Canonical Quantization of Spherically Symmetric Dust Collapse

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ABSTRACT

Quantum gravity effects are likely to play a crucial role in determining the outcome of gravitational collapse during its final stages. In this contribution we will outline a canonical quantization of the LeMaitre-Tolman-Bondi models, which describe the collapse of spherical, inhomogeneous, non-rotating dust. Although there are many models of gravitational collapse, this particular class of models stands out for its simplicity and the fact that both black holes and naked singularity end states may be realized on the classical level, depending on the initial conditions. We will obtain the appropriate Wheeler-DeWitt equation and then solve it exactly, after regularization on a spatial lattice. The solutions describe Hawking radiation and provide an elegant microcanonical description of black hole entropy, but they raise other questions, most importantly concerning the nature of gravity’s fundamental degrees of freedom.

Keywords: Gravitational Collapse, Quantum Gravity, Black Hole Thermodynamics.

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I. INTRODUCTION

It is a great pleasure to have been invited to contribute to a festschrift in honor of Joshua N. Goldberg. His gentle manner, his fine intelligence and his modest nature are recognized by all. They conceal his ardor in defending the moral ideals he possesses and loves.

So long as no generally agreed upon theory of quantum gravity exists, it is important to examine the quantization of particular models. This contribution addresses the quantization of the LeMaître-Tolman-Bondi (LTB) solutions, which describe the classical collapse of spherically symmetric, inhomogeneous, self-gravitating dust. The classical solutions were originally introduced by G. LeMaître to study cosmology, where it has found interesting applications. Our principal interest here will be to develop the Hamiltonian formalism for both the classical and quantum LTB models, as such we will describe a generalization of work by Kuchař, who developed a midisuperspace quantization of the Schwarzschild black hole. While it would be preferable to take a fundamental field (e.g., a scalar field) for the matter part, this would make the formalism much less tractable. Moreover, the relevant features of gravitational collapse already exhibit themselves for the dust model in the sense that the dust collapse may result classically in the formation of a black hole or of a naked singularity, depending on the initial conditions.

In section II we review classical LTB collapse and present the canonical formalism for these models. The hypersurface action yields two constraints, viz., the Hamiltonian constraint and the momentum constraint. We reconstruct the mass and time from the canonical data and this leads naturally to new variables viz., the mass function, the dust proper time, the physical radius and their conjugate momenta, which are introduced along with the generator of the canonical transformation from the old to the new variables. The momentum conjugate to the mass function may be eliminated in the Hamiltonian constraint using the momentum constraint. This leads to a new and simpler constraint that is able to take the place of the original Hamiltonian constraint. We apply Dirac’s quantization program to the new constraints in section III and obtain the Wheeler-DeWitt equation. We then introduce a lattice regularization of the functional equations and find exact solutions, which we use in section IV to describe Hawking radiation. In section V we show how black hole entropy can be recovered from a microcanonical ensemble of states and discuss some issues raised by our approach in section VI.

II. THE CLASSICAL LTB MODELS

The LTB models describe self-gravitating dust. The energy-momentum tensor reads $T_{\mu \nu} = \varepsilon(\tau, \rho)U_\mu U_\nu$, where $U^\mu = U^\mu(\tau, \rho)$ is the four-velocity vector of a dust particle with proper time $\tau$ and labeled by $\rho$ ($\rho$ thus labels the various shells that together form the dust cloud). The LTB line element is given by

$$ds^2 = d\tau^2 - \frac{(\partial_\rho R)^2}{1 + 2E(\rho)}d\rho^2 - R^2(\rho)d\Omega^2. \quad (1)$$
Inserting this expression into the Einstein equations yields

\[ \varepsilon(\tau, \rho) = \frac{\partial_\rho F}{R^2 \partial_\rho R} \quad \text{and} \quad (\partial_\tau R)^2 = \frac{F}{R} + 2E, \] (2)

where \( F(\rho) \) and \( E(\rho) \) are non-negative functions of the label coordinate and we have set \( 8\pi G = 1 \). The case of collapse is described by \( \partial_\tau R(\tau, \rho) < 0 \). There still exists the freedom to rescale the shell index \( \rho \), which we fix by demanding \( R(0, \rho) = \rho \), so that for \( \tau = 0 \) the label coordinate \( \rho \) is equal to the curvature radius \( R \). Now we can express the functions \( F(\rho) \) and \( E(\rho) \) in terms of the energy density \( \varepsilon(\tau, \rho) \) at \( \tau = 0 \). From (2),

\[ F(\rho) = \int_0^\rho \varepsilon(0, \tilde{\rho}) \tilde{\rho}^2 \, d\tilde{\rho}, \]
\[ E(\rho) = \frac{1}{2} \left[ \partial_\tau R(\tau = 0, \rho) \right]^2 - \frac{1}{\rho} \int_0^\rho \varepsilon(0, \tilde{\rho}) \tilde{\rho}^2 \, d\tilde{\rho}, \] (3)

so that \( F(\rho)/2 \) may be interpreted as the active gravitating mass inside of the shell labeled by \( \rho \) and \( E(\rho) \) as the total energy inside the shell. An analysis of the classical solutions for these models can be found in [3]. In the present work we discuss the canonical formalism for the so-called “marginally bound” models, defined by \( E(\rho) = 0 \) [6]. A generalization to non-marginally bound models is found in [7].

