Starting control of vibrations of points of a rectangular membrane

M T Tukhtasinov and G M Abdualimova
Department of Differential Equations and Mathematical Physics, National university of Uzbekistan, Tashkent, 700174, Uzbekistan
E-mail: mumin51@mail.ru

Abstract. The problem of starting control of the vibration of points of a rectangular membrane with fixed boundaries is considered. The concept of implemented functions for these membrane points is introduced. For each function being implemented, valid controls are defined that form the control sets. The structure of the control set is studied. The problem is solved in the game setting, the initial player controls the initial position of the membrane, the second, the initial speed. The criterion of the quality of control is determined by the control standards with which the functional is built. The saddle point of the functional and the equilibrium situation of the game are found.

1. Introduction
In [1], the problem of starting control of oscillations of a rectangular membrane was considered. It is shown that by choosing suitable initial conditions, any finite number of membrane points should be forced to describe independently arbitrary trajectories for a given time. To solve this problem, the Fourier method is applied and the Riesz basis is extracted from the vector family. In [2], the problem of boundary control of vibrations of a rectilinear-diaphragm membrane with a geometric constraint is considered, with the final state of the membrane being a small neighborhood of the resting point, and an explicit form of control is given. The work [3] is devoted to the study of the optimization of boundary controls for bringing a string into a given state. The explicit type of controls that solve the tasks. In a review paper [4], questions are studied about the complete stop of the membrane, plate, the existence of a boundary control with which it is brought to rest, and also about the estimation of the transition time. In the monograph [5], the problem of stopping the vibrations of a string of finite length is considered. The possibility of bringing the membrane with boundary control to rest at a finite time is proved. In this case, the moment method is used to study this problem, the essence of which is that the question of the solvability of this problem is replaced by the question of the existence of a solution to a countable system of integral equations. In the monograph [6], the issue of the complete stop of the membrane in a finite time in the case of a control distributed over the entire surface of the membrane was considered. An upper bound is found for optimal control time. In the thesis [7], the issue of approximate and exact boundary control of vibrations of rectangular membranes and plates was considered. It is proved that membranes and plates can be brought into a neighborhood of a state of rest for a finite time with boundary control in the case of a geometric border. In [8], the problem of controlling the transverse vibrations of a homogeneous
membrane fixed at the edges was considered. An iterative process for calculating the geometric multiplicity of the natural vibrations of a flat membrane is indicated. In [9–14], control problems of oscillatory systems are considered. In the present work, the problem of starting control of system oscillations is studied in the formulation of the conflict, while the controlling parameters of the conflicting parties are the initial position and speed of the membrane.

2. Basic concepts and theorem

2.1. Basic Information

Many physical processes of oscillation are described by an equation of hyperbolic type

\[ u_{tt} - \text{div} (a(x) \nabla_x u) + c(x)u = f(x,t), \]

where \( x \in \Omega, \ 0 < t < T, \ \Omega \) is some bounded region of the space \( R^n \), \( T \) – an arbitrary fixed point in time, \( a(\cdot) \in C^1(\bar{\Omega}), \ c(\cdot) \in C(\Omega), \ a(x) \geq a_0 > 0, \ x \in \Omega, \ \Omega \) – closed areas of \( \Omega \).

Let the function \( u(x,t), \ x \in \Omega, \ 0 \leq t \leq T \), satisfy both the initial

\[ u|_{t=0} = \mu(x), \]

\[ u_t|_{t=0} = \nu(x), \]

and boundary conditions

\[ u|_{\Gamma_T} = 0 \]

or

\[ \left( \frac{\partial u}{\partial n} + \sigma u \right) |_{\Gamma_T} = 0, \]

where \( \Gamma_T = \{(x,t) : x \in \partial \Omega, \ 0 < t < T \}, \ \sigma(\cdot) \in C(\partial \Omega), \ \sigma(x) \geq 0, \ x \in \partial \Omega. \)

Problems (1)–(4) and (1)–(3), (5) for \( \sigma \equiv 0 \) ((1)–(3), ((1)–(3), (5)) are called the first and second (third) mixed boundary tasks, respectively [15].

