Exact Black Hole and Cosmological Solutions in a Two-Dimensional Dilaton-Spectator Theory of Gravity

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Abstract

Exact black hole and cosmological solutions are obtained for a special two-dimensional dilaton-spectator ($\phi - \psi$) theory of gravity. We show how in this context any desired spacetime behaviour can be determined by an appropriate choice of a dilaton potential function $V(\phi)$ and a “coupling function” $l(\phi)$ in the action. We illustrate several black hole solutions as examples. In particular, asymptotically flat double- and multiple- horizon black hole solutions are obtained. One solution bears an interesting resemblance to the 2D string-theoretic black hole and contains the same thermodynamic properties; another resembles the 4D Reissner-Nordstrom solution. We find two characteristic features of all the black hole solutions. First the coupling constants in $l(\phi)$ must be set equal to constants of integration (typically the mass). Second, the spectator field $\psi$ and its derivative $\psi'$ both diverge at any event horizon. A test particle with “spectator charge” (i.e. one coupled either to $\psi$ or $\psi'$), will therefore encounter an infinite tidal force at the horizon or an “infinite potential barrier” located outside the horizon respectively. We also compute the Hawking temperature and entropy for our solutions. In 2D FRW cosmology, two non-singular solutions which resemble two exact solutions in 4D string-motivated cosmology are obtained. In addition, we construct a singular model which describes the 4D standard non-inflationary big bang cosmology ($\text{big} - \text{bang} \rightarrow \text{radiation} \rightarrow \text{dust}$). Motivated by the similarities between 2D and 4D gravitational field equations in FRW cosmology, we briefly discuss a special 4D dilaton-spectator action constructed from the bosonic part of the low energy heterotic string action and get an exact solution which contains dust and radiation behaviour.
1 Introduction

In two spacetime dimensions, the Einstein action is a topological invariant and so has no dynamical content. Over the years many attempts have been made to formulate a non-trivial theory of two-dimensional gravity (see, for example, [1]). Such a study is not only pedagogically rewarding for classical relativists [2], but can also yield insights into the behaviour of semi-classical and quantum gravity [3]. This is primarily because these theories are mathematically simpler than four-dimensional general relativity (GR) yet retain much of the conceptual complexity of (3 + 1) dimensional spacetime physics.

Recently two such theories, that of [3], referred to as the “$R = T$” theory, and the string-inspired theory of [4], have attracted some interest, due to primarily to the fact that their field equations admit exact black hole and cosmological solutions, making them an interesting arena for the study of gravitational effects in both classical and quantum regimes. The former theory ($R = T$) retains the Einsteinian “curvature = matter” notion; indeed this theory can be constructed by taking the $D \to 2$ limit of $D$-dimensional GR [5], or by taking the limit $\omega \to \infty$ for the Brans-Dicke parameter $\omega$ in the 2D Brans-Dicke theory [6]. In this sense the $R = T$ theory can be viewed as a two-dimensional version of GR. This viewpoint is further supported by results that indicate the theory has a number of features which are closely analogous to four-dimensional GR. These include a well-defined Newtonian limit, post-Newtonian expansion and gravitational collapse to a black hole [7]. Semi-classical calculations indicate that the black hole radiates spinor [8] and vector particles [9]. Its classical cosmological properties are also similar to the four-dimensional counter-parts [7,10].

In this paper, we search for new exact black hole and cosmological solutions in the above-mentioned $R = T$ theory. We start with the following two-dimensional generalized “dilaton- spectator” action:

$$S = S_G + S_M,$$

where

$$S_G = \int d^2x \sqrt{-g} \left( \psi R + \frac{1}{2} (\nabla \psi)^2 \right),$$

$$S_M = \int d^2x \sqrt{-g} (a\phi R + H(\phi)(\nabla \phi)^2 + h(\phi)F^2 + l(\phi)(\nabla \psi)^2 + V(\phi, \psi)).$$

$S_G$ is the gravitational action and $S_M$ is the matter action. The latter by definition is independent of the auxiliary field $\psi$ which has no effect on the classical evolution of the gravity/matter system [2,3]. In $S_M$, $\phi$ has couplings analogous to the usual dilaton field in other 2D theories; we shall therefore refer to it as the dilaton and to $\psi$ as the spectator field. $V(\phi, \psi)$ is the potential function for $\phi$ and $\psi$. $F^2 \equiv F_{\mu \nu} F^{\mu \nu}$ is the Maxwell contribution.

The action (1) contains several special cases. First of all, when $S_G = H(\phi) = l(\phi) = h(\phi) = 0$, $a = 1$ and $V(\phi, \psi) = 0$, one has the Jackiw-Teitelboim theory [11]. Second, if $S_G = H(\phi) = l(\phi) = 0$, $a = 1$ and $V(\phi, \psi) = V(\phi)$, one gets the kind of two-dimensional action which admits several black hole solutions dimensionally reduced from the three-dimensional “BTZ” charged and spinning black hole solution [12]. Third, when $S_G = H(\phi) = 0$, $a = 1$, one obtains the kind of action considered by Lechtenfeld and Nappi in the context of no hair theorem of black holes [13]. Fourth, just setting $S_G = 0$ yields the
action considered by Elizalde and Odintsov on the discussion of one-loop renormalization and charged black hole solutions [14]. In particular, if $a = 1$, $h(\phi) = 0$, $H(\phi) = \frac{1}{\phi}$, $l(\phi) = -\frac{2}{3}\phi$ and $V(\phi, \psi) = \lambda_1^2 e^{-2\phi} - \lambda_2^2 e^{-\frac{4}{3}\psi}$, then one gets the action considered in [15] in a study of $2D$ black hole radiation.

The aforementioned cases all have $S_G = 0$. Now if $S_G \neq 0$, we get the “$R=T$” theory. For this “$R=T$” theory, when $a = 1$, $H(\phi) = constant = 2b$, $h(\phi) = l(\phi) = 0$ and $V(\phi, \psi) = \Lambda e^{-2a\phi}$, (1) reduces to the action of a Liouville field coupled to $2D$ gravity, where exact black hole solutions have been found [16]. In this paper, we will consider the “$R=T$” theory (i.e. $S_G \neq 0$) with matter action $S_M$ such that $a = 0$, $H(\phi) = 2b$, and $V(\phi, \psi) = V(\phi)$. Hence we consider

$$S = S_G + \int d^2x \sqrt{-g}(2b(\nabla \phi)^2 + V(\phi) + l(\phi)(\nabla \psi)^2 + h(\phi)F^2). \quad (2)$$

Mathematically, $V(\phi)$ is a zero-form field, $\partial_\mu \psi$ is a one-form field (with dilaton coupling function $l(\phi)$) and $F_{\mu\nu}$ is a two-form field (with dilaton coupling function $h(\phi)$). Thus $S_M$ in (2) is of particular interest as it describes a $2D$ dilaton theory of gravity which (i) has couplings of the dilaton to all possible $n$-form matter fields in $2D$ and (ii) has its gravitational couplings manifest via the $2D$ “$curvature = matter$” notion described earlier.

It is worthwhile pointing out that four-dimensional two-scalar field gravitational theories are presently attracting much attention. In string gravity, in addition to the usual dilaton, a second scalar (called the modulus field) describes the radius of the compactified space [17]. Furthermore, a class of multiple fields scalar-tensor theories of gravity and cosmology has been studied in [18]. As pointed out by the authors in [13], the restriction to a single scalar is merely for simplicity. For a more general situation, one should consider a second scalar interacting with the usual dilaton.

We will show that the equations of motion of (2) in the static and spatially homogeneous cases are very easy to solve if one adopts a method which is the “inverse” of the usual approach. Usually one specifies the form of $l(\phi)$, $h(\phi)$, and $V(\phi)$ in the equations of motion and then solves for the metric, $\phi$ and $\psi$: that is, one first specifies the matter content, then solves the field equations to determine the behaviour of the fields of interest. In our approach we will do the opposite: we show that if $l(\phi)$ is non-vanishing, then one can first specify the metric and $\phi$ then solve for $l(\phi)$, $h(\phi)$, $V(\phi)$ and $\psi$. Indeed, the equations of motion permit almost any desired behaviour for the metric provided $l(\phi)$, $h(\phi)$, $V(\phi)$ and $\psi$ are appropriately chosen. The functions we obtain are $ad hoc$ in the sense that they are derived from desired behaviour of the metric, rather than from a two-dimensional field theory model. However, this stance is not unusual in other studies. For example, Ellis and Madsen adopted this approach in their studies of four-dimensional exact scalar field cosmology [19]. Trodden, Mukhanov and Brandenberger obtained a non-singular two-dimensional black hole solution in (1) for $S_G = 0$, $a = 1$, $H(\phi) = l(\phi) = h(\phi) = 0$ and $V(\phi, \psi) = V(\phi)$ [20]. They first specified the desired non-singular black hole metric and then solved for $V(\phi)$.

Although such an approach for the action (2) offers a rather straightforward way to obtain exact solutions, it has two (perhaps unattractive) characteristic features. First, for
any black hole solution, the spectator field ψ and the term \( l(\phi)(\nabla \psi)^2 \) in (2) both diverge at the event horizon. We will show in detail that if a test particle couples to ψ in its equation of motion (i.e. has “spectator charge”) it will encounter an infinite tidal force at the horizon within a finite proper time. Alternatively, if it couples to \( l(\phi)(\nabla \psi)^2 \), there exists an infinite potential barrier such that the test particle starting its journey from infinity towards the event horizon will be bounced back toward infinity before reaching the horizon. For a neutral particle (travelling along a geodesic), it will encounter no infinite tidal force (except at the center) or infinite potential barrier. Second, we find that no non-trivial exact solutions can be found unless (at least) one of the coupling constants in the coupling function \( l(\phi) \) functionally determines (at least one) an integration constant; typically this is the mass parameter of the black hole solution. A similar situation occurs in [21], in which a model of a two-dimensional universe has a cosmological constant in a 2D gravitational action appearing as a constant of integration in order to get the desired behaviour of the universe; namely, as the radius of the universe gets larger, the cosmological constant gets smaller. An analogous 4D situation was considered in [22] for a deSitter (or anti-deSitter) black hole in which the cosmological constant was (for whatever reason) set equal a function of the mass (and charge, when relevant) – these types of solutions (e.g. the extremal Schwarzschild-de Sitter solution) form a set of measure zero in the space of all possible solutions. In a more general approach, the 4D Einstein action may even be considered to have time-dependent coupling constants, namely the gravitational and cosmological constants [23].

The action is, of course, the primary quantity in any physical theory. The black hole solutions we obtain are therefore valid solutions to the field equations of the actions we consider only for black holes whose specific masses are determined by the parameters of those actions. The existence of other potentially interesting solutions associated with these actions remains an open question. Hence the approach we are using may be regarded as a method to suggest two dimensional actions that are of potential interest in 1+1 dimensions. If we restrict ourselves to discussions of a neutral test particle and adopt the aforementioned viewpoint regarding the black hole solutions we obtain, then our solutions are physically interesting since they admit double and multiple event horizons as examples, as well as a couple of interesting non-singular cosmological solutions.

Our paper is organized as follows. In section 2 the equations of motion associated with the action (2) are derived. We discuss their properties and see how to implement the desired behaviour of a given metric. In section 3 several interesting examples of black hole solutions which are asymptotically flat with double or multiple horizons are illustrated, and their quasi-local energy and mass are also calculated. In addition, we construct a special black hole solution which has a causal structure analogous to the Reissner-Nordstrom spacetime except that there are no singularities. Section 4 is devoted to a discussion of the motion of a test particle coupling to ψ or \( l(\phi)(\nabla \psi)^2 \). It is shown that the particle either encounters an infinite tidal force at the event horizon or is bounced back toward infinity before reaching it. The Hawking temperature and entropy for some of the black hole solutions are computed in section 5. One solution in particular resembles the 2D string black hole metric with the same thermodynamic behaviour. A couple of interesting non-singular FRW cosmological models are extracted in section 6. A singular universe
which is analogous to the 4D standard big bang model is also constructed. Motivated by the similarities in field equations between two and four-dimensional FRW cosmology, we also briefly discuss a two-scalar field cosmology obtained from 4D string theory. A dust/radiation solution is derived. We summarize our work in the final section. We set the 2D gravitational coupling constant to be 1 (i.e. mass has a dimension of inverse length) and the signature of the metric be $(-+)$. 

