Existence of Steady States for Over-the-Counter Market Models with Several Assets

Alain Bélanger\textsuperscript{a,}\textsuperscript{*,} Gaston Giroux\textsuperscript{a}, Ndoune Ndoune\textsuperscript{a}

\textsuperscript{a}Faculté d’administration, Université de Sherbrooke, Québec, J1K2R1, Canada

Abstract
We introduce and study three classes of over-the-counter markets specified by systems of Ordinary Differential Equations (ODE’s), in the spirit of Duffie-Gărleanu-Pedersen [1]. The key innovation is allowing for multiple assets. We compute the steady states for these ODE’s.

Keywords:
Non-linear ODE’s, Steady state, Market structure, Point processes

JEL: C30, D53, G10

1. Introduction
This article addresses the question of the existence of the stationary law in relatively opaque Over-the-Counter (OTC) markets with several traded assets. The three types of markets we study here are inspired by the widely cited and pioneering work of Duffie, Gărleanu and Pedersen (see Duffie et al. [1]). The financial crisis of 2008 brought significant concerns regarding the rôle of OTC markets, particularly from the viewpoint of liquidity and financial stability. Darrell Duffie’s recent monograph, Dark Markets (see Duffie [2]), documents some of the modelling efforts done to understand the effects of illiquidity associated with search and bargaining. Duffie also notes that this area is still underdeveloped in contrast with the vast literature available on central market mechanisms.

Our goal is to shed some light on foundational issues in asset pricing in OTC markets with several assets. In particular, we study three models of OTC markets described by ODE’s which have not yet appeared in the differential equations literature. For the specialists in financial economics, it is well known that in OTC markets, an investor who wishes to sell must search for a buyer, incurring opportunity and other costs until one is found (see for instance Duffie et al. [1]). For the case of one asset, when the bargaining results in all or
nothing, the evolution of an investor’s state can be described by a system of four quadratic differential equations, an overview is given in Chapter 4 of Duffie[2]. There the author develops a search-theoretic model of the cross-sectional distribution of asset returns, under the hypothesis that the eagerness of the investors are the same whether they have the asset or not. Here we study the more general case with several assets for three classes of extended models which are still described by systems of quadratic differential equations, but without the particular hypothesis. One should notice that without changes of positions the system would stop after a finite time and the market would become inefficient.

For the first extended model, we do not track the particular asset an investor wants to buy when she enters the market (it is called the non-segmented model/case); but the frequency at which she enters the market depends on that asset. For the second model, we do keep track of the asset an investor intends to purchase (it is called the partially-segmented model/case). In both of these models an investor’s position in an asset is binary: she owns it or she does not. In our third model, we allow investors to have different proportions of an asset or different mixes of assets as in Gärleanu’s portfolio model Gärleanu[3]. Unlike Gärleanu though we allow investors to trade with each other without the intervention of a market-maker. We will call this third model the heterogeneous position model.

In all our cases, the quantities of each asset do not have to be the same. Here we study these classes of markets in the spirit of Duffie et al.[1]. When there is only one traded asset, as in Duffie et al.[1], the first two cases collapse to the same model. Unlike DGP, we do not assume that the investors’ eagerness is the same whether they own the asset or not. The departure from this assumption in DGP requires us to use techniques from the theory of dynamical systems.

In such a framework, we describe our models and show the existence of a steady state (this steady state is designated, in the financial literature, by the equilibrium (time-invariant) cross-sectional variation in the distribution of ownership).

We show the existence of steady states for the first model by straightforward calculations, and for the other two models by using the Poncaré-Miranda theorem (see Kulpa[4]). The first two models always admit a unique steady state for every parameters of the system. For the model with heterogeneous positions, sufficient conditions are given for the existence of the steady state and a counterexample shows that some conditions on the parameters are necessary.

In Section 2, we describe the three classes of models. In Section 3.1, we show the existence of a steady state and compute it explicitly for the non-segmented case for any given number of assets. In Section 3.2, we do the same for the case of partially-segmented markets using the Poncaré-Miranda theorem in Kulpa[4]. In Section 3.3, we use again the Poncaré-Miranda theorem to show the existence of a steady state for the heterogeneous case under certain conditions and we give an example of a market without steady state. Finally, in section 4, we briefly describe an algorithm using the Poincaré-Miranda that enables us to calculate an approximation of the steady state to any desired level of accuracy.
2. Three classes of models

Duffie [2] and Duffie et al. [1] present their model of OTC market with one traded asset as a system of four linear ODE’s with two constraints which can be reduced to a system of two differential equations with two constraints. In this section, we describe two extensions of their model involving \( K \geq 1 \) assets. Before describing each model in details, we would like to set up a few general definitions.

The set of available assets will be denoted \( \mathcal{I} = \{1, \ldots, K\} \). Investors can hold at most one unit of any asset \( i \in \mathcal{I} \) and cannot short-sell. Time is treated continuously and runs forever. The market is populated by a continuum of investors. At each time, an investor is characterized by whether he owns the \( i \)-th asset or not, and by an intrinsic type which is either a ‘high’ or a ‘low’ liquidity state. Our interpretation of liquidity state is the same as in Duffie et al. [1]. For example, a low-type investor who owns an asset may have a need for cash and thus wants to liquidate his position. A high-type investor who does not own an asset may want to buy the asset if he has enough cash. Through time, investors’ ownerships will switch randomly because of meetings leading to trades, at a rate \( \lambda_i \), and the investor’s intrinsic type will change independently via an autonomous movement. This dynamics of investor’s type change is modeled by a (non-homogeneous) continuous-time Markov chain \( Z(t) \) on the finite set of states \( E \). This set \( E \) will be described in more details in each one of the following subsections since it depends on the model.

