Genuine High-Order Einstein-Podolsky-Rosen Steering

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Einstein-Podolsky-Rosen (EPR) steering [1, 2] demonstrates that two parties share entanglement even if the measurement devices of one party are untrusted [3]. Here, going beyond this bipartite concept, we develop a novel formalism to explore a large class of EPR steering from generic multipartite quantum systems of arbitrarily high dimensionality and degrees of freedom, such as graph states [4, 5] and hyperentangled systems [6, 7]. All of these quantum characteristics of genuine high-order EPR steering can be efficiently certified with a few measurement settings in experiments. We faithfully demonstrate for the first time such generality by experimentally showing genuine four-particle EPR steering and applications to universal one-way quantum computing [8, 9]. Our formalism provides a new insight into the intermediate type of genuine multipartite Bell non-locality [10, 11] and potential applications to quantum information tasks and experiments in the presence of untrusted measurement devices [8, 12, 13].

Einstein-Podolsky-Rosen (EPR) steering was originally introduced by Schrödinger in 1935 [1] to describe the EPR paradox [2]. Recently, it has been formulated to show a strict hierarchy among Bell non-locality, steering and entanglement [14, 17] and stimulated new applications to quantum communication [13]. The steering effect reveals that different ensembles of quantum states can be remotely prepared by measuring one particle of an entangled pair. We go a step further and consider the following question: how to experimentally observe genuine multipartite EPR steering? For instance, given an experimental output state $\rho_{\text{exp}}$, which is created according a target four-qubit cluster state of the form [10]

$$
|G_4\rangle = \frac{1}{2}(|+1\rangle_1 |02\rangle_2 |+3\rangle_4 |04\rangle + |+1\rangle_1 |12\rangle_2 |+3\rangle_4 |14\rangle \\
+ |+1\rangle_1 |02\rangle_2 |-3\rangle_4 |14\rangle + |+1\rangle_1 |12\rangle_2 |-3\rangle_4 |04\rangle),
$$

where $|\pm\rangle_k = (|0\rangle_k \pm |1\rangle_k)/\sqrt{2}$, how do we describe the effect of genuine multipartite EPR steering and then detect such steerability in the laboratory?

Inspired by the task-oriented formulation of bipartite steering [3], genuine multipartite EPR steering can be defined from an operational interpretation as the distribution of multipartite entanglement by uncharacterized (or untrusted) parties. Let us consider a system composed of $N$ parties and a source creating $N$ particles. Each party of the system can receive a particle from the source whenever a $N$-particle state is created. We divide the system into two groups, say $A_s$ and $B_s$, and assume that $A_s$ is responsible for sending particles from such a source to every party. Each time, after receiving particles, they measure their respective parts and communicate classically. Since $B_s$ does not trust $A_s$, $A_s$’s task is to convince $B_s$ that the state shared between them is entangled. $A_s$ will succeed in this task iff $A_s$ can prepare different ensembles of quantum states for $B_s$ by steering $B_s$’s state like the bipartite steering effect. Here we say a $N$-particle state generated from the source to possess genuine $N$-particle EPR steerability if by which $A_s$ succeed in the task for all possible bipartitions $A_s$ and $B_s$ of the $N$-particle system. This interpretation is consistent with the definition recently introduced by He and Reid [20]. In a wider scope, Schrödinger’s original concept can even be applied to quantum systems with many degrees of freedom (DOFs), e.g., hyperentangled systems, and, as will be shown presently, extended as genuine multi-DOF EPR steering. In this Letter we call these two sorts of quantum characteristic genuine high-order EPR steering.

