ON A PREDATOR-PREY SYSTEM WITH RANDOM SWITCHING THAT NEVER CONVERGES TO ITS EQUILIBRIUM

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Abstract. We study the dynamics of a predator-prey system in a random environment. The dynamics evolves according to a deterministic Lotka-Volterra system for an exponential random time after which it switches to a different deterministic Lotka-Volterra system. This switching procedure is then repeated. The resulting process switches between two deterministic Lotka-Volterra systems at exponential times and therefore is a Piecewise Deterministic Markov Process (PDMP). In the case when the equilibrium points of the two deterministic Lotka-Volterra systems coincide we show that almost surely the trajectory does not converge to the common deterministic equilibrium. Instead, with probability one, the densities of the prey and the predator oscillate between 0 and ∞. This proves a conjecture of Takeuchi et al (J. Math. Anal. Appl 2006).

The proof of the conjecture is a corollary of a result we prove about linear switched systems. Assume \((Y_t, I_t)\) is a PDMP that evolves according to \(\frac{dY_t}{dt} = A_{I_t}Y_t\) where \(A_0, A_1\) are \(2 \times 2\) matrices and \(I_t\) is a Markov chain on \(\{0, 1\}\) with transition rates \(k_0, k_1 > 0\). If the matrices \(A_0\) and \(A_1\) are not proportional and are of the form
\[
A_i := \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix},
\]
with \(a_i^2 + b_i c_i < 0\), then the average growth rates of \(\|Y_t\|\) are all equal and strictly positive. In particular, almost surely \(\lim_{t \to \infty} \|Y_t\| = +\infty\).

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1. Introduction and main results

One of the key issues in ecology is determining when species will persist and when they will go extinct. The randomness of the environment makes the dynamics of populations inherently stochastic. Therefore, one needs to take into account the combined effects of biotic interactions and environmental fluctuations. One way of doing this is by modelling the species as Markov processes and looking at the long-term behavior of these processes (see [Che00 ERSS13 EHS15 LES03 SLS09 SBA11 BEM07 BS09 BHS08 CM10 HNY17 HN17a]).

In order to allow for environmental fluctuations and their effect on the persistence or extinction of species one approach is to study the uniform persistence for non-autonomous differential equations ([Thi00 ST11 MSZ04]). A different approach is to consider systems that have random environmental perturbations. One way to do this is by studying stochastic differential equations ([ERS S13 SBA11 HNY17 HN17a HN17c HN17b]). The other possible way is looking at stochastic equations driven by a Markov chain. These systems are sometimes called Piecewise Deterministic Markov Processes (PDMP) or systems with telegraph noise.

PDMPs have been used recently to prove some very interesting facts about biological populations. In [BL16] the authors look at a two dimensional Lotka-Volterra system in a fluctuating environment. They show that the random switching between two environments that are both favorable to the same species can lead to the extinction of this favored species or to the coexistence of the two competing species (also see [MH16]). PDMPs are also used in [Cos16] where the author studies prey-predator communities where the predator population evolves much faster than the prey.

For a predator-prey system the classical deterministic example is the Lotka-Volterra model (see [Lot25] and [Vol28])

\[
\begin{align*}
\frac{dx(t)}{dt} & = x(t)(a - by(t)), \\
\frac{dy(t)}{dt} & = y(t)(-c + dx(t)),
\end{align*}
\]

(1.1)

where \(x(t), y(t)\) are the densities of the prey and the predator at time \(t \geq 0\) and \(a, b, c\) and \(d\) are positive constants. If one assumes that \(x(0) = x_0 > 0, y(0) = y_0 > 0\), so that both predator and prey are present, then the solutions of system (1.1) are periodic (see [GH75 HS98]). One should note that both the predator and the prey from (1.1) do not experience intraspecific competition. In particular, if the predator is not present (i.e. \(y_0 = 0\)) then the prey density blows up to infinity. [GH75, MHP14] are able to analyze the \(n\)-dimensional generalization of (1.1) i.e. the setting when one has one prey and \(n-1\) predators and each species interacts only with the adjacent trophic levels. Stochastic predator-prey models have been studied in the stochastic differential equation setting by [Rud03 HN17c HN17b]. However, we note that in all these studies one needed to assume that there exists intraspecific competition among the prey and the predators. This simplifies the analysis significantly because the predator and the prey densities get pushed towards the origin when they become too large.
In [TDHS06] the authors consider a random switching between two Lotka-Volterra prey-predator systems of the form (1.1). More precisely, for \( i \in E := \{0, 1\} \), let 
\[ F^i : \mathbb{R}_+^2 \to \mathbb{R}_+^2 \]
 denote the vector field

\[
F^i(x, y) = \begin{pmatrix}
    x(a_i - b_i y) \\
y(-c_i + d_i x)
\end{pmatrix}
\]

with \( a_i, b_i, c_i, d_i > 0 \). Let \((I_t)_{t \geq 0}\) be a continuous-time Markov Chain defined on some probability space \((\Omega, \mathcal{F}, P)\) and taking values in \( E := \{0, 1\}\). Suppose \( I_t \) has transition rates \( k_0, k_1 > 0 \). Throughout the paper we will let \( \mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\} \) and \( \mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\} \). We denote by \((X_t)_{t \geq 0} = (x_t, y_t)_{t \geq 0}\) the solution of