At any given time \( t \), let \( \mu_t(z) \) denote the proportion of investors in state \( z \in E \), i.e. for each \( t \geq 0 \), \( \mu_t \) is a probability law on \( E \).

Let \( m_i \) denote the proportion of asset \( i \), for all \( i \in \mathcal{I} \).

2.1. Non-segmented markets

In this simpler model, we recall that we do not track the particular asset an investor wants to buy when entering the market. Let \( l \) and \( h \) denote respectively a low liquidity and a high liquidity type and let \( o \) and \( n \) denote respectively whether an investor owns or does not own an asset. Then, the set of investors’ states is fully described as follows: \( E = \{(l, n), (h, n), (hi, o), (li, o)\} \).

As we said earlier, we do not assume the eagerness of investors is the same when they own the asset and when they don’t. For investors not-owning an asset, let us denote the switching intensity from low-type to high-type by \( \gamma_u \) and conversely the switching intensity from high-type to low-type by \( \gamma_d \). For investors owning asset \( i \), we will denote the switching intensity from low-type to high-type by \( \gamma_{ui} \) and conversely the switching intensity from high-type to low-type by \( \gamma_{di} \). In addition, investors meet each other at rate \( \lambda_i \), and an exchange of the asset occurs when an investor of type \( (li, o) \) (owns asset \( i \) but has a low liquidity state) meets one of type \( (h, n) \) (does not own an asset but has a high interest for acquiring one).

Hence, the dynamical system describing the evolution of the proportions of investors in a given state is the following system of \( 2K + 2 \) equations with \( K + 1 \)
constraints for $\mu_t(z)$ for each $z \in E$:

$$
\dot{\mu}_t(h, n) = -\mu_t(h, n) \sum_{i \in I} \lambda_i \mu_t(li, o) + \gamma_1 \mu_t(l, n) - \gamma_2 \mu_t(h, n) \tag{1}
$$

$$
\dot{\mu}_t(l, n) = \mu_t(h, n) \sum_{i \in I} \lambda_i \mu_t(li, o) - \gamma_1 \mu_t(l, n) + \gamma_2 \mu_t(h, n) \tag{2}
$$

$$
\dot{\mu}_t(h_i, o) = \lambda_i \mu_t(h, n) \mu_t(li, o) + \gamma_1 \mu_t(li, o) - \gamma_2 \mu_t(h_i, o), \forall i \in I \tag{3}
$$

$$
\dot{\mu}_t(li, o) = -\lambda_i \mu_t(h, n) \mu_t(li, o) - \gamma_1 \mu_t(li, o) + \gamma_2 \mu_t(h_i, o), \forall i \in I \tag{4}
$$

with the constraints

$$
\mu_t(h_i, o) + \mu_t(li, o) = m_i, \forall i \in I
$$

$$
\sum_{i \in I} m_i + \mu_t(h, n) + \mu_t(l, n) = 1
$$

This is a first generalized version of the system described in Duffie et al. [1].

A schematic of the dynamics between investors for this class of market with two assets is illustrated on Figure 1.

Since equation (2) and equation set (3) can be eliminated respectively by adding (2) to (1) and by adding each equation of (4) to (3), the initial system described by a set of $2 + 2K$ equations is reduced to the following set of $1 + K$
equations:
\[
\dot{\mu}_t(h,n) = -\mu_t(h,n) \sum_{i \in \mathcal{I}} \lambda_i \mu_t(li,o) + \gamma_{ui} \mu_t(l,n) - \gamma_d \mu_t(h,n)
\]
\[
\dot{\mu}_t(li,o) = -\lambda_i \mu_t(h,n) \mu_t(li,o) - \gamma_{ui} \mu_t(li,o) + \gamma_{di} \mu_t(li,o), \quad \forall i \in \mathcal{I}
\]
with the \(1 + K\) constraints
\[
\mu_t(hi,o) + \mu_t(li,o) = m_i, \quad \forall i \in \mathcal{I}
\]
\[
\sum_{i \in \mathcal{I}} m_i + \mu_t(h,n) + \mu_t(l,n) = 1
\]

Note that in the first set of constraints, \(m_i\) is the fraction of the investors’ population holding the \(i\)-th asset, with \(\sum_{i \in \mathcal{I}} m_i < 1\). The second constraint is the investors’ proportions normalisation. Moreover, since all parameters are positive, a minus sign in the system means an exit from the state and a positive sign means an entry in the state.

The system (5) is the Master Equation. It is non-linear but there is nevertheless for each initial law \(\mu_0\) a probability law \(P^{\mu_0}\) on the pure jump trajectories \(Z(t)\) on \(E\), which has the Markov property. We do not have, however, that this law \(P^{\mu_0}\) is the convex combination \(\sum_{z \in E} \mu_0(z) P^{\delta_z}\), where \(\delta_z\) are Dirac masses.

