RESIDUES, GROTHENDIECK POLYNOMIALS AND K-THEORETIC THOM POLYNOMIALS

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Abstract. Grothendieck polynomials were introduced by Lascoux and Schützenberger, and they play an important role in K-theoretic Schubert calculus. In this paper, we give a new definition of double stable Grothendieck polynomials based on an iterated residue operation. We illustrate the power of our definition by calculating the Grothendieck expansion of K-theoretic Thom polynomials of \( A_2 \) singularities. We present the expansion in two versions: one displays its expected stabilization property, while the other displays its expected finiteness property.

1. Introduction

From the point of view of enumerative geometry, the most important invariant of a subvariety \( X \) in a smooth variety \( M \) is its cohomological fundamental class \([X \subset M] \in \text{H}^{\text{codim}(X \subset M)}(M)\), obtained from the homology fundamental class by Poincaré duality. A general strategy to study this invariant is degeneracy loci theory (see e.g. [FP, BSz]), which reduces the problem of calculating fundamental classes to calculating \( G \)-equivariant fundamental classes

\[ [\eta \subset J] \in \text{H}^{\text{codim}(\eta \subset J)}_G(J) = \text{H}^{\text{codim}(\eta \subset J)}_G(\text{pt}) \]

of \( G \)-invariant subvarieties \( \eta \) of a \( G \)-representation \( J \).

We encounter this setup, for example, in modern Schubert calculus, where \( J \) is a representation vector space of a quiver and the fundamental class is called a quiver polynomial, see e.g. [KMS, BS, R1]. Another instance is global singularity theory, where \( J \) is the vector space of germs of maps acted upon by reparametrizations, and the fundamental class is called the Thom polynomial [I, R2] of the singularity.

In this paper we will be concerned with the notion of the \( G \)-equivariant K-theoretic fundamental class \([\eta \subset J] \in K_G(J) = K_G(\text{pt})\) of an invariant subvariety \( \eta \) of a \( G \)-representation \( J \). It turns out that there is some ambiguity in the definition of such an object (cf. Section 5), but, regardless, the cohomological fundamental class may always be recovered from the K-theoretic fundamental class via a limiting procedure.

In cohomological fundamental class theory, a natural basis is the Schur basis, in part because the Schur polynomials are related to the fundamental classes of Schubert varieties. It is thus natural to attempt to express K-theoretic fundamental classes in terms of Grothendieck polynomials introduced in [LS] which are similarly related to the K-theoretic fundamental classes of (the structure sheaves of) Schubert varieties.
There is a number of “flavors” of Schur and Grothendieck polynomials, and, in this article, we will focus on the so-called *double stable* polynomials, which are best adapted to the bivariant problems we study.

Finally, we would like to mention a central aspect of the theory: the various *positivity* results, which state that in a number of situations the coefficients of equivariant Poincaré duals in the Schur basis are nonnegative. This is true, for example, for the above mentioned Thom polynomials of singularities [PW], and quiver polynomials [KMS, BR]. In the K-theoretic setup, it seems that the corresponding notion is expressions with *alternating signs*. An example of this phenomenon is Buch’s result in [B3], which shows that K-theoretic Dynkin quiver polynomials may be expressed in terms of certain double stable Grothendieck polynomials; moreover, the coefficients in this expression are (conjecturally) alternating. 

In this paper, we present a residue calculus for double stable Grothendieck polynomials, which makes proving various properties, in particular, positivity of expansions, straightforward. Our formulas allow us to begin the study of K-theoretic Thom polynomials. Below, we give a quick introduction to these two subjects and explain the main results of the paper.

### 1.1. Grothendieck polynomials

In Section 2, we recall the original definition of double stable Grothendieck polynomials [FK1, FK2]. This involves first introducing *ordinary Grothendieck polynomials* \( G_w \), indexed by permutations, and defined by a recursion involving divided differences. Geometrically, the polynomials \( G_w \) represent torus-equivariant K-theoretic fundamental classes of Schubert varieties in full flag varieties. Next, *double stable Grothendieck polynomials* \( G_{\lambda}(\alpha; \beta) \) parametrized by partitions are defined by a limiting procedure from ordinary Grothendieck polynomials, and, finally, applying to these latter polynomials a set of certain *straightening laws*, one defines double stable Grothendieck polynomials \( G_I(\alpha; \beta) \) parametrized by arbitrary integer sequences. Another approach to double stable Grothendieck polynomials parametrized by partitions uses the combinatorics of set-valued tableaux [B2].

In §4.1, we propose a new formula for the most general integer-sequence parametrized double stable Grothendieck polynomials:

\[
G_I(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l) = \ \Res_{z_1=0, \infty} \ldots \Res_{z_r=0, \infty} \left( \prod_{j=1}^{r} (1-z_j) \prod_{i>j} (1-z_i/z_j) \prod_{j=1}^{r} \frac{\prod_{i=1}^{l} (1-z_j \beta_i)}{\prod_{i=1}^{l} (1-z_j \alpha_i)} \right)^{-k} \prod_{j=1}^{r} \frac{dz_j}{z_j}.
\]

This formula is analogous to the useful residue formula

\[
s_I(\bar{\alpha}_1, \ldots, \bar{\alpha}_k; \bar{\beta}_1, \ldots, \bar{\beta}_l) = \ (-1)^r \Res_{z_1=\infty} \ldots \Res_{z_r=\infty} \left( \prod_{j=1}^{r} z_j^l \prod_{j>i} (1-z_i/z_j) \prod_{j=1}^{r} \frac{\prod_{i=1}^{l} (1+\bar{\beta}_i/z_j)}{\prod_{i=1}^{l} (1+\bar{\alpha}_i/z_j)} \right)^{-k} \prod_{j=1}^{r} \frac{dz_j}{z_j}.
\]

\footnote{Using results of the present paper, Allman [A] showed stabilization properties of such expansions.}
for the double stable Schur polynomials (see e.g. [FR3, Lemma 6.1]). Note that in the case of Schur polynomials, the residues are taken only at infinity, while for Grothendieck polynomials, one takes the sum of the residues at 0 and infinity.

In addition to its simplicity and efficient computability, our formula (1) has the perfect form for computing the expansions of K-theoretic Thom polynomials.

1.2. K-theoretic Thom polynomials of singularities. The general reference for singularities of maps is [AVGL]. For a positive integer $N$, denote by $R^N(C^a)$ the algebra of $N$-jets of functions on $C^a$ at 0; this is the ring of polynomials in $a$ variables modulo monomials of degree at least $N+1$. Let $J^N(C^a, C^b)$ be the space of $N$-jets of maps $(C^a, 0) \to (C^b, 0)$ vanishing at 0. An element of $J^N(C^a, C^b)$ is given by a $b$-tuple of jets from the maximal ideal of $R^N(C^a)$. A singularity $\eta$ is an algebraic subvariety of $J^N(C^a, C^b)$ invariant under the group of formal holomorphic reparametrizations of $(C^a, 0)$ and $(C^b, 0)$ (cf. e.g. [BSz] §3).

An important set of examples of singularities, called contact singularities, is obtained as follows. A key reparametrization invariant of $N$-jets of function is the local algebra, defined for $h = (h_1(x_1, \ldots, x_a), \ldots, h_b(x_1, \ldots, x_a)) \in J^N(C^a, C^b)$ as the ideal quotient $R^N(C^a)/(h_1, \ldots, h_b)$. Then for a fixed finite-dimensional local commutative algebra $Q$ and nonnegative integers $a \leq b$, we can define the singularity $\eta_Q^{a \to b}$ as the Zariski closure of the set

$$\{ g \in J^N(C^a, C^b) : \text{the local algebra of } g \text{ is isomorphic to } Q \}.$$  

(We will omit the dimensions $a$ and $b$ from the notation when this causes no confusion.)

Denote the group of linear reparametrizations $GL_a(C) \times GL_b(C)$ by $GL[a \to b]$, and observe that the space $J^N(C^a, C^b)$ is equivariantly contractible, hence we have the identification with the symmetric polynomials:

$$H^*_{GL[a \to b]}(J^N(C^a, C^b)) = H^*_{GL[a \to b]}(pt) = \mathbb{Z}[\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b]^{S_a \times S_b},$$

$$K_{GL[a \to b]}(J^N(C^a, C^b)) = K_{GL[a \to b]}(pt) = \mathbb{Z}[\alpha_1^{\pm 1}, \ldots, \alpha_a^{\pm 1}, \beta_1^{\pm 1}, \ldots, \beta_b^{\pm 1}]^{S_a \times S_b},$$

where $\alpha_i$ ($\beta_j$), and $\alpha_i$, ($\beta_j$) are the cohomological and K-theoretic Chern roots of the standard representation of $GL_a(C)$ ($GL_b(C)$), and $S_m$ is the permutation group on $m$ elements .

In [5] we recall the definition of the equivariant Poincaré dual class $[X]$ of an invariant algebraic subvariety $X \subset V$ in a vector space acted upon by a Lie group. Using this notion, we define the Thom polynomial of the singularity $\eta$ as the equivariant Poincaré dual

$$Tp^a_{\eta} = [\eta] \in H^*_{GL[a \to b]}(J^N(C^a, C^b)).$$

The analogous K-theoretic notion

$$KTP^a_{\eta} = [\eta]^K \in K_{GL[a \to b]}(J^N(C^a, C^b))$$

is, in fact, somewhat problematic, and we will discuss its definition in detail in Section 5 as well.