**A. Hamiltonian formalism**

Begin with the general ansatz for a spherically-symmetric line element,

\[ ds^2 = -N^2 dt^2 + L^2 (dr + N^r dt)^2 + R^2 d\Omega^2, \] (4)

where \( N \) and \( N^r \) are the lapse and shift function, respectively. The canonical momenta are given by

\[ P_L = \frac{R}{N} \left( -\dot{R} + N^r R^r \right), \]
\[ P_R = \frac{1}{N} \left[ -L \dot{R} - \dot{L} R + (N^r LR)^r \right], \] (5)

where a dot denotes a derivative with respect to coordinate time \( t \), while a prime denotes a derivative with respect to \( r \). All variables are functions of \( t \) and \( r \). A Legendre transformation from the Einstein–Hilbert action then leads to

\[ S_{EH} = \int dt \int_0^\infty dr \left( P_L \dot{L} + P_R \dot{R} - NH^g - N^r H^g_r \right) + S_{\partial \Sigma}, \] (6)

in which the Hamiltonian and the diffeomorphism (momentum) constraint are given by

\[ H^g = -G \left( \frac{P_L P_R}{R} - \frac{L P^2_L}{2R^2} \right) + \frac{1}{G} \left[ -\frac{L}{2} - \frac{R^2}{2L} + \left( \frac{R R^r}{L} \right)^r \right], \]
\[ H^g_r = R' P_R - LP'_L, \] (7)
respectively, and the boundary action $S_{\partial \Sigma}$ is discussed below.

The total action is the sum of (6) and an action $S^d$ describing the dust. The canonical formalism for the latter was developed in [8], (see also [6]). The dust action reads

$$S^d = \int dt \int_0^\infty dr \left( P_r \dot{\tau} - N H^d - N^r H^d_r \right) ,$$

where

$$H^d = P_r \sqrt{1 + \frac{\tau^2}{L^2}}$$
and

$$H^d_r = \tau' P_r$$

are the dust Hamiltonian and momentum constraints respectively.

**B. Mass function in terms of canonical variables**

In the following we shall write the mass function $F(\rho)$, which was introduced in (3), in terms of the canonical data. This is essential for deriving consistent falloff conditions that are appropriate for a realistic collapse model. We begin by requiring the spacetime described by the metric (4) to be embedded in a LTB spacetime. Considering the LTB metric (9), a foliation described by functions $\tau(r,t)$ and $\rho(r,t)$ leads to

$$L^2 = R^2 \rho^2 - \tau^2 ,$$
$$N^r = \frac{R^2 \rho' - \dot{\tau} \dot{\tau'}}{L^2} ,$$
$$N = \frac{R'}{L} (\dot{\tau} \rho' - \dot{\rho} \tau') .$$

When these expressions for lapse function and shift vector into the expression for the canonical momentum $P_L$ in (5) and the equations (2) are used, we find

$$\frac{LP_L}{R} = R' \sqrt{1 - F} - F \tau' ,$$

where $F \equiv 1 - F/R$. Solving for $\tau'$ gives

$$\tau' = + \frac{1}{F} \left( R' \sqrt{1 - F} - \frac{LP_L}{R} \right)$$

and inserting this expression into (10) yields

$$L^2 = \left[ \frac{R^2}{F} - \frac{L^2 P_L^2}{R^2 F} \right] ,$$

which determines $F$ according to

$$F = \left[ \frac{R^2}{L^2} - \frac{P_L^2}{R^2} \right] .$$

We can thus express $F$ locally in terms of the canonical data as follows:

$$F = R \left[ 1 + \frac{P_L^2}{R^2} - \frac{R^2}{L^2} \right] .$$
This expression, although obtained here for marginal models, possesses a wider range of applicability, holding, in fact, for all cases \[7\]. Further, it turns out that the functions, \(P_F\), defined by

\[P_F = \frac{LP_L}{2RF}\]  \hspace{1cm} (16)

and the mass function, \(F\), form a conjugate pair of variables. Since \(R = F\) at the horizon, \(\mathcal{F} = 0\) there. We can check that though \(\mathcal{F}\) appears in the denominator of (12), \(\tau'\) is well behaved at the horizon:

\[\tau' \xrightarrow{\mathcal{F} \to 0} \frac{1}{2} \left( R' + L \right) .\]  \hspace{1cm} (17)

This is as it should be. We now make a canonical transformation in order to elevate the mass function \(F\) to a canonical coordinate. The canonical transformation, \((\tau, R, L, P_{\tau}, P_R, P_L) \rightarrow (\tau, R, F, P_{\tau}, \bar{P}_R, P_F)\), is generated by

\[G = \int_{0}^{\infty} dr \left[ LP_L - \frac{1}{2} R R' \ln \left| \frac{R R' + LP_L}{R R' - LP_L} \right| \right]\]  \hspace{1cm} (18)

and this gives

\[\bar{P}_R = P_R - \frac{LP_L}{2R} - \frac{LP_L}{2RF} - \frac{\Delta}{RL^2 \mathcal{F}} ,\]  \hspace{1cm} (19)

with

\[\Delta = (RR')(LP_L)' - (RR')(LP_L) .\]  \hspace{1cm} (20)

The action in the new canonical variables then reads

\[S_{EH} = \int dt \int_{0}^{\infty} dr \left( P_{\tau} \dot{\tau} + \bar{P}_R \dot{R} + P_F \dot{F} - NH - N^r H_r \right) + S_{\partial \Sigma} ,\]  \hspace{1cm} (21)

where the new constraints are

\[H = -\frac{1}{2L} \left( \frac{F' R'}{G \mathcal{F}} + 4G \mathcal{F} P_F P_R \right) + P_{\tau} \sqrt{1 + \frac{\tau'^2}{L^2}} ,\]

\[H_r = \tau' P_{\tau} + R' \bar{P}_R + F' P_F .\]  \hspace{1cm} (22)

We shall now discuss the boundary action \(S_{\partial \Sigma}\) in more detail.

