Heisenberg uncertainty principle and quantum Zeno effects in the linguistic interpretation of quantum mechanics

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Abstract

Recently we proposed measurement theory (i.e., quantum language, or the linguistic interpretation of quantum mechanics), which is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics. This turn from physics to language does not only extend quantum theory to classical theory but also yield the quantum mechanical world view (i.e., the (quantum) linguistic world view). Thus, we believe that the linguistic interpretation is the most powerful in all interpretations. Our purpose is to examine the power of measurement theory, that is, to try to formulate Heisenberg uncertainty principle (particulary, the relation between Ishikawa’s formulation and so called Ozawa’s inequality) and quantum Zeno effects in the linguistic interpretation. As our conclusions, we must say that our trials do not completely succeed. However, we want to believe that this does not imply that we must abandon our linguistic interpretation.

1 Measurement Theory (= Quantum language)

1.1 Overview

In this section, we shall mention the overview of measurement theory (or in short, MT).

It is well known (cf. [14]) that quantum mechanics is formulated in an operator algebra $B(H)$ (i.e., an operator algebra composed of all bounded linear operators on a Hilbert space $H$ with the norm $\|F\|_{B(H)} = \sup\{\|u\|_H = 1 \| Fu \|_H \}$) as follows:

(A) quantum mechanics

\[ = [\text{quantum measurement}] + [\text{causality}] \]

(A) quantum mechanics

\[ (\text{probabilistic interpretation}) \quad (\text{kinetic equation}) \]

Also, the Copenhagen interpretation due to N. Bohr (et al.) is characterized as the guide to the usage of quantum mechanics (A).

Measurement theory (cf. refs. [3]-[12]) is, by an analogy of the (A), constructed as the mathematical theory formulated in a certain $C^*$-algebra $\mathcal{A}$ (i.e., a norm closed subalgebra in $B(H)$, cf. [17]) as follows:

(B) measurement theory

\[ = [\text{measurement}] + [\text{causality}] \]

(B) measurement theory

\[ (\text{Axiom 1 in Section 1.2}) \quad (\text{Axiom 2 in Section 1.2}) \]

Note that this theory (B) is not physics but a kind of language based on “the mechanical world view” since it is a mathematical generalization of quantum mechanics (A). Thus, our linguistic interpretation is characterized as the guide to the usage of Axioms 1 and 2 in (B).

When $\mathcal{A} = B_c(H)$, the $C^*$-algebra composed of all compact operators on a Hilbert space $H$, the (B) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when $\mathcal{A}$ is commutative (that is, when $\mathcal{A}$ is characterized by $C_0(\Omega)$, the $C^*$-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space $\Omega$ (cf. [17, 18])), the (B) is called classical measurement theory.

Thus, we have the following classification:
1.2 Measurement theory (= Quantum language)

Measurement theory (B) has two formulations (i.e., the C*-algebraic formulation and the W*-algebraic formulation, cf. [6,7]). In this paper, we devote ourselves to the W*-algebraic formulation of the measurement theory (B).

Let \( \mathcal{A} \subseteq B(H) \) be a C*-algebra, and let \( \mathcal{A}^* \) be the dual Banach space of \( \mathcal{A} \). That is, \( \mathcal{A}^* = \{ \rho \mid \rho \) is a continuous linear functional on \( \mathcal{A} \} \), and the norm \( \| \rho \|_{\mathcal{A}^*} \) is defined by \( \sup \{ |\rho(F)| \mid F \in \mathcal{A} \} \). Define the mixed state \( \rho \in \mathcal{A}^* \) such that \( \| \rho \|_{\mathcal{A}^*} = 1 \) and \( \rho(F) \geq 0 \) for all \( F \in \mathcal{A} \) such that \( F \geq 0 \). And define the mixed state space \( \mathcal{S}^m(\mathcal{A}^*) \) such that

\[
\mathcal{S}^m(\mathcal{A}^*) = \{ \rho \in \mathcal{A}^* \mid \rho \) is a mixed state\}.
\]

A mixed state \( \rho \in \mathcal{S}^m(\mathcal{A}^*) \) is called a pure state if it satisfies \( \rho_n = \theta \rho_1 + (1-\theta)\rho_2 \) for some \( \rho_1, \rho_2 \in \mathcal{S}^m(\mathcal{A}^*) \) and \( 0 < \theta < 1 \). Put

\[
\mathcal{S}^p(\mathcal{A}^*) = \{ \rho \in \mathcal{S}^m(\mathcal{A}^*) \mid \rho \) is a pure state\}.
\]

which is called a state space. It is well known (cf. [17]) that \( \mathcal{S}^p(B_c(H)^*) = \{ |u\rangle\langle u| \) (i.e., the Dirac notation) \mid \| u \|_H = 1 \} \), and \( \mathcal{S}^p(C_0(\Omega)^*) = \{ \delta_\omega \mid \delta_\omega \) is a point measure at \( \omega \} \), where \( \int_\Omega f(\omega)\delta_\omega(d\omega) = f(\omega) \) \( \forall f \in C_0(\Omega) \). The latter implies that \( \mathcal{S}^p(C_0(\Omega)^*) \) can be also identified with \( \Omega \) (called a spectrum space or simply spectrum) such as

\[
\mathcal{S}^p(C_0(\Omega)^*) \ni \delta_\omega \leftrightarrow \omega \in \Omega \quad \text{(spectrum)}
\]

