Convergence of optimal control problems governed by second kind parabolic variational inequalities

Mahdi BOUKROUCHE¹, Domingo A. TARZIA²,
¹Lyon University, UJM F-42023, CNRS UMR 5208, Institut Camille Jordan, 23 Paul Michelon, 42023 Saint-Etienne Cedex 2, France. E-mail: Mahdi.Boukrouche@univ-st-etienne.fr;
²Departamento de Matemática-CONICET, FCE, Univ. Austral, Paraguay 1950, S2000FZF Rosario, Argentina. E-mail: DTarzia@austral.edu.ar

Abstract: We consider a family of optimal control problems where the control variable is given by a boundary condition of Neumann type. This family is governed by parabolic variational inequalities of the second kind. We prove the strong convergence of the optimal controls and state systems associated to this family to a similar optimal control problem. This work solves the open problem left by the authors in IFIP TC7 CSMO2011.

Keywords: Parabolic variational inequalities of the second kind, Aubin compactness arguments, Control border, Convergence of optimal control problems, Tresca boundary conditions, free boundary problems.

1 Introduction

The motivation of this paper is to prove the strong convergence of the optimal controls (borders) and state systems associated to a family of second kind parabolic variational inequalities. With this paper, we solve the open question, left in [11] and we generalize our work [10], to study the Control border.

To illustrate the problem considered, we consider in the following, just as examples, two free boundary problems which leads to second kind parabolic variational inequalities.

We assume that the boundary of a multidimensional regular domain Ω is given by ∂Ω = Γ₁ ∪ Γ₂ ∪ Γ₃ with \( \text{meas}(Γ₁) > 0 \) and \( \text{meas}(Γ₃) > 0 \). We consider a family of optimal control problems where the control variable is given by a boundary condition of Neumann type whose state system is governed by a free boundary problem with Tresca conditions on a portion Γ₂ of the boundary, with a flux \( f \) on Γ₃ as the control variable, given by:

**Problem 1.1**

\[
\frac{d}{dt} u - \Delta u = g \quad \text{in} \quad Ω \times (0, T),
\]

\[
\frac{\partial u}{\partial n} < q \quad \text{on} \quad Γ₂ \times (0, T),
\]

\[
\frac{\partial u}{\partial n} = q \quad \text{on} \quad Γ₁ \times (0, T),
\]

\[
-p \frac{\partial u}{\partial n} = f \quad \text{on} \quad Γ₃ \times (0, T),
\]

with the initial condition

\[
u(0) = u_b \quad \text{on} \quad Ω,
\]

and the compatibility condition on \( Γ₁ \times (0, T) \)

\[
u_b = b \quad \text{on} \quad Γ₁ \times (0, T),
\]

where \( q > 0 \) is the Tresca friction coefficient on \( Γ₂ \) (11, 2, [3]). We define the spaces \( F = L²(Ω \times Γ₃) \), \( V = H²(Ω) \), \( V₀ = \{ v ∈ V : v|Γ₁ = 0 \} \), \( H = L²(Ω) \), \( H = L²(0, T; H) \), \( V = L²(0, T; V) \) and the closed convex set \( K_b = \{ v ∈ V : v|Γ₁ = b \} \). Let given

\[
g ∈ H, \quad b ∈ L²(0, T; H^{1/2}(Γ₁)), \quad f ∈ F \quad \text{and} \quad q ∈ L²((0, T) × Γ₂), \quad q > 0, \quad u_b ∈ K_b.
\]

The variational formulation of **Problem 1.1** leads to the following parabolic variational problem:

**Problem 1.2**

Let given \( g, b, q, u₀ \) and \( f \) as in (1.1). Find \( u = u_f ∈ C(0, T, H) \cap L²(0, T; K_b) \) with \( u(0) = u_b \) and for \( t ∈ (0, T) \)

\[
\frac{d}{dt} u - \Delta u = g \quad \text{in} \quad Ω \times (0, T),
\]

\[
\frac{\partial u}{\partial n} < q \quad \text{on} \quad Γ₂ \times (0, T),
\]

\[
\frac{\partial u}{\partial n} = q \quad \text{on} \quad Γ₁ \times (0, T),
\]

\[
-p \frac{\partial u}{\partial n} = f \quad \text{on} \quad Γ₃ \times (0, T),
\]

with the initial condition

\[
u(0) = u_b \quad \text{on} \quad Ω,
\]

and the condition of compatibility on \( Γ₁ \times (0, T) \)

\[
u_b = b \quad \text{on} \quad Γ₁ \times (0, T).
\]
The variational formulation of the problem (1.3) leads to the following parabolic variational problem

