On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes

Pierpaolo Mastrolia
Max-Planck Institut für Physik, Föhringer Ring, 6, D-80805 München, Germany
Departamento de Física Teórica, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain
E-mail: ppaolo@mppmu.mpg.de

Giovanni Ossola
Physics Department, New York City College of Technology, City University Of New York, 300 Jay Street, Brooklyn NY 11201, USA.
The Graduate School and University Center, The City University of New York 365 Fifth Avenue, New York NY 10016, USA
E-mail: GOssola@citytech.cuny.edu

ABSTRACT: We propose a first implementation of the integrand-reduction method for two-loop scattering amplitudes. We show that the residues of the amplitudes on multi-particle cuts are polynomials in the irreducible scalar products involving the loop momenta, and that the reduction of the amplitudes in terms of master integrals can be realized through polynomial fitting of the integrand, without any a priori knowledge of the integral basis. We discuss how the polynomial shapes of the residues determine the basis of master integrals appearing in the final result. We present a four-dimensional constructive algorithm that we apply to planar and non-planar contributions to the 4- and 5-point MHV amplitudes in $\mathcal{N}=4$ SYM. The technique hereby discussed extends the well-established analogous method holding for one-loop amplitudes, and can be considered a preliminary study towards the systematic reduction at the integrand-level of two-loop amplitudes in any gauge theory, suitable for their automated semianalytic evaluation.
1. Introduction

Scattering amplitudes constitute the core of the perturbative structure of quantum field theories. The recent development of novel methods for computing them has been highly stimulated by a deeper understanding of the multi-channel factorization properties naturally emerging as *reactions* of the amplitudes under deformations of the kinematics in the complex plane dictated by on-shell \([1, 2]\) and generalized unitarity-cut conditions \([3, 4]\).

Analyticity and unitarity of scattering amplitudes \([5]\) have then been strengthened by the complementary classification of the mathematical structures present in the residues

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at the singularities, better understood after uncovering a quadratic recurrence relation for tree-level amplitudes, the so called BCFW-recursion [2], its link to the leading singularity of one-loop amplitudes [4], and a relation between numerator and denominators of one-loop Feynman integrals, yielding the multipole decomposition of Feynman integrands, stronghold of the by-now known as OPP method [6].

These new insights, which stem from a reinterpretation of tree-level scattering within the twistor string theory [7], have catalyzed the study of novel mathematical frameworks in the more supersymmetric sectors of quantum field theories, such as dual conformal symmetries [8], grassmanians [9], Wilson-loops/gluon-amplitudes duality [10], color/kinematic and gravity/gauge dualities [11,12], as well as on-shell [1,2,13,14] and generalised unitarity-based methods [3,4,15,20], and more generally the breakthrough advances in automating the evaluation of multi-particle scattering one-loop amplitudes, as demanded by the experimental programmes at hadron colliders.

In general, when a direct integration of Feynman integrals is prohibitive, the evaluation of scattering amplitudes beyond the leading order is addressed in two stages: i) the reduction in terms of an integral basis, and ii) the evaluation of the elements of such a basis, called master integrals (MI’s).

At one-loop, the advantage of knowing apriori that the basis of MI’s is formed by scalar one-loop functions [16], as well as the availability of their analytic expression [17], allowed the community to focus on the development of efficient algorithms for extracting the coefficients multiplying each MI’s. Improved tensor decomposition [18], complex integration and contour deformation [19], on-shell and generalised unitarity-based methods, and integrand-reduction techniques [6,20–22] led to results which only few years ago were considered inconceivable, and to such a high level of automation [21,23] that different scattering processes at the next-to-leading order accuracy can be handled by single, yet multipurpose, codes [24–28].

At higher-loop, and in particular at two-loop to begin with, the situation is different. The basis of MI’s is not known apriori. MI’s are identified at the end of the reduction procedure, and afterwards the problem of their evaluation arises. The most used multi-loop reduction technique is the well-known Laporta algorithm [29], based on the solution of algebraic systems of equations obtained through integration-by-parts identities [30]. The recent progress in evaluating amplitudes beyond one-loop has been necessarily accompanied by the improvement of mathematical methods dedicated to Feynman integrals, such as difference [29,31] and differential [32] equations, Mellin-Barnes integration [33], asymptotic expansions [34], sector decomposition [35], complex integration and contour deformation [36] – to list few of them.

In this paper we aim at extending the combined use of unitarity-based methods and integrand-reduction, in order to accomplish the semianalytic reduction of two-loop amplitudes to MI’s. The use of unitarity-cuts and complex momenta for on-shell internal particles turned unitarity-based methods into very efficient tools for computing scattering amplitudes. These methods exploit two general properties of scattering amplitudes, such as analyticity and unitarity: the former granting that amplitudes can be reconstructed
from the knowledge of their (generalised) singularity-structure; the latter granting that the residues at the singular points factorize into products of simpler amplitudes. Unitarity-based methods are founded on the underlying representation of scattering amplitudes as a linear combination of MI’s, and their principle is the extraction of the coefficients entering in such a linear combination by matching the cuts of the amplitudes onto the cuts of each MI.

In the past years, general criteria to determine the MI’s of arbitrary problems have been investigated [37], and very recently, a minimal basis for two-loop planar integrals has been identified [38], through an ameliorated solution of systems of equations involving integration-by-parts identities and supplementary Gram-determinant relations.

Cutting rules as computational tools have been introduced at two-loop in the context of supersymmetric amplitudes [39] and later applied to the case of pure QCD amplitudes [40]. The use of complex momenta for propagating particles to fulfill the multiple cuts of two-loop amplitudes has been proposed for extending the benefits of the one-loop quadruple-cut technique, to the octa-cut [41] and the leading singularity techniques [42], as well as the method of maximal cuts [43], all indicating the possibility of “reducing the computation of multi-loop amplitudes to the computation of residues (which end up being related to tree-amplitudes) and to the solution of linear systems” [42] – an idea we elaborate on hereby.

The multi-particle pole decomposition for the integrands of arbitrary scattering amplitudes emerges from the combination of analyticity and unitarity with the idea of a reduction under the integral sign.

The principle of an integrand-reduction method is the underlying multi-particle pole expansion for the integrand of any scattering amplitude, or, equivalently, the relation between numerator and denominators of the integrand: a representation where the numerator of each Feynman integral is expressed as a combination of products of the corresponding denominators, with polynomial coefficients.

The key element in the integrand-decomposition is represented by the shape of the residues on the multi-particle pole before integration: each residue is a (multivariate) polynomial in the irreducible scalar products (ISP’s) formed by the loop momenta and either external momenta or polarization vectors constructed out of them; the scalar products appearing in the residues are by definition irreducible, namely they cannot be expressed in terms of the denominators of the integrand – otherwise mutual simplifications may occur and the notion of residues to a specific set of vanishing denominators would become meaningless.

The polynomial structure of the multi-particle residues is a qualitative information that turns into a quantitative algorithm for decomposing arbitrary amplitudes in terms of MI’s at the integrand level. In the context of an integrand-reduction, any explicit integration procedure and/or any matching procedure between cuts of amplitudes and cuts of MI’s is replaced by polynomial fitting, which is a simpler operation.

Decomposing the amplitudes in terms of MI’s amounts to reconstructing the full polynomiality of the residues, i.e. it amounts to determining all the coefficients of each polynomial.
The main goal of this paper is to outline guiding criteria to constrain the polynomial form of the residue on each multiple-cut of an arbitrary two-loop amplitude. Unlike the one-loop case, where the residues of the multiple-cut have been systematized for all the cuts, in the two-loop case, their form is still unknown. Their existence is a prerequisite for establishing a relation between numerator and denominators of any two-loop integrand. Their implicit form can be given in terms of unknown coefficients, which are determined through polynomial fitting. As in the one-loop case, the full reconstruction of the polynomial residues is engineered via a projection technique based on the Discrete Fourier Transform [44], and requires only the knowledge of the numerator evaluated at explicit values of the loop momenta as many times as the number of the unknown coefficients.

Another feature of the integrand-reduction algorithm we are describing is that the determination of the polynomial form of the residues amounts to choose a basis of MI’s, which does not necessarily need to be known apriori. In fact, as we will see, each ISP appearing in the polynomial residues is the numerator of a potential MI which may appear in the final result (other than the scalar integrals). We remark that the set of MI’s which will emerge at the end of the integrand-reduction (as well as after applying any unitarity-based methods) is not necessarily the minimal set of basic integrals. Integration-by-parts identities, Lorentz-invariance identities, as well as Gram-determinant identities may not be detected in the framework of a cut-construction, and therefore constitute additional, independent relations which can further reduce the number of MI’s which have to be evaluated after the reduction stage.

