Spin–Statistics and CPT for Solitons

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Vienna, Preprint ESI 501 (1997)  

November 13, 1997

Supported by Federal Ministry of Science and Research, Austria
Available via http://www.esi.ac.at
Spin-Statistics and CPT for Solitons

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Abstract: It is analysed under which conditions the statistics of soliton sectors of massive two-dimensional field theories can be properly defined. A soliton field algebra is defined as a crossed product with the group of soliton sectors. In this algebra, the non-local commutation relations are determined and Weak Locality, Spin-Statistics and CPT theorems are proven. These theorems depart from their usual appearance due to the broken symmetry connecting the inequivalent vacua. An interpretation of these results in terms of modular theory is given. For the neutral subalgebra of the soliton algebra, the theorems hold in the familiar form, and Twisted Locality is derived.

1 Introduction

Solitons naturally appear in two-dimensional quantum field theory as superselection sectors of the observable algebra whenever there are several inequivalent vacuum sectors. The latter may arise due to the spontaneous breakdown of some (discrete) symmetry. In this generality, the notion of solitons is not related to integrability, and was first discussed by Fröhlich [10] and later, without reference to a broken symmetry, by Fredenhagen [6]. It is true that in all known models the degeneracy of the vacua is due to a broken abelian symmetry, and therefore we shall concentrate on this case.

It is the purpose of this letter to initiate a study of the fundamental theorems of quantum field theory such as Spin-Statistics and CPT for solitonic fields. In fact, we shall see that under suitable conservative assumptions, these theorems can be maintained but in a modified form. E.g., neither the spin nor the commutation relations of solitonic fields are determined by the statistics but depend also on other quantum numbers related to the broken symmetry.

It is of prime interest to control the departure from the naive form of the theorems in the most general case. But this is not the purpose of the present letter, and would inflate its dimensions. Instead, we want to emphasize the automatic appearance of modifications even in the most simple cases. This attitude should justify several simplifying conditions and assumptions made in the course of the arguments. Most of these are indeed easily proven in the case of \(Z_n\) or \(Z_N\)-symmetries, and presumably with some effort also for more general symmetry groups, as will be pointed out along the way. We shall also omit many technical details and rather concentrate on the necessary modifications of standard treatments of Spin-Statistics and CPT [1, 2, 9].

Our first topic will be the question whether the intrinsic notion of statistics due to Doplicher, Haag, and Roberts (DHR) [4, 5] extends to the case of soliton sectors at all. Having settled this issue, we proceed to the construction of the soliton algebra which extends the observable algebra and contains charged fields which interpolate between all soliton sectors of the observables. This canonical construction just copies the construction for simple DHR sectors [4]. But in contrast to the DHR case, the soliton algebra not only transforms under an unbroken gauge symmetry, but also inherits the broken symmetry of the observables, thereby giving rise to two sets of “quantum numbers”, one being superselected and the other not.

Our main results are the derivation of the space-like commutation relations for the soliton algebra, of the Spin-Statistics and CPT theorems and the relation of the latter to a modular structure reminiscent of the Bisognano-Wichmann theorem [1]. The formal difference from the standard theorems is an ubiquitous dependence on the broken quantum numbers. Especially for the commutation relations, this must have been anticipated since the soliton fields asymptotically implement the sym-
2 Statistics of solitons

In massive theories\(^1\) with Haag duality, it was shown by Müger \(^{12}\) that nontrivial DHR sectors do not exist. Hence all positive energy representations of the observable algebra \(A\) are either vacuum representations or, provided there are several vacua, soliton representations asymptotically connecting two different vacua. When the vacuum degeneracy is due to a spontaneously broken symmetry \(^{10}\), then any two vacuum representations are related as \(\pi_\beta = \pi_\alpha \sigma g\) where \(g \in G\) is a global discrete symmetry transformation commuting with the Poincaré group.

Then \(^{10, 15}\) for every pair \(\alpha, \beta\) there is a soliton representation which on the right complement of some double-cone coincides with \(\pi_\alpha\), and on its left complement coincides with \(\pi_\beta\).

All soliton representations can be modelled on the same Hilbert space of some reference vacuum sector \(H_\alpha\) with the help of soliton automorphisms \(g \leftarrow \rho \rightarrow h\) of \(A\) evaluated in the vacuum representation \(\pi_\alpha\).\(^2\) Among them are, of course, also the other vacua since \(g \leftarrow g \rightarrow g\) is a special case of a soliton automorphism. The automorphisms \(\rho\) can be freely composed.

The DHR approach to superselection sectors and statistics \(^{4, 5}\)\(^3\) addresses the question whether \(\rho_1 \rho_2\) is unitarily equivalent with \(\rho_2 \rho_1\). Looking at the asymptotic behaviour, it is clear that the group \(G\) must be necessarily abelian.

The issue is now whether the product of two soliton automorphisms is commutative whenever their interpolation regions are at space-like distance. Since for every \(\rho\), every local observable \(a\) and every \(k \in G\) one can choose an appropriately translated soliton automorphism \(\sigma\) which equals \(k\) on \(a\) and on \(\rho(a)\), requiring \(\rho\) to commute with \(\sigma\) at space-like distance implies that necessarily \(\rho k = k \rho\) on all \(a \in A\).

We therefore impose the condition on the soliton automorphisms (which also implies that \(G\) is abelian)

\[
(A) \quad \rho k = k \rho \quad (k \in G).
\]

\(^1\)to be precise: in theories satisfying the split property for wedges (SPW), SPW is expected - but was proven only in simple examples - to be a manifestation of a mass gap.

