Positivity of Symplectic Area for Perturbed $J$-holomorphic Curves

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Abstract
In this paper we will prove that for a compact, symplectic manifold $(M, \omega)$ and for $\omega$-compatible almost-complex structure $J$ any properly perturbed $J$-holomorphic curve has a non-negative symplectic area. This non-negative property provides us with a new obstruction to the bubbling off phenomenon and thus allows us to redefine the Floer symplectic homology. In particular, in subsequent papers, we will prove the Arnold conjecture in both degenerate and non-degenerate cases with integer coefficients for general, symplectic manifolds.

1 Introduction
Let us denote by $(M, \omega)$ a compact, symplectic manifold. Here $\omega$ is a closed, non-degenerate 2-form. If $H : \mathbb{R} \times M \to \mathbb{R}$ denotes a time-dependent Hamiltonian function then there is associated a time-dependent Hamiltonian vector fields $X_H$ on the symplectic manifold $M$ defined by the equality

$$\iota(X_H)\omega = dH.$$  \hfill (1)

We shall assume that the Hamiltonian function $H$ is of period 1 in time. The Arnold conjecture says that the number of 1-periodic solutions of the Hamiltonian equation

$$\dot{\gamma}(t) = X_H(\gamma(t))$$  \hfill (2)

is estimated from below by the number of critical points of a smooth function defined over the manifold $M$. (See \cite{A}, Appendix 9.)
In trying to prove the Arnold conjecture one is usually led to study the action functional
\[ a_H(\gamma) = -\int_D u^*\omega + \int_0^1 H_t(\gamma(t))dt \]
defined on the space of smooth, contractible loops \( \gamma : \mathbb{R} \to M \) with \( \gamma(t) = \gamma(t+1) \). See, for example, [CZ, F5, H, HZ, S] for some special cases for both nondegenerate and degenerate 1-periodic solutions of the equation (2).

In trying to extend the variational methods to the general case one encounters two difficulties:

1. The action functional \( a_H \) is not uniquely defined on the space of smooth, contractible loops of \( M \). Rather, it is well defined on the universal covering of the space of loops.

2. There is bubbling off phenomenon which causes difficulties with compactification of appropriate moduli spaces.

In [F2, F3, F4, HS, O1, O2, O3] these difficulties has been overcome in some more general cases.

In our approach to the Arnold conjecture which is valid for all compact symplectic manifolds we will restrict the action functional to a subset of its one special ‘branch’. Thus the first difficulty will be overcome in general. Fortunately, over the restricted set the bubbling problem will also disappear. Thus, in particular, we redefine the Floer symplectic homology to obtain a homology theory very similar to the finite dimensional Morse homology. Details of the construction of the new Floer homology and some of its consequences will be presented in subsequent papers.

In this paper we present the major tool in our construction of Floer homology which is the positivity of symplectic area of properly perturbed \( J \)-holomorphic curves. For a compact Riemann surface \((\Sigma, j)\) the map \( u : \Sigma \to M \) is said to satisfy a properly perturbed Cauchy-Riemann equation if
\[ du + J \circ du \circ j + P(u) = 0. \]
(See Def 3.2 for details.)

**Theorem 1.1** Let \( u : \Sigma \to M \) be a properly perturbed \( J \)-holomorphic curve. Then the symplectic area of \( u \) is non-negative:
\[ \int_\Sigma u^*\omega \geq 0. \]

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2 Example

Let $S^2$ be a Riemannian sphere $S^2 = \mathbb{C} \cup \{\infty\}$ with the standard Kaehler metric

$$dz \otimes \overline{dz} \left/ \left( |z|^2 + 1 \right)^2 \right.,$$

where $z = x + iy$ denotes a point of $S^2$. The induced standard symplectic form is of the form

$$\omega = \frac{dx \wedge dy}{(x^2 + y^2 + 1)^2},$$

and the induced metric by the standard complex structure on $S^2$ is of the form

$$\langle \hat{z}_1, \hat{z}_2 \rangle = \frac{\hat{x}_1 \hat{x}_2 + \hat{y}_1 \hat{y}_2}{(|z|^2 + 1)^2},$$

(3)

where $\hat{z}_j = \hat{x}_j + i \hat{y}_j$ denote a tangent vectors at $z = x + iy$.

We will consider a Hamiltonian function $H : \mathbb{R} \times S^2 \to \mathbb{R}$ of the form

$$H(s, z) = \psi(s) \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$  

The gradient vector field (depending on $s$) of the Hamiltonian function $H$ with respect to the metric (3) is $\nabla H(s, z) = 4\psi(s)z$.

Thus we will obtain the following equation for the properly perturbed holomorphic curve $u : \mathbb{R} \times S^1 \times S^2 \to S^2$ :

$$\frac{\partial u}{\partial s} + i \frac{\partial u}{\partial t} + 4\psi(s)u = 0.$$  

(4)

Let us consider solutions of the equation (4) of the form

$$u(s, t) = \exp \left\{ -4 \int_{-\infty}^{s} \psi(s')ds' \right\} \exp \{2\pi k(s + it)\}.$$  

(5)

Indeed, it is easy to verify that each function of the form (5), for which the function $\int_{-\infty}^{s} \psi(s')ds'$ is smooth, satisfies the perturbed Cauchy-Riemann equation (4).

Assume now that the function $\psi$ has a compact support. Under this condition we claim that the symplectic area of solutions (5) is non-negative i.e.

$$\int u^* \omega \geq 0.$$
Perhaps, the simplest way to see this is to homotop the curve \( u(s,t) \) to a holomorphic one. The homotopy can be done by considering the equation with a parameter \( \lambda, 0 \leq \lambda \leq 1 \):

\[
\frac{\partial u}{\partial s} + i \frac{\partial u}{\partial t} + 4\lambda \psi(s) u = 0.
\]

Then for each \( \lambda \) solutions of the above equation of the form

\[
u_\lambda(s,t) = \exp \left\{ -4\lambda \int_{-\infty}^{s} \psi(s') ds' \right\} \exp\{2\pi k(s + it)\}
\]

will provide us a homotopy between the original solution and the holomorphic curve \( u_0(s,t) = \exp 2\pi k(s + it) \). Now since the symplectic form \( \omega \) is closed and the symplectic area of the holomorphic curve is non-negative by Stokes theorem we obtain

\[
\int u^* \omega = \int u^* \omega_0 \geq 0.
\]

In fact, as we will see, the argument of deforming a properly perturbed Cauchy-Riemann equation to a non-perturbed one works in general and thus proving the positivity of symplectic area of perturbed \( J \)-holomorphic curves.

Note, however, that if we let the function \( \psi \) to be a nonzero constant: \( \psi(s) = \tau \) for all \( s \in \mathbb{R} \), then as it has been shown in [16] the function

\[
u_k(s,t) = \exp(4\tau s) \exp\{2\pi k(s + it)\}
\]

is a solution of the equation \( (4) \) whenever \( \pi k + 2\tau > 0 \). Moreover, for the symplectic area we have

\[
\int u_k^* \omega = \pi k.
\]

Thus, in particular, we obtain solutions \( u_k \) with a negative symplectic area if \( k \) is negative.

The key point why the function \( \psi \) with compact support produces only solutions with non-negative symplectic area is that the Hamiltonian part of the equation \( (4) \) represents a derivative of a global function on \( S^2 \). In the case of the function \( \psi \) being a nonzero constant this is not so.

### 3 Perturbed \( J \)-holomorphic Curves

Let \((M, \omega)\) denote a \( 2n \)-dimensional compact symplectic manifold with the symplectic form \( \omega \) and let \((\Sigma, j)\) denotes a closed connected Riemman surface
with a complex structure $j$ and with a fixed Kaehler metric. Let $J$ denote a smooth family of $\omega$-compatible almost-complex structure on $M$ depending on the parameter $z \in \Sigma$. We will denote the space of such families by $\mathcal{J}$.

Let $\mathcal{X} = \text{Map}(\Sigma, M; A)$ be the space of all smooth maps $u : \Sigma \to M$ which represent the homology class $A \in H_2(M)$ i.e. such that $u_*([\Sigma]) = A \in H_2(M)$, where $[\Sigma]$ denotes the fundamental class of the surface $\Sigma$ determined the orientation associated to the complex structure $j$. For simplicity in this paper we will consider only homology with integer coefficients and denote it $H_*(M)$. So in this situation we will use notation $[u] = A \in H_2(M)$.