The concept of verifying genuine high-order EPR steering leads us naturally to consider quantum scenarios based on genuine high-order entanglement in which $B_s$’s measurement apparatus are trusted, while $A_s$’s are not; see Fig. 1. One can imagine that such scenarios are distributed quantum processing networked by shared multipartite entanglement, like sharing intelligence between organizations, or running joint scientific tasks between remote computers. Demonstrations of genuine high-order EPR steerability guarantee faithful implementations of the quantum scenarios in the presence of uncharacterized parties. The present quantum strategies and protocols, however, requires prior knowledge of the measurement apparatus [8, 12, 13]. The certification of genuine high-order EPR steering is then not only a fundamentally scientific endeavour in itself, but is also the enabling technology for quantum applications when untrusted measurement devices are encountered. So far,
Genuine multipartite EPR steerability and applications. Generic entanglement-based quantum protocols rely on both characterized measurement devices and genuine multipartite entanglement, such as one-way quantum computation and multi-party quantum communications. Genuine multipartite EPR steerability enables them to perform quantum applications in the presence of untrusted measurement apparatus. Here, we develop quantum steering witnesses to ensure that an experimental state $\rho_{\text{exp}}$ of a $N$-partite quantum $d$-dimensional system ($N$ qudits) has such ability. For example, an experimental graph state $3, 4, 21$ for one-way computing. To implement gate operations such as circuits composed of one-qubit gate $H$ and two-qubit controlled-Z gate ($b$) and ($c$) for input states $\{|n\rangle, \ldots\}$, one needs to prepare chain-type ($d$), i.e., $|G_4\rangle$, and box-type ($e$) cluster states, respectively. By performing measurements $B_2^{(a)}$ and $B_2^{(b)}$ on qudits 2 and 3, respectively, the rest qudits 1, 4 together with the outcomes of measurements on qudits 2, 3 would provide a readout of the gate operations. In addition to releasing the assumptions about the measurement devices, genuine multipartite steerability promises high-quality one-way computation.

while verifying genuine tripartite steering becomes possible $21$, the fundamental problem such as the verification considered above and the cases for arbitrary large $N$ remains open.

Here we develop new quantum witnesses to observe a large class of genuine high-order EPR steering independent of the particle or DOF number. We start with a demonstration of the first experimental genuine four-partite EPR steerability for states close to the cluster state $|G_4\rangle$. In the four-partite steering task, we assume that two possible measurements can be performed on each particle ($m_k = 1, 2$ for particle $k = 1, 2, 3, 4$) and that each local measurement has two possible outcomes, $v_k^{(m_k)} \in \{0, 1\}$. We take the first measurement for each party who implements quantum measurements to a observable which has the nondegenerate eigenvectors $\{|0\rangle_{k,1} = |0\rangle_k, |1\rangle_{k,1} = |1\rangle_k\}$, and the second measurement to a observable with the eigenvectors $\{|0\rangle_{k,2} = |+\rangle_k, |1\rangle_{k,2} = |−\rangle_k\}$. The statistics of the experimental outcome sets for the two groups, say $\{v_a\}$ for $A_s$ and $\{v_b\}$ for $B_s$, then can be collected from their respective measurements.

The genuine four-partite EPR steerability of an ideal cluster state $|G_4\rangle$ can be revealed by using the relation of measurement results: $v_1^{(1)} + v_2^{(2)} + v_3^{(1)} = 0$ and $v_3^{(1)} + v_4^{(2)} = 0$, and the relation: $v_2^{(1)} + v_3^{(2)} + v_4^{(1)} = 0$ and $v_1^{(2)} + v_2^{(1)} = 0$, where $\equiv$ denotes equality modulo 2. When $A_s$ and $B_s$ share a state $|G_4\rangle$, $A_s$ can steer the states of $B_s$’s particles by measuring the particles held as the above connections described wherever the partition $A_s$ and $B_s$ is considered. Thus we construct the kernel of quantum steering witness in the form

$$W_4 \equiv P(v_1^{(1)} + v_2^{(2)} + v_3^{(1)} = 0, v_3^{(1)} + v_4^{(2)} = 0) + P(v_2^{(1)} + v_3^{(2)} + v_4^{(1)} = 0, v_3^{(2)} + v_2^{(1)} = 0),$$

where $P(\cdot)$ denotes the joint probability of obtaining the experimental outcomes satisfying the desired conditions. If $B_s$ has a pre-existing state known to $A_s$, rather than part of genuine multipartite entanglement shared with $A_s$, the maximum value of $W_4$ is

$$W_{4C} \equiv \max_{\{v_a\} \in \{A_s\}_C, \{v_b\} \in \{B_s\}_Q \text{QM}} W_4 = 1 + \frac{1}{\sqrt{2}} \sim 1.7071,$$