To simplify our notation, we will denote the Thom polynomial of the contact singularity $\eta_Q$ as $Tp_Q$ (and $KTP_Q$) when this causes no confusion. Consider the example of $Q = \mathcal{A}_2 = \mathbb{C}[x]/(x^3)$.
We will write formulas for $T_p A_2$ in terms of Schur functions $s_\lambda = s_\lambda(\bar{\alpha}_1, \ldots, \bar{\alpha}_a, \bar{\beta}_1, \ldots, \bar{\beta}_b)$ defined in [2], or equivalently, by the more standard definition $s_\lambda = \det(c_{\lambda(i)+j-i})$ with

$$1 + c_1 t + c_2 t^2 + \ldots = \frac{\prod_{i=1}^b (1 + \bar{\beta}_i t)}{\prod_{i=1}^a (1 + \bar{\alpha}_i)}.$$ 

The general formula due to Ronga [Ro] is as follows:

$$T_{p A_2}^{a \rightarrow a+l} = \sum_{i=0}^{l+1} 2^i s_{l+1+i, l+1-i}. \tag{3}$$

Here are the first few cases:

$T_{p A_2}^{a \rightarrow a} = s_{1,1} + 2s_{2,0}$, $T_{p A_2}^{a \rightarrow a+1} = s_{2,2} + 2s_{3,1} + 4s_{4,0}$, $T_{p A_2}^{a \rightarrow a+2} = s_{3,3} + 2s_{4,2} + 4s_{5,1} + 8s_{6,0}$.

Formula (3) illustrates three key features of cohomological Thom polynomials of contact singularities:

- (stability) The Thom polynomial $T_{p A_2}^{a \rightarrow b}$ only depends on the relative dimension $b - a$ (denoted by $l$), not on $a$ and $b$ individually.
- (l-stability) We obtain $T_{p A_2}^{a \rightarrow a+l}$ from $T_{p A_2}^{a \rightarrow a+l+1}$ by replacing each Schur polynomial $s_{a,b}$ by $s_{a-1,b-1}$ (note that $s_{a,-1} = 0$). The general statement of this property for arbitrary $Q$ may be found in [FR1, Theorems 2.1, 4.1].
- (positivity) The coefficients of Schur expansions of Thom polynomials of contact singularities are non-negative [PW].

In §8 we calculate the K-theoretic Thom polynomials $KT_{p A_2}^{a \rightarrow b}$ for all $a \leq b$, and in §10 we comment on the case of higher singularities.

In our calculations, we observe a feature new to K-theory: our K-theoretic Thom polynomials have two different types of expansions.

The first type begins with a formal infinite sum of Grothendieck polynomials indexed by integer sequences; this infinite sum has the $l$-stability property analogous to the $l$-stability of cohomological Thom polynomials, see Remark §3. Partially summing the series, one obtains a reduced series, whose all but finitely many terms vanish.

The second expansion, which we will call minimal (Theorem §4), expresses $KT_p$ as a finite sum of Grothendieck polynomials indexed by partitions. This expression is uniquely defined but it is not $l$-stable.

Let us give a visual explanation of the relation between the two Grothendieck expansions of $KT_{p A_2}$. Consider the rational function

$$f(x_1, x_2) = \frac{1}{1 - \frac{z_2}{z_1^2}} \bigg|_{z_1 = 1 - x_1, z_2 = 1 - x_2} = \frac{1 - 2x_1 + x_1^2}{x_2 - 2x_1 + x_1^2}.$$
The coefficients of its $|x_1| < |x_2|$ Laurent expansion are naturally arranged in the infinite grid

\[
\begin{array}{cccccccc}
1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & x_1^6 & x_1^7 & x_1^8 \\
1 & -2 & 1 & & & & & & \\
2 & -5 & 4 & -1 & & & & & \\
4 & -12 & 13 & -6 & 1 & & & & \\
8 & -28 & 38 & -25 & 8 & -1 & & & \\
16 & -64 & 104 & \ldots & \ldots & & & & \\
32 & & & & & & & & \\
\end{array}
\]

In the *formal stable version* of Grothendieck expansion of $\text{KTP}_{A_2}^{a\to a+1}$ (Theorem 8.2), these numbers are exactly the coefficients of the corresponding Grothendieck polynomials, for any $l$, with an appropriate shift. To obtain a finite expression, we sum these Grothendieck polynomials first in the vertical direction, and, as will we show, all but finitely many of these partial sums will vanish, giving a correct finite expression for $\text{KTP}_{A_2}$.

To obtain the *minimal version* of our formula, Theorem 8.4, the coefficients of $\text{KTP}_{A_2,a,a+l}$ for different $l$’s are obtained by different procedures from this grid of integers. For example, for $l = 1$ we “sweep up” all numbers from below the third row to the third row. That is, replace the $(3, k)$ entry with the sum of entries $(r, k)$ for $r \geq 3$ and then delete the rows from the 4th one down. This sweeping is illustrated by the framed entries in the picture. In the resulting table we get the numbers (reading along the diagonals) 1, 2, 4; −2, −5, −12 + 8 = −4; 1, 4, 13 − 28 + 16 = 1; −1, and then infinitely many 0’s. These are exactly the coefficients in the minimal Grothendieck expansion of $\text{KTP}_{A_2}^{a\to a+1}$, cf. (4). To get $\text{KTP}_{A_2}^{a\to a+2}$ we need to “sweep” the same table below the 4th row, for $l = 3$ we sweep from the 5th row, etc. The exact statement of this sweeping procedure is given in Theorem 8.4.

As a result, we obtain the following minimal expansions:

\[
\begin{align*}
\text{KTP}_{A_2}^{a\to a} &= (G_{1,1} + 2G_2) - (2G_{2,1} + G_3) + G_{3,1} \\
\text{KTP}_{A_2}^{a\to a+1} &= (G_{2,2} + 2G_{3,1} + 4G_4) - (2G_{3,2} + 5G_{4,1} + 4G_5) + (G_{4,2} + 4G_{5,1} + G_6) - G_{6,1} \\
\text{KTP}_{A_2}^{a\to a+2} &= (G_{3,3} + 2G_{4,2} + 4G_{5,1} + 8G_6) - (2G_{4,3} + 5G_{5,2} + 12G_{6,1} + 12G_7) \\
&+ (G_{5,3} + 4G_{6,2} + 13G_{7,1} + 6G_8) - (G_{7,2} + 6G_{8,1} + G_9) + G_{9,1}.
\end{align*}
\]
It is remarkable that the third key feature, the positivity of cohomological Thom polynomials extends to a rule of alternating signs for both of our expansions. This result will be proved in §9.

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2. Combinatorial definition of Grothendieck polynomials

In this section we will review the traditional definition of various versions of Grothendieck polynomials. We follow the references [LS, FK1, FK2, B1, B2, B3]. Our goal in Sections 2-4 is to replace these traditional definitions with the residue description of Definition 4.2. The reader not interested in the traditional definitions can take Definition 4.2 to be the definition of double stable Grothendieck polynomials and jump to Section 5.

We will use standard notations of algebraic combinatorics. A permutation \( w \in S_n \) will be represented by the sequence \( [w(1), w(2), \ldots, w(n)] \). The length of a permutation \( \ell(w) \) is the cardinality of the set \( \{i < j : w(i) > w(j)\} \). We will identify \( S_n \) with its image under the natural embedding \( S_n = \{w \in S_{n+1} : w(n+1) = n+1\} \).

2.1. Double Grothendieck polynomials. Double Grothendieck polynomials (in variables \( x_i, y_j \)) were introduced by Lascoux and Schutzenberger [LS]. In the present paper, following e.g. [B1], we perform the rational substitutions \( x_i = 1 - \frac{1}{\alpha_i} \) and \( y_i = 1 - \beta_i \) in those polynomials, and denote the resulting rational functions by \( G_w(\alpha, \beta) \). To keep the terminology simple, we will continue calling these functions “Grothendieck polynomials”.

The functions \( G_w(\alpha, \beta) \) are defined by the following recursion:

- For the longest permutation \( w_0 = [n, n - 1, \ldots, 1] \in S_n \), let
  \[
  G_w = \prod_{i+j \leq n} \left( 1 - \frac{\beta_i}{\alpha_j} \right),
  \]

- Let \( s_i \) be the \( i \)th elementary transposition. If \( \ell(ws_i) = \ell(w) + 1 \) then
  \[
  G_w = \pi_i(G_{ws_i}),
  \]
  where the isobaric divided difference operator \( \pi_i \) is defined by

  \[
  \pi_i(f) = \frac{\alpha_i f(\ldots, \alpha_i, \alpha_{i+1}, \ldots) - \alpha_{i+1} f(\ldots, \alpha_{i+1}, \alpha_i, \ldots)}{\alpha_i - \alpha_{i+1}} = \frac{f(\ldots, \alpha_i, \alpha_{i+1}, \ldots)}{1 - \alpha_{i+1}/\alpha_i} + \frac{f(\ldots, \alpha_{i+1}, \alpha_i, \ldots)}{1 - \alpha_i/\alpha_{i+1}}.
  \]

For example, here is the list of double Grothendieck polynomials for all \( w \in S_3 \)

\[
G_{321} = \left( 1 - \frac{\beta_1}{\alpha_1} \right) \left( 1 - \frac{\beta_2}{\alpha_1} \right) \left( 1 - \frac{\beta_1}{\alpha_2} \right), \quad G_{231} = \left( 1 - \frac{\beta_1}{\alpha_1} \right) \left( 1 - \frac{\beta_1}{\alpha_2} \right).
\]
\[ \mathcal{G}_{312} = \left(1 - \frac{\beta_1}{\alpha_1}\right) \left(1 - \frac{\beta_2}{\alpha_1}\right) \quad \mathcal{G}_{213} = 1 - \frac{\beta_1}{\alpha_1} \quad \mathcal{G}_{132} = 1 - \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2} \quad \mathcal{G}_{123} = 1. \]

2.2. **Stable versions.** For a permutation \( w \in S_n \) let \( 1^m \times w \in S_{m+n} \) be the permutation that is the identity on \( \{1, \ldots, m\} \) and maps \( j \mapsto w(j - m) + m \) for \( j > m \). The double stable Grothendieck polynomial \( G_w(\alpha, \beta) \) is defined to be

\[
G_w = \lim_{m \to \infty} \mathcal{G}_{1^m \times w}.
\]

For example, \( G_{21} = 1 - \frac{\beta_1 \beta_2 \beta_3 \cdots}{\alpha_1 \alpha_2 \alpha_3 \cdots} \). The precise definition of this limit may be found in [B1]: roughly, rewritten in the \( x \) and \( y \) variables mentioned above, each coefficient of \( \mathcal{G}_{1^m \times w} \) stabilizes with \( m \), and hence the limit is defined as a formal power series in \( x_i, y_j \) with the stabilized coefficients.

