A Framework for Verifiable Blind Quantum Computation

Theodoros Kapourniotis\textsuperscript{1}, Elham Kashefi\textsuperscript{2,3}, Dominik Leichtle\textsuperscript{3}, Luka Music\textsuperscript{4}, and Harold Ollivier\textsuperscript{5}

\textsuperscript{1} Department of Physics, University of Warwick, Coventry CV4 7AL, United Kingdom
\textsuperscript{2} School of Informatics, University of Edinburgh, 10 Crichton Street, Edinburgh EH8 9AB, United Kingdom
\textsuperscript{3} Laboratoire d’Informatique de Paris 6, CNRS, Sorbonne Université, 4 Place Jussieu, 75005 Paris, France
\textsuperscript{4} Quandela, 7 Rue Léonard de Vinci, 91300 Massy, France
\textsuperscript{5} INRIA, 2 rue Simone Iff, 75012 Paris, France

Abstract. With the recent availability of cloud quantum computing services, the question of the verifiability of quantum computations delegated by a weak quantum client to a powerful quantum server is becoming of practical interest. Over the last two decades, Verifiable Blind Quantum Computing (VBQC) has emerged as one of the key approaches to address this challenge. While many novel protocols have been proposed in recent years, each optimising one aspect of such schemes, a systematic study of required building blocks to achieve collectively the desired properties has been missing.

We present a framework that encompasses all known VBQC protocols and presents sufficient conditions to obtain composable security and noise robustness properties. We do this by providing a unified security proof within the Abstract Cryptography formalism that can be used for existing protocols as well as new ones that could be derived using the framework. While we choose Measurement-Based Quantum Computing (MBQC) as the working model for the presentation of our results, one could expand the domain of applicability of our framework via direct known translation between the circuit model and MBQC.

On a theoretical note, we uncover fundamental relations between verification and error detection and correction, a folklore belief in the field that has never been mathematically proven: (i) verification can be reduced to the task of error detection, and (ii) encoding of the target computation into an error-correcting code is necessary for exponential soundness. These simple yet fundamental facts set the design principles for further development towards practical VBQC protocols. Indeed, as a direct application we demonstrate how the framework can systematize the search for new verification techniques. We find completely new schemes that allow more efficient robust verification of $\mathsf{BQP}$ computations than current state of the art.

Keywords: Quantum Verification, Secure Delegated Computation.
1 Introduction

Secure delegated quantum computation is a long-standing topic of research where a client wants to perform a computation task on a remote server, without necessarily trusting it. In this context, a computation is deemed secure when the privacy of the data and algorithm is guaranteed, as well as the integrity of the computation. None of these criteria are specific to quantum computing as users have always needed to protect their data, their algorithmic know-how and ensure that no party can manipulate results beyond their ability to choose their inputs \cite{10,9}. However, this topic has gained further attention in the quantum realm due to the recent development of remotely accessible quantum computers, where no cryptographic guarantee is currently provided to clients delegating their computations to service providers. This, in turn, transformed a mostly theoretical question into a practical one.

An active line of research has been developed over the past two decades to provide increasingly powerful protocols for achieving secure verifiable delegated quantum computation (see recent review articles \cite{11,23}). The common building block in all such protocols is the computation via gate teleportation and in fact the measurement based quantum computing model (MBQC) \cite{22} has been considered extensively as it provides a rich mathematical tool kit to analyse all such protocols. MBQC can be seen as a set of classical instructions steering a quantum computation performed on a highly entangled quantum state. This allows the client to use relatively basic obfuscation techniques to prevent an untrusted operator, who is concretely implementing the MBQC, to access the abstract and meaningful level of the flow of information. One of the very first Verifiable Blind Quantum Computation (VBQC) protocols \cite{8} was described in this model and allows a verifier, restricted to operations on a single qubit at a time, to verify any efficiently describable quantum-input quantum-output computation. This is done by inserting known deterministic computations – called traps – among the qubits used for the computation, which allow to test the honesty of the server. Several optimised protocols based on this principle have since been described, e.g. \cite{16,15}.

However, none of the previous protocols could really be considered viable practical proposals, especially for current limited devices, as none are robust to even a polynomially low global amount of noise, which may remain even after error correction is integrated into the schemes. Furthermore, while most of these protocols keep the client simple they all have large hardware overheads on the server side. This situation changed recently with the introduction of a robust and efficient protocol for verifying BQP computations in \cite{17}. This protocol provides for the first time robustness to a constant amount of global noise, without any additional space overhead compared to the unverified MBQC computation. The only overhead is a polynomial number of repetitions of rounds, each of the same complexity as the unprotected computation. The key features in the protocol of \cite{17} are (i) the way in which the traps are inserted, and (ii) how the detection sensitivity is amplified. Inspired by this work, in the present paper we aim at revisiting trappification-based verification in the MBQC model by introducing
a general framework that not only encompasses previous approaches but also considerably simplifies and enlarges the set of techniques that can be used. Our motivation is to bring to light crucial elements for verification that are important from a theoretical standpoint, such as linking detection and verification, or the necessity of error correction for sensitivity amplification. As a result, the framework provides a systematic approach for the design of even more optimised protocols, a search that will benefit from our richer set of tools. As a direct application of this new methodology of protocol design we present a family of novel VBQC protocols that are optimal in terms of overhead while providing a higher robustness level and better security scaling in comparison to the state-of-the-art in this field.

Related Work. The possibility of classically verifying quantum computations [12] was also raised as a key question in [1] and remained an open question until the work of Mahadev that introduced an explicit protocol [18] for verifying $\text{BQP}$ computations by relying only on classical interactions between the prover and the verifier and a computational hardness assumption. Building up on this work, [2] was able to provide security in the context of parallel repetitions, thereby turning the protocol for classical verifier into a powerful primitive to devise more elaborate tasks such as in [4]. These lines of work however introduce a 1000-fold overhead on the server’s side due to the necessary lower bound on the size of the quantum circuit implementing the classical primitive such as LWE-based one-way functions, to ensure they remain secure against quantum adversaries. As such we are not exploring these class of protocols as part of our framework directly, focusing instead on information-theoretically secure protocols. However the applicability of our toolkit for further optimising this class of protocol remains an open question for future investigation. Moreover, to achieve practicality on limited hardware, the nonpermissive overhead above is fully avoided in our approach to VBQC as quantum communications between the client and the server is the only cryptographic primitive required in order to mask the computation from the server.

Overview. The paper is organised as follows. Section 3 defines the structures that are needed to detect deviations – namely trappified canvases and trappified schemes. Then, in Section 4 we give explicit conditions on the deviations which should be detected by a trappified scheme to allow the verification of a given class of computations, thereby intimately linking deviation detection and verification. The approach followed here is similar in spirit to [7] as it provides a generic proof of security, albeit with conditions which are more concrete and easier to manipulate. We also show that error correction is needed for verification with negligible correctness and security errors. In Section 6 we show how to use the framework to not only recover well-known traps but also construct new ones. Finally, using the latter, we prove in Section 7 that they allow to outperform previous state-of-the-art protocols for verifying $\text{BQP}$ computations [17] by reducing the time overhead and increasing their noise robustness.
2 Preliminaries

2.1 Measurement-Based Quantum Computation

The MBQC model of computation emerged from the gate teleportation principle. It was introduced in [22] where it was shown that universal quantum computing can be implemented using graphs-states as resources and adaptive single-qubit measurements. Therefore MBQC and gate-based quantum computations have the same power. The measurement calculus expresses the correspondence between the two models [6].

MBQC works by choosing an appropriate graph state, performing single-qubit measurements on a subset of this state and, depending on the outcomes, apply correction operators to the rest. Quantum computations can be easily delegated in this model by having the client supply the quantum input to the server and instruct it by providing measurement instructions, while the server is tasked with the creation of a large entangled state which is suitable for the client’s desired computation.

More precisely, we define the rotation operator around the $Z$ axis of the Bloch sphere by an angle $\theta$ as $Z(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$ and $|+\theta\rangle = Z(\theta) |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\theta} |1\rangle)$.

The rotation around the $X$ axis is then $X(\theta) = H Z(\theta) H$. While the discussions below hold for angles in $[0, 2\pi)$, if we settle for approximate universality it is sufficient to restrict ourselves to the set of angles $\Theta = \{k\pi/4\}_{k \in \{0,\ldots,7\}}$ [8].

The client’s computation is defined by a measurement pattern as follows.

Definition 1 (Measurement Pattern). A pattern in the Measurement-Based Quantum Computation model is given by a graph $G = (V, E)$, input and output vertex sets $I$ and $O$, a flow $f$ which induces a partial ordering of the qubits $V$, and a set of measurement angles $\{\phi(i)\}_{i \in O}$ in the $X-Y$ plane of the Bloch sphere.

Further details regarding the definition of the flow can be found in references [13,15].

Execution of MBQC pattern can then be delegated to servers, alleviating the need for the client to own a quantum machine using Protocol 1. Furthermore, if we are willing to allow the client to perform single qubit preparations and to use quantum communication, such delegation can be performed blindly [3], meaning that the Server does not learn anything about the computation besides the prepared graph $G$, the set of outputs $O$ and the order of measurements. To do so, the Client instead uses Protocol 2.

Note that in both protocols, if the output of the client’s computation is classical, the set $O$ is empty and the client only receives measurement outcomes. For MBQC, the measurement outcomes are sent directly to the client and represent the output of the computation itself. For the UBQC protocol, the measurement outcomes $b(i)$ sent by the Server need to be decrypted by the Client according to the equation $s(j) = b(j) \oplus r(j)$, thus preserving the confidentiality of the output of the computation.
Protocol 1 Delegated MBQC Protocol

Client’s Inputs: A measurement pattern \((G, I, O, \{\phi(i)\}_{i \in O^c}, f)\) and a quantum register containing the input qubits \(i \in I\).

Protocol:

1. The Client sends the graph’s description \((G, I, O)\) to the Server;
2. The Client sends its input qubits for positions \(I\) to the Server;
3. The Server prepares \(|+\rangle\) states for qubits \(i \in I^c\);
4. The Server applies a \(\text{CZ}\) gate between qubits \(i\) and \(j\) if \((i, j)\) is an edge of \(G\);
5. The Client sends the measurement angles \(\{\phi(i)\}_{i \in O^c}\) along with the description of \(f\) to the Server;
6. The Server measures the qubits \(i \in O^c\) in the order defined by \(f\) in the rotated basis \(|\pm \phi'(i)\rangle\) where

\[
\begin{align*}
\phi'_i &= (-1)^{s_X(i)}\phi(i) + s_Z(i)\pi, \\
s_X(i) &= \bigoplus_{j \in S_X(i)} s(j), \quad s_Z(i) = \bigoplus_{j \in S_Z(i)} s(j),
\end{align*}
\]

where \(s(j) \in \{0, 1\}\) is the measurement outcome for qubit \(j\), with 0 being associated to \(|+\phi'(j)\rangle\); and \(S_X(i)\) (resp. \(S_Z(i)\)) is the \(X\) (resp. \(Z\)) dependency set for qubit \(i\) defined by \(S_X(i) = f^{-1}(i)\) (resp. \(S_Z(i) = \{j : i \in N_G(f(j))\}\));
7. The Server performs the correction \(Z^{s_Z(i)}X^{s_X(i)}\) for output qubits \(i \in O\), which it sends back to the Client.

