Efficiency fluctuations in microscopic machines

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Nanoscale machines are strongly influenced by thermal fluctuations, contrary to their macroscopic counterparts. As a consequence, even the efficiency of such microscopic machines becomes a fluctuating random variable. Using geometric properties and the fluctuation theorem for the total entropy production, a “universal theory of efficiency fluctuations” at long times, for machines with a finite state space, was developed in [Verley et al., Nat. Commun. 5, 4721 (2014); Phys. Rev. E 90, 052145 (2014)]. We extend this theory to machines with an arbitrary state space. Thereby, we work out more detailed prerequisites for the “universal features” and explain under which circumstances deviations can occur. We also illustrate our findings with exact results for two non-trivial models of colloidal engines.

Understanding the functioning of machines on the micro- or nanoscale is of great interest because of their role in biological systems and their numerous technological applications [1–7]. Their small size makes this task a challenge, however, because of the effect that thermal fluctuations have on their operation. As a result, average values are no longer sufficiently informative, and fluctuations in heat, work, efficiency etc. must be taken into account to fully characterize a microscopic machine. Stochastic thermodynamics [8] provides a convenient framework for analyzing such systems by extending the notions of classical (ensemble-based) thermodynamics to individual realizations of a given process.

Before we begin our discussion of microscopic machines, consider first a macroscopic heat engine for which fluctuations are imperceptibly tiny and all quantities always assume their average values. Such a heat engine typically operates cyclically between two reservoirs at different temperatures $T_1 > T_2$ and performs work against an external load force. It is characterized by the (average) heat $Q_1$ and $Q_2$ exchanged with the two reservoirs and the (average) performed work $-W$. The Second law then implies that the efficiency

\[ \eta = \frac{-W}{Q_1}, \tag{1} \]

is universally bounded from above by the reversible or Carnot efficiency $\eta_C = 1 - T_2/T_1$.

The efficiency $\eta$ plays an equally pivotal role in quantifying the performance of microscopic machines; however, in these systems, due to thermal fluctuations, the value obtained in individual realizations can deviate significantly from the average behavior. We hence need to consider a distribution of efficiency values. Recently, in two seminal papers [9,10], Verley, Willaert, Van den Broeck, and Esposito (VWVE) developed a “universal theory of efficiency fluctuations” for machines with a finite state space. By characterizing the long-time behavior of the efficiency fluctuations in terms of their large-deviation function $J(\eta)$ (see below for more details), they found that the macroscopic efficiency (approached in the limit of long operational times) is the most likely and, for machines operating in a non-equilibrium steady state or under a time-symmetric periodic protocol, the reversible Carnot efficiency is the least likely one [11]. The VWVE theory has since been verified in numerous model systems with finite [12–19] but also even infinite [13,19–21] state spaces.

Nevertheless, there are a few examples of infinite state space systems at odds with the theory [22,23], in which the rate function $J(\eta)$ fails to be smooth and/or does not exhibit a unique maximum at the reversible efficiency. A clear understanding of why this is so, namely that some systems with infinite state space obey the “universal” theory while others do not, is lacking. In this Letter, we analyze under which circumstances the “universal” characteristics of efficiency fluctuations emerge and how the existence of an infinite state space can lead to modifications. Two examples of analytically solvable machines [24,25] serve as illustrations for our general findings.

We start by briefly summarizing the approach taken in [9,10] to analyze the efficiency fluctuations, which we then generalize to cases for which some of the (explicit and tacit) assumptions of the VWVE theory are not fulfilled. For all systems, the total work $W$ and heat $Q_1$ (as well as $Q_2$) grow extensively with increasing operational time $\tau$ of the machine. For microscopic systems they also naturally fluctuate due to thermal noise, leading to a distribution $p_\tau(q_1,w)$ for observing a heat absorption rate $q_1 = Q_1/\tau$ (the input power) and an output power $-w = -W/\tau$ over a process of duration $\tau$. For large times $\tau$, these rates will generically approach their average values $\langle w \rangle$ and $\langle q_1 \rangle$. Using the theory of large deviations [20], we can quantify the asymptotic decay of the probability $p_\tau(q_1,w)$ towards the delta-distribution peaked at
the expectation values $\langle q_1 \rangle$ and $\langle w \rangle$ by the large deviation or rate function $I(q_1, w)$,

$$p_r(q_1, w) \sim e^{-\tau I(q_1, w)} \quad (\tau \to \infty),$$

(2)

where $I(q_1, w) \geq 0$ and $I(\langle q_1 \rangle, \langle w \rangle) = 0$. Similarly, the stochastic efficiency $\eta = -w/q_1$ will tend towards the macroscopic efficiency $\hat{\eta} = \langle -w \rangle/q_1$. Again, we can describe this approach in terms of a rate function $J(\eta)$, providing an asymptotic relation for the probability distribution $p_r(\eta)$,

$$p_r(\eta) \sim e^{-\tau J(\eta)} \quad (\tau \to \infty).$$

(3)

It has been shown in [10] that $J(\eta)$ can be extracted from the scaled cumulant generating function (sCGF) of heat and work,

$$\phi(\lambda_Q, \lambda_W) := \lim_{\tau \to \infty} \frac{1}{\tau} \ln \langle e^{-\tau \lambda_Q q_1 - \tau \lambda_W w} \rangle \tau,$$

(4)

according to

$$J(\eta) = -\min_{\lambda} \phi(\eta \lambda, \lambda).$$

(5)

The angular brackets $\langle \cdots \rangle_\tau$ in (5) denote an average over all processes of duration $\tau$, weighted with the distribution $p_r(Q_1/\tau, W/\tau)$. As follows straightforwardly from the definition, $\phi(\lambda_Q, \lambda_W)$ is a convex function.

The relation (5) implies $\phi(0, 0) = 0 \leq J(\eta) \leq -\hat{\phi}$, where $\hat{\phi} := \min_{\lambda_Q, \lambda_W} \phi(\lambda_Q, \lambda_W)$ is the global minimum of $\phi$. Moreover, it has a nice geometric interpretation, illustrated in Fig. 1. For fixed $\eta$, the value $J(\eta)$ is obtained by minimizing $\phi(\lambda_Q, \lambda_W)$ along the line $\lambda_Q = \eta \lambda_W$ and inverting the sign. Denoting the point where this minimum is attained by $\hat{\lambda}(\eta) = (\lambda_Q(\eta), \lambda_W(\eta))$, the line $\lambda_Q = \eta \lambda_W$ is tangent to the iso-contour of $\phi$ at this point. As a function of $\eta$, $\hat{\lambda}(\eta)$ describes a curve in the $(\lambda_Q, \lambda_W)$-plane. The rate function $J(\eta)$ is thus given by the scaled cumulant generating function $\phi(\lambda_Q, \lambda_W)$ along this curve, $J(\eta) = -\phi(\hat{\lambda}(\eta), \hat{\lambda}(\eta))$.

