\textbf{C*-CORRESPONDENCE FUNCTIONALITY OF CUNTZ-PIMSNER ALGEBRAS}

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\textbf{Abstract.} We construct a functor that maps C*-correspondences to their Cuntz-Pimsner algebras. The objects in our domain category are C*-correspondences, and the morphisms are the isomorphism classes of C*-correspondences satisfying certain conditions. As an application, we recover a well-known result of Muhly and Solel. In fact, we show that functoriality leads us to a more generalized result: strongly Morita equivalent C*-correspondences have Morita equivalent Cuntz-Pimsner algebras.

1. Introduction

The study of C*-algebras arising from C*-correspondences has garnered a great deal of attention due to its epic variety of applications. We were mainly inspired by the fundamental work that Muhly and Solel presented in [9]. They developed a notion $^\text{SME}$ of strong Morita equivalence for given C*-correspondences $AX_A$ and $BY_B$, where there is an imprimitivity bimodule $AM_B$ so that

\[ X \otimes_B M \cong M \otimes_B Y \]

as $A - B$ correspondences. They proved that if two injective C*-correspondences are strongly Morita equivalent then the corresponding Cuntz-Pimsner algebras are strongly Morita equivalent in the sense of Rieffel. In [3], the authors presented an elegant proof to a more generalized result: they drop the assumption of injectivity. In this paper, we build a functor that will give us their result as an application. To that end, we first present two categories: the enchilada category and the enchilada category of C*-correspondences, which we denote the latter by $\text{ ECCor}$. The former category has C*-algebras as objects, and isomorphism classes of non-degenerate C*-correspondences as morphisms. One of the key points here is that the isomorphism of two objects in the enchilada category is equivalent to them being strongly Morita equivalent C*-algebras. In $\text{ ECCor}$ not only are the morphisms, but also the objects are C*-correspondences. We wish to construct this category so that two objects being isomorphic is equivalent to them being strongly Morita equivalent as C*-correspondences. For this purpose, we define morphisms in $\text{ ECCor}$ to be the isomorphism classes of non-degenerate regular (see Chapter 2 for the definition) C*-correspondences that satisfy a certain condition. Our main result is that any non-degenerate regular C*-correspondence that relates $AX_A$ and $BY_B$ in a certain way induces
an $\mathcal{O}_X - \mathcal{O}_Y$ correspondence, where $\mathcal{O}_X$ denotes the Cuntz-Pimsner algebra of $AX_A$. We do not put any condition on the objects in $ECCor$, $AX_A$ and $BY_B$, except non-degeneracy. It is worth mentioning that, when it comes to the application of our work, non-degeneracy does not cause an inconvenience. This is due to the fact that strong Morita equivalence automatically implies $C^*$-correspondences are non-degenerate.

Our functor maps any $C^*$-correspondence (an object in $ECCor$) to its Cuntz-Pimsner algebra. In [6], the authors follow a similar technique (working with quite different categories than ours) to give an alternate approach to some central results, such as the result of Hao and Ng: when the group $G$ is amenable there is an isomorphism $\mathcal{O}_{X \rtimes_G G} \cong \mathcal{O}_{X \rtimes_B G}$. We save the connection between our work and [6] as possible future study.

For the Cuntz-Pimsner algebra of a given $C^*$-correspondence, we use the characterization that Katsura presented in [7], which is based on the Fock representation. For the convenience of the reader, in Section 3, we briefly review his construction.

In Section 4, we present our first category: the enchilada category. We have studied this category in [4] in detail. However, in this paper, we give some properties of the enchilada category that are necessary for our work and that are not contained in [4]. These can be mainly seen as the factorizations of certain $C^*$-correspondences that are crucial for our computations. Once it is done, by the end of this section, we introduce our second category: the enchilada category of $C^*$-correspondences.

Section 5 is devoted to constructing our functor. During the construction process we had two formidable hurdles. The first one arose due to the fact that the Fock space $\mathcal{F}(X)$ of a given $C^*$-correspondence $AX_A$ can not, in general, be viewed as an $\mathcal{O}_X - A$ correspondence. We discovered that choosing morphisms in $ECCor$ to be regular $C^*$-correspondences is the key to conquer this problem. The second challenge was showing that our functor indeed preserves identity morphisms. We overcome this issue by proving the factorization

$$\tau_X(\mathcal{F}(X) \otimes \mathcal{T}_X) \tau_X \cong \tau_X(\mathcal{T}_X) \tau_X,$$

where $\mathcal{T}_X$ denotes the Toeplitz algebra of the $C^*$-correspondence $AX_A$. We present our main result in Theorem 5.18 and end the paper by discussing the application: a generalized version of the Muhly and Solel result [9].

2. Preliminaries

In this section we recall some basic definitions and general results on Hilbert modules and $C^*$-correspondences. Much of the information given below can be found in [11], [2], and [8].

A Hilbert $B$-module is a vector space $X$ equipped with a right $B$-module structure and a $B$-valued inner product, i.e., a positive-definite $B$-valued sesquilinear form $\langle \cdot, \cdot \rangle_B$ satisfying

$$\langle x, yb \rangle_B = \langle x, y \rangle_B b \quad \text{and} \quad \langle x, y \rangle_B^* = \langle y, x \rangle_B$$

for all $x, y \in X, b \in B$, and which is complete in the norm $\|x\| = \|\langle x, x \rangle_B\|^{1/2}$. The closed span of the inner products is an ideal $B_X$ of $B$, and $X$ is called full if $B_X = B$. The $B$-module operators $T$ on $X$ for which there is an operator $T^*$ satisfying

$$\langle Tx, y \rangle_B = \langle x, T^* y \rangle_B \quad \text{for all} \quad x, y \in X$$

are called adjointable operators on $X$. In this paper, we study the category $ECCor$ of $C^*$-correspondences and its objects $AX_A$. We use the notation $\mathcal{O}_X$ to denote the Cuntz-Pimsner algebra of $AX_A$.
form the \( C^* \)-algebra \( \mathcal{L}(X) \) of \textit{adjointable operators} with the operator norm, and the closed linear span of the \textit{rank-one operators} \( \theta_{x,y} \) given by

\[
\theta_{x,y}z = x\langle y, z \rangle_B
\]

is the closed ideal \( \mathcal{K}(X) \) of \textit{compact operators}.

By an \( A \rightarrow B \) \textit{correspondence} \( X \) we mean a Hilbert \( B \)-module \( X \) with a homomorphism\(^1\) \( \varphi_X : A \rightarrow \mathcal{L}(X) \), and we say the correspondence is \textit{non-degenerate} if \( AX = X \).\(^2\) As we mentioned in the introduction, \textit{all our correspondences will be non-degenerate by standing hypothesis}. That is, from now on when we use the term correspondence we will assume the non-degeneracy condition. An \( A \rightarrow B \) correspondence \( AX_B \) is \textit{regular} if the left action \( \varphi_X : A \rightarrow \mathcal{L}(X) \) is injective and \( \varphi_X(A) \subseteq \mathcal{K}(X) \). The \textit{identity correspondence} on \( A \) is the vector space \( A \) with the \( A \rightarrow A \) bimodule structure given by multiplication and the inner product \( \langle a, b \rangle_A = a^*b \).

An \textit{isomorphism} \( U : X \rightarrow Y \) of \( A \rightarrow B \) correspondences is a linear bijection such that

\[
U(ax) = aU(x) \quad \text{and} \quad \langle Ux, Uy \rangle_B = \langle x, y \rangle_B
\]

for all \( a \in A \), and \( x, y \in X \).

An \( A \rightarrow B \) bimodule \( X_0 \) is called a \textit{pre-correspondence} if it has a \( B \)-valued semi-inner product satisfying

\[
\langle x, y \cdot b \rangle_B = \langle x, y \rangle_B b, \quad \langle x, y \rangle_B^* = \langle y, x \rangle_B
\]

and \( \langle a \cdot x, a \cdot y \rangle_B \leq \|a\|^2 \langle x, y \rangle_B \) for all \( a \in A, b \in B \) and \( x, y \in X_0 \). The Hausdorff completion \( X \) of \( X_0 \) becomes an \( A \rightarrow B \) correspondence. Now let \( Z \) be an \( A \rightarrow B \) correspondence and assume there is a map \( \Phi : X_0 \rightarrow Z \) satisfying

\[
\Phi(a \cdot x) = \varphi_Z(a)\Phi(x) \quad \text{and} \quad \langle \Phi(x), \Phi(y) \rangle_B = \langle x, y \rangle_B.
\]

Then, \( \Phi \) extends uniquely to an injective \( A \rightarrow B \) correspondence homomorphism \( \Phi : X \rightarrow Z \).

The \textit{balanced tensor product} \( X \otimes_B Y \) of an \( A \rightarrow B \) correspondence \( X \) and a \( B \rightarrow C \) correspondence \( Y \) is formed as follows: the algebraic tensor product \( X \otimes Y \) is given the \( A \rightarrow C \) bimodule structure determined on the elementary tensors by

\[
a(x \otimes y)c = ax \otimes yc \quad \text{for} \quad a \in A, x \in X, y \in Y, c \in C,
\]

and the unique \( C \)-valued sesquilinear form whose values on elementary tensors are given by

\[
\langle x \otimes y, u \otimes v \rangle_C = \langle y, \langle x, u \rangle_Bv \rangle_C \quad \text{for} \quad x, u \in X, y, v \in Y.
\]

It is a well known fact that \( X \otimes Y \) is a \textit{pre-correspondence}. The Hausdorff completion is an \( A \rightarrow C \) correspondence \( X \otimes_B Y \). The term \textit{balanced} refers to the property

\[
xb \otimes y = x \otimes by \quad \text{for} \quad x \in X, b \in B, y \in Y.
\]

\(^1\)All homomorphisms are assumed to be \( * \)-homomorphisms.

\(^2\)Unless it is stated otherwise (as in, for example, Lemma \(^1\)) we denote the left action of a given \( C^* \)-correspondence \( AX_B \) by \( \varphi_X \).

\(^3\)Note that we actually mean \( AX = \{ax : a \in A, x \in X \} \) — by the Cohen-Hewitt factorization theorem this coincides with the closed span.
which is automatically satisfied.

