DESCENTS OF UNIPOTENT REPRESENTATIONS OF FINITE UNITARY GROUPS

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Abstract. Inspired by the Gan-Gross-Prasad conjecture and the descent problem for classical groups, in this paper we study the descents of unipotent representations of unitary groups over finite fields. We give the first descents of unipotent representations explicitly, which are unipotent as well. Our results include both the Bessel case and Fourier-Jacobi case, which are related via theta correspondence.

1. Introduction

In representation theory, a classical problem is to look for the spectral decomposition of a representation $\pi$ of a group $G$ restricted to a subgroup $H$. Namely, one asks for which representation $\sigma$ of $H$ has the property that

$$\text{Hom}_H(\pi, \sigma) \neq 0,$$

and what the dimension of this Hom-space is. In general such a restriction problem is hard and may not have reasonable answers. However when $G$ is a classical group defined over a local field and $\pi$ belongs to a generic Vogan $L$-packet, the local Gan-Gross-Prasad conjecture [GP1, GP2, GGP1] provides explicit answers and is one of the most successful examples concerning with those general questions. To be a little more precise, the multiplicity one property holds for this situation, namely

$$m(\pi, \sigma) := \dim \text{Hom}_H(\pi, \sigma) \leq 1,$$

and the invariants attached to $\pi$ and $\sigma$ that detect the multiplicity $m(\pi, \sigma)$ is the local root number associated to their Langlands parameters. In the $p$-adic case, the local Gan-Gross-Prasad conjecture has been resolved by J.-L. Waldspurger and C. Mœglin and J.-L. Waldspurger [W1, W2, W3, MW] for orthogonal groups, by R. Beuzart-Plessis [BP1, BP2] and W. T. Gan and A. Ichino [GI] for unitary groups, and by H. Atobe [Ato] for symplectic-metaplectic groups. On the other hand, D. Jiang and L. Zhang [JZ1] study the local descents for $p$-adic orthogonal groups, whose results can be viewed as a refinement of the local Gan-Gross-Prasad conjecture, and the descent method has important applications towards the global problem (see [JZ2]).

Motivated by the above works, in this paper which is the first one of a series, we will study the descent problem for unipotent representations of finite unitary groups. To begin with, we first set up some notations. Let $\mathbb{F}_q$ be an algebraic closure of a finite field $\mathbb{F}_q$, which is of characteristic $p > 2$. Let $G = U_n$ be an $\mathbb{F}_q$-rational form of $GL_n(\mathbb{F}_q)$, and $F$ be the corresponding Frobenius endomorphism, so that the group of $\mathbb{F}_q$-rational points is $G^F = U_n(\mathbb{F}_q)$. For convenience, we also occasionally write $G = G(\mathbb{F}_q) = GL_n(\mathbb{F}_q)$ for the $\mathbb{F}_q$-points of $G$, when no confusion arises. For an $F$-stable maximal torus $T$ of $G$ and a character $\theta$ of $T^F$, let $R_{T, \theta}^G$ be the virtual character of $G^F$ defined by P. Deligne and G. Lusztig in their seminal work [DL]. An irreducible representation $\pi$
is called unipotent if there is an $F$-stable maximal torus $T$ of $G$ such that $\pi$ appears in $R_T^G$. For two representations $\pi$ and $\pi'$ of a finite group $H$, define

\[ \langle \pi, \pi' \rangle_H := \dim \text{Hom}_H(\pi, \pi'). \]

Let $\pi$ and $\pi'$ be irreducible representations of $U_n(\mathbb{F}_q)$ and $U_m(\mathbb{F}_q)$ respectively, where $n \geq m$. The Gan-Gross-Prasad conjecture is concerned with the multiplicity

\[ m(\pi, \pi') := \langle \pi \otimes \nu, \pi' \rangle_{H(\mathbb{F}_q)} = \dim \text{Hom}_{H(\mathbb{F}_q)}(\pi \otimes \nu, \pi') \]

where the data $(H, \nu)$ is defined as in [GGP1, Theorem 15.1], and will be explained in details shortly. According to whether $n - m$ is odd or even, the above-Hom space is called the Bessel model or Fourier-Jacobi model. In [GGP2, Proposition 5.3], W. T. Gan, B. H. Gross and D. Prasad showed that if both $\pi$ and $\pi'$ are cuspidal, then

\[ m(\pi, \pi') \leq 1. \]

We should mention that our formulation of multiplicities differs slightly from that in the Gan-Gross-Prasad conjecture [GGP1], up to taking the contragradient of $\pi'$. This is more suitable for the purpose of descents, which will be clear from the discussion below. On the other hand, in this paper we will restrict our attention to unipotent representations of $U_n(\mathbb{F}_q)$, which are self-dual (see Proposition 6.4) and thus for $\pi$ unipotent the above Hom-space coincides with $\text{Hom}_{H(\mathbb{F}_q)}(\pi \otimes \pi', \nu)$.

Roughly speaking, for fixed $U_n(\mathbb{F}_q)$ and its representation $\pi$, the descent problem seeks the smallest member $U_m(\mathbb{F}_q)$ among a Witt tower which has an irreducible representation $\pi'$ satisfying $m(\pi, \pi') \neq 0$, and all such $\pi'$ give the first descent of $\pi$. To give the precise notion of descent, let us sketch the definition of the data $(H, \nu)$ following [GGP1] and [JZ1].

We first consider the Bessel model. Let $V_n$ be an $n$-dimensional space over $\mathbb{F}_{q^2}$ with a nondegenerate Hermitian form $(,)$. Consider pairs of Hermitian spaces $V_n \supset V_{n-2\ell-1}$ and the following partitions of $n$,

\[ \mathcal{P}_\ell = [2\ell + 1, 1^{n-2\ell-1}], \quad 0 \leq \ell < n/2. \]

Assume that $V_n$ has a decomposition

\[ V_n = X + V_{n-2\ell} + X^\vee \]

where $X + X^\vee = V_{n-2\ell}$ is a polarization. Let $\{e_1, \ldots, e_\ell\}$ be a basis of $X$, $\{e'_1, \ldots, e'_\ell\}$ be the dual basis of $X^\vee$, and let $X_i = \text{Span}_{\mathbb{F}_{q^2}}(e_1, \ldots, e_i)$, $i = 1, \ldots, \ell$. Let $P$ be the parabolic subgroup of $U_n$ stabilizing the flag

\[ X_1 \subset \cdots \subset X_\ell, \]

so that its Levi component is $M \cong \text{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \text{GL}_1 \times U_{n-2\ell}$. Its unipotent radical can be written as

\[ N_{\mathcal{P}_\ell} = \left\{ n = \begin{pmatrix} z & y & x \\ 0 & I_{n-2\ell} & y^* \\ 0 & 0 & z^* \end{pmatrix} : z \in U_{G_\ell} \right\}, \]

where the superscript $*$ means the conjugate transpose, and $U_{G_\ell}$ is the subgroup of unipotent upper triangular matrices of $G_\ell := \text{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \text{GL}_\ell$. Fix a nontrivial additive character $\psi_{\mathbb{F}_{q^2}}$ of $\mathbb{F}_{q^2}$. Pick up an anisotropic vector $v_0 \in V_{n-2\ell}$ and define a generic character $\psi_{\mathcal{P}, v_0}$ of $N_{\mathcal{P}}(\mathbb{F}_q)$ by

\[ \psi_{\mathcal{P}, v_0}(n) = \psi_{\mathbb{F}_{q^2}} \left( \sum_{i=1}^{\ell-1} z_{i,i+1} + (y_\ell, v_0) \right), \quad n \in N_{\mathcal{P}}(\mathbb{F}_q), \]
where $y$ is the last row of $y$. The stabilizer of $\psi_{\mathcal{P}_\ell, v_0}$ in $M(\mathbb{F}_q)$ is the unitary group of the orthogonal complement of $v_0$ in $V_{n-2\ell}$, which will be identified with $U_{n-2\ell-1}(\mathbb{F}_q)$. Put

$$H = U_{n-2\ell-1} \ltimes N_{\mathcal{P}_\ell}, \ \nu = \psi_{\mathcal{P}_\ell, v_0}.$$  

For irreducible representations $\pi$ and $\pi'$ of $U_n(\mathbb{F}_q)$ and $U_{n-2\ell-1}(\mathbb{F}_q)$ respectively, the uniqueness of Bessel models asserts that

$$m(\pi, \pi') = \dim \text{Hom}_{\mathbb{F}_q}(\pi \otimes \bar{\nu}, \pi') \leq 1.$$  

This was proved over $p$-adic local fields in [AGRS], and for cuspidal representations over finite fields in [GGP2]. It is clear that

$$\text{Hom}_{\mathbb{F}_q}(\pi \otimes \bar{\nu}, \pi') \cong \text{Hom}_{U_{n-2\ell-1}(\mathbb{F}_q)}(\mathcal{J}_\ell(\pi), \pi'),$$

where $\mathcal{J}_\ell(\pi)$ is the twisted Jacquet module of $\pi$ with respect to $(N_{\mathcal{P}_\ell}(\mathbb{F}_q), \psi_{\mathcal{P}_\ell, v_0})$. We simply define the notion of the $\ell$-th Bessel quotient of $\pi$ by

$$Q^B_\ell(\pi) := \mathcal{J}_\ell(\pi),$$

viewed as a representation of $U_{n-2\ell-1}(\mathbb{F}_q)$. Define the first occurrence index $\ell_0 := \ell^B_0(\pi)$ of $\pi$ in the Bessel case to be the largest nonnegative integer $\ell_0 < n/2$ such that $Q^B_{\ell_0}(\pi) \neq 0$. The $\ell_0$-th Bessel descent of $\pi$ is called the first Bessel descent of $\pi$ or simply the Bessel descent of $\pi$, denoted

$$D^{B}_{\ell_0}(\pi) := Q^B_{\ell_0}(\pi).$$

We next turn to the Fourier-Jacobi case. In the sequel we keep all the previous notations in the Bessel case, but view $V_n$ as a skew-Hermitian space by abuse of notation, which gives the unitary group $U_n(\mathbb{F}_q)$ (up to isomorphism) as well. Consider pairs of skew-Hermitian spaces $V_n \supset V_{n-2\ell}$ and partitions

$$\mathcal{P}_\ell = [n-2\ell, 1^{2\ell}], \ \ 0 \leq \ell \leq n/2.$$  

Note that if we let $P_\ell$ be the parabolic subgroup of $U_n$ stabilizing $X_\ell$ and let $N_\ell$ be its unipotent radical, then $N_{\mathcal{P}_\ell} = U_{G_\ell} \ltimes N_\ell$. Fix a nontrivial additive character $\psi$ of $\mathbb{F}_q$, and let $\omega_\psi$ be the Weil representation (see [Ger]) of $U_{n-2\ell}(\mathbb{F}_q) \ltimes \mathcal{H}_{n-2\ell}$, where $\mathcal{H}_{n-2\ell}$ is the Heisenberg group of $V_{n-2\ell}$. Roughly speaking, there is a natural homomorphism $N_\ell(\mathbb{F}_q) \rightarrow \mathcal{H}_{n-2\ell}$ invariant under the conjugation action of $U_{G_\ell}(\mathbb{F}_q)$ on $N_\ell(\mathbb{F}_q)$, which enables us to view $\omega_\psi$ as a representation of $U_{n-2\ell}(\mathbb{F}_q) \ltimes N_{\mathcal{P}_\ell}(\mathbb{F}_q)$. Let $\psi_\ell$ be the character of $U_{G_\ell}(\mathbb{F}_q)$ given by

$$\psi_\ell(z) = \psi \circ \text{Tr}_{\mathbb{F}_q^2/\mathbb{F}_q} \left( \sum_{i=1}^{\ell-1} z_{i,i+1} \right), \ z \in U_{G_\ell}(\mathbb{F}_q).$$

For the Fourier-Jacobi case, put

$$H = U_{n-2\ell} \ltimes N_{\mathcal{P}_\ell}, \ \nu = \omega_\psi \otimes \psi_\ell.$$  

For irreducible representations $\pi$ and $\pi'$ of $U_n(\mathbb{F}_q)$ and $U_{n-2\ell}(\mathbb{F}_q)$ respectively, the uniqueness of Fourier-Jacobi models asserts that

$$m(\pi, \pi') := \text{Hom}_{\mathbb{F}_q}(\pi \otimes \bar{\nu}, \pi') \leq 1.$$  

This was proved over $p$-adic local fields in [Su]. We observe that

$$\text{Hom}_{\mathbb{F}_q}(\pi \otimes \bar{\nu}, \pi') \cong \text{Hom}_{U_{n-2\ell}(\mathbb{F}_q)}(\mathcal{J}'_\ell(\pi \otimes \bar{\omega}_\psi), \pi'),$$

where $\mathcal{J}'_\ell(\pi \otimes \bar{\omega})$ is the twisted Jacquet module of $\pi \otimes \bar{\omega}_\psi$ with respect to $(N_{\mathcal{P}_\ell}(\mathbb{F}_q), \psi_\ell)$. Define the $\ell$-th Fourier-Jacobi quotient of $\pi$ to be

$$Q^F_{\ell}(\pi) := \mathcal{J}'_\ell(\pi \otimes \bar{\omega}),$$
viewed as a representation of $U_{n-2}(\mathbb{F}_q)$. Define the first occurrence index $\ell_0 := \ell_0^{FJ}(\pi)$ of $\pi$ in the Fourier-Jacobi case to be the largest nonnegative integer $\ell_0 \leq n/2$ such that $Q^{FJ}_{\ell_0}(\pi) \neq 0$. The $\ell_0$-th Fourier-Jacobi descent of $\pi$ is called the first Fourier-Jacobi descent of $\pi$ or simply the Fourier-Jacobi descent of $\pi$, denoted

$$D^{FJ}_{\ell_0}(\pi) := Q^{FJ}_{\ell_0}(\pi).$$

Recall from [LS] that irreducible unipotent representations of $U_n(\mathbb{F}_q)$ are parameterized by irreducible representations of $S_n$. It is well-known that the latter are parameterized by partitions of $n$. For a partition $\lambda$ of $n$, denote by $\pi_{\lambda}$ the corresponding unipotent representation of $U_n(\mathbb{F}_q)$. As is standard, we realize partitions as Young diagrams. Then our main result is the following.

**Theorem 1.1.** Let $\lambda$ be a partition of $n$ into $k$ rows, and $\lambda'$ be the partition of $n - k$ obtained by removing the first column of $\lambda$. Then the following hold.

(i) $\ell_0^B(\pi_{\lambda}) \leq k - \frac{1}{2}$, with equality hold if $k$ is odd, in which case $D^B_{\ell_0}(\pi_{\lambda}) = \pi_{\lambda'}$.

