Ten Lectures on the Moment Problem

August 31, 2020
If $\mu$ is a positive Borel measure on the line and $k$ is a nonnegative integer, the number

$$s_k \equiv s_k(\mu) = \int_{\mathbb{R}} x^k \, d\mu(x) \tag{0.1}$$

is called the $k$-th moment of $\mu$. For instance, if the measure $\mu$ comes from the distribution function $F$ of a random variable $X$, then the expectation value of $X^k$ is just the $k$-the moment,

$$E[X^k] = s_k = \int_{-\infty}^{+\infty} x^k dF(x),$$

and the variance of $X$ is $\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2 = s_2 - s_1^2$ (of course, provided the corresponding numbers are finite).

The moment problem is the inverse problem of “finding” the measure when the moments are given.

In order to apply functional-analytic methods it is convenient to rewrite the moment problem in terms of linear functionals. For a real sequence $s = (s_n)_{n \in \mathbb{N}_0}$ let $L_s$ denote the linear functional on the polynomial algebra $\mathbb{R}[x]$ defined by $L_s(x^n) = s_n$, $n \in \mathbb{N}_0$. Then, by the linearity of the integral, (0.1) holds for all $k$ if and only if

$$L_s(p) = \int_{-\infty}^{+\infty} p(x) d\mu(x) \quad \text{for} \quad p \in \mathbb{R}[x]. \tag{0.2}$$

That is, the moment problem asks whether a linear functional on $\mathbb{R}[x]$ admits an integral representation (0.2) with some positive measure $\mu$ on $\mathbb{R}$. This is the simplest version of the moment problem, the so-called Hamburger moment problem.

Moments and the moment problem have natural $d$-dimensional versions. For a positive measure $\mu$ on $\mathbb{R}^d$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, the $\alpha$-the moments is defined by

$$s_\alpha(\mu) = \int_{\mathbb{R}^d} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \, d\mu(x).$$
Let $K$ be a closed subset of $\mathbb{R}^d$. An important variant of the moment problem, called the $K$-moment problem, requires that the representing $\mu$ is supported on $K$. If only a finite number of moments are given, we have the truncated moment problem.

Moments were applied to the study of functions by the Russian mathematicians P.L. Chebychev (1874) and A.A. Markov (1884). However, the moment problem first appeared in a famous memoir (1894) of the Dutch mathematician T.J. Stieltjes. He formulated and solved this problem for the half-line and gave the first explicit example of an indeterminate problem. The cases of the real line and of bounded intervals were studied only later by H. Hamburger (1920) and F. Hausdorff (1920).

Because of the simplicity of its formulation it is surprising that the moment problem has deep interactions with many branches of mathematics (functional analysis, function theory, real algebraic geometry, spectral theory, optimization, numerical analysis, convex analysis, harmonic analysis and others) and a broad range of applications. The AMS volume “Moments in Mathematics”, edited by H.J. Landau (1987), and the book “Moments, Positive Polynomials and Their Applications” by J.P. Lasserre (2015) illustrate this in a convincing manner.

The following notes grew out from a series of lectures given at the

“School on Sums of Squares, Moment Problems, and Polynomial Optimization”,

which was organized from 1.1.2019 till 31.3.2019 at the

Vietnam Institute for Advanced Studies in Mathematics (VIASM)

in Hanoi. I would like to thank my Vietnamese colleague, Prof. Trinh Le Cong, for the kind invitation and his warm hospitality during this visit. The audience consisted of researchers, young scientists, and graduate students.

The first three lectures deal with one-dimensional full and truncated moment problems and contain well-known classical material. The topics of the other lectures were chosen according to the themes of the VIASM school. In Lectures 4–7, we investigate the multi-dimensional full moment problem with particular emphasis on the interactions with real algebraic geometry. Lecture 7 gives a digression into applications of moment methods and real algebraic geometry to polynomial optimization. The final Lectures 8–10 are devoted to the multidimensional truncated moment problem. In Lectures 4–10, I have covered selected advanced results as well as recent developments.

The following are smoothed and slightly extended versions of the handouts I had given to the audience. I have tried to maintain the casual style of the lectures and to illustrate the theory by many examples. The notes are essentially based on my book

“The Moment Problem”, Graduate Text in Mathematics, Springer, 2017,

which is quoted as [MP] in what follows. For some results complete proofs are given, while for most of them I referred to [MP]. These notes have no bibliography. The book [MP] contains additional material and also detailed references for the
main results. Finally, it should be emphasized that these notes are not a book on the moment problem. It is more appropriate to think of them as an entrance guide into some developments of the beautiful subject.
Contents

1 Integral representations of linear functionals ........................................ 9
   1.1 Positive linear functionals on adapted spaces .............................. 10
   1.2 Positive polynomials on intervals ........................................ 13
   1.3 Moment problems on intervals ........................................ 14

2 One-dimensional moment problem: determinacy ............................ 17
   2.1 An indeterminate measure: lognormal distribution .................. 17
   2.2 Carleman’s condition for the Hamburger moment problem .......... 18
   2.3 Krein’s condition for the Hamburger moment problem ............ 19
   2.4 Carleman and Krein conditions for the Stieltjes moment problem .. 20

3 The one-dimensional truncated moment problem on a bounded interval ........................................ 23
   3.1 Existence of a solution ........................................ 23
   3.2 The moment cone and its boundary points ........................................ 24
   3.3 Interior points and principal measures ........................................ 26

4 The moment problem on compact semi-algebraic sets .................. 29
   4.1 Basic notions from real algebraic geometry .......................... 29
   4.2 Strict Positivstellensatz and solution of the moment problem ...... 31
   4.3 Localizing functionals and Hankel matrices .......................... 33
   4.4 Moment problem criteria based on semirings ......................... 34
   4.5 Two technical results and the proof of Theorem 4.11 ............... 35

5 The moment problem on closed semi-algebraic sets: the fibre theorem .... 39
   5.1 Positive functionals which are not moment functionals ............. 39
   5.2 Properties (MP) and (SMP) and the fibre theorem .................... 40
   5.3 First application: cylinder sets with compact base .................. 43
   5.4 (SMP) for subsets of the real line .................................. 44
   5.5 Second application: the two-sided complex moment problem ....... 45
| Chapter | Title                                                                 | Pages   |
|---------|----------------------------------------------------------------------|---------|
| 6       | The moment problem on closed sets: determinacy and Carleman condition | 47      |
|         | 6.1 Various notions of determinacy                                   | 47      |
|         | 6.2 Determinacy via marginal measures                                | 48      |
|         | 6.3 The multivariate Carleman condition and Nussbaum’s theorem       | 49      |
|         | 6.4 Some operator-theoretic reformulations                           | 51      |
| 7       | Polynomial optimization and semidefinite programming                | 53      |
|         | 7.1 Semidefinite programming                                         | 53      |
|         | 7.2 Lasserre relaxations of polynomial optimization with constraints | 55      |
|         | 7.3 Polynomial optimization with constraints                          | 57      |
| 8       | Truncated multidimensional moment problem: existence via positivity  | 59      |
|         | 8.1 The Richter-Tchakaloff Theorem                                   | 60      |
|         | 8.2 Positive semidefinite 2n-sequences                               | 61      |
|         | 8.3 The truncated moment problem on projective space                 | 62      |
|         | 8.4 Stochel’s theorem                                                | 64      |
| 9       | Truncated multidimensional moment problem: existence via flat extensions | 67     |
|         | 9.1 Hankel matrices                                                  | 67      |
|         | 9.2 The full moment problem with finite rank Hankel matrix            | 69      |
|         | 9.3 Flat extensions and the flat extension theorem                   | 70      |
|         | 9.4 Hankel matrices of functionals with finitely atomic measures     | 71      |
| 10      | Truncated multidimensional moment problem: core variety and moment cone | 75      |
|         | 10.1 Strictly positive linear functionals                             | 75      |
|         | 10.2 The core variety                                                | 76      |
|         | 10.3 The moment cone                                                 | 79      |
Lecture 1
Integral representations of linear functionals

Abstract:
An integral representation theorem of positive functionals on Choquet’s adapted spaces is obtained. As applications, Haviland’s theorem is derived and existence results for moment problems on intervals are developed.

All variants of the moment problem deal with integral representations of certain linear functionals. We begin with a rather general setup.

Suppose \( X \) is a locally compact topological Hausdorff space. The Borel algebra \( \mathcal{B}(X) \) is the \( \sigma \)-algebra generated by the open subsets of \( X \).

**Definition 1.1.** A Radon measure on \( X \) is a measure \( \mu : \mathcal{B}(X) \to [0, +\infty] \) such that
\[
\mu(K) < \infty \quad \text{for each compact subset } K \text{ of } X \text{ and}
\]
\[
\mu(M) = \sup \{ \mu(K) : K \subseteq M, K \text{ compact} \} \quad \text{for all } M \in \mathcal{B}(X).
\] (1.1)

Thus, in our terminology Radon measures are always nonnegative!

Closed subsets of \( \mathbb{R}^d \) are locally compact in the induced topology of \( \mathbb{R}^d \).

Further, suppose \( E \) is a vector space of continuous real-valued functions on \( X \). In a very general and modern form, the moment problem is the following problem:

**Given a linear functional** \( L : E \to \mathbb{R} \) **and a closed subset** \( K \) **of** \( X \), when does there exist a Radon measure \( \mu \) **supported on** \( K \) **such that**
\[
L(f) = \int_X f(x) \, d\mu(x) \quad \text{for } f \in E?
\] (1.2)

In this case, \( L \) is called a \( K \)-moment functional or a moment functional if \( K = X \).

When we write equations such as (1.2) we always mean that the function \( f(x) \) is \( \mu \)-integrable and its \( \mu \)-integral is \( L(f) \).

Let \( \{ e_j : j \in J \} \) be a basis of the vector space \( E \). Then, for each real sequence \( s = (s_j)_{j \in J} \), there is unique linear functional \( L_s : E \to \mathbb{R} \) given by \( L_s(e_j) = s_j \) for \( j \in J \); this functional is called the Riesz functional associated with \( s \). Clearly, (1.2) holds for the functional \( L = L_s \) if and only if \( s_j = \int e_j(x) \, d\mu(x) \) for all \( j \in J \). In this case, the numbers \( s_j \) are called the generalized moment of \( \mu \).
The most important cases are: \( \mathcal{X} = \mathbb{R}^d \)

Case 1. \( E = \mathbb{R}_d[x] := \mathbb{R}[x_1, \ldots, x_d] \): classical full moment problem.

Case 2. \( E = \mathbb{R}_d[x]^m := \{ p \in \mathbb{R}_d[x] : \deg(p) \leq m \} \): truncated moment problem.

These two special cases lead already to an interesting theory with nice results and difficult problems! In both cases we can take a basis \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}, \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d \), of monomials; then the numbers \( \int x^\alpha d\mu(x) \) are the moments of the measure \( \mu \).

Why positive measures and not signed measures? It can be shown that in both Cases 1 and 2 each linear functional can be represented by some signed measure.

There is an obvious obstruction for a functional to be a moment functional. Set \( E_+ := \{ f \in E : f(x) \geq 0 \text{ for } x \in \mathcal{X} \} \).

Since integration of nonnegative functions by positive measures gives nonnegative numbers, each moment functional \( L \) is \( E_+ \)-positive, that is, \( L(f) \geq 0 \) for all \( f \in E_+ \).

In Case 1, the \( E_+ \)-positivity is also sufficient (by Haviland’s theorem below), in Case 2 it is not in general. Even if the \( E_+ \)-positivity is sufficient, one would need a description of \( E_+ \), which can be very difficult. We will return to this problem later.

### 1.1 Positive linear functionals on adapted spaces

The representation theorem proved below is based on the following notion, invented by G. Choquet, of an adapted space.

**Definition 1.2.** A linear subspace \( E \) of \( C(\mathcal{X}; \mathbb{R}) \) is called *adapted* if the following conditions are satisfied:

(i) \( E = E_+ - E_+ \).

(ii) For each \( x \in \mathcal{X} \) there exists an \( f_x \in E_+ \) such that \( f_x(x) > 0 \).

(iii) For each \( f \in E_+ \) there exists a \( g \in E_+ \) such that for any \( \varepsilon > 0 \) there exists a compact subset \( K_\varepsilon \) of \( \mathcal{X} \) such that \( |f(x)| \leq \varepsilon |g(x)| \) for all \( x \in \mathcal{X} \setminus K_\varepsilon \).

If condition (iii) is satisfied, we shall say that \( g \) dominates \( f \). Roughly speaking, this means that \( |f(x)|/g(x)| \to 0 \text{ as } x \to \infty \).

**Example 1.3.** Let \( \mathcal{X} \) be closed subset of \( \mathbb{R}^d \).

Then \( E = \mathbb{R}_d[x] \) is an adapted subspace of \( C(\mathcal{X}; \mathbb{R}) \). Indeed, condition (i) in Definition 1.2 follows from the relation \( 4p = (p + 1)^2 - (p - 1)^2 \). (ii) is trivial. If \( p \in E_+ \), then \( g = (x_1^2 + \cdots + x_d^2)f \) dominates \( f \), so condition (iii) is also fulfilled.

In contrast, \( E = \mathbb{R}_d[x]^m \) is not an adapted subspace of \( C(\mathcal{X}; \mathbb{R}) \), because (iii) fails.

From the technical side this is an important difference between the full moment problem and the truncated moment problem.
We choose a $C$ there exists a $g \in E_+$ such that $g(x) \geq f(x)$ for all $x \in X$.

**Proof.** Let $x \in X$. By Definition 1.2(ii) there exists a function $g_\ast \in E_+$ such that $g_\ast(x) > 0$. Multiplying $g_\ast$ by some positive constant we get $g_\ast(x) > f(x)$. This inequality remains valid in some neighborhood of $x$. By the compactness of $\text{supp} f$ there are finitely many $x_1, \ldots, x_n \in X$ such that $g(x) := g_{x_1}(x) + \cdots + g_{x_n}(x) > f(x)$ for $x \in \text{supp} f$ and $g(x) \geq f(x)$ for all $x \in X$.

We will use the following Hahn–Banach type extension result.

**Lemma 1.5.** Let $E$ be a linear subspace of a real vector space $F$ and let $C$ be a convex cone of $F$ such that $F = E + C$. Then each $(C \cap E)$-positive linear functional $L$ on $E$ can be extended to a $C$-positive linear functional $\tilde{L}$ on $F$.

**Proof.** Let $f \in F$. We define

$$q(f) = \inf \{L(g) : g \in E, g - f \in C\}. \quad (1.3)$$

Since $F = E + C$, there exist $g \in E$ and $c \in C$ such that $-f = -g + c$, so $c = g - f \in C$ and the corresponding set in (1.3) is not empty. It is easily seen that $q(f)$ is finite and that $q$ is a sublinear functional such that $L(g) = q(g)$ for $g \in E$. Therefore, by the Hahn–Banach theorem, there is an extension $\tilde{L}$ of $L$ to $F$ such that $\tilde{L}(f) \leq q(f)$ for all $f \in F$.

Let $h \in C$. Setting $g = 0, f = -h$ we have $g - f \in C$, so that $q(-h) \leq L(0) = 0$ by (1.3). Hence $\tilde{L}(-h) \leq q(-h) \leq 0$, so that $\tilde{L}(h) \geq 0$. Thus, $\tilde{L}$ is $C$-positive.

Our first main result is the following theorem.

**Theorem 1.6.** Suppose $E$ is an adapted subspace of $C(X; \mathbb{R})$. For any linear functional $L : E \to \mathbb{R}$ the following are equivalent:

(i) The functional $L$ is $E_+$-positive, that is, $L(f) \geq 0$ for all $f \in E_+$.

(ii) $L$ is a moment functional, that is, there exists a Radon measure $\mu$ on $X$ such that $L(f) = \int f \, d\mu$ for $f \in E$.

**Proof.** The implication (ii)$\rightarrow$(i) is obvious. We prove (i)$\rightarrow$(ii) and begin by setting

$$\tilde{E} := \{f \in C(X; \mathbb{R}) : |f(x)| \leq g(x), x \in X, \text{ for some } g \in E\}$$

and claim that $\tilde{E} = E + (\tilde{E})_+$. Obviously, $E + (\tilde{E})_+ \subseteq \tilde{E}$. Conversely, let $f \in \tilde{E}$. We choose a $g \in E_+$ such that $|f| \leq g$. Then we have $f + g \in (\tilde{E})_+, g \in E$ and $-f = -g + (g + f) \in E + (\tilde{E})_+$. That is, $\tilde{E} = E + (\tilde{E})_+$.

By Lemma 1.5 $L$ can be extended to an $(\tilde{E})_+$-positive linear functional $\tilde{L}$ on $\tilde{E}$. We have $C_c(X; \mathbb{R}) \subseteq \tilde{E}$ by Lemma 1.4 From the Riesz representation theorem it follows that there is a Radon measure $\tilde{\mu}$ on $X$ such that $\tilde{L}(f) = \int f \, d\tilde{\mu}$ for $f \in C_c(X; \mathbb{R})$. By Definition 1.2(i), $E = E_+ - E_+$. To complete the proof it therefore suffices to show that each $f \in E_+$ is $\tilde{\mu}$-integrable and satisfies $L(f) = \int f \, d\tilde{\mu}$. 
Thus, so

Example 1.8. Set \( E := C_c(\mathbb{R}; \mathbb{R}) + \mathbb{R} \cdot 1 \) and define a linear functional on \( E \) by

\[
L(f + \lambda \cdot 1) := \lambda \quad \text{for} \quad f \in C_c(\mathbb{R}; \mathbb{R}), \ \lambda \in \mathbb{R},
\]

where \( 1 \) is the constant function equal to 1. Then \( L \) is \( E_+ \)-positive, but it is not a moment functional. (Indeed, since \( L(f) = 0 \) for \( f \in C_c(\mathbb{R}; \mathbb{R}) \), the measure \( \mu \) would be zero. But this is impossible, because \( L(1) = 1 \).)

We can consider \( E \) as a subspace of \( C(\overline{\mathbb{R}}; \mathbb{R}) \), where \( \overline{\mathbb{R}} = \mathbb{R} \cup \{ \infty \} \) is the point compactification of \( \mathbb{R} \), by setting \( (f + \lambda \cdot 1)(\infty) = \lambda \). Then, \( L \) is given by the integral of \( \delta_\infty \), so \( L \) is a moment functional.

Now we give an important application. For a closed subset \( K \) of \( \mathbb{R}^d \) we set
Since \( E = \mathbb{R}_d[x] \) is an adapted subspace of \( C(\mathbb{K}, \mathbb{R}) \), as noted in Example 1.3, Theorem 1.6 gives the following result, which is called Haviland’s theorem.

**Theorem 1.9.** Let \( \mathbb{K} \) be a closed subset of \( \mathbb{R}^d \) and \( L \) a linear functional on \( \mathbb{R}_d[x] \). The following statements are equivalent:

(i) \( L(f) \geq 0 \) for all \( f \in \text{Pos}(\mathbb{K}) \).

(ii) \( L \) is a \( \mathbb{K} \)-moment functional, that is, there exists a Radon measure \( \mu \) on \( \mathbb{R}^d \) supported on \( \mathbb{K} \) such that \( L(f) = \int_{\mathbb{K}} f \, d\mu \) for all \( f \in \mathbb{R}_d[x] \).

### 1.2 Positive polynomials on intervals

To settle the existence problem for moment problems on intervals, by Haviland’s theorem it is natural to look for descriptions of positive polynomials on intervals.

Let \( p(x) \in \mathbb{R}[x] \) be a nonconstant polynomial. If \( \lambda \) is a non-real zero of \( p \) with multiplicity \( l \), so is \( \overline{\lambda} \). Clearly, \( (x - \lambda)^l(x - \overline{\lambda})^l = ((x - u)^2 + v^2)^l \), where \( u = \text{Re} \lambda \) and \( v = \text{Im} \lambda \). Therefore, by the fundamental theorem of algebra, \( p \) factors as

\[
p(x) = a(x - \alpha_1)^{n_1} \cdots (x - \alpha_r)^{n_r} ((x - u_1)^2 + v_1^2)^{l_1} \cdots ((x - u_k)^2 + v_k^2)^{l_k},
\]

where \( x - \alpha_i \) are pairwise different linear factors and \( (x - u)^2 + v^2 \) are pairwise different quadratic factors. Note that linear factors or quadratic factors may be absent.

Let \( \sum \mathbb{R}[x]^2 \) denote the set of finite sums of squares \( p^2 \), where \( p \in \mathbb{R}[x] \).

**Proposition 1.10.** (i) \( \text{Pos}(\mathbb{R}) = \sum \mathbb{R}[x]^2 \).

(ii) \( \text{Pos}([0, +\infty)) = \{ f + xg : f, g \in \sum \mathbb{R}[x]^2 \} \).

(iii) \( \text{Pos}([a, b]) = \{ f + (b - x)(x - a)g : f, g \in \sum \mathbb{R}[x]^2 \} \), where \( a, b \in \mathbb{R} \), \( a < b \).

**Proof.** It suffices to show that the sets on the left are subsets of the sets on the right.

(i): Let \( p \in \text{Pos}(\mathbb{R}) \), \( p \neq 0 \). Since \( p(x) \geq 0 \) on \( \mathbb{R}, a > 0 \) and the numbers \( n_1, \ldots, n_r \) in (1.5) are even. Hence \( p \) is a product of squares and of sums of two squares. Therefore, \( p \in \sum \mathbb{R}[x]^2 \).

(ii): Let \( p \in \text{Pos}([0, +\infty)) \), \( p \neq 0 \), and consider (1.5). Set \( Q := \sum \mathbb{R}[x]^2 + x \sum \mathbb{R}[x]^2 \). For \( f_1, f_2, g_1, g_2 \in \sum \mathbb{R}[x]^2 \), we have

\[
(f_1 + xg_1)(f_2 + xg_2) = (f_1f_2 + x^2g_1g_2) + x(f_1g_2 + g_1f_2) \in Q.
\]

Hence \( Q \) is closed under multiplication. Therefore, it suffices to show that all factors in (1.5) are in \( Q \). For products of quadratic factors and even powers of linear factors this is obvious. It remains to treat the constant \( a \) and linear factors \( x - \alpha_i \) with real zeros \( \alpha_i \) of odd multiplicities. Since \( p(x) \geq 0 \) on \( [0, +\infty) \), \( a > 0 \) by letting \( x \to +\infty \) and \( \alpha_i \leq 0 \), because \( p(x) \) changes its sign in a neighborhood of a zero with odd multiplicity. Hence \( a \in Q \) and \( x - \alpha_i = (-\alpha_i + x) \in \sum \mathbb{R}[x]^2 + x \sum \mathbb{R}[x]^2 = Q \).

(iii) follows by a similar, but slightly longer reasoning [MP, Proposition 3.3]. \( \square \)
The next result, called Markov–Lukacs theorem, sharpens Proposition 1.10(iii) by replacing sums of squares by single squares and adding degree requirements. Its proof is much more involved than that of Proposition 1.10(iii). However for the solution of the truncated moment problem on \([a,b]\) in Lecture 3 it suffices to have the weaker statement with sum of squares instead of single squares which is easy to prove [MP, Proposition 3.2].

**Proposition 1.11.** For \(a, b \in \mathbb{R}, a < b,\) and \(n \in \mathbb{N}_0,\)

\[
\text{Pos}([a,b])_{2n} = \{ p_n(x)^2 + (b-x)(a-x)q_{n-1}(x)^2 : p_n \in \mathbb{R}[x], q_{n-1} \in \mathbb{R}[x] \},
\]

\[
\text{Pos}([a,b])_{2n+1} = \{ (b-x)p_n(x)^2 + (a-x)q_n(x)^2 : p_n, q_n \in \mathbb{R}[x] \}.
\]

**Proof** (MP, Corollary 3.24). \(\square\)

### 1.3 Moment problems on intervals

In this section we combine Haviland’s theorem with the description of positive polynomials on intervals and derive the solutions for the classical one-dimensional moment problems.

Let \(s = (s_n)_{n \in \mathbb{N}_0}\) be a real sequence. The **Riesz functional** \(L_{s}\) is the linear functional on \(\mathbb{R}[x]\) defined by

\[
L_{s}(p_n(x)) = s_n, \quad n \in \mathbb{N}_0.
\]

The sequence \(s\) is called **positive semidefinite** if for all \(\xi_0, \xi_1, \ldots, \xi_n \in \mathbb{R}\) and \(n \in \mathbb{N}\) we have

\[
\sum_{k,l=0}^{n} s_{k+l} \xi_k \xi_l \geq 0. \quad (1.6)
\]

The set of positive semidefinite sequences is denoted by \(\mathcal{P}(\mathbb{N}_0)\).

A linear functional \(L\) on \(\mathbb{R}[x]\) is called **positive** if \(L(p^2) \geq 0\) for all \(p \in \mathbb{R}[x]\).

Further, we define the **Hankel matrix** \(H_n(s)\) and the **Hankel determinant** \(D_n(s)\) by

\[
H_n(s) = \begin{pmatrix}
0 & s_1 & s_2 & \cdots & s_n \\
s_1 & s_2 & s_3 & \cdots & s_{n+1} \\
s_2 & s_3 & s_4 & \cdots & s_{n+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
s_n & s_{n+1} & s_{n+2} & \cdots & s_{2n}
\end{pmatrix},
\]

\[
D_n(s) = \det H_n(s). \quad (1.7)
\]

The following result is **Hamburger’s theorem.**

**Theorem 1.12.** (Solution of the Hamburger moment problem)

For any real sequence \(s = (s_n)_{n \in \mathbb{N}_0}\) the following statements are equivalent:

(i) \(s\) is a Hamburger moment sequence, that is, there is a Radon measure \(\mu\) on \(\mathbb{R}\) such that

\[
s_n = \int_{\mathbb{R}} x^n d\mu(x) \quad \text{for} \quad n \in \mathbb{N}_0. \quad (1.8)
\]
1.3 Moment problems on intervals

(ii) \( s \in \mathcal{P}(\mathbb{N}_0) \), that is, the sequence \( s \) is positive semidefinite.

(iii) All Hankel matrices \( H_n(s) \), \( n \in \mathbb{N}_0 \), are positive semidefinite.

(iv) \( L_s(p^2) \geq 0 \) for \( p \in \mathbb{R}[x] \).

**Proof.** The main implication (iv) \( \Rightarrow \) (i) follows from Haviland’s Theorem 1.9 combined with Proposition 1.10(i). A straightforward computation shows that (i) implies (ii). Since the Hankel matrix \( H_n(s) \) is just the matrix associated with the quadratic form (1.6), (ii) and (iii) are equivalent. For \( p(x) = \sum_{j=0}^{n} a_n x^j \in \mathbb{R}[x] \) we compute

\[
L_s(p^2) = \sum_{k,l=0}^{n} a_k a_l L_s(x^{k+l}) = \sum_{k,l=0}^{n} a_k a_l s_{k+l}.
\]

Hence (ii) and (iv) are equivalent. \( \square \)

The next proposition answers the question of when \( s \) has a representing measure of finite support.

