On the Non-Real Roots of the Riemann Zeta Function $\zeta(s)$

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Abstract

This paper aims to solve a very difficult problem closely related to analytic number theory and specifically to prime numbers. This problem is the Riemann hypothesis according to which the prime numbers follow a very regular distribution among natural numbers. The statement of the Riemann hypothesis is: “All the non-real roots of the complex function $\zeta(s)$ have real part equal exactly to 1/2”. In this paper, we give an original proof to this hypothesis by contradiction. This proof is exceptional and more special because it is based on the relevant property that when $\eta(s)=0$ where $\eta(s)$ is the Dirichlet eta function, the sum of the first $m$-1 terms of $\eta(s)$ converges to -0.5 multiplied by the $m$th term with $m \to \infty$ and it is also based on the condition that when we want to replace the complex variable $z$ by $z'$ with $z' \neq z$ in the expression of $\eta(z)$ which is defined on the half-plane $\Re(z)>0$ by $\eta(z)=\sum_{n=1}^{\infty}(-1)^{n+1}n^{-z}$, we have to keep the same infinite number of $\eta$-terms beginning from $n=1$ to $n=\infty$ in order to converge exactly to the same function. After the exposure of the proof of the Riemann hypothesis, we give an original method combining two already existing for calculating the approximate value of the integral of any continuous real function $f$ over a closed interval $I=\{a, b\} \subseteq \mathbb{R}$ of course when the primitive of $f$ is not obvious by algebraic computation. This new numerical method is called “The Method of trapezes and half-ellipses THE” and its general approximation formula is defined by the following expression: $\int f(x)dx = \text{THE}_n = \int_{a}^{b} f(x)dx = \sum_{n} \left(1 - \frac{\pi}{4} R_{\eta} + \frac{\pi}{4} M_{\eta} \right)$ where $n$ is the number of subintervals obtained by regular subdivision of the interval $[a, b]$ and $y \in [0, 4\pi]$ and $T_{\eta}$ is the approximate value of $\int f(x)dx$ using the method of trapezes and $M_{\eta}$ is the approximate value of $\int f(x)dx$ using the method of rectangles with midpoint and comparing the margins of error, we note that The Method of trapezes and half-ellipses THE is more accurate than the following three known methods: The method of rectangles on the left $R_{\eta}^L$, The method of rectangles on the right $R_{\eta}^R$, and The method of trapezes $T_{\eta}$.

Keywords: Riemann’s hypothesis; Riemann Zeta function; Non-real roots of Riemann Zeta function; Dirichlet Eta function; Analytic number theory; Prime numbers; Numerical integration; The method of trapezes and half-ellipses THE

Introduction

The Riemann zeta function $\zeta(s)$ plays a very important role in analytic number theory; its importance comes essentially from the very close connection it has with prime numbers; this connection that the great German mathematician Georg Friedrich Bernhard Riemann (1826-1866) had shown in his famous manuscript published in 1859 (it is the same date when Charles Darwin (1809-1882) published his work "On the Origin of Species") when he gave an explicit formula linking the prime numbers with the roots of zeta function $\zeta(s)$, namely the solutions $s \in \mathbb{C}$ of the equation $\zeta(s)=0$, this explicit formula given by Riemann is $\pi(x) \sim \frac{x}{\log x} - \sum_{\rho} \frac{x^{\rho}}{\rho}$, where $\pi(x)$ is the function counting the prime numbers and $\log x$ is the Logarithmic integral defined by $\text{Li}(x) = \int_{2}^{x} \frac{1}{\log t} dt$ and $\rho$ represents all the non-real roots of $\zeta(s)$ and in his manuscript, Riemann claimed that it is very likely that all these non-real roots have real part equal exactly to 1/2.

The connection between the prime numbers and zeta function $\zeta(s)$ which is defined on the half-plane $\Re(s)>1$ by $\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^s}$ was established before Riemann by the celebrated Euler (1707-1783) product: $\sum_{n=1}^{\infty} \frac{1}{n^s} = \Pi(1 - \frac{r}{s})^{-1}$ where the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is absolutely convergent for $\Re(s)>1$ and the product $\Pi(1 - \frac{r}{s})^{-1}$ is being over the prime numbers $p \in P=\{2, 3, 5, 7, 11, \ldots\}$. So, in simple terms, the Riemann hypothesis says: “The non-real roots of $\zeta(s)$ have all real part equal exactly to 1/2.” This has been checked for the first 10,000,000,000,000 roots by experts and no counter-examples have been found. In practical terms, the Riemann hypothesis seems true, but theoretically, no proof to this moment has confirmed it. The validity of the Riemann hypothesis is equivalent to saying that the deviation of the number of the prime numbers from the mean $\text{Li}(x)$ is $\pi(x) = \text{Li}(x) + O(\sqrt{x}\log x)$. In this modest paper, we prove the absolute validity of the great Riemann hypothesis giving a very simple and rigorous proof which does not appeal to any complex theory and is very easy to understand.

Some Proven Results about the Non-Real Roots of $\zeta(s)$

- The non-real roots of the Riemann zeta function $\zeta(s)$ have all real part belonging to the critical strip $0, 1$ and they are symmetric with respect to $\Re(s)=1/2$.
- In 1914, the British mathematician Godfrey Harold Hardy (1877-1947) proved that there are infinitely many roots of $\zeta(s)$ on the critical line $\Re(s)=1/2$.
- The number of roots $s=\sigma + it$ of $\zeta(s)$ in the critical strip $0<\sigma<1$ is asymptotically equal to $N(T) = \frac{T}{2\pi}\log(T/2\pi) - (T/2\pi) + O(\log T)$.

On the other hand, Hardy and Littlewood (1885-1977) had proved in the 1920s a region without root of form $\Re(s)=1-k(\log(\log T))^{-1}$ for any $k>0$. The non-real roots of the Riemann zeta function $\zeta(s)$ have all real part equal exactly to 1/2. This has been checked for the first 10,000,000,000,000 roots by experts and no counter-examples have been found.

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The Riemann hypothesis implies that \( \zeta(s) \) and \( \zeta''(s) \) do not vanish in the strip \( 0 < \Re(s) < 1/2 \). There are still more proven results about the non-real roots of \( \{s\} \) apart from these results just mentioned, but all these results are not enough to say that the Riemann hypothesis is true. Fortunately, after several attempts we were able to solve this great problem and we believe that if there are many proofs to the Riemann hypothesis, our proof would probably be the simplest and the most beautiful.

On the Non-Real Roots of \( \zeta(s) \)

Key-question

There is an important question we have to ask ourselves, we wonder if any of the mathematicians who tried to prove the Riemann hypothesis and who failed if asked himself this question, it is the following key-question: "If \( \zeta(s) = 0 \) and \( \eta(1-s) = 0 \), \( \Re(s) < 1 \) and \( \eta(s) \neq 1/2 \), then the limit \( \lim_{n\to\infty} \zeta(1-\zeta(n)) \) has to be finite or infinite? It could be equal to 0? why not?". It is from this intelligent and relevant question and based on the explicit expression of \( \eta(s) \) which is described as an infinite complex series for \( \Re(s) > 0 \) and based on the special form of its general term \((-1)^n n^{-s}\) and referring to universally known formulas and proven theorems that we started our reflection and our reasoning and carefully following the logical and rational implications we were able to find the right answer to the previous question and then to prove the absolute validity of the Riemann hypothesis and it should be noted here that the key result of the concise proof below lies in the expression of \( \eta(s) \).

The proof of the Riemann hypothesis

We know that the Riemann zeta function \( \zeta(s) \) is the analytic function of the complex variable \( s \), defined on the half-plane \( \Re(s) > 1 \) by \( \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \) where the series \( \sum_{n=1}^{\infty} n^{-s} \) is absolutely convergent for \( \Re(s) > 1 \) and the product \( \prod (1-p^{-s})^{-1} \) is being over the prime numbers \( p \in \mathbb{P} = \{2, 3, 5, 7, 11, \ldots\} \) and the function \( \zeta(s) \) is defined in the complex plane \( \mathbb{C} \) by analytic continuation. As shown by Riemann, \( \zeta(s) \) can be continued analytically to \( \mathbb{C} \) as a meromorphic function and has a first order pole at \( s = 1 \) with residue 1. On the other hand, we know that the Riemann zeta function \( \zeta(s) \) is defined for any complex number \( s \) different from 1 and with real part strictly greater than 0 by \( \zeta(s) = \eta(s)/(1-21-s) \) [2] where \( \eta(s) \) is the Dirichlet etta function which is defined on the half-plane \( \Re(s) < 0 \) by \( \eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1}/n^s \) [3]. Noticing that \( 1/(1-21-s) \) \( x \rightarrow \infty \) (denote: we remark \( 0 = [0, i0, 0+i0] \) and \( oo = oo \) where \( i \in \mathbb{C}, i^2 = -1 \) at least for \( \Re(s) > 0 \), then we have to note the following result: if \( \zeta(s) = 0 \), then \( \eta(s) = 0 \), we thus have the Riemann hypothesis is equivalent to the following statement: "The non-real roots of the Dirichlet etta function \( \eta(s) \) belonging to the critical strip \( 0 < \Re(s) < 1 \) have all real part equal exactly to 1/2".

We also know that there is an important relationship between \( \zeta(s) \) and \( \zeta(1-s) \). This relationship is defined for any \( s \) in the complex plane \( \mathbb{C} \) by the following Riemann function equation: \( \xi(s) = \xi(1-s) \) [4] where \( \xi(s) = \eta(1/2-s) \Gamma(s/2) \zeta(s) \), \( \Gamma(s) \) being the Euler gamma function which is defined for any complex number \( s \) such that \( \Re(s) > 0 \) by \( \Gamma(s) = \int_{0}^{\infty} e^{-x}x^{s-1}dx \) [5]. So, the two equations [2] and [4] imply: \( \eta(1-s) \eta(s) = \Lambda(s)\Lambda(1-s) \) where \( \Lambda(s) = e^{\pi i s}/\Gamma(s/2) \). We also know that in 1896, the two mathematicians Jacques Salomon Hadamard (1865-1963) and Charles-Jean de LaVallee Poussin (1866-1962) independently proved that no root of \( \zeta(s) \) could be on the line \( \Re(s) = 1 \), and that all the non-real roots have to be in the interior of the critical strip \( 0 < \Re(s) < 1 \), and it is known that this demonstration was a key result in the first complete proof of the prime number theorem \( \{PNT: \lim_{n\to\infty}(\pi(n) - n/\ln(n)) = 0\} \). Now, let’s assume that there is a complex number \( s \) such that \( 0 < \Re(s) < 1 \) verifying \( \lim_{n\to\infty}(\Lambda(n) - 1/n) = 0 \). According to this assumption, we have to have: \( \lim_{n\to\infty}(\Lambda(n) - 1/n) = 0 \), so, we have if \( \Re(s) \neq 0, 1 \) and knowing that \( \Gamma(1-s) \) does not vanish for any \( s \) in \( \mathbb{C} \) and has an infinite simple poles with residue \( (-1)^{k}/k! \) at \( s = k \) where \( k = 0, 1, 2, 3, 4, 5, \ldots \) etc. then \( \lim_{n\to\infty}(\Lambda(n) - 1/n) = 0 \) \( \Rightarrow \Re(s) = 0 \). Therefore, we have to mention the following property: if \( \Re(s) = 0 \) and \( \Re(s) < 1 \), then \( \eta(1-s) = 0 \).
In general \( \lim_{x \to 0} f(x) = \lim_{y \to 0} g(y) \),

If \( \eta_{n}(z) = \sum (-1)^{m} n^{-m} \) then \( |\eta(z) - \eta_{n}(z)| \leq \lim_{n \to \infty} |\eta(z) - \eta_{n}(z)| \Rightarrow m = m \),

applying this condition we keep the same infinite number of \( \eta \)-terms in order to converge exactly to the same function in numerator and denominator,

To prove the Riemann hypothesis, it seemed to us that it is more convincing to prove the conjecture:

If \( \eta(s)=0 \) and \( 0 < \Re(s) < 1 \), then: \( \lim_{s \to 0} \eta(1-z) = 0 \), or \( \Re(s) = \frac{1}{2} \).

