Triple Cohomology of Lie–Rinehart Algebras and the Canonical Class of Associative Algebras

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Abstract We introduce a bicomplex which computes the triple cohomology of Lie–Rinehart algebras. We prove that the triple cohomology is isomorphic to the Rinehart cohomology [13] provided the Lie–Rinehart algebra is projective over the corresponding commutative algebra. As an application we construct a canonical class in the third dimensional cohomology corresponding to an associative algebra.

Key words: Lie–Rinehart algebra, Hochschild cohomology, cotriple.
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1 Introduction

Let A be a commutative algebra over a field K. A Lie–Rinehart algebra is a Lie K-algebra, which is also an A-module and these two structures are related in an appropriate way [9]. The leading example of Lie–Rinehart algebras is the set Der(A) of all K-derivations of A. Lie-Rinehart algebras are algebraic counterpart of Lie algebroids [9].

The cohomology $H^*_{Rin}(\mathcal{L}, \mathcal{M})$ of a Lie–Rinehart algebra $\mathcal{L}$ with coefficient in a Lie-Rinehart module $\mathcal{M}$ first was defined by Rinehart [13] and then was further developed by Huebschmann [6]. However these groups have good properties only in the case, when $\mathcal{L}$ is projective over A. In this paper following to [12] we introduce a bicomplexes $C^{**}(A, \mathcal{L}, \mathcal{M})$, whose cohomology $H^*(A, \mathcal{L}, \mathcal{M})$ is isomorphic to $H^*_{Rin}(\mathcal{L}, \mathcal{M})$ provided $\mathcal{L}$ is projective as an A-module. It turns out, that for general $\mathcal{L}$ the group $H^*(A, \mathcal{L}, \mathcal{M})$ is isomorphic to the triple cohomology of Barr-Beck [11] applied to Lie–Rinehart algebras. We also prove that for general $\mathcal{L}$, unlike to the Rinehart cohomology $H^*_{Rin}(\mathcal{L}, \mathcal{M})$, the groups $H^*(A, \mathcal{L}, \mathcal{M})$ in dimensions two and three classify all abelian and crossed extensions of $\mathcal{L}$ by $\mathcal{M}$.

As an application we consider the following situation. Let $S$ be an associative algebra over a field $K$. We let $H^*(S, S)$ be the Hochschild cohomology of $S$. It is well-known that $H^1(S, S)$ is a Lie $K$-algebra. It turns out that $H^1(S, S)$ is in fact a Lie-Rinehart algebra over A, where $A=H^0(S, S)$ is the center of $S$. Thus one can consider the cohomology $H^*(A, H^1(S, S), A)$. We construct an element

$$o(S) \in H^3(A, H^1(S, S), A)$$
which we call the canonical class of $S$. We prove that $o(S)$ is a Morita invariant. The construction of $o(S)$ uses crossed modules of Lie-Rinehart algebras introduced in [3].

2 Preliminaries on Lie-Rinehart algebras

The material of this section is well-known. We included it in order to fix terminology, notations and main examples. In what follows we fix a field $K$. All vector spaces are considered over $K$. We write $\otimes$ and $\text{Hom}$ instead of $\otimes_K$ and $\text{Hom}_K$.

2.1 Definitions, Examples

Let $A$ be a commutative algebra over a field $K$. Then the set $\text{Der}(A)$ of all $K$-derivations of $A$ is a Lie $K$-algebra and an $A$-module simultaneously. These two structures are related by the following identity

$$[D, aD'] = a[D, D'] + D(a)D', \quad D, D' \in \text{Der}(A).$$

This leads to the following notion, which goes back to Herz under the name "pseudo-algèbre de Lie" (see [5]) and which are algebraic counterpart of Lie algebroids [9].

**Definition 2.1** A Lie-Rinehart algebra over $A$ consists with a Lie $K$-algebra $L$ together with an $A$-module structure on $L$ and a map $\alpha: L \to \text{Der}(A)$ which is simultaneously a Lie algebra and $A$-module homomorphism such that

$$[X, aY] = a[X, Y] + X(a)Y.$$ 

Here $X, Y \in L$, $a \in A$ and we write $X(a)$ for $\alpha(X)(a)$ [6]. These objects are also known as $(K, A)$-Lie algebras [13] and $d$-Lie rings [11].

Thus $\text{Der}(A)$ with $\alpha = \text{Id}_{\text{Der}(A)}$ is a Lie-Rinehart $A$-algebra. Let us observe that Lie-Rinehart $A$-algebras with trivial homomorphism $\alpha: L \to \text{Der}(A)$ are exactly Lie $A$-algebras. Therefore the concept of Lie-Rinehart algebras generalizes the concept of Lie $A$-algebras. If $A=K$, then $\text{Der}(A)=0$ and there is no difference between Lie and Lie-Rinehart algebras. We denote by $\mathcal{LR}(A)$ the category of Lie-Rinehart algebras. One has the full inclusion

$$\mathcal{L}(A) \subset \mathcal{LR}(A),$$

where $\mathcal{L}(A)$ denotes the category of Lie $A$-algebras. Let us observe that the kernel of any Lie-Rinehart algebra homomorphism is a Lie $A$-algebra.
Example 2.2 If \( g \) is a \( K \)-Lie algebra acting on a commutative \( K \)-algebra \( A \) by derivations (that is there is given a homomorphism of Lie \( K \)-algebras \( \gamma : g \to \text{Der}(A) \)), then the transformation Lie-Rinehart algebra of \( (g, A) \) is \( \mathcal{L} = A \otimes g \) with the Lie bracket

\[
[a \otimes g, a' \otimes g'] := aa' \otimes [g, g'] + a\gamma(g)(a') \otimes g' - a'\gamma(g')(a) \otimes g
\]

and with the action \( \alpha : \mathcal{L} \to \text{Der}(A) \) given by \( \alpha(a \otimes g)(a') = a\gamma(g)(a') \).