We define the \(m\)-fold cut of a diagram as the set of on-shell conditions corresponding to the vanishing of \(m\) denominators present in that diagram. As in the maximal-cut method [43], we do not cut additional denominators which might arise from cutting one-loop sub-diagram.

We outline the driving principles for the cut-construction of the residues at the \(m\)-fold cut, and the use of self-consistency checks that ensure the correctness of the reconstructed polynomials, namely after the determination of their unknown coefficients. These checks, called *local* and *global* \((N = N)\)-tests, are analogous to the tests employed in the one-loop integrand-reduction [21,23], and monitor, respectively, the completeness of the polynomial residues, and the correctness of the final decomposition formula.

The values of the loop momenta used for the numerator sampling are chosen among the solutions of the corresponding \(m\)-fold cut.

We verify the integrand-reduction algorithm by applying its procedures to planar and non-planar contributions to the 4-point MHV [39] and 5-point MHV [47] amplitudes in \(\mathcal{N} = 4\) SYM, and derive an expression for the non-planar pentacross diagram in terms of master integrals.

This work can be considered as a first building block of a new technique, that once developed \textit{in toto}, would allow for the semianalytic reduction of multi-loop amplitudes.
2. Four-dimensional Reduction Algorithm

The reduction method hereby presented extends to two-loop the integrand-reduction procedures originally elaborated for arbitrary one-loop scattering amplitudes [6, 20].

An arbitrary two-loop \( n \)-point amplitude in the dimensional regularization scheme can be written as

\[
A_n = \int d^{4-2\epsilon} q \int d^{4-2\epsilon} k \ A(q, k),
\]

\[
A(q, k) = \frac{N(q, k)}{D_1 D_1 \cdots D_n},
\]

\[
D_i = (\alpha_i q + \beta_i k + p_i)^2 - m_i^2, \quad \alpha_i, \beta_i \in \{0, 1\}
\]

where \( \epsilon = (4 - d)/2 \), and \( d \) is the continuous-dimensional parameter. Extra-dimensional components of the loop momenta can be parametrized as pseudo-mass variables, \( \lambda_q \) and \( \lambda_k \), one for each loop momenta, according to the following scheme,

\[
q_{4-2\epsilon} = q_4 + i\lambda_q \gamma_5, \quad q_{4-2\epsilon}^2 = q_4^2 - \lambda_q^2,
\]

\[
k_{4-2\epsilon} = k_4 + i\lambda_k \gamma_5, \quad k_{4-2\epsilon}^2 = k_4^2 - \lambda_k^2,
\]

\[
\int d^{4-2\epsilon} q \int d^{4-2\epsilon} k = \int d^{-2\epsilon} \lambda_q \int d^{-2\epsilon} \lambda_k \int d^4 q_4 \int d^4 k_4.
\]

As in the one-loop case, \( \lambda_q^2 \) and \( \lambda_k^2 \) would appear as additional variables in the polynomial residues, and, consequently, would be responsible for the appearance of \( \Pi \)'s in higher dimensions [40]. The following discussion is limited to a purely four-dimensional reduction, in which the loop variables \( q_\mu \) and \( k_\mu \) are defined in four dimensions.

2.1 Integrand Decomposition

The stronghold of the integrand-reduction of an arbitrary two-loop \( n \)-point amplitude is the decomposition of the numerator \( N(q, k) \) in terms of denominators \( D_i \) for \( i = 1, \ldots, n \). Following the same pattern as in the one-loop case, a plausible ansatz reads,

\[
N(q, k) = \sum_{i_1 << i_8} \Delta_{i_1, \ldots, i_8}(q, k) \prod_{h \neq i_1, \ldots, i_8} D_h + \sum_{i_1 << i_7} \Delta_{i_1, \ldots, i_7}(q, k) \prod_{h \neq i_1, \ldots, i_7} D_h + \ldots + \sum_{i_1 << i_2} \Delta_{i_1, i_2}(q, k) \prod_{h \neq i_1, i_2} D_h,
\]

where \( i_1 << i_8 \) stands for a lexicographic ordering \( i_1 < i_2 < \ldots < i_7 < i_8 \), and where the \( \Delta \)'s are functions depending on the loop momenta. By using the decomposition (2.5) in Eq. (2.2), the multi-pole nature of the integrand of an arbitrary two-loop \( n \)-point amplitude becomes manifest,

\[
A(q, k) = \sum_{i_1 << i_8} \frac{\Delta_{i_1, \ldots, i_8}(q, k)}{D_{i_1} D_{i_2} \cdots D_{i_8}} + \sum_{i_1 << i_7} \frac{\Delta_{i_1, \ldots, i_7}(q, k)}{D_{i_1} D_{i_2} \cdots D_{i_7}} + \ldots + \sum_{i_1 << i_2} \frac{\Delta_{i_1, i_2}(q, k)}{D_{i_1} D_{i_2}}.
\]
The above expression, upon integration, yields the decomposition of the amplitude in terms of Master Integrals (MI’s), respectively associated to diagrams with 8-, 7-, \ldots, 2-denominators, namely down to the products of two 1-point functions (one tadpole for each loop). In Eq.\((2.6)\), each function \(\Delta(q,k)\) parametrizes the residue of the amplitude on the multi-particle cut that corresponds to the set of vanishing denominators it is sitting on. The four-dimensional decomposition in Eqs.(2.5,2.6) begins with 8-denominator terms. They correspond to the maximal singularities of two-loop amplitudes in four dimensions, accessed by freezing both integration momenta with the simultaneous vanishing of eight denominators. We expect that for dimensionally regulated amplitudes, due to the presence of additional degrees of freedom, \(\lambda_q\) and \(\lambda_k\), an extended integrand-decomposition formula should hold, where higher-denominator functions are accommodated. Extension of the presented method to two-loop amplitudes in dimensional regularization will be the subject of a future work.

2.2 Residues

We define the \(m\)-fold cut of a diagram as the set of on-shell conditions corresponding to the vanishing of \(m\) denominators present in that diagram. The calculation of a generic scattering amplitude amounts to the problem of extracting the coefficients of multivariate polynomials, generated at every step of the multiple-cut analysis. In fact, we will see that each \(\Delta(q,k)\) is polynomial in the scalar products of the loop momenta with either external momenta or polarization vectors constructed out of them. These scalar products cannot be expressed in terms of the denominators \(D_i\), and therefore are defined irreducible scalar products (ISP’s). In the case of the residue to an \(m\)-fold cut, \(\Delta_{i_{1} \ldots i_m}\), with \(m < 8\), the ISP’s correspond to the components of the loop momenta not frozen by the on-shell conditions; the eightfold-cut conditions of a two-loop amplitude freeze completely both integration momenta, like the quadruple-cut in the one-loop case.

The polynomial form of \(\Delta_{i_{1} \ldots i_m}\) depends on the number of independent external momenta of the \(n\)-point diagram identified by \(D_{i_{1}} \ldots D_{i_{n}}\). In an \(n\)-point diagram, due to momentum conservation, only \((n - 1)\) external momenta are independent. In four dimensions, there can be at most four independent external momenta. Therefore, despite the number of loops, we can trivially cast \(n\)-point amplitudes in two groups according to whether \(n\) is larger than 4 or not:

- In the case of \(n\)-point diagrams with \(n \geq 5\), four (out of \(n\)) external momenta can be chosen to form a real four-dimensional vector basis.

- In the case of \(n\)-point diagrams with \(n \leq 4\), we can choose only up to three (out of \(n\)) independent external momenta. But they are not sufficient, and additional elements, orthogonal to them, have to be taken into account to complete the four-dimensional basis. In this case, one can use complex polarization vectors as orthogonal complement, to form a complex basis.

- Each \(m\)-fold cut will be characterized by one of these two kinds of basis, and the loop momenta will be decomposed along the vectors forming it.
• The residue of an \( m \)-fold cut, \( \Delta_{i_1 \ldots i_m} \), is polynomial in the components of the loop momenta, and therefore it is polynomial in the ISP’s constructed from the loop momenta and the elements of the basis used to decompose them.

• The polynomial form of the residue \( \Delta_{i_1 \ldots i_m} \) determines the MI’s potentially appearing in the final decomposition.

### 2.2.1 Polynomial Structures and Master Integrals

Let’s consider \( q \) and \( k \) as the solutions of the \( m \)-fold cut identified by the vanishing of \( D_{i_1} \ldots D_{i_m} \). We can decompose the loop momentum \( q \) in terms of the basis \( \{ \tau_i \} \) and \( k \) along the basis \( \{ e_i \} \),

\[
q^\mu = -p_0^\mu + x_1 \tau_1^\mu + x_2 \tau_2^\mu + x_3 \tau_3^\mu + x_4 \tau_4^\mu ,
\]

\[
k^\mu = -r_0^\mu + y_1 e_1^\mu + y_2 e_2^\mu + y_3 e_3^\mu + y_4 e_4^\mu .
\]

where \( p_0 \) and \( r_0 \) are combinations of external momenta.