The multiplier algebra of $A$ is the $C^*$-algebra $M(A) = \mathcal{L}(AA_A)$, and we identify $A$ with its image under the left-module homomorphism $\phi_A : A \to M(A)$. In this way $A$ becomes an ideal of $M(A)$. More generally, it is a standard fact that $M(\mathcal{K}(X)) = \mathcal{L}(X)$. A homomorphism $\mu : A \to M(B)$ is non-degenerate if $\mu(A)B = B$, and non-degeneracy of a correspondence $AX_B$ is equivalent to non-degeneracy of the left-module homomorphism $\phi_X : A \to M(\mathcal{K}(X))$. Every non-degenerate homomorphism $\mu : A \to M(B)$ extends uniquely to a homomorphism $\bar{\mu} : M(A) \to M(B)$, and we typically drop the bar, just writing $\mu$ for the extension.

An $A-B$ Hilbert bimodule is an $A-B$ correspondence $X$ that is also equipped with an $A$-valued inner product $A(\cdot, \cdot)$, which satisfies

$$A(ax, y) = a_A(x, y) \quad \text{and} \quad A(x, y)^* = A(y, x)$$

for all $a \in A, x, y \in X$, as well as the compatibility property

$$A(x, y)z = x(y, z)_B \quad \text{for} \ x, y, z \in X.$$

A Hilbert bimodule $AX_B$ is left-full if the closed span of $A\langle X, X \rangle$ is all of $A$. The dual $B-A$ Hilbert bimodule $\tilde{X}$ is formed as follows: write $\tilde{x}$ when a vector $x \in X$ is regarded as belonging to $\tilde{X}$, define the $B-A$ bimodule structure by

$$b\tilde{x}a = \widetilde{a^*xb^*}$$

and the inner products by

$$B(\tilde{x}, \tilde{y}) = \langle x, y \rangle_B \quad \text{and} \quad (\tilde{x}, \tilde{y})_A = A\langle x, y \rangle$$

for $b \in B, x, y \in X, a \in A$. An $A-B$ imprimitivity bimodule is an $A-B$ Hilbert bimodule that is full on both the left and the right.

A representation $(\pi, t)$ of $AX_A$ on a $C^*$-algebra $B$ consists of a $*$-homomorphism $\pi : A \to B$ and a linear map $t : X \to B$ such that

$$\pi(a)t(x) = t(\varphi(a)(x)) \quad \text{and} \quad t(x)^*t(y) = \pi(\langle x, y \rangle_A),$$

for $a \in A$ and $x, y \in X$ and $\varphi$ is the homomorphism associated with $AX_A$. An application of the $C^*$-identity shows that $t(x)\pi(a) = t(xa)$ is also valid. For each representation $(\pi, t)$ of $AX_A$ on $B$, there exist a homomorphism $\Psi_t : \mathcal{K}(X) \to B$, such that

$$\Psi_t(\theta_x,y) = t(x)t(y)^*$$

for $x, y \in X$. The representation $(\pi, t)$ is called injective if $\pi$ is injective, in which case $t$ is an isometry and $\Psi_t$ is injective. We denote the $C^*$-algebra generated by the images of $\pi$ and $t$ in $B$ by $C^*(\pi, t)$.

3. The Fock representation

As mentioned in the introduction, for the Toeplitz and Cuntz-Pimsner algebras of given $C^*$-correspondence $AX_A$, we use the characterization that Katsura introduced in terms of Fock representations. We denote them by $\mathcal{F}_X$ and $\mathcal{O}_X$, respectively. The development of the construction can be found in [7]. Here we give a brief overview. Fix $X^{\otimes 0} = A$, etc.
Each \( X^n \) is a \( C^* \)-correspondence over \( A \) with
\[
\varphi_n(a)(x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A x_n) = \varphi(a)x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A x_n
\]
\[
(x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A x_n) \cdot a = x_1 \otimes_A x_2 \otimes_A \ldots \otimes_A (x_n \cdot a).
\]
The operator \( \varphi_0(a) \in \mathcal{K}(X^{\otimes 0}) \) is just the left multiplication operator on \( A \). Now, set \( t^0 = \pi \) and \( t^1 = t \). For \( n = 2, 3, \ldots \), define a linear map \( t^n : X^{\otimes n} \to C^*(\pi, t) \) by
\[
t^n(x \otimes_A x_{n-1}) = t(x)t^{n-1}(x_{n-1}),
\]
where \( x \in X \) and \( x_{n-1} \in X^{\otimes n-1} \). The pair \( (\pi, t^n) \) is a representation of \( A(X^{\otimes n})_A \). Hence, \( \Psi_{t^n} : \mathcal{K}(X^{\otimes n}) \to C^*(\pi, t) \) can be defined by \( \Psi_{t^n}(\theta_{\nu, \mu}) = t^n(\nu)t^n(\mu)^* \) for \( \nu, \mu \in X^{\otimes n} \). We have that
\[
C^*(\pi, t) = \text{span}\{t^n(x_n)t^m(y_m)^* : x_n \in X^{\otimes n}, m \in X^{\otimes m}, n, m \in \mathbb{N}\}.
\]
The Fock space of \( AX_A \) is the direct sum of all the Hilbert modules \( X^{\otimes n} \). It is a Hilbert \( A \)-module with the inner product
\[
\langle (x_n), (y_m) \rangle_A = \sum_n \langle x_n, y_n \rangle_A,
\]
for \( (x_n), (y_m) \in \mathcal{F}(X) \). It is full, since it contains \( A \) as a submodule.

Let \( x_n \in X^{\otimes n} \) for \( n \in \mathbb{N} \). For each \( m \in \mathbb{N} \), define the operator \( t_m^n(x_n) \in \mathcal{L}(X^{\otimes m}, X^{\otimes (n+m)}) \) by
\[
t_m^n(x_n)(y_m) := x_n \otimes_A y_m,
\]
where \( y_m \in X^{\otimes m} \). The adjoint \( t_m^n(x_n)^* \) of \( t_m^n(x_n) \) satisfies that
\[
t_m^n(x_n)^*(y_n \otimes_A z_m) = \varphi_m((x_n, y_n) A)z_m
\]
for \( y_n \in X^{\otimes n} \) and \( z_m \in X^{\otimes m} \).

The Fock representation \( (\varphi_\infty, t) \) of \( AX_A \) on \( \mathcal{L}(\mathcal{F}(X)) \) is defined by
\[
\varphi_\infty(a) = \sum_{m=0}^{\infty} \varphi_m(a), \quad t(x) = \sum_{m=0}^{\infty} t_m^1(x),
\]
for \( a \in A \), and \( x \in X \), where we use the strong operator topology for the infinite sum of elements in \( \mathcal{L}(\mathcal{F}(X)) \). The homomorphism \( \varphi_\infty \) is injective and defines a left action on \( \mathcal{F}(X) \). Therefore, \( \mathcal{F}(X) \) can be viewed as a \( C^* \)-correspondence over \( A \).

The ideal \( J_X \), called the Katsura ideal of \( A \), is defined as
\[
J_X = \varphi_\infty^{-1}(\mathcal{K}(X)) \cap (\text{Ker } \varphi_\infty)^\perp.
\]
A representation \( (\pi, t) \) is said to be covariant if \( \pi(a) = \Psi_t(\varphi_\infty(a)) \) for all \( a \in J_X \). The ideal \( \mathcal{K}(\mathcal{F}(X)J_X) = \text{span}\{\theta_{x_0,y} \in \mathcal{K}(\mathcal{F}(X)) : x, y \in \mathcal{F}(X), a \in J_X\} \) of \( \mathcal{L}(\mathcal{F}(X)) \) is contained in \( C^*(\varphi_\infty, t) \). Now, consider the quotient homomorphism \( \rho : \mathcal{L}(\mathcal{F}(X)) \to \mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X)J_X) \). Set \( \overline{\varphi} = \rho \circ \varphi_\infty \), and \( \overline{t} = \rho \circ t \). The pair \( (\overline{\varphi}, \overline{t}) \) is an

\[\text{We denote the set } \{0, 1, 2, \ldots \} \text{ of natural numbers by } \mathbb{N}.\]
injective covariant representation of $\mathcal{A}X$ on $\mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X)J_X)$. Katsura proves the isomorphisms $C^*(\varphi, t) \cong \mathcal{I}_X$ and $C^*(\mathcal{F}, \mathcal{F}) \cong \mathcal{O}_X$ (Proposition 6.5]). This allows us to identify $\mathcal{O}_X$ as the quotient $C^*$-algebra $\mathcal{I}_X/\mathcal{K}(\mathcal{F}(X)J_X)$, which is what we do throughout this paper.

We end this section with a quick review of so called cores of $C^*$-algebras generated by given representations. The detailed information can be found in [7]. For each $n \in \mathbb{N}$ set $B_n = \psi_n(\mathcal{K}(X^{\otimes n})) \subseteq C^*(\pi, t)$. Note that $B_0 := \pi(A)$ and that $B_n \cong \mathcal{K}(X^{\otimes n})$ when $(\pi, t)$ is injective. For $n, m \in \mathbb{N}$ with $n \geq 1$, we have

$$\overline{\text{span}}(t^n(X^{\otimes n})B_m t^n(X^{\otimes n})^*) = B_{n+m}$$

and $t^n(X^{\otimes n})B_{n+m} t^n(X^{\otimes n}) \subseteq B_m$. For $m, n \in \mathbb{N}$ with $m \leq n$, define $B[m, n] \subseteq C^*(\pi, t)$ by

$$B[m, n] = B_m + B_{m+1} + \ldots + B_n.$$ 

We denote $B[n, n]$ by $B_n$ for $n \in \mathbb{N}$. All $B[m, n]$'s are $C^*$-subalgebras of $C^*(\pi, t)$. In addition, $B[k, n]$ is an ideal of $B[m, n]$ for $m, k, n \in \mathbb{N}$ with $m \leq k \leq n$. In particular, $B_n$ is an ideal of $B[0, n]$ for each $n \in \mathbb{N}$. For $m \in \mathbb{N}$, define the $C^*$-subalgebra $B[m, \infty)$ of $C^*(\pi, t)$ by

$$B[m, \infty) = \bigcup_{n=m}^{\infty} B[m, n].$$

Notice that $B[m, \infty)$ is an inductive limit of the increasing sequence of $C^*$-algebras $\{B[m, n]\}_{n=m}^{\infty}$. The $C^*$-algebra $B[0, \infty)$ is called the core of the $C^*$-algebra $C^*(\pi, t)$.

4. The Categories

In the enchilada category, our objects are $C^*$-algebras, and the morphisms from $A$ to $B$ are the isomorphism classes of non-degenerate $A - B$ correspondences, with composition given by the balanced tensor product

$$[B Y C] \circ [A X B] = [A (X \otimes_B Y) C],$$

and identity morphisms given by identity correspondences. The paper [1] and the memoir [2] contain a development of all the theory, and [4] contains many properties of the enchilada category that we need.

