Explicit dyonic dilaton black holes of the four-dimensional Einstein-Maxwell-dilaton theory are known only for two particular values of the dilaton coupling constant $a = 1, \sqrt{3}$, while for other $a$ numerical evidence was presented earlier about existence of extremal dyons in theories with the discrete sequence of dilaton couplings $a = \sqrt{n(n+1)/2}$ with integer $n$. Apart from the lower members $n = 1, 2$, this family of theories does not have motivation from supersymmetry or higher dimensions, and so far the above quantization rule has not been derived analytically. We fill the gap showing that this rule follows from analyticity of the dilaton at the $AdS_2 \times S^2$ event horizon, with $n$ being the leading dilaton power in the series expansion. We also present its generalization for asymptotically anti-de Sitter dyonic black holes with spherical, planar and hyperbolic topology of the horizon.

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to Einstein-Maxwell (EM) system, which is the bosonic part of $N = 2$, $D = 4$ supergravity.
In this case the extremal dyons, defined geometrically as black holes with the degenerate
event horizon, saturate the supergravity Bogomol’nyi-Prasad-Sommerfield (BPS) bound and
have the $AdS_2 \times S^2$ horizon with zero Hawking temperature. For generic $a \neq 0$ only the static
purely electric or magnetic black holes are known analytically \cite{1-4}. In the non-extremal
case they have two horizons, the inner one being singular, so in the extremality limit the
event horizon becomes a null singularity with vanishing Beekenstein-Hawking entropy and
finite temperature (small black holes). For $a = (p/(p + 2))^{1/2}$ small black holes may be
interpreted as compactified regular non-dilatonic $p$-branes in the $(4 + p)$-dimensional theory
\cite{5}. The value $a = 1$ corresponds to $N = 4$, $D = 4$ supergravity or dimensionally reduced
heterotic string effective action, in this case the dyonic solutions are also known \cite{1, 2} which
are non-singular in the extremal limit and possess the $AdS_2 \times S^2$ horizons. The last partic-
ular case, $a = \sqrt{3}$, corresponds to dimensionally reduced $N = 2$, $D = 5$ supergravity; in this
case the static dyon solutions also have the $AdS_2 \times S^2$ horizon structure in the extremality
limit.

The rotating dyonic solutions are known analytically only for $a = 0$ and $a = \sqrt{3}$ \cite{6, 7}. In
the first case it is the Kerr-Newman solution of EM theory, while in the second these were
derived using the three-dimensional sigma-model on the symmetric space $SL(3, R)/SO(2, 1)$
corresponding to vacuum five-dimensional gravity. The EMD theories with these two values
of the dilaton coupling exhaust the set of models reducing to three-dimensional sigma-models
on coset spaces \cite{8}, so from this reasoning there are no indications on any particular status
of EMD theories with other $a$. Meanwhile, as was shown numerically by Poletti, Twamley
and Wiltshire \cite{9}, the values $a = 1, \sqrt{3}$ are just the two lowest members $n = 1, 2$ of the
“triangular” sequence of dilaton couplings

$$a_n = \sqrt{n(n + 1)/2}, \quad (1)$$

selecting EMD theories in which numerical non-extremal dyonic solutions exist with two
horizons and admit the extremal limit. This quantized sequence emerged as the result
of numerical fitting of the non-extremal solutions between the two horizons which looked
similar to solving a non-linear eigenvalue problem. Similar picture persists in the Gauss-
Bonnet gravity too \cite{10}. Meanwhile, no analytical justification of the rule (1) was given so
far.
Some attempts are known to explore possibility of fake supergravity embedding of the EMD theory with generic $a$. Using the Witten-Nester construction, Gibbons et al. [11] were able to derive the BPS-like inequality for arbitrary $a$. Meanwhile, as was later shown by Nozawa and Shiromizu [12], the corresponding Killing spinor equations do not imply the bosonic equations as the integrability condition, unless $a = 0, \sqrt{3}$. So it looks unlikely that the rule (1) may be interpreted as consequence of some hidden supersymmetry.

