QUADRATIC QUANDLES AND THEIR LINK INVARIANTS

R.A. LITHERLAND

Abstract. Carter, Jelsovsky, Kamada, Langford and Saito have defined an invariant of classical links associated to each element of the second cohomology of a finite quandle. We study these invariants for Alexander quandles of the form $\mathbb{Z}[t, t^{-1}] / (p, t^2 + \kappa t + 1)$, where $p$ is a prime number and $t^2 + \kappa t + 1$ is irreducible modulo $p$. For each such quandle, there is an invariant with values in the group ring $\mathbb{Z}[C_p]$ of a cyclic group of order $p$. We shall show that the values of this invariant all have the form $\Gamma^r p^s$ for a fixed element $\Gamma$ of $\mathbb{Z}[C_p]$ and integers $r \geq 0$ and $s > 0$. We also describe some machine computations, which lead us to conjecture that the invariant is determined by the Alexander module of the link. This conjecture is verified for all torus and two-bridge knots.

1. Quadratic Quandles

In [CJKLS], Carter, Jelsovsky, Kamada, Langford and Saito associate an invariant of classical links to each element of the second cohomology of a finite quandle. In this paper we study these invariants for a class of quandles we refer to as quadratic quandles, which will be described shortly. We assume familiarity with the definitions of quandle and quandle cohomology as given in [CJKLS]. Our notation generally follows that paper, except that we write the operation in a quandle as exponentiation (following Fenn and Rourke [FR]). We denote the ring $\mathbb{Z}[t, t^{-1}]$ of integer Laurent polynomials by $\Lambda$, and the ring $\mathbb{Z}/(n)[t, t^{-1}]$ by $\Lambda_n$. For any ring $R$, we denote its group of units by $R^\times$.

Recall that an Alexander quandle is a $\Lambda$-module with the quandle operation $a^b = ta + (1-t)b$. We let $p$ be a prime number, and denote the finite fields of orders $p$ and $p^2$ by $\mathbb{F}_p$ and $\mathbb{E}_p$ respectively. For $x \in \mathbb{E}_p$, we denote the conjugate of $x$ over $\mathbb{F}_p$ by $\bar{x}$ and its norm $N(x)$. We fix a generator $\theta$ of $\mathbb{E}_p$ over $\mathbb{F}_p$, and give $\mathbb{E}_p$ the structure of a $\Lambda$-module by setting $tx = \theta x$ ($x \in \mathbb{E}_p$). Thus $\mathbb{E}_p \cong \mathbb{E}_p[t, t^{-1}] / (t - \theta)$ as a $\Lambda$-module, so Theorem 2.2 of Mochizuki [M] gives the dimension of the quandle cohomology $H^2_Q(\mathbb{E}_p; \mathbb{E}_p)$ over $\mathbb{E}_p$, which is the same as that of $H^2_Q(\mathbb{E}_p; \mathbb{F}_p)$ over $\mathbb{F}_p$; the dimension is 1 if $\theta^{p+1} = 1$, and 0 otherwise. Since the non-trivial element of the Galois group of $\mathbb{E}_p$ over $\mathbb{F}_p$ is the Frobenius automorphism, $\theta^{p+1} = \theta = N(\theta)$. Hence we further require $\theta$ to have norm 1, and when this is so we call $\mathbb{E}_p$ with the structure of an Alexander quandle determined by $\theta$ a quadratic quandle. We denote the minimal polynomial of $\theta$ over $\mathbb{F}_p$ by $h$; it is an irreducible polynomial of the form $t^2 + \kappa t + 1$. An equivalent description is that a quadratic quandle is an Alexander quandle $\mathbb{E}_p / (h)$ where $h$ is such a polynomial.

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For the quadratic quandle $\mathbb{E}_p$ we have $H^2_Q(\mathbb{E}_p; \mathbb{F}_p) \cong \mathbb{F}_p$, and choosing a non-zero element gives a link invariant with values in the group ring $\mathbb{Z}[C_p]$ of a cyclic group of order $p$. To describe an explicit cocycle, we fix a non-zero element $i \in \mathbb{E}_p$, with $i = -i$. Define $\phi: \mathbb{E}_p^2 \to \mathbb{F}_p$ by $\phi(x, y) = i(x\bar{y} - \bar{x}y)$. Then $\phi$ is an alternating $\mathbb{F}_p$-bilinear form, and satisfies
\[
\phi(x, y, z) = N(z)\phi(x, y) \quad \text{for } x, y, z \in \mathbb{E}_p. \tag{1.1}
\]
Recall that the quandle chain group $C_2^Q(\mathbb{E}_p)$ is obtained from the free abelian group on $\mathbb{E}_p^2$ by quotienting out the subgroup generated by pairs $(x, x)$. Since $\phi(x, x) = 0$, $\phi$ determines a cochain in $C_2^Q(\mathbb{E}_p; \mathbb{F}_p)$, which we also denote by $\phi$.

**Lemma 1.1.** For the quadratic quandle $\mathbb{E}_p$, the function $\phi$ is a cocycle representing a non-zero element of $H^2_Q(\mathbb{E}_p; \mathbb{F}_p)$.

**Proof.** Let $x, y, z \in \mathbb{E}_p$. From (1.1) we have $\phi(\theta x, \theta y) = \phi(x, y)$ and so
\[
\phi(\theta x, (1 - \theta)y) = \phi(\theta x, y) - \phi(x, y) = \phi((\theta - 1)x, y).
\]
Hence
\[
\phi(x^\theta, y^\theta) = \phi(x + (1 - \theta)z, y + (1 - \theta)z)
= \phi(x, y) + \phi(\theta x, (1 - \theta)z) + \phi((1 - \theta)z, \theta y)
= \phi(x, y) + \phi((\theta - 1)(x - y), z)
= \phi(x, y) + \phi((\theta - 1)(x - y), z) - \phi(x, z)
= \phi(x, y) + \phi(x^\theta, z) - \phi(x, z)
\]
so
\[
\phi(\partial(x, y, z) = \phi(x, z) - \phi(x^\theta, z) - \phi(x, y) + \phi(x^\theta, y^\theta) = 0.
\]
Thus $\phi$ is a cocycle. To see that it is not a coboundary, we observe that $c = (1, -\theta) - (1, 0) - (\theta, 0) \in C_2^Q(\mathbb{E}_p)$ is a cycle and $\phi(c) = i(\theta - \theta) \neq 0$. 

2. **Their link invariants**

Let $K$ be an oriented classical link, and $Q(K)$ its fundamental quandle. The cocycle invariant $\Phi_\phi(K)$ defined in [CJKLS] involves a sum over all quandle homomorphisms from $Q(K)$ to $\mathbb{E}_p$. The number of such homomorphisms is determined in Theorem 1 of Inoue [I]. The proof of that result may be viewed in the following way. (See Example 2 in §6.4 of [FR].) We first recall some facts about the (one-variable) Alexander module of $K$. Let $X$ be the exterior of $K$, $\bar{X}$ its infinite cyclic cover, and $\bar{X}_0$ a fiber of $\bar{X}$. Then the Alexander module is $H_1(\bar{X}, \bar{X}_0)$, regarded as an $A$-module with $t$ acting by a generator of the group of covering transformations. We shall use the notation $A(K)$ for $H_1(\bar{X}, \bar{X}_0)$. We also set $\hat{A}(K) = H_1(\bar{X})$, and then there is a short exact sequence
\[
0 \to \hat{A}(K) \to A(K) \to \Lambda \to 0.
\]
(If $K$ is a knot, $\hat{A}(K)$ is the torsion submodule of $A(K)$.) Similarly, we have $A_p$-modules $A_p(K) = H_1(\bar{X}, \bar{X}_0; \mathbb{F}_p)$ and $\hat{A}_p(K) = H_1(\bar{X}; \mathbb{F}_p)$.

Now take a diagram $D$ of $K$. At each crossing $c$ let $\omega(c)$ be the overcrossing arc, $\lambda(c)$ the undercrossing arc to the left of $\omega(c)$, and $\rho(c)$ the one to the right. Then
A(K) has a $\Lambda$-module presentation with the arcs of $D$ as generators and a relation $\lambda(c) = tp(c) + (1-t)\omega(c)$ for each crossing. Also, the fundamental quandle of $K$ has a presentation with the arcs of $D$ as generators and a relation $\lambda(c) = \rho(c)\omega(c)$ for each crossing. Thus, when $A(K)$ is regarded as a quandle, there is a quandle homomorphism $Q(K) \to A(K)$ which is universal among homomorphisms from $Q(K)$ to Alexander quandles. Hence there is a one-to-one correspondence between quandle homomorphisms $Q(K) \to \mathbb{E}_p$ and $\Lambda$-module homomorphisms $A(K) \to \mathbb{E}_p$, which in turn correspond to $\Lambda$-module presentations of $\mathbb{E}_p$. We set $\text{Rep}(K, \mathbb{E}_p) = \text{Hom}_\Lambda(A_p(K), \mathbb{E}_p)$; this is a vector space over $\mathbb{E}_p$. Now $A_p$ is a PID, so we may consider the primary decomposition of $A_p(K)$. Let $\nu_h(K)$ be the number of $h$-torsion factors in this decomposition, and let $\nu_0(K)$ be the rank of $A_p(K)$.

