Tunneling of slow quantum packets through the high Coulomb barrier

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Abstract

We study the tunneling of slow quantum packets through a high Coulomb barrier. We show that the transmission coefficient can be quite different from the standard expression obtained in the plane wave (WKB) approximation (and larger by many orders of magnitude), even if the momentum dispersion is much smaller than the mean value of the momentum.

Keywords: Coulomb barrier, Quantum packets, Transmission coefficient

1. Introduction

One of the most striking properties of the quantum world is the tunneling effect [1, 2]. The well known quasiclassical formula for the transmission probability through the potential barrier \( U(x) \) reads

\[
D \sim \exp \left( -2 \int_{1}^{\infty} \sqrt{2m[U(x) - E]} dx / \hbar \right),
\]

where \( \hbar \) is the Planck constant, \( m \) is the mass of particle and \( E \) is its energy. But formula (1) was derived for idealized quantum states with a well defined energy, represented by the plane wave having infinite extension in the coordinate space. A more realistic approximation is to consider the tunneling of wave packets. Then one has to average the coefficient \( D(p) \) over the wave function \( \varphi(p) \) of the quantum packet in the momentum space, taking into account only the plane waves going in the direction of the barrier:

\[
T = \int_{0}^{\infty} D(p)|\varphi(p)|^2 dp.
\]

This was done for very narrow superpositions in the momentum space as far back as in Ref. [3]. On the other hand, it was shown later that various new effects can arise in scattering and tunneling of packets which are not extremely narrow in the momentum space (in particular, narrow in the coordinate space) [5, 6, 7, 8]. Here we consider this problem for the Coulomb potential barrier, which has numerous applications, especially for the fusion and radioactive decay phenomena. We make emphasis on the transmission probabilities of slow particles, because this regime attracted significant attention in attempts to explain experimental data related to low energy nuclear reactions [2, 10, 11, 12, 13, 14]. It seems obvious that the spread of the packet in momentum space should result in the increase of the barrier transparency, due to the enhanced contribution of the plane wave components with high values of momenta. What is not so obvious (at least, unexpected), it is the fact that even small dispersions of the momentum can result in increase of the transmission coefficient by many orders of magnitude. This is the motivation for writing this article.

2. Tunneling of wave packets

We confine ourselves to the idealized barrier in a single space dimension, \( U(x) = Ze^2 / x \) for \( x > 0 \) (and \( U(x) = 0 \) for \( x < 0 \)). The integral (1) is well known in this case (we omit a possible pre-exponential factor):

\[
D(p) = \exp (-a/p), \quad a = 2\pi Ze^2 m/\hbar,
\]

where \( p \) is the linear momentum. We know that the standard WKB approximate formula (1) needs modifications for the long-range potentials, such as the Coulomb one, but this is not essential for our purposes. Let us simply assume that \( D(p) \) is given by Eq. (2), and let us see what can happen for different wave packets.

If the packet is concentrated near the mean value \( p_0 \) and its spread can be well characterized by the momentum variance \( \sigma_p \), then integral (2) gives the value close to \( D(p_0) = \exp (-A) \) under the condition

\[
A \sqrt{B} \ll 1,
\]

where

\[
A = \frac{a}{p_0} = \frac{2\pi Ze^2}{\hbar v_0}, \quad B = \frac{\sigma_p^2}{p_0^2}
\]

\((v_0 = p_0/m)\) is the mean velocity of the wave packet). If parameter \( A \) is not very big (for example, \( A \sim 10 \) for deuterons with the kinetic energy of the order of 10 KeV), then the plane wave transmission formula can be used under the simple and obvious condition \( B \ll 1 \). But the situation is different for slow packets, when parameter \( A \) can be very large. For example, for deuterons with the kinetic energy of the order of few eV or smaller (i.e.,
for temperatures of the order of $10^3$ K, parameter $A$ can assume the values of the order of several hundred or thousand.

