Non-Abelian Vortex in Four Dimensions as a Critical Superstring

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1 Introduction

Confinement is not just ONE problem. It is TWO problems

- Understand the nature of confining strings
  
  What do we know?
  
  - Lattice
  
  - Supersymmetry: Seiberg-Witten solution of $\mathcal{N} = 2$ QCD. Abelian
  
  - Non-Abelian generalizations?
    
    Non-Abelian vortex strings
    
    Quarks condense $\Rightarrow$ monopoles are confined

- Quantize confining string outside critical dimension

  What do we know??????

  Shifman and Yung, 2015: Non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric QCD can behave as a critical superstring
Non-Abelian vortex strings

Non-Abelian strings were found in $\mathcal{N} = 2$ $U(N)$ QCD

*Hanany, Tong 2003*

*Auzzi, Bolognesi, Evslin, Konishi, Yung 2003*

*Shifman Yung 2004*

*Hanany Tong 2004*

$Z_N$ *Abelian string:* Flux directed in the Cartan subalgebra, say for

$SO(3) = SU(2)/Z_2$

$$\text{flux} \sim \tau_3$$

Non-Abelian string: *Orientational zero modes*

Rotation of color flux inside $SU(N)$. 
Idea:

Non-Abelian string has more moduli than Abrikosov-Nielsen-Olesen (ANO) string.

It has translational + orientational moduli.

We can fulfill the criticality condition: In $\mathcal{N} = 2$ QCD with $U(\mathcal{N} = 2)$ gauge group and $N_f = 4$ quark flavors.

- The solitonic non-Abelian vortex has six orientational moduli, which, together with four translational moduli, form a ten-dimensional space.

- For $N_f = 2N$ 2D world sheet theory on the string is conformal.
Most of solitonic strings are "thick".

Transverse size $= \frac{1}{m}$, where $m$ is the typical mass of bulk excitations.

$$S_{2D} = T \int d^2 \sigma \left\{ (\text{LE} \ \sigma-\text{model}) + O \left( \frac{\partial^n}{m^n} \right) \right\}$$

where $T$ is string tension

*Polchinski-Strominger, 1991: Without higher derivative terms*

the world sheet theory is not UV complete
Given that for non-Abelian vortex low energy world sheet theory is critical we conjecture that

\[ T \ll m^2 \]

is actually satisfied at strong coupling \( g_c^2 \sim 1 \).

\[ m(g) \rightarrow \infty, \quad g^2 \rightarrow g_c^2 \]

Higher derivative corrections can be ignored
2 Non-Abelian vortex strings

Bulk theory: 4D $\mathcal{N} = 2$ QCD with Fayet-Iliopoulos term.

For $U(N)$ gauge group in the bulk we have 2D $CP(N - 1)$ model on the string $CP(N - 1) \equiv U(1)$ gauge theory in the strong coupling limit

$$S_{CP(N-1)} = \int d^2x \left\{ |\nabla_\alpha n^P|^2 + \frac{e^2}{2} (|n^P|^2 - \beta)^2 \right\},$$

where $n^P$ are complex fields $P = 1, \ldots, N$,

Condition

$$|n^P|^2 = \beta \approx \frac{4\pi}{g^2},$$

imposed in the limit $e^2 \to \infty$
More flavors ⇒ semilocal non-Abelian string

The orientational moduli described by a complex vector \( n^P \) (here \( P = 1, \ldots, N \)), \( \tilde{N} = (N_f - N) \) size moduli are parametrized by a complex vector \( \rho^K \) \( (K = N + 1, \ldots, N_f) \).

The effective two-dimensional theory is the \( \mathcal{N} = (2, 2) \) weighted CP model

\[
S_{\text{WCP}} = \int d^2 x \left\{ |\nabla_{\alpha} n^P|^2 + |\tilde{\nabla}_{\alpha} \rho^K|^2 + \frac{e^2}{2} \left( |n^P|^2 - |\rho^K|^2 - \beta \right)^2 \right\},
\]

\( P = 1, \ldots, N \), \( K = N + 1, \ldots, N_f \).

