THE ENERGY CONSERVATION FOR WEAK SOLUTIONS TO THE RELATIVISTIC NORDSTRÖM-VLASOV SYSTEM

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Abstract. We study the Cauchy problem of the relativistic Nordström-Vlasov system. Under some additional conditions, total energy for weak solutions with BV scalar field are shown to be conserved.

1. Introduction. As a relativistic Lorentz invariant generalization of the Vlasov-Poisson system in the gravitational case, the three dimensional relativistic Nordström-Vlasov system consists of Vlasov’s equation for the particle density $f^\#$

$$\partial_t f^\# + v(p) \cdot \nabla_x f^\# - [(\partial_t \phi + v(p) \cdot \nabla_x \phi)p + (1 + p^2)^{-1/2} \nabla_x \phi] \cdot \nabla_p f^\# = 0$$

and the Nordström scalar gravitation theory (i.e., the gravitational effects are mediated by a scalar field $\phi$)

$$\partial_t^2 \phi - \Delta_x \phi = -e^{4\phi} \int_{\mathbb{R}^3} f^\# \frac{dp}{\sqrt{1 + p^2}},$$

where units are chosen such that all physical constant are equal to unity, $v(p) = \frac{p}{\sqrt{1 + p^2}}$ denotes the relativistic velocity of a particle with momentum $p$ and $t \in \mathbb{R}$ represents the time, $x \in \mathbb{R}^3$ the position and $p \in \mathbb{R}^3$ the momentum. For the purpose of the present investigation, it is convenient to rewrite the system in terms of the new unknowns $(f, \phi)$ where $f$ is given by

$$f(t, x, p) = e^{4\phi} f^\#(t, x, p),$$

and then we consider the following Nordström-Vlasov system (NV) (see [7, 6] for details):

$$\partial_t f + v(p) \cdot \nabla_x f - [(S\phi)p + (1 + p^2)^{-1/2} \nabla_x \phi] \cdot \nabla_p f = 4fS\phi, \quad (1)$$

$$\partial_t^2 \phi - \Delta_x \phi = -\mu, \quad (2)$$

$$\mu(t, x) = \int_{\mathbb{R}^3} f(t, x, p) \frac{dp}{\sqrt{1 + p^2}}, \quad (3)$$

where $S = \partial_t + v(p) \cdot \nabla_x$ is the free transport operator and $\phi(t, x)$ is the scalar gravitational field generated by the particles. In modern theories of classical and

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quantum gravity, the scalar fields is essential [19]. In recently years, the (NV) has been investigated extensively and readers can refer to [7, 6, 4, 5, 15, 11], where S. Calogero and G. Rein showed in [4] that global weak solutions for the (NV) in three dimensions exist and the energy of the system is bounded at all time by its initial value, but so far conservation of energy has not been obtained. Related with (NV), some classical systems such as the relativistic Vlasov-Maxwell system (RVM) and the Vlasov-Poisson system (VP), are also worthy of attention. For (RVM), the problem of the existence of global weak solutions was solved by R. J. Diperna and P. L. Lions in [8] and the same problem is revisited in [16]. R. Sospedra-Alfonso proved in [18] that the total energy for (RVM) is preserved. For (VP), G. Loeper established the uniqueness of global weak solution under the assumption of bounded spatial density [13] and further E. Miot improved the result in [14]. In this paper, by means of the methods introduced by R. J. Diperna and P. L. Lions in [9], we will show that under some additional conditions the total energy is preserved for almost all time. These methods were firstly applied to the Vlasov equation with BV coefficient in [2]. For BV functions, we refer the readers to [1]. For completeness, we recall the main conclusions in [4] in the following theorem.