Consider the pair \( [\mathcal{A}, \mathcal{N}]_{B(H)} \), called a basic structure. Here, \( \mathcal{A} \subseteq B(H) \) is a C*-algebra, and \( \mathcal{N} \subseteq B(H) \) is a particular C*-algebra (called a W*-algebra) such that \( \mathcal{N} \) is the weak closure of \( \mathcal{A} \) in \( B(H) \). Let \( \mathcal{N}^* \) be the pre-dual Banach space.

For example, we see (cf. [17]) that, when \( \mathcal{A} = B_c(H) \),

(i) \( \mathcal{A}^* = Tr(H) \) (=trace class), \( \mathcal{N} = B(H) \), \( \mathcal{N}_* = Tr(H) \).

Also, when \( \mathcal{A} = C_0(\Omega) \),

(ii) \( \mathcal{A}^* = \) “the space of all signed measures on \( \Omega \)”, \( \mathcal{N} = L^\infty(\Omega, \nu) \), \( \mathcal{N}_* = L^1(\Omega, \nu) \), where \( \nu \) is some measure on \( \Omega \) (cf. [17]).

For instance, in the above (ii) we must clarify the meaning of the “value” of \( F(\omega) \) for \( F \in L^\infty(\Omega, \nu) \) and \( \omega \in \Omega \). An element \( F(\Omega) \) is said to be essentially continuous at \( \rho_0(\in \mathcal{S}^p(\mathcal{A}^*)) \), if there uniquely exists a complex number \( \alpha \) such that

\[
\rho(G) \rightarrow \rho_0(G) \quad (\forall G \in \mathcal{A}(\subseteq \mathcal{N})),
\]

then \( \rho(F) \) converges to \( \alpha \).

And the value of \( \rho_0(F) \) is defined by the \( \alpha \).

According to the noted idea (cf. [1]), an observable \( \mathcal{O} := (X, \mathcal{F}, F) \) in \( \mathcal{N} \) is defined as follows:

(i) \( [\sigma\text{-field}] X \) is a set, \( F(\subseteq 2^X) \) is a \( \sigma\text{-field} \) of \( X \), that is, \( \mathcal{F} \subseteq \mathcal{N} \), \( \bigcup_{n=1}^\infty \Xi_n \in \mathcal{F} \), \( \Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F} \).

(ii) Countable additivity \( F \) is a mapping from \( \mathcal{F} \) to \( \mathcal{N} \), satisfying: (a): for every \( \Xi \in \mathcal{F} \), \( F(\Xi) \) is a non-negative element in \( \mathcal{N} \) such that \( 0 \leq F(\Xi) \leq I \), (b): \( F(\emptyset) = 0 \) and \( F(X) = I \), where \( 0 \) and \( I \) is the 0-element and the identity in \( \mathcal{N} \) respectively. (c): for any countable decomposition \( \Xi_1, \Xi_2, \ldots, \Xi_n, \ldots \) of \( \Xi \) (i.e., \( \Xi = \Xi_n \in \mathcal{F} \) \( n = 1, 2, 3, \ldots \)), \( \cup_{n=1}^\infty \Xi_n = \Xi, \Xi_i \cap \Xi_j = \emptyset \) \( (i \neq j) \), it holds that \( F(\Xi) = \sum_{n=1}^\infty F(\Xi_n) \) in the sense of weak* topology in \( \mathcal{N} \).
With any system $S$, a basic structure $[A,N]_{B(H)}$ can be associated in which the measurement theory (B) of that system can be formulated. A state of the system $S$ is represented by an element $\rho \in \mathcal{G}(\mathcal{A}^*)$ and an observable is represented by an observable $O := (X,F,F)$ in $N$. Also, the measurement of the observable $O$ for the system $S$ with the state $\rho$ is denoted by $M_N(0,S[\rho])$ (or more precisely, $M_N(O := (X,F,F), S[\rho])$). An observer can obtain a measured value $x \in X$ by the measurement $M_N(O,S[\rho])$.

The Axiom 1 presented below is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics (A). And thus, it is a statement without reality.

Now we can present Axiom 1 in the $W^*$-algebraic formulation as follows.

**Axiom 1 [Measurement].** The probability that a measured value $x \in X$ obtained by the measurement $M_N(O := (X,F,F), S[\rho])$ belongs to a set $\Xi \in F$ is defined by $\rho_0(F(\Xi))$ if $F(\Xi)$ is essentially continuous at $\rho_0(\in \mathcal{G}(\mathcal{A}^*))$.