The cocycle invariant may be defined as follows. (Our terminology and notation differs slightly from that of [CJKLS], but the definitions are equivalent.) Let $C_p$ be a (multiplicative) cyclic group of order $p$, with generator $u$. Let $\text{sign}(c) = \pm 1$ be the sign of the crossing $c$. For each $f \in \text{Rep}(K, \mathbb{E}_p)$, define the Boltzmann weight of $f$ to be

$$B(f) = \sum_c \text{sign}(c)\phi(f\rho(c), f\omega(c)) \in \mathbb{F}_p,$$

where the sum is taken over all crossings of the diagram $D$. Then

$$\Phi_\phi(K) = \sum_{f \in \text{Rep}(K, \mathbb{E}_p)} u^{B(f)} \in \mathbb{Z}[C_p].$$

The invariant depends only on the cohomology class of $\phi$; for quadratic quandles, it follows from Theorem 2.1 below that all non-zero classes give the same invariant. Now define $\eta$: $\text{Rep}(K, \mathbb{E}_p) \times \text{Rep}(K, \mathbb{E}_p) \to \mathbb{E}_p$ by

$$\eta(f, g) = \sum_c \text{sign}(c)i(f\rho(c)g\omega(c) - f\omega(c)g\rho(c)).$$

Then $\eta$ is a Hermitian form with respect to the involution $x \mapsto \bar{x}$ of $\mathbb{E}_p$, and $B(f) = \eta(f, f)$ for $f \in \text{Rep}(K, \mathbb{E}_p)$.

**Theorem 2.1.** Let $\Gamma_p = 1 + (p + 1)\sum_{i=1}^{p-1} u^i \in \mathbb{Z}[C_p]$. For the cocycle of Lemma 2.1, we have $\Phi_\phi(K) = \Gamma_p^{\nu}$. $\eta$ is Hermitian, where $r$ is the rank of the Hermitian form $\eta$ and $s = \nu_h(K) + \nu_0(K) - r$ is its nullity. Further, $s > 0$.

**Proof.** Let $n = \nu_h(K) + \nu_0(K)$, the dimension of $\text{Rep}(K, \mathbb{E}_p)$. We may choose a basis $\{f_1, \ldots, f_n\}$ of $\text{Rep}(K, \mathbb{E}_p)$ that is orthogonal with respect to $\eta$. Let $a_i = \eta(f_i, f_i) \in \mathbb{E}_p$. Then, for $x = (x_1, \ldots, x_n) \in \mathbb{E}_p^n$, we have $B(\sum_{i=1}^n x_if_i) = \sum_{i=1}^n a_iN(x_i)$, and so

$$\Phi_p(K) = \sum_{x \in \mathbb{E}_p^n} u^{\sum_{i=1}^n a_iN(x_i)} = \prod_{i=1}^n \sum_{x \in \mathbb{E}_p} u^{a_iN(x)}.$$

Since the norm restricts to a homomorphism of $\mathbb{E}_p^\times$ onto $\mathbb{F}_p^\times$, $\sum_{x \in \mathbb{E}_p} u^{a_iN(x)}$ is equal to $\Gamma_p$ if $a_i \neq 0$, and $p^2$ if $a_i = 0$. It only remains to prove that $s > 0$; that is, that the form $\eta$ is singular. This we do in Lemma 2.4 below.
Lemma 2.2. In the Alexander module of a link \( K \) with diagram \( D \), there is a relation

\[
(1 - t) \sum_c \text{sign}(c)(\rho(c) - \omega(c)) = 0.
\]

Proof. For each crossing \( c \), let \( \alpha(c) \) be the undercrossing arc oriented into \( c \), and \( \beta(c) \) the one oriented out. Thus, if \( \text{sign}(c) = 1 \) then \( \alpha(c) = \rho(c) \) and \( \beta(c) = \lambda(c) \), while if \( \text{sign}(c) = -1 \) then \( \alpha(c) = \lambda(c) \) and \( \beta(c) = \rho(c) \). We have

\[
(1 - t) \sum_c \text{sign}(c)(\rho(c) - \omega(c)) = \sum_c \text{sign}(c)(\rho(c) - \lambda(c))
\]

\[
= \sum_c (\alpha(c) - \beta(c))
\]

\[
= 0,
\]

because each arc of \( D \) is \( \alpha(c) \) for a single crossing \( c \), and \( \beta(c') \) for another crossing \( c' \).

Remark 2.3. If \( K \) is a knot, then \( A(K) \) has no \( (1 - t) \)-torsion, so we have the relation \( \sum_c \text{sign}(c)(\rho(c) - \omega(c)) = 0 \).

Let \( \text{Rep}_0(K, \mathbb{E}_p) \leq \text{Rep}(K, \mathbb{E}_p) \) be the annihilator of \( \hat{A}_p(K) \), a 1-dimensional subspace of \( \text{Rep}(K, \mathbb{E}_p) \). Because all arcs of \( D \), considered as elements of \( A_p(K) \), map to the same element of \( A_p(K)/\hat{A}_p(K) \), each element of \( \text{Rep}_0(K, \mathbb{E}_p) \) takes the same value on each arc. Thus elements of \( \text{Rep}_0(K, \mathbb{E}_p) \) correspond to constant colorings of \( D \), and have Boltzmann weight 0; however, more is true. We denote the radical of \( \eta \) by \( \text{rad} \eta \). Thus \( g \in \text{Rep}(K, \mathbb{E}_p) \) is in \( \text{rad} \eta \) iff \( \eta(f, g) = 0 \) for all \( f \in \text{Rep}(K, \mathbb{E}_p) \).

Lemma 2.4. \( \text{Rep}_0(K, \mathbb{E}_p) \) is contained in \( \text{rad} \eta \); in particular, \( \eta \) is singular.

Proof. Let \( f \in \text{Rep}(K, \mathbb{E}_p) \) and \( g \in \text{Rep}_0(K, \mathbb{E}_p) \), and let \( x \) be the common value of \( g \) on the arcs of \( D \). Then

\[
\eta(f, g) = \sum_c \text{sign}(c)i(f \rho(c) - f \omega(c))\bar{x} = ix f(a),
\]

where \( a = \sum_c \text{sign}(c)(\rho(c) - \omega(c)) \in A_p(K) \). By Lemma 2.2

\[
(1 - \theta)f(a) = f((1 - t)a) = 0.
\]

Since \( 1 - \theta \) is a non-zero element of the field \( \mathbb{E}_p \), \( f(a) = 0 \) and we are done.

Note that if \( \nu_0(K) = 1 \), this lemma implies that \( \text{rank} \eta \leq \nu_0(K) \). This is the case for all knots, and for all links considered in the rest of this paper.

3. Computations

We have carried out some machine computations of the invariants of knots associated to quadratic quandles. The program (written in C) proceeds as follows. The knots are taken from the tables distributed with Knotscape [HT], which describe minimal-crossing diagrams in Dowker-Thistlethwaite code. From this, a presentation matrix for \( A_p(K) \) is read off. This matrix is diagonalized, and the matrix expressing the original (arc) generators in terms of the new generators is computed. We find \( A_p(K)_h \) by finding the highest power of \( h \) dividing each non-zero entry; in particular, \( \nu_h(K) \) is just the number of non-zero entries divisible by \( h \). The form
$\eta'(f,g) = \eta(f,g) + \eta(g,f)$ is a symmetric $\mathbb{F}_p$-bilinear form on $\text{Rep}(K,\mathbb{E}_p)$ with rank $\eta' = 2\text{rank }\eta$. Using the change-of-basis matrix, a matrix for $\eta'$ can be found, and its rank determined, which gives rank $\eta$. (It may seem inefficient to use $\eta'$ instead of $\eta$ since this doubles the size of the matrix, but it is easier to work over $\mathbb{F}_p$ rather than $\mathbb{E}_p$.) Now $\Phi_K$ is determined by Theorem 2.1. Some computations of quandle invariants are given by Carter, Jelsovsky, Kamada and Saito [CJKS], and there is some overlap with ours. For $p = 2$ or $3$ there is just one choice for $h$, giving two quadratic quandles $\Lambda_2/(t^2 + t + 1)$ and $\Lambda_3/(t^3 + 1)$. In [CJKS], the values of the invariants for these quandles are computed for knots up to 9 crossings, and we have repeated these computations. For $p = 2$, the results agree, but for $p = 3$ there are some discrepancies, at least one of which is due to a typographical error in [CJKS]. To obtain our results from theirs:

1. in the list of knots with invariant $9(1 + 4t + 4t^2)$ replace $947$ by $948$;
2. from the list of knots with invariant $81$, delete $824$ [sic] and $946$, and add $86$, $93$ and $947$.

We have not extended the tabulation of invariants to higher crossing numbers; instead, we have counted the knots with a given combination of $A_2(K)_h$ and rank $\eta$. For example, with $p = 2$, of the 313230 knots of at most 15 crossings, 445 have $\nu_h(K)$ equal to 3. These may be further classified as in Table 1, where an entry $(a, b, c)$ in the first column means that $A_2(K)_h \cong \Lambda_2/(h^a) \oplus \Lambda_2/(h^b) \oplus \Lambda_2/(h^c)$. It will be noted that in these cases rank $\eta$ is equal to the number of ones in the triple $(a, b, c)$. In general, let $\nu'_h(K)$ be the number of factors in the primary decomposition of $A_2(K)$ of the form $\Lambda_2/(h)$.

Conjecture 1. For any link $K$, rank $\eta = \nu'_h(K)$.

If this is true, the invariants associated to quadratic quandles contain no more information than the Alexander module, which would be somewhat disappointing. We have verified it for $p = 2, 3, 5, 7$ and 11, for each possible choice of the coefficient $\kappa$ in $h$ (giving twelve invariants), and for all knots up to (and including) 16 crossings. In fact, a little more seems to be true. The number $\nu'_h(K)$ has a more conceptual description. Let $A'_2(K)$ be the submodule of all $a \in A_2(K)$ for which $ha = 0$, and let $\text{Rep}'(K,\mathbb{E}_p) \leq \text{Rep}(K,\mathbb{E}_p)$ be its annihilator.