Definition 2.3 A Lie-Rinehart module over a Lie-Rinehart \( A \)-algebra \( \mathcal{L} \) is a vector space \( M \) together with two operations

\[
\mathcal{L} \otimes M \to M, \quad (X, m) \mapsto X(m)
\]

and

\[
A \otimes M \to M \quad (a, m) \mapsto am,
\]

such that the first one makes \( M \) into a module over the Lie \( K \)-algebra \( \mathcal{L} \) in the sense of the Lie algebra theory, while the second map makes \( M \) into an \( A \)-module and additionally the following compatibility conditions hold

\[
(aX)(m) = a(X(m)),
\]

\[
X(am) = aX(m) + X(a)m.
\]

Here \( a \in A, \ m \in M \) and \( X \in \mathcal{L} \).

It follows that \( A \) is a Lie-Rinehart module over \( \mathcal{L} \) for any Lie-Rinehart algebra \( \mathcal{L} \). We let \( (\mathcal{L}, A)\text{-mod} \) be the category of Lie-Rinehart modules over \( \mathcal{L} \).

2.2 Rinehart cohomology of Lie-Rinehart algebras

Let \( M \) be a Lie-Rinehart module over \( \mathcal{L} \). Let us recall the definition of the Rinehart cohomology \( \mathcal{H}^{*}_{\text{Rin}}(\mathcal{L}, M) \) of a Lie-Rinehart algebra \( \mathcal{L} \) with coefficients in a Lie-Rinehart module \( M \) (see \(^{13}\) and \(^{11}\)). One puts

\[
C^{n}_{\Lambda}(\mathcal{L}, M) := \text{Hom}_{A}(\Lambda^{n}_{A} \mathcal{L}, M),
\]

where \( \Lambda^{n}_{A}(V) \) denotes the exterior algebra over \( A \) generated by an \( A \)-module \( V \). The coboundary map

\[
\delta : C^{n-1}_{\Lambda}(\mathcal{L}, M) \to C^{n}_{\Lambda}(\mathcal{L}, M)
\]

is given by

\[
(\delta f)(X_{1}, \ldots, X_{n}) = (-1)^{n} \sum_{i=1}^{n} (-1)^{(i-1)} X_{i}(f(X_{1}, \ldots, \hat{X}_{i}, \ldots, X_{n})).
\]
\((-1)^{n} \sum_{j<k} (-1)^{j+k} f([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_n).\)

Here \(X_1, \ldots, X_n \in \mathcal{L}, f \in C^{n-1}_{\mathcal{A}}(\mathcal{L}, M).\) By the definition \(H^*_{\text{Rin}}(\mathcal{L}, M)\) is the cohomology of the cochain complex \(C^*_\mathcal{A}(\mathcal{L}, M).\) One observes that if \(A = K,\) then this definition generalizes the classical definition of Lie algebra cohomology. For a general \(A\) by forgetting the \(A\)-module structure one obtains the canonical homomorphism

\[ H^*_{\text{Rin}}(\mathcal{L}, M) \rightarrow H^*_{\text{Lie}}(\mathcal{L}, M), \]

where \(H^*_{\text{Lie}}(\mathcal{L}, M)\) denotes the cohomology of \(\mathcal{L}\) considered as a Lie \(K\)-algebra.

It follows from the definition that one has the following exact sequence

\[ 0 \rightarrow H^0_{\text{Rin}}(\mathcal{L}, M) \rightarrow M \rightarrow \text{Der}_A(\mathcal{L}, M) \rightarrow H^1_{\text{Rin}}(\mathcal{L}, M) \rightarrow 0, \quad (1) \]

where \(\text{Der}_A(\mathcal{L}, M)\) consists with \(A\)-linear maps \(d : \mathcal{L} \rightarrow M\) which are derivations from the Lie \(K\)-algebra \(\mathcal{L}\) to \(M.\) In other words \(d\) must satisfy the following conditions:

\[
\begin{aligned}
&d(ax) = ad(x), \ a \in A, \ x \in \mathcal{L}, \\
&d([x,y]) = [x,d(y)] - [y,d(x)].
\end{aligned}
\]

For a Lie-Rinehart module \(M\) over a Lie-Rinehart algebra \(\mathcal{L}\) one can define the semi-direct product \(\mathcal{L} \rtimes M\) to be \(\mathcal{L} \oplus M\) as an \(A\)-module with the bracket

\[
[(X,m),(Y,n)] = ([X,Y], [X,m] - [Y,n]).
\]

**Lemma 2.4** Let \(\mathcal{L}\) be a Lie-Rinehart algebra over a commutative algebra \(A\) and let \(M \in (\mathcal{L}, A)\text{-mod}.\) Then there is a 1-1 correspondence between the elements of \(\text{Der}_A(\mathcal{L}, M)\) and the sections (in the category \(\mathcal{LR}(A)\)) of the projection \(p : \mathcal{L} \rtimes M \rightarrow \mathcal{L}.

**Proof.** Any section \(\xi : \mathcal{L} \rightarrow \mathcal{L} \rtimes M\) of \(p\) has the form \(\xi(x) = (x, f(x))\) and one easily shows that \(\xi\) is a morphism in \(\mathcal{LR}(A)\) iff \(f \in \text{Der}_A(\mathcal{L}, M).\) \(\Box\)

### 2.3 Abelian and crossed extensions of Lie-Rinehart algebras

**Definition 2.5** Let \(\mathcal{L}\) be a Lie-Rinehart algebra over a commutative algebra \(A\) and let \(M \in (\mathcal{L}, A)\text{-mod}.\) An abelian extension of \(\mathcal{L}\) by \(M\) is an exact sequence

\[
0 \longrightarrow M \overset{i}{\longrightarrow} \mathcal{L}' \overset{\partial}{\longrightarrow} \mathcal{L} \longrightarrow 0
\]

where \(\mathcal{L}'\) is a Lie-Rinehart algebra over \(A\) and \(\partial\) is a Lie-Rinehart algebra homomorphism. Moreover, \(i\) is an \(A\)-linear map and the following identities hold

\[
i([i(m), i(n)] = 0,
\]
\[ [i(m), X'] = (\partial(X'))m, \]

where \(m, n \in M\) and \(X' \in L'\). An abelian extension is called A-spit if \(\partial\) has an A-linear section.