\(^2\)The notation means that on the left and right space-like complements of some bounded interpolation region, \(\rho\) acts like \(g\) and like \(h\), respectively. The interpolation regions may be assumed to be double-cones, that is, open intersections of a forward and a backward light-cone.

\(^3\)We assume some familiarity with this theory and shall refrain from repeating the lengthy but standard arguments throughout this article. A rather compact introduction can be found in \(^8\).
It is not obvious that, given $G$ abelian, every soliton sector has a representative satisfying (A). A rather general proof is announced by Müger [13].

The next issue is transportability. For $\rho$ in a covariant sector, there is a unitary cocycle (with respect to the Poincaré group) of intertwiners $\gamma_\rho(x, \lambda) : \rho \rightarrow \rho_\lambda = \alpha_{x, \lambda} \rho \alpha_{x, \lambda}^{-1}$ (charge transporters). By Haag duality, these are localized operators, but for the application of the DHR theory a little more is required, namely that soliton automorphisms act trivially on charge transporters at space-like distance. Hence we need that charge transporters are invariant under $G$ ("neutral"). Indeed, we have:

**Lemma 1:** If the symmetry group $G$ commutes with the Poincaré group $P$ on the observables, and if $\rho$ commutes with $G$, then the charge transporters $\gamma_\rho(x, \lambda)$ for the translations ($\lambda = 1$) are neutral. If $G$ is finite, then also the charge transporters for the boosts are neutral.

**Proof:** $\hat{\rho}$ again commutes with $G$, and hence along with $\gamma_\rho(x, \lambda)$ also $k(\gamma_\rho(x, \lambda))$ is another cocycle with the same intertwining property. The two can therefore only differ by an $x, \lambda$- and $k$-dependent phase which must be a representation of both the Poincaré group and $G$, or put differently, a representation of $P$ with values in $\hat{G}$. If $G$ is finite, then $\hat{G}$ is discrete, and by continuity, the phase must be trivial. If $G$ is not finite, then for every $k \in G$, the phase is a one-dimensional representation of $P$. But all such representations are trivial on the translations. \hfill \Box

For $G$ not finite, possibly constructive methods [15] to obtain the cocycle give neutrality for the boosts also. In this section we shall need only the translations, but for later purposes, we shall have to impose the condition

(B) The cocycles $\gamma_\rho(\lambda)$ for the boosts are neutral.

It is now easy to show with the standard DHR arguments [5], using neutral charge transporters whenever necessary, that covariant soliton automorphisms satisfying condition (A) indeed commute at space-like distance, and commute up to unitary equivalence in general.

We want to proceed as usual [5] and define unitary statistics operators $\varepsilon(\rho_1, \rho_2) : \rho_1 \rho_2 \rightarrow \rho_2 \rho_1$ for any pair of covariant soliton automorphisms by first translating them with the help of unitary charge transporters until they are at space-like distance, then commuting them and translating back. Since this strategy involves the application of soliton automorphisms to auxiliary charge transporters at space-like distance, the independence on the choice of the latter is assured only if they are neutral. The statistics operator would be sensitive to any perturbation by a charge transporter which is not neutral.

It is now just a repetition of the DHR arguments [5, 8] to show that the statistics operators are well-defined if one admits only neutral charge transporters (which exist by Lemma 1), and that they depend at most on the orientation of the auxiliary regions at space-like distance chosen in the process. By convention, the possibility with $\rho_1$ transported to the right of $\rho_2$ defines $\varepsilon(\rho_1, \rho_2)$, and consequently the opposite choice yields $\varepsilon(\rho_2, \rho_1)^\dagger$.

One easily derives

**Proposition 1:** The statistics operators for soliton automorphisms satisfying (A) defined with neutral charge transporters are themselves neutral. They satisfy naturality, e.g.,

$$u_2 \varepsilon(\rho_1, \rho_2) = \varepsilon(\rho_1, \hat{\rho}_2) \rho_1(u_2)$$

with neutral intertwiners $u_2 : \rho_2 \rightarrow \hat{\rho}_2$ and multiplicativity, e.g.,

$$\varepsilon(\rho_1, \rho_2 \rho_3) = \rho_2(\varepsilon(\rho_1, \rho_3)) \varepsilon(\rho_1, \rho_2)$$

in both entries. They satisfy the braid group identities. The statistics of $\rho = k \in G$ is trivial:

$$\varepsilon(k, \rho_2) = \varepsilon(\rho_1, k) = 1.$$

The monodromy operator is given by

$$\varepsilon(\rho_1, \rho_2) \varepsilon(\rho_2, \rho_1) = \frac{\kappa_{\rho_1 \rho_2}}{\kappa_{\rho_1} \kappa_{\rho_2}} \cdot 1.$$

Here, since we are dealing with automorphisms, the statistics operator $\varepsilon(\rho, \rho)$ is just a complex phase called the statistics phase $\kappa_{\rho}$. It is invariant under perturbations of $\rho$ by any neutral unitary. For the braid group identity, cf. [8].