For simplicity, let us assume that \( \Delta_{i_1 \ldots i_m} \) is a one-dimensional polynomial. The multivariate extension follows the same principles. We can write \( \Delta_{i_1 \ldots i_m} \) in terms of scalar products \( (e_i \cdot (k + r_0)) \) as

\[
\Delta_{i_1 \ldots i_m} = \sum_j c_j (e_i \cdot (k + r_0))^j .
\]

Since \( \Delta_{i_1 \ldots i_m} \) is the polynomial sitting on the denominators \( D_{i_1} \ldots D_{i_m} \), the integral expression,

\[
\int d^4q \int d^4k \frac{\Delta_{i_1 \ldots i_m}}{D_{i_1} \ldots D_{i_m}} = \sum_j c_j \int d^4q \int d^4k \frac{(e_i \cdot (k + r_0))^j}{D_{i_1} \ldots D_{i_m}} ,
\]

(2.10)

can generate MI’s. In general, there are two possibilities which can be encountered:

\[ i) \quad \int d^4q \int d^4k \frac{(e_i \cdot (k + r_0))^j}{D_{i_1} \ldots D_{i_m}} \neq 0 ; \quad (2.11) \]

\[ ii) \quad \int d^4q \int d^4k \frac{(e_i \cdot (k + r_0))^j}{D_{i_1} \ldots D_{i_m}} = 0 . \quad (2.12) \]

In case \( i) \), the natural integral basis associated to the set of denominators \( D_{i_1} \ldots D_{i_m} \) is determined by the ISP \( (e_i \cdot (k + r_0)) \). According to the power \( j \), we identify the following MI’s:

\[ j = 0 : \quad \int d^4q \int d^4k \frac{1}{D_{i_1} \ldots D_{i_m}} \quad (\text{scalar MI}) \quad (2.13) \]

\[ j = 1 : \quad \int d^4q \int d^4k \frac{(e_i \cdot (k + r_0))}{D_{i_1} \ldots D_{i_m}} \quad (\text{linear MI}) \quad (2.14) \]

\[ j = 2 : \quad \int d^4q \int d^4k \frac{(e_i \cdot (k + r_0))^2}{D_{i_1} \ldots D_{i_m}} \quad (\text{quadratic MI}) \quad (2.15) \]

... ...
Option (ii) may occur in the case of an $m$–fold cut of an $n$-point diagram with $n \leq 4$, where the vector $e^i_\mu$ could be orthogonal to the independent vectors of the diagrams, yielding the vanishing of the integral in Eq. (2.12). Borrowing the terminology from the one-loop case, the ISP is *spurious* and does not generate any MI, other than the scalar one.

We observe that in the one-loop case the polynomial residues, from the single-cut up to the quadruple-cut, are written in terms of spurious ISP’s. Only the pentuple-cut could have been written in terms of non-spurious ISP’s. However, at the one-loop level there is no room for non-spurious ISP, because any scalar product between the loop variables and external momenta is reducible. In fact, this is another reason for the one-loop 5-point amplitude in four dimensions to admit a complete reduction in terms of 4-point integrals.

### 2.2.2 Numerator Sampling

As in the one-loop integrand-reduction, once the polynomial form of the residues is given in terms of unknown coefficients, the numerator decomposition in Eq. (2.5) becomes the engine for determining them.

Since the function on the *l.h.s* of Eq. (2.5), namely the numerator $N(q,k)$ of an arbitrary two-loop diagram is a known quantity, and the *r.h.s* of Eq. (2.5) has a canonical representation in terms of $\Delta$-polynomials and denominators $D_{i_1} \ldots D_{i_m}$, the determination of the unknown coefficients sitting in each of the $\Delta_{i_1, \ldots, i_j}$ can be achieved by solving a system of linear equations. This set of equations is obtained evaluating the *l.h.s* and the *r.h.s* of Eq. (2.5) for explicit values of the loop variables, as many times as the number of unknown coefficients to be determined. The numerator *sampling* can be performed either numerically or analytically.

Choosing the solutions of the $m$–fold cuts as sampling values is very convenient, because it allows a direct determination of the coefficients appearing in the corresponding residue $\Delta_{i_1, \ldots, i_m}$. Also, the polynomial fitting can be better achieved by a projection technique based on the Discrete Fourier Transform [44], instead of inverting a system.

### 2.2.3 Momentum Basis

Once the polynomial form of each $\Delta_{i_1, \ldots, i_m}$ is established, in terms of spurious or non-spurious ISP’s, one can use any basis to decompose the loop momenta $q$ and $k$ fulfilling the on-shell conditions $D_{i_1} = \ldots = D_{i_m} = 0$. We find it convenient to use always a complex four dimensional basis [6], which directly exposes the spurious character of ISP’s, should they be present.

For each cut, we decompose the loop momenta $q$ and $k$, by means of two specific basis of four massless vectors. We begin with the decomposition of the loop momentum $q^\mu$, flowing through an on-shell denominator carrying momentum $(q + p_0)$. Let us construct the massless vectors $\tau_1$ and $\tau_2$ as a linear combination of the two external legs, say $K_1$ and $K_2$,

\[
\tau_1^\mu = \frac{1}{\beta} \left( K_1^\mu + \frac{K_1^2}{\gamma} K_2^\mu \right), \quad \tau_2^\mu = \frac{1}{\beta} \left( K_2^\mu + \frac{K_2^2}{\gamma} K_1^\mu \right),
\]  

(2.16)
with
\[ \beta = 1 - \frac{K_1^2 K_2^2}{\gamma^2}, \quad \text{and} \quad \gamma = K_1 \cdot K_2 + \text{sgn}(1, K_1 \cdot K_2)\sqrt{(K_1 \cdot K_2)^2 - K_1^2 K_2^2}. \]

Then, one builds the massless vectors \( \tau_3 \) and \( \tau_4 \) from \( \tau_1 \) and \( \tau_2 \)
\[ \tau_3^\mu = \frac{(\tau_1 | \gamma^\mu | \tau_2)}{2}, \quad \tau_4^\mu = \frac{(\tau_2 | \gamma^\mu | \tau_1)}{2}, \]

such that
\[ \tau_i^2 = 0, \quad \tau_1 \cdot \tau_3 = \tau_1 \cdot \tau_4 = \tau_2 \cdot \tau_3 = \tau_2 \cdot \tau_4 = 0, \quad \tau_1 \cdot \tau_2 = -\tau_3 \cdot \tau_4. \]

The basis \( \tau_i \) can be used to decompose the loop-momentum \( q \), as
\[ q^\mu = -p_0^\mu + x_1 \tau_1^\mu + x_2 \tau_2^\mu + x_3 \tau_3^\mu + x_4 \tau_4^\mu, \]

We double this procedure, by introducing a second basis \( e_i \) to decompose the other loop momentum,
\[ k^\mu = -r_0^\mu + y_1 e_1^\mu + y_2 e_2^\mu + y_3 e_3^\mu + y_4 e_4^\mu. \]

where the basis \( e_i \) can be constructed through the same definitions as \( \tau_i \) by replacing \( (K_1, K_2) \) with another couple of external vectors.

The two basis \( \{\tau_i\} \) and \( \{e_i\} \) are adopted for decomposing the solutions of the multiple-cuts of the two-loop amplitude. In the following, the decompositions of \( q \) and \( k \) will be uniquely defined by specifying, case by case, \( p_0, \tau_1, \tau_2, \) and \( r_0, e_1, e_2, \) respectively.

### 2.3 Testing the Integrand-Decomposition

The integrand-reduction algorithm offers self-consistency checks that ensure the correctness of the reconstructed polynomials after the determination of their unknown coefficients. These checks, called local and global \( (N = N) \)-tests, are analogous to the tests employed in the one-loop integrand-reduction \cite{21, 23}. We simply recall here their definitions:

- The local \( (N = N) \)-test monitors the completeness of the polynomial residues, namely it can be used to verify that all the necessary ISP’s, and their powers, have been properly accounted for. Accordingly, after the determination of the polynomial coefficients of a given residue, \( \Delta_{i_1, \ldots, i_m} \), one can use any solutions of the corresponding \( m \)-fold cut-conditions other than the ones used for the determination of its coefficients, to verify the fulfillment of Eq. (2.5). This test can be implemented locally, cut-by-cut, and its failure means an incomplete parametrization of the residue.

We observe that there is only one case where the local \( (N = N) \)-test cannot be applied: the maximal-cut of a given loop, defined as the cut where all loop momenta are frozen by the on-shell cut-conditions. In this case, in fact, the number of solutions of the maximal cut is finite, and all of them might have been used for the determination of the polynomial coefficients.