It is a fundamental fact about the enchilada category that the invertible morphisms are precisely the isomorphism classes of imprimitivity bimodules (see, for example, [1, Proposition 2.6]).

**Lemma 4.1.** Let $A$, $B$ be $C^*$-algebras with $A \subseteq B$ and let $X$ be a $B - C$ correspondence. Then, $AAB \otimes_B X_C \cong AAX_C$.

**Proof.** Recall that the algebraic tensor product $A B \otimes X$ is a pre-correspondence. Now, denote the left action associated to the $C^*$-correspondence $B X_C$ by $\varphi$. Let $\Phi : AB \otimes X \to AX$ be the unique linear map defined by $\sum_{i=1}^{n} a_i b_i \otimes_B x_i \mapsto \sum_{i=1}^{n} \varphi(a_i b_i)x_i$. It suffices to use elementary tensors to make our computations, since the linear span of those elements
is dense in $AB \odot X$. We first show that $\Phi$ preserves the left action. Let $ab \otimes_B x \in AB \odot X$, and $c \in A$. Then we have

$$\Phi(c \cdot (ab) \otimes_B x) = \Phi(cab \otimes_B x) = \varphi(cab)x = \varphi(c)\varphi(ab)x = \varphi(c)\Phi(ab \otimes_B x).$$

Now, we observe that $\Phi$ preserves the inner product. Let $a, r \in A$, $b, s \in B$ and $x, y \in X$. We have

$$\langle \Phi(ab \otimes_B x), \Phi(rs \otimes_B y) \rangle_C = \langle \varphi(ab)(x), \varphi(rs)(y) \rangle_C$$

$$= \langle x, \varphi((ab, rs)_B)y \rangle_C$$

$$= \langle ab \otimes_B x, rs \otimes_B y \rangle_C.$$  

It remains to show that $\Phi$ is surjective. To that end, take any $a \cdot x \in AX$. Since $B$ acts on $X$ non-degenerately, we may find $b \in B$ and $x' \in X$ so that $x = b \cdot x'$. Then we obtain

$$a \cdot x = a \cdot (b \cdot x') = (ab) \cdot x' = \Phi(ab \otimes_B x').$$

Hence, $\Phi$ extends to a unique $C^*$-correspondence isomorphism $\tilde{\Phi} : AB \otimes_B X \to AX$. □

**Lemma 4.2.** Let $X$ be an $A \rightarrow B$ correspondence, and $I$ be a closed ideal of $A$ such that $I \subseteq \text{Ker}(\varphi_X)$. Then, we may view $X$ as an $A/I \leftarrow B$ correspondence.

**Proof.** Let $q : A \rightarrow A/I$ be the quotient map. Define a linear map $\Phi : A/I \rightarrow \mathcal{L}(X)$ so that $\Phi(q(a)) = \varphi_X(a)$, for any $a \in A$. It suffices to check that $\Phi$ is well defined. Let $a, b \in A$ satisfying $q(a) = q(b)$. Then, we have $q(a - b) = 0$, which means that $a - b \in I$. Since $I \subseteq \text{Ker}(\varphi_X)$, we obtain $\varphi_X(a - b) = 0$. Hence, $\varphi_X(a) = \varphi_X(b)$. □

**Remark 4.3.** Given an $A \rightarrow B$ correspondence $X$ and a closed ideal $J$ of $B$, let $q : B \rightarrow B/J$ and $\pi : X \rightarrow X/XJ$ be the quotient maps. Then, $X/XJ$ is a Hilbert $B/J$-module with the operations

$$\pi(x)q(b) := \pi(xb)$$

$$\langle \pi(x), \pi(y) \rangle_{B/J} := q(\langle x, y \rangle_B).$$

Now, letting $\varphi_X$ be the associated homomorphism with $A_XB$, define a linear map $\beta : \mathcal{L}_B(X) \rightarrow \mathcal{L}_{B/J}(X/XJ)$ by $\beta(T)(\pi(x)) := \pi(T(x))$. Then, we may view $X/XJ$ as an $A \leftarrow B/J$ correspondence with the non-degenerate left action $\beta \circ \varphi_X : A \rightarrow \mathcal{L}_{B/J}(X/XJ)$. So, we have $a \cdot \pi(x) = \pi(a \cdot x)$. Note here that $\mathcal{K}(XJ)$ is contained in $\text{Ker}(\beta)$.

**Corollary 4.4.** Given an $A \rightarrow B$ correspondence $X$, and a closed ideal $J$ of $B$, construct the $A \leftarrow B/J$ correspondence as above. For a closed ideal $I$ of $A$ such that $I \subseteq \text{Ker}(\beta \circ \varphi_X)$, we may view $X/XJ$ as an $A/I \leftarrow B/J$ correspondence.

**Proof.** Follows from Lemma 4.2 and Remark 4.3. □

**Lemma 4.5.** Let $A_XB$ be a $C^*$-correspondence and $I$ be a closed ideal of $B$. Then, we have the isomorphism $A(X \otimes_B B/I)_{B/I} \cong A(X/XI)_{B/I}$. 

Proof. Consider the quotient maps \( q : B \to B/I \) and \( \pi : X \to X/XI \). Let \( \Phi \) be the unique linear map from the pre-correspondence \( X \otimes B/I \) to the \( C^* \)-correspondence \( X/XI \), defined on the elementary tensors by
\[
x \otimes_B q(b) \mapsto \pi(x)q(b),
\]
where \( x \in X \) and \( b \in B \). For any \( a \in A \), we have
\[
\Phi(a \cdot (x \otimes_B q(b))) = \Phi(a \cdot x \otimes_B q(b)) = \pi(a \cdot x)q(b)
= a \cdot \pi(x)q(b)
= a \cdot \Phi(x \otimes_B q(b)).
\]

Thus, \( \Phi \) preserves the left action. Now, to see that \( \Phi \) preserves the inner product, take any two elementary tensors \( x \otimes_B b, x' \otimes_B b' \in X \otimes_B B/I \). We obtain
\[
\langle \pi(x)q(b), \pi(x')q(b') \rangle_{B/I} = \langle \pi(xb), \pi(x'b') \rangle_{B/I}
= q(\langle xb, x'b' \rangle_B)
= q(b^* \langle x, x' \rangle_B b')
= q(b^*q(\langle x, x' \rangle_bb'))
= \langle q(b), q(\langle x, x' \rangle_B)b' \rangle_{B/I}
= \langle x \otimes_B q(b), x' \otimes_B q(b') \rangle_{B/I}.
\]

It remains to show that \( \Phi \) is surjective. Let \( \pi(x) \in X/XI \). Let \( b \in B \) and \( x' \in X \) so that \( x = x' \cdot b \). Then we have
\[
\pi(x) = \pi(x' \cdot b) = \pi(x')q(b) = \Phi(x' \otimes_B b).
\]
Therefore, \( \Phi \) extends to an \( A-B/I \) correspondence isomorphism \( X \otimes_B B/I \to X/XI \).

Lemma 4.6. ([5] Lemma 2.7) Let \( X, Y \) be Hilbert \( A \)-modules, and \( \_M_B \) be an injective \( C^* \)-correspondence. Then, the mapping
\[
\otimes 1_M : \mathcal{L}(X, Y) \to \mathcal{L}(X \otimes_A M, Y \otimes_A M) : t \mapsto t \otimes 1_M
\]
is isometric. In particular, if \( \_M_B \) is regular then \( \mathcal{K}(X) \) is embedded into \( \mathcal{K}(X \otimes_A M) \).

Assume \( X_B \cong Y_B \) as Hilbert \( B \)-modules, and assume that there is a non-degenerate homomorphism \( \varphi_X : C \to \mathcal{L}(X) \). Then, \( Y \) can be viewed as a \( C-B \) correspondence as well, and \( C_X_B \cong C_Y_B \) : let \( U : X_B \xrightarrow{\cong} Y_B \). Then, we have the isomorphism
\[
AdU : \mathcal{L}(X_B) \to \mathcal{L}(Y_B).
\]
Now, letting \( \varphi_Y := AdU \circ \varphi_X \) allows us to view \( Y \) as a \( C-B \) correspondence. The isomorphism \( U \) preserves the left module structures automatically by the way we defined \( \varphi_Y \). Hence, \( C_X_B \cong C_Y_B \).

Theorem 4.7. There exists a category \( \text{ECCor} \) that has:
- \( C^* \)-correspondences as objects;
- isomorphism class of regular \( C^* \)-correspondences \( [\_M_B] : _M_A \to _M_B Y_B \) as morphisms, satisfying \( _M_A X \otimes_A M_B \cong _M_A M \otimes_B Y_B \).
in which the composition $[BNC] \circ [AM_B] \circ [AM_B]$ of $[AM_B]$: $AX \to BY$ and $[BN_C]$: $BY \to C\overline{Z_C}$ is the isomorphism class of the balanced tensor product $[A(M \otimes B)N]_B$, and

- the identity morphism on $AX_A$ is $[AA_A]$.

**Proof.** Let $\text{Mor}(AX_A, BY_B)$ denote the morphisms from $AX_A$ to $BY_B$ described as above. Take $[AM_B] \in \text{Mor}(AX_A, BY_B)$ and $[BN_C] \in \text{Mor}(BY_B, C\overline{Z_C})$. We first show that $M \otimes_B N$ is regular. The left action of $A$ on $M \otimes_B N$ is given by the homomorphism $a \mapsto \varphi_M(a) \otimes_B 1_N$. By Lemma 4.6, $\varphi_M(a) \otimes_B 1_N \in \mathcal{K}(M \otimes_B N)$, and $\varphi_M(a) \mapsto \varphi_M(a) \otimes_B 1_N$ is an isometric homomorphism. Since $AM_B$ is injective, we also have that $\varphi_M$ is injective. Hence, the map $a \mapsto \varphi_M(a) \otimes_B 1_N$ is injective. Note that

$$X \otimes_A (M \otimes_B N) \cong (X \otimes_A M) \otimes_B N$$

$$\cong (M \otimes_B Y) \otimes_C N$$

$$\cong M \otimes_B (Y \otimes_B N)$$

$$\cong M \otimes_B (N \otimes_C Z)$$

$$\cong (M \otimes_B N) \otimes_C Z$$

as $A - C$ correspondences. This implies

$$[BNC] \circ [AM_B] = [A(M \otimes_B N)_C] \in \text{Mor}(AX_A, C\overline{Z_C})$$

Now, let $D \in \mathcal{K}$ be a $C^*$-correspondence and let $[CK_D] \in \text{Mor}(C\overline{Z_C}, DR_D)$. The composition is associative:

$$([CR_D] \circ [BNC]) \circ [AM_B] = [AM \otimes_B (N \otimes_C R)_D]$$

(by definition)

$$= [A(M \otimes_B N) \otimes_C R_D]$$

$$= [CR_D] \circ ([BNC] \circ [AM_B]).$$

On the other hand, for each $C^*$-correspondence $AX_A$ over $A$, we have that

$$X \otimes_A A \cong X \cong A \otimes_A X,$$

and that $AA_A$ is regular. Hence, $[AA_A] \in \text{Mor}(AX_A, AX_A)$. Let $[AM_A] \in \text{Mor}(AX_A, AX_A)$. Then,

$$[AM_A] \circ [AA_A] = [A(A \otimes_A M)_A] = [AM_A].$$

Since $AM_A \cong (M \otimes_A A)_A$, it can be very similarly shown that $[AA_A] \circ [AM_A] = [AM_A]$. Thus, $[AA_A]$ is an identity morphism on $AX_A$.

