A smooth shift approach for a Ramanujan expansion

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to oncoming smooth-numbers aficionados

Abstract. All arithmetical functions $F$ satisfying Ramanujan Conjecture, i.e., $F(n) \ll n^\varepsilon$, and with $Q$–smooth divisors, i.e., with Eratosthenes transform $F' \equiv F * \mu$ supported in $Q$–smooth numbers, have a kind of unique Ramanujan expansion; also, these Ramanujan coefficients decay very well to 0 and have two explicit expressions (in the style of Carmichael and Wintner). This general result, then, is applied to the shift-Ramanujan expansions, i.e., the expansions for correlations with respect to the shift, whence the title.

1. Introduction, statements and proofs of the results.

In the following, we fix $Q \in \mathbb{N}$ and indicate the set of $Q$–smooth (positive) integers writing

$$(Q) \equiv \{n \in \mathbb{N} : n = 1 \text{ or } p|n \Rightarrow p \leq Q\}$$

(now on $p$ denotes a prime, eventually with subscripts) and writing (as usual $(a,b) \equiv g.c.d.(a,b)$ now on)

$$(Q') \equiv \{n \in \mathbb{N} : (n, \prod_{p \leq Q} p) = 1\}$$

the set of $Q$–sifted (positive) integers. See that $(Q) \cap (Q') = \{1\}$ and $n \in (Q), m \in (Q)$ implies $(n,m) = 1$.

We need to define the $Q$–smooth restriction of any $F : \mathbb{N} \to \mathbb{C}$ as

$$F_{(Q)}(n) \equiv \sum_{d|n, d \in (Q)} F'(d), \forall n \in \mathbb{N},$$

where as usual $F'$ is the Eratosthenes transform $[W]$ of $F$, namely $F' \equiv F * \mu$. (See $[T]$ for $*$, Dirichlet product, and $\mu$, Möbius function.)

Notice, in passing, that the Eratosthenes transform of our $F_{(Q)}$, namely $(F_{(Q)})'$, thanks to

$$F_{(Q)}(n) = \sum_{d|n} F'(d)1_{(Q)}(d), \forall n \in \mathbb{N},$$

is nothing else than $F' \cdot 1_{(Q)}$, with $1_A$ the characteristic function of the set $A$. (Here $A= \{Q$–smooth n.s.$\})$

See the similarity of notation with $F_Q$, which is the $Q$–truncation of our $F$, namely we truncate its divisors after $Q$, i.e., the Eratosthenes transform, now, has support in $\{1, \ldots, Q\}$ (compare $[C2], [CMS]$ and $[CM]$). Our $Q$–smooth restriction has an infinity of divisors, while of course $F_Q$ has only at most $Q$ of them!

While the $Q$–truncations (i.e., truncated divisor sums) are strictly connected to finite Ramanujan expansions (see section 5 of $[C2]$ and compare $[CMS], [CM]$), here the $Q$–restrictions (i.e., restricted divisor sums) are linked, as we’ll see in a moment, to infinite, pointwise converging Ramanujan expansions!

For $F : \mathbb{N} \to \mathbb{C}$ we define $[C2]$ Carmichael’s coefficients (provided following limits exist) and Wintner’s coefficients (if following series converge), with $\varphi(q)$ the Euler function and $c_q(n)$ the Ramanujan sum $[R],[M]$:

$$\text{Car}_q(F) \equiv \frac{1}{\varphi(q)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} F(n)c_q(n), \forall q \in \mathbb{N},$$

$$\text{Win}_q(F) \equiv \sum_{d \equiv 0(\text{mod }q)} F'(d)/d, \forall q \in \mathbb{N}.$$

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The main limit of our Theorem 1 [CM] is that we need a hypothesis (we may choose among three) for the finite Ramanujan expansion of (suitable) shifted convolution sums (SCS); now, we don’t need it, simply considering not the original SCS, but restricting its divisors, as above for the finite Ramanujan expansion of (suitable) shifted convolution sums (SCS); now, we don’t need it, simply (Compare, for a more rigorous definition, [C2].)

Let

Proposition 1.

Theorem 1 [CM]; in fact, following Proposition is already “implicit”, in [CM] Theorem 1 Proof.

We wish to prove, before general results in our Theorem 1, a Proposition that regards suitable SCS, also called correlations, that will be applied in our Corollary 1 (for correlations).

Its, say, Basic Hypothesis needs two definitions.