\[
\frac{dx_t}{dt} = x_t(a_{I_t} - b_{I_t} y_t), \quad \frac{dy_t}{dt} = y_t(-c_{I_t} + d_{I_t} x_t)
\]

for some initial condition \( X_0 = (x_0, y_0) \in \mathbb{R}_+^2 \). In particular, we note that there is no intraspecific competition for prey or predators. The process \((X, I) = (X_t, I_t)_{t \geq 0}\) is a Piecewise Deterministic Markov Process as introduced in [Dav84], and belongs to the more specific class of PDMPs recently studied in [BH12] and [BLBMZ15].

The process \((X, I)\) is constructed as follows: Suppose we start at \((X_0, I_0) = ((x_0, y_0), i)\). Then, the system evolves according to

\[
\frac{dx_i(t)}{dt} = x_i(t)(a_i - b_i y_i(t)), \quad \frac{dy_i(t)}{dt} = y_i(t)(-c_i + d_i x_i(t))
\]

for an exponential random time \( T_i \) with rate \( k_i \). After this time the Markov chain \( I \) jumps from state \( i \) to state \( j \in \{0, 1\} \setminus \{i\} \) and \( X_t \) evolves according to

\[
\frac{dx_j(t)}{dt} = x_j(t)(a_j - b_j y_j(t)), \quad \frac{dy_j(t)}{dt} = y_j(t)(-c_j + d_j x_j(t))
\]

for an exponential random time \( T_j \) with rate \( k_j \). This procedure then gets repeated. Intuitively our process follows an ODE for an exponential random time after which it switches to a different ODE, follows that one for an exponential random time and so on.
The generator $L$ of $(X, I)$ acts on functions $g : \mathbb{R}_+^2 \times E \to \mathbb{R}$ that are smooth in the first variable as

$$Lg(x, i) = \langle F^i(x), \nabla g(x) \rangle + k_i (g(x, 1 - i) - g(x, i)),$$

where $\langle \cdot, \cdot \rangle$ is the euclidean inner product on $\mathbb{R}^2$. As usual, for $x \in \mathbb{R}^2$ and $i \in E$, we denote by $P_{x,i}$ the law of the process $(X, I)$ when $(X_0, I_0) = (x, i)$ almost surely and by $E_{x,i}$ the associated expectation.

The vector field $F^i$ from [1.2] has a unique positive equilibrium $(p_i, q_i) = (c_i/d_i, a_i/b_i)$.

In [TDHS06] the authors look at the two cases

Case I. $p_0 = p_1 =: p$ and $q_0 = q_1 =: q$, i.e. common zero for $F^0$ and $F^1$,

Case II. $(p_0, q_0) \neq (p_1, q_1)$, i.e. different zeroes for $F^0$ and $F^1$.

We assume throughout this paper that $p_0 = p_1 =: p$ and $q_0 = q_1 =: q$. The vector fields $F^0$ and $F^1$ therefore have a common zero - this will allow us to use the recent results from [BS17]. We also assume that $F^0$ and $F^1$ are non collinear to avoid trivial switching.

In [TDHS06, Theorem 4.5] it is shown that only two long term behaviours are possible when the vector fields have a common zero: either $X_t$ converges almost surely to the common equilibrium $(p, q)$, or each coordinate oscillates between 0 and $+\infty$.

**Theorem 1.1** (Takeuchi et al., 2006). For any $(x_0, y_0) \in \mathbb{R}_+^2$, with probability 1, either

$$\lim_{t \to \infty} X_t = (p, q),$$

or

$$\limsup_{t \to \infty} x_t = \limsup_{t \to \infty} y_t = +\infty, \quad \liminf_{t \to \infty} x_t = \liminf_{t \to \infty} y_t = 0.$$

It was conjectured from simulations (see [TDHS06, Remark 5.1]) that only case 1.5 happens in the above theorem. Using Theorem 2.1 below and results from [BS17], we are able to prove this conjecture.

**Theorem 1.2.** There exist $\varepsilon > 0$, $b > 1$, $\theta > 0$ and $c > 0$ such that for all $x := (x_0, y_0) \in \mathbb{R}_+^2 \setminus \{(p, q)\}$ and $i \in E$,

$$E_{x,i}(b^{\tau^x}) \leq c (1 + \|x - (p, q)\|^{-\theta})$$

where $\tau^x := \inf\{t \geq 0 : \|X_t - (p, q)\| \geq \varepsilon\}$. In particular, for any $(x_0, y_0) \in \mathbb{R}_+^2 \setminus \{(p, q)\}$ we have with probability 1 that

$$\limsup_{t \to \infty} x_t = \limsup_{t \to \infty} y_t = +\infty, \quad \liminf_{t \to \infty} x_t = \liminf_{t \to \infty} y_t = 0.$$