Exploiting this geometrical picture, the aforementioned “universal theory” by VVWE [9, 10] establishes generic properties of $J(\eta)$ that are independent of system-specific details. As the main result they find that $J(\eta)$ is a smooth function with a unique minimum at the macroscopic efficiency $\hat{\eta}$, such that $J(\hat{\eta}) = 0$, and a unique maximum at some finite efficiency $\bar{\eta}$. For time-symmetric driving protocols, this “least likely” efficiency $\bar{\eta}$ coincides with the reversible efficiency $\eta_C$ [11], see the example in Fig. 1. The latter result is a direct consequence of the fluctuation theorem [8, 27], $p(\Delta S_{\text{tot}})/p(-\Delta S_{\text{tot}}) = \exp(\Delta S_{\text{tot}})$ for the total entropy production $\Delta S_{\text{tot}} = -Q_1/T_1 - Q_2/T_2 + \Delta S_{\text{sys}}$ (where $\Delta S_{\text{sys}}$ denotes the entropy change in the system itself), because it implies the symmetry

$$\phi(\lambda_Q, \lambda_W) = \phi(\lambda_Q^* - \lambda_Q, \lambda_W^* - \lambda_W),$$

(6)

of the sCGF $\phi$, with $\lambda_Q^* = \eta_C/T_2$ and $\lambda_W^* = 1/T_2$ (see below for more details). The minimum of $\phi$ is thus located at $(\lambda_Q^*/2, \lambda_W^*/2)$, $\hat{\phi} = \phi(\lambda_Q^*/2, \lambda_W^*/2)$, i.e. on the line $\lambda_Q = \eta_C \lambda_W$.

These results of the VVWE theory are based on a few assumptions, most notably: (i) The detailed fluctuation theorem $p(\Delta S_{\text{tot}})/p(-\Delta S_{\text{tot}}) = \exp(\Delta S_{\text{tot}})$ for the total entropy production is valid, (ii) $\phi(\lambda_Q, \lambda_W)$ is a smooth function of its arguments and the fluctuation theorem implies that it has the symmetry property (6), and (iii) the minimum $\hat{\phi}$ is unique. The validity of (i) is well-established for a large variety of different systems (or model classes) [8, 27] and we take it for granted here as well. However, we will demonstrate that it does not necessarily entail the validity of the symmetry (6) for all $(\lambda_Q, \lambda_W)$ as in (ii). Further, we discuss possible implications if, in addition, assumption (iii) does not hold either.

While assumption (ii) appears plausible for systems with a finite state space, it has been observed in certain models with infinite state space that the sCGF [1] can have a restricted domain of convergence $C_0$ [23, 24]. It has also been noticed that the symmetry property (6) need not necessarily hold [24, 28, 52]. The relationship between such a limited convergence domain and the symmetry relation (6), or lack thereof, has not been understood before in the manner we present it below, and hence we expand a little on this conceptual point. We begin by expressing the sCGF in terms of the individual time-extensive and -intensive contributions to the total entropy production,

$$\Delta S_{\text{tot}} = \frac{\eta_C}{T_2} Q_1 + \frac{1}{T_2} W + \Delta S_{\text{int}}.$$

(7)
While the work $W$ and heat exchange $Q_1$ are extensive in the time duration $\tau$, the term $\Delta S_{\text{int}} = -\frac{1}{T} \Delta E + \Delta S_{\text{sys}}$ collects the time-intensive contributions to the total entropy production. This term depends only on the initial and final states of the system, but not on the “path” connecting them. Here $\Delta E$ denotes the change in internal energy, which is, according to the First Law, connected to work and heat via $\Delta E = W + Q_1 + Q_2$.

We first write down the moment-generating function (MGF) for the combined probability distribution of the individual contributions from (7),

$\Psi_\tau(\lambda_Q, \lambda_W, \lambda_S) := \langle e^{-\lambda_Q Q_1 - \lambda_W W - \lambda_S \Delta S_{\text{int}}} \rangle_\tau.$

The fluctuation theorem for the total entropy production implies that $\Psi_\tau$ has the symmetry property $\Psi_\tau(\lambda_Q, \lambda_W, \lambda_S) = \Psi_\tau(\lambda_Q^*, -\lambda_Q, \lambda_W^*, -\lambda_W, 1 - \lambda_S)$ with $\lambda_Q^*$ and $\lambda_W^*$ as given above. Geometrically speaking, $\Psi_\tau$ is thus symmetric upon reflection through the point $(\lambda_Q^*/2, \lambda_W^*/2, 1/2)$. This symmetry is inherited by the three-dimensional sCGF

$\phi(\lambda_Q, \lambda_W, \lambda_S) := \lim_{\tau \to \infty} \frac{1}{\tau} \ln \Psi_\tau(\lambda_Q, \lambda_W, \lambda_S),

so that

$\phi(\lambda_Q, \lambda_W, \lambda_S) = \phi(\lambda_Q^* - \lambda_Q, \lambda_W^* - \lambda_W, 1 - \lambda_S).$

The sCGF (4) of heat and work alone is the restriction of that “total” sCGF to the $\lambda_S = 0$ plane, i.e. $\phi(\lambda_Q, \lambda_W) = \phi(\lambda_Q, \lambda_W, 0)$.

As a consequence of Eq. (10) we arrive at the central observation that this restricted sCGF fulfills a “symmetry” relation of the form $\phi(\lambda_Q, \lambda_W) = \phi(\lambda_Q, \lambda_W, 0) = \phi(\lambda_Q^* - \lambda_Q, \lambda_W^* - \lambda_W, 1)$, instead of the relation (6). However, (5) could still be valid if $\phi(\lambda_Q, \lambda_W, \lambda_S)$ were independent of $\lambda_S$ as in the VWVE theory [9 10], which is restricted to machines with a finite state space. In this case, both the internal energy $\Delta E$ and the system entropy $\Delta S_{\text{sys}}$ are bounded by $\tau$-independent constants, implying that the $\lambda_S$ dependence in (9) disappears in the $\tau \to \infty$ limit.

However, if fluctuations of the internal entropy production $\Delta S_{\text{int}}$ can become arbitrarily large, as is typically the case for machines with infinite state space [24 25 26], we cannot argue that $\phi$ is independent of $\lambda_S$. In this case too, the time-intensive contribution $\Delta S_{\text{int}}$ to the total entropy production, depending only on the initial and final state, is asymptotically independent of $\tau$. As a result, the $\lambda_S$ contributions drop out in Eq. (9) as $\tau \to \infty$, wherever $\Psi_\tau(\lambda_Q, \lambda_W, \lambda_S)$ remains real and finite. However, in contrast to the finite state-space case, the fluctuations of $\Delta S_{\text{sys}}$ can be arbitrarily large, such that this domain of convergence $C$ of all points $(\lambda_Q, \lambda_W, \lambda_S)$ with real and finite values for $\Psi_\tau$ will in general depend on $\lambda_S$. We can therefore conclude that $\phi(\lambda_Q, \lambda_W, \lambda_S)$ does not depend on $\lambda_S$ explicitly but may bear an implicit dependence via $C$. As a consequence of the fluctuation theorem symmetry obeyed by $\Psi_\tau$, $C$ is symmetric about the point $(\lambda_Q^*/2, \lambda_W^*/2, 1/2)$. Therefore, the restriction of $C$ to the $\lambda_S = 1/2$ plane satisfies a symmetry property like (6), but the domain of convergence $C_0$ of $\phi$ at $\lambda_S = 0$, will in general not obey this symmetry, and hence neither will $\phi(\lambda_Q, \lambda_W, 0)$. The situation just described is illustrated in Fig. 2.