Next, we explain Axiom 2. Let $[A_1,N_1]_{B(H_1)}$ and $[A_2,N_2]_{B(H_2)}$ be basic structures. A continuous linear operator $\Phi_{1,2} : N_2 \rightarrow N_1$ (with weak* topology) is called a Markov operator, if it satisfies that (i): $\Phi_{1,2}(F_2) \geq 0$ for any non-negative element $F_2$ in $N_2$, (ii): $\Phi_{1,2}(I_2) = I_1$, where $I_k$ is the identity in $N_k$, $(k = 1,2)$. In addition to the above (i) and (ii), in this paper we assume that $\Phi_{1,2}(A_2) \subseteq A_1$ and $\sup\{||\Phi_{1,2}(F_2)||_{A_1} | F_2 \in A_2\}$ such that $||F_2||_{A_2} \leq 1 = 1$.

It is clear that the dual operator $\Phi_{1,2}^* : A_1^* \rightarrow A_2^*$ satisfies that $\Phi_{1,2}^*(\mathcal{G}(\mathcal{A}^*)) \subseteq \mathcal{G}(\mathcal{A}^*)$.

Here note that, for any observable $O_2 := (X,F,F_2)$ in $N_2$, the $(X,F,\Phi_{1,2}F_2)$ is an observable in $N_1$.

Let $(T, \leq)$ be a tree, i.e., a partial ordered set such that “$t_1 \leq t_3$ and $t_2 \leq t_4$” implies “$t_1 \leq t_2$ or $t_2 \leq t_1$”. Put $T^2 = \{(t_1,t_2) \in T^2 | t_1 \leq t_2\}$. Here, note that $T$ is not necessarily finite.

Assume the completeness of the ordered set $T$. That is, for any subset $T' \subseteq T$ bounded from below (i.e., there exists $t' \in T$ such that $t' \leq t \forall t \in T'$), there uniquely exists an element $\inf(T') \in T$ satisfying the following conditions, (i): $\inf(T') \leq t \forall t \in T'$, (ii): if $s \leq t \forall t \in T'$, then $s \leq \inf(T')$.

The family $\{\Phi_{1,t_1} : N_2 \rightarrow N_1 | (t_1,t_2) \in T^2 \}$ is called a Markov relation (due to the Heisenberg picture), if it satisfies the following conditions (i) and (ii).

(i) With each $t \in T$, a basic structure $[A_t,N_t]_{B(H_t)}$ is associated.

(ii) For every $(t_1,t_2) \in T^2$, a Markov operator $\Phi_{t_1,t_2} : N_{t_2} \rightarrow N_{t_1}$ is defined. And it satisfies that $\Phi_{t_1,t_3}\Phi_{t_2,t_3} = \Phi_{t_1,t_3}$ holds for any $(t_1,t_2)$, $(t_2,t_3) \in T^2$.

When $\Phi_{t_1,t_2}^*(\mathcal{G}(\mathcal{A}^*_1)) \subseteq (\mathcal{G}(\mathcal{A}^*_2))$ holds for any $(t_1,t_2) \in T^2$, the Markov relation is said to be deterministic. Note that the classical deterministic Markov relation is represented by $\{\phi_{t_1,t_2} : \Omega_{t_1} \rightarrow \Omega_{t_2} | (t_1,t_2) \in T^2\}$, where the continuous map $\phi_{t_1,t_2} : \Omega_{t_1} \rightarrow \Omega_{t_2}$ is defined by $\phi_{t_1,t_2}(\delta_{\omega_1}) = \delta_{\phi_{t_1,t_2}(\omega_1)} \forall \omega_1 \in \Omega_1$.

Now Axiom 2 is presented as follows.

**Axiom 2 [Causality].** The causality is represented by a Markov relation $\{\Phi_{t_1,t_2} : N_{t_2} \rightarrow N_{t_1} | (t_1,t_2) \in T^2\}$.

1.3 The Linguistic Interpretation

Next, we have to study how to use the above axioms as follows. That is, we present the following interpretation (E) $[\text{E1} \cdots \text{E3}]$, which is characterized as a kind of linguistic turn of so-called Copenhagen interpretation (cf. [7–9]). That is, we propose:

(E1) Consider the dualism composed of “observer” and “system (=measuring object)”. And therefore, “observer” and “system” must be absolutely separated. In this sense, the interaction (or, measurement process such as $\ominus$ and $\bigcirc$ In Figure 2) should not be mentioned explicitly.

(E2) Only one measurement is permitted. And thus, the state after a measurement is meaningless since it can not be measured any longer. Thus, the collapse of the wavefunction is prohibited.

![Figure 2. Dualism in MT (cf. [7,8])](image-url)
We are not concerned with anything after measurement. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited.

(E3) Also, the observer does not have the space-time. Thus, the question: “When and where is a measured value obtained?” is out of measurement theory. And thus, Schrödinger’s cat is out of measurement theory,

and so on.