2 Field Equations

Varying (2) with respect to the auxiliary, metric, dilaton, spectator and Maxwell fields yields

$$\nabla^2 \tilde{\psi} - R = 0,$$

$$\frac{1}{2} \left( \nabla_\mu \tilde{\psi} \nabla_\nu \tilde{\psi} - \frac{1}{2} g_{\mu\nu} (\nabla \tilde{\psi})^2 \right) + g_{\mu\nu} \nabla^2 \tilde{\psi} - \nabla_\mu \nabla_\nu \tilde{\psi} = T_{\mu\nu},$$

$$-4b \nabla^2 \phi + \frac{d l}{d \phi} (\nabla \psi)^2 + \frac{d h}{d \phi} F^2 + \frac{d V}{d \phi} = 0,$$

$$\nabla^\mu (l(\phi) \nabla_\mu \psi) = 0,$$

and

$$\nabla^\mu (h(\phi) F_{\mu\nu}) = 0,$$

where

$$T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} V(\phi) - 2h(\phi) \left( F_{\mu\tau} F^{\nu\tau} - \frac{1}{4} g_{\mu\nu} F^2 \right) - 2b \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla \phi)^2 \right)$$

$$-l(\phi) \left( \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} g_{\mu\nu} (\nabla \psi)^2 \right).$$

The above form a complete set of eight differential equations. It is easy to see that $T_{\mu\nu}$ is conserved by taking the divergence of (4); along with the obvious gauge invariance of the Maxwell field, this shows that there are really only five independent field equations in (3)–(7). Hence given $V(\phi)$, $l(\phi)$ and $h(\phi)$, then these differential equations may be solved for the five unknown functions $\phi$, $\tilde{\psi}$, $\psi$, $F_{\mu\nu}$ and $g_{\mu\nu}$ where the latter two fields have respectively one independent function each due to gauge and co-ordinate invariance.

Insertion of the trace of (4) into (3) yields

$$R = T.$$  

The evolution of $\tilde{\psi}$ is then determined only by the traceless part of (4). Since $R$ completely determines the metric field in two dimensions [1] and $T$ is independent of $\tilde{\psi}$, we see that $\tilde{\psi}$ has no effect on the evolution of the gravity/matter system (although the converse is not true).

In two dimensions, a symmetric metric only has one degree of freedom [1]. Thus we can write the static metric in the form

$$ds^2 = -\lambda(x) dt^2 + \frac{dx^2}{\lambda(x)}.$$
Taking the form (10), the gravity-matter equation (9) reduces to

\[ \lambda'' = -V - \frac{2q^2}{h} \] (11a)

where the ' denotes an ordinary derivative with respect to \( x \) and \( q \) is the electric charge associated with the Maxwell field (see (11d)). One sees that the metric directly depends on the potential and Maxwell contribution, not on the spatial derivative of the dilaton or spectator. The spectator equation (7) becomes

\[ \psi' = \frac{C_0}{\lambda l}, \] (11b)

where \( C_0 \) is a constant of integration. The dilaton equation (5) is

\[ -4b(\lambda\phi')' + \frac{C_0^2}{\lambda l^2} \frac{dl}{d\phi} + \frac{d}{d\phi}\left(\frac{2q^2}{h} + V\right) = 0. \] (11c)

The Maxwell two-form is given by

\[ F_{\mu\nu} = \frac{q}{h} \epsilon_{\mu\nu}, \] (11d)

where \( q^2 \) is the square of magnitude of an electric charge and \( \epsilon_{\mu\nu} \) is the volume two-form \((\epsilon^2 = -2)\).

We are now left with three differential equations (11a)-(11c) from which one can solve simultaneously for \( \psi(x), \phi(x) \) and \( \lambda(x) \) for a given \( V(\phi), l(\phi) \) and \( h(\phi) \). Defining

\[ \tilde{V} = V + \frac{2q^2}{h}, \] (12)

(11a) and (11c) respectively become

\[ \lambda'' = -\tilde{V}, \] (13)

and

\[ -4b(\lambda\phi')' + \frac{C_0^2}{\lambda l^2} \frac{dl}{d\phi} + \frac{d\tilde{V}}{d\phi} = 0. \] (14)

The definition in (12) is possible since the Maxwell and the dilaton potential contributions are mathematically equivalent in two dimensions.

From now on, we will drop the tilde sign in (13) and (14). (3) always has the solution

\[ \tilde{\psi}' = -\frac{\lambda'}{\lambda} + C. \] (15)

\( C \) is an integration constant which will later be shown to be related to the mass parameter in our static solutions. If \( l(\phi) \) and \( V(\phi) \) are specified, one has to solve simultaneously three independent equations, (11b), (13) and (14) for \( \psi(x), \lambda(x) \) and \( \phi(x) \).
Differentiating both sides of (13) with respect to \( x \), multiplying by \( \lambda(x) \), and performing an integration (with respect to \( x \) again) yields

\[
\int \lambda dV = -\lambda\lambda'' + \frac{1}{2}\lambda'\lambda'.'
\] (16)

Provided \( \phi' \neq 0 \), (14) can always be transformed to

\[-4b(\lambda\phi')' + \frac{l'}{l^2} \frac{C_0^2}{\lambda\phi'} + \frac{V'}{\phi'} = 0.\] (17)

Integrating (17) yields (using (16))

\[
\frac{C_0^2}{l} = -2b(\lambda\phi')^2 - \lambda\lambda'' + \frac{1}{2}\lambda'\lambda' + C_1,
\] (18)

where \( C_1 \) is a constant of integration. In this paper, \( C \) and \( C_i \) \((i = 0, 1, 2, \ldots)\) denote integration constants in all the black hole and cosmological solutions. Equations (11b), (13) and (18) now form the complete set of differential equations to be solved. It is clear that if an invertible \( \phi = \phi(x) \) \((x = x(\phi))\) and a metric field \( \lambda(x) \) are both specified \( a\)-\textit{priori}, then (18) trivially yields \( l(\phi) \). Equation (13) then implies that \( V(\phi) \) can be determined from the given \( \phi(x) \) and \( \lambda(x) \). Here we take advantage of this arbitrariness to determine \( V(\phi) \) and \( l(\phi) \). In this sense we are determining which \( V(\phi) \) and \( l(\phi) \) are required to yield a static metric in the \( R = T \) theory of specified form. One can see from (18) that \( C_0 \) and \( C_1 \) are also coupling constants in \( l(\phi) \) for a given \( \lambda(x) \) and \( \phi(x) \). Hence in this approach the solutions we obtain are valid only when their constants of integration functionally depend upon the coupling constants \( C_0 \) and \( C_1 \) in \( l(\phi) \). In fact, \( C_0 \) is just a length scale with dimensions of inverse length and (if nonzero) has no physical significance. It can be scaled to 1 by adjusting the units. However, it will be later shown that \( C_1 \) and \( C \) are related to the mass parameter of a static black hole solution. Therefore (18) indicates that each \( l(\phi) \) we obtain yields a special class of black hole solutions whose specific mass parameters are determined by one of the coupling constants. Henceforth, unless otherwise stated, \( C \) and \( C_i \) \((i = 0, 1, 2, \ldots)\) will generally denote coupling constants in action (2) that determine the integration constants for the solutions we obtain via the above procedure; the rest of the constants are just coupling constants with no such functional dependence.

Although this procedure seems somewhat similar to prescribing a 4D metric, evaluating the resultant \( G_{\mu\nu} \) and then determining the stress-energy by setting it equal to this \( G_{\mu\nu} \) via Einstein’s equations \( i.e. \) one finds the matter required to give a desired geometric state), there is a subtle distinction in that there are always appropriate dilaton potentials which can be chosen to satisfy the field equations under the above constraints. The solutions we obtain are those for which the constants of integration functionally depend upon the coupling constants in these potentials – however other solutions to the field equations may exist for which no such dependence exists. Once a particular \( l(\phi) \) and \( V(\phi) \) have been determined, one can then regard the action containing these functions as a separate field theory in its own right and explore the solution space of its field equations (one point of which must contain the solution which originally led to this choice of \( l(\phi) \) and \( V(\phi) \)).
If $C_0$ vanishes, one can see in (18) that it is generally difficult (if not impossible) to integrate $\phi = \phi(x)$ and get its inverse if a desired $\lambda(x)$ is given. Finally $\psi(x)$ is determined by integrating (11b) or

$$C_0\psi' = -2b\lambda\phi'^2 - \lambda'' + \frac{1}{2}\frac{\lambda'^2}{\lambda} + \frac{C_1}{\lambda},$$

(19)

where (18) has been used. Generally speaking, there exists no general method for integrating (19) exactly. However, we see from the action (2) and its equations of motion that none of them depend explicitly or implicitly on $\psi(x)$ due to the fact that we chose the potential $V$ to depend only on $\phi$. This choice greatly simplifies our calculations and makes the exact integration of $\psi(x)$ unnecessary (although in the cases discussed below we find that $\psi'(x)$ can be integrated exactly). However, the price paid for this choice is that both $\psi(x)$ and $\psi'(x)$ diverge at the spatial point(s) where $\lambda(x) = 0$, that is, at the event (or cosmological) horizons. As a result, an infinite tidal force (for a test particle coupled to $\psi(x)$) is present at the event horizon or an infinite potential barrier (if it couples to $\psi'$) exists outside the horizon.

Before proceeding further we note the following. Besides the approach of obtaining exact solutions we mentioned in the last paragraph, one can also adopt an alternate approach: given a $V(\phi)$ and an invertible $\phi = \phi(x)$, (13) yields $\lambda(x)$. Now substituting the $\lambda(x)$ into (18) yields $l(\phi)$. This approach is less straightforward since $\lambda''(x)$ is not always integrable for a desired form of the potential. The third approach, (typically the most difficult), is the usual approach: given $l(\phi)$ and $V(\phi)$, one solves for $\lambda(x)$ and $\phi(x)$. Even in the present simple two dimensional context, the system of differential equations (13) and (18) are typically difficult to solve in this third approach. In this paper we adopt the first approach only.

3 Black Hole Solutions

In this section, we first specify the forms of $\lambda(x)$ and $\phi(x)$ and then solve for $l(\phi)$ and $V(\phi)$. To illustrate the procedure, we give some special and interesting examples of black hole solutions. We will not attempt to show all possible examples. We note that four-dimensional GR admits a static charged black hole solution which is asymptotically flat and has a double-horizon spacetime structure. In two-dimensional string-inspired theories of gravity, asymptotically flat charged black hole solutions are even more interesting: they have double as well as multiple horizons [24]. In $R = T$ theory, the situation is very different. When $V = \psi = \phi = 0$ and $h(\phi) = h_0$ in action (2), we have pure gravity with a Maxwell contribution. (11d) indicates that $F_{\mu\nu}$ is just a constant two form. The photon field $\tilde{A}_\mu$ ($F_{\mu\nu} = \partial_\mu \tilde{A}_\nu$) propagates no physical degrees of freedom but the solution of its field equation allows an arbitrary constant $q^2$ in the gravitational field equation (11a). Thus the Maxwell contribution (without $\phi$ and $\psi$) is equivalent to a constant $\frac{2q^2}{h_0}$ which appears as a constant of integration in the equations of motion of action (2). Now the metric is given by [25]

$$\lambda = -\frac{q^2}{h_0}x^2 + 2mx \pm 1.$$

(20)
is the mass parameter. Although (20) admits a double-horizon spacetime, it is not asymptotically flat. Indeed, \( q^2 \) plays the role as a 2D cosmological constant, with the sign of \( h_o \) determining whether the spacetime is asymptotically de Sitter or anti-de Sitter. When a dilaton with a potential is present, no exact asymptotically flat double-horizon black hole solutions are found either \[26\]. In order to see this, we use (18). We consider three types of the simplest metric forms (i) \( \lambda = A_1 + A_2 x + A_3 x \), (ii) \( \lambda = A_1 + A_2 x^2 + A_3 e^{-2kx} \), and (iii) \( \lambda = A_1 + A_2 e^{-kx} + A_3 e^{-2kx} \). \( A_j (j = 1, 2, 3) \) are constants. (i)-(iii) are asymptotically flat and admit double horizons for properly chosen signs and magnitude among \( A_j \). Note that in (i) the criterion for asymptotically flatness is slightly more general than in higher dimensions. One need only require that \( \lambda \to A_1 + A_3 x \) for large \( |x| \) since in this case the metric will become asymptotically like a Rindler space time; a Rindler transformation may then be applied locally to rewrite the metric in Minkowski form \[26\]. Now putting (i)-(iii) into (18) when \( C_0 \) is vanishing, it is not hard to check that \( \phi'(x) \) cannot be integrated exactly to close forms and therefore no inverse \( x = x(\phi) \) can be obtained. As a result, no exact \( V(x(\phi)) \) can be obtained from (13). More complicated forms of metrics with the desired properties may lead to an exact integrable \( \phi'(x) \) and its inverse \( x = x(\phi) \), but (i)-(iii) have the advantage that the outer and inner horizons can be expressed explicitly in terms of the \( A_j \). In addition, (ii) is the form found in both 2D string gravity and 4D GR and (iii) is found in 2D string gravity, it is interesting to use them as a basis of discussions in our present \( R = T \) case. As a conclusion, asymptotically flat double-horizon solutions (i)-(iii) can be obtained with a presence of a dilaton but no closed and invertible forms of the dilaton (and \( V(\phi) \)) can be found.

