A full characterization of Bertrand numeration systems

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Representing integers via an integer base sequence $U$

Representing real numbers via a real base $\beta$

Bertrand-Mathis’s work
Any $n \in \mathbb{N}$ can be decomposed in a unique way as

$$n = \sum_{i=1}^{\ell} a_i 3^{\ell-i}$$

where $a_i \in \{0, 1, 2\}$ and $a_1 \neq 0$. We write $\text{rep}_3(n) = a_1 \cdots a_\ell$.

The numeration language $\mathcal{N}_3$ is the set $0^*\text{rep}_3(\mathbb{N})$, which is simply $\{0, 1, 2\}^*$.
Representing real numbers in base 3

Any $x \in [0, 1)$ can be decomposed in a unique way as

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$$

where $a_i \in \{0, 1, 2\}$ and $a_i a_{i+1} a_{i+2} \cdots \neq 2^\omega$ for all $i$. We write $d_3(x) = a_1 a_2 a_3 \cdots$.

Define $D_3 = \{d_3(x) : x \in [0, 1)\}$.

The topological closure of $D_3$ is called the 3-shift:

$$S_3 = \{w \in \{0, 1, 2\}^\omega : \text{Fac}(w) \subseteq \text{Fac}(D_3)\} = \{0, 1, 2\}^\omega.$$  

Straightforward but crucial observation: $\text{Fac}(S_3) = \mathcal{N}_3$. 

Representing integers thanks to the Fibonacci sequence

We let $F_0 = 1$, $F_1 = 2$ and $F_{i+2} = F_{i+1} + F_i$ for $i \geq 0$.

Any $n \in \mathbb{N}$ can be decomposed in a unique way as

$$n = \sum_{i=1}^{\ell} a_i F_{\ell-i}$$

where $a_i \in \{0, 1\}$ and $a_1 \neq 0$ with the condition that $a_i a_{i+1} \neq 11$. We write $\text{rep}_F(n) = a_1 \cdots a_{\ell}$.

The numeration language $\mathcal{N}_F$ is the set $0^* \text{rep}_F(\mathbb{N})$.

This numeration system is called the Zeckendorf numeration system.
Representing real numbers in base $\varphi$

Let $\varphi = \frac{1+\sqrt{5}}{2}$ (the golden mean).

Any $x \in [0, 1)$ can be decomposed in a unique way as

$$x = \sum_{i=1}^{\infty} \frac{a_i}{\varphi^i}$$

where $a_i \in \{0, 1\}$, $a_ia_{i+1} \neq 11$ and $a_ia_{i+1}a_{i+2} \cdots \neq (10)^\omega$ for all $i$. We write $d_\varphi(x) = a_1a_2a_3\cdots$.

Define $D_\varphi = \{d_\varphi(x) : x \in [0, 1)\}$.

The topological closure of $D_\varphi$ is called the $\varphi$-shift:

$$S_\varphi = \{w \in \{0, 1\}^\omega : \text{Fac}(w) \subseteq \text{Fac}(D_\varphi)\} = \{0, 1\}^\omega \setminus \{0, 1\}^*11\{0, 1\}^\omega.$$ 

Straightforward but crucial observation: $\text{Fac}(S_\varphi) = N_F$. 

Let \( U = (U(i))_{i \geq 0} \) be an increasing integer sequence such that \( U(0) = 1 \) and

\[
C_U := \sup \{ i \geq 0 : \left\lceil \frac{U(i+1)}{U(i)} \right\rceil \} < \infty.
\]

We may represent any \( n \in \mathbb{N} \) by using the following greedy algorithm.

First, compute the least \( \ell \) such that \( n < U(\ell) \). Then for all \( i = 1, \ldots, \ell \), let \( a_i \) be the greatest integer such that

\[
\sum_{j=1}^{i-1} a_j U(\ell - j) + a U(\ell - i) \leq n.
\]

We get that

\[
\sum_{i=1}^{\ell} a_i U(\ell - i) = n.
\]

The finite word \( \text{rep}_U(n) = a_1 \cdots a_\ell \) is called the \( U \)-expansion of \( n \).

