DUALITY BETWEEN FRONT AND REAR MUTATIONS IN CLUSTER ALGEBRAS

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Abstract. We study the duality between the mutations and the rear mutations in cluster algebras, where the rear mutations are the transformations of rational expressions of cluster variables in terms of the initial cluster under the change of the initial cluster. In particular, we define the maximal degree matrices of the $F$-polynomials called the $F$-matrices and show that the $F$-matrices have the self-duality which is analogous to the duality between the $C$- and $G$-matrices.

1. Introduction

The cluster algebras are a class of commutative algebras introduced by [FZ02], which are generated by some distinguished elements called the cluster variables. The cluster variables are given by applying the mutations repeatedly starting from the initial cluster variables. Through the description of mutations as transformations on quivers or triangulations, the cluster algebras are applied in various mathematics such that representation theory of quivers [BMR06], hyperbolic geometry [FST08,FG09], etc.

Let us briefly recall the notion of mutations, which is essential in cluster algebra theory. For simplicity, we consider a cluster pattern without coefficients [FZ02]. Let $\mathbb{T}_n$ be the $n$-regular tree. Then, for each vertex $t \in \mathbb{T}_n$, cluster variables $x_t = (x_{1,t}, \ldots, x_{n,t})$ are attached such that for each adjacent $t$ and $t'$ in $\mathbb{T}_n$ those variables are related by a rational transformation called a mutation. Let $t_0 \in \mathbb{T}_n$ be given point in $\mathbb{T}_n$ called the initial point. Then, repeating mutations starting from the initial variables, we have the expression

$$x_{i;t} = \lambda_{i;t_0}^{t_0}(x_{i;t_0})$$

(1.1)

where the rational function $\lambda_{i;t_0}^{t_0}$ depends on $t$ and $t_0$. For a vertex $t'$ which is adjacent $t$, the function $\lambda_{i;t_0}^{t_0}$ transforms to $\lambda_{i;t_0}^{t_0}$ exactly by the mutation rule. On the other hand, for a vertex $t_1$ which is adjacent to $t_0$ in $\mathbb{T}_n$, the function $\lambda_{i;t_0}^{t_0}$ also transforms to some function $\lambda_{i;t_1}^{t_1}$. We call this transformation the rear mutations. (In contrast to the rear mutations, we call the ordinary mutations also as the front mutations in this paper.)

Thanks to the separation formulas by [FZ07], the cluster variables and the coefficients are described by the $C$-matrices, the $G$-matrices, and the $F$-polynomials, where the $C$- and $G$-matrices are the “tropical part”, while the $F$-polynomials are the “nontropical part”. The front and rear mutations of the cluster variables and the coefficients reduce to the ones for the $C$- and $G$-matrices and the $F$-polynomials.

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In [NZ12], the duality between the $C$- and $G$-matrices under the front and rear mutations was established (under the assumption of the sign-coherence of the $C$-matrices, which is established in [GHKK18]). In [RS18], an analogous duality was established also for the denominator vectors for some classes of cluster algebras. In this paper, we treat the nontropical part of the seeds, namely the $F$-polynomials. Since the $F$-polynomials are rather complicated, we consider the “degree matrices” of $F$-polynomials ($F$-matrices). Then, we establish the duality of those matrices under the front and rear mutations, which is parallel to the ones in [NZ12, RS18]. This is the main result of the paper.

We also exhibit the front and rear mutation formulas of the $C$- and $G$-matrices systematically, some of which are new in the literature. We stress that the duality of the front and rear mutations becomes manifest only after applying the sign-coherence of the $C$-matrices.

The organisation of the paper is as follows: In section 2, we start with recalling the definitions of the seed mutations and the cluster patterns according to [FZ07] and give the front mutations. Also, using the sign-coherence of the $C$-matrices, we obtain a reduced form of the front mutations. The formulas involving the $F$-matrices are new results. In section 3, the rear mutation formulas are given by using the $H$-matrices in [FZ07]. By the sign-coherence of the $C$-matrices and the duality of the $C$- and $G$-matrices of [NZ12], we prove the conjecture [FZ07, Conjecture 6.10], where the $H$-matrices are expressed by the $G$-matrices. Using this result, we obtain the duality of the $F$-matrices. In section 4, we consider the principal extension of the exchange matrices and present some properties. We show that these properties give alternative derivations of some equalities in previous sections.

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2. FRONT MUTATIONS

2.1. Seed mutations and cluster patterns. We start with recalling the definitions of the seed mutations and the cluster patterns according to [FZ07]. See [FZ07] for more information. A semifield $\mathbb{P}$ is an abelian multiplicative group equipped with an addition $\oplus$ which is distributive with the multiplication. We particularly make use of the following two semifields.

Let $\mathbb{Q}_{sf}(u_1,\ldots,u_\ell)$ be a set of rational functions in $u_1,\ldots,u_\ell$ which have subtraction-free expressions. Then, $\mathbb{Q}_{sf}(u_1, u_2,\ldots,u_\ell)$ is a semifield by the usual multiplication and addition. We call it the universal semifield of $u_1,\ldots,u_\ell$ ([FZ07, Definition 2.1]).

Let $\text{Trop}(u_1,\ldots,u_\ell)$ be the abelian semifield freely generated by the elements $u_1,\ldots,u_\ell$. Then, $\text{Trop}(u_1, u_2,\ldots,u_\ell)$ is a semifield by the following addition:

$$\prod_{j=1}^\ell u_j^{a_j} \oplus \prod_{j=1}^\ell u_j^{b_j} = \prod_{j=1}^\ell u_j^{\min(a_j,b_j)}.$$ (2.1)

We call it the tropical semifield of $u_1,\ldots,u_\ell$ ([FZ07, Definition 2.2]). We note that the universal semifield $\mathbb{Q}_{sf}(u_1,\ldots,u_\ell)$ has the universality. For any semifield $\mathbb{P}$ and
there exists a unique semifield homomorphism such that
\[
\pi : \mathbb{Q}_{sf}(y_1, \ldots, y_\ell) \to \mathbb{P} \\
y_i \mapsto p_i.
\]
For \( F(y_1, \ldots, y_\ell) \in \mathbb{Q}_{sf}(y_1, \ldots, y_\ell) \), we denote
\[
F|_\mathbb{P}(p_1, \ldots, p_\ell) := \pi(F(y_1, \ldots, y_\ell))
\]
and we call it the evaluation of \( F \) at \( p_1, \ldots, p_\ell \).

We fix a positive integer \( n \) and semifield \( \mathbb{P} \), which are called the rank and the coefficient semifield of the (forthcoming) cluster patterns. For a semifield \( \mathbb{P} \), let \( \mathbb{Z}\mathbb{P} \) be the group ring of \( \mathbb{P} \) as a multiplicative group. Since \( \mathbb{Z}\mathbb{P} \) is a domain ([FZ02, Section 5]), its total quotient ring is a field \( \mathbb{Q}(\mathbb{P}) \). We say that a field \( \mathcal{F} \) is an ambient field of the (forthcoming) cluster variables if \( \mathcal{F} \) is isomorphic to the field of the rational functions in \( n \) indeterminates with coefficients in \( \mathbb{Q}(\mathbb{P}) \).

A labeled seed with coefficients in \( \mathbb{P} \) is a triplet \((x, y, B)\), where
- \( x = (x_1, \ldots, x_n) \) is an \( n \)-tuple of elements of \( \mathcal{F} \) forming a free generating set of \( \mathcal{F} \).
- \( y = (y_1, \ldots, y_n) \) is an \( n \)-tuple of elements of \( \mathbb{P} \).
- \( B = (b_{ij}) \) is an \( n \times n \) integer matrix which is skew-symmetrizable, that is, there exists a positive diagonal matrix \( D \) such that \( DB \) is skew-symmetric.

We refer to \( x, y \) and \( B \) as the cluster variables, the coefficients and the exchange matrix, respectively. Also, we call \( D \) a skew-symmetrizer of \( B \).

Throughout the paper, for an integer \( b \), we use the notation \( [b]_+ = \max(b, 0) \). We note that
\[
b = [b]_+ - [-b]_+.
\]
Let \((x, y, B)\) be a labeled seed with coefficients in \( \mathbb{P} \), and let \( k \in \{1, \ldots, n\} \). The seed mutation \( \mu_k \) in direction \( k \) transforms \((x, y, B)\) into another labeled seed \( \mu_k(x, y, B) = (x', y', B') \) defined as follows:
- The entries of \( B' = (b'_{ij}) \) are given by
\[
b'_{ij} = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k, \\
b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik} [-b_{kj}]_+ & \text{otherwise}.
\end{cases}
\]
- The coefficients \( y' = (y'_1, \ldots, y'_n) \) is given by
\[
y'_j = \begin{cases} 
y_k^{-1} & \text{if } j = k, \\
y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{otherwise}.
\end{cases}
\]
- The cluster variables \( x' = (x'_1, \ldots, x'_n) \) is given by
\[
x'_j = \begin{cases} 
y_k \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+} = x_j \\
\frac{y_k \prod_{i=1}^{n} x_i^{[b_{ik}]_+} + \prod_{i=1}^{n} x_i^{[-b_{ik}]_+}}{(y_k \oplus 1)x_k} & \text{if } j = k, \\
x_j & \text{otherwise}.
\end{cases}
\]
We remark that (2.6) can be also expressed as follows:

\begin{equation}
\left\{\begin{array}{ll}
y_j' = \frac{1}{y_k} & \text{if } j = k, \\
y_j' = y_j k^{-\lfloor b_j \rfloor} + (y_k^{-1} + 1)^{\lfloor b_j \rfloor} & \text{otherwise}
\end{array}\right.
\end{equation}

Let \( T_n \) be the \( n \)-regular tree whose edges are labeled by the numbers 1, \ldots, \( n \) such that the \( n \) edges emanating from each vertex have different labels. We write \( t \overset{k}{\rightarrow} t' \) to indicate that vertices \( t, t' \in T_n \) are joined by an edge labeled by \( k \). We fix an arbitrary vertex \( t_0 \in T_n \), which is called the initial vertex.

A cluster pattern with coefficients in \( \mathbb{P} \) is an assignment of a labeled seed \( \Sigma_t = (x_t, y_t, B_t) \) with coefficients in \( \mathbb{P} \) to every vertex \( t \in T_n \) such that the seeds \( \Sigma_t \) and \( \Sigma_{t'} \) assigned to the endpoints of any edge \( t \overset{k}{\rightarrow} t' \) are obtained from each other by the seed mutation in direction \( k \). The elements of \( \Sigma_t \) are denoted as follows:

\begin{equation}
x_t = (x_{1:t}, \ldots, x_{n:t}), \quad y_t = (y_{1:t}, \ldots, y_{n:t}), \quad B_t = (b_{ij:t}).
\end{equation}

In particular, at \( t_0 \), we denote

\begin{equation}
x = x_{t_0} = (x_1, \ldots, x_n), \quad y = y_{t_0} = (y_1, \ldots, y_n), \quad B = B_{t_0} = (b_{ij}).
\end{equation}

We say that a cluster pattern \( \{ (x_t, y_t, B_t) \}_{t \in \mathbb{T}_n} \) has principal coefficients at the initial vertex \( t_0 \) if \( \mathbb{P} = \text{Trop}(y_1, \ldots, y_n) \) and \( y_{t_0} = (y_1, \ldots, y_n) \).