**Theorem 1.1.** For any initial data $0 \leq f(0) = f_0 \in L^1_{\text{kin}}(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ where the Banach space $L^1_{\text{kin}}(\mathbb{R}^6)$ is endowed with the norm

$$
\| \cdot \|_{L^1_{\text{kin}}(\mathbb{R}^6)} = \int_{\mathbb{R}^6} \sqrt{1 + |p|^2} \cdot |dpdx,
$$

and $\phi(0) = \phi_0 \in H^s(\mathbb{R}^3), \partial_t \phi(0) = \phi_1 \in H^{s-1}(\mathbb{R}^3)$ for some $s > 3/2$, there exists a global weak solution $(f, \phi)$ of the (NV) system (1)-(3) in distributional sense such that

$$
0 \leq f \in L^\infty(\mathbb{R}; L^1_{\text{kin}}(\mathbb{R}^6)) \cap L^\infty_{\text{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^6)), \phi \in L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}^3)),
$$

with

$$
\partial_t \phi \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3)), \nabla_x \phi \in L^\infty(\mathbb{R}; L^2(\mathbb{R}^3)), e^\phi \in H^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^3).
$$

In addition, the total energy

$$
\varepsilon(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{1 + |p|^2} f(t, x, p)dxdp + \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \phi(t, x)|^2 + |\nabla_x \phi(t, x)|^2 dx,
$$

is bounded at any time $t$ by its initial value.

**Remark 1.** (i) In the present case, let $(f, \phi)$ is a smooth solution of (1)-(3), the conservation law of the total energy is easily deduced by multiplying (1) by $\sqrt{1 + |p|^2}$ and integrating over the phase space to obtain

$$
\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{1 + |p|^2} f(t, x, p)dxdp = \int_{\mathbb{R}^3} \partial_t \phi dxdx, \tag{4}
$$

then multiplying (2) by $\partial_t \phi$ and integrating over the position space to obtain

$$
\frac{d}{dt} \frac{1}{2} \int_{\mathbb{R}^3} (\partial_t \phi)^2 + (\nabla_x \phi)^2 dx = -\int_{\mathbb{R}^3} \partial_t \phi dxdx,
$$

where we have used the divergence theorem. Then we add the two equalities and integrate from 0 to $t$, we obtain

$$
\varepsilon(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{1 + |p|^2} f(t, x, p)dxdp + \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t \phi(t, x)|^2 + |\nabla_x \phi(t, x)|^2 dx = \varepsilon(0).
$$
(ii) We notice that for the weak solution \( f \in L^\infty(R; L^{1}_{kin}(R^6)) \cap L^\infty_{loc}(R; L^\infty(R^6)) \) we can estimate \( \mu \) such that
\[
\| \mu(t) \|_{L^2}^2 = \| \int_{|p| \leq R} f(t,x,p) \frac{dp}{\sqrt{1 + |p|^2}} + \int_{|p| \geq R} f(t,x,p) \frac{dp}{\sqrt{1 + |p|^2}} \|_{L^2}^2 \\
\leq C \| f(t) \|_{L^\infty} \int_{R^3} \sqrt{1 + |p|^2} f(t,x,p) dx dp.
\]

In the present paper, we will show that a weak solution can also preserve the total energy under some specific conditions, more precisely, we have

**Theorem 1.2.** Assume that the initial data \((f_0, \phi_0, \phi_1)\) satisfy the same conditions of Theorem 1.1, and let \((f, \phi)\) be a weak solution of the \((NV)\) system with the initial data \((f_0, \phi_0, \phi_1)\). If \( \partial_t \phi \in L^1([0,T]; BV_{loc}(R^3)) \), \( \partial_x \phi \in L^1([0,T]; BV_{loc}(R^3)) \) and \( m_k(f)(t,x) := \int_{R^3} |p|^k f(t,x,p) dp \in L^\infty([0,T]; L^2(R^3)) \) for some \( k > 1 \), then the total energy for the \((NV)\) system satisfies \( \varepsilon(t) = \varepsilon(0) \) for almost all \( 0 \leq t < T \).

In this paper, for \( p \in [1, \infty] \), the norm of \( f(x) \in L^p(R^n) \) is denoted by \( \| f \|_{L^p}. \) \( C(R) \) or \( C_R \) denotes a positive constant depending on the parameter \( R \) and the notation \( C \) stands for generic constants whose values may change from line to line. \( C^i_c(R^p) \) denotes the space of compactly supported continuously differentiable functions. The rest of the paper is devoted to proving Theorem 1.2. In the next section, by establishing a lemma which is much more complicated than (RVM) where the vector field is divergence-free, we show the energy balance for the Vlasov equation which is analogous to (4). In section 3, we complete the proof of Theorem 1.2 by using the energy balance for the Vlasov equation obtained in section 2.