These words are written over the finite alphabet \( A_U = \{0, \ldots, C_U - 1\} \).
Representing real numbers via real bases $\beta > 1$

Let $\beta > 1$ be real number (called the base).

We may represent any $x \in [0, 1]$ by using the following greedy algorithm.

For all $i \geq 1$, let $a_i$ be the greatest integer $a$ such that

$$\sum_{j=1}^{i-1} \frac{a_j}{\beta^j} + \frac{a}{\beta^i} \leq x.$$ 

We get that

$$\sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = x.$$ 

The infinite word $d_\beta(x) = a_1 a_2 \cdots$ is called the $\beta$-expansion of $x$.

Only finitely many digits are used, namely $0, 1, \ldots, \lfloor \beta \rfloor$. 
Bertrand numeration systems

Let $U$ be a positional numeration system.

The set $\mathcal{N}_U = 0^* \text{rep}_U(\mathbb{N})$ is called the numeration language.

Two desired properties of $\mathcal{N}_U$ are:

- $\mathcal{N}_U$ is prefix-closed if all prefixes of words in $\mathcal{N}_U$ also belong to $\mathcal{N}_U$.
- $\mathcal{N}_U$ is prolongable if for all $w$ in $\mathcal{N}_U$, the word $w0$ also belongs to $\mathcal{N}_U$.

We say that $U$ is a Bertrand numeration system if $\mathcal{N}_U$ is both prefix-closed and prolongable.

Equivalently: $\forall w \in A^*_U, \ w \in \mathcal{N}_U \iff w0 \in \mathcal{N}_U$. 
State of the art

This form of the definition of Bertrand numeration systems, as well as their names after Bertrand-Mathis, was first given in

- Bruyère & Hansel 1997. Bertrand numeration systems and recognizability.

Then it was used in

- Point 2000. On decidable extensions of Presburger arithmetic: from A. Bertrand numeration systems to Pisot numbers
- Frougny 2002. Numeration systems. (Chapter 7 of Lothaire’s book "Algebraic combinatorics on words".)
- Lecomte & Rigo 2004. Real numbers having ultimately periodic representations in abstract numeration systems.
- Berthé & Rigo 2007. Odometers on regular languages.
- Charlier, Rampersad, Rigo & Waxweiler 2011. The minimal automaton recognizing $m\mathbb{N}$ in a linear numeration system.
- Massuiri, Pelomäki & Rigo 2019. Automatic sequences based on Parry or Bertrand numeration systems.
- Stipulanti 2019. Convergence of Pascal-like triangles in Parry-Bertrand numeration systems.
Other works considering Bertrand numeration systems are

- Loraud 1995. $\beta$-shift, systèmes de numération et automates.
- Frougny & Solomyak 1996. On representation of integers in linear numeration systems.
- Frougny 2003. On-line digit set conversion in real base.
- Frougny, Gazeau & Krejcar 2003. Additive and multiplicative properties of point sets based on beta-integers.
- Barat, Frougny & Pethö 2005. A note on linear recurrent Mahler numbers.
- Berthé & Siegel 2007. Purely periodic $\beta$-expansions in the Pisot non-unit case.
- Frougny & Sakarovitch 2010. Number representation and finite automata. (Chapter 2 of the book "Combinatorics, automata and number theory").
- Berthé, Frougny, Rigo & Sakarovitch 2020. The carry propagation of the successor function.
The $\beta$-shift

Before giving Bertrand-Mathis’s statement, we need one more notion on the real base side.

For $\beta > 1$, we let $D_\beta = \{d_\beta(x) : x \in [0, 1)\}$.