**Problem 1.4** Let given $g, b, u_0$ be the cost functional $J_h : \mathcal{F}_- \rightarrow \mathbb{R}^+$ is given by

$$J_h(f) = \frac{1}{2} \| u_h f \|_{\mathcal{H}}^2 + \frac{M}{2} \| f \|_{\mathcal{L}}^2 \ (M > 0, h > 0) \ (2.4)$$

and $u_{h f}$ is the unique solution of Problem 1.4 for a given $f \in \mathcal{F}_-$ and $h > 0$.

**Theorem 2.1** Under the assumptions $g \geq 0$ in $\Omega \times (0, T), b \geq 0$ on $\Gamma_1 \times (0, T)$ and $u_0 \geq 0$ in $\Omega$, we have the following properties:

(a) The cost functional $J$ is strictly convex on $\mathcal{F}_-$.
(b) There exists a unique optimal control $f_{\text{opt}} \in \mathcal{F}_-$ solution of the Neumann boundary optimal control Problem 2.1.

**Proof** We give some sketch of the proof, following [10] we generalize for parabolic variational inequalities of the second kind, given in Problem 1.2, the estimates obtained for convex combination between $u_{4}(\mu) = u_{f_{1}}, (1-\mu)_{f_{2}}$, and $u_{3}(\mu) = \mu u_{f_{1}} + (1-\mu)u_{f_{2}}$, for any two elements $f_{1}$ and $f_{2}$ in $\mathcal{F}$.

The main difficulty, to prove this result comes from the fact that the functional $\Phi$ is not differentiable. To overcome this difficulty, we use the regularization method and consider for $\varepsilon > 0$ the following approach of $\Phi$ defined by

$$\Phi_{\varepsilon}(u) = \int_{\Gamma_{3}} q \sqrt{\varepsilon^{2} + |u|^{2}} ds \quad \forall u \in V, \quad (2.5)$$

which is Gateaux differentiable, with

$$\langle \Phi_{\varepsilon}'(w), v \rangle = \int_{\Gamma_{3}} \frac{q u v}{\sqrt{\varepsilon^{2} + |u|^{2}}} ds \quad \forall (w, v) \in V^{2}.$$  

We define $w_{\varepsilon}$ as the unique solution of the corresponding parabolic variational inequality for all $\varepsilon > 0$. We obtain that for all $\mu \in [0, 1]$ we have $u_{3}(\mu) \leq u_{3}(\mu)$ for all $\varepsilon > 0$.

When $\varepsilon \rightarrow 0$ we have that: for $i = 1, \ldots , 4$.

$$u_{i} \rightarrow u_{i} \text{ strongly in } V \cap L^{\infty}(0, T; H). \quad (2.6)$$

As $f \in \mathcal{F}_-$, $g \geq 0$ in $\Omega \times (0, T), b \geq 0$ in $\Gamma_{1} \times (0, T)$ and $u_{0} \geq 0$ in $\Omega$, we obtain by the weak maximum principle that for all $\mu \in (0, 1]$ we have $0 \leq u_{3}(\mu)$, so following [10] we get

$$0 \leq u_{3}(\mu) \leq u_{3}(\mu) \quad \text{in } \Omega \times (0, T), \quad \forall \mu \in [0, 1]. \quad (2.7)$$

Then for all $\mu \in (0, 1)$, and for all $f_{1}, f_{2} \in \mathcal{F}_-$, and by using $f_{3}(\mu) = \mu f_{1} + (1-\mu) f_{2}$ we obtain that:

$$J(f_{3}) = \frac{1}{2} \| u_{3}(\mu) \|_{\mathcal{H}}^2 + \frac{M}{2} \| f \|_{\mathcal{L}}^2 \ (M > 0) \ (2.8)$$

Then $J$ is strictly convex functional on $\mathcal{F}_-$ and therefore there exists a unique optimal control $f_{\text{opt}} \in \mathcal{F}_-$ solution of the Neumann boundary optimal control Problem 2.1.