Apart from that, the local \( (N = N) \)-test is a very powerful tool for the classification of the polynomial structures characterizing the residue of the \( m \)-fold cut for arbitrary amplitudes, at any loop, for any \( m \) (other than maximal).
• The **global** \((\mathcal{N} = \mathcal{N})\)-test ensures the correctness of the overall integrand-decomposition. Accordingly, after the determination of all polynomial coefficients appearing in the \(r.h.s.\) of Eq.(2.3), the identity between \(l.h.s.\) and \(r.h.s.\) of Eq.(2.3) must hold for arbitrary values of the loop variables. One can therefore verify it: either \(i)\) at the end of the reduction procedure, or \(ii)\) during the reduction, in order to rule out any further contribution possibly coming from sub-diagram MI's. In the latter case, the failure of the test indicates that other contributions are missing, and the reduction should continue for their detection.

In the next sections we will present a series of examples that illustrate the integrand-reduction algorithm explicitly. The required spinor-algebra has been implemented in *Mathematica*, using the package \(S@M\) [45].

### 3. Four-gluon MHV Amplitude in \(\mathcal{N} = 4\) SYM

The 4-gluon MHV amplitude in \(\mathcal{N} = 4\) supersymmetric gauge theory was originally calculated in Ref. [39]. Two MI’s appearing in the result are the planar and the crossed double-box, shown in Fig.1.

![Figure 1: Two Master Integrals of the 4-gluon MHV amplitude in \(\mathcal{N} = 4\) SYM: the ladder (left) and crossed (right) s-channel double-box.](image)

#### 3.1 A contribution to the Ladder Amplitude

Let us consider the contribution to the (leading-color) 4-gluon MHV amplitude in \(\mathcal{N} = 4\) SYM coming from the helicity configuration depicted in Fig.2, where only gluons circulate in both loops. This case has been discussed in the context of generalised unitarity-based method in Ref. [41]. Here we reproduce the same result, and discuss its decomposition in terms of the planar double-box MI in Fig.1 (left), achieved by integrand-reduction.

**On-shell Solutions.** The solutions of the 4-point 7fold-cut, \(D_1 = \ldots = D_6 = D_7 = 0\) can be decomposed according to Eqs.(2.20, 2.21), with the following definitions:

\[
\begin{align*}
    p_0^\mu &= 0^\mu , & e_1^\mu &= p_1^\mu , & e_2^\mu &= p_2^\mu , \\
    r_0^\mu &= 0^\mu , & \tau_1^\mu &= p_3^\mu , & \tau_2^\mu &= p_4^\mu .
\end{align*}
\]  

In this case, the seven on-shell conditions, \(D_1 = \ldots = D_6 = D_7 = 0\), cannot freeze the loop momenta. The parametric solution reads,

\[
\begin{align*}
    q_{(7)}^\mu &= x_4 \tau_4^\mu , & k_{(7)}^\mu &= y_3 e_3^\mu ,
\end{align*}
\]
\[ D_1 = k^2 \]
\[ D_2 = (k + p_2)^2 \]
\[ D_3 = (k - p_1)^2 \]
\[ D_4 = q^2 \]
\[ D_5 = (q + p_3)^2 \]
\[ D_6 = (q - p_4)^2 \]
\[ D_7 = (q + k + p_2 + p_3)^2 . \]

**Figure 2:** 7-fold-cut of the 4-point ladder diagram (s-channel)

where one of the on-shell cut-conditions imposes a non-linear relation among \(x_4\) and \(y_3\), which can be implicitly written as \(x_4 = x_4(y_3)\). We choose \(y_3\), namely the component of \(k\) along \(e_3\), as the variable parametrizing the infinite set of solutions of the 7-fold-cut.

**Residue.** The residue of this 4-point 7-fold-cut is defined as,

\[ \Delta_{1234567}(q, k) = \text{Res}_{1234567}\{N(q, k)\} , \quad (3.4) \]

where \(\text{Res}_{1234567}\{N(q, k)\}\) is the product of the six 3-point tree-amplitudes sitting in the vertices exposed by the cuts. The definition of each 3-point tree-amplitude can be derived from the general expression,

\[ A_3^{\text{tree}}(1^-, 2^-, 3^+) = i \frac{(123)^3}{(23)(31)} \left( \frac{(23)}{(12)} \right)^a \quad (3.5) \]

with \(a = 0, 1, 2\) respectively for gluons, fermions, and scalars. In the case at hand, \(a = 0\).

To find the polynomial expression of \(\Delta_{1234567}\), we use the following criteria.

1. **Diagram topology and Vector basis.**

   In this 4-point double-box, only three external vectors are independent, therefore any possible basis involving them needs to be completed by an additional orthogonal vector \(\omega_3\), defined as,

   \[ \omega^\mu_3 = \frac{(p_2 \cdot \tau_3) \tau^\mu_4 - (p_2 \cdot \tau_4) \tau^\mu_3}{\tau_3 \cdot \tau_4} = \frac{(p_3 \cdot e_3) e^\mu_4 - (p_3 \cdot e_4) e^\mu_3}{e_3 \cdot e_4} . \quad (3.6) \]

2. **Irreducible Scalar Products.**

   The ISP’s which can be formed by the loop variables and the external momenta can be chosen to be \((q \cdot p_2)\) and \((k \cdot p_3)\). Due to the explicit expression of the on-shell solutions, on the cut, one has

   \[ (q \cdot p_2) \rightarrow (q(7) \cdot p_2) = (q(7) \cdot \omega_3) = -x_4(\tau_4 \cdot p_2) , \quad (3.7) \]
   \[ (k \cdot p_3) \rightarrow (k(7) \cdot p_3) = (k(7) \cdot \omega_3) = y_3(e_3 \cdot p_3) . \quad (3.8) \]
Therefore, $\Delta_{1234567}$ is as a polynomial in terms of $x_4$ and $y_3$, and can be written as

\[
\Delta_{1234567}(q,k) = c_{1234567,0} + \sum_{i=1}^{r} c_{1234567,i} (\omega_3 \cdot q)^i + \sum_{i=1}^{r} c_{1234567,i+r} (\omega_3 \cdot k)^i ,
\]

(3.9)

where $r$ is the maximum rank in the integration momenta. Due to the orthogonality between $\omega_3$ and the external momenta, we identify two classes of vanishing integrals,

\[
\int d^4q \int d^4k \frac{(\omega_3 \cdot q)^n}{D_1 \ldots D_7} = 0 ,
\]

(3.10)

\[
\int d^4q \int d^4k \frac{(\omega_3 \cdot k)^n}{D_1 \ldots D_7} = 0 .
\]

(3.11)

As a result, the ISP’s $(\omega_3 \cdot q)$ and $(\omega_3 \cdot k)$ are spurious, and the polynomial form in Eq.(3.9) does not generate any additional MI other than the scalar one, whose coefficient is $c_{1234567,0}$.

**Coefficients.** The $(2r+1)$ unknown coefficients $c_{1234567,i}$ can be determined by sampling Eq.(3.4) and Eq.(3.9) on $(2r+1)$ solutions of the 7fold-cut $(q(7), k(7))$ corresponding to $(2r+1)$ different values of $y_3$ (remember that $x_4$ also depends on $y_3$). In this case, we finally find that only $c_{1234567,0}$ is non-vanishing,

\[
c_{1234567,0} = -A_{\text{tree}}(1^-, 2^-, 3^+, 4^+) s_{12}^2 s_{23} ,
\]

(3.12)

\[
c_{1234567,i} = 0 \quad (1 \leq i \leq 2r+1) .
\]

(3.13)

This means that, after reconstruction, $\Delta_{1234567}$ is constant, $\Delta_{1234567}(q,k) = c_{1234567,0}$ , which is exactly the result of Ref. [41].

### 3.2 A contribution to the Crossed Amplitude

![Figure 3: 7fold-cut of the 4-point crossed diagram (s-channel)](image)

In this example, we consider the contribution to the (subleading-color) 4-gluon MHV amplitude in $\mathcal{N} = 4$ SYM coming from the helicity configuration depicted in Fig.3, where only gluons circulate in both loops. Its decomposition in terms of the crossed double-box MI in Fig.1 (right) is achieved by integrand-reduction.
On-Shell Solutions. The solutions of the 7fold-cut, $D_1 = \ldots = D_6 = D_7 = 0$, can be decomposed according to Eqs.(2.20,2.21), with the following definitions:

$$ p_0^\mu = 0^\mu , \quad e_1^\mu = p_1^\mu , \quad e_2^\mu = p_2^\mu , \quad (3.14) $$

$$ r_0^\mu = 0^\mu , \quad \tau_1^\mu = p_3^\mu , \quad \tau_2^\mu = p_4^\mu . \quad (3.15) $$

As before, the on-shell conditions, $D_1 = \ldots = D_6 = D_7 = 0$, cannot freeze the loop momenta, and one component is left over as a free variable. We choose $y_3$, namely the component of $k$ along $e_3$, as the variable parametrizing the infinite set of solutions of this 7fold-cut.

