VARIETIES OVER A FINITE FIELD WITH TRIVIAL
CHOW GROUP OF 0-CYCLES HAVE A RATIONAL
POINT

HÉLÈNE ESNAULT

1. Introduction

Let $X$ be a smooth projective variety of dimension $d$ over a field $k$. Let $\overline{k(X)}$ be the algebraic closure of its function field. If the Chow group of 0-cycles $CH_0(X \times_k \overline{k(X)})$ is equal to $\mathbb{Z}$, then S. Bloch shows in [4], Appendix to Lecture 1, that the diagonal $\Delta \in CH^d(X \times_k X) \otimes \mathbb{Q}$ decomposes. This means there are a $N \in \mathbb{N} \setminus \{0\}$, a 0-dimensional subscheme $\xi \subset X$, a divisor $D \subset X$, a dimension $d$ cycle $\Gamma \subset X \times D$ such that

$$N \cdot \Delta \equiv \xi \times X + \Gamma.$$

(1.1)

For sake of completeness, we briefly recall his argument. Using the norm, one sees that the kernel of $CH_0(X \times_k \overline{k(X)}) \to CH_0(X \times_k k(X))$ is torsion. Thus up to torsion, the cycle Spec $k(X)$ is equivalent in $CH_0(X \times_k \overline{k(X)})$ to a $k$-rational point of $X$, thus, up to torsion, to a $\xi$ as above. On the other hand, $CH_0(X \times_k k(X))$ is the inductive limit of $CH_0(X \times_k (X \setminus D))$ as $D$ runs over the divisors of $X$.

This has various consequences on the shape of de Rham or Hodge cohomologies in characteristic 0. Let $(\Delta) \in H^{2d}(X \times X)$ be the cycle class of $\Delta$. One applies the correspondence $[\Delta]_* = p_{2,*}((\Delta) \cup p_1^*) = [\xi \times X]_* + [\Gamma]_*$ to, for example, $H^i_{DR}(X)$. Then $[\xi \times X]_* H^i_{DR}(X) = 0$ for $i \geq 1$ as the correspondence factors through $H^i_{DR}(\xi) = 0$, while $[\Gamma]_* H^i_{DR}(X) \subset H^i_{DR}(X)$ dies via the restriction map $H^i_{DR}(X) \to H^i_{DR}(X \setminus D)$. Using the surjection $H^i_{DR}(X) \to H^i(X, \mathcal{O}_X)$ to lift classes, and the factorization $H^i_{DR}(X \setminus D) \to H^i(X, \mathcal{O}_X)$ coming from Deligne’s Hodge theory ([7]), one concludes that $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. S. Bloch developed this argument, and variants of it for étale cohomology, to kill the algebraic part of $H^2$ under the representability assumption of the Chow group of 0-cycles over $\overline{k(X)}$ (Mumford’s theorem).

Date: July 20 print, 2002.
The purpose of this note it to observe that P. Berthelot’s rigid cohomology \([1]\) has the required properties to make the above argument work in this framework. If \(k\) has characteristic \(p > 0\), let \(W(k)\) be its ring of Witt vectors, \(K\) be the quotient field of \(W(k)\). Let \(X\) be smooth proper over \(k\), \(Z \subset X\) be a closed subvariety, and \(U = (X \setminus Z)\) be its complement. The rigid cohomology, which coincides with the crystalline cohomology on \(X\), fulfills a localization sequence \((1.2)\)

\[ \cdots \rightarrow H^i_Z(X/K) \rightarrow H^i(X/K) \rightarrow H^i(U/K) \rightarrow \cdots \]

which is compatible with the Frobenius action \((3), \text{Theorem 2.4})\). By \([3]\) and \([13]\), the slope \([0 \ 1] - \text{part of } H^i(X/K)\) is \(H^i(X, W\mathcal{O}_X) \otimes_{W(k)} K\).

