Light Quark Condensates at Nonzero Chemical Potentials

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We show that the quark condensates for the two light up and down flavors can have significantly different values in the hadronic phase at nonzero temperature, baryon and isospin chemical potentials. We quantify this difference using a simple model.

I. INTRODUCTION

In recent years, an intense experimental and theoretical research effort has been devoted to the study of the strong interaction at nonzero temperature and density. These studies are important to get a better understanding of very different systems such as neutron stars, heavy-ion-collision experiments, or the early universe. Usually, theoretical studies are performed at nonzero baryon chemical potential, \( \mu_B \), and zero isospin chemical potential, \( \mu_I \). However, physical systems often have both nonzero baryon and isospin chemical potentials. Prime examples of such systems are neutron stars and heavy-ion-collision experiments. In neutron stars, \( \mu_I \neq 0 \) because of electric charge neutrality. In heavy-ion-collision experiments, the initial state corresponds to \( \mu_B \neq 0 \) and \( \mu_I \neq 0 \) and the interaction time is so short that both baryon number and isospin are conserved: the electroweak interactions are irrelevant, and the strong interactions determine the fate of the system. It is therefore phenomenologically important to study the more general problem where both baryon and isospin chemical potentials differ from zero. Such conditions are also obviously interesting on the theoretical perspective and a few models have been used to study the strong interaction at nonzero temperature, baryon and isospin chemical potentials.

The results obtained in some of these models are quite striking: the phase diagram in the \( (\mu_B, T) \) plane is qualitatively altered by the introduction of a small \( \mu_I \). In particular at small \( \mu_B \) and high \( T \), there are two phase transitions or crossovers between the hadronic phase and the quark-gluon-plasma phase when \( \mu_I \) is small. For fixed \( \mu_B \) and \( \mu_I \), the low temperature phase is the usual hadronic phase where both the up and down quark condensates, \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \), are large but different. When the temperature is increased, one of the condensates becomes small, whereas the other one remains large. Finally, if the temperature is further increased, both condensates become small. The existence of these different phases is possible since the light quark flavors have the same mass. However, the axial anomaly, which results in quark flavor mixing, might invalidate these results. Indeed, if the axial anomaly is important enough, it is not possible for one of the light quark condensates to be large while the other one is small. As a consequence, for a large enough axial anomaly, a small isospin chemical potential has little consequences for the phase diagram. In this article, we shall use a simple model based on chiral perturbation theory and the virial expansion to determine whether our current phenomenological understanding of the strong interaction in the hadronic phase simultaneously allows for a significant difference between the two light quark condensates at given temperature, baryon and isospin chemical potentials. Our phenomenology-based analysis provides a test whether the axial anomaly is strong enough to enforce \( \langle \bar{u}u \rangle \simeq \langle \bar{d}d \rangle \). We first describe our method and then evaluate quantitatively the difference between \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \) at nonzero temperature, baryon and isospin chemical potentials.

II. LIGHT QUARK CONDENSATES

In the hadronic phase, at low enough temperatures and chemical potentials, the physics of the strong interaction is dominated by the pseudo-Goldstone bosons due to the spontaneous breaking of chiral symmetry since they are the lightest excitations in the spectrum. This fact is at the basis of chiral perturbation theory. At finite temperature, it has been shown that the pseudo-Goldstone bosons are the most important modes below \( \sim 150 \text{ MeV} \). Above this temperature, massive modes have to be taken into account: although exponentially suppressed, their role starts to be significant.

The temperature dependence of the light quark condensates at zero chemical potentials has been thoroughly studied in up to three loops in chiral perturbation theory and included the contribution due to the massive modes. In this case, and for equal light quark masses, \( m_u = m_d \), the two corresponding condensates, \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \), are equal. However at nonzero baryon and isospin chemical potentials, the symmetry that enforces \( \langle \bar{u}u \rangle = \langle \bar{d}d \rangle \) is explicitly broken.
broken since $\mu_u \neq \mu_d$. In this article, we use a method similar to [8] to show that the difference between the light quark condensates can become significant at large enough temperature and chemical potentials.

The QCD partition function at nonzero quark chemical potentials is given by

$$Z_{\text{QCD}} = \int dA e^{-S_{\text{YM}}} \prod_f \det(i\gamma + m_f + \mu_f \gamma_0),$$

where $m_f$ and $\mu_f$ are the mass and the chemical potential related to the quark flavor $f$. In this article we will consider the case $m_u = m_d \neq m_s$, $\mu_u \neq \mu_d$, and $\mu_s = 0$. The quark condensate for the flavor $f$ is given by

$$\langle \bar{q}_f q_f \rangle = \lim_{V \rightarrow \infty} \frac{T}{V} \frac{\partial}{\partial m_f} \ln Z_{\text{QCD}}.$$  (2)

