1. Introduction

This paper arose from my earlier paper [14]. (See also the follow-up by Speyer [7].) The prototypical result in [14] is the following. Define

\[ S_n(x) = \prod_{i=0}^{n-1} \left( 1 + x^{2^i} + x^{2^{i+1}} \right). \]

Set \( S_n(x) = \sum_{k \geq 0} \binom{n}{k} x^k \) (a finite sum), with \( S_0(x) = 1 \), and define

\[ u_2(n) = \sum_{k \geq 0} \binom{n}{k}^2. \]

Then

\[ \sum_{n \geq 0} u_2(n)x^n = \frac{1 - 2x}{1 - 5x + 2x^2}. \]

Upon seeing this result and some similar ones, Doron Zeilberger asked what happens when \( 2^n \) is replaced by some other function satisfying a linear recurrence with constant coefficients, such as the Fibonacci numbers \( F_n \) (with initial conditions \( F_1 = F_2 = 1 \)). We will prove some results of this nature, but the data suggests that much more is true. We give a number of conjectures in this direction.

2. A Fibonacci Product

In this section we consider the product

\[ I_n(x) = \prod_{i=1}^{n} \left( 1 + x^{F_i+1} \right). \]

In particular, \( I_0(x) = 1 \) (the empty product) and \( I_1(x) = 1 + x \). Our main goal for this section is a proof of the following result.

\[ Date: \] October 1, 2021.
Theorem 2.1. Let $I_n(x) = \sum_{k \geq 0} c_n(k) x^k$, and set

$$v_2(n) = \sum_{k \geq 0} c_n(k)^2,$$

so $v_2(0) = 1$, $v_2(1) = 2$, $v_2(2) = 4$, $v_2(3) = 10$, etc. Then

$$\sum_{n \geq 0} v_2(n) x^n = \frac{1 - 2x^2}{1 - 2x - 2x^2 + 2x^3}.$$

The proof parallels the proofs in [14] of similar results by setting up a system of linear recurrences of order one. (In Section 5 we give another proof, the case $k = 2$ and $t = 1$ of Theorem 5.3). However, deriving these recurrences here is quite a bit more complicated. It simplifies somewhat the argument to replace $I_n(x)$ with another power series (with noninteger exponents). The justification for this replacement is provided by the following lemma. Part (b) is presumably known, though we couldn’t find this result in the literature.

Lemma 2.2. Let $\phi = \frac{1}{2}(1 + \sqrt{5})$.

(a) Suppose $\alpha = (a_0, a_1, \ldots)$ and $\beta = (b_0, b_1, \ldots)$ are sequences of 0’s and 1’s, with finitely many 1’s, such that

$$\sum_{i \geq 0} a_i \phi^i = \sum_{i \geq 0} b_i \phi^i.$$

Then $\alpha$ can be converted to $\beta$ by a sequence of operations that replace three consecutive terms 001 with 110, and vice versa.

(b) Suppose $\alpha = (a_0, a_1, \ldots)$ and $\beta = (b_0, b_1, \ldots)$ are sequences of 0’s and 1’s, with finitely many 1’s, such that

$$\sum_{i \geq 0} a_i F_{i+2} = \sum_{i \geq 0} b_i F_{i+2}.$$

Then $\alpha$ can be converted to $\beta$ by a sequence of operations that replace three consecutive terms 001 with 110, and vice versa.

Proof. (a) This is a simple consequence of the fact that $\phi$ is a zero of the irreducible polynomial $x^2 - x - 1$.

(b) Simple proof by induction on the largest $j$ for which $a_j = 1$ or $b_j = 1$. Details omitted. □

For a power series $P(x) = \sum_{i \geq 0} c_i x^{m_i}$ with real exponents $m_i \geq 0$, where each $c_i \neq 0$ and $m_0 < m_1 < \cdots$, we call the sequence $(c_0, c_1, \ldots)$ the sequence of coefficients of $P(x)$. It’s easy to see that Lemma 2.2 has the following consequence.
Corollary 2.3. Let \( G_n(x) = \prod_{i=0}^{n-1} \left( 1 + x^{\phi^i} \right) \), a “formal polynomial” whose exponents lie in the ring \( \mathbb{Z}[\phi] \). Then the sequence of coefficients of \( G_n(x) \) is equal to the sequence of coefficients of \( I_n(x) \). Moreover, if the coefficient of \( x^k \) in \( I_n(x) \) is 0, then \( k > \deg I_n(x) \).

To illustrate the next result, when we expand \( G_5(x) \) we obtain the following expression, where the terms are listed in increasing order of their exponents:

\[
G_5(x) = 1 + x + x^a + 2x^{a+1} + x^{a+2} + 2x^b + 2x^{b+1} + x^c + 3x^{c+1} + 2x^{c+2} + 2x^d + 3x^{d+1} + x^{d+2} + 2x^e + 2x^{e+1} + x^f + 2x^{f+1} + x^{f+2} + x^g + x^{g+1},
\]

for certain numbers \( a, b, \ldots, g \in \mathbb{Z}[\phi] \). Note that the terms come in groups (or strings) of length two or three, where within each string the exponents increase by one at each step.

Theorem 2.4. For \( n \geq 1 \), we can write \( G_n(x) \) as a sum \( G_n(x) = T_1(x) + T_2(x) + \cdots + T_k(x) \), where each \( T_i(x) \) has the form \( c_1 x^h + c_2 x^{h+1} \) or \( c_1 x^h + c_2 x^{h+1} + c_3 x^{h+2} \) for some positive integers \( c_1, c_2, c_3 \). Moreover, the largest exponent of a term in \( T_i(x) \) is less than the smallest exponent of a term in \( T_{i+1}(x) \). (As an aside, we have \( k = F_{n+1} \).)

Proof hint. The terms \( T_i(x) \) with two summands are of the form

\[
1 + x + c_1 x^{\phi + \phi^2 + a_3 \phi^3 + a_4 \phi^4 + \cdots} + c_2 x^{1 + \phi + \phi^2 + a_3 \phi^3 + a_4 \phi^4 + \cdots},
\]

where \( a_3, a_4, \ldots \) is a sequence of 0’s and 1’s with finitely many 1’s, and where \( c_1, c_2 \) are positive integers. Similarly, the terms \( T_i(x) \) with three summands are of the form

\[
c_1 x^{\phi + a_3 \phi^3 + a_4 \phi^4 + \cdots} + c_2 x^{\phi^2 + a_3 \phi^3 + a_4 \phi^4 + \cdots} + c_3 x^{1 + \phi^2 + a_3 \phi^3 + a_4 \phi^4 + \cdots}. \]

Note. Though we have no need of this result, let us mention that if \( d_n(i) \) denotes the number of terms (either two or three) of \( T_i(x) \) (coming from \( G_n(x) \)), then \( d_n(i) = d_n(F_{n+1} - i + 1) \) and

\[
d_n(i) = 1 + \lfloor i \phi \rfloor - \lfloor (i - 1) \phi \rfloor, \quad 1 \leq i \leq \left\lceil \frac{1}{2} F_{n+1} \right\rceil.
\]

Set \( d(i) = \lim_{n \to \infty} d_n(i) \). The sequence \( (d(1), d(2), \ldots) \) is obtained from sequence A014675 in OEIS by prepending a 1 and adding 1 to every term.

We now define an array analogous to Pascal’s triangle (or the arithmetic triangle) and Stern’s triangle of \([14]\). We call the resulting array the Fibonacci triangle \( \mathcal{F} \). (This definition is unrelated to some other definitions of Fibonacci triangle in the literature.)
Every row is a sequence of positive integers, together with a grouping of consecutive terms such that every string of the grouping has two or three terms. We will denote the grouping by a bullet (\(\bullet\)) between strings. The first row is the sequence 1, 1, which necessarily has a single element 1, 1 in its grouping. Regard the first entry in each row as preceded by a 0. Similarly, the last entry in each row is followed by a 0.