(ii) $\ell_0^{FJ}(\pi_{\lambda}) \leq \frac{k}{2}$, with equality hold if $k$ is even, in which case $D^{FJ}_{\ell_0}(\pi_{\lambda}) = \pi_{\lambda'}$.

Our result does not address the Bessel descent of $\pi_{\lambda}$ for $k$ even, nor the Fourier-Jacobi descent for $k$ odd. However, it is sufficient for some further applications (e.g. towards the wavefront set, cf. [JZ1, Conjecture 1.8]), noting that

$$k = \max\{2\ell_{0}^{B}(\pi_{\lambda}) + 1, 2\ell_{0}^{FJ}(\pi_{\lambda})\}.$$ 

As a special case, recall that $U_n(\mathbb{F}_q), n = k(k + 1)/2$ are the only unitary groups that possess unipotent cuspidal representations, and each of them has a unique unipotent cuspidal representation $\pi_k$, which corresponds to the partition $[k, k - 1, \ldots, 1]$. Then we obtain the following immediate consequence, which was communicated to us by L. Zhang.

**Corollary 1.2.** Let $\pi_k$ be the unique unipotent cuspidal representation of $U_n(\mathbb{F}_q), n = k(k + 1)/2$. Then the following hold.

(i) If $k$ is odd, then $\ell_0^B(\pi_k) = k - \frac{1}{2}$ and $D^B_{\ell_0}(\pi_k) = \pi_{k-1}$.

(ii) If $k$ is even, then $\ell_0^{FJ}(\pi_k) = \frac{k}{2}$ and $D^{FJ}_{\ell_0}(\pi_k) = \pi_{k-1}$.

We will prove the equivalent forms of Theorem 1.1 for the Bessel case and Fourier-Jacobi cases in Theorem 5.1 and Theorem 6.5 respectively. Let us outline the strategy of the proof. First of all, Proposition 5.2 and Proposition 6.3 show that parabolic induction preserves multiplicities, which are finite field analogs of Theorem 15.1 and Theorem 16.1 in [GGP1] respectively for unipotent representations. This reduces the calculation to the basic case. For Bessel models, in order to compute the right hand side of the equality

$$m(\pi, \pi') = (\pi \otimes \nu', \pi')_{H(\mathbb{F}_q)} = (I_{P}^{U_{n+1}}(\tau \otimes \pi'), \pi)_{U_n(\mathbb{F}_q)}$$

in Proposition 5.2, we first extend Reeder’s multiplicity formula in [R] for Deligne-Lusztig representations from connected simple algebraic groups to unitary groups. This is our main tool. However, explicit calculation with Reeder’s formula is still quite involved. It will be accomplished in Section 5, which is the most technical part of this paper. Following [GI, At], the Bessel case and Fourier-Jacobi case of the Gan-Gross-Prasad conjecture are related via theta correspondence. By see-saw arguments, we are able to resolve the descent problem for the latter from the former, using explicit theta correspondence between unipotent representations given in [AMR].

A few remarks are in order.
• A further delicate application of theta correspondence actually resolves the Gan-Gross-Prasad conjecture for unipotent representations of finite unitary groups. We have decided to postpone this part to a subsequent paper due to its different flavor. In that paper we will also study the Gan-Gross-Prasad conjecture and descent problem for unipotent cuspidal representations of finite orthogonal and symplectic groups. Overall, theta correspondence and Reeder’s formula will be our main tools.

• Using the method developed in [GRS], by composition of descents one should be able to read the wavefront set of representations of finite classical groups (cf. [JZ1, Conjecture 1.8] for the p-adic case). For unitary groups, the nilpotent orbits therein can be recovered from the largest parts of $p_j$ and $p_j'$ given by (1.1) and (1.5) respectively at first occurrence indices of the descents. For instance, the wavefront set of a unipotent representation $\pi_\lambda$ of $U_n(\mathbb{F}_q)$ is the singleton $\{\lambda\}$.

• In an ongoing joint work of the first named author with D. Jiang and L. Zhang, local descents are established for classical groups over local fields (cf. [JLZ]). In the p-adic case the spectral decomposition has not been fully understood yet. One ends up with a multiplicity free direct sum of square integrable representations whose Langlands parameters are not completely explicit. On the other hand, supercuspidal representations are compactly induced from certain cuspidal datum (see e.g. [Y]), which involve representations of finite Lie groups. Our results should have applications towards explicit local descents of supercuspidal representations.

We hope to address the above issues in our future works. This paper is organized as follows. In Section 2 we recall the description of $F$-stable maximal tori, and introduce the map from the set of $G^F$-conjugacy classes of $F$-stable maximal tori of $G$ to that of a subgroup of $G$ which we need in Reeder’s formula. In Section 3, we briefly recall the theory of Deligne-Lusztig characters. In Section 4 we extend Reeder’s formula to the case that $G^F = U_n(\mathbb{F}_q)$. In Section 5 we compute the multiplicities of Deligne-Lusztig representations of $U_n(\mathbb{F}_q)$, and thereby prove the Bessel case of our main theorem using the results in Section 4. In Section 6 we deduce the Fourier-Jacobi case from the Bessel case using the standard method of theta correspondence and see-saw dual pair.

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2. Remarks on maximal tori

The aim of this section is to recall the map $j_{G_s}$ used in the proof of Reeder’s multiplicity formula [R]. Moreover we will calculate the map $j_{G_s}$ for $G^F = U_n(\mathbb{F}_q)$, and show that it is injective for certain semisimple elements $s \in G^F$ of our concern.

2.1. Construction of $j_{G_s}$. Let $G$ be a connected reductive algebraic group over $\mathbb{F}_q$, $F$ be the corresponding Frobenius endomorphism, and $G^F$ be the group of rational points. Let $T$ be an $F$-stable maximal torus in $G$. Denote its normalizer in $G$ by $N_G(T)$ and the Weyl group $W_G(T) = N_G(T)/T$. Then $W_G(T)^F = N_G(T)^F/T^F$ by the Lang-Steinberg theorem.

Recall the classification of $F$-stable maximal tori in $G$. Fix an $F$-stable maximal torus $T_0$ in $G$ which is contained in an $F$-stable Borel subgroup of $G$, and for abbreviation write $N_G = N_G(T_0)$,
For any $F$-stable maximal torus $T$, there is $g \in G$ such that $^gT = T_0$. Since $T$ is $F$-stable, we have $gF(g^{-1}) \in N_G$. If $w$ is the image of $gF(g^{-1})$ in $W_G$, then we denote $T$ by $T_w$.

For a finite group $A$ with an $F$-action, $H^1(F, A)$ denotes the set of $F$-conjugacy classes in $A$. For $a, b \in A$, we say that $a$ is $F$-conjugate to $b$ if there is $c \in A$ such that $caF(c^{-1}) = b$. Let $[a] \in H^1(F, A)$ denote the $F$-conjugacy class of an element $a \in A$.

If $w := gF(g^{-1})T_0 \in N_G$ as above, then we obtain a class associate to $T$ given by

$$\text{cl}(T, G) := [w] \in H^1(F, W_G).$$

It is easy to check that the $F$-conjugacy class of $w$ is independent of the choice of $g$. Hence $\text{cl}(T, G)$ is well-defined.

Let $\mathcal{T}(G)$ denote the set of all $F$-stable maximal tori in $G$. Then $\mathcal{T}(G)$ is a finite union of $G^F$-orbits. For each $\omega \in H^1(F, W_G)$, the set

$$\mathcal{T}_\omega(G) := \{T \in \mathcal{T}(G) : \text{cl}(T, G) = \omega\}$$

is a single $G^F$-orbit in $\mathcal{T}(G)$, and all $G^F$-orbits are of this form. Thus, we have a partition of the set of $F$-stable maximal tori into $G^F$-orbits

$$\mathcal{T}(G) = \bigsqcup_{\omega \in H^1(F, W_G)} \mathcal{T}_\omega(G).$$

Let $s \in G^F$ be semisimple, $T_s$ be an $F$-stable maximal torus of $G_s$ contained in an $F$-stable Borel subgroup of $G_s$, and $W_{G_s}$ be the Weyl group of $T_s$ in $G_s$. As above, the set of $F$-stable maximal tori of $G_s$ has a partition into $G_s^F$-orbits given by

$$\mathcal{T}(G_s) = \bigsqcup_{\omega' \in H^1(F, W_{G_s})} \mathcal{T}_{\omega'}(G_s).$$

If $T \in \mathcal{T}(G)$, the set of $F$-stable maximal tori in $G_s$ which are $G^F$-conjugate to $T$ is a finite union (possibly empty) of $G_s^F$-orbits. We want to describe this union in terms of $F$-conjugacy classes in $W_{G_s}$. That is, given $\omega \in H^1(F, W_G)$, we have

$$\mathcal{T}_\omega(G) \cap \mathcal{T}(G_s) = \bigsqcup_{\omega' \in M_\omega} \mathcal{T}_{\omega'}(G_s)$$

for some subset $M_\omega \subset H^1(F, W_{G_s})$. In order to determine $M_\omega$, Reeder constructs a map

$$j_{G_s} : H^1(F, W_{G_s}) \to H^1(F, W_G)$$

such that

$$\mathcal{T}_\omega(G) \cap \mathcal{T}(G_s) = \bigsqcup_{\omega' \in j_{G_s}^{-1}(\omega)} \mathcal{T}_{\omega'}(G_s).$$

We now recall the construction of $j_{G_s}$. Note that $T_s$ is generally not contained in an $F$-stable Borel subgroup of $G$. Let $g \in G$ be such that $^gT_s = T_0$, and put $w_s := gF(g^{-1})T_0 \in W_G$. Then $\text{cl}(T_s, G) = [w_s] \in H^1(F, W_G)$,

and $\text{Ad}(g)$ gives a map from the $F$-conjugacy classes in $W_{G_s}$ to a subset $^gW_{G_s}$ of $w_s \circ F$-conjugacy classes in $W_G$. Define $j_{G_s} : H^1(F, W_{G_s}) \to H^1(F, W_G)$ to be the composition

$$j_{G_s} : H^1(F, W_{G_s}) \xrightarrow{\text{Ad}(g)} H^1(\text{w} \circ F, ^gW_{G_s}) \xrightarrow{\text{incl}} H^1(\text{w} \circ F, W_G) \xrightarrow{\tau_{w_s}} H^1(F, W_G).$$
where the middle map is induced by the inclusion \( g W_{G_s} \subseteq W_G \) and \( \tau_{w_s} \) is the twisting bijection given by \( \tau_{w_s}[x] = [x w_s] \). By \([R, (2.3)]\), we have
\[
(2.1) \quad \operatorname{cl}(T, G) = j_{G_s}(\operatorname{cl}(T, G_s)).
\]

2.2. The map \( j_{G_s} \) for \( U_n \). In this subsection we assume that \( G^F = U_n(\mathbb{F}_q) \), and we shall calculate \( j_{G_s} \) explicitly in this case.

Let us recall the classification of \( F \)-stable maximal tori of unitary groups. Denote by \( w_0 \in G \) the matrix which has entries one on the antidiagonal and zero elsewhere, and let \( T_0' \) be the maximal torus of diagonal matrices in \( G \). We know that \( T_0' \) is an \( F \)-stable maximal torus of \( G \) but it is not contained in an \( F \)-stable Borel subgroup. Let \( W_0' \) be the Weyl group of \( T_0' \). Then \( W_0' \cong S_n \), and the Frobenius map \( F \) acts on \( W_0' \) by
\[
F(w) = w_0 w w_0, \quad w \in W_0'.
\]
This means that for any \( w \in W_0' \), the \( F \)-conjugacy class of \( w w_0 \) consists of \( w' w_0 \) where \( w' \) is conjugate to \( w \) in \( W_0' \). Indeed, if \( g w g^{-1} = w' \) where \( g \in W_0' \), then
\[
g w_0 F(g^{-1}) = g w_0 w g^{-1} w_0 = g w_0 w = w' w_0.
\]
In this way, the \( F \)-conjugacy classes in \( W_0' \) are in one-to-one correspondence with the conjugacy classes in \( S_n \), which are parametrized by partitions of \( n \).

Take \( g_0 \in G \) such that \( g_0 F(g_0^{-1}) = w_0 \). Then we have a \( F \)-stable maximal torus \( T_0 = g_0^{-1} T_0' \) of \( G \) which is contained in an \( F \)-stable Borel subgroup. In the notations of the previous subsection, we have \( W_G = g_0 W_0' \). We now calculate the \( F \)-conjugacy classes in \( W_G \). If \( w w_0 \) and \( w' w_0 \) are in the same \( F \)-conjugacy classes of \( W_0' \) and \( g w = w' \), then
\[
(g_0 g) g_0 (w w_0) F(g_0 g)^{-1} = g_0 g w w_0 g^{-1} g_0^{-1} w_0 = g_0 (w' w_0).
\]
Hence the \( F \)-conjugacy classes in \( W_G \) are parametrized by partitions of \( n \) as well.

Let \( s \in G^F = U_n(\mathbb{F}_q) \) be a semisimple element conjugate to \( \operatorname{diag}(a_1, a_2, \ldots, a_{n-m}, 1, \ldots, 1) \), where \( a_i \neq a_j \neq 1 \) for \( 1 \leq i \neq j \leq n - m \). We only calculate \( j_{G_s} \) for this type of \( s \), since we will only need these semisimple elements in our further discussion. Suppose that \( g s \in T_0' \). Without loss of generality, we may assume that \( T_s = T_1 \times T_2 \), where \( T_2 \) is the torus of diagonal matrices in \( \operatorname{GL}_m(\mathbb{F}_q) \). Since \( s \in G^F \), we can require that \( T_1 \) and \( T_2 \) are \( F \)-stable maximal tori of \( \operatorname{GL}_{n-m}(\mathbb{F}_q) \) and \( \operatorname{GL}_m(\mathbb{F}_q) \), respectively. Hence \( g \) is of the following form
\[
g = \begin{pmatrix} g' & 0 \\ 0 & I_m \end{pmatrix}
\]
where \( g' F(g')^{-1} \) normalizes the torus of diagonal matrices in \( \operatorname{GL}_{n-m}(\mathbb{F}_q) \). Then we have
\[
G_s = T_1 \times \operatorname{GL}_m(\mathbb{F}_q) \quad \text{and} \quad G^F_s = T_1^F \times U_m(\mathbb{F}_q).
\]
It follows that the set \( H^1(F, W_{G_s}) \) is in bijection with partitions of \( m \). Suppose that \( T_1 \) corresponds to a partition \( \lambda \) of \( n - m \). For a partition \( \lambda' \) of \( m \), let \( T' \) be the \( F \)-stable maximal torus of \( \operatorname{GL}_m(\mathbb{F}_q) \) corresponding to \( \lambda' \), and let \( T' = T_1 \times T_2' \subseteq G_s \). Then
\[
(0.1) \quad j_{G_s}(\operatorname{cl}(T', G_s)) = [w_{\lambda, \lambda'}]
\]
where \( [w_{\lambda, \lambda'}] \) is the \( F \)-conjugacy class in \( W_G \) corresponding to the partition \( [\lambda, \lambda'] \) of \( n \). Hence
\[
(0.2) \quad j_{G_s}(\operatorname{cl}(T_1 \times T_{\lambda'}, G_s)) = j_{G_s}(\operatorname{cl}(T_1 \times T_{\lambda''}, G_s)) \iff \lambda' = \lambda''.
\]
We summarize the above discussion as follows.
Lemma 2.1. Assume that $G^F = U_n(F_q)$. Let $s \in G^F$ be a semisimple element of the above type. Then the map $j_{G_s}$ from $H^1(F,W_{G_s})$ to $H^1(F,W_G)$ is injective.