2.2 Abstract Cryptography

Abstract Cryptography (AC) is a security framework for cryptographic protocols that was introduced in [20,19]. The focus of the AC framework is to provide general composability. In this way, protocols that are separately shown to be secure within the framework can be composed in sequence or in parallel while keeping a similar degree of security. See [7] for further details.

On an abstract level, the AC framework considers resources and protocols. While a resource provides a specified functionality, protocols are essentially instructions how to construct resources from other resources. In this way, this framework allows the expansion of the set of available resources while ensuring general compatibility.

Technically, a quantum protocol \(\pi\) with \(N\) honest parties is described by \(\pi = (\pi_1, \ldots, \pi_N)\), where the combined actions of party \(i\), denoted \(\pi_i\), are called the converter of party \(i\) and consist in the quantum case of a sequence of efficiently implementable CPTP maps. A resource has interfaces with the parties that are allowed to exchange states with it. During its execution, it waits for all input interfaces to be initialised, then applies a CPTP map to all interfaces and its internal state, and finally transmits the states in the output interfaces back to the appropriate parties. This process may be repeated multiple times. Entirely classical resources can be enforced by immediate measurements of all input registers and the restriction of the output to computational basis states.

AC security is entirely based on the indistinguishability of resources. A protocol is considered to be secure if the resource which it constructs is indistinguish-
Protocol 2 UBQC Protocol

**Client’s Inputs:** A measurement pattern \((G, I, O, \{\phi(i)\}_{i \in O^c}, f)\) and a quantum register containing the input qubits \(i \in I\).

**Protocol:**

1. The Client sends the graph’s description \((G, I, O)\) and the measurement order to the Server;
2. The Client prepares and sends all the qubits in \(O^c \cup I\) to the Server:
   - for \(i \in I\), it chooses a random bit \(a(i)\) and a random \(\theta(i) \in \Theta\), applies \(Z(\theta(i))X^{a(i)}\) to the qubit \(i\) and sends it to the Server;
   - for \(i \in I^c\), it chooses a random \(\theta(i) \in \Theta\), prepares \(|+\theta(i)\rangle\) and sends it to the Server;
3. The Server applies a CZ gate between qubits \(i\) and \(j\) if \((i, j)\) is an edge of \(G\);
4. For all \(i \in O^c\), in the order specified by the flow \(f\), the Client computes the measurement angle \(\delta(i)\) and sends it to the Server, receiving in return the corresponding measurement outcome \(b(i)\):
   
   \[
   s(j) = b(j) \oplus r(j),
   \]

   \[
   s_X'(i) = \bigoplus_{j \in S_X(i)} s(j) \oplus a(i), \quad s_Z'(i) = \bigoplus_{j \in S_Z(i)} s(j) \oplus a(f^{-1}(i)),
   \]

   \[
   \delta(i) = (-1)^{s_X'(i)}\phi(i) + \theta(i) + (s_Z'(i) + r(i))\pi,
   \]

   where \(a\) has been extended so that \(a(i) = 0\) for \(i \in I^c\) and \(a(f^{-1}(i)) = 0\) for \(i \notin \text{range}(f)\);
5. The Server sends back the output qubits \(i \in O\);
6. The Client applies \(Z^{s_X'(i)}X^{s_Z'(i)}\) to the received qubits \(i \in O\).

Analogously, the correctness of a protocol is captured by the indistinguishability of the resource constructed by the protocol from the ideal resource when all parties are honest, i.e. they use their respective converters as specified from an ideal resource which encapsulates the desired security properties. Two resources with the same number of interfaces are called indistinguishable if no algorithm, called the distinguisher, can guess with good probability the resource that it is given access to. During this process, the distinguisher obtains access to all of the resource’s interfaces at the same time.

**Definition 2 (Indistinguishability of Resources).** Let \(\epsilon(\lambda)\) be a function of security parameter \(\lambda\) and \(\mathcal{R}_1\) and \(\mathcal{R}_2\) be two resources with same input and output interfaces. Then, these resources are called \(\epsilon\)-statistically-indistinguishable, denoted \(\mathcal{R}_1 \approx_{\text{stat},\epsilon} \mathcal{R}_2\), if for all (unbounded) distinguishers \(\mathcal{D}\) it holds that

\[
|\Pr[b = 1 \mid b \leftarrow \mathcal{D}\mathcal{R}_1] - \Pr[b = 1 \mid b \leftarrow \mathcal{D}\mathcal{R}_2]| \leq \epsilon
\]

Analogously, \(\mathcal{R}_1\) and \(\mathcal{R}_2\) are said to be computationally indistinguishable if this holds for all quantum polynomial-time distinguishers.

With this definition in mind, the correctness of a protocol is captured by the indistinguishability of the resource constructed by the protocol from the ideal resource when all parties are honest, i.e. they use their respective converters as specified from an ideal resource which encapsulates the desired security properties.
specified by the protocol. The security of the protocol against a set of malicious and collaborating parties is given by the indistinguishability of the constructed resource where the power of the distinguisher is extended to the transcripts of the corrupted parties. This is formally captured by Definition 3.

**Definition 3 (Construction of Resources).**

Let $\epsilon(\lambda)$ be a function of security parameter $\lambda$. We say that an $N$-party protocol $\pi \in \mathcal{E}$-statistically-constructs resource $S$ from resource $R$ against adversarial patterns $P \subseteq \varphi([N])$ if:

1. It is correct: $\pi R \approx \pi S$.
2. It is secure for all subsets of corrupted parties in the pattern $M \in P$: there exists a simulator (converter) $\sigma_M$ such that $\pi_M R \approx \pi_M S$.

Analogously, computational correctness and security is given for computationally bounded distinguishers as in Definition 2, and with a quantum polynomial-time simulator $\sigma_M$.

This finally allows us to formulate the General Composition Theorem at the core of the Abstract Cryptography framework.

**Theorem 1 (General Composability of Resources [20, Theorem 1]).**

Let $R$, $S$ and $T$ be resources, $\alpha$, $\beta$ and $id$ be protocols (where protocol $id$ does not modify the resource it is applied to). Let $\circ$ and $|$ denote respectively the sequential and parallel composition of protocols and resources. Then the following implications hold:

- The protocols are sequentially composable: if $\alpha R \approx \pi S$ and $\beta S \approx \pi T$ then $(\beta \circ \alpha) R \approx \pi T$.
- The protocols are context-insensitive: if $\alpha R \approx \pi S$ then $(\alpha \mid id)(R \mid T) \approx \pi S$.

A combination these two properties yields concurrent composability (where the distinguishing advantage accumulates additively as well).

We now describe the resource that captures the security properties of blind and verifiable delegated protocol for a given class of computation. Resource $\mathcal{R}$ allows a single Client to run a quantum computation on a Server so that the Server cannot corrupt the computation and does not learn anything besides a given leakage $l$. We recall the original definition from [7, Definition 4.2].

### 3 Detecting Deviations

#### 3.1 Deviation Detection Tools

We start by defining partial MBQC patterns in Definition 4 which fix only a subset of the measurement angles and flow conditions on a given graph. We constrain the flow such that the determinism of the computation is preserved on the partial pattern independently of the how the rest of the flow is specified.
Resource 1 Secure Delegated Quantum Computation

Public Information: Nature of the leakage \( l_\rho \).

Inputs:

- The Client inputs the classical description of a computation \( C \) and a quantum state \( \rho \) compatible with \( C \).
- The Server chooses whether or not to deviate. This interface is filtered by two control bits \((e,c)\) (set to 0 by default for honest behaviour).

Computation by the Resource:

1. If \( e = 1 \), the Resource sends the leakage \( l_\rho \) to the Server’s interface; if it receives \( c = 1 \), the Resource outputs \( |\perp\rangle \langle \perp| \otimes |\text{Rej}\rangle \langle \text{Rej}| \) at the Client’s output interface.
2. Otherwise it outputs \( C(\rho) \otimes |\text{Acc}\rangle \langle \text{Acc}| \) at the Client’s output interface.

Definition 4 (Partial MBQC Pattern).

Given a graph \( G = (V,E) \), a partial pattern \( P \) on \( G \) is defined by:

- \( G_P = (V_P, E_P = E \cap V_P \times V_P) \), a subgraph of \( G \);
- \( I_P \) and \( O_P \), the partial input and output vertices, with subspaces \( \Pi_{I_P} \) and \( \Pi_{O_P} \) defined on vertices \( I_P \) and \( O_P \) through bases \( B_{I_P} \) and \( B_{O_P} \) respectively;
- \( \{\phi(i)\}_{i \in V_P \setminus O_P} \), a set of measurement angles;
- \( f_P : V_P \setminus O_P \to V_P \setminus I_P \), a flow inducing a partial order \( \preceq_P \) on \( V_P \).

We now use this notion to define trapped canvases, i.e. partial patterns containing a subcomputation which, for a given input state, samples from a fixed distribution when its outputs are measured in the \( X \) basis. These traps will be used to detect deviations: whenever the trap computations are executed, they should provide outcomes that are compatible with the trap’s probability distribution. Failure to do so is a sign that the server deviated from the instructions given by the client.

Definition 5 (Trapped Canvas).

A trapped canvas \((T, \sigma, T, \tau)\) on a graph \( G = (V,E) \) consists of:

- \( T \), a partial pattern on a subset of vertices \( V_T \) of \( G \) with input and output sets \( I_T \) and \( O_T \);
- \( \sigma \), a tensor product of single-qubit states on \( \Pi_{I,T} \);
- \( T \), an efficiently classically computable probability distribution over binary strings;
- \( \tau \), an efficient classical algorithm that takes as input a sample from \( T \) and outputs a single bit and outputs a single bit;

such that that the \( X \)-measurement outcomes of qubits in \( O_T \) are drawn from probability distribution \( T \). Let \( \tau \) be such a sample, the outcome of the trapped canvas is given by \( \tau(t) \). By convention we say that it accepts whenever \( \tau(t) = 0 \) and rejects for \( \tau(t) = 1 \).
We will often abuse the notation and refer to the trappified canvas \((T, \sigma, T, \tau)\) as \(T\).

Note that the inputs and output qubits of a partial pattern may not be included in the input and output qubits of the larger MBQC graph. This gives us more flexibility in defining trappified canvases: during the protocol presented in the next section, the server will measure all qubits in \(O^c\) – which may include some of the trap outputs \(-\), while any measurement of qubits in \(O\) will be performed by the client. This allows the trap to catch deviations on the output qubits as well.

