On travelling waves of the non linear Schrödinger equation escaping a potential well

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Abstract

In this paper we consider the NLS equation with focusing nonlinearities in the presence of a potential. We investigate the compact soliton motions that correspond to a free soliton escaping the well created by the potential. We exhibit the dynamical system driving the exiting trajectory and construct associated nonlinear dynamics for untrapped motions. We show that the nature of the potential/soliton is fundamental, and two regimes may exist: one where the tail of the potential is fat and dictates the motion, one where the tail is weak and the soliton self interacts with the potential defects, hence leading to different motions.

1 Introduction

1.1 Setting of the problem

We consider in this paper the focusing nonlinear Schrödinger equation in the presence of a potential

$$i\partial_t u + \Delta u + V(x) u + |u|^{p-1} u = 0, \quad t \in \mathbb{R}, \; x \in \mathbb{R}^d,$$

in the $L^2$ sub-critical range $p < 1 + \frac{4}{d}$. This equation appears in a variety of physical models like the Ginzburg-Landau theory of superconductivity [15], the one-dimensional self-modulation of a monochromatic wave [50], stationary two-dimensional self-focusing of a plane wave [7], propagation of a heat pulse in a solid Langmuir waves in plasmas [42] and the self-trapping phenomena of nonlinear optics [25], see [5, 6, 13, 17, 41, 49] for more complete references.

The complete qualitative description of solutions to (1.1) is far from being complete. In the case of trivial external potential $V = 0$, various classes of solutions are known. First low energy scattering solutions which asymptotically in time $t \pm \infty$ behave as linear or modified linear waves are investigated in [4, 10–12, 21–24, 34, 39, 47, 48]. The ground state nonlinear soliton is a time periodic solution

$$u(t, x) = Q(x) e^{it}$$

where $Q$ is the ground state solution to

$$\Delta Q - Q + Q^p = 0. \quad (1.2)$$

Multisoliton like solutions which asymptotically behave like decoupled non trivial trains of solitons were first explicitly computed in the completely integrable case $d = 1$, $p = 3$, and then systematically constructed as compact solutions of the flow in [27, 28, 31]. As discovered in [26] for Hartree type nonlinearities and systematically computed in [30, 37], the interaction between two solitary waves can be computed through a two body problem dynamical system which describes their exchange of energy: untrapped hyperbolic like motion leading to asymptotically free non interacting bubbles, untrapped parabolic like regimes which is a threshold dynamic were solitary remain logarithmically close, and a priori trapped elliptic like motion. Note that the resonant parabolic motion can lead to spectacular exchanges of energy which may dramatically modify the size of the solitary waves and lead to finite or infinite time blow up mechanisms [30].

A fundamental open problem is to understand the possibility of trapped multitudes. While these are formally predicted by the two body problem, they are sometimes believed to exist, sometimes not, the instability mechanism being a subtle radiation phenomenon. We refer for example to [38] for a beautiful introduction to these problems for the Gross Pitaevski model with non vanishing density at $+\infty$.

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1.2 Potential/soliton interaction

We propose in this paper to start the investigation of such mechanisms for (1.1) with non trivial potential \( V \). Let us first recall that in \( L^2 \) sub-critical case, (1.1) is globally well-posed in \( H^1 \) for a wide class of potentials \( V \) (see, for example, Corollary 6.1.2 of [9]), but little is known on the long-time behaviour of solutions. The existence of standing wave solutions to (1.1) was investigated in [18]. The existence of low energy scattering for (1.1) in dimension \( d = 1 \) was studied in [14, 35, 36, 54]. The dynamics of solitons for NLS equation with certain external potentials were studied in [20]. In particular, it was shown that the dynamical law of motion of the soliton is close to Newton’s equation with some dissipation due to radiation of the energy. Long-time behaviour of solution for the perturbed NLS equation was also studied in [43–46, 51–53]. In [19] it was shown the existence of solutions for (1.1) that behaves as solitary waves for the free NLS equation for large but finite times.

Our aim in this paper is to start the investigation of compact soliton motions corresponding to a free soliton \( (V = 0) \) escaping the well created by \( V \). Our first task is to exhibit the dynamical system driving the exiting trajectory and construct associated nonlinear dynamics for untrapped motions. Interestingly enough, we shall see that the nature of the potential/soliton is fundamental, and two regimes may exist: one where the tail of the potential is fat and dictates the motion, one where the tail is weak and the soliton self interacts with the potential defects, hence leading to different motions.

1.3 Assumptions on the potential

The sharp structure of the potential is essential for the qualitative description of solutions. We exhibit below a large class of potentials for which we can construct compact non trapped solutions.

**Condition 1.1** We assume the following on \( V \):

1. Regularity and decay: \( V \in C^\infty (\mathbb{R}^d) \) is a real-valued, symmetric function that satisfies the decay estimate

\[
|V^{(k)}(|x|)| \leq C (1 + |x|)^{-\rho}, \quad x \in \mathbb{R}^d, \quad \rho > 0,
\]

for all derivatives of \( V^{(k)} \), \( k \geq 0 \).

2. Structure and monotonicity of the tail: there is \( r_0 \geq 0 \) such that the following is true for all \( r \geq r_0 \). The potential and its derivatives \( V^{(k)}(r) \), \( k \geq 0 \), are monotone. For any \( \lambda > 0 \), \( V \) has one of the forms

\[
V(r) = V_+(r) \text{ or } V(r) = V_-(r),
\]

where

\[
V_\pm(r) = k e^{-2^{\frac{d-1}{2}(d-1)\ln r} \pm H(r)},
\]

and \( H \geq 0 \) is such that \( H, H' \) are monotone and \( H''(r) \) is bounded. Moreover, the estimate

\[
|V^{(k)}(r)| \leq C_k e^{-2^{\frac{d-1}{2}(d-1)\ln r}}, \quad k \geq 0,
\]

with \( C_k > 0 \) is satisfied. In addition, in the case \( V = V_+ \), \( H \) is either \( H(r) = o(r) \) or \( 0 < cr \leq H(r) \leq 2r + o(r) \). Furthermore, in the case when \( H(r) = 2r + (d-1)\ln r + o(r) \), suppose that the potential \( V = e^{-h(r)} \) with \( h(r) \geq 0 \) satisfies \( V(r)^{N_0} \leq C |V''(r)| \) for some \( N_0 > 0 \). The function \( h \) is such that \( h(r) = o(r) \), \( h^{(k)}(r) \) are monotone for all \( k \geq 0 \) and

\[
h^{(k)} \left( \frac{r}{2} \right) \leq C_k \left| h^{(k)}(r) \right|, \quad k \geq 0,
\]

and

\[
h^{(k)}(r) \leq \frac{C}{r} \left| h^{(k-1)}(r) \right|, \quad k \geq 1,
\]

are valid.

In order to introduce the leading order dynamical system describing the motion of the center of the solitary wave escaping the well, we need to compare the tail of \( V \) with the one of the solitary wave. Recall that there is a unique positive, radial symmetric solution \( Q(x) \) to

\[
-\Delta Q - Q + Q^p = 0.
\]

(see Chapter 8 of [9]). Moreover, \( Q(x) = q(|x|) \) where \( q \) satisfies for some \( A > 0 \) and all \( |x| \geq 1 \)

\[
q(|x|) - A |x|^{-\frac{d+1}{2}} e^{-|x|} + q'(|x|) + A |x|^{-\frac{d+1}{2}} e^{-|x|} \leq C |x|^{-1-\frac{d+1}{2}} e^{-|x|}
\]

and, moreover

\[
|\nabla^k Q(x)| \leq C (1 + |x|)^{-\frac{d+1}{2}} e^{-|x|}, \quad k \geq 0.
\]
where $r$.

We observe that the force in (1.6) is given in terms of $\mathcal{C}$ where $\mathcal{C}$.

Therefore, in this case the force is given in terms of $\mathcal{V}$ itself.

The energy of the system (1.6) is given by

$$E_0 = \frac{|\chi^\infty|^2}{2} - \frac{U_{\lambda^-,\mathcal{V}}(\frac{\chi^\infty}{\lambda})}{\|Q\|^2_{L^2}}$$

where $r^\infty = |\chi^\infty|$. The asymptotic behaviour for large $t$ of the unbounded solutions $\chi^\infty$ to (1.6) depends on the regime.
$E_0 > 0$: hyperbolic motion with untrapped trajectory

$$r^\infty = K_{\text{hyp}} t + o(t) \text{ and } |\beta^\infty| = K'_{\text{hyp}} + o(1).$$

for some $K_{\text{hyp}}, K'_{\text{hyp}} > 0$.

$E_0 = 0$: parabolic motion with untrapped trajectory

$$\frac{\|Q\|_{L^2}}{\sqrt{2}} \int_{r_0}^{r^\infty} \frac{dr}{\sqrt{U_{r^\infty} (\frac{r}{r^\infty}) - (\frac{r}{r^\infty})^2}} = t + t_0, \quad |\beta^\infty| = \frac{\sqrt{U_{r^\infty} (\frac{x}{r^\infty}) - (\frac{x}{r^\infty})^2}}{\sqrt{2\|Q\|_{L^2}}},$$

where $\mu \geq 0$ is the angular momentum.

$E_0 < 0$: all the solutions to (1.6) remain bounded in space, this is the trapped regime.

Our main result is that both untrapped hyperbolic and parabolic regimes of the two body problem (1.6) can be reproduced as the leading order dynamics of the soliton centers for compact solutions to the full problem (1.1).

**Theorem 1.5** Let the potential $V$ satisfy Conditions 1.1 and 1.3. Suppose that $\lambda^\infty \in \mathbb{R}^+$ is such that $(\lambda^\infty)^2 \sup_{r \in \mathbb{R}} V(r) < 1$. Let $\Xi^\infty(t) = (\chi^\infty(t), \beta^\infty(t))$ be a solution to (1.6).

(i) **Positive energy.** If $E_0 > 0$, then there is a solution $u \in H^1$ for the perturbed NLS equation (1.1) and $\gamma(t) \in \mathbb{R}$, such that

$$\lim_{t \to +\infty} \left\| u(t, x) - \left( \lambda^\infty \right)^{-1} \overrightarrow{Q} \left( \frac{x - \chi^\infty(t)}{\lambda^\infty} \right) e^{-i \gamma(t) e^{i \beta^\infty(t) x}} \right\|_{H^1} = 0.$$

(ii) **Zero energy.** Suppose in addition that $V(r) \geq 0$, for all $r \geq r_0$, with some $r_0 > 0$. If $E_0 = 0$, then there is a solution $u \in H^1$ for the perturbed NLS equation (1.1) and $\chi(t) \in \mathbb{R}^d$, $\gamma(t) \in \mathbb{R}$, such that

$$\lim_{t \to +\infty} \left\| u(t, x) - \left( \lambda^\infty \right)^{-1} \overrightarrow{Q} \left( \frac{x - \chi(t)}{\lambda^\infty} \right) e^{-i \gamma(t) e^{i \beta^\infty(t) x}} \right\|_{H^1} = 0$$

and

$$\lim_{t \to +\infty} \left| \frac{\chi(t)}{\chi^\infty(t)} - 1 \right| = 0.$$

**Comments on the result.**

1. **Positivity condition on $V$** In the case $E_0 = 0$, the positivity condition on $V$ is made in order to ensure the existence of unbounded solutions for (1.6). Since by Condition 1.1 $V$ is monotone, there are no such solutions if this positivity assumption is not satisfied on $V$. Let us stress that as in [26], the parabolic regime is particularly difficult to close due to uncertainties on the trajectory of the centers and degeneracies in the control of the infinite dimensional part of the solution.

2. **Soliton/potential regimes.** Let us stress onto the fact that Theorem 1.5 covers both cases which are very different: one where the potential is "fat" at $\infty$ and where the potential tail drives the untrapped dynamics of the centers, one where the potential tail is "weak" and the soliton dynamics is driven by self interaction with the potential. The existence of such dynamics driven by a suitable leading order like two body problems was first predicted in [19], but for suitable transient times, while Theorem 1.5 ensures the existence of such global in time compact flows.

3. **Overview of the proof.** We adapt the method developed in [26]. We modulate the solitary wave by letting act the symmetries of the free NLS equation (see (2.3)). Then, we translate (1.1) into the stationary equation

$$\triangle W - W + |W|^{p-1} W = F,$$

where $F = F(\chi, \beta, \lambda)$, with $\chi$ the translation parameter, $\beta$ the Galilean drift and $\lambda$ the scaling parameter (see (2.42)). By adjusting the modulation parameters $\chi, \beta, \lambda$, we construct approximate solutions to (1.7) in such way that the error is uniformly bounded by a small enough constant. This is achieved in Lemmas 2.6 and 2.7. We separate the potentials in "fast" and "slow" decaying: $V(r) = O\left( e^{-cr} \right)$, $c > 0$, or $V(r) = O\left( e^{-h(r)} \right)$, $h(r) = o(r) \geq 0$, as $r \to \infty$, respectively. The both cases are delicate. On the one hand, this is due to the possible slow decay of the potential. On the other hand, if the potential decays very fast, the solitary waves dominates and it becomes complicated to control the error and to extract the leading order terms in the expansion for the speed parameter $\beta$. The construction of these approximate solutions for (1.7) yields modulation equations for $\chi, \beta, \lambda$. Then, by energy estimates applied in a neighborhood of the solitary wave, we obtain a priori bounds for the modulation parameters $\chi, \beta, \lambda$ and the error $\varepsilon(t, x)$ (see Section 3.2). Theorem 1.5 then follows from a compactness argument in Section 3 and the asymptotic behaviour of the approximate modulation parameters (see Lemma
There are two main open problems after this work. First to address the question of stability of the corresponding compact dynamics. This has been proved for two bubbles KdV like flows using remarkable dine monotonicity properties [27], but it is still an open problem for two bubbles in the Schrödinger case, and the case of the potential interaction seems a nice intermediate problem to investigate. The second main open problem is to address the case of trapped dynamics where bubbles are predicted to stay close to one another. Here new radiation mechanisms are expected which is a fundamental open problem in the field, see again[38] for beautiful related problems.

The rest of the paper is organized as follows. In Section 2, we construct approximate free solitary wave solutions for the perturbed NLS equation which we use to obtain approximate equations for the modulation parameters. Section 3 is devoted to the proof of Theorem 1.5. This proof depends on Lemma 3.2, which is stated in Section 3, but whose proof is deferred until Section 4. The asymptotics for modulation equation for the speed parameter are obtained in Section 5. In Section 6 we establish an invertibility result for the perturbation $L_V$ of the linearized operator for the equation (1.2) around $Q$. Finally, in Section 7 we prove a lemma that is used in the construction procedure of Section 3.

**Notations**

We denote by $L^p (\mathbb{R}^d)$, for $1 \leq p \leq \infty$, the usual (complex valued) Lebesgue spaces. $H^s (\mathbb{R}^d)$, $s \in \mathbb{R}$, is the usual (complex valued) Sobolev space. (See e.g. [1] for the definitions and properties of these spaces.) For any $f, g \in L^2$ we define the scalar product by

$$(f, g) := \text{Re} \int_{\mathbb{R}^d} f(x) \overline{g(x)} dx.$$ 

We adapt the Japanese brackets notation $\langle x \rangle = (1 + x^2)^{1/2}$.

Finally, the same letter $C$ may denote different positive constants which particular value is irrelevant.

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**2 Approximate solutions.**

**2.1 First order approximation.**

We recall that initial value problem for the free NLS equation

$$\begin{cases}
i \partial_t u + \Delta u + |u|^{p-1} u = 0, \\
u(0, x) = u_0(x).
\end{cases} \tag{2.1}$$

admits the following symmetries. Let $\Xi = (\chi, \beta, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$ be a vector of parameters and $\gamma \in \mathbb{R}$. Then, if $u_0(x) \rightarrow \lambda^{\frac{2}{p-1}} u_0(\lambda (x + \chi)) e^{-i\gamma t} e^{i\beta(x+\chi)}$, the solution to (2.1) is transformed as

$$u(t,x) \rightarrow \lambda^{-\frac{2}{p-1}} u_0(\lambda^{-2} t, \lambda^{-1} (x + \chi - \beta t)) e^{-i\gamma t} e^{\frac{\lambda}{2}(x+\chi - \frac{\beta}{2} t)} \tag{2.2}$$

Let $\Xi(t)$ encode the vector of parameters $\Xi(t) = (\chi(t), \beta(t), \lambda(t))$. We translate the solution $u$ by using a combination of the symmetries for the free NLS equation. We let

$$u(t,x) = \lambda^{-\frac{2}{p-1}}(t) v\left(t, \frac{x-\chi(t)}{\lambda(t)}\right) e^{-i\gamma(t) t} e^{i\beta(t)x}, \quad v = v(t,y), \quad y = \frac{x-\chi(t)}{\lambda(t)}. \tag{2.3}$$

It is convenient for us to rescale the potential $V$. We define

$$V(x) = V_{\lambda}(x) = V(\lambda x). \tag{2.4}$$

Also, we let

$$\Lambda = \frac{2}{p-1} + y \cdot \nabla.$$
Then
\[ i\partial_t u + \Delta u + |u|^{p-1} u + V(x) u = \frac{1}{\lambda^{2p-1}} \mathcal{E}(v)(t, \frac{x}{\lambda}) e^{-it\lambda} e^{i\beta(t) \cdot x}, \] (2.5)
where
\[ \mathcal{E}(v) = i\lambda^2 \partial_t v + \Delta v - v + |v|^{p-1} v - i\lambda\Lambda v - i\lambda(\chi - 2\beta) \cdot \nabla v - \lambda^3 (\beta \cdot y) v \\
+ \lambda^2 \left( \hat{\gamma} + \frac{1}{\lambda^2} - |\beta|^2 - \beta \cdot \chi \right) v + \lambda^2 V(|y + \frac{\chi}{\lambda}|) v. \]

We aim to show that there is a vector \( \Xi(t) \) such that \( v = Q + \varepsilon \) with \( \varepsilon(t, x) \in C((0, \infty); \mathcal{H}^1) \) solves the equation
\[ \mathcal{E}(v) = \mathcal{E}(Q + \varepsilon) = 0, \] (2.6)
in such way that
\[ \|\varepsilon(t, x)\|_{\mathcal{H}^1} \to 0 \text{ and } |\chi(t)| \to \infty, \]
as \( t \to \infty \). We also want to precise the asymptotic behaviour of \( \chi(t) \).

As a first step, we aim to construct approximate (in a suitable way) solution to the equation
\[ \mathcal{E}(W) = 0. \] (2.7)

Let us denote the approximate modulation equation for the speed parameter \( \hat{\beta} \) by \( B \). We decompose \( \mathcal{E}(v) \) as
\[ \mathcal{E}(v) = \tilde{\mathcal{E}}_{\text{apr}}(v) + \tilde{R}(v), \] (2.8)
with
\[ \tilde{\mathcal{E}}_{\text{apr}}(v) = \Delta v - v + |v|^{p-1} v + \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) v - \lambda^3 (B \cdot y) v, \] (2.9)
and
\[ \tilde{R}(v) = -i\lambda(\chi - 2\beta) \cdot \nabla v - i\lambda\Lambda v + \lambda^2 \left( \hat{\gamma} + \frac{1}{\lambda^2} - |\beta|^2 - \beta \cdot \chi \right) v - \lambda^3 \left( (\hat{\beta} - B) \cdot y \right) v. \] (2.10)

If we modulate the parameters \( \chi, \beta, \gamma \) in such a way that \( R \) is small, the main part in (2.8) comes from \( \tilde{\mathcal{E}}_{\text{apr}}(v) \). As \( Q \) solves (1.2), introducing \( v = Q + \varepsilon \) into (2.9) we have
\[ \tilde{\mathcal{E}}_{\text{apr}}(v) = \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) Q - \lambda^3 (B \cdot y) Q + r(\varepsilon), \] (2.11)
where
\[ r(\varepsilon) = \Delta \varepsilon - \varepsilon + V \left( |y + \frac{\chi}{\lambda}| \right) \varepsilon - (B \cdot y) \varepsilon + |Q + \varepsilon|^{p-1} (Q + \varepsilon) - |Q|^{p-1} Q. \]

The stability problem for one travelling wave solution suggests ([55],[29]) to adjust the modulation parameters in such way that \( \left( \tilde{\mathcal{E}}_{\text{apr}}(v), \nabla Q \right) \sim 0 \). Taking the scalar product of (2.11) with \( \nabla Q \) we get
\[ \left( \tilde{\mathcal{E}}_{\text{apr}}(v), \nabla Q \right) = \left( \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) Q - \lambda^3 (B \cdot y) Q, \nabla Q \right) + (r(\varepsilon), \nabla Q). \]
Then
\[ \left| \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) Q - \lambda^3 (B \cdot y) Q, \nabla Q \right| \leq C \left( \left| \left( \tilde{\mathcal{E}}_{\text{apr}}(v), \nabla Q \right) \right| + |(r(\varepsilon), \nabla Q)| \right). \]

Therefore, we put
\[ \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) Q - \lambda^3 (B \cdot y) Q, \nabla Q = 0. \]
Noting that \(- (B \cdot y) Q, \nabla Q = B \|Q\|_{L^2}^2 \) we arrive to
\[ B = B(\chi, \lambda) = -\frac{1}{2\lambda \|Q\|_{L^2}^2} \int V \left( |y + \frac{\chi}{\lambda}| \right) \nabla Q^2(y) dy. \] (2.12)

We want \( R \) to be small as \( t \to \infty \), approximately (2.10) and (2.12) yield the system
\[ \begin{cases} \\
\dot{\chi} = 2\beta, \\
\dot{\beta} = B(\chi, \lambda), \\
\dot{\gamma} = -\frac{1}{\lambda^2} + |\beta|^2 + \beta \cdot \chi. \end{cases} \] (2.13)
In order to understand the behaviour as $t \to \infty$ of the solutions to (2.13), we need to study the asymptotics of the integral

$$\mathcal{J} (\chi) = \mathcal{J}_V (\chi) = \int V (|y + \tilde{\chi}|) \nabla^2 Q (y) \, dy.$$  

This means that we need to compare the decay of the potential with the soliton $Q$. We consider a bounded function $V \in C^\infty$ satisfying

$$\left| \frac{d^k}{dr^k} V (r) \right| \leq C \left( 1 + r \right)^{-\rho}, \quad \rho > 0,$$

(2.15)

for all $k \geq 0$. Suppose that $V^{(k)} (r), k \geq 0$, are monotone for all $r \geq r_0 > 0$. We ask $V$ to have one of the forms

$$V (r) = V_+ (r) \quad \text{or} \quad V (r) = V_- (r),$$

(2.16)

where $V_+ (r) = \kappa e^{-2r - (d-1)\ln r \pm H (r)}$, and $H \geq 0$ is such that $H, H'$ are monotone and $H'' (r)$ is bounded, for all $r \geq r_0$. Moreover, we assume some control on the derivatives

$$\left| \left( \frac{d}{dr} \right)^k V_- (r) \right| \leq C_k e^{-2r - (d-1)\ln r}, \quad k \geq 0,$$

with $C_k > 0$. In addition, in the case $V = V_+$, $H$ is either $H (r) = o (r)$ or $0 < cr \leq H (r) \leq 2r + o (r)$.

We also need to estimate $V (|y + \tilde{\chi}|) Q (y)$, as $|\tilde{\chi}| \to \infty$. Therefore, we require some information about the behavior of the potential compared to the soliton $Q$. We suppose either one of the following asymptotics. Let $V_\pm^{(1)} = \kappa_1 e^{-r - \frac{1}{2} (d-1)\ln r \pm H_1 (r)}$, for $r \geq r_0$, with $H_1 \geq 0$, $H_1, H'_1$ are monotone, for all $r \geq r_0$, and

$$\left| \left( \frac{d}{dr} \right)^k V_-^{(1)} (r) \right| \leq C_k e^{-r - \frac{1}{2} (d-1)\ln r}, \quad k \geq 0,$$

(2.17)

with $C_k > 0$. Moreover, in the case $V_+^{(1)}$ we suppose that $H_1^{(j)}, j \geq 1$, are bounded. Then, we consider potentials $V$ of the form (2.16) that can be represented as

$$V = V_+^{(1)} \quad \text{or} \quad V = V_-^{(1)}.$$

(2.18)

Finally, in the case when $H_1 (r) = r + \frac{1}{2} (d-1)\ln r + o (r)$, we suppose that the potential $V$ is given by

$$V (r) = V_+ (r) = V_+^{(1)} (r) = V^{(2)} (r),$$

where

$$V^{(2)} (r) := e^{-h_1 (r)},$$

(2.19)

with $h_1 (r) \geq 0$ is such that $h_1 (r) = o (r)$, $h_1^{(k)}$ are monotone, $k \geq 0$ and

$$\left| h_1^{(k)} \left( \frac{r}{2} \right) \right| \leq C_k \left| h_1^{(k)} (r) \right|, \quad k \geq 0,$$

(2.20)

for all $r \geq r_0$. We also suppose that

$$\left| h_1^{(k)} (r) \right| \leq C r^{-1} h_1^{(k-1)} (r), \quad k \geq 1,$$

and that for some $N_0 > 0$,

$$|V (r)|^{N_0} \leq C |V'' (r)|$$

(2.21)

**Remark 2.1** We observe that since $V$ and $V$ are related by (2.4), the above assumptions on the potential are satisfied if Condition 1.1 holds.

We define

$$v (d) = \int_0^\infty e^{-2\eta^2} \eta^{d-2} \, d\eta,$$

(2.22)

and

$$C_{\pm} (\xi) = \xi^\frac{d-1}{2} \int_1^{\xi-1} (r (\xi - r))^{-\frac{d-1}{2}} e^{\pm H (r)} \, dr.$$  

(2.23)
If $K' = \lim_{r \to \infty} H'(r)$ exists, we set
\[ I = \int e^{(2-K')\frac{r}{H(r)}} Q^2(z) \, dz. \] (2.24)

If $V(r) = V_+(r)$ we define
\[ U_V(\xi) = \begin{cases} \frac{K}{2} A^2 v(d) C_+(\xi) e^{-2\xi \xi^{-d-1}}, & \text{if } H(r) = o(r), \\ IV(\xi), & \text{if } 0 < cr \leq H(r) \leq 2r. \end{cases} \] (2.25)

If $V(r) = V_-(r)$ we set
\[ U_V(\xi) = \begin{cases} \frac{1}{2} \left( \int (A^2 V(|z|) + K\kappa Q^2(z)) e^{\frac{2\xi z}{|z|}} dz \right) e^{-2\xi \xi^{-d-1}}, & \text{if } d \geq 4, \\ \frac{1}{2} A^2 v(d) \xi^{-2\xi \xi^{-d-1}}, & \text{if } d = 2, 3, \\ r^{-\frac{d-1}{2}} e^{-H(r)} \in L^1([1, \infty)), \end{cases} \] (2.26)

where we denote by $K = \lim_{r \to \infty} e^{-H(r)}$.

**Lemma 2.2** Let $V \in C^\infty$ have the form (2.16) where $H, H'$ are monotone and $H''(r)$ is bounded. If $V(r) = V_+(r)$ let $H(r) < 2r$. Then, the asymptotics
\[ \mathcal{J}(\chi) = -\frac{\chi}{|\xi|} (1 + o(1)) U_V(|\xi|), \] (2.27)
as $|\xi| \to \infty$ is true. If $V = V^{(2)}$, then
\[ \mathcal{J}(\chi) = -\frac{\chi}{|\xi|} \left( \left( \int Q^2(z) \, dz \right) V'(|\xi|) + r(|\xi|) + e^{-\frac{|\chi|}{r}} \right), \] (2.28)

where
\[ r(|\xi|) = O \left( \left( |h'(|\xi|)| \left( |h'(|\xi|)|^2 + \frac{|h''(|\xi|)|}{|\xi|} + |h''(|\xi|)| + |\chi|^{-2} \right) \right) + \frac{|h''(|\xi|)|}{|\xi|} \right) V(|\xi|). \]

**Proof.** See Lemmas 5.1 and 5.2 in Section 5. □

In the next lemma we compare the potential with the solitary wave $Q$. We denote
\[ \Theta(|\xi|) = \Theta_V(|\xi|) = \begin{cases} e^{-|\xi|} (1 + |\xi|)^{-\frac{d+1}{2}}, & \text{if } V(r) = V^{(1)}_+(1), \\ \frac{1}{|V(|\xi|)|}, & \text{if } V(r) = V^{(1)}_-(1). \end{cases} \] (2.29)

Observe that
\[ |U_V(|\xi|)| \leq C \Theta(|\xi|). \] (2.30)

We have the following result.

**Lemma 2.3** Let $V \in C^\infty$ be the form (2.16) and in addition may be represented as (2.18). Then, there is $C(V) > 0$ such that for any $|\xi| \geq C(V)$ the following hold. The estimate
\[ \left\| \left( \frac{d}{dr} \right)^k V \right\|_{L^\infty} \leq C_k \Theta(|\xi|), \quad k \geq 0, \quad C_k > 0, \] (2.31)
is valid for any $\tilde{Q}$ satisfying (1.4). Moreover, in the case when $V(r) = V^{(2)}(r)$ we have
\[ \left\| \left( \frac{d}{dr} \right)^k V \right\|_{L^\infty} \leq C_k \left( \left\| \left( \frac{d}{dr} \right)^k V \right\|_{L^\infty} + e^{-\frac{|\xi|}{\delta}} \right), \quad \delta > 0, \] (2.32)
for any $k \geq 0$.

**Proof.** Let us prove (2.31). If $V(r) = V^{(1)}_1$, using (1.4) and (2.17) we have
\[ \left\| \left( \frac{d}{dr} \right)^k V \right\| \leq C_k e^{-|\xi|} (1 + |y + \tilde{\chi}|)^{-\frac{d+1}{2}} e^{-\frac{|y + \tilde{\chi}|}{\delta}} \] (2.33)
and then
\[ \left| \left( \frac{d}{dr} \right)^k V \right| (|y + \hat{x}|) \hat{Q}(y) \right) \leq C_k e^{-|\hat{x}|} (1 + |\hat{x}|)^{-\frac{d}{2} (d-1)}, \quad C_k > 0. \]  
(2.34)

Suppose now that \( V(r) = V_+^{(1)} \). First, as all the derivatives \( H_j^{(j)}, j \geq 1 \), are bounded we have
\[ \left| \left( \frac{d}{dr} \right)^k V \right| (|y + \hat{x}|) \hat{Q}(y) \right) \leq C_k \left| V (|y + \hat{x}|) \hat{Q}(y) \right|. \]  
(2.35)

If \( |y + \hat{x}| \leq |\hat{x}| \), since \( H_1 \) is monotone similarly to (2.33) we get
\[ \left| V (|y + \hat{x}|) \hat{Q}(y) \right| \leq C \left| (1 + |y + \hat{x}|)^{-\frac{d}{2} (d-1)} e^{-|y + \hat{x}|} \hat{Q}(y) \right| e^{H_1(|\hat{x}|)} \leq C \left| V (|\hat{x}|) \right|. \]  
(2.36)

If \( |y + \hat{x}| \geq |\hat{x}| \), we have \( |y|^2 + 2\hat{x} \cdot y \geq 0 \). Then
\[ -(|y + \hat{x}| - |\hat{x}|) + (H_1(|y + \hat{x}|) - H_1(|\hat{x}|)) = -(1 - H'(\xi)) \frac{|y|^2 + 2\hat{x} \cdot y}{|y + \hat{x}| + |\hat{x}|}, \]
for some \( \xi \in [|\hat{x}|, |y + \hat{x}|] \). As \( V \) is bounded, \( H_1 \geq 0 \) and \( H'_1 \) is monotone, in particular \( H'_1 \leq 1 + o(1) \), as \( r \to \infty \). Then
\[ -(1 - H'(\xi)) \frac{|y|^2 + 2\hat{x} \cdot y}{|y + \hat{x}| + |\hat{x}|} \leq o(1) |y|, \]  
as \( |\hat{x}| \to \infty \), and hence by (2.33)
\[ \left| V (|y + \hat{x}|) \hat{Q}(y) \right| \leq C \left| V (|\hat{x}|) \right|. \]  
(2.37)

Therefore, using (2.36) and (2.37) in (2.35) we deduce
\[ \left| \left( \frac{d}{dr} \right)^k V \right| (|y + \hat{x}|) \hat{Q}(y) \right) \leq C_k \left| V (|\hat{x}|) \right|. \]  
(2.38)

Relation (2.31) follows from (2.34) and (2.38).

We now prove (2.32). For \( |y| \geq \frac{|\hat{x}|}{r} \), since \( h_1 (r) = o (r) \) we estimate
\[ \left| V' (|y + \hat{x}|) e^{-\delta|y|} \right| \leq C \left| V' \right|_{L^\infty} e^{-\frac{\delta|y|}{r}}. \]  
(2.39)

Using that \( h''_1 \) is bounded we get
\[ \left| h_1 (|\hat{x} + y|) - h_1 (|\hat{x}|) - h'_1 (|\hat{x}|) \left( \frac{\hat{x} \cdot y}{|\hat{x}|} \right) \right| \leq C \frac{1 + |y|^4}{|\hat{x}|} \]
for \( |y| \leq \frac{|\hat{x}|}{r} \). This implies
\[ \left| V (|\hat{x} + y|) - V (|\hat{x}|) e^{-h'_1 (|\hat{x}|) \frac{|y|^2}{|\hat{x}|^2}} \right| \leq \left| V (|\hat{x}|) \right| e^{-h'_1 (|\hat{x}|) \frac{1 + |y|^4}{|\hat{x}|}}, \text{ as } |\hat{x}| \to \infty. \]  
(2.40)

Since \( h_1 (r) = o (r) \) and \( h'_1 \) is monotone \( |h'_1 (r)| \leq \frac{\delta}{r} \), for all \( r \) sufficiently large. As for \( |y| \leq \frac{|\hat{x}|}{r} \), \( |\hat{x} + y| \geq \frac{|\hat{x}|}{2} \), using that \( h'_1 \) is monotone we get
\[ \left| V (|\hat{x} + y|) e^{-\delta|y|} \right| \leq C \left| V (|\hat{x}|) \right| e^{-\frac{\delta|y|}{r}}. \]  
(2.41)

Then, as for \( |y| \leq \frac{|\hat{x}|}{r} \), \( |\hat{x} + y| \geq \frac{|\hat{x}|}{2} \), using that \( h''_1 \) are monotone and relations (2.41), (2.20) we deduce
\[ \left| \left( \frac{d}{dr} \right)^k V \right| (|y + \hat{x}|) e^{-\delta|y|} \right) \leq C_k \left| \left( \frac{d}{dr} \right)^k V \right| \left( \frac{|\hat{x}|}{2} \right) e^{h'_1 (|\hat{x}|) V (|\hat{x}|)} \right| \leq C_k \left| \left( \frac{d}{dr} \right)^k V \right| (|\hat{x}|) \right|. \]

Combining the last estimate with (2.39), we get (2.32).
2.2 Refined approximation.

