Adelization of Automorphic Distributions and Mirabolic Eisenstein Series

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Dedicated to Gregg Zuckerman on his 60th birthday

Abstract. Automorphic representations can be studied in terms of the embeddings of abstract models of representations into spaces of functions on Lie groups that are invariant under discrete subgroups. In this paper we describe an adelic framework to describe them for the group $GL(n, \mathbb{R})$, and provide a detailed analysis of the automorphic distributions associated to the mirabolic Eisenstein series. We give an explicit functional equation for some distributional pairings involving this mirabolic Eisenstein distribution, and the action of intertwining operators.

1. Introduction

Ever since the Poisson integral formula, the principal of recovering an eigenfunction from its “boundary values” (which are in general distributions) has been a useful tool in analysis. For automorphic forms, which are eigenfunctions of a ring of invariant differential operators, the boundary values can alternatively be described in terms of embeddings of models of representations into spaces of functions, embeddings which share the invariance of the automorphic forms. These automorphic distributions then control an entire automorphic representation in terms of a single object.

In previous papers we have applied automorphic distributions to studying summation formulas and the analytic continuation of $L$-functions [18–20], mainly for the full level congruence subgroup $GL(n, \mathbb{Z}) \subset GL(n, \mathbb{R})$. In this paper we present automorphic distributions in an adelic setting, in order to use them for general congruence subgroups. We also provide a thorough treatment of the automorphic distributions for a special but prominent type of Eisenstein series, the mirabolic Eisenstein series for the congruence subgroup $\Gamma_0(N) \subset GL(n, \mathbb{Z})$. We derive a

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precise form of their Fourier expansions, which also gives the analytic continuation of this mirabolic series, and prove an intertwining relation that is analogous to a functional equation. We also show that these properties extend to a relevant automorphic pairing established in [22] that involves these mirabolic Eisenstein distributions. In our forthcoming paper [23] this pairing will be calculated as the exterior square $L$-function times a precise ratio of Gamma factors, thereby giving a new construction of this $L$-function that leads to a stronger analytic continuation than previously known, as well as a functional equation.

The notion of adelic automorphic distribution is designed so that the action of the $p$-adic groups $GL(n, \mathbb{Q}_p)$ matches its usual action on adelic automorphic forms. This has the advantage of being able to quote certain calculations, such as local integrals, that have already been performed in related problems. One could also attempt stronger generalizations, which more generally treat the boundary values on a finite number of $p$-adic groups simultaneously with those on the real group, or which extend to number fields and different groups.

Sections 2, 3, and 4 contain, respectively, some properties of cuspidal automorphic distributions, mirabolic Eisenstein distributions, and the pairings of automorphic distributions. These topics are then reconsidered in section 5 using adelic terminology, which re-expresses them in a different notation that is useful in many applications. We also include an appendix recalling the known description of the generic unitary dual of $GL(n, \mathbb{R})$, as well as Langlands’ recipe for defining the Gamma factors of the tensor product, symmetric square, and exterior square $L$-functions. Both are useful in analytic number theory, where one inputs the structure of a functional equation, and uses constraints on the shifts in the Gamma factors to obtain estimates.

It is a pleasure to dedicate this paper to Gregg Zuckerman on his 60th birthday, as his early work on Whittaker functions is essential to clarity with which we now understand the generic unitary dual. The first author in particular extends his appreciation to Zuckerman for his friendliness and helpfulness as a colleague at an early stage in his career. We also wish to thank Bill Casselman, Erez Lapid, and Freydoon Shahidi for helpful discussions, and the referee for a careful reading of the paper.

2. Automorphic Distributions

In this section we recall the notion of automorphic distribution. We let $G$ denote the group of real points of a reductive matrix group defined over $\mathbb{Q}$, and $\Gamma \subset G$ an arithmetic subgroup. The particular examples that will matter to us are $G = GL(n, \mathbb{R})$, and a rational conjugate of a congruence subgroup $\Gamma \subset GL(n, \mathbb{Z})$. We let $Z_G$ denote the center of $G$, and fix a unitary central character $\omega : Z_G \to \{ z \in \mathbb{C}^* \mid |z| = 1 \}$. (2.1)

Then $G$ acts unitarily, by right translation, on the Hilbert space

$$L^2_{\text{loc}}(\Gamma \backslash G) = \{ f \in L^2_{\text{loc}}(\Gamma \backslash G) \mid \int_{\Gamma \backslash G/Z_G} |f|^2 \, dg < \infty \text{ and } f(gz) = \omega(z)f(g), z \in Z_G \}. \quad (2.2)$$

The principal congruence subgroup $\Gamma(m) \subset GL(n, \mathbb{Z})$ is the kernel of the reduction map from $GL(n, \mathbb{Z})$ to $GL(n, \mathbb{Z}/m\mathbb{Z})$. A congruence subgroup is one which contains $\Gamma(m)$ for some $m$. For $n > 2$, they are precisely the finite index subgroups.
Automorphic distributions are associated to classical automorphic representations, i.e., to $G$-invariant unitary embeddings
\[ j : V \hookrightarrow L^2(\Gamma \backslash G) \] (2.3)
of an irreducible unitary representation $(\pi, V)$ of $G$. The space of $C^\infty$ vectors $V^\infty \subset V$ is dense in $V$, and carries a canonical Fréchet topology. The linear map
\[ \tau = \tau_j : V^\infty \longrightarrow \mathbb{C}, \quad \tau(v) = j(v)(e), \] (2.4)
is well defined and $\Gamma$-invariant because $j$ maps $V^\infty$ to $C^\infty(\Gamma \backslash G)$. It is also continuous with respect to the topology of $V^\infty$, and thus may be regarded as a $\Gamma$-invariant distribution vector for the dual unitary representation $(\pi', V')$,
\[ \tau \in \left( (V')^{-\infty} \right)^\Gamma. \] (2.5)
This is the automorphic distribution corresponding to the automorphic representation (2.3). The former determines the latter completely: for $v \in V^\infty$ and $g \in G$,
\[ j(v)(g) = j(\pi(g)v)(e) = \langle \tau, \pi(g)v \rangle = \langle \pi'(g^{-1})\tau, v \rangle, \] (2.6)
so one can reconstruct the functions $j(v)$, $v \in V^\infty$, in terms of $\tau$; because of the density of $V^\infty$ in $V$, $\tau$ determines $j(v) \in L^2(\Gamma \backslash G)$ for all vectors $v \in V$.

In the following, we shall also consider automorphic distributions that do not correspond to irreducible summands of $L^2(\Gamma \backslash G)$, as in (2.3). These are $\Gamma$-invariant distribution vectors for admissible representations of finite length which need not be unitary, in particular the distribution analogues of Eisenstein series.

Most traditional approaches to automorphic forms work with finite dimensional $K$-invariant spaces of automorphic functions, meaning collections of functions $\{j(v)\}$ with $v$ ranging over a basis of a finite dimensional, $K$-invariant subspace of $V$; here $K \subset G$ denotes a maximal compact subgroup. Finite dimensional, $K$-invariant subspaces necessarily consist of $C^\infty$ vectors, so these automorphic functions are smooth. When $(\pi, V)$ happens to be a spherical representation, it is natural to consider the single automorphic function $j(v_0)$ determined by the – unique, up to scaling – $K$-fixed vector $v_0 \in V$, $v_0 \neq 0$. In that case $j(v_0)$ can be interpreted as a $\Gamma$-invariant function on the symmetric space $G/K$. For non-spherical representations, typically no such canonical choice exists, and making a definite choice may in fact be delicate. In the theory of integral representations of $L$-functions, for example, a wrong choice may result in an integral being identically zero instead of the $L$-function one is interested in, or it may result in an archimedean integral that is more difficult to compute, possibly even not computable at all [2, §2.6]. By working directly with the automorphic distribution $\tau$, our approach avoids these issues; in particular it does not matter whether $(\pi, V)$ is spherical or not.

Results of Casselman [5] and Casselman-Wallach [6, 31] imply that $(V')^{-\infty}$ can be realized as a closed subspace of the space of distribution vectors for a not-necessarily-unitary principal series representation,
\[ (V')^{-\infty} \hookrightarrow V^{-\infty}_{\lambda,\delta}; \] (2.7)
the subscripts $\lambda, \delta$ refer to the parameters of the principal series and will be explained shortly. Thus
\[ \tau \in \left( V^{-\infty}_{\lambda,\delta} \right)^\Gamma \] (2.8)
\[ ^2 \text{As distinguished from adelic automorphic representations.} \]
becomes a $\Gamma$-invariant distribution vector for a principal series representation with parameters $(\lambda, \delta)$. The embedding (2.7) is equivalent to the representation $V$ being a quotient of the dual principal series representation $V^{-\lambda, \delta}$.

In describing the principal series, we specialize the choice of $G$ to keep the discussion concrete,

$$G = GL(n, \mathbb{R}).$$

(2.9)

Its two subgroups

$$B = \left\{ \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ * & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & b_n \end{pmatrix} \mid b_j \in \mathbb{R}^*, \ 1 \leq j \leq n \right\},$$

(2.10)

are, respectively, maximal solvable and maximal unipotent. The quotient

$$X = G/B$$

(2.11)

is compact, and is called the flag variety of $G$. Since $N$ acts freely on its orbit through the identity coset in $X = G/B$ and has the same dimension as $X$, one can identify $N$ with a dense open subset of the flag variety,

$$N \simeq N \cdot eB \hookrightarrow X.$$ (2.12)

This is the open Schubert cell in $X$.

The principal series is parameterized by pairs $(\lambda, \delta) \in \mathbb{C}^n \times (\mathbb{Z}/2\mathbb{Z})^n$. For any such pair, we define the character

$$\chi_{\lambda, \delta} : B \rightarrow \mathbb{C}^*,$$

$$\chi_{\lambda, \delta} \left( \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ * & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & b_n \end{pmatrix} \right) = \prod_{j=1}^n ((\text{sgn} b_j)^{\delta_j} |b_j|^{\lambda_j}).$$

(2.13)

The parametrization also involves the quantity

$$\rho = \left( \frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2} \right) \in \mathbb{C}^n.$$ (2.14)

Each pair $(\lambda, \delta)$ determines a $G$-equivariant $C^\infty$ line bundle $\mathcal{L}_{\lambda, \delta} \rightarrow X$, on whose fiber at the identity coset the isotropy group $B$ acts via $\chi_{\lambda, \delta}$. By pullback from $X = G/B$ to $G$, the space of $C^\infty$ sections becomes naturally isomorphic to a space of $C^\infty$ functions on $G$,

$$C^\infty(X, \mathcal{L}_{\lambda, \delta}) \simeq \{ f \in C^\infty(G) \mid f(gb) = \chi_{\lambda, \delta}(b^{-1})f(g) \text{ for } g \in G, b \in B \}.$$ (2.15)

This isomorphism relates the translation action of $G$ on sections of $\mathcal{L}_{\lambda, \delta}$ to left translation of functions. By definition,

$$V^\infty_{\lambda, \delta} = C^\infty(X, \mathcal{L}_{\lambda-\rho, \delta})$$ (2.16)

This convention differs slightly from our earlier papers [18, 19], where we had switched the role of $(\pi, V)$ and $(\pi', V')$ at this stage for notational convenience. However, that switch causes a notational inconsistency for our adelic automorphic distributions in section 5 that we have elected to avoid.
is the space of $C^\infty$ vectors of the principal series representation $V_{\lambda,\delta}$; the shift by $\rho$ serves the purpose of making the labeling compatible with Harish-Chandra's parametrization of infinitesimal characters. Analogously

$$V_{\lambda,\delta}^{-\infty} = C^{-\infty}(X, \mathcal{L}_{\lambda-\rho,\delta})$$

\[ \simeq \{ f \in C^{-\infty}(G) \mid f(gb) = \chi_{\lambda-\rho,\delta}(b^{-1})f(g) \text{ for } g \in G, b \in B \} \]  

(2.17)

is the space of distribution vectors. The isomorphism in the second line is entirely analogous to (2.15).

The group $N$, which we had identified with the open Schubert cell, intersects $B$ only in the identity. Thus, when the equivariant line bundle $\mathcal{L}_{\lambda-\rho,\delta} \to X$ is restricted to the open Schubert cell, it becomes canonically trivial, and distribution sections of the restricted line bundle become scalar-valued distributions,

$$C^{-\infty}(N, \mathcal{L}_{\lambda-\rho,\delta}) = C^{-\infty}(N).$$

(2.18)

This identification is $N$-invariant, of course. In particular any automorphic distribution

$$\tau \in (V_{\lambda,\delta}^{-\infty})^\Gamma = C^{-\infty}(X, \mathcal{L}_{\lambda-\rho,\delta})^\Gamma$$

restricts to a $\Gamma \cap N$-invariant distribution on the open Schubert cell:

$$\tau \in C^{-\infty}(\Gamma \cap N\backslash N).$$

(2.20)

Two comments are in order. Ordinarily, a distribution on a manifold is not completely determined by its restriction to a dense open subset. Since the $\Gamma$-translates of the open Schubert cell cover $X$, any automorphic distribution is determined by its restriction to $N$. The containment (2.20) should be interpreted in this sense. Secondly, when one views $\tau$ this way, the invariance under $\Gamma \cap N$ is directly visible. The invariance under any $\gamma \in \Gamma$ that does not lie in $N$ can be described in terms of an appropriate factor of automorphy.

The abelianization $N/[N,N]$ – i.e., the quotient of $N$ by the derived subgroup $[N,N]$ – is isomorphic to the additive group $\mathbb{R}^{n-1}$. Concretely, let

$$n(x) = \begin{pmatrix} 1 & x_1 & 0 & \cdots & 0 \\ 0 & 1 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \quad (x = (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}) \tag{2.21}$$

then $\mathbb{R}^{n-1} \cong N/[N,N]$ via

$$\mathbb{R}^{n-1} \ni x \mapsto \text{image of } n(x) \in N/[N,N].$$

(2.22)

A congruence subgroup $\Gamma \subset G$ intersects $N$ in a cocompact subgroup of $N$, and similarly $[N,N]$ in a cocompact subgroup of itself. This allows us to define

$$\tau_{\text{abelian}} = \frac{1}{\text{covol}(\Gamma \cap [N,N])} \int_{(\Gamma \cap [N,N]) \backslash [N,N]} \ell(n)\tau \, dn,$$

(2.23)

the sum of the abelian Fourier component of the automorphic distribution $\tau$, as in (2.19)–(2.20); $\ell(n)$ denotes left translation by $n$. Equivalently

$$\tau = \tau_{\text{abelian}} + \cdots,$$

(2.24)

where $\cdots$ refers to the sum of Fourier components of $\tau$ on which $[N,N]$ acts non-trivially. By construction, $\tau_{\text{abelian}} \in V_{\lambda,\delta}^{-\infty}$, and the restriction of $\tau_{\text{abelian}}$ to $N$ lies in $C^{-\infty}(([N,N] \cdot (\Gamma \cap N)) \backslash N)$. 

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The quotient \( ([N,N] \cdot (\Gamma \cap N))\setminus N \) is compact, connected, abelian, hence a torus. Like any distribution on a torus, \( \tau_{\text{abelian}} \) can be expressed as an infinite linear combination of characters. We may write

\[
\tau_{\text{abelian}}(n(x)) = \sum_{k \in \mathbb{Q}^{n-1}} c_k e(k_1 x_1 + k_2 x_2 + \cdots + k_{n-1} x_{n-1}) \tag{2.25}
\]

in which the coefficients \( c_k \) are tacitly assumed to vanish unless \( k \) lies inside \( M^{-1}\mathbb{Z} \), for some appropriate integer \( M \) (which takes into account the size of the torus). Here, as from now on, we use the notational convention

\[
e(z) := e^{2\pi i z}. \tag{2.26}
\]

In the case that \( \Gamma \) equals the full level congruence group \( GL(n, \mathbb{Z}) \), \( \Gamma \cap N = N(\mathbb{Z}) \) and \( k \) lies in \( \mathbb{Z}^{n-1} \), because the isomorphism (2.22) induces \([N,N] \cdot (\Gamma \cap N))\setminus N \simeq \mathbb{Z}^{n-1} \setminus \mathbb{R}^{n-1} \).

Recall the notion of a cuspidal automorphic representation: an automorphic representation in the same sense as (2.3), such that

\[
\int_{(\Gamma \cap N)\setminus N} j(v)(ng) \, dn = 0 \quad \text{for every } v \in V^\infty, \, g \in G, \tag{2.27}
\]

whenever \( N \subset G \) is the unipotent radical of a proper parabolic subgroup, defined over \( \mathbb{Q} \). We call an automorphic distribution \( \tau \in (V^{-\infty})^G \) cuspidal if the corresponding automorphic representation has that property; this is equivalent to

\[
\int_{N/(\Gamma \cap N)} \ell(n)\tau \, dn = 0 \tag{2.28}
\]

for every \( N \) as in [22, Lemma 2.16]. In our particular setting of \( GL(n) \) the cuspidality of \( \tau \) implies

\[
k \in \mathbb{Q}^{n-1}, \, k_j = 0 \quad \text{for at least one } j, \, 1 \leq j \leq n-1 \quad \implies \quad c_k = 0, \tag{2.29}
\]

as can be seen by averaging the \( u \)-translates of \( \tau \) over \( U_{j,n-j}(\mathbb{Z}) \setminus U_{j,n-j} \), the quotient of the unipotent radical of the \((j,n-j)\) parabolic modulo its group of integral points. However, the cuspidality of \( \tau \) cannot be characterized solely in terms of the vanishing of certain Fourier coefficients at each cusp; it also involves conditions “at infinity” – see, for example, [18, §5].

The Casselman embedding (2.27) does not necessarily determine the parameters \((\lambda, \delta)\) uniquely. For example, when \( V_{\lambda,\delta} \) is an irreducible principal series representation, \((\lambda, \delta)\) is determined only up to the action of the Weyl group. The abelian Fourier coefficients \( c_k, k \in \mathbb{Q}^{n-1} \), do depend on the choice of Casselman embedding. When \( \tau \) is cuspidal, one can introduce its renormalized Fourier coefficients

\[
a_{(k_1,k_2,\ldots,k_{n-1})} = \prod_{j=1}^{n-1} \left( (\text{sgn } k_j)^{\delta_1 + \delta_2 + \cdots + \delta_j} |k_j|^{\lambda_1 + \lambda_2 + \cdots + \lambda_j} \right) c_{(k_1,k_2,\ldots,k_{n-1})}, \tag{2.30}
\]

which have canonical meaning. The \( L \)-functions of \( \tau \) can be most naturally expressed in terms of the \( a_k \). For \( k \) coprime to a finite set of primes depending on \( \tau \), the \( a_k \) are actually the eigenvalues of certain Hecke operators \( T_k \) acting on the automorphic representation, provided the Hecke action preserves the automorphic representation. This applies to all \( k \) when \( \Gamma = GL(n, \mathbb{Z}) \), demonstrating that the \( a_k \) are independent of the particular Casselman embedding. This independence can also be shown directly, without reference to Hecke operators – meaning that this
defines an outer automorphism of also its contragredient. The map distribution – or equivalently, the corresponding automorphic representation – but \( L_t \) representational equations of their quotient \( \Gamma \). One then also has the following equality between distributions on \( G \) where

\[
\int_{\Gamma \cap \mathbb{Z}} \left( \prod_{j=1}^{n-1} |k_j|^{-1} \right)^{1/2} \omega_{\lambda,\delta}(D(k)g) \right. 
\]

In view of \( (2.25) \) and \( (2.30) \), the canonical extension of \( \tau_{\text{abelian}} \) to \( G \) can be written as

\[
\tau_{\text{abelian}}(g) = \sum_{k \in \mathbb{Q}^{n-1}} \left( \prod_{j=1}^{n-1} k_j^{1/2} \right) a_k \omega_{\lambda,\delta}(D(k)g). 
\]

One then also has the following equality between distributions on \( G \):

\[
\frac{1}{\text{covol}(\Gamma \cap \mathbb{N})} \int_{\Gamma \cap \mathbb{N}} \tau(ug) e(-k_1 u_{1,2} - \cdots - k_{n-1} u_{n-1,n}) du = \left( \prod_{j=1}^{n-1} k_j^{1/2} \right) a_k \omega_{\lambda,\delta}(D(k)g), 
\]

where \( u_{i,j} \) denote the entries of \( u \in \mathbb{N} \) and \( \text{covol}(\Gamma \cap \mathbb{N}) \) denotes the volume of the quotient \( \Gamma \cap \mathbb{N} \) under the Haar measure \( du \), normalized so that \( \text{covol}(\mathbb{N}(\mathbb{Z})) = 1 \).

A number of relations involving automorphic distributions, such as the functional equations of their \( L \)-functions, involve not only a particular automorphic distribution – or equivalently, the corresponding automorphic representation – but also its contragredient. The map

\[
g \mapsto \bar{g}, \quad \bar{g} = w_{\text{long}}(g'^{-1}) w_{\text{long}}^{-1}, \quad \text{with} \quad w_{\text{long}} = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix},
\]

defines an outer automorphism of \( G = GL(n, \mathbb{R}) \), which preserves the subgroups \( GL(n, \mathbb{Z}) \), \( B \) and \( N \). One easily checks that

\[
\bar{\tau}(g) =_{\text{def}} \tau(\bar{g}) \in (V_{\lambda,\delta}^{-\infty})^\Gamma, \quad \text{with} \quad \Gamma = \{ \gamma | \gamma \in \Gamma \},
\]
the contragredient of $\tau$, has abelian Fourier coefficients

$$\tilde{c}(k_1, k_2, \ldots, k_{n-1}) = c(-k_{n-1}, -k_{n-2}, \ldots, -k_1) \quad (2.37)$$

and principal series parameters

$$\tilde{\lambda} = (-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1), \quad \tilde{\delta} = (\delta_n, \delta_{n-1}, \ldots, \delta_1). \quad (2.38)$$

3. Mirabolic Eisenstein series for $GL(n)$

The Epstein zeta functions on $GL(n, \mathbb{R})$, which are sums of powers of the norms of lattice vectors in $\mathbb{R}^n$, were an early example of higher rank Eisenstein series. They have a functional equation and analytic continuation coming from Poisson summation, in complete analogy with the Riemann zeta function. Langlands, and later Jacquet and Shalika [10], studied mirabolic Eisenstein series, which are an adelic generalization involving homogenous functions other than the norm. They play a crucial role in the functional equation and analytic continuation of a number of integral representations of $L$-functions, e.g. [3, 4, 8, 25]. In this section we describe their distributional counterparts. Proposition 3.16 gives the analytic continuation and an explicit formula for their Fourier coefficients in terms of $L$-functions and arithmetic sums. These have direct applications elsewhere, most recently to string theory where they describe fine details of graviton scattering amplitudes (see, for example, [9, 24]). A functional equation is given in proposition 3.48. The analytic properties later transfer to the pairings in section 4. They are understood most easily in classical terminology; in section 5 we shall convert them into adelic expressions whose analytic properties rest on what is proven here. It is possible to recover the results here from [10], using sophisticated machinery of Casselman and Wallach. However, the translation between the two is somewhat lengthy and unenlightening, and so we have chosen to rederive them from basic principles instead, highlighting the role of degenerate principal series and intertwining operators.