Residue. The residue of this 7fold-cut is defined as,

$$ \Delta_{1234567}(q,k) = \text{Res}_{1234567}\{N(q,k)\} , \quad (3.16) $$

where $\text{Res}_{1234567}\{N(q,k)\}$ is the product of the six 3-point tree-amplitudes sitting in the vertices exposed by the cuts, which can be built out of the tree-level expressions in Eq.(3.3) used for the planar integrand.

To find the polynomial expression of $\Delta_{1234567}$, we use again our two criteria.

1. Diagram topology and Vector basis.

As for the planar case, in the considered 4-point double-box, only three external vectors are independent, therefore any possible basis involving them needs to be completed by an additional orthogonal vector $\omega_3$, previously defined in Eq.(3.6).

2. Irreducible Scalar Products.

As in the planar case, the ISP’s formed by the loop variables and the external momenta can be chosen to be $(q \cdot p_2)$ and $(k \cdot p_3)$. Due to the explicit expression of the on-shell solutions, unlike the planar case (see Eq.(3.8)), they both behave linearly in $y_3$. This reflects the fact that these two ISP’s are not linearly independent, as one can explicitly see using the set of denominators in Fig.3.

Therefore, $\Delta_{1234567}$ is a polynomial in $y_3$, and can be parametrized as

$$ \Delta_{1234567}(q,k) = \sum_{i=0}^{r} c_{1234567,i} (\omega_3 \cdot k)^i = \sum_{i=0}^{r} c_{1234567,i} ((e_3 \cdot p_3) y_3)^i , \quad (3.17) $$

where $r$ is the maximum rank in the integration momenta. The last equation holds because $(e_3 \cdot k) = (e_3 \cdot e_4) y_4$ and $y_4 = 0$, as required by the seven on-shell conditions.

Coefficients. One can determine the $(r + 1)$ unknown coefficients $c_{1234567,i}$, by solving a system of equations obtained from sampling Eq.(3.16) and Eq.(3.17) on $(r + 1)$ solutions of the 7fold-cut. The solution of the system finally reads,

$$ c_{1234567,0} = -A_{\text{tree}}^\mu (1^-, 2^-, 3^+, 4^+) s_{12} s_{23} , \quad (3.18) $$

$$ c_{1234567,i} \neq 0 \quad (i = 1, \ldots, 4) , \quad (3.19) $$

$$ c_{1234567,j} = 0 \quad (5 \leq j \leq r) . \quad (3.20) $$
By using these coefficients in the polynomial expression in Eq. (3.17) we verify that the local-\(N = N\) test is fulfilled. In practice, we verify the equivalence of Eq. (3.16) and the reconstructed polynomial Eq. (3.17) when evaluated in any solution of the 7fold-cut other than the ones used to determine the coefficients.

Notice that in this case the coefficients \(c_{1234567,i} (i = 1, \ldots, 4)\) are non-vanishing, but they do not affect the integrated result, because they multiply spurious integrals that vanish upon integration. Nevertheless their presence is important for the completeness of the polynomial expression in Eq. (3.17).

4. The MHV Pentabox in \(\mathcal{N} = 4\) SYM

\[
D_1 = k^2 \\
D_2 = (k + p_2)^2 \\
D_3 = (k - p_1)^2 \\
D_4 = q^2 \\
D_5 = (q + p_3)^2 \\
D_6 = (q - p_4)^2 \\
D_7 = (q - p_4 - p_5)^2 \\
D_8 = (q + k + p_2 + p_3)^2.
\]

**Figure 4:** 5-point pentabox diagram.

In the previous two examples we have discussed the 7fold-cut of a 4-point two-loop amplitude. In this section we apply the integrand-reduction to the decomposition in terms of MI’s of the 5-point two-loop pentabox in \(\mathcal{N} = 4\) SYM, shown in Fig. 4. The expression for the pentabox-integrand has been given in a very compact form in Ref. [46]. The decomposition involves three types of contributions, coming from: a 5-point 8fold-cut, two 4-point 7fold-cuts, and two 5-point 7fold-cuts.

4.1 Five-point Eightfold-Cut

**Figure 5:** 5-point 8fold-cut \(\Delta_{12345678}\).
On-Shell Solutions. The solutions of the 8fold-cut, \(D_1 = \ldots = D_8 = 0\), depicted in Fig.\ref{fig5}, can be decomposed according to Eqs.\((2.20,2.21)\), with the following definitions:

\[
\begin{align*}
    r_\mu^0 &= 0^\mu, & e_\mu^1 &= p_1^\mu, & e_\mu^2 &= p_2^\mu, \\
    p_\mu^0 &= 0^\mu, & \tau_\mu^1 &= p_3^\mu, & \tau_\mu^2 &= p_4^\mu.
\end{align*}
\]

The eight on-shell conditions \(D_1 = \ldots = D_8 = 0\) admit four solutions, where the loop momenta are completely frozen, namely they are expressed in terms of external kinematic variables.

Residue. The residue of the 8fold-cut is defined as,

\[
\Delta_{12345678}(q,k) = \text{Res}_{12345678} \left\{ N(q,k) \right\}
\]

where the numerator function \(N\) can be found in Table I (a) of \cite{46}, and can be parametrized as

\[
\Delta_{12345678}(q,k) = c_{12345678,0} + c_{12345678,1} (q \cdot p_1) + c_{12345678,2} (k \cdot p_4) + c_{12345678,3} (k \cdot p_5).
\]

To derive its expression, we considered that this 5-point two-loop integral in four dimensions can depend on four external momenta, hence four (out of five) legs can be chosen as a basis for decomposing the loop variables. Moreover, this pentabox diagram admits three ISP’s, and we have chosen \((q \cdot p_1)\), \((k \cdot p_4)\), and \((k \cdot p_5)\). Any other ISP can be expressed as a combination of them plus reducible scalar products.

The definition in Eq.\((4.3)\) is compatible with the existence of four MI’s with denominators \(D_1, \ldots, D_8\): the scalar, plus three rank-1 integrals, each carrying one of the chosen ISP in the numerator.

Coefficients. By sampling \(N(q,k)\) at the four solutions of this 8fold-cut we can determine the coefficients, and find that \(c_{12345678,0}\) and \(c_{12345678,1}\) are non-vanishing, while \(c_{12345678,2} = c_{12345678,3} = 0\). Therefore, only two (out of four) pentabox-like MI’s will appear in the decomposition of the two-loop 5-point amplitude in \(N = 4\) SYM. Our result is in agreement with the results of Ref. \cite{42,47}, although, in order to match the coefficients given in these references, one must use an modified expression for \(\Delta_{12345678}\), in which \(c_{12345678,1}\) multiplies \(((q - p_4 - p_5) \cdot p_1)\).

4.2 Five-point Sevenfold-Cut (i)

On-Shell Solutions. The solutions of the 7fold-cut, \(D_1 = \ldots = D_6 = D_8 = 0\), in Fig.\ref{fig6}, can be decomposed according to Eqs.\((2.20,2.21)\), by using,

\[
\begin{align*}
    p_\mu^0 &= 0^\mu, & e_\mu^1 &= p_1^\mu, & e_\mu^2 &= p_2^\mu, \\
    r_\mu^0 &= 0^\mu, & \tau_\mu^1 &= p_3^\mu, & \tau_\mu^2 &= p_4^\mu.
\end{align*}
\]

In this case, the seven on-shell conditions, \(D_1 = \ldots = D_6 = D_8 = 0\), cannot freeze the loop momenta, and the generic solution reads,

\[
q_\mu^{(7)} = x_4 \tau_4^\mu, \quad k_\mu^{(7)} = y_3 e_3^\mu,
\]
where one of the on-shell cut-conditions imposes a non-linear relation among \(x_4\) and \(y_3\), which can be implicitly written as \(x_4 = x_4(y_3)\). We choose \(y_3\), namely the component of \(k\) along \(e_3\), as the variable parametrizing the infinite set of solutions of the 7fold-cut.

**Residue.** The residue of this 7fold-cut is defined as,

\[
\Delta_{1234568}(q, k) = \text{Res}_{1234568} \left\{ \frac{N(q, k) - \Delta_{12345678}(q, k)}{D_7} \right\}.
\]

(4.8)

where \(\Delta_{12345678}(q, k)\) is the polynomial residue of the 8fold-cut, reconstructed in Eq.(4.4). To find the polynomial expression of \(\Delta_{1234568}\), we use the following criteria.