And therefore, in spite of Bohr’s realistic view, we propose the following linguistic view:

(F1) In the beginning was the language called measurement theory (with the linguistic interpretation (E)). And, for example, quantum mechanics can be fortunately described in this language. And moreover, almost all scientists have already mastered this language partially and informally since statistics (at least, its basic part) is characterized as one of aspects of measurement theory (cf. [5, 10–12]).

For completeness, we again note that,

(F2) The linguistic interpretation (E1)-(E3) is not only applied to quantum mechanics ([Q_m] in Figure 1) but also to whole MT.

This generalization has a merit such as the linguistic interpretation is determined “uniquely”, though so called Copenhagen interpretation has various variations. For example, the projection postulate of quantum mechanics can not be naturally extended to MT.

1.4 Sequential Causal Observable and Its Realization

For each \( k = 1, 2, \ldots, K \), consider a measurement \( M_N(0_k) \equiv (X_k, F_k, S_{[\rho]}(\cdot)) \). However, since the (E2) says that only one measurement is permitted, the measurements \( \{M_N(0_k, S_{[\rho]}(\cdot))\}_{k=1}^K \) should be reconsidered in what follows. Under the commutativity condition such that

\[
F_i(\Xi_i)F_j(\Xi_j) = F_j(\Xi_j)F_i(\Xi_i) \quad (\forall i \neq j, \forall \Xi_i, \Xi_j \in F_i, F_j)
\]

we can define the product observable \( \times_{k=1}^K O_k = (\times_{k=1}^K X_k, \times_{k=1}^K F_k, \times_{k=1}^K S_{[\rho]}(\cdot)) \) in \( N \) such that

\[
(\times_{k=1}^K F_k)(\times_{k=1}^K \Xi_k) = F_1(\Xi_1)F_2(\Xi_2) \cdots F_K(\Xi_K) \quad (\forall \Xi_k \in F_k, \forall k = 1, \ldots, K).
\]

Here, \( \times_{k=1}^K F_k \) is the smallest field including the family \( \{\times_{k=1}^K \Xi_k : \Xi_k \in F_k, k = 1, 2, \ldots, K\} \). Then, the above \( \{M_N(0_k, S_{[\rho]}(\cdot))\}_{k=1}^K \) is, under the commutativity condition (3), represented by the simultaneous measurement \( M_N(\times_{k=1}^K O_k, S_{[\rho]}(\cdot)) \).

Consider a tree \( (T = \{t_0, t_1, \ldots, t_n\}, \leq) \) with the root \( t_0 \). This is also characterized by the map \( \pi : T \setminus \{t_0\} \to T \) such that \( \pi(t) = \text{max}\{s \in T \mid s < t\} \). Let \( \{\Phi_{t,t'} : N_t \to N_{t'}\}_{t,t' \in T} \) be a causal relation, which is also represented by \( \{\Phi_{\pi(t),t} : N_{\pi(t)} \to N_t\}_{t \in T} \). Let an observable \( O_t \equiv (X_t, F_t, F_t) \) in \( N_t \) be given for each \( t \in T \). Note that \( \Phi_{\pi(t),t} \) is an observable in the \( N_{\pi(t)} \).

The pair \( (\mathcal{O}_T) = \{O_t \mid t \in T\} \) is a causal observable. For each \( s \in T \), put \( T_s = \{t \in T \mid t \leq s\} \). And define the observable \( \widehat{O}_s \equiv (\times_{t \in T_s} X_t, \times_{t \in T_s} F_t, F_t) \) in \( N_s \) as follows:

\[
\widehat{O}_s = \begin{cases} O_s & (\text{if } s \in T \setminus \pi(T)) \\ O_s \times (\times_{t \in \pi^{-1}(\{s\})} \Phi_{\pi(t),t} \widehat{O}_t) & (\text{if } s \in \pi(T)) \end{cases}
\]

if the commutativity condition holds (i.e., if the product observable \( O_s \times (\times_{t \in \pi^{-1}(\{s\})} \Phi_{\pi(t),t} \widehat{O}_t) \) exists) for each \( s \in \pi(T) \). Using (4) iteratively, we can finally obtain the observable \( \widehat{O}_{t_0} \) in \( N_{t_0} \). The \( \widehat{O}_{t_0} \) is called the realization (or, realized causal observable) of \( (\mathcal{O}_T) \).

1.5 Our motivation

Since MT has been shown to have a great power (cf. [3–12]), we believe that MT is superior to statistics as the language of science. However we now feel misgivings about the possibility that Heisenberg uncertainty principle (particularly, Ozawa’s inequality and quantum Zeno effects (cf. [13]) can not be formulated in MT. As our conclusions, we say that our trials partially succeed. This may imply that these should be understood in the other interpretations.

2 Heisenberg uncertainty principle
2.1 Preliminary

Let \([B_1(H), B_2(H)]_{B(H)}\) be the basic structure. Let \(A_i (i = 1, 2)\) be arbitrary self-adjoint operators on \(H\).

