On some new normed sequence spaces

G Pranajaya and E Herawati

1Department of Mathematics, University of Sumatera Utara, Indonesia
E-mail: 1gunturpranajaya13@gmail.com, 2herawaty.elv@gmail.com

Abstract. The sequence spaces $(c_0)_\Lambda$, $c_\Lambda$, and $(\ell_\infty)_\Lambda$ was introduced and studied by Mursaleen and Noman [11]. In the present paper, for $M$ is a generalization of Orlicz function, we extend the spaces Mursaleen and Noman’s to $[c_0(M)]_\Lambda$, $[c(M)]_\Lambda$, and $[\ell_\infty(M)]_\Lambda$, respectively, and investigate some topological properties of these spaces. Finally, we determine the necessary and sufficient conditions of an infinite matrix $A$ belonging to classes $(c_0(M), c_0(M))$, $(c(M), c(M))$, and $(\ell_\infty(M), \ell_\infty(M))$.

1. Introduction and Preliminaries

By $\omega$, we denote the space of all sequences of real or complex numbers. Any linear subspace of $\omega$ is called a sequence space. We shall write $c_0$, $c$, and $\ell_\infty$ for the spaces of all convergent to zero, convergent, and bounded sequences, respectively.

A sequence space $X$ is called a BK space provided $X$ is a complete normed space and a function $p_k : X \to \mathbb{R}$ defined by $x \mapsto p_k(x) = x_k$ is continuous for all $k \in \mathbb{N}$ (see [5]).

The sequence spaces $c_0$, $c$, and $\ell_\infty$ are BK spaces equipped with sup-norm $\|\cdot\|_\infty$ given by

$$\|x\| = \sup_{k \in \mathbb{N}} |x_k|.$$

Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$, and $X$, $Y$ be the sequence spaces. The map $A$ from $X$ into $Y$ is said to be matrix transformation if $Ax = (A_n(x))$ exists in $Y$ where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \text{ converges for all } n \in \mathbb{N} \text{ and all } x \in X. \quad (1)$$

We denote $(X, Y)$ as the class of all infinite matrices that map $X$ into $Y$. Thus, $A \in (X, Y)$ if and only if (1) hold, and $Ax \in Y$ for all $x \in X$. The theory of matrix transformation deals with establishing necessary and sufficient conditions on the entries of a matrix to map a sequence space $X$ into a sequence space $Y$.

For a sequence space $X$, the matrix domain of an infinite matrix $A$ in $X$ is a sequence space defined by

$$X_A = \left\{ x = (x_k) \in \omega : Ax \in X \right\}.$$

The idea of constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been studied by several authors, e.g., Altay and Başar [1],
Mursaleen et al. [2], Mursaleen and Noman [11, 12], Malkowsky [9], Malkowsky and Savas [10]. Mursaleen and Noman [11] introduced \( \Lambda \)-matrix and constructed the matrix domains on \( \Lambda \)-matrix in the classical sequence spaces \( c_0 \), \( c \), and \( \ell_\infty \). They examined some topological properties, established inclusion relations concerning with those spaces, determined their \( \alpha \)-, \( \beta \)-, \( \gamma \)-duals, and characterized some relate matrix classes.

On the other side, Lindenstrauss and Tzafriri [7] introduced the sequence space defined by Orlicz function as follows:

\[
\Omega = \{ x = (x_k) \in \omega : (\exists \rho > 0) \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \}
\]

which is called **Orlicz sequence space**. The space \( \Omega \) equipped with norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}
\]

becomes a Banach space.

Using the matrix domain \( X_\Lambda \) defined by Mursaleen and Noman [11], in this work, we introduce \( \Lambda \)-matrix domain for the sequences generated by a generalization of Orlicz function \( M \), denoted by \( [X(M)]_\Lambda \) where \( X \in \{ c_0, c, \ell_\infty \} \). Furthermore, we investigate some topological properties of these spaces over the norm spaces, and give the necessary and sufficient conditions on an infinite matrix \( A \) belonging to classes \( \{ c_0(M), c_0(M) \} \), \( \{ c(M), c(M) \} \), and \( \{ \ell_\infty(M), \ell_\infty(M) \} \).