**C. Boundary action**

Boundary terms are obtained from a careful discussion of the falloff conditions for the canonical variables, which were investigated in detail in \[3\]. It turns out that the only boundary term is obtained from the variation of the hypersurface action with respect to \(L\) and reads

\[\int dt \ N_+(t) \delta M_+(t) ,\]  \hspace{1cm} (23)

where \(N_+(t) \equiv N(t, r \to \infty)\) is the lapse function at infinity and \(M_+(t) \equiv F(r \to \infty)/2\) is the ADM mass. To avoid the conclusion that \(N_+(t)\) is constrained to vanish, which would
freeze the evolution at infinity, the boundary term has to be canceled by an appropriate boundary action. This can be achieved by adding the surface action

\[ S_{\partial \Sigma} = - \int dt N_+(t) M_+(t). \]  

(24)

Since varying \( N_+ \) would lead to a zero ADM mass, Kuchař has argued in [3] that \( N_+ \) has to be treated as a prescribed function. The lapse function gives the ratio of proper time to coordinate time in the direction normal to the foliation. Since \( N_r(r) \) vanishes for \( r \to \infty \), the time evolution at infinity is generated along the world lines of observers with \( r = \text{const.} \)

If we introduce the proper time, \( \bar{\tau} \), of these observers as a new variable, we can express the lapse function in the form \( N_+(t) = \dot{\bar{\tau}}_+(t) \). This leads to

\[ S_{\partial \Sigma} = - \int dt \dot{M}_+ \dot{\bar{\tau}}_+. \]  

(25)

and thus we have removed the necessity of fixing the lapse function at infinity. (In [3] this is called 'parametrization at infinities'.)

Our aim is to cast the homogeneous part of the action into Liouville form and to find a transformation to new canonical variables that absorb the boundary terms. This can be done by introducing the mass density \( \varGamma \equiv F' \) as a new canonical variable and using the boundary condition \( F(0) = 0 \) (which is appropriate for a collapse situation). Part of the Liouville form can then be rewritten as follows:

\[ \tilde{\theta} \equiv \int_0^\infty dr P_F \delta F - M_+ \delta \bar{\tau}_+ \]

\[ = \int_0^\infty dr \delta \varGamma \left( \frac{\bar{\tau}_+}{2} + \int_r^\infty dr' P_F(r') \right) - \delta (M_+ \bar{\tau}_+) \]  

(26)

(see [4] for details). From (26) we see that \( P_\varGamma = \bar{\tau}_+/2 + \int_r^\infty dr P_F \). Thus \( P_\varGamma(\infty) = \bar{\tau}_+/2 \).

Thus the new action reads

\[ S_{\text{EH}} = \int dt \int_0^\infty dr \left( P_\varGamma \dot{\varGamma} + \bar{P}_R \bar{R} + P_\varGamma \dot{\bar{\varGamma}} - NH' - N' H' \right). \]  

(27)

The constraints in the new variables are

\[ H = \frac{1}{2L} \left( \frac{\Gamma R'}{EF} - 4GFP_\varGamma \bar{P}_R \right) + P_\varGamma \sqrt{1 + \frac{\tau^2}{L^2}}, \]

\[ H_r = \tau' P_\varGamma + R' \bar{P}_R - \Gamma P_\varGamma'. \]  

(28)

For the Schwarzschild black hole, \( 2P_\varGamma \) is equal to the Killing time, \( T \), in the exterior, as shown in [3]. The Hamiltonian constraint can be greatly simplified if the momentum constraint is used to eliminate \( P_F \equiv -P_\varGamma' \) (see Appendix A of [7]). The constraints (28) can then be replaced by the following equivalent set,

\[ H = (P_\varGamma^2 + \bar{P}_R^2) - \frac{\varGamma^2}{4F} \approx 0, \]  

(29)
\[ H_r = \tau' P_\tau + R' P_R - \Gamma P_\Gamma' \approx 0. \] (30)

These equations will be used as the starting point for the quantization in Sec. III. Another useful relation follows from (12),

\[ \tau = T \pm \int dR \sqrt{\frac{1 - \mathcal{F}}{\mathcal{F}}} \] (31)

and relates the dust proper time, \( \tau \), to the Killing time, \( T \), in the exterior of the collapsing dust ball, where the mass function is constant. The positive sign refers to contracting clouds and the negative sign to expanding clouds.

D. Hamiltonian equations of motion

Here we shall give the Hamilton equations of motion and derive Einstein's equation (2) from them. The Hamiltonian equations are generally given by

\[
\dot{X} = \{ X, \mathcal{H}[N] + \mathcal{H}_r[N'] \}, \\
\dot{P}_X = \{ P_X, \mathcal{H}[N] + \mathcal{H}_r[N'] \},
\] (32)

where we have introduced the smeared constraints

\[ \mathcal{H}[N] = \int_0^\infty dr N(r) H(r), \quad \mathcal{H}_r[N'] = \int_0^\infty dr N'(r) H(r). \] (33)

Starting from the action (27), the Hamiltonian equations of motion are\(^3\)

\[
\dot{\tau} = 2N \tau' + N' \tau', \\
\dot{P}_\tau = (N_r P_\tau)', \\
\dot{R} = 2N \mathcal{F} P_R + N_r R', \\
\dot{P}_R = -N \left( \frac{\mathcal{F} R^2}{R^2} + \frac{\Gamma^2 \mathcal{F}}{4 \mathcal{F}^2 R^2} \right) + (N_r P_R)', \\
\dot{\Gamma} = (N' \Gamma)', \\
\dot{P}_\Gamma = N \frac{\Gamma}{2 \mathcal{F}} + N' P'_\Gamma + \int_\tau^\infty d\tilde{r} N(\tilde{r}) \left( \frac{P^2_{\tilde{R}}(\tilde{r})}{R(\tilde{r})} + \frac{\Gamma^2(\tilde{r})}{4 \mathcal{F}^2(\tilde{r}) R(\tilde{r})} \right). \] (34)

Consider now the momentum constraint in the form

\[ \tau' + \frac{R' P_R}{P_\tau} - \Gamma' \frac{P_\Gamma}{P_\tau} = 0 \] (35)

and use the Hamiltonian constraint

\[ \mathcal{P}_R = \pm \frac{P_\tau}{\mathcal{F}} \sqrt{\frac{\Gamma^2}{4 \mathcal{F}^2 - \mathcal{F}}} \] (36)

