Probe Branes, Time-dependent Couplings and Thermalization in AdS/CFT

Sumit R. Das\textsuperscript{a} \footnote{e-mail: das@pa.uky.edu}, Tatsuma Nishioka\textsuperscript{c} \footnote{e-mail: nishioka@hep-th.phys.s.u-tokyo.ac.jp} and Tadashi Takayanagi\textsuperscript{a} \footnote{e-mail: tadashi.takayanagi@ipmu.jp}

\textsuperscript{a}Institute for the Physics and Mathematics of the Universe (IPMU), University of Tokyo, Kashiwa, Chiba 277-8582, Japan
\textsuperscript{b}Department of Physics and Astronomy, University of Kentucky, Lexington, KY 40506, USA \textsuperscript{4}
\textsuperscript{c}Department of Physics, Faculty of Science, University of Tokyo, Tokyo 113-0033, Japan

Abstract

We present holographic descriptions of thermalization in conformal field theories using probe D-branes in $\text{AdS} \times S$ space-times. We find that the induced metrics on $D_p$-brane worldvolumes which are rotating in an internal sphere direction have horizons with characteristic Hawking temperatures even if there is no black hole in the bulk AdS. The AdS/CFT correspondence applied to such systems indeed reveals thermal properties such as Brownian motions and AC conductivities in the dual conformal field theories. We also use this framework to holographically analyze time-dependent systems undergoing a quantum quench, where parameters in quantum field theories, such as a mass or a coupling constant, are suddenly changed. We confirm that this leads to thermal behavior by demonstrating the formation of apparent horizons in the induced metric after a certain time.
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1 Introduction and Summary

The AdS/CFT correspondence \cite{1,2,3,4} has been useful in understanding time-dependent processes on both sides of the duality. On the gravity side, some insight has been obtained for time-dependent gravitational backgrounds like space-like singularities using the gauge theory dual \cite{5,6,7}. However, the correspondence has been most successful in the exploration of non-equilibrium properties of strongly coupled field theories at finite temperature and/or finite density using their gravity duals. The significant results in this area include the well known predictions for linear response functions \cite{8,9,10,11}. In recent years, however, this approach has been extended to the non-linear domain as well, e.g. in the derivation of non-linear fluid dynamics from gravity \cite{12} and in the study of thermalization in terms of the dual process of black hole formation \cite{13,14,15}.

A particularly interesting class of non-equilibrium problems involves time-dependent couplings or masses which pass through a critical point either at finite temperature or at zero temperature \cite{15}. This problem has experimental relevance in cold atom physics and has been studied theoretically in two extreme limits. In the first limit, a parameter is changed quickly through a critical point (a quantum quench) and the subsequent time evolution is studied \cite{16,17,18,19,20,21,22}. In the other limit, the parameter changes slowly across a critical point \cite{23}. In both cases, dynamical quantities have universal signatures of the critical point. There are, however, few conventional analytical tools to investigate such problems. It would be clearly interesting if gauge-gravity duality can be applied to such situations.

In this paper we take a small step in this direction. We investigate the dual bulk dynamics of a class of conformal field theories with time-dependent couplings, starting in an initial vacuum state. In known field theories, this is an interesting problem even when the couplings do not pass through a critical point. For example, if we start with a free massive field theory in its vacuum state and suddenly change the mass to a different value, the resulting real time correlation functions become thermal at late time \cite{17,19,20,21,22}. Similar behavior can be established in some interacting field theories as well \cite{19,22}, but the tools for investigating such phenomena in strongly coupled theories are rather limited.

When the field theory has a gravity (as opposed to full fledged string) dual, a time-dependent coupling corresponds to exciting a non-normalizable mode of the dual bulk field and calculating the time evolution. This is quite similar to the investigation of cosmological backgrounds in \cite{5}, forced fluid dynamics \cite{24} or black hole formation \cite{13,14}. However the class of field theories we consider are considerably simpler, and may have different applications. These are defect or flavored conformal field theories resulting from probe
D-branes in a background space-time of the form of $\text{AdS}_m \times S^n$. Such probe branes have been used to model flavor physics and quantum critical phenomena: this "top-down" approach is complementary to the "bottom-up" approach involving the physics of charged black holes in AdS. One positive feature of this top-down approach is that in this case the dynamics of the probe branes describe the strongly coupled dynamics of a known field theory. Furthermore, since the gravitational background is not altered in the probe limit, the entire dynamics is given by the world-volume dynamics of the probe brane. Solving this dynamics is usually easier than solving Einstein equations.

In the following, we will find nontrivial time-dependent classical solutions of several kinds of probe branes which represent time-dependent couplings of the dual defect conformal field theories. We find that the induced worldvolume metric on a large class of such classical solutions have apparent horizons characterized by a temperature in spite of the absence of black hole horizons in the AdS bulk metric. In fact, small fluctuations of the worldvolume fields around these solutions behave like fields in space-times defined by this induced metric. From the point of view of the dual field theory, this means that an initial vacuum state can evolve into a thermal state in the presence of time-dependent couplings. The emergence of apparent and event horizons on the worldvolume is somewhat similar to the emergence of an acoustic horizon for phonons around a supersonic fluid flow. Indeed, this kind of phenomenon can occur in a large class of non-linear field theories.

It is very important that in our probe approximation the dual field theory should be regarded as an open system. Consider such a setup in $\text{AdS}_5 \times S^5$ dual to the four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) which couples to a defect or flavor conformal field theory (see Fig. 1). Though the energy supplied by the time-dependent coupling first excites the defect or flavor sector, this energy will finally dissipate into the $\mathcal{N} = 4$ SYM sector due to the interactions between each other. In the bulk description this means that energy can flow from the probe brane to the bulk gravitational degrees of freedom. Indeed we find that typically the worldvolume apparent horizon evolves into an event horizon similar to that of a static black hole space-time only when the time derivative of the classical solution approaches a constant. In such a case, the long time correlators of fluctuations become thermal with the Hawking temperature of the event horizon. This

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5 Here we would like to mention earlier works on the presence of horizons in the induced metric. A general analysis of appearance such horizons on accelerated D-branes has been done in [36]. World-sheet horizons of F-string has already been discussed in [37–40]. In [41], the presence of horizons on time-dependent D-branes in flat space-time has been analyzed. In [42], horizons on D-branes from accelerated observers have been studied in pure AdS space-time.
can be explicitly demonstrated for the cases where the fluctuations can be decoupled. For most cases, these fluctuations are coupled in a non-minimal way. However, one expects that correlators will still inherit the thermal behavior implied by the background metric.

Figure 1: A schematic diagram for our non-equilibrium systems in the AdS/CFT. Here we concentrate on the example of a probe D7-brane in $AdS_5 \times S^5$ discussed in section 4. The other cases can be interpreted in the same way after obvious modifications.

In summary, our holographic setups can be regarded as an example of non-equilibrium steady states. One of the most important non-equilibrium properties of our systems is that there are two different temperatures: one is the bulk temperature $T_B$ which is vanishing for pure AdS and the other one is the Hawking temperature $T_H$ of the apparent horizon of the induced metric of probe D-branes.

The simplest example in our setups is a probe D-string or F-string extended along the radial direction of $AdS_5$. In the dual $\mathcal{N} = 4$ SYM theory this represents a monopole or a quark. In this case the relevant class of classical solutions can be found analytically. In these solutions, the end-point spins along a direction on the $S^5$ leading to a wave moving along the string towards the Poincare horizon. In the dual defect CFT this corresponds to a certain time-dependent coupling of the hypermultiplet fields with $\mathcal{N} = 4$ vector multiplet fields [45,46]. This generates a R-charge chemical potential only for the hypermultiplet and not for the $\mathcal{N} = 4$ SYM sector. If we start with a static string in the far past and turn on the spin, an apparent horizon develops on the worldsheet. When the value of the spin approaches a constant at late times, the induced metric coincides with that of the BTZ metric with an angular coordinate suppressed. The Hawking temperature of the event horizon is proportional to the final angular velocity $\omega$, given by simply $T_H = \frac{\omega}{2\pi}$. Fluctuations around this classical solution satisfy simple equations in this background metric leading to correlators which are thermal.
In particular, this leads to a Brownian motion of the end point. In the dual field theory this manifests itself as a Brownian motion of the monopole position as well as a Brownian motion in the internal space $SO(6)$. When the string extends all the way to the boundary, the mass of the monopole or the quark is infinite. However terminating the string on a suitable space-filling D7-brane makes the mass finite. It should be again emphasized that this is a non-equilibrium situation: it is only the monopole or quark which behaves thermally while the other $\mathcal{N} = 4$ degrees of freedom are still at zero temperature. $1/N$ corrections will eventually thermalize the latter - but that requires going beyond the probe approximation. Since we have an open system, energy can flow into the vector multiplet sector. In a closed system, we typically expect thermal behavior if the time dependence vanishes at late times. In contrast, in our case we need a steady time dependence to achieve thermality since energy can flow out of the defect CFT at a constant rate. Such Brownian motion behavior has been observed for fluctuations around static strings in the background of AdS black holes, i.e. when the field theory is already at some finite temperature [47,48]. In our case, the parent field theory is at zero temperature - an effective temperature is produced by a time-dependent coupling.

The physics of other probe D-branes is quite similar, but requires a combination of analytical and numerical tools. For D3, D5 and D7-branes in the $AdS_5 \times S^5$ geometry, we find classical solutions with angular momenta dual to R-charges which lead to induced worldvolume metrics with horizons. For a probe D3-brane, which is dual to a $(2 + 1)$-dimensional defect CFT, we calculate the conductivity by turning on a worldvolume gauge field. There is a finite AC conductivity which arises from dissipation into the $\mathcal{N} = 4$ degrees of freedom. A similar behavior has been observed earlier using probe D-branes at finite temperature, i.e. in AdS black hole backgrounds [29,33]. In contrast, we observe this behavior in a zero temperature field theory where a time-dependent coupling effectively produces a thermal state. In the D7-brane case, we can rotate a D7-brane in the standard D3-D7 system [26] dual to four-dimensional $\mathcal{N} = 4$ SYM with a massive flavor hypermultiplet. This setup has been already discussed in [49] (see also [50] for a similar analysis for D3-D5 system). This rotation excites a R-charge only in the hypermultiplet sector. We find that for a sufficiently large R-charge, the induced metric of the probe D7-brane has a horizon which surrounds a cusp-like singularity on the worldvolume. This structure looks very similar to the structure of the standard black holes in general relativity. As a final example, we holographically describe a time-dependent mass in the defect CFT sector dual to a probe D5-brane. We numerically show that an apparent horizon is formed under a time-dependent change of mass, thus describing thermalization of the dual CFT. In free field theory, it is known that the effective temperature after the
quench depends on the momenta of each modes since they are decoupled with each other in free field theory \[19,21\]. However, our holographic result predicts that this picture will be completely changed for the strongly interacting gauge theories. In the latter case the different momentum modes observe a common temperature.

It is also an intriguing problem to associate an entropy to such a D-brane with thermal horizon. We will show that one holographic candidate for such a entropy is divergent in general except in the lowest-dimensional case i.e. D1-brane case. We suggest that this might be interpreted as an entanglement entropy when we trace out the $\mathcal{N} = 4$ SYM sector, though we would like to leave its detail for future work.

This paper is organized as follows: In section 2, we consider probe Dp-branes which are extended in AdS space-time and rotating in a sphere direction. We show that an apparent horizon is always formed in the Poincare AdS space. We calculate the energy dissipation in such systems. In section 3, we study the thermal property of the rotating Dp-brane solutions constructed in section 2. In particular, we will study Brownian motion and thermal correlation functions for a probe D1-brane and the AC conductivity for a D3-brane. In section 4, we analyze the rotating D7-brane with a mass for flavor hypermultiplet. We will show that for a sufficiently fast rotation, the induced metric of the probe D7-brane has a horizon. In section 5, we will give a brief analysis of a holographic dual of a quantum quench caused by a rapid change of a mass parameter in a defect CFT. We consider a particular case of probe D5-brane and numerically show the appearance of an apparent horizon in the time-dependent induced worldvolume metric.