2. Results

2.1. The Sequence Space \([X(M)]_\Lambda\)

A function \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) is called a **generalization of Orlicz function** which is vanishing at zero, non decreasing, and continuous. A generalization of Orlicz function \( M \) is said to satisfy **\( \Delta_2 \)-condition** for all values of \( x \) if there exists a constant \( K > 0 \) such that \( M(2x) \leq KM(x) \) for \( x \geq 0 \). Furthermore, in [11], Mursaleen and Noman defined the infinite matrix \( \Lambda = (\lambda_{nk}) \) by

\[
\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} & ; 0 \leq k \leq n \\ 0 & ; k > n \end{cases}
\]

where \( \lambda = (\lambda_k) \) be a strictly increasing sequence of positive reals tending to infinity, that is, \( 0 < \lambda_0 < \lambda_1 < \cdots \) and \( \lambda_k \to \infty \) as \( k \to \infty \). By using (2), in the present section we define the sequence space \([X(M)]_\Lambda\) where \( X \in \{ c_0, c, \ell_\infty \} \) and \( M \) is a generalization of Orlicz function, and prove that these sequence spaces according to its norm are complete normed spaces. These sequence spaces are as follows:

\[
[c_0(M)]_\Lambda = \left\{ x = (x_k) \in \omega : (\exists \rho > 0) M \left( \frac{|A_{nk}(x)|}{\rho} \right) \to 0, n \to \infty \right\},
\]

\[
[c(M)]_\Lambda = \left\{ x = (x_k) \in \omega : (\exists \rho > 0, l \in \mathbb{R}) M \left( \frac{|A_{nk}(x)|}{\rho} \right) \to l, n \to \infty \right\}, \text{ and}
\]

\[
[\ell_\infty(M)]_\Lambda = \left\{ x = (x_k) \in \omega : (\exists \rho > 0) \sup_{n \in \mathbb{N}} M \left( \frac{|A_{nk}(x)|}{\rho} \right) < \infty \right\}.
\]

Now, we may begin with the following results which is essential in the text.
2.2. Linear Topological Structure of $[X(M)]_\Lambda$

In this section, we examine some topological properties of the sequence spaces defined above.

**Theorem 2.1.** If $M$ is a convex function, then the sequence space $[X(M)]_\Lambda$ for $X \in \{c_0, c, \ell_\infty\}$ is linear space over the set of real numbers $\mathbb{R}$.

**Proof.** We prove the theorem for $X = \ell_\infty$. Let $x, y \in [\ell_\infty(M)]_\Lambda$ and $\alpha, \beta \in \mathbb{R}$, then there exist some positive $\rho_1$ and $\rho_2$ such that

$$\sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(x)}{\rho_1} \right) < \infty \text{ and } \sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(y)}{\rho_2} \right) < \infty.$$  

Take $\rho = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$, then for a convex function $M$ we get

$$\sup_{n \in \mathbb{N}} M \left( \frac{|\Lambda_n(ax + \beta y)|}{\rho} \right) \leq \sup_{n \in \mathbb{N}} M \left( \frac{(|\alpha|\Lambda_n(x))}{2} + \frac{(|\beta|\Lambda_n(y))}{2} \right) \leq \sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(x)}{\rho_1} \right) + \frac{1}{2} \sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(y)}{\rho_2} \right) < \infty.$$  

This proves that $[\ell_\infty(M)]_\Lambda$ is linear space. \hfill \qed

It is easy to show that $[c_0(M)]_\Lambda$ and $[c(M)]_\Lambda$ are also linear spaces.