We also need the notion of crossed modules for Lie-Rinehart algebras introduced in [3]. The following definition is equivalent to one given in [3].

**Definition 2.6** A crossed module \(\partial : R \to L\) of Lie-Rinehart algebras over \(A\) consists of a Lie-Rinehart algebra \(L\) and a Lie-Rinehart module \(R\) over \(L\) together with an A-linear homomorphism \(\partial : R \to L\) such that for all \(r, s \in R, X \in L, a \in A\) the following identities hold:

1. \(\partial(X(r)) = [X, \partial(r)]\)
2. \((\partial(r))(s) + (\partial(s))(r) = 0\)
3. \(\partial(r)(a) = 0\).

It follows from this definition that \(R\) is a Lie A-algebra under the bracket \([r, s] = (\partial(r))(s)\) and \(\partial\) is a homomorphism of Lie \(K\)-algebras. Moreover, \(\text{Im}(\partial)\) is simultaneously a Lie \(K\)-ideal of \(L\) and an A-submodule, therefore \(\text{Coker}(\partial)\) is a Lie-Rinehart algebra. Furthermore \(\text{Ker}(\partial)\) is an abelian A-ideal of \(R\) and the action of \(L\) on \(R\) yields a Lie-Rinehart module structure of \(\text{Coker}(\partial)\) on \(\text{Ker}(\partial)\).

Let \(P\) be a Lie-Rinehart algebra and let \(M\) be a Lie-Rinehart module over \(P\). We consider the category \(\text{Cross}(P, M)\), whose objects are the exact sequences

\[ 0 \to M \to R \xrightarrow{\partial} L \xrightarrow{\nu} P \to 0 \]

where \(\partial : R \to L\) is a crossed module of Lie-Rinehart algebras over \(A\) and the canonical maps \(\text{Coker}(\partial) \to P\) and \(M \to \text{Ker}(\partial)\) are isomorphisms of Lie-Rinehart algebras and modules respectively. The morphisms in the category \(\text{Cross}(P, M)\) are commutative diagrams

\[
\begin{array}{ccccccccc}
0 & \to & M & \to & R & \xrightarrow{\partial} & L & \xrightarrow{\nu} & P & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & M' & \to & R' & \xrightarrow{\partial'} & L' & \xrightarrow{\nu'} & P' & \to & 0 \\
\end{array}
\]

where \(\beta\) is a homomorphism of Lie-Rinehart algebras, \(\alpha\) is a morphism of Lie A-algebras and for any \(r \in R, X \in L\) one has

\[ \alpha([X, r]) = [\beta(X), \alpha(r)]. \]

Furthermore, we let \(\text{Cross}_{A\text{-split}}(P, M)\) be the subcategory of \(\text{Cross}(P, M)\) whose objects and morphisms split in the category of A-modules, in other words one requires that the epimorphisms \(L \to P, R \to \text{Im}(\partial), L' \to P', R' \to \text{Im}(\partial'), L \to \text{Im}(\beta), L' \to \text{Coker}(\beta), R \to \text{Im}(\alpha), R' \to \text{Coker}(\alpha)\) have A-linear sections.

5
2.4 Main properties of Rinehart cohomologies

**Theorem 2.7**

i) If $L$ is projective as an $A$-module, then

$$H^n_{Rin}(L, M) \cong \text{Ext}^n_{(L, A)-\text{mod}}(A, M).$$

ii) If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence in the category $(L, A)$-\text{mod}, then one has a long exact sequence on cohomology

$$\cdots \to H^n_{Rin}(L, M_1) \to H^n_{Rin}(L, M) \to H^n_{Rin}(L, M_2) \to \cdots$$

provided $0 \to M_1 \to M \to M_2 \to 0$ splits in the category of $A$-modules, or $L$ is projective as $A$-module.

iii) The cohomology $H^2_{Rin}(L, M)$ classifies such abelian extensions

$$0 \to M \to L' \to L \to 0$$

of $L$ by $M$ in the category of Lie-Rinehart algebras which splits in the category of $A$-modules.

iv) For any Lie-Rinehart algebra $P$ which is projective as an $A$-module and any Lie-Rinehart module $M$ there exists a natural bijection between the classes of the connected components of the category $\text{Cross}_{A-spl}(P, M)$ and $H^3_{Rin}(P, M)$.

**Proof.** For the isomorphism of the part i) see Section 4 of [13]. The part ii) is trivial and for part iii) see Theorem 2.6 in [6]. Finally the part iv), which is in the same spirit as the classical result for group and Lie algebra cohomology (see [8] and [7]), was proved in [3].

Let $\mathfrak{g}$ be a Lie algebra over $K$ and let $M$ be a $\mathfrak{g}$-module. Then we have the Chevalley–Eilenberg cochain complex $C^n_{\text{Lie}}(\mathfrak{g}, M)$, which computes the Lie algebra cohomology (see [2]):

$$C^n_{\text{Lie}}(\mathfrak{g}, M) = \text{Hom}(\Lambda^n(\mathfrak{g}), M).$$

Here $\Lambda^*$ denotes the exterior algebra defined over $K$.

