ON PERIODIC SOLUTIONS IN THE WHITNEY’S INVERTED PENDULUM PROBLEM

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Dedicated to Professor Norman Dancer

Abstract. In the book “What is Mathematics?” Richard Courant and Herbert Robbins presented a solution of a Whitney’s problem of an inverted pendulum on a railway carriage moving on a straight line. Since the appearance of the book in 1941 the solution was contested by several distinguished mathematicians. The first formal proof based on the idea of Courant and Robbins was published by Ivan Polekhin in 2014. Polekhin also proved a theorem on the existence of a periodic solution of the problem provided the movement of the carriage on the line is periodic. In the present paper we slightly improve the Polekhin’s theorem by lowering the regularity class of the motion and we prove a theorem on the existence of a periodic solution if the carriage moves periodically on the plane.

1. Introduction. In the year 1941, in the first edition of the book “What is Mathematics?” Richard Courant and Herbert Robbins posed the following question suggested by Hassler Whitney: “Suppose a train travels from station A to station B along a straight section of track. The journey need not be of uniform speed or acceleration. The train may act in any manner, speeding up, slowing down, coming to a halt, or even backing up for a while, before reaching B. But the exact motion of the train is supposed to be known in advance; that is, the function $s = f(t)$ is given, where $s$ is the distance of the train from station $A$, and $t$ is the time, measured from the instant of departure. On the floor of one of the cars a rod is pivoted so that it may move without friction either forward or backward until it touches the floor. If it does touch the floor, we assume that it remains on the floor henceforth; this will be the case if the rod does not bounce. Is it possible to place the rod in such a position that, if it is released at the instant when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey form $A$ to $B$?” The question is illustrated in Figure 1. Assuming continuous dependence of the motion of the rod on its initial position, Courant and Robbins explained how the intermediate value theorem implies the positive answer. Moreover, as exercises they posed the problems: “the reasoning above may be generalized to the case when the journey is of infinite duration” and “generalize to the case where the motion of the train is along any curve in the plane.

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and the rod may fall in any direction” with a hint on nonexistence of a retraction of a disk onto its boundary (compare [10, pp. 319–321]). In the sequel we refer to the above question (either in finite or infinite time setting) as to the Whitney’s linear inverted pendulum problem (shorter: the problem or the Whitney’s problem). If the curve \( f \) lies in the plane, we call the problem “planar”.

In 1953 John E. Littlewood included the problem in his book “A Mathematician’s Miscellany” ([17, pp. 12–14]) and provided his own explanation (close to the Courant and Robbins’ one, actually). The fragment of [10] related to the problem was reproduced in [18, pp. 2412, 2413] (under the title “The Lever of Mahomet”) in the year 1960. In the paper [5] published in 1958 Arne Broman noticed that the assumption on continuity needs an explanation. He presented a comprehensive argument supporting the Courant and Robbins’ solution, although his proof lacks of formal rigor at some details.

In 1976 the continuity assumption was contested by Tim Poston in the article [21] in Manifold, a mimeographed magazine issued by the Warwick University. Basing on possible (according to him) phase portraits related to the problem, he claimed that the rod can come arbitrarily close to the floor of the car and then swing back causing discontinuity of its final position with respect to the initial one. The argument of Poston was replicated by Ian Stewart in the book “Game, Set and Math” ([26, pp. 63, 64, 68], 1989) and also in the Stewart’s comments to the second edition of [10] which appeared in 1996; see [11, pp. 505–507]. In the review of [11] published in The American Mathematical Monthly in 1998 (see [16]) Leonard Gillman opposed to the arguments of Poston and Stewart writing: “the acceleration of the train would have to be unbounded, which is not possible from physical locomotive” and presented a descriptive proof of the Courant and Robbins’ solution suggested to him by Charles Radin. In 2001, in another review of [11] Brian E. Blank also criticized the Stewart’s comment related to the problem by repeating the conclusion of the Gillman’s report (compare [3]).

The next comment contesting the continuity assumption appeared in the Vladimir Arnold’s short book “What is Mathematics?” published in 2002. Arnold writes (in my translation): “no continuous function – the finite position for a given initial position – can be seen immediately: it should be carefully defined (with the possibility of hitting the platform) and its continuity should be proved”. On the other hand, as the London Mathematical Society Newsletter reported in 2009 (see [12]), in the inaugural Christopher Zeeman Medal Award lecture entitled “The Strange Case of the Courant-Robbins Train” Stewart admitted that “Courant and Robbins were correct to assume continuity in the particular case where the carriage has a flat floor”. At that time Arnold still had objections towards the correctness of the solution from [10]. In the chapter “Courant’s Erroneous Theorems” of the book [2], after presentation of the problem and the solution (for the travel time from 0
to $T$ and the angle $\alpha \in [0, \pi]$ between the floor and the rod as a function of the initial position and time) he wrote: “many people disputed this (incorrect) proof, because even if a continuous function $\alpha(\cdot, T)$ of the initial position $\phi$ were defined, its difference from 0 to $\pi$ under the initial condition $\cdot = \phi$ would not imply that the angle $\alpha$ differs from 0 and $\pi$ at all intermediate moments of time $0 < t < T$”.

Finally, in 2014, 73 years from the announcement of the Whitney’s problem, in the paper [19] Ivan Polekhin provided a short rigorous proof of the Courant and Robbins’ solution based on the Ważewski retract theorem. Other proofs were published in [4] and [28]. A natural question on the existence of a non-falling $T$-periodic solution when the path $f$ of the car is $T$-periodic was also considered by Polekhin. In [19] he proved that if $f$ is of $C^3$-class then such a periodic solution exist in the linear problem. Moreover, in [20] he got the same conclusion in the planar problem, provided the rod moves with friction.

The main purpose of the present paper is to prove two theorems on periodic solutions in the Whitney’s problem. By lowering the regularity class of $f$, the first one provides a minor improvement to the corresponding result in [19].