Lemma 3.1. The codimension of $\text{Rep}'(K,\mathbb{E}_p)$ in $\text{Rep}(K,\mathbb{E}_p)$ is $\nu'_h(K)$.

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$A_2(K)_h$ & rank $\eta$ & knots \\
\hline
(1, 1, 1) & 3 & 217 \\
(1, 1, 2) & 2 & 114 \\
(1, 1, 3) & 2 & 110 \\
(1, 1, 4) & 2 & 2 \\
(1, 2, 2) & 1 & 1 \\
(1, 2, 3) & 1 & 1 \\
\hline
\end{tabular}
\end{center}

Table 1.

\footnote{Their $t$ is our $u$.}
Lemma 4.1. Let $R$ be a field of characteristic $p$ and $f$ be an irreducible element of $\mathbb{K}[t^\pm 1]$, and let $n$ be a positive integer. If $f \mid t^n - a^n$ then $\nu_f(t^n - a^n) = p^{\nu_p(n)}$.

Proof. Let $e = \nu_p(n)$, so that $n = p^e m$ where $p \nmid m$. Since $t^n - a^m$ is coprime to its derivative, it is square-free. Since $t^n - a^n = (t^m - a^m)^{p^e}$, the result follows.

One case of this lemma we need is when $\mathbb{K} = \mathbb{F}_p$, $a = 1$, and $f = h$. We denote the order of $\theta$ in $\mathbb{E}_p^\times$ by $\nu_p$. Then $h \mid t^n - 1$ iff $q \mid n$.

Let $T_{m,n}$ be the $(m,n)$ torus link. If $K$ is the mirror image of a link $K$, $\Phi_o(K) = \Phi_o(K)$, where $\Phi_o$ is a cocycle for the dual quandle to $\mathbb{E}_p$. Since quadratic quandles are self-dual (because $h$ is a symmetric polynomial), we may as well assume that $m$ and $n$ are both positive. Let $c$ be the number of components of $T_{m,n}$, which is the highest common factor of $m$ and $n$. We first determine the Alexander module of $T_{m,n}$.

Lemma 4.2. If $c = 1$, then $\hat{A}(T_{m,n}) \cong \Lambda/(\langle \frac{t^m - 1}{t - 1} \rangle)$. If $c \geq 2$, $\hat{A}(T_{m,n})$ is isomorphic to the direct sum of $c - 2$ copies of $\Lambda/(\langle t^m/c - 1 \rangle)$ and a module with presentation matrix

\[
\begin{bmatrix}
    \frac{t^m/c}{t - 1} & \frac{t^m/c - 1}{t - 1} \\
    0 & (t - 1) \frac{t^m/c - 1}{t^m - 1}
\end{bmatrix}
\]

(Here the columns correspond to generators and the rows to relations.)

Proof. The exterior $X$ of $T_{m,n}$ is the union of two solid tori $V_1$ and $V_2$, whose cores have linking numbers $m$ and $n$ respectively with $T_{m,n}$, and whose intersection consists of $c$ annuli. Let $\hat{V}_1$ and $\hat{V}_2$ be their inverse images in the infinite cyclic cover $\hat{X}$. Then $H_1(\hat{V}_1) = H_1(\hat{V}_2) = 0$, so $\hat{A}(T_{m,n})$ is isomorphic to the kernel of $H_0(\hat{V}_1 \cap \hat{V}_2) \to H_0(\hat{V}_1) \oplus H_0(\hat{V}_2)$. Further, $H_0(\hat{V}_1) \cong \Lambda/(t^m - 1)$, $H_0(\hat{V}_2) \cong \Lambda/(t^n - 1)$.
and $H_0(\tilde{V}_1 \cap \tilde{V}_2) \cong (\Lambda / (t^{mn/c} - 1))^c$, and we may compute the above map as follows. Take points $x_1, \ldots, x_c$, one from each component of $V_1 \cap V_2$, and for $2 \leq i \leq c$ take paths $\sigma_i$ from $x_{i-1}$ to $x_i$ in $V_1$ and $\tau_i$ from $x_{i-1}$ to $x_i$ in $V_2$. This may be done in such a way that $\sigma_i \tau_i$ has linking number $i - 1$ with $T_{m,n}$. Let $\tilde{x}_1$ be in the inverse image of $x_1$, and for $2 \leq x \leq c$ let $\tilde{\sigma}_i$ be the lift of $\sigma_i$ starting at $\tilde{x}_i$, let $\tilde{x}_i$ be its terminal point, and let $\tilde{\tau}_i$ be the lift of $\tau_i$ starting at $\tilde{x}_i$. Note that the terminal point of $\tilde{\tau}_i$ is $t^{i-1}\tilde{x}_1$. We take $\tilde{x}_1, \ldots, \tilde{x}_c$ as generators for $H_0(\tilde{V}_1 \cap \tilde{V}_2)$, and $\tilde{x}_1$ as a generator of each of $H_0(\tilde{V}_1)$ and $H_0(\tilde{V}_2)$. Each $\tilde{x}_i$ is homologous to $\tilde{x}_1$ in $\tilde{V}_1$ (by $\tilde{\sigma}_i$), and to $t^{i-1}\tilde{x}_1$ in $\tilde{V}_2$ (by $\tilde{\tau}_i$). Thus we have a commutative diagram

$$
\Lambda^c \xrightarrow{\gamma} \Lambda \oplus \Lambda \\
\downarrow \alpha \quad \downarrow \beta \\
H_0(\tilde{V}_1 \cap \tilde{V}_2) \longrightarrow H_0(\tilde{V}_1) \oplus H_0(\tilde{V}_2)
$$

in which $\alpha$ and $\beta$ are determined by our choice of generators and $\gamma(f_1, \ldots, f_c) = (\sum_{i=1}^c f_i, \sum_{i=1}^c t^{i-1}f_i)$. The case $c = 1$ follows easily (and is anyway well-known). Suppose that $c \geq 2$. Then $(f_1, \ldots, f_c) \in \text{Ker}(\beta \gamma)$ iff there exist $g_1, g_2 \in \Lambda$ such that

$$
\sum_{i=1}^c f_i = (t^m - 1)g_1 \quad \text{and} \quad \sum_{i=1}^c t^{i-1}f_i = (t^n - 1)g_2.
$$

These equations imply that

$$
f_1 = t^{m-1}g_1 / (t - 1) - t^n g_2 / (t - 1) + \sum_{i=3}^c t^{i-1}g_1 / (t - 1)
$$

and

$$f_2 = -t^{m-1}g_1 / (t - 1) + t^n g_2 / (t - 1) - \sum_{i=3}^c t^{i-1}g_1 / (t - 1).$$

Hence there is a surjection $\delta: \Lambda^c \twoheadrightarrow \text{Ker}(\beta \gamma)$ defined by $\delta(g_1, g_2, f_3, \ldots, f_c) = (f_1, f_2, f_3, \ldots, f_c)$, with $f_1$ and $f_2$ given by the above formulas. Further, we have $(g_1, g_2, f_3, \ldots, f_c) \in \text{Ker}(\alpha \delta)$ iff $f_i \equiv 0 \pmod{t^{mn/c} - 1}$ for $3 \leq i \leq c$ and there exist $h_1, h_2 \in \Lambda$ such that

$$
t^{m-1}g_1 / (t - 1) - t^n g_2 / (t - 1) = h_1(t^{mn/c} - 1)
$$

and

$$-t^{m-1}g_1 / (t - 1) + t^n g_2 / (t - 1) = h_2(t^{mn/c} - 1).$$

But then

$$g_1 = (h_1 + h_2) t^{mn/c} / (t - 1) \quad \text{and} \quad g_2 = (h_1 + t h_2) t^{mn/c} / (t - 1).$$

The rows of the matrix in the statement of the lemma are given by the cases $h_1 = 1$, $h_2 = 0$ and $h_1 = -1, h_2 = 1$, so the result follows.

Note that $\hat{A}_p(T_{m,n})$ is torsion, so the dimension of $\text{Rep}(T_{m,n}, E_p)$ is $\nu_h(T_{m,n}) + 1$, and rank $\eta \leq \nu_h(T_{m,n})$.

**Lemma 4.3.** If $c = 1$, then $\hat{A}_p(T_{m,n}) \cong \Lambda_p / ((t^m - 1)(t^n - 1))$. If $c \geq 2$, $\hat{A}_p(T_{m,n})$ has the same $h$-torsion as $\Lambda_p / ((t^m - 1) \oplus \Lambda_p / ((t^n - 1)) \oplus (\Lambda_p / (t^{mn/c} - 1))^{c-2}$. 

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Proof. The case $c = 1$ is obvious. If $c \geq 2$, we may assume (by symmetry in $m$ and $n$) that $p \nmid \frac{m}{c}$. Then $t - 1$ and $\frac{m \cdot n}{ce}$ are coprime in $\Lambda_p$ (by Lemma 4.3), and it follows that $A_p(T_{m,n})$ is isomorphic to the direct sum of $\Lambda_p/(\frac{m \cdot n}{ce} - 1)$, $\Lambda_p/(t - 1)\frac{m \cdot n}{ce} - 1)$, and $c - 2$ copies of $\Lambda_p/(\frac{m \cdot n}{ce} - 1)$. Since $\Lambda_p/(t - 1)\frac{m \cdot n}{ce} - 1)$ has the same $h$-torsion as $\Lambda_p/(\frac{m \cdot n}{ce} - 1)$, we are done. \hfill \Box

**Lemma 4.4.**

(0) If $q \nmid \frac{mn}{c}$ then $\nu_h(T_{m,n}) = \nu'_h(T_{m,n}) = 0$.