For example, it may satisfy that \([A_1, A_2] := A_1 A_2 - A_2 A_1 = i \hbar \sqrt{-1} I\). Let \(\mathbb{R}\) and \(\mathbb{B}\) be the real line and its Borel field respectively. Let \(O_{A_i} = (\mathbb{R}, \mathcal{B}, F_{A_i})\) be the spectral representation of \(A_i\), i.e., \(A_i = \int_0^\infty \lambda F_{A_i} (d\lambda)\), which is regarded as the observable in \(B(H)\).

Thus, two measurements:

\[(G_1) \ M_{B(H)}(O_{A_1} := (\mathbb{R}, \mathcal{B}, F_{A_1}), S_{\rho_u})\]

\[(G_2) \ M_{B(H)}(O_{A_2} := (\mathbb{R}, \mathcal{B}, F_{A_2}), S_{\rho_u})\]

be the real line and its Borel field respectively. Let \(O_{A_i} = (\mathbb{R}, \mathcal{B}, F_{A_i})\) be the spectral representation of \(A_i\), i.e., \(A_i = \int_0^\infty \lambda F_{A_i} (d\lambda)\), which is regarded as the observable in \(B(H)\). Let \(\rho_u = |u \rangle u\rangle\) be a state, where \(u \in H\) and \(\|u\| = 1\).

Thus, two measurements:

\[(H_1) \ M_{B(H \otimes K)}(O_{A_1 \otimes I}, S_{\rho_{us}})\]

\[(H_2) \ M_{B(H \otimes K)}(O_{A_2 \otimes I}, S_{\rho_{us}})\]

which is clearly equivalent to the above two \((G_1)\) and \((G_2)\) respectively.

Now we want to take these two measurements. However, the linguistic interpretation \((E_2)\) says that it is impossible, if \(A_1\) and \(A_2\) do not commute.

Let \(\hat{A}_i (i = 1, 2)\) be arbitrary self-adjoint operators on the tensor Hilbert space \(H \otimes K\), where it is assumed that

\[\hat{A}_1, \hat{A}_2 := \hat{A}_1 \hat{A}_2 - \hat{A}_2 \hat{A}_1 = 0\]  \hspace{1cm} (5)

Let \(\mathcal{O}_{\hat{A}_i} = (\mathbb{R}, \mathcal{B}, F_{\hat{A}_i})\) be the spectral representation of \(\hat{A}_i\), i.e., \(\hat{A}_i = \int_0^\infty \lambda F_{\hat{A}_i} (d\lambda)\), which is regarded as the observable in \(B(H \otimes K)\). Thus, we have two measurements as follows:

\[(I_1) \ M_{B(H \otimes K)}(O_{\hat{A}_1}, S_{\rho_{us}})\]

\[(I_2) \ M_{B(H \otimes K)}(O_{\hat{A}_2}, S_{\rho_{us}})\]

Note, by the commutative condition \((5)\), that the two can be realized as the simultaneous measurement

\[M_{B(H \otimes K)}(O_{\hat{A}_1} \times O_{\hat{A}_2}, S_{\rho_{us}})\]

where \(O_{\hat{A}_1} \times O_{\hat{A}_2} = (\mathbb{R}^2, \mathcal{B}^2, F_{\hat{A}_1} \times F_{\hat{A}_2})\).

Again note that any relation between \(A_i \otimes I\) and \(\hat{A}_i\) is not assumed. However, we want to regard this simultaneous measurement as the substitute of the above two \((H_1)\) and \((H_2)\). Putting

\[\hat{N}_i := \hat{A}_i - A_i \otimes I\]

\[(6)\]

we define the \(\Delta_{\hat{N}_i}^{u \otimes s}\) and \(\Xi_{\hat{N}_i}^{u \otimes s}\) such that

\[\Delta_{\hat{N}_i}^{u \otimes s} = \|\hat{N}_i(u \otimes s)\|\]

\[\Xi_{\hat{N}_i}^{u \otimes s} = \|\hat{N}_i - \langle u \otimes s, \hat{N}_i(u \otimes s) \rangle(u \otimes s)\|\]

where the following inequality:

\[\Delta_{\hat{N}_i}^{u \otimes s} \geq \Xi_{\hat{N}_i}^{u \otimes s}\]  \hspace{1cm} (8)

is common sense.