**Theorem 1.1.** If $f : \mathbb{R} \to \mathbb{R}$ is $T$-periodic and of $C^2$-class then the linear Whitney’s inverted pendulum problem has a $T$-periodic solution.

The second theorem is the main contribution of this research. In contrast to [20], it refers to the original planar problem with the frictionless movement of the rod.

**Theorem 1.2.** If $f : \mathbb{R} \to \mathbb{R}^2$ is $T$-periodic and of $C^3$-class then the planar Whitney’s inverted pendulum problem has a $T$-periodic solution.

Assuming the mass of the rod is concentrated at its top (i.e. it is a mathematical pendulum), both theorems are translated into theorems on periodic solutions of nonautonomous differential equations. In the proofs we apply a result from the paper [6] on the existence of periodic solutions by a continuation method. Problems related to periodic perturbations of inverted pendulum are considered also in control theory. In particular, in [7, 8, 9] modifications of results of [6] were applied in the proofs of theorems on the exact tracking problem.

The rest of the paper is organized as follows. In Section 2 we recall some standard notions corresponding to ordinary differential equations: dynamical system, evolution operator, Poincaré operator, etc. and also the notions of exit, entrance, and bound sets. The latter notion first appeared in a restricted context in [15, p. 42] and in full generality in [27]. Motivated by [15, p. 44] we introduce the notion of a curvature bound function which is used to construct bound sets. The main result of this section is Theorem 2.1 (which essentially is the same as [6, Corollary 3]), a sufficient condition for the existence of periodic solutions of a nonautonomous equation in terms of bound sets and the topological degree of a vectorfield homotopic to the right-hand side of the equation. We provide a direct proof of Theorem 2.1 based on the continuation of the fixed point index. In Section 3 we derive a second-order equation related to the linear problem by an elementary application of the Newton’s second law in the Cartesian coordinates, then we prove two lemmas related to the existence of bound sets for suitable modifications of the derived equation, and finally we apply Theorem 2.1 in a proof of Theorem 3.3, a reformulation of Theorem 1.1 in the Cartesian coordinates system. In exactly the same way we proceed in Section 4 on the planar problem; the main result here is Theorem 4.3 which reformulates Theorem 1.2 in the Cartesian coordinates. It should be noted, however, that in spite of similarities of the results in Sections 3 and 4, some proofs
in Section 4 are different and essentially more complex due to higher dimension of the phase space in the planar problem with respect to the dimension of the phase space in the linear one.

We use a standard vector notation in \( \mathbb{R}^n \). In particular, vectors are represented by columns, \( A^T \) denotes the transpose of \( A \), \( \text{diag}(A_1, \ldots, A_k) \) denotes the block diagonal matrix of square matrices \( A_1, \ldots, A_k \), and \( [x_1 \ \ldots \ \ x_k] \) denotes the matrix with columns \( x_1, \ldots, x_k \). The scalar product of vectors \( x \) and \( y \) is defined as \( x^T y \) and the norm of a vector \( x \) is given by \( |x| := \sqrt{x^T x} \). The derivative of a function \( f \) is denoted by \( Df \); if \( f \) is single-variable it is also denoted by \( \dot{f} \). The Hessian of a scalar function \( f \) is denoted by \( D^2 f \). The norm of a continuous \( T \)-periodic function \( f: \mathbb{R} \to \mathbb{R}^n \) is defined as \( ||f|| := \max_{t \in [0,T]} |f(t)| \).

2. A theorem on the existence of periodic solutions. Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and let \( v: \Omega \to \mathbb{R}^n \) be a vectorfield of \( C^1 \)-class. Denote by \( \phi \) the dynamical system generated by \( v \); recall that \( t \to \phi_t(x_0) \) is the maximal solution of the equation

\[
\dot{x} = v(x)
\]

with the initial value \( x(0) = x_0, \phi_0(x) = x \), and \( \phi_{s+t}(x) = \phi_s(\phi_t(x)) \). Let \( E \subset \Omega \). The \textit{entrance} and \textit{exit} sets of \( E \) are given, respectively, a

\[
E^+ := \{x \in E: \phi_{-\epsilon_n}(x) \notin E \text{ for some } \{\epsilon_n\}, \ 0 < \epsilon_n \to 0 \text{ as } n \to \infty\},
\]

\[
E^- := \{x \in E: \phi_{\epsilon_n}(x) \notin E \text{ for some } \{\epsilon_n\}, \ 0 < \epsilon_n \to 0 \text{ as } n \to \infty\}.
\]

Clearly, \( E^\pm \) are subsets \( \partial E \), the boundary of \( E \). A closed set \( E \) in \( \Omega \) is called a \textit{bound set} for the vectorfield \( v \) if for every \( \epsilon > 0 \) there is no \( x \in \partial E \) such that \( \phi_t(x) \in E \) for each \( t \in (-\epsilon, \epsilon) \).

**Proposition 1.** If \( E \) is closed and \( \partial E = E^+ \cup E^- \) then \( E \) is a bound set. \( \Box \)

Let \( U \) be an open subset of \( \Omega \) and let \( e: U \to \mathbb{R} \) be of \( C^2 \)-class. We call \( e \) a \textit{curvature bound function} for \( v \) if for each \( x \in U \) such that \( e(x) = 0 \),

\[
(De)v = 0 \implies v^T (D^2 e)v + (De)(Dv)v > 0.
\]

(2)

Here (and also in the sequel) we use an abbreviate notation: we write \( v \) instead of \( v(x) \), \( e \) instead of \( e(x) \), etc. whenever the choice of \( x \) is clear from the context.

**Proposition 2.** If \( e: U \to \mathbb{R} \) is a curvature bound function then for each \( x \in U \), \( e(x) = 0 \) there exists an \( \epsilon > 0 \) such that

- either \( e(\phi_t(x)) > 0 \) for \( t \in (-\epsilon, 0) \), \( e(\phi_t(x)) < 0 \) for \( t \in (0, \epsilon) \),
- or \( e(\phi_t(x)) < 0 \) for \( t \in (-\epsilon, 0) \), \( e(\phi_t(x)) > 0 \) for \( t \in (0, \epsilon) \),
- or else \( e(\phi_t(x)) > 0 \) for \( t \in (-\epsilon, 0) \cup (0, \epsilon) \).