**Lemma 2.8** Let $\mathfrak{g}$ be a Lie $K$-algebra acting on a commutative algebra $A$ by derivations and let $L$ be the transformation Lie-Rinehart algebra of $(\mathfrak{g}, A)$ (see Example 2.2). Then for any Lie-Rinehart $L$-module $M$ one has the canonical isomorphism of cochain complexes $C^n_{\text{Lie}}(\mathfrak{g}, M) \cong C^n_{L}(\mathfrak{g}, M)$ and in particular the isomorphism

$$H^n_{Rin}(L, M) \cong H^n_{\text{Lie}}(\mathfrak{g}, M).$$

**Proof.** Since $L = A \otimes \mathfrak{g}$ one has $\text{Hom}_A(\Lambda^n L, M) \cong \text{Hom}(\Lambda^n \mathfrak{g}, M)$ and Lemma follows.
3 The main construction

Thanks to Theorem 2.7 the cohomology theory $H^*_\text{Rin}(\mathcal{L}, -)$ has good properties only if $\mathcal{L}$ is projective as an $A$-module. In this section we introduce the bicomplex $C^{**}(A, \mathcal{L}, M)$, whose cohomology are good replacement of the Rinehart cohomology $H^*_\text{Rin}(\mathcal{L}, -)$ for general $\mathcal{L}$. The idea of the construction is very simple. First one observes that the transformation Lie-Rinehart algebras (see Example 2.2) are always free as $A$-modules, therefore the Rinehart cohomology of such algebras are the right objects. Secondly, for any Lie-Rinehart algebra $\mathcal{L}$ the two-sided bar construction $B_*(A, A, \mathcal{L})$ gives rise to a simplicial resolution of $\mathcal{L}$ in the category of Lie–Rinehart algebras. Since each term of this resolution is a transformation Lie-Rinehart algebra one can mix the Chevalley–Eilenberg complexes with the bar resolution to get our bicomplex.

3.1 A bicomplex for Lie–Rinehart algebras

Let $\mathcal{L}$ be a Lie–Rinehart algebra and let $M$ be a Lie–Rinehart module over $\mathcal{L}$. We have two cochain complex: the Rinehart complex $C^*(A, \mathcal{L}, M)$ and Chevalley–Eilenberg complex $C^*_{\text{Lie}}(\mathcal{L}, M)$. If one forgets $A$-module structure on $\mathcal{L}$, we get a Lie $K$-algebra acting on $A$ via derivations, thus the construction of Example 2.2 gives a Lie-Rinehart algebra structure on $A \otimes \mathcal{L}$. One can iterated this construction to conclude that $A^{\otimes n} \otimes \mathcal{L}$ is also a Lie–Rinehart algebra for any $n \geq 0$. The $A$-module structure comes from the first factor, while the bracket is a bit more complicated, for example for $n = 2$, one has

$$[a_1 \otimes a_2 \otimes X, b_1 \otimes b_2 \otimes Y] := a_1 b_1 \otimes a_2 b_2 \otimes [X, Y] + a_1 b_1 \otimes a_2 X(b_2) \otimes Y +$$

$$+ a_1 a_2 X(b_1) \otimes b_2 Y - a_1 b_1 \otimes b_2 Y(a_2) \otimes X - b_1 b_2 Y(a_1) \otimes a_2 \otimes X.$$

Let us also recall that the two-sided bar construction $B_*(A, A, \mathcal{L})$ is a simplicial object, which is $A^{\otimes n+1} \otimes \mathcal{L}$ in the dimension $n$, while the face maps are given by

$$d_i(a_0 \otimes \cdots \otimes a_n \otimes X) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes X,$$

if $i < n$ and

$$d_n(a_0 \otimes \cdots \otimes a_n \otimes X) = a_0 \otimes \cdots \otimes a_{n-1} \otimes \cdots \otimes a_n X,$$

if $i = n$. The degeneracy maps are given by

$$s_i(a_0 \otimes \cdots \otimes a_n \otimes X) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots a_n \otimes X.$$

In fact $B_*(A, A, \mathcal{L})$ is an augmented simplicial object in the category of Lie-Rinehart algebras, the augmentation $B_0(A, A, \mathcal{L}) = A \otimes \mathcal{L} \to \mathcal{L}$ is given by
(a, X) \mapsto aX. We can apply the functor $C^*_A(-, M)$ on $B_*(A, A, L)$ to get a cosimplicial object in the category of cochain complexes

$$[n] \mapsto C^*_A(A^\otimes n+1 \otimes L, M).$$

Finally we let $C^{**}(A, L, M)$ be the bicomplex associated to this cosimplicial cochain complex. We let $H^*(A, L, M)$ be the cohomology of the corresponding total complex. The augmentation $B_*(A, A, L) \to L$ yields the homomorphism

$$\alpha^n : H^n_{Rin}(L, M) \to H^*(A, L, M).$$

The bicomplex $C^{**}(A, L, M)$ has the following alternative description. According to Lemma 2.8 one has the isomorphism of complexes:

$$C^{pq}_*(A, L, M) \cong C^*_{Lie}(A^\otimes p \otimes L \otimes M),$$

where $M$ is considered as a module over $A^\otimes p \otimes L$ by

$$(a_1 \otimes \cdots \otimes a_p \otimes r)m := (a_1 \cdots a_p r)m.$$ 

To define the horizontal cochain complex structure one observes that elements of $C^{pq}$ can be identified with functions $f : A^\otimes pq \otimes L^\otimes q \to M$, which are alternative with appropriate blocks of variables. Then the corresponding linear map

$$d(f) : A^\otimes (p+1)q \otimes L^\otimes q \to M$$

is given by

$$df(a_{01}, \cdots, a_{0q}, a_{11}, \cdots, a_{1q}, \cdots a_{pq}, X_1, \cdots, X_q) =$$

$$a_{01} \cdots a_{0q} f(a_{11}, \cdots, a_{1q}, \cdots a_{pq}, X_1, \cdots, X_q) +$$

$$\sum_{0 \leq i < p} (-1)^{i+1} f(a_{01}, \cdots, a_{0q}, \cdots a_{i1}a_{i+1,1}, \cdots, a_{iq}a_{i+1,q}, \cdots a_{pq}, X_1, \cdots, X_q) +$$

$$(-1)^{p+1} f(a_{01}, \cdots, a_{0q}, \cdots, a_{p-1,1}, \cdots, a_{p-1,q}, a_{p1}X_1, \cdots, a_{pq}X_q).$$

**Theorem 3.1** i) The homomorphism

$$\alpha^n : H^n_{Rin}(L, M) \to H^n(A, L, M)$$

is isomorphism for $n = 0, 1$. The homomorphism $\alpha^2$ is a monomorphism. Moreover $\alpha^n$ is an isomorphism for all $n \geq 0$ provided $L$ is projective over $A$.

ii) If $0 \to M_1 \to M \to M_2 \to 0$ is an exact sequence in the category $(L, A)$-mod, then one has a long exact sequence on cohomology

$$\cdots \to H^n(A, L, M_1) \to H^n(A, L, M) \to H^n(A, L, M_2) \to \cdots.$$
iii) The cohomology $H^2(A, \mathcal{L}, M)$ classifies all abelian extensions

$$0 \to M \to \mathcal{L}' \to \mathcal{L} \to 0$$

of $\mathcal{L}$ by $M$ in the category of Lie-Rinehart algebras.

iv) For any Lie-Rinehart algebra $\mathcal{L}$ and any Lie-Rinehart module $M$ there exists a natural bijection between the classes of the connected components of the category $\text{Cross}(\mathcal{L}, M)$ and $H^3(A, \mathcal{L}, M)$.

