Defect Melting of Vortices in High-T_c Superconductors

Jürgen Dietel and Hagen Kleinert
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
(Dated: Received March 23, 2022)

We set up a melting model for vortex lattices in high-temperature superconductors based on the continuum elasticity theory of the vortex lattice. The model is Gaussian and includes defect fluctuations by means of a discrete-valued vortex gauge field. We derive the melting temperature of the lattice and predict the size of the Lindemann number. Our result agrees well with experiments on YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{7-δ}, but not on Bi\textsubscript{2}Sr\textsubscript{2}CaCu\textsubscript{2}O\textsubscript{6}. We also calculate the jumps in the entropy and the magnetic induction at the melting transition.

PACS numbers: 74.60.Ec, 74.60.Ge

Flux line lattice melting, proposed by Nelson in 1988 [1], has become an important topic in the phenomenology of high-temperature superconductors. The properties of this transition have been studied in a large number of theoretical as well as experimental papers. The simplest method to estimate the temperature where the transition takes place is adapted from the famous Lindemann criterion of three dimensional ordinary crystals [2]. The adaption by Houghton [3] to vortex lattices states that the vortex lattice undergoes a melting transition once the mean thermal displacement \( \langle u^2 \rangle^{1/2} \) becomes a certain fraction of the lattice spacing \( a \approx (\Phi_0/B)^{1/2} \) where \( \Phi_0 \) is the flux quantum, \( B \) the induction field. The size of the Lindemann number \( c_L = \langle u^2 \rangle^{1/2}/a \) is not predicted by Lindemann's criterion and must be extracted from experiments, and is usually found in the range \( c_L \approx 0.1-0.3 \). It has been shown in Ref. [2] that \( c_L \) is, in fact, roughly universal, with only a weak dependence on the induction of the system.

The most prominent examples of high-temperature superconductors exhibiting vortex lattice melting are the anisotropic compound YBa\textsubscript{2}Cu\textsubscript{3}O\textsubscript{7-δ} (YBCO), and the strongly layered compound Bi\textsubscript{2}Sr\textsubscript{2}CaCu\textsubscript{2}O\textsubscript{6} (BSCCO). Decoration experiments on BSCCO [4] show the formation of a triangular vortex lattice, on YBCO of a tilted square lattice of vortices close to the melting region, the latter being favored by the d-wave symmetry of the order parameter [5]. An explicit calculation of the Lindemann number \( c_L = \langle u^2 \rangle^{1/2}/a \) for YBCO can be found in Ref. [6] and for BSCCO in Ref. [7].

In this letter, we calculate the size of the Lindemann number \( c_L \) from a Gaussian model which takes into account the lattice elasticity and the defect degrees of freedom in the simplest possible way. The relevance of defect fluctuations to vortex melting has also been emphasized by Blatter [2]. For ordinary crystals, this is meanwhile textbook material: in Ref. [8], one of us has set up simple Gaussian lattice models based on linear elasticity and a fluctuating discrete-valued defect gauge field for square lattice crystals which clearly display melting transitions. These are usually of first order but, if rotational stiffness is sufficiently high, they may be split into two successive continuous transitions. An important virtue of these models is that in the first-order case, where fluctuations are small, they lead to a simple universal melting formula determining the melting point in terms of the elastic constants. The universal result is found from a lowest-order approximation, in which one identifies the melting point with the intersection of the high-temperature expansion of the free energy density dominated by defect fluctuations with the low-temperature expansion dominated by elastic fluctuations. The resulting universal formula for the melting temperature determines also the size of the Lindemann number. In two dimensions, this procedure was recently generalized from square to triangular lattices [9].

The intersection criterium was also used before to find the melting point of vortex lattices using the Abrikosov approximation of the Ginzburg-Landau model [10] useful for YBCO. Here we shall apply our defect model to calculate the melting curve of the vortex lattices in YBCO and BSCCO.

Due to the large penetration depth \( \lambda_{ab} \) in the layers in comparison to \( a \) we have to take into account the full non-local elasticity constants when integrating over the Fourier space, as emphasized by Brandt in Ref. [11]. For our Gaussian model, the partition function of the vortex lattice can be split into \( Z = Z_0 Z_\delta \) where \( Z_0 \) is the partition function of the rigid lattice and \( Z_\delta \) is thermally fluctuating part calculated via the elastic Hamiltonian

![Fig. 1: Melting curve \( B = B_m(T) \) for YBCO and BSCCO. The experimental values are from Ref. [8] for YBCO and Ref. [14] for BSCCO. The numbers at the vertices of the melting curve are the Lindemann numbers \( c_L \) calculated from \( c_L = \langle u^2 \rangle^{1/2}/a \) for YBCO and BSCCO.

\[ c_L \approx 0.09 \] for YBCO and \[ c_L \approx 0.1 \] for BSCCO.](image-url)
plus defects. Due to the translational invariance of the vortex lattice in the direction of the vortices, which we shall take to be the z-axis, we may simply extend the models on square and triangular and triangular lattices by a third dimension along the z-axis, which we artificially discretize to have a lattice spacing $a_3$, whose value will be fixed later. The elastic energy is

$$E_{el} = \frac{\nu}{2} \sum_x \int dz \left[ (c_{11} - 2c_{66})(\nabla_i u_i)^2 + c_{66}(\nabla_i u_j + \nabla_j u_i)/2 + c_{44}(\partial_z u_i)^2 \right].$$

(1) strain

Inserting the defect gauge field, it is extended to

$$\ln Z_{\text{eff}} = \det \left[ \frac{c_{66}}{4(c_{11} - c_{66})} \right]^{1/2} \det \left[ \frac{1}{2\pi \beta} \right]^{3/2} \prod_x \prod_{i \leq m} \int_{-\infty}^{\infty} d\sigma_{im} \left[ \prod_{m} \sum_{n_{m}(x) = -\infty}^{\infty} \right] \left[ \int_{-\infty}^{\infty} \frac{du}{a} \right] \exp \left\{ -\frac{1}{2\beta} \sum_{i < j} \sigma_{ij}^2 + \frac{1}{2} \sum_i \sigma_{ii}^2 - \left( \sum_i \sigma_{ii} \right) c_{11} - 2c_{66} \left( \sum_i \sigma_{ii} \right) + \left( \sum_i \sigma_{i3}^2 \right) c_{66} \left( \sum_i \sigma_{i3}^2 \right) \right\} e^{2\pi i \sum_x \left( \sum_{i \leq m} \nabla_m u_i \sigma_{im} + \sum_{i \leq j} D_{ij} \sigma_{ij} \right)}.$$  

(2)

In the canonical stress representation, the partition function becomes

$$Z_{\text{eff}} = \det \left[ \frac{c_{66}}{4(c_{11} - c_{66})} \right]^{1/2} \det \left[ \frac{1}{2\pi \beta} \right]^{3/2} \prod_x \prod_{i \leq m} \int_{-\infty}^{\infty} d\sigma_{im} \left[ \prod_{m} \sum_{n_{m}(x) = -\infty}^{\infty} \right] \left[ \int_{-\infty}^{\infty} \frac{du}{a} \right] \exp \left\{ -\frac{1}{2\beta} \sum_{i < j} \sigma_{ij}^2 + \frac{1}{2} \sum_i \sigma_{ii}^2 - \left( \sum_i \sigma_{ii} \right) c_{11} - 2c_{66} \left( \sum_i \sigma_{ii} \right) + \left( \sum_i \sigma_{i3}^2 \right) c_{66} \left( \sum_i \sigma_{i3}^2 \right) \right\} e^{2\pi i \sum_x \left( \sum_{i \leq m} \nabla_m u_i \sigma_{im} + \sum_{i \leq j} D_{ij} \sigma_{ij} \right)}.$$  

(3)

The subscripts $i, j (l, m, n)$ have the values 1, 2 (1, ..., 3), the the-dimensional vectors $u_i(x)$ are given the transverse displacements of the line elements of the vortex lines with coordinate $x$, and $\sigma_{im}(x)$ is the conjugate stress fields. We have suppressed the spatial arguments of the elasticity parameters, which are really functional matrices $c_{ij}(x, x') \equiv c_{ij}(x - x')$. In momentum space, the brackets are diagonal. Their precise forms were calculated by Brandt and Dietel.

The functional matrix $\beta$ is given by $\beta = \nu c_{66}/k_B T(2\pi)^2$ where $\nu$ is the volume $a^2 a_3$ (square) or $a^2 a_3 \sqrt{3}/2$ (triangular) of the fundamental cell. For a square lattice, the lattice derivatives $\nabla_i f(x)$ are given by $\nabla_i f(x) = [f(x + a e_i) - f(x)]/a$ and $\nabla_3 f(x) = [f(x + a_3 e_3) - f(x)]/a_3$. For a triangular lattice, the $xy$-part of the lattice has the link vectors $a e_{(m)}$ with $e_{(1,3)} = (\cos 2\pi/6, \pm \sin 2\pi/6, 0)$ and $e_{(2)} = (-1, 0, 0)$. The lattice derivatives around a plaquette are defined by $\nabla_{(1)} f(x) = [f(x + a e_{(1)}) - f(x)]/a$,$\nabla_{(2)} f(x) = [f(x) - f(x - a e_{(2)})]/a$, $\nabla_{(3)} f(x) = [f(x) - f(x + a e_{(1)})]/a$. From these we define discrete cartesian derivates used in the partition function $\nabla_i f(x) = (2/3)e_{(i)} \nabla_{(i)} f(x)$ and $\nabla_3 f(x) = [f(x + a_3 e_3) - f(x)]/a_3$.

We are now prepared to specify the matrix $D_{ij}(x)$ in Eq. (4). It is a discrete-valued local defect matrix composed of integer-valued defect gauge fields $n_1, n_2, n_2$ for square and triangular vortex lattices as follows:

$$D^\square_{ij} = \begin{pmatrix} n_1 & n_3 \\ n_3 & n_2 \end{pmatrix},$$

(4)

$$D^\triangle_{ij} = \begin{pmatrix} \frac{1}{\sqrt{3}} n_1 & \frac{2}{\sqrt{3}} n_3 \\ \frac{1}{\sqrt{3}} n_2 & -\frac{1}{\sqrt{3}} n_3 \end{pmatrix}.$$

(5)

The vortex gauge fields specify the Volterra surfaces in units of the Burgers vectors $[3 \pm 3]$. By summing over all $n_{1,2,3}(x)$, the partition function $Z_{\text{eff}}$ includes all defect
fluctuations, dislocations as well as disclinations. There is a constraint for a vortex lattice which does not exist for ordinary three-dimensional lattices. Dislocations in the vortex lattice can be both screw or edge type, but in either case the defect lines are confined in the plane spanned by their Burger’s vector and the magnetic field direction. The reason is that the flux lines in a vortex lattice cannot be broken. This results in the constraint $D_{11} = -D_{22}$ on the defect fields.

We now calculate the low-temperature and high-temperature expansions of the partition function $Z_0$ to lowest order, which includes only the $n_m = 0$-term. By carrying out the integration over the displacement fields $u_i(x)$ in $\Box$, we obtain, as in $\Box$, the leading term in the low-temperature expansion of the free energy

$$Z_0^{T \to 0} = \left( \frac{a_3}{a} \right)^{2N} \frac{1}{\det[(2\pi\beta)c_{44}/c_{66}]} e^{-N \sum_{i \in \{1,6\}} l_{ii}},$$

where

$$l_{ii} = \frac{1}{2} \int_{BZ} \frac{d^2kd\lambda}{\beta} \ln \left[ c_{44}^2 K_j^2 K_j + c^2 K_j^2 K_3 \right].$$

Here $K_m$ is the eigenvalue of $i \nabla_m$. The $k, k_3$-integrations in $\Box$ run over the Brillouin zone of the vortex lattice of volume $V_{BZ} = (2\pi)^3/v$, as indicated by the subscript BZ.

Next we calculate the high-temperature expansion $Z_0^{T \to \infty}$ to lowest order. By carrying out the integration over the displacement fields $u_i(x)$ in $\Box$ and further by summing over the defect fields $n_m$ under the consideration $D_{11} = -D_{22}$ mentioned above we obtain

$$Z_0^{T \to \infty} = \left( \frac{a_3}{a} \right)^{2N} C^N \frac{1}{2N \det[(2\pi\beta)c_{44}/c_{66}]} e^{-Nh},$$

with

$$h = \frac{1}{2} \int_{BZ} \frac{d^2kd\lambda}{\beta} \ln \left[ 1 + c_{44}^2 K_j^2 K_j + c^2 K_j^2 K_3 \right].$$

The constant $C$ has the values $C_{11} = 1$ for the square vortex lattice and $C_{12} = \sqrt{3}$ for the triangular lattice.

From the partition function $\Box$ with no defects ($n_m = 0$) we obtain for the Lindemann number $c_L = \langle u^2 \rangle^{1/2}/a$ the momentum integral

$$c_L^2 = \frac{e^2 k_B T}{a^2v V_{BZ}} \int_{BZ} \frac{d^2kd\lambda}{\beta} \frac{1}{c_{44}} \sum_{i=1,6} c_i^2 c_{44}^2 K_j^2 K_j + a^2 K_j^2 K_3.$$  

This expression can be simplified by taking into account that $c_{11}$ is much larger than $c_{06}, c_{44}$ in $\Box$. We shall calculate the melting temperature from the intersection of low- and high-temperature expansions, obtained by equating $Z_0^{T \to 0} = Z_0^{T \to \infty}$. By taking into account $\det[a_3^3 \nabla_3 \nabla_3] = 1$ we obtain $l_{11,6} \ll l_{06}$ and further that the summand for $i = 1$ in $\Box$ is much smaller than the summand for $i = 6$. In the following analytic discussion (but not in the numerical plots) we will neglect the corresponding terms. The temperature of melting is then given by

$$\frac{k_B T a_{44}^{1/2}}{e^{3/2}} \frac{1}{\det^{1/2}(c_{66})} C = e^{-l_{06}},$$

where $\det(c_{66})$ is the determinant of the $N \times N$ functional matrix $c_{66}$. The elastic moduli $c_{44}$ and $c_{66}$ at low reduced magnetic fields $k_B T/H_C < 0.25$ can be taken from Brandt’s paper $\Box$.

$$c_{66} = \frac{B\phi_0 \zeta}{(8\pi\lambda_{ab})^2},$$

where $\zeta$ is the penetration depth in the $xy$-plane, and $\zeta = 1$. $K_B$ is the boundary of the circular Brillouin zone $K_B^2 = 4\pi B/\phi_0$. At high fields ($b > 0.5$), $c_{66}$ is obtained by the factor $\zeta \approx 0.71(1 - b)$, and the penetration depths in $c_{66}, c_{44}$ are replaced by $\lambda^2 = \lambda^2(1 - b) - b \lambda$ denotes either $\lambda_{ab}$ or $\lambda_e$. In addition, the last two terms of $c_{44}$ are replaced by $B_0 \phi_0/(16\pi\lambda_{ab}^2)$. For YBCO we have $\Box$ $\Box$. $\Box$ $\Box$ $\Box$. The cutoff is due to thermal softening $\Box$, and becomes relevant for $\Box$ $\Box$. In $\Box$, we have used a momentum cutoff in the two-vortex interaction potential $k < 2/(a^2)^{1/2}$, and not the average of the correlation length $1/\xi_{44}$ in Ref. $\Box$. The cutoff is fulfilled in the melting regime of BSCCO, but not for YBCO.