By the commutative condition \((5)\) and \((6)\), we see that

\[\hat{N}_1, \hat{N}_2 + \hat{N}_1, A_2 \otimes I + [A_1 \otimes I, \hat{N}_2] = -[A_1 \otimes I, A_2 \otimes I]\]

(9)

Here we should note that the first term (or, precisely, \(\langle u \otimes s, A_1 \rangle \) the first term \(\langle u \otimes s, A_1 \rangle \)) of \((9)\) can be, by the Robertson uncertainty relation \((cf. [14])\), estimated as follows:

\[2\Delta_{\hat{N}_1}^{u \otimes s} \Xi_{\hat{N}_2}^{u \otimes s} \geq \|\hat{N}_1, \hat{N}_2(u \otimes s)\|\]  \hspace{1cm} (10)

Remark 1 There may be an opinion such that the physical meaning of \(\Delta_{\hat{N}_i}^{u \otimes s}\) (or, \(\Xi_{\hat{N}_i}^{u \otimes s}\)) is not clear. However, we do not worry about this problem. That is because our concern is not only quantum mechanics ([\(Q_m\)] in Figure 1) but also quantum system theory ([\(Q_s\)] in Figure 1). However, recalling \((F_2)\), in most cases, we can expect that

\[J\] \hspace{1cm} (J)

A (metaphysical) statement in quantum system theory is regarded as a (physical) statement in quantum mechanics, because both are formulated in the same mathematical structure, and moreover, are based on the linguistic interpretation.

2.2 Heisenberg uncertainty principle with the same average condition

In the previous section, any relation between \(A_i \otimes I\) and \(\hat{A}_i\) is not assumed. However, in this section we assume the following hypothesis:

Hypothesis 1 (The same average condition). We assume that

\[\langle u \otimes s, \hat{N}_i(u \otimes s) \rangle = 0 \hspace{1cm} (\forall u \in H, i = 1, 2)\]

or equivalently

\[\langle u \otimes s, \hat{A}_i(u \otimes s) \rangle = \langle u, A_i u \rangle \hspace{1cm} (\forall u \in H, i = 1, 2)\]
Thus we get (12).

**Remark 2** The existence of $\tilde{A}_i$ (with the conditions (5) and (11)) is guaranteed (cf. [2]). Also, we can assume that the (11) is equivalent to

$$\langle u \otimes s, \tilde{N}_i(v \otimes s) \rangle = 0 \quad (\forall u, v \in H, i = 1, 2)$$  \hspace{0.5cm} (13)

This is proved as follows:

$$0 = \langle (u + v) \otimes s, \tilde{N}_i((u + v) \otimes s) \rangle$$

$$= \langle u \otimes s, \tilde{N}_i(v \otimes s) \rangle + \langle v \otimes s, \tilde{N}_i(u \otimes s) \rangle$$

$$= 2 \text{[Real part]}((u \otimes s, \tilde{N}_i(v \otimes s)))$$

$$0 = \langle (u + \sqrt{-1}v) \otimes s, \tilde{N}_i((u + \sqrt{-1}v) \otimes s) \rangle$$

$$= 2\sqrt{-1} \text{[Imaginary part]}((u \otimes s, \tilde{N}_i(v \otimes s)))$$

Thus we get (13).

Using (13), we can calculate the second term (or, precisely, $\langle u \otimes s, "the second term"(u \otimes s) \rangle$) in (9) as follows:

$$\langle u \otimes s, [\tilde{N}_1, A_2 \otimes I](u \otimes s) \rangle$$

$$= \langle u \otimes s, \tilde{N}_1(A_2u \otimes s) \rangle - \langle A_2u \otimes s, \tilde{N}_1(u \otimes s) \rangle$$

$$= 0 \quad (\forall u \in H)$$  \hspace{0.5cm} (14)

Similarly, we calculate the third term in (9) as follows:

$$\langle u \otimes s, [A_1 \otimes I, \tilde{N}_2](u \otimes s) \rangle = 0 \quad (\forall u \in H)$$  \hspace{0.5cm} (15)

Also, it is clear that

$$\langle u \otimes s, [A_1 \otimes I, A_2 \otimes I](u \otimes s) \rangle$$

$$= \langle u, [A_1, A_2]u \rangle \quad (\forall u \in H)$$  \hspace{0.5cm} (16)

Summing up ((10), (12), (14), (15), (16)), we can conclude that

$$\Delta^u_{N_1} \cdot \Delta^u_{N_2} = (\Delta^u_{N_1} \cdot \Delta^u_{N_2})$$

$$\geq \frac{1}{2} \|\langle u, [A_1, A_2]u \rangle\| \quad (\forall u \in H \text{ such that } \|u\| = 1)$$  \hspace{0.5cm} (17)

which is Ishikawa’s formulation of Heisenberg’s uncertainty principle (cf. [2]).

**Remark 3** Assume that $[A_1, A_2] = \hbar \sqrt{-1}I$. If Hypothesis 1 is not assumed, we can say in what follows. That is, for any positive $\epsilon$, there exist $s \in K(\|s\| = 1)$, $\tilde{A}_i(i = 1, 2)$, $u \in H(\|u\| = 1)$ such that

$$\Delta^u_{N_1} < \epsilon, \quad \Delta^u_{N_2} < \epsilon$$

(cf. Remark 3 in ref. [2]). Thus, if we hope that Heisenberg uncertainty principle (17) holds, the same average condition is indispensable.