**Proposition 1.13.** For a positive semidefinite sequence \( s \) the following are equivalent:

(i) There is a number \( n \in \mathbb{N}_0 \) such that

\[
D_0(s) > 0, \ldots, D_{n-1}(s) > 0 \quad \text{and} \quad D_k(s) = 0 \quad \text{for} \ k \geq n. \quad (1.9)
\]

(ii) \( s \) is a moment sequence with a representing measure \( \mu \) with support of \( n \) points.

**Remark 1.14.** It was recently proved by Berg and Szwarc (2015) that the assumption “\( s \) is positive semidefinite” in Proposition 1.13 can be omitted.

To solve moment problems on intervals it is convenient to have the shifted sequence \( E_s \) defined by

\[
(E_s)_n = s_{n+1}, \quad n \in \mathbb{N}_0.
\]

Clearly, \( L_{E_s}(p(x)) = L_s(x p(x)) \) for \( p \in \mathbb{R}[x] \).

The next main result is **Stieltjes’ theorem.**

**Theorem 1.15.** (Solution of the Stieltjes moment problem)
For any real sequence \( s \) the following statements are equivalent:

(i) \( s \) is a Stieltjes moment sequence, that is, there is a Radon measure \( \mu \) on \([0, +\infty)\) such that

\[
s_n = \int_0^\infty x^n d\mu(x) \quad \text{for} \quad n \in \mathbb{N}_0. \quad (1.10)
\]

(ii) \( s \in \mathcal{P}(\mathbb{N}_0) \) and \( E_s \in \mathcal{P}(\mathbb{N}_0) \).

(iii) All Hankel matrices \( H_n(s) \) and \( H_n(E_s) \), \( n \in \mathbb{N}_0 \), are positive semidefinite.

(iv) \( L_s(p^2) \geq 0 \) and \( L_s(xq^2) \geq 0 \) for all \( p, q \in \mathbb{R}[x] \).
Proof. The proof is almost the same as the proof of Theorem 1.12 instead of Proposition 1.10(i) we use Proposition 1.10(ii).

Combining Haviland’s theorem with Proposition 1.10(iii) yields the following.

**Theorem 1.16. (Solution of the moment problem for a compact interval)**

Let \( a, b \in \mathbb{R}, a < b \). For a real sequence \( s \) the following are equivalent:

(i) \( s \) is an \([a, b]\)-moment sequence.

(ii) \( s \in \mathcal{P}(\mathbb{N}_0) \) and \( ((a + b)E_s - E(E_s) - ab s) \in \mathcal{P}(\mathbb{N}_0) \).

(iii) \( L_s(p^2) \geq 0 \) and \( L_s((b - x)(x - a)q^2) \geq 0 \) for all \( p, q \in \mathbb{R}[x] \).

Finally, we state two solvability criteria of moment problems that are not based on squares. The first result is easily derived from Bernstein’s theorem. It says that each polynomial \( p \in \mathbb{R}[x] \) such that \( p(x) > 0 \) on \([-1, 1]\) can be written as

\[
p(x) = \sum_{k,l=0}^{n} \alpha_{kl} (1-x)^k (1+x)^l, \quad \text{where } \alpha_{kl} \geq 0.
\]

**Theorem 1.17.** Let \( s = (s_n)_{n \in \mathbb{N}_0} \) be a real sequence and let \( L_s \) be its Riesz functional on \( \mathbb{R}[x] \). Then \( s \) is a \([-1, 1]\)-moment sequence if and only if

\[
L_s((1-x)^k (1+x)^l) \geq 0 \quad \text{for all } \quad k, l \in \mathbb{N}_0.
\]  

(1.11)

The next theorem is Hausdorff’s theorem. It can be obtained by a simple computation from Theorem 1.17 using the bijection \( x \mapsto \frac{1}{2}(1 + x) \) of \([-1, 1]\) onto \([0, 1]\).

**Theorem 1.18.** A real sequence \( s \) is a \([0, 1]\)-moment sequence if and only if

\[
((I - E)^n s)_k \equiv \sum_{j=0}^{k} (-1)^j \binom{n}{j} s_{k+j} \geq 0 \quad \text{for } \quad k, n \in \mathbb{N}_0.
\]  

(1.12)
Lecture 2
One-dimensional moment problem: determinacy

Abstract:
This lecture is concerned with the determinacy question for one-dimensional Hamburger and Stieltjes moment problems. The Carleman condition (a sufficient condition for determinacy) and the Krein condition (a sufficient condition for indeterminacy) are developed.

Let \( \mu \) be a Radon measure on \( \mathbb{R} \) with finite moments \( s_n = \int_{\mathbb{R}} x^n d\mu(x), n \in \mathbb{N}_0 \). If \( \mu \) has compact support, it is easily seen (using the Weierstrass approximation theorem on uniform approximation of continuous functions by polynomials) that \( \mu \) is uniquely determined by its moment sequence \( s = (s_n)_{n \in \mathbb{N}_0} \). Further, if \( c > 0 \), it can be shown that \( \mu \) is supported on the interval \([−c, c]\) if and only if

\[
\liminf_{n \to \infty} \frac{2}{\sqrt{2n}} s_{2n} \leq c.
\]

In particular, \( \mu \) is supported on \([−1, 1]\) if and only if the sequence \( s \) is bounded.

Measures with unbounded supports are not necessarily determined by their moment sequences. In this very short Lecture we will study when this happens.

Definition 2.1. A moment sequence \( s \) is called determinate if it has only one representing measure; otherwise \( s \) is called indeterminate.

Likewise, a Radon measure \( \mu \) with finite moments is called determinate (resp. indeterminate) if and only if its moment sequence has this property.

2.1 An indeterminate measure: lognormal distribution

The first example of an indeterminate moment sequence was discovered by T. Stieltjes (1894). He showed that the log-normal distribution \( d\mu = f(x)dx \) with density

\[
f(x) = \frac{1}{\sqrt{2\pi}} \chi_{(0, +\infty)}(x)x^{-1} \exp(-(\ln x)^2/2)
\]
has finite moments and the corresponding moment sequence is indeterminate. We reproduce this famous classical example here.

Let \( n \in \mathbb{Z} \). Substituting \( y = \ln x \) and \( t = y - n \), we compute

\[
\begin{align*}
 s_n &= \int_{\mathbb{R}} x^n \, d\mu(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x^{n-1} e^{-\left(\ln x\right)^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^y e^{-y^2/2} \, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(y-n)^2/2} e^{y^2/2} \, dy = e^{y^2/2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{t^2/2} \, dt = e^{n^2/2}.
\end{align*}
\]

This proves \( \mu \) that finite moments and its moment sequence is

\[
s_n = \left( e^{n^2/2} \right)_{n \in \mathbb{N}_0}.
\]

For \( c \in [-1, 1] \) we define a positive (!) measure \( \mu_c \) by

\[
d\mu_c(x) = [1 + c \sin(2\pi \ln x)] \, d\mu(x).
\]

Since \( \mu \) has finite moments, so has \( \mu_c \). For \( n \in \mathbb{Z} \), we compute

\[
\int_{\mathbb{R}} x^n \sin(2\pi \ln x) \, d\mu(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ny} \sin(2\pi y) e^{-y^2/2} \, dy \\
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-(y-n)^2/2} e^{ny/2} \sin 2\pi y \, dy = \frac{1}{\sqrt{2\pi}} e^{nym/2} \int_{\mathbb{R}} e^{-t^2/2} \sin 2\pi (t + n) \, dt = 0,
\]

where we used the fact that the function \( \sin 2\pi (t + n) \) is odd. By the definition of \( \mu_c \), it follows that any \( c \in [-1, 1] \) the measure \( \mu_c \) has the same moments as \( \mu \). (This is even true for all \( s_n \) with \( n \in \mathbb{Z} \).) Thus \( \mu \) is not determinate!

### 2.2 Carleman’s condition for the Hamburger moment problem

The following important result is the Carleman theorem.

**Theorem 2.2.** Let \( s = (s_n)_{n \in \mathbb{N}_0} \) be a Hamburger moment sequence. If \( s \) satisfies the Carleman condition

\[
\sum_{n=1}^{\infty} \frac{1}{s_{2n}} = +\infty,
\]

(2.1) then \( s \) is a determinate.

**Proof.** [MP,Theorem 4.3]. \( \Box \)

The standard proof of Theorem 2.2 derives the assertion from the Denjoy–Carleman theorem on quasi-analytic functions. An operator-theoretic proof based on Jacobi operators is developed in Section 6.4 of [MP].

Since \( s_{2n} = \int_{\mathbb{R}} x^{2n} \, d\mu \geq 0 \) for all \( n \in \mathbb{N}_0 \), we have either \( s_{2n}^{1/n} = 0 \) or \( s_{2n}^{1/n} > 0 \).

The Carleman condition (2.1) requires that the moment sequence is not growing to fast.
2.3 Krein’s condition for the Hamburger moment problem

**Corollary 2.3.** Let \( s = (s_n)_{n \in \mathbb{N}_0} \) be a Hamburger moment sequence. If there is a constant \( M > 0 \) such that
\[
s_{2n} \leq M^n (2n)! \quad \text{for} \quad n \in \mathbb{N},
\] (2.2)
then Carleman’s condition (2.1) holds and \( s \) is determinate.

**Proof.** It is obvious that \((2n)! \leq (2n)^{2n}\) for \( n \in \mathbb{N} \). Therefore, \([ (2n)! ]^{1/2n} \leq 2n\), so that \( \frac{1}{2n} \leq [ (2n)! ]^{-1/2n} \) and hence
\[
M^{-1/2} \frac{1}{2n} \leq M^{-1/2} [ (2n)! ]^{-1/2n} \leq s_{2n}, \quad n \in \mathbb{N}.
\]
Thus, Carleman’s condition (2.1) is satisfied, so Theorem 2.2(i) applies. \( \square \)

**Corollary 2.4.** Let \( \mu \) be a Radon measure on \( \mathbb{R} \). If there exists an \( \varepsilon > 0 \) such that
\[
\int_{\mathbb{R}} e^{\varepsilon |x|} d\mu (x) < \infty, \quad (2.3)
\]
then \( \mu \) has all moments, condition (2.2) holds, and \( \mu \) is determinate.

**Proof.** Clearly, \( e^{\varepsilon |x|} \geq (e \varepsilon)^{2n} \frac{1}{(2n)!} \) and hence \( x^{2n} e^{-\varepsilon |x|} \leq e^{-2n} (2n)! \) for \( n \in \mathbb{N}_0 \) and \( x \in \mathbb{R} \). Therefore,
\[
\int_{\mathbb{R}} x^{2n} d\mu (x) = \int_{\mathbb{R}} x^{2n} e^{-\varepsilon |x|} e^{\varepsilon |x|} d\mu (x) \leq e^{-2n} (2n)! \int_{\mathbb{R}} e^{\varepsilon |x|} d\mu (x) < \infty. \quad (2.4)
\]
This implies \( \int |x|^k d\mu (x) < \infty \) for \( k \in \mathbb{N}_0 \). From (2.4) it follows that (2.2) holds, so Corollary 2.4 gives the assertion. \( \square \)

As shown by C. Berg and J.P.R. Christensen, Carleman’s condition (2.1) implies also that the polynomials \( C[x] \) are dense in \( L^p (\mathbb{R}, \mu) \) for \( p \in [1, +\infty) \), where \( \mu \) is the unique representing measure of \( s \).

### 2.3 Krein’s condition for the Hamburger moment problem

The following Krein theorem shows that, for measures given by a density, the so-called Krein condition (2.5) is a sufficient condition for indeterminacy.

**Theorem 2.5.** Let \( f \) be a nonnegative Borel function on \( \mathbb{R} \) such that the measure \( \mu \) defined by \( d\mu = f(x) dx \) has finite moments \( s_n := \int x^n d\mu (x) \) for all \( n \in \mathbb{N}_0 \). If
\[
\int_{\mathbb{R}} \frac{\ln f(x)}{1 + x^2} dx > -\infty, \quad (2.5)
\]
then the moment sequence \( s = (s_n)_{n \in \mathbb{N}_0} \) is indeterminate and the polynomials \( C[x] \) are not dense in \( L^2 (\mathbb{R}, \mu) \).
Theorem 2.7. If $s = (s_n)_{n \in \mathbb{N}_0}$ is a Stieltjes moment sequence such that

$$\sum_{n=1}^{\infty} s_n \frac{1}{n^2} = +\infty,$$  \hspace{1cm} (2.7)

then $s$ is a determinate Stieltjes moment sequence.

Proof. [MP, Theorem 4.3(ii)]. \qed
Note that $s_n \geq 0$ for all $n \in \mathbb{N}_0$, because $s$ is a Stieltjes moment sequence.

The following two corollaries can be easily derived from Theorem 2.7.

**Corollary 2.8.** Let $s = (s_n)_{n\in\mathbb{N}_0}$ be a Stieltjes moment sequence. If there is a constant $M > 0$ such that

$$s_n \leq M^n(n!) \quad \text{for } n \in \mathbb{N},$$

then $s$ is a determinate Stieltjes moment sequence.

**Corollary 2.9.** Suppose that $\mu$ is a Radon measure supported on $[0, +\infty)$. If there exists an $\varepsilon > 0$ such that

$$\int_0^\infty e^{\varepsilon \sqrt{x}} d\mu(x) < \infty,$$

then $\mu$ has finite moments, condition (2.3) is satisfied, and the corresponding Stieltjes moment sequence is determinate.

In probability theory the two sufficient determinacy conditions (2.3) and (2.9) are called Cramer’s condition and Hardy’s condition, respectively.

The next theorem is about Krein’s condition for the Stieltjes moment problem.

**Theorem 2.10.** Let $f$, $\mu$, and $s$ be as in Theorem 2.5. If the measure $\mu \in \mathcal{M}_+(\mathbb{R})$ is supported on $[0, +\infty)$ and

$$\int_{\mathbb{R}} \ln f(x^2) \frac{dx}{1+x^2} = \int_{0}^{\infty} \ln f(x) \frac{dx}{1+x} \sqrt{x} > -\infty,$$

then $s$ is an indeterminate Stieltjes moment sequence.

**Proof.** [MP, Theorem 4.17].

**Example 2.11.** The Stieltjes moment problem for $d\mu = \chi_{[0,\infty)}(x)e^{-|x|^\alpha} dx$, $\alpha > 0$.

If $0 < \alpha < 1/2$, then (2.10) holds, so the Stieltjes moment sequence is indeterminate.

If $\alpha \geq 1/2$, then $2\alpha \geq 1$ and hence by (2.6),

$$s_n = \int_{0}^{\infty} x^n d\mu(x) = \int_{0}^{\infty} (x^2)^n d\mu(x^2) = \int_{0}^{\infty} x^{2n} e^{-|x|^{2\alpha}} dx \leq 4^n(2n)!.$$

By Corollary 2.8, the Stieltjes moment sequence is determinate for $\alpha \geq 1/2$.

In both Examples 2.6 and 2.11 the determinacy was decided for all parameter values $\alpha$ by using only the Carleman and Krein theorems. This indicates that both criteria are efficient and strong to cover even borderline cases.

**Remark 2.12.** Suppose $s$ is a determinate Stieltjes moment sequence. This means that $s$ has a unique representing measure $\mu$ supported on $[0, +\infty)$. Then it may happen (see, e.g., [MP, Example 8.11]) that $s$ is not determinate as a Hamburger moment sequence, that is, $s$ may have another representing measure which is not supported on $[0, +\infty)$. However, if $\mu(\{0\}) = 0$, then $s$ is also a determinate Hamburger moment sequence according to Definition 2.1 by [MP, Corollary 8.9].
Lecture 3
The one-dimensional truncated moment problem on a bounded interval

Abstract:
The truncated moment problem on a compact interval \([a, b]\) is treated. Basic results on the existence and the determinacy are obtained. For interior points of the moments cone principal measures and canonical measures are developed.

In contrast to the preceding lecture we study in this Lecture moment problems where only finitely many moments are given and the measures are supported on a bounded interval. More precisely, suppose \(a\) and \(b\) are real numbers such that \(a < b\) and \(m \in \mathbb{N}\). We consider the truncated moment problem on the interval \([a, b]\):

Given a real sequence \(s = (s_j)_{j=0}^m\), when is there a Radon measure \(\mu\) on \([a, b]\) such that \(s_j = \int_a^b x^j d\mu(x)\) for \(j = 0, \ldots, m\)?

In this case we say that \(s\) is a truncated \([a, b]\)-moment sequence and \(\mu\) is a representing measure for \(s\). We shall see that odd and even cases are different.

3.1 Existence of a solution

First we fix some notation. Suppose \(s = (s_j)_{j=0}^m\) is a real sequence. Let \(L_s\) be the Riesz functional on \(\mathbb{R}[x]_m := \{ p \in \mathbb{R}[x] : \deg p \leq m \}\) defined by \(L_s(x^j) = s_j, j = 0, \ldots, m\). The Hankel matrix \(H_k(s) := (s_{i+j})_{i,j=0}^{k}\) and the shifted sequence \(E_s\) is \(E_s := (s_1, \ldots, s_m) = (s_{j+1})_{j=0}^{m-1}\). Further, we define

\[
\text{Pos}([a, b])_m := \{ p \in \mathbb{R}[x]_m : p(x) \geq 0 \text{ on } [a, b]\}.
\]

The following notation differs between the two even and the odd cases \(m = 2n\):
The one-dimensional truncated moment problem on a bounded interval

\[ H_{2n} (s) := H_n (s) \equiv (s_{i+j})_{i,j=0}^n, \]

\[ \overline{H}_{2n} (s) := H_{n-1} ((b-E)(E-a)) s = ((a+b)s_{i+j+1} - s_{i+j+2} - abs_{i+j})_{i,j=0}^{n-1}, \]

\[ H_{2n+1} (s) := H_n (Es - as) \equiv (s_{i+j+1} - as_{i+j})_{i,j=0}^n, \]

\[ \overline{H}_{2n+1} (s) := H_n (bs - Es) \equiv (bs_{i+j} - s_{i+j+1})_{i,j=0}^n, \]

\[ L_n (s) := \text{det} H_n (s), \quad \overline{L}_n (s) := \text{det} \overline{H}_n (s). \]

The upper and lower bar notation allow us to treat the even and odd cases at once.

For \( f = \sum_{j=0}^k a_j x^j \in \mathbb{R}[x]_k \) let \( \bar{f} := (a_0, \ldots, a_k)^T \in \mathbb{R}^{k+1} \) denote the coefficient vector of \( f \). Then for \( p, q \in \mathbb{R}[x]_n \) and \( f, g \in \mathbb{R}[x]_{n-1} \) simple computations yield

\[
L_n (p q) = \bar{p}^T \overline{H}_{2n} (s) \bar{q}, \quad L_n ((b-x)(a-x)fg) = \bar{f}^T \overline{H}_{2n} (s) \bar{g}, \quad (3.1)
\]

\[
L_n ((x-a)p q) = \bar{p}^T H_{2n+1} (s) \bar{q}, \quad L_n ((b-x)p q) = \bar{p}^T \overline{H}_{2n+1} (s) \bar{q}. \quad (3.2)
\]

Then following two theorems settle the existence problem. The notation \( A \succeq 0 \) for a hermitian matrix \( A \) means that \( A \) is positive semidefinite.

**Theorem 3.1.** (Truncated \([a, b]\)-moment problem; even case \( m = 2n \))

For a real sequence \( s = (s_j)_{j=0}^{2n} \) the following statements are equivalent:

(i) \( s \) is a truncated \([a, b]\)-moment sequence.

(ii) \( L_n (p^2) \geq 0 \) and \( L_n ((b-x)(a-x)q^2) \geq 0 \) for \( p \in \mathbb{R}[x]_n \) and \( q \in \mathbb{R}[x]_{n-1} \).

(iii) \( H_{2n} (s) \succeq 0 \) and \( \overline{H}_{2n} (s) \succeq 0 \).

**Theorem 3.2.** (Truncated \([a, b]\)-moment problem; odd case \( m = 2n + 1 \))

For a real sequence \( s = (s_j)_{j=0}^{2n+1} \) the following are equivalent:

(i) \( s \) is a truncated \([a, b]\)-moment sequence.

(ii) \( L_n ((x-a)p^2) \geq 0 \) and \( L_n ((b-x)p^2) \geq 0 \) for all \( p \in \mathbb{R}[x]_n \).

(iii) \( H_{2n+1} (s) \succeq 0 \) and \( \overline{H}_{2n+1} (s) \succeq 0 \).

**Proofs of Theorems 3.1 and 3.2**

(i) \( \leftrightarrow \) (ii): We apply Proposition 1.7 to the subspace \( E = \mathbb{R}[x]_m \) of \( C([a, b]; \mathbb{R}) \). Then \( L_n \) is a truncated \([a, b]\)-moment functional if and only if \( L_n (p) \geq 0 \) for all \( p \in E_n = \text{Pos}([a, b])_m \). By Proposition 1.11 this is equivalent to condition (ii).

(ii) \( \leftrightarrow \) (iii) follows at once from the identities (3.1) and (3.2).

These theorems give characterize truncated moment sequences in terms of the positivity of two Hankel matrices. These are very useful criteria that can be verified by means of well-known criteria from linear algebra.

Note that since \( p^2 = (b-a)^{-1}[(b-x)p^2 + (x-a)p^2] \), condition (ii) in Theorem 3.2 implies in particular that \( L_n (p^2) \geq 0 \) for \( p \in \mathbb{R}[x]_n \).

**3.2 The moment cone and its boundary points**

Let \( \mathcal{M}_+ \) denote the Radon measures on \([a, b]\).
3.2 The moment cone and its boundary points

**Definition 3.3.** The moment cone $S_{m+1}$ and the moment curve $c_{m+1}$ are

\[ S_{m+1} := \{ s = (s_0, s_1, \ldots, s_m) : s_j = \int_a^b t^j \, d\mu(t), j = 0, \ldots, m, \mu \in \mathcal{M}_+ \}, \]

\[ c_{m+1} = \{ s(t) := (1, t, t^2, \ldots, t^m) : t \in [a, b] \} \]

If we identify the row vectors $s$ and $s(t)$ with the corresponding column vector $s^T$ and $s(t)^T$, then become $S_{m+1}$ and $c_{m+1}$ as subsets of $\mathbb{R}^{m+1}$. The curve $c_{m+1}$ is contained in $S_{m+1}$, since $s(t)$ is the moment sequence of the delta measure $\delta$. It is not difficult to show that the moment cone $S_{m+1}$ is a closed convex cone in $\mathbb{R}^{m+1}$ with nonempty interior and that it is the conic hull $C_{m+1}$ of the moment curve $c_{m+1}$.

Each moment sequence $s \in S_{m+1}, s \neq 0$, has a $k$-atomic representing measure

\[ \mu = \sum_{j=1}^k m_j \delta_{t_j}, \]

where $k \leq m + 1$ and $t_j \in [a, b]$ for all $j$. (This follows at once from the Richter-Tchakaloff theorem [8.1] proved in Lecture 8.)

That $\mu$ is $k$-atomic means that the points $t_j$ are pairwise distinct and $m_j > 0$. The numbers $t_j$ are called roots or atoms of $\mu$. Without loss of generality we assume

\[ a \leq t_1 < t_2 < \cdots < t_k \leq b. \]

To formulate our next theorem the following notion is convenient.

**Definition 3.4.** Let $s \in S_{m+1}, s \neq 0$. The index $\text{ind}(\mu)$ of the $k$-atomic representing measure (3.3) for $s$ is the sum

\[ \text{ind}(\mu) := \sum_{j=1}^k \epsilon(t_j), \ 	ext{where} \ \epsilon(t) := 2 \ 	ext{for} \ t \in (a, b) \ 	ext{and} \ \epsilon(a) = \epsilon(b) := 1. \]

The index $\text{ind}(s)$ of $s$ is the minimal index of all representing measures (3.3) for $s$.

The reason why boundary points and interior points are counted differently is the following fact which is used in the proofs: If $t_0 \in [a, b]$ is a zero of $p \in \text{Pos}(s)_{m}$ with multiplicity $k$, then $k \geq 2$ if $t_0 \in (a, b)$, while $k = 1$ is possible if $t_0 = a, b$.

The following theorem characterize the boundary points of the moment cone.

**Theorem 3.5.** For $s \in S_{m+1}, s \neq 0$, the following statements are equivalent:

(i) $s$ is a boundary point of the convex cone $S_{m+1}$.

(ii) $\text{ind}(s) \leq m$.

(iii) There exists a $p \in \text{Pos}(s)_{m}, p \neq 0$, such that $L_{s}(p) = 0$.

(iv) $\mathcal{D}_{m}(s) = 0 \text{ or } \overline{\mathcal{D}}_{m}(s) = 0$.

(v) $s$ has a unique representing measure $\mu \in \mathcal{M}_+$.

If $p$ is as in (iii) and $\mu$ is as in (v), then $\text{supp} \ \mu \subseteq \{ t \in [a, b] : p(t) = 0 \}$. 
Proof. [MP, Theorem 10.7].

This theorem shows that several important properties of \( s \) are equivalent. In particular, the last assertion is crucial: The atoms of \( \mu \) are contained in the zero set of the polynomial \( p \in \text{Pos}([a,b])_m \) satisfying \( L_s(p) = 0 \).

By (iii), the unique representing measure \( \mu \) has \( \text{ind}(\mu) = \text{ind}(s) \leq m \). Hence, if all \( t_j \) are in the open interval \( (a,b) \), then \( k \leq \frac{m}{2} \). If precisely one \( t_j \) is an end point, then \( k \leq \frac{m}{2} + 1 \) and if both end points are among the \( t_j \), then \( k \leq \frac{m}{2} + 1 \). The case \( k = \frac{m}{2} + 1 \) can only happen if \( m \) is even and both \( a \) and \( b \) are among the \( t_j \). Thus, all boundary points of \( S_{m+1} \) can be represented by \( k \)-atomic measures with \( k \leq \frac{m}{2} + 1 \).

### 3.3 Interior points and principal measures

Now we consider the interior points of the moment cone. Since \( s \in S_{m+1} \) is an interior point if and only if \( s \) is not a boundary point, Theorem 3.5 yields the following: \( s \in S_{m+1} \) is an interior point of \( S_{m+1} \) if and only if the Hankel matrices \( H_m(s) \) and \( \overline{H}_m(s) \) are positive definite, or equivalently, \( D_m(s) > 0 \) and \( \overline{D}_m(s) > 0 \).

Throughout this section, we suppose that \( s \) is an interior point of \( S_{m+1} \).

Then, by Theorem 3.5, we have \( \text{ind}(s) \geq m + 1 \). Thus, it is natural to ask whether there are representing measure which have the minimal possible index \( m + 1 \) and to describe these measures provided they exist.