It remains to determine the effective lattice spacing $a_3$ along the vortex lines. The melting condition $\Box$ for $a_3$ diverges for $a_3 \to 0$, whereas the displacement average $\Box$ remains finite in this limit. An elementary defect arises from an interchange of two vortex strings in the sample. This takes place over a typical length scale in the $z$-direction. To properly count the defect degrees of freedom in the partition function, the length $a_3$ should coincide with this length scale. The length $a_0$ is determined from the condition that the sum of elastic displacement energy and the energy required to stretch the line against the line tension is minimal $\Box$. Thus we insert the variational ansatz for the transverse displacement field $u_i = \delta_{i1} A_0 \exp[-2|z|/a_3]$ into the elastic energy $\Box$ and approximate $\Box$ $\Box$. The optimal length $a_3$ is chosen such that $E_{el}$ is minimal for a fixed amplitude $A_0 \approx a_3$ $\Box$.

In the following, we treat first the more isotropic square vortex crystal YBCO ($a = \sqrt{\phi_0/B}$). From $c_{66}$ and $c_{44}$ for YBCO, the optimal length is given by $\Box$.
\( a_3 = 4a_0 \lambda_{ab}/\lambda_c(1-b)^{1/2}.* \) When comparing the melting
criterion of the defect model in Eq. (11) with the Lindemann
criterion obtained by equating the parameter \( a \) to a fixed number, we get almost a precise coincidence of
the two criteria. An exact coincidence is reached when
taking into account that the integrand in (10) and in \( l_{06} \)
of Eq. (11) receive their main contribution from the region \( k \approx \sqrt{(k^2)_{B}} \approx K_{BZ}/2 \) where we can put \( k_3 \approx 0 \),
resulting in \( a^2 c_{66} / a^3 c_{44} \approx 2 \). Using this approximation in Eqs. (10) and (11), we can perform the integrals
numerically, adh. from the melting criterion to
precisely the Lindemann criterion with the Lindemann number \( a \) fixed to be
\[
\frac{k_BT_m}{4 \left[ \frac{c_{44}}{a^2} (\frac{K_{BZ}}{2}) \right] a} \approx c_L^{2} \approx (0.19)^2. \tag{14}
\]
Denoting by \( a_s \) the spacing between the CuO\(_2\) layers we obtain for the entropy jump per layer and vortex
\[
\Delta S_d = k_B T_m (\partial / \partial T_m) (a_s / a_3) \ln[Z^\gamma_{II} \rightarrow 0 / Z^\gamma_{III} \rightarrow 0]. \tag{15}
\]
Inserting (10) and (16) we obtain
\[
\Delta S_d \approx k_B T_m \left( \frac{a_s}{a_3} \ln \left[ \frac{k_BT_m / a^3}{c_{44} (\frac{K_{BZ}}{2}) (\frac{K_{BZ}}{2})} \right] \right)^{1/2}. \tag{16}
\]
Finally, we make use of the Clausius-Clapeyron equation
\[
\Delta S_d / a^2 a_s = -(dH_m / dT) \Delta B / 4\pi, \tag{18}
\]
relating the jump of the entropy density \( \Delta S_d / (a^2 s) \) to the jump of
the magnetic induction \( \Delta B \) across the melting transition. Here
\( H_m \) is the external magnetic field on the vortex.
From the Clausius-Clapeyron equation and Eqs. (14) and (16), we obtain the following relations near \( T_c \) (where \( b \) is small)
\[
B_m(T) \approx \frac{12 \zeta}{(1 - T/T_c)^{1/3}} \frac{C_T}{64\pi^2} \frac{k_B}{(k_B T)^2} \frac{\sigma^2}{\lambda_c(0)} \frac{\lambda_c^2}{(0)} \frac{1}{\gamma^2}(0), \tag{17}
\]
\[
\Delta S_d \approx \frac{\gamma s \lambda_c}{6a_3} \frac{k_B}{(1 - T_c / T_m)} \approx \frac{2.7 \cdot 10^{-3} \sigma^2}{\lambda_c(0)} \frac{\lambda_c^2}{(0)} \frac{1}{\gamma^2}(0), \tag{18}
\]
\[
\Delta B \approx \frac{\pi \zeta}{2a_0 \lambda_{ab}} \frac{k_B T_m}{\lambda_c(0)} \approx \frac{2.5 \cdot 10^{-2} (1 - T_m / T_c)^{2/3}}{\lambda_c(0)} \tag{19}
\]
These results agree with the scaling results in Ref. 11, where multiplicative prefactors are determined by fits to experimental curves (there is only a slight discrepancy because we use a different temperature dependence of the penetration depth).

Next, we calculate the corresponding expressions in the case of the more layered crystal BSCCO \( (a = 3^{1/4} / 2^{1/2} / \sqrt{\phi_0} / B) \). First, we have to determine the dislocation length \( a_3 \) in this case. For dislocation moves we have \( (c_{44}^2)^{1/2} = 1 / K_{BZ} \). This means that we can neglect the last two terms of \( c_{44} \) in (10), coming from the self-energy of the vortex line, for the determination of

where is the length of the dislocation. Rememer this we obtain

\[
\Delta S_d / a^2 a_s = -(dH_m / dT) \Delta B / 4\pi, \tag{18}
\]
relating the jump of the entropy density \( \Delta S_d / (a^2 s) \) to the jump of the magnetic induction \( \Delta B \) across the melting transition. Here
\( H_m \) is the external magnetic field on the vortex. From the Clausius-Clapeyron equation and Eqs. (14) and (16), we obtain the following relations near \( T_c \) (where \( b \) is small)
\[
B_m(T) \approx \frac{12 \zeta}{(1 - T/T_c)^{1/3}} \frac{C_T}{64\pi^2} \frac{k_B}{(k_B T)^2} \frac{\sigma^2}{\lambda_c(0)} \frac{\lambda_c^2}{(0)} \frac{1}{\gamma^2}(0), \tag{17}
\]
\[
\Delta S_d \approx \frac{\gamma s \lambda_c}{6a_3} \frac{k_B}{(1 - T_c / T_m)} \approx \frac{2.7 \cdot 10^{-3} \sigma^2}{\lambda_c(0)} \frac{\lambda_c^2}{(0)} \frac{1}{\gamma^2}(0), \tag{18}
\]
\[
\Delta B \approx \frac{\pi \zeta}{2a_0 \lambda_{ab}} \frac{k_B T_m}{\lambda_c(0)} \approx \frac{2.5 \cdot 10^{-2} (1 - T_m / T_c)^{2/3}}{\lambda_c(0)} \tag{19}
\]
These results agree with the scaling results in Ref. 11, where multiplicative prefactors are determined by fits to experimental curves (there is only a slight discrepancy because we use a different temperature dependence of the penetration depth).

Next, we calculate the corresponding expressions in the case of the more layered crystal BSCCO \( (a = 3^{1/4} / 2^{1/2} / \sqrt{\phi_0} / B) \). First, we have to determine the dislocation length \( a_3 \) in this case. For dislocation moves we have \( (c_{44}^2)^{1/2} \sim 1 / K_{BZ} \). This means that we can neglect the last two terms of \( c_{44} \) in (10), coming from the self-energy of the vortex line, for the determination of

The largest discrepancy of the melting curves for BSCCO lies in the high-field regime. Due to the large anisotropy of BSCCO, the layers decouple in this low-temperature regime into two dimensional lattices of nan-cake vortices [22] by Josephson decoupling [14, 15]. This process is ignored in the simple vortex lattice model set up in this paper. In the case of the entropy jump $\Delta S_d$ and magnetic field jump $\Delta B$ we obtain in the low-temperature regime a rough agreement of our theory with the curves of Kadowki et al. Ref. [24] but not with those of Zeldov et al. Ref. [22]. The discrepancy of the two experimental curves may be due to pinning [21]. For temperatures near $T_c$, the experiments give large values of $\Delta S_d$ and of $\Delta B$, which are not reproduced by our model where $\Delta S_d$ is almost constant at large temperatures. This must be due to system degrees of freedom not included in our vortex lattice model coming from thermally created vortex loops in addition to the magnetically created ones forming the lattice. These vortex loops screen the effective repulsive interaction between the magnetic induced vortex loops resulting in a rise of the magnetic jump $\Delta B$ [22].

FIG. 2: Entropy jump per layer per vortex $\Delta S_d$ (first row) and jump of magnetic induction field $\Delta B$ (second row) at the melting transition. The experimental values for YBCO are from Ref. [22] by torque measurements and Ref. [21] by squid experiments and Ref. [14] by torque measurements. The experimental values for BSCCO are from Ref. [22] (squid) and Ref. [14] (torque).

Discrepancies for YBCO can be explained by the simplicity of our model and the roughness of the approximations.

1. D. R. Nelson, Phys. Rev. Lett. 60, 1973 (1988) (readable online at www.physik.fu-berlin.de/~kleinert/re.html#b2)
2. A. Houghton et al., Phys. Rev. B 40, 6765 (1989); E. H. Brandt, Phys. Rev. Lett. 63, 1106 (1989).
3. P. Kim et al., Phys. Rev. Lett. 77, 5118 (1996).
4. B. Keimer et al., Phys. Rev. Lett. 73, 3459 (1994).
5. H. Won and K. Maki, Phys. Rev. B 53, 5927 (1996).
6. M. Tinkham, Introduction to Superconductivity, McGraw-Hill, New York, 1969.
7. G. Blatter, V. Geshkenbein, A. Larkin, and B. Feigelman, Phys. Rev. B 54, 73 (1996).
8. H. Kleinert, Gauge Fields in Condenser Matter, Vol. II Stresses and Defects, World Scientific, Singapore, 1989. (readable on http://www.physik.fu-berlin.de/~kleinert/re.html#b2)
9. J. Dietel and H. Kleinert, preparation.
10. S. E. Hikami, A. Fujita, and A. I. Larkin, Phys. Rev. B 44, 10400 (1991).
11. E. H. Brandt, Rep. Prog. Phys. 58, 1465 (1995).
Defect Melting of Vortices in High-\(T_c\) Superconductors

Jürgen Dietel and Hagen Kleinert

Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany

(Dated: Received March 23, 2002)

We set up a melting model for vortex lattices in high-temperature superconductors based on the continuum elasticity theory. The model is Gaussian and includes defect fluctuations by means of a discrete-valued vortex gauge field. We derive the melting temperature of the lattice and predict the size of the Lindemann number. Our result agrees well with experiments for \(\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}\), and with modifications also for \(\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_3\text{O}_8\). We calculate the jumps in the entropy and the magnetic induction at the melting transition.

PACS numbers: 74.60.Ec, 74.60.Ge

Flux line lattice melting, proposed by Nelson in 1988, has become an important topic in the phenomenology of high-temperature superconductors. The properties of this transition have been studied in a large number of theoretical as well as experimental papers. The simplest method to estimate the temperature where the transition takes place is adapted from the famous Lindemann criterion of three dimensional ordinary crystals.

The adaption by Houghton et al. and Brandt to vortex lattices states that the vortex lattice undergoes a melting transition once the mean thermal displacement \(\langle u^2 \rangle^{1/2}\) becomes a certain fraction of the lattice spacing \(a \approx (\Phi_0/B)^{1/2}\) where \(\Phi_0\) is the flux quantum, \(B\) is the induction field. The size of the Lindemann number \(c_L = \langle u^2 \rangle^{1/2}/a\) which should be independent of the magnetic field is not predicted by Lindemann’s criterion. It must be extracted from experiments, and is usually found in the range \(c_L \approx 0.1 - 0.3\). The most prominent examples of high-temperature superconductors exhibiting vortex lattice melting are the anisotropic compound \(\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}\) (YBCO), and the strongly layered compound \(\text{Bi}_2\text{Sr}_2\text{Ca}_2\text{Cu}_3\text{O}_8\) (BSCCO). Decoration experiments on BSCCO show the formation of a triangular vortex lattice, on YBCO of a tilted square lattice of vortices close to the melting region, the latter being favored by the \(d\)-wave symmetry of the order parameter and the anisotropy of the crystal. An explicit calculation of the Lindemann number \(c_L = \langle u^2 \rangle^{1/2}/a\) for YBCO can be found in Ref. 1 and for BSCCO in Ref. 1.

In this letter, we calculate the size of the Lindemann number \(c_L\) from a Gaussian model which takes into account the lattice elasticity and the defect degrees of freedom in the simplest possible way. The relevance of defect fluctuations to vortex melting has also been emphasized by many authors. For ordinary crystals, this is meanwhile textbook material: in Ref. 2 one of us has set up simple Gaussian lattice models based on linear elasticity and a fluctuating discrete-valued defect gauge field for square lattice crystals which clearly display first order melting transitions in three dimensions. An important virtue of these models is that in the first-order case, where fluctuations are small, they lead to a simple universal melting formula determining the melting point in terms of the elastic constants. The universal result is found from a lowest-order approximation, in which one identifies the melting point with the intersection of the high-temperature expansion of the free energy density dominated by defect fluctuations with the low-temperature expansion dominated by elastic fluctuations. The resulting universal formula for the melting temperature determines also the size of the Lindemann number. In two dimensions, this procedure was recently generalized from square to triangular lattices 3.

In our Gaussian model, the partition function of the vortex lattice can be split into \(Z = Z_0 Z_\text{H}\) where \(Z_0\) is the partition function of the rigid lattice and \(Z_\text{H}\) is thermally fluctuating part calculated via the elastic Hamiltonian plus defects. Due to the translational invariance of the vor-

![FIG. 1: Melting curve \(B = B_m(T)\) for YBCO and BSCCO. The experimental values are from Ref. 6 for YBCO and Ref. 11 for BSCCO. The numbers at the theoretical melting curves are the Lindemann numbers \(c_L\) calculated from \(\Phi_0/B \approx 0.1\).](image-url)
were calculated by Brandt of the elasticity parameters, which are really functional 

\[ E_{el} = \frac{v}{2} \sum_{x} (\nabla_{i} u_{i}) (c_{11} - 2c_{66})(\nabla_{i} u_{i}) \]  

(1) 

+ \frac{1}{2} \left[ (\nabla_{i} u_{j} + \nabla_{j} u_{i}) c_{66} (\nabla_{i} u_{j} + \nabla_{j} u_{i}) + (\partial_{z} u_{i}) c_{44} (\partial_{z} u_{i}) \right].

The subscripts \( i,j \) \((l,m,n)\) have values 1, 2 \((1,\ldots,3)\), the vectors \( u_{i}(x) \) are given by the transverse displacements of the line elements of the vortex lines with coordinate \( x \). We have suppressed the spatial arguments of the elasticity parameters, which are really functional matrices \( c_{ij}(x,x') \equiv c_{ij}(x-x') \). Their precise forms were calculated by Brandt and are given. The lattice cannot be broken. This results in the constraint

\[ \nabla_{i} f(x) = [f(x + a_{e_{(1)}}) - f(x)] / a \text{ and } \nabla_{3} f(x) = [f(x + a_{e_{3}}) - f(x)] / a. \]

For a triangular lattice, the \( x\text{-}y\)-part of the lattice has the link vectors \( a \mathbf{e}_{(m)} \) with \( e_{(1,3)} = (\cos 2\pi /6, \pm \sin 2\pi /6, 0) \) and \( e_{(2)} = (-1,0,0) \). The lattice derivates around a plaquette are defined by

\[ \nabla_{(1)} f(x) = [f(x + a_{e_{(1)}}) - f(x)] / a, \quad \nabla_{(2)} f(x) = [f(x) - f(x - a_{e_{(2)}})] / a, \quad \nabla_{(3)} f(x) = [f(x - a_{e_{(2)})} - f(x + a_{e_{(1)}})] / a. \]

From these we define discrete cartesian derivates used in the Hamiltonian

\[ \nabla_{i} f(x) = (2/3)\epsilon_{ij} \nabla_{j} f(x) \text{ and } \nabla_{3} f(x) = [f(x + a_{e_{3}}) - f(x)] / a \text{ transforming like the continuum derivates with respect to the symmetry group of the lattice.} \]

Therefore, the Hamiltonian has the full symmetry of the triangular lattice and the correct continuum elastic energy for zero lattice spacing.