\(^3\) Note that \( \delta F(r)/\delta \Gamma(\tilde{r}) = \delta (r - \tilde{r}). \)
so that

\[ \tau' = \mp \frac{R'}{F} \sqrt{\frac{\Gamma^2}{4F} - F + \frac{\Gamma P'_\tau}{P_\tau}} = 0. \]  

(37)

Now using

\[ P'_\tau = -P_F = -\frac{LP_L}{2RF}, \]  

(38)

(37) can be compared to (12) and shows that

\[ P_{\tau} = \frac{\Gamma}{2} \]  

(39)

so that

\[ \partial_{\tau} R = \frac{\dot{R}}{\tau} = \frac{2NF\dot{P}_R}{2NP_\tau} = \frac{2F\ddot{P}_R}{\Gamma}, \]  

(40)

which we can solve for \( \dot{P}_R \). Inserting this expression into the Hamiltonian constraint (29) gives

\[ 0 = H = \frac{\Gamma^2}{4} + \frac{\Gamma^2(\partial_{\tau} R)^2}{4F} - \frac{\Gamma^2}{4F} \]  

and solving for \( (\partial_{\tau} R)^2 \) leads to Einstein’s equation (3). Note that we did not have to specify the lapse function.

The algebra of the constraints is not of the standard form (given, for example, in [9]), because we have used the momentum constraint to eliminate \( P_F \) in the Hamiltonian constraint. In fact, a short calculation gives

\[ \{\mathcal{H}[N], \mathcal{H}[M]\} = 0, \]  

(42)

\[ \{\mathcal{H}_r[N^r], \mathcal{H}[N]\} = \mathcal{H}[N, N^r - N^r_r], \]  

(43)

\[ \{\mathcal{H}_r[N^r], \mathcal{H}_r[M^r]\} = \mathcal{H}_r[[N^r, M^r]]. \]  

(44)

The Poisson bracket of the Hamiltonian with itself vanishes, in contrast with the general case in which it closes on the momentum constraint. The other brackets coincide with the general case. The transformations generated by the Hamiltonian constraint can thus no longer be interpreted as hypersurface deformations. They are in general not orthogonal to the hypersurfaces, but act along the flow lines of dust.

III. QUANTIZATION

We shall now apply the quantization procedure proposed by Dirac and turn the classical constraints into quantum operators, cf. [4]. We begin with the expressions in (29). Poisson brackets are translated into commutators in the Schrödinger representation by substituting

\[ P_\tau(r) \rightarrow \frac{\hbar}{i} \frac{\delta}{\delta \tau(r)}, \quad \dot{P}_R(r) \rightarrow \frac{\hbar}{i} \frac{\delta}{\delta \dot{R}(r)}, \quad P_\Gamma(r) \rightarrow \frac{\hbar}{i} \frac{\delta}{\delta \Gamma(r)} \]  

(45)

and having them act on wave functionals. The Hamiltonian constraint (29) then leads to the WDW equation,

\[ -\hbar^2 \left( \frac{\delta^2}{\delta \tau(r)^2} + \frac{\delta^2}{\delta R(r)^2} + A(R, F)\delta(0) \frac{\delta}{\delta \dot{R}(r)} \right) \]
\[ + B(R, F) \delta(0)^2 - \frac{\Gamma^2}{4F} \] \[ \Psi[\tau, R, \Gamma] = 0, \quad (46) \]

where \( A \) and \( B \) are smooth functions of \( R \) and \( F \) that encapsulate the factor ordering ambiguities. We have introduced \( \delta(0) \) in order to indicate that the factor ordering problem is unsolved and can be dealt with only after some suitable regularization has been performed, cf. \( \text{[10]} \). That is, we choose the terms proportional to \( \delta(0) \) in such a way that the constraint algebra closes, which is usually called ‘Dirac consistency’. Quantizing the momentum constraint in \( \text{(29)} \) by using \( \text{(45)} \) gives

\[ \left\{ \tau' \frac{\delta}{\delta \tau(r)} + R' \frac{\delta}{\delta R(r)} - \Gamma \left( \frac{\delta}{\delta \Gamma(r)} \right) \right\} \Psi[\tau, R, \Gamma] = 0. \quad (47) \]

The next subsection is devoted to the application of a lattice regularization.

\section{Lattice regularization}

For solutions of the constraints, we make the ansatz

\[ \Psi[\tau, R, \Gamma] = \Psi^{(0)}[F] \exp \left[ -\frac{i}{2} \int dr \, \Gamma(r) \, W(\tau, R, F) \right], \quad (48) \]

where \( W(\tau, R, F) \) is some function to be determined. It automatically satisfies the diffeomorphism constraint. The Wheeler-DeWitt equation is second order in time derivatives so both positive and negative energy solutions exist, but we will confine our attention to the positive energy solutions above. It is worth noting that any functional

\[ \Psi[\tau, R, \Gamma] = U \left( -\frac{i}{2} \int dr \, \Gamma(r) \, W(\tau, R, F) \right) \quad (49) \]

would satisfy the diffeomorphism constraint provided that \( W \) has no explicit dependence on the label coordinate \( r \) except through the mass function, \( F(r) \). We have chosen \( U = \exp \) so that the wave-functional may also be factorizable on a spatial lattice, whose cell size we call \( \sigma \), taking \( \sigma \rightarrow 0 \) in the continuum limit. Diffeomorphism invariance requires that the continuum wave-functional and all physical results be independent of the cell size. On the lattice, the argument of the exponential function becomes \( \text{[7, 11]} \)

\[ \int dr \, \Gamma(r) \, W(\tau, R, F) \rightarrow \sigma \sum_j \Gamma_j \mathcal{W}(\tau_j, R_j, F_j), \quad (50) \]

where \( \Gamma_j = \Gamma(r_j) \), etc. This turns the wave-functional into a product state,

\[ \Psi[\tau, R, \Gamma] = \prod_j \psi_j(\tau_j, R_j, F_j) = \prod_j \psi_j^{(0)} \exp \left[ -\frac{i}{2} \sigma \sum_j \Gamma_j \mathcal{W}(\tau_j, R_j, F_j) \right] \quad (51) \]

