Counting spanning trees on fractal graphs and their asymptotic complexity

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Abstract
Using the method of spectral decimation and a modified version of Kirchhoff’s matrix-tree theorem, a closed form solution to the number of spanning trees on approximating graphs to a fully symmetric self-similar structure on a finitely ramified fractal is given in theorem 3.4. We show how spectral decimation implies the existence of the asymptotic complexity constant and obtain some bounds for it. Examples calculated include the Sierpiński gasket, a non-post critically finite analog of the Sierpiński gasket, the Diamond fractal, and the hexagasket. For each example, the asymptotic complexity constant is found.

Keywords: fractal graphs, spanning trees, spectral decimation, asymptotic complexity

1. Introduction

The problem of counting the number of spanning trees in a finite graph dates back more than 150 years. It is one of the oldest and most important graph invariants, and has been actively studied for decades. Kirchhoff’s famous matrix-tree theorem [32], appearing in 1847, relates properties of electrical networks and the number spanning trees. There are now a large variety of proofs for the matrix-tree theorem, for some examples see [9, 14, 28]. Counting spanning trees is a problem of fundamental interest in mathematics [8, 10, 13, 33, 56, e.g.] and physics [21, 22, 57–59, e.g.]. Its relation to probability theory was explored in [34, 36]. It has found applications in theoretical chemistry, relating to the enumeration of certain chemical isomers [12], and as a measure of network reliability in the theory of networks [19].

Recently, various authors have studied the number of spanning trees and the associated asymptotic complexity constants on regular lattices in [16–18, 42, 55]. A natural question is to also consider spanning trees on self-similar fractal lattices. Self-similar finitely ramified sets

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have been recently studied in physics and have connections with complex media, networks and even some biological processes. They exhibit scale invariance rather than translation invariance and this property becomes very useful when studying certain physical processes such as reaction-diffusion processes modeled by random or quantum walks [2, 37]. There have also been applications in engineering involving a fractal antenna [30].

In [15] Chang, Chen, and Yang calculated the number of spanning trees on the sequence of graph approximations to the Sierpiński gasket of dimension two, three, and four, as well as for two generalized Sierpiński gaskets ($SG_{2,3}(n)$ and $SG_{2,4}(n)$), and conjectured a formula for the number of spanning trees on the $d$-dimensional Sierpiński gasket at stage $n$, for general $d$. Their method of proof uses a decomposition argument to derive multi-dimensional polynomial recursion equations to be solved. Independently, that same year, Teuff and Wagner [47] give the number of spanning trees on the Sierpiński gasket of dimension two at stage $n$, using the same argument. In [48] they expand on this work, constructing graphs by a replacement procedure yielding a sequence of self-similar graphs (this notion of self-similarity is different than in [31]), which include the Sierpiński graphs. For a variety of enumeration problems, including counting spanning trees, they show that their construction leads to polynomial systems of recurrences and provide methods to solve these recurrences asymptotically. Using the same construction technique in [50], they give, under the assumptions of strong symmetry (see [50, section 2.2]) and connectedness, a closed form equation for the number of spanning trees [50, theorem 4.2]. This formulation requires calculating the resistance scaling factor and the tree scaling factor (defined in [50, theorem 4.1] and [49, 51]). In [51] they expand and generalize this work to a broader setting avoiding that symmetry assumption.

In this paper we develop a new methodology by studying the problem via a spectral approach. The central result of the present work, theorem 3.4, relies on the technique of spectral decimation to calculate in an analytic fashion the number of spanning trees of the sequence of graph approximations to self-similar fully symmetric finitely ramified fractals. Spectral decimation is widely studied independently of spanning tree enumeration in [4, 5, 23, 35, 40]. The idea behind our calculation is to use Kirchhoff’s matrix-tree theorem and explicit knowledge of the spectrum which is obtained from spectral decimation. Essentially, we calculate the product of the non-zero eigenvalues of the graph’s probabilistic Laplacian which can be thought of as a ‘determinant’ despite the matrix being singular. A drawback is that we require the full symmetry assumption to guarantee the use of spectral decimation which is a rather strong condition. The results of [49, 51], which are obtained in a more general setting, can be applied to more fractal graphs because they do not require this symmetry condition. However, a possible advantage of our calculation of this determinant is that it may have applications besides spanning tree enumeration as well as easily obtaining the number of spanning trees if one has previously studied the spectrum of a given fractal. There have also been investigations of this determinant with regards to its connection with spectral zeta functions and the regularized determinant of the Laplace operator in [17].

Section 2 of this work will set up some notation and preliminaries. In section 3 the main result of this work is presented. Theorem 3.4 allows one to write down a closed formula for the number of spanning trees on the class of fractal graphs considered. A nice corollary of this is the fact that such formulas remain simple. In section 4, we study the asymptotic complexity constant of the sequence of graphs and obtain a sharp lower bound for it and an upper bound involving only the number of contractions and the number of vertices on the first two approximations of the fractal graphs. We also give an alternate proof motivating as to why the asymptotic complexity constant exists. Section 5 includes a plethora of examples showing how to use theorem 3.4.

This work is an amalgamation and expansion of the first author’s work in [3] and the second author’s work in [52].
2. Background and preliminaries

Kirchhoff’s matrix-tree theorem relates a normalized product of the non-zero eigenvalues of the graph Laplacian to the number of spanning trees of a loopless connected graph. Fractal graphs are always connected and loopless so we will always make this assumption henceforth. However, using the method of spectral decimation one is only able to find the eigenvalues of the probabilistic graph Laplacian for a specified class of fractal graphs, so a suitable version of Kirchhoff’s theorem for probabilistic graph Laplacians must be used. For any graph $G = (V, E)$ having $n$ labelled vertices $v_1, v_2, \ldots, v_n$, with vertex set $V$ and edge set $E$, the graph Laplacian $L$ on $G$ is defined by $L = D - A$, where $D = (d_{ij})$ is the degree matrix on $G$ with $d_{ii} = \deg(v_i)$, and $A = ((a_{ij}))$ is the adjacency matrix on $G$ with $a_{ij}$ is the number of copies of $\{v_i, v_j\} \in E$. The probabilistic graph Laplacian of $G$ is defined by $L = -PL^{-1}$. The following theorem, originally shown in [39], is the version of the matrix-tree theorem that will be used in this work.

**Theorem 2.1** (Kirchhoff’s matrix-tree theorem for probabilistic graph Laplacians). For any connected, loopless graph $G$ with $n$ labelled vertices, the number of spanning trees of $G$ is

$$
\tau(G) = \left(\prod_{j=1}^{n} d_j \right)^{n-1} \left(\prod_{j=1}^{n} \lambda_j^{n-j} \right),
$$

where $\{\lambda_j\}_{j=1}^{n-1}$ are the non-zero eigenvalues of $P$ and $d_j$ is the degree of vertex $j$ in $G_n$.

