THE ALBANESE FUNCTOR COMMUTES WITH THE HODGE REALIZATION

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ABSTRACT. We prove that the embedding of the derived category of 1-motives up to isogeny into the triangulated category of effective Voevodsky motives, as well as its left adjoint functor $L\text{Alb}_Q$, commute with the Hodge realization. This result yields a new proof of the rational form of Deligne’s conjecture on 1-motives.

1. Introduction

1.1. Deligne’s Conjecture. In (D3) Deligne showed that the subcategory $MHS^}\mathbb{Z}_1 \subset MHS^\mathbb{Z}$ of the category of polarizable mixed Hodge structures consisting of torsion free objects of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ has a pure geometric description: $MHS^\mathbb{Z}_1$ is equivalent to the category $\mathcal{M}_1(\mathbb{C})$ of triples $[\Lambda, G, u]$, where $\Lambda$ is a free abelian group of finite rank, viewed as a discrete group scheme over $\mathbb{C}$, $G$ is a semi-abelian variety over $\mathbb{C}$ and $u : \Lambda \to G$ is a group homomorphism. The base field $\mathbb{C}$ in the definition of $\mathcal{M}_1(\mathbb{C})$ can be replaced by any field $k$; the resulted category $\mathcal{M}_1(k)$ is Deligne’s category of 1-motives. We will be primarily interested in the category $\mathcal{M}_1(k) \otimes \mathbb{Q}$ of 1-motives up to isogeny which is abelian. For $k \subset \mathbb{C}$, Deligne conjectured the existence of the dashed arrow in the following commutative diagram

$$
\begin{array}{ccc}
\text{Var}_k & \xrightarrow{H_1LAlb_\mathbb{Q}} & \mathcal{M}_1(k) \otimes \mathbb{Q} \\
\downarrow & & \downarrow \\
\text{Var}_\mathbb{C} & \xrightarrow{H_1LAlb_\mathbb{Q}} & MHS^\mathbb{Q}_1 = \mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q},
\end{array}
$$

where $\text{Var}_k$ denotes the category of varieties over $k$, vertical arrows are the base change functors and functor $H_1LAlb_\mathbb{Q}$ takes $X \in \text{Var}_\mathbb{C}$ to the maximal quotient of the $\mathbb{Q}$-Hodge structure $H_1(X)$ that belongs to $MHS^\mathbb{Q}_1$. The functor $H_1LAlb_\mathbb{Q}$ should be thought as a higher Albanese functor: for every smooth variety $X$, $H_1LAlb_\mathbb{Q}(X) = [0, Alb^0_X, 0]$, where $Alb^0_X$ is Serre’s Albanese variety. Deligne’s conjecture was proved by Barbieri-Viale, Rosenschon and Saito (BRS) and by Ramachandran (R2). In this paper, we shall show how the functor $H_1LAlb_\mathbb{Q}$ can be extended to a triangulated functor from the category of Voevodsky’s motives to the derived category of 1-motives fitting in a commutative diagram similar to (1.1).

1.2. Motivic Albanese functor. The first and the most important step in this direction was taken by Barbieri-Viale and Kahn who proved in [BK1] that for every perfect field $k$ the embedding ([Y1], [O])

$$
Tot_\mathbb{Q} : D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \hookrightarrow DM^\mathbb{Q}_{gm}(k; \mathbb{Q})
$$

\footnote{A more precise integral version of Deligne’s conjecture is not touched in the main body of this paper. See however [BRS].}
admits a left adjoint functor \( L\text{Alb}_k \colon DM^\text{eff}_{gm}(k; \mathbb{Q}) \rightarrow D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \).

For \( k \subset \mathbb{C} \), we can consider the composition of \( L\text{Alb}_k \) with the Hodge realization \( T^\text{Hodge}_k \colon D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \rightarrow D^b(\mathcal{M}_1(\mathbb{C}) \otimes \mathbb{Q}) \xrightarrow{\sim} D^b(M\text{HS}^Q_1) \).

The goal of this paper is to compute \( T^\text{Hodge}_k \circ L\text{Alb}_k \).

1.3. **Hodge Albanese functor.** To state our main result we define an analog of the functor \( L\text{Alb}_k \) on derived category of mixed Hodge structures. We shall say that a mixed \( \mathbb{Q} \)-Hodge structure (\( W. \subset V_\mathbb{Q}, F. \subset V_\mathbb{C} \)) is effective if \( F^1 = 0 \). Denote by \( M\text{HS}^Q_{eff} \subset M\text{HS}^Q \) the full subcategory of effective polarizable mixed Hodge structures. We prove in Proposition \( 3.1 \) that the induced functor

\[
\text{Tot}_k : D^b(\mathcal{M}^Q_1) \rightarrow D^b(M\text{HS}^Q_{eff}^{\text{eff}})
\]

is fully faithful and admits a left adjoint functor

\[
L\text{Alb}_k : D^b(M\text{HS}^Q_{eff}^{\text{eff}}) \rightarrow D^b(M\text{HS}^Q_1).
\]

Explicitly, \( L\text{Alb}_k \) is induced by the exact functor \( M\text{HS}^Q_{eff} \rightarrow M\text{HS}^Q \) that takes \( V \in M\text{HS}^Q_{eff} \) to its maximal quotient that belongs to \( M\text{HS}^Q_1 \).

1.4. **Main result.** For \( k \subset \mathbb{C} \), we denote by

\[
R^\text{Hodge}_k : DM^\text{eff}_{gm}(k; \mathbb{Q}) \rightarrow D^b(M\text{HS}^Q_{eff}^{\text{eff}})
\]

the homological Hodge realization (\( \text{Hu1}, \text{Hu2} \)). In Theorem \( 3 \) we construct isomorphism of triangulated functors

\[
\text{Tot}_k \circ T^\text{Hodge}_k \simeq R^\text{Hodge}_k \circ \text{Tot}_k;
\]

\[
L\text{Alb}_k \circ R^\text{Hodge}_k \simeq T^\text{Hodge}_k \circ L\text{Alb}_k.
\]

In particular, letting \( H_i L\text{Alb}_k(X) \) be the \( i \)-th homology of the complex \( L\text{Alb}_k(M(X)) \in D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \) we recover the commutative diagram (1.1). The advantage of the new approach to Deligne’s conjecture is the transparent universal property of \( L\text{Alb}_k(M(X)) \) similar to the universal property of classical Albanese variety. However, one needs the category of Voevodsky motives to state this universal property.

1.5. **Remarks on the proof.** The proof of (1.2) uses in an essential way the DG structure on the category of Voevodsky motives constructed in [BV]. The functors

\[
\text{Tot}_k \circ T^\text{Hodge}_k, R^\text{Hodge}_k \circ \text{Tot}_k : D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \rightarrow D^b(M\text{HS}^Q_1)
\]

take the abelian category \( \mathcal{M}_1(k) \otimes \mathbb{Q} \subset D^b(\mathcal{M}_1(k) \otimes \mathbb{Q}) \) to \( M\text{HS}^Q_1 \subset D^b(M\text{HS}^Q_1) \).

Using the Eilenberg-MacLane cube construction ([LP]) we define in [3.7,3.8] an isomorphism

\[
(\text{Tot}_k \circ T^\text{Hodge}_k)_{|\mathcal{M}_1(k)\otimes \mathbb{Q}} \simeq (R^\text{Hodge}_k \circ \text{Tot}_k)_{|\mathcal{M}_1(k)\otimes \mathbb{Q}}.
\]

Then, the idea is to use a categorical result from [Voe] (recalled in Theorem [4]), to deduce (1.2) from (1.5). To implement this idea we need to construct DG liftings of triangulated categories and functors appearing formula (1.4). The required work is

\footnote{As Barbieri-Viale and Kahn pointed out the existence of \( L\text{Alb}_k \) was announced by Voevodsky in his letter to Beilinson in 1992.}
done in \[\text{[2]}\] where we explain how the Hodge realization functor \(R^H_{\text{Hodge}}\) can be lifted to a DG quasi-functor to the derived DG category of Hodge structures.

Once the isomorphism \([1.2]\) is constructed we get a morphism \(\overline{L\text{Alb}}_Q \circ R^H_{\text{Hodge}} \to T^H_{\text{Hodge}} \circ L\text{Alb}_Q\) by adjunction. The computation of \(L\text{Alb}_Q(M(X))\) for smooth projective varieties from \([\text{BK1}]\) together with the Lefschetz (1, 1) Theorem completes the proof of \([1.3]\).

1.6. **Historical remarks.** The idea to use the functor \(L\text{Alb}_Q\) to give a more conceptual proof of Deligne’s conjecture is due to Barbieri-Viale and Kahn. In particular, the existence of a *canonical* isomorphism

\[
\overline{L\text{Alb}}_Q(H_i(X)) \simeq H_i(T^H_{\text{Hodge}} \circ L\text{Alb}_Q(M(X))).
\]

for a scheme \(X\) over \(\mathbb{C}\) was conjectured in \((\text{BK1}, \S 0.11)\).

A proof of Proposition \(3.1\) together with an isomorphism \(R^H_{\text{Hodge}} \circ \text{Tot}_Q \circ L\text{Alb}_Q \simeq \text{Tot}_Q \circ L\text{Alb}_Q \circ R^H_{\text{Hodge}}\) were independently obtained in \((\text{BK2}, \text{Theorems 17.2.1, 17.3.1})\).

1.7. **1-motives with integral coefficients.** The integral version of the isomorphism \([1.3]\) is more involved: the functor \(\text{Tot}_Z : D^b(\text{MHS}^Z_{\text{eff}}) \to D^b(\text{MHS}^Z_{1\text{eff}})\) does admit a left adjoint functor. To overcome this difficulty we consider the category of \(\text{MHS}^Z_{\text{en}}\), an enriched polarizable mixed Hodge structures \(V = (W^Z_0 \subset W^Z, F \subset W^Z)\) with the *weight filtration defined integrally*. An enriched Hodge structure is called effective if \(F^1 = 0\) and \(W^0 = W_2\). We introduce an exact structure on \(\text{MHS}^Z_{\text{en}}\) postulating that a sequence

\[
0 \to V^0 \to V^1 \to V^2 \to 0
\]

is exact if, for every \(i\), the sequence of pure integral Hodge structures

\[
0 \to \text{Gr}^i_W V^0 \to \text{Gr}^i_W V^1 \to \text{Gr}^i_W V^2 \to 0
\]
splits. Similarly, we define an exact structure on \(\mathcal{M}_1(k)\) postulating that a sequence

\[
0 \to [\Lambda^0, G^0, u_0] \to [\Lambda^1, G^1, u_1] \to [\Lambda^2, G^2, u_2] \to 0
\]
is exact if, for \(i = 0, -1, -2\), the sequence of pure 1-motives

\[
0 \to \text{Gr}^i_W [\Lambda^0, G^0, u_0] \to \text{Gr}^i_W [\Lambda^1, G^1, u_1] \to \text{Gr}^i_W [\Lambda^2, G^2, u_2] \to 0
\]
splits. The derived category of \(\mathcal{M}_1(k)\) equipped with the above exact structure is denoted by \(D^b(\mathcal{M}^Z_{\text{ar}}(k))\).

Let \(k \subset \mathbb{C}\) be an algebraically closed subfield. In an subsequent paper, we will construct a commutative diagram

\[
\begin{array}{ccc}
DM^Z_{\text{eff}}(k; Z) & \xrightarrow{L\text{Alb}_Z} & D^b(\mathcal{M}^Z_{\text{ar}}(k)) \\
\Big\downarrow R^H_{\text{Hodge}} & & \Big\downarrow \\
D^b(\text{MHS}_1^{Z, \text{eff}}) & \xrightarrow{L\text{Alb}_Z} & D^b(\text{MHS}_1^{Z, \text{en}}) \simeq D^b(\mathcal{M}^Z_{1\text{ar}}(\mathbb{C})).
\end{array}
\]

\[^3\]There is another much weaker exact structure on \(\mathcal{M}_1(k)\). See \((\text{BK1}, \S 1.4)\) or \(2.11\).
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2. Realizations of Voevodsky’s motives.

In this section we construct a DG structure on the category of étale Voevodsky motives and then lift Hodge and étale realization functors ([Hu1], [Hu2], [DG]) to DG quasi-functors. Our choice to work with étale motives rather than with Nisnevich ones is motivated only by possible future applications: with rational coefficients the two categories are equivalent.

2.1. DG categories. Let us begin by recalling some constructions from ([Dri], [BV] §1). Fix some universes $\mathcal{U} \in \mathcal{V}$. $\mathcal{U}$-small DG categories over commutative ring $A$ can be organized into a 2-category $\mathcal{DGcat}_\mathcal{U}$: for DG categories $\mathcal{C}_1, \mathcal{C}_2$, $\text{Mor}_{\mathcal{DGcat}_\mathcal{U}}(\mathcal{C}_1, \mathcal{C}_2)$ is the category $\mathcal{T}(\mathcal{C}_1, \mathcal{C}_2)$ of DG quasi-functors ([Dri], §16). Given $F \in \mathcal{T}(\mathcal{C}_1, \mathcal{C}_2)$ we will write $\text{Ho}(\mathcal{F}) : \text{Ho}(\mathcal{C}_1) \rightarrow \text{Ho}(\mathcal{C}_2)$ for the corresponding functor between the homotopy categories. Informally, the category of DG quasi-functors is obtained from the category of DG functors by inverting homotopy equivalences: a DG functor $\mathcal{F} : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is called a homotopy equivalence if $\text{Ho}(\mathcal{F})$ is an equivalence and, for every $X,Y \in \mathcal{C}_1$, the morphism $\text{Hom}_{\mathcal{C}_1}(X,Y) \rightarrow \text{Hom}_{\mathcal{C}_2}(\mathcal{F}(X),\mathcal{F}(Y))$ is a quasi-isomorphism. If $\mathcal{F}$ is a homotopy equivalence and $\mathcal{C}_3$ is another DG category then the composition with $\mathcal{F}$ induces an equivalence of categories:

$$\mathcal{T}(\mathcal{C}_2, \mathcal{C}_3) \sim \mathcal{T}(\mathcal{C}_1, \mathcal{C}_3).$$

A DG $\mathcal{U}$-category $\mathcal{C}$ is called cocomplete if its homotopy category is cocomplete. The DG $\mathcal{U}$-ind-completion of $\mathcal{C} \in \mathcal{DGcat}_\mathcal{U}$ will be denoted by $\mathcal{C}^{\text{perf}}_\mathcal{U}$. This is a pretriangulated cocomplete $\mathcal{V}$-small DG category satisfying the following universal property: for every pretriangulated cocomplete $\mathcal{V}$-category $\mathcal{C}' \in \mathcal{DGcat}_\mathcal{V}$:

$$\mathcal{T}^\circ(\mathcal{C}^{\text{perf}}_\mathcal{U}, \mathcal{C}') \sim \mathcal{T}(\mathcal{C}, \mathcal{C}').$$

where $\mathcal{T}^\circ(\mathcal{C}^{\text{perf}}_\mathcal{U}, \mathcal{C}')$ is the full subcategory of $\mathcal{T}(\mathcal{C}^{\text{perf}}_\mathcal{U}, \mathcal{C}')$ whose objects are quasi-functors $\mathcal{F}$ such that $\text{Ho}(\mathcal{F})$ commutes with arbitrary $\mathcal{U}$-small direct sums. The triangulated category $\text{Ho}(\mathcal{C}^{\text{perf}}_\mathcal{U})$ is the derived category of right DG modules over $\mathcal{C}$ and will be denoted by $\mathbb{D}(\mathcal{C})$.