2.2. Front mutations without sign-coherence of \( C \)-matrices. In this subsection, we will define the \( C \)-matrices, the \( G \)-matrices and the \( F \)-polynomials following [FZ07]. We also introduce the \( F \)-matrices which are new.

Throughout this paper, we use the following notations ([NZ12]). Let \( J_\ell \) denote the \( n \times n \) diagonal matrix obtained from the identity matrix \( I_n \) by replacing the \((\ell, \ell)\) entry with \(-1\). For a \( n \times n \) matrix \( B = (b_{ij}) \), let \([B]_+\) be the matrix obtained from \( B \) by replacing every entry \( b_{ij} \) with \([b_{ij}]_+\). Also, let \( B^k \) be the matrix obtained from \( B \) by replacing all entries outside of the \( k \)th row with zeros. Similarly, let \( B^{k}\) be the matrix replacing all entries outside of the \( k \)th column. Note that the maps \( B \mapsto [B]_+ \) and \( B \mapsto B^k \) commute with each other, and the same is true of \( B \mapsto [B]_+ \) and \( B \mapsto B^{k} \), so that the notations \([B]_+^k\) and \([B]_+^{k}\) make sense. Also, we have \( AB^k = (AB)^k \) and \( A^k B = (AB)^k \).

First, we recall the recursive definitions of the \( C \)-matrices, the \( G \)-matrices and the \( F \)-polynomials, following [FZ07]. For any initial exchange matrix \( B \) at \( t_0 \), let \( \{ B_t \}_{t \in \mathbb{T}_n} \) be the family of the exchange matrices in (2.9) with \( B_{t_0} = B \).

**Definition 2.1 ([FZ07], (5.9), Proposition 6.6)).** Let \( B \) be any initial exchange matrix at \( t_0 \). Then, the families of \( n \times n \) integer matrices \( \{ C_t^{B_{t_0}} \}_{t \in \mathbb{T}_n} \) and \( \{ G_t^{B_{t_0}} \}_{t \in \mathbb{T}_n} \) are recursively defined as follows:

(i) We set the initial condition,

\begin{equation}
C_{t_0}^{B_{t_0}} = I_n,
\end{equation}

and for any edge \( t \overset{\ell}{\rightarrow} t' \) in \( \mathbb{T}_n \), we set the recurrence relation,

\begin{equation}
C_{t'}^{B_{t_0}} = C_t^{B_{t_0}}(J_\ell + [\varepsilon B_t]_+^k) + [-\varepsilon C_t^{B_{t_0}}]_+^k B_t.
\end{equation}
(ii) We set the initial condition,

\[ G_{t_0}^{B;t_0} = I_n, \]

and for any edge \( t \xrightarrow{\ell} t' \) in \( \mathbb{T}_n \), we set the recurrence relation,

\[ G_{t'}^{B;t_0} = G_t^{B;t_0} (J_t + [\varepsilon B_t]_{+}^{\ell} ) - B[\varepsilon C_t^{B;t_0}]_{+}^{\ell}. \]

Here \( \varepsilon \in \{ \pm 1 \} \). The matrices \( C_t^{B;t_0} \) and \( G_t^{B;t_0} \) are called the \( C \)-matrix and the \( G \)-matrix at \( t \).

**Remark 2.2.** Because of (2.4), the right hand side of (2.12) does not depend on \( \varepsilon \). Meanwhile, the right hand side of (2.14) does not depend on \( \varepsilon \) due to the following equality ([FZ07, (6.14)]):

\[ G_t^{B;t_0} B_t = B C_t^{B;t_0} . \]

**Definition 2.3** ([FZ07, Proposition 5.1]). Let \( B \) be any initial exchange matrix at \( t_0 \) and \( y = (y_1, \ldots, y_n) \) be formal variables. Then, the family of polynomials \( F_{j;t_0}^{B;t_0}(y) \in \mathbb{Z}[y_1, \ldots, y_n] \) indexed by \( j \in \{1, \ldots, n\} \) and \( t \in \mathbb{T}_n \) are recursively defined as follows: We set the initial condition,

\[ F_{j;t_0}^{B;t_0}(y) = 1 \quad (j = 1, \ldots, n), \]

and for any edge \( t \xrightarrow{\ell} t' \) in \( \mathbb{T}_n \), we set the recurrence relation,

\[ F_{j;t'}^{B;t_0}(y) = \begin{cases} F_{j;t}^{B;t_0}(y)^{-1} \left( \prod_{i=1}^{n} y_i \left| c_{i;t_0}^{B;t_0} \right|^{+} \prod_{i=1}^{n} F_{i;t}^{B;t_0}(y) \left| b_{i,t_0}^{B;t} \right|^{+} \right. \\ \quad + \left. \prod_{i=1}^{n} y_i \left| c_{i,t_0}^{B;t_0} \right|^{+} \prod_{i=1}^{n} F_{i;t}^{B;t_0}(y) \left| b_{i,t_0}^{B;t} \right|^{+} \right) & \text{if } j = \ell, \\ F_{j;t}^{B;t_0}(y) & \text{if } j \neq \ell, \end{cases} \]

where \( c_{i,t_0}^{B;t_0} \) is the \((i, \ell)\) entry of \( C_t^{B;t_0} \) and \( b_{i,t_0}^{B;t} \) is the \((i, \ell)\) entry of \( B_t \). The polynomials \( F_{j;t}^{B;t_0}(y) \) are called the \( F \)-polynomials at \( t \).

The fact that seemingly rational functions \( F_{j;t}^{B;t_0}(y) \) are polynomials follows from the Laurent phenomenon of cluster variables ([FZ07, Proposition 3.6]). We also remark that for any \( j \in \{1, \ldots, n\} \) and \( t \in \mathbb{T}_n \), \( F_{j;t}^{B;t_0}(y) \) is an element of \( \mathbb{Q}(y_1, \ldots, y_n) \) because of (2.16) and (2.17).

In this paper, we refer to the recurrence relations (“mutations”) (2.12), (2.14) and (2.17) as the front mutations in contrast to the rear mutations appearing later. Abusing of notation, we denote \( C_{t'}^{B;t_0} = \mu_\ell(C_{t}^{B;t_0}), G_{t'}^{B;t_0} = \mu_\ell(G_{t}^{B;t_0}) \) and \( F_{j;t'}^{B;t_0}(y) = \mu_\ell(F_{j;t}^{B;t_0}(y)) \).

The \( C \)-matrices, the \( G \)-matrices, and the \( F \)-polynomials are important because they factorize cluster variables and coefficients by the following formulas:
**Proposition 2.4** (Separation formulas [FZ07], Proposition 3.13, Corollary 6.3). Let \( \{\Sigma\}_{\ell \in \mathbb{T}_n} \) be a cluster pattern with coefficient in \( \mathbb{P} \) with the initial seed (2.10). Then, for any \( t \in \mathbb{T}_n \) and \( j \in \{1, \ldots, n\} \), we have

\[
x_{jt} = \left( \prod_{k=1}^{n} \frac{g_{kjt}^{B_{t0}}}{g_{kjt}^{B_{t0}}} \right) \frac{F_{jt}^{B_{t0}}(x_1, \ldots, x_n)}{F_{jt}^{B_{t0}}(y_1, \ldots, y_n)}
\]

\[
y_{jt} = \prod_{k=1}^{n} \frac{g_{kjt}^{B_{t0}}}{g_{kjt}^{B_{t0}}} \prod_{k=1}^{n} \left( F_{kjt}^{B_{t0}}(x_1, \ldots, x_n) \right)^{y_{ktj}},
\]

where

\[
\hat{y}_t = y_t \prod_{j=1}^{n} x_{jt}^{b_{jt}}
\]

and \( g_{kjt}^{B_{t0}} \) and \( g_{kjt}^{B_{t0}} \) are the \((i, j)\) entry of \( G_{t0}^{B_{t0}} \) and \( C_{t0}^{B_{t0}} \), respectively. Also, the rational function \( F_{jt}^{B_{t0}}(x_1, \ldots, x_n) \) is the element of \( \mathcal{F} \) obtained by substituting \( \hat{y}_t \) for \( y_t \) in \( F_{jt}^{B_{t0}}(y_1, \ldots, y_n) \).

The following fact is well-known:

**Proposition 2.5.** (1) ([FZ07], (5.5)) For any \( j \in \{1, \ldots, n\} \) and \( t \in \mathbb{T}_n \), we have

\[
F_{jt}^{B_{t0}}|_{\text{Trop}(y_1, \ldots, y_n)}(y_1, \ldots, y_n) = 1.
\]

(2) ([FZ07], (2.13)) Let \( \{\Sigma\}_{\ell \in \mathbb{T}_n} \) be a cluster pattern which has principal coefficients at \( t_0 \). Then, for any \( j \in \{1, \ldots, n\} \) and \( t \in \mathbb{T}_n \), we have

\[
y_{jt} = \prod_{k=1}^{n} g_{kjt}^{B_{t0}}
\]

**Proof.** Because of (2.16) and (2.17), by using the tropicalization

\[
\pi : \mathbb{Q}^n(y_1, \ldots, y_n) \rightarrow \text{Trop}(y_1, \ldots, y_n)
\]

\[
y_t \mapsto y_t,
\]

we have (2.21). Moreover, setting \( \mathbb{P} = \text{Trop}(y_1, \ldots, y_n) \) in (2.19), we obtain (2.22) by (2.21).

Next, we introduce another family of matrices, which are the “degree matrices” of the \( F \)-polynomials.

**Definition 2.6.** Let \( B \) be any initial exchange matrix at \( t_0 \). For \( i \in \{1, \ldots, n\} \) and \( t \in \mathbb{T}_n \), let \( f_{i1:t}^{B_{t0}}, \ldots, f_{in:t}^{B_{t0}} \) be the maximal degrees of \( y_1, \ldots, y_n \) in the \( i \)th \( F \)-polynomial \( F_{ij:t}^{B_{t0}}(y_1, \ldots, y_n) \), respectively. We call the nonnegative integer vectors

\[
f_{i1:t}^{B_{t0}} = \begin{bmatrix} f_{i1:t}^{B_{t0}} \\ \vdots \\ f_{in:t}^{B_{t0}} \end{bmatrix}
\]

the \( f \)-vectors at \( t \). We also call the nonnegative integer \( n \times n \) matrix

\[
F_{t}^{B_{t0}}
\]

with columns \( f_{i1:t}^{B_{t0}}, \ldots, f_{in:t}^{B_{t0}} \) the \( F \)-matrix at \( t \).
We have the following description of the $F$-matrices: Consider a semifield homomorphism
\[ \pi : \mathbb{Q}_{st}(y_1, \ldots, y_n) \longrightarrow \text{Trop}(y_1^{-1}, \ldots, y_n^{-1}) \]
\[ y_i \longmapsto y_i. \]

Then, we have
\[ (2.23) \]
\[ \pi(F^{i}_{t_{0}}(y_1, \ldots, y_n)) = F^{i}_{t_{0}}|_{\text{Trop}(y_1^{-1}, \ldots, y_n^{-1})}(y_1, \ldots, y_n) = \prod_{i=1}^{n}(y_i^{-1})^{-f^{i}_{t_{0}}} = \prod_{i=1}^{n}y_i^{-f^{i}_{t_{0}}}. \]

The $F$-matrices are uniquely determined by the following recurrence relations:

**Proposition 2.7.** Let $B$ be any initial exchange matrix at $t_0$. Then, the $F$-matrices have the following recurrence: The initial condition is
\[ (2.24) \]
\[ F^{t_{0}}_{t_{0}} = O, \]
and for any edge $t \xleftarrow{\ell} t'$ in $\mathbb{T}_n$, we have the recurrence relation,
\[ (2.25) \]
\[ F^{t_{0}}_{t'} = F^{i}_{t_{0}}J_{t} + \max([C^{i}_{t_{0}}]_{++} + F^{i}_{t_{0}}[B]_{++}, [-C^{i}_{t_{0}}]_{++} + F^{i}_{t_{0}}[-B]_{++}). \]

Proof. Clearly, (2.24) is obtained by (2.16). Moreover, applying (2.23) to (2.17), we have (2.25).