2.3. **Truncated versions.** One usually considers specializations of double stable Grothendieck polynomials of the type

\[
G_{k,l}^w(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l) = G_w(\alpha_1, \ldots, \alpha_k, 1, 1, \ldots; \beta_1, \ldots, \beta_l, 1, 1, \ldots).
\]

In fact, \( G_{k,l}^w \) may be obtained by substituting \( \alpha_i = 1, i > k \), \( \beta_i = 1, i > l \) in \( \mathcal{G}_{1^m \times w} \) for \( m \gg k, l \). This way the truncated versions (6) may be calculated without the \( \lim_{m \to \infty} \) of (5).

Below, we will drop the superscripts \( k, l \) whenever they may be determined from the number of \( \alpha \) and \( \beta \) variables.

In the case \( l = 0 \), we will simply write \( G_w(\alpha_1, \ldots, \alpha_k) \).

2.4. **Stable Grothendieck polynomials parametrized by partitions.** As usual, a weakly decreasing sequence of nonnegative integers \( \lambda = (\lambda_1, \ldots, \lambda_r) \) will be called a partition. We will identify two partitions if they differ by a sequence of 0’s, and we define \( L(\lambda) \), the length of a partition \( \lambda \) to be the largest \( i \) for which \( \lambda_i > 0 \). The Grassmannian permutation associated to a partition \( \lambda \) with descent in position \( p \) is the permutation

\[
w_\lambda(i) = \begin{cases} w_\lambda(i) = i + \lambda_{p+1-i} & \text{for } i \leq p, \\ w_\lambda(i) < w_\lambda(i+1) & \text{unless } i = p. \end{cases}
\]

Note that necessarily \( p \geq L(\lambda) \).

We define the double stable Grothendieck polynomial \( G_\lambda \) of the partition \( \lambda \) as \( G_{w_\lambda}(\alpha; \beta) \). It is easy to show that this definition does not depend on the choice of \( p \) above.

2.5. **Stable Grothendieck polynomials parametrized by integer sequences.** The notion \( G_\lambda \) (with \( \lambda \) a partition) is extended to \( G_I \) where \( I \in \mathbb{Z}^r \) is any finite integer sequence—by repeated applications of the straightening laws

\[
G_{I,p,q,J} = \sum_{k=p+1}^{q} G_{I,q,k,J} - \sum_{k=p+1}^{q-1} G_{I,q-1,k,J} \quad \text{if } p < q,
\]

\[
G_{I,p} = G_{I,0} = G_I \quad \text{if } p < 0.
\]
3. Properties of Grothendieck polynomials

We will need the following three properties of Grothendieck polynomials.

**Proposition 3.1.** [FK2, B3 (2)] The polynomial

\[ G_w(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l) \]

is \( S_k \times S_l \)-supersymmetric, i.e. it is symmetric in the \( \alpha_i \) and the \( \beta_j \) variables separately, and satisfies

\[ G_w(\alpha_1, \ldots, \alpha_{k-1}, t; \beta_1, \ldots, \beta_{l-1}, t) = G_w(\alpha_1, \ldots, \alpha_{k-1}; \beta_1, \ldots, \beta_{l-1}) \]

In particular, the left hand side of this equality does not depend on \( t \).

The next statement is an easy application of the Fomin-Kirillov formulas [FK1], and also follows directly from the set-valued tableau description in [B1].

**Proposition 3.2.** Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be a partition with \( \lambda_r > 0 \) and let \( 0 < k < r \). Then

\[ G_\lambda(\alpha_1, \ldots, \alpha_k) = 0 \]

**Proposition 3.3.** Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be a partition with \( \lambda_r \geq 0 \). We have

\[ G_\lambda(\alpha_1, \ldots, \alpha_r) = \sum_{\sigma \in S_r} \prod_{i=1}^r \frac{1 - 1/\alpha_{\sigma(i)}}{1 - \alpha_{\sigma(i)}/\alpha_{\sigma(j)}} \]

\[ \lambda_i + r - i \]

\[ \prod_{i>j} \left(1 - \alpha_i/\alpha_j\right) \]

**Proof.** Consider the permutation

\( \bar{w}_\lambda = \lambda_1 + r, \lambda_2 + r - 1, \ldots, \lambda_r + 1, i_1, \ldots, i_s \)

where \( i_j < i_{j+1} \) for all \( j \), and \( s \) is sufficiently large to make this a permutation. The permutation \( \bar{w}_\lambda \) is a so-called dominant permutation. For dominant permutations the recursive definition of Section 2.1 can be solved explicitly ([LS], or see the diagrammatic description in [FK1]), and we obtain

\[ \mathcal{G}_{\bar{w}_\lambda}(\alpha_1, \ldots, \alpha_r) = \prod_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right)^{\lambda_i + r - i} \]

Observe that \( \bar{w}_\lambda \cdot w_0 = w_\lambda \), where \( w_0 \) is the longest permutation of \( 1, \ldots, r \). Hence

\[ \mathcal{G}_\lambda(\alpha_1, \ldots, \alpha_r) = \mathcal{G}_{w_\lambda}(\alpha_1, \ldots, \alpha_r) = \pi_{w_0(r)} \left( \prod_{i=1}^r \left(1 - \frac{1}{\alpha_i}\right)^{\lambda_i + r - i} \right) \]

where

\[ \pi_{w_0(r)}(f) = (\pi_1 \pi_2 \ldots \pi_{r-1})(\pi_1 \pi_2 \ldots \pi_{r-2}) \ldots (\pi_1)(f) = \sum_{\sigma \in S_r} \sigma \left( \frac{f}{\prod_{i>j}(1 - \alpha_i/\alpha_j)} \right) \]

The right-hand side of (9) is equal to the right-hand side of the displayed formula in the Proposition. If the number of \( \alpha \) variables is at least the length of the partition, then \( \mathcal{G}_\lambda(\alpha) = G_\lambda(\alpha) \), which concludes our proof. \( \square \)

Note that Proposition 3.3 may be used whenever the number of \( \alpha \) variables is larger than the length of the partition, because we can append 0’s to the end of \( \lambda \) to make the condition satisfied.
4. Grothendieck polynomials in residue form

In this section we introduce a residue calculus for Grothendieck polynomials and show how this new formalism helps to understand some of their properties.

Let \( z \) be a complex variable, and introduce the notation
\[
\text{Res}_{z=0} f(z) \, dz = \text{Res}_{z=0} f(z) \, dz + \text{Res}_{z=\infty} f(z) \, dz.
\]
The following property of \( \text{Res}_{z=0,\infty} \) is straightforward.

**Lemma 4.1.** Let \( 0 \leq a \leq s - r - 2 \) and let
\[
f(z) = z^a \cdot \frac{\prod_{i=1}^r (z - x_i)}{\prod_{i=1}^r (z - y_i)}
\]
for non-zero complex numbers \( x_i, y_i \). Then \( \text{Res}_{z=0,\infty} f(z) \, dz = 0 \). \( \square \)

4.1. Residue form of double stable Grothendieck polynomials. Let \( z_1, \ldots, z_r \) be complex variables. For nonnegative integers \( k, l \), define the differential form
\[
M_{k,l}(z_1, \ldots, z_r) = \prod_{j=1}^r \frac{\prod_{i=1}^l (1 - z_j \beta_i)}{\prod_{i=1}^l (1 - z_j \alpha_i)(1 - z_j)^{l-k}} \cdot \prod_{j=1}^r \frac{dz_j}{z_j}.
\]
When it causes no confusion, we will omit the indices \( k \) an \( l \), and denote the vector \((z_1, \ldots, z_r)\) by \( z \); thus we will write \( M(z) \) for \( M_{k,l}(z_1, \ldots, z_r) \).

**Definition 4.2.** For an integer sequence \( I \in \mathbb{Z}^r \), define the \( g \)-polynomial as
\[
g_I(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l) = \text{Res}_{z_1=0,\infty} \ldots \text{Res}_{z_r=0,\infty} \left( \prod_{j=1}^r (1 - z_j)^{l_j} \prod_{i>j} (1 - \frac{z_i}{z_j}) M_{k,l}(z_1, \ldots, z_r) \right).
\]

**Remark 4.3.** In general, iterated residue formulas are sensitive to the order in which one takes the residues \( \text{Res}_{z_i} \)—see for example [BSZ, K1, K2, FR2]—due to factors of the type \( z_i - z_j \) in the denominator. However, the denominators in (11) are linear factors each depending on a single variable, and hence the order in this case does not matter.

The following is evident from Definition 4.2.

**Lemma 4.4.** We have
\[
g_I(\alpha_1, \ldots, \alpha_k, 1; \beta_1, \ldots, \beta_l) = g_I(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l, 1) = g_I(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l).
\]
The function \( g_\lambda(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l) \) is supersymmetric: it is symmetric in the \( \alpha_i \) and the \( \beta_j \) variables separately, and we have
\[
g_\lambda(\alpha_1, \ldots, \alpha_{k-1}, t; \beta_1, \ldots, \beta_{l-1}, t) = g_\lambda(\alpha_1, \ldots, \alpha_k-1; \beta_1, \ldots, \beta_{l-1}).
\]
In particular, the left hand side does not depend on \( t \). \( \square \)

**Theorem 4.5.** For any integer sequence \( I \), and nonnegative integers \( k, l \), we have
\[
G_I(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l) = g_I(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_l).
\]
First we prove two lemmas.

