Exact diagonalization of the quantum supersymmetric $SU_q(n|m)$ model

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Abstract

We use the algebraic nested Bethe ansatz to solve the eigenvalue and eigenvector problem of the supersymmetric $SU_q(n|m)$ model with open boundary conditions. Under an additional condition that model is related to a multicomponent supersymmetric t-J model. We also prove that the transfer matrix with open boundary condition is $SU_q(n|m)$ invariant.

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1 Introduction

The integrability of two-dimensional lattice models with periodic boundary condition is a consequence of the Yang-Baxter equation\[1,2\],

\[
R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)
\]

(1)

where the R-matrix is the Boltzmann weight of the two-dimensional vertex model. As usual, \(R_{12}(u)\), \(R_{13}(u)\) and \(R_{23}(u)\) act in \(C^n \otimes C^n \otimes C^n\) with \(R_{12}(u) = R(u) \otimes 1, R_{23}(u) = 1 \otimes R(u)\), etc.

During the last years, much more attention has been paid on the investigation of integrable systems with nontrivial boundary conditions, which was initiated by Cherednik\[3\] and Sklyanin\[4\]. They have introduced a systematic approach to handle the boundary problem in which the reflection equations appear. In addition with the Yang-Baxter equation, the reflection equations ensure the integrability of open models. Using this approach, Sklyanin \[4\], Destri and de Vega \[5\] solved the spin-\(\frac{1}{2}\) XXZ model with general boundary conditions by generalizing quantum inverse scattering method. Under a particular choice of boundary conditions, the Hamiltonian is \(U_q[sl(2)]\) invariant \[6\]. In \[4\], Sklyanin assumed that the R-matrix is \(P^{-}\) and \(T^{-}\)-symmetric. Furthermore, the R-matrix satisfies unitarity and cross-unitarity properties. Because only few models satisfy these properties, Mezincescu and Nepomechie \[7\] extended Sklyanin’s formalism to the \(PT\)-invariant systems. Thus, all trigonometric R matrices listed by Bazhanov \[8\] and Jimbo \[9\] can be related to 1-dimensional quantum spin chains in this formulism. Using the unitarity and cross-unitarity properties of Belavin’s \(Z_n\) elliptic R-matrix, we have constructed the open boundary transfer matrix with one parameter \[10,11\].

On the other hand, the study of open boundary conditions in 2-dimensional field theory is related to the Sine-Gorden, Affine Toda and \(O(N)\) Sigma models \[12,13,14,15\]. Sklyanin generalized the hamiltonian to the case nonlinear partial differential equations with local boundary conditions \[15\]. The reflection matrix is consistent with the integrability of the systems.

Recently, Foerster and Karowski have used the nested Bethe ansatz method to find the eigenvalues and eigenvectors of the supersymmetric t-J model with open boundary conditions and proved its \(spl_q(2, 1)\) invariance \[16\]. Gonzalez-Ruiz also solved this problem with the general diagonal solutions of the reflection equation \[17\]. The investigated model is a graded 15-vertex model characterized by two bosons and one fermion. De Vega and Gonzalez-Ruiz have also generalized the nested Bethe ansatz to the case of \(SU_q(n)\)-invariant chains\[18\].

The graded vertex model was first proposed by Perk and Schultz \[19\]. In this model all variables take \(m+n\) different values and the weights favor ferroelectric or antiferroelectric configurations. In the references \[20\] and \[21\], the Bethe Ansatz equations and the exact free energy and excited spectrum of this model with periodic boundary condition are found. The finit size correction shows the central charge of the model being \(m+n-1\) (replacing \(n\) in ref \[20\] by \(m+n-1\)). In fact, the supersymmetric t-J model is a special
Perk-Schultz model \((m = 2, n = 1)\). Under an appropriate boundary condition, the model enjoys beautiful structure as quantum group symmetry. This motivates us to consider the general graded vertex model with open boundary condition.

In this paper, we use the nested Bethe ansatz method to find the eigenvalues and eigenvectors of the transfer matrix for a graded vertex model with open boundary conditions. The transfer matrix with fixed boundary conditions is proved to be \(SU_q(n|m)\) invariant. When \(m = 1\), the model reduces into the q-deformed version of the generalized supersymmetric t-J model with \(n\) components. The hamiltonian contains a spin hopping term, the nearest neighbour spin-spin interaction and the contribution of boundary magnetic fields (see equation (24)). Now, we outline the contents of this paper. In sect.2 we introduce the \(SU_q(n|m)\) vertex model. We find the matrices \(K^\pm\) which define boundary conditions and nontrivial boundary terms in the hamiltonian. The relation between the transfer matrix and the hamiltonian of the generalized supersymmetric t-J model is also discussed as an example. Sect.3 covers to the diagonalization and the energy spectrum of the model with open boundary conditions in framework of the nested Bethe Ansatz. In sect.4 we show that the vertex model is a realization of the quantum supergroup \(SU_q(n|m)\). A proof that the transfer matrix with open boundary conditions is \(SU_q(n|m)\) invariant is given. In sect.5 the summary of our main results is presented and some further problems are discussed. The appendix contains some detailed calculation.

2 The vertex model and integrable open boundary conditions

Our starting point is a graded vertex model which was introduced by Perk and Schultz [19]. The thermodynamics of the model with periodic boundary condition was studied in [20,21]. Some interesting application of this model in quantum field theory was considered by Babelon, de Vega and Viallet [22].

The model is defined by vertex weights \(R(u)\), whose non-zero elements are

\[
egin{align*}
R_{aa}^{aa}(u) &= \sin(\eta + (-1)^{\epsilon_a} u), \\
R_{ab}^{ab}(u) &= \sin(u)(-1)^{\epsilon_a \epsilon_b}, a \neq b \\
R_{ba}^{ab}(u) &= \sin(\eta) e^{i\pi \epsilon_c \epsilon_d (a-b)}, a \neq b.
\end{align*}
\]

(2)

where

\[
\epsilon_a = \begin{cases} 
0, & a = 1, \ldots, m \\
1, & a = m + 1, \ldots, m + n,
\end{cases}
\]

(3)

\(\eta\) is an anisotropy parameter, \(a, b\) are indices running from 1 to \(m + n\). For convenience, we denote
This model is an $m + n$ state vertex model characterized by $m$ bosons and $n$ fermions. If $m = 2$, $n = 1$, this model reduces to the one studied by Foerster, Karowski [16] and Gonzalez-Ruiz [17]. The $A_{m-1}$ vertex model studied by de Vega and Gonzalez-Ruiz [18] is just special case $n = 0$. When $m = n = 2$, we can get a new electronic strong interaction model which is a generalization of the model proposed by Essler, Korepin and Schoutens [22].

The R-matrix defined by (2,3) is a trigonometric solution of the Yang-Baxter equation (1). The local transition matrix which is the operator representation of Yang-Baxter equation is the $(m + n) \times (m + n)$ matrix $L(u)$ satisfying the following equation [19,20,21]:

\[
R_{12}(u - v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u - v)
\]  

where $L_1(u) = L(u) \otimes 1$, $L_2(u) = 1 \otimes L(u)$. The standard row-to-row monodromy matrix for an $N \times N$ square lattice is defined by

\[
T(u) = L_N(u) \cdots L_1(u) = R_{0N}(u) \cdots R_{01}(u),
\]

Throughout the paper, $L(u)$ is assumed to be in the fundamental representation. $T(u)$ also fulfills the Yang-Baxter equation

\[
R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v).
\]

The operator $T$ is an $(m + n) \times (m + n)$ matrix of the operators acting in the quantum space $V_n^{\otimes N}$.

We can see that the R-matrix does not satisfy the Sklyanin’s $P$- and $T$-symmetry, but fulfills $PT$ invariance

\[
P_{12}R_{12}(u)P_{12} = R_{12}^t(u).
\]

It also obeys the unitarity and cross-unitarity properties

\[
R_{12}(u)R_{21}(-u) = \sin(u + \eta)\sin(\eta - u) \cdot id,
\]

\[
R_{12}^t(u)M_1R_{12}^t(-u - d\eta)M_1^{-1} = -\sin(u)\sin(u + d\eta) \cdot id.
\]

where $d = m - n$ and $M$ is a $(m + n) \times (m + n)$ matrix

\[(4)\]
\[ M_{bc} = \delta_{bc} M_b, \]
\[ M_b = \begin{cases} 
\epsilon de^{i2\eta(b-1)}, & b \leq m \\
\epsilon de^{i2\eta(2m-b)}, & b > m
\end{cases} \tag{11} \]

This can be verified by straightforward calculation. One can also verify that \( M \) is a symmetry matrix of the R-matrix:

\[ [M \otimes M, R_{12}(u)] = 0 \tag{12} \]

Now, we can use Mezincescu and Nepomechie’s generalized formalism to construct integrable systems with open boundary conditions. In our case, the reflection equations take the following form \([7]\):

\[ R_{12}(u-v)K^-_1(u)R_{21}(u+v)K^-_2(v) = K^-_2(v)R_{12}(u+v)K^-_1(u)R_{21}(u-v) \tag{13} \]

\[ R_{12}(-u+v)K^+_1(u)t_1 M_1^{-1} R_{21}(-u-v-d\eta) M_1 K^+_2(v)t_2 
= K^+_2(v)t_2 M_1 R_{12}(-u-v-d\eta) M_1^{-1} K^+_1(u)t_1 R_{21}(-u+v) \tag{14} \]

Obviously, there is an isomorphism between \( K^+(u) \) and \( K^-(u) \).

\[ \phi : K^-(u) \rightarrow K^+(u) = K^-(-u - \frac{d\eta}{2}) \tag{15} \]

Therefore, given a solution \( K^-(u) \) of equation (13), we can also find a solution \( K^+(u) \) of equation (14). But in a transfer matrix of an integrable lattice, \( K^-(u) \) and \( K^+(u) \) need not satisfy equation (15). In this paper, we will take equation (15) to define \( K^+ \).