Row \(i+1\) is obtained from row \(i\) by the following recursive procedure. If a term \(a_j\) of row \(i\) ends a string (so \(a_{j+1}\) begins a string), then below \(a_j\) and \(a_{j+1}\) write in row \(i+1\) the 3-element string \(a_j, a_{j+1}, a_{j+1}\). If \(a_j\) in row \(i\) is the middle element of a 3-element string, then write in row \(i+1\) below \(a_j\) the 2-element string \(a_j, a_{j+1}\).

Note that according to this procedure, odd numbered rows will begin with a 2-element string 1, 1 (preceded by a 0) and end with a 2-element string 1, 1 (followed by a 0). On the other hand, even numbered rows will begin with the 3-element string 0, 1, 1 and end with the 3-element string 1, 1, 0. In all instances, entries equal to 0 are not regarded as actual entries of \(F\).

The first five rows of \(F\) look as follows:

\[
\begin{array}{ccccccccc}
1 & 1 & \bullet & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & \bullet & 1 & 2 & 1 & \bullet & 1 & 1 \\
1 & 1 & \bullet & 1 & 2 & 2 & \bullet & 1 & 2 \\
1 & 1 & \bullet & 1 & 2 & 2 & \bullet & 2 & 3 \\
1 & 1 & \bullet & 1 & 2 & 1 & \bullet & 1 & 1 \\
\end{array}
\]

Let \(\left[\begin{array}{c} n \\ k \end{array}\right]\) be the \(k\)th entry (beginning with \(k = 0\)) in row \(n\) (beginning with \(n = 1\)) of \(F\). Set \(H_n(x) = \sum_{k \geq 0} \left[\begin{array}{c} n \\ k \end{array}\right] x^k\). For instance,

\[
H_3(x) = 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6.
\]

The following result can be proved by induction.

**Theorem 2.5.** We have \(H_n(x) = I_n(x)\).

We now have all the ingredients for proving Theorem 2.1. Define

\[
\begin{align*}
\left[\begin{array}{c} n \\ k \end{array}\right]_1 &= \begin{cases} \left[\begin{array}{c} n \\ k \end{array}\right], & \text{if the } k\text{th entry in row } n \text{ of } F \text{ is the first entry of its string} \\
0, & \text{otherwise} \\
\end{cases} \\
\left[\begin{array}{c} n \\ k \end{array}\right]_2 &= \begin{cases} \left[\begin{array}{c} n \\ k \end{array}\right], & \text{if the } k\text{th entry in row } n \text{ of } F \text{ is the middle entry of its string} \\
0, & \text{otherwise} \\
\end{cases} \\
\left[\begin{array}{c} n \\ k \end{array}\right]_3 &= \begin{cases} \left[\begin{array}{c} n \\ k \end{array}\right], & \text{if the } k\text{th entry in row } n \text{ of } F \text{ is the last entry of its string} \\
0, & \text{otherwise.} \\
\end{cases}
\end{align*}
\]
Set

\[
A_1(n) = \sum_k \left[ \frac{n}{k} \right]_1^2 \\
A_2(n) = \sum_k \left[ \frac{n}{k} \right]_2^2 \\
A_3(n) = \sum_k \left[ \frac{n}{k} \right]_3^2 \\
A_{3,1}(n) = \sum_k \left[ \frac{n}{k} \right]_3 \left[ \frac{n}{k+1} \right]_1 \\
A_{1,2}(n) = \sum_k \left[ \frac{n}{k} \right]_1 \left[ \frac{n}{k+1} \right]_2 \\
A_{1,3}(n) = \sum_k \left[ \frac{n}{k} \right]_1 \left[ \frac{n}{k+1} \right]_3 \\
A_{2,3}(n) = \sum_k \left[ \frac{n}{k} \right]_2 \left[ \frac{n}{k+1} \right]_3.
\]

Using the definition of \( F \) one checks the following (all sums are over \( k \geq 0 \)):

\[
A_1(n+1) = \sum \left( \left[ \frac{n}{k} \right]_3^2 + \left[ \frac{n}{k} \right]_2^2 \right) \\
= A_2(n) + A_3(n)
\]

\[
A_2(n+1) = \sum \left( \left[ \frac{n}{k} \right]_3 + \left[ \frac{n}{k+1} \right]_1 \right)^2 \\
= A_1(n) + A_3(n) + 2A_{3,1}(n)
\]

\[
A_3(n+1) = \sum \left( \left[ \frac{n}{k} \right]_1^2 + \left[ \frac{n}{k} \right]_2^2 \right) \\
= A_1(n) + A_2(n)
\]

\[
A_{3,1}(n+1) = \sum \left( \left[ \frac{n}{k} \right]_1 \left[ \frac{n}{k+1} \right]_2 + \left[ \frac{n}{k} \right]_1 \left[ \frac{n}{k+1} \right]_3 + \left[ \frac{n}{k} \right]_2 \left[ \frac{n}{k+1} \right]_3 \right) \\
= A_{1,2}(n) + A_{1,3}(n) + A_{2,3}(n)
\]
\[ A_{1,2}(n+1) = \sum \binom{n}{k}^3 \left( \binom{n}{k} + \binom{n}{k+1} \right) \]
\[ = A_3(n) + A_{3,1}(n) \]
\[ A_{1,3}(n+1) = \sum \binom{n}{k}^2 \]
\[ = A_2(n) \]
\[ A_{2,3}(n+1) = \sum \left( \binom{n}{k}^3 + \binom{n}{k+1} \right) \binom{n}{k+1} \]
\[ = A_1(n) + A_{3,1}(n). \]

Let \( M \) denote the matrix
\[
M = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix},
\]
and let \( v(n) \) denote the column vector
\[
v(n) = [A_1(n), A_2(n), A_3(n), A_{3,1}(n), A_{1,2}(n), A_{1,3}(n), A_{2,3}(n)]^t
\]
(where \(^t\) denotes transpose). The recurrences above take the form \( v(n+1) = Mv(n) \). Hence, as in \[14\] \( \S 2 \), the seven functions \( A_\alpha(n) \) all satisfy a linear recurrence relation whose characteristic polynomial \( Q_2(x) \) is the characteristic polynomial \( \det(xI - M) \) of \( M \). Then \( \sum_{n \geq 0} A_\alpha(n)x^n \) is a rational function with denominator \( x^{\deg Q_2(x)} Q_2(1/x) \).

One computes \( Q_2(x) = x^2(x+1)^2(x^3-2x^2-2x+2) \). Taking into account the initial conditions for the case \( A_1(n) + A_2(n) + A_3(n) = v_2(n) \) yields Theorem [2.1]

Note that the factors \( x^2(x+1)^2 \) of \( Q_2(x) \) were spurious. This suggests that there should be a simpler argument involving a \( 3 \times 3 \) matrix rather than a \( 7 \times 7 \) matrix.

3. A Fibonacci triangle poset

This section assumes a basic knowledge of the combinatorics of partially ordered sets (posets) and symmetric functions such as that appearing in \[11\] Ch. 3 and \[10\] Ch. 3. It is unrelated to the rest of this paper.
There is a poset $\mathcal{F}$ that is naturally associated with the Fibonacci triangle $\mathcal{F}$. The elements $t_{nk}$ of $\mathcal{F}$ correspond to the entries $[n \choose k]$ of $\mathcal{F}$, with a bottom element 0 adjoined. The element $t_{n+1,k}$ covers $t_{nj}$ if the recurrence defining $t_{n+1,k}$ involves $t_{nj}$. Thus every element of $\mathcal{F}$ is covered by exactly two elements. The number of saturated chains from 0 to $t_{nk}$ is $[n \choose k]$. See Figure 1 which is drawn upside-down (as a poset) in order to agree with the way the Fibonacci triangle $\mathcal{F}$ is drawn. We call $\mathcal{F}$ the Fibonacci triangle poset or FT-poset. For another representation of this poset (considered as a “hyperbolic graph”), see Northshield [4, Fig. 4]. (There is already a poset called the Fibonacci poset [8]. For a further poset associated with Fibonacci numbers see [9, §5].)

