Discrete gravity as a local theory of the Poincaré group in the first-order formalism

Gabriele Gionti

Vatican Observatory Research Group, Steward Observatory, 933 North Cherry Avenue, The University of Arizona, Tucson, AZ 85721, USA
and
Specola Vaticana, V-00120 Cittá Del Vaticano, Vatican City State

E-mail: ggionti@as.arizona.edu

Received 12 July 2005, in final form 30 August 2005
Published 27 September 2005
Online at stacks.iop.org/CQG/22/4217

Abstract

A discrete theory of gravity, locally invariant under the Poincaré group, is considered as in a companion paper. We define a first-order theory, in the sense of Palatini, on the metric-dual Voronoi complex of a simplicial complex. We follow the same spirit as the continuum theory of general relativity in the Cartan formalism. The field equations are carefully derived taking in account the constraints of the theory. They look very similar to first-order Einstein continuum equations in the Cartan formalism. It is shown that in the limit of small deficit angles these equations have Regge calculus, locally, as the only solution. A quantum measure is easily defined which does not suffer the ambiguities of Regge calculus, and a coupling with fermionic matter is easily introduced.

PACS numbers: 04.60.Nc, 04.20.Cv

1. Introduction

In spite of many recent developments, in particular in string theory and loop quantum gravity, the problem of quantization of general relativity is still an open one both from mathematical and physical points of view. A key point of general relativity is its close resemblance (and yet not equivalence) to an ordinary gauge theory. This is very clear in the first-order (vierbein) formalism, where the dynamical variables are those suitable for a gauge theory of the Poincaré group. This has been the starting point of the loop quantum gravity approach too. This similarity led to various attempts to write down a regularized version of GR on a lattice. One of the most well known is Regge calculus [1], which is a natural and geometrically appealing way of discretizing gravity on a simplicial lattice. Recently, it has been an active field of research in connection with dynamical triangulations, two-dimensional quantum gravity and
therefore with conformal field theory and string theory. One of the weak points of the simplicial lattice is that it is naturally related to the second-order (metric) formalism of GR, so that its connection with gauge theory is rather obscure. The aim of this paper is to continue the work started by Caselle, D’Adda and Magnea [2], to reformulate Regge calculus in terms of dynamical variables belonging to the Poincaré group, so that it makes an explicit connection with gauge theory in general and Wilson-like gauge theory on a lattice [3] in particular. We have also found strict analogies between this approach and discrete Ashtekar variables [4]. We pursue this goal starting in section 2 by summarizing the main results of [2] in the second-order formalism which consists of assigning the rotational degrees of freedom (Lorentz connection) to the links of a dual metric complex (Voronoi complex) of the original simplicial complex, so that they are functions of the translational degrees of freedom. In sections 3 and 4, we formulate a first-order principle, in the sense of Palatini¹, in which the independent variables are the Lorentz connection and the normals to the $(n-1)$-faces. These normals are considered analogues of the $n$-bein in the continuum theory and the connection matrices as the connection one-form in GR. In section 5, we prove, in the case of small deficit angles, that Regge calculus is a solution of the first-order formalism. This result is not obvious if we vary independently the two sets of the variables above. Moreover, this equivalence for small deficit angles tells us that this first-order formalism has the effect of smoothing out some pathological configurations, like spikes, which affect Regge calculus and might prevent the theory from having a smooth continuum limit. These spike configurations are in fact in the region of large deficit angles, where the first-order and second-order formalisms are not equivalent. In section 6, we derive the general field equations for the connection matrices and for the $n$-bein. In section 7, a measure for the path integral of this discrete theory of gravity is introduced and is shown to be locally invariant under $SO(n)$. As a last step in section 7, we propose a coupling of this discrete theory of gravity with fermionic matter. This coupling is entirely performed by following the general prescription of the continuum theory. In other formulations of gravity (Regge calculus and dynamical triangulations), the coupling with fermionic matter is usually introduced ad hoc.

2. Second-order formalism

In 1961, Regge [1] (see also [7] for a recent and updated summary of Regge calculus and its alternative approaches) proposed a discretized version of general relativity now known as Regge calculus. It is, mainly, based on the idea of substituting a piecewise-linear (PL)-manifold for the differential manifold. In particular, these are simplicial manifolds built by gluing together two distinct $n$-dimensional flat simplices by one and only one $(n-1)$-dimensional face. The final product of this construction is called a simplicial complex, which owns the manifold structure if it has been made in such a way that each point of the simplicial complex has a neighbourhood homeomorphic to $\mathbb{R}^n$ [8]. The simplicial manifolds, we consider, are orientable. On each $(n-2)$-dimensional simplex $h$, called a hinge, a deficit angle $K(h)$ is defined as

$$K(h) = 2\pi - \sum_{\sigma_h^i \supset h} \theta(\sigma_h^i, h)$$

where $\sigma_h^i, i = 1, \ldots, p$ is one of the $p$ $n$-dimensional simplices incident on $h$ and $\theta(\sigma_h, h)$ are its dihedral angles on $h$. The dihedral angle of a $n$-simplex on the hinge is the angle between the $(n-1)$-dimensional faces that have the hinge in common. $K(h)$ and the volume of the

¹ The labelling of the first-order method as Palatini has been questioned by Hehl [5]. See also [6].
hinge $V(h)$ can be expressed [9] as functions of the squared length of the one-dimensional simplices (edges) of the complex. The Einstein–Regge action is, in analogy to the continuum case, a functional over the simplicial manifolds and depends on the incidence matrix of the simplicial complex [10] and on the (squared) lengths of the edges. Usually, in Regge calculus, the incidence matrix is fixed, so that the action can be written as

$$S_R = \sum_h K(h) V(h).$$

The corresponding Einstein equations [1] are derived by requiring that the action is stationary under the variation of the length of the edges.

The original aim of this theory was to give approximate solutions of the Einstein equations in the case in which the topology is known. The theory is, as stressed by Regge himself, completely coordinate independent.

It was pointed out in [2] that a theory of Regge calculus locally invariant under the Poincaré group can be formulated by choosing in every simplex an orthonormal reference frame. In this way every $n$-dimensional simplex, considered as a piece of $\mathbb{R}^n$, can be seen as a local inertial reference frame, being flat, and an $n$-bein base can be chosen in it. As in the continuum theory, we need a connection between the reference frames of two simplices that share a common $(n-1)$-dimensional simplex.