Observe that \(|\tilde{E}_{\text{apr}}(Q)| \leq C\Theta(|\chi|)|. This bound on the error is not good enough to close the estimates in Section 4. We can improve the estimate on \(\tilde{E}_{\text{apr}}\) if we consider a refined approximate solution \(W = Q + T\) and adjust the function \(T\) and the modulation parameters \(\Xi = (\chi, \beta, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}\) and \(\gamma \in \mathbb{R}\) in a suitable way. Indeed, motivated by [26], we search for a stationary function \(W = W(y, \Xi(t))\). Since \(W\) is stationary, the dependence of \(W\) on time is through the modulation vector \(\Xi(t)\). This yields the relation

\[
\partial_t W = \chi \cdot \nabla \chi W + \beta \cdot \nabla \beta W + \frac{\partial W}{\partial \lambda} \lambda.
\]

Using the last identity we get

\[
E(W) = DW - W + |W|^{p-1}W + i\lambda^2 \left( \chi \cdot \nabla \chi W + \beta \cdot \nabla \beta W + \frac{\partial W}{\partial \lambda} \lambda \right) - i\lambda \Lambda \Lambda W - i\lambda (\chi - 2\beta) \cdot \nabla W
- \lambda^2 \left( \beta \cdot y \right) W + \lambda^2 \left( \gamma + \frac{1}{\lambda^2} - \frac{1}{\beta} \chi \right) W + \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) W.
\]

(2.42)

Let \(M = M(\Xi(t))\) and \(B = B(\Xi(t))\) be modulation equations (to be defined) for the scaling and speed parameters \(\dot{\lambda}\) and \(\dot{\beta}\). Then, equation (2.42) takes the form

\[
E(W) = \mathcal{E}_{\text{apr}}(W) + R(W)
\]

(2.43)

where

\[
\mathcal{E}_{\text{apr}}(W) = DW - W + |W|^{p-1}W + \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) W
+ i\lambda^2 \left( 2\beta \cdot \nabla \chi W + B \cdot \nabla \beta W + \frac{\partial W}{\partial \lambda} \lambda \right) - i\lambda \Lambda \Lambda W - \lambda^2 (B \cdot y) W
\]

(2.44)

and

\[
R(W) = i\lambda^2 \left( (\chi - 2\beta) \cdot \nabla \chi W + \left( \beta - B \right) \cdot \nabla \beta W + (\dot{\lambda} - M) \frac{\partial W}{\partial \lambda} \lambda \right) - i\lambda \left( \dot{\lambda} - M \right) \Lambda W - i\lambda (\chi - 2\beta) \cdot \nabla W
- \lambda^2 \left( \left( \dot{\beta} - B \right) \cdot y \right) W + \lambda^2 \left( \frac{1}{\lambda^2} - \frac{1}{\beta} \chi \right) W.
\]

(2.45)

Observe that \(\mathcal{E}_{\text{apr}}(Q) = \tilde{E}_{\text{apr}}(Q) - i\lambda \Lambda \Lambda Q\). Introducing \(W = Q + T\), into (2.44) we get

\[
\mathcal{E}_{\text{apr}}(Q + T) = D(T) - T + \frac{p+1}{2} |Q|^{p-1} T + \frac{p-1}{2} |Q|^{p-1} \mathcal{T}
+ \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) Q - i\lambda \Lambda \Lambda Q - \lambda^2 (B \cdot y) Q
+ \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) T - i\lambda \Lambda \Lambda T - \lambda^2 (B \cdot y) T
+ i\lambda^2 \left( 2\beta \cdot \nabla \chi T + B \cdot \nabla \beta T + M \frac{\partial T}{\partial \lambda} \right) + \mathcal{N}(T),
\]

(2.46)

with

\[
\mathcal{N}(T) = |Q + T|^{p-1} (Q + T) - \left( Q^p + \frac{p+1}{2} |Q|^{p-1} T + \frac{p-1}{2} |Q|^{p-1} \mathcal{T} \right).
\]

Let

\[
L_+ u = -\Delta u + u - p|Q|^{p-1} u
\]

and

\[
L_- u = -\Delta u + u - Q^{p-1} u.
\]

Decomposing \(T = T_1 + iT_2\), with \(T_1, T_2\) real we get

\[
\mathcal{E}_{\text{apr}}(Q + T) = -L_+ T_1 + \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) T_1 + \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) Q - \lambda^3 (B \cdot y) Q
- i \left( L_- T_2 \right) + i\lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) T_2 - i\lambda \Lambda \Lambda Q + 2\lambda^2 \beta \cdot \nabla \chi T_1 + R_1,
\]

(2.47)

where

\[
R_1 = -i\lambda \Lambda \Lambda T - \lambda^3 (B \cdot y) T + i\lambda^2 \left( 2i\beta \cdot \nabla \chi T_2 + B \cdot \nabla \beta T + M \frac{\partial T}{\partial \lambda} \right) + \mathcal{N}(T).
\]

(2.48)

As a first step, we want to adjust the approximate modulation parameters \(B\) and \(M\) in such way that we can solve the equations

\[
\left( L_+ - \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) \right) T_1 = -\lambda^3 (B \cdot y) Q + \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) Q,
\]

(2.49)

\[
\left( L_- - \lambda^2 V \left( |y + \frac{\chi}{\lambda}| \right) \right) T_2 = -i\lambda \Lambda \Lambda Q + 2\lambda^2 \beta \cdot \nabla \chi T_1
\]

(2.50)

and \(R_1\) has a better decay than \(\mathcal{E}_{\text{apr}}(Q)\), as \(|\chi| \to \infty\). In this way, we get a first order approximation. Then, from (2.46), we recursively construct higher order approximations. This will be done separately for fast and slow decaying potentials in
Lemmas 2.6 and 2.7 below, respectively. Before we present this construction, we prepare two results that are involved in the construction. We denote by \( L : H^1 \to H^{-1} \) the linearized operator for the equation (1.2) around \( Q \):

\[
L f := -\Delta f + f - \frac{p+1}{2} Q^{p-1} f - \frac{p-1}{2} Q^{p-1} f, \quad f \in H^1.
\]  

(2.51)

Representing \( f \in H^1 \) as \( f = h + ig \), with real \( h \) and \( g \), we have

\[
\mathcal{L} f = L_+ h + iL_- g
\]

Then

\[
(\mathcal{L} f, f) = (L_+ h, h) + (L_- g, g),
\]

(2.52)

for real functions \( h, g \in H^1 \). Let us denote the perturbed operator

\[
\mathcal{L}_V = L - \lambda^2 V (|y + \tilde{\chi}|) .
\]

(2.53)

In order to solve (2.49) and (2.50), we need the following invertibility result for the operators \( \mathcal{L}_V \).

**Lemma 2.4** Suppose that \( 1 < p < 1 + \frac{4}{d} \) and \( V \in C^\infty \), \( |V^{(k)}(x)| \to 0 \), as \( |x| \to \infty \), for any \( k \geq 0 \). Let \( \lambda \geq \lambda_0 > 0 \) be such that \( \lambda^2 \sup_{r \in \mathbb{R}} V (r) < 1 \). Then, there is \( C(V) > 0 \) such that for any \( |\lambda| \geq C(V) \)

\[
\|\mathcal{L}_V f\|^2_{H^{-1}} \geq c \|f\|^2_{H^1} - \frac{1}{c} \left( |(f, iQ)|^2 + |(f, \nabla Q)|^2 \right),
\]

(2.54)

with some \( c > 0 \) and

\[
(\mathcal{L}_V f, f) \geq c \|f\|^2_{H^1} - \frac{1}{c} \left( |(f, Q)|^2 + |(f, xQ)|^2 + |(f, \lambda Q)|^2 \right).
\]

(2.55)

Moreover, for any \( |\lambda| \geq C(V) \) and \( F \in H^1 \cap C^\infty \) such that \( (F, \nabla Q) = 0 \) the equation \( L_+ u - \lambda^2 V (y + \tilde{\chi}) u = F, (u, \nabla Q) = 0 \), has a real-valued solution \( u \in H^1 \cap C^\infty \). If \( (F, Q) = 0 \), there is a solution \( w \in H^1 \cap C^\infty \) to \( L_- w - \lambda^2 V (y + \tilde{\chi}) w = F, (w, Q) = 0 \). Furthermore, if \( |F(x)| \leq C e^{-\eta |z|} \), for some \( 0 < \eta < 1 \), then

\[
|u(x)| + |w(x)| \leq C e^{-\eta |z|}.
\]

(2.56)

**Proof.** See Section 6. \( \blacksquare \)

We also present a lemma that follows from the properties of the Bessel potential \((1 - \Delta)^{-1}\).

**Lemma 2.5** Suppose that \( T \) solves

\[
\left( L_+ - \lambda^2 V \left( \left| \frac{\chi}{\tilde{\chi}} \right| \right) \right) T = f, \quad (T, \nabla Q) = 0,
\]

(2.57)

with \( f \) satisfying

\[
|e^{-\delta|z|} f (z)| \leq A_0 (|\lambda|),
\]

(2.58)

for some \( A_0 (|\lambda|) > 0 \) and \( 0 < \delta < 1 \). Then, the estimate

\[
\left\| e^{-\delta |z|} T (\cdot) \right\|_{L^\infty} \leq C (\delta, \delta') \left( A_0 (|\lambda|) + \|T\|_{L^\infty} \Theta (|\lambda|)^{\delta'} \right),
\]

(2.59)

with some \( C (\delta, \delta') > 0 \) and \( \delta' < \delta \) holds.

**Proof.** See Appendix. \( \blacksquare \)

We are in position to prove an approximation result for (2.7). For a vector of multi-indices \( K = (k, k_1, k_2, k_3) \), let us denote

\[
\mathcal{D} = \partial_{\lambda}^k \partial_{\lambda}^{k_1} \partial_{\lambda}^{k_2} \partial_{\beta}^{k_3}, \quad \mathcal{D}_1 = \partial_{\lambda}^k \partial_{\lambda}^{k_1} \partial_{\beta}^{k_2}, \quad \mathcal{D}_2 = \partial_{\lambda}^k \partial_{\lambda}^{k_1} \partial_{\beta}^{k_2}.
\]

Also let

\[
Y = \inf_{0 < \nu < 1} \inf_{0 < \delta < 1} \left\{ C (\nu, \delta) \Theta^{(1-\delta)} \left( \left| \frac{\chi}{\tilde{\chi}} \right| \right) e^{-\nu \delta |y|} \right\}, \quad 0 < C (\nu, \delta) < \infty.
\]

where \( 0 < C (\nu, \delta) < \infty \) is such that \( C (\nu, \delta) \leq C_{\nu_0, \delta_0} \), for all \( |\nu| \leq |\nu_0|, \quad |\delta| \leq |\delta_0| \) and \( C (\nu, \delta) \to \infty \), as \( \nu, \delta \to 1 \). Set

\[
p_1 = \min \{ p, 2 \}.
\]

(2.60)

We have the following.
Lemma 2.6 Let \( \Xi = (\chi, \beta, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \) be a vector of parameters with \( \lambda \geq \lambda_0 > 0 \) such that \( \lambda^2 \sup_{r \in \mathbb{R}} V(r) < 1 \). There is \( C(V) > 0 \) such that for any \( |\chi| \geq C(V) \) the following holds. For any \( n \geq 1 \), there are \( T^{(j)}, B_j, M_j \in L^\infty(\mathbb{R}^d), \ j = 1, \ldots, n \), such that

\[
|D T^{(j)}| \leq C_K \langle \lambda \rangle^{i_K} \langle \lambda^{-1} \rangle^{i_K} \langle |\beta| \rangle^{m_K} Y \left( \langle |\beta| + \Theta \left( \frac{k}{\lambda} \right) \rangle^{(p_1-1)(j-1)} \right),
\]

\[
|D_1 B_j | + |D_1 M_j | \leq C_K \langle \lambda \rangle^{i_K} \langle \lambda^{-1} \rangle^{i_K} \langle |\beta| \rangle^{m_K} \Theta \left( \frac{k}{\lambda} \right) \left( \langle |\beta| + \Theta \left( \frac{k}{\lambda} \right) \rangle^{(p_1-1)(j-1)} \right),
\]

for all \( i, l, m \) which imply (2.61), (2.62) for \( B \) (recall that \( k \)) and some \( i_K, l_K, m_K \geq 0 \) and \( C_K, \delta_K > 0 \). In addition, \( B_j \) and \( M_j \), \( j = 1, \ldots, n \), satisfy for all \( |k| + k_1 \leq 1 \)

\[
\left| \partial^k \partial^k \chi B \right| + \left| \partial^k \partial^k M \right| \leq C_K \langle \lambda \rangle^{i_K} \langle \lambda^{-1} \rangle^{i_K} \langle |\beta| \rangle^{m_K} \left( \left| U_V \left( \frac{\chi}{\lambda} \right) \right| \right)^{3/4} + \Theta^{3/2} \left( \frac{k}{\lambda} \right),
\]

\[
\left| \partial^k \partial^k B \right| + \left| \partial^k \partial^k M \right| \leq C_K \langle \lambda \rangle^{i_K} \langle \lambda^{-1} \rangle^{i_K} \left( \langle |\beta| + \Theta \left( \frac{k}{\lambda} \right) \rangle^{p_1} + \left| U_V \left( \frac{\chi}{\lambda} \right) \right|^{p_1} \right), \ j \geq 2.
\]

For the approximation \( Q + T \), the error \( \mathcal{E}_{\text{apr}} \) \((Q + T)\) satisfies the estimate

\[
|\mathcal{E}_{\text{apr}} \left( Q + T \right) | \leq C \langle \lambda \rangle^{i} \langle \lambda^{-1} \rangle^{l} \langle |\beta| \rangle^{m} \left( \left| B \right| + \Theta \left( \frac{k}{\lambda} \right) \right)^{A \left( n \right)} + \left| U_V \left( \frac{\chi}{\lambda} \right) \right|,
\]

for some \( i, l, m \geq 0 \) and \( C > 0 \), where \( A \left( n \right) = \min \{ n \left( p_1 - 1 \right), 2 \} \).

Proof. First, we solve equation (2.49) given by

\[
\left( L + \lambda^2 V \left( \left| y + \frac{\chi}{\lambda} \right| \right) \right) T_1 = f_1
\]

with

\[
f_1 = -\lambda^3 (B \cdot y) Q + \lambda^2 V \left( \left| y + \frac{\chi}{\lambda} \right| \right) Q.
\]

Taking \( B_1 = B (\chi, \lambda) \), with \( B (\chi, \lambda) \) defined by (2.12) we assure that the right-hand side of (2.64) is orthogonal to \( \nabla Q \). Noting that

\[
\partial^k V \left( \left| y + \frac{\chi}{\lambda} \right| \right) = \lambda^{-|k|} \partial^k y \left( \left| y + \frac{\chi}{\lambda} \right| \right)
\]

\[
\partial^k V \left( \left| y + \frac{\chi}{\lambda} \right| \right) = \partial^k V \left( \left| \lambda y + \chi \right| \right) = (\lambda y \cdot \nabla_y V) \left( \left| \lambda y + \chi \right| \right) = y \cdot \nabla_y V \left( \left| y + \frac{\chi}{\lambda} \right| \right)
\]

(recall that \( V \) is defined by (2.4)) we have

\[
\left| \partial^k \partial^k \chi \int V \left( \left| y + \frac{\chi}{\lambda} \right| \right) \nabla Q^2 \left( y \right) dy \right| \leq C_{k, k_1} \lambda^{-|k|} \left| \int V \left( \left| y + \frac{\chi}{\lambda} \right| \right) \left( \nabla_y y \right) k^1 \partial^k \left( \nabla Q^2 \left( y \right) \right) dy \right|.
\]

Then, using (2.27) we estimate

\[
\left| \partial^k \partial^k \chi \int V \left( \left| y + \frac{\chi}{\lambda} \right| \right) \nabla Q^2 \left( y \right) dy \right| \leq C_{k, k_1} \lambda^{-|k|} \left| U_V \left( \left| \frac{\chi}{\lambda} \right| \right) \right|.
\]

Thus, by (2.31) we have

\[
\left| \partial^k \partial^k \chi B \right| \leq C_{k, k_1} \langle \lambda^{-1} \rangle^{i_k} \left| U_V \left( \left| \frac{\chi}{\lambda} \right| \right) \right|,
\]

which imply (2.61), (2.62) for \( B_1 \). Also, by (2.31) we get

\[
\left| D_2 \left( V \left( \left| y + \frac{\chi}{\lambda} \right| \right) \right) Q \left( y \right) \right| \leq C_{1, K} \langle \lambda \rangle^{i_k} \langle \lambda^{-1} \rangle^{i_k} \inf_{0 < \delta < 1} \left\{ \left| V \left( \left| y + \frac{\chi}{\lambda} \right| \right) Q \left( y \right) \right|^{(1-\delta)} \right\} \left| Q \left( y \right) \right|^{\delta} \left| Q \left( y \right) \right|^\delta
\]

\[
\leq C_{1, K} \langle \lambda \rangle^{i_k} \langle \lambda^{-1} \rangle^{i_k} \left| \left| Q \left( y \right) \right| \right|.
\]

Hence, from (2.65) we see that

\[
|D_2 f_1 | \leq C_K \langle \lambda \rangle^{i_k} \langle \lambda^{-1} \rangle^{i_k} \left| \left| Q \left( y \right) \right| \right|.
\]

for all \( |k| + k_1 + |k_2| \geq 0 \). By Lemma 2.4, there exists a solution \( T_1 \in H^1 \) satisfying

\[
|T_1 | \leq C_0 \langle \lambda \rangle^{i_0} \langle \lambda^{-1} \rangle^{i_0} \left| \left| \left| Q \left( y \right) \right| \right| \right|.
\]
Differentiating equation (2.64) with respect to $\lambda$ we get
\[
(L_+ - \lambda^2 V \left( \left| y + \frac{X}{\lambda} \right| \right)) \partial_\lambda T_1 = \partial_\lambda f_1 + \partial_\lambda \left( \lambda^2 V \left( \left| y + \frac{X}{\lambda} \right| \right) \right) T_1. \tag{2.71}
\]
We write the above equation as
\[
\partial_\lambda T_1 = (-\Delta + 1)^{-1} \left( \left( p Q^{p-1} + \lambda^2 V \left( \left| y + \frac{X}{\lambda} \right| \right) \right) \partial_\lambda T_1 + \partial_\lambda f_1 + \partial_\lambda \left( \lambda^2 V \left( \left| y + \frac{X}{\lambda} \right| \right) \right) T_1 \right). \tag{2.72}
\]
Using the explicit expression for the kernel of the Helmhotz operator $-\Delta + 1$, we estimate
\[
|\partial_\lambda T_1 (y)| \leq C \Theta^{-(1-\delta)} e^{\delta |y|} \int_{\mathbb{R}^d} \frac{e^{-|y-z|}}{|y-z|^{d+2}} \left( Q^{p-1} (z) |\partial_\lambda T_1 (z)| + |\partial_\lambda f_1 (z)| + \left| \partial_\lambda \left( \lambda^2 V \left( \left| z + \frac{X}{\lambda} \right| \right) \right) T_1 (z) | \right) dz.
\]
Then, using (2.54) to control the first term in the right-hand side of the above equation, by (2.69) and (2.70) we have
\[
|\partial_\lambda T_1 | \leq C_1 \langle \lambda \rangle^{\mu_1} \langle \lambda^{-1} \rangle^{\mu_1} \mathbf{Y}. \tag{2.73}
\]
Moreover, from (2.72), by the regularity properties of $-\Delta + 1$ we show that (2.73) is valid for the derivatives $\partial_\lambda \partial_y^{k_2} T_1$ for all multi-indices $k_2$. Similarly we estimate $\partial_\chi^{k_1} \partial_y^{k_2} T_1$ for $|k| = 1$. By induction in $k, k_1$ we prove
\[
|D_2 T_1 | \leq C_K \langle \lambda \rangle^{\mu_k} \langle \lambda^{-1} \rangle^{\mu_k} \mathbf{Y}, \tag{2.74}
\]
for all $|k| + k_1 + |k_2| \geq 0$.

Next, we consider equation (2.50)
\[
\left( L_+ - \lambda^2 V \left( \left| y + \frac{X}{\lambda} \right| \right) \right) T_2 = f_2 \tag{2.75}
\]
with
\[
f_2 = -\lambda M_1 \Lambda Q + 2 \lambda^2 \beta \cdot \nabla \chi T_1. \tag{2.76}
\]
We fix $M_1$ in such way that
\[
(\lambda M_1 \Lambda Q - 2 \lambda^2 \beta \cdot \nabla \chi T_1, Q) = 0.
\]
That is
\[
M_1 = \frac{2\lambda}{(\Lambda Q, Q)} (\beta \cdot \nabla \chi T_1, Q). \tag{2.77}
\]
Observe that $(\Lambda Q, Q) = \left( -\frac{2}{p-1} - \frac{d}{2} \right) (Q, Q) \neq 0$. By (2.74) we have
\[
|D_1 M_1 | \leq C_K \langle \lambda \rangle^{\mu_k} \langle \lambda^{-1} \rangle^{\mu_k} \omega_K (\beta) \Theta, \tag{2.78}
\]
for all $|K| \geq 0$, where
\[
\omega_K (\beta) = \left\{ \begin{array}{ll}
|\beta|, & |k_3| = 0, \\
C_K, & |k_3| \geq 1.
\end{array} \right.
\]
Then, (2.61) and the relation in (2.62) for $M_1$ follow from (2.78). Observe by (2.31) that $f_1$ can be estimated as
\[
|D_2 f_1 | \leq C \left( \inf_{0 < \delta < 1} \left( e^{\delta |y|} |U_V (\left| \chi \right|) |^{1+\delta} \right) \right), \quad |k| + k_1 + |k_2| \geq 0. \tag{2.79}
\]
Then, using Lemma 2.5 and (2.74) we get
\[
\left\| e^{-\delta |\cdot|} D_2 T_1 (\cdot) \right\|_{L^\infty} \leq C_K (\delta, \delta') \left( \left| U_V (\left| \chi \right|) \right|^{1+\delta} + \Theta (\left| \chi \right|)^{1+\delta'} \right), \tag{2.80}
\]
with $0 < \delta' < \delta < 1$, for all $|k| + k_1 + |k_2| \geq 0$. Using the last estimate with $\delta > \delta' = \frac{1}{2}$ in (2.77) we get the second relation in (2.62). Similarly to (2.69), from (2.76) we deduce
\[
|D_2 f_2 | \leq C_K \langle \lambda \rangle^{\mu_k} \langle \lambda^{-1} \rangle^{\mu_k} \omega_K (\beta) \mathbf{Y}. \tag{2.81}
\]
for all $|K| \geq 0$. By Lemma 2.4, there exists a solution $T_2 \in H^1$ to (2.75). Moreover, similarly to (2.74) using (2.81) we prove that
\[
|D T_2 | \leq C_K \langle \lambda \rangle^{\mu_k} \langle \lambda^{-1} \rangle^{\mu_k} \omega_K (\beta) \mathbf{Y}. \tag{2.82}
\]
for all $|K| \geq 0$. We put 

$$T = T_1 + iT_2.$$ 

By (2.74) and (2.82) we get (2.61) for $T$. Using (2.68), (2.74), (2.78), (2.82) and 

$$|N(T)| \leq C |T|^{p_1},$$ 

(2.83) 

from (2.47) we get (2.63) with $n = 1$. Therefore, we constructed the first improved approximation $T^{(1)}$. Let us construct $T^{(2)}$. We write $T^{(2)} = T_1^{(2)} + iT_2^{(2)}$, and introduce $T = T^{(1)} + T^{(2)}$, $B = B_1 + B_2$, $M = M_1 + M_2$, into (2.46). Then, using (2.64) and (2.82) and setting 

$$\left( L_+ - \lambda^2 V \left( \left| y + \frac{\chi}{\lambda} \right| \right) \right) T_1^{(2)} = -\lambda^3 \left( B_2 \cdot y \right) Q + f_3$$ 

(2.84) 

with 

$$f_3 = \Re N \left( T^{(1)} \right)$$ 

(2.85) 

and 

$$\left( L_- - \lambda^2 V \left( \left| y + \frac{\chi}{\lambda} \right| \right) \right) T_2^{(2)} = -\lambda M_2 \lambda Q + f_4$$ 

(2.86) 

with 

$$f_4 = \Im N \left( T^{(1)} \right),$$ 

(2.87) 

we derive 

$$\mathcal{E}_{appr} (Q + T) = -i \lambda \Delta (B \cdot y) T + \lambda^2 \left( 2 \beta \cdot \nabla \chi \left( iT_2 + T^{(2)} \right) + B \cdot \nabla \beta + M \frac{\partial Q}{\partial \lambda} \right) + \Re N \left( T^{(1)} + T^{(2)} \right) - \Re N \left( T^{(1)} \right).$$ 

From (2.85) and (2.87), via (2.74), (2.82) and (2.83) we get 

$$|Df_j| \leq C \langle \lambda \rangle^{j+K} \langle \lambda^{-1} \rangle^{j-K} Y_{p_1}, \quad j = 3, 4,$$ 

(2.89) 

for $|K| \leq 1$. Note that the restriction on the order of derivatives $K$ is due to the nonlinear term in the definition of $f_j$. We put 

$$B_2 = \frac{\left( f_3, \nabla Q \right)}{\lambda^3 \|Q\|_{L^2}^2}$$ 

(2.90) 

and 

$$M_2 = \frac{\left( f_4, Q \right)}{\lambda \left( \Lambda Q, Q \right)}.$$ 

(2.91) 

so that by Lemma 2.4 there exist solution $T_1^{(2)}$ to equation (2.84). By (2.89) we derive (2.61) with $j = 2$ for $B_2$, $M_2$ and hence, for $T^{(2)}$ satisfies (2.61). Observe that thanks to the regularity of $(-\Delta + 1)^{-1}$, we can estimate the second derivative on the $y$ variable of $T^{(2)}$ by using (2.89) only. Let us prove (2.62). Using (2.82), (2.80) with $\delta = \frac{3-p_1}{2p_1}$ and $\delta'$ close to $\delta$, and (2.83) we estimate 

$$\left| \partial^k \Theta \left( \Re N \left( T^{(1)} \right), \nabla Q \right) \right| + \left| \partial^k \Theta \left( \Im N \left( T^{(1)} \right), Q \right) \right| \leq C (|\beta| \Theta)^{p_1} + C \left| U_V \left( \left| \frac{\chi}{\lambda} \right| \right) \right|^{1+\frac{p_1-1}{2}}, \quad k \geq 0.$$ 

Hence, from (2.90) and (2.91) we prove (2.62) for $j = 2$. Finally, as 

$$|N(a+b) - N(a)| \leq C |b| \left( |a|^{p_1-1} + |b|^{p_1-1} \right),$$ 

(2.92) 

using (2.61) and (2.62) with $j = 1, 2$, in (2.88) we obtain (2.63) with $n = 2$.

We now proceed by induction. Suppose that we have constructed $T^{(j)}, B_j, M_j, j = 1, ..., n$, for some $n \geq 3$, satisfying estimates (2.61), (2.62) and 

$$\left\| e^{-\frac{3}{n-1} |K|} D_2 T^{(j)} (\cdot) \right\|_{L^\infty} \leq C \left( (|\beta| + \Theta) + |U_V \left( \left| \frac{\chi}{\lambda} \right| \right) | \right),$$ 

(2.93) 

for $|K| \leq 2$ and $|k| + k_1 + |k_3| \leq 1$. Denote 

$$T^{(n)} = T_1^{(n)} + iT_2^{(n)}; \quad T_1^{(n)} = \sum_{j=1}^{n} T_1^{(j)}, \quad T_2^{(n)} = \sum_{j=1}^{n} T_2^{(j)},$$ 

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and
\[ B^{(n)} = \sum_{j=1}^{n} B_j, \quad M^{(n)} = \sum_{j=1}^{n} M_j. \]

Let us consider the equations
\[
\left( L_+ - \lambda^2 V \left( \| y + \frac{\chi}{\lambda} \| \right) \right) T^{(n+1)}_1 = f_{2n+1} = -\lambda^3 (B_{n+1} \cdot y) Q + \text{Re} \left( N \left( Y^{(n)} \right) - N \left( Y^{(n-1)} \right) \right) \tag{2.94}
\]
and
\[
\left( L_+ - \lambda^2 V \left( \| y + \frac{\chi}{\lambda} \| \right) \right) T^{(n+1)}_2 = f_{2n+2} = -\lambda M_{n+1} \Lambda Q + \text{Im} \left( N \left( Y^{(n)} \right) - N \left( Y^{(n-1)} \right) \right). \tag{2.95}
\]

We put
\[
B_{n+1} = -\frac{\left( \text{Re} \left( N \left( Y^{(n)} \right) - N \left( Y^{(n-1)} \right) \right), \nabla Q \right)}{\lambda^3 \| Q \|_{L^2}} \quad \text{and} \quad M_{n+1} = \frac{\left( \text{Im} \left( N \left( Y^{(n)} \right) - N \left( Y^{(n-1)} \right) \right), Q \right)}{\lambda \| A Q, Q \|}. \tag{2.96}
\]

Using (2.92) we have
\[
\left| N \left( Y^{(n)} \right) - N \left( Y^{(n-1)} \right) \right| \leq C \left| T^{(n)} \right| \left| Y^{(n-1)} \right| \| p_{1-1} \|. \tag{2.97}
\]

Then, by (2.61) we show
\[
|Df_i| \leq C \left( (\lambda^k (\lambda^{2k} (|\beta|)^m) \right) \tag{2.98}
\]

for \(|K| \leq 1\) and some \(k, l, m \geq 0\). Then, \(B_{n+1}\) and \(M_{n+1}\) satisfy (2.61) with \(j = n + 1\). By Lemma 2.4 we can solve (2.94) and (2.95) with \(T^{(n+1)}_1\) and \(T^{(n+1)}_2\) satisfying (2.61) with \(j = n + 1\). We put
\[
T^{(n+1)} = T^{(n+1)}_1 + iT^{(n+1)}_2. \tag{2.98}
\]

Then, from the equations (2.96), using (2.97) (with \(n\) replaced by \(n-1\)) and (2.93) we deduce (2.62) for \(j = n + 1\). Moreover, we prove
\[
e^{-\frac{3\pi^2}{2\| \beta \|}} T^{(2n+1)}_1 + e^{-\frac{3\pi^2}{2\| \beta \|}} T^{(2n+2)}_1 \leq C \left( (|\beta| + \Theta)^p + |U_1 (\left( \frac{\chi}{\lambda} \right))|^p \right), \quad \text{for} \quad |K| \leq 1. \tag{2.99}
\]

Hence, using (2.61) and (2.62) we prove (2.63) for \(n = N + 1\). ■

In the case when the potential does not decay fast enough, that is, \(V = V^{(2)}\), with \(V^{(2)}\) given by (2.19), the construction of Lemma 2.6 is not good enough to obtain a priori estimates on the modulation parameters of Section 4.1 (see Remark 4.1).

In order to cover also this case, we present a different construction in the following Lemma. We denote
\[
\Psi = \min \left\{ V' \left( \left( \frac{\chi}{\lambda} \right) \right), \frac{|V (\left( \frac{\chi}{\lambda} \right))|}{|\chi|} + e^{-\frac{|\lambda|}{\lambda^2}} + e^{-\frac{|\lambda|}{\lambda^2}}, V (\left( \frac{\chi}{\lambda} \right)) \right\},
\]

\[
Z = (|\beta| + V (\left( \frac{\chi}{\lambda} \right)) \}
\]

and
\[
\mathbf{e} = \mathbf{e} (y) = \inf_{0 < \delta < 1} \left\{ C (\delta) e^{-\delta |y|} \right\}, \quad 0 < C (\delta) < \infty.
\]

**Lemma 2.7** Suppose that \(V = V^{(2)}\). Let \(\Xi = (\chi, \lambda, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+\) be a vector of parameters with \(\lambda \geq \lambda_0 > 0\) such that \(\lambda^2 \sup_{r \in \mathbb{R}^d} V (r) < 1\). There is \(C (\mathcal{V}) > 0\) such that for any \(|\lambda| \geq C (\mathcal{V})\) the following holds. Given \(n \geq 0\), there exist \(\tilde{T}^{(j)}, \tilde{B}_j, \tilde{M}_j \in L^\infty (\mathbb{R}^d), j = 0, ..., n\) satisfying
\[
\left| \mathcal{D}_x^{k_1} \mathcal{D}_y^{k_2} \tilde{T}^{(0)} \right| \leq CC_{k_1 k_2 j} (\lambda) \left| V (\left( \frac{\chi}{\lambda} \right)) \right| \mathbf{e},
\]
\[
\left| \mathcal{D}_x^{k_1} \mathcal{D}_y^{k_2} \tilde{T}^{(0)} \right| \leq CC_{k_1 k_2 j} (\lambda) \Psi \mathbf{e}, \quad k_1, k_2 \geq 1,
\]
\[
\left| \mathcal{D}_x^{k_1} \mathcal{D}_y^{k_2} \tilde{T}^{(1)} \right| \leq CC_{j} (\lambda) \left( \lambda^{-1} \right)^{j_{x, \lambda}} \left( \lambda^{-1} \right)^{j_{y, \lambda}} \left| \beta \right| m_{x, \lambda} \Psi \mathbf{e},
\]
\[
\left| \mathcal{D}_x^{k_1} \mathcal{D}_y^{k_2} \tilde{T}^{(j)} \right| \leq CC_{j} (\lambda) \left( \lambda^{-1} \right)^{j_{x, \lambda}} \left( \lambda^{-1} \right)^{j_{y, \lambda}} \left| \beta \right| m_{x, \lambda} \left( \Psi^2 + \left| \beta \right| V'' (\left( \frac{\chi}{\lambda} \right)) \right) Z^{j-1} \mathbf{e}, \quad j \geq 2,
\]

\[\text{subject to } \tilde{T}^{(0)} = \frac{1}{\lambda^3 \| Q \|_{L^2}}, \text{ and } \tilde{T}^{(1)} = \frac{-1}{\lambda^3 \| Q \|_{L^2}} \Psi \mathbf{e}. \]
and

\[ |\partial_{\tilde{H}_1}^k \partial_{\tilde{H}_2}^l \tilde{B}_1| + \left| \partial_{\tilde{H}_1}^k \partial_{\tilde{H}_2}^l \tilde{M}_1 \right| \leq C_{j,K} \langle \lambda \rangle^{i_j,K} \langle |\beta| \rangle^{m_j,K} \Psi, \tag{2.100} \]

\[ |D_1 \tilde{B}_1| + |D_1 \tilde{M}_1| \leq C_{j,K} \langle \lambda \rangle^{i_j,K} \langle |\beta| \rangle^{m_j,K} \left( \Psi^2 + |V''(\langle |\chi| \rangle)| \right), \quad k \geq 1, \]

\[ |D_1 \tilde{B}_j| + |D_1 \tilde{M}_j| \leq C_{j,K} \langle \lambda \rangle^{i_j,K} \langle |\beta| \rangle^{m_j,K} \left( \Psi^2 + |V''(\langle |\chi| \rangle)| \right) \eta Z^{j-1}, \quad j \geq 2, \]

for all \( |K| \geq 0 \), with some \( i_j,K, l_j,K, m_j,K \geq 0 \) and \( C_{j,K}, \eta, \delta_j,K > 0 \). For the approximation \( Q + T \), with \( T = \sum_{j=0}^n \lambda^{T(j)} \), the error \( \mathcal{E}_{\text{apr}} (Q + T) \) satisfies the estimate

\[ |\mathcal{E}_{\text{apr}} (Q + T)| \leq C_n \langle \lambda \rangle^{i_n} \langle |\lambda| \rangle^{m_n} \psi \zeta e. \tag{2.101} \]

**Proof.** In the recursive construction of \( T^{(j)} \) presented in Lemma 2.6 we do not consider the terms in (2.46) containing derivatives of \( T \), treating them as an error. It is possible thanks to the correct decay of these terms. This is not the case now. We need to take into account the derivatives of \( T^{(j)} \) in their recurrent equations. As consequence, we must control high order derivatives \( \partial T^{(j)} \). This requires to control the derivatives of the nonlinear term, which may be not smooth enough. To avoid any issue, we approximate the nonlinear term by polynomials in the following way (see step 7 in the proof of Proposition 3.1 of [32]). For \( z \in \mathbb{C} \), we consider the function \( n(z) = 1 + z^{p-1} (1 + z) \). For \( j \geq 0 \), let \( P_j (z) \) be the homogeneous term of order \( j \) in the Taylor approximation of \( n(z) \) for \( |z| \leq \frac{1}{2} \). Then as \( |n(z)| \leq C \cdot |z|^p \), for all \( z \in \mathbb{C} \), we get

\[ |n(z) - \sum_{j=0}^m P_j(z)| \leq C_m |z|^{m+1}, \quad z \in \mathbb{C}, \quad m \geq 0. \]

Using the last relation we get

\[ |Q + T|^{p-1} (Q + T) - Q^p \sum_{j=0}^m P_j \left( \frac{T}{Q} \right) \leq C_m Q^{p-m-1} |T|^{m+1}. \tag{2.102} \]

In particular

\[ \left| N(T) - Q^p P_m \left( \frac{T}{Q} \right) \right| \leq C_m Q^{p-m-1} |T|^{m+1}. \tag{2.103} \]

where we denote

\[ P_m = \sum_{j=2}^m P_j, \quad m \geq 2. \]

Let us consider the equation

\[ L_4 T^{(0)} = \lambda^2 V \left( \frac{X}{\chi} \right) Q. \tag{2.104} \]

Since the right-hand side is orthogonal to \( \nabla Q \), it follows from (6.3) that there exists a solution \( T^{(0)} \) to (2.104) which satisfies

\[ \left| D_2 T^{(0)} \right| \leq C_0 (\lambda) \left| V^{(k)} \left( \frac{X}{\chi} \right) \right| e, \quad k + k_1 + k_2 \geq 0. \tag{2.105} \]

By the parity symmetry of equation (2.104), \( T^{(0)} \) may be chosen even. Actually, since \( L_4 (\Lambda Q) = -2Q, T^{(0)} \) is given explicitly by \( T^{(0)} = -\lambda^2 V \left( \frac{X}{\chi} \right) \frac{\Lambda Q}{Q} \).

