An Ideal Class to Construct Solutions for Skew Brownian Motion Equations

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Abstract
This paper contributes to the study of stochastic processes of the class $(\Sigma)$. First, we extend the notion of the above-mentioned class to càdlàg semi-martingales, whose finite variation part is considered càdlàg instead of continuous. Thus, we present some properties and propose a method to characterize such stochastic processes. Second, we investigate continuous processes of the class $(\Sigma)$. More precisely, we derive a series of new characterization results. In addition, we construct solutions for skew Brownian motion equations using continuous stochastic processes of the class $(\Sigma)$.

Keywords
Class $(\Sigma)$ - Skew Brownian motion - Balayage formula - Honest time - Relative martingales

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Introduction

This study investigates semi-martingales of the class \((\Sigma)\). We consider stochastic processes \(X\) of the form

\[ X = M + A, \]

where \(M\) is a local martingale and \(A\) is an adapted finite variation process such that \(dA_t\) is carried by \(\{t \geq 0 : X_t = 0\}\). Such processes are frequently encountered in stochastic analysis. Well-known examples of such processes include càdlàg local martingales, the absolute value of a continuous martingale, the positive and negative parts of a continuous martingale, solutions of skew Brownian motion equations starting from zero, and the drawdown of a càdlàg local martingale with only negative jumps. These processes play an important role in many probabilistic studies, e.g., the study of zeros of continuous martingales [1], the theory of Azéma–Yor martingales, the resolution of Skorokhod’s reflection equation and embedding problem, and the study of Brownian local times.

The notion of the class \((\Sigma)\) has been studied extensively (see [3, 6, 10–15, 19]), and its extensions have been presented in the literature. It was first introduced by Yor [19] for continuous nonnegative submartingales and subsequently extended [20] to continuous semi-martingales. Nikeghbali [14] and Cheridito et al. [3] proposed and studied an extension of this notion to càdlàg semi-martingales. However, we have highlighted some shortcomings of the above-mentioned developments, which we attempt to address in this study. First, in all the above-mentioned cases, the finite variation part of a process of the class \((\Sigma)\) is always considered continuous. Meanwhile, Nikeghbali presented two remarkable characterization results (Theorem 2.1 of [14] and Proposition 2.4 of [15]). The drawback of these results is that they only characterize nonnegative submartingales of the class \((\Sigma)\). Finally, we focus on the construction of solutions of the following skew Brownian motion equations:

\[ X_t = x + B_t + (2\alpha - 1)L_0^0(X) \]  \hspace{1cm} (0.1)

and

\[ X_t = x + B_t + \int_0^t (2\alpha(s) - 1)dL_0^0(X), \] \hspace{1cm} (0.2)

where \(B\) is a standard Brownian motion, \(x = 0\), and \(L_0^0(X)\) denotes the symmetric local time at 0 of the unknown process \(X\). Recall that (0.1) appeared in the seminal work of Itô and Mckean [8] as a natural generalization of the Brownian motion and (0.2) was introduced by Weinryb in [18]; since then, they have been studied extensively [2, 5, 7, 8, 16, 17]. In this study, we focus on the solution given by Bouhadou and Ouknine [2], which is of the form

\[ X_t = Z_{|B_t|}, \]
where $Z$ is the progressive process $Z^\alpha$ that we shall recall later in (2.1), $B$ is a standard Brownian motion, and $\gamma_t = \sup\{s \leq t : B_s = 0\}$. We remark that $|B|$ is an element of the class $(\Sigma)$. Furthermore, many results obtained on the processes of the class $(\Sigma)$ are generally extensions of the results initially proved for the Brownian motion. Hence, an intuitive question is to determine whether it is possible to construct solutions for the skew Brownian motion equations using other process of the class $(\Sigma)$ than $|B|$.

This study aims to contribute to the literature in the sense of the above-mentioned remarks. First, we present a general framework to study a larger class of càdlàg stochastic processes. More precisely, we propose extending the definition of Cheridito et al. [3] by weakening the continuity condition on the finite variation part of processes of the class $(\Sigma)$. Furthermore, many results obtained on the processes of the class $(\Sigma)$ are generally extensions of the results initially proved for the Brownian motion. Hence, an intuitive question is to determine whether it is possible to construct solutions for the skew Brownian motion equations using other process of the class $(\Sigma)$ than $|B|$.

**Definition 1** We say that a semi-martingale $X$ is of the class $(\Sigma)$ if it decomposes as $X = M + A$, where

1. $M$ is a càdlàg local martingale, with $M_0 = 0$;
2. $A$ is an adapted càdlàg predictable process with finite variations such that $A_0^- = A_0 = 0$;
3. $\int_0^t 1_{\{X_s \neq 0\}} dA_s = 0$ for all $t \geq 0$.

Note that $A_0^-$ denotes value of the process $(A_t^-)_t \geq 0$ when $t = 0$.

Hence, we explore the general properties obtained for the previous versions of the class $(\Sigma)$ in [3,14,15]. For instance, we study the positive and negative parts of processes of the class $(\Sigma)$. We prove that the product of processes of the class $(\Sigma)$ with vanishing quadratic covariation is also of the class $(\Sigma)$. Further, we show that every nonnegative process $X$ of the class $(\Sigma)$ admits a multiplicative decomposition. In other words, it can be decomposed as

$$X = CW - 1,$$

where $W$ is a positive local martingale with $W_0 = 1$ and $C$ is a non-decreasing process. This result is an extension of that obtained by Nikeghbali for nonnegative and continuous local submartingales [15]. Finally, we generalize a result of Nikeghbali (Theorem 2.1 of [14]) that gives a martingale characterization for nonnegative processes of the class $(\Sigma)$.

Second, we study continuous processes of the class $(\Sigma)$. To the best of our knowledge, this is the first study to present a series of results that permit characterization of all continuous processes (not necessarily nonnegative) of the class $(\Sigma)$. For instance, we extend the martingale characterization given in Theorem 2.1 of [14] as well as Proposition 2.4 of [15]. In addition, we obtain other characterization results using an interesting balayage formula given in Proposition 2.2 of [2] and subsequently derive some corollaries. Finally, we focus on the construction of solutions for homogeneous and inhomogeneous skew Brownian motion equations using continuous stochastic processes of the class $(\Sigma)$. More precisely, we generalize the construction of Bouhadou and Ouknine to all continuous processes of the class $(\Sigma)$. 
1 Processes of a New Extension of the Class $\mathcal{Q}$

In this section, we present a framework to study stochastic processes satisfying the conditions of Definition 1.

1.1 Preliminaries

Here, we explore some general properties of processes satisfying Definition 1. Hence, we start by studying the positive and negative parts of processes of the class $\mathcal{Q}$ in the following lemma.

Throughout, for any càdlàg process $X$, $X^c$ shall denote its continuous part and the process $(X_t^c)_{t \geq 0}$ is defined such that $X_{0-} = X_0$ and such that $\forall t > 0$, $X_{t-}$ is the left limit of $X$ at $t$.