And in the conjecture above, we denote: \( 0 = (0, 0, 0, 0) \) and \( \mathbb{C} = \{ \infty, 0, 0, 0, 0 \} \) where \( i \in \mathbb{C} \).

The proof of the conjecture [C] [algebraic proof]: We have previously shown that for every complex number \( s \) such that \( 0 < \Re(s) < 1 \), we have \( \lim_{s \to 0} \eta(1-s) = 0 \), and we have deduced according to this result that if we have \( \eta(s) = 0 \) and \( 0 < \Re(s) < 1 \), then we also must have \( \eta(1-s) = 0 \), and we know that \( \eta \)-function is defined on the half-plane \( \eta > 0 \) by \( \eta(s) = \sum \alpha_{s}(z) \) where \( \alpha_{s}(z) = \sum (-1)^{n} n^{s} \).

To prove the conjecture [C], we have to prove the following proposition:

If \( \eta(s) = 0 \), and \( \eta(1-s) = 0 \) and \( 0 < \Re(s) < 1 \), then

\[
\lim_{s \to 0} \eta(1-z) = \lim_{s \to 0} \eta(z) = \lim_{s \to 0} \alpha_{s}(z)
\]

So, let's prove the proposition [P].

**Proof:** We have:

\[
\eta(z) = \sum_{n=0}^{\infty} \alpha_{s}(z) + \sum_{n=0}^{\infty} \alpha_{s}(z), m \geq 2
\]

We denote:

\[
\eta_{n}(z) = \sum_{n=0}^{m} \alpha_{s}(z) \text{ and } \eta_{m}(z) = \sum \alpha_{s}(z)
\]

So, we have

\[
\lim_{s \to 0} \eta(1-z) = \lim_{s \to 0} \eta_{n}(z) + \eta_{m}(z) = \eta_{m}(z)
\]

and for any \( s \) in the complex plane \( \mathbb{C} \) with \( \Re(s) > 0 \), we have:

\[
\lim_{s \to 0} \eta_{n}(z) = 0 \quad \text{that is to say if then}
\]

\[
\lim_{s \to 0} \eta_{n}(z) = 0 \quad \text{we have:}
\]

\[
\lim_{s \to 0} q_{n}(z) = 0 \quad \text{where } q_{n}(z) = \frac{\eta_{n}(z)}{\eta_{n}(z)}
\]

but if \( \lim_{s \to 0} \eta_{n}(z) = 0 \), it will still remain \( \lim_{s \to 0} q_{n}(z) = 0 \). In the following, we are going to prove \( \lim_{s \to 0} \eta_{n}(z) = 0 \) then \( \lim_{s \to 0} q_{n}(z) = 0 \),

so in general we can simply write:

\[
\lim_{s \to 0} \eta(1-z) = \lim_{s \to 0} \eta(z) = \lim_{s \to 0} \alpha_{s}(z)
\]

If \( m_{*} = m \) (\( m_{*} \) is a necessary condition in the following calculation of limits), then we have:

\[
\eta(s) = \lim_{s \to 0} (\eta_{s}(z) + \eta_{m}(z)) = \lim_{s \to 0} \eta_{s}(z) + \lim_{s \to 0} \eta_{m}(z)
\]

And \( \forall s \geq 1 \) and \( \forall s > 0 \), we have:

\[
\alpha_{s}(z) = (1)^{s} \left( \frac{n}{n+m} \right)^{s} \alpha_{s}(z)
\]

So, for \( m = m_{*} \), we have

\[
\lim_{\eta \to 0} \sum_{j=1}^{\infty} \alpha_{s}(z) = \lim_{\eta \to 0} \sum_{j=1}^{\infty} \alpha_{s}(z) + \sum_{j=1}^{\infty} \alpha_{s}(z)
\]

\[
= \lim_{\eta \to 0} \sum_{j=1}^{\infty} \left( \alpha_{s}(z) + \sum_{j=1}^{\infty} (-1)^{j} \left( \frac{m}{m+j} \right)^{j} \right)
\]

It's trivial that if \( \lim_{\eta \to 0} f(x) \) and \( \lim_{\eta \to 0} g(x) \), then we can write:

\[
\lim_{\eta \to 0} f(x) g(x) = \lim_{\eta \to 0} f(x) \lim_{\eta \to 0} g(x)
\]

We have \( \lim_{\eta \to 0} \alpha_{s}(z) \), exists for \( \Re(s) > 0 \) and is equal to 0, but we have to prove that \( L \) also exists,

where:

\[
\lim_{\eta \to 0} \left[ 1 + \sum_{j=1}^{\infty} (-1)^{j} \left( \frac{m}{m+j} \right)^{j} \right]
\]

We have

\[
L = \lim_{\eta \to 0} \left[ 1 + \sum_{j=1}^{\infty} (-1)^{j} \left( \frac{m}{m+j} \right)^{j} \right] = \lim_{\eta \to 0} \left[ 1 + \sum_{j=1}^{\infty} \left( \left( \frac{m}{m+j} \right)^{j} \right) \right]
\]

Denoting \( \left( \frac{m}{m+j} \right)^{j} \) by \( x_{j}(z, m) \), we note that \( \forall j, m \geq 1 \)

\[
| x_{j}(z, m) | \leq 1
\]

That means \( \forall j, m \geq 1 \) verifying \( | x_{j}(z, m) | \leq 1 \) \( \iff x_{j}(z, m) \) is bounded and \( m \to \infty \) the \( x_{j}(z, m) \) tend to 0.

That is to say \( \forall j, j \geq 1 \) with \( j \neq j' \) when \( z, m \to s_{\infty} \),

\[
x_{j}(z, m) = x_{j}(z, m) = 1 + i 0
\]

And we know that \( \forall |x| < 1, \lim_{\eta \to 0} (-1)^{x} = \left( \frac{1}{1+i} \right)^{x} \)

We thus get,

\[
L = \lim_{\eta \to 0} \left[ 1 + \sum_{j=1}^{\infty} (-1)^{j} x_{j}(z, m) \right] = 1 + \sum_{j=1}^{\infty} (-1)^{j} x_{j}(z, m)
\]

\[
= 1 + \lim_{\eta \to 0} \frac{1}{1 + x_{j}(z, m)} - 1 = 1 + \frac{1}{1+1} - 1 = 0.5
\]

So, for \( m_{*} = m \) (we must not forget that \( m_{*} = m \) is a necessary condition in calculation of limits):

\[
L = \lim_{\eta \to 0} \alpha_{s}(z) = 0.5 \lim_{\eta \to 0} \sum_{j=1}^{\infty} (-1)^{j} x_{j}(z, m)
\]
That is to say if \( z, m \to s, \infty \) with \( \Re(s) > 0 \), then:
\[
\eta(z) = \eta_n(z) + 0.5\alpha_n(z)
\]
\((*)\)

So, if \( \eta(z) \to 0 \) when \( z, m \to s, \infty \), then we have:
\[
\eta_n(z) \to -0.5\alpha_n(z)
\]

We have to note that if \( \lim_{z \to s} \eta_n(1-z) = 0 \) and \( \lim_{z \to s} \eta_n(z) = 0 \) then,
\[
\lim_{z \to s} \frac{\eta(1-z)}{\eta(z)} = \lim_{z \to s} \frac{\eta_n(1-z)}{\eta_n(z)} = \frac{0}{0} = \frac{0}{0}
\]

So, according to \((*)\), it follows that if \( \eta(s) = 0 \) and \( \eta(l-s) = 0 \), then we must have:
\[
\lim_{z \to s} \frac{\eta(1-z)}{\eta(z)} = \lim_{z \to s} \frac{\eta_n(1-z)}{\eta_n(z)} = 0, z
\]

Thus it has been shown that:
\[
\eta(s) = 0 \quad \text{and} \quad 0 < \Re(s) < 1 \quad \text{implies the following result:}
\]
So, the proposition \([P]\) implies the following result:
\[
\eta(1-z) = \eta_n(1-z) = \alpha_n(z)
\]

In this verification we have taken:
\[
S1 = 1/2 + i14.13472514169931316355316469514012\ldots
\]
\[
S2 = 1/2 + i21.02203963887997875000696870947704\ldots
\]
\[
S3 = 1/2 + i25.01085758011307081110088141463319\ldots
\]
\[
S4 = 1/2 + i30.42442761865581748815209498218248\ldots
\]
\[
S5 = 1/2 + i32.93506158753386271624145502288709\ldots
\]

and using the first 5 roots of the Riemann zeta function \( \zeta(s) \) whose real parts are equal to 1/2 and the imaginary parts are positive and we should note that this numerical verification was done using Microsoft Office Excel 2007.

The first 5 roots of the Riemann zeta function \( \zeta(s) \) are:
\[
\left\{ S1 = 1/2 + i14.13472514169931316355316469514012\ldots, S2 = 1/2 + i21.02203963887997875000696870947704\ldots, S3 = 1/2 + i25.01085758011307081110088141463319\ldots, S4 = 1/2 + i30.42442761865581748815209498218248\ldots, S5 = 1/2 + i32.93506158753386271624145502288709\ldots \right\}
\]

Based on the following condition:
\[
\eta_n(s) = \sum_{n=1}^{\infty} \alpha_n(s) \quad \text{with} \quad \alpha_n(s) = \frac{1}{n^s} \quad \text{for} \quad s = \sigma + it
\]

It has been shown that if \( \eta(s) = 0 \) and \( \eta(l-s) = 0 \) when \( s, m \to s, \infty \), then we get:
\[
\lim_{s \to \infty} \eta(z) = \lim_{s \to \infty} \eta_n(z) = 0
\]

By this condition, we want to say that if someone wants to replace \( z \) by \( z' \neq z \) in the expression of \( \eta_n \), he has to keep the same number of \( \eta \)-terms beginning from \( n = 1 \) to \( n = m = \infty \) to keep exactly the same function and to change only the complex variable \( z \) in the expression of \( \eta_n \) and this condition is valid in the general case.