For \( j \geq 1 \) and \( m \geq 2 \), we define recursively \( T_j \) as an even solution of the equation

\[ L_4 T^{(j)} = F_j = \lambda^2 V \left( \frac{X}{\chi} \right) T^{(j-1)} + Q^p \left( P_m \left( Q^{-1} \sum_{i=0}^{j-1} T^{(i)} \right) - P_m \left( Q^{-1} \sum_{i=0}^{j-2} T^{(i)} \right) \right). \tag{2.106} \]

Due to the parity of \( T^{(0)} \), for \( j \geq 1 \) the right-hand side of (2.106) is orthogonal to \( \nabla Q \) and hence, such \( T^{(j)} \) exists. Observe that

\[ |P_m (a + b) - P_m (a)| \leq C_m \langle a \rangle^m \langle b \rangle^m |b| (|b| + |a|). \tag{2.107} \]
Then, as $|V(k)| \leq C(k)|V|$, for all $k \geq 0$, by induction in $j$ we estimate

$$\left| \partial_x^2 \partial_y^j T^{(j)} \right| \leq C_{k,k_1} (\lambda) \left| V \left( \frac{y}{x} \right) \right|^{1+j} e, \quad 1.$$  

$$\left| \mathcal{D}_2 T^{(j)} \right| \leq C_{k,k_1} (\lambda) \left| V' \left( \frac{y}{x} \right) \right| \left| V \left( \frac{y}{x} \right) \right| e, \quad k = 1,$$

$$\left| \mathcal{D}_2 T^{(j)} \right| \leq C_{k,k_1} (\lambda) \left( \left| V' \left( \frac{y}{x} \right) \right|^2 + \left| V'' \left( \frac{y}{x} \right) \right| \left| V \left( \frac{y}{x} \right) \right| \right) e, \quad k \geq 2.$$  

Let $j$ be such that $j_0 \geq 1$. Then, by (2.15)

$$\left| \partial_x^2 \partial_y^j T^{(j)} \right| \leq C_{k,k_1} (\lambda) \left| V \left( \frac{y}{x} \right) \right| |\chi|^{-j_0} e \leq CC_{k,k_1} (\lambda) \frac{|V|}{|\chi|} e.$$  

We put $\tilde{T}^{(0)} = \sum_{i=0}^j T^{(i)}$ and $B_0 = M_0 = 0$. By (2.108)

$$\left| \partial_x^2 \partial_y^j \tilde{T}^{(j)} \right| \leq CC_{k,k_1} (\lambda) \left| V \left( \frac{y}{x} \right) \right| e,$$

$$\left| \mathcal{D}_2 \tilde{T}^{(j)} \right| \leq C_{k,j} (\lambda) \left| V' \left( \frac{y}{x} \right) \right| e, \quad k = 1,$$

$$\left| \mathcal{D}_2 \tilde{T}^{(j)} \right| \leq C_{k,j} (\lambda) \left( \left| V' \left( \frac{y}{x} \right) \right|^2 + \left| V'' \left( \frac{y}{x} \right) \right| \right) e, \quad k \geq 2.$$  

In particular, we get (2.99) for $\tilde{T}^{(0)}$. From (2.46) we have

$$\mathcal{E}_{\text{apr}} \left( Q + \tilde{T}^{(0)} \right) = \lambda^2 \left( V \left( \frac{y}{x} \right) - V \left( \frac{y}{x} \right) \right) + \lambda^2 V \left( \frac{y}{x} \right) T^{(j)}$$

$$+ i \lambda^2 \left( 2 \beta \cdot \nabla_x \tilde{T}^{(0)} \right) + Q^p \left( P_m \left( Q^{-1} \tilde{T}^{(0)} \right) - P_m \left( Q^{-1} \sum_{i=0}^{j-1} T^{(i)} \right) \right) + N \left( \tilde{T}^{(0)} \right) - Q^p P_m \left( Q^{-1} \tilde{T}^{(0)} \right).$$  

Similarly to the proof of (2.32), by using (2.40) we show that

$$|V \left( \frac{y}{x} \right) - V \left( \frac{y}{x} \right)| e^{-\frac{|\lambda|^2}{|\chi|}} \leq C \left| \lambda \right|^i \left| \lambda^{-1} \right|^l \left| V' \left( \frac{y}{x} \right) \right| + \left| V \left( \frac{y}{x} \right) \right| \right| e^{-\frac{|\lambda|^2}{|\chi|}} , \quad i, l > 0.$$  

Hence, by (2.103), (2.107), (2.108), (2.109) and (2.111) we obtain

$$|\mathcal{E}_{\text{apr}} \left( Q + \tilde{T}^{(0)} \right)| \leq C \left( \Psi + \left| V \left( \frac{y}{x} \right) \right|^{m+1} \right) e.$$  

Taking

$$m = j + n$$

in (2.113) and using (2.15) we get

$$|\mathcal{E}_{\text{apr}} \left( Q + \tilde{T}^{(0)} \right)| \leq C \Psi \left( 1 + \left| V \left( \frac{y}{x} \right) \right|^{n} \right) e.$$  

In particular, we attain (2.101) for $n = 0$.

Let us consider now the equation

$$(L_+ - \lambda^2 V \left( \frac{y}{x} \right)) \tilde{T} = - \lambda^3 \left( B \cdot y \right) \left( Q + \tilde{T}^{(0)} \right) + \lambda^2 \left( V \left( \frac{y}{x} \right) - V \left( \frac{y}{x} \right) \right) \left( Q + \tilde{T}^{(0)} \right)$$

$$+ \lambda^2 V \left( \frac{y}{x} \right) T^{(j)} + Q^p \left( P_m \left( Q^{-1} \tilde{T}^{(0)} \right) - P_m \left( Q^{-1} \sum_{i=0}^{j-1} T^{(i)} \right) \right).$$  

Observe that

$$\left( \left( B \cdot y \right) \left( Q + \tilde{T}^{(0)} \right), \nabla Q \right) = - B \left( \left| Q \right|_{L^2}^2 - \left( y_1 \tilde{T}^{(0)}, q' \right) \right).$$

Moreover, by (2.110), for all $|\chi| \geq C > 0$, with $C$ large enough,

$$\left| Q \right|_{L^2}^2 - \left( y_1 \tilde{T}^{(0)}, q' \right) \geq \left| Q \right|_{L^2}^2 \frac{C}{2}.$$  

Then, since $\tilde{T}^{(0)}$ is even, taking

$$B_1 = \frac{\left( V \left( \frac{y}{x} \right) - V \left( \frac{y}{x} \right) \right) \left( Q + \tilde{T}^{(0)} \right), \nabla Q}{\lambda \left( \left| Q \right|_{L^2}^2 - \left( y_1 \tilde{T}^{(0)}, q' \right) \right)}.$$
we assure that the right-hand side of (2.115) is orthogonal to $\nabla Q$. Using (2.32), (2.110), (2.112) we deduce
\[
\left| \partial_{x}^{k} \tilde{B}_{1} \right| \leq C_{1,K} \langle \lambda \rangle^{i_{1},K} \langle \lambda^{-1} \rangle^{i_{1},K} \Psi
\]
\[
\left| \partial_{x}^{k} \partial_{y}^{k^{2}} \tilde{B}_{1} \right| \leq C_{1,K} \langle \lambda \rangle^{i_{1},K} \langle \lambda^{-1} \rangle^{i_{1},K} \left( \Psi^{2} + |V''(\|\hat{\chi}\|)| \right), \quad k \geq 1.
\]

Thus, (2.100) for $\tilde{B}_{1}$ follows. Moreover, similarly to (2.74), via (2.32) and (2.107), we estimate
\[
\left| \partial_{x}^{k} \partial_{y}^{k^{2}} \tilde{T}_{1} \right| \leq CC_{1,K} \langle \lambda \rangle^{i_{1},K} \langle \lambda^{-1} \rangle^{i_{1},K} \Psi e \quad \left| D_{2} \tilde{T}_{1} \right| \leq CC_{1,K} \langle \lambda \rangle^{i_{1},K} \langle \lambda^{-1} \rangle^{i_{1},K} \left( \Psi^{2} + |V''(\|\hat{\chi}\|)| \right) e, \quad k \geq 1.
\]

We now consider the equation
\[
\left( L_{-} - \lambda^{2} V \left( \|y + \frac{i}{\lambda}\| \right) \right) \tilde{T}_{2} = -\lambda MA \left( Q + \tilde{T}(0) \right) + \lambda^{2} \left( \tilde{M}_{1} \frac{\partial \tilde{T}(0)}{\partial \lambda} \right) + 2\lambda^{2} \beta \cdot \nabla \chi \left( T(0) + \tilde{T}_{1} \right).
\]

Since $(\Lambda Q, Q) \neq 0$, by (2.110) and (2.116), for all $|\chi| \geq C > 0$, with $C$ large enough,
\[
\left| \left( \Lambda \left( Q + \tilde{T}(0) \right) - \lambda \frac{\partial \tilde{T}(0)}{\partial \lambda}, Q \right) \right| \geq \frac{|(\Lambda Q, Q)|}{2}.
\]

Then, we define $\tilde{M}_{1}$ by
\[
\tilde{M}_{1} = \frac{2\lambda}{\left( \Lambda \left( Q + \tilde{T}(0) \right) - \lambda \frac{\partial \tilde{T}(0)}{\partial \lambda}, Q \right)} \left( \beta \cdot \nabla \chi \left( T(0) + \tilde{T}_{1} \right), Q \right).
\]

Using (2.110) and (2.116) we see that
\[
\left| \partial_{x}^{k} \partial_{y}^{k} \tilde{M}_{1} \right| \leq CC_{1,K} \langle \lambda \rangle^{i_{1},K} \langle \lambda^{-1} \rangle^{i_{1},K} \langle \|\beta\| \rangle^{m_{j,K}} \Psi,
\]
\[
\left| D_{2} \tilde{M}_{1} \right| \leq CC_{1,K} \langle \lambda \rangle^{i_{1},K} \langle \lambda^{-1} \rangle^{i_{1},K} \langle \|\beta\| \rangle^{m_{j,K}} \left( \Psi^{2} + |V''(\|\hat{\chi}\|)| \right), \quad k \geq 1,
\]

and thus, we get (2.100) for $\tilde{M}_{1}$. Moreover, we have
\[
\left| D \tilde{T}_{2} \right| \leq CC_{1,K} \langle \lambda \rangle^{i_{1},K} \langle \lambda^{-1} \rangle^{i_{1},K} \Psi e.
\]

We set $\tilde{T}_{1}^{(1)} = \tilde{T}_{1} + i\tilde{T}_{2}$. By (2.116) and (2.120) we deduce (2.99) for $\tilde{T}_{1}^{(1)}$. Using (2.106), (2.115) and (2.117) in (2.46) we get
\[
\epsilon_{\text{apr}} (T) = \epsilon_{\text{apr}} \left( Q + \tilde{T}(0) + \tilde{T}(1) \right) = -i\lambda \tilde{M}_{1} \Lambda \tilde{T}(1) - \lambda^{3} \left( \tilde{B}_{1} \cdot y \right) \tilde{T}(1) + \lambda^{2} \left( 2i\beta \cdot \nabla \chi \tilde{T}_{2} + i\tilde{B}_{1} \cdot \nabla \beta \tilde{T}_{2} + \tilde{M}_{1} \frac{\partial \tilde{T}(1)}{\partial \lambda} \right) + N \left( \tilde{T}(0) + \tilde{T}(1) \right) - Q \left( P_{m} \left( Q^{-1} \left( \tilde{T}(0) + \tilde{T}(1) \right) \right) \right).
\]

Therefore, as $m$ is given by (2.114), from (2.99), (2.103) and (2.107) we get
\[
\left| \epsilon_{\text{apr}} (T) \right| \leq C_{1} \langle \lambda \rangle^{i_{1}} \langle \lambda^{-1} \rangle^{i_{1}} \langle |\beta| \rangle^{m_{1}} \Psi \langle |\beta| + |V(\|\hat{\chi}\|)| \left( 1 + |V(\|\hat{\chi}\|)| \right)^{n-1} \rangle,
\]

and hence (2.101) with $n = 2$.

We now proceed by induction. Suppose that we have constructed $\tilde{T}_{1}^{(j)}, \tilde{B}_{j}, \tilde{M}_{j}, j = 0, ..., n$, for some $n \geq 3$, in such a way that (2.99), (2.100) and (2.101) hold for all $j = 0, ..., n$. Denote
\[
\tilde{T}^{(n)} = \tilde{T}^{(n)}_{1} + i\tilde{T}^{(n)}_{2}, \quad \text{with} \quad \tilde{T}^{(n)}_{i} = \sum_{j=1}^{n} \tilde{T}^{(j)}_{i}, \quad i = 1, 2,
\]

and
\[
\tilde{B}^{(n)} = \sum_{j=1}^{n} \tilde{B}_{j}, \quad \tilde{M}^{(n)} = \sum_{j=1}^{n} \tilde{M}_{j}.
\]
Let us consider the equations

\[ \left( L_+ - \lambda^2 V \left( \frac{y + \chi}{\chi} \right) \right) \tilde{T}_{1}^{(n+1)} = -\lambda^3 \left( \tilde{B}_{n+1} \cdot y \right) \left( Q + \tilde{T}^{(0)} \right) + \tilde{f}_{2n+1} \]  

(2.121)

with

\[ \tilde{f}_{2n+1} = -\lambda^3 \left( \tilde{B}(n) \cdot y \right) \tilde{T}_{1}^{(n)} - \lambda^3 \left( \tilde{B} \cdot y \right) \tilde{T}_{1}^{(n-1)} + \lambda \tilde{M}(n) \Lambda T_{2}^{(n)} + \lambda \tilde{M} \Lambda T_{2}^{(n-1)} \]

\[ -2\lambda^2 \left( \beta \cdot \nabla \chi \tilde{T}_{2}^{(n)} \right) - \lambda^2 \left( \tilde{B}(n) \cdot \nabla \beta \tilde{T}_{2}^{(n)} + \tilde{M}(n) \frac{\partial \tilde{T}_{2}^{(n)}}{\partial x} \right) - \lambda^2 \left( \tilde{B} \cdot \nabla \beta \tilde{T}_{2}^{(n-1)} + \tilde{M} \frac{\partial \tilde{T}_{2}^{(n-1)}}{\partial x} \right) \]

\[ + Q^p \text{Re} \left( P_m \left( Q^{-1} \tilde{T}_{n}^{(n)} \right) - P_m \left( Q^{-1} \tilde{T}_{n}^{(n-1)} \right) \right), \]

and

\[ \left( L_- - \lambda^2 V \left( \frac{y + \chi}{\chi} \right) \right) \tilde{T}_{2}^{(n+1)} = -\lambda \tilde{M}_{n+1} \Lambda \left( Q + \tilde{T}^{(0)} \right) + \lambda^2 \left( \tilde{M}_{n+1} \frac{\partial \tilde{T}_{2}^{(0)}}{\partial x} \right) + \tilde{f}_{2n+2} \]  

(2.123)

with

\[ \tilde{f}_{2n+2} = -\lambda \tilde{M}(n) \Lambda \tilde{T}_{1}^{(n)} - \lambda \tilde{M} \Lambda \tilde{T}_{1}^{(n-1)} - \lambda^3 \left( \tilde{B}(n) \cdot y \right) \tilde{T}_{2}^{(n)} - \lambda^3 \left( \tilde{B} \cdot y \right) \tilde{T}_{2}^{(n-1)} \]

\[ + 2\lambda^2 \left( \beta \cdot \nabla \chi \tilde{T}_{1}^{(n)} \right) + \lambda^2 \left( \tilde{B}(n) \cdot \nabla \beta \tilde{T}_{1}^{(n)} + \tilde{M}(n) \frac{\partial \tilde{T}_{1}^{(n)}}{\partial x} \right) + \lambda^2 \left( \tilde{B} \cdot \nabla \beta \tilde{T}_{1}^{(n-1)} + \tilde{M} \frac{\partial \tilde{T}_{1}^{(n-1)}}{\partial x} \right) \]

\[ + Q^p \text{Im} \left( P_m \left( Q^{-1} \tilde{T}_{n}^{(n)} \right) - P_m \left( Q^{-1} \tilde{T}_{n}^{(n-1)} \right) \right). \]

We put

\[ \tilde{B}_{n+1} = -\frac{\left( \tilde{f}_{2n+1}, \nabla Q \right)}{\lambda^3 \left( \left\| Q \right\|_{L^2}^2 - \left( y, \tilde{T}^{(0)} \cdot q' \right) \right)} \]

and

\[ \tilde{M}_{n+1} = -\frac{\left( \tilde{f}_{2n+2}, Q \right)}{\lambda \left( \Lambda \left( Q + \tilde{T}^{(0)} \right) - \lambda \frac{\partial \tilde{T}^{(0)}}{\partial x}, Q \right)} \]  

(2.125)

Using the hypothesis of induction we see that \( B_{n+1} \) and \( M_{n+1} \) satisfy (2.100) with \( j = n + 1 \). We recursively define \( T_{1}^{(n+1)} \) and \( T_{2}^{(n+1)} \) as solutions to equations (2.94) and (2.95), which exist thanks to (2.125) and Lemma 2.4. We put

\[ T^{(n+1)} = T_{1}^{(n+1)} + i T_{2}^{(n+1)}. \]

As (2.99) holds for all \( j = 0, ..., n \), and (2.100) is true for all \( j = 0, ..., n + 1 \), from (2.122) and (2.124) we deduce (2.99) for \( j = n + 1 \). Introducing \( T = Y^{(n)} + T^{(n+1)} \), \( B = B^{(n+1)} \), \( M = M^{(n+1)} \), into (2.46) we get

\[ \mathcal{E}_{appr} (Q + T) = -i\lambda \tilde{M}^{(n+1)} \Lambda T^{(n+1)} - i\lambda \tilde{M} \Lambda T^{(n)} - \lambda^3 \left( \tilde{B}(n+1) \cdot y \right) T^{(n)} - \lambda^3 \left( \tilde{B} \cdot y \right) T^{(n-1)} \]

\[ + i\lambda^2 \left( 2\beta \cdot \nabla \chi T^{(n+1)} + \tilde{B}(n+1) \cdot \nabla \beta T^{(n+1)} + \tilde{B}_{n+1} \cdot \nabla \beta Y^{(n)} + \tilde{M}(n+1) \frac{\partial T^{(n+1)}}{\partial x} + \tilde{M}_{n+1} \frac{\partial Y^{(n)}}{\partial x} \right) \]

\[ + \mathcal{N} \left( \left( T^{(n+1)} \right) - Q^p \left( P_m \left( Q^{-1} Y^{(n)} \right) \right) \right). \]

From the validity of (2.99) and (2.100), for any \( j = 0, ..., n + 1 \), using (2.103), we prove (2.101) with \( n \) replaced by \( n + 1 \).

Let us formulate the approximation result. Given a vector of parameters \( \Xi = (\chi, \beta, \lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \) and \( \gamma \in \mathbb{R} \) consider the approximate solutions \( T^{(j)} \) and \( T^{(j)} \) of Lemmas 2.6 and 2.7, respectively. For \( j \geq 0 \), we denote these solutions by \( T^{(j)} \) independently of the case. Let the approximate soliton solution to the perturbed NLS equation be

\[ W^{(N)}(t,x;\Xi) = \lambda^{-\frac{2}{\gamma^2}} W \left( \frac{x - \chi}{\lambda} \right) e^{-i\gamma \epsilon e^{i\beta x}}. \]  

(2.126)

with

\[ W = Q + \sum_{j=0}^{N} T^{(j)} \]  

(2.127)

We have the following.

**Lemma 2.8** Let \( \Xi(t) = (\chi(t), \beta(t), \lambda(t)) \) and \( \gamma(t) \) be \( C^1 \) functions on a time interval \( I = [t_0, t_1] \), \( t_1 \leq \infty \). Suppose that for \( t \in I \)

\[ 1 \leq \frac{|\chi(t_0)|}{2} \leq |\chi(t)|, \quad |\beta(t)| \leq 2 |\beta(t_0)|, \]

\[ 0 < \frac{\lambda(t_0)}{2} \leq \lambda(t) \leq 2 \lambda(t_0). \]
Let $N \geq 1$. Then, $W = W^{(N)}$ satisfies
\[i\partial_t W + \Delta W + |W|^{p-1} W + VW = \lambda^{-\frac{2}{p}} \left( E^{(N)} + R(W) \right) \left( \frac{x - \chi(t)}{\lambda(t)} \right) e^{-\gamma(t)} e^{i\beta(t) x},\]
(2.128)
where $R(W)$ is given by (2.45) and the error term $E^{(N)}$ satisfies the estimate
\[|E^{(N)}| \leq C_{N,\lambda,\beta} Y \left( (|\beta| + \Theta \left( \frac{|\lambda|}{\lambda} \right) \right)^{A(N)} + |UV \left( \frac{|\chi|}{\lambda} \right)|\],
(2.129)
with $A(N) = \min \{ N(p_1 - 1), 2 \}$ and $C_{N,\lambda,\beta} > 0$. In addition, in the case when $V = V^{(2)}$, the estimate
\[|E^{(N)}| \leq C_{N,\lambda,\beta} \Psi Z^N e\]
(2.130)
is true.

**Proof.** The result follows from (2.5), (2.43) and Lemmas 2.6, 2.7. \[\blacksquare\]

### 2.3 Approximate modulation parameters.

We want to construct approximate modulation equations for $\Xi (t)$ and $\gamma (t)$. For $\lambda^\infty \in \mathbb{R}^+$ let us consider the problem of the motion in the central field
\[
\left\{ \begin{array}{l}
\dot{\beta}^\infty = \frac{\chi^\infty}{2} - \frac{\lambda^\infty}{2\lambda} U_{V\lambda} \left( \frac{|\chi|}{\lambda} \right),
\end{array} \right.
(2.131)
where we emphasize the dependence of $V$ on $\lambda$ (recall (2.4)). The energy of the system is given by
\[E_0 = \frac{|\chi^\infty|^2}{2} - \frac{U_{V\lambda} \left( \frac{|\chi|}{\lambda} \right)}{\|Q\|_{L^2}^2} \]
(2.132)
where $r^\infty = |\chi^\infty|$. The unbounded solutions $\chi^\infty$ of (2.13) have the following behaviour for large $t$ depending on the regime. If $E_0 > 0$, for some $C_{\text{hyp}}$, $C_{\text{hyp}}' > 0$, we have
\[r^\infty = C_{\text{hyp}} t + o(t) \quad \text{and} \quad |\beta^\infty| = C_{\text{hyp}}' + o(1).
(2.133)
In the case $E_0 = 0$, the unbounded solutions, if they exist behave as
\[t = \frac{\|Q\|_{L^2}^2}{\sqrt{2}} \int_{t_0}^{t} \frac{dr}{\sqrt{U_{V\lambda} \left( \frac{|\chi|}{\lambda} \right) - \frac{\mu^2}{r^2}}} + t_0 \]
(2.134)
where $\mu \geq 0$ is the angular momentum, and
\[|\beta^\infty| = \frac{\sqrt{U_{V\lambda} \left( \frac{|\chi|}{\lambda} \right) - \frac{\mu^2}{(r^\infty)^2}}}{\sqrt{2} \|Q\|_{L^2}^2}.
(2.135)
Let $\lambda^\infty \in \mathbb{R}^+$ and $\Xi^\infty (t) = (\chi^\infty (t), \beta^\infty (t), \lambda^\infty)$ be a solution to (2.13) and $\gamma^\infty (t)$ be given by (2.14).

**Remark 2.9** Observe that in the case $E_0 = 0$, if the potential $U_{V\lambda} \left( \frac{|\chi|}{\lambda} \right)$ decays faster than the centrifugal energy term $\frac{\mu^2}{(r^\infty)^2}$, with $\mu > 0$, all the solutions are bounded. Therefore, in that case, the unbounded solutions are possible only if $\mu = 0$. If the potential decays slower than $r^2$, for instance if
\[|V_{\lambda^\infty} (r)| \geq c (r)^{-2+\nu} > 0,
(2.136)
for some $0 < \nu < 2$, then the solution to (2.131) with $E_0 = 0$ is given by
\[\chi^\infty (t) = r^\infty (t) \theta^\infty (t),
(2.137)
with $r^\infty (t)$ solving
\[\dot{r}^\infty = \sqrt{\frac{2}{\|Q\|_{L^2}^2} U_{V\lambda} \left( \frac{r^\infty}{\lambda^\infty} \right) - \frac{\mu^2}{(r^\infty)^2}}.
(2.138)
and \( \theta^\infty (t) \) satisfying
\[
|\dot{\theta}^\infty (t)| = \frac{\mu}{(r^\infty)^2}.
\] (2.139)

By (2.136), for any \( \theta^\infty \in \mathbb{S}^{d-1} \), there is a solution \( \theta^\infty (t) \) for (2.139) such that
\[
|\theta^\infty (t) - \theta_0^\infty| \leq \int_t^\infty \frac{\mu}{(r^\infty)^2} \, dr \leq C \int_t^\infty \frac{dr^\infty}{\sqrt{V(\xi^\infty) \, (r^\infty)^2}} \leq C \int_t^\infty \frac{dr^\infty}{(r^\infty)^{1+\nu/2}} \leq C (r^\infty)^{-\nu/2}.
\] (2.140)

We now define the approximate system of modulation parameters. The form of \( R(W) \) in (2.128) suggests to define the approximate system as follows. For \( j \geq 1 \), let \( B_j, M_j \) and \( \hat{B}_j, \hat{M}_j \) be defined by Lemmas 2.6 and 2.7, respectively. We omit the tilde and write \( B_j, M_j \) in both cases. Denote
\[
\mathbf{B}^{(N)} = \sum_{j=1}^N B_j \quad \text{and} \quad \mathbf{M}^{(N)} = \sum_{j=1}^N M_j.
\]

Consider the system
\[
\begin{cases}
\dot{\chi}^{(N)} = 2\beta^{(N)}, \\
\dot{\beta}^{(N)} = \mathbf{B}^{(N)}, \\
\dot{\lambda}^{(N)} = \mathbf{M}^{(N)}.
\end{cases}
\] (2.141)

Let us show that (2.141) may be solved from infinity with asymptotic behaviour given by \( \Xi^\infty (t) \). Observe that by Condition 1.1 the potential is given either by
\[
V (r) = V^{(1)} (r),
\] (2.142)
where \(|V^{(1)} (r)| \leq Ce^{-ct} \), for some \( c > 0 \), or else \( V (r) = V^{(2)} (r) \). Also by Condition 1.1
\[
|U_{\nu'}^{(2)} (r)| \leq C |U'_{\nu} (r)|.
\] (2.143)

By assumption (1.5) there are constants \( R_1, R_2 \in \mathbb{R} \) such that
\[
U'_{\nu} (r) \, \Upsilon (r, \lambda) = R_1 + o (1) \quad \text{and} \quad \frac{U'_{\lambda} (r)}{r} \Upsilon (r) = R_2 + o (1),
\] (2.144)
as \( r \to \infty \), where we denote
\[
\Upsilon (r, \lambda) = \left( \int_{r_0}^r \frac{d\tau}{\sqrt{U'_{\lambda} (\tau)} \, \Upsilon (\frac{\tau}{\lambda})} \right)^2.
\]

**Lemma 2.10** Let \( V \) satisfy by Condition 1.1. For any \( N \geq 1 \) there is a solution \( \Xi^{(N)} (t) = (\chi^{(N)} (t), \beta^{(N)} (t), \lambda^{(N)} (t)) \) to (2.141) on \([T_0, \infty)\), with \( T_0 \) large enough. This solution satisfies the following asymptotics depending on the energy \( E_0 \) given by (2.132). If \( E_0 > 0 \), then
\[
|\chi^{(N)} (t) - \chi^\infty (t)| + |\beta^{(N)} (t)|^{-1} |\chi^{(N)} (t) - \beta^\infty (t)| + |\lambda^{(N)} (t) - \lambda^\infty| = o (1).
\] (2.145)

If \( E_0 = 0 \), suppose in addition that \( V (r) \geq 0 \), for all \( r \) large enough. Then
\[
\left| \frac{|\chi^{(N)} (t)|}{|\chi^\infty (t)|} - 1 \right| + |\beta^\infty (t)|^{-1} |\beta^{(N)} (t) - \beta^\infty (t)| + |\lambda^{(N)} (t) - \lambda^\infty| = o (1).
\] (2.146)

**Proof.** The proof of (2.145) follows by a fixed point argument similarly to Lemma A.1 in [26]. We omit the proof.

We turn to relation (2.146). As we are interested in unbounded solutions, by Remark 2.9 we put
\[
\mu = 0, \quad \text{if} \quad |V (r)| \leq C (r)^{-2}.
\] (2.147)

Suppose first that \( V = V^{(1)} \). In this case we can precise the behaviour of \( |\chi^\infty| \). By the definition (2.25) and (2.26) of \( U_{\nu} (\xi) \), integrating by parts, for \( r_0 \) large enough we show that
\[
C'\frac{e^{r_0 + \frac{d+1}{2} \ln r_0}}{\sqrt{C_\pm (r_0^\infty)}} \leq \int_{r_0}^{r_\infty} \frac{dr}{\sqrt{U_{\lambda} \, (r)}} \leq C \frac{e^{r_\infty + \frac{d+1}{2} \ln r_\infty}}{\sqrt{C_\pm (r_\infty^\infty)}}, \quad \text{with some} \quad C, C' > 0,
\]
if \( V (r) = V_- (r) \) or \( V (r) = V_+ (r) \), with \( H (r) = o (r) \). Moreover, if \( V (r) = V_+ (r) \), with \( 0 < cr \leq H (r) < 2r \), we estimate

\[
C' e^{r \infty} + \frac{(d-1) \ln r \infty \frac{H (r \infty)}{2}}{2} \leq \int_{r_0}^{r \infty} \frac{dr}{\sqrt{U V (r)}} \leq C e^{r \infty} \frac{(d-1) \ln r \infty \frac{H (r \infty)}{2}}{2},
\]

Then, (2.147) and (2.134) imply

\[
r \infty = K (V) (1 + o (1)) \ln t, \text{ as } t \to \infty,
\]

with

\[
K (V) = \left\{ \begin{array}{ll}
1, & \text{if } V = V_- \text{ or } V = V_+, \text{ with } H (r) = o (r),
\\
\left( \lim_{r \to \infty} (1 - \frac{H (r)}{2r}) \right)^{-1}, & V = V_+, \text{ with } 0 < cr \leq H (r) < 2r.
\end{array} \right.
\]

Let us write \( \chi \infty = r \infty \theta \infty (t) \), with \( r \infty > 0 \) and \( \theta \infty (t) \in S^{d-1} \). Then, as in this case we put \( \mu = 0 \), we have \( \theta \infty (t) = \theta \infty_0 \) = constant and \( |\dot{\chi} \infty| = \dot{r} \infty \). By Lemma 2.6 we can rewrite (2.141) as follows

\[
\begin{cases}
\dot{\chi} = 2 \beta, \\
\dot{\beta} = B (\chi, \lambda) + f_1 (\chi, \beta, \lambda), \\
\dot{\lambda} = g_1 (\chi, \beta, \lambda),
\end{cases}
\]

(we omit the index \( N \) in \( \chi, \beta, \lambda \)) where

\[
|f_1 (\chi, \beta, \lambda)| \leq C (\chi, \beta, \lambda) \left( (|\beta| + \Theta (|\chi|)) \Theta (|\chi|) \right)^{p_1} + |U V (|\chi|)|^{p_1},
\]

and

\[
|g_1 (\chi, \beta, \lambda)| \leq C (\chi, \beta, \lambda) \left( (|\beta| + \Theta (|\chi|)) \Theta (|\chi|) \right) + |U V (|\chi|)|.\]

Let us consider the intermediate system

\[
\begin{cases}
\dot{\chi}_{\text{app}} = 2 \beta_{\text{app}}, \\
\dot{\beta}_{\text{app}} = B (\chi_{\text{app}}, \lambda_{\text{app}}).
\end{cases}
\]

Using Lemma 2.2 we have

\[
B (\chi_{\text{app}}, \lambda_{\text{app}}) = \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \frac{c_0}{4 \lambda_{\text{app}}} (1 + o (1)) U V (|\chi_{\text{app}}|).
\]

with \( c_0 = 2 \|Q\|_{L^2}^{-2} \). Then, from (2.153) we deduce

\[
\frac{d}{dt} |\chi_{\text{app}}|^2 = c_0 (1 + o (1)) \frac{d}{dt} U V (|\chi_{\text{app}}|). \tag{2.154}
\]