The grouping of the elements of row $n$ of the Fibonacci triangle into strings of size two and three is readily seen from the FT-poset. Consider the subposet consisting of ranks $n-1$ and $n$ (where the bottom element 0 has rank 0). The connected components of this subposet define the grouping.

Suppose that we label the edges of $\mathcal{F}$ as follows. The edges between ranks $2k$ and $2k+1$ are labelled alternately 0, $F_{2k+2}$, 0, $F_{2k+2}$, $\ldots$ from left to right. The edges between ranks $2k-1$ and $2k$ are labelled alternately $F_{2k+1}$, 0, $F_{2k+1}$, 0, $\ldots$ from left to right. See Figure 2. Then it is not difficult to show that if $t \in \mathcal{F}$ and rank($t$) = $n$, then the edge labels of all saturated chains from 0 to $t$ have the same sum $\sigma(t)$, and that these chains correspond to all ways to write $\sigma(t)$ as a sum of the elements of a subset of $\{F_2, F_3, \ldots, F_{n+1}\}$. 

Figure 1. The Fibonacci triangle poset $\mathcal{F}$
Now suppose that we label every point \( t \in \mathcal{F} \) by \( \sigma(t) \). Thus the labels at rank \( n \) consist of all integers \( 0, 1, \ldots, F_{n+3} - 2 \), since these are the exponents when we expand \( I_n(x) \). The sequence \( S(n) \) of labels at rank \( n \), read left-to-right, is a subsequence of the sequence of labels at rank \( n + 1 \), read left-to-right. Thus the sequences \( S(1), S(2), S(3), \ldots \) approach a limit, which is a dense linear order on the nonnegative integers that we denote by \( < \). For instance, from Figure 2 we see that \( S(4) = (7, 2, 10, 5, 0, 8, 3, 11, 6, 1, 9, 4) \).

The order \( < \) can be described as follows. Let \( 0 < m < n \). Every nonnegative integer has a unique representation as a sum of nonconsecutive Fibonacci numbers, where a summand equal to 1 is always taken to be \( F_2 \) (Zeckendorf’s theorem). Let \( m = F_{i_1} + \cdots + F_{i_r} \) and \( n = F_{j_1} + \cdots + F_{j_s} \) be such representations, with \( i_1 < \cdots < i_r \) and \( j_1 < \cdots < j_s \). Regard \( F_{i_r+1} = F_{j_s+1} = F_0 = 0 \). Let \( k \) be the least index for which \( i_k \neq j_k \). Then we have \( m < n \) precisely in the following cases.

- \( i_k \) and \( j_k \) are odd, and \( i_k < j_k \).
- \( i_k \) is odd and \( j_k \) is even.
- \( i_k \) is even and nonzero, \( j_k \) is even and nonzero, and \( i_k > j_k \).
- \( i_k = 0 \) and \( j_k \) is even.

For instance, let \( n \neq 0 \). Then \( m < n \) if \( j_1 \) is odd, while \( n > 0 \) if \( j_1 \) is even.

The poset \( \mathcal{F} \) is not “nice” in regard to its topological properties. For instance, the rank-selected subposets \( \mathcal{F}_{n-1,n}, n \geq 2 \), are not connected, so \( \mathcal{F} \) is not Cohen-Macaulay. Moreover, its flag \( h \)-vector \( \beta_\mathcal{F} \) [§3.13] can be negative, e.g., \( \beta_\mathcal{F}(1,2) = -1 \). Despite these shortcomings, \( \mathcal{F} \) does have some nice structural and enumerative properties which we now discuss.
A poset $P$ is called upper homogeneous or upho \cite{12} if for every $t \in P$, the dual principal order ideal $V_t = \{ s \in P : s \geq t \}$ is isomorphic to $P$. It is easily seen that $\mathcal{F}$ is upho. In fact, $\mathcal{F}$ has an especially simple structure. For $i, b \geq 2$ define the upho poset $P_{ib}$ by the following conditions:

- $P$ has a unique minimal element $\hat{0}$.
- Every element of $P_{ib}$ is covered by $i$ elements.
- $P_{ib}$ has a planar (i.e., no crossing edges) Hasse diagram such that if $u, u'$ are consecutive (reading the Hasse diagram from left-to-right) covers of $t$, then the elements $t, u, u'$ “extend to a $2b$-gon.” That is, there is an element $v > t$ for which the Hasse diagram of the interval $[t, v]$ contains $u, u'$ and looks like a $2b$-gon with no vertices or edges in its interior, and where $t$ and $v$ are antipodal edges (so the interval $[t, v]$ is graded). (See \cite{3, §4} for more information on planar upho posets.)

It’s not hard to see that $P_{ib}$ exists and is unique up to isomorphism. In particular $\mathcal{F} \cong P_{23}$, a surprisingly simple description of $\mathcal{F}$. Moreover, the poset corresponding to Pascal’s triangle (i.e., the product of two chains $t_0 < t_1 < \cdots$) is isomorphic to $P_{22}$, while the poset corresponding to Stern’s triangle \cite[p. 25] 16 is isomorphic to $P_{32}$.

Recall now that if $P$ is a finite graded poset with $\hat{0}$ and $\hat{1}$, then the Ehrenborg quasisymmetric function $E_P$ of $P$ \cite[Def. 4.1] 10 Exer. 7.48] is defined by

$$E_P = \sum_{\hat{0}=t_0 \leq t_1 \leq \cdots \leq t_k-1 < t_k=\hat{1}} x_1^{\rho(t_0, t_1)} x_2^{\rho(t_1, t_2)} \cdots x_k^{\rho(t_{k-1}, t_k)},$$

where $\rho(s, t)$ denotes the rank (length) of the interval $[s, t]$. (The sum ranges over all multichains from $\hat{0}$ to $\hat{1}$ of all possible lengths $k \geq 1$ such that $\hat{1}$ occurs with multiplicity one.) $E_P$ is a kind of generating function for the flag $h$-vector $\beta_P$ of $P$—knowing $E_P$ is equivalent to knowing $\beta_P$. If $P$ is infinite, graded, with $\hat{0}$ and with finitely many elements $q_n$ of each rank $n \geq 0$, then we can extend the definition of $E_P$ by setting

$$E_P = \sum_{t \in P} E_{[\hat{0}, t]}.$$

One nice feature of graded upho posets $P$ with finitely many elements of every rank is that $E_P$ is a symmetric function. If $P$ has $q_n$ elements of rank $n$ (so $q_0 = 1$), then in fact

$$E_P = \sum_{\lambda} q_{\lambda_1} q_{\lambda_2} \cdots m_{\lambda},$$
where \( \lambda \) ranges over all partitions \((\lambda_1, \lambda_2, \ldots)\) of all nonnegative integers, and where \( m_\lambda \) is a monomial symmetric function. Equivalently, define
\[
\Phi(P, x) = \sum_{n \geq 0} q_n x^n,
\]
the rank-generating function of \( P \). (The usual notation is \( F(P, q) \), but that might cause confusion with \( F_n^{(k)} \) as defined in §5.) Then \([3, \text{Lemma 2.3}]\)
\[
E_P = \Phi(P, x_1) \Phi(P, x_2) \Phi(P, x_3) \cdots ,
\]

Let us also note that if \( 1 \leq r_1 < r_2 < \cdots < r_k \), then the flag \( f \)-vector \( \alpha_P \) of \( P \) at \( S = \{r_1, r_2, \ldots, r_k\} \) is given by
\[
\alpha_P(S) = q_{r_1} q_{r_2-r_1} q_{r_3-r_2} \cdots q_{r_k-r_{k-1}},
\]
since there are \( q_{r_1} \) ways to choose an element \( t_1 \) of rank \( r_1 \), then \( q_{r_2-r_1} \) ways to choose an element \( t_2 \) of rank \( r_2 \) satisfying \( t_2 > t_1 \), etc.