After a long calculation, we find a solution of the reflection equation (13)

\[ K^-(u) = id \tag{16} \]

Correspondingly,

\[ K^+(u) = M \tag{17} \]

Taking Sklyanin’s formalism, the double-row monodromy matrix is defined as:

\[ U(u) = T(u)K^-(u)T^{-1}(-u) \tag{18} \]

where \( T^{-1}(u) \) is the inverse of \( T(u) \) in the auxiliary and quantum spaces, which explicitly is:

6
\[ T^{-1}(u) = L_1^{-1}(u) \cdots L_N^{-1}(u) = R_{01}^{-1}(u) \cdots R_{0N}^{-1}(u) \]  

(19)

With the help of the Yang-Baxter equation (7) and the reflection equation (13), one can prove that the double-row monodromy matrix satisfies the reflection equation

\[ R_{12}(u - v) U_1(u) R_{21}(u + v) U_2(v) = U_2(v) R_{12}(u + v) U_1(u) R_{21}(u - v) \]  

(20)

In this case, the transfer matrix is defined as:

\[ t(v) = tr K^+(v) U(v). \]  

(21)

Using the reflection equations (14,20) and the properties of the R-matrix (8-12), one can prove

\[ [t(u), t(v)] = 0. \]  

(22)

So the transfer matrix constitutes a one-parameter commutative family which ensures the integrability of the model. As indicated by Sklyanin, the transfer matrix is related to the hamiltonian of the quantum chain with nearest neighbour interaction and boundary terms

\[ t'(0) = 2 tr K^+(0) \frac{1}{\sin(\eta)} H - 2 N \cot \eta \]  

(23)

From equation (21), one can derive the explicit expression of the hamiltonian, which is omitted here because it is not used in the following discussion. In order to compare it with the \( SU_q(2|1) \) supersymmetric t-J model, we give the hamiltonian under \( m = 1 \), which is defined:

\[
    H = P \left\{ \sum_{j=1}^{N-1} \sum_s (c_{j,s}^\dagger c_{j+1,s} + c_{j+1,s}^\dagger c_{j,s}) \right\} P \\
    + \sum_{j=1}^{N-1} \cos \eta \left( \sum_{a=1}^m n_{j,a} n_{j+1,a} + \cos \eta \cdot (n_j + n_{j+1}) - \cos \eta \cdot (n_j n_{j+1}) \right) \\
    + \sum_{j=1}^{N-1} \left( \sum_{\alpha \in \Delta_+} (S_j^\alpha S_{j+1}^{-\alpha} + S_{j+1}^\alpha S_j^{-\alpha}) + isin \eta \cdot (n_j - n_{j+1}) \right) \\
    + \sum_{j=1}^{N-1} isin \eta \cdot (\sum_{a < b} n_{j,a} n_{j+1,b} - \sum_{a > b} n_{j,a} n_{j+1,b}),
\]  

(24)
where $c_j^\dagger (c_j)$ creates (annihilates) an electron with spin component $s, s = 1, 2, \cdots, n+m-1$ located $j$-th site. $n_j$ is the density operator, $S_j$ is the spin matrix at site $j$. $\Delta_+$ denotes the set of positive roots of the $su(m)$ algebra. $P$ is a operator projecting out doubly occupied states. The constraint is that more than one electron on each site is strictly prohibited. As we know [16,17], this hamiltonian is not hermitean, but it possesses real eigenvalues. We will show the hamiltonian to be $SU_q(n|m)$ invariant.

3 Nested Bethe ansatz for open boundary conditions

The graded vertex model with periodic boundary condition was investigated by de Vega and Lopes [20,21]. Based upon the Yang-Baxter equation, they obtain the Bethe Ansatz equations by using the nested Bethe ansatz method (periodic case). In this section, we want to generalize the nested Bethe ansatz method to solve the eigenvalue problem of the transfer matrix (21). In this case, the operator commutative relations are ruled by the reflection equation instead of the Yang-Baxter equation.

As we know, the double-row monodromy matrix satisfies the reflection equation. It is convenient to denote $u_- = u - v, \ u_+ = u + v$. We rewrite the equation (20) in the component form:

\[
R_{12}(u_-)^{a_1 a_2} R_{12}(u_+)^{d_1 d_2} U(v)^{d_2} a_2 c_2 R_{12}(u_-)^{a_1} a_1 R_{12}(u_+)^{d_1 d_2} U(v)^{d_2} b_2 \]

where the repeated indices sum over 1 to $m + n$. Next, we introduce a set of notations for convenience:

\[
A(v) = U(v)^{11}, \\
B_a(v) = U(v)^{1a}, \\
C_a(v) = U(v)^{a1}, \\
D_{ab}(v) = U(v)^{ab}, 2 \leq a, b \leq m + n. \tag{26}
\]

From equation (25) we will find the commutation relations. In order to simplify these relations, we introduce new operators:

\[
\tilde{D}_{ab}(v) = D_{ab}(v) - \delta_{ab} \frac{R_{12}(2v)^a_{1a}}{R_{12}(2v)^{11}} A(v) \tag{27}
\]

Considering the vertex model defined by equations (2,3), we rewrite equation (27) in an explicit form:
After some tedious calculation, we have found the commutation relations between $A(v)$, $\tilde{D}_{ab}(v)$ and $B_a(u)$ ($a, b = 2, \cdots, m + n$). The final results take the form (see Appendix A)

$$A(v)B_b(u) = \frac{a(u - v)b(u + v)}{a(u + v)b(u - v)}B_b(u)A(v)$$

- $\frac{c_+(u - v)b(2u)}{a(2u)b(u - v)}B_b(v)A(u)$

- $\frac{c_-(u + v)}{a(u + v)}B_c(v)\tilde{D}_{cb}(u)$

$$\tilde{D}_{a_1b_1}(u)B_{b_2}(v) = \frac{R_{12}(u + v + \eta)^{a_1c_2}_{c_1d_2}}{b(u - v)b(u + v + \eta)}R_{21}(u - v)^{d_1d_2}_{b_1b_2}B_{c_2}(v)\tilde{D}_{c_1d_1}(u)$$

- $R_{12}(2u + \eta)^{d_1d_2}_{b_1b_2}B_{d_1}(u)\tilde{D}_{d_2b_2}(v)$

All indices take values from 2 to $m + n$, and the repeated indices sum over 2 to $m + n$. The commutation relations presented above are only applicable to the cases $m \geq 1$. If $m=0$, the commutation relations change to the following form:

$$A(v)B_b(u) = \frac{w(u - v)b(u + v)}{w(u + v)b(u - v)}B_b(u)A(v)$$

- $\frac{c_+(u - v)b(2u)}{w(2u)b(u - v)}B_b(v)A(u)$

- $\frac{c_-(u + v)}{w(u + v)}B_c(v)\tilde{D}_{cb}(u)$

$$\tilde{D}_{a_1b_1}(u)B_{b_2}(v) = \frac{R_{12}(u + v - \eta)^{a_1c_2}_{c_1d_2}}{b(u - v)b(u + v - \eta)}R_{21}(u - v)^{d_1d_2}_{b_1b_2}B_{c_2}(v)\tilde{D}_{c_1d_1}(u)$$
\[-\frac{c_+(u-v)}{b(2u-\eta)b(u-v)} R_{12}(2u-\eta)^{a_1d_1} b_d \tilde{B}_d(v) \]
\[+ \frac{1}{b(2u-\eta) w(u+v) w(2v)} R_{12}(2u-\eta)^{a_1d_2} b_d A(v) \]  
(32)

The rule for indices is the same as the one in equations (29,30).

It is easy to find the so-called local vacuum $e_i^\pm$. We call the direct product of local vacuum a reference state or vacuum state. It takes the form:

$$|\text{vac} \rangle = \otimes^N \prod (1, 0, \ldots, 0)^t,$$

where $t$ denotes the transposition. One can find

$$A(u)|\text{vac} \rangle = \alpha(u)|\text{vac} \rangle,$$
$$C_a(u)|\text{vac} \rangle = 0,$$
$$B_a(u)|\text{vac} \rangle \neq 0,$$
$$\alpha(u) = [R(u)]^{11N}[R^{-1}(-u)]^{11N}.\]  
(34)

Next, let us calculate the action of $\tilde{D}_{ab}(u)$ on the vacuum state. We first recall the definition of $D_{ab}(u)$, and find

$$D_{ab}(u)|\text{vac} \rangle = T(u)_{a1} T^{-1}(-u)_{1b} |\text{vac} \rangle + T(u)_{ac} T^{-1}(-u)_{cb} |\text{vac} \rangle.\]  
(35)

The contribution of the first term can not be calculated directly. We will use the following method to find it. Taking $v = -u$ in the Yang-Baxter equation, we can get:

$$T_2^{-1}(-u) R_{12}(2u) T_1(u) = T_1(u) R_{12}(2u) T_2^{-1}(-u)\]  
(36)

Taking special indices in this relation and applying both sides of this relation to the vacuum state, we find:

$$T(u)_{a1} T^{-1}(-u)_{1b} |\text{vac} \rangle = \frac{c_+(2u)}{R(2u)^{11}} \left( \delta_{ab} \alpha(u) - T(u)_{ac} T^{-1}(-u)_{cb} \right) |\text{vac} \rangle.\]  
(37)

Substituting this relation to eq.(35), we have the result:

$$D_{ab}(u)|\text{vac} \rangle = \delta_{ab} \left\{ \frac{c_+(2u)}{R(2u)^{11}} \alpha(u) + \left( 1 - \frac{c_+(2u)}{R(2u)^{11}} \right) b^N(u) \tilde{b}^N(-u) \right\} |\text{vac} \rangle.\]  
(38)
So we have

\[ \tilde{D}_{ab}(u)|\text{vac} > = \delta_{ab} \left( 1 - \frac{c_+ (2u)}{R(2u)} \right) b^N(u)\tilde{b}^N(-u)|\text{vac} >, \quad (39) \]

where \( \tilde{b}(u) = R^{-1}(u)_{ab}^b, a \neq b \). In conclusion, the results of the action of \( A, B_a, C_a \) and \( \tilde{D}_{ab} \) on the vacuum state are listed as:

\[
A(u)|\text{vac} > = [R(u)^{\dagger}]^N[R^{-1}(u)^{\dagger}]^N|\text{vac} > = \alpha(u)|\text{vac} > \\
\tilde{D}_{ab}(u)|\text{vac} > = \delta_{ab} \left( 1 - \frac{c_+ (2u)}{R(2u)} \right) b^N(u)\tilde{b}^N(-u)|\text{vac} > = \delta_{ab}\beta(u)|\text{vac} > \\
C_a(u)|\text{vac} > = 0 \\
B_a(u)|\text{vac} > \neq 0. \quad (40)
\]

Note that the action of \( B_a(u) \) on the vacuum state is not proportional to the vacuum state.