Mirabolic Eisenstein series are induced from one dimensional representations of the so-called mirabolic subgroup of $GL(n)$, colloquially dubbed the “miraculous parabolic”4. In fact, the functional equation involves not just one, but two different mirabolic subgroups and Eisenstein series. The mirabolic subgroups and the “opposites” of their unipotent radicals are

$$P = \left\{ \begin{pmatrix} a & 0 & \ldots & 0 \\ \ast & \ast & \cdots & \ast \\ \vdots & \vdots & \ddots & \vdots \\ C \end{pmatrix} \bigg| C \in GL(n-1, \mathbb{R}), \ a \in \mathbb{R}^* \right\},$$

$$\tilde{P} = \left\{ \begin{pmatrix} C & \ast & \cdots & \ast \\ \ast & \ast & \cdots & \ast \\ \ast & \ast & \cdots & \ast \\ \ast & \ast & \cdots & \ast \end{pmatrix} \bigg| C \in GL(n-1, \mathbb{R}), \ a \in \mathbb{R}^* \right\}, \quad (3.1)$$

$$U = \left\{ \begin{pmatrix} 1 & \ast & \cdots & \ast \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} \right\}, \quad \tilde{U} = \left\{ \begin{pmatrix} 1 & \ast & \cdots & \ast \\ \ast & \ast & \cdots & \ast \\ \ast & \ast & \cdots & \ast \\ \ast & \ast & \cdots & \ast \end{pmatrix} \right\}.$$

4The terminology in the literature is not entirely consistent: some reserve the term “mirabolic” for the stabilizer of a line in $\mathbb{R}^n$, e.g. $\tilde{P}$, but not $P$. 
note that the outer automorphism \[ \text{(2.35)} \] relates \( P \) to \( \tilde{P} \) and \( U \) to \( \tilde{U} \). In analogy to the flag variety \( X = G/B, \)

\[
Y = G/P \quad \text{and} \quad \tilde{Y} = G/\tilde{P}
\]

(3.2)

are \textit{generalized flag varieties}. The former can be naturally identified with the projective space of hyperplanes in \( \mathbb{R}^n \), the latter with the projective space of lines. Since \( U \cap P = \tilde{U} \cap P = \{ e \} \), we can identify \( U \) and \( \tilde{U} \) with the open Schubert cells in these two spaces,

\[
U \simeq U \cdot eP \leftrightarrow Y, \quad \tilde{U} \simeq \tilde{U} \cdot e\tilde{P} \leftrightarrow \tilde{Y}.
\]

(3.3)

This is again entirely analogous to \[ \text{(2.12)}. \]

For \( \nu \in \mathbb{C} \) and \( \varepsilon \in \mathbb{Z}/2\mathbb{Z} \), we define

\[
\chi_{\nu,\varepsilon} : P \rightarrow \mathbb{C}^*, \quad \chi_{\nu,\varepsilon} \left( \begin{array}{cccc}
\alpha & 0 & \cdots & 0 \\
\star & \star & \cdots & \star \\
B & & & \\
\star & \star & \cdots & \star
\end{array} \right) = |a|^{\frac{(\alpha-1)\nu}{n}} (\text{sgn} \ a)^\varepsilon | \det B |^{-\hat{\nu}},
\]

(3.4)

\[
\tilde{\chi}_{\nu,\varepsilon} : \tilde{P} \rightarrow \mathbb{C}^*, \quad \tilde{\chi}_{\nu,\varepsilon} \left( \begin{array}{cccc}
0 & 0 & \cdots & 0 \\
B & & & \\
0 & \cdots & \star & \star
\end{array} \right) = | \det B |^{\hat{\nu}} |a|^{\frac{(\nu-1)\varepsilon}{n}} (\text{sgn} \ a)^\varepsilon.
\]

We study these two characters without any loss of generality, because they account for all characters of \( P \) and \( \tilde{P} \), up to tensoring by central characters. Taking these other choices amounts to multiplying our eventual Eisenstein distributions by \( \text{sgn}(\det g) \), and has no analytic impact. The quantity

\[
\rho_{\text{mir}} = \frac{n}{2}
\]

(3.5)

plays the role of \( \rho \) in the present context.

There exist unique \( G \)-equivariant \( C^\infty \) line bundles \( \mathcal{L}_{\nu,\varepsilon} \rightarrow Y \), \( \tilde{\mathcal{L}}_{\nu,\varepsilon} \rightarrow \tilde{Y} \), on whose fibers at the identity cosets the isotropy groups act by, respectively, \( \chi_{\nu,\varepsilon} \) and \( \tilde{\chi}_{\nu,\varepsilon} \). The group \( G \) acts via left translation on

\[
W_{\nu,\varepsilon}^\infty = C^\infty (Y, \mathcal{L}_{\nu-\rho_{\text{mir}},\varepsilon})
\]

\[
\simeq \{ f \in C^\infty (G) \mid f(gp) = \chi_{\nu-\rho_{\text{mir}},\varepsilon}(p^{-1})f(g) \text{ for } g \in G, p \in P \},
\]

\[
\tilde{W}_{\nu,\varepsilon}^\infty = C^\infty (\tilde{Y}, \tilde{\mathcal{L}}_{\nu-\rho_{\text{mir}},\varepsilon})
\]

\[
\simeq \{ f \in C^\infty (G) \mid f(g\tilde{p}) = \tilde{\chi}_{\nu-\rho_{\text{mir}},\varepsilon}(\tilde{p}^{-1})f(g) \text{ for } g \in G, \tilde{p} \in \tilde{P} \}.
\]

(3.6)

In particular, functions \( f \in W_{\nu,\varepsilon}^\infty \) and \( \tilde{f} \in \tilde{W}_{\nu,\varepsilon}^\infty \) obey the respective transformation laws

\[
f \left( g \left( \begin{array}{cc}
\alpha & B \\
\star & a
\end{array} \right) \right) = |a|^{n/2-\nu} (\text{sgn} \ a)^\varepsilon f(g) \quad \text{and}
\]

\[
\tilde{f} \left( g \left( \begin{array}{cc}
B & \star \\
\star & a
\end{array} \right) \right) = |a|^\nu-n/2 (\text{sgn} \ a)^\varepsilon \tilde{f}(g), \quad \text{provided } |a| | \det B | = 1.
\]

(3.7)

These are the spaces of \( C^\infty \) vectors for degenerate principal series representations \( W_{\nu,\varepsilon}, \tilde{W}_{\nu,\varepsilon} \).

As in the case of the principal series, the line bundle \( \mathcal{L}_{\nu-\rho_{\text{mir}},\varepsilon} \) is equivariantly trivial over the open Schubert cell \( U \subset Y \). Since \( \delta_\nu \in C^{-\infty}(U) \), the Dirac delta function at \( e \in U \), evidently has compact support in \( U \), we may regard it as a
distribution section of $L^{ν−ρ_{mir},ε}$, or in other words, as a vector in $W_{ν,ε}^{−∞}$. This makes
\[ δ_∞ \overset{\text{def}}{=} ℓ(w_{long})δ_e \in W_{ν,ε}^{−∞} \] 
well defined. By construction, $δ_∞$ is supported at $w_{long}P ∈ Y$, the unique fixed point of $U$, also known as the closed Schubert cell in $Y$. Similarly there exists a delta function $δ_∞ ∈ \widehat{W}_{ν,ε}^{−∞}$ supported on the closed Schubert cell $w_{long}P ∈ \widehat{Y}$.

Mirabolic Eisenstein series are globally induced from a character of $P$ or $\widehat{P}$. As for their analytic properties, it suffices to study them for the congruence subgroups
\[ \Gamma_0(N) = \left\{ γ ∈ GL(n,\mathbb{Z}) \mid γ ≡ \begin{pmatrix} * & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & * & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \pmod{N} \right\} \] 
(3.9)
or
\[ \Gamma(N) = \left\{ γ ∈ GL(n,\mathbb{Z}) \mid γ ≡ \begin{pmatrix} 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & * & 0 \\ 0 & \cdots & 0 & * \end{pmatrix} \pmod{N} \right\}, \] 
(3.10)
by means of a reduction we will discuss in section 5. Of course $\Gamma_0(N)$ and $\Gamma(N)$ are related by the outer automorphism (2.35). Any Dirichlet character $ψ$ modulo $N$ lifts to characters $α$ of $\Gamma_0(N)$ and $\tilde{α}$ of $\Gamma(N)$ defined through the formulas
\[ α(γ) = ψ(γ_{nm})^{-1} \quad \text{and} \quad \tilde{α}(γ) = ψ(γ_{11}), \quad γ = (γ_{ij}). \] 
(3.11)
The reason for the inverse is to ensure $\tilde{α}(\tilde{γ}) = α(γ)$, a property used below in (3.14). These characters are respectively trivial on the subgroups $Γ_1(N) ⊂ Γ_0(N)$ and $\widehat{Γ}_1(N) ⊂ \widehat{Γ}_0(N)$, which are defined by the congruence $γ_{nm} ≡ 1 \pmod{N}$ in the former case, and $γ_{11} ≡ 1 \pmod{N}$ in the latter case.

We let $Γ = \Gamma_0(N)$ and $Γ_∞ = Γ \cap w_{long}Pw_{long}$ denote its isotropy subgroup at $w_{long}P ∈ Y$. Because $−e ∈ Γ_∞$, we insist that $ψ(−1) = (−1)^e$ so that $Γ_∞$ acts trivially on $δ_e$. (Otherwise the Eisenstein series we presently define would be identically zero.) With this choice of parity parameter define
\[ E_{ν,ψ} = L(ν + \frac{e}{2}, ψ) \sum_{γ ∈ Γ/Γ_∞} α(γ) ℓ(γ)δ_∞ \in W_{ν,ε}^{−∞}. \] 
(3.12)
For $Re ν > ρ_{mir} = n/2$ this sum converges in the strong distribution topology. In the region $\{ Re ν > ρ_{mir} \}$, the resulting distribution vector depends holomorphically on $ν$ and satisfies the condition $ℓ(γ)E_{ν,ψ} = α(γ)^{-1}E_{ν,ψ}$ for all $γ ∈ Γ$. Entirely analogously, with $\tilde{Γ} = \tilde{Γ}_0(N)$,
\[ \tilde{E}_{ν,ψ} = L(ν + \frac{e}{2}, ψ) \sum_{γ ∈ \tilde{Γ}/\tilde{Γ}_∞} \tilde{α}(γ) ℓ(γ)δ_∞ \in \widehat{W}_{ν,ε}^{−∞}, \] 
(3.13)
converges and depends holomorphically on $ν$ in $\{ Re ν > ρ_{mir} = n/2 \}$. The two Eisenstein series are related by the involution (2.35):
\[ E_{ν,ψ}(g) = \tilde{E}_{ν,ψ}(\tilde{g}). \] 
(3.14)
The following proposition gives a simpler formula for these Eisenstein distributions when restricted to the open, dense Bruhat cells $U, w_{long}U ⊂ Y$, and $U, w_{long}U ⊂ \widehat{Y}$, respectively. Since both Eisenstein series are invariant under a congruence group, and the translates of any of these cells by that invariance group cover $Y$ and $\widehat{Y}$, respectively, restriction to either determines them completely. The statement involves the finite Fourier transform
\[ \tilde{ψ}(m) = \sum_{a \pmod{N}} ψ(a)e\left(\frac{am}{N}\right) \] 
(3.15)
of a Dirichlet character of modulus \(N\). (Note that \(\psi(a) = 0\) when \(a\) is not relatively prime to \(N\).

### 3.16. Proposition. (Analytic continuation and Fourier expansion of mirabolic Eisenstein distributions.) Let \(\Re \nu > n/2\). The restriction of the distribution \(E_{\nu, \psi}\) to \(U\) as well as the restriction of the distribution \(\tilde{E}_{\nu, \psi}\) to \(\tilde{U}\) are determined by the common formulas

\[
E_{\nu, \psi} \left( \begin{array}{ccc} 1 & -u_{n-1} & \cdots & -u_1 \\ 0 & 1 & 0 & \cdots \\ & & & & \\ 0 & \cdots & 1 \end{array} \right) = \tilde{E}_{\nu, \psi} \left( \begin{array}{ccc} 1 & 0 & u_1 \\ 0 & \cdots & 1 \\ & & & & \end{array} \right)
\]

\[
= \sum_{\nu \in \mathbb{Z}^n, v_1 > 0} \psi(\nu) v_1^{-\nu-n/2} \delta_{v_1/\nu_1}(u_1) \cdots \delta_{v_2/\nu_1}(u_n-1)
\]

\[
= N^{-\nu-n/2} \sum_{v \in \mathbb{Z}^n, v_2 > 0} v_1^{-\nu-n/2+1} \psi(\nu) v_1 \delta_{v_1/\nu_1}(u_2) \cdots \delta_{v_2/\nu_1}(u_n-1) \hat{\psi}(-\nu)
\]

\[
= \sum_{r \in \mathbb{Z}^{n-1}} a_r \psi(r_1 u_1 + \cdots + r_{n-1} u_{n-1}),
\]

where

\[
a_r = N^{-\nu-n/2} \sum_{d > 0, \nu_1 \cdots \nu_{n-1}} d^{-\nu-n/2-1} \hat{\psi}(-r_1/d).
\]

Their restrictions to \(w_{\text{long}} U\) and \(w_{\text{long}} \tilde{U}\) are determined by the common formula

\[
(\ell(w_{\text{long}}) E_{\nu, \psi}) \left( \begin{array}{ccc} 1 & u_{n-1} & \cdots & u_1 \\ 0 & 1 & 0 & \cdots \\ & & & & \\ 0 & \cdots & 1 \end{array} \right) = (\ell(w_{\text{long}}) \tilde{E}_{\nu, \psi}) \left( \begin{array}{ccc} 1 & u_1 \\ 0 & \cdots & 1 \end{array} \right)
\]

\[
= \sum_{\nu \in \mathbb{Z}^n, v_n > 0} \psi(\nu) v_n^{-\nu-n/2} \delta_{v_n/\nu_n}(u_1) \cdots \delta_{v_{n-1}/\nu_n}(u_n-1)
\]

\[
= \sum_{r \in \mathbb{Z}^{n-1}} c_r \psi(r_1 u_1 + \cdots + r_{n-1} u_{n-1})
\]

where

\[
c_r = \frac{1}{N^{n-1}} \sum_{d > 0, \nu_1 \cdots \nu_{n-1}} \psi(d) d^{-\nu-n/2-1}.
\]

These sums, and hence also both \(E_{\nu, \psi}\) and \(\tilde{E}_{\nu, \psi}\), can be holomorphically continued to \(\mathbb{C} - \{n/2\}\). They are entire if \(\psi\) is nontrivial, and have a simple pole at \(\nu = n/2\) otherwise.

**Proof:** Because of the relation [3.13] and the visible transformation properties of the asserted formulas, the formulas for \(E_{\nu, \psi}\) and \(\tilde{E}_{\nu, \psi}\) are equivalent. We shall thus work with \(\tilde{E}_{\nu, \psi}\), first deriving the formulas as sums of \(\delta\)-functions, then the alternative expressions in terms of Fourier series, and finally deduce the meromorphic continuation from these.

We begin with the second set of formulas, for the restriction to \(w_{\text{long}} \tilde{U}\). Letting \(\Gamma\) instead stand for \(w_{\text{long}} \tilde{\Gamma}_0(N)_{\text{long}}\), the expression for \(\tilde{E}_{\nu, \psi} \in \tilde{W}_{\nu, \psi}^{-\infty}\) in [3.13] may
be rewritten as
\[ \ell(u_{\text{long}})E_{v,\psi} = L(u + \frac{\nu}{2}, \psi) \sum_{\gamma \in \Gamma \cap \widehat{P}} \tilde{a}(u_{\text{long}} \gamma u_{\text{long}}) \ell(\gamma) \delta_{\xi}. \quad (3.17) \]

The last column of a matrix is unchanged, up to sign, after right multiplication by an element of \( \Gamma \cap \widehat{P} \). Moreover, every \( n \)-tuple of relatively prime integers occurs as the last column of some matrix in \( GL(n, \mathbb{Z}) \). Its subgroup \( \Gamma \) is defined by the congruence that all entries except for the final one in its last column are divisible by \( N \). Therefore, the cosets \( \Gamma / \Gamma \cap \widehat{P} \) are in bijection with the set

\[ \{ \text{vectors } v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \text{ with } GCD(v) = 1 \text{ and } N | v_1, \ldots, v_{n-1} \}/\{ \pm 1 \}. \]

Given \( v \in \mathbb{Z}^n \) whose entries are relatively prime and satisfy the above divisibility condition, we let \( \gamma_v \) denote a coset representative in \( \Gamma / \Gamma \cap \widehat{P} \).

When \( (3.17) \) is restricted to \( \tilde{U} \), some of the terms in the sum on the right hand side vanish because the \( \gamma \)-translate of \( \delta_{\xi} \) does not lie in the big cell. The nonvanishing terms are precisely those for which \( \gamma \in \Gamma \subset G \text{ projects into the big cell } \tilde{U} \subset \tilde{Y} = G / \widehat{P} \). A matrix whose final column is the vector \( v \) projects to the big cell \( \tilde{U} \) if and only if its last entry is nonzero; in this situation, applied to \( \gamma_v \), we have the explicit matrix decomposition

\[ \gamma_v = \left( I \ y \right) \left( \begin{array}{c} A \ 

v \n
\end{array} \right), \quad (3.18) \]

where \( u = \frac{1}{v_n}(v_1, \ldots, v_{n-1}) \in \mathbb{R}^{n-1} \) and \( A \) is a matrix with determinant \( \pm 1/v_n \). Therefore the range of summation in \( (3.17) \) is in bijection with

\[ \{ v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \text{ with } GCD(v) = 1, \text{ } N | v_1, \ldots, v_{n-1}, \text{ and } v_n > 0 \}. \quad (3.19) \]

The decomposition \( (3.18) \) allows us to compute the following action of \( \gamma_v^{-1} \) on the delta function \( \delta_{\gamma_v^{-1}}(\frac{v_1}{v_n}, \ldots, \frac{v_{n-1}}{v_n}) \) on \( \tilde{U} \):

\[ \ell(\gamma_v^{-1}) \delta_{\gamma_v^{-1}}(\frac{v_1}{v_n}, \ldots, \frac{v_{n-1}}{v_n}) = \ell\left( \gamma_v^{-1} \left( \begin{array}{c} 1 \\

0 \\

vdigitn-1/v_n \\

\end{array} \right) \right) \delta_{\xi} = \ell\left( \left( \begin{array}{c} A \ 

v \n
\end{array} \right)^{-1} \right) \delta_{\xi} = \delta_{\xi}, \quad (3.20) \]

In this last equation, the transformation rule \( (3.7) \) has provided a factor of \( (\text{sgn } v_n)^{\epsilon} |v_n|^{\nu-n/2} \), while the \( \delta \)-function identity \( \delta_{\xi}(\frac{4\pi}{v_n}) = |v_n|^n \delta_{\xi}(u) \) is responsible for the rest of the exponent. Using \( \tilde{a}(u_{\text{long}} \gamma u_{\text{long}}) = \psi(v_n) \), the summand for \( \gamma_v \) in \( (3.17) \) can be written as

\[ \tilde{a}(u_{\text{long}} \gamma_v u_{\text{long}}) \ell(\gamma_v) \delta_{\xi} = \psi(v_n) (\text{sgn } v_n)^{\epsilon} |v_n|^{-\nu-n/2} \delta_{\xi}(\frac{v_1}{v_n}, \ldots, \frac{v_{n-1}}{v_n}). \quad (3.21) \]

Summing this expression over the coset representatives from \( (3.19) \) gives, in terms of the coordinates \( (u_1, \ldots, u_{n-1}) \) on \( \tilde{U} \) in the second set of statements in the proposition, an expression similar to the one claimed there for \( \ell(u_{\text{long}})E_{v,\psi} \). They differ only in that the latter has no condition on \( GCD(v) \). However, the first set consists of scalar multiples, by positive integers relatively prime to \( N \), of the second set, and multiplication by the pre-factor \( L(\nu + \frac{\nu}{2}, \psi) \) in \( (3.17) \) – unused until now – accounts for the discrepancy. (Note that terms for which \( (v_n, N) > 1 \) vanish.)
At this point, we have established the δ-function formula for the restriction of \(\ell(w_{\text{long}})\tilde{E}_{\nu,\psi}\) to \(\tilde{U}\), and therefore also the one for the restriction of \(\ell(w_{\text{long}})E_{\nu,\psi}\) to \(U\), to which it is equivalent. Had we instead considered the series \(\tilde{E}_{\nu,\psi}\) instead of its \(w_{\text{long}}\)-translate, the last column of \(\gamma\) would have entries \((v_1, \ldots, v_1)\), the reverse of the situation we encountered above. The identical reasoning produces the same formula, but with \(v_j\) replaced by \(v_{n+1-j}\) in the summand – exactly the first claim of the proposition.

