Asymptotic properties of dual averaging algorithm for constrained distributed stochastic optimization∗

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Abstract

Considering the constrained stochastic optimization problem over a time-varying random network, where the agents are to collectively minimize a sum of objective functions subject to a common constraint set, we investigate asymptotic properties of a distributed algorithm based on dual averaging of gradients. Different from most existing works on distributed dual averaging algorithms that mainly concentrating on their non-asymptotic properties, we not only prove almost sure convergence and the rate of almost sure convergence, but also asymptotic normality and asymptotic efficiency of the algorithm. Firstly, for general constrained convex optimization problem distributed over a random network, we prove that almost sure consensus can be archived and the estimates of agents converge to the same optimal point. For the case of linear constrained convex optimization, we show that the mirror map of the averaged dual sequence identifies the active constraints of the optimal solution with probability 1, which helps us to prove the almost sure convergence rate and then establish asymptotic normality of the algorithm. Furthermore, we also verify that the algorithm is asymptotically optimal. To the best of our knowledge, it seems to be the first asymptotic normality result for constrained distributed optimization algorithms. Finally, a numerical example is provided to justify the theoretical analysis.

Key words. constrained distributed stochastic optimization, distributed dual averaging method, almost sure convergence, asymptotic normality, asymptotic efficiency

1 Introduction

Distributed algorithms for solving optimization problems that are defined over networks have been receiving increasing attention from researchers since the earlier seminal work [1, 3]. The

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most concerned problem among which is to optimize a sum of local objective functions of agents subject to the intersection of their local constraint sets, where the agents are connected through a communication network with each objective and constraint held privately. A large number of problems, such as multi-agent coordination [4], wireless networks [5, 6], machine learning [7], can be transformed into distributed optimization problems. In practice these problems are often random or large-scale, so they are very suitable to be solved by stochastic approximation (SA) based distributed algorithms. Over the last decades, numerous algorithms for distributed stochastic optimization have been developed and various scenarios have been considered, such as stochastic sub-gradient [8], distributed dual averaging [9], random gradient-free [10, 11], push-sum method [12]; or, with the same local constraint [13], with the different local constraint [14, 15], with asynchronous communications [16]. In most of the mentioned works, asymptotic convergence such as convergence in mean (and further the rate in mean) or almost sure convergence, or non-asymptotic properties in expectation, are commonly concerned.

Asymptotic normality and asymptotic efficiency are important topics of stochastic algorithms, which have been studied in SA for a long time. For centralized problem, the asymptotic normality of one-dimensional and multi-dimensional SA was provided in [17, 18] and [19], respectively. To archive asymptotic efficiency the so-called adaptive SA may be concerned, see e.g. [20], but it requires rather restrictive conditions to guarantee its convergence and optimality. On the other hand, the averaging technique introduced in [21] has been widely used. Recently, [22] gave the asymptotic efficiency of the dual average algorithm for solving linear constrained and nonlinear constrained optimization problems respectively. For decentralized problem, however, asymptotic normality and asymptotic efficiency results are rather limited. The asymptotic normality and asymptotic efficiency of a distributed stochastic approximation algorithm were proven in [23]; a distributed stochastic primal dual algorithm was proposed, and then whose asymptotic normality and asymptotic efficiency were provided in [24]. However, all of the above works on distributed optimization are concentrated on unconstrained problems. Inspired by [9, 22], we provide the asymptotic normality of distributed dual averaging algorithm for linear constrained problem.

The dual averaging algorithm was introduced by [25] in deterministic settings, and further analyzed and developed by many authors. For instance, [26] extended it to stochastic settings and composite optimization problem. [27] proved that the dual averaging algorithm can identify the optimal manifold with a high probability before finding the optimal solution, and provided a strategy to search for the optimal solution in the optimal manifold after identifying the active set. [22] showed that variants of Nesterov’s dual averaging algorithm guarantee almost sure finite time identification of active constraints in constrained stochastic optimization problems. The reason why the optimal manifold identification property is so concerned is that it contributes to prove algorithm’s asymptotic normality from a theoretical viewpoint, while it is also helpful to reduce the amount of computation and save storage space of data from a practical viewpoint, especially for sparsity problem.

The dual averaging algorithm was developed to solve distributed optimization problems in [9, 28], where it was shown how do the network size and topology influence sharp bounds on convergence rates in [9], and how do the delays in stochastic gradient information affect the convergence results in [28]. Applying the dual averaging algorithm to distributed optimization problems was concerned by many authors. For example, the effects of deterministic and probabilistic message quantization on distributed dual averaging algorithms for multi-agent optimization problem was considered in [29]. [30] extended the distributed algorithm based on
dual subgradient averaging to the online setting and provided an upper bound on regret as a function of connectivity in the underlying network. Recently, [31] proposed a distributed quasi-monotone sub-gradient algorithm, and proved this algorithm’s asymptotic convergence, where quasi-monotone algorithm introduced in [32] is a modification of dual averaging algorithm. However, these works are mostly focused on the non-asymptotic convergence analysis and asymptotic properties such as asymptotic normality have not been resolved for the distributed dual averaging algorithm.

In this paper, we investigate a dual averaging algorithm for the distributed stochastic optimization problem subject to a common constraint set over a time-varying random network. We first establish the almost sure consensus and almost sure convergence of the algorithm. And then in the linear constraint case we provide the almost sure active set identification, and with whose help we are able to analyze the almost sure convergence rate and prove the asymptotic normality as well as asymptotic efficiency of the algorithm. The main contributions of the paper are summarized as follows.

(a) Different from most existing works on distributed dual averaging algorithms that mainly focus on their non-asymptotic properties, we prove all agents’ estimates converge to the same optimal solution almost surely for general constrained optimization problem over time-varying random networks. In particular, the weight matrices are not restricted to be doubly stochastic, which are only required to be column stochastic in mean sense except for row stochasticity.

(b) Motivated by the idea of active set identification, we extend the method in [22] to distributed scenario, and show that the mirror map of the averaged dual sequence identifies the active set of the optimal solution after finite steps almost surely. As explained earlier, once the estimates enter into the optimal manifold, asymptotic convergence properties of the algorithm can be proved as unconstrained stochastic approximation algorithms. On this basis, we provide a novel result on almost convergence rate of the distributed dual averaging algorithm for the case of linear constrained convex distributed stochastic optimization.

(c) Different from [23, 24] that concentrate on unconstrained distributed optimization problem, we provide asymptotic normality and asymptotic efficiency of distributed dual averaging algorithms for linear constrained distributed optimization, which seems to be the first asymptotic normality result for constrained distributed optimization algorithms as far as we know.

The remainder of this paper is organized as follows. Section 2 introduces the distributed optimization problem model and a distributed dual averaging (DDA for short) algorithm. Section 3 gives not only the almost sure convergence of DDA algorithm for the convex optimization problem with general constraints, but also the almost sure convergence rate of DDA algorithm in the case objective function is restricted strong convex and constraints are linear. Section 4 proves the asymptotic normality and asymptotic efficiency of DDA algorithm. Section 5 presents a numerical example to justify these theoretic results.

Notations and basic definitions: Throughout this paper, we use the following notation. \( \mathbb{R}^d \) denotes the \( d \)-dimension Euclidean space with norm \( \| \cdot \| \) and \( \mathbb{R}_+^d := \{ x \in \mathbb{R}^d : x \geq 0 \} \). \( 1 := (1 1 \ldots 1)^T \in \mathbb{R}^m \), \( I_d \in \mathbb{R}^{d \times d} \) denotes the identity matrix and \( 0 \) denotes the zero matrix of compatible dimension, respectively. For a matrix \( A \), \( A^\dagger \) is its Moore-Penrose inverse and
\[ \|A\| = \sup_{\|x\| = 1} \|Ax\| \] is the spectral norm. For two matrices \( A \) and \( B \), \( A \otimes B \) stands for the Kronecker product. Given a set \( \mathcal{X} \subseteq \mathbb{R}^d \), \( 1_{\mathcal{X}} \) denotes the characteristic function of set \( \mathcal{X} \), which means that it equals 1 if \( x \in \mathcal{X} \), and 0 otherwise. \( \text{ri}(\mathcal{X}) \) denotes the set of relative interior of a non-empty convex set \( \mathcal{X} \). For a closed convex set \( \mathcal{X} \subseteq \mathbb{R}^d \), \( N_{\mathcal{X}}(x) \) denotes the normal cone and \( P_{\mathcal{X}}(z) \) denotes the projection operator, that is,

\[
N_{\mathcal{X}}(x) := \{ v \in \mathbb{R}^d : \langle v, y - x \rangle \leq 0, \forall y \in \mathcal{X} \}, \quad P_{\mathcal{X}}(z) = \arg\min_{x \in \mathcal{X}} \|x - z\|.
\]

For a sequence of random vectors \( \{\xi_k\} \) and a random vector \( \xi \), \( \xi_k \xrightarrow{a.s.} \xi \) and \( \xi_k \xrightarrow{d} \xi \) stand for \( \{\xi_k\} \) converges to \( \xi \) almost surely (a.s. for short) and in distribution, respectively.

### 2 Distributed optimization problem and dual averaging method

Consider the following distributed constrained stochastic optimization problem:

\[
\min_{x \in \mathcal{X}} f(x) = \sum_{j=1}^{m} f_j(x) \quad \text{s.t. } x \in \mathcal{X},
\]

where \( f_j(x) := \mathbb{E}[F_j(x; \xi_j)] \), \( j = 1, \ldots, m \) with \( \xi_j \) being a random vector defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) with support set \( \Xi_j \), \( \mathbb{E}[\cdot] \) denotes the expected value with respect to probability measure \( \mathbb{P} \) and \( \mathcal{X} \subseteq \mathbb{R}^d \) is a closed convex set.

In problem (1), each agent \( j \) shares the common constraint set \( \mathcal{X} \) but holds the private information on objective function \( f_j(x) \), such as, the value of sampled function or the corresponding gradient. But each agent can communicate with its immediate neighbors to cooperatively solve the constrained optimization problem (1). For convenience, denote by \( f^* = \inf_{x \in \mathcal{X}} f(x) \) the optimal value of problem (1), and by \( \mathcal{X}^* = \{ x \in \mathcal{X} : f(x) = f^* \} \) the optimal solution set.

The network over which the agents communicate at time \( k \) is represented by a directed graph \( G_k = (V, E_k) \), where \( V = \{1, 2, \ldots, m\} \) is the node set, and \( E_k \subseteq V \times V \) is associated with the weight matrix \( A_k \in \mathbb{R}^{m \times m} \) through

\[
E_k := \{(j, i) : [A_k]_{ij} > 0, i, j \in V \},
\]

where \( [A_k]_{ij} \) is the \( (i, j) \)-th entry of matrix \( A_k \). At time \( k \), \( N_{j,k} := \{i \in V : (i, j) \in E_k\} \) denotes the neighbors of agent \( j \).

The dual averaging method is proposed by Nesterov [25]. Consider the following optimization problem

\[
\min_{x \in \mathcal{X}} h(x),
\]

where \( h(x) : \mathbb{R}^d \to \mathbb{R} \) is a differentiable convex function, \( \mathcal{X} \subseteq \mathbb{R}^d \) is a closed convex set. The dual averaging method involves two alternate processes:

\[
z_k = z_{k-1} - \alpha_k \nabla h(x_k),
\]

\[
x_{k+1} = \arg\max_{x \in \mathcal{X}} \{ \langle z_k, x \rangle - \psi(x) \},
\]

### 2.1 Distributed optimization problem and dual averaging method
where $\psi: \mathcal{X} \rightarrow \mathbb{R}$ is called regularizer, which is a continuous and strongly convex function on $\mathcal{X}$, that is, there exists some $\sigma > 0$ such that

$$
\psi(\lambda x + (1 - \lambda) y) \leq \lambda \psi(x) + (1 - \lambda) \psi(y) - \frac{\sigma}{2} \lambda(1 - \lambda) \|x - y\|^2
$$

for all $x, y \in \mathcal{X}$ and $\lambda \in [0, 1]$. For example, the Euclidean regularization is mostly common used in the literature.

Recently, the dual averaging algorithm has been developed to solve distributed optimization problems in \cite{9, 28–31}. In this paper, we investigate a variant of the distributed dual averaging algorithm proposed in \cite{9} and focus on its asymptotic properties, which reads as the following.

**Algorithm 1** Distributed dual averaging algorithm

**Initialization:** For any $1 \leq j \leq m$, agent $j$ initializes its dual variable $z_{j,0} \in \mathbb{R}^d$ (possibly randomly).

**General step:** At time $k = 1, 2, \cdots$, update weighted matrix $A_k$ and stepsize $\alpha_k > 0$; agent $j$ maintains a pair of vectors $\{x_{j,k}, z_{j,k}\}$, exchanges $z_{j,k}$ between agents, and performs the following primal-dual iteration locally:

1. **Primal step:** Update the primal estimate by a projection defined by $\psi(x)$

   $$
x_{j,k} = \arg\max_{x \in \mathcal{X}} \{\langle z_{j,k-1}, x \rangle - \psi(x)\}.
   $$(2)

2. **Dual step:** Draw $\xi_{j,k} \overset{i.i.d.}{\sim} P$, compute $\nabla F_j(x_{j,k}; \xi_{j,k})$, update the dual estimate by

   $$
z_{j,k} = \sum_{i \in N_{j,k}} [A_k]_{ji} z_{i,k-1} - \alpha_k \nabla F_j(x_{j,k}; \xi_{j,k}).
   $$(3)

Throughout the paper, we define the filtration

$$
\mathcal{F}_k = \sigma\{z_{j,0}, \xi_{j,t}, A_t : j \in V, 1 \leq t \leq k - 1\}, \quad \mathcal{F}_1 = \sigma\{z_{j,0}, j \in V\}.
$$

It is obvious that $z_{j,k-1}, x_{j,k}$ is adapted to $\mathcal{F}_k$.

### 3 Almost sure convergence and convergence rate

In this section, we study the almost sure convergence of Algorithm 1. We show that each iteration $x_{j,k}$ converges almost surely to the same solution in $\mathcal{X}^*$, for the case where $f_j(\cdot)$ is convex for any $1 \leq j \leq m$. If $f(\cdot)$ is further restricted strong convex, we may provide an estimation of the almost sure convergence rate, which will be used to analyze the asymptotic normality of each estimate $x_{j,k}$ to the optimal solution.

#### 3.1 Almost sure convergence

We first introduce the conditions on objective functions, constraint set, network topology, step-size and sample.
**Assumption 1 (objective function).** For any $1 \leq j \leq m,$

(i) $F_j(\cdot; \xi_j)$ is differentiable convex function on $\mathcal{X}$ for any $\xi_j$;

(ii) $F_j(\cdot; \xi_j)$ is Lipschitz continuous on $\mathcal{X}$, that is,

$$|F_j(x; \xi_j) - F_j(y; \xi_j)| \leq L_{0,j}(\xi_j) \|x - y\|, \quad \forall x, y \in \mathcal{X},$$

where $L_{0,j}(\xi_j)$ is measurable and $\mathbb{E}[L_{0,j}^p(\xi_j)] < \infty$ for some $p \geq 2$.

The Lipschitz continuity of $F_j(\cdot; \xi_j)$ implies that $f_j(\cdot)$ is Lipschitz continuous, and that the gradient $\nabla F_j(x; \xi_j), \nabla f_j(x)$ are bounded by $L_{0,j}(\xi_j)$ and $\mathbb{E}[L_{0,j}^p(\xi_j)]$, respectively. For the convergence of Algorithm $\mathbf{1}$, the condition $p = 2$ in part (ii) of Assumption $\mathbf{1}$ is sufficient. When studying the asymptotic normality of the algorithm, $p > 2$ is needed to verify Lindeberg’s condition. Moreover, for ease of the notation, we denote the observation noise of gradient $\nabla f_j(x_{j,k})$ by

$$s_{j,k} := \nabla F_j(x_{j,k}; \xi_{j,k}) - \nabla f_j(x_{j,k}),$$

and

$$L_0 = \max_{1 \leq j \leq m} \mathbb{E}[L_{0,j}(\xi_j)], \quad L_0^p = \max_{1 \leq j \leq m} \mathbb{E}[L_{0,j}^p(\xi_j)]$$

throughout the paper.

We now turn to assumptions on the weight matrices $A_k$, which are commonly assumed to be doubly stochastic in most works (for instance $[8, 9, 14, 15]$). However, in practice it is rather easy to implement row-stochasticity ($A_k^1 = 1$) but hard to ensure column-stochasticity ($1^T A_k = 1^T$) since which implies more stringent restrictions on the network. Motivated by $[13]$ Assumption $\mathbf{1}$, we investigate Algorithm $\mathbf{1}$ under the relatively weaker conditions.

**Assumption 2 (weight matrices).** Let $A_k$ be the weight matrix at step $k$. Assume that

(i) $A_k$ is a sequences of matrix-valued random variables with nonnegative components and $A_k 1 = 1$, $1^T \mathbb{E}[A_k] = 1^T$, $\forall k \geq 1$.

(ii) $\rho_k$ denotes the spectral norm of matrix $\mathbb{E}\left[ A_k^T (I_m - \frac{1^T}{m} 1\cdot) A_k \right]$ and

$$\lim_{k \to \infty} k(1 - \rho_k) = \infty.$$  

(iii) Matrix $A_k$ is independent of $\sigma$-algebra $\mathcal{F}_k$.