We show that the eigenvectors of transfer matrix \( t(u) \) can be constructed by repeatedly applying operators \( B_b(v_i) \) on the vacuum state

\[ \Psi(v_1, \cdots, v_L) = B_{b_1}(v_1) \cdots B_{b_L}(v_L)|\text{vac} > F^{b_1 \cdots b_L} \quad (41) \]

Before using the Bethe ansatz method, let us introduce a set of notations that will be used in the following. We denote

\[ R_{21}(u)_{ab}^{cd}/R(u)^{\dagger}_{cd} = \tilde{R}_{21}(u)_{ab}^{cd}. \quad (42) \]

So from Appendix A, the commutation relations between \( B \)’s take the form

\[ B_{b_1}(u_1)B_{b_2}(u_2) = \tilde{R}_{12}(u_1 - u_2)_{b_2 b_1}^{d_2 d_1} B_{d_2}(u_2)B_{d_1}(u_1). \quad (43) \]

By repeatedly using this relation, we can commute \( B(v_k) \) with \( B(v_{k-1}), \cdots, B(v_1) \), respectively.

\[ B_{b_1}(v_1) \cdots B_{b_L}(v_L)|\text{vac} > = S(v_k, \{v_i\})_{b_1 \cdots b_L}^{d_1 \cdots d_L} B_{d_1}(v_k)B_{d_2}(v_1) \cdots B_{d_k}(v_k - 1)B_{d_{k+1}}(v_{k+1}) \cdots B_{d_L}(v_L)|\text{vac} > \quad (44) \]

where

\[ S(v_k, \{v_i\})_{b_1 \cdots b_L}^{d_1 \cdots d_L} = \prod_{j=1+k}^L \delta_{b_j d_j} \tilde{R}_{12}(v_1 - v_k)_{c_1 b_1}^{d_1 d_2} \tilde{R}_{12}(v_2 - v_k)_{c_2 b_2}^{d_2 d_3} \cdots \tilde{R}_{12}(v_k - 1 - v_k)_{b_k b_k - 1}^{c_{k-1} d_k} \quad (45) \]
Here $d_1$ and $b_k$ are considered as the "auxiliary space" indices, $b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_L$ and $d_2, \ldots, d_L$ are the "quantum space" indices. Notice that in the six vertex model, the B’s are commutable with each other. So one can use the symmetric argument that $v_k$ and $v_1$ are equivalent to each other. Now, we know that the B’s are not commutable with each other, but the relation (44) ensures that we can also use something like the symmetric argument. Actually, we can see from this relation that $v_k$ and $v_1$ are in an equivalent position if we omit the function $S$.

In the following, we deal with the case of $R(u)_{11} = a(u)$. It is convenient to introduce the notation:

$$L^{(1)}(\tilde{v}_1, \tilde{v}_1)_{a_1b_1} = R_{12}(\tilde{v}_1 + \tilde{v}_1)_{a_2b_2},$$

$$[L^{(1)}(\tilde{v}_1, \tilde{v}_1)^{-1}]_{a_1b_1} = \frac{R_{21}(-\tilde{v}_1 - \tilde{v}_1)_{a_1b_1}}{a(v_1 + v_1)a(-v_1 - v_1)},$$

$$R_{12}(\tilde{v}_1 + \tilde{v}_L)^{c_{qL-1dL}}R_{21}(\tilde{v}_1 - \tilde{v}_L)^{q_{qL-1bL}}\beta_{qL}(v_1)$$

$$= \frac{1}{a(\tilde{v}_1 - \tilde{v}_L)a(\tilde{v}_L - \tilde{v}_1)}[L^{(1)}(\tilde{v}_1, \tilde{v}_L)\beta(\tilde{v}_1)L^{(1)}(-\tilde{v}_1, \tilde{v}_L)^{-1}]^{c_{qL-1dL}}_{q_{qL-1bL}}.$$  \hspace{1cm} (46)

Here we have used the unitarity properties of R matrix, and $\tilde{v}_i = v_i + \eta/2$, $(\beta(\tilde{v}_1))_{ab} = \delta_{ab}\beta(\tilde{v}_1)$.

Now, let us evaluate the action of $A(u)$ on $\Psi$. Following the algebraic Bethe ansatz method, many terms will appear when we move $A(u)$ from the left hand side to the right hand side of $B_a$’s. They can be classified in two types: wanted and unwanted terms. The wanted terms in $A(u)\Psi$ can be obtained by repeatedly using the first term in relation (29), the unwanted terms arise from the second and third terms in relation (29), they are the types that $v_k$ is replaced by $u$. One unwanted term where $B(v_1)$ is replaced by $B(u)$ can be obtained by using first the second and third terms in relation (29), then repeatedly using the first terms in relation (29) and (30). Using this results we can obtain the general unwanted term where $B(v_k)$ is replaced by $B(u)$. So we can find the action of $A(u)$ on $\Psi$

$$A(u)B_{b_1}(v_1) \cdots B_{b_L}(v_L)|vac > F^{b_1 \cdots b_L}$$

$$= \prod_{j=1}^L \frac{a(v_j - u)b(v_j + u)}{b(v_j - u)a(v_j + u)} \alpha(u) \cdot B_{b_1}(v_1) \cdots B_{b_L}(v_L)|vac > F^{b_1 \cdots b_L}$$

$$+ \sum_{k=1}^L \frac{-c_+(v_k - u)b(2v_k)}{a(2v_k)b(v_k - u)} \prod_{j=1, \neq k}^L \frac{a(v_j - v_k)b(v_j + v_k)}{a(v_j + v_k)b(v_j - v_k)} \alpha(v_k)F^{b_1 \cdots b_L}.$$
Before calculating the action of the transfer matrix on Ψ, we should evaluate the action

\[ \text{where } u.t. \]

Here we rewrite relation (49) as:

Recalling the definition of the transfer matrix, we rewrite the transfer matrix as:

\[ \sum_{a=1}^{m+n} K_a^+(u) U(u)_{aa} \]

\[ = \sum_{a=2}^{m+n} K_a^+(u) D_{aa}(u) + \sum_{a=2}^{m+n} K_a^+(u) \left( \frac{c^+(2u)}{a(2u)} + K_a^I(u) \right) A(u) \]

\[ = \sum_{a=2}^{m+n} K_a^+(u) D_{aa}(u) + \frac{\sin(2u + \eta)}{\sin(2u + \eta)} e^{i(d-1)\eta} A(u) \]

Before calculating the action of the transfer matrix on Ψ, we should evaluate the action of \( K_a^+(u) D_{aa}(u) \) on it, which reads

\[ \sum_{a=2}^{m+n} K_a^+(u) D_{aa}(u) B_{b_1}(v_1) \cdots B_{b_L}(v_L) |\text{vac} > F^{b_1 \cdots b_L} \]

\[ = \sum_{a=2}^{m+n} K_a^+(u) F^{b_1 \cdots b_L} \beta(u) \prod_{j=1}^{L} \frac{1}{b(u - v_j) b(u + v_j + \eta)} \]

\[ \cdot R_{12}(u + v_1 + \eta)^{a_{d_1}} R_{21}(u - v_1)^{a_{b_1}} R_{12}(u + v_2 + \eta)^{p_{1d_2}} R_{21}(u - v_2)^{p_{2q_2}} \]

\[ \cdots R_{12}(u + v_{L + \eta})^{p_{Lq_2}} R_{21}(u - v_L)^{s_{Lq_L}} \]

\[ B_{d_1}(v_1) B_{d_2}(v_2) \cdots B_{d_L}(v_L) \delta_{p_{nq_n}} |\text{vac} > + u.t. \]

where \( u.t. \) stands for unwanted term. Using the definition of \( L^{(1)}(\bar{\bar{u}}, \bar{\bar{v}}) \) and its inverse, we rewrite relation (49) as:

\[ \sum_{a=2}^{m+n} K_a^+(u) F^{b_1 \cdots b_L} \prod_{j=1}^{L} \frac{a(u - v_j) a(v_j - u)}{b(u - v_j) b(u + v_j + \eta)} \beta(u) \]

\[ \left\{ \left( T^{(1)}(\bar{\bar{u}}, \{ \bar{\bar{v}}_1 \}) T^{(1)}(-\bar{\bar{u}}, \{ \bar{\bar{v}}_1 \}) \right)^{-1} \right\}_{a}^{b_1 \cdots b_L} \]

\[ B_{d_1}(v_1) B_{d_2}(v_2) \cdots B_{d_L}(v_L) |\text{vac} > + u.t. \]