Next we turn to the assertions about the Fourier expansions, starting first with the common expression for the \(w_{\text{long}}\) translates. It is periodic in each \(u_i\) with period \(N\), so the coefficient \(c_r\) is computed by the integral

\[
\frac{1}{N^{n-1}} \int_{(NZ\backslash \mathbb{G})^{n-1}} \sum_{\mathbf{v} \in \mathbb{Z}^n, \mathbf{v}_n > 0} \psi(v_n) v_n^{-\nu-n/2} e \left( -\frac{\sum_{i=1}^{n-1} r_i u_i}{N} \right) \times \\
\times \delta_{v_1/v_n}(u_1) \cdots \delta_{v_{n-1}/v_n}(u_{n-1}) \, du_1 \cdots du_{n-1}
\]

\[
= \frac{1}{N^{n-1}} \sum_{\mathbf{v}_n > 0} \sum_{\nu_1, \ldots, \nu_{n-1} \in \mathbb{Z}/Nv_n \mathbb{Z}} \psi(v_n) v_n^{-\nu-n/2} e \left( -\frac{\sum_{i=1}^{n-1} r_i v_i}{Nv_n} \right) \quad (3.22)
\]

\[
= \frac{1}{N^{n-1}} \sum_{d > 0} \sum_{\mathbf{v}_n \in \mathbb{Z}/d\mathbb{Z}} \psi(d) d^{-\nu-n/2} e \left( -\frac{\sum_{i=1}^{n-1} r_i v_i}{d} \right).
\]

The sum over any fixed \(v_j\), for \(1 \leq j \leq n-1\), equals \(d\) if \(d|v_j\), and zero otherwise. Therefore \(c_r\) is given by the formula stated in the proposition. The formula for \(a_r\) is computed by the same procedure. The hybrid formula for the restriction \(E_{\nu,\psi}\) or \(\tilde{E}_{\nu,\psi}\) which involves a Fourier series in \(u_1\), and δ-functions in the other variables, is proven by taking a Fourier integral only in the variable \(u_1\), and leaving the other \(u_j\) alone.

Finally we come to the analytic continuation, which is equivalent for each of the expressions involved. We therefore consider the last formula in the statement of the proposition. The coefficient \(c_r\) equals a finite sum which is entire in \(v\), unless \(r = (0, 0, \ldots, 0)\). In this exceptional case \(c_0 = N^{1-n} L(\nu-n/2+1, \psi)\), which is entire for all nontrivial characters \(\psi\), and has a simple pole at \(\nu = n/2\) when \(\psi\) is trivial. This establishes the asserted meromorphic continuation of the restriction of the Eisenstein series \(E_{\nu,\psi}\) to the open Schubert cell \(w_{\text{long}} U\). Since \(E_{\nu,\psi}\) is automorphic under \(\Gamma_0(N)\), and the \(\Gamma_0(N)\)-translates of \(w_{\text{long}} U\) cover \(Y = G/P\), the continuation is valid on all of \(Y\). Likewise, the identical meromorphic continuation applies to \(\tilde{E}_{\nu,\psi}\) because of (3.14).

We have now shown the analytic continuation of the mirabolic Eisenstein distributions. We next turn to their functional equations. The two degenerate principal series representations (3.6) are related by the standard intertwining operator

\[
I_\nu : W_{-\nu,\varepsilon}^\infty \longrightarrow \tilde{W}_{\nu,\varepsilon}^\infty,
\]

defined in terms of the realization by \(C^\infty\) functions by the integral

\[
(I_\nu f)(g) = \int_U f(g w_{\text{long}} u) \, du;
\]
recall the definition of \(w_{\text{long}}\) in (2.35). It is well known that the integral converges absolutely\(^{3}\) for \(\text{Re} \, \nu > n/2 - 1\), and we shall also see this directly. Two properties of \(I_\nu\) are crucial for our purposes:

a) \(I_\nu\) has a meromorphic continuation to all \(\nu \in \mathbb{C}\), and

b) it extends continuously to a linear operator \(I_\nu : \mathcal{W}_{-\nu,\varepsilon} \rightarrow \mathcal{W}_{-\nu,\varepsilon} ;\) see [12] for the former, and [3] for the latter.

We now give an explicit formula for the action of \(I_\nu\) in terms of the restriction of \(C^\infty\) functions to the open Schubert cells \(U \subset G/P\), \(\tilde{U} \subset G/\tilde{P}\), for \(\nu\) in the range of convergence – i.e., for \(\text{Re} \, \nu > n/2 - 1\).

**3.26. Proposition.** Let \(f \in \mathcal{W}_{-\nu,\varepsilon}\), and regard \(f\) as a function on \(U \cong \mathbb{R}^{n-1}\) via its restriction to \(U\) and the identification

\[
\mathbb{R}^{n-1} \ni x \mapsto u(x) = \begin{pmatrix} 1 & x_{n-1} & \cdots & x_1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \in U.
\]

Similarly, regard \(I_\nu f \in \mathcal{W}_{-\nu,\varepsilon}\) as a function on \(\tilde{U} \cong \mathbb{R}^{n-1}\) via the identification\(^4\)

\[
\mathbb{R}^{n-1} \ni y \mapsto \tilde{u}(y) = \begin{pmatrix} 1 & -y_1 & \cdots & 0 \\ 0 & 0 & \cdots & -y_{n-1} \end{pmatrix} \in \tilde{U}.
\]

Then, for \(\text{Re} \, \nu > n/2 - 1\), \((I_\nu f)(\tilde{u}(y))\) is given by the integral

\[
\int_{z \in \mathbb{R}^{n-1}} f(u(z)) \left| \sum_{j=2}^{n-1} y_j z_{n+1-j} - y_1 z_1 \right|^{\nu-n/2} \text{sgn}(\sum_{j=2}^{n-1} y_j z_{n+1-j} - y_1 z_1)^\varepsilon \, dz.
\]

**Proof:** By construction, the intertwining operator \(I_\nu\) is invariant under left translation by any \(g \in G\). To establish the assertion of the proposition, it therefore suffices to establish the integral expression for \(y = 0\), and then to check that it is compatible with translation from \(\tilde{u}(0) = e \mapsto \tilde{u}(y)\).

First the compatibility with translation. On the one hand, \((I_\nu f)(\tilde{u}(y)) = (\ell(\tilde{u}(-y)) (I_\nu f))(e) = (I_\nu \ell(\tilde{u}(-y)) f)(\tilde{u}(0))\); on the other,

\[
\int_{z \in \mathbb{R}^{n-1}} (\ell(\tilde{u}(-y)) f)(u(z)) \left| z_1 \right|^{\nu-n/2} \text{sgn}(-z_1)^\varepsilon \, dz = \int_{z \in \mathbb{R}^{n-1}} f(\tilde{u}(y) \cdot u(z)) \left| z_1 \right|^{\nu-n/2} \text{sgn}(-z_1)^\varepsilon \, dz.
\]

Since

\[
\tilde{u}(y) \cdot u(z) = \begin{pmatrix} 1 & z_{n-1} & \cdots & z_2 z_1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix},
\]

with \(z_1 = z_1 - y_1 + \sum_{2 \leq j \leq n-1} z_j y_{n+1-j}\), the transformation law (3.7) implies that the integral (3.27) coincides with the integral in the proposition.

---

\(^{3}\)For the sake of notational simplicity we are dropping the subscript \(\varepsilon\) for \(I_\nu\), since the action of the intertwining operator affects only \(\nu\), not \(\varepsilon\).

\(^{4}\)The minus signs are necessary to make (3.25) consistent with (3.14).
At this point, it suffices to treat the case $y = 0$. According to the definition of the intertwining operator,

$$(I_\nu f)(\bar{u}(0)) = \int_{z \in \mathbb{R}^{n-1}} f(w_{long} u(z)) \, dz$$

$$= \int_{z \in \mathbb{R}^{n-1}} f\left(u\left(\frac{1}{z_1}, \ldots, \frac{-z_{n-1}}{z_1}, \frac{1}{z_1}\right)\right) |z_1|^{-\nu-n/2} \text{sgn}(-z_1)^v \, dz$$

$$= \int_{z \in \mathbb{R}^{n-1}} f(u(z)) |z_1|^{-\nu-n/2} \text{sgn}(-z_1)^v \, dz;$$

at the second step, we have used the transformation law (3.7) and the matrix identity

$$w_{long} u(z) = \begin{pmatrix} 1 & -z_2/z_1 & \cdots & -z_{n-1}/z_1 & 1/z_1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 1/z_1 \end{pmatrix} \begin{pmatrix} -1/z_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ z_{n-1} & z_{n-2} & \cdots & 0 & 1 \end{pmatrix}, \quad (3.30)$$

and at the third step, the change of variables

$$(z_1, z_2, \ldots, z_{n-1}) \mapsto \left(\frac{1}{z_1}, \frac{-z_2}{z_1}, \ldots, \frac{-z_{n-1}}{z_1}, \frac{1}{z_1}\right). \quad (3.31)$$

The identity (3.29) completes the proof of the proposition. \qed

The identity (3.30) and the transformation law (3.7) directly imply a simple estimate: along the line \(x_2 = x_3 = \cdots = x_{n-1} = 0\), any \(f \in W_{\nu,\varepsilon}^\infty\) satisfies the bound \(|f(u(x))| = O(\|x\|^{-\Re \nu-n/2})\) as \(\|x\| \to \infty\); the implied constant depends on a bound for \(\ell(w_{long})f\) on a neighborhood of the origin. We consider \(SO(n-1)\) as a subgroup of \(GL(n)\) by embedding it into the bottom right corner. Then \(SO(n-1)\) acts transitively, by conjugation, on the set of lines in \(\mathbb{R}^{n-1} \cong U\). By compactness, the translates \(\ell(w_{long}m)f\), for \(m \in SO(n-1)\), are uniformly bounded on bounded subsets of \(\mathbb{R}^{n-1} \cong U\). Since \(f \in W_{\nu,\varepsilon}^\infty\) is invariant under right translation by elements of \(SO(n-1)\), the estimate we gave holds not on just a single line, but globally on \(U\):

$$f \in W_{\nu,\varepsilon}^\infty \implies \|f(u(x))\| = O(\|x\|^{-\Re \nu-n/2}) \text{ as } \|x\| \to \infty. \quad (3.32)$$

This bound and its derivation are valid for all \(\nu \in \mathbb{C}\). When \(\Re \nu > n/2 - 1\), it implies the convergence of the integral (3.29), both near the origin and at infinity. Since \(I_\nu\) is \(G\)-invariant, we have established that the integral (3.24) does converge for \(\Re \nu > n/2 - 1\) and any \(g \in G\), as was mentioned earlier.

In complete analogy to \(I_\nu : W_{\nu,\varepsilon}^\infty \to \tilde{W}_{\nu,\varepsilon}^\infty\) in (3.23, 3.24), one can define the operator \(\tilde{I}_\nu : \tilde{W}_{\nu,\varepsilon}^\infty \to W_{\nu,\varepsilon}^\infty\); this involves integrating over \(\tilde{U}\) instead of \(U\). Then \(I_\nu, \tilde{I}_\nu\) are dual to each other, in the sense that

$$\int_{\tilde{U}} I_\nu f_1(\bar{u}) \tilde{f}_2(\bar{u}) \, d\bar{u} = \int_{U} f_1(u) \tilde{I}_\nu \tilde{f}_2(u) \, du,$$

for all \(f_1 \in W_{\nu,\varepsilon}^\infty\) and \(\tilde{f}_2 \in \tilde{W}_{-\nu,\varepsilon}^\infty\); the integrals on the two sides implement the natural \(G\)-equivariant pairings between \(W_{\nu,\varepsilon}^\infty\) and \(W_{-\nu,\varepsilon}^\infty\), respectively \(W_{\nu,\varepsilon}^\infty\) and \(W_{\nu,\varepsilon}^\infty\). For \(\Re \nu > n/2 - 1\), i.e., when the integrals defining \(I_\nu\) and \(\tilde{I}_\nu\) converge, the identity follows from the explicit
formula for $I_\nu$ in proposition 3.26 and the analogous formula for $\tilde{I}_\nu$. Meromorphic continuation implies the identity for other values of $\nu$.

Since $I_\nu$ extends continuously to $I_{\nu}:W_{-\nu,\varepsilon}^{-}\to \tilde{W}_{\nu,\varepsilon}^{\infty}$, the identity (3.33) implies a concrete description of the effect of $I_{\nu}$ on distribution vectors,

$$\int_{U} I_\nu \tau(\bar{u}) \tilde{f}(u) du = \int_{U} \tau(u) \tilde{I}_\nu f(u) du,$$

for all $\tau \in W_{-\nu,\varepsilon}^{-}$, $\tilde{f} \in \tilde{W}_{\nu,\varepsilon}^{\infty}$.

Unlike in (3.33), the integrals in this identity have merely symbolic meaning: the pairings $\tilde{W}_{\nu,\varepsilon}^{\infty} \times \tilde{W}_{\nu,\varepsilon}^{\infty} \to \mathbb{C}$ and $W_{-\nu,\varepsilon}^{-} \times W_{\nu,\varepsilon}^{\infty} \to \mathbb{C}$ involve “integration” over $\tilde{Y} = G/P$ and $Y = G/P$, not only over the dense open cells $\tilde{U} \subset \tilde{Y}$, $U \subset Y$. The integrals as written do extend naturally to $\tilde{Y}$ and $Y$.

Let $E_{1,n} \in \mathfrak{g}(n, \mathbb{R})$ denote the matrix with the entry 1 in the $(1,n)$-slot, and zero entries otherwise. If $f \in W_{-\nu,\varepsilon}^{-}$ and $\Re \nu > 1 - n/2$, the estimate (3.32) shows that the integrals

$$J_{\nu} f(g) = \text{def} \int_{\mathbb{R}} f(g \exp(t E_{1,n})) dt \quad (f \in W_{\nu}^{\infty}, \ g \in G)$$

converge. For other values of $\nu$, $\nu \notin 1 - n/2 - \mathbb{Z}_{\geq 0}$, the integrals still make sense by meromorphic continuation (the unspecified integer in $\mathbb{Z}_{\geq 0}$ in fact has the same parity as $\varepsilon$ at any singularity). This can be seen by translating the point $\lim_{t \to \infty} \exp(t E_{1,n}) P \in Y$ to the origin.

3.36. Lemma. Suppose $I_{\nu}:W_{-\nu,\varepsilon}^{-} \to \tilde{W}_{\nu,\varepsilon}^{\infty}$ has no pole at $\nu$, $W_{\nu,\varepsilon}^{\infty}$ and $\tilde{W}_{\nu,\varepsilon}^{\infty}$ are irreducible, and $\nu \notin 1 - n/2 - \mathbb{Z}_{\geq 0}$. Then for any $f \in W_{\nu,\varepsilon}^{\infty}$, the integrals $J_{\nu} f(u)$ vanish for all $u \in U$ if and only if $I_{\nu} f \in \tilde{W}_{\nu,\varepsilon}^{\infty}$, viewed as $C^{\infty}$ section of the line bundle $\tilde{E}_{\nu-\nu,\varepsilon} \to \tilde{Y}$, vanishes on the entire complement of $\tilde{U}$ in $\tilde{Y}$.

Both representations are generically irreducible, and $I_{\nu}$ depends meromorphically on $\nu$, so the hypotheses are satisfied outside a discrete set of values of the parameter $\nu$. The automorphism (2.35) preserves the one parameter group $t \mapsto \exp(t E_{1,n})$. Since this automorphism switches the roles of $I_{\nu}$ and $\tilde{I}_{\nu}$, $W_{\nu,\varepsilon}^{\infty}$ and $\tilde{W}_{\nu,\varepsilon}^{\infty}$, etc., the lemma applies analogously to $\tilde{I}_{\nu}$.

The explicit formula for $I_{\nu} f$ – for $f \in C^{\infty}(U)$, so that convergence is not an issue – shows that $I_{\nu}$ cannot vanish. Because of the other hypotheses of the lemma, $I_{\nu}$ must then be one-to-one and have dense image. But the image is necessarily closed [6], hence in the situation of the lemma,

$$I_{\nu}:W_{-\nu,\varepsilon}^{-} \to \tilde{W}_{\nu,\varepsilon}^{\infty}$$

is a topological isomorphism.

3.37. Proof of Lemma 3.36. The $J_{\nu} f(u)$ depend meromorphically on $\nu$, provided $f \in W_{-\nu,\varepsilon}^{-}$ varies meromorphically with $\nu$. Evaluation of $I_{\nu} f$ at any particular point is also a meromorphic function of $\nu$. Thus, without loss of generality, we may suppose

$$\Re \nu \gg 0.$$  (3.38)

We shall relate $I_{\nu}$ and $J_{\nu}$ to the $GL(n-1)$-analogue of $I_{\nu}$. This requires a temporary change in notation: in this proof we write $W_{n,\nu}^{\infty}$, $I_{n,\nu}$, etc., to signify the dependence on $n$ (we omit the subscript $\varepsilon$ since it is fixed and does not play an essential role).
We define
\[ R_{n,\nu} : \tilde{W}_{n,\nu}^\infty \rightarrow \tilde{W}_{n-1,\nu-1/2}^\infty, \]
\[ (R_{n,\nu} \tilde{f})(g_1) = |\det g_1|^{n/2-\nu} \tilde{f} \begin{pmatrix} 0 & 1_{(n-1)\times(n-1)} & 0 \\ 0 & 1 & \cdots \\ 0 & 0 & \vdots \\ 0 & 0 & 0 & g_1 \end{pmatrix} \] \[ (\text{3.39}) \]
the fractional power of $|\det g_1|$ is necessary to relate the transformation law $\tilde{f} \in \tilde{W}_{n,\nu}^\infty$ to that for $R_{n,\nu} \tilde{f} \in \tilde{W}_{n-1,\nu-1/2}^\infty$. The first matrix factor in the argument of $\tilde{f}$ makes this restriction operator $GL(n-1)$-invariant relative to the tautological action on $\tilde{W}_{n-1,\nu-1/2}^\infty$ and the action on $\tilde{W}_{n,\nu}^\infty$ via the embedding $GL(n-1) \hookrightarrow GL(n)$ into the top left corner. This top left copy of $GL(n-1)$ acts transitively on the complement of $\tilde{U}$ in $\tilde{Y}$, hence
\[ R_{n,\nu} \tilde{f} \equiv 0 \iff \tilde{f} \text{ vanishes on the complement of } \tilde{U} \text{ in } \tilde{Y}. \] \[ (\text{3.40}) \]
Next we define
\[ A_{n,\nu} : W_{n,\nu}^\infty \rightarrow W_{n-1,\nu+1/2}^\infty, \]
\[ (A_{n,\nu} f)(g_1) = |\det g_1|^{n/2+\nu} J_{n,\nu} f \begin{pmatrix} g_1 & 0 \\ 0 & 1 \end{pmatrix}. \] \[ (\text{3.41}) \]
In this case, the power of $|\det g_1|$ reflects not only the discrepancy between the transformation laws \[ (\text{3.4}) \] for $n$ and $n-1$, but also the commutation of the appropriate factor across $\exp(tE_{1,1})$ in the defining relation \[ (\text{3.35}) \] for $J_\nu$. It is clear from the definition that $A_{n,\nu}$ relates the tautological action of $GL(n-1)$ on $W_{n-1,\nu+1/2}^\infty$ to that on $W_{*,\nu}^\infty$ via the embedding $GL(n-1) \hookrightarrow GL(n)$ into the top left corner.
We claim:
\[ A_{n,\nu} f \equiv 0 \iff J_{n,\nu} f(u) = 0 \text{ for all } u \in U. \] \[ (\text{3.42}) \]
Indeed, since $U$ is dense in $G/P$, $f$ vanishes identically if and only if $f$ vanishes on $U$.

We use the analogous assertion about $A_{\nu} f$, coupled with the following observation: let $U_1$ denote the intersection of $U$ with the image of $GL(n-1) \hookrightarrow GL(n)$; then $U_1 \cdot \{ \exp(tE_{1,1}) \} = U$.

The intertwining operators $I_{n,\nu}$, $I_{n-1,\nu-1/2}$ and the operators we have just defined constitute the four edges of a commutative diagram,
\[ \begin{array}{ccc}
W_{n,\nu}^\infty & \xrightarrow{I_{n,\nu}} & \tilde{W}_{n,\nu}^\infty \\
A_{n,\nu} \downarrow & & \downarrow R_{n,\nu} \\
W_{n-1,\nu+1/2}^\infty & \xrightarrow{I_{n-1,\nu-1/2}} & \tilde{W}_{n-1,\nu-1/2}^\infty.
\end{array} \] \[ (\text{3.43}) \]
The commutativity is a consequence of two matrix identities. The first,
\[ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1_{x_{n-1} \times 1} & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ 1 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, \] \[ (\text{3.44}) \]
implies a factorization of $I_{n,\nu}$ as the composition of $I_{n-1,\nu-1/2}$ with a certain intermediate operator, which involves an integration over the one parameter group.
\{\exp(t E_{2,n})\}$ instead of $\{\exp(t E_{1,n})\}$, as in the case of $J_\nu$. The second,
\[
\begin{pmatrix}
0 & 1_{(n-1)\times(n-1)} & 1 \\
0 & 0 & 0 \\
\otimes & \vdots \end{pmatrix}
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\end{pmatrix}
= (g_1 : 0)
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\end{pmatrix},
\]
relates this intermediate operator to $J_{n,\nu}$.

Under the hypotheses of the lemma $I_{n,\nu}$ is an isomorphism – recall (3.37). One can show that under the same hypotheses $I_{n-1,\nu-1/2}$ is also an isomorphism. Alternatively one can use the meromorphic dependence on $\nu$ to disregard the discrete set on which $I_{n-1,\nu-1/2}$ might fail to be an isomorphism. In any case, when both $I_{n-1,\nu-1/2}$ and $I_{n-1,\nu-1/2}$ are isomorphisms, (3.40), (3.42), and the commutativity of the diagram (3.43) imply the assertion of the lemma. □

The functional equation of the mirabolic Eisenstein series relates $E_{-\nu,\psi}$ to $\tilde{E}_{\nu,\psi-1}$ via the intertwining operator $L_\nu : W_{-\nu,\psi} \to \tilde{W}_{\nu,\psi}$. For the statement, we follow the notational convention
\[
G_\delta(s) = \int_{\mathbb{R}} e(x) (\text{sgn}(x))^\delta |x|^{s-1} \, dx = \begin{cases} 2(2\pi)^{-s} \Gamma(s) \cos \frac{\pi s}{2} & \text{if } \delta = 0 \\ 2(2\pi)^{-s} \Gamma(s) \sin \frac{\pi s}{2} & \text{if } \delta = 1 \end{cases} \tag{3.46}
\]
[47], which we shall also use later in this paper. Note that the integral converges, conditionally only, for $0 < \Re s < 1$, but the expression on the right provides a meromorphic continuation to the entire $s$-plane. The two cases on the right hand side of (3.46) can be written uniformly using $\Gamma$-function identities as
\[
G_\delta(s) = \iota^\delta \frac{\Gamma_\Re(s + \delta)}{\Gamma_\Re(1 - s + \delta)}, \quad \text{with } \Gamma_\Re(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \text{ and } \delta \in \{0, 1\}. \tag{3.47}
\]
We also need some notation pertaining to the finite harmonic analysis of Dirichlet characters. Let $\tau_\psi = \hat{\psi}(1) = \sum_{b (\text{mod } N)} \hat{\psi}(b) e\left(\frac{b}{N}\right)$ denote the Gauss sum for $\psi$, a Dirichlet character of modulus $N$ (cf. (3.15)). We let $\mathcal{L}(\mathbb{Z}/N\mathbb{Z})^*$ denote the group of characters of $\mathbb{Z}/N\mathbb{Z}^*$ and $\phi(N)$, the Euler $\phi$-function, its order.