**Theorem 2.2** Under the assumptions $g \geq 0$ in $\Omega \times (0, T), b \geq 0$ on $\Gamma_{1} \times (0, T)$ and $u_{0} \geq 0$ in $\Omega$, we have the following properties:

(a) The cost functional $J_{h}$ are strictly convex on $\mathcal{F}_-$, for all $h > 0$.
(b) There exists a unique optimal control $f_{\text{opt}} \in \mathcal{F}_-$ solution of the Neumann boundary optimal control Problem 2.2, for all $h > 0$.
3 Convergence when $h \to +\infty$

In this section we study the convergence of the Neumann optimal control Problem 2.2 to the optimal control Problem 2.1 when $h \to \infty$. For a given $f \in \mathcal{F}$ we have first the following result which generalizes [6, 7, 10, 16].

**Lemma 3.1** Let $u_{h,f}$ be the unique solution of the problem 1.4 and $u_f$ the unique solution of the problem 1.2, then

$$u_{h,f} \to u_f \in \mathcal{V}$$

strongly as $h \to +\infty$, for all $f \in \mathcal{F}$.

**Proof** Following [10], we take $v = u_f(t)$ in the variational inequality of the problem 1.4 where $u = u_{h,f}$, and recalling that $u_f(t) = b$ on $\Gamma_1 \times [0,T]$, taking $\phi_h(t) = u_{h,f}(t) - u_f(t)$ we obtain for $h > 1$, that $\|u_{h,f}\|_\mathcal{V}$ is also bounded for all $h > 1$ and for all $f \in \mathcal{F}$. Then there exists $\eta \in \mathcal{V}$ such that (when $h \to +\infty$)

$$u_{h,f} \to \eta$$

weakly in $\mathcal{V}$

and

$$u_{h,f} \to b$$

strongly on $L^2((0,T) \times \Gamma_1)$

so $\eta(0) = u_b$.

Let $\varphi$ be in $L^2(0,T,H^1_0(\Omega))$ and taking in the variational inequality of the problem 1.4 where $u = u_{h,f}$,

$$v = u_{h,f}(t) \mp \varphi(t),$$

we obtain as $\|u_{h,f}\|_\mathcal{V}$ is bounded for all $h > 1$, we deduce that $\|u_{h,f}\|_L^2(0,T,H^{-1}(\Omega))$ is also bounded for all $h > 1$. Then we conclude that

$$u_{h,f} \to \eta$$

weak, and in $L^\infty(0,T,H)$ weak star, and

$$u_{h,f} \to \eta$$

in $L^2(0,T,H^{-1}(\Omega))$ weak.

From the variational inequality of the problem 1.4 and taking $v \in K$ so $v = b$ on $\Gamma_1$, we obtain $a.e. \, t \in [0,T]$ and

$$\langle \dot{u}_{h,f}, v - u_{h,f} \rangle + a(u_{h,f}, v - u_{h,f}) - h \int_{\Gamma_1} |u_{h,f} - b|^2 \, ds \geq$$

$$\Phi(u_{h,f}) - \Phi(v) + (g, v - u_{h,f}) - \int_{\Gamma_3} f(v - u_{h,f}) \, ds,$$

for all $v \in K$, then as $h > 0$ we have $a.e. \, t \in [0,T]$,

$$\langle \dot{u}_{h,f}, v - u_{h,f} \rangle + a(u_{h,f}, v - u_{h,f}) \geq \Phi(u_{h,f}) - \Phi(v) + (g, v - u_{h,f}) - \int_{\Gamma_3} f(v - u_{h,f}) \, ds, \forall v \in K.$$  (3.2)

So using (3.1) and passing to the limit when $h \to +\infty$ we obtain

$$\langle \eta, v - \eta \rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) \geq (g, v - \eta) - \int_{\Gamma_3} f(v - \eta) \, ds, \forall v \in K \quad a.e. \, t \in [0,T],$$

and $\eta(0) = u_b$. Using the uniqueness of the solution of Problem 1.2 we get that $\eta = u_f$.