Proof. The first two possibilities come from the inequalities \( (De(x))v(x) < 0 \) and \( (De(x))v(x) > 0 \). If \( (De(x))v(x) = 0 \) then it follows by (2) the function \( t \to e(\phi_t(x)) \) has a strict local minimum at 0 (compare also [25], pp. 617, 618).

\( \Box \)

Let

\[
E = \bigcap_{k=1, \ldots, r} \{e_k \leq 0\}
\]

for some continuous functions \( e_k: \Omega \to \mathbb{R} \), hence \( E \) is closed in \( \Omega \). As an immediate consequence of Propositions 1 and 2 we get the following result.
Corollary 1. If for some (possibly empty) closed subsets \( Z_k \subset \partial E \), \( k = 1, \ldots, r \), (a) \( \epsilon_k \) is a curvature bound function for \( v \) in an open neighborhood of the set 
\[ \{ \epsilon_k = 0 \} \cap \partial E \setminus Z_k \]
(b) \( Z_k \subset E^- \cup E^+ \),
then \( E \) is a bound set for \( v \).

Let \( w: \mathbb{R} \times \Omega \to \mathbb{R}^n \) be of \( C^1 \)-class. The vector-field \( v: \mathbb{R} \times \Omega \to \mathbb{R} \times \mathbb{R}^n \) given by
\[ v(t, x) = \begin{bmatrix} 1 \\ w(t, x) \end{bmatrix}. \]
generates a dynamical system \( \phi \) on \( \mathbb{R} \times \Omega \) of the form
\[ \phi(t_0, x_0) = (t_0 + t, \Psi(t_0, t_0 + t, x_0)), \]
where \( t \to \Psi(t_0, t, x_0) \) is the maximal solution of the nonautonomous equation
\[ \dot{x} = w(t, x) \quad (4) \]
with the initial value \( x(t_0) = x_0 \). \( \Psi \) is called the evolution operator generated by \( w \). It satisfies \( \Psi(t, t, x) = x \) and \( \Psi(s, u, x) = \Psi(t, u, \Psi(s, t, x)) \).

Let \( T > 0 \) and assume \( t \to w(t, x) \) is \( T \)-periodic for every \( x \), hence
\[ \Psi(s + T, t + T, x) = \Psi(s, t, x). \]
The map \( P: x \to \Psi(0, T, x) \) is called the Poincaré operator. Denote by \( \text{Fix}(P) \) the set of all fixed points of \( P \).

Proposition 3. \( x_0 \in \text{Fix}(P) \) if and only if \( t \to \Psi(0, t, x_0) \) is a \( T \)-periodic solution of (4).

Let \( E \subset \mathbb{R} \times \Omega \). For \( t \in \mathbb{R} \) define \( E_t := \{ x \in \mathbb{R}^n; (t, x) \in E \} \). Assume \( E \) is a bound set for \( v \) and \( E_0 \) is compact. Set
\[ K := \{ x \in \text{Fix}(P); \Psi(0, t, x) \in E_t \forall t \in [0, T] \}, \]
\[ L := \{ x \in \text{Fix}(P) \cap E_0; \exists t \in (0, T); \Psi(0, t, x) \notin E_t \}. \]
Clearly, \( \text{Fix}(P) \cap E_0 = K \cup L \). Conditions imposed on \( E \) imply that both \( K \) and \( L \) are compact, \( K \subset \text{int} E_0 \), and \( K \cap L = \emptyset \). It follows that there exists open sets \( U, V \) in \( E_0 \), \( U \subset \text{int} E_0 \), such that \( U \cap V = \emptyset \) and
\[ U \cap \text{Fix}(P) = K, \quad (5) \]
\[ V \cap \text{Fix}(P) = L. \quad (6) \]
In particular, the fixed point index of \( P \) at \( U \) is defined; denote it by \( \text{ind}(P, U) \). We refer to [14] for its definition and properties. Actually, the excision property of the index imply that \( \text{ind}(P, U) \) does not depend on the choice of \( U \) satisfying (5).

Let \( \lambda \in [0, 1] \) and \( T > 0 \). Consider a continuous family of non-autonomous equations
\[ \dot{x} = w_\lambda(t, x), \quad (7) \]
where \( w_\lambda: \mathbb{R} \times \Omega \to \mathbb{R}^n \) is of \( C^1 \)-class and \( t \to w_\lambda(t, x) \) is \( T \)-periodic for every \( x \). Through remainder of this section we adopt the above notation concerning \( w \) to \( w_\lambda \) writing \( w_\lambda \) (as in (3)), \( P_\lambda, U_\lambda \), etc.

Proposition 4. Let \( E \subset \mathbb{R} \times \Omega \) and let \( E_0 \) be compact. If for every \( \lambda \in [0, 1] \), \( E \) is a bound set for \( v_\lambda \) then
\[ \text{ind}(P_0, U_0) = \text{ind}(P_1, U_1). \]
Proof. Fix $\lambda_0 \in [0, 1]$. There is an $\epsilon > 0$ such that

$$|P_{\lambda_0}(x) - x| > \epsilon$$

for $x \in E_0 \setminus (U_{\lambda_0} \cup V_{\lambda_0})$. Therefore, if $\lambda$ is sufficiently close to $\lambda_0$ then

$$\text{Fix}(P_{\lambda}) \cap E_0 \subset U_{\lambda_0} \cup V_{\lambda_0}$$

and, by the homotopy property of the index,

$$\text{ind}(P_{\lambda}, U_{\lambda_0}) = \text{ind}(P_{\lambda_0}, U_{\lambda_0}). \quad (8)$$

In order to finish the proof one should show that if $\lambda$ is close enough to $\lambda_0$ then

$$K_{\lambda} \subset U_{\lambda_0}, \quad K_{\lambda} \subset V_{\lambda_0}. \quad (9,10)$$

Indeed, in that case $K_{\lambda} \subset U_{\lambda_0}$, hence one can treat $U_{\lambda_0}$ as $U_{\lambda}$ in the equation (8) and therefore the function $\lambda \to \text{ind}(P_{\lambda}, U_{\lambda})$ is locally constant for $\lambda \in [0, 1]$, hence it is constant and the result follows.