Assumption $\mathbf{2}$ allows the broadcast gossip matrices and $\mathbf{(7)}$ holds if sup $\rho_k < 1$.

**Assumption 3 (step-size).** (i) $\alpha_k > 0$ is nonincreasing and $\sum_{k=1}^{\infty} \alpha_k = \infty$.

(ii) There exists $\beta > 0.5$ such that

$$\lim_{k \to \infty} k^\beta \alpha_k = 0,$$

$$\liminf_{k \to \infty} \frac{1 - \rho_k}{k^\beta \alpha_k} > 0.$$  

Note that $\mathbf{(8)}$ implies $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, which combines with Assumption $\mathbf{3}$ (i) is commonly used in SA. $\mathbf{(9)}$ means that the exchange of information between agents becomes rare as $k \to \infty$. When $A_k$ is an independent and identically distributed (i.i.d.) sequence, then $\rho_k \equiv \rho$ is constant, and both $\mathbf{(7)}$ and $\mathbf{(9)}$ hold if and only if $\rho < 1$ $[13]$. 
Assumption 4 (sample and $\sigma$-algebra). For any $1 \leq i, j \leq m$, (i) $\xi_{j,1}, \xi_{j,2}, \cdots$ is i.i.d. sample; (ii) $\xi_{i,k}$ and $\xi_{j,k}$ are conditionally independent given $F_k := \sigma(F_k \cup \sigma(A_k))$ when $i \neq j$; (iii) $\xi_{j,k}$ is conditionally independent of $A_k$ given $F_k$.

Above condition (i) and (ii) guarantee that the sequence of observation noise of gradient $\{s_{j,k}\}$ is a martingale difference sequence, that is,

$$E[s_{j,k}|F_k] = 0,$$

and the conditional covariance $\text{Cov}(\nabla F_j(x_{j,k}; \xi_{j,k}), \nabla F_i(x_{i,k}; \xi_{i,k}) | F_k) = 0$.

Combining condition (i) with Assumption 1 implies that

$$E[\|s_{j,k}\|^p | F_k] \leq \left( \left( E[\|\nabla f_j(x_{j,k})\|^p | F_k] \right)^{1/p} + \left( E[\|\nabla F_j(x_{j,k}; \xi_{j,k})\|^p | F_k] \right)^{1/p} \right)^p$$

$$\leq \left( (L_0^p)^{1/p} + (L_0^p)^{1/p} \right)^p = 2^p L_0^p,$$

where the Minkowski inequality and the fact that $\nabla F_j(x; \xi_j), \nabla f_j(x)$ are bounded by $L_0,j(\xi_j)$ and $E[L_0^p(\xi_j)] < \infty$ respectively have been involved.

Condition (iii) is similar with [13, Assumption 1 (c)], which ensures that weight matrix $A_k$ and $\xi_{j,k}$ are independent conditionally on the past.

For the regularizer $\psi(\cdot)$, recall the concepts of mirror map [33]

$$Q(z) := \arg\max_{x \in X} \{\langle z, x \rangle - \psi(x)\}$$

and Fenchel coupling

$$R(x,z) := \psi(x) + \psi^*(z) - \langle x, z \rangle, \forall x \in X, z \in R^d,$$

where $\psi^*(z) := \sup_{x \in X} \{\langle z, x \rangle - \psi(x)\}$ is the conjugate function of $\psi(x)$.

Assumption 5 (regularizer $\psi(\cdot)$). For any $x \in X$, $R(x,z_k) \to 0$ whenever $Q(z_k) \to x$.

Assumption 5 is called “reciprocity condition” [33, Assumption 3]. Most common regularizers such as the Euclidean and entropic regularizer satisfy this assumption, for details refer to [33, Examples 2.7 and 2.8].

In the next, we study the convergence of sequences $\{x_{j,k}\}$ generated by Algorithm 1. By definition (12), the first step of Algorithm 1 can be rewritten as $x_{j,k} = Q(z_{j,k-1})$. As a key step, we define two auxiliary sequences

$$\bar{z}_k := \frac{1}{m} \sum_{j=1}^{m} z_{j,k}, \quad \bar{x}_{k+1} := Q(\bar{z}_k)$$

(13)

as reference sequences to measure the agent disagreements. It is obvious that $\bar{z}_{k-1}$ and $\bar{x}_k$ are adapted to $F_k$. By [25, Lemma 1],

$$\|x_{j,k} - \bar{x}_k\| = \|Q(z_{j,k-1}) - Q(\bar{z}_{k-1})\| \leq \|z_{j,k-1} - \bar{z}_{k-1}\|/\sigma,$$

(14)

where $\sigma$ is the strongly convex parameter of $\psi(x)$. Then for any $1 \leq j \leq m$, we may study the consensus of $\{x_{j,k}\}$ by showing $z_{j,k} - \bar{z}_k \to 0$.  

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Lemma 1. Suppose Assumptions [1][2] hold. Then, for any $1 \leq j \leq m$,

(i) \[
\sup_k k^{2\beta} E \left[ \| \tilde{z}_k - z_{j,k} \|^2 \right] < \infty, \]  

where constant $\beta$ is defined in Assumption [3](ii).

(ii) Furthermore,

\[
\sum_{k=1}^{\infty} \| \tilde{z}_k - z_{j,k} \|^2 < \infty \text{ a.s.} \tag{16}
\]

and for any positive sequence $\{\gamma_k\}$ such that $\sum_{k=1}^{\infty} \gamma_k k^{-\beta} < \infty$,

\[
\sum_{k=1}^{\infty} \gamma_k \| \tilde{z}_k - z_{j,k} \| < \infty \text{ a.s.} \tag{17}
\]

(iii) If Assumption [3](ii) and Assumption [3] are replaced by

(a) $A_k, k = 1, 2, \cdots$ is i.i.d. and the spectral norm $\rho$ of matrix $E \left[ A_k^T (I_m - \frac{I_m}{m}) A_k \right]$ satisfies $\rho < 1$,

(b) $\alpha_k > 0$ is nonincreasing, $\sum_{k=1}^{\infty} \alpha_k = \infty$, $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, and $\lim_{k \to \infty} \frac{\alpha_k}{\alpha_{k+1}} = 1$,

respectively. Then

\[
\sup_k \alpha_k^{-2} E \left[ \| \tilde{z}_k - z_{j,k} \|^2 \right] < \infty, \]  

and thus [16] holds.

We provide the proof of Lemma 1 in Appendix A.

Lemma 1 shows that $z_{j,k} - \tilde{z}_k, \forall 1 \leq j \leq m$ converges to zero, which in turn implies the consensus of sequences $\{x_{j,k}\}, j = 1, \cdots, m$. Moreover, it also shows that $\tilde{z}_k - z_{j,k}$ tends to zero in the 2-nd mean at rate $O(k^{-2\beta})$ under Assumptions [2][3] and at rate $O(\alpha_k^2)$ under stronger conditions, which is the key results for analysing the convergence rate and asymptotic normality of sequences $\{x_{j,k}\}, j = 1, \cdots, m$.

Theorem 1. Suppose Assumptions [4][5] hold with $p = 2$ in Assumption [4](ii). Then $x_{j,k}, j = 1, \cdots, m$ and $\tilde{x}_k$ converge to some point in $X^*$ almost surely.

Proof. For any fixed $x^* \in X^*$, denote $R_k := R(x^*, \tilde{x}_k) \geq 0$. By [3] Lemma 3.2 (3.2b)],

\[
R_k \leq R_{k-1} + \langle Q(\tilde{z}_{k-1}) - x^*, \tilde{z}_k - \tilde{z}_{k-1} \rangle + \frac{1}{2\sigma} \| \tilde{z}_k - \tilde{z}_{k-1} \|^2
\]

\[
= R_{k-1} + \langle \bar{x}_k - x^*, \tilde{z}_k - \tilde{z}_{k-1} \rangle + \frac{1}{2\sigma} \| \tilde{z}_k - \tilde{z}_{k-1} \|^2,
\]

where $\sigma$ is the strongly convex parameter of the regularizer $\psi(x)$. Note that $\tilde{z}_{k-1}$ is adapted to $\mathcal{F}_k$, we have by taking conditional expectation on both sides of the above inequality with respect to $\mathcal{F}_k$ that

\[
E \left[ R_k | \mathcal{F}_k \right] \leq R_{k-1} + E \left[ \langle \bar{x}_k - x^*, \tilde{z}_k - \tilde{z}_{k-1} \rangle | \mathcal{F}_k \right] + \frac{1}{2\sigma} E \left[ \| \tilde{z}_k - \tilde{z}_{k-1} \|^2 | \mathcal{F}_k \right]. \tag{19}
\]
Firstly, we focus on the second term \( \mathbb{E} \left[ \langle \bar{x}_k - x^*, \tilde{z}_k - \check{z}_{k-1} \rangle \right| \mathcal{F}_k \) on the right-hand side of (19). By definitions, \( \bar{x}_k, x_{j,k} \) and \( \tilde{z}_{j,k-1} \) are adapted to \( \mathcal{F}_k \) and then

\[
\begin{align*}
\mathbb{E} \left[ \langle \bar{x}_k - x^*, \tilde{z}_k - \check{z}_{k-1} \rangle \right| \mathcal{F}_k &= \langle \bar{x}_k - x^*, \mathbb{E} [\tilde{z}_k - \check{z}_{k-1} \mid \mathcal{F}_k] \rangle \\
&= \langle \bar{x}_k - x^*, \mathbb{E} \left[ \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{i=1}^{m} [A_k]_{ij} - 1 \right) z_{j,k-1} - \frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla f_j(x_{j,k}; \xi_{j,k}) \right] \right| \mathcal{F}_k \rangle \\
&= \langle \bar{x}_k - x^*, \frac{1}{m} \sum_{j=1}^{m} \mathbb{E} \left[ \sum_{i=1}^{m} [A_k]_{ij} - 1 \right| \mathcal{F}_k \right] z_{j,k-1} - \frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla f_j(x_{j,k}) \rangle \\
&= \langle \bar{x}_k - x^*, -\frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla f_j(x_{j,k}) \rangle,
\end{align*}
\]

where the second equality follows from the definitions of \( \tilde{z}_k \) in (13) and \( z_{j,k} \) in (3), and the last equality follows from the fact that

\[
\mathbb{E} \left[ \sum_{i=1}^{m} [A_k]_{ij} \mid \mathcal{F}_k \right] = \mathbb{E} \left[ \sum_{i=1}^{m} [A_k]_{ij} \right] = 1, 1 \leq j \leq m, \tag{20}
\]

see Assumption 2(i) and 2(iii) for details. Moreover,

\[
\langle \bar{x}_k - x^*, -\nabla f_j(x_{j,k}) \rangle = (\nabla f_j(x_{j,k}), x^* - x_{j,k}) + (\nabla f_j(x_{j,k}), x_{j,k} - \bar{x}_k)
\]

\[
\leq f_j(x^*) - f_j(x_{j,k}) + \|\nabla f_j(x_{j,k})\| \|x_{j,k} - \bar{x}_k\|
\]

\[
\leq f_j(x^*) - f_j(\bar{x}_k) + f_j(\bar{x}_k) - f_j(x_{j,k}) + L_0 \|x_{j,k} - \bar{x}_k\|
\]

\[
\leq f_j(x^*) - f_j(\bar{x}_k) + 2L_0 \|x_{j,k} - \bar{x}_k\|
\]

\[
\leq f_j(x^*) - f_j(\bar{x}_k) + 2L_0 \|z_{j,k-1} - \check{z}_{k-1}\|/\sigma,
\]

where \( L_0 \) is defined in (6), the first inequality follows from the convexity of \( f_j(\cdot) \) and the Cauchy-Schwarz inequality, the second and the third inequalities follow from the Lipschitz condition (ii) of Assumption 1 and the last inequality follows from (14). Consequently,

\[
\mathbb{E} \left[ \langle \bar{x}_k - x^*, \tilde{z}_k - \check{z}_{k-1} \rangle \right| \mathcal{F}_k \] \leq \frac{\alpha_k}{m} (f^* - f(\bar{x}_k)) + \frac{2\alpha_k L_0}{m\sigma} \sum_{j=1}^{m} \|z_{j,k-1} - \check{z}_{k-1}\|. \tag{21}
\]

Next, we focus on the third term \( \frac{1}{2\sigma} \mathbb{E} \left[ \|\tilde{z}_k - \check{z}_{k-1}\|^2 \mid \mathcal{F}_k \right] \) on the right-hand side of (19),

\[
\frac{1}{2\sigma} \mathbb{E} \left[ \|\tilde{z}_k - \check{z}_{k-1}\|^2 \right| \mathcal{F}_k
\]

\[
= \frac{1}{2\sigma} \mathbb{E} \left[ \left\| \sum_{j=1}^{m} \frac{[A_k]_{ij}}{m} (z_{j,k-1} - \check{z}_{k-1}) - \frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla F_j(x_{j,k}; \xi_{j,k}) \right\|^2 \mid \mathcal{F}_k \right]
\]

\[
\leq \frac{1}{\sigma} \mathbb{E} \left[ \left\| \sum_{j=1}^{m} \frac{[A_k]_{ij}}{m} (z_{j,k-1} - \check{z}_{k-1}) \right\|^2 \mid \mathcal{F}_k \right] + \frac{1}{\sigma} \mathbb{E} \left[ \left\| \frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla F_j(x_{j,k}; \xi_{j,k}) \right\|^2 \mid \mathcal{F}_k \right]
\]

\[
\leq \frac{1}{\sigma} \sum_{j=1}^{m} \mathbb{E} \left[ \left\| \frac{[A_k]_{ij}}{m} \right\| \mathcal{F}_k \right] \|\tilde{z}_{k-1} - z_{j,k-1}\|^2 + \frac{\alpha_k^2}{m\sigma} \sum_{j=1}^{m} \mathbb{E} \left[ \left\| \nabla F_j(x_{j,k}; \xi_{j,k}) \right\|^2 \mid \mathcal{F}_k \right]
\]

\[
\leq \frac{1}{m\sigma} \sum_{j=1}^{m} \|\tilde{z}_{k-1} - z_{j,k-1}\|^2 + \frac{L_0^2}{\sigma} \alpha_k^2,
\]
where $L^2_0$ is defined in (6), the second inequality follows from the convexity of $\| \cdot \|^2$ and the fact that

$$[A_k]_{ij} \geq 0, \quad \sum_{j=1}^{m} \frac{\sum_{i=1}^{m}[A_k]_{ij}}{m} = 1,$$

the last inequality follows from (20) and the Lipschitz condition (ii) of Assumption 1.

Combining (19), (21) and (22), it follows that

$$E[R_k|\mathcal{F}_k] \leq R_{k-1} - \frac{\alpha_k}{m} (f(\bar{x}_k) - f^*) + \frac{2L_0\alpha_k}{m\sigma} \sum_{j=1}^{m} \|z_{j,k-1} - \bar{z}_{k-1}\| + \frac{1}{m\sigma} \sum_{j=1}^{m} \|\bar{z}_{k-1} - z_{j,k-1}\|^2 + \frac{L_0^2}{\sigma}\alpha_k^2.$$  

(23)

In what follows, we employ the supermartingale convergence theorem of Robbins and Siegmund (Lemma 6 in Appendix F) to study the convergence of $R_k$. For the consistency of the notations, denote

$v_k := R_k, \quad a_k := 0, \quad \phi_k := \frac{\alpha_k}{m} (f(\bar{x}_k) - f^*)$  
and

$b_k := \frac{2\alpha_k}{m\sigma} \sum_{j=1}^{m} L_0 \|z_{j,k-1} - \bar{z}_{k-1}\| + \frac{1}{m\sigma} \sum_{j=1}^{m} \|\bar{z}_{k-1} - z_{j,k-1}\|^2 + \frac{L_0^2}{\sigma}\alpha_k^2.$

(24)

Obviously, $v_k, a_k, b_k, \phi_k$ are nonnegative sequence and adapted to $\mathcal{F}_k$. Note that

$$\sum_{k=1}^{\infty} \alpha_k \|z_{j,k-1} - \bar{z}_{k-1}\| = \alpha_1 \|z_{j,0} - \bar{z}_0\| + \sum_{k=1}^{\infty} \alpha_{k+1} \|z_{j,k} - \bar{z}_k\|$$

$$\leq \alpha_1 \|z_{j,0} - \bar{z}_0\| + \sum_{k=1}^{\infty} \alpha_k \|z_{j,k} - \bar{z}_k\| < \infty \ a.s.,$$

where the inequality follows from the step-size $\alpha_k$ in nonincreasing by Assumption 3 and the summability follows from (17). Then by combining this with Assumption 3 and (16), we know that $\sum_{k=1}^{\infty} b_k < \infty$, and hence the conditions of Lemma 6 hold. By applying the lemma, we have that for any $x^* \in \mathcal{X}^*$, $R_k$ converges to a finite random variable $R_{\infty}$ almost surely and

$$\sum_{k=1}^{\infty} \alpha_k (f(\bar{x}_k) - f^*) < \infty \ a.s. \quad (24)$$

By [33] Lemma 3.2 (a)],

$$\|\bar{x}_k - x^*\|^2 = \|Q(\bar{z}_{k-1}) - x^*\|^2 \leq \frac{2}{\sigma} R_{k-1}$$

(25)
and then $\{\bar{x}_k\}$ is bounded almost surely. In addition, according to (24) and condition (i) of Assumption 3

$$\liminf_{k \to \infty} f(\bar{x}_k) - f^* = 0 \ a.s.$$

Consider a subsequence $\{\bar{x}_{k_t}\}$ such that $\lim_{t \to \infty} f(\bar{x}_{k_t}) = f^*$ and denote $\bar{x}$ as the limit point of $\{\bar{x}_{k_t}\}$. Since $f$ is continuous, we must have $f(\bar{x}) = f^*$, and hence $\bar{x} \in \mathcal{X}^*$. Fixing $x^* = \bar{x}$ in the definition of $R_k$. By Assumption 5, we see that for any subsequence of $\{\bar{x}_{k_t}\}$ that converges to $\bar{x}$, the corresponding subsequence of $R_{k_{t-1}}$ must converges to 0 almost surely, and thus $R_{\infty}$
equals to 0 almost surely. Consequently, (25) implies  \( \bar{x}_k \to \bar{x} \) almost surely. Note also that for any \( 1 \leq j \leq m, \)
\[
\|x_{j,k} - \bar{x}\| \leq \|x_{j,k} - \bar{x}_k\| + \|\bar{x}_k - \bar{x}\| \leq \frac{1}{\sigma} \|z_{j,k-1} - \bar{z}_{k-1}\| + \|\bar{x}_k - \bar{x}\|,
\]
where the second inequality follows from [14]. Then \( x_{j,k} \to \bar{x} \) almost surely as \( z_{j,k} \to \bar{z}_k \) and \( \bar{x}_k \to \bar{x} \) almost surely. The proof is completed.