We call a general arithmetic function \( g : \mathbb{N} \to \mathbb{C} \) “of range \( Q \)”, by definition, when \( g \) may be expressed through its Eratosthenes transform \( \hat{g} \), for a fixed \( Q \in \mathbb{N} \), as: (a truncated divisor sum!)

\[
g(m) \overset{\text{def}}{=} \sum_{d|m, d \leq Q} g'(d), \quad \forall m \in \mathbb{N}.
\]

(Compare, for a more rigorous definition, [C2].)

Once given a correlation (or Shifted Convolution Sum, SCS) of fixed \( f, g : \mathbb{N} \to \mathbb{C} \), i.e.,

\[
C_{f,g}(N, a) \overset{\text{def}}{=} \sum_{n \leq N} f(n)g(n+a), \quad \forall a \in \mathbb{N},
\]

where the “length”, \( N \in \mathbb{N} \), is fixed and the “shift”, \( a \in \mathbb{N} \), is our variable (so that the Eratosthenes transform of \( C_{f,g}(N, d) \) is \( C'_{f,g}(N, d) \overset{\text{def}}{=} \sum_{d|a} C_{f,g}(N, t)\mu(d/t) \), see Corollary 1), we say that

\[ C_{f,g}(N, a) \text{ is \,fair} \iff \text{dependence on the shift} \, a \text{\,is only inside} \, g^{'s} \, \text{argument} \, (n+a)
\]

(i.e. nor dependence on \( a \) inside \( f,g \), neither in their supports; esp., \( f_H \)'s correlation is not fair: [CM] end)

We prove very quickly a property of correlations, in “Basic Hypothesis”, i.e., the two hypotheses of Theorem 1 [CM]; in fact, following Proposition is already “implicit”, in [CM] Theorem 1 Proof.

Proposition 1. Let \( f, g : \mathbb{N} \to \mathbb{C} \) be such that

\[
\text{(BH)} \quad g \text{ is of range } Q \leq N \text{ and } C_{f,g}(N, a) \text{ is fair.}
\]

Then

(i) \( C_{f,g}(N, a) = \sum_{q \leq Q} \hat{g}(q) \sum_{n \leq N} f(n)c_q(n+a), \forall a \in \mathbb{N}, \) where \( \hat{g}(q) \overset{\text{def}}{=} \sum_{d \leq Q} \frac{g'(d)}{d}, \forall q \in \mathbb{N}; \)

(ii) \( C_{f,g}(N, a) \) is, with respect to \( a \in \mathbb{N}, \) periodic, whence bounded;

(iii) \( C_{f,g}(N, a) \) has coincident Carmichael and Wintner \( \ell-\)th coefficients: \( \hat{g}(q) \overset{\text{def}}{=} \sum_{n \leq N} f(n)c_q(n), \forall \ell \in \mathbb{N}. \)

Proof. Here, (i) follows from the \( g \) finite Ramanujan expansion \( g(n+a) = \sum_{q \leq Q} \hat{g}(q)c_q(n+a) \) of Ramanujan coefficients \( \hat{g}(q) \) as above, see [C2], beginning of section 5.

Then, from (i), together with fairness we get, since each \( c_q(n+a) \) is periodic modulo \( q \), with respect to \( a \), periodicity (with period dividing \( Q \overset{\text{def}}{=} \text{l.c.m.}(2, \ldots, Q) \), of course), w.r.t. \( a \), whence \( C_{f,g}(N, a) \) is bounded (w.r.t. \( a \)).

Finally, (iii) follows from the Delange 1987 Theorem [De87], in the equivalent form, given as Theorem 9 in [C2]: in fact, our \( C_{f,g}(N, a) \) is bounded, so bounded on average, as required by Th.9 first assumption and its second assumption is satisfied because \( C_{f,g}(N, a) \) has all the Carmichael coefficients, since by (i) and fairness we get (compare (CC) in [C2]), as \( \ell \)-th Carmichael coefficient of \( C_{f,g}(N, a) \),

\[
\frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} c_q(n) \sum_{q \leq Q} \hat{g}(q) \sum_{n \leq N} f(n)g(n+a) = \frac{1}{\varphi(\ell)} \sum_{q \leq Q} \hat{g}(q) \sum_{n \leq N} f(n) \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} c_q(n)g(n+a),
\]

whence (iii), applying the orthogonality of Ramanujan sums (proved by Carmichael in 1932 [Ca], see Theorem 1 in [M]).
We come to our main result. (Hereafter \(\omega(d)\) is the number of prime factors of \(d\).)

