Conformal Field Theory Approach to the 2-Impurity Kondo Problem: Comparison with Numerical Renormalization Group Results

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Abstract

Numerical renormalization group and conformal field theory work indicate that the two impurity Kondo Hamiltonian has a non-Fermi liquid critical point separating the Kondo-screening phase from the inter-impurity singlet phase when particle-hole (P-H) symmetry is maintained. We clarify the circumstances under which this critical point occurs, pointing out that there are two types of P-H symmetry. Only one of them guarantees the occurrence of the critical point. Much of the previous numerical work was done on models with the other type of P-H symmetry. We analyse this critical point using the boundary conformal field theory technique. The finite-size spectrum is presented in detail and compared with about 50 energy levels obtained using the numerical renormalization group. Various Green’s functions, general renormalization group behaviour, and a hidden $SO(7)$ are analysed.

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I. INTRODUCTION

While the single-impurity Kondo problem is by now rather well understood, the Kondo lattice model, of possible relevance to heavy fermion materials, presents additional difficulties. In particular there are two competing tendencies; the Kondo effect leads to a magnetic screening of the spins by conduction electrons while the RKKY interactions between the spins may lead to antiferromagnetism. The two-impurity Kondo problem provides a simple model in which to study these competing effects. Of course, in this model, true antiferromagnetism with static moments (i.e. spontaneous breaking of rotational symmetry) cannot occur. Rather the RKKY couplings tend to promote singlet formation by the two impurities.

A very general form of the Hamiltonian has two \( s = \frac{1}{2} \) spins, symmetrically located about the origin, interacting with a Fermi gas. The Hamiltonian is:

\[
H - \mu N = H_0 + H_K + H_{\text{self}}
\]

(1.1)

where \( H_0, H_K \) and \( H_{\text{self}} \) are the kinetic energy, Kondo interaction and impurity self interactions respectively:

\[
H_0 = \int d^3 \vec{k} \epsilon(\vec{k}) \psi^\dagger_{\alpha \vec{k}} \psi_{\alpha \vec{k}}
\]

\[
H_K = \int d^3 \vec{k}_1 \int d^3 \vec{k}_2 \psi^\dagger_{\alpha \vec{k}_1} \sigma_\alpha \psi_{\beta \vec{k}_2} \cdot [v(\vec{k}_1)^* v(\vec{k}_2) \vec{S}_1 + v(-\vec{k}_1)^* v(-\vec{k}_2) \vec{S}_2]
\]

\[
H_{\text{self}} = K \vec{S}_1 \cdot \vec{S}_2
\]

(1.2)

\( v(\vec{k}) \) is related to an Anderson model hybridization matrix element and \( K \) is the inter-impurity coupling.

The physics of this model becomes simple in the limits \( K \to \pm \infty \). When \( K \to +\infty \) the two impurities lock into a singlet state. Therefore the Kondo coupling has no effect and the electron gas is completely unaffected by the impurities. Conversely, when \( K \to -\infty \), the impurities become an effective single \( s = 1 \) impurity. As we shall see in the next section, the extended nature of this \( s = 1 \) impurity implies that two “channels” of conduction electrons interact with it. The two channel, \( s = 1 \) Kondo problem was studied by Nozières and Blandin. They concluded that, at low temperatures, the impurity gets magnetically screened by the conduction electrons. The remaining low energy conduction electron degrees of freedom are decoupled from the impurity but experience a \( \pi \) phase shift in both channels. This corresponds to a local Fermi liquid fixed point. The low energy electronic degrees of freedom are free electron-like; the many body interactions induced by the Kondo interaction lead to a simple phase shift at low energies.

The behaviour of the system at intermediate values of \( K \) is less obvious. In general, one might expect that a local Fermi liquid description holds at low temperatures for all \( K \). The phase shifts could vary continuously with \( K \). However, it was argued by Millis, Kotliar and Jones that this cannot happen if P-H symmetry is maintained while \( K \) is varied. In this case, the zero-energy phase shift, if it is well-defined, can only be 0 or \( \pi/2 \). Since the \( K \to \pm \infty \) fixed points are stable, it follows that there must be at least one point in the phase diagram not belonging to either phase, corresponding to some sort of phase transition.
Such a transition could, in principle, be first order. Alternatively, if it is continuous, the critical point cannot be characterizable by phase shifts. Numerical renormalization group (NRG) work indicates that the transition is indeed continuous. The critical point is of non-Fermi liquid type. On the other hand, quantum Monte Carlo (QMC) work has not seen such a critical point. NRG work on the Anderson model has seen the critical point only in the case of “energy-independent coupling constants”. In both cases P-H symmetry was maintained.

In the next section we re-examine the argument for a critical point in more detail. We show that there are actually two quite different types of P-H symmetry which may occur in models of this type; this is connected with the parity symmetry of the models. Only one of these P-H symmetries guarantees a phase transition. Much of the numerical work in Refs. (7,10) was done using models with the “wrong” type of P-H symmetry.

Two of us have recently developed a new boundary conformal field theory technique to study such non-trivial critical points. The essence of this method is that the local interactions can be replaced by a conformally invariant boundary conditions in the low energy effective Hamiltonian. One of the main purposes of this paper is to explain the application of the technique to this problem and elucidate its various predictions. A brief presentation of some of these results was made earlier. In Sec. III we review the reduction of the problem to a one-dimensional field theory and then discuss a convenient and somewhat unconventional “bosonization” of the model in terms of Wess-Zumino-Witten matrix fields and Ising model fields. In Section IV we solve for the non-trivial critical point by hypothesizing the corresponding boundary conditions (there are only a few possibilities). We calculate the resulting finite size spectrum using this boundary condition and compare it to the NRG results, obtaining excellent agreement. In Sec. V we discuss the various scaling operators at the non-trivial critical point, the stability of the fixed point (including the effects of P-H symmetry breaking) and the behaviour of various Green’s functions. The non-trivial fixed point possesses a remarkable hidden $SO(7)$ symmetry. We explain this in detail in Sec. VI and comment on the possible connection with a physical picture of the 2-impurity Kondo problem based on abelian bosonization and “fermionization”. In Sec. VII we summarize the disagreement with other calculations and the prospects for resolving them.

II. EXISTENCE OF THE NON-TRIVIAL FIXED POINT AND PARTICLE-HOLE SYMMETRY

We begin with a few comments on the Hamiltonian of Eq. (1.2). Often, one does not consider an explicit inter-impurity interaction but only the indirect RKKY interaction, of second and higher order in $J$. More generally an additional interaction could be generated by other exchange processes not involving the conduction electrons. Often one considers a $\delta$-function Kondo interaction, with the impurities at $\pm \vec{R}/2$ in which case:

$$v(\vec{k}) = v_0 e^{i\vec{k} \cdot \vec{R}/2}. \quad (2.1)$$

The more general form that we consider allows for longer range hybridization. Note that, if $\epsilon(-\vec{k}) = \epsilon(\vec{k})$, the Hamiltonian is invariant under the parity transformation:
corresponding to a reflection about a point in position space (the midpoint between the 2 impurities.) For a tight-binding model, the origin of parity symmetry may not correspond to a lattice point. Nonetheless, we define our Fourier transforms with respect to it.

For certain choices of the dispersion relation, \( \epsilon(\vec{k}) \) and hybridization matrix element, \( v(\vec{k}) \), the Hamiltonian of Eq. (1.2) will have particle-hole (P-H) symmetry. We consider invariance of \( H \) under a general P-H transformation of the form:

\[
\psi_\alpha \vec{k} \rightarrow \epsilon_{\alpha\beta} \psi_\beta \vec{k}', \\
\vec{S}_1 \rightarrow \vec{S}_1, \\
\vec{S}_2 \rightarrow \vec{S}_2.
\]

where \( \vec{k}' \) is some function of \( \vec{k} \). Invariance of \( H_0 \) requires:

\[
\epsilon(\vec{k}) = -\epsilon(\vec{k}').
\]

Invariance of the Kondo interaction requires (after a possible phase redefinition of \( v \) which doesn’t effect \( H_K \)):

\[
v(\vec{k}') = v(\vec{k})^*, \\
v(-\vec{k}') = v(-\vec{k})^* e^{i\alpha}
\]

where \( \alpha \) is a \( \vec{k} \)-independent phase. For instance, for a cubic lattice tight-binding model at half-filling,

\[
\vec{k}' = \vec{k}_0 - \vec{k},
\]

where \( \vec{k}_0 \) is the nesting wave-vector:

\[
\vec{k}_0 \equiv (\pi/a, \pi/a, \pi/a).
\]

Hence, for the \( \delta \)-function Kondo interaction of Eq. (2.1), and appropriate choice of the phase of \( v_0 \); Eq. (2.5) is obeyed with:

\[
\alpha = \vec{k}_0 \cdot \vec{R}.
\]

Assuming that \( \vec{R} \) (the vector connecting the two impurities) is a lattice vector, the phase \( \alpha \) can take on the values 0 or \( \pi \).

When particle-hole symmetry is present, there is also an exact \( SU(2) \) “isospin” symmetry, in addition to the normal spin symmetry. [For a general discussion of this see Sec. II of Ref. (19).] One of these isospin symmetry generators is simply the total electron number, corresponding to the \( z \)-component of isospin, \( I^z \). The lowering operator, \( I^- \) is proportional to:
\[
\int d^3 \vec{k} \psi_{\vec{k},+} \psi_{\vec{k},-},
\]

(2.9)

where \( \vec{k}' \) is the function of \( \vec{k} \) occurring in the particle-hole transformation.

The Hamiltonian of Eq. (1.2) can be reduced exactly to a one-dimensional one. To do this we define two fields:

\[
\psi_{\pm,E} \equiv \int d^3 \vec{k} \delta[\epsilon(\vec{k}) - E]v(\pm\vec{k}) \psi_{\vec{k}}.
\]

(2.10)

Note that, of the infinite set of fields of a given energy \( E \), (corresponding to all points on the constant energy surface in \( \vec{k} \)-space) only these two appear in the Kondo interaction. We may define parity even and odd orthonormal linear combinations of these two fields:

\[
\psi_{e,E} \equiv \frac{(\psi_{+,E} + \psi_{-,E})}{N_e(E)},
\]

\[
\psi_{o,E} \equiv \frac{(\psi_{+,E} - \psi_{-,E})}{N_o(E)},
\]

(2.11)

where:

\[
N_{e,o}(E) \equiv \int d^3 \vec{k} \delta[E - \epsilon(\vec{k})]|v(\vec{k})| \pm v(-\vec{k})|^2.
\]

(2.12)

The anti-commutators are normalized as follows:

\[
\{\psi_E, \psi_{E'}^\dagger\} = \delta(E - E').
\]

(2.13)

We may complete \( \psi_e \) and \( \psi_o \) into a complete orthonormal basis for each energy, \( E \); only these two fields appear in the Kondo interaction. Discarding the additional fields which decouple, we obtain an exact one-dimensional rewriting of the original Hamiltonian, with:

\[
H_0 = \int dE E [\psi_{e,E}^\dagger \psi_{e,E} + \psi_{o,E}^\dagger \psi_{o,E}]
\]

\[
H_K = \int dE dE' \left\{ (N_e(E)N_o(E'))\psi_{e,E}^\dagger \sigma_{e,E'} \psi_{o,E'} + N_o(E)N_o(E')\psi_{o,E}^\dagger \sigma_{o,E'} \psi_{e,E'} \right\} \cdot (\vec{S}_1 + \vec{S}_2)
\]

\[
+N_e(E)N_o(E') \left( \psi_{e,E}^\dagger \sigma_{o,E'} \psi_{o,E'} + \psi_{o,E}^\dagger \sigma_{e,E'} \psi_{e,E'} \right) \cdot (\vec{S}_1 - \vec{S}_2).
\]

(2.14)

If the original problem had the particle-hole symmetry of Eq. (2.3), then the one-dimensional problem also has P-H symmetry. The transformation of the fields \( \psi_{e,E} \) and \( \psi_{o,E} \) can be deduced from:

\[
\psi_{+,E} \rightarrow \psi_{+,E}^\dagger,
\]

\[
\psi_{-,E} \rightarrow e^{i\alpha} \psi_{-,E}^\dagger,
\]

(2.15)

which in turn follows from Eq. (2.10) using Eq. (2.3)-(2.5).

[We suppress the transformation of spin indices; it is the same as in Eq. (2.3).] In the case \( \alpha = 0 \) we have [ using Eq. (2.4) and (2.5)]:

\[
N_e(-E) = N_e(E)
\]

\[
N_o(-E) = N_o(E)
\]

(2.16)
and
\[ \psi_{e,E} \rightarrow \psi_{e,-E}^\dagger \]
\[ \psi_{o,E} \rightarrow \psi_{o,-E}^\dagger. \]  
(2.17)

For \( \alpha = \pi \), we find:
\[ N_e(-E) = N_o(E), \]  
(2.18)

and
\[ \psi_{e,E} \rightarrow \psi_{o,-E} \]
\[ \psi_{o,E} \rightarrow \psi_{e,-E}. \]  
(2.19)

It can be directly verified that the P-H transformation of Eq. (2.17) or (2.19) are symmetries of the one-dimensional Hamiltonian of Eq. (2.14) if \( N_e(E) \) and \( N_o(E) \) satisfy Eq. (2.16) or (2.18) respectively.