Wiltshire and collaborators have also investigated dyonic solutions with the cosmological constant [13, 14] and proved a no-go theorem for asymptotically $dS_4$ black holes with $\Lambda > 0$. On the contrary, numerical evidence was presented about existence of asymptotically $AdS_4$ black holes for $\Lambda < 0$ and certain values of $a$ which, however, were not found analytically.

In this paper we investigate the nature of the dilaton coupling quantization both for AF and asymptotically AdS dyons with various topologies of the horizon. We show that the rule (1) for AF solutions follow from analyticity of the dilaton at the $AdS_2 \times S^2$ event horizon and holds for all $n$ being the necessary condition of existence of extremal solutions. We also derive the generalization of the coupling quantization formula for asymptotically $AdS_4$ dyons with $AdS_2 \times \Sigma_k$ horizons, where $\Sigma_k = S^2, E_2$ or $H_2$. In all these cases the solutions exist in the families of theories with $a$ depending on the integer and the continuously varying $\Lambda$. The existence of global solutions, both with AF and AdS asymptotics, is confirmed numerically, an analytic proof will be given elsewhere.

II. SETUP

We choose the EMD lagrangian in the form

$$L = R - 2\Lambda - 2(\partial \phi)^2 - e^{-2a\phi} F^2,$$

and assume the static ansatz for the metric and the Maxwell one-form :

$$ds^2 = -e^{2\delta} N dt^2 + \frac{dr^2}{N} + R^2 d\Sigma_k^2,$$

$$A = -f dt - P \cos \theta \, d\phi,$$
where $P$ is the magnetic charge. The metric on the two-space $dΣ^2_k$ can be spherical, flat or hyperbolic:

$$dΣ^2_k = \begin{cases} 
  dθ^2 + \sinh^2 θdφ, & k = -1, \\
  dθ^2 + θ^2dφ, & k = 0, \\
  dθ^2 + \sin^2 θdφ, & k = 1,
\end{cases} \quad (5)$$

and the functions $f, \phi, N, R$ and $δ$ depend on the radial variable $r$ only. The equations of motions can be readily found from the reduced one-dimensional lagrangian

$$\mathcal{L} = 2ke^δ + 2e^{-δ} (e^{2δ} NR)' R' - 2 φ'^2 e^δ NR^2 - 2Λe^δ R^2 + 2 e^{-2aφ} \left( f'^2 R^2 e^{-δ} - \frac{P^2 e^δ}{R^2} \right). \quad (6)$$

Two particular gauges are relevant: one is $δ = 0$ in which some exact solutions are known, another is $R = r$, which is suitable for analytical derivation of the quantization rules and for numerical calculations. Choosing the gauge $R = r$ and solving Maxwell equations we find:

$$f' = \frac{Q e^{δ+2aφ}}{r^2},$$

where the electric charge $Q$ is an integration constant. The remaining dilaton and Einstein equations then read:

$$(Nϕ'e^δ r^2)' = 2ae^δ|PQ|\sinh(2aφ), \quad (7)$$

$$δ' = φ'^2 r, \quad (8)$$

$$e^{-δ} (e^δ Nr)' = k - Λr^2 - \frac{2|PQ|\cosh(2aφ)}{r^2}, \quad (9)$$

where the shifted dilaton function is introduced

$$φ = φ - (\ln z)/2a, \quad z = \frac{|P|}{|Q|}. \quad (10)$$

The third Einstein equation is related to (8-9) via the Bianchi identity. This system of equations possess a discrete electric-magnetic duality

$$P \leftrightarrow Q, \quad φ \leftrightarrow -φ, \quad (11)$$

so we can restrict $a ≥ 0$ without loss of generality.