Actually, thanks to Theorem 2.1, the first part of Theorem 1.2 can be generalised as follows. For $i \in E$, let $F^i$ be a vector field of class $C^2$ on $\mathbb{R}^2$, such that $F^i(0) = 0$. Also assume that for $i \in E$, $DF^i(0)$, the Jacobian matrix of $F$ at 0, has two purely imaginary eigenvalues. In this case, the equilibrium 0 is sometimes called a center. We now consider a Markov process $(X_t, I_t)_{t \geq 0}$ where $X_t$ is solution of

$$\frac{dX_t}{dt} = F^{I_t}(X_t)$$
and \( I_t \) is a jump process on \( E \) whose rates depend on \( X \)
\[
P(I_{t+s} = 1 - i | I_t = i, F_t) = k_{i, 1-i} (X_t) s + o(s),
\]
where \( F_t = \sigma((X_s, I_s) : s \leq t) \) and for all \( x \), \( (k_{ij}(x))_{i,j} \) is an irreducible matrix that is continuous in \( x \). The process \((X, I)\) is still a PDMP, with infinitesimal generator \( L \) acting on functions \( g : \mathbb{R}^2 \times E \to \mathbb{R} \) that are smooth in the first variable as
\[
L g(x, i) = \langle F^i(x), \nabla g^i(x) \rangle + k_{i, 1-i} (x) (g(x, 1-i) - g(x, i)).
\]

We can prove the following (see Remark 3.1) in this more general setting.

**Theorem 1.3.** Assume \( DF^0(0) \) and \( DF^1(0) \) are non proportional. Then there exist \( \varepsilon > 0, b > 1, \theta > 0 \) and \( c > 0 \) such that for all \( x := (x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) and \( i \in E \),
\[
\mathbb{E}_{x, i}(b^{\tau_{\varepsilon}}) \leq c(1 + \|x\|^{-\theta}),
\]
where \( \tau_{\varepsilon} = \inf\{t \geq 0 : \|X_t\| \geq \varepsilon\} \). In particular, for any \( (x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\} \), with probability one, \( X_t \) cannot converge to \((0, 0)\).

2. A result on linear switched systems

Let \( A_i \) denote the matrix
\[
A_i := \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix},
\]
for \( i = 0, 1 \), where \( a_i, b_i, c_i \) are real numbers satisfying
\[
a_i^2 + b_i c_i < 0.
\]
In this case, both matrices \( A_0, A_1 \) have purely imaginary eigenvalues.

We consider a random switching between the two dynamics given by \( A_0 \) and \( A_1 \). Let \((I_t)_{t \geq 0}\) be a continuous-time Markov Chain on \( E = \{0, 1\} \) with transition rates \( k_0, k_1 > 0 \). We denote by \((Y_t)_{t \geq 0}\) the solution of
\[
\frac{dY_t}{dt} = A_{I_t} Y_t
\]
\[Y_0 = y_0 \in \mathbb{R}^2 \setminus \{(0, 0)\}.
\]
The process \((Y_t, I_t)_{t \geq 0}\) is a PDMP living on \( \mathbb{R}^2 \setminus \{(0, 0)\} \times E \).

In this section, we show that \( \|Y_t\| \) converges exponentially fast to infinity with probability one. More precisely, we prove that there exists \( \lambda > 0 \) such that, for all \( y_0 \neq 0 \), almost surely
\[
\lim_{t \to \infty} \frac{1}{t} \log \|Y_t\| = \lambda.
\]
In order to do this we will use a *polar decomposition*. The use of polar decompositions to study Lyapunov exponents goes back to [Has60] in the case of stochastic differential equations. They have been used recently in the study of linear PDMPs (see [BLBMZ14, LMR14]) and more general PDMPs (see [BS17]).

Throughout the paper, we will denote by \( S^1 \subset \mathbb{R}^2 \) the circle with center at 0 and radius 1.
Whenever $y_0 \neq 0$ and $Y_t \neq 0$, setting $\Theta_t = Y_t/\|Y_t\|$ and $\rho_t = \|Y_t\|$, one can check using (2.3) that $(\rho_t, \Theta_t)_{t \geq 0}$ is the solution to

$$
\begin{align*}
\frac{d\Theta_t}{dt} &= A_t \Theta_t - \langle A_t \Theta_t, \Theta_t \rangle \Theta_t \\
\frac{d\rho_t}{dt} &= \rho_t \langle A_t \Theta_t, \Theta_t \rangle
\end{align*}
$$

(2.5)

with $\rho_0 = r_0 > 0$, $\rho_0 = \|y_0\|$. By the Ergodic Theorem and (2.5), proving (2.4) is equivalent to showing that the minimal average growth rate of $(\Theta_t)_{t \geq 0}$ is positive.

Recall from [BS17] that for any ergodic measure $\mu$ of the process $((\Theta_t, I_t))_{t \geq 0}$ (note that $((\Theta_t, I_t))_{t \geq 0}$ is a PDMP on $S^1 \times E$ - see [BS17]) the average growth rate with respect to $\mu$ is given by

$$
\Lambda(\mu) := \int \langle A_t \theta, \theta \rangle \mu(d\theta)di.
$$

The extremal average growth rates are the numbers defined by

$$
\Lambda^- := \inf \{ \Lambda(\mu) : \mu \in \mathcal{P}_{erg} \} \quad \text{and} \quad \Lambda^+ := \sup \{ \Lambda(\mu) : \mu \in \mathcal{P}_{erg} \},
$$

where $\mathcal{P}_{erg}$ is the set of ergodic measures of $((\Theta_t, I_t))_{t \geq 0}$ on $S^1 \times E$.