To prove the strong convergence, we take $v = u_f(t)$ in the variational inequality of the problem 1.4

$$\langle \dot{u}_{h,f}, u_f - u_{h,f} \rangle + a_h(u_{h,f}, u_f - u_{h,f}) + \Phi(u_f) - \Phi(u_{h,f}) \geq (g, u_f - u_{h,f}) + h \int_{\Gamma_1} b(u_f - u_{h,f}) \, ds$$

$$- \int_{\Gamma_3} f(u_f - u_{h,f}) \, ds,$$

a.e. $t \in [0,T]$, thus as $u_f = u_h$ on $\Gamma_1 \times [0,T]$, we put

$\phi_h = u_{h,f} - u_f$, so $a.e. \, t \in [0,T]$ we have

$$\langle \dot{\phi}_h, \Phi_h \rangle + a(\phi_h, \phi_h) + h \int_{\Gamma_1} |\phi_h|^2 \, ds + \Phi(u_{h,f}) - \Phi(u_f)$$

$$\leq \langle \dot{u}_f, \phi_h \rangle + a(u_f, \phi_h) + (g, \phi_h) - \int_{\Gamma_3} f\phi_h \, ds,$$

so

$$\frac{1}{2}\|\phi_h\|_{L^2(0,T,H)}^2 + \lambda_h \|\phi_h\|_{H^1_0}^2 + \Phi(u_{h,f}) - \Phi(u_f)$$

$$\leq - \int_0^T \langle \dot{\phi}_h(t), \phi_h(t) \rangle \, dt - \int_0^T a(u_f(t), \phi_h(t)) \, dt$$

$$+ \int_0^T (g(t), \phi_h(t)) \, dt - \int_0^T \int_{\Gamma_3} f\phi_h \, ds \, dt.$$

Using the weak semi-continuity of $\Phi$ and the weak convergence (3.1) the right side of the just above inequality tends to zero when $h \to +\infty$, then we deduce the strong convergence of $\phi_h = u_{h,f} - u_f$ to $0$ in $\mathcal{V} \cap L^\infty(0,T,H)$, for all $f \in \mathcal{F}$ and the proof holds.

We prove now the following lemma by using the Aubin compactness arguments. This Lemma 3.2 is very important and necessary which allow us to conclude this paper. Indeed this result is needed to pass to the limit exactly in the last term of the inequality (3.12) in the proof of the main Theorem 3.3.

**Lemma 3.2** Let $u_{h,f_{\partial \Omega}}$ the state system defined by the unique solution of Problem 1.4, where the flux $f$ is replaced by $f_{\partial \Omega}$, then, for $h \to +\infty$, we have

$$u_{h,f_{\partial \Omega}} \to u_f$$

in $L^2((0,T) \times \partial\Omega)$,  (3.3)

where $u_f$ is the the state system defined by the unique solution of Problem 1.2 with the flux $f$ on $\Gamma_3$.

**Proof** Let consider the variational inequality of Problem 1.4 with $u = u_{h,f_{\partial \Omega}}$ and $f = f_{\partial \Omega}$ i.e.

$$\langle \dot{u}_{h,f_{\partial \Omega}}, v - u_{h,f_{\partial \Omega}} \rangle + a_h(u_{h,f_{\partial \Omega}}, v - u_{h,f_{\partial \Omega}}) + \Phi(v) - \Phi(u_{h,f_{\partial \Omega}}) \geq (g, v - u_{h,f_{\partial \Omega}}) - \int_{\Gamma_3} f_{\partial \Omega}(v - u_{h,f_{\partial \Omega}}) \, ds$$

$$+ h \int_{\Gamma_1} b(v - u_{h,f_{\partial \Omega}}) \, ds, \quad \forall v \in \mathcal{V}.$$  (3.4)

and let $\varphi \in L^2(0,T;H^1_0(\Omega))$, and set $v = u_{h,f_{\partial \Omega}}(t) \pm \varphi(t)$ in (3.4), we get

$$\langle \dot{u}_{h,f_{\partial \Omega}}, \varphi \rangle = (g, \varphi) - a(u_{h,f_{\partial \Omega}}, \varphi).$$