Lemma 1 Let $X = M + A$ be a process of the class $\mathcal{Q}$. The following hold:

1. If $A$ has no negative jump and $\int_0^t 1_{\{X_s \neq 0\}} \, dA_s^c = 0$, then $X^+$ is a local submartingale.
2. If $A$ has no positive jump and $\int_0^t 1_{\{X_s \neq 0\}} \, dA_s^c = 0$, then $X^-$ is a local submartingale.
3. If $X$ has no positive jump, then $X^+$ is also of the class $\mathcal{Q}$.
4. If $X$ has no negative jump, then $X^-$ is also of the class $\mathcal{Q}$.

Proof 1. From Tanaka’s formula, we have

$$X^+_t = \int_0^t 1_{\{X_s > 0\}} \, dX_s + \sum_{0 < s \leq t} 1_{\{X_s = 0\}} X^+_s + \sum_{0 < s \leq t} 1_{\{X_s > 0\}} X^-_s + \frac{1}{2} L^0_t.$$  

However,

$$\int_0^t 1_{\{X_s > 0\}} \, dX_s = \int_0^t 1_{\{X_s > 0\}} \, dM_s + \int_0^t 1_{\{X_s > 0\}} \, dA_s$$

$$= \int_0^t 1_{\{X_s > 0\}} \, dM_s + \int_0^t 1_{\{X_s > 0\}} \, dA_s^c + \sum_{s \leq t} 1_{\{X_s > 0\}} \Delta A_s$$

$$= \int_0^t 1_{\{X_s > 0\}} \, dM_s + \int_0^t 1_{\{X_s > 0\}} \, dA_s^c + \sum_{s \leq t} 1_{\{X_s > 0\}} \Delta A_s$$

since $A^c$ is continuous. Then,

$$\int_0^t 1_{\{X_s > 0\}} \, dX_s = \int_0^t 1_{\{X_s > 0\}} \, dM_s + \sum_{s \leq t} 1_{\{X_s > 0\}} \Delta A_s$$
because $dA^c$ is carried by $\{t \geq 0 : X_t = 0\}$. Hence, we get

$$X_t^+ = \int_0^t 1_{\{X_s > 0\}} dM_s + \sum_{s \leq t} 1_{\{X_s > 0\}} \Delta A_s + \sum_{0 < s \leq t} 1_{\{X_s \leq 0\}} X_s^+ + \sum_{0 < s \leq t} 1_{\{X_s > 0\}} X_s^- + \frac{1}{2} L_t^0.$$

We know that $A$ has no negative jump. Thus, $(\sum_{s \leq t} 1_{\{X_s > 0\}} \Delta A_s ; t \geq 0)$ is an increasing process that is null at zero. Moreover, $(\sum_{0 < s \leq t} 1_{\{X_s \leq 0\}} X_s^+ + \sum_{0 < s \leq t} 1_{\{X_s > 0\}} X_s^- + \frac{1}{2} L_t^0 ; t \geq 0)$ is an increasing process that is vanishing at zero. Then, $X^+$ is a submartingale, since $M$ and $\int_0^t 1_{\{X_s > 0\}} dM_s$ are local martingales.

2. Now, we remark that $-X$ is also an element of the class $(\Sigma)$ and that its finite variation part, $-A$, has no negative jump when $A$ has no positive jump. Therefore, it follows that $X^- = (-X)^+$ is a submartingale.

3. We have

$$X_t^+ = \int_0^t 1_{\{X_s > 0\}} dX_s + \sum_{0 < s \leq t} 1_{\{X_s > 0\}} X_s^- + \frac{1}{2} L_t^0,$$

since $X$ has no positive jump. Moreover,

$$\int_0^t 1_{\{X_s > 0\}} dX_s = \int_0^t 1_{\{X_s > 0\}} dM_s + \int_0^t 1_{\{X_s > 0\}} dA_s.$$

Hence,

$$X_t^+ = \int_0^t 1_{\{X_s > 0\}} dM_s + \int_0^t 1_{\{X_s > 0\}} dA_s + \sum_{0 < s \leq t} 1_{\{X_s > 0\}} X_s^- + \frac{1}{2} L_t^0. \quad (1.1)$$

Now, let us set $Z_t = \sum_{0 < s \leq t} 1_{\{X_s > 0\}} X_s^-$. Since $M$ and $\int_0^t 1_{\{X_s > 0\}} dM_s$ are local martingales and $A$ is càdlàg, there exists a sequence of stopping times $(T_n ; n \in \mathbb{N})$ increasing to $\infty$ such that

$$E[(X_{T_n})^+] = E[(M_{T_n} + A_{T_n})^+] < \infty \quad \text{and} \quad E\left[\int_0^{T_n} 1_{\{X_s > 0\}} dM_s\right] = 0, \quad n \in \mathbb{N}.$$

It follows from (1.1) that $E[Z_{T_n}] \leq E[(X_{T_n})^+] < \infty$ for all $n \in \mathbb{N}$. Thus, by Theorem VI.80 of [4], there exists a right continuous increasing predictable process $V^Z$ such that $Z - V^Z$ is a local martingale vanishing at zero. Moreover, there exists a sequence of stopping times $(R_n ; n \in \mathbb{N})$ increasing to $\infty$ such that
E \left[ \int_{0}^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dV_s^Z \right] = E \left[ \int_{0}^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} d(V_s^Z - Z_s) + \int_{0}^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dZ_s \right] \\
= E \left[ \int_{0}^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dZ_s \right].

Hence,

\begin{align*}
E \left[ \int_{0}^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dV_s^Z \right] &= E \left[ \sum_{0 < s \leq t \wedge R_n} \left( 1_{\{X_s^+ \neq 0\}} 1_{\{X_s^- > 0\}} X_s^- \right) \right] \\
&= E \left[ \sum_{0 < s \leq t \wedge R_n} \left( 1_{\{X_s > 0\}} 1_{\{X_s^- > 0\}} X_s^- \right) \right].
\end{align*}

Thus,

\begin{align*}
E \left[ \int_{0}^{t \wedge R_n} 1_{\{X_s^+ \neq 0\}} dV_s^Z \right] &= 0,
\end{align*}

since \(1_{\{X_s > 0\}} X_s^- = 0\). This implies that \(\int_{0}^{t} 1_{\{X_s^+ \neq 0\}} dV_s^Z = 0\). Therefore, \(dV_s^Z\) is carried by \(\{t \geq 0; X_t^+ = 0\}\). Consequently,

\begin{align*}
X_t^+ &= \left( \int_{0}^{t} 1_{\{X_s^- > 0\}} dM_s + (Z_t - V_t^Z) \right) + \left( V_t^Z + \int_{0}^{t} 1_{\{X_s^- > 0\}} dA_s + \frac{1}{2} L_t^0 \right)
\end{align*}

is a stochastic process of the class \((\Sigma)\).

4. It is obvious that \((-X)\) is of the class \((\Sigma)\) and it has no positive jump. Then, from 3), \(X^- = (-X)^+\) is also of the class \((\Sigma)\).