**Definition 3.6.** A representing measure \( \mu \) of the form (3.3) for \( s \) is called
- **principal** if \( \text{ind}(\mu) = m + 1 \),
- **upper principal** if it is principal and \( b \) is an atom of \( \mu \),
- **lower principal** if it is principal and \( b \) is not an atom of \( \mu \).

Thus, for principal measures the index is equal to the number of prescribed moments. Then we have following existence theorem for such measures.

**Theorem 3.7.** Each interior point \( s \) of \( S_{m+1} \) has a unique upper principal representing measure \( \mu^+ \) and a unique lower principal representing measure \( \mu^- \).

**Proof.** [MP, Theorem 10.17].

Let \( t_j^\pm \) denote the roots of the principal measures \( \mu^\pm \). The location of these roots \( t_j^\pm \) in the even and odd cases are illustrated by the following scheme:

\[
\begin{align*}
  m = 2n, & \quad \mu^+: \quad a < t_1^+ < t_2^+ < \cdots < t_{n+1}^+ = b, \\
  m = 2n, & \quad \mu^-: \quad a = t_1^- < t_2^- < \cdots < t_{n+1}^- < b, \\
  m = 2n + 1, & \quad \mu^+: \quad a = t_1^+ < t_2^+ < \cdots < t_{n+1}^+ < t_{n+2}^+ = b, \\
  m = 2n + 1, & \quad \mu^-: \quad a < t_1^- < t_2^- < \cdots < t_{n+1}^- < b.
\end{align*}
\]

Further, we have
3.3 Interior points and principal measures

\[ m = 2n : \quad a = t_1^- < t_1^+ < t_2^- < t_2^+ < \cdots < t_n^- < t_{n+1}^- < t_{n+1}^+ = b, \]

\[ m = 2n + 1 : \quad a = t_1^+ < t_1^- < t_2^+ < t_2^- < \cdots < t_n^+ < t_{n+1}^- < t_{n+1}^+ = b. \]

These formulas show that the roots of \( \mu^+ \) and \( \mu^- \) are strictly interlacing.

**Definition 3.8.** A representing measure \( \mu \) of the form (3.3) for \( s \) is called canonical if \( \text{ind}(\mu) \leq m + 2 \).

While there exist precisely two principal measures by Theorem 3.7, there is a one-parameter family of canonical measures, as the following theorem shows.

**Theorem 3.9.** For each point \( \xi \in (a, b) \) there exists a unique canonical representing measure \( \mu_\xi \) of \( s \) which has \( \xi \) as an atom.

**Proof.** [MP, Corollary 10.13]. \( \Box \)

A deeper study of the set of representing measures can be found in the books by Karlin and Studden (1966) and by Krein and Nudelman (1977).
Lecture 4
The moment problem on compact
semi-algebraic sets

Abstract:
The moment problem for compact semi-algebraic sets is investigated and solved. The interplay between moment problem and Positivstellensätze of real algebraic geometry is discussed.

In this Lecture we enter the multidimensional moment problem by treating the moment problem for compact subsets of \( \mathbb{R}^d \) that are defined by means of finitely many polynomial inequalities. In Lecture 1 we have seen how the solvability criteria for intervals have been derived from descriptions of positive polynomials. This suggests that real algebraic geometry and Positivstellensätze might be useful tools for the multidimensional moment problem. For compact semi-algebraic sets this is indeed true and will be elaborated in this Lecture.

We begin by reviewing some concepts and results from real algebraic geometry. References are the excellent books by Marshall (2008) and Prestel and Delzell (2001).

### 4.1 Basic notions from real algebraic geometry

In this section, \( A \) denotes a unital commutative real algebra. The reader might always think of the polynomial algebra \( \mathbb{R}_d[x] \equiv \mathbb{R}[x_1, \ldots, x_d] \).

Positivity in \( A \) is described by means of the following notions.

**Definition 4.1.** A quadratic module of \( A \) is a subset \( Q \) of \( A \) such that

\[
Q + Q \subseteq Q, \quad 1 \in Q, \quad a^2Q \subseteq Q \quad \text{for all} \quad a \in A.
\]  

(4.1)

A quadratic module \( T \) is called a preordering if \( T \cdot T \subseteq T \).

**Example 4.2.** Obviously, \( Q = \sum \mathbb{R}_d[x]^2 + x_1 \sum \mathbb{R}_d[x]^2 + \cdots + x_d \sum \mathbb{R}_d[x]^2 \) is a quadratic module of \( \mathbb{R}_d[x] \). If \( d \geq 2 \), then product of elements of \( Q \) are not in \( Q \) in general, so \( Q \) is not a preordering.
Theorem 4.3 are given in the books by Marshall or by Prestel and Delzell; they are
A with the set
quotients
terms of

\text{Proof.}

Theorem 4.3.

Let \( K \) be a quadratic module of algebraic geometry. It describes nonnegative resp. positive polynomials on

\( Q \) be a quadratic module of

\[ \lambda \] is called the basic closed semi-algebraic set associated with

Another important concept is introduced in the following definition.

Definition 4.4. A quadratic module \( Q \) of \( A \) is called Archimedean if \( A \) coincides with the set \( A_b \) of bounded elements with respect to \( Q \), where

\[ A_b := \{ a \in A : \text{there exists a } \lambda > 0 \text{ such that } \lambda - a \in Q \text{ and } \lambda + a \in Q \}. \]

Lemma 4.5. Let \( Q \) be a quadratic module of \( A \) and \( a \in A \). Then we have \( a \in A_b \) if and only if \( \lambda^2 - a^2 \in Q \) for some \( \lambda > 0 \).

\[ \lambda^2 - a^2 = \frac{1}{2\lambda} \left( (\lambda + a)^2(\lambda - a) + (\lambda - a)^2(\lambda + a) \right) \in Q. \]

Conversely, if \( \lambda^2 - a^2 \in Q \) and \( \lambda > 0 \), then

\[ \lambda \pm a = \frac{1}{2\lambda} \left( (\lambda^2 - a^2) + (\lambda \pm a)^2 \right) \in Q. \]
4.2 Strict Positivstellensatz and solution of the moment problem

Lemma 4.6. $A_b$ is a unital subalgebra of $A$ for any quadratic module $Q$.

Proof. Clearly, sums and scalar multiples of elements of $A_b$ are again in $A_b$. It suffices to verify that this holds for the product of $a, b \in A_b$. By Lemma [4.5] there are $\lambda_1 > 0$ and $\lambda_2 > 0$ such that $\lambda_1^2 - a^2$ and $\lambda_2^2 - b^2$ are in $Q$. Then

$$(\lambda_1 \lambda_2)^2 - (ab)^2 = \lambda_2^2 (\lambda_1^2 - a^2) + a^2 (\lambda_2^2 - b^2) \in Q,$$

so that $ab \in A_b$ again by Lemma [4.5]. □

Corollary 4.7. For a quadratic module $Q$ of $R_d[\lambda]$ the following are equivalent:

(i) $Q$ is Archimedean.
(ii) There exists a number $\lambda > 0$ such that $\lambda - \sum_{k=1}^d x_k^2 \in Q$.
(iii) For any $k = 1, \ldots, d$ there exists a $\lambda_k > 0$ such that $\lambda_k - x_k^2 \in Q$.

Proof. (i)$\rightarrow$(ii) is clear by definition. If $\lambda - \sum_{j=1}^d x_j^2 \in Q$, then

$$\lambda - x_k^2 = \lambda - \sum_{j=1}^d x_j^2 + \sum_{j\neq k} x_j^2 \in Q.$$ 

This proves (ii)$\rightarrow$(iii). Finally, if (iii) holds, then $x_k \in R_d[\lambda]_b$ by Lemma [4.5] and hence $R_d[\lambda]_b = R_d[\lambda]$ by Lemma [4.6]. Thus, (iii)$\rightarrow$(i). □

Corollary 4.8. If the quadratic module $Q(f)$ of $R_d[\lambda]$ is Archimedean, then the set $K(f)$ is compact.

Proof. By Corollary [4.7] $\lambda - \sum_{j=1}^d x_j^2 \in Q(f)$ for some $\lambda > 0$. Therefore, since polynomials of $Q(f)$ are nonnegative on $K(f)$, the set $K(f)$ is compact. □

The converse of Corollary [4.8] does not hold. However, it does hold for the pre-ordering $T(f)$, as shown by Proposition [4.22] below.

The following separation result is used in the next section.

Proposition 4.9. Let $Q$ be an Archimedean quadratic module of $A$. If $a_0 \in A$ and $a_0 \notin Q$, there exists a $Q$-positive linear functional $\varphi$ on $A$ such that $\varphi(1) = 1$ and $\varphi(a_0) \leq 0$.

Proof. Let $a \in A$ and choose $\lambda > 0$ such that $\lambda \pm a \in Q$. If $0 < \delta \leq \lambda^{-1}$, then $\delta^{-1} \pm a \in Q$ and hence $1 \pm \delta a \in Q$. This shows that 1 is an internal point of $Q$. Therefore, Eidelheit’s separation theorem applies and there exists a $Q$-positive linear functional $\psi \neq 0$ on $A$ such that $\psi(a_0) \leq 0$. Since $\psi \neq 0$, we have $\psi(1) > 0$. (Indeed, if $\psi(1) = 0$, since $\psi$ is $Q$-positive, $\lambda \pm a \in Q$ implies $\psi(a) = 0$ for all $a \in A$ and so $\psi = 0$.) Then $\varphi := \psi(1)^{-1}\psi$ has the desired properties. □

4.2 Strict Positivstellensatz and solution of the moment problem

Throughout this section, $f = \{f_1, \ldots, f_k\}$ denotes a finite subset of $R_d[\lambda]$ such that the semi-algebraic set $K(f)$ is compact.

The following is the strict Positivstellensatz for compact semi-algebraic sets.
Theorem 4.10. Let \( h \in \mathbb{R}_d[x] \). If \( h(x) > 0 \) for all \( x \in K(f) \), then \( h \in T(f) \).

The next theorem contains the solution of the moment problem for compact semi-algebraic sets.

Theorem 4.11. Let \( L \) be a linear functional on \( \mathbb{R}_d[x] \). If \( L \) is \( T(f) \)-positive, then \( L \) is a \( K(f) \)-moment functional.

We next show that both theorems are equivalent and can be derived from each other. This emphasizes that there is a close interplay between real algebraic geometry and moment theory.

Proof of Theorem 4.11 (assuming Theorem 4.10):

Let \( h \in \mathbb{R}_d[x] \) and suppose \( h(x) \geq 0 \) on \( K(f) \). Then, for any \( \varepsilon > 0 \), \( h(x) + \varepsilon > 0 \) on \( K(f) \) and therefore \( h \in T(f) \) by Theorem 4.10. Hence \( L(h + \varepsilon) = L(h) + \varepsilon L(1) \geq 0 \) by the assumption. Then \( L(h) \geq 0 \) by letting \( \varepsilon \to 0 \). Therefore, \( L \) is a \( K(f) \)-moment functional by Haviland’s Theorem.

Proof of Theorem 4.10 (assuming Theorem 4.11 and Proposition 4.22):

Suppose \( h \in \mathbb{R}_d[x] \) and \( h(x) > 0 \) on \( K(f) \). Assume to the contrary that \( h \notin T(f) \). Since the preordering \( T(f) \) is Archimedean by Proposition 4.22, Proposition 4.9 applies, so there is a \( T(f) \)-positive linear functional \( L \) on \( \mathbb{R}_d[x] \) such that \( L(1) = 1 \) and \( L(h) \leq 0 \). By Theorem 4.11, \( L \) is a \( K(f) \)-moment functional, that is, there is a Radon measure \( \mu \) supported on \( K(f) \) such that \( L(p) = \int_{K(f)} p \, d\mu \) for \( p \in \mathbb{R}_d[x] \). But \( L(1) = \mu(K(f)) = 1 \) and \( h > 0 \) on \( K(f) \) imply that \( L(h) > 0 \). This is a contradiction, since \( L(h) \leq 0 \).

The proof of Theorem 4.11 is given by Proposition 4.23(ii) in Section 4.5. The crucial technical ingredient of this proof is Proposition 4.22 which states that the preorder \( T(f) \) is Archimedean if the semi-algebraic sets \( K(f) \) is compact. In the proof of this result assertion (i) of the Krivine-Stengle’s Positivstellensatz is used.

If we plug in these proofs and assume (!) the Archimedean property, then the assertions hold for a quadratic module rather than the preordering. The gives the following theorem. Assertion (i) is usually the Archimedean Positivstellensatz.

Theorem 4.12. Suppose the quadratic module \( Q(f) \) defined by (4.3) is Archimedean.

(i) If \( h \in \mathbb{R}_d[x] \) satisfies \( h(x) > 0 \) for all \( x \in K(f) \), then \( h \in Q(f) \).
(ii) Each \( Q(f) \)-positive linear functional \( L \) on \( \mathbb{R}_d[x] \) is a \( K(f) \)-moment functional.

The shortest and probably the most elegant approach to Theorems 4.12 and 4.11 is based on the multi-dimensional spectral theorem combined with the GNS construction; it is developed in [MP, Section 12.5].

We close the section with two examples.

Example 4.13. (d-dimensional compact interval \([a_1, b_1] \times \cdots \times [a_d, b_d]\))
Let \( a_j, b_j \in \mathbb{R} \), \( a_j < b_j \), and set \( f_{2j-1} := b_j - x_j \), \( f_{2j} := x_j - a_j \), for \( j = 1, \ldots, d \). The semi-algebraic set \( K(f) \) is the d-dimensional interval \([a_1, b_1] \times \cdots \times [a_d, b_d]\) and \( Q(f) \) is Archimedean by Lemma 4.6(ii). Hence, by Theorem 4.12(ii).
4.3 Localizing functionals and Hankel matrices

$L$ is a $K(f)$-moment functional if and only if it is $Q(f)$-positive, or equivalently, if $L_{f_1}, L_{f_2}, \ldots, L_{f_k}$ are positive functionals, that is,

$$L((b_j-x_j)^2) \geq 0 \text{ and } L((x_j-a_j)^2) \geq 0 \text{ for } j = 1, \ldots, d, \ p \in \mathbb{R}_d[x].$$

An immediate corollary of Theorem 4.11 is the following.

**Corollary 4.14.** Suppose that $\mathcal{I}$ is an ideal of $\mathbb{R}_d[x]$ such that the real algebraic set $V := Z(\mathcal{I}) = \{ x \in \mathbb{R}^d : f(x) = 0 \text{ for } f \in \mathcal{I} \}$ is compact. Then each positive linear functional on $\mathbb{R}_d[x]$ which annihilates $\mathcal{I}$ is a $V$-moment functional.

**Example 4.15. (Moment problem on unit spheres)**

Let $S^{d-1} = \{ x \in \mathbb{R}^d : x_1^2 + \cdots + x_d^2 = 1 \}$. Then a linear functional $L$ on $\mathbb{R}_d[x]$ is a $S^{d-1}$-moment functional if and only if

$$L(p^2) \geq 0 \text{ and } L((x_1^2 + \cdots + x_d^2 - 1)p) = 0 \text{ for } p \in \mathbb{R}_d[x].$$

### 4.3 Localizing functionals and Hankel matrices

The preceding result on the moment problem suggest the following question:

*How to verify the $Q(f)$-positivity or $T(f)$-positivity of a functional $L$ on $\mathbb{R}_d[x]$?*

Let $L$ be a linear functional on $\mathbb{R}_d[x]$ and $g \in \mathbb{R}_d[x]$. The linear functional $L_g$ defined by $L_g(p) = L(gp), p \in \mathbb{R}_d[x]$, is called the localization of $L$ at $g$.

For instance, suppose $L$ is a $K(f)$ moment functional and $g(x) \geq 0$ on $K(f)$. If $\mu$ is a representing measure for $L$, then $L_g(p) = \int p(x)g(x)d\mu, \ g \mu$ is a representing measure for $L_g$. This justifies the name "localization".

Let $s = (s_\alpha)_{\alpha \in \mathbb{N}_0^d}$ be the $d$-sequence given by $s_\alpha = L(x^\alpha)$ and write $g = \sum_\gamma g_\gamma x^\gamma$.

We define a $d$-sequence $g(E)s = ((g(E)s)_\alpha)_{\alpha \in \mathbb{N}_0^d}$ by

$$(g(E)s)_\alpha := \sum_\gamma g_\gamma s_{\alpha+\gamma}, \ \alpha \in \mathbb{N}_0^d,$$

and the localized Hankel matrix $H(gs) = (H(gs)_{\alpha,\beta})_{\alpha,\beta \in \mathbb{N}_0^d}$ with entries

$$H(gs)_{\alpha,\beta} := \sum_\gamma g_\gamma s_{\alpha+\beta+\gamma}, \ \alpha, \beta \in \mathbb{N}_0^d.$$  

Then, for $p(x) = \sum_\alpha a_\alpha x^\alpha \in \mathbb{R}_d[x]$, we obtain

$$L_g(gp^2) = \sum_{\alpha,\beta} a_\alpha a_\beta (g(E)s)_{\alpha+\beta} = \sum_{\alpha,\beta} a_\alpha a_\beta H(gs)_{\alpha,\beta}.$$  

By the preceding, $g(E)s$ is the sequence for the functional $L_g$ and $H(gs)$ is a Hankel matrix for the sequence $g(E)s$. From these formulas we easily derive the following.
Proposition 4.16. Let \( Q(g) \) be the quadratic module generated by \( g = \{ g_1, \ldots, g_m \} \). For a linear functional \( L \) on \( \mathbb{R}_d[x] \) the following are equivalent:

(i) \( L \) is a \( Q(g) \)-positive linear functional on \( \mathbb{R}_d[x] \).
(ii) \( L, L_{g_1}, \ldots, L_{g_m} \) are positive linear functionals on \( \mathbb{R}_d[x] \).
(iii) \( s, g_1(E)s, \ldots, g_m(E)s \) are positive semidefinite \( d \)-sequences.
(iv) \( H(s), H(g_1s), \ldots, H(g_ms) \) are positive semidefinite matrices.

Condition (iv) can be verified by using standard criteria from linear algebra. Thus, the \( Q(g) \)-positivity or \( T(g) \)-positivity of a functional are reasonable and familiar conditions in moment theory.

4.4 Moment problem criteria based on semirings

Definition 4.17. A semiring of \( A \) is a subset \( S \) satisfying

\[
S + S \subseteq S, \quad S \cdot S \subseteq S, \quad \lambda \cdot 1 \in S \quad \text{for} \quad \lambda \geq 0.
\]

A quadratic module is not necessarily invariant under multiplication, but a semiring is. While a quadratic module contains all squares, a semiring does not in general. Clearly, a quadratic module is a preordering if and only if it is a semiring.

The semiring \( S(f) \) generated by \( f_1, \ldots, f_k \in A \) is the set of all finite sums of terms

\[
\lambda f_1^{n_1} \cdot \cdots \cdot f_k^{n_k}, \quad \text{where} \quad n_1, \ldots, n_k \in \mathbb{N}_0, \quad \lambda \geq 0.
\]  

The next theorem applies to semi-algebraic sets contained in compact polyhedra.

Let \( f_1, \ldots, f_k \in \mathbb{R}_d[x] \) such that \( f_1, \ldots, f_m, m \leq k \), are linear. Setting

\[
\hat{f} = \{ f_1, \ldots, f_m \}, \quad f = \{ f_1, \ldots, f_k \}.
\]

Then \( K(\hat{f}) \) is a polyhedron such that \( K(f) \subseteq K(\hat{f}) \).

Theorem 4.18. Suppose the polyhedron \( K(\hat{f}) \) is compact and nonempty. Then a linear functional \( L \) on \( \mathbb{R}_d[x] \) is a \( K(f) \)-moment functional if and only if

\[
L(f_1^{n_1} \cdot \cdots \cdot f_k^{n_k}) \geq 0 \quad \text{for all} \quad n_1, \ldots, n_k \in \mathbb{N}_0.
\]  

Proof. [MP, Theorem 12.45]. □

Condition (4.6) means that \( L \) is nonnegative on the semiring \( S(f) \) defined by (4.5).

Note that the criterion in Theorem 1.17 in Lecture 1 was also a positivity condition for a semiring. Finally, we state three examples based on Theorems 4.12 and 4.18

Example 4.19. (Simplex in \( \mathbb{R}^d, d \geq 2 \))
Let \( f_1 = x_1, \ldots, f_d = x_d, f_{d+1} = 1 - \sum_{i=1}^{d} x_i \). The set \( K(f) \) is the simplex
A linear functional $L$ is a $K_d$–moment functional if and only if

$$L(x_i p^2) \geq 0, \ i = 1, \cdots, d, \text{ and } L((1 - (x_1 + x_2 + \cdots + x_d))p^2) \geq 0 \ \text{ for } p \in \mathbb{R}_d[x],$$

or equivalently,

$$L(x_1^{n_1} \cdots x_d^{n_d} (1 - (x_1 + \cdots + x_d))^{n_{d+1}}) \geq 0 \ \text{ for } n_1, \ldots, n_{d+1} \in \mathbb{N}_0.$$

**Example 4.20.** (Standard simplex $\Delta_d$ in $\mathbb{R}^d$)

Let $f_1 = x_1, \ldots, f_d = x_d, f_{d+1} = 1 - \sum_{i=1}^d x_i, f_{d+2} = -f_{d+1}$. Then $K(f)$ is

$$\Delta_d = \{ x \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, x_1 + \cdots + x_d = 1 \}.$$

A linear functional $L$ is a $\Delta_d$–moment functional if and only if

$$L(x_1^{n_1} \cdots x_d^{n_d}) \geq 0, \ L(x_1^{n_1} \cdots x_d^{n_d} (1-(x_1+\cdots+x_d))^q) = 0, \ n_1, \ldots, n_d \in \mathbb{N}_0, q \in \mathbb{N}.$$

**Example 4.21.** (Multidimensional Hausdorff moment problem on $[0,1]^d$)

Set $f_1 = x_1, f_2 = 1 - x_1, \ldots, f_{2d-1} = x_d, f_{2d} = 1 - x_d, k = 2d$. Then $K(f) = [0,1]^d$.

Let $s = (s_n)_{n \in \mathbb{N}_0^d}$ be a multisequence. We define the shift $E_j$ of the $j$–th index by

$$(E_j s)_m = s_{(m_1, \ldots, m_{j-1}, m_j+1, m_{j+1}, \ldots, m_d)}, \quad m \in \mathbb{N}_0^d.$$

Then $L_s$ is a $[0,1]^d$–moment functional on $\mathbb{R}_d[x]$ if and only if

$$L_s(x_1^{n_1} (1-x_1)^{m_1} \cdots x_d^{n_d} (1-x_d)^{m_d}) \geq 0 \ \text{ for } n, m \in \mathbb{N}_0^d.$$

**4.5 Two technical results and the proof of Theorem 4.11**

In this section, we reproduce from [MP] the proofs of the following results. Both results of interest in themselves. As noted above, they complete the proof of Theorem 4.11. Suppose $K(f)$ is compact. We write $a \preceq b$ if $b - a \in T(f)$.

**Proposition 4.22.** The preordering $T(f)$ is Archimedean.

**Proof.** Let $g \in \mathbb{R}_d[x]$ be a fixed polynomial and $\lambda > 0$. Suppose that $\lambda^2 > g(x)^2$ for all $x \in K(f)$. Our first aim to to show that there exists a $p \in T(f)$ such that

$$g^{2n} \leq \lambda^{2n+2} p \quad \text{for } n \in \mathbb{N}. \tag{4.7}$$

Indeed, by the Krivine–Stengle Positivstellensatz (Theorem 4.3(i)) there exist polynomials $p, q \in T(f)$ such that

$$p(\lambda^2 - g^2) = 1 + q. \tag{4.8}$$
Since \( q \in T(f) \) and \( T(f) \) is a quadratic module, \( g^{2n}(1 + q) \in T(f) \) for \( n \in \mathbb{N}_0 \). Therefore, using \((4.8)\) we conclude that
\[
g^{2n+2}p = g^{2n}\lambda^2 p - g^{2n}(1 + q) \leq g^{2n}\lambda^2 p.
\]
By induction it follows that
\[
g^{2n}p \leq \lambda^{2n}p. \tag{4.9}
\]
Since \( g^{2n}(q + pg^2) \in T(f) \), using first \((4.8)\) and then \((4.9)\) we derive
\[
g^{2n} \leq g^{2n} + g^{2n}(q + pg^2) = g^{2n}(1 + q + pg^2) = g^{2n}\lambda^2 p \leq \lambda^{2n+2}p.
\]
This completes the proof of \((4.7)\).

Now we put \( g(x) := (1 + x_1^2) \cdots (1 + x_k^2) \). Since \( g \) is bounded on the compact set \( K(f) \), we have \( \lambda^2 > g(x)^2 \) on \( K(f) \) for some \( \lambda > 0 \). Therefore, by the preceding there exists a \( p \in T(f) \) such that \((4.7)\) holds.

Further, for any multiindex \( \alpha \in \mathbb{N}_0^d, |\alpha| \leq k, k \in \mathbb{N} \), we obtain
\[
\pm 2x^\alpha \leq x^{2\alpha} + 1 \leq \sum_{|\beta| \leq k} x^{2\beta} = g^k. \tag{4.10}
\]
Hence there exist numbers \( c > 0 \) and \( k \in \mathbb{N} \) such that \( p \leq 2cg^k \). Combining the latter with \( g^{2n} \leq \lambda^{2n+2}p \) by \((4.7)\), we get \( g^{2k} \leq \lambda^{2k+2}2cg^k \) and so
\[
(g^k - \lambda^{2k+2}c)^2 \leq (\lambda^{2k+2}c - 1)^2.
\]
Hence, by Lemma \((4.5)\), \( g^k - \lambda^{2k+2}c \in \mathbb{R}_d[x]_b \) and so \( g^k \in \mathbb{R}_d[x]_b \) with respect to the preorder \( T(f) \). Since \( \pm x_j \leq g^k \) by \((4.10)\) and \( g^k \in \mathbb{R}_d[x]_b \), we obtain \( x_j \in \mathbb{R}_d[x]_b \) for \( j = 1, \ldots, d \). Now from Lemma \((4.6)\) it follows that \( T(f) \) is Archimedean. \( \square \)

**Proposition 4.23.** Suppose \( L \) is a \( T(f) \)-positive linear functional on \( \mathbb{R}_d[x] \).

(i) If \( g \in \mathbb{R}_d[x] \) and \( \|g\|_\infty \) denotes the supremum of \( g \) on \( K(f) \), then
\[
|L(g)| \leq L(1) \|g\|_\infty. \tag{4.11}
\]

(ii) \( L \) is a \( K(f) \)-moment functional.