We now consider the forms (i)-(iii) in the presence of a non-trivial \( \psi(x) \) and \( l(\phi) \). Before we do so, we need to know the equations to calculate the energy and mass for a static solution. General discussions on the concepts of quasilocal energy and mass in three and four dimensions can be found in \[27\]. More recently a general formula for the quasilocal energy and mass in two dimensions for an arbitrary 2D gravity theory has been derived \[28\]. For the theory described by (2), using the coordinates (10) we have

\[
\varepsilon(x) = \frac{1}{\sqrt{\lambda}} (\lambda' + C) - \varepsilon_o, \tag{21}
\]

for the quasilocal energy evaluated at a spatial position \( x \) and

\[
\mathcal{M}(x) = (\lambda' + C) - \mathcal{M}_o, \tag{22}
\]

for the mass. Here \( C \) is the integration constant in (15), which is determined by \( T_{\mu\nu} \) in (4) and (8), and \( \mathcal{M}_o \) and \( \varepsilon_o \) are the mass and energy for a chosen background static spacetime \( \lambda_o \). One can see that the energy \( \varepsilon = \frac{\mathcal{M}}{\sqrt{\lambda}} \) \( (\varepsilon_o = \frac{\mathcal{M}_o}{\lambda_o}) \) is simply the mass appropriately blueshifted. For an asymptotically flat spacetime one can take the limit \( x \to \infty \) in (22) to find the total mass \( m \equiv \mathcal{M}(\infty) \) associated with a given static spacetime. We shall use (22) to calculate the mass of the static solutions we obtain. Note that if the spatial co-ordinate \( x \) is replaced by \( r \equiv |x| \), then every static solution in terms of \( r \) remains a solution provided that either (i) it is considered to be a “dilaton-spectator vacuum” solution outside of a distribution of additional matter stress energy (such as a perfect fluid with pressure) or (ii)
an appropriate delta-function point-source of stress energy is inserted at the origin [26].
From now on, we will express every static solution in terms of \( r \). This choice is spatially
symmetric and renders the solutions somewhat more analogous to the four dimensional
spherically symmetric cases. In the following we consider four classes of examples, (3.1),
(3.2), (3.3) and (3.4) where they admit asymptotically flat black hole solutions with double
(or multiple) horizons, and one special black hole solution which has only one event horizon
but admits an infinite chain of universes connected by timelike wormholes. Note that in
all the examples, \( C \) and \( C_i \) will denote coupling constants which appear as integration
constants in action (2).

\[
(3.1) \quad \lambda(r) = \pm 1 + \frac{\beta^3 \Lambda}{r} + C_2 r, \quad \phi = -\ln\left(\frac{r}{\beta}\right).
\]

With this form of the metric and dilaton, equation (13) is satisfied if

\[
V(\phi) = -2\Lambda e^{3\phi}
\]

and (18) yields

\[
\frac{C_0^2}{l(\phi)} = \frac{3C_2}{\beta} e^\phi + \frac{3}{2\beta^2} e^{2\phi} + \Lambda e^{3\phi},
\]

\[
b = -\frac{3}{4}, \quad C_1 = -2C_2^2
\]

where we have set \( b = -\frac{3}{4} \) and \( C_1 = -2C_2^2 \) in (18) as an example. The equation \( \lambda(r) = 0 \)
can at most admit two real positive roots. Note that for the generalized metric \( \pm 1 + \frac{\beta^n + 2\Lambda}{r^2} + C_2 r \), a graphical analysis shows that \( \lambda(r) = 0 \) still has at most two real positive
roots for all \( n > 0 \). Recall that \( C \) and \( C_i, i = 0, 1, 2... \) denote the coupling constants in
the action (2) on which to constants of integration in the solutions we obtain functionally
depend. The rest of the constants, \( \beta \) and \( \Lambda \) are both coupling constants. In general, \( \psi'(r) \)
in (19) cannot be integrated exactly except for the choice of \( C_1 \) in (23b), in which case

\[
C_0 \psi = -\frac{1}{4L} \ln\left(\frac{1}{r} + \frac{L}{r^2} + C_2\right) + \frac{1}{r} + \frac{1 - 8LC_2}{2L(4LC_2 - 1)} \arctan\left(\frac{-1 + 2C_2r}{-1 + 4LC_2}\right), \quad (23c)
\]

where \( L = \beta^3 \Lambda \) and \( -4LC_0 \psi_o = \ln(r_o) \) is an integration constant. One can see that as
\( \lambda(r) \to 0 \), \( \psi(r) \) diverges, as we have mentioned this fact earlier. The existence of event
horizon(s) depends on the relative magnitude and signs of \( C_2 \) and \( \Lambda \). Therefore we have to
determine the mass parameter in \( \lambda(r) \) first. Naively, one expects \( C_2 \) to be related linearly
to the mass parameter since in the vacuum case \( (\Lambda = \phi = \psi = 0) \), the solution has the form
\( \lambda = \pm 1 + C_2 r \), for solution has \( C_2 \) as the mass parameter [26,28]. So we will choose the
background mass as the mass of the reference spacetime with \( C_2 = 0 \). It is straightforward
but lengthy to show that (4), (8), (15) and (23b) together imply

\[
C = \pm \sqrt{-2C_1} = \pm 2C_2.
\]

Now (22) yields

\[
m = -C_2, \quad \text{or} \quad m = 3C_2
\]
for the total mass as seen at infinity. If \( \Lambda = 0 \) in (23b), then \( \lambda = \pm 1 + C_2 r \) similar to the vacuum case discussed in [2,26,28]. However, \( C_2 = 2m \) in the vacuum case (see (20) with \( q^2 = 0 \)) rather than the relation in (23e). Therefore \( C_2 \) is “rescaled” or flips sign due to the nontrivial presence of \( l(\phi) \) in (23b).

Since \( m \) is related to \( C_1 = -2C_2^2 \) through (23e), the black hole solution admitted by \( l(\phi) \) in (23b) is a rather special case of the full range of solutions associated with this coupling function. Semi-classically when the black hole solution radiates, we expect the mass will change; as a consequence, one can accommodate this by either modifying the action (2) so that its coupling constants vary with time, or by performing a full semiclassical analysis of the action (2) with arbitrary fixed coupling constants to determine the complete evolution of the evaporating black hole whose mass is initially given in terms of these constants via (23e). Note that for a general \( C_1 \), the mass is given by

\[
m = \pm \sqrt{-2C_1 + C_2}. \tag{23f}
\]

It is obvious that if \( C_2 = \pm \sqrt{-2C_1} \), then one obtains an “massless” black hole solution provided that the delta-function point-source at the origin is removed \( (r = |x| \to x \text{ in the solution}) \). Note that regardless of whether or not the black hole has mass, there is a curvature singularity at the \( x = 0 \) due to the bad behaviour of the energy momentum tensor (8) there \( (T^\nu_\nu \text{ or the energy density of (8) diverges at } x = 0) \). This massless black hole has an advantage that \( l(\phi) \) is no longer depends on any integration constant. However, it is not clear how to do thermodynamical analysis on it since it has no mass. We will use (23e) in the following.

We demand that \( m \) in (23e) is always positive. The quasilocal energy is given by (21). It approaches the mass as \( r \to \infty \). Substituting (23f) into the metric yields either

\[
\lambda = \pm 1 + \frac{\beta^3 \Lambda}{r} - m r, \quad \text{or} \quad \lambda = \pm 1 + \frac{\beta^3 \Lambda}{r} + \frac{m}{3} r. \tag{23g}
\]

Due to the arbitrary sign of \( \Lambda \), we expect that (23g) generally admits both event and cosmological horizons. There are several cases of interest:

(a): \( \lambda = 1 - \frac{\beta^3 \Lambda}{r} + \frac{m}{3} r \)

We have set \(-\beta^3 \Lambda \equiv L > 0\). Since \( m \) is positive, \( \frac{m}{r} > 0 \). A simple graphical analysis shows that there exists only one event horizon in this case. As \( r \to +\infty \), \( \lambda \to +\infty \). As \( r \to 0^+ \), \( \lambda \to -\infty \). \( \lambda’ = \frac{L}{r} + \frac{m}{3} \) which is never zero. Since \( \lambda(r) \) is a continuous function of \( r \), it is easy to see that only one event horizon exists. The spatial co-ordinate of the event horizon is given by

\[
r_h = \frac{3}{2m} \left( -1 + \sqrt{1 + \frac{4mL}{3}} \right). \tag{23h}
\]

(b): \( \lambda = 1 - \frac{\beta^3 \Lambda}{r} - mr \)

As \( r \to +\infty \), \( \lambda \to -\infty \). As \( r \to 0^+ \), \( \lambda \to -\infty \). \( \lambda’ = \frac{L}{r^2} - m \) thus there exists one turning point. From the above information, it is easy to conclude that one may have one
event horizon and one cosmological horizon. Both horizons are given by

\[ r_{\pm} = \frac{1}{2m}(1 \pm \sqrt{1 - 4mL}), \quad (23i) \]

where + refers to the cosmological horizon and − refers to the event horizon.

(c): \( \lambda = -1 + \frac{L}{r} + \frac{m}{3}r \)

In this case we set and \( L = \beta^2 \Lambda > 0 \). Now, as \( r \to +\infty \), \( \lambda \to +\infty \). As \( r \to 0^+ \), \( \lambda \to +\infty \). Also, \( \lambda' = -\frac{L}{r^2} + \frac{m}{3} \) which implies there exists one turning point. \( \lambda(r) \) is a parabola-like curve opening upward. Therefore it may have inner and outer horizons given by

\[ r_{\pm} = \frac{3}{2m} \left( 1 \pm \sqrt{1 - \frac{4mL}{3}} \right). \quad (23j) \]

When \( 1 = \frac{4mL}{3} \), one gets an extremal case. If \( 1 < \frac{4mL}{3} \), then the singularity at \( r = 0 \) will be naked.

(d): \( \lambda = -1 - \frac{L}{r} + \frac{m}{3}r \)

This is the final case of the form (3.1) we discuss now. As \( r \to +\infty \) and \( 0^+ \), \( \lambda \to +\infty \) and \( -\infty \) respectively. In addition, \( \lambda' \) never becomes zero and so there exists exactly one event horizon which is located at

\[ r_h = \frac{3}{2m} \left( 1 + \sqrt{1 + \frac{4mL}{3}} \right). \quad (23k) \]

We will discuss the thermodynamics properties of the particular case (c) in the next section.

(3.2) \( \lambda = 1 - \frac{\Lambda_1 e^{-Kr}}{K^2} - \frac{\Lambda_2 e^{-2Kr}}{4K^2}, \phi = -\frac{K}{2}r, K > 0 \)

This form of solution was first obtained in a 2D string theory [24]. For appropriate choices of magnitude and signs of \( \Lambda_1 \) and \( \Lambda_2 \), this asymptotically flat metric admits black hole solutions. \( \lambda(r) \) has no curvature singularity at the origin apart from a delta-function singularity provided that \( \lambda'(0) \neq 0 \). Now, this solution gives \( b = -1 \),

\[ V(\phi) = \Lambda_1 e^{2\phi} + \Lambda_2 e^{4\phi}, \quad (24a) \]

and

\[ \frac{C_0^2}{l} = C_1 + \frac{K^2}{2} + \frac{3}{4} \Lambda_2 e^{4\phi} - \frac{1}{2K^2} \Lambda_1 \Lambda_2 e^{6\phi} - \frac{3}{32K^2} \Lambda_2^2 e^{8\phi}. \quad (24b) \]

\( \psi(r) \) in (19) is calculated to be

\[ C_0 \psi = -K^2 \left( \frac{3}{2} - \frac{2q_1^2}{q_2^2} \right) r + \left( \frac{3}{4} - \frac{q_1^2}{q_2^2} \right) + \frac{1}{2} \left( C_1 + \frac{K^2}{2} \right) \ln(e^{2Kr} - 2q_1 e^{Kr} + q_2^2) \]

\[ + K q_1 \left( q_2^2 \left( C_1 + \frac{K^2}{2} \right) - \frac{5}{2} - \frac{2q_1^2}{q_2^2}(q_2^2 - q_1^2)^{-\frac{1}{2}} \right) \arctan \left( \frac{e^{Kr} - q_1}{(q_2^2 - q_1^2)^{\frac{1}{2}}} \right), \quad (24c) \]
where we have set \( q_1 = \frac{\Lambda_1}{2K} \) and \( q_2^2 = -\frac{\Lambda_2}{4K} \). (4), (8), (15) and (24b) together imply

\[
C = \pm \sqrt{-2C_1}. \tag{24d}
\]

\( C \) has the same dependence on \( C_1 \) as in the previous case (3.1). Note that when \( C_0 = 0 \) and \( \Lambda_2 = 0 \), one gets back the black hole solution obtained in [26] but with \( C = K \); the mass may be shown to be equal to \( K \) when the background mass \( M_o \) is chosen to be zero [28]. As the solution (3.2) contains the black hole solution in [26] as a special case, we set \( \sqrt{-2C_1} = K \) in (24b) and choose \( M_o = 0 \) again. Now from (22) we get

\[
m = K = \sqrt{-2C_1}. \tag{24e}
\]

Since the mass \( m \) is positive, \( \lambda(r) \) is asymptotically flat. \( C_0, C_1 \) (\( = -\frac{K_2}{2} \)) and \( K \) are all coupling constants which appear as integration constants in (24b). \( \Lambda_1 \) and \( \Lambda_2 \) are coupling constants. Now when \( \Lambda_1 > 0 \) and \( \Lambda_2 < 0 \), \( \lambda(r) \) has an outer and an inner horizon. Their spatial co-ordinates are given by

\[
r_{\pm} = \frac{1}{m} \ln \left( 1 \pm \sqrt{1 + \frac{m^2 \Lambda_2}{\Lambda_1^2}} \right) + \frac{1}{m} \ln \frac{\Lambda_1}{2m^2}. \tag{24f}
\]

The extremal black hole is obtained when \( \Lambda_1^2 = -m^2 \Lambda_2 \). By rescaling the co-ordinates \( r \to \frac{\tilde{L}}{m} r \) and \( t \to \frac{\tilde{L}}{m} t \), one can take the limit \( m \to 0 \). The metric now is \( \lambda = -\left( \frac{\Lambda_1}{r^2} e^{-\tilde{L}r} + \frac{\Lambda_2}{4r^2} e^{-2\tilde{L}r} \right) \) and \( \phi = \tilde{L} r \). Since \( \Lambda_1 > 0 \) and \( \Lambda_2 < 0 \), the event horizon disappears but a cosmological horizon emerges.