We take \( \chi_{\text{app}} = r_{\text{app}} \theta_0 \infty \) of angular momentum \( \mu = 0 \) such that \( r_{\text{app}} \to \infty, \dot{r}_{\text{app}} \to 0 \) as \( t \to \infty \). Integrating (2.154) on \( [t, \infty) \) and using Lemma 2.2 we get

\[
\dot{r}_{\text{app}} = c_0 \left( (1 + o (1)) U V (|\chi_{\text{app}}|) \right)^{1/2},
\]

Hence

\[
t = c_0^{-1/2} \int_{r_0}^{r_{\text{app}}} \frac{dr}{\sqrt{(1 + o (1)) U V (|\chi_{\text{app}}|)}} + t_0. \tag{2.155}
\]

Similarly to (2.148) we show that \( r_{\text{app}} = K (V) (1 + o (1)) \ln t, \text{ as } t \to \infty \). Then,

\[
\left| \frac{r_{\text{app}}}{r \infty} - 1 \right| = o (1) \text{ and } |\beta_{\text{app}} - \beta \infty| = o (1) |\beta \infty|.
\]

Let us decompose \( \chi (t) = \chi_{\text{app}} (t) + \delta (t) \) and \( \lambda (t) = \lambda \infty + \mu (t) \). We aim to prove that for some \( T_0 > 0 \) sufficiently big the following a priori estimates are true

\[
|\delta (t)| \leq t^{-\frac{d}{2}}, \quad |\dot{\delta} (t)| \leq t^{-\left(1 + \frac{d}{2}\right)}, \quad |\mu (t)| \leq t^{-\frac{d}{4}}
\]

for all \( t \in [T_0, \infty) \). In view of relation (2.27) we write \( B (\chi, \lambda) = \frac{1}{|\chi|} b (|\chi|, \lambda) \). Linearizing \( B (\chi, \lambda) \) around \( (\chi_{\text{app}}, \lambda \infty) \) we get

\[
B (\chi, \lambda) = B (\chi_{\text{app}}, \lambda \infty) + (\theta_0 \infty \cdot \delta) \theta_0 \infty' b (|\chi_{\text{app}}|, \lambda \infty) + f_2
\]

22
where (2.143)

\[ f_2 = O \left( |\partial_x B (\chi_{\text{app}}, \lambda^\infty)| |\mu| + \frac{|\delta|}{|\chi_{\text{app}}|} |b (|\chi_{\text{app}}|, \lambda^\infty)| + |\delta|^2 |b' (|\chi_{\text{app}}|, \lambda^\infty)| \right) \]
\[ = O \left( |U'_V (r)| \left( |\mu| + \frac{|\delta|}{|\chi_{\text{app}}|} + |\delta|^2 \right) \right). \]

We use the Cartesian coordinates with the \( x_1 \)-axis directed along the vector \( \theta_0^\infty \). We decompose \( \delta = \sum_{j=1}^d \delta_j \vec{e}_j \), and \( \vec{e}_j \), \( j = 1, ..., d \) is the canonical basis in these coordinates. Then \( (\theta_0^\infty \cdot \delta) \theta_0^\infty = \delta_1 \) and by (2.150) we obtain

\[ \begin{cases} 
\ddot{\delta} = b' (|\chi_{\text{app}}|, \lambda^\infty) \delta_1 \vec{e}_1 + 2f_1 (\chi, \beta, \lambda) + 2f_2, \\
\dot{\lambda} = g_1 (\chi, \beta, \lambda).
\end{cases} \tag{2.158} \]

It follows from (2.138) that \( \dot{r}^\infty = \mathcal{X} \), where

\[ \mathcal{X} = \sqrt{U'_{V, \lambda^\infty} \left( \frac{r^\infty}{\lambda^\infty} \right)}. \tag{2.159} \]

Using (2.151), (2.152), (2.157) and (2.148) we estimate

\[ |f_1 (\chi, \beta, \lambda)| \leq C (\chi, \beta, \lambda) \mathcal{X}^{p_1} \leq Ct^{-2-p_1}, \]
\[ |f_2| \leq \frac{C}{t^2} \left( t^{-\frac{\xi}{2}} + t^{-\frac{d}{2}} (\ln t)^{-1} + t^{-p_1} \right) \]

and

\[ |g_1 (\chi, \beta, \lambda)| \leq C (\chi, \beta, \lambda) \mathcal{X} \leq Ct^{-2}. \]

Then, from (2.158), for some \( T_0 > 0 \) big enough we get

\[ |\delta_j| \leq \frac{1}{2t^{\frac{\xi}{4}}}, \quad |\delta_j| \leq \frac{1}{2t^{\frac{d}{4} + \frac{\xi}{4}}}, \quad j = 2, ..., d, \tag{2.160} \]

and

\[ |\mu (t)| \leq \frac{1}{2t^{\frac{\xi}{3}}}. \tag{2.161} \]

for \( t \in [T_0, \infty) \). Let us estimate \( \delta_1 \). Using Lemma 2.2 with \( V' \) instead of \( V \), we have

\[ b' (|\chi_{\text{app}}|, \lambda^\infty) = c_0 (\lambda^\infty) (1 + o (1)) U'_{V, \lambda^\infty} \left( \frac{r_{\text{app}}}{\lambda^\infty} \right). \]

Then

\[ \ddot{\delta}_1 = c_0 (1 + o (1)) U'_{V, \lambda^\infty} \left( \frac{r_{\text{app}}}{\lambda^\infty} \right) \delta_1 + O \left( t^{-2-p_1} \right). \]

By (2.144) and (2.155) we deduce

\[ U'_{V, \lambda^\infty} \left( \frac{r_{\text{app}}}{\lambda^\infty} \right) = (\mathcal{R}_1 + o (1)) c_0^{-1} (\lambda^\infty) t^{-2}. \]

Thus

\[ \ddot{\delta}_1 = \mathcal{R}_1 t^{-2} \delta_1 + O (o (1) t^{-2} |\delta_1| + t^{-2-p_1}) \tag{2.162} \]

If \( \mathcal{R}_1 = 0 \), we improve the estimate on \( \delta_1 \) and \( \dot{\delta}_1 \) (2.157) directly by integrating (2.162). If \( \mathcal{R}_1 \neq 0 \), the linear equation \( \ddot{\delta}_1 = \mathcal{R}_1 t^{-2} \delta_1 \) has two linearly independent solutions \( \delta_1^{(1)} , \delta_1^{(2)} \) such that \( |\delta_1^{(1)}|, |\delta_1^{(2)}| \leq Ct \). By variation of parameters we solve (2.162) and via (2.157) we estimate the solution as

\[ |\delta_1| \leq Co (1) \int_t^\infty (\tau^{-1} |\delta_1| + \tau^{-1-p_1}) \quad \text{and} \quad |\delta_1| \leq Co (1) (t^{-1} |\delta_1| + t^{-1-p_1}). \]

Hence, for \( T_0 > 0 \) big enough we obtain the bound \( |\delta_1| \leq \left( 2t^{\frac{\xi}{4}} \right)^{-1} \) and \( |\delta_1| \leq \left( 2t^{\frac{d}{4} + \frac{\xi}{4}} \right)^{-1} \) for \( t \in [T_0, \infty) \). Combining the last relation with (2.160) and (2.161) we strictly improve (2.157). Then, by a contraction argument, via (2.156), we prove the existence of a solution \( \Xi^{(2)} (\tau) \) for (2.150) with the asymptotics (2.146).

Let now \( V = V^{(2)} \). First we suppose that the potential \( V \) decays faster than \( |x|^{-2} \). Namely,

\[ |V (r)| \leq C |r|^{-2}. \tag{2.163} \]
We consider the intermediate system

\[
\begin{align*}
\dot{\beta}_{\text{app}} &= \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \dot{\chi}_{\text{app}} = 2\beta_{\text{app}}, \\
\end{align*}
\]

(2.164)

Recall that

\[
|h_1^{(k)}(r)| \leq Cr^{-1}h_1^{(k-1)}(r), \quad k \geq 1.
\]

(2.165)

Then

\[
|\dot{r} | \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) \leq C |h'(\frac{\chi_{\text{app}}}{|\chi_{\text{app}}|})| \left( |h'(\frac{\chi_{\text{app}}}{|\chi_{\text{app}}|})| + \frac{1}{|\chi_{\text{app}}|} V'(\frac{\chi_{\text{app}}}{|\chi_{\text{app}}|}) + |\chi_{\text{app}}|^{-2} V'(\frac{\chi_{\text{app}}}{|\chi_{\text{app}}|}) \right).
\]

Thus, it follows from (2.164) that

\[
\frac{d}{dt} |\dot{\chi}_{\text{app}}|^2 = \frac{d}{dt} \left( 2V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) (1 + O(h'(\chi_{\text{app}}))) \right).
\]

(2.166)

We take \(\chi_{\text{app}} = r_{\text{app}} \theta_{\text{app}}\) of angular momentum \(\mu = 0\) such that \(r_{\text{app}} \to \infty\), \(\dot{r}_{\text{app}} \to 0\) as \(t \to \infty\). Integrating (2.166) we get

\[
\dot{r}_{\text{app}} = \sqrt{2V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) (1 + O(h'(r_{\text{app}})))}.
\]

Then,

\[
t = \int_{r_0}^{r_{\text{app}}} \frac{dr}{\sqrt{2V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) (1 + O(h'(r)))}} + t_0
\]

(2.167)

Let \(z(t) = \frac{r_{\text{app}(t)}}{r^\infty(t)}\). We introduce

\[
F(z) = \int_{r_0}^{2r^\infty} \frac{dr}{\sqrt{2V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) (1 + O(h'(r)))}}.
\]

and

\[
F_0 = \int_{r_0}^{r^\infty} \frac{dr}{\sqrt{2V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right)}}.
\]

Expanding \(F(z)\) around \(z = 1\) we have

\[
F(z) = F_0 + O \left( \left( V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) \right)^{1/2} \right) + F'(1) (z - 1) + O \left( F''(1) (z - 1)^2 \right).
\]

By (2.134) \(t = F_0 + t_0\). Then, using (2.167) we deduce

\[
\frac{F(z)}{F_0} = 1 = 1 + \frac{O \left( \left( V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) \right)^{1/2} \right) + r^\infty F'(1) (z - 1) + O \left( r^\infty F''(1) (z - 1)^2 \right)}{F_0}.
\]

Hence,

\[
|z - 1| = \frac{O \left( \left( V \left( \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \right) \right)^{1/2} \right)}{r^\infty F'(1)} = O \left( \frac{1}{r^\infty} \right).
\]

Thus, we obtain

\[
\frac{|r_{\text{app}}(t)|}{r^\infty(t)} - 1 = o(1) \quad \text{and} \quad |\beta_{\text{app}} - \beta^\infty| = o(1) |\beta^\infty|.
\]

Then, arguing similarly to (2.157), we prove the existence of a solution \(\Xi^{(N)}(t)\) for (2.150) with the asymptotics (2.146) in the case when \(V\) decay faster than \(|x|^{-2}\).

Now, we consider the case of potentials that decay slower than \(|x|^{-2}\). We suppose that (2.136) holds. Let \(\chi^\infty = r^\infty \theta^\infty\) and \(\theta_{\text{app}}^\infty = \lim_{\theta \to \infty} \theta^\infty(t)\). Recall the definition (2.118) of \(M_1\) and estimate (2.119). Let us consider the intermediate system

\[
\begin{align*}
\dot{\chi}_{\text{app}} &= \frac{\chi_{\text{app}}}{|\chi_{\text{app}}|} \dot{\chi}_{\text{app}} = 2\beta_{\text{app}}, \\
\dot{\beta}_{\text{app}} &= \frac{1}{2\chi_{\text{app}} |\chi_{\text{app}}|} V'(\frac{\chi_{\text{app}}}{|\chi_{\text{app}}|}), \\
\dot{\lambda}_{\text{app}} &= M_1 (\lambda_{\text{app}}) = O (|\beta_{\text{app}}| \Psi).
\end{align*}
\]

(2.168)
Denote

\[ F_\mu (\xi) = \sqrt{2V\left(\frac{\xi}{\lambda}\right)} - \frac{\mu^2}{\xi^2}. \]

We search a solution to (2.168) in the form

\[ \chi_{\text{app1}} = r_{\text{app1}} \theta_0^\infty. \]

We denote \( \mu_{\text{app1}} = \lambda_{\text{app1}} - \lambda^\infty. \) Using (2.165) we get

\[ O \left( \mu_{\text{app1}} \left(V' \left( \frac{r_{\text{app1}}}{\lambda_{\text{app1}}} \right) + r_{\text{app1}} V'' \left( \frac{r_{\text{app1}}}{\lambda_{\text{app1}}} \right) \right) \right) = O \left( \mu_{\text{app1}} V'' \left( \frac{r_{\text{app1}}}{\lambda_{\text{app1}}} \right) \right) \]

and system (2.168) reads

\[
\begin{align*}
\dot{r}_{\text{app1}} &= \frac{1}{\lambda_{\text{app1}}} V' \left( \frac{r_{\text{app1}}}{\lambda_{\text{app1}}} \right) + O \left( \mu_{\text{app1}} V' \left( \frac{r_{\text{app1}}}{\lambda_{\text{app1}}} \right) \right), \\
\lambda_{\text{app1}} &= M_1 \left( \lambda_{\text{app1}} \right) = O \left( \dot{r}_{\text{app1}} \Psi \right),
\end{align*}
\]

(2.169)

Let us prove that for some \( r_{\text{app1}} \) there is \( T_0 > 0 \) such that

\[ |z(t) - 1| \leq \frac{1}{(r^\infty)^{\frac{2}{3}}} \text{ and } |\mu_{\text{app1}}| \leq (r^\infty)^{-\frac{2}{3}}, \text{ for } t \in [T_0, \infty), \]

(2.170)

where we denote \( z(t) = \frac{r_{\text{app1}}(t)}{r^\infty(t)}. \) Integrating the first equation we get

\[ \dot{r}_{\text{app1}} = F_0 \left( r_{\text{app1}} \right) + O \left( \mu_{\text{app1}} \sqrt{V \left( \frac{r_{\text{app1}}}{\lambda_{\text{app1}}} \right)} \right) \]

From (2.138) we get

\[ \dot{r}^\infty = F_\mu (r^\infty). \]

Then

\[ \dot{z} = \frac{1}{r^\infty} \left( \dot{r}_{\text{app1}} - z r^\infty \right) = \frac{1}{r^\infty} \left( F_0 \left( z r^\infty \right) - z F_\mu (r^\infty) \right) \]

\[ = \frac{1}{r^\infty} \left( (F_0 \left( z r^\infty \right) - z F_0 (r^\infty)) + (r^\infty) - F_\mu (r^\infty) \right) + O \left( \mu_{\text{app1}} \sqrt{V \left( \frac{r_{\text{app1}}}{\lambda_{\text{app1}}} \right)} \right). \]

We put \( w = z - 1. \) Linearizing \( F_0 \left( z r^\infty \right) \) around \( z = 1 \) we have

\[ \dot{w} = \left( F_0' (r^\infty) - \frac{F_0 (r^\infty)}{r^\infty} \right) w + w (r^\infty) \]

(2.171)

with

\[ w (r^\infty) = O \left( r^{\infty} F_0'' \left( r^\infty \right) r^2 \left( r^\infty \right)^{-3} F_0' \left( r^\infty \right) + \sqrt{V \left( \frac{r^\infty}{\lambda_{\text{app1}}} \right)} \right). \]

Using (2.136) we see that \( |h_1 (r^\infty)| \leq c \ln r^\infty. \) Then, by (2.170), by (2.165) we get

\[ w (r^\infty) = O \left( \frac{\dot{r}^\infty}{(r^\infty)^{\frac{1}{3} + \frac{\delta}{2}}} \right). \]

(2.172)

By variation of parameters we get

\[ w (t) = w_0^{-1} (t) \int_t^\infty w_0 (\tau) w (r^\infty (\tau)) \, d\tau \]

with

\[ w_0 (t) = e^{\int_t^T \left( F_0' (r^\infty) - \frac{F_0 (r^\infty)}{r^\infty} \right) \, d\tau}. \]

For any \( 0 < \delta < 1, \) there exists \( T_0 > 0 \) sufficiently big, such that

\[ \ln \left( \frac{(r^\infty)^{1-\delta}}{V^{1+\delta}} \right) \leq - \int_{T_0}^t \left( F_0' (r^\infty) - \frac{F_0 (r^\infty)}{r^\infty} \right) \leq \ln \left( \frac{(r^\infty)^{1+\delta}}{V^{1-\delta}} \right). \]

(2.173)
Then, using (2.172) and (2.15), and taking $\delta$ small enough we obtain

$$|w(t)| \leq C (V r^\infty)^{2\delta} \int_t^\infty \frac{r^{\infty}}{(r^\infty)^{1+\frac{\rho}{2}}} dr \leq C (r^\infty)^{2\delta(1-\frac{\rho}{2})} \leq \frac{1}{2 (r^\infty)^{\frac{\rho}{2}}} \tag{2.174}$$

Integrating the second equation in (2.169) we deduce $|\mu_{app1}| \leq \frac{1}{2} (r^\infty)^{-\frac{\rho}{2}}$. Therefore, we strictly improve (2.170). Thus, we show that there is a solution $\chi_{app1}$ to (2.168) such that

$$\left| \frac{r_{app1} (t)}{r^\infty (t)} - 1 \right| \leq \frac{1}{2 (r^\infty)^{\frac{\rho}{2}}} \quad |\beta_{app1} - \beta^\infty| = o(1) |\beta^\infty| \quad \text{and} \quad |\mu_{app1}| \leq (r^\infty)^{-\frac{\rho}{2}} \quad \text{for} \ t \in [T_0, \infty). \tag{2.175}$$

Let us now return to the full system (2.141). By Lemmas 2.2 and 2.7 we have

$$\begin{align*}
\dot{\beta} &= \frac{1}{\lambda M_1} \left( \frac{1}{2} \chi V' \left( \frac{\chi}{\lambda} \right) \right) + f_2 (\chi, \beta, \lambda), \\
\dot{\lambda} &= M_1 (\lambda) + g_2 (\chi, \beta, \lambda),
\end{align*} \tag{2.176}$$

(we omit the index $N$ in $\chi, \beta, \lambda$) where

$$|f_2 (\chi, \beta, \lambda)| \leq C (\chi, \beta, \lambda) \left( \Psi^2 + r \left( \left| \frac{\chi}{\lambda} \right| \right) + |\beta| \left| V'' \left( \frac{\chi}{\lambda} \right) \right| + e^{-\frac{|\chi|}{\lambda}} \right),$$

and

$$|g_2 (\chi, \beta, \lambda)| \leq C (\chi, \beta, \lambda) \Psi^2.$$}

We write $\chi = \chi_{app1} + \tilde{\delta}$ and $\lambda = \lambda_{app1} + \mu$. Then, as $\chi_{app1}$ and $\lambda_{app1}$ solve (2.168), from (2.176) we deduce

$$\begin{align*}
\tilde{\delta}'' &= -\frac{1}{\chi_{app1} r_{app1} (\delta \cdot \theta_0^\infty - \theta_0^\infty + \delta)} V'' \left( \frac{r_{app1}}{\chi_{app1}} \right) + (\delta \cdot \theta_0^\infty) \theta_0^\infty (\lambda_{app1})^{-2} V'' \left( \frac{r_{app1}}{\lambda_{app1}} \right) + f_3 (\chi, \beta, \lambda), \\
\dot{\mu} &= M_1' (\lambda_{app1}) \mu + O (\Psi^2).
\end{align*}$$

where

$$\begin{align*}
f_3 (\chi, \beta, \lambda) &= f_2 (\chi, \beta, \lambda) + O \left( \left( V'' \left( \frac{r_{app1}}{\chi_{app1}} \right) r_{app1} + V' \left( \frac{r_{app1}}{\chi_{app1}} \right) \right) |\mu| \right) \\
&\quad + O \left( V''' \left( \frac{r_{app1}}{\chi_{app1}} \right) + (r_{app1})^{-1} V'' \left( \frac{r_{app1}}{\chi_{app1}} \right) + (r_{app1})^{-2} V' \left( \frac{r_{app1}}{\chi_{app1}} \right) \right) |\delta|^2.
\end{align*}$$

We claim that for some $T_0 > 0$ sufficiently big,

$$|\delta (t)| \leq r_{app1}^{-\rho/4}, \quad \left| \dot{\delta} (t) \right| \leq r_{app1}^{-(1+\rho/4)}, \quad |\mu (t)| \leq r_{app1}^{-(1+\rho/4)}. \tag{2.177}$$

Observe that by (2.136) $|h_1 (r)| \leq C \ln r$. Then, using (2.15), (2.165) and (2.177) we estimate

$$\Psi \leq C \frac{\ln r_{app1}}{r_{app1}} \left| V \left( \frac{r_{app1}}{\lambda_{app1}} \right) \right|$$

and

$$|f_3 (\chi, \beta, \lambda)| \leq C \frac{(\ln r_{app1}) |V (r_{app1})|}{r_{app1}^2} \left( \frac{1}{r_{app1}} + \left| V \left( \frac{r_{app1}}{\lambda_{app1}} \right) \right|^{1/2} \right).$$

Arguing similarly to the case of (2.158) we show (2.177). Therefore, using (2.175), we prove the existence of a solution $\Xi^{(N)} (t)$ for (2.141) with the asymptotics (2.146). \blacksquare

3 Proof of Theorem 1.5.

For a solution $\Xi^{(N)} (t)$ given by Lemma 2.10 and

$$\gamma^{(N)} (t) = -\frac{1}{(\lambda^{(N)} (t))^2} + \left| \beta^{(N)} (t) \right|^2 + \dot{\beta}^{(N)} (t) \cdot \chi^{(N)} (t),$$

And the next line continues...
we define $W^{(N)}(t,x;\Xi^{(N)}(t))$ by (2.126) and (2.127). For $n \in \mathbb{N}$ let a sequence $T_n \to \infty$, as $n \to \infty$ and $u_n \in H^1$ be the solution to the NLS equation with initial data $u_n(T_n,x) = W^{(N)}(T_n,x;\Xi^{(N)}(T_n))$. That is

$$
\begin{align*}
\left\{ \begin{array}{l}
\partial_t u_n + \Delta u_n + |u_n|^{p-1} u_n + V(x) u_n = 0 \\
u_n(T_n,x) = W^{(N)}(T_n,x;\Xi^{(N)}(T_n)).
\end{array} \right.
\end{align*}
$$

(3.1)

Recall that for a given vector $\Xi = (\chi,\beta,\lambda) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$ and $\gamma \in \mathbb{R}$ (see (2.126) and (2.127))

$$
W(t,x;\Xi) = W^{(N)}(t,x;\Xi) = \lambda^{-\frac{4}{p-1}} W\left(\frac{x-\chi}{\lambda}\right) e^{-i\gamma t} e^{i\beta x}.
$$

(3.2)

By using the implicit function theorem in the following lemma we show that as long as the solution evolves close to the solitary-wave solution for the free NLS equation, it may be decomposed as

$$
u_n(t,x) = W^{(N)}(t,x;\tilde{\Xi}(t)) + \varepsilon(t,x),
$$

(3.3)

where the modulation parameters $\Xi(t) \in C^1(T_n-T, T_n+T)$, for some $T > 0$, are chosen in such a way that the remainder $\varepsilon$ satisfies the orthogonal conditions

$$
(\xi,W) = (\xi,yW) = (\xi,i\Delta W) = (\xi,i\nabla W) = 0
$$

(3.4)

with

$$
\xi(t,y) = \left(\lambda(t)\right)^{-\frac{2}{p-1}} \varepsilon(t,\lambda(t)y + \chi(t)) e^{i\gamma(t)} e^{-i\beta(t)}(\lambda(t)y+\chi(t)).
$$

(3.5)

\textbf{Lemma 3.1} Given $\Xi_0 = (\chi_0,\beta_0,\lambda_0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+$ and $\gamma_0 \in \mathbb{R}$, let $u(t,x)$ be a solution for the NLS equation with a potential (1.1) defined on the interval $[T,T_{in}]$, with some $0 < T < T_{in}$, with the initial value $u(T_{in},x) \in H^2$ satisfying

$$
\|u(T_{in},x) - W(T_{in},x;\Xi_0)\| \leq \delta,
$$

(3.6)

with $\delta > 0$. Suppose that $\varepsilon(T_{in},x) = u(T_{in},x) - W(T_{in},x;\Xi_0)$ satisfies (3.4). Then, there exist $A > 0$ sufficiently big and $\delta_0 > 0$ small enough such that for all $|\chi_0| \geq A$ and $\delta \leq \delta_0$ the following affirmation is true. There is an open interval $I(\delta) \ni T_{in}$, a unique vector $\Xi(t) = (\chi(t),\beta(t),\lambda(t)) \in C^1(I(\delta))$ and $\gamma(t) \in C^1(I(\delta))$ satisfying $\Xi(T_{in}) = \Xi_0$ and $\gamma(T_{in}) = \gamma_0$, such that $u$ decomposes as

$$
u(t,x) = W(t,x;\Xi(t)) + \varepsilon(t,x),
$$

(3.7)

where the error $\varepsilon$ satisfies (3.4) for any $t \in I(\delta)$.

\textbf{Proof.} Lemma 3.1 is proved similarly to Lemma 3 of [28] or Lemma 3 of [37]. We omit the details. $\blacksquare$

We now aim to compare the solution $(\Xi(t),\gamma(t))$ given by Lemma 3.1 with the solution $(\Xi^{(N)}(t),\gamma^{(N)}(t))$ of the approximate system (2.141). We have the following a priori estimates.

\textbf{Lemma 3.2} Let $\lambda^\infty \geq \lambda_0 > 0$ satisfy $(\lambda^\infty)^2 \sup_{r \in \mathbb{R}} V(r) < 1$. Suppose that $|V(r)| \leq C e^{-cr}$, for some $c > 0$. There exists $T_0 > 0$ large enough such that for all $t \in [T_0,T_n]$, the estimates

$$
\begin{align*}
\|\varepsilon(t)\|_{H^1} &\leq t^{-2}, \\
|\beta(t) - \beta^{(N)}(t)| + |\lambda(t) - \lambda^{(N)}(t)| &\leq t^{-1+\frac{4}{p-1}}, \\
|\chi(t) - \chi^{(N)}(t)| &\leq t^{-\frac{4}{p-1}}, \\
|\gamma(t) - \gamma^{(N)}(t)| &\leq t^{-\frac{4}{p-1}}.
\end{align*}
$$

(3.8)

are satisfied. If $V = V^{(2)}$, for some $T_0 > 0$ large enough the following a priori estimates are true

$$
\begin{align*}
\|\varepsilon(t)\|_{H^1} &\leq \lambda^N, \\
|\beta(t) - \beta^{(N)}(t)| + |\lambda(t) - \lambda^{(N)}(t)| &\leq (\Psi^2 + |V''(\lambda_0)|) \lambda^{\frac{N}{2}+2}, \\
|\chi(t) - \chi^{(N)}(t)| + |\gamma(t) - \gamma^{(N)}(t)| &\leq \Psi \lambda^{\frac{N}{2}+1}
\end{align*}
$$

(3.9)

for all $t \in [T_0,T_n]$.

\textbf{Remark 3.3} The condition on $\lambda^\infty$ in Lemma 3.2 is made in order to assure, via Lemma 2.10, that the assumptions of Lemmas 2.4, 2.6 and 2.7 are satisfied.
Lemma 3.2 is proved in Section 4 below. We now use the following corollary to prove Theorem 1.5.

**Corollary 3.4** For \( n \in \mathbb{N} \) let a sequence \( T_n \to \infty \), as \( n \to \infty \) and \( u_n \in H^1 \) be a solution to (3.1). Then there exists \( T_0 \) independent of \( n \) such that for all \( n \in \mathbb{N} \) and \( t \in [T_0, T_n] \),

\[
\left\| u_n(t) - W \left( t, x; \Xi^{(N)}(t) \right) \right\|_{H^1} \leq C \Theta_{\infty} \tag{3.10}
\]

where

\[
\Theta_{\infty} = \left\{ \begin{array}{ll}
t^{-\frac{n-1}{2}}, & \text{if } |V(r)| \leq Ce^{-cr}, \; c > 0, \\
\frac{\lambda}{\sqrt{2} + 1}, & \text{if } V = V(2).
\end{array} \right.
\]

**Proof.** We have

\[
\left\| u_n(t) - W \left( t, x; \Xi^{(N)}(t) \right) \right\|_{H^1} \leq \left\| u_n(t) - W(t, x; \Xi(t)) \right\|_{H^1} + \left\| W(t, x; \Xi(t)) - W \left( t, x; \Xi^{(N)}(t) \right) \right\|_{H^1} \leq C \Theta_{\infty}.
\]

By a compactness argument (see page 1525 of [26]) we now show that Theorem 1.5 is a direct consequence of the uniform backward estimate for the sequence \( u_n \) presented in Corollary 3.4. By (3.10) there is \( C > 0 \) such that for any \( n \in \mathbb{N} \),

\[
\left\| u_n(t) \right\|_{H^1} \leq C,
\]

for \( t \in [T_0, T_n] \). Then, by Lemma 3.4 of [26] there exists \( U_0 \in H^1 (\mathbb{R}^d) \) and a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) such that \( u_{n_k}(T_0) \to U_0 \) in \( L^2 (\mathbb{R}^d) \), as \( n_k \to \infty \). We consider the solution \( U \) to the initial value problem

\[
\begin{aligned}
i\partial_t u + \Delta u + |u|^{p-1} u + V(x) u &= 0, \\
(t, x) &\in \mathbb{R} \times \mathbb{R}^d, \; u(T_0) = U_0.
\end{aligned}
\]

Fix \( t \geq T_0 \). There is \( n_0 \) large enough such that \( T_n \geq t \), for \( n \geq n_0 \). By the continuous dependence of the solution of (1.1) on the initial data in \( L^2 (\mathbb{R}^d) \) (see Theorem 4.6.1 of [9]) \( u(t) \) is global and

\[
u_n(t) \to u(t) \quad \text{in } L^2 (\mathbb{R}^d), \quad \text{as } n \to \infty.
\]

Since \( u_{n_k}(t) - W(t, x; \Xi^{(N)}(t)) \) is uniformly bounded in \( H^1 (\mathbb{R}^d) \), it converges weakly to \( u(t) - W(t, x; \Xi^{(N)}(t)) \) in \( H^1 (\mathbb{R}^d) \), as \( n \to \infty \). Thus, by (3.10) we prove that

\[
\left\| u_{n_k}(t) - W \left( t, x; \Xi^{(N)}(t) \right) \right\|_{H^1} \leq C \Theta_{\infty},
\]

for all \( t \geq T_0 \). Therefore, Theorem 1.5 follows from the definition (2.126) of \( W \), Lemmas 2.6 and 2.7, the properties of \( \Xi^{(N)}(t) \) described by Lemma 2.10 and the relation \( U_{V, \infty} = U_{\lambda, \infty, V} \), that follows from Definition 1.2 and (2.4).