For a proof of (9) assume on the contrary that there exist $\lambda_n \to \lambda_0$ and $x_n \in V_{\lambda_0} \cap K_{\lambda_n}$. One can assume $x_n \to x_0$. Since $\Psi_{\lambda_0}(0, t, x_n) \in E_t$ for each $t \in [0, T]$ and $E$ is closed, $x_0 \in \overline{V_{\lambda_0} \cap K_{\lambda_0}}$, which is impossible. In a similar way the inclusion (10) follows.

Assume now that $w$ is $t$-independent, i.e. $w : \Omega \to \mathbb{R}^n$ is a $C^1$-class vectorfield. In this case the Poincaré operator $P$ associated with $T$-periodic solutions of (4) is equal to $\phi_T$, where $\phi$ is the dynamical system generated by $w$. Assume that $E \subset \Omega$ is a bound set for $v$. Following the notation introduced above we denote by $U$ be the open subset of int $E$ satisfying (5). By $\text{deg}(w, V, 0)$ we denote the topological degree at 0 of the $w$ in an open set $V$ (see [13] for the definition and properties).

Proposition 5. $\text{ind}(\phi_T, U) = (-1)^n \text{deg}(v, \text{int } E, 0)$.

Proof. Let $0 < \epsilon \leq T$. By assumptions and the argument in the proof of Proposition 4,

$$\text{ind}(\phi_T, U) = \text{ind}(\phi_\epsilon, U_\epsilon)$$

where $U_\epsilon \subset \text{int } E$ is an open neighborhood of $\epsilon$-periodic points such that their orbits are contained in the interior of $E$. If $\epsilon$ is small enough then [22, Theorem 5.1] implies

$$\text{ind}(\phi_\epsilon, U_\epsilon) = (-1)^n \text{deg}(v, \text{int } E, 0),$$

hence the result follows.

Now we formulate the key theoretical result for the proof of the existence of periodic solutions in the Whitney’s problem. As above, we consider the continuous family of equations (7), $\lambda \in [0, 1]$, where $w_{\lambda}$ is $T$-periodic with respect to $t$.

Theorem 2.1 (compare Corollary 3 in [6]). Let $B$ be a compact subset of $\Omega$. Assume $\mathbb{R} \times B$ is a bound set for $v_{\lambda}$, $0 \leq \lambda \leq 1$. Assume moreover that $w_0$ is $t$-independent and

$$\text{deg}(w_0, \text{int } B, 0) \neq 0.$$ 

Then the equation (7) for $\lambda = 1$ has a $T$-periodic solution with image contained in $B$.

Proof. This result is a direct consequence of Propositions 3, 4, and 5.
3. Periodic solutions in the linear Whitney’s problem. We begin with deriving the equation corresponding to the linear Whitney’s problem in the Cartesian coordinates. We assume that the whole mass $m$ of the rod is concentrated at its top. Let $x(t)$ and $y(t)$ denote the horizontal and, respectively, vertical position of the top of the rod with respect to the pivot at time $t$ and let $\ell$ be equal to the length of the rod, i.e. to the distance from the pivot to the top, thus
\[ y(t) = \sqrt{\ell^2 - x(t)^2}. \] (11)

The position of the pivot with respect to the origin at time $t$ is equal to $f(t)$, hence the acceleration of the top is equal to $\ddot{x}(t) + \ddot{f}(t)$. The constraint force is perpendicular to the arc $y = \sqrt{\ell^2 - x^2}$, hence the forces imposed on the top of the rod are given by the system of equations
\[
m(\ddot{x}(t) + \ddot{f}(t)) = \mu(t)x(t),
\]
\[
m\ddot{y}(t) = -mg + \mu(t)\sqrt{\ell^2 - x(t)^2},
\]
where $m > 0$ is the mass of the rod, $g > 0$ is the gravitational constant, and $\mu$ is an unknown function such that $\ell\mu(t)$ is equal to the magnitude of the constraint force at time $t$. It follows by (11) and (13),
\[
\mu = m \frac{g + \dot{y}}{\sqrt{\ell^2 - x^2}},
\]
\[
\ddot{y} = -\frac{x\ddot{x} + \dot{x}^2}{\sqrt{\ell^2 - x^2}} - \frac{x^2\ddot{x}^2}{(\ell^2 - x^2)^{3/2}},
\]

hence, by (12),
\[
\ddot{x} + \ddot{f}(t) = \frac{g}{\sqrt{\ell^2 - x^2}}x - \frac{x^2\ddot{x} + x\dot{x}^2}{\ell^2 - x^2} - \frac{x^2\ddot{x}^2}{(\ell^2 - x^2)^2}
\]
which is equivalent to the nonautonomous equation
\[
\ddot{x} = \left(\frac{g}{\ell^2} \sqrt{\ell^2 - x^2} - \frac{x^2\ddot{x} + x\dot{x}^2}{\ell^2 - x^2}\right) x - \frac{\ell^2 - x^2}{\ell^2} \ddot{f}(t). \tag{14}
\]

After changing the variable $x$ to $\frac{1}{\ell}x$ and rescaling $g$ and $\ddot{f}$ to
\[
G := \frac{g}{\ell}, \quad F := \frac{\ddot{f}}{\ell},
\]
the equation (14) assumes a simpler form
\[
\ddot{x} = \left(G\sqrt{1-x^2} - \frac{x^2\ddot{x} + x\dot{x}^2}{1-x^2}\right)x - (1-x^2)F(t). \tag{15}
\]