Let \( Q_T = \{(x,t) : x \in \Omega, \ 0 < t < T \} \) – mean a cylinder, \( Q_T = \{(x,t) : x \in \Omega, \ t = \tau \} \) – section of the cylinder by the plane \( t = \tau \). Function \( u(x,t) \), belonging to the space \( W_2^{(1)}(Q_T) \) is called textit generalized solution in \( Q_T \) of the first mixed problem (1)–(4) if it satisfies the initial (2), boundary (4) conditions and the equality

\[ \int_{Q_T} (a \nabla u \nabla \eta + c u \eta - u_t \eta_t) \ dx \ dt = \int_{\Omega_0} \nu \eta \ dx + \int_{Q_T} f \eta \ dx \ dt \]

for all \( \eta(\cdot) \in W_2^{(1)}(Q_T) \) for which conditions (4) are satisfied and

\[ \eta |_{\Omega_T} = 0, \]

where \( W_2^{(1)}(Q_T) \) – is the Sobolev space.

The function \( u(x,t) \) belonging to the space \( W_2^{(1)}(Q_T) \) is called the textit generalized solution in \( Q_T \) the third (the second for \( \sigma = 0 \)) of the mixed problem (1)–(3), (5) if it satisfies the initial condition (2) and the equality

\[ \int_{Q_T} (a \nabla u \nabla \eta + c u \eta - u_t \eta_t) \ dx \ dt + \int_{\Gamma_T} a \sigma u \eta \ ds \ dt = \]

\[ \int_{\Omega_0} \nu \eta \ dx + \int_{Q_T} f \eta \ dx \ dt \]
for all $\eta(\cdot) \in W^{2}_2(Q_T)$ for which condition (6) is satisfied.

By $\varphi_1, \varphi_2, ..., \lambda_1, \lambda_2, ...$ we denote the system of generalized proper functions and their corresponding eigen values of the first mixed boundary value problem, i.e. $\varphi_k(\cdot) \in \tilde{W}^{2}_2(\Omega)$ and

$$\int_{\Omega} (a \nabla \varphi_k \nabla \eta + c \varphi_k \eta) dx = \lambda_k \int_{\Omega} \varphi_k \eta dx$$

for all $\eta(\cdot) \in \tilde{W}^{2}_2(\Omega), \; k = 1, ..., or the third (second $\sigma = 0$) mixed boundary value problem, i.e. $\varphi_k(\cdot) \in \tilde{W}^{2}_2(\Omega)$

$$\int_{\Omega} (a \nabla \varphi_k \nabla \eta + c \varphi_k \eta) dx + \int_{\partial \Omega} a \sigma \varphi_k \eta ds = \lambda_k \int_{\Omega} \varphi_k \eta dx$$

for all $\eta(\cdot) \in \tilde{W}^{2}_2(\Omega), \; k = 1, 2, ...,$

As is known [15], the system of functions $\varphi_1, \varphi_2, ...$ is an orthonormal basis in the space $L_2(\Omega)$ and $\lambda_k > 0, \; \lambda_k \to \infty$. Besides,

$$f(x) = \sum_{k=1}^{\infty} f_k \varphi_k \left( f = \sum_{k=1}^{\infty} f_k \varphi_k \right),$$

for any function $f(\cdot) \in \tilde{W}^{2}_2(\Omega) (W^{2}_2(\Omega))$, and the inequality

$$\sum_{k=1}^{\infty} |\lambda_k| f_k^2 \leq \gamma \|f\|^2_{W^{2}_2(\Omega)} \left( \sum_{k=1}^{\infty} |\lambda_k| f_k^2 \leq \gamma \|f\|^2_{W^{2}_2(\Omega)} \right)$$

where $f_k = (f, \varphi_k)_{L_2(\Omega)}, \; k = 1, 2, ... -$ Fourier coefficients of the function $f(\cdot)$ in the system $\{\varphi_k\}$, and the constant $\gamma$ does not depend on $f$.

2.2. Theorem

Theorem 1. [15]. Let $f(\cdot) \in L_2(Q_T), \; \mu(\cdot) \in \tilde{W}^{2}_2(\Omega)$ and $\nu(\cdot) \in L_2(\Omega)$ in the case of the first mixed problem (1)–(4) and $\mu(\cdot) \in W^{2}_2(\Omega)$ in the case of the third (second) mixed problem (1)–(3), (5). Then a generalized solution $u(x, t), (x, t) \in Q_T$ of the corresponding problem exists and appears to converge in $W^{2}_2(Q_T)$ near

$$\sum_{k=1}^{\infty} u_k(t) \varphi_k(x),$$

where

$$u_k(t) = \mu_k \cos \omega_k t + \frac{\nu_k}{\omega_k} \sin \omega_k t + \frac{1}{\omega_k} \int_{0}^{t} f_k(\tau) \sin \omega_k (t - \tau) d\tau,$$

$$\mu_k = (\mu, \varphi_k)_{L_2(\Omega)}, \; \nu_k = (\nu, \varphi_k)_{L_2(\Omega)}, \; \omega_k = \sqrt{\lambda_k}, \; k = 1, 2, ... .$$