It remains to show that for each $[AN_B] \in \text{Mor}(AX_A, BY_B)$, we have

$$[AN_B] \circ [AA_A] = [AN_B] = [BN_B] \circ [AN_B].$$

But this follows directly from the fact that

$$A(N \otimes_B B)_B \cong AN_B \cong A(A \otimes_A N)_B,$$

which completes the proof. 

$\square$
One of the key observations for this work is that in ECCor, invertible morphisms are precisely the isomorphism classes of imprimitivity bimodules: noting that imprimitivity bimodules are regular, we have

\[ [A M_B] \in \text{Mor}(A X_A, B Y_B) \] invertible

\[ \iff A(M \otimes_B N)_A \cong A A_A \quad \text{and} \quad B(N \otimes_A M)_B \cong B B_B \quad \text{for some} \ [B N_A] \in \text{Mor}(B Y_B, A X_A) \]

\[ \iff A M_B \quad \text{is an imprimitivity bimodule.} \]

We will frequently drop the square brackets \([\cdot]\), since it will clean up the notation and no confusion will arise. We also find it beneficial to remind the reader that throughout the paper, we assume all \(C^*\)-correspondences to be non-degenerate.

5. The Functor

In this section, we prove that the passage from ECCor to the enchilada category is a functor. To that, we first construct an \(O_X - O_X\) correspondence for given two objects \(A X_A, B Y_B\) in ECCor and \(A M_B \in \text{Mor}(A X_A, B Y_B)\). In ECCor, morphisms being regular is crucial for us to make this construction. Let us start with some essential observations.

Let \((\varphi_\infty, t)\) denote the Fock representation of a given \(C^*\)-correspondence \(A X_A\). For \(x \in X^{\otimes n}, y \in X^{\otimes m}\), and \(a \in A\), we have

\[ t^n(x) t^m(y)^* = t^n(a \cdot x') t^m(y)^* \quad \quad x = a \cdot x'\quad \text{for some} \quad a \in A \quad \text{and} \quad x' \in X^n \]

\[ = \varphi_\infty(a) t^n(x') t^m(y)^* \quad \quad (\varphi_\infty, t^n) \quad \text{is a representation of} \quad X^{\otimes n}. \]

Therefore, \(T_X\) can be viewed as an \(A - T_X\) correspondence with the injective left action \(a \mapsto \varphi_\infty(a)\). By using the quotient map \(T_X \twoheadrightarrow T_X / \mathcal{K}(\mathcal{T}(X) J_X)\) we may view \(O_X\) as a \(T_X - O_X\) correspondence. In fact, it would be an epimorphism in the enchilada category \([\text{Proposition 3.4}]\). Last but not least, recall that since \(A\) acts on \(\mathcal{T}(X)\) non-degenerately via \(\varphi_\infty\), any element of \(\mathcal{T}(X)\) can be written as \(\varphi_\infty(a)x\) for some \(a \in A\) and \(x \in \mathcal{T}(X)\).

But we know that \(\varphi_\infty(a) \in T_X\). Thus, \(\mathcal{T}(X) \subseteq T_X \cdot \mathcal{T}(X)\), which implies that \(\mathcal{T}(X)\) can be viewed as a \(T_X - A\) correspondence as well.

**Lemma 5.1.** Given two objects \(A X_A, B Y_B\) in ECCor, and the morphism \(A M_B\) between them, we have

\[ \mathcal{T}(X) \otimes_A M \cong M \otimes_B \mathcal{T}(Y) \]

as \(A - B\) correspondences.

**Proof.** First, we show by induction that for any \(n \in \mathbb{N}\) we have the \(A - B\) correspondence isomorphism \(X^{\otimes n} \otimes_A M \cong M \otimes_B Y^{\otimes n}\). For \(n = 0\), the result follows from the fact that \(A(A \otimes_A M)_B \cong A(M \otimes_B B)_B\).

Assume \(X^{\otimes n} \otimes_A M \cong M \otimes_B Y^{\otimes n}\), for some \(n \geq 1\). We obtain

\[ A X^{\otimes n+1} \otimes_A M_B \cong A X \otimes_A X^{\otimes n} \otimes_B M_B \cong A X \otimes_A M_B \otimes_B Y^{\otimes n}_B \]

\[ \cong A M \otimes_B Y \otimes_B Y^{\otimes n}_B \quad (\quad \quad \quad [A M_B] \in \text{Mor}(A X_A, B Y_B) \quad )\]

\[ \cong A M \otimes_B Y^{\otimes n+1}_B. \]
Now we have
\[(\oplus X^{\otimes n}) \otimes_A M \cong \oplus (X^{\otimes n} \otimes_A M) \cong \oplus (M \otimes_B Y^{\otimes n}) \cong M \otimes_B (\oplus Y^{\otimes n})\]
as $A - B$ correspondences.

**Lemma 5.2.** For any $C^*$-correspondence $A X_A$, we have that $\mathcal{F}(X) \otimes_A \mathcal{F}(X) \cong \mathcal{F}(X)$ as $A - A$ correspondences.

**Proof.** Let $\Phi$ be the unique linear map $F(X) \otimes F(X) \to F(X)$ defined on the elementary tensors by $(x_n) \otimes_A (y_m) \mapsto (h_k)$ where $h_k := \sum_{k=n+m} x_n \otimes_A y_m$. Note that
\[
\sum_{k=n+m} a \cdot x_n \otimes_A y_m = \sum_{k=n+m} a \cdot (x_n \otimes_A y_m) = a \cdot \sum_{k=n+m} x_n \otimes_A y_m.
\]
This leads us to the conclusion that $\Phi$ preserves the left action:
\[a \cdot \left((x_n) \otimes_A (y_m)\right) = (a \cdot x_n) \otimes_A (y_m) \mapsto (a \cdot h_k) = a \cdot (h_k).
\]

We need to be careful when calculating the inner product $\langle(h_k), (h_k)\rangle_A$. We know that $\langle(h_k), (h_k)\rangle_A = \sum_k h_k \cdot h_k$, and that $\langle x_i, x_j \rangle_A = 0$ for $x_i \in X^{\otimes i}, \ x_j \in X^{\otimes j}$ with $i \neq j$. Therefore, we obtain
\[
\sum_k \langle h_k, h_k \rangle = \sum_m \langle y_m, \sum_n \langle x_n, x_n \rangle_A \cdot y_m \rangle_A.
\]

We now prove that $\Phi$ preserves the inner products. Let $(z_n), (t_m) \in \mathcal{F}(X)$ and let $\Phi((z_n) \otimes_A (t_m)) = (r_k)$, i.e.,
\[r_k = \sum_{k=n+m} z_n \otimes_A t_m
\]
for each $k \in \mathbb{N}$. Then we have
\[
\langle \Phi((x_n) \otimes_A (y_m)), \Phi((z_n) \otimes_A (t_m)) \rangle_A = \langle (h_k), (r_k) \rangle_A = \sum_k \langle h_k, r_k \rangle_A
\]
\[= \sum_m \langle y_m, \sum_n \langle x_n, z_n \rangle_A \cdot t_m \rangle_A
\]
\[= \langle y_m, \langle x_n, (z_n) \rangle_A \cdot (t_m) \rangle_A
\]
\[= \langle (x_n) \otimes_A (y_m), (z_n) \otimes_A (t_m) \rangle_A
\]

Notice that any $(z_k) \in \mathcal{F}(X)$ can be written as $(z_k) = (h_m) a = (h_m \cdot a)$, for some $a \in A$ and $(h_m) \in \mathcal{F}(X)$. Let $(a_n) := (a, 0, 0, 0, ...) \in \mathcal{F}(X)$. Now, one can easily observe that $z_k = \sum_{k=n+m} h_m \otimes_B a_n$ for each $k \in \mathbb{N}$, which implies that $\Phi$ is surjective. Hence, $\Phi$ extends to a $C^*$-correspondence isomorphism $\tilde{\Phi} : \mathcal{F}(X) \otimes_A \mathcal{F}(X) \to \mathcal{F}(X)$.

**Corollary 5.3.** The isomorphism above holds as $\mathcal{T}_X - A$ correspondences as well.

**Proof.** Let $(\varphi_\infty, t)$ be the Fock representation of $A X_A$. Since $\mathcal{T}_X$ is generated by the images of $\varphi_\infty$ and $t$, it suffices to check the behavior of the elements that are in the image of these
Let $\nu$ be the linear map defined in the proof of Lemma 5.2. Note that, for any $a \in A$ and $\mu \in \mathcal{F}(X)$, we have

$$\varphi_\infty(a) \cdot \mu = \varphi_\infty(a)\mu = a \cdot \mu.$$ 

Let $\nu \in \mathcal{F}(X)$. Since $\Phi$ preserves the left action of $A$, we automatically have

$$\Phi(\varphi_\infty(a) \cdot (\mu \otimes_A \nu)) = \varphi_\infty(a) \cdot \Phi(\mu \otimes_A \nu),$$

Now, let $x \in X$, and $(z_n), (y_m) \in \mathcal{F}(X)$. We obtain

$$\Phi[t(x) \cdot ((z_n) \otimes_A (y_m))] = \Phi[t(x)(z_n) \otimes_A (y_m)]$$

$$= \Phi[(x \otimes_A z_n) \otimes_A (y_m)]$$

$$= (h_k), \quad h_k = \sum_{k = n+m} x \otimes_A z_n \otimes_A y_m$$

$$= (h_k), \quad h_k = x \otimes_A \sum_{k = n+m} z_n \otimes_A y_m$$

$$= t(x) \cdot (h_k),$$

which proves that $\Phi$ preserves the left action of $\mathcal{F}_X$. \hfill \square

**Lemma 5.4.** Let $A^\otimes_n$ be a $C^*$-correspondence, and let $n \in \mathbb{N}$. Consider the linear map

$$\beta : X^\otimes_n \otimes \mathcal{F}(X) \to \mathcal{F}(X)$$

defined on the elementary tensors by

$$x_n \otimes_A (z_k) \mapsto (x_n \otimes_A z_k)_{k=0}^\infty.$$

Then, $\beta$ extends to a unique injective $A-A$ correspondence homomorphism from $X^\otimes_n \otimes_A \mathcal{F}(X)$ into $\mathcal{F}(X)$.