For positive cosmological constant this system does not admit the desired black hole solutions. In this case the cosmological horizon exists outside the event horizon, while the equations of motion imply zero value of some integral from a non-negatively defined
quantity over the space-like region between two horizons \[9\]. This argument holds only for dyons with non-trivial dilaton, but fails for the singly charged solutions \[15\] and for the non-dilatonic dyons with equal charges \(P = Q\). In this latter case one has the well-known solution \(\varphi \equiv 0, \delta \equiv 0\) and

\[N = k - \frac{\Lambda}{3} r^2 - \frac{2M}{r} + \frac{2Q^2}{r^2},\]  

(12)

where the mass \(M\) enters as an integration constant. For \(k = 1\) this is the Reissner-Nordström-de Sitter (anti-de Sitter) dyonic black hole for \(\Lambda > 0\) (\(\Lambda < 0\)) and the pure Reissner-Nordström dyon for \(\Lambda = 0\). For \(k = 0, -1\) and \(\Lambda < 0\) the Eq. \[12\] describes black holes with planar and hyperbolic horizons. The non-dilatonic solution with generic \(Q\) and \(M\) has an event horizon \(r = r_h\) which is the highest root of the equation \(N(r_h) = 0\). We are interested in extremal dyons with the horizons \(AdS_2 \times \Sigma_k\), whose radius \(r_h\) (finite in the chosen coordinates) satisfies the two equations:

\[N(r_h) = 0, \quad N'(r_h) = 0.\]  

(13)

Excluding the mass, we obtain an equation for \(r_h\):

\[(k - \Lambda r_h^2) r_h^2 = 2Q^2,\]  

(14)

while the second independent equation gives the mass in terms of \(r_h\):

\[M = r_h \left( k - \frac{2\Lambda}{3} r_h^2 \right).\]  

(15)

If \(Q^2 \neq P^2\), the dilaton is non-trivial, and then there is no black hole solutions with positive cosmological constant \[14\], so we are left with \(\Lambda \leq 0\).

### III. ASYMPTOTICALLY FLAT DILATONIC DYONS

For \(\Lambda = 0\) one has only the spherical geometry, \(k = 1\), and the asymptotic flatness (AF) implies \(N(\infty) = 1, \delta(\infty) = 0\) with the next leading terms

\[N \sim 1 - \frac{2M}{r} + \frac{\Sigma^2}{2r^2},\]  

(16)

\[e^{\delta} \sim 1 - \Sigma^2 \frac{2}{2r^2},\]  

(17)

\[\phi \sim \phi_\infty + \frac{\Sigma}{r},\]  

(18)
where $M$, $\Sigma$ and $\phi_\infty$ ($\varphi_\infty$ is related to $\phi_\infty$ by (10)) are free parameters of the local series solution. As expected, in global solutions the dilaton charge $\Sigma$ is not an independent quantity: integrating the Eq. (7) one obtains the sum rule:

$$\Sigma = 2a|QP| \int_{r_h}^{\infty} \frac{e^\delta \sinh 2a\varphi}{r^2} dr.$$  \hspace{1cm} (19)

It can be shown that for the AF solutions with the degenerate horizon satisfying (13), there is a second constraint on the charges, namely the no-force condition. First, from the equations of motion one can deduce that the quadratic form

$$I = \left(\frac{1}{2} N^2 e^{2\delta} \varphi'^2 + \frac{1}{8} e^{-2\delta} (N e^{2\delta})'^2\right) r^4 - |QP| Ne^{2\delta} \cosh(2a\varphi)$$  \hspace{1cm} (20)

is conserved on shell:

$$\frac{dI}{dr} = 0,$$  \hspace{1cm} (21)

(similar expression in the gauge $\delta = 0$ was given in [9]). Substituting $r = r_h$ one finds that in view of (13) this integral actually has zero value, $I = 0$. Then substituting the asymptotic expansions (16-18) we obtain from $I = 0$ the no-force condition:

$$M^2 + \Sigma^2 = Q^2_\infty + P^2_\infty,$$  \hspace{1cm} (22)

where $Q_\infty = Qe^{2a\phi_\infty}, P_\infty = Pe^{-2a\phi_\infty}$. For known exact solutions $\phi_\infty = 0$, so the asymptotic charges coincide with the initial ones. Constructing the numerical solutions, we will impose the same condition adjusting the horizon data.