In this case, the inequality

$$\|u\|_{W^{2}_2(Q_T)} \leq \gamma \left( \|\mu\|_{W^{2}_2(\Omega)} + \|\nu\|_{L_2(\Omega)} + \|f\|_{L_2(Q_T)} \right),$$

where the positive constant $\gamma$ is independent of the functions $\mu, \nu$ and $f$. 
3. Formulation of the problem and key Results

3.1. Problem Statement

It is known that the oscillation of a homogeneous rectangular membrane with fixed boundaries is described by the equation [15, 16]

\[ u_{tt} = a^2 (u_{xx} + u_{yy}) + f(x, t), \quad t > 0, \quad 0 < x < l, \]  

(7)

with boundary

\[ u(0, y, t) = u(l_1, y, t) = 0, \quad 0 \leq y \leq l_2, \quad t \geq 0, \]  

(8)

\[ u(x, 0, t) = u(x, l_2, t) = 0, \quad 0 \leq x \leq l_1, \quad t \geq 0, \]  

(9)

and initial

\[ u(x, y, 0) = \mu(x, y), \quad u_t(x, y, 0) = \nu(x, y), \quad 0 \leq x \leq l_1, \quad 0 \leq y \leq l_2 \]  

(10)

conditions where the function \( f(\cdot, \cdot) \) means the external force acting on the rectangular membrane. The control functions of the players are \( \mu(\cdot, \cdot) \in W^{(1)}_2(P) \), \( \nu(\cdot, \cdot) \in L_2(P) \) respectively.

In this paragraph, we study the conflict situation that occurs in the process described by problem (7)–(10). The control functions of the players are \( \mu(\cdot, \cdot) \in W^{(1)}_2(P) \), \( \nu(\cdot, \cdot) \in L_2(P) \) respectively.

Definition 1. The functions \( m_j(t), \quad 0 \leq t \leq T, \quad j = 1, \ldots, s \) are called realizable for points \( (x_1, y_1), \ldots, (x_s, y_s) \in P \) membranes if there are functions \( u(\cdot, \cdot), \nu(\cdot, \cdot) \) from the spaces \( W^{(1)}_2(P), L_2(P) \) respectively, such that the corresponding solution is \( u(x, y, t), \quad (x, y) \in P, \quad 0 \leq t \leq T, \) of problem (7) - (9) satisfies the equalities:

\[ u(x_j, y_j, t) = m_j(t), \quad 0 \leq t \leq T, \quad j = 1, \ldots, s. \]  

(11)

Let \( F_m, G_m \) denote the set of functions \( u(\cdot, \cdot) \in W^{(1)}_2(P), \nu(\cdot, \cdot) \in L_2(P) \), respectively, such that the solution \( u(x, y, t), \quad (x, y) \in P, \quad t \geq 0, \) of problem (7) - (9), satisfies equalities (11).

The conflict situation arises as follows: the first side seeks to minimize over the controls \( \mu(\cdot, \cdot) \in F_m \), and the second – to maximize over the controls \( \mu(\cdot, \cdot) \in F_m \), \( \nu(\cdot, \cdot) \in G_m \) of the following functional

\[ K(\mu, \nu) = \alpha \| \mu \|_1 - \beta \| \nu \|_2 + \gamma(\mu, \nu), \]  

where \( \alpha, \beta, \gamma \) are non-negative constants, \( \| \cdot \|_1, \| \cdot \|_2 \) – the norm in \( W^{(1)}_2(P) \) and \( L_2(P) \) respectively. Thus, we get a two-person game \( < F_m, G_m, K > \).

Problem 1. Given these functions \( m_j(\cdot, \cdot) \), \( j = 1, \ldots, s \) solve the game \( < F_m, G_m, K > \), i.e. find – the situation of equilibrium of the game, saddle points, and the value of the function (12).

3.2. Key Results

Before solving this problem 1, we study the structure of the sets \( F_m \) and \( G_m \).

Let \( \alpha, \beta (\beta \neq 0) \) be two real numbers. We say that they are commensurable (incommensurable), despite the fact that the ratio \( \alpha/\beta \) is a rational (irrational) number.

Further, we say that a pair of numbers \( (\alpha_1, \alpha_2) \) is commensurate with a pair of numbers \( (\beta_1, \beta_2) \), if either \( \alpha_1, \beta_1 \) is a number, or \( \alpha_2, \beta_2 \) numbers are comparable, or \( \alpha_1, \beta_1 \) and \( \alpha_2, \beta_2 \) numbers are comparable at the same time. In the future, assume that \( T \geq \max \left( \frac{2l_1}{a}, \frac{2l_2}{a} \right) \).