For a class $S$ of objects of a triangulated category $D$, the triangulated subcategory $< S >$ strongly generated by $S$ is the smallest strictly full triangulated subcategory of $D$ that contains $S$. If $D$ is cocomplete the triangulated subcategory $<< S >>$ generated by $S$ is the smallest strictly full triangulated subcategory of $D$ that contains $S$ and is closed under arbitrary $\mathcal{U}$-small direct sums. We will write $D^{\text{perf}} \subset D$ for the full subcategory of compact, a.k.a. perfect, objects of $D$. For a cocomplete DG category $\mathcal{C}$ we will denote by $\mathcal{C}^{\text{perf}} \subset \mathcal{C}$ the full DG subcategory that consists of objects compact in $\text{Ho}(\mathcal{C})$.

A DG structure on a triangulated category $D$ is a pretriangulated DG category $\mathcal{C}$ together with a triangulated equivalence between $D$ and $\text{Ho}(\mathcal{C})$. A DG structure
on $D$ induces a DG structure on any strictly full triangulated subcategory $I \subset D$: take for $C_I$ the strictly full DG subcategory of $C$ whose objects belong to the essential image of the functor $I \to D \xrightarrow{\sim} \text{Ho}(\mathcal{C})$. In particular, for a class $S \subset \text{Ob}(D)$ we refer to $C_{<S>} \subset C$ (resp. $C_{<S>^+} \subset C$) as the DG subcategory strongly generated (resp. generated) by $S$.

For a DG category $C \in \text{DGcat}_U$ its pretriangulated completion $C^{\text{pretr}} \in \text{DGcat}_U$ is a full subcategory of $C$ strongly generated by $\text{Ho}(C) \subset \text{Ho}(\mathcal{C})$. The category $C^{\text{pretr}}$ has the following universal property: for every pretriangulated DG category $C' \in \text{DGcat}_U$ we have

$$T(C^{\text{pretr}}, C') \xrightarrow{\sim} T(C, C').$$

For a full subcategory $B \subset C \in \text{DGcat}_U$ the DG quotient $C/B$ is a $U$-small DG category equipped with a DG quasi-functor $C \to C/B$ satisfying the following universal property: for every DG category $C' \in \text{DGcat}_U$ the functor

$$T(C/B, C') \to T(C, C').$$

is fully faithful embedding whose essential image consists of quasi-functors $F \in T(C, C')$ such that $\text{Ho}(F)(\text{Ho}(B)) = 0$. A DG quotient $C/B$ always exists (an unique isomorphism in $\text{DGcat}_U$). In particular, given an $A$-linear exact category $E$ we can define its bounded derived DG category $D^b_{dg}(E)$ to be the DG quotient of the DG category $C_{dg}^b(E)$ of bounded complexes by the subcategory of acyclic ones ([N], §1).

We will use the following result ([Vo], Theorem 1).

**Theorem 1.** Let $E$ be a $U$-small $A$-linear exact category and $E'$ a $U$-small abelian $A$-linear category. Assume that for every two objects $X,Y \in E$, the $A$-module $\text{Hom}(X,Y)$ is flat.

1. Let

$$F : D^b_{dg}(E) \to D^b_{dg}(E')$$

be a DG quasi-functor satisfying the following property:

(P) The functors

$$H^iF : E \to D^b_{dg}(E) \xrightarrow{F} D^b_{dg}(E') \xrightarrow{H^i} E'$$

are 0 for every $i < 0$ and effaceable (i.e., for every object $X \in E$, there is an admissible monomorphism $X \hookrightarrow Y$ such that the induced morphism $H^iF(X) \to H^iF(Y)$ is 0) for every $i > 0$.

Then the functor $F := H^0F : E \to E'$ is left exact, has a right derived DG quasi-functor ([Dri], §5)

$$RF : D^b_{dg}(E) \to D^b_{dg}(E'),$$

and there is a unique isomorphism $F \simeq RF$ such that the induced automorphism $F = H^0(F) \simeq H^0(RF) = F$ equals $\text{Id}$. Conversely, the right derived DG quasi-functor of any left exact functor $F : E \to E'$ satisfies property (P).

2. For every two DG quasi-functors $F,G \in T(D^b_{dg}(E), D^b_{dg}(E'))$ satisfying property (P) and every $i < 0$

$$\text{Hom}_{T(D^b_{dg}(E), D^b_{dg}(E'))}(F,G[i]) = 0,$$

In particular, if $B$ is nonempty and $\text{Ho}(C')$ does not have a zero object $T(C/B, C')$ is empty.
\[ \text{Hom}_{\mathsf{DGcat}}(\mathcal{C}, \mathcal{C}) \mathcal{F}, \mathcal{G} = \text{Hom}_{\mathsf{Fct} (\mathcal{C}, \mathcal{C}')} (\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathsf{DGcat} (\mathcal{C}, \mathcal{C}')} (H^0 \mathcal{F}, H^0 \mathcal{G}). \]

Here \( \mathsf{Fct} (\mathcal{C}, \mathcal{C}') \) denotes the category of all \( A \)-linear functors \( \mathcal{C} \rightarrow \mathcal{C}' \).

Having the 2-category \( \mathsf{DGcat} \) we define the notion of adjoint DG quasi-functors: given \( \mathcal{F} \in \mathcal{T} (\mathcal{C}, \mathcal{C}') \) a right adjunction datum \( (\mathcal{G}, \nu, \mu) \) consists of a quasi-functor \( \mathcal{G} \in \mathcal{T} (\mathcal{C}', \mathcal{C}) \) together with morphisms \( \nu : 1d \rightarrow \mathcal{G} \circ \mathcal{F}, \mu : \mathcal{F} \circ \mathcal{G} \simeq 1d \) such that the compositions

\[
\mathcal{F} \xrightarrow{\mathcal{F}(\nu)} \mathcal{F} \circ \mathcal{G} \circ \mathcal{F} \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}
\]

\[
\mathcal{G} \xrightarrow{\mathcal{G}(\nu)} \mathcal{G} \circ \mathcal{F} \circ \mathcal{G} \xrightarrow{\mathcal{G}(\mu)} \mathcal{G}
\]

are identity morphisms. It is a general property of 2-categories that the adjunction datum if it exists is unique up to a unique isomorphism ([Ben]). If this is the case we call \( \mathcal{G} \) the right adjoint quasi-functor. Similarly, one defines the notion of left adjoint quasi-functor.

**Lemma 2.1.** Let \( \mathcal{C}, \mathcal{C}' \) be pretriangulated DG categories and let \( \mathcal{F} : \mathcal{T} (\mathcal{C}, \mathcal{C}') \) be a DG quasi-functor. Assume that \( H^0 (\mathcal{F}) \) admits a right (resp. left) adjoint functor. Then \( \mathcal{F} \) admits a right (resp. left) adjoint quasi-functor.

**Proof.** For every \( \mathcal{F} \in \mathcal{T} (\mathcal{C}, \mathcal{C}') \) the induced functor \( \mathcal{F}^* : \mathcal{C} \rightarrow \mathcal{C}' \) admits a right adjoint quasi-functor \( \mathcal{F}^*_* \in \mathcal{T} (\mathcal{C}, \mathcal{C}') \) ([Dri], §14). If \( H^0 (\mathcal{G}) \) is a right adjoint to \( H^0 (\mathcal{F}) \), for every \( X \in \mathcal{C}' \), \( H^0 (\mathcal{F}^*)_*(X) \simeq H^0 (\mathcal{G}^*)(X) \) in \( H^0 (\mathcal{C}) \). Thus \( H^0 (\mathcal{F}^*) \) takes \( H^0 (\mathcal{C}) \) into the essential image of \( H^0 (\mathcal{C}^*_* \rightarrow H^0 (\mathcal{C}^*_*) \). Left adjoint functors are treated similarly. \( \square \)

**Conventions:** Unless stated otherwise, all categories assumed to be \( \mathcal{U} \)-small. We will write \( \mathsf{DGcat} \) for \( \mathsf{DGcat}_{\mathcal{U}} \). All fields, rings, modules that we will consider assumed to be \( \mathcal{U} \)-small.

### 2.2. DG category of Étale Voevodsky motives

Let \( A \) a commutative ring and let \( k \) be a of finite étale homological dimension with respect to \( A \), i.e. there exists an integer \( N \) such that, for every discrete \( A[Gal(\overline{k}/k)] \)-module \( M \) and every integer \( m > N \),

\[
H^m (Gal(\overline{k}/k), M) = 0.
\]

Let \( Sm_k \) be the category of smooth varieties over \( k \). We will write \( A_{tr}[Sm] = A_{tr}[Sm_k] \) for the category whose objects are smooth varieties and morphisms are finite correspondences with coefficients in \( A \) ([BV], §2.1.2). This an \( A \)-linear additive category and as such it can be viewed as a DG category over \( A \). Let \( PSh_{tr} \) be \( PSh_{tr}^A(Sm_k) \) the (abelian) category of presheaves of \( A \)-modules with transfers on \( Sm_k \) i.e., the category of contravariant \( A \)-linear functors \( A_{tr}[Sm] \rightarrow \text{Mod}(A) \), and let \( D(PSh_{tr}) \) be its derived category. We endow \( D(PSh_{tr}) \) with a DG structure

\[
Ho(A_{tr}[Sm]) \simeq \mathbb{D}(A_{tr}[Sm]) \simeq D(PSh_{tr}).
\]

**Remark 2.2.** The Yoneda embedding defines a homotopy equivalence \( A_{tr}[Sm] \rightarrow D_{dg}(PSh_{tr}) \) ([BV], §1.6.4, 1.8).
If $X \in Sm_k$ we write $A_{tr}[X]$ for the presheaf represented by $X$. Consider the following types of complexes in $PSh_{tr}$:

$(\Delta)$ The 2-term complexes $A_{tr}[X \times k^1] \to A_{tr}[X]$, where $X \in Sm_k$ and the differential comes from the projection $X \times k^1 \to X$.

(Et) The complexes of the form $A_{tr}[U] \to A_{tr}[X]$, where $U \to X$, $U_i \in Sm_k$, is a hypercovering for the étale topology [EGA4, V §7.3] and $A_{tr}[U]$ is the corresponding normalized complex.

Let $I^\Delta_{tr}$, $I^\circ_{tr}$, $I^\Delta_{tr}$ be the subcategories of $A_{tr}[Sm]$ generated by the objects of the corresponding types. We define the DG category $\mathcal{DM}^{ef}_{et}(k; A)$ of effective étale Voevodsky motives to be the DG quotient of $A_{tr}[Sm]$ by $I^\Delta_{tr}$. The triangulated category $Ho(\mathcal{DM}^{ef}_{et}(k; A)) \simeq D(PSh_{tr})/Ho(I^\Delta_{tr})$ is denoted by $\mathcal{DM}^{ef}_{et}(k; A)$. By abuse of notation we shall write $A_{tr}[X]$ for the object of the quotient category $\mathcal{DM}^{ef}_{et}(k; A)$ corresponding to the presheaf $A_{tr}[X]$ (i.e. the motive of $X$). The triangulated category $\mathcal{DM}^{ef}_{gm,et}(k; A)$ of effective geometric étale motives is the smallest Karoubian complete strictly full triangulated subcategory of $\mathcal{DM}^{ef}_{et}(k; A)$ that contains all objects of the form $A_{tr}[X]$, where $X \in Sm_k$. Being a full subcategory of category equipped with a DG structure $\mathcal{DM}^{ef}_{gm,et}(k; A)$ inherits a DG structure that we denote by $\mathcal{DM}^{ef}_{gm,et}(k; A)$.

The tensor structure on the category $A_{tr}[Sm]$ given by cartesian product of schemes induces a homotopy tensor structure on $\mathcal{DM}^{ef}_{gm,et}(k; A)$, $\mathcal{DM}^{ef}_{et}(k; A)$ ([BV], §1.9, 2.3). In particular, $\mathcal{DM}^{ef}_{gm,et}(k; A)$, $\mathcal{DM}^{ef}_{et}(k; A)$ are tensor categories.

**Lemma 2.3.** $\mathcal{DM}^{ef}_{et}(k; A)_{perf} = \mathcal{DM}^{ef}_{gm,et}(k; A)$.

**Proof.** The étale coverings define a Grothendieck topology on $A_{tr}[Sm]$ ([BV], §4.3). Let $Sh^{et}_{tr} \subset PSh^{et}_{tr}$ be the category of sheaves with transfers. According to ([BV], §1.11; 4.3) the sheafification functor $PSh^{et}_{tr} \to Sh^{et}_{tr}$ induces a homotopy equivalence

$$D_{dg}(PSh_{tr})/I^\circ_{tr} \simto D_{dg}(Sh^{et}_{tr}).$$

Hence,

$$\mathcal{DM}^{ef}_{et}(k; A) \simto D_{dg}(Sh^{et}_{tr})/I^\Delta_{tr}.$$ 

Let us check that, for every $X \in Sm_k$, the sheaf $A_{tr}[X]$ is a compact object of $D(Sh^{et}_{tr})$ i.e., for every $F_i \in D(Sh^{et}_{tr})$, $i \in I$, and every integer $m$, the canonical morphism

$$\bigoplus_I \text{Hom}_{D(Sh^{et}_{tr})}(A_{tr}[X], F_i[m]) \simeq \bigoplus \text{R}^m\Gamma(X, F_i) \to \text{R}^m\Gamma(X, \bigoplus F_i)$$

is an isomorphism. If each $F_i$ is a single sheaf the claim follows from ([EGA4], VII, 3.3) (and holds for any field $k$). On the other hand, under (2.4), for every sheaf $F$ of $A$-modules and every integer $l > N + 2dim X + 1 =: N'$, $H^l_{et}(X, F) = 0$ ([EGA4], X, 4.3). Thus, replacing each $F_i$ by its canonical truncation we may assume that $F_i$ is supported in homological degrees between $m - N'$ and $m$. Our claim follows by dévissage.

In particular, the subcategory $Ho(I^\Delta_{tr}) \subset D(Sh^{et}_{tr})$ is generated by compact objects. Hence, by ([BV], §1.4.2, Proposition (ii)) the sheaves $A_{tr}[X]$ remain compact in the quotient category $D(Sh^{et}_{tr})/Ho(I^\Delta_{tr})$. Part (i) of the same proposition completes the proof. □
Corollary 2.4. The category $DM^{eff}_{et}(k; A)$ is compactly generated.

Indeed, the category $DM^{eff}_{et}(k; A)$ is generated by motives of smooth varieties which are compact by the lemma.

Corollary 2.5. If $(DM^{eff}_{gm,et}(k; Z) \otimes Q)^{\kappa}$ denotes the homotopy idempotent completion ([BV], §1.6.2) of $DM^{eff}_{gm,et}(k; Z) \otimes Q$, the canonical morphism

$$(DM^{eff}_{gm,et}(k; Z) \otimes Q)^{\kappa} \to DM^{eff}_{gm,et}(k; Q)$$

is a homotopy equivalence.