Now we summarize and update some definitions and results of [2] that will be useful for our future discussions. Consider a hinge $h$ and let $\{P_1^h, \ldots, P_{n-1}^h\}$ be its vertices. Suppose that this hinge is shared by $k + 2$ $n$-simplices (see figure 1) $\{\alpha, \beta, \delta_1^h, \ldots, \delta_k^h\}$ whose vertices are labelled in this way

$$\alpha \equiv \{P_1^h, \ldots, P_{n-1}^h, Q_{\delta_1^h \alpha}, Q_{\alpha \beta}\}$$
$$\beta \equiv \{P_1^h, \ldots, P_{n-1}^h, Q_{\alpha \beta}, Q_{\beta \delta_1^h}\}$$
$$\cdots \equiv \cdots$$
$$\delta_k^h \equiv \{P_1^h, \ldots, P_{n-1}^h, Q_{\delta_1^h \delta_k^h}, Q_{\beta \delta_k^h}\}.$$

In each simplex $S_\alpha$, we can choose an origin and an orthogonal reference frame. In this frame, the vertices of the simplex $S_\alpha$ have the following coordinates:

$$P_i = \{x_i^a(\alpha)\}, \quad a = 1, \ldots, n \quad i = 1, \ldots, n - 1$$
$$Q_{\delta_1^h \alpha} = \{z^{\mu}_{\delta_1^h \alpha}(\alpha)\}$$
$$Q_{\alpha \beta} = \{z^{\mu}_{\alpha \beta}(\alpha)\}.$$
Given an \( n \)-dimensional simplicial complex, there exists a general procedure for defining the dual metric complex, called the Voronoi dual [2, 23, 24] (see also [25] for an easy treatment and calculation at the details). The \( n \)-dimensional Voronoi polyhedron dual to a vertex \( P \) of the simplicial complex is the set of the points of the simplicial complex closer to \( P \) than to any other vertex, using the standard flat metric in the simplex. It turns out that the \( k \)-simplex of the simplicial complex is dual to an \((n - k)\)-polyhedron in the dual Voronoi complex and the \( k \)-dimensional linear space identified by the \( k \)-simplex is orthogonal to the \((n - k)\)-dimensional space spanned by the corresponding polyhedron. In particular, the point dual to the \( n \)-simplex \( S \) is the centre of a \((d - 1)\)-dimensional sphere passing through all vertices of \( S \) (see figure 1).

We can associate uniquely with a dual Voronoi edge an element of the Poincaré group \( \Lambda(a, \beta) \equiv \{ \Lambda^a_b(a, \beta), \Lambda^a(a, \beta) \} \) by requiring that

\[
\Lambda^a_b(a, \beta) y^b(\beta) + \Lambda^a(a, \beta) z^b_{a\beta}(\beta) + \Lambda^a(a, \beta) = z^0_{a\beta}(\alpha).
\]  

(5)

In other words, we are embedding \( \alpha \) and \( \beta \) in \( \mathbb{R}^n \) and adopting the standard notion of parallel displacement in \( \mathbb{R}^n \). We move the origin of the reference frames of \( \beta \) to \( \alpha \). So that the position vectors in \( \beta \) of the vertices of the common face \( S_\alpha \cap S_\beta \) will become coincident with the position vectors of the same vertices in \( \alpha \). Thus, the matrix \( \Lambda^a_b(a, \beta) \) will be an orthogonal matrix which describes the change from the reference frame of \( \beta \) to \( \alpha \), considered now as two different reference frames of the same vector space [10]. Since the simplicial manifold is orientable we can choose the reference frame in \( \alpha \) and \( \beta \) in such a way that \( \Lambda^a_b(a, \beta) \) is a matrix of \( SO(n) \). This procedure determines a unique connection that is a torsion-free Levi-Civita or Regge connection [10]. We can assign to each vertex \( D_\alpha \) of the dual Voronoi complex, its coordinates \( x^a(\alpha) \) in the frame of \( \alpha \). Since we can choose arbitrarily the reference frames in \( \alpha \) and \( \beta \), then the theory is invariant under arbitrary Poincaré transformations \( U(\alpha) \),

\[
\Lambda(a, \beta) \mapsto U(a)\Lambda(a, \beta)U^{-1}(\beta) \quad x^a(\alpha) \mapsto U^a_b(\alpha)x^b(\alpha) + U^a(\alpha).
\]  

(6)

Now we choose, following [2], to put the origin of each reference frame in the dual Voronoi vertices, then \( x^a(\alpha) = 0 \) and the Poincaré group is restricted to the rotation matrices without translations. Then \( \Lambda(a, \beta) \) indicates only the rotation matrix. Without imposing this sort of gauge fixing, it might be possible to study a metric-affine theory of gravity on lattice and this could be the subject of further research. See [15] for more details on the metric-affine theory of gravity case as well as [16, 17] for the relative consequences in the continuum case.

The hinge \( h \) is in one-to-one correspondence with its dual two-dimensional Voronoi plaquette that we still label by \( h \). Now consider the following plaquette variable:

\[
W_h(h) = \Lambda(\alpha, \beta)\Lambda(\beta, \delta^1) \cdots \Lambda(\delta^k, \alpha).
\]  

(7)

As has been shown [9, 25], (7) is a rotation in the two-dimensional plane orthogonal to the \((n - 2)\)-dimensional hyperplane spanned by the hinge \( h \). The rotation angle is the deficit angle (1).

We consider \( n - 2 \) linear independent edge vectors of the hinge \( h \) defined in the following way:

\[
E^a_1(\alpha) \equiv y^a_1(\alpha) - y^a_{n-1}(\alpha), \ldots, E^a_{n-2}(\alpha) \equiv y^a_{n-2}(\alpha) - y^a_{n-1}(\alpha),
\]  

(8)

and the antisymmetric tensor related to the oriented volume of \( h \)

\[
\gamma^{ab}(\alpha) \equiv \frac{1}{(n - 2)!} \epsilon^{abc}_{1 \ldots n-2} E^c_1(\alpha) \cdots E^c_{n-2}(\alpha).
\]  

(9)
At this point, it seems natural to propose the following gravitational action \[2\]:

\[
I = -\frac{1}{2} \sum_h \left( W_{ab}^r (h) \gamma^h (a) \right).
\]

Let \( V(h) \) be the oriented volume of the hinge \( h \). The action (10) is equal \[2\] to

\[
I = \sum_h \sin K(h) V(h),
\]

which for small deficit angles \( K(h) \) reduces to the Regge action (2).