2. Energy balance for the Vlasov equation. Let \( \delta_{\varepsilon_1} \) and \( \delta_{\varepsilon_2} \) be two regularizing kernels respectively with respect to the space variable and the momentum with \( \varepsilon_1 > 0, \varepsilon_2 > 0 \), i.e.,

\[
\delta \in C^\infty_c(R^3), \quad \delta \geq 0, \quad \int_{R^3} \delta(x) dx = 1, \quad \delta \text{ even,}
\]

\[
\delta_{\varepsilon_1}(x) = \frac{1}{(\varepsilon_1)^3} \delta\left(\frac{x}{\varepsilon_1}\right), \quad \delta_{\varepsilon_2}(x) = \frac{1}{(\varepsilon_2)^3} \delta\left(\frac{p}{\varepsilon_2}\right).
\]

**Lemma 2.1.** Assume \((f, \phi)\) is a weak solution of (1)-(3) defined on \([0,T]\). If \( \partial_t \phi \in L^1([0,T]; BV_{loc}(R^3)) \), \( \partial_x \phi \in L^1([0,T]; BV_{loc}(R^3)) \), then there exist two sequences \( \varepsilon_{1}^n > 0, \varepsilon_{2}^n > 0, \varepsilon_{1}^n \to 0, \varepsilon_{2}^n \to 0 \) such that
\[
\begin{align*}
\varepsilon_{1}^n \int_{R^3} & v(p) \cdot \nabla_x (f \ast \delta_{\varepsilon_1} \delta_{\varepsilon_2}) \cdot (S(\phi)p + (1 + p^2)^{-1/2} \nabla_x \phi) \cdot \nabla_p (f \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2})) \\
& - (v(p) \cdot \nabla_x f) \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2}) + ((S(\phi)p + (1 + p^2)^{-1/2} \nabla_x \phi) \cdot \nabla_p f) \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2}) \\
& \to 0 \text{ in } L^1([0,T]; L^1_{loc}(R^3 \times R^3)).
\end{align*}
\]

**Proof.** For simplicity, we firstly define \( r_{\varepsilon_1, \varepsilon_2} := I^p_1 - I^p_2 + I^\phi_1 - I^\phi_2 \) with
\[
\begin{align*}
I^p_1 &= v(p) \cdot \nabla_x (f \ast \delta_{\varepsilon_1} \delta_{\varepsilon_2}), \\
I^\phi_1 &= \{[(S(\phi)p + (1 + p^2)^{-1/2} \nabla_x \phi) \cdot \nabla_p f] \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2}), \\
I^p_2 &= (v(p) \cdot \nabla_x f) \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2}), \\
I^\phi_2 &= [(S(\phi)p + (1 + p^2)^{-1/2} \nabla_x \phi) \cdot \nabla_p (f \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2}))).
\end{align*}
\]
In the rest of the proof, we also define any open set \( \Omega := B_R \times B_R \subset \mathbb{R}^3 \times \mathbb{R}^3 \) such that \( \Omega + \text{supp}(\delta_{\varepsilon_1} \delta_{\varepsilon_2}) \subset B_{R+1} \times B_{R+1} \). By the property of the regularization kernel and the divergence theorem, we can estimate \( I_1^p - I_2^p \) as follows:

\[
\|I_1^p - I_2^p\|_{L_1^p, x, p (0, T \times \Omega)} \\
= \|\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (v(p) - v(p - q))(f(t, x, y, p - q) - f(t, x, p)) \nabla_q \delta_{\varepsilon_1}(y) \delta_{\varepsilon_2}(q) dy dq\|_{L_1^p, x, p (0, T \times \Omega)} \\
\leq C_R \frac{\varepsilon^2_1}{\varepsilon_1} \|\int_{\mathbb{R}^3} \int_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x, p)\|_{L_1^p(0, T \times L_2^p, \Omega))} \cdot \|Dv(p)\|_{L_2^p(B_R)} \|f(t, x, p) - f(t, x, y, p - q)\|_{L_1^p(0, T \times L_2^p, \Omega))},
\]

where we have used \( \int |\varepsilon_1 \nabla_y \delta_{\varepsilon_1}(y)| dy = \int |\partial_q \delta(y)| dy \leq C \). For the term \( I_1^q \), we divide it into three parts and integrate by parts such that