**Theorem 1.** Let \(F : \mathbb{N} \to \mathbb{C}\) satisfy Ramanujan Conjecture and fix an integer \(Q > 1\). Then

(i) \(\text{Car}_t(F_Q) = \text{Win}_t(F_Q), \forall t \in \mathbb{N}\) and in particular \(\text{Car}_t(F_Q) = \text{Win}_t(F_Q) = 0, \forall t \not\in (Q)\);

(ii) \(\text{Car}_t(F_Q) = \prod_{p \leq Q} \left(1 - \frac{1}{p}\right) \sum_{t \in (Q)} F(t) c_t(t) \varphi(t), \forall t \in (Q), \) where \(\sum_{t \in (Q)} \frac{|F(t)c_t(t)|}{t} < \infty, \forall t \in \mathbb{N};\)

(iii) \(F_Q(a) = \sum_{t \in (Q)} \left(\prod_{p \leq Q} \left(1 - \frac{1}{p}\right) \sum_{t \in (Q)} F(t) c_t(t) \varphi(t)\right) c_a(a) = \sum_{t \in (Q)} \left(\sum_{d \in (Q) \atop \delta \equiv 0(\text{mod } t)} \frac{F'(d)}{d}\right) c_a(a), \forall a \in (Q), \)

whence, in particular,

\((RE) \quad F(a) = \sum_{t \in (Q)} \left(\prod_{p \leq Q} \left(1 - \frac{1}{p}\right) \sum_{t \in (Q)} F(t) c_t(t) \varphi(t)\right) c_a(a) = \sum_{t \in (Q)} \left(\sum_{d \in (Q) \atop \delta \equiv 0(\text{mod } t)} \frac{F'(d)}{d}\right) c_a(a), \forall a \in (Q);\)

(iv) the Ramanujan coefficients \(\hat{F}_Q^{(t)}(\ell) \overset{\text{def}}{=} \text{Car}_t(F_Q)\) satisfy \(\sum_{\ell = 1}^{\infty} 2^{\omega(t)} |\hat{F}_Q^{(t)}(\ell)| < \infty;\)

(v) \(F_Q(a) = \sum_{\ell = 1}^{\infty} R_Q(F, t) c_a(a), \forall a \in \mathbb{N}\) and (iv) holds for \(R_Q(F, t) \Rightarrow \text{Car}_t(F_Q) = \hat{F}_Q^{(t)}(\ell), \forall t \in \mathbb{N}.\)

**Proof.** Before going on, recall from the definition that the Wintner coefficients of \(F_Q\) are

\[\text{Win}_t(F_Q) = \sum_{d \in (Q) \atop \delta \equiv 0(\text{mod } t)} \frac{F'(d)}{d},\]

in which, of course, the condition \(\ell/d\) implies, since \(d \in (Q)\), that \(\ell \in (Q)\); otherwise, the coefficient vanishes. So we are left with the task to prove coincidence of Carmichael and Wintner \(\ell\)–th coefficients, for all \(t \in \mathbb{N}\). We start proving this part of (i). This can be done, proving the Delange Hypothesis

\[(DH) \quad \sum_{d = 1}^{\infty} \frac{2^{\omega(d)} |(F_Q)'(d)|}{d} < \infty\]

because Delange 1976 Theorem [De] infers from \((DH)\) both the identity of Carmichael & Wintner coefficients, i.e. (i) (for what we saw above), and the convergence, of corresponding Ramanujan expansion; thus proving, after we prove (ii), also (iii). In order to prove \((DH)\) above, recall \((F_Q)'(d) = F'(d) 1_{(Q)}(d)\) so (hereafter we use classic notation, Vinogradov’s \(\ll\) and Landau’s \(O\), like \(\pi(Q) \overset{\text{def}}{=} |\{p \leq Q\}|\), see [Da])

\[\sum_{d = 1}^{\infty} \frac{2^{\omega(d)} |(F_Q)'(d)|}{d} = \sum_{d \in (Q)} \frac{2^{\omega(d)} |F'(d)|}{d} \ll 2^{\pi(Q)} \sum_{d \in (Q)} \frac{|F'(d)|}{d} \ll_{Q, \varepsilon} \sum_{d \in (Q)} d^{\varepsilon - 1} < \infty,\]

where the Ramanujan Conjecture satisfied by \(F\) implies the same for \(F'\) and, then, we apply Lemma 3 (see next §2).