We shall denote by $\mathcal{X}^{1,p}_1$ the completion of the space $\mathcal{X}$ with respect to the Sobolev norm $W^{1,p}$. More precisely, $\mathcal{X}^{1,p}_1$ is the space of maps $u : \Sigma \to M$ whose first covariant derivatives with respect to Riemannian metric on $M$ are of class $L^p$. Since the manifold $M$ is compact the topology of this norm does not depend on the choice of the Riemannian metric. In order for the space $\mathcal{X}^{1,p}_1$ to be well-defined we must assume that $p > 2$. The tangent space $T_u \mathcal{X}^{1,p}_1$ at a smooth $u$ is the completion of the space $C^\infty(\Sigma, \Omega^0, 1 \otimes J u^* TM)$ of all smooth sections $\widehat{u} \in \Gamma(u^* TM)$ in the Sobolev norm.

For a family of almost complex structures $J \in \mathcal{J}$ let us consider the infinite dimensional vector bundle $\mathcal{E} \to \mathcal{X}^{1,p}$ where the fiber at $u$ is the space $\mathcal{E}_u = L^p(\Sigma, \Omega^{0,1} \otimes J u^* TM)$ of $L^p$-section of the vector bundle over $\Sigma$ whose fiber over a point $z \in \Sigma$ is the space of $C$-linear, with respect to $J(z, u(z))$, maps from $T_{z, 1} \Sigma$ to $(u^* TM)_z$.

We note that the zero set of the section

$$\overline{\partial}_J : \mathcal{X}^{1,p} \to \mathcal{E}$$

of the infinite dimensional vector bundle $\mathcal{E} \to \mathcal{X}^{1,p}$ given by the formula

$$\overline{\partial}_J(u) = du + J \circ du \circ j$$

is the set of all $J$-holomorphic curves in the class $A \in H_2(M)$.

We shall denote by $\Omega^{0,1, \Sigma} \otimes J TM$ the vector bundle over the space $\Sigma \times M$ whose fiber over a point $(z, m) \in \Sigma \times M$ is the space of $C$-antilinear maps from $T_{z, 1} \Sigma$ to $(TM)_m$ with respect to $J(z, m)$.

Now we want to introduce the general setting for introducing and dealing with the concept of perturbed $J$-holomorphic curves. Let $\Omega^1 \otimes C$ denote the space of all complex valued one form defined over the product $\Sigma \times M$. On the product $\Sigma \times M$ there is the almost-complex structure $j \times J$. With respect to this almost-complex structure we have the following decomposition of the space $\Sigma \times M$ into the direct sum

$$\Omega^1 \otimes C = \Omega^{1,0} \oplus \Omega^{0,1}$$
complex linear one forms and complex antilinear one forms.

Any one form in $\Omega^{1,0}$ can be uniquely written in the form

$$\alpha - i\alpha \circ (j \times J),$$

where $\alpha$ is a real valued one form in $\Omega^1$. Similarly, any one form in $\Omega^{0,1}$ can be uniquely written as

$$\alpha + i\alpha \circ (j \times J).$$

Our interest will be in following subspaces:

1. $\Omega^{1,0}_M = \text{subspace of all complex linear one form in } \Omega^{1,0}$ whose restriction to the horizontal subbundle $T_{\Sigma}(\Sigma \times M)$ of the tangent bundle $T(\Sigma \times M)$ is trivial.

2. $\Omega^{0,1}_\Sigma = \text{subspace of all complex linear one form in } \Omega^{0,1}$ whose restriction to the vertical subbundle $T_M(\Sigma \times M)$ of the tangent bundle $T(\Sigma \times M)$ is trivial.

We will use the $\bar{\partial}_\Sigma$ operator in the direction of $\Sigma$ in three different situations:

1. $\bar{\partial}_\Sigma : C^\infty(\Sigma \times M) \to \Omega^{0,1}_\Sigma$,

$$\bar{\partial}_\Sigma(f) = d_\Sigma f + i(d_\Sigma f) \circ j.$$

2. $\bar{\partial}_\Sigma : C^\infty(\Sigma \times M, \pi^*TM) \to C^\infty(\Sigma \times M, \Omega^{0,1}_\Sigma \otimes j TM)$,

$$\bar{\partial}_\Sigma(X) = d_\Sigma X + J \circ (d_\Sigma X) \circ j.$$

Here $d_\Sigma$ denote the partial derivative in the direction of $\Sigma$. Note that it make sense to define $d_\Sigma$ since all tangent spaces of the form $(\pi^*TM)_{(z,m)}$ are canonically identified with $TM_m$.

3. $\bar{\partial}_\Sigma : \Omega^{1,0}_M \to \Omega^{0,1}_\Sigma \otimes C^\infty \Omega^{1,0}_M$,

$$\bar{\partial}_\Sigma(\psi - i\psi \circ J) = \bar{\partial}_\Sigma(\psi) - i(\bar{\partial}_\Sigma(\psi)) \circ J,$$

where

$$\bar{\partial}_\Sigma(\psi) = d_\Sigma(\psi) + i(d_\Sigma(\psi)) \circ j$$

for any real valued one form $\psi$ which is zero on $T_{\Sigma}(\Sigma \times M)$.
We will also need the following maps:

1. \( \partial_M : C^\infty(\Sigma \times M) \to \Omega^{1,0}_M \),
   \[ \partial_M(f) = d_M f - i(d_M f) \circ J. \]

2. \( \partial_M : \Omega^{0,1}_\Sigma \to \Omega^{0,1}_\Sigma \otimes \Omega^{1,0}_M \),
   \[ \partial_M(\psi + i\psi \circ j) = \partial_M(\psi) + i(\partial_M(\psi)) \circ j, \]
   where
   \[ \partial_M(\psi) = d_M(\psi) - i(d_M(\psi)) \circ J \]
   for any real valued one form \( \psi \) which vanishes on \( T_M(\Sigma \times M) \).

We will often identify the tangent bundle \( \pi^*(TM) \) with that of \( \Omega^{1,0}_M \) using the metric corresponding to \( J \). The identification is given by the formula
\[ X \mapsto \Phi(X), \]
where
\[ \Phi(X)(Y) = \omega(X,Y) - i\omega(X,JY) = \langle X, JY \rangle_J + i \langle X, Y \rangle_J \] (6)
for any \( X, Y \in \pi^*(TM)_{(z,m)} \).

Note that the map \( \Phi \) is complex antilinear as it should be since the bundle \( \Omega^{1,0}_M \) of complex linear forms is a complex dual of the bundle \( \pi^*(TM) \).

Under this identification the following diagram commutes
\[ C^\infty(\pi^*(TM)) \xrightarrow{\overline{\partial}_\Sigma} C^\infty(\Omega^{0,1}_\Sigma \otimes_J TM) \quad \xrightarrow{\Phi} \quad C^\infty(\Omega^{1,0}_M) \]
\[ \Omega^{1,0}_M \xrightarrow{\overline{\partial}_\Sigma} \Omega^{0,1}_\Sigma \otimes \Omega^{1,0}_M. \]

This is because
\[ \overline{\partial}_\Sigma(X) \circ \Phi = \overline{\partial}_\Sigma \omega(X,.) - i(\overline{\partial}_\Sigma \omega(X,.)) \circ J \]
\[ = d_\Sigma \omega(X,.) + id_\Sigma \omega(X,.) \circ j \]
\[ = \omega(d_\Sigma X,.) - i\omega(d_\Sigma X,.) \circ J \]
\[ + i(\omega(d_\Sigma X,.) - i\omega(d_\Sigma X,.) \circ J) \circ j \]
\[ = \Phi(d_\Sigma X) + i\Phi(d_\Sigma X) \circ j \]
\[ = \Phi(d_\Sigma X + J \circ d_\Sigma X \circ j) \]
\[ = (id \otimes \Phi) \circ \overline{\partial}_\Sigma(X). \]
Definition 3.1  An element $P \in C^\infty(\Sigma \times M, \Omega^0_\Sigma \otimes_J TM)$ is said to be exact if there is a vector field $X \in C^\infty(\Sigma \times M, \pi^*TM)$ of the form

$$X = \Phi^{-1} \circ \partial_M f$$

such that

$$P = \bar{\partial}_\Sigma X = d_\Sigma + J \circ (d_\Sigma X) \circ j.$$  (9)

Definition 3.2  Let $\partial_{J,f} : X \to \mathcal{E}$ be a section of the form

$$\partial_{J,f}(u) = du + J \circ du \circ j + P(u),$$

where $P \in C^\infty(\Sigma \times M, \Omega^0_\Sigma \otimes_J TM)$ is exact, $P = \bar{\partial}_\Sigma \circ \Phi^{-1} \circ \partial_M f$, and $P(u)(z) = P(z, u(z))$.

The equation

$$\partial_{J,f}(u) = 0$$

will be called a perturbed Cauchy-Riemann equation. Solutions of the equation (11) will be called perturbed $J$-holomorphic curves.

Definition 3.3  The equation (11) is said to be a properly perturbed Cauchy-Riemann equation if there is a constant $z \in \mathbb{C}$ such that the function $f$ satisfies the equation

$$f = z g$$

for some real function $g \in C^\infty(\Sigma \times M, \mathbb{R})$.