An exact extremal dyon solution with $a = 1 \ (n = 1)$ in this gauge reads

$$e^{-2\delta} = 1 + \frac{\Sigma^2}{r^2}, \quad N = \left(1 - \frac{2M}{r^2} \sqrt{r^2 + \Sigma^2} + \frac{Q^2 + P^2 + \Sigma^2}{r^2}\right), \quad e^{2\varphi} = \left|\frac{Q}{P}\right| \sqrt{\frac{\sqrt{r^2 + \Sigma^2 + \Sigma}}{r^2 + \Sigma^2 - \Sigma}}.$$  \hspace{1cm} (23)

It has a horizon at $r_h = \sqrt{M^2 - \Sigma^2}$ and the dilaton charge satisfies

$$2M\Sigma = P^2 - Q^2.$$  \hspace{1cm} (24)

Another exact dyon solution is known for $n = 2, \ (a = \sqrt{3})$ [1,2]. The corresponding dilaton charge satisfies the following formula

$$\frac{Q^2}{\Sigma - aM} + \frac{P^2}{\Sigma + aM} = \frac{1 + a^2}{2a^2} \Sigma,$$  \hspace{1cm} (25)

which is valid both for $a = a_1, a_2$. 

Further information about higher $n$ dyons can be extracted from known exact solutions for singly charged black holes. This provides us with some knowledge about the limiting point $z = 0$ (an opposite limit $z = \infty$ can be explored via the discrete electric-magnetic duality (11)). These solutions look simpler in the gauge $\delta = 0$, the corresponding equations of motion being

$$
(N\varphi'R^2)' = \frac{2|PQ|}{R^2} \sinh(2a\varphi) , \tag{26}
$$

$$
R'' + \varphi'^2 R = 0 , \tag{27}
$$

$$
(NR)' = k - \Lambda R^2 - \frac{2|PQ| \cosh(2a\varphi)}{R^2} . \tag{28}
$$

The electrically charged solution valid for all $a$ in the general non-extremal case reads:

$$
R^2 = \rho^2 f_{-}^{1-\gamma} , \quad N = f_{+} f_{-}^\gamma , \quad e^{2a\varphi} = f_{-}^{\frac{2a}{1+a^2}} , \quad \gamma = \frac{1 - a^2}{1 + a^2} , \tag{29}
$$

with $f_{\pm} = 1 - r_{\pm}/\rho$, where we denoted the radial coordinate as $\rho$ to distinguish it from $r$ in the gauge used in the Eqs. (7-9). The mass, the electric charge and the dilaton charge are related to $r_{\pm}$ via

$$
r_{+}r_{-} = \frac{2Q^2}{1 + \gamma} , \quad 2M = r_{+} + \gamma r_{-} , \quad \Sigma = -\frac{a}{1 + a^2} r_{-} . \tag{30}
$$

Note that our system of equations has special points of two kinds: the zeroes of the function $R(\rho)$, which correspond to the curvature singularity, and the zeroes of $N(\rho)$ for which $R \neq 0$, where the space-time geometry is regular. Generic behavior of the metric functions and the dilaton near the curvature singularity is non-analytic (unless $a = 1$), while in the second case it is analytic. This is clearly seen from the singly charged solution (29) which has two zeroes of $N(\rho)$ at $\rho = r_{\pm}$, with $r_{-}$ being a zero of $R(\rho)$ too. The dilaton is analytic at the regular horizon $\rho = r_{+}$, but non-analytic in the singularity $\rho = r_{-}$. This can be expected for dyonic solutions as well. In the extremal limit $r_{+} = r_{-}$ the dilaton is therefore singular at the horizon, but this does not influence its asymptotic behavior. So the dilaton charge is still finite and given by (30). In the extremal limit it reads:

$$
\Sigma = -\frac{aQ}{\sqrt{1 + a^2}} . \tag{31}
$$

This expression is expected to match the corresponding limit of dyonic solutions.