In order to be useful, trappified canvases must contain enough empty space – vertices which have been left unspecified – to accommodate the client’s desired computation. Inserting this computation is done via an embedding algorithm as described in the following Definition.

**Definition 6 (Embedding Algorithm).**

Let \(\mathcal{C}\) be a class of quantum computations. An embedding algorithm \(E_{\mathcal{C}}\) for \(\mathcal{C}\) is an efficient classical probabilistic algorithm that takes as input:

- \(C \in \mathcal{C}\), the computation to be embedded;
- \(G = (V, E)\), a graph, and an output set \(O\);
- \(T\), a trappified canvas on graph \(G\);
- \(\preceq_G\), a partial order on \(V\) which is compatible with the partial order defined by \(T\);

and outputs a partial pattern \(C\) on \(V \setminus V_T\), with input and output vertices \(I_C \subset V \setminus V_T\) and \(O_C = O \setminus O_T\) and two subspaces \(\Pi_{I,C}\) and \(\Pi_{O,C}\) of \(I_C\) and \(O_C\) respectively such that, for all inputs \(\rho\) in \(\Pi_{I,C}\) expressed in basis \(B_{I,C}\), the partial pattern \(C\) transforms \(\rho\) into \(C(\rho)\) in basis \(B_{O,C}\). We say that the embedding is proper if the flow \(f_C\) of partial pattern \(C\) induces a partial order compatible with \(\preceq_G\) and \(f_C\) does not induce dependencies on vertices \(V_T\) of partial pattern \(T\).

If \(E_{\mathcal{C}}\) is incapable of performing the embedding, it aborts and outputs \(\bot\).

Given a computation \(C\) embedded into \(T\) using \(E_{\mathcal{C}}\), we call the corresponding completed pattern \(C \cup T\) a trappified pattern.

Note that the input and output qubits of the computation \(C\) might be constrained to be in (potentially strict) subspaces \(\Pi_{I,C}\) and \(\Pi_{O,C}\) of \(I_C\) and \(O_C\) respectively. This allows for error-protected inputs and outputs, without having to specify any implementation for the error correction scheme. In particular, it encompasses encoding classical output data as several, possibly noisy, repetitions which will be decoded by the client through a majority vote as introduced in [17]. It also allows to take into account the case where the trappified pattern comprises a fully fault-tolerant MBQC computation scheme for computing \(C\) using topological codes as described in [21].

**Remark 1.** Because the dependencies induced by \(f_C\) do not affect trap qubits \(V_T\) and the input of the trap is fixed, independently of the computation, the distribution of the trap measurement outcomes is also independent of the embedded
computation being performed on the rest of the graph as well as the input state of such computation.

For verification, our scheme must be able to cope with malicious behaviour: detecting deviations is useful for verification only so long as the server cannot adapt its behaviour to the traps that it executes. Otherwise, it could simply decide to deviate exclusively on non-trap qubits. This is achieved by executing the patterns in a blind way so that the server has provably no information about the location of the traps and cannot avoid them with high probability. To this end, we define blind-compatible patterns as those which share the same graph, output vertices and measurement order of their qubits. The UBQC Protocol described in Appendix 2.1 leaks exactly this information to the server, meaning that it cannot distinguish the executions of two different blind-compatible patterns.

**Definition 7 (Blind-Compatible Patterns).**

A set of patterns \( P \) is blind-compatible if all patterns \( P \in P \) share the same graph \( G \), the same output set \( O \) and there exists a partial ordering \( \preceq_P \) of the vertices of \( G \) which is an extension of the partial ordering defined by the flow of any \( P \in P \).

This definition can easily be extended to trappified canvases by considering the orderings induced by the flows of the partial patterns.

A single trap is usually not sufficient to catch deviations on more than a subset of positions of the graph. In order to catch all deviations, it is then necessary to randomise the blind delegated execution over multiple patterns. We therefore define a trappified scheme as a set of blind-compatible trappified canvases which can be efficiently sampled according to a given distribution, along with an algorithm for embedding computations from a given class into all the canvases.

**Definition 8 (Trappified Scheme).**

A trappified scheme \( (P, \preceq_G, P, E_C) \) over a graph \( G \) for computation class \( \mathcal{C} \) consists of:

- \( P \), a set of blind-compatible trappified canvases over graph \( G \) with common partial order \( \preceq_P \);
- \( \preceq_G \), a partial ordering of vertices \( V \) of \( G \) that is compatible with \( \preceq_P \);
- \( P \), a probability distribution over the set \( P \) which can be sampled efficiently;
- \( E_C \), an embedding algorithm for \( \mathcal{C} \);

such that for all \( C \in \mathcal{C} \) and all trappified canvases \( T \in P \), \( E_C(C, G, T, \preceq_G) \neq \bot \), i.e. any computation can be embedded in any trappified canvas.

Without loss of generality, in the following, the probability distribution used to sample the trappified canvases will generally be \( u(P) \), the uniform distribution over \( P \). The general case can be approximated from the uniform one with arbitrary fixed precision by having several copies of the same canvas in \( P \). We
take $T \sim P$ to mean that the trappified canvas is sampled according to the distribution $\mathcal{P}$ of trappified scheme $P$.

Note that while the first condition above ensures that the completed patterns obtained after running the embedding algorithm hide the location of the traps, the second one ensures that the order in which the qubits will be measured does not depend on the computation itself – which would otherwise break the blindness of the scheme.

### 3.2 Detecting Deviations Using Trappified Schemes

We can now describe the purpose of the objects described in the previous subsection, namely detecting the server’s deviations from their prescribed operations during a given delegated computation. We start by recalling that the blindness of UBQC Protocol is based on the Pauli twirling principle, which implies that any deviation can be reduced to a convex combination of Paulis. Then, we formally define Pauli deviation detection and insensitivity for trappified canvases and schemes. We show in the next section these key properties are sufficient for obtaining a verifiable delegated computation.

When a client delegates the execution of a pattern $P$ to a server using Protocol 2, the server can potentially deviate in an arbitrary way from the instructions it receives. By converting into quantum states both the classical instructions sent by the client – i.e. the measurement angles – and the measurement outcomes sent back by the server, all operations on the server’s side can be modelled as a unitary $F$ acting on all the qubits sent by the client and some ancillary states $|0\rangle_S$, before performing measurements in the computational basis to send back the outcomes $|b\rangle$ that the client expects from the server.

The instructions of server in an honest execution of the UBQC Protocol 2 correspond to:

1. entangling the received qubits corresponding to the vertices of the computation graph with operation $E_G = \bigotimes_{(i,j) \in E} CZ_{i,j}$;
2. performing rotations on non-output vertices around the $Z$ axis, controlled by the qubits which encode the measurement angles instructed by the client;
3. applying a Hadamard gate $H$ on all non-output vertices;
4. measuring non-output vertices in the $\{|0\rangle,|1\rangle\}$ basis.

The steps (i-iii) correspond to a unitary transformation $U_P$ that depends only on the public information that the server has about the pattern $P$ – essentially the computation graph $G$ and an order of its vertices compatible with the flow of $P$.

Hence, the unitary part $U_P$ of the honestly executed protocol for delegating $P$ can always be extracted from $F$, so that $F = F' \circ U_P$. Here, $F'$ is called a pure deviation and is applied right before performing the computational basis measurements for non-output qubits and right before returning the output qubits to the Client.

When the pattern is executed blindly using Protocol 2, the state in the server’s registers during the execution is a mixed state over all possible secret
parameters chosen by the client. It is shown in [14] that the resulting summation over the secret parameters which hide the inputs, measurement angles and measurement outcomes is equivalent to applying a Pauli twirl to the pure deviation $F'$. This effectively transforms it into a convex combination of Pauli operations applied after $U_P$.

Hence, any deviation by the server can be represented without loss of generality by choosing with probability $Pr[E]$ an operator $E$ in the Pauli group $\mathcal{G}_V$ over the vertices $V$ of the graph used to define $P$, and executing $E \circ U_P$ instead of $U_P$ for the unitary part of the protocol. By a slight abuse of notation, such transformation will be denoted $E \circ P$. Furthermore, if $P$ is a trappified pattern which contains a trap $T$ that samples $t = (t_1, \ldots, t_N)$ from the distribution $\mathcal{T}$, then in the presence of deviation $E$, it will sample from a different distribution denoted $E \circ \mathcal{T}$. For instance, whenever $E$ applies a $Z$ operator on a vertex, it can be viewed as an execution of a pattern where the angle $\delta$ for this vertex is changed into $\delta + \pi$. Whenever $E$ applies a $X$ operator on a vertex, $\delta$ is transformed into $-\delta$.

This has an interesting consequence for trappified canvases if one recalls Remark [11]—the trap $T$ samples from $\mathcal{T}$ independently of the computation embedded onto it. Indeed, for a completed trappified pattern $P$ obtained by embedding a computation $C$ onto a trappified canvas $T$, the action of $E$ on the vertices outside $V_T$ does not have an impact on the measurement outcomes of the vertices in $V_T$. That is $E \circ \mathcal{T}$ depends only on the trappified canvas. This is true because the flow for the corresponding partial pattern $C$ computing $C$ does not induce dependencies on vertices in $V_T$. As a consequence, it is possible to define what it means for a given trappified canvas to detect Pauli errors:

**Definition 9 (Pauli Detection).**

Let $T$ be a trappified canvas sampling from distribution $\mathcal{T}$. Let $\mathcal{E}$ be a subset of $\mathcal{G}_V$. For $\epsilon > 0$, we say that $T$ $\epsilon$-detects $\mathcal{E}$ if:

$$\forall E \in \mathcal{E}, \Pr_{t \sim E \circ T}[\tau(t) = 1] \geq 1 - \epsilon.$$  

We say that a trappified scheme $P$ $\epsilon$-detects $\mathcal{E}$ if:

$$\forall E \in \mathcal{E}, \sum_{T \in P} \Pr_{t \sim P}[\tau(t) = 1, T] \geq 1 - \epsilon,$$

**Definition 10 (Pauli Insensitivity).**

Let $T$ be a trappified canvas sampling from distribution $\mathcal{T}$. Let $\mathcal{E}$ be a subset of $\mathcal{G}_V$. For $\delta > 0$, we say that $T$ is $\delta$-insensitive to $\mathcal{E}$ if:

$$\forall E \in \mathcal{E}, \Pr_{t \sim E \circ T}[\tau(t) = 0] \geq 1 - \delta.$$  

We say that a trappified scheme $P$ is $\delta$-insensitive to $\mathcal{E}$ if:

$$\forall E \in \mathcal{E}, \sum_{T \in P} \Pr_{t \sim P}[\tau(t) = 0, T] \geq 1 - \delta.$$

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Above, the probability distribution stems both from the randomness of quantum measurements of the trap output qubits yielding the bit string $t$, and the potentially probabilistic nature of the decision function $\tau$.