Lemma 1. For each of the sets \( F_m \) and \( G_m \) to contain only one element, it is necessary and sufficient that at least one of the pairs of numbers \( (x_1, y_1), \ldots, (x_s, y_s) \) was incommensurable with a pair of numbers \( (l_1, l_2) \).
Necessity. Suppose the contrary, i.e. all pairs of numbers \((x_1, y_1), \ldots, (x_s, y_s)\) are commensurate with a pair of numbers \((l_1, l_2)\), but despite this the sets \(F_m\) and \(G_m\) contain at most one element.

Using twice the method of separation of variables, one can obtain a solution to problem (7)–(10) in the following form [15]

\[
\begin{align*}
  u(x, y, t) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \mu_{n,m} \cos(\omega_{n,m} at) + \nu_{n,m} \frac{1}{\omega_{n,m}} \sin(\omega_{n,m} at) + \right. \\
  &\left. \frac{4}{\omega_{n,m}} \int_0^t f_{n,m}(\tau) \sin(\omega_{n,m} a(t - \tau)) d\tau \right) \sqrt{\frac{4}{l_1l_2}} \sin \frac{n\pi}{l_1} x \sin \frac{m\pi}{l_2} y,
\end{align*}
\]

(13)

where \(\omega_{n,m} = \sqrt{\left(\frac{n\pi}{l_1}\right)^2 + \left(\frac{m\pi}{l_2}\right)^2}\) are multiples of the numbers \(q = q_1 \cdots q_k\) and \(r = r_1 \cdots r_\ell\), generally speaking, are arbitrary numbers (here \(q\) is the least common multiple of rational denominators numbers of the form \(\frac{x_1}{l_1}\), \(r\) is the least common multiple of the denominators of rational numbers of the form \(\frac{y_i}{l_2}\), \(k + \ell = s\), because for those values of the indices \(n, m\) in the corresponding term in (13) the values of the factors outside the bracket are equal to zero, which implies the non-uniqueness of the elements of the sets \(F_m\) and \(G_m\). This contradiction proves the necessity of the lemma.

Sufficiency. Suppose that at least one of the pairs of numbers \((x_1, y_1), \ldots, (x_s, y_s)\) is incommensurable with \(l_1, l_2\) and let this be for the prostate will be \((x_1, y_1)\), then obviously

\[
\sin \left(\frac{\pi n}{l_1} x_1\right) \sin \left(\frac{\pi m}{l_2} y_1\right) \neq 0, \ n, m = 1, 2, \ldots
\]

Due to this, the realizability of \(m_1(\cdot)\) and the linear independence of the functions \(\sin(\omega_{n,m}at), \cos(\omega_{n,m}at), \ n, m = 1, 2, \ldots\) the coefficients \(\mu_{n,m}, \nu_{n,m}\) from (14) for \(j = 1\). This implies the uniqueness of the initial functions \(\mu(\cdot, \cdot), \nu(\cdot, \cdot)\). Lemma 1 is proved.

Now consider the case when each of the sets \(F_m, G_m\) contains more than one element. By Lemma 1, this is equivalent to all pairs of numbers \((x_1, y_1), \ldots, (x_s, y_s)\) commensurate with a pair of numbers \((l_1, l_2)\). There may be various situations, for the prostate we consider the case when \(\frac{x_1}{l_1}, \ldots, \frac{x_s}{l_1}\) rational, \(\frac{y_1}{l_2}, \ldots, \frac{y_s}{l_2}\) irrational. Further, \(q\) denotes the least common multiple of the celebrities of the numbers \(\frac{x_1}{l_1}, \ldots, \frac{x_s}{l_1}\). Let \(\tilde{\mu}(\cdot, \cdot) \in W^{(1)}(0, \frac{l_1}{q}) \times (0, \frac{l_2}{r})\), \(\tilde{\nu}(\cdot, \cdot) \in L^2\left(0, \frac{l_1}{q}\right) \times (0, \frac{l_2}{r})\). Consider the functions of the form:

\[
\mu(x, y) = \begin{cases}
  \tilde{\mu}(x, y), & 0 < x < \frac{l_1}{q}, \ 0 < y < l_2 \\
  -\tilde{\mu}\left(\frac{2l_1}{q} - x, y\right), & \frac{l_1}{q} < x < \frac{2l_1}{q}, \ 0 < y < l_2
\end{cases}
\]