The existence of \(P^{\mu_0}\), on the pure jump trajectories, can be obtained by solving a martingale problem which is built with the intensity measure, \(m\), defined as follows \(\forall s \in [t, \infty)\):

\[
m(s, (h,n); (hi,o)) = \lambda_i \mu_s(li,o); \quad m(s, (li,o); (li,o)) = \lambda_i \mu_s(h,n);
\]
\[
m(s, (hi,o); (hi,o)) = \gamma_{ui}; \quad m(s, (hi,o); (li,o)) = \gamma_{di};
\]
\[
m(s, (l,n); (h,n)) = \gamma_u; \quad m(s, (h,n); (l,n)) = \gamma_d;
\]

for \(s \in [t, \infty)\), other terms being 0. This intensity measure satisfies the conditions of Theorem 2.1, page 216, of Stroock [5]. So, once we have solved the ODE system, for each initial condition \(\mu_0\), we see that there exists a probability measure \(P^{\mu_0}\). The fact that this law is supported by the set of pure jump trajectories can be proved as in Lemma 1, page 588, of Sznitman [6]. It is such a description that we use below to obtain an expression for the intrinsic value associated to the state of an investor at each time. Using the properties of this expression we can then evaluate the directly negotiated prices among investors in our relatively opaque market. One can also consult Appendix I of Duffie [7] for a review of the basic theory of intensity-based models.

It is worth noticing that the laws \(P^{\mu_0}\) can be obtained by a functional law of large numbers as in Ferland and Giroux [8] or by rewriting the system with the help of a single kernel and then using Theorem 1 of Bélanger and Giroux [9].

Weill [10] proposed a similar system with the assumption that the investors’ eagerness is the same for all assets.

### 2.2. Partially segmented markets

In this class of models, buyers who do not hold an asset enter the market with a specific asset they want to purchase. Hence, the set of investors’ type is given
by $E = \{(l, n), (hi, o), (hi, n), (li, o)\}_{i \in I}$. As before, the first letter designates
the investor’s intrinsic liquidity state and the second letter designates whether
the investor owns the asset or not.

In this case, the eagerness’ parametrization is the following: If an investor
initially does not own any asset and is a low-type, the switching intensity of
becoming a high-type is $\tilde{\gamma}_{ui}$ and it now depends on the asset type. If he initially
does not own any asset but is a high-type, he will seek to buy a specific asset $i$ and
his switching intensity of becoming a low-type is $\gamma_{ui}$. If he initially owns a specific asset $i$ and is a high-type (that is, he wants to keep his asset), the switching intensity
of becoming a low-type is $\gamma_{di}$. If he initially owns a specific asset $i$ but is a
low-type, the switching intensity of becoming a high-type is $\gamma_{ui}$. In addition,
investors meet each other at rate $\lambda_i$, but an exchange of the asset occurs only
if an investor of type $(li, o)$ meets one of type $(hi, n)$.

Hence, we have the following dynamical system of investors’ type proportions
measure $\mu_t(z)$ for each $z \in E$, which consists of $3K + 1$ equations with $K + 1$
constraints:

\[
\begin{align*}
\dot{\mu}_t(hi, n) &= -\lambda_i \mu_t(hi, n) \mu_t(li, o) + \tilde{\gamma}_{ui} \mu_t(l, n) - \tilde{\gamma}_{di} \mu_t(hi, n), \forall i \in I \\
\dot{\mu}_t(l, n) &= \sum_{i \in I} \lambda_i \mu_t(hi, n) \mu_t(li, o) - \sum_{i \in I} \tilde{\gamma}_{ui} \mu_t(l, n) + \sum_{i \in I} \tilde{\gamma}_{di} \mu_t(hi, n) \\
\dot{\mu}_t(hi, o) &= \lambda_i \mu_t(hi, n) \mu_t(li, o) + \gamma_{ui} \mu_t(li, o) - \gamma_{di} \mu_t(hi, o), \forall i \in I \\
\dot{\mu}_t(li, o) &= -\lambda_i \mu_t(hi, n) \mu_t(li, o) - \gamma_{ui} \mu_t(li, o) + \gamma_{di} \mu_t(hi, o), \forall i \in I
\end{align*}
\]

with the constraints

\[
\sum_{i \in I} m_i + \sum_{i \in I} \mu_t(hi, n) + \mu_t(l, n) = 1
\]

A schematic for the dynamics between investors for this class of models in a
two assets-market ($K = 2$) is illustrated on Figure 2.

Note that equation (9) of the previous system can be eliminated by adding
each equation of (8) to (9). Similarly, each equation of (10) can be eliminated
by adding it to the corresponding equation of (11). The system is then reduced
to the following system of $2K$ equations:

\[
\begin{align*}
\dot{\mu}_t(hi, n) &= -\lambda_i \mu_t(hi, n) \mu_t(li, o) + \tilde{\gamma}_{ui} \mu_t(l, n) - \tilde{\gamma}_{di} \mu_t(hi, n), \forall i \in I \\
\dot{\mu}_t(li, o) &= -\lambda_i \mu_t(hi, n) \mu_t(li, o) - \gamma_{ui} \mu_t(li, o) + \gamma_{di} \mu_t(hi, o), \forall i \in I
\end{align*}
\]

with the $1 + K$ constraints

\[
\sum_{i \in I} m_i + \sum_{i \in I} \mu_t(hi, n) + \mu_t(l, n) = 1
\]
The system (12) is our Master Equation and we define the intensity measure, \( m \), as follows \( \forall i \in I \):

\[
\begin{align*}
  m(s, (hi, n); (hi, o)) &= \lambda_i \mu(s, li, o) ; \quad m(s, (li, o); (l, n)) = \lambda_i \mu(s, hi, n) ; \\
  m(s, (li, o); (hi, o)) &= \gamma_{ui} ; \quad m(s, (hi, o); (li, o)) = \gamma_{di} ; \\
  m(s, (l, n); (hi, n)) &= \tilde{\gamma}_{ui} ; \quad m(s, (hi, n); (l, n)) = \tilde{\gamma}_{di} ;
\end{align*}
\]

for \( s \in [t, \infty) \), other terms being 0.

Vayanos and Wang [11] proposed a similar two asset market.