In the case of trappified schemes, the probability distribution for obtaining a given result for $\tau$ also depends on the choice of pattern $T \in P$, sampled according to the probability distribution $P$.

In the same spirit, there are physical deviations that nonetheless produce little effect on the computations embedded into trappified canvases and trappified schemes. When they occur, the computation is still almost correct.

**Definition 11 (Pauli Correctness).** Let $T$ be a trappified canvas and $E_\mathcal{E}$ an embedding algorithm. Let $P$ be the pattern obtained by embedding a computation $C \in \mathcal{E}$ on $T$ using $E_\mathcal{E}$. Let $E$ be a subset of $\mathcal{G}_V$. For $\nu \geq 0$, we say that $T$ is $\nu$-correct on $E$ if:

$$\forall E \in \mathcal{E}, \forall C \in \mathcal{C}, \| (E \circ P - P) \circ \Pi_{I,C} \|_0 \leq \nu,$$

where $\Pi_{I,C}$ is the projector onto the corresponding input space.

This is extended to a trappified scheme $P$ by requiring the bound to hold for all $T \in P$.

One might wonder why not just detecting all possible deviation rather than counting on the possibility that some have little impact on the actual computation. The answer is simple and specific to MBQC: there are always plenty of those. More precisely, following our convention to view all measurements as computational basis measurements preceded by an appropriate rotation, then any pure deviation that has $I$ and $Z$ operators does not change the measurement outcomes. Hence, such deviations have no effect on the actual result of the pattern.

In the following, sets of deviations that have little effect on the result of the computation according to diamond distance will be called *harmless*, while their complement are possibly harmful.

**Remark 2 (A Trappified Canvas is a Trappified Scheme).** Any trappified canvas $T$ can be seen as a trappified scheme $P = \{T\}$ and the trivial distribution. If the trappified pattern $\mathcal{E}$-detects $\mathcal{E}_1$ and is $\delta$-insensitive w.r.t. $\mathcal{E}_2$, so is the corresponding trappified scheme.

**Remark 3 (Simple Composition of Trappified Schemes).** Let $(P_i)_i$ be a sequence of trappified schemes with corresponding distributions $P_i$ such that $P_i$ $\epsilon_i$-detects $\mathcal{E}_1^{(i)}$ and is $\delta_i$-insensitive to $\mathcal{E}_2^{(i)}$. Let $(p_i)_i$ be a probability distribution.

Let $P = \bigcup_i P_i$ be the trappified scheme with the following distribution $P$:

1. Sample a trappified scheme $P_j$ from $(P_i)_i$ according to $(p_i)_i$;
2. Sample a trappified canvas from $P_j$ according to $P_j$.  

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Let \( \mathcal{E}_1 \subseteq \bigcup_i \mathcal{E}_1^{(i)} \) and \( \mathcal{E}_2 \subseteq \bigcup_i \mathcal{E}_2^{(i)} \). Then, \( \mathbf{P} \) \( \epsilon \)-detects \( \mathcal{E}_1 \) and is \( \delta \)-insensitive to \( \mathcal{E}_2 \) with:

\[
\epsilon = \min_{E \in \mathcal{E}_1} \sum_{i} p_i \epsilon_i, \quad \text{and} \quad \delta = \min_{E \in \mathcal{E}_2} \sum_{i} p_i \delta_i.
\]

In particular, \( \mathbf{P} \sum_i p_i \epsilon_i \)-detects \( \bigcap_i \mathcal{E}_1^{(i)} \) and is \( \sum_i p_i \delta_i \)-insensitive to \( \bigcap_i \mathcal{E}_2^{(i)} \).

**Remark 4 (Pure Traps).** A trappified scheme \( \mathbf{P} \) may only contain trappified canvases that cover the whole graph \( G = (V, E) \) if \( V_T = V \) for all \( T \in \mathbf{P} \). This corresponds to the special case where the trappified scheme cannot embed any computation, i.e. \( \mathcal{C} = \emptyset \) and the embedding algorithm applied to a canvas \( T \in \mathbf{P} \) always return \( T \). The detection, insensitivity and correctness properties also apply to this special case, although the correctness is trivially satisfied.

## 4 Verification as Deviation Detection

In this section, we formalise the following intuitive link between deviation detection and verification in the context of delegated computations. On one hand, if a delegated computation protocol is correct\(^6\) not detecting any deviation by the server from its prescribed sequence of operations should be enough to guarantee that the final result is correct. Conversely, detecting that some operations have not been performed as specified should be enough for the client to reject potentially incorrect results. Combining those two cases should therefore yield a verified delegated computation.

### 4.1 Blind Delegation of Trappified Schemes

Given a computation \( \mathcal{C} \), it is possible to delegate its trappified execution in a blind way. To do so, the Client simply chooses one trappified canvas from a scheme at random, inserts into it the computation \( \mathcal{C} \) using an embedding algorithm and blindly delegates the execution of the resulting trappified pattern to the Server. The steps are formally described in Protocol 3.

Note that this protocol offers blindness not only at the level of the chosen trappified pattern, but also at the level of the trappified scheme itself. More precisely, by delegating the chosen pattern, the Client reveals at most the graph of the pattern and the location of the output qubits of the pattern, if there are any, comprising computation and trap outputs. However, trappified patterns of a trappified scheme are compatible, that is they share the same graph and same set of output qubits. Therefore, the above protocol also hides which trappified pattern has been executed among all possible ones, hence concealing the location of traps.

---

\(^6\) Here we use correctness in a cryptographic setting, meaning that all parties execute as specified their part of the protocol.


Protocol 3 Deviation Detecting Delegated Blind Computation

Public Information:

– \( \mathcal{C} \), a class of quantum computations;
– \( G = (V, E) \), a graph with output set \( O \);
– \( P \), a trappified scheme on graph \( G \);
– \( \preceq_G \), a partial order on \( V \) compatible with \( P \).

Client’s Inputs: A computation \( C \in \mathcal{C} \) and a quantum state \( \rho \) compatible with \( C \).

Protocol:

1. The Client samples a trappified canvas \( T \) from the trappified scheme \( P \).
2. The Client runs the embedding algorithm \( E_C \) from \( P \) on its computation \( C \), the graph \( G \) with output space \( O \), the trappified pattern \( T \), and the partial order \( \preceq_G \). It obtains as output the trappified pattern \( C \cup T \).
3. The Client and Server execute the trappified pattern \( C \cup T \) on input state \( \rho \) blindly following the UBQC Protocol 2 with blinded outputs.
4. If the output set is non-empty (if there are quantum outputs), the Server returns the qubits in positions \( O \) to the Client
5. The Client measures the qubits in positions \( O \cap V_T \) in the \( X \) basis. It obtains the trap sample \( t \).
6. The Client checks the trap by computing \( \tau(t) \). If it outputs a value 0, the Client accepts the computation and sets as its output the output state of Protocol 2 on vertices \( O \setminus V_T \); it rejects otherwise.

4.2 Conditions for Verifiability

We now use the deviation detection capability of trappified schemes to perform verification. To this end, we show that a distinguisher cannot tell apart the concrete protocol execution and the Secure Delegated Quantum Computation Resource 1. This resource allows a Client to input a computation and a quantum state and either receives the correct outcome or an abort state depending on the Server’s choice, whereas the Server only learns a small amount of information contained in a leak \( l_\rho \). More precisely, we show that this holds with a bounded distinguishing advantage so long as the trappified scheme \( P \) detects efficiently a large fraction of deviations that are possibly harmful to the computation.

Theorem 2 (Detection Implies Verifiability).

Let \( P \) be a trappified scheme. Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be two sets of Pauli deviations such that \( \mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset \), and \( I \in \mathcal{E}_2 \). Let \( \epsilon, \delta, \nu \) be positive numbers. If \( P \):

– \( \epsilon \)-detects \( \mathcal{E}_1 \),
– is \( \delta \)-insensitive to \( \mathcal{E}_2 \),
– is \( \nu \)-correct on \( G_V \setminus \mathcal{E}_1 \),

In the original UBQC Protocol from [3], the outputs are prepared by the Server and are encrypted by the computation flow. Here we need to also hide the positions of trap output qubits among the output, which therefore need to be prepared by the Client.
then the Deviation Detecting Delegated Blind Computation Protocol for computing CPTP maps in $\mathcal{C}$ using $\mathcal{P}$ is $(\delta + \nu)$-correct and $\max(\epsilon, \nu)$-secure in the Abstract Cryptography framework, constructing the Secure Delegated Quantum Computation Resource for leak $l_\rho := (\mathcal{C}, G, \mathcal{P}, \preceq_G)$.

Proof. The correctness part of the theorem follows simply from the fact that in the honest case, the concrete and ideal settings will output different states only in the case where the protocol wrongly rejects the computation or outputs a wrong result despite the absence of errors. The distinguishing advantage measured in the diamond norm is bounded by $\delta$ for the first case as $l \in \mathcal{E}_2$, while it is bounded by $\nu$ for the second since $l \in \mathcal{G} \setminus \mathcal{E}_1$. Hence, the protocol is $(\delta + \nu)$-correct.

To prove the security of the protocol, we define as per Definition a Simulator $\sigma$ that has access to the Server’s interface of the Secure Delegated Quantum Computation Resource. The interaction involving either the Simulator or the real honest Client should be indistinguishable.

To do so, we note that when the protocol is run and a deviation is applied by the Server, the probability of accepting or rejecting the computation is dependent only on the deviation and not of the computation performed on the non-trap part of the pattern. This is a crucial property as this allows to simulate the behavior of the concrete protocol even when the computation performed is unknown. More precisely, we define a simulator in the following way:

**Simulator 1**

1. The Simulator request a leak from the Secure Delegated Quantum Computation Resource and receives in return $(\mathcal{C}, G, \mathcal{P}, \preceq_G)$.
2. It chooses at random any computation $C_S \in \mathcal{C}$ and an input which is compatible with $C$.
3. It performs the same tasks as those described by the Client’s side of the Deviation Detecting Delegated Blind Computation Protocol.
4. Whenever $\tau$ accepts, the Simulator sends $c = 0$ to the Secure Delegated Quantum Computation Resource, indicating that the honest Client should receive its output. If it rejects, the Simulator sends $c = 1$ Secure Delegated Quantum Computation Resource, indicating an abort.

We now show that the distinguisher cannot tell apart with high probability the simulation and the concrete protocol. The use of the UBQC Protocol guarantees that the only the Graph and order of measurement is leaked by the interaction. Therefore in total, only the trappified scheme is revealed to the Server in the real protocol before the final step. This exact same information is leaked by the Simulator and these steps are therefore indistinguishable. We focus in the following on the output state and the abort probability in both cases, which are the only remaining elements which the distinguisher can use to decide which setup it is interacting with.