If we have a group $G$ and a set $S$, we call a group action of $G$ on $S$ doubly transitive if for all pairwise distinct $x_1, x_2$ and $y_1, y_2$ there exists a $g \in G$ such that $gx_i = y_i$ for $i = 1, 2$. We will now introduce the fractal graphs we will be studying. As in [29], the definition of a fully symmetric finitely ramified self-similar structure is the following.

**Definition 2.2.** A fractal $K$ is a fully symmetric finitely ramified self-similar set if $K$ is a compact connected metric space with injective contraction maps $\{f_j\}_{j=1}^{m}$ such that

$$
K = f_1(K) \cup \cdots \cup f_m(K),
$$

and the following three conditions hold:

1. there exist a finite subset $V_0$ of $K$ such that $f_j(K) \cap f_k(K) = f_j(V_0) \cap f_k(V_0)$ for $j \neq k$ (this intersection may be empty);
2. if $v_0 \in V_0 \cap f_j(K)$ then $v_0$ is the fixed point of $f_j$;
3. there is a group $G$ of isometries of $K$ that has a doubly transitive action on $V_0$ and is compatible with the self-similar structure $\{f_j\}_{j=1}^{m}$, which means that for any $j$ and any $g \in G$ there exist a $k$ such that $g^{-1} \circ f_j \circ g = f_k$.

The set $V_0$ is called the boundary of $K$. For any self-similar set $K$, with respect to $\{f_1, f_2, \ldots, f_m\}$, there is a natural sequence of approximating graphs $G_n$ with vertex set $V_n$ defined as follows. For all $n \geq 0$ and for all $\omega \in W_n$, define $G_0$ as the complete graph with
vertices $V_0$, 
\[ V'_0 := \bigcup_{\omega \in W_0} V_\omega, \]
\[ V_c := \bigcup_{x \in V_0} V_c(x), \]

where $V_\omega := f_{d_0} \circ f_{d_{n-1}} \circ \cdots \circ f_{d_1}$ and $\omega = a_1 a_2 \cdots a_n$. Also, $x, y \in V_\omega$ are connected by an edge in $G_n$ if $f^{-1}_x(x)$ and $f^{-1}_y(y)$ are connected by an edge in $G_{n-1}$ for some $1 \leq i \leq m$. In what follows we denote $D_n = \prod_1^{V_0} d_i \left( \sum_{\omega \in W_0} |d_\omega| \right)$.

Notice that two non-isomorphic self-similar structures can have the same finitely ramified self-similar set, however the structures will not have the same sequence of approximating graphs $G_n$. Also, any two isomorphic self-similar structures whose compact metrizable topological spaces are finitely ramified self-similar sets will have approximating graphs which are isomorphic for all $n \geq 0$.

We note that our symmetry assumption (3) which guarantees spectral decimation is not always satisfied even for some common fractals. In [51] the weaker assumption of double homogeneity is used. An example of a fractal not admitting spectral decimation is the Pentagasket.

3. Counting spanning trees on fractal graphs

Let $K$ be a fully symmetric finitely ramified self-similar structure, $G_n$ be its sequence of approximating graphs, and $P_n$ denote the probabilistic graph Laplacian of $G_n$.

The next two propositions describe the spectral decimation process, which inductively gives the spectrum of $P_n$.

The $G_0$ network is the complete graph on the boundary set with cardinality $|V_0|$. Write $P_1$ in block form

\[ P_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \]

where $A$ is a square block matrix associated to the boundary points. Since the $G_1$ network never has an edge joining two boundary points, $A$ is the $|V_0| \times |V_0|$ identity matrix. The Schur Complement of $P_1$ is

\[ S(z) = (A - zI) - B(D - z)^{-1}C. \]

**Proposition 3.1** (Bajorin, et al [4]). For a given fully symmetric finitely ramified self-similar structure $K$ there are unique scalar valued rational functions $\phi(z)$ and $R(z)$ such that for $z \notin \sigma(D)$

\[ S(z) = \phi(z)(P_0 - R(z)). \]

Now $P_0$ has entries $a_{ii} = 1$ and $a_{ij} = \frac{-1}{|V_0| - 1}$ for $i \neq j$. Looking at specific entries of this matrix valued equation we get two scalar valued equations

\[ \phi(z) = -(|V_0| - 1)S_{1,2}(z) \quad \text{and} \quad R(z) = 1 - \frac{S_{1,1}}{\phi(z)}, \]

where $S_{i,j}$ is the $i, j$ entry of the matrix $S(z)$. 


In the electrical network approach of [51] the iterations of the renormalization map are studied. From the renormalization map it is possible to define the renormalization constant $\rho > 1$ which is connected to the spectral decimation function by the formula $\rho m = \frac{d}{dz} R(0)$.

Now, we let

$$E(P_0, P_1) := \sigma(D) \bigcup \{z : \phi(z) = 0\}$$

and call $E(P_0, P_1)$ the exceptional set.

Let $\text{mult}_m(z)$ be the multiplicity of $z$ as an eigenvalue of $D$, $\text{mult}_n(z)$ be the multiplicity of $z$ as an eigenvalue of $P_n$, $\text{mult}_P(z) = 0$ if and only if $z$ is not an eigenvalue of $P_n$, and similarly $\text{mult}_D(z) = 0$ if and only if $z$ is not and eigenvalue of $D$. Then we may inductively find the spectrum of $P_n$ with the following proposition.

**Proposition 3.2** (Bajorin, et al [4]). For a given fully symmetric finitely ramified self-similar structure $K$, and $R(z)$, $\phi(z)$, $E(P_0, P_1)$ as above, the spectrum of $P_n$ may be calculated inductively using the following criteria:

1. If $z \notin E(P_0, P_1)$, then
   $$\text{mult}_n(z) = \text{mult}_{n-1}(R(z))$$

2. If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at $z$ then,
   $$\text{mult}_n(z) = |V_{n-1}|$$

3. If $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at $z$, $R(z)$ has a removable singularity at $z$, and $\frac{d}{dz} R(z) \neq 0$, then
   $$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z) - |V_{n-1}| + \text{mult}_{n-1}(R(z))$$

4. If $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at $z$, and $\phi(z) = 0$, then
   $$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z) + \text{mult}_{n-1}(R(z))$$

5. If $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at $z$, and $\phi(z) = 0$, then
   $$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z) + |V_{n-1}| + \text{mult}_{n-1}(R(z))$$

6. If $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at $z$, $R(z)$ has a removable singularity at $z$, and $\frac{d}{dz} R(z) = 0$, then
   $$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z) - |V_{n-1}| + 2\text{mult}_{n-1}(R(z))$$

7. If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a pole at $z$, then $\text{mult}_n(z) = 0$.

8. If $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at $z$, $\phi(z) = 0$ and $R(z)$ has a pole at $z$, then
   $$\text{mult}_n(z) = m^{n-1} \text{mult}_D(z).$$
We can decompose the spectrum into two finite sets \( A \) and \( B \) of eigenvalues such that taking preiterates is not allowed and is allowed respectively and define for \( \alpha \in A, \alpha_n \equiv \text{mult}_n(\alpha) \) and for \( \beta \in B, \beta_n^k \equiv \text{mult}_n(R_{-k}(\beta)) \).