By integration in times for $t \in (0,T)$, we get

$$\int_0^T \langle \dot{u}_{h,f_{\partial \Omega}}, \varphi \rangle \, dt = \int_0^T (g, \varphi) \, dt - \int_0^T a(u_{h,f_{\partial \Omega}}, \varphi) \, dt$$

thus for $A = (c\|g\|_{\mathcal{H}} + \|u_{h,f_{\partial \Omega}}\|_{\mathcal{V}})$, we get

$$\left| \int_0^T \langle \dot{u}_{h,f_{\partial \Omega}}, \varphi \rangle \, dt \right| \leq A\|\varphi\|_{L^2(0,T;H^1_0(\Omega))},$$

where $c$ comes from the Poincaré inequality, and as in Lemma 3.1 we can obtain that $u_{h,f_{\partial \Omega}}$ is bounded in $\mathcal{V}$, so there exists a positive constant $C$ such that

$$\|u_{h,f_{\partial \Omega}}\|_{L^2(0,T;H^{-1}(\Omega))} \leq C.$$  (3.5)

Using now the Aubin compactness arguments, see for example [17] with the three Banach spaces $\mathcal{V}$, $H^\frac{1}{2}(\Omega)$ and $H^{-1}(\Omega)$, then

$$u_{h,f_{\partial \Omega}} \to u_f$$

$L^2(0,T;H^\frac{1}{2}(\Omega)).$
As the trace operator $\gamma_0$ is continuous from $H^\frac{1}{2}(\Omega)$ to $L^2(\partial\Omega)$, then the result follows. □

We give now, without need to use the notion of adjoint states [14, 18], the convergence result which generalizes the result obtained in [19] for a parabolic variational inequalities (see also [18, 20–23]). Other optimal control problems governed by variational inequalities are given in [24–26].

**Theorem 3.3** Let $u_{h\text{fop}} \in \mathcal{V}, f_{\text{op}} \in \mathcal{F}_-$ and $u_{\text{fop}} \in \mathcal{V}, f_{\text{op}} \in \mathcal{F}_-$ be respectively the state systems and the optimal controls defined in the problems (1.4) and (1.2). Then

$$\lim_{h \to +\infty} \|u_{h\text{fop}} - u_{\text{fop}}\|_\mathcal{F} = 0,$$

for all $v_{h\text{fop}} \in \mathcal{V}$.

As $u_{f_{\text{op}}}$ is bounded in $\mathcal{F}_-$, from (3.5) $u_{f_{\text{op}}}$ is bounded in $L^2(0, T; H^{-1}(\Omega))$, and $u_{hf_{\text{op}}}$ is also bounded in $\mathcal{V}$, and independently on $h$, so there exists a positive constant $\tilde{C}$ which does not depend on $h$ such that

$$\|\phi_h\|_\mathcal{V} = \|u_{h\text{fop}} - u_{\text{fop}}\|_\mathcal{V} \leq C, \quad \|\phi_h\|_{L^\infty(0, T; H)} \leq C,$$

and $(h - 1) \int_0^T \int_{\Gamma_1} |u_{hf_{\text{op}}} - b|^2 ds dt \leq C,$

then $\eta \in \mathcal{V}$ and

$$u_{hf_{\text{op}}} \to \eta \text{ in } \mathcal{V} \text{ and in } L^\infty(0, T, H) \text{ weak star (3.10)}$$

and

$$u_{hf_{\text{op}}} \to b \text{ in } L^2((0, T) \times \Gamma_1) \text{ strong (3.11)},$$

so $\eta(t) \in K_b$ for all $t \in [0, T]$.