More precisely, let $E \in \mathcal{G}_V$ be a Pauli deviation that models the deviation introduced by the distinguisher. First consider the case where $E \in \mathcal{E}_1$. Whenever the computation is rejected, the output state is identical in both setups. Whenever the computation is accepted, which happens with probability at most
\( \epsilon \) in both cases, the ideal resource will always output the correct state, while the concrete protocol outputs a potentially erroneous state. In the worst case this incorrect state is orthogonal to the honest state, meaning that the distinguisher could tell apart both settings with certainty.

Second, we consider the alternate case, where \( E \notin E_1 \). There again, the distinguisher can only tell apart both settings when the computation is accepted. While in general the probability for such an event is not constrained – and hence only upper bounded by 1 – the conditions of the theorem impose that the trappified scheme is \( \nu \)-correct for such error.

Putting both cases together, the diamond distance between the ideal Resource together with the Simulator and the concrete Protocol is thus upper bounded by \( \max(\epsilon, \nu) \).

### 4.3 Robustness

**Theorem 3 (Robust Detection Implies Robust Verifiability).** Let \( P \) be a trappified scheme. Let \( E_1 \) and \( E_2 \) be two sets of Pauli deviations satisfying the conditions of Theorem 2. We assume an execution of Protocol 3 with an honest-but-noisy Server which samples an error \( E \in G_V \) with a probability at least \( p \) that \( E \in E_2 \). Then, Protocol 3 accepts with probability at least \( p(1 - \delta) \), and the diamond distance between the implemented transformation and \( C \) is bounded by \( (\nu p(1 - \delta) + (1 - p))/ (1 - p\delta) \).

**Proof.** By construction, \( P \) is \( \delta \)-insensitive to \( E_2 \). Hence, it will accept deviations in \( E_2 \) with probability at least \( 1 - \delta \) which yields the acceptance probability. There are then two cases when the computation is accepted. Either the deviation is in \( E_2 \), then the diamond distance between \( C \) and the implemented transformation is bounded by \( \nu \), or the deviation is not in \( E_2 \) but is accepted and the diamond distance is bounded by 1. The first case happens with probability at least \( p \times (1 - \delta) / (1 - p\delta) \), while the second with probability at most \( (1 - p) / (1 - p\delta) \).

Using the convexity of the diamond norm, the result follows.

Whenever (i) a noise process generates deviations that are within \( E_2 \) with overwhelming probability, (ii) the embedding of the computation \( C \) within \( P \) adds redundancy in such a way that \( \nu \) is negligible, and (iii) \( P \) is \( \delta \)-insensitive to \( E_2 \) for a negligible \( \delta \), then the protocol will accept the computation almost all the time, and the computation will be very close to \( C \). We will see below that requirements (i-iii) can be achieved together with a negligible \( \epsilon \), thus allowing to design robust protocols with negligible security error.

### 4.4 Constraints on Trappified Schemes

We now present a consequence of Theorem 2 in the case where the correctness error \( \delta + \nu \) and the security error \( max(\epsilon, \nu) \) are negligible with respect to a security parameter \( \lambda \). We show that this correctness and security regime can only be achieved with a polynomial qubit overhead if the computation is error-protected. More precisely, we assume below that \( P(\lambda) \) is a sequence of trappified
schemes indexed by a security parameter $\lambda$, such that it $\epsilon(\lambda)$-detects a set $E_1(\lambda) \subseteq \mathcal{G}_V(\lambda)$ of Pauli deviations, is $\nu(\lambda)$-correct outside $E_1$, and is $\delta(\lambda)$-insensitive to $E_2(\lambda) \subseteq \mathcal{G}_V(\lambda) \setminus E_1(\lambda)$, for $\epsilon(\lambda)$, $\nu(\lambda)$ and $\delta(\lambda)$ negligible in $\lambda$.

**Error-Correction is Necessary to Avoid Resources Blow-up.** Let $C$ be a computation pattern which implements the client’s desired computation CPTP map $C \in \mathcal{C}$ on some input state $|\psi\rangle$. Here again, we assume that this computation will be executed using a sequence of trappified schemes indexed by $\lambda$ as introduced above. We consider the server’s qubit memory overhead introduced by implementing $C$ using $P(\lambda)$ for computation class $\mathcal{C}$ instead of the unprotected pattern $C$. This is expressed by the ratio $|G_P^\lambda|/|G_{\mathcal{C}}^\lambda|$ between the number of vertices in the graph common to all canvases in $P$ and the graph of $C$. For a trappified pattern $P$ obtained by using the embedding algorithm on a trappified canvas from $P(\lambda)$ we denote by $|O^P|$ the number of computation output qubits in $P$ and by $|O^C|$ the number of output qubits in $C$. Without loss of generality, we impose that $|O^C|$ is minimal, in the sense that given the set of possible inputs and $C$, the space spanned by all possible outputs is the whole Hilbert space of dimension $2^{|O^C|}$. This is always possible as one can add a compression phase at the end of any non-minimal pattern.

**Theorem 4.** Let $C$ be a minimal MBQC pattern implementing a CPTP map $C$. Let $P$ denote a trappified pattern implementing $C$ obtained from $P(\lambda)$. Further assume that $P(\lambda)$ has negligible security error $\max(\epsilon, \nu)$ with respect to $\lambda$.

If $|O^P|/|O^C| = 1$ for a non-negligible fraction of trappified patterns $P \in P(\lambda)$, then the overhead $|G_P^\lambda|/|G_{\mathcal{C}}^\lambda|$ is super-polynomial in $\lambda$.

The usefulness of this theorem comes from the contra-positive statement. Achieving exponential verifiability with a polynomial overhead imposes that $|O^P|/|O^C| > 1$ for an overwhelming fraction of the trappified patterns. This means that the computation is at least partially encoded into a larger physical Hilbert space. As will be apparent in the proof, this will serve to perform some error-correction.

**Proof.** Consider a trappified pattern $P$ for computing $C$ obtained from $P(\lambda)$ such that $|O^P| = |O^C|$. Let $\leq_0$ be a total ordering of the output qubits of the graph common to all trappified canvases in $P(\lambda)$. Let $op \in O^P$ be the first output position of the computation according to $\leq_0$. By definition, a bit-flip operation applied on position $op$ cannot be detected by $T$ since the outcome of the trap is independent of the computation. Additionally, because $C$ is minimal and $|O^P| = |O^C|$, we get that for some input states, the bit-flip deviation on $op$ will be harmful: there is a $\lambda_0$ such that, for all $\lambda \geq \lambda_0$, the diamond distance between $C$ and the bit-flipped version is greater than $\nu(\lambda)$. Hence, this bit flip must be in the set of $\epsilon$-detected deviations, meaning that attacking this position without being detected can happen for at most a negligible fraction $\eta(\lambda)$ of the trappified canvases in $P(\lambda)$. In other words, the position $op$ can only be the first output computation qubit for a negligible fraction $\eta(\lambda)$ of trappified patterns in $P(\lambda)$ that satisfy $|O^P| = |O^C|$.
We then define \( \tilde{P}(\lambda) = \{ P = E_\epsilon(C, P(\lambda)), \ |O^P| = |O^C| \} \), and the set of qubits \( O = \{ o_P, \ P \in P(\lambda) \} \). By construction we have \( \sum_{o \in O} |\{ P \in \tilde{P}, \ o_P = o \}| = |\tilde{P}| \). But, we just showed that \( |\{ P \in \tilde{P}, \ o_P = o \}|/|P(\lambda)| \) is upper-bounded by \( \eta \), negligible in \( \lambda \). Thus, \( |O| \) is lower-bounded by \( |\tilde{P}(\lambda)|/\eta \) which is super-polynomial in \( \lambda \) so long as \( |\tilde{P}(\lambda)|/\eta \) is not negligible in \( \lambda \). \( \blacksquare \)

Note that the situation where \( |O^P| > |O^C| \) is interesting only if the bit-flip deviation on qubit \( o_P \) does not alter the computation. Otherwise, the same reasoning as above is still applicable. This shows that enlarging the physical Hilbert space storing the output of the computation is useful only if it allows for some error-correction which transforms low-weight harmful errors into harmless ones.

5 Robust VBQC as a Deviation Detection Protocol

Theorem 2 presents a clear objective for traps: they should (i) detect harmful deviations while being insensitive to harmless ones. Yet a trap in a trappified pattern cannot detect deviations happening on the computation part of the pattern itself. To achieve exponential verifiability, one further needs to ensure that there are sufficiently many trappified patterns so that it is unlikely that a potentially harmful deviation hits only the computation part of the pattern, and that it is detected with high probability when it hits the rest. This is best stated by Theorem 4 which imposes to (ii) error-protect the computation so that hard-to-detect deviations are harmless while remaining harmful errors are easy to detect. Additionally, one further needs to (iii) find a systematic way to insert traps alongside computation patterns to generate these exponentially many trappified patterns.

The advantage of the Robust VBQC Protocol from [17] is that, by restricting the client to delegation of BQP computations, (i) (ii) and (iii) can be designed and analysed independently from one another. To make this more formal, we recall briefly the structure of the Robust VBQC Protocol and explain how it performs (ii) and (iii). The next sections will then focus on (i) and show how the performance of the protocol can be improved by better trap designs.

The Robust VBQC Protocol is based on the realisation that, because the output of the computation is classical, the error-protection required for exponential verification can be achieved by repetition followed by a majority vote. Therefore, if one considers a pattern \( C \) on a graph \( G \) implementing a computation \( C \), the error-protected version is obtained by running \( d \) times, in parallel or in sequence, the blind delegation protocol \( 2 \). Each of these \( d \) executions is called a computation round. Additionally, the protocol introduces \( s \) test rounds. Each of these test rounds is a pattern run on the same graph \( G \) so that it is blind-compatible with \( C \) (see Definition 7). In the original protocol, they are a collection of small deterministic computations that we will refer to as atomic traps\(^8\). The test round

\(^8\) In [17], these are called traps. We chose to rename them here to avoid confusion with our definition of trappified canvases (see Definition 5).
accepts whenever all atomic traps succeed, and fails otherwise. The collections of these test rounds constitute a trappified canvas according to Definition 5. It accepts if less than \( w \) test rounds fail. Now, because computation and trap patterns are executed using blind-compatatible patterns on the graph \( G \), the trap insertion (iii) can be achieved by interleaving at random the \( s \) test rounds with the \( d \) computation rounds.

The next lemma relates the parameters \( d, s, w \) with the deviation detection capability of the test rounds, thus showing that not only (i), (ii) and (iii) can be designed separately, but also analysed separately with regards to the security achieved by the protocol.