The $\beta$-shift is the topological closure of $D_\beta$:

$$S_\beta = \{w \in \{0, \ldots, \lceil \beta \rceil - 1\}^\omega : \text{Fac}(w) \subseteq \text{Fac}(D_\beta)\}.$$
Parry’s characterization of elements in the $\beta$-shift

In Parry’s theorem, the $\beta$-expansion and the quasi-greedy $\beta$-expansion of 1 play crucial roles.

The quasi-greedy $\beta$-expansion of 1 is

$$d^*_\beta(1) = \lim_{x \to 1^-} d_\beta(x).$$

If $d_\beta(1)$ does not end with a tail of zeros, we simply have $d^*_\beta(1) = d_\beta(1)$.

Otherwise, if $d_\beta(1) = t_1 \cdots t_n 0^\omega$ with $t_n \neq 0$, then $d^*_\beta(1) = (t_1 \cdots t_{n-1}(t_n - 1))^\omega$.

Theorem (Parry 1960)

$$S_\beta = \{w \in \{0, \ldots, \lceil \beta \rceil - 1\}^\omega : \forall i \geq 1, \; w_i w_{i+1} \cdots \leq_{\text{lex}} d^*_\beta(1)\}.$$
Parry’s descriptions of the 3-shift and the $\varphi$-shift

For $\beta = 3$, we get $d_3(1) = 30^\omega$ and $d_3^*(1) = 2^\omega$. So Parry’s theorem gives

$$S_3 = \{ w \in \{0, 1, 2\}^\omega : \forall i \geq 1, \ w_i w_{i+1} \cdots \leq_{\text{lex}} 2^\omega \}.$$ 

For $\beta = \varphi$, we get $d_\varphi(1) = 110^\omega$ and $d_\varphi^*(1) = (10)^\omega$. So Parry’s theorem gives

$$S_\varphi = \{ w \in \{0, 1\}^\omega : \forall i \geq 1, \ w_i w_{i+1} \cdots \leq_{\text{lex}} (10)^\omega \}.$$
Bertrand-Mathis’s statement

In 1989, Bertrand-Mathis stated that

\[ U \text{ is Bertrand if and only if } \exists \beta > 1 \text{ such that } N_U = \text{Fac}(S_\beta). \]

In this case, the following hold:

a. There is a unique such \( \beta \).

b. The alphabet \( A_U \) equals \( \{0, \ldots, \lceil \beta \rceil - 1\} \).

c. We have

\[ \forall i \geq 0, \quad U(i) = d_1 U(i - 1) + d_2 U(i - 2) + \cdots + d_i U(0) + 1 \]

and

\[ \lim_{i \to \infty} \frac{U(i)}{\beta^i} = \frac{\beta}{(\beta - 1) \sum_{i=1}^{\infty} i d_i \beta^{-i}} \]

where \( (d_i)_{i \geq 1} = d_\beta^*(1) \).

d. The system \( U \) has the dominant root \( \beta \), i.e., \( \lim_{i \to \infty} \frac{U(i+1)}{U(i)} = \beta \).
Bertrand-Mathis’s statement

In 1989, Bertrand-Mathis stated that

\[ \text{\(U\) is Bertrand if and only if} \ \exists \beta > 1 \text{ such that} \ N_U = \text{Fac}(S_\beta). \]

In this case, the following hold:

a. There is a unique such \(\beta\).

b. The alphabet \(A_U\) equals \(\{0, \ldots, \lceil\beta\rceil - 1\}\).

c. We have

\[ \forall i \geq 0, \quad U(i) = d_1 U(i - 1) + d_2 U(i - 2) + \cdots + d_i U(0) + 1 \]

and

\[ \lim_{i \to \infty} \frac{U(i)}{\beta^i} = \frac{\beta}{(\beta - 1) \sum_{i=1}^{\infty} i d_i \beta^{-i}} \]

where \((d_i)_{i \geq 1} = d^*_\beta(1)\).

d. The system \(U\) has the dominant root \(\beta\), i.e., \(\lim_{i \to \infty} \frac{U(i+1)}{U(i)} = \beta\).
For a real number $\beta > 1$, define

$$S'_{\beta} = \{ w \in \{0, \ldots, \lfloor \beta \rfloor \}^\omega : \forall i \geq 1, \ w_i w_{i+1} \cdots \leq_{\text{lex}} d_\beta(1) \}.$$ 

**Theorem (Charlier, Cisternino & Stipulanti 2022)**

A positional numeration system $U$ is Bertrand if and only if one of the following occurs.