### 2.3 Heisenberg uncertainty principle without the same average condition

We believe that Hypothesis 1 is very natural. However, in this section, we do not assume Hypothesis 1 (the same average condition).

Put $\sigma(A_1; u) = \| (A - \langle u, A_1u \rangle)u \|$. Using the Robertson uncertainty relation, we can estimate the second term (or, precisely, $\langle u \otimes s,"the second term"(u \otimes s) \rangle$) in (9) as follows:

$$2\Delta^u_{N_1} \cdot \sigma(A_2; u) \geq \| (u \otimes s, [\tilde{N}_1, A_2 \otimes I](u \otimes s)) \|$$

$$(\forall u \in H \text{ such that } \|u\| = 1)$$  \hspace{0.5cm} (18)

Similarly, we estimate the third term in (9) as follows:

$$2\Delta^u_{N_2} \cdot \sigma(A_1; u) \geq \| (u \otimes s, [A_1 \otimes I, \tilde{N}_2](u \otimes s)) \|$$

$$(\forall u \in H \text{ such that } \|u\| = 1)$$  \hspace{0.5cm} (19)

Summing up ((8), (10), (16), (18), (19)), we can conclude that

$$\Delta^u_{N_1} \cdot \Delta^u_{N_2} + \Delta^u_{N_1} \cdot \sigma(A_1; u) + \Delta^u_{N_2} \cdot \sigma(A_2; u)$$

$$\geq \frac{1}{2} \| (u, [A_1, A_2]u) \| \quad (\forall u \in H \text{ such that } \|u\| = 1)$$  \hspace{0.5cm} (20)

Since Hypothesis 1 is not assumed in this section, it is a matter of course that this (20) is more rough than the (17).

**Remark 4** (Ozawa’s inequality). In [15, 16], M. Ozawa tried to formulate Heisenberg’s γ-ray microscope thought experiment in his interpretation, and proposed the following inequality (so called Ozawa’s inequality):

$$\epsilon(A_1)\eta(A_2) + \eta(A_2)\sigma(A_1) + \epsilon(A_1)\sigma(A_2)$$

$$\geq \frac{1}{2} \| (u, [A_1, A_2]u) \|$$  \hspace{0.5cm} (21)

which is, by our notation, rewritten as follows.

$$\Delta^u_{N_1} \cdot \Delta^u_{N_2} + \Delta^u_{N_1} \cdot \sigma(A_1; u) + \Delta^u_{N_2} \cdot \sigma(A_2; u)$$

$$\geq \frac{1}{2} \| (u, [A_1, A_2]u) \| \quad (\forall u \in H \text{ such that } \|u\| = 1)$$  \hspace{0.5cm} (22)

Note that this (22) is mathematically the same as the above (20), but the (21) is not. Here, it should be noted that Ozawa’s assertion is just the (21), that is,
(K) the physical meanings of "error"(\(e(A_1) = \Delta_{\hat{N}_1}^{u_0}\)) and "disturbance"\(\eta(A_2) = \Delta_{\hat{N}_1}^{u_0}\)
are distinguished in Ozawa's inequality (21).

Therefore there is a great gap between Ozawa’s inequality (21) and the (20). In fact, the (20) is not the mathematical representation of Heisenberg’s \(\gamma\)-ray microscope thought experiment. Now we think that it may be impossible to formulate this (K) in the linguistic interpretation, since the \((E_2)\) says that anything after measurement can not be described. In fact, we are not successful yet. Therefore, we consider that the other interpretation is indispensable for the understanding of Ozawa’s inequality (21), and thus, the (20) and the (21) are different assertions.

3 Quantum Zeno effects

Let \([B,(H), B(H)]_{B(H)}\) be the basic structure. Let \(\mathbb{P} = [P_n]_{n=1}^{\infty}\) be the spectral resolution in \(B(H)\), that is, for each \(n\), \(P_n \in B(H)\) is a projection such that
\[
\sum_{n=1}^{\infty} P_n = I
\]

Define the \((\Psi_\varepsilon)_\ast : Tr(H) \rightarrow Tr(H)\) such that
\[
(\Psi_\varepsilon)_\ast (|u\rangle \langle u|) = \sum_{n=1}^{\infty} |P_n u\rangle \langle P_n u| \quad (\forall u \in H)
\]

Also, we define the Schrödinger time evolution \((\Psi_{\Delta t}^N)_\ast : Tr(H) \rightarrow Tr(H)\) such that
\[
(\Psi_{\Delta t}^N)_\ast (|u\rangle \langle u|) = |e^{-i\frac{\Delta t N}{\hbar}} u\rangle \langle e^{-i\frac{\Delta t N}{\hbar}} u| \quad (\forall u \in H)
\]