### 4 A priori estimates. Proof of Lemma 3.2.

#### 4.1 Control of the Modulation Parameters.

**Case I: Fast decaying potentials.**

Let the potential \( |V(r)| \leq Ce^{-cr} \), for some \( c > 0 \). Suppose that (3.8) is true for all \( t \in [T^*, T_n] \), with \( T_0 \leq T^* < T_n \). Observe that by (3.8), (2.145) and (2.146)

\[
\frac{|\chi(t)|}{|\chi^{(N)}(t)|} = 1 + o(1), \quad \frac{|\beta(t)|}{|\beta^{(N)}(t)|} = 1 + o(1) \quad \text{and} \quad \lambda(t) = \lambda^{(N)} + o(1),
\]

as \( t \to \infty \). Let \( \Phi \in C^\infty (\mathbb{R}^d) \) satisfy the estimate \( |\Phi(y)| \leq C (|y|^m Q(y) + Y) \) with some \( m \geq 0 \). We denote

\[
\Phi_1(t, x) = \lambda(t)^{-\frac{i}{\lambda(t)}} \Phi \left( \frac{x - \chi(t)}{\lambda(t)} \right) e^{-i\gamma(t)} e^{i\beta(t) \cdot x}.
\]

Let \( N \geq 1 \). For \( 1 \leq j \leq N \) and \( \Xi(t) \), let \( B_j = B_j (\Xi(t)) \), \( M_j = M_j (\Xi(t)) \) be the approximated modulation equations given by Lemma 2.6 corresponding to \( \Xi(t) \). Let

\[
B^{(N)}(t) = B^{(N)} (\Xi(t)) = \sum_{j=1}^N B_j (\Xi(t)) \quad \text{and} \quad M^{(N)}(t) = M^{(N)} (\Xi(t)) = \sum_{j=1}^N M_j (\Xi(t)).
\]
We put
\[ \mathcal{M}(t) = |\dot{\chi}(t) - 2\beta(t)| + \left| \frac{1}{\lambda^2(t)} - |\beta(t)|^2 - \dot{\beta}(t) \cdot \chi(t) \right| + \left| \dot{\beta}(t) - \mathcal{B}^{(N)}(\Xi(t)) \right| + \left| \dot{\lambda}(t) - \mathcal{M}^{(N)}(\Xi(t)) \right|. \]

Let us write the equation for \( \varepsilon \). Introducing the decomposition (3.7) into (1.1) and using (2.128) we obtain
\[ i\partial_t \varepsilon = -\Delta \varepsilon - V(x) \varepsilon - \left( \lambda^{-\frac{2}{\alpha^2}}(t) \left( \lambda^{-2}(t) N_0(W,\zeta) + \mathcal{E}^{(N)} + R(W) \right) \left( t, \frac{x - \chi(t)}{\lambda(t)} \right) e^{-i\gamma(t)} e^{i\beta(t) \cdot x} \right) \]
(4.2)
where we denote
\[ N_0(W,\zeta) = |W + \zeta|^p - 1 (W + \zeta) - |W|^p - 1 W. \]
with \( W \) given by (2.127) and \( \zeta \) defined by (3.5). Using (4.2) we have
\[ \frac{d}{dt} \text{Im} \int \varepsilon \Phi_1 = -\text{Re} \int i\partial_t \varepsilon \Phi_1 + \text{Re} \int \varepsilon (i\partial_t \Phi_1) \]
\[ = \text{Re} \int \varepsilon \left( i\partial_t \Phi_1 + \Delta \Phi_1 + V(x) \Phi_1 \right) + \text{Re} \int N_0(W,\varepsilon) \Phi_1(t,x) + \lambda^{-\frac{2}{\alpha^2}} \text{Re} \int (\mathcal{E}^{(N)} + R(W)) \Phi. \]  
(4.4)

Similarly to (2.5) we calculate
\[ (i\partial_t + \Delta + V(x)) \Phi_1 = \lambda^{-\frac{2}{\alpha^2}} \left( -\dot{\lambda} \Phi + \lambda^2 V \left( \left| y + \frac{x}{\lambda} \right| \right) \Phi - i\lambda \mathcal{M}^{(N)} \Phi - \lambda (\mathcal{B}^{(N)} \cdot y) \Phi \right) \]
\[ - \frac{p+1}{2} |W|^p - 1 \Phi - \frac{p-1}{2} |W|^{p-3} W^2 \Phi - i\lambda \left( \lambda - \mathcal{M}^{(N)} \right) \Phi - \lambda^3 \left( \left( \beta - \mathcal{B}^{(N)} \right) \cdot y \right) \Phi \]
\[ + \lambda^2 \left( \dot{\gamma} + \frac{1}{\lambda^2} - |\beta|^2 - \dot{\beta} \cdot \chi \right) \Phi \left( t, \frac{x - \chi}{\lambda} \right) e^{-i\gamma} e^{i\beta \cdot x}. \]

where we define
\[ \tilde{\mathcal{L}} f := -\Delta f + f - \frac{p+1}{2} |W|^p - 1 f - \frac{p-1}{2} |W|^{p-3} W^2 f, \quad f \in H^1. \]  
(4.5)
Then, using (2.62) to estimate the modification equations \( \mathcal{B}^{(N)}, \mathcal{M}^{(N)} \) and relation (2.129) from (4.4) we get
\[ \frac{d}{dt} \text{Im} \int \varepsilon \Phi_1 = \lambda^{-\frac{2}{\alpha^2}} \text{Re} \int R(W) \Phi - \text{Re} \int \varepsilon \lambda^{-\frac{2}{\alpha^2}} \left( \left( \tilde{\mathcal{L}} - \lambda^2 V \left( \left| y + \frac{x}{\lambda} \right| \right) \right) \Phi \left( t, \frac{x - \chi}{\lambda} \right) e^{-i\gamma} e^{i\beta \cdot x} + E_{N_1} + E_0 \right) \]
(4.6)
where
\[ N_1(W,\zeta) = N_0(W,\zeta) - \frac{p+1}{2} |W|^p - 1 \zeta - \frac{p-1}{2} |W|^{p-3} W^2 \zeta \]
(4.7)
\[ E_{N_1} = \text{Re} \int \lambda^{-\frac{2}{\alpha^2}} \left( N_1(W,\zeta) \Phi \left( t, \frac{x - \chi}{\lambda} \right) e^{-i\gamma} e^{i\beta \cdot x} \right) \]
and
\[ E_0 = O \left( \Theta \left( \left| |\beta| + \Theta \right| A^{(N)} + \right) + \left( \left| |\beta| + \Theta \right|^2 + U \right) + \mathcal{M}(t) \right) \parallel \varepsilon \parallel_{H^1} \]
with \( \Theta = \Theta \left( \frac{|\lambda|}{\lambda} \right) \) and \( U = |U_V(\left| \Phi \right|)| \). Let us estimate \( E_{N_1} \). Observe that
\[ |N_1(W,\zeta)| = O \left( \left| |\zeta|^p + |\zeta|^{p-\delta} + |\zeta|^2 \right|^\delta \right), \quad \delta > 0, \]
(4.8)
for \( p \neq 2 \) \( (N_1(W,\zeta) = O \left( \left| |\zeta|^2 \right|^\delta \right), \quad p = 2) \). If we estimate \( E_{N_1} \) by using (4.8) we have
\[ E_{N_1} = O \left( \left| \varepsilon \right|_{H^{\delta}}^p + \right) \left( \left| \varepsilon \right|_{H^{\delta}}^2 \right). \]
By (3.8) this gives the decay \( E_{N_1} \sim t^{-p+\delta} + t^{-2} \). In order to obtain the a priori estimates on modification parameters in (3.8), we need \( \mathcal{M}(t) \) to be integrable twice on \( [T_0, \infty) \). But, in the case when \( p < 2 \), we only have \( E_{N_1} \sim t^{-p+\delta} \). Hence, we need to obtain a better estimate on \( E_{N_1} \). We use an argument of [37] (see the proof of Proposition 10 on page 41). Let
\[ \Omega = \left\{ y \in \mathbb{R}^d : \max_{1 \leq j \leq N} \left| T^{(j)}(y) \right| \geq \frac{1}{2N} Q(y) \right\}. \]  
(4.9)
Then, as by (2.61) \( \left| T^{(j)} \right| \leq CY, \quad 1 \leq j \leq N \), we estimate
\[ |\Phi(y)| \leq C \left( \left| |y|^M Q(y) + Y \right| \leq C_N \inf_{0 < \delta < 1} \left( C(\delta) Q^{1-\delta}(y) \right) \left| y \right|^M e^{-\delta|y|} \right) + CY \leq CY, \quad y \in \Omega, \]
Thus, using the relation (4.8) and Sobolev embedding theorem to control the $L^p$ norm of $\zeta$, we estimate

$$
\int_{y \in \Omega} |(\mathcal{N}_1 (W, \zeta) \Phi) (y)| \, dy \leq C_N C (\delta) \| \zeta \|_{H^1}^{p_0} \Theta^{1-\delta}, \quad p_0 = \min \{ p - \delta, 2 \},
$$

(4.10)

for all $0 < \delta < 1$. Suppose now that $y \notin \Omega$. Using (4.8) we estimate

$$
|\mathcal{N}_1 (W, \zeta)| = O \left( |W|^{p-2} |\zeta|^2 \right).
$$

(4.11)

Thus, by Sobolev embedding theorem, we get

$$
\int_{y \notin \Omega} |(\mathcal{N}_1 (W, \zeta) \Phi) (y)| \, dy \leq C \| \zeta \|_{H^1}^2.
$$

(4.12)

Hence, from (4.10) and (4.12) we estimate

$$
\left| \Re \int \lambda^{-\frac{2p}{p-2}} (\mathcal{N}_1 (W, \zeta) \Phi) \left( t, \frac{x - \lambda}{\lambda} \right) e^{-i\gamma t} e^{i\beta x} \right| \leq C_N C (\delta) \| \zeta \|_{H^1}^{p_0} \Theta^{1-\delta} + C \| \zeta \|_{H^1}^2
$$

(4.13)

for all $0 < \delta < 1$.

Let us now calculate $\mathcal{L} \Phi$ for $\Phi = iW, iyW, \Lambda W, \nabla W$. We use (2.46) and Lemma 2.6 to derive

$$
\Delta W - W + |W|^{p-1} W + \lambda^2 V \left( |y + \frac{\lambda}{\chi}| \right) W = O (E_1)
$$

(4.14)

where

$$
E_1 = (|\beta| + \Theta + U) e^{-\delta |y|}, \quad \delta > 0.
$$

Letting act the group of symmetries of the free NLS equation on (4.14) and differentiating with respect to the symmetry parameters the resulting relation we calculate

$$
\left( \tilde{\mathcal{L}} - \lambda^2 (V \left( |y + \frac{\lambda}{\chi}| \right)) \right) (iW) = O (E_1),
$$

(4.15)

$$
\left( \tilde{\mathcal{L}} - \lambda^2 V \left( |y + \frac{\lambda}{\chi}| \right) \right) (iyW) = -2 \nabla W + O (E_1)
$$

(4.16)

$$
\left( \tilde{\mathcal{L}} - \lambda^2 V \left( |y + \frac{\lambda}{\chi}| \right) \right) (\Lambda W) = -2W + 2\lambda^2 V \left( |y + \frac{\lambda}{\chi}| \right) W + O \left( |V' \left( |y + \frac{\lambda}{\chi}| \right) (yW (y))| \right) + O (E_1),
$$

(4.17)

$$
\left( \tilde{\mathcal{L}} - \lambda^2 V \left( |y + \frac{\lambda}{\chi}| \right) \right) (\nabla W) = O \left( |V' \left( |y + \frac{\lambda}{\chi}| \right) W| \right) + O (E_1).
$$

(4.18)

We now use (4.6) with $\Phi = iW, iyW, \Lambda W, \nabla W$. By the orthogonal conditions (3.4), using (4.15)-(4.18) and (4.13) we obtain

$$
|M (t)| \leq C \Theta \left( (|\beta| + \Theta)^{A(N)} + U \right) + C (\delta) \left( |\beta| + \Theta + U + M (t) + \| \zeta \|_{H^1}^{p_0} \Theta^{1-\delta} + \| \zeta \|_{H^1}^{2} \right) \| \zeta \|_{H^1}^2
$$

$$
+ C (\| V \left( |y + \frac{\lambda}{\chi}| \right) - V \left( \left| \frac{\lambda}{\sqrt{\chi}} \right| \right) \| W \|_{L^2}^2 + \| V' \left( |y + \frac{\lambda}{\chi}| \right) (yW (y)) \|_{L^2}^2) \| \zeta \|_{H^1}^2,
$$

(4.19)

for all $0 < \delta < 1$. By (2.31) we estimate

$$
\| (V \left( |y + \frac{\lambda}{\chi}| \right) - V \left( \left| \frac{\lambda}{\sqrt{\chi}} \right| \right) \| W \|_{L^2}^2 + \| V' \left( |y + \frac{\lambda}{\chi}| \right) yW (y) \|_{L^2}^2 \leq C r^{\gamma} \sqrt{U}.
$$

(4.20)

Thus, from (4.19) we get

$$
|M (t)| \leq C \Theta \left( (|\beta| + \Theta)^{A(N)} + U \right) + C (\delta) \left( |\beta| + \Theta + r^{\gamma} \sqrt{U} + \| \zeta \|_{H^1}^{p_0} \Theta^{1-\delta} + \| \zeta \|_{H^1}^{2} \right) \| \zeta \|_{H^1}^2.
$$

(4.21)

By assumption, (3.8) is true for all $t \in [T^*, T_n]$. From (2.131) $|\beta^{\infty}| \leq C \chi X$, with $X = \sqrt{U V \left( \frac{\lambda^2}{\chi} \right)}$.

Let $N \geq \frac{2}{p_1 - 1}$, so that $A(N) \geq 2$. Relation (2.144) implies that $X \leq \mathcal{C} t^{-1}$. Then, as $\Theta \leq C \chi X$, from (4.1) and (4.21) with $\delta < \frac{p_1 - 1}{2}$

$$
|M (t)| \leq C t^{-5/2}
$$

(4.22)

for all $t \in [T^*, T_n]$. We are now in position to improve (3.8) for the modulation parameters by using (4.22). By (4.22) and (2.62) the vector $\Xi (t)$ solves

$$
\left\{ \begin{array}{l}
\dot{\chi} (t) = 2\beta (t) + O \left( t^{-5/2} \right), \\
\dot{\beta} (t) = B_1 (\Xi (t)) + O \left( t^{-5/2} + t^{-2p_1} \right), \\
\dot{\lambda} (t) = M_1 (\Xi (t)) + O \left( t^{-5/2} + t^{-2p_1} \right), \\
\dot{\gamma} (t) = -\frac{1}{\chi^2 (t)} |\beta (t)|^2 + \tilde{\beta} (t) \cdot \chi (t) + O \left( t^{-5/2} \right),
\end{array} \right.
$$

(4.23)
As \( \Xi(t) \) is only \( C^1 \), we cannot proceed directly as when we considered (2.157). We argue slightly different. Using (2.27) we write \( B_1(\Xi(t)) = \frac{d}{dt} b(\|\chi\|, \lambda) \). By (2.157) we deduce \( |\chi^{(N)}(t) - \frac{r_{app}}{\delta_0^\infty}t| \leq \frac{\mu}{r_{\infty}^2} \), for some constant vector \( \delta_0^\infty \in S^{d-1} \). Using (2.62) and (4.1) we estimate \( |B_1| + |\nabla \chi B_1| + |\partial_\lambda B_1| \leq C \lambda^2 \leq Ct^{-2} \). Then, the equation for \( \dot{\lambda}(t) \) in (4.23) takes the form

\[
\dot{\lambda}(t) = \delta_0^\infty b \left( \|\chi\|, \lambda^{(N)}(t) \right) + O \left( \left( \lambda - \lambda^{(N)} \right) \right) \left( \frac{|\chi - \chi^{(N)}(t)|}{|\lambda^{(N)}(t)|} + t^{-\frac{p_1}{2}} \right) t^{-2} + O \left( t^{-5/2} + t^{-2p_1} \right).
\]

Similarly, we have

\[
\dot{\beta}^{(N)}(t) = \delta_0^\infty b \left( \|\chi\|, \lambda^{(N)}(t) \right) + O \left( t^{-2 - \frac{p_1}{2}} \right).
\]

As in (2.158), we use the Cartesian coordinates with the \( x_1 \)-axis directed along the vector \( \delta_0^\infty \) and for a vector \( A \in \mathbb{R}^d \) we decompose \( A = \sum_{j=1}^d A_j e_j \), and \( e_j \), \( j = 1, \ldots, d \) is the canonical basis in these coordinates. Then, from (4.24) and (4.25) we deduce

\[
\dot{\beta}_1 = b \left( \|\chi\|, \lambda^{(N)}(t) \right) + O \left( \left( \|\mu\| + \frac{|\delta|}{r_{\infty}} + t^{-\frac{p_1}{2}} \right) t^{-2} + t^{-5/2} + t^{-2p_1} \right),
\]

\[
\dot{\beta}_j^{(N)} = b \left( \|\chi\|, \lambda^{(N)}(t) \right) + O \left( \left( \frac{|\delta|}{r_{\infty}} + t^{-\frac{p_1}{2}} \right) t^{-2} + t^{-5/2} + t^{-2p_1} \right),
\]

and

\[
\dot{\beta}_j - \dot{\beta}_j^{(N)} = O \left( \left( \frac{|\delta|}{r_{\infty}} + t^{-\frac{p_1}{2}} \right) t^{-2} + t^{-5/2} + t^{-2p_1} \right), \quad j = 2, \ldots, d,
\]

where we denote by \( \delta(t) = \chi(t) - \lambda^{(N)}(t) \) and \( \mu(t) = \lambda(t) - \lambda^{(N)}(t) \). Then, integrating (4.28) and using (3.8) we show that for any constant \( A > 0 \) there is \( T_0 > 0 \) such that

\[
|\delta_j(t)| \leq \frac{1}{A} t^{-\frac{p_1}{2}} \text{ and } |\dot{\delta}_j(t)| \leq \frac{1}{A t^{1 + \frac{p_1}{2}}} \text{, } t \in [T^*, T_n], \quad T^* \geq T_0,
\]

for \( j = 2, \ldots, d \). From (4.26) and (4.27), by using the equation for \( \chi(t) \) in (4.23) and integrating, via (3.8) and (4.29) we deduce

\[
\dot{\delta}_1 = (1 + o(1)) \left( 2 r_{\infty}^{-1} \right) b \left( r_{\infty}, \lambda^{(N)}(t) \right) \delta_1 + O \left( \frac{1}{A t^{1 + \frac{p_1}{2}}} \right).
\]

By (2.27) and (2.144) \( (i_{\infty})^{-1} b \left( r_{\infty}, \lambda^{(N)} \right) = \frac{K_1}{t} \left( 1 + o(1) \right) \), with \( K_1 > 1 \). Then

\[
\dot{\delta}_1 = \frac{K_2}{t} \delta_1 + O \left( \frac{1}{A t^{1 + \frac{p_1}{2}}} \right),
\]

where \( K_2 > 1 \). Thus, similarly to (2.174) we deduce that for some \( T_0 > 0 \)

\[
|\delta_j| \leq \frac{K_3}{A} t^{-\frac{p_1}{2}} \text{, } t \in [T^*, T_n], \quad K_3 > 1.
\]

Moreover, from (4.31) we get

\[
|\dot{\delta}_1| \leq \frac{K_4}{A} t^{-\frac{p_1}{2}}, \quad K_4 > 1.
\]

Taking \( A \geq 2 K_3 K_4 \) we deduce

\[
|\delta_j| \leq \frac{1}{2} t^{-\frac{p_1}{4}} \text{ and } |\dot{\delta}_1| \leq \frac{1}{2} t^{-(1 + \frac{p_1}{4})}, \quad t \in [T^*, T_n].
\]

Combining the last inequalities with (4.29) we prove

\[
|\delta| \leq \frac{1}{2} t^{-\frac{p_1}{4}} \text{ and } |\dot{\delta}_1| \leq \frac{1}{2} t^{-(1 + \frac{p_1}{4})}, \quad t \in [T^*, T_n].
\]

By (2.62) and (4.1) we estimate

\[
|M_1| + |\nabla \chi M_1| + |\partial_\lambda M_1| \leq C \lambda^2 \leq Ct^{-2} \quad \text{Moreover, by } (2.62)
\]

\[
\dot{\lambda}^{(N)}(t) = M_1 \left( \Xi^{(N)}(t) \right) + O \left( t^{-2p_1} \right).
\]

Then, by (3.8)

\[
\left| \dot{\lambda} - \dot{\lambda}^{(N)} \right| \leq \left| M_1 \left( \Xi(t) \right) - M_1^{(N)} \left( \Xi^{(N)}(t) \right) \right| + O \left( t^{-2p_1} \right)
\]

\[
\leq |\nabla \chi M_1| |\delta| + |\partial_\lambda M_1| |\mu| + O \left( t^{-2p_1} \right) \leq Ct^{-2 - \frac{p_1}{4}}.
\]
Finally, using (3.8), as \( |\dot{\beta}^{(N)}(t)| \leq C\lambda^{2} \leq Ct^{-2} \) and by (2.146), (2.148) \( |\chi^{(N)}(t)| \leq C \ln t \), we have

\[
\left| \dot{\gamma}(t) - \dot{\gamma}^{(N)}(t) \right| \leq C \left| \lambda(t) - \lambda^{(N)}(t) \right| + C \left| \beta(t) - \beta^{(N)}(t) \right| + C \left| \dot{\beta}^{(N)}(t) \right| \left| \chi(t) - \chi^{(N)}(t) \right| \\
+ C \left| \chi^{(N)}(t) \right| \left( \left| \ddot{\beta}^{(N)}(t) \right| + \left| \dot{\beta}(t) \right| \right) + C t^{-5/2} \leq Ct^{-\left(1+\frac{n-1}{k-n} \right)}.
\]

Hence, there is \( T_{0} > 0 \) such that

\[
\left| \lambda - \lambda^{(N)} \right| \leq \frac{t^{-\left(1+\frac{n-1}{k-n} \right)}}{2} \quad \text{and} \quad \left| \dot{\gamma}(t) - \dot{\gamma}^{(N)}(t) \right| \leq \frac{t^{-\frac{n-1}{k-n}}}{2}, \tag{4.33}
\]

for \( t \in [T^{*}, T_{n}] \). Therefore, from (4.32) and (4.33) we improve (3.8) for the modulation parameters \((\Xi(t), \gamma(t))\) on \([T^{*}, T_{n}]\).

By a continuity argument, we show that (3.8) for \((\Xi(t), \gamma(t))\) is satisfied on \([T_{0}, T_{n}]\).

**Remark 4.1** In the case when \( V = V^{(2)} \) the best bound on the error term in (2.129) that we are able to obtain by following the construction provided by Lemma 2.6 is \( \mathcal{E}^{(N)} \sim V \mathcal{E}^{(N)} \). For potentials that behave as \( V(r) \sim r^{-\rho}, \ 0 < \rho < 2 \), we cannot even integrate \( V \mathcal{E}^{(N)} \) two times on \([t, \infty)\). Then, the proof of (3.8) fails for the approximate modulations equations given in Lemma 2.6. Therefore, we use a different approximated modulation equations constructed in Lemma 2.7 to cover this case.

**Case II: Slow decaying potentials.**

We let now the potential \( V = V^{(2)} \). In this case we suppose that (3.9) is true for all \( t \in [T^{*}, T_{n}] \), with \( T_{0} \leq T^{*} \leq T_{n} \). By (3.9), (2.145) and (2.146) we see that (4.1) is true. Let \( N \geq 1 \). For \( 1 \leq j \leq N \) and \( \Xi(t) \), let \( B_{j} = B_{j} (\Xi(t)) \), \( M_{j} = M_{j} (\Xi(t)) \) be the approximated modulation equations given by Lemma 2.7 corresponding to \( \Xi(t) \). We omit the tilde and denote these equations by \( B_{j} = B_{j} (\Xi(t)) \), \( M_{j} = M_{j} (\Xi(t)) \). Let \( \Phi \in C^{\infty} (\mathbb{R}^{d}) \) satisfy the estimate \( |\Phi(y)| \leq C (|y|^{m} Q(y) + \Psi) \) with some \( m \geq 0 \). We denote

\[
\Phi_{1} (t, x) = \lambda(t)^{-\frac{d}{2} - \frac{d}{4}} \Phi \left( \frac{x - \lambda(t)}{\lambda(t)} \right) e^{-i\gamma(t)e^{i\beta(t)} x}.
\]

By using (2.100) and (2.130) instead of (2.62), (2.129), relation (4.6) takes the form

\[
\frac{d}{dt} \text{Im} \int \bar{\phi}_{1} = \lambda^{-\frac{d}{2}} \text{Re} \int \mathcal{R} \bar{\phi}_{1} - \text{Re} \int \varepsilon \lambda^{-\frac{d}{2}} \left( \left( \bar{\lambda} - \lambda^{2} V \right) \phi \right) \left( t, \frac{x}{\lambda} \right) e^{-i\gamma e^{i\beta} x} + \text{Re} \int \lambda^{-\frac{d}{2}} (\mathcal{N}_{1} (W, \zeta) \phi) \left( t, \frac{x}{\lambda} \right) e^{-i\gamma e^{i\beta} x} + O \left( \Psi Z^{N} + \Psi \| \varepsilon \|_{H^{1}} \right).
\]

By (4.8) and Sobolev embedding theorem we get

\[
\left| \int \lambda^{-\frac{d}{2}} (\mathcal{N}_{1} (W, \zeta) \phi) \left( t, \frac{x}{\lambda} \right) e^{-i\gamma e^{i\beta} x} \right| \leq C \| \varepsilon \|_{H^{1}}^{p-\delta}.
\]

Using (2.46) and Lemma 2.7 we obtain (4.14) with \( E_{1} \) replaced by \( O (Z \Psi) \). Then, (4.15)-(4.18) are true with \( O (Z \Psi) \) instead of \( E_{1} \). Therefore, similarly to (4.19), by using (4.34) with \( \Phi = iW, iyW, \Lambda W, \nabla W \), via (4.35) we get

\[
|\mathcal{M}(t)| \leq C \Psi \| \varepsilon \|_{H^{1}} + C (\delta) \left( \Psi^{1-\delta} \left( \| \mathcal{M}+ \| \zeta \|_{H^{1}}^{p-1} \right) + C \| \zeta \|_{H^{1}} \right) \| \zeta \|_{H^{1}} \]

\[
+ C \left( \| V \left( |y + \frac{x}{\lambda} | \right) - V \left( |\frac{x}{\lambda} | \right) \|_{L^{2}} + \| V' \left( |y + \frac{x}{\lambda} | \right) \|_{L^{2}} \right) \| \zeta \|_{H^{1}}.
\]

By (2.32) and (2.112)

\[
\| V \left( |y + \frac{x}{\lambda} | \right) - V \left( |\frac{x}{\lambda} | \right) \|_{L^{2}} + \| V' \left( |y + \frac{x}{\lambda} | \right) \|_{L^{2}} \leq C \Psi.
\]

Hence, we obtain

\[
|\mathcal{M}(t)| \leq C \Psi Z^{N} + C (\delta) \left( \Psi^{1-\delta} \| \zeta \|_{H^{1}}^{p-1} + \Psi + \| \zeta \|_{H^{1}} \right) \| \zeta \|_{H^{1}}.
\]

By assumption, (3.9) is true for all \( t \in [T^{*}, T_{n}] \). Recall that \( |\beta^{\infty}| \leq C \zeta \), where \( \zeta = \sqrt{U_{V} \left( \frac{1}{2n} \right)} = \sqrt{V \left( \frac{1}{2n} \right)}, \) and \( \zeta \leq C \zeta \).
Taking $|N - 8|\rho > 4$, we estimate
\[ \chi^{N/4} \leq \Psi. \quad (4.37) \]

Then
\[ |\mathcal{M}(t)| \leq C (\Psi^{\gamma} + \chi^{2N}) \leq C\Psi \chi^{N}. \quad (4.38) \]

We now use (4.38) to improve (3.9). By (2.100) we estimate
\[ \left| \nabla \chi \mathbf{M}^{(N)} \right| + \left| \nabla \chi \mathbf{B}^{(N)} \right| \leq C \left( \Psi^2 + |V''(\chi)| \right) \]
and
\[ \left| \nabla_\beta \mathbf{M}^{(N)} \right| + \left| \nabla_\beta \mathbf{B}^{(N)} \right| + \left| \partial_\alpha \mathbf{M}^{(N)} \right| + \left| \partial_\alpha \mathbf{B}^{(N)} \right| \leq C \Psi. \quad (4.40) \]

From the equations for $\beta(N), \lambda(N)$ we get
\[ \left| \dot{\lambda} - \dot{\lambda}(N) \right| + \left| \dot{\beta} - \dot{\beta}(N) \right| \leq |\mathcal{M}(t)| + |\mathbf{M}^{(N)} (\Xi(t)) - \mathbf{M}^{(N)} (\Xi^{(N)}(t))| + |\mathbf{B}^{(N)} (\Xi(t)) - \mathbf{B}^{(N)} (\Xi^{(N)}(t))| \]
\[ \leq |\mathcal{M}(t)| + C \left( \left| \nabla \chi \mathbf{M}^{(N)} \right| + \left| \nabla \chi \mathbf{B}^{(N)} \right| \right) |\chi - \chi(N)| \]
\[ + C \left( \left| \nabla_\beta \mathbf{M}^{(N)} \right| + \left| \nabla_\beta \mathbf{B}^{(N)} \right| \right) |\beta - \beta(N)| + C \left( \left| \partial_\alpha \mathbf{M}^{(N)} \right| + \left| \partial_\alpha \mathbf{B}^{(N)} \right| \right) |\lambda - \lambda(N)| \quad (4.41) \]
Recall that by (2.21) there is $N_0 > 0$ such that $\chi^{2N_0} \leq C |V''(\chi^{N_0})|$. Taking $N > 4 (N_0 + 1)$ we get
\[ \chi^{N} \leq \chi^{2N_0} \chi^{\frac{N}{2} + 2} \leq C |V''(\chi^{N_0})| \chi^{\frac{N}{2} + 2}. \quad (4.42) \]
Then, by (4.38)
\[ |\mathcal{M}(t)| \leq C \left( \Psi^2 + |V''(\chi^{N_0})| \right) \Psi \chi^{\frac{N}{2} + 2}. \quad (4.43) \]
Thus, using (4.39) and (4.40), by (3.9) and (4.37) from (4.41) we have
\[ \left| \dot{\lambda} - \dot{\lambda}(N) \right| + \left| \dot{\beta} - \dot{\beta}(N) \right| \leq C \left( \Psi^2 + |V''(\chi^{N_0})| \right) \Psi \chi^{\frac{N}{2} + 1}. \quad (4.44) \]

As $V, V'$ are monotone, $V', V''$ are of a definite sign. Integrating (4.44) on $[t, \infty)$ we deduce
\[ |\lambda(t) - \lambda^{(N)}(t)| + |\beta(t) - \beta^{(N)}(t)| \]
\[ \leq C \left( \Psi^2 + |V''(\chi^{N_0})| \right) \left( \int_t^\infty V'(r) V^{\frac{N}{2}}(r) dr + (\rho \infty)^{\rho \frac{N}{2}} \chi^{\frac{N}{2} + 2} \int_{r^\infty}^{\infty} \frac{dr}{r^{1 + \rho \frac{N}{2}}} \right) \]
\[ \leq \frac{C}{\rho N} \left( \Psi^2 + |V''(\chi^{N_0})| \right) \chi^{\frac{N}{2} + 2}, \quad (4.45) \]
for all $t \in [T^*, T_n]$. Using (4.43) and (4.45) we obtain
\[ \left| \dot{\lambda} - \lambda^{(N)}(t) \right| \leq C \Psi \chi^{N} + C \left| \beta(t) - \beta^{(N)}(t) \right| \leq C \left( \Psi^2 + |V''(\chi^{N_0})| \right) \Psi \chi^{\frac{N}{2} + 2} + \frac{C}{\rho N} \left( \Psi^2 + |V''(\chi^{N_0})| \right) \chi^{\frac{N}{2} + 2}, \]
and then,
\[ \left| \lambda(t) - \lambda^{(N)}(t) \right| \leq \frac{C}{\rho N} \Psi \chi^{\frac{N}{2} + 1}. \quad (4.46) \]
for all $t \in [T^*, T_n]$. Finally, by (4.43), (4.44), (4.45) and (4.46) we deduce
\[ \left| \dot{\gamma}(t) - \gamma^{(N)}(t) \right| \leq C \left( \Psi^2 + |V''(\chi^{N_0})| \right) \Psi \chi^{\frac{N}{2} + 2} + \frac{C}{\rho N} \left( \Psi^2 + |V''(\chi^{N_0})| \right) \chi^{\frac{N}{2} + 2} + \frac{C}{\rho N} \Psi \chi^{\frac{N}{2} + 1} \left| \dot{\beta}(N)(t) \right| \]
\[ + C \left| \gamma^{(N)}(t) \right| \left( \Psi^2 + |V''(\chi^{N_0})| \right) \Psi \chi^{\frac{N}{2} + 1}. \]
Hence,
\[ \left| \gamma(t) - \gamma^{(N)}(t) \right| \leq \frac{C}{\rho N} \Psi \chi^{\frac{N}{2} + 2}, \quad t \in [T^*, T_n]. \quad (4.47) \]
Therefore, by (4.45), (4.46) and (4.47), taking $N$ sufficiently large we prove that
\[ |\lambda(t) - \lambda^{(N)}(t)| + |\beta(t) - \beta^{(N)}(t)| \leq \frac{1}{2} \left( \Psi^2 + |V''(\chi^{N_0})| \right) \chi^{\frac{N}{2} + 2}, \]
\[ |\lambda(t) - \lambda^{(N)}(t)| + |\gamma(t) - \gamma^{(N)}(t)| \leq \frac{1}{2} \Psi \chi^{\frac{N}{2} + 1}, \]
on $t \in [T^*, T_n]$, which strictly improve (3.9) for the modulation parameters $(\Xi(t), \gamma(t))$. Again, by a continuity argument, we show that (3.9) for $(\Xi(t), \gamma(t))$ is true on $[T_0, T_n]$.
4.2 Control of the error $\varepsilon$.

Let us consider the energy, the mass and the momentum of $u_n$. By using the orthogonal conditions (3.4) we have

$$E (u_n) = E (W + \varepsilon) = E (W) + \frac{1}{2} \int |\nabla \varepsilon|^2 - \frac{1}{2} \int |\mathcal{V}|^2 - \frac{1}{1 + p} \left( \int |W + \varepsilon|^{p+1} - \int |W|^{p+1} \right) - \Re \int \mathcal{W} \overline{\varepsilon} - \Re \int \mathcal{V} \overline{\varepsilon},$$

$$M (u_n) = \text{Im} \int \nabla (W + \varepsilon) \overline{(W + \varepsilon)} = \text{Im} \int \nabla W \overline{W} + \text{Im} \int \nabla \varepsilon \overline{\varepsilon}$$

and

$$\int |u_n|^2 = \int |W|^2 + \int |\varepsilon|^2.$$ (4.49)

Moreover, by (2.128) and (3.4) we have

$$\text{Re} \int (\mathcal{W} + \mathcal{V} \mathcal{W}) \overline{\varepsilon} = - \Re \int \left( |\mathcal{W}|^{p-1} \mathcal{W} + \lambda - \frac{2}{\lambda^2} \mathcal{E}^{(N)} \left( \frac{x - \chi(t)}{\lambda(t)} \right) e^{-i\gamma(t)} e^{i\beta(t) x} \right) \overline{\varepsilon}.$$ (4.48)

Then, from (4.48)

$$E (u_n) = E (W + \varepsilon) = E (W) + \frac{1}{2} \int |\nabla \varepsilon|^2 - \frac{1}{2} \int |\mathcal{V}|^2 - \frac{1}{1 + p} \left( \int |W + \varepsilon|^{p+1} - \int |W|^{p+1} - (p + 1) |W|^{p-1} \Re \int \mathcal{W} \overline{\varepsilon} \right) + \text{Er},$$

with

$$\text{Er} = \Re \int \left( \lambda - \frac{2}{\lambda^2} \mathcal{E}^{(N)} \left( \frac{x - \chi(t)}{\lambda(t)} \right) e^{-i\gamma(t)} e^{i\beta(t) x} \right) \overline{\varepsilon}.$$ (4.51)

Note that by Lemma 2.8 Er is small.

Case I: Fast decaying potentials.

Let the potential $|V (r)| \leq C e^{-cr}$, for some $c > 0$. Suppose that (3.8) is true for all $t \in [T^*, T_n]$, with $T_0 \leq T^* < T_n$. Let $\psi_{\lambda\infty} \in C^\infty (\mathbb{R})$ be such that $0 \leq \psi \leq 1$, $\psi (x) = 1$ for $|x| \leq \frac{1}{\lambda^{1/2}}$ and $\psi (x) = 0$ for $|x| \geq \frac{1}{\lambda^{1/2}}$. Set

$$\psi_{\lambda} (x) = \psi_{\lambda\infty} \left( \frac{8 (x - \chi(t))}{\lambda(t) \ln t} \right).$$

We introduce the following conserved quantity of $u_n$ which is a combination of the three conservation laws (4.48)-(4.50) for the NLS equation with a potential (1.1)

$$K_{\text{tot}} = E (u_n) - \beta (t) \cdot \text{Im} \int \psi_{\lambda} \nabla u_n \overline{\varepsilon} + \frac{1}{2} \left( \lambda^{-2} (t) + |\beta (t)|^2 \right) \int |u_n|^2.$$ (4.52)

Observe that we localize the momentum in a neighborhood of the potential. We compare $K_{\text{tot}}$ with

$$K_{\text{sol}} = E (W) - \beta (t) \cdot \text{Im} \int \psi_{\lambda} \nabla W \overline{W} + \frac{1}{2} \left( \lambda^{-2} (t) + |\beta (t)|^2 \right) \int |W|^2.$$ (4.53)

Then, from (4.49), (4.50), (4.51) we deduce

$$K_{\text{tot}} - K_{\text{sol}} = \mathcal{G} (t) + \text{Er}$$

where

$$\mathcal{G} (\varepsilon (t)) = \mathcal{G}_W (\varepsilon (t)) = \frac{1}{2} \int |\nabla \varepsilon|^2 + \frac{1}{2} \left( \lambda^{-2} (t) + |\beta (t)|^2 \right) \int |\varepsilon|^2 - \frac{1}{2} \int |\mathcal{V}|^2 - \frac{1}{1 + p} \int \left( |W + \varepsilon|^{p+1} - |W|^{p+1} - (1 + p) |W|^{p-1} \Re \int \overline{\mathcal{W} \varepsilon} \right) - \beta (t) \cdot \text{Im} \int \psi_{\lambda} \nabla \varepsilon.$$ (4.54)

($\mathcal{V}$ is related to $V$ by (2.4)). We now study the properties of $\mathcal{G} (\varepsilon (t))$. We have the following result.
Lemma 4.2. There exist $T_0$ large enough such that for $t \in [T^*, T_n]$, $T^* \geq T_0$, the following hold:

i) (Coercivity of the linearized energy) There is a constant $c_0 > 0$ such that

$$
G(\varepsilon(t)) \geq c_0 \|\varepsilon\|^2_{H^1}.
$$

ii) (Energy estimate on $\varepsilon$) For some $N$ large enough

$$
\left| \frac{d}{dt} G(\varepsilon(t)) \right| \leq \frac{C}{t^{3\ln t}}.
$$

Before proving Lemma 4.2 we improve (3.8) for $\varepsilon(t)$. Integrating (4.53) we have

$$
G(\varepsilon(t)) \leq \frac{C}{t^{2\ln t}}.
$$

Then, using (4.52) we prove

$$
\|\varepsilon\|^2_{H^1} \leq \frac{C}{c_0^2 \ln t}.
$$

Thus, taking $T_0 > e^{2C/c_0}$ we strictly improve (3.8) for $\varepsilon(t)$. By a continuity argument, we conclude that (3.9) for $\varepsilon(t)$ is true on $[T_0, T_n]$.