In string-inspired theories, the mass parameter appears as a coefficient \( \frac{2m}{K} \) in front of the term \( e^{-Kr} \) in the metric, and the square of the electric charge appears linearly in front of \( e^{-2Kr} \) [24,26], \( K \) being related to a coupling constant in the string theories instead of a coupling constant that determines an integration constant. In the present case, \( \Lambda_2 \) can be interpreted as the square of a charge (recall that in (12), the Maxwell contribution is absorbed into the second term of \( V(\phi) \)); therefore only \( \Lambda_1 \) in \( V(\phi) \) is the coupling constant. However, the mass parameter is given by \( K = m \) (see (24e)). This fact leads to significantly different black hole thermodynamic behaviour. It is obvious that one may adjust \( C_1 \) in (22), (24b) and (24d) in order to make the mass parameter appear linearly in front of \( e^{-Kr} \) in the metric as in the string case in [24].

Since \( C_1 \) is the coupling in (2) that determines the integration constant, we have the freedom to “fix” it. In particular, if one sets \( \sqrt{-2C_1} = \frac{\Lambda_1}{\tilde{L}} \) (which implies \( \Lambda_1 = \tilde{L} m \), see (24e)), where \( \tilde{L} > 0 \) and has the same dimension as \( m \) and is a coupling constant, then the mass is now linear in front of \( e^{-Kr} \). More precisely, the metric is now given by

\[
\lambda = 1 - \frac{m \tilde{L}}{K^2} e^{-Kr} - \frac{\Lambda_2}{4K^2} e^{-2Kr}. \tag{24g}
\]

In this case the mass parameter depends on both \( C_1 \) and \( \Lambda_1 \), the latter two quantities no longer being independent. \( K \) now is a coupling constant. In the present situation not
only \( l(\phi) \) explicitly depends on \( m \) but also \( V(\phi) \). In the presence of a non-trivial \( l(\phi) \), the “position” of the mass in the metric can be adjusted by varying \( C_1 \) in \( l(\phi) \). It is not surprising since different \( C_1 \)’s corresponding to different theories for action (2). We see that in order for the \( R = T \) theory to “reproduce” the results in string theory [24], not only are an additional scalar \( \psi \) and a coupling function \( l(\phi) \) required in the matter action, but the constants of integration in the solution are functions of the coupling constants in the action. Finally, if we set \( C_1 = 0 \), for example, then we will get a massless black hole metric provided that \( r \to x \) (i.e. the delta-function point-source is removed). In this situation, \( l(\phi) \) and \( V(\phi) \) will no longer depend on the mass.

\[
(3.3) \quad \lambda = 1 - \frac{\beta^3 \Lambda_1}{2r} - \frac{\beta^4 \Lambda_2}{6r^2}, \quad \phi = -\ln\left(\frac{r}{\beta}\right)
\]

The choice of these forms for \( \lambda(r) \) and \( \phi(r) \) is particularly interesting since the same form is obtained in a modified version of 2D string theory [29]. In 4D general relativity, the Reissner-Nordstrom solution also has the same metric form. Now (13) is satisfied if

\[
V(\phi) = \Lambda_1 e^{3\phi} + \Lambda_2 e^{4\phi} \tag{25a}
\]

whereas (18) implies that

\[
\frac{C_0^2}{l(\phi)} = \frac{1}{6} \Lambda_1 \Lambda_2 \beta^2 e^{5\phi} + \left( -\frac{1}{3} \Lambda_2 + \frac{5}{8} \Lambda_1^2 \beta^2 \right) e^{4\phi} - 3 \Lambda_1 e^{3\phi} + \frac{4}{\beta^2} e^{2\phi} + C_1, \tag{25b}
\]

where we have set \( b = -2 \) in (19) for simplicity (note that an extra term \( e^{6\phi} \) will appear in \( l(\phi) \) when \( b \neq -2 \)). \( \psi(r) \) can be integrated exactly and so

\[
C_0 \psi = \left( \frac{2q_1(q_2^2 - q_1^2)}{q_2^4} + C_1 q_1 \right) \ln\left( 1 - \frac{2q_1}{r} + \frac{q_2^2}{r^2} \right) + 2C_1 q_1 \ln\left( \frac{r}{\beta} \right) - \frac{2(q_1^2 + q_2^2)}{q_2^2} \frac{1}{r} + \frac{2q_1}{r^2}
\]

\[
+ C_1 r + \frac{2q_1^2 - q_2^2}{(q_2^2 - q_1^2)^2} \left( C_1 - \frac{2}{q_2^4} (q_2^2 - q_1^2) \right) \arctan\left( \frac{r - q_1}{(q_2^2 - q_1^2) \beta} \right), \tag{25c}
\]

where for simplicity we have set \( q_1 = \frac{\beta^3 \Lambda_1}{4} \) and \( q_2 = -\frac{\beta^4 \Lambda_2}{6} \). Now, one can show that

\[
C = \pm \sqrt{-2C_1}, \tag{25d}
\]

in (15) by using (4), (8) and (25b). Here \( C \) has the same dependence on \( C_1 \) as in the previous cases (3.1) and (3.2). As previously mentioned, different \( C_1 \)’s correspond to different choices of theory. Similar to the previous case, we set \( M_o = 0 \). We further set \( \sqrt{-2C_1} = \frac{\Lambda_1}{4L_1} \), where \( L_1 > 0 \) and is a coupling constant with a dimension of inverse length. This choice for \( C_1 \) is interesting since it yields a Reissner-Nordstrom metric form (25f) with the same thermodynamic properties. Now \( C_0, C_1 \) and \( \Lambda_1 \) are no longer independent, and the integration constant \( m \) depends on the remaining independent combination of these. \( \frac{\Lambda_1}{6} \) can be interpreted as the square of electric charge (see (25f)). Therefore only \( \beta \) and \( \bar{L} \)
are the only remaining independent coupling constants. Using (25d), where the positive sign is being used, we obtain

\[ m = \frac{\Lambda_1}{4L_1}. \]  

(25e)

The metric is now given by

\[ \lambda = 1 - \frac{2\beta^3 L_1 m}{r} - \frac{\beta^4 \Lambda_2}{6r^2}. \]  

(25f)

Now the metric is exactly the Reissner-Nordstrom metric (with the same thermodynamics properties) obtained in 2D string theory and 4D GR.

Note that the presence of \( \beta^3 \) and \( \beta^4 \) in the \( r^{-1} \) and \( r^{-2} \) terms are due to the fact that mass and charge both have dimensions of inverse length in two dimensions, instead of length in four dimensions. As a result, extra length scales \( \beta^3 \) and \( \beta^4 \) are required. If \( \Lambda_2 < 0 \) in (25f), then we have outer and inner horizons. We comment that \( V(\phi) \) and \( l(\phi) \) both explicitly depend on the mass parameter \( m \) similar to the situation in the last section. However, if one sets \( C_1 = 0 \) instead, one will get a massless black hole.

(3.4) Multiple Horizons

In this section, we illustrate two examples which admit multiple-horizon spacetime structures. Multiple-horizon structures are possible in two-dimensional gravity due to simpler field equations. Our first example is

\[ \lambda = 1 - \frac{\Lambda_1}{K^2} e^{-Kr} - \frac{\Lambda_2}{4K^2} e^{-2Kr} - \frac{\Lambda_3}{9K^2} e^{-3Kr}, \quad \phi = -\frac{K}{2}r. \]  

(26a)

This form of \( \lambda(r) \) and \( \phi(r) \) was obtained in [24,29]. Now (13), (18) and (26a) yield

\[ V = \Lambda_1 e^{2\phi} + \Lambda_2 e^{4\phi} + \Lambda_3 e^{6\phi}, \]

\[ \frac{C_0^2}{l} = C_1 + \frac{K^2}{2} + \frac{3}{4} \Lambda_2 e^{4\phi} + \left(-\frac{1}{2K^2}\Lambda_1 \Lambda_2 + \frac{8}{9}\Lambda_3\right) e^{6\phi} - \frac{1}{K^2} \left(\frac{3}{32}\Lambda_2^2 + \frac{2}{3}\Lambda_1 \Lambda_3\right) e^{8\phi} - \frac{1}{6K^2} \Lambda_2 \Lambda_3 e^{10\phi} - \frac{4}{81K^2} \Lambda_3^2 e^{12\phi} \]  

(26b)

with \( b = -1 \). Generally speaking, one can have \( 1 + \sum_{n=1} e^{-nKr} \) in (26a) by adjusting \( V(\phi) \) and \( l(\phi) \) in (13) and (18) respectively. For simplicity, we just illustrate the first three terms. Similar to case (3.2), where only the first two terms were considered, \( C \) is still given by (24d). We again set \( \sqrt{-2C_1} = \frac{\Lambda_1}{L} \) in (26b) and obtain exactly the three-horizon black hole obtained in string theory [24,29]. If one instead sets \( C_1 = 0 \), one gets a massless black hole. For the former choice \( C_0, C_1 \) and \( \Lambda_1 \) are all coupling constants in the action (2), on which the integration constants for the solution (26a) functionally depend. \( \Lambda_2 \) can be interpreted as square of an electric charge, and \( K, L \) and \( \Lambda_3 \) are coupling constants.

Regardless of whether we choose a massless black hole or not, it can be shown that

\[ e^{-Kr_i} = -\frac{3\Lambda_2}{4\Lambda_3} \left(1 - 2\sqrt{1 - \frac{16}{3}\frac{\Lambda_1 \Lambda_3}{\Lambda_2^2}} \cos \theta_i\right) \]
\[
\cos(3\theta_i) = -\left(1 - \frac{16}{3} \frac{\Lambda_1 \Lambda_3}{\Lambda_2^2}\right)^{-\frac{3}{2}} \left(1 - 8 \frac{\Lambda_1 \Lambda_3}{\Lambda_2^2} - \frac{32}{3} \frac{\Lambda_3^2}{\Lambda_2^3} \Lambda^2 \right).
\]

(26c)

Here \( r_1, r_2 \) and \( r_3 \) are the three possible roots of the cubic equation \( \lambda(r) = 0 \) in (26a). When \( \Lambda_1 = 0 \), (26c) indicates that there exists three real positive roots provided that \( 0 < \frac{K^2 \Lambda_2^2}{\Lambda_2^2} \leq \frac{3}{64} \). When all \( \Lambda_1, \Lambda_2, \Lambda_3 \) are non-vanishing, a graphical analysis gives us the following conditions on \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \) for the cubic equation \( \lambda(r) = 0 \) to admit three positive real roots. As \( r \to \infty, \lambda \to 1 \). If \( K^2 < \Lambda_1 + \frac{\Lambda_2}{9} + \frac{\Lambda_3}{3} \), then \( \lambda < 0 \) as \( r \to 0^+ \). In addition, from the equation \( \lambda' = 0 \) and the continuity of \( \lambda \), one can check that there are two turning points for \( \lambda(r) \) (one is the maxima, the other one is the minima) provided that \( 1 > \frac{16}{3} \frac{\Lambda_1 \Lambda_3}{\Lambda_2^2} > 0 \), \( \Lambda_1 > 0 \) and \( \Lambda_2 < 0 \). The above conditions on \( \Lambda_i \) yield three horizons. The Penrose diagram of such black hole with three horizons is called the 2D lattice shown in [24].

