Non-linear electrostatic waves in Born-Infeld plasmas

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Abstract

Motivated by the suggestion that Born-Infeld plasmas could have significance for electron acceleration in neutron star crusts, we obtain an upper bound on the amplitude of electrostatic waves propagating parallel to a longitudinal magnetic field in a Born-Infeld plasma.

1 Introduction

Born-Infeld electrodynamics has attracted considerable interest over recent years. It was originally introduced in the 1930s [1] as an attempt to describe the classical electron entirely in terms of its electromagnetic field, but interest in it soon waned in favour of quantum theory. However, during the first superstring revolution in the mid-1980s it was shown that Born-Infeld-type theories are a feature of low energy string field theory [2], and this discovery led to the resurgence of interest in Born-Infeld electrodynamics seen recently [3–7].

Furthermore, among the family of non-linear generalizations of Maxwell electrodynamics, it has long been known that Born-Infeld theory possesses a number of highly attractive features; in particular, like the vacuum Maxwell equations, the Born-Infeld equations exhibit zero birefringence and its solutions have exceptional causal behaviour [8,9]. The vacuum Maxwell and Born-Infeld field equations are the only physical theories with a single light cone obtainable from a local Lagrangian constructed solely from the two Lorentz invariants associated with the electromagnetic field strength tensor and the metric tensor.

Any self-consistent theory describing a large collection of charged particles must include all electromagnetic forces between the particles. However, the notorious problem of determining the classical force on a single accelerating point charge due to its own electromagnetic field has stimulated research for over a century and remains unresolved. The structure of an isolated single electron is currently beyond observation and one often proceeds classically by associating the electron with a singularity in the electromagnetic field described by Maxwell’s equations in vacuo. Following Dirac [10], an equation of motion for the electron may be obtained by appealing to conservation of the total energy-momentum of the electron and its electromagnetic field (see [11] for a recent discussion). In order to remove singularities in the equation of motion, Dirac made “natural assumptions” about the origin of the electron mass. The resulting Lorentz-Dirac equation of motion contains third order proper time derivatives of the electron’s world line, and possesses solutions that violate intuition. In particular, unless special conditions are adopted for the final state of the electron, it predicts that a free electron in vacuo can self-accelerate; furthermore the equations possess solutions in which the electron experiences a sudden acceleration before it enters a region of space containing a non-vanishing external electrostatic field (see [12] for a recent discussion).

The search for a complete dynamical theory of a point charge within Born-Infeld electrodynamics is ongoing [13], and this theory is expected to provide a resolution to the radiation-reaction problem. In particular, the difficulties associated with the Lorentz-Dirac equation are thought to have their origin in the electron’s singular total mass-energy in classical Maxwell electrodynamics; however, the electric field of a Born-Infeld electron at rest is non-singular and its total mass-energy is finite.

Some of the most extreme conditions ever encountered in a terrestrial laboratory are created when high-power laser pulses interact with matter. The laser pulse immediately vaporizes the matter to form an intense

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laser-plasma providing novel avenues for generating intense bursts of coherent electromagnetic radiation for a wide range of applications in biological and material science \[14\]. Furthermore, laser-plasmas permit controllable investigation of matter in extreme conditions that only occur naturally away from the Earth. It is expected that the next generation of ultra-intense lasers will, for the first time, allow controllable access to regimes where a host of different quantum electrodynamic phenomena will be evident. In particular, the challenge of extending Schwinger’s classic analysis \[15\] of vacuum breakdown in a static external electric field to breakdown in an intense laser-plasma is on-going \[16\]. However, the radiation-reaction problem is sufficiently strong motivation for exploring whether a Born-Infeld-type theory can yield experimental signatures before quantum effects become significant \[6\]. A sufficiently short and intense laser pulse propagating through a plasma may create a travelling longitudinal plasma wave whose phase velocity is approximately the same as the laser pulse’s group velocity. However, it is not possible to sustain arbitrarily large electric fields; substantial numbers of plasma electrons become trapped in the wave and are accelerated, which dampens the wave (the wave ‘breaks’). Early theoretical investigation of non-linear plasma waves was undertaken in the mid 1950s by Akhiezer and Polovin \[17\], and later expounded by Dawson \[18\] in the context of wave-breaking. Furthermore, this acceleration mechanism was recently employed \[19\] to explain the emission of energetic electrons from within the interiors of pulsars; such electrons are necessary for the formation of the electron-positron plasma populating a pulsar’s magnetosphere.

Wave-breaking is a fundamentally non-linear phenomenon, and it is natural to explore the properties of Born-Infeld plasmas from this perspective. Moreover, the magnetic fields found in neutron stars are typically \(\sim 10^8\) T, whilst those in magnetars may be two orders of magnitude higher and such fields have energy densities commensurate with the Schwinger limit (i.e. commensurate with a static electric field of strength \(\sim 10^{18}\) V/m).

The following is a brief summary of our recent analysis of Born-Infeld plasma waves near breaking. The present work is an application of the approach established in \[7\], generalized here to include the presence of a background magnetic field. The notation and conventions used here are identical to those in \[7\], to which we refer the reader for further details. In particular, we use units in which the speed of light \(c = 1\) and the permittivity of the vacuum \(\varepsilon_0 = 1\).