The main result of this section is the following theorem.

**Theorem 2.1.** Assume $A_0$ and $A_1$ are non proportional matrices of the form (2.1) with coefficients satisfying (2.2). Then all the average growth rates are equal and strictly positive

$$
\Lambda^+ = \Lambda^- > 0.
$$

As shown in the following lemma, non proportionality is not required to prove the uniqueness and nonnegativity of the average growth rate.

**Lemma 2.2.** The process $(\Theta_t, I_t)$ admits a unique invariant probability measure $\mu$ on $S^1 \times E$. Furthermore, $\Lambda(\mu) \geq 0$.

**Proof.** The uniqueness follows from [BS17, Proposition 2.10 and Example 2.11]. Indeed, since we study a two dimensional system, a sufficient condition is that at least one matrix $A_i$ has no real eigenvalue. This is the case for both $A_1$ and $A_2$. Since $A_0$ and $A_1$ have zero trace, [BS17, Corollary 2.6] implies that $\Lambda(\mu) = \Lambda^+ \geq 0$. $\square$

### 2.1. Lyapunov exponents and Bougerol’s theorem.

In order to prove Theorem 2.1 we will use results from [Bou88] on Lyapunov exponents. These numbers give the exponential growth rate of a linear Random Dynamical System (see Arnold [Arn98] for definition). In [BS17], the authors show that the process $Y$ from (2.3) together with the canonical shift is a linear random dynamical system satisfying the integrability conditions of Oselede’t’s Multiplicative Ergodic Theorem (see [Arn98, Theorem 3.4.1] or [CM, Proposition 3.12]). Thus, according to this theorem, there exist $d \in \{1, 2\}$
numbers $\lambda_1 > \lambda_d$ called the Lyapunov exponents, a Borel set $\tilde{\Omega} \subset \Omega$ with $P(\tilde{\Omega}) = 1$, and for each $\omega \in \tilde{\Omega}$ distinct vector spaces

$$\{0\} = V_{d+1}(\omega) \subset V_d(\omega) \subset V_1(\omega) = \mathbb{R}^2$$

such that

$$\lim_{t \to \infty} \frac{1}{t} \log \|Y_t\| = \lambda_i$$

for all $y_0 \in V_1(\omega) \setminus V_{i+1}(\omega)$.

**Remark 2.3.** By [BS17] Proposition 2.4 and Corollary 2.6 one can note that $\sum_i \lambda_i = 0$ and $\lambda_1 = \Lambda^+$. Therefore, proving that $\Lambda^+ > 0$ is equivalent to showing that $d = 2$.

In order to prove that $d = 2$, we will use [Bou88, Theorem 1.7]. We start by presenting the abstract framework of [Bou88].

Let $(\sigma_t)_{t \geq 0}$ be a stationary Markov process on some metric space $E$, and $(M_t)_{t \geq 0}$ a process with values in $GL_2(\mathbb{R})$, the set of invertible $2 \times 2$ matrices with real coefficients. We introduce the following definition, which is [Bou88, Definition 1.1 and 1.2].

**Definition 2.4.** Let $\pi$ be a probability measure on $E$. We say that $(M, \sigma, \pi)$ is a multiplicative system, if:

(i): The process $(M, \sigma)$ is Markovian with semigroup $(P_t)_{t \geq 0}$;

(ii): For any Borel subset $A \subset E$ (resp. $B \subset GL_2(\mathbb{R})$), $t \geq 0$, $C \in GL_2(\mathbb{R})$ and $i \in E$, one has

$$P_t ((C, i); A \times BC) = P_t ((\text{Id}, i); A \times B),$$

where $BC = \{NC; N \in B\}$;

(iii): $\pi$ is an ergodic measure for $\sigma$ and $\sup_{0 \leq t \leq 1} \mathbb{E}_{\text{Id}, \pi} \left[ \log^+ \|M_t\| + \log^+ \|M_t^{-1}\| \right] < \infty$.

We recall that the semigroup of a Markov process $(M, \sigma)$ is a family of measures defined for $t \geq 0$ and $(A, i) \in GL_2(\mathbb{R}) \times E$ by

$$P_t ((A, i); \cdot) = \mathbb{P}_{A, i} ((M_t, \sigma_t) \in \cdot).$$

Equivalently, the semigroup can be seen as a family $(P_t)_{t \geq 0}$ of operators which act on bounded measurable functions $f : GL_2(\mathbb{R}) \times E \to \mathbb{R}$ according to

$$P_t f ((A, i)) = \mathbb{E}_{A, i} [f((M_t, \sigma_t))], \ A \in GL_2(\mathbb{R}), i \in E.$$

We denote by $U$ the first order resolvent of $(P_t)_{t \geq 0}$. This is defined via

$$U = \int_0^{+\infty} e^{-t} P_t dt.$$

For $i \in E$ let $D_i$ be the support of $U((\text{Id}, i), \cdot)$ and $S_i = \{A \in GL_2(\mathbb{R}) : (i, A) \in D_i\}$. We will also let $K$ be the first order resolvent of the semigroup $(R_t)_{t \geq 0}$ of the Markov process $\sigma$.