**Proof of Lemma 4.2.** Proof of item i. We decompose $G(\varepsilon(t))$ as

$$
G(\varepsilon(t)) = G_1 + G_2 + G_3,
$$

where

$$
G_1 = \frac{1}{2} \int \left( |\nabla \varepsilon|^2 + \lambda - 2 \varepsilon^2 - V |\varepsilon|^2 - \frac{e^{+1}}{2} |W|^{-1} |\varepsilon|^2 - \frac{p-1}{2} |W|^{p-3} \text{Re}(W \overline{\varphi})^2 \right),
$$

$$
G_2 = -\frac{1}{1+p} \left( \int \left( |W + \varepsilon|^{p+1} - |W|^{p+1} - (1 + p) |W|^{p-1} \text{Re}(W \overline{\varphi}) \right) - \frac{1}{2} \text{Re} \left( \frac{p+1}{2} |W|^{p-1} |\varepsilon|^2 + \frac{p-1}{2} |W|^{p-3} (W \overline{\varphi})^2 \right) \right)
$$

and

$$
G_3 = -\beta(t) \cdot \text{Im} \int \psi \lambda \nabla \varepsilon + \frac{1}{2} |\beta(t)|^2 \int |\varepsilon|^2 = O \left( \|\beta\| \|\varepsilon\|^2_{H^1} \right).
$$

By Sobolev embedding theorem we estimate

$$
G_2 = O \left( \|\varepsilon\|^{p+1-\delta}_{H^1} \right), \quad 0 < \delta < 1.
$$

Using (3.5) and (2.126) we have

$$
G_1 = \frac{1}{2} (\lambda(t))^{d-2(1+p)} (L \varphi \zeta, \zeta) + G_{11}
$$

with $L \varphi$ defined by (2.53) and

$$
G_{11} = -(\lambda(t))^{d-2(1+p)} \int \left( \frac{p+1}{2} \left( |W|^{-1} - Q^{-1} \right) |\zeta|^2 + \frac{p+1}{2} \text{Re} \left( \left( |W|^{-3} W^2 - Q^{-1} \right) \zeta^2 \right) \right).
$$

By Lemma 2.6 we show that $G_{11} = O \left( \Theta^{1-\delta} \|\varphi\|^2_{H^1} \right)$ and

$$
|\langle \zeta, W - Q \rangle | + |\langle \zeta, x (W - Q) \rangle | + |\langle \zeta, iA (W - Q) \rangle | = O \left( \Theta^{1-\delta} \|\varphi\|^2_{H^1} \right).
$$

Then, from (2.55), by using (3.4) we conclude that there is $c > 0$ such that

$$
G_1 \geq c \|\varphi\|^2_{H^1} + O \left( \Theta^{1-\delta} \|\varphi\|^2_{H^1} \right).
$$

Hence, from (4.55), (4.56), (4.57), via (3.8), we conclude that for $T_0$ large enough there is $c_0 > 0$ such that for $t \in [T^*, T_n]$, $T^* \geq T_0$ (4.52) is true.

Proof of item ii. Let us make the change of variables $\varepsilon_1 = e^{-i\gamma_1(t)} \varepsilon$ and $W_1 = e^{-i\gamma_1(t)} W$, with

$$
\gamma_1(t) = \int_{T_0}^{t} \left( \lambda^{-2} (\tau) + |\beta(t)|^2 \right) d\tau.
$$
Then $G_W(\varepsilon(t)) = G_{W_1}(\varepsilon_1(t))$. Differentiating $G_{W_1}(\varepsilon_1(t))$ we have
\begin{equation}
\frac{d}{dt}G_{W_1}(\varepsilon_1(t)) = G_{11}(t) + G_{12}(t) + G_{13}(t),
\end{equation}
where
\begin{equation}
G_{11}(t) = -\left(\hat{W}_1, N_1(W_1, \varepsilon_1)\right),
\end{equation}
\begin{equation}
G_{12}(t) = \left(\hat{\varepsilon}_1, -\Delta \varepsilon_1 + \left(\lambda^{-2}(t) + |\beta(t)|^2\right)\varepsilon_1 - \mathcal{V}\varepsilon_1 - N_0(W_1, \varepsilon_1)\right),
\end{equation}
and
\begin{equation}
G_{13}(t) = -\hat{\beta} \cdot \text{Im} \int \psi_\chi \nabla \varepsilon_1 \tilde{\varepsilon}_1 - \beta \cdot \text{Im} \int \psi_\chi \nabla \varepsilon_1 \tilde{\varepsilon}_1 + 2\beta \cdot \text{Im} \int \psi_\chi \nabla \tilde{\varepsilon}_1 \varepsilon_1.
\end{equation}

Let us consider $G_{11}(t)$. By (2.126) we have
\begin{equation}
\hat{W}_1 = -\lambda^{-\frac{3}{2}} \left(\frac{\lambda}{\chi} \Lambda W + \left(\frac{\lambda}{\chi} \cdot \nabla W + 2i|\beta|^2 W\right)\right) \left(\frac{\chi - \lambda}{\lambda}\right) e^{-i(\gamma(t) + \nu(t)) + i\beta(t)s} \cdot x,
\end{equation}
\begin{equation}
G_{11}^{(0)}(t) = -\lambda^{-\frac{3}{2}} \lambda \left(1 - \hat{\phi}\right) \hat{W}_1 N_1(W_1, \varepsilon_1) dx
\end{equation}
and
\begin{equation}
G_{11}^{(1)}(t) = \lambda^{-\frac{3}{2}} \lambda \lambda \left(1 - \hat{\phi}\right) \hat{W}_1 N_1(W_1, \varepsilon_1) dx,
\end{equation}
with
\begin{equation}
W = \lambda^{-\frac{3}{2}} \left(\frac{\lambda}{\chi} \cdot \nabla W + 2i|\beta|^2 W\right) \left(\frac{\chi - \lambda}{\lambda}\right) e^{-i(\gamma(t) + \nu(t)) + i\beta(t)s} \cdot x.
\end{equation}

Let $\phi$ be the indicator function of the complement of the set $\Omega$ defined by (4.9). In particular, $|T(y)| \leq \frac{1}{2}Q(y)$ when $\phi = 1$. Moreover, as by (2.61) $|T^{(i)}| \leq CY$, $Q(y) \leq CY$ whereas $\phi = 0$. Using (4.59) we decompose
\begin{equation}
G_{11}(t) = \text{Re} \int W N_1(W_1, \varepsilon_1) dx + G_{11}^{(0)}(t) + G_{11}^{(1)}(t),
\end{equation}
with
\begin{equation}
\hat{W}_1 = \lambda^{-\frac{3}{2}} \left(\frac{\lambda}{\chi} \cdot \nabla W + 2i|\beta|^2 W\right) \left(\frac{\chi - \lambda}{\lambda}\right) e^{-i(\gamma(t) + \nu(t)) + i\beta(t)s} \cdot x,
\end{equation}
\begin{equation}
G_{11}^{(0)}(t) = -\lambda^{-\frac{3}{2}} \lambda \left(1 - \hat{\phi}\right) \hat{W}_1 N_1(W_1, \varepsilon_1) dx
\end{equation}
and
\begin{equation}
G_{11}^{(1)}(t) = \lambda^{-\frac{3}{2}} \lambda \lambda \left(1 - \hat{\phi}\right) \hat{W}_1 N_1(W_1, \varepsilon_1) dx,
\end{equation}
with
\begin{equation}
W = \lambda^{-\frac{3}{2}} \left(\frac{\lambda}{\chi} \cdot \nabla W + 2i|\beta|^2 W\right) \left(\frac{\chi - \lambda}{\lambda}\right) e^{-i(\gamma(t) + \nu(t)) + i\beta(t)s} \cdot x.
\end{equation}

Therefore, using (4.59), and Lemma 2.6 to control $\Lambda W, yW, W$, we get
\begin{equation}
\left|W_1(y)\right| \leq C(\delta_1) \left(\|\beta\| + M(t) + \|B^{(N)}\| + \|M^{(N)}\|\right) (Q^{1-\delta_1}(y) + Y), \quad y \in \mathbb{R}^d,
\end{equation}
for $0 < \delta_1 < 1$. Thus, by (4.8), (4.63), Sobolev embedding theorem and (3.8), as $Q(y) \leq CY$ on the support of $1 - \phi$, taking $\delta_1$ and $\delta$ small enough, we estimate
\begin{equation}
\left|G_{11}^{(0)}(t)\right| \leq C \left(\|\beta\| + M(t) + \|B^{(N)}\| + \|M^{(N)}\|\right) Y^{1-\delta_1} \left(\|\varepsilon_1\|_{H^1}^{p-\delta} + \|\varepsilon_1\|_{H^1}^2\right) \leq CT^{-3} \left(\frac{\|\varepsilon_1\|}{\sqrt{\varepsilon_1}}\right),
\end{equation}
for $t \in [T^*, T_n]$. On the support of $\phi$ we have
\begin{equation}
|W(y)| \geq 2^{-1}Q(y).
\end{equation}
Using (4.11) and (4.65), and taking $\delta < \frac{p+1}{2}$ we estimate
\begin{equation}
\phi e^{-(1-\delta)|y|} |N_1(W, \zeta)| \leq Ce^{-(1-\delta)|y|} \|W\|_{H^{p-2}}^2 \|\zeta\|^2 \leq Ce^{-(1-\delta)|y|} Q^{p-2}(y) \|\zeta\|^2 \leq Ce^{-\frac{p+1}{2}|y|} \|\zeta\|^2.
\end{equation}

(Here $\zeta$ is related to $\varepsilon_1$ by (3.5)). By Lemma 2.6 we get
\begin{equation}
\left(\frac{\lambda}{\chi} \cdot \nabla W + i(\gamma + \lambda^{-2} - |\beta|^2 - \hat{\beta} \cdot \chi) W - i\lambda(\hat{\beta} \cdot y) W + \hat{W}\right) \leq C \left(\|\beta\| + M(t) + \|B^{(N)}\| + \|M^{(N)}\|\right) e^{-(1-\delta)|y|},
\end{equation}

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with $0 < \nu + \delta_1 < 1$. Then, from (4.66) and (3.8), for $\nu + \delta_1 < \frac{p-1}{2}$ we get

$$|G_{11}^{(1)}(t)| \leq C \left( |\beta| \Theta^{\nu} + M(t) + \left| B^{(N)} \right| + \left| M^{(N)} \right| \right) ||\varepsilon_1||_{H^1}^2 \leq Ct^{-3-\nu},$$

(4.67)

for $t \in [T^*, T_n]$, with $0 < \nu < \frac{p-1}{2}$. If $||\varepsilon_1|| \leq \frac{|W_1|}{2}$ we expand

$$\mathcal{N}_1(W_1, \varepsilon_1) = \mathcal{N}(W_1, \varepsilon_1) + O \left( |W_1|^{-1+\delta} |\varepsilon_1|^{p+1-\delta} \right), \quad \delta > 0,$$

(4.68)

where

$$\mathcal{N}(W_1, \varepsilon_1) = \frac{p-1}{2} |W_1|^{p-3} \frac{|W_1 \varepsilon_1|^2}{2} + \frac{p^2 - 1}{4} |W_1|^{p-5} |W_1\varepsilon_1|^2 + \frac{(p-1)(p-3)}{2} |W_1|^{p-5} W_1 \text{Re}(W_1\varepsilon_1)^2.$$

If $||\varepsilon_1|| \geq \frac{|W_1|}{2}$ then $|\mathcal{N}_1(W_1, \varepsilon_1)| = O \left( |W_1|^{-1+\delta} |\varepsilon_1|^{p+1-\delta} \right), \quad \delta > 0$. Hence

$$\mathcal{N}_1(W_1, \varepsilon_1) = \mathcal{N}(W_1, \varepsilon_1) + O \left( |W_1|^{-1+\delta} |\varepsilon_1|^{p+1-\delta} \right), \quad \delta > 0.$$

(4.69)

Using (4.69) in (4.60) we obtain

$$G_{11}(t) = \text{Re} \int \mathcal{W} \mathcal{N}(W_1, \varepsilon_1) + G_{11}^{(0)}(t) + G_{11}^{(1)}(t) + G_{11}^{(2)}(t),$$

(4.70)

where

$$G_{11}^{(2)}(t) = O \left( \int \mathcal{W} \left( |W_1|^{-1+\delta} |\varepsilon_1|^{p+1-\delta} \right) dx \right).$$

Using Lemma 2.6 and (3.8) we control

$$|\mathcal{W}| \leq C \left( |\chi| + |\beta|^2 \right) e^{-(1-\delta)|\varepsilon_1|} \leq Ct^{-1} e^{-(1-\delta)|\varepsilon_1|},$$

for $t \in [T^*, T_n]$. Then, from (4.65), via Sobolev embedding theorem, taking $\delta$ sufficiently small we deduce

$$|G_{11}^{(2)}(t)| \leq Ct^{-1} ||\varepsilon_1||_{H^1}^{p+1-\delta} \leq Ct^{-3-\nu},$$

(4.71)

Gathering (4.64), (4.67) and (4.71), from (4.70) we get

$$G_{11}(t) = \text{Re} \int \mathcal{W} \mathcal{N}(W_1, \varepsilon_1) + O \left( t^{-3-\nu} \right), \quad 0 < \nu < \frac{p-1}{2}$$

(4.72)

for $t \in [T^*, T_n]$.

We turn now to $G_{12}(t)$. Using (4.2) we obtain the equation for $\varepsilon_1$

$$i \partial_t \varepsilon_1 = -\Delta \varepsilon_1 + \left( \lambda^{-2}(t) + |\beta(t)|^2 \right) \varepsilon_1 - \mathcal{V}_1 - \mathcal{N}_0(W_1, \varepsilon_1) - \left( \lambda^{-\frac{2}{\nu+1}}(t) \left( \mathcal{E}^{(N)} + R(W) \right) \left( t, \frac{x - \chi(t)}{\lambda(t)} \right) e^{-i(\gamma(t \tau) + \gamma(t \tau))} e^{i\beta(t \tau)}}. \right)$$

Then

$$G_{12}^{(1)}(t) = - \left( \mathcal{R}, i \left( -\Delta \varepsilon_1 + \left( \lambda^{-2}(t) + |\beta(t)|^2 \right) \varepsilon_1 - \mathcal{V}_1 - \mathcal{N}_0(W_1, \varepsilon_1) \right) \right),$$

with

$$\mathcal{R} = \lambda^{-\frac{2}{\nu+1}}(t) \left( \mathcal{E}^{(N)} + R(W) \right) \left( t, \frac{x - \chi(t)}{\lambda(t)} \right) e^{-i(\gamma(t \tau) + \gamma(t \tau))} e^{i\beta(t \tau)}}.$$

Using (3.5) and (2.129), as

$$|R(W)| \leq Ce^{-/(1-\delta)|\varepsilon_1|},$$

(4.74)

we get

$$G_{12}(t) = G_{12}^{(1)}(t) + G_{12}^{(2)}(t)$$

where

$$G_{12}^{(1)}(t) = - \lambda^{-2} \frac{1}{\nu+1} \left( R(W), i \left( -\Delta \varepsilon_1 + \left( -\Delta \varepsilon_1 + \lambda^{-2}(t) \varepsilon_1 - \mathcal{V}_1 - \mathcal{N}_0(W, \varepsilon_1) \right) \right) \right)$$

and

$$G_{12}^{(2)}(t) = O \left( \left( \Theta^{3-\delta} \frac{|\chi|}{\lambda} + |\beta| \mathcal{M}(t) \right) ||\varepsilon_1||_{H^1} \right).$$
We decompose
\[
\mathcal{G}_{12}^{(1)} (t) = -\lambda^{-2} - \frac{1}{\lambda (t)} (t) \left( \left( \hat{L} - \lambda^2 (t) V \left( \left| + \frac{\chi}{\lambda} \right| \right) R (W), i \zeta \right) + \lambda^{-2} - \frac{1}{\lambda (t)} (t) (R (W), i \mathcal{N}_1 (W, \zeta)) \right)
\]
where $\hat{L}$ is defined by (4.5). Using (4.8) and Sobolev embedding theorem, we estimate the second term in the right-hand side of the last relation as $O \left( \mathcal{M} (t) \| \varepsilon_1 \|_{H^2}^{\rho_1 + \delta} \right)$. By the orthogonal conditions (3.4), using (4.15)-(4.18) and (4.20) we control
\[
\left| \mathcal{G}_{12}^{(1)} (t) \right| \leq Cr^\infty \sqrt{U} \mathcal{M} (t) \| \varepsilon_1 \|_{H^1}.
\]
Hence, by (3.8), we get
\[
| \mathcal{G}_{12} (t) | \leq C \left( (\ln t) t^{-1} \mathcal{M} (t) + t^{-3 + \delta} \right) \| \varepsilon_1 \|_{H^1}.
\] (4.75)

Next, we consider $\mathcal{G}_{13} (t)$. The first term in the right-hand side of $\mathcal{G}_{13} (t)$ is estimated by Cauchy-Schwartz as
\[
\left| \hat{\beta} \cdot \mathop{\text{Im}} \int \psi_x \nabla \varepsilon_1 \varepsilon_1 \right| \leq C \| \hat{\beta} \| H^1 \| \varepsilon_1 \|_{H^1}. \quad (4.76)
\]
Since
\[
\left| \psi_x (x) \right| \leq \left( 1 + \left| \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right| \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right) \left| \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right| \left| \psi' \left( \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right) \right|
\]
and
\[
| \nabla \psi_x | \leq \frac{1}{\lambda (t)} \left| \psi' \left( \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right) \right|
\] (4.77) we estimate
\[
\left| \beta \cdot \mathop{\text{Im}} \int \psi_x \nabla \varepsilon_1 \varepsilon_1 \right| \leq C \| \beta \| \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \| \varepsilon_1 \|_{H^1}. \quad (4.79)
\]
and
\[
\left| \beta \cdot \mathop{\text{Im}} \int \nabla \psi_x \varepsilon_1 \varepsilon_1 \right| \leq C \| \beta \| \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \| \varepsilon_1 \|_{H^1}. \quad (4.80)
\]

Using (4.73), (4.74), (2.129) and integrating by parts we have
\[
\mathop{\text{Im}} \int \psi_x \nabla \varepsilon_1 \varepsilon_1 = I_1 + I_2 + I_3 + O \left( \left( \Theta^{3 - \delta} \left( \frac{|x|}{\lambda} \right) + \mathcal{M} (t) \right) \| \varepsilon_1 \|_{H^1} \right).
\]
where
\[
I_1 = \mathop{\text{Re}} \int \nabla \psi_x \nabla \varepsilon_1 \varepsilon_1 \left( -\Delta \varepsilon_1 + \left( \lambda^{-2} (t) + |\beta (t)| \right) \varepsilon_1 - V \varepsilon_1 \right),
\]
\[
I_2 = -\mathop{\text{Re}} \int \psi_x V \varepsilon_1 \varepsilon_1^2
\]
and
\[
I_3 = \mathop{\text{Re}} \int \psi_x \nabla \varepsilon_1 \mathcal{N}_0 (W_1, \varepsilon_1) .
\]

By (4.78) we have
\[
|I_1| \leq \frac{C}{\lambda (t)} \| \varepsilon_1 \|_{H^1}^2 . \quad (4.82)
\]
By (3.8), there is $T_0 > 0$ such that $|\lambda (t)| \leq 2\lambda^\infty$, $t \in [T^*, T_n]$, $T^* \geq T_0$. Then
\[
\| \psi \left( \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right) \nabla \varepsilon_1 \|_{L^\infty} \leq C \| V' \left( \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right) \|_{L^\infty} \leq C \left( \frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)} \right) \| \varepsilon_1 \|_{H^1} \geq C t^{-\frac{\Theta (\Theta (t) - \lambda (t))}{\lambda (t)}},
\]
t \in [T^*, T_n]. By (4.7) we write
\[
\mathcal{N}_0 (W_1, \varepsilon_1) = \mathcal{N}_L (W_1, \varepsilon_1) + \mathcal{N}_1 (W_1, \varepsilon_1),
\]
where
\[
\mathcal{N}_L (W_1, \varepsilon_1) = \frac{p+1}{2} \| W_1 \|_{L^\infty} \| \varepsilon_1 \|_{L^\infty} + \frac{p-1}{2} \| W_1 \|_{L^\infty} \varepsilon_1 \|_{L^\infty} .
\]
We decompose
\[
I_3 = I_{31} + I_{32} \quad (4.84)
\]
with

\[ I_{31} = \text{Re} \int \psi_\chi \nabla \varepsilon \nabla L (W_1, \varepsilon_1) \]

and

\[ I_{32} = \text{Re} \int \psi_\chi \nabla \varepsilon \nabla I_1 (W_1, \varepsilon_1). \]

Integrating by parts we have

\[ I_{31} = - \text{Re} \int \psi_\chi (\nabla W_1) \nabla (W_1, \varepsilon_1) + I_{31}^{(1)}, \]  

with

\[ I_{31}^{(1)} = - \text{Re} \int (\nabla \psi_\chi) \left( \frac{p+1}{4} |W_1|^{p-1} |\varepsilon_1|^2 + \frac{p-1}{4} \text{Re} \left( |W_1|^{p-3} W_1^2 (\varepsilon_1^2) \right) \right). \]  

By (4.8) we estimate

\[ |I_{32}| \leq C \|\varepsilon_1\|_{H^{1+\delta}}. \]  

Using that (4.78) we control

\[ |I_{31}^{(1)}| \leq C \frac{1}{\lambda(\varepsilon_1)} \|\varepsilon_1\|_{H^1}^2. \]  

By Lemma 2.6 we have

\[ Q(y) - \max_{1 \leq j \leq N} |T^{(j)}(y)| \geq Q(y) \left( 1 - t^{-1/2} \right), \quad |y| \leq \frac{\eta t}{4}. \]

Then, there is \( T_0 > 0 \) such that

\[ \max_{1 \leq j \leq N} |T^{(j)}(y)| < (2N)^{-1} Q(y), \]

for all \( |y| \leq \frac{\eta t}{4} \) and \( t \in [T^*, T_n], T^* \geq T_0 \). In particular, \( \psi_\chi \phi = \psi_\chi \) and then

\[ \int \psi_\chi (\nabla W_1) \nabla (W_1, \varepsilon_1) = \int \phi (\nabla W_1) \nabla (W_1, \varepsilon_1) + \int \phi (\psi_\chi - 1) (\nabla W_1) \nabla (W_1, \varepsilon_1). \]

From (4.88) we get \( |W(y)| \geq 2^{-1}Q(y) \). Then, as

\[ |\nabla (W_1, \varepsilon_1)| \leq C |\nabla W_1|^{p-2} |\varepsilon_1|^2 \]

we obtain

\[ |\phi (\psi_\chi - 1) (\nabla W_1) \nabla (W_1, \varepsilon_1)| \leq Ct^{-\frac{p-1}{4}} |\varepsilon_1|^2. \]

Hence

\[ \int \psi_\chi (\nabla W_1) \nabla (W_1, \varepsilon_1) = \int \phi (\nabla W_1) \nabla (W_1, \varepsilon_1) + O \left( t^{-\frac{p-1}{4}} \|\varepsilon_1\|_{H^1}^2 \right). \]

Using the last relation in (4.85), from (4.87) we deduce

\[ I_{31} = - \text{Re} \int \phi (\nabla W_1) \nabla (W_1, \varepsilon_1) + O \left( t^{-\frac{p-1}{4}} + \frac{1}{\lambda(\varepsilon_1)} \|\varepsilon_1\|_{H^1}^2 \right). \]

Then, by (4.84) and (4.86) we get

\[ I_3 = - \text{Re} \int \phi (\nabla W_1) \nabla (W_1, \varepsilon_1) + O \left( t^{-\frac{p-1}{4}} + \frac{1}{\lambda(\varepsilon_1)} \|\varepsilon_1\|_{H^1}^2 + \|\varepsilon_1\|_{H^{1+\delta}} \right), \]

with \( 0 < \delta < 1 \). Thus, from (4.81), (4.82) and (4.83), via (3.8) it follows that

\[ \text{Im} \int \psi_\chi \nabla \varepsilon \nabla \varepsilon_1 = - \text{Re} \int \phi (\nabla W_1) \nabla (W_1, \varepsilon_1) + O \left( \frac{1}{t^2 \ln t} \right). \]

From (2.126) we have

\[ \nabla W_1 = \lambda^{-\frac{2}{p-1}} (\lambda^{-1} \nabla W + i \beta W) \left( \frac{x - \chi}{\lambda} \right) e^{-i(\gamma(t) + \gamma(t))} e^{i\beta(t) \varepsilon}. \]

Then, using (4.88) and (4.89) we get

\[ 2\beta \cdot \int \phi (\nabla W_1) \nabla (W_1, \varepsilon_1) = \int \nabla W (W_1, \varepsilon_1) + O \left( M(t) \|\varepsilon_1\|_{H^1}^2 \right). \]

Therefore, by (4.76), (4.79), (4.80), (4.90), (3.8) taking into account (4.61) we derive

\[ \mathcal{G}_{13} (t) = - \int \nabla W (W_1, \varepsilon_1) + O \left( \frac{1}{t^3 \ln t} \right). \]

for \( t \in [T^*, T_n] \). Finally, gathering together (4.72), (4.75), (4.92), from (4.58) we attain (4.53).
Case II: Slow decaying potentials.

Let now the potential \( V = V^{(2)} \) and suppose that (3.9) is true for all \( t \in [T^*, T_n] \), with \( T_0 \leq T^* < T_n \). Let \( \varphi \in C^\infty (\mathbb{R}^d) \) be such that \( 0 \leq \varphi \leq 1 \), \( \varphi (x) = 1 \) for \( |x| \leq \frac{1}{4\lambda \infty} \) and \( \varphi (x) = 0 \) for \( |x| \geq \frac{1}{2\lambda \infty} \). Set

\[
\varphi x (x) = \varphi \left( \frac{x - \chi (t)}{\lambda (t) |\chi (t)|} \right).
\]

We consider

\[
\begin{align*}
\tilde{G} (\varepsilon (t)) &= \tilde{G}_W (\varepsilon (t)) = \frac{1}{2} \int |
\nabla \varepsilon|^2 + \frac{1}{2} \left( \lambda^{-2} (t) + |\beta (t)|^2 \right) \int |\varepsilon|^2 - \frac{1}{2} \int \mathcal{V} |\varepsilon|^2 \\
&- \frac{1}{1 + \rho} \int \left( |\mathcal{W} + \varepsilon|^p - |\mathcal{W}|^p + (1 + p) |\mathcal{W}|^{p-1} \Re \mathcal{V} \mathcal{W} \right) \\
&- \beta (t) \cdot \Im \int \varphi x \nabla \varepsilon \varepsilon.
\end{align*}
\]

As in the case of Lemma 4.2 we show that for some \( T_0 > 0 \) there is a constant \( c_0 > 0 \) such that

\[
\tilde{G} (\varepsilon (t)) \geq c_0 \|\varepsilon\|^2_{H^1} \tag{4.93}
\]

for any \( t \geq T_0 \). Suppose that for some \( N \) large enough

\[
\frac{d}{dt} \|\varepsilon\|^2_{H^1} \leq C \left( \left( \frac{1}{(1 + \rho)} \right)^{-1} + |\mathcal{V}' (\varepsilon |x|^{1/4})| + \Psi \right) \lambda^{2N} \tag{4.94}
\]

for \( t \in [T^*, T_n] \), \( T^* \geq T_0 \), with constant \( C \) independent on \( N \). Integrating (4.94) and using (4.93) we get

\[
\|\varepsilon\|^2_{H^1} \leq \frac{C}{c_0 N} \lambda^{2N}.
\]

with some constant \( C \) independent on \( N \). Then for some \( N \) big enough we strictly improve (3.9) for \( \varepsilon \). Hence, by continuity we conclude that (3.9) for \( \varepsilon (t) \) is true on \([T_0, T_n] \).

Therefore we need to prove (4.94). Similarly to (4.58) we decompose

\[
\frac{d}{dt} \tilde{G}_{W_1} (\varepsilon_1 (t)) = \tilde{G}_{11} (t) + \tilde{G}_{12} (t) + \tilde{G}_{13} (t), \tag{4.95}
\]

where

\[
\tilde{G}_{11} (t) = - \left( \tilde{W}_1, \mathcal{N}_1 (W_1, \varepsilon_1) \right),
\]

\[
\tilde{G}_{12} (t) = \left( \dot{\varepsilon}_1, -\Delta \varepsilon_1 + \left( \lambda^{-2} (t) + |\beta (t)|^2 \right) \varepsilon_1 - \mathcal{V} \varepsilon_1 - \mathcal{N}_0 (W_1, \varepsilon_1) \right),
\]

and

\[
\tilde{G}_{13} (t) = - \beta \cdot \Im \int \varphi x \nabla \varepsilon_1 \varepsilon_1 - \beta \cdot \Im \int \varphi x \nabla \varepsilon_1 \varepsilon_1 + \beta \cdot \Im \int \nabla \varphi x \varepsilon_1 \varepsilon_1 + 2 \beta \cdot \Im \int \varphi x \nabla \varepsilon_1 \varepsilon_1. \tag{4.96}
\]

Let us consider \( \tilde{G}_{11} (t) \). We define

\[
\varphi x (y) = \varphi \left( \frac{y}{4 \ln \lambda^{-2N}} \right). \tag{4.97}
\]

Using (4.59) we split

\[
\tilde{G}_{11} (t) = \Re \int \tilde{W} \mathcal{N}_1 (W_1, \varepsilon_1) dx + \tilde{G}_{11}^{(0)} (t) + \tilde{G}_{11}^{(1)} (t), \tag{4.98}
\]

with

\[
\tilde{W} = \lambda^{-\frac{3}{2}} \left( \psi x \left( \frac{x}{\lambda} \cdot \nabla W + 2 i |\beta|^2 W \right) \right) \left( \frac{x - \chi}{\lambda} \right) e^{-i \left( \gamma (t) \varphi (t) \right)} e^{i \beta x},
\]

\[
\tilde{G}_{11}^{(0)} (t) = - \lambda^{-\frac{3}{2}} \Re \int (1 - \varphi x) \tilde{W}_1 \mathcal{N}_1 (W_1, \varepsilon_1) dx
\]

and

\[
\tilde{G}_{11}^{(1)} (t) = \lambda^{-\frac{3}{2}} \Re \int \mathcal{N}_1 (W_1, \varepsilon_1) \tilde{W}_1 dx,
\]

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where 
\[ \tilde{W}_1 = \left( \tilde{\varphi}_\lambda \right) \left( \frac{\lambda}{2} \Lambda W + i \left( \gamma + \lambda^2 - |\beta|^2 - \vec{\beta} \cdot \chi \right) W - i \lambda \left( \vec{\beta} \cdot y \right) W + \tilde{W} \right) \frac{1}{\chi} e^{-i(\gamma(t) + \tau(t))} e^{i\beta \cdot x}. \]

Using (4.59) and Lemma 2.7 we obtain (4.63). Thus, by (4.8), (4.63), Sobolev embedding theorem and (3.9) we estimate
\[ \left| \tilde{G}_{11}^{(0)} (t) \right| \leq C\mathcal{X}^{N} \left( \|\varepsilon_1\|_{H^\delta}^p + \|\varepsilon_1\|_{H^1}^2 \right) \leq C \mathcal{X}^{N} \left( \mathcal{X}^{(p-\delta)N} + \mathcal{X}^{2N} \right). \quad (4.99) \]

Using (4.69) in (4.98) we obtain
\[ \tilde{g}_{11} (t) = \text{Re} \int \tilde{W} \left( W_1, \varepsilon_1 \right) + \tilde{g}_{11}^{(0)} (t) + \tilde{g}_{11}^{(1)} (t) + \tilde{g}_{11}^{(2)} (t), \quad (4.100) \]

where
\[ \tilde{g}_{11}^{(2)} (t) = O \left( \int \tilde{W} \left( \|W_1\|^{-1+\delta} \|\varepsilon_1\|^{p+1-\delta} \right) dx \right). \]

By (3.9), there is \( T_0 > 0 \) such that \( |\lambda(t)| \leq \frac{1}{2} \lambda^{\infty}, t \in [T^*, T^*], T^* \geq T_0. \) From Lemma 2.7 it follows
\[ |W (y)| \geq Q (y) - |T (y)| \geq Q (y) \left( 1 - |V \left( \frac{|\lambda|}{\lambda} \right)| \mathcal{X}^{-2\delta N} \right), \quad |y| \leq 4 \ln \mathcal{X}^{-2N}. \]

From (3.9) it follows that \( |V \left( \frac{|\lambda|}{\lambda} \right)| \mathcal{X}^{-2\delta N} \leq C \mathcal{X}, \) for \( \delta < (8N)^{-1}. \) Then, there is \( C_0 > 0 \) such that
\[ |W (y)| \geq 2^{-1} Q (y), \quad (4.101) \]

for all \( |y| \leq 4 \ln \mathcal{X}^{-2N} \) and \( |\lambda| \geq C_0. \) Using (4.11) and (4.101) we estimate
\[ e^{-\left(1-\delta\right)|y|} |V_1 (W_1, \varepsilon_1)| \leq C e^{-\left(1-\delta\right)|y|} |W_1|^{p-2} |\varepsilon_1|^2 \leq C e^{-\left(1-\delta\right)|y|} |Q^{p-2} (y) |\varepsilon_1|^2 \leq C |\varepsilon_1|^2, \]

for all \( |y| \leq 4 \ln \mathcal{X}^{-2N}. \) By (4.62), using Lemma 2.7 to control \( \Lambda W, W, yW, \tilde{W}, \) we get
\[ \left| \tilde{g}_{11}^{(1)} (t) \right| \leq C \left( \mathcal{M} (t) + \left| B^{(N)} \right| + \left| M^{(N)} \right| + |\beta| |\Psi| \right) \|\varepsilon_1\|_{H^1}^2 \quad (4.102) \]

Moreover, via Sobolev embedding theorem we deduce
\[ \left| \tilde{g}_{11}^{(2)} (t) \right| \leq C |\chi| \left( \|\varepsilon_1\|_{H^\delta}^{2+\delta} + \|\varepsilon_1\|_{H^1}^{p+1-\delta} \right), \quad (4.103) \]

with \( 0 < \delta < p - 1. \) Using (4.99), (4.102) and (4.103) in (4.100) we get
\[ \tilde{g}_{11} (t) = \text{Re} \int \tilde{W} \left( W_1, \varepsilon_1 \right) + C \mathcal{X}^{N} \left( \mathcal{X}^{(p-\delta)N} + \mathcal{X}^{2N} \right) \]
\[ + O \left( \mathcal{M} (t) + \left| B^{(N)} \right| + \left| M^{(N)} \right| + |\beta| |\Psi| \right) \|\varepsilon_1\|_{H^1}^2 + |\chi| \left( \|\varepsilon_1\|_{H^\delta}^{2+\delta} + \|\varepsilon_1\|_{H^1}^{p+1-\delta} \right). \quad (4.104) \]