**Proof.** (i): Fix \( \varepsilon > 0 \) and put \( \lambda := \|g\|_\infty + \varepsilon \). We define a real sequence \( s = (s_n)_{n \in \mathbb{N}_0} \) by \( s_n := L(g^n) \). Then \( L_q(y) = L(q(g)) \) for \( q \in \mathbb{R}[y] \). For any \( p \in \mathbb{R}[y] \), we have \( p(g)^2 \in \sum \mathbb{R}_d[x]^2 \subseteq T(f) \) and hence \( L_q(p(y)^2) = L(p(g)^2) \geq 0 \), since \( L \) is \( T(f) \)-positive. Thus, by Hamburger’s theorem, there exists a Radon measure \( \nu \) on \( \mathbb{R} \) such that \( s_n = \int_\mathbb{R} t^n d\nu(t), n \in \mathbb{N}_0 \).

For \( \gamma > \lambda \) let \( \chi_\gamma \) denote the characteristic function of the set \((-\infty, -\gamma] \cup [\gamma, +\infty)\). Since \( \lambda^2 - g(x)^2 > 0 \) on \( K(f) \), we have \( g^{2n} \leq \lambda^{2n+2}p \) by equation \((4.7)\) in the proof of Proposition \((4.22)\) Using the \( T(f) \)-positivity of \( L \) we derive
4.5 Two technical results and the proof of Theorem 4.11

\[ \gamma^{2n} \int_{\mathbb{R}} \chi_\gamma(t) \, d\nu(t) \leq \int_{\mathbb{R}} t^{2n} \, d\nu(t) = s_{2n} = L(g^{2n}) \leq \lambda^{2n+2} L(p) \quad (4.12) \]

for all \( n \in \mathbb{N} \). Since \( \gamma > \lambda \), (4.12) implies that \( \int_{\mathbb{R}} \chi_\gamma(t) \, d\nu(t) = 0 \). Hence \( \text{supp} \nu \subseteq [-\lambda, \lambda] \). Therefore, applying the Cauchy–Schwarz inequality for \( L \), we derive

\[
|L(g)|^2 \leq L(1) L(g^2) = L(1) s_2 = L(1) \int_{-\lambda}^{\lambda} t^2 \, d\nu(t) \\
\leq L(1) \nu(\mathbb{R}) \lambda^2 = L(1)^2 \lambda^2 = L(1)^2 (\|g\|_\infty + \varepsilon)^2.
\]

Letting \( \varepsilon \to +0 \), we get \( |L(g)| \leq L(1) \|g\|_\infty \).

(ii): Suppose that \( g \in \mathbb{R}_d[\chi] \) and \( g \geq 0 \) on \( K(f) \). Then, we conclude easily that \( \|1 \cdot |g|\|_\infty - 2g\|_\infty = \|g\|_\infty \). Using this equality and (4.11) we obtain

\[
L(1) \|g\|_\infty - 2L(g) = L(1) \|g\|_\infty - 2g \leq L(1) \|1 \cdot |g|\|_\infty - 2g = L(1) \|g\|_\infty,
\]

which in turn implies that \( L(g) \geq 0 \). Therefore, by Haviland’s theorem, \( L \) is a \( K(f) \)-moment functional. \( \square \)
Lecture 5
The moment problem on closed semi-algebraic sets: the fibre theorem

Abstract:
This main result of this lecture is the fibre theorem about the existence for the moment problem on (unbounded) semi-algebraic sets. This theorem reduces moment properties of a preordering \( T(f) \) to those for fibre preorderings built by fibres of bounded polynomials on the set \( K(f) \).

The moment problem on compact semi-algebraic sets has rather satisfactory results. There are useful existence theorems (Theorems 4.11, 4.12 and 4.18) and the solution is always unique (by the Weierstrass theorem on uniform approximation by polynomials). Both facts are longer true in the noncompact case. The moment problem on unbounded sets leads to new principal difficulties concerning the existence of a solution and also concerning the uniqueness of the representing measure.

For noncompact semi-algebraic sets, only in rare cases the positivity of a linear functional on the preordering is sufficient for being a moment functional. The fibre theorem is a powerful and deep general result from which almost all known affirmative results can be derived.

5.1 Positive functionals which are not moment functionals

Each real polynomial in one variable which is nonnegative on \( \mathbb{R} \) is a sum of squares. This result was the crucial step main to the simple criteria for the one-dimensional Hamburger moment problem. As already noted by Hilbert (1888), there are nonnegative polynomials in two variables which cannot be represented as finite sums of squares of polynomials. A prominent example is Motzkin’s polynomial

\[
p_c(x_1,x_2) := x_1^2 x_2^2 (x_1^2 + x_2^2 - c) + 1, \quad 0 < c \leq 3.
\]  

(5.1)

It can be shown that the cone \( \sum \mathbb{R}[x]^2 \) of sums of squares of polynomials is closed in the finest locally convex topology on the vector space \( \mathbb{R}[x] \). Hence, since
The moment problem on closed semi-algebraic sets: the fibre theorem

Let $\sum R_2(x)^2$, the separation theorem of convex sets implies that there is a linear functional $L$ on $R_2(x)$ such that $L$ is nonnegative on $\sum R_2(x)^2$ and $L(p_c) < 0$. Thus $L$ is a positive linear functional on the algebra $R_2(x)$. But, since $p_c \geq 0$ on $R^2$, $L$ cannot be represented as an integral by a positive measure on $R^2$. This indicates that the existence problem for the moment problem on noncompact sets is much more difficult than in the compact case.

A similar example for the Stieltjes moment problem in $R^2$ is

$$q_c(x_1, x_2) := x_1 x_2 (x_1 + x_2 - c) + 1, \ c \in (0, 3].$$

(5.2)

Since $q_c(x_1^2, x_2^2) = p_c(x_1, x_2)$, $q_c \geq 0$ on the positive quarter plane $R^2_+$ and $q_c$ does not belong to the preordering

$$T = \sum R[x_1, x_2]^2 + x_1 \sum R[x_1, x_2]^2 + x_2 \sum R[x_1, x_2]^2 + x_1 x_2 \sum R[x_1, x_2]^2.$$ 

Hence the functional $L'$ defined by $L'(f) = L(f(x_1^2, x_2^2)), \ f \in R[x_1, x_2]$, is a $T$-positive linear functional, which cannot be given by a Radon measure on $R^2_+$.

5.2 Properties (MP) and (SMP) and the fibre theorem

In this Lecture, $A$ is a finitely generated commutative real unital algebra.

Then $A$ is (isomorphic to) a quotient algebra $R_d[x] / J$ for some ideal $J$ of $R_d[x]$ and the character set $\hat{A}$ of $A$ is the zero set $Z(J)$ of $J$. Hence $\hat{A}$, equipped with the weak topology, is a locally compact Hausdorff space and the elements of $\hat{A}$ become continuous functions on this space.

For a quadratic module $Q$ of $A$ we define

$$K(Q) := \{ x \in \hat{A} : f(x) \geq 0, f \in Q \}$$

Let $\mathcal{M}_+(\hat{A})$ denote the Radon measures $\mu$ on $\hat{A}$ such that each $f \in A$ is $\mu$-integrable.

Our main concepts are introduced in the following definition.

Definition 5.1. A quadratic module $Q$ of $A$ has the

- moment property (MP) if each $Q$-positive linear functional $L$ on $A$ is a moment functional, that is, there exists a Radon measure $\mu \in \mathcal{M}_+(\hat{A})$ such that

$$L(f) = \int_{\hat{A}} f(x) \ d\mu(x) \ \text{for all} \ f \in A,$$

(5.3)

- strong moment property (SMP) if each $Q$-positive linear functional $L$ on $A$ is a $K(Q)$-moment functional, that is, there is a Radon measure $\mu \in \mathcal{M}_+(\hat{A})$ such that $\text{supp} \ \mu \subseteq K(Q)$ and (5.3) holds.

As noted in Section 5.1 (MP) fails for the preordering $\sum R_d(x)^2$, $d \geq 2$. Obviously, (SMP) implies (MP). The converse is not true, as shown by the next example.
We denote by $\mathcal{K}(T_1) = \{0, +\infty\}$. By Hamburger’s theorem each $T_1$-positive linear functional can be given by a Radon measure on $\mathbb{R}$, so $T_1$ satisfies (MP). It can be shown (see [MP, Example 13.7]) that there exists a $T_2$-positive functional on $A$ which has no representing measure supported on $[0, +\infty)$. This means that (SMP) fails for the preordering $T_2$.

Set $T_1 = \sum \mathbb{R}[x]^2 + x\sum \mathbb{R}[x]^2$. Then, by Stieltjes theorem, each $T_1$-positive linear functional has a representing measure supported on $[0, +\infty)$, so $T_1$ has (SMP).

Note that $\mathcal{K}(T_3) = \mathcal{K}(T_1) = [0, +\infty)$. Thus, whether a preordering $T$ has (SMP) is not determined by the semi-algebraic set $\mathcal{K}(T)$. It depends essentially on the “right” generators of $T$.

Now we begin with the preparations for the fibre theorem.

Suppose $T$ is a finitely generated preordering of $A$ and $f = \{f_1, \ldots, f_k\}$ is a set of generators of $T$. We consider an $m$-tuple $h = (h_1, \ldots, h_m)$ of elements $h_k \in A$. Let

$\lambda = (\lambda_1, \ldots, \lambda_r) \in h(\mathcal{K}(T)) := \{(h_1(x), \ldots, h_m(x)) : x \in \mathcal{K}(T)\}$. (5.4)

We denote by $\mathcal{K}(T)_{\lambda}$ the subset of $\hat{A}$ given by

$\mathcal{K}(T)_{\lambda} = \{x \in \mathcal{K}(T) : h_1(x) = \lambda_1, \ldots, h_m(x) = \lambda_m\}$,

and by $T_{\lambda}$ the preordering of $A$ generated by the sequence

$f(\lambda) := \{f_1, \ldots, f_k, h_1 - \lambda_1, \lambda_1 - h_1, \ldots, h_m - \lambda_m, \lambda_m - h_m\}$. 

Clearly, $\mathcal{K}(T)$ is the disjoint union of fibre set $\mathcal{K}(T)_{\lambda}$, where $\lambda \in h(\mathcal{K}(T))$.

Let $\mathcal{I}_{\lambda}$ denote the ideal of $A$ generated by $h_1 - \lambda_1, \ldots, h_m - \lambda_m$. Then the preordering $(T + \mathcal{I}_{\lambda})/\mathcal{I}_{\lambda}$ of the quotient algebra $A/\mathcal{I}_{\lambda}$ is generated by

$\pi_{\lambda}(f) := \{\pi_{\lambda}(f_1), \ldots, \pi_{\lambda}(f_k)\}$,

where $\pi_{\lambda} : A \rightarrow A/\mathcal{I}_{\lambda}$ denotes the canonical map.

Let us illustrate this in the simple case of a strip $[a, b] \times \mathbb{R}$ in $\mathbb{R}^2$.

Example 5.3. Let $a, b \in \mathbb{R}$, $a < b$, $d = 2$, and $f_1 = \{f_1 := (x_1 - a)(b - x_1)\}$, Then

$T := T(f_1) = \sum \mathbb{R}[x_1, x_2]^2 + (x_1 - a)(b - x_1) \sum \mathbb{R}[x_1, x_2]^2$, $\mathcal{K}(T) = [a, b] \times \mathbb{R}$. 

Set $h = (h_1)$, where $h_1(x_1, x_2) := x_1$. Clearly, $h(\mathcal{K}(T)) = [a, b]$. Fix $\lambda \in [a, b]$.

Then the fibre $\mathcal{K}(T)_{\lambda}$ is the line $\{\lambda\} \times \mathbb{R}$, the ideal $\mathcal{I}_{\lambda}$ is generated by $x_1 - \lambda$ and $\pi_{\lambda}(f_1) = (\lambda - a)(b - \lambda)$ is a constant. Since $x_1 = \lambda$ in the quotient algebra by $\mathcal{I}_{\lambda}$, we obtain $A/\mathcal{I}_{\lambda} = \mathbb{R}[x_2]$ and $(T + \mathcal{I}_{\lambda})/\mathcal{I}_{\lambda} = \sum \mathbb{R}[x_2]^2$.

Now we add the polynomial $f_2(x_1, x_2) = x_2$ to $f_1$, that is, we replace $f_1$ by $f_2 = \{f_1 := (x_1 - a)(b - x_1), f_2 := x_2\}$. Then we have $\mathcal{K}(T) = [a, b] \times [0, +\infty)$. As above, let $h = (h_1)$. Let $\lambda \in [a, b]$. As in the case of $f_1$, we get $A/\mathcal{I}_{\lambda} = \mathbb{R}[x_2]$. But now $\mathcal{K}_{\lambda} = [\lambda] \times [0, +\infty)$ and the preordering is $(T + \mathcal{I}_{\lambda})/\mathcal{I}_{\lambda} = \sum \mathbb{R}[x_2]^2 + x_2 \sum \mathbb{R}[x_2]^2$. 

In both cases, the fibre preorderings have (SMP) and (MP) by the solutions of the Hamburger and Stieltjes moment problems, respectively.

Finally, let us consider the case \( f_3 = \{ f_1 = (x_1 - a)(b - x_1), f_2 := x_2^3 \} \). Then we obtain \( (T + I_\lambda)/I_\lambda = \sum \mathbb{R}[x_2]^2 + x_3^3 \sum \mathbb{R}[x_2]^2 \). This fibre preordering has (MP), but (SMP) fails, as noted in Example 5.2.

The following fibre theorem is the main result of this Lecture. It was proved by the author first (2003) for \( A = \mathbb{R}_d[x] \) and later (2015) in the general case.

**Theorem 5.4.** Let \( A \) be a finitely generated commutative real unital algebra and let \( T \) be a finitely generated preordering of \( A \). Suppose \( h_1, \ldots, h_m \) are elements of \( A \) that are bounded on the set \( K(T) \). Then the following are equivalent:

(i) \( T \) satisfies property (SMP) (resp. (MP)) in \( A \).

(ii) \( (T + I_\lambda)/I_\lambda \) satisfies (SMP) (resp. (MP)) in \( A/I_\lambda \) for all \( \lambda \in \hat{h}(K(T)) \).

**Proof.** [MP, Theorem 13.10]. □

The main implication (ii) \( \rightarrow \) (i) of this theorem allows us to derive the properties (SMP) or (MP) for \( T \) from the corresponding properties of fibre preorderings \( T_\lambda \). Thus, if the algebra \( A \) contains elements \( h_j \) that are bounded on the set \( K(T) \), then the moment properties of \( T \) can be reduced to the preorderings \( T_\lambda \) which have (in general) “lower dimensional” fibre sets \( K(T_\lambda) \).

The proof of (i) \( \rightarrow \) (ii) is not difficult, while the proof of (ii) \( \rightarrow \) (i) is very long and technically involved (see [MP, Section 13.10]). It uses the Krivine–Stengle Positivstellensatz. There is also a version of the fibre theorem for quadratic modules, but this is much weaker and it is stated in [MP, Theorem 13.12].

**Remark 5.5.** Let us consider the special case \( A = \mathbb{R}_d[x] \), \( T = T(f) \), and assume that the semi-algebraic sets \( K(T(f)) \) is compact. Since the coordinate functions \( x_j \) are bounded on \( K(T(f)) \), they can be taken as \( h_j \). Then, all fibre algebras \( A/I_\lambda \) are equal to \( \mathbb{R} \) and \( (T + I_\lambda)/I_\lambda \) is \([0, +\infty)\), so it has obviously (SMP). Therefore, by Theorem 5.4 \( T = T(f) \) obeys (SMP) in \( \mathbb{R}_d[x] \). This is the assertion of Theorem 4.11 of Lecture 3.

**Theorem 5.4** is useful if the algebra contains many bounded elements. This is often the case for algebras of rational functions as the following example shows.

**Example 5.6.** Let \( A \) be the real algebra of rational functions on \( \mathbb{R}^d \) generated by the polynomial algebra \( \mathbb{R}_d[x] \) and \( q_j(x) = (1 + x_j^2)^{-1}, j = 1, \ldots, d \). Let \( T = \sum A^2 \).

Then \( \hat{A} \) is given the evaluations at points of \( \mathbb{R}^d \). Hence \( K(T) = \hat{A} = \mathbb{R}^d \). Clearly, the elements \( h_j := q_j \) and \( h_{d+j} = x_jq_j, j = 1, \ldots, d, \) of \( A \) are bounded on \( K(T) \).

Let \( \lambda \in \hat{h}(K(T)) \). Since \( \lambda_j = q_j(\lambda) \neq 0, \lambda_{d+j} = \lambda_jq_j(\lambda) \), we have \( x_j = \lambda_{d+j}/\lambda_j^{-1}, j = 1, \ldots, d \), in the fibre algebra \( A_\lambda \). Thus, \( A_\lambda = \mathbb{R} \), so it is trivial that \( \sum (A_\lambda)^2 \) has (MP). Therefore, \( \sum A^2 \) obeys (MP) by Theorem 5.4. This means that each positive linear functional on the algebra \( A \) is given by some Radon measure on \( \hat{A} = \mathbb{R}^d \).

This conclusion and the preceding reasoning remain valid if \( A \) is replaced by the algebra generated by \( \mathbb{R}_d[x] \) and the single function \( q(x) = (1 + x_1^2 + \cdots + x_d^2)^{-1} \).
5.3 First application: cylinder sets with compact base

The following simple lemmas are needed for the applications in the next sections.

**Lemma 5.7.** If $\sum A^2$ has property (MP) in $\mathbb{A}$ and $\mathcal{I}$ is an ideal of $\mathbb{A}$, then $\sum (A/\mathcal{I})^2$ has (MP) in $\mathbb{A}/\mathcal{I}$.

**Proof.** Let $\rho$ denote the canonical map of $\mathbb{A}$ into $\mathbb{A}/\mathcal{I}$. Suppose $\mathbb{L}$ is a positive linear functional on $\mathbb{A}/\mathcal{I}$. Then $\mathbb{L} := \mathbb{L} \circ \rho$ is a positive functional on $\mathbb{A}$. Since $\sum A^2$ has (MP) by assumption, $\mathbb{L}$, hence $\mathbb{L}$, is given a Radon measure on $\mathbb{A}$. Because $\mathbb{L}$ annihilates $\mathcal{I}$, the measure is supported on the zero set of $\mathcal{I}$ and so on $\hat{\mathbb{A}}/\mathcal{I}$. This means that $\sum (A/\mathcal{I})^2$ has (MP). \qed

**Lemma 5.8.** If a real algebra $\mathbb{A}$ has a single generator, then $\sum A^2$ obeys (MP).

**Proof.** Because $\mathbb{A}$ is single generated, it is a quotient algebra $\mathbb{R}[y]/\mathcal{I}$ for some ideal $\mathcal{I}$ of $\mathbb{R}[y]$. Since $\sum \mathbb{R}[y]^2$ has (MP) by Hamburger’s theorem, it follows from Lemma 5.7 that $\sum A^2 = \sum (\mathbb{R}[y]/\mathcal{I})^2$ has (MP). \qed

5.3 First application: cylinder sets with compact base

The moment problem for cylinders with compact base fits nicely to the fibre theorem, as shown by the following Theorem.

**Theorem 5.9.** Let $\mathcal{C}$ be a compact set in $\mathbb{R}^{d-1}$, $d \geq 2$, and let $f$ be a finite subset of $\mathbb{R}_+[x]$. Suppose the semi-algebraic subset $\mathcal{K}(f)$ of $\mathbb{R}^d$ is contained in the cylinder $\mathcal{C} \times \mathbb{R}$. Then the preordering $T(f)$ has (MP). If $\mathcal{C}$ is a semi-algebraic set in $\mathbb{R}^{d-1}$ and $\mathcal{K}(f) = \mathcal{C} \times \mathbb{R}$, then $T(f)$ satisfies (SMP).

**Proof.** Define $h_j(x) = x_j$ for $j = 1, \ldots, d - 1$. Since $\mathcal{K}(f) \subseteq \mathcal{C} \times \mathbb{R}$ and $\mathcal{C}$ is compact, the polynomials $h_j$ are bounded on $\mathcal{K}$, so the assumptions of Theorem 5.4 are fulfilled. Then all fibres $\mathcal{K}(f)_\lambda$ are subsets of $(\lambda_1, \ldots, \lambda_{d-1}) \times \mathbb{R}$, the preordering $T(f)_\lambda$ contains $\sum \mathbb{R}[x_d]^2$, and the quotient algebra $\mathbb{R}[x_d]/\mathcal{I}_\lambda$ is an algebra of polynomials in the single variable $x_d$. By Lemma 5.8, $\sum (\mathbb{R}[x_d]/\mathcal{I}_\lambda)^2$ obeys (MP) and so obviously does the larger preordering $T(f)_\lambda/\mathcal{I}_\lambda$. Therefore, $T(f)$ has (MP) by Theorem 5.4 (ii) $\Rightarrow$ (i).

If $\mathcal{K}(f) = \mathcal{C} \times \mathbb{R}$, the fibres for $\lambda \in h(\mathcal{K}(f))$ are equal to $(\lambda_1, \ldots, \lambda_{d-1}) \times \mathbb{R}$ and $T(f)_\lambda = \sum \mathbb{R}[x_d]^2$. Hence the $T(f)_\lambda$ satisfy (SMP) and so does $T(f)$. \qed

We return to the case of a strip in $\mathbb{R}^2$ and derive a classical result due to Devinatz (1955).

**Example 5.10.** (Example 5.3 continued)

Let us retain the notation of Example 5.3. As noted therein, all fibre preorderings for $f_1$ and $f_2$ obey (SMP), so do $T(f_1)$ and $T(f_2)$ by Theorem 5.4.

We state this result for $T(f_1)$: Given a linear functional $L$ on $\mathbb{R}[x_1, x_2]$, there exists a Radon measure $\mu$ on $\mathbb{R}^2$ supported on $[a, b] \times \mathbb{R}$ such that $\mu$ is $\mu$-integrable and...
The difficulty of getting the "right support" of the representing measure appears
Then if
Example 5.14. $f_4(x) = x_1, f_2(x) = 1 - x_1, f_2(x) = x_2^2 - x_1^2, f_2(x) = 4 - x_1 x_2$.
Then $h_1(x) = x_1$ is bounded and $h_1(K(f)) = [0, 1]$. The fibres for $\lambda \in (0, 1]$ are compact, so the preordering $T_2$ has (SMP). The fibre set at $\lambda = 0$ is $\{0\} \cup [1, +\infty)$ and the sequence $\pi_0(f)$ is $\{0, 1, x_2^2 - x_1^2, 4\}$. Since $\pi_0(f)$ does not contain multiples of all natural choice generators for $\{0\} \cup [1, +\infty)$, $T(f)$ does not have (SMP).
Example 5.15. \( f_1(x) = x_1, f_2(x) = 1 - x_1, f_3(x) = 1 - x_1x_2, f_4(x) = x_1^2. \)

Again let \( h_1(x) = x_1, \) so that \( h_1(K(f)) = [0, 1]. \) All fibres at \( \lambda \in [0, 1] \) are compact, so they obey (SMP). The fibre set at \( \lambda = 0 \) is \([0, +\infty)\) and the sequence \( \pi_0(f) = \{0, 1, 0, x_2^2\} \) does not contain a multiple of the natural choice generator \( x_2 \) for \([0, +\infty)\). Hence \( T(f) \) does not satisfy (SMP). However, if we replace \( f_4 \) by \( \tilde{f}_4(x) = x_2, \) then \( T(f) \) has (SMP). \( \circ \)

5.5 Second application: the two-sided complex moment problem

Let \( B = \mathbb{C}[z, \overline{z}, z^{-1}, \overline{z}^{-1}] \) be the \(*\)-algebra of complex Laurent polynomials in \( z \) and \( \overline{z}. \) The involution \( f \mapsto f^\dagger \) of \( B \) is the complex conjugation. Note that \( B \) is \((\ast\)-isomorphic to\) the semigroup \(*\)-algebra for the \(*\)-semigroup \( \mathbb{Z}^2 \) with involution \((m,n)^\dagger = (n,m)\).

Suppose \( s = (s_{m,n})_{(m,n) \in \mathbb{Z}^2} \) is a complex sequence and \( L_s \) is the \( \mathbb{C} \)-linear functional on \( B \) defined by \( L_s(z^m\overline{z}^n) = s_{m,n}, (m,n) \in \mathbb{Z}^2. \)

The **two-sided complex moment problem** is the following question:

*When does there exist a Radon measure \( \mu \) on \( \mathbb{C}^\times := \mathbb{C} \setminus \{0\} \) such that the function \( z^m\overline{z}^n \) on \( \mathbb{C}^\times \) is \( \mu \)-integrable and

\[
\int_{\mathbb{C}^\times} z^m\overline{z}^n d\mu(z) \quad \text{for} \quad (m,n) \in \mathbb{Z}^2,
\]

or equivalently,

\[
L_s(p) = \int_{\mathbb{C}^\times} p(z,\overline{z}) d\mu(z) \quad \text{for} \quad p \in B ?
\]

In this case we call \( s \) a moment sequence and \( L_s \) a moment functional.

The next result is **Bisgaard’s theorem** (1989). In terms of \(*\)-semigroups it means that each positive semidefinite sequence on \( \mathbb{Z}^2 \) is a moment sequence on \( \mathbb{Z}^2 \). The main assertion of Theorem 5.16 says that each positive functional on \( B \) is a moment functional. Since \( B \) has the character set \( \mathbb{R}^2 \setminus \{0\}, \) this result is really surprising.

**Theorem 5.16.** A linear functional \( L \) on \( B \) is a moment functional if and only if \( L \) is a positive functional, that is, \( L(f^+f) \geq 0 \) for all \( f \in B. \)

**Proof.** It clearly suffices to prove the if part.

First we describe the \(*\)-algebra \( B \) in terms of generators. A vector space basis of \( \mathbb{C}[z, \overline{z}, z^{-1}, \overline{z}^{-1}] \) is \( \{z^k\overline{z}^l : k,l \in \mathbb{Z}\}. \) Writing \( z = x_1 + ix_2 \) with \( x_1, x_2 \in \mathbb{R} \) we get

\[
z^{-1} = \frac{x_1 - ix_2}{x_1^2 + x_2^2} \quad \text{and} \quad \overline{z}^{-1} = \frac{x_1 + ix_2}{x_1^2 + x_2^2}.
\]

Hence, as a complex unital algebra, \( B \) is generated by the four real functions
The moment problem on closed semi-algebraic sets: the fibre theorem

\[ x_1, x_2, y_1 := \frac{x_1}{x_1^2 + x_2^2}, \quad y_2 := \frac{x_2}{x_1^2 + x_2^2}. \]  \hspace{1cm} (5.5)

The Hermitean part \( A := \{ f \in B : f^* = f \} \) of the complex \( * \)-algebra \( B \) is a real algebra and its character set \( \hat{A} \) is given by the point evaluations \( \chi_x \) at \( x \in \mathbb{R}^2 \setminus \{0\} \).

(Obviously, \( \chi_x \) is a character for \( x \in \mathbb{R}^2 \setminus \{0\} \). Since \( (y_1 + iy_2)(x_1 - ix_2) = 1 \), there is no character \( \chi \) on \( A \) for which \( \chi(x_1) = 0 \) and \( \chi(x_2) = 0 \).)