A DDA algorithm is proposed by Duchi et al. [9] where the convergence rate of gap between the functional value of local average and the optimal values have been established. In comparison, Theorem 1 establishes the almost sure convergence of the solutions \( x_{j,k}, j = 1, \cdots, m \) and \( \bar{x}_k \) generated by DDA algorithm.

### 3.2 Almost sure convergence rate

Let \( x^* \) be the limit point of sequence \( \{\bar{x}_k\} \) in Theorem 1. In this subsection, we study the convergence rate of \( \|\bar{x}_k - x^*\| \) to zero. Hereafter, we consider the case that the constraint set \( \mathcal{X} \) in problem (1) is defined by linear inequalities,
\[
\mathcal{X} = \{x \in \mathbb{R}^d : Bx - b \leq 0, \ Cx - c \leq 0\}
\]
and the regularizer in (13) is \( \psi(x) = \frac{1}{2}\|x\|^2 \), where \( B \in \mathbb{R}^{d_1 \times d}, b \in \mathbb{R}^{d_1}, C \in \mathbb{R}^{d_2 \times d} \) and \( c \in \mathbb{R}^{d_2} \). For simplicity, we assume that \( Bx^* - b = 0, \ Cx^* - c < 0, \) that is, \( Bx - b \leq 0 \) is the active constraint on \( x^* \) while the other is inactive, and denote
\[
\mathcal{Y} = \{x : Bx = 0\}, \quad U = (u_1, u_2, \cdots, u_d) \in \mathbb{R}^{d \times d},
\]
where \( \mathcal{Y} \) is a \( r \)-dimension subspace of \( \mathbb{R}^d \), \( u_1, u_2, \cdots, u_r \) and \( u_{r+1}, u_{r+1}, \cdots, u_d \) are the standard orthogonal basis of \( \mathcal{Y} \) and its orthogonal subspace respectively. Moreover, the two auxiliary sequences defined in (13) read as follows:
\[
\bar{z}_k = \frac{1}{m} \sum_{j=1}^{m} z_{j,k}, \quad \bar{x}_{k+1} = \arg\min_{x \in \mathcal{X}} \{(-\bar{z}_k, x) + \frac{1}{2}\|x\|^2\}.
\]

The following assumptions are needed.

**Assumption 6 (strengthened Assumption)** (i) Assumption 7 holds. (ii) For any \( 1 \leq j \leq m, \) there exists a constant \( L > 0 \) such that
\[
\|\nabla f_j(x) - \nabla f_j(y)\| \leq L\|x - y\|, \forall x, y \in \mathcal{X}.
\]
There exist constants \( c_0, \epsilon \in (0, \infty) \) such that for \( x \in \mathcal{X} \cap \{x : \|x - x^*\| \leq \epsilon\}, \)
\[
\|\nabla f(x) - \nabla f(x^*) - \nabla^2 f(x^*)(x - x^*)\| \leq c_0\|x - x^*\|^2.
\]
(iii) There exists \( \mu > 0 \) such that for any \( x \) in the critical tangent cone \( \mathcal{T}_\mathcal{X}(x^*) \),
\[
x^T \nabla^2 f(x^*) x \geq \mu\|x\|^2.
\]
Assumption 6(iii) is the standard second-order sufficiency (or restricted strong convexity) condition [34], which guarantees the uniqueness of minimizer of function $f(\cdot)$ over $\mathcal{X}$. Moreover, it implies that [34, Theorem 3.2(i)]: there exists $\epsilon' > 0$ such that
\[
\langle \nabla f(x), x - x^* \rangle \geq f(x) - f(x^*) \geq \epsilon' \min \{\|x - x^*\|^2, \|x - x^*\|\} \quad \forall x \in \mathcal{X}.
\]

(31)

Assumption 7 (constraint qualification). [22, Assumption B] The vector $\nabla f(x^*)$ satisfies
\[
-\nabla f(x^*) \in \text{ri} N_{\mathcal{X}}(x^*),
\]

(32)

where $\text{ri} N_{\mathcal{X}}(x^*)$ is the relative interior of normal cone $N_{\mathcal{X}}(x^*)$.

The nondegeneracy condition (32) is common in manifold identification analysis [22, 27]. As we assumed that $Bx^* = b$ and $Cx^* < c$, the norm cone in Assumption 7 and critical tangent cone in Assumption 6 are
\[
N_{\mathcal{X}}(x^*) = \{y : B^T \lambda = y, \lambda \in \mathbb{R}^d_+\}, \quad T_{\mathcal{X}}(x^*) = \{x : Bx = 0\}.
\]

We need stronger assumptions on weight matrix $A_k$ and step-size $\alpha_k$.

Assumption 8 (stronger conditions on weight matrix). (i) $A_k, k = 1, 2, \cdots$ is doubly stochastic matrix with nonnegative components; (ii) $A_k, k = 1, 2, \cdots$ is i.i.d. and the spectral norm $\rho$ of matrix $E[A_k^T(I_m - \frac{11^T}{m})A_k]$ satisfies $\rho < 1$; (iii) Assumption 2(iii) holds.

Assumption 9 (stronger conditions on step-size). The step-size $\alpha_k = \frac{a}{k^\alpha}$ with $\alpha \in (\frac{2}{3}, 1), a > 0$.

The following lemma studies the active set identification of dual averaging algorithm [1] which is an extension of [22, Theorem 3] to distributed optimization setting.

Lemma 2. Suppose Assumptions 4, 6-9 hold. Then with probability one, there exists some (random) $K < \infty$ such that when $k \geq K$,
\[
B\bar{x}_k = b, \quad C\bar{x}_k < c.
\]

The proof is presented in Appendix B.

Define
\[
P_B := I_d - B^T(BB^T)^\dagger B
\]

(33)
as the projection operator onto subspace $\mathcal{Y}$ (26) and
\[
H := \frac{1}{m}P_B \nabla^2 f(x^*)P_B.
\]

(34)

Lemma 2 implies
\[
P_B(\bar{x}_k - x^*) = \bar{x}_k - x^* \quad \text{a.s.},
\]
when $k$ is large enough. Therefore, we may study the convergence rate of $\|\bar{x}_k - x^*\|$ through $\|P_B(\bar{x}_k - x^*)\|$. For easy of the notation, we denote
\[
\Delta_k := P_B(\bar{x}_k - x^*)
\]

(35)
throughout the paper.

The following lemma provides the recursive formula of $\Delta_k$, whose proof is provided in Appendix C.
Lemma 3. Suppose Assumptions 6-8 hold. Then
\[
\triangle_{k+1} = \triangle_k - \alpha_k H \triangle_k + \alpha_k (\zeta_k + \eta_k + s_k + \epsilon_k)
\] (36)
or
\[
\triangle_{k+1} = [I_d - \alpha_k (H + D_k)] \triangle_k + \alpha_k (\eta_k + s_k + \epsilon_k),
\] (37)
where
\[
\begin{align*}
\zeta_k &= -\frac{1}{m} P_B \left[ \nabla f(\bar{x}_k) - \nabla f(x^*) - \nabla^2 f(x^*)(\bar{x}_k - x^*) \right], \\
\eta_k &= \frac{1}{m} \sum_{j=1}^{m} P_B [\nabla f_j(\bar{x}_k) - \nabla f_j(x_{j,k})], \\
\epsilon_k &= \frac{1}{\alpha_k} P_B C^T (\mu_{k-1} - \mu_k) + \frac{1}{m} P_B \nabla^2 f(x^*)(P_B - I_d)(\bar{x}_k - x^*), \\
s_k &= -\frac{1}{m} \sum_{j=1}^{m} P_B s_{j,k}, \\
D_k &= -\zeta_k \frac{\triangle^T_k}{\|\triangle_k\|^2}.
\end{align*}
\]

Lemma 3 provides two kind of recursive formulas of \(\triangle_k\), where (37) will be used to analyse the almost sure convergence rate in Theorem 2 and asymptotic normality of Algorithm 1 in Theorem 3 and (36) will be used to analyse the asymptotic efficiency of Algorithm 1 in Theorem 4.

The following technical results will help us to study the rate of convergence of \(\|\bar{x}_k - x^*\|\) by focusing on the subspace \(Y\) determined by the active constraints on the optimal solution \(x^*\).

Lemma 4. Recall \(Y, U\) and \(P_B\) have been defined in (26) and (33) respectively. Then

(i) \(U^T: Y \rightarrow \mathbb{R}^r \times 0\) is a bijection, where \(0^T = (0, 0, \cdots, 0)\), and \(\times^d\) is the Cartesian Product.

(ii) For any \(y \in Y\) and \(H \in \mathbb{R}^{d \times d}\),
\[
U^T P_B H y = \begin{pmatrix} G_1 y_1 \\ 0 \end{pmatrix},
\] (39)
where \(y_1 \in \mathbb{R}^r\), \(G_1\) is the \(r\)-order sequential principal minor of \(U^T H U\). Moreover, if there exists a constant \(\mu > 0\) such that
\[
y^T H y \geq \mu \|y\|^2, \forall y \in Y,
\]
then \(G_1\) is a positive definite matrix.

The proof is presented in Appendix D.

Theorem 2. Suppose Assumptions 4, 6-9 hold with \(p = 2\) in Assumption 1(ii). Then for any \(\delta \in (0, 1 - 1/(2\alpha))\),
\[
\|\triangle_k\| = o(\alpha_k^\delta) \quad \text{a.s.}
\] (40)
Proof. We employ [33] Lemma 3.1.1 (Lemma 7 in Appendix 7) to prove (40). We reformulate the recursion \( \frac{\Delta_{k+1}}{\alpha_{k+1}^\delta} \) in the form of (99) in Lemma 7 first.

Dividing \( \alpha_k^{\delta_{k+1}} \) on both sides of equation (37), we have

\[
\frac{\Delta_{k+1}}{\alpha_k^{\delta_{k+1}}} = \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta \left[ I_d - \alpha_k (H + D_k) \right] \frac{\Delta_k}{\alpha_k^{\delta_k}} + \alpha_k \left( \frac{\eta_k}{\alpha_k^{\delta_k}} + \frac{s_k}{\alpha_k^{\delta_k}} + \frac{\epsilon_k}{\alpha_k^{\delta_k}} \right)
\]

where the second inequality follows from (29) and the fact \( \bar{\alpha} \parallel H \parallel \). On both sides of equation (37), we have

\[
\frac{\Delta_k}{\alpha_k^{\delta_k}} = \left[ I_d - \alpha_k (H + C_k) \right] \frac{\Delta_k}{\alpha_k^{\delta_k}} + \alpha_k \left( \frac{\eta_k}{\alpha_k^{\delta_k}} + \frac{s_k}{\alpha_k^{\delta_k}} + \frac{\epsilon_k}{\alpha_k^{\delta_k}} \right)
\]

where

\[
C_k := \frac{1}{\alpha_k} \left( 1 - \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta \right) I_d + \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta H + \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta D_k
\]

and \( H_k := H + C_k \). Note that by Assumption 9, \( \alpha_k = a/k^\alpha, \alpha \in (2/3, 1) \), we obtain

\[
\left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta \rightarrow 1, \quad \frac{1}{\alpha_k} \left( 1 - \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta \right) = \frac{k^\alpha}{a} \left( 1 - \left( 1 + \frac{1}{k} \right)^\alpha \right) \rightarrow 0.
\]

Note also that

\[
\| D_k \| \leq \frac{\| \zeta_k \|}{\| \Delta_k \|} \leq \frac{c \| P_B \| \| \bar{x}_k - x^* \|}{\| \Delta_k \|} = \frac{c \| P_B \| \| \bar{x}_k - x^* \|}{\| \bar{x}_k - x^* \|} = c \| P_B \| \| \bar{x}_k - x^* \| \rightarrow 0, \quad a.s.,
\]

where the second inequality follows from (29) and the fact \( \bar{x}_k \rightarrow x^* \) almost surely. Then \( C_k \rightarrow 0 \) almost surely which implies \( H_k = H + C_k \rightarrow H \) almost surely. By definitions of \( \Delta_k, H \) and \( D_k \) in (34), (35) and (38) respectively,

\[
\Delta_k = P_B \Delta_k, \quad H = P_B H, \quad D_k = P_B D_k.
\]

Then

\[
H_k \frac{\Delta_k}{\alpha_k^{\delta_k}} = \left( H + C_k \right) \frac{\Delta_k}{\alpha_k^{\delta_k}}
\]

where

\[
\frac{\Delta_k}{\alpha_k^{\delta_k}} = \left( \frac{1}{\alpha_k} \left( 1 - \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta \right) \right) \frac{\Delta_k}{\alpha_k^{\delta_k}} + \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta \left( H \frac{\Delta_k}{\alpha_k^{\delta_k}} + \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta D_k \frac{\Delta_k}{\alpha_k^{\delta_k}} \right)
\]

and

\[
P_B \frac{\Delta_k}{\alpha_k^{\delta_k}} = \frac{\Delta_k}{\alpha_k^{\delta_k}} \left( I_d + \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta \right) H + \left( \frac{\alpha_k}{\alpha_{k+1}} \right)^\delta D_k \frac{\Delta_k}{\alpha_k^{\delta_k}}
\]

Subsequently,

\[
\frac{\Delta_{k+1}}{\alpha_k^{\delta_{k+1}}} = \left[ I_d - \alpha_k P_B H_k \right] \frac{\Delta_k}{\alpha_k^{\delta_k}} + \alpha_k \left( \frac{\eta_k}{\alpha_k^{\delta_k}} + \frac{s_k}{\alpha_k^{\delta_k}} + \frac{\epsilon_k}{\alpha_k^{\delta_k}} \right). \quad (41)
\]
Left multiplying $U^T$ on both sides of equation [41], we have

\[
U^T \frac{\Delta_k + 1}{\alpha_{k+1}} = U^T \left[ I_d - \alpha_k P_B H_k \right] \frac{\Delta_k}{\alpha_k} + \alpha_k U^T \left( \frac{\eta_k}{\alpha_{k+1}^\delta} + \frac{s_k}{\alpha_{k+1}^\delta} + \frac{\epsilon_k}{\alpha_{k+1}^\delta} \right).
\]

Since $\Delta_k, \eta_k, s_k, \epsilon_k \in \mathcal{Y}$, Lemma [4] implies

\[
\begin{pmatrix}
\frac{\Delta_{k+1}'}{\alpha_{k+1}^\delta} \\
0
\end{pmatrix} = \begin{pmatrix}
\frac{\Delta_k'}{\alpha_k^\delta} \\
\frac{G_k \Delta_k'}{\alpha_k^\delta}
\end{pmatrix} - \alpha_k \begin{pmatrix}
\frac{\eta_k}{\alpha_{k+1}^\delta} \\
\frac{s_k}{\alpha_{k+1}^\delta} \\
\frac{\epsilon_k}{\alpha_{k+1}^\delta}
\end{pmatrix} + \alpha_k \begin{pmatrix}
\frac{\eta_k}{\alpha_{k+1}^\delta} \\
\frac{s_k}{\alpha_{k+1}^\delta} \\
\frac{\epsilon_k}{\alpha_{k+1}^\delta}
\end{pmatrix},
\]

where

\[
\Delta_k' = (U^T)^{(r)} \Delta_k, \quad \eta_k' = (U^T)^{(r)} \eta_k, \quad s_k' = (U^T)^{(r)} s_k, \quad \epsilon_k' = (U^T)^{(r)} \epsilon_k,
\]

$(U^T)^{(r)}$ is a $r \times d$-matrix composed of first $r$ row vectors of $U^T$ and $G_k$ is the $r$-order sequential principal minor of $U^T H_k U$. Obviously, we only need to focus on the linear recurrence

\[
\frac{\Delta_{k+1}'}{\alpha_{k+1}^\delta} = (I_r - \alpha_k G_k) \frac{\Delta_k'}{\alpha_k^\delta} + \alpha_k \begin{pmatrix}
\frac{\eta_k'}{\alpha_{k+1}^\delta} \\
\frac{s_k'}{\alpha_{k+1}^\delta} \\
\frac{\epsilon_k'}{\alpha_{k+1}^\delta}
\end{pmatrix}.
\]

Denote

\[
y_k = \frac{\Delta_k'}{\alpha_k^\delta}, \quad F_k = -G_k, \quad e_k = \frac{s_k'}{\alpha_{k+1}^\delta}, \quad \nu_k = \frac{\eta_k'}{\alpha_{k+1}^\delta} + \frac{\epsilon_k'}{\alpha_{k+1}^\delta},
\]

[44] can be rewritten as

\[
y_{k+1} = y_k + \alpha_k F_k y_k + \alpha_k (e_k + \nu_k),
\]

which is in the form (99) of Lemma [7].