Another example in this section is the “extension” of the Reissner-Nordstrom solution. It is given by

\[
\lambda = 1 - \frac{\beta^3 \Lambda_1}{2} \frac{1}{r} - \frac{\beta^4 \Lambda_2}{6} \frac{1}{r^2} - \frac{\beta^5 \Lambda_3}{12} \frac{1}{r^3}, \quad \phi = -\ln \left(\frac{r}{\beta}\right).
\]

(26d)

This form of \( \lambda(r) \) and \( \phi(r) \) can be found in [24,29]. As usual, \( V(\phi) \) and \( l(\phi) \) can be determined from the given \( \lambda(r) \) and the invertible \( \phi(r) \). For (26d), (13) and (18) show that for \( b = -2 \) as in section (3.3), they are given by

\[
V = \Lambda_1 e^{3\phi} + \Lambda_2 e^{4\phi} + \Lambda_3 e^{5\phi},
\]

\[
\frac{C_0^2}{l} = C_1 + \frac{4}{\beta^2} e^{2\phi} - 3\Lambda_1 e^{3\phi} + \left(\frac{5}{8} \beta^2 \Lambda_1^2 - \frac{1}{3} \Lambda_2\right) e^{4\phi} + \left(\frac{1}{6} \beta^2 \Lambda_1 \Lambda_2 + \frac{1}{3} \Lambda_3\right) e^{5\phi}
\]

\[
-\frac{1}{8} \beta^2 \Lambda_1 \Lambda_3 e^{6\phi} - \frac{1}{18} \beta^2 \Lambda_2 \Lambda_3 e^{7\phi} - \frac{288}{7} \beta^2 \Lambda_3^2 e^{8\phi}.
\]

(26e)

More terms \( r^{-4}, r^{-5}\ldots \) can be added to (26d) by adjusting \( V(\phi) \) and \( l(\phi) \). \( C \) is still given by \( C = \pm \sqrt{-2C_1} \), and one can have the mass parameter related to \( \Lambda_1 \) as in (25e) in case (3.3). Now we obtain the three-horizon black hole obtained in [29]. The three possible roots for \( \lambda = 0 \) in (26d) are given by

\[
\frac{1}{r_i} = \frac{16}{3} \frac{\Lambda_2}{\Lambda_3} \left(1 - 2 \sqrt{1 - \frac{9}{2} \frac{\Lambda_1 \Lambda_3}{\Lambda_2^2} \cos \theta_i}\right)
\]

\[
\cos 3\theta_i = -\left(1 - \frac{9}{2} \frac{\Lambda_1 \Lambda_3}{\Lambda_2^2}\right)^{-\frac{3}{2}} \left(1 - \frac{27}{4} \frac{\Lambda_1 \Lambda_3}{\Lambda_2^2} - \frac{81}{4\beta^2} \frac{\Lambda_3^2}{\Lambda_2^3}\right).
\]

(26f)

If \( \Lambda_1 = 0 \), then (26f) indicates that there exist three positive roots provided that \( 0 < \frac{\Lambda_1^2}{\Lambda_2^2} \leq \frac{2}{81} \) is satisfied. If all \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \) are non-vanishing, a simple graphical analysis shows that there exist three real positive roots provided that \( 1 > \frac{9}{2} \frac{\Lambda_1 \Lambda_3}{\Lambda_2^2} > 0, A_2 < 0, \) and \( A_1 > 0 \).
Finally we illustrate an example which has only one event horizon but admits interesting causal structure. The metric is given by

\[
    ds^2 = -\left(1 - \frac{s}{1 + \cosh(Qx)}\right)dt^2 + \frac{dx^2}{\left(1 - \frac{s}{1 + \cosh(Qx)}\right)}, \quad \infty > s > 2. \tag{27}
\]

This metric was previously obtained in [30,31] for a finite \(s\) (a real number) as a solution to 2D string gravity with a non-zero tachyon field. Re-writing it in non-Schwarzchildian co-ordinates and taking the limit \(s \to \infty\) it reduces to the string metric in (3.2) in the same co-ordinates. It is believed that it solves the β-function equations exactly – this has been confirmed by explicit computation to four-loop order (see discussions in [30]). \(Q\) is a coupling constant and has an inverse length. (27) is symmetric and asymptotically flat as \(x \to \pm \infty\). Note that unlike previous sections, we have used \(x\) instead of \(r = |x|\). The horizon is located at \(Qx_H = \cosh^{-1}(s - 1)\). Note that the dilaton for the solution (27) in [30] or [31] is not invertible. Since the classical geometry of the spacetime is solely determined by the metric (27) and is not affected by the form of dilaton, we will choose an invertible dilaton, \(\phi(x) = \ln(\cosh(Qx) + 1)\), which simplifies the calculations, in order to obtain \(l(\phi)\) and \(V(\phi)\). With that form of dilaton, the potential and coupling functions are

\[
    V = sQ^2(e^{-\phi} - 3e^{-2\phi}),
\]

\[
    \frac{C_0^2}{l} = Q^2 \left(-3s + 4)e^{-\phi} + s \left(\frac{3}{2}s + 5\right)e^{-2\phi} - 2s^2 e^{-3\phi} + \frac{C_1}{Q^2} + 2\right). \tag{28}
\]

\(C\) in (15) is again given by

\[
    C = \sqrt{-2C_1}. \tag{29}
\]

If we set \(\frac{2L^2}{s} = -2C_1\), where \(\tilde{L}\) is an integration constant with an inverse length, and choose \(M_o = 0\), then (22) yields the ADM mass

\[
    m = \tilde{L}\sqrt{\frac{2}{s}}. \tag{30}
\]

which is exactly the mass formula obtained in [31]. Now \(C_0\) and \(C_1\) (or \(\tilde{L}\)) are coupling constants on which the integration constants in (28) functionally depend. Both \(R = T\) and 2D string gravity (with a tachyon field) yield the same classical black hole geometry with the same thermodynamic properties.

If we perform a co-ordinate transformation, \(Qx \to \cosh^{-1}\left(\frac{(s-2)\tilde{x} + s}{2}\right)\), where \(\beta\) is a length scale, then (27) becomes

\[
    ds^2 = 2(s - 2)\left(-f(\tilde{x})dt^2 + \frac{\beta^2 d\tilde{x}^2}{4(\tilde{x}^2 - \beta^2)}\right),
\]

\[
    f(\tilde{x}) = \left(\frac{\tilde{x} + \beta}{\tilde{x} - \beta} - \frac{2\beta}{s}\right)^{-1}, \tag{31}
\]
which was obtained in [32] with $\beta$ scaled to be 1. Using the chosen dilaton above we can obtain $l(\phi)$ and $V(\phi)$ in the $\tilde{x}$ co-ordinate. In fact, (27) only covers part of the spacetime of (31) [30]. It was shown in [30] that the maximally extended exact black hole geometry corresponds to an infinite sequence of asymptotically flat regions linked by wormholes (it is reminiscent of the causal structure of the Reissner-Nordstrom black hole, except that the singularities are removed). Such an unusual causal structure has no analog in $D > 2$ GR but it is possible in the present $R = T$ theory ($D \to 2$ limit of 4D GR).

Before ending this section, we want to comment that the above approach in constructing exact static solutions is very straightforward in two spacetime dimensions due to the fact that (10) has only one degree of freedom and the field equations are much simpler than their higher dimensional counterparts. In higher dimensions, such an approach is possible [19] in the $FRW$ single scalar field cosmology since the metric only has one dynamical degree of freedom, namely the scale factor $a(t)$. However, higher dimensional static spherically symmetric solutions in any gravitational theory are typically not obtained in the approach adopted here. We also note that $l(\phi)$ in all the above solutions is generally a function of $r$. It may have different signs for different ranges of $r$ outside (inside) the event (cosmological) horizon. When $l(\phi)$ is negative (positive), $\psi$ behaves as a massless scalar (ghost). Although the overall quasilocal energy and mass (or $ADM$) are positive and finite for all $r$, the ambiguity of the transitions scalar $\leftrightarrow$ ghost and the stability of all the above static solutions are open questions.

4 Equations of Motions of Test Particles

The third term in (19), when performing the integration over $r$, becomes $\frac{1}{2}(\lambda^'\ln\lambda - \int \ln\lambda d\lambda^')$. At an event or cosmological horizon $\lambda = 0$ and so $\psi$ generally diverges at the horizon (see (23c), (24c), (25c) for examples). One might argue that since the action (2) and its equations of motion do not explicitly depend on $\psi$, its divergence may not be physically pathological. However the term $l(\phi)(\nabla\psi)^2$ in (2) (which is equal to $C_0\psi^'$ from (19) in metric (10) with spatial $r$), also diverges at the horizons. In this section, we will concentrate our discussions only on event horizons and show that a particle with “spectator charge” (i.e. one linearly coupled to $\psi^'$ in the coordinates we use) will encounter an infinite potential barrier before hitting the event horizon. In four dimensions the static “Bekenstein black hole” considered in [33] is a solution of coupled Einstein-Maxwell conformal scalar field equation and encounters a similar problem. The conformal scalar, which appears explicitly in the action, diverges at the event horizon. However, it was shown in [33] that the infinity in the scalar field need not to be physically pathological due to completeness of the trajectory and the absence of both infinite tidal forces and an infinite potential barrier at the event horizon for a test particle linearly coupled to the conformal scalar. In the $2D$ black hole metrics considered at the end of the last section, the string dilaton obtained in [30] or [31] also diverges at the event horizon.

In the following, we will briefly investigate the motion of a test particle linearly coupled to $\psi$ and $\psi^'$. For simplicity, in the following we will restrict ourselves to the double-horizon metrics in cases (3.2) and (3.3) considered in the last section. These two cases have the properties that as $r \to r_h$, $\lambda(r) \to 0$ and $\lambda^'(r)$ and $\lambda^{''}(r)$ are both finite. When $r \to \infty$,
\( \lambda(r) \to 1 \) and all its derivatives vanish. For the case (3.1), as \( r \to \infty \), \( \lambda(r) \) diverges. However, a local Rindler transformation may be applied to rewrite the asymptotic metric in Minkowski form, as we have mentioned previously. For simplicity, we will use the \( r-t \) coordinate in (10) in the following and consider cases (3.2) and (3.3) only. Every conclusion drawn from cases (3.2) and (3.3) should be valid for the case (3.1). We will closely follow the computations done in [33] since almost all its computations in the \( r-t \) co-ordinates for the radial motion of a test particle are also valid in our two-dimensional cases.

The simplest parameter-invariant for a particle of a unit rest mass coupled linearly to a function \( F(\psi, l(\phi)(\nabla \psi)^2) \) is

\[
S = -\int (1 + \eta F) \left( -g_{\alpha\beta} \frac{dx^\alpha}{d\gamma} \frac{dx^\beta}{d\gamma} \right) \frac{1}{2} d\gamma, \tag{32}
\]

where \( \eta \) is a coupling constant, \( \gamma \) is a parameter and \( x^\alpha(\gamma) \) is the trajectory of the particle. The term proportional to 1 is the action for a free particle; that proportional to \( \eta F(\psi, l(\phi)(\nabla \psi)^2) \) is the interaction term for a particle with spectator charge. We will choose \( F(\psi, l(\phi)(\nabla \psi)^2) = \frac{l(\phi)(\nabla \psi)^2}{C_0} \) or \( \psi \) in our discussion.

Since the action is invariant under a change of parameter \( \gamma \), we are free to impose a condition on \( -g_{\mu\nu} U^\mu U^\nu \), \( U^\mu = \frac{dx^\mu}{d\tau} \) to fix the choice of \( \gamma \). A useful choice which simplifies the equation of motion is

\[
(1 + \eta F)^2 = -g_{\mu\nu} U^\mu U^\nu = \left( \frac{d\tau}{d\gamma} \right)^2, \tag{33}
\]

where \( \tau \) is the proper time of the test particle. Now \( \gamma \) is an affine parameter. The equation of motion which follows from variation of \( S \) with respect to \( x^\mu \) now becomes

\[
\frac{D^2 x^\nu}{d\gamma^2} = -\eta(1 + \eta F) \nabla^\nu F, \tag{34}
\]

where \( \frac{D^2 x^\nu}{d\gamma^2} = U^\mu \nabla_\mu U^\nu \). The energy \( E \) is a constant of the motion and is given by

\[
E = \lambda \left( \frac{dt}{d\gamma} \right), \tag{35}
\]

from which (33) becomes

\[
\frac{dr}{d\gamma} = \pm \left( E^2 - \lambda(1 + \eta F)^2 \right)^{\frac{1}{2}}. \tag{36}
\]

Here + and − refer to outgoing and ingoing trajectories respectively. Using (33) and (36), we get

\[
(1 + \eta F) \frac{dr}{d\tau} = \pm \left( E^2 - \lambda(1 + \eta F)^2 \right)^{\frac{1}{2}}. \tag{37}
\]

Now we set \( F = \psi' \) since this is the term in the action which diverges at the horizon. (19) indicates that as \( r \to \infty \), \( \psi' \to -\frac{2b\phi'}{C_0} + \frac{C_1}{C_0} \). When the test particle starts its journey from \( r \to \infty \), (36) becomes

\[
\frac{dr}{d\gamma} = - \left( E^2 - \left( 1 + \frac{\eta}{C_0} \left( -2b\phi'^2 + C_1 \right) \right)^2 \right)^{\frac{1}{2}}. \tag{38}
\]
In all the black hole solutions $\phi'$ is finite as $r \to \infty$. We can always choose some proper value of $\frac{\eta}{C_0}$ such that the right hand side of (38) is real and negative for a given $E$. As $r \to r_h^+$, the event horizon, $\psi' \to \frac{1}{2Cc_0\lambda}(\lambda' + 2C_1)$. Thus (36) now becomes

$$\frac{dr}{d\gamma} = -\left(E^2 - \frac{\eta^2}{4C_0^2\lambda}(\lambda' + 2C_1)^2\right)^{\frac{1}{2}}. \tag{39}$$

It is obvious that the right hand side of (39) is now complex and diverging. It is clear that as $r \to \infty$, the function inside the bracket of square root of the right hand side of (36) can be chosen to be positive, while as $r \to r_h^+$, it must be negative and diverging. Since the function is continuous for $\infty > r > r_h$, there must exist a point $r_o$ between infinity and the event horizon which is a turning point; that is,

$$\frac{dr}{d\gamma} = 0, \quad r = r_o. \tag{40}$$

(40) shows that with respect to the affine parameter $\gamma$, there is something like an infinite potential barrier at $r = r_o$. The particle must stop at $r_o$ and rebound due to the fact that it is not travelling along a geodesic (i.e. it is coupled to $\psi'(r)$). The non-gravitational force generated by $\psi'(r)$ in the right hand side of (34) (which vanishes if $\eta = 0$ in which case the trajectory is a geodesic) produces a repulsive effect. The effect is strong enough to prevent the particle from reaching the horizon. One can set $\psi(r)$ to be a constant to get rid of this repulsive effect but we cannot get a closed form for the dilaton and its potential. More precisely, we can still have an asymptotically flat double-horizon black hole metric but the form of $\phi(r)$ and $V(\phi)$ are not closed; that is, the solution is not exact (see discussions on section 3).