The fields \( n^P \) and \( \rho^K \) have charges +1 and −1 with respect to the auxiliary U(1) gauge field

\( e^2 \to \infty \)

Global group

\[
SU(N) \times SU(\tilde{N}) \times U(1)
\]
3 From non-Abelian vortices to critical strings

String theory

\[
S = \frac{T}{2} \int d^2 \sigma \sqrt{h} h^{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x_\mu \\
+ \int d^2 \sigma \sqrt{h} \left\{ h^{\alpha \beta} \left( \tilde{\nabla}_\alpha \bar{n}_P \nabla_\beta n^P + \nabla_\alpha \bar{\rho}_K \tilde{\nabla}_\beta \rho^K \right) \\
+ \frac{e^2}{2} \left( |n^P|^2 - |\rho^K|^2 - \beta \right)^2 \right\} + \text{fermions},
\]

where \( h^{\alpha \beta} \) is the world sheet metric. It is independent variable in the Polyakov formulation.
Criticality conditions

- Conformal invariance

\[ b_{WCP} = N - \tilde{N} = 0 \Rightarrow N = \tilde{N}, \quad N_f = 2N \]

- Critical dimension = 10

Number of orientational + size degrees of freedom

\[ = 2(N + \tilde{N} - 1) = 2(2N - 1) \]

\[ 4 + 2(2N - 1) = 4 + 6 = 10, \quad \text{for } N = 2 \]

Our string is BPS so we have \( \mathcal{N} = (2, 2) \) supersymmetry on the world sheet.

For these values of \( N \) and \( \tilde{N} \) the target space of the weighted \( CP(2, 2) \) model is a non-compact Calabi-Yau manifold studied by Candelas, Witten and Vafa, namely conifold.
Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

For closed string moving on Calabi-Yau manifold $\mathcal{N} = (2, 2)$ world sheet supersymmetry ensures $\mathcal{N} = 2$ supersymmetry in 4D.

This is expected since we started with 4D QCD with $\mathcal{N} = 2$ supersymmetry.

Type IIB string is a chiral theory and breaks parity while Type IIA string theory is left-right symmetric and conserves parity.

Our bulk theory conserves parity $\Rightarrow$ we have Type IIA superstring
We conjectured that the string becomes thin \( m \to \infty \) at \( g^2 \to g_c^2 \sim 1 \).

\[ g^2 \iff \beta \]

4D coupling  2D coupling

It is natural to expect that

\[ g_c^2 \iff \beta = 0 \]

\textit{D}-term condition in weighted CP(2,2) model

\[ |n^P|^2 - |\rho^K|^2 = \beta, \quad P = 1, 2, \quad K = 1, 2 \]

At \( \beta = 0 \) conifold develops conical singularity.
4 4D massless states

Our goal:

Study states of closed string propagating on

$$R_4 \times Y_6, \quad Y_6 = \text{conifold}$$

and interpret them as hadrons in 4D $\mathcal{N} = 2$ QCD.

Massless states = Deformations of 10D metric preserving Ricci flatness

Massless 4D graviton

Constant wave functions over conifold

Non-normalizable on non-compact $Y_6$.

No 4D graviton == good news!

We do not have gravity in our 4D $\mathcal{N} = 2$ QCD
Kahler form deformations

Kahler form deformations = variations of 2D coupling $\beta$

$D$-term condition in weighted CP(2,2) model

$$|n^P|^2 - |\rho^K|^2 = \beta, \quad P = 1, 2, \quad K = 1, 2$$

Resolved conifold

$\beta$ - non-normalizable mode
5 Deformation of the complex structure

$D$-term condition

$$|n^P|^2 - |\rho^K|^2 = \beta, \quad P = 1, 2, \quad K = 1, 2$$

Construct $U(1)$ gauge invariant "mesonic" variables

$$w^{PK} = n^P \rho^K.$$ 

$$\det w^{PK} = 0$$

**Take $\beta = 0$**

Complex structure deformation $\Rightarrow$ **Deformed conifold**

$$\det w^{PK} = b$$
$b$ – complex modulos

The effective action for $b(x)$ is

$$S(\beta) = T \int d^4x \ h_b (\partial_\mu b)^2,$$

where

$$h_b = \int d^6y \sqrt{g} g^{li} \left( \frac{\partial}{\partial b} g_{ij} \right) g^{jk} \left( \frac{\partial}{\partial b} g_{kl} \right)$$

Using explicit Calabi-Yau metric on deformed conifold we get

$$h_b = (4\pi)^3 \frac{4}{3} \log \frac{T^2 L^4}{|b|}$$

For Type IIA string $b$ should be a part of hypermultiplet.
6 Non-Abelian vortex and Little String Theory

For $\beta = 0$ supergravity approximation does not work.