Proof. It suffices to check that for every $X, Y \in Sm$ the map

$$Hom_{DM^{eff}_{et}(k; Z)}(Z_{tr}[X], Z_{tr}[Y]) \otimes Q \to Hom_{DM^{eff}_{et}(k; Z)}(Z_{tr}[X], Q_{tr}[Y])$$

is an isomorphism. Write $Q_{tr}[Y]$ as the direct limit of the diagram $Z_{tr}[Y] \to Z_{tr}[Y] \to Z_{tr}[Y] \to \cdots$. Since $Z_{tr}[X]$ is perfect, $Hom_{DM^{eff}_{et}(k; Z)}(Z_{tr}[X], Q_{tr}[Y])$ is the direct limit of

$$Hom_{DM^{eff}_{et}(k; Z)}(Z_{tr}[X], Q_{tr}[Y]) \to Hom_{DM^{eff}_{et}(k; Z)}(Z_{tr}[X], Z_{tr}[Y]) \to \cdots.$$ 

\[\square\]

Remark 2.6. Voevodsky’s category $DM^{eff}_{et}(k; A)$ is the image $D^-(Sh^{et}_{tr})$ in $DM^{eff}_{et}(k; A)$.

Replacing in the definition of $DM^{eff}_{et}(k; A)$ étale hypercoverings by Nisnevich ones we obtain the DG category $DM^{eff}(k; A)$ of effective Voevodsky motives ([BV]). The DG quasi-functor

$$DM^{eff}(k; A) \to DM^{eff}_{et}(k; A)$$

is a homotopy equivalence for every $A \subset Q$. This is derived from the equivalence $Sh^{et}_{tr}(Sm_k) \sim Sh^{Nis}_{tr}(Sm_k)$ ([V1], §3.3).

2.3. Base change. Let $k \subset k'$ be a field extension. The functor

$$(2.6) \quad A_{tr}[Sm_k] \to A_{tr}[Sm_{k'}]$$

that takes a smooth $k$-scheme $X$ to $k'$-scheme $X \times_{spec \ k} spec \ k'$ induces DG quasi-functors

$$(2.7) \quad f^*: DM^{eff}(k; A) \to DM^{eff}(k'; A), \quad f^*: DM^{eff}_{et}(k; A) \to DM^{eff}_{et}(k'; A).$$

We want to describe explicitly the effect $f^*$ on a complex of presheaves with transfers. Consider the left Kan extension of $f^*$

$$f^{-1} : PSh_{tr}(Sm_k) \to PSh_{tr}(Sm_{k'}).$$

If $E \in PSh_{tr}(Sm_k)$ is a presheaf on the category $A_{tr}[Sm_k]$, then

$$f^{-1} E(X) = \colim_{(Y, g \in cor_k(X, Y))} F(Y),$$

where $X$ is a smooth scheme over $k'$ and the colimit is taken over the category $C$ all pairs $(Y, g \in cor_k(X, Y) = cor_k(X, Y \times spec \ k' spec \ k'))$ with $Y \in A_{tr}[Sm_k'; Hom_{C}((Y, g), (Y', g')) = \{ h \in cor_k(Y, Y') : h \circ g = g' \}$. One can easily check that the category $C$ is cofiltrant ([KS], §3.1). As a consequence, the functor $f^{-1}$ is exact.
Remark 2.7. The subcategory $C' \subset C$ of pairs $(Y, g)$ where $Y \in Sm_k, \ g$ is a morphism $X \to Y$ of $k$-schemes, and $\text{Hom}_{C'}((Y, g), (Y', g')) = \{ h \in Mor_k(Y, Y') : h \circ g = g' \}$ is co-cofinal in $C ([KS], \S2.5)$. In particular,

$$f^{-1}E(X) = \text{colim}_{g : X \to Y} F(Y).$$

As an exact functor $f^{-1}$ extends in the obvious way to a DG quasi-functor between the DG derived categories of presheaves and this extension fits into the following commutative diagram

$$\begin{array}{ccc}
DM^{eff}_c(k; A) & \leftarrow & A_{tr}[Sm_k] \cong D_d(PSh_{tr}(Sm_k)) \\
\downarrow f^* & & \downarrow f^{-1} \\
DM^{eff}_c(k'; A) & \leftarrow & A_{tr}[Sm_{k'}] \cong D_d(PSh_{tr}(Sm_{k'})),
\end{array}$$

where $\circ = et$ or $Nis$, the left vertical arrow is induced by (2.6), and the horizontal homotopy equivalences are induced by the Yoneda embeddings.

2.4. **h-topology.** We shall recall below a different description of the category $DM^{eff}_c(k; A)$ that does not involve finite correspondences. This description will be used in our construction of the realization functors. In the rest of this section char $k = 0$.

Let $A[Sm_k] = A[Sm]$ be the category, whose objects are smooth varieties over $k$ and whose morphisms are defined by the formula

$$\text{Hom}_{A[Sm]}(A[U], A[Y]) = \oplus_i A[Mor(X_i, Y)],$$

where $X_i$ are connected varieties. Denote by $PSh$ the category of contravariant $A$-linear functors $A[Sm] \to Mod(A)$. Consider the pair of adjoint DG quasi-functors

$$\Phi^* : A[Sm] \to A_{tr}[Sm],$$

$$\Phi_* : A_{tr}[Sm] \to A[Sm]$$

induced by $\Phi : A[Sm] \to A_{tr}[Sm]$. The triangulated functor

$$\text{Ho}(\Phi_*) : \text{Ho}(A_{tr}[Sm]) \simeq D(PSh_{tr}) \to D(PSh) \simeq \text{Ho}(A[Sm])$$

is induced by the forgetful functor $PSh_{tr} \to PSh$; $\text{Ho}(\Phi^*)$ is left adjoint to $\text{Ho}(\Phi_*)$.

Let $I^h \subset A[Sm]$ (resp. $I^h_{tr} \subset A_{tr}[Sm]$) be the DG subcategory generated by objects of the form

(h) $A[U] \to A[X]$ (resp. $A_{tr}[U] \to A_{tr}[X]$),

where $U \to X, U_i \in Sm_k, \ i$ is a hypercovering for the h-topology ([SV], §10), and let $I^{h, \Delta} \subset A[Sm]$ (resp. $I^{h, \Delta}_{tr} \subset A_{tr}[Sm]$) be the subcategory generated by $I^h$ (resp. $I^h_{tr}$) and objects of the form $A[X \times \Delta^1] \to A[X]$ (resp. of type $(\Delta)$.) The functor $\Phi^*$ takes $I^h$ into $I^h_{tr}$ and, hence, descends to a DG quasi-functor:

$$\mathcal{F}^* : A[Sm]/I^h \to A_{tr}[Sm]/I^h_{tr}.$$

The following proposition is essentially a reformulation of a result proven by Voevodsky ([VI], Theorem 4.1.12).
Proposition 2.8. a) The quasi-functor $\Phi$ is a homotopy equivalence.
b) The composition $Ho(I^h_{tr}) \to D(PSh_{tr}) \to DM^eff_{et}(k; A)$ is zero.
c) The functor $\Phi$ yields a homotopy equivalences:
$$A[Sm]/I^h,\Delta \xrightarrow{\sim} A_{tr}[Sm]/I^h_{tr},\Delta \xrightarrow{\sim} DM^eff_{et}(k; A).$$

Proof. We will derive the proposition from Voevodsky’s results. Let $Sh$ be the category of sheaves of $A$-modules on smooth schemes equipped with the $h$-topology, and let $A^h[X]$ be the $h$-sheaf associated with $A[X]$. Then

$$A^h[X] \xrightarrow{\sim} A_{tr}[X].$$

This follows by combining (V2, Theorem 3.3.5, Proposition 3.3.6) and (BV, 2.1.3) with the observation that the category of $h$-sheaves on smooth schemes is equivalent to the one on all $k$-schemes of finite type. The last assertion holds because every scheme of finite type over $k$ admits a smooth $h$-cover.

It follows from (2.8) that the forgetful functor $Ho(\Phi_*) : D(PSh_{tr}) \to D(PSh)$ takes every complex in $Ho(I^h_{tr})$ to an object of $Ho(I^h)$ (BV, 1.11). Consider the induced functor:

$$Ho(\Phi_*) : D(PSh_{tr})/Ho(I^h_{tr}) \to D(PSh)/Ho(I^h).$$

We claim that $Ho(\Phi_*)$ and $Ho(\Phi)$ are inverse one to the other. Indeed, (2.8) implies that $Ho(\Phi_*) \circ Ho(\Phi) \simeq Id$. Next, since $Ho(\Phi_*)$ is left adjoint to $Ho(\Phi_*)$, $Ho(\Phi)$ is left adjoint to $Ho(\Phi)$. This implies that $Ho(\Phi_*)$ is fully faithful. Since $Ho(\Phi_*)$ is obviously surjective, the first part of the proposition is proved. Part b) follows from (VI, Theorem 4.1.12). The last part is a corollary of a) and b).

2.5. Compactifications. Let $A[Sm]$ be the category whose objects are triples $(X, \overline{X}, j)$, where $\overline{X}$ is smooth proper scheme over $\mathbb{C}$ and $j : X \to \overline{X}$ is an open dense embedding such that the complement $\overline{X} \setminus X$ is a divisor with normal crossings. The space of morphisms between two such triples $(X, \overline{X}, j)$ and $(X', \overline{X}', j')$ with connected $\overline{X}$ is freely generated over $A$ by pairs $(f : X_i \to X', g : \overline{X}_i \to \overline{X}')$ such that $j'f = gj$. In general,

$$Hom_{A[Sm]}(A[(\cup X_i, \cup \overline{X}_i, j)], X') = \oplus Hom_{A[Sm]}(A[(X_i, \overline{X}_i, j)], X'),$$

where $\overline{X}_i$ are connected and $X' = A[(X', \overline{X}', j')]$. Let $N$ be the DG subcategory of $A[Sm]$ generated by complexes of the form

$$(N) \xrightarrow{(id, g)} A[X, \overline{X}, j].$$

The functor $A[Sm] \to A[Sm]$ that takes a triple $(X, \overline{X}, j)$ to $X$ induces a DG quasi-functor

$$(2.9) \xrightarrow{A[Sm]/N \to A[Sm]}$$

Lemma 2.9. The functor $(2.9)$ is a homotopy equivalence.

Proof. Denote by $S \subset Mor(A[Sm])$ the set of morphisms of the form

$$(2.10) \xrightarrow{(id, g)} A[X, \overline{X}, j].$$

It follows from the Hironaka theorem on resolution of singularities that the set $S$ is a left multiplicative system and that, if $A[Sm]_S$ is the localization of $A[Sm]$ by $S$ (KS, §7), the functor $A[Sm]_S \to A[Sm]$ induced by $A[Sm] \to A[Sm]$ is an equivalence of
categories. The rest of the proof is a bit of abstract nonsense. It will suffice to prove
that for any additive category $\mathcal{A}$ and a left multiplicative system $\mathcal{S} \subset Mor(\mathcal{A})$ the
functor
$$T : Ho(\mathcal{A}/N) \to Ho(\mathcal{A}/S)$$
is an equivalence of triangulated categories. Here $N$ denotes the DG subcategory of $\mathcal{A}$
generated by complexes of the form $X' \to X$, where $X, X' \in \mathcal{A}$ and $s \in \mathcal{S}$. For
every $X, Y \in \mathcal{A}, i \in \mathbb{Z}$ we have
$$Hom_{Ho(\mathcal{A}/N)}(X,Y[i]) \cong Hom_{Ho(\mathcal{A}_{\text{perf}}/N_{\text{perf}})}(X,Y[i]) \cong \text{colim} \ Hom_{Ho(\mathcal{A})}(C,Y[i]),$$
where the colimit is taken over the cofiltrant category $Q$ of pairs $(C \in Ho(\mathcal{A}_{\text{perf}}), g : C \to X)$
with $\text{cone}(g) \in Ho(N_{\text{perf}})$. The subcategory $Q' \subset Q$ formed $(X' \in \mathcal{A}, s : X' \to X)$ with $s \in \mathcal{S}$ is cofinal in $Q$. Hence
$$\text{(2.1)}
\begin{equation}
Hom_{Ho(\mathcal{A}/N)}(X,Y[i]) \cong \text{colim} \ Hom_{Ho(\mathcal{A})}(X', Y[i]) \to Hom_{Ho(\mathcal{A}/S)}(X,Y[i]).
\end{equation}
$$
(In particular, all groups are 0 for $i \neq 0$.) Since objects of $\mathcal{A}$ generate $Ho(\mathcal{A})$,
$Ho(\mathcal{A}/N)$ and are compact in the both categories ([BV], Proposition 1.4.2) formula
\text{(2.1)} implies that $T$ is fully faithful.

It remains to check that $T$ is essentially surjective. Indeed, $T$ provides an equivalence
of $Ho(\mathcal{A}/N)$ with a full triangulated subcategory of $Ho(\mathcal{A}/S)$ which is closed
under under small direct sums and contains $\mathcal{A}_S$. Every subcategory with these
properties coincides with $Ho(\mathcal{A}_S)$. \hfill \Box

2.6. DG realizations of $\mathcal{D}M_{eff}^{cyl}(k; A)$. Let $\mathcal{C}$ be a pretriangulated cocomplete DG
category over $A$, and let $\mathcal{F} : A[Sm] \to \mathcal{C}$ (resp. $\mathcal{F} : A[Sm] \to \mathcal{C}$) be a DG quasi-
functor. According to formula \text{(2.1)} $\mathcal{F}$ extends uniquely (up to a unique isomorphism)
to a DG quasi-functor $\tilde{\mathcal{F}} : A[Sm] \to \mathcal{C}$ (resp. $\tilde{\mathcal{F}} : A[Sm] \to \mathcal{C}$) such that $Ho(\tilde{\mathcal{F}})$
commutes with arbitrary direct sums. Let $\mathcal{T}^h,\Delta(A[Sm], \mathcal{C})$ (resp. $\mathcal{T}^h,\Delta,\mathcal{N}(A[Sm], \mathcal{C})$ )
be a full subcategory of $\mathcal{T}(A[Sm], \mathcal{C})$ (resp. $\mathcal{T}(A[Sm], \mathcal{C})$) whose objects are quasi-
functors $\mathcal{F}$ such that $Ho(\tilde{\mathcal{F}})$ takes complexes of type $(h), (\Delta)$ (resp. $(h), (\Delta), (\mathcal{N})$) to
0 in $Ho(\mathcal{C})$. The following result is the basic tool for constructing various realization
functors.

\textbf{Theorem 2.} The functor $\overline{\Phi}$ induces an equivalence of triangulated categories
$$\mathcal{T}^c(\mathcal{D}M_{ct}^{cyl}(k; A), \mathcal{C}) \xrightarrow{\sim} \mathcal{T}^h,\Delta(A[Sm], \mathcal{C}) \xrightarrow{\sim} \mathcal{T}^h,\Delta,\mathcal{N}(A[Sm], \mathcal{C})$$
where $\mathcal{T}^c(\mathcal{D}M_{ct}^{cyl}(k; A), \mathcal{C})$ is the full subcategory of $\mathcal{T}(\mathcal{D}M_{ct}^{cyl}(k; A), \mathcal{C})$ formed by
quasi-functors $\mathcal{F}$ such that $Ho(\mathcal{F})$ commutes with arbitrary direct sums.