**Corollary:** $\kappa_{\rho} = \kappa_{\rho^{-1}} = \kappa_{\rho \cdot k}$ for $k \in G$.

The present results do not exclude the possibility that the group of soliton automorphisms generated by some $\rho$ contains some $\hat{\rho}$ which is unitarily equivalent to $\rho$ but not related by a neutral
unitary. E.g., imagine $\rho^2 = \text{Ad } u$ where $u$ is not neutral. (Since $\rho^2$ commutes with $G$, it is only guaranteed that $g(u)u^* = \text{a scalar}.)$ Therefore, the statistics within any group of soliton automorphisms containing $\rho$ might not be an invariant of the sector ($\rho^3 = \text{Ad } u \rho \rho$ will have different statistics from $\rho$).

We exclude this possibility by picking one representative $g \rightarrow \rho_{g,h} \rightarrow h$ for every pair of vacuum sectors, with the property

(C) The assignment $(g, h) \mapsto \rho_{g,h}$ is a group homomorphism $G \times G \rightarrow \text{Aut}(A)$, and $\rho_{g,\varpi} = g \in G$.

This property entails condition (A) if $G$ is abelian. It is easily fulfilled for $G = \mathbb{Z}$ by choosing $\rho_{g,\varpi}$ to be powers of some generating element commuting with $G$, and putting $\rho_{g,\varpi} = g \rho_{e,\varpi}^{-1}$. M"uger informed us that he can in fact construct such a homomorphism also in the general case [13].

Within the group of soliton automorphisms $\{\rho_{g,h}\}$, the statistics has all the properties of the statistics of low-dimensional simple DHR sectors [5, 8]. As a consequence of property (C), all statistics operators between $\rho_{g,h}$ are just numerical phases since the $\rho_{g,h}$ commute with each other, and as a consequence of Prop. 1 they are multiplicative in both entries as well as invariant under shifts by $k \in G$. We summarize some further properties for later reference.

**Lemma 2:** (i) Let $g_i \rightarrow \rho_i \rightarrow h_i$, $i = 1, 2$. Then

$$\epsilon(k_1, k_2) := \epsilon(\rho_{e, k_1}, \rho_{e, k_2}) = \epsilon(\rho_1, \rho_2)$$

depends only on $k_i = h_i g_i^{-1}$. This function on $G \times G$ is a two-cocycle, that is

$$\epsilon(k_1, k_2)\epsilon(k_1 k_2, k_3) = \epsilon(k_1, k_2 k_3)\epsilon(k_2, k_3).$$

(ii) If $\epsilon(k_1, k_2)$ is a coboundary (which is automatic if $G$ has trivial second cohomology), that is

$$\epsilon(k_1, k_2) = \frac{\mu(k_1 k_2)}{\mu(k_1)\mu(k_2)},$$

then one can choose the complex phases $\mu(k)$ to satisfy

$$\mu(\epsilon) = 1 \quad \text{and} \quad \mu(k) = \mu(k^{-1})$$

$$\mu(k)^2 = \epsilon(k, k) \equiv \kappa_{g,h} \quad \text{if} \quad k = hg^{-1}. $$

**Proof:** The first statement in (i) is $G$-invariance, and the second statement follows from multiplicativity of the statistics operators (Prop. 1).

The coboundary property in (ii) already implies $\mu(\epsilon) = 1$ for $k_2 = \epsilon$, and $\mu(k)\mu(k^{-1}) = \epsilon(k, k)$ for $k_1 k_2 = \epsilon$. The possibility to choose $\mu(k) = \mu(k^{-1})$ is seen as follows. Let $G$ be generated by commuting elements $a_i$. We choose a square root $\mu_i$ of $\epsilon(a_i, a_j)$ for every generator and put

$$\mu(k) := \prod_i \mu_i^{p_i(k)} \prod_{i<j} \epsilon(a_i, a_j)^{p_i p_j}$$

if $k = \prod_i a_i^{p_i}$ is a representation in terms of the generators, with the qualification that $p_i$ has to be chosen even if $a_i$ is of odd order. This ensures the definition to be independent of the choice of $p_i$ since $\mu_i^{2N} = 1 = \epsilon(a_i, a_j)^N$ whenever $a_i^N = 1$. Furthermore, direct computation shows that the coboundary of $\mu$ thus defined yields $\epsilon$, using $\epsilon(a_1, a_2) = \epsilon(a_2, a_1)$ (being a coboundary) and multiplicativity of $\epsilon$ in both entries (Prop. 1).

### 3 The soliton algebra

In this and the next sections, the conditions (A)-(C) shall be understood.

Property (C) is of structural importance since it permits to define the soliton algebra as the crossed product of the observables with the group of soliton sectors $G \times G$ acting through the automorphisms $\rho_{g,h}$ [4], or equivalently as a global section through the field bundle [5], as follows.

All superselection sectors of the observables are realized by $\pi_\rho \epsilon \rho$ on the representation space of a reference vacuum sector $\pi_\varpi$, and $\rho = \rho_{g,h}$ exhaust the irreducible equivalence classes. Calling the Hilbert space equipped with these representations $H_{g,h}$, we define the soliton algebra $B$ acting irreducibly on the direct sum $H = \bigoplus_{g,h} H_{g,h}$. It is generated by its elements $F = (\rho_{g,h}, b) \; (b \in A)$ acting on each subspace by

$$(\rho, b)(\pi, \Phi) = (\pi \rho, \pi(b) \Phi).$$

It contains the observables $a \equiv (\text{id}, a)$ as well as unital field operators which implement the automorphisms $\rho_{g,h}$ of the observables.