where $\{A_s\}_C$ denotes the index set of $A_s$ for all possible partitions and $\{v_a\} \in \{A_s\}_C$ indicates that the outcome set $\{v_a\}$ is derived from such pre-existing-state scenario. $B_s$’s outcome set $\{v_b\} \in \{B_s\}_Q \text{QM}$ is obtained by performing quantum measurements on the pre-existing quantum states. Hence we posit that if an experimental state $\rho_{\text{exp}}$ shows that

$$W_4(\rho_{\text{exp}}) > 1 + \frac{1}{\sqrt{2}},$$

then $\rho_{\text{exp}}$ can exhibit genuine four-partite EPR steerability close to $|G_4\rangle$. This rules out all the possibilities of results mimicked by tripartite steerability including all possible mixtures of them.

Four important features and implications of the steering witness (1) are worthy of paying attention to. Firstly,
FIG. 2. Experimental set-up. A pulse (5 ps) of UV light with a central wavelength of 355 nm and an average power of 200 mW at repetition rate of 80 MHz double passes a twocrystal structured BBO to produce polarization entangled photon pairs either in the forward direction or in the backward direction. To create desired entangled pairs in mode $R_A$, $R_B$ and in $L_A$, $L_B$, two quarter wave plates (QWPs) are properly tilted along their optic axis. Half wave plates (HWPs), polarizing beam splitters (PBSs) and eight single-photon detectors are used as polarization analyzers for the output states. 3-mm bandpass filters (IFs) with central wave-length 710 nm are placed in front of them. We use three different setups to implement spatial mode analysis. When performing the measurement $m_k = 2$ for $k = 3$ (4), a beam splitter is set at the intersection of modes $R_{A(B)}$ and $L_{A(B)}$. Furthermore, combined with a $\alpha$-phase shifter, measurement $B_k^{(\alpha)}$ can be realized for one-way quantum computing (Figs. 1, d,e and Methods). Whereas for the measurement $m_k = 1$, they are left empty. If a PBS is placed at the mode intersection, one can realize the measurements $m_{1,2} = 1$ and $m_{3,4} = 1$ simultaneously.

the states satisfying this witness enables quantum protocols based on genuine four-partite entanglement to be faithfully implemented when uncharacterized measurement apparatus are unavoidably used (Fig. 1). Secondly, $W_4(\rho_{\text{exp}})$ can be used to estimate the computation fidelity of one-way computation. Steerability can be verified by performing one-way computation as well. See Fig. 3 and Supplementary Information. Thirdly, the state fidelity of a generated state with respect to the target state, $F_S(\rho_{\text{exp}}) \equiv \langle G_4 | \rho_{\text{exp}} | G_4 \rangle$, can be estimated from $W_4(\rho_{\text{exp}})$. Furthermore, $F_S(\rho_{\text{exp}})$ is also an indicator showing genuine multipartite steerability (Methods). Finally, implanting the witness is experimentally efficient. Only the minimum two local measurement settings, $\{m_1\} : \{1,2,1,2\}$ and $\{2,1,2,1\}$, are sufficient to show the steerability.

To experimentally observe the genuine-four partite EPR steerability, we utilize the technique developed in previous experiments to generate a two-photon four-qubit ($d = 2$) cluster state source entangled both in polarization and spatial modes. As illustrated in Fig. 2, an ultraviolet (UV) pulse passes twice through two contiguous type-I beta barium borate (BBO) to produce polarization entangled photon pairs in the forward (spatial modes $R_{A,B}$) and the backward ($L_{A,B}$) directions. Through perfect temporal overlaps of modes $R_A$ and $L_A$ and of modes $R_B$ and $L_B$, one can create a four-qubit state

\[
|G'_4\rangle = \frac{1}{2} \left[ (|H\rangle_A |H\rangle_B + |V\rangle_A |V\rangle_B) |R_A\rangle |R_B\rangle + (|H\rangle_A |H\rangle_B - |V\rangle_A |V\rangle_B) |L_A\rangle |L_B\rangle \right],
\]