(1) If $q \mid \frac{mn}{c}$, $p \mid c$ and $p \nmid \frac{m}{c}$ then

$$\nu_h(T_{m,n}) = \begin{cases} c, & \text{if } q \nmid n \text{ and either } q \nmid m \text{ or } p \nmid \frac{m}{c}; \\ c - 2, & \text{if } q \mid m, q \mid n \text{ and } p \nmid \frac{m}{c}; \\ c - 1, & \text{otherwise} \end{cases}$$

and $\nu'_h(T_{m,n}) = 0$.

(2) If $q \mid \frac{mn}{c}$, $p \mid m$ and $p \nmid n$ then

$$\nu_h(T_{m,n}) = \begin{cases} c, & \text{if } q \mid m; \\ c - 1, & \text{if } q \mid m \end{cases}$$

and $\nu'_h(T_{m,n}) = \begin{cases} 1, & \text{if } p = 2, 3 \mid n \text{ and } m \equiv 2 \pmod{4}; \\ 0, & \text{otherwise}. \end{cases}$

(3) If $q \mid \frac{mn}{c}$ and $p \nmid mn$ then

$$\nu_h(T_{m,n}) = \nu'_h(T_{m,n}) = \begin{cases} c, & \text{if } q \nmid m \text{ and } q \nmid n; \\ c - 2, & \text{if } q \mid m \text{ and } q \nmid n; \\ c - 1, & \text{otherwise}. \end{cases}$$

Note that by symmetry in $m$ and $n$ (and the fact that $p$ cannot divide both $\frac{m}{c}$ and $\frac{n}{c}$), we may always assume that we are in one of the four cases of this lemma. The condition $3 \mid n$ appears in case 2 because $q = 3$ when $p = 2$.

Proof. By the previous lemma, $A_p(T_{m,n})_h \cong \bigoplus_{i=1}^c \Lambda/(h^{e_i})$, where if $c = 1$ then $e_1 = v_h(t^m - 1) - v_h(t^{mn/c} - 1)$, $e_2 = v_h(t^{mn/c} - 1) - v_h(t^n - 1)$, and $e_i = v_h(t^{mn/c} - 1)$ for $i \geq 3$. The number of non-zero $e_i$ is $\nu_h(T_{m,n})$, and the number equal to 1 is $\nu'_h(T_{m,n})$. The rest of the proof is a simple, if tedious, case-by-case check using Lemma 4.3. \hfill \Box

If $a$ and $b$ are coprime integers, let $\text{ord}_a(b)$ denote the order of $b$ in $\mathbb{Z}/(a)^\times$.

**Theorem 4.5.** For a torus link, rank $\eta = \nu'_h(T_{m,n})$, except perhaps if $q \mid \frac{mn}{c}$, $p \mid mn$ and $4 \mid \text{ord}_c(p)$.

For a torus knot, the theorem applies for any $p$. More generally, this is true if every odd prime factor of $c$ is congruent to 3 (mod 4) and 16 $\nmid c$, since then $\mathbb{Z}/(c)^\times$ has no elements of order 4. For $p = 2$, the smallest link not covered by the theorem is $T_{5,15}$. In [CJKS, Remark 4.1], the invariant of this link is given as $544 + 480a = 4(1 + 3a)^4$. Thus rank $\eta = 4$, which is in fact equal to $\nu'_h(T_{5,15})$.

**Corollary 4.6.** For a torus link, $\text{rad} \eta = \text{Rep}'(T_{m,n}, \mathbb{E}_p)$, except perhaps if $q \mid \frac{mn}{c}$, $p \mid mn$ and $4 \mid \text{ord}_c(p)$.
Proof. This follows from Theorem 4.5 and Lemma 3.2 except when $p = 2$, $3 \mid n$, $m \equiv 2 \pmod{4}$, and $2 \nmid n$; this case will be dealt with in Lemma 4.10 below. \hfill \Box

In case 0 of Lemma 4.4, the proof of the theorem is trivial: $\nu_h(T_{m,n}) = 0$, so $\operatorname{rank} \eta = \nu_h'(T_{m,n}) = 0$. We assume from now on that $q \mid \frac{mn}{c}$, and use the method of §5 of CJKS. Let $\delta$ be the (oriented) $m$-string braid $\sigma_{m-1}\sigma_{m-2}\cdots\sigma_1$. (The 3-string braids shown in Figure 1 should make our conventions sufficiently clear.)

Let $\vec{x} = (x_1, \ldots, x_m) \in \mathbb{E}_p^m$, and represent endomorphisms of $\mathbb{E}_p^m$ by $m \times m$ matrices acting on the left. If the top arcs of $\delta$ are assigned the colors $x_1, \ldots, x_m$ (from left to right), the colors on the bottom arcs are determined by the vector $M_m(\vec{x})$, where

$$M_m = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
\theta & 0 & 0 & \cdots & 0 & 1 - \theta \\
0 & \theta & 0 & \cdots & 0 & 1 - \theta \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \theta & 0 & 1 - \theta \\
0 & 0 & \cdots & 0 & \theta & 1 - \theta
\end{bmatrix}.$$ (This is the transpose of the matrix in CJKS because we use a left action.) Since $T_{m,n}$ is the closure of $\delta^n$, $\vec{x}$ determines a coloring of $T_{m,n}$ iff $\vec{x}$ is fixed by $M_m^n$, so we may identify $\operatorname{Rep}(T_{m,n}, \mathbb{E}_p)$ with the space $V_{m,n} = \{ \vec{x} \in \mathbb{E}_p^m \mid M_m^n(\vec{x}) = \vec{x} \}$.

With this identification, the form $\eta$ is given as follows. Let $\vec{x}$ and $\vec{y}$ be in $V_{m,n}$. Then

$$\eta(\vec{x}, \vec{y}) = \sum_{i=0}^{n-1} \sum_{j=1}^{m-1} i \left( M_m^i(\vec{x})_j M_m^i(\vec{y})_m - M_m^i(\vec{x})_m M_m^i(\vec{y})_j \right). \quad (4.1)$$

**Lemma 4.7.**

1. The characteristic polynomial $\det(IT_m - M_m) \in \mathbb{E}_p[T]$ of $M_m$ is equal to $(T - 1)^{m-1}T_m - \theta^m$.
2. If $\vec{x} \in \mathbb{E}_p^m$ is an eigenvector of $M_m$, then $x_m \neq 0$.
3. The eigenspaces of $M_m$ are one-dimensional.
4. If $\vec{x} \in \mathbb{E}_p^m$ is an eigenvector of $M_m$ of eigenvalue $\xi$, then $\vec{x} \in V_{m,n}$ iff $\xi^n = 1$. 

**Figure 1.**
Proof. Since each row of \( M_m \) sums to 1, each row of \( TI_m - M_m \) sums to \( T - 1 \), and it follows that \( \det(TI_m - M_m) = (T - 1)\Delta_m \), where
\[
\Delta_m = \det \begin{bmatrix} T & 0 & 0 & \cdots & 0 & 1 \\ -\theta & T & 0 & \cdots & 0 & 1 \\ 0 & -\theta & T & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\theta & T & 1 \\ 0 & 0 & \cdots & 0 & -\theta & 1 \end{bmatrix},
\]
Expanding along the first row gives \( \Delta_m = T\Delta_{m-1} + \theta^{m-1} \), and part (1) follows by induction. Suppose that \( \vec{x} \in \mathbb{E}_p^m \) and \( M_m(\vec{x}) = \xi\vec{x} \). Then \( \theta x_{i-1} + (1 - \theta)x_m = \xi x_i \) for \( 2 \leq i \leq m \), and it follows that \( x_m = 0 \) implies \( \vec{x} = 0 \). This proves parts (2) and (3). Also \( M_m(\vec{x}) = \xi\vec{x}^m \), which gives (4).

We make \( \mathbb{E}_p^m \) into a module over \( \mathbb{E}_p[T, T^{-1}] \) by setting \( T\vec{x} = M_m(\vec{x}) \); a generator for the order ideal is \( \det(TI_m - M_m) \). Then \( V_{m,n} \) is a submodule, and is annihilated by the greatest common divisor of \( \det(TI_m - M_m) \) and \( T^n - 1 \). Define \( \sigma, \pi : V_{m,n} \to \mathbb{E}_p \) by \( \sigma(\vec{x}) = \sum_{i=1}^m x_i \) and \( \pi(\vec{x}) = x_m \). Note that in the inner sum of (4.1), we may let \( j \) range from 1 to \( m \) (as the extra terms cancel), so the equation may be rewritten as
\[
\eta(\vec{x}, \vec{y}) = i \sum_{i=0}^{n-1} \left( \sigma(T^i\vec{x})\pi(T^i\vec{y}) - \pi(T^i\vec{x})\sigma(T^i\vec{y}) \right). \tag{4.2}
\]
Note also that
\[
\sigma(T\vec{x}) = \theta\sigma(\vec{x}) + m(1 - \theta)x_m. \tag{4.3}
\]
For \( f \in \mathbb{E}_p[T, T^{-1}] \), set \( f^* = \hat{f}(T^{-1}) \). Note that \( T \) preserves the form \( \eta \), so that \( \eta(f\vec{x}, \vec{y}) = \eta(\vec{x}, f^*\vec{y}) \) for \( f \in \mathbb{E}_p[T, T^{-1}] \) and \( \vec{x}, \vec{y} \in V_{m,n} \).

Lemma 4.8. Suppose that \( \vec{x}, \vec{y} \in V_{m,n} \) have orders \( f, g \in \mathbb{E}_p[T, T^{-1}] \), respectively, and that \( f^* \) is coprime to \( g \). Then \( \eta(\vec{x}, \vec{y}) = 0 \).