It’s not hard to see (since every element of \( P_{ib} \) is covered by \( i \) elements, and every element of rank \( n-b \) is the bottom element of \( i-1 \) \( 2b \)-gons whose top element has rank \( n \)) that
\[
\Phi(P_{ib}, x) = \frac{1}{1 - ix + (i-1)x^b}.
\]

Equivalently, \( q_n \) satisfies the initial conditions and recurrence
\[
q_0 = 1, \quad q_1 = i, \quad q_2 = i^2, \ldots, \quad q_{b-1} = i^{b-1}, \quad q_n = iq_{n-1} - (i-1)q_{n-b}, \quad n \geq b.
\]

In particular, for \( \mathfrak{F} \cong P_{23} \) we have
\[
\Phi(\mathfrak{F}, x) = \frac{1}{1 - 2x + x^3} = \frac{1}{(1-x)(1-x-x^2)} = \sum_{n \geq 0} (F_{n+3} - 1)x^n.
\]

In other words, the Fibonacci triangle \( \mathcal{F} \) has \( F_{n+3} - 1 \) elements in row \( n \). Of course this is easy to see by a more direct argument.

The symmetric functions \( E_{P_{ib}} \) have “nice” expansions in terms of the power sum symmetric functions \( p_\lambda \) and the forgotten symmetric functions \( f_\lambda = \omega m_\lambda \), where \( \omega \) is the standard involution on symmetric functions. Define \( \tilde{q}_n \) by
\[
\tilde{q}_0 = 1, \quad \tilde{q}_1 = i, \quad \tilde{q}_2 = i^2, \ldots, \quad \tilde{q}_{b-1} = i^{b-1}, \quad \tilde{q}_b = i^b - b(i-1), \quad \tilde{q}_n = iq_{n-1} - (i-1)q_{n-b}, \quad n \geq b+1.
\]
Thus \( \tilde{q}_n \) satisfies the same recurrence as \( q_n \), but beginning at \( n = b + 1 \), not \( n = b \), and with different initial conditions. We use notation such as \( b^31^4 \) to denote the partition \( (b, b, 1, 1, 1, 1) \).

**Theorem 3.1.**

(a) We have

\[
E_{P_{b^2}} = \sum_{\lambda} z^{-\lambda} q_{\lambda_1} q_{\lambda_2} \cdots p_{\lambda},
\]

where \( \lambda \) ranges over all partitions \( (\lambda_1, \lambda_2, \ldots) \) of all \( n \geq 0 \).

(b) We have

\[
E_{P_{b^2}} = \sum_{n \geq 0} \sum_{j \geq 0} (-1)^j (i-1)^j j^{n-j} \delta_{0b^31^4-n-jb},
\]

where we set \( \delta_{0b^31^4-n-jb} = 0 \) if \( jb > n \).

**Proof.**

(a) By equations (3.1) and (3.2) we have

\[
E_{P_{b^2}} = \frac{1}{\prod_{m \geq 1} (1 - ix_m + (i-1)x_m^b)}.
\]

Let \( 1 - ix + (i-1)x^b = \prod_{h=1}^b (1 - \alpha_h x) \), where \( \alpha_h \in \mathbb{C} \). Then

\[
\log E_{P_{b^2}} = -\sum_{m} \sum_{h} \log (1 - \alpha_h x_m)
\]

\[
= \sum_{m} \sum_{h} \sum_{k \geq 1} \frac{\alpha_h^k x_m^k}{k}
\]

\[
= \sum_{k \geq 1} \left( \sum_{h} \alpha_h^k \right) \frac{p_k}{k},
\]

where \( p_k = \sum_m x_m^k \), the \( k \)th power sum symmetric function of the \( x_m \)'s. Now for any polynomial \( \sum_{h} \beta_h x^h \), we have

\[
\sum_{k \geq 1} \left( \sum_{h} \beta_h^k \right) x^k = \frac{-xQ'(x)}{Q(x)}.
\]

Letting \( Q(x) = 1 - ix + (i-1)x^b \) gives

\[
\sum_{k \geq 1} \left( \sum_{h} \alpha_h^k \right) x^k = \frac{i x - (i-1)bx^b}{1 - ix + (i-1)x^b}
\]

\[
= ix + i^2 x^2 + \cdots + i^{b-1} x^{b-1}
\]

\[
+ (i^b - (i-1)b)x^b + \cdots.
\]

It follows easily that \( \sum_h \alpha_h^k = \tilde{q}_k \).
Now apply the exponential function $\exp$ to (3.5). By e.g. [10, Prop. 7.7.4], we obtain equation (3.3).

(b) Let $R(x)$ be any power series with constant term 1, and let $\Gamma = R(x_1)R(x_2)\cdots$. If $\omega$ is the standard involution on symmetric functions, then an elementary argument gives

$$\omega \Gamma = \frac{1}{R(-x_1)R(-x_2)\cdots}.$$ 

Hence by equation (3.4),

$$\omega E_{P_{ib}} = \prod_m (1 + i x_m + (-1)^b(i - 1)x_m^b)$$

$$= \sum_n \sum_{j \geq 0} (-1)^b(i - 1)^j i^{n-j} b_{m_{ib}, n-j}.$$ 

Since $\omega$ interchanges $m_\lambda$ and $f_\lambda$, the proof follows. \qed

Note. Fix $i$ and $b$. Let $t_{nk}$ be the $k$th element from the left in the $n$th row, beginning with $k = 0$, of $P_{ib}$. Write $[n \atop k]$ for the number of saturated chains from $0$ to $t_{nk}$, and as usual let $q_n$ be the number of elements of $P_{ib}$ of rank $n$. It is immediate from the recurrence $q_n = iq_{n-1} - (i - 1)q_{n-b}$ and the initial conditions for $q_0, \ldots, q_{i-1}$ that $q_n - q_{n-1}$ is divisible by $i - 1$. Set $r_n = (q_n - q_{n-1})/(i - 1)$. Then it can be shown that

$$\sum_k [n \atop k] x^k = \prod_{j=1}^n (1 + x^{r_j} + x^{2r_j} + \cdots + x^{(i-1)r_j}).$$

It might be interesting to further investigate the posets $P_{ib}$. For the case $P_{32}$ (Stern’s poset), see Yang [16].

4. SOME GENERALIZATIONS

There are several ways we can try to generalize Theorem 2.1. In this section we will consider generalizing the product $I_n(x)$ and the function $v_2(n)$. However, we continue to deal with Fibonacci numbers. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{m-1}) \in \mathbb{N}^m$ (where $\mathbb{N} = \{0, 1, \ldots\}$), and define

$$v_\alpha(n) = \sum_{k \geq 0} \left[\begin{array}{c} n \\ k \end{array}\right]^{\alpha_0} \left[\begin{array}{c} n \\ k+1 \end{array}\right]^{\alpha_1} \cdots \left[\begin{array}{c} n \\ k+m-1 \end{array}\right]^{\alpha_{m-1}}.$$ 

This definition is completely analogous to the definition of $u_\alpha(n)$ in [14]. As in [14], we write $v_{\alpha_0, \ldots, \alpha_{m-1}}$ as short for $v(\alpha_0, \ldots, \alpha_{m-1})$.

Our proof of Theorem 2.1 carries over to the following result. The argument is analogous. We just have to ascertain that we don’t end up
with a system of infinitely many equations. This is proved in the same way as in \cite{14}, Thm. 2.