**Lemma 4.6.** Let $I$ and $J$ be integer sequences. Then we have

\[(13)\quad g_{I,p,q,J} = \sum_{k=p+1}^{q} g_{I,q,k,J} - \sum_{k=p+1}^{q-1} g_{I,q-1,k,J}\quad \text{if } p < q,
\]

and

\[(14)\quad g_{I,p,J} = g_{I}\quad \text{if } p \leq 0.
\]

**Proof.** For simplicity of notation, we assume that $I = J = \emptyset$. The general case is treated similarly. For $p < q$ consider

\[g_{p,q} - g_{q-1,q} = \Res_{z_1=0,\infty} \Res_{z_2=0,\infty} \left((1 - z_1)^{p-1}(1 - z_2)^{q-2} - (1 - z_1)^{q-2}(1 - z_2)^{q-2}\right) \left(1 - \frac{z_2}{z_1}\right) \cdot M(z_1, z_2).
\]

Applying the identities

\[
\left(1 - \frac{z_2}{z_1}\right) = -\frac{z_2}{z_1} \left(1 - \frac{z_1}{z_2}\right)
\]

and

\[
\left((1 - z_2)^{q-2}(1 - z_1)^{p-1} - (1 - z_2)^{q-2}(1 - z_1)^{q-2}\right) \left(-\frac{z_2}{z_1}\right) = \sum_{k=p+1}^{q-1} (1 - z_2)^{q-1}(1 - z_1)^{k-2} - \sum_{k=p+1}^{q-1} (1 - z_2)^{q-2}(1 - z_1)^{k-2},
\]

we obtain that $g_{p,q} - g_{q-1,q}$ equals

\[
\Res_{z_1=0,\infty} \Res_{z_2=0,\infty} \left(\sum_{k=p+1}^{q-1} (1 - z_2)^{q-1}(1 - z_1)^{k-2} - \sum_{k=p+1}^{q-1} (1 - z_2)^{q-2}(1 - z_1)^{k-2}\right) \left(1 - \frac{z_1}{z_2}\right) \cdot M(z_1, z_2).
\]

Using the definition of $g$ with the role of $z_1$ and $z_2$ switched, we obtain

\[g_{p,q} - g_{q-1,q} = \sum_{k=p+1}^{q-1} g_{q,k} - \sum_{k=p+1}^{q-1} g_{q-1,k}.
\]

This is equivalent to (13) up to the easy equality

\[g_{q-1,q} = g_{q,q},
\]

whose proof we leave to the reader.

Formula (14) immediately follows from the fact that, for $p \leq 0$,

\[\Res_{z_r=0} (1 - z_r)^{p-r} \prod_{i=1}^{r-1} \left(1 - \frac{z_r}{z_i}\right) M_{k,i}(z_r) = 1,
\]

while the residue of this expression at $z_r = \infty$ vanishes. □
Lemma 4.7. Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) be a partition with \( \lambda_r > 0 \). Then for \( k < r \), we have \( g_\lambda(\alpha_1, \ldots, \alpha_k) = 0 \).

Proof. First note that, according to Lemma 4.4, the case \( k = r - 1 \) implies the case \( k < r \).

Now assume \( k = r - 1 \), and introduce the temporary notation \( \omega \) for the differential form in (11). Assume that the values of the \( \alpha_i \)'s are all different.

We calculate the first residue \( \text{Res}_{z_r = 0, \infty} \omega \), taking into account Remark 4.3 and applying the 1-variable Residue Theorem. The exponent \( I_r - r + k - l \) of the factor \( (1 - z_r \beta_i) \) is nonnegative, since \( I_r = \lambda_r > 0 \), \( k = r - 1 \), and \( l = 0 \), and hence there is no pole at \( z_r = 1 \). The remaining poles are thus the points \( z_r = 1/\alpha_i, i = 1, \ldots, k \), and each of these poles is simple. The residue at the simple pole \( z_r = 1/\alpha_i \), up to a factor of \(-\alpha_i\) is obtained by omitting the factor \((1 - \alpha_i z_r)\) in the denominator, and then substituting into the remainder \( z_r = 1/\alpha_i \). Continuing the application of residues in (11), we obtain a sum over all choices of indices \( 1 \leq i_j \leq k, j = 1, \ldots, r \), of terms of the following form

\[
\prod_{j=1}^r (1 - \alpha_{i_j})^\epsilon \prod_{m>j} \left(1 - \frac{\alpha_{i_m}}{\alpha_{i_j}}\right) \tilde{M},
\]

where \( \epsilon \geq 0 \) and \( \tilde{M} \) is some rational expression in the \( \alpha \)'s. The relevant factor in the product is the second one, which vanishes as long as \( i_m = i_j \) for some \( 1 \leq j < m \leq r \). As \( k < r \), this is certainly the case, and this completes the proof. \( \square \)

Now we are ready to prove Theorem 4.5.

Proof. Since both \( g \) and \( G \) are supersymmetric (Proposition 3.1 and Lemma 4.4), it is sufficient to prove \( G_\lambda = g_\lambda \) for the \( \beta_1 = \beta_2 = \ldots = 1 \) substitution. For that substitution, both \( g_\lambda \) and \( G_\lambda \) vanish if the number of \( \alpha \)'s is less than the length of \( \lambda \) (see Proposition 3.2 and Lemma 4.7).

Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \) and consider formula (11) for \( k = r, l = 0 \). We will apply the Residue Theorem for each residue \( \text{Res}_{z_i = 0, \infty} \), i.e. we replace \( \text{Res}_{z_i = 0, \infty} \) by \(-\sum_p \text{Res}_{z_i = p} \) where sum runs over all poles different from 0 and \( \infty \). We claim that the only such poles are at \( z_i = 1/\alpha_j \). Indeed the substitution \( \beta_i = 1 \) makes the exponent of \((1 - z_i)\) in the formula equal to \( \lambda_i - i + r \), which is nonnegative.

The only nonzero finite residues hence correspond to permutations \( \sigma \in S_r \): \( z_i = 1/\alpha_{\sigma(i)} \). Straightforward calculation shows that the \(-\text{Res}_{z_i = 1/\alpha_{\sigma(i)}} \) operation yields the term corresponding to \( \sigma \in S_r \) in Proposition 3.3. This proves the theorem. \( \square \)

4.2. Consequences of the \( g = G \) theorem. Grothendieck polynomials have a rich algebraic structure and they display beautiful finiteness and alternating-sign properties. We believe that the residue form for the stable Grothendieck polynomials above sheds light on many of those properties. We will illustrate this in Section 8 in a so-far unexplored situation—the Thom polynomials of singularities. Here we will just sketch a simple example showing how the multiplication structure of Grothendieck polynomials is encoded in their residue form.
4.3. Multiplication. Consider the concrete example of calculating the \( g \)-expansion of the product \( g_2 \cdot g_2 \) (here “2” in the subscript is a length 1 partition). We have

\[
g_2 \cdot g_2 = \operatorname{Res}_{z=0,\infty} (1 - z) M(z) \cdot \operatorname{Res}_{u=0,\infty} (1 - u) M(u) = \operatorname{Res}_{z,u=0,\infty} (1 - z)(1 - u) M(z, u) =
\]

\[
\operatorname{Res}_{z,u=0,\infty} (1 - z)(1 - u) \left( \frac{1}{1 - \frac{z}{u}} \right) M(z, u) =
\]

\[
\operatorname{Res}_{z,u=0,\infty} (1 - z)(1 - u) \left( \sum_{i=0}^{2} \frac{(1 - z)^i}{(1 - u)^{i+1}} - \sum_{i=1}^{2} \frac{(1 - z)^i}{(1 - u)^i + \frac{u(1 - z)^3}{(z - u)(1 - u)^3}} \right) \left( 1 - \frac{u}{z} \right) M(z, u).
\]

The term involving \( u(1 - z)^3/(z - u)(1 - u)^3 \) has \( u \)-residue 0, because of Lemma 4.1. Hence we further obtain

\[
g_2 \cdot g_2 = \operatorname{Res}_{z,u=0,\infty} \left( \left( \sum_{i=0}^{2} \frac{(1 - z)^i}{(1 - u)^{i+1}} - \sum_{i=1}^{2} \frac{(1 - z)^i}{(1 - u)^i} \right) (1 - \frac{u}{z}) M(z, u) \right)
\]

\[
= g_{2,2} + g_{3,1} + g_{4,0} - g_{3,2} - g_{4,1}.
\]

In general the calculation of products of arbitrary Grothendieck polynomials is similar, see [AR]. Namely, to find an explicit expression for \( g_i \cdot g_j \) as sums of Grothendieck polynomials, one considers

\[
\prod_{i} (1 - z_i)^{l_i} \prod_{j} (1 - u_j)^{J_j} \prod_{i,j} \frac{1}{1 - \frac{u_j}{z_i}},
\]

and replaces \( 1/(1 - u_j/z_i) \) with an appropriate initial sum of its Laurent series at \( z_i = u_j = 1 \). This should be done in such a way that the remainder multiplied by \( \prod(1 - z_i)^{l_i} \prod(1 - u_j)^{J_j} \) has 0 residue.

Remark 4.8. The consideration above shows that the product of two Grothendieck polynomials (parametrized by integer sequences) is a finite sum of Grothendieck polynomials parametrized by integer sequences with coefficients with alternating signs. Proving the analogous statement for Grothendieck polynomials parametrized by partitions needs extra considerations (cf. [AR]). We will perform a similar analysis for Thom polynomials in Section 9.

5. Fundamental class in cohomology and K-theory

5.1. The cohomology fundamental class. Let \( X \) be a subvariety of codimension \( d \) in a smooth projective variety \( M \). Then \( X \) has a well-defined fundamental class \( [X] \in H^{2d}(M, \mathbb{Q}) \), satisfying

\[
\int_X \iota^* \omega = \int_M [X] \cdot \omega,
\]

where \( \iota : X \to M \) is the embedding, and \( \omega \in H^*(M, \mathbb{Q}) \) is arbitrary, cf. [GH].

There is a natural extension of this notion to the equivariant setting, which plays a fundamental role in enumerative geometry. Let \( V \) be a complex vector space acted upon by a complex torus
Then a $T$-invariant affine subvariety $X$ has a fundamental class $[X]_T \in H^d_T(V) = H^d_T(pt)$, $d = \text{codim}(X)$, which satisfies the equivariant version of (15):

$$\int_X t^* \omega = \int_V [X]_T \cdot \omega,$$

where $\omega$ is any equivariantly closed, compactly supported form on $V$.