1. **For all** $i \geq 0$, $U(i) = i + 1$.

2. There exists a real number $\beta > 1$ such that $N_U = \text{Fac}(S_{\beta})$.

3. There exists a real number $\beta > 1$ such that $N_U = \text{Fac}(S'_{\beta})$. 

Moreover, in Case 2 (resp. Case 3), the following hold:

a. There is a unique such $\beta$.

b. The alphabet $A_U$ equals $\{0, \ldots, \lceil \beta \rceil - 1\}$ (resp. $\{0, \ldots, \lfloor \beta \rfloor \}$).

c. We have

$$\forall i \geq 0, \quad U(i) = a_1 U(i - 1) + a_2 U(i - 2) + \cdots + a_i U(0) + 1$$

and

$$\lim_{i \to \infty} \frac{U(i)}{\beta^i} = \frac{\beta}{(\beta - 1) \sum_{i=1}^{\infty} i a_i \beta^{-i}}$$

where $(a_i)_{i \geq 1}$ is $d_\beta^*(1)$ (resp. $d_\beta(1)$).

d. The system $U$ has the dominant root $\beta$, i.e., $\lim_{i \to \infty} \frac{U(i+1)}{U(i)} = \beta$. 
Non-canonical Bertrand systems and non-canonical $\beta$-shifts

Let $\beta$ be a simple Parry number, i.e., such that $d_{\beta}(1)$ ends with a tail of zeroes. In this case, $d^*_{\beta}(1) \neq d_{\beta}(1)$, and hence there are two Bertrand numeration systems associated with $\beta$.

- The **canonical** Bertrand system is built from the digits of $d^*_{\beta}(1)$.
- The **non-canonical** Bertrand system is built from the digits of $d_{\beta}(1)$.

Similarly,

- The set $S_{\beta} = \{w \in \{0, \ldots, \lceil \beta \rceil - 1\}^\omega : \forall i \geq 1, \ w_i w_{i+1} \cdots \leq_{\text{lex}} d^*_{\beta}(1)\}$ is called the **canonical** $\beta$-shift.
- The set $S'_{\beta} = \{w \in \{0, \ldots, \lfloor \beta \rfloor\}^\omega : \forall i \geq 1, \ w_i w_{i+1} \cdots \leq_{\text{lex}} d_{\beta}(1)\}$ is called the **non-canonical** $\beta$-shift.
Canonical Bertrand numeration system associated with 3

Since $d^*_\beta(1) = 2^\omega$, the canonical Bertrand system associated with 3 is given by

$$\forall i \geq 0, \ U(i) = 2U(i - 1) + 2U(i - 2) + \cdots + 2U(0) + 1.$$ 

Thus, $U(0) = 1$ and for all $i \geq 0$, one has

$$U(i + 1) - U(i) = (2U(i) + 2U(i - 1) + \cdots + 2U(0) + 1)$$

$$- (2U(i - 1) + 2U(i - 2) + \cdots + 2U(0) + 1)$$

$$= 2U(i).$$ 

Hence $U(i + 1) = 3U(i)$ for all $i \geq 0$.

We see that this is precisely the integer base 3 numeration system $U = (3^i)_{i \geq 0}$. 
Non-canonical Bertrand system associated with 3

Since $d_3(1) = 30^ω$, the non-canonical Bertrand system associated with 3 is given by

$$∀i \geq 0, \ U(i) = 3U(i - 1) + 1.$$ 

We have $U = (1, 4, 13, 40, 121, \ldots)$. 