To see what can happen if condition (4) is broken, we consider the family of packets

$$|\varphi(p)|^2 = \frac{N}{\sigma_p} \exp \left[ -\beta \left( \frac{P - P_0}{\sigma_p} \right)^2 \right], \quad (6)$$

$$N = \frac{\gamma \Gamma(3/\gamma)^{1/2}}{2\Gamma(1/\gamma)^{3/2}}, \quad \beta = \frac{(3/\gamma)^{1/2}}{\Gamma(1/\gamma)},$$

where $\gamma$ is some positive constant and $\Gamma(z)$ is the Gamma function. In particular, $\beta = \sqrt{2}$ for $\gamma = 1$ and $\beta = 1/2$ for $\gamma = 2$.

Putting function (6) in Eq. (2), we obtain the total transmission probability

$$T(A, B) = \frac{N}{\sqrt{B}} \int_0^{\infty} \exp \left[ -\frac{A}{\gamma} - \beta \left( \frac{y - 1}{B} \right)^{1/2} \right] dy. \quad (7)$$

The results of numerical calculations of the integral (7) for $\gamma = 2$ (Gaussian packets) and $\gamma = 1$ are demonstrated in Figs. 1 and 2. We see that the transmission coefficient can be many orders of magnitude bigger than the plane wave approximation value for $B \sim 10^{-3}$ (if $\gamma = 2$) and even for $B \sim 10^{-5}$ (if $\gamma = 1$).

For $A \gg 1$, we can look for an approximate analytical expression for the integral (7), using the steepest descent method. The saddle point $y_s > 1$ is the solution to the equation

$$\frac{G}{y^\gamma} = (y - 1)^{\gamma - 1}, \quad G = AB^{1/2}/(\gamma \beta). \quad (8)$$

An approximate solution to Eq. (8) can be obtained for $G \gg 1$:

$$y_s \approx G^{(\gamma + 1)/\gamma} + \frac{\gamma - 1}{\gamma + 1}. \quad (9)$$

Then standard formulas of the steepest descent method lead to the approximate expression

$$T_s \approx N \sqrt{\frac{2\pi}{\gamma + 1}} (\gamma \beta)^{-\gamma/2} \left( \frac{B}{A^2} \right)^{1/(2\gamma)} \left( \gamma + 1 \right)^{\gamma/2}. \quad (10)$$

The ratio $R = T_s/T$ of approximate and exact numerical values of the transmittivity of packets with $A = 700$ and different values of parameter $\gamma$. We see that for $\gamma > 3/2$ and $A = 700$, the steepest descent method gives a good approximation, provided $B \gtrsim 1$. This is the consequence of the condition $G^{\gamma/(\gamma + 1)} \gg 1$: see Eq. (9). For example, for $A = 700$ and $B = 0.1$ we have $G \approx 70^{1/2} \approx 4$, which is not big number.

On the other hand, formula (10) gives an excellent approximation to the numerical results for $\gamma = 1$ [when (9) is the exact solution to Eq. (8)], even for very small values of parameter $B$. To understand why this happens, we notice that for $\gamma = 1$ and $A \gg 1$, the absolute value $|y - 1|$ in the argument of the integral (7) can be replaced by the difference $y - 1$ in the whole interval.
$0 < y < \infty$. The error due to incorrect values of this function in the interval $0 < y < 1$ is strongly suppressed by the factor $\exp(-A/y)$ in this interval. Then integral (7) can be calculated exactly, in view of formula 3.324.1 from [15]:

$$
\int_0^\infty \exp(-\xi y - \eta y) \, dy = 2\sqrt{\xi/\eta}K_1(2\sqrt{\xi\eta}).
$$

(11)

where $K_1(z)$ is the Bessel function of the first kind. Using the known asymptotical formula $K_1(z) \approx \sqrt{\pi/(2z)} \exp(-z)$ for $z \gg 1$ and calculating all coefficients, we arrive at the formula

$$
T(A, B) \approx N \sqrt{\pi} \left[ A^2/(8B) \right]^{1/8} \exp \left[ -2 \left( \frac{A^2}{B} \right)^{1/4} + \sqrt{\frac{2}{B}} \right],
$$

(12)

which coincides exactly with (10) for $\gamma = 1$.