We have proved both (i) and (iii), once we prove (ii), too.

For (ii) we start proving the absolute convergence:

\[\sum_{t \in (Q)} \frac{|F(t)c_t(t)|}{t} \ll_{\varepsilon} \sum_{t \in (Q)} (\ell, t)t^{\varepsilon - 1} \ll_{\varepsilon} \sum_{\ell \in (Q) \atop d|\ell} d \sum_{t \in (Q) \atop \ell \equiv 0(\text{mod } d)} t^{\varepsilon - 1} \ll_{\varepsilon} \sum_{d \in (Q) \atop \ell \equiv 0(\text{mod } d)} d^{\varepsilon} \sum_{K \in (Q)} K^{\varepsilon - 1} \ll_{Q, \varepsilon, \ell} 1,\]
using the inequality $|c_\ell(t)| \leq (\ell, t)$ (see Lemma A.1 in [CM]) and Lemma 3 at §2. We have left to prove the formula for Carmichael $\ell$–th, once $\ell \in (Q)$. Starting with the definition for these coefficients and adding Möbius switch, namely Lemma 1 in next §2,

$$\text{Car}_\ell(F_{(Q)}) = \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \left[ \sum_{a \leq x} F(t \ell) a \left( \frac{1}{x} \right) \right]$$

true $\forall \ell \in \mathbb{N}$; however, assuming $\ell \in (Q)$ now, we use the fact that $K \in (Q)$ to get $(\ell, K) = 1$, whence $c_\ell(tK) = c_\ell(t)$, $\forall t \in \mathbb{N}$, getting from the count in Lemma 2 (§2)

$$\text{Car}_\ell(F_{(Q)}) = \frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \left[ \sum_{t \leq x} F(t)c_\ell(t) \cdot \frac{1}{t} \left( 1 - \frac{1}{p} \right) \prod_{p \leq Q} \left( \frac{1}{p} \right) \right],$$

in which, using the absolute convergence just proved, we have convergence of main term, i.e.

$$\frac{1}{\varphi(\ell)} \lim_{x \to \infty} \frac{1}{x} \left[ \sum_{t \leq x} F(t)c_\ell(t) \cdot \frac{1}{t} \left( 1 - \frac{1}{p} \right) \prod_{p \leq Q} \left( \frac{1}{p} \right) \right] = \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \frac{1}{\varphi(\ell)} \sum_{t \leq x} F(t)c_\ell(t) \cdot \frac{1}{t},$$

so, we’re left with proving that remainders don’t count, as next term is infinitesimal with $x \to \infty$:

$$\sum_{t \leq x} F(t)c_\ell(t)O_Q \left( \frac{1}{x} \right) \ll_{Q, \varepsilon} x^{-\varepsilon - 1} \sum_{t \leq x} (\ell, t) \ll_{Q, \varepsilon} x^{-\varepsilon - 1} \sum_{t \leq x} d \ll_{Q, \varepsilon} x^{-\varepsilon - 1} \sum_{t \leq x} d \left( \frac{x}{d} \right)^\varepsilon \ll_{Q, \varepsilon, t} x^{2\varepsilon - 1},$$

applying, in penultimate step, the bound of Lemma 2 (§2) and recalling $\varepsilon > 0$ is small, finally proving $(ii)$.

We come to proving $(iv)$, now.

Since

$$\sum_{t \leq x} 2^{\omega(\ell)}|F_{(Q)}(\ell)| = \sum_{t \leq x} 2^{\omega(\ell)} \left\| \sum_{d \mid (0)} \frac{F'(d)}{d} \right\| \leq 2 \pi(Q) \sum_{t \leq x} \sum_{d \mid (0)} \frac{\left\| F'(d) \right\|}{d} \ll_{Q, \varepsilon} \sum_{t \leq x} \sum_{K \in (Q)} K^{\varepsilon - 1},$$

we prove, here, even more than $(iv)$, thanks to Lemma 3 (at §2), again.

We need only to prove $(v)$, a kind of “uniqueness”, for the Ramanujan expansion we found (with Carmichael coefficients = Wintner coefficients). It follows from Theorem 4 of [C2], recalling $(iv)$ is a kind of “Dual Delange”, as we call it in [C2], assumption.

An immediate consequence, thanks also to our previous Proposition, is the following Corollary.