The perturbation term (9) is said to be properly exact if the function $f$ in the equation (8) satisfies the equation (12).

4  Properties of Perturbed $J$-holomorphic Curves

Lemma 4.1  Let $P = \bar{\partial}_\Sigma \circ \Phi^{-1} \circ \partial_M f$ be an exact perturbation. Then it can be written as

$$P = (id \otimes \Phi)^{-1} \circ \partial_M \circ \bar{\partial}_\Sigma f.$$  (13)

On the other hand, any perturbation of the form (13) is exact.
Proof: Let us first verify that the following diagram commutes
\[
\begin{array}{ccc}
C^\infty(\Sigma \times M) & \xrightarrow{\partial M} & \Omega^0_{M} \\
\downarrow \partial M & & \downarrow \partial M \\
\Omega^1_{M} & \xrightarrow{\bar{\partial}_\Sigma} & \Omega^1_{M} \otimes \Omega^0_{M}.
\end{array}
\] (14)

Indeed, we may assume that the function \( f \) is real. Then
\[
\partial M(\bar{\partial}_\Sigma f) = \partial M(d_\Sigma f + i(d_\Sigma f) \circ j)
\]
\[
= \partial M(d_\Sigma f) + i(\partial M(d_\Sigma f)) \circ j
\]
\[
= d_M(d_\Sigma f) - i(d_M(d_\Sigma f)) \circ J
\]
\[
+ i(d_M(d_\Sigma f)) \circ j + (d_M(d_\Sigma f)) \circ J \circ j
\]

and
\[
\bar{\partial}_\Sigma(\partial M f) = \bar{\partial}_\Sigma(\partial M f - i(\partial M f) \circ J)
\]
\[
= d_\Sigma(d_M f) + i(d_\Sigma(d_M)) \circ j
\]
\[
- i(d_\Sigma(d_M)) \circ J + (d_\Sigma(d_M f)) \circ j \circ J.
\]

Since partial derivatives commutes the commutativity of the diagram follows. To finish the proof of the lemma combine the two commutative diagrams (7) and (14).

**Proposition 4.2** Let \( \Sigma = S^2 \) be the Riemannian sphere and let \( \psi \in \Omega^0_{\Sigma} \). Then the perturbation
\[
(id \otimes \Phi)^{-1} \circ \partial_M \psi
\]
is exact.

Proof: Using the Lemma 4.1 it is enough to show that \( \psi = \bar{\partial}_\Sigma(f) \) for some function \( f \in C^\infty(\Sigma \times M) \). For every \( m \in M \) we have \( \psi_m = \psi(., m) \in \Lambda^0(S^2) \). Since \( \Lambda^0(S^2) = 0 \), \( \bar{\partial}\psi_m = 0 \). Moreover, since \( H^0(S^2) = 0 \) and \( H^0(S^2) = \mathbb{C} \) there is the unique function \( f_m : S^2 \to \mathbb{C} \) such that \( \bar{\partial}f_m = \psi_m \) and \( f_m(z_0) = 0 \) for the fixed point \( z_0 \in S^2 \). Define \( f \) by the formula
\[
f(z, m) = f_m(z).
\]

Then one verifies that \( \psi = \bar{\partial}_\Sigma(f) \) and this finishes the proof of the Proposition.

We note that if \( g : \mathbb{R} \times S^1 \to S^2 \), \( g(s, t) = z = s + it \) is a holomorphic coordinate system and a function \( H : \mathbb{R} \times S^1 \times M \to \mathbb{R} \) has a compact support then the Proposition 4.2 implies that the perturbation
\[
P = \nabla H ds - J \nabla H dt
\]

(15)
is exact. This is so because

\[
\begin{align*}
    \text{id} \otimes \Phi(\nabla H ds - J \nabla H dt) &= (\text{id}H + dH \circ J) \otimes (ds - dt) \\
    &= i(dH - idH \circ J) \otimes ds + (dH - idH \circ J) \otimes dt \\
    &= \partial_M((iH ds + H dt)).
\end{align*}
\]

In our construction of a new Floer symplectic (co)homology it will be very important to note that the Hamiltonian perturbation term is in fact properly exact. It follows from the equation

\[
\bar{\partial}_\Sigma(i g) = iH ds + H dt,
\]

where the function \( g \) is defined as

\[
g(z, m) = \int_{(\infty, m)}^{(z, m)} H ds + H dt,
\]

where the integration is taken over any smooth path contained in the set \( \Sigma \times m \).

**Theorem 4.3** Let \( u : \Sigma \to M \) be a smooth map from a Riemannian surface \( \Sigma \) to a symplectic manifold \( M \) with compatible family of almost-complex structures \( J \). Assume that \( du(z_0) \neq 0 \) and that \( du(z_0) \) is complex anti-linear at a single point \( z_0 \in \Sigma \).

Then \( u \) is not a perturbed \( J \)-holomorphic curve.

**Proof:** Assume that \( u \) is a perturbed \( J \)-holomorphic curve. Then by the Definition 3.2 and the Lemma 4.1 it satisfies the equation

\[
du + J \circ du \circ j + P(u) = 0,
\]

where

\[
P = (\text{id} \otimes \Phi)^{-1} \circ \partial_M \circ \bar{\partial}_\Sigma f.
\]

Consider the tensor \((\text{id} \otimes \Phi) \circ P \in \Omega_{\Sigma}^{0,1} \otimes \Omega_{M}^{1,0}\). For any two vectors \( v, w \in T_{z_0} \Sigma \) we have the following formula

\[
\theta(v, u) := (\text{id} \otimes \Phi) \circ P(v \otimes du(w))
\]

\[
= -\frac{1}{2} \langle \bar{\partial}_J u(v), J \circ \bar{\partial}_J u(w) \rangle - \frac{1}{2} i \langle \bar{\partial}_J u(v), \bar{\partial}_J u(w) \rangle.
\]
This is because of the formula (6) and the fact that, since $du(z_0)$ is complex anti-linear, at $z_0$ we have

$$\overline{\partial}_J u(z_0) = -P(u)(z_0) = 2du(z_0).$$

In particular, since vectors $\overline{\partial}_J u(v)$ and $J \circ \overline{\partial}_J u(v)$ are orthogonal we obtain that the quadratic function

$$\theta(v,v) = -\frac{1}{2} i \langle \overline{\partial}_J u(v), \overline{\partial}_J u(v) \rangle$$

is purely imaginary. Moreover, one easily verifies that

$$\theta(j \circ v, j \circ v) = \theta(v,v). \quad (18)$$

Therefore, because of (17) the expression $\partial_M \circ \overline{\partial}_\Sigma f(v, du(v))$ is also purely imaginary at $z_0$. Thus one computes at $z_0$

$$\partial_M \circ \overline{\partial}_\Sigma f(v, du(v)) = i(d_j v d_{du(v)} f_1 + d_v d_{du(jv)} f_1)$$

$$+ i(d_v d_{du(v)} f_2 - d_j v d_{du(jv)} f_2),$$

where we have written $f = f_1 + i f_2$.

Now, one can easily see that at $z_0$

$$\partial_M \circ \overline{\partial}_\Sigma f(v, du(v)) = -\partial_M \circ \overline{\partial}_\Sigma f(j \circ v, du(j \circ v)).$$

This is a contradiction because of the equation (18). This proves that $u$ can not be a perturbed $J$-holomorphic curve.

**Remark 4.4** The Theorem 4.3 implies that the antipodal map $u : S^2 \rightarrow S^2$ where $S^2$ is equipped with the standard complex structure is not a perturbed $J$-holomorphic curve.

## 5 Hermitian Structures

Here we will review basic properties of Hermitian structures on almost-complex manifolds and apply them in our context. Apart from other sections of these notes we will consider almost-complex manifold $(M, J)$ with a fixed almost-complex structure $J$.

Let $E \rightarrow M$ be a complex vector bundle over $M$. Then the almost-complex structure $J$ induces the following decomposition

$$\Omega^k(M, E) = \bigoplus_{p+q=k} \Omega^{p,q}(M, E),$$

11
where $\Omega^k(M, E)$ denotes the space $\Omega^k(M) \otimes_C E$ of smooth $E$-valued $k$-forms and $\Omega^{p,q}(M, E)$ denotes its subset of all $k$-form complex linear with respect to $p$ arguments and complex anti-linear with respect to $q$ arguments.

Let $\nabla : C^\infty(M, E) \to \Omega^1(M, E)$ be a covariant derivative on the vector bundle $E$. We can decompose $\nabla$ as

$$\nabla = \partial\nabla + \overline{\partial}\nabla,$$  \hfill (19)

into complex linear part and complex anti-linear part, respectively.