provided that \( U = \exp \).
Before proceeding further it is necessary to define what is meant by a functional derivative when functions are defined on a lattice. The defining equations can be understood by analogy with the simplest properties of functional derivatives of the functions $J(x)$

$$\frac{\delta J(y)}{\delta J(x)} = \delta(y - x),$$

$$\frac{\delta}{\delta J(x)} \int dy J(y) = 1$$

and from these definitions follows

$$\frac{\delta}{\delta J(x)} \int dy J(y) \phi(y) = \phi(x).$$

On a lattice we define, for the lattice intervals $x_i$ and $x_j$,

$$\frac{\delta J(x_i)}{\delta J(x_j)} = \Delta(x_i - x_j) = \lim_{\sigma \to 0} \delta_{ij} \sigma,$$

where $r_i$ labels the $i^{th}$ lattice site and $\delta_{ij}$ is the Kronecker $\delta$, equal to zero when the lattice sites $x_i$ and $x_j$ are different and one when they are the same. Just as $\delta(y - x)$ is only defined as an integrand in an integral, so $\Delta(x_i - x_j)$ should also be considered defined only as a summand in a sum over lattice sites. Hence

$$\lim_{\sigma \to 0} \frac{\delta}{\delta J(r_j)} \sigma \sum_i J(r_i) = \lim_{\sigma \to 0} \sigma \sum_i \delta J(r_i) \delta J(r_j) = 1$$

and

$$\frac{\delta}{\delta J(r_j)} \sigma \sum_i J(r_i) \phi(r_i) = \lim_{\sigma \to 0} \sigma \sum_i \Delta(r_i - r_j) \phi(r_i) = \phi(r_j).$$

It follows that

$$\frac{\delta}{\delta J(x_j)} \to 1 \lim_{\sigma \to 0} \frac{\partial}{\partial J_j},$$

where $J_j = J(x_j)$. This is compatible with the formal (continuum) definition of the functional derivative.

### B. Collapse Wave Functionals

When (54) and (57) are applied to the Wheeler-DeWitt equation in (46) and $\Psi[\tau, R, \Gamma]$ is taken to be a product state, one obtains an equation describing the wave functions at each lattice point

$$\left[ \frac{\partial^2}{\partial \tau_j^2} + F_j \frac{\partial}{\partial R_j^2} + A_j \frac{\partial}{\partial R_j} + B_j + \frac{\sigma^2 \Gamma_j^2}{4} \right] \psi_j \approx 0,$$

but there is a further restriction arising from the diffeomorphism constraint. Inserting the ansatz in (54) into (59), we find

$$\frac{\sigma^2 \Gamma_j^2}{4} \left[ \left( \frac{\partial \mathcal{W}_j}{\partial \tau_j} \right)^2 + F_j \left( \frac{\partial \mathcal{W}_j}{\partial R_j} \right)^2 - 1 \right]$$
\[
\frac{\sigma \Gamma_j}{2} \left[ \frac{\partial^2 W_j}{\partial \tau_j^2} + \mathcal{F}_j \left( \frac{\partial^2 W_j}{\partial R_j^2} + A_j \frac{\partial W_j}{\partial R_j} \right) \right] + B_j = 0,
\]
(59)

which must be satisfied \textit{independently} of \(\sigma\). This is only possible if the following three equations are simultaneously satisfied at each lattice site \cite{7},

\[
\begin{align*}
\left[ \left( \frac{\partial W_j}{\partial \tau_j} \right)^2 + \mathcal{F}_j \left( \frac{\partial W_j}{\partial R_j} \right)^2 - \frac{1}{\mathcal{F}_j} \right] &= 0, \\
\left[ \frac{\partial^2 W_j}{\partial \tau_j^2} + \mathcal{F}_j \frac{\partial^2 W_j}{\partial R_j^2} + A_j \frac{\partial W_j}{\partial R_j} \right] &= 0, \\
B_j &= 0.
\end{align*}
\]
(60)

Moreover, it is straightforward that the Hamiltonian constraint is Hermitean if and only if

\[
A_j = \mathcal{F}_j \partial_{R_j} \ln(\mathcal{m}_j |\mathcal{F}_j|),
\]
(61)

where \(\mathcal{m}_j\) is the Hilbert space measure.

Unique solutions to the equations in (60) and having the form given in (51) have been obtained in all, even the non-marginally bound, cases \cite{7}. For the marginally bound models the solution for the phase \(W_j\) in the exterior, \textit{i.e.}, for shells that lie outside the apparent horizon \((R_j > F_j)\), is

\[
W_j^{(\pm)} = \tau_j \pm 2F_j \left[ z_j - \tanh^{-1} \frac{1}{z_j} \right], \quad z_j > 1,
\]
(62)

where \(z_j = \sqrt{R_j/F_j}\). The positive sign refers to ingoing waves, traveling toward the horizon and the negative sign to outgoing waves, as can be seen from the signature of the phase velocity,

\[
\dot{z}_j = \mp \frac{z_j^2 - 1}{2F_j z_j^2},
\]
(63)

keeping in mind that \(z_j > 1\). The phase velocity approaches zero at the horizon. In the interior, \textit{i.e.}, for shells that lie inside the apparent horizon \((R_j < F_j)\), the solution is

\[
W_j^{(\pm)} = \tau_j \pm 2F_j \left[ z_j - \tanh^{-1} z_j \right], \quad z_j < 1,
\]
(64)

but here the the positive sign refers to outgoing waves and the negative sign to ingoing waves, traveling toward the central singularity, again as determined by the phase velocity. Furthermore, as shown in Appendix B of \cite{7}, the system in (60) determines not only \(W_j\) but the Hilbert space measure, \(\mathcal{m}_j\), as well. For the marginal models under consideration, \(\mathcal{m}_j\) is regular everywhere and given by

\[
\mathcal{m}_j = z_j
\]
(65)