Now, we shall show that the product of processes of the class \((\Sigma)\) with vanishing quadratic covariation is also of the class \((\Sigma)\).

**Lemma 2** Let \((X_t^1)_{t \geq 0}, \ldots, (X_t^n)_{t \geq 0}\) be processes of the class \((\Sigma)\) such that \([X^i, X^j] = 0\) for \(i \neq j\). Then, \((\Pi_{i=1}^n X_t^i)_{t \geq 0}\) is also of the class \((\Sigma)\).

**Proof** Since \([X^1, X^2] = 0\), integration by parts yields

\begin{align*}
X_t^1 X_t^2 &= \int_{0}^{t} X_s^1 dX_s^2 + \int_{0}^{t} X_s^2 dX_s^1,
\end{align*}

i.e.,

\begin{align*}
X_t^1 X_t^2 &= \left[ \int_{0}^{t} X_s^1 dM_s^2 + \int_{0}^{t} X_s^2 dM_s^1 \right] + \left[ \int_{0}^{t} X_s^1 dA_s^2 + \int_{0}^{t} X_s^2 dA_s^1 \right].
\end{align*}
It is easy to see that $M_t = \int_0^t X^1_s \, dM^2_s + \int_0^t X^2_s \, dM^1_s$ is a càdlàg local martingale. Furthermore, the process $A_t = \int_0^t X^1_s \, dA^2_s + \int_0^t X^2_s \, dA^1_s$ is a finite variation process such that

$$dA_t = X^1_t \, dA^2_t + X^2_t \, dA^1_t$$

is carried by $\{ t \geq 0 : X^1_t X^2_t = 0 \}$. Therefore, $X^1 X^2$ is of the class $(\Sigma)$. If $n \geq 3$, then $[X^1 X^2, X^3] = 0$, and we obtain the result by induction. \(\square\)

In the next lemma, we derive a new property using the balayage formula for càdlàg semi-martingales.

**Lemma 3** Let $X = M + A$ be a process of the class $(\Sigma)$ and denote $\gamma_t = \sup\{ s \leq t : X_s = 0 \}$. Then, for any bounded predictable process $K$, $K_{\gamma} X$ is an element of the class $(\Sigma)$ and its finite variation part is given by $\int_0^t K_{\gamma_s} dA_s$.

**Proof** We obtain the following by applying the balayage formula to the càdlàg case:

$$K_{\gamma_t} X_t = K_{\gamma_0} X_0 + \int_0^t K_{\gamma_s} dX_s = \int_0^t K_{\gamma_s} dM_s + \int_0^t K_{\gamma_s} dA_s.$$  

Since $dA_t$ is carried by $\{ t \geq 0 : X_t = 0 \}$, we have the identity $K_{\gamma_s} dA_s = K_s dA_s$. Therefore,

$$K_{\gamma_t} X_t = \int_0^t K_{\gamma_s} dM_s + \int_0^t K_s dA_s.$$  

It is easy to see that $\int_0^t K_{\gamma_s} dM_s$ is a local martingale and that $K_t dA_t$ is carried by $\{ t \geq 0 : K_{\gamma_t} X_t = 0 \}$. This completes the proof. \(\square\)

**Corollary 1** Let $X = M + A$ be a process of the class $(\Sigma)$ and $f$ be a bounded Borel function. Then, the process

$$(f(A_t) X_t - F(A_t) : t \geq 0)$$

is a local martingale, where $F(A_t) = \int_0^t f(A_s) dA_s$.

In Proposition 2.1 of [15], Nikeghbali showed that every continuous nonnegative local submartingale $Y$ with $Y_0 = 0$ decomposes as

$$Y = MC - 1,$$

where $M$ is a continuous nonnegative local martingale with $M_0 = 1$ and $C$ is a continuous increasing process with $C_0 = 1$. We extend this result to nonnegative stochastic processes satisfying Definition 1 in the following corollary.
Corollary 2  Let $X = M + A$ be a nonnegative process of the class $(\Sigma)$. Then, there exist a càdlàg non-decreasing process $C$ and a càdlàg positive local martingale $W$ with $W_0 = 1$ such that $\forall t \geq 0,$

$$X_t = C_t W_t - 1.$$ 

Proof  According to Lemma 1, $X$ is a submartingale and $A$ is a non-decreasing process. Since the function $f$ defined by $f(x) = e^{-x}$ is a bounded Borel function on $[0, +\infty[$, it follows from Corollary 1 that

$$\left(e^{-A_t} (X_t + 1) - 1 : t \geq 0 \right)$$

is a càdlàg local martingale that is null at zero. Then,

$$W = \left(e^{-A_t} (X_t + 1) : t \geq 0 \right)$$

is a positive local martingale with $W_0 = 1$. Therefore, taking $C_t = e^{A_t}$, we obtain that $\forall t \geq 0,$

$$X_t = C_t W_t - 1.$$ 

This completes the proof. $\square$

1.2 Extension of the Martingale Characterization

In this subsection, we generalize some known results subsequent to the martingale characterization of processes of the class $(\Sigma)$. Let us begin with those of Lemma 2.3 of [3].

Theorem 1  Let $X = M + A$ be a process of the class $(\Sigma)$ and $A^c$ be the continuous part of $A$. For every $C^1$ function $f$ and a function $F$ defined by $F(x) = \int_0^x f(z)dz$, the process

$$\left(F(A_t^c) - f(A_t^c) X_t + \sum_{s \leq t \ delta} [f(A_s^c) - f'(A_s^c) X_s] \Delta A_s ; t \geq 0 \right)$$

is a càdlàg local martingale.

Proof  Through integration by parts, we get

$$f(A_t^c)X_t = \int_0^t f(A_s^c) dX_s + \int_0^t f'(A_s^c) X_s \, dA_s^c.$$
However, we have

\[ \int_0^t f'(A_s^c)X_s \, dA_s^c = \int_0^t f'(A_s^c)X_s \, dA_s^c \]

since \( A^c \) is a continuous process. Hence,

\[ f(A_t^c)X_t = \int_0^t f(A_s^c) \, dX_s + \int_0^t f'(A_s^c)X_s \, dA_s^c, \]

i.e.,

\[ f(A_t^c)X_t = \int_0^t f(A_s^c) \, dX_s + \int_0^t f'(A_s^c)X_s \, dA_s - \sum_{s \leq t} f'(A_s^c)X_s \Delta A_s \]

because \( A = A^c + \sum_{s \leq t} \Delta A_s \). Furthermore, we have \( \int_0^t f'(A_s^c)X_s \, dA_s = 0 \) since \( dA \) is carried by \( \{ t \geq 0 : X_t = 0 \} \). Therefore, it follows that

\[ f(A_t^c)X_t = \int_0^t f(A_s^c) \, dM_s + \int_0^t f'(A_s^c)X_s \, dA_s + \sum_{s \leq t} [f(A_s^c) - f'(A_s^c)X_s] \Delta A_s. \]