Following [Bou88], we will say that the semigroup $(P_t)_{t \geq 0}$ is Feller if for any bounded continuous map $f : GL_2(\mathbb{R}) \times E \to \mathbb{R}$, and for all $t \geq 0$, the function $P_t f$ is continuous.
Definition 2.5. We say that a multiplicative system \((M, \sigma, \pi)\) satisfies hypothesis \(H\) if the following conditions hold

(i): The space \(E\) is a complete metric space.

(ii): The semigroup \((P_t)_{t \geq 0}\) is Feller.

(iii): The support of \(\pi\) is \(E\). If \(h\) is a bounded measurable function which is a fixed point for the first order resolvent of \(\sigma\), i.e.

\[ \text{Kh} = h \]

then \(h\) is continuous.

One has the following result (see [Bou88, Theorem 1.7]).

Theorem 2.6 (Bougerol, 1988). Assume \((M, \sigma, \pi)\) is a multiplicative system satisfying hypothesis \(H\). Assume furthermore that

(i): There exists a matrix in \(S_1\) with two eigenvalues with different modulus.

(ii): There does not exist some finite union \(W\) of one-dimensional vector spaces such that, for all matrices \(M\) in \(S_1\), \(MW = W\).

Then \(d = 2\).

Remark 2.7. Theorem 2.6 is a reformulation of [Bou88, Theorem 1.7], which is given for the numbers \(\gamma_i\) that are the Lyapunov exponents for the external power of \(M\) (see [Bou88, Proposition 2.2] or [Arn98, Theorem 3.3.3] for details). The numbers \(\gamma_1\) and \(\gamma_2\) are the numbers \(\lambda_i\) counted with multiplicity. (see [Arn98, Definition 3.3.8 and Theorem 3.4.1]).

We show that we can use Theorem 2.6 in our context. Let \((M_t)_{t \geq 0}\) with \(M_t \in GL_2(\mathbb{R}), t \geq 0\) be the solution of the matrix equation

\[ \frac{dM_t}{dt} = A_t M_t \quad \text{with} \quad M_0 \in GL_2(\mathbb{R}). \]

The process \((M_t, I_t)\) is a PDMP living on \(GL_2(\mathbb{R}) \times E\). One can note that when \(M_0 = \text{Id}\), the identity matrix, then for all \(y \in \mathbb{R}_+^2\) the process \(Y\) from (2.3) can be written as

\[ Y_t = M_t y \]

if \(Y_0 = y\).

Lemma 2.8. Set \(\pi = (q_1/(q_0 + q_1), q_0/(q_0 + q_1))\) and \(\mathcal{E} = E = \{0, 1\}\). Then \((M, I, \pi)\) is a multiplicative system satisfying \(H\).

Proof. First we show that \((M, I, \pi)\) is a multiplicative system. \((M, I)\) is a PDMP thus a Markov process. In addition, if we denote by \(M^N\) the process \(M\) when \(M_0 = N\) almost surely, one can easily check that \(M^N = M^{\text{Id}}N\) almost surely. As a result we see that point (ii) of definition 2.4 is satisfied. Straightforward computations show that \(\pi\) is the
unique invariant distribution of $I$ and is therefore ergodic. Let $K$ be a constant such that $\|A_i\| \leq K$ for $i \in E$. Then from

$$\frac{dM_t}{dt} = A_t M_t,$$

$$\frac{dM_t^{-1}}{dt} = -M_t^{-1} A_t,$$

together with $M_0 = M_0^{-1} = Id$ and Gronwall’s Lemma, one can show that for all $t \geq 0$

$$\|M_t\|, \|M_t^{-1}\| \leq e^{Kt}.$$ 

This proves point (iii).

Now we show that $(M, I, \pi)$ satisfy hypothesis $H$. In our case, $E = E$ is a finite set, thus points (i) and (iii) of Definition 2.5 are straightforward. To prove that $(M, I)$ is Feller, we use [BLBMZ15, Proposition 2.1] where the authors show that for a PDMP remaining in a compact set, the semigroup maps every continuous function to a continuous function. Their proof adapts verbatim to the case where the process does not remain in a compact set with the additional assumption that the continuous function is bounded and provided the jump rates are bounded - in our setting the jump rates are constant. This concludes the proof. □

2.2. Proof of Theorem 2.1. We start by showing that it suffices to prove Theorem 2.1 for a specific class of matrices.

**Lemma 2.9.** Assume Theorem 2.1 holds when $A_0$ is of the special form

$$A_0 = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}.$$ 

Then Theorem 2.1 holds for any $A_0$.