**Definition 12 (RVBQC Compiler).**

Let \( P \) be trappified scheme on a graph \( G = (V, E) \), and let \( d, s, w \in \mathbb{N} \) and \( n = d + s \). Let \( \mathcal{C} \) be the class of computations with classical inputs that can be evaluated by an MBQC pattern on \( G \) using an order \( \preceq_G \) which is compatible with the order \( \preceq_P \) induced by \( P \). We define the RVBQC Compiler that turns \( P \) into a trappified scheme \( P' \) on \( G^n \) for computation class \( \mathcal{C} \) as follows:

- The trappified canvases \( T' \in P' \) and their distribution is given by the following sampling procedure:
  1. Randomly choose a set \( S \subset [1, n] \) of size \( s \). These will be the test rounds;
  2. For each \( j \in S \), independently sample a trappified canvas \( T_j \) from the distribution of \( P \).

- For each trappified canvas \( T' \) defined above and an output \( t = (t_j)_{j \in S} \), the decision function \( \tau' \) by thresholding over the outputs of the decision functions \( \tau_j \) of individual trappified canvases. More precisely:
  \[ \tau'(t) = 0 \text{ if } \sum_{j \in S} \tau_j(t_j) < w, \text{ and } 1 \text{ otherwise} \]

- The partial ordering of vertices of \( G^n \) in \( P' \) is given by the ordering \( \preceq_G \) on every copy of \( G \).

- Let \( C \in \mathcal{C} \) and \( C \) the pattern on \( G \) which implements the computation \( C \). Given a trappified canvas \( T' \in P' \), the embedding algorithm \( E_C \) sets to \( C \) the pattern of the \( d \) graphs that are not in \( S \).

**Lemma 1 (From Constant to Exponential Detection and Insensitivty Rates).** Let \( P \) be a trappified scheme on graph \( G \) which \( \epsilon \)-detects the error set \( \mathcal{G}_V \setminus \{I\} \) and is \( \delta \)-insensitive to \( \{I\} \). For \( d, s, w \in \mathbb{N} \) and \( n = d + s \), let \( P' \) be the trappified scheme resulting from the compilation defined in Definition 12.

For \( E \in \mathcal{G}_V \), let \( \text{wt}(E) \) be defined as the number of copies of \( G \) on which \( E \) does not act as the identity. Let \( \mathcal{E}_{\geq k} = \{ E \in \mathcal{G}_V \mid \text{wt}(E) \geq k \} \), and \( \mathcal{E}_{< k} \) be defined analogously.
Let $k_1 > nw/(s\epsilon)$ and $k_2 > nw/(s\delta)$. Then, $P'$ $\epsilon'$-detects $\mathcal{E}_{\geq k_1}$ and is $\delta'$-insensitive to $\mathcal{E}_{\leq k_2}$ where:

$$
\epsilon' = \min_{\chi \in [0, \frac{k_1}{n} - \frac{w}{s\epsilon}]} \exp(-2\chi^2 s) + \exp\left(-2\left(\frac{k_1}{n} - \chi\right)s\epsilon - \frac{w^2}{s}\right),
$$

$$
\delta' = \min_{\chi \in [0, \frac{k_2}{n} - \frac{w}{s\delta}]} \exp(-2\chi^2 s) + \exp\left(-2\left(\frac{k_2}{n} - \chi\right)s\delta - \frac{w^2}{s}\right).
$$

**Proof.** The argument resembles the proof presented in [17].

Let $E \in \mathcal{E}_{\geq k_1}$. Let $X$ be a random variable describing the number of affected test rounds, where the probability is taken over the choice of the trappified canvas $P'$. By definition of $E$ and construction of $P'$, it then holds for all $\chi \geq 0$ that

$$
\Pr\left[X \leq \left(\frac{k_1}{n} - \chi\right)s\right] \leq \exp(-2\chi^2 s),
$$

using tail bounds for the hypergeometric distribution. Let $Y$ be a random variable counting the number of test rounds for which the decision function rejects. Note that $Y$, conditioned on a lower bound $x$ for $X$, is lower-bounded in the usual stochastic order by a $(x, \epsilon)$-binomially distributed random variable $\tilde{Y}$. Hoeffding’s inequality for the binomial distribution then implies that:

$$
\Pr[Y < w \mid X \geq x] \leq \exp\left(-2\frac{(x\epsilon - w)^2}{x}\right).
$$

All in all, we conclude for $\chi \leq \frac{k_1}{n} - \frac{w}{s\epsilon}$ that:

$$
\Pr[Y < w] \leq \exp(-2\chi^2 s) + \exp\left(-2\left(\frac{k_1}{n} - \chi\right)s\epsilon - \frac{w^2}{s}\right).
$$

This concludes the first statement.

Let now $E \in \mathcal{E}_{\leq k_2}$. Let $X$ be a random variable describing the number of affected test rounds, where the probability is taken over the choice of the trappified canvas. By definition of $E$ and construction of $P'$, it then holds for all $\chi \geq 0$ that

$$
\Pr\left[X \geq \left(\frac{k_2}{n} + \chi\right)s\right] \leq \exp(-2\chi^2 s),
$$

using tail bounds for the hypergeometric distribution. Let $Y$ be a random variable counting the number of test rounds for which the decision function rejects. Note, that $Y$, conditioned on an upper bound $x$ for $X$, is upper-bounded in the usual stochastic order by a $(x, \delta)$-Binomially distributed random variable $\tilde{Y}$. Hoeffding’s inequality for the Binomial distribution then implies that

$$
\Pr[Y \geq w \mid X \leq x] \leq \exp\left(-2\frac{(x\delta - w)^2}{x}\right).
$$
All in all, we conclude for $\chi \leq \frac{k_2}{n} - \frac{w_s}{s^3}$ that

$$\Pr [Y \geq w] \leq \exp (-2\chi^2 s) + \exp \left(-2\frac{(\frac{k_2}{n} - \chi) s \delta - w}{(\frac{k_2}{n} - \chi) s}\right). \quad \blacksquare$$

Remark 5. The statement of Lemma 1 can be straightforwardly generalised to the case when $P$ detects a smaller error set $E$. Then, $P'$ detects the error set $E_{\geq k} = \{ E \in G_{V^n} \mid \text{wt}(E) \geq k \}$ where the error weight $\text{wt}(E)$ is generalized to the number of copies of $G$ on which $E$'s projection is contained in $\mathcal{E}$. All bounds remain identical. The same generalisation holds analogously for the error-insensitivity of $P$ and $P'$.

The consequence of the above lemma is that whenever the trappified schemes are constructed by interleaving computation rounds with test rounds chosen at random from a given set, the performance of the resulting Robust VBQC Protocol crucially depends on the ability of these test rounds to detect harmful errors.

The following section precisely addresses this issue. It recovers the traps used in the original protocol, and constructs new ones by changing both the prepared states for each test round and the binary decision algorithm that accepts or rejects a computation.

Remark 6. Note that we do not make use in Definition 12 of the embedding function or computation class associated with the trappified scheme $P$. In fact the initial scheme can even consist of pure traps as described in Remark 4. This is the case for the scheme describe in the next section. Furthermore, since we are concerned only with amplifying the detection and insensitivity rates and not the correctness of the computation, we do not need to specify here how the $d$ computation are recombined to yield the output of the computation, meaning that the output subspace is not specified in the definition of $E_C$. As is apparent from the lemma, in order to recover exponential verifiability as well, one needs to recombine the computations in a way such that errors of weight lower than $k_1$ are corrected. If the output is classical, the multiple executions form a natural repetition code, meaning that a simple majority vote can correct up to $d/2$ errors. Finding such a distillation procedure in the quantum case is left an open question.

6 Designing Traps for Robust VBQC

6.1 Standard Traps

Following existing VBQC protocols, we impose here that all traps outcomes are deterministic for all possible deviations. Given the specific structure of the Robust VBQC Protocol outlined in the previous section, this means that each test round is also equipped with a binary decision algorithm $\tau$ that accepts or rejects deterministically once a deviation is chosen. Additionally, Theorem 2
requires that the scheme accepts in the absence of deviations. We here therefore impose that all test rounds accept for honest executions of the protocol. To understand what this means in practice, we first focus on the simpler case where the decision algorithm $\tau$ for each test round is such that $\tau(t) = t_0$ where $t_0$ is the first bit of measurement outcome $t$ of the test round. In other words the test round accepts if the outcome $t_0 = 0$, which corresponds to obtaining outcome $|0\rangle$ for, say, qubit $q_0$, while all other measurements outcomes $t_i$ for $i > 0$ are ignored. For the test round to be deterministic, qubit $q_0$ must be equal to $|0\rangle$ in absence of deviation before the computational basis measurement. In other words, the state of $q_0$ then is an eigenstate of $Z_0$, the Pauli $Z$ matrix for qubit $q_0$. By commuting $Z_0$ towards the initialisation of the qubits – through the entangling operations defined by the graph $G$ used to implement trapped pattern $P$, we conclude that determinism and acceptance of deviation-less test rounds implies that the initial state of the qubits before running the protocol is an eigenstate of $X_0 \bigotimes_{i \in N_G(0)} Z_i$, where $N_G(0)$ denotes the indices of the qubits which are neighbours of qubit $q_0$.

Now, to be effective in our model of computation, the initial state of such test round must be compatible with the allowed operations on the client’s side, i.e. the state must be a tensor product of single qubit states. The following lemma shows that it is always possible.

**Lemma 2 (Tensor Product Preparation of a State in a Stabiliser Subspace).** Let $P$ be an element of the Pauli group over $N$ qubits, such that $P^2 \neq -1$. Then, there exists $|\psi\rangle = \bigotimes_{i=1}^N |\psi_i\rangle$ such that $|\psi\rangle = P |\psi\rangle$, and $\forall i, |\psi_i\rangle \in \{|b\rangle\}_{b \in \{0,1\}} \cup \{|\pm k\pi/2\rangle\}_{k \in [0,3]}$.

**Proof.** Without loss of generality, one can write $P = s \bigotimes_i P_i$ with $s = \pm 1$ and $P_i \in \{I, X, Y, Z\}$. Then by construction, $P \in \langle S \rangle$, where $\langle S \rangle$ denotes the multiplicative group generated by the set $S = \{sP_i \bigotimes_{j \neq i} 1_j\}_{i \geq 1} \cup \{P_i \bigotimes_{j \neq i} 1_j\}_{i \geq 1}$. Now, consider the state that is obtained by taking the tensor product of single qubit states that are $+1$ eigenstates of $sP_0$ and $\{P_i\}_{i \geq 1}$. The above shows that it is a $+1$ eigenstate of all operators in $\langle S \rangle$, and in particular of $P$, which concludes the proof as eigenstates of single-qubit Pauli operators are precisely the desired set.