Consequently,

\[ f(A_t^c)X_t = \int_0^t f(A_s^c) \, dM_s + F(A_t^c) + \sum_{s \leq t} [f(A_s^c) - f'(A_s^c)X_s] \Delta A_s. \]

This implies that

\[ F(A_t^c) + \sum_{s \leq t} [f(A_s^c) - f'(A_s^c)X_s] \Delta A_s - f(A_t^c)X_t = -\int_0^t f(A_s^c) \, dM_s. \]

This completes the proof.

\[ \square \]

**Corollary 3** Let \( X = M + A \) be a process of the class \( (\Sigma) \) such that the continuous part \( A^c \) of the process \( A \) satisfies the following: \( \forall t \geq 0, \int_0^t 1_{\{X_s \neq 0\}} dA_s^c = 0 \). Then, \( f(A^c)X \) is also of the class \( (\Sigma) \). Further, its finite variation part is given by

\[ V_t = F(A_t^c) + \sum_{s \leq t} f(A_s^c) \Delta A_s. \]
Proof According to Theorem 1, the process $W_t$ defined by

$$W_t = F(A_t^c) - f(A_t^c)X_t + \sum_{s \leq t} [f(A_s^c) - f'(A_s^c)X_s] \Delta A_s$$

is a càdlàg local martingale. Since $dA$ is carried by $\{t \geq 0 : X_t = 0\}$, we get

$$\int_0^t X_s dA_s = \int_0^t X_s dA_s^c + \sum_{s \leq t} X_s \Delta A_s = 0,$$

i.e., $\sum_{s \leq t} X_s \Delta A_s = 0$ since $\int_0^t X_s dA_s^c = 0$. Thus,

$$\sum_{s \leq t} f'(A_s^c)X_s \Delta A_s = 0.$$ 

Therefore,

$$W_t = F(A_t^c) - f(A_t^c)X_t + \sum_{s \leq t} f(A_s^c) \Delta A_s.$$ 

Consequently,

$$f(A_t^c)X_t = -W_t + F(A_t^c) + \sum_{s \leq t} f(A_s^c) \Delta A_s.$$ 

This gives the result. \qed

Remark 1 Theorem 1 and Corollary 3 are natural extensions of Lemma 2.3 obtained by Cheridito et al. [3] for continuous $A$.

Now, we shall present an extension of the martingale characterization to nonnegative submartingales (Theorem 2.1 of [14]).

Theorem 2 Let $X = M + A$ be a nonnegative semi-martingale. Then, the following are equivalent:

1. $X \in (\Sigma)$;
2. There exists a non-decreasing predictable process $V$ such that for any $F \in C^2$, the process

$$\left(F(V_t^c) - F'(V_t^c)X_t + \sum_{s \leq t} [F'(V_s^c) - F''(V_s^c)X_s] \Delta V_s; t \geq 0 \right)$$

is a càdlàg local martingale and $V \equiv A.$
Proof (1) ⇒ (2) Let us take \( V = A \). Hence, from Theorem 1, we get that
\[
\left( F(A_t^c) - F'(A_t^c)X_t + \sum_{s \leq t} [F'(A_s^c) - F''(A_s^c)X_s] \Delta A_s; \ t \geq 0 \right)
\]
is a càdlàg local martingale.

(2) ⇒ (1) First, let us take \( F(x) = x \). Then, the process \( W \) defined by
\[
W_t = V_t^c + \sum_{s \leq t} \Delta V_s - X_t = V_t - X_t
\]
is a local martingale. Hence, by the uniqueness of the Doob–Meyer decomposition, we get \( V = A \). Next, we take \( F(x) = x^2 \). Then, the process \( B \) defined by
\[
B_t = (V_t^c)^2 - 2V_t^cX_t + 2 \sum_{s \leq t} V_s^c \Delta V_s - 2 \sum_{s \leq t} X_s \Delta V_s
\]
is a local martingale. However, through integration by part, it follows that
\[
B_t = 2 \int_0^t V_s^c dV_s - 2 \int_0^t V_s^c dX_s - 2 \int_0^t X_s dV_s + 2 \sum_{s \leq t} V_s^c \Delta V_s - 2 \sum_{s \leq t} X_s \Delta V_s
\]
\[
= 2 \int_0^t V_s^c d \left( V_s^c + \sum_{u \leq s} \Delta V_u - X_s \right) - 2 \int_0^t X_s d \left( V_s^c + \sum_{u \leq s} \Delta V_u \right)
\]
\[
= 2 \int_0^t V_s^c dW_s - 2 \int_0^t X_s dV_s.
\]
Consequently, we must have
\[
\int_0^t X_s dV_s = 0.
\]
In other words, \( dA \) is carried by the set \( \{ t \geq 0 : X_t = 0 \} \).

2 Contribution to Continuous Semi-martingales of the Class (\( \Sigma \))

Now, we study continuous processes of the class (\( \Sigma \)). First, we state some new characterization results. Second, we construct solutions for skew Brownian motion equations. A well-known continuous process of the class (\( \Sigma \)) is the absolute value of a Brownian motion \(|B|\). Bouhadou and Ouknine [2] constructed a solution for the inhomogeneous skew Brownian motion equation using the process \(|B|\). Our contribution to this topic is to extend the construction of Bouhadou and Ouknine to all continuous processes of the class (\( \Sigma \)).

For the readers’ benefit, we first recall some useful results and terminologies.
2.1 Recalling Useful Results

We begin by defining two stochastic processes that are important for the present study. First, we remark that for any continuous semi-martingale \( Y \), the set \( \mathcal{W} = \{ t \geq 0; Y_t = 0 \} \) cannot be ordered. However, the set \( \mathbb{R}_+ \setminus \mathcal{W} \) can be decomposed as a countable union \( \bigcup_{n \in \mathbb{N}} J_n \) of intervals \( J_n \). Each interval \( J_n \) corresponds to some excursion of \( Y \). In other words, if \( J_n = ]g_n, d_n[ \) and \( Y_{g_n} = Y_{d_n} = 0 \). For any constant \( \alpha \in [0, 1] \), we consider a sequence \((\xi_n)\) of i.i.d. Bernoulli variables such that

\[
P(\xi_n = 1) = \alpha \quad \text{and} \quad P(\xi_n = -1) = 1 - \alpha.
\]

Now, let us define the process \( Z^\alpha \) as follows:

\[
Z^\alpha_t = \sum_{n=0}^{+\infty} \zeta_n 1_{]g_n, d_n[}(t).
\] (2.1)

If we assume that \( \alpha \) is a piecewise constant function associated with a partition \((0 = t_0 < t_1 < \cdots < t_{n-1} < t_m)\), i.e., \( \alpha \) is of the form

\[
\alpha(t) = \sum_{i=0}^{m} \alpha_i 1_{[t_i, t_{i+1})}(t),
\]

where \( \alpha_i \in [0, 1] \) for all \( i = 0, 1, \cdots, m \), then we shall consider the process

\[
Z_i^\alpha = \sum_{n=0}^{+\infty} \sum_{i=0}^{m} \zeta_n^i 1_{]g_n, d_n[ \cap [t-i, t_{i+1})}(t),
\] (2.2)

where \((\xi_n^i)_{n \geq 0}, i = 1, 2, \cdots, m\), are \( m \) independent sequences of independent variables such that

\[
P(\xi_n^i = 1) = \alpha_i \quad \text{and} \quad P(\xi_n^i = -1) = 1 - \alpha_i.
\]

The balayage formula, especially the balayage formula for continuous semi-martingales in the progressive case (See Proposition 2.1 of \cite{2} or Theorem 1 of \cite{9}), is a crucial tool in this study. We recall it below.