**Proof.** First we show that a linear change of coordinates does not change the value of $\Lambda^+$. Let $G \in GL_2(\mathbb{R})$, and set, for all $t \geq 0$, $Z_t = G Y_t$. Then $(Z_t, I_t)$ is a PDMP with $Z$ solution of

$$\frac{dZ_t}{dt} = B_t Z_t,$$

where $B_t = GA_t G^{-1}$. Due to $\lambda_1 = \frac{\log \|Y_t\|}{t} = \lim \frac{\log \|Z_t\|}{t}$ one can see that the maximal growth rates of $Y_t$ and $Z_t$ are equal.

Next, since the eigenvalues of $A_0$ are $\pm i \omega_0$ for $\omega_0 := \sqrt{-(a_0^2 + b_0 c_0)}$, a classical result in linear algebra (see for example [HS74, Chapter 4, Theorem 3]) states that there exists a matrix $G \in GL_2(\mathbb{R})$ such that

$$B_0 = GA_0 G^{-1} = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}.$$ 

Thus if the result is shown for a matrix $A_0$ of this form, it will be proven for every matrix $A_0$ with purely complex eigenvalues because $A_0$ and $A_1$ are proportional if and only if $B_0$ and $B_1$ are. □
Proof of Theorem 2.1. It suffices to show that (i) and (ii) of Theorem 2.6 are satisfied.

We first show that (i) holds.

According to Lemma 2.9, it suffices to show the assumptions are satisfied for \( A_0 \) of the form
\[
A_0 = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}.
\]

By standard computations, one can show that for all \( t \geq 0 \) and \( i \in E \),
\[
e^{tA_i} = \cos(\omega_i t) \text{Id} + \frac{1}{\omega_i} \sin(\omega_i t) A_i,
\]
where \( \omega_i := \sqrt{-(a_i^2 + b_i c_i)} \). In particular, since \( \text{Tr}(A_i) = 0 \), one has that for all \( s, t \geq 0 \)
\[
\varphi(s, t) := \text{Tr}(e^{sA_0} e^{tA_1}) = 2 \cos(\omega_0 s) \cos(\omega_1 t) + \frac{1}{\omega_0 \omega_1} \sin(\omega_0 s) \sin(\omega_1 t) \text{Tr}(A_0 A_1).
\]

On the other hand, since \( \text{Tr}(A_i) = 0 \), one has \( \det(e^{sA_0} e^{tA_1}) = 1 \). Thus, denoting by \( \mu_1, \mu_2 \) the eigenvalues of \( e^{sA_0} e^{tA_1} \) one can see that \( \mu_1 \mu_2 = 1 \) or equivalently \( \mu_1 = 1/\mu_2 \). In order to apply Theorem 2.6 we need to have \( |\mu_1| > |\mu_2| \). This cannot happen if \( \mu_1 = \mu_2 \) or if \( \mu_1 = \mu_2 \).

Due to the fact that \( \mu_1 + \mu_2 = \text{Tr}(e^{sA_0} e^{tA_1}) \), the condition \( |\mu_1| > |\mu_2| \) is equivalent to \( |\varphi(s, t)| > 2 \). By studying the derivatives of \( \varphi(s, t) \), one sees that its extremal values are reached at points \( (s^*, t^*) \) of the form \( (\pi/\omega_0, \pi/\omega_1) \) or \( (\pi/2\omega_0, \pi/2\omega_1) \) modulo \( \pi \). From this we note that the extremal values are \( \varphi(s^*, t^*) = \pm 2 \) and
\[
\varphi(s^*, t^*)^2 = \frac{1}{\omega_0^2 \omega_1^2} \text{Tr}(A_0 A_1)^2 = \frac{(b_1 - c_1)^2}{\omega_1^2}.
\]

Therefore
\[
(2.9) \quad \varphi(s^*, t^*)^2 > 4 \iff a_1 > 0 \text{ or } b_1 + c_1 > 0 \iff A_1 \text{ is not proportional to } A_0.
\]

By assumption, \( A_1 \) and \( A_0 \) are not proportional. Therefore, using (2.9) one infers that the matrix \( N(t_0, t_1) = N := e^{t_0 A_0} e^{t_1 A_1} \) has two eigenvalues with different moduli for \( (t_0, t_1) = (\pi/2\omega_0, \pi/2\omega_1) \).

In order to conclude that assumption (i) from Theorem 2.6 is satisfied, we show that the matrix \( N \) lies in \( S_1 \).