The equation (15) induces the system
\[
\dot{t} = 1, \quad \tag{16}
\]
\[
\dot{x} = p, \quad \tag{17}
\]
\[
\dot{p} = \left(G\sqrt{1-x^2} - \frac{p^2}{1-x^2}\right)x - \lambda(1-x^2)F(t) \quad \tag{18}
\]

with parameter $\lambda \in [0, 1]$. Following the notation used in Section 2 we denote by $w_\lambda$ be the right-hand side of (17),(18) and by $v_\lambda$ the right-hand side of (16),(17),(18).
We assume $F$ is continuous and $T$-periodic. We are looking for a bound set for $v_\lambda$ of the form $\mathbb{R} \times \Gamma_a \cap \Delta_b$, where $0 < a < 1$, $b > 0$, and

$$\Gamma_a := \{(x,p) \in \mathbb{R} \times \mathbb{R}: |x| \leq a\}, $$

$$\Delta_b := \{(x,p) \in \mathbb{R} \times \mathbb{R}: |x| < 1, \ b|x| + |p| \leq b\}.$$  

**Lemma 3.1.** There exists an $a_0 \in (0,1)$ such that if $a_0 \leq a < 1$ then $\mathbb{R} \times \Gamma_a$ is a bound set for $v_\lambda$ for all $\lambda \in [0,1]$.

**Proof.** We apply Proposition 1. Let $t_0 \in \mathbb{R}$ and $0 < a < 1$. If $p_0 > 0$ then $(t_0,a,p_0) \in (\mathbb{R} \times \Gamma_a)^-$ since $\dot{x}(t_0) > 0$ for the solution $t \to (x(t),p(t))$ of the system (17),(18) with the initial value $(x(t_0),p(t_0)) = (a,p_0)$. Similarly, if $p_0 < 0$ then $(t_0,a,p_0) \in (\mathbb{R} \times \Gamma_a)^+$. Now let $t \to (x(t),p(t))$ be the solution of (17),(18) with the initial value $(x(t_0),p(t_0)) = (a,0)$. We assert that if

$$Ga\sqrt{1-a^2} - \lambda(1-a^2)F(t_0) > 0$$

then $(t_0,a,0) \in (\mathbb{R} \times \Gamma_a)^- \cap (\mathbb{R} \times \Gamma_a)^+$. Indeed, if (19) holds then $\dot{p}(t_0) > 0$, hence there exists $\epsilon > 0$ such that $p(t) < 0$, hence $\dot{x}(t) < 0$, for $t \in (t_0-\epsilon,t_0)$ and $p(t) > 0$, hence $\dot{x}(t) > 0$, for $t \in (t_0,t_0 + \epsilon)$. This means $t \to x(t)$ has a strict local minimum at $t_0$ and the assertion follows. In a similar way we treat the points of the form $(t_0,-a,p_0)$; in the case $p_0 = 0$ the strict local minimum at $t_0$ is guaranteed if

$$-Ga\sqrt{1-a^2} - \lambda(1-a^2)F(t_0) < 0.$$  

(20)

It is clear that (19) and (20) are satisfied for all $\lambda \in [0,1]$ if

$$Ga - \|F\|\sqrt{1-a^2} > 0$$

which implies the conclusion. \hfill \Box

Through reminder of this section we assume $a$ satisfies Lemma 3.1.

**Lemma 3.2.** There exists $b > 0$ such that the set $\mathbb{R} \times \Gamma_a \cap \Delta_b$ is a bound set for $v_\lambda$ for all $\lambda \in [0,1]$.

**Proof.** Actually, we prove that if

$$b^2 > \frac{(1+a)\|F\|}{1-a},$$

(21)

then the conclusion holds. Indeed, by Lemma 3.2 it is enough to consider $(t,x,p)$ such that $b|x| + |p| = b$ and $|x| \leq a$. At first we assume $0 \leq x \leq a$ and $p \geq 0$, hence $p = b(1-x)$. We estimate the scalar product of $w_\lambda$ at $(t,x,a)$ with the vector $[b \ 1]^T$ perpendicular to the line $bx + p = b$ and directed outward from $\Delta_b$. As a consequence of (21) we get

$$w_\lambda(t,x,p)^T \begin{bmatrix} b \\ 1 \end{bmatrix} = bp + Gx\sqrt{1-x^2} - \frac{x^2}{1-x^2} - \lambda(1-x^2)F(t)$$

$$= b^2(1-x) \left(1 - \frac{x}{1+x} - \lambda \frac{(1+x)F(t)}{b^2}\right) + Gx\sqrt{1-x^2}$$

$$> b^2(1-a) \left(1 - a - \frac{(1+a)\|F\|}{b^2}\right) > 0,$$

hence $(t,x,p) \in (\mathbb{R} \times \Delta_b)^-$. The same inequality (21) implies $(t,x,p) \in (\mathbb{R} \times \Delta_b)^-$ if $-a \leq x \leq 0$ and $p \leq 0$, and $(t,x,p) \in (\mathbb{R} \times \Delta_b)^+$ if $0 \leq x \leq a$ and $p \leq 0$ or $-a \leq x \leq 0$ and $p \geq 0$, as it is sketched out in Figure 2, hence the result follows by Proposition 1. \hfill \Box
Theorem 3.3. If $F$ is continuous and $T$-periodic then (15) has a $T$-periodic solution.

Proof. The equation (15) is equivalent to the system (17), (18) for $\lambda = 1$. Assume first $F$ is of $C^1$-class; we prove that there exists a $T$-periodic solution with image contained in $\Gamma_a \cap \Delta_b$. By Theorem 2.1 and Lemma 3.2 it remains to prove that the degree of $w_0$ in the interior of $\Gamma_a \cap \Delta_b$ is not equal to zero. The origin $(0,0)$ is the only zero of $w_0$ and

$$(Dw_0)(0,0) = \begin{bmatrix} 0 & 1 \\ G & 0 \end{bmatrix},$$

hence

$$\deg(w_0, \text{int}(\Gamma_a \cap \Delta_b), 0) = \text{sign} \det(Dw_0)(0,0) = -1$$

and the proof is complete in the $C^1$-class case. Since $\Gamma_a \cap \Delta_b$ is compact, a standard approximation argument provides a proof if $F$ is continuous.