3. Deligne-Lusztig characters

Let $G$ be a connected reductive algebraic group over $F_q$. In their celebrated paper [DL], P. Deligne and G. Lusztig defined a virtual representation $R^G_{T,\theta}$ of $G^F$, associated to a character $\theta$ of $T^F$. The construction of Deligne-Lusztig characters makes use of the theory of $\ell$-adic cohomology. Here we only review some standard facts on these characters which will be used in this paper (cf. [C, Chapter 7]).

If $T$ is contained in an $F$-stable Borel subgroup $B$ of $G$, then $\theta$ lifts to a character of $B^F$ and in this case

$$R^G_{T,\theta} = \text{Ind}_{B^F}^G \theta.$$  

In general, if $y = su$ is the Jordan decomposition of an element $y \in G$, then

$$(3.1) \quad R^G_{T,\theta}(y) = \frac{1}{|C^0(y)|} \sum_{g \in G, s^g \in T} \theta(s^g)Q^G_{sT}(u)$$

where $C^0(s) = C^0_G(s)$ is the connected component of the centralizer of $s$ in $G$, and $Q^G_{sT}(u)$ is the Green function of $C^0(s)$ associated to $sT$. Note that $s^g = g^{-1}sg \in T$ if and only if $gT = gTg^{-1} \subset C^0(s)$.

The group $G^F_s$ acts on $N_G(s,T)^F$ by left multiplication, and we set

$$\bar{N}_G(s,T)^F := G^F_s \backslash N_G(s,T)^F.$$  

By [R, Corollary 2.3], we get an explicit formula for $|\bar{N}_G(s,T)^F|$:

Lemma 3.1. Let $\omega \in H^1(F,W_G)$ and $T \in T_\omega(G)$. Then the set $N_G(s,T)^F$ is nonempty if and only if the fiber $j^{-1}_{G_s}(\omega)$ is nonempty, in which case we have

$$|\bar{N}_G(s,T)^F| = \sum_{\omega' \in j^{-1}_{G_s}(\omega)} \left| \frac{|W_G(T)^F|}{|W_{G_s}(T_{\omega'})^F|} \right|,$$

where for each $\omega' \in j^{-1}_{G_s}(\omega)$, the torus $T_{\omega'}$ can be chosen arbitrarily in $T_{\omega'}(G_s)$.

Thus, we may rewrite (3.1) as

$$R^G_{T,\theta}(y) = \sum_{\gamma \in \bar{N}_G(s,T)^F} \theta(s^\gamma)Q^G_{sT}(u).$$  

In our future computations with $R^G_{T,\theta}$ in Reeder’s formula, it will be useful to let $s$ vary in $G^F$ in such a way that $G_s$ is unchanged. Let $Z(G_s)$ denote the center of $G_s$. For $\omega' \in j^{-1}_{G_s}(\omega)$, the function

$$\theta_{\omega'}(z) := \sum_{\gamma \in O_{\omega'}} \theta(z^\gamma)$$

is well-defined on $Z(G_s)^F$, where $O_{\omega'}$ is the $W_G(T)^F$-orbit in $\bar{N}_G(s,T)^F$ corresponding to $\omega' \in j^{-1}_{G_s}(\omega)$ as in Lemma 3.1. Then we have

$$R^G_{T,\theta}(zu) = \sum_{\omega' \in j^{-1}_{G_s}(\omega)} \theta_{\omega'}(z)Q^G_{sT}(u), \quad \text{if } G_z = G_s.$$
More generally, for an $F$-stable Levi subgroup $L$ which is not necessarily contained in an $F$-stable parabolic subgroup, and a representation $\pi$ of $L^F$, one has the generalized Deligne-Lusztig induction $R_L^G(\pi)$. In particular, if $L$ is contained in an $F$-stable parabolic subgroup $P$, then $R_L^G(\pi)$ coincides with the parabolic induction

\begin{equation}
I_L^G(\pi) := \text{Ind}_{P^F}^{G^F} \pi.
\end{equation}

For instance, recall that if $L = T$ is contained in an $F$-stable Borel subgroup $B$, then

\[ I_B^G(\theta) = R_T^G(\theta). \]

The following result is standard.

**Proposition 3.2.** Let $Q \subset P$ be $F$-stable parabolic subgroups of $G$, and $M \subset L$ be their Levi subgroups respectively. Then

\[ I_P^G \circ I_Q^{G \cap L} = I_M^G. \]

An $F$-stable maximal torus $T$ is said to be minisotropic if $T$ is not contained in any $F$-stable proper parabolic subgroup of $G$. Then a representation $\pi$ of $G^F$ is cuspidal if and only if

\[ \langle \pi, R_T^G(\theta) \rangle_{G^F} = 0 \]

whenever $T$ is not minisotropic, for any character $\theta$ of $T^F$ (see [S1, Theorem 6.25]). Note that if $G^F = \text{GL}_n(\mathbb{F}_q)$, then $T$ is said to be minisotropic when $T^F \cong \text{GL}_1(\mathbb{F}_q)$.

Let $\theta \in T^F$, $\theta' \in T'^F$ where $T$, $T'$ are $F$-stable maximal tori. The pairs $(T, \theta)$, $(T', \theta')$ are said to be geometrically conjugate if for some $n \geq 1$, there exists $x \in G^{F^n}$ such that

\[ xT^{F^n} = T'^{F^n} \quad \text{and} \quad x(\theta \circ N) = \theta' \circ N \]

where $N$ is the norm map. By [?, p. 378], in any geometrically conjugate class $\kappa$, there is only one regular character $\pi_{\kappa}^{reg}$ appears in some $R_T^G(\theta)$ where $(T, \theta) \in \kappa$; and any regular character only appears in one geometrically conjugate class. Moreover

\[ \pi_{\kappa}^{reg} = \sum_{(T, \theta) \in \kappa \mod G^F} \frac{\varepsilon_G \varepsilon_T R_T^G(\theta)}{\langle R_T^G(\theta), \theta \theta' \rangle_{G^F}}. \]

The above equation implies that $\pi_{\kappa}^{reg}$ appears in $R_T^G(\theta)$ for each pair $(T, \theta) \in \kappa$. Thus $\pi_{\kappa}^{reg}$ is cuspidal if and only if $T$ is minisotropic and $\theta$ is regular for every pair $(T, \theta) \in \kappa$. Here $\theta$ regular means that

\[ x \theta = \theta, \quad x \in W_G(T)^F \]

if and only if $x = 1$.

In particular, if $\tau$ is an irreducible cuspidal representation of $\text{GL}_n(\mathbb{F}_q)$, then there is a pair $(T, \theta)$ with $T$ an $F$-stable minisotropic maximal torus and $\theta$ regular such that $\tau = \pm R_T^G(\theta)$.

We now review the classification of representations of $U_n(\mathbb{F}_q)$. The classification of the representations of $U_n(\mathbb{F}_q)$ was given by Lusztig and Srinivasan in [LS]. Denote by $W_n \cong S_n$ the Weyl group of the diagonal torus in $U_n$.

**Theorem 3.3.** Let $\sigma$ be an irreducible representation of $S_n$. Then

\[ R_{U_n}^{V_n} := \frac{1}{|W_n|} \sum_{w \in W_n} \sigma(w w_0) R_{T_{w,1}}^{U_n} \]

is (up to sign) a unipotent representation of $U_n(\mathbb{F}_q)$ and all unipotent representations of $U_n(\mathbb{F}_q)$ arise in this way.
By the above theorem, we know that every irreducible representation of $S_n$ corresponds to a unipotent representation of $U_n(\mathbb{F}_q)$. It is well known that irreducible representations of $S_n$ are parametrized by partitions of $n$. For a partition $\lambda$ of $n$, denote by $\sigma_\lambda$ the corresponding representation of $S_n$, and let $R^\lambda_{U_n} := R^\lambda_{\sigma_\lambda}$ be the corresponding unipotent representation of $U_n(\mathbb{F}_q)$. By Lusztig’s result [L2], $R^\lambda_{U_n}$ is (up to sign) a unipotent cuspidal representation of $U_n(\mathbb{F}_q)$ if and only if $n = \frac{k(k+1)}{2}$ for some positive integer $k$ and $\lambda = [k, k-1, \cdots, 1]$.

4. Reeder’s formula

As before, let $G$ be a connected reductive algebraic group over $\mathbb{F}_q$. Let $H \subset G$ be a connected reductive subgroup of $G$ over $\mathbb{F}_q$, and $S$ be an $F$-stable maximal torus of $H$. Let $B$ and $B_H$ be Borel subgroups of $G$ and $H$, respectively, and let $\delta$ be the minimum codimension of a $B_H$-orbit in $G/B$.

In [R], Reeder gives a formula for the multiplicity $\langle R^G_{T, \theta}, R^H_{S, \theta'} \rangle_{HF}$ when $G$ and $H$ are simple. More precisely, by [R, Theorem 1.4] there is a polynomial $M(t)$ of degree at most $\delta$ whose coefficients depend on the characters $\theta$ and $\theta'$ of $T^F$ and $S^F$ respectively, and an integer $m \geq 1$ such that

$$\langle R^G_{T, \theta'}, R^H_{S, \theta'} \rangle_{HF^\nu} = M(q^\nu)$$

for all positive integers $\nu \equiv 1 \mod m$, where $\theta' = \theta \circ N^T_\nu$ with $N^T_\nu : T^F \to T^F$ the norm map. And the degree $\delta$ is optimal. Moreover, [R, Proposition 7.4] gives an explicit formula for the leading coefficient in $M(t)$, which we denote by $A$ below.

In order to calculate $\langle I^U_{F^n+1}(\tau \otimes \pi'), \pi_\lambda \rangle_{U_n(\mathbb{F}_q)}$ using Reeder’s method, it is necessary to extend his result from connected simple algebraic groups to unitary groups. In particular, we will show that $\deg M(t) = 0$, hence $\langle I^U_{F^n+1}(\tau \otimes \pi'), \pi_\lambda \rangle_{U_n(\mathbb{F}_q)} = A$. In this case, the explicit formula for $A$ will be given in Proposition 4.4.

In the sequel, we will first recall some results in [R, Sections 3-5] without proof, which do not require that $G$ is simple. Then we extend [R, Sections 6-7] to the case of $U_n$.

4.1. A partition of $S$. We first recall the notations in [R]. Let $I(S)$ be an index set for the following set of subgroups of $G$,

$$\{G_s : s \in S\}$$

where $G_s := C_G(s)^\circ$. Note that if $G^F = U_n(\mathbb{F}_q)$, then $I(S)$ is finite. For $\iota \in I(S)$, denote by $G_\iota$ be the corresponding connected centralizer, and put

$$S_\iota := \{s \in S : G_s = G_\iota\}.$$

The $F$-action on $S$ induces a permutation of $I(S)$, and let $I(S)^F$ be the $F$-fixed points in $I(S)$. Note that if $S_\iota^F$ is nonempty, then $\iota \in I(S)^F$.

For $\iota \in I(S)$, set

$$H_\iota := (H \cap G_\iota)^\circ.$$

We observe that if $G_s = G_\iota$, then

$$s \in H_\iota \subset G_\iota.$$

Put $Z_\iota := Z(G_\iota) \cap S$. Then it is easy to check that

$$Z_\iota \subset Z(H_\iota),$$
$$S_\iota \subset Z_\iota \subset S,$$
$$G_\iota = C_G(Z_\iota)^\circ.$$
Define a partial ordering on $I(S)$ by

$$
t' \leq t \iff G_{t'} \subset G_t \iff Z_{t'} \subset Z_t,
$$

and set

$$I(t, S) := \{ t \in I(S) : t' < t \}.$$ 

For a subset $J \subset I(S)$, put

$$Z_J := \bigcap_{t \in J} Z_t.$$ 

4.2. **Multiplicity as a polynomial.** Let $f_J(t)$ be the polynomial of degree $\dim Z_J$ in $[R, \text{Section 5.6}]$. Let $Q_{v,u}^G(t)$ and $Q_{\varsigma,u}^H(t)$ be the Green polynomials in $[R, \text{Section 5.4}]$ and $P_{v,u}(t)$ be the polynomial given by $[R, (5.11)]$. Then we have

$$\det P_{v,u}(t) = \dim C_H(u), \quad \deg f_I(t) = \dim Z_I.$$ 

We set

$$\chi_v := \frac{1}{|W_G(T)|} \sum_{x \in W_G(T)} (^{x}\chi)|Z_v \quad \text{and} \quad \eta_{\varsigma} := \frac{1}{|W_H(S)|} \sum_{y \in W_H(S)} (^{y}\varsigma)|Z_{\varsigma},$$

Denote the set of unipotent elements of $H_{\varsigma}$ by $U(H_{\varsigma})$. Let $\Theta_{\alpha}(t)$ be the rational function given by

$$\Theta_{\alpha}(t) := \langle \chi_v, \eta_\varsigma \rangle_{Z^F_v} + \sum_{J \subset I(t, S)} (-1)^{|J|} \langle \chi_v, \eta_\varsigma \rangle_{Z^F_J} \frac{f_J(t)}{f_I(t)},$$

and $\Psi_{\alpha}(t)$ be the rational function given by

$$\Psi_{\alpha}(t) := f_I(t) \cdot \frac{Q_{v,u}^G(t)Q_{\varsigma,u}^H(t)}{|N_H(t, S)|^F \cdot |P_{v,u}(t)|}.$$ 

The following result is given by $[R, (5.18)]:$

$$\langle R_{T,\chi}^G, R_{S,\varsigma}^H \rangle_{H^F} = \sum_{\alpha} \Psi_{\alpha}(q) \Theta_{\alpha}(q)$$

where $\alpha$ runs over quadruples $(t, u, v, \varsigma)$, with

$$t \in I(S)^F, \quad [u] \in U(H_{\varsigma})^F, \quad v \in j_{G_{\varsigma}}^{-1}(\cl(T, G)), \quad \varsigma \in j_{H_{\varsigma}}^{-1}(\cl(S, H)).$$

4.3. **Degree of $\Psi_{\alpha}(t)$.** From now on, assume that

$$\langle G^F, H^F \rangle = (U_{n+1}(\mathbb{F}_q), U_n(\mathbb{F}_q)).$$

Recall that in this case $G(\mathbb{F}_q) = \text{GL}_{n+1}(\mathbb{F}_q)$. Let $B_G$ be the variety of Borel subgroups of $G$, and let $B_G^u$ be the variety of fixed points of $u$. Set

$$d_G(u) := \dim B_G^u.$$ 

Steinberg proved that

$$2d_G(u) = \dim C_G(u) - \overline{rk} G,$$

where $\overline{rk} G$ is the absolute rank of $G$.