**Proof.** We first observe that $\beta$ preserves the left action. For any $a \in A$, we have

$$\beta(a \cdot x_n \otimes_A (z_k)) = (\varphi_n(a)x_n \otimes_A z_k)_{k=0}^\infty = \varphi_\infty(a)(x_n \otimes_A z_k)_{k=0}^\infty = \varphi_\infty(a)\beta(x_n \otimes_A (z_k)).$$

On the other hand, for $x_n, y_n \in X^\otimes_n$ and $(z_k), (h_k) \in \mathcal{F}(X)$, we obtain

$$\langle \beta(x_n \otimes_A (z_k)), \beta(x_n \otimes_A (h_k)) \rangle_A = \langle (x_n \otimes_A z_k)_k, (y_n \otimes_A h_k)_k \rangle_A$$

$$= \sum_k \langle x_n \otimes_A z_k, y_n \otimes_A h_k \rangle_A$$

$$= \sum_k \langle z_k, \varphi_k((x_n, y_n)_A)h_k \rangle_A$$

$$= \langle (z_k), \varphi_\infty((x_n, y_n)_A)(h_k) \rangle_A$$

$$= \langle x_n \otimes_A (z_k), y_n \otimes_A (h_k) \rangle_A.$$

Thus, $\beta$ extends to an injective $A-A$ correspondence homomorphism $X^\otimes_n \otimes_A \mathcal{F}(X) \to \mathcal{F}(X)$. \hfill \square
As a preparation for the next lemma, consider the $C^*$-correspondence $AX_A$ and its Fock representation $(\varphi_\infty,t)$. Let $x = (x_n) \in \mathcal{F}(X)$. Notice that for any $n \in \mathbb{N}$, and $z = (z_m) \in \mathcal{F}(X)$, we have
\[ t^n(x_n)(z) = (x_n \otimes_A z_m)_{m=0}^{\infty} = \beta(x_n \otimes_A (z_m)) \in \mathcal{F}(X). \]
It is clear that for any $k \in \mathbb{N}$, the finite sum $\sum_{n=0}^{k} t^n(x_n)$ is an element of $\mathcal{I}_X$. This operator behaves as
\[ \sum_{n=0}^{k} t^n(x_n)(z) = (h_t), \quad \text{where} \quad h_t = \sum_{t=n+m}^{\infty} x_n \otimes_A z_m, \]
for each $t \in \mathbb{N}$.

**Lemma 5.5.** The operator $\sum_{n=0}^{\infty} t^n(x_n)$ is an element of the Toeplitz algebra $\mathcal{I}_X$.

**Proof.** We prove that $\sum_{n=0}^{\infty} t^n(x_n)$ converges in the operator norm. It suffices to show that
\[ \left\| \sum_{n=m}^{k} t^n(x_n) \right\| \to 0 \]
as $m \to \infty$ and $m \leq k$. Let $z = (z_i) \in \mathcal{F}(X)$ with $\|z\|_A = 1$. We have
\[
\left\| \sum_{n=m}^{k} t^n(x_n)(z) \right\|_A^2 = \left\| \sum_{n=m}^{k} t^n(x_n)(z), \sum_{n=m}^{k} t^n(x_n)(z) \right\|_A^2 \\
= \left\| \sum_{r,s \in [m,k]} \langle t^r(x_r)(z), t^s(x_s)(z) \rangle_A \right\|_A \\
= \left\| \sum_{r,s \in [m,k]} \langle \beta(x_r \otimes_A z), \beta(x_s \otimes_A z) \rangle_A \right\|_A \\
= \left\| \sum_{r,s \in [m,k]} \langle x_r \otimes_A z, x_s \otimes_A z \rangle_A \right\|_A \quad (\beta \text{ is an isometry}) \\
= \left\| \sum_{r,s \in [m,k]} \langle z, \varphi_\infty(\langle x_r, x_s \rangle_A)(z) \rangle_A \right\|_A \\
= \left\| \sum_{n=m}^{k} \langle z, \varphi_\infty(\langle x_n, x_n \rangle_A)(z) \rangle_A \right\|_A \quad (\langle x_r, x_s \rangle_A = 0 \text{ for } r \neq s)
\]
Denote $\sum_{n=m}^{k} \langle x_n, x_n \rangle_A \in A$ by $a$. Then we may continue the above computation as
\[= \|\langle z, \varphi_\infty(a)(z)\rangle_A\| \quad (\varphi_\infty \text{ is a homomorphism})
\leq \|\varphi_\infty(a)\| \quad (\varphi_\infty \text{ is injective})
= \|a\|.
\]

But since \(a = \sum_{n=0}^{\infty} (x_n, x_n)_A\) converges, we know that
\[\left\| \sum_{n=m}^{k} (x_n, x_n)_A \right\| \to 0,
\]
which completes the proof. \(\square\)

Recall that we have a \(T_X - A\) correspondence isomorphism \(\tilde{\Phi} : \mathcal{F}(X) \otimes_A \mathcal{F}(X) \to \mathcal{F}(X)\) defined on the elementary tensors by \((x_n) \otimes_A (y_m) \mapsto (h_k)\), where
\[h_k := \sum_{n+m} x_n \otimes_A y_m.
\]

Now, let \(\xi = (x_n), \nu = (y_n), \mu = (z_n) \in \mathcal{F}(X)\). Consider the bounded linear map \(T : \mathcal{F}(X) \to \mathcal{L}(\mathcal{F}(X))\) defined as
\[T(\xi)(\nu) := \tilde{\Phi}(\xi \otimes_A \nu).
\]
The adjoint \(T(\xi)^*\) of \(T(\xi)\) satisfies
\[T(\xi)^*(\tilde{\Phi}(\nu \otimes_A \mu)) = \varphi_\infty(\langle \xi, \nu \rangle_A)\mu. \quad (5.1)
\]
Indeed, let \(z = (z_n) \in \mathcal{F}(X)\). We have
\[\langle T(\xi)(\nu), \tilde{\Phi}(\mu \otimes_A z) \rangle_A = \langle \tilde{\Phi}(\xi \otimes_A \nu), \tilde{\Phi}(\mu \otimes_A z) \rangle_A
= \langle \xi \otimes_A \nu, \mu \otimes_A z \rangle_A \quad (\tilde{\Phi} \text{ is isometric})
= \langle \nu, \varphi_\infty(\langle \xi, \mu \rangle_A)z \rangle_A \quad (A \text{ acts on } \mathcal{F}(X) \text{ via } \varphi_\infty).
\]

Notice that this also gives us the equality
\[T(\xi)^* T(\nu) = \varphi_\infty(\langle \xi, \nu \rangle_A)
\]
for any \(\xi, \nu \in \mathcal{F}(X)\). The linear span of the elements \((y_n) \otimes_A (z_n)\) is dense in \(\mathcal{F}(X) \otimes_A \mathcal{F}(X)\). So, we may take any \((x_n) \in \mathcal{F}(X)\) to be the image (under \(\tilde{\Phi}\)) of some finite sum of such elements. Therefore, Formula \((5.1)\) fully characterizes \(T(\xi)^*\) for any \(\xi \in \mathcal{F}(X)\).

**Corollary 5.6.** Let \((\varphi_\infty, t)\) be the Fock representation of \(A X_A\), and \(x = (x_n) \in \mathcal{F}(X)\). Then,
\[
\sum_{n=0}^{k} t^n(x_n) \to T((x_n))
\]
in operator norm, as \(k \to \infty\).
Proof. First, note that for any \((z) = (z_m) \in \mathcal{F}(X)\), we obtain

\[
T((x_n))(z) - \sum_{n=0}^{k} t^n(x_n)(z) = (h_t),
\]

where \(h_t = 0\) for \(t \leq k\), and \(h_t = \sum_{n=k+m}^{t} x_n \otimes_A z_m\) for \(t > k\). We follow the same path with the proof of Lemma 5.2 when computing \(\langle (h_t), (h_t) \rangle_A\):

\[
\langle (h_t), (h_t) \rangle_A = \sum_t \langle h_t, h_t \rangle_A
\]

\[
= \sum_{n=k+1}^{\infty} \sum_{m=0}^{\infty} \langle x_n \otimes_A z_m, x_n \otimes_A z_m \rangle_A
\]

\[
= \sum_{n=k+1}^{\infty} \sum_{m=0}^{\infty} \langle z_m, \varphi_m((x_n, x_n)), z_m \rangle_A
\]

\[
= \sum_{n=k+1}^{\infty} \langle z, \varphi_{\infty}((x_n, x_n)), z \rangle_A
\]

Now, let \(z = (z_m) \in \mathcal{F}(X)\) with \(\|z\| \leq 1\). Then, by using the above information we obtain

\[
\left\| T((x_n))(z) - \sum_{n=0}^{k} t^n(x_n)(z) \right\|_A^2 \leq \| \varphi_{\infty}(\sum_{n=k+1}^{\infty} (x_n, x_n)) \|
\]

\[
= \left\| \sum_{n=k+1}^{\infty} (x_n, x_n) \right\|,
\]

which converges to 0, since \(\sum_n (x_n, x_n)_A\) converges. \(\square\)

Corollary 5.6 tells us that \(T((x_n))\) is an element of \(\mathcal{J}_X\), which is the key fact for our next proposition.

**Proposition 5.7.** Let \(A \mathcal{X} A\) be a \(C^*\)-correspondence with the Fock representation \((\varphi_{\infty}, t)\). Then, \(\mathcal{F}(X) \otimes_A \mathcal{J}_X\) and \(\mathcal{J}_X\) are isomorphic as \(\mathcal{J}_X - \mathcal{J}_X\) correspondences.