## 2 Born-Infeld plasma

Let \((\mathcal{M}, g)\) be a spacetime. The electromagnetic sector of a Born-Infeld plasma may be expressed in terms of the Lagrangian \(\mathcal{L}_{BI}(X, Y)\),

\[
\mathcal{L}_{BI}(X, Y) = \frac{1}{\kappa^2} \left(1 - \sqrt{1 - \kappa^2 X - \kappa^4 Y^2/4}\right),
\]

where the invariants \(X\) and \(Y\) are

\[
X = \star(F \wedge \star F), \quad Y = \star(F \wedge F),
\]

with \(\star\) the Hodge map associated with the spacetime metric \(g\) and \(\kappa\) is a constant that characterizes the self-interaction of the electromagnetic field. The relationship between the excitation 2-form \(G\) and the Maxwell 2-form \(F\) is

\[
G = 2 \left(\frac{\partial \mathcal{L}_{BI}}{\partial X} F - \frac{\partial \mathcal{L}_{BI}}{\partial Y} \star F\right)
= \frac{1}{\sqrt{1 - \kappa^2 X - \kappa^4 Y^2/4}} \left(F - \frac{\kappa^2 Y}{2} \star F\right)
\]

and Maxwell electrodynamics (where \(G = F\)) is recovered in the limit \(\kappa \to 0\).

For simplicity, the plasma electrons are represented as a cold relativistic fluid; their worldlines are trajectories of a unit normalized future-pointing timelike 4-vector field \(V\) satisfying

\[
\nabla_V \bar{V} = \frac{q}{m} \bar{v} V, \quad g(V, V) = -1
\]
where $q < 0$ is the charge on the electron, $m$ is the electron rest mass, $q_1 V F$ is the Lorentz 4-force acting on the electron fluid, $\iota V$ is the interior product with respect to $V$ and $\nabla$ is the Levi-Civita connection on $\mathcal{M}$. The 1-form $\tilde{V}$ is the metric dual of the vector field $V$, i.e. the 1-form $\tilde{V}$ satisfies $\tilde{V}(U) = g(V, U)$ for all vector fields $U$ on $\mathcal{M}$.

We are interested in the evolution of a plasma over timescales during which the motion of the ions is negligible in comparison with the motion of the electrons. Here the ions are specified as a background and their worldlines are trajectories of the prescribed future-pointing timelike 4-vector field $N_{\text{ion}}$ (the ion number 4-current) on $\mathcal{M}$.

Maxwell’s equations are

$$dF = 0, \quad d \ast G = -q_n \ast \tilde{V} - q_{\text{ion}} \ast \tilde{N}_{\text{ion}}$$

where the 0-form $n$ is the electron proper number density and $q_{\text{ion}} = Z|q|$ is the charge on an ion, with $Z$ the multiplicity of the ionization. The 1-form $\tilde{N}_{\text{ion}}$ is the metric dual of the vector field $N_{\text{ion}}$.

3 Non-linear electrostatic waves

Particle acceleration in non-linear electrostatic waves close to breaking has recently been proposed as a possible mechanism for explaining how energetic electrons are ejected from within the interiors of pulsars [19].

If an atom is immersed in a uniform background magnetic field whose strength is much greater than $\sim 10^{15}\text{T}$ then the corresponding magnetic force on the electrons is much greater than the atom’s Coulombic forces [20]. The atom settles into the ground Landau level, limiting the electrons’ spatial displacement transverse to the magnetic field lines. Thus, electrons are conducted preferentially along the direction of the magnetic field lines, and one may approximate the bulk electron motion as 1-dimensional [19]. Moreover, the magnetic field lines in the iron crust of a neutron star are expected to run parallel to its surface and to be strongly curved near the poles, where they emerge normal to the star’s surface. The magnetic curvature near the poles is expected to lead to variations in electron number density and excite electrostatic waves in the electron ‘gas’ within the iron crust [19].

To obtain some insight into the behaviour of a Born-Infeld plasma in this context, in the following we explore properties of solutions to (4), (5) that describe large-amplitude longitudinal electrostatic waves propagating in a uniform background ion density in a flat spacetime. It is useful to envisage the electrons in the plasma as belonging to one or other of two families. The first family and the ion background are represented by the triple $\{V, n, N_{\text{ion}}\}$ and constitute the bulk plasma; those electrons and the background ions form the electrostatic wave. The members of the second family are the rest of the electron population, some of which are trapped in the wave’s potential; we do not attempt to include the second family in the simple model explored here.

Let $(x^a)$ be an inertial coordinate system on Minkowski spacetime $(\mathcal{M}, g)$ where $x^0$ is the proper time of observers at fixed Cartesian coordinates $(x^1, x^2, x^3)$. The metric tensor $g$ has the form

$$g = \eta_{ab} \, dx^a \otimes dx^b$$

with

$$\eta_{ab} = \begin{cases} 
-1 & \text{if } a = b = 0, \\
1 & \text{if } a = b \neq 0, \\
0 & \text{if } a \neq b
\end{cases}$$

and the Hodge map $\ast$ is induced from the 4-form $\ast 1$ on $\mathcal{M}$ where

$$\ast 1 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$ (8)

The ion background is given as the number 4-current $N_{\text{ion}} = n_{\text{ion}} \partial / \partial x^0$, where $n_{\text{ion}}$ is a constant.