It turns out that the model behaves very differently depending on which type of P-H symmetry is present. With the first type, a phase transition must separate the Kondo-screened and inter-impurity singlet phases; with the second type no transition is required or expected to occur. We reiterate the argument for a transition with the first type of P-H symmetry.

In a Fermi liquid phase, it should be possible to characterize the zero-temperature fixed point by phase shifts for the even and odd channels at \( E = 0 \). These amount to boundary conditions relating incoming and outgoing operators:
\[ \psi_{e,E}^{\text{out}} = e^{2i\delta_e} \psi_{e,E}^{\text{in}} \]
\[ \psi_{o,E}^{\text{out}} = e^{2i\delta_o} \psi_{o,E}^{\text{in}}. \]  
(2.20)

The Hermitian conjugate fields, \( \psi_{e,o}^\dagger \) obey the same conditions with \( \delta_{e,o} \rightarrow -\delta_{e,o} \). Hence, if the first type of P-H symmetry is obeyed, both \( \delta_e \) and \( \delta_o \) can only take the values 0 or \( \pi/2 \); [from Eq. (2.17) at \( E = 0 \), right at the Fermi energy]. [Note that the boundary conditions of Eq. (2.20) only depend on \( 2\delta \) (mod \( 2\pi \)) so that \( \delta = 0 \) or \( \pi \) are equivalent as are \( \delta = \pm \pi/2 \).] \( \delta_e = \delta_o = 0 \) in the inter-impurity singlet phase and \( \delta_e = \delta_o = \pi/2 \) in the Kondo-screening phase. Furthermore, both of these fixed points are absolutely stable (no relevant or marginal operators.) It then follows that there must be some sort of phase transition separating these two phases. It could be first order, or correspond to a non-Fermi liquid critical point at which the phase shifts are not defined. On the other hand, if we have the second type of P-H symmetry, \( \delta_e \) and \( \delta_o \) may take on arbitrary values subject only to the condition \( \delta_e = -\delta_o \). Thus they may vary continuously from \((0,0)\) in the limit of \( \infty \) antiferromagnetic inter-impurity coupling to \((\pi/2, -\pi/2)\) in the opposite limit of \( \infty \) ferromagnetic inter-impurity coupling. No transition is necessary in this case.

The same argument can be made by considering possible potential scattering terms in the Hamiltonian. While the original Hamiltonian has no such terms, they will be generated in higher orders of perturbation theory. In general these take the form:
\[ H_{PS} = \int dE dE'[V_e(E, E')\psi_{e,E}^{\dagger}\psi_{e,E'} + V_o(E, E')\psi_{o,E}^{\dagger}\psi_{o,E'}]. \]  
(2.21)
The phase shifts at zero energy contain terms proportional to \( V_{e}(0,0) \) and \( V_{o}(0,0) \) respectively, for weak potential scattering, ignoring the Kondo interactions. With the first type of P-H symmetry:

\[
V_{e,o}(E, E') = -V_{e,o}(-E', -E). \tag{2.22}
\]

This implies \( V_{e,o}(0,0) = 0 \); no phase shift at zero energy. On the other hand the second type of P-H symmetry implies:

\[
V_{e}(E, E') = -V_{o}(-E', -E). \tag{2.23}
\]

At zero energy this gives: \( V_{e}(0,0) = -V_{o}(0,0) \), allowing (equal and opposite) phase shifts at zero energy.

In Sec. IV we explicitly study the stability of the non-trivial critical point under potential scattering. Our analysis shows that \( V_{e}(0,0) = -V_{o}(0,0) \neq 0 \) is a relevant perturbation.

### III. ONE DIMENSIONAL BOSE-ISING REPRESENTATION

In this section we derive a representation of the two impurity Kondo problem which is convenient for studying the non-trivial critical point. The one-dimensional Hamiltonian of Eq. (2.14) is treated using a generalized bosonization method based on the Goddard-Kent-Olive coset construction. We end up representing the fermions by three Wess-Zumino-Witten non-linear \( \sigma \)-models together with an Ising model.

Taylor-expanding \( N_{e,o}(E) \) in Eq. (2.14) around \( E = 0 \), we obtain the non-derivative interactions:

\[
H_{K} = \frac{1}{2} \int dEdE' \left\{ (J_{e}\psi_{e,E}^\dagger \bar{\sigma} \psi_{e,E'} + J_{o}\psi_{o,E}^\dagger \bar{\sigma} \psi_{o,E'}) \cdot (\vec{S}_{1} + \vec{S}_{2}) \right. \\
\left. + J_{m}(\psi_{e,E}^\dagger \bar{\sigma} \psi_{o,E'} + \psi_{o,E}^\dagger \bar{\sigma} \psi_{e,E'}) \cdot (\vec{S}_{1} - \vec{S}_{2}) \right\} \tag{3.1}
\]

where

\[
J_{e} \equiv 2N_{e}(0)^{2}, \quad J_{o} \equiv 2N_{o}(0)^{2}, \quad J_{m} \equiv 2N_{o}(0)N_{o}(0). \tag{3.2}
\]

Sums over spin indices are implied. We effectively obtain a two channel one-dimensional model. The origin of the two channels is simply the fact that the two impurities are at different spatial locations. In general, with \( n \) impurities at different locations, we would obtain an \( n \)-channel one-dimensional model at long wavelengths.

The higher terms in the Taylor expansion of \( N_{e,o}(E) \) give various derivative interactions. These are irrelevant at the weak coupling (unstable) critical point, assuming the first type of particle-hole symmetry discussed in Sec. II. That is to say, they are irrelevant assuming they don’t generate any potential scattering terms; this is the case with the first type of P-H symmetry. From our analysis of the non-trivial critical point we will conclude that these extra terms are also irrelevant at that critical point, assuming the first type of P-H symmetry.
We now see that, in the limit of an infinite ferromagnetic inter-impurity interaction \( K \), the model reduces to a single-impurity two-channel \( s = 1 \) Kondo problem, with different couplings, \( J_e \) and \( J_o \) to the two channels. This difference is known to be irrelevant.

For analysing the critical point, we actually find it somewhat more convenient to introduce a different orthonormal basis:

\[
\psi_{1,2} \equiv (\psi_e \pm \psi_o)/\sqrt{2}
\] (3.3)

The kinetic energy remains diagonal, as in Eq. (2.14) and the Kondo interaction becomes:

\[
H_K = \frac{1}{2} \int dE dE' \left\{ J_+ [\psi_{1,E}^\dagger \sigma \psi_{1,E'} + \psi_{2,E}^\dagger \sigma \psi_{2,E'}] \cdot [\vec{S}_1 + \vec{S}_2] + J_m [\psi_{1,E}^\dagger \sigma \psi_{1,E'} - \psi_{2,E}^\dagger \sigma \psi_{2,E'}] \cdot [\vec{S}_1 - \vec{S}_2] \right\}
\] (3.4)

where

\[
J_\pm \equiv (J_e \pm J_o)/2
\] (3.5)

Note that \( \psi_{1,2} \) are not, in general, the same as \( \psi_\pm \) since \( N_e \neq N_o \). We emphasize that this is the most general Hamiltonian consistent with particle-hole symmetry, up to operators which are irrelevant at the (unstable) zero-coupling fixed point, i.e. in ordinary perturbation theory.

The conserved isospin operators referred to above take simple forms in the effective one-dimensional theory:

\[
I^z \equiv \frac{1}{2} \int dE [\psi_{1,E}^\dagger \sigma \psi_{1,E} + \psi_{2,E}^\dagger \sigma \psi_{2,E}]
\]

\[
I^- \equiv \int dE [\psi_{1,E}^\dagger \sigma \psi_{-1,E} - \psi_{2,E}^\dagger \sigma \psi_{-2,E}]
\] (3.6)

We note that when \( J_- = 0 \) there is even more symmetry. Now the charges of the 1 and 2 species of fermions are separately conserved and in fact we have two commuting sets of isospin generators:

\[
I_{1}^z \equiv \frac{1}{2} \int dE \psi_{1,E}^\dagger \sigma \psi_{1,E}
\]

\[
I_{1}^- \equiv \int dE \psi_{1,E}^\dagger \sigma \psi_{1,E}
\]

(3.7)

and similarly for \( I_2 \). While the isospin generators \( \vec{I}_1 \) and \( \vec{I}_2 \), obey the usual \( SU(2) \) commutation relations, they commute with each other and also with the normal spin rotation generators, \( \vec{J} \) (not to be confused with the coupling constants).

In position space the one-dimensional problem can be defined in terms of left-movers only:

\[
\psi_L(t,x) = \frac{1}{\sqrt{v}} \int_{-D}^D dE e^{-iE(t+x/v)} \psi_E,
\]

(3.8)

for \(-\infty < x < \infty\). Here \( D \) is a bandwidth cut-off and \( v \) is an arbitrary velocity parameter which defines the scale of length in the effective one-dimensional problem. We adopt an
unconventional normalization for our one-dimensional position space fermion fields so that they obey:

\[ \{ \psi^\dagger_L(x), \psi_L(y) \} = 2\pi \delta(x - y) \] (3.9)

Alternatively, we may define right and left-movers on the half-line, \( x > 0 \), with

\[ \psi_R(t, x) \equiv \psi_L(t, -x), \quad (x > 0) \] (3.10)

and \( \psi_L(x, t) \) defined as in Eq. (3.8). Note that Eq. (3.10) would follow automatically from defining \( \psi_R(x, t) \) to be the right-moving field obeying the boundary condition \( \psi_R(t, 0) = \psi_L(t, 0) \). (Henceforth we drop \( L, R \), subscripts. Fields are generally left-movers.)

A key feature in the conformal field theory approach to this problem is bosonization. Actually, we use a somewhat generalized version of bosonization in which fermion fields are represented in terms of boson fields and Ising fields. A conventional abelian bosonization approach would involve introducing four bosons for the spin and charge degrees of freedom of each channel, \( \phi_{s,i}, \phi_{c,i} \) with \( i = 1, 2 \). In fact we only introduce the left-moving components of these bosons, or equivalently introduce left and right movers on the half-line with an appropriate boundary condition. The bosonization formulas take the form:

\[ \psi_{\uparrow i} \propto e^{i\sqrt{2\pi}(\phi_{c,i} + \phi_{s,i})}, \quad \psi_{\downarrow i} \propto e^{i\sqrt{2\pi}(\phi_{c,i} - \phi_{s,i})} \] (3.11)

As discussed elsewhere, these bosons should be regarded as periodic variables since only their exponentials (or derivatives) occur in physical quantities. Adopting a fixed normalization for the free boson Lagrangian density:

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \] (3.12)

we define a “compactification radius” by identifying:

\[ \phi \equiv \phi + 2\pi R \] (3.13)

The above bosonization formulas, Eq. (3.11), identify the compactification radius for the spin boson as \( R = 1/\sqrt{2\pi} \). At this radius, the free spin boson theory exhibits \( SU(2) \) symmetry and is, in fact equivalent to an \( SU(2) \) Wess-Zumino-Witten (WZW) non-linear \( \sigma \) model with Kac-Moody central charge \( k = 1 \). The spin factors occurring in Eq. (3.11) can be written in terms of the left-moving factors of the WZW fields, \( g_{ai} \):

\[ \psi_{ai} \propto e^{i\sqrt{2\pi}\phi_{c,i}} g_{ai} \] (3.14)

In fact, the charge bosons have the same compactification radius. This is no accident. It reflects the fact that the free fermion theories for each channel have an \( SU(2) \times SU(2) \) symmetry. This corresponds to separate spin and isospin symmetry [see Eq. (3.7)] for each channel. This is equivalent to \( O(4) \). The \( O(4) \) symmetry is manifest if the two spin components are each written in terms of hermitean and anti-hermitean parts, giving a total of four hermitean fermion fields, for each channel. The Kondo interactions will break the separate spin symmetries down to the diagonal subgroup; they have a similar effect on the
separate isospin symmetries except in the special case, $J_- = 0$. Thus we may introduce two more $k = 1$ WZW field, $(h_i)_A$ corresponding to the two charge fields. Here $A$ is an isospin index whereas $i$ labels the two species. The bosonization formulas of Eq. (3.11) then take the form:

$$\psi_{ai} \propto (h_i)_1 g_{ai}$$

(3.15)

The left-moving $k = 1$ WZW fields obey:

$$g^{\alpha \dagger} = \epsilon^{\alpha \beta} g_{\beta}$$

(3.16)

and similarly for the $h_i$'s, as can be seen by comparing the abelian and non-abelian bosonization formulas. As we shall see, the charge bosons play a purely passive spectator role in the 2-impurity Kondo effect and it is not necessary to replace the charge bosons by the non-abelian $h$-fields in what follows.

