The $k$-independent graph of a graph

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Abstract

Let $G = (V, E)$ be a simple graph. A set $I \subseteq V$ is an independent set, if no two of its members are adjacent in $G$. The $k$-independent graph of $G$, $I_k(G)$, is defined to be the graph whose vertices correspond to the independent sets of $G$ that have cardinality at most $k$. Two vertices in $I_k(G)$ are adjacent if and only if the corresponding independent sets of $G$ differ by either adding or deleting a single vertex. In this paper, we obtain some properties of $I_k(G)$ and compute it for some graphs.

Keywords: independence number; $k$-independent graph; reconfiguration.

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1 Introduction

Given a simple graph $G = (V, E)$, a set $I \subseteq V$ is an independent set of $G$, if there is no edge of $G$ between any two vertices of $I$. A maximal independent set is an independent set that is not a proper subset of any other independent set. A maximum independent set is an independent set of greatest cardinality for $G$. This cardinality is called independence number of $G$, and is denoted by $\alpha(G)$. Reconfiguration problems have been studied often in recent years. These

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arise in settings where the goal is to transform feasible solutions to a problem in a step-by-step manner, while maintaining a feasible solution throughout.

For the study of dominating set reconfiguration problem: given two dominating sets $S$ and $T$ of a graph $G$, both of size at most $k$, is it possible to transform $S$ into $T$ by adding and removing vertices one-by-one, while maintaining a dominating set of size at most $k$ throughout? Regarding to this dominating set reconfiguration problem, recently the $k$-dominating graph of a graph $G$ has defined in [9]. The $k$-dominating graph of $G$, $D_k(G)$, is defined to be the graph whose vertices correspond to the dominating sets of $G$ that have cardinality at most $k$. Two vertices in $D_k(G)$ are adjacent if and only if the corresponding dominating sets of $G$ differ by either adding or deleting a single vertex. Authors in [9], gave conditions that ensure $D_k(G)$ is connected. In [1] authors proved that if $G$ is a graph without isolated vertices of order $n \geq 2$ and with $G \cong D_k(G)$, then $k = 2$ and $G = K_{1,n-1}$ for some $n \geq 4$. It is also proved that for a given $r$ there exist only a finite number of $r$-regular, connected dominating graphs of connected graphs ([1]).

One of the most well-studied problem in reconfiguration problems, is the reconfiguration of independent sets. For a graph $G$ and integer $k$, the independent sets of size at least/exactly $k$ of $G$ form the feasible solutions. Independent sets are also called token configurations, where the independent set vertices are viewed as tokens [4]. Deciding for existence of a reconfiguration between two $k$-independent sets with at most $\ell$ operations is strongly NP-complete ([10]). Bonamy and Bousquet in [3] have considered the $k$-TAR reconfiguration graph, $TAR_k(G)$, as follows:

A $k$-independent set of $G$ is a set $S \subseteq V$ with $|S| \geq k$, such that no two elements of $S$ are adjacent. Two $k$-independent sets $I$ and $J$ are adjacent if they differ on exactly one vertex. This model is called the Token Addition and Removal (TAR). Authors in [3] provided a cubic-time algorithm to decide whether $TAR_k(G)$ is connected when $G$ is a graph which does not contain induced paths of length 4. Their work solves an open question in [4]. Also they described a linear-time algorithm which decides whether two elements of $TAR_k(G)$ are in the same connected component. As usual we denote the complete graph, path and cycle of order $n$ by $K_n$, $P_n$ and $C_n$, respectively. Also $K_{1,n}$ is the star graph with $n + 1$ vertices.

In the next section, we study the $k$-independent graph of a graph $G$. In Section 3, we study the $\alpha$-independent graph of a graph. Finally in Section 3, we exclude the empty set from the family set of independent sets of $G$, denote the new $k$-independent graph of $G$ by $I_k^*(G)$ and study its connectedness.
2 The $k$-independent graph of a graph

In this section we shall study the $k$-independent graph of a graph $G$. First let to rewrite the definition of the reconfiguration graph $\text{TAR}_k(G)$, as follows. For a graph $G$ and a non-negative integer $k$, the $k$-independent graph of $G$, $I_k(G)$, is defined to be the graph whose vertices correspond to the independent sets of $G$ that have cardinality at most $k$. Two vertices in $I_k(G)$ are adjacent if and only if the corresponding independent sets of $G$ differ by either adding or deleting a single vertex. As an example, Figure 1 shows $I_3(K_{1,3})$.