The complex linear part $\partial\nabla : C^\infty(M, E) \to \Omega^{1,0}(M, E)$ is given by the formula

$$\partial\nabla = \frac{1}{2} (\nabla - i \nabla \circ J)$$  \hfill (20)

and the complex anti-linear part $\overline{\partial}\nabla : C^\infty(M, E) \to \Omega^{0,1}(M, E)$ is given by the formula

$$\overline{\partial}\nabla = \frac{1}{2} (\nabla + i \nabla \circ J).$$  \hfill (21)

**Definition 5.1** An operator $D'' : C^\infty(M, E) \to \Omega^{0,1}(M, E)$ is said to be a Cauchy-Riemann operator if

$$D''(fs) = \overline{\partial} f \otimes s + f \overline{D''} s,$$  \hfill (22)

for every $f \in C^\infty(M, \mathbb{C})$ and any $s \in C^\infty(M, E)$.

**Lemma 5.2** The complex anti-linear part $\overline{\partial}\nabla$ of the covariant derivative $\nabla$ is a Cauchy-Riemann operator on $E$.

**Definition 5.3** A Hermitian metric $h$ in $E$ is a smooth family of Hermitian inner products in the fibers of the vector bundle $E$.

As a example, let $(M, J)$ be a symplectic manifold and let $J$ be $\omega$-compatible almost-complex structure on $M$. Then the tensor $\langle \cdot, \cdot \rangle$ given by the formula

$$\langle u, v \rangle = \omega(Ju, v) + i \omega(u, v)$$  \hfill (23)

defines a Hermitian metric on the tangent bundle $TM$ with respect to the almost-complex structure $J$. Compare this with the formula (6).

Any Hermitian metric $h$ on $E$ determines the Hermitian connection $\nabla_h = \nabla$ on $E$. It is a unique connection

$$\nabla : C^\infty(M, E) \to \Omega^1(M, E)$$
which preserves both the complex structure of $E$ and the metric $Re h$ (the real part of $h$) induced by the Hermitian structure $h$.

Alternatively, the Hermitian connection is a unique connection $\nabla$ such that

$$ dh(u, v) = h(\nabla u, v) + h(u, \nabla v). \quad (24) $$

Here is the basic fact about Hermitian connections:

**Proposition 5.4** For every Cauchy-Riemann operator $D'' : \mathcal{C}^\infty(M, E) \to \Omega^{0,1}(M, E)$ there exists a unique Hermitian connection $\nabla$ such that its complex anti-linear part

$$ \overline{\partial}_h = \frac{1}{2}(\nabla + i \nabla \circ J) $$

is equal to $D''$.

A Hermitian metric $h$ on a complex bundle $E$ induces the Hermitian metric $h^*$ on the complex dual bundle $E^*$. Namely, If $e = (e_i)$ is a unitary frame for $E$, $e^* = (e^*_j)$ the dual frame for $E^*$, then set

$$ h^*(e_i^*, e_j^*) = \delta_{ij}. $$

If we identify $E$ with $E^*$ (in complex anti-linear way) via formula

$$ s \mapsto s^* = h(\cdot, s) \quad (25) $$

then the formula (24) implies that the Hermitian connection $\nabla^*$ on the dual bundle $E^*$ is uniquely determined by the requirement:

$$ d \langle t, s \rangle = \langle \nabla^* t, s \rangle + \langle t, \nabla s \rangle \quad (26) $$

for $t \in \mathcal{C}^\infty(M, E^*)$ and $s \in \mathcal{C}^\infty(M, E)$.

This is so because under the identification (25) the connection $\nabla$ corresponds to a connection which is both preserving the metric and the complex structure on $E^*$ and thus it corresponds to $\nabla^*$

**Theorem 5.5** Let $(M, J)$ be an almost-complex manifold and let $h$ be a Hermitian metric on the tangent bundle $E = TM$. Then the complex anti-linear part

$$ \overline{\partial}_{h^*} : \Omega^{1,0}(M) \to \Omega^{0,1}(\Omega^{1,0}(M)) \cong \Omega^{0,1}(M) \otimes \Omega^{1,0}(M) $$

of the dual Hermitian connection $\nabla^*$ is given by the following formula

$$ \iota_X(\overline{\partial} \eta) = -\frac{1}{2}(\iota_X(d \eta) + i \iota_J X(d \eta)) \quad (27) $$

for any vector field $X$. 

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Note that we have identified the tangent space $TM$ with bundle $\Omega^{1,0}(M)$ of complex linear forms via formula (25). Proof: The formula (26) implies that for any complex linear form $\eta \in \Omega^{1,0}(M)$ and for any vector field $X$ we have
\[ d(\iota_X \eta) = \iota_X (\nabla^* \eta) + \eta(\nabla X). \] (28)

Let us fixed an arbitrary point $m \in M$ and let $X_m \in T_m(M)$ and $Y_m \in T_m(M)$ be any two tangent vectors at $m$. Choose a smooth map
\[ \sigma : (-\epsilon, \epsilon)^3 \to M \]
with $\sigma(0,0,0) = m$ satisfying the following three conditions:
1. $d\sigma_{(0,0,0)}([1,0,0]) = X_m, \quad d\sigma_{(0,0,0)}([0,1,0]) = JX_m$,
2. $d\sigma_{(0,0,0)}([0,0,1]) = Y_m$,
3. $\eta(d\sigma_{(s,t,0)}([0,0,1])) = \text{const}$.

Let $X_1, X_2,$ and $Y$ denote vector fields $d\sigma([1,0,0]), d\sigma([0,1,0]),$ and $d\sigma([0,0,1])$, respectively. With this notation the condition 3. above implies that
\[ d(\iota_Y \eta)(X_m) = d(\iota_Y \eta)(JX_m) = 0. \]

Therefore combining general identities for exterior derivative
\[ \iota_{X_i}(d\eta)(Y) = d\eta(X_i,Y) = d(\iota_Y \eta)(X_i) - d(\iota_X \eta)(Y) - \eta([X_i,Y]) \]
with the identity (28) and noting that commutators $[X_i,Y]$ are trivial we obtain
\[ \iota_{X_i}(d\eta)(Y) = -\iota_{X_i}(\nabla^* \eta)(Y) - \eta(\nabla X_i)(Y) \]
for any tangent vector $Y$ at $m$.

Thus at the point $m$ we have
\[ \iota_{X_m}(d\eta) = -\iota_{X_m}(\nabla^* \eta) - \eta(\nabla X_m) \] (29)
and
\[ \iota_{JX_m}(d\eta) = -\iota_{JX_m}(\nabla^* \eta) - \eta(\nabla (JX_m)). \] (30)

Now notice that the expression $\eta(\nabla X)$ is complex linear with respect to the variable $X$ since the metric $\nabla$ is Hermitian and the 1-form $\eta$ is chosen to be complex linear. Therefore combining identities (25) and (30), at the point $m \in M$ we obtain the equation
\[ \iota_{X_m}(\overline{\partial} \eta) = -\frac{1}{2}(i\iota_{X_m}(d\eta) + i\iota_{JX_m}(d\eta)). \]

Since this equality is true for any point $m \in M$ and any tangent vector $X_m \in T_mM$ the theorem follows.
Proposition 5.6 Let \((M, \omega)\) be a symplectic manifold of the dimension \(2n\) with a compatible almost-complex structure \(J\). Denote by \(h'\) a Hermitian metric on the tangle bundle \(TM\) by the equation (23) and by \(\Phi : TM \cong \Omega^{1,0}(M)\) the isomorphism given by the equation (24). Let \(f : M \to \mathbb{C}\) be a complex function of the form \(f = z\zeta\) for some complex number \(z\) and a real valued function \(\zeta\) such that \(\partial(f)(m) \neq 0\) for \(m \in M\). Then there is an open set \(U\) with \(m \in U\) and a Hermitian metric \(h\) on the set \(U\) with the properties:

\begin{itemize}
  \item \(h(X_1, X_2) = h'(X_1, X_2)\) for any two vector fields \(X_i, i = 1, 2\) with \(h'(X_1, \Phi^{-1} \circ \partial(f)) \equiv 0\) on \(U\). Moreover, \(h'(X, \Phi^{-1} \circ \partial(f)) = 0\) if and only if \(h(X, \Phi^{-1} \circ \partial(f)) = 0\) for any vector field \(X\) on \(U\).
  \item The map

\[
U \to \mathbb{R}
\]

\[
u \mapsto h(\Phi^{-1} \circ \partial(f), \Phi^{-1} \circ \partial(f)) = h(\Phi^{-1} \circ \partial(f)(m), \Phi^{-1} \circ \partial(f)(m))
\]

is constant.