In what follows, we verify the conditions of [35] Lemma 3.1.1.

Firstly, we show that $F_k$ converges to a stable matrix $F$. Note that $G_k$ is the $r$-order sequential principal minor of $U^T H_k U$ and $H_k \to H$ almost surely, $F_k$ converges to $-G$, where $G$ is the $r$-order sequential principal minor of $U^T H U$. By Lemma [4](iii) it follows from Assumption [6](iii) that the $r$-order sequential principal minor of $U^T H U$ is a positive definite matrix, which implies the stability of the limit of $\{F_k\}$.

Next, we show $\nu_k \to 0$ almost surely, where it is sufficient to prove

\[
\frac{\epsilon_k'}{\alpha_{k+1}^\delta} \to 0, \quad \frac{\eta_k'}{\alpha_{k+1}^\delta} \to 0.
\]

On the one hand, recall the definition (38)

\[
\epsilon_k = \frac{1}{\alpha_k} P_B C^T (\mu_{k-1} - \mu_k) + \frac{1}{m} P_B \nabla^2 f(x^*)(P_B - I_d)(\bar{x}_k - x^*).
\]

By Lemma [2] $\epsilon_k = 0$ almost surely when $k$ is large enough as $\mu_k = \mu_{k+1} = 0$ and $(P_B - I_d)(\bar{x}_k - x^*) = 0$ when $k \geq K$, where $K < \infty$ is specified in Lemma [2]. Then

\[
\frac{\epsilon_k}{\alpha_{k+1}^\delta} = \frac{(U^T)^{(r)} \epsilon_k}{\alpha_{k+1}^\delta} = 0
\]
almost surely. On the other hand, note that
\[
\mathbb{E}[\|\eta'_k\|^2] = \mathbb{E}[\|\langle U^T \rangle^{(r)} \eta_k\|^2] = \mathbb{E}\left[\left\|\langle U^T \rangle^{(r)} \frac{1}{m} \sum_{j=1}^{m} P_B (\nabla f_j(x_{j,k}) - \nabla f_j(\bar{x}_k))\right\|^2\right]
\leq \frac{\|\langle U^T \rangle^{(r)}\|^2}{m} \mathbb{E} \left[\sum_{j=1}^{m} \|\nabla f_j(x_{j,k}) - \nabla f_j(\bar{x}_k)\|^2\right] \leq \frac{\|\langle U^T \rangle^{(r)}\|^2}{m} P_B^2 \mathbb{E} \sum_{j=1}^{m} \|x_{j,k} - \bar{x}_k\|^2,
\]
where the last inequality follows from the Lipschitz continuity of \( \nabla f_j(\cdot) \). By using the fact
\[
\left(\frac{\alpha_k}{\alpha_{k+1}}\right)^\delta = \left(1 + \frac{1}{k}\right)^{\alpha_\delta} \leq 2^{\alpha_\delta},
\]
and denoting \( c_m = 4^\delta \|\langle U^T \rangle^{(r)}\|^2 P_B^2 L^2 / m \), we have
\[
\sum_{k=1}^{\infty} \mathbb{E}\left[\left\|\frac{\eta'_k}{\alpha_{k+1}^{\delta}}\right\|^2\right] \leq c_m \sum_{k=1}^{\infty} \sum_{j=1}^{m} \mathbb{E}\left[\left\|\frac{x_{j,k} - \bar{x}_k}{\alpha_{k}^{\delta}}\right\|^2\right] \\
\leq c_m \sum_{k=1}^{\infty} \sum_{j=1}^{m} \mathbb{E}\left[\left\|\frac{z_{j,k} - \bar{z}_{k-1}}{\alpha_{k}^{\delta}}\right\|^2\right] = c_m \sum_{k=1}^{\infty} \sum_{j=1}^{m} \mathbb{E}\left[\left\|\frac{z_{j,k} - \bar{z}_k}{\alpha_{k+1}^{\delta}}\right\|^2\right] + c_m \mathbb{E}\left[\left\|\frac{z_{j,0} - z_0}{\alpha_{1}^{\delta}}\right\|^2\right] \\
\leq c_m \sum_{k=2}^{\infty} \frac{\alpha_k^2}{\alpha_{k+1}^{2\delta}} + c_m \mathbb{E}\left[\left\|\frac{z_{j,0} - z_0}{\alpha_1^{\delta}}\right\|^2\right] = c_m \sum_{k=2}^{\infty} \frac{\alpha_k^2 - 2\alpha_k^{1-\delta}}{\alpha_1^{2\delta}} + c_m \mathbb{E}\left[\left\|\frac{z_{j,0} - z_0}{\alpha_1^{\delta}}\right\|^2\right] < \infty,
\]
where the second inequality follows from (14), the third one from (18), the last one from \( 2\alpha_1(1 - \delta) > 1 \) by the definition, and \( c_m \) is a constant. Then by monotone convergence theorem,
\[
\sum_{k=0}^{\infty} \left\|\frac{\eta'_k}{\alpha_{k+1}^{\delta}}\right\|^2 < \infty \text{ a.s.,}
\]
which implies \( \frac{\eta'_k}{\alpha_{k+1}^{\delta}} \to 0 \) almost surely. Therefore, \( \nu_k \to 0 \) almost surely.

We are left to verify
\[
\sum_{k=1}^{\infty} \alpha_k e_k < \infty \text{ a.s.} \tag{47}
\]
Denote
\[
e'_k = \left(\frac{\alpha_k}{\alpha_{k+1}}\right)^\delta \langle U^T \rangle^{(r)} s_k.
\]
Obviously, \( \{e'_k, \mathcal{F}_{k+1}\} \) is a martingale difference sequence since \( \{s_k, \mathcal{F}_{k+1}\} \) is a martingale difference sequence. Then
\[
sup_k \mathbb{E}[\|e'_k\|^2 | \mathcal{F}_k] = sup_k \mathbb{E}\left[\left\|\frac{\alpha_k}{\alpha_{k+1}}\right\|^\delta \langle U^T \rangle^{(r)} s_k\right\|^2 | \mathcal{F}_k]
\leq 4^\delta \|\langle U^T \rangle^{(r)}\|^2 sup_k \mathbb{E}[\|s_k\|^2 | \mathcal{F}_k] = 4^\delta \|\langle U^T \rangle^{(r)}\|^2 sup_k \mathbb{E}\left[\left\|\frac{1}{m} \sum_{j=1}^{m} P_B s_j, k_0\right\|^2 | \mathcal{F}_k\right]
\leq 4^\delta \|\langle U^T \rangle^{(r)}\|^2 P_B^2 sup_k \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}[\|x_j, k\|^2 | \mathcal{F}_k] \leq 4^\delta \|\langle U^T \rangle^{(r)}\|^2 P_B^2 A_{l_0}^2 < \infty,
\]

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where the first inequality follows from (45), the second one from the convexity of $\| \cdot \|^2$, and the last one from Assumptions 4 and 6 which imply
\[
\mathbb{E} \left[ \| s_{j,k} \|^2 | \mathcal{F}_k \right] = \mathbb{E} \left[ \| \nabla f_j(x_{j,k}) - \nabla F_j(x_{j,k}; \xi_{j,k}) \|^2 | \mathcal{F}_k \right] \leq 4L_0^2,
\]
and $L_0^2$ is defined as in (6). Since
\[
\sum_{k=1}^{\infty} \alpha_k^{2(1-\delta)} = \sum_{k=1}^{\infty} a_k^{2(1-\delta)} < \infty,
\]
then by the convergence theorem for martingale difference sequences \cite[Appendix B.6, Theorem B 6.1]{35},
\[
\sum_{k=1}^{\infty} \alpha_k e_k = \sum_{k=1}^{\infty} \alpha_k^{1-\delta} e_k' < \infty.
\]
Then employing \cite[Lemma 3.1.1]{35} yields $y_k = \frac{\Delta'_{k}}{\alpha_k^\delta} \to 0$ almost surely. By the definition of $\Delta_{k+1}'$ in (43), we conclude that $\| \Delta_{k+1} \| = o(\alpha_k^\delta)$ almost surely. The proof is completed. $\Box$

The almost sure convergence rate in terms of the step-size of stochastic approximation algorithms for root-finding problems have been well studied, see \cite[35, 36]{35}. More recently, \cite[37, 38]{35} study the convergence rate of consensus problem when stochastic approximation method is used. To the best of our knowledge, Theorem 2 seems to be the first result on almost convergence rate of stochastic approximation method for distributed constrained stochastic optimization problems. As we will see, this result is useful for establishing asymptotic normality of the DDA algorithm.

## 4 Asymptotic normality and asymptotic efficiency

Asymptotic normality and asymptotic efficiency of stochastic algorithms can be traced back to the works on 1950s \cite[17, 18]{17}. More recently, \cite[23, 24]{23} study the asymptotic normality and asymptotic efficiency of stochastic algorithms for distributed unconstrained optimization problem. In this section, we focus on these asymptotic properties of Algorithm 1 for distributed constrained optimization problems.

We first present the asymptotic normality of Algorithm 1.

**Theorem 3.** Suppose Assumptions 4, 6-9 hold with $p > 2$ in Assumption 7(ii). Let $x^*$ be the limit point of sequence $\{ \bar{x}_k \}$. The covariance matrix mapping $\sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j))$ is continuous at point $x^*$. Then for any $1 \leq j \leq m$,
\[
\frac{x_{j,k} - x^*}{\sqrt{\alpha_k}} \xrightarrow{d} N(0, \Sigma),
\]
where
\[
\Sigma = U \left( \begin{array}{cc}
\Sigma_1 & 0 \\
0 & 0
\end{array} \right) U^T,
\]
\[
\Sigma_1 = \int_0^\infty e^{(-G)^t (U^T)^{(r)} P_B \Sigma B (U^T)^{(r)} T e^{(-G^T)^t) t} dt, \quad \Sigma = \frac{1}{m^2} \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j)),
\]

\[ \Sigma_1 = \int_0^\infty e^{(-G)^t (U^T)^{(r)} P_B \Sigma B (U^T)^{(r)} T e^{(-G^T)^t) t} dt, \quad \Sigma = \frac{1}{m^2} \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j)), \]
\((U^T)^{(r)} \in \mathbb{R}^{r \times d}\) is composed by first \(r\) row vectors of \(U^T\), \(G\) is the \(r\)-order sequential principal minor of \(U^T H U\) and \(H\) is defined as in (34).

Proof. We employ [35, Theorem 3.3.1] (Lemma 8 in Appendix F) to prove (48). By definition (34),
\[
H = \frac{1}{m} P_B \nabla^2 f(x^*) P_B = \frac{1}{m} P_B^2 \nabla^2 f(x^*) P_B = P_B H.
\]
Then (36) can be reformulated as
\[
\Delta_{k+1} = \left[I_d - \alpha_k P_B H \right] \Delta_k + \alpha_k \left( \zeta_k + \eta_k + s_k + \epsilon_k \right). \tag{51}
\]
Left multiplying \(U^T\) on both side of (51), Lemma 4 implies
\[
\begin{pmatrix} \Delta'_{k+1} \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta'_{k} \\ 0 \end{pmatrix} - \alpha_k \begin{pmatrix} G \Delta'_{k} \\ 0 \end{pmatrix} + \alpha_k \begin{pmatrix} \zeta_k' \\ \eta_k' \\ s_k' \\ \epsilon_k' \end{pmatrix},
\]
where
\[
\Delta'_k = (U^T)^{(r)} \Delta_k, \quad \zeta_k' = (U^T)^{(r)} \zeta_k, \quad \eta_k' = (U^T)^{(r)} \eta_k, \quad s_k' = (U^T)^{(r)} s_k, \quad \epsilon_k' = (U^T)^{(r)} \epsilon_k,
\]
\((U^T)^{(r)}\) is a \(r \times d\)-matrix composed by first \(r\) row vectors of \(U^T\), and \(G\) is the \(r\)-order sequential principal minor of \(U^T H U\). Define
\[
\Delta''_{k+1} := (I_r - \alpha_k G) \Delta''_k + \alpha_k \left( \zeta_k' + s_k' + \epsilon_k' \right), \tag{52}
\]
where the initial \(\Delta''_0 \in \mathbb{R}^r\) is arbitrary. Consequently,
\[
\frac{\Delta'_{k+1} - \Delta''_{k+1}}{\sqrt{\alpha_{k+1}}} = \sqrt{\frac{\alpha_k}{\alpha_{k+1}}} \left(I_r - \alpha_k G\right) \frac{\Delta'_{k} - \Delta''_{k}}{\sqrt{\alpha_k}} + \frac{\alpha_k}{\sqrt{\alpha_{k+1}}} \eta_k' \tag{53}
\]
where
\[
G'_k := \left(1 - \frac{1}{\sqrt{\alpha_k \alpha_{k+1}}} \right) I_r + \sqrt{\frac{\alpha_k}{\alpha_{k+1}}} G.
\]
For \(k \geq t\), denote
\[
\Psi^k_t := \left(I_r - \alpha_k G'_k\right) \cdots \left(I_r - \alpha_t G'_t\right), \quad \Psi^t_{t+1} = I_r.
\]
Recursively, we can reformulate (53) as
\[
\frac{\Delta'_{k+1} - \Delta''_{k+1}}{\sqrt{\alpha_{k+1}}} = \Psi^k_t \frac{\Delta'_{1} - \Delta''_{1}}{\sqrt{\alpha_1}} + \sum_{t=1}^{k} \Psi^k_{t+1} \frac{\alpha_t}{\sqrt{\alpha_{t+1}}} \eta_t'. \tag{54}
\]
By the Assumption 9 and the definition of \(G'_k\), it is easy to get that \(\lim_{k \to \infty} G'_k = G\). Since \(-G\) is stable, by [35, Inequality (3.1.8) in Lemma 3.1.1], there exist constants \(b_1, b_2 > 0\) such that
\[
\left\| \Psi^k_t \right\| \leq b_1 \exp(-b_2 \sum_{l=t}^{k} \alpha_l), \quad \forall k \geq t. \tag{55}
\]
Obviously, (55) implies the first term on the right-hand side of (54) tends to zero almost surely. Next, we show that the second term on the right-hand side of (54) tends to 0 in probability.

Note that
\[
\frac{\alpha_t}{\sqrt{\alpha_{t+1}}} = \left(1 + \frac{1}{t}\right) \frac{\alpha}{2} \sqrt{\alpha_t} \leq \sqrt{a} \left(\frac{3}{2}\right)^\frac{\alpha}{2} \sqrt{\alpha_t},
\]
and
\[
\mathbb{E}\left[\|\eta_t\|\right] = \mathbb{E}\left[\|\left(U^T\right)^{(r)} \frac{1}{m} \sum_{j=1}^{m} P_B(\nabla f_j(x_{j,t}) - \nabla f_j(\bar{x}_t))\|\right]
\leq \frac{\|\left(U^T\right)^{(r)}\|\|P_B\|}{m} \sum_{j=1}^{m} \mathbb{E}\left[\|x_{j,t} - \bar{x}_t\|\right]
\leq \frac{\|\left(U^T\right)^{(r)}\|\|P_B\|}{m} \sum_{j=1}^{m} \mathbb{E}\left[\|z_{j,t-1} - \bar{z}_{t-1}\|\right] \leq b_3 \alpha_{t-1},
\]
where the second inequality follows from (14) and the last one from (18). We obtain the estimate
\[
\mathbb{E}\left[\left\|\sum_{t=1}^{k} \Psi_{t+1}^{k} \frac{\alpha_t}{\sqrt{\alpha_{t+1}}} \eta_t\right\|\right] \leq \sum_{t=1}^{k} \left\|\Psi_{t+1}^{k}\right\| \frac{\alpha_t}{\sqrt{\alpha_{t+1}}} \mathbb{E}\left[\left\|\eta_t\right\|\right]
\leq b_3 \sqrt{a} \left(\frac{3}{2}\right)^\frac{\alpha}{2} \sum_{t=1}^{k} \left\|\Psi_{t+1}^{k}\right\| \sqrt{\alpha_t \alpha_{t-1}}
\leq b_3 \sqrt{a} \left(\frac{3}{2}\right)^\frac{\alpha}{2} \sum_{t=1}^{k} \alpha_t \left\|\Psi_{t+1}^{k}\right\|^\frac{3}{2} \left\|\Psi_{t+1}^{k}\right\|^\frac{1}{2} o(1),
\]
where \(o(1) = \frac{\alpha_{t-1}}{\sqrt{\alpha_t}} \to 0\) as \(t \to \infty\). By (55) and (35) Inequality (3.3.6) in Lemma 3.3.2], the term on right-hand side of the last inequality of (56) tends to 0, which implies the second term on the right-hand side of (54) tends to 0 in probability. Therefore (54) tends to 0 in probability, which implies \(\frac{\Delta_{k+1}}{\sqrt{\alpha_{k+1}}} = \frac{\Delta_{k+1}}{\sqrt{\alpha_{k+1}}}\) have same limit distribution.