In short, the most interesting result proved in this paper is the following theorem:

If \( \zeta(s) = 0 \) and \( \Re(s) < 0 \), then \( \Re(s) = 1/2 \) and \( \Im(s) = \pi/\log y \) where \( s \in \mathbb{C} \), and \( y \in ]0, 1[ \cup ]1, +\infty[ \).

**Numerical verification:** We have made the numerical verification for \((*)\) and \((***)\) using the first 5 roots of the Riemann zeta function \( \zeta(s) \) whose real parts are equal to 1/2 and the imaginary parts are positive and we should note that this numerical verification was done using Microsoft Office Excel 2007.

The first 5 roots of the Riemann zeta function \( \zeta(s) \) are:
\[
\left\{ S1 = 1/2 + i14.13472514169931316355316469514012\ldots, S2 = 1/2 + i21.02203963887997875000696870947704\ldots, S3 = 1/2 + i25.01085758011307081110088141463319\ldots, S4 = 1/2 + i30.42442761865581748815209498218248\ldots, S5 = 1/2 + i32.93506158753386271624145502288709\ldots \right\}
\]

For example, for \( t \) being the imaginary part of a non-real root of \( \zeta(s) \) and belonging to the critical line \( \Re(s) = 1/2 \) and \( k \in \mathbb{Z}^* \), we have:
\[
\lim_{s \to \infty} \frac{\alpha_n(z)}{\alpha_n(z)} = \lim_{s \to \infty} \frac{\alpha_n(z)}{\alpha_n(z)} = \frac{0}{0} = \frac{0}{0}
\]

If we look for more precision and rigor, we have to take into consideration in our calculation of limits the following obvious condition (note: this condition is respected in the previous calculation of limits) [6]:
\[
\eta_n(z) = \sum_{n=1}^{\infty} (-1)^{n-1}a_n \quad \text{and} \quad \eta(z) = \sum_{n=1}^{\infty} \alpha_n(z)
\]

It has been shown that if \( \eta(s) = 0 \) and \( \eta(l-s) = 0 \) when \( s, m \to s, \infty \), then we get:
and referring to this result, it has been deduced that:

\[ \lim_{n \to \infty} \frac{\eta(1-z)}{\eta(z)} = \lim_{n \to \infty} \frac{\eta_n(1-z)}{\eta_n(z)} = \lim_{n \to \infty} \frac{-0.5\alpha_n(1-z)}{\alpha_n(z)} = \lim_{n \to \infty} \frac{\alpha_n(1-z)}{\alpha_n(z)} \]

and referring to this result, it has been deduced that:

\[ \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m} \frac{\eta(1-z)}{\eta(z)} = \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m} \frac{\eta_n(1-z)}{\eta_n(z)} = \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m} \frac{-0.5\alpha_n(1-z)}{\alpha_n(z)} = \lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m} \frac{\alpha_n(1-z)}{\alpha_n(z)} \]

Figure 1: The paces of the two quotients \( 1/\eta_m(s) \sum \eta(s) \eta_m(s) \) for \( m \) ranging from 10 to 1000000.

Figure 2: Chart explaining the existence and purpose of method of Trapezoids and Half-ellipses T.H.E.

The Method of Trapezoids and Half-Ellipses T.H.E

Purpose

This method is original and its main purpose is the approximate calculation of the integral of any continuous real function \( f \) on a closed interval \([a, b]\) with \( a < b \) replacing each arc of \( f \)-curve \((M_jM_{j+1})\) by an elliptical half-arc (H E) and we remind that this method can be used
| $m$ | $\Re (\eta_m(s_1))$ | $\Im (\eta_m(s_1))$ | $\frac{\Re (\eta_m(s_1))}{\Im (\eta_m(s_1))}$ |
|-----|---------------------|---------------------|----------------------------------|
| 10  | 0.192867143         | -0.134784134        | -1.43,09,33,577                  |
| 50  | 0.01263537          | -0.044150483        | -0.286188715                     |
| 100 | -0.029184638        | 0.063666475         | -0.458398835                     |
| 150 | -0.003723617        | 0.011241483         | -0.331239537                     |
| 200 | 0.030322087         | -0.06178251         | -0.490787555                     |
| 250 | -0.027422519        | 0.055639584         | -0.492860063                     |
| 300 | 0.013530651         | -0.028224466        | -0.479394402                     |
| 350 | 0.01216527          | -0.023342749        | -0.521158498                     |
| 400 | -0.024727443        | 0.049543065         | -0.499110077                     |
| 450 | -0.001542723        | 0.001944584         | -0.690493699                     |
| 500 | 0.022164627         | -0.044384495        | -0.499377699                     |
| 550 | 0.007500051         | -0.014478126        | -0.518026366                     |
| 600 | -0.015631457        | 0.031554882         | -0.495373648                     |
| 650 | -0.017808192        | 0.035419543         | -0.502778706                     |
| 700 | -0.001687111        | 0.002992738         | -0.563734948                     |
| 750 | 0.0141525           | -0.028510481        | -0.496396395                     |
| 800 | 0.017224078         | -0.034363969        | -0.501224931                     |
| 850 | 0.007995398         | -0.015732746        | -0.505821048                     |
| 900 | -0.005300603        | 0.010845709         | -0.488728123                     |
| 950 | -0.014373582        | 0.028849992         | -0.498217885                     |
| 1000| -0.015351898        | 0.030640875         | -0.501026749                     |
| 2000| 0.009107231         | -0.018166118        | -0.501330609                     |
| 3000| 0.009108325         | -0.018212105        | -0.50012478                     |
| 4000| -0.004315441        | 0.008606909         | -0.50138266                     |
| 5000| 0.003783442         | -0.007549603        | -0.501144497                     |
| 6000| -0.005834998        | 0.01166299          | -0.500300352                     |
| 7000| 0.005184651         | -0.010374924        | -0.499729058                     |
| 8000| 0.001131601         | -0.002253545        | -0.502162233                     |
| 9000| -0.005238931        | 0.010478471         | -0.49997094                     |
| 10000| -0.000950073       | 0.00189316          | -0.501845638                     |
| 20000| -0.0006394         | 0.001281242         | -0.499047018                     |
| 30000| -0.001044122       | -0.002086959        | -0.500307864                     |
| 40000| 0.001316711        | -0.002634157        | -0.499660487                     |
| 50000| -0.001201397       | 0.002403316         | -0.4998914                      |
| 60000| 5.52E-01           | -1.5162E-05         | -0.47912419                     |
| 70000| 0.001548127        | -0.003096024        | -0.500037144                     |
| 80000| -0.001414379       | 0.0028828936        | -0.499968539                     |
| 90000| -0.00087016        | 0.001740092         | -0.500065514                     |
| 100000| 0.001276813       | -0.002537352        | -0.49997533                     |
| 200000| -0.001080988      | 0.002161994         | -0.499995837                     |
| 300000| -0.000629301      | 0.001258633         | -0.499987685                     |
| 400000| 0.00078539        | -0.001570776        | -0.500001273                     |
| 500000| -0.000701411       | 0.001402818         | -0.500001426                     |
| 600000| 0.000584699       | -0.001169403        | -0.49997862                     |
| 700000| -0.000101445      | 0.000202903         | -0.499967965                     |
| 800000| -0.000493971      | 0.000878937         | -0.500002531                     |
| 900000| 0.000289333      | -0.000578773        | -0.499993953                     |
| 1000000| 0.000438865     | -0.000877726        | -0.500002279                     |

Table 1: Calculation of numerical verification 1.1.
| $m$ | $\Re(\eta_m(s_1))$                  | $\Im(\eta_m(s_1))$                  | $\Im(\nu_m(s_1))$                  | $\Re(\nu_m(s_1))$                  |
|-----|-----------------------------------|------------------------------------|-----------------------------------|-----------------------------------|
| 10  | 0.09,94,02,356                    | -0.28,60,65,093                    | -0.34,74,81,599                   |
| 50  | -0.07,67,349                      | 0.13,43,53,023                    | -0.52,60,28,283                   |
| 100 | 0.04,09,08,291                    | -0.07,71,14,071                   | -0.53,04,90,616                   |
| 150 | 0.04,07,68,907                    | -0.080872098                      | -0.50,41,15,857                   |
| 200 | -0.01,83,10,429                   | 0.03,43,93,626                    | -0.53,23,78,558                   |
| 250 | 0.01,58,37,008                    | -0.030070566                      | -0.52,66,61,454                   |
| 300 | -0.02,553,643                     | 0.00,50,36,579                    | -0.50,70,19,348                   |
| 350 | 0.023824468                       | -0.04,80,85,954                   | -0.49,54,55,825                   |
| 400 | 0.00,38,11,944                    | -0.00,67,44,234                   | -0.56,52,15,264                   |
| 450 | -0.02,35,47,985                   | 0.04,71,00,327                    | -0.49,99,53,748                   |
| 500 | -0.00,305,4444                    | 0.00,54,78,744                    | -0.55,75,08,071                   |
| 550 | 0.01,99,69,559                    | -0.04,01,06,928                   | -0.49,79,07,967                   |
| 600 | 0.01,31,42,658                    | -0.02,25,90,282                   | -0.50,73,83,289                   |
| 650 | -0.00,82,35,604                   | 0.01,68,49,851                    | -0.48,87,64,203                   |
| 700 | -0.01,88,30,511                   | 0.03,76,77,778                    | -0.49,97,77,641                   |
| 750 | -0.01,15,45,216                   | 0.02,28,14,158                    | -0.05,06,05,488                   |
| 800 | 0.00,40,06,358                    | -0.00,83,13,703                   | -0.48,18,98,139                   |
| 850 | 0.01,51,78,428                    | -0.03,04,78,702                   | -0.49,80,01,129                   |
| 900 | 0.01,58,06,735                    | -0.03,15,19,545                   | -0.50,14,89,949                   |
| 950 | 0.00,75,30,681                    | -0.01,48,42,827                   | -0.50,73,61,637                   |
| 1000| -0.00,380,2202                    | 0.00,78,19,003                    | -0.48,62,77,087                   |
| 2000| 0.00,06,48,777                    | -0.01,30,38,104                   | -0.49,76,00,725                   |
| 3000| 0.00,06,21,364                    | -0.00,12,85,525                   | -0.48,33,54,272                   |
| 4000| -0.00,66,24,574                   | 0.01,32,63,526                    | -0.49,94,57,987                   |
| 5000| 0.00,59,74,167                    | -0.01,19,58,407                   | -0.49,95,78,832                   |
| 6000| -0.00,276,0977                    | 0.00,55,35,461                    | -0.49,87,97,957                   |
| 7000| -0.00,29,72,586                   | 0.00,59,34,484                    | -0.50,09,005                     |
| 8000| 0.00,54,74,619                    | -0.01,09,50,887                   | -0.49,99,24,709                   |
| 9000| 0.00,05,77,008                    | -0.00,11,45,756                   | -0.50,36,04,607                   |
| 1000| -0.00,490,0935                    | 0.00,9819162                      | -0.49,99,43,949                   |
| 2000| 0.00,347,7281                     | -0.006954022                      | -0.500038826                     |
| 3000| 0.00,269,1335                     | -0.00,53,83,116                   | -0.49,99,58,574                   |
| 4000| -0.00,21,25,171                   | 0.00,42,48,849                    | -0.50,00,58,002                   |
| 5000| 0.00,18,85,592                    | -0.00,37,71,481                   | -0.50,00,47,594                   |
| 6000| -0.00,20,41,243                   | 0.00,40,82,467                    | -0.50,00,02,327                   |
| 7000| 0.00,10,83,862                    | -0.00,21,68,029                   | -0.49,99,26,966                   |
| 8000| 0.00,10,60,449                    | -0.00,21,20,641                   | -0.50,00,60,595                   |
| 9000| -0.00,14,21,483                   | 0.00,2843095                      | -0.49,99,77,313                   |
| 10000| -0.00,93,261                      | 0.00,18,65,034                    | -0.50,00,49,865                   |
| 20000| 0.00,0285425                      | -0.00,05,70,773                   | -0.50,00,67,452                   |
| 30000| 0.00,06,61,298                    | -0.00,13,22,565                   | -0.05,00,17,12                    |
| 40000| 9.03502E-05                       | -0.00,01,80,728                   | -0.49,99,23,642                   |
| 50000| -8.95746E-05                      | 0.00,01,79,169                    | -0.49,99,44,745                   |
| 60000| -0.00,02,73,486                   | 0.00,05,46,957                    | -0.50,00,13,712                   |
| 70000| 0.00,05,86,941                    | -0.00,17,788                      | -0.50,00,00,849                   |
| 80000| -0.00,02,61,711                   | 0.00,05,23,431                    | -0.49,99,91,403                   |
| 90000| -0.000440494                      | 0.000880864                      | -0.05,00,00,277                   |
| 100000| 0.00,02,39,578                    | -0.00,47,594                      | -0.49,99,93,739                   |