entangled both in spatial modes $(R,L)$ and horizontal $(H)$ and vertical $(V)$ polarizations. By encoding logical qubits as $|H(V)\rangle_{A/B} \equiv |0(1)\rangle_{1/2}$ and $|R(L)\rangle_{A/B} \equiv |0(1)\rangle_{3/4}$, $|G'_4\rangle$ is equivalent to the cluster state $|G_4\rangle$ up to a transformation $\hat{H}_2 \otimes \hat{H}_3$ where $\hat{H}_k$ is defined by $\hat{H}_k |0(1)\rangle_k \equiv |(+(-))_k\rangle$. The witness kernel for this target state has a corresponding change in measurement settings by $W'_4 \equiv P(v_1^{(1)} + v_2^{(1)} + v_3^{(2)} + v_4^{(2)} = 0, v_3^{(2)} + v_4^{(2)} = 0) + P(v_2^{(2)} + v_3^{(1)} + v_4^{(1)} = 0, v_1^{(2)} + v_2^{(2)} = 0)$ but keeps the same bound of the witness as (2).

In the experiment, we obtain a high generation rate of cluster state about $1.2 \times 10^4$ per second with 200 mW UV pump. To identify the steerability of our experimental state, we measure the two joint probabilities in the witness kernel with designed measurement settings (Fig. 2) and have

\[
P(v_1^{(1)} + v_2^{(1)} + v_3^{(2)} + v_4^{(2)} = 0, v_3^{(2)} + v_4^{(2)} = 0) = 0.9490 \pm 0.0022,
\]

\[
P(v_2^{(2)} + v_3^{(1)} + v_4^{(1)} = 0, v_1^{(2)} + v_2^{(2)} = 0) = 0.9339 \pm 0.0027.
\]

Then we observe genuine four-partite steerability in the experimental state $\rho_{\text{exp}}$ verified by the value of the witness kernel

\[
W'_4(\rho_{\text{exp}}) = 1.8829 \pm 0.0049,
\]

which is clearly larger than the maximum value the preexisting-state scenario can achieve, $W'_4(\rho_{\text{cr}})$, by $\sim 36$ standard deviations. Compared with the ideal target state $|G'_4\rangle$, the state $\rho_{\text{exp}}$ has high similarity by the estimate of the state fidelity

\[
0.8829 \pm 0.0049 \leq F_S(\rho_{\text{exp}}) \leq 0.9415 \pm 0.0025,
\]

which provides another evidence of genuine four-partite steerability (Methods).

The experimental cluster state can serve as a high-quality source for implementing one-way quantum computing with partial uncharacterized measurement devices. Two sorts of quantum circuits are realized (Figs. 1b,c and Methods) and objectively evaluated by three important performance indicators. As shown in Fig. 3, our experimental gates are capable of performing faithful gate operations by genuine four-partite steerability.

The idea of the criterion (1) can be directly applied to quantum states with complex structures. One of the main general extensions is to certify steerability of a general $d$-dimensional $N$-partite and $q$-colorable graph state.
We proceed to show how we observe EPR steering in our experimental demonstration and enables implementations for all the quantum protocols based on graph states (Fig. 1). The detections are robust against noise (Fig. 4) and capable of being efficiently implemented in experiments with only q local measurement settings, regardless of the number of qudits. The state fidelity can also be considered as a good indicator showing genuine high-order EPR steering (see Methods for experimental examples). We remark that, for a target state with genuine N-partite entanglement that does not belong to the above state types, for example, the $W$ states [27], useful steering witnesses still could be derived from one’s knowledge to this desired state (Supplementary Information).

We proceed to show how we observe EPR steering in all DOFs under consideration. The main existing experimentally created N-DOF hyperentangled state is illustrated in Fig. 5. $A_k$ ($B_k$) denotes the kth DOF of the subsystem $A$ ($B$) with $d_k$ dimensions. The entangled state in each DOF is a two-qudit graph state. Since this state has EPR steerability for all DOFs, the steering therein is called genuine N-DOF EPR steering. To identify such new steerability (Fig. 5), it is crucial to recognize the difference between N-DOF and (N − k)-DOF quantum steering for 1 ≤ k < N. The latter involves k pre-existing-state scenarios respectively for different k DOFs. To rule out all of these simulations, the following witness is introduced to certify genuine N-DOF EPR steerability: $\prod_{k=1}^{N} W^{(k)}(2, d |\rho_{\text{exp}})/2 > (1+1/\sqrt{d})/2$, where $d = \min\{d_k\}$ and $W^{(k)}(2, d)$ is the witness kernel for the kth DOF as (4). Alternatively, one can identify $\rho_{\text{exp}}$ as truly N-DOF EPR steerable if the state fidelity satisfies the condition $F_S(\rho_{\text{exp}}) > 1/\sqrt{d}$. See Methods for their experimental illustrations and applications.