Proof. This is a standard argument. Let \( f_1, g_1 \in \mathbb{E}_p[T, T^{-1}] \) be such that \( f_1f^* + g_1g = 1 \). Then
\[
\eta(\vec{x}, \vec{y}) = \eta(\vec{x}, (f_1f^* + g_1g)\vec{y}) = \eta(f\vec{x}, f_1\vec{y}) + \eta(\vec{x}, g_1g\vec{y}) = 0.
\]

We next prove Theorem 4.5 in cases 1 and 2 of Lemma 4.4.

Lemma 4.9. If \( q \mid \frac{ap}{m} \) and \( p \mid m \), then \( \text{rank} \eta = \nu_q(T_{m,n}) \).

Proof. Let \( \vec{x}, \vec{y} \in V_{m,n} \). Since \( p \mid m \), equation (4.3) simplifies to \( \sigma(T\vec{x}) = \theta\sigma(\vec{x}) \), so (4.2) gives
\[
\eta(\vec{x}, \vec{y}) = \sum_{i=0}^{n-1} \left( \theta^i\sigma(\vec{x})\pi(T^i\vec{y}) - \pi(T^i\vec{x})\theta^i\sigma(\vec{y}) \right) = \sum_{i=0}^{n-1} \left( \sigma(\vec{x})\pi(\theta^{i+1}T^i\vec{y}) - \pi(\theta^{-i-1}T^{-i}\vec{x})\sigma(\vec{y}) \right) = \sum_{i=0}^{n-1} \sigma(\vec{x})\left( \theta^{-n+1}T^n - \theta^n \right) \pi\left( \theta^{-n+1}T^n - \theta^n \frac{T^n - \theta^n}{T - \theta} \frac{T^n - \theta^n}{T - \theta} \sigma(\vec{y}) \right).
\]

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Suppose that \( q \nmid n \), so that \( \theta^n \neq 1 \). Then \( \sigma(\vec{x}) = \sigma(T^n\vec{x}) = \theta^n\sigma(\vec{x}) \), and hence \( \sigma(\vec{x}) = 0 \) for all \( \vec{x} \in V_{m,n} \). In this case, \( \text{rank} \eta = 0 \), as required.

Suppose then that \( q \mid n \). Consider first case 1 of Lemma 4.4, where \( p \mid c \) and \( p \nmid \frac{m}{2} \). Then \( \nu_p(m) \leq \nu_p(n) \). By Lemma 4.1, \( \nu_{T-\theta}(\det(TM_m - M_m)) = p^{\nu_p(m)} - 1 \), while \( \nu_{T-\theta}(T^n - 1) = p^{\nu_p(n)} \). It follows that \( V_{m,n} \) is annihilated by \( T_{T-\theta}^{-\theta^n} \), and again we find \( \text{rank} \eta = 0 \).

Finally, suppose \( p \nmid n \). Let \( V' \) be the submodule of \( V_{m,n} \) consisting of elements whose order is a power of \( T - \theta \), and \( V'' \) the submodule of elements whose order is prime to \( T - \theta \). Then \( \text{rank} \eta = 0 \), as required. Because \( T - \theta \) is not a repeated factor of \( T^n - 1 \), every non-zero element of \( V' \) is an eigenvector, so \( V' \) is one-dimensional. If \( \vec{x} \in V'' \) has order \( f(T) \), then \( 0 = \sigma(f(T)\vec{x}) = f(\theta)\sigma(\vec{x}) \), and since \( f(\theta) \neq 0 \), \( \sigma(\vec{x}) = 0 \). It follows that \( \eta \) is identically zero on \( V'' \times V'' \), and hence that \( \eta \) is zero or \( 1 \) according as \( \eta(\vec{x},\vec{x}) \) is or is not 0 for \( \vec{x} \) an eigenvector of eigenvalue \( \theta \). For such \( \vec{x} \), we have

\[
\eta(\vec{x},\vec{x}) = m_i \left( \sigma(\vec{x})x_m - x_m\sigma(\vec{x}) \right).
\]

For \( 1 \leq i < m \), \( \theta x_{i+1} = (Tx)_{i+1} = \theta x_i + (1 - \theta)x_m \), so \( x_i - x_{i+1} = (1 - \theta)x_m \), and \( \sigma(\vec{x}) = \sum_{i=1}^{m-1} 1(x_i - x_{i+1}) = \frac{m(m-1)}{2} \). If \( p \neq 2 \), then \( \frac{m(m-1)}{2} \equiv 0 \) (mod \( p \)), so \( \sigma(\vec{x}) = 0 \) and \( \text{rank} \eta = 0 \). If \( p = 2 \), we have \( \theta = 1 \), and so \( \sigma(\vec{x}) = (m/2)\sigma(x_m) \) and \( \eta(\vec{x},\vec{x}) = 0 \). In this case, \( V_{m,n} \) is annihilated by \( \sigma(\vec{x}) = 0 \), and the proof is complete.

**Lemma 4.10.** Suppose that \( p = 2, 3 \mid n \), \( m \equiv 2 \) (mod \( 4 \)), and \( 2 \nmid n \). Then \( \text{rad} \eta = \text{Rep}'(T_{m,n}, \mathbb{F}_p) \).

**Proof.** We have \( \text{rank} \eta = \nu_q(T_{m,n}) = 1 \), and it is enough to show that \( \text{rad} \eta \leq \text{Rep}'(T_{m,n}) \). From the proof of Lemma 4.4, \( \text{rad} \eta \) is the kernel of \( \sigma \). From the proof of Lemma 3.1, it is enough to exhibit a single element \( a \) of \( A_2(T_{m,n}) \) such that \( ha = 0 \), \( a \notin hA_2(T_{m,n}) \), and \( \vec{x}(a) = 0 \) for all \( \vec{x} \in \text{rad} \eta \). In the diagram of \( T_{m,n} \), corresponding to the braid \( \delta^n \), let \( a_{i,1}, \ldots, a_{i,m} \) be the arcs at the top of the \( i^{th} \) copy of \( \delta \), for \( 1 \leq i \leq n \). Then \( A_2(T_{m,n}) \) is generated by \( \{ a_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m \} \), subject to the relations

\[
\begin{align*}
a_{i,1} &= a_{i-1,m} & \text{for } 2 \leq i \leq n, \\
a_{i,j} &= ta_{i-1,j-1} + (1 - t)a_{i-1,m} & \text{for } 2 \leq i \leq n \text{ and } 2 \leq j \leq m, \\
a_{1,1} &= a_{n,m}, \\
\text{and } a_{1,j} &= ta_{n,j-1} + (1 - t)a_{n,m} & \text{for } 2 \leq j \leq m.
\end{align*}
\]

We may eliminate all the generators except \( a_{1,1}, \ldots, a_{1,m} \), and the relations between these may be expressed as follows. Let \( N_m \) be the \( m \times m \) matrix over \( K_2 \) that results from replacing \( \theta \) by \( t \) in \( M_m \), and \( N_m^T \) its transpose. Then the relations are \( a_{1,i} = \sum_{j=1}^{m} (N_m)^{i,j}a_{1,j} \) for \( 1 \leq i \leq m \). For \( \vec{f} = (f_1, \ldots, f_m) \in A_2^m \), let \( \vec{f} = \sum_{i=1}^{m} f_i a_{1,i} \). Then \( (N_m f_1)^n = \vec{f}^n \). Let \( f_0 = (1, 1, \ldots, 1) \in A_2^n \). Then (because \( m \) is even) \( N_m f_0 = t f_0^m \), so \( (t^n - 1)f_0 = 0 \). Since \( 3 \mid n \) and \( 2 \nmid n \), we have \( \nu_h(t^n - 1) = 1, \) so \( t^n - 1 = gh \) for some \( g \in K_2 \) which is coprime to \( h \), and
hence has $g(\theta) \neq 0$. Let $a = [g_0 f] \in A_2(T_{m,n})$. Then $ha = 0$ and, for $x \in V_{m,n}$, $\bar{x}(a) = g(\theta)(x_1 + \cdots + x_m)$. Hence $\bar{x}(a) = 0$ iff $x \in \text{rad} \eta$. Because there exists an $x \in V_{m,n}$ with $\bar{x}(a) \neq 0$, $a \notin hA_2(T_{m,n})$, and we are done.

We let $\bar{E}_p$ be the splitting field of $T^{c_1} - 1$ over $E_p$.

**Lemma 4.11.** Suppose that $p \nmid c$ and $4 \nmid \text{ord}_c(p)$. Then the involution $x \mapsto \bar{x}$ of $E_p$ extends uniquely to an involution of $\bar{E}_p$.

**Proof.** Because $[\bar{E}_p : E_p]$ is even, there is a unique non-trivial involution $\alpha$ of $\bar{E}_p$. Let $K$ be the finite field of order $p^d$. Since $p \nmid c$, $T^{c_1} - 1$ splits over $K$ if and only if $K$ contains $c$ distinct $c^{th}$ roots of unity. Since $K^x$ is cyclic of order $p^d - 1$, this happens if and only if $c \mid p^d - 1$; i.e., if $\text{ord}_c(p) \mid d$. Hence the splitting field $\bar{F}_p$ of $T^{c_1} - 1$ over $E_p$ has $[\bar{F}_p : E_p] = \text{ord}_c(p)$. If $\text{ord}_c(p)$ is odd, then $[\bar{E}_p : E_p] = \text{ord}_c(p)$, while if $\text{ord}_c(p)$ is even then $\bar{E}_p = \bar{E}_p$ and $[\bar{E}_p : E_p] = \text{ord}_c(p)/2$. Since $4 \nmid \text{ord}_c(p)$, $[\bar{E}_p : E_p]$ is odd in either case. Hence the fixed field of $\alpha$ does not contain $E_p$, so the restriction of $\alpha$ to $E_p$ is non-trivial. Since $x \mapsto \bar{x}$ is the only non-trivial automorphism of $E_p$, the result follows.