\textbf{Proof.} This is an immediate corollary of part c) of Proposition \text{[2,4]} \hfill \Box

2.7. Betti realization. $k = \mathbb{C}$. Let $D_{dg}(Mod(A))$ be the derived DG category of
$A$-modules i.e., the DG quotient of the category $C(Mod(A))$ of unbounded complexes
of $A$-modules by the subcategory of acyclic ones. We apply Theorem\text{[2]} to the quasi-
functor $C_A^{sing} : A[Sm] \to C(Mod(A)) \to D_{dg}(Mod(A))$ that takes a smooth variety $X$
to its singular chain complex $C^{sing}(X(\mathbb{C})) \otimes A$. Since singular homology are homotopy
invariant and satisfy the descent property for the $h$-topology $C_A^{sing}$ yields a DG quasi-
functor
$$R_A^{Betti} : \mathcal{D}M_{ct}^{cyl}(\mathbb{C}; A) \to D_{dg}(Mod(A)).$$
2.8. Hodge realization. Let $\mathbb{Z} \subset A \subset \mathbb{Q}$ be a subring, and let $MHS^A$ be the category of polarizable mixed $A$-Hodge structures ([D3]). We shall say that a Hodge structure $(V_A, W, V_Q, F^- \subset V_C) \in MHS^A$ is effective if

$$F^1 = 0.$$  

(2.12)

Note that the condition (2.12) implies that $W_0 = V_Q$.

Denote by $MHS^A_{\text{eff}} \subset MHS^A$ the full subcategory of effective Hodge structures. We shall construct a DG quasi-functor,

$$R^b_{\text{Hodge}}: DM_{gm,et}^A(\mathbb{C}; A) \to D^b_{dg}(MHS^A_{\text{eff}}),$$

(2.13)

together with an isomorphism of quasi-functors $F \circ R^b_{\text{Hodge}} \simeq R^b_{\text{Betti}}: DM_{gm,et}^A(\mathbb{C}; A) \to D^b_{dg}(\text{Mod}(A))$.

Here $F : D^b_{dg}(MHS^A_{\text{eff}}) \to D^b_{dg}(\text{Mod}(A))$ is the forgetful functor.

Beilinson associated with every variety $X$ over $\mathbb{C}$ an element of the derived category $D^b(MHS^A)$ whose cohomology are Deligne’s mixed Hodge structures on (co)homology groups of $X$. In fact, his construction gives a DG quasi-functor $A[\mathfrak{m}] \to D^b_{dg}(MHS^A)$. Let us explain this. In ([Bei], §3.9-4.1) Beilinson introduced auxiliary triangulated categories $K^b_{\text{H}_p}, D^b_{\text{H}_p}$ of $\hat{p}$-Hodge complexes. We define DG structures on these categories: consider a DG category $C^b_{\text{H}_p}$ whose objects are $\hat{p}$-Hodge complexes and the group $\text{Hom}((\tilde{\mathcal{F}}, \delta), (\tilde{\mathcal{G}}, \delta))$ of degree $n$ morphisms between $\hat{p}$-Hodge complexes

$$(\tilde{\mathcal{F}}, \delta) = \tilde{\mathcal{F}}_A \overset{\alpha}{\leftarrow} \tilde{\mathcal{F}}_Q \overset{\beta}{\leftarrow} (\tilde{\mathcal{F}}_Q, \tilde{W}_Q) \overset{\gamma}{\leftarrow} (\tilde{\mathcal{F}}_Q, \tilde{W}_Q) \overset{\delta}{\leftarrow} (\tilde{\mathcal{F}}_Q, \tilde{W}_Q)$$

(\tilde{\mathcal{G}}, \delta) = \tilde{\mathcal{G}}_A \overset{\alpha}{\leftarrow} \tilde{\mathcal{G}}_Q \overset{\beta}{\leftarrow} (\tilde{\mathcal{G}}_Q, \tilde{W}_Q) \overset{\gamma}{\leftarrow} (\tilde{\mathcal{G}}_Q, \tilde{W}_Q) \overset{\delta}{\leftarrow} (\tilde{\mathcal{G}}_Q, \tilde{W}_Q)$$

is a subgroup of $\prod \text{Hom}(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ formed by all $\tilde{h} \in \prod \text{Hom}(\tilde{\mathcal{F}}, \tilde{\mathcal{G}})$ such that

a) $h$ commutes with $\alpha, \beta, \gamma$ and $\delta$ (but not necessarily with the differential)

b) $h$ preserves the Hodge filtration

c) $h$ shifts the weight filtration: $h(\tilde{W}_Q) \subset \tilde{W}_Q$.

d) $d_{\delta} \circ h = (-1)^n h \circ d_F$.

The formula $d(h) := d_{\delta} \circ h = (-1)^n h \circ d_F$ defines a differential on $\text{Hom}((\tilde{\mathcal{F}}, \delta), (\tilde{\mathcal{G}}, \delta))$. By definition, Beilinson’s category $C^b_{\text{H}_p}$ is the homotopy category of $C^b_{\text{H}_p}$. We have a DG functor

$$C^b(MHS^A) \to C^b_{\text{H}_p}$$

(2.14)

that takes a complex $V^- = (V_A, W, V_Q, F^- \subset V_C)$ of Hodge structures to the $\hat{p}$-Hodge complex $\tilde{\mathcal{F}}$ with $\tilde{\mathcal{F}}_A = V_A$, $\tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q = \tilde{\mathcal{F}}_Q$, and $W_i(\tilde{\mathcal{F}}_Q) = W_{i+j}(V_Q)$.

Denote by $D^b_{dg} C^b_{\text{H}_p}$ the DG quotient of $C^b_{\text{H}_p}$ by the subcategory acyclic complexes.

Then the functor (2.14) descends to a homotopy equivalence ([Bei], Lemmas 2.1 and 3.11):

$$D^b_{dg}(MHS^A) \sim D^b_{dg} C^b_{\text{H}_p}.$$

\[\text{Warning: a } \hat{p}\text{-Hodge complex } (\tilde{\mathcal{F}}, \delta) \text{ is called acyclic if } \tilde{\mathcal{F}}_A \text{ is acyclic. The subcomplexes } \tilde{W}_Q \text{ need not be acyclic.}\]
Next, with every \((X, \overline{X}, j) \in A[\overline{Sm}]\) Beilinson associated a canonical \(\tilde{p}\)-Hodge complex \(\mathcal{D}[X, \overline{X}]\). Set \(\bar{R}^{Hoddge}_A((X, \overline{X}, j)) = \mathcal{H}om(\mathcal{D}[X, \overline{X}], A)\). We obtain DG quasi-functors:

\[
\bar{R}^{Hoddge}_A : A[\overline{Sm}] \to C^b_{DG},
\]

\[
\bar{R}^{Hoddge}_A : A[\overline{Sm}] \to D_{dg}(MHS^A)^{\sim} \xrightarrow{P} D_{dg}(Ind(MHS^A)),
\]

where \(Ind(MHS^A)\) denotes the ind-completion of \(MHS^A\) and \(P\) is induced (via (2.1)) by the embedding \(MHS^A \to Ind(MHS^A)\). By construction, the composition of \(\bar{R}^{Hoddge}_A\) with the forgetful functor \(F : D_{dg}(Ind(MHS^A)) \to D_{dg}(Mod(A))\) is isomorphic to \(A[\overline{Sm}] \to D_{dg}(Ind(MHS^A))\). Now \(Ho(\bar{R}^{Hoddge}_A)\) carries complexes of type \((h), (\Delta), (N)\) to 0, because \(Ho(F)\) is conservative. Theorem \(\mathbf{2}\) gives

\[
R^{Hoddge}_A : DM_{et}^{eff}(\mathbb{C}; A) \to D_{dg}(Ind(MHS^A)).
\]

The functor \(Ho(R^{Hoddge}_A)\) takes every geometric motive to the essential image of the fully faithful embedding \(D_{b}(MHS^A) \to D(Ind(MHS^A))\). This yields

\[
R^{Hoddge}_A : DM_{et}^{eff}(\mathbb{C}; A) \to D_{dg}(MHS^A).
\]

We will show in Proposition \(\mathbf{3}\) that the functor \(D_{b}(MHS^A_{eff}) \to D_{b}(MHS^A)\) induced by the embedding \(MHS^A_{eff} \hookrightarrow MHS^A\) is fully faithful. Since the mixed Hodge structures on the homology groups of any variety are effective, it follows that the DG quasi-functor \(R^{Hoddge}_A\) factors uniquely through \(D_{dg}(MHS^A_{eff})\). This gives (2.13).

If \(k \subset \mathbb{C}\), we will write \(R^{Hoddge}_A\) for the composition

\[
(2.15) \quad DM_{et}^{eff}(k; A) \xrightarrow{f^*} DM_{et}^{eff}(\mathbb{C}; A) \to D_{dg}(MHS^A_{eff}),
\]

where \(f^*\) is the base change functor from (2.3).

2.9. Étale realization. Fix an algebraic closure \(k \subset \overline{k}\). Denote by \(Mod(G, A)\) the category of discrete \(A\)-modules over the Galois group \(G := Gal(\overline{k}/k)\) and by \(D_{dg}(G, A)\) its derived DG category. Let \(C^A : D_{dg}(PSH^A_{et}(\text{Sm}_{\overline{k}})) \to D_{dg}(PSH^A_{et}(\text{Sm}_{\overline{k}}))\) be the \(\mathbb{A}^1\)-homotopy localization endofunctor (BV, §3.1.1). Consider a DG quasi-functor

\[
\tilde{S}_A : D_{dg}(PSH^A_{et}(\text{Sm}_{\overline{k}})) \to D_{dg}(G, A)
\]

that takes a complex of presheaves \(F\) to

\[
\tilde{S}_A(F) = C^A(F)\text{(spec } k') := \underset{k' \subset \overline{k}}{\colim} C^A(F)\text{(spec } k'),
\]

where \(k'\) runs through all finite extensions of \(k\) in \(\overline{k}\). According to the Suslin-Voevodsky theorem (SM) for \(X\) one has a canonical isomorphism of \(G\)-modules

\[
(2.16) \quad H_i(\tilde{S}_{Z/n}(\mathbb{Z}/n\mathbb{V}[X])) \simeq H^\ell_i(X \times \text{spec } \overline{k}, \mathbb{Z}/n).
\]

Here is a more precise statement.

---

7Recall that a \(G\)-module \(M\) is called discrete if the stabilizer of every element \(m \in M\) is open in \(G\).
Proposition 2.10. We have
\begin{equation}
Ho(\tilde{S}_A)(Ho(I^{et,\Delta}_{tr})) = 0.
\end{equation}
Thus, \(\tilde{S}_A\) induces a DG quasi-functor
\[ S_A : DM^{eff}_{et}(k;A) \to D_{dg}(G,A). \]
The functor \(S_{\mathbb{Z}/n}\) is a homotopy equivalence:
\begin{equation}
Ho(S_{\mathbb{Z}/n}) : DM^{eff}_{et}(k;\mathbb{Z}/n) \xrightarrow{\sim} D_{dg}(G,\mathbb{Z}/n).
\end{equation}
Proof. Assume, first, \(A = \mathbb{Z}/n\). In this case, formula (2.17) is proved in ([MVW], Theorem 9.35). It is also shown there that \(Ho(\tilde{S}_{\mathbb{Z}/n})\) induces an equivalence of the bounded from above categories:
\[ DM^{eff}_{et}(k;\mathbb{Z}/n) \xrightarrow{\sim} D^-(G,\mathbb{Z}/n). \]
This together with (2.16) implies
\[ Ho(S_{\mathbb{Z}/n}) : DM^{eff}_{et}(k;\mathbb{Z}/n)^{perf} \xrightarrow{\sim} D(G,\mathbb{Z}/n)^{perf}. \]
Since \(Ho(S_{\mathbb{Z}/n})\) commutes with arbitrary direct sums the equivalence (2.18) follows from Corollary 2.4.

Let us prove the first claim of the proposition. It suffices to check that \(Ho(\tilde{S}_A)(M) = 0\), for every complex \(M \in Ho(I^{et,\Delta}_{tr})\) of type \((\Delta)\) and of type \((Et)\) (with \(A = \mathbb{Z}\)). In either case \(M\) is a complex of torsion-free presheaves. Consider the distinguished triangle
\[ Ho(\tilde{S}_A)(M) \otimes \mathbb{Q}/\mathbb{Z}[-1] \to Ho(\tilde{S}_A)(M) \to Ho(\tilde{S}_A)(M) \otimes \mathbb{Q} \to Ho(\tilde{S}_A)(M) \otimes \mathbb{Q}/\mathbb{Z}. \]
The complex \(Ho(\tilde{S}_A)(M) \otimes \mathbb{Q}\) is acyclic by ([MVW], Theorem 9.35). The complex \(Ho(\tilde{S}_A)(M) \otimes \mathbb{Q}/\mathbb{Z}\) is a colimit of acyclic complexes \(Ho(\tilde{S}_A)(M) \otimes \mathbb{Q}/\mathbb{Z}\) over a filtrant category. This implies that \(Ho(\tilde{S}_A)(M)\) is acyclic.

Let \(l\) be a prime number. Denote by \(Mod(G,\mathbb{Z}/l')\) the abelian category whose objects are inverse systems \(\cdots \to N_i \xrightarrow{\phi_i} N_{i-1} \to \cdots \xrightarrow{\phi_1} N_1\), where \(N_i \in Mod(G,\mathbb{Z}/l')\) and \(\phi_i\) are morphisms of \(G\)-modules. A morphism \(N \to N'\) is a compatible system of homomorphisms \(N_i \to N'_i\). Denote and \(D_{dg}(G,\mathbb{Z}/l')\) the derived DG category of \(Mod(G,\mathbb{Z}/l')\). Let \(Mod(G,\mathbb{Z}) \to Mod(G,\mathbb{Z}/l')\) be the functor that takes \(N \in Mod(G,\mathbb{Z})\) to the inverse system \(N := N \otimes \mathbb{Z}/l'\), and let
\[ l \otimes \mathbb{Z}/l' : D_{dg}(G,\mathbb{Z}) \to D_{dg}(G,\mathbb{Z}/l') \]
be its left derived DG quasi-functor ([Dri], §5). Consider the composition of quasi-functors
\begin{equation}
R^t_{\mathbb{Z}/l} : DM^{eff}_{et}(k;\mathbb{Z}) \xrightarrow{S_A} D_{dg}(G,\mathbb{Z}) \xrightarrow{l \otimes \mathbb{Z}/l'} D(G,\mathbb{Z}/l').
\end{equation}

Proposition 2.11. There is a morphism of DG quasi-functors
\begin{equation}
S_{\mathbb{Z}} \to R^B_{\mathbb{Z}} : DM^{eff}_{et}(\mathbb{C};\mathbb{Z}) \to D_{dg}(Mod(\mathbb{Z}))
\end{equation}
such that for every integer \(n\) the induced morphism
\[ S_{\mathbb{Z}} \otimes \mathbb{Z}/n \to R^B_{\mathbb{Z}} \otimes \mathbb{Z}/n\]
is an isomorphism. In particular,

\[ R_{Z/l}^t \simeq R_{Z}^{Betti} \otimes \mathbb{Z}/l'. \]

**Proof.** For the first statement, according to Theorem 2 it suffices to construct a morphism between DG quasi-functors

\[ \tilde{S}_{Z} \to C_{Z}^{sing} : Z[Sm] \to D_{dg}(Mod(Z)). \]

Consider a third functor \( C_{DT} : Z[Sm] \to D_{dg}(Mod(Z)) \) that takes a variety \( X \) to its Dold-Thom complex \( \text{Hom}_{top}(\Delta_{top} \cup q S^dX(\mathbb{C}))^+ \) (SV, §1). We have canonical morphisms

\[ C_{Z}^{sing} \to C_{DT} \leftarrow \tilde{S}_{Z} \]

with the first arrow being a quasi-isomorphism ([DT]; [SV], Theorem 8.2). This yields (2.20).