3.48. Proposition (Functional Equation).

\[
I_\nu E_{-\nu,\psi} = \langle -1 \rangle^\nu N^{2\nu - \frac{\delta}{2} + \frac{1}{2}} G_\varepsilon(\nu - \frac{\delta}{2} + 1) \phi(N) \sum_{(\text{mod } N)} \hat{\psi}(a) \xi(a)^{-1} \ell(w_{\text{long}}) \ell \left(\begin{array}{c} N \\
N \end{array} \right)_1 \tilde{E}_{\nu,\xi}.
\]
Consequently, if $\psi$ is a primitive Dirichlet character of modulus $N$, then
\[
I_\nu E_{-\nu,\psi} = \langle -1 \rangle^\nu \tau_\psi N^{2\nu - \frac{\delta}{2} + \frac{1}{2}} G_\varepsilon(\nu - \frac{\delta}{2} + 1) \ell(w_{\text{long}}) \ell \left(\begin{array}{c} N \\
N \end{array} \right)_1 \tilde{E}_{\nu,\psi}.
\]
In particular
\[
I_\nu E_{-\nu,1} = G_0(\nu - \frac{\delta}{2} + 1) \tilde{E}_{\nu,1},
\]
where $1$ is the trivial Dirichlet character of conductor $N = 1$. 
Thus, applying the lemma in reverse, we find

$$\tilde{I}_\nu$$ converges. Because of (3.34), the proposition is equivalent to the equality

$$\frac{1}{(-1)^n N^{2\nu -\frac{n}{2}} + \frac{n}{2} + 1} \int_U E_{-\nu, \psi}(u) \tilde{I}_\nu \tilde{f}(u) du = \frac{1}{\phi(N)} \sum_{a \pmod{N}} \psi(a) \xi(a)^{-1} \int_{\tilde{U}} \ell(1_{n-1} N) \ell(w_{\text{long}}) \tilde{E}_{\nu, \xi}(\tilde{u}) \tilde{f}(\tilde{u}) d\tilde{u},$$  

(3.50)

for all $\tilde{f} \in \tilde{W}_\nu$. Both $E_{-\nu, \psi}$ and $\tilde{E}_{\nu, \xi}$ are invariant under congruence subgroups of $GL(n, \mathbb{Z})$, and $\tilde{I}_\nu : \tilde{W}_\nu \simeq \tilde{W}_\nu$ by (3.37). It therefore suffices to establish this equality when $I_\nu \tilde{f}$ has – necessarily compact – support in the open cell $U \subset Y$,

$$\text{supp}(\tilde{I}_\nu \tilde{f}) \text{ is compact in } U.$$  

(3.51)

We shall make one other assumption, namely

$$\int_R \tilde{I}_\nu \tilde{f}(u(x_1, x_2, \ldots, x_{n-1})) dx_1 = 0, \text{ for all } x_2, \ldots, x_{n-1} \in \mathbb{R}.$$  

(3.52)

Indeed, if (3.50) were to hold subject to the condition (3.52), the restriction to $\tilde{U}$ of the difference between $I_\nu E_{-\nu, \psi}$ and the formula we have asserted it is equal to could be expressed as a Fourier series

$$\sum_{r_2, \ldots, r_{n-1} \in \mathbb{Z}} a_{r_2, \ldots, r_{n-1}} e(r_2 y_2 + \cdots + r_{n-1} y_{n-1}),$$  

(3.53)

without dependence on $y_1$. But no such expression can be the restriction to $\tilde{U}$ of a distribution vector invariant under a congruence subgroup $\Gamma$: any generic $\gamma \in \Gamma$ will transform the expression (3.53) to a distribution that does depend non-trivially on $y_1$. This justifies the additional hypothesis (3.52).

In effect, the integrals (3.50) coincide with the integrals $J_{-\nu}(\tilde{I}_\nu \tilde{f})(u)$, as in (3.35), for $u \in U$. Consequently lemma (3.36) implies the vanishing of $I_{-\nu} \circ \tilde{I}_\nu \tilde{f}$ on the complement of $\tilde{U}$. But our hypotheses ensure that $I_{-\nu} \circ \tilde{I}_\nu$ is a multiple of the identity, so

$$\tilde{f} \text{ vanishes on the complement of } \tilde{U} \text{ in } \tilde{Y}.$$  

(3.54)

Having compact support in $U$, $\tilde{I}_\nu \tilde{f}$ surely vanishes on the complement of $U$ in $Y$. Thus, applying the lemma in reverse, we find

$$\int_R \tilde{f}(\tilde{u}(y_1, y_2, \ldots, y_{n-1})) dy_1 = 0, \text{ for all } y_2, \ldots, y_{n-1} \in \mathbb{R}.$$  

(3.55)

We shall also need the estimate

$$|P\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-1}}\right) \tilde{f}(\tilde{u}(y))| = O(||y||^{-\nu - n/2}) \text{ as } ||y|| \to \infty,$$  

(3.56)

for all constant coefficient differential operators $P\left(\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-1}}\right)$. It follows from (3.32), combined with the fact that the elements of the Lie algebra $\tilde{u}$ of $\tilde{U}$ act on
\( \tilde{W} \) by constant coefficient vector fields on \( \tilde{U} \cong \mathbb{R}^{n-1} \). In view of (3.49), (3.50) implies the decay of \( \tilde{f}(\tilde{u}(y)) \) and all its derivatives.

We compute the integral on the right hand side of (3.50) using the last restriction formula in proposition 3.16:

\[
\int_{\tilde{U}} \xi(1_{\tilde{U}} \cdot \tilde{u}) d\tilde{u} = \frac{N^{(1/2-\nu/n)(n-1)}}{N^{n-1}} \sum_{r \in \mathbb{Z}^{n-1}} \xi(d) d^{-\nu+n/2-1} \int_{\mathbb{R}^{n-1}} \tilde{f}(\tilde{u}(y)) e(r \cdot y) dy;
\]

(3.57)

here we have used the fact that \( (1_{\tilde{U}})^{-1} \tilde{u}(y) = \tilde{u}(Ny) (1_{\tilde{U}})^{-1} \), and the transformation law (3.6) to pull out the power of \( N \) in the numerator. The terms corresponding to \( r_1 = 0 \) have been dropped because of (3.55). The sum in (3.57) is absolutely convergent because of the derivative bound (3.56).

Let us now consider the finite sum over \( a \) and \( \xi \) to its left in (3.50). By orthogonality of characters

\[
\frac{1}{\phi(N)} \sum_{a \equiv (\text{mod } N)} \hat{\psi}(a) \xi(a)^{-1} \xi(d) = \begin{cases} 
0, & (d, N) > 1 \\
\hat{\psi}(d), & (d, N) = 1.
\end{cases}
\]

(3.58)

Therefore the right hand side of (3.50) is equal to

\[
N^{(1-n)(1/2+\nu/n)} \sum_{r \in \mathbb{Z}^{n-1}} \hat{\psi}(d) d^{-\nu+n/2-1} \int_{\mathbb{R}^{n-1}} \tilde{f}(\tilde{u}(y)) e(r \cdot y) dy.
\]

(3.59)

The compact support of \( \tilde{I}_\nu \tilde{f} \) and (3.52) imply the analogous expression for the integral on the other side of (3.50), but using the hybrid formula for the restriction of \( E_{-\nu, \psi} \) to \( U \) in proposition 3.16:

\[
\int_{\tilde{U}} E_{-\nu, \psi}(u) \tilde{I}_\nu f(u) du = N^{\nu-n/2} \sum_{v \in \mathbb{Z}^n} \hat{\psi}(v) v_1^{\nu-n/2+1} \times
\]

\[
\times \int_{\mathbb{R}^{n-1}} \tilde{I}_\nu \tilde{f}(\tilde{u}(x)) e(v_1 v_2 x_1) \delta_{v_n-1/v_1} (x_2) \cdots \delta_{v_2/v_1} (x_{n-1}) dx.
\]

(3.60)

It is important to note that this sum converges absolutely. Indeed,

\[
\sum_{v_2, \ldots, v_n \in \mathbb{Z}} \left| \int_{\mathbb{R}} \phi(x_1, v_2, \ldots, v_n) e(v_1 v_n x_1) dx_1 \right| \leq C v_1^{n-2} \sup_{x_2, \ldots, x_{n-1} \in \mathbb{R}} \left| \int_{\mathbb{R}} \phi(x_1, x_2, \ldots, x_{n-1}) e(v_1 v_n x_1) dx_1 \right|,
\]

(3.61)

for any \( \phi \in C^\infty_c(U) \) such as \( \phi = \tilde{I}_\nu \tilde{f} \), with \( C \) depending only on the diameter of the support of \( \phi \); the supremum on the right decays faster than any negative power of \( |v_1 v_n| \).
In view of (3.59) and (3.60), a notation change reduces (3.50) to the following assertion: under the hypotheses (3.49) and (3.51–3.52),

\[
(-1)^{c} G_{c}(\nu - \frac{n}{2} + 1) \sum_{r \in \mathbb{Z}^{n-1}} \hat{\psi}(d) d^{n/2 - \nu - 1} \int_{\mathbb{R}^{n-1}} \tilde{f}(\bar{u}(y)) e\left( \sum_{j} d r_{j} y_{j} \right) dy
\]

\[
= \sum_{d > 0 \atop k \neq 0} \hat{\psi}(d) k^{\nu - n/2 + 1} \sum_{r_{2}, \ldots, r_{n-1} \in \mathbb{Z}} \int_{\mathbb{R}} I_{\nu} \tilde{f}(x_{1}, \frac{r_{1}}{k}, \ldots, \frac{r_{n-1}}{k}) e(dx_{1}) dx_{1}.
\]

(3.62)

The explicit formula for \( I_{\nu} \) in proposition 3.26 – or more accurately, the analogous formula for \( I_{\nu} \) – implies

\[
\int_{\mathbb{R}} I_{\nu} \tilde{f}(x_{1}, \frac{r_{1}}{k}, \ldots, \frac{r_{n-1}}{k}) e(dx_{1}) dx_{1} =
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \tilde{f}(\bar{u}(z)) e(dx_{1}) |\sum_{j \geq 2} \frac{r_{j} z_{n+1-j}}{k} - z_{1} - x_{1}|^{\nu - n/2} \times
\]

\[
\times \text{sgn}(\sum_{j \geq 2} \frac{r_{j} z_{n+1-j}}{k} - z_{1} - x_{1})^{\epsilon} dz \, dx_{1}
\]

\[
= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \tilde{f}(\bar{u}(z)) e(dx_{1}) + d \sum_{j \geq 2} r_{j} z_{n+1-j} - dk z_{1}) \times
\]

\[
\times |x_{1}|^{\nu - n/2} \text{sgn}(-x_{1})^{\epsilon} dz \, dx_{1}
\]

(3.63)

\[
= \int_{\mathbb{R}} |x_{1}|^{\nu - n/2} \text{sgn}(-x_{1})^{\epsilon} e(dx_{1}) \times
\]

\[
\times \int_{\mathbb{R}^{n-1}} \tilde{f}(\bar{u}(z)) e(-dk z_{1} + d \sum_{j \geq 2} r_{j} z_{j}) dz.
\]

The change of variables \( x_{1} \mapsto x_{1} - z_{1} + d^{-1} \sum r_{j} z_{n+1-j} \) at the second step depends on interchanging the order of the two integrals. The \( z \)-integral is an ordinary, convergent integral, whereas the \( x_{1} \)-integral is that of a distribution against a \( C^\infty \) function. It can be turned into an ordinary, convergent integral by repeated integration by parts near \( x_{1} = \infty \) to bring down the real part of the exponent \( \nu - n/2 \).

Away from infinity the \( x_{1} \)-integral already is an ordinary convergent integral since \( \text{Re} \nu > 0 \); the two phenomena must be separated by a suitable cutoff function. Our paper \([17]\) describes these techniques in detail. They apply equally to the evaluation of the integral

\[
\int_{\mathbb{R}} |x_{1}|^{\nu - n/2} \text{sgn}(-x_{1})^{\epsilon} e(dx_{1}) dx_{1} = (-1)^{\epsilon} |dk|^{n/2 - \nu - 1} G_{c}(\nu - \frac{n}{2} + 1),
\]

(3.64)

reducing it to (3.46) in the convergent range. Identifying \( k \) with \( r_{1} \) and summing over \( d > 0 \) and \( r \in \mathbb{Z}^{n-1}, r_{1} \neq 0 \), gives the identity (3.62), and hence completes the proof.

The parameter \( \nu \) is natural from the representation theoretic point of view. In applications to functional equations, we set

\[
\nu = ns - \rho_{\text{mir}} = ns - 1/2,
\]

(3.65)

which has the effect of translating the symmetry \( \nu \mapsto -\nu \) into \( s \mapsto 1 - s \).
4. Pairing of Distributions

In this section we discuss some pairings of automorphic distributions that were constructed in [22], and how the analytic continuation and functional equations of Eisenstein distributions carry over to these pairings. In some cases the pairings can be computed as a product of shifts of the functions $G_{\delta}$ defined in (3.46), times certain $L$-functions. This gives a new construction of these $L$-functions, and a new method to directly study their analytic properties. In particular the results here are used crucially in our forthcoming paper [23] to give new results about the analytic continuation that were not available by the two existing methods, the Rankin-Selberg and Langlands-Shahidi methods.

We begin with a discussion of the distributional pairings in [22], though not in the same degree of generality as in that paper. We consider the semidirect product $G \cdot U$ of a real linear group $G$ with a unipotent group $U$. We suppose that $G \cdot U$ acts on flag varieties or generalized flag varieties $Y_j$ of real linear groups $G_j$, $1 \leq j \leq r$, in each case either by an inclusion $G \cdot U \hookrightarrow G_j$, or via $G \hookrightarrow G_j$ composed with the quotient map $G \cdot U \to G$. Then $G \cdot U$ acts on the product $Y_1 \times \cdots \times Y_r$. We suppose further that $G \cdot U$ has an open orbit $\mathcal{O} \subset Y_1 \times \cdots \times Y_r$, and at points of $\mathcal{O}$ the isotropy subgroup of $G \cdot U$ coincides with $Z_{G_j} = \text{center of } G_j$, (4.1)

so that $\mathcal{O} \cong (G \cdot U)/Z_G$, and that

the conjugation action of $G$ on $U$ preserves Haar measure on $U$. (4.2)

We let $\Gamma \subset G$, $\Gamma_U \subset U$, $\Gamma_j \subset G_j$ denote arithmetically defined subgroups such that $\Gamma \cdot \Gamma_U \hookrightarrow \Gamma_1 \times \cdots \times \Gamma_r$.

Our theorem also involves automorphic distributions $\tau_j \in C^{-\infty}(Y_j, L_j)_{\Gamma_j}$, in other words, $\Gamma_j$-invariant distribution sections of $G_j$-equivariant $C^\infty$ line bundles $L_j \to Y_j$, $1 \leq j \leq r$. The exterior tensor product

$L_1 \boxtimes \cdots \boxtimes L_r \to Y_1 \times \cdots \times Y_r$ (4.3)

restricts to a $G \cdot U$-equivariant line bundle over $\mathcal{O} \cong (G \cdot U)/Z_G$. If the isotropy group $Z_G$ acts trivially on the fiber of $L_1 \boxtimes \cdots \boxtimes L_r$ at points of $\mathcal{O}$, (4.4)

as we shall assume from now on, the restriction of the line bundle (4.3) to the open orbit $\mathcal{O}$ is canonically trivial. We can then regard

$$\tau = \text{restriction of } \tau_1 \boxtimes \cdots \boxtimes \tau_r \text{ to } \mathcal{O}$$ (4.5)

as a scalar valued distribution on $(G \cdot U)/Z_G$, a $\Gamma \cdot \Gamma_U$-invariant distribution, since the $\tau_j$ are $\Gamma_j$-invariant:

$$\tau \in C^{-\infty}((\Gamma \cdot \Gamma_U)/(G \cdot U)/Z_G).$$ (4.6)

As the final ingredient, we fix a character $\chi : U \to \{z \in \mathbb{C}^* \mid |z| = 1\}$ such that $\chi(gug^{-1}) = \chi(u)$ for all $g \in G$, $u \in U$, and $\chi(\gamma) = 1$ for all $\gamma \in \Gamma_U$. (4.7)

Since $\Gamma_U \backslash U$ is compact,

$$\left\{ g \mapsto \int_{\Gamma_U \backslash U} \chi(u) \tau(u g) \, du \right\} \in C^{-\infty}(\Gamma \backslash G/Z_G)$$ (4.8)
is a well defined distribution on $G/Z_G$ – a $\Gamma$-invariant scalar valued distribution because of (4.11). Finally, we require that at least one of the $\tau_i$ is cuspidal.

4.10. Theorem. [22 Theorem 2.29] Under the hypotheses just stated, for every test function $\phi \in C^\infty_c(G)$, the function

$$g \mapsto F_{\tau,\chi,\phi}(g) = \int_{h \in G} \int_{\Gamma_U \backslash U} \chi(u) \tau(ugh) \phi(h) \, du \, dh$$

is a well defined $C^\infty$ function on $G/Z_G$, invariant on the left under $\Gamma$. This function is integrable over $\Gamma \backslash G/Z_G$, and the resulting integral

$$P(\tau_1, \ldots, \tau_r) = \int_{\Gamma \backslash G/Z_G} \int_{h \in G} \int_{\Gamma_U \backslash U} \chi(u) \tau(ugh) \phi(h) \, du \, dh \, dg$$

does not depend on the choice of $\phi$, provided $\phi$ is normalized by the condition $\int_G \phi(g) \, dg = 1$. The $r$-linear map $(\tau_1, \ldots, \tau_r) \mapsto F_{\tau,\chi,\phi} \in L^1(\Gamma \backslash G/Z_G)$ is continuous, relative to the strong distribution topology, in each of its arguments, and relative to the $L^1$ norm on the image. If any one of the $\tau_j$ depends holomorphically on a complex parameter $s$, then so does $P(\tau_1, \ldots, \tau_r)$.

At first glance, the hypothesis (4.1) does not seem to include the hypothesis (2.4b) in [22]. However, since $Z_G$ acts trivially on the orbit $\mathcal{O}$, the hypothesis (2.4b) does hold if we replace $G$ by its derived group. Thus, instead of integrating over $\Gamma \backslash G/Z_G$, we could integrate over $(\Gamma \cap [G,G]) \backslash [G,G]/Z_G$. The hypotheses (4.1)[2][2] are therefore sufficient to apply the results of [22].

We shall now describe two interesting cases of this pairing that both involve a similar setup of flag varieties and the mirabolic Eisenstein series as a factor. Because we shall work with more than one group and flag variety, we use subscripts: $G_k$ will denote $GL(k,\mathbb{R})$ and $X_k = G_k/B_k$ its flag variety; cf. (2.10)–(2.11). The Eisenstein distributions $E_{u,\psi}$ from (3.12) are $\Gamma_1(N)$-invariant sections of the line bundle $L_{u,\psi}$ over the generalized flag variety $Y_n \cong \mathbb{R}^{p_n-1}$. In addition to these series and representations $W_{\nu,\xi}$ and $\tilde{W}_{\nu,\xi}$, we also consider their products with the character $\text{sgn}(\det)^\eta$, $\eta \in \mathbb{Z}/2\mathbb{Z}$ (see the remark above (3.5)). Our two particular pairings depend crucially on the following geometric fact:

$G_n$ acts on $X_n \times X_n \times Y_n$ with a dense open orbit; the action on this open orbit is free modulo the center, which acts trivially. (4.11)

Indeed, the diagonal action of $G_n$ on $X_n \times X_n$ has a dense open orbit. At any point in the open orbit, the isotropy subgroup consists of the intersection of two opposite Borel subgroups – equivalently, a $G_n$-conjugate of the diagonal subgroup. That group has a dense open orbit in $Y_n$, and only $Z_n = \text{center of } G_n$ acts trivially.

In the first example, which represents the Rankin-Selberg $L$-function for automorphic distributions $\tau_1$, $\tau_2$ on $GL(n,\mathbb{R})$, the integer $r = 3$, $U = \{e\}$, $Y_1 = Y_2 = X_n$, and $Y_3 = \mathbb{R}^{p_n-1}$. We require both $\tau_1$ and $\tau_2$ to be cuspidal, but impose no such condition on $\tau_3$, which is taken to be the mirabolic Eisenstein distribution.