Here
where a tedious calculation based upon the similar considerations as in the quantum space. The unwanted terms in equation (49) take two forms. After some long tedious calculation based upon the similar considerations as in the $A(u)$ case, we can get the following expression After long tedious calculation

$$u.t. = \sum_{k=1}^{L} S(u_k, \{v_i\})_{b_1 \ldots b_L}^{d_1 \ldots d_L} F_{b_1 \ldots b_L}^{c_1 \ldots c_L} \cdot \alpha(v_k) e^{i \eta}$$

$$\frac{c_+(u + v_k)b(2v_k+b(2u+\eta))}{a(u+v_k)a(2v_k+b(2u+\eta))} \prod_{j=i, \neq k}^{L} a(v_j - v_k)b(v_j + v_k) \cdot B_{d_1}(u) B_{d_2}(v_1) \cdots B_{d_k}(v_{k-1}) B_{d_{k+1}}(v_{k+1}) \cdots B_{d_L}(v_L) |vac > + \sum_{k=1}^{L} S(u_k, \{v_i\})_{c_1 \ldots c_L}^{b_1 \ldots b_L} F_{b_1 \ldots b_L}^{c_1 \ldots c_L} \cdot \beta(v_k) e^{i \eta}$$

$$\frac{-c_+(u - v_k)b(2v_k+b(2u+\eta))}{b(u-v_k)b(2u+\eta)} \prod_{j=i, \neq k}^{L} a(v_j - v_k)a(v_k - v_j)$$

$$\left(L^{(1)}(\tilde{v}_k, \tilde{(v)}_1) \cdots L^{(1)}(\tilde{v}_k, \tilde{v}_{k-1}) L^{(1)}(\tilde{v}_k, \tilde{v}_{k+1}) \cdots \right)$$

$$\cdots L^{(1)}(\tilde{v}_k, \tilde{v}_L) L^{(1)}(-\tilde{v}_k, \tilde{v}_L)^{-1} \cdots L^{(1)}(-\tilde{v}_k, \tilde{v}_{k+1})^{-1}$$

$$L^{(1)}(-\tilde{v}_k, \tilde{v}_{k-1})^{-1} \cdots L^{(1)}(-\tilde{v}_k, \tilde{v}_1)^{-1} \right)_{c_1 \cdots c_L}^{d_1 \cdots d_L}$$

$$B_{d_1}(u) B_{d_2}(v_1) \cdots B_{d_k}(v_{k-1}) B_{d_{k+1}}(v_{k+1}) \cdots B_{d_L}(v_L) |vac > .$$

In order to simplify equations (47) and (52), we need an important relation

$$\left\{L^{(1)}(\tilde{v}_k, \tilde{(v)}_1) \cdots L^{(1)}(\tilde{v}_k, \tilde{v}_{k-1}) L^{(1)}(\tilde{v}_k, \tilde{v}_{k+1}) \cdots \right\}_{c_1 \cdots c_L}^{d_1 \cdots d_L}$$

$$\cdot L^{(1)}(-\tilde{v}_k, \tilde{v}_L)^{-1} \cdots L^{(1)}(-\tilde{v}_k, \tilde{v}_{k+1})^{-1} L^{(1)}(-\tilde{v}_k, \tilde{v}_{k-1})^{-1} \cdots$$

$$\cdot L^{(1)}(-\tilde{v}_k, \tilde{v}_1)^{-1} \right\}_{c_1 \cdots c_L}^{d_1 \cdots d_L} S(v_k, \{v_i\})_{b_1 \ldots b_L}^{c_1 \ldots c_L}$$

$$= \frac{\sin(\eta)}{\sin(2v_k+d\eta)} e^{-i \eta} S(v_k, \{v_i\})_{b_1 \ldots b_L}^{c_1 \ldots c_L} \tau^{(2)}(\tilde{v}_k, \{\tilde{v}_i\})_{b_1 \ldots b_L}^{c_1 \ldots c_L}$$

where

$$\tau^{(2)}(u, \{\tilde{v}_i\})_{b_1 \ldots b_L}^{c_1 \ldots c_L}$$

$$= \sum_{a=2}^{m+n} K_a^{+}(u) \left\{L^{(1)}(u, \tilde{v}_1) \cdots L^{(1)}(u, \tilde{v}_L) \right\}_{b_1 \ldots b_L}^{c_1 \ldots c_L}$$

$$= \sum_{a=2}^{m+n} K_a^{+}(u) \left\{L^{(1)}(u, \tilde{v}_1) \cdots L^{(1)}(u, \tilde{v}_L) \right\}_{b_1 \ldots b_L}^{c_1 \ldots c_L}$$
\[ \cdot L^{(1)}(-u, \tilde{v}_L)^{-1} \cdots L^{(1)}(-u, \tilde{v}_1)^{-1} \}_{b_1 \cdots b_L}^{(1)} \]

\[ = \sum_{a=2}^{m+n} K^+_{a}(u) \left\{ \left( T^{(1)}(u, \{ \tilde{v}_i \}) T^{(1)}(-u, \{ \tilde{v}_i \})^{-1} \right)_{b_1 \cdots b_L} \right\}_{aa} \]  

(54)

Using the equations (48) and (53), we then obtain the action of \( t(u) \) on \( \Psi \)

\[ t(u)\Psi = \alpha(u) e^{i(d-1)\eta} \frac{\sin(2u + d\eta)}{\sin(2u + \eta)} \prod_{j=1}^{L} \frac{a(v_j - u)v_j + u)}{b(v_j - u)a(v_j + u)} \]

\[ F^{b_1 \cdots b_L}B_{b_1}(v_1) \cdots B_{b_L}(v_L)|vac > \]

\[ + \prod_{j=1}^{L} \frac{a(v_j - u)a(v_j + u)}{b(v_j - u)b(v_j + u + \eta)} \beta(u) \tau^{(2)}(\tilde{u}, \{ \tilde{v}_i \})^{d_1 \cdots d_L}_{b_1 \cdots b_L} \]

\[ F^{b_1 \cdots b_L}B_{d_1}(v_1) \cdots B_{d_L}(v_L)|vac > \]

\[ + \sum_{k=1}^{L} \left( \frac{-c_+(v_k - u)}{b(v_k - u)} e^{-i\eta} + \frac{c_+(v_k + u)}{a(v_k + u)} \right) \frac{\sin(2u + d\eta)b(2v_k)}{\sin(2u + \eta)a(2v_k)} e^{id\eta} \]

\[ \cdot \prod_{j=1}^{L} \frac{a(v_j - v_k)b(v_j + v_k)}{b(v_j - v_k)a(v_j + v_k)} \alpha(v_k)S(v_k, \{ v_i \})^{d_1 \cdots d_L}_{b_1 \cdots b_L} \]

\[ \cdot B_{d_1}(u)b_{d_2}(v_1) \cdots B_{d_k}(v_{k-1})B_{d_{k+1}}(v_{k+1}) \cdots B_{d_L}(v_L)|vac > \]

(55)

\[ - \sum_{k=1}^{L} \left( \frac{c_+(u + v_k)}{a(u + v_k)} e^{-i\eta} + \frac{c_+(v_k - u)}{b(v_k - u)} \right) \frac{\sin(2u + d\eta)}{\sin(2u + \eta)} \cdot \prod_{j=1, j \neq k}^{L} \frac{a(v_k - v_j)b(v_j + v_k + \eta)}{b(v_k - v_j)b(v_j + v_k + \eta)} \]

\[ \cdot \tau^{(2)}(\tilde{v}_k, \{ \tilde{v}_i \})^{c_1 \cdots c_L}_{b_1 \cdots b_L} F^{b_1 \cdots b_L} \]

\[ B_{d_1}(u)b_{d_2}(v_1) \cdots B_{d_k}(v_{k-1})B_{d_{k+1}}(v_{k+1}) \cdots B_{d_L}(v_L)|vac > \]

From the above equation, one can see that the function \( \Psi \) is not the eigenstate of \( t(u) \) unless \( F \)’s are the eigenstates of \( \tau^{(2)} \) and the sum of the third and the fourth term in the above equation is zero, which will give a restriction on the \( L \) spectrum parameters \( \{ v_i \} \). So, we have the following results:

If \( F \) is the eigenstate of \( \tau^{(2)} \) with the eigenvalue \( \Lambda^{(2)} \) satisfying equation (57), then \( \Psi \) is the eigenstate of \( t(u) \) with the eigenvalue \( \Lambda^{(1)} \),

\[ \Lambda^{(1)}(u) = \alpha(u) e^{i(d-1)\eta} \frac{\sin(2u + d\eta)}{\sin(2u + \eta)} \prod_{j=1}^{L} \frac{a(v_j - u)v_j + u)}{b(v_j - u)a(v_j + u)} \]

\[ + \beta(u) \prod_{j=1}^{L} \frac{a(v_j - u)v_j + u)}{b(u - v_j)b(v_j + u + \eta)} \Lambda^{(2)}(u, \{ v_i \}) \]  

(56)
where
\[
\tau^{(1)}(u, \{v_i\}) F = \Lambda^{(2)}(u, \{v_i\}) F
\]
\[
\Lambda^{(2)}(v_j, \{v_i\}) = \frac{\alpha(v_k)b(2v_k)\sin(2v_k + d\eta)}{\beta(v_k)a(2v_k)\sin(\eta)} e^{i(d-1)\eta} \prod_{j=1,\neq_k}^L \frac{b(v_j + v_k)}{a(v_k - v_j)}
\]

Therefore, the diagonalization of \(t(u)\) is reduced to finding the eigenvalue of \(\tau^{(2)}\). The explicit expression of \(\tau^{(2)}\) (see equation (55)) implies that \(\tau^{(2)}\) can be considered as the transfer matrix of an \(L\)-sites quantum chain, in which every spin takes \(m + n - 1\) values. The related Yang-Baxter equation is the same as the one of \(t(u)\), exception \(R\) being an \((m + n - 1)^2 \times (m + n - 1)^2\) matrix. Hence, we can use the same method to find the eigenvalue of \(\tau^{(2)}\). Repeating the procedure \(m\) times, one can reduce to a subsystem \(\tau^{(m+1)}\) which is an \(n \times n\) matrix in auxiliary space. The related Yang-Baxter equation is also defined by equation (2), but one should notice that in this case all \(\epsilon_a = -1\) due to \(m = 0\).