**Proof.** i) The statement is obvious for $n = 0, 1$. For $n = 2$ it follows from the part iii) below and part iii) of Theorem 2.7. It remains to prove the last assertion. It is well-known that the augmentation $B_*(A, A, \mathcal{L}) \to \mathcal{L}$ is a homotopy equivalence in the category of simplicial A-modules and therefore, for each $s$ a weak equivalence in the category of simplicial A-modules. Assume now $\mathcal{L}$ is projective as an A-module, then $B_*(A, A, \mathcal{L}) \to \mathcal{L}$ is a homotopy equivalence in the category of simplicial A-modules and therefore, for each $k \geq 0$ the induced map $\Lambda^k_A(B_*(A, A, \mathcal{L})) \to \Lambda^k_A(\mathcal{L})$ is a homotopy equivalence in the category of simplicial A-modules, which implies that the same is true after applying the functor $\text{Hom}_A(\mathcal{L}, M)$. Thus for each $k \geq 0$ the induced map $C^k_A(\mathcal{L}, M) \to C^k_A(B_*(A, A, \mathcal{L}))$ is a weak equivalence of cosimplicial objects and the comparison theorem for bicomplexes yields the result.

ii) Since $\text{Hom}$ and exterior powers involved in $C^m_{\text{Lie}}(g, M)$ are taken over $K$ it follows that for each $p$ and $q$ the functor $C^q_{\text{Lie}}(\Lambda^p \otimes \mathcal{L}, -)$ is exact and the result follows.

iii) Thanks to a well-known fact from topology we can use the normalized (in the simplicial direction) cochains to compute $H^*(A, \mathcal{L}, M)$. Having this in mind we have $H^2(A, \mathcal{L}, M) = Z^2/B^2$, where $Z^2$ consists with pairs $(f, g)$ such that $f : \Lambda^2(\mathcal{L}) \to M$ is a Lie 2-cocycle and $g : A \otimes \mathcal{L} \to M$ is a linear map such that $g(1, X) = 0,$

$$ag(b, X) - g(ab, X) + g(a, bX) = 0$$

and

$$abf(X, Y) - f(ax, by) = aXg(b, Y) - bYg(a, X) - g(ab, [X, Y]) - g(aX(b), Y) + g(bY(a), X).$$

Here $a, b \in A$ and $X, Y \in \mathcal{L}$. Moreover $(f, g)$ belongs to $B^2$ iff there exist a linear map $h : \mathcal{L} \to M$ such that $f(X, Y) = Xh(Y) - h([X, Y]) - Yh(X)$ and $g(a, X) = ah(X) - h(aX).$ Starting with $(f, g) \in Z^2$ we construct an abelian extension of $\mathcal{L}$ by $M$ by putting $\mathcal{P} = M \oplus \mathcal{L}$ as a vector space. An A-module structure on $\mathcal{P}$ is given by $a(m, X) = (am + g(a, X), aX)$, while a Lie bracket on $\mathcal{P}$ is given by $[(m, X), (n, Y)] = (Xn - Ym + f(X, Y), [X, Y]).$ Conversely, given an abelian extension $(\mathcal{P})$ and a $K$-linear section $h : \mathcal{L} \to \mathcal{P}$ we put $f(X, Y) =$
$[h(X), h(Y)] - h([X, Y])$ and $g(a, X) := h(aX) - ah(X)$. One easily checks that $(f, g) \in Z^2$ and one gets iii).

iv) Similarly, we have $H^3(A, \mathcal{L}, M) = Z^3/B^3$. Here $Z^3$ consists with triples $(f, g, h)$ such that $f : \Lambda^3(\mathcal{L}) \to M$ is a Lie 3-cocycle, $g : \Lambda^2(A \otimes \mathcal{L}) \to M$ and $h : A \otimes A \otimes \mathcal{L} \to M$ are linear maps and the following relations hold:

$$f(aX, bY, cZ) - abc \cdot f(X, Y, Z) =$$

$$= aX \cdot g(b, c, Y, Z) - bY \cdot g(a, c, X, Z) + cZ \cdot g(a, b, X, Y) - g(ab, c, [X, Y], Z) +$$

$$+ g(aX(b), c, Y, Z) - g(bY(a), c, X, Z) + g(ac, b, [X, Y], Y) - g(aX(c), b, Z, Y) +$$

$$+ g(cZ(a), b, X, Y) - g(bc, a, [Y, Z], X) + g(bY(c), a, Z, X) - g(cZ(b), a, Y, X)$$

and

$abX \cdot h(c, d, Y) - cdX \cdot h(a, b, X) - h(ac, bd, [X, Y]) - h(ac, bX(d), Y) -$

$$- h(abX(c), d, Y) + h(ac, dY(b), X) - h(cdY(a), b, X) =$

$$= abg(c, d, X, Y) - g(ac, bd, X, Y) + g(a, b, cX, dY).$$

Moreover, $(f, g, h)$ belongs to $B^3$ iff there exist a linear maps $m : \Lambda^2(\mathcal{L}) \to M$ and $n : A \otimes \mathcal{L} \to M$ such that