\item \(\nabla(\Phi^{-1} \circ \partial(f)) \in TM \otimes_J \Omega^{1,0}(U)\), where \(\nabla\) denotes the Hermitian connection of the metric \(h\).
\end{itemize}

Proof: We first note that the first two conditions determine the metric \(h\) uniquely. Thus we only need to show that \(\nabla(\Phi^{-1} \circ \partial(f))\) is complex linear. Without loss of generality we may assume that \(f\) is a purely imaginary function: \(f = i\zeta\). Next, we note that

\[
h'(\cdot, \Phi^{-1} \circ \partial(f)) = \partial \zeta.
\]

Let

\[
\eta := h(\cdot, \Phi^{-1} \circ \partial(f)).
\]

By the construction the forms \(\partial \zeta\) and \(\eta\) differ only by a factor of real function and thus have the same zero sets. Using the Theorem 5.5 it is enough to show that

\[
d(\eta) = 0.
\]

Define function \(g : M \to \mathbb{R}\)

\[
g(x) = \begin{cases} 
0 & \text{if } x \in \zeta^{-1}(m) \\
t & \text{where } t \text{ is a time needed to travel from the level } \zeta^{-1}(m) \text{ to } x \text{ along the flow of } \Phi^{-1} \circ \partial(f). 
\end{cases}
\]
By construction of the form \( \eta \) if \( \eta(Y) = 0 \) then the vector \( Y \) is tangent to a level set of the function \( g \). Now since \( g \) is a real function we have
\[
\oint_{\gamma} \partial g = 0
\]
over any closed (contractible) loop \( \gamma \). This implies that the function \( \rho : M \to \mathbb{C} \)
\[
\rho(x) = \int_{m}^{x} \partial g
\]
is well defined in a (contractible) neighborhood of \( m \). Moreover simple computation shows that
\[
d\rho = \eta.
\]
This finishes the proof of the Proposition since we have now
\[
d\eta = dd\rho = 0.
\]

For vector field whose connection is of type \((1, 0)\) we have the following:

**Lemma 5.7** Let \((M, J)\) be an almost-complex manifold of dimension \(2n\) and let \( h \) be a Hermitian metric defined on the tangle bundle \( TM \). Let \( X \) be a vector field on \( M \). Assume that
\[
\nabla(X) \in TM \otimes \Omega^{1,0}(M).
\]
Then for any point \( m \in M \) and any tangent vector \( Y_m \in T_m(M) \) the torsion \( T(X, Y_m) \) is trivial.

**Proof:** Choose a vector field \( Y \) which agree with the tangent vector \( Y_p \) at the point \( m \) such that commutators \([X, Y]\) and \([JX, Y]\) are trivial. Using the fact that the Torsion tensor \( T(X, Y) \) is complex anti-linear with respect to the two variables \( X \) and \( Y \) (see the Section 7) we have
\[
JT(X, Y_m) = -T(JX, Y) = -\nabla_{JX}(Y) + \nabla_{Y}(JX) = -J(\nabla_{X}(Y) - \nabla_{Y}(X)) = -JT(X, Y_m).
\]
This shows that \( T(X, Y_m) = 0 \).
6 Compactness

Let us consider the following weak version of the Gromov’s Compactness Theorem [G]

**Theorem 6.1** Let \((M, \omega)\) be a compact symplectic manifold and let \(J_k\) be a sequence of \(\omega\)-tame almost complex structures which converge to \(J_\infty\) in \(C^\infty\)-topology. Then for any sequence \(u_k: \Sigma \to M\) of \(J_k\)-holomorphic curves with uniformly bounded energy there are subsequence (still denoted by \(u_k\)), a finite collection \((u^1, \ldots, u^m)\) of \(J_\infty\)-holomorphic spheres \(u^i: S^2 \to M\), and a \(J_\infty\)-holomorphic curve \(u^\infty: \Sigma \to M\) such that for corresponding homology classes in \(H_2(M)\) we have

\[
[u_k] = [u^\infty] + [u^1] + \ldots + [u^m]
\]

for all \(k\) large enough.

We want to prove a similar theorem for perturbed \(J\)-holomorphic curves satisfying the equation (11). Let us describe an extension of Gromov’s nice trick to perturbed \(J\)-holomorphic curves. Consider a solution \(u: \Sigma \to M\) of the differential equation (11). To such \(u\) we associate a map \(\bar{u}: \Sigma \to \Sigma \times M\) given by the formula

\[
\bar{u}(z) = (z, u(z)),
\]

for \(z = (s, t) \in \Sigma\). Then \(\bar{u}\) satisfies the nonlinear Cauchy-Riemann equation

\[
\overline{\partial}_j(\bar{u}) = d\bar{u} + \bar{J} \circ d\bar{u} \circ j = 0,
\]

where \(\bar{J}\) is almost-complex structure on the product \(\Sigma \times M\) given by the formula

\[
\bar{J} = \begin{bmatrix} j & 0 \\ -Pj & J \end{bmatrix}.
\]

To check that \(\bar{u}\) satisfies the equation (11), note that

\[
\overline{\partial}_j(\bar{u}) = \begin{bmatrix} 0 \\ \overline{\partial}_j(u) + P(u) \end{bmatrix}.
\]

Choose a symplectic form \(\omega_0\) on \(\Sigma\) such that the complex structure \(j\) on \(\Sigma\) is compatible with \(\omega_0\) and such that \(\omega_0([\Sigma]) = 1\). Define a symplectic structure \(\bar{\omega}\) on \(\Sigma \times M\) by the formula

\[
\bar{\omega} = N\omega_0 + \omega,
\]

where \(N\) is a positive number.
Lemma 6.2 If $N$ is large enough, then the almost-complex structure $\tilde{J}$ is $\tilde{\omega}$-tame.

Proof: Define
\[ f = ||\omega||_{L^\infty} \sup_{(z,m)\in \Sigma \times M} ||P(z, m)|| \]
and compute
\[
\tilde{\omega} \left( \tilde{J} \left[ \begin{array}{c} a \\ v \end{array} \right] , \left[ \begin{array}{c} a \\ v \end{array} \right] \right) = N\omega_0(j(a), a) + \omega(P(a) + Jv, v)
\geq N|a|^2 - |a|f|v|_J + |v|^2_J
= ((N)^{1/2}|a| - |v|)^2 + (2(N)^{1/2} - f)|a||v|
\geq (2(N)^{1/2} - f)|a||v|.
\]
Here we have used the notation
\[ |v|^2 = |v|_J^2 = \sup_z \omega(J(z)v, v). \]
To finish the proof of the lemma it is enough to choose $N > \left( \frac{f}{4} \right)^2$.

So we choose $N$ large enough so that the almost-complex structure $\tilde{J}$ is $\tilde{\omega}$-tame. In this situation there is a Riemannian metric defined on the product $\Sigma \times M$ via the formula
\[ \langle v, w \rangle_J = \frac{1}{2} \left( \tilde{\omega}(\tilde{J}v, w) + \tilde{\omega}(\tilde{J}w, v) \right), \]
for tangent vectors $v, w$.

The energy of $\tilde{u}$ is defined as
\[ E(\tilde{u}) = \frac{1}{2} \int \int |D\tilde{u}|^2_J. \]
In fact the energy depends only on a homology class $[\tilde{u}]$ of $\tilde{u}$ and we have
\[ \tilde{\omega}(\tilde{u}) = E(\tilde{u}) \geq 0. \]

Theorem 6.3 Let $(M, \omega)$ be a compact symplectic manifold and let $J_\nu$ be a sequence in $\mathcal{J}$ of $\omega$-compatible families of almost complex structures which converge to $J_\infty$ in $C^1$-topology. Let $P_\nu$ be a sequence of perturbations of the form
\[ P_\nu \in C^\infty(\Sigma \times M, \Omega^0_{\Sigma} \otimes J_\nu TM) \]
which converge to $P_\infty$ in $C^1$-topology. Then for any sequence $u_\nu : \Sigma \to M$ of solutions of the equation (11) with $P = P_\nu$, $J = J_\nu$ such that $[u_\nu] = A \in H_2(M)$ there is subsequence (still denoted by $u_\nu$) and a finite collection $(u_\infty; C_1, \ldots, C_m)$, where $u_\infty$ is a of solution of the equation (11) with $P = P_\infty$ and $J = J_\infty$ and $C_i$ are $J_\infty(z_i)$-holomorphic spheres $C_i : S^2 \to M$ such that for corresponding homology classes in $H_2(M)$ we have

$$A = [u_\infty] + [C_1] + \ldots + [C_m]. \tag{32}$$

Proof: Consider corresponding elements $\tilde{u}_\nu$, $\tilde{J}_\nu$, and $\tilde{J}_\infty$. Since $J_\nu \to J_\infty$, $P_\nu \to P_\infty$, and all $u_\nu$ represent the same homology class $A$ the energy of the sequence $\tilde{u}_\nu$ is uniformly bounded for almost all $\nu$ and we may apply the Gromov’s theorem 6.1. Thus there is a collection $(\tilde{u}^i : S^2 \to \Sigma \times M)$, $i = 1, \ldots, m$, of $\tilde{J}_\infty$-holomorphic spheres $\tilde{J}_\infty$-holomorphic curve $\tilde{u}_\infty : \Sigma \to \Sigma \times M$ such that

$$A + [\Sigma] = [\tilde{u}_\nu] = [\tilde{u}_\infty] + \sum_{i=1}^{m} [\tilde{u}^i]$$

in $H_2(\Sigma) \oplus H_2(M) \subseteq H_2(\Sigma \times M)$. For every $i$, we can write a unique decomposition

$$[\tilde{u}^i] = A^i + B^i,$$

and also a unique decomposition

$$[\tilde{u}_\infty] = A^\infty + B^\infty,$$

where $A^i, A^\infty \in H_2(M)$ and $B^i, B^\infty \in H_2(S^2)$. In particular, we have

$$A^\infty + \sum_{i=0}^{m} A^i = A, \quad B^\infty + \sum_{i=0}^{m} B^i = [\Sigma].$$