### 2.3. Heterogeneous positions markets

In this model, investors take multiple positions in the assets and maintain the possibility to trade those assets with each other without necessarily having to transact with a market-maker. The set of asset positions is denoted by \( I = \{0, 1, 2, ..., M\} \) and it represents all the price in tick size of a portfolio of assets. The set of investors’ state in this market, \( E = \{(h, j), (l, j)|0 \leq j \leq M\} \), is specified by its liquidity status and the price of the portfolio of assets it holds.

We will restrict ourselves to the case where \( M = 2 \) and we will assume we have \( N \) investors in the market. The state of the market is initially given by the vector \( X(0) = (X_1(0), X_2(0), ..., X_N(0)) \) where \( X_i(0) = (Y_i(0), Z_i(0)) \) with \( Y_i(0) \) either \( h \) or \( l \); and \( Z_i(0) \) taking the values 0, 1 or 2. After an exponential time, \( T \), the state of the market, \( X(T) = (X_1(T), X_2(T), ..., X_N(T)) \), will be different form \( X(0) \) by only one of its \( N \) components if an investor has had an autonomous change in its liquidity status or \( X(T) \) will differ from \( X(0) \) components if
a trade occurred between two investors. More specifically, the independent changes from \((l, i)\) to \((h, i)\) and conversely, from \((h, i)\) to \((l, i)\) happen with intensities respectively of \(\gamma_{ui}\) and \(\gamma_{di}\). Independent of these movements there are also transactions occurring in the market. We have the following transactions occurring with intensity \(\lambda\): \((h, 0) \rightarrow (l, 1)\) and symmetrically \((l, 1) \rightarrow (h, 0)\); and \((h, 1) \rightarrow (l, 0)\) and symmetrically \((l, 0) \rightarrow (h, 1)\). Finally, we have the following transactions with intensity \(\lambda a\) and \(\lambda b\) respectively for \(a + b = 1\): \((h, 0) \rightarrow (h, 1)\) and symmetrically \((l, 2) \rightarrow (h, 0)\); and \((h, 0) \rightarrow (l, 2)\) and symmetrically \((l, 0) \rightarrow (h, 2)\). Figure 3 gives a schematic representation of the market evolution. This is the dynamics of a continuous-time, non-linear Markov chain. Classical (i.e. linear) continuous-time Markov chains are treated at length in many books like Norris [12], for instance. It is convenient to take \(\lambda_N = \lambda \frac{N-1}{2}\).

When \(N\) tends to infinity, we have from Ferland et Giroux Ferland and Giroux [8] a functional law of large numbers which enables us to use the following dynamical system to describe the time-evolution of the investors’ proportions in the market just as we did for the other two models.
\[
\begin{align*}
\frac{d\mu_t(h, 0)}{dt} &= -\mu_t(h, 0)\mu_t(l, 1) - \mu_t(h, 0)\mu_t(l, 2) - \mu_t(h, 0)c_0 + \mu_t(l, 0)d_0 \quad (1) \\
\frac{d\mu_t(h, 1)}{dt} &= \mu_t(h, 0)\mu_t(h, 1) + \mu_t(h, 0)\mu_t(l, 2)a - \mu_t(h, 1)\mu_t(l, 1) - \mu_t(h, 1)c_1 + \mu_t(l, 1)d_1 \quad (2) \\
\frac{d\mu_t(h, 2)}{dt} &= \mu_t(h, 0)\mu_t(l, 2)b + \mu_t(h, 1)\mu_t(l, 1) + \mu_t(h, 1)\mu_t(l, 2) - \mu_t(h, 2)c_2 + \mu_t(l, 2)d_2 \quad (3) \\
\frac{d\mu_t(l, 0)}{dt} &= \mu_t(h, 0)\mu_t(l, 1) + \mu_t(h, 0)\mu_t(l, 2)b + \mu_t(l, 1)\mu_t(h, 1) + \mu_t(h, 0)c_0 - \mu_t(l, 0)d_0 \quad (4) \\
\frac{d\mu_t(l, 1)}{dt} &= \mu_t(h, 0)\mu_t(l, 2)a - \mu_t(l, 1)\mu_t(h, 0) + \mu_t(l, 2)\mu_t(h, 1) - \mu_t(h, 1)\mu_t(l, 1)c_1 - \mu_t(l, 1)d_1 \quad (5) \\
\frac{d\mu_t(l, 2)}{dt} &= -\mu_t(l, 2)\mu_t(h, 0)a - \mu_t(l, 2)\mu_t(h, 1) + \mu_t(h, 2)c_2 - \mu_t(l, 2)d_2 \quad (6)
\end{align*}
\]

With the following constraints: \(\mu_t(h, 0) + \mu_t(h, 1) + \mu_t(h, 2) + \mu_t(l, 0) + \mu_t(l, 1) + \mu_t(l, 2) = 1\) et \(\mu_t(h, 1) + \mu_t(l, 1) + 2(\mu_t(h, 2) + \mu_t(l, 2)) = s\), where \(0 \leq s \leq 2\).