up to a constant scaling.
IV. Hawking Radiation

In this section we will argue that the states described above yield Hawking radiation. Our first approach will closely parallel Hawking’s original work \[12\]. First we need to introduce the concept of a black hole into the formalism above, which, following \[13\], we do by taking the mass function to be of the form

\[ F(r) = 2M\Theta(r) + f(r), \]  

where \( \Theta(r) \) is the Heaviside function and \( f(r) \) represents a dust perturbation \( (f(r)/2M \ll 1) \). It is easy to see that with this choice of mass function, the black hole state factors out in \(51\) and the remaining state then assumes the same form with \( F \) replaced by \( 2M \) and \( \Gamma \) replaced by \( f'(r) \),

\[ \psi[\tau, R, \Gamma] = e^{\pm i M W_0} \times \exp \left[ \pm \frac{i}{2} \int dr f'(r) W_f(\tau, R, M) \right] \overset{\text{def}}{=} \Psi_{\text{bh}} \times \Psi_f, \]  

where \( W_0 = W(\tau(0), R(0), F(0)) \) and the first exponent represents the black hole at the origin and the second, up to order \( f(r) \), represents the matter distribution that propagates in this background if we take \( F(r) \approx 2M \) in \( W_f \).

Next, we must identify those quantum states that correspond to the ingoing and outgoing modes, respectively and evaluate an appropriate inner product. Since the description should refer to observers at infinity, the inner product will be evaluated on hypersurfaces of constant Killing (Schwarzschild) time \( T \), and not on hypersurfaces of constant dust time (which corresponds to freely falling observers). It is not difficult to show that for contracting clouds the an infalling wave is given by

\[ \Psi_f = \exp \left[ -i \int dr f'(r) \left( T + 8M \left( z - \tanh^{-1} \frac{1}{z} \right) \right) \right] \]  

being approximately

\[ \Psi_f^- \approx \exp \left[ -i \int dr f'(r) \left( T + 8Mz \right) \right] \]  

when \( T \to -\infty \) and \( z \to \infty \) and

\[ \Psi_f^+ \approx \exp \left[ -i \int dr f'(r) \left( T - 8M \tanh^{-1} \frac{1}{z} \right) \right] \]  

when \( T \to +\infty \) and \( z \to 1 \). Thus, the simple looking phase on \( \Im^- \) scatters through the geometry to turn into the complicated looking phase on \( \Im^+ \) near the horizon. This is similar to what happens in the geometric optics approximation.

Equation \(69\) represents infalling waves. We can think of it as a product over plane waves, one at each label \( r \), as follows:

\[ \Psi^-_\omega = \prod_j e^{-i \omega_j T_j + 8M z_j}. \]
This should represent a complete set of infalling modes at each label \( j \), if we think of the \( \omega_j \) as the frequency of the modes. A complete set of outgoing modes on \( \mathbb{I}^+ \) would likewise be given by the functional

\[
\Psi_\omega^+ = \prod_j e^{-i\omega_j[T_j - 8Mz]} .
\]  

(72)

We now ask: what is the projection of our solution (70) on the negative frequency modes of the outgoing basis on \( \mathbb{I}^+ \). For this purpose we must consider the inner product of states on a hypersurface of constant Schwarzschild time \( T \). It turns out that Hawking’s thermal radiation is recovered if take the metric in the \((\tau, R)\) plane is given by the quadratic term in the Hamiltonian constraint (29) instead of (65). We speculate that this has to do with the essentially classical role played by the black hole, described by \( \Psi_{bh} \). Transforming to the metric in \((R, T)\) coordinates we find

\[
g_{RR} = \left( \frac{R}{R - 2M} \right)^2 .
\]

The required Bogoliubov coefficient is then given by the following inner product on a constant \( T \) hypersurface,

\[
\beta(f, \omega) = \langle \Psi_\omega \mid \Psi_f^+ \rangle = \prod_j \int \sqrt{g_{RR}} \, dR_j \, \Psi_\omega^+ \Psi_f^+ ,
\]  

(73)

which represents the negative frequency modes present in (70). A straightforward computation then yields

\[
\langle \text{in} \mid \hat{N}_{\text{out}} \mid \text{in} \rangle = |\beta(f, \omega)|^2 \approx \prod_j \frac{2\pi M}{\Delta f_j} \left[ \frac{1}{e^{8\pi M \Delta f_j} - 1} \right] ,
\]  

(74)

which is interpreted as the eternal black hole being in equilibrium with a thermal bath at the Hawking temperature \((8\pi M)^{-1}\). Thus we have a functional Schrödinger picture for dust Hawking radiation (see [14] for a generalization to the non-marginal models).

An alternative approach, one that is better adapted to quantum collapse, was considered in [15]. By matching the shell wave functions describing gravitational collapse across the apparent horizon, it was shown that an ingoing wave on one side of the apparent horizon is necessarily accompanied by an outgoing wave on the other side. Furthermore, the relative amplitude of the outgoing wave is suppressed by the square root of the Boltzmann factor at the “Hawking” temperature of the shell. Strictly speaking the Hawking temperature, \( T_H = (8\pi GM)^{-1} \), refers to an eternal black hole of mass \( M \). The temperature appearing in the Boltzmann factor from matching shell wave functions across the horizon is \( T_H = (4\pi GF)^{-1} \), where \( F \) is the mass function and represents twice the mass contained within the shell. Diffeomorphism invariant wave functionals describing the collapse can also be matched and yields the same picture, but now the relative amplitude of the outgoing wave functional to the ingoing one is given by \( e^{-S/2} \), where \( S \) is the entropy of the final state black hole.
V. BLACK HOLE ENTROPY

As mentioned in the previous section, black holes with ADM mass parameter $M$ are special cases of the solution in (1), obtained when the mass function is constant, $F = 2GM$, and the energy function is vanishing. This can be shown directly by a coordinate transformation of (1) from the comoving system $(\tau, \rho)$ to static coordinates $(T, R)$, in which the metric has the standard form,

$$ds^2 = \mathcal{F}(R)dT^2 - \mathcal{F}^{-1}(R)dR^2 - R^2d\Omega^2.$$  

(75)