In conclusion, we have developed a novel formalism to explore genuine high-order EPR steering and experimentally demonstrated such generality and applications with photonic cluster states. Being capable of revealing genuine high-order EPR steering pushes beyond the capability of bipartite steering and may enable new applications and experiments. One can probe more sorts of steerability, for example, in resonating valence-bond states, which would allow an analogue quantum simulator [12] to run without fully characterized quantum measurements. Similarly, other quantum strategies based on both characterized measurements and genuine multipartite entanglement, like quantum metrology [13], can be benefited from it as well. Moreover, since genuine multipartite EPR steerability can not be mimicked by

\[ W_N(q, d |\rho_{\text{exp}}) > \frac{1}{2}q + \sqrt{\frac{2(q^2 - 2q + 2) + \gamma_q}{d}}, \]

(4)
FIG. 4. Robustness of steering witness. A state mixed with white noise, \( \rho_{\text{noise}} = \rho_{\text{noise}}I/d^N + (1 - \rho_{\text{noise}}) \ket{G} \bra{G} \), where \( I \) denotes the identity operator, is identified to possess truly high-order EPR steerability by the witness (4) if \( \rho_{\text{noise}} < p \). For large dimensionality \( d \), the criterion is very robust and the noise tolerance is up to \( \rho_{\text{noise}} < 1/2 \), independent of the number of qudits and the types of graph states. In addition, as illustrated by detecting states close to two-color \( (q = 2) \) star [Greenberger-Horne-Zeilinger (GHZ) state] (a) and chain (cluster state) graph states (b), our criterion is robust for the cases of finite \( d \) as well.

parts of the whole system, such ability together with steering witness could facilitate multipartite secret sharing \([14]\) in generic one-sided device-independent mode \([18,20]\). It is interesting to investigate further our method from the all-versus-nothing (AVN) point of view. Subtle AVN proof for steering and their experimental demonstration are given for special classes of two qubits \([28,29]\). We note that, after finishing this work, we became aware of two experimental demonstrations of tripartite steering, \[arXiv:1412.7212\] and \[arXiv:1412.7730\].

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METHODS

Experimental quantum gates. For a given cluster state, a quantum computing is defined by consecutive single-qudit measurements in basis \( B_{q}^{(a)} = \{ \ket{+}_k, \ket{-}_k \} \) and their results of measurements, where \( \ket{\pm}_k = (\ket{0}_k \pm e^{i\alpha} \ket{1}_k)/\sqrt{2} \) for any real \( \alpha \). A measurement outcome of \( \ket{+}_k \) denotes \( s_k = 0 \) while \( \ket{-}_k \) means \( s_k = 1 \). This measurement basis determines a gate operation \( R_{z}^{(a)} = \exp(-iaZ/2) \) where \( Z = \ket{0}\bra{0} - \ket{1}\bra{1} \), followed by a transformation \( \hat{H} \). A chain-type graph (horseshoe cluster) state \( \ket{G}_4 \) (Fig. 1b) can realize a two-qubit controlled-Z gate \( \ket{UCZ} \) (Fig. 1d): \( UCZ \ket{j} k = (-1)^{jk} \ket{j} k \), \( j, k = 0, 1 \). When measuring along basis \( B_{q}^{(a)} \) and \( B_{q}^{(b)} \), the output state on qubits 1 and 4 would be \( \ket{\text{Out}_{I}^{(a)} \beta} = (X^{s_2} \otimes X^{s_3}) (H \otimes H) \ket{\text{In}_{I}^{(a)} \beta} \), where the input state is \( \ket{\text{In}_{I}^{(a)} \beta} = R_{z}^{(-a)} \otimes R_{z}^{(-b)} \ket{+} \ket{+} \). For \( (a, \beta) \in \{0, \pi\} \), the output states are entangled (Fig. 3a). To demonstrate this gate in one-way realization, we have created states close to \( \ket{G}_4 \) [Eq. (2)], which is equivalent to the cluster state \( \ket{G}_4 \) up to a transformation. Furthermore, with the wave plate sets together with BS and PBS (Fig. 2), we perform the required measurements \( B_{q}^{(a)} \) as well. Similarly, our created states can be directly used for creating a gate composed of two controlled-Z gates (Fig. 1c), since the box-cluster state (Fig. 1e) needed for realizing such a gate operation distinguishes only the state \( \ket{G}_4 \) from an \( \hat{H} \) transformation on every qubit and swap between qubits 2 and 3.