Table 2: Calculation of numerical verification 1.2.
Table 3: Calculation of numerical verification 1.3.

| m     |  \( \eta_m(s_1) \) |  \( a_m(s_1) \) |  \( \Re(\eta_m(s_1)) \) |
|-------|---------------------|-----------------|--------------------------|
| 10    | -0.298803655        | 0.090320283     | -3.30826748              |
| 50    | 0.06833117          | -0.12002282     | -0.569318113             |
| 100   | -0.039062783        | 0.08364493      | -0.46876397              |
| 150   | 0.000838921         | -0.007372696    | -0.113787548             |
| 200   | -0.006958822        | 0.01022426      | -0.68018646              |
| 250   | -0.030995955        | 0.062370315     | -0.49696466              |
| 300   | 0.025512949         | -0.049970481    | -0.510560405             |
| 350   | -0.02218031         | 0.043391872     | -0.51112782              |
| 400   | 0.024164864         | -0.047925056    | -0.50422192              |
| 450   | -0.021729465        | 0.043839399     | -0.49566065              |
| 500   | 0.005463041         | -0.011826955    | -0.461914415             |
| 550   | 0.016570237         | -0.032600321    | -0.50824474              |
| 600   | -0.01650938         | 0.033415728     | -0.494060162             |
| 650   | -0.009686153        | 0.018808426     | -0.51499009              |
| 700   | 0.016332111         | -0.032931023    | -0.49594093              |
| 750   | 0.011016676         | -0.021613651    | -0.50970165              |
| 800   | -0.011529711        | 0.023400347     | -0.492715386             |
| 850   | -0.015702739        | 0.031220641     | -0.502960173             |
| 900   | 0.000767843         | -0.001923818    | -0.399124553             |
| 950   | 0.015024203         | -0.030172433    | -0.497944697             |
| 1000  | 0.012184744         | -0.024148842    | -0.504568459             |
| 2000  | -0.0101168          | 0.020280562     | -0.498842192             |
| 3000  | 0.002094356         | -0.004250569    | -0.49272368              |
| 4000  | -2.51383E-05        | 8.73E-01        | -2.8803618               |
| 5000  | -0.007089382        | 0.014138711     | -0.500001874             |
| 6000  | 0.000570098         | -0.010125745    | -0.500713577             |
| 7000  | -0.004304464        | 0.008596143     | -0.500743182             |
| 8000  | 0.005075981         | -0.010145473    | -0.500319798             |
| 9000  | -0.005128103        | 0.01025875      | -0.499876008             |
| 10000 | 0.001998329         | -0.00406189     | -0.498810464             |
| 20000 | 0.002344438         | -0.004685918    | -0.500303249             |
| 30000 | -0.002882405        | 0.005765873     | -0.499994536             |
| 40000 | -0.002395292        | 0.004799029     | -0.499963994             |
| 50000 | 0.000686115         | -0.001371329    | -0.500328513             |
| 60000 | 0.000755687         | -0.001512452    | -0.499782472             |
| 70000 | -0.000869896        | 0.00174029      | -0.49985692              |
| 80000 | 0.000253446         | -0.000507351    | -0.49954765              |
| 90000 | 0.000830553         | -0.001660765    | -0.500102664             |
| 100000| -0.001569306        | 0.003138584     | -0.500007647             |
| 200000| 0.000590745         | -0.001181589    | -0.499958107             |
| 300000| 0.000308187         | -0.000616313    | -0.500049488             |
| 400000| 0.000433172         | -0.000866309    | -0.500020201             |
| 500000| 0.000583069         | -0.001186153    | -0.499993569             |
| 600000| -0.000642894        | 0.001286785     | -0.500001167             |
| 700000| 0.000586993         | -0.001173981    | -0.50000213              |
| 800000| -0.000553097        | 0.00106195      | -0.49999548              |
| 900000| 0.000362932         | -0.000725874    | -0.49993112              |
| 1000000| 8.3178E-05          | -0.000166333    | -0.500031864             |
| $m$  | $\Re(\zeta(s))$     | $\Im(\zeta(s))$   | $\left|\frac{\zeta(s)}{\zeta(0)}\right|$ |
|------|---------------------|--------------------|------------------------------------------|
| 10   | -0.1292896          | 0.303504857        | -0.426621112                             |
| 50   | 0.024809322         | -0.074796528       | -0.331690824                             |
| 100  | 0.031856819         | -0.054744001       | -0.58192347                              |
| 150  | -0.040985555        | 0.081316112        | -0.50402748                              |
| 200  | -0.034758978        | 0.069967596        | -0.496786798                             |
| 250  | 0.006559889         | -0.010485409       | -0.625620708                             |
| 300  | 0.01359563          | -0.028918581       | -0.470134755                             |
| 350  | -0.01496219         | 0.031213581        | -0.479477642                             |
| 400  | 0.006501864         | -0.014254439       | -0.45612907                              |
| 450  | 0.00181916         | -0.017300011       | -0.52982744                              |
| 500  | -0.0216997          | 0.043129145        | -0.50313309                              |
| 550  | 0.013436983         | -0.027484557       | -0.488892108                             |
| 600  | 0.012024249         | -0.023453269       | -0.512689681                             |
| 650  | -0.017064307        | 0.034419539        | -0.495773839                             |
| 700  | -0.0095258          | 0.018550449        | -0.513507786                             |
| 750  | 0.014568941         | -0.029430994       | -0.495020352                             |
| 800  | 0.013409517         | -0.028503279       | -0.50596904                              |
| 850  | -0.006910842        | 0.014203596        | -0.486555799                             |
| 900  | -0.016654744        | 0.033277771        | -0.500476549                             |
| 950  | -0.006132047        | 0.011927105        | -0.514127024                             |
| 1000 | 0.010083883         | -0.020416498       | -0.493908554                             |
| 2000 | 0.004762884         | -0.009418005       | -0.505721116                             |
| 3000 | -0.008886052        | 0.017755731        | -0.500461062                             |
| 4000 | -0.007906176        | 0.015811386        | -0.500030548                             |
| 5000 | 0.000170479         | -0.00031122        | -0.547776493                             |
| 6000 | 0.003995544         | -0.008088493       | -0.498913341                             |
| 7000 | -0.004149507        | 0.005304425        | -0.499240706                             |
| 8000 | 0.002342311         | -0.004697805       | -0.498596998                             |
| 9000 | 0.001217342         | -0.002242635       | -0.502486755                             |
| 10000| -0.004583444        | 0.009162448        | -0.50024293                              |
| 20000| 0.002646545         | -0.005295486       | -0.49977377                              |
| 30000| -0.000158785        | 0.000315547        | -0.503205545                             |
| 40000| -0.000716001        | -0.001430734       | -0.500443129                             |
| 50000| 0.002128214         | -0.004256695       | -0.499968638                             |
| 60000| -0.001896133        | 0.003791986        | -0.50003692                              |
| 70000| 0.001677717         | -0.003355161       | -0.500040684                             |
| 80000| -0.00174951         | 0.003498942        | -0.500011146                             |
| 90000| -0.001444982        | -0.002890151       | -0.499967649                             |
| 100000| -0.000193108       | 0.00038654         | -0.499575723                             |
| 200000| -0.000949222       | 0.001898338        | -0.500016856                             |
| 300000| 0.000859276        | -0.001718573       | -0.499998399                             |
| 400000| 0.000661334        | -0.00132269        | -0.499991684                             |
| 500000| 0.000420039        | 0.00080054         | -0.500014999                             |
| 600000| -5.79E+00          | 0.000115863        | -0.499902471                             |
| 700000| 0.000112173        | -0.000224364       | -0.49995887                              |
| 800000| 8.11E+00           | -0.00016227        | -0.500046219                              |
| 900000| -0.00382175        | 0.000764342        | -0.500005233                              |
| 1000000| 0.000493034       | -0.0008607         | -0.499998986                              |