\[
I_1^{q1} = -3f(\partial_t \phi) * (\delta_{\varepsilon_1} \delta_{\varepsilon_2}) + (\partial_t \phi \cdot \phi) * (\delta_{\varepsilon_1} \nabla_p \delta_{\varepsilon_2}), \\
I_1^{q2} = \{f \nabla_p \cdot [p(v(p) \cdot \nabla_x \phi)] \} * (\delta_{\varepsilon_1} \delta_{\varepsilon_2}) + \{f(v(p) \cdot \nabla_x \phi) \} * (\delta_{\varepsilon_1} \nabla_p \delta_{\varepsilon_2}), \\
I_1^{q3} = -\{f \nabla_p \cdot [f(1 + p^2)^{-1/2} \nabla_x \phi] \} * (\delta_{\varepsilon_1} \delta_{\varepsilon_2}) + \{f(1 + p^2)^{-1/2} \nabla_x \phi \} * (\delta_{\varepsilon_1} \nabla_p \delta_{\varepsilon_2}).
\]

For \( I_2^p \), in the same way, we have

\[
I_2^p = \partial_t \phi \cdot [f * (\delta_{\varepsilon_1} \nabla_p \delta_{\varepsilon_2})] + (\partial_t \phi \cdot \phi) * (\delta_{\varepsilon_1} \nabla_p \delta_{\varepsilon_2}) \\
+ (1 + p^2)^{-1/2} \nabla_t \phi(t, x) [f * (\delta_{\varepsilon_1} \nabla_p \delta_{\varepsilon_2})] := I_2^{p1} + I_2^{p2} + I_2^{p3}.
\]

Now we need to estimate each term for \( I_1^p - I_2^p \). Firstly for \( I_1^{p1} - I_2^{p1} \), we obtain that

\[
I_1^{p1} - I_2^{p1} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\partial_t \phi(t, x, y) - \partial_t \phi(t, x, p)| \nabla_y \delta_{\varepsilon_2}(y) \delta_{\varepsilon_1}(x) dy dp dq dt \\
(f(t, x, y, p - q) - f(t, x, p)) \cdot \nabla_q \delta_{\varepsilon_2}(q) \delta_{\varepsilon_1}(y) dy dq \\
+ 3f(t, x, p) \cdot (\partial_t \phi * (\delta_{\varepsilon_1} \delta_{\varepsilon_2})) - 3(\partial_t \phi \cdot f) * (\delta_{\varepsilon_1} \delta_{\varepsilon_2}) \\
:= J_1(t, x, p) + J_2(t, x, p).
\]

For \( J_1 \), it follows by the assumption that

\[
\|J_1(t, x, p)\|_{L_1^p, x, p (0, T \times \Omega)} \\
\leq \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\partial_t \phi(t, x, y) - \partial_t \phi(t, x, p)| |f(t, x, y, p - q) - f(t, x, p)| \\
\nabla_q \delta_{\varepsilon_2}(q) \delta_{\varepsilon_1}(y) dy dp dq dt \\
+ \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\partial_t \phi(t, x) \varepsilon_2| |f(t, x, y, p - q) - f(t, x, p)| \\
\nabla_q \delta_{\varepsilon_2}(q) \delta_{\varepsilon_1}(y) dy dp dq dt \\
\leq \frac{C_R}{\varepsilon_2} \|\partial_t \phi(t, x, y) - \partial_t \phi(t, x, p)\|_{L_1^p, x, p (0, T \times B_R)} + C_R \|\partial_t \phi\|_{L^\infty(0, T \times L_2^2)} \\
\sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x, y, p - q) - f(t, x, p)\|_{L_2^p(\Omega)} + C_R \|\partial_t \phi\|_{L^\infty(0, T \times L_2^2(\Omega))},
\]
where we note that according to the assumption, we have

\[
\|f(t, x, y, p - q) - f(t, x, p)\|_{L^1([0,T]; L^2_y(\Omega))} \leq C_R \varepsilon^2 \sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x, y, p - q) - f(t, x, p)\|_{L^\infty_x, y, p(\Omega)}
\]

\[
+ C_R \|\partial_t \phi\|_{L^\infty([0,T]; L^2_y(\Omega))} \cdot \sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x, y, p - q) - f(t, x, p)\|_{L^1([0,T]; L^2_y(\Omega))},
\]