The next crucial step is to rewrite the bosonized theory in terms of a total spin boson and some leftover degrees of freedom, describing relative spin fluctuations. The obvious way of doing this, which is not the one we use, consists of changing variables to the sum and difference of the spin bosons, $\phi_{s,1}$ and $\phi_{s,2}$. The reason that we do not follow this procedure is that it does not explicitly maintain the total $SU(2)$ symmetry of ordinary spin. Instead we use a procedure based on the Sugawara form of the Hamiltonian.

Before explaining this in detail, we pause to review some basic properties of conformal field theories and Sugawara Hamiltonians. We work with left-movers only (so that all operators are functions of $x + vt$ only) and scale a factor of $v/2\pi$ out of the Hamiltonian for convenience, writing:

$$H = \frac{v}{2\pi} \int_{-l}^{l} dx \mathcal{H}$$

(3.17)

Assuming that $\mathcal{H}(-l) = \mathcal{H}(l)$, we define the Fourier transform:

$$L_n \equiv \frac{l}{2\pi^2} \int dx e^{ln\pi i/l} \mathcal{H}(x)$$

(3.18)

For a general conformally invariant Hamiltonian, these Fourier modes generate the Virasoro algebra:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12\pi} n(n^2 - 1) \delta_{n+m,0}$$

(3.19)

where $c$ is the conformal anomaly parameter. One class of conformally invariant theories, relevant to the present discussion, has a Hamiltonian density quadratic in the currents, $J^a(x)$ ($a = 1, 2, 3, ... n$) of some group of dimension $n$. In the present case, only the group $SU(2)$, with $n = 3$, is relevant. The Fourier modes of the currents obey the Kac-Moody algebra:

$$[J^a_n, J^b_m] = i \epsilon^{abc} J^c_{n+m} + \frac{1}{2} k n \delta_{n+m,0}$$

(3.20)

Here $k$ is the Kac-Moody central charge (or level). It must be a positive integer. The Hamiltonian then takes the Sugawara form:
\[ \mathcal{H}(x) = \frac{1}{2 + k} : \vec{J}(x) \cdot \vec{J}(x) : \]  

(3.21)

or, in momentum space:

\[ L_n = \frac{1}{2 + k} \sum_{-\infty}^{\infty} : \vec{J}_{-n} \cdot \vec{J}_{n+m} : \]  

(3.22)

It follows from the Kac-Moody algebra that the Sugawara Hamiltonian obeys the Virasoro algebra with conformal anomaly,

\[ c = \frac{3}{2 + k} \]  

(3.23)

The Hamiltonian generates time-translation of the currents implying the additional commutation relations:

\[ [L_n, J^a_m] = -m J^a_{n+m} \]  

(3.24)

This fixes the normalization of the Hamiltonian in Eq. (3.21).

We now return to the problem at hand, considering at first the zero coupling, free theory. The free theory of two channels of left-moving fermions with spin can be written in Sugawara form. Altogether there are four commuting terms in the Hamiltonian, quadratic in the charge and spin currents for channels 1 and 2. The charge parts can actually be written more symmetrically as quadratic forms in the Isospin currents defined in Eq. (3.6) but this is not of great importance for what follows. The two sets of commuting spin currents for each channel obey the Kac-Moody algebra at level \( k = 1 \), and the associated Hamiltonians, \( \mathcal{H}_{s1} \) and \( \mathcal{H}_{s2} \), each have conformal anomaly \( c = 1 \). It is natural to rewrite the theory in terms of the total spin currents:

\[ \vec{J}(x) \equiv \vec{J}_1(x) + \vec{J}_2(x) \]  

(3.25)

These obey the Kac-Moody algebra with \( k = 2 \). The associated Hamiltonian, constructed from \( \vec{J} \) using Eq. (3.21) with \( k = 2 \), \( \mathcal{H}_s \), has \( c = 3/2 \). A crucial point is that \( \mathcal{H}_{s1} + \mathcal{H}_{s2} \) and \( \mathcal{H}_s \) both generate time-translation for the total spin currents. I.e., they both obey the commutation relations of Eq. (3.24) with the total spin currents. Since \( \mathcal{H}_s \) is itself quadratic in these total spin currents it follows that

\[ [\mathcal{H}_{s1}(x) + \mathcal{H}_{s2}(x) - \mathcal{H}_s(x), \mathcal{H}_s(y)] = 0 \]  

(3.26)

Thus the spin Hamiltonian, \( \mathcal{H}_{s1} + \mathcal{H}_{s2} \), can be written as a sum of two commuting pieces, \( \mathcal{H}_s \) and a remainder. This is an example of the Goddard-Kent-Olive coset construction. The remainder is associated with the coset \([SU(2) \times SU(2)]/SU(2)_D\) in this case. (“D” represents the diagonal subgroup.) The value of the conformal anomaly for the coset Hamiltonian is \( c = 1 + 1 - 3/2 = 1/2 \). The coset Hamiltonian obeys the Virasoro algebra with this value of \( c \). There is a unique unitary conformal field theory with this value of \( c \), namely the Ising model.
Thus rather than replacing the two spin bosons for each channel by a total spin boson and a difference boson we replace them by a $k = 2$ Kac-Moody conformal field theory and an Ising model. The former theory can be considered to be the left-moving part of a $k = 2$ WZW model. i.e., we may construct the various operators out of the left-moving factor of a unitary matrix, $g_\alpha$. The value of $k$ for this matrix field is the coefficient of the Wess-Zumino topological term in the Lagrangian. We may now write down representations for the various operators in the free fermion theory as products of charge (or isospin) bosons, the total spin boson, $g_\alpha$ and the Ising field. The $k = 2$ WZW model has primary fields of spin $j = 0$ (identity operator, 1), $j = 1/2$ (fundamental field, $g_\alpha$) and $j = 1$ (denoted $\phi$). The $k = 1$ WZW model only has the identity operator and the $j = 1/2$ primary, $h_A$. Their scaling dimension is given by the general formula:

$$x = \frac{j(j+1)}{2+k}$$

(3.27)

There are three primary fields in the Ising model: the identity operator, 1, the Ising order parameter, $\sigma$ of dimension $x = 1/16$ and the energy operator $\epsilon$ with dimension $x = 1/2$. [These are dimensions of left-moving factors only. Thus the order parameter has total dimension $1/8$ corresponding to a correlation exponent, $\eta = 2x = 1/4$. The energy operator has total dimension $x = 1$ corresponding to a thermal exponent, $\nu = 2 - x = 1$.] The fermion field is written in this representation as:

$$\psi_{\alpha i} \propto (h_i)_1 g_\alpha \sigma$$

(3.28)

Note that the three factors have dimensions which add up correctly to that of the fermion operator: $x = 1/4 + 3/16 + 1/16 = 1/2$. This determines uniquely the representation. Other operator representations can be determined using the operator product expansion (OPE). For the $SU(2)$ WZW fields, $h_i$ and $g$, the OPE of two primary fields of spin $j$ and $j'$ gives each primary field with spin from $|j - j'|$ up to the minimum of $j + j'$ and $k - j - j'$. For the Ising model the OPE gives:

$$\sigma \times \sigma \rightarrow 1 + \epsilon, \quad \sigma \times \epsilon \rightarrow \sigma, \quad \epsilon \times \epsilon \rightarrow 1$$

(3.29)

This OPE is equivalent to that of the $k = 2$ WZW model with the identification of Ising and WZW primary fields:

$$\sigma \leftrightarrow g, \quad \epsilon \leftrightarrow \phi$$

(3.30)

Using the OPE, symmetry considerations and consistency of scaling dimensions, we can determine the representation of any operator in the free fermion theory. An important point is that not all products of isospin, spin and Ising operators occur in the representation of free fermions. When taking the OPE of a product of operators representing a free fermion operator, only a subset of all operators in the products of the OPE’s for each factor occurs. This type of representation of free fermions is known as a conformal embedding. The complete set of products of primary fields that occurs in the representation of free fermions is given in Table I. Note that we only list primary fields. Other operators such as $\vec{J}$ are descendents.
A subtlety arises concerning parity. Parity takes $\psi_o \rightarrow -\psi_o$ in the one-dimensional theory. In the alternative basis of Eq. (3.3) it interchanges $\psi_1$ and $\psi_2$. It thus follows that it interchanges the two isospin fields, $h_1$ and $h_2$. However, that is not the whole story as we see by considering the operator $\psi_1^\dagger \vec{\sigma} \psi_1 - \psi_2^\dagger \vec{\sigma} \psi_2$. This is odd under parity but has the representation $\vec{\phi}$, independent of the isospin fields. To obtain a consistent definition of parity we define it to also take $\vec{\phi} \rightarrow -\vec{\phi}$.

In the Bose-Ising representation the Kondo interaction is written as:

$$H_K = \tilde{J}_+ \vec{J}(0) \cdot (\vec{S}_1 + \vec{S}_2) + \tilde{J}_m \vec{\epsilon}(0) \cdot (\vec{S}_1 - \vec{S}_2) + \tilde{J}_- (h_1^\dagger A(0) h_2 A(0)) \vec{\phi}(0) \cdot (\vec{S}_1 + \vec{S}_2)$$

(3.31)

Here the three coupling constants, $\tilde{J}_\pm$, $\tilde{J}_m$ are proportional to the ones defined in Eq. (3.4). Note that while $\tilde{J}_+$ only couples the total spin field, $\tilde{J}_m$ also couples the Ising field and $\tilde{J}_-$ also couples the charge fields. Since $\tilde{J}_-$ will turn out to be irrelevant for the particle-hole symmetric case being considered here, it follows that the relevant part of the Kondo interaction only involves the total spin and Ising fields.

We may write the finite-size spectrum in terms of the “bosonized” representation. Here we take the one-dimensional effective theory and impose convenient boundary conditions on it on a line of finite length. Because of the rather unsymmetrical way that the dimensional reduction takes place [see Sec. II], these boundary conditions would not arise from any simple or natural ones on the original three-dimensional problem. Nevertheless we are interested in considering the finite system for two reasons. The first is that the Wilson numerical renormalization group method essentially studies a finite one-dimensional system. We will make detailed comparisons of the finite-size spectrum with the results of this method. The second reason is that there is an intimate connection between the finite-size spectrum and the operator content (for the infinite system) which we exploit. Thus we restrict the left-moving fermions to the interval:

$$-l < x < l$$

(3.32)

and impose the boundary condition:

$$\psi_L(l) = -\psi_L(-l)$$

(3.33)

We note that in the formalism with left and right movers on the positive $x$-axis [see Eq. (3.10)] the boundary condition becomes:

$$\psi_L(l) + \psi_R(l) = 0$$

(3.34)

The free fermion spectrum is obtained by populating the single fermion momentum eigenstates with momentum $k = (n + 1/2)\pi/l$. Note that with these boundary conditions, the groundstate is unique with all negative $k$ states filled and positive $k$ states empty. ($k$ is measured from $k_F$.) In general states have energies (measured from that of the ground-state) of the form $E = (\pi/l)x$ where $x$ is an integer or half-integer. There is a one-to-one correspondence between states and operators with these boundary conditions; the value of $x$ corresponding to the dimension of the operator. For instance, $x = 1/2$ corresponds to
a single particle or hole state. It is created from the groundstate by a single application of the fermion field. Just as the operators can be written as products of charge, spin and Ising operators, the states can be written as direct products of states from each sector. In each sector, the states are grouped into several conformal towers with energies of the form $E = (\pi/l)(x+n)$ where $x$ is fixed and fractional and $n$ is an integer. The conformal towers are in one-to-one correspondence with the set of primary fields, with $x$ being the scaling dimension. The set of products of conformal towers from each sector which can occur in the free fermion spectrum corresponds to the set of products of primary fields that occur in representing free fermion operators. Thus it can be read off from Table I.

IV. FINITE-SIZE SPECTRUM

In this section we postulate a critical theory of the unstable multicritical point which occurs in the particle-hole symmetric case at a critical value of the inter-impurity coupling, $K$. We derive the corresponding finite-size spectrum and compare it to the results of the NRG.

A fundamental assumption behind our approach is that the critical point can be described as a conformally invariant boundary condition. We expect this to be generally true for a wide class of critical phenomena involving quantum impurities. Indeed this assumption is very analogous to the one which is widely made in studying bulk critical phenomena in two-dimensional [or (1+1)-dimensional] systems. There it is assumed that a wide class of critical phenomena exhibits conformal invariance. Although such an assumption can rarely be proven, it is consistent in the following sense.