## 2 Worldvolume Horizons and Mini Black Holes from Rotating D-branes in AdS

Typical setups of AdS/CFT correspondence are $AdS_{d+2} \times X^q$ backgrounds in string theory or M-theory, where $AdS_{d+2}$ is the $(d + 2)$-dimensional AdS space in Poincare coordinates and $X^q$ is a $q$-dimensional Einstein manifold $[1,4]$. In this paper we mainly assume the simplest case $X^q = S^q$. We can write their metric as follows (we set the AdS radius $R = 1$ throughout this paper, for simplicity of presentation.)

$$ds^2 = -r^2 dt^2 + r^2 \sum_{i=1}^{d} dx_i^2 + \frac{dr^2}{r^2} + (d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\Omega_{q-2}^2).$$  \hspace{1cm} (2.1)

This clearly includes the celebrated example of the $AdS_5 \times S^5$ in type IIB string theory by setting $d = 3$ and $q = 5$, which is dual to the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory.
The D-brane systems we would like to consider now are Dp-branes rotating in the $S^q$ directions. These rotations are dual to non-vanishing R-charges in CFT$_{d+1}$. The equations of motion require that such D-branes should rotate along the largest circles in $S^q$. Below we will show that such D-brane systems can be regarded as mini black holes in AdS spaces and they describe locally thermal vacua in a probe limit.

Since the analysis of rotating D1-branes is much simpler we will first study this and then proceed to the analysis of general Dp-branes.

All the rotating solutions that will be constructed in this section and their relation to dual gauge theory are summarized in Fig. 2.

![Figure 2: We present the schematic pictures of (a) a thermal D1-brane solution extended in Poincare AdS$_5$, (b) a suspended D1-brane solution in Poincare AdS$_5$, (c) a D1-brane solution in global AdS$_5$, and their CFT duals: (a'),(b') and (c'). The D1-branes are all rotating along a great circle in the $S^5$.](image-url)
2.1 Rotating D1-branes

Consider a probe D1-brane (or a F-string equivalently owing to S-duality) in AdS$_{d+2} \times S^q$ \([2.1]\). The D1-brane extends in both \(t\) and \(r\) direction, infinitely and localized at \(x_i = 0\). In addition, it is spinning in the \(\varphi\) direction. Therefore, the D1-brane world-volume is specified by the function
\[
\varphi = \varphi(t, r),
\]
(2.2)
at constant \(\theta(\equiv \theta_*)\).

In this setup, the DBI action of the D1-brane is
\[
S_{DBI} = -T_{D1} \int dt dr L ,
\]
(2.3)
where \(T_{D1} = \frac{1}{2\pi\alpha'g_s}\) is the tension of a D-string. The equation of motion reads
\[
\frac{\partial}{\partial r} \left( \frac{r^2 \sin^2 \theta_* \varphi'}{L} \right) = \frac{\partial}{\partial t} \left( \frac{\sin^2 \theta_* \varphi}{r^2 L} \right).
\]
(2.5)
Consider solutions of the form
\[
\varphi = \omega t + g(r) , \quad g'(r) = \sqrt{\frac{1 - \omega^2/r^2}{Ar^4 - r^2}} ,
\]
\[
\theta_* = \frac{\pi}{2} ,
\]
(2.6)
(2.7)
where \(A\) is an arbitrary constant. The Lagrangian density for this solution is
\[
L = \sqrt{\frac{A(r^2 - \omega^2)}{Ar^4 - 1}} .
\]
(2.8)
In order to avoid a negative sign inside the square root, we need to require
\[
A = \frac{1}{\omega^2} .
\]
(2.9)
In this way, we find the rotating D1-brane is given by the solution
\[
\varphi = \omega t - \frac{\omega}{r} + \varphi_0 ,
\]
(2.10)
up to an integration constant \(\varphi_0\).

The induced metric on this D1-brane worldsheet is
\[
ds^2 = -(r^2 - \omega^2)dt^2 + 2\frac{\omega^2}{r^2}dt dr + \left( \frac{1}{r^2} + \frac{\omega^2}{r^4} \right) dr^2 .
\]
(2.11)
We can rewrite this by defining a new time coordinate
\[
\tau \equiv t - \frac{1}{r} - \frac{1}{2\omega} \log \frac{r - \omega}{r + \omega}, \tag{2.12}
\]
as follows
\[
ds^2 = -(r^2 - \omega^2)d\tau^2 + \frac{dr^2}{r^2 - \omega^2}. \tag{2.13}
\]
Notice that \(\tau\) becomes identical to \(t\) as we approach the boundary \(r \to \infty\).

Interestingly, the induced metric (2.13) coincides with the BTZ black hole with the angular coordinate suppressed. Therefore it has a horizon at \(r = \omega\) and we can read off its Hawking temperature \(T_H\)
\[
T_H = \frac{\omega}{2\pi}, \tag{2.14}
\]
by Wick-rotating \(\tau\) into a Euclidean time (see the picture (a) in Fig. 2).

If we take into account the back-reaction of this solution to the supergravity background, it is natural to expect that such D1-branes produce a very small black hole in the bulk AdS localized in the \(R^d\) direction (we may call it a mini black hole).

This shows that the rotating D1-brane describes a thermal object with temperature \(T_H\) in the dual CFT. Notice that the bulk of AdS is at zero temperature. If we concentrate on the most important example of \(AdS_5 \times S^5\), our system is dual to \(\mathcal{N} = 4\) super Yang-Mills theory coupled to an infinitely heavy monopole. The \(\mathcal{N} = 4\) gauge theory itself is at zero temperature, while the monopole is at finite temperature \(T_H\). Therefore such systems are in non-equilibrium steady states.

The rotating D1 brane solution corresponds to a time dependent coupling in the \(\mathcal{N} = 4\) theory coupled to hypermultiplets living on the zero dimensional defect. The D1-D3 system is 1/4-BPS and the D1-D3 open strings lead to the two complex scalars \((Q, \tilde{Q})\) of hypermultiplets which belong to the fundamental and anti-fundamental representations of the color \(SU(N)\) gauge group. Let us express the three complex adjoint scalar fields in the \(\mathcal{N} = 4\) super Yang-Mills by \((\Phi_1, \Phi_2, \Phi_3)\). These correspond to cartesian coordinates in the transverse \(C^3\) composed of \((r, \Omega_5)\) where \(\Omega_5\) represents the 5-sphere. We choose \(\Phi_3\) such that its phase rotation describes the one in the \(\varphi\) direction and that \(\theta = \pi/2\) is equivalent to \(\Phi_1 = \Phi_2 = 0\). We now argue that the time dependent coupling term which corresponds to a uniformly rotation D1-brane is given by
\[
\int dt \left[ \overline{Q} \left[ \text{Im}(\Phi_3 e^{-i\omega t}) \right]^2 Q + \tilde{Q} \left[ \text{Im}(\Phi_3 e^{-i\omega t}) \right]^2 \overline{\tilde{Q}} \right]. \tag{2.15}
\]
To see this consider going to the Coulomb branch where \(\langle \Phi_3 \rangle \neq 0\). This corresponds to separating out some number \(k\) of the original \(N\) D3-branes. Apart from generating
masses for the off diagonal vector multiplet fields, the coupling in (2.15) generates masses for appropriate components of the hypermultiplet fields. Consider the simplest situation where only one D3-brane is separated in the \( \theta = \pi/2 \) hyperplane, so that we only have \( \langle \Phi_3 \rangle = \text{diag}[\Phi_3, 0, 0, \cdots] \). In this case the topmost component of \( Q \) would become massive. This mass should be equal to the length of the shortest open string joining the D1-brane with this separated brane (upto a factor of the string tension). The rotating D1 described above is described by \((0, 0, z e^{i\varphi(t)} = z e^{i\omega t})\) in the transverse \( C^3 \), where \( z \) takes any real value. The shortest distance is then given by \( |\text{Im}(\Phi_3 e^{-i\omega t})| \), leading to a mass of the scalar fields \((Q, \tilde{Q})\) exactly as predicted by (2.15). In the above discussion we have taken \( \langle \Phi_1 \rangle = \langle \Phi_2 \rangle = 0 \) to extract the main time-dependent part of the interaction. However the result can be generalized to a generic point in the Coulomb branch, where other terms involving the \( \Phi_1 \) and \( \Phi_2 \) would be involved.

As a further check on consistency, note that the operator in (2.15) has dimension 1. As we will see in a later section the field \( \varphi \) is a massless minimally coupled scalar on the worldsheet. The usual AdS/CFT formula (now applied to \( AdS_2 \) on which the D1-brane wraps) relating bulk masses to conformal dimensions then predict that the dual operator should have dimension 1.

We thus see that the AdS/CFT correspondence shows that if we consider a system with such time-dependent interactions, the system gets thermal (see Fig. [1]).

### 2.2 General F1/D1 solutions

It turns out that there are more general purely right-moving or purely left-moving solutions of D1 or F1 branes in the background of the class of metrics which include \( AdS \times S \) as well as black branes or black holes in Poincare or global coordinates. Consider a metric of the form

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + \sum_{i=1}^{d} dx_i^2 + (d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\Omega_{q-2}^2) , \tag{2.16}
\]

and introduce the outgoing and ingoing Eddington-Finkelstein coordinates

\[
u = t - \int \frac{dr}{f(r)} , \quad v = t + \int \frac{dr}{f(r)} . \tag{2.17}
\]

The DBI action then becomes

\[
S = -T_{D1} \int d\nu d\tau f(r) \left[ 1 - \frac{4}{f(r)} \partial_u \varphi \partial_v \varphi \right]^{1/2} . \tag{2.18}
\]
The equation of motion then becomes

\[
\partial_u \partial_v \phi + \frac{2}{L} \partial_v \phi \partial_u \left( \frac{\partial_u \phi \partial_v \phi}{f(r)} \right) + \frac{2}{L} \partial_u \phi \partial_v \left( \frac{\partial_u \phi \partial_v \phi}{f(r)} \right) = 0 ,
\] 

(2.19)

where

\[
L = 1 - \frac{4}{f(r)} \partial_u \phi \partial_v \phi .
\]

(2.20)

It is easy to see that any \( \phi \) which satisfies either \( \partial_u \phi = 0 \) or \( \partial_v \phi = 0 \) is a solution of (2.19). In particular, solutions which are functions of \( v \) alone may be thought of the retarded effect of a boundary value of \( \phi \).

The induced metric produced by such a retarded solution is given by

\[
ds_{ind}^2 = -f(r) du dv + (\partial_v \phi)^2 dv^2 = 2 dr dv - [f(r) - (\partial_v \phi)^2] dv^2 .
\]

(2.21)

This is a two-dimensional AdS Vaidya metric which shows the formation of an apparent horizon at \( f(r) = (\partial_v \phi)^2 \), provided this equation has a solution for real \( r \). The dual CFT description is captured by the interaction (2.15) with \( \omega t \) is replaced with the general function \( \phi(t) \) as is clear from the derivation in (2.15).

Special cases of the background metric (2.16) include Poincare patch \( AdS_5 \), with \( f(r) = r^2 \) and black 3-branes with \( f(r) = r^2(1 - \frac{A}{r^4}) \). With the replacement of \( \sum_i (dx_i)^2 \to d\Omega_3^2 \) this can include global \( AdS_5 \) as well as AdS-Schwarzschild. We will discuss these solutions further in the following sections.

In the special case where the function \( \phi(v) \) approaches \( \phi(v) \to \omega v \) at late retarded times, the apparent horizon asymptotes to an event horizon at \( f(r) = \omega^2 \) which signals thermalization. In dual CFT, this system consists of two sectors with different temperatures.

### 2.3 Rotating Dp-branes

We can extend the previous system to the rotating higher-dimensional Dp-branes. Consider a rotating Dp-brane for \( p \leq d + 1 \) in \( AdS_{d+2} \times S^q \) space-time which extends in \( (t, r, x_1 \ldots, x_{p-1}) \) directions of AdS and rotates in \( \phi \) direction at \( \theta = \pi/2 \). We again assume \( \phi \) depends on only \( t \) and \( r \) directions.

The DBI action is given by

\[
S_{DBI} = -T_p \int dt dr d^{p-1}x \sqrt{1 + \left( r^2 \phi'^2 - \frac{\phi'^2}{r^2} \right)} .
\]

(2.22)
It follows that we obtain the equation of motion:

\[
\partial_t \left( \frac{r^{p-3} \dot{\varphi}}{\sqrt{1 + (r^2 \dot{\varphi}^2 - \frac{\dot{\varphi}^2}{r^2})}} \right) = \partial_r \left( \frac{r^{p+1} \varphi'}{\sqrt{1 + (r^2 \varphi'^2 - \frac{\varphi'^2}{r^2})}} \right).
\]  
(2.23)

We now consider the rotating solution with angular velocity \( \omega \) along \( \varphi \) direction, and we thus assume the following form of the solution:

\[
\varphi(t, r) = \omega t + g(r).
\]  
(2.24)

Substituting the ansatz into the equation of motion the function \( g(r) \) looks like

\[
g' = \sqrt{\frac{A^2(r^2 - \omega^2)}{r^4(r^{2p} - A^2)}},
\]  
(2.25)

where \( A \) is the integration constant, and should be set to \( A = \omega^p \) to avoid a divergence of the function \( g(r) \) at \( r = \omega \).