Now taking $v \in K$ in Problem (1.4) where $u = u_{hf_{\text{op}}}$ and $f = f_{\text{op}}$ so

$$\langle \tilde{u}_{hf_{\text{op}}}, v - u_{hf_{\text{op}}} \rangle + a_h(u_{hf_{\text{op}}}, v - u_{hf_{\text{op}}}) + \Phi(v) - \Phi(u_{hf_{\text{op}}}) \geq (f_{\text{op}}, v - u_{hf_{\text{op}}}) + h \int_{\Gamma_1} b(v - u_{hf_{\text{op}}}) ds$$

$$- \int_{\Gamma_b} f_{\text{op}}(v - u_{hf_{\text{op}}}) ds, \quad \text{a.e. } t \in [0, T].$$

Thus

$$\langle \tilde{u}_{hf_{\text{op}}}, u_{f_{\text{op}}} - v \rangle + a(u_{hf_{\text{op}}}, u_{f_{\text{op}}} - v)$$

$$+ \Phi(u_{hf_{\text{op}}}) - \Phi(v) \leq -(g, v - u_{hf_{\text{op}}})$$

$$- \int_{\Gamma_b} f_{\text{op}}(v - u_{hf_{\text{op}}}) ds, \quad \text{a.e. } t \in [0, T].$$

Using Lemma 3.2, (3.9) and (3.10), we deduce that [3.27]

$$\langle \eta, v - \eta \rangle + a(\eta, v - \eta) + \Phi(v) - \Phi(\eta) \geq (f, v - \eta)$$

$$- \int_{\Gamma_b} \tilde{f}(v - \eta) ds, \quad \forall v \in K, \text{ a.e. } t \in [0, T],$$

so also by the uniqueness of the solution of Problem (1.2) we obtain that

$$\tilde{\eta} = \eta.$$  (3.13)

We prove that $\tilde{f} = f_{\text{op}}$. Indeed we have

$$J(\tilde{f}) = \frac{1}{2} \eta^2 + \frac{M}{2} \tilde{f}^2$$

$$\leq \liminf_{h \to +\infty} \left\{ \frac{1}{2} \|u_{h\text{fop}}\|^2_\mathcal{H} + \frac{M}{2} \|f_{\text{op}}\|^2_\mathcal{F} \right\}$$

$$\leq \liminf_{h \to +\infty} J_h(f_{\text{op}}),$$

so using now the strong convergence $u_{hf_{\text{op}}} \to u_f$ as $h \to +\infty$, and $f \in \mathcal{F}_-$ (see Lemma 3.1), we obtain that

$$J(\tilde{f}) \leq \liminf_{h \to +\infty} J_h(f_{\text{op}}) \leq \frac{1}{2} \|u_f\|^2_\mathcal{H} + \frac{M}{2} \|f\|^2_\mathcal{F}$$
\[ = J(f), \quad \forall f \in \mathcal{F} \] 
\[ \text{(3.14)} \]

then by the uniqueness of the optimal control Problem (1.2) we get
\[ \tilde{f} = f_{\text{op}}. \] 
\[ \text{(3.15)} \]

Now we prove the strong convergence of \( u_{h_{\text{fop}}}, \) \( \eta = u \) \( f \in \mathcal{V} \cap L^\infty(0, T; H) \cap L^2(0, T; L^2(\Gamma_1)), \) indeed taking \( v = \eta \) in Problem (1.4) where \( u = u_{h_{\text{fop}}} \) and \( f = f_{\text{op}}, \) \( \eta(t) \in K \) \( \forall t \in [0, T], \) so \( \eta = b \) on \( \Gamma_1, \) we obtain
\[ \frac{1}{2} \| u_{h_{\text{fop}}} - \eta \|_{L^\infty(0, T; H)}^2 + \lambda_1 \| u_{h_{\text{fop}}} - \eta \|^2 + \int_0^T \{ \Phi(u_{h_{\text{fop}}} - \Phi(\eta)) \} dt + \tilde{h} \| u_{h_{\text{fop}}} - \eta \|^2 \]
\[ \leq \int_0^T (g, u_{h_{\text{fop}}}(t) - \eta(t)) dt - \int_0^T (\eta, u_{h_{\text{fop}}} - \eta) dt + \int_0^T \alpha(\eta(t), (t) - u_{h_{\text{fop}}}(t)) - f_{\text{op}}(u_{h_{\text{fop}}} - \eta) ds dt, \]
\[ \text{(3.16)} \]
where \( \tilde{h} = h - 1. \)