Here, the reader familiar with the line of work following [8] should note that we have indeed recovered their atomic traps. The lemma above states that one should prepare $q_0$ as an eigenstate of $X$, while its neighbours in the graph underlying the pattern $P$ should all be prepared as an eigenstate of $Z$. Such atomic traps are then able to detect all deviations which do not commute with the $Z_0$ measurements of $q_0$, i.e. $X$ and $Y$ deviations on this qubit.

---

9 Recall that throughout the paper, our convention is to view rotated $\{|\pm \theta\rangle\}$ measurements as $Z$ rotations followed by a Hadamard gate and a measurement in the computational basis.
Additionally, within each test round, it is possible to include several such atomic traps as long as their initial states can be prepared simultaneously – i.e. they can at most overlap on qubits that need to be prepared as eigenstates of $Z$. More precisely, take $H$ to be the indices of an independent set of qubits with respect to $G$ (see Definition 13). Consider the decision algorithm $\tau$ to be $\sum_{i \in H} t_i$. That is, the test round accepts whenever all outcomes $Z$ measurements for qubits $q_i, i \in H$ are 0. Following the same line of argument as above, one can show that in absence of deviation, the state of $q_i$ is $|0\rangle$ for all $i \in H$ before the measurement, or equivalently, is an eigenstate of $Z_i$. Commuting these operators towards the initialisation of the qubits shows that the qubits in $H$ must be prepared in the state $|+\rangle$, and $|0\rangle$ for qubits in $N_G(H)$. Other qubits can be prepared in any allowed state.

Various types of test rounds can be generated easily by changing the independent set $H$, and can be combined to form a single trap of a trappified canvas according to Definition 5. This is done by using the RVBQC Compiler of Definition 12 i.e. choosing $s$ test rounds at random from all possible ones and interleaving them at random with computation rounds. Its input locations $I_T$ are, for each test round, all the qubits of the underlying independent set and their neighbours. The output locations $O_T$ are equal to the independent set $H$. The decision algorithm for the trappified canvas $\tau'$ is equal to $\tau'(t) = \sum_j \tau_j(t) \leq w$ where $j$ is an index over the various test rounds, $\tau_j$ their respective decision function described in the previous paragraph and $w$ the threshold value defined by the client running the protocol. In the following, these traps will be called standard traps.

Note that, by varying the independent sets used to construct the test rounds in such a way, each qubit should be in at least one of the independent sets. Then one can conclude that all $X$ and $Y$ deviations have a non-zero probability of being detected, while $I$ and $Z$ deviations are never detected, but are also harmless since applying $Z$ before a computational basis measurement does not change the outcome. As a consequence, one can apply Lemma 11 to prove that this scheme provides exponential detection for all harmful deviations.

### 6.2 General Traps

Above, the traps we obtained are a consequence of determinism, insensitivity to harmless deviations and the specific form of the decision algorithms $\tau$ for the individual test rounds. Now, to construct generalised traps, we simply change the latter ingredient and consider $\tau$ as being the parity of some measurement outcomes for qubits located in a subset $H$ of the vertices of the underlying graph.

---

[i] This is valid in the case of classical outputs when all qubits are measured. In the quantum output case the Client recovers the output which include traps as well, performs the decryption and measures these traps in the computational basis. This decryption operation performs a second twirling of the deviation on the trap qubits, which may permute the Paulis as well. The $Z$ deviations then become $X$ or $Y$ and can be detected as well.
of trappified pattern $P$, i.e. $\tau(t) = \bigoplus_{i \in H} t_i$. Here, $H$ is not constrained to be an independent set of $G$ anymore. This means that, for the test round to accept, the state of these qubits needs to be in the $+1$ eigenspace of the operator $\bigotimes_{i \in H} Z_i$.

From there, one can apply the same steps as for the ones used to construct standard traps. Commuting this operator to the initialisation imposes to prepare a $+1$ eigenstate of $\bigotimes_{i \in H_{\text{even}}} X_i \otimes \bigotimes_{j \in H_{\text{odd}}} Y_j \otimes \bigotimes_{k \in N^{\text{odd}}(H)} Z_k$, where $H_{\text{even}}$ (resp. $H_{\text{odd}}$) are the qubits of even (resp. odd) degree within $H$, and $k \in N^{\text{odd}}(H)$ means $k$ is in the odd neighbourhood of $H$. Again, applying Lemma 2 allows us to find in the eigenspace of this operator a state that can be obtained as a tensor product of single-qubit states. Defining a global trap can thus be done by randomly choosing $s$ sets $H_k$ of qubits and by constructing $s$ test rounds following the procedure described above – one per chosen set of qubits. The decision algorithm $\tilde{\tau}$ is chosen as for standard traps, accepting whenever less than $w$ test rounds reject. The only difference is that the decision algorithm for each test round is the parity of the measurement outcomes for the qubits defining the set $H = \cup_k H_k$ in the construction above.

It is easy to see that these traps detect all deviations that anticommute with $\bigotimes_{i \in H} Z_i$, that is deviations that have an odd number of $X$ or $Y$ for qubits in $H$. These traps are called general traps.

Note that, for a random choice of $H$ and any deviation $E$ with an $X$ or a $Y$, the corresponding test round has a non-negligible probability of detecting $E$. As a consequence, one can apply Lemma 1 to prove that this scheme provides exponential verifiability, since deviations with only $I$ and $Z$ are harmless.

7 Optimising the Distribution of Traps

In this section, we examine the influence of different trap designs on the performance of the whole verification protocol. Below, the performance will be measured by the qubit memory overhead required to obtain verification for a given computation when compared to the same unprotected blind delegation scheme. More specifically, we will seek to optimise, with respect to the protocol’s overhead, the choice of test rounds constructed in the previous section as well as how they are sampled to form standard and generalised traps.

The background in graph theory and graph colourings necessary for this section can be found in Section A.

7.1 Standard Traps

We start with the case of standard traps. Recall that these are obtained by sampling test rounds themselves being the combination of atomic traps – i.e. single isolated qubits prepared in the $|+\rangle$-state, measured in the $X$-basis, and surrounded by $Z$-eigenstates. Test rounds are therefore defined by the location of the $|+\rangle$-states, that is by the choice of a set $H \in \mathcal{I}(G)$, where $\mathcal{I}(G)$ denotes the independent sets of $G$. 
As we have argued before (see Lemma 1), the overhead will be governed by the detection probability of individual test rounds with respect to $X$ deviations. In other words, the performance of the scheme will vary depending on the choice of probability distribution over the independent set $\mathcal{I}(G)$ and the detection capability of each individual test round.

A test round, and therefore its corresponding trappified canvas, will detect a Pauli error if and only if at least one of the $|+\rangle$-states is hit by a local $X$ or $Y$ deviation.

**Lemma 3 (Detection Rate).** Let $G = (V,E)$ be an undirected graph. Let $\mathcal{D}$ be a probability distribution over $\mathcal{I}(G)$, giving rise to the trappified scheme $\mathcal{P}$ where every element of $\mathcal{I}(G)$ describes one trappified canvas. Then, we call:

$$p_{\text{det}}(\mathcal{D}, G) = \min_{M \subseteq V, M \neq \emptyset} \Pr_{H \sim \mathcal{D}}[M \cap H \neq \emptyset]$$

the detection rate of $\mathcal{D}$ over $G$, and $\mathcal{P}$ $p_{\text{det}}(\mathcal{D}, G)$-detects the error set $\mathcal{E} = \{I, X, Y\} \otimes V \setminus \{I \otimes V\}$.

In the definition above, $H$ corresponds to a choice of test round, while $M$ is the set of qubits that are affected by to-be-detected $X$ and $Y$ deviations.

To obtain the lowest overhead, the distribution $\mathcal{D}$ should be chosen such that it maximises the detection probability $p_{\text{det}}(\mathcal{D}, G)$ for a given graph $G$. The following characterisation of the detection rate is going to be useful to determine upper bounds on $p_{\text{det}}$.

**Remark 7.** For any graph $G$ and any distribution $\mathcal{D}$ over $\mathcal{I}(G)$ it holds that

$$p_{\text{det}}(\mathcal{D}, G) = \min_{M} \Pr_{H \sim \mathcal{D}}[M \cap H \neq \emptyset]$$

where the minimum ranges over distributions $M$ over $\wp(V) \setminus \{\emptyset\}$.

There is a natural relation between the detection rates of these standard traps and graph colourings.

**Lemma 4.** For every (non-null) graph $G$ there exists a distribution $\mathcal{D}$ over $\mathcal{I}(G)$ such that $p_{\text{det}}(\mathcal{D}, G) \geq \frac{1}{\chi(G)}$, where $\chi(G)$ is the chromatic number of $G$.

**Proof.** Let $G = (V,E)$ and $(H_1, \ldots, H_k)$ be a proper $k$-colouring of $G$ for $k \geq 1$. Let $\mathcal{D}$ be the uniform distribution over the set $\{H_1, \ldots, H_k\}$. Then, for all $M \subseteq V, M \neq \emptyset$, there exists an $l \in \{1, \ldots, k\}$ such that $M \cap H_l \neq \emptyset$. It also holds that $\Pr_{H \sim \mathcal{D}}[M \cap H \neq \emptyset] \geq \frac{1}{k}$, and hence $p_{\text{det}}(\mathcal{D}, G) \geq \frac{1}{k}$. Choosing a colouring with $k = \chi(G)$ concludes the proof. ■

Lemma 4 mirrors the results from [17], showing that verification schemes can be designed with security bounds depending on the chromatic number of the underlying graph.

We will now show how, analogously, upper bounds on $p_{\text{det}}(D, G)$ are naturally related to cliques of $G$. 26
Lemma 5. For every (non-null) graph $G$ and every distribution $D$ over $\mathcal{I}(G)$ it holds that $p_{\text{det}}(D, G) \leq \frac{1}{\omega(G)}$, where $\omega(G)$ is the clique number of $G$.

Proof. Let $k \in \mathbb{N}$, $k \geq 2$, and $G = (V, E)$ be a graph with a $k$-clique $C \subseteq V$. Let further $\mathcal{M} = \mathcal{U}(\{M \subseteq C \mid |M| = 1\})$ be the uniform distribution over the 1-element subsets of this clique. Further note that for any $T \in \mathcal{I}(G)$ it holds that $|T \cap C| \leq 1$, because $T$ is an independent set. It then follows for any $T \in \mathcal{I}(G)$ that $\Pr_{M \sim \mathcal{M}}[M \cap T \neq \emptyset] \leq \frac{1}{k}$. Remark 4 finally implies that for any distribution $D$ over $\mathcal{I}(G)$,

$$p_{\text{det}}(D, G) \leq \max_{t \in \mathcal{I}(G)} \Pr_{M \sim \mathcal{M}}[M \cap T \neq \emptyset] \leq \frac{1}{k}.$$ 

Letting $k = \omega(G)$ concludes the proof. ■

Lemma 5 and Lemma 6 put the best achievable detection rate by standard traps for a graph $G$ in the interval $\left[\frac{1}{\chi(G)}, \frac{1}{\omega(G)}\right]$. Note that the two graph invariants $\chi(G)$ and $\omega(G)$ are dual in the sense that they are integer solutions to dual linear programs. It turns out that both bounds can be improved to depend on the solutions of the relaxations of the respective linear programs. This closes the integrality gap between the chromatic number and the clique number.