\[
\nu(x, y) = \begin{cases}
  \tilde{\nu}(x, y), & 0 < x < \frac{l_2}{r}, \ 0 < y < l_2 \\
  -\tilde{\nu}\left(\frac{2l_2}{r} - x, y\right), & \frac{l_2}{r} < x < \frac{2l_2}{r}, \ 0 < y < l_2
\end{cases}
\]


\begin{equation}
\nu(x, y) = \begin{cases} 
\bar{\nu}(x, y), & 0 < x < \frac{l_1}{q} \quad 0 < y < l_2 \\
-\bar{\nu}(\frac{2l_1}{q} - x, y), & 0 < x < \frac{2l_1}{q}, \; 0 < y < l_2.
\end{cases}
\end{equation}

Next, we consider these functions continued with the period \( \frac{2l_1}{q} \), along the x axis for the segment \([0, l_1]\).}

\textbf{Lemma 2.} The sets \( F_m \) and \( G_m \) contain functions, the difference of any two of which has the form (15) respectively (extended with the period \( \frac{2l_1}{q} \)).

\textbf{Proof.} Let \( x_i = \frac{p_i}{q_i}, \; i = 1, ..., s, \; \mu^0(\cdot, \cdot), \; \mu^1(\cdot, \cdot) \in F_m, \; \nu^0(\cdot, \cdot), \; \nu^1(\cdot, \cdot) \in G_m. \) By \( u^0(x, y, t), \; u^1(x, y, t) \) we denote the solutions to problem (7)–(10) with \( w \) from wind wow - wow -

If we denote the difference \( u(x, y, t) = u^0(x, y, t) - u^1(x, y, t) \), then we have

\[
u(x, y, t) = 0, \; 0 \leq t \leq T, \; j = 1, ..., s.
\]

Hence, for \( 0 \leq t \leq T, \; j = 1, ..., s \) we have

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( (\mu^0_{n,m} - \mu^1_{n,m}) \cos(\omega_{n,m} at) + (\nu^0_{n,m} - \nu^1_{n,m}) \frac{1}{\omega_{n,m} a} \sin(\omega_{n,m} at) + \right.
\]

\[
\frac{4}{\omega_{n,m} a} \int_0^t f_{n,m}(\tau) \sin(\omega_{n,m} a(t - \tau)) d\tau \right) \frac{1}{l_1l_2} \sin \frac{n\pi}{l_1} x_j \sin \frac{m\pi}{l_2} y_j = 0.
\]

Due to the fact that the system of functions \( \sin(\omega_{n,m} at), \cos(\omega_{n,m} at), \; n, m = 1, 2, ..., \) are linearly independent on \([0, T]\), from the last equality we have \( (\mu^0_{n,m} - \mu^1_{n,m}) \sin \frac{n\pi}{l_1} x_j =
\]

\( (\nu^0_{n,m} - \nu^1_{n,m}) \frac{1}{\omega_{n,m} a} \sin \frac{n\pi}{l_1} x_j = 0 \; j = 1, ..., s, \; n, m = 1, 2, ..., \). From here, there can be either \( \mu^0_{n,m} \neq \mu^1_{n,m}, \) or \( \nu^0_{n,m} \neq \nu^1_{n,m} \) if and only if \( \frac{n\pi}{l_1} x_j = 0, \) for all \( j = 1, ..., s, \) and this is possible only for \( n = q, 2q, ..., \). Therefore, the functions \( \mu^0(\cdot, \cdot), \mu^1(\cdot, \cdot) \) \( (\nu^0(\cdot, \cdot), \nu^1(\cdot, \cdot)) \) differ apart from each other when expanding them into Fourier series only with coefficients \( \mu_{n,m} (\nu_{n,m}), \)

for \( n = q, 2q, ..., \) i.e. on functions of the form \( \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} c_{k,m} \sin \frac{k\pi}{l_1} x_j \sin \frac{m\pi}{l_2} y_j, \; 0 \leq t \leq T. \) But such expansions have the sum of a function of the form (15). Lemma 2 is proved.