1. **Diagram topology and Vector basis.** We observe that this 5-point diagram depends on four external momenta and, as in the pentabox case, its integrand does not contain any spurious ISP.

2. **Irreducible Scalar Products.** The ISP’s which can be formed by the loop variables and the external momenta can be chosen to be \((q \cdot p_1), (q \cdot p_2), (k \cdot p_3), (k \cdot p_4)\). Due to the explicit expression of the on-shell solutions, on the cut, one has

\[
(q \cdot p_1) \rightarrow (q(7) \cdot p_1) = -x_4(\tau_4 \cdot p_1),
\]

(4.9)

\[
(q \cdot p_2) \rightarrow (q(7) \cdot p_2) = -x_4(\tau_4 \cdot p_2),
\]

(4.10)

\[
(k \cdot p_3) \rightarrow (k(7) \cdot p_3) = y_3(e_3 \cdot p_3),
\]

(4.11)

\[
(k \cdot p_4) \rightarrow (k(7) \cdot p_4) = y_3(e_3 \cdot p_4).
\]

(4.12)

Therefore, \(\Delta_{1234567}\) is polynomial in \(x_4\) and \(y_3\), and can be written as

\[
\Delta_{1234568}(q, k) = c_{1234568,0} + \sum_{i=1}^{r} c_{1234568,i} (p_2 \cdot q)^i + \sum_{i=1}^{r} c_{1234568,i+r} (p_3 \cdot k)^i,
\]

(4.13)

where \(r\) is the maximum rank in the integration momenta.

**Coefficients.** As by-now understood, the unknowns \(c_{1234568,i}\) are found by sampling Eq.(4.8) and Eq.(4.13) on \((2r + 1)\) solutions of the 7fold-cut. Also in this case we find that \(\Delta_{1234568}(q, k)\) is a trivial polynomial, namely just a constant, because only \(c_{1234568,0}\) is non-vanishing.

Let us finally remark that integrands with numerator \((p_2 \cdot q)^i\), and \((p_3 \cdot k)^i\) are non-vanishing, and would be MI’s. They do not show up in this case because they are multiplied by null coefficients.
4.3 Five-point Sevenfold-Cut (ii)

\[ k - q - p_4 \]

Figure 7: 5-point 7fold-cut \( \Delta_{1234678} \)

This case is specular to the previous one and can be treated by relabeling the external momenta \((1 \leftrightarrow 2, 3 \leftrightarrow 5)\). As before only \( c_{1234678,0} \) is non-vanishing, therefore only the scalar integral contribute.

4.4 Four-point Sevenfold-Cut (i)

\[ k - q - p_4 \]

Figure 8: 4-point 7fold-cut \( \Delta_{1234578} \)

The 7fold-cut in Fig. 8 is equivalent to the ladder case treated in Sec. 3.1. The solutions of \( D_1 = \ldots = D_5 = D_7 = D_8 = 0 \) can be decomposed according to Eqs. (2.20, 2.21), by using

\[
\begin{align*}
\tau^\mu_0 &= 0^\mu, \\
\epsilon^\mu_1 &= p^\mu_1, \\
\epsilon^\mu_2 &= p^\mu_2, \\
p^\mu_6 &= 0^\mu, \\
\tau^\mu_1 &= p^\mu_3, \\
\tau^\mu_2 &= \frac{P^\mu_{45}}{2P_{45}} - \frac{s_{45}}{P_{45} \cdot \tau_1^\mu}.
\end{align*}
\]

As in Sec. 3.1, the seven on-shell conditions are not sufficient to freeze the loop momenta, and one component is left over as a free variable. We choose \( y_3 \), namely the component of \( k \) along \( e_3 \), as the variable parametrizing the infinite set of solutions.

The residue of the 7fold-cut, \( D_1 = \ldots = D_5 = D_7 = D_8 = 0 \), is defined as,

\[
\Delta_{1234578}(q, k) = \text{Res}_{1234578} \left\{ \frac{N(q, k) - \Delta_{12345678}(q, k)}{D_6} \right\},
\]

where \( \Delta_{12345678}(q, k) \) is the polynomial residue of the 8fold-cut, reconstructed in Eq. (4.4). The polynomial expression of \( \Delta_{1234578} \) is equivalent to the one given in Eq. (3.9), where \( \omega^\mu_3 \), defined in Eq. (3.6), must be constructed with the basis vector \( \{e_i\} \) used in Eq. (4.15).

We determine the unknown coefficients and find that only \( c_{1234578,0} \) is non-vanishing.
4.5 Four-point Sevenfold-Cut \( (ii) \)

Also the case in Fig. 9 falls in the two-loop 4-point category discussed in Sec. 3.1 and in the previous section. The solutions of the 7fold-cut, \( D_1 = \ldots = D_3 = D_5 = \ldots = D_8 = 0 \), can be decomposed according to Eqs. (4.20, 4.21), with the following definitions:

\[
\begin{align*}
p_0^\mu &= 0^\mu, & e_1^\mu &= p_1^\mu, & e_2^\mu &= p_2^\mu, \\
r_0^\mu &= -p_4^\mu, & \tau_1^\mu &= p_5^\mu, & \tau_2^\mu &= P_{34}^\mu - \frac{s_{34}}{2P_{34} \cdot \tau_1^\mu}.
\end{align*}
\]

Again, we find it convenient to use \( y_3 \) as free variable to parametrize the infinite set of solutions.

The residue, defined as

\[
\Delta_{1235678}(q, k) = \text{Res}_{1235678} \left\{ \frac{N(q, k) - \Delta_{12345678}(q, k) D_4}{D_4} \right\},
\]

with \( \Delta_{12345678}(q, k) \) being the the 8fold-cut polynomial in Eq. (4.4), can be written as in Eq. (3.9), where \( \omega_3^\mu \) must be replaced by

\[
\omega_5^\mu = \frac{(p_5 \cdot e_3)e_4^\mu - (p_5 \cdot e_4)e_3^\mu}{e_3 \cdot e_4}.
\]

By polynomial sampling we find that only \( c_{1234578,0} \) is non-vanishing.

4.6 Reconstructed Integrand

Combining the results of the previous sections, the numerator of the planar 5-point pentabox diagram can be decomposed as,

\[
N(q, k) = \Delta_{12345678}(q, k) + \\
\quad + \Delta_{1234568}(q, k)D_7 + \Delta_{1234578}(q, k)D_6 + \\
\quad + \Delta_{1234678}(q, k)D_5 + \Delta_{1235678}(q, k)D_4 = \\
\quad = c_{12345678,0} + c_{12345678,1} (q \cdot p_1) + \\
\quad + c_{1234568,0}D_7 + c_{1234578,0}D_6 + \\
\quad + c_{1234678,0}D_5 + c_{1235678,0}D_4,
\]

which corresponds to a decomposition of the integral in terms of six MI’s,
\[ N(q,k) = c_{12345678,0} + c_{12345678,1} + c_{1234568,0} + c_{1234578,0} + c_{1234678,0} + c_{1235678,0} + (4.22) \]

We checked the correctness of the Eq.(4.21) through the \textit{global-}(N = N) test, namely by verifying the identity of the \textit{l.h.s} and of the \textit{r.h.s.} for arbitrary values of \( q \) and \( k \).

5. The MHV Pentacross in \( \mathcal{N} = 4 \) SYM

\[ D_1 = k^2 \]
\[ D_2 = (k + p_2)^2 \]
\[ D_3 = (k + q - p_4 - p_5)^2 \]
\[ D_4 = q^2 \]
\[ D_5 = (q + p_3)^2 \]
\[ D_6 = (q - p_4)^2 \]
\[ D_7 = (q - p_4 - p_5)^2 \]
\[ D_8 = (q + k + p_2 + p_3)^2 . \]

\textbf{Figure 10:} 5-point Pentacross (non-planar)

In this section we present the result of the integrand reduction of the MHV pentacross diagram in \( \mathcal{N} = 4 \) SYM, depicted in Fig.[10]. The expression for the its integrand has been given in Table I (b) of Ref. [46], and happens to have the same expression of the planar diagram, although it is sitting on a different set of denominators, listed in Fig.[10].

The integrand-reduction follows the same pattern as described in the Section 3 and 4. In particular the expressions of the polynomial residues of the 5-point 8−fold cut, and 4-point 7−fold cut, can be obtained following the same procedures as in Sec. 4.1 and Sec. 3.2 respectively. The 5-point 7−fold cut deserves a dedicated discussion.
5.1 Five-point Sevenfold-Cut

**On-Shell Solutions.** The solutions of the 5point 7-fold-cut, \( D_1 = \ldots = D_6 = D_8 = 0 \), depicted in Fig. 11, can be decomposed according to Eqs.(2.20,2.21), with the following definitions:

\[
\begin{align*}
    p_0^\mu &= 0^\mu, \\
    e_1^\mu &= p_1^\mu, \\
    e_2^\mu &= p_2^\mu, \\
    r_0^\mu &= 0^\mu, \\
    \tau_1^\mu &= p_3^\mu, \\
    \tau_2^\mu &= p_4^\mu.
\end{align*}
\]

(5.1)

(5.2)

The on-shell conditions, \( D_1 = \ldots = D_6 = D_8 = 0 \), cannot freeze the loop momenta, and we choose \( y_3 \) to parametrize the infinite set of solutions.