The three functions

\[ h_1(x) = x_1y_1 = \frac{x_1^2}{x_1^2 + x_2^2}, \quad h_2(x) = x_2y_2 = \frac{x_2^2}{x_1^2 + x_2^2}, \quad h_3(x) = x_1y_2 = x_2y_1 = \frac{x_1x_2}{x_1^2 + x_2^2} \]

are elements of \( A \) and they are bounded on \( \hat{A} \).

Consider a nonempty fibre set given by \( h_j(x) = \lambda_j \), where \( \lambda_j \in \mathbb{R} \) for \( j = 1, 2, 3 \).
Since \( \lambda_1 + \lambda_2 = 1 \), we can assume without loss of generality that \( \lambda_1 \neq 0 \). In the quotient algebra, \( A/I_{\lambda} \) we have \( x_1y_1 = \lambda_1 \neq 0 \), so \( y_1 = \lambda_1x_1^{-1} \), and \( x_2y_1 = x_1y_2 = \lambda_3 \), so \( y_2 = \lambda_3x_1^{-1} \) and \( x_2 = \lambda_3\lambda_1^{-1}x_1 \). Hence the algebra \( A/I_{\lambda} \) is generated by \( x_1, x_1^{-1} \), so it is a quotient of the algebra \( \mathbb{R}[x_1, x_1^{-1}] \) of Laurent polynomials.

The character set of \( \mathbb{R}[x, x^{-1}] \) is the set of nonzero reals and the corresponding moment problem is the two-sided Hamburger moment problem. Since, as easily shown, each nonnegative \( f \in \mathbb{R}[x, x^{-1}] \) is a sum of squares, \( \sum \mathbb{R}[x, x^{-1}]^2 \) obeys (MP) by Haviland’s Theorem and so does the preordering \( \sum (A/I_{\lambda})^2 \) of its quotient algebra \( A/I_{\lambda} \) by Lemma 5.7. Therefore, by Theorem 5.4 \( \sum \mathbb{R}^2 \) satisfies (MP). This means that each positive functional on \( A \), hence on \( B \), is a moment functional. \( \square \)
Lecture 6
The moment problem on closed sets: determinacy and Carleman condition

Abstract:
In this lecture we discuss various forms of determinacy in the multidimensional case. The multivariate Carleman condition leads to a far-reaching existence and uniqueness theorem for the moment problem.

In Lecture 3 we have noted that the moment problem for noncompact semi-algebraic sets leads to new difficulties concerning the existence of a solution. In this Lecture we will see that the same is true for the problem of determinacy.

6.1 Various notions of determinacy

Let us begin with the following classical result due to M. Riesz (1923) for the determinacy of the one-dimensional Hamburger moment problem.

**Theorem 6.1.** Suppose \( \mu \) is a Radon measure on \( \mathbb{R} \) such that all moments are finite and let \( s \) be its moment sequence. Then the measure \( \mu \), or equivalently, its moment sequence \( s \), is determinate if and only if the set of polynomials \( \mathbb{C}[x] \) is dense in the Hilbert space \( L^2(\mathbb{R}, (1 + x^2) d\mu) \).

In particular, if \( \mu \) is determinate, then \( \mathbb{C}[x] \) is dense in \( L^2(\mathbb{R}, \mu) \).

Proof (MP, Theorem 6.10 and Corollary 6.11).

That \( \mathbb{C}[x] \) is dense in the Hilbert space \( L^2(\mathbb{R}, (1 + x^2) d\mu) \) means that, given a \( f \in L^2(\mathbb{R}, (1 + x^2) d\mu) \) and \( \varepsilon > 0 \), there exists a polynomial \( p \in \mathbb{C}[x] \) such that

\[
\int_{\mathbb{R}} |f(x) - p(x)|^2 (1 + x^2) d\mu(x) < \varepsilon.
\]

The use of complex polynomials here is only of technical nature, because Hilbert spaces such as \( L^2(\mathbb{R}, (1 + x^2) d\mu) \) are usually over the complex field.
In higher dimensions $d \geq 2$ the determinacy problem is much more subtle and the equivalence stated in Theorem 6.1 is no longer valid.

Let $\mathcal{M}_+(\mathbb{R}^d)$ denote the set of Radon measures on $\mathbb{R}^d$ for which all moments are finite. For $\mu \in \mathcal{M}_+(\mathbb{R}^d)$ let $\mathcal{M}_\mu$ be the set of measures $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ which have the same moments as $\mu$, or equivalently, which satisfy

$$\int_{\mathbb{R}^d} p(x) d\nu(x) = \int_{\mathbb{R}^d} p(x) d\mu(x) \quad \text{for all} \quad p \in C_d[\mathbb{R}^d].$$

We write $\nu \equiv \mu$ if $\nu \in \mathcal{M}_\mu$. Obviously, “$\equiv$” is an equivalence relation in $\mathcal{M}_+(\mathbb{R}^d)$.

**Definition 6.2.** For a measure $\mu \in \mathcal{M}_+(\mathbb{R}^d)$ we shall say that

- $\mu$ is *determinate* if $\mathcal{M}_\mu$ is a singleton, that is, if $\nu \in \mathcal{M}_\mu$ implies $\mu = \nu$,
- $\mu$ is *strictly determinate* if $\mu$ is determinate and $C_d[\mathbb{R}^d]$ is dense in $L^2(\mathbb{R}^d, d\mu)$,
- $\mu$ is *strongly determinate* if $C_d[\mathbb{R}^d]$ is dense in $L^2(\mathbb{R}^d, (1+|x|^2) d\mu)$ for $j=1,\ldots,d$,
- $\mu$ is *ultradeterminate* if $C_d[\mathbb{R}^d]$ is dense in $L^2(\mathbb{R}^d, (1+\|x\|^2) d\mu)$.

Suppose $\mu \in \mathcal{M}_+(\mathbb{R}^d)$. Let $s$ be the moment sequence of $\mu$ and $L$ the moment functional of $\mu$. Then we say that $s$, or $L$, is *determinate, strongly determinate, strictly determinate, ultradeterminate*, if $\mu$ has this property.

Note that all four determinacy notions are defined in terms of the measure $\mu$!

If $\mu$ has a compact support, it follows from the Weierstrass theorem that it is ultradeterminate, so all four determinacy notions are valid in this case.

Further, in the case $d = 1$ it follows at once from Theorem 6.1 that the four concepts (determinacy, strict determinacy, strong determinacy, ultradeterminacy) are equivalent. However, in dimension $d \geq 2$ all of them are different! Counterexamples can be found in [MP].

More surprisingly, as shown by Berg and Thill (1991), there exist determinate measures $\mu \in \mathcal{M}_+(\mathbb{R}^d)$, $d \geq 2$, such that $C_d[\mathbb{R}^d]$ is not dense in $L^2(\mathbb{R}^d, d\mu)$. Such measures are not strictly determinate.

By definition strict determinacy implies determinacy. Theorem 6.10 below shows that if $\mu$ is strongly determinate it is strictly determinate and hence determinate. Since the norm of $L^2(\mathbb{R}^d, (1+\|x\|^2) d\mu)$ is obviously stronger than that of $L^2(\mathbb{R}^d, (1+|x|^2) d\mu)$, ultradeterminacy implies strong determinacy. Thus we have the following implications:

ultradeterminate $\Rightarrow$ strongly determinate $\Rightarrow$ strictly determinate $\Rightarrow$ determinate.

### 6.2 Determinacy via marginal measures

Suppose $\mu$ is a Radon measure on $\mathbb{R}^d$. Let $\pi_j(x_1, \ldots, x_d) = x_j$ denote the $j$-th coordinate mapping of $\mathbb{R}^d$ into $\mathbb{R}$. The $j$-th marginal measure $\pi_j(\mu)$ is the Radon measure
on $\mathbb{R}$ defined by $\pi_j(\mu)(M) := \mu(\varphi^{-1}(M))$ for any Borel set $M$ of $\mathbb{R}$. Then the transformation formula

$$\int_{\mathbb{R}} f(y) \, d\pi_j(\mu)(y) = \int_{\mathbb{R}^d} f(x_j) \, d\mu(x)$$

(6.1)

holds for any function $f \in L_1(\mathbb{R}, \pi_j(\mu))$.

The following basic result is Petersen’s theorem (1982).

**Theorem 6.3.** Let $\mu \in \mathcal{M}_+(\mathbb{R}^d)$. If all marginal measures $\pi_1(\mu), \ldots, \pi_d(\mu)$ are determinate, then $\mu$ itself is determinate.

**Proof.** [MP, Theorem 14.6]. $\square$

The converse of Theorem 6.3 does not hold. That is, there exists a determinate measure of $\mu \in \mathcal{M}_+(\mathbb{R}^2)$ such that $\pi_1(\mu)$ and $\pi_2(\mu)$ are not determinate (see, e.g., [MP, Exercise 14.7]).

### 6.3 The multivariate Carleman condition and Nussbaum’s theorem

From Lecture 2 we recall the Carleman condition for a positive semidefinite 1-sequence $(t_n)_{n \in \mathbb{N}_0}$:

$$\sum_{n=1}^{\infty} t_n^{2n} = +\infty.$$ (6.2)

For a $d$-sequence $s = (s_n)_{n \in \mathbb{N}_0^d}$ the 1-sequences

$$s^{[1]} := (s_{(n,0,...,0)})_{n \in \mathbb{N}_0}, s^{[2]} := (s_{(0,n,...,0)})_{n \in \mathbb{N}_0}, \ldots, s^{[n]} := (s_{(0,...,0,n)})_{n \in \mathbb{N}_0}$$ (6.3)

are called marginal sequences of $s$. If $s$ is the moment sequence of $\mu \in \mathcal{M}_+(\mathbb{R}^d)$, then it follows from formula (6.1) that

$$s^{[j]}_n = \int_{\mathbb{R}^d} x_j^n \, d\mu(x_1, \ldots, x_d) = \int_{\mathbb{R}} y^n \, d\pi_j(\mu)(y), \quad n \in \mathbb{N}_0, j = 1, \ldots, d,$$

that is, the $j$-th marginal sequence $s^{[j]}$ is just the moment sequence of the $j$-th marginal measure $\pi_j(\mu)$ of $\mu$. Thus, Petersen’s Theorem 6.3 says that a $d$-moment sequence $s$ is determinate if all marginal sequences $s^{[1]}, \ldots, s^{[d]}$ are determinate.

A $d$-sequence $s = (s_n)_{n \in \mathbb{N}_0^d}$ is called positive semidefinite if

$$\sum_{n,m \in \mathbb{N}_0^d} s_{n+m} \xi_n \xi_m \geq 0$$

for all finite real multisequences \((\xi_n)_{n \in \mathbb{N}_0}\). It is easily verified that \(s\) is positive semidefinite if and only if its Riesz functional \(L_s\) is positive, that is, \(L_s(p^2) \geq 0\) for \(p \in \mathbb{R}_d[\mathbf{x}]\).

**Definition 6.4.** Let \(s\) be a positive semidefinite \(d\)-sequence. We shall say that \(s\), and equivalently the functional \(L_s\), satisfy the **multivariate Carleman condition** if all marginal sequences \(s[x_1], \ldots, s[x_d]\) satisfy Carleman’s condition \((6.2)\), that is, if

\[
\sum_{n=1}^{\infty} (s[j])^2 n^{-1} = \sum_{n=1}^{\infty} L_s(x_j^{2n}) n^{-1/2} = +\infty \quad \text{for} \quad j = 1, \ldots, d. \tag{6.4}
\]

The following fundamental result is **Nussbaum’s theorem** (1965).

**Theorem 6.5.** Each positive semidefinite \(d\)-sequence \(s = (s_n)_{n \in \mathbb{N}_0}\) satisfying the multivariate Carleman condition is a strongly determinate moment sequence.

Theorem 6.5 follows at once from the following more general result.

**Theorem 6.6.** Suppose \(s = (s_n)_{n \in \mathbb{N}_0}\) is a positive semidefinite \(d\)-sequence such that the first \(d-1\) marginal sequences \(s[x_1], \ldots, s[x_{d-1}]\) fulfill Carleman’s condition \((6.2)\). Then \(s\) is a moment sequence.

If in addition the sequence \(s[x_d]\) satisfies Carleman’s condition \((6.2)\) as well, then the moment sequence \(s\) is strongly determinate.

**Proof.** [MP, Theorem 14.6]. □

Theorem 6.5 is a strong and very useful result. It shows that for any positive semi-definite \(d\)-sequence the multivariate Carleman condition implies the existence and the uniqueness of a solution of the moment problem!

We illustrate the usefulness of Theorem 6.5 by the following application.

**Corollary 6.7.** Let \(\mu \in \mathcal{M}_+(\mathbb{R}^d)\). Suppose that there exists an \(\varepsilon > 0\) such that

\[
\int_{\mathbb{R}^d} e^{\varepsilon |\mathbf{x}|} \, d\mu(x) < +\infty.
\]

Then \(\mu \in \mathcal{M}_+(\mathbb{R}^d)\) and \(\mu\) is strongly determinate.

**Proof.** The proof is similar to the proof of Corollary 2.4 in Lecture 2. Let \(j \in \{1, \ldots, d\}\) and \(n \in \mathbb{N}_0\). Then \(x_j^2 e^{-\varepsilon |x_j|} \leq e^{-2\varepsilon (2n)!}\) for \(x_j \in \mathbb{R}\) and hence

\[
\int_{\mathbb{R}^d} x_j^{2n} \, d\mu = \int_{\mathbb{R}} x_j^{2n} e^{-\varepsilon |x_j|} e^{\varepsilon |x_j|} \, d\mu \leq e^{-2\varepsilon (2n)!} \int_{\mathbb{R}^d} e^{\varepsilon |\mathbf{x}|} \, d\mu < +\infty. \tag{6.5}
\]

Let \(p \in \mathbb{R}[\mathbf{x}]\). Then \(p(x) \leq c (1 + x_1^{2n} + \cdots + x_d^{2n})\) on \(\mathbb{R}^d\) for some \(c > 0\) and \(n \in \mathbb{N}\), so \((6.5)\) implies that \(p\) is \(\mu\)-integrable. Thus \(\mu \in \mathcal{M}_+(\mathbb{R}^d)\).

Let \(s\) be the moment sequence of \(\mu\). By \((6.5)\) there is a constant \(M > 0\) such that
Theorems 6.5 and 6.1.

Suppose the Riesz functional $L$ on operator $T$. Therefore, $s$ and $\mu$ are strongly determinate by combining Theorems 6.5 and 6.1.

Carleman’s condition can be also used to localize the support of representing measures, as the following result of Lasserre (2013) shows.

**Theorem 6.8.** Let $s$ be a real $d$-sequence and $f = \{f_1, \ldots, f_k\}$ a finite subset of $\mathbb{R}_d[x]$. Suppose the Riesz functional $L_f$ is $Q(f)$-positive (that is, $L_f(p^2) \geq 0$ and $L_f(f_j p^2) \geq 0$ for $j = 1, \ldots, k$, $p \in \mathbb{R}_d[x]$) and satisfies the multivariate Carleman condition. Then the unique representing measure of the determinate moment sequence $s$ is supported on the semi-algebraic set $K(f)$.

**Proof.** [MP, Theorem 14.25].

**6.4 Some operator-theoretic reformulations**

Apart from real algebraic geometry, the operator theory on Hilbert space is another powerful tool for the study of the multidimensional moment problem. This connection is not treated in these lectures. In this final section we briefly touch a few operator-theoretic points. First we recall some basic notions.

Suppose $\mathcal{H}$ is a complex Hilbert space with scalar product $\langle \cdot, \cdot \rangle$. A symmetric operator on $\mathcal{H}$ is a linear mapping $T$ of a linear subspace $D(T)$, called the domain of $T$, into $\mathcal{H}$ such that $\langle T\varphi, \psi \rangle = \langle \varphi, T\psi \rangle$ for $\varphi, \psi \in D(T)$.

An operator $T$ is called closed if for each sequence $(\varphi_n)$ from $D(T)$ such that $\varphi_n \to \varphi$ and $T\varphi_n \to \psi$ in $\mathcal{H}$ then $\varphi \in D(T), \psi = T\varphi$. If $T$ has a closed extension, it has a smallest closed extension, called the closure of $T$ and denoted by $\overline{T}$.

Suppose that $D(T)$ is dense in $\mathcal{H}$. Then the adjoint operator $T^*$ of $T$ is defined: Its domain $D(T^*)$ is the set of vectors $\psi \in \mathcal{H}$ for which there exists an $\eta \in \mathcal{H}$ such that $\langle T\varphi, \psi \rangle = \langle \varphi, \eta \rangle$ for all $\varphi \in D(T)$; in this case $T^*\psi = \eta$.

A densely defined operator $T$ is called self-adjoint if $T = T^*$ and essentially self-adjoint if its closure $\overline{T}$ is self-adjoint, or equivalently, if $T = T^*$.

We recall the following well-known fact from operator theory.

**Lemma 6.9.** A densely defined symmetric operator $T$ is essentially self-adjoint if and only if the subspaces $(T - i)D(T)$ and $(T + i)D(T)$ are dense in $\mathcal{H}$.

Now we reformulate Theorem 6.1. Suppose $\mu \in M_+(\mathbb{R})$. Let $X$ denote the multiplication operators by the variable $x$ on the Hilbert space $L^2(\mathbb{R}, \mu)$, that is, $(Xp)(x) = xp(x)$ for $p \in D(X) := C[x]$. It is easily checked that $X$ is symmetric.

For a function $f$ of $\mathbb{R}$ we set $g_\pm := (x \pm i)f$. Then, for $p \in C[x]$,
\begin{equation}
\int_{\mathbb{R}} |g_\pm(x) - (x \pm i)p(x)|^2 d\mu(x) = \int_{\mathbb{R}} |f(x) - p(x)|^2 (1 + x^2) d\mu(x) \tag{6.6}
\end{equation}

Since \( g_\pm \in L^2(\mathbb{R}, \mu) \) if and only if \( f \in L^2(\mathbb{R}, (1 + x^2) d\mu) \), it follows from (6.6) that \((X \pm i)\mathcal{D}(X)\) is dense in \( L^2(\mathbb{R}, \mu) \) if and only if \( C[x] \) is dense in \( L^2(\mathbb{R}, (1 + x^2) d\mu) \). Therefore, by Lemma 6.9, the measure \( \mu \) is determinate if and only if the multiplication operator \( X \) on \( L^2(\mathbb{R}, \mu) \) is essentially self-adjoint. This is an interesting operator-theoretic characterization of determinacy.

Strong determinacy has also a nice operator-theoretic interpretation. For a measure \( \mu \in M_+(\mathbb{R}^d) \), let \( X_j \) denote the multiplication operator by the variable \( x_j \) with domain \( \mathcal{D}(X_j) = C_d[\bar{x}] \) in the Hilbert space \( L^2(\mathbb{R}^d, \mu) \).

**Theorem 6.10.** Suppose \( L \) is a moment functional on \( C_d[\bar{x}] \) and \( \mu \) is a representing measure for \( L \). Then \( \mu \) is strongly determinate if and only if the symmetric operators \( X_1, \ldots, X_d \) are essentially self-adjoint. If this holds, then \( \mu \) is strictly determinate, that is, \( \mu \) is determinate and \( C_d[\bar{x}] \) is dense in \( L^2(\mathbb{R}^d, \mu) \).

**Proof.** [MP, Theorem 4.2]. \( \square \)

Finally, we relate Carleman’s condition to the notion of a \textit{quasi-analytic vector}.

Let \( T \) be a symmetric linear operator on a Hilbert space \( \mathcal{H} \) and \( \varphi \in \bigcap_{n=1}^{\infty} \mathcal{D}(T^n) \). Since the operator \( T \) is symmetric, it is easily verified that the real sequence

\[ t = (t_n := \langle T^n \varphi, \varphi \rangle)_{n \in \mathbb{N}_0} \]

is positive semidefinite and hence a moment sequence by Hamburger’s theorem [1.12]. The vector \( \varphi \) is called \textit{quasi-analytic} for \( T \) if

\[ \sum_{n=1}^{\infty} \| T^n \varphi \|^{-1/n} = +\infty. \]

Note that \( t_{2n} = \langle T^{2n} \varphi, \varphi \rangle = \| T^n \varphi \|^2 \) for \( n \in \mathbb{N}_0 \). Hence the vector \( \varphi \) is quasi-analytic for \( T \) if and only if the sequence \( t \) satisfies Carleman’s condition (6.7).

Now suppose \( \mu \in M_+(\mathbb{R}^d) \). Let \( s \) be the moment sequence of \( \mu \) and \( L_s \) the corresponding Riesz functional on \( R_d[\bar{x}] \). On the Hilbert space \( L^2(\mathbb{R}^d, \mu) \) we have

\[ s^{[j]}_{2n} = L_s(x_j^{2n}) = \int x_j^{2n} d\mu(x) = \langle X_j^{2n} 1, 1 \rangle = \| X_j^n 1 \|^2, \quad j = 1, \ldots, d, n \in \mathbb{N}_0. \]

Therefore, from the preceding discussion it follows that the sequence \( s \) fulfills the \textit{multivariate Carleman condition} (according to Definition [6.4]) if and only if the constant function \( \varphi = 1 \) is a \textit{quasi-analytic vector} for the multiplication operators \( X_1, \ldots, X_d \) by the variable \( x_1, \ldots, x_d \), respectively.
Abstract:
Semidefinite programming is introduced. Positivstellensätze and the moment problem provide methods for determining the minimum of a polynomial over a semi-algebraic set via Lasserre relaxation. From the Archimedean Positivstellensatz convergence results are obtained.

7.1 Semidefinite programming

Semidefinite programming is a generalization of linear programming, where the linear constraints are replaced by linear matrix constraints.

Let $\text{Sym}_n$ denote the real symmetric $n \times n$-matrices and $\langle A, B \rangle := \text{Tr} AB$. $A \succeq 0$ means that the matrix $A$ is positive semidefinite and $A \succ 0$ that $A$ is positive definite.

We define a semidefinite program and its dual program. Suppose $b \in \mathbb{R}^m$ and $m + 1$ matrices $A_0, \ldots, A_m \in \text{Sym}_n$ are given.

The primal semidefinite program (SDP) is the following:

$$p_* = \inf_{y \in \mathbb{R}^m} \{ b^T y : A(y) := A_0 + y_1 A_1 + \cdots + y_m A_m \succeq 0 \}. \quad (7.1)$$

That is, one minimizes the linear function $b^T y = \sum_{j=1}^m b_j y_j$ in a vector variable $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$ subject to the linear matrix inequality (LMI) constraint

$$A(y) := A_0 + y_1 A_1 + \cdots + y_m A_m \succeq 0. \quad (7.2)$$

The set of points $y \in \mathbb{R}^m$ satisfying $(7.2)$ is called a spectrahedron. Since $A(y) \succeq 0$ if and only if its principal minors are nonnegative and each such minor is a polynomial, a spectrahedron is a closed semi-algebraic set. By $(7.2)$, a spectrahedron is also convex. A convex closed semi-algebraic set is not necessarily a spectrahedron.
If all matrices \( A_j \) are diagonal, the constraint \( A(y) \succeq 0 \) consists of linear inequalities, so (7.1) is a linear program. Conversely, each linear problem is a semidefinite program by writing the linear constraints as an LMI with a diagonal matrix.

The dual program associated with (7.1) is defined by

\[
p^* = \sup_{Z \in \text{Sym}_n} \left\{ -\langle A_0, Z \rangle : Z \succeq 0 \quad \text{and} \quad \langle A_j, Z \rangle = b_j, \ j = 1, \ldots, m \right\}.
\]

Thus, one maximizes the linear function \(-\langle A_0, Z \rangle = -\text{Tr}A_0Z\) in a matrix variable \(Z\) subject to the constraints \(Z \succeq 0\) and \(\langle A_j, Z \rangle = \text{Tr}A_jZ = b_j\). It can be shown that the dual program (7.3) is also a semidefinite program.

A vector \(y \in \mathbb{R}^m\) resp. a matrix \(Z \in \text{Sym}_n\) is called feasible for (7.1) resp. (7.3) if it satisfies the corresponding constraints. If there are no feasible points, we set \(p_* = +\infty\) resp. \(p^* = -\infty\). A program is called feasible if it has a feasible point.

**Proposition 7.1.** If \(y\) is feasible for (7.1) and \(Z\) is feasible for (7.3), then

\[
b^T y \geq p_* \geq p^* \geq -\langle A_0, Z \rangle.
\]

**Proof.** Since \(y\) is feasible for (7.1) and \(Z\) is feasible for (7.3), \(A(y) \succeq 0\) and \(Z \succeq 0\). Therefore, \(\langle A(y), Z \rangle \geq 0\), so using (7.3) we derive

\[
b^T y = \sum_{j=1}^m y_j \langle A_j, Z \rangle = \langle A(y), Z \rangle - \langle A_0, Z \rangle \geq -\langle A_0, Z \rangle.
\]

Taking the infimum over \(y\) and the supremum over \(Z\) we obtain (7.4). \(\square\)

In contrast to linear programming, \(p_*\) is not equal to \(p^*\) in general. The number \(p_* - p^*\) is called the duality gap. A simple example of a semidefinite program with positive duality gap is the following: Minimize \(x_1\) subject to the constraint

\[
\begin{pmatrix}
0 & x_1 & 0 \\
x_1 & x_2 & 0 \\
0 & 0 & x_1 + 1
\end{pmatrix} \succeq 0.
\]

In this case, we have \(p_* = 0\) and \(p^* = -1\).

The next proposition shows that, under the stronger assumption of strict feasibility, the duality gap is zero.

**Proposition 7.2.** (i) If \(p_* > -\infty\) and (7.1) is strictly feasible (that is, there is a \(y \in \mathbb{R}^m\) such that \(A(y) > 0\)), then \(p_* = p^*\) and the supremum in (7.3) is a maximum.

(ii) If \(p^* < +\infty\) and (7.3) is strictly feasible (that is, there is a \(Z \in \text{Sym}_n\) such that \(Z > 0\) and \(\langle A_j, Z \rangle = b_j, j = 1, \ldots, m\)), then \(p_* = p^*\) and the infimum in (7.1) is a minimum.

**Proof.** The proof is based on separation of convex sets, see [MP, Proposition 16.2]. \(\square\)
We give two examples where semidefinite programs appears.