What are the consequences of the finite domain of convergence of $C_0$ and its lack of symmetry, for the large deviation function $J(\eta)$? The answer depends on whether the minimizing curve $\tilde{\lambda}(\eta)$ is completely contained inside $C_0$ or whether it touches or hits the boundary of $C_0$. We illustrate this difference using the isothermal work-to-work converter from Ref. 24. This machine consists of a Brownian particle in contact with a single heat bath at temperature $T$ and two additional white-noise forces, interpreted as a load and drive force, respectively. Identifying the work done by the drive force with the absorbed heat $Q_1$ from the hot reservoir and the work done by the load force with $W$, we can calculate the sCGF $\phi(\lambda_Q, \lambda_W)$ exactly, and from that the curve $\tilde{\lambda}(\eta)$ and the resulting rate function $J(\eta)$ (see 24 and the Supplemental Material [41] for details).

In the first case, when $\tilde{\lambda}(\eta)$ lies completely inside $C_0$, the existence of singular points for $\phi(\lambda_Q, \lambda_W)$ is irrelevant, resulting in a $J(\eta)$ that has exactly the properties and “universal shape” predicted by the VWVE theory (see the top panels in Fig. 3). In particular, the reversible efficiency $\eta_C = 1$ is still the least likely efficiency, because the global minimum $\phi$ of $\phi$ is still attained at the point $(\lambda_Q^*/2, \lambda_W^*/2)$ despite the “asymmetry” of $C_0$. By contrast, in the second case, $\phi$ takes its minimal value on the boundary of $C_0$ (lower left panel in Fig. 3). The minimizing curve $\tilde{\lambda}(\eta)$ thus follows the boundary of $C_0$.

Figure 2. Contour plots of a typical $\phi(\lambda_Q, \lambda_W)$ for $\lambda_S = 0$, 1/2, 1 along with the domain of convergence $C$ and the domain of convergence $C_0$ for $\phi$ at $\lambda_S = 0$. The functional form of $\phi(\lambda_Q, \lambda_W, \lambda_S)$ is the same in all $\lambda_S = \text{const}$ planes, but the limited domain of convergence leads to cutoffs whose location changes as a function of $\lambda_S$. The symmetry around the point $(\lambda_Q^*/2, \lambda_W^*/2, 1/2)$ is a consequence of the fluctuation theorem for the total entropy production.
for some range of \( \eta \) values and becomes non-smooth at the points where it hits this boundary, leading to kinks in \( J(\eta) \) that are visible in its first derivative (lower right panel in Fig. 3). We conclude that the appearance of cutoffs in \( \phi(\lambda_Q, \lambda_W) \) can lead to discontinuities or “kinks” in \( J(\eta) \) or its derivatives. In general (see also [41]), \( J(\eta) \) is a smooth function of \( \eta \) if and only if \( \phi(\lambda_Q, \lambda_W) \) is smooth along the curve \( \tilde{\lambda}(\eta) \). Note that in the second example of Fig. 3 (lower panels), the minimal value of \( \phi \) along the curve \( \tilde{\lambda}(\eta) \) (corresponding to the least likely efficiency in \( J(\eta) \)) is attained at a point on the line \( \lambda_Q = \eta C \lambda_W \), with the consequence that the least likely efficiency is still \( \eta C \), even though \( \phi(\lambda_Q, \lambda_W) \) does not obey the symmetry \( \eta \rightarrow -\eta \). However, this need not be the case in general, and the least likely efficiency need no longer be identical with the reversible efficiency, because the minimal \( \phi \)-value on the boundary of \( C_0 \) (and thus on the curve \( \tilde{\lambda}(\eta) \)) may not be located on the line \( \lambda_Q = \eta C \lambda_W \).

Next, we investigate the situation when assumption (iii) fails to hold and the global minimum \( \tilde{\phi} = \min_{\lambda_Q, \lambda_W} \phi(\lambda_Q, \lambda_W) \) is not unique but rather degenerate, i.e., there exist multiple points \( (\lambda_Q, \lambda_W) \) in the set \( \tilde{\mathcal{R}} := \{ (\lambda_Q, \lambda_W) \phi(\lambda_Q, \lambda_W) = \tilde{\phi} \} \). Due to the convexity of \( \phi(\lambda_Q, \lambda_W) \), this set will be a connected region in the \( (\lambda_Q, \lambda_W) \)-plane. In this case, \( J(\eta) \) assumes its maximal value \( -\phi \) for all \( \eta \) for which the line \( \lambda_Q = \eta C \lambda_W \) intersects the region \( \tilde{\mathcal{R}} \), leading to a plateau of degenerate maxima. The reversible efficiency \( \eta_C \) is one of these maximizing efficiencies if and only if the line \( \lambda_Q = \eta C \lambda_W \) intersects the region \( \tilde{\mathcal{R}} \) within the domain of convergence \( C_0 \) (see also [11]).

We illustrate this situation with the example of the “Brownian gyrator” from Refs. [25, 42]. This heat engine consists of a colloidal particle in two dimensions, immersed in a fluid environment and experiencing thermal fluctuations of different intensity along two perpendicular directions (temperatures \( T_1 \) and \( T_2 \), friction coefficients \( \gamma_1 \) and \( \gamma_2 \); see [43, 44] for experimental realizations). The particle is trapped in a harmonic potential whose principal axes with stiffnesses \( u_1 \) and \( u_2 \), respectively, are rotated by an angle \( \alpha \) with respect to the preferred axes of the heat baths. As a consequence, the particle experiences a net torque that lets it rotate around the origin on average [25]. Applying a linear external “load torque” with slope \( f_{ext} \), the system operates as a stationary, minimal heat engine [42] (see also [11] for details). The resulting scGF \( \phi(\lambda_Q, \lambda_W) \) and rate function \( J(\eta) \) can be computed exactly [11] using path-integral techniques [45, 46] and are shown for two different configurations in Fig. 4. In this system, the degenerate minimum of \( \phi(\lambda_Q, \lambda_W) \) arises due to \( \phi(\lambda_Q, \lambda_W) \) becoming a function of only \( \lambda_Q - \eta C \lambda_W \) within the domain of convergence \( C_0 \). This is related to the tight coupling between work and heat, as shown in an earlier work [47] for another model. As a result, the iso-contours of \( \phi \) are parallel lines with slope \( 1/\tilde{\eta} \), and also the set \( \tilde{\mathcal{R}} \) is one of these straight lines (see left panels of Fig. 4). The resulting plateaus for \( J(\eta) \) are clearly visible in the right panels. For the configuration in the top panels, the region \( \tilde{\mathcal{R}} \) intersects the \( \lambda_Q \)-axis, so that the plateau of \( J(\eta) \) extends to \( \pm \infty \). Moreover, in this configuration the line \( \lambda_Q = \eta C \lambda_W \) intersects \( \tilde{\mathcal{R}} \), so that the Carnot efficiency lies at the edge of the plateau region of degenerate maxima of \( J(\eta) \). In contrast, for the second configuration in the lower panels \( \tilde{\mathcal{R}} \) does not intersect the \( \lambda_Q \)-axis and the plateau is restricted to a finite region of \( \eta \) values. Furthermore, this plateau does not contain the Carnot efficiency \( \eta_C \). We note that in both cases \( J(\eta) \) has kinks resulting from the
The minimizing curve $\tilde{\lambda}(\eta)$ hitting the boundary of the domain of convergence $C_0$. An efficiency distribution similar to that in the top right panel of Fig. 4 has also been obtained in [22] for a closely related model.