The induced metric on the rotating Dp-brane is

\[
\begin{align*}
\text{ds}_{ind}^2 &= -(r^2 - \omega^2)dt^2 + \left( \frac{1}{r^2} + g'^2 \right)dr^2 + 2\omega g'dtdr + r^2 d\bar{x}_{p-1}^2 \\
&= -(r^2 - \omega^2)d\tau^2 + \frac{r^{2(p-1)}}{r^{2p} - \omega^{2p}} dr^2 + r^2 d\bar{x}_{p-1}^2,
\end{align*}
\]  
(2.26)

where the new time coordinate \( \tau \) is defined as

\[
\tau = t - \int dr \frac{\omega^{p+1}}{r^2 \sqrt{(r^{2p} - \omega^{2p})(r^2 - \omega^2)}}.
\]  
(2.27)

This metric [2.26] clearly has a horizon at \( r = \omega \). After Wick rotation of \( \tau \) we can determine its Hawking temperature by demanding smoothness at \( r = \omega \):

\[
T = \frac{\sqrt{p} \omega}{2\pi}.
\]  
(2.28)

### 2.4 Back-reactions and Charge/Energy Dissipations

In the above analysis, back-reaction to the supergravity sector has been neglected since we employed the probe approximation. We would now like to see how much and when this is justified. Since the essential issues are the same, we concentrate on a rotating D1-brane located at \( x_i = 0 \).

The total energy of our rotating D1-brane conjugate to time \( t \) is given by

\[
E = \int_0^\infty dr \frac{1 + r^2 \dot{\varphi}^2}{\sqrt{1 + r^2 (\dot{\varphi}')^2 - \frac{1}{r^2} \dot{\varphi}^2}} = \int_0^\infty dr \left( 1 + \frac{\omega^2}{r^2} \right).
\]  
(2.29)
This shows that the energy density blows up at \( r = 0 \). Thus in the IR limit we cannot neglect the gravitational back-reaction. This is intuitively understood if we plot the form of D1-brane profile (2.10) at some given time in the \((r \sin \varphi, r \cos \varphi)\) plane, as in Fig. 3. Notice that the D1-brane wraps infinitely many times at \( r = 0 \). Similar behavior is present for the rotating D-brane in AdS black holes.

![Figure 3: The profile of a rotation D1-brane in AdS at some given time in the \((x^4 = r \sin \varphi, x^5 = r \cos \varphi)\) plane. The time evolution is obtained by rotating the profile.](image)

It is reasonable to conclude that the large back-reaction will result in the formation of a black hole in the bulk of AdS, centered at \( x_i = r = 0 \). The size of this black hole should grow as the energy is pumped into it from the D1-brane constantly. To obtain this energy flux we need to calculate the energy-stress tensor [51] for the probe D1-brane. We define the energy-stress tensor by

\[
T^a_b = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{ac}} g_{cb},
\]

where \( g_{ab} \) denotes the bulk metric (the index \( a \) and \( b \) runs all ten (or eleven) coordinates of our space-time). This satisfies the equation of motion

\[
\nabla_a T^a_b = 0,
\]

which is reduced to

\[
\partial_a (T^a_t \sqrt{-g}) = 0,
\]
for static space-times. This leads to the conserved energy

\[ E = \int dr \sqrt{-g} \, T^t_t. \] (2.33)

In our rotating D1, the explicit expressions for \( T^a_t \) are

\[
\sqrt{-g} \, T^t_t = T_{D1} \frac{1 + \frac{r^2}{2} \phi'^2}{\sqrt{1 + r^2 \phi'^2 - \frac{\phi'^2}{r^2}}} = T_{D1} \left(1 + \frac{\omega^2}{r^2}\right),
\]

\[
\sqrt{-g} \, T^r_t = T_{D1} \frac{r^2 \phi' \phi'}{\sqrt{1 + r^2 \phi'^2 - \frac{\phi'^2}{r^2}}} = T_{D1} \omega^2,
\]

\[
\sqrt{-g} \, T^r_r = -T_{D1} \frac{1 - \frac{\phi'^2}{r^2}}{\sqrt{1 + r^2 \phi'^2 - \frac{\phi'^2}{r^2}}} = -T_{D1} \left(1 - \frac{\omega^2}{r^2}\right). \] (2.34)

Thus we can calculate the time evolution of the total energy as

\[
\frac{dE}{dt} = \frac{d}{dt} \int dr \sqrt{-g} T^t_t = \int dr \partial_r (\sqrt{-g} T^t_t) = \sqrt{-g} T^t_t \bigg|_{r=\infty} = T_{D1} \omega^2 - T_{D1} \omega^2 = 0. \] (2.35)

Even though the total energy does not depend on time, this result shows that the energy \( \frac{dE}{dt} = T_{D1} \omega^2 \) per unit time is injected at the boundary \( r = \infty \) by some external system (reservoir) and the same energy \( \frac{dE}{dt} = T_{D1} \omega^2 \) is dissipated from the IR region into the bulk AdS as summarized in the picture \( (a') \) in Fig. 2. This dissipation from the D1-brane to the bulk AdS will create a localized black hole in AdS. The dissipation via the AdS horizon is very similar to the phenomenon called drag force, which is dual to the jet quenching [52–54].

One may wonder how we can explicitly understand the external injection of energy in our stationary solutions as we argued. To see this, let us introduce UV and IR cut off into our rotating D1-brane system so that it extends from \( r = r_{IR} \) to \( r = r_{UV}(\gg r_{IR}) \). In this setup it is clear that the energy flux \( T_{D1} \omega^2 \) is coming from \( r = r_{UV} \) and the same energy is going out at \( r = r_{IR} \). The presence of this energy flux is clear from the non-vanishing value of \( T^r_t \) in (2.34) at \( r = r_{IR} \) and \( r = r_{UV} \). At \( r = r_{IR} \), the energy is not reflected back but its back-reaction will make a localized black hole, which absorbs the injected energy. Another way to see the energy flow explicitly is to consider general time-dependent solutions presented in section 2.2. In this case we can choose the D1-brane profile such that the energy is injected like a pulse and this travels into the IR region. In this case, \( T^t_t \) gets non-vanishing at the UV cut off only during a certain time interval. Our stationary rotating D1-brane solution can be regarded as an infinitely many collections of such pluses and therefore it requires the constant energy injection at the AdS boundary.

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Similarly we can calculate the dissipation of angular momentum which is dual to a R-charge in the CFT. The angular momentum of a rotating D1 is given by

\[ Q_R = \frac{\delta S}{\delta \dot{\varphi}} = T_{D1} \int dr \frac{\omega}{r^2} , \tag{2.36} \]

which is divergent like its energy.

The emission toward the horizon \( r = 0 \) for the Poincare AdS can be estimated as

\[ \frac{dQ_R}{dt} \bigg|_{r=0} = -T_{D1} \frac{r^2 \dot{\varphi}'}{\sqrt{1 + r^2 \dot{\varphi}'^2}} \bigg|_{r=0} = -T_{D1} \omega . \tag{2.37} \]

This amount of angular momentum (=-R-charge) comes from the boundary \( r = \infty \) and flows into the horizon \( r = 0 \). For higher-dimensional rotating Dp-branes, the situations are almost the same. From the viewpoint of the boundary CFT which lives on \( R^{1,d} \), only the codimension \( d - p + 1 \) part, defined by the intersection between the AdS boundary and the Dp-brane, is thermalized at the temperature \( T_H \), while the other part is at zero temperature in the probe approximation. The energy dissipation we found in (2.35) corresponds to the energy flow and the R-charge flow from the thermal part to the cold region in the bulk of \( R^{1,d} \). Therefore the temperature of the bulk region should constantly increase. This is dual to a growing black hole once back-reaction is taken into account.

### 2.5 Rotating D-branes in Global AdS

One way to suppress a large IR back-reaction is to replace the Poincare AdS\(_{d+2} \) with the global AdS\(_{d+2} \). Again we will concentrate on D1-branes just for simplicity. Then we need to take \( f(r) = r^2 + 1 \) in the equation (2.16). The rotating D1-brane is described by the profile

\[ \varphi = \omega (t + \arctan r) . \tag{2.38} \]

Notice that the D1-brane in the global AdS pierces the center of the AdS and thus has two end points at the boundary, which are identified with the north-pole and south-pole of the boundary \( S^3 \), respectively, as shown in the picture (c) in Fig. 2. This is conveniently described by extending the range of \( r \) in (2.38) to \(-\infty < r < \infty \).

The induced metric on the worldsheet is given by

\[ ds_{ind}^2 = -(r^2 + 1 - \omega^2) d\tau^2 + \frac{dr^2}{r^2 + 1 - \omega^2} , \tag{2.39} \]

where \( \tau = t + h(r) \) and \( h'(r) = \frac{\omega^2}{(1 + r^2)(r^2 + 1 - \omega^2)} \). Therefore, a horizon exists only when \( |\omega| > 1 \). This gap is analogous to the Hawking-Page transition between the global AdS...
and the AdS black hole \[55\]. In our case, it is dual to the confinement/deconfinement phase transition of the hypermultiplets from D3-D1 strings. The Hawking temperature is given by \( T_H = \frac{\sqrt{\omega^2 - 1}}{2\pi} \).

The total energy of the D1-brane is given by

\[
E = T_{D1} \int_{-\infty}^{\infty} dr \left( 1 + \frac{\omega^2}{r^2 + 1} \right). \tag{2.40}
\]

Its divergent part which is independent of \( \omega \) can be canceled by the standard holographic renormalization. The total angular momentum becomes finite for the D1 in global AdS as given by

\[
Q_R = T_{D1} \int_{-\infty}^{\infty} dr \frac{\omega}{r^2 + 1} = \pi T_{D1} \omega. \tag{2.41}
\]

Thus we can conclude that the back-reaction of the probe D1-brane to the metric is not large even when \( \omega \) is non-zero. This is rather different from the rotating D1-brane in Poincare AdS.

Just as in the Poincare patch solutions in the previous sections, there are general purely left moving or purely right moving solutions. The solutions corresponding to the retarded effect of specified boundary conditions are of the form

\[
\varphi = \varphi(t + \arctan r), \tag{2.42}
\]

for arbitrary choice of the function \( \varphi(v) \). The R-charge current shows the behavior

\[
\frac{dQ_R}{dt} = (r^2 + 1) \varphi'\big|_{\infty} = \dot{\varphi}(t + \pi/2) - \dot{\varphi}(t - \pi/2) \equiv j_1 - j_2. \tag{2.43}
\]

This means that in the dual CFT, the current \( j_1 \) starts from the north pole \( P_1 \), it flows into the bulk of \( S^3 \) and is eventually absorbed at the south pole \( P_2 \) as the current \( j_2 \) (see the picture \((c')\) in Fig. 2). The time delay \( \Delta t = \pi \) is just the propagation of this current at the speed of light from \( P_1 \) to \( P_2 \). Notice that in this global AdS, there is no energy dissipation toward the bulk (i.e. \( \mathcal{N} = 4 \) SYM sector) at planar order, which will be because the bulk system is in the confinement phase. The current is carried by flowing R-charged particles. Also notice that the effective temperature for \( P \) and \( Q \) are \( \frac{j_1}{2\pi} \) and \( \frac{j_2}{2\pi} \).

### 2.6 Suspended Rotating D1-branes in Poincare AdS

As we have seen, in the global AdS space, the rotating D1-brane always has two end points at the boundary. This is because we cannot have a single source of charge current
in a compact space. In the Poincare AdS case, this is no longer true since the space is non-compact. However, still it is interesting to consider a system where there are two point sources $P_1$ and $P_2$ in the boundary $R^{1,3}$. In this case, there are two possibilities of rotating D1-branes. The first one is two disconnected rotating D1-branes infinitely stretching in the radial direction. As we already analyzed, this has a thermal horizon at $r = \omega$ and has the temperature $T = \frac{\omega}{2\pi}$. We can take the temperatures of two points independently, denoted by $T_1$ and $T_2$. Then the total R-charge current flow from $P_1$ to $P_2$ in the dual CFT is given by

$$\frac{dQ_R}{dt} = 2\pi T_{D1}(T_1 - T_2) ,$$

and the energy flow is

$$\frac{dE}{dt} = 4\pi^2 T_{D1}(T_1^2 - T_2^2) .$$

At the two points, the system gets thermalized and the carrier possesses the energy $\omega$ per a unit R-charge. They will strongly interact with the bulk gauge theory even though it is at zero temperature. Notice that in this case there is no correlation between two points and the physics does not depend on the distance $L$ between them.