The corresponding numeration language $\mathcal{N}_U$ is equal to $\text{Fac}(S'_3)$ where the non-canonical 3-shift is

$$S'_3 = \{w \in \{0, 1, 2, 3\}^ω : ∀i \geq 1, \ w_iw_{i+1} \cdots \leq_{\text{lex}} 30^ω\}.$$ 

It is accepted by the DFA

![ DFA diagram ]

From this DFA, we can see that $U$ is Bertrand, i.e., that $\mathcal{N}_U$ is prefix-closed and prolongable.
Canonical Bertrand system associated with $\varphi$

Since $d^\ast_\varphi(1) = (10)^\omega$, the canonical Bertrand system associated with $\varphi$ is given by

$$\forall i \geq 0, \quad U(i) = \begin{cases} U(i - 1) + U(i - 3) + \cdots + U(1) + 1, & \text{if } i \text{ is even;} \\ U(i - 1) + U(i - 3) + \cdots + U(0) + 1, & \text{if } i \text{ is odd.} \end{cases}$$

Thus, $U(0) = 1$, $U(1) = U(0) + 1 = 2$ and for all $i \geq 0$, one has

$$U(i + 2) - U(i) = U(i + 1).$$

Hence $U(i + 2) = U(i + 1) + U(i)$ for all $i \geq 0$.

We see that this is precisely the Zeckendorf system $F = (1, 2, 3, 5, 8, 13, \ldots)$. 
Non-canonical Bertrand system associated with $\varphi$

Since $d_\varphi(1) = 110^\omega$, the non-canonical Bertrand system associated with $\varphi$ is given by

$$\forall i \geq 0, \ U(i) = U(i - 1) + U(i - 2) + 1,$$

i.e.,

$$U(0) = 1, \ U(1) = U(0) + 1 = 2, \text{ and } \forall i \geq 0, \ U(i + 2) = U(i + 1) + U(i) + 1.$$  

We have $U = (1, 2, 4, 7, 12, 20, 33, 54, \ldots)$.

The corresponding numeration language $\mathcal{N}_U$ is equal to $\text{Fac}(S'_\varphi)$ where the non-canonical $\varphi$-shift is

$$S'_\varphi = \{ w \in \{0, 1\}^\omega : \forall i \geq 1, \ w_i w_{i+1} \cdots \leq_{\text{lex}} 110^\omega \}.$$  

It is accepted by the DFA

![DFA Diagram](image)

From this DFA, we can check that $U$ is indeed a Bertrand numeration system.
Intermediate $\beta$-representations of 1

At first, our guess was that there could be other kinds of Bertrand numeration systems, namely any $U$ defined by

$$\forall i \geq 0, \quad U(i) = a_1 U(i - 1) + a_2 U(i - 2) + \cdots + a_i U(0) + 1$$

with the sequence of coefficients given by

$$(a_i)_{i \geq 1} = (t_1 \cdots t_{n-1}(t_n - 1))^k t_1 \cdots t_n 0^\omega$$

for any $k \in \mathbb{N} \cup \{\infty\}$.

In fact, what we get is that only the cases $k = 0$ or $k = \infty$ are possible.
Intermediates are not Bertrand

Let \((a_i)_{i \geq 1} = 230^\omega\). We have \(\frac{2}{3} + \frac{3}{2^2} = 1\).

Define \(U\) by

\[
U(0) = 1,
U(1) = 2U(0) + 1 = 3,
U(i) = 2U(i - 1) + 3U(i - 2) + 1, \quad i \geq 2.
\]

We get \(U = (1, 3, 10, 30, 91, \ldots)\).

This system is not Bertrand since for example, \(30 \in \mathcal{N}_U\) but \(3, 300 \notin \mathcal{N}_U\), showing that \(\mathcal{N}_U\) is neither prefix-closed nor prolongable.