The presence in the action of \( \sin K(h) \) instead of \( K(h) \) is only a lattice artefact \[2\]. The useful regime of this theory is \( \sin K(h) \approx K(h) \). In fact, Fröhlich \[10\] has conjectured, as claimed also in \[2\], that this action converges to the Einstein–Hilbert action, and Cheeger, Muller and Schrader in \[11\] have proved that the Einstein–Regge action (2) converges to the Einstein–Hilbert action. They proved that the convergence of the action is in the sense of measure. This applies when, roughly speaking, the number of the hinges of the simplicial manifold increases along with the incident number of the simplices at each hinge. Then the triangulations will be finer and finer and the difference, in modulus, between the Einstein–Hilbert action on a manifold and the Regge–Einstein one, on the triangulations of the same manifold, becomes smaller and smaller.

Let us consider the \((n-1)\)-dimensional face \( f_{αβ} = \{ P_1, \ldots, Q_{αβ} \} \) between the simplices \( α \) and \( β \). The vertex coordinates of \( f_{αβ} \) are \( x_1(α), \ldots, x_n(α) \). The following vector \[2\]

\[

b_{αβ} (α) = \epsilon_{ab \ldots b_{n-1}} (x_1(α) - x_n(α))^b_1 \cdots (x_{n-1}(α) - x_n(α))^b_{n-1},
\]

is normal to the face \( f_{αβ} \), and it is assumed to point outward from the interior of the simplex \( α \). Simple considerations of linear algebra show that \( b_{αβ} (α) \) is an element of the inverse \( n \times n \) matrix built using the components of \( n \) linearly independent edge vectors of the simplex \( α \) multiplied by the determinant of the direct matrix (mathematically, the adjoint elements, modulo the determinant, of the matrix of the \( n \)-independent vectors). The analogous vector \( b_{βα} (β) \) in the reference frame of \( β \) is related to the previous one by

\[
b_{αβ} (α) = \Lambda_{αβ}^b (α, β) b_{βα} (β).
\]

It can be easily proved that (9) can be written as a bivector

\[
γ_{c^i} (α) = \frac{1}{n!(n-2)!} V(α) \left( b_{αβ}^{c_i} (α) b_{αβ}^{c_j} (α) - b_{αβ}^{c_j} (α) b_{αβ}^{c_i} (α) \right),
\]

where \( V(α) \) is the oriented volume of the simplex \( α \).

We can consider the action as written on the dual Voronoi complex of the original simplicial complex. The dual of the hinge \( h \) is a two-dimensional plaquette whose vertices will be \( \{ α, β, δ^h_1, \ldots, δ^h_k \} \). These vertices are the dual of the \( n \)-simplices incident on the hinge \( h \).

If, briefly, we indicate \( \Lambda(α, β) \) as \( \Lambda_{αβ} \), and so on, the holonomy matrix (7) around the plaquette is

\[
U_{αα}^h \equiv \Lambda_{αβ} \Lambda_{βδ^h_1} \cdots \Lambda_{δ^h_k α},
\]

so that the action can be written in term of the bivectors \( b_{αβ} (α) \),

\[
S = -\frac{1}{2} \sum_h \text{Tr}(U_{αα}^h γ^h (α)).
\]

As remarked in \[2\], this sum (i) does not depend on the starting point of each two-dimensional plaquette chosen and (ii) it is a functional on the two-dimensional plaquette of the Voronoi complex.
3. First-order set-up

Up to now we have dealt with a second-order formalism of discrete general relativity. More precisely the connection matrices $\Lambda_{\alpha\beta}$ and the $b_{\alpha\beta}$ are both functions of the coordinates of the edges of the simplicial complex. Now we introduce a first-order formalism in which $\Lambda_{\alpha\beta}$ and $b_{\alpha\beta}(\alpha)$ are independent variables. We set only the following constraints:

$$b_{\alpha\beta}^{\mu}(\alpha) = \Lambda_{\alpha\beta}^{\mu} b_{\rho\mu}(\beta),$$

which fix $n$ independent conditions for each face, not enough to determine the $\frac{n(n-1)}{2}$ degrees of freedom of $\Lambda_{\alpha\beta}$.

On $b_{\alpha\beta}(\alpha)$ there is a further constraint since the $n+1$ normals to the $(n-1)$-dimensional faces of an $n$-simplex $\alpha$ are linearly dependent,

$$\sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha) = 0.$$ (18)

In this first-order formalism of general relativity [13, 14], we will consider torsion-free, metric-compatible connection matrices more general than the Levi-Civita one defined by equation (5) (for a discussion on a possible weakening of the metricity condition see [15]. Regge calculus with torsion has been studied by Drummond [18]). There is a technical problem we would like to point out. The gravitational action in the form (16) could be dependent on the starting simplex. So we need to define the following antisymmetric tensor on the plaquette $h$:

$$W(h)^{c_1c_2} = \frac{1}{k_h} + 2 (\gamma^{c_2}_{\alpha} + \Lambda_{\alpha\beta} \gamma^{c_2}_{\beta}) \Lambda_{\beta\alpha} + \cdots + \Lambda_{\alpha\beta} \cdots \Lambda_{\gamma_{n-3}\delta_h} W^{c_2}_{\delta_h} (\delta_h) \Lambda_{\delta_h \delta_h} \cdots \Lambda_{\beta\alpha},$$ (19)

The action then is

$$S = -\frac{1}{2} \sum_{h} \text{Tr} (U_{\alpha\alpha}^{h} W^{h}(\alpha)).$$ (20)

The action (20) coincides with the action (16) in the second-order formalism, and, since (in matrix notation)

$$W^{h}(\alpha) = \Lambda_{\alpha\beta} \cdots \Lambda_{\delta_h \delta_h} W^{h}(\delta_h) \Lambda_{\delta_h \delta_h} \cdots \Lambda_{\beta\alpha},$$ (21)

equation (20) is independent of the starting simplex.