The system above is our Master Equation of this section and we define the intensity measure, \(m\), as follows for all \(i \in \{0, 1, 2\}\):

\[
\begin{align*}
&\quad m(s, (h, 1); (h, 2)) = \lambda \mu_s(l, 1); \quad m(s, (l, 1); (l, 0)) = \lambda [\mu_s(h, 0) + \mu_s(h, 1)]; \\
&\quad m(s, (h, 0); (h, 1)) = \lambda [\mu_s(l, 1) + \lambda \mu_s(l, 2)]; \quad m(s, (l, 2); (l, 1)) = \lambda [\mu_s(h, 1) + \lambda \mu_s(h, 0)]; \\
&\quad m(s, (l, 2); (l, 0)) = \lambda \mu_s(l, h, 0); \quad m(s, (h, 0); (h, 2)) = \lambda b \mu_s(l, 2); \\
&\quad m(s, (h, i); (l, i)) = \gamma_{ii} = c_i; \quad m(s, (l, i); (h, i)) = \gamma_{di} = d_i;
\end{align*}
\]

for \(s \in [0, 2]\), other terms being 0.

\[3. \text{ The steady state of ODE systems}\]

We have a steady state when the left hand side of our systems (5) and (12) are equal to zero. That is, when there is no longer any dependence on time. We first state the Poincaré-Miranda theorem which will get used later. Let \(x_i\) be the \(i\)th coordinate of the vector \(x \in \mathbb{R}^n\). Let \(I = [0, 1]^n\) be the \(n\)-dimensional cube in the Euclidean space \(\mathbb{R}^n\). For each \(i \leq n\), let us denote \(I_i^- = \{x \in I | x_i = 0\}\) and \(I_i^+ = \{x \in I | x_i = 1\}\).

**Theorem 3.1.** (Poincaré-Miranda) Let \(f = (f_1, f_2, ..., f_n) : I \rightarrow \mathbb{R}^n\) be a continuous map such that for all \(i \leq n\) \(f_i(I_i^-) \subset [\infty, 0]\) and \(f_i(I_i^+) \subset [0, \infty]\). Then there exists a point \(\omega\) in \(I\) such that \(f(\omega) = 0\).
3.1. Non-segmented markets

The following proposition asserts the existence, uniqueness of the steady state for non-segmented markets under arbitrary specification of parameters.

**Proposition 3.1.** For each non-segmented market described above, there is a unique solution satisfying the system (5)-(7).

**Proof.**

Here, we need to solve the following system of equations:

\[
0 = -\mu(h, n) \sum_{i \in I} \lambda_i \mu(l, o) + \gamma_u \mu(l, o) - \gamma_d \mu(h, n) \tag{15}
\]

\[
0 = -\lambda_i \mu(h, n) \mu(l, o) - \gamma_u \mu(l, o) + \gamma_d \mu(hi, o), \quad \forall \in I \tag{16}
\]

First, note that we can eliminate \(\mu(h, n)\) in (15) using the constraint equation (7). Thus, (15) becomes

\[
0 = -\mu(h, n) \sum_{i \in I} \lambda_i \mu(l, o) + \gamma_u \left(1 - \sum_{i \in I} m_i - \mu(h, n)\right) - \gamma_d \mu(h, n)
\]

\[
= -\mu(h, n) \sum_{i \in I} \lambda_i \mu(l, o) + \gamma_u \left(1 - \sum_{i \in I} m_i\right) - \gamma \mu(h, n) \tag{17}
\]

where \(\gamma \equiv \gamma_d + \gamma_u\). Moreover, to simplify the last equation, we then subtract the \(K\) equations of (16) to (17) to have

\[
0 = -\mu(h, n) \sum_{i \in I} \lambda_i \mu(l, o) + \sum_{i \in I} \left[\lambda_i \mu(h, n) \mu(l, o) + \gamma_u \mu(l, o) - \gamma_d \mu(hi, o)\right]
\]

\[
+ \gamma_u \left(1 - \sum_{i = 1}^{K} m_i\right) - \gamma \mu(h, n)
\]

\[
= \sum_{i \in I} \gamma_u \mu(l, o) - \sum_{i \in I} \gamma_d \mu(hi, o) + \gamma_u \left(1 - \sum_{i \in I} m_i\right) - \gamma \mu(h, n)
\]

By using the constraint equation (6) to replace each \(\mu(hi, o)\) in the last equation, we then have

\[
0 = \sum_{i \in I} \gamma_u \mu(l, o) - \sum_{i \in I} \gamma_d (m_i - \mu(l, o)) + \gamma_u \left(1 - \sum_{i \in I} m_i\right) - \gamma \mu(h, n)
\]

\[
= \sum_{i \in I} \gamma_i \mu(l, o) - \sum_{i \in I} \gamma_d m_i + \gamma_u \left(1 - \sum_{i \in I} m_i\right) - \gamma \mu(h, n) \tag{18}
\]

where \(\gamma_i \equiv \gamma_d + \gamma_u\).
Furthermore, each of the $K$ equations in (16) gives us the identity

$$\mu(li,o) = \frac{\gamma_di m_i}{\lambda_i \mu(h,n) + \gamma_i}$$

which can be substituted into (18) to have

$$F(\mu(h,n)) \triangleq \sum_{i \in I} \gamma_i \frac{\gamma_di m_i}{\lambda_i \mu(h,n) + \gamma_i} - \sum_{i \in I} \gamma_di m_i + \gamma_u \left(1 - \sum_{i \in I} \mu(h,o)\right) - \gamma_d \mu(h,n).$$

(20)

Then, one need to solve $F(x) = 0$ for $x \triangleq \mu(h,n)$. Hence we get $\mu(h,n)$ from which we get by (6) $\mu(l,n) = 1 - \mu(h,n) - \sum_{i \in I} m_i$, each $\mu(li,o)$ by identity (19) and finally, each $\mu(hi,o) = m_i - \mu(li,o)$, by (7).

The challenge here is to solve for $F(x) = 0$. First, note that we have

1. $F(0) = \gamma_u \left(1 - \sum_{i \in I} m_i\right) > 0$ since $\gamma_u > 0$ and $\sum_{i \in I} m_i < 1$;
2. $F \left(1 - \sum_{i \in I} m_i\right) < -\gamma_d \left(1 - \sum_{i \in I} m_i\right) < 0$;
3. $F(x)$ is a decreasing function for $x \geq 0$.