Since \( G_n \) is connected \( \text{mult}_n(0) = 1 \) for all \( n \geq 0 \). Again from [4], we get that

\[
\sigma(P_n) \setminus \{0\} = \bigcup_{\alpha \in A} \{\alpha\} \bigcup_{\beta \in B} \left[ \bigcup_{k=0}^{n} \{R_{-k}(\beta) : \beta_n^k \neq 0\} \right].
\]

Hence the non-zero eigenvalues of \( P_n \) are the zeros of polynomials or preiterates of rational functions. To be able to use theorem 2.1, we need to know how to take the product of preiterates of rational functions of a particular form. The proof of theorem 3.4 will show that \( R(z) \) satisfies the assumptions of the next Lemma, and use this information to be able to calculate the number of spanning trees on the fractal graphs under consideration.

**Lemma 3.3.** Let \( R(z) \) be a rational function such that \( R(0) = 0, \deg(R(z)) = d, R(z) = \frac{P(z)}{Q(z)} \) with \( \deg(P(z)) > \deg(Q(z)) \). Let \( P_d \) be the leading coefficient of \( P(z) \). Fix \( \alpha \in \mathbb{C} \). Let \( \{R_{-n}(\alpha)\} \) be the set of \( n \)th preiterates of \( \alpha \) under \( R(z) \). By convention, \( R_{(0)}(\alpha) \equiv \{\alpha\} \). Then for \( n \geq 0, \)

\[
\prod_{z \in \{R_{-n}(\alpha)\}} z = \alpha \left( \frac{-Q(0)}{P_d} \right)^{\frac{d-n}{d}}.
\]

**Proof of lemma 3.3.** For \( n = 0 \), the result is clear. For \( n = 1 \), we note

\[
\{R_{-1}(\alpha)\} = \{z : R(z) = \alpha\} = \{z : P(z) - \alpha Q(z) = 0\} = \{z : P_d z^d + \cdots - Q(0) \alpha = 0\},
\]

where \( Q(0) \) is the constant term of \( Q(z) \). As the product of the roots of a polynomial is equal to the constant term over the coefficient of the highest degree term, we have that

\[
\prod_{z \in \{R_{-1}(\alpha)\}} z = \frac{-\alpha Q(0)}{P_d}.
\]

Assume our equation holds for \( n \). Then for \( n + 1 \) we have

\[
\{w : w \in R_{-(n+1)}(\alpha)\} = \{R_{-1}(w) : w \in R_{-n}(\alpha)\}.
\]

Hence

\[
\prod_{w \in \{R_{-(n+1)}(\alpha)\}} w = \prod_{w \in \{R_{-n}(\alpha)\}} \left( \prod_{z \in \{R_{-1}(w)\}} z \right) = \prod_{w \in \{R_{-n}(\alpha)\}} \left( \frac{-w Q(0)}{P_d} \right),
\]

with the second equality following from the \( n = 1 \) case.
Since $|R_{\{n\}}(\alpha)| = d^n$ (not necessarily distinct) this equality becomes

$$\prod_{w \in [R_{\{n\}}(\alpha)]} w = \left(\frac{-Q(0)}{P_0}\right)^{\alpha} \prod_{w \in [R_{\{n\}}(\alpha)]} w$$

$$= \left(\frac{-Q(0)}{P_0}\right)^{\alpha} \cdot (\alpha) \left(\frac{-Q(0)}{P_0}\right)^{\frac{d^n-1}{d-1}}$$

$$= \alpha \left(\frac{-Q(0)}{P_0}\right)^{\frac{d^n+1}{d-1}},$$

as desired. ∎

The following theorem is the main result of this paper.

**Theorem 3.4.** For a given fully symmetric self-similar structure on a finitely ramified fractal $K$, let $G_n$ denote its sequence of approximating graphs and let $P_n$ denote the probabilistic graph Laplacian of $G_n$. Arising naturally from the spectral decimation process, there is a rational function $R(z)$, which satisfies the conditions of lemma 3.3, finite sets $A, B \subset \mathbb{R}$ such that for all $\alpha \in A, \beta \in B$, and integers $n, k \geq 0$, there exist functions $\alpha_n$ and $\beta_n$ such that the number of spanning trees of $G_n$ is given by

$$\tau(G_n) = D_n \prod_{\alpha \in A} \alpha^{\alpha_n} \prod_{\beta \in B} \left(\beta^{\sum_{k=0}^{\infty} \beta_n^k} \left(-\frac{Q(0)}{P_0}\right)^{\sum_{k=0}^{\infty} \beta_n^k \left(\frac{d^n-1}{d-1}\right)} \right)$$

where $d$ is the degree of $R(z)$, $P_d$ is the leading coefficient of the numerator of $R(z)$ and $|V_n|$ is the number of vertices of $G_n$.

**Proof of theorem 3.4.** From Kirchhoff’s matrix-tree theorem for probabilistic graph Laplacians (theorem 2.1), we know that

$$\tau(G_n) = D_n \prod_{j=1}^{\vert V_n \vert-1} \lambda_j,$$

where $\lambda_j$ are the non-zero eigenvalues of $P_n$.

Existence and uniqueness of the rational function $R(z)$ is given proposition (3.1). After carrying out the inductive calculations using proposition (3.2) items (1)–(8), we get the sets $A$ and $B$, and the functions $\alpha_n$ and $\beta_n$.

To see that the sets $A$ and $B$ are finite. Recall that the functions $R(z)$ and $\phi(z)$ from proposition (3.2) are rational, thus $R(z)$, $\phi(z)$, and $R(z)\phi(z)$ have finitely many zeroes, poles, and removable singularities. Also, since the matrix $D$, from writing $P_1$ in block form to define the Schur Complement, is finite, $\sigma(D)$ is finite. Following items (1)–(8) of proposition (3.2) these observations imply that $A$ and $B$ are finite sets.

From proposition (3.2) we know that

$$\{\lambda_j\}_{j=1}^{\vert V_n \vert-1} = \bigcup_{\alpha \in A} \{\alpha\} \bigcup_{\beta \in B} \left[\bigcup_{k=0}^{n} \{R^{-k}(\beta) : \beta_n^k \neq 0\}\right],$$

where the multiplicities of $\alpha \in A$ are given by $\alpha_n$ and the multiplicities of $\{R^{-k}(\beta)\}$ are given by $\beta_n^k$. We also have that $\lambda_{\vert V_n \vert}=0$. 