There is a number of definitions of this notion (cf. [BSz, §3] for a discussion); below we recall one due to Joseph [J]. We begin with introducing some necessary notation.

- Let $\exp : \text{Lie}(T) \to T$ be the exponential map; the pull-back of a function from $f : T \to \mathbb{C}$ to $\text{Lie}(T)$ via this map will be denoted by $\exp^* f$.
- For a character $\alpha \in \text{Hom}(T, \mathbb{C}^*)$, we will write $\bar{\alpha}$ for the corresponding weight in the weight lattice $W_T \subset \text{Lie}(T)^\vee$. We will thus have the following equality of functions on $\text{Lie}(T)$:

  $$\exp^* \alpha = e^{\bar{\alpha}},$$

where factor of $2\pi i$ is considered to be absorbed in the definition of the exponential, and will be ignored in what follows.
- Fix a $\mathbb{Z}$-basis $\beta_1, \ldots, \beta_r : T \to \mathbb{C}^*$ of $\text{Hom}(T, \mathbb{C}^*)$. We then have

  $$H^*_{T}(V) = H^*_{T}(pt) = \mathbb{Z}[\bar{\beta}_1, \ldots, \bar{\beta}_r].$$

- Let $x_j$, $j = 1, \ldots, N$ be a set of coordinates on $V$, corresponding to a basis of eigenvectors of the $T$ action, and denote by $\eta_j \in \text{Hom}(T, \mathbb{C}^*)$, $j = 1, \ldots, N$, the corresponding characters: for $t \in T$, we have $t \cdot x_j = \eta_j(t)^{-1}x_j$. For what follows, it is convenient to make the following assumption.

**Assumption 5.1.** All the weight vectors of the vector space $V$ lie in an open half-space of the weight lattice $\mathcal{W}_T \subset \text{Lie}(T)^\vee$, i.e. there exists an element $Z \in \text{Lie}(T)$ such that we have

$$\langle \bar{\eta}_j, Z \rangle > 0, \quad j = 1, \ldots, N.$$

One can carry out the constructions of the theory without this assumption as well, but this is more technical, and this case is sufficient for our purposes.

Recall that for a finite-dimensional representation $W$ of $T$ with a diagonal basis

$$W = \bigoplus_{i=1}^m \mathbb{C} w_i, \quad t \cdot w_i = \alpha_i(t) \cdot w_i,$$

we have $\text{Tr} \{ t \mid W \} = \sum_{i=1}^m \alpha_i$, for $t \in T$.

This function on $T$ is called the **character of $W$**.

Now let $X \subset V$ be a $T$-invariant subvariety, and denote by $RX$ the ring of algebraic functions on $X$. The character

$$\chi_X(t) = \text{Tr} \{ t \mid RX \}, \quad t \in T$$

of $RX$ considered as a $T$-representation is only a formal series since $RX$ is infinite-dimensional whenever the dimension of $X$ is positive. Under Assumption 5.1, however, this series converges in a domain in $T$, and $\chi_X(t)$ makes sense as a rational function on $T$. 
For example, $RV = \mathbb{C}[x_1, \ldots, x_N]$ is the ring of polynomial functions on $V$, and we have

\begin{equation}
\chi_V = \prod_{j=1}^{N} \frac{1}{1 - \eta_j t},
\end{equation}

as can be seen by expanding this function in an appropriate domain in $T$.

The following theorem is a consequence of the Hilbert’s syzygy theorem (cf. also [MS, §4.3]).

**Theorem 5.2.** Let $X \subset V$ be a $T$-invariant subvariety of codimension $d$. Then $\chi_X$ is a function on $T$ defined whenever $\chi_V$ is defined (cf. (16)), and has the form of a finite integral linear combination of $T$-characters multiplied by $\chi_V$:

\begin{equation}
\chi_X = \chi_V \cdot \sum_{j=1}^{M} a_j \bar{\theta}_j,
\end{equation}

where $a_j \in \mathbb{Z}$, $\theta_j \in \text{Hom}(T, \mathbb{C}^*)$.

Moreover, expanding the function $\exp^*(\chi_X/\chi_V) = \sum_{j=1}^{M} a_j \bar{\theta}_j$ on $\text{Lie}(T)$ around the origin, we obtain a power series with lowest degree terms in degree $d$:

\begin{equation}
\sum_{j=1}^{M} a_j \exp \bar{\theta}_j = \frac{1}{d!} \sum_{j=1}^{M} a_j \bar{\theta}_j^d + \rho_{d+1} \quad \text{with } \rho_{d+1} \in \mathfrak{m}^{d+1},
\end{equation}

where $\mathfrak{m}$ is the maximal ideal of analytic functions vanishing at the origin in $\text{Lie}(T)$.

The last part of the theorem states that, after the expansion, the terms up to degree $d - 1$ cancel.

**Definition 5.3.** Let $X \subset V$ be a $T$-invariant subvariety of codimension $d$. We define the $T$-equivariant fundamental class of $X$ in $V$ as the degree-$d$ (leading) term on the right hand side of (18) interpreted as an element of $H^*_T(V)$:

\begin{equation}
[X]_T = (-1)^d \sum_{j=1}^{M} a_j \bar{\theta}_j^d.
\end{equation}

**Example 5.4.** Let $V = \mathbb{C}^2$ be endowed with a diagonal action of $T = \mathbb{C}^*$ with weight 1 on each of the two coordinate functions $x$ and $y$, and let $X = \{xy = 0\}$. Then $X$ is $T$-invariant, and there is a short exact sequence of $RV$-modules

\begin{equation}
0 \rightarrow RV[2] \rightarrow RV \rightarrow RX \rightarrow 0,
\end{equation}

where $RV[2]$ stands for the free module of rank 1, generated by a single element of degree 2, whose image is the function $xy$. This implies

\begin{equation}
\chi_V = \frac{1}{(1 - \beta^{-1})^2}, \quad \text{and} \quad \chi_X = \frac{1 - \beta^{-2}}{(1 - \beta^{-1})^2} = \frac{1 + \beta^{-1}}{1 - \beta^{-1}}.
\end{equation}

Now we substitute $\beta = e^\bar{\beta}$, and we see that modulo $\bar{\beta}^3$, we have $\chi_X/\chi_V = 1 - \beta^{-2} = 2\bar{\beta}$, and hence $[X]_T = 2\bar{\beta}$.
5.2. Equivariant \(K\)-theoretic fundamental classes. It is not immediately obvious what one should take as the appropriate definition of the equivariant fundamental class in \(K\)-theory.

In our setup, we have

\[
K_T(pt) = \mathbb{Z}\text{Hom}(T, \mathbb{C}^*) = \mathbb{Z}[\beta_1^{\pm 1}, \beta_2^{\pm 1}, \ldots, \beta_r^{\pm 1}],
\]

and thus for a \(T\)-invariant \(X \subset V\), it would seem natural to define as this fundamental class the linear combination of torus characters \(\chi_X/\chi_V\) in (17), which naturally lies in this space. This invariant is very difficult to calculate, however (cf. [K] for a more detailed discussion), and, in fact, there are some alternatives.

**Proposition 5.5.** Let \(X \subset V\) be a \(T\)-invariant subvariety in the vector space \(V\) satisfying Assumption 5.1. Then the cohomology groups of the structure sheaf \(H^i(Y, \mathcal{O}_Y)\) for a smooth \(T\)-equivariant resolution \(\pi : Y \to X\) are independent of the choice of \(Y\), and thus are invariants of \(X\). In particular,

\[
\tilde{\chi}_X(\tau) \overset{\text{def}}{=} \sum_{i=0}^{\dim Y} (-1)^i \text{Tr} [\tau \mid H^i(Y, \mathcal{O}_Y)]
\]

is an invariant of \(X\), which coincides with \(\chi_X\) if \(X\) has only rational singularities. Moreover, \(\chi_X/\chi_V\) and \(\tilde{\chi}_X/\chi_V\) have the same leading term in the sense of (17) and (18) in Theorem 5.2.

These statements are fairly standard—see for example [MS, H]—hence we only give a sketch of the proof to emphasize the key ideas involved. First we recall that for two smooth resolutions \(Y_1 \to X \leftarrow Y_2\), there exists a resolution \(Y \to X\) which dominates \(Y_1, Y_2\). This fact reduces the theorem to the case when both \(X\) and \(Y\) are smooth and \(\pi\) is birational. In this case, the first statement may be found in [H, Chapter III].

The statement on rational singularities is essentially a tautology: for an affine variety \(X\), having rational singularities means precisely that for any smooth resolution \(Y \to X\), we have \(H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)\) and \(H^i(Y, \mathcal{O}_Y) = 0\) for \(i > 0\).

Finally, note that the cohomology groups \(H^i(Y, \mathcal{O}_Y)\) are the sections over \(X\) of the derived push-forward sheaves \(R^i\pi_*\mathcal{O}_Y\). Applying the flat base change for the smooth locus in \(X\), we see that for \(i > 1\), these sheaves are supported on the singular locus of \(X\), which is of higher codimension than \(X\) itself. For such a sheaf then, the corresponding leading term will be of higher degree than \(d\), the codimension of \(X\) (see [MS]), and this completes the proof.

**Definition 5.6.** Let \(X\) be a \(T\)-invariant subvariety of the vector space \(V\) endowed with a \(T\)-action and satisfying Assumption 5.1. Then we define the \(K\)-theoretic fundamental class \([X]_K\) of \(X\) in \(V\) as the character \(\tilde{\chi}_X/\chi_V\), where \(\tilde{\chi}_X\) is given by the formula (19).