We can calculate the R-charge conductance from (2.44). We consider the setup of type IIB string on $AdS_5 \times S^5$ and recover the radius of the AdS space $R = (4\pi g_s N \alpha'^2)^{\frac{1}{4}}$. The chemical potential at the two points are identified with $\omega_1$ and $\omega_2$. Thus we find the conductance, which is the ratio of voltage $V$ to the current $I$, is given by

$$I \frac{V}{\omega_1 - \omega_2} = T_{D1} R^2 = \frac{1}{2\pi} \frac{N}{\sqrt{\lambda}} ,$$

where $\lambda = N g_s$ is the 't Hooft coupling of the dual $\mathcal{N} = 4$ SYM. Notice that we here normalized the R-charge such that a transverse complex scalar $\Phi_3$ in the $\mathcal{N} = 4$ has +1 i.e. $R(\Phi_3) = 1$. For (1+1)-dimensional fermi liquids, the value of the conductance is quantized into $\frac{T}{2\pi}$ times integers ($T$ is the transmission probability), known as the Landauer formula [56]. On the other hand, in our case, it is quantized in the unit of $\frac{1}{\sqrt{\lambda}}$ and we may estimate $T \sim \frac{1}{\sqrt{\lambda}}$ for the $\mathcal{N} = 4$ SYM media.

The second solution is a suspended rotating D1-brane which connects the boundary two points $P$ and $Q$ as depicted as the picture (b) in Fig. 2. We describe its profile by

$$\varphi = \omega t + g(r) ,
$$

$$x \equiv x_1 = x(r) .$$

The action looks like

$$S_{DBI} = -T_{D1} \int dt dr \sqrt{K} ,$$

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where

$$K = \frac{r^2 - \omega^2}{r^2} + r^2(r^2 - \omega^2)x' + r^2g' . \tag{2.49}$$

Using the symmetry under constant translations of $x$ and $g$, we can simply reduce the equation of motion into

$$g'(r) = \frac{B(r^2 - \omega^2)}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}}, \tag{2.50}$$

$$x'(r) = \frac{1}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}},$$

where $A$ and $B$ are integration constants.

Define $r_*$ as a solution to $B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1 = 0$. The divergence of $x'$ at $r = r_*$ shows that $r = r_*$ is the turning point of the suspended D1-brane. Thus the D1-brane starts from $r = \infty$ and turns around $r = r_*$ and gets back to the boundary $r = \infty$. The length $L$ in $x$ direction of this suspended D1-brane is

$$L = 2\int_{r_*}^{\infty} \frac{dr}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}}. \tag{2.51}$$

We can find that $g(r) \simeq \text{const} - \frac{1}{Ar} + \cdots$ in the $r \to \infty$ limit. Thus the current flow is

$$\frac{dQ_R}{dt} = \frac{T_{D1}}{A} - \frac{T_{D1}}{A} = 0 , \tag{2.52}$$

and the energy flow is

$$\frac{dE}{dt} = \frac{T_{D1}\omega}{A} - \frac{T_{D1}\omega}{A} = 0 . \tag{2.53}$$

The reason we find that $\frac{dQ_R}{dt} = \frac{dE}{dt} = 0$ is that we employed the simple ansatz and if we want to violate this we need to consider truly time-dependent solutions which are too complicated. The parameters which describe this system are $\omega, A$ and $L$. Remember that in the previous solution of thermal D1-branes, we had the constraint from the regularity of the solutions and the the R-charge current is determined from the total R-charge. On the other hand, in the connected D1-brane solution, they are independent as there is no thermal equilibrium even in the flavor sector.

Moreover, because $r_* > \omega$, all of these suspended rotating D1-branes do not have any horizons in the induced metric

$$ds_{\text{ind}}^2 = -(r^2 - \omega^2)dt^2 + 2\omega g'dt'dr + \left(\frac{1}{r^2} + r^2x'^2 + g'^2\right)dr^2. \tag{2.54}$$

We can compute the time delay along the null geodesic in the induced metric similarly to the global AdS case. The induced metric on the suspended solution can be rewritten
as follows
\[ ds^2_{\text{ind}} = -(r^2 - \omega^2) \left( dt - \frac{B\omega}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}} dr \right)^2 + \frac{A^2B^3(r^2 - \omega^2)}{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1} dr^2 \]
\[ = -(r^2 - \omega^2)dudv, \quad (2.55) \]

where
\[ u = t - \int dr \frac{B(Ar^2 + \omega)}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}} , \]
\[ v = t + \int dr \frac{B(Ar^2 - \omega)}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}} . \quad (2.56) \]

Thus the time delay of massless scalar field propagating from a boundary \( Q \) to another boundary \( P \) is given by
\[ \Delta t = - \int_{r_*}^{\infty} dr \frac{B(Ar^2 - \omega)}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}} + \int_{r_*}^{\infty} dr \frac{B(Ar^2 + \omega)}{r^2\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}} \]
\[ = \int_{r_*}^{\infty} dr \frac{2AB}{\sqrt{B^2(r^2 - \omega^2)(A^2r^2 - 1) - 1}} . \quad (2.57) \]

Actually, we can show the fluctuations in \( \theta \) direction around the D1-brane are described by massless scalar fields only when \( A = \frac{1}{\omega} \). We therefore find
\[ \Delta t = 2 \int_{r_*}^{\infty} \frac{dr}{\sqrt{(r^2 - \omega^2)^2 - \frac{\omega^2}{B^2}}} , \quad (2.58) \]
which is much larger than \( L \) if \( B \ll \frac{1}{\omega} \). Thus, as opposed to the global AdS case, we can have a large time delay \( \Delta t \gg L \) and this is due to the strong interactions with the deconfined \( \mathcal{N} = 4 \) SYM.

3 Properties of Thermal Vacua from Rotating D-branes in AdS

In this section we calculate various physical quantities from our rotating D-branes and confirm that they are indeed thermalized as we expected from the induced metric.

3.1 Action of Fluctuations

First we need to show that the fluctuations on the rotating D-branes see the horizon which we found from the induced metric. Consider a Dp-brane in the background metric \( (2.16) \).
We choose the gauge where the worldvolume coordinates \( \xi^a = x^a, \ a = 0 \ldots p \). The transverse coordinates are \( x^I, I = p + 1 \ldots 9 \). Let us write the metric (3.59) as
\[
 ds^2 = g_{ab}(x^a, x^I)dx^adx^b + G_{IJ}(x^a, x^K)dx^I dx^J. \quad (3.59)
\]
Expanding around a classical solution \( x_I^0(x^a) \),
\[
x^I(x^a) = x_I^0(x^a) + y^I(x^a), \quad (3.60)
\]
we find that up to terms of order \( O(y^2) \) the induced metric is
\[
 \gamma_{ab} = \gamma^{(0)}_{ab} + M_{ab} + N_{ab}, \quad (3.61)
\]
where \( \gamma^{(0)}_{ab} \) is the induced metric for \( x_I^0 \). The matrices \( M \) and \( N \) are given by
\[
 M_{ab} = (\partial_I g_{ab}) y^I + G_{I,J}(\partial_a y^I \partial_b x_J^0 + \partial_b y^I \partial_a x_J^0) + (\partial_K G_{IJ})(\partial_a x^K_0 \partial_b x_J^0) y^K, \\
 N_{ab} = G_{IJ}(\partial_a y^I \partial_b y^J + (\partial_K G_{IJ})(\partial_a y^I \partial_b x_J^0 + \partial_b y^I \partial_a x_J^0) y^K + \\
 \frac{1}{2}(\partial_K \partial_L G_{IJ}) y^K y^L \partial_a x^K_0 \partial_b x_L^0 + \frac{1}{2}(\partial_I \partial_J g_{ab}) y^I y^J. \quad (3.62)
\]
The DBI action is then
\[
 S_{DBI} = -T_p \int d^{p+1}\xi \sqrt{\det \gamma_{ab}} \\
 = S_0 + \frac{T_p}{2} \int d^{p+1}\xi \sqrt{-\gamma_0} \left[ \gamma_0^{ab} N_{ab} - \frac{1}{2} \gamma_0^{ab} \gamma_0^{cd} M_{da} + \frac{1}{4} (\gamma_0^{ab} M_{ba})^2 + O(y^3) \right], \quad (3.63)
\]
where \( S_0 \) is the action of the background classical solution \( x_I^0 \).

### 3.2 Fluctuations around Rotating Strings

In general the fluctuations are coupled with each other and have complicated actions. For fluctuations around the rotating D1 or F1 solution considered above the action simplifies drastically. In the \((u, v)\) coordinates introduced in (2.17), the matrix elements of \( M \) and \( N \) yield the following simple result
\[
 M_{uu} = 0, \quad M_{vv} = 2(\partial_v y^r)(\partial_v \varphi_0), \quad M_{uv} = (\partial_u y^r)(\partial_v \varphi_0), \quad (3.64)
\]
and
\[
 N_{ab} = G_{I,J}(\partial_a y^I \partial_b y^J) - (\partial_a \varphi_0 \partial_b \varphi_0)(y^a)^2. \quad (3.65)
\]
Plugging this into (3.63) we see that in the first term only the first term of (3.65) survives since $\gamma_0^{ab}\partial_a\phi_0\partial_b\phi_0 = 0$ while the contributions from the second and third terms precisely cancel each other, leading to the following action for quadratic fluctuations

$$S_2 = \frac{T_{D1}}{2} \int d^2\xi \sqrt{-\gamma_0^{ab}} G_{IJ}(\xi^a, x_0^I) \partial_a y^I \partial_b y^J.$$ (3.66)

In particular, the fluctuations of $\varphi, \theta$ i.e. all fluctuations in $S^5$ directions are minimally coupled massless scalars on the worldsheet, while the fluctuations of the boundary gauge theory spatial directions $x^i, i = 1 \cdots 3$ have an additional factor of $r^2$ coming from the fact that $G_{ij} = r^2 \delta_{ij}$ along these directions.

For a D1 or F1 rotating with a constant angular velocity, it is easy to see that all fluctuations perceive a horizon at $r = \omega$ and its temperature is $T_H = \frac{\omega}{2\pi}$. For example, the action of the fluctuations in $x^i$ direction, denoted by $X$, is given by

$$S = -T_{D1} \int dt dr \sqrt{1 - \frac{r^2}{r^2 - \omega^2} (\partial_r X)^2 + r^2 (r^2 - \omega^2)(X')^2}.$$ (3.67)

### 3.3 Retarded Green’s Functions from Rotating D1-branes

As we discussed above, a rotating D1 or F1 corresponds to a source in the $(0 + 1)$-dimensional defect CFT. In this subsection we will discuss the retarded Green’s function of dual operators to the worldvolume fields.

For the transverse scalars in the $S^5$ directions, this can be done using the standard bulk-boundary relation [2, 3] in the presence of horizon [8], since these are minimally coupled fields on the worldsheet. Consider for example the worldsheet field $\varphi$. This is a massless scalar. Let us, however, generalize the analysis and consider a scalar field $\Phi$ with arbitrary mass $m$ which propagates in the induced metric (2.21). They correspond to the infinitely many stringy excitations of open strings. The equation of motion is given by

$$((r^2 - \varphi'(v)^2)\Phi')' + 2\Phi' - m^2 \Phi = 0.$$ (3.68)

where prime denotes derivative with respect to $r$ and dot denotes derivative with respect to $v$. In particular, we consider the constant angular velocity case, i.e., $\varphi'(v) = \omega$. Its exact solutions with energy $\Omega$ look like

$$\Phi(v, r) = e^{-i\Omega v} \left[ A \left( \frac{r + \omega}{r - \omega} \right)^{-\frac{\Omega}{\omega}} P^\Omega_\kappa(r/\omega) + B \left( \frac{r + \omega}{r - \omega} \right)^{-\frac{\Omega}{\omega}} Q^\Omega_\kappa(r/\omega) \right],$$ (3.69)

where $\kappa = \frac{\sqrt{1 + 4m^2} - 1}{2}$; $P^\Omega_\kappa(x)$ and $Q^\Omega_\kappa(x)$ are the first and second kind of the Legendre bi-function. By imposing the ingoing boundary condition at the horizon $r = \omega$, we simply...
find $B = 0$. The retarded Green’s function of the operator $O$ dual to the scalar field can be computed from the ratio $\frac{\beta}{\alpha}$ when we expand the scalar field as $\Phi \sim \alpha r^\kappa + \beta r^{-1-\kappa}$ in the boundary limit $r \to \infty$. Finally we thus find that it is given by the ratio of gamma functions as follows

$$
\langle O(\Omega)O(-\Omega) \rangle_R = \omega^{2\kappa+1} \frac{\Gamma(\kappa + 1 - i\Omega/\omega)}{\Gamma(-\kappa - i\Omega/\omega)}.
$$

(3.70)

up to some numerical factor independent of $\Omega$. Indeed, for generic values of $m$, it has non-trivial poles only in the lower half plane and has the correct property of retarded Green’s functions of thermal systems. Particularly for a massless scalar field, the dependence on $\omega$ drops out in (3.70) by chance and the thermal effect is not explicit. However, the same is true for the retarded Green’s functions in massless free field theory at finite temperature and thus we can still conclude that the flavor sector dual to the our probe D-string is thermalized. For example, this issue is clear by comparing the time-ordered two point function (A.142) with the retarded two point function (A.144) of a harmonic oscillator given in the Appendix A. This thermal property will also be shown more clearly in the Brownian motion analysis given in the next subsection.