---

*This argument actually shows that a multiplicative two-cocycle is a coboundary if and only if it is symmetric.*
The algebraic structure of $B$ is determined by the multiplication law

\[(\rho_1, b_1)(\rho_2, b_2) = (\rho_2\rho_1, \rho_2(b_1)b_2)\]

which shows that $B$ is a crossed product of $A$ with the group of soliton sectors. The adjoint in $B$ is

\[(\rho, b)^* = (\rho^{-1}, \rho^{-1}(b^*))\]

and makes $B$ a C* algebra.

Let the representatives $\rho$ be localized in some region $O$. Then $F = (\rho, b)$ is localized in $\lambda O + x$ if $\gamma_\rho(x, \lambda)b \in A(\lambda O + x)$. This definition of localization guarantees solitonic commutation relations with the observables at space-like distance, and is preserved by the * operation.

The soliton algebra $B$ is Poincaré covariant with

\[a_{x,\lambda}(\rho, b) = (\rho, \gamma_\rho(x, \lambda)^*a_{x,\lambda}(b)),\]

extending the Poincaré transformations of $A \subset B$.

The soliton algebra possesses two (commuting) internal symmetries: first there is an unbroken dual gauge symmetry

\[F \mapsto \eta(g, h) \cdot F\]

if $F$ belongs to the sector $(g, h)$, where $\eta \in \hat{G} \times \hat{G}$ is a character of $\hat{G} \times \hat{G}$. Second, the original broken symmetry extends to $B$ by the action of $k \in G$

\[(\rho, b) \mapsto (\rho, k(b)).\]

Both symmetries preserve localization and commute with * . Actually, the $G$-symmetry is now inner: it is implemented by the unitaries $Y_k$ where $Y_k = (k, 1) \in \bigcap O B(O)$ connect the vacua by $Y_k \Omega_\alpha = \Omega_{\alpha \cdot k}$. We shall call $\chi \in \hat{G}$ the “character” of a field operator whenever the latter transforms like

\[k(F) = \langle \chi, k \rangle \cdot F.\]

The dual symmetry commutes with the Poincaré group. The $G$-symmetry commutes with the translations, and also with the boosts only provided the cocycle is neutral (cf. condition (B) and Lemma 1). Otherwise $a_{x,\lambda}$ and $k a_{x,\lambda}$ would differ by a continuous dual gauge transformation.

Now, repeating the reasoning as in the case of DHR sectors [8], one finds the commutation relations at space-like distance:

**Proposition 2:** (Commutation Relations) Let $F_i = (\rho_i, b_i)$ be localized in $O_i$ at space-like distance, with $g_i \leftrightarrow \rho_i \rightarrow h_i$. Then

\[F_1 F_2 = \varepsilon(\rho_1, \rho_2) \cdot g_1^{-1}(F_2) h_2(F_1) \quad \text{if} \quad O_2 < O_1\]

and similar, exchanging $\varepsilon_{1,2}$ with $\varepsilon_{2,1}$ and $h_i$ with $g_i$, if $O_1 < O_2$.

The new feature in these commutation relations are the sector-dependent transformations under the broken symmetry, due to the non-triviality of the soliton automorphisms on space-like complements. They contribute extra phase factors $\langle \chi_1, h_2 \rangle/\langle \chi_2, g_1 \rangle$ to the exchange coefficient between two fields of given character.

Applying the commutation relations repeatedly, one obtains (with Lemma 2 (i))

**Proposition 3:** (Weak Locality) Let $F_i$ with soliton charge $(g_i, h_i)$ and character $\chi_i$ be localized in $O_n < \ldots < O_1$. Then

\[F_1 \cdots F_n = \prod_{i<j} \varepsilon_{ij} \prod_i \langle \chi_i, f_i \rangle \cdot F_n \cdots F_1\]

where $\varepsilon_{ij} = \varepsilon(k_i, k_j)$, $k_i = h_i g_i^{-1}$, and $f_i = \prod_{i<n} g_n^{-1} \prod_{i<n} h_n$. If $O_1 < \ldots < O_n$, then a similar law holds, exchanging $\varepsilon_{ij}$ with $\varepsilon_{ji}^{-1}$ and $h_i$ with $g_i$.

In particular, let $\Omega_\alpha$ be any of the vacuum vectors in $H$, and $F_i$ with soliton charge $(g_i, h)$ and character $\chi_i$ be localized in $O_2 < O_1$. Then the two-point function satisfies

\[\langle F_1 \Omega_\alpha, F_2 \Omega_\alpha \rangle = \frac{1}{\kappa_{\rho_\alpha, b}} \langle \chi_2, g \rangle \cdot (F_2^* \Omega_\alpha, F_1^* \Omega_\alpha).\]

It is crucial to note here, that the character is not superselected due to the spontaneous breakdown of the symmetry, that is, vacuum correlations between fields of different character need not vanish. If they did, then only operators of trivial character (neutral operators) could have a vacuum expectation and consequently the vacuum states $\omega_\alpha$

\[5\] Here and in the sequel we write $x < y$ (and accordingly for sets) if the difference vector $x - y$ lies in the left space-like wedge $W_- = \{ x : -x^1 > |x^0| \} = -W_+$. 

were all equal. On the contrary, soliton operators of any given character will create dense subspaces of $H_{g,h}$, and these subspaces are disjoint but not orthogonal.