In order to diagonalize \(\tau^{(m+1)}\), we need the definition of \(\tilde{D}\) by the second equation (28). The elements of \(T^{(m)}\) satisfy equations (31) and (32). Following the same procedure, one can further reduce the \(\tau^{(m+1)}\) into the \(\tau^{(m+2)}\) subsystem. The later has the same structure as the former. In this case one finally obtains the eigenvalue of \(\tau^{(m+n-1)}\). This is the well-known nested Bethe Ansatz. Because the wave-functions are not needed in this paper, we omit them here. The eigenvalue and the constraint on the spectral parameters read as

\[
\Lambda^{(k)}(u, \{v_i^{(k-1)}\}, \{v_i^{(k)}\}) = \alpha^{(k)}(u, \{v_i^{(k-1)}\}) e^{i(d+k-2)\eta} \frac{\sin(2u + d\eta)}{\sin(2u + k\eta)}
\]
\[
\cdot \prod_{j=1}^{P_k} \frac{a(v_j^{(k)} - u)b(u + v_j^{(k)}) + (k - 1)\eta)}{b(v_j^{(k)} - u)a(u + v_j^{(k)}) + (k - 1)\eta)}
\]
\[
+ \beta^{(k)}(u, \{v_i^{(k-1)}\}) \prod_{j=1}^{P_k} \frac{a(v_j^{(k)} - u)a(u - v_j^{(k)})}{b(u - v_j^{(k)})b(u + v_j^{(k)} + k\eta)}
\]
\[
\cdot \Lambda^{(k+1)}(u, \{v_i^{(k)}\}, \{v_i^{(k+1)}\})
\]

\((1 \leq k \leq m)\)
\[ \Lambda^{(k)}(u, \{v_i^{(k-1)}\}, \{v_i^{(k)}\}) = \alpha^{(k)}(u, \{v_i^{(k-1)}\}) e^{i(d+2m-k)\eta} \frac{\sin(2u + d\eta)}{\sin(-2u + (k - 2m)\eta)} \]

\[
\cdot \prod_{j=1}^{P_k} \frac{w(v_j^{(k)} - u)b(u + v_j^{(k)} + (2m - k + 1)\eta)}{b(v_j^{(k)} - u)w(u + v_j^{(k)} + (2m - k + 1)\eta)}
\]

\[
+ \beta^{(k)}(u, \{v_i^{(k-1)}\}) \prod_{j=1}^{P_k} \frac{a(v_j^{(k)} - u)a(u - v_j^{(k)})}{b(u - v_j^{(k)})b(u + v_j^{(k)} + (2m - k)\eta)}
\]

\[
\cdot \Lambda^{(k+1)}(u, \{v_i^{(k)}\}, \{v_i^{(k-1)}\})
\]

\[(m + 1 \leq k \leq n) \tag{59}\]

and

\[
\Lambda^{(k+1)}(v_i^{(k)}, \{v_i^{(k)}\}, \{v_i^{(k+1)}\}) = \frac{\alpha^{(k)}(v_i^{(k)}, \{v_i^{(k-1)}\})}{\beta^{(k)}(v_i^{(k)}, \{v_i^{(k-1)}\})} \frac{\sin(2v_i^{(k)} + d\eta)}{\sin(\eta)} e^{i(d+k-2)\eta}
\]

\[
\cdot \frac{\sin(2v_i^{(k)} + (k-1)\eta)}{\sin(2v_i^{(k)} + k\eta)} \prod_{j=1, \neq i}^{P_k} \frac{\sin(v_i^{(k)} + v_j^{(k)} + (k-1)\eta)}{\sin(v_j^{(k)} - v_i^{(k)} - \eta)}
\]

\[(1 \leq k \leq m) \tag{60}\]

\[
\Lambda^{(k+1)}(v_i^{(k)}, \{v_i^{(k)}\}, \{v_i^{(k+1)}\}) = \frac{\alpha^{(k)}(v_i^{(k)}, \{v_i^{(k-1)}\})}{\beta^{(k)}(v_i^{(k)}, \{v_i^{(k-1)}\})} e^{i(d+2m-k)\eta}
\]

\[
\frac{\sin(2v_i^{(k)} + d\eta)\sin(2v_i^{(k)} + (2m - k + 1)\eta)}{\sin(\eta)\sin(2v_i^{(k)} + (2m - k)\eta)}
\]

\[
\cdot \prod_{j=1, \neq i}^{P_k} \frac{\sin(v_i^{(k)} + v_j^{(k)} + (2m - k + 1)\eta)}{\sin(v_i^{(k)} - v_j^{(k)} + \eta)}
\]

\[(m + 1 \leq k \leq m + n - 1) \tag{61}\]
where

\[
\alpha^{(k)}(u, \{v_i^{(k-1)}\}) = \begin{cases} 
\prod_{j=1}^{P_{k-1}} \frac{\sin(u + v_j^{(k-1)} + k\eta)}{\sin(v_j^{(k-1)} - u + \eta)}, & 1 \leq k \leq m \\
\prod_{j=1}^{P_{k-1}} \frac{\sin(u + v_j^{(k-1)} + (2m - k)\eta)}{\sin(v_j^{(k-1)} - u - \eta)}, & m + 1 \leq k \leq n \\
\prod_{j=1}^{P_{k-1}} \frac{\sin(u + v_j^{(k-1)} + (k - 1)\eta)\sin(u - v_j^{(k-1)})}{\sin(v_j^{(k-1)} - u + \eta)\sin(u - v_j^{(k-1)} + \eta)} \\
\cdot \frac{\sin(2u + (k - 1)\eta)}{\sin(2u + k\eta)}e^{-in}, & 1 \leq k \leq m \\
\prod_{j=1}^{P_{k-1}} \frac{\sin(u + v_j^{(k-1)} + (2m - k - 1)\eta)\sin(u - v_j^{(k-1)})}{\sin(v_j^{(k-1)} - u + \eta)\sin(u - v_j^{(k-1)} + \eta)} \\
\cdot \frac{\sin(2u + (2m - k - 1)\eta)}{\sin(2u + (2m - k)\eta)}e^{-in}, & m + 1 \leq k \leq n
\end{cases}
\]

(62)

In the above representation, we have assumed \(v_j^{(1)} = v_j, v_j^{(0)} = 0, P_0 = N, P_1 = L\) and \(\Lambda^{(m+n+1)} = 1\). Notice that \(\beta^{(k)}(u, \{v_i^{(k-1)}\})\) vanishes at the special points \(v_i^{(k-1)}\) due to the factor \(\sin(u - v_i^{(k-1)})\) appearing in \(\beta^{(k)}\). Taking \(u = v_i^{(k-1)}\) in formulae (58) and (59), we can get another kind of constraints on \(\Lambda^{(k)}\)

\[
\Lambda^{(k)}(v_i^{(k-1)}, \{v_i^{(k-1)}\}, \{v_i^{(k)}\}) = \alpha^{(k)}(v_i^{(k-1)}, \{v_i^{(k-1)}\})e^{i(d+k-2)\eta} \frac{\sin(2v_i^{(k-1)} + d\eta)}{\sin(2v_i^{(k-1)} + k\eta)} \\
\prod_{j=1}^{P_k} \frac{\sin(v_j^{(k)} - v_j^{(k-1)} + \eta)\sin(v_j^{(k)} + v_j^{(k-1)} + (k - 1)\eta)}{\sin(v_j^{(k)} - v_j^{(k-1)})\sin(v_j^{(k)} + v_j^{(k-1)} + k\eta)}
\]

(63)

\[
\Lambda^{(k)}(v_i^{(k-1)}, \{v_i^{(k-1)}\}, \{v_i^{(k)}\}) = -\alpha^{(k)}(v_i^{(k-1)}, \{v_i^{(k-1)}\})e^{i(d+2m-k)\eta} \frac{\sin(2v_i^{(k-1)} + d\eta)}{\sin(2v_i^{(k-1)} + (2m - k)\eta)} \\
\prod_{j=1}^{P_k} \frac{\sin(v_j^{(k)} - v_j^{(k-1)} - \eta)\sin(v_j^{(k)} + v_j^{(k-1)} + (2m - k + 1)\eta)}{\sin(v_j^{(k)} - v_j^{(k-1)})\sin(v_j^{(k)} + v_j^{(k-1)} + (2m - k)\eta)}
\]

(64)