**Theorem 2.2.** If $M$ satisfy $\Delta_2$-condition, then the space $[X(M)]_\Lambda$ for $X \in \{c_0, c, \ell_\infty\}$ is complete normed space equipped with the norm defined by

$$\|x\|_{[X(M)]_\Lambda} = \inf \left\{ \rho > 0 : \sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(x)}{\rho} \right) \leq 1 \right\} \tag{3}$$

**Proof.** We prove the theorem for $X = \ell_\infty$ and the other cases will follow similarly. Let $x, y \in [\ell_\infty(M)]_\Lambda$. It is easily seen that $\|x\|_{[\ell_\infty(M)]_\Lambda} \geq 0$. Next, if $x = 0$, then obviously $\|x\|_{[\ell_\infty(M)]_\Lambda} = 0$. Conversely, suppose $\|x\|_{[\ell_\infty(M)]_\Lambda} = 0$, then for every $\epsilon > 0$ we get $\|x\|_{[\ell_\infty(M)]_\Lambda} < \epsilon$. This implies there exists some $\rho_0$ with $0 < \rho_0 < \epsilon$ such that

$$\sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(x)}{\epsilon} \right) < \sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(x)}{\rho_0} \right) \leq 1.$$  

Since $M$ is a generalization of Orlicz function, it follows that for every $\epsilon > 0$ and for every $n \in \mathbb{N}$,

$$|\Lambda_n(x)| = \left| \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})x_k \right| = 0.$$  

Under the assumption that $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive real numbers, it is easy to check by mathematical induction that $x_k = 0$ for every $k \in \mathbb{N}$. Thus, $x = 0$.

Furthermore, let $x \in [\ell_\infty(M)]_\Lambda$ and $\alpha \in \mathbb{R}$. If $\alpha = 0$, it is clear that the homogeneous property of the norm holds. Assume $\alpha \neq 0$, we get

$$\|\alpha x\|_{[\ell_\infty(M)]_\Lambda} = |\alpha| \inf \left\{ \frac{\rho}{|\alpha|} > 0 : \sup_{n \in \mathbb{N}} M \left( \frac{\Lambda_n(x)}{\rho} \right) \leq 1 \right\}.$$
This gives $\|ax\|_{(\ell_\infty(M))_A} = |\alpha|\|x\|_{(\ell_\infty(M))_A}$.

Now, let $x, y \in [\ell_\infty(M)]_A$, then there exists some $\rho_1, \rho_2 > 0$ such that

$$\sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x)|}{\rho_1}\right) \leq 1 \quad \text{and} \quad \sup_{n \in \mathbb{N}} M\left(\frac{|A_n(y)|}{\rho_2}\right) \leq 1.$$ 

Hence, if we choose $\rho = \rho_1 + \rho_2$, then by the properties of $M$, we have $M\left(\frac{|A_n(x+y)|}{\rho}\right) \leq M\left(\frac{|A_n(x)|}{\rho_1}\right) + M\left(\frac{|A_n(y)|}{\rho_2}\right)$ for all $n \in \mathbb{N}$. Consequently, $\|x + y\|_{(\ell_\infty(M))_A} \leq \|x\|_{(\ell_\infty(M))_A} + \|y\|_{(\ell_\infty(M))_A}$. Hence, $[\ell_\infty(M)]_A$ is a normed space.

Now, suppose that $(x^i)$ be any Cauchy sequence in $[\ell_\infty(M)]_A$. Then, for each $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ and $\rho_0$ where $0 < \rho_0 < \epsilon$ such that

$$\sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x^j - x^i)|}{\epsilon}\right) < \sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x^j - x^i)|}{\rho_0}\right) \leq 1.$$ 