4 Spin-Statistics and CPT

We want to prove Spin-Statistics and CPT theorems in terms of non-local Wightman fields associated with the soliton algebra. For this purpose, we have to assume that non-local Wightman fields $\varphi(x)$ are associated with the soliton algebra in the sense that there are limits of operators $F$ of point-like localization which are then translated to obtain $\varphi(x)$ (as unbounded operator-valued distributions, that is, the limits need to exist only after smearing with a test function). These fields of course inherit the Poincaré and symmetry transformations as well as the * operation and the commutation relations described by Prop. 2. Their quantum numbers are the soliton charge $(g,h)$ and the character $\chi$, the latter not being superselected. The field $\varphi^*(x)$ carries the charge $(g^{-1},h^{-1})$ and has character $\chi^{-1}$.

By appropriate projections, we may assume these fields to have definite spin $s \in \mathbb{R}$, that is, to transform as

$$V(t)\varphi(x)V(t)^{-1} = e^{i\lambda x} \varphi^*(x)$$

under the boost $\lambda = \left( \begin{array}{cc} \cosh t & \sinh t \\ \sinh t & \cosh t \end{array} \right)$. This assumption requires implicitly the validity of condition (B) since the discussion of Lemma 1 shows that otherwise the character would change continuously under Ad $V(t)$. Clearly, $\varphi$ and $\varphi^*$ have the same spin.

To summarize, we shall assume:

(D) There are sufficiently many covariant non-local Wightman fields $\varphi$ with quantum numbers $(g,h)$ and $\chi$ so that the vectors $\varphi(f)\Omega_\alpha$ span a dense subspace of $H_{g,h}$.

The Reeh-Schlieder theorem applies to these fields, hence also the fields localized in some region $O$ generate dense subspaces.

We cannot give convincing arguments on the validity of this assumption, but they are certainly the best one may expect. A priori there is no reason why soliton automorphisms should not be localizable in arbitrarily narrow interpolation regions, but the existence of limits is certainly a nontrivial dynamical problem. We refer to [7] for the analogous problem in scale invariant models.

**Proposition 4:** (Spin-Statistics) The spin $s \in \mathbb{R}$ of a field $\varphi$ in the sector $g \leftarrow \rho \rightarrow h$ and with character $\chi$ is determined up to integers by

$$e^{2\pi is} = \kappa_\rho \cdot \langle \chi, h g^{-1} \rangle$$

which depends only on $h g^{-1}$ and $\chi$.

**Proof:** Standard arguments of Wightman theory, exploiting the spectrum condition for the translations, tell that $V(t)$ can be analytically continued in $t$ to positive resp. negative imaginary parts on (improper) vectors $\varphi(x)\Omega_\alpha$ provided $x$ lies in the right resp. left space-like wedge. For $\varphi_i$ both in the sector $(g,h)$ and $x < 0 < y$ it follows that

$$(\varphi_1(x)\Omega_\alpha, \varphi_2(y)\Omega_\alpha) = (V(-i\pi)\varphi_1(x)\Omega_\alpha, V(i\pi)\varphi_2(y)\Omega_\alpha) = e^{i\pi(s_1+s_2)}(\varphi_1(-x)\Omega_\alpha, \varphi_2(-y)\Omega_\alpha) = \omega_{12} \cdot (\varphi_2(-y)^*\Omega_\alpha, \varphi_1(-x)^*\Omega_\alpha).$$

In the last step Weak Locality was used, and

$$\omega_{12} = e^{i\pi(s_1+s_2)} \frac{1}{\kappa_\rho} \langle \chi_2, g \rangle.$$ 

The two-point function on the very right of this equation equals $\langle \varphi_2(x)^*\Omega_\alpha, \varphi_1(y)^*\Omega_\alpha \rangle$ by translation invariance. Especially for $\varphi_1 = \varphi_2$, both two-point functions on the very left and on the very right are positive distributions in $x-y$ satisfying the spectrum condition. This is only possible if they are equal and the diagonal coefficient $\omega_{11}$ equals 1.

Note that the spin is not determined by the sector alone, and that fields of spin not differing by an integer may have non-vanishing vacuum correlations since the character is not superselected.

If we do the analogous computation leading to $\omega_{12}$ in the proof of Prop. 4 with $y < 0 < x$ rather than $x < 0 < y$ we obtain another phase factor differing from $\omega_{12}$ by the substitutions $(s, \kappa, h) \leftrightarrow (-s, \kappa^{-1}, g)$. By the Spin-Statistics theorem, both
expressions coincide. By translation invariance, the same relation between two-point functions

\[
(\varphi_1(x)\Omega_0, \varphi_2(y)\Omega_0) = \omega_{12} \cdot (\varphi_2(-y)^*\Omega_0, \varphi_1(-x)^*\Omega_0)
\]

therefore holds for all space-like \( x - y \).

We now make a choice for the sign of \( \mu(k) := \sqrt{\epsilon(k,k)} \) for each \( k \), invariant under \( k \leftrightarrow k^{-1} \), hence \( \mu(hg^{-1})^2 = \kappa_{g,h} \). If \( \epsilon(k_1,k_2) \) is a coboundary, we choose \( \mu(k) \) as in Lemma 2 (ii). For every field \( \varphi \) with quantum numbers \((g,h)\), \( \chi \) and spin \( s \) we define a complex phase

\[
\omega_\varphi := e^{i\pi s}(\chi, h^{-1})/\mu(hg^{-1})
\]

Clearly, \( \omega_\varphi = \omega_{\varphi^*} \). Using the Spin-Statistics theorem, one easily verifies that

\[
\omega_{12} = \omega_{\varphi^*}/\omega_{\varphi^*}.
\]

In view of the above relations between two-point functions, this means that all inner products between the (improper) vectors \( \omega_\varphi \varphi(-x)^*\Omega_0 \) are the complex conjugates of those between \( \varphi(x)\Omega_0 \).