In conclusion, we have demonstrated how deviations from the “universal” VWVE theory of efficiency fluctuations can arise when certain prerequisites are violated. Exact solutions of two non-trivial models of colloidal engines support our observations. The first central observation is that $J(\eta)$ itself may still follow the “universal” behavior. It develops singular points, and hence deviates from the “universal” shape, if and only if $\phi(\lambda_Q, \lambda_W)$ is non-smooth along the curve $\tilde{\lambda}(\eta)$. In the latter case, the Carnot efficiency need not be the least likely efficiency. It would be very interesting to understand this behaviour further by studying initial-state dependence or finite-time effects. A second crucial observation is that $\phi$ can have degenerate minima, which will typically lead to a plateau of maximal values in the rate function $J(\eta)$. The model of the Brownian Gyator provides a concrete example of this situation. A third general insight is that the reversible efficiency $\eta_C$ is the “least likely” efficiency (or one of several least likely plateau efficiencies) if and only if the line $\lambda_Q = \eta_C \lambda_W$ intersects the region $R$ of minimizing points $(\lambda_Q, \lambda_W)$ for $\phi(\lambda_Q, \lambda_W)$. The fluctuation theorem symmetry $[6]$, which is usually violated in the presence of cutoffs, is sufficient, but not necessary for the least likelihood of $\eta_C$. It would be very interesting if there are other models, studied theoretically or experimentally [48, 49], which show some of the behaviour we detail here.

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Figure 4. Scaled cumulant generating functions (left) and efficiency rate functions (right) for two configurations of the Brownian gyator. Dashed black lines mark the macroscopic efficiency $\tilde{\eta}$, dashed orange lines the reversible efficiency $\eta_C$. The minimizing curve $\tilde{\lambda}(\eta)$ is shown in red in the left panels. The efficiencies at the edges of the plateau region are marked by dotted blue lines in the right panels. Top: $u_1 = 5$, $u_2 = 1$, $\alpha = \pi/4$, $f_{\text{ext}} = -1/2$, $k_B T_1 = 1$, $k_B T_2 = 1/3$. Bottom: $u_1 = 4$, $u_2 = 2$, $\alpha = \pi/4$, $f_{\text{ext}} = -1/2$, $k_B T_1 = 2$, $k_B T_2 = 1/10$. In all plots, $\gamma_1 = \gamma_2 = 1$. 

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SUPPLEMENTAL MATERIAL

This document provides further details of the calculations behind the results presented in the manuscript “Efficiency fluctuations in microscopic machines”. In the first section, we further elaborate the relationship between properties of the scaled cumulant generating function of heat and work \( \phi(\lambda_Q, \lambda_W) \) and properties of the efficiency rate function \( J(\eta) \). In the second section, we summarize the essential findings (relevant for our analysis) of Ref. [24] about the isothermal work-to-work converter used as an illustrative example in the main text. In the third section, we introduce the Brownian gyrator model, which served as a second illustrative example in the main text, and present details on the derivation of its scaled cumulant generating function of heat and work.

RELATION BETWEEN PROPERTIES OF \( \phi(\lambda_Q, \lambda_W) \) AND PROPERTIES OF \( J(\eta) \)

In this section, we formalize how certain prerequisites for the scaled cumulant generating function (sCGF) of heat and work \( \phi(\lambda_Q, \lambda_W) \) lead to properties of the efficiency rate function \( J(\eta) \), namely, smoothness, the least likely efficiency and plateaus.

Before we turn to the specific observations from the main text, we collect a few basic properties of the sCGF \( \phi(\lambda_Q, \lambda_W) \). By definition (see Eq. (4)), \( \phi(\lambda_Q, \lambda_W) \) is a convex function, notably meaning that the sublevel sets

\[ A_r := \{ (\lambda_Q, \lambda_W) \in \mathbb{R}^2 : \phi(\lambda_Q, \lambda_W) \leq r \} \]  

are convex. Moreover, it satisfies the normalization condition \( \phi(0, 0) = 0 \). As \( J(\eta) \) is obtained from \( \phi(\lambda_Q, \lambda_W) \) by minimizing along lines \( \lambda_Q = \eta \lambda_W \) through the origin and inverting the sign (see Eq. (6)), this immediately implies \( J(\eta) \geq 0 \). It also implies that the curve \( \tilde{\lambda}(\eta) \) is contained in the sublevel set \( A_0 \).

Furthermore, for large deviation theory to be applicable to the problem at all, we need that the rate function \( I(q_1, w) \) from Eq. (2) is well-defined and, in particular, \( I(0, 0) \) has a finite value. This in turn implies, according to the Gärtner-Ellis theorem \( I(q_1, w) = \max_{\lambda_Q, \lambda_W} [\lambda_Q q_1 + \lambda_W w - \phi(\lambda_Q, \lambda_W)] \), that \( \phi(\lambda_Q, \lambda_W) \) is bounded from below, so that there is a value \( \hat{\phi} \in \mathbb{R} \) with \( \phi(\lambda_Q, \lambda_W) \geq \hat{\phi} \) for all \( \lambda_Q, \lambda_W \). All these properties will be taken for granted in the following.

Smoothness

As argued in the main text, \( J(\eta) \) is a smooth function of \( \eta \) if and only if \( \phi(\lambda_Q, \lambda_W) \) is smooth along the curve \( \tilde{\lambda}(\eta) \). This follows immediately from the definition [5] of \( J(\eta) \) and from the definition of \( \tilde{\lambda}(\eta) \), implying \( J(\eta) = -\tilde{\phi}(\tilde{\lambda}(\eta)) \). However, this criterion is not very “practical” because it generally becomes quite complicated to determine the curve \( \tilde{\lambda}(\eta) \) in the presence of singular points for \( \phi(\lambda_Q, \lambda_W) \). A more accessible (albeit weaker) characterization is as follows:

If the global minimum of \( \phi(\lambda_Q, \lambda_W) \) is unique and there exists an open region \( U \subseteq \mathbb{R}^2 \) with \( A_0 \subseteq U \) such that \( \phi(\lambda_Q, \lambda_W) \) is smooth in \( U \) and the Hessian matrix of \( \phi(\lambda_Q, \lambda_W) \) is positive definite in \( U \), then \( J(\eta) \) is smooth.