Within the elastic approximation the displacement fields are restricted to values within the fundamental cell. In order to contain also defect degrees of freedoms one has to put in \( 3 \) integer valued defect gauge fields. In the canonical stress representation, the partition function containing these fields becomes

\[ Z_{H} = \det \left[ \frac{c_{66}}{4(c_{11} - c_{66})} \right]^{1/2} \left[ \frac{1}{2\pi\beta} \right]^{5/2} \prod_{x} \prod_{i \leq m} \int_{-\infty}^{\infty} d\sigma_{im} \left[ \prod_{m} \prod_{n_{m}(x) = -\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{du}{a} \right] \exp \left\{ -\sum_{x} \frac{1}{2\beta} \left[ \sum_{i < j} \sigma_{ij} + \frac{1}{2} \sum_{i} \sigma_{ii} - \left( \sum_{i} \sigma_{ii} \right) \frac{c_{11} - 2c_{66}}{4(c_{11} - c_{66})} \left( \sum_{i} \sigma_{ii} \right) + \sum_{i} \sigma_{ij} \frac{c_{66}}{c_{44}} \sigma_{ij} \right] \right\} e^{2\pi i \sum_{x} (\sum_{i < m} \nabla_{m} u_{i} \sigma_{im} + \sum_{i < j} D_{ij} \sigma_{ij})}. \]  

(2)

The functional matrix \( \beta \) is given by \( \beta = v c_{66} / k_{B} T(2\pi)^{2} \). \( \sigma_{ij} \) represent stress fields. We are now prepared to specify the matrix \( D_{ij}(x) \) in Eq. (10). It is a discrete-valued local defect matrix composed of integer-valued defect gauge fields \( n_{1}, n_{2}, n_{3} \) for square and triangular vortex lattices as follows:

\[ D_{ij}^{\square} = \begin{pmatrix} n_{1} & n_{2} \\ n_{3} & n_{2} \end{pmatrix}, \]

(3)

\[ D_{ij}^{\triangle} = \begin{pmatrix} \frac{1}{2}n_{1} & \frac{1}{2}n_{1} \\ \frac{1}{\sqrt{3}}(n_{1} - n_{2}) & \frac{2}{\sqrt{3}}n_{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}n_{1} & \frac{1}{2}n_{1} \\ \frac{1}{\sqrt{3}}(n_{1} - n_{2}) & \frac{2}{\sqrt{3}}n_{2} \end{pmatrix}. \]

(4)

The vortex gauge fields specify the Volterra surfaces in units of the Burgers vectors. By summing over all \( n_{1,2,3}(x) \), the partition function \( Z_{H} \) includes all defect fluctuations, dislocations as well as disclinations. There is a constraint for a vortex lattice which does not exist for ordinary three-dimensional lattices. Dislocations in the vortex lattice can be both screw or edge type, but in either case the defect lines are confined in the plane spanned by their Burger’s vector and the magnetic field. The reason is that the flux lines in a vortex lattice cannot be broken. This results in the constraint \( D_{11} = -D_{22} \) on the defect fields.
tion $D_{11} = -D_{22}$ mentioned above we obtain

$$Z_{\mathbf{H}}^{T \to \infty} = \left( \frac{a_{3}}{a} \right)^{2N} \frac{C^{N}}{2^{N}} \frac{1}{\det \left[ (2\pi \beta)^{2} c_{44}/c_{66} \right]} e^{-Nh} \quad (7)$$

with

$$h = \frac{1}{2} \frac{1}{V_{BZ}} \int_{BZ} d^{2}kd\xi \log \left[ 1 + \frac{c_{11} - c_{66} K_{1}^{2} K_{3}^{2}}{c_{44} K_{1}^{2} K_{3}^{2}} \right]. \quad (8)$$

The constant $C$ has the values $C_{\lambda} = 1$ for the square vortex lattice and $C_{\Delta} = \sqrt{3}$ for the triangular one.

From the partition function (4) with no defects ($n_{m} = 0$) we obtain for the Lindemann number $c_{L} = \langle u^{2} \rangle^{1/2}/a$ the momentum integral

$$c_{L}^{2} = \frac{a_{3}^{2}}{a^{2}v} \frac{k_{B}T}{V_{BZ}} \int_{BZ} d^{2}kd\xi \sum_{i=1,6} \frac{1}{c_{44}} c_{i}^{2} K_{1}^{2} K_{3}^{2} + a_{3}^{2} K_{1}^{2} K_{3}^{2} \quad (9)$$

This expression can be simplified by taking into account that $c_{11}$ is much larger than $c_{66}, c_{44}$. We shall calculate the melting temperature from the intersection of low-and high-temperature expansions, obtained by equating $Z_{T=0}^{T} = Z_{T \to \infty}^{T}$. By taking into account $\det[\alpha_{0}^{2} \nabla_{\mathbf{z}}^{2}] = 1$ we obtain $h_{i=1,6} \ll l_{66}$ and further that the summation for $i = 1$ in $c_{44}$ is much smaller than the summation for $i = 6$. In the following analytic discussion (but not in the numerical plots) we will neglect the corresponding terms. The temperature of melting is then given by

$$k_{B}T = \frac{1}{v} \frac{1}{\det[1^{N}/c_{66}]} C \approx \frac{e^{-\lambda_{a}}}{\pi}, \quad (10)$$

where $\det[c_{66}]$ is the determinant of the $N \times N$ functional matrix $c_{66}$. The elastic moduli $c_{44}$ and $c_{66}$ at low reduced magnetic fields $b \equiv B/H_{c2} < 0.25$ can be taken from Brandt’s paper \[1\]

$$c_{66} = \frac{B_{0} \phi_{0} \zeta}{(8\pi \lambda_{ab})^{2}}, \quad (11)$$

$$c_{44} = \frac{B^{2}}{4\pi(1 + \lambda^{2} K_{1}^{4} K_{3}^{4} + 2\pi \beta \lambda_{ab} K_{1}^{2} K_{3}^{2})} + \frac{B_{0} \phi_{0}}{\pi} \ln \left[ 1 + \frac{(2\lambda^{2})^{1/2}}{1 + \lambda_{ab} K_{1}^{2} K_{3}^{2}} \right] + \frac{B_{0} \phi_{0}}{32\pi^{2} \lambda_{ab} K_{1}^{2} K_{3}^{2}} \ln \left[ 1 + \frac{\lambda_{ab} K_{1}^{2}}{1 + \lambda_{ab} K_{1}^{2}} \right]. \quad (12)$$

where $\lambda_{c}$ is the penetration depth in the $xy$-plane, and $\zeta = 1$. $K_{BZ}$ is the boundary of the circular Brillouin zone $K_{BZ}^{2} = 4\pi B/\phi_{0}$. At high fields ($b > 0.5$), $c_{66}$ is altered by the factor $\zeta \approx 0.71(1 - b)$, and the penetration depths in $c_{66}, c_{44}$ are replaced by $\lambda^{2} = \lambda_{c}^{2}/(1 - b)$, where $\lambda$ denotes either $\lambda_{ab}$ or $\lambda_{c}$. In addition, the last two terms of $c_{44}$ are replaced by $B_{0} \phi_{0} / 16 \pi^{2} \lambda^{2}$. For YBCO we have

$$4/\sqrt{\pi} \approx 1.7 = \xi(0) \approx 1/(T/T_{c})^{1/3}, \quad \xi(\lambda(T)) = \xi(0) \approx 1/(T/T_{c})^{1/3} \text{ for BSCCO}, \quad \xi(T) = \xi(0) \approx 1/(T/T_{c})^{1/3}. \quad (13)$$

For the calculation of $c_{44}$ in (12) we have used a momentum cutoff in the two-vortex interaction potential $k \leq 2/(a^{2})^{1/2}$, and not the inverse of the correlation length $1/\xi_{a}$ as in Ref. \[9\]. The cutoff is due to thermal softening $^{6}$, and becomes relevant for $u^{2}/(2 \pi \beta) \xi_{a} \gg 1$, or equivalently for

$$\sqrt{\pi} \frac{2 \pi ^{2} \beta}{2 \pi \beta \xi_{a}} \chi(H) \xi_{a} \gg 1, \quad (8)$$

where $\chi(H) \xi_{a}$ is the line tension for the vortex lattice (for example an interchanging of two vortex strings) takes place over a typical length scale in the $z$-direction which is determined from the condition that the sum of elastic displacement energy and the energy required to stretch the line against the line tension is minimal. A dislocation network containing this length spacing in $z$-direction has to take into account in the continuum elastic Hamiltonian for small lattice deformations. This dislocation network consists of integer fields coupling to the displacement fields of length scale larger or equal to the length spacing of the dislocation network.

The displacement fields of smaller length scales can be easily integrated out giving an additive correction term to the free energy not relevant for the melting transition properties. The relevant part of the free energy is given by the discretized free energy $-\log(Z_{0} / Z_{H})/\beta$ with $a_{3}$ given by the length spacing of the dislocation network in $z$ direction. To determine $a_{3}$ we insert the variational ansatz for the transverse displacement field

$$u_{i} = \delta_{i1} A_{0} \exp[-2|z|/a_{3}], \quad (3)$$

in Eq. (10). At high fields ($\delta_{i1} A_{0}$) we receive their main contribution from the region where the average $\langle u \rangle$ was taken with respect to a circular Brillouin zone. The optimal length $a_{3}$ is chosen such that $E_{a}$ is minimal for a fixed amplitude $A_{0} \approx a_{3}$ corresponding to a typical defect elongation. $^{*}$

In the following, we treat first the more isotropic square vortex crystal YBCO ($a = \sqrt{\phi_{0} / B}$). From $c_{66}$ and $c_{44}$ for YBCO, the optimal length is given by $a_{3} = \alpha a_{66} / \lambda_{c} \zeta (1 - b)^{1/2}$. When comparing the melting criterium of the defect model in Eq. (10) with the Lindemann criterium obtained by equating the parameter $k_{B}T$ to a fixed number, we get coincidence when taking into account that the integrand in $c_{44}$ and in $c_{66}$ of Eq. (10) receive their main contribution from the region $k \approx \sqrt{k^{2}} \approx K_{BZ}/\sqrt{2}$. We can put $k_{3} \approx 0$ in this region resulting in $a_{3}^{2} c_{66} / a^{2} c_{44} \approx 4/\pi$. Using this approximation in Eqs. (11) and (12), we can perform the integrals numerically. Then we obtain from the melting condition (10) precisely the Lindemann criterium with the Lindemann number $\xi_{a}$ fixed to be

$$k_{B}T_{m} \approx \frac{1}{4} \left[ \frac{c_{44}(K_{BZ}^{2} / \sqrt{2}, 0) c_{66}(K_{BZ}^{2} / \sqrt{2}, 0)}{c_{44}(K_{BZ}^{2} / \sqrt{2}, 0) c_{66}(K_{BZ}^{2} / \sqrt{2}, 0)} \right]^{1/2} a^{3} \approx c_{L}^{2} \approx (0.18)^{2}. \quad (13)$$

Denoting by $a_{66}$ the spacing between the CuO$_{2}$ double layers we obtain for the entropy jump per double layer and vortex

$$\Delta S_{l} \approx k_{B}T_{m}(\partial / \partial T_{m})(a_{s} / a_{3}) \ln[Z_{\mathbf{H}}^{T \to \infty} / Z_{\mathbf{H}}^{T \to 0}]. \quad (14)$$

Sie meinen das hier?
Inserting (10) and (11) we obtain
\[ \Delta S_l \approx \frac{k_B T_m a_s}{\lambda_b} \frac{\partial}{\partial T_m} \ln \left[ \frac{k_B T_m / a^3}{c_{44}(K_{\text{eff}}/a^2,0)c_{46}(K_{\text{eff}}/a^2,0)} \right]. \]  
(15)

Finally, we make use of the Clausius-Clapeyron equation \[ \Delta S_l/a_3 = -(dH_m/dT) \Delta B/4\pi, \] relating the jump of the entropy density \( \Delta S_l/a_3 \) to the jump of the magnetic induction \( \Delta B \) across the melting transition. Here \( H_m \) is the external magnetic field on the melting line. From the Clausius-Clapeyron equation and Eqs. (9) and (10), we obtain the following relations near \( T_c \):
\[ B_m(T) \approx \frac{12 \zeta}{16 \pi^3} (1 - T/T_c)^{3/4} \lambda_2^{3/2} \phi_0^5, \]
\[ \Delta S_l = \frac{\sqrt{\zeta} s}{6 a} \frac{k_B}{\lambda_b (1 - T_m/T_c)^{3/4}} \approx 2.7 \times 10^{-3} \frac{c_l^2}{\phi_0^3} s, \]
\[ \Delta B = \frac{\sqrt{\zeta} \pi \lambda_c}{2 a \phi_0 \lambda_b T_m} \approx \frac{2.5}{10^2} (1 - T_m/T_c)^{3/2} \phi_0 \lambda_c. \]
(16)

These results agree with the scaling results in Ref. [18], where multiplicative prefactors are determined by fits to experimental curves (there is only a slight discrepancy because we use a different temperature dependence of the penetration depth).

Next, we calculate the corresponding expressions in the case of the more layered crystal BSCCO (\( a = (2^{1/2}/3^{1/4}) \sqrt{\phi_0/B} \)). First, we have to determine the dislocation length \( a_3 \) in this case. For dislocation moves we have \( (a^2)^{1/2} \sim 1/K_{\text{BZ}} \). This means that we can neglect the last two terms of \( c_{44} \) in (12), coming from the self-energy of the vortex line, for the determination of the length of the dislocation. Remembering this we obtain by a similar procedure as for YBCO the dislocation length \( a_3 \approx 4 a \sqrt{2} \lambda_b / \lambda_c \sqrt{\pi} \). From this we find \( a_3^2 \approx b_4 / c_{44} \) with \( b_4 < 1 \) and resulting in \( a_3 \approx 0 \). By taking into account that \( B \pi^2 \lambda_2^{3/2} / 32 \phi_0 \approx 1 \) on the melting line we obtain that \( c_{44}(k, k_3) \) for \( |k_3| < \pi / a_3 \) is dominated by the last term in (12). Then we obtain
\[ c_{44}(k, k_3) \approx \frac{B \phi_0}{32 \pi^2 \lambda_2 (1 + \lambda_2 K_{\text{BZ}})} \text{ for } k_3 < \frac{1}{\lambda_2}, \]
\[ c_{44}(k, k_3) \approx \frac{B \phi_0}{32 \pi^2 \lambda_2 (1 + \lambda_2 K_{\text{BZ}})} \text{ for } k_3 > \frac{1}{\lambda_2}. \]
(17)

By using these values we obtain by numerical integration of
\[ c_l^2 \approx \frac{k_B T_m \cdot 0.36}{a^3 c_{46}(K_{\text{eff}}/a^2,0)c_{44}(K_{\text{eff}}/a^2,0)} + \frac{k_B T_m a_3 \cdot 1.60}{a^3 c_{44}(K_{\text{eff}}/a^2,0)} \]
\[ \approx \frac{k_B T_m a_3^2}{\phi_0 \lambda_b} \frac{159}{a^3} \sqrt{1 + \lambda_2 K_{\text{BZ}}} + \frac{k_B T_m a_3^2}{\phi_0 a^3 \log(1/c_l^2)} \lambda_c. \]
(18)

The first term comes from the integration region \( |k_3| < 1/\lambda_2 \) the second from the region \( 1/\lambda_2 < |k_3| < \pi / a_3 \) in (17).
a good approximation in these ranges is well known. At the low temperature side the discrepancy comes mainly from Josephson decoupling of the layers \cite{Glazman1} most pronounced for the large anisotropic BSCCO superconductor which results then also on large pinning effects \cite{Blasius1}.