One has

**Theorem 1.1.** Let \(X\) be a smooth projective variety over a perfect field \(k\) of characteristic \(p > 0\). If the Chow group of 0-cycles \(CH_0(X \times_k k(X))\) is equal to \(\mathbb{Z}\), then the slope \([0 \ 1]\) part of \(H^i(X/K)\) is vanishing for \(i > 0\).

On the other hand, if one now assumes that \(k = \mathbb{F}_q\) is a finite field, with \(q = p^n\), the Lefschetz trace formula for crystalline cohomology (see e.g. \([11]\), section II.1)

\[ |X(k)| = \sum_i (-1)^i \text{Trace(Frob}^n|H^i(X/K)) \]

implies in particular that if all the slopes of Frobenius are \(\geq 1\) and \(X\) is geometrically connected, then

\[ |X(k)| \equiv 1 \mod q. \]

Thus one has

**Corollary 1.2.** Let \(X\) be a smooth, projective, geometrically connected variety over a finite field \(k\). If the Chow group of 0-cycles \(CH_0(X \times_k k(X))\) is equal to \(\mathbb{Z}\), then \(X\) has a rational point over \(k\).

An example of application is provided by Fano varieties. A variety \(X\) is said to be Fano if it is smooth, projective, geometrically connected and the dual of the dualizing sheaf \(\omega_X\) is ample. By \([15]\), Theorem V. 2.13, Fano varieties on any algebraically closed field are chain rationally connected \((15), \text{Definition IV. 3.2})\) in the sense that any two rational points can be joined by a chain of rational curves. This implies in particular that \(CH_0(X \times_k k(X)) = \mathbb{Z}\). Thus one concludes

**Corollary 1.3.** Let \(X\) be a Fano variety over a finite field \(k\), or more generally, let \(X\) be a smooth, projective, geometrically connected variety.
over a finite field $k$, which is chain rationally connected over $\overline{k(X)}$. Then $X$ has a rational point.

This corollary answers positively a conjecture by S. Lang [16] and Yu. Manin [17]. This is the reason why we write down the argument for Theorem 1.1, while it is a direct adaption of S. Bloch’s argument [4] to crystalline and rigid cohomologies.

That a study of crystalline cohomology should yield via the congruence (1.3) the existence of a rational point on Fano varieties over finite fields is entirely due to M. Kim. His idea was to kill the whole cohomology $H^i(X, W\omega_X)$ for $i < \dim(X)$, using solely the structure of crystalline cohomology on $X$, together with its Verschiebung and Frobenius operators, and using the ampleness of $\omega_X^{-1}$. The point of Corollary 1.2 is that Kollár-Miyaoka-Mori’s and Campana’s theorem ([13], loc. cit.), which is anchored in geometry, together with Bloch’s type Chow group argument, force (weaker) cohomological consequences. In a way, the difficult theorem is the geometric one.

Acknowledgements. This note relies on P. Berthelot’s work on rigid cohomology, with which the author is not familiar. It is a pleasure to thank P. Berthelot, S. Bloch and O. Gabber for their substantial help and for their encouragement. I thank the IHES for support during the preparation of this work.

2. Proof of theorem 1.1

In this section we prove Theorem 1.1. Thus $X$ is a smooth projective variety over $k$. For any codimension $d$ cycle $Z \subset X \times X$, the correspondence

$$[Z]_* = p_{2,*}((Z) \cup p_1^*)$$

is well defined on $H^i(X/K)$. One needs for this the existence of the cycle class

$$Z \in H^{2d}((X \times X)/K)$$

which is provided by [1], Corollaire 5.7, (ii), and by [18], 6.2, for the factorization through the Chow group, the contravariance for $p_1^*$ and the covariance for $p_{2,*}$, applied to crystalline cohomology of smooth proper varieties. In particular, this correspondence factors through $H^i(\xi/K)$ if $Z = (\xi \times X)$, which shows via formula (1.1) that

$$N[\Delta]_* = [\Gamma]_*$$
on \( H^i(X/K) \) for \( i > 0 \). On the other hand, since \( \Gamma \subset X \times D \), \([\Gamma]_*\), seen as a correspondence of \( X \) to \((X \setminus D)\), is trivial. One has

\[
(2.4) \quad [\Gamma]_* (H^i(X/K)) \subset \text{Ker}(H^i(X/K) \to H^i((X \setminus D)/K))
\]

(by \([\mathcal{L}]\)) \( \text{Im}(H^i_D(X/K) \subset H^i(X/K)) \).