Now, changing the index \(k\) into \(k + 1\) in the above formulae, we can obtain constrains on \(\Lambda^{(k+1)}\). Comparing these with equations (60) and (61), one can derive out the following
Bethe ansatz equations

\[
\begin{align*}
P_{k-1} & \prod_{j=1}^{k-1} \frac{\sin(v_i^{(k)} - v_j^{(k-1)} + \eta) \sin(v_i^{(k)} + v_j^{(k-1)} + k\eta)}{\sin(v_i^{(k)} - v_j^{(k-1)} + (k-1)\eta) \sin(v_i^{(k)} + v_j^{(k-1)} + k\eta)} \\
& \cdot \prod_{j=1}^{k+1} \frac{\sin(v_i^{(k)} - v_j^{(k+1)} - \eta) \sin(v_i^{(k)} + v_j^{(k+1)} + k\eta)}{\sin(v_i^{(k)} - v_j^{(k+1)} + (k+1)\eta) \sin(v_i^{(k)} + v_j^{(k+1)} + k\eta)} \\
& = \prod_{j=1,\not=j}^{P_k} \frac{\sin(v_i^{(k)} - v_j^{(k)} + \eta) \sin(v_i^{(k)} + v_j^{(k)} + (k+1)\eta)}{\sin(v_i^{(k)} - v_j^{(k)} + (k-1)\eta) \sin(v_i^{(k)} + v_j^{(k)} + (k+1)\eta)}
\end{align*}
\]

(65)

\[
\begin{align*}
P_{m-1} & \prod_{j=1}^{m-1} \frac{\sin(v_i^{(m)} - v_j^{(m-1)} + \eta) \sin(v_i^{(m)} + v_j^{(m-1)} + m\eta)}{\sin(v_i^{(m)} - v_j^{(m-1)} + (m-1)\eta) \sin(v_i^{(m)} + v_j^{(m-1)} + m\eta)} \\
& \cdot \prod_{j=1}^{m+1} \frac{\sin(v_i^{(m)} - v_j^{(m+1)} + \eta) \sin(v_i^{(m)} + v_j^{(m+1)} + (m+1)\eta)}{\sin(v_i^{(m)} - v_j^{(m+1)} + (m+1)\eta) \sin(v_i^{(m)} + v_j^{(m+1)} + (m+1)\eta)} \\
& = 1
\end{align*}
\]

(66)

\[
\begin{align*}
P_{k-1} & \prod_{j=1}^{k-1} \frac{\sin(v_i^{(k)} - v_j^{(k-1)} - \eta) \sin(v_i^{(k)} + v_j^{(k-1)} + (2m-k)\eta)}{\sin(v_i^{(k)} - v_j^{(k-1)} + (2m-k+1)\eta) \sin(v_i^{(k)} + v_j^{(k-1)} + (2m-k+1)\eta)} \\
& \cdot \prod_{j=1}^{k+1} \frac{\sin(v_i^{(k)} - v_j^{(k+1)} + \eta) \sin(v_i^{(k)} + v_j^{(k+1)} + (2m-k-1)\eta)}{\sin(v_i^{(k)} - v_j^{(k+1)} + (2m-k)\eta) \sin(v_i^{(k)} + v_j^{(k+1)} + (2m-k)\eta)} \\
& = \prod_{j=1,\not=j}^{P_k} \frac{\sin(v_i^{(k)} - v_j^{(k)} + \eta) \sin(v_i^{(k)} + v_j^{(k)} + (2m-k)\eta)}{\sin(v_i^{(k)} - v_j^{(k)} + (2m-k-1)\eta) \sin(v_i^{(k)} + v_j^{(k)} + (2m-k+1)\eta)}
\end{align*}
\]

(67)

The above Bethe ansatz equations are very complicated, but they can be simplified by introducing the following new variables

\[
v_j^{(k)} = \begin{cases} 
    w_i^{(k)} - k\eta/2, & 1 \leq k \leq m \\
    w_i^{(k)} - (2m-k)\eta/2, & m + 1 \leq k \leq m + n
\end{cases}
\]

(68)

The Bethe Ansatz equations then take the form
The function $\Lambda^{(1)}(u, \cdots)$ must not be singular at $u = v_j^{(k)}$ ($1 \leq j \leq p_k, 1 \leq k \leq m+n-1$) since the transfer matrix $t(u)$ is an analytic function of $u$. In fact, the equation (57) comes from the condition under which the unwanted term vanishes. One can understand this constraint from another point of view: From equation (56), we know that $u = v_j = v_j^{(1)}$ is a pole of $\Lambda^{(1)}(u)$. In order to keep the analyticity of $\Lambda^{(1)}(u)$, one should need the residue of $\Lambda^{(1)}(u)$ at $v_j$ vanishing, which also gives the constraint (57). So, $\Lambda^{(1)}(u)$ is analytic at $v_j$. Similarly, equations (69) ensure the analyticity of $\Lambda^{(1)}(u)$ at all $v_j^{(k)}$. Therefore, the eigenvalues of the transfer matrix are analytic functions if the previous Bethe ansatz equations are satisfied.

The energy spectrum of 1-dimensional quantum system defined by equation (23) can be derived from $\Lambda^{(1)}(u)$. It is

$$E = - \sum_{k=1}^{P_k=L} \frac{1}{\cos(\eta + 2v_k) - \cos(\eta)} + \frac{\sin[(d-1)\eta]}{4\sin^3(\eta)}$$ (71)

In order to comparing our results with the Bethe Ansatz equations given in references [20,21], we introduce new variables

$$\lambda_j^k = \begin{cases} v_j^{(k)}, & 1 \leq j \leq P_k \\ -v_{2P_k-j+1}^{(k)}, & P_k + 1 \leq j \leq 2P_k \end{cases}$$

Then equations (69) and (70) deduce to the Bethe Ansatz equations (see, for example, equation (4) in ref. [20]) up to a phase. In this sense, the Bethe Ansatz equations for the system with quantum group symmetry is the duoble of the ones for the same system with periodic boundary condition. One should note the constrain in the right hand side of equation (69), which will contribute a non-zero term to the free energy.
4 Quantum group structure of the model

In this section we will show that the vertex model under consideration is a realization of quantum supergroup $SU_q(n|m)$, and we will also prove that the transfer matrix for open boundary conditions is $SU_q(n|m)$ invariant.

Firstly, denoting $x = e^{iv}$, $q = e^{i\eta}$, the Yang-Baxter equation becomes

$$R_{12}(x/y)T_1(x)T_2(y) = T_2(y)T_1(x)R_{12}(x/y).$$  \hfill (72)

We write the R-matrix as

$$R(x) = xR_+ - x^{-1}R_-,$$  \hfill (73)

similarly, the $L$ operators can be written as

$$L(x) = xL_+ - x^{-1}L_-.$$  \hfill (74)

From the definition of R-matrix, $L_\pm$ can be written in the following form.

$$L_+^i = \begin{cases} q^{w_i}, & i \leq m, \\ \sigma_i q^{-w_i}, & m < i \leq m + n, \end{cases}$$  \hfill (75)

$$L_+^{i+1} = \begin{cases} (q - q^{-1})q^{-\frac{1}{2} \sum_{j \neq i, i+1} w_j} f_i, & i < m, \\ (q - q^{-1})q^{-\frac{1}{2} \sum_{j \neq m, m+1} w_j - w_{m+1}} \sigma_m f_m, & i = m, \\ (q - q^{-1})q^{\frac{1}{2} \sum_{j \neq i, i+1} w_j} \sigma_i f_i, & m < i \leq m + n, \end{cases}$$  \hfill (76)

$$L_-^i = \begin{cases} q^{-w_i}, & i \leq m, \\ \sigma_i q^{w_i}, & i > m, \end{cases}$$  \hfill (77)

$$L_-^{i+1} = \begin{cases} -(q - q^{-1})e_i q^{\frac{1}{2} \sum_{j \neq i, i+1} w_j}, & i < m, \\ -(q - q^{-1})e_m q^{\frac{1}{2} \sum_{j \neq m, m+1} w_j + w_{m+1}}, & i = m, \\ -(q - q^{-1})\sigma_i e_i q^{\frac{1}{2} \sum_{j \neq i, i+1} w_j}, & i > m. \end{cases}$$  \hfill (78)

Here $L_\pm$ are lower and upper triangular matrices with $L_{++} = L_{--} = 0$, if $i < j$, $w_i, i = 1, \ldots, m + n$; $e_i, f_i, i = 1, \ldots, m + n - 1$, are the generators of the $SU(n|m)$ superalgebra in the graded Cartan-Chevalley basis; the definition of the matrices $\sigma_i$ and the details of the classical simple Lie algebra $SU(n|m)$ are given in Appendix B. Recall the definition of the monodromy matrix $T(x)$: In the limit $x \to \infty, 0$, we find the leading terms $T_\pm$ of the monodromy matrix $T(x)$ to take the form,
Here \( T_\pm \) are lower and upper triangular matrices with \( T_{+j}^i = T_{-i}^j = 0 \), if \( i < j \), \( \alpha_\pm = q^{\pm 1/2}(q - q_{-1}) \), and

\[
T_{+,i}^i = \begin{cases} 
q^{-N/2}q^{w_i}, & i \leq m, \\
q^{-N/2}\sigma_iq^{-W_i}, & m < i \leq m + n, 
\end{cases} \tag{79}
\]

\[
T_{+i+1}^{i+1} = \begin{cases} 
\alpha_-q^{-\frac{1}{2}\sum_{j\neq i+1}W_j}F_i, & i < m, \\
\alpha_-q^{-\frac{1}{2}\sum_{j\neq, m+1}W_j-W_{m+1}}\tilde{\sigma}_mF_m, & i = m, \\
q^{-N}\alpha_+q^{\frac{1}{2}\sum_{j\neq, i+1}W_j}\tilde{\sigma}_iF_i, & i > m, 
\end{cases} \tag{80}
\]

\[
T_{-,i}^i = \begin{cases} 
q^{N/2}q^{-W_i}, & i \leq m, \\
q^{N/2}\sigma_iq^{W_i}, & m < i \leq m + n, 
\end{cases} \tag{81}
\]

\[
T_{-,i+1}^{-i+1} = \begin{cases} 
-\alpha_+E_iq^{\frac{1}{2}\sum_{j\neq, i+1}W_j}, & i \leq m, \\
-\alpha_+E_mq^{\frac{1}{2}\sum_{j\neq, m+1}W_j+W_{m+1}}, & i = m, \\
-q^{N}\alpha_-\tilde{\sigma}_iE_iq^{-\frac{1}{2}\sum_{j\neq, i+1}W_j}, & m < i \leq m + n. 
\end{cases} \tag{82}
\]