The longitudinal electrostatic waves considered here propagate parallel to the $x^3$-axis with phase velocity $v$ (with $0 < v < 1$) in the ion rest frame. We introduce the pair $\{e^1, e^2\}$

$$e^1 = v dx^3 - dx^0, \quad e^2 = dx^3 - v dx^0$$

(9)
and note that the orthonormal co-frame \( \{ \gamma e^1, \gamma e^2, dx^1, dx^2 \} \) is adapted to observers moving at velocity \( v \) along \( x^3 \) (observers in the ‘wave frame’), where the Lorentz factor \( \gamma = 1/\sqrt{1 - v^2} \).

We seek a unit normalized 4-velocity field \( V \) of the form

\[
\tilde{V} = \mu(\zeta) e^1 - \sqrt{\mu(\zeta)^2 - \gamma^2} e^2
\]

where \( \zeta = x^3 - vx^0 \) is the wave’s phase and \( e^2 = d\zeta \). We have adopted the so-called ‘quasi-static approximation’; the pointwise dependence of \( \mu \) is on \( \zeta \) only. The velocity of the bulk plasma electrons observed in the wave frame is \( \sqrt{\mu^2 - \gamma^2}/\mu \) in the direction of decreasing \( x^3 \), so the electrons move slower than the wave propagates. The maximum amplitude wave arises for oscillations during which \( \mu \) is arbitrarily close to \( \gamma \) (i.e. the bulk plasma electrons catch the wave).

The Maxwell 2-form \( F \) is

\[
F = E(\zeta) dx^0 \wedge dx^3 - B dx^1 \wedge dx^2
\]

where \( E \) is the \( x^3 \) component of the electric field and its pointwise dependence is on \( \zeta \) only. The constant \( B \) is the component of the magnetic field along \( x^3 \), and the remaining components of the electric and magnetic field vanish.

Employment of (6)-(11) reduces (3)-(5) to a second-order ordinary differential equation for \( \mu \), viz.

\[
\frac{d}{d\zeta} \left( \frac{E}{\sqrt{1 - \kappa^2 E^2}} \right) = \frac{q Z n_{\text{ion}} \gamma^2}{\sqrt{1 + \kappa^2 B^2}} \left( \frac{v \mu}{\sqrt{\mu^2 - \gamma^2}} - 1 \right),
\]

where

\[
E = \frac{1}{\gamma^2} \frac{md\mu}{q d\zeta}
\]

Inspection of (12) reveals that, for oscillatory solutions, the electric field \( E \) has a turning point where \( \mu = \gamma^2 \). Further investigation reveals that this turning point is a minimum of \( E \) (for \( q < 0 \)), and an upper bound \( E_{\text{max}} \) (‘wave-breaking limit’) on \( E \) may be obtained by evaluating the first integral of (12) between the points \( (\mu = \gamma, E = 0) \) and \( (\mu = \gamma^2, E = -E_{\text{Bmax}}) \). Restoring the speed of light \( c \) and permittivity of the vacuum \( \varepsilon_0 \) yields

\[
E_{\text{Bmax}} = \frac{1}{\kappa} \sqrt{1 - \frac{\kappa^2 E_{\text{APmax}}^2}{2\sqrt{1 + \kappa^2 c^2 B^2}} + 1}^{-2}
\]

where \( E_{\text{APmax}} \) is the maximum electric field for a relativistic cold Maxwell plasma,

\[
E_{\text{APmax}} = \frac{m\omega_p c}{|q|} \sqrt{2(\gamma - 1)}
\]

first obtained by Akhiezer and Polovin [17], and the constant \( \omega_p \) is the plasma frequency

\[
\omega_p = \sqrt{\frac{q^2 Z n_{\text{ion}}}{m\varepsilon_0}}
\]

If \( \kappa = 10^{-18} \text{m/V} \) then \( \kappa cB = 1 \) corresponds to \( B = 10^9 \text{T} \), which is within the range of the surface magnetic fields of rotation-powered radio pulsars [20]. The relationship between \( \kappa E_{\text{Bmax}} \) and \( \kappa E_{\text{APmax}} \) is shown in figure 1 for \( B = 0 \) and \( B = 1/\kappa c \).

### 4 Acknowledgments

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Figure 1: A graph of the maximum amplitude $E_{\text{BI}}^{\text{max}}$ of an electrostatic wave in a Born-Infeld plasma versus the maximum amplitude $E_{\text{AP}}^{\text{max}}$ of a Maxwell plasma. Both field strengths are scaled by the Born-Infeld constant $\kappa$. The dashed curve corresponds to $B = 0$ and the solid curve corresponds to $B = 1/(\kappa c)$. The dotted line lies along the diagonal.

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