Moreover, the action (20) is invariant under the following set of transformations:

$$\Lambda_{\alpha\beta} \mapsto O(\alpha) \Lambda_{\alpha\beta} O^{-1}(\beta) \quad b_{\alpha\beta}(\alpha) \mapsto O(\alpha) b_{\alpha\beta}(\alpha)$$ (22)

where $O(\alpha)$ and $O(\beta)$ are two elements of $SO(n)$. This can be interpreted as the diffeomorphism invariance of this discrete theory of gravity. As we remarked before, without gauge fixing, the affine group would have been the invariance group. This would have been the starting point to look for a description of gravity, in the connection formalism, as a Yang–Mills theory of the affine group [19] in the discrete case. We highlighted this problem as a possible subject for a future investigation.

4. n-bein on the simplicial complex

It has been pointed out [10, 2] that $b_{\alpha\beta}$ is the $n$-bein in the reference frame of each simplex. We are going to stress a feature which will be useful in the following discussions. It has been noted in [10] that the way in which we have defined the coordinates, and, as a consequence, $b_{\alpha\beta}$
itself, does not take into account the symmetry of each simplex. A better definition consists in introducing affine or ‘barycentric’ coordinates. Each point \( P \) in the interior of a simplex \( \beta \) can be considered as a barycentre of \( n+1 \) masses assigned at the vertices of the simplex (see [10, 12] for more details). In particular, if we choose in the \( n \)-simplex \( \beta \) a reference frame in which the origin coincides with the geometric barycentre \( [2] \), the coordinates \( z^a_i(\beta) \) of the vertices, in this reference frame, by the definition of the geometric barycentre, satisfy the condition

\[
\sum_{i=0}^{n} z^a_i(\beta) = 0
\]

which is analogous to (18). In barycentric coordinates \( [2] \), we may write \( b_{a\beta} \) as

\[
b_{a\beta}(\beta) = \frac{1}{(n-1)!} \epsilon_{a_{1}...a_{n-1}}^{b_{1}...b_{n-1}} z_{j_1}^{a_1}(\beta) \cdot \cdot \cdot z_{j_{n-1}}^{a_{n-1}}(\beta).
\]

The meaning of this formula can be understood, geometrically, in an easy way if we look at figure 2. \( b_3(\beta) \) is also normal to a face which belongs to \( \gamma \). So it can be evaluated by solving (18), considered as an equation in \( \gamma \), with respect to \( b_3(\beta) \). Equation (24) does not give preference to the coordinates of any vertex. Thus, the symmetry of the simplex is preserved as desired. The relation (24) can be inverted with respect to \( z^a_i \) and, as in [2], we obtain

\[
z^a_i(\beta) = \frac{n}{(n+1)!V(\alpha)(n-2)} \epsilon_{a_{1}...a_{n-1}}^{b_{1}...b_{n-1}} b_{j_1}^{a_1}(\beta) \cdot \cdot \cdot b_{j_{n-1}}^{a_{n-1}}(\beta).
\]

It could be seen that (25) is related to the dual barycentric base as explained in [10, 12], but we do not explicitly discuss this link, since it is not relevant for the purpose of this paper.

5. First-order field equations for small deficit angles

As remarked in [2], in the second-order formalism the action (16) is equivalent to the Regge action for small deficit angles \( K(h) \). In our first-order formalism we do not have angles \( K(h) \). The variables related to the deficit angles are the connection matrices \( \Lambda_{a\beta} \). Then we assume, by definition, that the small deficit angle approximation in the first-order formalism is the passage from the group variables \( \Lambda_{a\beta} \) of \( SO(n) \) to the algebraic variables \( \phi_{a\beta} \) of \( so(n) \). As a consequence, the connection matrices can be written in the form

\[
\Lambda_{a\beta} = I + \epsilon \phi_{a\beta} + o(\epsilon).
\]
In order to avoid the technical complications, which we shall discuss in the following section, we now substitute the constraint $b_{\alpha \beta}(\alpha) = \Lambda_{\alpha \beta} b_{\beta \alpha}(\beta) + o(\epsilon)$. (27)

A straightforward explanation of the action up to first order shows that

$$S = -\frac{1}{2} \epsilon \sum_h \text{Tr} (\phi_{\alpha \beta} + \phi_{\beta \alpha})^0 W^h(\alpha(\epsilon)) + o(\epsilon)$$

(28)

where $^0W^h_{c_1c_2}$ is the $(2, 0)$ antisymmetric tensor (19) at zeroth order, in the $\epsilon$-expansion. Here, we have used the approximation (26). For each Voronoi edge we have solved the constraint (17) up to the first order (27).

We also impose the requirement that the action is stationary under variation with respect to $\phi_{\alpha \beta}$,

$$\frac{\delta S}{\delta \phi_{\alpha \beta} c_1c_2} = \epsilon \sum_{h \in (\alpha \beta)} ^0W^h_{c_1c_2} + o(\epsilon) = 0.$$ (29)

These are the $\frac{n(n-1)}{2}$ implicit equations for $\phi_{\alpha \beta}$, which has $\frac{n(n-1)}{2}$ independent components. Since

$$\frac{\partial}{\partial \phi_{\alpha \beta} c_1c_2} (\phi_{\rho \sigma}, b_i) = -\epsilon n!(n-2)! V(\beta) \left[ \delta^{c_1}_{\beta_1} \delta^{c_2}_{\beta_2} b_{\alpha \beta}^{\beta 1} b_{\beta 2}^{\beta 2} b_{\alpha \beta}^{\beta 1} b_{\alpha \beta}^{\beta 2} \right]$$ (30)

the determinant of this matrix is non-zero (note that matrix (30) has dimension $\frac{n(n-1)}{2}$). By the inverse function theorem (also labelled Dini’s theorem), locally, we can invert equation (29), so that, again locally, there is one solution giving $\phi_{\alpha \beta}$ as a function of the $n$-bein $b_i$. We do not know anything about the uniqueness of this solution globally. We are, in some sense, in a situation analogous to that described in [14]. Barrett found, locally, the uniqueness of the solution, but has showed that this result is not valid globally. Locally, the unique solution can be found by using the Levi-Civita–Regge connection. We determine the matrix connection by equations (5), in which the translation vector is a known quantity, once we put the origins of the reference frames in the barycentre of the simplices. Equations (5) determine the connection matrices as functions of the coordinates of the vertices. In barycentric coordinates these are in one-to-one relation with the $b_i$. Then, we determine the connection matrices as functions of the $b_i$. The identity