Next, we focus on investigating the limit distribution of \(\frac{\Delta_{k+1}}{\sqrt{\alpha_{k+1}}}\). Denote
\[
y_{k+1} = \Delta_{k+1}, \quad F_k = -G, \quad e_k = s_k', \quad \nu_k = \zeta_k' + \epsilon_k', \quad \alpha_k = \alpha_k.
\]
(52) can be rewritten as
\[
y_{k+1} = y_k + \alpha_k F_k y_k + \alpha_k (e_k + \nu_k),
\]
which is in the form (99) of Lemma 8 Then we may employ (35) Theorem 3.3.1] to study the limit distribution of \(\frac{\Delta_{k+1}}{\sqrt{\alpha_{k+1}}}\). In what follows, we verify the conditions of Lemma 8 in Appendix F. By Assumption 9
\[
\alpha_{k+1}^{-1} - \alpha_k^{-1} \to 0,
\]
which implies condition (i) of Lemma 8. Note also that \(F_k = -G\) is stable, condition (ii) of Lemma 8 holds. Then we focus on condition (iii) of Lemma 8. On the one hand, we may show
that \( \nu_k = \epsilon'_k + \zeta'_k = o(\sqrt{\alpha_k}) \) almost surely. In fact, for \( \epsilon'_k \), recall the definition of \( \epsilon_k \) in (38). By Lemma \( \epsilon_k = 0 \) and then \( \epsilon'_k = (UT)^{(r)}\epsilon_k = 0 \) almost surely when \( k \) is large enough. For \( \zeta'_k \), when \( k \) is large enough

\[
\|\zeta'_k\| = \left\| -\frac{1}{m} (UT)^{(r)} P_B \left[ \nabla f(\bar{x}_k) - \nabla f(x^*) - \nabla^2 f(x^*)(\bar{x}_k - x^*) \right] \right\|
\leq \frac{1}{m} \left\| (UT)^{(r)} P_B \left\| \nabla f(\bar{x}_k) - \nabla f(x^*) - \nabla^2 f(x^*)(\bar{x}_k - x^*) \right\| \right.
\leq \frac{c_0}{m} \left\| (UT)^{(r)} P_B \right\| \|\bar{x}_k - x^*\|^{2}
= \frac{c_0}{m} \left\| (UT)^{(r)} P_B \right\| \|\Delta_k\|^{2} = \frac{c_0}{m} \left\| (UT)^{(r)} P_B \right\| o\left(\alpha_k^{2/5}\right) \text{ a.s.},
\]

where the second inequality follows from (29) in Assumption \( \alpha \) and \( \bar{x}_k \to x^* \) almost surely, the second equality follows from Lemma \( \epsilon_k \) and the last equality follows from Theorem \( \epsilon_k \). Therefore,

\[
\nu_k = \epsilon'_k + \zeta'_k = o\left(\alpha_k^{2/5}\right) \leq o(\sqrt{\alpha_k}) \text{ a.s.}
\]
as \( \delta \in [1/4, 1 - 1/(2\alpha)] \).

On the other hand, we verify (100)-(102) of Lemma \( \epsilon_k \) for the term \( \epsilon_k = s'_k \). By definition \( s'_k = (UT)^{(r)} s_k \), it is easy to verify that

\[
E \left[ s'_k | \mathcal{F}_k \right] = 0, \sup_k E \left[ \|s'_k\|^2 | \mathcal{F}_k \right] \leq \left\| (UT)^{(r)} \right\|^{2} \sup_k E \left[ \|s_k\|^2 | \mathcal{F}_k \right] \leq \|P_B\|^2 \left\| (UT)^{(r)} \right\|^{2} 4L_0^2, \quad (57)
\]

and hence (100) of Lemma \( \epsilon_k \) holds. By the definition of \( s_k \),

\[
E \left[ s_k s_k^T | \mathcal{F}_k \right] = E \left[ \left( \frac{1}{m} P_B \sum_{j=1}^{m} s_{j,k} \right) \left( \frac{1}{m} P_B \sum_{j=1}^{m} s_{j,k} \right)^T \right] \mathcal{F}_k \]

\[
= \frac{1}{m^2} P_B \left( \sum_{1 \leq i, j \leq m} E \left[ s_{j,k}(s_{j,k})^T | \mathcal{F}_k \right] \right) P_B \]

\[
= \frac{1}{m^2} P_B \left( \sum_{1 \leq i, j \leq m} E \left[ \nabla F_i(x_i; \xi_{i,k}) - \nabla f_i(x_i) \right] \left[ \nabla F_j(x_j; \xi_{j,k}) - \nabla f_j(x_j) \right]^T | \mathcal{F}_k \right) P_B \quad (58)
\]

\[
= \frac{1}{m^2} P_B \left( \sum_{j=1}^{m} E \left[ \nabla F_j(x_j; \xi_{j,k}) - \nabla f_j(x_j) \right] \left[ \nabla F_j(x_j; \xi_{j,k}) - \nabla f_j(x_j) \right]^T | \mathcal{F}_k \right) P_B \]

\[
= \frac{1}{m^2} P_B \left( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x; \xi_j)) | x = x_{j,k} \right) P_B,
\]

where the fourth equality follows from that \( \xi_{j,k} \) is independent of \( \xi_{i,k} \) for any \( i \neq j \), \( \text{Cov}(\nabla F_j(x; \xi_j)) | x = x_{j,k} \) means the value of covariance matrix \( \text{Cov}(\nabla F_j(x; \xi_j)) \) with respect to \( \xi_j \) taking at the point \( x = x_{j,k} \). Since for any \( 1 \leq j \leq m \), \( x_{j,k} \to x^* \) almost surely and the \( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x; \xi_j)) \) is
continuous at point $x^*$,

$$\lim_{k \to \infty} \mathbb{E} \left[ s_k s_k^T | \mathcal{F}_k \right] = \lim_{k \to \infty} \frac{1}{m^2} P_B \left( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x; \xi_j))|_{x=x_{j,k}} \right) P_B$$

$$= \frac{1}{m^2} P_B \left( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j)) \right) P_B \quad \text{a.s.}$$

(59)

Note that $\sup_k \mathbb{E}[\|s_k\|^2 | \mathcal{F}_k] \leq \|P_B\|^2 4L_0^2$. Then according to dominated convergence theorem,

$$\lim_{k \to \infty} \mathbb{E} \left[ s_k s_k^T \right] = \mathbb{E} \left[ \lim_{k \to \infty} \mathbb{E} \left[ s_k s_k^T | \mathcal{F}_k \right] \right] = \frac{1}{m^2} P_B \left( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j)) \right) P_B. \quad (60)$$

Moreover, by the definition of $s_k'$ and (59)-(60), we have

$$\lim_{k \to \infty} \mathbb{E} \left[ s_k'(s_k')^T | \mathcal{F}_{k-1} \right] = \lim_{k \to \infty} (U^T)^{(r)} \mathbb{E} \left[ s_k s_k^T | \mathcal{F}_{k-1} \right] (U^T)^{(r)T}$$

$$= \frac{1}{m^2} (U^T)^{(r)} P_B \left( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j)) \right) P_B (U^T)^{(r)T} \quad \text{a.s.,}$$

(59)

$$\lim_{k \to \infty} \mathbb{E} \left[ s_k'(s_k')^T \right] = \frac{1}{m^2} (U^T)^{(r)} P_B \left( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j)) \right) P_B (U^T)^{(r)T},$$

which shows (101) in Lemma 8.

By Chebyshev’s inequality and (57)

$$\mathbb{P}(\|s_k\| > N) \leq \frac{\mathbb{E}[\|s_k\|^2]}{N^2} \leq \frac{\|P_B\|^2 \|U^T\|^2}{N^2} \frac{4L_0^2}{N^2}.$$  

Furthermore, for $p > 2$ given in Assumption 1(ii) and $q > 0$ such that $2/p + 1/q = 1$,

$$\mathbb{E} \left[ \|s_k\|^2 \mathbb{1}_{\{\|s_k\| > N\}} \right] \leq \left( \mathbb{E} \left[ \|s_k\|^{(p/2)} \right] \right)^{2/p} \left( \mathbb{E} \left[ \mathbb{1}_{\{\|s_k\| > N\}} \right] \right)^{1/q}$$

$$= \left( \mathbb{E} \left[ \left( (U^T)^{(r)} P_B \frac{1}{m} \sum_{j=1}^{m} s_{j,k} \right)^p \right] \right)^{2/p} \left( \mathbb{P}(\|s_k\| > N) \right)^{1/q}$$

$$\leq \|U^T\|^2 \|P_B\|^2 \left( \frac{1}{m} \sum_{j=1}^{m} \mathbb{E} \left[ \|s_{j,k}\|^p \right] \right)^{2/p} \left( \mathbb{P}(\|s_k\| > N) \right)^{1/q}$$

$$\leq \|U^T\|^2 \|P_B\|^2 4(L_0^p)^{2/p} \left( \mathbb{P}(\|s_k\| > N) \right)^{1/q}$$

$$\leq \|U^T\|^2 \|P_B\|^2 2^{2/q} \frac{16(L_0^p)^{2/p}(L_0^q)^{1/q}}{N^{2/q}},$$

where the first inequality follows from the Hölder inequality, the second inequality follows from the convexity of $\| \cdot \|_p$ and the third inequality follows from (11). Then we have

$$\lim_{N \to \infty} \sup_k \mathbb{E} \left[ \|s_k\|^2 \mathbb{1}_{\{\|s_k\| > N\}} \right] \leq \lim_{N \to \infty} \|U^T\|^2 \|P_B\|^2 2^{2/q} \frac{16(L_0^p)^{2/p}(L_0^q)^{1/q}}{N^{2/q}} = 0, \quad (61)$$
which verifies (102) in Lemma 8. Therefore, by Lemma 8,
\[
\frac{\Delta_k''}{\sqrt{\alpha_k}} \xrightarrow{d} N(0, \Sigma_1),
\]
(62)
where \( \Sigma_1 \) is defined in (50), and \((U^T)^{(r)} \in \mathbb{R}^{r \times d}\) is composed by first \( r \) row vectors of \( U^T \).

Note that \( \Delta_k = U \left( (\Delta_k')^T 0^T \right)^T \) and \( \frac{\Delta_k}{\sqrt{\alpha_k}} \) has the same limit distribution with \( \frac{\Delta_k''}{\sqrt{\alpha_k}} \). Therefore,
\[
\frac{\Delta_k}{\sqrt{\alpha_k}} \xrightarrow{d} N(0, \Sigma),
\]
(63)
where \( \Sigma \) is defined in (49). Recall that Lemma 2 implies that \( \Delta_k = \bar{x}_k - x^* \) when \( k \) is large enough and hence
\[
\mathbb{E} \left[ \left\| \bar{x}_k - x_{j,k} \right\| \right] = O(\sqrt{\alpha_k}) \to 0, \quad \forall 1 \leq j \leq m,
\]
by Lemma 1 and (14). Therefore, an application of Slutsky’s theorem yields (48). The proof is completed. \( \square \)

Theorem 3 presents the asymptotic normality of Algorithm 1 with the rate \( \sqrt{\alpha_k} \). Note that \( \alpha_k = ak^{-m}, \alpha \in (2/3, 1) \), the convergence given by (48) implies that \( \delta \) in the convergence rate \( x_{j,k} - x^* = o(\alpha_k^2) \) cannot be improved to 1/2. Next, we employ the averaging technique introduced in [21] to derive the asymptotic efficiency of Algorithm 1.

For simplicity, we present a technical result first.

**Lemma 5.** Suppose Assumptions 4, 6-9 hold with \( p > 2 \) in Assumption 7(ii). Then
\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \left\| \Delta_t \right\|^2 \to 0 \quad \text{a.s.,}
\]
(64)
where the projected error \( \Delta_t \) is defined in (35).

The proof is presented in Appendix E.

**Theorem 4.** Suppose Assumptions 4, 6-9 hold with \( p > 2 \) in Assumption 7(ii). Let \( x^* \) be the limit point of sequence \( \{ \bar{x}_k \} \). The covariance matrix mapping \( \sum_{j=1}^{m} \text{Cov}(\nabla F_j(\cdot; \xi_j)) \) is continuous at point \( x^* \). Then for any \( 1 \leq j \leq m \),
\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} (x_{j,t} - x^*) \xrightarrow{d} N(0, \Sigma^*),
\]
(65)
where
\[
\Sigma^* = H^\dagger P_B \Sigma P_B H^\dagger, \quad \Sigma = \frac{1}{m^2} \sum_{j=1}^{m} \text{Cov}(\nabla F_j(x^*; \xi_j)),
\]
\( H^\dagger \) is the Moore-Penrose inverse of \( H \), \( P_B \) and \( H \) are defined in (33) and (34) respectively.
Proof. Lemma 2 has shown that \( \Delta_k = \bar{x}_k - x^* \) almost surely when \( k \) is large enough. Then
\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} (\bar{x}_t - x^*) \quad \text{and} \quad \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \Delta_t
\]
have the same limit distribution. Note also that, for any \( 1 \leq j \leq m \),
\[
\mathbb{E} \left\| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} (x_{j,t} - x^*) - \frac{1}{\sqrt{k}} \sum_{t=1}^{k} (\bar{x}_t - x^*) \right\| = \mathbb{E} \left\| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} (x_{j,t} - \bar{x}_t) \right\|
\]
and
\[
\mathbb{E} \left\| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} (x_{j,t} - \bar{x}_t) \right\| \leq \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \mathbb{E} \left\| x_{j,t} - \bar{x}_t \right\| \leq \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \mathbb{E} \| z_{j,t-1} - \bar{z}_{t-1} \|
\]
where the last inequality follows from (18) and \( c > 0 \) is a constant. In addition, by the Kronecker lemma and the fact \( \sum_{t=1}^{\infty} \frac{1}{\sqrt{t}} \alpha_t = \sum_{t=1}^{k} \frac{\alpha t}{t^{1/2+\alpha}} < \infty \), \( \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \alpha_t \to 0 \). Thus, it follows from (66) that, for any \( 1 \leq j \leq m \), \( \frac{1}{\sqrt{k}} \sum_{t=1}^{k} (x_{j,t} - x^*) \) and \( \frac{1}{\sqrt{k}} \sum_{t=1}^{k} (\bar{x}_t - x^*) \) have the same limit distribution. Therefore, it is sufficient to show that
\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \Delta_t \stackrel{d}{\to} N(0, \Sigma^*). \tag{67}
\]

In what follows, we employ [22, Proposition 2] to prove (67). Recall \( \Delta_{k+1} \) in (36) of Lemma 3
\[
\Delta_{k+1} = \Delta_k - \alpha_k H \Delta_k + \alpha_k \left( \zeta_k + \eta_k + s_k + \epsilon_k \right)
\]
and
\[
\Delta_{k+1} = \Delta_k - \alpha_k P_B H P_B \Delta_k + \alpha_k P_B \left( \zeta_k + \eta_k + s_k \right) + \alpha_k \epsilon_k, \tag{68}
\]
where the second equality follows from the (34) and the fact that \( \zeta_k, \eta_k, s_k \) defined in (38) are all in subspace \( \mathcal{Y} \). With a slight abuse of notation, define
\[
\zeta'_k := \zeta_k + \eta_k, \quad \epsilon'_k := \alpha_k \epsilon_k. \tag{69}
\]
Identifying \( P_B, s_k, \zeta'_k, \) and \( \epsilon'_k \) to \( P_T, \xi_k, \zeta_k, \) and \( \epsilon_k \), respectively, then (68) falls into the form [22, (34)]. We are left to verify Assumptions F and G of [22, Proposition 2].