Using (4.74), \( |N_0 (W_1, \varepsilon_1)| \leq C |\varepsilon_1| \) and (2.130), via Sobolev theorem we control
\[ \left| \tilde{g}_{12} (t) \right| \leq C \left( \Psi \mathcal{X}^{N} + \mathcal{M} (t) \right) \|\varepsilon_1\|_{H^1}. \quad (4.105) \]

Next, we consider \( \tilde{g}_{13} (t). \) Let us estimate the last term in the right-hand side of (4.96). Using (4.73), (4.74), (2.130) and integrating by parts we have
\[ \text{Im} \int \varphi_\lambda \nabla \tilde{\varphi}_\lambda \tilde{\varepsilon}_1 = \tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + O \left( \left| \Psi \mathcal{X}^{N} + \mathcal{M} (t) \right| \|\varepsilon_1\|_{H^1} \right). \quad (4.106) \]

where
\[ \tilde{I}_1 = \text{Re} \int \nabla \varphi_\lambda \nabla \left( -\Delta \varepsilon_1 + \left( \lambda^{-2} (t) + |\beta (t)|^2 \right) \varepsilon_1 - \mathcal{V} \varepsilon_1 \right), \]
\[ \tilde{I}_2 = - \text{Re} \int \varphi_\lambda \nabla |\varepsilon_1|^2 \]
and
\[ \tilde{I}_3 = \text{Re} \int \varphi_\lambda \nabla \varepsilon_1 N_0 (W_1, \varepsilon_1). \]
By (4.78) we have
\[ |\tilde{I}_1| \leq \frac{C}{|\chi(t)|} \|\varepsilon_1\|^2_{H^1}. \] (4.107)

Noting that \( \|\varphi \left( \frac{x - \chi(t)}{\lambda(t)(t)} \right) \nabla \nu(t) \|_{L^\infty} \leq C \| \nu' \left( \frac{|x(t)|}{4} \right) \| \), we get
\[ |\tilde{I}_2| \leq C \left( \| \nu' \left( \frac{|x(t)|}{4} \right) \| \right) \|\varepsilon_1\|^2_{H^1}. \] (4.108)

Observe that there is \( C_0 > 0 \) such that \( \varphi \chi \tilde{\varphi}_X = \tilde{\varphi}_X \), with \( \tilde{\varphi}_X \) given by (4.97). By (4.7) we write
\[ N_0 (W_1, \varepsilon_1) = N_L (W_1, \varepsilon_1) + N_1 (W_1, \varepsilon_1), \]
where
\[ N_L (W_1, \varepsilon_1) = \frac{p+1}{2} |W_1|^{p-1} \varepsilon_1 + \frac{p-1}{2} |W_1|^{p-3} \tilde{W}_1^2 \varepsilon_1. \]

We decompose \( \tilde{I}_3 = \tilde{I}_{31} + \tilde{I}_{32} \) with
\[ \tilde{I}_{31} = \text{Re} \int \tilde{\varphi}_X \nabla \tilde{\varphi}_X N_L (W_1, \varepsilon_1) \]
and
\[ \tilde{I}_{32} = \text{Re} \int \varphi (1 - \tilde{\varphi}_X) \nabla \tilde{\varphi}_X N_0 (W_1, \varepsilon_1) + \text{Re} \int \tilde{\varphi}_X \nabla \tilde{\varphi}_X N_1 (W_1, \varepsilon_1). \]

Integrating by parts we have
\[ \tilde{I}_{31} = - \text{Re} \int \tilde{\varphi}_X (\nabla W_1) N (W_1, \varepsilon_1) + \tilde{I}_{31}^{(1)}, \]
with
\[ \tilde{I}_{31}^{(1)} = - \text{Re} \int (\nabla \tilde{\varphi}_X) \left( \frac{p+1}{2} |W_1|^{p-1} \varepsilon_1^2 + \frac{p-1}{2} \text{Re} \left( |W_1|^{p-3} \tilde{W}_1^2 \varepsilon_1^2 \right) \right). \]

Using that \( |N_0 (W_1, \varepsilon_1)| \leq C \left( |W_1|^p + |\varepsilon_1|^p \right) \), as \( \left( |\nabla \tilde{\varphi}_X| W_1 (y) \right) + \left| \left( 1 - \tilde{\varphi}_X \right) W_1 (y) \right| \leq CA^{2(1-\delta)N} \) and \( |N_1 (W_1, \varepsilon_1)| \leq C \left( |\varepsilon_1|^{p_1} + |\varepsilon_1|^{p_1-\delta} \right), \delta > 0 \), \( (p_1 = \min \{ p, 2 \}) \) we get
\[ \left| \int (\nabla \tilde{\varphi}_X) \left( \frac{p+1}{2} |W_1|^{p-1} \varepsilon_1^2 + \frac{p-1}{2} \text{Re} \left( |W_1|^{p-3} \tilde{W}_1^2 \varepsilon_1^2 \right) \right) \right| + |I_{32}| \leq C \left( A^{2(1-\delta)N(p+1) \|\varepsilon_1\|^2_{H^1} + \|\varepsilon_1\|^p_{H^1}} \right). \]

Hence
\[ \tilde{I}_3 = - \text{Re} \int \tilde{\varphi}_X (\nabla W_1) N (W_1, \varepsilon_1) + O \left( \left( A^{2(1-\delta)N(p+1) \|\varepsilon_1\|^2_{H^1} + \|\varepsilon_1\|^p_{H^1}} \right) \right). \] (4.109)

Using (4.107), (4.108) and (4.109) in (4.106) we get
\[ 2\beta \cdot \text{Im} \int \varphi \nabla \tilde{\varphi}_X \varepsilon_1 = -2\beta \cdot \text{Re} \int \tilde{\varphi}_X (\nabla W_1) N (W_1, \varepsilon_1) + \text{Er}_1 \] (4.110)

with
\[ \text{Er}_1 = O \left( \beta \left( \left| \chi(t) \right|^{-1} + \| \nu' \left( \frac{|x(t)|}{4} \right) \| \right) \|\varepsilon_1\|^2_{H^1} \right) \]
\[ + O \left( \beta \left( A^{2(1-\delta)N(p+1) \|\varepsilon_1\|^2_{H^1} + \|\varepsilon_1\|^p_{H^1}} \right) \right). \]

Similarly to (4.76), (4.79) and (4.80) we estimate the first three terms in (4.96) by \( O \left( \left( \beta \right) + \left( \beta \frac{|\varepsilon_1|}{|\chi(t)|} \right) \|\varepsilon_1\|^2_{H^1} \right). \) Therefore, from (4.110) we obtain
\[ \tilde{G}_{13} (t) = -2\beta \cdot \text{Re} \int \hat{\varphi}_X (\nabla W_1) N (W_1, \varepsilon_1) + \text{Er}_2. \]
Lemma 5.1

Let \( K \) the behavior of (5.1), as \( \theta \) where

By (4.101) \( |\tilde{\varphi} x N (W_1, \varepsilon_1)| \leq C |\varepsilon_1|^2 \). Then, using (4.91) we get

\[
\hat{G}_{13} (t) = - \text{Re} \int \hat{W} N (W_1, \varepsilon_1) + \text{Em} + O \left( M (t) \| \varepsilon_1 \|_{H^1}^2 \right).
\]

Using (4.104), (4.105), (4.111) in (4.95) we arrive to

\[
\begin{aligned}
\frac{d}{dt} \hat{G}_W (\varepsilon (t)) &= \frac{d}{dt} \hat{G}_W (\varepsilon (t)) + \text{Em} + C \chi^N (A^{(p-\delta)N} + A^{2N}) \\
&+ \left( \frac{M (t)}{\| \varepsilon_1 \|_{H^1}^2} + \| \varepsilon_1 \|_{H^1}^2 + |\chi| \right) \| \varepsilon_1 \|_{H^1}^2 + |\chi| \left( \| \varepsilon_1 \|_{H^1}^{2+\delta} + \| \varepsilon_1 \|_{H^1}^{p+1-\delta} \right).
\end{aligned}
\]

Finally, using (4.36) and (3.9), for \( N \) sufficiently big such that \( \lambda^{N(p-1)} \leq C \left( (r^\infty)^{-1} + |\varphi (r^\infty)| + \psi \right) \) we prove (4.94).

5 Asymptotics of \( J (\chi) \).

Let us study the asymptotics as \( |\chi| \to \infty \) of the integral

\[
J (\chi) = \int G (|y + \chi|) \nabla Q^2 (y) dy = \int G (|z|) \nabla Q^2 (\chi - z) dz
\]

where \( G \in C^\infty \). We consider \( G \in C^\infty \) of the form (2.16). That is

\[
G (r) = V_+ (r) \quad \text{or} \quad G (r) = V_- (r).
\]

Recall that \( v (d) \) and \( C_{\pm} (\lambda) \) are defined by (2.22) and (2.23), respectively. First we study the case when \( \nabla Q^2 \) determines the behavior of (5.1), as \( |\chi| \to \infty \). That is

\[
G (r) = V_- (r), \quad H \geq 0.
\]

We denote by \( K \) the limit \( e^{-H} \to K \). We prove the following.

Lemma 5.1 Let \( G \in C^\infty \) be as in (5.3), where \( H, H' \) are monotone. Then, the following is true. If \( d \geq 4 \), the asymptotics

\[
J (\chi) = \frac{\chi}{|\chi|} e^{-2|\chi| |\chi|^{-d-1}} \left( \int (A^2 G (|z|) + K_\kappa Q^2 (z)) e^{2\chi^2 \nabla^2} dz + o (1) \right)
\]

as \( |\chi| \to \infty \) holds. Suppose that \( d = 2 \) or \( d = 3 \). If \( r \frac{d-1}{d-2} e^{-H (r)} \in L^1 (|1, \infty|) \), then

\[
J (\chi) = \frac{\chi}{|\chi|} e^{-2|\chi| |\chi|^{-d-1}} \left( A^2 \int G (|z|) e^{2\chi^2 \nabla^2} dz + o (1) \right)
\]

as \( |\chi| \to \infty \). In the case when \( r \frac{d-1}{d-2} e^{-H (r)} \notin L^1 (|1, \infty|) \), the expansion

\[
J (\chi) = \frac{\chi}{|\chi|} \kappa A^2 v (d) (1 + o (1)) C_- (|\chi|) e^{-2|\chi| |\chi|^{-d-1}},
\]

as \( |\chi| \to \infty \) takes place.

Proof. First, we note that \( J (\chi) \) is directed along the vector \( \chi \). Indeed, we introduce the polar coordinate system, where the \( x_1 \)-axis is directed along the vector \( \chi \). Then, \( \chi = |\chi| (1, 0, \ldots, 0) \) and \( y = |y| (\cos \theta, \sin \theta \cos \theta_1, \ldots, \sin \theta \sin \theta_1 \ldots \sin \theta_{d-2}) \), where \( \theta \) is the angle between \( \chi \) and \( z \). Thus

\[
J (\chi) = \int_0^\infty \int_0^\pi \int_0^{2\pi} G \left( \left| \sqrt{|\chi|^2 + r^2 + 2 |\chi| r \cos \theta} \right| \right) (q^2)^d (r) r^{d-1} y_\theta d\Omega \]

\[
= C \frac{\chi}{|\chi|} \int_0^\infty \int_0^\pi \int_0^{2\pi} G \left( \left| \sqrt{|\chi|^2 + r^2 + 2 |\chi| r \cos \theta} \right| \right) \cos \theta \left( q^2 \right)^d (r) r^{d-1} d\theta dr,
\]

with \( y_\theta = (\cos \theta, \sin \theta \cos \theta_1, \ldots, \sin \theta \sin \theta_1 \ldots \sin \theta_{d-2}) \) and \( d\Omega = \sin^{d-2} \theta \sin^{d-3} \theta_1 \ldots \sin \theta_{d-3} d\theta d\theta_1 \ldots d\theta_{d-2} \).
Using (1.4) we estimate

\[ \left| \int_{|z| \geq |x|/2} G(|z|) \nabla Q^2(\chi - z) \, dz \right| \leq C \int_{|z| \geq |x|/2} e^{-2|z|-(d-1)\ln|z|-H(|z|)} \left| \nabla Q^2(\chi - z) \right| \, dz \]

\[ \leq C e^{-3|x|} \left( \int \left| \nabla Q^2(\chi - z) \right| \, dz \right). \]

Therefore

\[ \left| \int_{|z| \geq |x|/2} G(|z|) \nabla Q^2(\chi - z) \, dz \right| \leq C e^{-3|x|}. \] (5.8)

Next, we consider the region \(|x|/2 \leq |z| \leq 3|x|/2\). Since for \(|z| \leq |x|/2\), the inequality \(|x|/2 \leq |z| \leq 3|x|/2\) holds, we write

\[ \int_{|x|/2 \leq |z| \leq 3|x|/2} G(|z|) \nabla Q^2(\chi - z) \, dz = -\int_{|x|/2 \leq |z| \leq 3|x|/2} G(|z - x|) \nabla Q^2 (z) \, dz \]

\[ = -Ke^{-2|x|} \int_{|z| \leq |x|/2} e^{-2(|z-x|-|x|+|z|)} \left| \chi - z \right|^{-(d-1)} e^{-H(|x-z|)} \left( \frac{\partial_z e^{2|z|\partial_z q^2 (|z|)}}{e^{2|z|\partial_z q^2 (|z|)}} \right) \, dz \]

\[ -\int_{|x|/2 \leq |z-x| \leq 3|x|/2} G(|z - x|) \nabla Q^2 (z) \, dz. \] (5.9)

We now present the following estimates. Using the coordinate system in (5.7) we have

\[ \left| \frac{z}{|z|} - \frac{x}{|x|} \right| \leq C \left( (1 - \cos \theta) + \sin \theta \right). \] (5.10)

By

\[ |y + x| - |x| + |y| = \frac{2|x| |y| (1 + \frac{x^2}{|x|^2})}{|y + x| + |x| - |y|} \] (5.11)

we get

\[ e^{-2(|z-x|-|x|+|z|)} - e^{-2\frac{x|z|(1-\cos \theta)}{|x|-|z|}} \leq C e^{-|z|(1-\cos \theta)} \left| \frac{z}{|x|} \right|^{2(1 - \cos \theta)^2}. \] (5.12)

Also, note that

\[ \left| \frac{z}{|z|} - \left( \frac{x}{|x|} \right) \right| \leq C \frac{|z|(1 - \cos \theta)}{|x|^2}, \text{ for } |z| \leq \frac{|x|}{2}. \] (5.13)

Suppose first that \(d \geq 4\). Let \(e^{-H} \to K\) and \(\psi \in L^\infty(\mathbb{R})\) such that \(\psi (r) = 1\), for \(0 \leq r \leq 1\), and \(\psi (r) = 0\), for \(r > 1\). Using (1.4), (5.12) and (5.13) we have

\[ \left| \int_{|z| \leq |x|/2} e^{-2(|z-x|-|x|+|z|)} \left| \chi - z \right|^{-(d-1)} e^{-H(|x-z|)} \left( \frac{z}{|z|} e^{2|z|\partial_z q^2 (|z|)} \right) \, dz - I_1 \right| \leq r_1 \] (5.14)

with

\[ I_1 = K \int \psi \left( \frac{2|z|}{|x|} \right) e^{-2\frac{x|z|(1-\cos \theta)}{|x|-|z|}} \left( \frac{z}{|z|} e^{2|z|\partial_z q^2 (|z|)} \right) \, dz \]

and

\[ r_1 = C |x|^{-(d-1)} \left( \sup_{|z| \leq |x|/2} e^{-H(|x-z|)} - K \right) \left| \frac{z}{|z|} \right| \int_{|z| \leq |x|/2} e^{-\frac{1}{|z|}(1-\cos \theta)} \left| \frac{z}{|z|} \right|^{-(d-1)} \, dz. \]

By (1.4)

\[ \psi \left( \frac{2|z|}{|x|} \right) e^{-2\frac{x|z|(1-\cos \theta)}{|x|-|z|}} \left( \frac{z}{|z|} e^{2|z|\partial_z q^2 (|z|)} \right) \leq C |x|^{-(d-1)} e^{-\frac{1}{|z|}(1-\cos \theta)} \left| \frac{z}{|z|} \right|^{-(d-1)}. \]
Let us show that \( e^{-\frac{1}{2}(1-\cos \theta)} |z|^{-(d-1)} \) is integrable if \( d \geq 4 \). Using the polar coordinate system in (5.7) we have

\[
\int e^{-\frac{1}{2}(1-\cos \theta)} |z|^{-(d-1)} \, dz \leq C \int_0^\infty \left( \int_0^\pi e^{-\frac{1}{2}(1-\cos \theta)} \theta^{d-2} \, d\theta \right) \, dr
\]

\[
\leq C \int_0^1 \left( \int_0^\pi e^{-\frac{1}{2}(1-\cos \theta)} \theta^{d-2} \, d\theta \right) \, dr + C \int_1^\infty r^{-\frac{d-4}{2}} \left( \int_0^\pi \sqrt{\theta} e^{-\frac{1}{2} \theta^{d-2}} \, d\theta \right) \, dr
\]

\[
\leq C + C \left( \int_0^\infty e^{-\frac{1}{2} \theta^{d-2}} \, d\theta \right) \int_1^\infty r^{-\frac{d-4}{2}} \, dr \leq C.
\]

Thus, by the dominated convergence theorem,

\[
I_1 = K |\chi|^{-(d-1)} \int \frac{z}{|z|} e^{\frac{2}{|\chi|} \partial z \cdot g^2 (|z|)} \, dz + |\chi|^{-(d-1)} o(1),
\]

as \( |\chi| \to \infty \). Integrating by parts we get

\[
I_1 = -K \frac{\chi}{|\chi|} |\chi|^{-(d-1)} \int e^{\frac{2}{|\chi|} Q^2} (z) \, dz + |\chi|^{-(d-1)} o(1).
\]

(5.16)

Also from (5.15) it follows that

\[
r_1 = |\chi|^{-(d-1)} o(1).
\]

(5.17)

To estimate the last term in the right-hand side of (5.9) we decompose

\[
\int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|} \text{, } \frac{1}{|\chi|} \leq |z| \leq \frac{1}{|\chi|}} G (|\chi - z|) \nabla Q^2 (z) \, dz
\]

\[
= \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|} \text{, } \frac{1}{|\chi|} \leq |z| \leq \frac{1}{|\chi|}} G (|\chi - z|) \nabla Q^2 (z) \, dz
\]

\[
+ \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|} \text{, } \frac{1}{|\chi|} \leq |z| \leq \frac{1}{|\chi|}} G (|\chi - z|) \nabla Q^2 (z) \, dz.
\]

Using (1.4) and passing to the polar system as in (5.7) we get

\[
\left| \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|} \text{, } \frac{1}{|\chi|} \leq |z| \leq \frac{1}{|\chi|}} G (|\chi - z|) \nabla Q^2 (z) \, dz \right|
\]

\[
\leq C e^{-2 |\chi|} |\chi|^{-(d-1)} e^{-H \left( \frac{1}{|\chi|} \right)} \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{1}{|\chi|}} e^{-\frac{1}{2} (1-\cos \theta)} |z|^{-(d-1)} \, dz
\]

\[
\leq C e^{-2 |\chi|} |\chi|^{-(d-1)} e^{-H \left( \frac{1}{|\chi|} \right)} |\chi|^{\frac{1}{2} \frac{d-4}{2}} \frac{d-4}{2}
\]

(5.18)

and

\[
\left| \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|} \text{, } \frac{1}{|\chi|} \leq |z| \leq \frac{1}{|\chi|}} G (|\chi - z|) \nabla Q^2 (z) \, dz \right| \leq C e^{-2 |\chi|} |\chi|^{-(d-1)} e^{-H \left( \frac{1}{|\chi|} \right)} |\chi|^{\frac{1}{2} \frac{d-4}{2}}.
\]

Hence

\[
\left| \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|} \text{, } \frac{1}{|\chi|} \leq |z| \leq \frac{1}{|\chi|}} G (|\chi - z|) \nabla Q^2 (z) \, dz \right| \leq C e^{-2 |\chi|} |\chi|^{-(d-1)} e^{-H \left( \frac{1}{|\chi|} \right)} |\chi|^{\frac{1}{2} \frac{d-4}{2}}.
\]

(5.19)

Making use of (5.14), (5.16), (5.17) and (5.19) in (5.9) we arrive to

\[
\int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|}} G (|z|) \nabla Q^2 (\chi - z) \, dz = K \frac{\chi}{|\chi|} e^{-2 |\chi|} |\chi|^{-(d-1)} \int e^{\frac{2}{|\chi|} Q^2} (z) \, dz + o \left( e^{-2 |\chi|} |\chi|^{-(d-1)} \right).
\]

(5.20)

Next, we consider the cases \( d = 2 \) and \( d = 3 \). If \( H' (r) \) does not tend to 0, as \( r \to \infty \), then as \( H \geq 0 \), we see that \( H (r) \geq cr \), for some \( c > 0 \). In this case by (1.4) we have

\[
\left| \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|}} G (|z|) \nabla Q^2 (\chi - z) \, dz \right|
\]

\[
\leq C |\chi|^{-(d-1)} e^{-2 |\chi|} e^{-\frac{c |\chi|}{2}} \int_{\frac{1}{|\chi|} \leq |z| \leq \frac{2}{|\chi|}} e^{-\frac{c |\chi|}{2}} |\chi - z|^{-(d-1)} \, dz \leq C |\chi|^{-(d-1)} e^{-2 |\chi|} e^{-\frac{c |\chi|}{2}}.
\]
Suppose now that $H' \to 0$. We take the following estimate into account
\[ |e^{-H(|x-z|)} - e^{-H(|x|-|z|)}| \leq C |H' (|x| - |z|)| e^{-H(|x|-|z|)} |z| (1 - \cos \theta). \] (5.21)

Using (1.3), (5.10), (5.12), (5.13) and (5.21) we estimate
\[ \left| \int_{|z| \leq \frac{1}{24}} e^{-2(|z-x|-|x+z|)} |x-z|^{-(d-1)} e^{-H(|x-z|)} \left( \frac{z}{|z|} e^{2|z| \partial_{|z|} \eta^2 (|z|)} \right) \right| dz + I_2 \leq r_2 \] (5.22)
with
\[ I_2 = A^2 \frac{x}{|x|} \int_{|z| \leq \frac{1}{24}} e^{-2(|z-x|-|x+z|)} |x-z|^{-(d-1)} e^{-H(|x|-|z|)} |z|^{-(d-1)} dz \]
and
\[ r_2 = C |x|^{-(d-1)} \int_{|z| \leq \frac{1}{24}} e^{-\frac{|z|^2}{2(1-\cos \theta)}} |H' (|x|-|z|)| e^{-H(|x|-|z|)} |z|^{-(d-1)} dz \]
\[ + C |x|^{-(d-1)} e^{-\frac{H(|x|)}{2}} \int_{|z| \leq \frac{1}{24}} e^{-\frac{|z|^2}{2(1-\cos \theta)}} \left( (1 + |z|)^{-(d-1)} \right) dz. \]

Passing to the polar system as in (5.7) we get
\[ \left| \int_{|z| \leq \frac{1}{24}} e^{-\frac{2|x_z|}{|x|}} \left( |x| - |z| \right)^{-(d-1)} e^{-H(|x|-|z|)} |z|^{-(d-1)} dz \right| - \sigma \int_1^{\frac{1}{24}} \left( |x| - r \right)^{-(d-1)} e^{-H(|x|-r)} \left( \int_0^{\pi} e^{-\frac{2|x_z| r^2}{|x|}} \theta d\theta \right) dr \leq C |x|^{-(d-1)} e^{-H\left(\frac{|x|}{2}\right)} \left( 1 + \int_{|z| \leq \frac{1}{24}} e^{-\frac{|z|^2}{2(1-\cos \theta)}} |z|^{-(d-1)} dz \right), \] (5.23)
\[ \sigma = \frac{2\pi \frac{d-1}{2}}{\Gamma \left( \frac{d-1}{2} \right)} \]

Making the change $\theta \rightarrow \left( \frac{|x| - |x_z|}{|x|} \right)^{\frac{d-1}{2}} \eta$ we obtain
\[ \int_1^{\frac{1}{24}} \left( |x| - r \right)^{-(d-1)} e^{-H(|x|-r)} \left( \int_0^{\pi} e^{-\frac{2|x_z| r^2}{|x|}} \theta d\theta \right) dr = v (d) |x|^{-\frac{d+1}{2}} \int_{\frac{1}{24}}^{\frac{1}{24}} r^{-\frac{d+1}{2}} e^{-H(r)} dr \]
\[ - |x|^{-\frac{d+1}{2}} \int_1^{\frac{1}{24}} \left( r \left( |x| - r \right) \right)^{-\frac{d+1}{2}} e^{-H\left( \frac{|x|}{2} \right)} \left( \int_0^{\pi} e^{-2\eta^2 \eta \theta^2 - 2d\eta} dr \right) \] (5.24)
where $v (d)$ is given by (2.22). The second term in the right-hand side of (5.24) is estimated by using
\[ |x|^{-\frac{d+1}{2}} \int_1^{\frac{1}{24}} \left( r \left( |x| - r \right) \right)^{-\frac{d+1}{2}} \left( \int_0^{\infty} e^{-2\eta \eta \eta^2 - 2d\eta} dr \right) \leq C |x|^{-(d-1)} \int_1^{\infty} r^{-\frac{d+1}{2}} e^{-\frac{\eta^2 \eta^2 - 2d\eta} r} dr \leq C |x|^{-(d-1)}. \] (5.25)

From (5.24) and (5.25) we get
\[ \int_1^{\frac{1}{24}} \left( |x| - r \right)^{-(d-1)} e^{-H\left( \frac{|x|}{2} \right)} \left( \int_0^{\pi} e^{-\frac{2|x_z| r^2}{|x|}} \theta d\theta \right) dr \]
\[ = v (d) |x|^{-\frac{d+1}{2}} \int_{\frac{1}{24}}^{\frac{1}{24}} r^{-\frac{d+1}{2}} e^{-H\left( \frac{|x|}{2} \right)} dr + O \left( |x|^{-(d-1)} e^{-H\left( \frac{|x|}{2} \right)} \right). \] (5.26)

Passing to the polar system as in (5.7) we estimate
\[ \int_{|z| \leq \frac{1}{24}} e^{-\frac{|z|^2}{2(1-\cos \theta)}} |H' (|x|-|z|)| e^{-H\left( \frac{|x|}{2} \right)} |z|^{-(d-1)} dz \]
\[ \leq C \left| H' \left( \frac{|x|}{2} \right) \right| \int_1^{\frac{1}{24}} r^{-\frac{d+1}{2}} e^{-H\left( \frac{|x|}{2} \right)} \left( \int_0^{\pi} e^{-\frac{2\eta \eta \eta^2 - 2d\eta} r} dr \right) \] (5.27)
\[ \leq C \left| H' \left( \frac{|x|}{2} \right) \right| \int_{\frac{1}{24}}^{\frac{1}{24}} \left( |x| - r \right)^{-\frac{d+1}{2}} e^{-H\left( \frac{|x|}{2} \right)} dr \]
and
\[
\int_{|z| \leq \frac{1}{2^{d-1}}} e^{-\frac{|z|^2}{2}} (1-\cos \theta)^\left(\frac{1}{2} + \frac{1}{2}\right) \left(1 + |z|^{-\frac{d-1}{2}}\right) |z|^{-d+1} \, dz \\
\leq C \left(1 + \int_{1}^{\frac{1}{2^{d-1}}} r^{-\frac{d}{2}} \left(\int_{0}^{\pi} e^{-\frac{r^2}{2}} (1+\theta)^{d-1} \, d\theta\right) \right) \leq C \left\{ \begin{array}{l} \ln |\chi|, \quad d = 2, \\
1, \quad d = 3. \end{array} \right.
\]
(5.28)

Thus, using (5.26) and (5.28) in (5.23) we obtain
\[
\left| J_2 - \mathcal{A}^2 \frac{X}{|\chi|} v(d) |\chi|^{-\frac{d-1}{2}} \int_{\frac{1}{2^{d-1}}}^{1} (|\chi| - r)^{-\frac{d-1}{2}} e^{-H(r)} \, dr \right| \\
\leq C |\chi|^{-(d-1)} e^{-H\left(\frac{1}{2}\right)} \left\{ \begin{array}{l} \ln |\chi|, \quad d = 2, \\
1, \quad d = 3. \end{array} \right.
\]
(5.29)

Moreover (5.27) and (5.28) imply
\[
|\chi|^{-\frac{d-1}{2}} \int_{\frac{1}{2^{d-1}}}^{1} (|\chi| - r)^{-\frac{d-1}{2}} \, dr \geq C |\chi|^{-(d-1)} \left\{ \begin{array}{l} \sqrt{|\chi|}, \quad d = 2, \\
\ln |\chi|, \quad d = 3, \end{array} \right.
\]
(5.31)

via (5.9), (5.19), (5.22), (5.29) and (5.30) we arrive to
\[
\int_{\frac{1}{2^{d-1}}}^{1} G \left(\frac{|z|}{|\chi|}\right) \nabla Q^2 (\chi - z) \, dz \\
= \kappa \mathcal{A}^2 v(d) \left(\frac{x}{|\chi|} + o(1)\right) e^{-2|x|/|\chi|} \int_{\frac{1}{2^{d-1}}}^{1} (|\chi| - r)^{-\frac{d-1}{2}} e^{-H(r)} \, dr.
\]
(5.32)

Let us consider now the region $|z| \leq \frac{1}{2^{d-1}}$. Note that
\[
\left| \frac{\chi - z}{|\chi| - z} - \frac{\chi}{|\chi|} \right| \leq C \frac{|\chi - z|}{|\chi|}.
\]
(5.33)

Then, using (1.3), (5.12) and (5.13) we have
\[
\left| \int_{|z| \leq \frac{1}{2^{d-1}}} G \left(\frac{|z|}{|\chi|}\right) \nabla Q^2 (\chi - z) \, dz - \kappa \mathcal{A}^2 \frac{X}{|\chi|} e^{-2|x|} \int_{|z| \leq \frac{1}{2^{d-1}}} e^{-2|x|/|\chi|} |\chi|^{-(d-1)} e^{-H\left(\frac{|z|}{|\chi|}\right)} \, dz \right| \leq r_3,
\]
(5.34)

with
\[
r_3 = C e^{-2|x|} \int_{|z| \leq \frac{1}{2^{d-1}}} e^{-\frac{2|x| |z| (1-\cos \theta)}{|\chi| |z|}} |\chi|^{-(d-1)} e^{-H\left(|z|\right)} \, dz.
\]
(5.35)

If $d \geq 4$, by (5.15) the integral in the right-hand side of (5.35) exists. Then by the dominated convergence theorem
\[
\int_{|z| \leq \frac{1}{2^{d-1}}} e^{-2|x|/|\chi|} |z|^{-(d-1)} e^{-H\left(|z|\right)} \, dz = |x|^{-(d-1)} \int e^{-2|x| (1-\cos \theta)} |z|^{-(d-1)} e^{-H\left(|z|\right)} \, dz + o \left(|x|^{-(d-1)}\right).
\]
Thus, from (5.34) it follows
\[
\int_{|z| \leq \frac{1}{2^{d-1}}} G \left(\frac{|z|}{|\chi|}\right) \nabla Q^2 (\chi - z) \, dz = \mathcal{A}^2 \frac{X}{|\chi|} e^{-2|x|} \int_{|z| \leq \frac{1}{2^{d-1}}} e^{-\frac{2|x| |z| (1-\cos \theta)}{|\chi| |z|}} |z|^{-(d-1)} e^{-H\left(|z|\right)} \, dz + o \left(|x|^{-(d-1)}\right).
\]
(5.36)

In the case of dimensions $d = 2$ or $d = 3$. If $r^{-\frac{d-1}{2}} e^{-H(r)} \in L^1 ([1, \infty))$, $r_3 = O \left(e^{-2|x|} |x|^{-\frac{d+1}{2}}\right)$, similarly to the case $d \geq 4$ we obtain (5.36). If $r^{-\frac{d-1}{2}} e^{-H(r)} \notin L^1 ([1, \infty))$ similarly to (5.29) via (5.28), (5.25), (2.22) we have
\[
\left| \int_{|z| \leq \frac{1}{2^{d-1}}} e^{-2|x|/|\chi|} |z|^{-(d-1)} e^{-H\left(|z|\right)} \, dz - v(d) \int_{1}^{\frac{1}{2^{d-1}}} (|\chi| - r)^{-\frac{d-1}{2}} e^{-H\left(|z|\right)} \, dr \right| \\
\leq C |\chi|^{-(d-1)} \int_{1}^{\frac{1}{2^{d-1}}} r^{-\frac{d}{2}} e^{-H\left(|z|\right)} \, dr.
\]
(5.37)
Moreover
\[ r_3 \leq C e^{-2|\chi| |\chi|^{-d+\frac{1}{2}}} \int_{1}^{\frac{|\chi|}{2}} r^{-\frac{d-1}{2}} e^{-rH(r)} dr \leq C e^{-2|\chi| |\chi|^{-(d-1)}}. \]

Then as \(|\chi|^{-\frac{d+1}{2}} \int_{1}^{\frac{|\chi|}{2}} (|\chi| - r) \frac{d}{dr} e^{-rH(r)} dr \geq |\chi|^{-d+\frac{1}{2}} \int_{1}^{\frac{|\chi|}{2}} r^{-\frac{d-1}{2}} e^{-rH(r)} dr,\)
(5.34) implies
\[
\int_{|z| \leq \frac{3|\chi|}{2}} G(|z|) \nabla Q^2(\chi - z) dz = o(1) \left( e^{-2|\chi| |\chi|^{-d+\frac{1}{2}}} \int_{1}^{\frac{|\chi|}{2}} (|\chi| - r) \frac{d}{dr} e^{-rH(r)} dr \right). \tag{5.38}
\]

We now conclude as follows. If \(d \geq 4\), gathering together (5.8), (5.20) and (5.36) we get (5.4). In the case of dimensions \(d = 2\) or \(d = 3\), suppose first that \(r^{-\frac{d-1}{2}} e^{-rH(r)} \in L^1([1, \infty))\). Then, (5.32) implies that
\[
\int_{\frac{3|\chi|}{2} \leq |z| \leq |\chi|} G(|z|) \nabla Q^2(\chi - z) dz = o(1) \left( e^{-2|\chi| |\chi|^{-d+\frac{1}{2}}} \int_{1}^{\frac{|\chi|}{2}} (|\chi| - r) \frac{d}{dr} e^{-rH(r)} dr \right).
\]

Therefore, from (5.8) and (5.36) we attain (5.5). If \(r^{-\frac{d-1}{2}} e^{-rH(r)} \notin L^1([1, \infty))\), by (5.8), (5.32) and (5.38) we deduce (5.6).

We now turn to the study of the case when \(G\) determines the behavior of (5.1). We consider bounded function \(G \in C^\infty(\mathbb{R}^d)\) of the form
\[ G(r) = V_+(r), \quad H \geq 0. \tag{5.39} \]

We prove the following.