Hence we have the first part of

**Proposition 5:** (CPT) The antilinear operator \( \Theta \) densely defined by

\[
\Theta \varphi(x)\Omega_0 = \omega_{\varphi} \cdot \varphi(-x)^*\Omega_0
\]

is involutive and anti-unitary on its domain of definition and therefore extends to an anti-unitary involution \( \Theta \) on \( H \).

The CPT operator \( \Theta \) transforms a field \( \varphi \) of soliton charge \((g,h)\) as follows:

\[
\Theta \varphi(x) |_{H_{k,l}} = \zeta \cdot (kl)^{-1} (\varphi \varphi(-x)^*)
\]

with a prefactor \( \zeta = \epsilon(hg^{-1}, ik^{-1})^{-1} \mu(g^{-1}h^{-1}ik^{-1}) \mu(hg^{-1}) \mu(k^{-1}) \).

If \( \epsilon(k_1,k_2) \) is a coboundary (automatic for, e.g., \( G = \mathbb{Z} \) or \( \mathbb{Z}_N \)) and hence \( \zeta \) is trivial, then

\[
\Theta B(W_{\pm}) = Y B(W_{\mp}) Y
\]

where the unitary and involutive operator \( Y \) equals \( Y_{(kl)^{-1}} \) on each subspace \( H_{k,h} \).

The proof of the last statements will be given after Prop. 6.

The rest of the discussion of CPT follows standard reasoning (e.g., \([1, 2, 9]\)). We call \( \mu \) and \( \kappa = \mu^2 \) the unitary diagonal operators with eigenvalues \( \mu(hg^{-1}) \) and \( \kappa_{g,h} \) on \( H_{g,h} \), respectively, and compute (for \( \varphi \) with soliton quantum numbers \((g,h)\))

\[
\begin{align*}
\Theta V(i\pi) \varphi(x)\Omega_0 &= \mu^* h(\varphi(x)^*)\Omega_0 \quad (x > 0) \\
\Theta V(-i\pi) \varphi(x)\Omega_0 &= \mu g(\varphi(x)^*)\Omega_0 \quad (x < 0),
\end{align*}
\]

This suggests the introduction of antilinear densely defined unbounded operators by

\[
\begin{align*}
\hat{S}_+ F\Omega_0 := h(F^*)\Omega_0 & \quad (F \in B(W_+)) \\
\hat{S}_- F\Omega_0 := g(F^*)\Omega_0 & \quad (F \in B(W_-))
\end{align*}
\]

if \( F \) carries soliton charge \((g,h)\).

By Weak Locality, one sees that \( \hat{S}_\pm \) are defined on the domains of \( \hat{S}_\mp \) where they coincide with the latter up to a unitary factor: \( \hat{S}_\pm \subset \hat{K} \cdot \hat{S}_\mp \). Thus they are closable and the closures \( \hat{S}_\pm \) possess unique polar decompositions. By comparison, and since \( V(\pm i\pi) \) are positive operators, one gets

**Proposition 6:** (Modular Structure) The closures of \( \hat{S}_\pm \) satisfy

\[
\hat{S}_\pm^* = \kappa^\pm 1 \cdot \hat{S}_\mp
\]

and have polar decompositions

\[
\hat{S}_\pm = \mu^\pm 1 \cdot V(\pm i\pi).
\]

The following commutation relations with boosts and translations hold:

\[
\begin{align*}
\Theta V(\pm i\pi) &= V(\mp i\pi) \Theta \\
\Theta V(t) &= V(t) \Theta \\
\Theta U(x) &= U(-x) \Theta.
\end{align*}
\]

Proof of the commutation relations: The first relation obtains from \( \hat{S}_\pm = \kappa \hat{S}_\pm^* \). The second one follows since \( V(t) \) is an imaginary power of \( V(i\pi) \). The last one follows from the definition of \( \Theta \). □

Proof of the CPT transformation law (Prop. 5): We compute the improper vector \( \Theta \varphi(x)\varphi_1(y)\Omega_0 \) where \( \varphi_1 \) belongs to the sector \((k^{-1}, l^{-1})\). E.g., for \( 0 < x < y \) where \( V(i\pi) \) is defined, we substitute \( \Theta = \mu S_- V(i\pi) \) and apply the definitions as well as the commutation relations of Prop. 2 to restore the operator order inverted by \( S_- \). The result is proportional to \( \varphi(-x)^* \Theta \varphi_1(y)\Omega_0 \). The accumulating
phase factors are seen, using the Spin-Statistics theorem, to give rise to \( \zeta \) as in the proposition. The twist by \( Y \) removes the symmetry transformation from the transformation law, and (if \( \zeta = 1 \))

\[
Y \Theta \varphi(x) \Theta Y = \omega_\varphi \cdot Y_{g,h} \varphi(-x)^*.
\]

The image is a field with soliton charge \((h,g)\) if \( \varphi \) has charge \((g,h)\).