The fluctuations in the $x^i$ direction have an additional factor of $r^2$ in front of the Lagrangian, and leads to the following purely ingoing solution which coincides with the one in [47]

$$
X^i(t, r) = \frac{r + i\Omega}{r} \left( \frac{r - \omega}{r + \omega} \right)^{\frac{i\Omega}{\pi \omega}}.
$$

(3.71)

For large values of $r$, the leading term is a constant, but the coefficient of the subleading $1/r$ term vanishes. In this case, however, the retarded Green’s function cannot be simply read off from the ratio of the subleading to the leading term, since the equation of motion is different, and the analysis of [8] has to be redone.

### 3.4 Brownian Motion of Endpoints for Rotating D1-branes

The thermal nature of the state produced by time dependence becomes clear from a slightly different calculation - the fluctuation of the end-point of the string. In [47] and [48] it has been shown that the fluctuations of a string suspended from the horizon of a AdS black brane ended at a flavor D-brane near the boundary of AdS are dual to Brownian motion of the corresponding quark in the hot $\mathcal{N} = 4$ gauge theory. In this case the bulk black brane metric induces a worldsheet metric which has a horizon. The fluctuations then reflect Hawking radiation from the worldsheet horizon. The results in [47] and [48] are consistent with the fluctuation-dissipation theorem.
In the D-brane solutions considered above, the bulk metric has no horizon. However due to the motion of the D-brane, the induced metric on the worldvolume can develop a horizon. Since the fluctuations of \[17\text{48}\] comes purely from properties of the induced metric it is natural to expect that a similar phenomenon appears in our case.

We therefore consider a rotating D1 (or F1) brane ending on a flavor D-brane situated close to the boundary at \( r = r_c \). One example of such a flavor D-brane is the rotating D7-brane which will be discussed in the next section and in this case we need to require \( \omega < r_c \). As usual a finite \( r_c \) means that the dual description of the string is a finite mass monopole (or quark), \( m = T_{D1} r_c \). Following \[47\] we now compute the normal ordered Green’s functions of the end-point of the string,

\[
\langle 0, U | : [y^I(t, r_c) y^J(t', r_c)] : | 0, U \rangle
\]

in a Unruh vacuum state \(|0, U\rangle\) of the worldsheet theory. This quantity is equal to the mean square fluctuation of the end-point with a trivial divergence due to zero point energy subtracted. We will consider the case of a constant spin.

For constant spin, \( \varphi_0(v) = \omega v \) the induced metric is

\[
ds^2_{\text{ind}} = (\gamma_0)_{ab} d\xi^a d\xi^b = 2dr dv - (f(r) - \omega^2) dv^2 = -(f(r) - \omega^2) d\bar{u} dv , \tag{3.72}
\]

where

\[
\bar{u} = v - 2r_*, \quad r_* = \int \frac{dr}{f(r) - \omega^2} . \tag{3.73}
\]

The coordinate \( r_* \) is the tortoise coordinate on the worldsheet, where the boundary is at \( r_* = 0 \) and the horizon is at \( r_* = -\infty \). The UV cutoff corresponding to the D7-brane means that the range of \( r_* \) is now \( -\infty < r_* < r_{*,c} \), and we need to impose Neumann conditions on the fluctuation field \( y^\varphi \) at \( r_* = r_{*,c} \).

The \( \varphi \) fluctuations behave as minimally coupled massless scalars in the induced metric produced by the rotating string solution. It turns out that for these fluctuations the UV cutoff can be removed and we can treat the problem in the full range \( -\infty < r_* < 0 \). the mode expansion is given by

\[
y^\varphi(\bar{t}, r_*) = \int \frac{dv}{2\pi \sqrt{2\nu}} \cos(\nu r_*) [a_{\nu} e^{-i\nu \bar{t}} + \text{h.c.}] , \tag{3.74}
\]

where

\[
\bar{t} \equiv \frac{1}{2}(\bar{u} + v) . \tag{3.75}
\]

Note that on the boundary \( \bar{t} = t \). The oscillators \( a_{\nu}^\dagger \) create "Schwarzschild" particles from the Schwarzschild vacuum \(|0, S\rangle : a_{\nu} |0, S\rangle = 0 \).

Consider now the case where the background is \( AdS_5 \times S^5 \) in the Poincare patch, i.e. \( f(r) = r^2 \). In the Unruh vacuum the correlators of \( a_{\nu} \) are thermal with a temperature
\[ T = 1/\beta = \omega/2\pi, \text{ i.e.} \]
\[ \langle 0, U|a_\nu^\dagger a_{\nu'}|0, U \rangle = \frac{\delta(\nu - \nu')}{e^{\beta \nu} - 1}. \quad (3.76) \]

Using this it is straightforward to calculate the mean square displacement in the \( \varphi \) direction,
\[
\langle (\Delta y_\varphi(t - t'))^2 \rangle = \frac{2}{\pi} \int \frac{d\nu}{\nu} \sin^2(\nu(t - t')/2) \frac{e^{\beta \nu} - 1}{e^{\beta \nu} - 1},
\]
\[
= \frac{1}{2\pi} \log \left( \frac{\sinh(\pi(t - t')/\beta)}{(\pi(t - t')/\beta)} \right). \quad (3.77)
\]

Therefore
\[
\langle (\Delta y_\varphi(t - t'))^2 \rangle \sim \frac{\pi(t - t')^2}{12\beta^2}, \quad \pi(t - t') \ll \beta,
\]
\[
\langle (\Delta y_\varphi(t - t'))^2 \rangle \sim \frac{(t - t')^2}{2\beta} - \frac{1}{2\pi} \log[2\pi(t - t')/\beta], \quad \pi(t - t') \gg \beta, \quad (3.78)
\]

exactly as in Brownian motion. The transition from the short time ballistic behavior to the long time diffusive behavior happens at a time scale set by the inverse temperature, which is the only scale in the problem. The result is identical for the \( y^0 \) fluctuations, as well as fluctuations in the \( S^3 \) directions.

The fluctuations in the \( \vec{x} \) directions are not conformally coupled scalars. The equation is in fact identical to the equation for fluctuations of the end point of a string in the background of a BTZ black hole, as in [47]. In this case, the UV cutoff at \( r = r_c \) is essential. The final result is [47]
\[
\langle (\Delta y^i(t - t'))^2 \rangle \sim \frac{(t - t')^2}{m^2 \beta}, \quad t \ll m^2 \beta,
\]
\[
\sim \beta |t - t'|, \quad t \gg m^2 \beta. \quad (3.79)
\]

Unlike fluctuations in the internal \( SO(6) \) space, Brownian motion in the physical space on which the gauge theory lives is present only if the mass of the monopole or quark is finite. This is of course expected. Furthermore the time scale which controls the transition from a ballistic behavior is not simply the inverse temperature, but involves the mass in an essential way.

As argued above a rotating D1-brane corresponds to a time-dependent coupling in the defect \( (0 + 1) \)-dimensional CFT. Brownian motion is a manifestation of thermalization of
this defect CFT, or equivalently the hypermultiplet sector of the theory. This Brownian motion is a physical consequence of the Hawking radiation on the world-sheet and this corresponds to the thermalization of the hypermultiplet sector (or defect sector). Notice that this is not directly related to the Hawking radiation in the bulk AdS which comes from the thermalization of the $\mathcal{N} = 4$ super Yang-Mills sector. Indeed their temperatures can be chosen differently.

### 3.5 Conductivity from Rotating D3-branes

For rotating D$p$-branes of section (2.3), there are several other response functions which carry signatures of thermalization. A good example is a probe D3-brane configuration obtained by taking T-dual of the rotating D1 in $x, y$ direction. This is a non-supersymmetric configuration in $AdS_5 \times S^5$. The rotating D3-brane solution is obtained from (2.24) and (2.25) by setting $p = 3$. Its induced metric is given by the black hole geometry (2.26) at $p = 3$ and its Hawking temperature is $T_H = \frac{\sqrt{3} \omega}{2\pi}$.

The worldvolume can now contain a nontrivial gauge field, and the DBI action of the D3-brane looks like

$$L = \sqrt{r^4 + r^6 \varphi'^2 - r^2 \dot{\varphi}^2 - (1 + r^2 \varphi'^2) F_{tx}^2 - r^4 F_{tr}^2 + (r^2 - \dot{\varphi}^2) r^2 F_{xT}^2 + 2r^2 F_{rx} F_{tx} \varphi \varphi'} ,$$

$$= \sqrt{\frac{r^8}{r^4 + r^2 \omega^2 + \omega^4} - \frac{r^4}{r^6 - \omega^6} F_{tx}^2 - r^4 F_{tr}^2 + r^2 (r^2 - \omega^2) F_{xT}^2 ,}$$

(3.80)

This shows that the gauge field is propagating in a space-time with a horizon.

We would like to calculate the electric conductivity in the probe D3-brane background by applying the method developed in AdS black holes [8, 11]. We regard the abelian gauge field on the probe D3-brane as an external gauge field for which we define the AC conductivity $\sigma(\nu)$, where $\nu$ is the frequency. In other words, here we are thinking the charge with respect to the global flavor symmetry as the electric charge to define the conductivity.

The DBI action (3.80) can be expressed using the induced metric $g_{\text{ind}}$:

$$L_{D3} = \sqrt{- \det g_{\text{ind}}} \sqrt{1 + \frac{F_{\alpha \beta} F^{\alpha \beta}}{2}} .$$

(3.81)

It follows that the equations of motion are

$$\nabla_\alpha \left( \frac{F^{\alpha \beta}}{\sqrt{1 + F^2/2}} \right) = 0 .$$

(3.82)
Before computing the conductivity, we have to find a configuration of gauge field with non zero gauge potential \( A_a = (A_t(r) = \Phi(r), 0, 0, 0) \) which obeys the following equation:

\[
\partial_r \left( \frac{r^2 \sqrt{r^4 + r^2 \omega^2 + \omega^4} \Phi'}{\sqrt{1 - r^4/r_4 \omega^2 + \omega^4 \Phi'^2}} \right) = 0 .
\]  

(3.83)

We can easily find a solution with the boundary condition \( \Phi(r) = 0 \) at \( r = \omega \):

\[
\Phi(r) = \int_\omega^r ds s^2 \frac{s}{\sqrt{(1 + Cs^4)(s^4 + s^2 \omega^2 + \omega^4)}} .
\]  

(3.84)

The gauge potential behaves near \( r \sim \infty \) like

\[
\Phi(r) = \mu - \frac{\rho}{r} + \ldots ,
\]  

(3.85)

and \((\mu, \rho)\) stand for the chemical potential and the electric density, respectively. Accordingly, we can compute the electric density using (3.84) as follows:

\[
\rho = \lim_{r \to \infty} \Phi'(r) r^2 = C^{-1/2} .
\]  

(3.86)

Now we consider a small perturbation \( A_x = A_x(r) e^{-i\nu t} \) on this background. We use \( z \equiv 1/r \) coordinate for simple calculation in the following. The equation we solve is that

\[
A_x''(z) - \frac{z^3(6z^6 + z^4 + z^2 - 2) + \tilde{C}z(z^2 + z^2 + 4)}{(1 - z^6)(z^4 + \tilde{C})} A_x' + \tilde{\nu}^2 A_x = 0 ,
\]  

(3.87)

where we rescaled \( z \to z/\omega \), and define \( \tilde{C} = C\omega^4 \) and \( \tilde{\nu} = \nu/\omega \). The gauge field is an ingoing wave near the horizon \( z = 1 \), and it thus can be expanded as follows:

\[
A_x(z) = (1 - z)^{-i\tilde{\nu} \sqrt{\tilde{C}}}(1 + a_1(1 - z) + a_2(1 - z)^2 + \ldots) ,
\]  

(3.88)

while it behaves at the boundary \( z \sim 0 \) as

\[
A_x(z) = A_x^{(0)} + A_x^{(1)} z + \ldots .
\]  

(3.89)

We can obtain the conductivity

\[
\sigma(\nu) = -\frac{iA_x^{(1)}}{\nu A_x^{(0)}} .
\]  

(3.90)

In Fig. 4 we plot the conductivity obtained by solving (3.87) numerically. The behavior that \( \text{Re} \sigma(\nu) \) approaches to a finite constant is known to common in (2+1)-dimensional
critical theories [11]. As opposed to the purely bulk calculation of the conductivity from the AdS charged black holes, where we have Re $\sigma(\nu) \propto \delta(\nu)$, instead we observe a smooth Drude-like peak both in Re $\sigma(\nu)$ and Im $\sigma(\nu)$ near $\nu = 0$ in Fig. 4. This is because our probe approximation introduces dissipation into the bulk and has the finite DC conductivity. A very similar behavior has been already obtained in [29,33] from the probe analysis of D-branes in AdS black holes. Here we managed to get a similar physics by using a rotating D-branes in pure AdS.