This provides a sufficient (but not necessary) condition on \( \phi(\lambda_Q, \lambda_W) \) for \( J(\eta) \) to be smooth. To prove this, assume that \( \phi(\lambda_Q, \lambda_W) \) is smooth in some open region \( U \) containing the sublevel set \( A_0 \). Moreover, assume that the Hessian matrix of \( \phi(\lambda_Q, \lambda_W) \) is positive definite on \( U \), implying that the function is strictly convex on \( U \). Smoothness means that \( \phi(\lambda_Q, \lambda_W) \) is infinitely differentiable for all \( (\lambda_Q, \lambda_W) \in U \). For every \( \eta \) denote by \( \tilde{\lambda}(\eta) \equiv (\tilde{\lambda}_Q(\eta), \tilde{\lambda}_W(\eta)) \) a point with \( \tilde{\lambda}_Q(\eta) = \eta \tilde{\lambda}_W(\eta) \) that minimizes [7], i.e. \( \phi(\tilde{\lambda}_Q(\eta), \tilde{\lambda}_W(\eta)) = \min_{\lambda_Q, \lambda_W} \phi(\eta \lambda, \lambda) \). As observed above, \( J(\eta) \geq 0 \) for all \( \eta \), so that \( \tilde{\lambda}(\eta) \in A_0 \subseteq U \). Obviously, \( J(\eta) = -\tilde{\phi}(\tilde{\lambda}(\eta)) \), meaning that the function \( J(\eta) \) is determined by the values of \( \phi(\lambda_Q, \lambda_W) \) on the curve \( \eta \mapsto \tilde{\lambda}(\eta) \) with \( \eta \in \mathbb{R} \). It suffices to show that this mapping is smooth. The smoothness of \( \phi(\lambda_Q, \lambda_W) \) in \( U \) then implies that \( J(\eta) = \tilde{\phi}(\tilde{\lambda}(\eta)) \) is smooth as well.

Smoothness and convexity of \( \phi(\lambda_Q, \lambda_W) \) imply that the line \( \lambda_Q = \eta \lambda_W \) is tangent to the iso-contour \( \phi(\lambda_Q, \lambda_W) = J(\eta) \) in the point \( \tilde{\lambda}(\eta) \). Due to smoothness, the \( \phi(\lambda_Q, \lambda_W) = r \) iso-contour is \( \partial A_r \), the boundary of the sublevel set \( A_r \) from Eq. (11). Since \( \phi(\lambda_Q, \lambda_W) \) is strictly convex, so are the sublevel sets \( A_r \), and consequently the point \( \tilde{\lambda}(\eta) \) is unique for all \( \eta \). The fact that the line of efficiency \( \eta \) is tangent to an iso-contour of \( \phi(\lambda_Q, \lambda_W) \) in \( \tilde{\lambda}(\eta) \) means that the ray vector \( (\eta, 1) \) of the line \( \lambda_Q = \eta \lambda_W \) is orthogonal to the gradient of \( \phi(\lambda_Q, \lambda_W) \) in \( \tilde{\lambda}(\eta) \). Thus

\[
\eta \frac{\partial \phi}{\partial \lambda_Q}(\tilde{\lambda}_Q(\eta), \tilde{\lambda}_W(\eta)) + \frac{\partial \phi}{\partial \lambda_W}(\tilde{\lambda}_Q(\eta), \tilde{\lambda}_W(\eta)) = 0.
\]  

(12)
This relation along with $\tilde{\lambda}_Q(\eta) = \eta \tilde{\lambda}_W(\eta)$ implicitly defines $\tilde{\lambda}(\eta)$ as a function of $\eta$. More precisely, we consider the function

$$f(\eta, \lambda) := \eta \frac{\partial \phi}{\partial \lambda_Q}(\eta \lambda, \lambda) + \frac{\partial \phi}{\partial \lambda_W}(\eta \lambda, \lambda).$$

(13)

Assume that we have a particular solution $\lambda_*$ of Eq. (12) for some given $\eta_*$, so that $f(\eta_*, \lambda_*) = 0$. Due to the assumed positive definiteness of the Hessian matrix of $\phi(\lambda_Q, \lambda_W)$, we have

$$\frac{\partial f}{\partial \lambda}(\eta_*, \lambda_*) = (\eta_* 1) \left( \begin{array}{c} \frac{\partial^2 \phi}{\partial \lambda^2_Q}(\eta_*, \lambda_*) - \frac{\partial^2 \phi}{\partial \lambda_Q \partial \lambda_W}(\eta_*, \lambda_*) \\ \frac{\partial^2 \phi}{\partial \lambda_W}(\eta_*, \lambda_*) \end{array} \right) (\eta_*) > 0. \quad (14)$$

By the implicit function theorem, there exists a function $\eta \mapsto \hat{\lambda}(\eta)$ on an open interval $I \subseteq \mathbb{R}$ with $\eta_* \in I$ and such that $f(\eta, \hat{\lambda}(\eta)) = 0$ for all $\eta \in I$. Moreover, this function is of the same differentiability class as $f(\eta, \lambda)$. Put differently, the parameter function $\eta \mapsto \hat{\lambda}(\eta)$ implicitly defined by Eq. (12) is well-defined and smooth in $I$. As this holds everywhere in $U$, we conclude that $\hat{\lambda}(\eta) = (\hat{\eta} \hat{\lambda}(\eta), \hat{\lambda}(\eta))$ and thus $J(\eta) = -\phi(\hat{\lambda}_Q(\eta), \hat{\lambda}_W(\eta))$ is smooth.

Least likely efficiency

The “least likely” efficiency, i.e. the value $\hat{\eta}$ that maximizes $J(\eta)$, is directly related to the global minimum of $\phi(\lambda_Q, \lambda_W)$. Indeed, if $\phi(\lambda_Q, \lambda_W)$ assumes its minimal value at $(\hat{\lambda}_Q, \hat{\lambda}_W)$, then $J(\eta)$ will become maximal for $\hat{\eta} = \lambda_Q / \hat{\lambda}_W$ by Eq. (3). As observed in the main text and investigated in more detail in the next section of this Supplemental Material, the global minimum of $\phi(\lambda_Q, \lambda_W)$ need not be unique in general, so that $J(\eta)$ can have a degenerate maximum ("plateau") as well. In any case, the reversible efficiency $\eta_C$ maximizes $J(\eta)$ if and only if the global minimum of $\phi(\lambda_Q, \lambda_W)$ lies on the line $\lambda_Q = \eta_C \lambda_W$.

As observed in Refs. [2][10], the symmetry property [6] provides a sufficient (but not necessary) condition for the least likelihood of the reversible efficiency $\eta_C$. Indeed, if [8] holds then the iso-contour lines of $\phi(\lambda_Q, \lambda_W)$ are invariant under reflection through the point $(\lambda_Q^*, \lambda_W^*)$. By convexity, $\phi(\lambda_Q, \lambda_W)$ must therefore attain its minimal value in the reflection point, so that $\phi(\lambda_Q^*/2, \lambda_W^*/2) = \hat{\phi}$. Hence $J(\eta)$ assumes the maximum possible value for the line with slope $\hat{\eta} = \lambda_Q^*/\lambda_W^* = \eta_C$.