**Corollary 1.** Given $f : \mathbb{N} \to \mathbb{C}$ and $g : \mathbb{N} \to \mathbb{C}$, satisfying the Basic Hypothesis (BH) above, then, for $G_{(Q), f,g,N}(a) = G(a) = \sum_{d \mid a} C_{f,g}(N, d) 1_{(Q)}(d), \forall a \in \mathbb{N}$, we have the following RAMANUJAN EXPANSION

$$G(a) = \sum_{t \in (Q)} \left( \sum_{d \mid (0)} \frac{C_{f,g}(N, d)}{d} \right) c_\ell(a) = \sum_{t \in (Q)} \left( \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \frac{1}{\varphi(\ell)} \right) \sum_{t \in (Q)} \left( 1 - \frac{1}{p} \right) \frac{1}{\varphi(\ell)} \sum_{t \in (Q)} \frac{C_{f,g}(N, t)c_\ell(t)}{t},$$

whence, in particular, $\forall a \in (Q)$,

$$\sum_{t \in (Q)} \frac{C_{f,g}(N, d)}{d} c_\ell(a) = \sum_{t \in (Q)} \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \frac{1}{\varphi(\ell)} \sum_{t \in (Q)} \frac{C_{f,g}(N, t)c_\ell(t)}{t}.$$
The “Dual Delange” assumption on the Ramanujan coefficients, say \( \hat{G}(\ell) \), of \( G \):

\[(DD) \quad \sum_{\ell=1}^{\infty} 2^{\omega(\ell)}|\hat{G}(\ell)| < \infty \]

holds for the coefficients above (which vanish outside of \( Q \)-smooth numbers) and ONLY for them: IF \( \hat{G}(a) = \sum_{\ell=1}^{\infty} \hat{G}(\ell)c_\ell(a), \forall a \in \mathbb{N} \) AND \( (DD) \) HOLDS

THEN

\[ \hat{G}(\ell) = \sum_{\substack{d \in (Q) \\
 d \equiv 0 \bmod \ell}} \frac{C'_{f,g}(N,d)}{d} = \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \frac{1_{(Q)}(\ell)}{\varphi(\ell)} \sum_{t \in (Q)} \frac{C_{f,g}(N,t)c_t(t)}{t}, \quad \forall \ell \in \mathbb{N}. \]

Furthermore,

\[ \sum_{\substack{d \in (Q) \\
 d \equiv 0 \bmod \ell}} \frac{C'_{f,g}(N,d)}{d} = \hat{g}(\ell) \sum_{n \leq N} f(n)c_\ell(n) - \sum_{\substack{d \notin (Q) \\
 d \equiv 0 \bmod \ell}} \frac{C'_{f,g}(N,d)}{d}, \quad \forall \ell \in \mathbb{N}, \]

which, in particular, gives

\[ \sum_{\substack{d \notin (Q) \\
 d \equiv 0 \bmod \ell}} \frac{C'_{f,g}(N,d)}{d} = \hat{g}(\ell) \sum_{n \leq N} f(n)c_\ell(n), \quad \forall \ell \notin Q. \]

**Proof.** The first part (i.e., up to the uniqueness) comes from Theorem 1 with \( F(a) = C_{f,g}(N,a) \), since this \( F \) is bounded, from \((ii)\) of Proposition 1 thanks to \((BH)\).

Then we apply again \((BH)\), implying

\[ \sum_{d \equiv 0 \bmod \ell} \frac{C'_{f,g}(N,d)}{d} = \hat{g}(\ell) \sum_{n \leq N} f(n)c_\ell(n), \quad \forall \ell \in \mathbb{N}, \]

from \((iii)\) of Proposition 1. Then, we may separate \( d \in (Q) \) and \( d \notin (Q) \) series, thanks to absolute convergence in Wintner coefficients, say, with Eratosthenes transform restricted to \( Q \)-smooth numbers, i.e.,

\[ (*) \quad \sum_{\substack{d \in (Q) \\
 d \equiv 0 \bmod \ell}} \frac{|C'_{f,g}(N,d)|}{d} < \infty; \]

in fact, \( C_{f,g}(N,a) \) bounded \( \Rightarrow C'_{f,g}(N,d) \ll_{N,Q,\varepsilon} d^\varepsilon \), whence:

\[ \sum_{\substack{d \in (Q) \\
 d \equiv 0 \bmod \ell}} \frac{|C'_{f,g}(N,d)|}{d} \ll_{N,Q,\varepsilon} \sum_{\substack{d \in (Q) \\
 d \equiv 0 \bmod \ell}} d^{\varepsilon-1} \ll_{N,Q,\varepsilon} \ell^{\varepsilon-1} \sum_{K \in (Q)} K^{\varepsilon-1} \ll_{N,Q,\varepsilon} \ell^{\varepsilon-1} \ll_{N,Q,\varepsilon} 1, \]

implying \((*)\) above, from Lemma 3 at next §2. \( \square \)