From Section 5.4 and equation (5.9) in $[R]$, we find that

$$\deg \Psi_{\alpha}(t) \leq \dim Z_i + d_{G_i}(u) + d_{H_i}(u) - \dim C_{H_i}(u)$$

$$= \dim Z_i + \frac{1}{2}(\dim C_{G_i}(u) - \dim C_{H_i}(u) - \overline{rk} G - \overline{rk} H).$$
Define
\[ \delta_\iota := \dim Z_\iota + \dim \mathcal{B}_{G_\iota} - \dim \mathcal{B}_{H_\iota} - \dim S \]
\[ = \dim Z_\iota + \frac{1}{2}(\dim C_{G_\iota}(1) - \dim C_{H_\iota}(1) - \ell \dim \overline{G} - \ell \dim \overline{H}). \]

**Lemma 4.1.** Assume (4.4). Then \( \deg \Psi_\alpha(t) \leq \delta_\iota \) with equality hold only if \( u = 1 \).

**Proof.** By [R, (6.3)], for a simple group \( G' \) we have
\[ (4.6) \quad \dim C_{G'_\iota}(u) - \dim C_{H'_\iota}(u) \leq \dim C_{G'_\iota}(1) - \dim C_{H'_\iota}(1). \]
Since \( \text{SL}_{n+1}(\overline{F}_q) \) is simple, it follows that
\[ (4.7) \quad \dim C_{\text{GL}_{n+1,\iota}}(u) - \dim C_{\text{GL}_{n,\iota}}(u) \]
\[ = \dim C_{\text{SL}_{n+1,\iota}}(u) + 1 - (\dim C_{\text{SL}_{n,\iota}}(u) + 1) \]
\[ = \dim C_{\text{SL}_{n+1,\iota}}(u) - \dim C_{\text{SL}_{n,\iota}}(u) \leq \dim C_{\text{SL}_{n+1,\iota}}(1) - \dim C_{\text{SL}_{n,\iota}}(1) \]
\[ = \dim C_{\text{GL}_{n+1,\iota}}(1) - \dim C_{\text{GL}_{n,\iota}}(1). \]
Thus, (4.6) also holds for \( (\text{GL}_{n+1}(\overline{F}_q), \text{GL}_n(\overline{F}_q)) \). Then the desired inequality follows from (4.5). Since equality holds in (4.5) for \( (\text{SL}_{n+1}(\overline{F}_q), \text{SL}_n(\overline{F}_q)) \) if and only if \( u = 1 \) (see [R, lemma 6.4]), it follows easily that equality holds in (4.6) for \( (\text{GL}_{n+1}(\overline{F}_q), \text{GL}_n(\overline{F}_q)) \) if and only if \( u = 1 \) as well. \( \square \)

For a fixed \( \iota \), let \( s \in S_\iota \) be a semisimple element in \( \text{GL}_{n+1}(\overline{F}_q) \). Suppose that \( s \) is conjugate to \( \text{diag}(a_1, a_2, \ldots, a_{n+1-m}, 1, \ldots, 1) \) with \( a_i \neq 1 \) for \( i = 1, \ldots, n+1-m \). Let \( n_1, \ldots, n_r \) be the multiplicities of distinct eigenvalues among \( a_1, \ldots, a_{n+1-m} \). Then \( G_s \) and \( H_s \) are conjugate to \( \text{GL}_{n_1}(\overline{F}_q) \times \cdots \times \text{GL}_{n_r}(\overline{F}_q) \times \text{GL}_m(\overline{F}_q) \) and \( \text{GL}_{n_1}(\overline{F}_q) \times \cdots \times \text{GL}_{n_r}(\overline{F}_q) \times \text{GL}_{m-1}(\overline{F}_q) \), where the latter are block diagonal subgroups of \( \text{GL}_{n+1}(\overline{F}_q) \) and \( \text{GL}_n(\overline{F}_q) \) respectively. It follows that
\[ \dim \mathcal{B}_{G_\iota} - \dim \mathcal{B}_{H_\iota} = \dim \mathcal{B}_{\text{GL}_{m}(\overline{F}_q)} - \dim \mathcal{B}_{\text{GL}_{m-1}(\overline{F}_q)} = m - 1 \]
and
\[ \dim Z_\iota - \dim S = r - n. \]
We obtain that \( \delta_\iota = r - n - 1 + m \leq 0 \), with equality hold only if \( r = n + 1 - m \). In other words, \( \delta_\iota = 0 \) if and only if \( a_i \neq a_j \neq 1 \) for any \( 1 \leq i \neq j \leq n+1-m \).

In summary, we obtain the following result.

**Lemma 4.2.** Assume (4.4). Then \( \deg \Psi_\alpha(t) \leq 0 \), with equality hold only if \( u = 1 \) and there is \( s \in S_\iota \) conjugate to \( \text{diag}(a_1, a_2, \ldots, a_{n+1-m}, 1, \ldots, 1) \) with \( a_i \neq a_j \neq 1 \) for any \( 1 \leq i \neq j \leq n+1-m \).

4.4. The leading term of \( \Theta_\alpha(q) \). We have known that \( \deg \Theta_\alpha(t) \leq 0 \) (see [R, (5.17)]). We now show that in fact there is only one term in \( \Theta_\alpha(q) \) of degree zero.

**Lemma 4.3.** Assume (4.4), \( \iota' < \iota \) and \( \delta_\iota = 0 \). Then \( \dim Z_{\iota'} < \dim Z_\iota \).

**Proof.** Suppose that \( \iota' \in S_{\iota'} \) is conjugate to \( \text{diag}(a_1, a_2, \ldots, a_{n+1-m}, 1, \ldots, 1) \) as above, with \( n_1, \ldots, n_r \) the multiplicities of distinct eigenvalues among \( a_1, \ldots, a_{n+1-m} \). Then \( G_{\iota'} \) is conjugate to \( \text{GL}_{n_1}(\overline{F}_q) \times \cdots \times \text{GL}_{n_r}(\overline{F}_q) \times \text{GL}_m(\overline{F}_q) \), and \( Z_{\iota'} = Z(G_{\iota'}) \cap S \) is conjugate to \( \text{diag}(\overline{F}_q^{x \cdot I_{n_1}}, \ldots, \overline{F}_q^{x \cdot I_{n_r}}) \cdot I_m \). It follows that \( \dim Z_{\iota'} = r \).

Since \( \iota' < \iota \), we have \( G_\iota \subset G_{\iota'} \). Thus, \( s \in S_\iota \) is a semisimple element of \( G_{\iota'} \). Suppose that \( s \) is conjugate to \( \text{diag}(s_1, s_2, \ldots, s_r, 1, \ldots, 1) \), where \( s_i \in \text{GL}_{n_i}(\overline{F}_q) \) has eigenvalues \( s_{i,1}, s_{i,2}, \ldots, s_{i,n_i} \).
Since $\delta_i = 0$, from the above discussion we can assume that $s_{ij}$ are all distinct. Then $\dim Z_i \geq r$. It is easy to check that $\dim Z_i = r$ if and only if $n_i = 1, i = 1, \ldots, r$. However in this case $r = \nu'$. □

Recall that $f_j(t)$ is a polynomial of degree $\dim Z_J$. The above lemma implies that $\deg f_j(t) < \deg f_i(t)$. Recall the formula (4.2)

$$\Theta_\alpha(t) = \langle \chi_v, \eta_e \rangle_{Z^+_F} + \sum_{J \subseteq I(t, S)^F} (-1)^{|J|} \langle \chi_v, \eta_e \rangle_{Z^+_F} \frac{f_j(t)}{f_i(t)}.$$

Then the only degree zero term of $\Theta_\alpha(q)$ is $\langle \chi_v, \eta_e \rangle_{Z^+_F}$.

### 4.5. The leading term of $M(t)$

We shall find an explicit and effective formula of $M(t)$. Recall that

$$\langle R^G_{T_x, R^H_{S,y}} \rangle_{H^F} = \sum_\alpha \Psi_\alpha(q) \Theta_\alpha(q).$$

Assuming (4.4), we have known that $\deg \Psi_\alpha(q) \leq 0$ and $\deg \Theta_\alpha(q) \leq 0$. So the polynomial $M(t)$ is a constant, which equals the coefficient of the leading term of $\sum_\alpha \Psi_\alpha(q) \Theta_\alpha(q)$.

We now calculate $M(t)$ by the definitions of $\Psi_\alpha(q)$ and $\Theta_\alpha(q)$ in (4.2) and (4.3) respectively. By Lemmas 4.1 and Lemma 4.2, only quadruples $\alpha$ with $u = 1$ and $\delta_i = 0$ contribute to the leading term; henceforth we assume that $\alpha$ is of this form. If we take $u = 1$, then by [R, (5.10)],

$$Q_{\epsilon, 1}^{G_F}(t) = \epsilon_G(v) t^{\deg(1)} + \text{lower powers of } t,$$

where $\epsilon_G(v) = (-1)^{rk(G)+rk(T)}$. From [R, Section 5.5], we know that $P_{\epsilon, u}(t)$ is of the form $A_{\epsilon}(u)$ times a monic polynomial in $\mathbb{Z}[t]$, where $A_{\epsilon}(u)$ is the centralizers in $H_\epsilon$ of all unipotent elements $u \in H^F_\epsilon$ for every $\epsilon \in I(S)^F$. By [R, Section 5.6], for any subset $J \subseteq I(S)$,

$$f_J(q^m) = |Z^F_T|^m, \quad \text{for all } \nu \equiv 1 \text{ mod } m,$$

where $m$ is a positive integer divisible by the exponent of the finite group $W_G \rtimes \langle \vartheta \rangle$. Here the actions of $\vartheta$ as automorphisms on the set of roots $\Phi(T_0, G)$ as well as $W_G$, are induced by $F$.

As a power series in $t$, the leading coefficient of $\Psi_\alpha(t)$ is

$$\frac{(-1)^{rk(G)+rk(H)} \cdot \nu^{|N_H(\epsilon, S)^F|}}{|N_H(\epsilon, S)^F|},$$

and the leading coefficient of $\Theta_\alpha(t)$ is

$$\langle \chi_v, \eta_e \rangle_{Z^+_F}.$$

Combining the above calculations and Lemma 2.1, we obtain that

**Proposition 4.4.** Assume (4.4). Then

$$\langle R^G_{T_x, R^H_{S,y}} \rangle_{H^F} = \sum_{\epsilon \in I(S)^F} \frac{(-1)^{rk(G)+rk(H)+rk(T)+rk(S)}}{|N_H(\epsilon, S)^F|} \langle \chi_v, \eta_e \rangle_{Z^+_F}.$$

where $v = j^{-1}_{G^-}(\text{cl}(T, G))$ and $\varsigma = j^{-1}_{H^-}(\text{cl}(S, H))$ for some $\epsilon$ such that $j^{-1}_{G^-}(\text{cl}(T, G))$ and $j^{-1}_{H^-}(\text{cl}(S, H))$ are not empty.
5. Branching laws for $U_n(\mathbb{F}_q)$

In this section we study the branching of unipotent representations of finite unitary groups. We will prove the following result, which is the Bessel case of Theorem 1.1.

**Theorem 5.1.** Assume that $n > m$ and $n - m$ is odd. Let $\lambda$ be a partition of $n$ into $k$ rows, and $\lambda'$ be the partition of $n - k$ obtained by removing the first column of $\lambda$. Let $\pi'$ be an irreducible representation of $U_m(\mathbb{F}_q)$. Then the following hold.

(i) If $m < n - k$, then

$$m(\pi_\lambda, \pi') = 0.$$  

(ii) If $k$ is odd and $m = n - k$, then

$$m(\pi_\lambda, \pi') = \begin{cases} 
1, & \text{if } \pi' \cong \pi_{\lambda'}, \\
0, & \text{otherwise}.
\end{cases}$$

5.1. Basic case. In this subsection we first show that parabolic induction preserves multiplicities, and thereby make a reduction to the basic case. Then we explore Reeder’s multiplicity formula in details in this case. Put

$$G_\ell := \text{Res}_{\mathbb{F}_{q^{2\ell}}/\mathbb{F}_q} \text{GL}_\ell,$$

so that $G_\ell(\mathbb{F}_q) = \text{GL}_\ell(\mathbb{F}_{q^{2\ell}})$. Let $P$ be an $F$-stable maximal parabolic subgroup of $U_n$ with Levi factor $G_\ell \times U_m$ (so that $n+1 = m+2\ell$), and $\sigma$ be an irreducible cuspidal representation of $\text{GL}_\ell(\mathbb{F}_{q^{2\ell}})$. From [GGP2, Proposition 5.3], we know that parabolic induction preserves multiplicities between cuspidal representations, namely,

$$\langle \pi \otimes \bar{\nu}, \pi' \rangle_{H(\mathbb{F}_q)} = \langle I^U_{P \times U_m} (\sigma \otimes \pi'), \pi \rangle_{U_n(\mathbb{F}_q)}$$

for cuspidal representations $\pi$ and $\pi'$ of $U_n(\mathbb{F}_q)$ and $U_m(\mathbb{F}_q)$ respectively. In the same manner, we have the following analog when $\pi$ is unipotent, which reduces the calculation to the basic case.

**Proposition 5.2.** Let $\pi$ be an irreducible unipotent representation of $U_n(\mathbb{F}_q)$, and $\pi'$ be an irreducible representation of $U_m(\mathbb{F}_q)$ with $n > m$ but $m \neq n$ mod 2. Let $P$ be an $F$-stable maximal parabolic subgroup of $U_{n+1}$ with Levi factor $G_\ell \times U_m$ (so that $m+2\ell = n+1$), and $\sigma$ be an irreducible cuspidal representation of $\text{GL}_\ell(\mathbb{F}_{q^{2\ell}})$. Then we have

$$m(\pi, \pi') = \langle \pi \otimes \bar{\nu}, \pi' \rangle_{H(\mathbb{F}_q)} = \langle I^U_{P \times U_m} (\sigma \otimes \pi'), \pi \rangle_{U_n(\mathbb{F}_q)},$$

where the data $(H, \nu)$ is given by (1.2).