Hence, for every $\epsilon > 0$, $M\left(\frac{|A_n(x^j - x^i)|}{\rho}\right) < \epsilon$. Consequently, $|x^j - x^i| < \epsilon$ for each $\epsilon > 0$, every $j \geq i \geq i_0$, and every $k \in \mathbb{N}$, where $\lambda = (\lambda_k)$ is a strictly increasing sequence of positive real numbers. We see that $(x^k)$ is Cauchy sequences of real numbers. Since $\mathbb{R}$ is complete, there exists $x_k \in \mathbb{R}$ such that $x^j \rightarrow x_k$ as $j \rightarrow \infty$ for all $k \in \mathbb{N}$. Using these limits, we define $x = (x_k)$ and show that $x \in [\ell_\infty(M)]_A$ and $x^j \rightarrow x$ as $i \rightarrow \infty$ in $[\ell_\infty(M)]_A$. From (3), we have for all $i \geq i_0$

$$\sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x^i - x)|}{\rho_0}\right) = \lim_{j \rightarrow \infty} \sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x^j - x^i)|}{\rho_0}\right) \leq 1.$$ 

We obtain $\|x^i - x\| < \epsilon$ for every $i \geq i_0$. This shows that $x^i \rightarrow x$ as $i \rightarrow \infty$ in $[\ell_\infty(M)]_A$.

Since $x^i \in [\ell_\infty(M)]_A$ and $M$ satisfy $\Delta_2$-condition, we have

$$\sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x)|}{\rho}\right) \leq \frac{K_1}{2} \sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x-x^i)|}{\rho}\right) + \frac{K_2}{2} \sup_{n \in \mathbb{N}} M\left(\frac{|A_n(x^i)|}{\rho}\right) < \infty$$ 

for some $K_1, K_2 > 0$. It is show that $x \in [\ell_\infty(M)]_A$.

Since $(x^i)$ was an arbitrary Cauchy sequence in $[\ell_\infty(M)]_A$, this proves completeness of $[\ell_\infty(M)]_A$. \hfill $\square$

Now, the following result is immediate by Theorem 2.2.

**Theorem 2.3.** $[X(M)]_A$ is a BK space where $X \in \{c_0, c, \ell_\infty\}$ and $[X(M)]_A$ is a AK space where $X = c_0$.

### 2.3. The Certain Classes of Matrix Transformations $(X(M), X(M))$

**Theorem 2.4.** $A \in (c(M), c(M))$ if and only if for $\rho > 0$ the following conditions are held:

(i) $\sup_{n \in \mathbb{N}} M\left(\frac{\sum_{k=0}^{\infty}|a_{nk}|}{\rho}\right) < \infty$,

(ii) $\lim_{n \rightarrow \infty} M\left(\frac{|a_{nk}|}{\rho}\right) = \alpha_k$ exists for each $k \in \mathbb{N}$, and

(iii) $\lim_{n \rightarrow \infty} M\left(\frac{\sum_{k=0}^{\infty}a_{nk}}{\rho}\right) = \alpha$ exist.
Proof. For proving the necessity, suppose that \( A \in (c(M), c(M)) \). If we choose \( x = (x_k) \) by \( x_k = \text{sgn}(a_{nk}) \) for all \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \), then \( \sup_{n \in \mathbb{N}} M \left( \sum_{k=0}^{\infty} \frac{|a_{nk}|}{\rho} \right) = \sup_{n \in \mathbb{N}} M \left( \sum_{k=0}^{\infty} a_{nk} x_k \right) < \infty \). This shows that the condition (i) holds. Furthermore, if we take \( x = (x_k) \) where for all \( j \in \mathbb{N} \) and \( k \in \mathbb{N} \),

\[
x_j = e_j^{[k]} = \begin{cases} 1 & : j = k \\ 0 & : j \neq k \end{cases}
\]

Since \( Ax \in c(M) \) for every \( x \in c(M) \), there exists \( \rho > 0 \) such that \( \lim_{n \to \infty} M \left( \frac{\sum_{j=0}^{\infty} a_{nj} x_j}{\rho} \right) \) exist. Hence, for all \( k \in \mathbb{N} \), \( \lim_{n \to \infty} M \left( \frac{\sum_{j=0}^{\infty} a_{nj} x_j}{\rho} \right) = \lim_{n \to \infty} M \left( \frac{|a_{nk}|}{\rho} \right) \). It is shown that \( \lim_{n \to \infty} M \left( \frac{|a_{nk}|}{\rho} \right) \) exists for each \( k \in \mathbb{N} \), and the condition (ii) holds. Next, since \( x = (x_k) = (1, 1, 1, \cdots) \) belongs to \( c(M) \), the condition (iii) holds.