Near $T \approx T_c$ our model does not include the increase of the entropy by thermal creation of vortices beside the magnetical ones forming the lattice \cite{Ryu1}. For YBCO, also order parameter fluctuations become important \cite{Ginzburg1}.

\begin{thebibliography}{24}

[1] D. R. Nelson, Phys. Rev. Lett. 60, 1973 (1988).
[2] A. Houghton et al., Phys. Rev. B 40, 6763 (1989); E. H. Brandt, Phys. Rev. Lett. 63, 1106 (1989).
[3] P. Kim et al., Phys. Rev. Lett. 77, 5118 (1996).
[4] B. Keimer et al., Phys. Rev. Lett. 73, 3459 (1994).
[5] H. Won and K. Maki, Phys. Rev. B 53, 5927 (1996).
[6] M. Tinkham, Introduction to Superconductivity, McGraw-Hill, New York, 1969.
[7] G. Blatter, V. Geshkenbein, A. Larkin, and V. Vinokur, Phys. Rev. B 54, 73 (1996).
[8] H. Kleinert, Gauge Fields in Condensed Matter, Vol. II, Stresses and Defects, World Scientific, Singapore, 1989. (readable online at \url{www.physik.fu-berlin.de/~kleinert/re.html}).
[9] J. Dietel and H. Kleinert, in preparation.
[10] S. E. Hikami, A. Fujita, and A. I. Larkin, Phys. Rev. B 44, 10400 (1991).
[11] E. H. Brandt, Rep. Prog. Phys. 58, 1465 (1995).
[12] M. C. Marchetti and D. R. Nelson, Phys. Rev. B 41, 1910 (1990); J. Kierfeld and V. Vinokur, Phys. Rev. B 61, R14928 (2000).
[13] R. Labusch, Physics Letters 22, 9 (1966).
[14] L. I. Glazman and A. E. Koshelev, Phys. Rev. B 43, 2835 (1991).
[15] M. J. Dodson et al., Phys. Rev. Lett 80, 837 (1998).
[16] S. Kamal et al., Phys. Rev. Lett. 73, 1845 (1994).
[17] A. Schilling et al., Nature (London) 382, 791 (1996).
[18] U. Welp et al., Phys. Rev. Lett. 76, 4809 (1996).
[19] M. Willemen et al., Phys. Rev. Lett. 81, 4236 (1998).
[20] E. Zeldov et al., Nature (London) 375, 791 (1995).
[21] K. Kadawki and K. Kimura, Phys. Rev. B 57, 11674 (1998).
[22] T. Blasius et al., Phys. Rev. Lett. 82, 4926 (1999).
[23] S. Ryu and D. Stroud, Phys. Rev. B 57, 14476 (1998).
[24] V. L. Ginzburg, Fiz. Tverd. Tela 2, 2031 (1960) [Sov. Phys. Solide State 2, 1824 (1961)].
\end{thebibliography}
Defect Induced Melting of Vortices in High-\(T_c\) Superconductors

Jürgen Dietel and Hagen Kleinert

Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany

(Dated: Received March 23, 2002)

We set up a melting model for vortex lattices in high-temperature superconductors based on the continuum elasticity theory. The model is Gaussian and includes defect fluctuations by means of a discrete-valued vortex gauge field. We derive the melting temperature of the lattice and predict the size of the Lindemann number. Our result agrees well with experiments for \(\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}\), and with modifications also for \(\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8\). We calculate the jumps in the entropy and the magnetic induction at the melting transition.

PACS numbers: 74.25.Qt, 74.72.-h

I. INTRODUCTION

The magnetic flux lattices in high-temperature superconductors can undergo a melting transition as was first suggested by Nelson in 1988 \[1\] \[2\] previously Brezin et al. \[3\] had calculated a first-order liquid to solid phase transition by renormalization group methods \[4\]. Since then detailed properties of this transition have been studied in various theoretical and experimental papers.

Most prominent are computer simulations methods based on the Langevin equation \[5\] for the dynamics of the vortices or Monte Carlo simulations on the XY model \[6\] coupled to an external magnetic field. Analytic approaches are based mainly on the Ginzburg-Landau model \[7\] or elastic models of the vortex lattice. The simplest estimates for the transition temperature in the vortex lattice came from an adaption of the famous Lindemann criterion of three-dimensional ordinary crystals \[8\]. Due to Houghton et al. and Brandt \[9\], the criterion states that the vortex lattice undergoes a melting transition once the mean thermal displacement \(\langle u^2 \rangle^{1/2}\) reaches a certain fraction of the lattice spacing \(a \approx \langle \Phi_0/B \rangle^{1/2}\), where \(\Phi_0\) is the flux quantum and \(B\) the magnetic induction. The ratio \(c_L \equiv \langle u^2 \rangle^{1/2}/a\) is the characteristic Lindemann number, which should be independent of \(B\). Its value is not predicted by Lindemann’s criterion. It must be extracted from experiments, and is usually found to lie in the range \(c_L \approx 0.1 - 0.3\). The most prominent examples of high-temperature superconductors which exhibit vortex lattice melting are the anisotropic compound \(\text{YBa}_2\text{Cu}_3\text{O}_{7-\delta}\) (YBCO), and the strongly layered compound \(\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_8\) (BSCCO). Decoration experiments on BSCCO \[10\] show the formation of a triangular vortex lattice, neutron scattering on YBCO of a tilted square lattice of vortices \[11\] close to the melting region, the latter being favored by the \(d\)-wave symmetry of the order parameter and the anisotropy of the crystal \[12\]. An explicit calculation of the Lindemann number \(c_L \equiv \langle u^2 \rangle^{1/2}/a\) for YBCO can be found in Ref. \[13\] and for BSCCO in Ref. \[14\].

In this paper, we present a theory which is capable of specifying the size of the Lindemann number \(c_L\). Basis is a simple Gaussian model which takes into account both lattice elasticity and defect degrees of freedom in the simplest possible way. The relevance of defect fluctuations for the understanding of melting transitions is well-known. For ordinary crystals, this is textbook material \[15\]. In the context of vortex melting it was emphasized in Ref. \[16\]. For ordinary crystals, the size of the Lindemann number has been calculated successfully by means of Gaussian lattice models with elastic and defect fluctuations, which clearly display first-order melting transitions in three dimensions and both first-order or sequence of continuous transitions in two dimensions.

An important quality of these models is that in the first-order case, where fluctuations are small, they lead to a simple universal melting formula determining the melting point in terms of the elastic constants. The universal result is found from a lowest-order approximation, in which one identifies the melting point with the intersection of the high-temperature expansion of the free energy density, dominated by defect fluctuations with the low-temperature expansion dominated by elastic fluctuations. The resulting universal formula for the melting temperature determines also the size of the Lindemann number. Recently, the results of Ref. \[17\] for square crystals were successfully extended to face-centered and body-centered cubic lattices in three dimensions \[18\] and also to two-dimensional triangular lattices \[19\]. A similar intersection criterium was also used for the melting point of vortex lattices in the Abrikosov approximation of the Ginzburg-Landau model \[20\] useful for YBCO. Here we shall apply our defect model to calculate the melting curves, the entropy jumps, and magnetic induction jumps of the vortex lattices in YBCO and BSCCO. We do not discuss in this work effects on the vortex lattice from other sources than defect fluctuations which can give rise to tricritical points and glass transitions \[21\].

Theoretical work on this subject can be found in Refs. \[18\] \[19\] \[20\]. There is in principle no problem of adding pinning in our formalism. For simplicities, we shall confine our discussion to the defect mechanism of melting.

The melting criterium will be derived in Section II. The calculation of the melting temperatures, the entropy jumps, and the jumps of magnetic induction jumps for YBCO and BSCCO is carried out in Section III.
II. MELTING CRITERION

Due to the large penetration depth $\lambda_{ab}$ in the layers in comparison to $a$ we have to take into account the full non-local elasticity constants when integrating over the Fourier space, as emphasized by Brandt in Ref. [21]. For our Gaussian model, the partition function of the vortex lattice can be split into $Z = Z_0 Z_h$ where $Z_0$ is the partition function of the rigid lattice and $Z_h$ is thermally fluctuating part calculated via the elastic Hamiltonian plus defects. Due to the translational invariance of the vortex lattice in the direction of the vortices, which we shall take to be the $z$-axis, we may simply extend the models on square [2] and triangular lattices [12] by a third dimension along the $z$-axis, which we artificially discretize to have a lattice spacing $a_3$, whose value will be fixed later. The elastic energy is

$$E_{el} = \frac{v}{2} \sum_x (\nabla_i u_i)(c_{11} - 2c_{66})(\nabla_i u_i)$$

$$+ \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) c_{66} (\nabla_i u_j + \nabla_j u_i) + (\partial_z u_i) c_{44} (\partial_z u_i).$$

The subscripts $i, j$ values 1, 2 and $l, m, n$ have values 1, $\ldots$, 3. The vectors $u_i(x)$ are given by the transverse displacements of the line elements of the vortex lines with coordinate $x$. We have suppressed the spatial arguments of the elasticity parameters, which are really functional matrices $c_{ij}(x,x') \equiv c_{ij}(x-x')$. Their precise forms were calculated by Brandt [21]. The volume of the fundamental cell $v$ is equal to $a_2 a_3 (\text{square})$ or $a_2 a_3 \sqrt{3}/2 (\text{triangular})$. For a square lattice, the lattice derivative $\nabla_i$ in (1) are given by $\nabla_i f(x) = [f(x + a e_i) - f(x)]/a$ and $\nabla_3 f(x) = [f(x + a_3 e_3) - f(x)]/a_3$. For a triangular lattice, the $xy$-part of the lattice has the link vectors $\pm a e_{(m)}$ with $e_{(3)} = (cos 2\pi/6, \pm \sin 2\pi/6, 0)$ and $e_{(2)} = (-1, 0, 0)$. The lattice derivatives around a plaquette are defined by $\nabla_1 f(x) = [f(x + a e_{(1)}) - f(x)]/a, \nabla_2 f(x) = [f(x - f(x - a e_{(2)})]/a, \nabla_3 f(x) = [f(x - a e_{(3)}) - f(x + a e_{(1)})]/a$. From these we define discrete cartesian lattice derivatives used in the Hamiltonian (1) $\nabla_i f(x) = (2/3) e_{(i)} \nabla_i f(x)$ and $\nabla_3 f(x) = [f(x + a_3 e_3) - f(x)]/a_3$ transforming like the continuum derivative with respect to the symmetry group of the lattice [17]. Therefore, the Hamiltonian (1) has the full symmetry of the triangular lattice and the correct continuum elastic energy for zero lattice spacing.

Within the elastic approximation the displacement fields are restricted to values within the fundamental cell. In order to contain also defect degrees of freedoms one has to insert into (1) integer valued defect gauge fields [7]. In the canonical stress representation, the partition function containing these fields becomes

$$Z_h = \det \left[ \frac{c_{66}}{4(c_{11} - c_{66})} \right]^{1/2} \frac{1}{(2\pi \beta)^{\frac{3}{2}} \prod_{i \leq m} \int_{-\infty}^{\infty} d\sigma_{im}} \left[ \prod_{m, n_m(x) = -a} \sum_{\sigma_{im}} \int_{-\infty}^{\infty} \frac{du}{a} \exp \left\{ -\frac{1}{4\beta} \sum_{x} \left( \sum_{i \leq m} \nabla_i u_i \sigma_{im} + \sum_{i \leq j} D_{ij} \sigma_{ij} \right) \right\} \right] e^{2\pi i \sum_{x} \left( \sum_{i \leq m} \nabla_i u_i \sigma_{im} + \sum_{i \leq j} D_{ij} \sigma_{ij} \right)}.$$

The parameter $\beta$ is given by $\beta = v c_{66}/k_B T(2\pi)^2$, $\sigma_{ij}$ represent stress fields \[7\]. The matrix $D_{ij}(x)$ in Eq. (4) is a discrete-valued local defect matrix composed of integer-valued defect gauge fields $n_1, n_2, n_3$ for square \[2\] and triangular vortex fields \[12\] as follows:

$$D_{ij}^\square = \begin{pmatrix} n_1 & n_3 \\ n_3 & n_2 \end{pmatrix},$$

$$D_{ij}^\triangle = \begin{pmatrix} 1/2 n_1 & 1/2 n_2 \\ 1/2 n_2 & 1/2 n_3 \end{pmatrix} \begin{pmatrix} 1/2 n_1 - 1/2 n_2 + 1/2 n_3 \\ 1/2 n_1 + 1/2 n_2 - 1/2 n_3 \end{pmatrix}.$$

The vortex gauge fields specify the Volterra surfaces in units of the Burgers vectors. By summing over all $n_{1,2,3}(x)$, the partition function $Z_h$ includes all defect fluctuations, dislocations as well as disclinations. There is a constraint for a vortex lattice which does not exist for ordinary three-dimensional lattices. Dislocations in the vortex lattice can be both screw or edge type, but in either case the defect lines are confined in the plane spanned by their Burgers vector and the magnetic field \[13, 22\]. The reason is that the flux lines in a vortex lattice cannot be broken. This results in the constraint $D_{11} = D_{22}$ on the defect fields.

We now calculate the low-temperature expansion of the partition function $Z_h$ to lowest order, which includes only the $n_m = 0$-term. By carrying out the integration over the displacement fields $u_i(x)$ in (2) we obtain, as in \[7\], the leading term in the low-temperature expansion of the free energy

$$Z_h^{\rightarrow 0} \equiv \left( \frac{a_3}{a} \right)^{2N} \frac{1}{\det[(2\pi \beta)c_{44}/c_{66}]} e^{-N \sum_{i \in \{1, 3\}} l_{ii}},$$

where

$$l_{ii} = \frac{1}{2} \frac{1}{V_{BZ}} \int_{BZ} d^2 k \log \left[ \frac{c_{ii} a_3^2}{c_{44}} K_i K_j + a_3^2 K_i^2 K_3 \right].$$

(6)
Here $K_m$ is the eigenvalue of $i\nabla_n$. The momentum integrations in (6) run over the Brillouin zone of the vortex lattice whose volume is $V_{BZ} = (2\pi)^3/v$, as indicated by the subscript BZ.

Next we calculate the high-temperature expansion $Z^{T\rightarrow\infty}_\Delta$ to lowest order. By carrying out the integration over the displacement fields $u_m(x)$ in (6) and further by summing over the defect fields $n_m$ under the condition $D_{11} = -D_{22}$ mentioned above, it turns out that the stress fields $\sigma_{12}$ and $\sigma_u \equiv \sigma_{11} - \sigma_{22}$ can have only integer numbers. Doing the integrals over the stress fields $\sigma_{12}$ and $\sigma_u \equiv \sigma_{11} + \sigma_{22}$, we obtain, in the lowest order high temperature limit, corresponding to $\sigma_{12} = 0$ and $\sigma_u = 0$:

$$Z^{T\rightarrow\infty}_\Delta = \left(\frac{a_3}{a}\right)^{2N} \frac{C_N}{2^N} \frac{1}{\det \left[ (2\pi\beta)^2 c_{44} / c_{66} \right]} e^{-N\Delta}$$

with

$$h = \frac{1}{2} \frac{1}{V_{BZ}} \int_{BZ} d^2 k d^3 k_3 \log \left[ 1 + \frac{c_{11} - c_{66}}{c_{44}} \frac{K_1^* K_3}{K_3^* K_3} \right].$$

The constant $C$ has the values $C_{\square} = 1$ for the square vortex lattice and $C_{\triangle} = \sqrt{3}$ for the triangular one.