$$f(X, Y, Z) = Xm(Y, Z) - Ym(X, Z) + Zm(X, Y) -$$

$$-m([X, Y], Z) + m([X, Z], Y) - m([Y, Z], X),$$

$$g(a, b, r, s) = abm(X, Y) - m(aX, bY) - aXn(b, Y) +$$

$$bYn(a, X) + n(ab, [X, Y]) + n(aX(b), Y) + n(bY(a), X)$$

and

$$h(a, b, X) = an(b, X) - n(ab, X) + n(a, bX).$$

Let

$$0 \to M \to \mathcal{R} \xrightarrow{\partial} \mathcal{P} \xrightarrow{\pi} \mathcal{L} \to 0$$

be a crossed extension. We put $V := \text{Im}(\partial)$ and consider $K$-linear sections $p : \mathcal{L} \to \mathcal{P}$ and $q : V \to \mathcal{R}$ of $\pi : \mathcal{P} \to \mathcal{L}$ and $\partial : \mathcal{R} \to V$ respectively. Now we define $t : \mathcal{L} \otimes \mathcal{L} \to \mathcal{R}$ and $s : A \otimes \mathcal{L} \to \mathcal{R}$. by $t(X, Y) := q([p(X), p(Y)]) - p([X, Y])$ and $s(a, X) := q(ap(X) - p(aX))$. Finally we define three functions as follows. The function $f : \Lambda^3(\mathcal{L}) \to M$ is given by

$$f(X, Y, Z) := p(X)g(Y, Z) - p(Y)g(X, Z) + p(Z)g(X, Y) -$$

$$-g([X, Y], Z) + g([X, Z], Y) - g([Y, Z], X).$$

The function $g : \Lambda^2(A \otimes \mathcal{L}) \to M$ is given by

$$g(a, b, X, Y) := p(aX)s(b, Y) - p(bY)s(a, X) - p(ab, [X, Y]) -$$
while the function and $h : A \otimes A \otimes \mathcal{L} \to M$ is given by

$$h(a, b, X) := a s(b, X) - s(ab, X) + s(a, bX).$$

Then $(f, g, h) \in Z^3$ and the corresponding class in $H^3(A, R, M)$ depends only on the connected component of a given crossed extension. Thus we obtain a well-defined map $\text{Cros}(A, R, M) \to H^3(A, R, M)$ and a standard argument (see [7]) shows that it is an isomorphism.

## 4 Triple cohomology of Lie–Rinehart algebras

In this section we prove that the cohomology theory developed in the previous section in canonically isomorphic to the triple cohomology of Barr-Beck [1] applied to Lie-Rinehart algebras.

### 4.1 Cotriples and cotriple resolutions

The general notion of (co)triples (or (co)monads, or (co)standard construction) and (co)triple resolutions due to Godement [4] and then was developed in [1]. Let $\mathcal{C}$ be a category. A cotriple on $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\epsilon : T \to 1_\mathcal{C}$ and $\delta : T \to T^2$ satisfying the counit and the coassociativity properties. Here $T^n = T \circ T$ and a similar meaning has $T^n$ for all $n \geq 0$. For example, assume $U : \mathcal{C} \to \mathcal{B}$ is a functor which has a left adjoint functor $F : \mathcal{B} \to \mathcal{C}$. Then there is a cotriple structure on $T = FU : \mathcal{C} \to \mathcal{C}$ such that $\epsilon$ is the counit of the adjunction. Given a cotriple $T$ and an object $C$ one can associate a simplicial object $T^\ast C$ in the category $\mathcal{C}$, known as Godement or cotriple resolution of $C$. Let us recall that $T_n C = T^{n+1} C$ and the face and degeneracy operators are given respectively by $\partial_i = T^i \epsilon T^{n-i}$ and $s_i = T^i \delta T^{n-i}$. To explain why it is called resolution, consider the case when $T = FU$ is associated to the pair of adjoint functors. Then firstly $\epsilon$ yields a morphism $T_\ast C \to C$ from the simplicial object $T_\ast C$ to the constant simplicial object $C$ and secondly the induced morphism $U(T_\ast C) \to U(C)$ is a homotopy equivalence in the category of simplicial objects in $\mathcal{B}$. The cotriple cohomology is now defined as follows. Let $M$ be an abelian group object in the category $\mathcal{C}/\mathcal{C}$ of arrows $X \to C$ then $\text{Hom}_{\mathcal{C}/\mathcal{C}}(T_\ast C, M)$ is a cosimplicial abelian group, which can be seen also as a cochain complex. Thus $H^\ast(\text{Hom}_{\mathcal{C}/\mathcal{C}}(T_\ast C, M))$ is a meaningful and they are denoted by $H^\ast_\ast(C, M)$. Of the special interest is the case, when $T = FU$ is associated to the pair of adjoint functors and the functor $U : \mathcal{C} \to \mathcal{B}$ is tripliable [1]. In this case the category $\mathcal{C}$ is completely determined by the triple $E = U F : \mathcal{B} \to \mathcal{B}$. Because of this fact $H^\ast_\ast(C, M)$ in this case are known as triple cohomology of $\mathcal{C}$ with coefficients in $M$. 

11
4.2 Free Lie–Rinehart Algebras

We wish to apply these general constructions to Lie–Rinehart algebras. One has the functor

$$U : \mathcal{LR}(A) \to \text{Vect}/\text{Der}(A)$$

which assigns $$\alpha : \mathcal{L} \to \text{Der}(A)$$ to a Lie–Rinehart algebra $$\mathcal{L}$$. Here Vect/ Der(A) is the category of $$K$$-linear maps $$\psi : V \to \text{Der}(A)$$, where $$V$$ is a vector space over $$K$$. A morphism $$\psi \to \psi_1$$ in Vect/ Der(A) is a $$K$$-linear map $$f : V \to V_1$$ such that $$\psi = \psi_1 \circ f$$. Now we construct the functor

$$F : \text{Vect}/\text{Der}(A) \to \mathcal{LR}(A)$$

as follows. Let $$\psi : V \to \text{Der}(A)$$ be a $$K$$-linear map. We let $$\mathcal{L}(V)$$ be the free Lie $$K$$-algebra generated by $$V$$. Then one has the unique Lie $$K$$-algebra homomorphism $$\mathcal{L}(V) \to \text{Der}(A)$$ which extends the map $$\psi$$, which is still denoted by $$\psi$$. Now we can apply the construction from Example 2.2 to get a Lie–Rinehart algebra structure on $$A \otimes \mathcal{L}(V)$$. We let $$F(\psi)$$ be this particular Lie–Rinehart algebra and we call it the free Lie–Rinehart algebra generated by $$\psi$$. In this way we obtain the functor $$F$$, which is the left adjoint to $$U$$.