Let \( V \) be a neighborhood of \( N \) in \( GL_2(\mathbb{R}) \). Then, by continuity, there exists \( \varepsilon > 0 \) such that for all \( u \in [t_0 - \varepsilon, t_0 + \varepsilon], s \in [t_1 - \varepsilon, t_1 + \varepsilon] \) and \( \delta \leq \varepsilon \), the matrix \( N_{s,u,\delta} := e^{\delta A_1} e^{u A_0} e^{s A_1} \) is in \( V \). Let \( V_\varepsilon \) be the set of the matrices \( N_{s,u,\delta} \) for \( s, u \) and \( \delta \) as before. Let \( (U_n)_{n \geq 1} \) denote the sequence of interjump times of the process \( I \). Then, on the event \( B_{t,\varepsilon} = \{ U_1 \in [t_0 - \varepsilon, t_0 + \varepsilon]; U_2 \in [t_1 - \varepsilon, t_1 + \varepsilon]; t - (U_1 + U_2) \leq \varepsilon; U_1 + U_2 + U_3 \geq t \}, \)
\( I_t = 1 \) and \( M_t \in V_\varepsilon \). Thus one has
\[
\mathbb{P}_{\text{Id},1} ((M_t, I_t) \in V \times \{1\}) \geq \mathbb{P}_{\text{Id},1} ((M_t, I_t) \in V_\varepsilon \times \{1\}) \geq \mathbb{P}_{\text{Id},1} (B_{t,\varepsilon}).
\]
This last probability is positive for all $\varepsilon > 0$ and $t \in [t_0 + t_1 - 2\varepsilon, t_0 + t_1 + 3\varepsilon]$. Hence

$$U((\text{Id}, 1), V \times \{1\}) = \int_0^{+\infty} e^{-t} \mathbb{P}_{\text{Id}, 1}((M_t, I_t) \in V \times \{1\}) \, dt > 0.$$ 

This is true for all neighborhoods of $N$, so $N \in S_1$ and point (i) is shown.

Using similar arguments, one can show that the family of matrices $(e^{tA_1})_{t \geq 0}$ is in $S_1$. Since $A_1$ has two complex eigenvalues, one cannot find a finite union of one dimensional vector spaces invariant by the family $(e^{tA_1})_{t \geq 0}$. This proves that assumption (ii) of Theorem 2.6 holds.

\[\square\]

3. Proof of Theorem 1.2

Let $A_i$ denote the Jacobian matrix of the vector field $F^i$ at $(p, q)$. Then

$$A_i = \begin{pmatrix} 0 & -b_i p \\ d_i q & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_i \\ \beta_i & 0 \end{pmatrix},$$

where $\alpha_i = b_i p$ and $\beta_i = d_i q$. The linear PDMP $(Y, I)$ where $Y$ is the solution of

$$\frac{dY_t}{dt} = A_{I_t} Y_t,$$

is a particular case of the systems studied in Section 2.

To apply Theorem 2.1, we have to check that $A_0$ and $A_1$ are non collinear. This is equivalent to showing that $\alpha_1 \beta_0 \neq \alpha_0 \beta_1$. Assume that $\alpha_1 \beta_0 = \alpha_0 \beta_1$. Then since $\alpha_i = b_i p$ and $\beta_i = d_i q$, we get $b_1 d_0 = b_0 d_1$. Moreover, since $p_0 = p_1$ and $q_0 = q_1$, one has $c_0 d_1 = c_1 d_0$ and $a_0 b_1 = a_1 b_0$. If we set $\gamma = b_1 / b_0$, we note that $k \kappa_1 = \gamma \kappa_0$ for $\kappa = a, b, c, d$, which implies $F^1 = \gamma F^0$. This contradicts the assumption that the vector fields $F^0$ and $F^1$ are non collinear.

As a result, $A_0$ and $A_1$ cannot be collinear. We can therefore apply Theorem 2.1 and conclude that $\Lambda^- > 0$.

Now, even though the process $X_t$ does not remain in a compact set [BS17, Theorem 3.2, (iii)] is still true because it is a local result. We prove this fact.

Let $K \subset \mathbb{R}_{++}^2$ be a compact set containing $(p, q)$ in its interior. Let $\varphi_K : \mathbb{R}_{++}^2 \to [0, 1]$ be a smooth function such that $\varphi_K = 1$ on $K^\delta$ and $\varphi_K = 0$ on the complement of $K^{2\delta}$. Here $K^\delta = \{x \in \mathbb{R}_{++}^2 : d(x, K) < \delta\}$ is the $\delta$ - neighbourhood of $K$ and $\delta > 0$ is such that $K^{2\delta} \subset \mathbb{R}_{++}^2$. For $i \in E$, set $F^i_K = \varphi_K F^i$. Note that $F^i_K = F^i$ on $K^\delta$. In particular, $(p, q)$ is a common zero of $F^0_K$ and $F^1_K$ and $DF^i_K((p, q)) = DF^i((p, q)) = A_i$. Now consider the PDMP $(X^K, I)$, with $(X^K_{I_t})_{t \geq 0}$ solution of

$$\frac{dX^K_t}{dt} = F^i_K(X^K_t).$$

Then we have the two following facts. First, denote by $\tau_K = \inf\{t \geq 0 : X_t \notin K\}$ the exit time of $K$ for $X_t$. Then if $X_0 = X^K_0 = x \in K$, for all $t \leq \tau_K$, $X_t = X^K_t$ almost surely. Next, since $DF^K_0((p, q)) = A_i$, the average growth rate $\Lambda^-_K$ of $(X^K, I)$ is equal to $\Lambda^-$. Now, since $X^K_t$ remains in the compact set $K^{2\delta}$ and $\Lambda^-_K = \Lambda^- > 0$, one can
apply [BS17, Theorem 3.2, (iii)]. According to this theorem, since $\Lambda^{-}_K > 0$, there exist $\varepsilon > 0$, $\theta > 0$, $b > 1$ and $c > 0$ such that for all $x \in \mathbb{R}^2_{++} \setminus \{(p, q)\}$ and $i \in E$,