Firstly, by the definition of \( H \) in (34)
\[
y^T H y = (P_B y)^T \left( \frac{1}{m} \nabla^2 f(x^*) \right) P_B y = y^T \left( \frac{1}{m} \nabla^2 f(x^*) \right) y \geq \frac{\mu}{m} \| y \|^2, \quad \forall y \in \mathcal{Y},
\]
where the inequality follows from Assumption 6. Secondly, \( \{ s_k, \mathcal{F}_{k+1} \} \) is a martingale difference sequence and
\[
\mathbb{E} \left[ \| s_k \|^2 | \mathcal{F}_k \right] = \mathbb{E} \left[ \left\| \frac{1}{m} \sum_{j=1}^{m} s_{j,k} \right\|^2 | \mathcal{F}_k \right] \leq \frac{1}{m} \sum_{j=1}^{m} \mathbb{E} \left[ \| s_{j,k} \|^2 | \mathcal{F}_k \right] \leq 4L_0^2. \tag{70}
\]
For validation of Assumption F, we may employ Lemma 3.3.1 to prove that

\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} s_t \overset{d}{\to} \mathcal{N}(0, \Sigma).
\]

In fact, since \(\{s_k, F_{k+1}\}\) is a martingale difference sequence satisfying (70) and (58)-(60), and also the fact

\[
\lim_{N \to \infty} \sup_k \mathbb{E} [\|s_k\|^2 1_{\{\|s_k\| > N\}}] \leq \lim_{N \to \infty} \|P_B\|^{2+2/q} \frac{16(L_0^p)^{2/p} (L_0^2)^{1/q}}{N^{2/q}} = 0,
\]

which is similar to the analysis of (61), then identifying \(s_i/\sqrt{k}\) to \(\xi_{k,i}\) in Lemma 3.3.1, we can derive the desired argument.

Next, we verify Assumption G. Recall (38) and (69), we have

\[
\zeta'_t = \zeta_k + \eta_k = -\frac{1}{m} P_B \left[ \nabla f(\bar{x}_k) - \nabla f(x^*) - \nabla^2 f(x^*)(\bar{x}_k - x^*) \right] + \frac{1}{m} \sum_{j=1}^{m} P_B (\nabla f_j(x_{j,k}) - \nabla f_j(\bar{x}_k)).
\]

Then

\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \|P_B \zeta'_t\| \leq \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \|\zeta_t\| + \frac{1}{m \sqrt{k}} \|P_B\| \sum_{t=1}^{k} \sum_{j=1}^{m} \|\nabla f_j(x_{j,t}) - \nabla f_j(\bar{x}_t)\|
\]

\[
\leq \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \|\zeta_t\| 1_{\{\|\bar{x}_t - x^*\| > \epsilon\}} + \frac{1}{m \sqrt{k}} \|P_B\| \sum_{t=1}^{k} \|\bar{x}_t - x^*\|^2 + \frac{L \|P_B\|}{m \sqrt{k}} \sum_{t=1}^{k} \sum_{j=1}^{m} \|x_{j,t} - \bar{x}_t\|
\]

\[
\leq \frac{1}{\sqrt{k}} \sum_{t=1}^{k} \|\zeta_t\| 1_{\{\|\bar{x}_t - x^*\| > \epsilon\}} + \frac{1}{m \sqrt{k}} \|P_B\| \sum_{t=1}^{k} \|\bar{x}_t - x^*\|^2 + \frac{L \|P_B\|}{m \sqrt{k}} \sum_{t=1}^{k} \sum_{j=1}^{m} \|z_{j,t-1} - \bar{z}_{t-1}\|
\]

where the second inequality follows from (29) in Assumption 6 and the Lipschitz continuity of \(\nabla f_j(\cdot)\) in Assumption 6, the third inequality follows from (14).

We need to show that all terms on the right-hand side of inequality (71) converge to 0 almost surely. Evidently, the first term on the right-hand side of inequality (71) converge to 0 almost surely as \(\bar{x}_t \to x^*\). Note that \(\Delta_t = \bar{x}_t - x^*\) when \(t\) is large enough, and hence the second term converges to 0 almost surely by Lemma 5, while the third term converges to 0 in probability by (66). Therefore,

\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \|P_B \zeta'_t\| \to 0 \quad \text{a.s.}
\]

Note also that by Lemma 2, \(\epsilon'_k = \alpha_k \epsilon_k = 0\) when \(k\) is large enough and Lemma 5 implies

\[
\frac{1}{\sqrt{k}} \sum_{t=1}^{k} \|\bar{x}_t - x^*\|^2 \to 0,
\]

hence Assumption G holds. Then an application of Proposition 2 yields (67). The proof is completed.
5 Numerical simulation

In this section, we give a numerical example to justify the theoretical analysis. We carry out simulations on the distributed parameter estimation problem [24, 39]. Over a connected network consisting of \( m \) agents, we want to estimate a real vector \( x^* \) in a distributed manner. Each agent \( j = 1, \ldots, m \) at time \( k \) has access to its real scalar measurement \( d_{j,k} \) given by the following linear time-varying model

\[
d_{j,k} = u_{j,k}^T x^* + v_{j,k},
\]

where \( u_{j,k} \in \mathbb{R}^d \) is the regression vector accessible to agent \( j \), and \( v_{j,k} \) is the observation noise of agent \( j \). Assume that \( \{u_{j,k}\} \) and \( \{v_{j,k}\} \) are mutually independent i.i.d. Gaussian sequences with distributions \( \mathcal{N}(0, R_{u,j}) \) and \( \mathcal{N}(0, \sigma_{v,j}^2) \) respectively. Then the problem can be reformulated as follows:

\[
\min_{x \in \mathbb{R}^d} f(x) = \sum_{j=1}^{m} f_j(x) \quad \text{s.t.} \quad x \in \mathcal{X}, \tag{72}
\]

where each agent’s cost function

\[
f_j(x) := \mathbb{E} [(u_{j,k}^T x - d_{j,k})^2] = (x - x^*)^T R_{u,j} (x - x^*) + \sigma_{v,j}^2.
\]

![Figure 1](a) Trajectories of \( x_{j,k} \) and \( \sum_{j=1}^{50} x_{j,k} / 50 \)

![Figure 1](b) Active set identification of \( \bar{x}_k \)

Figure 1: Convergence properties of selected agents’ estimates \( x_{j,k}, \sum_{j=1}^{50} x_{j,k} / 50 \), and \( \bar{x}_k \).

In the numerical test, we set the optimal solution \( x^* = (1, 2)^T \),

\[
\mathcal{X} := \{(x^{(1)}, x^{(2)})^T \in \mathbb{R}^2 : -2x^{(1)} + x^{(2)} \leq 0, \ x^{(1)} \leq 5, \ x^{(2)} \geq 0\}
\]

and the subspace corresponds to \( (26) \) is

\[
\mathcal{Y} = \{x : -2x^{(1)} + x^{(2)} = 0\}. \tag{73}
\]

\( R_{u,j}, j = 1, \ldots, m \) is randomly generated semi-positive definite matrix in \( \mathbb{R}^{2 \times 2} \). Moreover, the regularizer is \( \psi(x) = \frac{1}{2} \|x\|^2 \). For each implement, the step-size \( \alpha_k = 5/k^{0.67} \), the initial point is random generated in set \([0, 5] \times [0, 5]\).

In the first simulation, we set the number of agents \( m = 50 \), and the weigh matrix is generated by the broadcast gossip scheme, which is not doubly stochastic but \( 1^T \mathbb{E}(A_k) = 1^T \) [24].

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Figure 2: The histograms and limit distributions for $\frac{x_{1,k} - x^*}{\sqrt{\alpha_k}}$ and $\frac{1}{\sqrt{k}} \sum_{t=1}^{k} (x_{1,t} - x^*)$. 
To demonstrate the path-wise convergence properties of the algorithm, the trajectories with \( k \leq 1000 \) of selected agents’ estimates \( x_{j,k} \), which are picked randomly from three of 50 agents, and averaged estimate \( \sum_{j=1}^{50} x_{j,k} / 50 \) are shown in Fig. 1(a). The simulation results are consistent with Theorem 1.

To show the result of active set identification, the points of \( \bar{x}_k \) generated by Algorithm 1 (denoted by DDA) and distributed projection stochastic gradient (DPG) algorithm are plotted in phase plane respectively. It can be seen from Fig. 1(b) that the DPG algorithm fails to identify the active constraint (73), while the DDA algorithm identifies it.

In the second simulation, the weigh matrix is generated by the pairwise gossip scheme, which is doubly stochastic [23]. Algorithm 1 is run for 1000 times independently.

Fig. 2 demonstrates the asymptotic normality and asymptotic efficiency of Algorithm 1. On the one hand, Fig. 2(a) shows the histograms for each component and all component of \( x_{1,k} - x^* \sqrt{\alpha_k} \) at time \( k = 2000 \) respectively. We use the normal distribution to fit the 1000 samples for \( x_{1,k}^{(1)} - x^*(1) \sqrt{\alpha_k} \), \( x_{1,k}^{(2)} - x^*(2) \sqrt{\alpha_k} \) with \( k = 2000 \). It is shown that the data set are fitted with the normal distribution, which verifies the asymptotic normality result of Theorem 3. Moreover, the left bottom figure in Fig. 2 shows that almost all the points lies on the subspace \( Y \) defined by (73), which is consistent with active-set identification result of Lemma 2. On the other hand, Fig. 2(b) presents the histograms of averaged estimate \( \frac{1}{\sqrt{k}} \sum_{t=1}^{2000} (x_{1,t} - x^*) \). In order to eliminate the impact of non-identification points of active-set at the beginning iterations, we take the average of the last 500 of the 2000 iterations, that is, \( \frac{1}{\sqrt{500}} \sum_{t=1501}^{2000} (x_{1,t} - x^*) \). It is shown from Fig. 2(b) that the averaged estimates have small variances compared to the left counterparts, which is coherent with the asymptotic efficiency result given in Theorem 4.

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Appendix

A  Proof of Lemma

Proof. Let
\[ \hat{A}_k := A_k \otimes I_d, \quad J := \frac{1}{m} 11^T \otimes I_d, \quad J_\perp := I_{md} - J, \]
where \( \otimes \) denotes the Kronecker product. Denote
\[ Z_k := \begin{pmatrix} z_{1,k} \\ z_{2,k} \\ \vdots \\ z_{m,k} \end{pmatrix} \in \mathbb{R}^{md}, \quad \bar{Z}_k := JZ_k = \begin{pmatrix} \bar{z}_k \\ \bar{z}_k \\ \vdots \\ \bar{z}_k \end{pmatrix} \in \mathbb{R}^{md}, \quad Z_{k,\perp} := J_\perp Z_k = Z_k - \bar{Z}_k. \]
(75)

We prove the lemma by investigating the recursion of disagreement vector \( Z_{k,\perp} \). Recall the following recursion in Algorithm 1
\[ z_{j,k} = \sum_{i \in N_{j,k}} [A_k]_{ji} z_{i,k} - 1 - \alpha_k \nabla F_j(x_{j,k}; \xi_{j,k}), \]
which reduces to
\[ Z_k = \hat{A}_k Z_{k-1} - \alpha_k G_k \text{ with } G_k = (\nabla F_1(x_{1,k}; \xi_{1,k})^T, \ldots, \nabla F_m(x_{m,k}; \xi_{m,k})^T)^T, \]
(76)
by using the notation (74). Hence, we obtain the recursion for \( Z_{k,\perp} \)
\[ Z_{k,\perp} = J_\perp \hat{A}_k Z_{k-1,\perp} - \alpha_k J_\perp G_k = J_\perp \hat{A}_k Z_{k-1,\perp} - \alpha_k J_\perp G_k, \]
(77)
where the second equality follows from the fact \( J_\perp \hat{A}_k J_\perp = J_\perp \hat{A}_k \). Introducing an auxiliary matrix
\[ W_k := \hat{A}_k J^2_\perp \hat{A}_k = \left( A^T_k (I_m - \frac{11^T}{m}) A_k \right) \otimes I_d, \]
it follows from (77) that
\[ \|Z_{k,\perp}\|^2 = Z_{k-1,\perp}^T W_k Z_{k-1,\perp} + \alpha_k^2 G_k^T J_\perp G_k - 2\alpha_k G_k^T J_\perp \hat{A}_k Z_{k-1,\perp} \]
(78)
\[ = Z_{k-1,\perp}^T W_k Z_{k-1,\perp} + \alpha_k^2 G_k^T J_\perp G_k - 2\alpha_k G_k^T J_\perp \hat{A}_k Z_{k-1,\perp} \]
where the second equality follows from the fact \( J^2_\perp = J_\perp \).

Note that \( W_k \) is independent of \( F_k \) and \( G_k \) by Assumption 2 and 4. Taking conditional expectation on both side of (78) with respect to \( F_k \) and \( G_k \), we have
\[ \mathbb{E} [\|Z_{k,\perp}\|^2 | F_k, G_k] \leq \rho_k \|Z_{k-1,\perp}\|^2 + 2\alpha_k \sqrt{m} \|J_\perp\| \|Z_{k-1,\perp}\| \|G_k\| + \alpha_k^2 \|J_\perp\| \|G_k\|^2. \]
where the bound $\|\tilde{A}_k\| = ||A_k \otimes I_d|| = \|A_k\| \le \sqrt{m}$ is obtained by the row stochasticity of $A_k$. Taking expectation on both sides, we arrive at

$$
\mathbb{E} \left[ \|\tilde{Z}_{k,\perp}\|^2 \right] \\
\le \rho_k \mathbb{E} \left[ \|\tilde{Z}_{k-1,\perp}\|^2 \right] + 2\alpha_k \sqrt{m} \|J\| \mathbb{E} \left[ \|\tilde{Z}_{k-1,\perp}\| \|G_k\| \right] + \alpha_k^2 \|J\| \mathbb{E} \left[ \|G_k\|^2 \right] \\
\le \rho_k \mathbb{E} \left[ \|\tilde{Z}_{k-1,\perp}\|^2 \right] + 2\alpha_k \sqrt{m} \|J\| \mathbb{E} \left[ \|\tilde{Z}_{k-1,\perp}\| \right] \mathbb{E} \left[ \|G_k\|^2 \right] + \alpha_k^2 \|J\| m L_0^2
$$

(79)

where the second inequality follows from the Cauchy-Schwarz inequality and the last inequality follows from the fact

$$
\sqrt{\mathbb{E} \left[ \|G_k\|^2 \right]} = \sqrt{\mathbb{E} \left[ \left( \nabla F_1(x_{1,k-1}; \xi_{1,k-1})^T, \nabla F_2(x_{2,k-1}; \xi_{2,k-1})^T \ldots, \nabla F_m(x_{m,k-1}; \xi_{m,k-1})^T \right)^T \right]}
$$

$$
= \sqrt{\mathbb{E} \left[ \sum_{j=1}^m \|\nabla F_j(x_{j,k-1}; \xi_{j,k})\|^2 \right]} \le \sqrt{m L_0^2},
$$

and $L_0^2$ is defined in (6).

Define $u_k = \mathbb{E} \left[ \|\tilde{Z}_{k,\perp}\|^2 \right], M = \max \{2m \|J\| \sqrt{T_0^2}, \|J\| mL_0^2 \}$. Then (79) can be rewritten as

$$
u_k \le \rho_k u_{k-1} + M \alpha_k \sqrt{u_{k-1}} + M \alpha_k^2.
$$

(80)

We now apply [23, Lemma 3] to prove the lemma. For this, we need to validate conditions (22)-(25) of [23, Lemma 3]. First of all, the step-size $\alpha_k$ fulfills the requirement and (80) can be viewed as a special case of (22)-(23) of [23, Lemma 3] with $v_k \equiv 0$. Then, we verify the bound $\limsup_k \phi_k u_k$ for two scenarios of the lemma.

(i) Taking $\phi_k = k^{2\beta}$, by Assumption 3 (ii), we have

$$
\begin{align*}
\limsup_k \left( \alpha_k \sqrt{\phi_k} + \frac{\phi_{k-1}}{\phi_k} \right) &= \limsup_k \left( \alpha_k k^{\beta} + \frac{1}{k^{\beta}} \right) = 1 < \infty \\
\liminf_k \left( \alpha_k \sqrt{\phi_k} \right)^{-1} \left( \frac{\phi_{k-1}}{\phi_k} - \rho_k \right) &= \liminf_k \left( \frac{1}{k^{\beta}} - \frac{\rho_k}{\alpha_k k^{\beta}} \right) = 0
\end{align*}
$$

(81)

hence all conditions of [23, Lemma 3] are satisfied, we obtain (15).