It was supposed some time ago [16], that the transmission probability for the so called correlated wave packets (with nonzero correlation coefficient between the coordinate and momentum $r = \sigma_{\rho p}/\sqrt{\sigma_p \sigma_{\rho p}}$) can be higher than for uncorrelated packets, and that the increase of this probability can be described (at least approximately) by means of replacing the true Planck constant by the effective constant $\hbar_{eff} = \hbar/\sqrt{1 - r^2}$. This question was studied later in [5, 11, 12, 13, 17]. The results of that papers show that in majority of cases, the nonzero correlation coefficient does increase the probability of tunneling (although there exist specific configurations, where the probability can diminish [5]). But the concept of effective Planck constant seems to not work for the Coulomb potential (at least in the case of tunneling of free wave packets, considered in this article). Indeed, for the Gaussian packets ($y = 2$) with a fixed coordinate variance $\sigma_x$, the momentum variance increases with the correlation coefficient as $\sigma_p = \sigma_{\rho p}/\sqrt{1 - r^2}$. Strongly correlated states ($r \to 1$) have large momentum variances. In this case, Eq. (10) yields

$$
T \sim \exp \left[ -(3/2) \left( a^2/\sigma_p \right)^{1/3} \right] = [T(r = 0)](1 - r^2)^{1/3},
$$

and the last expression is rather different from the formula $T(r) \sim [T(r = 0)](1 - r^2)^{1/2}$, suggested in [12, 13] on the basis of the concept of effective Planck constant.

3. Conclusions

Our main conclusion is that for high Coulomb barriers and slow particles, it is impossible to evaluate the transmission probability without precise knowledge of the concrete shape of wave packet (for example, the value of parameter $\gamma$ in the simplest cases). The plane wave approximation can result in underestimating the real transparency by many orders of magnitude. For some families of packets (such as (6) with $\gamma \approx 1$), the transmission coefficient can be much bigger than that given by the plane wave approximation, even for very small ratios $B = \sigma_p/\rho_p \approx 10^{-5}$.

Note that in the regimes of validity of approximation (10) (especially for $\gamma \approx 0$), the transmission probability depends mainly on the ratio $A^2/B = a^2/\sigma_p$, so that it depends not on the mean value of momentum $\rho_B$ or total particle energy ($E = (\rho_B^2 + \sigma_p^2)/(2m)$, but on the momentum dispersion only. This result seems to be quite unexpected.

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References

[1] G. Gamow, Z. Phys. 51 (1928) 204.
[2] R. W. Gurney, E. W. Condon, Phys. Rev. 33 (1929) 127.
[3] L. A. MacColl, Phys. Rev. 40 (1932) 621.
[4] M. A. Andreata, V. V. Dodonov, J. Phys. A 35 (2002) 8373.
[5] V. V. Dodonov, A. B. Klimov, V. I. Man’ko, Phys. Lett. A 220 (1996) 41.
[6] B. B. Kadomtsev, M. B. Kadomtsev, Phys. Lett. A 225 (1997) 303.
[7] V. K. Ignatovich, Phys. Lett. A 322 (2004) 36.
[8] M. A. Andreata, V. V. Dodonov, J. Phys. A 37 (2004) 2423.
[9] A. B. Balantekin, N. Takigawa, Rev. Mod. Phys. 70 (1998) 77.
[10] E. Storms, Naturwissenschaften 97 (2010) 861.
[11] V. I. Vysotskii, S. V. Adamenko, Zh. Tekh. Fiz. 80 (2010) 23 [ Tech. Phys. 55 (2010) 613].
[12] V. I. Vysotskii, M. V. Vysotsky, S. V. Adamenko, Zh. Exp. Theor. Fiz. 141 (2012) 276 [ J. Exp. Theor. Phys. 114 (2012) 243].
[13] V. I. Vysotskii, M. V. Vysotsky, Eur. Phys. J. A 49 (2013) 99.
[14] B. Ivlev, Phys. Rev. C 87 (2013) 034619.
[15] I. S. Gradshtein, I. M. Ryzhik, Tables of Integrals, Series and Products, Academic, New York, 1994.
[16] V. V. Dodonov, E. V. Kurnyshov, V. I. Man’ko, Phys. Lett. A 79 (1980) 150.
[17] V. V. Dodonov, A. V. Dodonov, arXiv:1401.4160 (2014).