Here \( T_\pm \) are lower and upper triangular matrices with \( T_{+j}^i = T_{-i}^j = 0 \), if \( i < j \), \( \alpha_\pm = q^{\pm 1/2}(q - q_{-1}) \), and

\[
\tilde{\sigma}_i = \sigma_i \otimes \sigma_i \otimes \cdots \otimes \sigma_i, \quad i = m + 1, \ldots, m + n, \\
q^{\pm W_i} = q^{\pm w_i} \otimes \cdots \otimes q^{\pm w_i}, \quad i = 1, \ldots, m + n, \\
X_i = \sum_{j=1}^N q^{-h_i/2} \otimes \cdots \otimes q^{-h_i/2} \otimes x_i^{j_{th}} \otimes q^{h_i/2} \otimes \cdots \otimes q^{h_i/2}, \quad i < m, \tag{83}
\]

\[
X_m = \sum_{j=1}^N q^{-h_m/2} \otimes \cdots \otimes q^{-h_m/2} \otimes x_m^{j_{th}} \otimes (\sigma_{m+1}q^{h_m/2}) \otimes \cdots \otimes (\sigma_{m+1}q^{h_m/2}), \\
X_i = \sum_{j=1}^N q^{h_i/2} \otimes \cdots \otimes q^{h_i/2} \otimes x_i^{j_{th}} \otimes (\sigma_{m+1}\sigma_{m+2} \cdots \sigma_iq^{-h_i/2}) \\
\otimes \cdots \otimes (\sigma_{m+1}\sigma_{m+2} \cdots \sigma_iq^{-h_i/2}), \quad m < i \leq m + n.
\]

where \( X_i = E_i, F_i, x_i = e_i, f_i \), respectively. \( h_i = w_i - w_{i+1}, i \neq m \), and \( h_m = w_m + w_{m+1} \). In the case of \( N = 2 \), these formulae define the coproduct of a Hopf algebra. From this point of view, the equation (83) can be written as

\[
X_i = \Delta^{N-1}(x_i) = (\Delta \otimes id)\Delta^{N-2}(x_i) \tag{84}
\]

In the following we will discuss the algebraic relations of \((q^{w_i}, X_i)\). Taking the appropriate limits of the R-matrix and the row-to-row monodromy matrix \( T \), we have

22
\[
\lim_{x \to 0} x R(x/y) = -y R_-
\]
\[
\lim_{x \to \infty} x^{-1} R(x/y) = \frac{1}{y} R_+
\]
and
\[
\lim_{x \to \infty} x^N T(x) = -T_-
\]
\[
\lim_{x \to \infty} x^{-N} T(x) = T_+
\]
In the limits \(x \to 0, \infty\), the Yang-Baxter equation gives:
\[
R_\pm T_1 \pm T_2(y) = T_2(y) T_1 \pm R_\pm
\]
and
\[
R_\pm T_1 \pm T_2 \varepsilon = T_2 \varepsilon T_1 \pm R_\pm
\]
with \(\varepsilon = \{+, -\}\). These spectral-parameter independent Yang-Baxter relations govern \(q\)-(anti)commutation rules and \(q\)-Serre relations for the quantum supergroup \(SU_q(n|m)\).

Substituting the definition of \(R_\pm, T_\pm\) into equation (90), we get
\[
q^{H_i} q^{H_j} = q^{H_j} q^{H_i},
\]
\[
q^{H_i} F_j q^{-H_i} = q^{a_{ij}} F_j,
\]
\[
q^{H_i} E_j q^{-H_i} = q^{-a_{ij}} E_j,
\]
\[
[F_i, E_i] = \frac{q^{H_i} - q^{-H_i}}{q - q^{-1}}, i \neq m,
\]
\[
[F_m, E_m]_+ = \frac{q^{H_m} - q^{-H_m}}{q - q^{-1}}
\]
\[
E_m^2 = F_m^2 = 0, [F_i, E_j] = 0, i \neq j,
\]
\[
(F_i)^2 F_{i+1} - (q + q^{-1}) F_i F_{i+1} F_i + F_{i+1}(F_i)^2 = 0,
\]
\[
(E_i)^2 E_{i+1} - (q - q^{-1}) E_i E_{i+1} E_i + E_{i+1}(E_i)^2 = 0.
\]
\[
(t(y), T_\pm) = 0
\]
where \(H_i = W_i - W_{i+1}\), if \(i \neq m\), \(H_m = W_m + W_{m+1}\), and \(a_{ij}\) is a component of the Cartan matrix which is given in Appendix B. The generators \(H_i, E_i, F_i, i = 1, \ldots, m + n - 1\), and relations listed above provide a definition of the quantum supergroup \(SU_q(n|m)\). In the remaining part of this section, we will verify that the transfer matrix \(t(y)\) with open boundary conditions is \(SU_q(n|m)\) invariant. The entries of lower and upper triangular matrix \(T_\pm\) are elements of \(SU_q(n|m)\). So, it is not necessary to compute commutators of \(t(y)\) with individual \(SU_q(n|m)\) generators. If the relation
\[
[t(y), T_\pm] = 0
\]
is correct, we are led to the conclusion that the transfer matrix $t(u)$ is $SU_q(n|m)$ invariant. From eq. (89) we have the result

$$[R_{\pm} T_{1\pm}, T_2(y) T_2^{-1}(y^{-1})] = 0$$

(93)

Recall the relation (12), we have

$$[R_{\pm}, M_1 M_2] = 0$$

(94)

Similarly, from the unitarity and cross-unitarity relations (9,10), with the help of $PT$ invariance of R-matrix, we find

$$R_{\pm} R_{\mp}^{t_1 t_2} = 1$$

$$R_{\pm}^{t_1} M_1 R_{\mp}^{t_2} M_1^{-1} = 1$$

(95)

So we have the identity $R_{\pm}^{t_1} = (R_{\pm}^{-1})^{t_2}$ which implies that the following relation

$$M_1^{-1}(R_{\pm}^{-1})^{t_2} M_1 R_{\pm}^{t_2} = 1$$

(96)

is correct. Notice that we choose $K_- = 1$ in this paper, so the transfer matrix can be written as $t(y) = tr M T (y) T^{-1}(y^{-1})$. Now, let us prove relation (92)

$$T_1 \pm t(y) = tr_2 T_{1\pm} M_2 T_2(y) T_2^{-1}(y^{-1})$$

$$= tr_2 M_2 R_{\pm}^{-1} R_{\pm} T_{1\pm} T_2(y) T_2^{-1}(y^{-1})$$

(97)

here we have added an identity $R_{\pm}^{-1} R_{\pm}$ in the relation, then using the relations (95) and (96), we find

$$\cdots = tr_2 M_1^{-1} R_{\mp}^{-1} M_1 (M_2 T_2(y) T_2^{-1}(y^{-1})) R_{\pm} T_{1\pm}$$

$$= tr_2 \{ M_1^{-1} R_{\mp}^{-1} M_1 \}^{t_2} \{ (M_2 T_2(y) T_2^{-1}(y^{-1})) R_{\pm} T_{1\pm} \}^{t_2}$$

$$= tr_2 M_2 T_2(y) T_2^{-1}(y^{-1}) T_{1\pm}$$

$$= t(y) T_{1\pm}.$$

(98)

Thus, we have proved that the transfer matrix with a particular choice of open boundary conditions is quantum supergroup $SU_q(n|m)$ invariant.

5 Summary

In this paper, we have diagonalized the graded vertex model with open boundary condition by using the generalized algebraic Bethe ansatz method. In order to get the energy
spectrum of 1-dimensional quantum system defined by equation (23), we assume \( v_j^{(0)} \) to be zero in equations (58) and (60). However, one would as will assume \( v_j^{(0)} \neq 0 \). In this case, equations (56), (69) and (70) lead to the solution of inhomogeneous graded vertex model. Formally, one can also define a 1-dimensional quantum system by equation (23). Generally, the Hamiltonian is not represented in the nearest neighbour interaction form. We also show the \( SU_q(m|n) \) invariance of the quantum spin chain (equivalent to a graded vertex model). Thus, the generators of \( SU_q(m|n) \) commute with the infinite number of conserved quantities. The Hilbert space of the system can be classified according to the irreducible representations of \( SU_q(m|n) \). We hope that it will be help to solve the Bethe ansatz equations.

In order to find the free energy of the system, one should to solve the Bethe ansatz equations. Following the method given in reference [18], we can deduce the Bethe ansatz equations into those of the periodic case on \( 2N \) sites with an additional source factor (see ref. [20]). The free energy contains two terms. One is the known bulk free energy, another is the surface free energy which is the correction of the open boundary conditions (that keeps the quantum group symmetry). This was pointed by de Vega and Gonzalez-Ruiz in \( SU(n) \) case.