From now on we assume that $q \nmid mn$, $p \nmid mn$ and $4 \nmid \text{ord}_c(p)$; in particular, Lemma 4.11 applies. That lemma having been established, no confusion will result if we write the involution of $\bar{E}_p$ as $x \mapsto \bar{x}$. We consider the action of the matrix $M_m$ on $\bar{E}_p^m$, and set

$$\hat{V}_{m,n} = \{ \bar{x} \in \bar{E}_p^m \mid M_m^n(\bar{x}) = \bar{x} \}.$$ 

Then $\hat{V}_{m,n}$ is the $\bar{E}_p$-subspace of $\bar{E}_p^m$ generated by $V_{m,n}$, and in particular has dimension $\nu_h(T_{m,n}) + 1$. The action of $M_m$ makes $\bar{E}_p^m$ into a module over $\bar{E}_p[T, T^{-1}]$ with $V_{m,n}$ as a submodule. Extend $\sigma$ and $\pi$ to $\sigma, \pi : \hat{V}_{m,n} \to \bar{E}_p$ using the same formulas as before, and extend $\eta$ to a Hermitian form $\eta$ on $\hat{V}_{m,n}$ given by the right-hand side of (4.2). Then rank $\eta = \text{rank} \eta$, and Lemmas 4.7 and 4.8 hold with $E_p$ replaced by $\bar{E}_p$. For $x \in \bar{E}_p$, set $x^* = \bar{x}^{-1}$; note that $x = x^*$ iff $N(x) = 1$.

**Lemma 4.12.** Suppose that $q \nmid mn$, $p \nmid mn$ and $4 \nmid \text{ord}_c(p)$. Then the number of distinct eigenvalues of $T|\hat{V}_{m,n}$ in $\bar{E}_p$ is $\nu_h(T_{m,n}) + 1$, and if $\xi$ is an eigenvalue, so is $\xi^*$. Further, $\theta$ is not an eigenvalue.

**Proof.** Let $E$ be the set of eigenvalues; by Lemma 4.7, this is the set of common roots of the polynomials $(T - 1)\frac{T^m - \theta^m}{T^m - \bar{\theta}^m}$ and $T^n - 1$ in $\bar{E}_p$. Let $E'$ be the set of common roots of $T^m - \theta^m$ and $T^n - 1$. The symmetric difference of $E$ and $E'$ contains at most $1$ and $\theta$. Since $p \nmid m$, $\theta$ is a root of $T^m - \theta^m$ but not of $\frac{T^m - \theta^m}{T^m - \bar{\theta}^m}$; hence $\theta \notin E$, while $\theta \in E'$ iff $q \mid n$. On the other hand, $1 \in E$, while $1 \in E'$ iff $q \mid m$. Hence

$$|E| = \begin{cases} |E'| + 1, & \text{if } q \nmid m \text{ and } q \nmid n; \\ |E'| - 1, & \text{if } q \mid m \text{ and } q \mid n; \\ |E'|, & \text{otherwise.} \end{cases}$$

Let $c_1$ be the highest common factor of $c$ and $q$. Since $q \mid mn$, we have $q = c_1m_1n_1$ where $c_1m_1 \mid m$ and $c_1n_1 \mid n$, and then $m_1$ and $n_1$ are coprime. Let $m = c_1m_2n_2$ and $n = c_1n_1n_2$. By the Chinese Remainder Theorem, there is an integer $i$ with $i \equiv 0 \pmod{m_1}$ and $i \equiv 1 \pmod{n_1}$. Set $\xi = \theta^i$. Since $ic_1n_1 \equiv 0 \pmod{q}$, $\xi^n = \theta^{ic_1n_1n_2}$.
\((\theta^{ic_1m_1})^{n_2} = 1\), and since \(ic_1m_1 \equiv c_1m_1 \pmod{q}\), \(\xi^m = (\theta^{ic_1m_1})^{n_2} = (\theta^{c_1m_1})^{m_2} = \theta^m\). Thus \(\xi \in E'\). Since \(E'\) is non-empty, \(|E'|\) is equal to the number of \(\mathbb{E}_p^n\) elements of unity in \(\mathbb{E}_p\), which is \(c\). By case 3 of Lemma 4.4, we do have \(|E| = \nu_h(T_{m,n}) + 1\).

It was observed above that \(\theta \not\in E\). It is clear that \(\xi \in E'\) iff \(\xi^\bullet \in E'\); since \(1^\bullet = 1\) and \(\theta^\bullet = \theta\), the same is true of \(E\), and we are done.

The next lemma gives all remaining cases of Theorem 4.5.

Lemma 4.13. Suppose that \(q \mid \frac{m_n}{p}\), \(p \nmid mn\) and \(4 \nmid \text{ord}_c(p)\). Then \(\text{rank} \eta = \nu'_h(T_{m,n})\).

Proof. By Lemma 4.12, \(\hat{V}_{m,n}\) is the direct sum of the eigenspaces of \(T\), which are all one-dimensional, and the eigenvalues with norm different from 1 come in pairs \(\xi\) and \(\xi^\bullet\). An eigenvector of eigenvalue \(\xi\) has order \((T - \xi)^\ast = T^{-1} - \xi = -T^{-1}\xi(T - \xi^\bullet)\), so Lemma 4.8 shows that:

1. an eigenspace corresponding to an eigenvalue of norm 1 is orthogonal to the other eigenspaces;
2. for a pair of eigenvalues \(\xi\) and \(\xi^\bullet\) of norm different from 1, the sum of the corresponding eigenspaces is orthogonal to all other eigenspaces, and \(\hat{\eta}\) is zero on each of these spaces.

Hence, \(\text{rank} \hat{\eta}\) is the number of eigenvalues \(\xi\) such that \(\hat{\eta}(\bar{x}, \bar{y}) \neq 0\) for eigenvectors \(\bar{x}\) and \(\bar{y}\) with eigenvalues \(\xi\) and \(\xi^\bullet\). By case 3 of Lemma 4.4, \(\nu'_h(T_{m,n}) = \dim \hat{V}_{m,n} - 1\), so we must show that there is just one eigenvalue for which this condition does not hold.

Let \(\bar{x}\) be an eigenvector of eigenvalue \(\xi\), and \(\bar{y}\) an eigenvector of eigenvalue \(\xi^\bullet\). From equation (4.2),

\[
\hat{\eta}(\bar{x}, \bar{y}) = ni(\sigma(\bar{x})\bar{y}_m - x_m\bar{\sigma}(\bar{y})).
\]

If \(\xi = 1\), \(\bar{x}\) and \(\bar{y}\) are multiples of \((1, 1, \ldots, 1)\), so \(\hat{\eta}(\bar{x}, \bar{y}) = 0\). Suppose \(\xi \neq 1\). Equation (4.3) gives

\[
\xi\sigma(\bar{x}) = \theta\sigma(\bar{x}) + m(1 - \theta)x_m
\]

and

\[
\xi^\bullet\sigma(\bar{y}) = \theta\sigma(\bar{y}) + m(1 - \theta)y_m.
\]

Hence

\[
(\xi - \theta)(\xi^\bullet - \theta)\hat{\eta}(\bar{x}, \bar{y}) = ni((\xi^\bullet - \theta)(\xi - \theta)\sigma(\bar{x})\bar{y}_m - (\xi - \theta)x_m(\xi^\bullet - \theta)\sigma(\bar{y}))
\]

\[
= ni((\xi^\bullet - \theta)m(1 - \theta)x_m\bar{y}_m - (\xi - \theta)x_m(m(1 - \theta)y_m)
\]

\[
= mnix_m\bar{y}_m(\xi^\bullet(\theta - \xi)(1 - \theta) - (\xi - \theta)(1 - \bar{\theta}))
\]

\[
= mnix_m\bar{y}_m(\xi - \theta)(1 - \bar{\theta})(\xi^\bullet - 1),
\]

which is non-zero by part (2) of Lemma 4.7. This completes the proof.

5. Two-Bridge Knots

Let \(M_+\) and \(M_-\) be the \(2 \times 2\) matrices over \(\Lambda\) given by

\[
M_{\pm} = \begin{bmatrix} t_{\pm 1} & 1 - t_{\pm 1} \\ t_{\pm 1} - 1 & 2 - t_{\pm 1} \end{bmatrix}.
\]
Lemma 5.1. For \( n \in \mathbb{Z} \), we have
\[
M_n^\pm = \begin{bmatrix}
 n(t^\pm 1 - 1) + 1 & n(1 - t^\pm 1) \\
n(t^\mp 1 - 1) & n(1 - t^\pm 1) + 1
\end{bmatrix}.
\]

Proof. The desired equation is equivalent to \( M_n^\pm = nM_\pm - (n - 1)I \). Since \( M_\pm \) has determinant 1 and trace 2, \( M_\pm^2 - 2M_\pm + I = 0 \). Using this in the form \( M_\pm^2 = 2M_\pm - I \), the cases \( n \geq 1 \) follow by an upward induction, and a downward induction using \( M_{n-1}^- = 2I - M^- \) gives the other cases.