Still can be used for massless states = chiral primary operators (4D BPS states)

Protected

Consider massive states

Ghoshal, Vafa, 1995; Giveon Kutasov 1999

Critical string on a conifold is equivalent to non-critical $c = 1$ string

$$\mathcal{R}^4 \times \mathcal{R}_\phi \times S^1,$$

$\mathcal{R}_\phi$ is a real line associated with the Liouville field $\phi$ and the theory has a linear in $\phi$ dilaton, such that string coupling is given by

$$g_s = e^{-\frac{Q}{2} \phi}.$$
String theories with this behavior of the dilaton are holographic – ”Little String Theories”

Non-trivial dynamics is localized on the $\mathcal{R}^4$ boundary

This is exactly what we want!

We expect that LST in our case is 4D $\mathcal{N} = 2$ supersymmetric QCD at the self-dual value of the gauge coupling $g^2 = 4\pi$ (in the hadronic description)

$$T_{--} = -\frac{1}{2} \left[ (\partial_z \phi)^2 + Q \partial^2_z \phi + (\partial_z Y)^2 \right]$$

$$Y \sim Y + 2\pi Q \quad Q = \sqrt{2}, \quad c_{\phi+Y}^{SUSY} = 3 + 3Q^2 = 9$$

Liouville interaction

$$\delta L = b \int d^2 \theta \ e^{-\frac{\phi+iY}{Q}}$$
Mirror description: $SL(2, R)/U(1)$ WZNW model at level $k = 1$.

Bosonic part is 2D Witten’s black hole with target space forming semi-infinite cigar.

Liouville field $\phi$ – motion along the cigar.

The spectrum of primary operators was computed exactly.

*Dixon, Peskin, Lykken, 1989; Mukhi, Vafa, 1993; Evans, Gaberdiel, Perry, 1998*

$$V_{j,m} \approx \exp \left( \sqrt{2} j \phi + i \sqrt{2} m Y \right), \quad \phi \to \infty$$

- Normalizable states – discrete series with $j \leq -\frac{1}{2}$
- No negative norm states

$$j = -\frac{1}{2}, \quad m = \pm \left\{ \frac{1}{2}, \frac{3}{2}, \ldots \right\}$$

$$j = -1, \quad m = \pm \left\{ 1, 2, \ldots \right\}$$
10D "tachyon"

\[ V_{j,m}^{S}(p_{\mu}) = e^{-\varphi} e^{ip_{\mu}x^{\mu}} V_{j,m}, \quad j = -\frac{1}{2}, \quad m = \pm \left\{ \frac{1}{2}, \frac{3}{2}, \ldots \right\} \]

\[ \frac{(M^{S})^{2}}{8\pi T} = -\frac{p_{\mu}p^{\mu}}{8\pi T} = m^{2} - \frac{1}{2} - j(j + 1) = m^{2} - \frac{1}{4} = 0, 2, 6, \ldots \]

Massless state at \( m = \pm \frac{1}{2} \) – b-state

Spin-2 states

\[ V_{j,m}^{G}(p_{\mu}) = \xi_{\mu\nu} \psi_{L}^{\mu} \psi_{R}^{\nu} e^{-\varphi} e^{ip_{\mu}x^{\mu}} V_{j,m}, \quad j = -1, \quad m = \pm \left\{ 1, 2, \ldots \right\} \]

\[ \frac{(M^{G})^{2}}{8\pi T} = m^{2} = 1, 4, 9, \ldots \]

No massless graviton
Global group of the 4D QCD:

\[ SU(2) \times SU(2) \times U(1) \]

U(1) - "baryonic" symmetry.

\[ Q_B = 4m \]
7 Supermultiplet structure

Lowest states:

- Massless state $b$
  \[ j = -\frac{1}{2}, \quad m = \pm \frac{1}{2} \]
  
  Short BPS multiplet

Hypermultiplet = $4_{scalar} + \text{fermions}$

- $j = -\frac{1}{2}, \quad m = \pm \frac{3}{2}$

\[
\frac{(M_{j=-\frac{1}{2}, m=\pm\frac{3}{2}})^2}{8\pi T} = 2
\]

Two long non-BPS vector supermultiplets

\[ (\mathcal{N} = 2)_{\text{vector}} = 1_{\text{vector}} + 5_{\text{scalar}} + \text{fermions} \]
\( j = -1, m = \pm 1 \)

\[
\frac{(M_{j=-1,m=\pm1})^2}{8 \pi T} = 1
\]

\( (j = -1) \text{ states} = 2 \times (N = 2)_{\text{spin}-2} + 4 \times (N = 2)_{\text{vector}} \)

where

\( (N = 2)_{\text{spin}-2} = 1_{\text{spin}-2} + 6_{\text{vector}} + 1_{\text{scalar}} + \text{fermions} \)
Strings in the $U(N)$ theories are stable; they cannot be broken. Thus, we deal with the closed string.