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From items (1)–(8) of proposition (3.2) it follows that for all \( z \in \{R_{\pm k}(\beta)\} \) the multiplicity of \( z \) depends only on \( n \) and \( k \) and thus we get

\[
\prod_{j=1}^{\lfloor \frac{V_n - 1}{0} \rfloor} \lambda_j = \left( \prod_{\alpha \in A} \alpha^\alpha \right) \left( \prod_{j \in B} \left( \prod_{k=0}^{n} \left( \prod_{z \in \{R_{\pm k}(\beta)\}} z^{|z|} \right) \right) \right).
\]

From lemma 4.9 in [35], \( R(0) = 0 \). From corollary 1 in [29], it follows that, if we write \( R(z) = \frac{P(z)}{Q(z)} \) where \( P(z) \) and \( Q(z) \) are relatively prime polynomials, then \( \deg(P(z)) > \deg(Q(z)) \). Thus, the conditions of lemma 3.3 are satisfied, and applying this theorem gives

\[
\prod_{j=1}^{\lfloor \frac{V_n - 1}{0} \rfloor} \lambda_j = \left( \prod_{\alpha \in A} \alpha^\alpha \right) \left( \prod_{j \in B} \left( \prod_{k=0}^{n} \left( \beta^{d-1} \frac{Q(0)}{P_d} \right)^{d^k} \right) \right)
\]

\[
= \left( \prod_{\alpha \in A} \alpha^\alpha \right) \left( \prod_{j \in B} \left( \beta^{\sum_{k=0}^{n} d^k} \left( \frac{Q(0)}{P_d} \right)^{d^k} \right) \right).
\]

Applying Kirchhoff’s matrix-tree theorem for probabilistic graph Laplacians (theorem 2.1), we verify the result.

In [15], the authors derived multidimensional polynomial recursion equations to solve explicitly for the number of spanning trees on \( SG_2(n) \) with \( d \) equal to two, three and four, and on \( SG_{2,d}(n) \) with \( d \) equal to two and three. They note in that work that it is intriguing that their recursion relations become more and more complicated as \( b \) and \( d \) increase, but the solutions remain simple, and comment that with their methods, they do not have a good explanation for this. This can be explained by theorem 2 and remark 2 of [51]. Alternatively, it can also be seen from the following corollary.

**Corollary 3.5.** For a given fully symmetric self-similar structure on a finitely ramified fractal \( K \), with approximating graphs \( G_n \), there exist a finite set of primes \( \{p_k\}_{k=1}^{r} \) and functions \( \{f_k : \mathbb{N}_0 \rightarrow \mathbb{N}_0\}_{k=1}^{r} \) such that

\[
\tau(G_n) = \prod_{k=1}^{r} p_k^{f_k(n)}.
\]

**Proof of corollary 3.5.** Since \( \tau(G_n) \) is a non-negative integer, it can be factorized into prime numbers. Observing equation (1), we see that the sets \( A \) and \( B \) are fixed, and self-similarity gives that for any \( n \geq 2 \) the only prime factors of \( \prod_{k=1}^{r} d_i \) are the prime factors of \( \prod_{k=1}^{r} d_i \).

**4. Asymptotic complexity**

Let \( T_n \) for \( n \geq 0 \) be a sequence of finite graphs, \( |V_n| \) the number of vertices in \( T_n \), and \( \tau(T_n) \) denote the number of spanning trees of \( T_n \). We call \( \tau(T_n) \) the *complexity* of \( T_n \). The asymptotic
complexity of the sequence \( T_n \) is defined as
\[
\lim_{n \to \infty} \log(\tau(T_n)) / |T_n|.
\]
When this limit exists, it is called the *asymptotic complexity* constant, or the *tree entropy* of \( T_n \).

For any two, finite, connected graphs \( G_1, G_2 \), let \( G_1 \cup_{x_1} G_2 \) denote the graph formed by identifying the vertex \( x_1 \in G_1 \) with vertex \( x_2 \in G_2 \). Then for all \( x_1 \in G_1, x_2 \in G_2 \), it is clear that
\[
\tau(G_1 \cup_{x_1} G_2) = \tau(G_1) \cdot \tau(G_2).
\] (2)

If we were to drop the assumption of full symmetry, we lose the spectral decimation process, but still have the following theorem.

**Theorem 4.1.** For a given self-similar structure on a finitely ramified fractal \( K \), let \( G_n \) denote its sequence of approximating graphs. Let \( m \) denote the number of 0-cells of the \( G_1 \) graph.

1. If \( G_1 \) is a tree, then \( \tau(G_n) = 1 \) for all \( n \geq 0 \).
2. If \( G_1 \) is not a tree and \(|V_0| > 2\), then \( \log(\tau(G_n)) \in \Theta(|V_0|) = \theta(m^a) \). Specifically, we have that
\[
\frac{(|V_0| - 2)\log|V_0|}{|V_0| - 1} \leq \lim \inf_{n \to \infty} \frac{\log \tau(G_n)}{|V_0|}
\]
and
\[
\lim \sup_{n \to \infty} \frac{\log \tau(G_n)}{|V_0|} \leq \log \left( \frac{(m - 1)|V_0|(|V_0| - 1)}{|V_0| - |V_0|} \right).
\]

**Proof of Theorem 4.1.** From [50], we have the formula \(|V_n| = m|V_{n-1}| - m|V_0| + |V_0|\) from which we can derive that
\[
|V_n| = m^a(|V_0| - |V_0|) + m|V_0| - |V_0|.
\]
Thus we see that \( \lim_{n \to \infty} \frac{|V_n|}{m^a} = \frac{|V_0| - |V_0|}{m^a - 1} \) which, for convenience, we denote as \(|V_n| \sim m^a\). If \( G_1 \) is a tree, then \( K \) is a fractal string. Hence, for all \( n \geq 0 \), \( G_n \) is a tree. In the case that it is not a tree, then \( G_n \) is \( m^n \) copies of the \( G_0 \) graph and we have that \( \tau(G_n) \geq \tau(G_0 \cup_{x,y} m^n) \), where \( G_0 \cup_{x,y} G_0 \) denotes \( m^n \) copies of \( G_0 \) each identified to each other at some vertex \( x \in V_0 \). Then, since the \( G_0 \) graph is the complete graph on \( |V_0| \) vertices, by Cayley’s formula we have that \( \tau(G_0) = |V_0|^{(|V_0|-2)} \), and we see
\[
\tau(G_0 \cup_{x,y} m^n) G_0) = |V_0|^{(|V_0|-2)m^n}
\]
and
\[
\tau(G_n) \geq |V_0|^{(|V_0|-2)m^n}.
\]
So for \( n \geq 0 \),
\[
\log(\tau(G_n)) \geq m^a(|V_0| - 2)\log(|V_0|). \] (3)
Now, regarding the number of vertices, we have the following bound
\[ |V_n| \leq m^n(|V_0| - 1) + 1. \]
This follows due to the fact that the \(G_n\) graph is \(m^n\) copies of the \(G_0\) one and therefore we obviously have that \( |V_n| \leq m^n|V_0| \). However, due to connectivity, some vertices need to overlap. At minimum, one vertex from each 0-cell will overlap which would mean that
\[ |V_n| \leq m^n|V_0| - 1 - 1 - ... - 1 \]
with the number of \(-1\) being as many times as the cells minus one, namely \(m^n - 1\) which would give us \( |V_n| \leq m^n|V_0| - m^n + 1 \).