Proof of quantum steering witnesses. The steering witness kernel for a general \( d \)-dimensional \( N \)-partite and \( q \)-colorable graph state, \( \ket{G} \), is of the form

\[
W_N(q,d) \equiv \sum_{m=1}^{q} P(v_{j}^{(2)}) + \sum_{i \in \{j\}} v_{i}^{(1)} = 0 |v_{j} \in Y_m|
\]
and performing the above maximization, we have
\[
B_k = \frac{\sum_{i=0}^{d-1} \omega^i |v_k^i\rangle}{\sqrt{d}}.
\]

The steering witness operator for detecting steerability. Let us assume that this witness operator is of the form, \(W_G = \delta \mathbb{I} - |G\rangle\langle G|\). The unknown parameter \(\delta\) can be determined by showing \(W_G - \gamma W_G \geq 0\), where \(\gamma\) is some positive constant. This shows that when a state is detected by \(W_G\), it is certified by \(W_G\) as well. When setting \(\gamma = 1/2\), the above relation holds if \(\delta = W_{NC}(q,d)/2\). Hence, from \(W'_{NC}\) we have the criterion on state fidelity
\[
F_S(\rho_{\text{exp}}) > \frac{1}{4}(q + \sqrt{\frac{2(q^2 - 2q + 2) + 2q}{d}}).
\]

For \(q = 2\), such as GHZ (star graph) and cluster states, the criterion reads \(F_S(\rho_{\text{exp}}) > 1/2(1+1/\sqrt{d})\). This is especially useful when one already has the state fidelity information. One is allowed to evaluate the existing experimental results for steerability, while they did not measure the steering witness kernel before. For instances, the \(N\)-qubit entangled ions for \(N = 2,3,...,6\) created in the experiment [28] can be identified as genuinely \(N\)-partite steerable when considering their fidelities. Similarly, genuine tripartite steering can be confirmed in a superconducting circuit as well [29]. In addition, with the condition \(W_G - W_{2G}/2 \geq 0\), we also have \(|G\rangle\langle G| \leq W_N(q,d)/2\), implying the upper bound of the state fidelity. By using a similar approach, one can get \(W_N(q,d) - \mathbb{I} \leq |G\rangle\langle G|\), which serves as the lower bound of the experimental state fidelity. In summary, the experimental state fidelity is estimated by
\[
W_N(q,d|\rho_{\text{exp}}) - 1 \leq F_S(\rho_{\text{exp}}) \leq \frac{1}{2}W_N(q,d|\rho_{\text{exp}}).
\]

The method introduced above can be applied to the hyperentangled systems directly. As reported in the pioneering experiment [3], the experimental polarization-spatial modes hyperentangled state \((d_1 = d_2 = 2)\) has the experimental state fidelity, \(F_S(\rho_{\text{exp}}) = 0.955(2)\). This satisfies the condition on the state fidelity for \(d = 2\), \(F_S(\rho_{\text{exp}}) > 1/\sqrt{2} \approx 0.7071\), which supports the existence of genuine two-DOF EPR steerability in their experimental states. On the other hand, the proposed criteria also can detect genuine \(N\)-pair EPR steering of \(2N\) individual systems, such as dimer-covering states of a spin-1/2 tetramer (\(N = 2\)). Their recent experimental demonstrations [12] can show such steerability by satisfying the above condition on state fidelity (see Fig. 4 in ref 12).