Quarks are condensed in 4D theory. Therefore, monopoles are confined.

In $U(N)$ gauge theories the confined monopoles are junctions of two non-Abelian vortex strings.

Monopole-antimonopole meson  Monopole-monopole baryon
9 Conclusions

- In $\mathcal{N} = 2$ supersymmetric QCD with gauge group $U(2)$ and $N_f = 4$ quark flavors, non-Abelian BPS vortex behaves as a critical superstring.

- Massless closed string state $b$ associated with deformations of the complex structure of the conifold $\Rightarrow$ monopole-monopole baryon.

- Successful tests of our gauge-string duality:
  - $\mathcal{N} = 2$ supersymmetry in 4D QCD
  - Absence of graviton and unwanted vector fields.

- Spectrum of lowest massive baryons is calculated using "Little String Theory" description

We calculate hadron spectrum from first principles!
Higher derivative terms at weak coupling, \( g \ll 1 \)

\[
O \left( \frac{\partial^n}{m^n} \right), \quad m \sim g \sqrt{T}
\]

At \( J \sim 1 \) \( \partial \rightarrow \sqrt{T} \)

Thus higher derivative terms

\[
\rightarrow \left( \frac{T}{m^2} \right)^n
\]

blow up at weak coupling!

*Polyakov*: string surface become "crumpled".

4D interpretation: String grows short and thick.

\[
L^2 \sim \frac{J}{T} \lesssim \frac{1}{m}, \quad \text{for } J \sim 1
\]
There is self-duality in 4D bulk theory

\[ \tau \rightarrow \tau_D = -\frac{1}{\tau}, \quad \tau = \frac{4\pi i}{g^2} + \frac{\theta_{AD}}{2\pi}, \]

We conjectured that the string becomes thin at \( g^2 \rightarrow g_c^2 \sim 1 \).

It is natural to expect that \( g_c^2 = 4\pi = \) self-dual point.

\[ m^2 \rightarrow T \times \begin{cases} 
  g^2, & g^2 \ll 1 \\
  \infty, & g^2 \rightarrow 4\pi \\
  16\pi^2/g^2, & g^2 \gg 1
\end{cases} \]

In 2D theory on the string self-dual point is \( \beta = 0 \)

Conifold develops conical singularity.
QUESTION:

Can we find any example of a 4D field theory which supports thin vortex strings?

Non-Abelian vortex in $\mathcal{N} = 2$ QCD with U(2) gauge group and $N_f = 4$ flavors is critical.
\( \mathcal{N}^c = 2 \) supersymmetric QCD with gauge group \( U(N) \) and \( N_f \) quark flavors

(Scalar) quarks condense ⇒ monopoles are confined

Strings in the \( U(N) \) theories are stable; they cannot be broken.

In \( U(N) \) gauge theories the confined monopoles are junctions of two non-Abelian vortex strings.

Example

Monopole-antimonopole meson

Constituent quark = monopole
Physical nature of non-normalizable modes

*Gukov, Vafa, Witten 1999:* Non-normalizable moduli = coupling constants in 4D

- 4D metric do not fluctuate. It is fixed to be flat. "Coupling constants."
- 2D coupling $\beta$ is related to 4D coupling $g^2$. Fixed. Non-dynamical.

Another option:

Large $y_i \Rightarrow$ large $n^P$ and $\rho^K$

Non-normalizable modes are not localized on the string.

Unstable states. Decay into massless perturbative states.

Higgs branch: $\dim \mathcal{H} = 4N \tilde{N} = 16$. 
Strong coupling

Global group of the 4D QCD:

\[ SU(2) \times SU(2) \times U(1) \]

U(1) - "baryonic" symmetry.

\( b \)-hypermultiplet: (1, 1, 2)

Logarithmically divergent norm == Marginal stability at \( \beta = 0 \)

\( b \)-state can decay into massless bi-fundamental (screened) quarks living on the Higgs branch.