**Lemma 4.1** Let $$\mathcal{L}$$ be a free Lie–Rinehart algebra generated by $$\psi : V \to \text{Der}(A)$$ and let $$M$$ be any Lie–Rinehart module over $$\mathcal{L}$$. Then

$$H^i_{\text{Rin}}(\mathcal{L}, M) = 0, \ i > 1.$$  

**Proof.** By our construction $$\mathcal{L}$$ is a transformation Lie–Rinehart algebra of $$(\mathcal{L}(V), A)$$. Thus one can apply Lemma 2.8 to get an isomorphism $$H^i_{\text{Rin}}(\mathcal{L}, M) \cong H^i_{\text{Lie}}(\mathcal{L}(V), M)$$ and then one can use the well-known vanishing result for free Lie algebras. $\square$

4.3 The cohomology $$H^*_{LR}(\mathcal{L}, M)$$

Since we have a pair of adjoint functors we can take the composite

$$T = FU : \mathcal{LR}(A) \to \mathcal{LR}(A)$$

which is a cotriple. Thus for any Lie–Rinehart algebra $$\mathcal{L}$$ one can take the cotriple resolution $$T_*(\mathcal{L}) \to \mathcal{L}$$. It follows from the construction of the cotriple resolution that each component of $$T_*(\mathcal{L})$$ is a free Lie–Rinehart algebra. Moreover, according to the general properties of the cotriple resolutions the natural augmentation $$T_*(\mathcal{L}) \to \mathcal{L}$$ is a homotopy equivalence in the category of simplicial vector spaces. It follows that $$T_*(\mathcal{L}) \to \mathcal{L}$$ is a weak homotopy equivalence in the category of $$A$$-modules.
Let $M$ be a $\mathcal{L}$-module. Then $M$ is also a module over $T_n(\mathcal{L})$ for any $n \geq 0$ thanks to the augmentation morphism $T_*(\mathcal{L}) \to \mathcal{L}$. Thus one can form the following bicomplex

$$C^*_A(T_*(\mathcal{L}), M)$$

which is formed by the degreewise applying the Rinehart cochain complex. The cohomology of the total complex of the bicomplex $C^*_A(T_*(\mathcal{L}), M)$ is denoted by $H^*_{LR}(\mathcal{L}, M)$.

**Lemma 4.2** For any Lie–Rinehart algebra $\mathcal{L}$ and any Lie–Rinehart module $M$ one has a natural isomorphism

$$H^*(A, \mathcal{L}, M) \cong H^*_{LR}(\mathcal{L}, M).$$

*Proof.* We denote by $C^*(A, \mathcal{L}, M)$ the total complex associated to the bicomplex $C^{**}(A, \mathcal{L}, M)$. Recall that it comes with a natural cochain map

$$C^*_A(\mathcal{L}, M) \to C^*(A, \mathcal{L}, M)$$

which is quasi-isomorphism provided $\mathcal{L}$ is projective as an $A$-module. Let us apply $C^*(A, -, M)$ on $T_*(\mathcal{L})$ degreewise. Then we obtain the morphism of bicomplex

$$C^*_A(T_*(\mathcal{L}), M) \to C^*(A, T_*(\mathcal{L}), M)$$

which is quasi-isomorphism because each $T_n(\mathcal{L})$ is free as an $A$-module. It remains to show that the augmentation $T_*(\mathcal{L}) \to \mathcal{L}$ yields the quasi-isomorphism

$$C^*(A, \mathcal{L}, M) \to C^*(A, T_*(\mathcal{L}), M).$$

To this end, one observes that $T_*(\mathcal{L}) \to \mathcal{L}$ is a quasi-isomorphism thanks to the general properties of cotriple resolutions and therefore is a homotopy equivalence in the category of simplicial vector spaces. Thus the same is true for $\Lambda^n(T_*(\mathcal{L})) \to \Lambda^n(\mathcal{L})$ and therefore $C^n(A, \mathcal{L}, M) \to C^n(A, T_*(\mathcal{L}), M)$ is also a homotopy-equivalence for each $n$ and the result follows from the comparison theorem of bicomplexes.

### 4.4 Triple cohomology and $H^*_{LR}(\mathcal{L}, M)$

According to the Backs triplibility criterion the functor $U : \mathcal{LR}(A) \to \text{Vect}/\text{Der}(A)$ is tripliable, so we have also the triple cohomology theory for Lie–Rinehart algebras. Let $\mathcal{L}$ be a Lie–Rinehart algebra. There is an equivalence from the category of Lie–Rinehart modules over $\mathcal{L}$ to the category of abelian group objects in $\mathcal{LR}(A)/\mathcal{L}$, which assigns the projection $\mathcal{L} \times M \to \mathcal{L}$ to $M \in (\mathcal{L}, A)\text{-mod}$. Having in mind this equivalence, Lemma 4.1 says that for any object $\mathcal{P} \to \mathcal{L}$ of $\mathcal{LR}(A)/\mathcal{L}$ the homomorphisms from $\mathcal{P} \to \mathcal{L}$ to $\mathcal{L} \times M \to \mathcal{L}$ in the category of abelian group objects in $\mathcal{LR}(A)/\mathcal{L}$ is nothing else, but $\text{Der}_A(\mathcal{P}, M)$. Therefore the triple cohomology $H^*_T(\mathcal{L}, M)$ is the same as $H^q(\text{Der}_A(T_*(\mathcal{L}), M))$. 