We note, however, that one crucial point of the Bisognano-Wichmann result [1] for local Wightman fields is missing. Namely, our operators \( S_\pm \) are not Tomita operators \( F^t \Omega \rightarrow F^t \Omega \ (F \in B(W_\pm)) \) defined by the adjoint alone, but involve the symmetry transformations. Since the latter do not preserve the reference vacuum vector, \( S_\pm \) are also not related by unitaries to a Tomita operator, and therefore its polar factor \( \Theta \) cannot be unitarily related with the Tomita conjugation \( J \). As \( J \) takes the respective algebras into their commutants, the image of \( \Theta \) is not a twisted commutant. In view of the CPT transformation law in Prop. 5, even if there is no cohomological obstruction, we conclude that the soliton algebras of the two wedges are not their mutual twisted commutants with any unitary twist operator.

The present CPT operator \( \Theta \) takes a given sector \((g,h)\) into its conjugate sector \((g^{-1}, h^{-1})\). This is not precisely what one expects from a CPT operator which should rather exchange the two asymptotic directions and therefore connect \((g,h)\) with \((h,g)\). Furthermore, as one might have already remarked, \( S_\pm \) depend on the choice of the reference vacuum sector \( \Omega_a \). E.g., on \( H_a \) they coincide with the Tomita operators \( \pi_a(a)\Omega_a = \pi_a(a^*)\Omega_a \ (a \in A(W_\pm)) \) for this vacuum sector, but other vacuum sectors \( H_{a,ck} \) are mapped into \( H_{a,ck^{-1}} \).

Both these unpleasant features can be amended by a trivial trick: One changes \( S_\pm \) and \( \Theta \) into \( Y S_\pm Y = S_\pm \) and \( Y \Theta = \Theta Y \) where \( Y \mid H_{g,h} = Y_{(gh)^{-1}} \). Since \( Y \) is a unitary involution on \( H \) and commutes with \( \mu \), the polar decomposition formulae are not affected by this change. The new operators map \( H_{g,h} \) into \( Y_{g,h} H_{g^{-1},h^{-1}} = H_{g,h} \) as desired, and one can convince oneself that the action of the \( Y S_\pm \) on the entire Hilbert space, albeit defined relative to some reference vacuum vector, no longer depends on the choice of the latter. In particular, they coincide with the Tomita operators resp. the CPT operator of the observables on each of the vacuum sectors simultaneously.

As a bonus, the modified CPT operator \( Y \Theta \) maps the opposite wedge algebras directly into each other, without a twist (cf. Prop. 5).

## 5 Neutral solitons

If it were not for the option (cf. discussion at the end of the previous section) to have \((h,g)\) along with \((g,h)\) among the sectors, one can as well define the “right-oriented” soliton algebra \( B_R \subset B \) generated by the operators with soliton charge \( \rho_{e,h} \) only. It acts irreducibly on each of the subspaces \( H_{k,R} = \bigoplus_k H_{k,h} \) which contain only one vacuum sector \( H_\beta, \ \omega_\beta = \omega_{e,ck} \), each. On \( B_R \) the dual symmetry reduces to \( \hat{G} \). The \( G \)-symmetry (no longer inner) is again broken and connects the representations on \( H_{k,R} \).

The entire discussion leading to Props. 3–6 remains valid for \( B_R \) on every single \( H_{k,R} \), putting all left vacuum quantum numbers \( g = \epsilon \). Only the operator \( Y \) in Prop. 5 cannot be defined on \( H_{k,R} \).

The right-oriented soliton algebra is completely satisfactory if one is interested in the superselection sectors of the neutral observables \( A^0 \). By condition (A), the automorphisms \( \rho_{g,h} \) restrict on \( A^0 \) and turn into DHR automorphisms since the asymptotic transformations under \( G \) become trivial. Furthermore, the restrictions of \( \rho_{g,h} \) and \( \rho_{g,h} \) coincide. Therefore, the restricted automorphisms \( \rho_{e,h} \) exhaust the DHR sectors of \( A^0 \).

We emphasize that the vacuum sectors of \( A \) not only become equivalent upon restriction to \( A^0 \), but also remain irreducible. This fact is due to the spontaneous breakdown, and implies that \( A^0 \) is not Haag-dual in its vacuum representation. This explains why \( A^0 \) can possess DHR sectors, in view of M"uger’s result [12] on the absence of DHR sectors in Haag-dual massive (that is: SPW) theories. The field algebra \( B^0 \) for \( A^0 \) is generated by operators

\[
F = (\rho_{e,h}, b) \quad (b \in A^0)
\]

which are neutral with respect to \( G \) but carry a DHR charge, with the same localization and *
structures as described for $B$ in Sect. 3. It is clear that this field algebra equals the neutral subalgebra

$$B^0 = (BR)^G$$

of the right-oriented soliton algebra. Note that the subalgebra $B^0$ is preserved by the Poincaré group only if the cocycles for the boosts are neutral (condition (B)).

One can again repeat the entire discussion of Sect. 3 and derive Props. 3–6 for $B^0$. The difference is now that all fields involved are neutral, that is, all characters and transformations under $G$ are trivial. Thus, the spin is indeed determined by the sector alone:

$$e^{2\pi i s} = \kappa_p$$

according to Prop. 4, and is superselected mod $\mathbb{Z}$.

Furthermore, the restriction of $\Theta$ is the true CPT operator (provided $\zeta = 1$ in Prop. 5)

$$\Theta B^0(W_{\pm}) \Theta = B^0(W_{\mp}),$$

and the restricted operators $S_{\pm}$ are the Tomita operators $F \Theta \rightarrow F^* \Theta$ ($F \in B^0(W_{\pm})$). Hence

$$\mu_{\pm 1} \Theta = J_{\pm}$$

are the Tomita conjugations of the right and left wedge algebras $B^0(W_{\pm})$ with respect to the vacuum vector, which take the respective algebras into their commutants.