**Theorem 4.1.** For any $\alpha \in \mathbb{N}^m$, the generating function

$$J_{\alpha}(x) = \sum_{n \geq 0} v_{\alpha}(n)x^n$$

is rational.

We used the Maple package gfun to “guess” the rational function $J_{\alpha}(x)$ for some small $\alpha$. Gfun finds the “simplest” rational function fitting the data, which consists of values of $v_{\alpha}(n)$ for small $n$ (typically around $0 \leq n \leq 36$). Subsequently Zeilberger \cite{17} developed a Maple package \texttt{SternCF.txt} that can make such computations using variants of the linear algebra method of Section \ref{2}. Thus he obtains rigorous proofs of results like the following examples, where $\alpha = (r)$. No guesswork using gfun is necessary.

\begin{align*}
J_3(x) &= \frac{1 - 4x^2}{1 - 2x - 4x^2 + 2x^3} \\
J_4(x) &= \frac{1 - 7x^2 - 2x^4}{1 - 2x - 7x^2 - 2x^4 + 2x^5} \\
J_5(x) &= \frac{1 - 11x^2 - 20x^4}{1 - 2x - 11x^2 - 8x^3 - 20x^4 + 10x^5} \\
J_6(x) &= \frac{1 - 17x^2 - 88x^4 - 4x^6}{1 - 2x - 17x^2 - 28x^3 - 88x^4 + 26x^5 - 4x^6 + 4x^7} \\
J_7(x) &= \frac{1 - 26x^2 - 311x^4 - 84x^6}{1 - 2x - 26x^2 - 74x^3 - 311x^4 + 34x^5 - 84x^6 + 42x^7}.
\end{align*}

Note that the denominators all have odd degree, and the numerator is the even part of the denominator. This behavior has been verified empirically (not rigorously) for $n \leq 17$. For $8 \leq n \leq 17$, the denominator degrees are $9, 7, 9, 9, 13, 11, 13, 11, 13, 13$, respectively. See Conjecture \ref{5.5} for a generalization.
Here are some examples where $\alpha$ has at least two terms:

\[
\begin{align*}
J_{1,1}(x) &= \frac{x + x^2}{1 - 2x - 2x^2 + 2x^3} \\
J_{1,0,1}(x) &= \frac{2x^2 + x^3 - x^4}{(1 - x)(1 - 2x - 2x^2 + 2x^3)} \\
J_{2,1}(x) &= \frac{x + x^2}{1 - 2x - 4x^2 + 2x^3} \\
J_{1,3}(x) &= \frac{x + x^2 + x^3 + x^4}{1 - 2x - 7x^2 - 2x^4 + 2x^5} \\
J_{2,2}(x) &= \frac{x + x^2 - x^3 - x^4}{1 - 2x - 7x^2 - 2x^4 + 2x^5} \\
J_{2,3}(x) &= \frac{x + x^2 - x^3 - x^4}{1 - 2x - 11x^2 - 8x^3 - 20x^4 + 10x^5} \\
J_{1,1,1}(x) &= \frac{2x^2 + 2x^3 - 2x^4}{(1 - x)(1 - 2x - 4x^2 + 2x^3)} \\
J_{1,0,2}(x) &= \frac{2x^2 + x^3 - 2x^4 + x^5}{(1 - x)^2(1 - 2x - 4x^2 + 2x^3)} \\
J_{2,1,1}(x) &= \frac{2x^2 + 2x^3 - 4x^4 + 4x^5}{(1 - x)^2(1 - 2x - 7x^2 - 2x^4 + 2x^5)} \\
J_{1,2,1}(x) &= \frac{2x^2 + 4x^3 - 2x^4}{(1 - x)(1 - 2x - 7x^2 - 2x^4 + 2x^5)}
\end{align*}
\]

It appears that $J_\alpha(x)$ has a denominator of the form $(1 - x)^c \cdot D_r(x)$, where $c_\alpha \geq 0$, $r = \sum \alpha_i$, and $D_r(x)$ is the denominator of $J_r(x)$. This heuristic observation is in complete analogy to [14, Thm. 3] and presumably has a similar proof.

We can also generalize the definition of $I_n(x)$. In analogy to [14, Thm. 4] we have the following conjecture.

**Conjecture 4.2.** Let $h \geq 1$, $(a_1, \ldots, a_h) \in \mathbb{C}^h$, and $P(x) \in \mathbb{C}[x]$. Set

\[
I_{h,P,n}(x) = P(x) \prod_{i=1}^{n} \left(1 + a_1 x^{F_i} + a_2 x^{F_{i+1}} + \cdots + a_h x^{F_{i+h-1}}\right).
\]
Regarding $h, P$ as fixed, let $c_n(p)$ denote the coefficient of $x^p$ in $I_{h,P,n}(x)$. For $\alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^m$ define

$$v_{h,P,\alpha}(n) = \sum_{p \geq 0} c_n(p)^{\alpha_0} c_n(p+1)^{\alpha_1} \cdots c_n(p+m-1)^{\alpha_{m-1}}.$$  

Then the generating function $\sum_{n \geq 0} v_{h,P,\alpha}(n) x^n$ is rational.

Let us consider one simple special case of this conjecture. Let $t$ be any complex number (or an indeterminate), and define $I_{n,t}(x) = \prod_{i=1}^{n} (1 + tx^{F_{i+1}})$.

We now get a triangle $F(t)$ with the same grouping into strings of length two or three as in $F$, but the first row is $1, t$. Row $i+1$ is obtained from row $i$ by the following recursive procedure. If a term $a_j$ of row $i$ ends a string (so $a_{j+1}$ begins a string), then below $a_j, a_{j+1}$ write in row $i+1$ the 3-element string $a_j, ta_j + a_{j+1}, ta_{j+1}$. If $a_j$ in row $i$ is the middle element of a 3-element string, then write in row $i+1$ below $a_j$ the 2-element string $a_j, ta_j$.

The following result now is proved in complete analogy with the proof of Theorem 2.1.

**Theorem 4.3.** Let $v_{2,t}(n)$ denote the sum of the squares of the coefficients of $I_{n,t}(x)$. Then

$$\sum_{n \geq 0} v_{2,t}(n) x^n = \frac{1 - t(t^2 + 1)x^2}{1 - (t^2 + 1)x - t(t^2 + 1)x^2 + t(t^4 + 1)x^3}.$$  

The polynomial $I_{n,-1}(x) = \prod_{i=1}^{n} (1 - x^{F_{i+1}})$ has been considered before. It was shown by Yufei Zhao [18] that all its nonzero coefficients are equal to $\pm 1$. Thus $v_{2,-1}(n)$ is equal to the number of nonzero coefficients of $I_{n,-1}(x)$, with generating function

$$\sum_{n \geq 0} v_{2,-1}(n) x^n = \frac{1 + 2x^2}{1 - 2x + 2x^2 - 2x^3}.$$  

This fact is stated (in equivalent form) in the OEIS [3]. Note that we can also directly compute, using the technique in the proof of Theorem 2.1, that $v_{4,-1} = v_{2,-1}$. This gives a new proof (albeit involving a cumbersome computation) of Zhao’s result.

**Example 4.4.** As a somewhat random special case of Conjecture 4.2 let $h = 3$, $(a_1, a_2, a_3) = (0, 1, 1), P(x) = 1$, and $\alpha = (2)$. Thus we are considering the sum $w(n)$ of the squares of the coefficients of the
product $\prod_{i=1}^{n} (1 + x^{F_{i+1}} + x^{F_{i+2}})$. Then gfun suggests and Zeilberger [17, p. 15] confirms that

$$\sum_{n \geq 0} w(n)x^n = \frac{1 - 4x - 5x^2 + 24x^3 + 4x^4 - 34x^5 + 2x^6 + 10x^7 - 4x^8}{1 - 7x + x^2 + 47x^3 - 32x^4 - 84x^5 + 50x^6 + 34x^7 - 18x^8}.$$  

In fact, Zeilberger is able to compute the generating function for $\alpha = (r)$ when $1 \leq r \leq 6$. For $r = 6$ the denominator has degree 405.