**Proposition 1** Let \( Y \) be a continuous semi-martingale and \( \gamma'_t = \sup\{s \leq t : Y_s = 0\} \). Let \( k \) be a bounded progressive process, where \( \mathcal{p} k \) denotes its predictable projection. Then,

\[
k_{\gamma'_t} Y_t = k_0 Y_0 + \int_0^{\gamma'_t} \mathcal{p} k_{\gamma'_t} dY_s + R_t,
\]

where \( R \) is an adapted, continuous process with bounded variations such that \( dR_t \) is carried by the set \( \{Y_s = 0\} \).
Proposition 1 is a powerful and interesting tool. However, the fact that we know nothing about the form of the process $R$ can be limiting. The processes $Z^\alpha$ and $\mathcal{Z}^\alpha$ are critical to this study. Bouhadou and Ouknine [2] identified the process $R$ of Proposition 1 when the progressive process $k$ is equal to $Z^\alpha$ or $\mathcal{Z}^\alpha$. We recall these results below.

**Proposition 2** (Ouknine and Bouhadou [2]) Let $Y$ be a continuous semi-martingale and $Z^\alpha$ be the process defined in (2.1). Then,

$$Z^\alpha_t Y_t = \int_0^t Z^\alpha_s dY_s + (2\alpha - 1)L_0^0(Z^\alpha Y),$$

where $L_0^0(Z^\alpha Y)$ is the local time of the semi-martingale $Z^\alpha Y$.

**Proposition 3** (Ouknine and Bouhadou [2]) Let $Y$ be a continuous semi-martingale and $Z^\alpha$ be the process defined in (2.1). Then,

$$\mathcal{Z}^\alpha_t Y_t = \int_0^t \mathcal{Z}^\alpha_s dY_s + \int_0^t (2\alpha(s) - 1)dL_s^0(\mathcal{Z}^\alpha Y),$$

where $L_s^0(\mathcal{Z}^\alpha Y)$ is the local time of the semi-martingale $\mathcal{Z}^\alpha Y$.

We conclude this subsection by recalling an important theorem of [6], i.e., a result that enables us to characterize stochastic processes of the class $(\Sigma)$.

**Theorem 3** Let $X$ be a continuous process that vanishes at zero. Then,

$$X \in (\Sigma) \iff |X| \in (\Sigma).$$

### 2.2 New Characterization Results for Continuous Semi-martingales of the Class $(\Sigma)$

Now, we shall state new characterization results for all continuous processes of the class $(\Sigma)$. We begin by extending Theorem 2.1 of [14], which characterizes only nonnegative submartingales of the class $(\Sigma)$.

**Proposition 4** Let $X$ be a continuous semi-martingale. The following are equivalent:

1. $X \in (\Sigma)$;
2. For every locally bounded Borel function $f$, the process

$$(f(L_t)|X_t| - F(L_t); t \geq 0)$$

is a local martingale, where $L$ is the local time of $X$ at level zero and $F(x) = \int_0^x f(z)dz$. 

\[ \square \] Springer
Proof According to Theorem 3, we have $X \in (\Sigma)$ if, and only if $|X|$ is a nonnegative submartingale of the class $(\Sigma)$. However, from the martingale characterization of Nikeghbali [14], this is equivalent to the fact that the process

$$(f(L_t)|X_t| - F(L_t); t \geq 0)$$

is a local martingale. This completes the proof. \square

The following proposition extends another result characterizing the nonnegative submartingales of the class $(\Sigma)$ (see Proposition 2.4 of [15]).

**Proposition 5** Let $X$ be a continuous semi-martingale. The following are equivalent:

1. $X \in (\Sigma)$;
2. There exists a unique strictly positive, continuous local martingale $M$, with $M_0 = 1$, such that

$$|X_t| = \frac{M_t}{I_t} - 1,$$

where

$$I_t = \inf_{s \leq t} M_s.$$

The local martingale $M$ is given by

$$M_t = (1 + |X_t|) \exp(-L_t).$$

Proof From Theorem 3, we have $X \in (\Sigma)$ if, and only if $|X|$ is a continuous nonnegative submartingale of the class $(\Sigma)$. Consequently, we obtain the result by using Proposition 2.4 of [15]. \square

Next, we present a new method to characterize stochastic processes of the class $(\Sigma)$ using the progressive process $Z^\alpha$ defined in (2.1).

**Theorem 4** Let $X$ be a continuous semi-martingale and $Z^\alpha$ be the process defined in (2.1) with respect to $X$. The following are equivalent:

1. $X \in (\Sigma)$.
2. $\forall \alpha \in [0, 1], Z^\alpha X \in (\Sigma)$.
3. $\exists \alpha \in [0, 1]$ such that $Z^\alpha X \in (\Sigma)$.

Proof 1 $\Rightarrow$ 2) Let $X = M + V$ be an element of the class $(\Sigma)$. From Proposition 2.2 of [2], we have

$$Z^\alpha_t X_t = \int_0^t Z^\alpha_s dX_s + (2\alpha - 1)L^0_t(Z^\alpha X)$$

$$= \int_0^t Z^\alpha_s dM_s + \int_0^t Z^\alpha_s dV_s + (2\alpha - 1)L^0_t(Z^\alpha X).$$
However, we know that $\int_0^t Z_\alpha^X dV_t = 0$ since $dV_t$ is carried by $\{ t \geq 0; X_t = 0 \}$ and $X_t = 0 \iff Z_\alpha^X = 0$. Hence,

$$Z_\alpha^X X_t = \int_0^t Z_\alpha^X dM_s + (2\alpha - 1)L_t^0(Z_\alpha^X).$$

(2.3)

Then, $Z_\alpha^X \in (\Sigma)$ since $(2\alpha - 1)dL_t^0(Z_\alpha^X)$ is carried by $\{ t \geq 0; Z_\alpha^X X_t = 0 \}$ and $\left( \int_0^t Z_\alpha^X dM_s : t \geq 0 \right)$ is a local martingale.

2 $\Rightarrow$ 3) If we assume that $\forall \alpha \in [0, 1], Z_\alpha^X \in (\Sigma)$. In particular, it follows that $\exists \alpha \in [0, 1]$ such that $Z_\alpha^X \in (\Sigma)$.