So far our approach has been to couple the particle to quantities which are diverging at the event horizon and see what happens. We set $F = \psi'(r)$ (recall that $C_0\psi' = l(\phi)(\nabla\psi)^2$) in the above. Apart from being the simplest form, $\psi'(r)$ explicitly appears in the field equations and it diverges at the horizon. Although the resultant equations of motion of the test particle do not follow from the field equations, it is natural to consider $F = \psi'(r)$ from the above point of view. In addition $\psi(r)$, although it does not appear explicitly or implicitly in the field equations, diverses at the horizon. Hence we consider dependence of $F$ on $\psi$ as well.

If one sets $F(\psi, \psi') = \psi$, the situation changes. As $r \to r_h^+$, $C\psi \to \lambda'\ln\lambda$. Thus (36) now becomes

$$\frac{dr}{d\gamma} \to -\left(E^2 - \frac{\eta^2}{C_0^2\lambda}(\lambda' + 2C_1)^2(\ln\lambda)'^2\right)^{\frac{1}{2}}. \tag{41}$$

Since $\lambda'(r)$ is finite at $r = r_h$, and $\lambda(\ln\lambda)^2 \to 0$ as $r \to r_h^+$, we have $\frac{dr}{d\gamma} \to -E$ at the horizon. As $r \to \infty$, $\psi(r)$ in (24c) and (25c) diverge (except when $2C_1 = -K^2$ in (24c) and $C_1 = 0$ in (25c)). However, we can always let the particle starts from a finite spatial point $r_1$ such that by choosing some proper value of $\frac{\eta}{C_0}$, $\frac{dr}{d\gamma}$ at $r_1$ is negative and real. (24c) and (25c) indicate that none of the $\psi(r)$ are monotonic decreasing/increasing functions of $r$. We no longer conclude that $\frac{dr}{d\gamma}$ will become zero at a point $r_o$ between $r_1$ and the
event horizon. It is also not possible to exactly solve the equation \( \frac{dτ}{dγ} = 0 \) in (36) to get the \( r_o \) due to the presence of the logarithm and arctangent functions. Despite the lack of information about the existence of an infinite potential barrier, we shall now show that even if there is no such barrier, particles linearly coupled to \( ψ \) will encounter an infinite tidal force at the horizon in a finite proper time.

A criterion for a physical singularity is that the relative tidal acceleration between pairs of nearby trajectories of some particles should become unbounded as the trajectories approach the point in question. Consider a family of trajectories \( x^α(γ, ν) \) parametrized by \( γ \), an affine parameter along the trajectories, and \( ν \), a parameter labelling the trajectories. The equation of motion is given by (34). Defining \( y^α = \frac{∂x^α}{∂ν} \), which is the separation vector, the relative acceleration is given by

\[
\frac{D^2y^μ}{dγ^2} = R^μ_{αβσ}U^αU^βy^σ + y^β\nabla_βN^μ,
\]

where \( N^β = -η(1 + ηF)\nabla^βF \) from the right hand side of (34). Equation (42) is the generalization of the geodesic deviation equation for particles with an additional force term [33]. Using (33) and (34), (42) becomes

\[
\frac{D^2y^μ}{dτ^2} = R^μ_{αβσ} \frac{dx^α}{dτ} \frac{dx^β}{dτ} y^σ \\
-η(1 + ηF)^{-1}(y^β\nabla_β\nabla^μF + η(1 + ηF)^{-1}y^β\nabla_βF\nabla^μF + (\nabla_βF)\frac{dx^β}{dτ} \frac{dy^μ}{dτ}).
\]

We need to express the components of \( \frac{D^2y^μ}{dτ^2} \) in an instantaneously comoving inertial frame to calculate the local relative acceleration. The dyad \( e^μ_{a} \) (where \( a \) is the dyad index) is constructed by taking \( e^μ_{0} \) as the two-velocity of an infalling timelike geodesic in the metric (10), and taking the other component orthogonal to this basis vector and parallel transported along the geodesic. Also, it is required that the dyad be instantaneously comoving with the moving test particle at \( r = r_p \), where \( r_1 ≥ r_p ≤ r_h \). The dyad satisfying the above conditions is given by [33]

\[
e^μ_{0} = \pm \frac{E}{λ}(1 + ηF)^{-1}\delta^μ_{t} - (E^2(1 + ηF)^{-2} - λ)^{\frac{1}{2}}\delta^μ_{r},
\]

\[
e^μ_{1} = \frac{1}{λ}(E^2(1 + ηF)^{-2} - λ)^{\frac{1}{2}}\delta^μ_{t} ± E(1 + ηF)^{-1}\delta^μ_{r},
\]

where \( e^μ_{a}e^ν_{b}g_{μν} = n_{ab} \) and \( e^μ_{0}\nabla_μe^ν_{b} = 0 \). This set of basis vectors has the same expression as the \( r - t \) part of the tetrads considered in [33] where only radial motion was considered. The components of the relative acceleration in the inertial frame are now given by the right hand side of (43) provided that the indices are dyad indices \( (a, b = 0, 1) \), and the contractions are performed with the Minkowski metric \( n_{ab} \). The components \( R^α_{bcd} \) are bounded since the geometries are all well-behaved at the event horizon for the metrics discussed in the last section. \( \frac{dx^α}{dτ} = δ^α_{0} \) by definition so it is finite as well.
Now we need to calculate \((1 + \eta F)^{-1} \nabla_a F\) and \((1 + \eta F)^{-1} \nabla_a \nabla_b F\) in (43). When \(F = \psi(r)\), using (24c) and (25c) it is lengthy but straightforward to check that these quantities all diverge as \(r \to r_h\). Using (37) we get near the horizon that \(\delta \tau \to 0\) for two points \(r_h + \delta r\) and \(r_h\) (except for extremal cases, where an infinite throat exists). This indicates that an ingoing test particle coupled to \(\psi\) will hit the horizon within a finite proper time provided there is no infinite potential barrier outside the horizon.

To summarize, a test particle linearly coupled to either \(\psi'(r)\) or \(\psi(r)\) may encounter an infinite potential barrier outside the horizon or an infinite tidal force at the horizon for an asymptotically flat spacetime which admits a double- or even multiple-horizon structure. However, one may argue that apart from being the simplest form, the action (32) is an artifact since it does not follow from any field equation and therefore the non-geodesic motion and singularities are “artificial” (unlike the analogous action in [33], where for \(F = \psi\) (the conformal scalar) the action is singled out by its conformal invariance properties due to the conformal invariance of the Einstein-Maxwell-conformal scalar equations). Since any form of equation of motion does not follow from the field equations, one has the freedom to choose a special form \(F(\psi, \psi')\) (still diverging at the horizons) in which a test particle with spectator charge encounters no singularities throughout its motion. We have shown that at least some forms of \(F = \psi(r)\), \(\psi'(r)\) yield the two kind of infinities mentioned above for a test particle, and shall not pursue this issue further.

5 Hawking Temperature and Entropy

An important thermodynamic quantity in a static black hole solution is the Hawking temperature \(T_H\). The existence of a meaningful temperature presupposes the existence of both an event horizon and a well-defined value of the surface gravity at the horizon and is given by [25]

\[
4\pi T_H = \lambda'(r_h).
\]

The entropy \(S\) associated with a two-dimensional black hole is not as straightforwardly obtained as for higher-dimensional cases, as the event horizon has no area. It may be deduced from the Noether charge technique [34] or alternatively from the black hole analog of the thermodynamic equation \(dm = TdS\) when all other integration constants are fixed [25]. So the entropy is given by integrating the following equation

\[
\frac{\partial S}{\partial m} = \frac{1}{T_H}.
\]

Note that \(T_H\) and \(S\) are quantities measured at spatial infinity and (46) does not hold for an extremal black hole in which \(T_H = 0\) (the entropy for an extremal black hole and a finite-space formulation of black hole thermodynamics require seperated investigations). We are going to calculate \(T_H\) and \(S\) in several examples. For the case (3.1c), with the choice of mass in (23f), (45) yields the temperature,

\[
4\pi T_H = \frac{m}{3} \frac{1 - \frac{4}{3} Lm + \sqrt{1 - \frac{4}{3} Lm}}{1 - \frac{2}{3} Lm + \sqrt{1 - \frac{4}{3} Lm}}.
\]
Recall that $L = \beta^3 \Lambda > 0$ in (3.1c). For the extremal case where $4Lm = 3$, $T_H = 0$. When $L = 0$, (47) reduces to the equation obtained in [25] for a vacuum black hole (apart from the factor in front of $m$). The entropy in (46) can be calculated by integrating (47):

$$S = 12\pi \left( \ln \left( \frac{2L}{L_{o} \sqrt{1 - \frac{4}{3}Lm + 1}} \right) + \ln \left( \frac{m}{M_{o}} \right) \right),$$

where $S_o = -\ln(L_{o}M_{o})$ which is an integration constant. We note that in (48) above, the second term is the entropy contribution from a vacuum black hole in [25]. The first term in (48) can be interpreted as the entropy contribution from the potential (23a).

For the case (3.2), (45) implies that

$$4\pi T_H = \frac{2m \sqrt{1 + \frac{\Lambda_2 m^2}{\Lambda_1^2}}}{1 + \sqrt{1 + \frac{\Lambda_2 m^2}{\Lambda_1^2}}},$$

where the choice of mass (24e) is used. Again $T_H = 0$ for the extremal case where $\Lambda_1^2 = -\Lambda_2 m^2$. When $\Lambda_2 = 0$ ($\Lambda_2 = 0$ and (24e) imply $l(\phi) = 0$ in (24b)), (49) reduces to the relation obtained in [26] for a Liouville field coupled to gravity. Therefore the thermodynamics of the Louville black hole in [26] is the same as the vacuum black hole obtained in [25]. When $l(\phi)$ and $\psi(r)$ are non vanishing, $T_H$ will be altered according to (49). Now the entropy in (46) is

$$S = 2\pi \ln \left( 1 - \sqrt{1 + \frac{\Lambda_2 m^2}{\Lambda_1^2}} \right) + S_o.$$

When $\Lambda_2 \to 0$, $S \to 2\pi \ln \left( \frac{m}{M_{o}} \right)$ which can further be written as $S \to 4\pi \ln \left( \frac{m}{M_{o}} \right)$ where the contribution from vanishing $\Lambda_2$ has been absorbed into $S_o$. Therefore the entropy reduces to the case considered in [26]. For the Reissner-Nordstrom case in (3.3), the thermodynamic quantities $T_H$ and $S$ were widely studied. We will not repeat the calculations here.

Finally, we comment that by adjusting $C_1$ in (18) (i.e., adjusting the theory for action (2)), one can construct alternate thermodynamic behaviours for the black hole solutions in section 3. As the final example, we consider metric (24g) with $\Lambda_2 = 0$ in case (3.2). Using (45) we get the temperature

$$T_H = K.$$

$T_H$ is independent of the mass. This result was previously obtained in 2D string black hole solutions. One sees that with the presence of $\psi(r)$ and $l(\phi)$, we can have different black hole solutions with different thermodynamical properties. Other examples of interest can be constructed. We will not discuss them further.
6 FRW Cosmology

In this section, we consider non-static solutions to action (2), namely the FRW-type cosmological solutions. 2D FRW cosmological models were previously considered in [10] for the $R = T$ theory with the matter action replaced by a general perfect fluid with the equation of state $p = (\gamma - 1)\mu$, where $p$ is the pressure and $\mu$ is the energy density. In addition, cosmological singularities were also discussed. It was shown that a singularity in the energy density can occur in both FRW and tilted models if certain energy conditions for $T_{\mu\nu}$ are satisfied. Exact scalar field cosmologies which arise from the action (2) when there are no $\psi$ and $F_{\mu\nu}$ fields in the matter action [35] have also been studied. A series of exact solutions were derived, including a non-singular one. This non-singular solution is possible since in the presence of a scalar field, certain energy conditions are violated. $R = T$ FRW cosmology can also be a useful model for studying phase transitions and topological defects in two dimensions, and such investigations may turn out be important for the understanding of some of the distinct features of 2D gravity [36].

To this end, we study FRW cosmologies which follow from the action (2) when there are non-trivial $\psi$ and $l(\phi)$. Similar to the static case, it will be shown that there are $l(\phi)$ and $V(\phi)$ which can lead to almost any desired behaviour for the metric. That means one can find the “cosmological fluid”, which is described by $\phi$ and $\psi$ in the matter action in (2), required to give a desired cosmological behaviour.