**Proof.** It can be proved in the same way as [GGP1, Theorem 15.1]. The cuspidality assumption of $\pi$ in [GGP2, Proposition 5.3] was used to obtain the following statement: for an $F$-stable maximal parabolic subgroup $P'$ of $U_n$ with Levi factor $G_\ell \times U_{m-1}$,

$$\langle I^U_{P'} (\sigma \otimes (\pi'|_{U_{m-1}(\mathbb{F}_q)})), \pi \rangle_{U_n(\mathbb{F}_q)} = 0.$$  

Since in our case $\pi$ is unipotent, this multiplicity is nonzero if and only if $\pi$ and $\pi'|_{U_{m-1}(\mathbb{F}_q)}$ are both unipotent. It is well known that $\text{GL}_\ell(\mathbb{F}_{q^{2\ell}})$ has no unipotent cuspidal representations. By assumption $\sigma$ is cuspidal, it is not unipotent. Therefore the above multiplicity is zero. The rest of the proof is the same as that of [GGP1, Theorem 15.1].
This result shows that to calculate \( (\pi_\lambda \otimes \tilde{\nu}, \pi')_H(F_q) \), we only need to calculate \( \langle I_{\pi}^{U_n}(\tau \otimes \pi'), \pm R^\lambda_{U_n} \rangle_{U_n}(F_q) \) (cf. Theorem 3.3). We now compute

\[
\langle R^G_{T_1 \times T_2, \theta \otimes 1}, R^H_{S, 1} \rangle_{H}(F_q) = \sum_{\iota \in I(S)^F} \frac{(-1)^{rk(G)_{\iota}+rk(H)_{\iota}+rk(T)+rk(S)}}{|N_H(\iota, S)^F|} \langle (\theta \otimes 1)_{\iota}, 1 \rangle_{Z^F_{T}}
\]

where \( v = j_{G_{\iota}}^{-1}(cl(T_1 \times T_2, G)) \) and \( \zeta = j_{H_{\iota}}^{-1}(cl(S, H)) \) for some \( \iota \) such that \( j_{G_{\iota}}^{-1}(cl(T_1 \times T_2, G)) \) and \( j_{H_{\iota}}^{-1}(cl(S, H)) \) are not empty. We note that by our selection of \( \iota \in I(S)^F \), we always have \( j_{G_{\iota}}^{-1}(cl(T_1 \times T_2, G)) \neq \emptyset \). In (2.1), we have known that \( j_{G_{\iota}}^{-1}(cl(T_1 \times T_2, G)) \neq \emptyset \) if and only if there is a \( F \)-stable maximal torus \( T' \) of \( G_\iota \) such that \( T'^F \cong (T_1 \times T_2)^F \).

We now calculate the pairing \( \langle (\theta \otimes 1)v, 1\zeta \rangle_{Z^F_{T}} \). We may conjugate \( T_1 \times T_2 \) and \( S \) to ensure that \( Z_{\iota} \subset (T_1 \times T_2) \cap S \). Then

\[
\langle (\theta \otimes 1)v, 1\zeta \rangle_{Z^F_{T}} = \frac{1}{|W_{G_\iota}(T_1 \times T_2)^F|} \sum_{x \in W_{G_\iota}(T_1 \times T_2)^F} x(\theta \otimes 1)|_{Z_{\iota}},
\]

\[
1\zeta = \frac{1}{|W_{H_\iota}(S)^F|} \sum_{y \in W_{H_\iota}(S)^F} (y)|_{Z_{\iota}} = \frac{|W_{H_\iota}(S)^F|}{|W_{H_\iota}(S)^F|} \cdot 1.
\]

Recall Lemma 3.1 that \( |N_H(\iota, S)^F| = |W_{H_\iota}(S)^F|/|W_{H_\iota}(S)^F| \). Rewrite (5.3) as

\[
\langle R^G_{T_1 \times T_2, \theta \otimes 1}, R^H_{S, 1} \rangle_{H}(F_q) = \sum_{\iota \in I(S)^F} \frac{(-1)^{rk(G)_{\iota}+rk(H)_{\iota}+rk(T)+rk(S)}}{|N_H(\iota, S)^F|} \langle (\theta \otimes 1)_{\iota}, 1 \rangle_{Z^F_{T}}
\]

\[
= \sum_{\iota \in I(S)^F} \frac{(-1)^{rk(G)_{\iota}+rk(H)_{\iota}+rk(T)+rk(S)}}{|W_{T_1 \times T_2, \theta \otimes 1, \iota}|} |W_{G_\iota}(T_1 \times T_2)^F| \langle (\theta \otimes 1)v, 1\zeta \rangle_{Z^F_{T}} = 1\}
\]

where

\[
W_{T_1 \times T_2, \theta \otimes 1, \iota} := \{ x \in W_{G_\iota}(T_1 \times T_2)^F \mid x(\theta \otimes 1)|_{Z^F_{T}} = 1 \}.
\]

To compute (5.4), we rearrange the summation and use semisimple elements in \( H \) as indices. Let \( I(G) \) be an index set for the set of subgroups

\[
\{ G_s : s \text{ lies in an } F \text{-stable maximal torus of } H \}\.
\]

For an \( F \)-stable stable maximal torus \( S \) of \( H \), if \( \iota \in I(G) \) and \( \iota' \in I(S) \) such that \( G_\iota = G_{\iota'} \), then \( \iota' \) can be regarded as an element of \( I(G) \) and thus can be identified with \( \iota \in I(G) \). In this manner
we embed $I(S)$ as a subset of $I(G)$. For two $F$-stable stable maximal tori $S_1$ and $S_2$ of $H$, and $\nu_i \in I(S_i)$, $i = 1, 2$, if $G_{\nu_1} = G_{\nu_2}$, then we say that $\nu_1 = \nu_2$ as elements of $I(G)$. Thus we have

$$\bigcup_{S \in H} I(S) = I(G)$$

and moreover

$$\bigcup_{S \in H} I(S)^F = I(G)^F.$$

For two elements $\nu_1, \nu_2 \in I(G)^F$, we say that $\nu_1 \sim \nu_2$ if $G_{\nu_1}^F$ and $G_{\nu_2}^F$ are in the same $H^F$-conjugacy class. Denote the equivalence class of $\nu$ by $[\nu]$. Let $[I(G)^F]$ be the set of equivalence classes in $I(G)^F$, and $[I(S)^F]$ be the subset of equivalence classes $[\nu] \in [I(G)^F]$ such that $\nu \sim \nu'$ for some $\nu' \in I(S)^F$. It is easy to check that for $\nu_1, \nu_2 \in [\nu]$, we have

$$\delta_{\nu_1} = \delta_{\nu_2},$$

$$(-1)^{\text{rk}(G_{\nu_1}) + \text{rk}(H_{\nu_1})} \left| \frac{W_{T_1 \times T_2, \Theta \otimes 1, \nu_1}}{W_{G_{\nu_1}}(T_1 \times T_2)^F} \right| = (-1)^{\text{rk}(G_{\nu_2}) + \text{rk}(H_{\nu_2})} \left| \frac{W_{T_1 \times T_2, \Theta \otimes 1, \nu_2}}{W_{G_{\nu_2}}(T_1 \times T_2)^F} \right|,$$

and

$$j_{H_{\nu_1}}^{-1}(\text{cl}(S, H)) \neq \emptyset \iff j_{H_{\nu_2}}^{-1}(\text{cl}(S, H)) \neq \emptyset \text{ for any } S,$$

$$j_{G_{\nu_1}}^{-1}(\text{cl}(T, G)) \neq \emptyset \iff j_{G_{\nu_2}}^{-1}(\text{cl}(T, G)) \neq \emptyset \text{ for any } T.$$

This means that the properties of $\nu$ involved in the sum of (5.4) do not change when $\nu$ varies over an equivalence class. If we denote a typical summand of (5.4) by $X_\delta$, then (5.4) can be rewritten as

$$\sum_{\nu \in I(S)^F} X_\delta = \sum_{[\nu] \in [I(S)^F]} \# [\nu] \cdot X_\delta.$$

Recall Lemma 4.2 that if $\delta = 0$, then there is $s \in S_\nu$ conjugate to $\text{diag}(a_1, \ldots, a_{n+1-m}, 1, \ldots, 1)$, with $a_i \neq a_j \neq 1$ for $i \neq j$. Moreover, there is $g \in G^F$ such that $g \in T'_1 \times T'_2$, where $T'_1$ and $T'_2$ are $F$-stable maximal tori in $U_{n+1-m}(\mathbb{F}_q)$ and $U_m(\mathbb{F}_q)$ respectively. Thus,

$$G_{\nu}^F \cong (g G_{\nu})^F = (g G_s)^F = G_{\nu}^F = T'_1^F \times U_m(\mathbb{F}_q).$$

Hence the set

$$\{[\nu] \in [I(G)^F] : \delta = 0 \}$$

is parameterized by pairs $(m, \lambda)$, where $m \leq n$ is a nonnegative integer and $\lambda$ is a partition of $n + 1 - m$. Note that only the terms satisfying $j_{G_{\nu}}^{-1}(\text{cl}(T_1 \times T_2, G)) \neq \emptyset$ and $j_{H_{\nu}}^{-1}(\text{cl}(S, H)) \neq \emptyset$ contribute to (5.3). The following lemma gives an explicit description of these terms, which follows from Lemma 2.1 and (2.1).

**Lemma 5.3.** Assume that $[\nu] \in [I(G)^F]$, $\delta = 0$, corresponds to a partition $\mu'$ of $m$.

(i) Let $T$ be an $F$-stable maximal torus of $G$ which corresponds to a partition $\mu$ of $n + 1$. Then $j_{G_{\nu}}^{-1}(\text{cl}(T, G)) \neq \emptyset$ if and only if $\{\mu'_i \} \subset \{\mu_1 \}$. 

(ii) Let $S$ be an $F$-stable maximal torus of $H$ which corresponds to a partition $\lambda$ of $n$. Then $j_{G_{\nu}}^{-1}(\text{cl}(S, H)) \neq \emptyset$ if and only if $\{\mu'_i \} \subset \{\lambda_1 \}$. Moreover $j_{H_{\nu}}^{-1}(\text{cl}(S, H)) \neq \emptyset$ if and only if $\nu \sim \nu'$ for some $\nu' \in I(S)^F$, i.e. $[\nu] \in [I(S)^F]$.

In the above, $\{\mu'_i \} \subset \{\mu_i \}$ means the containment of multisets of integers. From this lemma we obtain that

$$\{[\nu] \in [I(G)^F] : j_{H_{\nu}}^{-1}(\text{cl}(S, H)) \neq \emptyset \} = [I(S)^F].$$
In particular,
\[ \sum_{[i] \in [I(S)^F]} \# [i] = \sum_{[i] \in [I(G)^F]} \# [i]. \]

Denote \#([i] \cap I(S)^F) by \( C_{S,i} \). If \( S \) and \([i]\) correspond to partitions \( \lambda \) and \( \mu \) respectively, then we also write \( C_{S,i} \) as \( C_{\lambda,\mu} \). We now give a formula for \( C_{\lambda,\mu} \). By the choice of \( i \), \( \{ \mu_i \} \subset \{ \lambda_i \} \). Let us write
\[ (5.5) \quad \lambda = [n_1^{a_1}, \ldots, n_l^{a_l}] \quad \text{and} \quad \mu = [n_1^{b_1}, \ldots, n_l^{b_l}], \]
where \( n_i \)'s are distinct so that \( 0 \leq b_i \leq a_i, i = 1, \ldots, l \). If \( i \in I(S)^F \) with \( \delta_i = 0 \), then by Lemma 4.2, \( G_{i}^F \cong T^F \times U_m(F_q) \), where \( T \) corresponds to a partition \( \lambda' = (\lambda_i') \) and \( m = n - |\lambda'| \). We have \( \{ \lambda_i' \} \subset \{ \lambda_i \} \). Note that if \( i \sim i' \), then \( \{ \lambda_i' \} = \{ \lambda_i' \} \). It follows that
\[ C_{\lambda,\mu} = \prod_{i=1}^{l} \left( \frac{a_i}{b_i} \right). \]

For example, \( C_{[2,1^2],[2,1]} = 2 \). We have that
\[ \langle R_{T_1 \times T_2, \theta \otimes 1}, R_{S,1}^H \rangle_{H(F_q)} = \sum_{i \in I(S)^F} \frac{(-1)^{\mathrm{rk}(G_i) + \mathrm{rk}(H_i) + \mathrm{rk}(T) + \mathrm{rk}(S)}}{|N_H(i, S)^F|} \sum_{v, \xi} \langle \theta \otimes 1 \rangle_{v, \xi} Z^F_{\xi}. \]

If \( j_{G_i}^{-1}(\mathrm{cl}(T_1 \times T_2, G)) \neq \emptyset \) and \( G_{i}^F \cong T_i^F \times U_{n'}(F_q) \), then there are two cases for \( i \):

Case (1): \( T_{i}^F \cong T_{i}^F \times T_{i}^F \) for some torus \( T_{i}^F \);
Case (2): \( T_{i}^F \cong T_{i}^F \times T_{i}^F \) for any torus \( T_{i}^F \).

Accordingly, for a fixed \( T_2 \) we have
\[ (5.6) \quad \sum_{[i] \in [I(G)^F]} \frac{(-1)^{\mathrm{rk}(G_i) + \mathrm{rk}(H_i) + \mathrm{rk}(T) + \mathrm{rk}(S)}}{|N_H(i, S)^F|} \sum_{v, \xi} \langle \theta \otimes 1 \rangle_{v, \xi} Z^F_{\xi} \]
\[ = \sum_{[i] \in [I(G)^F]} \frac{(-1)^{\mathrm{rk}(G_i) + \mathrm{rk}(H_i) + \mathrm{rk}(T) + \mathrm{rk}(S)}}{|N_H(i, S)^F|} \sum_{v, \xi} \langle \theta \otimes 1 \rangle_{v, \xi} Z^F_{\xi} \]
\[ + \sum_{[i] \in [I(G)^F]} \frac{(-1)^{\mathrm{rk}(G_i) + \mathrm{rk}(H_i) + \mathrm{rk}(T) + \mathrm{rk}(S)}}{|N_H(i, S)^F|} \sum_{v, \xi} \langle \theta \otimes 1 \rangle_{v, \xi} Z^F_{\xi}. \]

For convenience, abbreviate \( \sum_{[i] \in [I(G)^F]} \langle \theta \otimes 1 \rangle_{v, \xi} Z^F_{\xi} \) by \( \sum_{[i]} \) for \( i = 1, 2 \).