$$\sum_{h \in (\alpha \beta)} ^h_{c_1c_2}(\alpha) = 0,$$ (31)

can be easily proved if we express $\gamma^{(h)}_{c_1c_2}(\alpha)$ as in (14), and use the constraint $\sum_{\beta=1}^{n+1} b_{\alpha \beta}^{\beta} (\alpha) = 0$. The statement that this is the Levi-Civita–Regge connection implies

$$\gamma^{(h)}(\alpha) = \Lambda_{\alpha \beta} \gamma^{(h)}(\beta) \Lambda_{\rho \alpha} = \cdots = \Lambda_{\alpha \beta} \cdots \Lambda_{\gamma_{n+1} \alpha} \gamma^{(h)}(\gamma_{n+1}) \Lambda_{\gamma_{n+1} \gamma_{n+1-1}} \cdots \Lambda_{\rho \alpha},$$ (32)

so that to zero order

$$\gamma^{(h)}(\alpha) = \gamma^{(h)}(\beta) + O(\epsilon) = \cdots = \gamma^{(h)}(\gamma_{n+1}) + O(\epsilon).$$ (33)

These facts imply that

$$\epsilon \sum_{h \in (\alpha \beta)} ^0W^h_{c_1c_2} = \sum_{h \in (\alpha \beta)} (\epsilon)^h_{c_1c_2}(\alpha) + O(\epsilon) = 0 + O(\epsilon^2).$$ (34)

that is to say the Levi-Civita–Regge connection is the solution of our first-order equations for the connection matrices in the limit of small deficit angles.
6. First-order field equation: the general case

In the previous section we have seen that, in the case of small deficit angles, Regge calculus is the solution of the first-order field equations. Now we are going to deal with the general problem. We would like to derive the equation of motion by varying the action with respect to $\Lambda_{\alpha\beta}$ and $b_{\alpha\beta}$. First, we have to take into account the constraints (17) and (18). So, in order to perform independent variations of $\Lambda_{\alpha\beta}$ and $b_{\alpha\beta}$, it is necessary to put Lagrange multipliers in the action. Then the action, in the Palatini first order, is

$$S \equiv -\frac{1}{2} \sum_{h} \text{Tr}(U_{\alpha}^{h} W^{h}(\alpha)) + \sum_{(\alpha\beta)} \lambda_{\alpha\beta}(b_{\alpha\beta}(\alpha) - \Lambda_{\alpha\beta} b_{\beta\alpha}(\beta))$$

$$+ \sum_{(\alpha\beta)} \text{Tr}(\lambda^{(\alpha\beta)}(\Lambda_{\alpha\beta}\Lambda_{\alpha\beta}^T - I)) + \sum_{\alpha} \mu(\alpha) \left( \sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha) \right), \quad (35)$$

where $\lambda^{(\alpha\beta)}$ and $\mu(\alpha)$ are $n$-dimensional vectors and Lagrange multipliers, respectively, and $\lambda^{(\alpha\beta)}$ is an $n \times n$ matrix. The constraint $\Lambda_{\alpha\beta}\Lambda_{\alpha\beta}^T = I$ is introduced to restrict the variation of $\Lambda_{\alpha\beta}$ on the group $SO(n)$.

We introduce the following lemma:

**Lemma 1.** The action, in matrix notation,

$$S' \equiv \text{Tr}(\Lambda A) + \text{Tr}(\lambda(\Lambda\Lambda^T - I))$$

(36)

gives the equation of motion (if we assume that the variation with respect to $\Lambda$ is stationary)

$$(\Lambda A) = (\Lambda A)^T. \quad (37)$$

**Proof.** If we consider the variation of the action, using the property that $\text{Tr}(M) = \text{Tr}(M^T)$ and the action is stationary with respect to $\Lambda$ and $\Lambda^T$, we have

$$0 \equiv \frac{\delta S'}{\delta \Lambda} = A + \Lambda^T \lambda, \quad 0 \equiv \frac{\delta S'}{\delta \Lambda^T} = A^T + \lambda \Lambda. \quad (38)$$

Multiplying the first equation for $\Lambda$ on the left and the second for $\Lambda^T$ on the right, and subtracting these two equations term by term, we have (37). □

Applying the lemma to the action (35) for the variation with respect to $\Lambda_{\alpha\beta}$, we obtain the field equations

$$\sum_{h} \left( U_{\alpha\beta}^{h} W^{h}(\alpha) \right)_{ij} - \lambda_{\alpha\beta} \cdot b_{\alpha\beta}(\alpha)_{j} = \sum_{h} \left( U_{\alpha\beta}^{h} W^{h}(\alpha) \right)^{T}_{ij} - \lambda_{\alpha\beta} \cdot b_{\alpha\beta}(\alpha)_{i}. \quad (39)$$

The next step will be to determine the field equations for the variations of $b_{\alpha\beta}$. For this it is necessary to determine the quantity $\frac{\partial V(\alpha)}{\partial b_{\alpha\beta}}$. As remarked in [2], we have the following identity:

$$V(\alpha)^{n-1} = \frac{1}{n!} \epsilon_{a_{1}.a_{n}} \epsilon^{j_{1}..j_{n}} \epsilon^{i_{1}..i_{n}} b_{i_{1}}^{a_{1}}(\beta) \cdots b_{i_{n}}^{a_{n}}(\beta). \quad (40)$$

We can write this formula in a way which is not dependent on the chosen index $j$. Equivalently, equation (40) can be written as