The second example, which represents the exterior square $L$-function of a cuspidal automorphic distribution $\tau$ on $GL(2n,\mathbb{R})$, involves a nontrivial unipotent group, has $r = 2$, and only a single cusp form $\tau_1 = \tau$ (the mirabolic Eisenstein distribution). The decomposition $\mathbb{R}^{2n} = \mathbb{R}^r \oplus \mathbb{R}^n$ induces embeddings

$$G_n \times G_n \hookrightarrow G_{2n}, \quad X_n \times X_n \hookrightarrow X_{2n}.$$ (4.12)
The translates of $X_n \times X_n$ under the abelian subgroup

$$U = \left\{ \begin{pmatrix} I_n & A \\ 0_n & I_n \end{pmatrix} \bigg| A \in M_{n \times n}(\mathbb{R}) \right\} \subset G_{2n} \tag{4.13}$$

sweep out an open subset of $X_{2n}$; moreover the various $U$-translates are disjoint, so that

$$U \times X_n \times X_n \hookrightarrow X_{2n}. \tag{4.14}$$

Let $\tau \in C^{-\infty}(X_{2n}, L_{\lambda-\rho,\delta})^\Gamma$ be a cuspidal automorphic distribution as in (2.19), and $du$ be the Haar measure on $U$ identified with the standard Lebesgue measure on $M_{n \times n}(\mathbb{R})$. The group of integral matrices $U(\mathbb{Z})$ lies in the kernel of the character

$$\theta : U \rightarrow \mathbb{C}^*, \quad \theta \left( \begin{pmatrix} I_n & A \\ 0_n & I_n \end{pmatrix} \right) = e(\text{tr} A), \tag{4.15}$$

and because $\Gamma \cap U(\mathbb{Z})$ has finite index in $U(\mathbb{Z})$, the integral

$$S_{\theta \tau} \overset{\text{def}}{=} \frac{1}{\text{covol}(\Gamma \cap U(\mathbb{Z}))} \int_{\Gamma \cap U(\mathbb{Z}) \backslash U} \theta(u) \ell(u) \tau \, du \in C^{-\infty}(X_{2n}, L_{\lambda-\rho,\delta}) \tag{4.16}$$

is well defined, even if $\Gamma$ is replaced by a finite index subgroup. It restricts to a distribution section of $L_{\lambda-\rho,\delta}$ over the image of the open embedding (4.14). As such, it is smooth in the first variable, since $\ell(u) S_{\theta \tau} = \theta(u)^{-1} S_{\theta \tau}$ for $u \in U$. We can therefore evaluate this distribution section at $e \in U$, and define

$$S_{\theta \tau} \big|_{X_n \times X_n} \in C^{-\infty}(X_n \times X_n, L_{\lambda-\rho,\delta}|_{X_n \times X_n})_{\Gamma_n} \tag{4.17}$$

Here $\Gamma_n$ is a congruence subgroup of $G_n(\mathbb{Z})$ whose diagonal embedding into $G_n \subset G_{2n}$ leaves $\tau$ invariant under the left action, and preserves $\Gamma \cap U(\mathbb{Z})$ by conjugation. The superscript signifies invariance under the diagonal action of $\Gamma_n$ on $X_n \times X_n$. This invariance is a consequence of the fact that conjugation by the diagonal embedding of any $\gamma \in \Gamma_n$ also preserves the character $\theta$ as well as $U$, without changing the measure.

We restrict the product of the $G_n$-equivariant line bundles $L_{\lambda-\rho,\delta}|_{X_n \times X_n}$ and $L_{\nu-\rho_{\min},\varepsilon} \rightarrow Y_n$ to the open orbit and pull it back to $G_n/Z_n$ ($Z_n = Z_{G_n} = \text{center of } G_n$), resulting in a $G_n$-equivariant line bundle $L \rightarrow G_n/Z_n$; $S_{\theta \tau} \cdot E_{\nu,\psi}$ is then a $\Gamma'$-invariant distribution section of $L$ for

$$\Gamma' = \Gamma_n \cap \Gamma_1(N). \tag{4.18}$$

The center $Z_n$ acts on the fibers of $L$ by the restriction to $Z_n$ of the character $\chi_{\lambda-\rho,\delta} \cdot \chi_{\nu-\rho_{\min},\varepsilon} \cdot \text{sgn}(\det)^\eta$, where $\eta \in \mathbb{Z}/2\mathbb{Z}$; recall (2.13) and (3.4), and note that $\chi_{\lambda-\rho,\delta}$ takes values on $Z_n$ via its diagonal embedding into $Z_{2n} \subset G_{2n}$. We shall assume that $Z_n$ lies in the kernel of $\chi_{\lambda-\rho,\delta} \cdot \chi_{\nu-\rho_{\min},\varepsilon} \cdot \text{sgn}(\det)^\eta$ – equivalently,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{2n} = 0, \quad \delta_1 + \delta_2 + \cdots + \delta_{2n} \equiv e + n \eta \pmod{2}. \tag{4.19}$$

The first of these conditions involves no essential loss of generality, since twisting an automorphic representation by a central character does not affect the automorphy. The character $\chi_{\nu-\rho_{\min},0}$ takes the value 1 on $Z_n$ regardless of the choice of $\nu$, hence (4.19) makes $L \rightarrow G_n/Z_n$ a $G_n$-equivariantly trivial line bundle. In this situation, $S_{\theta \tau} \cdot E_{\nu,\psi}$ becomes a $\Gamma'$-invariant scalar valued distribution on $G_n/Z_n$,

$$S_{\theta \tau} \cdot E_{\nu,\psi} \in C^{-\infty}(G_n/Z_n)^{\Gamma'}. \tag{4.20}$$

Theorem 4.10 applies to this specific setting and states...
4.21. Corollary ([22]). Under the hypotheses just stated, for every test function \( \phi \in C_c^\infty(G_n) \)

\[
P(\tau, E_{\nu, \psi}) = \int_{\Gamma \backslash G_n / Z_n} \int_{h \in G_n} (S\tau \cdot E_{\nu, \psi})(gh) \phi(h) dh \, dg
\]

does not depend on the choice of \( \phi \), provided \( \phi \) is normalized by the condition \( \int_{G_n} \phi(g) \, dg = 1 \). The function \( \nu \mapsto P(\tau, E_{\nu, \psi}) \) is holomorphic for \( \nu \in \mathbb{C} - \{n/2\} \), with at most a simple pole at \( \nu = n/2 \).

To make (4.20) concrete, we identify \( X_{2n} \cong G_{2n} / B_{2n} \), \( Y_n \cong G_n / P_n \) as before. We regard \( \tau \) and \( E_{\nu, \psi} \) as scalar distributions on \( G_{2n} \) and \( G_n \) respectively, with \( \tau \) left invariant under \( \Gamma \subseteq G_{2n}(\mathbb{Z}) \), transforming according to \( \chi_{\lambda - \rho, \delta} \) on the right under \( B_{2n} \), and \( E_{\nu, \psi} \) left invariant under \( \Gamma_1(N) \subseteq G_n(\mathbb{Z}) \), transforming according to \( \chi_{\nu - \rho_{\text{pair}}, \epsilon} \) on the right under \( P_n \). The averaging process (4.10) makes sense also on this level. When we choose \( f_1, f_2, f_3 \in G_n \) so that \( (f_1 B_n, f_2 B_n, f_3 P_n) \) lies in the open orbit, we obtain an explicit description of \( S\tau \cdot E_{\nu, \psi} \):

\[
S\tau \cdot E_{\nu, \psi}(g) = \frac{1}{\text{covol}(\Gamma \cap U(\mathbb{Z}))} \int_{\Gamma \cap U(\mathbb{Z}) \backslash U} \theta((u) \tau) \left( \begin{smallmatrix} g & \varepsilon \\ 0 & 1 \end{smallmatrix} \right) E_{\nu, \psi}(gf_3) \, du. \tag{4.22}
\]

We note that the \( f_j \) are determined up to simultaneous left translation by some \( f_0 \in G_n \) and individual right translation by factors in \( B_n \), respectively \( P_n \). Translating the \( f_j \) by \( f_0 \) on the left has the effect of translating \( S\tau \cdot E_{\nu, \psi} \) by \( f_0^{-1} \) on the right; it does not change the value of \( P(\tau, E_{\nu, \psi}) \) because the ambiguity can be absorbed by \( \phi \). Translating any one of the \( f_j \) on the right by an element of the respective isotropy group affects both \( S\tau \cdot E_{\nu, \psi} \) and \( P(\tau, E_{\nu, \psi}) \) by a multiplicative factor – a non-zero factor depending on \( (\lambda, \delta) \) in the case of \( f_1 \) or \( f_2 \), and the factor \( \chi_{\nu - \rho_{\text{pair}}, \epsilon}(p^{-1}) \) when \( f_3 \) is replaced by \( f_3 p, p \in P_n \).

One can eliminate the potential dependence on \( \nu \) in this factor by requiring \( f_3 \in U_n \); cf. (3.1). Specifically, in the following, we choose

\[
f_1 = I_n, \quad f_2 = \left( \begin{smallmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{smallmatrix} \right), \quad \text{and} \quad f_3 = \left( \begin{smallmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{n-1} \end{smallmatrix} \right), \tag{4.23}
\]

which do determine a point \( (f_1 B_n, f_2 B_n, f_3 P_n) \in X_n \times X_n \times Y_n \) lying in the open orbit. Note that \( f_3 \in U_n \) and \( f_2 = w_{\text{long}} \), in the notation of (2.35).

The pairing \( P(\tau, E_{\nu, \psi}) \) inherits a functional equation from that of \( E_{\nu, \psi} \), which involves the contragredient automorphic distribution \( \tilde{\tau} \) defined in (2.37). The argument we give below for it works mutatis mutandis to provide an analogous statement for the Rankin-Selberg pairing as well.
4.24. Proposition.

\[ P(\tau, E_{-\nu, \psi}) = \]
\[ (-1)^{\epsilon+\delta_{n+1}+\cdots+\delta_{2n}} N^{2\nu-\frac{n}{2}-\frac{1}{2}} \prod_{j=1}^{n} G_{\delta_{n+j}+\delta_{n+1-j}+\eta(\lambda_{n+j} + \lambda_{n+1-j} + \frac{\nu}{n} + \frac{1}{2})} \times \]
\[ \sum_{a \mod (N)} \frac{1}{\delta(N)} \hat{\psi}(a) \xi^{-1} \left( \ell \left( \begin{pmatrix} -w_{\text{long}} & w_{\text{long}} \\ \ell & 1 \end{pmatrix} \begin{pmatrix} N & I_{n-1} \\ I_{n-1} & 1 \end{pmatrix} \right) \right) \cdot \tilde{\tau}, E_{\nu, \xi} \right). \]

The pairings on the right hand side are integrations over the quotient \( \Gamma^* \backslash G_n / Z_n \), where

\[ \Gamma^* = w_{\text{long}} \left( \begin{pmatrix} N & I_{n-1} \\ I_{n-1} & 1 \end{pmatrix} \right) \tilde{\Gamma}' \left( \begin{pmatrix} N & I_{n-1} \\ I_{n-1} & 1 \end{pmatrix} \right)^{-1} \]

is the subgroup that \( \tilde{S}\tilde{T} \cdot E_{\nu, \xi} \) is naturally invariant under (cf. \( (4.18) \)). In the special case that \( \tau \) is invariant under \( GL(2n, \mathbb{Z}) \), \( N = 1 \), \( \psi = 1 \) is the trivial Dirichlet character, and \( \varepsilon \equiv \eta \equiv 0 \pmod{2} \), the relation simplifies to

\[ P(\tau, E_{\nu, 1}) = \]
\[ (-1)^{\delta_1+\cdots+\delta_n} \prod_{j=1}^{n} G_{\delta_{n+j}+\delta_{n+1-j}+\eta(\lambda_{n+j} + \lambda_{n+1-j} - \frac{\nu}{n} + \frac{1}{2})} P(\tilde{\tau}, E_{-\nu, 1}). \] (4.26)

A similar formula using the second displayed line in proposition 3.48 of course also gives a simplified functional equation when \( \psi \) is primitive, though we will not need to use this formula in what follows.

Proof: In analogy to \( \tilde{S}\tilde{T} \cdot E_{\nu, \psi} \) in (4.22), one can define a product \( \tilde{S}\tilde{\tau} \cdot \bar{\rho} \) of \( \tilde{S}\tilde{\tau} \) and any distribution section \( \bar{\rho} \) of \( \mathcal{L}_{-\nu, -\rho_{\text{mir}}, \varepsilon} \rightarrow \tilde{Y} \) as

\[ S\tilde{T} \cdot \bar{\rho}(g) = \]
\[ \frac{1}{\text{covol}(\Gamma \cap U(\mathbb{Z}))} \int_{\Gamma \cap U(\mathbb{Z})} \theta(u) \left( \ell(u) \begin{pmatrix} g \tilde{f}_2 & 0_n \\ 0_n & g \tilde{f}_1 \end{pmatrix} \right) \bar{\rho}(g \tilde{f}_3) \, du. \] (4.27)

Here we have applied the outer automorphism (2.35) to the base points \( f_1 B_n, f_2 B_n, f_3 P_n \), and also switched the order of the two factors \( X_n \). This choice of base points is in effect only when we multiply \( \tilde{S}\tilde{T} \), or \( \bar{\rho} \), by a section of \( \mathcal{L}_{-\nu, -\rho_{\text{mir}}, \varepsilon} \rightarrow \tilde{Y} \) such as \( \tilde{E}_{\nu, \xi} \) or \( I_{\nu} E_{-\nu, \psi} \), rather than by \( E_{-\nu, \psi} \); it is used internally in this proof, but not elsewhere in the paper.

Though corollary 4.21 as stated does not apply to (4.27) when \( \bar{\rho} = \tilde{E}_{\nu, \xi} \) or \( I_{\nu} E_{-\nu, \psi} \), its conclusions apply so long as \( \Gamma' \) is appropriately modified to take into account the invariance group of \( \bar{\rho} \). This can be seen either as a consequence of the general statement theorem 4.10, or alternatively deduced directly from corollary 4.21 using the outer automorphism (2.35). Let \( \phi \in C_\infty^\infty(G_n) \) have \( \int_{G_n} \phi(h) \, dh = 1 \). The proof of the proposition involves computing the integral

\[ \mathcal{I} = \int_{\Gamma^* \backslash G_n / Z_n} \int_{G_n} (S\tilde{T} \cdot I_{\nu} E_{-\nu, \psi})(gh) \phi(h) \, dh \, dg \] (4.28)
in two different ways. The first involves inserting the formula for \( I, E_{-\nu, \psi} \) from proposition 3.48 obtaining

\[
I = (-1)^v N^{2v-1} \tilde{\tau} \cdot \mathcal{G}_\xi (\nu - \frac{n}{2} + 1) \frac{1}{\phi_{\text{Euler}}(N)} \times \\
\sum_{a \pmod{N}} \tilde{\psi}(a) \xi(a)^{-1} \int_{I'(\mathbb{Z})} \int_{I'(\mathbb{Q})} \int_{I'(\mathbb{Z})} \theta(u) \times \\
\times \left( \ell(u) \tau \left( \begin{array}{cc} gh_2 & 0 \\ 0 & gh_f \end{array} \right) \right) \tilde{E}_v \xi \left( w_{\text{long}} \left( I_{n-1}^{-1} \right) gh_f \right) \frac{1}{N} \phi(h) \frac{dh}{dg}.
\]

(4.29)

(we have denoted the Euler \( \phi \)-function as \( \phi_{\text{Euler}} \) here in order to avoid confusing it with the smooth function \( \phi \) in the integrand). The integral can be written as

\[
\frac{1}{\text{covol}(\Gamma \cap U(\mathbb{Z}))} \int_{I'(\mathbb{Z})} \int_{I'(\mathbb{Q})} \int_{I'(\mathbb{Z})} \theta(u) \times \\
\times \left( \ell(u) \tau \left( \begin{array}{cc} gh_2 & 0 \\ 0 & gh_f \end{array} \right) \right) \tilde{E}_v \xi \left( w_{\text{long}} \left( I_{n-1}^{-1} \right) gh_f \right) \frac{1}{N} \phi(h) \frac{dh}{dg}.
\]

(4.30)

We now change variables \( g \mapsto \tilde{g}, h \mapsto \tilde{h} \), and then apply identities (2.36) and (3.11), after which we must replace \( \Gamma' \) by \( \Gamma' \): the integral becomes

\[
\frac{1}{\text{covol}(\Gamma \cap U(\mathbb{Z}))} \int_{I'(\mathbb{Z})} \int_{I'(\mathbb{Q})} \int_{I'(\mathbb{Z})} \theta(u) \times \\
\times \left( \tilde{\tau} \left( \begin{array}{cc} g_h & 0 \\ 0 & g_h \end{array} \right) \right) \tilde{E}_v \xi \left( w_{\text{long}} \left( N^{I_{n-1}} \right) gh_f \right) \frac{1}{N} \phi(\tilde{h}) \frac{d\tilde{h}}{dg}.
\]

(4.31)

The above expression is unchanged if both instances of \( \Gamma \) are replaced by any finite index subgroup, in particular the principal congruence subgroup \( \Gamma(m) = \{ \gamma \in G_{2n}(\mathbb{Z}) | \gamma \equiv I_{2n} \pmod{m} \} \) for some \( m \) (and hence any positive multiple of it). The change of variables \( u \mapsto \tilde{u}^{-1} = w_{\text{long}} \tilde{u} w_{\text{long}} \) preserves \( \Gamma(m), U(\mathbb{Z}), U, \) the character \( \theta \), and the Haar measure \( du; \) it allows us to rewrite (4.31) as

\[
\frac{1}{\text{covol}(\Gamma(m) \cap U(\mathbb{Z}))} \int_{I'(\mathbb{Z})} \int_{I'(\mathbb{Q})} \int_{I'(\mathbb{Z})} \theta(u) \times \\
\times \left( \tilde{\tau} \left( \begin{array}{cc} g_h & 0 \\ 0 & g_h \end{array} \right) \right) \tilde{E}_v \xi \left( w_{\text{long}} \left( N^{I_{n-1}} \right) gh_f \right) \frac{1}{N} \phi(\tilde{h}) \frac{d\tilde{h}}{dg}.
\]

(4.32)

We may freely replace \( \phi(h) \) with \( \phi(\tilde{h}) \) because corollary 4.21 guarantees that this substitution of smoothing functions does not affect the overall value. Since

\[
\tilde{\tau} \left( \begin{array}{cc} I_n & A \\ 0_n & I_n \end{array} \right) \left( \begin{array}{cc} g_1 & 0 \\ 0 & g_2 \end{array} \right) = \tilde{\tau} \left( \begin{array}{cc} -I_n & 0 \\ 0 & I_n \end{array} \right) \left( \begin{array}{cc} A & 0 \\ 0 & I_n \end{array} \right) \left( \begin{array}{cc} -I_n & 0 \\ 0 & I_n \end{array} \right)^{-1} \left( \begin{array}{cc} g_1 & 0 \\ 0 & g_2 \end{array} \right) \right)
\]

(4.33)

we may replace \( u \) in the argument of \( \tilde{\tau} \) by \( u^{-1} \), so long as we left translate it by \( -I_{n-1} \) and multiply the overall expression by \( (-1)^{\delta_{n+1} + \cdots + \delta_{2n}} \). Replacing \( g \)

by \( g \mapsto \left( N^{I_{n-1}} \right)^{-1} w_{\text{long}} g \) converts \( \Gamma' \) into \( \Gamma^* \), and nearly converts (4.32) into

\[
(-1)^{\delta_{n+1} + \cdots + \delta_{2n}} P \left( \tilde{\tau} \left( \begin{array}{cc} -w_{\text{long}} & \left( N^{I_{n-1}} \right) \end{array} \right) \right) \tilde{\tau}, \tilde{E}_v \xi \left( w_{\text{long}} \left( N^{I_{n-1}} \right) \right) \frac{1}{N} \phi(\tilde{h}) \frac{d\tilde{h}}{dg}.
\]

(4.34)

the only difference is that the \( u \)-integration is changed by the presence of these two matrices that left-translate \( \tilde{\tau} \). The compensating change of variables in \( u \) that undoes this conjugation preserves the character \( \theta \), but alters \( \Gamma(m) \) because some non-diagonal entries are multiplied or divided by \( N \). Were \( m \) replaced by \( mN \) in
this conjugate would still be a subgroup of $\Gamma$, and hence its normalized $u$-integration would have the same value. We conclude that

$$I = (-1)^{c + \delta_{n+1} + \cdots + \delta_n} N^{2\nu - \frac{n}{2} - \frac{N}{2}} G_\epsilon(\nu - \frac{n}{2} + 1) \frac{1}{\phi(N)} \times$$

$$\sum_{(a \mod N)} \hat{t}(a) \xi(a)^{-1} P \left( \ell \left( \left( \frac{w_{\text{long}}}{w_{\text{long}}} \right) \left( \begin{smallmatrix} I_n & \frac{1}{N} \\ N & I_n \end{smallmatrix} \right) \right) \right) \bar{\nu}, E_{\nu, \xi}. \quad (4.35)$$

The proof of the proposition now reduces to demonstrating that

$$I = \frac{G_\epsilon(\nu - \frac{n}{2} + 1)}{\prod_{j=1}^n G_{\delta_{n+j} + \delta_{n+1-j} + \eta}(\lambda_{n+j} + \lambda_{n+1-j} + \frac{\nu}{n} + \frac{1}{2})} P(\nu, E_{-\nu, \xi}) \quad (4.36)$$

By combining (4.27) and (4.28), $I$ can be written as

$$I = \frac{1}{\text{covol}(\Gamma \cap U(Z))} \int_{\Gamma \cap U(Z)} \int_{G_n / Z_n} \int_{\Gamma \cap U(Z) \setminus U} \theta(u) \times$$

$$\times \left( \ell(u) \tau \right) \left( \begin{smallmatrix} g_{h\ell}f_1 & 0_n \\ 0_n & g_{h\ell}f_2 \end{smallmatrix} \right) I_{\nu, E_{-\nu, \xi}}(g_{h\ell}f_3) \, du \, dh \, dg. \quad (4.37)$$

Right translating $h$ by $w_{\text{long}}$ converts $h_{f_1} = h$ to $hw_{\text{long}} = hf_2$, and $h_{f_2} = hw_{\text{long}}$ to $h = h_{f_1}$. It also changes $\phi(h)$ to $\phi(hw_{\text{long}})$; however, this change can be undone by replacing $\phi(g)$ with $\phi(gw_{\text{long}})$, as both functions have the same total integral over $G_n$. Hence $I$ can be expressed as

$$I = \frac{1}{\text{covol}(\Gamma \cap U(Z))} \int_{\Gamma \cap U(Z)} \int_{G_n / Z_n} \int_{\Gamma \cap U(Z) \setminus U} \theta(u) \times$$

$$\times \left( \ell(u) \tau \right) \left( \begin{smallmatrix} g_{h\ell}f_1 & 0_n \\ 0_n & g_{h\ell}f_2 \end{smallmatrix} \right) I_{\nu, E_{-\nu, \xi}}(ghw_{\text{long}}f_3) \, du \, dh \, dg. \quad (4.38)$$

We shall now use the definition (3.21) of the intertwining operator $I_{\nu}$. Since this involves an integral over the non-compact manifold $U_n$, it might seem that the formula cannot be applied to the distribution $E_{-\nu, \xi}$. However, the self-adjointness property (3.33) justifies the calculations we are about to present. In effect, the calculations with $E_{-\nu, \xi}$ reflect legitimate operations on the dual side. This is completely analogous to applying the calculus of differential operators to distributions as if they were functions. The duality depends on interpreting $\phi$ as a $C^\infty$ section of a line bundle over $X_n \times X_n \times Y_n$, the mirror image of viewing the distribution section $S_{\tau} \cdot E_{-\nu, \xi}$ as a scalar distribution on $G_n$. In effect, we interpret the $h$-integration as the pairing of a distribution section of one line bundle against a smooth section of the dual line bundle, tensored with the line bundle of differential forms of top degree, by integration over the compact manifold $X_n \times X_n \times Y_n$. In a slightly different setting, this process is carried out in the proof of lemma 3.9 in [22]. What matters is that $G_n$ acts on $X_n \times X_n \times Y_n$ with an open orbit. In any case, applying the definition (3.21) of $I_{\nu}$, the notation $u(x)$ in proposition 5.20 and

---

7Strictly speaking, we should work with a smoothing function $\phi \in C^\infty(G_n / Z_n)$ instead of $\phi \in C^\infty(G_n)$, but this makes little difference for the rest of the argument.
the definition \((2.35)\) of the automorphism \(g \mapsto \tilde{g}\), we find

\[
I, E_{-\nu, \psi}(gh u(x))
\]

\[
= \int_{\mathbb{R}^{n-1}} E_{-\nu, \psi}(gh f_3(x)) dx
= \int_{\mathbb{R}^{n-1}} E_{-\nu, \psi}(gh u) \left| 1 - \sum_j x_j \right|^{-\nu-n/2} \text{sgn}(1 + \sum_j x_j)^\nu dx
\]

the equality at the second step follows from the transformation law \((3.7)\) and the matrix identity

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
= \begin{pmatrix}
1 & x_{n-1} & \cdots & x_1 \\
0 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]

and the third step in \((4.39)\) from the change of coordinates \(x_j \mapsto x_j/(1 + \sum_j x_j)^{-1}\).