3 $\Rightarrow$ 1) Now, assume that $\exists \alpha \in [0, 1]$ such that $Z_\alpha^X \in (\Sigma)$. Then, according to Theorem 3, $|Z_\alpha^X| \in (\Sigma)$. However, $\forall t \geq 0$, $Z_\alpha^X \in \{-1, 0, 1\}$ and $Z_\alpha^X = 0 \iff X_t = 0$. Therefore,

$$|Z_\alpha^X| = |X|.$$ 

Consequently,

$$|X| \in (\Sigma).$$ 

This completes the proof.

Remark 2 In the above-mentioned theorem, we have proved that when $X \in (\Sigma)$, we have $\forall \alpha \in [0, 1]$,

$$Z_\alpha^X X_t = \int_0^t Z_\alpha^X dM_s + (2\alpha - 1)L_t^0(Z_\alpha^X).$$

Now, as an application of Theorem 4, we have the following corollary, which gives a new martingale characterization of the class $(\Sigma)$.

Corollary 4 Let $X$ be a continuous semi-martingale and $Z_\alpha^X$ be the process defined in (2.1) with respect to $X$. The following holds:

$$X \in (\Sigma) \iff \exists \alpha \in [0, 1] \text{ such that } Z_\alpha^X \text{ is a local martingale}.$$ 

Proof $\Rightarrow$) It follows from Remark 2 that $\forall \alpha \in [0, 1]$,

$$Z_\alpha^X X_t = \int_0^t Z_\alpha^X dM_s + (2\alpha - 1)L_t^0(Z_\alpha^X).$$

Hence, in particular, for $\alpha = \frac{1}{2}$, we obtain

$$Z_{\frac{1}{2}}^X X_t = \int_0^t Z_{\frac{1}{2}}^X dM_s.$$
Therefore, $Z^{\alpha}X$ is a local martingale.

$\Leftarrow$ Now, if we assume that $\exists \alpha \in [0, 1]$ such that $Z^{\alpha}X$ is a local martingale, it follows that $Z^{\alpha}X \in (\Sigma)$. Then, it follows from Theorem 4 that $X \in (\Sigma)$. \hfill \square

It is well known that the absolute value $|M|$ of a continuous local martingale $M$ is an element of the class $(\Sigma)$. In the next corollary, we show that for any stochastic process $X$ of the class $(\Sigma)$, there exists a local martingale $M$ that has the same absolute value as $X$.

**Corollary 5** Let $X$ be a continuous semi-martingale. Then, $X$ is an element of the class $(\Sigma)$ if and only if there exists a local martingale $M$ such that $|X| = |M|$. 

**Proof** $\Rightarrow$) Assume that $X$ is an element of the class $(\Sigma)$ and define $Z^{\alpha}$ with $\alpha = \frac{1}{2}$. Hence, it follows from Corollary 4 that $M = Z^{\alpha}X$ is a continuous local martingale. Then, $|X| = |M|$ since $|Z^{\alpha}X| = |X|$. 

$\Leftarrow$) Now, assume that there exists a continuous martingale $M$ such that $|X| = |M|$. From Tanaka’s formula, we get

$$|X_t| = |M_t| = \int_0^t \text{sign}(M_s) dM_s + L^0_0(M).$$

However, $L^0_0(M) = L^0_0(X)$ and $dL^0_0(X)$ is carried by $\{t \geq 0 : X_t = 0\}$. Thus, $|X| \in (\Sigma)$. Consequently, it follows from Theorem 3 that $X \in (\Sigma)$. \hfill \square

### 2.3 Construction of Solutions for Skew Brownian Motion Equations

This subsection is devoted to the construction of solutions for skew Brownian motion equations. More precisely, we construct solutions from continuous processes of the class $(\Sigma)$ for the following equations:

$$X_t = x + B_t + (2\alpha - 1)L^0_t(X) \quad (2.4)$$

and

$$X_t = x + B_t + \int_0^t (2\alpha(s) - 1)dL^0_s(X), \quad (2.5)$$

where $B$ is a standard Brownian motion and $x = 0$.

#### 2.3.1 Construction of Solutions with Processes Whose Local Martingale Part is a Brownian Motion

First, we use stochastic processes of the class $(\Sigma)$ whose local martingale part is a Brownian motion. Many such processes can be found in the literature. For instance, we have $|B|$, $(\sup_{s \leq t} B_s - B_t)_{t \geq 0}$ or solutions of Eqs. (2.4) and (2.5) that start from zero. Our solutions are constructed as follows. For any process $X$ of the class $(\Sigma)$ with
a Brownian motion as its local martingale part in its Doob–Meyer decomposition, we set $Y^\alpha = Z^\alpha X$, $|Y^\alpha| = Z^\alpha |X|$, $\mathcal{Y}^\alpha = \mathcal{Z}^\alpha X$, and $|\mathcal{Y}^\alpha| = \mathcal{Z}^\alpha |X|$, where $Z^\alpha$ and $\mathcal{Z}^\alpha$ are, respectively, given in (2.1) and (2.2) and constructed with respect to $X$. These solutions are inspired by the construction of Bouhadou and Ouknine [2]. In fact, their solution is a particular case of the solutions given in the present study.

When $\alpha$ is a constant, we have the following result.

**Proposition 6** Let $X = M + A$ be a process of the class $(\Sigma)$ such that its local martingale part is a standard Brownian motion and $Z^\alpha$ be the process defined in (2.1) with respect to $X$. Then, $Y^\alpha = Z^\alpha X$ and $|Y^\alpha| = Z^\alpha |X|$ are weak solutions of (2.4) with the parameter $\alpha$ and starting from 0.

**Proof** From Remark 2, we have

$$Y^\alpha_t = W_t + (2\alpha - 1)L^0_t(Y^\alpha)$$

with $W_t = \int_0^t Z^\alpha_s dM_s$. Now, let us define a process $k$ as follows:

$$k_t = \sum_{n=0}^{+\infty} \zeta_n 1_{[g_n,d_n]}(t).$$  \hspace{1cm} (2.6)

First, we remark that for $\gamma_t = \sup\{s \leq t : X_s = 0\}$, we have

$$k_{\gamma_t}X_t = Z^\alpha_t X_t.$$

Meanwhile, we obtain the following from Proposition 1:

$$k_{\gamma_t}X_t = \int_0^t \nu k_{\gamma_s} dX_s + R_t,$$

where $R$ is an adapted, continuous process with bounded variations such that $dR_t$ is carried by the set $\{t \geq 0 : X_t = 0\}$. Since $k$ is a càdlàg process, we obtain

$$k_{\gamma_t}X_t = \int_0^t k_{\gamma_s} dX_s + R_t.$$

Finally, from the continuity of $X$, we get

$$k_{\gamma_t}X_t = \int_0^t k_s dX_s + R_t.$$

However,

$$\langle W, W \rangle_t = \langle Y^\alpha, Y^\alpha \rangle_t = \langle k_{\gamma_t}X, k_{\gamma_t}X \rangle_t.$$  \hspace{1cm} (123)
Then,
\[ \langle W, W \rangle_t = \int_0^t (k_s)^2 d\langle X, X \rangle_s = \langle X, X \rangle_t = \langle M, M \rangle_t \]
since \( k_s \in \{-1, 1\} \). This implies that \( \langle W, W \rangle_t = t \) because \( M \) is a Brownian motion. Thus, \( W \) is a Brownian motion. Consequently, \( Y^\alpha \) is a weak solution of (2.4). Meanwhile, from Theorem 3, we have that \( |X| \) is a continuous submartingale of the class \( (\Sigma) \). Moreover,

\[ |X_t| = |Z_\alpha X_t| = \int_0^t \text{sgn}(Z_\alpha X_s) d(Z_\alpha X_s) + L_0^0(Z^\alpha X), \]
i.e.,

\[ |X_t| = \int_0^t Z_\alpha^\alpha \text{sgn}(Z_\alpha X_s) dM_s + (2\alpha - 1) \int_0^t \text{sgn}(Z_\alpha X_s) dL_\alpha^0(Z^\alpha X) + L_0^0(Z^\alpha X). \]