Now let us look for extremal dyon solutions with the regular horizon for arbitrary $a$. It turns out that already the local power series solution in the vicinity of the degenerate horizon
implies a constraint on $a$. Coming back to the gauge $R = r$, in which curvature singularity is at $r = 0$, and assuming (13) we will have in the leading order:

$$
N = \nu x^2 + O(x^3), \quad x = (r - r_h)/r_h, \quad (32)
$$

$$
\varphi = \varphi_h + \mu x^n + O(x^{n+1}), \quad (33)
$$

where $\mu$ and $\nu$ are dimensionless parameters, and $n$ is an integer. Substituting this into the Eq. (8) we find

$$
\delta = \delta_h + \frac{\mu^2 n^2}{(2n - 1)} x^{2n - 1} + O(x^{2n}), \quad (34)
$$

and therefore $e^\delta$ is finite and continuous at the horizon. So the leading term of the l.h.s. of the Eq. (7) is

$$
\left( N \varphi' e^\delta r^2 \right)' = \nu \mu n (n + 1) e^\delta r_h^2 x^n + O(x^{n+1}). \quad (35)
$$

This is zero at $x = 0$ for any $n$, so looking at the r. h. s. of Eq. (7) we immediately find that one must have

$$
\varphi_h = 0. \quad (35)
$$

Then the linear in $x$ term at the r. h. s. of (7) agrees with that at the l.h.s. and we find:

$$
\nu n (n + 1) = 4a^2|QP|/r_h^2. \quad (36)
$$

Now consider the equation (9) in the vicinity of $r \sim r_h$. One sees that the l.h.s. is linear in $x$, so the r.h.s. has to be linear in $x$ too. Vanishing of the constant terms for $\Lambda = 0, k = 1$ gives an equation for $r_h$:

$$
r_h^2 = 2|QP|, \quad (37)
$$

while expanding $r^{-2} = r_h^{-2}(1 - 2x) + ...$ and equating the linear in $x$ terms we obtain $\nu = 1$. Substituting this and (37) into (36), we arrive at

$$
a^2 = a_n^2 = \frac{n(n + 1)}{2}. \quad (38)
$$

This is the necessary condition for existence of the AF regular extremal dilatonic dyons which coincides with (1). It shows, in particular, that such solutions do not exist for $a < 1$, except for $a = 0$.

The main characteristic of new solutions is the dependence of the dilaton charge $\Sigma$ on $z$. This can be extracted from the numerical solutions which must fit the no-force condition.
To construct numerical solutions of the system (7)-(9) on the interval \( r_h < r < \infty \) we have to set initial conditions at some tiny step away from the horizon \( r_h \) which is the singular point of the system of differential equations. To perform this with a given accuracy we need to keep more terms in the series expansions:

\[
N = \sum_{k=2}^{2n} (-1)^k (k-1) x^k - n(\mu^2 + 2) x^{2n+1} + O(x^{2n+2}),
\]

\[
\varphi = \mu x^n - n\mu x^{n+1} + \frac{n(n+1)}{2} x^{n+2} + O(x^{2n}),
\]

\[
\delta = \delta_h + \frac{\mu^2 n^2}{(2n-1)} x^{2n-1} + O(x^{2n}),
\]

This local solution has two free parameters: \( \mu \) and \( \delta_h \), from which the latter is trivial because of the symmetry of the Eqs. (7-9) under translations of \( \delta \). At infinity we have asymptotic expansions in terms of the physical mass \( M \), the dilaton charge and the electric and magnetic charges (16)-(18). Actually, the mass can be extracted from the no-force condition (22), and one of the charges may be set equal to one defining the length (mass) scale. So the only remaining parameter which has to be computed from numerical solutions is the dilaton charge as a function of the ratio \( z = |P/Q| \). In view of the electric-magnetic duality it will be enough to consider the interval \( z \in [0, 1] \), i.e. the electrically dominated solutions. For them the dilaton charge \( \Sigma \) is negative.