Figure 4: The frequency dependence of the conductivity. Four lines are depicted for $\rho = 10, \sqrt{10}, 1$ and $1/\sqrt{10}$, respectively when $\omega = 1$. The real part of the conductivity approaches to Re $\sigma(\nu) = 1$ line as $\rho$ decreases, while the imaginary part becomes zero.

3.6 Comments on Entropy for Rotating D-branes

Before we finish this section, we would like to comment on a possible definition of entropy of rotating D-branes. Since we find a thermal horizon in the induced metric, it is natural to expect that there exists non-trivial entropy in such a system in spite of the absence of real black holes.

There is one natural candidate of such an entropy for probe Dp-branes: it is calculated from their free energy by Wick rotating the time coordinate $\tau$ in (2.26). Consider first a rotating D1-brane in $AdS_5 \times S^5$. The free energy is simply calculated from the DBI action as follows

$$F_1 = T_{D1} R^2 \int_\omega^\infty dr ,$$

where we revive the AdS radius $R$ for convenience. The entropy is obtained by taking
derivative w.r.t. $T_H = \frac{\omega}{2\pi}$

$$S_1 = -\frac{\partial F_1}{\partial T_H} = 2\pi R^2 T_{D1} = 2\pi \sqrt{\frac{N}{\pi g_s}} = 2\sqrt{\pi} \frac{N}{\sqrt{\lambda}}.$$ (3.92)

The linear dependence of the rank $N$ of gauge group agrees with what we expect for the flavor contribution in the planar limit.

Let us move on to the higher-dimensional D-branes. As one such example, we would like to consider the rotating D3-brane in $AdS_5 \times S^5$ discussed in the previous subsection. Then its free energy looks like

$$F_3 = T_{D3} R^4 V_2 \int_{\omega}^{r_\infty} \frac{dr}{\sqrt{r^4 + \omega^2 r^2 + \omega^2}},$$

$$= T_{D3} R^4 V_2 \left( \frac{r_\infty^3}{3} - \frac{1}{2} r_\infty \omega^2 + k \omega^3 \right),$$ (3.93)

where $k \simeq 0.135$ is a numerical constant; $V_2$ is the area of non-compact space defined by the intersection of the D3 with the AdS boundary. We also kept the UV cut off $r_\infty (\to \infty)$ explicitly. By taking the derivative with respect to the temperature $T_H = \frac{\sqrt{3} \omega}{2\pi}$, we get the following entropy

$$S_3 = -\frac{\partial F_3}{\partial T_H} = \frac{2\pi}{\sqrt{3}} T_{D3} R^4 L^2 \left( r_\infty \omega - 3 b \omega^2 \right) = \frac{2}{3} N V_2 \left( \frac{T_H}{a} - \tilde{k} T_H^2 \right),$$ (3.94)

where $a = 1/r_\infty$ is the lattice spacing (or UV cut off) of the dual CFT and $\tilde{k} = 2\pi \sqrt{3} k \simeq 1.47$ is a numerical constant. Thus this entropy is UV divergent and this behavior is always true for rotating D$p$-branes with $p \geq 1$. One may think this is unnatural because the entropy which we calculated in this way should be the thermal entropy of the hypermultiplet, which should be finite. This might suggest that the above way of computation of entropy via our naive Wick rotation is not correct. However, we would like to propose a possible interpretation of this divergent entropy. Recall that in our probe calculation, the interactions with the $\mathcal{N} = 4$ SYM sector were already incorporated as the supergravity background. This means that the $\mathcal{N} = 4$ SYM sector was already traced out in the density matrix. Therefore the entropy which we calculated in the above should be regarded as the entanglement entropy between the flavor sector and the $\mathcal{N} = 4$ SYM sector. In thermal equilibrium, we can neglect the entanglement. However, in our non-equilibrium systems where the two different temperatures coexist, there may be non-trivial entanglement entropy which can be UV divergent. Notice that the UV divergence of the standard entanglement entropy is holographically explained by the infinite volume of AdS space and the origin of the divergence here looks similar. It is a quite intriguing future problem to confirm this from field theory calculations.
4 Rotating D7-branes in $AdS_5 \times S^5$

So far we have assumed that the rotating D$p$-brane is point-like in $S^q$. However, this class of D-branes misses important supersymmetric configurations because the latter often wrap on $S^n (n \geq 1)$ in $S^q$. One such example is the probe D7-branes in $AdS_5 \times S^5$, which is dual to the $\mathcal{N} = 2$ flavor hypermultiplets coupled to the $\mathcal{N} = 4$ super Yang-Mills [26–28]. Notice that in this case the space-time of the dual field theory is $(3 + 1)$-dimensional even for the probe D7-brane sector. Motivated by this, in this section we will study the rotating D7-branes in $AdS_5 \times S^5$. This system has two independent parameters i.e. the distance $m$ between the D7-brane and the background D3-branes and the angular velocity $\omega$. This setup has been already studied in [49,50]. Though our D7-brane solution presented below is essentially the same as the one in [49], we will study the solution from a different viewpoint focusing on the presence of the thermal horizon in the induced metric.

4.1 D7-brane World-Volume

The D7-brane world-volume coordinates are extending in the $AdS_5$ and $S^3$ included in $S^5$ with a rotation in $\varphi$ direction. This is specified by

$$\theta = \theta(r) \ , \ \varphi = \omega t + g(r) \ . \quad (4.95)$$

The DBI action now becomes

$$L = -r^3 \cos^3 \theta \sqrt{L} \ , \quad (4.96)$$

where

$$L = 1 + r^2 \sin^2 \theta \varphi'^2 - \frac{\omega^2 \sin^2 \theta}{r^2} + r^2 \theta'^2 - \omega^2 \sin^2 \theta \theta'^2 . \quad (4.97)$$

The equation of motion for $\varphi$ reads

$$\frac{\partial}{\partial r} \left( \frac{r^5 \cos^3 \theta \sin^2 \theta \varphi'}{\sqrt{L}} \right) = 0 . \quad (4.98)$$

The equation of motion for $\theta$ is given by

$$3r^3 \cos^2 \theta \sin \theta \sqrt{L} - r^3 \cos^4 \theta \sin \theta \frac{r^2 \varphi'^2}{\sqrt{L}} - \omega^2 \theta'^2 + \partial_r \left[ r^3 \cos^3 \theta \frac{(r^2 - \omega^2 \sin^2 \theta)\theta'}{\sqrt{L}} \right] = 0 . \quad (4.99)$$

The solutions to the equation (4.98) should be either

$$g'(r) = 0 , \quad (4.100)$$
or
\[ g'(r) = \frac{1}{r^2 \sin \theta} \sqrt{\frac{(r^2 - \omega^2 \sin^2 \theta)(1 + r^2 \theta^2)}{(A^2 r^8 \cos^6 \theta \sin^2 \theta - 1)}}. \] (4.101)

where \( A \) is an integration constant.

Let us first study the case (4.100). When \( \omega = 0 \), we can find the solution \( r \sin \theta = m \).
This corresponds to the D7-brane separated from D3-branes by the distance \( m \), which is dual to the mass of hypermultiplets. Now consider solving the equation (4.99) with a boundary condition \( \theta = \pi/2 \) at \( r = a \) where \( a \) is an arbitrary constant. The function \( r(\theta) \) for \( \theta = \pi/2 + \delta \) for an infinitesimally small \( \delta \) is then determined by the equation of motion (4.99),
\[ r(\pi/2 - \delta) \simeq a + \frac{4a^3 - 3\omega^2 a}{8(a^2 - \omega^2)} \delta^2 + \cdots. \] (4.102)

Using this we can solve for the function \( r(\theta) \) numerically for \( 0 < \omega < a \). In this case the solutions does not differ qualitatively from the \( \omega = 0 \) solution and there is no horizon, as can been from the left graphs in the Fig. 5. If we increase \( \omega \) such that \( \omega > a \), we cannot obtain smooth solutions from this ansatz (4.100).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{The shapes of the rotating D7-brane with \( \omega = 1 \). In the left panel where we consider \( g' = 0 \) case, we depict the curves for \( a = 1.01, 1.1, 1.2 \) and 1.5. In the right panel where we consider \( g' \neq 0 \), we depict them for \( \alpha = \pi/2, \pi/2.1, \pi/2.5, \pi/3, \pi/4, \pi/10 \). The dotted regions are inside the horizon.}
\end{figure}

Next we would like to consider the case when (4.101) is satisfied. The denominator \( A^2 r^8 \cos^6 \theta \sin^2 \theta - 1 \) can become negative. To avoid the resulting singularity in the function \( L \), we need to require that \( r^2 - \omega^2 \sin^2 \theta \) and \( A^2 r^8 \cos^6 \theta \sin^2 \theta - 1 \) vanish at the same time i.e.
\[ a^2 = \omega^2 \sin^2 \alpha, \quad A^2 a^8 \cos^6 \alpha \sin^2 \alpha = 1, \] (4.103)
where we set \( r = a \) and \( \theta = \alpha \) at this point. This point will turn out to be the horizon. If we expand the solution near this point, we find from the equation of motion (4.99)

\[
\theta(r) \simeq \alpha + \beta(r - a) + \cdots ,
\]

(4.104)

where the (negative) constant \( \beta(< 0) \) is determined by

\[
3 + 2\beta \omega \cos \alpha - 3\beta^2 \omega^2 \sin^2 \alpha = 0 .
\]

(4.105)

The solution to this equation for \( \beta \) is

\[
\beta = \frac{1}{3\omega \sin^2 \alpha} (\cos \alpha - \sqrt{\cos^2 \alpha + 9 \sin^2 \alpha}) .
\]

(4.106)

Thus if we input the values \((a, \omega)\) we can uniquely find the solution numerically by solving (4.99) starting from the thermal horizon point \( r = a \). They are shown in the right graphs of Fig. 5. Notice that the two solutions (4.100) and (4.101) of the D7-brane match at \( \omega = a \) (or \( \alpha = \pi/2 \)) as depicted in Fig. 6.

![Figure 6: The D7-brane is shaped like almost flat brane for \( \omega < a \), and becomes singular when \( \omega = a \). Then it always have a horizon on its worldvolume for \( \omega > a \).](image)

4.2 Induced Metric and Thermal Horizon

The induced metric reads

\[
ds^2 = -(r^2 - \omega^2 \sin^2 \theta)dt^2 + 2\omega \sin^2 \theta g'(r)dt dr + \cos^2 \theta d\Omega_3^2 \\
+ \left(\frac{1}{r^2} + \theta'^2 + \sin^2 \theta g'^2\right) dr^2 + r^2(dx^2 + dy^2 + dz^2) .
\]

(4.107)

After the coordinate change \( t = \tau + h(r) \) with

\[
h'(r) = \frac{\omega \sin^2 \theta g'}{r^2 - \omega^2 \sin^2 \theta} ,
\]

(4.108)
we can rewrite the metric as follows

\[
 ds^2 = -(r^2 - \omega^2 \sin^2 \theta)dr^2 + \left( \frac{1}{r^2} + \frac{r^2 \sin^2 \theta g^2}{r^2 - \omega^2 \sin^2 \theta} \right) dr^2 \\
+ r^2(dx^2 + dy^2 + dz^2) + \cos^2 \theta d\Omega^2_3. 
\] (4.109)

Thus the horizon is identified with \( r = \omega \sin \theta \) as already mentioned. The topology of horizon is given by \( R^3 \times S^3 \).

The Hawking temperature can be found from this metric

\[
 T = \frac{\omega}{2\pi} \sqrt{\frac{\sin \alpha \tan \alpha (4 \sin \alpha - \beta \omega (3 + \beta \omega \cos 3\alpha))}{1 + \beta^2 \omega^2 \sin^2 \alpha}}. 
\] (4.110)

We can see that \( T \) is monotonically increasing function of \( a \) when we fix \( \omega \). At \( a = \omega \), we find \( T = \infty \), while at \( a = 0 \), we have \( T = 0 \). The flavor mass \( m \equiv \lim_{r \to \infty} r \sin \theta \) of this solutions is plotted as a function of \( a \) in the left panel of Fig. 7. The dependence of the temperature on the mass is also presented in the right panel of Fig. 7.