Therefore, \( \text{sgn}(Z_\alpha^\alpha X_s) = Z_\alpha^\alpha \text{sgn}(X_s) \). Hence,

\[ |X_t| = \int_0^t (Z_\alpha^\alpha)^2 \text{sgn}(X_s) dM_s + (2\alpha - 1) \int_0^t Z_\alpha^\alpha \text{sgn}(X_s) dL_\alpha^0(Z^\alpha X) + L_0^0(Z^\alpha X). \]

Thus,

\[ |X_t| = \int_0^t \text{sgn}(X_s) dM_s + L_0^0(Z^\alpha X) \]
since \( Z^\alpha \) is defined on the complementary set of \( \{ t \geq 0 : X_t = 0 \} = \{ t \geq 0 : Z_\alpha^\alpha X_t = 0 \} \) and \( dL_\alpha^0(Z^\alpha X) \) is carried by \( \{ t \geq 0 : X_t = 0 \} = \{ t \geq 0 : Z_\alpha^\alpha X_t = 0 \} \). However, we remark that the local martingale part \( W_t = \int_0^t \text{sgn}(X_s) dM_s \) satisfies the following: \( \forall t \geq 0, \)

\[ \langle W, W \rangle_t = \int_0^t (\text{sgn}(X_s))^2 d\langle M, M \rangle_s = \langle M, M \rangle_t = t. \]

In other words, \( W \) is a Brownian motion. Consequently, from above, it follows that \( |Y^\alpha| \) is also a weak solution of (2.4).

When \( \alpha \) is a piecewise constant, we propose the following solutions.

**Proposition 7** Let \( X = M + A \) be a process of the class \( (\Sigma) \) such that \( M \) is a standard Brownian motion and \( Z^\alpha \) be the process defined in (2.1) with respect to \( X \). Then, \( Y^\alpha = Z^\alpha X \) and \( |Y^\alpha| = Z^\alpha |X| \) are weak solutions of (2.5) with piecewise constant parameter \( \alpha \) and starting from 0.
Proof By applying Proposition 2, we get

\[ Y_\alpha(t) = \int_0^t \mathcal{Z}_\alpha dX_s + \int_0^t (2\alpha(s) - 1) dL_s^0(\mathcal{Y}^\alpha) \]

Hence, we get

\[ Y_\alpha(t) = \int_0^t \mathcal{Z}_\alpha dM_s + \int_0^t \mathcal{Z}_\alpha dV_s + \int_0^t (2\alpha(s) - 1) dL_s^0(\mathcal{Y}^\alpha), \]

since \( \mathcal{Z}_\alpha \) is defined on the complementary set of the zero set of \( X \) and \( dV_t \) is carried by \( \{ t \geq 0 : X_t = 0 \} \). Now, let

\[ k_t = \sum_{n=0}^{+\infty} \sum_{i=0}^{m} \zeta_n^i 1_{[g_n,d_n]\cap[t-i,t_{i+1}]}(t). \]

We can see that \( \forall t \geq 0, Y_\alpha(t) = k_\gamma X_t \). From Proposition 1, we have

\[ k_\gamma X_t = \int_0^t p_k_{\gamma s} dX_s + R_t, \]

where \( R \) is an adapted, continuous process with bounded variations such that \( dR_t \) is carried by the set \( \{ t \geq 0 : X_t = 0 \} \). Hence, we obtain

\[ k_\gamma X_t = \int_0^t k_s dX_s + R_t, \]

since \( k \) is a càdlàg process. Finally, from the continuity of \( X \), we get

\[ k_\gamma X_t = \int_0^t k_s dX_s + R_t. \]

Therefore, by letting \( W_t = \int_0^t \mathcal{Z}_\alpha dM_s \), we have

\[ \langle W, W \rangle_t = \langle \mathcal{Y}_\alpha, \mathcal{Y}_\alpha \rangle_t = \langle k_\gamma X, k_\gamma X \rangle_t, \]

i.e.,

\[ \langle W, W \rangle_t = \int_0^t (k_s)^2 d\langle X, X \rangle_s = \langle X, X \rangle_t = \langle M, M \rangle_t, \]

since \( k_s \in \{-1, 1\} \). This implies that \( \langle W, W \rangle_t = t \) because \( M \) is a Brownian motion. Thus, \( W \) is a Brownian motion. Consequently, \( \mathcal{Y}^\alpha \) is a weak solution of (2.5).
Meanwhile, from above, we can say that $|X| = m + V$ is a continuous process of the class $(\Sigma)$ and its local martingale part is a Brownian motion. Consequently, from above, $|\mathcal{Y}^\alpha|$ is also a weak solution of (2.5).

**Remark 3** Recall that the solution proposed by Bouhadou and Ouknine [2] is $|\mathcal{Y}^\alpha| = \mathcal{Z}^\alpha|B|$, where $B$ is a standard Brownian motion and $\mathcal{Z}^\alpha$ is constructed with respect to $|B|$. It is now clear that this solution is a particular case of the one we presented in Proposition 7 since $|B|$ is a process of the class $(\Sigma)$ and its local martingale part $(\int_0^t \text{sgn}(B_s)dB_s)_{t \geq 0}$ is a Brownian motion.

### 2.3.2 Construction of General Solutions from the Class $(\Sigma)$

Now, we shall construct solutions of (2.4) and (2.5) from continuous processes of the class $(\Sigma)$ whose local martingale part is not necessarily a Brownian motion. For this purpose, we propose the following constructions. We consider a continuous process $X = M + A$ of the class $(\Sigma)$. We define $\tau_t = \inf\{s \geq 0 : \langle M, M \rangle_s > t\}$, $Y_t = X_{\tau_t}$, and we construct $Z^\alpha$ and $\mathcal{Z}^\alpha$ with respect to $Y$. Hence, our solutions are constructed as follows:

$$\forall t \geq 0, Y^\alpha_t = Z^\alpha_t Y_t, \quad |Y^\alpha_t| = Z^\alpha_t |Y_t|, \quad \mathcal{Y}^\alpha_t = \mathcal{Z}^\alpha_t Y_t \quad \text{and} \quad |\mathcal{Y}^\alpha_t| = \mathcal{Z}^\alpha_t |Y_t|.$$  

First, we consider the case of constant $\alpha$.