We first of all assume scale invariance at the critical point. This could be taken as a definition of criticality. We then argue that the critical theory should have $SO(2)$ or Lorentz [$SL(1,1)$] invariance. Although this is not a symmetry of most underlying microscopic theories it can be seen that the operators which break this symmetry down to whatever subgroup exists in the microscopic theory are irrelevant. In the case of a two-dimensional theory defined on the square lattice, the subgroup of $SO(2)$ is the symmetry group of the square lattice. This permits non-rotationally invariant derivative terms in a Landau-Ginsburg theory such as $(\partial \phi/\partial x)^4 + (\partial \phi/\partial y)^4$. All such terms are irrelevant. For a one-dimensional quantum fermion system the dispersion relation is not usually exactly linear. However, the additional terms in the Hamiltonian reflecting the non-linearity involve at least two derivatives and hence are usually irrelevant. (Of course the situation changes if the linear term vanishes.) Once we have convinced ourselves that our critical theory is scale invariant and Lorentz invariant it follows immediately that it must be conformally invariant.

A quantum impurity problem cannot be invariant under the full conformal group. Indeed it is not even Lorentz invariant since a special point is singled out; or a special line, $x = 0$ in the space-time description. The maximal symmetry that we could hope for is the subgroup of the conformal group which leaves this line fixed. In the imaginary time formulation, introducing the complex co-ordinate, $z = v\tau + ix$, the full conformal group is all analytic transformations $z \rightarrow w(z)$. The subgroup leaving the line $x = 0$, (the real axis) invariant is the set of transformations for which $w(\tau) \in \mathcal{R}$. Taylor expanding the analytic function:
we see that while, for a general conformal transformation the $a_n$'s can be arbitrary complex numbers, in the presence of a boundary they must all be real. Thus, the (infinite) number of symmetry generators is reduced by a factor of 2. This infinite set of symmetries corresponds to time translations and rescalings of space and time which may be performed independently at each space-time point.

We generally expect that a system which exhibits conformal invariance far from a boundary will exhibit boundary conformal invariance as the boundary is approached. Such a system could be a two-dimensional statistical system at its bulk critical temperature with a boundary. It could also be a one-dimensional quantum system in a gapless phase with linear dispersion relation in the presence of a quantum impurity. Note that we have managed to formulate the two-impurity Kondo model as such a system. In the left-right formalism the model is defined on the positive $x$-axis with both impurities at $x = 0$. Of course, the universal critical behavior only emerges in the scaling limit. If we calculate correlation functions we only expect universal behavior when all points are well separated. Furthermore all points must be far from the boundary compared to microscopic scales. Since the ratios of distances from the boundary to distances between the points remain as free parameters it is still possible to observe critical behavior which is affected by the boundary. Generally we can also make universal predictions when the points are close to the boundary but far from each other. However in this case only exponents, not amplitudes are universal. Again if we assume invariance under time translations and global rescaling of space or time the local invariance follows.

The specification of conformally invariant boundary conditions can be conveniently formulated in terms of the finite-size spectrum. We consider a system defined on a cylinder of length $l$ and circumference $\beta$ (ie. a quantum system at temperature $T = 1/\beta$) with some conformally invariant boundary conditions $A$ and $B$ at the two ends. Conformal invariance implies that the spectrum can be specified in terms of the conformal towers of the periodic left-moving system on an interval of length $2l$. The partition function can only be a sum of partition functions for each conformal tower with a multiplicity factor:

$$Z(l/\beta) = \sum_i n_{iAB}^i Z_i(l/\beta)$$

Various pairs of conformally invariant boundary conditions correspond to various sets of integers $n_{iAB}^i$. It turns out that not all possible choices of integers $n_{iAB}^i$ correspond to a pair of conformally invariant boundary conditions. Cardy derived a set of powerful consistency equations that these integers must obey\[[4]\] In the particular case where the theory is a product of several decoupled theories, as for all versions of the Kondo problem, the set of conformal towers in Eq. (4.2) must be summed over all products of conformal towers from each sector. For the two-impurity Kondo problem, there are four sectors: the two isospins, total spin and Ising. The number of conformal towers in each sector is two for each isospin (labelled by the isospin of the primary field, $i = 0$ or $i = 1/2$) three for spin ($j = 0, 1/2, 1$) and three for Ising ($1, \sigma$ and $\epsilon$). Thus altogether there are 36 products of conformal towers and 36 integers to specify. For the free fermion spectrum discussed in the previous section,
six of these integers have the value 1 and the rest are 0. The 6 products of conformal towers that occur are given in Table I.

It turns out that a useful way of generating new conformally invariant spectra from old ones is “fusion” \[^3\]. This corresponds to a particular mapping of each conformal tower into a set of other ones. The fusion rules are not obviously related to boundary critical phenomena per se but come from the bulk operator product expansion. If we consider the OPE of two bulk primary operators, \( \mathcal{O}_i \) and \( \mathcal{O}_j \), then it will contain the complete set of primary fields \( \mathcal{O}_k \) with multiplicities \( N_{ij}^k \). The new spectrum is obtained by the replacement:

\[
n^i \to \sum_k N_{jk}^i n^k
\]

for any fixed \( j \). This procedure gives a new solution of Cardy’s consistency equations for any choice of the conformal tower, \( j \). The fusion rule coefficients are a property of each sector of the theory independently. They were given for the Ising sector in Eq. \((3.29)\).

As a trivial example of a new conformally invariant spectrum generated by fusion we consider fusion with the \( j = 1 \) field in the spin sector, \( \vec{\phi} \). We see from the fusion rules that this simply interchanges the \( j = 0 \) and \( j = 1 \) conformal towers. The number of products of conformal towers remains six and is given in Table II. Notice that in this case the groundstate is 16-fold degenerate. This spectrum corresponds to free fermions with a \( \pi/2 \) phase shift. I.e. the Fermi level now sits at one of the single-particle energy levels. Since each of the four species of Fermions (two channel times two spins) can have this level filled or empty, this accounts for the 16-fold degeneracy. In fact precisely this spectrum describes the Kondo fixed point which occurs for large ferromagnetic inter-impurity interaction. We also note, that the same spectrum is obtained by fusion with either the \( \epsilon \) field in the Ising sector or the \( i_1 = i_2 = 1/2 \) field in the isospin sector.

The other possible fusion in the Ising sector is with the \( \sigma \) field. This gives the non-trivial critical point in the two-impurity Kondo problem. The resulting spectrum is given in Table III. In this case there are eight products of conformal towers. Note that the same spectrum can be obtained by beginning with the spectrum after \( \vec{\phi} \)-fusion, of Table II and then performing \( \sigma \)-fusion. Thus the non-trivial fixed point is symmetric with respect to the zero phase shift and \( \pi/2 \) phase shift fixed points. We now wish to compare this in detail with the NRG results. This is done by multiplying together the various conformal towers. A general state will have energy \( E = (\pi v/l)(x + n) \). Here \( x \) is the energy of the primary (or highest weight) state and the integer \( n \) is the descendent level. Both \( x \) and \( n \) are obtained by summing over the four sectors. The values of \( x \) that occur for the eight products of conformal towers are given in Table III. We will measure energies from that of the groundstate; thus \( x - 1/16 \) occurs. We will content ourselves with working out the energies of all states with \( x + n - 1/16 < 2 \). This implies \( n = 0 \) or \( 1 \) only. Thus we only need know the first descendents in each conformal tower. Altogether there are 8 different conformal towers in the three inequivalent component theories (\( k = 1 \) WZW, \( k = 2 \) WZW and Ising). All first descendents of all 8 conformal towers are given in Table IV. We now obtain the complete spectrum at the non-trivial fixed point with \( x + n - 1/16 < 2 \) by taking either the primary state in each sector or taking a first descendent in one sector and primaries in the other three. This gives the set of states shown in Table V. Recall that parity interchanges the two isospins and also multiplies by \((-1)\) for the \( \vec{\phi} \) conformal tower. The spectrum is
symmetric between the two isospins. By taking symmetric or antisymmetric products of the two isospins we obtain states of definite parity.

The general particle-hole symmetric Hamiltonian ((2.6) or (2.9), with arbitrary coefficients) has three symmetries: total isospin $\vec{I} = \vec{I}_1 + \vec{I}_2$, total spin, and (exchange) parity. When the coupling constant $J_\perp = 0$, a further symmetry develops, and one can describe the eigenstates with an alternative set of three quantum numbers: the two isospins, $i_1$ and $i_2$, and total spin. This was actually the case in the NRG calculation to which we compare the conformal field theory (CFT) spectrum. Eigenstates within the same $i_1$, $i_2$ isospin-parity multiplets are seen in Table V to be degenerate to better than one part in $10^4$, an indication of the accuracy to which the symmetry holds in the NRG calculations.

In Table V we express the lowest eigenstates via both sets of quantum numbers. Columns one and two constitute a complete set: the two isospins $(i_1, i_2)$ and the total spin $(j)$. To facilitate comparison with the NRG, we have also resolved the spectrum into states of definite $i$ (total isospin), $j$ (total spin), and $P$, parity. (Note that resolving states of definite $i$ picks out states of definite symmetry with respect to interchanging the two isospins and hence determines states of definite parity. In particular the $i = 0$ part of an $(i_1, i_2) = (1/2, 1/2)$ multiplet is antisymmetric and the $i = 1$ part is symmetric. An additional intrinsic odd parity must be included for all states whose spin factor is in the $j = 1$ conformal tower.) Column five of Table V indicates $i$ and $P$ ($+ \equiv$ even, $- \equiv$ odd) in this alternative formulation. Altogether we obtain 49 multiplets with definite $(i, j, P)$. In the last column in Table V we give the energies of the lowest states of the same quantum numbers obtained using the NRG.

Very briefly, numerical renormalization group calculations iteratively diagonalize a Hamiltonian, bringing in successive low-energy degrees of freedom until asymptotically the ground state is reached. After each iteration, the set of energy levels is ordered and truncated to keep only a fixed number of the lowest-energy states. Truncation errors are the main source of inaccuracy in the energy levels, and although the truncation approximation is expected to be a good one for the very lowest energies, the uncertainty (typically a few percent at most) increases with increasing energy. For the results in column six of Table V, approximately 1100 states were kept at each iteration. The coupling constant $\nu J = 0.18$ (c.f. Eqs. (2.6, 2.7)). For this value of $J$, the unstable fixed point is fully realized by 14 iterations.

To determine the critical point, the ratio of RKKY coupling to Kondo temperature was varied until the region of unchanging energy levels was maximized. In practice, of course, numerically one can never exactly sit at the critical point; since the critical point occurs for a finite (probably irrational) value of the coupling parameters, it is in practice only possible to choose initial parameters such that the flows are asymptotically close to the critical point. And, in fact, no matter how carefully one adjusts the parameters, at any iteration there are always some energy levels that are not completely flat. Since for an inexact choice of parameters the critical region does not extend all the way to $T = 0$, one must choose a temperature (iteration number) at which the approach to the critical point is closest. These two facts, that one is never right at the unstable point in parameter space, and that even with a very good initial parameter choice, it is not clear which iteration to choose to best represent the unstable fixed point, thus introduce additional uncertainty in the numerical
energy values in Table V.

The agreement with NRG nonetheless seems very satisfactory. The 49 NRG states shown in Table V consist of numbers 1−47, 68−69 in a consecutive numbering by increasing energy. The overall energy scale, set by \( v \), is not determined by the NRG so we have adjusted it to fit the CFT spectrum. In a few cases the CFT predicts several multiplets with identical quantum numbers and the same energy, so the matching with the NRG levels involves some arbitrary choices. Otherwise, there are no free parameters. The agreement is to within about 2\% for the first 17 levels but gets progressively worse at higher energies. At the highest energies shown the disagreement is about 5\% for most of the levels. The two sets of multiplets involving the \( \frac{1}{2}^\prime \) descendent in the spin sector (column 2 in Table V) have unusually large disagreements with CFT compared to other states at the same energy, 11\% and 19\% respectively. It may be possible to understand this in terms of the contribution of the leading irrelevant operator to finite-size effects.

We regard Table V as rather convincing evidence that we have identified the correct boundary CFT critical point for this problem and that the NRG method is highly accurate.

It seems clear that the spectrum of Table V could not be obtained from simply imposing linear boundary conditions, respecting spin-rotation symmetry, on the free fermion fields, and filling up the corresponding levels using Fermi statistics. Note, for example, the strange ratios of energy gaps: 3 : 4 : 7 : 8 and the peculiar 7-fold degeneracy of the states with \( x - 1/16 = 1/2 \). Thus we refer to this as a “non-Fermi liquid spectrum”. Nonetheless, it is possible to describe this spectrum in terms of free Majorana Fermion field and their spinor representations, using the alternative Ising ×SO(7) \(_1\) bosonization scheme of Ref. (14). These fields are related in a non-local way to the original conduction electron degrees of freedom. A related physical picture can also be found in Ref.’s (15,16). We explain this hidden symmetry in detail in Sec. V.