We call the recurrence relation (2.25) the front mutations for the $F$-matrices. As with the $C$-, $G$-matrices and the $F$-polynomials, we denote $F^{t_{0}}_{t_{0}} = \mu_{t}(F^{t_{0}}_{t_{0}})$.

Under the exchange of the initial exchange matrices $B$ and $-B$ at $t_0$, we have the simple relations between the $C$-, $G$- and $F$-matrices.

**Proposition 2.8.** We have the following relations:
\[ (2.26) \]
\[ C^{t_{0}}_{t} = C^{t_{0}}_{t} + F^{t_{0}}_{t}B_{t}, \]
\[ (2.27) \]
\[ G^{t_{0}}_{t} = G^{t_{0}}_{t} + BF^{t_{0}}_{t}, \]
\[ (2.28) \]
\[ F^{t_{0}}_{t} = F^{t_{0}}_{t}. \]

Proof. Let $\{\Sigma^{B}_{t} = (x_{t}, y_{t}, B_{t})\}_{t \in \mathbb{T}_n}$ be a cluster pattern with coefficients in any semifield $\mathbb{F}$ with the initial seed $(x, y, B)$. Also, let $\{\Sigma^{-B}_{t} = (x'_{t}, y'_{t}, B'_{t})\}_{t \in \mathbb{T}_n}$ be a cluster pattern with coefficients in $\mathbb{F}$ with the initial seed $(x, y^{-1}, -B)$. Then, by the definition of the mutation (2.5), (2.6) and (2.7), we have
\[ (2.29) \]
\[ x'_{t} = x_{t}, \quad y'_{t} = y_{t}^{-1}, \quad B'_{t} = -B_{t} \]
\[ ([FZ07], \text{Proof of Proposition 5.3})]. \]

We also note that for the initial seed $\Sigma^{-B}_{t_0} = (x, y^{'}, B^{'}) = (x, y^{-1}, -B)$, we have
\[ (2.30) \]
\[ y'_i := y'_i \prod_{j=1}^{n}x'_j^{b'_{ij}} = y_i^{-1} \prod_{j=1}^{n}x_j^{-b_{ij}} = y_i^{-1}. \]
Now we set $\mathbb{P} = \text{Trop}(y_1^{-1}, \ldots, y_n^{-1})$ and apply the separation formulas (2.18) and (2.19) to (2.29). Then, we obtain

\[(2.31)\]
\[
\left( \prod_{k=1}^{n} x_k^{-g_{kj;\ell}} \right) F_{j:t}^{-B;\ell} | x(y_1^{-1}, \ldots, y_n^{-1}) = \left( \prod_{k=1}^{n} x_k^{-g_{kj;\ell}} \right) F_{j:t}^{-B;\ell} | \text{Trop}(y_1^{-1}, \ldots, y_n^{-1})(y_1, \ldots, y_n),
\]

\[(2.32)\]
\[
\left( \prod_{k=1}^{n} (y_k^{-1}) c_{kij;\ell}^{-1} \right)^{-1} = \prod_{k=1}^{n} y_k c_{kij;\ell} \prod_{k=1}^{n} (F_{k:t}^{-B;\ell} | \text{Trop}(y_1^{-1}, \ldots, y_n^{-1})(y_1, \ldots, y_n))^{b_{kij;\ell}},
\]

by (2.21) and (2.22). Applying (2.23) to (2.32), we obtain

\[(2.33)\]
\[
c_{ij;\ell}^{-B;\ell} = c_{ij;\ell}^{B;\ell} + \sum_{i=1}^{n} b_{ij;\ell} f_{ij;\ell}^{B;\ell},
\]

thus we have (2.26). To show (2.27) from (2.31), let us set the $\mathbb{Z}^n$-gradings in $\mathbb{Z}[x_1^{-1}, \ldots, x_n^{-1}, y_1, \ldots, y_n]$ as follows ([FZ07 (6.1), (6.2)]):

\[(2.34)\]
\[
\deg(x_i) = e_i, \quad \deg(y_i) = -b_i,
\]

where $e_i$ is the $i$th column vector of $I_n$ and $b_i$ is the $i$th column vector of $B$. Then, we have

\[(2.35)\]
\[
\deg(y_i) = 0.
\]

Hence comparing the $\mathbb{Z}^n$-gradings of both sides of (2.31), we obtain

\[(2.36)\]
\[
g_{ij;\ell}^{-B;\ell} = g_{ij;\ell}^{B;\ell} + \sum_{i=1}^{n} f_{ij;\ell}^{B;\ell} b_i.
\]

Therefore, we have (2.27). Let us prove (2.28). Substituting $x_i = 1$ ($i = 1, \ldots, n$) for (2.31), we have

\[(2.37)\]
\[
F_{j:t}^{-B;\ell}(y_1^{-1}, \ldots, y_n^{-1}) = \frac{F_{j:t}^{B;\ell}(y_1, \ldots, y_n)}{F_{j:t}^{B;\ell} \mid \text{Trop}(y_1^{-1}, \ldots, y_n^{-1})(y_1, \ldots, y_n)}.
\]

([FZ07 (5.6)]). Under the exchange of $B$ and $-B$ in (2.37), we also have

\[(2.38)\]
\[
F_{j:t}^{B;\ell}(y_1, \ldots, y_n) = \frac{F_{j:t}^{-B;\ell}(y_1^{-1}, \ldots, y_n^{-1})}{F_{j:t}^{-B;\ell} \mid \text{Trop}(y_1^{-1}, \ldots, y_n^{-1})(y_1^{-1}, \ldots, y_n^{-1})}.
\]

Hence combining (2.37) with (2.38), we obtain

\[(2.39)\]
\[
F_{j:t}^{-B;\ell} \mid \text{Trop}(y_1, \ldots, y_n)(y_1^{-1}, \ldots, y_n^{-1}) F_{j:t}^{B;\ell} \mid \text{Trop}(y_1^{-1}, \ldots, y_n^{-1})(y_1, \ldots, y_n) = 1.
\]

Comparing exponents of $y_i$ of both sides of (2.39), we have

\[(2.40)\]
\[
-f_{j:t}^{-B;\ell} + f_{j:t}^{B;\ell} = 0,
\]

thus we obtain (2.28).

Thanks to the relation (2.26), we have the following alternating expression of the front mutations of the $F$-matrices:
Proposition 2.9. Let \( \varepsilon \in \{ \pm 1 \} \). For any edge \( t \xymatrix{ & t' } \) in \( \mathbb{T}_n \), the matrices \( F_{t_{B^{3}A}} \) and \( F_{t'_{B^{3}A}} \) are related by

\[
(2.41) \quad F_{t'_{B^{3}A}} = F_{t_{B^{3}A}}(J_t + [-\varepsilon B_t]^\ell) + [-\varepsilon C_t^{B^{3}A}]^\ell + \varepsilon C_{t_{B^{3}A}}^\ell.
\]

Proof. Firstly, we prove the case of \( \varepsilon = 1 \) in (2.41). By (2.4), (2.25) and (2.26), we have

\[
F_{t'_{B^{3}A}} = F_{t_{B^{3}A}}(J_t + [-B_t]^\ell) + \max([C_t^{B^{3}A}]^\ell + F_{t_{B^{3}A}} B_t^\ell, [-C_t^{B^{3}A}]^\ell)
\]

\[
= F_{t_{B^{3}A}}(J_t + [-B_t]^\ell) + \max([C_t^{B^{3}A}]^\ell + (C_t^{B^{3}A})^\ell - (C_t^{B^{3}A})^\ell, [-C_t^{B^{3}A}]^\ell)
\]

\[
= F_{t_{B^{3}A}}(J_t + [-B_t]^\ell) + \max([-C_t^{B^{3}A}]^\ell + (C_t^{B^{3}A})^\ell, [-C_t^{B^{3}A}]^\ell)
\]

\[
= F_{t_{B^{3}A}}(J_t + [-B_t]^\ell) + [-C_t^{B^{3}A}]^\ell + [C_t^{B^{3}A}]^\ell
\]

as desired. Secondly, we prove the case of \( \varepsilon = -1 \) in (2.41). By the same way as \( \varepsilon = 1 \), we have

\[
F_{t'_{B^{3}A}} = F_{t_{B^{3}A}}(J_t + [B_t]^\ell) + \max([-C_t^{B^{3}A}]^\ell + F_{t_{B^{3}A}} B_t^\ell, [-C_t^{B^{3}A}]^\ell)
\]

\[
= F_{t_{B^{3}A}}(J_t + [B_t]^\ell) + \max([-C_t^{B^{3}A}]^\ell + (C_t^{B^{3}A})^\ell - (C_t^{B^{3}A})^\ell, [-C_t^{B^{3}A}]^\ell)
\]

\[
= F_{t_{B^{3}A}}(J_t + [B_t]^\ell) + [-C_t^{B^{3}A}]^\ell + [C_t^{B^{3}A}]^\ell
\]

as desired. \( \square \)

2.3. Front mutations with sign-coherence of \( C \)-matrices. In this subsection, we reduce the front mutation formulas by applying the sign-coherence of the \( C \)-matrices.

Definition 2.10. Let \( A \) be an (not necessarily square) integer matrix. We say that \( A \) is \textit{column sign-coherent} (resp. \textit{low sign-coherent}) if for any column (resp. low) of \( A \), its entries are all nonnegative or nonpositive, and not all is 0.

When \( A \) is column sign-coherent (resp. low sign-coherent), we can define its \( \ell \)th \textit{column sign} \( \varepsilon_{\ast}(A) \) (resp. \textit{low sign} \( \varepsilon_{\ell}(A) \)) as the sign of nonzero entries of the \( \ell \)th column (resp. low) of \( A \). We have the following fundamental and nontrivial result:

Theorem 2.11 ([GHKK18 Corollary 5.5]). For any initial exchange matrix \( B \), every \( C \)-matrix \( C_t^{B_{3}A} \) in \( \mathbb{T}_n \) is column sign-coherent.

The column signs of the \( C \)-matrix \( C_t^{B_{3}A} \) are called the \textit{tropical signs} due to (2.22). By using them, the following reduced expression of the front mutations of the \( C \)- and \( G \)-matrices are obtained:

Proposition 2.12 ([NZ12 Proposition 1.3]). For any edge \( t \xymatrix{ & t' } \) in \( \mathbb{T}_n \), we have

\[
(2.42) \quad C_{t'_{B^{3}A}} = C_{t_{B^{3}A}}(J_t + [\varepsilon_{\ast}(C_t^{B_{3}A})B_t]^\ell),
\]

\[
(2.43) \quad G_{t'_{B^{3}A}} = G_{t_{B^{3}A}}(J_t + [-\varepsilon_{\ell}(C_t^{B_{3}A})B_t]^\ell).
\]
As a result, they are obtained from (2.12) and (2.14) by setting $\varepsilon = \varepsilon_* (C_t^{B,0})$ and $\varepsilon = -\varepsilon_* (C_t^{B,0})$, respectively.

