crit}} \to \infty \) at \( x \to 3/4 \) at \( n = 0 \) and \( x \to .6824 \) at \( n = 1 \).

A. Classical example

Inverse \( \beta \)-decay reactions in dense degenerate matter of white dwarf stars lead to nuclei transformations \((A, Z) \to (A, Z - 2)\) ("neutronization"), and to density jump with \( q = Z/(Z - 2) < 1.5 \). \( M - \rho_c \) curves for equilibrium cold white dwarfs in classical paper by T.Hamada, E.Salpeter, Ap.J. 134 (1961) 669 are incorrect, as:

a) central density \( \rho_c \) is not continious variable, and b) mass maximum is not at point \( P_c = P_0 \), but at some \( P_c > P_0 \).
B. General Relativity

In GR, the critical value of energy density jump $q = \varepsilon_2/\varepsilon_1$ is equal to $3/2 \times (1 + P_0/\varepsilon_1)$ for stability loss at $P_c = P_0$ for any EoS.

IV. TWO-INCOMPRESSIBLE-PHASE STAR

Here are some important features of this model:

- At $q \leq 1.5$ there is no unstable equilibrium states

- At $q > 1.5$ recover of stability occurs (at point $P_c = P_2$) for relative radius of new-phase core, $x = r_{\text{core}}/R_{\text{star}}$, defined by relation:

\[
    f(q, x) = (q - 1)^2 x^4 + 4 (q - 1) x + 3 - 2q = 0
\]

(1)

Star with $x > \sqrt{2} - 1$ is stable for arbitrary large $q$

- Frequency squared of the small adiabatic radial pulsations of the lowest mode for star in slow rotation with angular velocity $\Omega$:

\[
    \omega_0^2 = \omega_0^2 + \Delta_\Omega(q, x);
\]

\[
    \omega_0^2 = \frac{4\pi G \rho_1 f(q, x)}{3(q - 1)(1 - x)};
\]

(2)

\[
    \Delta_\Omega(q, x) = \frac{2}{3} \Omega^2 \left( \frac{5x(1 - x)(1 + x)^2}{1 + (q - 1)x^5} - \frac{1 + (q - 1)x}{(q - 1)(1 - x)} \right).
\]

(3)

At small cores ($x \to 0$), $\Delta_\Omega$ is negative -- rotation reduces the stability of star with PT1. In general, $\Delta_\Omega$ may be of both signs, e.g., at $q > 2.11$, rotation leads to decreasing of value of $x_{\text{crit}}$ from Eq.(1).
A. Non-linear pulsations

In the next approximation, pulsations are non-harmonic.

Writing down $R = R_{eq} + z$, $|z| << R_{eq}$, $R_{eq}$ being radius of equilibrium model with the same mass, we get in next-to-zeroth approximation the following equation of motion:

$$\ddot{z} + \omega_0^2 z + C z^2 + D \dot{z}^2 = 0,$$

(4)

where $\omega_0^2$ is as in Eq. (2), while constants $C$ and $D$ are some functions of $q$ and of equilibrium value of $x$. The solution of Eq. (4) with accuracy up to $a^2$, ($a$ being an amplitude of $z$) is as follows:

$$z(t) = -\frac{1}{2}(\frac{C}{\omega_0^2} + D)a^2 + a \times \cos \omega_0 t + \frac{1}{6}(\frac{C}{\omega_0^2} - D)a^2 + a^2 \times \cos 2\omega_0 t.$$

(5)

In this approximation the frequency does not differ from one in zero’th approximation, $\omega_0$, that is period $T = T_0 = 2\pi/\omega_0$, while pulsations are non-sinusoidal.

An amplitude of a star’s expansion is larger than an amplitude of a star’s compression and star spends more time with $R > R_{eq}$ than in state with $R < R_{eq}$. A character of non-harmonicity - a slow large-amplitude ”expansion” and rapid ”contraction” with smaller amplitude - should be the same for all equilibrium stable models at $\lambda > 3/2$.

Damping of pulsations of star with PT1 depends largely on relation between velocity of motion $v_p$ and a sound velocity $v_s$ in the region of phase transition.

In general, the larger $v_p/v_s$ the larger damping effect due to PT.

B. General Relativity

Full analytical investigation of equilibrium and stability of two-incompressible-phase model with PT1 is possible by static and variational methods.

In first post-Newtonian approximation, $(P_0/\varepsilon_1 \ll 1)$ critical value of relative ”radius” of core, at which stability recover occurs, is equal to:
\[ x_{\text{crit}}(q) = x + \Delta_{PN}(P_0/\varepsilon_1), \]  

with \( x \equiv x_N \) defined in Eq. (1) and:

\[
\Delta_{PN} = \frac{9 - 7q + 27(q - 1)x + (4q^2 - 33q + 27)x^2 + (q - 1)(9 - 4q)x^3}{2(q - 1)(1 + (q - 1)x^3)^3}
\]  

At \( q \to 3/2 \), \( x_{PN} \to q - 3/2 - 3/2 P_0/\varepsilon_1 \), so that \( x_{PN} = 0 \) at \( q = 3/2 (1 + P_0/\varepsilon_1) \), which coincides with result for any EoS and for any, not only for small values of \( P_0/\varepsilon_1 \). In general, first post-Newtonian correction to \( x_{\text{crit}} \) may be of both signs, negative at \( q < 1.89 \) and positive for larger \( q \)'s. At \( q = 4 + \sqrt{8} \), \( x = \sqrt{2} - 1 + (59 - 41\sqrt{2})/2 P_0/\varepsilon_1 \).

In fact, dependence of both corrections, due to GR and rotation, on \( x \) and \( q \) is rather complicated, and Fig. 2 presents only part of \( (q - x) \)-plane with lines, on which \( \Delta_{\Omega}(q, x) = 0 \) and \( \Delta_{PN}(q, x) = 0 \). Also shown is the curve \( f(q, x) = 0 \) from Eq. (1) which marks a boundary between stable equilibrium states (at right) from unstable ones (at left).

\[ \text{FIG. 2. Three important curves at } (q - x)\text{-plane: critical state of stability recover, see Eq. 1, and GR and Rotational corrections equal to zero, see Eq. (3) and Eq. (7) respectively.} \]

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