At low temperatures, the dominant contribution to the light quark condensates comes from the pseudo-Goldstone bosons. The physics of these mesons is described by the chiral perturbation theory partition function, $Z_{\text{ChPT}}$. However, since chiral perturbation theory contains only mesons, $Z_{\text{ChPT}}$ at $\mu_s = 0$ depends only on the isospin chemical potential, $\mu_I = \mu_d - \mu_u$, not on the baryon chemical potential $\mu_B = \frac{1}{2}(\mu_u + \mu_d)$. Notice that for $\mu_s = 0$, the strangeness chemical potential $\mu_S = -\frac{1}{3} \mu_B$. Therefore, for equal light quark masses, $m_u = m_d$, the pseudo-Goldstone bosons contribute equally to $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$, since $Z_{\text{ChPT}}(\mu_I) = Z_{\text{ChPT}}(-\mu_I)$. We thus conclude that in the hadronic phase, the difference between the two light quark condensates comes solely from the massive modes. In the hadronic phase, the contribution of a hadron of mass $m_H$ is exponentially suppressed by the Boltzmann factor $\sim \exp(-(m_H/T))$, and its interaction with another hadron of mass $m'_H$ is damped by a factor $\sim \exp(-(m_H + m'_H)/T)$. Interactions between hadrons and pions are further suppressed because pions are pseudo-Goldstone modes and therefore interact weakly at rest. In the hadronic phase, where temperatures never exceed $\sim 200$ MeV, it should therefore be sufficient to treat the massive modes in the free gas approximation.

In the free gas approximation, the difference between the light quark condensates is given by

$$\delta = \langle \bar{u}u - \bar{d}d \rangle = \sum_H \frac{g_H^2}{\langle \bar{q}q \rangle}\sqrt{\frac{m_H T^3}{8\pi^3}} \left( \frac{\partial m_H^2}{\partial m_u} - \frac{\partial m_H^2}{\partial m_d} \right) e^{-m_H/T} \cosh \left( \frac{(B_H + \frac{1}{3} S_H) \mu_B - I_{H3} \mu_I}{T} \right),$$

where $\langle \bar{q}q \rangle = \langle \bar{d}d \rangle$ is the light quark condensate at zero temperature and chemical potentials. The sum in the equation above is over massive hadrons, $H$, with spin degeneracy $g_H$, mass $m_H$, baryon number $B_H$, third component of isospin $I_{H3}$, and strangeness $S_H$. We have used that $\mu_S = -\frac{1}{3} \mu_B$ when $\mu_s = 0$.

We now have to determine $\partial m_H^2/\partial m_f$. Any hadron mass can be written as

$$m_H^2 = m_H^2 + C_H I_{H3} (m_u - m_d),$$

where $m_H$ is the hadron mass at $m_u = m_d$ and includes the contributions of the strange and heavier quarks. Since the up and down quarks are light, it is sufficient to use a linear expansion in $\delta$: the linear term dominates over other terms that depend on $m_u - m_d$. The quantity $C_H$ comes from isovector interactions and is related to the energy cost to change the third component of isospin by one unit, i.e. to replace a $u$ quark by a $d$ quark, within an isospin multiplet. Therefore we find that

$$\frac{\partial m_H^2}{\partial m_u} - \frac{\partial m_H^2}{\partial m_d} = 2C_H I_{H3},$$

and the difference between the light quark condensates reads

$$\delta = \sum_H C_H I_{H3} g_H \sqrt{\frac{m_H T^3}{2\pi^3}} e^{-m_H/T} \cosh \left( \frac{(B_H + \frac{1}{3} S_H) \mu_B - I_{H3} \mu_I}{T} \right).$$

Equivalently, if we sum over each isospin multiplet we get that

$$\delta = \sum_{\text{isospin multiplets}} C_H g_H \sqrt{\frac{2m_H T^3}{\pi^3}} e^{-m_H/T} \sinh((B_H + \frac{1}{3} S_H) \mu_B/T) \sum_{I_{H3} \geq 0} I_{H3} \sinh(I_{H3} \mu_I/T),$$

where we have used that isospin symmetry at $m_u = m_d$ implies that $m_H$ and $C_H$ do not change inside an isospin multiplet. Therefore, since on general grounds $C_H > 0$, we have shown that the light quark condensates do differ in the hadronic phase at nonzero temperature, baryon and isospin chemical potentials. Notice that (7) implies that only baryons with nonzero third component of isospin contribute to the difference between the light quark condensates.
III. QUANTITATIVE RESULTS

In this section, we evaluate the importance of the difference between the light quark condensates at nonzero temperature, baryon and isospin chemical potentials. It is clear from the above analysis that the quantitative difference between the condensates depends on the size of \( C_H \), and that this difference will grow exponentially when either the temperature, the baryon or the isospin chemical potentials are increased. The three lightest isospin multiplets that contribute to \( \delta \) are the \( N(939) \), the \( \Sigma(1193) \), and the \( \Delta(1232) \).