In the low-temperature expansion representing the solid phase, defect field configurations $n_m \neq 0$ correspond to dislocations and disclinations giving finite temperature corrections to the free energy $-\log(Z) / k_B T$. These corrections are exponentially small with an exponent proportional to $-\beta$. In contrast to this corrections to the high-temperature expansion in the fluid phase corresponding to stress configurations $\sigma_{12} \neq 0$ and $\sigma_u \neq 0$ of integer values result in temperature corrections to the free energy which are also exponentially small with an exponent proportional to $-1/\beta$. The structure of the high- and low-temperature corrections to the partition function is extensively discussed in Refs. [7, 12] for ordinary crystals, and can be easily transferred to our case of vortex lattices. It was shown in Refs. [7, 12] for the two dimensional square and triangular as well as the three dimensional square crystal that the exponentially vanishing higher order corrections to the low- and high-temperature expansion of the free energy are negligible in the determination of the transition temperature. This is particularly true for the three dimensional crystal (see p. 1082 in [7]) which we take as a justification to restrict our calculation to lowest order in this paper.

From the partition function (6) with no defects ($n_m = 0$) we obtain for the Lindemann number $c_{44} = \langle u^2 \rangle^{1/2} / a$ the momentum integral

$$c_{44}^2 = \frac{a_3^2 k_B T}{a^2 v V_{BZ}} \int_{BZ} d^2 k d^3 k_3 \frac{1}{c_{44}} \sum_{i = 1, 2} \frac{c_{i1} c_{i2}}{c_{44}} K_i^* K_j + a_3^2 K_3^* K_3.$$

This can be simplified by taking into account that $c_{44}$ is much larger than $c_{66}, c_{44}$ in the relevant regime [21, 22]. As announced, we find the melting temperature from the intersection of low- and high-temperature expansions, obtained by equating $Z^{T\rightarrow\infty}_\Delta = Z^{T\rightarrow\infty}_{\infty}$. By taking into account $\det[a_3^2 \nabla_n^2 \nabla_n^2] = 1$ we obtain $h, c_{11} \ll c_{66}$, and further that the $i = 1$-term in (9) is much smaller than the $i = 6$-term. In the following analytic discussion (but not in the numerical plots) we neglect these small contributions. The temperature of melting is then given by the simple formula

$$\frac{k_B T}{v c_{66}} \frac{1}{\det^{1/N} \left[ c_{66} \right]} C = e^{-\frac{\ln 2}{\pi}},$$

where $\det[c_{66}]$ is the determinant of the $N \times N$ functional matrix $c_{66}$. The elastic moduli $c_{44}$ and $c_{66}$ at low reduced magnetic fields $b = B/H_{c2} < 0.25$ can be taken from Brandt’s paper [21]

$$c_{66} = \frac{B \phi_0 \zeta}{(8\pi \lambda_{ab})^2},$$

$$c_{44} = \frac{B^2}{4\pi (1 + \lambda_{c}^2 k^2 + \lambda_{ab}^2 k^2) } + \frac{B \phi_0}{32\pi^2 \lambda_{ab} K_{BZ}} \ln \frac{1 + \lambda_{c}^2 k^2 / (1 + \lambda_{c}^2 K_{BZ})}{1 + \lambda_{ab}^2 k^2 / (1 + \lambda_{ab}^2 / (u^2))}.$$

where $\lambda_{c}$ is the penetration depth in the $xy$-plane, $\zeta = 1$, and $K_{BZ}$ is the boundary of the circular Brillouin zone $K_{BZ} = 4\pi B / \phi_0$. At high fields ($b > 0.5$), $c_{66}$ is altered by a factor $\zeta \approx 0.71 (1 - b)$, and the penetration depths in $c_{66}, c_{44}$ are replaced by $\lambda_{bc} = \lambda^2 / (1 - b)$, where $\lambda$ denotes either $\lambda_{ab}$ or $\lambda_{abc}$. In addition, the last two terms in $c_{44}$ are replaced by $B \phi_0 / 16\pi \lambda_{c}^2$. For YBCO we have [24] $\lambda(T) = \lambda(0) \left[ 1 - \left( T/T_c \right) \right]^{-1/2}$. For BSCCO [27] $\lambda(T) = \lambda(0) \left[ 1 - \left( T/T_c \right) \right]^{-1/2}$. For $T = \xi(0) \left[ 1 - \left( T/T_c \right) \right]^{-1/2}$. When calculating $c_{44}$ we have used a momentum cutoff in the two-vortex interaction potential $k \leq 2/(u^2)^{1/2}$, and not the inverse of the correlation length $1/\xi$ as in Ref. [21]. The cutoff is due to thermal softening [7], and becomes relevant for $\langle u^2 \rangle^{1/2} / \xi \gg 1$, or equivalently for $c_{44} V_{BZ} H_{c2} / B \gg 1$, which is fulfilled in the melting regime of BSCCO, but not for YBCO.

It remains to determine the effective lattice spacing $a_3$ along the vortex lines. An elementary defect in the vortex lattice (arising for example from a crossing of two vortex strings) takes place over a typical length scale in the $z$-direction determined from the condition that the sum of elastic displacement energy and the energy required to stretch the line against the line tension is minimal. It is the elastic energy of this smallest defect that has to be taken into account for in our model. The energy of an ensemble of dislocations is determined by the interplay of elastic energy of small displacements and integer-valued defect fields. The relevant part of the free energy is given by the discretized free energy $-\log(Z_{\Delta} Z_{\pi}) k_B T$ in which $a_3$ is equal to the above length scale in the $z$-direction. To determine this, we insert the variational ansatz for the transverse displacement field $u_i = \delta_{i1} A_0 \exp(-2|z|/a_3)$ into
the continuum version (in $z$ direction) of the elastic energy and approximate $-\nabla_{z}^2 \approx (K^2) \approx K_{BZ}/4$ in $E_{d1}$ and $K^2 \approx (K^2) \approx K_{BZ}/2$ in the elastic constants, where the average $\langle \ldots \rangle$ was taken with respect to a circular Brillouin zone. The optimal length scale $a_{5}$ is chosen such that $E_{d1}$ is minimal for a fixed amplitude $A_{0} \approx a_{3}$ corresponding to a typical defect elongation.

### III. APPLICATION TO YBCO AND BSCCO

In the following, we treat first the more isotropic square vortex crystal YBCO ($a = \sqrt{\phi_{0}/B}$). From $c_{46}$ and $c_{44}$ for YBCO, the optimal length scale is given by $a_{3} = 4a_{3}a_{4}/\lambda_{c}(1-b)^{1/2}$. When comparing the melting criterion of the defect model in Eq. (10) with the Lindemann criterion obtained by equating the parameter (9) to a universal number, we obtain identical results when taking into account that the integrand in (9) and in $l_{66}$ of Eq. (14) receive their main contribution from the region $k \approx \sqrt{k_{BZ}/\sqrt{2}}$. We can approximate $k_{3} \approx 0$ in this region, resulting in $a_{3}^{2}c_{46}/a_{3}c_{44} \approx 4/\pi$. With the same approximation in Eqs. (6) and (9), we can perform the integrals numerically. Then we obtain from the melting condition (10) precisely the Lindemann criterion in which the Lindemann number (9) is predicted to be

$$\frac{k_{B}T_{m}}{4 \left[c_{44}(\frac{K_{B}}{\sqrt{2}}, 0) c_{66}(\frac{K_{B}}{\sqrt{2}}, 0)\right]^{1/2}} \approx c_{L}^{2}(\approx 0.18)^{2}. \quad (13)$$

Denoting the spacing between the CuO$_{2}$ double layers by $a$, we obtain for the entropy jump per double layer and vortex

$$\Delta S_{I} \approx k_{B}T_{m}(\partial/\partial T_{m})(a_{3}/a_{3}) \ln[Z_{h}^{T \rightarrow \infty}/Z_{h}^{T \rightarrow 0}]. \quad (14)$$

Inserting (9) and (14), this becomes

$$\Delta S_{I} \approx k_{B}T_{m}a_{3} \frac{\partial}{\partial T_{m}} \ln \left[\frac{k_{B}T_{m}}{c_{44}(\frac{K_{B}}{\sqrt{2}}, 0) c_{66}(\frac{K_{B}}{\sqrt{2}}, 0)}\right]. \quad (15)$$

Finally, we make use of the Clausius-Clapeyron equation which relates the jump of the entropy density across the melting transition to the jump of the magnetic induction by $\Delta S_{I}a_{3}/\partial a_{3} = -\langle dH_{m}/dT \rangle \Delta B/4\pi$. Here $H_{m}$ is the external magnetic field on the melting line. Combining the Clausius-Clapeyron equation with Eqs. (12) and (15), we obtain, with the abbreviation $\tau_{m} \equiv T_{m}/T_{c}$, the following relations near $T_{c}$:

$$B_{m}(T) \approx \frac{12\zeta(1-(T/T_{c})^{4/3})}{16\zeta^{4}(k_{B}T)^{2}a_{3}^{2}\lambda_{ab}(0)\phi_{0}^{2}} \quad (16)$$

$$\Delta S_{I} \approx \frac{\sqrt{\zeta}}{6a} \frac{a_{3}}{\lambda_{ab}(1-\tau_{m})} \approx \frac{2.7}{10^{8} T_{c}(1-\tau_{m})^{1/3} \lambda_{ab}(0)}.$$  

$$\Delta B \approx \frac{\sqrt{\zeta}}{2a_{3}\lambda_{ab}k_{B}T_{m}} \approx \frac{2.5}{10^{2}} (1-\tau_{m})^{2/3} \frac{c_{L}^{2}\phi_{0}^{2}}{\lambda_{ab}(0)}. \quad (16)$$

These results agree with the general scaling results in Ref. (26), with the advantage that here the prefactors are predicted whereas those in (26) had to be determined by fits to experimental curves (there is only a slight discrepancy because we use a different temperature dependence of the penetration depth).

Next, we calculate the corresponding expressions in the case of the more layered crystal BSCCO ($a = (21/2)/3^{1/4}\sqrt{\phi_{0}/B}$). First, we have to determine the dislocation length scale $a_{3}$ in this case. For dislocation moves we have $\langle a_{3}^{2}\rangle^{1/2} \approx 1/K_{BZ}$. This means that we can neglect the last two terms of $c_{44}$ in (12), coming from the self-energy of the vortex line, when determining $a_{3}$. Remembering this we obtain by a similar procedure as for YBCO the dislocation length scale $a_{3} \approx 4a_{3}\sqrt{2}\lambda_{ab}/\lambda_{c}\sqrt{\pi}$. From this we find $a_{3}c_{66}/a_{3}^{2}c_{44} \ll 1$, resulting in $l_{66} \approx 0$. By taking into account that $B_{m}^{3}\lambda_{ab}^{2}/32\phi_{0} \lesssim 1$ on the melting line we obtain that $c_{44}(k_{3}, k_{3})$ for $|k_{3}| < \pi/a_{3}$ is dominated by the last term in (12). Then we obtain

$$c_{44}(k_{3}, k_{3}) \approx \frac{B\phi_{0}}{32\pi^{2}\lambda_{ab}^{2}(1+\lambda_{ab}^{2}K_{BZ}^{2}),} \quad (17)$$

$$c_{44}(k_{3}, k_{3}) \approx \frac{B\phi_{0} \ln(1+2BA_{3}(\phi_{0})^{2})}{32\pi^{2}\lambda_{ab}^{2}k_{3}^{2}}, \quad (18)$$

$$\frac{k_{B}T_{m}a_{3}}{\phi_{0}^{2} a} \sqrt{1 + \lambda_{ab}^{2}K_{BZ}^{2}} + \frac{k_{B}T_{m}a_{3}^{2} \lambda_{ab}^{2} \lambda_{ab}^{2} - 137 \lambda_{ab}}{\phi_{0}^{2} a^{2} \ln(1/c_{L}^{2})} \lambda_{c}. \quad (19)$$

The first term comes from the integration region $|k_{3}| < 1/\lambda_{c}$, the second from the region $1/\lambda_{c} < |k_{3}| < \pi/a_{3}$ in (19).

From our melting criterion (10) and the Clausius-Clapeyron equation (where $dH_{m}/dT \approx dB_{m}/dT$ due to $B_{m}(T) \approx H_{c1}(T)$ (23)), we obtain for BSCCO

$$B_{m}(T) \approx \frac{1}{192 \sqrt{3\pi}} (1-\pi^{2}) \lambda_{c}(0) \lambda_{ab}^{2}(0) \lambda_{ab}^{2}(0) T_{m}(1-\tau_{m})a_{3}^{2}\phi_{0}^{2}, \quad (16)$$

$$\Delta S_{I} \approx \frac{\sqrt{\zeta}}{6a} \frac{a_{3}}{\lambda_{ab}(1-\tau_{m})} \approx \frac{2.7}{10^{8} T_{c}(1-\tau_{m})^{1/3} \lambda_{ab}(0)}.$$  

$$\Delta B \approx \frac{\sqrt{\zeta}}{2a_{3}\lambda_{ab}k_{B}T_{m}} \approx \frac{2.5}{10^{2}} (1-\tau_{m})^{2/3} \frac{c_{L}^{2}\phi_{0}^{2}}{\lambda_{ab}(0)}. \quad (16)$$

Parameter values for optimal doped YBCO (BSCCO) are given by (24) $\lambda_{ab}(0) \approx 1186 \AA$ ($\lambda_{ab}(0) \approx 2300 \AA$), $\epsilon_{ab}(0) \approx 15 \AA$ ($\epsilon_{ab}(0) \approx 30 \AA$), the CuO$_{2}$ double layer spacing $a_{3} = 12 \AA$ ($a_{3} = 14 \AA$), $T_{c} = 92.7 \, \text{K}$ ($T_{c} = 90 \, \text{K}$) and the anisotropy parameter $\gamma = \lambda_{c}/\lambda_{ab} \approx 5$ ($\gamma \approx 200$).

We now calculate numerically the melting curves, the associated Lindemann parameter $c_{L}$, the entropy and the magnetic induction jumps $\Delta S_{I}$ and $\Delta B$ from the intersection criterion of the full low- and high-temperature
curves are the Lindemann numbers \cite{27} for BSCCO. The numbers on the theoretical melting curve at low temperatures \cite{34} resulting in a decrease of entropy and the magnetic induction jumps in Fig. 2 are in reasonable agreement with the experimental curves, except at the end point of the melting line at low temperatures and near \( T \approx T_c \). It could have been anticipated that our vortex lattice model is not a good approximation in these regimes. At low temperature, the discrepancy comes mainly from Josephson decoupling of the layers \cite{32}, most pronounced for the strongly anisotropic BSCCO superconductor, which leads also to large pinning effects \cite{33}. We think that this is also the reason for the difference in the curves of Kadowki et al. \cite{31} and of Zeldov et al. \cite{27} shown in Fig. 2. Pinning has the largest influence on the form of the melting curve at low temperatures \cite{34} resulting in a decrease of \( \Delta S_I \) and \( \Delta B \) \cite{20} in the limit of low temperatures shown by the curves of Zeldov et al. Near \( T \approx T_c \), our model does not include the increase of the entropy by thermal creation of vortices in addition to the ones caused by the external magnetic field which form the lattice \cite{32}. For YBCO, also order parameter fluctuations become important \cite{30} which are ignored here.

Summarizing, we obtain within our formalism the melting curve, the entropy and the magnetic jump within a single theory. This is not possible by the widely used Lindemann rule. Our curves agree well with the experimental curves for YBCO and BSCCO except at the endpoints. We gave analytical approximations for the various quantities. Our theory is a direct generalization of the elastic theory of vortex displacements by taking into account the defect degrees of freedom of the lattice which makes it possible to obtain the physics of the liquid phase of the vortex lattice beside the solid phase.