**Note.** There is an alternative way of describing the nonzero coefficients of the polynomial $I_n(x) = \prod_{i=1}^{n} (1 + x^{F_{i+1}})$. Let $A_n$ denote the set of all words of length $n$ in the letters $a,b$, so $\#A_n = 2^n$. Define $\pi, \sigma \in A_n$ to be **equivalent** if $\sigma$ can be obtained from $\pi$ by a sequence of substitutions (on three consecutive terms) $baa \rightarrow abb$ and $abb \rightarrow baa$, an obvious equivalence relation $\sim$. For instance, when $n = 5$ one of the equivalence classes is $\{baaaa, abbaa, ababb\}$. The quotient monoid of the free monoid generated by $a,b$ modulo $\sim$ is called the **Fibonacci monoid** in [15], though other monoids are also called the Fibonacci monoid. Here we are interested not in the monoid itself, but rather the sizes of its equivalence classes. It follows easily from Lemma 2.2(b) that the multiset $M_n$ of equivalence class sizes of $\sim$ on $A_n$ coincides with the multiset of (nonzero) coefficients of $I_n(x)$. Thus if $u_n^*(r) = \sum_{j \in M_n} j^r$ ($r \in \mathbb{N}$), then the generating function $\sum_{n \geq 0} u_n^*(r)x^n$ is rational. What other equivalence relations on $A_n$ obtained by substitutions of words of equal length yield rational generating functions? For instance, the substitutions $ab \leftrightarrow ba$ do not give rational generating functions for $r \geq 2$. For $r = 2$ the generating function is algebraic but not rational, while for $r \geq 3$ it is D-finite but not algebraic [10, Exer. 6.3, 6.54]. Thus we can also ask in general when we get algebraic and D-finite generating functions.

**Note.** It is a nice exercise to show that if $f_1, f_2, \ldots$ is a sequence of positive integers satisfying $f_1 \neq f_2$ and $f_{i+1} = f_i + f_{i-1}$ for all $i \geq 2$, then for all $n \geq 1$ the sequence of nonzero coefficients of the polynomial $\prod_{i=1}^{n} (1 + x^{f_i})$ depends only on $n$.

### 5. Generalizing the Fibonacci numbers

What happens if we replace $F_{i+1}$ in the definition (2.1) and its generalizations with some other sequence? We consider only sequences $f_1, f_2, \ldots$ satisfying linear recurrences with constant integer coefficients, called **C-finite sequences** by Zeilberger [17]. Note that if $f_{i+1} \geq 2f_i$ for all $i$, then the nonzero coefficients of $\prod_{i=1}^{n} (1 + x^{f_i})$ are all equal to 1, which is not so interesting. One class of sequences that have more
interesting behavior is given for fixed \( k \geq 1 \) by
\[
F^{(k)}_{i+1} = F^{(k)}_i + F^{(k)}_{i-1} + \cdots + F^{(k)}_{i-k+1},
\]
say with initial conditions \( F^{(k)}_1 = F^{(k)}_2 = \cdots = F^{(k)}_k = 1 \). Thus \( F^{(2)}_i = F_i \).

We conjecture that Conjecture 4.2 has a direct \( F^{(k)} \)-analogue.

**Conjecture 5.1.** Let \( k \geq 2, \ h \geq 1, \ (a_1, \ldots, a_h) \in \mathbb{C}^h \), and \( P(x) \in \mathbb{C}[x] \). Set
\[
I^{(k)}_{h,P,n}(x) = P(x) \prod_{i=1}^n \left( 1 + a_1 x^{F^{(k)}_i} + a_2 x^{F^{(k)}_{i+1}} + \cdots + a_h x^{F^{(k)}_{i+h-1}} \right).
\]
Regarding \( h, P, k \) as fixed, let \( c_n(p) \) denote the coefficient of \( x^p \) in \( I^{(k)}_{h,P,n}(x) \). For \( \alpha = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{N}^m \) define
\[
v^{(k)}_{h,P,\alpha}(n) = \sum_{p \geq 0} c_n(p)^{\alpha_0} c_n(p+1)^{\alpha_1} \cdots c_n(p+m-1)^{\alpha_{m-1}}.
\]
Then the generating function \( \sum_{n \geq 0} v^{(k)}_{h,P,\alpha}(n)x^n \) is rational.

For the special case
\[
I^{(k)}_{n,P,n}(x) = \prod_{i=1}^n \left( 1 + t x^{F^{(k)}_{i+k-1}} \right),
\]
we can prove this conjecture by a combinatorial technique. When \( k = 2 \) this gives a new proof of Theorem 2.1.

To give this proof, for \( k \geq 2 \) define \( \mathcal{M}^{(k)} \) to be the set of all pairs \( \pi \) of finite binary sequences of the same length, say \( n \), denoted
\[
\pi = \left( \begin{array}{cccc}
a_1 & a_2 & \cdots & a_n \\
b_1 & b_2 & \cdots & b_n
\end{array} \right),
\]
such that
\[
\sum_{i=1}^n a_i F^{(k)}_{i+k-1} = \sum_{i=1}^n b_i F^{(k)}_{i+k-1}.
\]
It is easily seen that if \( \pi \in \mathcal{M}^{(k)} \) (where \( \pi \) is given by equation (5.1)) and if
\[
\sigma = \left( \begin{array}{cccc}
c_1 & c_2 & \cdots & c_p \\
d_1 & d_2 & \cdots & d_p
\end{array} \right) \in \mathcal{M}^{(k)}
\]
then the concatenation
\[
\pi \sigma = \left( \begin{array}{cccc}
a_1 & a_2 & \cdots & a_n & c_1 & c_2 & \cdots & c_p \\
b_1 & b_2 & \cdots & b_n & d_1 & d_2 & \cdots & d_p
\end{array} \right)
\]
also belongs to $M^{(k)}$. Thus $M^{(k)}$ is a monoid under concatenation. (The empty array is the identity element.)

For a binary letter $a = 0, 1$ let $a^j$ denote a sequence of $j$ $a$’s. For instance, $0^4 = 0000$. Given $k \geq 2$, let $G^{(k)}$ be the set of all pairs of binary sequences equal to

\[(5.2) \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 1 \end{pmatrix},\]

or equal to one of the two forms (which differ by interchanging the rows)

\[(5.3) \quad \begin{pmatrix} 1^k \ast 1^{k-1} \ast 1^{k-1} \ast \ldots \ast 1^{k-1} \ast 0 \\ 0^k \ast 0^{k-1} \ast 0^{k-1} \ast \ldots \ast 0^{k-1} \ast 1 \end{pmatrix}, \text{ or} \]

\[\begin{pmatrix} 0^k \ast 0^{k-1} \ast 0^{k-1} \ast \ldots \ast 0^{k-1} \ast 1 \\ 1^k \ast 1^{k-1} \ast 1^{k-1} \ast \ldots \ast 1^{k-1} \ast 0 \end{pmatrix},\]

where $\ast$ can be 0 or 1, but two $\ast$’s in the same column must be equal. It’s easy to see that $G^{(k)} \subset M^{(k)}$. Then the following key lemma is fairly straightforward to prove.

**Lemma 5.2.** The set $G^{(k)}$ freely generates $M^{(k)}$. That is, every element $\pi$ of $M^{(k)}$ can be written uniquely as a product of words in $G^{(k)}$.

We can now state the main (nonconjectural) result of this section.