Plateau

We have illustrated in the main text that there could be cases where the global maximum of $J(\eta)$ is not unique, meaning that there may be an entire “plateau region” where $J(\eta)$ assumes its maximal value. As stated in the main text,

The maximum of $J(\eta)$ is unique if and only if all minimizing points $(\lambda_Q, \lambda_W)$ of $\phi$ lie on a line $\lambda_Q = \hat{\eta} \lambda_W$ through the origin with fixed slope $\hat{\eta}$. Below, we elaborate on this feature.

Let us first assume that there exist $\lambda^{(1)}, \lambda^{(2)} \in \mathcal{A}_\phi$ with $\lambda^{(1)} \neq \lambda^{(2)}$ and $\lambda_Q^{(1)} / \lambda_W^{(1)} \neq \lambda_Q^{(2)} / \lambda_W^{(2)}$, so that the minimizing points of $\phi(\lambda_Q, \lambda_W)$ do not lie on a single line through the origin. (Recall that $\mathcal{A}_\phi$ denotes the set of all $\lambda = (\lambda_Q, \lambda_W)$ with $\phi(\lambda_Q, \lambda_W) = \hat{\phi}$, c.f. Eq. (11)). The latter condition ensures that the slopes $\eta^{(1)} = \lambda_Q^{(1)} / \lambda_W^{(1)}$ of the lines connecting the origin with $\lambda^{(1)}$ and $\lambda^{(2)}$, respectively, are different, meaning that $\eta^{(1)} \neq \eta^{(2)}$. But since both lines cut through the global minimum, it follows that $J(\eta^{(1)}) = J(\eta^{(2)}) = -\hat{\phi}$, establishing the degeneracy of $J(\eta)$. Moreover, by convexity of $\phi(\lambda_Q, \lambda_W)$, all lines with slopes $\eta$ between $\eta^{(1)}$ and $\eta^{(2)}$ will also cross the global minimum, so that all plateau efficiencies are connected. In other words, there cannot be two separate plateaus in disjoint intervals of the extended(!) real line $\mathbb{R} \cup \{\infty\}$. (However, the plateau may be connected through the point $\eta = \pm \infty$, corresponding to the line $\lambda_W = 0$.) We remark that the emergence of plateaus need not necessarily be due to “tight coupling” as in the Brownian gyration example presented in the main text (Fig. 4). The tight coupling case, where $\phi(\lambda_Q, \lambda_W) = \phi(\lambda_Q - \hat{\eta} \lambda_W)$ is a special case exhibiting a degenerate global minimum.

To show the converse direction, assume that there is a unique $\hat{\eta}$ such that all points $\hat{\lambda} = (\hat{\lambda}_Q, \hat{\lambda}_W)$ with $\phi(\hat{\lambda}_Q, \hat{\lambda}_W) = \hat{\phi}$ satisfy $\hat{\lambda}_Q / \hat{\lambda}_W = \hat{\eta}$. Then all such points $\hat{\lambda}$ lie on the line $\lambda_Q = \hat{\eta} \lambda_W$, while for all $\eta \neq \hat{\eta}$ and all $\lambda \in \mathbb{R}$, $\phi(\eta \lambda, \lambda) > \hat{\phi}$. Hence $J(\eta)$ has a unique maximum at $\hat{\eta}$. 


EXAMPLE 1: ISOTHERMAL WORK-TO-WORK CONVERTER ENGINE [24]

In this section, we provide details about the example of an isothermal work-to-work converter by briefly summarizing the main results from Ref. [24] that are relevant for our purposes.

The model consists of a Brownian particle of mass $m$ in a fluid environment at temperature $T$ with instantaneous velocity $v(t)$. By the fluctuation-dissipation theorem, the coupling to the heat bath gives rise to a fluctuating force $\sqrt{2k_BT}\gamma \eta(t)$ as well as a frictional force $-\gamma v(t)$, where $\gamma$ is the friction coefficient and $\eta(t)$ is a Gaussian white-noise process with $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t-t')$. In addition, the particle is subject to two more fluctuating forces $f_1(t)$ and $f_2(t)$ with Gaussian white-noise statistics, independent of each other as well as of the white-noise, i.e. $\langle f_i(t)f_j(t') \rangle = \delta_{ij} f_i^2 \delta(t-t')$ and $\langle f_i(t)\eta(t') \rangle = 0$. The resulting equation of motion thus reads

$$m\ddot v(t) = -\gamma v(t) + f_1(t) + f_2(t) + \sqrt{2k_BT}\gamma \eta(t).$$

(15)

The relative strength of the three fluctuating forces with respect to each other is parameterized by the positive parameters $\theta$ and $\alpha$ such that $f_1^2 = 2k_BT\gamma\theta$ and $f_2^2 = 2k_BT\gamma\theta\alpha^2$. The force $f_1(t)$ and $f_2(t)$ are interpreted as a load and drive force, respectively. The work done by them is given by

$$W_i = \frac{1}{T} \int_0^T dt \, f_i(t) v(t).$$

(16)

Translated to the setting in the main text, we thus identify $W_1$ with $W$ and $W_2$ with $Q_1$. The resulting moment-generating function for $W_1$ and $W_2$ was found in Ref. [21] to satisfy the asymptotic relation

$$\Psi_r(\lambda_1, \lambda_2) = \langle e^{-\lambda_1 W_1 - \lambda_2 W_2} \rangle_r \sim g(\lambda_1, \lambda_2) e^{\tau \mu(\lambda_1, \lambda_2)} \quad (\tau \to \infty),$$

(17)

where

$$\mu(\lambda_1, \lambda_2) = \frac{1}{2} [1 - \nu(\lambda_1, \lambda_2)],$$

(18)

$$\nu(\lambda_1, \lambda_2) = [1 + 4\theta \left\{ \lambda_1(1 - \lambda_1) + \alpha^2 \lambda_2(1 - \lambda_2) - \alpha^2 \theta(\lambda_1 - \lambda_2)^2 \right\}]^{\frac{1}{2}},$$

(19)

$$g(\lambda_1, \lambda_2) = \frac{2^{\sqrt{-4\alpha^2\theta^2(\lambda_1 - \lambda_2)^2 - 4\theta(\alpha^2(\lambda_2 - 1)\lambda_1 + (\lambda_1 - 1)\lambda_2) + 1}}}{\sqrt{-4\alpha^2\theta^2(\lambda_1 - \lambda_2)^2 + 4\theta(\alpha^2(\lambda_2 - 1)\lambda_1 - \lambda_1^2 + \lambda_2) + 1} + 2\theta(-\lambda_1^2(\alpha^2\theta + 1) + \alpha^2(\lambda_1 - \lambda_2)(\lambda_1(\alpha^2\theta + 1) + 1) + 1)}.$$  

(20)

From this, the scaled cumulant generating function $\phi(\lambda_1, \lambda_2) \equiv \phi(\lambda_Q, \lambda_W)$ can be extracted straightforwardly.

EXAMPLE 2: BROWNIAN GYRATOR

In this section, we give a detailed definition of the Brownian gyrator model adapted from Ref. [25] and provide the exact solution of its scaled cumulant generating function $\phi(\lambda_Q, \lambda_W)$.