Actually, in the above proof we do not need to assume first that $F$ is of $C^1$-class. That assumption was required since Theorem 2.1 was formulated in the $C^1$-class setting applied for the whole Section 2. A general formulation of that result given in [6] is valid even for equations satisfying the Carathéodory conditions.

The phase portrait shown in Figure 2 indicates an alternative proof of Theorem 3.3 by an application of [23, Theorem 1] (see also [24, Corollary 7.4]). As it was mentioned in Section 1, Theorem 3.3 is the reformulation of Theorem 1.1 in the Cartesian coordinates system, hence the latter theorem is also proved.

4. Periodic solutions in the planar Whitney’s problem. Now we consider the planar Whitney’s problem. We proceed in an analogous way as in Section 3. At first we derive the corresponding differential equations. The top of the rod at time
t has the horizontal coordinates \( x(t) = (x_1(t), x_2(t)) \) with respect to the pivot, its distance from the pivot is equal to \( \ell \), and the position of the pivot is represented by \( f(t) = (f_1(t), f_2(t)) \). The vertical position is equal to \( y(t) \), hence

\[
y(t) = \sqrt{\ell^2 - |x(t)|^2}. \tag{22}
\]

The constraint force is perpendicular to the half-sphere \( y = \sqrt{\ell^2 - |x|^2} \). Denote by \( m \) the mass of the rod; we assume it is concentrated at the top. Let \( g \) be the gravitational constant and let \( \mu(t) \) refers to the magnitude of the constraint force as in (12),(13), hence

\[
m(\ddot{x}(t) + \ddot{f}(t)) = \mu(t)x(t), \tag{23}
\]

\[
m\ddot{y} = -mg + \mu(t)\sqrt{\ell^2 - |x(t)|^2}. \tag{24}
\]

The equations (22) and (24) imply

\[
\mu = m \frac{g + \ddot{y}}{\sqrt{\ell^2 - |x|^2}},
\]

\[
\ddot{y} = -\frac{x^T\dot{x} + |\dot{x}|^2}{\sqrt{\ell^2 - |x|^2}} - \frac{(x^T\dot{x})^2}{(\ell^2 - |x|^2)^{3/2}},
\]

hence, by (23), the system

\[
(\ell^2 - x_2^2)\ddot{x}_1 + x_1x_2\ddot{x}_2 = \left(g\sqrt{\ell^2 - |x|^2} - \frac{(x^T\dot{x})^2}{\ell^2 - |x|^2} - |\dot{x}|^2 \right) x_1 - (\ell^2 - |x|^2)\ddot{f}_1,
\]

\[
x_1x_2\ddot{x}_1 + (\ell^2 - x_1^2)\ddot{x}_2 = \left(g\sqrt{\ell^2 - |x|^2} - \frac{(x^T\dot{x})^2}{\ell^2 - |x|^2} - |\dot{x}|^2 \right) x_2 - (\ell^2 - |x|^2)\ddot{f}_2
\]

which resolves to the equation

\[
\ddot{x} = \frac{1}{\ell^2} \left(g\sqrt{1 - |x|^2} - \frac{(x^T\dot{x})^2}{1 - |x|^2} - |\dot{x}|^2 \right) x + \frac{x^T\dot{f}(t)}{\ell^2} x - \ddot{f}(t). \tag{25}
\]

After changing the variable \( x \) to \( \frac{1}{\ell}x \), the equation (25) becomes

\[
\ddot{x} = \left(G\sqrt{1 - |x|^2} - \frac{(x^Tp)^2}{1 - |x|^2} - |p|^2 \right) x + (x^TF(t))x - F(t), \tag{26}
\]

where \( G := \frac{g}{\ell} \) and \( F := \frac{1}{\ell}f \). We associate with (26) the autonomous system

\[
\dot{t} = 1, \tag{27}
\]

\[
\dot{x} = p, \tag{28}
\]

\[
\dot{p} = \left(G\sqrt{1 - |x|^2} - \frac{(x^Tp)^2}{1 - |x|^2} - |p|^2 \right) x + \lambda((x^TF(t))x - F(t)) \tag{29}
\]

with parameter \( \lambda \in [0, 1] \). Set

\[
R := G\sqrt{1 - |x|^2} - \frac{(x^Tp)^2}{1 - |x|^2} - |p|^2,
\]

\[
\Phi := \lambda((x^TF(t))x - F(t)).
\]

Assume \( F \) is \( T \)-periodic and of \( C^1 \)-class. As in Section 3, we denote by \( w_\lambda \) be the right-hand side of (28),(29) and by \( v_\lambda \) the right-hand side of (27),(28),(29);

\[
w_\lambda(t, x, p) := \begin{bmatrix} p \\ R_x + \Phi \end{bmatrix}, \quad v_\lambda(t, x, p) := \begin{bmatrix} 1 \\ p \\ R_x + \Phi \end{bmatrix}.
\]
Lemma 4.2. There exists $v$.

Proof. Let $a$.

The derivative of $v$ is given by

$$Dv_\lambda = \begin{bmatrix} 0 & 0 & 0 \\ \Phi & R + x \frac{\partial R}{\partial x} + \frac{\partial \Phi}{\partial x} & x \frac{\partial R}{\partial p} \end{bmatrix},$$

hence

$$(Dv_\lambda)v_\lambda = \begin{bmatrix} 0 \\ Rx + \Phi \\ \frac{\partial \Phi}{\partial t} + R + x \frac{\partial R}{\partial x} + \frac{\partial \Phi}{\partial x} + R + x \frac{\partial R}{\partial p} + x \frac{\partial R}{\partial p} \Phi \end{bmatrix}.$$ (30)

Similarly as it was done in Section 3 in the 2-dimensional setting, for $0 < a < 1$ and $b > 0$ we define subsets of $\mathbb{R}^3$,

$$\Gamma_a := \{(x, p) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x| \leq a\},$$

$$\Delta_b := \{(x, p) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x| < 1, b|x| + |p| \leq b\}.$$

Let $m_a(t, x, p) := \frac{1}{2} |x|^2 - \frac{1}{2} a^2$, hence

$$\{m_a \leq 0\} = \mathbb{R} \times \Gamma_a.$$

Lemma 4.1. There exists an $a_0 \in (0, 1)$ such that if $a_0 \leq a < 1$ then $m_a$ is a curvature bound function for $v_\lambda$ for all $\lambda \in [0, 1]$.