**Theorem 5.3.** Let $v_2^{(k)}(n, t)$ denote the sum of the squares of the coefficients of the polynomial $\prod_{i=1}^{n} (1 + t x^{F^{(k)}_{i+k-1}})$. Then

\[
\sum_{n \geq 0} v_2^{(k)}(n, t) x^n = \frac{1 - t^{k-1}(1 + t^2)x^k}{1 - (1 + t^2)x - t^{k-1}(1 + t^2)x^k + t^{k-1}(1 + t^4)x^{k+1}}.
\]

**Proof.** Write $\ell(\pi)$ for the length of $\pi \in M^{(k)}$, and write $N(\pi)$ for the total number of 1’s in $\pi$. Note that $\ell(\pi\sigma) = \ell(\pi) + \ell(\sigma)$ and $N(\pi\sigma) = N(\pi) + N(\sigma)$. Define

\[
G^{(k)}(x) = \sum_{\pi \in G^{(k)}} t^{N(\pi)} x^{\ell(\pi)}.
\]

By a standard simple argument (see [11, §4.7.4]),

\[
\sum_{n \geq 0} v_2^{(k)}(n, t) x^n = \frac{1}{1 - G^{(k)}(x)}.
\]

We can use Lemma 5.2 to compute $G^{(k)}(x)$. The two generators in equation (5.2) contribute $(1 + t^2)x$ to $G^{(k)}(x)$. Now consider the
generators $\pi$ and $\sigma$ of equation (5.3). The two generators differ only by switching rows, so consider just $\pi$. Suppose there are $j \geq 0$ columns of *'s. The number of 1's in the remaining columns is $k + j(k - 1) + 1$. The length of $\pi$ is $(j + 1)k + 1$. Since each of the $j$ columns of *'s has zero or two 1's, the contribution to $G^{(k)}(x)$ of all generators (5.3) is $t j^{(k-1)+k+1}(1 + t^2)j x^{jk+k+1}$. The same is true of the second generator $\sigma$. Hence

$$G^{(k)}(x) = (1 + t^2)x + 2 \sum_{j \geq 0} t j^{(k-1)+k+1}(1 + t^2)j x^{jk+k+1} = (1 + t^2)x + \frac{2 t^{k+1} x^{k+1}}{1 - t^{k-1}(1 + t^2)x^k}.$$ 

It follows that

$$\sum_{n \geq 0} v_2^{(k)}(n, t)x^n = \frac{1}{1 - (1 + t^2)x - \frac{2 t^{k+1} x^{k+1}}{1 - t^{k-1}(1 + t^2)x^k}} = \frac{1 - t^{k-1}(1 + t^2)x^k}{1 - (1 + t^2)x - t^{k-1}(1 + t^2)x^k + t^{k-1}(1 + t^4)x^{k+1}}.$$ 

Naturally we can ask how the statement and proof of Theorem 5.3 can be extended. For any $r \geq 2$ let $v_r^{(k)}(n, t)$ denote the sum of the $r$th powers of the coefficients of the polynomial $\prod_{i=1}^{n} (1 + tx^{F^{(k)}_{i,k-1}})$. Define the monoid $M^{(k)}(r)$ analogously to $M^{(k)}$ by letting the elements of $M^{(k)}(r)$ be $r$-tuples of binary words of the same length such that $\sum_{i=1}^{n} a_i F^{(k)}_{i,k-1}$ is the same for all the $r$ words $a_1 a_2 \cdots a_n$. It’s easy to see that $M^{(k)}(r)$ is a free monoid, basically because if $\pi$ and $\sigma$ are $r$-tuples of binary words such that $\pi \in M^{(k)}(r)$ and $\pi \sigma \in M^{(k)}(r)$, then $\sigma \in M^{(k)}(r)$. However, finding the free generators of $M^{(k)}(r)$ seems complicated for $r \geq 3$, and we have not tried to do so. For $r = 3$ we have the following conjecture, due to Zeilberger [17] (verified by him for $k \leq 5$), a correction of the conjecture in the original version of the present paper. We use the notation $u = t^{k-1}$ and

$$J^{(k)}_r(t, x) = \sum_{n \geq 0} v_r^{(k)}(n, t)x^n.$$

**Conjecture 5.4.** We have

$$J^{(k)}_3(t, x) = \frac{1 - u(u + 1)(t^3 + 1)x^k + u^3(t^3 - 1)^2 x^{2k}}{D^{(k)}_3(t, x)},$$

where
We have Conjecture 5.5.

Note that the numerator in the above conjecture consists of the terms in the denominator with $x$-exponents $0, k, 2k$. For higher values of $r$ the coefficients seem to be more complicated. For instance, it seems that the coefficient of $x^4$ in the denominator of $J_4^{(k)}(t, x)$ is $t^2(t^{12} + t^{10} - t^8 - 4t^6 - t^4 + t^2 + 1)$. The factor $t^{12} + t^{10} - t^8 - 4t^6 - t^4 + t^2 + 1$ is irreducible over $\mathbb{Q}$. We give below some conjectures when $t = 1$.

**Conjecture 5.5.** We have

$$J_4^{(k)}(1, x) = \frac{1 - 7x^k - 2x^{2k}}{1 - 2x - 7x^k - 2x^{2k} + 2x^{2k+1}}$$

$$J_5^{(k)}(1, x) = \frac{1 - 11x^k - 20x^{2k}}{1 - 2x - 11x^k - 8x^{k+1} - 20x^{2k} + 10x^{2k+1}}$$

$$J_6^{(k)}(1, x) = \frac{1 - 17x^k - 88x^{2k} - 4x^{3k}}{D_6^{(k)}(1, x)}$$

$$J_7^{(k)}(1, x) = \frac{1 - 26x^k - 311x^{2k} - 84x^{3k}}{D_7^{(k)}(1, x)},$$

where

$$D_6^{(k)}(1, x) = 1 - 2x - 17x^k - 28x^{k+1} - 88x^{2k} + 6x^{2k+1} - 4x^{3k} + 4x^{3k+1}$$

and

$$D_7^{(k)}(1, x) = 1 - 2x - 26x^k - 74x^{k+1} - 311x^{2k} + 34x^{2k+1} - 84x^{3k} + 42x^{3k+1}.$$
and where \( a_i(t, u) \) is a polynomial in \( t \) and \( u \) (depending on \( r \) but independent from \( k \)) such that \( a_{2m+1}(t, t^k - 1) \neq 0 \), and possibly even \( a_i(t, t^k - 1) \neq 0 \) for all \( 0 \leq i \leq 2m + 1 \). Moreover (in order to account for the odd denominator degrees in equation (4.1)), the largest index \( j \) for which \( a_j(1, 1) \neq 0 \) is odd.

What other sequences satisfying linear recurrences with constant coefficients have interesting behavior related to this paper? We were unable to find any further recurrences with “nice” behavior. For instance, gfun fails to find rational generating functions (using the values for \( 0 \leq n \leq 40 \)) for the sum of the squares of the coefficients of \( \prod_{i=1}^n (1 + x f_i + 2) \), when either \( f_{i+1} = f_i + f_{i-2} \) or \( f_{i+1} = f_{i-1} + f_{i-2} \), with initial conditions \( f_1 = f_2 = f_3 = 1 \). Zeilberger, however, is much more adept at calculations, and he informs me (private communication, 24 March 2021), that for \( f_{i+1} = f_i + f_{i-2} \), the sum of the squares of the coefficients of \( \prod_{i=1}^n (1 + x f_{i+1}) \) has a generating function \( P(x)/Q(x) \) where \( \deg P(x) = 24 \) and \( \deg Q(x) = 25 \). For the sum of the cubes, the denominator has degree 88. Similarly, for \( f_{i+1} = f_{i-1} + f_{i-2} \) the denominator of the generating function for sum of the squares has degree 73.