Then we have the following
\[ \frac{1}{|V_n|} \geq \frac{1}{m^n(|V_0| - 1) + 1} = \frac{1}{m^n(|V_0| - 1 + \frac{1}{m^n})}. \]

Then by the inequality (3), we have that
\[ \frac{\log(\tau(G_n))}{|V_n|} \geq \frac{m^n(|V_0| - 2)\log(|V_0|)}{m^n(|V_0| - 1 + \frac{1}{m^n})}, \]
and thus
\[ \liminf_{n \to \infty} \frac{\log(\tau(G_n))}{|V_n|} \geq \frac{(|V_0| - 2)\log|V_0|}{|V_0| - 1}. \]

Now for the upper bound. It is known from [27] that an upper bound on the number of spanning trees of a graph \(G\) is
\[ \tau(G) \leq 1 \left( \frac{2|E|}{|V|} \right)^{|V|-1} \]
and thus the asymptotic complexity constant is bounded by \( \log \kappa \) where \( \kappa = \lim_{n \to \infty} \frac{2E_n}{|V_n|} \) is the so-called effective coordination number. For the fractal graphs, if we denote \( E_n \) the cardinality of the edge set of \(G_n\), then we have that \( E_n = m^n \left( \frac{|V_0|}{2} \right) \) which can be seen from the self-similarity of the graph and the fact that \(G_0\) is the complete graph on \(V_0\) vertices. The upper bound can then be obtained by the fact that \( \lim_{n \to \infty} \frac{|V_n|}{m^n} = \frac{|V_0| - 1}{m - 1} \).

In [33] the existence of the asymptotic complexity constant was shown if the sequence of graphs approximates an infinite graph, in a certain sense. That result is quite general and covers many cases, including those of the fractal graphs. In the next theorem we show that the assumption of full symmetry, and thus the ability to perform spectral decimation, can also imply the existence of the asymptotic complexity constant for fractal graphs, thereby giving an alternate proof which is in spirit closer to analysis on fractals.

**Theorem 4.2.** For any fully symmetric self similar fractal, \(K\), the asymptotic complexity constant of its sequence of approximating graphs exists.

Before the proof, we state the Stolz–Cesàro Lemma, which will be used.

**Lemma 4.3.** Let \((a_n)\) and \((b_n)\) be sequences of real numbers such that \((b_n)_n\) is strictly monotone and divergent to \(+\infty\) or \(-\infty\). If we have that the following limit exists
then we have that \( \lim_{n \to \infty} \frac{a_n}{b_n} = c \).

We now present the proof of the theorem 4.2.

**Proof.** We want to prove the existence of the limit of the sequence \( \log \tau(G_n) \). We already have from the theorem above that the sequence is bounded. Therefore it suffices to check that we do not have any oscillatory behavior. By the full symmetry assumption, we can perform spectral decimation and \( \tau(G_n) \) is given by equation (1) and thus we obtain that

\[
\log \tau(G_n) = \log D_n + \log \left( \prod_{\alpha \in A} \alpha^{\alpha_n} \right) + \log \left( \prod_{\beta \in B} \beta^{\sum_{k=1}^{n} \beta_{n+1}^k} \right) + \log \left( \prod_{\beta \in B} \frac{-Q(0)}{P_d} \right) \sum_{k=0}^{n} \beta_{n+1}^k \frac{d^{k-1}}{d_k}.
\]

Therefore it suffices to prove that the limit \( \lim_{n \to \infty} \frac{\log \tau(G_n)}{\log |V_n|} \) exists and for each \( \alpha \in A \) and \( \beta \in B \) the limits

\[
\lim_{n \to \infty} \frac{\alpha_n}{|V_n|} \quad \text{and} \quad \lim_{n \to \infty} \frac{\sum_{k=0}^{n} \beta_{n+1}^k}{|V_n|}
\]

also exist. Summing the multiplicities of the eigenvalues of \( P_n \), we have

\[
\sum_{\alpha \in A} \alpha_n + \sum_{\beta \in B} \sum_{k=0}^{n} \beta_{n+1}^k + 1 = |V_n|.
\]

Since \( \alpha_n, \beta_{n+1}^k \) are non-negative integers we see that for each \( \alpha \in A \) and \( \beta \in B \) that \( \frac{\alpha_n}{|V_n|} \), \( \frac{\sum_{k=0}^{n} \beta_{n+1}^k}{|V_n|} \) must be bounded and thus the same holds for \( \frac{\sum_{k=0}^{n} \beta_{n+1}^k}{|V_n|} \) and \( \frac{\sum_{k=0}^{n} \beta_{n+1}^k d_k^{k-1}}{|V_n|} \). Now, for a given \( \alpha \in A \), we have by the definition of the finite set \( A \) that the multiplicities \( \alpha_n = \text{mult}_\alpha(\alpha) \), which can be found from proposition 1.3, above depend only on the eigenvalue \( \alpha \) and the level \( n \) and in each of the cases of the proposition we have convergence as \( |V_n| \sim m^n \). Now for the remaining limits. Take \( \beta \in B \) and \( \beta_{n+1}^k = \text{mult}_\beta(R_n(\beta)) \). By the general algorithm of the spectral decimation methodology, we have that every pre-iterate of the spectral decimation rational function preserves the multiplicity of the eigenvalues. Therefore we have that \( \beta_{n+1}^{k+1} = \beta_{n+1}^k \) for \( 1 \leq k \leq n + 1 \) and thus the sum of multiplicities at level \( n + 1 \) must be the sum of the multiplicities at level \( n \) along with those with generation of birth \( n + 1 \). This is just the following formula

\[
\sum_{k=0}^{n+1} \beta_{n+1}^k = \sum_{k=1}^{n+1} \beta_{n+1}^k + \beta_{n+1}^0 = \sum_{k=0}^{n+1} \beta_{n+1}^k + \beta_{n+1}^0 = \sum_{k=0}^{n+1} \beta_{n+1}^k + \beta_{n+1}^0
\]

and

\[
\sum_{k=0}^{n+1} \beta_{n+1}^k d_k = \sum_{k=1}^{n+1} \beta_{n+1}^k d_k + \beta_{n+1}^0 = \sum_{k=1}^{n+1} \beta_{n+1}^k d_k + \beta_{n+1}^0 = d \sum_{k=0}^{n+1} \beta_{n+1}^k + \beta_{n+1}^0.
\]
By taking into account that \( V_{n+1} \xrightarrow{1} m \) and by looking at the proposition 1.3, we have a list of possible choices for the term \( \beta^0_{n+1} \) and similarly to the case of the eigenvalues in the set \( A \) before it must be that \( \frac{\beta^0_{n+1}}{V_{n+1}} \) converges to a finite positive constant, which we can call \( c \).