**Residue.** The residue is defined as,

\[
\Delta_{1234568}(q,k) = \text{Res}_{1234568} \left\{ \frac{N(q,k) - \Delta_{12345678}(q,k)}{D_7} \right\}.
\]

(5.3)

The function \( \Delta_{12345678} \) is the 8-fold-cut polynomial of the pentacross. To find the polynomial expression of \( \Delta_{1234568} \), we use the following criteria.

1. **Diagram topology and Vector basis.** We observe that this 5-point diagram depends on four external momenta, therefore its integrand does not contain any spurious ISP.

2. **Irreducible Scalar Products.** The ISP’s formed by the loop variables and the external momenta can be chosen to be \((q \cdot p_1), (q \cdot p_2), (k \cdot p_3), (k \cdot p_4)\). Due to the explicit expression of the on-shell solutions, they all behave linearly in \( y_3 \). This reflects the fact that although irreducible, these four ISP’s are not linearly independent.

Therefore, \( \Delta_{1234568} \) is as a polynomial in \( y_3 \), and can be written as

\[
\Delta_{1234568}(q,k) = c_{1234568,0} + \sum_{i=1}^{r} c_{1234568,i} (p_1 \cdot q)^i,
\]

(5.4)

where \( r \) is the maximum rank in the integration momenta.

**Coefficients.** As by-now understood, the unknowns \( c_{1234568,i} \) are found by sampling Eq.(5.3) and Eq.(5.4) on \((r + 1)\) solutions of the 7-fold-cut. Also in this case we find that \( c_{1234568,0} \) is the only non-vanishing coefficient.

Let us finally remark that integrands with numerator \((p_1 \cdot q)^i\) would generate MI’s. They do not show up in the final result because they are multiplied by null coefficients.
5.2 Reconstructed Integrand

The integrand-decomposition for the MHV pentacross diagram given in Table I (b) of Ref. [46] has the same structure of the planar pentabox,

\[ N(q, k) = \Delta_{12345678}(q, k) + \]
\[ + \Delta_{1234568}(q, k)D_7 + \Delta_{1234578}(q, k)D_6 + \]
\[ + \Delta_{1234678}(q, k)D_5 + \Delta_{1235678}(q, k)D_4 = \]
\[ = c_{12345678,0} + c_{12345678,1}(q \cdot p_1) + \]
\[ + c_{1234568,0}D_7 + c_{1234578,0}D_6 + \]
\[ + c_{1234678,0}D_5 + c_{1235678,0}D_4, \]

(5.5)

and contains the same coefficients of Eq.(4.21). The above decomposition has been verified to fulfill the global-(\(N = N\)) test. Therefore, the final expression of the pentacross diagram in terms of MI’s reads,

\[ N(q, k) = c_{12345678,0} + c_{12345678,1}(q \cdot p_1) + \]
\[ + c_{1234568,0}D_7 + c_{1234578,0}D_6 + \]
\[ + c_{1235678,0}D_5 + c_{1235678,0}D_4, \]

(5.6)

6. Conclusions

We illustrated a first implementation of the integrand-reduction method for two-loop scattering amplitudes. We have shown that the residues of the amplitudes on the multi-particle cuts are polynomials written in terms of independent irreducible scalar products formed by loop momenta and either external momenta or polarization vectors built out of them. The independence conditions among irreducible scalar products can be investigated through their polynomial behavior in terms of the components of the loop momenta still undetermined after imposing the on-shell cut-conditions.

The reduction of the amplitudes in terms of master integrals can be realized through polynomial fitting of the integrand, without any need of an apriori knowledge of the integral basis. We discussed how the polynomial shapes of the residues determine the basis of master integrals appearing in the final result. In particular, we have found that the multiparticle residues of amplitudes with less then five external legs can eventually be written in terms of spurious irreducible scalar products which do not generate any master integral upon integration.
We applied the integrand-reduction algorithm to cases of modest complexity, such as planar and non-planar contributions to the 4-point MHV and 5-point MHV amplitudes in $\mathcal{N} = 4$ SYM. We worked out the polynomials parametrizing the residues at the 4-point 7fold-cut, 5-point 7fold-cut, 5-point 8fold-cut which contribute to the decomposition of the considered amplitudes.

In this work we identified general principles which could guide in the classification of all polynomial residues required by the complete integrand-reduction of arbitrary two-loop amplitudes. The technique we have presented extends the well-established analogous method for one-loop amplitudes, and can be considered a preliminary study towards the systematic reduction at the integrand-level of multi-loop amplitudes in any gauge theory, suitable for their automated semianalytic evaluation.

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References

[1] F. Cachazo, P. Svrcek and E. Witten, JHEP 0409 (2004) 006 [arXiv:hep-th/0403047].
[2] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 715 (2005) 499 [arXiv:hep-th/0412308].
   R. Britto, F. Cachazo, B. Feng and E. Witten, Phys. Rev. Lett. 94 (2005) 181602
   [arXiv:hep-th/0501052].
[3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 425 (1994) 217;
   Nucl. Phys. B 435 (1995) 59.
[4] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 725 (2005) 275 [arXiv:hep-th/0412103].
[5] L. D. Landau, Nucl. Phys. 13, 181 (1959).
   S. Mandelstam, Phys. Rev. 112, 1344 (1958), Phys. Rev. 115, 1741 (1959).
   R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).
   R. J. Eden, et al., The Analytic S Matrix (Cambridge University Press, 1966).
   M. G. J. Veltman, Physica 29 (1963) 186.
   E. Remiddi, Helv. Phys. Acta 54 (1982) 364.
[6] F. del Aguila and R. Pittau, JHEP 0407 (2004) 017 [arXiv:hep-ph/0404120].
   G. Ossola, C. G. Papadopoulos and R. Pittau, Nucl. Phys. B 763 (2007) 147
   [arXiv:hep-ph/0609007]; JHEP 0707 (2007) 085 [arXiv:0704.1271 [hep-ph]]; JHEP 0805
   (2008) 004 [arXiv:0802.1876 [hep-ph]].
[7] E. Witten, Commun. Math. Phys. 252 (2004) 189 [arXiv:hep-th/0312171].
[8] J. M. Drummond, J. Henn, V. A. Smirnov and E. Sokatchev, JHEP 0701 (2007) 064 [arXiv:hep-th/0607160].
Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower and V. A. Smirnov, Phys. Rev. D 75 (2007) 085010 [arXiv:hep-th/0610248].
J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, Nucl. Phys. B 828 (2010) 317 [arXiv:0807.1095 [hep-th]].
A. Brandhuber, P. Heslop and G. Travaglini, Phys. Rev. D 78 (2008) 125005 [arXiv:0807.4097 [hep-th]].
[9] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP 1003 (2010) 020 [arXiv:0907.5418 [hep-th]].
L. J. Mason and D. Skinner, JHEP 0911 (2009) 045 [arXiv:0909.0250 [hep-th]].
N. Arkani-Hamed, F. Cachazo and C. Cheung, JHEP 1003 (2010) 036 [arXiv:0909.0483 [hep-th]].
[10] L. F. Alday and J. M. Maldacena, JHEP 0706 (2007) 064 [arXiv:0705.0303 [hep-th]].
[11] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D 78 (2008) 085011 [arXiv:0805.3993 [hep-ph]].
Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. Lett. 105 (2010) 061602 [arXiv:1004.0476 [hep-th]].
[12] S. D. Badger, E. W. N. Glover, V. V. Khoze and P. Svrcek, JHEP 0507 (2005) 025 [arXiv:hep-th/0504159].
Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Rev. D 73 (2006) 065013 [arXiv:hep-ph/0507005].
[13] Z. Bern and A. G. Morgan, Nucl. Phys. B 467 (1996) 479 [arXiv:hep-ph/9511336].
R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D 72 (2005) 065012 [arXiv:hep-ph/0503132].
R. Britto, B. Feng and P. Mastrolia, Phys. Rev. D 73 (2006) 105004 [arXiv:hep-ph/0602178]; Phys. Rev. D 78 (2008) 025031 [arXiv:0803.1989 [hep-ph]].
P. Mastrolia, Phys. Lett. B 678 (2009) 246 [arXiv:0905.2909 [hep-ph]].
N. E. J. Bjerrum-Bohr, D. C. Dunbar and W. B. Perkins, JHEP 0804 (2008) 038 [arXiv:0709.2086 [hep-ph]].
P. Mastrolia, Phys. Lett. B 644 (2007) 272 [arXiv:hep-th/0611091].
D. Forde, Phys. Rev. D 75 (2007) 125019 [arXiv:0704.1835 [hep-ph]].
E. W. Nigel Glover and C. Williams, JHEP 0812 (2008) 067 [arXiv:0810.2964 [hep-th]].
R. Britto and B. Feng, Phys. Lett. B 681 (2009) 376 [arXiv:0904.2766 [hep-th]].
R. Britto and E. Mirabella, JHEP 1101 (2011) 135 [arXiv:1011.2344 [hep-th]].
A. Brandhuber, S. McNamea, B. J. Spence and G. Travaglini, JHEP 0510 (2005) 011 [arXiv:hep-th/0506068].
C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. B 645 (2007) 213 [arXiv:hep-ph/0609191]; JHEP 0703 (2007) 111 [arXiv:hep-ph/0612277].
S. D. Badger, JHEP 0901 (2009) 049 [arXiv:0806.4600 [hep-ph]].
[16] G. Passarino and M. J. G. Veltman, Nucl. Phys. B 160 (1979) 151.
G. ’t Hooft, and M. Veltman, Nucl. Phys. B 153 (1979) 365.