We imagine therefore that the eternal black hole is a single shell and represented by the mass function

$$F(r) = 2M\Theta(r)$$  

(76)

($G = 1$), where $M$ is the mass at label $r = 0$ and $\Theta(r)$ is the Heaviside function. The mass density function is therefore

$$\Gamma(r) = 2M\delta(r)$$  

(77)

and, because of the $\delta-$distributional mass density, the wave-functional in (48) turns into the wave-function

$$\Psi[\tau, R, \Gamma] = e^{-\frac{i}{2}\int_0^\infty dr\Gamma(r)W(\tau(r), R(r), F(r))} = e^{-iMW_0(\tau, R, F)}$$  

(78)

where $\tau = \tau(0)$, $R = R(0)$ and $F = F(0)$. The Wheeler-DeWitt equation now becomes the Klein-Gordon equation describing the shell. Taking into account the factor ordering ambiguities and absorbing the $M$ dependent term, which now renormalizes the potential, into the function $B(R, F)$ we have

$$\left[\frac{\partial^2}{\partial \tau^2} + \mathcal{F}\frac{\partial^2}{\partial R^2} + A\frac{\partial}{\partial R} + B\right]e^{-iMW_0(\tau, R, F)} = 0,$$  

(79)

In contrast with the case in which the mass density is a smooth function over some set of non-zero measure, no regularization is necessary here. This means that no further conditions must be met and therefore that the measure as well as the functions $A(R, F)$ and $B(R, F)$ will remain undetermined although the function $A(R, F)$ will continue to be related to the measure according to (65). Thus two conditions are required to proceed with the quantization of the black holes as described above.

The first condition we impose is one on the measure appropriate to the Hilbert space of wave-functions. In the previous section we obtained the Hawking evaporation of a collapsing dust cloud surrounding a pre-existing black hole by taking the dust as a small perturbation to the black hole mass function in (76). The calculation proceeded by evaluating the Bogoliubov coefficient in the near horizon limit outside the horizon and crucial to obtaining the correct Hawking temperature is the choice of measure appropriate for eternal black holes. The measure was obtained from the DeWitt supermetric, $\gamma_{ab}$, on the configuration space $(\tau, R)$ and can be read directly from the Hamiltonian constraint

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\mathcal{F} \end{pmatrix}.$$  

(80)

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It gives \( \mu = 1/\sqrt{|F|} \), i.e.,

\[
\langle \Psi_1, \Psi_2 \rangle = \int \frac{dR}{\sqrt{|F|}} \Psi_1^\dagger \Psi_2
\]

(81)
as well as the function \( A(R, F) \) via the hermiticity condition (65). As long as \( F \neq 0 \) the Wheeler DeWitt equation can now be written as

\[
\left[ \frac{\partial^2}{\partial \tau^2} \pm \frac{\partial^2}{\partial R^2} + B \right] \Psi = 0,
\]

(82)

where the positive sign in the above equation refers to the exterior, while the negative sign refers to the interior and \( R_* \) is defined by

\[
R_* = \pm \int dR \sqrt{|F|}.
\]

(83)
The second condition arises because we are describing a single shell in this simple quantum mechanical model of an eternal black hole and because \( B(R, F) \) represents an interaction of the shell with itself. We simply demand there are no self interactions, i.e., that \( B(R, F) = 0 \).

The quantum evolution is then described by the free wave equation in the interior, but by an elliptic equation in the exterior. This signature change has been noted in other models [16, 17] and occurs because of the behavior of \( F \), which passes from positive outside the horizon to negative inside. For the black hole, it means that its wave function is supported in its interior. The spectrum will be determined by the proper radius, \( L_h \), of the horizon,

\[
L_h(M) = \int_0^{R_h} \frac{dR}{\sqrt{|F|}},
\]

(84)

where \( R_h \) is its area radius. If we extend the coordinate \( R_* \) to range over \((-\infty, \infty)\), thereby avoiding any issues related to a boundary at the center, this simple model of a quantum black hole effectively describes a dust shell in a “box” of radius \( 2L_h(M) \), which itself depends on its total ADM mass. The stationary states describe a spectrum of the form (reintroducing \( \hbar \) and \( G \))

\[
4GM_j L_h,j = A_{Pl} \left( j + \frac{1}{2} \right),
\]

(85)

where \( j \) is a whole number and \( A_{Pl} = \hbar G \) is the Planck area. It is straightforward to show that \( L_h = \pi GM \), so that

\[
\frac{A_j}{4} = A_{Pl} \left( j + \frac{1}{2} \right),
\]

(86)

where \( A_j \) is the horizon area [18].

As we show below, the equispaced area spectrum predicted by our simplified model of a quantum black hole implies that the entropy obeys the Bekenstein-Hawking area law provided that the area quanta are assumed distinguishable. The entropy therefore also admits a discrete and equispaced spectrum. What is the origin of the degeneracy that leads to the black hole entropy? Although the black hole is treated as a single shell with the spectrum in (85), this single shell is in fact the end state of many shells that have collapsed
to form the black hole. Regardless of their history, we assume that each of the shells then occupies only the levels of (85), contributing some multiple of the Planck area to the total horizon area of the final state. A black hole microstate is thus a particular distribution of collapsed shells among the available levels. If the distribution of shells is such that \( \mathcal{N}_j \) shells occupy level \( j \), the black hole’s total horizon area becomes

\[
\frac{A}{4} = A_{\text{Pl}} \sum_j \left( j + \frac{1}{2} \right) \mathcal{N}_j
\]

and the (single shell) solution in (75) is to be interpreted as an excitation by \( \mathcal{N} = \sum_j \mathcal{N}_j \) collapsed shells.