IV. DISCUSSION

Our approximate analytic results \cite{14} and \cite{15} for YBCO and BSCCO turn out to give practically the same curves. For comparisons, we show in both figures the experimental curves for YBCO of Ref. \cite{28, 29, 30} and for BSCCO of Ref. \cite{27, 31}. We see in Fig. 1 and Eq. \cite{13} that the Lindemann number is independent of the magnetic field for YBCO for the entire temperature range. For BSCCO we obtain a magnetic field dependence of the Lindemann parameter near \( T_c \). This comes mainly from the second term in \cite{13}, having its largest variation near \( T_c \). The theoretical melting curves in Fig. 1 and the entropy and the magnetic induction jumps in Fig. 2 are in reasonable agreement with the experimental curves, except at the end point of the melting line at low temperatures and near \( T \approx T_c \). It could have been anticipated that our vortex lattice model is not a good approximation in these regimes. At low temperature, the discrepancy comes mainly from Josephson decoupling of the layers \cite{32}, most pronounced for the strongly anisotropic BSCCO superconductor, which leads also to large pinning effects \cite{33}. We think that this is also the reason for the difference in the curves of Kadowki et al. \cite{31} and of Zeldov et al. \cite{27} shown in Fig. 2. Pinning has the largest influence on the form of the melting curve at low temperatures \cite{34} resulting in a decrease of \( \Delta S_I \) and \( \Delta B \) \cite{20} in the limit of low temperatures shown by the curves of Zeldov et al. Near \( T \approx T_c \), our model does not include the increase of the entropy by thermal creation of vortices in addition to the ones caused by the external magnetic field which form the lattice \cite{32}. For YBCO, also order parameter fluctuations become important \cite{30} which are ignored here.

Summarizing, we obtain within our formalism the melting curve, the entropy and the magnetic jump within a single theory. This is not possible by the widely used Lindemann rule. Our curves agree well with the experimental curves for YBCO and BSCCO except at the endpoints. We gave analytical approximations for the various quantities. Our theory is a direct generalization of the elastic theory of vortex displacements by taking into account the defect degrees of freedom of the lattice which makes it possible to obtain the physics of the liquid phase of the vortex lattice beside the solid phase.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{Melting curve \( B = B_m(T) \) for YBCO and BSCCO. The experimental values are from Ref. \cite{28} for YBCO and Ref. \cite{27} for BSCCO. The numbers on the theoretical melting curves are the Lindemann numbers \( c_L \) calculated from \cite{9}.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Entropy jump per double layer per vortex \( \Delta S_I \) (first row) and jump of magnetic induction field \( \Delta B \) (second row) at the melting transition. The experimental values for YBCO are from Ref. \cite{28} for \( \Delta S_I \), Ref. \cite{24} for \( \Delta B \) by SQUID experiments (SQUID), and Ref. \cite{29} by torque measurements (Torque). The experimental values for BSCCO are from Ref. \cite{31} by SQUID measurements (SQUID) of the magnetic field and Ref. \cite{27} by microscopic Hall sensors (Hall).}
\end{figure}

expressions \cite{13} and \cite{14} without further approximation. To accomplish this, we use the elastic constant \( c_{11} \) given by Brandt in Ref. \cite{21}. The intersection criterium of the low and high temperature expansion of the partition function is then at least in the case of BSCCO a complicated integral equation via the dependence of \( c_{44} \) on the Lindemann parameter \( c_L \). One can solve this integral equation by numerical methods. The results are shown in Fig. 1 and Fig. 2.

\begin{thebibliography}{10}
\bibitem{1} D. R. Nelson, Phys. Rev. Lett. \textbf{60}, 1973 (1988).
\bibitem{2} D. R. Nelson and H. S. Seung, Phys. Rev. B \textbf{39}, 9153 (1989).
\bibitem{3} E. Brezin, D. R. Nelson, and A. Thiaville, Phys. Rev. B \textbf{31}, 7124 (1985).
\bibitem{4} A. van Otterlo, R. T. Scalettar, and G. T. Zimnyi, Phys. Rev. Lett. \textbf{81}, 1497 (1998).
\bibitem{5} Y. Li and S. Teitel, Phys. Rev. Lett. \textbf{66}, 3301; R. E. Het-
\end{thebibliography}
zal, A. Sudbø, and D. A. Huse, Phys. Rev. Lett. 69, 518; [6] S. E. Hikami, A. Fujita, and A. I. Larkin, Phys. Rev. B 44, 10400 (1991); D. Li and B. Rosenstein, Phys. Rev. Lett. 86, 3618 (2001).

[7] H. Kleinert, Gauge Fields in Condensed Matter, Vol. II Stresses and Defects, World Scientific, Singapore, 1989. (readable online at www.physik.fu-berlin.de/~kleinert/re.html#b2)

[8] A. Houghton, R. A. Pelcovits and A. Sudbø, Phys. Rev. B 40, 6763 (1989); E. H. Brandt, Phys. Rev. Lett. 63, 1106 (1989).

[9] P. Kim , Z. Yao, and C. M. Lieber, Phys. Rev. Lett. 86, 448 (1991).

[10] B. Keimer , W. Y. Shih, R. W. Erwin, J. W. Lynn3, F. Dogan, and I. A. Aksay, Phys. Rev. Lett. 73, 3459 (1994); S. P. Brown, D. Charalambous, E. C. Jones, E. M. Forgan, P. G. Kealey, A. Erb, and J. Kohlbrecher, Phys. Rev. Lett. 92, 067004 (2004).

[11] H. Won and K. Maki, Phys. Rev. B 53, 5927 (1996).

[12] G. Blatter, V. Geshkenbein, A. Larkin, and H. Norborg, Phys. Rev. B 54, 73 (1996).

[13] M. C. Marchetti and D. R. Nelson, Phys. Rev. B 41, 1910 (1990); J. Kierfeld and V. Vinokur, Phys. Rev. B 61, R14928 (2000); J. Lidmar, Phys. Rev. Lett 91, 097001 (2003).

[14] H. Kleinert and Y. Jiang, Phys. Lett. A 313, 152 (2003).

[15] J. Dietel and H. Kleinert, Phys. Rev. B 73, 024113 (2006).

[16] F. Bouquet, C. Marcenat, E. Steep, R. Calemczuk, W. K. Kwok, U. Welp, G. W. Crabtree, R. A. Fisher, N. E. Phillips, A. Schilling, Nature (London) 411, 448 (2001).

[17] N. Avraham, B. Khaykovich, Y. Myasoedov, M. Rappaport, H. Shtrikman, D. E. Feldman, T. Tamegai, P. H. Kes, M. Li, M. Konczykowski, K. van der Beek , E. Zeldov, Nature (London) 411, 451 (2001); C. J. van der Beek, S. Colson, M. V. Indenbom, and M. Konczykowski, Phys. Rev. Lett. 84, 4196 (2000); H. Beidenkopf, N. Avraham, Y. Myasoedov, H. Shtrikman, E. Zeldov, B. Rosenstein, E. H. Brandt and T. Tamegai, Phys. Rev. Lett. 95, 257004 (2005).

[18] D. S. Fisher, M. P. A. Fisher, and D. A. Huse, Phys. Rev. B 43, 130 (1991).

[19] T. Giamarchi and P. L. Doussal, Phys. Rev. B 55, 6577 (1997); G. M. Mikitik and E. H. Brandt, Phys. Rev. B 64, 184514 (2001); G. Menon, Phys. Rev. B 65, 104527 (2001). Y. Radzyner et al., Phys. Rev. B 65, 100513 (2002); G. M. Mikitik and E. H. Brandt, Phys. Rev. B 68, 054509 (2003); J. Kierfeld and V. Vinokur, Phys. Rev. B 69, 024501 (2004).

[20] D. Li and B. Rosenstein, Phys. Rev. Lett. 90, 167004 (2003).

[21] E. H. Brandt, Rep. Prog. Phys. 58, 1465 (1995); see also G. Blatter, M. V. Feigel’man, V. B. Geshkenbein, A. I. Larkin, and V. M. Vinokur, Rev. Mod. Phys. 66, 1125 (1994).

[22] R. Labusch, Physics Letters 22, 9 (1966).

[23] By using the elastic constants from Ref. 21 we obtain that $c_{11} \gg c_{44}, c_{66}$ almost everywhere on the melting line except in a small vicinity near $T^* \approx T_c$. Here we take into account that $|k_3| < \pi/a_3 \approx \pi a_1/\lambda_{ab}$ and further that $B \lambda_{ab}^2/\varphi_0 \gg 1/16\pi$ is fulfilled almost everywhere on the melting line (except in a region where $|T/T_c - 1| < 0.02$ for BSCCO, the corresponding region for YBCO is even smaller).

[24] S. Kamal, D. A. Bonn, N. Goldenfeld, P. J. Hirschfeld, R. Liang, and W. N. Hardy, Phys. Rev. Lett. 73, 1845 (1994).

[25] M. Tinkham, Introduction to Superconductivity, McGraw-Hill, New York, 1996.

[26] M. J Dodgson, V. B. Geshkenbein, H. Nordborg, and G. Blatter, Phys. Rev. Lett 80, 837 (1998).

[27] E. Zeldov, D. Majer, M. Konczykowski, V. B. Geshkenbein, V. M. Vinokur, and H. Shtrikman, Nature (London) 375, 791 (1995).

[28] A. Schilling, R. A. Fisher, and G. W. Crabtree, Nature (London) 382, 791 (1996).

[29] U. Welp, J. A. Fendrich, W. K. Kwok, G. W. Crabtree, and B. W. Veal, Phys. Rev. Lett. 76, 4899 (1996).

[30] M. Willemim, A. Schilling, H. Keller, C. Rossel, J. Hofer, U. Welp, W. K. Kwok, R. J. Olsson, and G. W. Crabtree, Phys. Rev. Lett. 81, 4236 (1998).

[31] K. Kadowki and K. Kimura, Phys. Rev. B 57, 11674 (1998).

[32] L. I. Glazman and A. E. Koshelev, Phys. Rev. B 43, 2835 (1991).

[33] T. Blasius, Ch. Niedermayer, J. L. Tallon, D. M. Poole, A. Golnik, and C. Bernhard, Phys. Rev. Lett. 82, 4926 (1999).

[34] B. Khaykovich, M. Konczykowski, E. Zeldov, R. A. Doyle, D. Majer, P. H. Kes and T. W. Li, Phys. Rev. B 56, R517 (1997).

[35] S. Ryu and D. Stroud, Phys. Rev. B 57, 14476 (1998).

[36] V. L. Ginzburg, Fiz. Tverd. Tela 2, 2031 (1960) [Sov. Phys. Solide State 2, 1824 (1961)].
Defect induced melting of vortices in high-Tc superconductors: A model based on continuum elasticity theory

Jürgen Dietel and Hagen Kleinert
Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
(Dated: Received March 23, 2022)

We set up a melting model for vortex lattices in high-temperature superconductors based on the continuum elasticity theory. The model is Gaussian and includes defect fluctuations by means of a discrete-valued vortex gauge field. We derive the melting temperature of the lattice and predict the size of the Lindemann number. Our result agrees well with experiments for YBa$_2$Cu$_3$O$_{7-δ}$, and with modifications also for Bi$_2$Sr$_2$CaCu$_2$O$_{8+δ}$. We calculate the jumps in the entropy and the magnetic induction at the melting transition.

PACS numbers: 74.25.Qt, 74.72.-h

I. INTRODUCTION

The magnetic flux lattices in high-temperature superconductors can undergo a melting transition as was first suggested by Nelson in 1988 [1, 2]. Previously, Brezin et al. [3] had calculated a first-order liquid to solid phase transition by renormalization group methods [3]. Since then detailed properties of this transition have emerged from various theoretical and experimental papers.

Most prominent are computer simulations of the Langevin equation [4] for the dynamics of the vortices, or Monte Carlo simulations of XY-type model [5] coupled to an external magnetic field. Analytic approaches are based mainly on the Ginzburg-Landau model [6], or on elastic models of the vortex lattice. The simplest estimates for the transition temperature in the vortex lattice came from an adaption of the famous Lindemann criterion of three-dimensional ordinary crystals [7]. In the formulation of Houghton et al. and Brandt [8], the criterion states that the vortex lattice undergoes a melting transition once the mean thermal displacement $\langle u^2 \rangle^{1/2}$ reaches a certain fraction of the lattice spacing $a \approx (\Phi_0/B)^{1/2}$, where $\Phi_0$ is the flux quantum and $B$ the magnetic induction. The ratio $c_L \equiv \langle u^2 \rangle^{1/2}/a$ is the characteristic Lindemann number, which should be independent of $B$. Its value is not predicted by Lindemann’s criterion. It must be extracted from experiments, and is usually found to lie in the range $c_L \approx 0.1 - 0.3$.

The most prominent examples of high-temperature superconductors which exhibit vortex lattice melting are the anisotropic compound YBa$_2$Cu$_3$O$_{7-δ}$ (YBCO), and the strongly layered compound Bi$_2$Sr$_2$CaCu$_2$O$_8$ (BSCCO). Decoration experiments on BSCCO [9] show the formation of a triangular vortex lattice, neutron scattering on YBCO of a tilted square lattice of vortices close to the melting region, the latter being favored by the d-wave symmetry of the order parameter and the anisotropy of the crystal [10]. An explicit calculation of the Lindemann number $c_L = \langle u^2 \rangle^{1/2}/a$ for YBCO can be found in Ref. [8] and for BSCCO in Ref. [12].

In this paper, we present a theory which is capable of specifying the size of the Lindemann number $c_L$, and predicting corrections to the criterium. Our theor is based on a simple Gaussian model which takes into account both lattice elasticity and defect degrees of freedom in the simplest possible way. The relevance of defect fluctuations for the understanding of melting transitions is well-known. For ordinary crystals, this is textbook material [7]. In the context of vortex melting it was emphasized in Ref. [13]. For ordinary crystals, the size of the Lindemann number has been calculated successfully by means of Gaussian lattice models with elastic and defect fluctuations, which clearly display first-order melting transitions in three dimensions and both first-order or sequence of continuous transitions in two dimensions.

An important quality of these models is that in the first-order case, where fluctuations are small, they lead to a simple universal melting formula determining the melting point in terms of the elastic constants. The universal result is found from a lowest-order approximation, in which one identifies the melting point with the intersection of the high-temperature expansion of the free energy density, dominated by defect fluctuations with the low-temperature expansion dominated by elastic fluctuations. The resulting universal formula for the melting temperature determines also the size of the Lindemann number. Recently, the results of Ref. [7] for square crystals were successfully extended to face-centered and body-centered cubic lattices in three dimensions [14] and also to two-dimensional triangular lattices [15]. Similar intersection criterium was also used for the melting point of vortex lattices in the Abrikosov approximation of the Ginzburg-Landau model [6] useful for YBCO. Here we shall apply our model to calculate the melting curves, the entropy jumps, and magnetic induction jumps of the vortex lattices in YBCO and BSCCO. We do not discuss in this work effects on the vortex lattice from other sources than defect fluctuations which can give rise to tricritical points and glass transitions [16, 17].

Theoretical work on this subject can be found in Refs. [18, 19, 20]. There is in principle no problem of adding pinning in our formalism. For simplicities, we shall confine our discussion to the defect mechanism of melting.

The melting criterium will be derived in Section II. The calculation of the melting temperatures, the entropy
jumps, and the jumps of magnetic induction for YBCO and BSCCO is carried out in Section III.