Using (3.10) and the weak semi-continuity of \( \Phi \) we deduce that
\[ \lim_{h \to +\infty} \| u_{h_{\text{fop}}} - \eta \|_{L^\infty(0, T; H)} = \lim_{h \to +\infty} \| u_{h_{\text{fop}}} - \eta \|_{L^2(0, T)} = 0, \]
and with (3.13) and (3.15) we deduce (3.6). Then from (3.14) and (3.15) we can write
\[ J(f_{\text{op}}) = \frac{1}{2} \| f_{\text{op}} \|_{H}^2 + \frac{M}{2} \| f_{\text{op}} \|_{\mathcal{F}}^2 \leq \liminf_{h \to +\infty} J_h(f_{\text{op}}) \]
\[ = \liminf_{h \to +\infty} \left\{ \frac{1}{2} \| u_{h_{\text{fop}}} \|_{H}^2 + \frac{M}{2} \| f_{\text{op}} \|_{\mathcal{F}}^2 \right\} \]
\[ \leq \lim_{h \to +\infty} J_h(f_{\text{op}}) = J(f_{\text{op}}) \] 
\[ \text{(3.16)} \]
and using the strong convergence (3.6), we get
\[ \lim_{h \to +\infty} \| f_{\text{op}} \|_{\mathcal{F}} = \| f_{\text{op}} \|_{\mathcal{F}}. \] 
\[ \text{(3.17)} \]
Finally as
\[ \| f_{\text{op}} - f_{\text{op}} \|_{\mathcal{F}}^2 = \| f_{\text{op}} \|_{\mathcal{F}}^2 + \| f_{\text{op}} \|_{\mathcal{F}}^2 - 2(f_{\text{op}}, f_{\text{op}}) \] 
\[ \text{(3.18)} \]
and by the first part of (3.9) we have
\[ \lim_{h \to +\infty} (f_{\text{op}}, f_{\text{op}}) = \| f_{\text{op}} \|_{\mathcal{F}}^2, \]
so from (3.17) and (3.18) we get (3.7). This ends the proof. \( \square \)

**Corollary 3.4** Let \( u_{h_{\text{fop}}}, v_{\text{fop}} \) be \( \mathcal{V}, f_{\text{fop}} \) in \( \mathcal{F}, u_{\text{fop}} \) in \( \mathcal{V} \) and \( f_{\text{fop}} \) in \( \mathcal{F} \) be respectively the state systems and the optimal controls defined in the problems (1.4) and (1.2). Then \( \lim_{h \to +\infty} (J_h(f_{\text{op}}) - J(f_{\text{op}})) = 0. \)

**Proof** It follows from the definitions (2.1) and (2.2), and the convergences (3.6) and (3.7). \( \square \)

## 4 Conclusion

The main difference here with our work [10] where the control variable was the function \( g, \) is that we consider here as a control variable the function \( f \) given by the Neumann boundary condition on \( \Gamma_3. \) This change induce in the variational problems 1.2 and 1.4, and also in the proofs of Lemma 3.1 and Theorem 3.3, a new integral term on \( \Gamma_3. \) The main difficulty here is in Section 3 and the question is exactly how to pass to the limit for \( h \to +\infty \) in the last integral term on \( \Gamma_3 \) in (3.12). To overcome this main difficulty we have introduced the new Lemma 3.2, which is the key of our problem. The idea of Lemma 3.1 and Theorem 3.3 and their proofs are indeed similar to those of our work [10] with the differences and difficulties mentioned just above.

**Acknowledgements**

This paper was partially sponsored by the Institut Camille Jordan ST-Etienne University for first author and the project PICTO Austral # 73 from ANPCyT and Grant AFOSR FA9550-10-1-0023 for the second author.

**References**

[1] A.Amassad, D.Chenais, C.Fabre. Optimal control of an elastic contact problem involving Tresca friction law. Nonlinear Analysis, 2002, 48: 1107-1135.

[2] M.Boukrouche, R.El Mir. On a non-isothermal, non-Newtonian lubrication problem with Tresca law: Existence and the behavior of weak solutions. Nonlinear Analysis: Real World Applications, 2008, 9 (2): 674-692.

[3] G.Duvaut, J.L.Lions. Les inéquations en Mécanique et en Physique, Paris : Dunod, 1972.

[4] D.Kinderlehrer, G.Stampacchia. An introduction to variational inequalities and their applications. New York : Academic Press, 1980.