Consider \(t = 0, 1\). Putting \(\Delta t = \frac{1}{N}, H = H_0 = H_1\), we can define the \((\Phi_{0,1}^{(N)})_\ast : Tr(H_0) \rightarrow Tr(H_1)\) such that
\[
(\Phi_{0,1}^{(N)})_\ast = ((\Psi_{\Delta t}^{1/N})_\ast(\Psi_\varepsilon)_\ast)^N
\]

which induces the Markov operator \(\Phi_{0,1}^{(N)} : B(H_1) \rightarrow B(H_0)\) as the dual operator \(\Phi_{0,1}^{(N)} = ((\Phi_{0,1}^{(N)})_\ast)_\ast\). Let \(\rho = |\psi\rangle \langle \psi|\) be a state at time 0. Let \(O_1 := (X, F, F)\) be an observable in \(B(H_1)\). Thus, we have a measurement:

\[
M_{B(H_0)}(\Phi_{0,1}^{(N)} O_1, S_{|\psi\rangle \langle \psi|})
\]

( or more precisely, \(M_{B(H_0)}(\Phi_{0,1}^{(N)} O := (X, F, \Phi_{0,1}^{(N)} F), S_{|\psi\rangle \langle \psi|})\) ). Here, Axiom 1 says that

\[
tr(|\psi\rangle \langle \psi| \cdot \Phi_{0,1}^{(N)} F(\Xi)) \quad (23)
\]

Now we shall explain "quantum Zeno effect" in the following example.

**Example 1** Let \(\psi \in H\) such that \(|\psi\rangle = 1\). Define the spectral resolution
\[
\mathbb{P} = [P_1 = |\psi\rangle \langle \psi|, P_2 = I - P_1] \quad (24)
\]

And define the observable \(O_1 := (X, F, F)\) in \(B(H_1)\) such that
\[
X = \{x_1, x_2\}, \quad F = 2^X
\]

and
\[
F(x_1) = |\psi\rangle \langle \psi| = P_1, F(x_2) = I - |\psi\rangle \langle \psi| = P_2,
\]

Now we can calculate (23) (i.e., the probability that a measured value \(x_1\) is obtained) as follows.

\[
(23) = \langle \psi, ((\Psi_{\Delta t}^{1/N})_\ast(\Psi_\varepsilon)_\ast)^N (\psi |\psi\rangle |\psi\rangle |\psi\rangle \psi\rangle \\
\geq |\langle \psi, e^{-i\frac{\Delta t N}{\hbar}} \psi, e^{i\frac{\Delta t N}{\hbar}} \psi\rangle |^N \\
\approx (1 - \frac{1}{N^2} (||\frac{\mathcal{M}}{\hbar}\psi||^2 - |\langle \psi, (\frac{\mathcal{M}}{\hbar})\psi\rangle|^2))^N \rightarrow 1 \\
(N \rightarrow \infty)
\]

(25)

Thus, if \(N\) is sufficiently large, we see that

\[
M_{B(H_0)}(\Phi_{0,1}^{(N)} O_1, S_{|\psi\rangle \langle \psi|}) = M_{B(H_0)}(\Phi_1 O_1, S_{|\psi\rangle \langle \psi|})
\]

(where \(\Phi_1 : B(H_1) \rightarrow B(H_0)\) is the identity map)
or, we say, roughly speaking in terms of the Schrödinger picture, that the state \(|\psi\rangle \langle \psi|\) does not move.

**Remark 5** The above argument is motivated by B. Misra and E.C.G. Sudarshan [13]. However, the title of their paper: "The Zeno’s paradox in quantum theory" urges us to guess that

\[
(\Phi) \quad \text{the probability that the measured value obtained by the measurement belongs to } \Xi(\in \mathcal{F}) \text{ is given by}
\]

\[
tr(|\psi\rangle \langle \psi| \cdot \Phi_{0,1}^{(N)} F(\Xi)) \quad (23)
\]

If this (M) is their assertion, we can not understand "quantum Zeno effect". That is because the linguistic interpretation require the commutative condition (4) should be satisfied, however, \(\mathbb{P}\) and \(\Psi_{\Delta t}^N\mathbb{P}\) do not commute. In the sense of Example 1, this effect should be called "brake effect" and not "watched pot effect".
4 Conclusions

In this paper, we point out the possibility that (N) two nice ideas (K) and (M) can not be understood in the linguistic interpretation.

That is because our theory is not concerned with any influence after a measurement. In spite of the difficulty such as (N), we do not give up to assert the linguistic interpretation, since it has a great power of description (cf. refs. [3]-[12]).

It is always interesting to find a phenomenon that can not be explained in MT. Thus, in spite of our conjecture (N), we earnestly hope that the readers investigate the following problem:

(O) Describe Ozawa’s inequality (21) in the linguistic interpretation!

This problem is very important in quantum mechanics. That is because it is generally believed that the difference of interpretations is usually negligible in practical problems. If the formulation of Heisenberg uncertainty principle depends on quantum interpretations, our next problem may be to investigate ”What is the most certain interpretation?” And we believe that the linguistic interpretation is quite hopeful. However, it should be examined from various points of view.

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