V. OPERATOR CONTENT

The operator content at the non-trivial fixed point is determined by “double fusion” with \( \sigma \). This is because we must effectively consider the spectrum on a strip with the same boundary condition at each side. The result in shown in Table VI. To understand the stability of the non-trivial fixed point we must carefully consider all relevant and marginal boundary operators \((x \leq 1)\) which are allowed by symmetry to appear in the Hamiltonian. (As usual in critical phenomena we expect that “anything that can happen will happen”.)

Apart from the identity operator, there are seven other multiplets of relevant operators with \( x = 1/2 \). If we assume that \( \vec{J}, \vec{I}_1 \) and \( \vec{I}_2 \) are all conserved then only the seventh operator in Table VI, which we may write simply as \( \epsilon \), may appear in the Hamiltonian. The \( \vec{J}_- \) coupling constant of Eq. (3.31) breaks isospin down to the diagonal subgroup. This might seem to permit an element of the eighth operator multiplet in Table VI: \((h_1^1)^A(h_2)_A^B\). Using Eq. (3.16), this can be rewritten as \((h_1)_A^1(h_2)_B\). Since we may regard the fields \( h_1 \) and \( h_2 \) as commuting, it follows that this operator is odd under parity. This conclusion also follows from the form of the Hamiltonian in Eq. (3.31); since \( \vec{\phi} \) is odd under parity, so must be \((h_1^1)^A(h_2)_A^B\). Similarly the various marginal primary operators in Table VI are not allowed by symmetry. There are also marginal descendent operators, the currents \( \vec{J} \) and \( \vec{I}_i \). These
are forbidden by spin and (diagonal) isospin symmetries.

Hence we reach the important conclusion that there is only one relevant operator permitted, $\epsilon$. It follows that the non-trivial critical point can be reached, in general, by tuning one parameter to its critical value. We could vary any of the coupling constants in the problem to reach the critical point. The physical picture is clearest if we vary the RKKY coupling $K$, as explained in the introduction. We expect that varying $K$ away from its critical value, $K_c$, drives the system to one of the stable fixed points with either a $\pi/2$ phase shift in both channels or zero phase shift in both channels. The coefficient of $\epsilon$ in the effective Hamiltonian should be proportional to $K - K_c$. In particular, it is important to note that for the case of particle-hole symmetry, tuning such that $J = 0$, that is, setting even and odd-parity couplings equal, is not necessary for obtaining the critical point as long as $K = K_c$.

There is a formal analogy between this renormalization group flow and one that occurs in the classical Ising model. Consider a two dimensional Ising model at its critical temperature, defined on a half-plane with free boundary conditions. Then apply a weak magnetic field at the boundary only, pointing up or down. The free boundary condition represents an unstable fixed point; applying the magnetic field drives the system to the stable boundary condition with spins pointing up or down at the boundary. The local magnetic field corresponds to the $\epsilon$ operator of dimension $1/2$. The three different boundary conditions, spin up, spin down and free are related to each other by fusion. $\epsilon$ fusion takes spin up into spin down and vice versa; $\sigma$ fusion takes spin up or down into free.

We may approach the critical point in various ways; one is by setting $T = 0$ and letting $K$ approach $K_c$; another is by setting $K = K_c$ and letting $T$ go to zero. If we tune $K$ to $K_c$, then the corrections to scaling behaviour are governed by the leading irrelevant operator. None of the other primary operators in Table VI are allowed by symmetry. The leading irrelevant operator is the $x = 3/2$ first descendent $\epsilon'$. It can be checked that the various other dimension $3/2$ operators are not allowed by symmetry. In particular, $\vec{J}_- \cdot \vec{\phi}$, the leading irrelevant operator in the multi-channel, single-impurity Kondo effect is not allowed by parity.

We may define the coefficient of the $\epsilon'$ term in the effective Hamiltonian as $1/\sqrt{T_K}$, where $T_K$ is the Kondo temperature. Thus, at low temperatures and for $K$ close to $K_c$, we must consider the following Hamiltonian:

$$H = H_{FP} + \alpha \frac{(K - K_c)}{\sqrt{T_K}} \epsilon + \frac{1}{\sqrt{T_K}} \epsilon', \quad (5.1)$$

where $\alpha$ is a constant and $H_{FP}$ is the Hamiltonian at the fixed point, with the non-trivial boundary condition. We first consider the properties of $H_{FP}$ itself, assuming that $K = K_c$ and ignoring the effects of the irrelevant operator. We then consider a non-zero $\epsilon'$ with $\epsilon$ still zero, corresponding to approaching the critical point by setting $K = K_c$ and $T \to 0$. Finally we consider $\epsilon$ and $\epsilon'$ both non-zero corresponding to a general approach to the critical point.

A quantity which characterizes the critical point is the residual impurity entropy. This is defined by taking the length of the system to $\infty$ first and then taking the temperature to zero. In general the impurity entropy, i.e. the part which does not scale with length, but rather goes to a constant as the length goes to $\infty$, goes to a finite non-zero constant as $T \to 0$. This quantity is always the log of an integer for a Fermi liquid fixed point, the
integer simply being the degeneracy of the decoupled impurity. In the two-impurity Kondo problem, we expect zero degeneracy at the stable fixed points since the impurity spins are always in singlet states. At a non-Fermi liquid fixed point, the residual impurity entropy, $S(0) \equiv \ln g$ where $g$ need not be an integer, in general. When the fixed point is obtained by fusion, a general expression for $g$ has been derived in terms of the modular S-matrix. This gives:

$$g = \frac{S_\sigma}{S_I} = \sqrt{2}. \quad (5.2)$$

Note that $g$ decreases under renormalization from the unstable critical point to the stable ones, in accord with the “$g$-theorem”.

Another striking signature of the non-trivial critical point is provided by the zero-energy one-particle S-matrix, $S(1)$. This is the amplitude for a single electron to scatter off the impurity and produce a single electron final state. If the system exhibited Fermi liquid behavior then $S(1)$ would have unit amplitude since all multi-particle final states would have vanishing probability at zero energy. Thus $S(1)$ would reduce to a phase shift, $e^{2i\delta}$. However, at a non-Fermi liquid fixed point, the probability of inelastic scattering (one particle into many) does not vanish at zero energy so $S(1)$ can have non-unit amplitude without violating unitarity. $1 - |S(1)|$ provides a direct measurement of how “non-Fermi liquid like” the critical point is. It is a universal quantity characterizing the various fixed points. It is defined precisely by the asymptotic behaviour of the single electron Green’s function. In the absence of any impurity couplings, this would have the non-interacting form:

$$\langle \psi_i^{\alpha \dagger}(z) \psi_{\beta j}(z') \rangle \rightarrow \frac{\delta_{\beta i} \delta_{\alpha j}}{z - z'}. \quad (5.3)$$

This dependence on spin indices will always be of the above form, due to spin-rotation symmetry, so we will suppress it in what follows. Ignoring the irrelevant, $J_-$ term in Eq. (3.4), the same is true of the dependence on the channel indices. If $x$ and $x'$ have the same sign, the form of Eq. (3.3) holds even in the presence of impurity interactions. This is a general feature of conformally invariant boundary conditions. On the other hand, when the two fields at $x$ and $x'$ straddle the impurity, then different asymptotic behaviour can emerge, which depends on the universality class of the boundary interactions. The most general form consistent with conformal invariance and spin and channel symmetries is simply:

$$\langle \psi_i^{\alpha \dagger}(z) \psi_{\beta j}(z') \rangle \rightarrow \frac{\delta_{\beta i} \delta_{\alpha j} S(1)}{z - z'}. \quad (5.4)$$

where $S(1)$ is a universal amplitude, which we can interpret as the S-matrix element for one-particle into one-particle scattering. In the non-interacting case, $S(1) = 1$. This should also be the case at the stable Fermi liquid fixed point where the two impurities form a singlet. On the other hand, at the Fermi liquid fixed point where both impurities are Kondo screened, $S(1) = -1$, corresponding to a $\pi/2$ phase shift. As mentioned in Sec. I, $S(1)$ must be real in all cases by particle-hole symmetry. $S(1)$ is given by the matrix element of the $\sigma$ operator (since this is the Ising factor of the fermion operator, $\psi$) between the boundary state and the vacuum state. Since the boundary state is obtained by fusion with the $\sigma$ primary field, this gives the result:
where \( S^\text{I}_i \) is the modular S-matrix (which has no obvious connections with the scattering matrix) of the Ising model. It turns out that \( S^\text{I}_\sigma = 0 \), while the other matrix elements are non-zero. Hence, we conclude that:

\[
S^{(1)} = 0 
\]  

(5.5)

at the non-trivial fixed point. This implies that the scattering is entirely inelastic at zero energy. Thus, is a sense, this fixed point is as non-Fermi liquid like as possible. The fact that \( S^{(1)} = 0 \) can be understood as a consequence of the symmetry between the trivial fixed points with \( S^{(1)} = 1 \) and \( S^{(1)} = -1 \). As mentioned in Sec. III, the non-trivial fixed point can be reached by \( \sigma \)-fusion starting from either trivial fixed point. This implies that \( S^{(1)} \) at the non-trivial fixed point is proportional to its value at both trivial fixed points with the same constant of proportionality. Since \( S^{(1)} \) has opposite sign at the two trivial fixed points, it follows that it must vanish at the non-trivial one.

If we relax the assumption that \( \tilde{J} = 0 \), then the Green’s function need no longer be diagonal in channel indices. We may also relax the assumption of particle-hole symmetry. In general, at a Fermi liquid fixed point we may write:

\[
< \psi_i^{\alpha \dagger}(z) \psi_{j}(z') > \rightarrow \frac{\delta_{\alpha \beta} S^{(1)}_{ij}}{z - z'},
\]

(5.7)

where \( S^{(1)}_{ij} \) is a general unitary matrix. It is the non-unitarity of \( S^{(1)} \) which signals non-Fermi liquid behaviour.

Ignoring all (relevant or irrelevant) perturbations about the critical point, the impurity specific heat and the uniform impurity susceptibility, which is the response of the system to a conserved quantity, vanish. This occurs for the same reason as in the single impurity case and we refer the reader to Ref. (11). On the other hand the staggered susceptibility does not vanish. This is defined in terms of the correlation function of \( \vec{S}_1 - \vec{S}_2 \). This operator is odd under parity so we expect that it reduces to the primary field \( \vec{\phi} \) at the critical point. This operator has dimension 1/2 giving a correlation function:

\[
< [\vec{S}_1 - \vec{S}_2](\tau) \cdot [\vec{S}_1 - \vec{S}_2](0) > \propto \frac{1}{\tau}
\]

(5.8)

This is the same as the impurity spin correlation function in the overscreened two-channel single impurity case mentioned above. Fourier transforming at finite temperature and analytically continuing to real frequency, we obtain [see third reference in [3] the staggered susceptibility. The real part behaves as:

\[
\text{Re} \chi_s(\omega, T) \propto \frac{1}{T_K} \ln(T_K/T) \quad (\omega << T)
\]

(5.9)

\[
\propto \frac{1}{T_K} \ln(T_K/\omega) \quad (T << \omega).
\]

(5.10)

The imaginary part behaves as:
\[ \text{Im} \chi_s \propto \tanh(\beta \omega / 2). \] (5.11)

The electron spin operator with the same symmetry, \( \psi_1^\dagger \sigma_1 \psi_1 - \psi_2^\dagger \sigma_2 \psi_2 \) should also couple to \( \phi \) and exhibit the same decay, close to the boundary. The universal crossover function from bulk to boundary behaviour could be calculated following the techniques of Ref. (3). Various other operators exhibit the same \(|\tau|^{-1}\) decay due to coupling to other \( x = 1/2 \) boundary operators in Table VI, \( \epsilon \) and \( (h_1)_A (h_2)_B \). As already remarked, the various interaction terms in the Hamiltonian such as \( \vec{S}_1 \cdot \vec{S}_2 \) couple to \( \epsilon \) and so should have this decay. Similarly, the spin-singlet, channel symmetric pair operator, \( (\psi_{a_1} \psi_{J_2} + \psi_{a_2} \psi_{J_2}) \epsilon^{a_1 b} \) should couple to \((h_1)_1 (h_2)_1\).

Now we consider the effect of the leading irrelevant operator, \( (1/\sqrt{T_K}) \epsilon' \), still assuming that \( K = K_c \). We can calculate physical properties by doing perturbation theory in \( 1/\sqrt{T_K} \). This is similar to the calculations for the two-channel single impurity, \( s = 1/2 \) Kondo effect, where the leading irrelevant operator also has dimension 3/2. However, in this case the leading irrelevant operator is a Virasoro descendent, rather than a primary. Indeed, we may consider it to be the derivative of the primary field, \( \epsilon' \approx d \epsilon / d \tau \). Consequently the finite temperature two-point function has a different form than in the two-channel case and we find that there is no logarithmic singularity in the specific heat. I.e.,

\[ c_{\text{imp}} \propto T / T_K. \] (5.12)

Note that the uniform susceptibility [ie. the response to a field coupling to the conserved total spin of impurities plus conduction electrons] obtains no contribution to second order in perturbation theory in the \( \epsilon' \) term and therefore also has no logarithmic singularity. This is again unlike the two-channel case where another dimension 3/2 irrelevant operator, \( \vec{J}_{-1} \cdot \vec{\phi} \) is present in the effective Hamiltonian. It is absent in the two-impurity case by parity. We expect a finite impurity susceptibility as \( T \to 0 \), of \( O(1/T_K) \), coming from the dimension two irrelevant operator \( \vec{J}^2 \).