The spectrum in (87) represents the “area ensemble” and the number of black hole microstates giving the “area” \( A \) will depend on assumptions concerning the degeneracy of the microstates. Assuming the shells to be distinguishable, the number of states can be written in terms of the total number of area quanta, \( Q \), and the total number of shells, \( \mathcal{N} \), as

\[
\Omega(Q, \mathcal{N}) = \frac{(\mathcal{N} + Q - 1)!}{(\mathcal{N} - 1)! Q!},
\]

where

\[
Q = \frac{A}{A_{\text{Pl}}} - \frac{\mathcal{N}}{2}.
\]

Holding \( A \) fixed and extremizing the microcanonical entropy, \( S_{\text{micro}} = k_B \ln \Omega \), with respect to the number of shells gives

\[
S_{\text{micro}} = (2k_B \coth^{-1} \sqrt{5}) \frac{A}{4A_{\text{Pl}}},
\]

which is in excellent (better than 96%) agreement with the Bekenstein-Hawking entropy. In addition to the exponential growth in the number of states, the area quantization in (87) ensures that the entropy is effectively quantized in units of the Planck area, as originally proposed by Bekenstein [19].

Note that it is quite a simple matter to show that had the shells been assumed indistinguishable then the entropy would depend on the square-root of the area [20]. The fact that the area degrees of freedom must be treated as distinguishable runs contrary to our intuition for elementary degrees of freedom in quantum field theory and calls into question whether “area” is a fundamental quantity in quantum gravity.

VI. DISCUSSION

In our contribution we have described a canonical quantization of collapsing, inhomogeneous dust and some of its implications. While here we have confined ourselves to four dimensions and a vanishing cosmological constant, our set up can be extended to describe dust collapse in any dimension both with and without a cosmological constant and in all cases it is possible to show that static black holes will radiate at the appropriate Hawking temperature of the black hole [21, 22], at least in the semi-classical approximation in which
the mass function is taken to be of the form given in (66). Two, likely related, features of this description remain to be understood. Firstly, why does the Hawking picture rely on the measure derived from the classical Hamiltonian constraint to describe eternal black holes? Secondly how does the semiclassical picture hold up during collapse, i.e., when the dust is not taken to be a perturbation on the background of a massive black hole? In principle, the second question could be addressed by pursuing the approach of [15], but it still remains to be done. Another related problem concerns the regularization used to define the functional Wheeler-DeWitt equation (16). The lattice regularization we have used is an ad hoc regularization in which the divergent terms have to cancel each other. It leads directly to (10), implying that the Hamilton Jacobi solution is exact and, if factorizability on the lattice is required, making (63) and (64) unique as well. However, regularization independence of the results described here remains an open question and factorizability on a spatial lattice may be too strong a condition to impose.

Fundamental questions concerning the nature of the quantum gravitational degrees freedom remain, and these are most manifest in the assumptions that must be made concerning the statistics obeyed by them. For the BTZ black hole in 2+1 dimensions it is necessary to employ Bose statistics instead of Boltzmann statistics in order to recover the entropy [23]. This corresponds to the fact that the BTZ black hole admits an equispaced mass spectrum (as opposed to an area spectrum) and must be treated in the energy ensemble. Its heat capacity is positive, whereas it is negative for the Schwarzschild black hole. The actual computation of the entropy is similar to that employing the AdS/CFT correspondence [24] and the entropy is found to depend on two quantities, viz., the energy of what is taken to be the vacuum solution, $\Delta_0$, and a constant, $M_0$, arising out of a boundary contribution at the origin.4 A comparison between the result from the canonical theory and the result obtained via the AdS/CFT correspondence yields an effective central charge

$$c_{\text{eff}} = \frac{1}{2} \left[ 1 - \frac{48l}{h} (M_0 - \Delta_0) \right] ,$$

(91)

which must be set to $3l/2G\hbar$ to achieve agreement with the Bekenstein-Hawking entropy. This is the central charge of the Liouville theory induced at spatial infinity by 2+1 dimensional gravity [26].

In three or more spatial dimensions, with a negative cosmological constant, the spectrum of states describing eternal black holes has a more complicated description [27]. For small black holes, by which we mean black holes whose horizon radii are much smaller than the AdS length, an approximate area spectrum, similar to (86), is obtained and the entropy is recovered by working in an area ensemble in which the degrees of freedom are assumed distinguishable. In this case, the black hole heat capacity is negative. On the other hand, for large AdS black holes, i.e., black holes whose horizon radii are much larger than the

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4 Whereas a boundary contribution from the origin in 3+1 dimensional collapse would represent a singular initial configuration and is therefore set to zero, in 2+1 dimensions a non-vanishing contribution is essential to allow for an initial velocity profile that vanishes there. This does not lead to singular initial data and the boundary contribution does not have the interpretation of a point mass situated at the center [25].
AdS length, it is the mass that is quantized in integer units and the (mass) spectrum turns out to be independent of the gravitational constant, $G$. This is similar to what happens with the BTZ black hole (for which, however, there is no “small black hole” limit). The entropy is now obtained in an energy ensemble and the degrees of freedom must be assumed indistinguishable. It can then be shown that the thermodynamics in the large black hole limit is inextricably connected with the thermodynamics in the opposite limit by a duality of the Bose partition function. The gravitational constant, $G$, absent in the mass spectrum, reemerges in the thermodynamic description via this duality and the black hole heat capacity is positive. It appears that the Hawking-Page transition [28] separates the particle-like degrees of freedom of large black holes, which must be counted as indistinguishable, from the geometric degrees of freedom of small black holes, which are counted as distinguishable. This remains rather mysterious and seems worth pursuing further, but that is complicated by the fact that it is difficult to obtain closed form solutions for the spectrum of AdS black holes between the two limits described.

In conclusion, we now seem to be in a position to address some of the issues that have surrounded Black Hole thermodynamics for decades, such as the information loss problem, questions about singularity avoidance and the radiation from a naked singularity, at least within the context of a special class of models. We expect progress in addressing these and the new issues that have arisen since developing the quantization described here to be made in the near future.

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