13
Theorem 4.3 For any Lie–Rinehart algebra \( \mathcal{L} \) and any \( \mathcal{L} \)-module \( M \) there is a natural isomorphism:

\[
H_{LR}^{q+1}(\mathcal{L}, M) \cong H_{LR}^q(\mathcal{L}, M), \quad q > 0.
\]

In other words the cotriple cohomology of \( \mathcal{L} \) with coefficients in \( M \) is isomorphic to the cohomology \( H_{LR}^*(\mathcal{L}, M) \) up to shift in the dimension.

Proof. As usual with bicomplex we have a spectral sequence

\[
E^2_{pq} \Rightarrow H^*_LR(\mathcal{L}, M)
\]

where \( E^2_{pq} \) is obtained in two steps: First one takes \( p \)-th homology in each \( C^*(T_q(\mathcal{L}), M), q \geq 0 \) and then one takes the \( q \)-th homology. But \( C^*(T_q(\mathcal{L}), M) \) is just the Rinehart complex of \( T_q(\mathcal{L}) \). Since \( T_q(\mathcal{L}) \) is free we can use Lemma 4.1 to conclude that \( E^1_{pq} = 0 \) for all \( p \geq 2 \). According to the exact sequence (1) one has also an exact sequence

\[
0 \to E^1_{0q} \to M \to \text{Der}_A(T_q(\mathcal{L}), M) \to E^1_{1q} \to 0
\]

One observes that \( E^1_{0*} \) and \( M \) are constant cosimplicial vector spaces and therefore \( E^2_{0q} = 0 \) for all \( q > 0 \). Thus we get

\[
H_{LR}^{q+1}(\mathcal{L}, M) \cong E^2_{1q} \cong H^q(\text{Der}_A(T^*(\mathcal{L}), M)), \quad q > 0.
\]

5 The canonical class of associative algebras

Let \( S \) an associative algebra over \( K \). We let \( A \) be the center of \( S \). As an application of our results we construct a canonical class \( o(S) \in H^3(A, H^1(S, S), A) \), where \( S \) is an associative algebra and \( H^*(S, S) \) denotes the Hochschild cohomology of \( S \).

Let us first recall the definitions of the zero and the first dimensional Hochschild cohomology involved in this construction. Let \( S \) be an associative \( K \)-algebra. A \( K \)-derivation \( D : S \to S \) is a \( K \)-linear map, such that \( D(ab) = D(a)b + aD(b) \). We let \( \text{Der}(S) \) be the set of all \( K \)-derivations. It has a natural Lie \( K \)-algebra structure, where the bracket is defined via the commutator \( [D, D_1] = DD_1 - D_1D \).

There is a canonical \( K \)-linear map

\[
ad : S \to \text{Der}(S)
\]

given by \( \text{ad}(s)(x) = sx - xs, s, x \in S \). Then the zero and the first dimensional Hochschild cohomology groups are defined via the exact sequence:

\[
0 \to H^0(S, S) \to S \xrightarrow{\text{ad}} \text{Der}(S) \to H^1(S, S) \to 0
\]
It follows that $A = H^0(S, S)$ is the center of $S$. We claim that $\text{Der}(S)$ is a Lie–Rinehart algebra over $A$. Indeed, the action of $A$ is defined by $(aD)(s) = aD(s)$, $D \in \text{Der}(S)$, $s \in S$, $a \in A$, while the homomorphism $\alpha : \text{Der}(S) \to \text{Der}(A)$ is just the restriction. To see that $\alpha$ is well-defined, it suffices to show that $D(A) \subset A$ for any $D \in \text{Der}(S)$. To this end, let us observe that for any $s \in S$ and $a \in A$ one has

$$D(a)s - sD(a) = (D(as) - aD(s)) - (D(sa) - D(s)a) = 0$$

and therefore $D(a) \in A$. On the other hand the commutator $[s, t] = st - ts$ defines a Lie $A$-algebra structure on $S$ and $\text{ad} : S \to \text{Der}(S)$ is a Lie $K$-algebra homomorphism. Actually more is true: $\text{ad}$ is a crossed module of Lie–Rinehart algebras over $A$, where the action of the Lie–Rinehart algebra $\text{Der}(S)$ on $S$ is given by $(D, s) \mapsto D(s)$. It follows that $H^1(S, S) = \text{Coker}(\text{ad} : S \to \text{Der}(S))$ is also a Lie–Rinehart algebra over $A$ and $A = \ker(\text{ad} : S \to \text{Der}(S))$ is a Lie–Rinehart module over $H^1(S, S)$. In particular the groups $H^n(A, H^1(S, S), A)$ are well-defined. According to Theorem 3.1 the vector space $H^3(A, H^1(S, S), A)$ classifies the crossed extension of $H^1(S, S)$ by $A$. By our construction the exact sequence (2) is one of such extension and therefore it defines a canonical class $o(S) \in H^3(A, H^1(S, S), A)$.

**Lemma 5.1** $o(S)$ is a Morita invariant.

**Proof.** Let $R$ be the $K$-algebra of $n \times n$ matrices. We have to prove that $o(S) = o(R)$. Let $D$ be a derivation of $S$. We let $g(D)$ be the derivation of $R$ which is componentwise extension of $D$. Furthermore, for an element $s \in S$ we let $f(s)$ be the diagonal matrix with $s$ on diagonals. Then one has the following commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\text{ad}} & \text{Der}(S) \\
\downarrow f & & \downarrow g \\
R & \xrightarrow{\text{ad}} & \text{Der}(R)
\end{array}
$$

in the category $\mathcal{LR}(A)$ and the result follows from the fact that Hochschild cohomology is a Morita invariant. $\square$

Let us observe that if $S$ is a smooth commutative algebra, then $A = S$ and $H^3(A, H^1(S, S), A)$ is isomorphic to the de Rham cohomology of $S$ (of course $o(S) = 0$ in this case). So, in general one can consider the groups $H^3(A, H^1(S, S), A)$ as a sort of noncommutative de Rham cohomology.

By forgetting $A$-module structure, one obtains an element

$$o'(S) \in H^3_{\text{Lie}}(H^1(S, S), A).$$

These groups and probably the corresponding elements can be compute in many cases using the results of Strametz [14].
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