Thus, using \([\mathcal{L}]\), Theorem 2.4, Theorem 1.1 is a consequence of the following

Lemma 2.1. (P. Berthelot) Let \( X \) be a smooth, geometrically connected, quasi-compact and separated scheme over a perfect field \( k \) of characteristic \( p > 0 \) and let \( Z \subset X \) be a non-empty subvariety of codimension \( \geq 1 \). Then the slopes of \( H^i_Z(X/K) \) are \( \geq 1 \).

Proof. Let \( \ldots \subset Z_i \subset Z_{i-1} \subset \ldots \subset Z_0 = Z \) be a finite stratification by closed subsets such that \( Z_{i-1} \setminus Z_i \) is smooth. The localization \([\mathcal{L}], 2.5.1\)

\[
(2.5) \quad \ldots \to H^i_{Z_i}(X/K) \to H^i_{Z_{i-1}}(X/K) \to H^i_{Z_{i-1} \setminus Z_i}((X \setminus Z_i)/K) \to \ldots
\]

allows to reduce to the case where \( Z \) is smooth. If \( X \) is affine, then the Gysin isomorphism (purity) \( H^{i-2 \cdot \text{codim}(Z)}(Z) \xrightarrow{\sim} H^i_Z(X) \) commutes to \( F_{\text{codim}(Z)} \). Frob on \( H^{i-2 \cdot \text{codim}(Z)}(Z/K) \) and Frob on \( H^i_Z(X/K) \) \([\mathcal{L}], \text{Theorem 2.4}\). Since the slopes on \( H^i(Z/K) \) are all \( \geq 0 \), we conclude that the slopes of the cohomology with support are \( \geq 1 \). In general, one considers a finite affine covering \( X = \bigcup_{i=0}^N U_i \). The spectral sequence

\[
(2.6) \quad E^a_b = H^a(\ldots \to H^b_Z(U^a/K) \to H^b_Z(U^a/K) \to H^b_Z(U^{a+1}/K) \to \ldots)
\]

converges to \( H^{a+b}(X/K) \). Here the open sets \( U^a \) are the \((a+1)\) by \((a+1)\) intersections of the \( U_i \) and the maps are the restriction maps. If \( H^b_Z(U^a/K) \neq 0 \), then \( Z \) meets \( U^a \) and its slopes are \( \geq 1 \) by the previous case. Thus the spectral sequence has only contributions with \( \geq 1 \) slopes. This finishes the proof. \( \square \)

3. Comments

Corollary \([\mathcal{L}]\) can be compared to the main theorems of \([\mathcal{L}], \text{resp.} \), where the finite field is replaced by \( k = F(\text{curve}) \) for \( F = \mathbb{C}, \text{resp.} \) an algebraically closed field in characteristic \( p > 0 \). There the authors show the existence of a rational point on a smooth Fano variety defined over \( k \). In the latter case, one has to add “separably” to “chain rationally connected”.

On the other hand, if \( X \) is a hypersurface \( \subset \mathbb{P}^n \) of degree \( \leq n \) over a field \( k \) of characteristic \( 0 \), then \( CH_0(X \times_k k(X)) = \mathbb{Z} \) by Roitman’s
CHOW GROUP AND RATIONAL POINT

5

theorem ([19]), whether \( X \) is smooth or not. It suggests that perhaps there is a stronger version of Theorem 1.1 requiring \( X \) projective but not smooth. This would be compatible with the congruence results of Ax and Katz [4], and their Hodge theoretic counter-part ([8], [9], [10]). In this case it says \( H^i(X', \mathcal{O}_{X'}) = 0 \), where \( X' = \pi^{-1}(X)_{\text{red}} \) and \( \pi : \mathbb{P} \to \mathbb{P}^n \) is a birational map with \( \mathbb{P} \) smooth, such that \( X' \) is a normal crossings divisor. The method used here does not apply to the singular situation.