[17] G. J. van Oldenborgh and J. A. M. Vermaseren, Z. Phys. C 46 (1990) 425.
G. J. van Oldenborgh, Comput. Phys. Commun. 66 (1991) 1.
R. K. Ellis and G. Zanderighi, JHEP 0802 (2008) 002 [arXiv:0712.1851 [hep-ph]].
T. Binoth, J. P. Guiliet, G. Heinrich, E. Pilon and T. Reiter, Comput. Phys. Commun. 180 (2009) 2317 [arXiv:0810.0992 [hep-ph]].
A. van Hameren, arXiv:1007.4716 [hep-ph].
T. Hahn and M. Rauch, Nucl. Phys. Proc. Suppl. 157 (2006) 236 [arXiv:hep-ph/0601248].
A. Denner and S. Dittmaier, Nucl. Phys. B 844 (2011) 199 [arXiv:1005.2076 [hep-ph]].
G. Cullen, J. P. Guillet, G. Heinrich, T. Kleinschmidt, E. Pilon, T. Reiter and M. Rodgers, Comput. Phys. Commun. 182 (2011) 2276 [arXiv:1101.5595 [hep-ph]].

[18] T. Binoth, J. P. Guillet, G. Heinrich, E. Pilon, C. Schubert, JHEP 0510 (2005) 015.
[hep-ph/0504267].
A. Denner and S. Dittmaier, Nucl. Phys. B 734 (2006) 62 [arXiv:hep-ph/0509141].
J. Gluza, K. Kajda, T. Riemann and V. Yundin, Eur. Phys. J. C 71 (2011) 1516
[arXiv:1010.1667 [hep-ph]].
F. Campanario, arXiv:1105.0920 [hep-ph].

[19] D. E. Soper, Phys. Rev. D 62 (2000) 014009 [arXiv:hep-ph/9910292].
Z. Nagy and D. E. Soper, Phys. Rev. D 74 (2006) 093006 [arXiv:hep-ph/0610028].

[20] R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP 0803 (2008) 003 [arXiv:0708.2398 [hep-ph]].
W. T. Giele, Z. Kunszt and K. Melnikov, JHEP 0804 (2008) 049 [arXiv:0801.2237 [hep-ph]].
R. K. Ellis, W. T. Giele, Z. Kunszt and K. Melnikov, Nucl. Phys. B 822 (2009) 270
[arXiv:0806.3467 [hep-ph]].

[21] P. Mastrolia, G. Ossola, T. Reiter and F. Tramontano, JHEP 1008 (2010) 080
[arXiv:1006.0710 [hep-ph]].

[22] G. Heinrich, G. Ossola, T. Reiter and F. Tramontano, JHEP 1010 (2010) 105
[arXiv:1008.2441 [hep-ph]].

[23] G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP 0803, 042 (2008) [arXiv:0711.3596
[hep-ph]].
C. F. Berger et al., Phys. Rev. D 78, 036003 (2008) [arXiv:0803.4180 [hep-ph]].
W. T. Giele and G. Zanderighi, JHEP 0806, 038 (2008) [arXiv:0805.2152 [hep-ph]].
A. Lazopoulos, arXiv:0812.2998 [hep-ph].

[24] T. Hahn, PoS ACAT2010, 078 (2010). [arXiv:1006.2231 [hep-ph]].

[25] A. van Hameren, C. G. Papadopoulos and R. Pittau, JHEP 0909 (2009) 106
[arXiv:0903.4665 [hep-ph]].

[26] G. Bevilacqua et al., Nucl. Phys. Proc. Suppl. 205-206 (2010) 211 [arXiv:1007.4918
[hep-ph]].
G. Bevilacqua, M. Czakon, C. G. Papadopoulos, R. Pittau and M. Worek, JHEP 0909
(2009) 109 [arXiv:0907.4723 [hep-ph]].
[27] V. Hirschi, R. Frederix, S. Frixione, M. V. Garzelli, F. Maltoni and R. Pittau, JHEP 1105 (2011) 044 [arXiv:1103.0621 [hep-ph]].

[28] G. Cullen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, G. Ossola, T. Reiter, and F. Tramontano, GoSam, in preparation; see G. Ossola’s contribution to EPS-HEP 2011 Europhysics Conference on High-Energy Physics Grenoble, July 21-27, 2011.

[29] S. Laporta, Int. J. Mod. Phys. A 15 (2000) 5087 [arXiv:hep-ph/0102033]. Phys. Lett. B 504 (2001) 188 [arXiv:hep-ph/0102032].

[30] F. V. Tkachov, Phys. Lett. B 100 (1981) 65.

[31] R. N. Lee, A. V. Smirnov and V. A. Smirnov, Nucl. Phys. Proc. Suppl. 205-206 (2010) 308 [arXiv:1005.0362 [hep-ph]].

[32] A. V. Kotikov, Phys. Lett. B 267 (1991) 123.

[33] V. A. Smirnov, Nucl. Phys. Lett. B 460 (1999) 397 [arXiv:hep-ph/9905323].

[34] M. Beneke and V. A. Smirnov, Nucl. Phys. B 522 (1998) 321 [arXiv:hep-ph/9711391].

[35] T. Binoth and G. Heinrich, Nucl. Phys. B 585 (2000) 741 [arXiv:hep-ph/0004013]. Nucl. Phys. B 680 (2004) 375 [arXiv:hep-ph/0305234].

[36] C. Anastasiou, S. Beerli and A. Daleo, JHEP 0705 (2007) 071 [arXiv:hep-ph/0703282].

[37] M. Y. Kalmykov and B. A. Kniehl, arXiv:1105.5319 [math-ph].

[38] D. Bardin, J.-P. Guillet, and E. Remiddi, Comput. Phys. Commun. 182, 1566-1581 (2011). [arXiv:1011.5493 [hep-ph]].
[38] J. Gluza, K. Kajda and D. A. Kosower, Phys. Rev. D 83 (2011) 045012 [arXiv:1009.0472 [hep-th]].

[39] Z. Bern, J. S. Rozowsky and B. Yan, Phys. Lett. B 401 (1997) 273 [arXiv:hep-ph/9702424].
Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B 530 (1998) 401 [arXiv:hep-th/9802162].

[40] Z. Bern, L. J. Dixon and D. A. Kosower, JHEP 0001 (2000) 027 [arXiv:hep-ph/0001001].

[41] E. I. Buchbinder and F. Cachazo, JHEP 0511 (2005) 036 [arXiv:hep-th/0506126].

[42] F. Cachazo, arXiv:0803.1988 [hep-th].

[43] Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower, Phys. Rev. D 76 (2007) 125020 [arXiv:0705.1864 [hep-th]].
Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. D 78 (2008) 105019 [arXiv:0808.4112 [hep-th]].
Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. Lett. 105 (2010) 061602 [arXiv:1004.0476 [hep-th]].

[44] P. Mastrolia, G. Ossola, C. G. Papadopoulos and R. Pittau, JHEP 0806 (2008) 030 [arXiv:0803.3964 [hep-ph]].

[45] D. Maitre, P. Mastrolia, Comput. Phys. Commun. 179 (2008) 501-574. [arXiv:0710.5559 [hep-ph]].

[46] J. J. M. Carrasco and H. Johansson, arXiv:1106.4711 [hep-th].

[47] Z. Bern, M. Czakon, D. A. Kosower, R. Roiban and V. A. Smirnov, Phys. Rev. Lett. 97 (2006) 181601 [arXiv:hep-th/0604074].