**Proof.** Denote by \(\theta\) the unique linear map defined on the elementary tensors from the algebraic tensor product \(\mathcal{F}(X) \otimes \mathcal{J}_X\) to \(\mathcal{J}_X\) by

\[
\xi \otimes A S \mapsto T(\xi)S
\]

for \(\xi = (x_n) \in \mathcal{F}(X)\) and \(S \in \mathcal{J}_X\). We first observe that \(\theta\) preserves the inner product. Let \(\xi = (x_n), \nu = (y_n) \in \mathcal{F}(X)\) and \(S_1, S_2 \in \mathcal{J}_X\). Then we have

\[
\langle \langle \xi \otimes A S_1, \nu \otimes A S_2 \rangle_{\mathcal{J}_X} \rangle_X = \langle S_1, \varphi_{\infty}(\langle \xi, \nu \rangle_A) S_2 \rangle_{\mathcal{J}_X}
\]

\[
= S_1^* \varphi_{\infty}(\langle \xi, \nu \rangle_A) S_2
\]
This allows us to conclude that $\theta$ preserves the left action of $\mathcal{T}_X$: let $r, S \in \mathcal{T}_X$, and $\nu \in \mathcal{F}(X)$. We have

$$\theta(r \cdot (\nu \otimes_A S)) = \theta(r(\nu \otimes_A S)) = T(r(\nu))S = r \cdot T(\nu)S = r \cdot \theta(\nu \otimes_A S).$$

It remains to show that $\theta$ is surjective. Any element of $\mathcal{T}_X$ can be written as $\varphi_\infty(a)S$, for some $a \in A$ and $S \in \mathcal{T}_X$. Let $(x_n) \in \mathcal{F}(X)$, where $x_0 = a$, and $x_n = 0$ for $n \geq 1$. Then we have

$$\theta((x_n) \otimes_A S) = T((x_n))S = \lim_{k \to \infty} \sum_{n=0}^k t^n(x_n)S = t^0(x_0)S \quad (t^n(x_n) = 0 \text{ for } n \geq 1) = \varphi_\infty(a)S \quad (t^0 = \varphi_\infty, \text{ by definition}).$$

We may now conclude that $\theta$ extends to a unique $\mathcal{T}_X - \mathcal{T}_X$ correspondence isomorphism $\tilde{\theta}: \mathcal{F}(X) \otimes_A \mathcal{T}_X \to \mathcal{T}_X$. $\square$

**Lemma 5.8.** Let $A_{MB}, B_{YB}$ be $C^*$-correspondences and let $T \in \mathcal{L}(M \otimes_B \mathcal{K}(Y))$. If the operator $T \otimes_{\mathcal{X}(Y)} 1_Y \in \mathcal{L}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y)$ is compact, then $T \in \mathcal{K}(M \otimes_B \mathcal{K}(Y))$.

**Proof.** First, notice that $\mathcal{X}(Y)_{YB}$ is an injective correspondence. Therefore, by Lemma 4.6, $T \otimes_{\mathcal{X}(Y)} 1_Y$ is an isometric homomorphism from $\mathcal{L}(M \otimes_B \mathcal{K}(Y))$ into $\mathcal{L}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y)$. Now let $\{u_\lambda\}$ be an approximate identity for $\mathcal{K}(M \otimes_B \mathcal{K}(Y))$, and let $y \in Y$. Since $\mathcal{K}(M \otimes_B \mathcal{K}(Y))$ acts on $M \otimes_B \mathcal{K}(Y)$ non-degenerately, any $\xi \in M \otimes_B \mathcal{K}(Y)$ can be written as $K(\xi')$ for some $K \in \mathcal{K}(M \otimes_B \mathcal{K}(Y))$, and $\xi' \in M \otimes_B \mathcal{K}(Y)$. Then we have

$$\lim_{\lambda}(u_\lambda \otimes_{\mathcal{X}(Y)} 1_Y)\xi \otimes_B y = \lim_{\lambda}(u_\lambda \otimes_{\mathcal{X}(Y)} 1_Y)K(\xi') \otimes_B y = \lim_{\lambda} u_\lambda K(\xi') \otimes_B y = K\xi' \otimes_{\mathcal{X}(Y)} 1_Y = \xi \otimes_B y.$$

Since any element of $M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y$ is some finite sum of elementary tensors, we may conclude that

$$(u_\lambda \otimes_{\mathcal{X}(Y)} 1_Y) \xrightarrow{\text{sot}} 1$$
on $\mathcal{L}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y)$. In fact,

$$(u_\lambda \otimes_{\mathcal{X}(Y)} 1_Y) \overset{\text{strictly}}{\rightarrow} 1$$

in $\mathcal{L}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y) = \mathcal{M}((\mathcal{K}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y)) : \text{take } \mu, \nu \in M \otimes_B K(Y) \otimes_{\mathcal{X}(Y)} Y$. Then,

$$(u_\lambda \otimes_{\mathcal{X}(Y)} 1_Y)\theta_{\mu,\nu} = \theta_{(u_\lambda \otimes_{\mathcal{X}(Y)} 1_Y)\mu,\nu} \rightarrow \theta_{\mu,\nu}.$$  

Now, assume $T \otimes_{\mathcal{X}(Y)} 1_Y \in \mathcal{L}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y)$ is compact. As we discussed at the beginning of the proof, we have that $T \mapsto T \otimes_{\mathcal{X}(Y)} 1_Y$ is an isometric homomorphism from $\mathcal{L}(M \otimes_B \mathcal{K}(Y))$ into $\mathcal{L}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y)$. Then,

$$0 = \lim_\lambda \|T \otimes_{\mathcal{X}(Y)} 1_Y - (u_\lambda \otimes_{\mathcal{X}(Y)} 1_Y)(T \otimes_{\mathcal{X}(Y)} 1_Y)\|$$

$$= \lim_\lambda \|T - u_\lambda T\|.$$  

Since $\{u_\lambda T\}$ is in $\mathcal{K}(M \otimes_B \mathcal{K}(Y))$, we conclude $T \in \mathcal{K}(M \otimes_B \mathcal{K}(Y))$. \hfill \Box

**Corollary 5.9.** Consider the C*-correspondences $A M_B$ and $B Y_B$. For any $a \in A$, if $\varphi_M(a) \otimes_B 1_Y \in \mathcal{K}(M \otimes_B Y)$, then $\varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)} \in \mathcal{K}(M \otimes_B \mathcal{K}(Y))$. In addition, if the map $a \mapsto \varphi_M(a) \otimes_B 1_Y$ is injective, then so is the map $a \mapsto \varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)}$.

**Proof.** Denote the $A - B$ correspondence isomorphism $M \otimes_B Y \to M \otimes_B K(Y) \otimes_{\mathcal{X}(Y)} Y$ by $U$. Then,

$$AdU : \mathcal{L}(M \otimes_B Y) \to \mathcal{L}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y)$$

is an isomorphism satisfying

$$AdU(\varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)}) = \varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)} \otimes_{\mathcal{X}(Y)} 1_Y$$

for any $a \in A$, since $U$ preserves the left action. Therefore, if $\varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)} \in \mathcal{K}(M \otimes_B Y)$, we must have

$$\varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)} \otimes_{\mathcal{X}(Y)} 1_Y = AdU(\varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)}) \in \mathcal{K}(M \otimes_B \mathcal{K}(Y) \otimes_{\mathcal{X}(Y)} Y).$$

By Lemma 5.8, this implies $\varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)} \in \mathcal{K}(M \otimes_B \mathcal{K}(Y))$.

Now, assume that the map $a \mapsto \varphi_M(a) \otimes_B 1_Y$ is injective, and let $\varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)} = 0 \in \mathcal{L}(M \otimes_B \mathcal{K}(Y))$. Then,

$$AdU(\varphi_M(a) \otimes_B 1_Y) = \varphi_M(a) \otimes_B 1_{\mathcal{X}(Y)} \otimes_{\mathcal{X}(Y)} 1_Y = 0.$$  

Since $AdU$ is an isomorphism, this implies that $\varphi_M(a) \otimes_B 1_Y = 0$. Finally, the assumption $a \mapsto \varphi_M(a) \otimes_B 1_Y$ being injective gives us that $a = 0$. \hfill \Box

**Remark 5.10.** Let $A M_B$ be any C*-correspondence and $I$ be a non-zero ideal of $A$. The Hilbert $B$-module $IM$ can be viewed as a $I - B$ correspondence with the left action $\varphi_{IM} : I \to \mathcal{L}(IM)$ defined by $\varphi_{IM}(a)(jm) = \varphi_M(a)m$, for any $a, j \in I$, and $m \in M$.

**Proposition 5.11.** Let $A X_A$, $B Y_B$ be objects in $\text{ECCor}$ with $J_X \neq \{0\}$ and $A M_B$ be the morphism between them. Let $(\phi, t)$ be the Fock representation of $B Y_B$, and let $B[0, \infty)$ denote the core of $B Y_B$. Then, $J_X(J_X M \otimes_B B_1)_{B_1}$ is regular.
Proof. Recall that since $A_B$ is a morphism in $\text{ECCor}$, we have that $A(X \otimes_A M)_B \cong A(M \otimes_B Y)_B$. The left action is defined on $X \otimes_A M$ by $a \to \varphi_X(a) \otimes_A 1_M$, and on $M \otimes_B Y$ by $\varphi_M(a) \otimes_B 1_Y$. Denote the isomorphism $X \otimes_A M \to M \otimes_B Y$ by $U$. Then,

$$AdU : \mathcal{L}(X \otimes_A M) \to \mathcal{L}(M \otimes_B Y)$$

is an isomorphism such that $AdU \circ (\varphi_X(a) \otimes_A 1_M) = \varphi_M(a) \otimes_B 1_Y$. Since $M$ is regular, for any $k \in \mathcal{K}(X)$, we have $k \otimes_A 1_M \in \mathcal{K}(X \otimes_A M)$ by Lemma 4.6. This implies that $AdU \circ k \in \mathcal{K}(M \otimes_B Y)$. Hence, for $a \in J_X$, $\varphi_M(a) \otimes_B 1_Y \in \mathcal{K}(M \otimes_B Y)$. In other words, $\varphi_{J_X M}(a) \otimes_B 1_Y \in \mathcal{K}(J_X M \otimes_B Y)$, where $\varphi_{J_X M}$ is defined as in Remark 5.10. Hence,

$$\varphi_{J_X M}(a) \otimes_B 1_{J_X(Y)} \in \mathcal{K}(J_X M \otimes_B \mathcal{K}(Y))$$

by Corollary 5.9. Recall that $B_1 \cong \mathcal{K}(Y)$ as $C^*$-algebras since $\phi$ is injective. So, for any $a \in A$, we may consider the map $\varphi_{J_X M}(a) \otimes_B 1_{J_X(Y)}$ as an element of $\mathcal{K}(J_X M \otimes_B B_1)$. It remains to show that the map $a \mapsto \varphi_{J_X M}(a) \otimes_B 1_{J_X(Y)}$ is injective. Since $\varphi_X$ is injective on $J_X$, by Lemma 4.6, the left action of $A$ on $X \otimes_B M$ given by

$$a \mapsto \varphi_X(a) \otimes_B 1_M$$

is injective on $J_X$ as well. Since $AdU$ is an isomorphism, this tells us that the map $a \mapsto \varphi_X(a) \otimes_B 1_Y$ is injective. Hence, the left action

$$a \mapsto \varphi_{J_X M}(a) \otimes_B 1_Y$$

of $J_X$ on $J_X M \otimes_B Y$ is injective. The result now follows from the second part of Corollary 5.9. □

Since $B_1 \subseteq B[1, \infty) = \bigcup_{n=1}^{\infty} B[1, n]$, we may construct a $B_1 - B[1, \infty)$ correspondence $\iota(B_1)B[1, \infty)$ by using the embedding $\iota : B_1 \to M(B[1, \infty))$. Now, $\iota$ maps $B_1$ injectively into $B[1, \infty) \cong \mathcal{K}(B[1, \infty))$, which means $B_1 \iota(B_1)B[1, \infty)B[1, \infty)$ is regular. Note here that $B_1 \otimes_B B_1 B[1, \infty) \cong B_1 B[1, \infty)$ as $B - B[1, \infty)$ correspondences (Lemma 4.1). Hence, the $J_X - B[0, \infty)$ correspondence

$$J_X M \otimes_B B_1 \otimes_B B_1 B[1, \infty) \cong J_X M \otimes_B B_1 B[1, \infty)$$

is regular.