Let \( K(P, Q) \) be the two-bridge knot whose 2-fold branched cyclic cover is the lens space \( L(P, Q) \). Here \( P \) is odd, \( Q \) is coprime to \( P \), and we may assume that \( 0 \leq Q < P \). Since the mirror image of \( K(P, Q) \) is \( K(P, P - Q) \), we may further assume that \( Q \) is even. Then \( \frac{P}{Q} \) has a continued fraction expansion of the form
\[
2m_1 - \frac{1}{2n_1 - \frac{1}{2m_2 - \cdots - \frac{1}{2n_k}}}, \quad (5.1)
\]
Hence \( K(P, Q) \) has the diagram shown in Figure 2, in which each box represents the indicated number of full right-hand twists.

If \( D \) is any oriented tangle diagram, we may define \( A(D) \) and \( A_p(D) \) to be the modules over \( \Lambda \) and \( \Lambda_p \), respectively, with generators the arcs of \( D \) and a relation for each crossing just as for link diagrams. Then, setting \( \text{Rep}(D, \mathbb{E}_p) = \text{Hom}_{\Lambda_p}(A_p(D), \mathbb{E}_p) \), we may define \( B(f) \in \mathbb{F}_p \) for \( f \in \text{Rep}(D, \mathbb{E}_p) \) as before.

Lemma 5.2. Let \( D \) be the tangle diagram of Figure 3(a). In \( A(D) \), we have the relations
\[
c = n(t - 1)(a - b) + a
\]
and
\[
d = n(t - 1)(a - b) + b.
\]
For \( f \in \text{Rep}(D, \mathbb{E}_p) \), \( B(f) = n\bar{\theta} - \theta N(f(a - b)) \).
Figure 4.

**Proof.** By Lemma 5.1, the claimed relations in $A(D)$ are equivalent to the matrix equation

$$
\begin{bmatrix}
c \\
d
\end{bmatrix} = M^n_+ \begin{bmatrix} a \\ b \end{bmatrix}.
$$

It is enough to check the cases $n = \pm 1$, and this is left to the reader. For the second part, let the images of $a$, $b$, $c$ and $d$ under $f$ be $\alpha$, $\beta$, $\gamma$ and $\delta$. When $n = 1$, both crossings are negative, and we have $B(f) = -\phi(\alpha, \beta) - \phi(\delta, \gamma) = \phi(\gamma, \delta) - \phi(\alpha, \beta)$. When $n = -1$, the crossings are positive, and we have $B(f) = \phi(\beta, \alpha) + \phi(\gamma, \delta) = \phi(\gamma, \delta) - \phi(\alpha, \beta)$. Hence $B(f) = \phi(\gamma, \delta) - \phi(\alpha, \beta)$ for any $n$. From the first part,

$$
\gamma \bar{\delta} - \bar{\gamma} \delta = n(\theta - 1)(\alpha - \beta)(\bar{\beta} - \bar{\alpha}) + n(\bar{\theta} - 1)(\bar{\alpha} - \bar{\beta})(\alpha - \beta) + \alpha \bar{\beta} - \bar{\alpha} \beta
$$

$$
= n(\bar{\theta} - \theta)N(\alpha - \beta) + \alpha \bar{\beta} - \bar{\alpha} \beta.
$$

Hence

$$
\phi(\gamma, \delta) - \phi(\alpha, \beta) = i(\gamma \bar{\delta} - \bar{\gamma} \delta - \alpha \bar{\beta} + \bar{\alpha} \beta)
$$

$$
= ni(\bar{\theta} - \theta)N(\alpha - \beta),
$$

as required. \qed

**Lemma 5.3.** Let $D$ be the tangle diagram of Figure 3(b). In $A(D)$, we have the relations

$$
c = n(t^{-1} - 1)(a - b) + a
$$

and

$$
d = n(t^{-1} - 1)(a - b) + b.
$$

For $f \in \text{Rep}(D, E_p)$, $B(f) = 0$.

**Proof.** The first part is proved as in Lemma 5.2, using $M_-$ in place of $M_+$. In the Boltzmann weight, the contributions from the crossings cancel in pairs. \qed

We set $\nabla = -(t-1)(t^{-1} - 1) = t - 2 + t^{-1} \in \Lambda$.

**Lemma 5.4.** Let $D$ be the tangle diagram of Figure 4. In $A(D)$, we have the relations

$$
\begin{bmatrix}
a' - b' \\
b' - c'
\end{bmatrix} = \begin{bmatrix} 1 + mn \nabla & -n(t^{-1} - 1) \\ m(t-1) & 1 \end{bmatrix} \begin{bmatrix} a - b \\ b - c \end{bmatrix}.
$$

**Proof.** From Lemmas 5.2 and 5.3 we have

$$
a' = m(t - 1)(a - b) + a,$$

$$
d = m(t - 1)(a - b) + b,$$

$$
b' = n(t^{-1} - 1)(d - c) + d,$$

and

$$
c' = n(t^{-1} - 1)(d - c) + c.$$
a\Lambda\text{-module on similar expressions for } a\Lambda. Put differently, the initial presentation (given by all arcs and crossings) of \( \text{(5.2)} \) is understood that the coefficients of the polynomials are taken modulo \( \alpha_i \).

Let \( m = m_i n_i x \) be the tangle diagram of Figure 5. It is clear that \( A(D) \) has a presentation with generators \( a \) and \( b \), and one relation \( \alpha_\ell \in Z[\alpha] \).

**Lemma 5.5.** Let \( D \) be the tangle diagram of Figure 3. In \( A(D) \), we have the relations

\[
\begin{bmatrix}
    a_i - b_i \\
    b_i - c_i
\end{bmatrix} =
\begin{bmatrix}
    \alpha_i(\nabla) & -(t^{-1} - 1)\beta_i(\nabla) \\
    (t - 1)\gamma_i(\nabla) & \delta_i(\nabla)
\end{bmatrix}
\begin{bmatrix}
    a_0 - b_0 \\
    b_0 - c_0
\end{bmatrix}
\]

**Proof.** Using Lemma 5.4 and the fact that the matrix

\[
\begin{bmatrix}
    \alpha_i(\nabla) & -(t^{-1} - 1)\beta_i(\nabla) \\
    (t - 1)\gamma_i(\nabla) & \delta_i(\nabla)
\end{bmatrix}
\]

is equal to

\[
\begin{bmatrix}
    1 + m_i n_i \nabla & -n_i(t^{-1} - 1) \\
    m_i(t - 1) & 1
\end{bmatrix}
\begin{bmatrix}
    \alpha_{i-1}(\nabla) & -(t^{-1} - 1)\beta_{i-1}(\nabla) \\
    (t - 1)\gamma_{i-1}(\nabla) & \delta_{i-1}(\nabla)
\end{bmatrix},
\]

this follows by induction. \( \square \)

**Lemma 5.6.** Let \( K \) be the two-bridge knot \( K(P,Q) \), where \( \frac{P}{Q} \) has the continued fraction expansion \( \text{(5.1)} \). Then \( A(K) \) has a presentation with generators \( a \) and \( d \) and one relation \( \alpha_k(\nabla) d = 0 \). For \( f \in \text{Rep}(K,E) \),

\[
B(f) = i(\bar{\theta} - \theta)N(f(d)) \sum_{i=1}^{k} m_i \alpha_{i-1}(\nabla(\theta))^2.
\]

In the expression \( \alpha_{i-1}(\nabla(\theta)) \), and similar expressions occuring later, it is to be understood that the coefficients of the polynomials are taken modulo \( p \).

**Proof.** Let \( D \) be the tangle diagram of Figure 3. It is clear that \( A(D) \) is the free \( \Lambda \)-module on \( a_0, b_0 \) and \( c_0 \). Let \( c_k = f_1 a_0 + f_2 b_0 + f_3 c_0 \), where \( f_1, f_2, f_3 \in \Lambda \). Then similar expressions for \( a_k \) and \( b_k \) are determined by the case \( i = k \) of the relations \( \text{(5.2)} \). Put differently, the initial presentation (given by all arcs and crossings) of

\[
\begin{bmatrix}
    a_0 & b_0 & c_0 \end{bmatrix}
\begin{bmatrix}
    n_1 & c_1 \\
    b_1 & m_2 \\
    a_1 & \cdots & \cdots \end{bmatrix}
\begin{bmatrix}
    a_k \\
    b_k \\
    c_k
\end{bmatrix}
\]

**Figure 5.**
$A(D)$ is equivalent to a presentation with generators $a_0$, $b_0$, $c_0$, $a_k$, $b_k$ and $c_k$, and the relations
\begin{align*}
    a_k - b_k &= \alpha_k(\nabla)(a_0 - b_0) - (t^{-1} - 1)\beta_k(\nabla)(b_0 - c_0), \\
    b_k - c_k &= (t - 1)\gamma_k(\nabla)(a_0 - b_0) + \delta_k(\nabla)(b_0 - c_0), \\
    c_k &= f_1a_0 + f_2b_0 + f_3c_0.
\end{align*}

The diagram of $K$ in Figure 2 is obtained from $D$ by joining the free ends without introducing any more crossings. Consider instead the diagram where one curl is introduced in each connecting arc. This gives a presentation of $A(K)$ which is obtained from the initial presentation of $A(D)$ by adding a relation for each of the three extra crossings. However, in this presentation, any relation is a consequence of the others, so we need only add
\[ b_0 = c_0, \quad a_k = b_k. \] (5.4)

Hence $A(K)$ has a presentation with generators $a_0$, $b_0$, $c_0$, $a_k$, $b_k$ and $c_k$, and the relations (5.3) and (5.4). In the presence of relations (5.4), relations (5.3) are equivalent to
\begin{align*}
    0 &= \alpha_k(\nabla)(a_0 - b_0), \\
    a_k - c_k &= (t - 1)\gamma_k(\nabla)(a_0 - b_0), \\
    c_k &= f_1a_0 + (f_2 + f_3)b_0.
\end{align*}

We may now eliminate four of the generators to arrive at a presentation of $A(K)$ with generators $a_0$ and $b_0$ and the single relation $\alpha_k(\nabla)(a_0 - b_0) = 0$. Setting $a = a_0$ and $d = a_0 - b_0$ gives the claimed presentation.