Now, the class $[\Sigma]$ is indecomposable in $H_2(\Sigma)$. Therefore $B^\infty = [\Sigma]$ and $B^i = 0$. Thus the map $pr_1 \circ \tilde{u}_\infty : \Sigma \to \Sigma$, where $pr_1 : \Sigma \times M \to \Sigma$ is the natural projection on the first factor, is a holomorphic of degree one. Eventually reparametrizing the map $\tilde{u}_\infty$ we may assume that the map $pr_1 \circ \tilde{u}_\infty$ is the identity map on $\Sigma$. With this reparametrization define

$$u_\infty = pr_2 \circ \tilde{u}_\infty,$$

where $pr_2 : \Sigma \times M \to M$ is the natural projection on the second factor. It is easy to verify now, that $u_\infty$ such defined satisfies the equation (11) with $P = P_\infty$ and $J = J_\infty$. 

Consider now maps $\tilde{u}^i$, for $i = 1, \ldots, m$. The projections $pr_1 \circ \tilde{u}^i$ are holomorphic maps from $S^2$ to $\Sigma$ of degree zero. Therefore they must be constant:

$$pr_1 \circ \tilde{u}^i(S^2) = z_i \in \Sigma.$$ 

This easily implies that $pr_2 \circ \tilde{u}^i : S^2 \to M$ is $J(z_i)$-holomorphic sphere. Define

$$C_i = pr_2 \circ \tilde{u}^i : S^2 \to M.$$ 

This finishes the proof of the theorem since the equation (32) is obvious now.

**Remark 6.4** If $\Sigma$ is a Riemann sphere $S^2$ then the above theorem is true if we replace homology classes in $H_2(M)$ by homotopy classes in $\pi_2(M)$.

**Remark 6.5** Note that we have the following identity:

$$\omega(C_i) = \tilde{\omega}(\tilde{u}^i).$$

### 7 Linearization

We will examine the moduli space

$$\mathcal{M}(A, J, P)$$

of all solutions of the equation (11) which represent a given homology class $A \in H^2(M)$. In order to do it we need study the linearization of the perturbed Cauchy-Riemann operator.

In general, there is no unique way to construct such linearization since we have to define a way to identify, for each fixed $z \in \Sigma$, the fiber

$$L^p \left( \Omega^{0,1}(z) \otimes J(z) T_{u(z)} M \right)$$

with that of the form

$$L^p \left( \Omega^{0,1}(z) \otimes J(z) T_m M \right)$$

for any $m$ close to $u(z)$.

Perhaps, it will be the best for us if we choose such identification based on the family of Hermitian connections $\nabla(z)$ relative to the family of almost complex structures $J(z)$. Recall that a Hermitian connection is a connection that preserves a Hermitian metric with respect to the $J(z)$. In contrast to
the Levi-Civita connection its torsion tensor $T$ is, in general, nontrivial and, moreover, it is complex anti-linear in two variables, i.e.

$$T(J\xi,\eta) = T(\xi, J\eta) = -JT(\xi, \eta).$$

To describe the linearization based on Hermitian connections let us choose $u \in \mathcal{M}(A, J, P)$ and $z \in \Sigma$. Since the hermitian connection $\nabla(z)$ preserves the almost-complex structure $J(z)$ the map

$$\Phi_u(z) : L^p((\exp(\hat{u}))^*(\Omega^{0,1} \otimes J TM)) \to L^p(\Omega^{0,1} \otimes J u^*TM)$$

induced by the parallel transport along the geodesic curve $t \to \exp(t\hat{u})$ corresponding to the Hermitian metric at $z$ on $M$ is well-defined. Thus in the neighborhood of $u$ the perturbed Cauchy-Riemann $\partial J,P$ is represented by the map

$$F : W^{1,p}(u^*TM) \to L^p(\Omega^{0,1} \otimes J u^*TM)$$

defined by

$$F(\hat{u}) = \Phi_u(z)(\overline{\partial}_{J,P}(\exp(\hat{u}))).$$

The linearization at $u$

$$D_u : W^{1,p}(u^*TM) \to L^p(\Omega^{0,1} \otimes J u^*TM)$$

is defined as $D_u(\hat{u}(z)) = dF(0)(\hat{u}(z))$. It is not hard to compute $D_u$ (See [MS] for $J$-holomorphic curves).

**Proposition 7.1** If $u : \Sigma \to M$ satisfies the equation (11) with the exact perturbation term $P = \overline{\partial}_{\Sigma}X$ and $\hat{u} \in C^\infty(u^*TM)$ then the operator $D_u(\hat{u})$ can be written as

$$D_u\hat{u} = \nabla^*(\hat{u}) + T(du, \hat{u}) + \nabla_{\hat{u}}(d\Sigma X) + J \circ (\nabla^*(\hat{u}) + T(du, \hat{u}) + \nabla_{\hat{u}}(d\Sigma X)) \circ j,$$

where $\nabla^*$ denotes the induced connection on $u^*(TM)$ via the map $(id, u)$.

**Lemma 7.2** For $u \in \mathcal{M}(A, J, P)$ the operator $D_u : W^{1,2}(u) \to L^2(u)$ is elliptic. Here $W^{1,2}(u)$ and $L^2(u)$ denote spaces $W^{1,2}(u^*TM)$ and $W^{0,2}(\Omega^{0,1} \otimes J u^*TM)$, respectively. Thus the operator $D_u$ is Fredholm.

**Proof:** Since the vector bundles $W^{1,2}(u^*TM)$ and $W^{0,2}(\Omega^{0,1} \otimes J u^*TM)$ are defined over the compact manifold $\Sigma$ it is enough to notice that main symbol of the operator $F(u)$ is elliptic. From general theory of elliptic operators defined on vectors bundles over a compact manifold follows that $F(u)$ is also Fredholm.
Remark 7.3 By the elliptic regularity it follows that the map
\[ D_u : W^{1,p}(u^*TM) \to L^p(\Omega^{0,1} \otimes_f u^*TM) \]
is also Fredholm.

Here is the basic fact about properly exact perturbations.

**Theorem 7.4** Let the perturbation term \( P \) be properly exact, \( P = \partial_\Sigma X \).

Then there exists a family of Hermitian connections such that
\[ D_u((id, u)^* X) \]
is closed to \( (id, u)^*(\partial_\Sigma X) \). so \( (id, u)^*(\partial_\Sigma X) \) is in the range of \( D_u \).

**Proof:** If \( X(z, u(z)) = 0 \) then choose a Hermitian connection \( \nabla(z) \) corresponding to the Hermitian metric determined by \( (J(z), \omega) \). If \( \|X(z, u(z))\| > \epsilon \) then choose the perturbed Hermitian connections (in the neighborhood of \( u(z) \)) given by the Proposition 5.6. For the latter connections the Proposition 5.6 and the Lemma 5.7 imply that
\[ J \nabla_Y(X) = \nabla_{J_Y}(X). \]

Therefore for these connections we have
\[ D_u((id, u)^* X) = (id, u)^*(\partial_\Sigma X) + \nabla_{du + Jodu + P(u)}(X). \]

The last expression is zero. Now make \( \epsilon \) small enough and choose Hermitian connections for \( z \) satisfying \( 0 < \|X(z, u(z))\| < \epsilon \) so that we get a smooth bounded (by a constant independent of the choice of \( \epsilon \)) family of Hermitian connections satisfying the conclusion of the Theorem.

## 8 Compactness Properties of the space \( V \)

We will study the space of smooth functions (real valued, for simplicity, but all applies to the space of function of the form \( zf \) where \( z \) is a fixed complex number and \( f \) is arbitrary real valued function) defined over the set \( \Sigma \times M \).