We give the Lemmas used above, in next §2; then, in §3 a kind of new orthogonality relations for Ramanujan sums provides a new approach to Theorem 1, see Proposition 2; which, in turn, suggests a new Conjecture (for the Reef [C2]), implying Hardy-Littlewood one [C1], in §4; finally, §5 gives further remarks.
2. Lemmas.

We give a page of Lemmas for our proofs.

First Lemma is “Möbius switch”.

**Lemma 1.** For any \( F : \mathbb{N} \to \mathbb{C} \) we have \( \sum_{d \mid n} F'(d) = \sum_{t \mid Q} F(t) \), \( \forall a \in \mathbb{N} \).

**Proof.** From the definition of Eratosthenes transform,

\[
F'(d) = \sum_{t \mid d} F(t) \mu \left( \frac{d}{t} \right) \Rightarrow \sum_{d \mid Q} F'(d) = \sum_{t \mid Q} F(t) \sum_{K \mid (Q/t)} \mu(K).
\]

The characteristic function of \( Q \) is \( 1_Q \), multiplicative, so the thesis comes from

\[
\sum_{K \mid (Q/t)} \mu(K) = \sum_{K \mid n} \mu(K) 1_Q(K) = \prod_{p \mid n} (1 - 1_{Q_p}(p)) = 1_{Q,n}(n), \ \forall n \in \mathbb{N}.
\]

Our next Lemma bounds the \( n \in \mathbb{N} \) that are \( Q \)-smooth and counts those which are \( Q \)-sifted.

**Lemma 2.** As \( x \to \infty \), \( \sum_{n \leq x, n \in (Q)} 1 \ll_{Q,x} x^\varepsilon \) and \( \sum_{n \leq x, n \in (Q)} 1 = \prod_{p \leq Q} (1 - 1/p) x + O_Q(1) \).

**Proof.** We may represent (in a unique way) any \( n \in (Q) \) as \( n = p_1^{K_1} \cdots p_r^{K_r} \), where \( 2 = p_1 < p_2 < \cdots < p_r \) are consecutive prime numbers, \( K_j \geq 0 \) are integers \( \forall j \leq r \) and this \( r \) is \( \pi(Q) \) (number of \( p \leq Q \)). Then, by “Rankin’s trick”, \( \forall \varepsilon > 0 \) we have

\[
\sum_{n \in (Q)} 1 \leq \sum_{n \in (Q)} \frac{x^\varepsilon}{n^\varepsilon} \ll x^\varepsilon \sum_{K_1=0}^\infty \cdots \sum_{K_r=0}^\infty (p_1^{-\varepsilon})^{K_1} \cdots (p_r^{-\varepsilon})^{K_r} = x^\varepsilon \prod_{p \leq Q} \frac{1}{1-p^{-\varepsilon}} \ll_{Q,x} x^\varepsilon.
\]

This proves the bound.

On the other side, abbreviating \( P_Q := \prod_{p \leq Q} P \), the condition \( (n, P_Q) = 1 \) is detected by \( \sum_{d \mid n, d \mid P_Q} \mu(d) \):

\[
\sum_{n \in (Q)} 1 = \sum_{d \mid P_Q} \mu(d) \left[ \frac{x}{d} \right] = \sum_{d \mid P_Q} \mu(d) \frac{x}{d} + O \left( \sum_{d \mid P_Q} \mu^2(d) \right) = \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) x + O_Q(1).
\]

Our last Lemma, the core of our arguments, gives an estimate for a series restricted to \( Q \)-smooth numbers (badly diverging, without restrictions), that we’ll use many times. (As usual, we assume \( \varepsilon > 0 \).)