**Example 7.3. (Largest eigenvalue of a symmetric matrix)**

Let \( \lambda_{\text{max}}(B) \) denote the largest eigenvalue of \( B \in \text{Sym}_n \). Then

\[
\lambda_{\text{max}}(B) = \min_{y \in \mathbb{R}} \{ y : (yI - B) \succeq 0 \}
\]

gives \( \lambda_{\text{max}}(B) \) by a semidefinite program. Since this program and its dual are strictly feasible (take \( y \in \mathbb{R} \) such that \( (yI - B) \succ 0 \) and \( Z = I \)), Proposition 7.2 yields

\[
\lambda_{\text{max}}(B) = \max_{Z \in \text{Sym}_n} \{ \langle B, Z \rangle : Z \succeq 0, \langle I, Z \rangle = 1 \}.
\]

There is also a semidefinite program for the sum of the \( j \) largest eigenvalues. \( \diamond \)

**Example 7.4. (Sos representation of a polynomial)**

A polynomial \( f(x) = \sum_{\alpha} f_{\alpha} x^{\alpha} \in \mathbb{R}_d[\mathbb{X}]_{2n} \) is in \( \sum_{\alpha} |\alpha| = n \) if and only if there exists a positive semidefinite matrix \( G \) such that

\[
f(x) = (f_n) \, G, \quad \text{where} \quad f_n = (1, x_1, \ldots, x_d, x_1^2, x_1 x_2, \ldots, x_2^2, x_3^2, \ldots, x_1^2, \ldots, x_n^2)^T.
\]

If we write \( f_n(f_n)^T = \sum \alpha A_{\alpha} x^{\alpha} \) with \( A_{\alpha} \in \text{Sym}_{d(n)} \) and compare coefficients in the equation \( f(x) = (f_n)^T G f_n \), we obtain \( \langle G, A_{\alpha} \rangle = \text{Tr} \, GA_{\alpha} = f_{\alpha} \) for \( \alpha \in \mathbb{N}_0^n, |\alpha| \leq 2n \). Therefore, \( f \in \sum |\alpha| \) if and only if there exists a matrix \( G \in \text{Sym}_{d(n)} \) such that

\[
G \succeq 0 \quad \text{and} \quad \langle Q, A_{\alpha} \rangle = f_{\alpha} \quad \text{for} \quad \alpha \in \mathbb{N}_0^n, |\alpha| \leq 2n.
\]

Hence \( f \) is a sum of squares if and only if the feasibility condition (7.5) of the corresponding semidefinite program is satisfied. \( \diamond \)

### 7.2 Lasserre relaxations of polynomial optimization with constraints

We fix \( f = \{ f_0, \ldots, f_k \} \), where \( f_j \in \mathbb{R}[\mathbb{X}], f_j \neq 0, f_0 = 1, \) and \( p \in \mathbb{R}[\mathbb{X}], p \neq 0 \).

Our aim is to minimize the polynomial \( p \) over the semi-algebraic set \( K(f) \):

\[
p^{\text{min}} := \inf \{ p(x) : x \in K(f) \}. \tag{7.6}
\]

Clearly,

\[
p^{\text{min}} = \sup \{ \lambda \in \mathbb{R} : p - \lambda \geq 0 \quad \text{on} \quad K(f) \} \tag{7.7}.
\]

\[
p^{\text{min}} = \inf \{ L(p) : L \in \mathbb{R}[\mathbb{X}]_0^+, L(1) = 1, L(f) \geq 0 \quad \text{if} \quad f \geq 0 \quad \text{on} \quad K(f) \}. \tag{7.8}
\]

The idea is to weaken the constraints in (7.7) and (7.8). We require in (7.7) that \( p - \lambda \) is a certain weighted sum of squares which is nonnegative on \( K(f) \) and consider in
functions $L$ which are nonnegative only on certain weighted sums of squares.

For $n \in \mathbb{N}_0$ let $n_j$ be the largest integer such that $n_j \leq \frac{1}{2}(n - \deg(f_j))$. Set

$$Q(f)_n := \left\{ \sum_{j=0}^k f_j \sigma_j : \sigma_j \in \sum \mathbb{R}_{d}[\mathbb{X}]^2_{n_j} \right\},$$

$$Q(f)_n^* := \{ L \in \mathbb{R}_{d}[\mathbb{X}]^n : L(1) = 1 \text{ and } L(g) \geq 0, g \in Q(f)_n \}.$$

Now we define two Lasserre relaxations by

$$p_{\text{mom}}^n := \inf \{ L(p) : L \in Q(f)_n^* \},$$

$$p_{\text{sos}}^n := \sup \{ \lambda \in \mathbb{R} : p - \lambda \in Q(f)_n \},$$

where we set $p_{\text{sos}}^n = -\infty$ if there is no $\lambda \in \mathbb{R}$ such that $p - \lambda \in Q(f)_n$.

Both relaxations (7.9) and (7.10) can be reformulated in terms of a semidefinite program and its dual. Let us begin with (7.9).

A linear functional $L$ on $\mathbb{R}_{d}[\mathbb{X}]_n$ is completely described by the $\binom{d+n}{n}$ variables $y_\alpha := L(x^\alpha)$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq n$. By definition, $L \in Q(f)_n^*$ if and only if $L(1) = 1$ and

$$L(f_j y) \geq 0 \quad \text{for} \quad q \in \mathbb{R}_{d}[\mathbb{X}]_{n_j}, \; j = 0, \ldots, k. \quad (7.11)$$

We reformulate these conditions in terms of the variables $y_\alpha$. Clearly, $L(1) = 1$ means $y_0 = 1$. Let $f_j = \sum_\alpha f_{j,\alpha} x^\alpha$. We denote by $H_{n_j}(f_j y)$ the $\binom{d+n}{n_j} \times \binom{d+n}{n_j}$-matrix with entries

$$H_{n_j}(f_j y)_{\alpha,\beta} := \sum_\gamma f_{j,\gamma} y_{\alpha+\beta-\gamma}, \quad |\alpha| \leq n_j, |\beta| \leq n_j. \quad (7.12)$$

Thus $H_{n_j}(f_j y)$ is a localized Hankel matrix. For $q = \sum_\alpha a_\alpha x^\alpha \in \mathbb{R}_{d}[\mathbb{X}]_{n_j}$, we compute

$$L(f_j q^2) = \sum_\alpha f_{j,\alpha} a_\alpha a_\beta L(x^\alpha + \beta) = \sum_{\alpha,\beta} H_{n_j}(f_j y)_{\alpha,\beta} a_\alpha a_\beta,$$

so (7.11) holds if and only if the matrices $H_{n_j}(f_j y)$, $j = 1, \ldots, k$, are positive semidefinite. Let $H(f)(y)$ denote the block diagonal matrix with diagonal blocks $H_{n_0}(f_0 y), H_{n_1}(f_1 y), \ldots, H_{n_k}(f_k y)$. Then (7.11) is satisfied if and only if $H(f)(y) \succeq 0$. The matrix $H(f)(y)$ has type $N \times N$, where $N := \sum_{j=0}^k \binom{d+n}{n_j}$.

By (7.12), all entries of $H(f)(y)$ are linear in the variables $y_\alpha$. Hence, inserting $y_0 = 1$, there are (constant!) real symmetric $N \times N$-matrices $A_\alpha$ such that

$$H(f)(y) = A_0 + \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \leq n} y_\alpha A_\alpha. \quad (7.13)$$

Let $p(x) = \sum_\alpha p_\alpha x^\alpha$. Then $L(p) = p_0 + \sum_{\alpha \neq 0} p_\alpha y_\alpha$. By (7.9), $p_{\text{mom}}^n$ is the infimum of $L(p)$ in the $M := \binom{d+n}{n} - 1$ variables $y_\alpha$ subject to $H(f)(y) \succeq 0$. 


7.3 Polynomial optimization with constraints

Summarizing, the relaxation \(7.9\) leads to the \textbf{primal semidefinite program}

\[
p_{\text{sos}}^n - p_0 = \inf_{y_{\alpha} \in \mathbb{R}^m} \left\{ \sum_{0 < |\alpha| \leq n} p_{\alpha} y_{\alpha} : A_0 + \sum_{0 < |\alpha| \leq n} y_{\alpha} A_{\alpha} \succeq 0 \right\}. \tag{7.14}
\]

Next we show that \(7.10\) leads to the corresponding dual program. Suppose that \(\lambda \in \mathbb{R}\) and \(p - \lambda \in Q(f)_n\), that is,

\[
p - \lambda = \sum_{j=0}^k f_j \sigma_j, \quad \sigma_j \in \sum \mathbb{R}_d[x]_{n_j}^2.
\tag{7.15}
\]

Then \(\sigma_j \in \sum \mathbb{R}_d[x]_{n_j}^2\) if and only if there is a matrix \(Z(j) \succeq 0\) such that

\[
\sigma_j(x) = \sum_{|\alpha|, |\beta| \leq n} Z(j)_{\alpha, \beta} x^{\alpha + \beta}.
\]

Let \(Z\) be the block diagonal \(N \times N\)-matrix with blocks \(Z(0), \ldots, Z(k)\). Clearly, \(Z \succeq 0\) if and only if \(Z(j) \succeq 0\) for all \(j\).

Recall that \(f_j = \sum_{\alpha} f_{j, \alpha} \alpha^\alpha\). Equating coefficients in \(7.15\) yields

\[
p_0 - \lambda = \sum_{j=0}^k f_{j,0} Z(j)_{00} = \text{Tr} A_0 Z = \langle A_0, Z \rangle,
\]

\[
p_{\alpha} = \sum_{j=0}^k \sum_{\beta + \gamma + \delta = \alpha} f_{j, \delta} Z(j)_{\beta, \gamma} = \text{Tr} A_{\alpha} Z = \langle A_{\alpha}, Z \rangle, \quad \alpha \neq 0.
\]

Clearly, taking the supremum of \(\lambda\) in \(7.10\) is equivalent to taking the supremum of \(\lambda - p_0 = -\langle A_0, Z \rangle\) subject to the conditions \(p_{\alpha} = \langle A_{\alpha}, Z \rangle, 0 < |\alpha| \leq n\).

Thus, the second relaxation \(7.10\) leads to the corresponding \textbf{dual program}

\[
p_{\text{sos}}^n - p_0 = \sup_{Z \in \text{Sym}_n} \left\{ -\langle A_0, Z \rangle : Z \succeq 0, \ p_{\alpha} = \langle A_{\alpha}, Z \rangle \text{ for } 0 < |\alpha| \leq n \right\}. \tag{7.16}
\]

\[7.3\] Polynomial optimization with constraints

First we note that for \(n \in \mathbb{N}\),

\[
p_{\text{sos}}^n \leq p_{\text{sos}}^{n+1} \quad \text{and} \quad p_{\text{mom}}^n \leq p_{\text{mom}}^{n+1} \tag{7.17}
\]

\[
p_{\text{sos}}^n \leq p_{\text{mom}}^n \leq p_{\text{min}}^{n+1}. \tag{7.18}
\]

We give the simple proofs of these inequalities. Since \(Q(f)_n\) is a subspace of \(Q(f)_{n+1}\), \(p - \lambda \in Q(f)_n\) implies that \(p - \lambda \in Q(f)_{n+1}\), so \(p_{\text{sos}}^n \leq p_{\text{sos}}^{n+1}\). Since restrictions of functionals from \(Q(f)_n^*\) belong to \(Q(f)_{n+1}^*\), it follows that \(p_{\text{mom}}^n \leq p_{\text{mom}}^{n+1}\).
Polynomials of $Q(f)$ are nonnegative on $\mathcal{K}(f)$. Hence each point evaluation at $x \in \mathcal{K}(f)$ is in $Q(f)_{+}$, so $p_{n}^{\text{mom}} \leq p(x)$ for $x \in \mathcal{K}(f)$. This implies $p_{n}^{\text{mom}} \leq p_{\text{min}}$.

Let $L \in Q(f)_{+}$. If $p - \lambda L \in Q(f)_{n}$, then $L(p - \lambda) = L(p) - \lambda \geq 0$, that is, $\lambda \leq L(p)$. Taking the supremum over $\lambda$ and the infimum over $L$ we get $p_{n}^{\text{sos}} \leq p_{n}^{\text{mom}}$.

As an application of the Archimedean Positivstellensatz we show that for an Archimedean module $Q(f)$ both relaxations converge to the minimum of $p$.

**Theorem 7.5.** Suppose the quadratic module $Q(f)$ is Archimedean. Then the set $\mathcal{K}(f)$ is compact, so $p$ attains its minimum over $\mathcal{K}(f)$, and we have

$$\lim_{n \to \infty} p_{n}^{\text{sos}} = \lim_{n \to \infty} p_{n}^{\text{mom}} = p^{\text{min}}. \quad (7.19)$$

**Proof.** We have noted that $Q(f)$ is Archimedean implies that $\mathcal{K}(f)$ is compact.

Let $\lambda \in \mathbb{R}$ be such that $\lambda < p^{\text{min}}$. Then $p(x) - \lambda > 0$ on $\mathcal{K}(f)$. Therefore, $p - \lambda \in Q(f)$ by the Archimedean Positivstellensatz. This means that $p - \lambda = \sum_{j} f_{j} \sigma_{j}$ for some elements $\sigma_{j} \in \sum_{d} \mathbb{R}[x]^{2}$. We choose $n \in \mathbb{N}$ such that $n \geq \deg(f_{j} \sigma_{j})$ for all $j$. Then $p - \lambda = \sum_{j} f_{j} \sigma_{j} \in Q(f)_{n}$ and hence $p_{n}^{\text{sos}} \geq \lambda$ by the definition of $p_{n}^{\text{sos}}$. Since $\lambda < p^{\text{min}}$ was arbitrary and $p_{n}^{\text{sos}} \leq p_{n}^{\text{mom}} \leq p^{\text{min}}$ and $p_{n}^{\text{sos}} \leq p_{n+1}^{\text{sos}}$ by (7.17) and (7.18), this implies (7.19). $\square$

The following propositions deal with two special situations. The first shows that the supremum in (7.16) is attained if $\mathcal{K}(f)$ has interior points.

**Proposition 7.6.** Suppose that $\mathcal{K}(f)$ has a nonempty interior. Then $p_{n}^{\text{sos}} = p_{n}^{\text{mom}}$ for $n \in \mathbb{N}$. If $p_{n}^{\text{mom}} > -\infty$, then the supremum in (7.16) is a maximum.

**Proof (MP, Proposition 16.7).** $\square$

The quadratic module $Q(f)$ is called stable if for $n \in \mathbb{N}_{0}$ there is a $k(n) \in \mathbb{N}_{0}$ such that each $q \in Q(f)_{n}$ can be written as $q = \sum_{j} f_{j} \sigma_{j}$ with $\sigma_{j} \in \sum_{d} \mathbb{R}[x]^{2}$, $\deg(f_{j} \sigma_{j}) \leq k(n)$. (This is a restriction on degree cancellations for elements of $Q(f)$.)

**Proposition 7.7.** Suppose that the quadratic module $Q(f)$ is stable. Then there exists an $n_{0} \in \mathbb{N}_{0}$, depending only on $\deg(p)$, such that $p_{n}^{\text{sos}} = p_{n_{0}}^{\text{sos}}$ for all $n \geq n_{0}$.

**Proof.** [MP, Proposition 16.8]. $\square$

It can be shown that if $\dim \mathcal{K}(f) \geq 2$, then the assumptions of Theorem 7.5 and Proposition 7.7 exclude each other.

It is natural to look for conditions which imply finite convergence for the limit

$$\lim_{n \to \infty} p_{n}^{\text{mom}} = p^{\text{min}}. \quad (7.20)$$

The flat extension theorem of Lecture 5 gives such a result:

Let $m := \max \{ \deg(f_{j}) : j = 0, \ldots, k \}$ and $n > m$. If the infimum in (7.9) is attained at $L$ and $\text{rank} H_{n-m}(L) = \text{rank} H_{n}(L)$, then $p_{k}^{\text{mom}} = p^{\text{min}}$ for all $k \geq n$. 

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Abstract:
The theorem of Richter-Tchakaloff is proved. Existence criteria by positivity conditions are formulated and discussed. Stochel’s theorem is stated.

The next three Lectures are concerned with the multidimensional truncated moment problem. We consider the following general setup:
- $\mathcal{X}$ is a locally compact Hausdorff space,
- $E$ is a finite-dimensional vector space of continuous real-valued functions on $\mathcal{X}$,
- $K$ is a closed subset of $\mathcal{X}$.

That $E$ has finite dimension is crucial! Further, we assume the following:

There exists a function $e \in E$ such that $e(x) \geq 1$ for $x \in \mathcal{X}$. \hfill (8.1)

Let us recall the following fundamental notion.

**Definition 1.** A linear functional $L : E \rightarrow \mathbb{R}$ is called a $K$-moment functional if there exists a Radon measure $\mu$ on $\mathcal{X}$ supported on $K$ such that

$$L(f) = \int_{\mathcal{X}} f(x) d\mu(x) \quad \text{for } f \in E?$$

In the case $K = \mathcal{X}$ we say that $L$ is a moment functional.

Our guiding example and most important case is the following:
- $\mathcal{X} = \mathbb{R}^d$ and $E = \mathbb{R}_d[x]_m := \{ p \in \mathbb{R}_d[x] : \deg(p) \leq m \}$.

In this case we have the classical multi-dimensional truncated moment problem. Assumption (8.1) is satisfied with $e(x) := 1$.

Further, $\delta_x$ denotes the Delta measure and $\delta_x$ is the point evaluation functional on $E$ at $x \in \mathcal{X}$. For $k \in \mathbb{N}$, a $k$-atomic measure is a measure $\mu = \sum_{r=1}^{k} c_r \delta_{x_r}$, where all $c_r > 0$ and the $x_r$ are pairwise different points of $\mathcal{X}$. We consider the zero measure as a 0-atomic measure.
8.1 The Richter-Tchakaloff Theorem

The following theorem is the most important general result on truncated moment problems. It implies that \( K \)-moment functionals on \( E \) have always finitely atomic representing measures and are finite linear combinations of point evaluations. Thus, convex analysis comes up as a useful technical tool to study moment problems.

The theorem was proved in full generality by Hans Richter (1957). It is often called Tchakaloff theorem in the literature and Richter–Tchakaloff theorem in [MP]. From its publication data and its generality it seems to be fully justified to call it Richter theorem. We reproduce the proof given in [MP].

**Theorem 8.1.** Suppose \( (\mathcal{Y}, \mu) \) is a measurable space and \( V \) is a finite-dimensional linear space of \( \mu \)-integrable measurable real-valued functions on \( (\mathcal{Y}, \mu) \). Let \( L^\mu \) denote the functional on \( V \) defined by \( L^\mu(f) = \int f \, d\mu, \ f \in V \). Then there is a \( k \)-atomic measure \( \nu = \sum_{j=1}^k m_j \delta_{x_j} \) on \( \mathcal{Y} \), \( k \leq \dim V \), such that \( L^\mu = L^\nu \), that is,

\[
\int f \, d\mu = \int f \, d\nu = \sum_{j=1}^k m_j f(x_j), \quad f \in V.
\]

**Proof.** Let \( C \) be the convex cone in the dual space \( V^* \) of all nonnegative linear combinations of point evaluations \( l_x \), where \( x \in \mathcal{Y} \), and let \( \overline{C} \) be the closure of \( C \) in \( V^* \). We prove by induction on \( m := \dim V \) that \( L^\mu \in C \).

First let \( m = 1 \) and \( V = \mathbb{R} \cdot f \). Set \( c := \int f \, d\mu \). If \( c = 0 \), then \( \int (\lambda f) \, d\mu = 0 \cdot l_x (\lambda f) \), \( \lambda \in \mathbb{R} \), for any \( x_1 \in \mathcal{Y} \). Suppose now that \( c > 0 \). Then \( f(x_1) > 0 \) for some \( x_1 \in \mathcal{Y} \). Hence \( m_1 := c f(x_1)^{-1} > 0 \) and \( \int (\lambda f) \, d\mu = m_1 l_x (\lambda f) \) for \( \lambda \in \mathbb{R} \). The case \( c < 0 \) is treated similarly.

Assume that the assertion holds for vector spaces of dimension \( m-1 \). Let \( V \) be a vector space of dimension \( m \). By standard approximation of \( \int f \, d\mu \) by integrals of simple functions it follows that \( L^\mu \in \overline{C} \). We now distinguish between two cases.

- **Case 1:** \( L^\mu \) is an interior point of \( \overline{C} \).
  
  By a basic result of convex analysis, the convex set \( C \) and its closure \( \overline{C} \) have the same interior points. Hence we have \( L^\mu \in C \) in Case 1.

- **Case 2:** \( L^\mu \) is a boundary point of \( \overline{C} \).

Then there exists a supporting hyperplane \( F_0 \) for the cone \( \overline{C} \) at \( L^\mu \), that is, \( F_0 \) is a linear functional on \( V^* \) such that \( F_0 \neq 0 \), \( F_0(L^\mu) = 0 \) and \( F_0(L) \geq 0 \) for all \( L \in \overline{C} \). Because \( V \) is finite-dimensional, there is a function \( f_0 \in V \) such that \( F_0(L) = L(f_0) \), \( L \in V^* \). For \( x \in \mathcal{Y} \), we have \( l_x \in C \) and hence \( F_0(l_x) = l_x(f_0) = f_0(x) \geq 0 \). Clearly, \( F_0 \neq 0 \) implies that \( f_0 \neq 0 \). We choose an \((m-1)\)-dimensional linear subspace \( V_0 \) of \( V \) such that \( \mathcal{Y} = V_0 \oplus \mathbb{R} \cdot f_0 \). Let us set \( \mathcal{Z} := \{ x \in \mathcal{Y} : f_0(x) = 0 \} \). Since \( 0 = F_0(L^\mu) = L^\mu(f_0) = \int f_0 \, d\mu \) and \( f_0(x) \geq 0 \) on \( \mathcal{Y} \), it follows that \( f_0(x) = 0 \mu\text{-a.e. on } \mathcal{Y} \), that is, \( \mu(\mathcal{Y} \setminus \mathcal{Z}) = 0 \). Now we define a measure \( \tilde{\mu} \) on \( \mathcal{Z} \) by \( \tilde{\mu}(M) = \mu(M \cap \mathcal{Z}) \). Then

\[
L^\mu(g) = \int_{\mathcal{Y}} g \, d\mu = \int_{\mathcal{Z}} g \, d\mu = \int_{\mathcal{Z}} g \, d\tilde{\mu} = L^{\tilde{\mu}}(g) \quad \text{for} \quad g \in V_0.
\]
We apply the induction hypothesis to the functional $L^\mu$ on $V_0 \subseteq L^1(\mathbb{Z}, \tilde{\mu})$. Since $L^\mu = L^{\tilde{\mu}}$ on $V_0$, there exist $\tilde{\lambda}_j \geq 0$ and $x_j \in \mathbb{Z}$, $j = 1, \ldots, n$, such that for $f \in V_0$,

$$L^\mu(f) = \sum_{j=1}^n \tilde{\lambda}_j f(x_j). \quad (8.2)$$

Since $f_0 = 0$ on $\mathbb{Z}$, hence $f_0(x_j) = 0$, and $L^\mu(f_0) = 0$, (8.2) holds for $f = f_0$ as well and so for all $f \in V$. Thus, $L^\mu \in C$. This completes the induction proof.

The set $C$ is a cone in the $m$-dimensional real vector space $V^\ast$. Since $L^\mu \in C$, it follows from Carathéodory’s theorem that there is a representation (8.2) with $n \leq m$. This means that $L^\mu$ is the integral of the measure $\nu = \sum_{j=1}^n \tilde{\lambda}_j \delta_{x_j}$. Clearly, $\nu$ is $k$-atomic, where $k \leq n \leq m$. (We only have $k \leq n$, since some numbers $\tilde{\lambda}_j$ in (8.2) could be zero and the points $x_j$ are not necessarily different.)

Recall that Carathéodory’s theorem (see, e.g., [MP, Proposition A.35]) says the following: If $X$ is a subset of a $d$-dimensional real vector space $V$, then each element of the cone generated by $X$ is a nonnegative combination of $d$ points of $X$.

For the truncated moment problem this theorem has the following corollary.

**Corollary 8.2.** Each moment functional $L$ on $E$ has a $k$-atomic representing measure $\nu$, where $k \leq \dim E$. If $\mu$ is a representing measure of $L$ and $\mathcal{Y}$ is a Borel subset of $X$ such that $\mu(\mathcal{X} \setminus \mathcal{Y}) = 0$, then all atoms of $\nu$ can be chosen from $\mathcal{Y}$.

**Proof.** Apply Theorem 8.1 to the measure space $(\mathcal{Y}, \mu(\mathcal{Y})$ and $V = E$. $\square$

### 8.2 Positive semidefinite 2n-sequences

If the space $X$ is compact, each $E_+\ast$-positive linear functional on $E$ is a moment functional by Proposition 1.7 of Lecture 1. If $X$ is not compact, this is no longer true. In this section, we discuss this for the truncated Hamburger moment problem of $E = \mathbb{R}[x]_{2n}$, $X = \mathbb{R}$.

**Theorem 8.3.** For a real sequence $s = (s_j)^{2n}_{j=0}$, the following are equivalent:

(i) $s$ is positive semidefinite, that is,

$$\sum_{k,l=0}^n s_{k+l}c_kc_l \geq 0 \text{ for all } (c_0, \ldots, c_n)^T \in \mathbb{R}^{n+1}, n \in \mathbb{N}.$$

(ii) The Hankel matrix $H_n(s) = (s_{l+j})^n_{l,j=1}$ is positive semidefinite.

(iii) $L_n(p^2) \geq 0$ for all $p \in \mathbb{R}[x]_n$.

(iv) $L_n(q) \geq 0$ for all $q \in \mathbb{R}[x]_{2n}$ such that $q(x) \geq 0$ on $\mathbb{R}$.

(v) There exists a Radon measure $\mu$ on $\mathbb{R}$ and a number $a \geq 0$ such that

$$s_j = \int_{\mathbb{R}} x^j \, d\mu(x) \text{ for } j = 0, \ldots, 2n - 1, \text{ and } s_{2n} = a + \int_{\mathbb{R}} x^{2n} \, d\mu(x). \quad (8.3)$$
We study truncated moment problems on the \(d\)-dimensional real projective space \(\mathbb{P}^d(\mathbb{R})\) and apply this to the truncated Hankel moment problem for closed sets in \(\mathbb{R}^d\).

The points of \(\mathbb{P}^d(\mathbb{R})\) are equivalence classes of \((d+1)\)-tuples \((t_0, \ldots, t_d) \neq 0\) of real numbers under the equivalence relation

\[
\frac{a_1 t_0 + \cdots + a_d t_d}{a_{d+1} t_0 + \cdots + a_{d+d} t_d}
\]

**Proof.** (iii) \(\rightarrow\)(v): The proof is based on Proposition 1.7 from Lecture 1. Let \(\mathcal{X} := \mathbb{R} \cup \{\infty\}\) denote the one point compactification of \(\mathbb{R}\). The functions

\[
u_j(x) := x^j (1 + x^{2n})^{-1}, \quad j = 0, \ldots, 2n,
\]

are continuous functions on \(\mathcal{X}\) by \(\nu_j(\infty) := 0, j = 0, \ldots, 2n - 1\), and \(\nu_{2n}(\infty) := 1\). Now we define a linear functional \(L\) on

\[
F := \text{Lin}\{u_j : j = 0, \ldots, 2n\} \quad \text{by} \quad L(u_j) = L_x(x^j), j = 0, \ldots, 2n.
\]

Using that positive polynomials of \(\mathbb{R}[x]\) are sums of squares one easily verifies that \(L\) is \(F_+\)-positive. Hence, by Proposition 1.7, \(L\) has a representing Radon measure \(\hat{\mu}\) on \(\mathcal{X}\).