Table 4: Calculation of numerical verification 1.4.
| $m$   | $\Re (\eta_j(s))$       | $a_j(s)$          | $\Re(\eta_j(s))$ / $a_j(s)$ |
|-------|-------------------------|------------------|-------------------------------|
| 10    | 0.448284561             | -0.159819003     | -2.804951555                  |
| 50    | -0.071854242            | 0.127115345      | -0.565268041                  |
| 100   | -0.01904003             | 0.048915773      | -0.389241115                  |
| 150   | 0.037357774             | -0.076882011     | -0.485910469                  |
| 200   | 0.031022457             | -0.059585908     | -0.520634124                  |
| 250   | 0.031161654             | -0.062681924     | -0.497139399                  |
| 300   | -0.009305261            | 0.016286624      | -0.57134376                   |
| 350   | -0.010228407            | 0.02217931       | -0.46166855                   |
| 400   | 0.014023613             | -0.02926023      | -0.478686578                  |
| 450   | -0.009241699            | 0.019663369      | -0.469995706                  |
| 500   | -0.002259912            | 0.003402611      | -0.66419957                   |
| 550   | 0.016122602             | -0.031579255     | -0.51054490                   |
| 600   | -0.019793606            | 0.039763414      | -0.49778437                   |
| 650   | 0.003576233             | -0.007892821     | -0.453360465                  |
| 700   | 0.016879458             | -0.03341943      | -0.504890129                  |
| 750   | -0.010664078            | 0.021809242      | -0.488970593                  |
| 800   | -0.01388918             | 0.027414235      | -0.50652177                   |
| 850   | 0.009887957             | -0.020178073     | -0.490034752                  |
| 900   | 0.014835989             | -0.029446734     | -0.503824601                  |
| 950   | -0.004109635            | 0.008628723      | -0.476273836                  |
| 1000  | -0.015808874            | 0.03161724       | -0.50008034                   |
| 2000  | -0.000363142            | 0.000865888      | -0.419386803                  |
| 3000  | 0.006228512             | -0.012511409     | -0.49726584                   |
| 4000  | 0.007872022             | -0.015738314     | -0.50018204                   |
| 5000  | 0.005802183             | -0.011623931     | -0.499158417                  |
| 6000  | -0.004449285            | 0.008786266      | -0.501120894                  |
| 7000  | 0.000276627             | -0.000531904     | -0.520069411                  |
| 8000  | 0.000846807             | -0.001710832     | -0.49467945                   |
| 9000  | 0.000229212             | -0.000443778     | -0.51650149                   |
| 10000 | -0.002612648            | 0.005214494      | -0.501035767                  |
| 20000 | -0.003116677            | 0.006235361      | -0.499839063                  |
| 30000 | 0.002814184             | -0.005627784     | -0.50051885                   |
| 40000 | 0.001051196             | -0.002100961     | -0.500340558                  |
| 50000 | 0.002028101             | -0.004055711     | -0.50060532                   |
| 60000 | 0.000568791             | -0.001138395     | -0.499642918                  |
| 70000 | -0.001586411            | 0.003173177      | -0.499944062                  |
| 80000 | 0.001644039             | -0.00328827      | -0.499970805                  |
| 90000 | -0.001401291            | 0.002802825      | -0.499956651                  |
| 100000| 0.000747314             | -0.001494973     | -0.499884613                  |
| 200000| -0.000953273            | 0.00190647       | -0.500019932                  |
| 300000| 0.000273893             | -0.000547713     | -0.50006641                   |
| 400000| -0.000451122            | 0.000902284      | -0.499977834                  |
| 500000| 0.000606E-05            | -0.000133966     | -0.500131777                  |
| 600000| 0.000625865             | -0.001251737     | -0.499997204                  |
| 700000| -0.000533734            | 0.001067459      | -0.500004216                  |
| 800000| 0.000439992             | -0.000879973     | -0.50000625                   |
| 900000| 0.000470137             | 0.000940267      | -0.500003722                  |
| 1000000| 0.000499695            | -0.0009993       | -0.5                          |

Table 5: Calculation of numerical verification 1.5.
| m   | \( \beta_3(s) \) | \( \eta_1(s) \) | \( \phi_4(s) \) |
|-----|----------------|----------------|----------------|
| 10  | 0.063087093    | -0.272869724  | -0.231198581   |
| 50  | -0.014944382   | 0.061981361   | -0.241110904   |
| 100 | 0.046798444    | -0.087219534  | -0.53659207    |
| 150 | -0.016981693   | 0.027492236   | -0.6176905     |
| 200 | 0.017195077    | -0.038072556  | -0.45169368    |
| 250 | -0.005785644   | 0.00842475    | -0.686743702   |
| 300 | -0.027378675   | 0.055390245   | -0.49426945    |
| 350 | 0.024730603    | -0.048633538  | -0.508592331   |
| 400 | -0.02073001    | 0.040518429   | -0.511619293   |
| 450 | 0.02170702     | -0.042824601  | -0.505672252   |
| 500 | -0.022265473   | 0.044591728   | -0.499290635   |
| 550 | 0.01397341     | -0.02865192   | -0.487695414   |
| 600 | 0.005046997    | -0.009246853  | -0.545019583   |
| 650 | -0.019293777   | 0.038420892   | -0.502168898   |
| 700 | 0.008520338    | -0.017631694  | -0.483239897   |
| 750 | 0.0148299      | -0.02928635   | -0.506375837   |
| 800 | -0.010952313   | 0.022326121   | -0.49058497    |
| 850 | -0.014020797   | 0.027736545   | -0.505496045   |
| 900 | 0.007607843    | -0.01562053   | -0.487041285   |
| 950 | 0.015698822    | -0.031275817  | -0.501947623   |
| 1000| 0.000493631    | -0.000591706  | -0.834250455   |
| 1000| 0.011176058    | -0.022343908  | -0.50013674    |
| 3000| 0.006674901    | 0.01329654    | -0.502002852   |
| 4000| 0.000734637    | -0.001518383  | -0.48382852    |
| 5000| -0.004042275   | 0.008055075   | -0.501829592   |
| 6000| -0.004677003   | 0.009372121   | -0.49903357    |
| 7000| 0.005986961    | -0.011940445  | -0.4999671     |
| 8000| -0.005525843   | 0.011048668   | -0.500136578   |
| 9000| 0.005265628    | -0.01053158   | -0.49984618    |
| 1000| -0.004263257   | 0.008532822   | -0.499630368   |
| 2000| 0.001669324    | -0.03334707   | -0.500590906   |
| 3000| 0.000643305    | -0.001288945  | -0.49909422    |
| 4000| 0.002268274    | -0.04537176   | -0.49930794    |
| 5000| 0.00094173     | -0.001884465  | -0.49973346    |
| 6000| -0.001960402   | 0.003620551   | -0.500032266   |
| 7000| 0.001027013    | -0.002053451  | -0.500140008   |
| 8000| -0.000649735   | 0.001298953   | -0.500199006   |
| 9000| 0.000902318    | -0.00180424   | -0.500109741   |
| 10000| -0.001393389   | 0.002786585   | -0.50003463    |
| 20000| -0.000584187   | 0.001168492   | -0.499949508   |
| 30000| 0.000870814    | -0.00174165   | -0.49993684    |
| 40000| 0.000649223    | -0.001298416  | -0.500011553   |
| 50000| 0.000703926    | -0.001407854  | -0.4999929     |
| 60000| -0.000157985   | 0.000315945   | -0.500039564   |
| 70000| -0.000268832   | 0.000537684   | -0.499981402   |
| 80000| 0.00034483     | -0.000689672  | -0.4999913     |
| 90000| -0.00023822    | 0.000476454   | -0.49985308    |
| 100000| -0.87141E-05   | 3.74162E-05   | -0.500160358   |

Table 6: Calculation of numerical verification 1.6.
| $m$  | $\Re (\eta_m(s))$ | $\Im (s_m)$ | $\frac{\Re (\eta_m(s_m))}{\Im (s_m)}$ |
|------|-------------------|-------------|-----------------------------------------|
| 10   | 0.743881692       | -0.186295877 | -3.993012105                            |
| 50   | 0.058896264       | -0.13246976  | -0.444801575                            |
| 100  | -0.008037138      | 0.030587692  | -0.26275255                             |
| 150  | 0.000843704       | 0.06583153   | 0.12816108                              |
| 200  | -0.021972495      | 0.039421248  | -0.55736951                             |
| 250  | -0.004620945      | 0.005397286  | -0.856160856                            |
| 300  | -0.022151837      | 0.042272429  | -0.524025648                            |
| 350  | -0.016905337      | 0.035522259  | -0.475908275                            |
| 400  | 0.025014556       | -0.049851106 | -0.501785357                            |
| 450  | -0.020871699      | 0.040929649  | -0.509940826                            |
| 500  | 0.019049601       | -0.037331041 | -0.510288502                            |
| 550  | -0.020296719      | 0.040180224  | -0.505141756                            |
| 600  | 0.020103667       | -0.040348224 | -0.498254074                            |
| 650  | -0.01246367       | 0.02561283   | -0.48661823                             |
| 700  | -0.003702041      | 0.006592795  | -0.561528305                            |
| 750  | 0.017265557       | -0.03426361  | -0.503903617                            |
| 800  | -0.011751598      | 0.023989606  | -0.489862068                            |
| 850  | -0.009247696      | 0.017967146  | -0.514700331                            |
| 900  | 0.015357509       | -0.030917047 | -0.496732725                            |
| 950  | 0.058108575       | -0.009863586 | -0.525223226                            |
| 1000 | -0.014941446      | 0.03002631   | -0.497611794                            |
| 2000 | 0.003748574       | -0.007655995 | -0.489625973                            |
| 3000 | 0.001040701       | -0.002173143 | -0.478892093                            |
| 4000 | 0.004176745       | -0.008301795 | -0.503113483                            |
| 5000 | 0.000352638       | -0.000662262 | -0.532475063                            |
| 6000 | 0.004564823       | -0.000910665 | -0.50129476                             |
| 7000 | 0.004131647       | -0.008281728 | -0.498887068                            |
| 8000 | -0.005554139      | 0.011105475  | -0.500126199                            |
| 9000 | 0.004476607       | -0.008943536 | -0.500541061                            |
| 10000| -0.004068925      | 0.008128786  | -0.500557525                            |
| 20000| 0.003396641       | -0.006794685 | -0.499896758                            |
| 30000| 0.002517775       | -0.005036938 | -0.499862218                            |
| 40000| -0.000945091      | 0.00189193  | -0.499538038                            |
| 50000| -0.001742758      | 0.003486351  | -0.499880247                            |
| 60000| -0.000320479      | 0.000641978  | -0.49920558                             |
| 70000| 0.001872554       | -0.003744983 | -0.500016689                            |
| 80000| -0.000869719      | 0.001738848  | -0.50016965                             |
| 90000| 0.000120573       | -0.000240583 | -0.501170074                            |
| 100000| -1.32844E-05    | 2.60877E-05  | -0.509220821                            |
| 200000| 0.000883075       | -0.001766044 | -0.50003011                             |
| 300000| 0.000829724       | -0.001659407 | -0.500012354                            |
| 400000| -0.000767542      | 0.001535097  | -0.49995766                             |
| 500000| 0.000682612       | 0.001365212  | -0.500004395                            |
| 600000| -0.000574894      | 0.001149802  | -0.49993912                             |
| 700000| 0.0002836        | -0.000567177 | -0.500020276                            |
| 800000| 0.000231324       | -0.000462667 | -0.499979467                            |
| 900000| -0.000402352      | 0.000804714  | -0.499993787                            |
| 1000000| 0.000401575      | -0.000803159 | -0.499994397                            |