\textbf{Theorem 2.} Problem 1 has a solution:

\[
\mu_0(x, y) = \begin{cases} 
M(x, y), & 0 < x < \frac{l_1}{q}, \; 0 \leq y \leq l_2, \\
M_1(x, y) - M(x, y), & \frac{l_1}{q} < x < 2\frac{l_1}{q}, \; 0 \leq y \leq l_2, \\
M_2(x, y) - M_1(x, y) + M(x, y), & 2\frac{l_1}{q} < x < 3\frac{l_1}{q}, \; 0 \leq y \leq l_2, \\
\vdots \\
M_{q-1}(x, y) - M_{q-2}(x, y) + ... + (-1)^{q-1} M(x, y), & (q - 1)\frac{l_1}{q} < x < l_1, \; 0 \leq y \leq l_2,
\end{cases}
\]
\[ \nu_0(x, y) = \begin{cases} 
N(x, y), & 0 < x < \frac{l_1}{q}, \ 0 \leq y \leq l_2, \\
N_1(x, y) - N(x, y), & \frac{l_1}{q} < x < \frac{2l_1}{q}, \ 0 \leq y \leq l_2, \\
N_2(x, y) - N_1(x, y) + N(x, y), & \frac{2l_1}{q} < x < \frac{3l_1}{q}, \ 0 \leq y \leq l_2, \\
\ldots \\
N_{q-1}(x, y) - N_{q-2}(x, y) + \ldots + (-1)^{q-1}N(x, y), (q-1)\frac{l_1}{q} < x < l_1, \ 0 \leq y \leq l_2, 
\end{cases} \]

Where

\[ M(x, y) = \sum_{i=1}^{q-1} \left( \frac{-1}{q} \right)^{q-1-i} M_{qi}(x, y), \quad N(x, y) = \sum_{i=1}^{q-1} \left( \frac{-1}{q} \right)^{q-1-i} N_{qi}(x, y), \]

\( M_i(\cdot, \cdot), N_i(\cdot, \cdot), i = 1, \ldots, q - 1 \) known functions expressed in terms of the functions \( m_j(\cdot), f_j(\cdot, 0), j = 1, \ldots, s \) and their derivatives.

**Proof.** Let \((x_0, y_0)\) be one of the pairs \((x_i, y_i), i = 1, \ldots s.\) Then from the solution of problem (7)–(10) of the form (13), it is easy to obtain the following relation

\[ u(x_0 + x, y_0, t) + u(x_0 - x, y_0, t) = \dot{\bar{m}}(t + \frac{x}{a}) + \bar{m}(t - \frac{x}{a}) + N(x, y_0, t), 0 \leq x \leq x_0, \ 0 \leq t \leq T. \] (16)

Here

\[ \bar{m}(t) = m(t) - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4}{\omega_{n,m} a} \int_0^t f_{n,m}(\tau) \sin(\omega_{n,m} a (t - \tau)) d\tau \sqrt{\frac{4}{l_1 l_2}} \sin \frac{n \pi}{l_1} x_0 \sin \frac{m \pi}{l_2} y_0, \]

\[ N(x, y, t) = 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{4}{\omega_{n,m} a} \int_0^t f_{n,m}(\tau) \sin(\omega_{n,m} a (t - \tau)) d\tau \times \]

\[ \sqrt{\frac{4}{l_1 l_2}} \sin \frac{n \pi}{l_1} x_0 \cos \frac{n \pi}{l_1} x \sin \frac{m \pi}{l_2} y, 0 \leq t \leq T, \ 0 \leq x \leq x_0. \]

Note that the function \( \bar{m}(t), \ 0 \leq t \leq \frac{2l_1}{a} \) can be periodically extended to the whole line \((-\infty, \infty)\) with the period \(\frac{2l_1}{a}\).

By virtue of equalities (9), from (16) for \( t = 0 \) we have

\[ \mu(x_0 + x, y_0) + \mu(x_0 - x, y_0) = \bar{m}(x/a) + \bar{m}(-x/a) + N(x, y_0, 0), \] (17)

\[ \nu(x_0 + x, y_0) + \nu(x_0 - x, y_0) = \dot{\bar{m}}(x/a) + \dot{\bar{m}}(-x/a) + N'(x, y_0, 0). \] (18)

It follows from Lemma 1 that if \( \frac{x_1}{l_1}, \ldots, \frac{x_s}{l_1} - \) are rational numbers and \( \frac{y_1}{l_2}, \ldots, \frac{y_s}{l_2} - \) are irrational numbers, then the sets \( F_m \) and \( G_m \) contain more than one element. Noting that \( q \) means the least multiple of the denominators of the numbers \( \frac{x_1}{l_1}, \ldots, \frac{x_s}{l_1} - \), it is easy to establish that \( \frac{l_1}{q} - \) is the largest uniform step in dividing the segment \([0, l_1]\) passing through the points \( x_1, \ldots, x_s \).
Using the obvious arguments, we can get \( \mu(x, y_0) = M(x, y_0), \) \( \nu(x, y_0) = N(x, y_0), \) for \( x = \frac{l_1}{q}, \frac{2l_1}{q}, \ldots, (q - 1) \frac{l_1}{q} \), where \( M(x, y_0), N(x, y_0), \) \( 0 \leq x \leq l_1 \) – well-known functions expressed using the functions \( \bar{m}(\cdot), N(\cdot, \cdot, \cdot) \) (see (17), (18)) and their derivatives. We have the relations