The following fact is known:

**Proposition 3.13** ([FZ07, Conjecture 5.5, Proposition 5.6]). For any initial exchange matrix $B$, the following are equivalent:

(i) The sign-coherence of the $C$-matrices holds.

(ii) Every polynomial $F_{x,t}^{B,0}(y)$ has constant term 1.

(iii) Every polynomial $F_{x,t}^{B,0}(y)$ has a unique monomial of maximal degree. Furthermore, this monomial has coefficient 1, and it is divisible by all the other occurring monomials.

**Remark 2.14.** In the definition of the (column) sign-coherence in [FZ07], the nonzero vector property of column vectors are not assumed. However, this property can be easily recovered, since $\det C_t^{B,0} = \pm 1$ due to (2.42).

By Theorem 2.11 and Proposition 2.13, we have the following description of the $f$-vectors:

**Corollary 2.15.** The $f$-vector $f_{x,t}^{B,0}$ is the exponent vector of the unique monomial with maximal degree of $F_{x,t}^{B,0}(y)$. In other words, the unique monomial with maximal degree of $F_{x,t}^{B,0}(y)$ is given by $y_1^{f_{x,t}^{B,0}} \ldots y_n^{f_{x,t}^{B,0}}$.

Now, let us give the reduced expression of the front mutations of the $F$-matrices by using the tropical signs.

**Proposition 2.16.** For any edge $t \rightarrow t'$ in $T_n$, and $\varepsilon \in \{\pm 1\}$, we have

$$F_{y,t}^{B,0} = F_{x,t}^{B,0}(J_t + \left[ \varepsilon_* (C_t^{B,0}) \varepsilon B_t \right] t) + \left[ \varepsilon_* (C_t^{B,0}) C_t^{B,0} \right] t. \tag{2.44}$$

**Proof.** Substituting $\varepsilon = -\varepsilon_* (C_t^{B,0})$ or $\varepsilon = \varepsilon_* (C_t^{B,0})$ for (2.41), we obtain (2.44). \qed

## 3. Rear mutations

### 3.1. Rear mutations of functions $\mathcal{Y}$ and $\mathcal{X}$

We introduce the concept of the **rear mutations** which appears in [FZ07, NZ12, RS18] in the following way. Let $Q_{sf}(y)$ be the universal semifield with formal variables $y = (y_1, \ldots, y_n)$ in Section 2.1. Let $\{\Sigma_t\}_{t \in T_n}$ be the cluster pattern with coefficients in $Q_{sf}(y)$ where the initial coefficients $y_{i,t}$ are taken as the above formal variables $y$. Then, by recursively applying the mutations (2.6) or (2.8) from the initial coefficients, $y_{i,t}$ are written as a rational function of $y$:

$$y_{i,t} = \mathcal{Y}_{i,t}^{B,0}(y) \in Q_{sf}(y). \tag{3.1}$$

Similarly, by recursively applying the mutations (2.7), $x_{i,t} \in \mathcal{X}$ are written as a rational function of the initial cluster variables $x_{i,0} = x$ with coefficients in $Q(Q_{sf}(y))$:

$$x_{i,t} = \mathcal{X}_{i,t}^{B,0}(x) \in Q(Q_{sf}(y))(x). \tag{3.2}$$

Then, for any cluster pattern $\{\Sigma_t\}_{t \in T_n}$ with coefficients in $\mathbb{P}$, we recover $x_{i,t}$ and $y_{i,t}$ by the specialisation $\pi : Q_{sf}(y) \rightarrow \mathbb{P}$ with $y_i$ setting to be the initial coefficients of...
The matrix for any Definition 3.1 (\cite[(6.16)]{FZ07}) introduce the $F$-functions $Y_{i,t}^{B_{i,t}}(y)$ and $Y_{i,t}^{B_i;1t}(y)$ are related by
\begin{equation}
Y_{i,t}^{B_i;1t}(y) = \rho_k(Y_{i,t}^{B_i,t}(y)),
\end{equation}
where $\rho_k$ is the semifield automorphism of $\mathbb{Q}_{sf}(y)$ defined by
\begin{equation}
\rho_k : \mathbb{Q}_{sf}(y) \to \mathbb{Q}_{sf}(y)
\end{equation}
y_j \mapsto \begin{cases} y_j^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]^+} (y_k \oplus 1)^{-b_{kj}} & \text{otherwise.} \end{cases}
Similarly, the rational functions $X_{i,t}^{B_{i,t}}(x)$ and $X_{i,t}^{B_i;1t}(x)$ are related by
\begin{equation}
X_{i,t}^{B_i;1t}(x) = \rho_k(X_{i,t}^{B_i,t}(x)),
\end{equation}
where $\rho_k$ is the field automorphism of $\mathbb{Q}(\mathbb{Q}_{sf}(y))(x)$ defined by
\begin{equation}
\rho_k : \mathbb{Q}(\mathbb{Q}_{sf}(y))(x) \to \mathbb{Q}(\mathbb{Q}_{sf}(y))(x)
\end{equation}
y_j \mapsto \begin{cases} y_j^{-1} & \text{if } j = k, \\ y_j y_k^{[b_{kj}]^+} (y_k \oplus 1)^{-b_{kj}} & \text{otherwise.} \end{cases}
x_j \mapsto \begin{cases} y_j \prod_{i=1}^n x_i^{[b_{ki}]^+} + \prod_{i=1}^n x_i^{-[b_{ki}]^+} \\ (y_k \oplus 1)x_k & \text{if } j = k, \\ x_j & \text{otherwise.} \end{cases}
We call them the rear mutations of the functions $Y$ and $X$.

3.2. Rear mutations without sign-coherence of $C$-matrices. We use the notations in Section 3.1 continuously. By Proposition 24 we have
\begin{equation}
X_{j,t}^{B_{j,t}}(x) = \left(\prod_{k=1}^n x_k^{B_{k,t}}\right)^{\frac{F_{j,t}^{B_{j,t}}(\hat{y}_1, \ldots, \hat{y}_n)}{F_{j,t}^{B_{j,t}}(y_1, \ldots, y_n)}},
\end{equation}
\begin{equation}
Y_{j,t}^{B_{j,t}}(y) = \prod_{k=1}^n x_k^{B_{k,t}} \prod_{k=1}^n (F_{k,j}^{B_{j,t}}(y_1, \ldots, y_n))^{b_{kj}}.
\end{equation}
As with the front mutation, we will define the rear mutations in direction $k$ of the $C$-matrices (resp. the $G$-matrices, the $F$-polynomials, the $F$-matrices) as transformations from $C^{B_{j,t}}$ to $C^{B_i;1t}$ (resp. from $G^{B_{j,t}}$ to $G^{B_i;1t}$, from $F^{B_{j,t}}$ to $F^{B_i;1t}$, from $F^{B_{j,t}}$ to $F^{B_i;1t}$). We will deduce the rear mutations of the $C^-$, $G$-matrices, the $F$-polynomials and the $F$-matrices. In order to describe these rear mutations, we introduce the $H$-matrices according to \cite{FZ07}.

Definition 3.1 (\cite[(6.16)]{FZ07}). Let $B$ be any initial exchange matrix at $t_0$. Then, for any $t$, the $(i,j)$ entry of $H^{B_{j,t}} = (H^{B_{j,t}}_{ij})$ is given by
\begin{equation}
u^{B_{j,t}}_{ij} = F_{j,t}^{B_{j,t}} |_{\text{trop}(u)}(u^{-[b_{ji}]^+}, \ldots, u^{-1}, \ldots, u^{-[b_{ij}]^+})(u^{-1} \text{ in the } i\text{th position}).
\end{equation}
The matrix $H^{B_{j,t}}$ is called the $H$-matrix at $t$. 

\begin{align}
\Sigma_{t_0}. \text{ Let } t_1 \in \mathbb{T}_n \text{ be the vertex with } t_0 \overset{k}{\longrightarrow} t_1 \text{ and let } B_1 = \mu_k(B). \text{ Then, the rational functions } Y_{i,t}^{B_{i,t}}(y) \text{ and } Y_{i,t}^{B_i;1t}(y) \text{ are related by}
\end{align}
The following fact holds ([FZ07, Proof of Proposition 6.8]).

**Lemma 3.2.** We have the following equality:

\[
y_k^{B_{t_0}} = F_{j,t_0}^{B_{t_0}} |_{\text{Trop}(y_1', \ldots, y_n')}(y_1, \ldots, y_n),
\]

where \((y_1', \ldots, y_n')\) is coefficients at \(t_1\) connected with \(t_0\) by an edge labeled \(k\).

**Proof.** Consider the cluster pattern with coefficients in \(\text{Trop}(y_1', \ldots, y_n')\). Let \(y = (y_1, \ldots, y_n)\) be the coefficients at \(t_0\). Then, \(y\) and \(y'\) have the following relation:

\[
y_i = \begin{cases} y_k^{-1} & \text{if } i = k, \\ y'_k & \text{if } i \neq k. \end{cases}
\]

Therefore, for any \(j \in \{1, \ldots, n\}\), we have

\[
F_{j,t_0}^{B_{t_0}} |_{\text{Trop}(y_1', \ldots, y_n')}(y_1, \ldots, y_n) = F_{j,t_0}^{B_{t_0}} |_{\text{Trop}(y_1', \ldots, y_n')}(y'_1 y_k^{b_{k}^{+}}, \ldots, y'_n y_k^{b_{k}^{+}})
\]

where \((3.16)\) was given in [FZ07, Proposition 6.8]:

\[
F_{j,t_0}^{B_{t_0}} |_{\text{Trop}(y_1', \ldots, y_n')}(y_1, \ldots, y_n) = y_k^{h_{k,j,t}^{B_{t_0}}}. 
\]

\[\square\]

The rear mutations of the \(C\)- and \(G\)-matrices are given as follows, where the latter was given in [FZ07 Proposition 6.8]:

**Proposition 3.3.** Let \(t_0 \xrightarrow{k} t_1 \in \mathbb{T}_n, \mu_k(B) = B_1\) and \(\varepsilon \in \{\pm 1\}\). Then, for any \(t\), we have

\[
C_{t}^{B_{t_1}; t_1} = (J_k + [-\varepsilon B]_k^*) C_{t}^{B_{t_0}; t_0} + H_t(\varepsilon) k B_t, 
\]

\[
C_{t}^{B_{t_1}; t_1} = (J_k + [\varepsilon B]_k^*) G_{t}^{B_{t_0}; t_0} - B H_t(\varepsilon) k, 
\]

where \(H_t(+) = H_t^{B_{t_0}; t_0}, H_t(-) = H_t^{B_{t_1}; t_1}\).