**Proposition 8** The processes $Y^\alpha$ and $|Y^\alpha|$ are weak solutions of (2.4) with piecewise constant parameter $\alpha$ and starting from 0.

**Proof** From Proposition 2.1, we have

$$Y^\alpha_t = \int_0^t Z^\alpha_s dY^\alpha_s + (2\alpha - 1)L^0_t(Y^\alpha),$$

i.e.,

$$Y^\alpha_t = \int_0^t Z^\alpha_s dM_s + \int_0^t Z^\alpha_s dA_{\tau_s} + (2\alpha - 1)L^0_t(Y^\alpha).$$

Hence,

$$Y^\alpha_t = \int_0^t Z^\alpha_s dM_{\tau_s} + (2\alpha - 1)L^0_t(Y^\alpha)$$

since $dA_{\tau_s}$ is carried by $\{s \geq 0 : X_{\tau_s} = 0\} = \{s \geq 0 : Z^\alpha_s = 0\}$. Then,

$$Y^\alpha_t = \int_0^{\tau_t} Z^\alpha_{(M, M)} s dM_s + (2\alpha - 1)L^0_t(Y^\alpha),$$
i.e.,
\[ Y_\alpha^t = W_t + (2\alpha - 1)L_0^t (Y_\alpha), \]

with \( W_t = \int_0^{\tau_t} Z_{(M,M)}^\alpha dM_s \).

Now, we construct the process \( k \) with respect to \( Y \). More precisely,
\[ k_t = \sum_{n=0}^{+\infty} \zeta_n \mathbf{1}_{[s_n, d_n]}(t). \]

We can see that \( \forall t \geq 0, Y_\alpha^t = k_{Y_\gamma} Y_t \). From Proposition 1, we have
\[ k_{Y_\gamma} Y_t = \int_0^t p_{Y_\gamma} dY_s + R_t, \]

where \( R \) is an adapted, continuous process with bounded variations such that \( dR_t \) is carried by the set \( \{ t \geq 0 : Y_t = 0 \} \). Hence, we obtain
\[ k_{Y_\gamma} Y_t = \int_0^t k_s dY_s + R_t, \]

since \( k \) is a càdlàg process. It follows from the continuity of \( Y \) that
\[ k_{Y_\gamma} Y_t = \int_0^t k_s dY_s + R_t. \]

Therefore, we have
\[ \langle W, W \rangle_t = \langle Y_\alpha, Y_\alpha \rangle_t = \langle k_{Y_\gamma} Y, k_{Y_\gamma} Y \rangle_t, \]
i.e.,
\[ \langle W, W \rangle_t = \int_0^{\tau_t} (k_{(M,M)}^2) d\langle X, X \rangle_s = \langle X, X \rangle_{\tau_t} = \langle M, M \rangle_{\tau_t}, \]

since \( k_{(M,M)}^2 \in \{-1, 1\} \). This implies that \( \langle W, W \rangle_t = t \). Thus, \( W \) is a Brownian motion. Consequently, \( Y_\alpha \) is a weak solution of (2.4) with the parameter \( \alpha \) and starting from 0.

Meanwhile, from Theorem 3, we have \( |X| \in (\Sigma) \). Then, from above, \( |Y_\alpha| \) is also a weak solution of (2.4) with the parameter \( \alpha \) and starting from 0.

Now, we propose solutions for (2.5) in the following proposition.

**Proposition 9** \( Y_\alpha^t \) and \( |Y_\alpha^t| \) are weak solutions of (2.5) with the parameter \( \alpha \) and starting from 0.
Proof From Proposition 2, we have
\[
\mathcal{Y}^\alpha_t = \int_0^t \mathcal{X}^\alpha_s \, dY_s + \int_0^t (2\alpha(s) - 1) \, dL^0_s(\mathcal{Y}^\alpha)
\]
\[
= \int_0^t \mathcal{X}^\alpha_s \, dM_{\tau_s} + \int_0^t \mathcal{X}^\alpha_s \, dV_{\tau_s} + \int_0^t (2\alpha(s) - 1) \, dL^0_s(\mathcal{Y}^\alpha).
\]
This implies that
\[
\mathcal{Y}^\alpha_t = \int_0^t \mathcal{X}^\alpha_s \, dM_{\tau_s} + \int_0^t (2\alpha(s) - 1) \, dL^0_s(\mathcal{Y}^\alpha),
\]
since $\mathcal{X}^\alpha$ is defined on the complementary set of the zero set of $Y$ and $dV_{\tau_t}$ is carried by $\{ t \geq 0 : Y_t = 0 \}$. Now, we consider the following process $k$:
\[
k_t = \sum_{n=0}^{+\infty} \sum_{i=0}^m \xi_n^i 1_{\{ g_n, d_n \cap [t-i, t+i+1) \}}(t),
\]
which is defined with respect to $Y$. We can see that $\forall t \geq 0$, $\mathcal{Y}^\alpha_t = k_{\gamma_t} Y_t$. From Proposition 1, we have
\[
k_{\gamma_t} Y_t = \int_0^t p_{k_{\gamma_s}} \, dY_s + R_t,
\]
where $R$ is an adapted, continuous process with bounded variations such that $dR_t$ is carried by the set $\{ t \geq 0 : Y_t = 0 \}$. Hence, we obtain
\[
k_{\gamma_t} Y_t = \int_0^t k_s \, dY_s + R_t,
\]
since $k$ is a càdlàg process. It follows from the continuity of $Y$ that
\[
k_{\gamma_t} Y_t = \int_0^t k_s \, dY_s + R_t.
\]
Therefore, by letting $W_t = \int_0^t \mathcal{X}^\alpha_s \, dM_{\tau_s}$, we have
\[
\langle W, W \rangle_t = \langle \mathcal{Y}^\alpha, \mathcal{Y}^\alpha \rangle_t = \langle k_{\gamma_t} Y_t, k_{\gamma_t} Y_t \rangle_t,
\]
i.e.,
\[
\langle W, W \rangle_t = \int_0^{\tau_t} (k_{\langle M, M \rangle_s})^2 \, d\langle X, X \rangle_s = \langle X, X \rangle_{\tau_t} = \langle M, M \rangle_{\tau_t},
\]
since $k_{\langle M, M \rangle_s} \in \{-1, 1\}$. This implies that $\langle W, W \rangle_t = t$. Thus, $W$ is a Brownian motion. Consequently, $\mathcal{Y}^\alpha$ is a weak solution of (2.5).
Meanwhile, recall from Theorem 3 that $|X|$ is a continuous semi-martingale of the class $(\Sigma)$. Therefore, from above, we can say that $|\mathcal{Y}^\alpha|$ is also a weak solution of (2.5). \hfill \Box

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