By Lemmas (5.2) and (5.3), $B(f) = \sum_{i=1}^k m_i(\bar{\theta} - \theta)N(f(a_i - b_i))$. In $A(K)$, we have $a_i - b_i = \alpha_i(\nabla)d_i$, so $f(a_i - b_i) = \alpha_i(\nabla)\theta f(d_i)$ and (since $\nabla(\theta) \in \mathbb{F}_p$) $N(f(a_i - b_i)) = \alpha_i(\nabla(\theta))^2N(f(d_i))$, giving the required formula for $B(f)$. \hfill \Box

**Theorem 5.7.** Let $K$ be a two-bridge knot. Then $\text{rad} \, \eta = \text{Rep}^* (K, \mathbb{E}_p)$.

**Proof.** Recall that $h = t^2 + \kappa^2 + 1$ for some $\kappa \in \mathbb{F}_p$. Set $\lambda = \nabla(\theta) = -\kappa - 2$, and note that $\lambda \neq 0$ because $h$ is irreducible. Suppose that $K$ is determined by the continued fraction (5.2). Since $\nu_h(K) \leq 1$ and $\nu_0(K) = 1$, Lemma 3.2 shows we need only prove that $\text{rank} \, \eta = \nu_h^*(K)$. Since $\text{rank} \, \eta \leq \nu_h(K)$ and $\nu_h(K) \leq \nu_h(K)$, this is trivially true if $\nu_h(K) = 0$. Suppose for the rest of the proof that $\nu_h(K) = 1$; by Lemma 5.6, this is equivalent to $\alpha_k(\lambda) = 0$. It is enough to show that $\nu^h_0(K) = 0$ iff $\lambda$ is a repeated root of $\alpha_k$, which is true iff $\lambda$ is a root of the derivative $\partial_k$. Also, rank $\eta = 0$ iff $B(f) = 0$ for all $f \in \text{Rep}(K, \mathbb{E}_p)$, and by Lemma 5.6 again, this is true iff $\epsilon_k(\lambda) = 0$ where $\epsilon_k = \sum_{i=1}^k m_i \alpha_{i-1}^2 \in \mathbb{Z}[x]$. Thus we are reduced to proving that $\partial_k(\lambda) = 0$ iff $\epsilon_k(\lambda) = 0$.

If $p \mid n_k$, we may replace $n_k$ by 0 without changing $\alpha_k$ or $\epsilon_k$ modulo $p$, and then the new $K$ has a shorter continued fraction. Thus we may assume that $p \nmid n_k$.

We have $\alpha_k \beta_k - x\beta_k \gamma_k = 1$, as one may sec directly from the definitions, or more easily by using the matrix form of the recursion from the proof of Lemma 5.3. Hence $\gamma_k(\lambda) \neq 0$. From the definitions, we have $\alpha_k = \alpha_{k-1} + n_k x \gamma_k$, and hence $\alpha_{k-1}(\lambda) \neq 0$. We also have $\alpha_k = \alpha_{k-1} + n_k x \sum_{i=1}^k m_i \alpha_{i-1}$, and hence $\alpha_{k-1}(\lambda) \neq 0$. Using this recursive relation, we prove in an appendix that $\partial_k \alpha_{k-1} - \alpha_k \partial_{k-1} = n_k \epsilon_k$, and it follows that $\partial_k(\lambda) = 0$ iff $\epsilon_k(\lambda) = 0$, completing the proof. \hfill \Box
Appendix A. A Polynomial Identity

It will be convenient to replace the integers $m_1, n_1, \ldots$ of \S3 by indeterminates. Thus, let $R$ be the ring of integer polynomials in variables $x, y_1, z_1, y_2, z_2, \ldots$. Define polynomials $f_0, f_1, f_2, \ldots \in R$ inductively by $f_0 = 1$ and

$$f_k = f_{k-1} + xz_k \sum_{i=1}^{k} y_i f_{i-1}$$

for $k \geq 1$. Comparing this with the recursion for the polynomials $\alpha_i$ from the proof of Theorem 5.7, one sees that Lemma A.2 below is what is needed to complete that proof.

Let $M \subset R$ be the set of monomials of the form

$$\mu = y_{i_1} z_{j_1} y_{i_2} z_{j_2} \cdots y_{i_l} z_{j_l}, \quad 1 \leq i_1 \leq j_1 < i_2 \leq j_2 < \cdots < i_l \leq j_l.$$  

We define the length of $\mu$ to be $l(\mu) = l$, and its height to be 0 if $l = 0$ (when $\mu = 1$) and $j_l$ if $l > 0$. For $k \geq 1$, let $M_k$ be the set of elements of $M$ of height less than $k$.

Lemma A.1.

For $k \geq 0$, $f_k = \sum_{\mu \in M_{k+1}} x^{l(\mu)} \mu$; \hfill (A.1)

for $k \geq 1$, $f_k = \sum_{\mu \in M_k} x^{l(\mu)} \mu + \sum_{i=1}^{k} \sum_{\mu \in M_i} x^{l(\mu) + 1} \mu y_i z_k$. \hfill (A.2)

Proof. Clearly (A.1) holds for $k = 0$. For $k \geq 1$, the right-hand sides of (A.1) and (A.2) are equal because $M_{k+1}$ is the disjoint union of $M_k$ and the sets $\{ \mu y_i z_k \mid \mu \in M_i \}$ for $1 \leq i \leq k$. The result follows by induction on $k$. \hfill $\square$

Lemma A.2. Let $\hat{f}_k$ be the derivative of $f_k$ with respect to $x$. Then, for $k \geq 1$,

$$\hat{f}_k f_{k-1} - f_k \hat{f}_{k-1} = z_k \sum_{i=1}^{k} y_i f^2_{i-1}.$$  

Proof. Let $S = \{ (i, \mu, \nu) \mid 1 \leq i < k, \mu \in M_i, \nu \in M_k - M_i \}$. By Lemma A.1, we have

$$\hat{f}_k f_{k-1} = \left( \sum_{\mu \in M_k} l(\mu) x^{l(\mu) - 1} \mu \right) + \sum_{i=1}^{k} \sum_{\mu \in M_i} (l(\mu) + 1) x^{l(\mu)} \mu y_i z_k \left( \sum_{\nu \in M_k} x^{l(\nu)} \nu \right)$$

$$= \sum_{\mu, \nu \in M_k} l(\mu) x^{l(\mu) + l(\nu) - 1} \mu \nu + \sum_{i=1}^{k} \sum_{\mu \in M_i} (l(\mu) + 1) x^{l(\mu) + l(\nu)} \mu y_i z_k$$

$$= \sum_{\mu, \nu \in M_k} l(\mu) x^{l(\mu) + l(\nu) - 1} \mu \nu + \sum_{i=1}^{k} \sum_{\mu, \nu \in M_i} (l(\mu) + 1) x^{l(\mu) + l(\nu)} \mu y_i z_k$$

$$+ \sum_{(i, \mu, \nu) \in S} (l(\mu) + 1) x^{l(\mu) + l(\nu)} \mu y_i z_k.$$
Similarly,

\[
f_k \hat{f}_{k-1} = \left( \sum_{\mu' \in M_k} x^{l(\mu')} \mu' + \sum_{i=1}^{k} \sum_{\mu' \in M_i} x^{l(\mu') + 1} \mu' y_i z_k \right) \left( \sum_{\nu' \in M_k} l(\nu') x^{l(\nu') - 1} \nu' \right) \\
= \sum_{\mu, \nu \in M_k} \mu \nu x^{l(\mu) + l(\nu) - 1} + \sum_{i=1}^{k} \sum_{\mu, \nu \in M_i} \mu \nu x^{l(\mu) + l(\nu)} \mu' y_i z_k \\
+ \sum_{(i', \mu', \nu') \in S} \mu' y_i z_k.
\]

(In the first two sums of the final expression, we have set \( \mu = \nu' \) and \( \nu = \mu' \). Let \((i, \mu, \nu) \in S\). Then \( \nu \) can be written uniquely in the form \( \nu = \mu y_i z_j \) with \( 1 \leq i' \leq j < k \) and \( \mu' \in M_{i'} \). Since \( \nu \notin M_i \), we also have \( i \leq j \). Set \( \nu' = \mu y_i z_j \). Then \( \nu' \in M_{k - M_i} \), so \((i', \mu', \nu') \in S\). The function \((i, \mu, \nu) \mapsto (i', \mu', \nu')\) is an involution of \( S \). Since \( l(\mu') = l(\nu) - 1 \), \( l(\nu') = l(\mu) + 1 \), and \( \mu' y_i = \mu y_i \), it follows that

\[
\sum_{(i, \mu, \nu) \in S} (l(\mu) + 1) x^{l(\mu) + l(\nu)} \mu y_i z_k = \sum_{(i', \mu', \nu') \in S} l(\nu') x^{l(\mu') + l(\nu')} \mu' y_i z_k.
\]

Hence

\[
f_k \hat{f}_{k-1} - f_k \hat{f}_{k-1} = \sum_{i=1}^{k} \sum_{\mu, \nu \in M_i} x^{l(\mu') + l(\nu)} \mu y_i z_k \\
= z_k \sum_{i=1}^{k} \left( \sum_{\mu \in M_i} x^{l(\mu)} \mu \right)^2 y_i \\
= z_k \sum_{i=1}^{k} y_i f_{i-1}^2,
\]

as required. \( \square \)

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**Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803**

**E-mail address:** lither@math.lsu.edu