II. MELTING CRITERION

Due to the large penetration depth $\lambda_{ab}$ in the layers in comparison to $a$ we have to take into account the full non-local elasticity constants when integrating over the Fourier space, as emphasized by Brandt in Ref. [21]. For our Gaussian model, the partition function of the vortex lattice can be split into $Z = Z_0 Z_\phi$ where $Z_0$ is the partition function of the rigid lattice and $Z_\phi$ is thermally fluctuating part calculated via the elastic Hamiltonian plus defects. Due to the translational invariance of the vortex lattice in the direction of the vortices, which we shall take to be the $z$-axis, we may simply extend the models on square [7] and triangular lattices [15] by a third dimension along the $z$-axis, which we artificially discretize to have a lattice spacing $a_3$, whose value will be fixed later. The elastic energy is

$$E_{\text{el}} = \frac{1}{2} \sum_x (\nabla_i u_i)(\sigma_{11} - 2 c_{66}) (\nabla_i u_i)$$

(1)

$$+ \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) c_{66} (\nabla_i u_j + \nabla_j u_i)$$

$$+ (\nabla_3 u_i) c_{44} (\nabla_3 u_i).$$

The subscripts $i, j$ have values 1, 2 and $l, m, n$ have values 1, $\ldots$, 3. The vectors $u_i(x)$ are given by the transverse displacements of the line elements of the vortex lines with coordinate $x$. We have suppressed the spatial arguments of the elasticity parameters, which are really functional matrices $c_{ij}(x, x') \equiv c_{ij}(x - x')$. Their precise forms were calculated by Brandt [21]. The volume of the fundamental cell $v$ is equal to $a^2 a_3$ (square) or $a^2 a_3 \sqrt{3}/2$ (triangular). For a square lattice, the lattice derivative $\nabla_i$ in (1) are given by $\nabla_i f(x) = [f(x + a e_i) - f(x)]/a$ and $\nabla_3 f(x) = [f(x + a_3 e_3) - f(x)]/a_3$. For a triangular lattice, the $xy$-part of the lattice has the link vectors $\pm a e_{(1,3)}$ with $e_{(1,3)} = (\cos 2\pi/6, \pm \sin 2\pi/6, 0)$ and $e_{(2)} = (-1, 0, 0)$. The lattice derivatives around a plaquette are defined by $\nabla_{(1)} f(x) = [f(x + a e_{(1)}) - f(x)]/a$, $\nabla_{(2)} f(x) = [f(x) - f(x - a e_{(2)})]/a$, $\nabla_{(3)} f(x) = [f(x - a e_{(2)}) - f(x + a e_{(1)})]/a$. From these we define discrete cartesian lattice derivatives used in the Hamiltonian $\nabla_i f(x) = (2/3) e_{(i)} \nabla_{(i)} f(x)$ and $\nabla_3 f(x) = [f(x + a_3 e_3) - f(x)]/a_3$ transforming like the continuum derivative with respect to the symmetry group of the lattice [11]. Therefore, the Hamiltonian [11] has the full symmetry of the triangular lattice and the correct continuum elastic energy for zero lattice spacing.

The quadratic approximation to the energy [11] is so far only appropriate for the the low-temperature classical thermodynamic behaviour. It is possible to extend the Hamiltonian at the quadratic level in such a way that the range of applicability extends beyond the melting transition. This is possible by the introduction of integer-valued defect gauge fields. We observe that the displacement fields in [11] are restricted to values within the fundamental cell. The defect gauge fields enter to characterize the jumps of the displacement fields across the Volterra surface [7, 13]. As usual for gauge fields we choose a minimal coupling to the lattice displacements. On account of the three lattice derivates per fundamental cell and further two dimensions for the displacements there are six independent integer-valued gauge fields per fundamental cell corresponding to the various defect configurations. One can eliminate two of them (we choose the defect fields corresponding to jumps in the $z$-direction) by relaxing the restriction of the displacement fields to the fundamental cell (elimination of gauge freedom). See the discussion in Ref. [7] for square lattices. A similar consideration was also carried out in [15] for the two-dimensional triangular lattice where the elimination of the gauge degrees of freedom is more complicated due to the absence of the $z$-direction. Finally, we can eliminate one more integer-valued defect field since the elastic energy in [11] depends only on the displacement fields $u_i$ via the strain tensor $\nabla_i u_j + \nabla_j u_i$. In summary, only three independent integer-valued fields have to be included in [11] to obtain the elastic energy of the vortex lattice including defects. By taking into account the above considerations one can then easily determine the partition function including defects for the square vortex lattice by using the considerations in Ref. [7] and for the triangular ones by using Ref. [15].

By using a Hubbard-Stratonovich decoupling of the quadratic displacement terms in [11], the stress representation [7] of the partition function becomes

$$Z_{\phi} = \left[ \frac{c_{66}}{4(c_{11} - c_{66})} \right]^{1/2} \left[ \frac{1}{2 \pi \beta} \right]^{5/2} \prod_{i \leq m} \int_{-\infty}^{\infty} d\sigma_{im} \left[ \prod_{m} \sum_{m(x) = -\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{du}{a} \right] \exp \left\{ - \sum_{x} \frac{1}{2 \beta} \left[ \sum_{i} \sigma_{ii}^2 + \frac{1}{2} \sum_{i,j} \sigma_{ij}^2 + \left( \sum_{i} \sigma_{ij} \right) \frac{c_{11} - 2 c_{66}}{4(c_{11} - c_{66})} \left( \sum_{i} \sigma_{ij} \right) + \sum_{i} \sigma_{13} \frac{c_{66}}{4 c_{44}} \sigma_{i3} \right] \right\} e^{2 \pi i \sum_{x} (\sum_{i \leq m} \nabla_m u_i \sigma_{im} + \sum_{i \leq j} D_{ij} \sigma_{ij})} \right].$$

(2)

The parameter $\beta$ is given by $\beta = v c_{66}/k_B T(2\pi)^2$. $\sigma_{ij}$ represent stress fields [7]. The matrix $D_{ij}(x)$ in Eq. (2) is
a discrete-valued local defect matrix composed of integer-valued defect gauge fields $n_1, n_2, n_3$ for square and triangular vortex lattices \[ \text{eqn} \] as follows:

\[
D_{ij}^\square = \begin{pmatrix}
n_1 & n_3 \\ n_3 & n_2
\end{pmatrix},
\]

\[
D_{ij}^\triangle = \begin{pmatrix}
\frac{1}{\sqrt{3}}(n_1-n_2) + \frac{2}{\sqrt{3}}n_3 \\ \frac{1}{\sqrt{3}}(n_1-n_2) - \frac{2}{\sqrt{3}}n_3
\end{pmatrix}.
\]

The vortex gauge fields specify the Volterra surfaces in units of the Burgers vectors. By summing over all $n_{1,2,3}(\mathbf{x})$, the partition function $Z_\square$ includes all defect fluctuations, dislocations as well as disclinations. There is a constraint for a vortex lattice which does not exist for ordinary three-dimensional lattices. Dislocations in the vortex lattice can be both screw or edge type, but in either case the defect lines are confined in the plane spanned by their Burger's vector and the magnetic field. The reason is that the flux lines in a vortex lattice cannot be broken. This results in the constraint $D_{11} = -D_{22}$ on the defect fields.

We now calculate the low-temperature expansion of the partition function $Z_\square$ to lowest order, which includes only the $n_m = 0$-term. By carrying out the integration over the displacement fields $u_i(\mathbf{x})$ in \[ \text{eqn} \] we obtain, as in \[ \text{eqn} \], the leading term in the low-temperature expansion of the free energy

\[
Z_{\square}^{T \to 0} = \left( \frac{a_3}{a} \right)^{2N} \frac{1}{\det\left( (2\pi\beta)c_{44}/c_{66} \right)} e^{-N \sum_i \epsilon(i,\mathbf{K}) l_i},
\]

where

\[
l_i = \frac{1}{2} \frac{1}{V_{BZ}} \int_{BZ} d^2k dk_3 \log \left[ c_{4i}a_3^2 c_{4i} + a_3^2 K_{ij}^2 K_3^2 \right].
\]

The constant $C$ has the values $C_{\square} = 1$ for the square vortex lattice and $C_{\triangle} = \sqrt{3}$ for the triangular one.

In the low-temperature expansion representing the solid phase, defect field configurations $n_m \neq 0$ correspond to dislocations and disclinations giving finite temperature corrections to the free energy $-\log(Z)/k_BT$. These corrections are exponentially small with an exponent proportional to $-\beta$ \[ \text{eqn} \]. In contrast to this corrections to the high temperature expansion in the fluid phase corresponding to stress configurations $\sigma_{12} \neq 0$ and $\sigma_{ij} \neq 0$ of integer values result in temperature corrections to the free energy which are also exponentially small with an exponent proportional to $-1/\beta$. The structure of the high- and low-temperature corrections to the partition function is extensively discussed in Refs. \[ \text{eqn} \] for ordinary crystals, and can be easily transferred to our case of vortex lattices. It was shown in Refs. \[ \text{eqn} \] for the two dimensional square and triangular as well as the three dimensional square crystal that the exponentially vanishing higher order corrections to the low- and high-temperature expansion of the free energy are negligible in the determination of the transition temperature. This is particularly true for the three dimensional crystal (see p. 1082 in \[ \text{eqn} \]) which we take as a justification to restrict our calculation to lowest order in this paper.

From the partition function \[ \text{eqn} \] with no defects ($n_m = 0$) we obtain for the Lindemann number $c_L = \langle u^2 \rangle^{1/2}/a$ the momentum integral

\[
c_L^2 = \frac{a_3^2}{a^2 v} \frac{k_B T}{V_{BZ}} 4 \int_{BZ} d^2k dk_3 \frac{1}{c_{4i}} \sum_{i=1,2,3} \frac{1}{c_{4i}^2} \epsilon(i,\mathbf{K})^2 K_{ij}^2 K_3^2 + a_3^2 K_{ij}^2 K_3^2
\]

This can be simplified by taking into account that $c_{11}$ is much larger than $c_{66}, c_{44}$ in the relevant region. As announced, we find the melting temperature from the intersection of low- and high-temperature expansions, obtained by equating $Z_{\square}^{T \to 0} = \hat{Z}_{\square}^{T \to \infty}$. By taking into account $\det[3^3 V_3^3 V_3^3] = 1$ we obtain $h_{\text{L1}} \ll l_{66}$, and further that the $i = 1$-term in \[ \text{eqn} \] is much smaller than the $i = 6$-term. In the following analytic discussion (but not in the numerical plots) we neglect these small contributions. The temperature of melting is then given by the simple formula

\[
k_B T = \frac{1}{v \det 1/N(c_{66})} C = e^{-\lambda_{\text{L}}},
\]

where $\det[c_{66}]$ is the determinant of the $N \times N$ functional matrix $c_{66}$. The elastic moduli $c_{44}$ and $c_{66}$ at low reduced magnetic fields $b \equiv B/H_\lambda < 0.25$ can be taken from Brandt’s paper \[ \text{eqn} \].

\[
c_{66} = \frac{B \phi_0 \zeta}{(8\pi \lambda_{ab})^2},
\]

\[
c_{44} = \frac{B^2}{4\pi(1+\lambda_{ab}^2 k_3^2)} \left( \frac{1+2\lambda_{ab}^2 k_3^2/1+\lambda_{ab}^2 k_3^2}{1+2\lambda_{ab}^2 k_3^2} + \frac{B \phi_0}{2\pi^2 \lambda_{ab}^2 k_3^2} \ln \frac{1+\lambda_{ab}^2 k_3^2}{(1+2\lambda_{ab}^2 k_3^2)} + \frac{B \phi_0}{2\pi^2 \lambda_{ab}^2 k_3^2} \ln \frac{1+\lambda_{ab}^2 k_3^2}{(1+2\lambda_{ab}^2 k_3^2)} \right).
\]
where $\lambda_c$ is the penetration depth in the $xy$-plane, $\zeta = 1$, and $K_{BZ}$ is the boundary of the circular Brillouin zone $K_{BZ} = 4\pi B/\phi_0$. At high fields ($b > 0.5$), $c_{66}$ is altered by a factor $\zeta \approx 0.71(1-b)$, and the penetration depths in $c_{66}$, $c_{44}$ are replaced by $\tilde{\lambda}^2 = \lambda^2/(1-b)$, where $\lambda$ denotes either $\lambda_{ab}$ or $\lambda_c$. In addition, in the last two terms in $c_{44}$ are replaced by $B\phi_0/16\pi^2 \tilde{\lambda}^2$. For YBCO we have $\lambda(T)/\lambda_c = \lambda_0/[1 - T/\tau_m]^{-1/2}$, $\zeta(T) = \zeta_0/[1 - T/\tau_m]^{-1/2}$, $\zeta(T) = \zeta_0/[1 - T/\tau_m]^{-1/2}$, where $\lambda_c$ is replaced by $K_{BZ}$ and $K_{BZ}$ is altered by a factor $K_{BZ}/2$.

To determine the effective lattice spacing $a_3$ along the vortex lines. An elementary defect in the vortex lattice (arising for example from a crossing of two vortex strings) takes place over a typical length scale in the $z$-direction determined from the condition that the sum of elastic displacement energy and the energy required to stretch the line against the line tension is minimal. It is the elastic energy of this small-defect that has to be taken into account for in our model. The energy of an ensemble of dislocations is determined by the interplay of elastic energy of small displacements and integer-valued defect fields. The relevant part of the free energy is given by the discretized free energy $-\log(Z_0 Z_B) k_B T$ which is taken into the continuum version (in $z$ direction) of the elastic energy $\lambda_0 [1 - T/\tau_m]^{-1/2}$, and approximate $-\nabla^2 \sim \approx (K_{BZ}^2/4)$ in $E_{cl}$ and $K_{BZ}^2 \approx (K_{BZ}^2/2)$ in the elastic constants, where the average ... was taken with respect to a circular Brillouin zone. The optimal length scale $a_3$ is chosen such that $E_{cl}$ is minimal for a fixed amplitude $A_0 \approx a$ corresponding to a typical defect elongation.

III. APPLICATION TO YBCO AND BSCCO

In the following, we treat first the more isotropic square vortex crystal YBCO ($a = \sqrt{\phi_0/B}$). From $c_{66}$ and $c_{44}$ for YBCO, the optimal length scale is given by $a_3 = 4\lambda_0/\lambda_c(1-b)^{1/2}$. When comparing the melting criterium of the defect model in Eq. 10 with the Lindemann criterium obtained by equating the parameter $B_{\phi_0}$ to a universal number, we obtain identical results when taking into account that the integrand in $B_{\phi_0}$ and in $c_{66}$ of Eq. 5 receive their main contribution from the region $k \approx \sqrt{k^2} \sim K_{BZ}/\sqrt{2}$. We can approximate $k_3 \approx 0$ in this region, resulting in $a_3^2 c_{66}/a_2 c_{44} \approx 4/\pi$. With the same approximation in Eqs. 9 and 11, we can perform the integrals numerically. Then we obtain from the melting condition $[10]$ precisely the Lindemann criterium in which the Lindemann number $[9]$ is predicted to be

$$\begin{align*}
\frac{k_B T_m}{[c_{44}(K_{BZ}/\sqrt{2}, 0) c_{66}(K_{BZ}/\sqrt{2}, 0)]^{1/2}} &\approx c_L^2 \approx (0.18)^2. \quad (13)
\end{align*}$$

Denoting the spacing between the CuO$_2$ double layers by $a_s$ we obtain for the entropy jump per double layer and vortex

$$\Delta S \approx \frac{k_B T_m (\partial/\partial T_m)(a_s/a_m)}{a_3^2} \ln \left| \frac{Z_{m}}{Z_{0}} \right|^{2} = \frac{k_B T_m(a_3^2)}{a_3} \ln \left| \frac{Z_{m}}{Z_{0}} \right|^{2}. \quad (14)
$$

Inserting $5$ and $11$, this becomes

$$\Delta S \approx \frac{k_B T_m a_3 \partial}{\partial T_m} \ln \left| \frac{c_{44}(K_{BZ}/\sqrt{2}, 0) c_{66}(K_{BZ}/\sqrt{2}, 0)}{a_3} \right|^2. \quad (15)
$$

Finally, we make use of the Clausius-Clapeyron equation which relates the jump of the entropy density across the melting transition to the jump of the magnetic induction by $\Delta S / a_3 / a_m = -(dH_m/dT) \Delta B / 4\pi$. Here $H_m$ is the external magnetic field on the melting line. Combining the Clausius-Clapeyron equation with Eqs. 12 and 15 we obtain, with the abbreviation $\tau_m \equiv T_m/T_c$, the following relations near $T_c$: $B_{m}(T) \approx \frac{12}{(1-T/T_c)^{1/3}} (k_B T^2 c_{66}/a_3^3) \lambda_0^2 (a_3^2)^{1/3}$, $\Delta S \approx \frac{\sqrt{\zeta} a_3 \tau_m}{2 \alpha \lambda_{ab} c_{66} k_B T m} \approx \frac{2.5 (1 - \tau_m^{2/3}) c_{66}^2 \phi_0}{10^2 \lambda_{ab}^2} \lambda_0^2$. (16)

These results agree with the general scaling results in Ref. 26, with the advantage that here the prefactors are predicted whereas those in 26 had to be determined by fits to experimental curves (there is only a slight discrepancy because we use a different temperature dependence of the penetration depth).