Note that in these two cases, the unique real zeros \( \psi \) of the corresponding characteristic polynomials \( x^3 - x^2 - 1 \) and \( x^3 - x - 1 \) are PV numbers, i.e., they are real algebraic integers greater than 1 all of whose conjugates are less than 1 in absolute value. Similarly the unique positive real zeros of \( x^k - x^{k-1} - x^{k-2} - \cdots - x - 1, \ k \geq 2 \), are PV numbers. Thus Zeilberger [17, p. 14] conjectures that his algorithms for computing \( \sum u_\alpha(n)x^n \), corresponding to a C-finite sequence, terminates for all \( \alpha \) if and only if the the largest zero of the characteristic polynomial of the recurrence is a PV number. We can make the somewhat stronger conjecture that this condition on the recurrence is necessary and sufficient for \( \sum u_\alpha(n)x^n \) to be a rational function for all \( \alpha \).

6. Congruence properties

For \( 0 \leq a < m \), let \( g_{m,a}(n) \) denote the number of coefficients of \( S_n(x) \) (defined by equation (1.1)) that are congruent to \( a \) modulo \( m \). Reznick [6] showed that the generating function

\[
G_{m,a}(x) = \sum_{n \geq 0} g_{m,a}(n)x^n
\]

is rational. See also [13, pp. 28–37], where some open questions are on page 32. In particular, the denominator of \( G_{m,a}(x) \) has quite a bit
of factorization that remains unexplained. (For some small progress related to the denominator factorization, see Bogdanov [1].) For the proof that \( G_{m,a}(x) \) is rational, it is necessary to introduce auxiliary generating functions \( G_{m,a,b}(x) = \sum_{n \geq 0} g_{m,a,b}(n) x^n \), where \( g_{m,a,b}(n) \) is equal to the number of integers \( 0 \leq k \leq \deg S_n(x) \) for which \( \lfloor n \rfloor k \equiv a \pmod{m} \) and \( \lfloor n \rfloor k+1 \equiv b \pmod{m} \).

We can do something analogous for the Fibonacci triangle. For \( 0 \leq a < m \), let \( h_{m,a}(n) \) denote the number of coefficients of \( I_n(x) \) (defined by equation (2.1)) that are congruent to \( a \) modulo \( m \). Define

\[
H_{m,a}(x) = \sum_{n \geq 0} h_{m,a}(n) x^n.
\]

The proof sketched in [13] that \( G_{m,a}(x) \) is rational carries over, mutatis mutandis, to \( H_{m,a}(x) \). As in the proof for \( G_{m,a}(x) \), we need to introduce some auxiliary generating functions that take into account consecutive coefficients of \( I_n(x) \). However, we need also specify whether these coefficients are the beginning, middle, or end of a string (as defined in Section 2). Thus we will have numbers like \( g^{(3,1)}_{m,a,b}(n) \) which count the number of integers \( 0 \leq k \leq \deg I_n(x) \) for which \( \lfloor n \rfloor k \) ends a string and satisfies \( \lfloor n \rfloor k \equiv a \pmod{m} \), while \( \lfloor n \rfloor k+1 \) begins a string and satisfies \( \lfloor n \rfloor k+1 \equiv b \pmod{m} \). When these procedures are carried out we obtain the following result.

**Theorem 6.1.** The generating function \( H_{m,a}(x) \) is rational.

Naturally we would like to say more about \( H_{m,a}(x) \) than just its rationality. Here are some values suggested by gfun. None have been proved.
\[ H_{2,0}(x) = \frac{x^3(1 - 2x^2)}{(1 - x)(1 - x - x^2)(1 - 2x + 2x^2 - 2x^3)} \]
\[ H_{2,1}(x) = \frac{1 + 2x^2}{1 - 2x + 2x^2 - 2x^3} \]
\[ H_{3,0}(x) = \frac{2x^5(1 - 2x^2)}{(1 - x)(1 - x - x^2)(1 - 2x + 2x^2 - 3x^3 + 4x^4 - 4x^5)} \]
\[ H_{3,1}(x) = \frac{1 - 2x + 4x^2 - 6x^3 + 8x^4 - 10x^5 + 8x^6 - 6x^7}{(1 - x)(1 - x + x^2)(1 - 2x + 2x^2 - 3x^3 + 4x^4 - 4x^5)} \]
\[ H_{3,2}(x) = \frac{x^3(1 + 2x^4)}{(1 - x)(1 - x + x^2)(1 - 2x + 2x^2 - 3x^3 + 4x^4 - 4x^5)} \]
\[ H_{4,0}(x) = \frac{x^6(1 - 2x^2)(1 - 3x^2 + 4x^3 - 4x^4)}{(1 - x)(1 - x - x^2)(1 - x^2 + 2x^4)(1 - 2x + 2x^2 - 2x^3)^2} \]
\[ H_{4,1}(x) = \frac{(1 - x - 2x^2 + 5x^3 - 8x^4 - 10x^5 + 12x^6 - 12x^7)}{(1 - x)(1 - x^2 + 2x^4)(1 - 2x + 2x^2 - 2x^3)^2} \]
\[ H_{4,2}(x) = \frac{x^3(1 + x^2)(1 - 2x^2)}{(1 - x^2 + 2x^4)(1 - 2x + 2x^2 - 2x^3)^2} \]
\[ H_{4,3}(x) = \frac{2x^5(1 + x^2)}{(1 - x)(1 - 2x + 2x^2 - 2x^3)(1 - x + 2x^2 - 2x^3 + 2x^4)} \]

Note that just as for \( G_{m,a}(x) \), there is a lot of denominator factorization. Moreover, some of the numerators of \( H_{m,a}(x) \) have only two terms, in analogy to some numerators of \( G_{m,a}(x) \) having just one term.

We can try to extend Theorem 6.1 to

\[ I_n^{(k)}(x) = \prod_{i=1}^{n} \left( 1 + x^{r_{i+k-1}} \right). \]

For \( 0 \leq a < m \), let \( h_{m,a}^{(k)}(n) \) denote the number of coefficients of \( I_n^{(k)}(x) \) that are congruent to \( a \) modulo \( m \). Define

\[ H_{m,a}^{(k)}(x) = \sum_{n \geq 0} h_{m,a}^{(k)}(n)x^n. \]

**Conjecture 6.2.** The generating function \( H_{m,a}^{(k)}(x) \) is rational.

We have some scanty evidence for a “congruence analogue” of Conjecture 5.6. For \( (m, a) = (2, 1) \) we found enough evidence to conjecture the following.
Conjecture 6.3. We have

\[ H_{2,1}^{(k)}(x) = \frac{1 + 2x^k}{1 - 2x + 2x^k - 2x^{k+1}}. \]

For \((m, a) = (3, 1)\) gfun suggests the following:

\[ H_{3,1}^{(2)}(x) = \frac{1 - 2x + 4x^2 - 6x^3 + 8x^4 - 10x^5 + 8x^6 - 6x^7}{1 - 4x + 8x^2 - 12x^3 + 16x^4 - 20x^5 + 19x^6 - 12x^7 + 4x^8}, \]

\[ H_{3,1}^{(3)}(x) = \frac{1 - 2x + 4x^3 - 6x^4 + 8x^6 - 10x^7 + 8x^9 - 6x^{10}}{D(x)}, \]

where

\[ D(x) = 1 - 4x + 4x^2 + 4x^3 - 12x^4 + 8x^5 + 8x^6 - 20x^7 + 11x^8 + 8x^9 - 12x^{10} + 4x^{11}. \]

The connection between the two numerators is obvious. Note the denominator coefficients of \(H_{3,1}^{(2)}(x)\) are obtained from those of \(H_{3,1}^{(3)}(x)\) by adding the coefficients of the pairs \((x^2, x^3)\), \((x^5, x^6)\) and \((x^8, x^9)\), keeping the other coefficients unchanged. It shouldn’t be difficult to come up with more general conjectures. Even better, of course, would be some theorems!

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