![Figure 1: Graphs $I_3(K_{1,3})$ and $I_2(P_3)$, respectively.](image)

Note that $k$-dominating and $k$-independent graph are similar to recent work in graph colouring, too. Given a graph $H$ and a positive integer $k$, the $k$-colouring graph of $H$, denoted $G_k(H)$, has vertices corresponding to the (proper) $k$-vertex-colourings of $H$. Two vertices in $G_k(H)$ are adjacent if and only if the corresponding vertex colourings of $G$ differ on precisely one vertex. Authors in [5, 6, 7, 8] studied the connectedness of $k$-colouring graphs. Also they studied their hamiltonicity. Let to introduce a notation. Let $A$ and $B$ be independent sets of $G$ of cardinality at most $k$. We use the notation $A \leftrightarrow B$, if there is a path in $I_k(G)$ joining $A$ and $B$. It is easy to see that for every $A, B \in I_k(G)$, $A \leftrightarrow B$ if and only if $B \leftrightarrow A$ and if $A \supseteq B$, then $A \leftrightarrow B$ and $B \leftrightarrow A$.

The following theorem, gives some properties of the $k$-independent graph of a graph:

**Theorem 2.1**

(i) If $G$ is a graph of order $n$, then $I_1(G) \cong K_{1,n}$.

(ii) For every graph $G$ and every $0 \leq k \leq \alpha(G)$, the independent graph $I_k(G)$ is connected and $\Delta(I_k(G)) = |V(G)|$.

(iii) For every graph $G$, the independent graph $I_k(G)$ is a bipartite graph.
(iv) If $G \not\cong \overline{K_n}$, then $I_k(G)$ is not a regular graph.

(v) If $G \not\cong \overline{K_n}$ then $I_k(G)$ is not a vertex-transitive graph, and so is not a Cayley graph.

Proof.

(i) It follows from the definition.

(ii) It is straightforward.

(iii) Let $X$ be the set of independent sets of size less than $k + 1$ of $G$ with odd cardinality and $Y$ be the set of independent sets of size less than $k + 1$ with even cardinality. It is clear that $X \cup Y = V(I_k(G))$ and $X \cap Y = \phi$. Suppose that $A, B \in X$, then $(A \setminus B) \cup (B \setminus A)$ cannot be a vertex of $I_k(G)$. Because $|A| = |B|$ or $|A| - |B| \geq 2$. So $AB$ is not an edge of $I_k(G)$ and with similar argument we have this for two vertices in $Y$. Therefore $I_k(G)$ is a bipartite graph with parts $X$ and $Y$.

(iv) Let $G$ be a graph of order $n$. The empty set is an independent set of $G$ which has degree $n$ in $I_k(G)$. Let $I_1$ be an independent set of $G$ with $|I_1| = \alpha(G)$. We know that $I_1$ is adjacent to $\alpha$ independent sets. Since $G \not\cong \overline{K_n}$, we have $\alpha(G) \neq n$. Therefore $I_k(G)$ is not a regular graph.

(v) It follows from Part (iv). □

Theorem 2.2  (i) Let $G$ be a graph of order $n$. There is no integer $k$, such that $I_k(G) \cong G$.

(ii) If $G \not\cong K_n$, then the girth of $I_k(G)$ is 4.

(iii) Let $G \neq K_n$ be a graph. Then for all integers $k \geq 2$, $I_k(G)$ is not a tree.

Proof.

(i) Since for every integer number $k \geq 1$, $|V(I_k(G))| \geq n + 1$, so we have the result.

(ii) Let $v_1$ and $v_2$ be two non-adjacent vertices of graph $G$. So $\{v_1\}$ and $\{v_2\}$ are two independent sets of $G$ and therefore two vertices of $I_k(G)$. Now $\emptyset$, $\{v_1\}$, $\{v_1, v_2\}$, $\{v_2\}$ is a cycle in $I_k(G)$ and this is the shortest cycle in $I_k(G)$. Therefore the girth of $I_k(G)$ is 4.

(iii) It follows from Part (ii). □
3 The $\alpha$-independent graph of some graphs

Let $G$ be a simple graph with independence number $\alpha$. Looks that in the among of $k$-independent graph of $G$, the $\alpha$-independent graph of $G$ is more important. In this section, we study the $\alpha$-independent graph of some graphs. To study the $\alpha$-independent graph of $G$, we are interested to know the order of $I_\alpha(G)$. Let $i_k$ be the number of independent sets of cardinality $k$ in $G$. The polynomial

$$I(G, x) = \sum_{k=0}^{\alpha(G)} i_k x^k,$$

is called the independence polynomial of $G$ ([2]). Obviously $I(G, 1)$ gives the number of all independent sets of a graph $G$. In other words, $|V(I_\alpha(G))| = I(G, 1)$. Since $I(K_n, x) = 1 + nx$, we have $I(K_n, 1) = n + 1$. Therefore we have the following easy result:

**Theorem 3.1** For any integer $k > 1$, there is some connected graph $G$ such that $|V(I_\alpha(G))| = k$.

The following theorem is about the $\alpha$-independent graph of stars:

**Theorem 3.2**

(i) The $n$-independent graph of $K_{1,n}$, i.e., $I_n(K_{1,n})$, is a bipartite graph with parts $X$ and $Y$, with $|X| = 2^{n-1}$ and $|Y| = 2^{n-1} + 1$.