Proof. Let $m_a(t, x, p) = 0$, i.e. $|x| = a$. Assume

$$(Dm_a)v_\lambda = 0$$ (31)
at $(t, x, p)$. Since

$$Dm_a = \begin{bmatrix} 0 & x^T \\ 0 & 0 \end{bmatrix}, \quad D^2m_a = \text{diag}(0, I, 0),$$

the equations (30) and (31) imply $x^T p = 0$ and

$$v_\lambda^T (D^2m_a)v_\lambda + (Dm_a)(Dv_\lambda)v_\lambda = |p|^2 + R |x|^2 + x^T \Phi \geq |p|^2 + Ga^2 \sqrt{1 - a^2} - |p|^2 a^2 - \lambda a(1 - a^2) |F(t)|$$

$$\geq (1 - a^2) |p|^2 + a \sqrt{1 - a} \left(Ga \sqrt{1 + a} - \lambda(1 + a) \|F\| \sqrt{1 - a}\right).$$

Clearly, there exists $a_0$ close to 1 such that if $a$ satisfies $a_0 \leq a < 1$ then the right-hand side is positive for all $\lambda \in [0, 1]$, hence the result follows.

In the sequel we fix an $a$ satisfying the conclusion of Lemma 4.1.

Lemma 4.2. There exists $b > 0$ such that the set $\mathbb{R} \times \Gamma_\alpha \cap \Delta_b$ is a bound set for $v_\lambda$ for all $\lambda \in [0, 1]$.

Proof. Let $n_b(t, x, p) := b |x| + |p| - b$, hence

$$\{n_b \leq 0\} = \mathbb{R} \times \Delta_b.$$

It follows the set $\mathbb{R} \times \Gamma_\alpha \cap \Delta_b$ is of the form required in Corollary 1 for $e_1 = m_a$ and $e_2 = n_b$. By Lemma 4.1 and the choice of $a$, in order to apply the theorem it is enough to examine the part of the boundary corresponding to $n_b = 0$. Set $Z := \{(t, 0, p) : t \in \mathbb{R}, |p| = b\}$. Clearly, $n_b$ is of $C^2$-class in a neighborhood of each point $(t, x, p) \in (\mathbb{R} \times \partial \Delta_b) \setminus Z$. At first we find an estimate on $b$ which guarantee that $Z$ is contained in the exit set of $\mathbb{R} \times \Delta_b$. Let $(t, 0, p) \in Z$. It suffices to find
b such that \( v_\lambda(t, 0, p) \notin C \), where \( C \) denotes the cone at \( (t, 0, p) \) of vectors directed to \( \mathbb{R} \times \Delta_b \), which means

\[
\begin{bmatrix}
1 \\
p \\
-\lambda F(t)
\end{bmatrix} \neq \mu \begin{bmatrix}
u \\
x \\
-p
\end{bmatrix}
\]

for each \( \mu > 0 \), \( u \in \mathbb{R} \), and \( |x| \leq 1 \). This is satisfied if

\[b^2 > \|F\|.
\] (32)

It remains to find values of \( b \) for which the implication (2) with \( e = n_b \) and \( v = v_\lambda \) is satisfied at each point \( (t, x, p) \in (\mathbb{R} \times \Gamma_\alpha \cap \partial \Delta_a) \setminus Z \), i.e. at each \( (t, x, p) \) such that \( t \in \mathbb{R}, 0 < |x| \leq a \), and \( |p| = b(1 - |x|) \). Direct calculations show

\[
Dn_b = \begin{bmatrix}
0 & b \\
\frac{b}{|p|} x^T & \frac{1}{|p|} p^T
\end{bmatrix}, \quad D^2n_b = \text{diag}(0, A, B),
\]

where

\[
A := \frac{b}{|x|^3} \begin{bmatrix}
x_2^2 & -x_1x_2 \\
-x_1x_2 & x_1^2
\end{bmatrix}, \quad B := \frac{1}{|p|^3} \begin{bmatrix}
p_2^2 & -p_1p_2 \\
p_1p_2 & p_1^2
\end{bmatrix},
\]

hence

\[
(Dn_b)v_\lambda = \left( \frac{b}{|x|} + \frac{G}{|p|} \sqrt{1 - |x|^2} - \frac{(x^T p)^2}{|p|(1 - |x|^2)} - |p| \right) x^T p + \lambda \frac{x^T F(t)x^T p - F(t)^T p}{|p|}
\] (33)

and, by (30),

\[
v_\lambda^T (D^2n_b)v_\lambda + (Dn_b)(Dv_\lambda)v_\lambda
\]

\[
= \frac{b}{|x|^3} \left( \det [x \quad p] \right)^2 + \frac{1}{|p|^3} \left( \det [p \quad Rx + \Phi] \right)^2
\]

\[
+ bR|x| + R|p| + \frac{1}{|p|} \left( \frac{\partial R}{\partial x} p + R \frac{\partial R}{\partial p} x \right) x^T p
\]

\[
+ \frac{b}{|x|} x^T \Phi + \frac{1}{|p|} p^T \left( \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} p \right) + \frac{x^T p \partial R}{|p|} \Phi.
\] (34)