We conclude for the neutral soliton algebra:

**Proposition 7:** (Twisted Duality) If $\ell(k_1, k_2)$ is a coboundary, then the algebra $B^0(W_{\pm})$ of the left wedge equals the twisted commutant of the algebra $B^0(W_{\mp})$ of the right wedge, that is:

$$B^0(W_{\pm}) = \Theta B^0(W_{\mp}) \Theta = \mu_{-1} B^0(W_{\mp}) \mu$$

where the twist operator $\mu$ is a square root of $e^{2\pi i s}$.

## 6 Discussion

Let us consider the square of inclusions

$$B_R \subset B^0 \quad \cup \quad A \subset A^0$$

where the horizontal inclusions are fixpoints under the broken $G$-symmetry and the vertical inclusions are fixpoints under the dual $\hat{G}$-symmetry.

The theories in the same row live on the same Hilbert space, while the vertical extensions require an extension of the Hilbert space.

The net $A^0$ violates Haag duality, while $A$ was assumed to be Haag dual. Therefore, $A$ is the dual net canonically associated with $A^0$. The net $B^0$ was shown to satisfy twisted duality (Prop. 7) while nothing of the sort could be shown for $B_R$.

The reason is clear: $B^0$ being already twisted dual, it cannot possess any twisted dual extension on the same Hilbert space.

A comparison suggests itself with the situation studied by M"uger [11]

$$\hat{F} \subset F \quad \cup \quad A^d \subset A$$

where $F$ was assumed to satisfy (fermionic) duality with an unbroken symmetry $G$ with fixpoints $A$. The fixpoints then violate duality, and M"uger constructed the dual net $A^d$ by first extending $F$ to a non-local net $\hat{F}$ including “disorder operators” which implement the symmetry of $F$ on the left complement of some region and leave invariant the elements of $F$ in the right complement. (For the existence of such operators, the assumption of the split property for wedges (SPW) is essential).

He showed that $\hat{F}$ carries a dual symmetry under $\hat{G}$ which is spontaneously broken and commutes with the original symmetry (actually, in the non-abelian case, $G$ and $\hat{G}$ together form the quantum double). The dual net $A^d$ is then obtained as the fixpoints of $\hat{F}$ under the original symmetry.

The two scenarios under discussion obviously describe the same situation, by identifying the diagrams and exchanging the roles of $G$ and its dual. The difference consists in the circumstance that we construct the “top right” net from the “bottom left” net, passing via the “top left” non-local net, while the analysis in [11] goes the opposite way. The latter point of view is more general as it admits non-abelian unbroken groups, while ours is more general as $B^0$ needs not to be fermionic but rather may turn out to be anyonic.
To complete the comparison, we should be able to describe the disorder operators \[11\] in our setting. Indeed, they are given by \( U_\eta = (id, u) \in A \) where \( u \) is a local unitary of character \( \eta \). By Prop. 2, \( U_\eta F U_\eta^* \) equals \( F \) if \( F \in B^0 \) is localized at right space-like distance from \( U_\eta \), and equals \( \eta(F) \) if \( F \) is localized at left space-like distance from \( U_\eta \). It is clear that the unitaries \( U_\eta \) along with the neutral operators in \( B^0 \) generate \( B_R \).

As an instructive example which is perfectly consistent with the picture, we want to match it with our semi-classical and perturbative knowledge of the Sine-Gordon vs. massive Thirring model \[3\]. Let \( A \) denote the Sine-Gordon model generated by its scalar field \( \phi \) which carries the broken \( \mathbb{Z} \)-symmetry \( \phi \leftrightarrow \phi + 2\pi n \). The fixpoints \( A^0 \) are the fields \( \partial \phi \) and \( \sin \phi \), \( \cos \phi \). The solitons of the Sine-Gordon model are known to be the Thirring fermions \( \psi \), therefore the extension \( B_R \) by the solitons should be a theory containing both fields \( \phi \) and \( \psi \) with complicated commutation relations and both the \( \mathbb{Z} \)-symmetry of the Sine-Gordon model and the \( U(1) \)-symmetry of the Thirring model. Passing to the fixpoints under \( \mathbb{Z} \) eliminates the field \( \phi \) but retains the derivative and trigonometric fields which are identified \[3\] with the quadratic gauge-invariant combinations of the Thirring fermion. Therefore, \( B^0 \) is just the Thirring model. Its fixpoints under \( U(1) \) are generated by the quadratic invariants, hence coincide with \( A^0 \). This completes the square of inclusions.

Note that this example is both abelian, and yields a fermionic theory \( B^0 \). It thereby belongs to the intersection of models covered by Möger’s and by our approach, respectively.

The departure from the standard form of the fundamental theorems will indeed be seen in this familiar model as soon as one considers “mixed” fields in the style of \( \psi \exp i a \phi \), which involve both Thirring fermions and Sine-Gordon fields of non-trivial character.

Acknowledgments: This work has been done at the Werner-Heisenberg-Institut (MPI), München, and at the Erwin Schrödinger International Institute (ESI), Vienna. To both institutions I am indebted for hospitality and for financial support. Many thanks go also to M. Niedermaier, M. Möger, D. Schlingemann and others for stimulating discussions.

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