Table 7: Calculation of numerical verification 1.7.
| $m$  | $\zeta(\eta_m(s_i))$       | $b_i(s)$ | $\zeta(\eta_m(s_i)) / b_i(s)$ |
|------|---------------------------|----------|-----------------------------|
| 10   | 0.16006872                | -0.255526606 | -0.626426823                |
| 50   | -0.045711611              | 0.049515278  | -0.923181952                |
| 100  | 0.050075594               | -0.095207106 | -0.525964879                |
| 150  | 0.041096814               | -0.081383836 | -0.50497514                 |
| 200  | -0.027886389              | 0.058702344  | -0.475042783                |
| 250  | -0.031374863              | 0.063014834  | -0.497896484                |
| 300  | -0.018605809              | 0.039323976  | -0.473141602                |
| 350  | 0.020757479               | -0.039941357 | -0.51968893                 |
| 400  | 0.000890712               | -0.003855806 | -0.25436821                 |
| 450  | -0.011008302              | 0.023387733  | -0.470687005                |
| 500  | 0.011750631               | -0.024625056 | -0.4771819                  |
| 550  | -0.006584136              | 0.014273394  | -0.461287343                |
| 600  | -0.003622867              | 0.062199298  | -0.58261244                 |
| 650  | 0.015158478               | -0.029705967 | -0.51023944                 |
| 700  | -0.018543516              | 0.037217019  | -0.498253662                |
| 750  | 0.005966035               | -0.012622931 | -0.472634684                |
| 800  | 0.013217734               | -0.025971115 | -0.508939797                |
| 850  | -0.014452169              | 0.029217328  | -0.494643781                |
| 900  | -0.006492962              | 0.012496829  | -0.52111646                 |
| 950  | 0.01537939                | -0.039086601 | -0.497573637                |
| 1000 | 0.005190006               | -0.009920722 | -0.523148013                |
| 2000 | -0.010535022              | 0.021098182  | -0.501448462                |
| 3000 | -0.009070078              | 0.018127625  | -0.500345633                |
| 4000 | 0.006712936               | -0.013456605 | -0.49885807                 |
| 5000 | 0.007062856               | -0.014126621 | -0.49953669                 |
| 6000 | 0.004564295               | -0.009151298 | -0.498759302                |
| 7000 | -0.004318159              | 0.008618012  | -0.501062078                |
| 8000 | -0.000635297              | 0.001291671  | -0.491841189                |
| 9000 | 0.0002781975              | -0.005578913 | -0.498568968                |
| 10000| -0.002906056              | 0.005824331  | -0.498951038                |
| 20000| -0.000981404              | 0.001957716  | -0.501326615                |
| 30000| -0.001412192              | 0.002821806  | -0.5004568                  |
| 40000| 0.000314493               | -0.004628339 | -0.500807                  |
| 50000| 0.001401015               | -0.002809957 | -0.50015142                 |
| 60000| 0.002015935               | -0.004031691 | -0.50002199                 |
| 70000| 0.000254944               | -0.000510702 | -0.499203058                |
| 80000| -0.001539029              | 0.003078377  | -0.49948187                 |
| 90000| 0.001662304               | -0.00332484  | -0.49995187                 |
| 100000| -0.001581087              | 0.00316217   | -0.50000632                 |
| 200000| 0.000685697               | -0.001371528 | -0.499951149                |
| 300000| 0.000308648               | -0.000761381 | -0.49994418                 |
| 400000| 0.000189421               | -0.000378783 | -0.500077881                |
| 500000| -0.000184504              | 0.000399049  | -0.49994452                 |
| 600000| 0.000939337               | -0.000587045 | -0.5000247                  |
| 700000| 0.000526306               | -0.001052084 | -0.49999297                 |
| 800000| -0.00050891               | 0.001017811  | -0.500004421                |
| 900000| 0.000340428               | -0.000680842 | -0.500010281                |
| 1000000| -0.000297889             | 0.000595765  | -0.50001091                 |

Table 8: Calculation of numerical verification 1.8.
| $m$  | $\Re (\eta_j(s))$     | $\Im (\eta_j(s))$  | $\Im (\eta_j(s)) / \Re (\eta_j(s))$ |
|------|-----------------------|---------------------|-----------------------------------|
| 10   | 0.96,21,75,573        | -0.28,64,35,644     | -33,59,13,352                     |
| 50   | -0.07204829           | 0.14,13,22,356      | -0.50,98,15,234                   |
| 100  | 0.03,85,10,014        | -0.06,40,90,757     | -0.60,08,66,893                   |
| 150  | 0.000725 118          | 0.00,7,49,855       | 0.09,67,01,096                    |
| 200  | 0.00,21,22,301        | -0.01,00,08,362     | -0.21,20,52,781                   |
| 250  | 0.02,88,55,067        | -0.05,91,30,526     | -0.48,79,89,351                   |
| 300  | 0.02,22,05,736        | -0.0046269956       | -0.47,99,16,947                   |
| 350  | -0.00,85,13,308       | 0.01,45,94,976      | -0.58,33,04,008                   |
| 400  | -0.00,20,19,761       | 0.04,15,16,874      | -0.48,64,91,589                   |
| 450  | 0.02,34,56,676        | -0.04,6,35,379      | -0.50,92,10,028                   |
| 500  | -0.020232304          | 0.00,39,77,083      | -0.50,77,22,197                   |
| 550  | 0.01,92,83,981        | -0.03,7,69,523      | -0.50,78,80,518                   |
| 600  | -0.02,01,37,142       | 0.04,00,38,679      | -0.50,29,42,217                   |
| 650  | 0.01,85,40,022        | -0.03,7,67,727      | -0.49,61,50,649                   |
| 700  | -0.009683806          | 0.02,01,13,366      | -0.48,14,61,233                   |
| 750  | -0.00,59,21,684       | 0.01,10,75,533      | -0.53,46,61,763                   |
| 800  | 0.01,72,37,036        | -0.034285655         | -0.50,27,47,753                   |
| 850  | -0.01,04,26,495       | 0.02,13,66,588      | -0.48,79,81,282                   |
| 900  | -0.00,94,81,895       | 0.00,18,45,065      | -0.51,39,05,743                   |
| 950  | 0.01,49,85,393        | -0.03,01,69,727     | -0.46,70,29,27                    |
| 1000 | 0.00,42,88,876        | -0.08,07,219        | -0.53,13,15,046                   |
| 2000 | 0.00,60,46,856        | -0.01,22,46,231     | -0.49,37,72,819                   |
| 3000 | 0.00,89,30,818        | -0.017880408        | -0.49,94,75,068                   |
| 4000 | -0.00760822           | 0.01,56,25,417      | -0.49,97,12,744                   |
| 5000 | -0.004344305          | 0.00,86,51,331      | -0.50,21,54,524                   |
| 6000 | -0.00,52,10,786       | 0.01,04,00,146      | -0.50,10,30,082                   |
| 7000 | -0.00,50,17,103       | 0.00,10,94,907      | -0.49,92,60,429                   |
| 8000 | 0.00,43,39,471        | -0.00,86,64,127     | -0.50,08,54,962                   |
| 9000 | -0.00,07,92,091       | 0.00,15,65,065      | -0.50,61,07,414                   |
| 10000| -0.000884296          | 0.00178475          | -0.495473316                      |
| 20000| 0.0003005846          | -0.006014678        | -0.499751774                      |
| 30000| 0.002808479           | -0.005616177        | -0.50009531                      |
| 40000| -0.002400136          | 0.004799665         | -0.50063234                      |
| 50000| -0.000490229          | 0.000979016         | -0.500736454                      |
| 60000| -0.000977535          | 0.001964077         | -0.500254064                      |
| 70000| -0.001872691          | 0.003745487         | -0.49995983                      |
| 80000| 0.000767782           | -0.001534864        | -0.500215003                      |
| 90000| 0.000473951           | 0.000948484         | -0.49963195                      |
| 100000| 0.000914615         | 0.001829651         | -0.499884951                      |
| 200000| 0.001110497         | -0.00221013         | -0.499995723                      |
| 300000| 0.000714685         | -0.001429306        | -0.50022388                      |
| 400000| 0.000593546         | 0.001187049         | -0.500018122                      |
| 500000| 0.000152119         | -0.00304283         | -0.499926056                      |
| 600000| -3.79E-05           | 7.94E-05            | -0.500222665                      |
| 700000| -0.000570392         | 0.001140077         | -0.50003945                      |
| 800000| 6.44E-06            | -1.29E-05           | -0.50088333                      |
| 900000| 0.000349926         | -0.000699866        | -0.49989998                      |
| 1000000| -0.000434838        | 0.000869679         | -0.49999576                      |

Table 9: Calculation of numerical verification 1.9.
| $m$  | $\alpha(\eta_m(s))$ | $b_m(s)$ | $\beta(\eta_m(s))$ |
|-----|---------------------|----------|------------------|
| 10  | -0.0999279099       | -0.13,90,94,857 | 0.7409 17235 |
| 50  | 0.02,14,81,704      | 0.00,52,90,727  | 4.06,02,55,613 |
| 100 | 0.03,15,987        | -0.07,67,61,806 | -0.43,19,83,974 |
| 150 | 0.04,11,35,797      | -0.08,13,04,603 | -0.50,59,46,717 |
| 200 | -0.03,35,45,701     | 0.0699988050   | -0.50,65,37,362 |
| 250 | -0.01,31,81,946     | 0.02,24,40,609  | -0.58,74,14,807 |
| 300 | -0.01,18,55,146     | 0.03,45,31,501  | -0.53,72,32,946 |
| 350 | -0.02,53,85438      | 0.05,14,211     | -0.49,36,77,459 |
| 400 | 0.01,47,95,353      | -0.02,78,63,043 | -0.51,00,277   |
| 450 | 0.00,25,89,303      | -0.00,68,62,125 | -0.37,62,35,974|
| 500 | -0.00,95,75,928     | 0.0204541922    | -0.46,82,16,532|
| 550 | 0.00,91,37,597      | -0.01,94,03,534 | -0.47,09,24,369|
| 600 | -0.00,34,38,645     | 0.00,79,73,131  | -0.50,64,71,913|
| 650 | -0.00,8436337       | 0.01,19,21,176  | -0.53,99,07,801|
| 700 | 0.01,62,42,538      | -0.03,20,00,374 | -0.50,07,33,882|
| 750 | -0.01,17,28,151     | 0.03,47,94,624  | -0.49,66,71,842|
| 800 | 0.00,39,63,916      | -0.008630982    | -0.45,92,65,933|
| 850 | 0.01,36,26,793      | -0.02,68,31,688 | -0.50,78,81,935|
| 900 | -0.01,37,15,647     | 0.02,77,61,207  | -0.49,40,58,021|
| 950 | -0.00,6230058       | 0.0119,33,95    | -0.52,20,44,922|
| 1000| 0.01,52,24,928      | -0.03,05,75,149 | -0.49,79,51,065|
| 2000| -0.09,40,613        | 0.01,87,09,084  | -0.50,27,57,377|
| 3000| -0.00,18,94,788     | 0.00,36,91,117  | -0.51,33,37,291|
| 4000| 0.00,12,412         | -0.00,24,17,921 | -0.51,33,33,562|
| 5000| -0.00,55,79,657     | 0.01,11,87,246  | -0.49,87,51,614|
| 6000| -0.00,38,10,271     | 0.00,76,48,766  | -0.49,81,54,997|
| 7000| 0.00,32,47,415      | -0.06,47,096    | -0.50,18,44,394|
| 8000| 0.00,35,24,353      | -0.007066322    | -0.49,87,53,524|
| 9000| -0.00,52,10,759     | 0.010424091     | -0.49,98,76,584|
| 10000| 0.001289763         | -0.000259037    | -0.50002879   |
| 20000| -0.001861508        | 0.003718016     | -0.500674202  |
| 30000| 0.00,06,67,771      | -0.00,13,38,615 | -0.49,88,52,172|
| 40000| -0.00,06,99,591     | 0.00,14,01,147  | -0.49,92,98,789|
| 50000| -0.00,02,18,168     | 0.00,36,366    | -0.49,99,65,625|
| 60000| -0.00,17,91,963     | 0.00,35,84,445  | -0.49,99,27,604|
| 70000| 0.00,02,53,936      | -0.00,50,699    | -0.50,00,86,984|
| 80000| 0.00,15,92,344      | -0.00,31,84,995 | -0.49995 1805 |
| 90000| -0.00,15,97,863     | 0.00,31,95,542  | -0.50,00,82,797|
| 100000| 0.00,12,89,763      | -0.00257922     | -0.50,00,93,32|
| 200000| -0.00,01,29,611     | 0.00,02,59,037  | -0.50,03,57,092|
| 300000| 0.000567943         | -0.00,11,35,966 | -0.49,99,64,788|
| 400000| -0.00,05,22,211     | 0.00,10,44,469  | -0.49,99,77,501|
| 500000| -0.000690552        | 0.00,13,81,091  | -0.50,00,04,706|
| 600000| -0.00,06,44,275     | 0.00,12,88,549  | -0.50,00,00,388|
| 700000| -0.00,01,78,315     | 0.000356656     | -0.49,99,63,35 |
| 800000| 0.00,05,56,979      | -0.00,01,11,796 | -0.49,99,99,106|
| 900000| -0.00,03,94,412     | 0.000788225     | -0.50,00,99,515|
| 1000000| 0.000246815        | -0.00,04,93,617 | -0.50,00,13,168|