For a 2D FRW cosmology, the non-static metric is given by [7,10],

$$ds^2 = -dt^2 + a^2(t)dx^2,$$

where $a(t)$ is the cosmic scale factor. In virtue of (53), (3), (4) and (8), we get

$$\frac{2}{a} = V.$$

The dot denotes an ordinary derivative with respect to time $t$. Following similar algebraic procedures to the static case, (5) becomes

$$\frac{C_0^2}{la^2} = -\frac{C_2}{a^2} - 2b\dot{\phi}^2 - 2\left(\frac{\dot{t}}{a}\right),$$

where we have used the fact that (6) is given by

$$\dot{\psi} = \frac{C_0}{la}.$$

$C_0$ is an integration constant. It is just a length scale with a dimension of inverse length with no physical significance. (55) is in fact the 2D version of Friedmann equation. In 4D FRW cosmology, the analogous constant $C_2$ can be scaled to 0, ±1 corresponding to open flat space, closed sphere and closed hyperboloid respectively. However from (53) we see that the openness/closedness of spacetime has no effect on the evolution of the scale factor. In fact, $C_2$ has no relationship with the one-dimensional spatial geometry [10]. On the
other hand, $C_2$ in $4D$ FRW cosmology is related to the intrinsic curvature of the three-dimensional spatial geometry. However, the intrinsic curvature of the one-dimensional space is zero and thus $C_2$ “decouples” from it. We will leave it as a coupling constant that determines an integration constant in $l(\phi)$. In order to obtain an exact solution, one has to solve (54) and (55). $\dot{\psi}(t)$ in (56) can be re-written as

$$C_0 \dot{\psi} = 2 \frac{\ddot{a}^2}{a} - 2 \ddot{a} - 2b \dot{\phi}^2 a,$$

which is not necessarily integrable. None of the equations of motion and other physical quantities (e.g., energy density and pressure in (59) and (60)) implicitly or explicitly depend on $\psi(t)$. It is sufficient to calculate $\dot{\psi}(t)$ in the following cases.

Now we see that when a desired $a(t)$ and an invertible $\phi(t)$ are given, they together yield $V(\phi)$ and $l(\phi)$ in (54) and (55). We emphasize that especially in $4D$ single scalar cosmologies, this kind of approach (finding the matter to give a desired geometric state) has attracted some attention (see [19,37] for discussions) due to the fact that the metric field has only one degree of freedom (the scale factor) and it depends on one variable $t$ only, making the field equations relatively easy to handle. However with just one scalar this approach (in two or four dimensions) yields exact $V(\phi)$ and $\phi(t)$ only for certain choices of the scale factors. In two dimensions, we see from (54) and (55) that when $\psi$ and $l(\phi)$ are present, any choice of $a(t)$ and invertible $\phi(t)$ is possible.

For non-static solutions, the definition of mass (a conserved charge) is no longer meaningful. Rather we treat the cosmological fluid as a perfect fluid – this can always be done in an FRW model. When putting $T_{\mu\nu}$ in (8) in the form

$$T_{\alpha\beta} = (p + \mu) U_\alpha U_\beta + pg_{\alpha\beta},$$

where $U^\alpha$ is the normalized ($U^\alpha U_\alpha = -1$) two-velocity, the energy density ($\mu$) and pressure ($p$) are given by

$$\mu = -b \dot{\phi}^2 - \frac{1}{2} V - \frac{1}{2} l \dot{\psi}^2,$$

and

$$p = -b \dot{\phi}^2 + \frac{1}{2} V - \frac{1}{2} l \dot{\psi}^2.$$

From (54), (55) and (56), (59) becomes

$$\mu = -\left(\frac{\dot{a}}{a}\right)^2 + \frac{C_2}{2a^2},$$

and the pressure (60) can be written as

$$\mu - p = -2 \frac{\ddot{a}}{a}.$$

One sees that if $C_2 \leq 0$, then $\mu \leq 0$. The weak energy condition (WEC) is always violated. We will later see that in $4D$ FRW cosmology, $\mu(t)$ always respects the WEC even if the
analogous $C_2$ vanishes. We express the pressure in terms of $\mu - p$, which plays the role as a gravitational mass. When the strong energy condition SEC (positivity of gravitational mass) is respected, we see that $\ddot{a} < 0$.

We illustrate three interesting examples. First of all, we consider a non-singular scale factor

$$a = \left(1 + \frac{t^2}{A^2}\right)^{\frac{1}{2}}, \quad A^2 > 0. \quad (63)$$

This scale factor is an exact solution in a string-motivated single scalar-field with a two-term exponential potential cosmology in four dimensions [38]. Instead of adopting the fact that $\phi \propto \ln\left(1 + \frac{t^2}{A^2}\right)$ obtained in [38], we first consider $\phi = kt$ for simplicity and illustrating the freedom of choosing any invertible $\phi(t)$ for a given $a(t)$. Now (54) is satisfied if

$$V = \frac{2k^2}{(1 + \phi^2)^2}, \quad (64)$$

where we have set $A^2k^2 = 1$. From (55) $l(\phi)$ is

$$\frac{C_0}{\ell} = 2k^2\left(\frac{-1 + \phi^2 - b(1 + \phi^2)^2}{1 + \phi^2}\right) - C_2. \quad (65)$$

We can rewrite $a(t)$ in the form

$$a(t) = (1 + \phi^2)^{\frac{1}{2}}. \quad (66)$$

When $k = C_2 = 0$, $V(\phi)$, $\phi(t)$, $\mu(t)$ and $p(t)$ (see (67)) vanish; $a_o$ also vanishes and implying $\psi(t)$ is a constant. Thus once $\phi(t)$ is “switched off”, the spacetime becomes a 2D Minkowski space. When $\psi(t)$ is switched off, one must have $C_0 = 0$. Then (65) implies either $k = C_2 = 0$ or $\phi$ is a constant satisfying $2k^2(-1 + \phi^2 - b(1 + \phi^2)^2) = C_2(1 + \phi^2)$. In either case, the metric becomes Minkowski. Therefore switching $\phi(t)$ or $\psi(t)$ off may lead to a Minkowski spacetime. Using (61) and (62), the energy density and pressure are easily calculated to be

$$\mu = \frac{k^2t^2(-2k^2 + C_2) + C_2}{(1 + k^2t^2)^2}, \quad \mu - p = -\frac{2k^2}{(1 + k^2t^2)^2}. \quad (67)$$

$\mu(t)$ and $p(t)$ are similar to those obtained in [38] in terms of $t$ since they can be calculated from the scale factor directly in two and four dimensions (see (85) and (86) later). We will assume that $C_2 > 2k^2$ since the WEC is respected for this inequality. The SEC is always violated and therefore one gets an anti-gravitational effect. It is clear that $\mu$ and $p$ are finite for all $-\infty < t < \infty$. As $t \to \pm \infty$, $p = \mu \to \frac{-2k^2 + C_2}{k^2}$ which is the equation of state of radiation in two dimensions [7,10]. At $t = 0$, $\mu = C_2$ and $p = 2k^2 + C_2$. There is no initial curvature and density singularity.

Now if we choose $\phi = \epsilon_1\ln\left(1 + \frac{t^2}{A^2}\right)$ as in [38], $\mu(t)$ and $p(t)$ will be the same as before but with $k^2 \to A^{-2}$ in (67). $V(\phi)$ now becomes

$$V = \frac{2}{A^2}e^{-\frac{\epsilon_1}{A^2}\phi}. \quad (68)$$
Comparing with the 4D case [38], \( V(\phi) \) no longer depends on two exponential terms but just one exponential term. \( l(\phi) \) is given by

\[
\frac{C_2^2}{l} = \frac{2}{A^2} (1 - 4b\epsilon_1^2) - \frac{4}{A^2} (1 - 2b\epsilon_1^2)e^{-\frac{\phi}{C_1}} - C_2. 
\]  

(69)

Therefore, this 2D non-singular model has a similar scale factor, energy density and pressure in terms of \( t \) to that of the 4D non-singular model. However, it has just a single exponential potential term rather than two, and an additional spectator field with the non-trivial coupling function. In (63), when \( A^{-2} \neq 0 \), the birth of the universe may be viewed as a quantum tunneling effect from \( a = 0 \) (at an imaginary time), to a bounce point \( a = 1 \) (at \( t = 0 \) where \( \mu = C_2 \) and \( p = 2A^{-2} + C_2 \)), beyond which the universe evolves according to (63).

Note that without \( l(\phi) \) and \( \psi(t) \), one has to solve for \( \phi(t) \) (i.e., given a desired \( a(t) \) in (55), we no longer have the freedom to set \( \phi(t) \) to whatever we want to simplify the calculations – it must be solved for). Simple calculations show that \( \dot{\phi} \) can exactly be integrated in (55) for \( a(t) \) in (63), but the resultant \( \phi(t) \) is not invertible.

Another case of non-singular universe model is described by the following scale factor obtained in [38]:

\[
a = 1 + \frac{t^2}{2A^2}, \quad A^2 > 0. 
\]  

(70)

Again we assume that \( \phi = kt \). Now (54) yields

\[
V = \frac{4k^2}{2 + \phi^2}, 
\]  

(71)

where we have set \( A^2k^2 = 1 \). Now the coupling function \( l(\phi) \) in (55) is given by

\[
\frac{C_0^2}{l} = k^2((1 - 2b)\phi^2 - 2(1 + b) - \frac{b\phi^4}{2}) - C_2. 
\]  

(72)

The energy density and pressure in (61) and (62) become

\[
\mu = \frac{2C_2 - 4k^4t^2}{(2 + k^2t^2)^2}, \quad \mu - p = -\frac{4k^2}{(2 + k^2t^2)^2}. 
\]  

(73)

This model is less interesting than the previous one since the WEC is only respected for the range \( \sqrt{\frac{C_2}{2}} \frac{1}{k^2} > t > -\sqrt{\frac{C_2}{2}} \frac{1}{k^2} \). Note that the WEC is respected in both models for all \( t \) in four dimensions [38]. \( \mu(t) \) and \( p(t) \) are finite for all \( t \). As \( t \to \pm\infty \), \( p = \mu \to -\frac{4}{t^2} \).

When \( t = 0 \), \( \mu = \frac{C_2}{2} \) and \( p = k^2 + C_2 \). The universe is non-singular in every physical and geometrical property and the SEC is violated. The universe may quantum mechanically evolve from \( a(t) = 0 \) to \( a(t) = 1 \). All properties of this universe are similar to the previous one except the WEC is only respected at a certain range of \( t \).

Now if we choose \( \phi = \epsilon_2ln(1 + \frac{t^2}{2A^2}) \) as in the case in [37], we get

\[
V = \frac{2}{A^2} e^{-\frac{\phi}{C_2}}, \quad \frac{1}{C_2} = \sqrt{\frac{1}{2}} \frac{1}{k^2}. 
\]  

(74)
and
\[ \frac{C_0^2}{l} = \frac{2}{A^2} (1 - 2b\epsilon_2^2) e^\frac{2}{A^2} + \frac{4}{A^2} (b\epsilon_2^2 - 1) - C_2. \] (75)

Thus the potential depends only on one exponential term. The simpler \( V(\phi) \) in (68) and (74) compared with the two-term potential in four dimensions in [38] is due to the presence of \( l(\phi) \) in (65) and (72) and the simpler field equation (54) in two dimensions. If \( l(\phi) \) and \( \psi(t) \) are constant, it can be checked that \( \dot{\phi}(t) \) can still be integrable in (55) for \( a(t) \) given in (70). However, the resultant \( \phi(t) \) is not invertible. We note that a previous attempt on construction of singularity-free 2D cosmologies was considered in [38], where the authors used the same kind of action (but a different potential) they considered in [20] for a construction of a non-singular black hole metric (see the discussion in the introductory section in this paper). In the cosmological case, they obtained FRW solutions that are non-singular and asymptotically approach a dust-dominated universe at late time. However, explicit solution for the scale factor is not possible. In our cases, all \( a(t), \phi(t), l(\phi) \) and \( V(\phi) \) are exact (it can be checked that \( \psi(t) \) are exact in the above two cases as well). No approximations and asymptotic limits are taken.