The second arrow in (2.22) induces a quasi-isomorphism \( S_{Z}^{L} \otimes \mathbb{Z}/n \to C_{DT}^{L} \otimes \mathbb{Z}/n \) by a key result of Suslin and Voevodsky ([SV], Theorem 8.3).

Let \( \text{Mod}(G, Z_l) \) be the category of all \( Z_l \)-modules over \( G \), and let

\[ \lim : \text{Mod}(G, Z_l') \to \text{Mod}(G, Z_l) \]

be the inverse limit functor.

**Corollary 2.12.** For every geometric motive \( M \in DM^{eff}_{gm, et}(C; Z) \) and integer \( i \) one has a natural isomorphism

\[ \lim H_i(R_{Z/l}^t (M)) \simeq H_i(R_{Z}^{Betti}(M)) \otimes \mathbb{Z}/l. \]

Indeed, for every geometric motive \( M \), the complex \( R_{Z}^{Betti}(M) \) is perfect, i.e. quasi-isomorphic to a finite complex of finitely generated free abelian groups. Hence, by ([AM], pp 107-109)

\[ \lim H_i(R_{Z}^{Betti}(M) \otimes \mathbb{Z}/l') \simeq H_i(R_{Z}^{Betti}(M)) \otimes \mathbb{Z}/l. \]

**2.10. Relation to Huber’s approach.** Huber’s Hodge realization functor ([Hu1], Corollary 2.3.5) is the dual to the functor \( Ho(R_{Q}^{Hodge}) \) defined above. This can be seen as follows. By Proposition 2.8 the functor

\[ Ho(\Phi^*) : Ho(\mathbb{Q}[Sm]) \to DM^{eff}(k; \mathbb{Q}) \]

induced by \( \mathbb{Q}[Sm] \to \mathbb{Q}_{tr}[Sm] \) exhibits \( DM^{eff}(k; \mathbb{Q}) \) as a quotient of \( Ho(\mathbb{Q}[Sm]) \). Thus, giving an isomorphism between two triangulated realization functors on \( DM^{eff}(k; \mathbb{Q}) \) is equivalent to giving an isomorphism between their compositions with (2.23). The comparison result follows by inspection.

**Remark 2.13.** To construct the Hodge realization functor, Huber defined in ([Hu2]) a functor \( Ho(\Phi^*) : DM^{eff}(k; \mathbb{Q}) \to DS_{mk} \), where \( DS_{mk} \) is a certain quotient of the category \( Ho(\mathbb{Q}[Sm]) \) such that \( Ho(\Phi^*) \circ Ho(\Phi^*) \) is the projection. In Remark on p. 197 of loc. cit. she conjectured that \( Ho(\Phi^*) \) is an equivalence of categories. In fact, her conjecture follows from ([SV], Lemma 5.16). Note that Huber does not use explicitly Voevodsky’s results on \( h \)-topology. The integral theory (i.e., \( Ho(R_{Z}^{Hodge}) \)) is missing in ([Hu1], [Hu2]).
2.11. 1-motives. All the results and constructions in this subsection are borrowed from [BK1], [D3], [O]. Let \( k \) be a field of characteristic 0. A 1-motive over \( k \) is a complex of group schemes

\[
M = [\Lambda \xrightarrow{u} G],
\]

where \( \Lambda \) is a \( k \)-lattice (i.e., a group scheme such that \( \Lambda(\overline{k}) \) is a discrete \( Gal(\overline{k}/k) \)-module isomorphic to \( \mathbb{Z}^r \) as an abelian group and the canonical morphism \( \Lambda(\overline{k}) \times spec \ k \rightarrow \Lambda \) is an isomorphism) and \( G \) is a semi-abelian \( k \)-scheme. Morphisms between 1-motives are given by commutative squares. The (additive) category of 1-motives is denoted by \( \mathcal{M}_1 = \mathcal{M}_1(k) \). The category \( \mathcal{M}_1 \otimes \mathbb{Q} \) is abelian.

For an abelian group scheme \( H \) over \( k \) we denote by \( \underline{H} \) the corresponding étale sheaf on \( Spec \ k \)

\[
\underline{H}(X) := Hom(X, H).
\]

If the neutral component \( H^0 \) is quasi-projective the sheaf \( \underline{H} \) has a unique structure of a sheaf with transfers ([BK1], Lemma 1.3.2). Let

\[
(2.24) \quad Tot : C^b(\mathcal{M}_1) \rightarrow C^b(Sh_{et}^{rig})
\]

be the DG functor that takes a 1-motive \([\Lambda \xrightarrow{u} G] \in \mathcal{M}_1 \subset C^b(\mathcal{M}_1)\) to the complex

\[
[\Lambda \xrightarrow{u} G].
\]

As a complex, \( \Lambda \) is placed in degree 0 and \( G \) in degree 1. We introduce a structure of an exact category on \( \mathcal{M}_1 \): a complex \( M \in C^b(\mathcal{M}_1) \) is said to be acyclic if \( Tot(M) \) is acyclic. The homotopy category of acyclic complexes is strongly generated by short exact sequences. Let \( D^b_{dg}(\mathcal{M}_1) \) be the derived DG category of \( \mathcal{M}_1 \). We get a DG quasi-functor

\[
(2.25) \quad D^b_{dg}(\mathcal{M}_1) \rightarrow D^b_{dg}(Sh_{et}^{rig}) \xrightarrow{(2.24)} D^c_{et}(k; \mathbb{Z}).
\]

Let \( d_{\leq 1}D^c_{et}(k; \mathbb{Z}) \) be the homotopy idempotent completion of the full subcategory of \( D^c_{gm,et}(k; \mathbb{Z}) \) strongly generated by motives of smooth curves. Then, according to ([BK1], Theorem 2.1.2) the functor \( (2.25) \) induces a homotopy equivalence between \( D^b_{dg}(\mathcal{M}_1) \) and \( d_{\leq 1}D^c_{gm,et}(k; \mathbb{Z}) \). In particular, the functor \( (2.25) \) factors through the subcategory of geometric motives:

\[
Tot_2 : D^b_{dg}(\mathcal{M}_1) \rightarrow D^c_{gm,et}(k; \mathbb{Z}).
\]

For a subring \( A \subset \mathbb{Q} \) we set

\[
Tot_A : D^b_{dg}(\mathcal{M}_1 \otimes A) \xrightarrow{\text{2.1}} D^b_{dg}(\mathcal{M}_1) \otimes A \xrightarrow{Tot} D^c_{gm,et}(k; A).
\]

The left arrow is a homotopy equivalence ([BK1], Corollary 1.6.2). As in the integral case, \( Tot_A \) is a homotopy equivalence between \( D^b(\mathcal{M}_1 \otimes A) \) and \( d_{\leq 1}D^c_{gm,et}(k; A) \subset D^c_{gm,et}(k; A) \). For \( A = \mathbb{Q} \) this result was announced by Voevodsky and proved by Orgogozo [O].

According to ([BK1], §5) the triangulated functor \( Ho(Tot_{\mathbb{Q}}) \) has a left adjoint functor; thus by Lemma 2.1 so does the functor \( Tot_{\mathbb{Q}} \). We denote the left adjoint DG quasi-functor by

\[
LA_{b_{\mathbb{Q}}} : D^c_{gm,et}(k; \mathbb{Q}) \rightarrow D^b_{dg}(\mathcal{M}_1 \otimes \mathbb{Q}).
\]

\footnote{Replacing in (2.24) the étale topology by Zariski one we obtain a stronger exact structure defined in [17].}
In (D3, §10.1) Deligne constructed an equivalence of categories
\[(2.26) \quad \mathcal{M}_1(C) \sim MHS^\mathbb{Z}_1.\]
where $MHS^\mathbb{Z}_1$ is the full subcategory of the category $MHS^\mathbb{Z}$ of mixed polarizable Hodge structures that consists of torsion-free objects of type \{(0, 0), (0, -1), (-1, 0), (-1, -1)\}.
As a subcategory of an abelian category $MHS^\mathbb{Z}_1$ inherits an exact structure. It follows easily from (BK1, Proposition 1.4.1) that \[(2.26)\] is, in fact, an equivalence of exact categories. For $k \subset \mathbb{C}$, $A \subset \mathbb{Q}$, we set
\[T^H_{\mathbb{Z}^1} : D^b_k(M_1(C)) \to D^b_k(M_1(C)) \to D^b(MHS^\mathbb{Z}_1),\]
\[T^H_A : D^b_k(M_1(k) \otimes A) \to D^b(MHS^A_1).\]
The l-adic realization of 1-motives (BK1, C.6) will be denoted by
\[T^{eff}_{\mathbb{Z}/l} : D^b_d(M_1(k)) \to D^b_d(G, \mathbb{Z}/l').\]

3. Proofs

3.1. Functor $L\text{Alb}_Q$. Fix a subring $\mathbb{Z} \subset A \subset \mathbb{Q}$. Recall that a mixed polarizable Hodge structure $(V, W, \subset V, F' \subset V)$ is in $MHS^A$ is called effective if $F^1 = 0$. We will write $MHS^A_{eff} \subset MHS^A$ for the full subcategory of effective Hodge structures. Let
\[T_{\text{eff}} : D^b_d(MHS^A_{eff}) \to D^b(MHS^A_{eff})\]
be the functor induced by the embedding $MHS^A_{1} \subset MHS^A_{eff}$.

**Proposition 3.1.** The functor $Ho(T_{\text{eff}})$ is fully faithful. The functor $\text{Tot}_Q$ has a left adjoint functor
\[L\text{Alb}_Q : D^b(MHS^Q_{eff}) \to D^b(MHS^Q_1),\]
which is t-exact i.e., for every $M \in MHS^Q_{eff}$, $Ho(L\text{Alb}_Q)(M)$ is isomorphic to an object of $MHS^Q_1 \subset D^b(MHS^Q_1)$.

**Proof.** It is known that the category $MHS^A$ of polarizable mixed Hodge structures has homological dimension 1 (i.e., $Ext^i(M, N) = 0$ for every $M, N \in MHS^A$ and every $i > 1$. See Bc, Corollary 1.10). The first claim of the proposition follows from this fact and the next lemma.

**Lemma 3.2.** Let $B$ be an abelian category of homological dimension $\leq 1$ and let $A \subset B$ be a full abelian subcategory closed under extensions. Then
a) homological dimension of $A$ is at most 1
b) the functor $D^b(A) \to D^b(B)$ is fully faithful.

**Proof.** a) It is enough to check that for every $M_1, M_2, M_3 \in A$ the Yoneda product
\[Ext^1_A(M_1, M_2) \times Ext^2_A(M_2, M_3) \to Ext^3_A(M_1, M_3)\]
is 0. In turn, this is equivalent to showing that for every extensions
\[(3.1) \quad 0 \to M_2 \to N_1 \to M_1 \to 0\]
\[0 \to M_3 \to N_2 \to M_2 \to 0\]
there exists an object $P \in A$ with a 3 step filtration $0 \subset M_3 \subset N_2 \subset P$ such that the extension
\[0 \to N_2/M_3 = M_2 \to P/M_3 \to P/N_2 \to 0\]
is isomorphic to (3.1). Since $\text{Ext}_B^2(M_1, M_3) = 0$ there exists $P \in B$ with these properties and since $\mathcal{A} \subset \mathcal{B}$ is closed under extensions $P \in \mathcal{A}$.

b) It is enough to show that for every $M_1, M_2 \in \mathcal{A}$ and for every $i \geq 0$ the morphism

$$\text{(3.2)} \quad \text{Ext}_{\mathcal{A}}^i(M, N) \to \text{Ext}_B^i(M, N)$$

is an isomorphism. This is true for $i = 0$ because $\mathcal{A} \subset \mathcal{B}$ is a full subcategory and for $i = 1$ because $\mathcal{A}$ is closed under extensions. If $i > 1$ both groups in (3.2) are trivial by part a).

To prove the existence of $\overline{\text{LAlb}}_\mathbb{Q}$ we first show that the embedding of abelian categories

$$\text{tot} : \text{MHS}_1^\mathbb{Q} \subset \text{MHS}_{\text{eff}}^\mathbb{Q}$$

has a left adjoint functor

$$\overline{\text{lab}} : \text{MHS}_{\text{eff}}^\mathbb{Q} \to \text{MHS}_1^\mathbb{Q}.$$  

The functor $\overline{\text{lab}}$ is going to be the composition

$$\text{MHS}_{\text{eff}}^\mathbb{Q} \xrightarrow{\text{w}_{\geq -2}} \text{MHS}_{-2, \text{eff}}^\mathbb{Q} \xrightarrow{\delta} \text{MHS}_1^\mathbb{Q}, \quad \overline{\text{lab}} := \delta \circ \text{w}_{\geq -2},$$

where $\text{MHS}_{-2, \text{eff}}^\mathbb{Q}$ is a full subcategory of $\text{MHS}_{\text{eff}}^\mathbb{Q}$ consisting of mixed Hodge structures that have weights greater or equal than $-2$, $\text{w}_{\geq -2}$ is the functor that takes a Hodge structure $V$ to the quotient $V/W_{-3}V$. The second functor $\delta$ is defined as follows. Given $V \in \text{MHS}_{-2, \text{eff}}^\mathbb{Q}$ there exists a unique decomposition of the pure Hodge structure $W_{-2}V$ into the direct sum of a Hodge-Tate substructure and a substructure $P \subset W_{-2}V$ that has no Hodge-Tate subquotients. The existence follows from the semi-simplicity of the category of pure polarizable Hodge structures. Set $\delta(V) = V/i(P)$. We leave it to the reader to check that that $\delta$ is indeed a functor and that the projection $V \to \delta(V)$ extends to a morphism of functors $\text{Id} \to \delta$. Composing it with the natural morphism $\text{Id} \to \text{w}_{\geq -2}$ we get

$$\overline{\eta} : \text{Id} \to \overline{\text{tot}} \circ \overline{\text{lab}}.$$  

Together with the obvious isomorphism

$$\overline{\eta} : \overline{\text{lab}} \circ \overline{\text{tot}} \simeq \text{Id}$$

the triple $(\overline{\text{lab}}, \overline{\eta}, \overline{\eta})$ is an adjunction datum: the compositions

$$\xrightarrow{\overline{\text{lab}}} \xrightarrow{\overline{\text{lab}}} \overline{\text{lab}} \circ \overline{\text{tot}} \circ \overline{\text{lab}} \xrightarrow{\overline{\eta}} \overline{\text{lab}}$$

$$\xrightarrow{\overline{\eta}} \overline{\text{lab}} \circ \overline{\text{tot}} \circ \overline{\text{lab}} \circ \overline{\text{tot}} \xrightarrow{\overline{\eta}} \overline{\text{lab}} \circ \overline{\text{tot}}$$

are identity morphisms. Furthermore, the functors $\text{w}_{\geq -2}$, $\delta$ are exact and so is the composition $\overline{\text{lab}}$. The functor $\overline{\eta}$ is also exact. Thus, $(\overline{\text{lab}}, \overline{\eta}, \overline{\eta})$ automatically extends to an adjunction datum $(\overline{\text{LAlb}}_\mathbb{Q}, \nu, \mu)$ for the derived DG categories. The $t$-exactness of $\overline{\text{LAlb}}_\mathbb{Q}$ is clear.