For a general first order linear recurrence \( S_{n+1} = f_n S_n + g_n \) we know that it has solution
\[
S_n = \left( \prod_{k=0}^{n-1} f_k \right) \left( A + \sum_{m=0}^{n-1} \frac{g_m}{\prod_{k=0}^{m} f_k} \right),
\]
where \( A \) is a constant. From the arguments above, we have that \( V_{n+1} = y_n V_n \) where \( y_n \) is a sequence such that \( y_n \to m \) and \( \frac{\beta^0_{n+1}}{y_n} = c + x_n \) with \( x_n \) being a sequence such that \( x_n \to 0 \).

Then for \( S_n = \sum_{i=0}^{n-1} \frac{\beta^0_{i+1}}{y_n} \) we obtain that \( S_{n+1} = \frac{d}{y_n} S_n + c + x_n \). Since we know that \( S_n \) is bounded, it must be that \( \frac{d}{y_n} \leq 1 - \epsilon \) for some \( \epsilon > 0 \) and large \( n \). Then
\[
S_n = \left( \prod_{k=0}^{n-1} \frac{d}{y_k} \right) \left( A + \sum_{i=0}^{n-1} \frac{c + x_i}{\prod_{k=0}^{i} \frac{d}{y_k}} \right).
\]

We care about the limit of \( n \to \infty \) so the constant part becomes 0 and we are left with
\[
\frac{\sum_{i=0}^{n-1} \frac{\beta^0_{i+1}}{y_n}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} + \frac{\sum_{i=0}^{n-1} \frac{c + x_i}{\prod_{k=0}^{i} \frac{d}{y_k}}}{\prod_{k=0}^{n-1} \frac{y_k}{d}}
\]
The second summand goes to 0 as can be seen by the Stolz–Cesàro lemma in the following way. Due to the fact that \( \frac{\beta^0_{n+1}}{y_n} \leq 1 - \epsilon \) we have that \( \prod_{k=0}^{n-1} \frac{y_k}{d} \) is a strictly increasing sequence diverging to \( +\infty \). Then
\[
\frac{\sum_{i=0}^{n-1} \frac{\beta^0_{i+1}}{y_n}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} - \frac{\sum_{i=0}^{n-1} \frac{c + x_i}{\prod_{k=0}^{i} \frac{d}{y_k}}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} = \frac{\sum_{i=0}^{n-1} \frac{\beta^0_{i+1}}{y_n}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} = \frac{x_n}{\prod_{k=0}^{n-1} \frac{y_k}{d}} \to 0
\]
since \( y_n \to m \) and \( x_n \to 0 \).

The first summand is just \( \frac{\sum_{i=0}^{n-1} d^{n-i-1} \prod_{k=0}^{i} \frac{1}{y_k}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} \) which is a positive series and since \( S_n \) is bounded, it must be that it converges. Thus we get existence of \( \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \frac{\beta^0_{i+1}}{y_n}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} \). By an exact similar argument, or more easily by the Stolz–Cesàro lemma, we have the existence of the limit \( \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \frac{c + x_i}{\prod_{k=0}^{i} \frac{d}{y_k}}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} \) and thus also we get that \( \lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} \frac{\beta^0_{i+1}}{y_n}}{\prod_{k=0}^{n-1} \frac{y_k}{d}} \) exists.

We also have that \( |V_n|^{-1} \log D_n \) is bounded and that \( \lim_{n \to \infty} \frac{\log |V_n| d}{|V_n|} = 0 \). Moreover the limit \( \lim_{n \to \infty} \frac{\log |V_n| d}{|V_n|} \) cannot oscillate due to the symmetry of the fractal graph and thus exists, as it is bounded. Thus all the required limits exist and we obtain our result.

Combining the theorem and proposition above we obtain that for fully symmetric self-similar fractal graphs with \( |V_0| > 2 \) we have that the asymptotic complexity constant satisfies
\[
\frac{(|V_0| - 2) \log |V_0|}{|V_0| - 1} \leq c \leq \log \left( \frac{(m - 1)|V_0|(|V_0| - 1)}{|V_0| - |V_0|} \right).
\]
Remark 4.4. We have that \(|V_0|\) is an integer strictly greater than two and the function \(f : [3, +\infty) \to \mathbb{R}, \ f(x) = \frac{(x - 2)\log x}{x - 1}\) has a global minimum at \(x = 3\) which gives us that the asymptotic complexity constant must always be at least \(\frac{\log 3}{2}\).

If we consider the \(m\)-Tree fractal we have \(|V_0| = 1 + (m - 1)m^n\) and therefore by Cayley’s formula \(\tau(G_n) = m^{m-2}\) and thus \(\tau(G_n) = m^{(m-2)m^n}\). Then the asymptotic complexity constant is \(\frac{(m-2)\log m}{m-1}\) which shows the following things. First there is no universal upper bound on the asymptotic complexity constant. Secondly, by evaluating our upper bound we find it to be \(\log m\) which is asymptotically the same as the real value of the asymptotic complexity constant and thus we cannot hope for a significantly better upper bound for the general case. Finally, the asymptotic complexity constant is exactly equal to the lower bound which means that it is sharp.

5. Examples

In this section we calculate explicitly the number of spanning trees of some fractal graphs. For the Sierpiński gasket, these formulas have been previously obtained in [15]. As for the rest, to our knowledge, the number of spanning trees has not been specifically calculated previously in the literature. These examples are a good way of demonstrating how to apply theorem 3.4 in practice.

5.1. Sierpiński gasket

The Sierpiński gasket has been extensively studied (in [5, 6, 20, 23, 31, 38, 41, 43, 44], among others.) It can be constructed as a p.c.f. fractal, in the sense of Kigami [31], in \(\mathbb{R}^2\) using the contractions

\[ f_i(x) = \frac{1}{2}(x - q_i) + q_i, \]

for \(i = 1, 2, 3\), where the points \(q_i\) are the vertices of an equilateral triangle.