Figure 7: [Left] The dependence of the function \( m \equiv \lim_{r \to \infty} r \sin \theta \) on \( a \) in the second solution (4.101). We set \( \omega = 1 \) and connected the two solutions (4.100) and (4.101) at \( a = 1 \). [Right] The dependence of the temperature on the mass \( m \) at \( \omega = 1 \). For \( m \gtrsim 1.15 \), there is no horizon.

### 4.3 Dual CFT Interpretation

When the D7-brane does not rotate, our setup is the same as the standard D3-D7 system. The dual CFT is described by the four-dimensional \( \mathcal{N} = 2 \) \( SU(N) \) SQCD which is defined by the \( \mathcal{N} = 4 \) SYM coupled to a \( \mathcal{N} = 2 \) flavor hypermultiplet in the large \( N \) limit. The time-dependence of \( \varphi(t) = \omega t \) (4.95) at the boundary \( r = \infty \) is dual to the time-dependent mass

\[
 m(t) = me^{i\varphi(t)} = me^{i\omega t}. 
\] (4.111)
In terms of the superpotential we have

\[ W = \text{Tr}[\Phi_1, \Phi_2] \Phi_3 + \bar{Q} \Phi_3 Q + m(t) \bar{Q} Q , \]  

(4.112)

where \( \Phi_1, \Phi_2, \Phi_3 \) are the transverse complex scalar fields on the \( N \) D3-branes. \( (Q, \bar{Q}) \) are the hypermultiplets from D3-D7 open strings. We will also denote the fermions in the hypermultiplet by \( (\psi_Q, \bar{\psi}_Q) \). The original \( SU(4) = SO(6) \) R-charge of \( \mathcal{N} = 4 \) super Yang-Mills theory is now broken to \( SO(4) \times U(1)_R = SU(2)_R \times SU(2)_L \times U(1)_R \).

(4.113)

The \( SO(4) \) rotates the \( R^4 \) described by \( \Phi_1 \) and \( \Phi_2 \). \( U(1)_R \) corresponds to the rotation of the remaining \( R^2 \) i.e. \( \Phi_3 \). The R-symmetry of \( \mathcal{N} = 2 \) super-Yang Mills is given by \( U(2)_R = SU(2)_R \times U(1)_R \). If we denote the spins under \( SU(2)_R \times SU(2)_L \times U(1)_R \) by \( (j_R, j_L, q_{U(1)}) \), the complex scalar fields \( (Q, \bar{Q}) \) in the hypermultiplet belongs to \( (1/2, 0, 0) \). On the other hand, fermions \( (\psi_Q, \bar{\psi}_Q) \) belong to \( (0, 0, \pm 1) \). The transverse scalar fields \( \Phi_1, \Phi_2 \) belong to \( (1/2, 1/2, 0) \), while \( \Phi_3 \) to \( (0, 0, 1) \).

Now we would like to come back to the superpotential (4.112). After integrating the fermionic coordinates of the superfield, we find the time-dependent bosonic potential

\[ \int dx^4 \left[ \bar{Q}(\Phi_3 - me^{i\omega t})(\Phi_3 - me^{i\omega t})Q + \bar{Q}(\Phi_3 - me^{i\omega t})(\Phi_3 - me^{i\omega t}) \bar{Q} \right] . \]  

(4.114)

as well as the time-dependent fermion mass term

\[ \int dx^4 \ me^{i\omega t} \bar{\psi}_Q \psi_Q + (h.c.) . \]  

(4.115)

The AdS/CFT claims that the above time-dependent system will lead to a thermal vacuum. These time-dependent potentials triggers a physics similar to that in quantum quench and strong interactions in the hypermultiplets will be expected to lead to a thermalization. However, since the hypermultiplet is coupled to the \( \mathcal{N} = 4 \) SYM sector, the energy eventually will dissipate, as we have explained in the simpler model of rotating D1-brane previously, schematically summarized in Fig. 1. It will be a quite interesting future problem to confirm this prediction from the field theory calculations.

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6 One may notice that if we redefine \( \Phi_3 \) as \( \hat{\Phi}_3 = \Phi_3 e^{i\varphi(t)} \), then the system is equivalent to the standard flavored \( \mathcal{N} = 2 \) SYM with the R-charge chemical potential \( A_t = \omega \) only for the field \( \hat{\Phi}_3 \) without any time-dependence. However, from this viewpoint we need to take a time-dependent state and its time evolution will be described by this time-independent Hamiltonian. This is because we know from AdS/CFT that the hypermultiplet is initially exited and has non-zero R-charge instead of the \( \mathcal{N} = 4 \) SYM sector.
5 Quantum Quench and Thermalization via AdS/CFT

So far we have mainly studied stationary configurations of the Dp-branes. We would now like to discuss truly time-dependent D-branes in AdS spaces and its interpretation via AdS/CFT correspondence. The goal of this section is to holographically model quantum quench - the time evolution of a system following a sudden change of a parameter in a given quantum field theory, such as the mass of scalar fields. The phenomenon of quantum quench (see e.g. [16, 17, 19–22] for approach from conformal field theories) has been intensively studied in condensed matter theory recently.

5.1 Quantum Quench in Free Field Theory

Some interesting aspects of the effect of quantum quench can be in fact observed in a free quantum field theory [19, 21]. Let us consider a $d$-dimensional massive bosonic free field theory with Hamiltonian

$$H = \int d^d k \left( \frac{1}{2} \pi_k \pi_{-k} + \frac{1}{2} \omega_k^2 \phi_k \phi_{-k} \right),$$

(5.116)

with a dispersion relation $\omega_k^2 = k^2 + m^2$.

Each momentum mode is independent of each other, and the propagator is just that of a single harmonic oscillator. We therefore consider a single harmonic oscillator with momentum $k$ that initially lies in a thermal state with temperature $\beta_0$, and then we quench its frequency from $\omega_{0k}$ to $\omega_k$ by changing mass from $m_0$ to $m$ at $t = 0$. At a later time $t$ after the quench, the scalar field is given by $\phi_k(t) = \phi_k(0) \cos \omega_k t + \pi_k(0) \sin \omega_k t/\omega_k$.

The time ordered correlator will be [21]

$$C_{\beta_0}(k; t_1, t_2) \equiv T\langle \phi_k(t_1) \phi_{-k}(t_2) \rangle = \frac{1}{2\omega_k} e^{-i\omega_k|t_1 - t_2|} + \left[ \frac{\omega_{0k}}{4} \left( \frac{1}{\omega_{0k}^2} + \frac{1}{\omega_k^2} \right) \coth \frac{\beta_0 \omega_{0k}}{2} \frac{1}{\omega_k^2} \right] \cos \omega_k (t_1 - t_2)
+ \frac{\omega_{0k}}{4} \left( \frac{1}{\omega_{0k}^2} - \frac{1}{\omega_k^2} \right) \coth \frac{\beta_0 \omega_{0k}}{2} \cos \omega_k (t_1 + t_2).$$

(5.117)

This result (5.117) directly follows from the two point function (A.146) in a harmonic oscillator as explained in the Appendix A of the present paper.

In real space, the correlator is obtained by taking the Fourier transform of (5.117). The important fact is that the third term in (5.117), which violates time translation invariance, turns out to decrease and eventually vanish after a long time under quite general conditions [19, 21]. The time one has to wait to see this effect depends on the
wave-number \( k \) and increases as \( |k| \) decreases. Notice that the density matrix of this time-dependent system is always pure since we start from a pure state at zero temperature and it evolves in a unitary way under the time-dependent Hamiltonian. Thus to find a thermal behavior we need a process of coarse graining.

In this way, we can ignore this part and compare the others to the thermal propagator with temperature \( \beta \) (see \( \text{(A.142)} \) in Appendix A for an elementary derivation)

\[
G_\beta(k; t_1, t_2) = \frac{1}{2\omega_k} e^{-i\omega_k|t_1-t_2|} + \frac{\cos \omega_k(t_1 - t_2)}{\omega_k(e^{\beta \omega_k} - 1)}. \tag{5.118}
\]

We can read off the effective temperature by equating the coefficient of \( \cos \omega_k(t_1 - t_2) \)

\[
\beta_{\text{eff}}(k) = \frac{1}{\omega_k} \log \left( \frac{\omega_k - \omega_{0k}}{\omega_k + \omega_{0k}} \right)^2 + \frac{e^{\beta \omega_{0k}}(\omega_k + \omega_{0k})^2}{\omega_k(e^{\beta \omega_{0k}} - 1)} . \tag{5.119}
\]

It follows from this result that the effective temperature depends on each momentum mode.

It is clear that an effective temperature is obtained even when the initial state is the vacuum state rather than a thermal state. In this case

\[
\beta_{\text{eff},0}(k) = \frac{2}{\omega_k} \log \frac{\omega_k + \omega_{0k}}{\omega_k - \omega_{0k}} . \tag{5.120}
\]

It is easy to extend the above calculation to a situation where the mass starts from a value \( m_0 \) at \( t = 0 \), changes to a value \( m \neq m_0 \) for \( 0 < t < T \) and then finally changes back to \( m = m_0 \) for \( t > T \). Interestingly the late time correlation functions for \( t \gg T \) are again thermal with a \( k \)-dependent effective temperature. This is consistent with the fact that once the system thermalizes, it remains thermal.

### 5.2 Exact Toy Model from D1-branes

The framework considered in this paper can be used to construct a very simple toy model for a quantum quench in a strongly interacting theory. This is provided by the exact time-dependent solution of D1-brane whose induced metric is given by the AdS Vaidya metric studied generally in \( \text{(2.21)} \). In our current example, its two-dimensional induced metric looks like

\[
ds^2 = 2drdv - (r^2 - \varphi'(v)^2)dv^2 . \tag{5.121}
\]

To describe a quantum quench we can for example take \( v \equiv t - 1/r \)

\[
\varphi(v) = \varphi_0(1 + \tanh kv) , \tag{5.122}
\]
which approaches a step function in the large $k$ limit. Then the apparent horizon is situated at

$$r = \varphi'(v) = \frac{k \varphi_0}{\cosh^2 k v}.$$  \hfill (5.123)

In this model, the thermalization is occurred due to the time-dependent coupling \hfill (2.15) as we discussed before.

When the apparent horizon is slowly evolving, it is possible to define a local temperature by considering the analytic continuation of outgoing modes across the apparent horizon \hfill [58]. In our case this leads to a “local” temperature given by $\varphi'(v)$. While this expression is not strictly valid when the evolution is fast, we can still use this to get a rough estimate of the temperature. In an eikonal approximation, the Hawking particles emitted at some point $(r,t)$ on the apparent horizon will travel to the boundary along a line of constant $u = t + 1/r$ and will therefore reach the boundary at time $t = u - 1/r = v + 2/r$. Therefore the temperature perceived at the boundary at time $t$ is given by

$$T_H(t) = \varphi'(v), \quad t = v + \frac{2}{\varphi'(v)}.$$  \hfill (5.124)

where we need to first express $v(t)$ using the second equation of (5.124) and then substitute this in the first equation to obtain $T_H(t)$.

For a $\varphi(v)$ of the form \hfill (5.122) this is plotted in Fig. 8. The temperature is highest at $t = \varphi'(0) = \varphi_0$ and decreases to zero at $t = \infty$. Notice that the late time behavior is given by

$$T_H(t) \sim \frac{2}{t}.$$  \hfill (5.125)

This is a consequence of the conformal invariance of our D3-D1 system.

Quantum quench calculations in the literature have often been done in a closed environment. In such a case, a system got excited suddenly at some time by an external force and will reach an equilibrium which are static. On the other hand, in our system we find that the effective temperature decreases to zero finally. This is because the hypermultiplet sector, which are initially excited, is strongly coupled to the $\mathcal{N} = 4$ SYM sector and thus the energy will dissipate.