Table 10: Calculation of numerical verification 1.10.
| m     | $\Re(q_m(s'))$  | $q_m(s')$  | $\frac{\Re(q_m(s'))}{\Im(q_m(s'))}$ |
|------|-----------------|-----------|----------------------------------------|
| 10   | -0.001870233    | 0.04367938| -0.042825574                           |
| 50   | -0.006962498    | 0.01237139| -0.562789958                           |
| 100  | -0.003286542    | 0.00618569| -0.531313039                           |
| 150  | 0.000543644     | -0.000884624| -0.614548102                        |
| 200  | -0.001594816    | 0.003092649| -0.515646254                           |
| 250  | 0.001950134     | -0.00390792| -0.499020963                           |
| 300  | 0.000246483     | -0.00442312| -0.557260486                           |
| 350  | -0.001358336    | 0.002723944| -0.498665171                           |
| 400  | -0.000785308    | 0.001546425| -0.507821589                           |
| 450  | 0.000423631     | -0.000866907| -0.488669488                        |
| 500  | 0.000976023     | -0.00195396| -0.499510225                           |
| 550  | 0.000729462     | -0.001448551| -0.503580475                        |
| 600  | 0.00011691      | -0.000221156| -0.528631373                        |
| 650  | -0.000425434    | 0.000859106| -0.495205481                           |
| 700  | -0.000680077    | 0.001361972| -0.499332585                           |
| 750  | -0.000634749    | 0.001266128| -0.501330829                           |
| 800  | -0.00038963     | 0.000773212| -0.503910958                           |
| 850  | -7.20E-05       | 0.000137773| -0.522902165                           |
| 900  | 0.000214271     | -0.000433453| -0.494335026                        |
| 950  | 0.000407904     | -0.000818535| -0.49834219                           |
| 1000 | 0.000488251     | -0.00097698| -0.499755369                           |
| 2000 | 0.000244185     | -0.00048849| -0.499877173                           |
| 3000 | -0.000158371    | 0.000316532| -0.50033172                           |
| 4000 | 0.000122108     | -0.000244245| -0.499940533                        |
| 5000 | -2.34E-05       | 4.69E-05   | -0.498173531                           |
| 6000 | -7.92E-05       | 0.000158266| -0.500166176                           |
| 7000 | 1.03E-05        | -2.07E-05  | -0.49872573                           |
| 8000 | 6.8E-05         | -0.000122122| -0.49997134                        |
| 9000 | 3.66E-05        | -7.31E-05  | -0.50031666                           |
| 10000| -1.17E-05       | 2.35E-05   | -0.49906235                           |
| 20000| -5.86E-06       | 1.17E-05   | -0.499545032                           |
| 30000| 1.16E-05        | -2.32E-05  | -0.499931104                           |
| 40000| -2.93E-06       | 5.87E-06   | -0.499770813                           |
| 50000| -7.72E-06       | 1.54E-05   | -0.500043397                           |
| 60000| 5.81E-06        | -1.16E-05  | -0.499964691                           |
| 70000| 5.91E-06        | -1.18E-05  | -0.500027091                           |
| 80000| -1.47E-06       | 2.93E-06   | -0.499887536                           |
| 90000| -5.36E-06       | 1.07E-05   | -0.499997201                           |
| 100000| -3.86E-06      | 7.72E-06   | -0.500021374                          |
| 200000| -1.93E-06      | 3.86E-06   | -0.500013063                          |
| 300000| 5.68E-07       | -1.14E-06  | -0.50002023                           |
| 400000| -9.65E-07      | 1.93E-06   | -0.500004664                          |
| 500000| 9.09E-07       | -1.82E-06  | -0.4999978                            |
| 600000| 2.84E-07       | -5.68E-07  | -0.500011434                          |
| 700000| -6.19E-07      | 1.24E-06   | -0.499999193                          |
| 800000| -4.82E-07      | 9.65E-07   | -0.500002591                          |
| 900000| 1.03E-07       | -2.05E-07  | -0.499987818                          |
| 1000000| 4.54E-07     | -9.09E-07  | -0.49999945                           |

Table 11: Calculation of numerical verification 1.11.
| $m$  | $\Im(\eta_m(s'))$ | $\Re(b_m(s'))$ | $\arctan(\Im/\Re)$ |
|------|-----------------|----------------|-------------------|
| 10   | 0.05,87,75,434  | -0.08,99,60,265 | -0.65,33,48,831   |
| 50   | -0.0,73,75,109  | 0.01,57,14,596  | -0.46,93,15,851   |
| 100  | -0.0,03,80,811  | 0.00,78,57,298  | -0.48,46,58,976   |
| 150  | 0.00,33,01,521  | -0.0066077 14   | -0.49,96,46,474   |
| 200  | -0.0,01,93,419  | 0.00,39,28,649  | -0.49,23,29,551   |
| 250  | -0.0,04,62,944  | 0.00,08,53,326  | -0.54,25,17,162   |
| 300  | 0.00,16,51,342  | -0.00,33,03,857 | -0.49,92,82,486   |
| 350  | 0.00,04,49,355  | -0.00,08,62,203 | -0.52,11,70,766   |
| 400  | -0.00,09,74,629 | 0.00,19,64,325  | -0.49,16,14,844   |
| 450  | -0.00,01,02,858 | 0.00,20,46,153  | -0.50,26,89,682   |
| 500  | -0.000222401    | 0.00,04,26,663  | -0.52,12,56,823   |
| 550  | 0.00,05,43,959  | -0.00,10,98,857 | -0.49,50,22,555   |
| 600  | 0.00,08,25,817  | -0.00,16,51,929 | -0.49,91,01,701   |
| 650  | 0.00,06,41,609  | -0.00,12,76,245 | -0.50,27,31,842   |
| 700  | 0.00,02,20,114  | -0.00,04,31,101 | -0.51,05,85,686   |
| 750  | -0.00,02,05,298 | 0.00,04,17,968  | -0.49,11,81,143   |
| 800  | -0.00,04,89,198 | 0.00,08,82,162  | -0.49,80,82,801   |
| 850  | -5,84E-04       | 0.00,11,88,376  | -0.49,79,49,59    |
| 900  | -0.00,05,12,914 | 0.00,10,23,077  | -0.50,13,44,474   |
| 950  | -0.00,03,33,047 | 0.00,06,61,841  | -0.50,32,13,007   |
| 1000 | -0.00,01,08,933 | 0.00,02,13,331  | -0.51,06,29,023   |
| 2000 | -5,38997E-05    | 0.00,01,06,666  | -0.50,53,12,846   |
| 3000 | -6,20156E-05    | 0.00,01,04,492  | -0.49,77,95,047   |
| 4000 | -2,68081E-05    | 5,33329E-05     | -0.50,26,55,959   |
| 5000 | 9,72E-05        | -1,94E-04       | -0.50,01,58,426   |
| 6000 | -2,61E-05       | 0,00,00,52,246  | -0.49,88,97,523   |
| 7000 | -7,07E-05       | 1,41E-04        | -0.50,00,82,773   |
| 8000 | -1,34E-05       | 2,66664E-05     | -0.50,13,27,513   |
| 9000 | 4,18E-05        | -8,37E-05       | -0.49,80,81,81    |
| 10000| 4,86E-05        | -9,72E-05       | -0.50,00,79,213   |
| 20000| 2,43E-05        | -4,86E-05       | -0.50,00,40,121   |
| 30000| -4,20E-05       | 2,39E-05        | -0.50,08,36,34    |
| 40000| 1,22E-05        | -2,43E-05       | -0.50,20,57,55    |
| 50000| -6,36E-06       | 1,27E-05        | -0.49,99,51,233   |
| 60000| -5,96E-06       | 1,20E-05        | -0,0,04,18,22     |
| 70000| 4,02E-06        | -8,03E-06       | -0.49,99,26,43    |
| 80000| 6,08E-06        | -1,22E-05       | -0.50,00,10,699   |
| 90000| 1,46E-06        | -2,92E-06       | -0.50,00,94,092   |
| 100000| -3,18E-06       | 6,36E-06        | -0.49,99,87,48    |
| 200000| -1,59E-06       | 3,18E-06        | -0.49,99,85,84    |
| 300000| 1,57E-06        | -3,13E-06       | -05               |
| 400000| -7,95E-07       | 1,59E-06        | -0.49,99,94,66    |
| 500000| -4,17E-07       | 8,34E-07        | -0.50,00,10,795   |
| 600000| 7,83E-07        | -1,57E-06       | -0.49,99,26,32    |
| 700000| 3,56E-07        | -7,12E-07       | -0.50,00,06,323   |
| 800000| -3,97E-07       | 7,95E-07        | -0.49,99,37,49    |
| 900000| -5,46E-07       | 1,09E-06        | -0.50,00,02,747   |
| 1000000| -2,08E-07      | 4,17E-07        | -0.50,00,05,99    |