(ii) The $n$-independent graph $I_n(K_{1,n})$ is not Hamiltonian.

**Proof.**

(i) Let $X$ be the set of independent sets of $K_{1,n}$ with even cardinality and $Y$ be the set of independent sets of odd cardinality. By Theorem 2.1(iii), $I_n(K_{1,n})$ is a bipartite graph with parts $X$ and $Y$. Obviously $|X| = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}$ and since the number of independent sets of $K_{1,n}$ is $I(K_{1,n}, 1) = 2^n + 1$, we have $|Y| = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k-1}$. Therefore we have the result.

(ii) Since a bipartite graph with different number of vertices in its parts is not a Hamiltonian graph, so the $n$-independent graph $I_n(K_{1,n})$ is not a Hamiltonian graph. $\square$

Here we consider the $\alpha$-independent of some another graphs. Figure 1 shows the $I_2(P_3)$.

**Theorem 3.3** For every $n \in \mathbb{N}$, $\delta(I_\alpha(P_n)) = \lfloor \frac{n}{2} \rfloor$.
Proof. The minimum degree of vertices of $I_{\lceil \frac{n}{2} \rceil}(P_n)$ is due to maximal independent sets of $P_n$ with minimum cardinality. These vertices are adjacent to $n - \lceil \frac{n}{2} \rceil = \lfloor \frac{n}{2} \rfloor$ of independent sets with less cardinality. □

Here we shall obtain information on the Hamiltonicity of $\alpha$-independent of some specific graphs. Using the value of the independence polynomial at $-1$, we have $I(G; -1) = i_0 - i_1 + i_2 - \ldots + (-1)^n i_\alpha = f_0(G) - f_1(G)$, where $f_0(G) = i_0 + i_2 + i_4 + \ldots$, $f_1(G) = i_1 + i_3 + i_5 + \ldots$ are equal to the numbers of independent sets of even size and odd size of $G$, respectively. $I(G, -1)$ is known as the alternating number of independent sets. We need the following theorem:

**Theorem 3.4** For $n \geq 1$, the following hold:

(i) $I(P_{3n-2}; -1) = 0$ and $I(P_{3n-1}; -1) = I(P_{3n}; -1) = (-1)^n$;

(ii) $I(C_{3n}; -1) = 2(-1)^n$, $I(C_{3n+1}; -1) = (-1)^n$ and $I(C_{3n+2}; -1) = (-1)^{n+1}$;

(iii) $I(W_{3n+1}; -1) = 2(-1)^n - 1$ and $I(W_{3n+2}; -1) = I(W_{3n}; -1) = (-1)^n - 1$.

**Corollary 3.5** For all positive integer $n$, the graphs $I_\alpha(P_{3n-1})$, $I_\alpha(P_{3n})$, $I_\alpha(C_n)$ and $I_\alpha(W_n)$ are not Hamiltonian.

Proof. We know that $I_\alpha(P_n)$, $I_\alpha(C_n)$ and $I_\alpha(W_n)$ are bipartite graphs with parts containing the independent sets of even and odd cardinality. By Theorem 3.4, theses bipartite graphs have parts with different cardinality. Therefore we have the result. □

4. Connectedness of $I_k^*(G)$

As we have seen in the Section 2, since the empty set is an independent set of any graph, so the $k$-independent graph $I_k(G)$ is a connected graph. Let us to do not consider empty set in the study of $k$-independent graph.

Suppose that $\mathcal{I}$ is a family of all independent sets of graph $G$. If we put $V(I_k(G)) = \mathcal{I} \setminus \emptyset$, then we denote the $k$-independent graph of $G$, by $I_k^*(G)$. Note that in this case, for some $k$ and $G$, $I_k^*(G)$ is disconnected and for some $k$ and $G$ is connected. For example, the Figure 2 shows $I_3^*(K_{1,3})$ and $I_2^*(C_4)$, which are disconnected graphs with two components. Also Figure 3 shows $I_3^*(W_5)$ and $I_3^*(P_5)$, respectively. Observe that $I_3^*(P_5)$ is connected and $I_3^*(W_5)$ is disconnected with three components. Theorem 2.2 implies that for any graph $G \neq K_n$, and for all integers $k \geq 2$, $I_k(G)$ is not a tree, but as we see in the Figure 3 the graph $I_k^*(G)$ can be a forest. This naturally raises the question: For which graph $G$, the component of $I_k^*(G)$ is a forest? What is the number of components?

The following theorem is a sufficient condition for disconnectedness of $I_\alpha^*(G)$. 


Theorem 4.1  If a graph $G$ of order $n$ has a vertex of degree $n - 1$, then $I_0^*(G)$ is disconnected.