**Proof.** We start with the proof of \((3.14)\). By [FZ07, Proposition 6.8], for any \(j \in \{1, \ldots, n\}\), we have

\[
g'_{ij,t} = \begin{cases} -g_{kj,t} & \text{if } i = k, \\ g_{ij,t} + [b_{ik}] + g_{kj,t} - b_{ik} h_{kj,t} & \text{if } i \neq k, \end{cases}
\]

and interchanging \(t_0\) and \(t_1\), we also have

\[
g'_{ij,t} = \begin{cases} -g_{kj,t} & \text{if } i = k, \\ g_{ij,t} + [b_{ik}] + g_{kj,t} - b_{ik} h'_{kj,t} & \text{if } i \neq k, \end{cases}
\]

where \(G_{t}^{B_{t_0}} = (g_{ij,t}), G_{t}^{B_{t_1}; t_1} = (g'_{ij,t}), H_{t_0}^{B_{t_1}; t_1} = (h_{ij,t})\) and \(H_{t_0}^{B_{t_1}; t_1} = (h'_{ij,t})\). Therefore, we get the desired equality. Let us show \((3.13)\). Consider the same cluster pattern as in the proof of Lemma 3.2. Then, applying \((2.19)\) and \((2.22)\) to any coefficient \(y_{j,t}\), we have

\[
\prod_{t=1}^{n} y_{ij,t}^{C_{ij,t}} = \prod_{t=1}^{n} y_{ij,t}^{C_{ij,t}} \prod_{t=1}^{n} F_{t_0}^{B_{t_0}} |_{\text{Trop}(y_1', \ldots, y_n')}(y_1, \ldots, y_n)^{b_{ij,t}}.
\]
where \( C_{B_{d_0}} = (c_{ij};t) \) and \( C_{B_{1;t_1}}^t = (c_{ij};t) \). Substituting \((3.12)\) for \((3.17)\) and using \((3.11)\), we have

\[
\prod_{i=1}^{n} y_i^{c_{ij};t} = \left( \prod_{i \neq k} y_i^{c_{ij};t} y_k^{-b_{ki}+c_{ij};t} \right) y_k^{-c_{kj};t} \prod_{i=1}^{n} y_i^{h_{kij};b_{ij};t}.
\]

Comparing exponents of \( y_i' \) of the both sides of \((3.18)\), we obtain

\[
c_{ij};t = \begin{cases} 
-c_{kj};t + \sum_{\ell=1}^{n} [-b_{k\ell}] + c_{ij};t + \sum_{\ell=1}^{n} h_{k\ell;0} b_{ij};t & \text{if } i = k, \\
c_{ij};t & \text{if } i \neq k.
\end{cases}
\]

Thus, we have \((3.13)\). \(\square\)

The rear mutations of the \( F \)-polynomials were given in [FZ07, (6.21)] as follows:

**Proposition 3.4** ([FZ07, (6.21)]). Let \( t_0 \xrightarrow{k} t_1 \) in \( T_n \) and \( \mu_k(B) = B_1 \). Then, for any \( j \in \{1, \ldots, n\} \) and \( t \in T_n \), the polynomials \( F^{B_{1,t};t_0}_{j;0}(y) \) and \( F^{B_{1,t};t_1}(y) \) are related by

\[
F^{B_{1,t};t_1}_{j;0}(y_1, \ldots, y_n) = (1 + y_k)^{B_{1,t};t_0} y_k^{-h_{kj};t_0} \times F^{B_{1,t};0}_{j;0}(y, y_1)^{-b_{1,1}+1} y_2, \ldots, y_{k-1}, \ldots, y_n y_k^{-b_{kn}+1} (y_k + 1)^{b_{kn}},
\]

where \( y_k^{-1} \) is in the \( k \)th position.

The rear mutation of the \( F \)-matrices are deduced from Proposition 3.4 as follows:

**Proposition 3.5.** Let \( t_0 \xrightarrow{k} t_1 \) in \( T_n \), \( \mu_k(B) = B_1 \) and \( \epsilon \in \{ \pm 1 \} \). Then for any \( t \), the matrices \( F^{B_{t};t_0}_{k} \) and \( F^{B_{1,t};t_1}_{k} \) are related by

\[
F^{B_{1,t};t_1}_{t_0} = (J_k + [\epsilon B]_{+}^{k_+}) F^{B_{t};0}_{t} + (\epsilon G_{t}^{B_{t};t_0} - H_{t}^{B_{t};t_0}(\epsilon))^{k_+} - H_{t}^{B_{t};t_0}(\epsilon),
\]

\[
F^{B_{1,t};t_1}_{t_0} = (J_k + [-\epsilon B]_{+}^{k_+}) F^{B_{t};0}_{t} + (\epsilon G_{t}^{B_{t};t_0} - H_{t}^{B_{t};t_0}(\epsilon))^{k_+} - H_{t}^{B_{t};t_0}(\epsilon).
\]
Proof. Let us show (3.22) in case of $\varepsilon = 1$. Substituting (3.21) for $y_i = y'_i$ and evaluating (3.21) at $\operatorname{Trop}(y'_1, \ldots, y'_n)$, we have

$$F_{j,t}^{B;t} \big|_{\operatorname{Trop}(y'_1, \ldots, y'_n)} (y'_1, \ldots, y'_n)$$

$$= (1 \oplus y'_k)^{g_{kJ}} y'_k - h_{kJ}^{g_{kJ}}$$

$$\times F_{j,t}^{B;0} \big|_{\operatorname{Trop}(y'_1, \ldots, y'_n)} (y'_1 y''_k + (y'_k + 1)^{bk_1}, \ldots, y'_n y''_k + (y'_k + 1)^{bk_n})$$

$$= \left(1 \oplus y'_k\right)^{g_{kJ}} y'_k - h_{kJ}^{g_{kJ}} 
\times F_{j,t}^{B;0} \big|_{\operatorname{Trop}(y'_1, \ldots, y'_n)} (y'_1 y''_k + (y'_k + 1)^{bk_1}, \ldots, y'_n y''_k + (y'_k + 1)^{bk_n})$$

$$= y'_k^{g_{kJ}} y'_k - h_{kJ}^{g_{kJ}} F_{j,t}^{B;0} \big|_{\operatorname{Trop}(y'_1, \ldots, y'_n)} (y'_1 y''_k + (y'_k + 1)^{bk_1}, \ldots, y'_n y''_k + (y'_k + 1)^{bk_n})$$

$$= y'_k^{g_{kJ}} y'_k - h_{kJ}^{g_{kJ}} F_{j,t}^{B;0} \big|_{\operatorname{Trop}(y'_1, \ldots, y'_n)} (y'_1 y''_k + (y'_k + 1)^{bk_1}, \ldots, y'_n y''_k + (y'_k + 1)^{bk_n})$$

$$= y'_k^{g_{kJ}} y'_k - h_{kJ}^{g_{kJ}} \prod_{i \neq k} \left(y'_{i} y''_k + (y''_k + f_{ij}) y'_k - f_{kj}\right).$$

Comparing the exponent of both sides, we have

$$f_{ij,t}^{B;0} = \begin{cases} g_{kJ}^{B;0} - h_{kJ}^{B;0} - h_{kJ}^{B;0} + \sum_{i=1}^{n} [b_{ki}] + f_{ij,t}^{B;0} - f_{ij,t}^{B;0} & \text{if } i = k, \\ f_{ij,t}^{B;0} & \text{if } i \neq k. \end{cases}$$

Hence we obtain the desired equality (3.22). Also replacing $B$ with $-B$ in (3.22) and applying (2.28) to it, we get (3.23).

### 3.3. Rear mutations with sign-coherence of $C$-matrices

In this subsection, we reduce the rear mutation formulas by applying the sign-coherence of the $C$-matrices. Let us introduce a duality between the $C$-matrices and the $G$-matrices, which is a result in [NZ12], and give the reduced form of the rear mutations of the $C$- and $G$-matrices.

Under the sign-coherence of the $C$-matrices (Theorem 2.11), we have the following result:

**Proposition 3.6.** (1) For any exchange matrix $B$ and $t_0, t \in \mathbb{T}_n$, we have

$$\left(G_t^{B;t_0}\right)^T = C_t^{B;t_0}.$$
Proof. Equalities (3.25) and (3.27) are the results in [NZ12, (1.13), (4.1)]. Also, (3.28) is obtained by combining (3.25) with [NZ12, (1.16), (2.7)]. Using (3.25), we have (3.26) by (2.14). We note that the $G$-matrices have the low sign-coherence by (3.25). By substituting $\varepsilon = \varepsilon_k(G_t^{B;t_0})$ for (3.27), we obtain (3.29). □

Through the duality (3.25), we can find out the dual equalities between the unreduced form of the front and rear mutations, (2.14) and (3.26), (2.12) and (3.27), respectively. Similarly, the reduced form of the front and rear mutations, (2.42) and (3.28), (2.43) and (3.28) are dual equalities, respectively.

Using Proposition 3.6, we prove the conjecture [FZ07, Conjecture 6.10], which is the relation between the $H$-matrices and the $G$-matrices as follows:

**Proposition 3.7.** [FZ07, Conjecture 6.10] For any $t \in \mathbb{T}_n$, we have the following relation:

\[
H_t^{B;t_0} = -[-G_t^{B;t_0}]_+.
\]

**Proof.** We assume the sign-coherence of the $C$-matrices. Then, comparing (3.14) with (3.27) and setting $\varepsilon = 1$, we get

\[
B[-G_t^{B;t_0}]_+ = -B(H_t^{B;t_0})^{k\bullet}.
\]

Since $k$ is arbitrary, we have

\[
B[-G_t^{B;t_0}]_+ = -BH_t^{B;t_0}.
\]

If $B$ have no zero column vector, then choosing $i$ which satisfies $b_{ik} \neq 0$, we have $[-g_{k;j}]_+ = -h_{k;j}$ and thus we have (3.30) as desired. We prove the case that $B$ have $m(\neq 0)$ zero column vectors. Permuting labels of $n$-regular tree $\mathbb{T}_n$, we can assume

\[
B = \begin{bmatrix}
B' & O \\
O & O
\end{bmatrix},
\]

where $B'$ is $(n - m) \times (n - m)$ matrix without zero column vector. Under this assumption, for $t_0 \overset{i_1}{\longrightarrow} \cdots \overset{i_s}{\longrightarrow} t$ in $\mathbb{T}_n$, we have

\[
\mu_{i_1} \cdots \mu_{i_k} \mu_{i_1}(G_t^{B;t_0})_{|n-m} = \begin{bmatrix}
\mu_{i'_1} \cdots \mu_{i'_k} \mu_{i'_1}(G_t^{B;t_0})_O \\
O
\end{bmatrix},
\]

\[
\mu_{i_1} \cdots \mu_{i_k} \mu_{i_1}(H_t^{B;t_0})_{|n-m} = \begin{bmatrix}
\mu_{i'_1} \cdots \mu_{i'_k} \mu_{i'_1}(H_t^{B;t_0})_O \\
O
\end{bmatrix},
\]

where $(i'_1, \ldots, i'_s)$ is a sequence which is obtained by removing $n - m + 1, \ldots, n$ from $(i_1, \ldots, i_s)$, and $|_{n-m}$ means taking the left $n \times (n - m)$ submatrix. Also about the other diagonal entries of $B$, we have the similar equalities. Thus we have

\[
G_t^{B;t_0} = G_t'^{B';t_0} \oplus G_{t_1}^{(0);t_0} \oplus \cdots \oplus G_{t_m}^{(0);t_0},
\]

\[
H_t^{B;t_0} = H_t'^{B';t_0} \oplus H_{t_1}^{(0);t_0} \oplus \cdots \oplus H_{t_m}^{(0);t_0},
\]

where $t'$ is a vertex of $\mathbb{T}_{m-n}$ which satisfies $\Sigma_{t'} = \mu_{i'_1} \cdots \mu_{i'_s}(\Sigma_{t_0})$ and

\[
t_j = \begin{cases}
t_0 & \text{if the number of } n - m + j \text{ in } (i_1, \ldots, i_s) \text{ is even}, \\
t'_0 & \text{if the number of } n - m + j \text{ in } (i_1, \ldots, i_s) \text{ is odd},
\end{cases}
\]
in $T_1$: $t_0 \xrightarrow{j} t'_0$ for any $j \in \{1, \ldots, m\}$. Explicitly, we have

\[
G_{t,t_0}^{B':t_0} = \begin{bmatrix}
G_{t,t_0}^{B':t_0} & 0 \\
(-1)^{N_1} & \cdots & (-1)^{N_m}
\end{bmatrix},
\]

\[
H_{t,t_0}^{B':t_0} = \begin{bmatrix}
H_{t,t_0}^{B':t_0} & 0 \\
-[(1)^{N_1}] & \cdots & -[(1)^{N_m}]
\end{bmatrix},
\]

where $N_j$ is the number of $n - m + j$ in $(i_1, \ldots, i_s)$. Since $B'$ has no zero column vectors and $[\begin{bmatrix}
B_{t,t_0}^{B':t_0} + O_N \end{bmatrix}]_+ = [\begin{bmatrix}
-G_{t,t_0}^{B':t_0} + O_N \end{bmatrix}]_+$ holds by (3.32), we have $[\begin{bmatrix}
-G_{t,t_0}^{B':t_0} + O_N \end{bmatrix}]_+ = [-H_{t,t_0}^{B':t_0}]_+$ for all $j$. Therefore, we have (3.30) as desired. □

We point out the following equivalence.