After receiving this note, G. Faltings and O. Gabber explained to me that one could use a similar argument in \( \acute{e} \text{tale} \) cohomology in order to obtain the same conclusion, and M. Kim replaced the use of rigid cohomology in the localisation argument presented here by the one of de Rham-Witt cohomology with logarithmic poles. Since it does not yield a stronger result, we do not develop their arguments.

Finally let us observe that, replacing \( D \) in the argument by a subvariety of codimension \( \kappa \geq 1 \), replaces the congruence \((1 \mod q)\) by \((1 \mod q^\kappa)\). But it is hard to understand what are the geometric conditions which guarantee higher codimension. Again, hypersurfaces of very low degree fulfill the correct congruence ([14]), but I do not know whether it is reflected by this strong Chow group property. Without this, the method presented here gives a different proof of the main result of [4] in the smooth case when \( \kappa = 1 \).

References

[1] Berthelot, P.: Finitude et pureté en cohomologie rigide, Invent. math. 128 (1997), 329-377.
[2] Berthelot, P.: Dualité de Poincaré et formule de Künneth en cohomologie rigide, CRAS 325 (1997), 493-498.
[3] Bloch, S.: Algebraic K-theory and crystalline cohomology, Publ. Inst. Hautes Ét. Sc. 47 (1977), 187-268.
[4] Bloch, S.: Lecture on Algebraic cycles, Duke University Mathematics Series, IV, (1980).
[5] Chiarellotto, B.: Weights in rigid cohomology and applications to unipotent \( F \)-isocrystals. Ann. Sci. École Norm. Sup. (4) 31 (1998), no. 5, 683–715.
[6] de Jong, J.; Starr, J.: Every rationally connected variety over the function field of a curve has a rational point, preprint 2002.
[7] Deligne, P.: Théorie de Hodge II, Publ. Math. IHES 40 (1972), 5-57.
[8] Deligne, P.; Dimca, A.: Filtrations de Hodge et par l’ordre du pôle pour les hypersurfaces singulières, Ann. Sci. Éc. Norm. Supér. (4) 23 (1990), 645-656.
[9] Esnault, H.: Hodge type of subvarieties of \( \mathbb{P}^n \) of small degrees, Math. Ann. 288 (1990), no. 3, 549-551.
[10] Esnault, H.; Nori, M.; Srinivas, V.: Hodge type of projective varieties of low degree, Math. Ann. 293 (1992), no. 1, 1-6.
[11] Etesse, J.-Y.: Rationalité et valeurs de fonctions $L$ en cohomologie cristalline, Ann. Sc. de l’Inst. Fourier 38 4 (1988), 33-92.
[12] Graber, T.; Harris, J.; Starr, J.: Families of Rationally Connected Varieties, preprint 2002.
[13] Illusie, L.: Complexe de de Rham-Witt et cohomologie cristalline, Ann. Sci. École Norm. Sup. (4) 12 (1979), 501-661.
[14] Katz, N.: On a theorem of Ax. Amer. J. Math. 93 (1971), 485-499.
[15] Kollár, J.: Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 32 (1996), Springer-Verlag, Berlin, 1996.
[16] Lang, S.: Cyclotomic points, very anti-canonical varieties, and quasi-algebraic closure, preprint 2000.
[17] Manin, Yu.: Notes on the arithmetic of Fano threefolds, Compos. math. 85 (1993), 37-55.
[18] Petrequin, D.: Classes de Chern et classes de cycles en cohomologie rigide, preprint 2001, 69 pages, to appear in Bull. Soc. Math. de France.
[19] Roitman, A.: Rational equivalence of zero-dimensional cycles. Mat. Zametki 28 (1980), no. 1, 85-90.

Mathematik, Universität Essen, FB6, Mathematik, 45117 Essen, Germany

E-mail address: esnault@uni-essen.de