Assume that \( T_2 \) corresponds to a partition \( \mu \) of \( m \). Then we have two types of \( T_2 \):

(A) \( \mu \) only contains odd parts..
(B) \( \mu \) contains an even part.

To simplify the presentation, let us first give details for a special case of type (A), namely,
\[(A_0) \mu = [1^m].\]

The general situation will be similar. Then in this special case \(T_1 \times T_2\) corresponds to the partition \([2\ell, 1^m]\). If \(\iota\) is in Case (1), then by Lemma 5.3, the toral part \(T'\) of \(G_1\) corresponds to \([2\ell, 1^k]\) for some \(k \leq m\). Thus \(T''\) corresponds to \((1^k)\). Note that

\[Z_{1} \cong T_1 \times T'' \times I_{n'} \hookrightarrow T_1 \times T_2.\]

Then \(W_G(T_1 \times T_2, \theta \otimes 1)^F = \emptyset\) because \(x(\theta \otimes 1)|_{T_1} \neq 1\). In other words, we obtain that \(\sum_{(1)} = 0\). For \(\iota\) in Case (2), we have \(T_1^F \subset U_{n'}(F_q)\) for some \(n'\), and thus

\[G_{\iota}^F \cong U_{n'}(F_q) \times T^F = U_{n'}(F_q) \times U_1(F_q) \times \cdots \times U_1(F_q),\]

\[T_1^F \times T_2^F\]

If \(T^F\) is isomorphism to a product of \(k\) copies of \(U_1(F_q)\), then we denote \(\iota\) by \(\iota_k\). Note that \([\iota_k] = [\iota_{k'}]\) if and only if \(k = k'\). Then in Case (2), \(j_{G_{\iota}^{-1}}(\cl(T_1 \times T_2, G_{\iota})) \neq \emptyset\) if and only if \([\iota] = [\iota_k]\) for some \(k = 0, 1, \ldots, m\). Hence we obtain that

\[\sum_{(2)} (-1)^{\text{rk}(G_{\iota}) + \text{rk}(H_{\iota}) + \text{rk}(T) + \text{rk}(S)} C_{S, \iota} \frac{|W_{T_1 \times T_2, \theta \otimes 1}|}{|W_G(T_1 \times T_2)|^F} = \sum_{k=0}^{m} \sum_{[\iota_k] \in [T(G)^F]} (-1)^{\text{rk}(G_{\iota_k}) + \text{rk}(H_{\iota_k}) + \text{rk}(T) + \text{rk}(S)} C_{S, \iota_k} \frac{|W_G(T_1 \times T_2)|^F}{|W_{G_{\iota_k}}(T_1 \times T_2)|^F},\]

noting that \(x(\theta \otimes 1)|_{T_1} = 1\) for any \(x \in W_G(T_1 \times T_2)^F\).

Using the relation

\[\text{rk}(U_n(F_q)) - \text{rk}(U_{n-1}(F_q)) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even,} \end{cases}\]

we deduce that

\[(-1)^{\text{rk}(G_{\iota_k}) + \text{rk}(H_{\iota_k})} = (-1)^{n-k}.\]

Recall that \(W_n = S_n\) is the Weyl group of the diagonal torus in \(U_n\). For a partition \(\lambda\) of \(n\), we denote \(w \sim \lambda\) if the conjugacy class of \(w\), hence the \(F\)-conjugacy class of \(ww_0\), corresponds to \(\lambda\), where \(w_0\) is the element of maximal length. In this case we also write \(T_{\lambda} = T_{w_0}.\)
Substituting (5.7) into (5.2), we compute that

\[ \langle R_{T_1 \times T_2, \theta \otimes 1}^G, R_{H}^H \rangle_{H(F_q)} = \frac{1}{|W_n|} \sum_{w \in W_n} \sigma_\lambda(w w_0) \langle R_{T_1 \times T_2, \theta \otimes 1}^G, R_{T_w, \theta}^H \rangle_{H(F_q)} \]

\[ = \frac{1}{|W_n|} \sum_{k=0}^{m} \sum_{\substack{w \in W_n \\ j^{-1}_{H, k}(\text{cl}(T_w, H)) \neq \emptyset}} (-1)^{rk(G_k) + rk(H_k) + rk(T) + rk(T_w)} \sigma_\lambda(w w_0) C_{T_w, t_{k}} \frac{|W_G(T_1 \times T_2)_F|}{|W_{G_k}(T_1 \times T_2)_F|} \]

\[ = \frac{(-1)^{rk(T)}}{|W_n|} \sum_{k=0}^{m} (-1)^{n-k} \frac{|W_G(T_1 \times T_2)_F|}{|W_{G_k}(T_1 \times T_2)_F|} \sum_{w \in W_n \ \ j^{-1}_{H, k}(\text{cl}(T_w, H)) \neq \emptyset} C_{T_w, t_{k}} \text{sgn}(w w_0) \sigma_\lambda(w) \]

where \( \mu' \) runs over partitions of \( n - k \).

We now turn to the general case. Assume that \( T_2 \) corresponds to a partition \( \mu = [\mu_1, \cdots, \mu_d] \) of \( m \). For a subset \( D = \{d_1, d_2, \cdots, d_{|D|}\} \) of \( d := \{1, 2, \cdots, d\} \), if

\[ G^F_t \cong T'^F \times U_{n+1-|D|}(F_q) \]

where \( T' \) is a torus corresponding to the partition

\[ \mu_D := [\mu_{d_1}, \cdots, \mu_{d_{|D|}}], \]

then we denote \( \iota \) by \( \iota_{\mu, D} \). Note that \( \mu = \mu_D \) in the above notation. By Lemma 5.3, \( \iota \) is in Case (2) if and only if \( \iota = \iota_{\mu, D} \) for some \( \mu \) and \( D \) as above. Note that \( D \) can be empty. For the fixed partition \( \mu \), we say that two subsets \( D \) and \( D' \) of \( d \) are equivalent if \( \mu_D = \mu_{D'} \). For example, if \( \mu = [2, 1, 1] \), then \( D = \{1, 2\} \) and \( D' = \{1, 3\} \) are equivalent. Denote by \( [D] \) the equivalence class of \( D \).

We now consider general type (A). In the same manner, one shows that only Case (2) contributes to \( \langle R_{T_1 \times T_2, \theta \otimes 1}^G, R_{H}^H \rangle_{H(F_q)} \). By Lemma 5.3, there is a bijection between the set of \( [\iota] \) in Case (2) satisfying \( j^{-1}_{G, F}(\text{cl}(T_1 \times T_2, G)) \neq \emptyset \) and the set of equivalence classes of subsets of \( \{1, \cdots, d\} \). Applying the arguments for type (A0), we similarly have

\[ \langle R_{T_1 \times T_2, \theta \otimes 1}^G, R_{S, 1}^H \rangle_{H(F_q)} \]

\[ = \sum_{(2)} \frac{(-1)^{rk(G_{\mu, D}) + rk(H_{\mu, D}) + rk(T) + rk(S)}}{|N_H(\iota_{\mu, D}, S)_F|} \langle (\theta \otimes 1)_\iota, 1_L \rangle_{Z_{\mu, D}^F} \]

\[ = \sum_{\substack{[D] \\ j^{-1}_{H, \mu, D}(\text{cl}(S, H)) \neq \emptyset}} \frac{(-1)^{rk(G_{\mu, D}) + rk(H_{\mu, D}) + rk(T) + rk(S)}}{|W_G(T_1 \times T_2)_F|} C_{S, \iota_{\mu, D}} \frac{|W_{G_{\mu, D}}(T_1 \times T_2)_F|}{|W_{G_{\mu, D}}(T_1 \times T_2)_F|}. \]
Note that \( \text{rk}(G_{\mu,D}) + \text{rk}(H_{\mu,D}) \) is even (resp. odd) if \( n - |\mu_D| \) is even (resp. odd). Substituting (5.8) in to (5.2) gives that

\[
\langle R^G_{T_1 \times T_2, \theta \otimes 1}, R^H_\lambda \rangle_{H(\mathbb{F}_q)} = \frac{(-1)^{\text{rk}(T)}}{|W_n|} \sum_{[D]} (-1)^{n-|\mu_D|} \frac{|W_G(T_1 \times T_2)^F|}{|W_{G_{\mu,D}}(T_1 \times T_2)^F|} \times \sum_{\mu'} \sum_{w \in W_n, w \sim [\mu_D, \mu']} C_{[\mu_D, \mu'], \mu_D} \text{sgn}(w) \sigma(\lambda(w)),
\]

where \([D]\) runs over equivalence classes of subsets of \( d \) and \( \mu' \) runs over partitions of \( n - |\mu_D| \).

We next consider type (B). In Case (2) we have the same discussion as in type (A). For Case (1), assume that \( T_2^F \cong T_1^F \times \cdots \times T_1^F \times T_3^F \), where \( T_3^F \) has no component equal to \( T_1^F \). Then there is \( x \in W_G(T_1 \times T_2)^F \) such that \( x(\theta \otimes 1)|_{Z_1} = 1 \). For example, if \( G_{\theta}^F \cong T_1^F \times U_m(\mathbb{F}_q) \) and \( T_1 \times T_2 \cong T_1 \times T_1 \times T_3 \), then there is \( x \in W_G(T_1 \times T_2)^F \) such that \( x(\theta \otimes 1 \otimes 1) = 1 \otimes \theta \otimes 1 \). Thus \( x(\theta \otimes 1 \otimes 1)|_{Z_1} = 1 \). Then we have

\[
\sum_{(1)} = \sum_{j^{-1}_{\mu_D} \in \text{cl}(\lambda, \theta)} \frac{(-1)^{\text{rk}(G_{\mu,D}) + \text{rk}(H_{\mu,D})} + \text{rk}(T) + \text{rk}(S)}{C_{\mu_D, \mu_D} |W_{G_{\mu,D}}(T_1 \times T_2)^F|},
\]

where in the first sum, \([D]\) runs over equivalence classes of subsets of \( d \) such that \( \mu_D \) contains an even part.

Combining the previous calculations, it follows that

\[
\langle R^G_{T_1 \times T_2, \theta \otimes 1}, R^H_\lambda \rangle_{H(\mathbb{F}_q)} = \sum_{(1)} + \sum_{(2)} = \frac{(-1)^{\text{rk}(T)}}{|W_n|} \sum_{[D]} (-1)^{n-|\mu_D|} \frac{|W_G(T_1 \times T_2, \theta \otimes 1, \mu_D)|}{|W_{G_{\mu,D}}(T_1 \times T_2)^F|} \times \sum_{\mu'} \sum_{w \in W_n, w \sim [\mu_D, \mu']} C_{[\mu_D, \mu'], \mu_D} \text{sgn}(w) \sigma(\lambda(w)),
\]

where in the first sum \( D \) runs over equivalence classes of subsets of \( d \) and in the second sum \( \mu' \) runs over partitions of \( n - |\mu_D| \).

In summary, we have

**Proposition 5.4.** Let \( T = T_1 \times T_2 \) be an \( F \)-stable maximal torus of \( U_{n+1} \), where \( T_1 \) is an \( F \)-stable minisotropic maximal torus of \( G_{\ell} \), and \( T_2 \) is an \( F \)-stable maximal torus of \( U_m \) corresponding to a partition \( \mu = [\mu_1, \ldots, \mu_d] \) of \( m := n - 2\ell + 1 \). Let \( \theta \) be a regular character of \( T_1^F \) and \( \lambda \) be a partition of \( n \). Then

\[
\langle R^U_{T_1 \times T_2, \theta \otimes 1}, R^U_\lambda \rangle_{U_n(\mathbb{F}_q)} = \frac{(-1)^{\text{rk}(T)}}{|W_n|} \sum_{[D]} (-1)^{n-|\mu_D|} \frac{|W_G(T_1 \times T_2, \theta \otimes 1, \mu_D)|}{|W_{G_{\mu,D}}(T_1 \times T_2)^F|} \times \sum_{\mu'} \sum_{w \in W_n, w \sim [\mu_D, \mu']} C_{[\mu_D, \mu'], \mu_D} \text{sgn}(w) \sigma(\lambda(w)),
\]

where \([D]\) runs over equivalence classes of subsets of \( d \), \( \mu' \) runs over partitions of \( n - |\mu_D| \).

Moreover, we can make a slight refinement and obtain the multiplicity \( \langle R^U_{T_1 \times T_2, \theta \otimes 1}, R^U_\lambda \rangle_{U_n(\mathbb{F}_q)} \) for any pair \((T_1, \theta)\) with \( 1 \notin (T_1, \theta) \). Here given a pair \((T, \theta)\) for \( U_m \), we say that \( 1 \notin (T, \theta) \) if \((T, \theta)\) is not of the form \((T_1^\ell \times T_2^\ell, \theta \otimes 1)\).
Proposition 5.5. Let $T = T_1 \times T_2$ be an $F$-stable maximal torus of $U_{n+1}$, where $T_1$ is an $F$-stable maximal torus of $U_{n+1-m}$, and $T_2$ is an $F$-stable maximal torus of $U_m$ corresponding to a partition $\mu = [\mu_1, \cdots, \mu_d]$ of $m < n$. Assume that $1 \notin (T_1, \theta)$ and $\lambda$ is a partition of $n$. Then

$$
(R_{T_1}^{U_{n+1}}, R_{\lambda}^{U_n})_{U_n(\mathbb{F}_q)} = \frac{(-1)^{|\text{T}(T)|}}{|W_n|} \sum_{[D]} (-1)^{n-|\mu_D|} \frac{|W_{T_1}^{T_2, \theta \otimes 1, \mu_D}|}{|W_{G_D, D}(T_1 \times T_2)^F|} \times \sum_{\mu'} \sum_{w \in W_n, w \sim [\mu_D, \mu']} C_{[\mu_D, \mu'], \mu_D} \text{sgn}(w) \sigma_\lambda(w)
$$

where $[D]$ runs over equivalence classes of subsets of $d$, and $\mu'$ runs over partitions of $n - |\mu_D|$.

5.2. Proof of Theorem 5.1. Recall that for a pair of partitions $\lambda, \mu$ as in (5.5), one has

$$
C_{\lambda, \mu} = \prod_{i=1}^l \left( \frac{a_i}{b_i} \right).
$$

We keep the notations $T_1, T_2$ and $\mu$ of Proposition 5.5. For a fixed subset $D$ of $d$, let $T_3$ be the $F$-stable maximal torus of $U_{|\mu_D|}$ corresponding to the partition $\mu_D$. Then

$$
|W_{T_1 \times T_2, \theta \otimes 1, \mu_D}| = C_{\mu_D, \mu_D} |W_{\mu_D}(T_3)^F| \cdot |W_{\mu_D}(T_1 \times T_2)^F|.
$$

In particular, if $D = d$, then

$$
|W_{T_1 \times T_2, \theta \otimes 1, \mu_d}| = |W_{\mu_d}(T_2)^F| \cdot |W_{n+1-\mu_d} (T_1)^F|.
$$

By the Littlewood-Richardson rule, we have the following result.