**Proposition 3.8.** The following statements are equivalent:

(i) The C-matrices have the column sign-coherence.

(ii) The G-matrices have the low sign-coherence, and equality (3.30) holds.

**Proof.** We proved (i)⇒(ii) by (3.25) and Proposition 3.7. We give the proof of (ii)⇒(i). To prove it, let us show (3.25). We note that we can not use (3.25) directly because we do not assume the sign-coherence of the C-matrices now. By assumption, we have $H_{t}^{(\epsilon)^{k}} = [-\epsilon G_{t,t_0}^{B':t_0}]_+$. Substituting it for (3.13) and (3.14), we have (3.26) and (3.27). Let us show (3.25) by the induction on the distance between $t$ and $t_0$.

If $(G_{t,t_0}^{B':t_0})^T = C_{t_0}^{B':t}$ holds for some $t \in T_n$, then for $t' \in T_n$ such that $t \xrightarrow{\ell} t'$, using (3.27) and (2.14), we have

\[
(C_{t,t_0}^{B':t'})^T = \left( J_{t'} + [-\epsilon B_{t'}]_{+} \right) C_{t_0}^{B':t} - [-\epsilon C_{t_0}^{B':t}]_{+} B^T
\]

Thus by the low sign-coherence of the G-matrices, we get the sign-coherence of the C-matrices immediately. □

By Theorem 2.11 and Proposition 3.8, the (low) sign-coherence the G-matrices and Proposition 3.7 hold. Furthermore, applying the sign-coherence of the C-matrices is equivalent to applying the sign-coherence of the G-matrices and (3.30). By using it, let us give a reduced expression of the rear mutations of F-matrices.
Proposition 3.9. (1) Let \( t_0 \overset{k}{\rightarrow} t_1 \) in \( \mathbb{T}_n \), \( \mu_k(B) = B_1 \) and \( \varepsilon \in \{\pm 1\} \). Then, we have
\[
(3.35) \quad F_{t_0}^{B_{t_1; t_1}} = (J_k + [\varepsilon B]_+^{[k]} \varepsilon) F_{t_0}^{B_{t_1; t_1}} + [\varepsilon G_{t_0}^{B_{t_1; t_1}}]_+ + [\varepsilon G_{t_0}^{B_{t_1; t_1}}]_+.
\]
(2) We have a reduced form of the rear mutations as follows:
\[
(3.36) \quad F_{t_0}^{B_{t_1; t_1}} = (J_k + [\varepsilon k(G_{t_0}^{B_{t_1; t_1}})(\varepsilon B)]_+^{[k]} \varepsilon) F_{t_0}^{B_{t_1; t_1}} + [\varepsilon k(G_{t_0}^{B_{t_1; t_1}})G_{t_0}^{B_{t_1; t_1}}]_+.
\]

Proof. (1) Thanks to Proposition 3.7, we can substitute \( H_{t_0}^{B_{t_1; t_1}}(\varepsilon)k = [-\varepsilon G_{t_0}^{B_{t_1; t_1}}]_+ \) and \( H_{t_0}^{B_{t_1; t_1}}(\varepsilon)k = [-\varepsilon G_{t_0}^{B_{t_1; t_1}}]_+ \) for \( H_{t_0}^{B_{t_1; t_1}}(\varepsilon)k = [-\varepsilon G_{t_0}^{B_{t_1; t_1}}]_+ \). Then, we have
\[
(3.37) \quad F_{t_0}^{B_{t_1; t_1}} = (J_k + [\varepsilon B]_+^{[k]} \varepsilon) F_{t_0}^{B_{t_1; t_1}} + [\varepsilon G_{t_0}^{B_{t_1; t_1}}]_+ + [\varepsilon G_{t_0}^{B_{t_1; t_1}}]_+ \]
as desired.
(2) Substituting \( \varepsilon = \varepsilon k(G_{t_0}^{B_{t_1; t_1}}) \) or \( \varepsilon = -\varepsilon k(G_{t_0}^{B_{t_1; t_1}}) \) for \( (3.35) \), we obtain \( (3.36) \).

Like the duality between the front and rear mutations of the \( C \)- and \( G \)-matrices, \( (2.41) \) and \( (3.35) \), \( (2.44) \) and \( (3.36) \) are dual equalities, respectively. We show the self-duality of the \( F \)-matrices, which is analogous to the duality \( (3.25) \) between the \( C \)- and \( G \)-matrices. This is the main theorem in this paper.

Theorem 3.10. For any exchange matrix \( B \) and \( t, t_0 \) in \( \mathbb{T}_n \), we have
\[
(3.37) \quad (F_{t_0}^{B_{t_1; t_1}})^T = F_{t_0}^{B_{t_1; t_1}}.
\]

Proof. We prove \( (3.37) \) by the induction on the distance between \( t \) and \( t_0 \) in \( \mathbb{T}_n \).
When \( t = t_0 \), we have \( (F_{t_0}^{B_{t_1; t_1}})^T = O = F_{t_0}^{B_{t_1; t_1}} \) as desired. We show that if \( (3.37) \) holds for some \( t \in \mathbb{T}_n \), then it also holds for \( t' \in \mathbb{T}_n \) such that \( t \overset{\ell}{\rightarrow} t' \). By the inductive assumption, \( (2.44), (3.9) \) and \( (3.25) \), we have
\[
(F_{t_0}^{B_{t_1; t_1}})^T = (J_k + [\varepsilon \ell(G_{t_0}^{B_{t_1; t_1}})^T]_+^{[k]} \varepsilon) F_{t_0}^{B_{t_1; t_1}} + [\varepsilon \ell(G_{t_0}^{B_{t_1; t_1}})G_{t_0}^{B_{t_1; t_1}}]_+^{[k]} \varepsilon
\]
as desired.

3.4 Examples. We introduce an example for the front and rear mutations in the case of \( A_2 \). Let \( n = 2 \), and consider a tree \( \mathbb{T}_2 \) whose edges are labeled as follows:
\[
\cdots \overset{1}{\rightarrow} t_0 \overset{2}{\rightarrow} t_1 \overset{1}{\rightarrow} t_2 \overset{2}{\rightarrow} t_3 \overset{1}{\rightarrow} t_4 \overset{2}{\rightarrow} t_5 \overset{1}{\rightarrow} \cdots
\]
We set \( B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) as the initial exchange matrix at \( t_0 \). Then, the coefficients, the cluster variables, the \( C \)-, \( G \)- and \( F \)-matrices are given by Table 1 and Table 2 [FZ07, Example 2.10].
We show the expressions of the coefficients and the cluster variables at \( t_0 \) in Type \( A_2 \) in Table 3, and its counterpart the \( C-, G- \) and \( F- \)matrices in Table 4.
DUALITY BETWEEN FRONT AND REAR MUTATIONS IN CLUSTER ALGEBRAS

Table 3. Expressions of coefficients and cluster variables at $t_0$ in type $A_2$

| $t$ | $Y_{B}^{t_{0}:t}$ | $X_{B}^{t_{0}:t}$ |
|-----|------------------|------------------|
| 0   | $y_1$  $y_2$     | $x_1$  $x_2$     |
| 1   | $y_1(y_2 \oplus 1)$  $1/y_2$ | $x_1$  $y_2x_1 + 1/(y_2 \oplus 1)x_2$ |
| 2   | $y_1y_2 \oplus y_2 \oplus 1/y_1$  $1/y_2(y_1 \oplus 1)$ | $y_1x_2 + 1/(y_1 \oplus 1)x_1$  $y_1y_2x_2 + y_2 + x_1/(y_1y_2 \oplus y_2 \oplus 1)x_1x_2$ |
| 3   | $y_1 \oplus 1/y_1y_2$  $y_2/y_1y_2 \oplus y_1 \oplus 1$ | $(y_1y_2 \oplus y_1 \oplus 1)x_1x_2$  $y_1 + x_2/(y_1 \oplus 1)x_1$ |
| 4   | $1/y_2$  $y_1y_2/y_2 \oplus 1$ | $y_2 + x_1/(y_2 \oplus 1)x_2$  $x_1$ |
| 5   | $y_2$  $y_1$     | $x_2$  $x_1$     |

Table 4. $C$-, $G$- and $F$-matrices in type $A_2$ (moving the initial vertex)

| $t$ | $C_{B}^{t_{0}:t}$ | $G_{B}^{t_{0}:t}$ | $F_{B}^{t_{0}:t}$ |
|-----|------------------|------------------|------------------|
| 0   | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ |
| 1   | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ |
| 2   | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ |
| 3   | $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ |
| 4   | $\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ |
| 5   | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ |

Comparing Table 2 with Table 4, we can see the duality of the $C$-, $G$- and $F$-matrices in (3.25) and (3.37).
4. Properties on Principal Extension of Exchange Matrices

In this section, we present some properties of the principal extension of the exchange matrices. These properties yield alternative derivations of some equalities in previous sections. Also, we believe that they are useful and important in cluster algebra theory.