$$V(\alpha)^{n-1} = \frac{1}{n!} \frac{1}{n+1} \sum_{j=1}^{n+1} \epsilon_{a_{1}..a_{n}} \epsilon^{j_{1}..j_{n}} b_{i_{1}}^{a_{1}}(\beta) \cdots b_{i_{n}}^{a_{n}}(\beta), \quad (41)$$
so that it is clearly independent of any index. Thus, we have
\[
V(\alpha) = \left(1 - \frac{1}{n+1} \sum_{j=1}^{n+1} e_{\alpha,1...\alpha_j} e^{h_{j\alpha}} b_{\alpha}^{n+1}(\alpha) \cdots b_{\alpha_j}^{n+1}(\alpha) \right)^{\frac{1}{2}}. \tag{42}
\]
Evaluating its derivative with respect to \( b_{\alpha}^i(\alpha) \), we obtain
\[
\frac{\partial}{\partial b_{\alpha}^i(\alpha)} \left( \frac{1}{n+1} \sum_{j=1}^{n+1} e_{\alpha,1...\alpha_j} e^{h_{j\alpha}} b_{\alpha}^{n+1}(\alpha) \cdots b_{\alpha_j}^{n+1}(\alpha) \right) = -\frac{1}{n!} \frac{(n-1)!}{n!} V^2(\alpha) z_{\alpha}^i(\alpha). \tag{43}
\]
Now we are ready to derive the field equations for the variations of \( b_{\alpha\beta} \). So let us consider the action (35). It is straightforward to write it as
\[
S = -\frac{1}{4} \sum_h \text{Tr}(U_{\alpha\alpha}^h W^h(\alpha) + U^T_{\alpha\alpha}^h W^T^h(\alpha)) + \sum_{(\alpha\beta)} \lambda_{\alpha\beta}(b_{\alpha\beta}(\alpha) - \Lambda_{\alpha\beta} b_{\beta\alpha}(\beta)) + \sum_{(\alpha\beta)} \mu(\alpha) \left( \sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha) \right) \tag{44}
\]
Let us define
\[
U_{\alpha\alpha}^h ij - U^T_{\alpha\alpha}^h ji = \Omega_{\alpha\alpha}^h ij, \tag{45}
\]
the variation with respect to \( b_{\alpha\beta}^i(\alpha) \) is
\[
\frac{\partial S}{\partial b_{\alpha\beta}^i(\alpha)} = \sum_h \frac{1}{4n!(n-2)!} V(\alpha) b_{\alpha\beta}^j(\alpha) \left( \Omega_{\alpha\alpha}^h ij - \frac{2}{n!} \Omega_{\alpha\alpha}^h klj b_{\alpha\beta}^k(\alpha) z_{\alpha}^l(\alpha) \right) + \lambda_{\alpha\beta} - \mu_i(\alpha) = 0. \tag{46}
\]
The field equations (46) and (39) are the analogues of the first-order Einstein equations in the \( n \)-bein formalism (see [20] for a strict analogy). The connection matrices, \( \Lambda_{\alpha\beta} \), the \( n \)-bein, \( b_{\alpha\beta}(\alpha) \), and the Lagrange multipliers \( \lambda_{\alpha\beta} \) and \( \mu_i(\alpha) \) must be determined by solving equations (46) and (39) and constraints (17), (18) as well the orthogonal condition of the connection matrices.

A nice feature of these equations is that if \( \{ \Lambda_{\alpha\beta}, b_{\alpha\beta}(\beta), \lambda_{\alpha\beta}, \mu_i(\alpha) \} \) is a solution of the equations of motion, the map
\[
\Lambda_{\alpha\beta} \mapsto \Lambda_{\alpha\beta}, \quad b_{\alpha\beta}(\beta) \mapsto \omega b_{\alpha\beta}(\beta), \quad \lambda_{\alpha\beta} \mapsto \omega^{-\frac{1}{n-1}} \lambda_{\alpha\beta}, \quad \mu_i(\alpha) \mapsto \omega^{-\frac{n-1}{n-2}} \mu_i(\alpha), \tag{47}
\]
will provide another solution, as it is easy to verify, of the discrete Einstein equations. This map, if we restrict to the \( b \)s only, shows that the discrete Einstein equations have a conformal symmetry, anyway in the \textit{solutions} not in the \textit{action}. This is the same feature as the Regge equations [1].

7. The measure

In this section, we shall discuss the quantum measure to associate with the previous classical action. The action [2] is invariant under the action of the group \( SO(n) \), that is the gauge group (22).

We will use the following notation:
\[
\mu(b_{\alpha\beta}(\alpha)) \equiv \mu_1 b_{\alpha\beta}^1(\alpha) \cdots \mu_n b_{\alpha\beta}^n(\alpha), \tag{48}
\]
and let
\[ \mu(\Lambda_{a\beta}) \] (49)
be the Haar measure on \( SO(n) \). The partition function for this theory is
\[ Z = \int \exp \left( \frac{1}{2} \sum_{h} \text{Tr}(U_{h}^{\dagger} W^{h}(\alpha)) \right) \prod_{\alpha} \delta \left( \sum_{\beta=1}^{n+1} b_{a\beta}(\alpha) \right) \]
\[ \times \prod_{a\beta} \delta \left( b_{a\beta}(\alpha) - \Lambda_{a\beta} b_{\beta a}(\beta) \right) \mu(\Lambda_{a\beta}) \mu(b_{a\beta}). \] (50)

Here, the product \( \prod_{\alpha} \) is a product over all vertices of the dual complex as well as the product \( \prod_{a\beta} \) over all the Voronoi links which have \( \alpha \) as one of their vertices. It is straightforward to see that the measure is invariant under the gauge transformation (22). In fact, if we perform the gauge transformation (22), the modifications to the measure are
\[ \delta \left( \sum_{\beta=1}^{n+1} b_{a\beta}(\alpha) \right) \delta \left( b_{a\beta}(\alpha) - \Lambda_{a\beta} b_{\beta a}(\beta) \right) \mu(\Lambda_{a\beta}) \mu(b_{a\beta}). \] (51)

But \( \mu(b_{a\beta}) \) is equal to \( \text{det}(O(\alpha)) \mu(b_{a\beta}) \), and reduces to \( \mu(b_{a\beta}) \) as well. The Haar measure of \( SO(n) \) is right and left invariant. So \( \mu(\Lambda_{a\beta}') = \mu(\Lambda_{a\beta}) \). Again, by using equations (22) the two deltas of (51) can be written in the following form:
\[ \delta \left( O(\alpha) \sum_{\beta=1}^{n+1} b_{a\beta}(\alpha) \right), \delta \left( O(\alpha)(b_{a\beta}(\alpha) - \Lambda_{a\beta} b_{\beta a}(\beta)) \right). \] (52)

Then, the properties of the delta function along with the last considerations prove that equation (51) can be written as
\[ \delta \left( \sum_{\beta=1}^{n+1} b_{a\beta}(\alpha) \right) \delta \left( b_{a\beta}(\alpha) - \Lambda_{a\beta} b_{\beta a}(\beta) \right) \mu(\Lambda_{a\beta}) \mu(b_{a\beta}). \] (53)

This establishes the invariance of the measure under gauge transformations.