\[
\mu(x, y_0) + \mu(2 \frac{l_1}{q} - x, y_0) = M_1(x, y_0), \quad \mu(2 \frac{l_1}{q} - x, y_0) + \mu(2 \frac{l_1}{q} + x, y_0) = M_2(x, y_0), \ldots
\]

\[
\mu((q - 1) \frac{l_1}{q} - x, y_0) + \mu((q - 1) \frac{l_1}{q} + x, y_0) = M_{q-1}(x, y_0),
\]

\[
\nu(x, y_0) + \nu(2 \frac{l_1}{q} - x, y_0) = N_1(x, y_0), \quad \nu(2 \frac{l_1}{q} + x, y_0) = N_2(x, y_0), \ldots, \nu((q - 1) \frac{l_1}{q} - x, y_0) + \nu((q - 1) \frac{l_1}{q} + x, y_0) = N_{q-1}(x, y_0), 0 \leq x \leq \frac{l_1}{q},
\]

where \( q - 1 \) is odd (for even \( q \), a similar study is carried out).

Let \( \mu(\cdot, y_0) \in F_m \), then we have

\[
\int_0^{l_1} \mu^2(x, y_0) dx = \frac{l_1}{q} \int_0^{l_1/q} \mu^2(x, y_0) dx + \frac{2l_1}{q} \int \mu^2(x, y_0) dx + \ldots + \frac{l_1}{q} \int (q - 1) \frac{l_1}{q} \mu^2(x, y_0) dx = \int \mu^2(x, y_0) dx
\]

Hence

\[
\int_0^{l_1} \mu^2(x, y_0) dx = \frac{l_1}{q} \int_0^{l_1/q} \mu^2(x, y_0) dx - 2 \frac{l_1}{q} \int (M_1(x, y_0) - M_2(x, y_0) + M_1(x, y_0) + 
\]

\[
\ldots + M_1(x, y_0)) \mu(x, y_0) dx + \frac{l_1}{q} \int M(x, y_0) dx, \quad \text{where} \ M(x, y_0) = M_1^2(x, y_0) + (M_2(x, y_0) - M_1(x, y_0))^2 + \ldots + (M_{q-1}(x, y_0) - M_{q-2}(x) + \ldots + (-1)^q M_1(x, y_0))^2. \] If the last equality is rewritten as

\[
\int_0^{l_1} \mu^2(x, y_0) dx = \frac{l_1}{q} \int_0^{l_1/q} (q \mu^2(x, y_0) - 2(M_1(x, y_0) - M_2(x, y_0) + M_1(x, y_0) + 
\]

\[
\ldots + M_1(x, y_0)) \mu(x, y_0) + M(x, y_0)) dx, \quad \text{it will become clear that by solving the problem}
\]

\[
\min \| \mu(\cdot, y_0) \|, \quad (\nu(\cdot, y_0) = \nu(\cdot, y_0), \quad \mu(\cdot, y_0) \in F_m, \quad \nu(\cdot, y_0) \in G_m)
\]

are functions

\[
\mu^0(x) = \sum_{i=1}^{q-1} \frac{(-1)^{q-1-i} M_{qi}(x, y_0)}{q}, \quad \nu^0(x, y_0) = \sum_{i=1}^{q-1} \frac{(-1)^{q-1-i} N_{qi}(x, y_0)}{q},
\]

for \( x \in \left[ \frac{0}{l_1}, \frac{l_1}{q} \right] \) are determined from relations (19).

We verify the following equalities

\[
(\mu^0, \nu^0) = (\mu, \nu), \quad \mu(\cdot, y_0) \in F_m, \quad \nu(\cdot, y_0) \in G_m.
\]


notice, that

\[
  I_1 \int_0^1 \mu^0(x, y_0) \nu^0(x, y_0) \, dx = \frac{l_1}{l_1/q} \int_0^{l_1/q} \left[ \int_0^1 \left( M_1(x, y_0)N_1(x, y_0) + (M_2(x, y_0) - M_1(x, y_0)) \right) N_2(x, y_0) - \right.