Let us now consider the behavior for \( K \) slightly displaced from \( K_c \). As mentioned, we expect the system to renormalize to one or the other of the stable Fermi liquid fixed points. At these fixed points the impurity spins either form an inter-impurity singlet or else are Kondo-screened. There are various leading irrelevant operators of dimension 2. One such operator comes from the Ising sector; it is the Ising energy-momentum tensor, \( T_I \). When \( K \) is very close to \( K_c \) the system does not renormalize away from the unstable critical point until a very low energy scale is reached of order \( (K - K_c)^2 / T_K \). This plays the role of the lowest effective Kondo temperature at the stable fixed point. Thus we expect a term of the form

\[ \delta H = \frac{T_K}{(K - K_c)^2} T_I \] (5.13)
in the effective Hamiltonian at the stable fixed point. The other leading irrelevant operators, such as \( \vec{J}^2 \) in the spin sector, presumably have coefficients which remain finite as \( K \to K_c \). We expect this because the delayed flow away from the unstable critical point only occurs in the Ising sector. The uniform impurity susceptibility goes to a constant as \( T \to 0 \) and can be determined by first order perturbation theory in the leading irrelevant operators. It
should remain finite as $K \to K_c$, because it doesn’t involve any Ising sector fields and is determined by the coefficient of $\vec{J}^2$. On the other hand, the specific heat does get a first order contribution from the irrelevant operator of Eq. (5.13). Therefore, we expect a specific heat slope,

$$\gamma \propto \frac{T_K}{(K - K_c)^2}. \quad (5.14)$$

In general, for finite $T$ and finite $K - K_c$, we expect the specific heat to be given by some universal scaling function:

$$c_{\text{imp}} = \frac{T}{T_K} f \left( \frac{T}{T_K}, \frac{K - K_c}{T_K} \right), \quad (5.15)$$

where the scaling function $f(x, y)$ has the asymptotic behaviour:

$$f(x, 0) \to \text{constant} \quad (x \to 0)$$

$$f(0, y) \to \text{constant} \cdot \frac{y^2}{y} \quad (y \to 0). \quad (5.16)$$

These scaling results are consistent with those of the NRG, which, at $T = 0$ find that $C_{\text{imp}}/T$ diverges as $1/(K - K_c)^2$, and that the uniform susceptibility is non-singular as $K \to K_c$ in the presence of particle-hole symmetry.

We now wish to consider the effect of particle-hole symmetry breaking. The first type of P-H symmetry, defined in Sec. II corresponds to:

$$\psi_{\alpha i} \to \epsilon_{\alpha \beta} \psi_{\beta i}^\dagger \quad (5.17)$$

In the bosonized representation, this has the effect $(h_i)_1 \to (h_i)^\dagger$ or equivalently:

$$(h_i)_A \to \epsilon_{AB} (h_i)_B \quad (5.18)$$

The relevant particle-hole symmetry breaking interactions correspond to potential scattering terms of the form:

$$\delta H = V_e \psi_e^\dagger \psi_e + V_o \psi_o^\dagger \psi_o \quad (5.19)$$

(Here, the operators are evaluated at the origin in position space.) In terms of the fields, $\psi_i$, this becomes:

$$\delta H = \frac{V_e + V_o}{2} [\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2] + \frac{V_e - V_o}{2} [\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1] \quad (5.20)$$

Upon bosonizing, this gives:

$$\delta H \propto (V_e + V_o)(I_1^z + I_2^z) + \text{constant} \cdot (V_e - V_o) \text{tr}(h_1)^\dagger \tau^3(h_2) \epsilon \quad (5.21)$$

The first term produces the marginal current operator. In terms of an abelian total charge boson, $\phi$, this is proportional to $\partial \phi / \partial x$. We expect this to be exactly marginal and to
produce a line of fixed points with continuously varying phase shifts connected to the non-trivial critical point. This is similar to adding potential scattering to the two-channel $s = 1/2$ critical point. We expect the second operator to produce the relevant operator $\text{tr}(h_1)^{\dagger}\tau^3(h_2)$ at the non-trivial critical point. There are various ways of seeing that this should occur. One is by observing that, after double fusion, $\epsilon$ can turn into the identity operator. Another is simply to observe that this relevant operator transforms the same way as the second term in Eq. (5.21) under all the symmetries of the problem. If $V_e = V_o$ and $\tilde{J}_- = 0$, then the relevant operator is not allowed by symmetry, since the original Hamiltonian has two independent symmetries $I^3_1$ and $I^3_2$ whereas the relevant term only respects the diagonal symmetry $I^3_1 + I^3_2$. If $\tilde{J}_- \neq 0$ then the relevant term is allowed by symmetry even if $V_e = V_o$. In this case we expect it to appear. Thus we conclude that potential scattering is only relevant if $V_e \neq V_o$ or $\tilde{J}_- \neq 0$. We expect that the relevant potential scattering term will drive the system to a Fermi liquid fixed point, corresponding to a unitary $S(1)$ of the general form of Eq. (5.7). In the limit where the relevant potential scattering term, $\text{tr}(h_1)^{\dagger}\tau^3(h_2)$ has a very small coefficient, the flow to the stable, Fermi liquid, fixed point occurs at a very low energy scale, so we again expect that one of the leading irrelevant operators at the stable fixed point will have a large coefficient of $O[T_K/(V_e - V_o)^2]$ (assuming that $\tilde{J}_- = 0$). This time, the large coupling constant should occur in the isospin sector, rather than the Ising sector, as above. Thus the relevant potential scattering term should produce a diverging specific heat coefficient $\propto 1/(V_e - V_o)^2$. We do not expect uniform or staggered spin susceptibilities to diverge as $V_e - V_o \to 0$ in this case, since the spin operators do not involve the isospin fields. However, divergences should occur in charge and pair correlation functions.

Earlier NRG calculations are all in agreement with these predictions of CFT regarding particle-hole symmetry breaking. When $J_- = 0$ and $V_e = V_o$, the two Fermi liquid fixed points and the non-Fermi liquid critical non-trivial critical point are each extended into a line of fixed points with continuously varying phase shift. Potential scattering is thus irrelevant in this case. When $J_- \neq 0$ or $V_e \neq V_o$, all $T = 0$ flows are to a single line of Fermi liquid fixed points, connecting the antiferromagnetic RKKY singlet at $\delta = 0$ to the two-impurity Kondo point at $\delta = \pi/2$. Potential scattering is relevant, and the non-trivial critical point is washed out in this case.

Note that the first type of P-H symmetry is necessary to stabilize the non-trivial fixed point. The second type allows potential scattering with $V_e = -V_o$. This permits the relevant interaction discussed above [the second term in Eq. (5.21)] so the non-trivial critical point should not occur in this case.

Finally, we show that an anisotropy of the Kondo exchange coupling constants $J_\pm, J_m$ gives rise only to an irrelevant perturbation of the non-Fermi-liquid fixed point. This shows that the exchange-anisotropic version of the 2-impurity Kondo Hamiltonian gives rise to the same universal low temperature physics as the isotropic one, described above. More precisely, we consider an anisotropic Kondo exchange coupling, such that the Hamiltonian is only invariant under rotations about the $z$-axis in $SU(2)$-spin space (so far, it has been invariant under rotations about any such rotation axis). To show the irrelevance of the exchange anisotropy, we proceed as in Ref. (13). Since the exchange anisotropy breaks the total spin $SU(2)$ symmetry, some operators may occur in the fixed point Hamiltonian which were not allowed in the isotropic case on symmetry grounds. We will now show that the only additional such operators are irrelevant. To see this we simply need to consult Table
VI. of boundary operators which exist at the non-Fermi-liquid fixed point: Operators of spin \( j = 1/2 \) are not allowed since rotations about \( 2\pi \) about any axis in \( SU(2) \)-spin space leaves the anisotropic hamiltonian invariant, but multiplies these operators by \((-1)\). Thus we are left with the spin \( j = 1 \) operators in Table VI: Amongst those, the only relevant or marginal new operators (scaling dimension \( x \leq 1 \)) that could occur in the anisotropic case are (i): \( \phi^z \), with quantum numbers \((i_1, i_2, j, I_s) = (0, 0, 1, 1)\), (ii): \( \phi^\dagger \epsilon \), with quantum numbers \((0, 0, 1, \epsilon)\), as well as (iii) \( (h_1^\dagger A(h_2)A\phi^z \) with quantum numbers \((1/2, 1/2, 1, 1)\) and (iv): the spin-descendant \( J^z \), with quantum numbers \((0, 0, 1', 1)\). However, operators (i) and (ii) are odd under parity, and are thus not not allowed to occur since the microscopic (exchange anisotropic) Hamiltonian is parity invariant. Operator (iii) changes sign under rotations by \( \pi \) about the \( x \)- or \( y \)-axis in spin space whereas the microscopic Hamiltonian does not. Hence (iii) cannot occur. Operator (iv) is odd under time reversal [this can be seen e.g. from Eq. (3.31)] and thus cannot occur either (the latter does, in any case, not renormalize, and could not destroy the non-Fermi-liquid fixed point for that reason). Hence we conclude that no relevant operators can occur at the fixed point even in the presence of exchange anisotropy.

VI. HIDDEN \( SO(7) \) SYMMETRY

An interesting feature of the spectrum at the non-trivial fixed point, discussed in Sec. III, is various apparently “accidental” degeneracies. For instance the second and third conformal tower in Table III have the same value of \( x - 1/16 = 3/8 \) although they have different values of \((i_1, i_2, j)\). This suggests the presence of some type of symmetry which is higher than the spin and isospin symmetries discussed so far. A related observation was made in Ref. (24): there, an additional conserved quantity was noticed at the non-trivial fixed point, namely, 

\[ \int dx \left( \vec{J}_e - \vec{J}_o \right) \]

where,

\[ \vec{J}_e - \vec{J}_o \equiv \psi_e^\dagger \vec{\sigma} \psi_e - \psi_o^\dagger \vec{\sigma} \psi_o = \psi_1^\dagger \vec{\sigma} \psi_2 + \psi_2^\dagger \vec{\sigma} \psi_1. \]

(6.1)

In fact, a higher symmetry has been found in Ref. (14) and corresponds to the group \( SO(7) \) which contains \( SU(2)^{i_1} \times SU(2)^{i_2} \times SU(2)^j \) as a subgroup. [This is in fact a conformal embedding, \( SO(7)_1 \rightarrow SU(2)^{i_1}_1 \times SU(2)^{i_2}_1 \times SU(2)^j_1 \).] In this section we spell out this symmetry in more detail.

We emphasize at the outset that this symmetry is rather different and more mysterious than the isospin and spin symmetries discussed so far in that it does not become manifest for any choice of the Kondo couplings in Eq. (3.4), except the trivial case where all couplings are zero. In this case the full symmetry is actually \( SO(8) \). This follows because we have 4 species of complex or Dirac fermions, equivalent to 8 Hermitean or Majorana fermions. This \( SO(8) \) symmetry is also present at the Kondo screened fixed point (where the RKKY interaction is strongly ferromagnetic) since the phase shift boundary condition does not break the symmetry between the 8 Majorana fermions. Remarkably, an \( SO(7) \) subgroup of this \( SO(8) \) also remains at the non-trivial fixed point. The key to understanding this symmetry is the realization that the non-trivial critical point is related to the free fermion one by fusion in the Ising sector. Thus what we must do is to represent the free fermions as a
product of an Ising model and another sector of maximal possible symmetry. This symmetry will be preserved under fusion in the Ising sector. The maximally symmetric representation corresponds to a conformal embedding or alternative non-abelian bosonization scheme where the 8 free fermions are represented by an \( SO(7) \) Kac-Moody conformal field theory together with an Ising conformal field theory. This preserves the value of the conformal anomaly or Virasoro central charge parameter \( c \), since the KM algebra \( SO(7) \) has \( c = 7/2 \) and the Ising model, \( c = 1/2 \):

\[
c = 4 = 7/2 + 1/2. \tag{6.2}
\]

There are actually two inequivalent ways of constructing a conformal embedding depending on the \( SO(7) \) representations into which the 8 Majorana fermions, transforming under the vector representation of \( SO(8) \), decompose. The trivial embedding corresponds to:

\[
SO(8)_1 \to SO(7)_1 \times Is \quad 8 \to (7, 1) + (1, \epsilon). \tag{6.3}
\]