4.1. Principal extension. For an exchange matrix $B_t$, $\tilde{B}_t = \left[ \begin{array}{c} B_t \\ C_t^{B,t_0} \end{array} \right]$ ($t \in T_n$) is the extended exchange matrix \cite{FZ02} Section 5]. We fix the initial vertex $t_0 \in T_{2n}$. Let us regard $T_n$ as subtree of $T_{2n}$ consisting of vertices which are reachable from $t_0$ via edges labeled by $1, \ldots, n$ and edges which connect these vertices. Let us consider the following “full extension of $B$”. We set

$$\mathcal{B} := \left[ \begin{array}{cc} B & -I_n \\ I_n & O \end{array} \right].$$

Note that $\mathcal{B}$ is regular. Then, we obtain a family of $2n \times 2n$ skew-symmetrizable matrices $\{\mathcal{B}_t\}_{t \in T_{2n}}$ such that $\mathcal{B} = \mathcal{B}_{t_0}$ ($t_0 \in T_n \subset T_{2n}$) and they are related by the mutation (2.5). Then, the left half of $\mathcal{B}_t$ is $\tilde{B}_t$ for $t \in T_n$, and also $\mathcal{B}_t$ is regular. We call $\{\mathcal{B}_t\}_{t \in T_{2n}}$ the principal extension of $\{B_t\}_{t \in T_n}$. The following proposition gives the explicit expression of $\mathcal{B}_t$ ($t \in T_n$):

Proposition 4.1. Let $\{\Sigma_t = (x_t, y_t, B_t)\}_{t \in T_n}$ be a cluster pattern with the initial vertex $t_0$. For any mutation series $\mu_{i_m} \mu_{i_{m-1}} \cdots \mu_{i_1}$ ($i_1, \ldots, i_m \in \{1, \ldots, n\}$) of $T_n$, we set $t_1, \ldots, t_{m-1}, t_m, t$ as $t_0 \overset{i_1}{\longrightarrow} t_1 \overset{i_2}{\longrightarrow} \cdots \overset{i_{m-1}}{\longrightarrow} t_{m-1} \overset{i_m}{\longrightarrow} t_m = t$ in $T_n$. Then, the following equality holds:

$$\mathcal{B}_t = \left[ \begin{array}{c} B_t \\ C_t^{B,t_0} \end{array} \right] \sum_{s=0}^{m-1} \left( C_{t_s}^{B,t_0} [-D^{-1}(C_{t_s}^{B,t_0})^T D \left[ -D^{-1}(C_{t_s}^{B,t_0})^T D \right]_{s+1}^{i_s+1} - [-C_{t_s}^{B,t_0}]_{s+1}^{i_{s}+1} D^{-1}(C_{t_s}^{B,t_0})^T D \right) \right],$$

where $D$ is a skew-symmetrizer of $B$. In particular, applying the sign-coherence of the $C$-matrices to (4.2), we have

$$\mathcal{B}_t = \left[ \begin{array}{c} B_t \\ C_t^{B,t_0} \end{array} \right] \left[ -D^{-1}(C_{t}^{B,t_0})^T D \right].$$

Proof. We prove (4.2) by the induction on the distance between $t$ and $t_0$ in $T_n$. When $m = 1$, we have (4.2) by direct calculation. It remains to show that if (4.2) holds for some $t \in T_n$, then it also holds for $t' \in T_n$ such that $t \overset{\ell}{\longrightarrow} t'$. The left $2n \times n$ submatrix of $\mathcal{B}_t$ regards as the extended matrix $\tilde{B}_t$ in \cite{FZ02} Section 5]. Therefore, it suffices to prove the right $2n \times n$ submatrix of $\mathcal{B}_t$. For any $1 \leq i \leq n$
and \( n + 1 \leq j \leq 2n \), we have

\[
\begin{align*}
    b_{ij,t'} &= \begin{cases} 
        -b_{ij,t} + b_{i\ell,t} \{ b_{\ell j,t} \} \hspace{1em} & \text{if } i = \ell, \\
        b_{ij,t} + b_{i\ell,t} \{ b_{\ell j,t} \} + [-b_{\ell \ell,t}] + b_{ij,t} & \text{otherwise,}
    \end{cases}
    \\
    &= \begin{cases} 
        d^{-1}_i c_{j-n,i,\ell,t} d_j & \text{if } i = \ell, \\
        -d^{-1}_i c_{j-n,i,\ell,t} d_j + b_{i\ell,t} [-d^{-1}_\ell c_{j-n,\ell,\ell,t} d_j] + [-b_{\ell \ell,t}] + d^{-1}_\ell c_{j-n,\ell,\ell,t} d_j & \text{otherwise,}
    \end{cases}
    \\
    &= \begin{cases} 
        d^{-1}_i c_{j-n,i,\ell,t} d_j & \text{if } i = \ell, \\
        -d^{-1}_i c_{j-n,i,\ell,t} d_j + b_{i\ell,t} [-c_{j-n,\ell,\ell,t} + c_{j-n,\ell,\ell,t}] d_j & \text{otherwise,}
    \end{cases}
    \\
    &= -d^{-1}_i c_{j-n,i,\ell,t} d_j.
\end{align*}
\]

Since

\[
(C_t^{B_{t:0}} - D^{-1}(C_t^{B_{t:0}})^T D)^{t_{m+1}} - [C_t^{B_{t:0}}]^{t_{m+1}} D^{-1}(C_t^{B_{t:0}})^T D)_{1-n,j-n} = c_{i-n,\ell,t} [-d^{-1}_\ell c_{j-n,\ell,\ell,t} d_j] + [-c_{i-n,\ell,t}] + d^{-1}_\ell c_{j-n,\ell,\ell,t} d_j
\]

holds by the inductive assumption, for any \( n + 1 \leq i, j \leq 2n \), we have

\[
(4.4) \quad b_{ij,t'} = b_{ij,t} + b_{i\ell,t} \{ b_{\ell j,t} \} + [-b_{\ell \ell,t}] + b_{ij,t}
\]

\[
= \sum_{s=0}^{m-1} \left( c_{i-n,i_{s+1};t_{s}} [-d^{-1}_{i_{s+1}} c_{j-n,i_{s+1};t_{s}} d_j] + [-c_{i-n,i_{s+1};t_{s}}] + d^{-1}_{i_{s+1}} c_{j-n,i_{s+1};t_{s}} d_j \right)
\]

\[
+ c_{i-n,\ell,t} [-d^{-1}_\ell c_{j-n,\ell,\ell,t} d_j] + [-c_{i-n,\ell,t}] + d^{-1}_\ell c_{j-n,\ell,\ell,t} d_j
\]

\[
= \sum_{s=0}^{m} \left( c_{i-n,i_{s+1};t_{s}} [-d^{-1}_{i_{s+1}} c_{j-n,i_{s+1};t_{s}} d_j] + [-c_{i-n,i_{s+1};t_{s}}] + d^{-1}_{i_{s+1}} c_{j-n,i_{s+1};t_{s}} d_j \right)
\]

\[
= \sum_{s=0}^{m} (C_t^{B_{t:0}} - D^{-1}(C_t^{B_{t:0}})^T D)^{t_{m+1}} - [C_t^{B_{t:0}}]^{t_{m+1}} D^{-1}(C_t^{B_{t:0}})^T D)_{1-n,j-n},
\]

where \( t_m = t \) and \( i_{m+1} = \ell \). Thus we have (4.2) and also have (4.3) by (4.4) and the sign-coherence of the \( C \)-matrices immediately. \( \square \)

Let us study the \( C \)-, \( G \)- and \( F \)-matrices for the initial matrix \( \overline{B} \) in (4.1), especially for the subtree \( T_n \) in \( T_{2n} \).

**Theorem 4.2.** Let \( \{ \Sigma_t = (x_t, y_t, B_t) \}_{t \in T_n} \) be a cluster pattern with the initial vertex \( t_0 \). For any mutation series \( \mu_1, \mu_{m-1} \ldots \mu_i \ldots \mu_i \) \( i_1, \ldots, i_m \in \{1, \ldots, n\} \) of \( T_n \), we set \( t_1, \ldots, t_{m-1}, t_m, t \) as \( t_0 \xrightarrow{i_1} t_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{m-1}} t_{m-1} \xrightarrow{i_m} t_m = t \) in \( T_n \). Then, the
following equalities hold:

\[(4.5)\]

\[
C_t^{B_{t_0}} = \left[ C_t^{B_{t_0}} \sum_{s=0}^{m-1} \left( C_t^{B_{t_0}} [D^{-1}(C_t^{B_{t_0}})^T D]_{s+1}^+ \right) - \left[ C_t^{B_{t_0}} \right]_+^{s+1} D^{-1}(C_t^{B_{t_0}})^T D \right],
\]

\[(4.6)\]

\[
G_t^{B_{t_0}} = \left[ G_t^{B_{t_0}} \begin{bmatrix} O & I_n \end{bmatrix},
\right.
\]

\[(4.7)\]

\[
F_{\ell t}^{B_{t_0}}(y) = \begin{cases} 
F_{\ell t}^{B_{t_0}}(y) & \text{if } \ell \in \{1, \ldots, n\}, \\
1 & \text{if } \ell \in \{n + 1, \ldots, 2n\}.
\end{cases}
\]

In particular, applying the sign-coherence of the $C$-matrices to (4.5), we have

\[(4.8)\]

\[
C_t^{B_{t_0}} = \left[ C_t^{B_{t_0}} \begin{bmatrix} O & I_n \end{bmatrix}
\right.
\]

Proof. Firstly, we prove (4.5) by the induction on the distance between $t$ and $t_0$ in $T_n$. When $m = 1$, we have (4.5) by direct calculation. It remains to show that if (4.5) holds for some $t \in T_n$, then it also holds for $t' \in T_n$ such that $t - t'$. By (2.12), we have

\[
C_t^{B_{t_0}}(J_t + [-B_t]_+^{(\star)}) + \left[ C_t^{B_{t_0}} \right]_+^{s} B_t
\]

\[
= \left[ C_t^{B_{t_0}} \begin{bmatrix} O & X \end{bmatrix} \begin{bmatrix} J_t & O \\
0 & I_n \end{bmatrix} + \begin{bmatrix} [-B_t]_+^{(\star)} & \left[ D^{-1}(C_t^{B_{t_0}})^T D \right]_+^{(\star)} \end{bmatrix} \right]
\]

\[
+ \begin{bmatrix} C_t^{B_{t_0}} \right]_+ [O & \begin{bmatrix} B_t & D^{-1}(C_t^{B_{t_0}})^T D \end{bmatrix} Y
\end{bmatrix}
\]

\[
= \left[ C_t^{B_{t_0}} \begin{bmatrix} J_t & X \end{bmatrix} + \begin{bmatrix} [-B_t]_+^{(\star)} \end{bmatrix} + \begin{bmatrix} C_t^{B_{t_0}} \right]_+ B_t Z
\end{bmatrix}
\]

\[
= \left[ C_t^{B_{t_0}} \begin{bmatrix} O & I_n \end{bmatrix}
\right.
\]

where

\[
X = \sum_{s=0}^{m-1} (C_t^{B_{t_0}} [D^{-1}(C_t^{B_{t_0}})^T D]_{s+1}^+ \right) - \left[ C_t^{B_{t_0}} \right]_+^{s+1} D^{-1}(C_t^{B_{t_0}})^T D),
\]

\[
Y = \sum_{s=0}^{m-1} (C_t^{B_{t_0}} [-D^{-1}(C_t^{B_{t_0}})^T D]_{s+1}^+ \right) - \left[ C_t^{B_{t_0}} \right]_+^{s+1} D^{-1}(C_t^{B_{t_0}})^T D),
\]

\[
Z = X + C_t^{B_{t_0}} [D^{-1}(C_t^{B_{t_0}})^T D]_+^{(\star)} - \left[ C_t^{B_{t_0}} \right]_+^{(\star)} D^{-1}(C_t^{B_{t_0}})^T D).
\]