N_1(x, y_0) + \cdots + (M_{q-1}(x, y_0) - M_{q-2}(x, y_0) + \cdots + (-1)^q M_1(x, y_0) \times

(1 - 1) M_2(x, y_0) = (M_1(x, y_0) - M_1(x, y_0)) N_1(x, y_0) + (M_2(x, y_0) - M_1(x, y_0)) + \mu^0(x, y_0) \right) \left( N_{q-1}(x, y_0) - N_{q-2}(x, y_0) + \cdots + (-1)^q N_1(x, y_0) \right) \, dx

Further, given that \( \nu(\cdot, y_0) \in G_m \) we have

\[
  I_1 \int_0^1 \mu^0(x, y_0) \nu(x, y_0) \, dx = \frac{l_1}{l_1/q} \int_0^{l_1/q} \left[ \int_0^1 \left( (q \mu^0(x, y_0) - M_1(x, y_0) + M_2(x, y_0) - M_3(x, y_0) + \ldots \right) \right.

+ (M_1(x, y_0) - M_{q-2}(x, y_0) + \cdots + (-1)^q M_1(x, y_0)) \mu +

(M_2(x, y_0) - M_1(x, y_0)) N_1(x, y_0) + (M_2(x, y_0) - M_1(x, y_0) + \mu^0(x, y_0)) \times

(1 - 1) M_2(x, y_0) = (M_1(x, y_0) - M_1(x, y_0)) N_1(x, y_0) + (M_2(x, y_0) - M_1(x, y_0)) + \mu^0(x, y_0) \right) \left( N_{q-1}(x, y_0) - N_{q-2}(x, y_0) + \cdots + (-1)^q N_1(x, y_0) \right) \, dx

Since the expression in the first bracket (the coefficient before \( \mu(x, y_0) \)) is equal to zero, then after simple transformations we have

\[
  I_1 \int_0^1 \mu^0(x, y_0) \nu(x, y_0) \, dx = \frac{l_1}{l_1/q} \int_0^{l_1/q} \left[ \int_0^1 \left( M_1(x, y_0)N_1(x, y_0) + (M_2(x, y_0) - M_1(x, y_0)) \right) \right.

N_2(x, y_0) = (M_1(x, y_0) - M_{q-2}(x, y_0) + \cdots + (-1)^q M_1(x, y_0)) \left( N_{q-1}(x, y_0) - N_{q-2}(x, y_0) + \cdots + (-1)^q N_1(x, y_0) \right) \, dx

In the same way, we obtain an expression of the form (21) for the pair

\[
(\mu(x, y_0), \nu^0(x, y_0), \mu(\cdot, y_0) \in F_m. Now, bearing in mind these expressions, we obtain equality

(20), which proves Theorem 2.

Remark 1. In other cases of commensurability or incommensurability of pairs of numbers

\( (x_1, y_1), \ldots, (x_s, y_s) \) with a pair of numbers \( (l_1, l_2) \) you can get similar results.

Remark 2. The implemented functions \( m_j(t), 0 \leq t \leq T; \ j = 1, \ldots, s \) can be found, for example, experimentally, i.e., observation over time \( T \), the laws of oscillation of the points

\( (x_1, y_1), \ldots, (x_s, y_s) \), or their isolation from the solution of the corresponding problem.

References

[1] Avdonin S A, Ivanov S V and Yo I 1990 Automatika 6 68-71
[2] Akulenko L D 1981 Prir. mat. i mez. 45(6) 135–145
[3] Ilyin B A and Moiseev E I 2005 Dokl.RAN 400(5) 587–591
[4] Lions J L 1988 SIAM Review 1 1–168
[5] Butkovsky A G 1975 Control methods for systems with distributed parameters Moskow: Nauka
[6] Chernous’ko L F, Ananievsky I M and Reshilin S A 2006 Methods of control of nonlinear mechanical systems Moskow: Fizmatlit
[7] Romanov I V 2011 Diss. For the candidate of physical and mathematical sciences Moskow States university
[8] Kopets M M 2014 Kybernetics and Computer Engineering 177 28–42
[9] Okhezin S P 1992 Izv. RAN Texnicheskaya kibernetika 1 204–206
[10] Satimov N Yu and Tukhtasinov M 2004 Math. Notes 75(5) 669–675
[11] Tukhtasinov M 2003 Computational Technologies 8(6) 93–97
[12] Tukhtasinov M 1998 Uzbek Mathematical journal 3 73–80
[13] Tukhtasinov M 2004 Ukrainian Mathematical Journal 56(1) 228–238
[14] Chabakauri G D 2001 Differential Equations 37(12) 1655–1663
[15] Mikhailov V P 1976 Differential equations in partial equations (Moskow: Nauka)
[16] Mandelstam L I 1972 Lectures on the theory of oscillations (Moskow: Nauka)