In fact, we have

\[
U(i + 1) = \begin{cases} 
3U(i), & \text{if } i \text{ is odd;} \\
3U(i) + 1, & \text{if } i \text{ is even.} 
\end{cases}
\]
Intermediates are not Bertrand

Let \((a_i)_{i \geq 1} = 10110^\omega\). We have \(\frac{1}{\varphi} + \frac{1}{\varphi^3} + \frac{1}{\varphi^4} = 1\).

Define \(U\) by

\[
U(0) = 1, \\
U(1) = U(0) + 1 = 2, \\
U(2) = U(1) + 1 = 3, \\
U(3) = U(2) + U(0) + 1 = 5, \\
U(i) = U(i - 1) + U(i - 3) + U(i - 4) + 1, \quad i \geq 4.
\]

We get \(U = (1, 2, 3, 5, 9, 15, 24, 39, \ldots)\).

This system is not Bertrand since for example, \(1100, 11000 \in \mathcal{N}_U\) but \(11, 110, 110000 \notin \mathcal{N}_U\), showing that \(\mathcal{N}_U\) is neither prefix-closed nor prolongable.

In fact, we have

\[
U(i + 2) = \begin{cases} 
U(i + 1) + U(i), & \text{if } i \equiv 2, 3 \pmod{4}; \\
U(i + 1) + U(i) + 1, & \text{if } i \equiv 0, 1 \pmod{4}.
\end{cases}
\]
Proposition (Hollander 1998)

Let $U$ be a positional numeration system such that $\lim_{i \to \infty} \frac{U(i+1)}{U(i)} = \beta > 1$.

- If $\beta$ is not a simple Parry number, then
  \[ \lim_{i \to \infty} \text{rep}_U(U(i) - 1) = d_\beta(1). \]

- If $d_\beta(1) = t_1 \cdots t_n$ with $t_n \neq 0$, then for all $\ell \geq 0$, there exists $l \geq 0$ such that for all $i \geq l$, there exists $k \geq 0$ such that
  \[ \text{Pref}_\ell(\text{rep}_U(U(i) - 1)) = \text{Pref}_\ell((t_1 \cdots t_{n-1}(t_n - 1))^k t_1 \cdots t_n 0^\omega). \]

Proposition (Charlier, Cisternino & Stipulanti 2022)

Let $U$ be a positional numeration system such that $\lim_{i \to \infty} \frac{U(i+1)}{U(i)} = \beta > 1$.

If $\lim_{i \to \infty} \text{rep}_U(U(i) - 1)$ exists, then it is either $d^*_\beta(1)$ or $d_\beta(1)$. 
Another characterization of Bertrand numeration systems

**Theorem (Charlier, Cisternino & Stipulanti 2022)**

A positional numeration system $U$ is Bertrand if and only if one of the following conditions is satisfied.

1. We have $\text{rep}_U(U(i) - 1) = \text{Pref}_i(10^\omega)$ for all $i \geq 0$.
2. There exists $\beta > 1$ such that $\text{rep}_U(U(i) - 1) = \text{Pref}_i(d_\beta^*(1))$ for all $i \geq 0$.
3. There exists $\beta > 1$ such that $\text{rep}_U(U(i) - 1) = \text{Pref}_i(d_\beta(1))$ for all $i \geq 0$. 
Understanding the non-canonical $\beta$-shift

A subshift (i.e., a subset of $A^\omega$ that is topologically closed and shift-invariant) is said to be sofic if its factors form a language that is accepted by a finite automaton.

A Parry number is a real number $\beta > 1$ such that $d_\beta(1)$ is ultimately periodic (or equivalently, $d_\beta^*(1)$ is ultimately periodic).

**Theorem (Bertrand-Mathis 1986)**

For $\beta > 1$, the subshift $S_\beta$ is sofic if and only if $\beta$ is a Parry number.