The model consists of a Brownian particle in two dimensions at position $x = (x_1, x_2)$, sketched in Fig. 5. The particle is immersed in a fluid environment and simultaneously coupled to two (effective) heat baths at different temperatures $T_1 > T_2$ that only act in the $x_1$ and $x_2$ directions, respectively. For example, the colder temperature $T_2$ may be the temperature of the surrounding fluid, while there are additional fluctuations in the $x_1$ directions due to external fields or an irradiating heat bath leading to a higher effective temperature $T_1$ [25], see [43] for an experimental realization or [44] for an equivalent electric circuit system.

By the fluctuation-dissipation theorem, the coupling to the environments leads, in both directions, to fluctuating forces $\sqrt{2k_BT_i}\gamma_i \eta_i(t)$ on the one hand and frictional forces $-\gamma_i \dot x_i$ on the other hand, where $\gamma_i$ are the respective friction coefficients and $\xi_i(t)$ are independent Gaussian white-noise processes with $\langle \eta_i(t) \rangle = 0$ and $\langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\delta(t-t')$. The particle is confined by a parabolic potential $U(x)$ with stiffnesses $u_1$ and $u_2$ along its principal axes, which are tilted by an angle $\alpha$ with respect to the coordinate axes:

$$U(x) = \frac{1}{2} x^T R_\alpha^T u \, R_\alpha \, x,$$

$$R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}.$$  

(22)

Due to the asymmetry of the thermal and restoring forces (for $T_1 \neq T_2$, $u_1 \neq u_2$, and $\alpha \neq \pi n/4$, $n \in \mathbb{Z}$), the particle reaches a non-equilibrium steady state and rotates around the origin on average [25]. It thereby exerts a torque on
the environment and can thus work as a microscopic heat engine. To quantify the work done, we generalize the model studied in [25] by introducing an additional external force $f_{\text{ext}}$ (green).

$$ f_{\text{ext}}(x) = -f_{\text{ext}} \epsilon x, \quad \text{where } \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

(23)

is the two dimensional anti-symmetric tensor. In the overdamped limit the dynamics of the Brownian Gyrorator is then described by the equations of motion

$$ \dot{x}(t) = -A x(t) + B \eta(t), \quad \text{where} $$

$$ A = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad K = R_\alpha^T u R_\alpha + f_{\text{ext}} \epsilon, \quad B = \begin{pmatrix} \sqrt{2k_B T_1 / \gamma_1} & 0 \\ 0 & \sqrt{2k_B T_2 / \gamma_2} \end{pmatrix}. $$

(25)

For the range of parameter values where the matrix $A$ is positive definite, the system reaches a steady state with probability distribution,

$$ p_{\text{st}}(x) = \frac{1}{4\pi \sqrt{\text{det} \Sigma(\infty)}} \exp(-\frac{1}{2} x \Sigma^{-1}(\infty) x), $$

(26)

where $\Sigma(\infty)$ is obtained as a solution of

$$ A\Sigma(\infty) + \Sigma(\infty)A^T = 2D, \quad D = \frac{1}{2} BB^T. $$

(27)

The work done by the external load force $f_{\text{ext}}$ as well as the heat taken from the hot reservoir over a process of time duration $\tau$ can be obtained using the standard definitions of stochastic thermodynamics as

$$ W[x(\cdot)] = \sum_{i,j} \int_0^\tau Y^{W}_{ij} x_j \, dx_i, \quad Q_1[x(\cdot)] = \sum_{i,j} \int_0^\tau Y^{Q_1}_{ij} x_j \, dx_i $$

(28)

with

$$ Y^{W} = -f_{\text{ext}} \epsilon, \quad Y^{Q_1} = \begin{pmatrix} K_{11} & K_{12} \\ 0 & 0 \end{pmatrix}. $$

(29)
where can be obtained as, with $f$ denoting the Onsager-Machlup path weight \[50–52\]. Since all the terms in the exponent of the RHS of Eq. (30) are quadratic in $x$, the efficiency rate function $J(q)$ has an infinite plateau as in the top panel of Fig. 4 from the main text. Parameter values: $k_B T_1 = 1$, $k_B T_2 = 1/3$, $u_1 = 5$, $u_2 = 1$, $\alpha = \pi/4$, $\gamma_1 = \gamma_2 = 1$.

In Fig. 6, we display the resulting average input and output powers $w$, $q_1$, and $q_2$ as a function of the load amplitude $f_{ext}$ for a certain choice of parameters. It illustrates that the system indeed works as a heat engine for moderate loads with $f_{ext} = -1 \ldots 0$.

Now using the path integral formalism, the moment generating function (MGF) of $Q_1$ and $W$ at arbitrary times can be obtained as,

$$
\Psi_t(\lambda_Q, \lambda_W) = \langle e^{-\lambda_Q Q_1 - \lambda_W W} \rangle = \int dx_0 p_{xt}(x_0) \int dx_\tau \int_{x_0}^{x_\tau} Dx[\cdot] P[x[\cdot]] e^{-\lambda_Q Q_1[x[\cdot]] - \lambda_W W[x[\cdot]]}, \tag{30}
$$

where

$$
P[x[\cdot]] \propto \exp \left( - \int_0^T dt \left( \dot{x}(t) + A x(t) \right)^T \frac{1}{2D} \left( \dot{x}(t) + A x(t) \right) \right) \tag{31}
$$

denotes the Onsager-Machlup path weight \[50\]. Since all the terms in the exponent of the RHS of Eq. (30) are quadratic in $x_1$, $x_2$ and their derivatives, we can write this as \[45\].

$$
\langle e^{-\lambda_Q Q_1[x]} - \lambda_W W[x] \rangle = \int dx_0 \int dx_\tau \int_{x_0}^{x_\tau} Dx[\cdot] \exp \left( \int_0^T x(t) \hat{O}_{\lambda_Q, \lambda_W} x(t) + \text{Boundary terms} \right) \tag{32}
$$

$$
= \sqrt{\frac{\det \hat{O}_{0,0}}{\det \hat{O}_{\lambda_Q, \lambda_W}}} \tag{33}
$$

Here the operator $\hat{O}$ is a matrix with differential operators as its entries \[52\], and the determinants that appear in Eq. (33) are functional determinants. For our problem, it can be shown that the matrix $\hat{O}$ has the form,

$$
\hat{O} = \begin{bmatrix}
-a \frac{d^2}{dt^2} + b, & c \frac{dt}{d\gamma} + d \\
-c \frac{dt}{d\gamma} + d & -e \frac{d^2}{dt^2} + f
\end{bmatrix}, \tag{34}
$$
where,

\[
\begin{align*}
    a &= \frac{1}{4D_{11}}, \\
    b &= \frac{1}{2} \left( \frac{A_{11}^2}{2D_{11}} + \frac{A_{21}^2}{2D_{22}} \right), \\
    c &= \frac{1}{2} \left( -\frac{A_{12}}{2D_{11}} + \frac{A_{21}}{2D_{22}} \right) - \lambda_{Q} \frac{A_{12}}{2} + \lambda_{W} f_{\text{ext}}, \\
    d &= \frac{1}{2} \frac{A_{11} A_{12}}{2D_{11}} + \frac{1}{2} \frac{A_{21} A_{22}}{2D_{22}}, \\
    e &= \frac{1}{4D_{22}}, \\
    f &= \frac{1}{2} \left( \frac{A_{12}^2}{2D_{11}} + \frac{A_{22}^2}{2D_{22}} \right).
\end{align*}
\]