which will be used in the following, and where \(o(\varepsilon_1) \to 0\) when \(\varepsilon_1 \to 0\). On the other hand, by the property of mollifiers, the first term in \(J_2\) converges to \(3f(t, x, p) \cdot \partial_t \phi\) and the second one in \(J_2\) converges to \(-3f(t, x, p) \cdot \partial_t \phi\) in \(L^1([0,T) \times B_R]\) with \(\varepsilon_1, \varepsilon_2 \to 0\), which implies that \(J_2\) tends to 0 in \(L^1([0,T]; L^1_{loc}(R^3 \times R^3))\).

Secondly, for \(I^{12} - I^{22}\), we have by integration by parts

\[
I^{12} - I^{22} = \int_{R^3} \int_{R^3} (f(t, x, y, p - q) - f(t, x, p)) [(v(p - q) \cdot \nabla_y \phi(t, x - y))
\cdot (\nabla_q \delta_{\varepsilon_2}(q)) - (v(p) \cdot \nabla_x \phi(t, x))p
f(t, x, y, p - q) \nabla_q \delta_{\varepsilon_2}(q) \delta_{\varepsilon_1}(y) dy dq
+ \{(\nabla_p \cdot v(v(p) \cdot \nabla_x \phi)) \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2}) - f \ast (\nabla_p \cdot v(v(p) \cdot \nabla_x \phi)) \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2})\}
= J_3(t, x, p) + J_4(t, x, p).
\]

Similar to \(J_1\), we can estimate \(J_3\) by (8) and the Hölder equality such that

\[
\|J_3(t, x, p)\|_{L^1_{x,y,p}(0,T]} \leq C_R \varepsilon^2 \sup_{|y| \leq |q| \leq \varepsilon_2} \|f(t, x, y, p - q) - f(t, x, p)\|_{L^\infty_x, y, p(0,T \times \Omega)}
\]

\[
+ (R + \varepsilon_2) \|\partial_x \phi\|_{L^2_x([0,T] \times B_{R_1+1})} \|Dv(p)\|_{L^2_y(B_{R+1})} \cdot \sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x, y, p - q) - f(t, x, p)\|_{L^2_y(x, p)(0,T \times \Omega)}
\]

\[
+ C_R \|\partial_x \phi\|_{L^2_y([0,T] \times B_{R+1})} \cdot \sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x, y, p - q) - f(t, x, p)\|_{L^2_y(x, p)(0,T \times \Omega)}.
\]

Finally, the same way implies that

\[
I^{13} - I^{23} = \int_{R^3} \int_{R^3} (f(t, x, y, p - q) - f(t, x, p)) [(1 + (p - q)^2)^{-1/2} \nabla_x \phi(t, x - y)
\cdot (\nabla_q \delta_{\varepsilon_2}(q) \delta_{\varepsilon_1}(y) dy dq
+ f \ast (\nabla_p \cdot (1 + p^2)^{-1/2} \nabla_x \phi)) \ast (\delta_{\varepsilon_1} \delta_{\varepsilon_2})]
\]

\[
= J_5(t, x, p) + J_6(t, x, p).
\]
For $J_5$, by the same method of estimating $J_3$ we have that
\begin{align}
    &\|J_5(t, x, p)\|_{L^1_t(x,p)(0,T|\times \Omega)} \\
    &\leq C_R \frac{o(\varepsilon_1)}{\varepsilon_2} \sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x - y, p - q) - f(t, x, p)\|_{L^\infty_t(x,p)(0,T|\times \Omega)} \\
    &\quad + C_R \sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x - y, p - q) - f(t, x, p)\|_{L^2_t(x,p)(0,T|\times \Omega)} \\
    &\quad \|\partial^2_\phi\|_{L^2_t(x,p)(0,T|\times \Omega)}. \tag{10}
\end{align}

Now, by the assumption of the Lemma, we obtain that
\begin{align}
    &\sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x - y, p - q) - f(t, x, p)\|_{L^\infty_t(x,p)(0,T|\times \Omega)} \\
    &\leq C(\|f\|_{L^\infty_t(x,p)(0,T|\times \Omega)}), \tag{11}
\end{align}

and the translation continuity in $L^2$ implies that as $\varepsilon_1, \varepsilon_2 \to 0$
\begin{align}
    &\sup_{|y| \leq \varepsilon_1, |q| \leq \varepsilon_2} \|f(t, x - y, p - q) - f(t, x, p)\|_{L^2_t(x,p)(0,T|\times \Omega)} \to 0. \tag{12}
\end{align}