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SUPPLEMENTARY INFORMATION

Detecting steerability with one-way quantum computing

To show that genuine multipartite EPR steerability can be revealed by performing one-way quantum computing, we use the information about input and output of a computation to construct the steering witness. We derive this from the two-qubit controlled-Z gate, Fig. 1b, and the witness operator \( \hat{W}_C = \hat{W}_{4C}^{-1} \hat{W}_4 \), where \( \hat{W}_4 \) is an operator form of the kernel \( W_4 \). We assume that this witness has the form \( \hat{W}_{CZ} = \delta I - \hat{W}_{CZ} \), where

\[
\hat{W}_{CZ} = \sum \langle \alpha_+|^\alpha \rangle_{32} \langle \beta_+|^\beta \rangle_{33} \langle \hat{C}_Z \rangle \langle \hat{C}_Z \rangle \\hat{U}_{CZ} = \hat{U}_{CZ},
\]

where \( \hat{U}_{CZ} = (\hat{H} \otimes \hat{H})\hat{U}_{CZ} \). The operator \( \hat{W}_{CZ} \) is also a quantum steering witness if \( \hat{W}_{CZ} - \gamma \hat{W}_{CZ} \geq 0 \). When \( \gamma = 1/2 \), we determine the parameter as \( \delta = (\hat{W}_{4C}^{-1}/2 + 1) \). Similarly, one can construct a steering witness which is based on the quantum gate, Fig. 1c, realized in the one-way mode, Fig. 1e: \( \hat{W}_{CZ} = (\hat{W}_{4C}^{-1} + 1)I - \hat{W}_{CZ} \), where

\[
\hat{W}_{CZ} = \sum \langle \alpha_+|^\alpha \rangle_{32} \langle \beta_+|^\beta \rangle_{33} \langle \hat{C}_Z \rangle \langle \hat{C}_Z \rangle \\hat{U}_{CZ} = \hat{U}_{CZ},
\]

and \( \hat{U}_{CZ} = (\hat{H} \otimes \hat{H})\hat{U}_{CZ} \). Measuring the above witness operators is equivalent to performing one-way computations by inputting two sets of input states into the gates and evaluating their results with respect to the target ones. Here the input sets \( \{|\alpha, \beta \rangle \rangle \alpha, \beta = 0, \pi \rangle \rangle \).
and \( \{|\alpha,\beta\rangle \} \) are complementary to each other \([23]\). This idea can be directly applied to generic quantum circuit of one-way quantum computing. Hence one-way quantum computation can be used to verify genuine multipartite steerability.

**Experimental computation fidelity**

The average computation fidelity is defined by

\[
F_{\text{comp}} \equiv \frac{1}{8} \sum_{(\alpha,\beta)\in\{0,\pi\}} \langle \text{In}_\alpha \rangle U_{\text{CZ}} E_{\text{CZ}} \langle \text{In}_\beta \rangle U_{\text{CZ}} \langle \text{In}_\beta \rangle,
\]

where \( E_{\text{CZ}} \) denotes the experimental gate operations. Eight input states are used to consider the gate performance. The average computation fidelity can be obtained by the expectation value of the witness operator \( W_{\text{CZ}} \):

\[
F_{\text{comp}} \times \frac{1}{4} = \frac{\langle W_{\text{CZ}} \rangle}{8}.
\]

The factor 8 is for the eight input states and 1/4 is attributed to the assumption that the ideal probability \( P(\alpha,\beta) = 1/4 \) is assigned to all settings of \((\alpha,\beta)\). From this relation together with \( W_{\text{CZ}} - \gamma W_4 \geq 0 \) (i.e., \( W_{\text{CZ}} \leq W_4/2 + 1 \)), one can derive the upper bound of \( F_{\text{comp}} \) from the experimental value \( \langle W_4 \rangle \equiv W_4(\rho_{\text{exp}}) : F_{\text{comp}} \leq W_4(\rho_{\text{exp}})/4 + 1/2 \). Similarly, by using the same approach we have another relation: \( 2(W_4 - 1) \leq W_{\text{CZ}} \), which provides the lower bound of \( F_{\text{comp}} \). Hence we conclude the following estimate of \( F_{\text{comp}} \):

\[
W_4(\rho_{\text{exp}}) - 1 \leq F_{\text{comp}} \leq \frac{1}{4} W_4(\rho_{\text{exp}}) + \frac{1}{2}.
\]