[Here and in the following we label \( SO(n) \) representations by their dimensions and conformal towers by the representation of their highest weight state.] This simply takes advantage of the fact that the Ising model is itself equivalent to a free Majorana fermion. We may define the \( SO(7) \) transformation to rotate 7 of the fermions and leave the 8th invariant. However, this is not the appropriate embedding; ie. this Ising model does not appear in the representation of the fermions in the same way as the one which we discussed above. Rather the correct representation corresponds to an alternative embedding in which:

\[
SO(8)_1 \to SO(7)_1 \times Is \quad 8 \to (8, \sigma). \tag{6.4}
\]

Here, the 8 on the right-hand-side of this equation is the 8-dimensional spinor representation of \( SO(7) \). [See any book on Lie groups, e.g. Ref. (28), regarding the properties of the orthogonal groups.] In general, the scaling dimension, \( x \), of a Kac-Moody primary field transforming under a representation \( \rho \) of a group is given by:

\[
x_\rho = \frac{C_\rho}{C_A + k}, \tag{6.5}
\]

where \( C_\rho \) is the quadratic Casimir invariant for the representation \( \rho \), \( C_A \) is the quadratic Casimir for the adjoint representation and \( k \) is the Kac-Moody central extension (level). For the spinor (8), vector (7) and adjoint (21) representations of \( SO(7) \), the Casimirs are:

\[
C_7 = 3 \quad C_8 = 21/8 \quad C_{21} = 5. \tag{6.6}
\]

Taking \( k = 1 \), we obtain for the scaling dimensions:

\[
x_7 = 1/2 \quad x_8 = 7/16. \tag{6.7}
\]
We see that a consistent conformal embedding represents the 8 Majorana fermions, $\chi_A$, of $SO(8)_1$, in terms of the $SO(7)_1$ spinor field, $S_A$, ($A = 1, 2, 3, ... 8$.) and the Ising order parameter field, $\sigma$:

$$\chi_A \propto S_A \cdot \sigma. \quad (6.8)$$

Since $x_\sigma = 1/16$, the dimension adds up correctly to $1/2$ for a free fermion. Comparing to our previous bosonization formula of Eq. (3.28) we note that the Ising factor, $\sigma$, is common to both formulae and that the $SO(7)$ spinor field has replaced the product of $SU(2)$ spin and isospin fields. The free fermion spectrum of Table I can be rewritten in terms of $SO(7)_1 \times$ Ising conformal towers. We may now use the conformal embedding of $SO(7)_1$ conformal towers [labeled by dimensions] into $SU^{i_1}(2)_1 \times SU^{i_2}(2)_1 \times SU^j(2)_2$ conformal towers [labeled by angular momenta]:

$$7 \rightarrow (1/2, 1/2; 0) + (0, 0; 1)$$
$$8 \rightarrow (1/2, 0; 1/2) + (0, 1/2; 1/2), \quad (6.9)$$

(Note that both KM algebras have the same conformal anomaly parameter: $c = 7/2 = 1 + 1 + 3/2$.) From the equation above and TABLE I, we find for the combinations of conformal towers of the free fermion spectrum: $(SO(7)_1, \text{Ising}) = (1, \sigma), (8, \epsilon), (7, \epsilon)$. It is now immediately evident that the $SO(7)$ symmetry will be preserved in any new spectrum obtained from this one by fusion with an Ising field only. In particular, the combinations of conformal towers at the non-trivial fixed point are given in this notation by: $(SO(7)_1, \text{Ising}) = (1, \sigma), (8, 1), (8, \epsilon), (7, \sigma)$. Note that the two-fold degenerate sets of representations in Table III at $x - 1/16 = 3/8, 1/2$ and $7/8$ collapse to single representations of $SO(7)$ in all three cases. The conformal tower at $x - 1/16 = 1$ in Table III actually corresponds to a descendent of $(1, \sigma)$ from the $SO(7)$ point of view. We may also simplify Table V by constructing the spectrum up to $x - 1/16 < 2$ in this new basis. The first descendents of the $SO(7)$ conformal towers are given by:

$$SO(7)_1-\text{primary} \rightarrow SO(7)_1-\text{descendant}$$
$$1 \rightarrow 21'$$
$$7 \rightarrow 7', 35'$$
$$8 \rightarrow 8', 48'. \quad (6.10)$$

We find that all states in Table V can be grouped into $SO(7)$ representations and this explains all “accidental” degeneracies. (Of course other degeneracies persist which arise from the fact that the spacing of levels of all conformal towers in any conformal field theory is $\delta x = 1$.)

We remark that there is an isomorphism of the fusion rules of the $SO(7)_1$ and the Ising conformal field theories, corresponding to the following mapping of $SO(7)_1$ and Ising conformal towers:

$$SO(7)_1 \leftrightarrow \text{Ising}$$
$$1 \leftrightarrow 1$$
$$8 \leftrightarrow \sigma$$
$$7 \leftrightarrow \epsilon \quad (6.11)$$
Note that since the free fermion spectrum corresponds to a sum of products of the three pairs of identified conformal towers, namely \((SO(7)_1, Is) = (1, 1), (8, \sigma), (7, \epsilon)\), we can go from the free fermion to non-trivial fixed point by fusion either with \(\sigma\) in the Ising sector or, equivalently, with 8 in the \(SO(7)_1\) sector.

The decomposition of the adjoint representation of \(SO(7)\), under which the \(SO(7)_1\) current [a KM descendant of the \(SO(7)_1\) identity operator] transforms, gives information about the additional conserved quantities:

\[
SO(7)_1 \rightarrow SU^{i_1}(2) \times SU^{i_2}(2) \times SU^{j}(2)
\]

\[
21' \rightarrow (1', 0; 0) + (0, 1'; 0) + (0, 0; 1') + (1/2, 1/2; 1).
\]

At the zero coupling, infinite coupling and non-trivial fixed points, there are 21 (Hermitean) conserved quantities with these quantum numbers. The first 9 clearly correspond to the currents, \(\vec{I}_1, \vec{I}_2\) and \(\vec{J}\). The remaining 12 operators are uniquely identified by their quantum numbers as \(\psi_1^\dagger \sigma \psi_2, \psi_1 \sigma^2 \psi_2\) together with the hermitean conjugates of these operators. We see that one linear combination of these corresponds to the extra conserved quantity identified previously in the NRG work, Eq. (6.1).

The 28 fermion bilinears that can be formed from 8 Majorana fermions, which transform under the adjoint representation of \(SO(8)\) and represent the \(SO(8)_1\) KM current operator, decompose into the 21 and 7 dimensional representation of \(SO(7)\). Thus under the conformal embedding \(SO(8)_1 \rightarrow SO(7)_1 \times Is\), 21 of the fermion bilinears become the \(SO(7)\) current and the remaining 7 become the product of the vector (7) and \(\epsilon\) primaries. It can be seen that both types of bilinears appear in the Kondo interactions. As stated above, these interactions do not manifestly preserve the \(SO(7)\) symmetry for any non-zero value of the couplings; rather this symmetry is “dynamically restored” at the non-trivial critical point.

This (non-abelian) bosonization scheme of the 4 species of Dirac Fermions in terms of an \(SO(7)_1\) Wess-Zumino-Witten theory and an Ising model [first pointed out in Ref. (14)], appears to be closely related to a physical picture for the 2-impurity Kondo problem proposed more recently by Sire et al. and Gan. In these papers, the fermions are first transformed by ordinary abelian bosonization. Certain linear combinations of bosons are defined and then it is observed that a subset of the boundary operators can be re-expressed in terms of some different fermions, a transformation sometimes referred to as “refermionization”. For a special, anisotropic, value of the Kondo couplings, up to irrelevant operators, it is found that only 1 of the new Majorana fermion couples to the impurity spins, the other 7 completely decoupling. This theory has an obvious \(SO(7)\) symmetry. This procedure apparently gives an explicit realization of the \(SO(7)\) bosonization scheme, with the added bonus that the \(SO(7)\) symmetry at the non-trivial critical point becomes manifest.

**VII. DISCREPANCY WITH OTHER CALCULATIONS**

Finite temperature Monte Carlo (MC) calculations by Fye and Hirsch have so far discovered no evidence for the non-trivial critical point discussed here. It was suggested that the non-trivial critical point may be an artifact of the energy-independent coupling constant approximation used in the NRG work. Furthermore, NRG calculations by Sakai et al. on the related 2-impurity Anderson model, while reproducing the results of Ref. (9) in
the energy (and parity) independent coupling constant case, did not find such a critical point using energy-dependent coupling constants or a “parity-splitting” term. In this section we discuss the apparent discrepancy between these calculations and the ones presented in the present paper.

Our most important point was derived in detail in Sec. II and Sec. V; it is the existence of two different types of P-H symmetry, only one of which leads to a non-trivial fixed point. As explained in Sec. II, when the first type of particle-hole symmetry is maintained, some sort of phase transition as a function of inter-impurity coupling, $K$, is inevitable. The reason is that for large antiferromagnetic $K$ there is no phase shift and for large ferromagnetic $K$ the phase shift is $\pi/2$ in both channels. Furthermore, because of this type of particle-hole symmetry, the phase shifts, if they are well-defined, can only be 0 or $\pi/2$. Both of these phases are absolutely stable (no relevant or marginal operators). Thus they must be separated by (at least) one point which is not in either phase. In principle, this could either be a first order transition point or a critical point. This argument seems to us very general and convincing. It should not, for instance, depend on ignoring energy-dependence of the coupling constants, discussed in Sec. II. It only relies on the rather well-established local Fermi liquid picture of the single-impurity Kondo effect developed by Nozières and others. The statement that the phase shift is $\pi/2$ for some range of parameters has a precise mathematical meaning. It is a statement about the asymptotic behaviour of the electron Green’s function, discussed in Sec. V. If the Green’s function did not have this behaviour for a range of ferromagnetic $K$ this would be rather shocking and would presumably indicate a breakdown of the standard theory of the single-impurity Kondo effect. Likewise for a range of antiferromagnetic $K$. We think that measuring the quantity, $S_{(1)}$, governing the asymptotic behaviour of the electron Green’s function, defined in the previous section, would be the easiest way to verify the existence of the non-trivial critical point. It ought to be possible to show that $S_{(1)} = 1$ for sufficiently large antiferromagnetic inter-impurity coupling and $S_{(1)} = -1$ for sufficiently large ferromagnetic inter-impurity coupling. A numerical study of how and where it passes between 1 and $-1$ would give information about the non-trivial critical point. We emphasize that $S_{(1)} \neq \pm 1$ corresponds to non-Fermi liquid behaviour. At $T = 0$ some sort of phase transition must exist between these two Fermi liquid cases. An analysis of scaling with temperature will be required to study this transition.

Of course, this argument says nothing about the nature of the phase transition separating the two stable phases. It could, for example, be first order, in which case there would not be a critical point. To determine the nature of the phase transition requires other methods. The NRG and CFT methods provide complementary approaches, which taken together, suggest rather clearly that the two phases are separated by the non-trivial critical point discussed in this paper. The CFT approach allows a systematic classification of possible critical points, given the assumption of conformal invariance at the critical point and the various symmetries of the problem. The NRG results allow a determination of which of the possible critical points actually occurs. Having identified a candidate critical point, and shown that it does occur in a particular microscopic formulation of the model (corresponding to the NRG with energy-independent coupling constants), we can then determine whether it will be stable under changes in the microscopic Hamiltonian using the RG. The complete set of relevant and marginal operators at the proposed critical point are those in Table VI, together with the marginal current operators, $\tilde{J}_i$, $\tilde{I}_i$. If the microscopic Hamiltonian has the
first type of particle-hole symmetry (together with parity and at least a $U(1)$ subgroup of the
spin-rotation symmetry) then all but one of these operators are forbidden to appear in the
effective Hamiltonian of Eq. (3.4). The presence of the first type of particle-hole symmetry
implies the diagonal isospin symmetry, $\vec{I}$. This diagonal isospin symmetry together
with spin symmetry [or at least a $U(1)$ subgroup of it] forbids all these operators except $\epsilon$
and an element of the eighth operator in Table VI, which is forbidden by parity. (See the
beginning of Sec. V.) Thus we reach the crucial conclusion that the non-trivial critical point
can be reached by adjusting only one parameter, the coupling constant corresponding to $\epsilon$
in the effective Hamiltonian, assuming the first type of particle-hole symmetry. By varying the
inter-impurity coupling, $K$, or any other parameter, so as to pass from the inter-impurity
singlet phase to the Kondo-screened phase, we can make this coupling constant pass through
zero. Thus this is a conventional critical point; it is reached by varying only one parameter,
assuming the first type of particle-hole symmetry.

This analysis indicates that allowing energy-dependent coupling constants should not
make any difference, provided that the first type of particle-hole symmetry is maintained.
No lowering of symmetry ensues from allowing energy-dependent coupling constants, as long
as the first type of P-H symmetry is maintained, so no additional operators are permitted
in the effective Hamiltonian describing the critical point. Of course, the actual location of
the critical point, as a function of the various Kondo and RKKY couplings can change.