Set \(a = \hat{\mu}(\{\infty\})\) and define measures \(\hat{\mu}\) and \(\mu\) on \(\mathbb{R}\) by \(\hat{\mu}(M) := \hat{\mu}(M)\) for \(M \subseteq \mathbb{R}\) and \(d\mu := (1 + x^{2n})^{-1} d\hat{\mu}\). Then, for \(j = 0, \ldots, 2n,

\[
s_j = L_x(x^j) = \int_{\mathcal{X}} \nu_j(x) d\hat{\mu} = a u_j(\infty) + \int_{\mathbb{R}} x^j d\hat{\mu} = a \delta_{j, 2n} + \int_{\mathbb{R}} x^j d\mu,
\]

(v) \(\rightarrow\)(ii): Let \((c_0, \ldots, c_n)^T \in \mathbb{R}^{n+1}\). Using (8.3) we derive

\[
\sum_{k,l=0}^n s_{k+l} c_k c_l = a c_0^2 + \sum_{k=0}^n \int_{\mathbb{R}} c_k c_l x^{k+l} d\mu = a c_0^2 + \int_{\mathbb{R}} \left(\sum_{k=0}^n c_k x^k\right)^2 d\mu \geq 0, \quad (8.4)
\]

since \(a \geq 0\). This proves (ii).

The other implications are trivial or easily checked. \(\square\)

**Example 8.4.** \(s = (0, 0, 1)\) is positive semidefinite. Since \(s_0 = 0\), \(s\) cannot be given by a positive measure on \(\mathbb{R}\), but \(s\) has a representation (8.3) with \(\mu = 0, a = 1\). \(\odot\)

This simple example shows that the \(E_+\)-positivity of a linear functional or the positive semidefiniteness of a sequence \(s\) are not sufficient for representing it by a positive measure. Roughly speaking, "there may be atoms at infinity".

But if there is a polynomial \(p \in \mathbb{R}[x]\) of degree \(n\) such that \(L_x(p^2) = 0\), it follows from (8.4) that \(a = 0\), hence \(L_x\) is indeed a moment functional. The notion of the Hankel rank can be used to characterize truncated Hamburger moment functionals, see [MP, Section 9.5].

### 8.3 The truncated moment problem on projective space

We study truncated moment problems on the \(d\)-dimensional real projective space \(\mathbb{P}^d(\mathbb{R})\) and apply this to the truncated Hankel moment problem for closed sets in \(\mathbb{R}^d\).

The points of \(\mathbb{P}^d(\mathbb{R})\) are equivalence classes of \((d+1)\)-tuples \((t_0, \ldots, t_d) \neq 0\) of real numbers under the equivalence relation
Then we have
\[(t_0, \ldots, t_d) \sim (t'_0, \ldots, t'_d) \quad \text{if} \quad (t_0, \ldots, t_d) = \lambda (t'_0, \ldots, t'_d) \quad \text{for} \quad \lambda \neq 0. \tag{8.5}\]
The equivalence class is \([t_0 : \ldots : t_d]\). Thus, \(\mathbb{P}^d(\mathbb{R}) = (\mathbb{R}^{d+1}\setminus \{0\}) / \sim\). The map
\[\varphi : \mathbb{R}^d \ni (t_1, \ldots, t_d) \mapsto [1 : t_1 : \cdots : t_d] \in \mathbb{P}^d(\mathbb{R})\]
is injective. We identify \(t \in \mathbb{R}^d\) with \(\varphi(t) \in \mathbb{P}^d(\mathbb{R})\). Then \(\mathbb{R}^d \subseteq \mathbb{P}^d(\mathbb{R})\). The complement of \(\mathbb{R}^d\) in \(\mathbb{P}^d(\mathbb{R})\) is the hyperplane \(H^d_{\infty} = \{[0 : t_1 : \cdots : t_d] \in \mathbb{P}^d(\mathbb{R})\}\).

We denote by \(\mathcal{H}_{d+1,2n}\) the homogeneous polynomials of \(\mathbb{R}[x_0, x_1, \ldots, x_d]\) of degree \(2n\). The map
\[\phi : p(x_0, \ldots, x_d) \mapsto \tilde{p}(x_1, \ldots, x_d) := p(1, x_1, \ldots, x_d)\]
is a bijection of the vector spaces \(\mathcal{H}_{d+1,2n}\) and \(\mathbb{R}[x][x]_{2n}\).

The projective space \(\mathbb{P}^d(\mathbb{R})\) is a compact space and each \(q \in \mathcal{H}_{d+1,2n}\) can be considered as a continuous function \(\tilde{q}\) on this space by
\[\tilde{q}(t) := \frac{q(t_0, \ldots, t_d)}{(t_0^2 + \cdots + t_d^2)^n}, \quad t = [t_0 : \cdots : t_d] \in \mathbb{P}^d(\mathbb{R}). \tag{8.6}\]
We set \(E := \{\tilde{q} : q \in \mathcal{H}_{d+1,2n}\}\). For \(e(x) := (x_1^2 + \cdots + x_2^2)^n\), we have \(\tilde{e}(t) = 1\) for \(t \in \mathbb{P}^d(\mathbb{R})\), so assumption (8.1) is fulfilled. Since \(\mathbb{P}^d(\mathbb{R})\) is compact, each \(E\)-positive linear functional on \(E\) is a moment functional. Then the representing measure is a Radon measure on \(\mathbb{P}^d(\mathbb{R})\).

Let us return to the \(K\)-moment problem for \(\mathbb{R}[x][x]_{2n}\) for a closed set \(K \subseteq \mathbb{R}^d\).

The closure \(\overline{K}^\circ\) of \(K\) in \(\mathbb{P}^d(\mathbb{R})\) is the disjoint union of \(K\) and \(K_m := \overline{K^\circ} \cap H^d_{\infty}\).

We want to characterize linear functionals on \(\mathbb{R}[x][x]_{2n}\) which are nonnegative on \(\overline{K}\).

\[\text{Pos}(K)_{2n} := \{ p \in \mathbb{R}[x][x]_{2n} : p(x) \geq 0 \quad \text{for} \quad x \in K \}. \tag{8.7}\]

For \(p \in \mathbb{R}[x][x]_{2n}\) let \(p_{2n}\) denote its homogeneous part of degree \(2n\) and put
\[\tilde{p}_{2n}(t) := \frac{p_{2n}(t_1, \ldots, t_d)}{(t_1^2 + \cdots + t_d^2)^n} \quad \text{for} \quad t = (0 : t_1 : \cdots : t_d) \in H^d_{\infty} \cong \mathbb{P}^{d-1}(\mathbb{R})\]

The new ingredient is that measures on \(K_m\) give \(\text{Pos}(K)_{2n}\)-positive functionals.

**Lemma 8.5.** If \(\mu_m\) is a Radon measure on \(\mathbb{H}^d_{\infty}\) supported on \(K_m\), then
\[L_{\mu_m}(p) := \int_{K_m} \tilde{p}_{2n}(t) \, d\mu_m(t), \quad p \in \mathbb{R}[x][x]_{2n}, \tag{8.8}\]
defines a linear functional on \(\mathbb{R}[x][x]_{2n}\) such that \(L_{\mu_m}(p) \geq 0\) for \(p \in \text{Pos}(K)_{2n}\).

The next theorem characterizes \(\text{Pos}(K)_{2n}\)-positive linear functionals on \(\mathbb{R}[x][x]_{2n}\).

**Theorem 8.6.** Suppose \(K \subseteq \mathbb{R}^d\) is closed and \(L\) is a linear functional on \(\mathbb{R}[x][x]_{2n}\). Then we have
\[ L(p) \geq 0 \quad \text{for} \quad p \in \text{Pos}(K)_{2n} \quad (8.9) \]

if and only if there are Radon measures \( \mu \) on \( \mathbb{R}^d \) supported on \( K \) and \( \mu_\infty \) on \( \mathbb{H}_\infty \) supported on \( K_\infty \) such that

\[ L(p) = \int_K p(t) \, d\mu(t) + \int_{K_\infty} \tilde{p}_{2n}(t) \, d\mu_\infty(t) \quad \text{for} \quad p \in \mathbb{R}_d[X]_{2n}. \quad (8.10) \]

**Proof.** [MP, Theorem 17.3]. \( \square \)

If the set \( K \) in Theorem 8.6 is compact, then \( K_\infty \) is empty, so the second summand in (8.10) does not occur and we recover the known result that for compact sets \( K \) the positivity condition characterizes \( K \)-moment functionals.

The moment problem on closed subsets \( K \) of \( \mathbb{P}_d(\mathbb{R}) \) has several advantages. First, \( K \) is compact, so \( K \)-moment functionals are characterized by positivity conditions. Secondly, homogeneous polynomials are more convenient to deal with and additional technical tools such as the apolar scalar product (see [MP, Section 9.1]) are available.

As an easy application of Theorem 8.6 we show that \( K \)-moment functionals for closed subsets \( K \) of \( \mathbb{R}^d \) can be characterized be the following extension property.

**Theorem 8.7.** Let \( K \) be a closed subset of \( \mathbb{R}^d \) and \( L_0 \) a linear functional on \( \mathbb{R}_d[X]_{2n-2} \). Then \( L_0 \) is a \( K \)-moment functional on \( \mathbb{R}_d[X]_{2n-2} \) if and only if \( L_0 \) admits an extension to a \( \text{Pos}(K)_{2n} \)-positive linear functional \( L \) on \( \mathbb{R}_d[X]_{2n} \):

\[ L(p) \geq 0 \quad \text{for} \quad p \in \text{Pos}(K)_{2n}. \quad (8.11) \]

**Proof.** Let \( L_0 \) be a \( K \)-moment functional on \( \mathbb{R}_d[X]_{2n-2} \). By Richter’s theorem, \( L_0 \) has a finitely atomic representing measure \( \mu \). Then it suffices to define \( L \) by \( L(p) = \int f \, d\mu \), \( f \in \mathbb{R}_d[X]_{2n} \).

Conversely, assume (8.11) holds. By Theorem 8.6 \( L \) is of the form (8.10). If \( p \in \mathbb{R}_d[X]_{2n-2} \), then \( p_{2n} = 0 \). Hence the second summand in (8.10) vanishes and we obtain \( L_0(p) = \int_K p(t) \, d\mu(t) \) for \( p \in \mathbb{R}_d[X]_{2n-2} \). \( \square \)

### 8.4 Stochel’s theorem

The following interesting result is **Stochel’s theorem** (2001). It says that if for a functional \( L \) on \( \mathbb{R}_d[X] \) the truncated \( K \)-moment problem on \( \mathbb{R}_d[X]_{2n} \) is solvable for all \( n \in \mathbb{N} \), then \( L \) is a \( K \)-moment functional on \( \mathbb{R}_d[X] \).

**Theorem 8.8.** Let \( K \) be a closed subset of \( \mathbb{R}^d \) and \( L \) a linear functional on \( \mathbb{R}_d[X] \). Suppose for each \( n \in \mathbb{N} \) the restriction \( L_n := L \big| \mathbb{R}_d[X]_{2n} \) is a \( K \)-moment functional on \( \mathbb{R}_d[X]_{2n} \), that is, there exists a Radon measure \( \mu_n \) on \( \mathbb{R}^d \) supported on \( K \) such that

\[ L_n(p) \equiv L(p) = \int p(x) \, d\mu_n(x) \quad \text{for} \quad p \in \mathbb{R}_d[X]_{2n}. \]
Then $L$ is a $K$-moment functional on $\mathbb{R}_d[x]$.

*Proof.* [MP, Theorem 17.3].

In terms of sequences Theorem 8.8 can be rephrased as follows: Let $s = (s_\alpha)_{\alpha \in \mathbb{N}_0^d}$ be a real multisequence. If for each $n \in \mathbb{N}$ the truncation $s^{(n)} := (s_\alpha)_{\alpha \in \mathbb{N}_0^d, |\alpha| \leq 2n}$ has a representing measure supported on $K$, then the sequence $s$ does as well.
Lecture 9
Truncated multidimensional moment problem: existence via flat extensions

Abstract:
Hankel matrices are introduced and their basic properties are investigated. The flat extension theorem of Curto and Fialkow is treated.

Hankel matrices appeared already in Lectures 1 and 3. They are also an important technical tool for the study of multidimensional truncated moment problems. The flat extension developed in Section 9.3 is essentially based on rank conditions of Hankel matrices.

9.1 Hankel matrices

Throughout this Lecture, we suppose that \( \mathbb{N} \) be a nonempty finite subset of \( \mathbb{N}_0^d \) and \( \mathcal{A} = \text{Lin}\{x^\alpha : \alpha \in \mathbb{N}\} \). We shall consider the moment problem on the (finite-dimensional) linear subspace \( \mathcal{A}^2 \) (not of \( \mathcal{A} \)) of the polynomials \( \mathbb{R}[x_1, \ldots, x_d] \):

\[
\mathcal{A}^2 := \text{Lin}\{pq : p, q \in \mathcal{A}\} = \text{Lin}\{x^\beta : \beta \in \mathbb{N} + \mathbb{N}\}.
\]

Suppose \( L \) is a linear functional on the real vector space \( \mathcal{A}^2 \).

**Definition 9.1.** \( pq \in \mathbb{R}[x_1, \ldots, x_d] \) The Hankel matrix of \( L \) is the symmetric \( |\mathbb{N}| \times |\mathbb{N}| \)-matrix

\[
H(L) = (h_{\alpha, \beta})_{\alpha, \beta \in \mathbb{N}}, \quad \text{where} \quad h_{\alpha, \beta} := L(x^{\alpha+\beta}), \ \alpha, \beta \in \mathbb{N}.
\]

The number \( \text{rank} L := \text{rank} H(L) \) is called the rank of \( L \).

We fix an ordering of the index set \( \mathbb{N} \). For \( f = \sum_{\alpha \in \mathbb{N}} f_{\alpha} x^\alpha \in \mathcal{A} \) we denote by \( \vec{f} = (f_{\alpha})^T \in \mathbb{R}^{|\mathbb{N}|} \) the coefficient vector of \( f \) according to the ordering of \( \mathbb{N} \). Define
This Hilbert space is another useful tool in treating the moment problem. Since $L$ is a moment functional on $A$, the symmetric bilinear form $\langle \cdot, \cdot \rangle$ implies $\text{supp} \, (\cdot)$: Since the functional $L$ is positive, the Cauchy–Schwarz inequality holds. Thus, $\text{supp} \, \mu \subseteq \mathcal{V}_L$ for each representing measure $\mu$.

**Proof.** (i): Let $f = \sum_{\alpha \in \mathbb{N}} f_\alpha x^\alpha \in A$ and $g = \sum_{\alpha \in \mathbb{N}} g_\alpha x^\alpha \in A$. Then

$$L(fg) = \sum_{\alpha, \beta \in \mathbb{N}} f_\alpha g_\beta L(x^{\alpha+\beta}) = \sum_{\alpha, \beta \in \mathbb{N}} h_{\alpha, \beta} f_\alpha g_\beta = (f_t)^T H(L)(g_t) = f_t^T H(L)g_t.$$

(iii): $\text{rank} \, H(L) = \dim \mathbb{R}^{\mathbb{N}} - \dim \ker H(L) = \dim A - \dim N_L = \dim (A/N_L)$.

(v): If $p \in N_L$, then $L(p^2) = \int p(t)^2 d\mu = 0$. Since $p(x)$ is continuous on $\mathbb{R}^d$, this implies $\text{supp} \, \mu$ is contained in the zero set of $p$. Thus, $\text{supp} \, \mu \subseteq \mathcal{V}_L$. □

Now suppose that the functional $L$ is positive, that is, $L(f^2) \geq 0$ for $f \in A$. Then the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the quotient space $D_L := A/N_L$ defined by

$$\langle f + N_L, g + N_L \rangle := L(fg), \quad f, g \in A.$$  

(9.4)

is nondegenerate and positive definite, so it is an inner product. By Proposition 9.2(iii), $\dim D_L = \text{rank} \, L$. Thus, $(D_L, \langle \cdot, \cdot \rangle)$ is a real finite-dimensional Hilbert space. This Hilbert space is another useful tool in treating the moment problem. Since element of $A^2$ are sums of products $fg$ with $f, g \in A$, the functional $L$ on $A^2$ can be recovered from the space $(D_L, \langle \cdot, \cdot \rangle)$ by using equation (9.4).

The next proposition shows that $N_L$ obeys some ideal-like properties.

**Proposition 9.3.** Let $p \in N_L$ and $q \in A$.

(i) If $L$ is a positive functional and $pq^2 \in A$, then $pq \in N_L$.

(ii) If $L$ is a $K$-moment functional and $pq \in A$, then $pq \in N_L$.

**Proof.** (i): Since the functional $L$ is positive, the Cauchy–Schwarz inequality holds. Using that $pq^2 \in A$ and $L(p^2) = 0$ we obtain
so that \( pq \in \mathcal{N}_L \) by Proposition 9.2(iv).

(ii) Since \( L \) is a \( K \)-moment functional, it has a representing measure \( \mu \), which is supported on \( K \). Then \( \text{supp} \, \mu \subseteq \mathcal{V}_L \) by Proposition 9.2(v). Therefore, since \( p \in \mathcal{N}_L \) and hence \( p(x) = 0 \) on \( \mathcal{V}_L \), we get

\[
L((pq)^2) = \int_K (pq)^2(x) \, d\mu(x) = \int_{\mathcal{V}_L \cap K} p(x)^2 q(x)^2 \, d\mu(x) = 0.
\]

Thus, \( pq \in \mathcal{N}_L \) again by Proposition 9.2(iv). \( \square \)

We restate the preceding results in the case of our standard example.

**Corollary 9.4.** Let \( L \) be a linear functional on \( A = \mathbb{R}_d[x]_{2n} \). Suppose that \( p \in \mathcal{N}_L \) and \( q \in \mathbb{R}_d[x]_{n} \).

(i) If \( L \) is a positive functional and \( pq \in \mathbb{R}_d[x]_{n-1} \), then \( pq \in \mathcal{N}_L \).

(ii) If \( L \) is a \( K \)-moment functional and \( pq \in \mathbb{R}_d[x]_{n} \), then \( pq \in \mathcal{N}_L \).

**Proof.** (i): Proposition 9.3(i) yields the assertion in the case \( q = x_j, \, j = 1, \ldots, d \). Since \( \mathcal{N}_L \) is a vector space, repeated applications give the general case.

(ii) follows from Proposition 9.3(ii). \( \square \)

Corollary 9.4 show an important difference between positive functionals and moment functionals: The assertion \( pq \in \mathcal{N}_L \) holds for a moment functional if \( pq \in \mathbb{R}_d[x]_{n} \), while for a positive functional it is assumed that \( pq \in \mathbb{R}_d[x]_{n-1} \).

### 9.2 The full moment problem with finite rank Hankel matrix

Definition 9.1 of the Hankel matrix \( H(L) \) extends verbatim to linear functionals \( L \) on the polynomial algebra \( \mathbb{R}_d[x] \). In this case we have the full moment problem and in the. If this Hankel matrix \( H(L) \) has finite rank, there is the following result.

**Theorem 9.5.** Suppose that \( L \) is a positive linear functional on \( \mathbb{R}_d[x] \) such that \( \text{rank} \, H(L) = r, \, where \, r \in \mathbb{N} \). Then \( L \) is a moment functional with unique representing measure \( \mu \). This measure \( \mu \) has \( r \) atoms and \( \text{supp} \, \mu = \mathcal{V}_L \).

**Proof.** [MP, Theorem 17.29]. \( \square \)

Idea of proof: The GNS representation \( \pi_L \) acts on a finite-dimensional Hilbert space. Then \( \pi_L(x_1), \ldots, \pi_L(x_d) \) are commuting self-adjoint operators and their spectral measure leads to the measure \( \mu \).
9.3 Flat extensions and the flat extension theorem

Recall that $N$ is a finite subset of $\mathbb{N}_0^d$ and $\mathcal{A} = \text{Lin}\{x^\alpha : \alpha \in N\}$. Let $N_0$ be a proper subset of $N$ and $\mathcal{B} = \text{Lin}\{x^\alpha : \alpha \in N_0\}$.

**Definition 9.6.** A linear functional $L$ on $\mathcal{A}^2$ is called flat with respect to $\mathcal{B}^2$ if

$$\text{rank} H(L) = \text{rank} H(L_0),$$

where $L_0$ denotes the restriction to $\mathcal{B}^2$ of $L$.

To motivate this definition we write the Hankel matrix $H(L)$ as a block matrix

$$H(L) = \begin{pmatrix} H(L_0) & X_{12} \\ X_{21} & X_{22} \end{pmatrix}.$$

Then $H(L)$ is a flat extension of the matrix $H(L_0)$ if $\text{rank} H(L) = \text{rank} H(L_0)$. This is the useful notion of flatness for block matrices.

**Proposition 9.7.** Let $L$ be a linear functional on $\mathcal{A}^2$ which is flat with respect to $\mathcal{B}^2$. If $L(p^2) \geq 0$ for all $p \in \mathcal{B}$, then $L(q^2) \geq 0$ for all $q \in \mathcal{A}$.

**Proof.** [MP, Proposition 17.34]. ⊓ ⊔

Though there are general versions of the flat extension theorem (see [?]), we restrict ourselves to the important case where $\mathcal{B} = \mathbb{R}_d[x]_{n-1}$ and $\mathcal{A} = \mathbb{R}_d[x]_n$.

Let $L$ be a linear functional on $\mathbb{R}_d[x]_{2n}$. The Hankel matrix $H_n(L)$ has the entries $h_{\alpha,\beta} := L(x^{\alpha+\beta})$, where $\alpha, \beta \in \mathbb{N}_0^d, |\alpha|, |\beta| \leq n$.

The Hankel matrix $H_{n-1}(L)$ of its restriction to $\mathbb{R}_d[x]_{2n-2}$ has the entries $h_{\alpha,\beta}$ with $|\alpha|, |\beta| \leq n - 1$.

The main result of this Lecture is the following flat extension theorem of R. Curto and L. Fialkow (1996). It says that if a functional on $\mathbb{R}_d[x]_{2n}$ is flat with respect to $\mathbb{R}_d[x]_{2n-2}$ and positive on $\mathbb{R}_d[x]_{2n-2}$, then it is a moment functional.

**Theorem 9.8.** Suppose $L$ is a linear functional on $\mathbb{R}_d[x]_{2n}, n \in \mathbb{N}$, such that

$$L(p^2) \geq 0 \quad \text{for} \quad p \in \mathbb{R}_d[x]_{n-1} \quad \text{and} \quad r := \text{rank} H_n(L) = \text{rank} H_{n-1}(L).$$

Then $L$ is a moment functional with a unique representing measure. This measure is $r$-atomic and we have $r \leq \binom{n-1+d}{d} = \text{dim} \mathbb{R}_d[x]_{n-1}$.

**Proof.** [MP, Theorem 17.37]. ⊓ ⊔
**Idea of proof:** The flatness is used to extend \( L \) to a functional \( \tilde{L} \) on \( \mathbb{R}^d \) which is flat with respect to \( \mathbb{R}^d \). This is the crucial part of the proof. By Proposition 9.7, \( \tilde{L} \) is positive. Since \( \text{rank} \tilde{L} \) is finite, \( \tilde{L} \) is a moment functional by Theorem 9.5.

The next theorem deals with the \( K \)-moment problem, where

\[
K := K(f) = \{ x \in \mathbb{R}^d : f_1(x) \geq 0, \ldots, f_k(x) \geq 0 \}. 
\]

**Theorem 9.9.** Let \( L \) be linear functional on \( \mathbb{R}^d \) and \( r := \text{rank} H_n(L) \). Then the following are equivalent:

(i) \( L \) is a \( K \)-moment functional which has an \( r \)-atomic representing measure with all atoms in \( K \).

(ii) \( L \) extends to a linear functional \( \tilde{L} \) on \( \mathbb{R}^d \) such that

\[
\text{rank} H_{n+m}(\tilde{L}) = \text{rank} H_n(L) \quad \text{and} \quad \tilde{L}(f_j p^2) \geq 0
\]

for \( p \in \mathbb{R}^d \) and \( j = 0, \ldots, k, f_0 = 1, m := \max \{ 1, \deg(f_j) : j = 1, \ldots, k \} \).

**Proof.** [MP, Theorem 17.38]. \( \square \)

This result says that all \( K \)-moment functionals on \( \mathbb{R}^d \) can be obtained by flat extensions to some appropriate larger space \( \mathbb{R}^d \).

### 9.4 Hankel matrices of functionals with finitely atomic measures

By the Richter-Tchakaloff theorem [8,1] each moment functional on \( \mathcal{A}^2 \) has a finitely atomic representing measure. Therefore, the corresponding Hankel matrices are of particular interest.

In this Section, we suppose \( \mu \) is a finitely atomic measure

\[
\mu = \sum_{j=1}^{k} c_j \delta_{x_j}, \quad \text{where} \quad c_j > 0, \ x_j \in \mathbb{R}^d \quad \text{for} \quad j = 1, \ldots, k, \quad (9.6)
\]

and \( L \) is the corresponding moment functional on \( \mathcal{A}^2 \):

\[
L(f) = \int f(x) \, d\mu = \sum_{j=1}^{k} c_j f(x_j), \quad f \in \mathcal{A}^2. \quad (9.7)
\]

Clearly, \( \mu \) is \( k \)-atomic if and only if the points \( x_j \) are pairwise distinct.

For \( x \in \mathbb{R}^d \), let \( s_N(x) = s(x) \) denote the column vector \( (x^\alpha)_{\alpha \in \mathbb{N}} \). Note that \( s_N(x) \) is the moment vector of the delta measure \( \delta_x \) for \( \mathcal{A} \), not for \( \mathcal{A}^2 \)!

**Proposition 9.10.** For the Hankel matrix \( H(L) \) we have
\[ H(L) = \sum_{j=1}^{k} c_j s_N(x_j) s_N(x_j)^T, \quad (9.8) \]

\[ \text{rank } H(L) \leq k = |\text{supp } \mu| \leq |\mathcal{V}_L|, \quad (9.9) \]

\( \text{rank } H(L) = k \text{ if and only if the vectors } s_N(x_1), \ldots, s_N(x_k) \text{ are linearly independent.} \)

**Proof.** For \( x \in \mathbb{R}^d \) the \((\alpha, \beta)\)-entry of \( s(x)s(x)^T \) is \( x^{\alpha+\beta} \). Thus, \( s(x)s(x)^T \) is the Hankel matrix of the point evaluation \( l_x \) on \( \mathcal{A}^2 \). Since \( L = \sum_j c_j l_{x_j} \), this gives (9.8).