$\square$
3.2. The main theorem.

Theorem 3. Let $k$ be a field of characteristic 0 of finite étale homological dimension and $l$ a prime number. Then

\[(3.3) \quad T^\text{et}_{Z/l} \cong R^\text{et}_{Z/l} \circ \text{Tot}_Z \]

(1) Let $k$ be a subfield of $\mathbb{C}$ and $A$ a subring of $\mathbb{Q}$. Then

\[(3.4) \quad \overline{\text{Tot}_A \circ T^\text{Hodge}} \cong R^\text{Hodge}_A \circ \text{Tot}_A\]

and the morphism of functors

\[(3.5) \quad \overline{\text{Alb}_Q \circ R^\text{Hodge}_Q} \to \text{Tot}_Q \circ T^\text{Hodge}_Q \cong R^\text{Hodge}_Q \circ \text{Tot}_Q \circ \text{Alb}_Q\]

is an isomorphism.

The rest of this section is devoted to a proof of the theorem.

3.3. Proof of (3.3). We shall show that, for every 1-motive $M = [\Lambda \overset{u}{\to} G]$, the modules $H^i(R^\text{et}_{Z/l} \circ \text{Tot}(M)) \in \text{Mod}(G, Z/l)$ are 0 for $i \neq 0$ and canonically isomorphic to Deligne's $l$-adic realization of $M$ for $i = 0$. The formula (3.3) would follow from Theorem 4.

Since presheaves $\Lambda, G$ are homotopy invariant the canonical morphism

\[C^\Delta(\text{Tot}(M)) \to \text{Tot}(M)\]

is a quasi-isomorphism. Thus

\[R^\text{et}_{Z/l} \circ \text{Tot}(M) \cong (\Lambda(\overline{k}) \overset{u}{\to} G(\overline{k})) \overset{L_{Z/l}}{\cong} (\Lambda(\overline{k}) \overset{u+v}{\to} G(\overline{k}) \oplus \Lambda(\overline{k}) \otimes Q) \overset{L_{Z/l}}{\cong} (G(\overline{k}) \oplus \Lambda(\overline{k}) \otimes Q) \overset{L_{Z/l}}{\cong} \ker(G(\overline{k}) \oplus \Lambda(\overline{k}) \otimes Q) \overset{L_{Z/l}}{\cong} G(\overline{k}) \oplus \Lambda(\overline{k}) \otimes Q)/Z.

Here $v : \Lambda(\overline{k}) \to \Lambda(\overline{k}) \otimes Q$ is the canonical embedding, the morphism $\alpha$ sending $\Lambda(\overline{k}) \otimes Q$ to zero is a quasi-isomorphism because $Q \otimes Z/l \cong 0$. The module at the right-hand side of the formula is Deligne's $l$-adic realization of $M$. This completes the proof of (3.3).

3.4. Compatibility of $\text{Alb}_Q$ with base change. When proving the remaining part of the theorem, we may assume that $k = \mathbb{C}$. Indeed, we have the following general result.

Proposition 3.3. Let $k \subset k'$ be any extension, $A = \mathbb{Z}$ or $\mathbb{Q}$, and let

\[f^* : D^b_{dg}(M_1(k) \otimes A) \to D^b_{dg}(M_1(k') \otimes A),\]

be the corresponding pullback functors. Then

\[(3.7) \quad f^* \circ \text{Tot}_A \cong \text{Tot}_A \circ f^* : D^b_{dg}(M_1(k) \otimes A) \to D^b_{gm}(M_1(k') \otimes A),\]

\[(3.8) \quad f^* \circ \text{Alb}_Q \cong \text{Alb}_Q \circ f^* : D^b_{gm}(M_1(k') \otimes \mathbb{Q}) \to D^b_{dg}(M_1(k') \otimes \mathbb{Q}).\]
Lemma 3.4. The morphism  
\( f : \text{DM}(P) \rightarrow \text{Tot}_Q \)  
right-orthogonal complement  
To prove that \( \alpha \) we have canonical isomorphisms  
(3.9)  
\[ \alpha : f^* \circ \text{LAlb}_Q \rightarrow \text{LAlb}_Q \circ f^* \]
To prove that \( \alpha \) is an isomorphism we shall need the following general property of \( f^* \). Recall from ([V1], Proposition 3.2.8) that for every \( E, G \in \text{DM}^\text{eff}(k; A) \) there is inner Hom-object \( \text{Hom}(E, G) \in \text{DM}^\text{eff}(k; A) \). Set \( f^*_{tr} = \text{Ho}(f^*) \).

Lemma 3.4. The morphism  
\( f^*_{tr}(\text{Hom}(E, G)) \rightarrow \text{Hom}(f^*_{tr}(E), f^*_{tr}(G)) \)  
defined by the distinguished element of the group  
\[ \text{Hom}(f^*_{tr}(\text{Hom}(E, G)) \otimes f^*_{tr}(E), f^*_{tr}(G)) \leftrightarrow \text{Hom}(\text{Hom}(E, G) \otimes E, G). \]
is an isomorphism.

Proof. Recall from ([BV], §4.4) that the projection  
\( P : D(PSh_{tr}(Sm_{k})) \rightarrow \text{DM}^\text{eff}(k; A) \)  
has a right adjoint functor \( C^M \) that identifies the category \( \text{DM}^\text{eff}(k; A) \) with the right-orthogonal complement  
\[ I^\text{Zar,\Delta}_1(k) \subset D(PSh_{tr}(Sm_{k})). \]
We shall first show that the functor \( f^{-1} \) takes \( I^\text{Zar,\Delta}_1(k) \) into \( I^\text{Zar,\Delta}_1(k') \subset D(PSh_{tr}(Sm_{k'})) \). In fact,  
(3.11)  
\[ f^{-1}(I^\text{Zar,\Delta}_1(k)) \subset I^\text{Zar,\Delta}_1(k'), \quad f^{-1}(I^\Delta_1(k')) \subset I^\Delta_1(k'). \]
Let us just check the first inclusion. Let \( G \in I^\text{Zar,\Delta}_1(k), X \in Sm_{k'}, U_1 \cup U_2 = X \) an open covering of a smooth scheme over \( k' \), and let  
\[ MV(U_1, U_2, X) := Z_{tr}[U_1 \cap U_2] \rightarrow Z_{tr}[U_1] \oplus Z_{tr}[U_2] \rightarrow Z_{tr}[X] \]
be the Mayer-Vietoris complex. We have to show that  
\[ \text{Hom}_{D(PSh_{tr}(Sm_{k'}))}(MV(U_1, U_2, X), f^*G) = 0. \]
There exist \( Z \in Sm_{k'}, \) an open covering \( \bar{U}_1 \cup \bar{U}_2 = Z, \) and a \( k \)-morphism \( h : X \rightarrow Y \) such that \( U_i = h^{-1}(U_i) \). We have  
\[ \text{Hom}_{D(PSh_{tr}(Sm_{k'}))}(MV(U_1, U_2, X), f^*G) = \]
\[ \colim_{X \cong Y \cong Z} \text{Hom}_{D(PSh_{tr}(Sm_{k'}))}(MV(g'^{-1}(\bar{U}_1), g'^{-1}(\bar{U}_2), Y), G) = 0. \]
Here the colimit is taken over the category of triples \( (Y \in Sm_{k}, g, g') \) with \( g \circ g' = h \). Proof of the second inclusion in (3.11) is similar.

As a consequence we see that the morphism \( f^{-1} \circ C^M \rightarrow C^M \circ f^*_{tr} \) induced by \( P \circ f^{-1} \circ C^M \rightarrow f^*_{tr} \) is an isomorphism. Let us also observe a natural isomorphism  
\[ C^M \text{Hom}_{\text{DM}^\text{eff}(k; A)}(P(F), G) \approx \text{Hom}_{D(PSh_{tr}(Sm_{k}))}(F, C^M(G)) \]
coming from the monoidal structure on the functor \( P \).
Now we are ready to prove the Lemma. We shall check that \( C^M \) applied to the morphism (3.10) is an isomorphism. Choosing \( F \in D(PSh_{tr}(Sm_k)) \) with \( P(F) = E \) and using the above remarks we reduce our problem to proving that morphism (3.12)

\[
\overline{f^{-1} \text{Hom}_{D(PSh_{tr}(Sm_k))}(F,C^M(G))} \to \text{Hom}_{D(PSh_{tr}(Sm_k))}(f^{-1}(F),f^{-1}C^M(G))
\]

is an isomorphism \(^{10}\). Moreover, it will suffice to show this for \( F = A_{tr}[Z] \), where \( Z \in Sm_k \). For any \( X \in Sm_k \), we have

\[
\text{Hom}(A_{tr}[X],f^{-1} \overline{\text{Hom}(A_{tr}[Z],C^M(G))}) \simeq \text{colim}_{X \times \Delta Y} \text{Hom}(A_{tr}[Y \times Z],C^M(G)) \simeq \text{colim}_{X \times_k Z \times \Delta Y} \text{Hom}(A_{tr}[Y'],C^M(G)) \]

\[
\simeq \text{Hom}(A_{tr}[X], \overline{\text{Hom}(f^{-1}(A_{tr}[Z]),f^{-1}C^M(G)))}.
\]

\[\square\]

Let us prove that, for every \( M \in DM^{eff}_g(k; \mathbb{Q}) \), morphism \( Ho(\alpha)(M) \) (see [BM]) is an isomorphism. Since the functor \( \text{Hom}(\cdot, \mathbb{Q}(1)) \) is fully faithful on the subcategory \( d_{\leq 1}DM^{eff}_g(k; \mathbb{Q}) \subset DM^{eff}_g(k; \mathbb{Q}) \) (BK1, Proposition 4.4.1) it is enough to prove that the morphism (3.13)

\[
\text{Hom}(Ho(f^* \circ \text{Tot}_{Q} \circ LAlb_Q)(M), \mathbb{Q}(1)) \to \text{Hom}(Ho(\text{Tot}_{Q} \circ LAlb_Q \circ f^*)(M), \mathbb{Q}(1))
\]

is an isomorphism. By (BK1, Cor. 6.2.1), for any geometric effective motive \( N \), the morphism

\[
\text{Hom}(\text{Tot}_{Q} \circ LAlb_Q)(N), \mathbb{Q}(1)) \to \text{Hom}(N, \mathbb{Q}(1))
\]

induced by \( N \to Ho(\text{Tot}_{Q} \circ LAlb_Q)(N) \) is an isomorphism. Applying this to \( N = Ho(f^*)(M) \) we see that (3.13) is an isomorphism if and only if so is the morphism (3.14)

\[
\text{Hom}(Ho(f^* \circ \text{Tot}_{Q} \circ LAlb_Q)(M), \mathbb{Q}(1)) \to \text{Hom}(Ho(f^*)(M), \mathbb{Q}(1))
\]

given by \( M \to Ho(\text{Tot}_{Q} \circ LAlb_Q)(M) \). Lemma (3.4) identifies (3.14) with the pullback of the isomorphism

\[
\text{Hom}(Ho(\text{Tot}_{Q} \circ LAlb_Q)(M), \mathbb{Q}(1)) \simeq \text{Hom}(M, \mathbb{Q}(1)).
\]

\[\square\]

3.5. **Beginning of the proof of (3.4).** It will suffice to construct isomorphism (3.4) for \( A = \mathbb{C} \). By Proposition (3.3) we may assume that \( k = \mathbb{C} \). Corollary (2.12) together with formula (3.3) imply that for every \( M \in M_1(\mathbb{C}) \)

\[
H^i(R^j_{\text{Hodge}} \circ \text{Tot}_{\mathbb{Z}}(M)) = 0, \quad i \neq 0.
\]

Thus, by Theorem (1) it is enough to construct an isomorphism (3.16)

\[
H^0(R^j_{\text{Hodge}} \circ \text{Tot}_{\mathbb{Z}}) \simeq H^0(\overline{\text{Tot}_{\mathbb{Z}} \circ T^j_{\text{Hodge}}}) : M_1(\mathbb{C}) \to MH^{3g}_c.
\]

Every such isomorphism extends uniquely to (3.4).

\(^{10}\)The proof below shows that \( C^M(G) \) can be replaced by an arbitrary complex of presheaves with transfers.
3.6. Computation of $H_0(R^B_{Z} \circ Tot_Z)$. We shall construct a functorial isomorphism of abelian groups:

\begin{equation}
\Theta_M : H^0(R^B_{Z} \circ Tot_Z(M)) \cong H^0(Tot_Z \circ T^B_{Z}(M)) = \ker(\Lambda(\mathbb{C}) \oplus g \xrightarrow{w \circ exp} G(\mathbb{C})),
\end{equation}

where $M = [\Lambda \xrightarrow{\gamma} G] \in M_1(\mathbb{C})$, $g$ is the Lie algebra of $G$, and $exp : g \to G(\mathbb{C})$ is the exponential map. In the next subsection we check that $\Theta_M$ is compatible with the Hodge structures.

To construct $\Theta_M$ we consider an auxiliary functor from the category of 1-motives to the category of complexes of abelian groups that takes $M$ to the complex

\begin{equation}
C^\text{sing}_Z(M) := \text{cone}(\Lambda(\mathbb{C}) \to NMaps(\Delta^1, G(\mathbb{C})))[-1],
\end{equation}

where $NMaps(\Delta^1, G(\mathbb{C}))$ is the normalized chain complex of the simplicial abelian group $Maps(\Delta^1, G(\mathbb{C}))$. Recall that for every abelian Lie group $P$ the complex $NMaps(\Delta^1, P)$ computes the homotopy groups of $P$. In particular,

\begin{equation}
H^i(NMaps(\Delta^1, G(\mathbb{C}))) = \begin{cases} 
0 & \text{if } i \neq -1 \\
\pi_1(G(\mathbb{C})) & \text{otherwise.}
\end{cases}
\end{equation}

We construct a functorial quasi-isomorphism

\[ \phi : C^\text{sing}_Z(M) \to \text{cone}(\Lambda(\mathbb{C}) \oplus g \to G(\mathbb{C}))[-1] \]

as follows.

\begin{align*}
\rightarrow & \quad N_2Maps(\Delta^1, G(\mathbb{C})) \\
\rightarrow & \quad \Lambda \oplus N_1Maps(\Delta^1, G(\mathbb{C})) \\
\rightarrow & \quad N_0Maps(\Delta^1, G(\mathbb{C}))
\end{align*}

\begin{align*}
\begin{bmatrix} d_1 & d_0 \end{bmatrix} \\
\begin{bmatrix} 0 & \phi_0 \end{bmatrix} \\
\begin{bmatrix} 0 & \text{id} \end{bmatrix}
\end{align*}

Here the map $\phi_0$ is the sum of the identity map on $\Lambda(\mathbb{C})$ and the homomorphism

\begin{equation}
N_1Maps(\Delta^1, G(\mathbb{C})) \to g
\end{equation}

defined as follows. An element of $N_1Maps(\Delta^1, G(\mathbb{C}))$ is a continuous map $\gamma : \Delta^1 = [0, 1] \to G(\mathbb{C})$ such that $\gamma(0) = 0$. Let $\hat{\gamma} : [0, 1] \to g$ be the lifting of $\gamma$ such that $\hat{\gamma}(0) = 0$. The map (3.18) takes $\gamma$ to $\hat{\gamma}(1)$.