So there is a positive root between 0 and $1 - \sum_{i \in I} m_i$ which can always be calculated numerically. Thus, there always exists a stationary solution $\mu(h,n)$ for any $K$.

3.2. Partially segmented markets

The following proposition asserts the existence, uniqueness and the stationarity of the steady state for the partially segmented markets.

**Proposition 3.2.** For each partially segmented market previously described, there is a unique solution satisfying the system (12)-(14).

**Proof**

From our Master equation (12), we need to solve the following system of equations:

$$0 = -\lambda_i \mu(hi,n) \mu(li,o) + \bar{\gamma}_{ui} \mu(l, n) - \bar{\gamma}_{di} \mu(hi,n), \; \forall i \in I$$

(21)

$$0 = -\lambda_i \mu(hi,n) \mu(li,o) - \gamma_{ui} \mu(li,o) + \gamma_{di} \mu(hi,o), \; \forall i \in I$$

(22)

with the constraints

$$\mu(hi,o) + \mu(li,o) = m_i, \; \forall i \in I$$

(23)

$$\sum_{i \in I} m_i + \sum_{i \in I} \mu(hi,n) + \mu(l,n) = 1$$

(24)

Using each of the constraint (23) and substituting them in each equation of (22) for $\mu(hi,o)$, we get

$$0 = -\lambda_i \mu(hi,n) \mu(li,o) - \gamma_{ui} \mu(li,o) - \gamma_{di} \mu(li,o) + \gamma_{di} m_i, \; \forall i \in I$$

11
and thus
\[ \mu(li,o) = \frac{\gamma_{di} m_i}{\lambda_i \mu(hi,n) + \gamma_i}, \forall i \in I \] (25)

where \( \gamma_i \triangleq \gamma_{ui} + \gamma_{di} \).

Now, subtracting each (22) to each (21) and using constraint (24) to substitute for \( \mu(l,n) \), we get:
\[ 0 = \tilde{\gamma}_ui \left[ 1 - \sum_{i \in I} m_i - \sum_{i \in I} \mu(hi,n) \right] - \tilde{\gamma}_{di} \mu(hi,n) + \gamma_{ui} \mu(li,o) - \gamma_{di} \mu(hi,o), \forall i \in I \]

\[ \Rightarrow \tilde{\gamma}_i \mu(hi,n) = \tilde{\gamma}_ui \left[ 1 - \sum_{i \in I} m_i \right] - \tilde{\gamma}_ui \sum_{j \neq i} \mu(hj,n) + \gamma_{ui} \mu(li,o) - \gamma_{di} \mu(hi,o), \forall i \in I \]

Using constraint (23) to substitute for \( \mu(hi,o) \) and substituting (25) for \( \mu(li,o) \), we finally get:
\[ \mu(hi,n) = -\frac{\tilde{\gamma}_ui}{\gamma_i} \sum_{j \neq i} \mu(hj,n) + \frac{\gamma_i \gamma_{di} m_i}{\tilde{\gamma}_i (\lambda_i x_i + \gamma_i)} \]
\[ - \frac{\gamma_{di} m_i}{\gamma_i} - \frac{\tilde{\gamma}_ui}{\gamma_i} \left( 1 - \sum_{i \in I} m_i \right), \forall i \in I \] (26)

Hence, we have to solve a nonlinear system of \( K \) equations in \( K \) unknowns \( \mu(hi,n) \). Once we have solved for \( \mu(hi,n) \), we can get \( \mu(li,o) \) by (25) and deduce that \( \mu(hi,o) = m_i - \mu(li,o) \), by (23), and that \( \mu(l,n) = 1 - \sum_{i \in I} m_i = \sum_{i \in I} \mu(hi,n) \), by (24).

Since the case \( K = 1 \) is the same whether the market is non-segmented or partially segmented, we have the result by the previous subsection. We will prove the general case in the following.

We set \( x_i = \mu(h_i,n), \) with \( 1 \leq i \leq K \). Let \( I = [0,1]^K \) and \( f = (f_1,f_2,...,f_K) : I \longrightarrow \mathbb{R}^K \), defined by:

\[ f_i(x) = x_i + \frac{\tilde{\gamma}_ui}{\gamma_i} \sum_{j \neq i} x_j - \frac{\gamma_i \gamma_{di} m_i}{\tilde{\gamma}_i (\lambda_i x_i + \gamma_i)} \]
\[ + \frac{\gamma_{di} m_i}{\gamma_i} - \frac{\tilde{\gamma}_ui}{\gamma_i} \left( 1 - \sum_{i \in I} m_i \right), \forall i \in I \] (27)

Because each \( f_i \), \( 1 \leq i \leq K \) is continuous map, then \( f \) is a continuous map.

Let \( i \) such that \( 1 \leq i \leq K \).
\[ f_i(I_i^-) = \frac{\tilde{\gamma}_ui}{\gamma_i} \sum_{j \neq i} x_j - \frac{\gamma_i \gamma_{di} m_i}{\tilde{\gamma}_i \gamma_i} \]
\[ + \frac{\gamma_{di} m_i}{\gamma_i} - \frac{\tilde{\gamma}_ui}{\gamma_i} \left( 1 - \sum_{i \in I} m_i \right), \forall i \in I \] (28)
Thus $f_i(I^+_i) = -\tilde{\gamma}_i \left(1 - \sum_{j \in I} m_i - \sum_{j \neq i} x_j \right)$ which is a negative quantity.