The NRG work on the Anderson model considers several different Hamiltonians. The
model with energy-independent coupling constants, reduces, in the Kondo limit of the An-
derson model, to the same model studied previously, in this case apparently the same
non-trivial critical point is observed. The other models considered, with energy-dependent
coupling constants or a “parity-splitting” term do not have the first type of particle-hole
symmetry. Thus, by the discussion, in Sec. II, we do not expect them to exhibit the non-
trivial critical point, and indeed they do not. Some of these models do have the second
type of particle-hole symmetry. This symmetry is not enough preserve the critical point as
we discussed in Sec. II, V. However, it should be possible to study models with energy-
dependent coupling constants which do not break the first type of particle-hole symmetry,
and which therefore should exhibit the non-trivial fixed point. This corresponds to choosing
couplings $N_{e,o}(E)$ [corresponding essentially to $W_{0,1}(E)$ in the notation of Ref. (11)]
which are non-trivial even functions of energy, $E$, for both parities, $|p = 0, 1|$. (The other param-
eters in the Hamiltonian must also be chosen to preserve particle-hole symmetry.) Further
calculations with such models would be valuable in settling this controversy.

Most of the QMC work of Ref. (7) was done on the one-dimensional tight-binding model
at half-filling with $\delta$-function Kondo interactions, of the form:

$$J[\psi^\dagger(R/2)\vec{\sigma}\psi(R/2)\cdot\vec{S}_1 + \psi^\dagger(-R/2)\vec{\sigma}\psi(-R/2)\cdot\vec{S}_2],$$

where $R = na$, $a$ being the lattice spacing. [In the case $n$ odd, the origin is midway between
2 sites.] The various cases, $n = 1, 2, 4, 8$ were studied. We see from Eq. (2.8) that the first
type of P-H symmetry occurs for $n$ even and the second for $n$ odd. A systematic search for
the non-trivial critical point was only made in the case $n = 1$, where it is not expected to
occur! The reason for this was connected with the fact that these authors did not put in an
inter-impurity coupling by hand. They expect (based on weak-coupling perturbation theory)
that the RKKY coupling will be ferromagnetic for \( n \) even and antiferromagnetic for \( n \) odd. The NRG work of Ref. (3) suggests that the non-trivial critical point should occur for antiferromagnetic inter-impurity coupling, of \( O(T_K) \). From another point of view, it is quite easy to see that these models are generally in the Kondo-screening phase at strong coupling, so if a transition is to be encountered they had better be in the inter-impurity singlet phase at weak coupling. A necessary condition for this is presumably an antiferromagnetic RKKY coupling.

There is another potential problem with the even \( n \) models. In this case \( N_e(0) = 0 \), [see Eqs. (2.11), (2.12)], so the even channel has only irrelevant Kondo couplings. If we assume the even channel decouples, we obtain a single-channel Kondo effect with a spin-1 impurity for ferromagnetic inter-impurity coupling. This leads to a Fermi liquid fixed point, with an underscreened \( s = 1/2 \) effective impurity. However, we expect that higher orders of perturbation theory will generate a non-zero Kondo couplings to the even channel at \( E = 0 \). Such a coupling to the leftover \( s = 1/2 \) impurity would be relevant, provided that it is antiferromagnetic. However, if it is ferromagnetic, the underscreened Fermi liquid fixed point would be stable and the symmetry argument given above for the existence of the non-trivial critical point may fail.

Thus it seems quite likely that the non-trivial critical point will not occur for any value of \( n \) and \( J \) in this model. However, it should be possible to see the critical point by generalizing the model somewhat, while maintaining the desired symmetry. It may be sufficient to take an even \( n \) model (say \( n = 2 \)) and add a direct inter-impurity coupling, \( K \). To avoid the potential difficulty mentioned in the previous paragraph, it may also be necessary to modify the Kondo couplings somewhat while preserving the first type of P-H symmetry. It can be seen that this symmetry occurs at half-filling provided that there is site-parity, ie. reflection symmetry about a site as occurs for \( n \) even. The second type occurs if there is link-parity, ie. reflection about a link. One example would be to make the replacement in Eq. (7.1):

\[
\psi(R/2) \rightarrow v_0\psi(0) + iv_1\psi(a) \\
\psi(-R/2) \rightarrow v_0\psi(0) + iv_1\psi(-a).
\]

This gives, in wave-vector space:

\[
v(k) = v_0 + iv_1e^{iak}.
\]

The first type of P-H symmetry holds provided that the \( v_i \)'s are real. For \( v_0 \) and \( v_1 \) both non-zero, \( N_e(0) \) and \( N_o(E) \) are both non-zero. However, we find that the RKKY coupling is always ferromagnetic for all the models of this type that we have considered with the first type of P-H symmetry. Thus it may be necessary to add a direct inter-impurity coupling which is varied to pass between the two stable phases.

It may still be difficult to see the non-trivial critical point by studying the staggered susceptibility since it only shows a logarithmic divergence. It may be necessary to choose couplings very close to the critical value and very low temperatures to see the anomalous behavior. As discussed above, the single-particle Green’s function would likely exhibit a much clearer signal of the critical point.

In conclusion, we expect that further numerical work, along the lines of Refs. (10) and (9) should be able to see the critical point provided that:
1) the first type of particle-hole symmetry is maintained,
2) a parameter is varied to pass from the inter-impurity singlet to Kondo screened phase
and
3) sufficiently low temperatures and finely tuned parameters are obtained.
The last condition should be much easier to achieve if the single-particle Green’s function is
measured instead of the staggered susceptibility.

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Foundation.
### TABLE I. Conformal towers occurring for free fermions with the boundary conditions of Eq. \(3.33\).

| \(i_1\) | \(i_2\) | \(j\) | Ising | \(x\) |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 |
| 1/2 | 0 | 1/2 | \(\sigma\) | 1/2 |
| 0 | 1/2 | 1/2 | \(\sigma\) | 1/2 |
| 0 | 0 | 1 | \(\epsilon\) | 1 |
| 1/2 | 1/2 | 1 | 1 | 1 |
| 1/2 | 1/2 | 0 | \(\epsilon\) | 1 |

### TABLE II. Conformal towers occurring for free fermions with a \(\pi/2\) phase shift, corresponding to \(\vec{\phi}\) [or \(\epsilon\) or \((1/2, 1/2)\)] fusion.

| \(i_1\) | \(i_2\) | \(j\) | Ising | \(x\) |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1/2 |
| 1/2 | 0 | 1/2 | \(\sigma\) | 1/2 |
| 0 | 1/2 | 1/2 | \(\sigma\) | 1/2 |
| 0 | 0 | 0 | \(\epsilon\) | 1/2 |
| 1/2 | 1/2 | 0 | 1 | 1/2 |
| 1/2 | 1/2 | 1 | \(\epsilon\) | 3/2 |

### TABLE III. Conformal towers occurring after \(\sigma\) fusion.

| \(i_1\) | \(i_2\) | \(j\) | Ising | \(x - \frac{1}{16}\) |
|---|---|---|---|---|
| 0 | 0 | 0 | \(\sigma\) | 0 |
| 1/2 | 0 | 1/2 | 1 | 3/8 |
| 0 | 1/2 | 1/2 | 1 | 3/8 |
| 0 | 0 | 1 | \(\sigma\) | 1/2 |
| 1/2 | 1/2 | 0 | \(\sigma\) | 1/2 |
| 1/2 | 0 | 1/2 | \(\epsilon\) | 7/8 |
| 0 | 1/2 | 1/2 | \(\epsilon\) | 7/8 |
| 1/2 | 1/2 | 1 | \(\sigma\) | 1 |
TABLE IV. All first descendents of all conformal towers. The subscript in the first row labels the Kac-Moody level, $k$. $SU(2)$ descendents are labelled by their spin. All descendents will be marked $'$.

| (0)$_1$ | $\frac{1}{2}$ | (0)$_2$ | $\frac{1}{2}$ | (1)$_2$ | 1 | $\sigma$ | $\epsilon$ |
|--------|-------------|--------|-------------|--------|---|--------|--------|
| $1'$   | $\frac{1}{2}'$ | $1'$   | $\frac{1}{2}'$ | $\frac{3}{2}'$ | $0', 1'$ | $-$ | $\sigma'$ | $\epsilon'$ |
| \((i_1, i_2)\) | \(j\) | Ising | \(x - \frac{1}{16}\) | \(i^P\) | \(E_{\text{NRG}}\) |
|--------------|--------|-------|----------------|--------|-------------|
| (0,0)        | 0      | \(\sigma\) | 0              | 0\(^+\) | 0.00000     |
| \(\frac{1}{2}, 0\) | \(\frac{3}{2}\) | 1     | \(\frac{3}{8}\) | \(\frac{1}{2}\) | 0.37761     |
| (0,0)        | 1      | \(\sigma\) | \(\frac{1}{2}\) | 0\(^-\) | 0.50454     |
| \(\frac{1}{2}, \frac{1}{2}\) | 0     | \(\sigma\) | \(\frac{3}{2}\) | 0\(^-\) | 0.50724     |
| \(\frac{1}{2}, 0\) | \(\frac{3}{2}\) | \(\epsilon\) | \(\frac{7}{8}\) | \(\frac{1}{2}\) | 0.88696     |
| (0,0)        | 1'     | \(\sigma\) | 1              | 0\(^+\) | 0.99952     |
| \(\frac{1}{2}, \frac{1}{2}\) | 1     | \(\sigma\) | 1              | 1\(^-\) | 1.00298     |
| \(1', 0\), \(0, 1'\) | 0     | \(\sigma\) | 1              | 1\(^-\) | 1.00642     |
| (0,0)        | 0      | \(\sigma'\) | 1              | 0\(^+\) | 1.01078     |
| \(\frac{1}{2}, 0\) | \(\frac{3}{2} \) | 1     | \(\frac{13}{8}\) | \(\frac{1}{2}\) | 1.39056     |
| \(\frac{1}{2}, \frac{1}{2}\) | \(\frac{1}{2}\) | 1     | \(\frac{13}{8}\) | \(\frac{1}{2}\) | 1.39321     |
| \(1', \frac{1}{2}\), \(\frac{1}{2}, 1'\) | \(\frac{1}{2}\) | 1     | \(\frac{13}{8}\) | \(\frac{1}{2}\) | 1.39699     |
| \(\frac{1}{2}, 0\) | \(\frac{1}{2}\) | 1     | \(\frac{13}{8}\) | \(\frac{1}{2}\) | 1.39700     |
| (0,0)        | 1'     | \(\sigma\) | \(\frac{13}{7}\) | 0\(^-\) | 1.50437     |
| \(\frac{1}{2}, \frac{1}{2}\) | 0     | \(\sigma\) | \(\frac{13}{7}\) | 0\(^-\) | 1.51090     |
| \(1', 0\), \(0, 1'\) | 1     | \(\sigma\) | \(\frac{13}{7}\) | 1\(^-\) | 1.51094     |
| (0,0)        | 0'     | \(\sigma\) | \(\frac{13}{7}\) | 0\(^+\) | 1.55755     |
| \(\frac{1}{2}, \frac{1}{2}\) | 1'     | \(\sigma\) | \(\frac{13}{7}\) | 0\(^-\) | 1.55755     |
| \(1', 0\), \(0, 1'\) | 1     | \(\sigma\) | \(\frac{13}{7}\) | 1\(^-\) | 1.57224     |
| (0,0)        | 1      | \(\sigma'\) | \(\frac{13}{7}\) | 0\(^-\) | 1.61452     |
| \(\frac{1}{2}, \frac{1}{2}\) | 0      | \(\sigma'\) | \(\frac{13}{7}\) | 0\(^-\) | 1.62305     |
| \(\frac{1}{2}, 0\) | \(\frac{1}{2}\) | \(\epsilon\) | \(\frac{13}{7}\) | \(\frac{1}{2}\) | 1.92017     |
| (0,0)        | \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) | 1.92029     |
| (0,0)        | \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) | \(\frac{1}{2}\) | 1.97512     |
| $i_1$ | $i_2$ | $j$ | Ising | $x$ |
|-------|-------|-----|-------|-----|
| 0     | 0     | 0   | 1     | 0   |
| 1/2   | 0     | 1/2 | $\sigma$ | 1/2 |
| 1/2   | 0     | 1/2 | $\sigma$ | 1/2 |
| 0     | 1/2   | 1/2 | $\sigma$ | 1/2 |
| 0     | 0     | 1   | 1     | 1/2 |
| 0     | 0     | 0   | $\epsilon$ | 1/2 |
| 1/2   | 1/2   | 0   | 1     | 1/2 |
| 0     | 1     | 1   | $\epsilon$ | 1 |
| 1/2   | 1/2   | 1   | 1     | 1 |
| 1/2   | 1/2   | 0   | $\epsilon$ | 1 |
| 1/2   | 1/2   | 1   | $\epsilon$ | 3/2 |

TABLE VI. Operator content at the non-trivial critical point.
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