$$\mathbb{E}_{x,i}(b^{\tau_K}) \leq c(1 + \|x - (p, q)\|^{-\theta}),$$

where $\tau_K = \inf\{t \geq 0 : \|X^K_t - (p, q)\| \geq \varepsilon\}$. Without loss of generality, we can assume that the ball of center $(p, q)$ and radius $\varepsilon$ is included in the interior of $K$. Let $\tau^\varepsilon = \inf\{t \geq 0 : \|X_t - (p, q)\| \geq \varepsilon\}$. Now if $\|x - (p, q)\| \geq \varepsilon$, $\tau^\varepsilon = 0$. If $\|x - (p, q)\| < \varepsilon$, then since $X_t = X^K_t$ for all $t \leq \tau_K$, one gets that $\tau^\varepsilon = \tau_K \leq \tau_K$. In particular, for all $x \in \mathbb{R}^2_{++} \setminus \{(p, q)\}$ and $i \in E$,

$$\mathbb{E}_{x,i}(b^{\tau^\varepsilon}) \leq c(1 + \|x - (p, q)\|^{-\theta}).$$

We claim that because of (3.1), $X_t$ cannot converge to $(p, q)$. We argue by contradiction. Let $x \in \mathbb{R}^2_{++}, i \in E$ and assume that $X_t$ converges to $(p, q)$ almost surely under $\mathbb{P}_{x,i}$. Define two stopping times by

$$\tau^\varepsilon_{in,1} = \inf\{t \geq 0 : \|X_t - (p, q)\| \leq \varepsilon/2\}$$

and

$$\tau^\varepsilon_{out,1} = \inf\{t > \tau^\varepsilon_{in,1} : \|X_t - (p, q)\| \geq \varepsilon\}.$$

Since $X_t$ converges to $(p, q)$ almost surely, one has $\mathbb{P}_{x,i}(\tau^\varepsilon_{in,1} < \infty) = 1$. Using the strong Markov property at $\tau^\varepsilon_{in,1}$, one gets

$$\mathbb{P}_{x,i}(\tau^\varepsilon_{out,1} < \infty) = \mathbb{E}_{x,i}(\mathbb{P}_{X_{\tau^\varepsilon_{in,1}}}(\tau^\varepsilon < \infty)) = 1.$$

Construct recursively a family of stopping times

$$\tau^\varepsilon_{in,k} = \inf\{t > \tau^\varepsilon_{out,k-1} : \|X_t - (p, q)\| \leq \varepsilon/2\}$$

and

$$\tau^\varepsilon_{out,k} = \inf\{t > \tau^\varepsilon_{in,k} : \|X_t - (p, q)\| \geq \varepsilon\},$$

by repeating the above procedure. Then one gets that for all $k \geq 1$, $\tau^\varepsilon_{in,k}$ and $\tau^\varepsilon_{out,k}$ are finite almost surely. This contradicts the fact that $X_t$ converges to $(p, q)$. As a result we have shown that $X_t$ cannot converge to $(p, q)$.

**Remark 3.1.** The proof of Theorem 1.2 above extends verbatim to the proof of Theorem 1.5. The fact that the jump rates now depend on the position does not affect the result because when it comes to the linear system in Theorem 2.7, one just has the constants $k_{ij}(0)$ as jump rates (see [BS17, Section 2] for details).

4. Future research

Using some of the methods developed in [BS17] we were able to prove a conjecture from [TDHS06] and show that if one switches between two deterministic Lotka-Volterra systems with a common equilibrium point at $(p, q)$ then the resulting PDMP can never converge to this equilibrium. We reduced the analysis from the non-linear Lotka-Volterra PDMP to the study of a linear PDMP (a linearization of the original PDMP around the equilibrium point).
Recently, there have been several studies about randomly switched linear systems in dimension 2 (see [BLBMZ14], [LMR14] and [Lag16]). In these studies, the authors show that the growth rate is positive for some switching rates by a direct computation of the invariant measure of the process \((\Theta, I)\) (this is the process that arises as the angular part when doing the polar decomposition). One could try a similar method in our setting. However, the integral expression we obtain for the growth rate does not easily yield the sign of the growth rate. Nonetheless, it could be interesting to investigate this integral expression, possibly through numerical simulations.

Another interesting direction for the future is finding out whether the process \(X_t\) defined by (1.3) is transient, null-recurrent of positive recurrent. The simulations done in [TDHS06] seem to suggest the following conjecture.

**Conjecture 4.1.** Suppose \(X_t = (x_t, y_t)\) is the process defined by (1.3) together with the initial condition \(X_0 = (x_0, y_0) \in \mathbb{R}_t^2++\). Then, almost surely

\[
\lim_{t \to \infty} \left( x_t + y_t + \frac{1}{x_t} + \frac{1}{y_t} \right) = \infty
\]

and the process \(X_t\) is transient.

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