For sufficiency, let \( x_k \to r \) as \( k \to \infty \) and let the conditions (i), (ii), and (iii) hold. We write

\[
\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} a_{nk} (x_k - r) + r \sum_{k=0}^{\infty} a_{nk} \text{ for all } n \in \mathbb{N}.
\]

Since \( M \) satisfy \( \Delta_2 \)-condition, then for some \( \rho > 0 \) we get

\[
M \left( \frac{\sum_{k=0}^{\infty} a_{nk} x_k}{\rho} \right) \leq \frac{K_0}{2} M \left( \frac{\sum_{k=0}^{\infty} a_{nk} (x_k - r)}{\rho} \right) + \frac{K_1^{m_0+1}}{2} M \left( \frac{\sum_{k=0}^{\infty} a_{nk}}{\rho} \right)
\]

for all \( n \in \mathbb{N} \) and for some \( K_0, K_1 > 0 \). By (iii), we get \( \lim_{n \to \infty} \frac{K_1^{m_0+1}}{2} M \left( \frac{\sum_{k=0}^{\infty} a_{nk}}{\rho} \right) = \alpha K^{m_0+1} \).

Further, since \( x_k \to r \) as \( k \to \infty \), we get

\[
\frac{K_0}{2} M \left( \frac{\sum_{k=0}^{\infty} a_{nk} (x_k - r)}{\rho} \right) + \frac{K_1^{m_0+1}}{2} M \left( \frac{\sum_{k=0}^{\infty} a_{nk}}{\rho} \right) \to \alpha_1 \text{ as } n \to \infty
\]

where \( \alpha_1 = \frac{\alpha K^{m_0+1}}{2} \). It is shown that \( M \left( \frac{\sum_{k=0}^{\infty} a_{nk} x_k}{\rho} \right) \to \alpha_1 \) as \( n \to \infty \). Thus, \( Ax \in c(M) \). Since for each \( x \in c(M) \) implies \( Ax \in c(M) \), we conclude that \( A \in (c(M), c(M)) \), which proves the theorem.

\[\textbf{Theorem 2.5.} \ A \in (c_0(M), c_0(M)) \text{ if and only if for } \rho > 0 \text{ the following conditions are held :}
\]

(i) \( \sup_{n \in \mathbb{N}} M \left( \sum_{k=0}^{\infty} \frac{|a_{nk}|}{\rho} \right) < \infty \), and

(ii) \( \lim_{n \to \infty} M \left( \frac{|a_{nk}|}{\rho} \right) = 0 \) for each \( k \in \mathbb{N} \).

Proof. We first derive the necessary conditions (i) and (ii). Since \( x = (x_j) = (e_j^{[k]}) \) belongs to \( c_0(M) \), then

\[
\lim_{n \to \infty} M \left( \frac{|a_{nk}|}{\rho} \right) = \lim_{n \to \infty} M \left( \frac{\sum_{j=0}^{\infty} a_{nj} x_j}{\rho} \right) = 0 \text{ for each } k \in \mathbb{N}.
\]

It is shown that (ii) holds. Further, we define \( x = (x_k) \) by \( x_k = \text{sgn}(a_{nk}) \) for all \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \). Thus, for some \( \rho > 0 \) we have

\[
\sup_{n \in \mathbb{N}} M \left( \sum_{k=0}^{\infty} \frac{|a_{nk}|}{\rho} \right) = \sup_{n \in \mathbb{N}} M \left( \sum_{k=0}^{\infty} a_{nk} \right) = \sup_{n \in \mathbb{N}} M \left( \sum_{k=0}^{\infty} a_{nk} x_k \right).
\]
Since $\lim_{n \to \infty} M \left( \frac{\sum_{k=0}^{\infty} a_{nk} x_k}{\rho} \right)$ exists, then sequence $M \left( \frac{\sum_{k=0}^{\infty} a_{nk} x_k}{\rho} \right)$ is bounded. Thus, $\sup_{n \in \mathbb{N}} M \left( \frac{\sum_{k=0}^{\infty} a_{nk} x_k}{\rho} \right) < \infty$, which yields (i) holds.