For the nucleon, the Feynman–Hellman theorem implies that \( C_N = 2m_N \langle p|\bar{u}u - \bar{d}d|p \rangle \). This can be estimated by using \( SU(3) \) flavor symmetry [9]:

\[
\frac{C_N}{\langle 0|\bar{q}q|0 \rangle} \simeq \frac{m_\Sigma^2 - m_N^2}{(m_u - \hat{m})(m_d - \hat{m})} \simeq \frac{2(m_\Sigma^2 - m_N^2)}{(\frac{m_u}{\hat{m}} - 1)m_\pi^2 F_\pi^2},
\]

where \( \hat{m} = (m_u + m_d)/2 \) and we have used the Gell-Mann–Oakes–Renner relation: \( m_\pi^2 F_\pi^2 = 2\hat{m} \langle 0|\bar{q}q|0 \rangle \). Numerically, if we use that \( m_u \approx 25\hat{m} \) [10], we get that \( C_N/\langle 0|\bar{q}q|0 \rangle \approx (1.3 \times 10^{-2}\text{ MeV}^{-1})^2 \). Much less is known about \( C_\Sigma \) and \( C_\Delta \). We shall assume that \( C_N \simeq C_\Sigma \simeq C_\Delta \), which is true in the large \( N_c \) limit. The results depicted in Fig. 1 were produced using these values for \( C_H \) in [1] with the \( N, \Sigma \) and \( \Delta \) isospin multiplets.

![Fig. 1: The difference between the up and down quark condensates, \( \delta = \langle \bar{u}u - \bar{d}d \rangle/\langle 0|\bar{q}q|0 \rangle \) as a function of temperature at fixed baryon and isospin chemical potentials. The baryon chemical potential is set to 250, 500, and 750 MeV in the first, second, and third graphs, respectively. In each graph, the dotted line corresponds to \( \mu_I = 30 \text{ MeV} \), the dash-dotted line to \( \mu_I = 60 \text{ MeV} \), the dashed line to \( \mu_I = 90 \text{ MeV} \), and the solid line to \( \mu_I = 120 \text{ MeV} \). The light gray line corresponds to the critical temperature determined by the peak of a flavor insensitive susceptibility [2].](image)

As shown in Fig. 1, we find that the difference between the up and down quark condensates can be relatively large compared to the value of these condensates in the vacuum: up to \( \sim 50\% \) for \( \mu_B = 750 \text{ MeV} \) and \( \mu_I = 120 \text{ MeV} \). Of course, these are quite large chemical potentials, but our study demonstrates that, as a matter of principle, there can be a sizeable difference between \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \) at nonzero baryon and isospin chemical potentials. The size of the difference depends mainly on \( C_H \). The uncertainty on this quantity is rather large, and our quantitative results cannot be very precise. At large enough \( \mu_I \), the largest contribution to \( \delta \) comes from \( \Delta(1232) \), which dominates over the nucleon because of its larger third component of isospin and its larger spin. In comparison, the contribution of the \( \Delta(1600) \) resonance is negligible compared to that of the nucleon. To check the validity of the free gas approximation we have calculated the next-to-leading term in the virial expansion taking into account pion-nucleon interactions. We have found that this term is negligible compared to the leading term in the virial expansion, i.e. the free gas approximation. This stems from the fact that pions are pseudo-Goldstone bosons and thus are weakly interacting particles. In [11], it was shown that heavier resonances can have a significant impact when taken collectively into account. In our case, we only used the lightest contributing modes in the spectrum to evaluate \( \delta \). Heavier baryons with nonzero third component of isospin will also contribute to \( \delta \), with the same sign. Therefore, the contributions from heavier resonances will add up and increase \( \delta \) for any temperature, baryon and isospin chemical potentials. However, because of the lack of precision on the value of \( C_H \) for heavier baryons, we decided not to evaluate their contributions and restrict ourselves to the three lightest isospin multiplets which should represent the dominant contribution.
IV. CONCLUSIONS AND OUTLOOK

We have shown that the up and down quark condensates can significantly differ at nonzero temperature, baryon and isospin chemical potentials. We have evaluated their difference in a simple model. We have found that the difference between the light quark condensates increases exponentially with the temperature, baryon or isospin chemical potentials. This difference becomes significant for large enough chemical potentials.

These results imply that there are two phase transitions or crossovers between the hadronic phase and the quark-gluon-plasma phase at nonzero baryon and isospin chemical potentials, as different models have predicted \[1, 2\]. In the hadronic phase, both \( \langle \bar{u}u \rangle \) and \( \langle \bar{d}d \rangle \) are large but different. In the quark-gluon-plasma phase, both light quark condensates are very small. Our results show that, at nonzero baryon and isospin chemical potentials, there is an intermediate phase where only one of the light quark condensates is large while the other one is very small. As has been discussed in \[4, 5\], the existence of this intermediate phase leads to important qualitative changes for the QCD phase diagram and for the nature of the critical endpoint, which might have interesting signatures in heavy-ion-collision experiments.

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