Table 12: Calculation of numerical verification 1.12.
| \( m \) | \( \frac{\zeta_m(s)}{\zeta_m(s')} \) | \( \frac{\zeta_m(s)}{\zeta_m(s')} \) |
|---|---|---|
| 10 | 3.689741821 | 3.162277858 |
| 50 | 7078600484 | 7071076839 |
| 100 | 9.989939317 | 99.99995 |
| 150 | 12.23517792 | 12.24744686 |
| 200 | 14.12972581 | 14.142136 |
| 250 | 1579934786 | 15.81130597 |
| 300 | 17.30893587 | 17.32048684 |
| 350 | 18.69717121 | 18.70825462 |
| 400 | 19.9893288 | 2.00,00,048 |
| 450 | 21.20294133 | 21.21320595 |
| 500 | 22.35083149 | 22.3605654 |
| 550 | 23.44257321 | 23.4520597 |
| 600 | 24.4857063 | 24.4948882 |
| 650 | 26.44858426 | 25.49512053 |
| 700 | 27.377337 | 26.45730969 |
| 750 | 28.27604478 | 27.38598194 |
| 800 | 28.27604478 | 28.2842539 |
| 850 | 29.14678157 | 29.15478354 |
| 900 | 29.9920262 | 29.999915 |
| 950 | 30.8144809 | 30.8205444 |
| 1000 | 31.61533501 | 31.6227842 |
| 2000 | 44.71598015 | 44.72157446 |
| 3000 | 54.76782958 | 54.77287894 |
| 4000 | 63.2413589 | 63.24535762 |
| 5000 | 70.70715187 | 70.7105232 |
| 6000 | 77.4568376 | 77.4598321 |
| 7000 | 83.66307805 | 83.66605719 |
| 8000 | 89.44000195 | 89.44244327 |
| 9000 | 94.86559748 | 94.86848147 |
| 10000 | 99.99764024 | 99.9999 |
| 20000 | 141.495571 | 14,14.214 |
| 30000 | 173.2036131 | 173.2047748 |
| 40000 | 199.2429627 | 200.0112009 |
| 50000 | 223.5514486 | 223.7048923 |
| 60000 | 244.9163998 | 244.461809 |
| 70000 | 264.5124602 | 264.6271471 |
| 80000 | 282.6427349 | 281.7625014 |
| 90000 | 300.0219743 | 300.024477 |
| 100000 | 316.1599247 | 316.1638253 |
| 200000 | 447.1197902 | 447.1210085 |
| 300000 | 546.781404 | 548.259266 |
| 400000 | 632.3240857 | 632.3295467 |
| 500000 | 707.0654563 | 707.058613 |
| 600000 | 774.9132585 | 773.1643492 |
| 700000 | 836.577005 | 836.5733484 |
| 800000 | 894.7034584 | 894.223084 |
| 900000 | 948.6862736 | 950.3575028 |
| 1000000 | 1000.352186 | 999.9355062 |

**Table 13:** Calculation of numerical verification 1.13.
like all other numerical methods when the primitive off is not obvious by algebraic computation (Figure 2) [7,8].

**Hypothesis**

- It is assumed here that the function- arc (M,M) is concave on [x, X],
- (H E) is the elliptical half-arc passing through the points Mi, M+i and Pj; with: N is the point of intersection of the curve-arc (M,M) and the vertical line whose equation is x=m, where \( m = \frac{X + Y}{2} \)
- \( P \) is the orthogonal projection of \( M_i \) on the axis \( Y \),
- \( X \) is the axis passing through the points \( M_i \) and \( M_{i+1} \), and \( X \perp Y \) where \( X \) and \( Y \) are the two axes of the half-ellipse (H E),
- \( a_1 \) and \( b_1 \) are the two parameters of the half-ellipse (H E),
- \( O \left( \frac{m}{2}, f(x) \right) \) is the origin of the reference \( (x,y) \),
- \( Q \) is the point whose coordinates \( x=X \) and \( y=f(x) \),
- \( \alpha = M_1 \), \( M \) and \( \beta = N \hat{N} O \),
- \( c = |N \hat{N} P| \) and \( d = \left( f(m_i) - f(x_{i+1}) \right) \) is the error committed in \([a, b]\) using the method of trapezes.

According to Pythagoras' theorem, we have:

\[
\frac{h}{2} \int \frac{f(x) + f(x+1)}{2} dx = \frac{f(x) + f(x+1)}{2} dx
\]

This implies \( c = d \left( f(x) + f(x+1) - \frac{f(x) - f(x+1)}{2} \right) \)

So, \( h \cdot \Delta x = \left( \frac{f(x) - f(x+1)}{2} \right) - \left( \frac{f(x) - f(x+1)}{2} \right) \)

**Approximate calculation of \( \int_a^b f(x) dx \):**

Let, \( s = \frac{\pi a h}{2} = \frac{\pi h}{2} \left( f(m_i) - f'(x_{i+1}) \right) \)

\[
\int_a^b f(x) dx = h \left( f(x) + f'(x_{i+1}) \right) + s = \left( 1 - \frac{\pi}{4} \right) \left( f(x) + f'(x_{i+1}) \right) + \left( \frac{\pi}{4} \right) h f(m_i)
\]

**Approximation formula**

Doing the summation for \( i \) ranging from 0 to \( n-1 \) with \( x_{i+1} = a \) and \( x_n = b \), we obtain the following approximation formula:

\[
\int_a^b f(x) dx = \text{THE} = \left( 1 - \frac{\pi}{4} \right) \left( f(a) + f'(b) \right) + s = \left( 1 - \frac{\pi}{4} \right) \left( f(a) + f'(b) \right) + \left( \frac{\pi}{4} \right) h \sum_{i=1}^n f(m_i)
\]

Note: \( h \left( f(a) + f'(b) \right) \frac{1}{2} \sum_{i=1}^n f(x_i) \) = \( T_n \). The approximate value of the integral of on \([a, b]\) using the method of trapezes.

\[
\frac{h}{2} \sum_{i=1}^n f(m_i) = M_n : \text{The approximate value of the integral of } f \text{ on } [a, b] \text{ using the method of rectangles with midpoint.}
\]

So, the approximation formula of The Method of trapezoid 8 half-ellipses is:

\[
\frac{h}{8} \int_a^b f(x) dx = \text{THE}_n = \left( 1 - \frac{\pi}{4} \right) \left( f(a) + f'(b) \right) + \left( \frac{\pi}{4} \right) M_n
\]

**Calculation of the theoretical maximum error \( e(n) \)**

Let’s call \( e(n) \) the error committed in the subinterval \([x, x_{i+1}]\). Thus, according to the approximation formula previously established, we have:

\[
e(n) = \left( 1 - \frac{\pi}{4} \right) \frac{h}{8} \int_a^b f(x) dx = \left( 1 - \frac{\pi}{4} \right) \left( f(a) + f'(b) \right) + \left( \frac{\pi}{4} \right) \left( f(a) - f(b) \right)
\]

\[
\left( 1 - \frac{\pi}{4} \right) \frac{h}{8} \int_a^b f(x) dx = \left( 1 - \frac{\pi}{4} \right) \left( f(a) + f'(b) \right) + \left( \frac{\pi}{4} \right) \left( f(a) - f(b) \right)
\]

**Note**:

Knowing that \( 2T_n = T_{n+} + M_n \), we can write:

\[
\text{THE}_n = \left( 1 - \frac{\pi}{4} \right) T_n + \left( \frac{\pi}{4} \right) M_n
\]

The error committed in \([x, x_{i+1}]\) using the method of trapezoids with midpoint, so:

\[
e_{\text{trapeze}} = \left( 1 - \frac{\pi}{4} \right) \frac{h}{8} \int_a^b f(x) dx = \left( \frac{\pi}{4} \right) \left( f(a) - f(b) \right)
\]

**The error committed in \([x, x_{i+1}]\) using the method of rectangles with midpoint, so**:

\[
e_{\text{rectangle}} = \left( \frac{\pi}{4} \right) \left( f(a) - f(b) \right)
\]

We assume that \( f \) is class \( C^2 \) on the interval \([a, b]\), so we know that 003A.

\[
\left| f(x) \right|, \left| f'(x) \right|, \left| f''(x) \right| \leq \left| \frac{\lambda}{12} \right|
\]

where \( \lambda = \sup \left| f(x') \right| x \in \left[ x_i, x_{i+1} \right] \)

\( X_{i+1} \), then we have:

\[
\left| e(n) \right| \leq \left( 1 - \frac{\pi}{8} \right) \frac{h}{12} \left| \lambda \right|
\]

and denoting \( \lambda = \sup \left( \lambda \right) = \sup \left| f(x') \right| x \in \left[ a, b \right] \), we get:

\[
\left| e(n) \right| \leq \left( 1 - \frac{\pi}{8} \right) \frac{h}{12} \left| \lambda \right|
\]

**Note**: If someone replace the elliptical half-arc studied in the previous case by the quarter of the circumference of the ellipse whose parameters are \( h = \frac{b-a}{n} \) and \( \delta = f(x_{i+1}) - f(x_i) \) he gets:

\[
\int_a^b f(x) dx = \text{RQE}_n = \left( 1 - \frac{\pi}{4} \right) k_1 + \left( \frac{\pi}{4} \right) k_1
\]

[RQE: Rectangles and Quarters of Ellipses].

where:
• $R_n^{(l)}$ is the approximate value of the integral of on $[a, b]$ using the method of rectangles on the left,
• $R_n^{(r)}$ is the approximate value of the integral of on $[a, b]$ using the method of rectangles on the right,

and assuming that $f$ is class $C^1$ on the interval $[a, b]$, we get:

$$\left| \int_a^b f(x)dx - RQE_n \right| \leq \frac{(b-a)^2}{2} \sup_{[a,b]} |f'|$$

**Conclusion 2**

Comparing the margins of error, we note that The Method of trapezes and half-ellipses THE is more accurate and more precise than the following 3 methods:

• The method of rectangles on the left $R_n^{(l)}$,
• The method of rectangles on the right $R_n^{(r)}$ and
• The method of trapezes $T_n$.

The approximation formula of The Method of trapezes and half-ellipses given above can be considered as a special case of the following general formula:

$$\int_a^b f(x)dx \approx \text{THE}_n = \left(1 - \frac{\gamma}{4}\right) T_n + \left(\frac{\gamma}{4}\right) M_n$$

$$0 \leq \gamma \leq \frac{4}{\pi}$$

$|\varepsilon(f, n)| \leq \left(1 - \frac{\gamma}{4}\right) \sup_{[a,b]} |f'|$ of course if $f$ is $C^2$ on $[a, b]$.

For $\gamma=0$: THE$_n=T_n$

For $\gamma=\frac{4}{\pi}$: THE$_n=M_n$

So, we can consider The method of trapezes and The method of rectangles (on the left, on the right and with midpoint) as special cases of the general method which is called The Method of trapezes and half-ellipses THE.

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