Hence, we can appropriately choose two sequences $\varepsilon^n_1, \varepsilon^n_2 \to 0$ with $o(\varepsilon^n_1)/\varepsilon^n_2 \leq \frac{1}{n}$, as $n \to \infty$ such that for some sufficient large $n$, (6), the second term in (7), the second term and the third one in (9), and the second term in (10) are less than $1/n$ by (12). Then by (11), the right hand sides of (7), (9) and (10) tend to 0 with $n \to \infty$. In addition, just as $J_2$, we can easily show that $\|J_4\|_{L^1_t(x,p)(0,T|\times \Omega)}$ and $\|J_6\|_{L^1_t(x,p)(0,T|\times \Omega)}$ also tend to zero when $n \to \infty$, which implies $r_{\varepsilon_1^n, \varepsilon_2^n} \to 0$ in $L^1([0,T]; L^1_{loc}(R^3 \times R^3))$. So we complete the proof. \qed

**Proposition 1.** In addition to assumptions of Lemma 2.1, we further assume $m_k(f)(t, x) = \int_{R^3} |p|^k f(t, x, p) dp \in L^\infty([0,T]; L^2(R^3))$ for some $k > 1$. Then for almost all $t \in [0, T],$
\begin{align}
    &\int_{R^3} \int_{R^3} \sqrt{1 + p^2} f dx dp = \int_{R^3} \int_{R^3} \sqrt{1 + p^2} f_0 dx dp + \int_0^t \int_{R^3} \partial_s \phi \cdot \mu(s, x) dx ds.
\end{align}

**Proof.** We regularize the Vlasov equation (1) with $\delta_{\varepsilon_1^n}$ and $\delta_{\varepsilon_2^n}$ such that
\begin{align}
    \partial_t f^n + v(p) \cdot \nabla_x f^n - [(S\phi)p + (1 + p^2)^{-1/2} \nabla_x \phi] \cdot \nabla_p f^n &= v(p) \cdot \nabla_x f^n - [(S\phi)p + (1 + p^2)^{-1/2} \nabla_x \phi] \cdot \nabla_p f^n - (v(p) \nabla_x f) * (\delta_{\varepsilon_1^n} \delta_{\varepsilon_2^n}) \\
    + [(S\phi)p + (1 + p^2)^{-1/2}] \cdot \nabla_p f) * (\delta_{\varepsilon_1^n} \delta_{\varepsilon_1^n}) + (4S\phi) * (\delta_{\varepsilon_1^n} \delta_{\varepsilon_2^n}) \nonumber \\
    &:= r_n + (4S\phi) * (\delta_{\varepsilon_1^n} \delta_{\varepsilon_2^n}),
\end{align}