**Lemma 3.** For all \( 0 < \varepsilon < 1 \) we get

\[
\sum_{m \in (Q)} m^{\varepsilon - 1} \ll_{Q,x} 1.
\]

**Proof.** Representing as above the \( m \in (Q) \),

\[
\sum_{m \in (Q)} m^{\varepsilon - 1} = \sum_{K_1=0}^\infty \cdots \sum_{K_r=0}^\infty (p_1^{\varepsilon - 1})^{K_1} \cdots (p_r^{\varepsilon - 1})^{K_r} = \prod_{p \leq Q} \frac{1}{1-p^{-\varepsilon}} \ll_{Q,x} 1.
\]
3. Smooth-Twisted Orthogonality.
We give a kind of orthogonality relations (for Ramanujan sums) which are, so to speak, smooth-twisted, i.e., they contain a kind of twist, namely the indicator function of smooth numbers (with a factor at the denominator); see that the two variables expressing the orthogonality have both to live in smooth numbers. In fact, with this restriction, the LHS (left hand side) in next result is meaningful.

We state and prove the “Smooth-Twisted Orthogonality”. It provides another approach to Theorem 1.

**Proposition 2.** Let \( q, \ell \in (Q) \). Then

\[
\frac{1}{t} \sum_{t \in (Q)} \frac{c_q(t)c_{\ell}(t)}{t} = \varphi(\ell) 1_{q=\ell}.
\]

**Remark.** We ask diligent readers to prove 4 exercise : absolute convergence in LHS with Ramanujan sums.

**Proof.** Representing the denominator in LHS as

\[
\sum_{t \in (Q)} \frac{1}{t} = \sum_{K_1=0}^{\infty} \cdots \sum_{K_r=0}^{\infty} (p_1^{-1})^{K_1} \cdots (p_r^{-1})^{K_r} = \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right)^{-1} = \left( \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \right)^{-1},
\]

from the representation of numbers \( t \in (Q) \), compare Lemma 2 proof, we are left with proving

\[
\prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \sum_{t \in (Q)} \frac{c_q(t)c_{\ell}(t)}{t} = 1_{q=\ell} \varphi(\ell).
\]

This is a straight task, applying elementary properties like (see [M] and [T])

\[
c_q(t) = \sum_{q^{|t} \mid q} q' \mu(q/q') \quad \text{and} \quad \sum_{d \mid n} \varphi(d) = n,
\]

with \( n = (\ell', q') \), in the following, so to get

\[
\prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \sum_{t \in (Q)} \frac{c_q(t)c_{\ell}(t)}{t} = \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \sum_{q' \mid q} \mu \left( \frac{q}{q'} \right) \sum_{\ell' \mid \ell} \mu(\ell'/\ell) \sum_{K \in (Q)} \frac{c_{q'K}}{K} =
\]

\[
= \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \sum_{q' \mid q} \mu \left( \frac{q}{q'} \right) \sum_{\ell' \mid \ell} \mu(\ell'/\ell) \sum_{K \in (Q)} \frac{1}{K} =
\]

\[
= \prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \sum_{q' \mid q} \mu \left( \frac{q}{q'} \right) \sum_{\ell' \mid \ell} \mu(\ell'/\ell) \sum_{K \in (Q)} \frac{1}{K} =
\]

\[
= \sum_{q' \mid q} \mu \left( \frac{q}{q'} \right) \sum_{\ell' \mid \ell} \mu(\ell'/\ell) \sum_{K \in (Q)} \frac{1}{K} =
\]

\[
= \sum_{q' \mid q} \mu \left( \frac{q}{q'} \right) \sum_{\ell' \mid \ell} \mu(\ell'/\ell) \sum_{K \in (Q)} \frac{1}{K} =
\]

\[
= \varphi(\ell) 1_{\ell \mid q} \sum_{q' \mid q} \mu \left( \frac{q}{q'} \right) = \varphi(\ell) 1_{\ell \mid q} \sum_{q' \mid q} \mu \left( \frac{q}{q'} \right) = 1_{q=\ell} \varphi(\ell),
\]

since [T] Möbius inversion \( \sum_{d \mid n} \mu(d) = 1_{(1)}(n) \) is applied twice. Thus (**) is completely proved. □
4. Twisted Orthogonality Conjecture.

Previous Proposition 2 suggests in a very natural way the following “Twisted Orthogonality Conjecture”.