To ensure convergence of the integral – or rather, of the corresponding integral on the dual side – we suppose \(\text{Re } \nu > n/2 - 1\).

We now combine \((4.38)\) with \((4.39)\). The resulting expression for \(I\) involves four integrals: the integrals over \(\mathbb{R}^{n-1}\) and \((\Gamma \cap U(\mathbb{Z})) \backslash U\) on the inside – in either order, since they are independent – then the \(h\)-integral, and finally the integral over \(G_n(\mathbb{Z}) \backslash G_n/\mathbb{Z}_n\) on the outside. We claim that we can interchange the order of integration, to put the integration over \(\mathbb{R}^{n-1}\) on the outside.\(^8\) We can use partitions of unity to make the integrands for all the integrals have compact support. Then, using the definition of operations on distributions using the duality between distributions and smooth functions, the expression is converted into one for which Fubini’s theorem applies. In terms of our specific choice of flags \((4.23)\), this means

\[
I = \frac{1}{\text{covol}(\Gamma \cap U(\mathbb{Z}))} \int_{\mathbb{R}^{n-1}} \int_{\Gamma \cap U(\mathbb{Z}) \backslash U} \int_{G_n/\mathbb{Z}_n} \int_{\Gamma \cap U(\mathbb{Z}) \backslash U} \theta(u) \times
\]

\[
\times \left( \ell(u) \tau \left( \begin{pmatrix}
g & 0 \\
on & 0
\end{pmatrix} gh u(x) \right) \right) \left| 1 + \sum_j x_j \right|^{-\nu-n/2}
\]

\[
\times \text{sgn}(1 + \sum_j x_j)^\nu \phi(h) dh \, dg \, dx.
\]

Neglecting a set of measure zero, we may integrate over \((\mathbb{R}^*)^{n-1}\) instead of \(\mathbb{R}^{n-1}\).

For \(x \in (\mathbb{R}^*)^{n-1}\), \(u(x)\) is conjugate to \(f_3\) under the diagonal Cartan subgroup of \(G_n\),

\[
u(x) = a_x f_3 a_x, \quad \text{with } a_x = \begin{pmatrix} 1 & 0 \\ 0 & x_2 & \cdots & x_n \end{pmatrix}.
\]

\(^8\)The integration over \(U(\mathbb{Z}) \backslash U\) must remain on the inside; it is necessary to make sense of \(S_{\pi}^\gamma\) as a distribution section over \(X_n \times X_n\).
We now change variables to replace $h$ by $h a_x$. The identity

$$E_{-\nu, \psi}(g h a_x u(x)) = \prod_{j=1}^{n-1} x_j^{\nu - n/2} \text{sgn} \left( \prod_{j=1}^{n-1} x_j \right) E_{-\nu, \psi}(g h f_3).$$

(4.43)

follows from the transformation law [3,3] because the representation $W_{n, \delta}$ has been tensored by $\text{sgn}(\text{det}(-))^n$ (see the comments between (4.17) and (4.19)). Similarly the identity

$$w_{\text{long}}^{-1} a_x w_{\text{long}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

(4.44)

and the transformation law (2.4) imply

$$\left( \ell(u) \right) \begin{pmatrix} g h a_x & 0_n \\ 0_n & g h a_x w_{\text{long}} \end{pmatrix} = \prod_{j=1}^{n-1} \left[ x_j^{-\lambda_n + j - \lambda_{n+1-j} (\text{sgn} x_j) \delta_n + j + \delta_{n+1-j}} \right] \left( \ell(u) \right) \begin{pmatrix} g h & 0_n \\ 0_n & g h w_{\text{long}} \end{pmatrix}.$$  

(4.45)

Therefore these characters of the $x_j$ may be moved to the outermost integral in (4.11). The only remaining instance of $x$ in the inner three integrations is in the argument of the test function, $\phi(h a_x)$. By the same reasoning as before, $h \mapsto \phi(h a_x)$ has total integral one, just like $\phi$. Since these inner three integrations define the pairing $P(\tau, E_{-\nu, \psi})$, they depend only on this total integral, and hence their value is unchanged if $a_x$ is removed from the argument of $\phi$. The $x$-integral in (4.11) splits off to give

$$\mathcal{I} = \mathcal{H} \times P(\tau, E_{-\nu, \psi}),$$

(4.46)

with

$$\mathcal{H} = \int_{\mathbb{R}^{n-1}} \left| 1 + \sum_{j=1}^{n-1} x_j^{\nu - n/2} \text{sgn} \left( 1 + \sum_{j=1}^{n-1} x_j \right)^\varepsilon \times \prod_{j=1}^{n-1} \left[ x_j^{-\lambda_n + j - \lambda_{n+1-j} - \nu/n} (\text{sgn} x_j) \delta_n + j + \delta_{n+1-j} + \eta \right] \right| dx.$$ 

(4.47)

This integral can be explicitly evaluated: according to lemma 4.50 below,

$$\mathcal{H} = (-1)^{\delta_2 + \cdots + \delta_{n-1} + (n-1)\eta} \times \frac{G_\varepsilon(\nu - \frac{n}{2} + 1) \prod_{j=1}^{n-1} G_{\delta_n + j + \delta_{n+1-j} + \eta} (-\lambda_n + j - \lambda_{n+1-j} - \frac{n}{\nu} + \frac{1}{2})}{G_{\varepsilon + \delta_2 + \cdots + \delta_{n-1} + (n-1)\eta} (\nu - \frac{n}{2} + 1 - \lambda_2 + \cdots - \lambda_{n-1} - \frac{n}{\nu} + \frac{1}{2})}.$$ 

(4.48)

At this point, the hypothesis (4.19) and the identity

$$G_\delta(s) G_\delta(1 - s) = (-1)^\delta$$

(4.49)

(which follows directly from (3.47)), establish (4.36) and hence the proposition. $\square$
4.50. Lemma. For \( t \in \mathbb{R}^n, t_n \neq 0 \), the integral
\[
\int_{\mathbb{R}^{n-1}} \left| t_n - \sum_{j=1}^{n-1} t_j \right|^{\beta_0 - 1} \text{sgn} \left( t_n - \sum_{j=1}^{n-1} t_j \right)^{\eta_0} \times \prod_{j=1}^{n-1} \left| t_j \right|^{\beta_j - 1} \text{sgn}(t_j)^{\eta_j} dt_1 \cdots dt_{n-1},
\]
converges absolutely when the real parts of \( 1 - \beta_0 - \beta_1 - \cdots - \beta_{n-1} \) and of the \( \beta_j \) are all positive. As a function of the \( \beta_j \) it extends meromorphically to all of \( \mathbb{C}^n \), and equals
\[
\frac{G_{\eta_0}(\beta_0)G_{\eta_1}(\beta_1)\cdots G_{\eta_{n-1}}(\beta_{n-1})}{G_{\eta_0+\eta_1+\cdots+\eta_{n-1}}(\beta_0 + \beta_1 + \cdots + \beta_{n-1})} |t_n|^{\beta_0 + \beta_1 + \cdots + \beta_{n-1} - 1} \text{sgn}(t_n)^{\eta_0 + \eta_1 + \cdots + \eta_{n-1}}.
\]

Proof: First we show that absolute convergence implies the formula we want to prove. We let \( I(t_n) \) denote the value of the integral. Changing variables appropriately one finds
\[
I(t_n) = |t_n|^{\beta_0 + \beta_1 + \cdots + \beta_{n-1} - 1} \text{sgn}(t_n)^{\eta_0 + \eta_1 + \cdots + \eta_{n-1}} I(1).
\]
(4.51)
Recall the defining formula (3.46). Integration of the right hand side of the equality (4.51) against the function \( e(t_n) \) results in the expression
\[
G_{\eta_0+\eta_1+\cdots+\eta_{n-1}}(\sum_{j=0}^{n-1} \beta_j) I(1),
\]
(4.52)
whereas multiplication of the actual integral with \( e(t_n) \), subsequent integration with respect to \( t_n \), interchanging the order of integration, and the change of variables \( t_j \mapsto t_j \) for \( 1 \leq j \leq n - 1 \), \( t_n \mapsto \sum t_j \), result in the integral
\[
\int_{\mathbb{R}^n} e(t_1 + \cdots + t_n) |t_n|^{\beta_0 - 1} \text{sgn}(t_n)^{\eta_0} \prod_{j=1}^{n-1} \left| t_j \right|^{\beta_j - 1} \text{sgn}(t_j)^{\eta_j} dt_1 \cdots dt_n.
\]
(4.53)
Strictly speaking these integrals converge only conditionally, in the range \( \text{Re} \beta_j \in (0, 1) \). They can be turned into convergent integrals by a partition of unity argument and repeated integration by parts; for details see [17]. The integral (4.53) splits into a product of integrals of the type (3.46). The explicit formula for this integral, equated to the expression (4.52), gives the formula we want for \( I(1) \), and hence for \( I(t_n) \). Absolute convergence of \( I(t_n) \) in the range \( \text{Re} \beta_j > 0 \), \( \text{Re} (\sum \beta_j) < 1 \) can be established by induction on \( n \). For \( n = 2 \), the assertion follows from direct inspection. For the induction step, one integrates out one variable first and uses the induction hypothesis, coupled with the explicit formula for the remaining integral in \( n - 2 \) variables. \( \square \)

5. Adelization of Automorphic Distributions

The definition of automorphic distribution in section 2 used classical language, as it is better suited for describing the necessary analysis of distributions on Lie groups. However, modern automorphic forms heavily uses the language of adeles to simplify and organize calculations, especially for general congruence subgroups \( \Gamma \). In this section, we extend the notions there to the adeles by illustrating two different methods. In the first, we use strong approximation to derive an adelization of cuspidal automorphic distributions, analogous to the usual procedure of
adelizing automorphic forms; in the second, we construct adelic Eisenstein distributions directly. Both constructions can be adapted to either case, and rely on the analysis in earlier sections at their core; it should be emphasized that the role of the adeles here is nothing more significant than a bookkeeping mechanism. However, there are deeper generalizations of this adelization which simultaneously take into account embeddings of several components of an automorphic representation. Such distributions are more complicated, and are useful for extending our theory to nonarchimedean places and number fields. The section concludes with the adelic analog of the pairing of the previous section.

For the sake of clarity, we have chosen to give an explicit, detailed discussion of this adelization for the linear algebraic group $GL(n)$ over $\mathbb{Q}$; this suffices for the application in [23]. However, the method generalizes to adelic automorphic representations for arbitrary connected, reductive linear algebraic groups defined over arbitrary number fields. We will make comments about the general case after describing the specifics for $GL(n)$ over $\mathbb{Q}$.

We for the most part use standard notation: $\mathbb{A}$ refers to the adeles of $\mathbb{Q}$, and $\mathbb{A}_f$ denotes the finite adeles, i.e., the restricted direct product of all $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, $p < \infty$. If $H$ denotes a group defined over $\mathbb{Z}$ such as $G = GL(n)$ or the unit upper triangular matrices $N$, we use the notation $H(R)$ to represent its $R$-points for the rings $R = \mathbb{Z}, \mathbb{Q}, \mathbb{Q}_p, \mathbb{R}, \mathbb{A},$ and $\mathbb{A}_f$. The maximal compact subgroup $\prod_{p < \infty} G(\mathbb{Z}_p)$ of $G(\mathbb{A}_f)$ will be denoted by $K_f$. We often stress membership in one of these groups with an appropriate subscript; for example, the general adel $g_h \in G(\mathbb{A}_f)$ can be decomposed as the product $g_h = g_{\infty} \times g_2 \times g_3 \times g_5 \times \cdots$, or more concisely as $g_{\infty} \times g_f$, where the finite part $g_f \in G(\mathbb{A}_f)$ is the remaining product over the primes. The group $G(\mathbb{Q})$ sits inside each $G(\mathbb{Q}_p)$, and so at the same time embeds diagonally into $G(\mathbb{A}_f)$. In order to avoid confusion here we shall use $G_\mathbb{Q}$ to denote this diagonally-embedded image; likewise, we let $H_\mathbb{Q} \subset G_\mathbb{Q}$ denote the diagonally embedded image of the rational points of an algebraic subgroup $H \subset G$ defined over $\mathbb{Z}$. Thus strong approximation, for example, asserts that $G(\mathbb{A}_f) = G_\mathbb{Q} G(\mathbb{R}) K_f$.

Suppose now that $\pi = \otimes_{p \leq \infty} \pi_p$ is an irreducible, cuspidal adelic automorphic representation of $G(\mathbb{A}_f)$, with representation space $U \subset L^2(G_\mathbb{Q} \backslash G(\mathbb{A}_f))$ under the right action of $G(\mathbb{A}_f)$. Here $\omega$ denotes a character of the center $Z(\mathbb{A}_f)$, which we may assume is a finite order character after twisting $\pi$ by a character of the determinant. Each function $\phi_h \in U$ restricts to a function $\phi_\mathbb{R}$ on $G(\mathbb{R}) \subset G(\mathbb{A}_f)$. Since the representation $\pi$ acts continuously, $\phi_\mathbb{R}$ is stabilized by a congruence subgroup $K$ of $K_f$. At the same time it is invariant on the left under $G_\mathbb{Q}$; since the $K_f$ factor commutes across the $G(\mathbb{R})$ factor, we conclude that $\phi_\mathbb{R}$ is left-invariant under a congruence subgroup $\Gamma$ of $G(\mathbb{Z})$. The same holds true (with different $K$ and $\Gamma$) if we restrict $\phi_h$ to a different section of $G(\mathbb{R})$ inside $G(\mathbb{A}_f)$, for example one of the form $G(\mathbb{R}) \times \{g_f\}$; this is simply the restriction to $G(\mathbb{R})$ of $\pi(g_f) \phi_h$. By strong approximation and the left invariance of $\phi_h$ under $G_\mathbb{Q}$, this is tantamount to left translating $\phi_\mathbb{R} = \phi_h \mid_{G(\mathbb{R})}$ by a rational, real matrix whose inverse approximates $g_f$. Thus adelic automorphic forms are functions from $G(\mathbb{A}_f)$ to smooth automorphic forms on $G(\mathbb{R})$. We shall use this vantage point as a template for adelizing automorphic distributions.

We now assume, as we may, that $\phi_h$ corresponds to a nonzero pure tensor for $\pi = \otimes_{p \leq \infty} \pi_p$ that is furthermore a smooth vector for $\pi_\infty$. Right translation by $G(\mathbb{R})$ commutes with the above correspondence, so $\phi_\mathbb{R}$ sits inside a classical automorphic
representation equivalent to $\pi_\infty$. It is therefore the image of an embedding of the form $\mathfrak{g}_A$. By connecting these two constructions, an automorphic distribution $\tau$ now defines an embedding $J$ of $(\pi_\infty, V_\infty)$ into a subspace $U_\infty$ of $U$: the closure of the subspace spanned by right $G(\mathbb{R})$-translates of $\phi_A$.

Again as in section 2, $\tau$ is a distribution vector for $\pi'_\infty$, and hence may be viewed as a distribution on $G(\mathbb{R})$ once a principal series embedding $\pi'_\infty \rightarrow V_{\lambda, \delta}$ has been chosen (cf. 2.7). In what follows we fix such an embedding. The above procedure of course associates a distribution in $C^{-\infty}(G(\mathbb{R}))$ to any right translate of $\phi_A$ by $g_f \in G(\mathbb{A}_f)$, a distribution which is left invariant under a discrete group that depends on $g_f$. Assembling these together, we form a map from $G(\mathbb{A}_f)$ to $C^{-\infty}(G(\mathbb{R}))$ which we call an “adelic automorphic distribution” for the automorphic representation $\pi$. More concretely, $\tau_A(g_h) = \tau_A(g_\infty \times g_f)$ is defined to be the automorphic distribution in the variable $g_\infty$ which describes the embedding of $(\pi_\infty, V_\infty)$ into the space $\{\text{restrictions of functions in } \pi(g_f)U_\infty \rightarrow G(\mathbb{R})\}$.

The fixed principal series embedding for $\pi'_\infty$ naturally exhibits $\pi_\infty$ as the quotient of the dual principal series $V_{-\lambda, \delta}$. In particular, we may regard the pairing between $\tau(g_\infty \times g_f)$ and smooth vectors $v(g_\infty)$ in $V_\infty$ as integration in $g_\infty$ over a flag variety. We shall use the following notation generalizing (2.6):

$$J(v)(h_\infty \times h_f) = \langle \tau_A(g_\infty \times h_f), \pi_\infty(h_\infty)v(g_\infty) \rangle$$

$$= \langle \tau_A(h_\infty g_\infty \times h_f), v(g_\infty) \rangle,$$  \hspace{0.5cm} (5.1)

where $g_\infty$ is again the variable of integration in the pairing.

By convention $\tau_A$ behaves like a function under diffeomorphisms and is dual to smooth, compactly supported measures in the $g_\infty$ variable. Right translation of $\tau_A$ by $G(\mathbb{A}_f)$ corresponds to right translation of functions in $U$. The group $G(\mathbb{A})$ also acts on $\tau_A$ by left translation,

$$\langle \ell(h_\infty \times h_f), g_f \rangle = \tau_A(h_\infty g_f).$$  \hspace{0.5cm} (5.2)

This action on $\tau_A$, restricted to $G(\mathbb{R})$, is consistent with (2.6) and (5.1), but note however that its restriction to $G(\mathbb{A}_f)$ acts on the left (as opposed to on the right, as it does for functions in $U$). Because the purpose of (5.2) is merely notational, this discrepancy will be harmless. Conjugates of the congruence subgroup $K \subset K_f$ that stabilizes $\phi_A$ also stabilize $\tau_A$:

$$\langle \ell(k) \tau_A(g_\infty \times g_f), g_f \rangle = \tau_A(g_\infty g_f) \quad \text{for each } k \in g_fKg_f^{-1}.\hspace{0.5cm} (5.3)$$

We claim that $G_Q$ acts trivially on $\tau_A$ under $\ell$, i.e.,

$$\tau_A(\gamma g_A) = \tau_A(g_A) \quad \text{for each } \gamma \in G_Q.\hspace{0.5cm} (5.4)$$

Indeed, writing $\gamma$ as $\gamma_\infty \times g_f$, this amounts to checking that

$$\langle \tau_A(\gamma_\infty g_\infty \times g_f), v(g_\infty) \rangle = \langle \tau_A(g_\infty \times g_f), v(g_\infty) \rangle,$$  \hspace{0.5cm} (5.5)

or equivalently,

$$J(v)(\gamma_\infty \times g_f) = J(v)(g_f)\hspace{0.5cm} (5.6)$$

for arbitrary $g_f \in \mathbb{A}_f$ and smooth vectors $v \in V_\infty$. The left hand side, $J(v)(\gamma g_f)$, equals the right hand side because the function $J(v) \in U$ is automorphic under $G_Q$.

Let us now briefly indicate how this adelization works for a general connected, reductive linear algebraic group defined over a number field $F$ and its adele ring $\mathbb{A} = \mathbb{A}_F$ (we refer to [1] as a general reference for the definition, and facts quoted below). Let $\phi_A$ again denote a smooth vector for an automorphic representation...
$\pi = \otimes_v \pi_v$ of $G(\mathbb{A})$, where $v$ runs over all places of $F$. The function $\phi_\lambda$ on $G(\mathbb{A})$ is left invariant under the diagonally embedded $G_F$, and is right invariant under a congruence subgroup $K$ of $K_f$, the product of maximal compact subgroups of $G(F_v)$ over all nonarchimedean places $v$ of $F$. Though strong approximation fails in this setting (even for $G = GL(n)$ when the class number of $F$ is greater than 1), the restriction of $\phi_\lambda$ to $G_\infty$, the product of $G(F_v)$ over all archimedean places $v$, is left invariant under
\[
\Gamma = \{ \gamma \in G(F) \mid \gamma_f \in K \},
\]
regarded as a subgroup of $G_\infty$. Since $\Gamma$ is an arithmetic subgroup of $G_\infty$, the quotient $Z_\infty \Gamma \backslash G_\infty$ has finite volume, where $Z$ is the maximal $F$-split torus of the center of $G$, and $Z_\infty$ denotes the product of $Z(F_v)$ over all archimedean places $v$. Automorphic representations are assumed to transform according to a character of the adelic points $Z(\mathbb{A})$ of $Z$. Thus, as before, the restriction of a vector in the adelic automorphic representation gives rise to a classical automorphic representation of the real group $G_\infty$, and hence an automorphic distribution on $G_\infty$ (this uses the fact that the Casselman-Wallach embedding theorem holds for arbitrary real reductive groups). Right translation by $G(\mathbb{A}_f)$ then allows us to construct an adelic automorphic distribution $\tau_\lambda$ following the same procedure as before.