**Lemma 5.2** Let \(G \in C^\infty(\mathbb{R}^d)\) be bounded and of the form (5.39), where \(H, H'\) are monotone. If \(H(r) = o(r)\), as \(r \to \infty\), the asymptotics
\[ J(\chi) = \frac{\chi}{|\chi|} \kappa A^2 v(d) \left( 1 + o(1) \right) C_+ (|\chi|) e^{-2|\chi| |\chi|^{-(d-1)}} \tag{5.40} \]
as \(|\chi| \to \infty\) are valid with \(C_+ (|\chi|)\) given by (2.23). If \(0 < c_\epsilon H (r) < 2r\) suppose in addition that \(H'' (r)\) is bounded. Then
\[ J(\chi) = T \frac{\chi}{|\chi|} \left( 1 + O \left( \frac{K' H'(|\chi|)}{|\chi|} \right) \right) \left( 2 - H'(|\chi|) \right) G(|\chi|), \tag{5.41} \]
where \(K' = \lim_{r \to \infty} H'(r)\) and \(T\) is defined by (2.24). Finally, if \(V = V^{(2)}\),
\[ J(\chi) = -\frac{\chi}{|\chi|} \left( \left( \int Q^2(z) dz \right) V'(|\chi|) + r(|\chi|) + e^{-\frac{1}{2}} \right), \tag{5.42} \]
with
\[ r(|\chi|) = O \left( \left( |H'(|\chi|)| \right) \left( |H''(|\chi|)|^2 + \frac{|H'(|\chi|)|^2 + |H''(|\chi|)| + |\chi|^{-2}}{|\chi|} + \frac{|H''(|\chi|)|}{|\chi|} + |H''(|\chi|)| \right) V(|\chi|) \right). \tag{5.43} \]

**Proof.** Suppose first that \(H(r) = o(r)\), as \(r \to \infty\). In this case, as \(H'\) is monotone, \(H'(r) = o(1)\), as \(r \to \infty\). Similarly to (5.8) we estimate
\[
\left| \int_{|z| \geq \frac{3|\chi|}{2}} G(|\chi - z|) \nabla Q^2(z) dz \right| \leq C e^{-3|\chi|} \int |G(|\chi - z|)| dz \leq C e^{-3|\chi|}. \tag{5.44} \]

Next, we decompose
\[
\int_{\frac{3|\chi|}{2} \leq |z| \leq |\chi|} G(|\chi - z|) \nabla Q^2(z) dz = -\int_{\frac{3|\chi|}{2} \leq |z| \leq |\chi|} G(|z|) \nabla Q^2(\chi - z) dz
= -\int_{|z| \leq \frac{3|\chi|}{2}} G(|z|) \nabla Q^2(\chi - z) dz - r_4, \tag{5.45} \]
with
\[ r_4 = \int_{\frac{3|\chi|}{2} \leq |z| \leq |\chi|} G(|z|) \nabla Q^2(\chi - z) dz. \]
To estimate $r_4$ we write
\begin{equation*}
    r_4 = \int_{\frac{|z|}{2}}^{|z|} G(|z|) \nabla Q^2(\chi - z) \, dz
    + \int_{\frac{|z|}{2}}^{|z|} e^{-|z|} \, G(|z|) \nabla Q^2(\chi - z) \, dz
    + \int_{\frac{|z|}{2}}^{|z|} e^{-|z|} \, G(|z|) \nabla Q^2(\chi - z) \, dz.
\end{equation*}
where $A = \max \left\{ \frac{|z|}{2}, H(|\chi|) \right\}$. Using (1.4) and (5.11), as $H(r) = o(r)$, similarly to (5.18) we get
\begin{equation*}
    |r_4| \leq C |\chi|^{-(d-1)} e^{-2|\chi|} \int_{\frac{|z|}{2}}^{|z|} \frac{r^{-\frac{d-1}{2}} e^{-H(r)}}{r} \, dr \leq C |\chi|^{-(d-1)} e^{-2|\chi|} o(1) \int_{1}^{\frac{|z|}{2}} r^{-\frac{d-1}{2}} e^{-H(r)} \, dr.
\end{equation*}
(5.46)

Introduce the polar coordinate system as in (5.7). By (1.3), (5.12), (5.13) and (5.33) we have
\begin{equation*}
    \left| \int_{|z| \leq \frac{|z|}{2}} G(|z|) \nabla Q^2(\chi - z) \, dz - A^2 \frac{Y}{|\chi|} e^{-2|\chi|} \int_{|z| \leq \frac{|z|}{2}} e^{-\frac{2|z| |z| (1 - \cos \theta)}{|z| + |z|} e^{H(|z|)}} |z|^{-(d-1)} (|\chi| - |z|)^{-(d-1)} \, dz \right| \leq r_5
\end{equation*}
with
\begin{equation*}
    r_5 = C |\chi|^{-(d-1)} e^{-2|\chi|} \int_{|z| \leq \frac{|z|}{2}} e^{-\frac{|z| (1 - \cos \theta)}{|z| + |z|} e^{H(|z|)}} (1 + |z|)^{-\frac{d}{2}} |z|^{-(d-1)} \, dz.
\end{equation*}
Proceeding similarly to (5.37) and (5.28) we obtain
\begin{equation*}
    \left| \int_{|z| \leq \frac{|z|}{2}} e^{-\frac{2|z| |z| (1 - \cos \theta)}{|z| + |z|} e^{H(|z|)}} |z|^{-(d-1)} (|\chi| - |z|)^{-(d-1)} \, dz - v(d) |\chi|^{-(d-1)} \int_{1}^{\frac{|z|}{2}} r^{-(d-1)} e^{H(r)} \, dr \right| \leq C |\chi|^{-(d-1)} \int_{1}^{\frac{|z|}{2}} r^{-\frac{d-1}{2}} e^{H(r)} \, dr \leq C o(1) |\chi|^{-(d-1)} \int_{1}^{\frac{|z|}{2}} r^{-\frac{d-1}{2}} e^{H(r)} \, dr
\end{equation*}
and
\begin{equation*}
    r_5 \leq C o(1) |\chi|^{-(d-1)} \int_{1}^{\frac{|z|}{2}} r^{-\frac{d-1}{2}} e^{H(r)} \, dr.
\end{equation*}
Hence
\begin{equation*}
    \int_{|z| \leq \frac{|z|}{2}} G(|z|) \nabla Q^2(\chi - z) \, dz = v(d) A^2 \frac{Y}{|\chi|} (1 + o(1)) e^{-2|\chi|} |\chi|^{\frac{d-1}{2}} \int_{1}^{\frac{|z|}{2}} r^{-(d-1)} e^{H(r)} \, dr,
\end{equation*}
and thus from (5.45) and (5.46) we deduce
\begin{equation*}
    \int_{|z| \leq \frac{|z|}{2}} G(|z|) \nabla Q^2(\chi - z) \, dz = -v(d) A^2 \frac{Y}{|\chi|} (1 + o(1)) e^{-2|\chi|} |\chi|^{\frac{d-1}{2}} \int_{1}^{\frac{|z|}{2}} r^{-(d-1)} e^{H(r)} \, dr.
\end{equation*}
(5.47)

and thus from (5.45) and (5.46) we deduce
\begin{equation*}
    \int_{\frac{|z|}{2}}^{2|\chi|} G(|z|) \nabla Q^2(\chi - z) \, dz = -v(d) A^2 \frac{Y}{|\chi|} (1 + o(1)) e^{-2|\chi|} |\chi|^{\frac{d-1}{2}} \int_{1}^{\frac{|z|}{2}} r^{-(d-1)} e^{H(r)} \, dr.
\end{equation*}
(5.48)

Note that
\begin{equation*}
    |H(|\chi| - z| - H(|\chi| - |z|)| \leq CH'(|\chi| - |z|)|z|(1 - \cos \theta).
\end{equation*}
Then, similarly to (5.47) (see also the proof of (5.32)) we show that
\begin{equation*}
    \int_{|z| \leq \frac{|z|}{2}} G(|\chi - z|) \nabla Q^2(\chi - z) \, dz = -v(d) A^2 \frac{Y}{|\chi|} (1 + o(1)) e^{-2|\chi|} |\chi|^{\frac{d-1}{2}} \int_{1}^{\frac{|z|}{2}} r^{-(d-1)} e^{H(r)} \, dr.
\end{equation*}
(5.49)

Therefore, as
\begin{equation*}
    \mathcal{J}(\chi) = -\int G(|\chi - z|) \nabla Q^2(z) \, dz
\end{equation*}
(5.50)
by (5.44), (5.48), (5.49) we prove (5.40).

Suppose now $0 < cr \leq H(r) \leq 2r + o(r)$ for all $r$ sufficiently large. First we note that
\begin{equation*}
    \int_{|z| \geq 2|\chi|} G(|\chi - z|) \nabla Q^2(z) \, dz \leq C |\chi|^{-(d-1)} e^{-|\delta| |\chi|} \int_{|z| \geq |\delta| |\chi|} e^{-|\delta| |z|} \, dz \leq C |\chi|^{-(d-1)} e^{-|\delta| |\chi|},
\end{equation*}
(5.51)
Then, from (5.50), (5.51) and (5.55) we attain (5.41).

\[
\left| x + y \right| - \left| x \right| - \frac{x \cdot y}{\left| x \right|} \leq C \frac{|y|^2}{|x|}, \tag{5.52}
\]

and

\[
\left| x + y \right|^{(d-1)} - \left| x \right|^{(d-1)} \leq C \frac{|y|}{|x|^d}. \tag{5.53}
\]

Since \(0 < cr \leq H(r) < 2r\) and \(H'\) is monotone for all \(r\) sufficiently large. Then, using that \(H''\) is bounded we get

\[
\left| H(|x - z|) - H(|x|) + H'(|x|) \left( \frac{x \cdot z}{|x|} \right) \right| \leq C \frac{1 + |z|^4}{|x|}, \tag{5.54}
\]

for \(|z| \leq \delta |x|\). In particular, (5.52)-(5.54) imply

\[
\left| G(|x - z|) - G(|x|) e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} \right| \leq |G(|x|)| e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} \frac{1 + |z|^4}{|x|}, \text{ as } |x| \to \infty.
\]

Using the last estimate we deduce

\[
\left| \int_{|z| \leq \delta |x|} G(|x - z|) \nabla Q^2(z) \, dz - G(|x|) \int_{|z| \leq \delta |x|} e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} \nabla Q^2(z) \, dz \right| \leq C \frac{|G(|x|)|}{|x|} r_6 \tag{5.55}
\]

where

\[
r_6 = \int e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} \left( 1 + |z|^4 \right) \left| \nabla Q^2(z) \right| \, dz.
\]

Integrating by parts we have

\[
\int e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} \nabla Q^2(z) \, dz = - (2 - H'(|x|)) \frac{x}{|x|} \int e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} Q^2(z) \, dz.
\]

Since \(|2 - H'(|x|)| \leq \max\{2 - c_1, 1\} < 2\), for all \(|x|\) sufficiently large, it follows from (1.4) that \(r_6 < \infty\),

\[
\int e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} \nabla Q^2(z) \, dz \leq Ce^{-c_1'\delta |x|}, \quad 0 < \delta' < \delta,
\]

and

\[
\left| \int \left( e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} - e^{(2-H'(|x|))\frac{x \cdot z}{|x|}} \right) Q^2(z) \, dz \right| \leq C |K' - H'(|x|)|.
\]

Then, from (5.50), (5.51) and (5.55) we attain (5.41).

Finally, we consider the case \(V = V^{(2)}\). Recall that in this case \(V(r) = e^{-h_1(r)}\), with \(h_1(r) \geq 0\), \(h_1(r) = o(r)\), \(h_1^{(k)}\) is even, we get

\[
\int V_1(z, \chi) Q^2(z) \, dz = 0.
\]

Then, expanding the potential \(V'\) we have

\[
V'(|x - z|) = V'(|x|) + V_1(z, \chi) + r(|x|) |z|^2,
\]

for \(|z| \leq \frac{1}{|x|} \), where \(V_1(z, \chi)\) denotes the linear in \(z\) polynomial in the Taylor expansion of \(V'\). Since \(Q^2(z)\) is even, we get

\[
V_1(z, \chi) Q^2(z) \, dz = 0.
\]

Then, we have

\[
\int_{|z| \leq \frac{1}{|x|}} \nabla V(|x - z|) Q^2(z) \, dz = \frac{x}{|x|} \left( V'(|x|) \int Q^2(z) \, dz + r(|x|) \right).
\]

Hence, as \(\int_{|z| \geq \frac{1}{|x|}} V(|x - z|) Q^2(z) \, dz \leq Ce^{-\frac{|x|}{|x|}}\), we attain (5.42).
6 Invertibility of $\mathcal{L}_V$. Proof of Lemma 2.4

The second statement of Lemma 2.4 is a direct consequence of (2.54). Indeed, assuming (2.54), the invertibility of the operators $L_{\pm} - \lambda^2 V (|y + \hat{x}|)$ follows from Fredholm alternative applied to $(-\Delta + 1)^{-1} (L_{\pm} - \lambda^2 V (|y + \hat{x}|))$. The regularity and exponential decay of the solution follow from the properties of the elliptic operators ([2]). See Lemma 2.4 of [26] for the proof in the case of the Hartree equation.

Therefore, we need to prove (2.54). We recall the positivity estimates for $L_{\pm}$,

$$\ker L_{+} = \text{span}\{\nabla Q\}, \quad L_{+}|_{\{Q\}^\perp} \geq 0 \quad \text{and} \quad L_{-}|_{\{Q\}^\perp} > 0$$

(see, for example, Lemmas 2.1 and 2.2 of [33]). Then, from (2.52) we get

$$(\mathcal{L} f, f) \geq c \|f\|_{H^1}^2, \quad c > 0,$$  \hspace{1cm} (6.1)

for all $f \in H^1$ such that $|(f, Q)| + |(f, iQ)| + |(f, \nabla Q)| = 0$. We claim that (6.1) and identity

$$L_{+} Q = - (p - 1) (-\Delta + 1) Q$$

imply

$$\|f\|_{H^1} \leq C \left( \|\mathcal{L} f\|_{H^1} + |(f, iQ)| + |(f, \nabla Q)| \right),$$  \hspace{1cm} (6.3)

for all $f \in H^1$. Indeed, let $f = h + ig$, with real $h$ and $g$. We write $h = (h, Q) \|Q\|_{L^2}^{-1} Q + (h, \nabla Q) \|\nabla Q\|_{L^2}^{-1} \nabla Q + h^\perp$ and $g = (g, Q) \|Q\|_{L^2}^{-1} g^\perp$, $(h^\perp, Q) = (h^\perp, \nabla Q) = 0$ and $(g^\perp, Q) = 0$. Then

$$L_{+} h = (h, Q) \|Q\|_{L^2}^{-1} L_{+} Q + L_{+} h^\perp$$

and $L_{-} g = L_{-} g^\perp$.

Since

$$(Q, \nabla Q) = (h^\perp, \nabla Q) = 0 \quad \text{and} \quad \ker L_{+} = \text{span}\{\nabla Q\},$$

$L_{+} Q$ and $L_{+} h^\perp$ are linearly independent. Moreover, there is $0 < \delta < 1$, such that

$$\left| \|L_{+} Q, L_{+} h_{n}^\perp\|_{H^1} \right| \leq |1 - \delta| \|L_{+} Q\|_{H^1} \left\|L_{+} h_{n}^\perp\right\|_{H^1},$$  \hspace{1cm} (6.5)

uniformly on $h$. Otherwise, there is a sequence $\{h_{n}^\perp\}_{n \in \mathbb{N}}, \|h_{n}^\perp\|_{H^1} = 1$, satisfying $(h_{n}^\perp, Q) = (h_{n}^\perp, \nabla Q) = 0$, such that

$$\left| \|L_{+} Q, L_{+} h_{n}^\perp\|_{H^1} \right| = \mu_{n} \|L_{+} Q\|_{H^1} \left\|L_{+} h_{n}^\perp\right\|_{H^1},$$

with $\mu_{n} \to 1$, as $n \to \infty$. Since $h_{n}^\perp$ converges weakly to a function $h_{\infty}^\perp \in H^1$, we have $\lim_{n \to \infty} \left( L_{+} Q, L_{+} h_{n}^\perp \right)_{H^1} = \left( L_{+} Q, L_{+} h_{\infty}^\perp \right)_{H^1}$ and $\|L_{+} h_{\infty}^\perp\|_{H^1} \leq \lim_{n \to \infty} \|L_{+} h_{n}^\perp\|_{H^1}$. Then

$$\left| \|L_{+} Q, L_{+} h_{\infty}^\perp\|_{H^1} \right| = \|L_{+} Q\|_{H^1} \left( \lim_{n \to \infty} \|L_{+} h_{n}^\perp\|_{H^1} \right).$$

Hence, as $\left| \|L_{+} Q, L_{+} h_{\infty}^\perp\|_{H^1} \right| \leq \|L_{+} Q\|_{H^1} \left\|L_{+} h_{\infty}^\perp\right\|_{H^1}$, we get

$$\left| \|L_{+} Q, L_{+} h_{\infty}^\perp\|_{H^1} \right| = \|L_{+} Q\|_{H^1} \left\|L_{+} h_{\infty}^\perp\right\|_{H^1},$$

which means that $L_{+} \left( \gamma Q - h_{\infty}^\perp \right) = 0$, for some $\gamma \neq 0$. Then by (6.4), $\gamma Q - h_{\infty}^\perp = 0$. Multiplying the last equality by $Q$, we get $\gamma = 0$, a contradiction. Hence, (6.5) holds. Then, by (6.1) and (6.2)

$$\|L_{+} h\|_{H^1}^2 = (h, Q)^2 \|L_{+} Q\|_{H^1}^2 + \|L_{+} h^\perp\|_{H^1}^2 + 2 (h, Q) \left( L_{+} Q, L_{+} h_{\infty}^\perp \right)_{H^1} \geq c \left( (h, Q)^2 \|L_{+} Q\|_{H^1}^2 + \|L_{+} h^\perp\|_{H^1}^2 \right) \geq b \|h\|_{H^1}^2 - \frac{1}{b} |(h, \nabla Q)|^2,$$

and

$$\|L_{-} g\|_{H^1}^2 = \|L_{-} g^\perp\|_{H^1}^2 \geq |g^\perp|_{H^1}^2 \geq b \|g\|_{H^1}^2 - \frac{1}{b} |(g, Q)|^2$$

for some $b > 0$. Therefore, (6.3) follows.

Let us show that (6.3) remains true for the perturbed operator

$$\mathcal{L}_V = \mathcal{L} - \lambda^2 V (|y + \hat{x}|)$$
for all $\lambda \geq \lambda_0 > 0$ such that $\lambda^2 \sup_{r \in \mathbb{R}} V(r) < 1$ and $|\chi|$ sufficiently big. Let $\rho \in C^\infty (\mathbb{R}^d)$ be such that $0 \leq \rho \leq 1$, $\rho(x) = 1$ for $|x| \leq \frac{1}{4}$ and $\rho(x) = 0$ for $|x| \geq \frac{1}{2}$. For $f \in H^1$ and $\sigma > 0$, we decompose $f = f_1 + f_2$, where $f_1(y) = \rho_1 \left( \frac{y}{\sigma} \right) f(y)$, $f_2(y) = \rho_2 \left( \frac{y}{\sigma} \right) f(y)$, $\rho_1(y) = 1 - \rho(y)$ and $\rho_2(y) = \rho(y)$. We estimate

$$\|\mathcal{L}_V f\|_{H^{-1}} = \|\mathcal{L}_V f_1\|_{H^{-1}} + \|\mathcal{L}_V f_2\|_{H^{-1}} + 2 (\mathcal{L}_V f_1, \mathcal{L}_V f_2)_{H^{-1}} \geq \left\| \left( -\Delta + 1 - \lambda^2 V (|y + \bar{\chi}|) \right) f_1 \right\|_{H^{-1}}^2 + \|\mathcal{L}_V f_2\|_{H^{-1}} + 2 (\mathcal{L}_V f_1, \mathcal{L}_V f_2)_{H^{-1}} + r,$$

(6.6)

with

$$r = -2 \left| \left( -\Delta + 1 - \lambda^2 V (|y + \bar{\chi}|) \right) f_1 - \frac{p+1}{2} Q^{p-1} f_1 - \frac{p-1}{2} Q^{p-1} f_1 \right|_{H^{-1}} - 2 \lambda^2 \| (\mathcal{L}_V f_2, V (|y + \bar{\chi}|) f_2)_{H^{-1}} \|

$$

Observe that

$$\int Q^{p-1} |f_1| |g| \leq \int Q^{p-1} (y) \rho_1 \left( \frac{y}{\sigma} \right) f(y) |g(y)| dy \leq \lambda^p \left( \frac{\sigma}{4} \right) \|f\| \|g\|

$$

and

$$\int |V (y + \bar{\chi})| |f_2| |g| \leq \int |V (y + \bar{\chi})| \rho_2 \left( \frac{y}{\sigma} \right) f(y) |g(y)| dy \leq \|V (y + \bar{\chi}) \rho_2 \left( \frac{y}{\sigma} \right) \|_{L^\infty} \|f\| \|g\|.

$$

Then

$$|r| \leq K_1 (V) \left( \left\| V (|y + \bar{\chi}|) \rho_2 \left( \frac{y}{\sigma} \right) \right\|_{L^\infty} + \lambda^p \left( \frac{\sigma}{4} \right) \right) \|f\|_{H^{-1}}^2, \ K_1 (V) > 0.

$$

(6.9)

Using that $\nabla f_j(y) = \sigma^2 \rho_j \left( \frac{y}{\sigma} \right) f(y) + \rho_j \left( \frac{y}{\sigma} \right) \nabla f(y)$, $j = 1, 2$ we have

$$(\mathcal{L}_V f_1, \mathcal{L}_V f_2)_{H^{-1}} = \left( \rho_1 \left( \frac{y}{\sigma} \right) \nabla f, \rho_2 \left( \frac{y}{\sigma} \right) \nabla f \right) + (f_1, f_2) + r_1 + r_2 + r_3

$$

(6.10)

where

$$r_1 = \frac{1}{\sigma^2} \left( \rho_1 \left( \frac{y}{\sigma} \right) f(y), \rho_2 \left( \frac{y}{\sigma} \right) f(y) \right) + \frac{1}{\sigma} \left( \frac{y}{|y|} \rho_1 \left( \frac{y}{\sigma} \right) f(y), \rho_2 \left( \frac{y}{\sigma} \right) \nabla f(y) \right) - \frac{1}{\sigma} \left( \rho_1 \left( \frac{y}{\sigma} \right) \nabla f(y), \frac{y}{|y|} \rho_2 \left( \frac{y}{\sigma} \right) f(y) \right)

$$

$$r_2 = - \left( (-\Delta + 1) f_1 \chi^2 + \frac{p+1}{2} Q^{p-1} f_2 + \frac{p-1}{2} Q^{p-1} f_1 \bar{\chi}^2 + \lambda^2 V (|y + \bar{\chi}|) f_2 \right)_{H^{-1}} - \left( \frac{p+1}{2} Q^{p-1} f_1 + \frac{p-1}{2} Q^{p-1} f_2 \right)_{H^{-1}} + \left( \lambda^2 V (|y + \bar{\chi}|) f_1, (-\Delta + 1) f_2 \right)_{H^{-1}}

$$

$$r_3 = \left( \lambda^2 V (|y + \bar{\chi}|) f_1, -\frac{p+1}{2} Q^{p-1} f_2 + \frac{p-1}{2} Q^{p-1} f_1 \right)_{H^{-1}}.

$$

We estimate $r_1$ as

$$|r_1| \leq \frac{1}{\sigma} (2 + \sigma^{-1}) \left( 1 + \|\rho\|_{L^\infty}^2 \right) \|f\|_{H^{-1}}.

$$

(6.11)

Using (6.7) and (6.8) we control $r_2$ by

$$|r_2| \leq 2pq^{p-1} \left( \frac{\sigma}{4} \right) \|f\|_{H^{-1}}^2 + 2 \lambda^2 \left\| V (|y + \bar{\chi}|) \rho_2 \left( \frac{y}{\sigma} \right) \|_{L^\infty} \|f\|_{H^{-1}}^2

$$

$$+ \frac{p+1}{2} L^{-1} \left( \frac{\sigma}{4} \right) \left( p \|Q\|_{L^\infty}^2 + \lambda^2 \|V\|_{L^\infty}^2 \right) \|f\|_{H^{-1}}^2

$$

$$+ \lambda^2 \left\| V (|y + \bar{\chi}|) \rho_2 \left( \frac{y}{\sigma} \right) \|_{L^\infty} \|V\|_{L^\infty} \|f\|_{H^{-1}}^2.

$$

(6.12)

Finally, we estimate $r_3$. We have

$$|r_3| \leq \lambda^2 \left\| V (|y + \bar{\chi}|) \left( \frac{\sigma}{4} \right)^{-2} \right\|_{L^\infty} \|f\|_{H^{-1}} \left( \frac{\sigma}{4} \right)^2 \left( \frac{p+1}{2} Q^{p-1} f_2 + \frac{p-1}{2} Q^{p-1} f_1 \right)_{H^{-1}}.

$$

Since

$$\left\| V (|y + \bar{\chi}|) \left( \frac{\sigma}{4} \right)^{-2} \right\|_{L^\infty} \leq \|V (|y + \bar{\chi}|)\|_{L^\infty(|y| \leq \frac{3}{4})} + \left( 1 + \left| \frac{\chi}{4} \right\|^2 \right)^{-1} \|V\|_{L^\infty}

$$

and

$$\left\| \left( \frac{\sigma}{4} \right)^2 \left( \frac{p+1}{2} Q^{p-1} f_2 + \frac{p-1}{2} Q^{p-1} f_1 \right)_{H^{-1}} \right\| \leq K_2 \|f\|_{H^{-1}}, \ K_2 > 0,
we deduce
\[ |r_3| \leq K_2 \lambda^2 \left( \left\| V \left( |y + \bar{x}| \right) \right\|_{L^\infty(|y| \leq |y' I|)} + \left( 1 + |y|^2 \right)^{-1} \left\| V \right\|_{L^\infty} \right) \left\| f \right\|_{H^1}^2. \] (6.13)

As by assumption \( \lambda^2 \sup_{r \in R} V (r) < 1 \), we have
\[ \left( (-\Delta + 1 - \lambda^2 V (\cdot + \bar{x})) f_1, f_1 \right) \geq \left( (-\Delta + c_1) f_1, f_1 \right) \geq b_1 \| f_1 \|_{H^1}^2, \]
with some \( b_1, c_1 > 0 \). Then, using (6.3) and (6.10) in (6.6) we have
\[ \| L V f \|_{H^{-1}}^2 \geq b_3 \| f \|_{H^1}^2 - \frac{2}{b_3} \left( \left( f_2, iQ \right) + \left( f_2, \nabla Q \right) \right) + r + 2 \left( r_1 + r_2 + r_3 \right), \quad b_3 > 0. \] (6.14)

Since
\[ \left( f_1, iQ \right)^2 + \left( f_1, \nabla Q \right)^2 \leq 2 \left( q + |q'| \right) \left( \frac{c}{4} \right) \left( \int Q + |\nabla Q| \right) \| f \|_{H^1}^2, \]
we estimate
\[ \left( f_2, iQ \right)^2 + \left( f_2, \nabla Q \right)^2 \leq 2 \left( |f, iQ| \right)^2 + 2 \left( |f, \nabla Q| \right)^2 + r_4. \] (6.15)

with
\[ r_4 = 4 (q + |q'|) \left( \frac{c}{4} \right) \left( \int Q + |\nabla Q| \right) \| f \|_{H^1}^2. \]

Then, (6.14) takes the form
\[ \| L V f \|_{H^{-1}}^2 \geq b_3 \| f \|_{H^1}^2 - \frac{4}{b_3} \left( \left( f, iQ \right)^2 + \left( f, \nabla Q \right)^2 \right) + r + 2 \left( r_1 + r_2 + r_3 \right) - \frac{2}{b_3} r_4. \] (6.16)

We choose \( \sigma > 0 \) and \( C (V) > 0 \) in (6.9), (6.11), (6.12), (6.13) such that \( |r| + 2 \left( |r_1| + |r_2| + |r_3| \right) + \frac{2}{b_3} r_4 \leq \frac{3b_2}{4} \| f \|_{H^1}^2, \) for all \( |x| \geq C (V) \). Therefore, from (6.14) we deduce (2.54).

In order to complete the proof, we need to show that (2.55) holds. We use the coercivity property of the unperturbed operator \( L \) that follows, for example, from Lemma 2.2 of [33] (see also [29])
\[ (L f, f) \geq c \| f \|_{H^1}^2 - \frac{1}{c} \left( (f, Q)^2 + (f, x Q)^2 + f, i \Lambda Q^2 \right), \quad f \in H^1, \]
for some \( c > 0 \) independent of \( f \). Then, decomposing \( (L_{\sigma} f, f) \) similarly to (6.6) and arguing as in the proof of (2.54), we deduce (2.55). This completes the proof of Lemma 2.4.

7 Appendix.

Proof of Lemma 2.5.

Recall that the kernel \( G (x) \) of the Bessel potential \( (1 - \Delta)^{-1} \) behaves asymptotically as (see pages 416-417 of [3])
\[ G (x) = 2 \frac{\pi}{d} \frac{\Gamma (\frac{d}{4})}{\Gamma (\frac{d}{2})} |x|^{-\frac{d}{4}} e^{-|x| \left( 1 + o (1) \right)}, \quad \text{as } |x| \to \infty, \] (7.1)
and
\[ G (x) = \begin{cases} \frac{\pi}{\Gamma (\frac{d}{2})} \left( 1 + o (1) \right), & d = 2, \\ \frac{\pi}{\Gamma (\frac{d}{2})} \left( 1 + o (1) \right), & d \geq 3, \end{cases} \quad \text{as } |x| \to 0, \] (7.2)
where \( \Gamma \) denotes the gamma function. For \( 0 < \delta < 1 \), let \( G_\delta (x) = e^{\delta |x|} G (x) \). Using (7.1) and (7.2) we estimate
\[ |G_\delta (x)| \leq C e^{-|x|^{1-\delta}} \langle x \rangle^{-d+1} \langle |x|^{-2-\nu} \rangle, \quad \nu > 0. \] (7.3)

We decompose
\[ \left| e^{-\delta |y|} T (y) \right| \leq C_0 (I_1 + I_2 + I_3), \] (7.4)
where
\[ I_1 = \int_{\mathbb{R}^d} G_\delta (y - z) e^{-\delta |z|} Q^{p-1} (z) |T (z)| dz, \]
\[ I_2 = \int_{\mathbb{R}^d} G_{\delta} (y - z) e^{-\delta|z|} V \left( \left| \frac{z}{\lambda} \right| \right) |T(z)| \, dz \]

and
\[ I_3 = \int_{\mathbb{R}^d} G_{\delta} (y - z) e^{-\delta|z|} |f(z)| \, dz, \]

for some \( C_0 > 0 \). Let \( \psi \in C^\infty \) be such that \( \|\psi]\|_{L^\infty} \leq 1 \), \( \psi = 1 \) for \( |x| \leq 1 \) and \( \psi = 0 \) for \( |x| \geq 2 \). For \( a > 0 \) we decompose
\[ I_1 = I_{11} + I_{12}, \]

with
\[ I_{11} = \int_{\mathbb{R}^d} G_{\delta} (y - z) e^{-\delta|z|} Q^{p-1} (z) \psi \left( \frac{z}{a} \right) |T(z)| \, dz \]

and
\[ I_{12} = \int_{\mathbb{R}^d} G_{\delta} (y - z) e^{-\delta|z|} Q^{p-1} (z) \left( 1 - \psi \left( \frac{z}{a} \right) \right) |T(z)| \, dz. \]

By (2.54)
\[ \|\psi T\|_{H^1}^2 \leq C_1 \left( \|\mathcal{L}_V (\psi T)\|_{H^{-1}}^2 + |(\psi T, \nabla Q)|^2 \right), \quad C_1 > 0. \]

Noting that \( [\mathcal{L}_V, \psi \left( \frac{z}{a} \right)] = -a^{-2} (\Delta \psi) \left( \frac{z}{a} \right) - a^{-1} (\nabla \psi) \left( \frac{z}{a} \right) \nabla \), as \( (T, \nabla Q) = 0 \), we have
\[ \|\psi T\|_{H^1}^2 \leq C_1 \left( \|\psi f\|^2 + a^{-1} |(\nabla\psi) \nabla T|^2 + a^{-2} \|\Delta\psi T\|^2 + |(1 - \psi) T, \nabla Q)|^2 \right) \leq C_1 \left( \|\psi f\|^2 + C_2 a^{-1} |T|_{H^1}^2 \right), \quad C_2 > 0. \]

Then, taking \( a \geq A_0^{-2} (|\chi|) + 1 \) and using (2.58) we get
\[ \|\psi T\|_{H^1}^2 \leq C_1 \left( \|\psi e^{\delta|z|} e^{-\delta|z|} f\|^2 + C_2 A_0^2 (|\chi|) |T|_{H^1}^2 \right) \leq C_3 A_0^2 (|\chi|) \]

with \( C_3 > 0 \). Thus, by (7.3) and Young’s inequality we get
\[ \| I_{11} \|_{L^2} \leq C_0 \| G_{\delta} (\cdot) \|_{L^1} \| \psi \|_{L^2} \leq C_4 (\delta) A_0 (|\chi|), \quad C_4 (\delta) > 0. \]

Moreover, we have
\[ \| I_{12} \|_{L^2} \leq C a^{-1} \| G_{\delta} (\cdot) \|_{L^1} \| |z| Q^{p-1} (z) \|_{L^\infty} \| T \|_{L^2} \leq C a^{-1} \leq CC_4 (\delta) A_0^2 (|\chi|). \]

Hence, \( \| I_1 \|_{L^2} \leq CC_4 (\delta) A_0 (|\chi|) \). By (2.31) and (2.70)
\[ \|I_2\|_{L^2} \leq C \|T\|_{L^\infty} \left| e^{-\delta|z|} V \left( \frac{z}{\lambda} + \frac{1}{\lambda} \chi \right) \right|_{L^2} \leq C \|T\|_{L^\infty} \left| e^{\delta F} e^{-\delta|z|} \right|_{L^2} \leq CC_5 (\delta, \delta') \|T\|_{L^\infty} \|\chi\|_\delta, \quad C_5 (\delta, \delta') > 0, \]

for any \( \delta' < \delta \). By using (2.58) we control \( \|I_3\|_{L^2} \leq C_0 (\delta) A_0 (|\chi|), \quad C_0 (\delta) > 0 \). Using the estimates for \( I_1, I_2, I_3 \) in (7.4) we deduce
\[ \left| e^{-\delta T} \right|_{L^2} \leq C (\delta, \delta') \left( A_0 (|\chi|) + \|T\|_{L^\infty} \|\chi\|_\delta \right). \quad (7.5) \]

Since \( G_{\delta} \in L^p \), for any \( 1 < p < \frac{d}{d-2} \), from the equation (2.57), via Young’s inequality, we control the \( L^{\frac{2d}{d-2}} \) norm of \( e^{-\delta|z|} T (\cdot) \) by the right-hand side of (7.5). Note that \( \frac{dp}{d-2} > 2 \). Iterating the last argument a finite number of times, we attain (2.59). Lemma 2.5 is proved.
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