For each given \( z \) there is a set of \( \mu \) yielding asymptotically flat solutions with different values of the dilaton at infinity, from which we extract one corresponding to \( \phi_\infty = 0 \), or in terms of \( \varphi \):

\[
\varphi_\infty = \frac{1}{2a} \ln \left| \frac{Q}{P} \right| .
\]

(39)

In this case the physical charge \( Q_\infty \) coincides with the initial \( Q \), so the no-force condition (22) reduces to \( M^2 + \Sigma^2 = Q^2 + P^2 \). A series of lower-\( n \) numerical solutions for the dilaton \( \varphi(r) \) are shown on Fig. 1. The metric functions \( N(r) \) and \( \delta(r) \) are monotonous, increasing from zero values at the horizon to unity at infinity, so we do not show them here. The solutions with various \( z \) are then used to calculate \( \Sigma(z) \) via the sum rule (19). The resulting curves for lower \( n \) are shown on Fig. 2. They interpolate between the \( z = 0 \) values \( \Sigma_n(0) \) given by (31) with \( a \) equal to (1) and zero value at \( z = 1 \). Note that the sequence \( \Sigma_n(0) \) converges to \(-Q\) as \( n \to \infty \).
IV. ASYMPTOTICALLY ADS DYONS

For \( \Lambda < 0 \) we keep the topological parameter \( k = 1, 0, -1 \) arbitrary. Then the asymptotic behavior in the leading order is given by:

\[
N \sim \frac{r^2}{l^2} + k - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2},
\]

(40)

\[
\phi \sim \phi_\infty + \frac{\hat{\Sigma}}{r^3},
\]

(41)

\[
e^\delta \sim 1 - \frac{3\hat{\Sigma}^2}{2r^6},
\]

(42)

where the AdS curvature radius is given by

\[
l^2 = -\frac{3}{\Lambda}.
\]

(43)

The modified dilaton charge \( \hat{\Sigma} \) has the dimension of \((\text{length}^3)\) and satisfies the sum rule

\[
\hat{\Sigma} = \frac{2a|QP|}{\Lambda} \int_{r_h}^{\infty} e^\delta \sinh 2a\varphi \frac{dr}{r^2}.
\]

(44)

To generalize the coupling quantization condition \((38)\) we follow the same strategy keeping track for \( \Lambda \) and \( k \) terms the Eq. \((9)\). Then we obtain the following equation:

\[
r_h^2(k - \Lambda r_h^2) = 2|PQ|,
\]

(45)

leading to

\[
r_h^2 = \frac{1}{2\Lambda} \left(-k \pm \sqrt{k^2 - 8\Lambda|PQ|}\right),
\]

(46)

where we have to choose the upper sign. Note that for \( k = -1 \) the horizon radius remains finite if one of the charges or both of them are zero:

\[
r_h^2 = l^2 / 3.
\]

(47)

This case may be therefore attributed either to vacuum solution with the degenerate horizon, previously mentioned in \([16]\), or to a singly charged planar black hole.

Equating the linear in \( x \) terms in \((9)\) we get

\[
\nu = k - 2\Lambda r_h^2.
\]

(48)

Substituting this to the Eq. \((36)\), we obtain

\[
a^2 = \frac{n(n + 1)}{2} \left(1 - \frac{\Lambda r_h^2}{k - \Lambda r_h^2}\right),
\]

(49)
where \( r_h \) is the solution of (45). The novel feature of this condition is the dependence of the critical \( a \) on the ratio of the horizon radius to the AdS curvature radius \( l: |\Lambda| r_h^2 = 3 r_h^2 / l^2 \), except for the planar case \( k = 0 \). Actually the dilaton constant is a parameter of the theory within which we are looking for dyonic solutions. So this relation should be regarded as determining the horizon radius for given \( \lambda \) and \( a \). It has multiple branches labeled by \( n \). Substituting (46) into (49) we find another useful formula

\[
a^2 = n(n + 1) \left( 1 + \frac{k}{\sqrt{k^2 + 24|QP|/l^2}} \right)^{-1}.
\]

In the limit \( l^2 \to \infty \) we come back to the triangular number (1). In the planar case \( k = 0 \) we obtain

\[
a^2 = n(n + 1),
\]

which is also the limit \( |QP|/l^2 \to \infty \) for the spherical case \( k = 1 \). For \( n \geq 2 \) the branches for \( k = 1 \) with different \( n \) start to overlap thus allowing for spherically symmetric extremal dyons for any \( a \geq a_2 \). In the hyperbolic case \( k = -1 \) the Eq. (49) exhibits divergence at \( r_h^2 \) given by (47), corresponding to the uncharged (or singly charged) solution.