Now let us revisit Example 5.4. Denote by \(Y\) the normalization of \(X\), which is the union of two nonintersecting lines. Then \(H^0(Y, \mathcal{O}_Y)\) is two copies of a polynomial ring in one variable, and \(H^0(X, \mathcal{O}_X) \subset H^0(Y, \mathcal{O}_Y)\) is the subset of those pairs of polynomials whose constant terms

\footnote{This polynomial is called the \(K\)-polynomial in [MS] for this reason.}
coincide. We have
\[ \tilde{\chi}_X = \chi_Y = \frac{2}{1 - \beta^{-1}}, \quad \chi_Y = \frac{1}{(1 - \beta^{-1})^2}, \quad \text{and hence} \quad [X]^K_T = \frac{\tilde{\chi}_X}{\chi_Y} = 2(1 - \beta^{-1}). \]

It is instructive to verify directly the last statement of Proposition 5.5 even in this simple case. When we used \( \chi_X \) instead of \( \tilde{\chi}_X \), we obtained a different answer:
\[ \frac{\chi_X}{\chi_Y} = \frac{(1 + \beta^{-1})/(1 - \beta^{-1})}{1/(1 - \beta^{-1})^2} = 1 - \beta^{-2}. \]
Yet, after substituting \( \beta = e^\delta \), we see that, modulo \( (\delta^3) \) we have the equality:
\[ \frac{\tilde{\chi}_X}{\chi_Y} = \frac{\chi_X}{\chi_Y} = 2\delta \mod (\delta^3), \]
recovering the cohomological fundamental class of Example 5.4.

**Remark 5.7.** For a holomorphic map between complex manifolds \( g : M^a \to P^b \), one can consider the \( \eta \)-singularity points
\[ \eta(g) = \{ x \in M : \text{the N-jet of } g \text{ at } x \text{ belongs to } \eta \}. \]
Thom’s principle on cohomological Thom polynomials states that if \( g \) satisfies certain transversality properties then
\[ [\eta(g)] = \text{Tp}_{\eta}^{a \to b}(\text{Chern roots of } TM, \text{Chern roots of } g^*(TP)). \]
This powerful statement relies on the fact that the notion of “cohomological fundamental class” is consistent with pullback morphisms. The way we set up the notion of K-theoretic fundamental class in Definition 5.6 is not consistent with pullback morphisms (rather, it is consistent with push-forward morphisms), hence Thom’s principle does not hold for our K-theoretic Thom polynomials. The interesting project of studying another version of K-theoretic fundamental class of singularities—one for which Thom’s principle holds—is started in [K].

We end this section with an observation addressing the situation when the group \( G \) acting on \( V \) is a general reductive group with maximal torus \( T \). For a reductive group \( G \), we have \( K_G(\text{pt}) = K_T(\text{pt})^W \) (the Weyl-invariant part). For a \( G \)-invariant \( X \subset V \), the class \( [X]^K_T \) will be in this Weyl-invariant part, and hence we can define \( [X]_G^K = [X]^K_T \).

In the rest of the paper, if the group that acts is obvious, we will drop the subscript and use the notation \( [X] = [X]_G \), \( [X]^K = [X]^K_G \) for the cohomological and K-theoretic fundamental class.

### 6. Singularities and their Thom polynomials

Recall the notion of contact singularities and their Thom polynomials from §1.2. Let us see a few examples.

**Example 6.1.**
- The simplest case is \( Q = \mathbb{C} \), also known as the \( \mathcal{A}_0 \)-algebra. In this case, we have
  \[ \eta_{\mathcal{A}_0}^{a \to b} = J_N(\mathbb{C}^a, \mathbb{C}^b), \]
  which is essentially the inverse function theorem.
• When the algebra $Q$ is $A_1 = \mathbb{C}[x]/(x^2)$, the set $\eta_{A_1}^{a\to b}$ is the set of singular map-jets, i.e. those whose derivative at 0 is not injective.

• For $r > 0$, consider $Q = \mathbb{C}[x_1, \ldots, x_r]/(x_1, \ldots, x_r)^2$. In this case, $\eta_Q^{a\to b}$ is the set of those map-jets whose linear part has corank at least $r$ (also known as the $\Sigma_r$ singularity).

• The contact singularities corresponding to the algebra $Q = A_r = \mathbb{C}[x]/(x^{r+1})$ are called Morin singularities. A generic element of $\eta_{A_2}^{2\to 2}$ may be represented as $(x, y) \mapsto (x^3 + xy, y)$; it is called the cusp singularity.

6.1. The model. By a model for a singularity $\eta \subset J(\mathbb{C}^a, \mathbb{C}^b)$, we mean a $\text{GL}(\mathbb{C}^a) \times \text{GL}(\mathbb{C}^b)$-equivariant commutative diagram

$$
\begin{array}{ccc}
X & \overset{i}{\longrightarrow} & M \times J(\mathbb{C}^a, \mathbb{C}^b) \\
\pi \downarrow & & \pi_1 \downarrow \\
M & \overset{\pi_2}{\longrightarrow} & J(\mathbb{C}^a, \mathbb{C}^b) \\
\rho & \underset{p_M}{\longrightarrow} & pt,
\end{array}
$$

where

- $M$ is a smooth compact manifold,
- $\pi : X \to M$ is a subbundle of the trivial bundle $\pi_1 : M \times J(\mathbb{C}^a, \mathbb{C}^b) \to M$,
- $\rho = \pi_2 \circ i$ is birational to $\eta$,
- and $p_M$ is the map from $M$ to a point $pt$.

Let $\nu$ be the quotient bundle of $\pi_1 : M \times J(\mathbb{C}^a, \mathbb{C}^b) \to M$ by $X \to M$. It follows that for such a model for the singularity $\eta$ one has

$$
T_{p_M} \eta = p_M!(e(\nu)),
$$

where $e$ stands for the (equivariant) Euler class. Indeed, we have

$$
(20) \quad T_{p_M} \eta = \rho_* \pi_2^* (i_* (1)) = \pi_2^* (e(\nu)) = p_M! (e(\nu)).
$$

The advantage of our definition of K-theoretic fundamental class in Section 5 is that the argument (20) goes through without change to the K-theoretic setting, and we have

$$
\text{KTP}_{p_M} \eta = p_M!(e(\nu)),
$$

where $e$ is now the K-theoretic (equivariant) Euler class, and $p_M$ is the K-theoretic push-forward map.

6.2. Integration in K-theory using residues. In what follows we will use residue calculus for the push-forward map in K-theory.

Let the torus $T$ act on the smooth variety $X$ with finitely many fixed points. Let $W$ be a rank-$d$ equivariant vector bundle over $X$, and let $\omega_1, \ldots, \omega_w$ be its Chern roots (i.e. virtual line bundles whose sum is $W$). Let $p : \text{Gr}(r, W) \to X$ be the Grassmannization of $W$, that is an equivariant bundle whose fiber over $x \in X$ is the Grassmannian $\text{Gr}(r, W_x)$ of dimension $r$ linear subspaces of the fiber $W_x$ of $W$ over $x$. Let $S$ be the tautological subbundle over $\text{Gr}(r, W)$,
and let $\sigma_1, \ldots, \sigma_r$ be its Chern roots. A symmetric Laurent polynomial $g(\sigma_1, \ldots, \sigma_r)$ is hence an element of $K_T(\text{Gr}(r, W))$.

**Lemma 6.2.** We have

\[
p_t(g(\sigma_1, \ldots, \sigma_r)) = \text{Res}_{z_1=0, \infty} \ldots \text{Res}_{z_r=0, \infty} \left( \prod_{i>j} \left( 1 - \frac{z_i}{z_j} \right) \frac{g(z_1, \ldots, z_r)}{\prod_{i=1}^r \prod_{j=1}^w \left( 1 - \frac{z_i}{\omega_j} \right) \prod_{i=1}^r dz_i} \right).
\]

**Proof.** Consider first the special case when $X$ is a point. Then the equivariant localization formula for the push-forward map is

\[
p_t(f(\sigma_1, \ldots, \sigma_r)) = \sum_I f(\omega_{I_1}, \ldots, \omega_{I_r}) \prod_{i \in I} \prod_{j \in \bar{I}} \left( 1 - \frac{\omega_i}{\omega_j} \right),
\]

where the summation is over $r$-element subsets $I$ of $\{1, \ldots, n\}$, and $\bar{I}$ is the complement of $I$. Applying the Residue Theorem for the right hand side of (21), for $z_1, z_2, \ldots$ gives the same expression. This proves the lemma when $X$ is a point.

The general case is shown applying this special case to $W$ restricted to fixed points. □

When $G$ is a connected algebraic group $G$, Lemma 6.2 may be applied to the maximal torus $T \subset G$, and since $K_G(X)$ is the Weyl-invariant part of $K_T(X)$, formula (21) holds without change.

### 7. $\Sigma^r$ singularities

#### 7.1. The model for $\Sigma^r$. The obvious model for the

$\Sigma^r = \Sigma^r(\mathbb{C}^a, \mathbb{C}^b) = \{ g \in J^1(\mathbb{C}^a, \mathbb{C}^b) : \text{dim ker } g \geq r \}$

singularity is $M = \text{Gr}(r, \mathbb{C}^a)$, and

$X = \{(V, g) \in \text{Gr}(r, \mathbb{C}^a) \times J^1(\mathbb{C}^a, \mathbb{C}^b) : g|_V = 0 \}$.

Let the tautological rank $r$ bundle over $\text{Gr}(r, \mathbb{C}^a)$ be $S$. The bundle $\pi : X \to \text{Gr}(r, \mathbb{C}^a)$ can be identified with $J^1(\mathbb{C}^a / S, \mathbb{C}^b)$, hence the normal bundle is $\nu = J^1(S, \mathbb{C}^b)$. Thus $K_Tp_{\Sigma^r} = p_t(e(J^1(S, \mathbb{C}^b)))$ for the map $p : \text{Gr}(r, \mathbb{C}^a) \to \text{pt}$.