Choose a decreasing sequence \( \epsilon_k > 0 \) and consider the subspace \( C^\infty_c(\Sigma \times M) \) of smooth functions \( f \in C^\infty(\Sigma \times M) \) which satisfy
\[ \|f\|_\epsilon^2 = \sum_{k=0}^{\infty} \epsilon_k \left\langle \nabla^k f, \nabla^k f \right\rangle < \infty, \]
where $\nabla^k$ denotes $k$-th hermitian covariant derivative determined by the metric on the manifold $\Sigma \times M$. This defines a separable Hilbert space of the subspace of smooth functions defined on $\Sigma \times M$ and induces topology on the space $C^\infty(\Sigma \times M)$. For a given sequence $\epsilon_k$ we will call this topology the $\epsilon$-topology, and the space $C^\infty(\Sigma \times M)$ with the $\epsilon$-topology will be denoted by $V$. Following Floer \cite{F} one can choose a sequence $\epsilon_k$ such that the space $V$ is a dense subset of $L^p(\Sigma \times M)$, for $p > 2$.

**Proposition 8.1** For every positive number $K$ the $\epsilon$-open set

$$V(K) = \{X \in V \mid \|f\|_\epsilon < K\}$$

is relatively compact in the $C^\infty$-topology. Moreover, if $f_k \to f_\infty$ in $C^\infty$-topology and $\|f_k\|_\epsilon \leq K$ for all $k$, then $\|f_\infty\|_\epsilon \leq K$.

**Proof:** To show that the set $V(K)$ is relatively compact in the $C^\infty$-topology we will use the method of diagonal subsequence. For each natural $n$ let us introduce a norm $\|\cdot\|_{\epsilon, n}$ by the formula

$$\|f\|_{\epsilon, n}^2 = \sum_{k=0}^{n} \epsilon_k \langle \nabla^k f, \nabla^k f \rangle.$$

Each norm $\|\cdot\|_{\epsilon, n}$ is equivalent to the corresponding Sobolev norm. Since all functions have support in the compact set $\Sigma \times M$ then the natural embedding $W(\epsilon, n) \to W(\epsilon, m)$ is compact if $n > m$. Here $W(\epsilon, n)$ denotes the completion in the norm $\|\cdot\|_{\epsilon, n}$.

Let $f_k$ be a sequence of smooth functions such that $\|f_n\|_\epsilon < K$ for every natural number $k$. Then it is bounded in the $\|\cdot\|_{\epsilon, 2}$-norm and by the above remark there exists a subsequence $f_{k}^1$ convergent in the $\|\cdot\|_{\epsilon, 1}$-norm. Next, the sequence $f_{k}^1$ is bounded in $\|\cdot\|_{\epsilon, 3}$-norm so we can choose a subsequence $f_{k}^2$ of the sequence $f_{k}^1$ convergent in the $\|\cdot\|_{\epsilon, 2}$-norm.

Continuing this process we will obtain for each $l > 1$ a subsequence $f_{k}^l$ of the sequence $f_{k}^{l-1}$ which is convergent in the $\|\cdot\|_{\epsilon, l}$-norm. Choose the diagonal subsequence $f_{k}^k$. It is convergent in the $\|\cdot\|_{\epsilon, l}$-norm for every $l$. Therefore it is convergent in the $C^\infty$-topology. This proves relative compactness in the $C^\infty$-topology.

To show that if $f_k \to f_\infty$ in $C^\infty$-topology and $\|f_k\|_\epsilon \leq K$ for all $k$, then $\|f_\infty\|_\epsilon \leq K$, it is enough to notice that

$$\|f\|_\epsilon^2 = \lim_{n \to \infty} \|f\|_{\epsilon, n}^2$$

for every $f$. This proves the proposition.
9 Universal Moduli Space

We will study the universal moduli space

\[ M(A, J) = \{(u, f) \in X^{1,p} \times V \mid \bar{\partial}_{J,f}(u) = 0\}, \]

where \( u \) is solution of the equation (11) corresponding to homology class \( A \in H_2(M) \) and to the family of almost complex structures \( J \). Here \( V \) denotes the space of functions \( f \) as described in the Proposition 8.1.

Consider the infinite dimensional vector bundle \( E \to X^{1,p} \times V \) whose fiber at the point \( (u, f) \) is the space \( E(u, f) = L^p(\Omega^1 \otimes J^* u^* TM) \) of \( L^p \)-sections of the vector bundle \( \Omega^1 \otimes J^* (u^* TM) \) over \( \Sigma \). Then the moduli space \( M(A, J) \) is a zero set \( F^{-1}(0) \) of the section of the vector bundle given by the formula

\[ F(u, f) = \bar{\partial}_{J,f}(u). \]

If the point \( (u, f) \) is zero of the section \( F \) then the differential at this point

\[ D(F)(u, X) : W^{1,p}(u^* TM) \times V \to E(u, f), \]

is given by the formula

\[ D(F)(u, f)(\hat{u}, \hat{f}) = D_u\hat{u} + (id, u)^* \left( \bar{\partial}_{\Sigma}(\Phi^{-1} \circ \partial_M \hat{f}) \right), \quad (33) \]

since \( \bar{\partial}_{\Sigma} \) is linear.

Let \( L \) denote the codimension one subspace of the space \( V \) orthogonal to the vector \( f \) and define the operator

\[ D_L(F)(u, f) : W^{1,p}(u^* TM) \times L \to E(u, f) \]

as the restriction of the operator \( D(F)(u, f) \) to the subspace \( W^{1,p}(u^* TM) \times L \).

**Proposition 9.1** If the point \( (u, f) \) is zero of the section \( F \) and \( u \) is a map with \( u^*(\omega) \neq 0 \) then the linear operator \( D_L(F)(u, f) \) is onto for suitable chosen family of Hermitian connections.

**Proof:** We claim that the operators \( D(F)(u, f) \) and \( D_L(F)(u, f) \) have the same range. Indeed, let \( \xi = D_u(\hat{u}) + \bar{\partial}_{\Sigma}(\Phi^{-1} \circ \partial_M \hat{f}) \). Write \( \hat{f} = tf + g \), where \( g \in L \). By the Theorem [7,4] we have

\[ D_u((id, u)^* \Phi^{-1} \circ \partial_M f) = (id, u)^* (\bar{\partial}_{\Sigma} \Phi^{-1} \circ \partial_M f). \]
Therefore,
\[
\xi = D_u(\hat{u}) + (id, u)^*(\overline{\partial}_\Sigma(t\Phi^{-1} \circ \partial_M f)) + (id, u)^*(\overline{\partial}_\Sigma(\Phi^{-1} \circ \partial_M g))
\]
\[
= D_u(\hat{u}) + (id, u)^*(t\Phi^{-1} \circ \partial_M f)) + (id, u)^*(\overline{\partial}_\Sigma(\Phi^{-1} \circ \partial_M g))
\]
\[
= D_L(\mathcal{F})(u, f)(\hat{u} + (id, u)^*(t\Phi^{-1} \circ \partial_M f), g).
\]

Thus it is enough to show that the range of \(D(\mathcal{F})(u, f)\) is equal to \(L^p(\Omega^0, \otimes J u^*TM)\). Since \(D_u\) is a Fredholm operator, by the Remark 7.3 the operator \(D(\mathcal{F})\) has a closed range and it is enough to prove that the range is dense.

Using the Hahn-Banach Theorem it is enough to show that if \(\eta \in L^q(\Omega^0, \otimes J u^*TM)\) with \(\frac{1}{p} + \frac{1}{q} = 1\) satisfies
\[
\int \langle \eta, D_u \hat{u} \rangle = 0
\]
and
\[
\int \langle \eta, \overline{\partial}_\Sigma(\Phi^{-1} \circ \partial_M f) \rangle = 0 \tag{34}
\]
for every \(\hat{u} \in W^{1,p}(u^*TM)\) and every \(\hat{f} \in V\), then \(\eta \equiv 0\). From the first equation we obtain that \(\eta\) is a week solution of \(D_u^*\eta = 0\). However, the coefficients of the first order terms of \(D_u\) are of class \(C^\infty\) and the same is true for the adjoint \(D_u^*\). Thus by elliptic regularity \(\eta\) satisfies the equation \(D_u^*\eta = 0\) in the strong sense and, moreover, \(\eta\) is of class \(C^\infty\). Hence we can write
\[
D_u D_u^*\eta = \Delta \eta + \text{lower order terms} = 0
\]
and using the Aronszajn’s theorem [41] it is enough to show that \(\eta\) vanishes at some open set.