Theorem 5.1. The number of spanning trees on the Sierpiński gasket at level \(n\) is given by

\[ \tau(G_n) = 2^L \cdot 3^{s_n} \cdot 5^{h_n}, \quad n \geq 0, \]

where

\[ f_n = \frac{1}{2}(3^n - 1), \ g_n = \frac{1}{4}(3^{n+1} + 2n + 1), \quad \text{and} \quad h_n = \frac{1}{4}(3^n - 2n - 1). \]

Proof of theorem 5.1. Before applying theorem 3.4, we make the following observations. It is known that the \(G_n\) network of the Sierpiński gasket has for \(n \geq 0\)

\[ |V_n| = \frac{3^{n+1} + 3}{2} \]

vertices, three of which have degree 2 and the remaining vertices have degree 4. Hence

\[ D_n = 2^{3^{n+1} - 1} \cdot 3^{-n+1}. \]
In [4], they use a result from [5] to carry out spectral decimation for the Sierpiński gasket. In our language, they showed that

\[ A = \left\{ \frac{3}{2} \right\}, \quad B = \left\{ \frac{3}{4}, \frac{5}{4} \right\}. \]

(I) For \( \alpha = \frac{3}{2} \), \( \alpha_n = \frac{3^n + 3}{2}, \quad n \geq 0. \)

(II) For \( \beta = \frac{3}{4} \) and \( n \geq 1 \) we have that \( \beta_n^k = \frac{3^{n-k} + 3}{2} \) for \( k = 0, \ldots, n - 1 \) and \( \beta_n^k = 0 \) for \( k = n. \)

(III) For \( \beta = \frac{5}{4} \), and \( n \geq 2 \) we have that \( \beta_n^k = \frac{3^{n-k} - 1}{2} \) for \( k = 0, \ldots, n - 2 \) and \( \beta_n^k = 0 \) for \( k = n - 1, n, \)

and \( R(z) = z(5 - 4z). \) So \( d = 2, Q(0) = 1 \) and \( P_d = -4. \)

Applying the above to theorem 3.4 we get

\[ \tau(G_n) = 2^{\ell_n} \cdot 3^{h_n} \cdot 5^{h_n} \quad n \geq 2, \]

as desired. The formula can also easily be verified for all \( n \geq 0. \)

As in [15], we immediately have the following corollary.

**Corollary 5.2.** The asymptotic growth constant for the Sierpiński gasket is

\[ c = \frac{\log 2}{3} + \frac{\log 3}{2} + \frac{\log 5}{6}. \]

5.2. **A Non-p.c.f. analog of the Sierpiński gasket**

As described in [5, 7, 46], this fractal is finitely ramified but not p.c.f. in the sense of Kigami. It can be constructed as a self-affine fractal in \( \mathbb{R}^2 \) using 6 affine contractions. One affine contraction has the fixed point \( (0, 0) \) and the matrix

\[
\begin{pmatrix}
\frac{1}{2} & \frac{1}{6} \\
\frac{1}{2} & \frac{1}{4}
\end{pmatrix},
\]

and the other five affine contractions can be obtained though combining this one with the symmetries of the equilateral triangle on vertices \( (0, 0), (1, 0) \) and \( \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \). Its \( G_1 \) graph can be seen in figure 1.

**Theorem 5.3.** The number of spanning trees on the non-p.c.f. analog of the Sierpiński gasket at level \( n \) is given by

\[ \tau(G_n) = 2^{\ell_n} \cdot 3^{h_n} \cdot 5^{h_n}, \quad n \geq 0, \]
where

\[ f_n = \frac{2}{25} (11 \cdot 6^n - 30n - 11), \quad g_n = \frac{1}{5} (2 \cdot 6^n + 3), \quad \text{and} \]
\[ h_n = \frac{1}{25} (4 \cdot 6^n + 30n - 4). \]

Before the proof, we need the following result of which we omit the proof.

**Lemma 5.4.** The \( G_n \) network of the non-p.c.f. analog of the Sierpiński gasket, for \( n \geq 0 \), has \( \frac{4 \cdot 6^n + 11}{5} \) vertices. Among these vertices,

(i) 3 have degree \( 2^{n+1} \),
(ii) \( 6^{k-1} \) have degree \( 3 \cdot 2^{n-k+2} \) for \( 1 \leq k \leq n \), and
(iii) \( 3 \cdot 6^{k-1} \) have degree \( 2^{n-k+2} \) for \( 1 \leq k \leq n \).

We are now ready for the proof of the main theorem in this section.

**Proof of theorem 5.3.** We apply theorem 3.4. In [5], they use a result from [4] to carry out spectral decimation for the non-p.c.f. analog of the Sierpiński gasket. In our language, they showed that

\[ A = \left\{ \frac{3}{2} \right\}, \quad \text{and} \quad B = \left\{ \frac{3}{4}, \frac{5}{4}, \frac{1}{2}, 1 \right\}. \]

This means that for \( n \geq 2 \) the following hold:

(I) \( \alpha = \frac{3}{2}, \quad \alpha_n = 6^{n-1} + 1. \)

(II) For \( \beta = \frac{3}{4} \) and \( \beta = \frac{5}{4} \), we have that \( \beta_n^k = 6^{n-k-2} + 1 \) for \( k = 0, \ldots, n-2 \), \( \beta_{n-1}^k = 2 \) and \( \beta_0^k = 0. \)

(III) For \( \beta = \frac{1}{2} \), we have \( \beta_n^k = \frac{11 \cdot 6^{n-k-2}}{5} \) for \( k = 0, \ldots, n-2 \) and \( \beta_n^k = 0 \) for \( k = n-1, n. \)

(IV) For \( \beta = 1 \), we have \( \beta_n^k = \frac{6^{n-k-2} - 6}{5} \) for \( k = 0, \ldots, n-2 \) and \( \beta_n^k = 0 \) for \( k = n-1, n. \)
Moreover, we have
\[
R(z) = \frac{-24z(z - 1)(2z - 3)}{14z - 15}.
\]
So \(d = 3\), \(Q(0) = -15\) and \(P_2 = -48\). Applying theorem 3.4 we obtain that
\[
\tau(G_n) = 2^{d-1} \cdot 3^n \cdot 5^h, \quad n \geq 2,
\]
where \(f_n\), \(g_n\), and \(h_n\) are as claimed.