Moreover, if \( W_4(\rho_{\text{exp}}) > W_{4C} \) is experimentally ensured, we will have

\[
\frac{1}{4} W_{4C} + \frac{1}{2} < F_{\text{comp}} \leq \frac{1}{4} W_4(\rho_{\text{exp}}) + \frac{1}{2}.
\]

These results are applicable to the experimental gate operation \( E_{\text{CZ}} \) aimed to the target quantum gate, Fig. 1c.

**Experimental quantum process and average quantum state fidelities**

The quantum process fidelity is a comparison between an experimental process, say \( \mathcal{E} \), and a target quantum process: \( F_{\text{process}} \equiv \text{Tr}[\chi_{\text{ideal}} \chi_{\text{exp}}] \), where \( \chi_{\text{ideal}} \) and \( \chi_{\text{exp}} \) represent the ideal and experimental process matrices \([24]\), respectively. A process matrix completely characterizes a quantum process \( \mathcal{E} \). While it is experimentally measurable, the required measurement settings increase exponentially with the number of qubits participated in a process. Fortunately, a good estimate can be efficiently measured by the average computation fidelity \([23]\):

\[
F_{\text{process}} \geq 2F_{\text{comp}} - 1,
\]

where two measurement settings corresponding to two complementary sets of input states are sufficient. With this efficient criterion and the connection between \( F_{\text{comp}} \) and \( W_4(\rho_{\text{exp}}) \) introduced above, we have

\[
F_{\text{process}} \geq \frac{1}{2} W_4(\rho_{\text{exp}}).
\]

Furthermore, this lower bound of process fidelity also helps derive the average quantum state fidelity \([24]\),

\[
F_{\text{avg}} \equiv \text{mean}_{\text{all} |\psi\rangle} \text{Tr}[\mathcal{E}(|\psi\rangle\langle\psi|)U |\psi\rangle \langle U|],
\]

where \( U \) is the target quantum transformation (gate operation). It is known that \( F_{\text{process}} \) can be converted to \( F_{\text{avg}} \) by \( F_{\text{avg}} = (MF_{\text{process}} + 1)/(M + 1) \ ([24]\), where \( M \) is the dimension of the gate \( U \). Therefore we get the lower bound of average quantum state fidelity

\[
F_{\text{avg}} \geq \frac{1}{5} [2W_4(\rho_{\text{exp}}) + 1].
\]

**Quantum steering witnesses derived from full state knowledge**

Let us assume that the target state, say \( |\psi\rangle \), is pure. One can design the witness kernel as:

\[
W_{\psi} = \sum_{v_1^{(1)} \ldots v_N^{(n)}} c(v_1^{(1)} \ldots v_N^{(n)}) P(v_1^{(1)}) \ldots P(v_N^{(n)}),
\]

where the parameters \( c(v_1^{(1)} \ldots v_N^{(n)}) \) are captured from the tomographic decomposition of state \( |\psi\rangle \):

\[
|\psi\rangle \langle \psi| = \sum_{v_1^{(1)} \ldots v_N^{(n)}} c(v_1^{(1)} \ldots v_N^{(n)}) |v_1^{(1)}\rangle \ldots |v_N^{(n)}\rangle.
\]

The full information about the target state is used for \( W_{\psi} \). Each orthonormal basis \( \{ |v_k^{(m)}\rangle \}_{k,m} \) is composed of eigenvectors of an observable for the measurement \( m_k \). Here, since the quantum state tomography is used, take \( d = 2 \) for example, the observables can be chosen as the identity operator and three Pauli matrices. The maximum value \( W_{\psi} \) that is achieved by the pre-existing state scenario is determined by using the same method to find \( W_{NC}(q,d) \) (Methods). As a concrete illustration of this scheme, the steering witness for the three-qubit \( W \) state, \( W_{\psi} > (1 + \sqrt{2})/3 \approx 0.8047 \), can detect the experimental state as truly tripartite steerable \([30]\).