We are looking for \( b \) such that the value of the right-hand side of (34) is positive provided \((Dn_b)v_\lambda = 0\). Since

\[
\frac{b}{|x|} - \frac{(x^T p)^2}{|p|(1 - |x|^2)} - |p| \geq \frac{b}{|x|} - \frac{|x|^2|p|^2}{|p|(1 - |x|^2)} - |p|
\]

\[
= \frac{b}{|x|} - \frac{|p|}{1 - |x|^2} = \frac{b}{|x|} - \frac{b}{1 + |x|} = \frac{b}{|x|(1 + |x|)} > 0,
\]

by (33) the equation \((Dn_b)v_\lambda = 0\) implies

\[
|x^T p| \leq \frac{|x|(1 + |x|)(1 + |x|^2)}{b} \|F\| \leq \frac{4\|F\||x|}{b}.
\] (35)

Assume \( b \) satisfies

\[
b^4 > \frac{16\|F\|^2}{(1 - a)^3}.
\] (36)
It follows, in particular, that (32) also holds. As a consequence of (35) and (36) we get

\[
\frac{b}{|x|^3} (\det [x \\ p])^2 + bR|x| + R|p| = \frac{|p|^2}{|x|} \frac{b}{|x|^3} (x^T p)^2 + bR
\]

\[
> \frac{b}{|x|} \left( (1 - |x|)|p|^2 - \frac{16\|F\|^2 b^2}{b^2} \right) - \frac{16\|F\|^2 |x|^2}{(1 - |x|^2)b} 
\]

\[
\geq \frac{b}{|x|} \left( (1 - a^3)b^2 - \frac{16\|F\|^2 a^2}{b^2} \right) - \frac{16\|F\|^2 a^2}{(1 - a^2)b} \geq K b^3 - L \frac{1}{b}, \quad (37)
\]

where

\[
K := \frac{(1 - a)^3}{a}, \quad L := 16\|F\|^2 \left( \frac{1}{a} + \frac{a^2}{1 - a^2} \right).
\]

Now we examine the remaining terms of the right-hand side of (34). The term \( \frac{1}{|p|^2} (\det [p \ Rx + \Phi])^2 \) is always non-negative. Since

\[
\frac{\partial R}{\partial x} = -\frac{G}{\sqrt{1 - |x|^2}} x^T - \frac{2x^T p}{1 - |x|^2} p^T - \frac{2(x^T p)^2}{(1 - |x|^2)^2} x^T,
\]

\[
\frac{\partial R}{\partial p} = -\frac{2x^T p}{1 - |x|^2} x^T - 2p^T,
\]

as an application of (35) we get

\[
\left| \frac{1}{|p|} \left( \frac{\partial R}{\partial x} x^T p + \frac{\partial R}{\partial p} x^T p \right) \right| 
\]

\[
\leq \frac{G}{|p| \sqrt{1 - |x|^2}} (x^T p)^2 + \frac{2|p|}{1 - |x|^2} (x^T p)^2 + \frac{2}{|p|(1 - |x|^2)^2} |x^T p|^3 
\]

\[
+ \frac{2G |x|^2}{|p| \sqrt{1 - |x|^2}} (x^T p)^2 + \frac{2|p|^2}{|p|(1 - |x|^2)^2} (x^T p)^4 + \frac{2|p| |x|^2}{1 - |x|^2} (x^T p)^2 
\]

\[
+ \frac{2 \sqrt{1 - |x|^2}}{|p|} (x^T p)^2 + \frac{2}{|p|(1 - |x|^2)} (x^T p)^4 + 2|p|(x^T p)^2 
\]

\[
\leq M \left( \frac{1}{b} + \frac{1}{b^3} + \frac{1}{b^5} \right) \quad (38)
\]

for some non-negative constant \( M \) depending on \( a, \|F\|, \) and \( G \). Furthermore,

\[
\left| \frac{b}{|x|} x^T \Phi \right| \leq \|F\|b, \quad (39)
\]

\[
\left| \frac{1}{|p|} p^T \Phi \right| \leq 2\|\Phi\|. \quad (40)
\]

Since

\[
\frac{\partial \Phi}{\partial x} = \lambda((x^T F(t))I + xF^T),
\]

one gets

\[
\left| \frac{1}{|p|} p^T \frac{\partial \Phi}{\partial x} \right| \leq 2\|F\||x||p| < 2\|F\|b. \quad (41)
\]
Finally, (35) implies
\[
\left| \frac{x^T p \partial R}{|p|} \right| \Phi \leq 2 \left( \frac{|x^T p| |x^T F(t)|| x |^2}{1 - |x|^2} + \frac{|x^T p| |x^T F(t)|}{1 - |x|^2} + |x^T p| |x^T F(t)| + |p^T F(t)| \right)
\]
\[
\leq N \left( \frac{1}{b} + \frac{1}{b^2} \right)
\] (42)
for some non-negative constant \(N\) depending on \(a\) and \(\|F\|\). Since the constant \(K\) is positive, the equation (34) together with the estimates (37)–(42) imply
\[
v^T (D^2 n_0) v + (Dn_0)(Dv_\lambda) v > K b^3 - 3 \|F\| b - 2 \|\dot{F}\| - (L + M + N) \frac{1}{b}
\]
\[
- N \frac{1}{b^2} - M \left( \frac{1}{b^4} + \frac{1}{b^5} + \frac{1}{b^6} \right) > 0
\]
provided \(b\) satisfying (36) (hence also satisfying (32)) is large enough. Now the conclusion is an immediate consequence of Corollary 1.

**Theorem 4.3.** If \(F\) is \(T\)-periodic and of \(C^1\)-class then (26) has a \(T\)-periodic solution.

**Proof.** We apply the same argument as in the proof of Theorem 3.3. The equation (26) is equivalent to the system (28),(29) for \(\lambda = 1\). Since
\[
(Dw_0)(0, 0) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ G & 0 & 0 & 0 \\ 0 & G & 0 & 0 \end{bmatrix}
\]
and (0, 0) is the only zero of \(w_0\),
\[
\deg(w_0, \text{int}(\Gamma_a \cap \Delta_b), 0) = \text{sign} \det(Dw_0)(0, 0) = 1,
\]
hence the result is a consequence of Theorem 2.1 and Lemma 4.2.

By proving Theorem 4.3 we simultaneously provided a proof of Theorem 1.2, the main result of the present paper.

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