The modified dilaton charge \( \hat{\Sigma} \) can be found numerically using the same strategy as in the AF case. The initial data are obtained from the near-horizon expansions:

\[
N(r) = -\frac{a^2 r_h^2 \Lambda}{a^2 - a_n^2} x^2 + \sum_{m=3}^{2n} (-1)^{m+1} \frac{1}{3} \frac{\Lambda r_h^2 (2a^2 + n(n + 1)(m - 1)}{a^2 - a_n^2} x^m + O(x^{2n+1}),
\]

\[
\varphi(r) = \mu x^n + \sum_{l=n+1}^{2n} \frac{\mu \Pi_l(a)}{a_l^i} x^l + O(x^{2n}),
\]

\[
\delta = \delta_h + \frac{2\mu^2 n^2}{2n - 1} x^{2n-1} + O(x^{2n}),
\]

where \( \Pi_l(a) \) are polynomials of the degree \( l \). Boundary conditions at infinity are specified by the expansions (40)-(42). Contrary to the AF case, the system no longer has the scaling symmetry, which makes numerical procedure more tedious.

The results are shown on Figs. (3-5). Note that for the hyperbolic horizon the dilaton charge \( \hat{\Sigma} \) tends to \(-\infty\) at \( z = 0 \). This follows directly from the formula (49). Since the radius of the horizon tends to \(-1/\Lambda\), the denominator in brackets explodes, which leads to the divergence of \( \hat{\Sigma} \) through the sum rule (44).
V. CONCLUSIONS

Our main result is an analytic derivation of the “triangular” quantization rules for the dilaton coupling $a$ in EMD($\Lambda$) theory as necessary condition of existence of regular extremal dilatonic dyons. These rules follow from the purely local analysis of analyticity of the dilaton at the $AdS_2 \times S^2$ event horizon. The integer $n$ is shown to be the leading power index in the Taylor expansion of the dilaton function. In the AF ($\Lambda = 0$) case our result reproduces the discrete sequence found by Poletti, Twamley and Wiltshire [9] in a slightly different setting. In the asymptotically AdS case our formulas are entirely new and applicable to topological solutions as well. Dyons with hyperbolic topology are shown to make contact with the uncharged solutions with AdS asymptotic [10].

We expect possible generalization of our results to dyonic branes [17]. A challenging task is to find analytical formulas for triangular dyons, some evidence in favor of such a possibility was presented in [10].

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FIG. 1: Behavior of the dilaton function $\varphi(r)$ of asymptotically flat electrically dominated dyons for $n = 1, 2, 3, 4$. The curves start from zero and approach the values given by Eq. (39).

FIG. 2: Dilaton charges $\Sigma$ defined by Eq. (19) for electrically dominated AF dyons as functions of $z = |P/Q|$ for $n = 1, 2, 3, 4$. The curves start with the values given by Eq. (31) at $z = 0$ and tend to zero at $z = 1$, corresponding to non-dilatonic solutions.
FIG. 3: Modified dilaton charges $\hat{\Sigma}$ [44] for electrically dominated asymptotically AdS dyons with spherical horizons, $k = 1$.

FIG. 4: Modified dilaton charges $\hat{\Sigma}$ [44] for electrically dominated asymptotically AdS dyons with planar horizons, $k = 0$. 
FIG. 5: Modified dilaton charges $\hat{\Sigma}$ (44) for electrically dominated asymptotically AdS dyons with hyperbolic horizon, $k = -1$. 