Lemma 5.6. Let $\lambda = [\lambda_1, \cdots, \lambda_l]$ be a partition of $n$. If $m > \lambda_1$, then

$$
\langle \sigma_\lambda, 1 \rangle_{W_m} = 0.
$$

Let $\lambda' = [\lambda_2, \cdots, \lambda_l]$ be the partition of $n - \lambda_1$ obtained by removing the first row of $\lambda$, and $\sigma$ be an irreducible representation of $W_{n-\lambda_1}$. Then

$$
\langle \sigma_\lambda, \sigma \otimes 1 \rangle_{W_{n-\lambda_1} \times W_{\lambda_1}} = \begin{cases} 1, & \text{if } \sigma = \sigma_{\lambda'}, \\ 0, & \text{otherwise}. \end{cases}
$$

In particular,

$$
\langle \sigma_\lambda, 1 \rangle_{W_{\lambda_1}} = 1.
$$

Proof. It can be proved in the same way as [AM, Proposition 3.4]. It should be mentioned that our notation of the representation of $\sigma_\lambda$ differs from that of [AM], where the representation $\sigma_\lambda$ in [AM] is equal to $\sigma_\lambda$ in our paper.

Now we can explicitly calculate the multiplicity formula.

Proposition 5.7. Keep the notations and assumptions in Proposition 5.5. Let $\lambda$ be a partition of $n$ into $k$ rows, and $\lambda'$ be the partition of $n - k$ obtained by removing the first column of $\lambda$. Then the following hold.

(i) If $|\mu_D| < n - k$, then

$$
\sum_{\mu'} \sum_{w \in W_n, w \sim [\mu_D, \mu']} C_{[\mu_D, \mu'], \mu_D} \text{sgn}(w) \sigma_\lambda(w) = 0.
$$
(ii) If $|\mu| = n - k$, then

$$\langle R_{T_1 \times T_2, \theta \otimes 1}^{U_{n+1}}, R_{\lambda}^{U_n}\rangle_{U_n(F_q)} = (-1)^{r_k(T)+1} \sigma_{\lambda}(w_{\mu}),$$

where $w_{\mu}$ is an element of $W_m$ in the conjugacy class corresponding to $\mu$.

**Proof.** Denote by $[w]$ the conjugacy class of a element $w$ of $W_n$. We have that

$$\sum_{\mu'} \sum_{w \sim [\mu, \mu']} C_{[\mu, \mu'], [\mu']} \cdot \#[w] \sigma_{\lambda}(w)$$

$$= \sum_{\mu'} \sum_{w \sim [\mu, \mu']} C_{[\mu, \mu'], [\mu']} \cdot \#[w] \sigma_{\lambda}(w)$$

$$= \sum_{\mu'} \sum_{w \sim [\mu, \mu']} C_{[\mu, \mu'], [\mu']} \cdot \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sigma_{\lambda}(w)$$

$$= \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sum_{\mu'} \sum_{[w] \sim [\mu, \mu']} \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sigma_{\lambda}(w)$$

$$= \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sum_{\mu'} \sum_{w \sim [\mu, \mu']} \#[w] \sigma_{\lambda}((w_{\mu}, 1)) \sigma_{\lambda}((1, w))$$

Then (i) follows easily from Lemma 5.6.

If $|\mu| = n - k$, then $|\mu_D| \geq n - k$ if and only if $\mu_D = \mu$. It follows that

$$\langle R_{T_1 \times T_2, \theta \otimes 1}^{U_{n+1}}, R_{\lambda}^{U_n}\rangle_{U_n(F_q)}$$

$$= (-1)^{r_k(T)+1} \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sum_{\mu'} \sum_{w \sim [\mu, \mu']} C_{[\mu, \mu'], [\mu']} \cdot \#[w] \sigma_{\lambda}(w)$$

$$= (-1)^{r_k(T)+1} \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sum_{\mu'} \sum_{w \sim [\mu, \mu']} \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sigma_{\lambda}(w)$$

$$= (-1)^{r_k(T)+1} \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sum_{\mu'} \sum_{w \sim [\mu, \mu']} \frac{|W_{n}^{F}|}{|W_{[\mu']}|} \sigma_{\lambda}(w)$$

$$= (-1)^{r_k(T)+1} \sigma_{\lambda}(w_{\mu})$$
which gives (ii).

Combining previous results, we are now ready to give the main result of this subsection.

**Proposition 5.8.** Keep the notations and assumptions in Proposition 5.5. Let \( \lambda \) be a partition of \( n \) into \( k \) rows, and \( \lambda' \) be the partition of \( n - k \) obtained by removing the first column of \( \lambda \). Then the following hold.

(i) If \( m < n - k \), then

\[
\langle R_{T_1 \times T_2, \theta \otimes 1}^{U_n+1} \rangle_{U_n(F_q)} = 0.
\]

In particular, in this case if \( P \) is an \( F \)-stable maximal parabolic subgroup of \( U_{n+1} \) with Levi factor \( G \), \( \xi \), and \( \tau \) is an irreducible cuspidal representation of \( GL_n(F_q) \), then for any representation \( \pi' \) of \( U_m(F_q) \),

\[
\langle I_{P}^{U_{n+1}}(\tau \otimes \pi'), R_{\lambda}^{U_{n}} \rangle_{U_n(F_q)} = 0.
\]

(ii) If \( k = 2\ell - 1 \) so that \( m = n - k \), then for an irreducible representation \( \pi \) of \( U_{m}(F_q) \),

\[
\langle I_{P}^{U_{n+1}}(\tau \otimes \pi'), \pi\lambda \rangle_{U_n(F_q)} = \begin{cases} 1, & \text{if } \pi' \cong \pi\lambda', \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** (i) follows from Proposition 5.7. For (ii), if \( \pi' = \pi_\nu \) is unipotent with a partition \( \nu \) of \( m \), then by Proposition 5.7 (ii) we obtain that

\[
\langle I_{P}^{U_{n+1}}(\tau \otimes \pi'), R_{\lambda}^{U_{n}} \rangle_{U_n(F_q)} = \pm \langle I_{P}^{U_{n+1}}(\tau \otimes R_\nu), R_{\lambda}^{U_{n}} \rangle_{U_n(F_q)}
\]

\[
= \pm \frac{1}{|W_m|} \sum_{w \in W_m} \sigma_\nu(ww_0) \langle R_{T_1 \times T_2, \theta \otimes 1}^{U_n+1} \rangle_{U_n(F_q)}
\]

\[
= \pm \frac{1}{|W_m|} \sum_{w \in W_m} \sigma_\nu(w) \langle R_{T_1 \times T_2, \theta \otimes 1}^{U_n+1} \rangle_{U_n(F_q)}
\]

\[
= \pm \frac{1}{|W_m|} \sum_{\mu} \sum_{w \in W_m \atop w \sim \mu} \sigma_\nu(w) \langle R_{T_1 \times T_2, \theta \otimes 1}^{U_n+1} \rangle_{U_n(F_q)}
\]

\[
= \pm \frac{1}{|W_m|} \sum_{\mu} \# \{ w_\mu \} \sigma_\nu(w_\mu) \langle R_{T_1 \times T_2, \theta \otimes 1}^{U_n+1} \rangle_{U_n(F_q)}
\]

\[
= \pm \frac{1}{|W_m|} \sum_{\mu} \# \{ w_\mu \} \sigma_\nu(w_\mu) \text{sgn}(w_\mu) \sigma_{\lambda'}(w_\mu)
\]

\[
= \pm \langle \sigma_{\nu}, \sigma_{\lambda'} \rangle_{W_m}
\]

\[
= \begin{cases} 1, & \text{if } \nu = \lambda', \\ 0, & \text{otherwise}, \end{cases}
\]

where \( \mu \) runs over partitions of \( m \). Suppose that \( \pi' \) is not unipotent. Then there is a geometric conjugacy class \( \kappa \) of \( U_m \) such that \( \pi' \in R_{T_1, \theta}^{G} \) only if pair \((T, \theta) \in \kappa \). Since \( \pi' \) is not unipotent, there is an integer \( m' < m \) such that every pair \((T, \theta) \in \kappa \) has the form \((T, \theta) = (T'_1 \times T'_2, \theta' \otimes 1)\) with \( 1 \notin (T'_1, \theta') \) where \( T'_1 \) is an \( F \)-stable maximal torus of \( U_{m'} \), and \( T_2 \) is an \( F \)-stable maximal torus of \( U_{n+1-m'} \). Hence for any pair \((T, \theta) \in \kappa \),

\[
\langle R_{T_1 \times T_2, \theta \otimes (\theta' \otimes 1)}^{U_n+1} \rangle_{U_n(F_q)} = \langle R_{(T_1 \times T_2, \theta' \otimes 1) \times T_2, \theta' \otimes 1}^{U_n+1} \rangle_{U_n(F_q)} = 0
\]
noting that \( 1 \notin ((T_1 \times T_1') (\theta \otimes \theta')) \) and \( T_1 \times T_1' \) an \( F \)-stable maximal torus of \( U_{k+m'} \). Since \( \pi' \) is a linear combination of \( R_{T_1 \times (T_2' \times T_2'), \theta \otimes (\theta' \otimes 1)}^{U_{n+1}} \) for pairs \( (T_1' \times T_2', \theta' \otimes 1) \in \kappa \), we obtain that
\[
\langle I_{F_n^{n+1}}(\tau \otimes \pi'), \pi \rangle_{U_n(F_q)} = 0.
\]
This finishes the proof of (ii). \( \square \)

Finally, Theorem 5.1 follows from Proposition 5.2 and Proposition 5.8.

6. Fourier-Jacobi case

We have established the Bessel descents of unipotent representations of finite unitary groups. In this section we deduce the Fourier-Jacobi case from the Bessel case by the standard arguments of theta correspondence and see-saw dual pairs, which are used in the proof of local Gan-Gross-Prasad conjecture (see [GI, Ato]).

6.1. Weil representation and theta lifting. Let \( \omega_{\text{Sp}_{2N}} \) be the character of the Weil representation (cf. [Ger]) of the finite symplectic group \( \text{Sp}_{2N}(\mathbb{F}_q) \), which depends on a nontrivial additive character \( \psi \) of \( F \). Let \( (G, G') \) be a reductive dual pair in \( \text{Sp}_{2N} \), and write \( \omega_{G, G'} \) for the restriction of \( \omega_{\text{Sp}_{2N}} \) to \( G^F \times G'^F \). Then it decomposes into a direct sum
\[
\omega_{G, G'} = \bigoplus_{\pi, \pi'} m_{\pi, \pi'} \pi \otimes \pi'
\]
where \( \pi \) and \( \pi' \) run over irreducible representations of \( G^F \) and \( G'^F \) respectively, and \( m_{\pi, \pi'} \) are nonnegative integers.. We can rearrange this decomposition as
\[
\omega_{G, G'} = \bigoplus_{\pi} \pi \otimes \Theta_{G, G'}(\pi)
\]
where \( \Theta_{G, G'}(\pi) = \bigoplus_{\pi'} m_{\pi, \pi'} \pi' \) is a (not necessarily irreducible) representation of \( G'^F \), called the (big) theta lifting of \( \pi \) from \( G^F \) to \( G'^F \). Write \( \pi' \subset \Theta_{G, G'}(\pi) \) if \( \pi \otimes \pi' \) occurs in \( \omega_{G, G'} \), i.e. \( m_{\pi, \pi'} \neq 0 \). We remark that in general one only has
\[
\pi \subset \Theta_{G, G'}(\Theta_{G, G'}(\pi)),
\]
where the equality does not necessarily hold.

Consider a dual pair of unitary groups \( (G, G') = (U_n, U_{n'}) \) in \( \text{Sp}_{2nm'} \). Denote \( \omega_{G, G'} \) by \( \omega_{n, n'} \), and \( \Theta_{G, G'} \) by \( \Theta_{n, n'} \). In particular, denote by \( \omega_n \) the restriction of \( \omega_{\text{Sp}_{2n}} \) to \( U_n(\mathbb{F}_q) \). By [AM, Theorem 3.5], theta lifting between unitary groups sends unipotent representations to unipotent representations, and we will recall the explicit correspondence below.

We say that two partitions \( \mu = [\mu_i] \) and \( \mu' = [\mu'_i] \) are close if \( |\mu_i - \mu'_i| \leq 1 \) for every \( i \), and that \( \mu \) is even if \( \# \{ i | \mu_i = j \} \) is even for any \( j > 0 \), i.e. every part of \( \mu \) occurs with even multiplicities. Let
\[
\mu \cap \mu' = [\mu_i | \{ i | \mu_i = \mu'_i \}]
\]
be the partition formed by the common parts of \( \mu \) and \( \mu' \). Following [AMR], we say that \( \mu \) and \( \mu' \) are 2-transverse if they are close and \( \mu \cap \mu' \) is even. In particular, if \( \mu \) and \( \mu' \) are close and \( \mu \cap \mu' = \emptyset \), then \( \mu \) and \( \mu' \) are 2-transverse, and in this case we say that they are transverse. For example, let \( \lambda = [\lambda_1, \ldots, \lambda_k] \) be a partition of \( n \), and let \( \lambda_* = [\lambda_2, \ldots, \lambda_k] \) be the partition of \( n - \lambda_1 \) obtained by removing the first row of \( \lambda \). Then \( ^t \lambda \) and \( ^t \lambda_* \) are transverse. Moreover, \( \lambda_* \) is the unique partition of \( n - \lambda_1 \) such that \( ^t \lambda \) and \( ^t \lambda_* \) are 2-transverse.
For partitions \( \lambda \) and \( \lambda' \) of \( n \) and \( n' \) respectively, denote the multiplicity of \( \pi_\lambda \otimes \pi_{\lambda'} \) in \( \omega_{n,n'} \) by \( m_{\lambda,\lambda'} \). By [AMR] Theorem 4.3, Lemma 5.3 and Lemma 5.4, we have

**Proposition 6.1.** With above notations,

\[
m_{\lambda,\lambda'} = \begin{cases} 
1, & \text{if } \^t \lambda \text{ and } \^t \lambda' \text{ are 2-transverse}, \\
0, & \text{otherwise}.
\end{cases}
\]

In other words,

\[
\Theta_{n,n'}(\pi_\lambda) = \bigoplus_{\text{\( \lambda \) and \( \lambda' \) are 2-transverse}} \pi_{\lambda'}
\]

In particular, \( \Theta_{n,n'}(\pi_\lambda) = 0 \) if \( n' < n - \lambda_1 \).