**Conjecture 1.** Let $q, \ell \in (Q)$ and $n \in \mathbb{Z}$. Then

$$\frac{1}{\ell} \sum_{t \in (Q)} \frac{c_t(n + \ell)c_{\ell}(t)}{t} = 1_{q=t}c_{\ell}(n).$$

**Remark.** If we take any $n \equiv 0(\text{mod } q)$ we get previous result on Ramanujan sums.

However, this seemingly innocent Conjecture has deep consequences on correlations and their Ramanujan expansions; in fact, it is a Big Conjecture, since it implies in two lines (see next Proposition), say, in a crystal clear way, the Reef: which is, see [C2], the most important formula for correlations. This Reef we get from Conjecture 1, actually, is a weaker version of the one we found in [CM] (and that we study in [C2]), i.e., it is an example of “restricted Reef”, see [C2]; however its strength is rather unaltered by restricting (by the way to $Q$—smooth shifts) as in fact it still implies (in case of von Mangoldt function $f = g = \Lambda$) the famous Hardy-Littlewood Conjecture (see [C1] for this)!

We get the following “Fair Correlations Conjecture”.

**Conjecture 2.** Let $f, g$ satisfy (BH). Then we have for their correlation the $Q$—smooth restricted Reef

$$(Q) – \text{REEF} : 
C_{f,g}(N, a) = \sum_{\ell \leq Q} \left( \frac{\bar{g}(\ell)}{\varphi(\ell)} \sum_{n \leq N} f(n)c_{\ell}(n) \right) c_{\ell}(a), \forall a \in (Q).$$

It is very easy to prove that Conjecture 1 implies Conjecture 2. We now prove this, as follows.

**Proposition 3.** Let $f, g$ satisfy (BH) and let Conjecture 1 hold. Then we have the $(Q)–\text{REEF}$.

**Proof.** Use (i) of Proposition 1 to expand $C_{f,g}(N, t)$ in its $\ell$—th coefficient in Corollary 1 (SCS):

$$\prod_{p \leq Q} \left( 1 - \frac{1}{p} \right) \frac{1}{\varphi(\ell)} \sum_{t \in (Q)} C_{f,g}(N, t)c_{\ell}(t) = \frac{1}{\ell} \sum_{t \in (Q)} \frac{1}{\varphi(\ell)} \sum_{q \leq Q} \bar{g}(q) \sum_{n \leq N} f(n) \sum_{t \in (Q)} \frac{c_q(n + t)c_{\ell}(t)}{t},$$

proving: $(Q)–\text{REEF}$ (where $\ell \leq Q$ comes from $\bar{g}$ support), thanks to Conjecture 1, because the finite product is the reciprocal of the $Q$—smooth reciprocity series, see first line of Proposition 2 proof.

The restriction of the Reef to $Q$—smooth shifts, actually, has no effect on the beginning of the Proof of Corollary 2 in [C1], whence our Conjecture 2 with $f = \Lambda, g = \Lambda_N$ (the $N$—truncation of $\Lambda$ : see [C1]) can prove Hardy-Littlewood Conjecture, in the same style of [C1] Corollary 2 proof. This time, the only condition these $f, g$ have to satisfy is (BH) (no more (DH), too, like in [C1]), which is “for free”, as we know: so, the only requirement is that Conjecture 1 holds.

Following “conditional Proof of Hardy-Littlewood Conjecture”, in fact, is proved with Proposition 3.

**Proposition 4.** Under Conjecture 1, the Hardy-Littlewood Conjecture follows, like in Corollary 2 of [C1].

As you can see, Conjecture 1 is really strong ! Maybe, too strong and we may doubt about it. The first attempts trying to prove it gave no answer, as the same proof we gave for Proposition 2 of course doesn’t work (because of the additional $n$); however, maybe we could get something trying to rewrite it in terms of exponential sums (that’s the richness of Ramanujan sums: having at least two expressions living in different worlds)!

Of course, two reasons supporting Conjecture 1 are the Remark after it and the similarity with the analogue orthogonality relations due to Carmichael, but with the additional shift $n$ in one of the two Ramanujan sums, compare Theorem 1 [M] for this !
5. Further remarks.
The present results have, of course, applications to our study in [C1], [C2], [CL] and in the series of papers starting with [CMS], [CM]. In particular, they may be applied to averages of correlations (see [CL]) and to single correlations [C2], [CM], with a more expected success (for reasons that we’ll explain in future papers) for the averages (having, see [CL], a big impact on moments of the Riemann \(\zeta\)–function on the critical line).

For a more extensive discussion on these arguments, compare especially Generations [CL] and [CM]. For remarks on the Ramanujan expansion coefficients and their decay see [C2] and [CM]. Last but not least, for applications to conditional proofs of Hardy-Littlewood Conjecture, compare [C1].

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P.S.: I apologize as previous version had 7 pages, not 9 like in comments but now, never too late, it is true!