On the other hand, we have:

$$f_i(I^+_i) = 1 + \frac{\tilde{\gamma}_i}{\gamma_i} \sum_{j \neq i} x_j - \frac{\gamma_i \gamma_i m_i}{\gamma_i (\Lambda_i + \gamma_i)}$$

$$+ \frac{\tilde{\gamma}_i}{\gamma_i} m_i - \frac{\tilde{\gamma}_i}{\gamma_i} \left(1 - \sum_{i \in I} m_i\right), \forall i \in I$$

Let $L_i = -\frac{\gamma_i \gamma_i m_i}{\gamma_i (\Lambda_i + \gamma_i)} + \frac{\tilde{\gamma}_i}{\gamma_i} m_i$

and $P_i = 1 + \frac{\tilde{\gamma}_i}{\gamma_i} \sum_{j \neq i} x_j - \frac{\gamma_i}{\gamma_i} \left(1 - \sum_{i \in I} m_i\right)$.

We need to show that $L_i$ et $P_i$ are positive numbers. We have that $L_i = \frac{\gamma_i \gamma_i m_i}{\gamma_i (\Lambda_i + \gamma_i)}$, which is positive.

Because $\frac{\tilde{\gamma}_i}{\gamma_i} \leq 1$, we have that $P_i = 1 - \frac{\tilde{\gamma}_i}{\gamma_i} (x_i + \mu(l, n)) \geq 0$. Therefore, $P_i$ is positive. Hence $f_i(I^+_i) \geq 0$.

The existence of a steady state is then a consequence of the Poincaré-Miranda theorem, so there exists a point $\omega$ de $[0, 1]^K$ so that $f(\omega) = 0$. For each of the functions $f_i$, $1 \leq i \leq K$, there is a variable for which $f_i$ is a contraction in each one of the other variables. Since $f_i$ is continuous on a compact set it is uniformly continuous and the Hadamard-Perron theorem tells us that $f_i$ is a contraction in the variable $x_i$, so $f$ itself is a contraction and we have uniqueness.

3.3. Heterogeneous positions markets

We study the existence of steady state for the case $M = 2$. Let us denote by $x := \mu_i(h, 0)$, $y := \mu_i(h, 1)$, $z := \mu_i(h, 2)$, $u := \mu_i(l, 0)$, $v := \mu_i(l, 1)$ et $w := \mu_i(l, 2)$. By a random change of time, let us assume that $\lambda = 1$. Let also denote $\gamma_{ui}$ and $\gamma_{di}$ respectively by $d_i$ and $c_i$. The steady states are solutions of the following system.

\[
\begin{cases}
-xv - xw - xc_0 + ud_0 = 0 & (1) \\
-xv + xwa - yv - yw - yc_0 + vd_1 = 0 & (2) \\
xwb + yv + yw - zc_2 + wd_2 = 0 & (3) \\
xv + xwb + yv + xc_0 - ud_0 = 0 & (4) \\
xwa - xv + yw - yv - yc_0 - vd_1 = 0 & (5) \\
-xw - yw + zc_2 - wd_2 = 0 & (6)
\end{cases}
\]

This market model moreover satisfies the following two constraints: $x + y + z + u + v + w - 1 = 0$ et $y + v + 2(z + w) - s = 0$ avec $0 \leq s \leq 2$.

We will find a solution to this system in $[0, 1]^6$ and we will show that it is unique.

We have that $(1) + (4)$ is equivalent to $xwa = yv$. If we substitute this relation in each one of the equations of the homogeneous system, we get the
following system which includes the constraints:

\[
\begin{align*}
  x_v + x_w + xc_0 - ud_0 &= 0 \quad (1) \\
  x_v - yw - yc_1 + vd_1 &= 0 \quad (2) \\
  x_w + yw - zc_2 + wd_2 &= 0 \quad (3) \\
  yv - xwa &= 0 \quad (4) \\
  x + y + z + u + v + w - 1 &= 0 \quad (5) \\
  y + v + 2(z + w) - s &= 0 \quad (6)
\end{align*}
\]

With \(x, y, z, u, v, w\) belonging to \([0, 1]\) and \(0 \leq s \leq 2\).

In contrast with the other two models, we can choose parameters for which the corresponding model with heterogeneous positions does not have a steady state.

Let \(c_0 = 0, d_0 = 1, a = 1, c_1 = 0, d_1 = 0, b = 0, c_2 = 1, d_2 = 0\). So the system becomes:

\[
\begin{align*}
  x_v + x_w - u &= 0 \quad (1) \\
  x_v - yw &= 0 \quad (2) \\
  x_w + yw - z &= 0 \quad (3) \\
  yv - xwa &= 0 \quad (4) \\
  x + y + z + u + v + w - 1 &= 0 \quad (5) \\
  y + v + 2(z + w) - s &= 0 \quad (6)
\end{align*}
\]

From (2) and (4) we get \(y^2 w = x^2 w\). If we take \(y = x\) then we get \(v = w\). Then equations (1) and (3) give us, in turn, \(u = 2xv = z\).

If we substitute in (5) we get \(v = 1 - 2x^2 + 4x^3\). And then substituting in (5) we get that \(x\) must satisfy the equation \(4X^2 + 4sX + 2s - 3 = 0\), with positive discriminant \(\Delta = 16(s^2 - 2s + 3)\). The only root to consider is \(x = (-s + \sqrt{s^2 - 2s + 3})/2\), since the other root is negative for all values of \(s\).

But \(x\) is also negative if \(\frac{s}{2} \leq s \leq 2\), so there is no steady state for these values. But there are sufficient conditions on the parameters for the existence of a steady state. Once again, we will obtain it by an appeal to the Poincaré-Miranda theorem.