**Theorem 7.1.** We have

\[
K_Tp_{\Sigma^r} = \text{Res}_{z_1=0, \infty} \ldots \text{Res}_{z_r=0, \infty} \left( \prod_{i>j} \left( 1 - \frac{z_i}{z_j} \right) \prod_{i=1}^r \prod_{j=1}^b \left( 1 - \frac{z_i}{\beta_j} \right) \prod_{i=1}^r dz_i \right).
\]

**Proof.** We have

\[
K_Tp_{\Sigma^r} = p_t(e(J^1(S, \mathbb{C}^b))) = p_t \left( \prod_{i=1}^r \prod_{j=1}^b \left( 1 - \frac{\sigma_i}{\beta_j} \right) \right),
\]

and applying Lemma 6.2 proves the Theorem. □
Comparing expression (22) with the residue formula for Grothendieck polynomials (Definition 4.2), we obtain
\[ \text{KTP}_{2^r} = \sum_{r} \prod_{\alpha} \left( 1 - \frac{x}{\alpha} \right) \left( \prod_{\beta} \left( 1 - \frac{x}{\beta} \right) \right) \].

This result is known in Schubert calculus [LS] as the K-theoretic Giambelli-Thom-Porteous formula.

8. **A}_2 SINGULARITIES

8.1. **The model for **A}_2. Consider the tautological exact sequence \( S \to \mathbb{C}^a \to Q \) over \( \text{Gr}(1, \mathbb{C}^a) \). Let \( M = \text{Gr}(1, S^{\otimes 2} \oplus Q) \) be the projectivization of the vector bundle \( S^{\otimes 2} \oplus Q \) over \( \text{Gr}(1, \mathbb{C}^a) \), and denote the tautological line bundle over \( M \) by \( D \).

According to [BSz, K2] there is a model for the \( A}_2 \) singularity with this \( M \), and normal bundle \( \nu = \text{Hom}(S \oplus D, \mathbb{C}^b) \).

8.2. **Residue formula for **KTP}_{A}_2.

**Theorem 8.1.** We have
\[ \text{KTP}_{A}_2^{a \to b} = \text{Res}_{z_2=0, \infty} \text{Res}_{z_1=0, \infty} \left( \frac{1 - \frac{z_2}{z_1}}{1 - \frac{z_2}{\sigma}} \prod_{j=1}^{b} \frac{1 - \frac{z_2}{\beta_j}}{1 - \frac{z_2}{\omega_j}} \frac{dz_2dz_1}{z_2z_1} \right) \].

Note that the order of taking residues is important here: first we take residues with respect to \( z_2 \), then with respect to \( z_1 \).

**Proof.** We know that \( \text{KTP}_{A}_2 = p_{M!}(e(\text{Hom}(D \oplus S, \mathbb{C}^b))) \). Let the Chern roots of the bundle \( Q \) be \( \omega_1, \ldots, \omega_{a-1} \), and let the class of \( S \) be \( \sigma \), and the class of \( D \) be \( \tau \). We have
\[ e(\nu) = \prod_{j=1}^{b} \left( 1 - \frac{\sigma}{\beta_j} \right) \prod_{j=1}^{b} \left( 1 - \frac{\tau}{\beta_j} \right) \].

Pushing forward this class to \( \text{Gr}(1, \mathbb{C}^a) \), using Lemma 6.2 we get
\[ \text{Res}_{z_2=0, \infty} \left( \frac{\prod_{j} \left( 1 - \frac{\sigma}{\beta_j} \right) \prod_{j} \left( 1 - \frac{z_2}{\omega_j} \right) dz_2}{\prod_{j} \left( 1 - \frac{z_2}{\sigma} \right) z_2} \right) \].

Using the fact that \( S \to \mathbb{C}^a \to Q \) is an exact sequence, this is further equal to
\[ \text{Res}_{z_2=0, \infty} \left( \frac{\prod_{j} \left( 1 - \frac{\sigma}{\beta_j} \right) \prod_{j} \left( 1 - \frac{z_2}{\omega_j} \right) \left( 1 - \frac{z_2}{\sigma} \right) dz_2}{\prod_{j} \left( 1 - \frac{z_2}{\omega_j} \right) z_2} \right) \].
Pushing this class further from \( \text{Gr}(1, \mathbb{C}^2) \) to a point, using Lemma 8.23 we obtain
\[
\text{Res}_{\zeta_1 = 0, \infty} \text{Res}_{\zeta_2 = 0, \infty} \left( \prod_j \left( 1 - \frac{z_1}{\beta_j} \right) \prod_j \left( 1 - \frac{z_2}{\alpha_j} \right) \frac{dz_2}{z_2} \frac{dz_1}{z_1} \right),
\]
which is what we wanted to prove. \( \square \)

8.3. \( \text{KTP}_{A_2} \) in terms of Grothendieck polynomials—the stable expansion. Let
\[
\frac{1}{1 - z_2 / z_1^2} = \sum_{r,s} d_{r,s} (1 - z_1)^r (1 - z_2)^s
\]
be the Laurent expansion of the named rational function on the \(|1 - z_1| < |1 - z_2|\) region. Equivalently, after substituting \( x_1 = 1 - z_1, x_2 = 1 - z_2 \), let
\[
\frac{1 - 2 x_1 + x_1^2}{x_2 - 2 x_1 + x_1^2} = \sum_{r,s} d_{r,s} x_1^r x_2^s
\]
be the Laurent expansion of the named rational function on the \(|x_1| < |x_2|\) region. Based on the calculation
\[
(23) \quad \frac{1}{x_2 - 2 x_1 + x_1^2} = \frac{1}{x_2} \cdot \frac{1}{1 - (2 x_1 - x_1^2)/x_2} = \sum_{k=1}^{\infty} \frac{1}{x_2^2} (2 x_1 - x_1^2)^k = \frac{1}{x_2^2} \sum_{k=1}^{\infty} \left( \prod_{r=0}^{k-1} (2 x_1 - x_1^2)^k \right)
\]
we have that
\[
d_{r,s} = (-1)^{r+s+1} \left( 2^{2s-2-r} \left( \begin{array}{c} -s - 1 \\ -2s - r - 2 \end{array} \right) + 2^{-2s-r} \left( \begin{array}{c} -s - 1 \\ -2s - r - 1 \end{array} \right) + 2^{-2s-r} \left( \begin{array}{c} -s - 1 \\ -2s - r \end{array} \right) \right)
\]
for \( r = 0, 1, \ldots, s = -r - 1, \ldots, -\lfloor r/2 \rfloor \). In particular, the sign of \( d_{r,s} \) is \((-1)^{r+s+1}\).

For the values of \( d_{r,s} \) for small (absolute value) \( r, s \) see the table in [Table2].

**Theorem 8.2** (Grothendieck expansion of \( \text{KTP}_{A_2} \): the stable version). Let \( l = b - a \), and \( N > 2l + 2 \). Then
\[
(24) \quad \text{KTP}_{A_2}^{a \rightarrow b} = \sum_{r=0}^{N} \sum_{s=-r-1}^{-\lfloor \frac{s}{2} \rfloor} d_{r,s} G_{r+l+1,s+l+2}(\alpha_1^{-1}, \ldots, \alpha_a^{-1}; \beta_1^{-1}, \ldots, \beta_b^{-1}).
\]

Note that for a given \( r \), the set of non-zero \( d_{r,s} \) coefficients are exactly those between \( s = -r - 1 \) and \( s = -\lfloor r/2 \rfloor \), hence, in the summation above, \( s \) runs through all its relevant values.

**Remark 8.3.** Since \( N \) may be arbitrarily large in (24), it is tempting to phrase Theorem 8.2 informally as
\[
(25) \quad \text{KTP}_{A_2}^{a \rightarrow b} = \sum_{r,s} d_{r,s} G_{r+l+1,s+l+2}(\alpha_1^{-1}, \ldots, \alpha_a^{-1}; \beta_1^{-1}, \ldots, \beta_b^{-1}).
\]
This series does not converge, however.

**Proof.** The finite expansion of $1/(1 - z_2/z_1^2)$ with respect to $z_1$, around $z_1 = 1$, with remainder term is

$$
\frac{1}{1 - z_2/z_1^2} = \sum_{r=0}^N \left( \sum_s d_{r,s}(1 - z_2)^s \right) (1 - z_1)^r + R_N(z_1, z_2)
$$

where the $s$-summation is finite. A quick calculation shows that the remainder term may be expressed as

$$
R_N(z_1, z_2) = -\left( \frac{1 - z_1}{1 - z_2} \right)^{N+1} \frac{z_1 q_N(z_2) + p_N(z_2)}{1 - z_1^2/z_2^2}.
$$

where

$$
p_N(z) = \sum_{i=0}^{\lfloor N+1/2 \rfloor} \binom{N+1}{2i} z^i, \quad q_N(z) = \sum_{i=0}^{\lfloor N/2 \rfloor} \binom{N+1}{2i+1} z^i.
$$

According to Theorem 8.1, we have the following expression for $\text{KTP}_{A_2^{a \rightarrow b}}$:

$$
\text{KTP}_{A_2^{a \rightarrow b}} = \text{Res}_{z_1=0, \infty \ z_2=0, \infty} \left( (1 - z_1)^l (1 - z_2)^l \frac{1}{1 - z_2/z_1^2} \times \right.
$$

$$
\times \left( 1 - \frac{z_2}{z_1} \right)^2 \prod_{i=1}^N \prod_{a_j=1}^{b_j} \frac{1 - \frac{z_i}{\beta_j}}{1 - \frac{z_i}{\alpha_j}} \frac{dz_2 dz_1}{z_2 z_1} \right)
$$

Substituting (26), we obtain

$$
\text{KTP}_{A_2^{a \rightarrow b}} = \text{Res}_{z_1=0, \infty \ z_2=0, \infty} \left( \sum_{r=0}^N \left( \sum_s d_{r,s}(1 - z_2)^s+l \right) (1 - z_1)^{r+l} \times \right.
$$

$$
\left. \times \left( 1 - \frac{z_2}{z_1} \right)^2 \prod_{i=1}^N \prod_{a_j=1}^{b_j} \frac{1 - \frac{z_i}{\beta_j}}{1 - \frac{z_i}{\alpha_j}} \frac{dz_2 dz_1}{z_2 z_1} \right) + \text{Res}_{z_1=0, \infty \ z_2=0, \infty} \left( R_N(z_1, z_2) \left( 1 - \frac{z_2}{z_1} \right)^2 \prod_{i=1}^N \prod_{a_j=1}^{b_j} \frac{1 - \frac{z_i}{\beta_j}}{1 - \frac{z_i}{\alpha_j}} \frac{dz_2 dz_1}{z_2 z_1} \right).
$$

According to the residue expression for Grothendieck polynomials (Definition 4.2) the first term equals

$$
\sum_{r=0}^N \sum_s d_{r,s} G_{r+l+1,s+l+2}(\alpha_1^{-1}, \ldots, \alpha_n^{-1}, \beta_1^{-1}, \ldots, \beta_b^{-1})
$$
and we claim that the second term vanishes for large $N$. Indeed, using the form \((27)\) of the remainder term $R_N(z_1, z_2)$, we can see that for large $N$, the rational form

\[
R_N(z_1, z_2) \left(1 - \frac{z_2}{z_1}\right)^2 \prod_{i=1}^2 \frac{\prod_{j=1}^b \left(1 - \frac{z_2}{\beta_j}\right)}{\prod_{j=1}^a \left(1 - \frac{z_2}{\alpha_j}\right)} \frac{dz_2 dz_1}{z_2 z_1}
\]

satisfies the conditions of Lemma 4.1 in $\text{z}_2$. This means that already applying the first residue operation $\text{Res}_{z_2 = 0, \infty}$ results in 0. This completes the proof.