The next result shows that theta lifting and parabolic induction are compatible.

**Proposition 6.2.** Let \( \tau \) be an irreducible cuspidal representation of \( \text{GL}_\ell(F_{q^2}) \), \( \pi \) be an irreducible representation of \( \text{U}_n(F_q) \), and \( \pi' := \Theta_{n,n'}(\pi) \). Then we have

\[
\Theta_{n+2\ell,n'+2\ell}(R_{U_n \times U_n}^{U_{n+2\ell}}(\tau \otimes \pi)) = R_{U_{n'} \times U_{n'}^{2\ell}}(\tau \otimes \pi').
\]

**Proof.** Let \( ^* R_{G_x \times U_n}^{U_{n+2\ell}} \) be the Jacquet functor, which is adjoint to the parabolic induction \( R_{G_x \times U_n}^{U_{n+2\ell}} \). By [MVW, Chap. 3, IV, th. 5],

\[
(^* R_{G_x \times U_n}^{U_{n+2\ell}} \otimes 1)(\omega_{n+2\ell,n'+2\ell}) = \bigoplus_{i=0}^{\ell} R_{U_n \times (G_{\ell-i} \times G_i) \times G_i \times U_{n'+2(\ell-i)}}(\omega_{n,n'+2(\ell-i)} \otimes 1_{G_{\ell-i}(F_q)} \otimes R_{G_i}),
\]

where \( R_{G_i} \) is the regular representation of \( G_i(F_{q^2}) = \text{GL}_i(F_q^2) \). Since \( \tau \) is cuspidal, \( \tau \otimes \pi \) only appears in the summand with \( i = \ell \), and the proposition follows. \( \square \)

**6.2. See-saw dual pairs.** Recall the general formalism of see-saw dual pairs. Let \( (G,G') \) and \( (H,H') \) be two reductive dual pairs in a symplectic group \( \text{Sp}(W) \) such that \( H \subset G \) and \( G' \subset H' \). Then there is a see-saw diagram

\[
\begin{array}{ccc}
G & \rotatebox{90}{\text{H}} & G' \\
\rotatebox{90}{\text{H}} & & \\
H & & G'
\end{array}
\]

and the associated see-saw identity

\[
(\Theta_{G',G}(\pi_{G'}), \pi_H)_H = (\pi_{G'}, \Theta_{H,H'}(\pi_H))_{G'},
\]

where \( \pi_H \) and \( \pi_{G'} \) are representations of \( H \) and \( G' \) respectively.

In our case, if we put

\[
G = U_n \times U_n, \quad G' = U_n \times U_1, \quad H = U_n, \quad \text{and} \quad H' = U_{n+1},
\]

then the left-hand side of the see-saw identity concerns the basic case of Fourier-Jacobi model whereas the right-hand side concerns the basic case of Bessel model. In general, we need Proposition 5.2 and the following result.
Proposition 6.3. Let \( \pi \) be an irreducible unipotent representation of \( U_n(\mathbb{F}_q) \), and \( \pi' \) be an irreducible representation of \( U_m(\mathbb{F}_q) \) with \( n > m \) and \( m \equiv n \mod 2 \). Let \( P \) be an \( F \)-stable maximal parabolic subgroup of \( U_m \) with Levi factor \( G_\ell \times U_m \) (so that \( m + 2\ell = n \)) and let \( \tau \) be an irreducible cuspidal representation of \( GL_\ell(\mathbb{F}_q^2) \). Then we have
\[
m(\pi, \pi') = \langle \pi \otimes \bar{\nu}, \pi' \rangle_H(\mathbb{F}_q) = \langle \pi \otimes \omega_n, I^U_n(\tau \otimes \pi') \rangle_U(\mathbb{F}_q),
\]
where the data \((H, \nu)\) is given by (1.6).

Similar to Proposition 5.2, the proof of Proposition 6.3 is an adaptation of that of [GGP1, Theorem 16.1] to unipotent representations, which will be omitted here. As mentioned in the Introduction, our formulation of multiplicities differs from that in the Gan-Gross-Prasad conjecture by taking contragradient of \( \pi \), but we may take the advantage that unipotent representations are self-dual. This fact should be well-known to the experts, but we are not able to find a reference so we provide a short proof using theta correspondence.

Proposition 6.4. If \( \pi \) is a unipotent representation of \( U_n(\mathbb{F}_q) \), then \( \pi \cong \pi^\vee \).

Proof. It is well-known that Weil representation is self-dual. For instance by [Ger, Proposition 1.4], its character \( \omega \) takes positive value, hence \( \omega = \omega^\vee \). It follows that \( \Theta(\pi^\vee) \cong \Theta(\pi)^\vee \). We shall prove the proposition by induction on \( n \), which is clear if \( n = 1 \). Assume that \( n > 1 \) and \( \pi = \pi_\lambda \), where \( \lambda = [\lambda_1, \ldots, \lambda_k] \) is a partition of \( n \). If \( \lambda = [n] \) or \([1^n]\), then \( \pi = 1 \) or \( St_n \) is self-dual. So let us assume that it is neither of the two cases. By Proposition 6.1 and the discussion above it, for the dual pair \( U_n \times U_{n-\lambda_1} \) we have
\[
\Theta_{n,n-\lambda_1}(\pi_\lambda) = \pi_{\lambda_*},
\]
where \( \lambda_* = [\lambda_2, \ldots, \lambda_k] \) is the partition obtained by removing the first row of \( \lambda \). By induction hypothesis, \( \pi_{\lambda_*} \) is self-dual hence
\[
\Theta_{n,n-\lambda_1}(\pi_\lambda^\vee) = \pi_{\lambda_*}.
\]
Since \( \Theta_{n-\lambda_1,n}(\pi_{\lambda_*}) \) is a direct sum of unipotent representations, \( \pi_{\lambda_*}^\vee \subset \Theta_{n-\lambda_1,n}(\pi_{\lambda_*}) \) is unipotent and thus \( \pi_{\lambda_*} \cong \pi_{\mu} \) for some partition \( \mu \) of \( n \). Similarly write \( \mu = [\mu_1, \ldots, \mu_{k'}] \) and \( \mu_* = [\mu_2, \ldots, \mu_{k'}] \). By Proposition 6.1 again,
\[
\Theta_{n,n'}(\pi_\mu) = 0
\]
for \( n' < n - \mu_1 \), hence \( \mu_1 \geq \lambda_1 \). On the other hand for \( n' < n - \lambda_1 \) one has
\[
\Theta_{n,n'}(\pi_\mu)^\vee = \Theta_{n,n'}(\pi_\lambda) = 0,
\]
from which one deduces that \( \mu_1 \leq \lambda_1 \). It follows that \( \mu_1 = \lambda_1 \) and
\[
\Theta_{n,n-\lambda_1}(\pi_{\mu_*}) = \pi_{\mu_*},
\]
which gives that \( \pi_{\mu_*} = \pi_{\lambda_*} \). Hence \( \mu_* = \lambda_* \) and \( \mu = \lambda \).

Finally we are ready to prove the Fourier-Jacobi case of Theorem 1.1.

Theorem 6.5. Assume that \( n > m \) and \( n - m \) is even. Let \( \lambda \) be a partition of \( n \) into \( k \) rows, and \( \lambda' \) be the partition of \( n - k \) obtained by removing the first column of \( \lambda \). Let \( \pi' \) be an irreducible representation of \( U_m(\mathbb{F}_q) \). Then the following hold.

(i) If \( m < n - k \), then
\[
m(\pi_\lambda, \pi') = 0.
\]

(ii) If \( k \) is even and \( m = n - k \), then
\[
m(\pi_\lambda, \pi') = \begin{cases} 1, & \text{if } \pi' \cong \pi_{\lambda'}, \\ 0, & \text{otherwise}. \end{cases}
\]
Proof. By Proposition 6.3, we only need to compute
\[ \langle \pi_\lambda \otimes \omega_n, I^{U_n}(\tau \otimes \pi') \rangle_{U_n(F_q)}, \]
where \( \tau \) is an irreducible cuspidal representation of \( \text{GL}_\ell(F_{q^2}) \) with \( n - m = 2\ell \). Here and in the sequel, we drop the subscripts of various \( F \)-stable parabolic subgroups from the induced representations, which should be clear from the context.

(i) Following the proof of Proposition 6.4, write \( \lambda = [\lambda_1, \ldots, \lambda_k] \) and put \( \lambda^* = [\lambda_2, \ldots, \lambda_k] \), so that we have \( \Theta_{n,n-\lambda_1}(\pi_\lambda) = \pi_{\lambda^*} \). Consider the see-saw diagram

\[
\begin{array}{c}
U_n \times U_n \\
\text{crossing} \\
U_n \quad U_{n-\lambda_1} \times U_1
\end{array}
\]

By the see-saw identity, Proposition 6.2 and Proposition 6.3,
\[
\langle \pi_\lambda \otimes \omega_n, I^{U_n}(\tau \otimes \pi') \rangle_{U_n(F_q)} \\
\leq \langle \Theta_{n-\lambda_1,n}(\pi_{\lambda^*}) \otimes \omega_n, I^{U_n}(\tau \otimes \pi') \rangle_{U_n(F_q)} \\
= \langle \pi_{\lambda^*}, \Theta_{n,n-\lambda_1+1}(I^{U_n}(\tau \otimes \pi')) \rangle_{U_{n-\lambda_1}(F_q)} \\
= \langle \pi_{\lambda^*}, I^{U_{n-\lambda_1+1}}(\tau \otimes \Theta_{m,m-\lambda_1+1}(\pi')) \rangle_{U_{n-\lambda_1}(F_q)} \\
= m(\pi_{\lambda^*}, \Theta_{m,m-\lambda_1+1}(\pi')).
\]

If \( m < n - k \), then by Theorem 5.1 (i) the above multiplicity vanishes, noting that
\[ m - \lambda_1 + 1 < n - \lambda_1 - (k - 1). \]

(ii) Assume that \( k \) is even and \( m = n - k \). If \( \pi' \) is not unipotent, then \( \Theta_{m,m-\lambda_1+1}(\pi') \) has no unipotent components, hence by Theorem 5.1 (ii) we have
\[ \langle \pi_\lambda \otimes \omega_n, I^{U_n}(\tau \otimes \pi') \rangle_{U_n(F_q)} \leq m(\pi_{\lambda^*}, \Theta_{m,m-\lambda_1+1}(\pi')) = 0. \]

Assume that \( \pi' = \pi_\mu \) is unipotent, where \( \mu \) is a partition of \( m \). Let \( \mu_1 \) be the largest part of \( \mu \), and take an arbitrary \( \mu_0 \geq \max\{\lambda_1, \mu_1\} \). Put \( \mu^* = [\mu_0, \mu] \). Then \( \Theta_{m+\mu_0,m}(\pi_{\mu^*}) = \pi_\mu \). Consider the see-saw diagram

\[
\begin{array}{c}
U_n \times U_n \\
\text{crossing} \\
U_n \quad U_{n+\mu_0+1} \times U_1
\end{array}
\]
Similar to the proof of (i), one has

\[
\langle \pi_\lambda \otimes \omega_n, I_{U_n}^m (\tau \otimes \pi_\mu) \rangle_{U_n(F_q)} = (\langle \pi_\lambda, I_{U_n}^m (\tau \otimes \pi_\mu) \rangle_{U_n(F_q)} = \langle \pi_\lambda, \Theta_{n+\mu_0, n} \left( I_{U_n}^{m+\mu_0} (\tau \otimes \pi_{\mu^*}) \right) \otimes \omega_n \rangle_{U_n(F_q)} = \langle \Theta_{n,n+\mu_0+1}(\pi_\lambda), I_{U_n}^{m+\mu_0} (\tau \otimes \pi_{\mu^*}) \rangle_{U_n+\mu_0}(F_q) = m(\Theta_{n,n+\mu_0+1}(\pi_\lambda), \pi_{\mu^*}).
\]

By Proposition 6.1,

\[
\Theta_{n,n+\mu_0+1}(\pi_\lambda) = \bigoplus_{\lambda \text{ and } \tilde{\lambda} \text{ are 2-transverse}} \pi_{\lambda}.
\]

Thus

\[
m(\Theta_{n,n+\mu_0+1}(\pi_\lambda), \pi_{\mu^*}) = \sum_{\lambda \text{ and } \tilde{\lambda} \text{ are 2-transverse}} m(\pi_{\lambda}, \pi_{\mu^*}).
\]

If \(\lambda\) and \(\tilde{\lambda}\) are 2-transverse, then \(\tilde{\lambda}\) has at most \(k + 1\) parts. Since \(m + \mu_0 = (n + \mu_0 + 1) - (k + 1)\), applying Theorem 5.1 gives that

\[
m(\pi_{\lambda}, \pi_{\mu^*}) = \begin{cases} 1, & \text{if } \mu^* = \tilde{\lambda}', \\ 0, & \text{otherwise}, \end{cases}
\]

where \(\tilde{\lambda}'\) is the partition of \(n + \mu_0 + 1\) obtained by removing the first column of \(\tilde{\lambda}\). Note that if \(m(\pi_{\lambda}, \pi_{\mu^*}) \neq 0\) then necessarily \(\lambda\) has \(k + 1\) parts. Clearly

\[
\lambda^* := [k + 1, \mu^*]
\]

is the unique partition \(\lambda^*\) of \(n + \mu_0 + 1\) satisfying that \(\tilde{\lambda}' = \mu^*\), if such partition exists at all. Namely, \(\lambda^*\) is obtained by adding a column of \(k + 1\) squares to the left of \(\mu^*\). Thus we may rephrase (6.1) as

\[
m(\pi_{\lambda}, \pi_{\mu^*}) = \begin{cases} 1, & \text{if } \lambda = \lambda^*, \\ 0, & \text{otherwise}. \end{cases}
\]

Noting that the first row of \(\lambda^*\) has \(\mu_0 + 1\) squares, i.e. \(\lambda^*\) has \(\mu_0 + 1\) columns, it is not hard to see that if \(\lambda\) and \(\tilde{\lambda}\) are 2-transverse, then necessarily

\[
\lambda^* = [\mu_0 + 1, \lambda],
\]

which together with \((\lambda^*)' = \mu^*\) imply that \(\lambda' = \mu\). In summary, we conclude that

\[
m(\pi_\lambda, \pi_\mu) = m(\Theta_{n,n+\mu_0+1}(\pi_\lambda), \pi_{\mu^*}) = \begin{cases} 1, & \text{if } \mu = \lambda', \\ 0, & \text{otherwise}, \end{cases}
\]

which finishes the proof of (ii).  \(\square\)

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DESCENTS OF UNIPOTENT REPRESENTATIONS OF FINITE UNITARY GROUPS 29

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