By the assumption there is an open set \(U \subset \Sigma\) such that the map \(u\) restricted to the set \(U\) is an embedding. Choose \(z_0 \in U\) such that \(\eta(z_0) \neq 0\). Let \(Y\) be a vector field on \(M\) with support in a small neighborhood of \(u(z_0)\). Choose polar coordinates \(z = r \exp(2\pi i \theta)\) on \(\Sigma\) with the property that if \(Y(r \exp(2\pi i \theta)) \neq 0\) then \(r_1 < r < r_2\) for some positive \(r_1\) and \(r_2\). Next, choose a function \(g : (0, \infty) \to \mathbb{R}\) with a compact support such that \(\int_0^\infty g(r)dr = 0\), and \(g(r) > 0\) for \(r_1 < r < r_2\).

Let \(f(r) = \int_0^r g(s)ds\). After complexification it means \(\overline{\partial}_\Sigma(f) = gd\overline{\zeta}\) and \(\overline{\partial}_\Sigma(Y f) = gd\overline{\zeta} \otimes J Y\). We may assume that if \(gd\overline{\zeta} \otimes J Y(z, u(z)) \neq 0\) then
\[
\langle \eta, gd\overline{\zeta} \otimes J Y \rangle (z) > 0.
\]

Thus if (34) holds \(\eta\) must be zero.

Now we are ready to prove the following theorem
Theorem 9.2 Let \((u, f)\) with \(||f||_\varepsilon = K\) satisfies \(F(u, X) = 0\) and \(du\) is of maximum rank at some point. Then there is a pair \((u_1, f_1)\) with \(||f_1||_\varepsilon < K\) such that \(F(u_1, f_1) = 0\).

Proof: We will need the following version of the implicit function theorem for Banach spaces.

Theorem 9.3 Let \(f : E_1 \times E_2 \to F\) be a smooth map between Banach spaces. Assume that the partial derivative \(D_1 f\) is surjective at the point \((e_1, e_2) \in E_1 \times E_2\) and admits a bounded right inverse. Then for every \(f_2 \in E_2\) near \(e_2\) there exists \(f_1 \in E_1\) such that \(f(e_1, e_2) = f(f_1, f_2)\).

Let \(L\) denote a tangent space at \(X\) to the sphere \(S(K)\) of radius \(K\) in the Hilbert space \(V\). Then by the Proposition 9.1 the partial derivative of \(F\) at \((u, f)\) in the direction \(W_{1,p}(u^*TM) \times L\) is onto. It has also a bounded right inverse since its restriction to the space \(W_{1,p}(u^*TM)\) is Fredholm by the Remark 7.3. Thus we apply the Implicit Function Theorem 9.3 to obtain a pair \((u_1, f_1)\) with \(||f_1||_\varepsilon < K\) such that \(F(u_1, f_1) = 0\).

Definition 9.4 The extended universal moduli space is the space
\[
\mathcal{M}_{ex}(A, J) \subset W^{1,p}(u^*TM) \times V \times \mathbb{R}
\]
of all triples \((u, f, \lambda)\) with \(f \in V\) such that \((u, f)\) is a solution of the equation \(F(u, f) = 0\) such that homology class \([u]\) is equal to \(A\) and \(||f||_\varepsilon = \lambda\).

The Theorem 9.2 implies

Proposition 9.5 Let \(\Lambda(A, J)\) be a set defined as the image of the projection
\[
\pi : \mathcal{M}_{ex}(A, J) \to \mathbb{R}
\]
on the third factor. Then \(\Lambda(A, J)\) is open in the set of all real numbers \(\mathbb{R}\).

10 Nonnegative Properties of Symplectic Area

In this section we will prove the main theorem of these notes. Consider a solution \(u\) of the nonlinear partial differential equation
\[
du + J \circ du \circ j + \partial X(u) = 0,
\]
with properly exact perturbation term.
Theorem 10.1 Let \( u \) be a solution of the equation (35) for some \( J \in \mathcal{J} \). Then the symplectic form \( \omega \) evaluated on the class \([u]\) is nonnegative:

\[
\int_{\Sigma} u^* \omega \geq 0.
\]

Proof: We will prove the theorem by arriving at a contradiction. Thus assume that

\[
\int_{\Sigma} u^* \omega < 0.
\]

Without loss of generality we may assume that \(|f|_\epsilon = 1\). Let us choose a constant \( h \) such that \( \omega(C) > h \) for any \( J(z)-\)holomorphic sphere \( C \) in \( M \).

Consider the set

\[
\Lambda_0 = \Lambda(A, J) \cap [0, 1].
\]

where \( A \) is a homology class of \( u \), \( A = [u] \). (See notation of the Proposition 9.5). Then the set \( \Lambda_0 \) is not empty since \( 1 \in \Lambda_0 \). By the Proposition 9.5 it is also open in \([0, 1]\). Let

\[
\lambda_1 = \inf \Lambda_0.
\]

By the definition of \( \lambda_1 \) there exists a sequence \((u_n, f_n, \lambda_n)\) of elements of the extended universal moduli space \( \mathcal{M}_{cx}(A, J) \) (Definition 9.4) such that \( \lambda_n \to \lambda_1 \).

By the compactness properties (Proposition 8.1) we can assume that \( f_n \to f^1 \), where \( f^1 \) satisfies

\[
||f^1||_\epsilon \leq \lambda_1.
\]

By the compactness Theorem 6.3 there exists a subsequence of \((u_n, f_n, \lambda_n)\) and the collection \((u^1; C^1_1, ..., C^1_{r(1)})\) where \( u^1 \) is a solution of the equation (35) with \( ||f^1||_\epsilon \leq \lambda_1 \) and \( C^1_1, ..., C^1_{r(1)} \) are \( J(z_i)-\)holomorphic spheres. We have

\[
[u] = [u^1] + [C^1_1] + ... + [C^1_{r(1)}]
\]

so

\[
\tilde{\omega}([\tilde{u}]) = \tilde{\omega}([\tilde{u}^1]) + \omega([C^1_1]) + ... + \omega([C^1_{r(1)})].
\]

See the Remark 6.5. We have

\[
\omega([u^1]) \leq \omega([u]) < 0.
\]

For the energy of \( \tilde{u}^1 \), \( \tilde{\omega}([\tilde{u}^1]) \), we obtain the following equality:

\[
\tilde{\omega}([\tilde{u}^1]) = \tilde{\omega}(A) - \omega([C^1_1]) - ... - \omega([C^1_{r(1)})].
\]
We claim that $\lambda_1 > 0$ since otherwise $u^1$ there would be a holomorphic curve with negative symplectic area which is impossible. Moreover we claim that $r(1)$, the number of $J(z_i)$-holomorphic spheres, is strictly positive since otherwise because of the inequality

$$||f^1||_\epsilon \leq \lambda_1$$

the Theorem 9.2 would imply that there was an element of the universal moduli space

$$\mathcal{M}_{ex}(A, J)$$

corresponding to the parameter $\lambda^1$. But this would contradict the fact that $\Lambda_0$ is an open set.

Assume that for a natural number $k \geq 1$ we have the following data which we will call $k$-data:

1. For each $1 \leq i \leq k$ there exists a triple

$$(u^i, f^i, \lambda^i) \in \mathcal{M}_{ex}(A^i, J).$$

2. For each $1 < i \leq k$ numbers $\lambda^i > 0$ satisfies

(a) $$\lambda^{i-1} > \lambda^i,$$

(b) $$\lambda^i = \inf \Lambda_{i-1},$$

where

$$\Lambda_{i-1} = \Lambda(A^{i-1}, J) \cap [0, 1],$$

3. For each $1 < i \leq k$ there exists a natural number $r(i) > 0$ and a sequence $(C^i_1, C^i_2, ..., C^i_{r(i)})$ of non constant $J(z)$-holomorphic spheres such that:

(a) $$\omega(A^{i-1}) = \omega(A^i) + \omega([C^i_1]) + ... + \omega([C^i_{r(i)}]),$$

(b) $$\tilde{\omega}(\tilde{u}^i) = \tilde{\omega}(A^{i-1}) - \omega([C^i_1]) - ... - \omega([C^i_{r(i)})].$$
If $\lambda^k > 0$ then, using the above method, we can produce the $(k + 1)$-data.
Assume that this process never stops i.e. for any natural number $n \in \mathbb{N}$ there is the $n$-data.

Then, by the definition of the constant $h$ and by the condition (3) of $n$-data, we obtain
\[
\tilde{\omega}(\tilde{u}^n) \leq \tilde{\omega}(A) - \omega(C^1) - \cdots - \omega(C^n) < \tilde{\omega}(A) - nh.
\]
Choose
\[
n > \frac{\tilde{\omega}(A)}{h}.
\]
Then we obtain that $\tilde{\omega}(\tilde{u}^n) < 0$. But this is a contradiction since the energy of a holomorphic curve can never be negative. Therefore, the process must stop somewhere i.e.
\[
\lambda^m = 0
\]
for some $m$.

However this can not happen by the same reason as $\lambda^1$ could never be zero and we have arrived at a contradiction. Therefore, $\int_\Sigma u^* \omega \geq 0$, and this completes the proof of the theorem.

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