Proof. Let $v$ be a vertex of degree $n - 1$. Obviously $\{v\}$ is a non-empty independent set of $G$, and so is an isolated vertex of $I_0^*(G)$. □

Note that the converse of Theorem 4.1 is not true. For example $I_2^*(C_4)$ has two components, but $C_4$ is 2-regular (Figure 3). Now, we state the following theorem:

Theorem 4.2  Let $K_{n_1,n_2,...,n_m}$ be a complete $m$-partite graph, then $I^*_\alpha(K_{n_1,n_2,...,n_m})$ has $m$ connected components.

Proof. Let $X_1$ and $X_2$ be two arbitrary parts of $K_{n_1,n_2,...,n_m}$. Suppose that $I_1$ contains all nonempty subsets of part $X_1$ and $I_2$ contains all nonempty sets of part $X_2$. Obviously, each member of $I_1$ and each member of $I_2$ are independent sets of $K_{n_1,n_2,...,n_m}$ and so they are vertices of $I^*_\alpha(K_{n_1,n_2,...,n_m})$. No member of $I_1$ is adjacent to a member of $I_2$ in $I^*_\alpha(K_{n_1,n_2,...,n_m})$. So $I^*_\alpha(K_{n_1,n_2,...,n_m})$ is a disconnected graph. Since the members of $I_1$ (and the members of $I_2$) form a connected graph, therefore we have $m$ components. □

It is obvious that, for all graph $G$ with $\alpha(G) = 2$, $I_2^*(G)$ is a forest.

Theorem 4.3  For all graph $G$ with $\alpha(G) > 2$, the components of $I_k^*(G)$, $2 \leq k \leq \alpha$, are not forest.

Proof. We consider two following cases:

Case 1. If $k = 2$. Let $\{v_1,v_2,v_3\}$ be an independent set of $G$. So $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_1,v_2\}$, $\{v_1,v_3\}$ and $\{v_2,v_3\}$ are independent sets of $G$ and vertices of
$I_k^*(G)$. Therefore $\{v_1\}, \{v_1, v_2\}, \{v_2\}, \{v_2, v_3\}, \{v_1, v_3\}, \{v_1\}$ make a cycle in $I_k^*(G)$.

Case 2. If $k > 2$. Let $\{v_1, v_2, v_3\}$ be an independent set of $G$. So $\{v_1\}, \{v_1, v_2\}$ and $\{v_1, v_3\}$ are independent sets of $G$ and vertices of $I_k^*(G)$. Therefore $\{v_1\}, \{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_1, v_3\}, \{v_1\}$ make a cycle in $I_k^*(G)$ and so $I_k^*(G)$ is not a forest. □

Note that if $G$ is a graph of order $n$ with $\alpha(G) > 2$, then similar to Theorem 4.3 $I_k^*(G)$ cannot be a path, cycle and a chordal graph.

Figure 3: Graphs $I_2^*(W_5)$ and $I_3^*(P_5)$, respectively.

**Theorem 4.4** Let $G$ be a (non complete) bipartite graph of order $n > 4$. Then $I_k^*(G)$ is connected.

**Proof.** Let $I_1$ and $I_2$ be two independent sets of $G$ and $|I_1|, |I_2| \leq k$, so $I_1$ and $I_2$ are two vertices of $I_k(G)$. If $I_1 \cap I_2 \neq \phi$ then $I_1 \leftrightarrow I_1 \cap I_2 \leftrightarrow I_2$. If $I_1 \cap I_2 = \phi$, we consider two following cases:

Case 1. There are $v_1 \in I_1$ and $v_2 \in I_2$ such that $v_1$ and $v_2$ are not adjacent then $I_1 \leftrightarrow \{v_1\} \leftrightarrow \{v_1, v_2\} \leftrightarrow \{v_2\} \leftrightarrow I_2$.

Case 2. For all $v_1 \in I_1$ and $v_2 \in I_2$, $v_1$ is adjacent to $v_2$. So $I_1 \subset A$ and $I_2 \subset B$, where $A$ and $B$ are two parts of $G$. Since $G$ is not complete bipartite graph so $I_1 \neq A$ and $I_2 \neq B$ and there are $v_3 \in A$ and $v_4 \in B$ such that $v_3 \notin I_1$ and $v_3$ is not adjacent to $v_4$. We put $I_3 = (I_1 \setminus \{v_1\}) \cup \{v_3\}$. So $|I_3| = |I_1|$ and $I_1 \leftrightarrow I_1 \setminus \{v_1\} \leftrightarrow I_3$ and $I_3 \leftrightarrow \{v_3\} \leftrightarrow \{v_3, v_4\} \leftrightarrow \{v_4\} \leftrightarrow I_2$. Therefore $I_1 \leftrightarrow I_2$.

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