**Proposition 3.5.** The morphism $\phi$ is a quasi-isomorphism.

**Corollary 3.6.** There is a functorial isomorphism

\[ H^0(\overline{\phi}) : H^0(C^\text{sing}_Z(M)) \xrightarrow{\sim} H^0(Tot_Z \circ T^B_{Z}(M)). \]

**Proof.** First, we have to show that $\phi$ is a morphism of complexes.

(i) $\phi_0 \cdot d_1 = 0$. Indeed, for every $\nu \in N_1Maps(\Delta^1, G(\mathbb{C}))$ the map $d_1(\nu) : [0, 1] \to G(\mathbb{C})$ takes the boundary points 0 and 1 to 0. Moreover, the induced map $S^1 \to G(\mathbb{C})$ is contractible. It follows that $d_1(\nu)(1) = 0$.

(ii) $d_0 = (u \oplus exp) \phi_0$. Indeed, for every $(\lambda \oplus \gamma) \in \Lambda \oplus N_1 Maps(\Delta^1, G(\mathbb{C}))$

\[ d_0(\lambda \oplus \gamma) = u(\lambda) \cdot \gamma(1) = (u \oplus exp) \phi_0(\lambda \oplus \gamma). \]

To show that $\phi$ is a quasi-isomorphism consider the morphism

\begin{equation}
NMaps(\Delta^1, g) \xrightarrow{exp} NMaps(\Delta^1, G(\mathbb{C}))
\end{equation}

induced by the homomorphism $exp : g \to G(\mathbb{C})$. Let

\[ NMaps(\Delta^1, g) \xrightarrow{\lambda} C^\text{sing}_Z(M) \]
be the composition of \( \exp_s \) and the morphism \( NMaps(\Delta', G(\mathbb{C})) \to C_{Z}^{\text{sing}}(M) \). It is easy to see that \( \phi \) factors through the morphism

\[
\hat{\phi} : \text{cone}(s) \to T_Z^{\text{Betti}}(M).
\]

Observe that the map

\[
N_i Maps(\Delta \cdot, g) \xrightarrow{\exp} N_i Maps(\Delta \cdot, G(C)) \to C_{\text{sing}} Z(M).
\]

is an isomorphism for every \( i > 0 \) (for every continuous map \( \Delta' \to G(C) \) sending all the faces but one to 0 lifts uniquely to a map \( \Delta' \to g \) with the same property). It follows that \( \hat{\phi} \) is a quasi-isomorphism. Since the complex \( NMaps(\Delta, g) \) is acyclic (for its cohomology groups are the homotopy groups of \( g \)), \( \phi \) is a quasi-isomorphism as well.

It remains to construct a functorial quasi-isomorphism

\[
H^0(RBetti Z \circ Tot Z)(M) \xrightarrow{\sim} H^0(C_{Z}^{\text{sing}}(M)).
\]

With rational coefficients (3.20) can be easily derived from the Eilenberg-MacLane cube construction. We explain this short proof in §3.7. The integral statement is more involved. Consider the double complex of presheaves with transfers

\[
\begin{array}{ccc}
\mathbb{Z}_{tr}[A \times A] & \xrightarrow{p_1 + p_2 - m} & \mathbb{Z}_{tr}[A] \\
\downarrow_{u \times u} & & \downarrow_{u} \\
\mathbb{Z}_{tr}[G \times G] & \xrightarrow{p_1 + p_2 - m} & \mathbb{Z}_{tr}[G],
\end{array}
\]

where \( p_1, m : \mathbb{Z}_{tr}[A \times A] \to \mathbb{Z}_{tr}[A] \) (resp. \( p_1, m : \mathbb{Z}_{tr}[G \times G] \to \mathbb{Z}_{tr}[G] \)) are the maps induced by the projections and the addition operation on \( A \) (resp. \( G \)). Denote by \( \tilde{M} \) the associated total complex shifted so that \( \mathbb{Z}_{tr}[G] \) is in cohomological degree 1. We shall use the same notation \( \tilde{M} \) for the corresponding motive. The construction of \( \tilde{M} \) is functorial: sending \( M \) to \( \tilde{M} \) we get a (nonadditive) functor from the category of 1-motives to the triangulated category of étale Voevodsky motives. Next, we have a functorial morphism:

\[
\tilde{M} \to Tot Z(M),
\]

induced by the map of double complexes:

\[
\begin{array}{ccc}
\mathbb{Z}_{tr}[A \times A] & \to & \mathbb{Z}_{tr}[A] \\
\downarrow_{u \times u} & & \downarrow_{u} \\
\mathbb{Z}_{tr}[G \times G] & \to & \mathbb{Z}_{tr}[G],
\end{array}
\]

Consider the induced map of Betti realizations

\[
Ho(R_{Z}^{\text{Betti}}(\tilde{M})) \to Ho(R_{Z}^{\text{Betti}} \circ Tot Z)(M).
\]

Since the complex \( R_{Z}^{\text{Betti}} \circ Tot Z(M) \) has trivial cohomology in negative degrees (see (3.15)) the map (3.22) canonically factors through the morphism

\[
\tau_{\geq 0} Ho(R_{Z}^{\text{Betti}}(\tilde{M})) \xrightarrow{\psi} Ho(R_{Z}^{\text{Betti}} \circ Tot Z)(M).
\]

**Proposition 3.7.** The morphism \( \psi \) is a quasi-isomorphism.

**Corollary 3.8.** We have a functorial isomorphism

\[
H^0(\psi) : H^0(R_{Z}^{\text{Betti}}(\tilde{M})) \xrightarrow{\sim} H^0(R_{Z}^{\text{Betti}} \circ Tot Z)(M).
\]
Proof. We shall first prove the proposition in the following two special cases.

(i) $\Lambda = 0$. In this case the proposition is equivalent to the exactness of the sequence
\[ \mathbb{Z}[\Lambda(\mathbb{C}) \times \Lambda(\mathbb{C})] \to \mathbb{Z}[\Lambda(\mathbb{C})] \to \Lambda(\mathbb{C}) \to 0. \]

(ii) $\Lambda = 0$. The exact triangle
\[ \mathbb{Z}_{tr}[G \times G][-1] \to \mathbb{Z}_{tr}[G][-1] \to \tilde{M} \to \mathbb{Z}_{tr}[G \times G] \]
yields a long exact sequence
\[ \to H_1(G(\mathbb{C}) \times G(\mathbb{C})) \xrightarrow{p_1 \cdot p_2 - m_*} H_1(G(\mathbb{C})) \xrightarrow{\alpha} H^0(\mathbb{R}_{\text{Betti}}^G(\tilde{M})) \to \]
\[ H_0(G(\mathbb{C}) \times G(\mathbb{C})) \xrightarrow{p_1 \cdot p_2 - m_*} H_0(G(\mathbb{C})) \to H^1(\mathbb{R}_{\text{Betti}}^G(\tilde{M})) \to 0. \]

The map $p_1 \cdot p_2 - m_*$ is 0 on $H_1$ and an isomorphism on $H_0$. It follows that the cohomology groups of $\mathbb{R}_{\text{Betti}}^G(\tilde{M})$ are trivial in positive degrees and identified (via the map $\alpha$) with $H_1(G(\mathbb{C}))$ in degree 0. The following result completes the proof.

Lemma 3.9. The morphism $i : \mathbb{Z}_{tr}[G] \to \mathcal{G}$ induces a quasi-isomorphism
\[ H_1(G(\mathbb{C}))[1] \simeq R_{\text{Betti}}^G(\mathcal{G}). \]

Proof. By Corollary 2.12 it suffices to show that, for every prime $l$, $i$ induces a quasi-isomorphism
\[ H_1(\mathbb{R}_{\text{Betti}}^G(\mathbb{Z}_{tr}[G]))[1] \simeq R_{\text{Betti}}^G(\mathcal{G}). \]

By definition of l-adic realization (2.19), this amounts to showing that the map
\[ H_1(C^\Lambda(\mathbb{Z}_{tr}[G])(\mathbb{C}) \otimes \mathbb{Z}/l^n)[1] \to G(\mathbb{C}) \otimes \mathbb{Z}/l^n \simeq \ker(G(\mathbb{C}) \xrightarrow{\iota_n} G(\mathbb{C})) \]
is a quasi-isomorphism. The latter is a part of generalized Roitman’s theorem ([BK1], Theorem 14.2.5).\]
with exact rows (and injective map \( \beta \)). Since \( \psi_i \) are isomorphisms so is the map \( \psi \).

Each term of the complex \( \tilde{M} \) is a direct sum of representable presheaves. It follows, that the Betti realization of \( \tilde{M} \) is canonically isomorphic to the total complex \( C^\text{sing}_Z(M) \) of the following double complex

\[
\begin{array}{ccc}
Z[\Lambda(\mathbb{C}) \times \Lambda(\mathbb{C})] & \overset{p_1 + p_2 - m}{\rightarrow} & Z[\Lambda(\mathbb{C})] \\
\downarrow_{u \times u} & & \downarrow_{u} \\
C^\text{sing}_Z(G(\mathbb{C}) \times G(\mathbb{C})) & \overset{p_1 + p_2 - m}{\rightarrow} & C^\text{sing}_Z(G(\mathbb{C})),
\end{array}
\]

where \( C^\text{sing}(\cdot) \) denotes the singular chain complex of a topological space. Define a morphism

\[
(3.26) \quad R^\text{Betti}_Z(\tilde{M}) = C^\text{sing}_Z(\tilde{M}) \to C^\text{sing}_Z(M)
\]

Since the complex \( C^\text{sing}_Z(M) \) is acyclic in positive degrees the morphism \( (3.26) \) factors through

\[
\tau_{\geq 0} C^\text{sing}_Z(\tilde{M}) \xrightarrow{c} C^\text{sing}_Z(M).
\]

**Proposition 3.10.** The morphism \( \rho \) is a quasi-isomorphism.

**Corollary 3.11.** There is functorial isomorphism

\[
(3.27) \quad H^0(\rho) \circ H^0(\psi)^{-1} : H^0(R^\text{Betti}_Z \circ \text{Tot}_Z)(M) \xrightarrow{\sim} H^0(C^\text{sing}_Z(M)).
\]

**Proof.** The argument is parallel to the proof of Proposition 3.7. We only explain the analog of Lemma 3.9.

**Lemma 3.12.** The morphism \( C^\text{sing}_Z(G(\mathbb{C})) \to \text{NMaps}(\Delta', G(\mathbb{C})) \) induces a quasi-isomorphism

\[
H_1(G(\mathbb{C}))[1] \simeq \text{NMaps}(\Delta', G(\mathbb{C})).
\]

**Proof.** The map

\[
H_1(G(\mathbb{C}))[1] \to \text{NMaps}(\Delta', G(\mathbb{C})) \simeq \pi_1(G(\mathbb{C}))[1]
\]

is the left inverse to the Hurewicz isomorphism \( \pi_1(G(\mathbb{C}))[1] \to H_1(G(\mathbb{C}))[1] \).

We define \( \Theta_M \) to be the composition

\[
\Theta_M = H^0(\phi) \circ H^0(\rho) \circ (H^0(\psi))^{-1}.
\]
3.7. Remark: $\text{Tot}_Q$ via additivization. The complex defining $\hat{M}$ is a truncation of the Eilenberg-MacLane cube construction. In fact, one can use the whole Eilenberg-MacLane complex to give a short conceptual proof of \cite[3.2.2; 13.2.6]{LP} that the Eilenberg-MacLane cube complex

$$
\cdots \to Q^2 \to Q^1 \to Q^0
$$

is a complex over $Fct(\text{Mod}(\mathbb{Z}), \text{Mod}(\mathbb{Z}))$ whose terms are functors of the form $F(V) = \mathbb{Z}[V^{2^n}]$, where $\mathbb{Z}[V^{2^n}]$ denotes the free abelian group generated by the set $V^{2^n}$, and whose $i$-th homology functor $H_i(Q^2)$ is isomorphic to the stable homology of the Eilenberg-MacLane spaces:

$$
H_i(Q^2)(V) = H_{i+n}(K(V,n)), \quad n \geq i + 1.
$$

In particular, for positive $i$, $H_i(Q^2)(V)$ is a torsion group and $H_0(Q^2)(V)$ is canonically isomorphic to $V$. It follows that the complex $Q^2 = Q^2 \otimes \mathbb{Q}$ is a resolution of the functor $F(V) = V \otimes \mathbb{Q}$. Let $[\Lambda \to G] \in \mathcal{M}_1 \subset C^b(\mathcal{M}_1)$ be a 1-motive. The complex of representable presheaves $Q^2(M)$

$$
Q^2(M)(X) := \text{cone}(Q^2(\Delta(X)) \to Q^2(G(X)))[−1], \quad X \in Sm
$$

is a resolution of the complex $cone(\Delta \to G)[−1] \otimes \mathbb{Q}$. Consider the sheafification $Q^{2,h}(M)$ of $Q^2(M)$ for the h-topology on $Sm$. The sheafification of a representable presheaf $\mathbb{Q}[Y]$ is canonically isomorphic to $Q_{tr}[Y]$ (\cite[V2, Theorem 3.3.5, Proposition 3.3.6]{V2}; $\Delta, G$ are already sheaves (\cite[V2, Theorem 3.2.9]{V2}). Thus, by Proposition \ref{2.8}

$$
Q^{2,h}(M) \simeq \text{Tot}_Q(M)
$$

in $\text{DM}^{eff}(\mathbb{C}, \mathbb{Q})$. On the other hand, as every term of $Q^2(M)$ is a representable presheaf we can compute $R^1_{Q^{2,h}} \circ Q^{2,h}(M)$ by applying $C^{\text{sing}}_Q$ to $Q^2(M)$ termwise. The resulted complex is isomorphic to the total complex of the simplicial complex

$$
\text{cone}(Q^2(\text{Maps}(\Delta, \Lambda(\mathbb{C}))) \to Q^2(\text{Maps}(\Delta, G(\mathbb{C}))))[−1]
$$

Applying \ref{3.28} again we find that $R^1_{Q^{2,h}} \circ Q^{2,h}(M)$ is canonically quasi-isomorphic to $C^{\text{sing}}_Q(M)$. Summarizing,

$$
R^1_{Q^{2,h}} \circ \text{Tot}_Q(M) \simeq C^{\text{sing}}_Q(M).
$$

The above argument expresses the idea that the functor $\text{Tot}_Q$ is the additivization of the functor that takes a 1-motive $M$ to the motive $cone(Q_{tr}[\Lambda] \to Q_{tr}[G])[−1]$. We refer the reader to [\cite{K}] for the definition of additivization.

3.8. Computation of $H^0(R^1_{\text{Hodge}} \circ \text{Tot}_Z(M))$. The abelian groups $H^0(R^1_{\text{Betti}} \circ \text{Tot}_Z(M))$, $H^0(\text{Tot}_Z \circ T^1_{\text{Betti}}(M))$ underlie the Hodge structures $H^0(R^1_{\text{Hodge}} \circ \text{Tot}_Z(M))$, $H^0(\text{Tot}_Z \circ T^1_{\text{Hodge}}(M))$ respectively. Let us show that the isomorphism \ref{3.17} is compatible with the Hodge structures. It will suffice to check that $\Theta_{\text{str}}^{-1}$ preserves the weight and Hodge filtrations.