Challenged by [21] we can argue that the metric structure we are considering is not as peculiar as the metric written as a function of the edge lengths in Regge calculus. In first-order formalism, we neither have transition functions which depend on the metric structure, the \( n \)-bein in our case, nor do we sum over a metric that is gauge fixed. From the form of the constraint equations, those in the argument of the delta functions, and from the calculations we have performed, we expect that this measure, once we have integrated over the deltas, is really highly not local as has been discovered for Regge calculus.

8. Coupling with matter

In the continuum theory on Riemannian manifolds with torsion-free connection, the coupling with fermionic matter (for the same case in the presence of torsion see [17]) is given by the Lagrangian density
\[ \mathcal{L} = \frac{i}{2} \left( \bar{\psi} e_{\mu}^{a} \gamma^{a} \nabla_{\mu} \psi - e_{\mu}^{a} \nabla_{\mu} \bar{\psi} \gamma^{a} \psi \right) - m \bar{\psi} \psi. \] (54)

The \( \gamma^{a}, a = 1, \ldots, n \) are the Dirac matrices satisfying the Clifford algebra
\[ \gamma^{a} \gamma^{b} + \gamma^{b} \gamma^{a} = 2\delta^{ab}, \] (55)
whereas $\psi$ is the $n$-dimensional Dirac spinor field ($\bar{\psi} \equiv \psi^1 \gamma^1$), $\nabla_\mu$ is the covariant derivative and $e^\mu_a$ is the $n$-beins on the tangent space of the Riemannian manifold $(M, g)$, where the Lagrangian density is defined (54), such that

$$g^{\mu\nu}(x) = e^\mu_a(x)e^\nu_b(x)\delta^{ab}. \hspace{1cm} (56)$$

We are assuming that the Riemannian manifold $(M, g)$ in question has a spin structure, that is its second Stiefel–Whitney class is zero.

Now we have all the ingredients to define the coupling of gravity with fermionic matter on the lattice in analogy with the continuum case. Let $2^\nu = n$ or $2^\nu + 1$ (depending on whether $n$ is even or odd) and consider the $2^\nu$-dimensional representation of the two-fold covering group of $SO(n)$ [10]. So, instead of considering the connection matrices $\Lambda_{\alpha\beta}$, we will deal with the $2^\nu \times 2^\nu$ connection matrices $D_{\alpha\beta}$ such that

$$D_{\alpha\beta}\gamma^a D_{\alpha\beta}^{-1} = (\Lambda_{\alpha\beta})^a_b\gamma^b. \hspace{1cm} (57)$$

Given $D_{\alpha\beta}$ we can determine $\Lambda_{\alpha\beta}$. Furthermore, if we know $\Lambda_{\alpha\beta}$, we can determine $D_{\alpha\beta}$ except for a sign. In particular, from (57), we can write $\Lambda_{\alpha\beta}$ as [10]

$$\Lambda_{\alpha\beta} = \frac{1}{2^\nu} \text{Tr}(\gamma^a D_{\alpha\beta}\gamma^b D_{\alpha\beta}^{-1}). \hspace{1cm} (58)$$

In the discrete theory, we assume that the spinor field is a $2^\nu$ complex vector defined at each vertex of the dual Voronoi complex, that is to say a map that with each vertex $\alpha$ associates the $2^\nu$ complex vector $\psi(\alpha)$.

In order to define the covariant derivative on a lattice we have to derive the distance $|\alpha\beta|$ between the two neighbouring circumcentres in $\alpha$ and $\beta$. Our reasoning concerning the barycentres can be extended to the circumcentres too. The distance $\Delta h_1$ of the circumcentre in $\alpha$ from the face $\alpha\beta$ can be determined by calculating the volume of the $n$-dimensional simplex obtained by joining the circumcentre with the $n$-vertices of the face $\alpha\beta$, and dividing it by $n$ and by the volume of the face itself. So we have

$$\Delta h_1 = \frac{1}{n^3} \sum_{i=1}^n b^{a}_\alpha(\alpha)z^i_\alpha(\alpha). \hspace{1cm} (59)$$

in which $z^i_\alpha$ are the circumcentric coordinates of the vertices of the face $\alpha\beta$ and $|b_{\alpha\beta}(\alpha)|$ the module of $b_{\alpha\beta}(\alpha)$ written as a function of $z^i_\alpha$, as in (24).

In the same manner, we have

$$\Delta h_2 = \frac{1}{n^3} \sum_{i=1}^n b^{a}_\beta(\beta)z^i_\beta(\beta). \hspace{1cm} (60)$$

Finally, we have $|\alpha\beta| = \Delta h_1 + \Delta h_2$.

We are ready to define the covariant derivative $(\nabla_\mu \psi)(\alpha)$ on a lattice

$$(\nabla_\mu \psi)(\alpha) = \frac{D_{\alpha\beta}\psi(\beta) - \psi(\alpha)}{|\alpha\beta|}. \hspace{1cm} (61)$$

So far the discrete version of the action for the coupling between gravity and fermionic matter can be written in the form (see also [22])

$$S_F = \sum_\alpha \left( \frac{1}{2|\alpha\beta|} \left( \bar{\psi}(\alpha)b^{a}_{\alpha\beta}\gamma_a D_{\alpha\beta}\psi(\beta) \right) - D_{\alpha\beta}\bar{\psi}(\beta)b^{a}_{\alpha\beta}\gamma_a \psi(\alpha) - mV(\alpha)\bar{\psi}(\alpha)\psi(\alpha) \right). \hspace{1cm} (62)$$
Then the quantum measure, which also includes fermionic matter, can be written as
\[
Z = \int e^{-\left(S + S_F\right)} \prod_\alpha \mu(\psi(\alpha)) \mu(\bar{\psi}(\alpha)) \delta \left( \sum_{\beta=1}^{n+1} b_{\alpha\beta}(\alpha) \right) \times \prod_{\alpha\beta} \delta(b_{\alpha\beta}(\alpha) - \Lambda_{\alpha\beta} b_{\beta\alpha}(\beta)) \mu(D_{\alpha\beta}) \mu(b_{\alpha\beta})
\]
where $S$ is the action for pure gravity (20), $\mu(D_{\alpha\beta})$ is the Haar measure on the two-fold covering group of $SO(n)$, while $\mu(\psi(\alpha)) = d\psi(\alpha)$ is the standard measure on $C^{2^n}$.