By (9.8), \( H(L) \) is a sum of \( k \) rank one matrices \( c_j s(x_j)s(x_j)^T \), so \( \text{rank } H(L) \leq k \).

Now we prove the last assertion.

If \( f = \sum_{\alpha \in \mathbb{N}} f_\alpha x^\alpha \in \mathcal{A} \) and \( e_\alpha = (\delta_{\alpha, \beta})_{\beta \in \mathbb{N}} \) is the \( \alpha \)-th basis vector, then

\[ H(L)f = \sum_{\alpha \in \mathbb{N}} f_\alpha H(L)e_\alpha = \sum_{\alpha \in \mathbb{N}} f_\alpha \sum_{j=1}^{k} c_j x^\alpha s(x_j) = \sum_{j=1}^{k} c_j l_{x_j}(f)s(x_j). \quad (9.10) \]

Since \( \text{im } H(L) \) is contained in the span of \( s(x_1), \ldots, s(x_k) \), it follows from (9.10) that \( \text{rank } H(L) = \dim \text{im } H(L) = k \) if and only if the point evaluations \( l_{x_1}, \ldots, l_{x_k} \) on \( \mathcal{A} \) are linearly independent. Clearly, \( \sum_j a_j l_{x_j} = 0 \) if and only if \( \sum_j a_j x_j = 0 \) on \( \mathcal{A} \). Hence \( \text{rank } H(L) = k \) if and only if \( s(x_1), \ldots, s(x_k) \) are linearly independent. \( \square \)

**Corollary 9.11.** For each moment functional \( L \) on \( \mathcal{A}^2 \) we have

\[ \text{rank } H(L) \leq |\mathcal{V}_L|. \quad (9.11) \]

**Proof.** By Theorem 8.1 and Proposition 9.2(v), \( L \) has a finitely atomic representing measure \( \mu \) such that \( \text{supp } \mu \subseteq \mathcal{V}_L \). Then \( \text{rank } H(L) \leq |\text{supp } \mu| \) by (9.9), which yields (9.11). \( \square \)

**Example 9.12.** Let \( d = 1, N = \{0,1\} \), so that \( \mathcal{A} = \{a+bx : a, b \in \mathbb{R}\} \). The functionals \( l_-1, l_0, l_1 \) are linearly independent on \( \mathcal{A}^2 \), but they are linearly dependent on \( \mathcal{A} \), since \( 2l_0 = l_-1 + l_1 \) on \( \mathcal{A} \). For \( L = l_-1 + l_0 + l_1 \) we have \( \text{rank } H(L) = 2 \). \( \circ \)

The last theorem in this lecture is only a slight variation of Theorem 9.9 in the case \( \mathcal{K} = \mathbb{R}^d \). It says that all moment functionals on \( \mathbb{R}_d[\lambda]_{2n} \) have extensions to some \( \mathbb{R}_d[\lambda]_{2n+2k} \) which are flat with respect to \( \mathbb{R}_d[\lambda]_{2n+2k-2} \).

**Theorem 9.13.** A linear functional \( L \) on \( \mathbb{R}_d[\lambda]_{2n} \) is a moment functional if and only if there exist a number \( k \in \mathbb{N} \) and an extension of \( L \) to a positive linear functional \( \tilde{L} \) on \( \mathbb{R}_d[\lambda]_{2n+2k} \) such that \( \text{rank } H_{n+k}(\tilde{L}) = \text{rank } H_{n+k-1}(L) \).

**Proof.** The if part follows from the flat extension Theorem 9.8. Conversely, suppose \( L \) is moment functional. By Theorem 8.1 it has a finitely atomic representing measure \( \mu \). Define \( \tilde{L}(f) = \int f d\mu \) on \( \mathbb{R}_d[\lambda] \). Then \( \text{rank } H_m(\tilde{L}) \leq |\text{supp } \mu| \) by (9.9). Hence \( \text{rank } H_m(L) = \text{rank } H_{m-1}(L) \) for some \( m > n \).
We summarize some of the preceding results. The following are necessary conditions for a linear functional $L$ on $\mathcal{A}^2$ to be a moment functional:

- **the positivity condition:**
  \[ L(f^2) \geq 0 \quad \text{for} \quad f \in \mathcal{A}, \]  
  \( (9.12) \)

- **the rank-variety condition:**
  \[ \text{rank} H(L) \leq |\mathcal{V}_L|, \]  
  \( (9.13) \)

- **the consistency condition:**
  \[ p \in \mathcal{N}_L, q \in \mathcal{A} \quad \text{and} \quad pq \in \mathcal{A} \quad \text{imply} \quad pq \in \mathcal{N}_L. \]  
  \( (9.14) \)

Conditions \( (9.13) \) and \( (9.14) \) follow from \( (9.11) \) and Proposition \( 9.3 \).

**Problem:** When are these (and may be, additional) conditions sufficient for being a moment functional?

One result in this direction is the following: Curto and Fialkow (2005) have shown for the polynomials $R_2[x_1^2]$ in 2 variables that conditions \( (9.12) \) and \( (9.14) \) are sufficient if there is a polynomial of degree at most 2 in the kernel $\mathcal{N}_L$.

**Definition 9.14.** A moment functional on $\mathcal{A}^2$ is called **minimal** if \( \text{rank} H(L) = |\mathcal{V}_L| \).

It is not difficult to show that each minimal moment functional has a unique representing measure and this measure is rank $H(L)$-atomic.

**Example 9.15.** (An example for which $|\mathcal{V}_L| > \text{rank} H(L)$)

Suppose that $d = 2, n \geq 3$, and $\mathcal{A} = \mathbb{R}[x_1, x_2]_n$. Set

\[ p(x_1, x_2) = (x_1 - \alpha_1) \cdots (x_1 - \alpha_n), \quad q(x_1, x_2) = (x_2 - \beta_1) \cdots (x_2 - \beta_n), \]

where $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$. Then $\mathcal{Z}(p) \cap \mathcal{Z}(q) = \{ (\alpha_i, \beta_j) : i, j = 1, \ldots, n \}$.

Define an $n^2$-atomic measure $\mu$ such that the atoms are the points of $\mathcal{Z}(p) \cap \mathcal{Z}(q)$ and a moment functional on $\mathcal{A}^2$ by $L(f) = \int f \, d\mu$. Since $L(p^2) = L(q^2) = 0$, we have $p, q \in \mathcal{N}_L$, so that \( \text{rank} H(L) = \dim \langle \mathcal{A} / \mathcal{N}_L \rangle \leq \binom{n+2}{2} - 2 \). Then $|\mathcal{V}_L| = n^2$, so

\[ |\mathcal{V}_L| - \text{rank} H(L) \geq n^2 - \left( \binom{n+2}{2} - 2 \right) + 2 = \binom{n-1}{2}. \]  
  \( (9.15) \)

This shows that the difference $|\mathcal{V}_L| - \text{rank} H(L)$ can be arbitrarily large.
Lecture 10
Truncated multidimensional moment problem: core variety and moment cone

Abstract:
The core variety is defined and basic results about the core variety are obtained. Results on the structure of the moment cone are discussed.

In this Lecture, we retain the setup stated at the beginning of Lecture 8. Recall that $E$ is a finite-dimensional vector space of real-valued continuous functions on a locally compact space $X$.

10.1 Strictly positive linear functionals

As noted in Lecture 8 (see Example 8.4 therein), $E_+$-positive functionals are not necessarily moment functionals. But strictly positive functionals are, as Proposition 10.2 below shows.

**Definition 10.1.** A linear functional $L$ on $E$ is called strictly $E_+$-positive if

$$L(f) > 0 \quad \text{for all} \quad f \in E_+, \ f \neq 0. \quad (10.1)$$

It is not difficult to verify that $L$ is strictly positive if and only if $L$ is an inner point of the dual wedge $(E_+)^\circ$ in $E^*$.

**Proposition 10.2.** Suppose $L$ a strictly $E_+$-positive linear functional on $E$. Then $L$ is a moment functional. Further, for each $x \in X$, there is a finitely atomic representing measure $\nu$ of $L$ such that $\nu(\{x\}) > 0$.

**Proof.** [MP, Theorem 1.30].
10.2 The core variety

Suppose $L$ is a moment functional on $E$. A natural question is to describe the set of points of $\mathcal{X}$ which are atoms of some representing measure of $L$. The idea to tackle this problem is the following simple fact.

**Lemma 10.3.** Let $\mu$ be a representing measure for a moment functional $L$. If $f \in E$ satisfies $f(x) \geq 0$ on $\text{supp} \, \mu$ and $L(f) = 0$, then $\text{supp} \, \mu$ is contained in the zero set $\mathcal{Z}(f) := \{x \in \mathcal{X}: f(x) = 0\}$ of $f$.

**Proof.** Suppose $x_0 \in \mathcal{X}$ and $x_0 \notin \mathcal{Z}(f)$. Then $f(x_0) > 0$. Since $f$ is continuous, there are an open neighborhood $U$ of $x_0$ and an $\varepsilon > 0$ such that $f(x) \geq \varepsilon$ on $U$. Then

$$0 = \int_{\mathcal{X}} f(x) \, d\mu \geq \int_{U} f(x) \, d\mu \geq \varepsilon \mu(U) \geq 0,$$

so that $\mu(U) = 0$. Therefore, $x_0 \notin \text{supp} \, \mu$. □

This core variety is defined by a repeated application of this idea. It was invented by L. Fialkow (2015).

Suppose $L$ is an arbitrary linear functional on $E$ such that $L \neq 0$. We define inductively cones $\mathcal{N}_k(L)$, $k \in \mathbb{N}$, of $E$ and subsets $\mathcal{V}_j(L)$, $j \in \mathbb{N}_0$, of $\mathcal{X}$ by $\mathcal{V}_0(L) = \mathcal{X}$,

$$\mathcal{N}_k(L) := \{p \in E: L(p) = 0, ~ p(x) \geq 0 \text{ for } x \in \mathcal{V}_{k-1}(L)\}, \quad (10.2)$$

$$\mathcal{V}_j(L) := \{x \in \mathcal{X}: p(x) = 0 \text{ for } p \in \mathcal{N}_j(L)\}. \quad (10.3)$$

**Definition 10.4.** The core variety $\mathcal{V}(L)$ of the functional $L$, $L \neq 0$, on $E$ is

$$\mathcal{V}(L) := \bigcap_{j=0}^\infty \mathcal{V}_j(L). \quad (10.4)$$

If $\mathbb{R} = \mathcal{X}$ and $E$ is a subset $\mathbb{R}_d[\mathbb{x}]$, then $\mathcal{V}(L)$ is the zero set of real polynomials, that is, $\mathcal{V}(L)$ is a real algebraic set.

For instance, if $L$ is strictly positive, then $\mathcal{N}_1(L) = \{0\}$ and therefore $\mathcal{V}(L) = \mathcal{X}$.

Some basic facts on these sets are collected in the next proposition.

**Proposition 10.5.** (i) $\mathcal{N}_{j-1}(L) \subseteq \mathcal{N}_j(L)$ and $\mathcal{V}_j(L) \subseteq \mathcal{V}_{j-1}(L)$ for $j \in \mathbb{N}$.

(ii) If $\mu$ is a representing measure of $L$, then $\text{supp} \, \mu \subseteq \mathcal{V}(L)$.

(iii) There exists a $k \in \mathbb{N}_0$ such that

$$\mathcal{X} = \mathcal{V}_0(L) \supseteq \mathcal{V}_1(L) \supseteq \ldots \supseteq \mathcal{V}_k(L) = \mathcal{V}_{k+1}(L) = \mathcal{V}(L), \quad j \in \mathbb{N}. \quad (10.5)$$

For a moment functional $L$ let $k(L)$ denote the number $k$ in equation (10.5), that is, $k(L)$ is the smallest $k$ such that $\mathcal{V}_k(L) = \mathcal{V}(L)$. It is an interesting problem to characterize the set of moment sequences $L$ for which $k(L)$ is a fixed number $k$. Further, it is likely to expect that for each number $n \in \mathbb{N}$ there exists a moment functional $L$ such that $n = k(L)$. 
The importance of the core variety stems from the following three theorems. The first theorem is due to Blekherman and Fialkow, the two other are from my recent joint paper with Ph. di Dio.

**Theorem 10.6.** A linear functional $L \neq 0$ on $E$ is a moment functional if and only if $L(e) \geq 0$ and the core variety $\mathcal{V}(L)$ is not empty.

**Proof.** [MP, Theorem 18.22].

For a moment functional $L$ on $E$ we define the set of possible atoms:

$$\mathcal{W}(L) := \{ x \in X : \mu(\{x\}) > 0 \text{ for some representing measure } \mu \text{ of } L \}. \quad (10.6)$$

The second theorem says that the core variety is just the set of possible atoms. In particular, it implies that $\mathcal{W}(L)$ is a closed subset of $X$.

**Theorem 10.7.** Let $L$ be a truncated moment functional on $E$. Then

$$\mathcal{W}(L) = \mathcal{V}(L). \quad (10.7)$$

Each representing measure $\mu$ of $L$ is supported on $\mathcal{V}(L)$. For each point $x \in \mathcal{V}(L)$ there is a finitely atomic representing measure $\mu$ of $L$ which has $x$ as an atom.

**Proof.** [MP, Theorem 18.21].

A moment functional is called *determinate* if it has a unique representing measure.

**Theorem 10.8.** For any moment functional $L$ on $E$ the following are equivalent:

(i) $L$ is determinate.
(ii) $|\mathcal{V}(L)| \leq \dim(E \setminus \mathcal{V}(L))$.

**Proof.** [MP, Theorem 18.23].

Thus, in particular, $L$ is not determinate if $|\mathcal{V}(L)| > \dim E$.

We close this section with three illustrating examples.

**Example 10.9.** ($\mathcal{V}(L) \neq \emptyset$ and $L$ is not a truncated moment functional)

Let $d = 1$, $N = \{0, 2\}$. Then $E = \{ a + bx^2 : a, b \in \mathbb{R} \}$. Define a linear functional $L$ on $E$ by $L(a + bx^2) = -a$. Clearly, $L$ is not a moment functional, because $L(1) = -1$.

Then $E_+ = \{ a + bx^2 : a \geq 0, b \geq 0 \}$, so $\mathcal{N}_1(L) = \mathbb{R}_+ \cdot x^2$ and $\mathcal{V}_1(L) = \{0\}$. Hence

$$\{ f \in E : f(x) \geq 0 \text{ for } x \in \mathcal{V}_1(L) \} = \{ a + bx^2 : a \geq 0, b \in \mathbb{R} \}.$$

Therefore, $\mathcal{N}_2(L) = \mathbb{R} \cdot x^2$ and $\mathcal{V}_2(L) = \{0\}$, so that $\mathcal{V}(L) = \{0\} \neq \emptyset$. 
Example 10.10. (A truncated moment functional with $\mathcal{V}(L) = \mathcal{V}_2(L) \neq \mathcal{V}_1(L)$)

Let $d = 1$ and $N = \{0, 2, 4, 5, 6, 7, 8\}$. Then $E = \text{Lin} \{1, x^2, x^3, x^6, x^7, x^8\}$. We fix a real number $\alpha > 1$ and define $\mu = \delta_{-1} + \delta_1 + \delta_{\alpha}$. For the corresponding moment functional $L = \mathcal{L} = l_1 + l_1 + l_\alpha$ on $E$ we have

$\mathcal{W}(L) = \mathcal{V}(L) = \mathcal{V}_2(L) = \{1, -1, \alpha\} \subset \{1, -1, \alpha, -\alpha\} = \mathcal{V}_1(L)$. \hfill (10.8)

Let us prove (10.8). Put $p(x) := (x^2 - 1)^2(x^2 - \alpha^2)^2$. Clearly, $p \in E, L(p) = 0$ and $p \in \text{Pos}(\mathbb{R})$, so $p \in \mathcal{N}_1(L)$. Conversely, let $f \in \mathcal{N}_1(L)$. Then $L(f) = 0$ implies that $f(\pm 1) = f(\alpha) = 0$. Since $f \in \text{Pos}(\mathbb{R})$, the zeros $1, -1, \alpha$ have even multiplicities. Hence $f(x) = (x - 1)^2(x + 1)^2(x - \alpha)^2(ax^2 + bx + c)$ with $a, b, c \in \mathbb{R}$. Since $x$ and $\alpha$ are not in $A$, the coefficients of $x^3$ vanish. This yields $ax^2 + bx + c = a(x + \alpha)^2$ with $a \geq 0$. Thus $\mathcal{N}_1(L) = \mathbb{R}_{+}p$. Hence $\mathcal{V}_1(L) = \mathcal{Z}(p) = \{1, -1, \alpha, -\alpha\}$.

Now we set $q(x) = x^4(1-x^2)(\alpha - x)$. Then $q \in E$. Since $q(\pm 1) = q(\alpha) = 0$ and $q(-\alpha) = \alpha^4(\alpha^2 - 1) > 0$, we have $q(x) \geq 0$ on $\mathcal{V}_1(L)$ and $L(q) = 0$. Thus, $q \in \mathcal{N}_2(L)$ and hence $\mathcal{V}_2(L) \subseteq \mathcal{V}_1(L) \cap \mathcal{Z}(q) = \{1, -1, \alpha\}$.

Since $1, -1, \alpha$ are atoms of $\mu$, $\{1, -1, \alpha\} \subseteq \mathcal{W}(L)$. Now (10.8) follows. The moment functional $L$ is determinate.

Example 10.11. (An example based on the Robinson polynomial)

Let $H_{3,6}$ denote the homogeneous polynomials in 3 variables of degree 6 and $\mathbb{P}^2(\mathbb{R})$ the 2-dimensional projective space. The polynomial $R \in H_{3,6}$, defined by

$R(x, y, z) := x^6 + y^6 + z^6 + 3x^2y^2z^2 - x^4y^2 - x^4z^2 - x^2y^4 - y^4z^2 - x^2z^4 - y^2z^4,$

is called the Robinson polynomial. It has a number of interesting properties:

**Proposition 10.12.**

(i) $R(x, y, z) \geq 0$ for $(x, y, z) \in \mathbb{R}^3$.

(ii) $R$ is not a sum of squares in $\mathbb{R}[x, y, z]$.

(iii) $R$ has exactly 10 zeros, $t_1, \ldots, t_{10}$, given by (10.9)–(10.10), in $\mathbb{P}^2(\mathbb{R})$.

(iv) If $p \in H_{3,6}$ vanishes on $t_1, \ldots, t_{10}$, then $p = \lambda R$ for $\lambda \in \mathbb{R}$.

**Proof.** [MP, Proposition 19.19].

Assertion (i) follows from the identity

$$(x^2 + y^2)R = x^2z^2(x^2 - z^2)^2 + y^2z^2(y^2 - z^2)^2 + (x^2 - y^2)^2(x^2 + y^2 - z^2)^2.$$ 

The zeros of $R$ in the projective space $\mathbb{P}^2(\mathbb{R})$ are:

$$(t_1 = (1, 1, 1), t_2 = (1, 1, -1), t_3 = (1, -1, 1), t_4 = (1, -1, -1), t_5 = (1, 0, 1), \quad (10.9)$$

$$(t_6 = (0, 1, 0), t_7 = (1, 1, 0), t_8 = (1, -1, 0), t_9 = (0, 1, 1), t_{10} = (0, 1, -1). \quad (10.10)$$

Fix $t_0 \in \mathbb{P}^2(\mathbb{R})$ such that $t_0 \neq t_j, j = 1, \ldots, 10$. Then $R(t_0) \neq 0$. Put

$$\nu = \sum_{j=1}^{10} m_j \delta_{t_j}, \quad \mu = \nu + m_0 \delta_{t_0}, \quad \text{where} \quad m_j > 0.$$
10.3 The moment cone

Let \( L^\nu \) and \( L^\mu \) be the moment functionals given by the atomic measures \( \mu \) and \( \nu \), respectively, on the vector space \( \mathcal{H}_{3,6} \) of continuous functions on the compact topological space \( \mathbb{P}^2(\mathbb{R}) \). From Proposition 10.12(iv) we obtain:

- \( L^\mu \) is strictly positive. Hence \( \mathcal{V}(L^\mu) = \mathbb{P}^2(\mathbb{R}) \) and each \( x \in \mathbb{P}^2(\mathbb{R}) \) is atom of a representing measure.
- For \( L^\nu \) only the points \( t_1, \ldots, t_{10} \) are atoms of some representing measure. It can be shown that \( \nu \) is determinate and \( \mathcal{V}(L) = \{t_1, \ldots, t_{10}\} \).

The measure \( \mu \) differs from \( \nu \) by a single atom, but the corresponding core varieties are opposite extreme cases, one is the whole space and the other is discrete.

10.3 The moment cone

From now on, \( A := \{a_1, \ldots, a_m\} \) denotes a fixed basis of the vector space \( E \).

There is a one-to-one correspondence between linear functionals \( L \) on \( E \) and vectors \( s = (s_1, \ldots, s_m) \in \mathbb{R}^m \) given by \( L(a_j) = s_j, j = 1, \ldots, m \). For \( s \in \mathbb{R}^m \) the corresponding functional is the Riesz functional \( L_s \) of \( s \).

Let \( \mathcal{M}_+(E) \) denote the set of Radon measures on \( \mathcal{X} \) for which all functions of \( E \) are \( \mu \)-integrable. For \( \mu \in \mathcal{M}_+(E) \),

\[
L^\mu(f) = \int f(x) d\mu(x), \quad f \in E, \tag{10.11}
\]

is equivalent to

\[
s_j(\mu) = \int a_j(x) d\mu(x), \quad j = 1, \ldots, m. \tag{10.12}
\]

Recall that the functional \( L_s \) in (10.11) is called the moment functional of \( \mu \) and the vector \( s(\mu) = (s_1(\mu), \ldots, s_m(\mu)) \) given by (10.12) is the moment vector of \( \mu \). Thus, \( s \in \mathbb{R}^m \) is moment vector if and only if \( L_s \) is a moment functional.

**Definition 10.13.** The moment cone \( \mathcal{S} \) is the cone of all moment sequences, that is,

\[
\mathcal{S} := \{(s_1(\mu), \ldots, s_m(\mu)) : \mu \in \mathcal{M}(E) \};
\]

and \( \mathcal{L} \) denotes the cone of all moment functionals in \( E^* \), that is,

\[
\mathcal{L} := \{L^\mu \in E^* : L^\mu = \int f d\mu, f \in E, \text{ where } \mu \in \mathcal{M}_+(E)\}.
\]

For \( x \in \mathcal{X} \), the moment vector of the delta measure \( \delta_x \) is

\[
s_A(x) := (a_1(x), \ldots, a_m(x)).
\]
Since each moment functional has a finitely atomic representing measure by Theorem 8.1, each vector of $S$ is a nonnegative linear combination of vectors $s_A(x)$ for $x \in X$, that is, $S$ is the convex cone in $R^m$ generated by the vectors $s_A(x), x \in X$.

The map $s \mapsto L_s$ is a bijection of $S$ and $L$. Further, we have

$$R^m = S - S \quad \text{and} \quad E^* = L - L.$$

In general, both cones $S$ and $L$ are not closed, as the following example shows.

**Example 10.14.** Let $A = \{1, x, x^2\} \cup X$ on $X = R$. For $\mu_n = n^{-2} \delta_n$ we have $s(\mu_n) = (n^{-2}, n^{-1}, 1)$. But $s := \lim_n s(\mu_n) = (0, 0, 1)$ is not in $S$. \hfill $\diamond$

For a cone $C$ in a real vector space $V$, its dual cone is defined by

$$C^\wedge = \{ \varphi \in V^* : \varphi(c) \geq 0 \quad \text{for} \quad c \in C \}.$$

Let $\overline{C}$ denote the closure of the cone $C$ in $E^*$.

**Proposition 10.15.** $L \subseteq (E_+)^\wedge = \overline{L}$ and $L^\wedge = E_+ = (E_+)^\wedge$.

If the space $X$ is compact, then the cone $L$ is closed in the norm topology of the dual space $E^*$ and we have $L = (E_+)^\wedge$.

**Proof.** [MP, Propositions 1.26, 1.27]. \hfill $\square$

Finally, we turn to the supporting hyperplanes of the cone $L$ of moment functionals. Suppose $L \in \mathcal{L}$. Recall from (10.4) that $\mathcal{N}_1(L) = \{f \in E_+ : L(f) = 0 \}$.

The next proposition shows that the nonzero elements of $\mathcal{N}_1(L)$ correspond to the supporting hyperplanes of the cone $L$.

**Proposition 10.16.** (i) Let $p \in \mathcal{N}_1(L), p \neq 0$. Then $\varphi_p(L') = L'(p), L' \in E^*$, defines a supporting functional $\varphi_p$ of the cone $L$ at $L$. Each supporting functional of $L$ at $L$ is of this form.

(ii) $L$ is a boundary point of the cone $L$ if and only if $\mathcal{N}_1(L) \neq \{0\}$.

(iii) $L$ is an inner point of the cone $L$ if and only if $\mathcal{N}_1(L) = \{0\}$.

**Proof.** [MP, Proposition 1.42]. \hfill $\square$

An exposed face of a cone $C$ in a finite-dimensional real vector space is a subcone of the form $F = \{f \in C : \varphi(f) = 0\}$ for some functional $\varphi \in C^\wedge$.

Since $L^\wedge = E_+$, each $\varphi \in L^\wedge$ is of the form $\varphi_p(L') = L'(p), L' \in E^*$, for some $p \in E_+$. Hence the exposed faces of the cone $L$ in $E^*$ are precisely the sets the sets

$$F_p := \{L' \in L : \varphi_p(L') \equiv L'(p) \equiv 0\}, \quad \text{where} \quad p \in E_+ \quad (10.13)$$

Let $L \in \mathcal{L}$. Since $L \subseteq (E_+)^\wedge, \mathcal{N}_1(L)$ is an exposed face of the cone $E_+$. If $X$ is compact, then $(E_+)^\wedge = L$ by Proposition 10.15, so each exposed face of $E_+$ is of this form. Thus, in this case the subcones $\mathcal{N}_1(L)$ are precisely the exposed faces of $E_+$.

Recall that for inner points of $L$ we have $\mathcal{W}(L) = X$ (by Proposition 10.2) and hence $\mathcal{W}(L) = \mathcal{V}_1(L)$. In general, $\mathcal{W}(L) \neq \mathcal{V}_1(L)$, as shown by Example 10.10.

The next result characterizes those boundary points for which $\mathcal{W}(L) = \mathcal{V}_1(L)$.
Proposition 10.17. Let $L$ be a boundary point of $\mathcal{L}$. Then $W(L) = V(L)$ if and only if $L$ lies in the relative interior of an exposed face of the cone $\mathcal{L}$.

Proof. [MP, Theorem 1.45].