Our final example is a singular universe with some interesting properties. The desired scale factor is given by
\[ a = ABt^{\frac{1}{2}} + At^{\frac{3}{2}} = At^\frac{1}{2} (B + t^{\frac{1}{2}}), \quad A, B > 0. \quad (76) \]

As \( t^{\frac{1}{2}} \ll B, a(t) \to ABt^{\frac{1}{2}} \) which is the scale factor for a radiation-dominated universe in four dimensions. As \( t^{\frac{1}{2}} \gg B, a(t) \to At^{\frac{3}{2}} \) which is the scale factor of a dust-dominated universe in four dimensions. Physically this means that for early enough times, the 2D universe behaves as a 4D radiation universe and for sufficiently late times it behaves like a 4D dust universe. Thus we have a 2D cosmological model which resembles the classical evolution (without any kind of inflation) of a 4D radiation/dust universe. We need \( \phi \) to complete our solution. For simplicity, we set \( \phi = \ln\left(\frac{t}{t_0}\right) \) and \(-4b = \frac{41}{18}\). Now, \( V(\phi) \) in (54) is given by
\[ V(\phi) = -\frac{Bt_0^{\frac{1}{2}} e^\phi}{2t_0} + \frac{4A^2 t_0^{\frac{3}{2}} e^{\frac{3}{2}\phi}}{Bt_0^{\frac{3}{2}} e^\phi + t_0^{\frac{3}{2}} e^{\frac{3}{2}\phi}}. \] (77)

\( l(\phi) \) in (55) is given by
\[ \frac{C_0^2}{l} = -\frac{5A^2B^2}{36t_0} e^{-\phi} + \frac{7A^2}{36t_0} e^{-\frac{2}{3}\phi} - C_2. \] (78)

The energy density and pressure become
\[ \mu = -\frac{(B + \frac{4}{3}t^{\frac{1}{2}})^2}{4(B + t^{\frac{1}{2}})^2 t^2} + \frac{C_2}{2A^2t(B + t^{\frac{1}{2}})^2}, \quad (79) \]
\[ \mu - p = \frac{B + \frac{8}{3}t^{\frac{1}{2}}}{2(B + t^{\frac{1}{2}}) t^2} \quad (80) \]
Both $\mu(t)$ and $p(t)$ diverge at $t = 0$. The curvature scalar diverges as well. When $B \gg t^\frac{1}{6}$, $\mu \to -\frac{1}{4t^2} + \frac{C_2}{2A^2B^2t^4}$, $p = -\frac{3}{4t^2} + \frac{C_2}{2A^2B^2t^4}$. When $B \ll t^\frac{1}{6}$, $\mu \to -\frac{4}{9t^2} + \frac{C_2}{2A^2t^4}$ and $p = -\frac{8}{9t^2} + \frac{C_2}{2A^2t^4}$. A graphical analysis of (79) shows that the $\mu < 0$ in the range $0 < t < t_w$ (one sets $\mu = 0$ in (79) and the real positive root will be the $t_w$; however due to the cubic power in $t$, one cannot generally get the expression $t_w$ explicitly in terms of $A$, $B$ and $C_2$) and $\mu > 0$ for $t_w > 0$. Thus, for this “big bang” model the WEC is respected only at a certain range of time. The SEC is not violated, making the active gravitational mass positive, and therefore, the above “big – bang $\to$ radiation $\to$ dust” singular universe possible.

Motivated by the similarities in terms of field equations between 2D and 4D FRW cosmological models, we briefly discuss a two-scalar FRW model in four dimensions. We expect that due to the more complicated nature of the field equations, it is more difficult to get exact solutions even through we follow the above approach to solve the field equations. However, we can still get an exact dust/radiation solution. The 4D action we consider is

$$S = \int d^4x \sqrt{-g} \left( R - \frac{1}{2} l(\phi)(\nabla \psi)^2 - 2(\nabla \phi)^2 - 2V(\phi, \psi) \right). \quad (81)$$

We will consider the choice $l = e^{4\phi}$ in the above action where it corresponds to the bosonic part of the low energy heterotic string action to zero order in the string tension $\alpha'$ [40]. To order $\alpha'$, higher order curvature terms such as Gauss-Bonnet curvature will be present. For simplicity, we only consider zero order in $\alpha'$ and ignore these terms. Again $\psi(t)$ is the spectator and $\phi(t)$ is the dilaton. The exact form of the dilaton-spectator potential $V(\phi, \psi)$ is not known in string theory [40]. We will derive the form of it for a desired scale factor instead of assuming it a-priori. In order to simplify the calculations, we assume $V(\phi, \psi) = V(\phi)$. The cosmological fluid (e.g. dust and radiation) is described by $\phi(t)$ and $\psi(t)$.

From (81) the field equations and conservation equations are

$$3H^2 + \frac{3C_2}{a^2} = \dot{\phi}^2 + \frac{1}{4} e^{4\phi} \dot{\psi}^2 + V. \quad (82)$$

$$3\ddot{a} = V - 2\dot{\phi}^2 - \frac{1}{2} e^{4\phi} \dot{\psi}^2, \quad (83)$$

where

$$\dot{\psi} = \frac{2C_0}{a^3} e^{-4\phi}, \quad (84a)$$

is the spectator equation and the dilaton equation is

$$\ddot{\phi} + 3H \dot{\phi} + \frac{1}{2} \frac{dV}{d\phi} - \frac{1}{8} \frac{dl}{d\phi} \dot{\psi}^2 = 0. \quad (84b)$$

The first one is the Friedmann equation. The second one is Raychaudhuri’s equation. $H = \frac{\dot{a}}{a}$ is the Hubble parameter. The third one is the spectator equation. The constant $C_2$, which has dimensions of inverse length squared has the following significance. For either a
dust- or radiation-dominated universe, when $C_2$ is positive, the universe is spatially closed and it starts with a big bang and ends at a big crunch, while a vanishing (or negative) $C_2$ implies a forever-expanding spatially open universe. In two dimensions, however, $C_2$ has no relationship with the one-dimensional spatial geometry. It only relates to the positivity of the energy density in (61). Provided that $\dot{\phi} \neq 0$, the last equation follows from the conservation of stress-energy, $\nabla^\mu T_{\mu\nu} = 0$, where $T_{\mu\nu}$ is in the perfect fluid form with energy density and pressure given by

$$\mu = \dot{\phi}^2 + \frac{1}{4} e^{4\phi} \dot{\psi}^2 + V = 3 \frac{C_2}{a^2} + 3 \left( \frac{\ddot{a}}{a} \right)^2. \quad (85)$$

$$p = \dot{\phi}^2 + \frac{1}{4} e^{4\phi} \dot{\psi}^2 - V = -\left( \frac{C_2}{a^2} + 2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right). \quad (86)$$

Comparing (85) with (61), we see that it is “easier” for the 2D models to violate the WEC for a given scale factor and $C_2$ (with arbitrary signs). If we obtain a solution of (82), (83) and (84) with $\dot{\phi} \neq 0$, the local conservation equation $\nabla^\mu T_{\mu\nu} = 0$ will necessarily be satisfied. Again $\dot{\psi}$ in (84) is not required to be integrable, since we have assumed $V(\psi, \phi) = V(\phi)$. To specify a model, we note that twice (82) plus (83) gives

$$V = 2H^2 + \frac{2C_2}{a^2} + \frac{\ddot{a}}{a} \quad \text{while the combination} \ (82)-(83) \ \text{gives}$$

$$\dot{\phi}^2 = -\frac{C_0^2}{a^6} e^{-4\phi} + \frac{C_2}{a^2} - \dot{H}. \quad (88)$$

These two equations together are equivalent to the system (82)–(84). It is generally not trivial to get exact solutions for (87) and (88). It is particularly interesting to see whether (87) and (88) can yield dust/radiation scale factors. For this purpose we try the ansatz

$$a = At^n. \quad (89)$$

Now, $H = \frac{n}{t}$, $\dot{H} = -\frac{n}{t^2}$ and $\frac{\ddot{a}}{a} = \frac{n(n-1)}{t^2}$. In order to solve (87) and (88), we choose $\phi = \alpha \ln \left( \frac{t}{A} \right)$. An exact solution can be obtained when $C_2 = 0$ and

$$(-1 + n)(1 - 9n) = \frac{4C_0^2 \beta^{4\alpha}}{A^6}, \quad 2\alpha = 1 - 3n. \quad (90)$$

From this equation, we see that we must have $1 > n > \frac{1}{9}$. The potential in (87) is given by

$$V = \frac{n(3n - 1)}{\beta^2} e^{-\frac{2}{3} \phi}. \quad (91)$$

(85) and (86) imply that $\mu = \frac{3n^2}{t^2}$ and $p = \frac{n(2-3n)}{t^2}$. When $n = \frac{1}{2}$ and $\frac{2}{3}$, one obtains radiation and dust scale factor respectively. The potentials are $\frac{1}{43\pi} e^{8\phi}$ for the radiation
and \( \frac{2}{35} e^{4\phi} \) for dust. Earlier attempts on modelling radiation/dust universe using a single scalar can be found in [19]. Here we obtain the same result but the differences are that the action (81) is inspired from low energy string theory (except that the potential is arbitrary) and we have two scalars instead of one.

Finally we comment that if we assume a general coupling function \( l(\phi) \) instead of \( e^{4\phi} \) in action (81) (i.e., we no longer consider string theory), then using (87) and (88) it is easy to see that we can have any desired \( a(t) \) and invertible \( \phi(t) \) by adjusting \( l(\phi) \) and \( V(\phi) \), similar to the 2D solutions we discussed above. More precisely, given any \( a(t) \) and an invertible \( \phi(t) \), one uses (87) and (88) to trivially obtain \( V(\phi) \) and \( l(\phi) \) respectively. Therefore this 4D model is very useful for constructing the cosmological fluid required to obtain desired features of a 4D universe. For example, it is interesting to implement the big bang model \( a(t) \) in (76) in the 4D action (81). Further discussion on this point is beyond the scope of this paper. In [19], where \( C_0 = 0 \) in (88), the authors could only solve for \( \phi(t) \) for limited choices of \( a(t) \) in (88). The reason is that \( \phi(t) \) may not be solvable, or even if solvable, is non-invertible for a desired \( a(t) \) when \( C_0 = 0 \) in (88). Nevertheless it would be of interest and to consider a two- or even multiple- scalar system as a cosmological fluid and then investigate under what conditions the desired features of our universe can be reproduced.

6 Conclusions

The common means of determining the classical metric of a given spacetime is to begin with a prescribed action (that is typically motivated by some set of fundamental physical principles and symmetry requirements) and then find exact solutions to its associated field equations for boundary conditions of physical interest.

In this paper we have taken what is essentially the inverse approach: within the context of the \((1 + 1)\) dimensional \( R = T \) theory we have shown that it is possible to find the matter required to give a desired geometry of spacetime. The matter contains the dilaton \( \phi \) and spectator \( \psi \) fields with a coupling function \( l(\phi) \) and a dilaton potential \( V(\phi) \) in (2). In the usual approach, treating \( \phi \) and \( \psi \) as the matter source does not determine the dynamics until \( V(\phi) \) and \( l(\phi) \) are known; in our approach we take the advantage of this arbitrariness to determine the \( V(\phi) \) and \( l(\phi) \) necessary to generate our desired solution. In essence we are ‘engineering’ the black hole solutions we want, as opposed to ‘discovering’ them as solutions to an \textit{a-priori} action. Although this procedure does not produce generic restrictions on \( V(\phi) \) and \( l(\phi) \), it does necessitate that the masses of the desired black holes are determined in terms of the coupling constants in the potentials, as opposed to being fully arbitrary. Such black holes are similar to discrete bound state solutions, in contrast with black holes in ordinary general relativity. The \( V(\phi) \) and \( l(\phi) \) so determined may therefore have other interesting solutions – we have not searched for these in this paper.

The black hole metrics we have considered in this paper are of physical interest insofar as they model interesting spacetime structures of physical relevance in higher dimensions or spacetimes that arise from low-energy effective string-inspired actions. Specifically, we have been able to construct exact black hole solutions with double and multiple horizons and have computed their Hawking temperature and entropy. One of the solutions resembles
the 2D string black hole with the same thermodynamics. A 2D version of the Reissner-Nordstrom solution was also constructed. However, there are two features of all the black hole solutions. As mentioned above, the solution must be such that at least some of the coupling constants in \( l(\phi) \) (or \( V(\phi) \) in some cases) functionally determine an integration constant (namely the mass parameter). Second, when a test particle is linearly coupled to \( \psi \) and \( \psi' \), it will respectively encounter an infinite tidal force at the horizon or an infinite potential barrier outside the horizon. In the 2D string black holes, no such infinities are present.

In the context of FRW cosmology, we represented the cosmological fluid by \( \phi \) and \( \psi \) again. Given any desired \( a(t) \) and invertible \( \phi \), the field equations are exactly solvable by adjusting \( V(\phi) \) and \( l(\phi) \). We illustrated two non-singular solutions which respectively resemble two 4D string-motivated cosmological solutions. A singular model which is analogous to 4D standard big-bang/radiation/dust was also constructed. In addition, we solved the equations of motion for a 4D low energy string action (81) and obtained a dust and a radiation solutions. We note that if the coupling function in (81) is left arbitrary instead of \( -\frac{1}{2} e^{4\phi} \), one can have any desired \( a(t) \) as an exact cosmological solution by adjusting the dilaton potential and the coupling functions, similar to the two-dimensional cases.

Finally we comment on two facts. First of all, due to the simplicity of gravitational field equations in two dimensions, we see that one can have two different 2D gravitational theories (\( R = T \) and string) that both yield the same classical black hole geometry and thermodynamical properties. It is tempting to see whether one can construct a new 2D gravitational theory which can reproduce the various 2D string black hole solutions but is free of dilaton divergences and setting integration constants equal to coupling constants. The new theory may serve as another toy model for investigation of black hole information loss. Second, given an asymptotically flat black hole solution which has double or multiple horizons, it is natural to ask whether or not exact mass inflation can happen. Mass inflation is a well-known phenomenon in four dimensions. More recently, mass inflation has been shown to occur in 2D and 3D string black hole solutions (see e.g [41]). For the black hole solutions in this paper, which share the same classical geometry and thermodynamics properties with the string cases, it is very difficult, if not impossible, to solve the non-static field equations in the Eddington-Finkelstein ingoing co-ordinates [42]. Even if the field equations could be solved exactly, it is difficult to interpret the mass-inflating parameter in action (2). It was claimed in [43] that mass inflation is a generic phenomenon, and does not depend on the particular details of the theory. It would be interesting to see whether or not the double-horizon black hole solutions in this paper may serve as counter-examples. We intend to relate further details elsewhere.

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