Then the determinant ratio that appears in Eq. (33) can be computed using a technique described in [33] and recently used in [45], which is based on the spectral-\(\zeta\) functions of Sturm-Liouville type operators. Applying this method to the model at hand, it can be shown that this ratio can be obtained in terms of a characteristic polynomial function \(F\) as,

\[
\begin{align*}
    &\left\langle e^{-\lambda_{Q}Q_{1}[x(\cdot)]-\lambda_{W}W[x(\cdot)]} \right\rangle_{\tau} = \sqrt{\frac{F(0,0)}{F(\lambda_{Q}, \lambda_{W})}}, \\
    &F(\lambda_{Q}, \lambda_{W}) \equiv \text{Det} [M + NH(\tau)].
\end{align*}
\]

Here \(H\) is the matrix of suitably normalized fundamental solutions \(x^{1}(t), \ldots, x^{4}(t)\) of the homogeneous equation \(\hat{O} \mathbf{x} = 0,\)

\[
H(t) = \begin{bmatrix}
    x^{1}(t) & x^{2}(t) & x^{3}(t) & x^{4}(t) \\
    x^{1}(t) & x^{2}(t) & x^{3}(t) & x^{4}(t) \\
    x^{1}(t) & x^{2}(t) & x^{3}(t) & x^{4}(t) \\
    x^{1}(t) & x^{2}(t) & x^{3}(t) & x^{4}(t)
\end{bmatrix},
\]

\(H(0) = I_{4}.\)

and \(M\) and \(N\) contain information about the boundary conditions from Eq. (32), for which we require

\[
\begin{align*}
    &M \left[ \begin{array}{c}
        x(0) \\
        \dot{x}(0)
    \end{array} \right] = 0, \\
    &N \left[ \begin{array}{c}
        x(\tau) \\
        \dot{x}(\tau)
    \end{array} \right] = 0.
\end{align*}
\]

A derivation of Eq. (36), applicable to a class of driven Langevin systems with quadratic actions is given in [45]. We stress that the expression given in Eq. (36) is valid within the domain \(C_{\lambda_{Q}}, \lambda_{W}\) for which the operator \(\hat{O}\) doesn’t have negative eigenvalues. The MGF is not convergent outside this domain.

For the Brownian gyrator problem, we obtain the four independent solutions as

\[
\begin{align*}
    x^{1}_{1}(t) &= \exp( \pm t \sqrt{\frac{\sigma_{x} f^{2} - 2abc f - 2ac^{2} \lambda_{W} f + 4ad^{2} e + b^{2} c^{2} e^{2} - 2bc^{2} e + c^{4}}{a e} } + \frac{b}{a} - \frac{c^{2}}{a e} + \frac{f}{e} ) \\
    x^{2}_{1}(t) &= x^{1}_{1}(t) \left( (c^{2} d - a d f) + c (a f - c^{2}) x^{1}_{1}(t) - a c e x^{1}_{1}(t) - a d e x^{1}_{1}(t)(t) \right) \\
    x^{3}_{1}(t) &= 0 \\
    x^{4}_{1}(t) &= 0 
\end{align*}
\]

The matrices \(M\) and \(N\) are given by,

\[
\begin{align*}
    M &= \begin{pmatrix}
        2D_{11} \theta_{11} A_{11} + A_{11} - 2D_{11} \Sigma_{11} & 2D_{12} \theta_{12} A_{12} + A_{12} - 2D_{12} \Sigma_{12} - D_{11} \Sigma_{12} \\
        -A_{12} - 2D_{12} \Sigma_{12} & 2D_{22} \theta_{22} A_{22} + A_{22} - 2D_{22} \Sigma_{22} - D_{22} \Sigma_{22}
    \end{pmatrix} \\
    N &= \begin{pmatrix}
        0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0
    \end{pmatrix}.
\end{align*}
\]
Using these, the MGF can be computed exactly using Eq. (36). Notice that the solution is valid for arbitrary time $\tau$. Various interesting finite time aspects of this solution will be discussed in a future publication. Here we focus on the large time, leading order form of the MGF given by,

$$\Psi_\tau(\lambda_Q, \lambda_W) \sim g(\lambda_Q, \lambda_W) \ e^{\tau \ \phi(\lambda_Q, \lambda_W)}.$$  \hspace{1cm} (43)

Using the exact solution obtained from Eq. (36), the large time functional form given above can be obtained by performing an asymptotic expansion using Mathematica. We provide here the exact functional forms for the completion of the discussion.

$$\phi(\lambda_Q, \lambda_W) = \frac{1}{8} \sqrt{D_3^2 \left( 16 D_1^1 \left( 2 \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - A_1 e^2 \right) + 2 A_1 A_2 D_1 D_2 - A_1^2 D_1^2 \right) - \sqrt{\left( - \frac{A_1^2 e^2}{2 A_2} - 2 A_1 \lambda e^2 + \frac{3 A_2}{d} \lambda W \right)_{\lambda \rightarrow 0}}} + \sqrt{\left( \frac{3 M}{4} - \frac{d e^2}{a e^2} + \frac{b}{e^2} + \frac{f}{e^2} \right)}.$$ \hspace{1cm} (44)

In terms of the function $\Gamma(\lambda_Q, \lambda_W)$ defined as

$$\Gamma(\lambda_Q, \lambda_W) = \frac{\lambda_{41} (ad t - bc d) \sqrt{\frac{2 \lambda e^2}{a} + a f + b g}}{\lambda_{41} \sqrt{2 \lambda e^2 + a f + b g}} \left[ \sqrt{-2 \lambda e^2 \lambda_{41} (ad t - bc d) \left( 4 e \left( a \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) + d \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) \right) + \lambda_{41} (d e^2 - c d e f + d f e)} \right] + \right.

\left. 4 \left( 4 e \left( a \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) + d \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) \right) + \lambda_{41} (d e^2 - c d e f + d f e) \right) \right] + \right.

\left. 4 \left( 4 e \left( a \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) + d \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) \right) + \lambda_{41} (d e^2 - c d e f + d f e) \right) \right] + \right.

\left. 4 \left( 4 e \left( a \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) + d \left( \sqrt{\frac{2 \lambda e^2}{a} + a f + b g} - \lambda_{41} \right) \right) + \lambda_{41} (d e^2 - c d e f + d f e) \right) \right].$$

where

$$\begin{pmatrix} m & n & a \\ p & l & s \end{pmatrix} = \begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{41} & N_{42} & N_{44} \end{pmatrix}, \quad \begin{pmatrix} i & u & v \\ w & x & y \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{24} \end{pmatrix},$$

\hspace{1cm} (45)

we obtain

$$g(\lambda_Q, \lambda_W) = \sqrt{\frac{\Gamma(0, 01)}{\Gamma(\lambda_Q, \lambda_W)}}.$$ \hspace{1cm} (46)