For proving the sufficiency, let us take any $x \in c_0(M)$. Since $M$ is continuous, then $x_k \to 0$ as $k \to \infty$. Thus, $\lim_{n \to \infty} M \left( \frac{\sum_{k=0}^{\infty} a_{nk} x_k}{\rho} \right) = 0$. It is shown that $Ax \in c_0(M)$. Since for each $x \in c(M)$ implies $Ax \in c(M)$, then the infinite matrix $A$ belongs to the class $(c_0(M), c_0(M))$, which completes the proof. \hfill \Box

**Theorem 2.6.** $A \in (\ell_{\infty}(M), \ell_{\infty}(M))$ if and only if for $\rho > 0$

$$\sup_{n \in \mathbb{N}} M \left( \frac{\sum_{k=0}^{\infty} |a_{nk}|}{\rho} \right) < \infty.$$  

**Proof.** For proving the necessity, suppose that $A \in (\ell_{\infty}(M), \ell_{\infty}(M))$, that is, for each $x \in \ell_{\infty}(M)$ implies $Ax \in \ell_{\infty}(M)$. Thus, $\sup_{n \in \mathbb{N}} M \left( \frac{\sum_{k=0}^{\infty} |a_{nk}| x_k}{\rho} \right) < \infty$. Then, define $x = (x_k)$ by $x_k = \text{sgn}(a_{nk})$ for all $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Thus, for some $\rho > 0$ and for every $n \in \mathbb{N}$, we have

$$M \left( \frac{\sum_{k=0}^{\infty} |a_{nk}| x_k}{\rho} \right) = M \left( \frac{\sum_{k=0}^{\infty} a_{nk} x_k}{\rho} \right).$$

Since $\sup_{n \in \mathbb{N}} M \left( \frac{\sum_{k=0}^{\infty} |a_{nk}| x_k}{\rho} \right) < \infty$, then $\sup_{n \in \mathbb{N}} M \left( \frac{\sum_{k=0}^{\infty} |a_{nk}|}{\rho} \right) < \infty$.

For sufficiency, suppose $\sup_{n \in \mathbb{N}} M \left( \frac{\sum_{k=0}^{\infty} |a_{nk}|}{\rho} \right) < \infty$ and take any $x \in \ell_{\infty}(M)$. Since $M$ is non-decreasing, there exists $N_1 > 0$ such that $|x_k| \leq N_1 = \rho N_0$ for all $k \in \mathbb{N}$. Hence, by using Hölder inequality [8], we get

$$\sup_{n \in \mathbb{N}} M \left( \frac{|A_n(x)|}{\rho} \right) \leq \sup_{n \in \mathbb{N}} M \left( \frac{\sup_{k \in \mathbb{N}} |x_k| \sum_{k=0}^{\infty} |a_{nk}|}{\rho} \right).$$

Further, by using the Archimedean property, since $N_1 \in \mathbb{R}$, then there exists $m_0 \in \mathbb{N}$ such that $N_1 \leq 2^{m_0}$. Since $M$ satisfy $\Delta_2$-condition, we have

$$\sup_{n \in \mathbb{N}} M \left( \frac{|A_n(x)|}{\rho} \right) \leq K^{m_0} \sup_{n \in \mathbb{N}} M \left( \frac{\sum_{k=0}^{\infty} |a_{nk}|}{\rho} \right) < \infty$$

for some $K > 0$. It is show that $Ax \in \ell_{\infty}(M)$. Since for each $x \in \ell_{\infty}(M)$ implies $Ax \in \ell_{\infty}(M)$, then the infinite matrix $A \in (\ell_{\infty}(M), \ell_{\infty}(M))$, and the proof is complete. \hfill \Box

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