For \(n = 6\) et define \(f = (f_1, f_2, ..., f_6)\) par \(f_1(x, y, z, u, v, w) = x_v + x_w + xc_0 - ud_0; f_2(x, y, z, u, v, w) = yv - xwa\); \(f_3(x, y, z, u, v, w) = y + v + 2(z + w) - s; f_4(x, y, z, u, v, w) = x + y + z + u + v + w - 1; f_5(x, y, z, u, v, w) = x_v - yw - yc_1 + vd_1\) and \(f_6(x, y, z, u, v, w) = x_w + yw - zc_2 + wd_2\).

We have \(f_i(I^-_i) \subset ]-\infty, 0]\) for \(i \in \{1, 2, 3, 4, 5, 6\}\) and \(f_i(I^+_i) \subset [0, \infty]\) for
$i \in \{3, 4\}$. In the other cases we have to impose the following conditions:

$$\begin{align*}
& v + w + c_0 - ud_0 \geq 0 \quad (1) \\
& v - xwa \geq 0 \quad (2) \\
& x - yw - yc_1 + d_1 \leq 0 \quad (3) \\
& x + y - zC_2 + d_2 \leq 0 \quad (4)
\end{align*}$$

We denote by (P) the conjunction of these four conditions.

**Proposition 3.3.** If the set of parameters used in a market with heterogeneous positions satisfies (P) then there is a unique steady state.

Proof When the market satisfies the condition (P), a straightforward calculation allows for an approximation of the steady state.

4. An approximation algorithm

The Poincaré-Miranda theorem allows for more than the mere existence of the steady state. By successive applications of the theorem to smaller rectangles within $L_0 = [0, 1]^M$, where $M$ is the number of assets, we can approximate the steady state to any level of desired accuracy. Let $\varepsilon > 0$ be such a level of accuracy. Let $\mathcal{H}_i^+$ be the hyperplane parallel to the support of the face $I_i^+$ of the rectangle $L_0 = [0, 1]^M$, and passing through the centroid $J_1$ of $L_0$. The subdivision of $L_0$ by the hyperplanes $\mathcal{H}_i^+$ gives $2^M$ rectangles, each having a volume equal to $\frac{1}{2^M}$. According to the Poincaré-Miranda theorem applied to each of the $2^M$ rectangles, the steady state belongs exactly to one of these rectangles, which we will call the active rectangle.

In the following description, $k$ stands for the iteration index, $L_k$ denotes the active rectangle and $\mathbb{P}_k$ is the set of all rectangles obtained by the subdivision of the active rectangle $L_{k-1}$. The point $J_k$ is the centroid of $L_{k-1}$ and $\mathcal{H}_i^+[k]$ is the hyperplane parallel to $I_i^+$ and passing through the centroid $J_k$ of $L_{k-1}$.

- Set $k = 0$, $L_0 = [0, 1]^n$, $\omega \in L_0$ such that $f(\omega) = 0$.

- $k = 1$,
  if $\omega = J_1$, end;
  if not, subdivide $L_0$ by the hyperplanes $\mathcal{H}_i^+$; pick the rectangle $L_1$ such that $\omega \in L_1$.

- Repeat;
  R1: Pick $L_{k-1}$ such that $\omega \in L_{k-1}$.
  R2: if $\omega = J_k$, end;
  R3: if not, subdivide $L_{k-1}$ by the hyperplanes $\mathcal{H}_i^+[k]$;
  R4: pick the rectangle $L_k$ such that $\omega \in L_k$.
  Until $\text{Volume}(L_k) < \varepsilon$. 

15
The above iterative algorithm provides a procedure which zooms on the steady state. The volume of the rectangle $L_k$ is equal to $\frac{1}{2^k M}$, hence we iterate $k$ times for $k = \text{roundup}\left(\frac{\ln(\epsilon)}{M \ln(0.5)}\right)$.

**Acknowledgement:** This research is supported in part by a team grant from Fonds de Recherche du Québec - Nature et Technologies (FRQNT grant no. 180362).

## 5. Bibliography

[1] D. Duffie, N. Gârleanu, L. H. Pedersen, Over-the-counter markets, Econometrica 73 (2005) 1815–1847.

[2] D. Duffie, Dark Markets: Asset pricing and information transmission in over-the-counter markets, 2, Princeton University Press, 2012.

[3] N. Gârleanu, Portfolio choice and pricing in illiquid markets, Journal of Economic Theory 144 (2009) 532–564.

[4] W. Kulpa, The poincaré-miranda theorem, The American Mathematical Monthly (Springer) 104 (1997) 545–550.

[5] D. W. Stroock, Diffusion processes associated with lévy generators, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32 (1975) 209–244.

[6] A. S. Sznitman, équations de type Boltzmann spatialement homogènes, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 66 (1984) 209–244.

[7] D. Duffie, Dynamic Asset Pricing Theory, 3 ed., Princeton University Press, 2001.

[8] R. Ferland, G. Giroux, Law of large numbers for dynamic bargaining markets, Journal of Applied Probability 45 (2008) 45–54.

[9] A. Bélanger, G. Giroux, Some new results on information percolation, Stochastic Systems 3 (2013) 1–10.

[10] P. O. Weill, Liquidity premia in dynamic bargaining markets, Journal of Economic Theory 140 (2008) 66–96.

[11] D. Vayanos, T. Wang, Search and endogeneous concentration of liquidity in asset markets, Journal of Economic Theory 136 (2007) 66–104.

[12] J. R. Norris, Markov chains, 12, Cambridge University Press, 1998.