The weight filtration on $H^0(\text{Tot}_Z \circ T^1_{\text{Hodge}}(M))$ is induced by a filtration on $M = [\Lambda \to G]$ by submotives: if $G$ is an extension of an abelian variety $A$ by a torus $T$, $W_{-2}M = (0, T) \subset W_{-1}M = (0, G) \subset W_0M = M$. The Hodge structure $H^0(R^1_{\text{Hodge}}(\mathcal{F})[−1])$ is pure of weight $−2$ and the Hodge structure $H^0(R^1_{\text{Betti}}(\mathcal{G})[−1])$
has weights −2, −1. Thus, the functoriality of isomorphism (3.17) implies that \( \Theta_M \) preserves the weight filtration.

Let us show that \( \Theta_M \) preserves the Hodge filtration. A choice of a basis for \( \Lambda(\mathbb{C}) \), \( \Lambda = \mathbb{Z}[S] \), yields a lifting

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{\delta} & \mathbb{Z}_{tr}[G] \\
\downarrow & & \downarrow \text{Id} \\
\mathbb{Z}_{tr}[\Lambda] & \xrightarrow{u} & \mathbb{Z}_{tr}(G),
\end{array}
\]  

(3.30)

where the homomorphism \( \delta : \Lambda = \mathbb{Z}[S] \xrightarrow{u} \mathbb{Z}[\Lambda] = \mathbb{Z}_{tr}[\Lambda] \) is defined on generators \( s \in S \subset \Lambda \) by the formula

\[
\delta(s) = [s] - [0] \in \mathbb{Z}[\Lambda].
\]  

(3.31)

Set \( M' = cone(\tilde{\alpha})[-1] \). The diagram (3.30) yields a morphism \( M' \to \hat{M} \).

**Lemma 3.13.** The morphisms \( M' \to \hat{M} \to Tot_{\mathbb{Z}}(M) \) induce

\[
H^0(\mathbb{R}_Z^Betti(M')) \xrightarrow{\alpha} H^0(\mathbb{R}_Z^Betti(\hat{M})) \xrightarrow{\beta} H^0(\mathbb{R}_Z^Betti \circ Tot_{\mathbb{Z}}(M)),
\]

\[
H^0(\mathbb{R}_Z^{Hodge}(M')) \xrightarrow{\beta} H^0(\mathbb{R}_Z^{Hodge} \circ Tot_{\mathbb{Z}}(M)).
\]

**Proof.** We have already proved in Proposition 3.7 that \( \beta \) is an isomorphism. Thus, it is enough to show that the composition \( \beta\alpha \) has the same property. Let us apply the functor \( \mathbb{R}_Z^Betti \) to the morphism of exact triangles

\[
\begin{array}{ccc}
\mathbb{Z}_{tr}[G][-1] & \to & M' \to \Lambda \xrightarrow{\tilde{\alpha}} \mathbb{Z}_{tr}[G] \\
\downarrow & & \downarrow \\
\mathbb{Q}[-1] & \to & Tot_{\mathbb{Z}}(M) \to \Lambda \to \mathbb{Q},
\end{array}
\]

The key observation is that the connecting homomorphism

\[
\tilde{\alpha}_* : \mathbb{Z}[S(\mathbb{C})] = H^0(\mathbb{R}_Z^Betti(\Lambda)) \to H^0(\mathbb{R}_Z^Betti(\mathbb{Z}_{tr}[G])) = \mathbb{Z}
\]

is 0. The rest of the proof is an easy diagram chasing combined with the formula (3.28). □

By the lemma is suffices to check that the isomorphism

\[
\Theta' = (\Theta_M \circ \beta \circ \alpha) \otimes \text{Id} : H^0(\mathbb{R}_Z^Betti(M')) \otimes \mathbb{C} \xrightarrow{\sim} T_Z^Betti(M) \otimes \mathbb{C}
\]

is strictly compatible with the Hodge filtration. To do this we construct a smooth \( G \)-equivariant compactification \( \overline{G} \) of \( G \). Recall that \( G \) is an extension of an abelian variety \( A \) by an algebraic torus \( T \). Choose an isomorphism \( T \cong G_r \). Then \( G \), viewed as a principal \( T \)-bundle over \( A \), gives rise to a vector bundle \( E \to A \) (which is a direct sum of \( r \) line bundles). Define

\[
\overline{G} := \mathbb{P}(E \oplus 1).
\]

The complement \( \overline{G} - G \) is a normal crossing divisor on \( \overline{G} \). Thus, we can view \( M' \) as a complex over \( \mathbb{Z}[S \mathbb{m}] \) (4.25). The (cohomological) de Rham complex \( R^*_Dh(M') \) is identified with

\[
cone(\mathbb{R}\Gamma(\overline{G}, \Omega_{\text{log}}) \to \mathbb{C}[S])
\]

\[\text{(3.17)}\]

\[\text{(3.30)}\]

\[\text{(3.23)}\]

\[\text{(3.17)}\]

\[\text{(3.23)}\]

\[\text{(3.17)}\]
and the Hodge filtration on $R^i_{DR}(M')$ is induced by the stupid filtration on the logarithmic de Rham complex $\Omega_{log}$. In particular, $F^1H^0R^i_{DR}(M')$ is identified with the space $\Gamma(\overline{\mathcal{G}}, \Omega^1_{log})$ of global 1-forms on $\overline{\mathcal{G}}$ with logarithmic singularities ([D2], Theorem 3.2.5).

The translation action of $G$ on itself extends to a $G$-action on $\overline{\mathcal{G}}$. Every global form $\omega \in \Gamma(\overline{\mathcal{G}}, \Omega_{log})$ is $G$-invariant (for $G$ acts trivially on its de Rham cohomology). Conversely, every invariant 1-form on $G$ extends to a 1-form on $\overline{\mathcal{G}}$ with logarithmic singularities (for $\text{dim} \ G = \text{dim} F^1H^1_{DR}(G(\mathbb{C}))$). Summarizing, we get an identification $\gamma$ of $F^1H^0(R^*_{DR}(M'))$ with the cotangent space $\mathfrak{g}^*$. The canonical pairing

$$\ker (\Lambda \oplus \mathfrak{g} \xrightarrow{\text{w} \text{exp}} G(\mathbb{C})) \otimes \mathfrak{g}^* \xrightarrow{\Theta^{-1} \otimes \gamma^{-1}} H^0(R^1_{\text{Betti}}(M')) \otimes F^1(H^0 R^*_{DR}(M')) \to \mathbb{C}$$

is given by the formula

$$(\lambda \oplus \theta) \otimes \eta \mapsto <\theta, \eta>.$$

It follows that, the Hodge filtration $F^{-1} \subset H^0(R^1_{\text{Betti}}(M'))$ is carried over by $\Theta'$ to the kernel of the projection

$$\ker (\Lambda \oplus \mathfrak{g} \xrightarrow{\text{w} \text{exp}} G(\mathbb{C})) \otimes \mathbb{C} \to \mathfrak{g}.$$

This completes the proof of (3.4).

3.9. Proof of formula (3.3). Set

$$d_{\leq 1} = Ho(Tot_Q \circ LAlb_b) : DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q}) \to d_{\leq 1}DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q}) \subset DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q}),$$

$$\overline{d}_{\leq 1} = Ho(Tot_Q \circ LAlb_b) : D^b(MHS^{Q}_{eff}) \to \overline{d}_{\leq 1}D^b(MHS^{Q}_{eff}) \subset D^b(MHS^{Q}_{eff}),$$

$$R^H_{Q,tr} = Ho(R^H_{Q}) : DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q}) \to D^b(MHS^{Q}_{eff}),$$

where $d_{\leq 1}DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q})$ (resp. $\overline{d}_{\leq 1}D^b(MHS^{Q}_{eff})$) is the smallest strictly full subcategory of $DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q})$ (resp. $D^b(MHS^{Q}_{eff})$) that contains the image of $Tot_Q$ (resp. $TQ$). The functor $d_{\leq 1}$ (resp. $\overline{d}_{\leq 1}$) is left adjoint to $d_{\leq 1}DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q}) \hookrightarrow DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q})$ (resp. $\overline{d}_{\leq 1}D^b(MHS^{Q}_{eff}) \rightarrow D^b(MHS^{Q}_{eff})$). We have to show that for every $M \in DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q})$ the morphism

$$(3.32) \quad \overline{d}_{\leq 1} \circ R^H_{Q,tr}(M) \xrightarrow{\alpha} R^H_{Q,tr} \circ d_{\leq 1}(M)$$

is an isomorphism. It suffices to do this in the case when $M$ is the motive of a smooth connected projective variety. Let $X$ be such a variety. Recall from ([BK1]) the structure of the Albanese motive $d_{\leq 1}(Q_{tr}[X])$. Let $X \to A_X$ be the canonical morphism from $X$ to the extended Albanese scheme of $X$ ([L1], Section 1). $A_X$ is a group scheme fitting into the following exact sequence

$$0 \to A_X^0 \to A_X \to \mathbb{Z} \to 0,$$

where $A_X^0$ is Serre’s Albanese abelian variety and $\mathbb{Z}$ is viewed as the discrete group scheme over $\mathbb{C}$. The sheaf with transfers $A_X$ represented by $A_X$ defines an object of the Voevodsky category $DM^{eff}_{gm,et}(\mathbb{C}, \mathbb{Q})$. We shall write $A_X \otimes \mathbb{Q}$ for its image in $DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q})$. It is clear that $A_X \otimes \mathbb{Q} \subset d_{\leq 1}DM^{eff}_{gm}(\mathbb{C}, \mathbb{Q})$. Consider the exact triangles

$$P \to Q_{tr}[X] \xrightarrow{\text{u}} A_X \otimes \mathbb{Q} \to P[1],$$

$$d_{\leq 1}(P) \xrightarrow{\text{u}} d_{\leq 1}(Q_{tr}[X]) \xrightarrow{\text{u}} A_X \otimes \mathbb{Q} \to d_{\leq 1}(P)[1],$$

where $Q_{tr}$ is the truncated cohomology of $X$.
where \( u \) is defined as the composition \( \mathbb{Q}_{tr}[X] \to \mathbb{Q}_{tr}[A_X] \to A_X \otimes \mathbb{Q} \). According to ([3, K1], Theorem 10.3.2) the second triangle yields a commutative diagram

\[
\begin{align*}
\text{Hom}(d_{\leq 1}(\mathbb{Q}_{tr}[X]), \mathbb{Q}(1)[2]) & \xrightarrow{\varphi^*} \text{Hom}(\mathbb{Q}(1)[2]) \\
\text{Pic}(X) & \simeq \text{Hom}(\mathbb{Q}_{tr}[X], \mathbb{Q}(1)[2]) & \to & NS_X \otimes \mathbb{Q}
\end{align*}
\]

(3.33)

where \( NS_X \) is the Néron-Severi group of \( X \) and the map at the bottom line equals the canonical projection \( \text{Pic}(X) \to NS_X \). Moreover, the induced morphism

\[
d_{\leq 1}(P) \to \text{Hom}(NS_X, \mathbb{Q}(1)[2])
\]

is an isomorphism.

It is enough to prove that (3.32) is an isomorphism for \( M = P \). By Lemma 3.9

\[
H_i \text{Hodge}^X(P) = 0 \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad H_i \text{Hodge}^X(P) \xrightarrow{\sim} H_i \text{Hodge}(Q_{tr}[X]) \quad \text{for} \quad i > 1.
\]

It follows, that \( \mathcal{E}_{\leq 1} \circ \text{Hodge}(P) = \mathcal{E}_{\leq 1}(H_2(X))[2] \) is of type \((-1, -1)\). Thus, we only need to show that morphism \( \alpha^* \) in the commutative diagram below is an isomorphism.

\[
\begin{align*}
\text{Hom}(R_{Q, tr} \circ d_{\leq 1}(P), \mathbb{Q}(1)[2]) & \xrightarrow{\alpha^*} \text{Hom}(\mathcal{E}_{\leq 1} \circ R_{Q, tr}^X(P), \mathbb{Q}(1)[2]) \\
\text{Hom}(P, \mathbb{Q}(1)[2]) & \xrightarrow{R_{Q, tr}^X} \text{Hom}(R_{Q, tr}^X(P), \mathbb{Q}(1)[2]) & \to & \text{Hom}_{\text{Hodge}}(H_2(X), \mathbb{Q}(1))
\end{align*}
\]

(3.34)

The next lemma shows that map \( c_1 : NS_X \otimes \mathbb{Q} = \text{Pic}(X)/\text{Pic}^0(X) \otimes \mathbb{Q} \to \text{Hom}_{\text{Hodge}}(H_2(X), \mathbb{Q}(1)) \) is induced by the first Chern class. Therefore, by the Lefschetz (1, 1) Theorem, \( c_1 \) and \( \alpha^* \) are isomorphisms.

**Lemma 3.14.** The diagram below is commutative.

\[
\begin{align*}
\text{Hom}(\mathbb{Q}_{tr}[X], \mathbb{Q}(1)[2]) & \xrightarrow{R_{Q, tr}^X} \text{Hom}(R_{Q, tr}^X(\mathbb{Q}_{tr}[X]), \mathbb{Q}(1)[2]) \\
\text{Pic}(X) \otimes \mathbb{Q} & \xrightarrow{c_1} \text{Hom}_{\text{Hodge}}(H_2(X), \mathbb{Q}(1)),
\end{align*}
\]

(3.35)

Here \( c_1 \) denotes the first Chern class map.

**Proof.** As \( X \) is projective, \( \text{Pic}(X) \) is generated by very ample line bundles. This reduces the proof to the case when \( X = \mathbb{P}^n_\mathbb{C} \). Moreover, since \( H^2(\mathbb{P}^n_\mathbb{C}, \mathbb{Q}) \xrightarrow{\sim} H^2(\mathbb{P}^1_\mathbb{C}, \mathbb{Q}) \), it suffices to prove the commutativity of the diagram for \( X = \mathbb{P}^1_\mathbb{C} \) and \( v \in \text{Hom}(\mathbb{Q}_{tr}[\mathbb{P}^1_\mathbb{C}], \mathbb{Q}(1)[2]) \) being the projection to the direct summand. In this case the lemma is true by definition. \( \square \)

The main theorem is proven.

**Notation.** DGcat, Ho(C), Ho(F), T(C_1, C_2), C′_p, C_{perf}, C_{prec}, D(C), D_{dg}(E), C_{dg}(E), Sm_k, A_{tr}[Sm_k], PSh_{tr}, A_{tr}[X], I^\alpha_{tr}, I^\beta_{tr}, I^\gamma_{tr}, D_{M eff}, D_{M eff}, D_{M eff}, D_{M eff}, D_{M eff}, D_{M eff}, D_{M eff}, D_{M eff}, A_{[Sm_k]}, I^\alpha_{tr} A[S_{M}], I^\beta_{tr} A[S_{M}], A_{[Sm_k]}, C^\text{sing}, R_{\text{Betti}}(\mathcal{E}), \text{MHS}^A, \text{MHS}_{\text{et}}^A, R_{A_{\text{Hodge}}}, C_{\Delta}, D_{dg}(G, A), S_A, D_{dg}(G, \mathbb{Z}/l), R_{et}, \text{M}_{1}(k), D_{dg}(\text{M}_{1}(k)), \text{Tot}_A, \text{LAlb}_Q, T_{A_{\text{Hodge}}}, T_{Z_{l/\ell}}, \text{LAlb}_Q, \text{Tot}_A, \text{LAlb}_Q, \text{Tot}_A, \text{LAlb}_Q, \text{Tot}_A
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