[5] J.F.Rodrigues. Obstacle problems in mathematical physics. Amsterdam: North-Holland, 1987.

[6] E.D.Tabacman, D.A.Tarzia. Sufficient and or necessary condition for the heat transfer coefficient on \( \Gamma_1, \) and the heat flux on \( \Gamma_2 \) to obtain a steady-state two-phase Stefan problem. J. Diff. Equations, 1989, 77 (1): 16-37.

[7] D.A.Tarzia. Una familia de problemas que converge hacia el caso estacionario del problema de Stefan a dos fases, Math. Notae, 1979, 26:157-165.

[8] H.Brezis. Problèmes unilatéraux. J. Math. Pures Appl., 1972, 51: 1-162.

[9] M.Chipot. Elements of nonlinear Analysis. Birkhäuser Advanced Texts, 2000.

[10] M.Boukrouche, D.A.Tarzia. Convergence of distributed optimal controls for second kind parabolic variational inequalities. Nonlinear Analysis: Real World Applications, 2011, 12: 2211-2224.

[11] M.Boukrouche, D.A.Tarzia. On existence, uniqueness, and convergence, of optimal control problems governed by parabolic variational inequalities. Accepted in Springer IFIP Series. 25th IFIP TC7 Conference on System Modeling and Optimisation 2011.

[12] S.Kesavan, M.Thukuram. Low-cost control problems on perforated and non-perforated domains. Proc. Indian Acad. Sci. (Math. Sci.), 2008, 118 (1) : 133-157.

[13] S.Kesavan, J.Saint Jean Paulin. Optimal control on perforated domains. J. Math. Anal. Appl., 1997, 229 : 563-586.

[14] J.L.Lions. Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles, Paris : Dunod, 1968.

[15] F.Troltzsch. Optimal control of partial differential equations: Theory, methods and applications. Providence : American Math. Soc., 2010.

[16] M.Boukrouche, D.A.Tarzia. Convergence of distributed optimal controls for elliptic variational inequalities. Computational Optimization and Applications, (2012) 53:375393.

[17] C.Foias, O.Manley, R.Rosa, R.Temam. Navier-Stokes equations and turbulence. Cambridge University Press, 2001.
[18] C.M. Gariboldi, D.A. Tarzia. Convergence of boundary optimal controls problems with restrictions in mixed elliptic Stefan-like problems. *Adv. Diff. Eq. and Control Processes*, 2008, (1): 113-132.

[19] J.L. Menaldi, D.A. Tarzia. A distributed parabolic control with mixed boundary conditions. *Asymptotic Anal.*, 2007, 52: 227-241.

[20] N. Arada, H. El Fekih, J.P. Raymond. Asymptotic analysis of some control problems. *Asymptotic Analysis*, 2000, 24: 343-366.

[21] F. Ben Belgacem, H. El Fekih, H. Metoui. Singular perturbation for the Dirichlet boundary control of elliptic problems. *ESAIM: M2AN*, 2003, 37: 833-850.

[22] F. Ben Belgacem, H. El Fekih, J.P. Raymond. A penalized Robin approach for solving a parabolic equation with nonsmooth Dirichlet boundary conditions. *Asymptotic Analysis*, 2003, 34: 121-136.

[23] C.M. Gariboldi, D.A. Tarzia. Convergence of distributed optimal controls on the internal energy in mixed elliptic problems when the heat transfer coefficient goes to infinity. *Appl. Math. Optim.*, 2003, 47 (3): 213-230.

[24] V. Barbu. Optimal control of variational inequalities. Research Notes in Mathematics, 100. Boston: Pitman (Advanced Publishing Program), 1984.

[25] J.C. De Los Reyes. Optimal control of a class of variational inequalities of the second kind. *SIAM J. Control Optim.*, 2011, 49: 1629-1658.

[26] F. Mignot. Contrôle dans les inéquations variationnelles elliptiques. *J. Functional Anal.*, 1976, 22 (2): 130-185.

[27] D.A. Tarzia. Etude de l’inéquation variationnelle proposée par Duvaut pour le problème de Stefan à deux phases, I. *Boll. Unione Mat. Italiana*, 1982, 1B: 865-883.