8.4. $K_{T\mu A_2}$ in terms of Grothendieck polynomials – the minimal expansion.

**Theorem 8.4** (Grothendieck expansion of $K_{T\mu A_2}$, the minimal version). We have the following expression for $K_{T\mu A_2}$ in Grothendieck polynomials indexed by partitions:

$$K_{T\mu A_2}^{a \rightarrow b} = \sum_{r=0}^{2l+2-\left\lfloor \frac{b}{2} \right\rfloor} \sum_{s=-r-1}^{-\left\lfloor \frac{b}{2} \right\rfloor} d_{r,s} G_{r+l+1, s+l+2}(\alpha_1^{-1}, \ldots, \alpha_a^{-1}; \beta_1^{-1}, \ldots, \beta_b^{-1}),$$

where $l = b - a$, and

$$D_{r,s,t} = \begin{cases} d_{r,s} & \text{if } s > -l - 2 \\ \sum_{t=-r-1}^{-l-2} d_{r,t} = \sum_{t=-\infty}^{-l-2} d_{r,t} & \text{if } s = -l - 2. \end{cases}$$

**Proof.** It follows from Theorem 8.2 that for large $N$

$$K_{T\mu A_2}^{a \rightarrow b} = \sum_{r=0}^{N} \sum_{s=-r-1}^{-\left\lfloor \frac{b}{2} \right\rfloor} d_{r,s} G_{r+l+1, s+l+2}.$$

For notational simplicity we omit the arguments $\alpha_i^{-1}, \beta_i^{-1}$ of the Grothendieck polynomials. Consider the sum

$$\sum_{s=-r-1}^{-\left\lfloor \frac{r}{2} \right\rfloor} d_{r,s} G_{r+l+1, s+l+2}$$

for a given $r$. In it, the occurring Grothendieck polynomials have the same first index $r+l+1$, but varying second index $s+l+2$. Notice that if $r > 2l + 2$ then all $s+l+2$ indexes are non-positive. Indeed, if $r > 2l + 2$ then $s \leq -\left\lfloor \frac{r}{2} \right\rfloor < -\left\lfloor (2l + 2)/2 \right\rfloor = -l - 1$ and hence $s+l+2 < 0$. Then using the straightening law $G_{1,0} = G_{1,-1} = G_{1,-2} = \ldots$ (see \([8]\) or Lemma 4.6) we have that

$$\sum_{s=-r-1}^{-\left\lfloor \frac{r}{2} \right\rfloor} d_{r,s} G_{r+l+1, s+l+2} = \left(\sum_{s=-r-1}^{-\left\lfloor \frac{r}{2} \right\rfloor} d_{r,s}\right) G_{r+l+1, 0}.$$

Plugging in $z_2 = 0$ into $1/(1 - z_2/z_1^2)$ results 1, hence for $r > 0$ we have $\sum_{s=-r-1}^{-\left\lfloor \frac{r}{2} \right\rfloor} d_{r,s} = 0$, and in turn, the expression \((30)\) is 0. This proves that in \((29)\) the number $N$ can be chosen to be as small as $2l+2$. The same statement may be obtained from a careful analysis of the vanishing of the residues of \((28)\).
Now let \( r \leq 2l + 2 \). Using the same straightening law of Grothendieck polynomials we obtain

\[
-\left\lfloor \frac{r}{2} \right\rfloor \sum_{s=-r-1}^{s=-l-1} d_{r,s} G_{r+l+1,s+l+2} = \left( \sum_{s=-r-1}^{s=-l-1} d_{r,s} \right) G_{r+l+1,0} + \sum_{s=-l-1}^{s=-l-1} d_{r,s} G_{r+l+1,s+l+2},
\]

completing the proof. \( \square \)

**Remark 8.5.** The expansion in Theorem 8.4 is minimal in the sense that each occurring Grothendieck polynomial is parametrized by a partition (with non-negative components), and hence can not be simplified by the straightening laws (7)-(8) (or Lemma 4.6).

9. **Alternating signs**

The coefficients of the Grothendieck polynomials in both the stable and the minimal Grothendieck polynomial expansions of \( K_{TP A_2} \) have alternating signs. 

**Theorem 9.1.** The coefficient of \( G_{a,b}(\alpha_1^{-1}, \ldots, \alpha_t^{-1}, \beta_1^{-1}, \ldots, \beta_b^{-1}) \) in both the expansion of Theorem 8.2 and the expansion of Theorem 8.4 has sign \((-1)^{a+b}\).

**Proof.** The statement for the expansion in Theorem 8.2 is equivalent to \( d_{r,s} \) having sign \((-1)^{r+s+1}\), which follows from the explicit formula for \( d_{r,s} \) in Section 8.3.

The statement for the expansion in Theorem 8.4 is equivalent to \( D_{r,s,l} \) having sign \((-1)^{r+s+1}\) for any \( l \). For this we need to additionally prove that

\[
\text{the sign of } \sum_{t=-\infty}^{t=-l-2} d_{r,t} \text{ is } (-1)^{r+s+1}
\]

for any \( l \).

To prove (31) consider \( f = (1-2x_1+x_1^2)/(x_2-2x_1+x_2^2) = \sum_{r,s} d_{r,s} x_1^r x_2^s \) (as before, \(|x_1| < |x_2|\)), and let \( g = (-1+f)/(1-x_2) \). On the one hand \( g = 1/(x_2-2x_1+x_2^2) \) (from the explicit form of \( f \)). On the other hand

\[
g = \left( -1 + \sum_{r,s} d_{r,s} x_1^r x_2^s \right) \left( 1 + x_2 + x_2^2 + \ldots \right) = \sum_{r,s} \left( \sum_{t=-\infty}^{t=-l-2} d_{r,t} \right) x_1^r x_2^s.
\]

Here we used that \( d_{0,-1} = 1 \) and \( d_{0,s} = 0 \) for all \( s \neq -1 \).

Comparing the two forms of \( g \) we find that statement (31) is equivalent to the the property that the coefficient of \( x_1^r x_2^s \) in the expansion of \( 1/(x_2-2x_1+x_2^2) \) has sign \((-1)^{r+s+1}\). This latter claim follows from the calculation (23). \( \square \)

10. **Remarks on higher singularities**

For singularities higher than \( A_2 \), it is difficult to carry out our program. There are no practical models for \( A_d \)-singularities for \( d \geq 7 \), but even in the case of \( A_3 \), where the model is very simple ([BSZ, K2]), the combinatorial problems we face are rather complicated. A proof analogous to that of Theorem 8.4 in this case yields the following statement.
Theorem 10.1. We have

\[ \text{KT}_{P_{A_3}}^{a \rightarrow b} = \text{Res}_{z_1=0, \infty} \text{Res}_{z_2=0, \infty} \text{Res}_{z_3=0, \infty} \left( \frac{1 - \frac{z_2}{z_1}}{1 - \frac{z_3}{z_1}} \right) \left( \frac{1 - \frac{z_3}{z_2}}{1 - \frac{z_3}{z_1}} \right) \left( \frac{1 - \frac{z_3}{z_1}}{1 - \frac{z_3}{z_2}} \right) \prod_{j=1}^{3} \frac{\prod_{j=1}^{b} \left( 1 - \frac{z_j}{\alpha_j} \right)}{\prod_{j=1}^{a} \left( 1 - \frac{z_j}{\beta_j} \right)} dz_3 dz_2 dz_1 \right). \]

This formula suggests that to obtain the Grothendieck expansion of \( \text{KT}_{P_{A_3}} \), we ought to consider the expansion

\[ \frac{1}{(1 - z_2/z_1^2)(1 - z_3/z_1^2)(1 - z_3/z_1 z_2)} = \sum_{r,s,t} d_{r,s,t}(1 - z_1)^r(1 - z_2)^s(z - z_3)^t, \]

valid in the region \(|1 - z_1| < |1 - z_2| < |1 - z_3|\), and then find an appropriate way to resum the series

(32) \[ \sum_{r,s,t} d_{r,s,t} G_{r+l+1,s+l+2,t+l+3}(\alpha_1^{-1}, \ldots, \alpha_a^{-1}; \beta_1^{-1}, \ldots, \beta_b^{-1}). \]

to obtain finite expressions. The concrete form of the resummation procedure and the resulting finite expression is not clear at the moment.

It seems even more difficult to find the analogue of Theorem 8.4 (the minimal Grothendieck expansion) for \( A_3 \). To achieve the Grothendieck expansion of Theorem 8.4 from that of Theorem 8.2 we needed to work only with one of the straightening laws, namely (8). However, to “straighten” the partitions in (32) one is forced to use the other straightening law, namely (7), and this seems much more complex. It would be interesting to develop the residue calculus or another analytic tool which replaces the combinatorics of (7).

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