We now return to some features of the earlier discussion about $G = GL(n)$ over $F = \mathbb{Q}$, starting with a description of the adelic version of the Whittaker distribution $w_{\lambda,\delta}$ from (2.31). Let $\psi_+$ denote the standard choice of additive character on $\mathbb{Q}\backslash \mathbb{A}$: the unique such character whose archimedean component maps $x \mapsto e^{2\pi ix}$. (What we say below needs to be modified slightly if a different nontrivial character of $\mathbb{Q}\backslash \mathbb{A}$ is chosen instead.) There is a standard group homomorphism $c$ defined on the group of unipotent upper triangular matrices $N$, given by summing the entries just above the diagonal:
\[
c : (n_{ij}) \mapsto n_{1,2} + n_{2,3} + \cdots.
\]
The composition $\psi_+ \circ c$ is a nondegenerate character of $N(\mathbb{Q}) \backslash N(\mathbb{A})$, and is used to define global Whittaker integrals on the automorphic representation $\pi$:
\[
W_{\phi_\lambda}(g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \phi_\lambda(ng) \psi_+(c(n))^{-1} \, dn, \quad \phi_\lambda \in U.
\]
Here, as usual, $dn$ denotes Haar measure on $N(\mathbb{A})$, normalized to give the quotient $N(\mathbb{Q}) \backslash N(\mathbb{A})$ volume equal to 1. Likewise, we define an analogous adelic Whittaker integral for $\pi$ using $\tau_\lambda$:
\[
w(g) = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \tau_\lambda(ng) \psi_+(c(n))^{-1} \, dn,
\]
or more succinctly
\[
w = \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \ell(n) \tau_\lambda(\psi_+(c(n))) \, dn.
\]
Like $\tau_\lambda$, $w(g) = w(g_\infty \times g_f)$ should be thought of as a function of $g_f \in G(\mathbb{A}_f)$ with values in $C^{-\infty}(G(\mathbb{R}))$. Indeed, for any fixed $g_f \in G(\mathbb{A}_f)$, (4.4) shows that $\tau_\lambda$ is stabilized by a finite index subgroup of $K_f \cap N(\mathbb{A}_f)$; strong approximation then shows this integration is therefore actually over a finite cover of the compact quotient $N(Z) \backslash N(\mathbb{R})$. Hence it reduces to (2.34) and gives a valid distribution in the $g_\infty$ variable. If $v$ is a smooth vector for $V_\infty$ and $\phi_\lambda = J(v)$, then it is easily
seen that the distribution \( w \) embeds \( v \) to \([5.11]\). This is because the pairing between distributions and vectors here involves integration on the right, whereas the above integrations take place on the left.

When \( \phi_A \) is a pure tensor for \( \pi = \bigotimes_{p \leq \infty} \pi_p \), the integral \([5.9]\) factors into a product of local Whittaker functions:

\[
W_{\phi_A}(g_{\infty} \times g_f) = W_{\infty}(g_{\infty}) W_f(g_f), \quad W_f(g_f) = \prod_{p < \infty} W_p(g_p). \tag{5.12}
\]

Here the \( W_p \) lie in the Whittaker model \( W_p \) for \( \pi_p \), and are constrained to be the standard spherical Whittaker function (i.e., \( W_p|G(\mathbb{Z}_p) \equiv 1 \)) for almost all primes \( p \).

Importantly, by varying the pure tensor \( \phi_A \), the \( W_p \) can be chosen arbitrarily in \( W_p \) for any given finite set of primes. Were we to instead start with such a modified choice of \( \phi_A \in U \) and construct \( \tau_A \) from it as above, its adelic Whittaker integral \([5.10]\) would have a similar factorization:

\[
w(g_{\infty} \times g_f) = w_{\infty}(g_{\infty}) W_f(g_f). \tag{5.13}
\]

The distribution \( w_{\infty} \in C^{\infty}(G(\mathbb{R})) \) coincides with a nonzero multiple of the distribution \( w_{\lambda,\delta} \) from \([2.31]\), where \( (\lambda, \delta) \) are the principal series parameters for the Casselman embedding of \( \pi_{\infty}' \). The paper \([27]\) provides a rather complete study of the connection between the archimedean Whittaker distributions \( w_{\infty} \) and Whittaker functions for general Lie groups. The remaining product over primes is itself naturally related to the coefficient in \([2.33]\).

We have therefore shown the following fact, which is useful in constructing adelic automorphic distributions with prescribed behavior at finite places.

**5.14. Proposition.** Let \( \pi = \otimes \pi_p \) be a cuspidal automorphic representation of \( GL(n)/\mathbb{Q} \), and \( S \) any finite set of primes. For each \( p \in S \) chose a function \( W_p \) in the Whittaker model for \( \pi_p \), and set \( W_p \) equal to the standard spherical vector for each prime \( p \notin S \). Then there exists a pure tensor \( \phi_A \) for \( \pi \) whose corresponding adelic automorphic distribution \( \tau_A \) satisfies \([77.13]\).

A famous theorem independently proven by Piatetski-Shapiro and Shalika \([26, 27]\) states that a smooth vector \( \phi_A \in U \) can be reconstructed as the sum of left translates of its global Whittaker function \([5.9]\) by coset representatives \( C \) for \( N^{(n-1)} \backslash GL(n-1)_{\mathbb{Q}} \), where \( N^{(n-1)} = \{ (n-1) \times (n-1) \text{ unit upper triangular matrices} \} \). The analogous formula

\[
\tau_A(g) = \sum_{\gamma \in C} w((\gamma_1)g) \tag{5.15}
\]

holds for \( \tau_A \), as a consequence of the above relationships between embeddings of smooth vectors \( v \in V_{\infty} \). It can also be proven using Fourier analysis on the nilpotent group \( N(\mathbb{A}) \), following along the lines of the original argument in \([20, 27]\). In particular, integrating \([5.15]\) over \( N'(\mathbb{Q}) \backslash N'(\mathbb{A}) \), where \( N' = [N, N] \) is the derived subgroup of \( N \), gives the following formula for the adelization of \( \tau_{\text{abelian}} \):

\[
\tau_{\text{abelian}}(g) = \sum_{k \in \mathbb{Q}^{n-1}} w(D(k)g), \tag{5.16}
\]

where \( D(k) \in G_{\mathbb{Q}} \) is the matrix defined just after \([2.31]\). It is evident that \( W_f(D(k)g) \) from \([5.13]\) must equal the ratio multiplying \( w_{\lambda,\delta} D(k)g \) in \([2.31]\). This observation also demonstrates that the normalized coefficients \( a_k \) are independent of the chosen Casselman embedding.
Next we turn to the adelic version of the mirabolic Eisenstein series distributions that were defined and analytically continued in section 3. Though these can be constructed as a special case of the adelic automorphic distributions just described, it is more useful to construct them directly, and then verify that they match the earlier construction. Jacquet and Shalika studied adelic mirabolic Eisenstein series as part of their integral representations of the Rankin-Selberg $L$-functions on $GL(n) \times GL(n)$ 10 and the exterior square $L$-functions on $GL(2n)$ 11. As we commented earlier, it is also possible to derive the results here from theirs, using sophisticated machinery of Casselman and Wallach.

Our adelic construction involves modifying the archimedean data in the Jacquet-Shalika construction in order to mimic the $\delta$-function that is averaged in (3.12), but leaving the nonarchimedean data intact. We begin by recalling the Schwartz-Bruhat space of $\mathbb{Q}_p^n$, which is the usual Schwartz space in $n$ real variables when $p = \infty$, and is the space of locally constant, compactly supported functions when $p < \infty$. The latter are precisely the finite linear combinations of characteristic functions of sets of the form $v + p^N \mathbb{Z}_p^n$, where $v \in \mathbb{Q}_p^n$ and $N \in \mathbb{Z}$; for $v$ fixed one need only consider $N$ large, because of overlap among these sets. The global adelic Schwartz-Bruhat space consists of all finite linear combinations of functions which are global products $\Phi(g) = \prod_{p \leq \infty} \Phi_p(g_p)$ of Schwartz-Bruhat functions $\Phi_p$ on $\mathbb{Q}_p^n$, in which all but a finite number of functions $\Phi_p$ are constrained to be the “standard unramified choice” of the characteristic function of $\mathbb{Z}_p^n$.

The adelic Eisenstein series distributions are designed to have central character $\omega^{-1}$, the inverse of the central character of $\tau_h$; this is done in anticipation of the pairing between these objects at the end of the section. Strong approximation for $\mathbb{A}^*$ equates the double cosets $\mathbb{Q}^* \backslash \mathbb{A}^*/\mathbb{R}^*_0$ to the inverse limit of all $(\mathbb{Z}/NZ)^*$, $N \in \mathbb{N}$. Therefore any Dirichlet character $\psi$, in particular the one in (3.12), has an adelization to a global character $\psi_{\mathbb{A}} = \prod_{p \leq \infty} \psi_p$ of $\mathbb{A}^*$ that is trivial on $\mathbb{Q}^*$. We assume for the rest of the paper that

$$\psi_{\mathbb{A}} = \chi^n \omega^{-1},$$

(5.17)

where $\chi$ is also a finite order character of $\mathbb{Q}^* \backslash \mathbb{A}^*$ of parity $\eta \in \mathbb{Z}/2\mathbb{Z}$, consistent with (4.19).

Set $P'$ equal to the $(n - 1, 1)$ standard parabolic subgroup of $G$, so that $P' = w_{\text{long}} P w_{\text{long}}$ (cf. (3.11)). Jacquet-Shalika form their Eisenstein series as averages of the function

$$I(g,s) = \chi(\det g)^{-1} |\det g|^s \int_{\mathbb{A}^*} \Phi(c_n tg) |t|^{ns} \psi_{\mathbb{A}}(t)^{-1} d^* t,$$

(5.18)

where $c_n = (0, 0, \ldots, 0, 1)$ is the $n$-dimensional elementary basis row vector. Our construction of the Eisenstein distribution differs in that we modify the archimedean component $\Phi_\infty$ of each summand of $\Phi$ to be the $\delta$-function of a nonzero point in $\mathbb{R}^n$. To emphasize this distinction, we sometimes refer to Jacquet-Shalika’s choice as $\Phi_{\text{JS, } \infty}$ and ours as $\Phi_{\text{JS, } \infty}$. When $\Phi(g) = \prod_{p \leq \infty} \Phi_p(g_p)$ is a pure tensor, the integral (5.18) splits as a product of local integrals over $\mathbb{Q}_p$, $p \leq \infty$, so that $I(g,s) = I_\infty(g_\infty, s) I_f(g_f, s)$, $I_f(g_f, s)$ being the product over all $p < \infty$. The computation of $I_f(g_f, s)$ is unchanged from the setting of Jacquet-Shalika, but the

---

9Please note this identification between Dirichlet and global characters is inverse to the one used by Jacquet-Shalika.
archimedean integral

\[ I_\infty(g_\infty, s) = \text{sgn}(\det g_\infty)^n |\det g_\infty|^s \int_{\mathbb{R}^*} \Phi_\infty(e_n t g_\infty) |t|^{ns} \text{sgn}(t)^s d^* t \quad (5.19) \]

differs in that it defines a distribution on \( G \) instead of a smooth function when \( \Phi_\infty = \Phi_{D, \infty} \). The local integrals obey the transformation law

\[ I_p((B_a^* g_p) = \psi_p(a) \chi_p(a)^{-1} |a|^{-(n-1)s} \chi_p(\det B)^{-1} |\det B|^s I_p(g_p), \quad (5.20) \]
as can be seen by the change of variables \( t \mapsto t/a \) in the integral.

We shall now describe how the respective local integrals \( I_{JS, \infty} \) and \( I_{D, \infty} \) are related by right smoothing. If \( \phi \) is any smooth, compactly supported function on \( G(\mathbb{R}) \), we may choose

\[ \Phi_{JS, \infty}(v) = \int_{G(\mathbb{R})} \Phi_{D, \infty}(v h) \phi(h) \text{sgn}(\det h)^n |\det h|^s dh, \quad v \in \mathbb{R}^n, \quad (5.21) \]
since the integral defines a smooth function of compact support in \( v \). The respective local integrals \( (5.19) \) of \( \Phi_{JS, \infty} \) and \( \Phi_{D, \infty} \) are related by

\[ I_{JS, \infty}(g_\infty, s) = \int_{G(\mathbb{R})} I_{D, \infty}(g_\infty h, s) \phi(h) dh. \quad (5.22) \]

It follows that right convolution of our distributional \( I(g, s) \) over \( G(\mathbb{R}) \) results in an instance of Jacquet-Shalika's \( (5.18) \).

We now consider the computation of \( I(g, s) \) for a particular type of pure tensor \( \Phi \), namely when \( \Phi_\infty \) is the \( \delta \)-function supported at \( e_1 = (1, 0, \ldots, 0) \) and \( \Phi_p \) is the characteristic function of \( e_n + p^n \mathbb{Z}_p^n \), where \( N = \prod p^{N_p} \) is the factorization of a positive integer \( N \). Then \( I_\infty(g, s) \) is supported on \( P'(\mathbb{R})w_{\text{long}} = w_{\text{long}} P(\mathbb{R}) \) by construction, and is in fact a constant multiple of the distribution

\[ \delta_\infty \in W_{\nu_\infty} \otimes \text{sgn}(\det)^n \quad (5.23) \]
defined in \( (3.3) \), with \( \nu = n(s - 1/2) \) (cf. \( (3.65) \)). The local integral for \( p < \infty \) is computed as

\[ I_p(g_p, s) = \chi_p(\det g_p)^{-1} |\det g_p|^s \int_{t \in \mathbb{R}_p^n} |t|^{ns} \psi_p(t)^{-1} d^* t, \quad (5.24) \]

where \( v \) is the bottom row of \( g \). The transformation law \( (5.20) \) reduces the computation to \( g_p \in GL(n, \mathbb{Z}_p) \), a set of coset representatives for the subgroup of upper triangular matrices, so that in particular we may assume \( v \in \mathbb{Z}_p^n, p \nmid v \). In the case that \( N_p = 0 \), the set in the second constraint is simply \( \mathbb{Z}_p^n \), and the integration is over \( 0 < |t| \leq 1 \). The integral is then a Tate integral for \( L(s, \psi) \): it represents \( (1 - \psi(p)p^{-ns})^{-1} \) if \( \psi_p \) is unramified at \( p \), and zero otherwise. If \( N_p \geq 1 \), the second constraint reads \( tv_j \equiv 0 (\text{mod } p^{N_p}) \) for \( j < n \), while \( tv_n \equiv 1 (\text{mod } p^{N_p}) \). This forces \( v_n \in \mathbb{Z}_p - p \mathbb{Z}_p \), and the range of integration to \( t \in v_n^{-1} + p^{N_p} \mathbb{Z}_p \). The integral vanishes if the ramification degree of \( \psi_p \) exceeds \( N_p \), and equals a constant times \( \psi(v_n) \) otherwise.

In particular, if \( \gamma \in GL(n, \mathbb{Z}) \), then \( I(\gamma g_\infty, s) \) is the product of a constant, \( \delta_\infty(g_\infty) \), \( L(ns, \phi) \), and the characteristic function of \( \Gamma_0(N) \). The sum of this over all cosets for \( P'(\mathbb{Z}) \setminus G(\mathbb{Z}) \) is precisely the Eisenstein series in \( (3.12) \), up to a constant multiple. The coset space \( P'(\mathbb{Z}) \setminus G(\mathbb{Q}) \) is in bijective correspondence with \( P'(\mathbb{Q}) \setminus G(\mathbb{Q}) \) via the inclusion map \( G(\mathbb{Z}) \hookrightarrow G(\mathbb{Q}) \), because of the fact that every
invertible rational matrix can be decomposed as an upper triangular rational matrix times an invertible integral one. We conclude that with this particular choice of local data,

\[ E_h(g_h, s) = \sum_{\gamma \in F'_0 \backslash G_0} I(\gamma g_h, s) \]  \hspace{5em} (5.25)

is a constant multiple of \((4.12)\) when \(g_h \in G(\mathbb{R})\), and in particular converges in the strong distributional topology for \(\text{Re} \, s > 1\). Strong approximation reduces the evaluation of the general \(g_h \in G(\mathbb{A})\) to this case, so the sum makes sense in general for \(\text{Re} \, s > 1\) and defines an adelic automorphic distribution: a map from \(G(\mathbb{A}_f)\) to automorphic distributions in \(C^\infty_c(G(\mathbb{R}))\). Because of \((5.22)\), the right smoothing of \(E_h(g_h, s)\) over \(G(\mathbb{R})\) results in a smooth Eisenstein series on \(G(\mathbb{Q}) \backslash G(\mathbb{A})\) considered by Jacquet-Shalika. Thus \(E_h\) is also an automorphic distribution in the earlier sense of a distribution which embeds into smooth automorphic forms.

The general choice of local data involves broader choices in two respects: \(\Phi_\infty\) may be a \(\delta\)-function supported at another nonzero point, and \(\Phi_p\) may be the characteristic function of \(v + p^N \mathbb{Z}_p^n, \, N\) large. Right translating \(E_h(g_h, s)\) by some \(h \in GL(n, \mathbb{A})\) has the effect of replacing \(\Phi(v)\) by \(\Phi(v h)\). Since \(GL(n)\) acts with two orbits on \(n\)-dimensional vectors, this means the general \(\delta\)-function for \(\Phi_\infty\) can be reduced to the case above, and that the characteristic functions for \(\Phi_p\) can be reduced to the situation that \(v = 0\) or \(v = e_n\). Since \(e_n + \mathbb{Z}_p^n = \mathbb{Z}_p^n\), the sets \(e_n + p^N \mathbb{Z}_p^n\) for \(N \geq 0\) we considered above indeed cover all possibilities. Thus the analytic properties of the general instance of \((5.25)\) for linear combinations of such pure tensors \(\Phi\) reduce to those we have just considered. In particular they have a meromorphic continuation to \(s \in \mathbb{C} - \{1\}\), with at most a simple pole at \(s = 1\) that occurs only when \(\psi\) is trivial.

Finally, we conclude by writing the general form of the automorphic pairing in terms of adeles, generalizing \((4.10)\). We need to slightly adapt the notation there to the adelic setting. Let \(U\) denote the algebraic group

\[ U = \left\{ \begin{pmatrix} I_n & A \\ 0_n & I_n \end{pmatrix} \right\} \subset GL(2n) \]  \hspace{5em} (5.26)

whose real points were previously denoted by \(U\) in \((4.13)\). The character \(\theta\) from \((4.15)\) has a natural adelic extension,

\[ \theta : U(\mathbb{A}) \longrightarrow \mathbb{C}^\times, \hspace{1em} \theta \left( \begin{pmatrix} I_n & A \\ 0_n & I_n \end{pmatrix} \right) = \psi_+ (\text{tr} \, A), \]  \hspace{5em} (5.27)

where \(\psi_+\) is the additive character defined just above \((5.8)\). Let \(du\) denote the Haar measure on \(U(\mathbb{A})\) which gives the quotient \(U(\mathbb{Q}) \backslash U(\mathbb{A})\) volume 1.

With \(f_1, \, f_2, \, \text{and} \, f_3\) still standing for flag representatives in \(G(\mathbb{R})\) and \(\psi \in C^\infty_c(G(\mathbb{R}))\) having total integral 1, the general adelic pairing is defined as

\[ P(\tau_h, E_h(s)) = \int_{\mathbb{Z}(\mathbb{A})G_0 \backslash G(\mathbb{A})} \int_{G(\mathbb{R})} \int_{U(\mathbb{Q}) \backslash U(\mathbb{A})} \tau_h \left( u \left( \begin{pmatrix} g h f_1 \\ g h f_2 \end{pmatrix} \right) \right) \overline{\theta(u)} du \times E_h(g h f_3, s) \psi(h) dh \, dg. \]  \hspace{5em} (5.28)

Several comments are in order to explain why the above makes sense. Firstly, for the same reason as in \((5.10)\), the bracketed inner integration is over a finite cover of the compact quotient \(U(\mathbb{Z}) \backslash U(\mathbb{R})\), and so defines a map from \(G(\mathbb{A}_f)\) to distributions in \(G(\mathbb{R})\) that corresponds to \((4.10)\). This map is left invariant under the
diagonal rational subgroup $G_Q$ because of (5.3), and because conjugation through $u$ changes neither $\theta(u)$ nor the measure $du$. It is also invariant under $Z(\mathbb{A})$ because (4.19) ensures that the central characters of $\tau_\lambda$ and $E_\lambda$ are inverse to each other. The invariance under both $G_Q$ and $Z(\mathbb{A})$ is not affected by the second integration, which only involves $h$ on the right. The second integration simultaneously smooths both the bracketed expression and $E_p(ghf_3)$ over $G(\mathbb{R})$: it gives a map from $G(\mathbb{A})$ to smooth automorphic functions on $G(\mathbb{R})$. According to corollary 4.21 these restrictions to $G(\mathbb{R})$ are each integrable over their fundamental domain. Because of (5.3) and strong approximation, the last integration takes place on a finite cover of $Z(\mathbb{R})G(\mathbb{Z})\backslash G(\mathbb{R})$ – again by the same reasoning used for the bracketed inner integration in (5.28), and for (5.10) before it. Corollary 4.21 shows that the last integral is independent of the choice of $\psi$, assuming its normalization $\int_{G(\mathbb{R})} \psi(g)dg = 1$.

The above pairing inherits the meromorphic continuation to $s \in \mathbb{C} - \{1\}$ that its classical counterpart possesses (corollary 4.21), as well as a functional equation from (4.24):

$$P(\tau_\lambda, E_\lambda(1-s)) = N^{2n(s-n)} \prod_{j=1}^{n} G_{\delta_{n+j} + \delta_{n+1-j} + \eta}(s + \lambda_{n+j} + \lambda_{n+1-j}) P(\tau'_\lambda, E'_\lambda(s)), \quad (5.29)$$

where $\tau'_\lambda$ and $E'_\lambda$ correspond to the translated contragredient cusp form $\tilde{\tau}$ and sum of the remaining Eisenstein data, respectively, from the right hand side in proposition (4.21). This formula simplifies when both $\pi_\rho$ and the Eisenstein data $\Phi_\rho$ are unramified at all $p < \infty$ (which put us in the situation that $N = 1$). If $\Phi_\infty$ is the delta function at $e_1 \in \mathbb{R}^n$, then $E_\lambda(s) = (-1)^{\delta_1 + \cdots + \delta_n} E_\lambda(s)$ and $\tau'_\lambda = \tilde{\tau}_\lambda$ (cf. (4.20)).