Corollary 5.5. The asymptotic growth constant for the non-p.c.f. analog of the Sierpiński gasket is
\[
c = \frac{11 \log 2}{10} + \frac{\log 3}{2} + \frac{\log 5}{5}
\]
(6)

5.3. Diamond fractal

The diamond self-similar hierarchical lattice appeared as an example in several physics works, including [24, 25] and [26]. In [4], the authors modify the standard results for the unit interval \([0, 1]\) to develop the spectral decimation method for this fractal, hence theorem 3.4 still applies. Its \(G_1\) and \(G_2\) graph is shown in figure 2.

Theorem 5.6. The number of spanning trees on the Diamond fractal at level \(n\) is given by
\[
\tau(G_n) = 2^{4^{n-1}} \quad n \geq 1.
\]

We will need the following result which can be proved combinatorially.

Lemma 5.7. The \(G_n\) network of the diamond fractal, for \(n \geq 1\), has
\[
\frac{4 + 2 \cdot 4^n}{3}
\]
vertices. Among these vertices,

(i) \(2 \cdot 4^{n-k}\) have degree \(2^k\) for \(1 \leq k \leq n - 1\)

(ii) \(4\) have degree \(2^n\).

Figure 2. The \(G_1\) and \(G_2\) network of the Diamond fractal.
Remark 5.8. In [4], the number of vertices of $V_n$ is incorrect as stated in theorem 7.1(ii).

Corollary 5.9. For the $G_n$ network of the Diamond fractal, we have

$$D_n = 2^{\frac{1}{2}(3.4^{n+1} - 6n - 17)}.$$

We now give the proof of the theorem.

Proof of theorem 5.6. We apply theorem 3.4. In [4], they carry out spectral decimation for the Diamond fractal. In our language, they showed that $A = \{2\}$ and $B = \{1\}$. For $n \geq 1$, the following hold:

(I) For $\alpha = 2$, $\alpha_n = 1$.

(II) For $\beta = 1$, we have that $\beta^k_n = \frac{a^{n-k+2}}{3}$ for $k = 0, \ldots, n - 1$ and $\beta^0_n = 0$.

And

$$R(z) = 2z(2 - z).$$

So $d = 2$, $Q(0) = 1$, and $P_0 = -2$. We can then use equation (1) in theorem 3.4 to calculate that $\tau(G_n) = 2^{\frac{1}{2}(4n-1)}$, for $n \geq 1$ as desired. \hfill \Box

Corollary 5.10. The asymptotic growth constant for the Diamond fractal is $c = \log 2$.

5.4 Hexagasket

The hexagasket, is also known as the Hexakun, a Polygasket, a 6-gasket, or a $(2, 2, 2)$-gasket, see [1, 5, 11, 31, 43, 46, 53, 54]. The $G_1$ network of the hexagasket is shown in figure 3.

Theorem 5.11. The number of spanning trees on the hexagasket at level $n$ is given by

$$\tau(G_n) = 2^L \cdot 3^S \cdot 7^H, \quad n \geq 0,$$
where
\[ f_n = \frac{1}{225} \cdot (27 \cdot 6^n - 100 \cdot 4^n - 60n - 62), \]
\[ g_n = \frac{1}{25} \cdot (4 \cdot 6^n + 5n + 1), \]
\[ h_n = \frac{1}{25} \cdot (6^n + 5n - 1). \]

**Proof of theorem 5.11.** We apply theorem 3.4. From [5] it is known that
\[ V_n = \frac{(6 + 9 \cdot 6^n)}{5} \quad n \geq 0, \]
of these vertices, \( \frac{6(6^n - 1)}{5} \) have degree 4, and the remaining \( \frac{(12 + 3 \cdot 6^n)}{5} \) have degree 2. So we compute
\[ D_n = 2^{3 \cdot 6^n - n - 1} \cdot 3^{-(n+1)} \]
for \( n \geq 0 \).

In [5], they use a result from [4] to carry out spectral decimation for the hexagasket. We note that in [5] theorems 6.1 and 6.2, the bounds on \( k \) should be \( \pm \) and in (vii) the bounds should be \( 0 \leq k \leq n - 2 \). This can be verified using table 2 in the same paper. In our language they showed that
\[ A = \left\{ \frac{3}{2} \right\}, \quad \text{and} \quad B = \left\{ \frac{1}{4}, \frac{3 + \sqrt{2}}{4}, \frac{3 - \sqrt{2}}{4} \right\}, \]
and for \( n \geq 2 \) the following hold:

(I) For \( \alpha = \frac{3}{2} \), we have \( \alpha_0 = \frac{6 + 4 \cdot 6^n}{5} \).

(II) For \( \beta = 1 \), we have \( \beta_n^k = 1 \) for \( k = 0, \ldots, n - 1 \) and \( \beta_n^n = 0 \).

(III) For \( \beta = \frac{1}{2} \), we have \( \beta_n^k = \frac{6 + 4 \cdot 6^{n-1}}{5} \) for \( k = 0, \ldots, n - 1 \) and \( \beta_n^n = 0 \).

(IV) For \( \beta = \frac{3 + \sqrt{2}}{4} \), \( \frac{3 - \sqrt{2}}{4} \), we have \( \beta_n^k = \frac{6^{n-1} - 1}{5} \) for \( k = 0 , \ldots, n - 2 \) and \( \beta_n^n = 0 \), for \( k = n - 1, n \),
\[ R(z) = \frac{2z(z-1)(7-24z+16z^2)}{(2z-1)}. \]

So \( d = 4 \), \( Q(0) = -1 \) and \( P_d = 32 \).

By using equation (1) in theorem 3.4 we obtain
\[ \tau(G_n) = 2^4 \cdot 3^3 \cdot 7^h_n, \quad n \geq 2, \]
where \( f_n, g_n, \) and \( h_n \) are as claimed. Furthermore, it is easy to verify that the formula holds for all \( n \geq 0 \).

**Corollary 5.12.** The asymptotic growth constant for the hexagasket is
\[ c = \frac{2 \log 2}{5} + \frac{8 \log 3}{15} + \frac{\log 7}{45}. \]

There exist many other self-similar fractals to which our approach could be applied. As long as we have spectral decimation, then theorem 3.4 is valid. In the figure below are the \( G_1 \)
networks of three such examples of fractals admitting spectral decimation. For the two-dimensional Sierpiński gaskets of level-$k$, denoted as $SG_k$, we can see from inequality (5) that
\[
\frac{\log 3}{2} \leq c_k \leq \log \frac{6(k + 2)}{k + 4}.
\]
The last example in figure 4 is a variation of the unit interval called the $pq$-model studied in [45]. While there is obviously always only one spanning tree, a different Laplacian is defined which also admits spectral decimation and thus the product of its non-zero eigenvalues can be evaluated from theorem 3.4. One way to come up with more examples is by noting that any nested self-similar set satisfying definition 2.2 and whose boundary set has two or three fixed points admits spectral decimation. We refer the reader to [35] for further details regarding spectral decimation and our symmetry assumption.

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