We get the analogous result:

**Proposition**

For $\beta > 1$, the subshift $S'_\beta$ is sofic if and only if $\beta$ is a Parry number.
The entropy of a subshift $S$ of $A^\omega$ is

$$\lim_{i \to \infty} \frac{1}{i} \log(\text{Card}(\text{Fac}(S) \cap A^i)).$$

**Theorem**

*For all $\beta > 1$, the $\beta$-shift $S_\beta$ has entropy $\log(\beta)$.***

We have the analogous result:

**Proposition**

*For all $\beta > 1$, the subshift $S'_\beta$ has entropy $\log(\beta)$.***
Some negative results

A subshift $S$ is said to be of finite type if there exists a finite set $X \subset A^*$ such that $S = \{w \in A^\mathbb{N} : \text{Fac}(w) \cap X = \emptyset\}$.

**Theorem**

*For all $\beta > 1$, the $\beta$-shift $S_\beta$ is of finite type if and only if $\beta$ is a simple Parry number.*

However:

**Proposition**

*For any simple Parry number $\beta > 1$, the subshift $S'_\beta$ is not of finite type.*
A subshift $S$ is said to be *coded* if there exists a prefix code $Y \subset A^*$ such that $\text{Fac}(S) = \text{Fac}(Y^*)$.

**Theorem**

*For all $\beta > 1$, the canonical $\beta$-shift $S_\beta$ is coded.*

In order to show that $S'_\beta$ is not coded, we prove the stronger statement that $S'_\beta$ is not irreducible.

A subshift $S$ is said to be *irreducible* if for all $u, v \in \text{Fac}(S)$, there exists $w \in \text{Fac}(S)$ such that $uwv \in \text{Fac}(S)$.

**Proposition**

*For any simple Parry number $\beta$, the non-canonical $\beta$-shift $S'_\beta$ is not irreducible.*
A relation between the number of words of length $i$ in the canonical and the non-canonical $\beta$-shifts.

Suppose that $\beta > 1$ is a real number such that $d_\beta(1) = t_1 \cdots t_n 0^\omega$ with $n \geq 1$ and $t_n \neq 0$, and let $U$ and $U'$ respectively be the canonical and non-canonical Bertrand numeration systems associated with $\beta$.

Thanks to our characterization of Bertrand systems, we know that for all $i \geq 0$,

- the number of words of length $i$ in $\text{Fac}(S_\beta)$ is $U(i)$
- the number of words of length $i$ in $\text{Fac}(S'_\beta)$ is $U'(i)$.

**Proposition**

*For all $i \geq 0$, one has $U'(i + n) = U(i + n) + U'(i)$.*
\[ U'(i + n) = U(i + n) + U'(i) \text{ for all } i \geq 0 \]

For \( \beta = 3 \), we have \( d_3(1) = 30^\omega \), hence \( n = 1 \).

We have seen that
\[
U(i) = 3^i \quad \forall i \geq 0
\]

and that
\[
U'(0) = 1, \quad U'(i + 1) = 3U'(i) + 1 \quad \forall i \geq 0.
\]

| \( i \)  | 0  | 1  | 2   | 3   | 4   | 5   | \cdots |
|----------|----|----|-----|-----|-----|-----|--------|
| \( U(i) \) | 1  | 3  | 9   | 27  | 81  | 243 |        |
| \( U'(i) \) | 1  | 4  | 13  | 40  | 121 | 364 |        |
\[ U'(i + n) = U(i + n) + U'(i) \] for all \( i \geq 0 \)

For \( \beta = \varphi \), we have \( d_{\varphi}(1) = 110^\omega \), hence \( n = 2 \).

We have seen that

\[
U(0) = 1, \quad U(1) = 2, \quad U(i + 2) = U(i + 1) + U(i) \quad \forall i \geq 0
\]

and that

\[
U'(0) = 1, \quad U'(1) = 2, \quad U'(i + 2) = U(i + 1) + U(i) + 1 \quad \forall i \geq 0.
\]

| \( i \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \cdots |
|--------|---|---|---|---|---|---|---|---|---|--------|
| \( U(i) \) | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 |
| \( U'(i) \) | 1 | 2 | 4 | 7 | 12 | 20 | 33 | 54 | 88 |
Thank you!
Merci !