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6 Appendix A

The starting point for commutation relations is reflection equation (25). Let \( a_1 = a_2 = b_2 = 1, \ b_1 = b \neq 1 \), we find:

\[
A(v)B_b(u) = \frac{R_{12}(u_-)^{11}R_{21}(u_+)^{b1}}{R_{12}(u_+)^{11}R_{21}(u_-)^{b1}} B_b^*(u)A(v)
- \frac{R_{12}(u_+)^{1b}R_{21}(u_-)^{b1}}{R_{12}(u_+)^{11}R_{21}(u_-)^{b1}} B_b^*(v)A(u)
- \frac{R_{12}(u_+)^{1c}B_c(v)D_{cb}(u)}{R_{12}(u_+)^{11}}
\]

(99)

Due to eq.(27), it can be checked that the following relation is always true for \( R(u)^{11}_{11} = \sin(\eta \pm u) \).

\[
A(v)B_b(u) = \frac{R_{12}(u_-)^{11}R_{21}(u_+)^{b1}}{R_{12}(u_+)^{11}R_{21}(u_-)^{b1}} B_b^*(u)A(v)
\]
Case 1:

The results can be written in a simple form, the main calculation results are as follows.

Substituting (27) to (A.3), we obtain

\[
\begin{align*}
- \frac{R_{12}(u^-)_{i_1}R_{12}(2u)_{i_2}}{R_{12}(2u)_{i_1}R_{12}(u^-)_{i_2}} B_a(v)A(u) \\
- \frac{R_{12}(u^+)_{a_1}c_{1_1}B_c(v)\tilde{D}_{cb}(u)}{R_{12}(u^+)_{a_1}B_c(v)\tilde{D}_{cb}(u)}
\end{align*}
\]

(100)

Obviously, commutation relations (29,31) can be obtained from (100).

Next, let \( a_2 = 1, a_1, b_1, b_2 \neq 1 \), one can get:

\[
D_{a_1b_1}(u)B_{b_2}(v) = \frac{R_{12}(u^+)_{a_1}c_{1_1}}{R_{12}(u^+)_{a_1}c_{1_1}R_{12}(u^+)_{a_1}B_{c_2}(v)D_{c_1d_1}(u)}
\]

(101)

Substituting (27) to (A.3), we obtain

\[
\tilde{D}_{a_1b_1}(u)B_{b_2}(v) = \frac{1}{R_{12}(u^-)_{a_1}R_{12}(u^+)_{a_1}R_{12}(2u)_{a_1}R_{12}(2u)_{a_1}R_{12}(u^-)_{a_1}}
\]

(102)

\[
F = R_{12}(u^+)_{a_1}c_{1_1}R_{12}(u^-)_{b_1b_2} \frac{R_{12}(2u)_{c_1}c_{1_1}}{R_{12}(2u)_{c_1}c_{1_1}} B_{c_2}(v)A(u)
\]

(103)

In the following we will calculate the function \( F \) for the case of \( R(u)_{11} = a(u) = \sin(u+\eta) \).

The results can be written in a simple form, the main calculation results are as follows.

Case 1: \( a_1 \neq b_1 \)

\[
F = \delta_{a_1b_2} \frac{\sin(\eta)\sin(u+v)\sin(u+v+\eta)\sin(v+\eta)}{\sin(u+v+\eta)\sin(v+\eta)} B_{b_2}(v)A(v)
\]

(104)
Case 1: $a_1 = b_1 = b_2 = a$

$$F = \frac{\sin(\eta) e^{i(u+v)} \sin(u+v) \sin(u-v) \sin(2v) \sin(\eta + \epsilon_a(2u + \eta))}{\sin(2u + \eta) \sin(2v + \eta) \sin(u + v + \eta)} B_a(u) A(v)$$

$$- \frac{\sin^2(\eta) \sin(\eta + \epsilon_a(u - v))}{\sin(u + v + \eta)} B_c(v) \tilde{D}_{ca}(u)$$

$$+ \frac{\sin(u - v + \eta) \sin^2(\eta) e^{i(u-v)}}{\sin(2u + \eta)} B_c(u) \tilde{D}_{ca}(v)$$

(105)

Case 2: $a_1 = b_1 = b_2 = a$

$$F = \frac{\sin(\eta) e^{i(u+v)} \sin(u+v) \sin(u-v) \sin(2v) \sin(\eta + \epsilon_a(2u + \eta))}{\sin(2u + \eta) \sin(2v + \eta) \sin(u + v + \eta)} B_a(u) A(v)$$

$$- \frac{\sin^2(\eta) R_{21}(u-v)^{2a_1 b_2}}{\sin(u + v + \eta)} B_c(v) \tilde{D}_{cb_2}(u)$$

$$+ \frac{\sin(u - v + \eta) \sin^2(\eta) e^{i(u-v)}}{\sin(2u + \eta)} B_c(u) \tilde{D}_{cb_2}(v)$$

(106)

Case 3: $a_1 = b_1 \neq b_2$

Correspondingly, in the case of $R(u)_{a_a}^a = \sin(\eta - u) = w(u), a = 1, \cdots, n$, the main calculation results are presented in the following form.

Case 1: $a_1 \neq b_1$

$$F = \delta_{a_1 b_2} \frac{\sin(\eta) e^{i(u+v)} \sin(2v) \sin(v + v) \sin(u - v)}{\sin(\eta - u - v) \sin(\eta - 2v)} B_{b_1}(u) A(v)$$

$$- \delta_{a_1 b_2} \frac{\sin^2(\eta) \sin(u - v)}{\sin(\eta - u - v)} B_c(v) \tilde{D}_{cb_1}(u)$$

(107)

Case 2: $a_1 = b_1 = b_2 = a$

$$F = \frac{\sin(\eta) e^{i(u+v)} \sin(u+v) \sin(u-v) \sin(2v) \sin(\eta + \epsilon_a(2u + \eta))}{\sin(\eta - 2v) \sin(\eta - 2u) \sin(\eta - u - v)} B_a(u) A(v)$$

$$- \frac{\sin^2(\eta) \sin(\eta - u + v)}{\sin(\eta - u - v)} B_c(v) \tilde{D}_{ca}(u)$$

$$+ \frac{\sin^2(\eta) e^{i(u-v)} \sin(\eta - u + v)}{\sin(\eta - 2u)} B_c(u) \tilde{D}_{ca}(v)$$

(108)

Case 3: $a_1 = b_1 \neq b_2$

$$F = \frac{\sin(u + v) \sin(u - v) \sin(2v) \sin(\eta) e^{i(u+v)} R_{12}(2u - \eta)^{a_1 b_2}}{\sin(2u - \eta) \sin(\eta - 2v) \sin(\eta - u - v)} B_{b_2}(u) A(v)$$

27
\[
\begin{align*}
-\frac{\sin^2(\eta)R_{21}(u-v)b_{a_1}a_1}{\sin(\eta - u - v)}B_c(v)\tilde{D}_{cb}(u) \\
+ \frac{\sin(\eta - u + v)\sin^2(\eta)\sin(u-v)}{\sin(2u - \eta)}B_c(u)\tilde{D}_{cb}(v)
\end{align*}
\] (109)

Though we have already simplified the results, they still seem to be too complicated to be dealt with. Fortunately, we have found that the results (104-109) can be summarized as a concise form which indicates the commutation rules between \(\tilde{D}_{a_1b_1}(u)\) and \(B_{b_2}(v)\). The explicit commutation relations are written in sect. 3. One can prove it by expanding relations (30, 32) according to different cases mentioned above. Thus, we have obtained the commutation relations (29-32). It is also necessary to calculate the commutation relations between \(B_a(u)\) and \(B_b(v)\).

Let \(a_1 = a_2 = 1, b_1, b_2 \neq 1\), we have the results:

\[
B_{b_1}(u)B_{b_2}(v) = \frac{R_{12}(u_+)^{c_2}R_{21}(u_-)^{d_1c_2}}{R_{12}(u_-)^{b_1b_2}R_{21}(u_+)^{d_1c_2}}B_{c_2}(v)B_{d_1}(u)
\] (110)

7 Appendix B

The classical simple graded Lie algebra \(SU(n|m)\) is defined by generators \(h_i, e_i, f_i, i = 1, \cdots, m + n - 1\) and the following relations:

\[
\begin{align*}
[h_i, h_j] &= 0, \\
[h_i, f_j] &= a_{ij}f_j, [h_i, e_j] = -a_{ij}e_j, \\
[f_i, e_i] &= h_i, i \neq m, \\
[f_i, e_m]_+ &= h_m, \\
[f_i, e_j] &= 0, i \neq j, \\
f_i^2 &= e_m^2 = 0, \\
e_i f_{i\pm 1} - f_i e_{i\pm 1} &+ f_{i\pm 1} f_i^2 = 0 \\
e_i^2 &+ e_{i\pm 1} - 2e_i e_{i\pm 1}f_i + e_{i\pm 1}f_i^2 = 0
\end{align*}
\] (111)

The last two relations are the so called Serre relations, which are compatible conditions for \(SU(n|m)\), \(a_{ij}\) is the component of the graded Cartan matrix A defined by:

\[
\begin{align*}
a_{ii} &= 2, i \neq m, \\
a_{mm} &= 0, \\
a_{i+1, i} &= -1, \\
a_{i, i+1} &= -1, i \neq m \\
a_{m, m+1} &= 1
\end{align*}
\] (112)
the other elements being equal to zero. We define $\sigma_i$ as:

$$\sigma_i = diag(1, 1, \cdots, 1, -1, 1, \cdots, 1)$$  \hspace{1cm} (113)

where -1 is the $i$th element.

The fundamental representation of the generators takes the form

$$w_i = E_{i,i}, \; i = 1, \cdots, m+n,$$
$$f_i = E_{i,i+1}, \; i = 1, \cdots, m+n-1,$$
$$e_i = E_{i+1,i}, \; i = 1, \cdots, m+n-1,$$
$$h_i = w_i - w_{i+1}, \; i \neq m,$$
$$h_m = w_m + w_{m+1}.$$  \hspace{1cm} (114)

Here $E_{ij}$ are $(m+n) \times (m+n)$ matrices with the element in $i$-row $j$-column equal to 1, all other elements being zero.

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