Appendix A. Archimedean components of automorphic representations on $GL(n, \mathbb{R})$

Recall from section 2 that we study automorphic distributions in terms of the embedding (2.7) of $\pi_\infty$ into principal series representations $V_{\lambda, \delta}$. These embeddings are not unique. For full principal series representations, the parameters $(\lambda, \delta)$ are determined only up to simultaneous permutation of the $\lambda_j$ and $\delta_j$. In general, there is a smaller choice of embedding parameters. On the other hand, the Gamma factors predicted by Langlands also depend on the nature of the archimedean component of the automorphic representation in question. We use this connection between multiple embeddings and Gamma factors to exclude unwanted poles of $L$-functions.

In this appendix we collect the relevant results about embeddings into principal series and Langlands Gamma factors. All of these are well known to experts, but do not appear in the literature – at least not in convenient form.

A.1. The Generic unitary dual of $GL(n, \mathbb{R})$ and embeddings into the principal series. The possible real representations of $GL(n, \mathbb{R})$ that can occur as the archimedean component $\pi_\infty$ of a cuspidal automorphic representation $\pi$ are extremely limited by a number of local and global constraints. The latter are extremely subtle, and hence a complete classification seems hopeless at present. In this subsection we will instead describe the representations that satisfy perhaps the most well known local constraints for $\pi_\infty$, namely those that are unitary and generic (i.e., have a Whittaker model).
The unitary dual for $GL(n, \mathbb{R})$ was first described by Vogan [30], and later by Tadić [28] using different methods. Tadić describes the unitary dual as certain parabolically induced representations from an explicit set $\mathcal{B}$ of representations of $GL(n', \mathbb{R})$, $n' \leq n$. He also proves that permuting the order of the induction data yields the same irreducible representation of $GL(n, \mathbb{R})$. His set $\mathcal{B}$ is defined in terms of not only induced representations of square integrable (modulo the center) representations of $GL(1, \mathbb{R})$ and $GL(2, \mathbb{R})$, but also certain irreducible quotients. These quotients, however, are not “large” in the sense of [29], and hence neither are any representations induced from them. It is a result of Casselman, Zuckerman, and Kostant (see [14]) that all generic representations of $GL(n, \mathbb{R})$ are large, and conversely that all large representations are generic.

Hence Tadić’s list gives a description of the generic unitary dual, once these quotients are removed from $\mathcal{B}$. We now summarize this description, after making further simplifications using transitivity of induction. Let $n = n_1 + \cdots + n_r$ be a partition of $n$, and let $P \subseteq G = GL(n, \mathbb{R})$ be the standard parabolic subgroup of block upper triangular matrices corresponding to this partition. The Levi subgroup $M$ of $P$ is isomorphic to $GL(n_1, \mathbb{R}) \times \cdots \times GL(n_r, \mathbb{R})$. Let $\sigma_i$ denote an irreducible, square integrable (modulo the center) representation of $GL(n_i, \mathbb{R})$. This forces $n_i$ to equal 1 or 2, and $\sigma_i$ to be one of the following possibilities:

1. If $n = 1$, $\sigma_i$ is either the trivial representation of $GL(1, \mathbb{R}) \simeq \mathbb{R}^*$, or else the sign character $\text{sgn}(x)$.
2. If $n = 2$, $\sigma_i$ is a discrete series representation $D_k$ (indexed to correspond to holomorphic forms of weight $k$, $k \geq 2$).

These representations are self dual. For each $1 \leq i \leq r$ and $s_i \in \mathbb{C}$, the twist $\sigma_i[s_i] = \sigma_i \otimes |\det(\cdot)|^{s_i}$ defines a representation of $GL(n_i, \mathbb{R})$. The tensor product of these twists defines a representation of $M$ which extends to $P$ by allowing the unipotent radical of $P$ to act trivially. Let $I(P; \sigma_1[s_1], \ldots, \sigma_r[s_r])$ denote the representation of $G$ parabolically induced from this representation of $P$, where the induction is normalized to carry unitary representations to unitary representations. In order to be consistent with the conventions of [10], the group action in this induced representation operates on the right, on functions which transform under $P$ on the left.

We now give the constraints on the parameters $s_i$ that govern precisely when $I(P; \sigma_1[s_1], \ldots, \sigma_r[s_r])$ is irreducible, generic, and unitary according to the results of Casselman, Kostant, Tadić, Vogan, and Zuckerman mentioned above. We assume that this representation is normalized to have a unitary central character, as we of course may by tensoring with a character of the determinant.

- **Unitarity constraint:** the multisets $\{\sigma_i[s_i]\}$ and $\{\sigma_i[-s_i]\}$ must be equal, i.e., these lists are equal up to permutation (recall the $\sigma_i$ are self dual). This is because the representation dual to $I(P; \sigma_1[s_1], \ldots, \sigma_r[s_r])$ is $I(P; \sigma_1[-s_1], \ldots, \sigma_r[-s_r])$.

- **Unitary dual estimate:** $|\text{Re } s_i| < 1/2$. In the case of the principal series, this is commonly called the “trivial bound”.

- **Permutation of order:** for any permutation $\tau \in S_r$, the induced representations $I(P; \sigma_1[s_1], \ldots, \sigma_r[s_r])$ and $I(P^\tau; \sigma_{\tau(1)}[s_{\tau(1)}], \ldots, \sigma_{\tau(r)}[s_{\tau(r)}])$ are equal, where $P^\tau$ is the standard parabolic whose Levi component is $GL(n_{\tau(1)}, \mathbb{R}) \times \cdots \times GL(n_{\tau(r)}, \mathbb{R})$. 


The principal series representations $V_{\lambda,\delta}$ in (2.10) are induced representations, but induced from a lower triangular Borel subgroup (2.10). Our convention is well-suited for studying automorphic distributions, but induction from an upper triangular Borel subgroup is the more common convention in the literature on Langlands' classification of representations of real reductive groups [16] (e.g., his prediction of $\Gamma$-factors for automorphic $L$-functions). Using the Weyl group element $w_{\text{long}}$ from (2.35) and the inverse map between the two, it is straightforward to show that $V_{\lambda,\delta}$ is equivalent to $I(B_+; \sgn \lambda_n[\lambda_1], \ldots, \sgn \lambda_1[\lambda_1])$, where $B_+$ is the upper triangular Borel subgroup of $GL(n, \mathbb{R})$. More generally, induction on the right from a lower triangular parabolic involves reversing the order of the inducing data, though the order is irrelevant for the representations in Tadić's classification of the unitary dual anyhow.

Embeddings into principal series are of course tautological for $n = 1$, where all irreducible representations are one dimensional. When $n = 2$, the discrete series representation $D_k$ is a subrepresentation of the principal series representation $V_{\lambda,\delta}$ with parameters $\lambda = (-\frac{k-1}{2}, \frac{k-1}{2})$ and $\delta = (k, 0)$. This embedding is not unique: actually $D_k \otimes \sgn \cong D_{k+1}$, so $\delta = (k+1, 1)$ is an equally valid parameter. An irreducible principal series representation $V_{\lambda_1,\lambda_2}(\delta_1,\delta_2)$ embeds not only into itself, but also into $V_{\lambda_2,\lambda_1}(\delta_2,\delta_1)$. However, $D_k$ is not a subrepresentation, but instead a quotient, of the representation $V_{\lambda_1,\lambda_2}(0,k)$. If $\rho_1 \rightarrow \rho_2$, then $\rho_1[s] \leftrightarrow \rho_2[s]$. The twist $V_{\lambda,\delta}[s]$ is the principal series representation $V_{\lambda+(s,s,\ldots,s),\delta}$, so $D_k[s]$ embeds both into $V_{(s-\frac{k+1}{2},s+\frac{k+1}{2}),0}$ and also $V_{(s-\frac{k+1}{2},s+\frac{k+1}{2}),1}$. The description above shows that these are the only types of unitary generic representations of $GL(2, \mathbb{R})$.

Next we move to $GL(n, \mathbb{R})$ and consider a unitary, generic representation $\pi_\infty = I(P; \sigma_1[s_1], \ldots, \sigma_r[s_r])$ as above. Embeddings for $\pi_\infty'[s] = I(P; \sigma_1[-s_1], \ldots, \sigma_r[-s_r])$ may be deduced from the previous paragraph, using the principle of transitivity of induction as follows. Let $k_i$ denote the weight of the discrete series in block $i$ (provided $n_i = 2$, of course). Now define vectors $\lambda_i \in \mathbb{C}^n$ and $\delta_i \in (\mathbb{Z}/2\mathbb{Z})^n$ in the following manner. If the integer $1 \leq j \leq n$ is contained in the $i$-th block $n_i$ of the partition $n = (n_1, \ldots, n_r)$, set $\lambda_j$ to be

$$
\lambda_j = \begin{cases} 
-s_i, & n_i = 1; \\
-s_i - \frac{k_i-1}{2}, & n_i = 2 \; \text{and} \; j = n_1 + \ldots + n_{i-1} + 1; \\
-s_i + \frac{k_i-1}{2}, & n_i = 2 \; \text{and} \; j = n_1 + \ldots + n_{i-1} + 2. 
\end{cases}
$$

Similarly, set

$$
\delta_j = \begin{cases} 
\varepsilon, & n_i = 1 \; \text{and} \; \sigma_i = \sgn(\cdot)^r; \\
k_i, & n_i = 2 \; \text{and} \; j = n_1 + \ldots + n_{i-1} + 1; \\
0, & n_i = 2 \; \text{and} \; j = n_1 + \ldots + n_{i-1} + 2.
\end{cases}
$$

One may alternatively replace $k_i$ and 0 in the last two cases by $k_i+1$ and 1, respectively. In other words, $\lambda$ and $\delta$ are formed by concatenating the corresponding vectors which describe the embedding parameters for the $\sigma_i[-s_i], 1 \leq i \leq r$. By transitivity of induction, $\pi_\infty'[s] = I(P; \sigma_1[-s_1], \ldots, \sigma_r[-s_r])$ is a subrepresentation of $V_{\lambda,\delta}$.

**A.2. Langlands’ $\Gamma$-factors.** The $\Gamma$-factors which accompany an automorphic $L$-function $L(s, \pi, \rho)$ in its functional equation are conjectured to always be products, with shifts, of the functions

$$
\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2) \quad \text{and} \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s) = \Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{R}}(s+1). \quad (A.3)
$$
Langlands [14] gives a procedure to compute this archimedean factor \( L_\infty(s, \pi, \rho) \) in terms of his description of \( \pi_\infty \) as a subquotient of an induced representation, along with a calculation involving the L-group representation \( \rho \) and the Weil group. When dealing with the group \( GL(n) \), however, it is much more convenient to avoid the Weil group, and instead describe these \( \Gamma \)-factors in terms of the (freely permuted) induction data. We give a description of this for some notable examples, following the description in [13].

It is convenient to use Langlands’ isobaric notation [15] for induced representations

\[
\pi_\infty = I(P; \sigma_1[s_1], \ldots, \sigma_r[s_r]) = \sigma_1[s_1] \boxplus \cdots \boxplus \sigma_r[s_r],
\]

in which the operation \( \boxplus \) on the right hand side should be thought of as a formal, abelian addition. Recall that the classification in section A.1 shows that every generic unitary representation of \( GL(n, \mathbb{C}) \) is an isobaric sum of the form (A.4), independent of the order. We use these formal sums here only as a bookkeeping device used to define \( \Gamma \)-factors; they do not always correspond to irreducible, archimedean components of cuspidal automorphic representations. This formal addition satisfies the following two properties. First, two isobaric sums \( \Pi_1, \Pi_2 \) may themselves be concatenated into a longer isobaric sum \( \Pi_1 \boxplus \Pi_2 \). Second, an isobaric sum can be twisted by the rule (A.10) (A.11)

\[
\Gamma(s, \Pi) = \Gamma_\Pi(s) = \Gamma_\Pi(s + 1),
\]

Next are rules for isobaric sums and twists:

\[
L(s, \Pi[s]) = L(s + s', \Pi) \quad \text{and} \quad L(s, \Pi_1 \boxplus \Pi_2) = L(s, \Pi_1) \cdot L(s, \Pi_2).
\]

Therefore \( L(s, \Pi) \), for a general isobaric sum \( \Pi = \sigma_1[s_1] \boxplus \cdots \boxplus \sigma_r[s_r] \), is given by

\[
L(s, \Pi) = \prod_{i=1}^r L(s + s_i, \sigma_i),
\]

and is explicitly determined by the definitions (A.5) (A.6).

Let now \( \Pi = \Pi_1 \boxplus \Pi_2 \boxplus \cdots \boxplus \Pi_r \) be an isobaric representation of \( GL(n, \mathbb{R}) \), and \( \Pi' = \Pi_1' \boxplus \Pi_2' \boxplus \cdots \boxplus \Pi_r' \), be an isobaric representation of \( GL(m, \mathbb{R}) \). The isobaric sum for the Rankin-Selberg tensor product representation \( \Pi \times \Pi' \) of \( GL(nm, \mathbb{R}) \) is given by

\[
\Pi \times \Pi' = \boxplus_{j=1}^r \boxplus_{k=1}^r \Pi_j' \times \Pi_k',
\]

where now the meaning of \( \Pi_j \times \Pi_k' \) must be explained. It is in general not the usual tensor product of two representations (more on this below). One has the relations

\[
\Pi[s] \times \Pi'[s'] = (\Pi \times \Pi')[s + s']
\]

and

\[
\Pi \times \Pi' = \Pi' \times \Pi,
\]
which along with \([A.3]\) may be regarded as formal rules for the calculation of tensor product on isobaric representations. They boil the general calculation down to the examples of \(\sigma \times \sigma'\), where \(\sigma, \sigma' \in \{\text{triv}, \text{sgn}, D_k \mid k \geq 2\}\). First, if \(\sigma\) or \(\sigma'\) is one of the representations \text{triv} or \text{sgn}, then the Rankin-Selberg product corresponds to the usual tensor product. The only other case is when \(\sigma\) and \(\sigma'\) are both discrete series representations of \(GL(2, \mathbb{R})\). In this situation one has \(D_k \times D_\ell = D_{k+\ell-1} \oplus D_{|k-\ell|+1}\).

In summary \(\sigma \times \sigma'\) is given by the following table:

| \(\sigma\) \(\setminus \sigma'\) | \text{triv} | \text{sgn} | \(D_k\) |
|-------------------------------|----------------|-------------|----------------|
| \text{triv}  | \text{triv} | \text{sgn} | \(D_k\) |
| \text{sgn}  | \text{sgn} | \text{triv} | \(D_k\) |
| \(D_\ell\)  | \(D_\ell\) | \(D_{k+\ell-1} \oplus D_{|k-\ell|+1}\) |

If \(k = \ell\) there is no representation \(D_1\), yet we use the convention \([A.6]\) to write \(L(s, D_1) = \Gamma_\infty(s)\). In light of \([A.3]\), it is equivalent to regard \(D_1\) as \text{triv} \(\oplus\) \text{sgn}.

We now come to the exterior square representation \(Ext^2\) that maps \(GL(n) \rightarrow GL(\frac{(n+1)(n-1)}{2})\). It satisfies the following formal rules:

\[
Ext^2 (\pi \oplus \pi') = (Ext^2 \pi \oplus \pi') \oplus (\oplus_{1 \leq j < k \leq r} (\pi_j \times \pi_k)) \quad \text{(A.12)}
\]

and

\[
Ext^2 (\pi[s]) = (Ext^2 \pi) [2s]. \quad \text{(A.13)}
\]

Similarly to the above situation of tensor products, it is completely determined by the table:

| \(\sigma\) | \text{triv} | \text{sgn} | \(D_k\) |
|----------------|----------------|-------------|----------------|
| \text{Ext}^2\sigma | \emptyset | \emptyset | \text{sgn}^k |

The notation \(\emptyset\) here indicates not to include a corresponding term in the formal sum; equivalently, \(L(s, \emptyset) = 1\).

As a final example, consider the symmetric square representation \(Sym^2\) that maps \(GL(n) \rightarrow GL(\frac{(n+1)n}{2})\). It satisfies both rules \([A.12]\) and \([A.13]\), with the substitution of \(Sym^2\) for \(Ext^2\), and is completely determined by the table:

| \(\sigma\) | \text{triv} | \text{sgn} | \(D_k\) |
|----------------|----------------|-------------|----------------|
| \(Sym^2\sigma\) | \text{triv} | \text{sgn} | \(D_{2k-1} \oplus \text{sgn}^{k+1}\) |

To illustrate, we will conclude by explicitly calculating \(L_\infty(s, \pi, Ext^2 \otimes \chi)\) when \(\pi\) is a cuspidal automorphic representation of \(GL(n)\) over \(\mathbb{Q}\), and \(\chi\) is a Dirichlet character. We write \(\pi_\infty\) as the isobaric sum

\[
\Pi = (\bigoplus_{i=1}^{r_1} \text{sgn}^s [s_i]) \oplus (\bigoplus_{j=1}^{r_2} D_{k_j}[s_{r_1+j}]), \quad \text{(A.14)}
\]

as this is its most general form according to the description in section \([A.1]\). The rules \([A.12], A.13]\) show that

\[
Ext^2 \Pi = \Pi_1 \oplus \Pi_2 \oplus \Pi_3 \oplus \Pi_4 \oplus \Pi_5, \quad \text{(A.15)}
\]

where

\[
\begin{align*}
\Pi_1 & = (\bigoplus_{i=1}^{r_1} (Ext^2 \text{sgn}^s_i)[2s_i]) = \bigoplus_{i=1}^{r_1} \emptyset[2s_i] = \emptyset, \\
\Pi_2 & = (\bigoplus_{j=1}^{r_2} (Ext^2 D_{k_j})[2s_{r_1+j}]) = \bigoplus_{j=1}^{r_2} \text{sgn}^{k_j}[2s_{r_1+j}], \\
\Pi_3 & = (\bigoplus_{j \leq r_1} (\text{sgn}^s \times D_{k_j})[s_i + s_{r_1+j}]) = \bigoplus_{j \leq r_1} D_{k_j}[s_i + s_{r_1+j}], \\
\Pi_4 & = (\bigoplus_{1 \leq i < k \leq r_1} (\text{sgn}^s \times \text{sgn}^s_k)[s_i + s_k]) = \bigoplus_{1 \leq i < k \leq r_1} \text{sgn}^{s_i + s_k}[s_i + s_k],
\end{align*}
\]

(A.16)
If we choose \( \varepsilon \text{ and } \varepsilon' \) respectively, then
\[
\prod_{1 \leq j < \ell \leq r_2} (D_{k_j} \times D_{k_{\ell}})[s_{r_1+j} + s_{r_1+\ell}]
\]
\[
= \prod_{1 \leq j < \ell \leq r_2} (D_{k_j+k_{\ell}} \times D_{k_{|\ell|-1}})[s_{r_1+j} + s_{r_1+\ell}) \oplus D_{|k_j-k_{\ell}|+1}[s_{r_1+j} + s_{r_1+\ell}]).
\]
(A.17)

If we choose \( \varepsilon_{ik} \) and \( \varepsilon'_j \in \{0, 1\} \) to be congruent to \( \varepsilon_i + \varepsilon_k \) and \( k_j \) modulo 2, respectively, then
\[
L(s, \Pi_1) = 1,
\]
\[
L(s, \Pi_2) = \prod_{j=1}^{r_2} \Gamma_R(s + 2s_{r_1+j} + \varepsilon'_j),
\]
\[
L(s, \Pi_3) = \prod_{\begin{smallmatrix} i \leq r_1 \cr j \leq r_2 \end{smallmatrix}} \Gamma_C(s + s_i + s_{r_1+j} + \frac{k_i-1}{2}),
\]
\[
L(s, \Pi_4) = \prod_{1 \leq i < k \leq r_1} \Gamma_C(s + s_i + s_k + \varepsilon_{ik}), \quad \text{and}
\]
\[
L(s, \Pi_5) = \prod_{1 \leq j < \ell \leq r_2} \Gamma_C(s + s_{r_1+j} + s_{r_1+\ell} + \frac{k_j+k_{\ell}-2}{2})\Gamma_C(s + s_{r_1+j} + s_{r_1+\ell} + \frac{|k_j-k_{\ell}|}{2}).
\]
(A.18)

Consequently, \( L_\infty(s, \pi, Ext^2) = L(s, Ext^2\Pi) = L(s, \Pi_2)L(s, \Pi_3)L(s, \Pi_4)L(s, \Pi_5) \) is the product of these factors.

The archimedean component \( \chi_\infty \) of the character \( \chi \) is \( sgn^\eta \), where \( \eta \) is the parity parameter of \( \chi \) defined by \( \chi(-1) = (-1)^\eta \). The isobaric decomposition of \( Ext^2\pi_\infty \otimes \chi_\infty = \bigoplus_{j=1}^{5} (\Pi_j \otimes sgn^\eta) \)
(A.19)

may be computed using the tensoring rules above. These imply that \( \Pi_1, \Pi_3, \) and \( \Pi_5 \) are unchanged by tensoring with \( \chi_\infty \), and that \( \Pi_2 \) and \( \Pi_4 \) change by adding \( \eta \) to their exponents of \( \text{sgn} \). The result is that
\[
L_\infty(s, \pi, Ext^2 \otimes \chi) = L(s, \Pi_1)L(s, \Pi_2 \otimes sgn^\eta)L(s, \Pi_3)L(s, \Pi_4 \otimes sgn^\eta)L(s, \Pi_5),
\]
(A.20)

where
\[
L(s, \Pi_2 \otimes sgn^\eta) = \prod_{j=1}^{r_2} \Gamma_R(s + 2s_{r_1+j} + \varepsilon'_j),
\]
\[
L(s, \Pi_4 \otimes sgn^\eta) = \prod_{1 \leq i < k \leq r_1} \Gamma_R(s + s_i + s_k + \varepsilon_{ik\eta}),
\]
(A.21)

and \( \varepsilon'_j \) and \( \varepsilon_{ik\eta} \in \{0, 1\} \) are congruent to \( \varepsilon'_j + \eta \equiv k_j + \eta \) and \( \varepsilon_{ik} + \eta \equiv \varepsilon_i + \varepsilon_k + \eta \) (mod 2), respectively.

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