9. Conclusions

In this paper, we have studied a discrete theory of gravity in its first-order formalism. Following an earlier previous paper [2], we have chosen an orthonormal reference frame in each simplex and have defined a connection matrix as a transformation matrix between the reference frames of two $n$-dimensional simplices that share a common $(n-1)$-face, considered as two distinct reference frames at the same point. This defines a Levi-Civita–Regge connection. These matrices allow us to write a holonomy matrix around each hinge $h$. We define the action as the sum over the hinges of the traces of the holonomy matrices multiplied by the oriented volumes of the hinges. This action can be written on the dual Voronoi complex of the original simplicial complex. We can express the action as a function of the connection matrices and of the vectors $b_{\alpha\beta}$ which are normal to the faces of the simplices and whose modulo is proportional to the volume of the faces themselves.

The action is very similar to the Wilson action of lattice gauge theory. Here, $b_{\alpha\beta}$ have the same role as the $n$-bein in the continuum theory. Moreover, the action is locally invariant under the action of the Poincaré group.

On the dual Voronoi metric complex, the first-order formalism is implemented by considering the dynamical variables $\Lambda_{\alpha\beta}$ and $b_{\alpha\beta}$ independent. It is shown that in the limit of small deficit angles the Levi-Civita or Regge connection is a solution of the equations of motion. As in Barrett [14], the solution is unique locally, by Dini’s theorem. However, we do not know what is going to happen globally.

The general equations for $\Lambda_{\alpha\beta}$ and $b_{\alpha\beta}$ are derived by using Lagrange multipliers, in order to take into account the constraints of the first-order formalism.

A quantum measure and the relative partition function have been defined. They are locally invariant under the action of $SO(n)$. We have introduced a coupling of gravity with fermionic matter on the Voronoi complex as well. This coupling seems as natural as for the continuum theory. In particular, it seems that the Voronoi complex allows us to introduce the coupling with matter, avoiding all the troubles we had in Regge calculus. We hope that this first-order formalism might be useful for numerical simulations. In particular its first-order character, numerically, could have more advantages than the usual (Regge) second-order formulation in implementing the evolution of equations of motion.

Acknowledgments

I would like to thank Alessandro D’Adda for the early and determinant collaboration on this topic and Ruth Williams for discussions and helpful comments on this work. This research has been, partially, supported by Fondazione della Riccia while I was at UC Irvine. I would like to express my gratitude to Fr Bill Stoeger SJ for directing, encouraging and
advising me to complete this work, as well the moral support of Fr Secondo Bongiovanni SJ, Fr Gianluigi Brena SJ and Fr Giuseppe Pirola SJ. It is a pleasure to thank Fr George Coyne SJ and Fr Francesco Tata SJ for contributions to the last stages of this research.

References

[1] Regge T 1961 General relativity without coordinates *Nuovo Cimento* 19 559–71
[2] Caselle M, D’Adda A and Magnea L 1989 Regge calculus as a local theory of the Poincaré group *Phys. Lett.* B 232 457–61
[3] Wilson K T 1974 Confinement of quarks *Phys. Rev.* D 10 2445–59
[4] Immirzi G 1997 Quantum gravity and Regge calculus *Nucl. Phys. Proc. Suppl.* 57 65–72
[5] Hehl F W 1979 *Four Lectures on Poincaré Gauge Field Theory* (Erice: Erice Cosmology Inst.) p 40 footnote
[6] Ferraris et al 1982 *Gen. Rel. Grav.* 14 243
[7] Regge T and Williams R M 2000 Discrete structures in gravity *J. Math. Phys.* 41 3964–84
[8] Seifert H and Threlfall W 1980 *Seifert and Threlfall: A Textbook of Topology* (New York: Academic)
[9] Hamber H W 1986 Simplicial quantum gravity *Critical Phenomena, Random Systems, Gauge Theories* (Les Houches Session XLII 1984) ed K Osterwalder and R Stora (Amsterdam: Elsevier)
[10] Fröhlich J 1992 Regge calculus and discretized gravitational functional integrals *Non-Perturbative Quantum Field Theory—Mathematical Aspects and Applications* (Singapore: World Scientific)
[11] Cheeger J, Muller W and Schrader R 1984 On the curvature of piecewise flat manifolds *Commun. Math. Phys.* 92 405–54
[12] Sorkin R 1975 The electromagnetic field on a simplicial net *J. Math. Phys.* 16 2432–40
[13] Palatini A 1919 Deducizione invariante delle equazioni gravitazionali dal principio di Hamilton *R. C. Circ. Mat. Palermo* 43 203–12
[14] Barrett J 1994 First order Regge calculus *Class. Quantum Grav.* 11 2723–30
[15] Hehl F W, McCrea J D, Mielke E W and Neeman Y 1995 Metric-affine gauge theory of gravity: field equation, Noether identities, world spinors, and breaking of dilaton invariance *Phys. Rep.* 258 1–171
[16] Mielke E W 2002 Chern–Simons solution of chiral teleparallelism constraints of gravity *Nucl. Phys.* B 622 457–71
[17] Mielke E W 2001 Beautiful gauge field equations in Clifford *Int. J. Theor. Phys.* 40 171–89
[18] Drummond I T 1986 Regge–Palatini calculus *Nucl. Phys.* B 273 125–36
[19] Tresguerres R and Mielke E W 2000 Gravitational Goldstone fields from affine gauge theory *Phys. Rev.* D 62 044004
[20] Esposito G, Gionti G and Stornaiolo C 1995 Space-time covariant form of Ashtekar’s constraints *Nuovo Cimento B* 110 1137–52
[21] Menotti P and Peirano P P 1996 Functional integration on two-dimensional Regge geometries *Nucl. Phys.* B 473 426–54
[22] Ren H 1988 Matter fields in lattice gravity *Nucl. Phys.* B 301 661–84
[23] Senechal M 1995 *Quasicrystal and Geometry* (Cambridge: Cambridge University Press)
[24] Okabe A, Boots B and Sugihara K 1992 *Spatial Tessellation Concepts and Applications of Voronoi Diagrams* (New York: Wiley)
[25] Gionti G 1998 Discrete approaches toward the definition of a quantum theory of gravity *PhD Thesis* SISSA (Preprint gr-qc/9812080)