Lemma 5.12. Let $A_X B$ and $B M_C$ be $C^*$-correspondence and $J$ be a non-zero ideal of $B$ such that $B_X \subseteq J$ and the left action of $J$ on $M_C$ is non-degenerate. Then,

$$X \otimes_J M \cong X \otimes_B M$$

as $A - C$ correspondences.

Proof. We need to prove that the balancing relations determined by $\otimes_J$ and $\otimes_B$ coincide. Let $x \in X$, $a \in A$, and $m \in M$. Since $B_X \subseteq J$, we have that $X$ has a right Hilbert $J$-module structure (Lemma 3.2). Therefore, we can choose $x' \in X, j \in J$ so that $x = x' j$. Then, the following computation in $X \otimes_J M$ suffices:

$$xa \otimes m = x' ja \otimes m = x \otimes (ja) \cdot m \quad (ja \in J)$$
Proposition 5.13. Let $A X_A$, $B Y_B$ be objects in $\text{ECCor}$, and let $[A M_B] \in \text{Mor}(A X_A, B Y_B)$. Let $B[0, \infty)$ represent the core of $B Y_B$. Then, for any $k \in \mathcal{K}(\mathcal{F}(X) J_X)$, the map given by $k \mapsto k \otimes J_X 1$, where $1$ denotes the identity morphism on $J_X M \otimes_B B_1 B[1, \infty)$, defines an embedding into $\mathcal{K}(\mathcal{F}(X) \otimes_A M \otimes_B B[1, \infty))$.

**Proof.** Let $\xi : \mathcal{K}(\mathcal{F}(X) J_X) \mapsto \mathcal{L}(\mathcal{F}(X) \otimes_A M \otimes_B B[1, \infty))$ denote the map defined by $k \mapsto k \otimes J_X 1$ as above. Since the correspondence $J_X (J_X M \otimes_B B_1 B[1, \infty)) B[1, \infty)$ is regular, by Lemma 4.6 we know that $\xi$ is injective. In addition, for any $k \in \mathcal{K}(\mathcal{F}(X) J_X)$ we have

$$\xi(k) \in \mathcal{K}(\mathcal{F}(X) J_X M \otimes_B B_1 B[1, \infty)) \quad \text{(Lemma 4.6)}$$

$$\cong \mathcal{K}(\mathcal{F}(X) J_X M \otimes_B B_1 B[1, \infty)) \quad \text{(Lemma 5.12)}$$

Since $\mathcal{F}(X) J_X \subseteq \mathcal{F}(X)$, $J_X M \subseteq M$, and $B_1 B[1, \infty) \subseteq B[1, \infty)$, any finite rank operator on the Hilbert module $\mathcal{F}(X) J_X \otimes_A J_X M \otimes_B B_1 B[1, \infty)$ can be considered as an operator on $\mathcal{F}(X) \otimes_A M \otimes_B B[1, \infty)$. Thus, we may conclude that

$$\xi(k) \in \mathcal{K}(\mathcal{F}(X) \otimes_A M \otimes_B B[1, \infty)),$$

which completes the proof. \qed

**Lemma 5.14.** Consider the $C^*$-correspondence $A X_A$ and its Fock representation $(\varphi_\infty, t)$, where the core of $\mathcal{T}_X$ denoted by $B[0, \infty)$. We have $B[0, \infty)/B[1, \infty) \cong A$ as $C^*$-algebras.

**Proof.** We first prove by induction that $B_0 \cap B[1, n] = \{0\}$, for any positive integer $n$. We have by [7, Proposition 3.3] that $B_0 \cap B_1 = \{0\}$. Assume we have $B_0 \cap B[1, n - 1] = \{0\}$ for some $n > 2$. Now, take $\varphi_\infty(a) \in B[1, n]$. Then, $\varphi_\infty(a) = k + k'$ for some $k \in B[1, n - 1]$ and $k' \in B_n$. So, $k' = \varphi_\infty(a) - k \in B[0, n - 1]$. Hence, $k' \in B[0, n - 1] \cap B_n = \{0\}$ by [7, Proposition 5.12]. This leads us to the conclusion that $B_0 \cap B[1, \infty) = \{0\}$.

On the other hand, we have the fact that $B[0, \infty) = B_0 + B[1, \infty)$, and that $B[1, \infty)$ is an ideal of $B[0, \infty)$. Now, define $\Phi : B_0 + B[1, \infty) \to A$ via

$$\sum_{i \in I} \varphi_\infty(a_i) + k_i \mapsto \sum_{i \in I} a_i$$

where $I \subseteq \mathbb{N}$ is finite, $a_i \in A$ and $k_i \in B[1, \infty)$. We first show that $\Phi$ is well defined. Assume $\sum_{i=1}^N \varphi_\infty(a_i) + r_i = \sum_{i=1}^M \varphi_\infty(b_i) + s_i$, for $a_i, b_i \in A$ and $r_i, s_i \in B[1, \infty)$. Then we have

$$\sum_{i=1}^N \varphi_\infty(a_i) - \sum_{i=1}^M \varphi_\infty(b_i) = \sum_{i=1}^M s_i - \sum_{i=1}^N r_i \in B_0 \cap B[1, \infty).$$
But \( B_0 \cap B[1, \infty) = \{0\} \). Therefore, we must have \( \sum_{i=1}^{N} \varphi_\infty(a_i) - \sum_{i=1}^{M} \varphi_\infty(b_i) = 0 \), which implies the equality
\[
\sum_{i=1}^{N} a_i = \sum_{i=1}^{M} b_i.
\]

Surjectivity of \( \Phi \) follows from the fact that any \( a \in A \) can be written as \( a + 0 = \Phi(\varphi_\infty(a) + 0) \).

It is a homomorphism, since \( \varphi_\infty \) is. It remains to prove that \( \text{Ker} \Phi = B[1, \infty) \).

Showing \( \text{Ker} \Phi \subseteq B[1, \infty) \) suffices. Take any \( m \in B_0 + B[1, \infty) \) with \( \Phi(m) = 0 \). We have \( m = \sum_{i \in I} \varphi_\infty(a_i) + k_i \) for some finite \( I \subseteq \mathbb{N} \), \( a_i \in A \) and \( k_i \in B[1, \infty) \). And,
\[
0 = \Phi(\sum_{i \in I} \varphi_\infty(a_i) + k_i) = \sum_{i \in I} a_i.
\]

This implies that \( m = \sum_{i \in I} k_i \in B[1, \infty) \). The desired isomorphism now follows from [10, Theorem 3.1.6].

For the rest of the paper we have the following assumptions:

1. \( A X_A \) and \( B Y_B \) are objects in \( \text{ECCor} \).
2. \( A M_B \) \( \in \text{Mor}(A X_A, B Y_B) \).
3. \( B[0, \infty) \) represents the core of \( B Y_B \).

**Theorem 5.15.** The quotient vector space \( (\mathcal{F}(X) \otimes_A M \otimes_B B[0, \infty])/(\mathcal{F}(X) \otimes_A M \otimes_B B[1, \infty]) \) is an \( \mathcal{O}_X - B \) correspondence.

**Proof.** Since \( \mathcal{F}(X) \) is a \( \mathcal{T}_X - A \) correspondence, we may view \( \mathcal{F}(X) \otimes_A M \otimes_B B[0, \infty) \) as a \( \mathcal{T}_X - B[0, \infty) \) correspondence with the embedding
\[
\iota : \mathcal{T}_X \to \mathcal{L}(\mathcal{F}(X) \otimes_A M \otimes_B B[0, \infty)).
\]

Denote this correspondence by \( Z \). Then, by Remark 4.3, \( \mathcal{Z}/Z B[1, \infty) \) can be viewed as a \( \mathcal{T}_X - B[0, \infty)/B[1, \infty) \) correspondence with the left action \( \Phi := \beta \circ \iota \), where \( \beta \) is the homomorphism \( \mathcal{L}(Z) \to \mathcal{L}(\mathcal{Z}/Z B[1, \infty)) \).

Proposition 5.13 tells us that we can embed \( \iota(\mathcal{K}(\mathcal{F}(X) J_X)) \cong \mathcal{K}(\mathcal{F}(X) J_X) \) into \( \mathcal{K}(Z B[1, \infty)) \), which is certainly contained in \( \text{Ker} \beta \).

Therefore, we conclude that \( \mathcal{K}(\mathcal{F}(X) J_X) \) is contained in \( \text{Ker}(\Phi) \).

By Corollary 4.4, we may now view \( \mathcal{Z}/Z B[1, \infty) \) as a \( \mathcal{T}_X / \mathcal{K}(\mathcal{F}(X) J_X) - B[0, \infty)/B[1, \infty) \) correspondence. This completes the proof, since \( \mathcal{T}_X / \mathcal{K}(\mathcal{F}(X) J_X) \cong \mathcal{O}_X \) and \( B[0, \infty)/B[1, \infty] \cong B \).

**Corollary 5.16.** The Hilbert module \( \mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Y \) is an \( \mathcal{O}_X - \mathcal{O}_Y \) correspondence.