## 5.3 D3-D5 System with Time-dependent Mass

Here we would like to turn to a more interesting example: a holographic dual of quantum quench induced by time-dependent masses of scalar and spinor fields. For this purpose we consider a probe D5-brane wrapped on $AdS_4 \times S^2$ in $AdS_5 \times S^5$. This setup is supersymmetric and was first considered in \hfill [25]. It is dual to the four-dimensional $\mathcal{N} = 4$ SYM
Figure 8: The time-dependence of the effective temperature $T_H(t)$ in the D1-brane model under an eikonal approximation. We set $k = 4$ and $\varphi_0 = 1$.

coupled to a three-dimensional defect CFT. If we write the metric of $AdS_5 \times S^5$ as
\[
ds^2 = r^2(-dt^2 + \sum_{i=1}^{3} dx_i^2) + \frac{dr^2}{r^2} + d\theta^2 + \cos^2 \theta d\Omega_2^2 + \sin^2 \theta d\tilde{\Omega}_2^2,
\]
then the D5 is wrapped on $\Omega_2$ and $(t, x_1, x_2, r)$ directions, while it is trivially localized in $\tilde{\Omega}_2$ and $x_3$ direction. However, in actual numerical calculations, it is useful to perform the following coordinate transformation
\[
\eta = r \sin \theta, \quad z = \frac{1}{r \cos \theta},
\]
for which the $AdS_5 \times S^5$ metric looks like
\[
ds^2 = (\eta^2 + z^{-2})(-dt^2 + \sum_{i=1}^{3} dx_i^2) + \frac{d\eta^2}{\eta^2 + z^{-2}} + \frac{dz^2}{z^2 + z^2 \eta^2} + \frac{d\Omega_2^2}{1 + z^2 \eta^2} + \frac{\eta^2 d\tilde{\Omega}_2^2}{\eta^2 + z^{-2}}.
\]
In this new coordinate, we can specify the profile of the D5-brane by the function $\eta = \eta(t, z)$. Its DBI action reads
\[
S = -T_{D5} V_4 \int dt dz \frac{1}{z^4} \sqrt{1 + z^4 \eta^2 - \frac{\dot{\eta}^2}{(\eta^2 + z^{-2})^2}},
\]
where $V_4$ is the (infinite) volume of the $S^2 \times R^2$. Its equation of motion looks like
\[
\frac{\partial}{\partial z} \left( \frac{\eta'}{\sqrt{K}} \right) - \frac{\partial}{\partial t} \left( \frac{\dot{\eta}}{(1 + z^2 \eta^2)^2 \sqrt{K}} \right) - \frac{2\eta \dot{\eta}^2}{z^4(z^{-2} + \eta^2)^3 \sqrt{K}} = 0,
\]
37
where \( K \equiv 1 + z^4 \eta'^2 - \frac{\dot{\eta}^2}{(\eta^2 + z^{-2})^2} \).

When \( \eta \) can be treated as a small perturbation so that it satisfies

\[
z^2 \eta' \ll 1, \quad \text{and} \quad z^2 \dot{\eta} \ll 1 ,
\]

we can approximate the equation of motion (5.130) by the simple wave equation i.e. \( \eta'' - \dot{\eta} = 0 \). Therefore, if we require that the defect sector in the dual CFT has the time-dependent mass \( m(t) \), we can find the simple solution under this approximation

\[
\eta(t, z) = m(t - z) .
\]

However, in the interested regions where horizons appear, the condition (5.131) is inevitably violated. This is clear from the expression of \( g_{tt} \) component in the induced metric

\[
g_{tt}|_{\text{ind}} = -(\eta^2 + z^{-2}) + \frac{\dot{\eta}^2}{\eta^2 + z^{-2}} ,
\]

which should change its sign at apparent horizons. Since in our particular setup, the area of a codimension two surface for fixed values of \( t \) and \( z \) is always only a function of \( z \) (proportional to \( z^{-2} \)), the apparent horizon for our induced metric, which is defined by the vanishing area change along null geodesics, actually coincides with the points where \( g_{tt} \) vanishes.

Thus we numerically integrate the equation of motion (5.130) under the two boundary conditions presented below. The first boundary condition is imposed at the AdS boundary with a UV cut off \( z = z_0 \ll 1 \):

\[
\eta(t, z_0) = m(t - z_0) , \quad \frac{\partial \eta(t, z)}{\partial z} \bigg|_{z = z_0} = -\dot{m}(t - z_0) .
\]

The second one is at a initial time \( t = t_0 \):

\[
\eta(t_0, z) = m(t_0 - z) , \quad \frac{\partial \eta(t, z)}{\partial t} \bigg|_{t = t_0} = \dot{m}(t_0 - z) .
\]

Below we would like to analyze a specific choice of the mass function \( m(t) \) defined by

\[
m(t) = m_0(1 + \tanh kt) ,
\]

which describes a sudden quench for large \( k \).

In both of the above conditions we employed the approximated solution (5.132). For the first one (5.134) this is clearly justified because \( z_0 \) is infinitesimally small. On the other hand, for the next one (5.135), the condition (5.131) requires \( t_0 \ll \frac{1}{\sqrt{\text{max}}} \) in our particular case (5.136).
Figure 9: The function $\eta(t, z)$ for the choice $k = 10, m_0 = 0.1, t_0 = 0.01$ and $z_0 = 0.001$. We plotted the region defined by $0.01 < t < 2$ and $0.001 < z < 3$.

Following this strategy, we plotted the function $\eta(t, z)$ in the Fig. 9 for the particular choice of the parameters $k = 10, m_0 = 0.1, t_0 = 0.01$ and $z_0 = 0.001$. As is obvious from this graph, the kink, generated by the sudden mass change (5.136) in the UV, propagates at the speed of light into the IR region. This is qualitatively similar to our previous exact D1-brane solutions discussed in the previous subsection. To show the existence of the apparent horizon, we need to examine the sign of $g_{tt}$ of the induced metric. We plotted $-g_{tt}$ in Fig. 10 only when it is positive. We can confirm a hairpin-shape boundary near the line $t = z$, where $g_{tt}$ vanishes. Therefore we can conclude that our time-dependent mass at the boundary generates a thermal horizon after a certain time (estimates as $\sim \frac{1}{\sqrt{m_0 k}}$). Its temperature from an observer at the boundary decreases as time passes and eventually goes to zero due to the dissipation of energy into the bulk. From the dual CFT viewpoint, this clearly describes the process of the thermalization in the open system when we change the mass of scalar fields and fermions at some time.

Finally it is instructive to compare our result with that of free field theories. In the free scalar field theory, after a sudden change of scalar mass, the system should reach a sort of thermal equilibrium state. However, as we have reviewed briefly, different momentum modes have different temperatures (5.119) [19,21]. This momentum dependent temperature arises because the different momentum modes are not mixed with each other in the free field theory. A similar issue has been known to occur in integrable systems [18,21], where the final equilibrium state is described by the generalized Gibbs ensemble, due to the infinite number of conserved quantities. However, our system is a strongly coupled
Figure 10: The plot of $-g_{tt}(t,z)$ of the induced metric for the same choice as the previous plot. We concentrated on the region defined by $0.5 < t < 1.5$ and $0.8 < z < 2$. We plotted the graph only when $-g_{tt} > 0$ and thus the hairpin-shape boundary in the middle represents the positions of the apparent horizons.

quantum field theory as we work in the supergravity regime and we expect substantial mixings between states with different momenta. Indeed, our holographic analysis shows that the effective temperatures for all momentum modes in the dual CFT are originated from the single apparent horizon at a fixed time. Therefore our holographic description predicts that the final temperatures for each momenta at the equilibrium should be the same in strongly coupled field theories.

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\section{Thermal Correlation Functions and Mass Quench in Free Theories}

\subsection{Thermal Correlation Functions}

Here we would like to give an elementary derivation of thermal correlation functions for a harmonic oscillator and a free scalar field theory. We start with the standard Hamiltonian of a harmonic oscillator

\[ H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 = \omega \left( a^\dagger a + \frac{1}{2} \right), \quad (A.137) \]

where

\[ a^\dagger = \frac{1}{\sqrt{2}} \left( \sqrt{\omega} x - i \hat{p} \sqrt{\omega} \right), \quad a = \frac{1}{\sqrt{2}} \left( \sqrt{\omega} x + i \hat{p} \sqrt{\omega} \right). \quad (A.138) \]

The vacuum \(|0\rangle\) is defined by \(a|0\rangle = 0\) and we can define the normalized number state \(|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle\) as usual.

Using the commutation relation \([a, a^\dagger] = 1\), we obtain

\[ \hat{x}(t) = \frac{1}{\sqrt{2}\omega} \left( ae^{-i\omega t} + a^\dagger e^{i\omega t} \right), \quad (A.139) \]

The finite temperature density matrix for the inverse temperature \(\beta\) is given by

\[ \rho = e^{-\beta H} = \sum_{n=0}^{\infty} \frac{e^{-\left(n+\frac{1}{2}\right)\beta\omega} |n\rangle\langle n|}{Z_0}, \quad \left( Z_0 \equiv \frac{1}{2 \sinh \frac{\beta\omega}{2}} \right). \quad (A.140) \]

We can calculate various two-point functions as follows. First of all, we obtain the Wightman function

\[ \langle \hat{x}(t) \hat{x}(0) \rangle \equiv \text{Tr}[e^{-\beta H} \hat{x}(t) \hat{x}(0)] = \frac{\cosh \left( \frac{\beta\omega}{2} - i\omega t \right)}{2\omega \sinh \frac{\beta\omega}{2}}. \quad (A.141) \]

Next, the time ordered two-point function is given by

\[ T\langle \hat{x}(t) \hat{x}(0) \rangle = \frac{\cosh \left( \frac{\beta\omega}{2} - i\omega |t| \right)}{2\omega \sinh \frac{\beta\omega}{2}} = \frac{1}{2\omega} e^{-i\omega |t|} + \frac{\cos \omega t}{\omega(e^{\beta\omega} - 1)}. \quad (A.142) \]

It is also useful to calculate the commutator

\[ \langle [\hat{x}(t), \hat{x}(0)] \rangle \equiv \text{Tr}[e^{-\beta H} [\hat{x}(t), \hat{x}(0)]] = -\frac{i}{\omega} \sin \omega t, \quad (A.143) \]
which is actually independent of the temperature. This leads to the retarded Green function

\[ G_R = \langle [\hat{x}(t), \hat{x}(0)] \rangle_R \equiv \theta(t) \text{Tr}[e^{-\beta H}[\hat{x}(t), \hat{x}(0)]] = -\frac{i}{\omega} \theta(t) \sin \omega t . \] (A.144)

In the free scalar field theory (5.116) with a mass \( m \), if we work in the momentum basis \( k \), the calculations are identical to Harmonic oscillators with the \( k \)-dependent frequency \( \omega_k = \sqrt{m^2 + k^2} \). Therefore the time ordered two point function is found to be (5.118) as is clear from (A.142).

### A.2 Mass Quench

Now we would like to analyze the setup of the quantum quench due to the sudden change of frequency \( \omega \) in a harmonic oscillator (A.137) as considered in [21]. Let us assume for \( t < 0 \) the frequency is given by \( \omega_0 \) and the system is in a thermal equilibrium at temperature \( T_0 = 1/\beta_0 \). At \( t = 0 \), the frequency suddenly is changed into \( \omega \) by a certain external force and we are interested in physics when \( t > 0 \). Since the \( \hat{x} \) and \( \hat{p} \) are continuous at \( t = 0 \), the creation and annihilation operators change at \( t = 0 \) from \((a_0, a_0^\dagger)\) to \((a, a^\dagger)\) via the following Bogoliubov transformation

\[ a^\dagger = \cosh \zeta \ a_0^\dagger + \sinh \zeta \ a_0 , \quad a = \sinh \zeta \ a_0^\dagger + \cosh \zeta \ a_0 , \]

\[ \cosh \zeta \equiv \frac{1}{2} \left( \frac{\omega}{\omega_0} + \sqrt{\frac{\omega_0}{\omega}} \right) . \] (A.145)

Using (A.145) and (A.139), we can obtain the time-ordered two point function after the quench [21]

\[ T\langle \hat{x}(t_1)\hat{x}(t_2) \rangle \equiv T \left[ \text{Tr}[e^{-\beta_0 H_0}\hat{x}(t_1)\hat{x}(t_2)] \right] \]

\[ = \frac{1}{2\omega} e^{-i\omega|t_1-t_2|} + \left[ \frac{\omega_0}{4} \left( \frac{1}{\omega^2} + \frac{1}{\omega_0^2} \right) \coth \frac{\beta_0 \omega_0}{2} - \frac{1}{2\omega} \right] \cos \omega(t_1 - t_2) \]

\[ + \frac{\omega_0}{4} \left( \frac{1}{\omega_0^2} - \frac{1}{\omega^2} \right) \coth \frac{\beta_0 \omega_0}{2} \cos \omega(t_1 + t_2) . \] (A.146)

In section 5.1 we discuss the quantum quench in the free massive scalar field theory, where the mass is suddenly changed from \( m_0 \) to \( m \). In this case we can simply replace \( \omega \) and \( \omega_0 \) with the \( k \)-dependent frequency \( \omega_k = \sqrt{k^2 + m^2} \) and \( \omega_{0k} = \sqrt{k^2 + m_0^2} \), respectively. This leads to the two point function (5.117).
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