Calculating $Z$, we have

\[
Z = \sum_{s=0}^{m} (C_t^{B_{t_0}} [D^{-1}(C_t^{B_{t_0}})^T D]_{s+1}^+ \right) - \left[ C_t^{B_{t_0}} \right]_+^{s+1} D^{-1}(C_t^{B_{t_0}})^T D).
\]
where $t_m = t$ and $i_m + \ell$. Thus we have (4.5). Furthermore, since the $(i,j)$ entry of
\[
C_t^{B:0} [D^{-1}(C_t^{B:0})^T D]^{t_+} - [C_t^{B:0}]^{t_+} D^{-1}(C_t^{B:0})^T D
\]
is
\[
d_t^{-1}(c_{i:t}c_{j:t}) - [c_{i:t}]c_{j:t}d_{ij},
\]
we have $Z = O$ by applying the sign-coherence of the $C$-matrices. Hence we have (4.8). Secondly, we prove (4.6) by the same way as (4.5). When $m = 1$, we have (4.6) by direct calculation. By the inductive assumption, we have
\[
G_t^{B:0} = G_t^{B:0}(J_t + [B_t]^{t_+}) - B[C_t^{B:0}]^{t_+}
\]
\[
= G_t^{B:0}(J_t + [B_t]^{t_+}) - B(C_t^{B:0})^{t_+} O
\]
\[
= (G_t^{B:0}(J_t + [B_t]^{t_+}) - B(C_t^{B:0})^{t_+} O
\]
as desired. Finally, we prove (4.7) by the same way as (4.5) or (4.6). When $m = 1$, we have (4.7) by direct calculation. Let $C_t^{B:0} = (c_{ij:t})$ and $B_t = (b_{ij:t})$, and we abbreviate $F_t^{B:0} = F_{i:t}$ and $F_t^{B:0} = F_{i:t}$. By the inductive assumption and Proposition 4.1 and (4.5), we have
\[
F_{i:t'} = F_{i:t} = F_{i:t'} \quad \text{if } i \neq \ell,
\]
\[
F_{i:t'} = \prod_{j=1}^{2n} y_j^{e_{ij:t}} \prod_{j=1}^{2n} F_{i:t}^{e_{ij:t}} + \prod_{j=1}^{2n} y_j^{-e_{ij:t}} \prod_{j=1}^{2n} F_{i:t}^{-e_{ij:t}}
\]
\[
F_{i:t'} = \prod_{j=1}^{n} y_j^{e_{ij:t}} \prod_{j=1}^{n} F_{i:t}^{e_{ij:t}} + \prod_{j=1}^{n} y_j^{-e_{ij:t}} \prod_{j=1}^{n} F_{i:t}^{-e_{ij:t}}
\]
\[
= F_{i:t'}
\]
as desired. □

Remark 4.3. The equality (4.8) is the specialisation of [CL] (2). Indeed, setting
\[
m = n \quad \text{and substituting } B, I_n, -I_n \quad \text{and } O \quad \text{for } B_1, B_2, B_3 \quad \text{and } B_4 \quad \text{in } \tilde{B} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \\ I_n & 0 \\ 0 & I_m \end{pmatrix}
\]
appearing in [CL] Proof of Theorem 4.5, we have (4.8) by [CL] (2).

Remark 4.4. We did not use the sign-coherence of the $C$-matrices when we show (4.6) and (4.7). Furthermore, (4.7) always holds when $B$ is a $2n \times 2n$ matrix whose the upper left $n \times n$ submatrix is $B$. 
By Theorem 4.2 and Remark 4.4 we have the following corollary:

**Corollary 4.5.** Let \( \{ \Sigma_t = (x_t, y_t, B_t) \}_{t \in T_n} \) be a cluster pattern with the initial vertex \( t_0 \). Then, for any \( t \in T_n \), the following equalities hold:

\[
F_t^{B_{t_0}} = \begin{bmatrix}
F_t^{B_{t_0}} & O \\
O & O
\end{bmatrix},
\]

\[
H_t^{B_{t_0}} = \begin{bmatrix}
H_t^{B_{t_0}} & O \\
O & O
\end{bmatrix}.
\]

**Proof.** We have (4.9) immediately by (4.7). Except for the lower left \( n \times n \) submatrix, (4.10) is obtained from the definition of the \( H \)-matrices and (4.7). For \( i \in \{ n + 1, \ldots, 2n \} \) and \( j \in \{ 1, \ldots, n \} \), we have

\[
|_{T_{\text{top}}(u)}(u^{[-b_{1,1}]+}, \ldots, u^{-1}, \ldots, u^{[-b_{n,2n}]+}) (u^{-1} \text{ in the } i\text{th position})
\]

Hence we obtain \( h_{ij:t} = 0 \). \( \square \)

### 4.2. Alternative derivations of Propositions 2.9, 3.5, 3.7 and Theorem 3.10

By using the principal extension, we give alternative derivations of Proposition 2.9, Proposition 3.5, Proposition 3.7 and Theorem 3.10.

**Alternative derivation of Proposition 2.9.** For any edge \( t \xrightarrow{\ell} t' \) in \( T_n \), by using (2.14) and (2.27), \( G_{t'}^{B_{t_0}} \) is indicated two ways shown by the following diagram:

Applying (2.27) to \( G_t^{B_{t_0}} \) before applying the front mutation, we have

\[
G_t^{B_{t_0}} = G_{t'}^{B_{t_0}} + BF_{t}^{B_{t_0}}
\]

\[
= G_{t'}^{B_{t_0}}(J_t + [\varepsilon B_t]_{\ell}^+) - B[\varepsilon C_t^{B_{t_0}}]_{\ell}^+ + BF_{t}^{B_{t_0}}.
\]

On the other hand, applying the front mutation to \( G_t^{B_{t_0}} \) before applying (2.27), we have

\[
G_{t'}^{B_{t_0}} = G_t^{B_{t_0}}(-B)_{\ell}^+ + BF_{t'}^{B_{t_0}}(J_t + [-\varepsilon B_t]_{\ell}^+) + B[\varepsilon C_t^{B_{t_0}}]_{\ell}^+.
\]

Comparing these two expressions, we obtain

\[
BF_{t'}^{B_{t_0}} = G_t^{B_{t_0}}(-B)_{\ell}^+ + BF_{t'}^{B_{t_0}}(J_t + [-\varepsilon B_t]_{\ell}^+) + B[\varepsilon C_t^{B_{t_0}}]_{\ell}^+.
\]

Hence if \( B \) is invertible, we have

\[
F_{t'}^{B_{t_0}} = B^{-1}G_t^{B_{t_0}}(-B)_{\ell}^+ + F_{t'}^{B_{t_0}}(J_t + [-\varepsilon B_t]_{\ell}^+) + [\varepsilon C_t^{B_{t_0}}]_{\ell}^+.
\]
Since $B^{-1}G_t^{B;4} = C_t^{B;4}B_t^{-1}$, the first term of the right hand side of (4.12) is $(-\varepsilon C_t^{B;4})^\bullet$. Moreover, we note that we have

$(-\varepsilon C_t^{B;4})^\bullet + [\varepsilon C_t^{B;4}]_+^\bullet = [-\varepsilon C_t^{B;4}]_+^\bullet$.

Thus we have (2.41) as desired. When $B$ is not invertible, substituting $\overline{B}$ for $B$ in (4.11), we have the equality which is substituted $\overline{B}$ for $B$ in (4.12). Thanks to (4.5) and (4.9), comparing the upper left $n \times n$ submatrix of it, we have (2.41) as desired.

Alternative derivation of Proposition 3.7. We prove (3.22). By using (3.13) and (2.26), $C_t^{-B;1}t_1$ is indicated two ways shown by the following diagram:

$$
\begin{array}{c}
C_t^{B;0} \xrightarrow{(2.26)} C_t^{-B;0} \\
\text{rear mutation (3.13)} \\
\end{array}
\begin{array}{c}
C_t^{B;1}t_1 \xrightarrow{(2.26)} C_t^{-B;1}t_1. \\
\text{rear mutation (3.13)} \\
\end{array}
$$

Applying (2.26) to $C_t^{B;0}$ before applying the rear mutation, we have

$$
C_t^{-B;1}t_1 = (J_k + [-\varepsilon(-B)]_+^\bullet)C_t^{-B;0} + H_t^{-B;0}(\varepsilon)^k \cdot (-B_t) = (J_k + [\varepsilon B]^k_+)C_t^{B;1} + F_t^{B;1}B_t - H_t^{-B;0}(\varepsilon)^k \cdot B_t.
$$

On the other hand, applying the rear mutation to $C_t^{B;0}$ before applying (2.26), we have

$$
C_t^{-B;1}t_1 = C_t^{B;1} + F_t^{B;1}B_t = (J_k + [-\varepsilon B]^k_+)C_t^{B;1} + H_t^{B;0}(\varepsilon)^k \cdot B_t + F_t^{B;1}B_t.
$$

Comparing these two expressions, we have

$$
F_t^{B;1}B_t = (\varepsilon B)^k \cdot C_t^{B;0}B_t^{-1} + (J_k + [\varepsilon B]^k_+)F_t^{B;1}B_t - H_t^{-B;1}B_t - H_t^{B;0}(\varepsilon)^k \cdot B_t - H_t^{B;0}(\varepsilon)^k \cdot B_t.
$$

When $B$ is invertible, all $B_t$ also is. Therefore, we have

$$
F_t^{B;1}B_t = (\varepsilon B)^k \cdot C_t^{B;0}B_t^{-1} + (J_k + [\varepsilon B]^k_+)F_t^{B;1}B_t - H_t^{-B;1}B_t - H_t^{B;0}(\varepsilon)^k \cdot B_t.
$$

By (2.15), $BC_t^{B;0} = G_t^{B;0}B_t$ for any $t$, the first term of right hand side of (4.14) is $(\varepsilon G_t^{B;1})^k$. Hence we have desired equality. On the other hand, when $B$ is not invertible, we consider replacing $B$ with $\overline{B}$ in (4.14). Thanks to (4.6) and Corollary 4.5, we have (3.22).

Alternative derivation of Proposition 3.7. We get (3.31) by the same way as a previous proof. If $B$ is invertible, then we have $[-G_t^{B;4}]_+^\bullet = (H_t^{B;4})^k$. Applying this equality to all $k = 1, \ldots, n$, we have $H_t^{B;0} = [-G_t^{B;4}]_+^\bullet$. If $B$ is not invertible, replacing $B$ with $\overline{B}$ in (3.31), we have $[-G_t^{B;4}]_+^\bullet = (H_t^{B;4})^k$. Comparing the upper left $n \times n$ submatrices of both sides of it, we obtain $[-G_t^{B;4}]_+^\bullet = (H_t^{B;4})^k$. □
Alternative derivation of Theorem 3.10. By the equalities which substitute $B^T_t$, $t_0$ and $t$ for $B$, $t_0$ and $t$ respectively in (2.26), (2.27) and (3.25), we have

$$B(F^{B,t_0}_t)^T = BF^{B,T,t}_t.$$  

When $B$ is invertible, we get (3.37) immediately. We prove the case that $B$ is not invertible. Replacing $B$ with $B$ in (4.15), we have

$$F^{B,t_0}_t = F^{B,T,t}_t$$

because $B$ is invertible. Since we have

$$(F^{B,t_0}_t)^T = \begin{bmatrix} (F^{B,t_0}_t)^T & O \\ O & O \end{bmatrix}$$

and

$$F^{B,T,t}_t = F^{B,T,t}_t = \begin{bmatrix} F^{B,T,t}_t & O \\ O & O \end{bmatrix}$$

by (4.7) and Remark 4.4, we have (3.37). \hfill $\square$

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