**Proof.** We know that \( \mathcal{T}_X = \mathcal{O}_X \), whenever \( J_X = \{0\} \). Hence, the result is trivial in that case. So, assume \( J_X \neq \{0\} \).

We have
\[
\tau_X(\mathcal{F}(X) \otimes_A M)_{B} \cong \tau_X(\mathcal{F}(X) \otimes_A M \otimes_B B)_{B}
\]
\[
\cong \tau_X(\mathcal{F}(X) \otimes_A M \otimes_B B[0, \infty)/B[1, \infty))_{B} \quad \text{(Lemma 5.14)}
\]
Since we have \( B(B[0, \infty)/B[1, \infty))_{B} \cong B[0, \infty) \otimes_{B[0, \infty)} (B[0, \infty)/B[1, \infty))_{B} \) by Lemma 4.5, we may continue the above calculation as
\[
\cong \tau_X(\mathcal{F}(X) \otimes_A M \otimes_B B[0, \infty)) \otimes_{B[0, \infty)} B[0, \infty)/B[1, \infty)_{B}
\]
Theorem 5.17. Given $A X_A$, we have the isomorphism
\[ (\mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_X)_{\mathcal{O}_X} \cong (\mathcal{F}(X) \otimes_A \mathcal{O}_X)_{\mathcal{O}_X}, \]

where $\mathcal{O}_X$ is an isomorphic copy of the $C^*$-correspondence over $\mathcal{O}_X$. Therefore, we may consider $\mathcal{O}_X(\mathcal{F}(X) \otimes_A \mathcal{O}_X)_{\mathcal{O}_X}$ as an isomorphic copy of the $C^*$-correspondence $\mathcal{O}_X(\mathcal{F}(X) \otimes_A A \otimes_A \mathcal{O}_X)_{\mathcal{O}_X}$.

**Theorem 5.18.** The assignments $A X_A \rightarrow \mathcal{O}_X$ and $M \rightarrow \mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Y$ define a functor $\mathcal{E}$ from ECCor to the enchilada category.

**Proof.** Let $A M_B \in \text{Mor}(AX_A, BY_B)$ and $BN_C \in \text{Mor}(AY_B, CZ_C)$ in ECCor. These morphisms are mapped to $\mathcal{O}_X(\mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_X}$ and $\mathcal{O}_Y(\mathcal{F}(Y) \otimes_B N \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z}$, respectively. We first prove that the composition is preserved:

\[ \tau_X(\mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (\mathcal{F}(Y) \otimes_B N \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z} \]

which completes the proof. 

The assignments $A X_A \rightarrow \mathcal{O}_X$ and $M \rightarrow \mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Y$ define a functor $\mathcal{E}$ from ECCor to the enchilada category. 

**Proof.** Let $A M_B \in \text{Mor}(AX_A, BY_B)$ and $BN_C \in \text{Mor}(AY_B, CZ_C)$ in ECCor. These morphisms are mapped to $\mathcal{O}_X(\mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Y)_{\mathcal{O}_X}$ and $\mathcal{O}_Y(\mathcal{F}(Y) \otimes_B N \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z}$, respectively. We first prove that the composition is preserved:

\[ \tau_X(\mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} (\mathcal{F}(Y) \otimes_B N \otimes_C \mathcal{O}_Z)_{\mathcal{O}_Z} \]
\[ \cong \tau_X \left( \mathcal{F}(X) \otimes_A M \right) \otimes_B \left( \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}(Y) \otimes_B \mathcal{O}_Z \right)_{\mathcal{O}_Z} \]
\[ \cong \tau_X \left( \mathcal{F}(X) \otimes_A M \right) \otimes_B \mathcal{F}(Y) \otimes_B \mathcal{O}_Z \otimes_{\mathcal{O}_Z} \mathcal{O}_Z \] 
\[ \cong \tau_X \left( \mathcal{F}(X) \otimes_A M \otimes_B \mathcal{F}(Y) \otimes_B \mathcal{O}_Z \mathcal{O}_Z \right) \] 
\[ \cong \tau_X \left( \mathcal{F}(X) \otimes_A M \otimes_B \mathcal{O}_Z \right)_{\mathcal{O}_Z} \] 
\[ \cong \tau_X \left( \mathcal{F}(X) \otimes_A M \right) \cong \tau_X \left( \mathcal{O}_X \right)_{\mathcal{O}_X} \] 

Denote the last module by \( V_2 \), and the module on line (1) by \( V_1 \). We have shown above that they are isomorphic as \( \mathcal{T}_X - \mathcal{O}_Z \) correspondences. Now we have
\[ \tau_X (V_1)_{\mathcal{O}_Z} \cong \tau_X (\mathcal{O}_X \otimes_{\mathcal{O}_X} (V_1)_{\mathcal{O}_Z}) \] 
\[ \cong \tau_X (V_2)_{\mathcal{O}_Z} \] 
\[ \cong \tau_X (\mathcal{O}_X \otimes_{\mathcal{O}_X} (V_2)_{\mathcal{O}_Z}). \]

Since \( \tau_X (\mathcal{O}_X)_{\mathcal{O}_X} \) is an epimorphism (left cancelative), we obtain \( \mathcal{O}_X (V_1)_{\mathcal{O}_Z} \cong \mathcal{O}_X (V_2)_{\mathcal{O}_Z} \). This means
\[ \mathcal{E}(\lfloor B N_C \rfloor \circ \lfloor A M_B \rfloor) = \mathcal{E}(\lfloor A M \otimes_B N_C \rfloor) \]
\[ = \left[ \mathcal{O}_X \left( \mathcal{F}(X) \otimes_A M \otimes_B N \otimes_{\mathcal{C}} \mathcal{O}_Z \right)_{\mathcal{O}_Z} \right] \]
\[ = \mathcal{E}(\lfloor B N_C \rfloor) \circ \mathcal{E}(\lfloor A M_B \rfloor). \]

It remains to show that the identity morphisms are preserved. The passage \( \mathcal{E} \) maps the identity morphism \( A A_A \) on \( A X_A \) to \( \mathcal{O}_X \left( \mathcal{F}(X) \otimes_A A \otimes_{\mathcal{O}_X} \mathcal{O}_X \right)_{\mathcal{O}_X} \cong \mathcal{O}_X \left( \mathcal{F}(X) \otimes_A \mathcal{O}_X \right)_{\mathcal{O}_X} \). Now we have
\[ \left[ \mathcal{O}_X \left( \mathcal{F}(X) \otimes_A A \otimes_{\mathcal{O}_X} \mathcal{O}_X \right)_{\mathcal{O}_X} \right] \]
\[ = \left[ \mathcal{O}_X \left( \mathcal{F}(X) \otimes_A \mathcal{O}_X \right)_{\mathcal{O}_X} \right] \]
\[ = \left[ \mathcal{O}_X \left( \mathcal{O}_X \right)_{\mathcal{O}_X} \right] \] 

which is the identity morphism on \( \mathcal{O}_X \) in the enchilada category. \( \square \)

**Remark 5.19.** Our technique can be used to study the functoriality of Toeplitz algebras, as well. More specifically, the functor \( \mathcal{E} \) can be defined by the assignments
\[ A X_A \rightarrow \mathcal{T}_X \quad \text{and} \quad [A M_B] \rightarrow [\tau_X (\mathcal{F}(X) \otimes_A M \otimes_A \mathcal{T}_Y)_{\mathcal{T}_Y}]. \]
In fact, the construction process works more smoothly. The reason for this is that since \( \mathcal{F}(X) \) is a \( \mathcal{T}_X - A \) correspondence, we do not need to deal with quotient correspondences (as in Theorem 5.17) in order to reach the \( \mathcal{T}_X - \mathcal{T}_Y \) correspondence \( \mathcal{F}(X) \otimes_A M \otimes_A \mathcal{T}_Y \). Therefore, we may even drop the regularity condition of the morphisms in ECCor. Showing \( \mathcal{E} \) preserves the identity morphism would be the upmost hurdle here. However, we have already provided a proof for that (Proposition 5.7). Following the method we used in the proof above, it is not difficult to see that \( \mathcal{E} \) preserves the composition.

As mentioned in the introduction, Muhly and Solel (9) proved that strongly Morita equivalent injective \( C^* \)-correspondences have Morita equivalent Cuntz-Pimsner algebras. In 3, the authors give a proof for possibly non-injective \( C^* \)-correspondences. By using
functoriality, we recover this result as an application. In order to do this, first recall that the invertible morphisms are precisely the isomorphism classes of imprimitivity bimodules in the enchilada category (see [4] Proposition 3.6 and Corollary 4.6 for details). Therefore, in the enchilada category, \([AM_B]\) is an isomorphism if and only if \(AM_B\) is an imprimitivity bimodule. On the other hand, as we discussed at the end of Section 4, in \(ECCor\), we have that
\[
[AM_B]: AX_A \rightarrow BY_B \text{ is invertible } \iff AM_B \text{ is an imprimitivity bimodule.}
\]
This implies that \([AM_B]\) is an isomorphism in \(ECCor\) if and only if \(AM_B\) is an imprimitivity bimodule and
\[
AX \otimes A M \cong AM \otimes BY_B,
\]
i.e., \(AX_A\) and \(BY_B\) are strong Morita equivalent. Now, take two \(C^*\)-correspondences \(AX_A\) and \(BY_B\) that are strong Morita equivalent via \(AM_B\). Then, \([AM_B]\) is an isomorphism in \(ECCor\). Since any functor has to preserve isomorphisms, we must have that
\[
E([AM_B]) = [\mathcal{O}_X (\mathcal{F}(X) \otimes A M \otimes B Y) \mathcal{O}_Y]
\]
is an isomorphism in the enchilada category. Then, \(\mathcal{F}(X) \otimes A M \otimes B Y\) is an imprimitivity bimodule, which means that \(\mathcal{O}_X\) and \(\mathcal{O}_Y\) are Morita equivalent \(C^*\)-algebras. Very similarly, the functor that is described in Remark 5.19 provides the Morita equivalence of \(TX\) and \(TY\).

We want to note here that there is an alternative (non-functorial) way to see the Morita equivalence of Toeplitz algebras. Very roughly, one can show that \(C^*\)-correspondences \(\tau_X (\mathcal{F}(X) \otimes A M \otimes B Y) \tau_Y\) and \(\tau_Y (\mathcal{F}(Y) \otimes B N \otimes A T_X) \tau_X\) are each others inverse, where \(BN_A\) denotes the inverse of \(AM_B\). This can be achieved by using Prop 5.7 and some \(C^*\)-correspondence factorizations (for example Lemma 4.1 and Corollary 5.3) that we have given in Sections 4 and 5.

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