Lemma 6. For every (non-null) graph $G$ there exists a distribution $D$ over $\mathcal{I}(G)$ such that $p_{\text{det}}(D, G) \geq \frac{1}{\chi_f(G)}$, where $\chi_f(G)$ is the fractional chromatic number of $G$ (see Definition 16).

Proof. Let $D$ be a distribution over $\mathcal{I}(G)$ such that for all $v \in V$ it holds that $\Pr_{H \sim D}[v \in H] \geq \frac{1}{k}$. For all $M \subseteq V, M \neq \emptyset$, then $\Pr_{H \sim D}[M \cap H \neq \emptyset] \geq \frac{1}{k}$ and therefore $p_{\text{det}}(D, G) \geq \frac{1}{k}$. By Lemma 9 we can find such a distribution $D$ for any $k \geq \chi_f(G)$. ■

We finally arrive at an improved version of the upper bound from Lemma 5 related to fractional cliques.

Lemma 7. For every (non-null) graph $G$ and every distribution $D$ over $\mathcal{I}(G)$ it holds that $p_{\text{det}}(D, G) \leq \frac{1}{\omega_f(G)}$, where $\omega_f(G)$ is the fractional clique number of $G$ (see Definition 17).

As a consequence, this shows that the robust VBQC protocol described in [17] can sometimes be improved by constructing additional test rounds that would allow to have a probability of detection greater than the reported $1/\chi(G)$. In fact, this proves that the best possible (and achievable) detection rate by standard traps is equal to $1/\chi_f(G)$.

Yet, this leaves a dependency of the protocol’s efficiency on graph invariants, meaning that depending on the chosen computation, the protocol could perform poorly. The next section shows how to overcome this obstacle, as long as the client is willing to use more generalised traps.
7.2 Generalised Traps

We now turn to the analysis of the detection probability for generalised traps. Recall that such traps are based on test rounds that are defined by a set $H \subseteq V$ of qubit locations. The test round is accepting whenever the parity of outcomes of $Z$-measurements on the qubits of $H$ is even.

Again, looking at a given deviation $E$, we conclude that a test-round defined by $H$ detects $E$ if and only if $|E \cap H|$ is odd – here $E$ denotes the set of qubits where $E$ is equal to $X$ or $Y$. Note, that the testing set $H$ can be chosen freely here, and does not need to be independent as in the construction of standard traps. Now, if $H$ is sampled uniformly at random from $\wp(G)$, then $\Pr_{H \sim \mathcal{U}(\wp(G))}[|E \cap H| \equiv 1 \mod 2] = 1/2$, and this is valid for any $E \neq I$.

As a conclusion, we obtain that the probability of detection for this scheme is equal to $1/2$, which is independent of the graph $G$, and generally will beat the upper bound obtained in the previous section through standard traps.

7.3 A Linear Programming Problem for Further Customisation

In particular situations, it might be useful to have more granular control of the design and error-detecting capabilities of the test rounds.

For instance, because of hardware constraints or ease of implementation, it might be favourable to restrict the set of tests one is willing to perform to only a subset of the tests resulting from generalised traps. As one example, one might desire to avoid the preparation of dummy states and therefore restrict the set of feasible tests to those requiring the preparation of quantum states in the $X$-$Y$-plane only. It might also not be necessary for the employed tests to detect all possible Pauli errors because of inherent robustness of the target computation. In such cases, we can expect better error detection rates if we (i) allow for more types of tests, or (ii) remove deviations from the set of errors that are required to be detected. To this end, we present a linear programming formulation of the search for more efficient tests in Problem 1.

Remark 8. While efficient algorithms exist to find solutions to such real-valued constrained linear problems, in this case the number of constraints grows linearly with the number of errors that need to be detected, and therefore generally exponentially in the size of the graph.

Remark 9. Solutions to the dual problem of Problem 1 are distributions of deviations applied to the test rounds. An optimal solution to the dual gives therefore an optimal attack, i.e. a distribution of deviations that achieves a minimal detection rate with the tests at hand.

8 Discussion and Future Work

We provided a general framework that encompasses all previous trap-based blind verification schemes in the prepare-and-send MBQC model, which covers the
**Problem 1** Optimisation of the distribution of tests

**Given**

- a set of errors $\mathcal{E}$ to be detected,
- a set of feasible tests $\mathcal{H}$,
- a relation between tests and errors describing whether a test detects an error, $R : \mathcal{H} \times \mathcal{E} \to \{0, 1\}$,

**find** an optimal distribution $p : \mathcal{H} \to [0, 1]$ **maximising** the detection rate $\epsilon \in [0, 1]$

**subject to** the following conditions:

- $p$ describes a probability distribution, i.e. $\sum_{H \in \mathcal{H}} p(H) \leq 1$,
- all concerned errors are detected at least with the target detection rate, i.e.

$$\forall E \in \mathcal{E} : \sum_{\substack{H \in \mathcal{H} \\colon \ \ R(H,E) = 1}} p(H) \geq \epsilon.$$

---

majority of proposed protocols from the literature. In addition, we believe that all results mentioned here also apply to receive-and-measure MBQC protocols. We leave this question for future work.

On the theoretical side, we then uncovered a natural connection between error detection and verification through an alternate generic composable security proof in the AC framework. We also provided a proof for the requirement of error correction if one hopes to have negligible correctness and security errors with polynomial overhead when comparing unprotected and unverified computations and their secure counterparts. In doing so it further emphasises the link between error detection and verification.

From a practical standpoint, we used our framework to describe new trapped schemes that improve the overhead of state of the art verification protocols, thus making it more appealing for experimental realisation and possibly for integration into future quantum computing platforms.

Overall, we strongly suspect the link between error detection and verification can be further developed and yield new trapped schemes with not only more efficient implementations but also additional capabilities.

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A  Graph Colourings

In this section, we introduce graph colourings and recall some known related results that are useful to our theory.

Definition 13 (Independent Set). Let $G = (V, E)$ be a graph. Then a set of vertices $t \subseteq V$ is called an independent set of $G$ if

$$\forall v_1, v_2 \in t : \{v_1, v_2\} \notin E.$$ 

The size of the largest independent set of $G$ is called the independence number of $G$ and denoted by $\alpha(G)$. The set of all independent sets of $G$ is denoted $I(G)$.

Definition 14 (Graph Colouring). Let $G = (V, E)$ be a graph. Then a collection of $k$ pairwise disjoint independent sets $H_1, \ldots, H_k \subseteq V$ such that $\bigcup_{j=1}^{k} H_j = V$ is called a (proper) $k$-colouring of $G$. The smallest number $k \in \mathbb{N}_0$ such that $G$ admits a $k$-colouring is called the chromatic number of $G$ and denoted by $\chi(G)$.

Definition 15 (Clique). Let $G = (V, E)$ be a graph. Then a complete subgraph $C \subseteq V$ of size $k$ is called a $k$-clique of $G$. The largest number $k \in \mathbb{N}_0$ such that $G$ admits a $k$-clique is called the clique number of $G$ and denoted by $\omega(G)$.

Lemma 8. For any graph $G$ it holds that $\omega(G) \leq \chi(G)$. For any $n \in \mathbb{N}$, there exists a graph $G_n$ such that $\chi(G_n) - \omega(G_n) \geq n$.

Definition 16 (Fractional Graph Colouring). Let $G = (V, E)$ be a graph. For $b \in \mathbb{N}$, a collection of independent sets $H_1, \ldots, H_k \subseteq V$, such that for all $v \in V : |\{1 \leq j \leq k \mid v \in H_j\}| = b$, is called a $k:b$-colouring of $G$. The smallest number $k \in \mathbb{N}_0$ such that $G$ admits a $k:b$-colouring is called the $b$-fold chromatic number of $G$ and denoted by $\chi_b(G)$. Since $\chi_b(G)$ is subadditive we can define the fractional chromatic number of $G$ as

$$\chi_f(G) = \lim_{b \to \infty} \frac{\chi_b(G)}{b} = \inf_{b \in \mathbb{N}} \frac{\chi_b(G)}{b}.$$ 

Note that $k:1$-colourings are $k$-colourings and therefore $\chi_1(G) = \chi(G)$ which in turn implies that for all $b \in \mathbb{N}$ it holds that

$$\chi_f(G) \leq \chi_b(G) \leq \chi(G).$$

Lemma 9. Let $G = (V, E)$ be a graph. Then $\chi_f(G)$ equals the smallest number $k \in \mathbb{R}_+^*$ such that there exists a probability distribution $D$ over the independent sets $I(G)$ such that for all $v \in V$ it holds that

$$\Pr_{H \sim D}[v \in t] \geq \frac{1}{k}.$$ 

Definition 17 (Fractional Clique). Let $G = (V, E)$ be a graph. For $b \in \mathbb{N}$, a function $f : V \to \mathbb{N}_0$, such that for all $H \in I(G) : \sum_{v \in H} f(v) \leq b$ and
\[ \sum_{v \in V} = k, \text{ is called a } k:b\text{-clique of } G. \text{ The biggest number } k \in \mathbb{N}_0 \text{ such that } G \text{ admits a } k:b\text{-clique is called the } b\text{-fold clique number of } G \text{ and denoted by } \omega_b(G). \]

Since \( \chi_b(G) \) is superadditive we can define the fractional clique number of \( G \) as

\[ \omega_f(G) = \lim_{b \to \infty} \frac{\omega_b(G)}{b} = \sup_{b \in \mathbb{N}} \frac{\omega_b(G)}{b}. \]

Note that \( k:1\)-cliques are \( k\)-cliques and therefore \( \omega_1(G) = \omega(G) \) which in turn implies that for all \( b \in \mathbb{N} \) it holds that \( \omega(G) \leq \omega_b(G) \leq \omega_f(G) \).

**Lemma 10.** Let \( G = (V,E) \) be a graph. Then \( \omega_f(G) \) equals the biggest number \( k \in \mathbb{R}_0^+ \) such that there exists a probability distribution \( D \) over the vertices \( V \) such that for all \( H \in I(G) \) it holds that

\[ \Pr_{v \sim D}[v \in H] \leq \frac{1}{k}. \]

Both the fractional clique number \( \omega_f(G) \) and the fractional chromatic number \( \chi_f(G) \) are rational-valued solutions to dual linear programs. By the strong duality theorem, the two numbers must be equal.

**Lemma 11.** For any graph \( G \) it holds that \( \omega_f(G) = \chi_f(G) \).