(ii) Noticing $\|\tilde{Z}_{k,\perp}\|^2 = \sum_{j=1}^m \|\tilde{z}_k - z_{j,k}\|^2$ and (15), there exists a constant $d$ such that for any $j \in V$

$$
\sum_{k=1}^\infty \mathbb{E} \left[ \|\tilde{z}_k - z_{j,k}\|^2 \right] \le d^2 \sum_{k=1}^\infty k^{-2\beta} < \infty.
$$

By the monotone convergence theorem, we have

$$
\sum_{k=0}^\infty \|\tilde{z}_k - z_{j,k}\|^2 < \infty \quad \text{a.s.}
$$
Similarly,
\[
\sum_{k=1}^{\infty} \gamma_k \mathbb{E}[\|z_k - z_{j,k}\|] \leq \sum_{k=1}^{\infty} \gamma_k \sqrt{\mathbb{E}[\|z_k - z_{j,k}\|^2]} \leq d \sum_{k=1}^{\infty} \gamma_k k^{-\beta} < \infty,
\]
and hence we obtain that for any \(j \in V\)
\[
\sum_{k=1}^{\infty} \gamma_k \|z_k - z_{j,k}\| < \infty \quad \text{a.s.}
\]

(iii) If \(A_k, k = 1, 2, \ldots\) satisfies (a) and the step-size satisfies (b), then by taking \(\phi_k = \alpha_k^{-2}\), we can also show in a similar way to (81) that all conditions of [23, Lemma 3] are satisfied, and hence (18) holds. \(\square\)

B Proof of Lemma 2

Proof. By the iteration (27)
\[
\bar{x}_{k+1} = \arg\min_{x \in \{Bx \leq b, Cx \leq c\}} \left\{ \langle \nabla f(x^*), x \rangle + \langle v_k, x \rangle + \frac{m}{2\alpha_k} \|x\|^2 \right\}, \tag{82}
\]
where
\[
v_k = -\frac{m\bar{z}_k}{\alpha_k} - \nabla f(x^*), \quad \tilde{\alpha}_k := \sum_{t=1}^{k} \alpha_t.
\]
According to the Karush-Kuhn-Tucker (KKT) conditions of problem (82), there exist \(\lambda_k, \mu_k \geq 0\) such that
\[
\nabla f(x^*) + v_k + \frac{m\bar{x}_{k+1}}{\alpha_k} + B^T \lambda_k + C^T \mu_k = 0.
\]
Note that \(\bar{x}_k \to x^*\) almost surely and \(\tilde{\alpha}_k := \sum_{t=1}^{k} \alpha_t \to \infty\), which implies
\[
\frac{m\bar{x}_{k+1}}{\alpha_k} \to 0 \quad \text{a.s.}
\]
If \(v_k \to 0\) almost surely, it is easy to show in a similar way to [22, part 12.1] that
\[
B\bar{x}_k = b, \ C\bar{x}_k < c \quad \text{a.s.}
\]
when \(k\) is large enough.

Next, we show \(v_k \to 0\) almost surely. For convenience of notation, we denote
\[
\nabla f^* := \begin{pmatrix} \nabla f(x^*) \\ \nabla f(x^*) \\ \ldots \\ \nabla f(x^*) \end{pmatrix}, \quad \nabla f_{\alpha g}^k := \begin{pmatrix} \nabla f_1(x^*) \\ \nabla f_2(x^*) \\ \ldots \\ \nabla f_m(x^*) \end{pmatrix}, \quad \nabla f_t := \begin{pmatrix} \nabla f_1(x_{1,t}) \\ \nabla f_2(x_{2,t}) \\ \ldots \\ \nabla f_m(x_{m,t}) \end{pmatrix}, \quad S_t := \begin{pmatrix} s_{1,t} \\ s_{2,t} \\ \ldots \\ s_{m,t} \end{pmatrix}.
\]
Recall the definition of \(\bar{Z}_k\) in (75),
\[
\|v_k\|^2 = \left\| -\frac{m\bar{z}_k}{\tilde{\alpha}_k} - \nabla f(x^*) \right\|^2 = \frac{1}{m} \left\| -\frac{m\bar{z}_k}{\tilde{\alpha}_k} - \nabla f^* \right\|^2.
\]
Then it is sufficient to show \( \frac{m \tilde{Z}_k}{\alpha_k} - \nabla f^* \) converges to 0 almost surely. Recall the definitions of (75) and (76). Note also that

\[
\tilde{Z}_k = JZ_k = J \left( \hat{A}_k Z_{k-1} - \alpha_k G_k \right) = J \hat{A}_k Z_{k-1} - \alpha_k J G_k = J Z_{k-1} - \alpha_k J G_k \ldots = \tilde{Z}_0 - \sum_{t=1}^k \alpha_t J G_t,
\]

where the third equality follows from the fact that \( \hat{A}_k \) is doubly stochastic. Without loss of generality, we set \( \tilde{Z}_0 = 0 \). Then by the fact \( G_t = \nabla f_t + S_t \)

\[
\tilde{Z}_k = -\sum_{t=1}^k \alpha_t J G_t = -\sum_{t=1}^k \alpha_t J \nabla f_t - \sum_{t=1}^k \alpha_t J S_t.
\]

Thus

\[
\left\| \frac{m \tilde{Z}_k}{\alpha_k} - \nabla f^* \right\|^2 \leq 2 \left\| \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} m J \nabla f_t - \nabla f^* \right\|^2 + 2 \left\| \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} m J S_t \right\|^2,
\]

where the inequality due to the fact \( \|a + b\|^2 \leq 2\|a\|^2 + 2\|b\|^2 \). We left to show that the two terms on the right-hand side of above inequality converge to 0.

Note that

\[
\left\| \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} m J \nabla f_t - \nabla f^* \right\|^2 = \left\| \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} m J \left( \nabla f_t - \nabla f^*_{ag} \right) \right\|^2 \leq m^2 \|J\|^2 \left\| \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} \left( \nabla f_t - \nabla f^*_{ag} \right) \right\|^2 \leq m^2 \|J\|^2 \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} \| \nabla f_t - \nabla f^*_{ag} \|^2 \leq m^2 \|J\|^2 \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} \sum_{j=1}^m \| \nabla f_j(x_{j,t}) - \nabla f_j(x^*) \|^2 \leq m^2 \|J\|^2 L \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} \sum_{j=1}^m \|x_{j,t} - x^*\|^2,
\]

where the first equality follows from the fact \( \nabla f^* = m J \nabla f^*_{ag} \), the second inequality follows from the convexity of \( \| \cdot \|^2 \) and the fact \( \sum_{t=1}^k \frac{\alpha_t}{\alpha_k} = 1 \), the third inequality follows from the Lipschitz
continuity of $\nabla f_j(\cdot)$. Moreover,

$$
\| \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} mJ \nabla f_t - \nabla f^* \|_2^2 \leq m^2 \| J \|_2^2 \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} \sum_{j=1}^{m} \| x_{j,t} - x^* \|_2^2
$$

$$
\leq 2m^2 \| J \|_2^2 \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} \sum_{j=1}^{m} \| x_{j,t} - \bar{x}_t \|_2^2 + 2m^3 \| J \|_2^2 \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} \| x^* - x^* \|_2^2
$$

$$
\leq \frac{2m^2 \| J \|_2^2}{\sigma} \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} \sum_{j=1}^{m} \| z_{j,t-1} - \bar{z}_{t-1} \|_2^2 + 2m^3 \| J \|_2^2 \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} \| x^* - x^* \|_2^2
$$

$$
\leq \frac{2m^2 \| J \|_2^2}{\sigma} \frac{1}{\alpha_k} \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} \sum_{j=1}^{m} \| z_{j,t-1} - \bar{z}_{t-1} \|_2^2 + 2m^3 \| J \|_2^2 \frac{1}{\alpha_k} \sum_{t=1}^{k} \alpha_k \| x^* - x^* \|_2^2,
$$

where the third inequality follows from (14). By (16) in Lemma 1

$$
\sum_{t=1}^{\infty} \alpha_t \sum_{j=1}^{m} \| z_{j,t-1} - \bar{z}_{t-1} \|_2^2 \leq \alpha_1 \sum_{t=1}^{\infty} \sum_{j=1}^{m} \| z_{j,t-1} - \bar{z}_{t-1} \|_2^2 < \infty, \quad \text{a.s.}
$$

On the other hand, by [22, Lemma 9.5] and (24),

$$
\sum_{k=1}^{\infty} \alpha_k \| x^* - x^* \|_2^2 \leq \frac{1}{c} \sum_{k=1}^{\infty} \alpha_k (f(\bar{x}_k) - f(x^*)) < \infty, \quad \text{a.s.}
$$

where $c$ is a random positive constant that depends on the bound $M := \sup_x \| \bar{x} - x^* \| \vee 1 < \infty$. Therefore, we can argument that $\| \sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} mJ \nabla f_t - \nabla f^* \|_2^2$ converges to 0 almost surely as $\alpha_k \to \infty$.

Next, we show $\sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} mJS_t$ converges to 0 almost surely. For this purpose, by the Kronecker lemma, it is sufficient to show that

$$
\sum_{t=1}^{\infty} \frac{\alpha_t}{\alpha_k} mJS_t < \infty, \quad \text{a.s.}
$$

Note that $\{\sum_{t=1}^{k} \alpha_t mJS_t, F_{k+1}\}$ is a martingale sequence as $\{S_k, F_{k+1}\}$ is a martingale difference sequence. Moreover,

$$
\sum_{t=1}^{\infty} \frac{1}{\alpha_t} \mathbb{E} \left[ \alpha_t mJS_t \right] \leq \sum_{t=1}^{\infty} \frac{1}{\alpha_t^2} \alpha_t^2 m^2 \| J \|_2^2 \mathbb{E} \left[ \sum_{j=1}^{m} \| s_{j,t} \|_2^2 \right] F_t \right] \leq \sum_{t=1}^{\infty} \frac{1}{\alpha_t^2} 4L_0^2 m^3 \| J \|_2^2 \alpha_t^2 < \infty,
$$

where the second inequality follows from (11). Then the convergence theorem for martingale difference sequences [35, Appendix B.6, Theorem B.6.1] implies $\sum_{t=1}^{k} \frac{\alpha_t}{\alpha_k} mJS_t$ converges almost surely. This proof is completed. \hfill $\square$

## C Proof of Lemma 3

Proof. By the definition (27), $\bar{x}_{k+1}$ satisfies the following KKT condition

$$
\bar{x}_{k+1} - \bar{z}_k + B^T \lambda_k + C^T \mu_k = 0,
$$

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where \( \lambda_k \geq 0 \) and \( \mu_k \geq 0 \) are the corresponding Lagrange multipliers. Then

\[
\bar{x}_{k+1} - x^* = \bar{x}_k - x^* + (\bar{z}_k - \bar{z}_{k-1}) + B^T(\lambda_{k-1} - \lambda_k) + C^T(\mu_{k-1} - \mu_k). \tag{84}
\]

By the definition \( \bar{z}_k \) in [27],

\[
\bar{z}_k = \frac{1}{m} \sum_{j=1}^{m} z_{j,k} = \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{i=1}^{m} [A_k]_{ji} z_{i,k-1} - \alpha_k \nabla F_j(x_{j,k}; \xi_{j,k}) \right) \\
= \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{m} [A_k]_{ji} z_{i,k-1} - \alpha_k \sum_{j=1}^{m} \nabla F_j(x_{j,k}; \xi_{j,k}) \\
= \bar{z}_{k-1} - \frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla F_j(x_{j,k}; \xi_{j,k}),
\]

where the fourth equality follows from that \( A_k \) is doubly stochastic matrix. Then

\[
\bar{z}_k - \bar{z}_{k-1} = -\frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla F_j(x_{j,k}; \xi_{j,k}) \\
= -\frac{\alpha_k}{m} \sum_{j=1}^{m} \nabla f_j(x_{j,k}) - \frac{\alpha_k}{m} \sum_{j=1}^{m} s_{j,k} \\
= \frac{\alpha_k}{m} \sum_{j=1}^{m} [\nabla f_j(\bar{x}_k) - \nabla f_j(x_{j,k})] - \frac{\alpha_k}{m} \left[ \nabla f(\bar{x}_k) - \nabla f(x^*) \right] \\
- \nabla^2 f(x^*)(\bar{x}_k - x^*) - \frac{\alpha_k}{m} \sum_{j=1}^{m} [\nabla f(x^*) + \nabla^2 f(x^*)(\bar{x}_k - x^*)] - \frac{\alpha_k}{m} \sum_{j=1}^{m} s_{j,k}.
\]

By left multiplying \( P_B \) on both side of formula above,

\[
P_B(\bar{z}_k - \bar{z}_{k-1}) = \frac{\alpha_k}{m} \sum_{j=1}^{m} P_B [\nabla f_j(\bar{x}_k) - \nabla f_j(x_{j,k})] - \frac{\alpha_k}{m} P_B [\nabla f(\bar{x}_k) - \nabla f(x^*)] \\
- \nabla^2 f(x^*)(\bar{x}_k - x^*) - \frac{\alpha_k}{m} P_B \nabla^2 f(x^*)(\bar{x}_k - x^*) - \frac{\alpha_k}{m} \sum_{j=1}^{m} P_B s_{j,k},
\]

where the equality follows from the fact \( P_B \nabla f(x^*) = 0 \). By the definition of \( \Delta_k \) in (35) and the fact \( P_B B^T = 0 \), we have by left multiplying \( P_B \) on both side of (84) that

\[
\Delta_{k+1} = \Delta_k + P_B(\bar{z}_k - \bar{z}_{k-1}) + P_B C^T(\mu_{k-1} - \mu_k) \\
= \Delta_k - \frac{\alpha_k}{m} P_B \nabla^2 f(x^*) P_B(\bar{x}_k - x^*) - \frac{\alpha_k}{m} P_B [\nabla f(\bar{x}_k) - \nabla f(x^*)] \\
- \nabla^2 f(x^*)(\bar{x}_k - x^*) + \frac{\alpha_k}{m} \sum_{j=1}^{m} P_B [\nabla f_j(\bar{x}_k) - \nabla f_j(x_{j,k})] \\
+ \frac{\alpha_k}{m} P_B \nabla^2 f(x^*)(P_B - I_d)(\bar{x}_k - x^*) + P_B C^T(\mu_{k-1} - \mu_k) - \frac{\alpha_k}{m} \sum_{j=1}^{m} P_B s_{j,k} \\
= \Delta_k - \alpha_k H \Delta_k + \alpha_k (\xi_k + \eta_k + \epsilon_k + s_k),
\]

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where $H$ is defined in (34) and $\zeta_k, \eta_k, \epsilon_k, s_k$ are defined in (38). Obviously, formula above can be rewritten as
\[
\triangle_{k+1} = (I_d - \alpha_k (H + D_k)) \triangle_k + \alpha_k (\eta_k + s_k + \epsilon_k),
\]
where $D_k = -\zeta_k \frac{\triangle_k}{\parallel \triangle_k \parallel^2}$. The proof is completed.

\section{Proof of Lemma 4}

\textit{Proof.} Part (i) is the well known result in linear algebra and we only prove part (ii).

By definition
\[
U^T P_B U = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix},
\]
where $0_1 \in \mathbb{R}^{r \times (d-r)}$, $0_2 \in \mathbb{R}^{(d-r) \times r}$, $0_3 \in \mathbb{R}^{(d-r) \times (d-r)}$. Then for any $y \in \mathcal{Y}$, we have
\[
U^T P_B H y = U^T P_B H P_B y = U^T P_B (U U^T) H (U U^T) P_B (U U^T) y = (U^T P_B U) (U^T H U) (U^T P_B U)^T y = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix} U^T H U \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}^T y.
\]
Let $U^T H U = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix}$. Then,
\[
U^T P_B H y = \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix} \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{pmatrix} I_r & 0_1 \\ 0_2 & 0_3 \end{pmatrix}^T y = \begin{pmatrix} G_1 & 0_1 \\ G_2 & 0_3 \end{pmatrix} U^T y = \begin{pmatrix} G_1 y_1 \\ 0 \end{pmatrix},
\]
where $y_1 \in \mathbb{R}^r$ determined by $U^T y = (y_1^T, 0^T)^T$, which means equality (39) holds.

For any nonzero vector $y_1 \in \mathbb{R}^r$, let $y := U (y_1^T, 0^T)^T$. By the definition of matrix $U$, we have that $y$ is a nonzero vector and $y \in \mathcal{Y}$. Then
\[
y_1^T G_1 y_1 = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}^T \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \begin{pmatrix} y_1 \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ 0 \end{pmatrix}^T U^T H U \begin{pmatrix} y_1 \\ 0 \end{pmatrix} = (U^T y)^T U^T H U (U^T y) = y^T (U U^T) H (U U^T) y = y^T H y \geq \mu \|y\|^2 > 0.
\]
Therefore $G_1$ is positive definite. The proof is completed. \qed
E Proof of Lemma 5

Proof. Note that when $\tilde{x}_k, \tilde{x}_{k-1} \in \{Bx = b, Cx < c\}$, $\tilde{x}_k - \tilde{x}_{k-1}$ can be expressed as

$$\tilde{x}_k - \tilde{x}_{k-1} = P_B[\tilde{x}_k - \tilde{x}_{k-1}] = P_B[(\tilde{z}_{k-2} - \tilde{z}_{k-1}) + B^T(\lambda_{k-2} - \lambda_{k-1})] = P_B(\tilde{z}_{k-2} - \tilde{z}_{k-1}),$$

and hence we obtain the recursion

$$\tilde{x}_k = \tilde{x}_{k-1} + P_B(\tilde{z}_{k-2} - \tilde{z}_{k-1}).$$  \hspace{1cm} (87)

For $\epsilon > 0$ specified in Assumption 6(ii), define the event

$$\mathcal{Y}_{l,k} = \{\|\Delta_j\| \leq \epsilon, B\tilde{x}_j = b, C\tilde{x}_j < c, l \leq j \leq k\},$$

then $\mathcal{Y}_{l,k} \in \mathcal{F}_k$. Define $V_{l,k} = \|\Delta_k\|1_{\mathcal{Y}_{l,k}}$ and note that $1_{\mathcal{Y}_{l,k}} \leq 1_{\mathcal{Y}_{l,k-1}}$, it follows from (87) that

$$V_{l,k}^2 \leq \|\Delta_k\|^21_{\mathcal{Y}_{l,k}} \leq \|\Delta_k\|^21_{\mathcal{Y}_{l,k-1}} = \|\Delta_{k-1} + P_B(\tilde{z}_{k-2} - \tilde{z}_{k-1})\|^21_{\mathcal{Y}_{l,k-1}}$$

$$\leq V_{l,k-1}^2 + 2\|\Delta_{k-1}\|1_{\mathcal{Y}_{l,k-1}}P_B(\tilde{z}_{k-2} - \tilde{z}_{k-1}) + \|\tilde{z}_{k-2} - \tilde{z}_{k-1}\|^2,$$  \hspace{1cm} (88)

where the non-expansiveness property of $P_B$ is involved in the last inequality of (88).