Next, we calculate the corresponding expressions in the case of the more layered crystal BSCCO ($a = (2^{1/2}/3^{1/4}) \sqrt{\phi_0/B}$). First, we have to determine the dislocation length scale $a_3$ in this case. For dislocation moves we have $\langle u^2 \rangle^{1/2} \sim 1/K_{BZ}$. This means that we can neglect the last two terms of $c_{44}$ in 12, coming from the self-energy of the vortex line, when determining $a_3$. Remembering this we obtain by a similar procedure as for YBCO the dislocation length scale $a_3 \approx 4\alpha \sqrt{4\lambda_{ab}/\lambda_c \sqrt{\pi}}$. From this we find $\lambda_{ab}^2 \approx c_{44} \\ c_{66}^2 \approx 1$, resulting in $\lambda_{ab}^2 \approx 1$. By taking into account that $B_{\phi_0} \lambda_{ab}^2/32 \phi_0 \ll 1$ on the melting line we obtain that $c_{44}(k, k_3) \approx |k_3| \pi/a_3$ is dominated by the last term in 12. Then we obtain

$$c_{44}(k, k_3) \approx \frac{B_{\phi_0} \ln (1+2B_{\phi_0}^2 c_{66}^2)}{32\pi \lambda_{ab}^2 k_3^2} \quad \text{for } k_3 \gg 1.$$
FIG. 1: Melting curve $B = B_m(T)$ for YBCO and BSCCO. The experimental values are from Ref. [28] for YBCO and Ref. [27] for BSCCO. The numbers on the theoretical melting curves are the Lindemann numbers $c_L$ calculated from (9).

By using these values we obtain by numerical integration of (9)

$$c_L^2 \approx \frac{k_B T_m \cdot 0.36}{a^2 \sqrt{c_6(\frac{K_m}{\sqrt{2}},0)c_{44}(\frac{K_m}{\sqrt{2}},0)}} + \frac{k_B T_m \cdot 1.60}{a^4 c_{44}(\frac{K_m}{\sqrt{2}},1/a_3)}$$

$$\approx \frac{k_B T_m \lambda_{ab}^2 \cdot 138}{\phi_B a} \sqrt{1 + \lambda_{ab}^2 K_{BZ}^2} + \frac{k_B T_m \lambda_{ab}^2 \lambda_{c}^2 \cdot 137 \lambda_{ab}}{\phi_B a^3 \log(1/c_L^2) \lambda_c}$$

The first term comes from the integration region $|k_3| < 1$, the second term from the region $1 \lambda_{ab} < |k_3| < \pi/\alpha_3$ in (9).

From our melting criterion (10) and the Clausius-Clapeyron equation (where $dH_m/dT \approx dB_m/dT$ due to $B_m(T) \approx H_{c1}(T)$) [21], we obtain for BSCCO

$$B_m(T) \approx \frac{1}{192} \sqrt{3 \pi^2} \frac{\lambda_{c}^2(0) \lambda_{ab}^2(0)}{(k_B T)^2},$$

$$\Delta S_l \approx \frac{\sqrt{\pi} a_b k_B \lambda_c}{4 \sqrt{2} a} \frac{1 + 3 \pi^4}{4 \lambda_{ab} 1 - \pi^4} \approx 2.9 \frac{a_b (1 + 3 \pi^4) \phi_0}{10^4 \lambda_{ab}(0)},$$

$$\Delta B \approx \frac{\pi^{3/2}}{2 \sqrt{2} a} \frac{\lambda_c k_B T_m \approx 1.8 (1 - \pi^4 \phi_0)}{10^4 \lambda_{ab}(0)}.$$

Parameter values for optimal doped YBCO (BSCCO) are given by [24] $\lambda_{ab}(0) \approx 1186 \AA$ ($\lambda_{ab}(0) \approx 2300 \AA$), $\xi_{ab}(0) \approx 15 \AA$ ($\xi_{ab}(0) \approx 30 \AA$), the CuO$_2$ double layer spacing $a_s = 12 \AA$ ($a_s = 14 \AA$), $T_c = 92.7 K$ ($T_c = 90 K$) and the anisotropy parameter $\gamma = \lambda_c/\lambda_{ab} \approx 5$ ($\gamma \approx 200$).

We now calculate numerically the melting curves, the associated Lindemann parameter $c_L$, the entropy and the magnetic induction jumps $\Delta S_l$ and $\Delta B$ from the intersection criterion of the full low- and high-temperature expressions (5) and (7) without further approximation. To accomplish this, we use the elastic constant $c_{11}$ given by Brandt in Ref. [21]. The intersection criterion of the low and high temperature expansion of the partition function is then at least in the case of BSCCO

FIG. 2: Entropy jump per double layer per vortex $\Delta S_l$ (first row) and jump of magnetic induction field $\Delta B$ (second row) at the melting transition. The experimental values for YBCO are from Ref. [28] for $\Delta S_l$, Ref. [29] for $\Delta B$ by SQUID experiments (SQUID), and Ref. [30] by torque measurements (Torque). The experimental values for BSCCO are from Ref. [31] by SQUID measurements (SQUID) of the magnetic field and Ref. [32] by microscopic Hall sensors (Hall).

a complicated integral equation via the dependence of $c_{44}$ on the Lindemann parameter $c_L$. One can solve this integral equation by numerical methods. The results are shown in Fig. 1 and Fig. 2.

IV. DISCUSSION

Our approximate analytic results [12] and [13] for YBCO and BSCCO turn out to give practically the same curves. For comparison, we show in both figures the experimental curves for YBCO of Ref. [28] and [29] and for BSCCO of Ref. [27] and [31]. The good agreement in Fig. 1 with the theoretical curves based on Eq. (13) shows that the Lindemann number is independent of the magnetic field for YBCO for the entire temperature range. For BSCCO the agreement is good for smaller B-fields, where the second term in (18) introduces some dependence of the Lindemann on B, the largest near B = 0. There is some disagreement in Fig. 1 at larger B and in Fig. 2 at small B. This is not surprising since our vortex lattice model cannot be a good approximation in these regimes. At high B, the discrepancy comes mainly from Josephson decoupling of the layers [32], most pronounced for the strongly anisotropic BSCCO superconductor, which leads also to large pinning effects [32]. We think that this is also the reason for the difference in the curves of Kadowaki et al. in Ref. [31] and of Zeldov et al. in Ref. [27] shown in Fig. 2. Pinning has the largest influence on the form of the melting curve at high B [34], resulting in a decrease of $\Delta S_l$ and $\Delta B$ [20] in the limit of low temper-
atures shown by the curves of Zeldov et al. Near $B = 0$, our model does not include the increase of the entropy coming from the thermal creation of vortices, in addition to the ones caused by the external magnetic field which form the lattice \[35\]. Moreover, in YBCO order parameter fluctuations become important \[36\] which are ignored here.

Summarizing, we have obtained the melting curve, the entropy, and the magnetic jump from a simple lattice defect model, and derived the Lindemann rule, including the size of the Lindemann number, and corrections to it. The determination of jump quantities over the phase transition cannot be obtained by the simple Lindemann rule. Our curves agree well with the experimental curves for YBCO and BSCCO except at zero and large magnetic fields. The simplicity of the model has allowed us to derive all results via analytic approximations. Our defect model is the simplest extension of the linear elasticity theory of vortex displacements. We have merely added integer-values defect gauge fields which introduce into the elasticity theory the rich physics of other phases of the vortex lattice caused by defect fluctuations, in particular the liquid phase and the associated melting transition.

---

[1] D. R. Nelson, Phys. Rev. Lett. **60**, 1973 (1988).
[2] D. R. Nelson and H. S. Seung, Phys. Rev. B **39**, 9153 (1989).
[3] E. Brezin, D. R. Nelson, and A. Thiaville, Phys. Rev. B **31**, 7124 (1985).
[4] A. van Otterlo, R. T. Scalettar, and G. T. Zimanyi, Phys. Rev. Lett. **81**, 1497 (1998).
[5] Y. H. Li and S. Teitel, Phys. Rev. Lett. **66**, 3301 (1991); R. E. Hetzel, A. Sudbø, and D. A. Huse, Phys. Rev. Lett. **69**, 518 (1992).
[6] S. Hikami, A. Fujita, and A. I. Larkin, Phys. Rev. B **44**, 10400 (1991); D. Li and B. Rosenstein, Phys. Rev. Lett. **86**, 3618 (2001).
[7] H. Kleinert, *Gauge Fields in Condensed Matter*, Vol. II *Stresses and Defects*, World Scientific, Singapore, 1989. (readable online at www.physik.fu-berlin.de/~kleinert/re.html#b2)
[8] A. Houghton, R. A. Pelcovits and A. Sudbo, Phys. Rev. B **40**, 6763 (1989); E. H. Brandt, Phys. Rev. Lett. **63**, 1106 (1989).
[9] P. Kim, Z. Yao, and C. M. Lieber, Phys. Rev. Lett. **77**, 5118 (1996).
[10] B. Keimer, W. Y. Shih, R. W. Erwin, J. W. Lynn, F. Dogan, and I. A. Aksay, Phys. Rev. Lett. **73**, 3459 (1994); S. P. Brown, D. Charalambous, E. C. Jones, E. M. Forgan, P. G. Kealey, A. Erb, and J. Kohlbrecher, Phys. Rev. Lett. **92**, 067004 (2004).
[11] H. Won and K. Maki, Phys. Rev. B **53**, 5927 (1996).
[12] G. Blatter, V. Geshkenbein, A. Larkin, and H. Nordborg, Phys. Rev. B **54**, 72 (1996).
[13] M. C. Marchetti and D. R. Nelson, Phys. Rev. B **41**, 1910 (1990); J. Kierfeld and V. Vinokur, Phys. Rev. B **61**, R14928 (2000); J. Lidmar, Phys. Rev. Lett. **91**, 097001 (2003).
[14] H. Kleinert and Y. Jiang, Phys. Lett. A **313**, 152 (2003).
[15] J. Dietel and H. Kleinert, Phys. Rev. B **73**, 024113 (2006).
[16] F. Bouquet, C. Marcenat, E. Steep, R. Calenyczuk, W. K. Kwok, U. Welp, G. W. Crabtree, R. A. Fisher, N. E. Phillips, A. Schilling, Nature (London) **411**, 448 (2001).
[17] N. Avraham, B. Khaykovich, Y. Myasoedov, M. Rapaport, H. Shtrikman, D. E. Feldman, T. Tamegai, P. H. Kes, M. Li, M. Konczykowski, K. van der Beek, E. Zeldov, Nature (London) **411**, 451 (2001); C. J. van der Beek, S. Colson, M. V. Indenbom, and M. Konczykowski, Phys. Rev. Lett. **84**, 4196 (2000); H. Biedenkopf, N. Avraham, Y. Myasoedov, H. Shtrikman, E. Zeldov, B. Rosenstein, E. H. Brandt and T. Tamegai, Phys. Rev. Lett. **95**, 257004 (2005).
[18] D. S. Fisher, M. P. A. Fisher, and D. A. Huse, Phys. Rev. B **43**, 130 (1991).
[19] T. Giamarchi and P. LeDoussal, Phys. Rev. B **55**, 6577 (1997); G. P. Mikitik and E. H. Brandt, Phys. Rev. B **64**, 184514 (2001); G. I. Menon, Phys. Rev. B **65**, 104527 (2001). Y. Radzyner, A. Shaulov, Y. Yeshurun, Phys. Rev. B **65**, 100513(R) (2002); G. P. Mikitik and E. H. Brandt, Phys. Rev. B **68**, 054509 (2003); J. Kierfeld and V. Vinokur, Phys. Rev. B **69**, 024501 (2004).
[20] D. Li and B. Rosenstein, Phys. Rev. Lett. **90**, 167004 (2003).
[21] E. H. Brandt, Rep. Prog. Phys. **58**, 1465 (1995); see also G. Blatter, M. V. Feigel’man, V. B. Geshkenbein, A. I. Larkin, and V. M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).
[22] R. Labusch, Physics Letters **22**, 9 (1966).
[23] By using the elastic constants from Ref. [21] we obtain that $c_{11} \gg c_{44}, c_{66}$ almost everywhere on the melting line except in a small vicinity near $T \approx T_c$. Here we take into account that $|k_x| < \pi/a_x \approx \pi \lambda_c/\lambda_{aB} > 0$ and further that $B \lambda_{aB}/\phi_0 \approx 1/16\pi$ is fulfilled almost everywhere on the melting line (except in a region where $|T/T_c| - 1| \lesssim 0.02$ for BSCCO, the corresponding region for YBCO is even smaller).
[24] S. Kamal, D. A. Bonn, N. Goldenfeld, P. J. Hirschfeld, R. Liang, and W. N. Hardy, Phys. Rev. Lett. **73**, 1845 (1994).
[25] M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, New York, 1996.
[26] M. J. W. Dodgson, V. B. Geshkenbein, H. Nordborg, and G. Blatter, Phys. Rev. Lett. **80**, 837 (1998).
[27] E. Zeldov, D. Majer, M. Konczykowski, V. B. Geshkenbein, V. M. Vinokur, and H. Shtrikman, Nature (London) **375**, 791 (1995).
[28] A. Schilling, R. A. Fisher, and G. W. Crabtree, Nature (London) **382**, 791 (1996).
[29] U. Welp, J. A. Hendrich, W. K. Kwok, G. W. Crabtree, and B. W. Veal, Phys. Rev. Lett. **76**, 4809 (1996).
[30] M. Willemin, A. Schilling, H. Keller, C. Rossel, J. Hofer, U. Welp, W. K. Kwok, R. J. Olsson, and G. W. Crabtree, Phys. Rev. Lett. **81**, 4236 (1998).
[31] K. Kadowaki and K. Kimura, Phys. Rev. B **57**, 11674
(1998).

[32] L. I. Glazman and A. E. Koshelev, Phys. Rev. B 43, 2835 (1991).

[33] T. Blasius, Ch. Niedermayer, J. L. Tallon, D. M. Poole, A. Golnik, and C. Bernhard, Phys. Rev. Lett. 82, 4926 (1999).

[34] B. Khaykovich, M. Konczykowski, E. Zeldov, R. A. Doyle, D. Majer, P. H. Kes and T. W. Li, Phys. Rev. B 56, R517 (1997).

[35] S. Ryu and D. Stroud, Phys. Rev. B 57, 14476 (1998).

[36] V. L. Ginzburg, Fiz. Tverd. Tela 2, 2031 (1960) [Sov. Phys. Solide State 2, 1824 (1961)].