Taking the conditional expectation,

$$\mathbb{E}[V_{l,k}^2|\mathcal{F}_{k-1}] \leq V_{l,k-1}^2 + 2\mathbb{E}[(P_B^2\Delta_{k-1}1_{\mathcal{Y}_{l,k-1}}, \tilde{z}_{k-2} - \tilde{z}_{k-1})|\mathcal{F}_{k-1}] + \mathbb{E}[\|\tilde{z}_{k-2} - \tilde{z}_{k-1}\|^2|\mathcal{F}_{k-1}]
$$

$$= V_{l,k-1}^2 + 2\mathbb{E}[(\Delta_{k-1}1_{\mathcal{Y}_{l,k-1}}, \tilde{z}_{k-2} - \tilde{z}_{k-1})|\mathcal{F}_{k-1}] + \mathbb{E}[\|\tilde{z}_{k-2} - \tilde{z}_{k-1}\|^2|\mathcal{F}_{k-1}],$$  \hspace{1cm} (89)

where the equality follows from the fact $P_B\Delta_{k-1} = \Delta_{k-1}$ due to $\Delta_{k-1} \in \{x : Bx = 0\}$. Next we analyse the last two terms on the right-hand side of the equality of (89).

For the third term, by (85) and Assumption 7(ii), we have

$$\mathbb{E} \left[\|\tilde{z}_{k-2} - \tilde{z}_{k-1}\|^2|\mathcal{F}_{k-1}\right] = \mathbb{E} \left[\left\|\frac{\alpha_{k-1}}{m} \sum_{j=1}^{m} \nabla F_j(x_{j,k-1}; \xi_{j,k-1})\right\|^2|\mathcal{F}_{k-1}\right]\leq \frac{\alpha_{k-1}^2}{m} \sum_{j=1}^{m} \mathbb{E} \left[\|\nabla F_j(x_{j,k-1}; \xi_{j,k-1})\|^2|\mathcal{F}_{k-1}\right]\leq L_{0}^2\alpha_{k-1}^2,$$

where $L_{0}^2$ is defined in (6).

For the second term, substituting the following expression

$$\tilde{z}_{k-2} - \tilde{z}_{k-1} = -\frac{\alpha_{k-1}}{m} \sum_{j=1}^{m} \nabla f_j(\tilde{x}_{k-1}; \xi_{j,k-1})$$

$$= -\frac{\alpha_{k-1}}{m} \nabla f(\tilde{x}_{k-1}) - \frac{\alpha_{k-1}}{m} \sum_{j=1}^{m} \left[\nabla f_j(x_{j,k-1}) - \nabla f_j(\tilde{x}_{k-1})\right] - \frac{\alpha_{k-1}}{m} \sum_{j=1}^{m} s_{j,k-1},$$

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Taking iterations down to \( \epsilon \) for some constant \( k \) where the last inequality follows from the fact \( \exp(-\Delta) \) by the restricted strongly convex property (31), we find that
\[
E[(\Delta_{k-1})^2 | F_{k-1}] 
\leq E\left[\left( \Delta_{k-1} 1_{\mathbb{I}_{k-1}}, -\frac{\alpha_{k-1}}{m} \nabla f(\bar{x}_{k-1}) + \frac{\alpha_{k-1}}{m} \sum_{j=1}^{m} \nabla f_j(x_{j,k-1}) - \nabla f_j(\bar{x}_{k-1}) \right)^2 \right] | F_{k-1} \]
\leq \frac{\alpha_{k-1}}{m} (f(x^*) - f(\bar{x}_{k-1})) 1_{\mathbb{I}_{k-1}} + \frac{\alpha_{k-1}}{m} \sum_{j=1}^{m} \|\nabla f_j(x_{j,k-1}) - \nabla f_j(\bar{x}_{k-1})\| \tag{91}
\leq \frac{\alpha_{k-1}}{m} (f(x^*) - f(\bar{x}_{k-1})) 1_{\mathbb{I}_{k-1}} + \frac{\alpha_{k-1}}{m} L \sum_{j=1}^{m} \|\bar{z}_{k-1} - z_{j,k-1}\|.
\]
Substituting (90) and (91) into (89), it follows that
\[
E[V_{l,k}^2 | F_{k-1}] \leq V_{l,k-1}^2 + \frac{2\alpha_{k-1}}{m} (f(x^*) - f(\bar{x}_{k-1})) 1_{\mathbb{I}_{k-1}} + \frac{2\epsilon L \alpha_{k-1}}{m} \sum_{j=1}^{m} \|\bar{z}_{k-1} - z_{j,k-1}\| + L_0^2 \alpha_{k-1}^2.
\]
By the restricted strongly convex property (31), we find that
\[
E[V_{l,k}^2 | F_{k-1}] \leq \left(1 - \frac{2\epsilon L \alpha_{k-1}}{m}\right) V_{l,k-1}^2 + \frac{2\epsilon L \alpha_{k-1}}{m} \sum_{j=1}^{m} \|\bar{z}_{k-1} - z_{j,k-1}\| + L_0^2 \alpha_{k-1}^2
\]
for some constant \( \epsilon' > 0 \). Taking expectation on both sides of the above inequality yields
\[
E[V_{l,k}^2] \leq \left(1 - \frac{2\epsilon' \alpha_{k-1}}{m}\right) E[V_{l,k-1}^2] + \frac{2\epsilon L \alpha_{k-1}}{m} \sum_{j=1}^{m} E[\|\bar{z}_{k-1} - z_{j,k-1}\|] + L_0^2 \alpha_{k-1}^2
\]
\[
\leq \exp\left(- \frac{2\epsilon' \alpha_{k-1}}{m}\right) E[V_{l,k-1}^2] + (2\epsilon L D + L_0^2) \alpha_{k-1}^2,
\]
where the last inequality follows from the fact \( \exp(-x) \geq (1 - x), x \in (0, 1) \) and \( D \) is a constant specified in Lemma (iii) such that
\[
\sup_k \alpha_k^{-1} E[\|\bar{z}_k - z_{j,k}\|] \leq \sup_k \sqrt{\alpha_k^{-2} E[\|\bar{z}_k - z_{j,k}\|^2]} \leq D < \infty. \tag{92}
\]
Taking iterations down to \( l = \lfloor k/2 \rfloor \) in such way, here \( \lfloor x \rfloor \) denotes the integer part of \( x \), we obtain
\[
E[V_{l,k}^2] \leq \exp\left(- \frac{2\epsilon' \sum_{t=\lfloor k/2 \rfloor}^{k-1} \alpha_t}{m}\right) E[\|\Delta_{\lfloor k/2 \rfloor}\|^2] + (2\epsilon L D + L_0^2) \sum_{t=\lfloor k/2 \rfloor}^{k-1} \alpha_t^2 \exp\left(- \frac{\sum_{t=t+1}^{k-1} \alpha_t}{m}\right). \tag{93}
\]
In what follows, we prove that \( \sup_k \mathbb{E}[\|\Delta_k\|^2] < \infty \). In fact, taking expectation on both sides
of (23) and noting that the regularizer $\psi(x) = \frac{1}{2} \|x\|^2$ is 1-strong convex, we find that

$$
\mathbb{E}[R_k] \leq \mathbb{E}[R_{k-1}] - \frac{\alpha_k}{m} \mathbb{E}[(f(\bar{x}_k) - f(x^*))] + \frac{2L_0\alpha_k}{m} \sum_{j=1}^{m} \mathbb{E}[\|z_{j,k-1} - \bar{z}_{k-1}\|] \\
+ \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}[\|\bar{z}_{k-1} - z_{j,k-1}\|^2] + L_0^2\alpha_k^2 \\
\leq \mathbb{E}[R_{k-1}] - \frac{\alpha_k}{m} \mathbb{E}[(f(\bar{x}_k) - f(x^*))] + \frac{2L_0\alpha_k}{m} \sum_{j=1}^{m} \sqrt{\mathbb{E}[\|z_{j,k-1} - \bar{z}_{k-1}\|^2]} \\
+ \frac{1}{m} \sum_{j=1}^{m} \mathbb{E}[\|\bar{z}_{k-1} - z_{j,k-1}\|^2] + L_0^2\alpha_k^2 \\
\leq \mathbb{E}[R_{k-1}] - \frac{\alpha_k}{m} \mathbb{E}[(f(\bar{x}_k) - f(x^*))] + 2L_0\sqrt{D}\alpha_k\alpha_{k-1} + D^2\alpha_k^2 + L_0^2\alpha_k^2 \\
\leq \mathbb{E}[R_{k-1}] - \frac{\alpha_k}{m} \mathbb{E}[(f(\bar{x}_k) - f(x^*))] + \left(2L_0\sqrt{D} + D^2 + L_0^2\right)\alpha_k^2,
$$

where the second inequality follows from the Cauchy-Schwarz inequality, the third inequality from (92), and the last inequality from $\alpha_k$ being nonincreasing. Summing the above inequality from 1 to $k$ yields

$$
\mathbb{E}[R_k] \leq \mathbb{E}[R_0] - \frac{\alpha_k}{m} \sum_{t=1}^{k} \mathbb{E}[(f(\bar{x}_t) - f(x^*))] + \left(2L_0\sqrt{D} + D^2 + L_0^2\right)\sum_{t=1}^{k-1} \alpha_t^2,
$$

which implies $\sup_k \mathbb{E}[R_k] < \infty$. Therefore, we have

$$
\sup_k \mathbb{E}[\|\Delta_k\|^2] \leq \|P_B\|^2 \sup_k \mathbb{E}[\|\bar{x}_k - x^*\|^2] \leq 2\|P_B\|^2 \sup_k \mathbb{E}[R_k] < \infty,
$$

where the second inequality follows from (25). Denote $D_1 = \sup_k \mathbb{E}[\|\Delta_k\|^2]$. Then by (93) and the fact that there has a constant $D_2$ such that $\sum_{t=[k/2]}^{k-1} \alpha_t > D_2k^{1-\alpha}$, we have

$$
\mathbb{E}[\|\Delta_k\|^21_{[k/2],k}] \leq D_1 \exp(-D_2k^{1-\alpha}) \\
+ \left(2\epsilon DL + L_0^2\right) \sum_{t=[k/2]}^{k-1} \alpha_t^2 \exp(\epsilon DL + L_0^2), \tag{94}
$$

By Theorem 2 and Lemma 2 for any given $a > 0$,

$$
\mathbb{P}\left\{ \sup_{2k_0 \leq t < \infty} \|\Delta_t\| < \epsilon, \ K < k_0 \right\} > 1 - a, \tag{95}
$$

if $k_0$ is sufficiently large, where $K$ is a finite random integer specified in Lemma 2. Summing (94) from $2k_0$ to $k$ yields

$$
\sum_{t=2k_0}^{k} \frac{1}{\sqrt{t}} \mathbb{E}[\|\Delta_t\|^21_{[t/2],t}] \leq D_1 \sum_{t=2k_0}^{k} \frac{1}{\sqrt{t}} \exp(-D_2\epsilon^{1-\alpha}) + \left(2\epsilon DL + L_0^2\right) \sum_{t=2k_0}^{k} \frac{1}{\sqrt{t}} \log t \frac{1}{t^{a+1/2}}.
$$
which follows from \cite[Lemma 15.5, Part 15]{22}. Let $k \to \infty$, we have
\[
\sum_{t=2k_0}^{\infty} \mathbb{E} \left[ \frac{1}{\sqrt{t}} \| \Delta_t \|^2 1_{[t/2,t]} \right] < \infty,
\]
and by the monotone convergence theorem,
\[
\sum_{t=2k_0}^{\infty} \frac{1}{\sqrt{t}} \| \Delta_t \|^2 1_{[t/2,t]} < \infty \text{ a.s.} \tag{96}
\]
which means that
\[
\left\{ \sup_{t \geq 2k_0} \| \Delta_t \| < \epsilon, K < k_0 \right\} \subset \left\{ \sup_{t \geq 2k_0} \| \Delta_t \| < \epsilon, B \bar{x}_t = b, C \bar{x}_t < c, \forall t \geq 2k_0 \right\}
\subset \left\{ \sum_{t=2k_0}^{\infty} \frac{1}{\sqrt{t}} \| \Delta_t \|^2 < \infty \right\} \tag{97}
\]
Combining (95) with (97) shows that
\[
\mathbb{P} \left\{ \sum_{t=2k_0}^{\infty} \frac{1}{\sqrt{t}} \| \Delta_t \|^2 < \infty \right\} > 1 - a,
\]
or equivalently
\[
\mathbb{P} \left\{ \sum_{t=1}^{\infty} \frac{1}{\sqrt{t}} \| \Delta_t \|^2 < \infty \right\} > 1 - a.
\]
This verifies
\[
\sum_{t=1}^{\infty} \frac{1}{\sqrt{t}} \| \Delta_t \|^2 < \infty \text{ a.s.}
\]
because $a > 0$ can be arbitrarily small. Finally, an application of the Kronecker lemma implies (64). This complete the proof.

\section*{F Results on stochastic approximation}

For ease of reading, we recall some results on stochastic approximation from \cite{40} and \cite{35}.

\textbf{Lemma 6.} \cite{40} Let $\{F_k\}$ be an nondecreasing sequence of $\sigma$-algebra and $\{v_k\}, \{a_k\}, \{b_k\}, \text{ and } \{\phi_k\}$ be the four nonnegative sequence adopted to $F_k$. Assume that for all $k$,
\[
\mathbb{E}[v_{k+1}|F_k] \leq (1 + a_k)v_k + b_k - \phi_k.
\]
If $\sum_{k=1}^{\infty} a_k < \infty$ and $\sum_{k=1}^{\infty} b_k < \infty$ almost surely. Then $\{v_k\}$ converges to a finite random variable $v_\infty$ and $\sum_{k=1}^{\infty} \phi_k < \infty$ almost surely.
Lemma 7. [35, Lemma 3.1.1] Suppose \( d \times d \)-dimension matrix \( F_k \rightarrow F \), \( F \) is a stable matrix, that is, every eigenvalue of \( F \) has strictly negative real part. If step-size \( \alpha_k \) satisfies
\[
\alpha_k > 0, \alpha_k \xrightarrow{k \to \infty} 0, \sum_{k=1}^{\infty} \alpha_k = \infty,
\]
and \( d \)-dimension vectors \( \{e_k\}, \{v_k\} \) satisfy the following conditions
\[
\sum_{k=1}^{\infty} \alpha_k e_k < \infty, \ v_k \to 0,
\]
then \( \{y_k\} \) defined by the following recursion with arbitrary initial value \( x_0 \) tends to zero:
\[
y_{k+1} = y_k + \alpha_k F_k y_k + \alpha_k (e_k + v_k).
\]

Lemma 8. [35, Theorem 3.3.1] Let \( \{y_k\} \) be given by (99) with an arbitrarily given initial value. Assume the following conditions holds:

(i) \( \alpha_k > 0, \alpha_k \to 0 \) as \( k \to \infty \), \( \sum_{k=1}^{\infty} \alpha_k = \infty \), and
\[
\alpha_k - 1 \to a \geq 0 \text{ as } k \to \infty;
\]

(ii) \( F_k \to F \) and \( F + \frac{a}{2} I \) is stable;

(iii)
\[
\nu_k = o(\sqrt{\alpha_k}), \quad e_k = \sum_{t=0}^{\infty} C_t s_{k-t}, s_t = 0 \text{ for } t < 0,
\]
where \( C_t \) are \( d \times d \) constant matrices with \( \sum_{t=0}^{\infty} ||C_t|| < \infty \) and \( \{s_k, F_k\} \) is a martingale difference sequence of \( d \)-dimension satisfying the following conditions
\[
\mathbb{E}[s_k|F_{k-1}] = 0, \ \sup_k \mathbb{E}[||s_k||^2|F_{k-1}] \leq \sigma \text{ with } \sigma \text{ being a constant},
\]
\[
\lim_{k \to \infty} \mathbb{E}[s_k^T s_k|F_{k-1}] = \lim_{k \to \infty} \mathbb{E}[s_k^T s_k] := S_0
\]
and
\[
\lim_{N \to \infty} \sup_k \mathbb{E}[||s_k||^21_{\{||s_k|| > N\}}] = 0.
\]

Then \( \frac{y_k}{\sqrt{\alpha_k}} \) is asymptotically normal:
\[
\frac{y_k}{\sqrt{\alpha_k}} \xrightarrow{k \to \infty} N(0, S),
\]
where
\[
S = \int_0^{\infty} e^{(F+a/2) t} \sum_{k=0}^{\infty} C_k S_0 \sum_{k=0}^{\infty} C_k^T e^{(F^T+a/2) t} dt.
\]