where $r_n = r_{\varepsilon_1^n, \varepsilon_2^n}$ is defined in Lemma 2.1. Now, let $\zeta(x, p)$ be the six dimensional mollifier such that $\zeta \equiv 1$ on the ball $B_1$ and suppor $\zeta \subseteq B_2$. For any $R \geq 1$, we multiply the above equality by $\sqrt{1 + p^2} \zeta_R(x, p) := \sqrt{1 + p^2} \zeta(\frac{x}{R}, \frac{p}{R})$ and integrate with respect to $t, x, p$, then we obtain
\begin{align}
    &\int_{R^3 \times R^3} \zeta_R(x, p) \sqrt{1 + p^2} f^n(t, x, p) dp dx \\
    &\quad - \int_{R^3 \times R^3} \zeta_R(x, p) \sqrt{1 + p^2} f^n(0, x, p) dp dx
\end{align}
\begin{align}
    &= \int_0^t \int_{R^3} \int_{R^3} \nabla_x \zeta_R(x, p) \sqrt{1 + p^2} v(p) f^n(s, x, p) dp dx ds.
\end{align}
For the second term, it follows that
\[
\int_{t}^{t_0} \int_{R^3} \frac{1}{1 + p^2} \zeta R \partial_s \phi f^n dx dpds
\]
\[
+ \int_{t}^{t_0} \int_{R^3} \int_{R^3} \int_{R^3} \frac{1}{1 + p^2} \zeta R \partial_s \phi f^n dx dpds
\]
\[
+ \int_{t}^{t_0} \int_{R^3} \int_{R^3} (\partial_p \zeta R \cdot p) \sqrt{1 + p^2} \partial_s \phi f^n dx dpds
\]
\[
+ \int_{t}^{t_0} \int_{R^3} \int_{R^3} 4 \sqrt{1 + p^2} \zeta R (f \partial_s \phi) \cdot \delta_{\varepsilon_1^2} dx dpds
\]
\[
- \int_{t}^{t_0} \int_{R^3} \int_{R^3} 4 \sqrt{1 + p^2} \zeta R f^n \partial_s \phi dx dpds
\]
\[
+ [ -4 \int_{t}^{t_0} \int_{R^3} \int_{R^3} \zeta R (p \cdot \nabla_x \phi) f^n dx dpds
\]
\[
+ 4 \int_{t}^{t_0} \int_{R^3} \int_{R^3} \sqrt{1 + p^2} \zeta R (f \cdot (v(p) \cdot \nabla_x \phi)) \cdot \delta_{\varepsilon_1^2} dx dpds
\]
\[
- \int_{t}^{t_0} \int_{R^3} \int_{R^3} (\nabla_p \zeta R \cdot p) \cdot (p \cdot \nabla_x \phi) f^n dx dpds
\]
\[
- \int_{t}^{t_0} \int_{R^3} \int_{R^3} \nabla_p \zeta R \nabla_x \phi f^n dx dpds.
\]

Let \( n \rightarrow \infty \), it follows from Lemma 2.1, the property of mollifier and (13) that
\[
\int_{R^3} \int_{R^3} \zeta R(x, p) \sqrt{1 + p^2} f(t, x, p) dx dp
\]
\[- \int_{R^3} \int_{R^3} \zeta R(x, p) \sqrt{1 + p^2} f_0(x, p) dx dp
\]
\[- \int_{t}^{t_0} \int_{R^3} \int_{R^3} \frac{1}{1 + p^2} \zeta R \partial_s \phi f dx dpds
\]
\[
= \int_{0}^{t} \int_{R^3} \int_{R^3} \nabla_x \zeta R(x, p) \sqrt{1 + p^2} v(p) f(s, x, p) dx dpds
\]
\[
+ \int_{0}^{t} \int_{R^3} \int_{R^3} (\partial_p \zeta R \cdot p) \sqrt{1 + p^2} \partial_s \phi f dx dpds
\]
\[
- \int_{0}^{t} \int_{R^3} \int_{R^3} (\nabla_p \zeta R \cdot p) \cdot (p \cdot \nabla_x \phi) f dx dpds - \int_{0}^{t} \int_{R^3} \int_{R^3} \nabla_p \zeta R \nabla_x \phi f dx dpds.
\]

Now we estimate each term on the right hand side of (14), for the first term we have
\[
\left| \int_{0}^{t} \int_{R^3} \int_{R^3} \nabla_x \zeta R(x, p) \sqrt{1 + p^2} v(p) f(s, x, p) dx dpds \right| = \frac{C}{R} \int_{0}^{t} \int_{R^3} \int_{R^3} \sqrt{1 + p^2} f(s, x, p) dx dpds \leq \frac{C}{R}. \quad (15)
\]
For the second term, it follows that
\[
\left| \int_{0}^{t} \int_{R^3} \int_{R^3} (\partial_p \zeta R \cdot p) \sqrt{1 + p^2} \partial_s \phi f dx dpds \right| \leq \frac{C}{R} \int_{0}^{t} \int_{R^3} |\partial_s \phi(s, x)| \int_{1 \leq R \leq |p| \leq 2R} |p|^2 f(s, x, p) dx dp ds.
In order to complete the proof, we take a family of smooth cut-off functions \( \zeta_R = \zeta (\frac{x}{R}) \), where \( R \geq 1 \), \( \zeta \in C_c^\infty (\mathbb{R}^3), \zeta \geq 0 \) and \( \zeta = 1 \) on \( B_1 \) such that \( \text{supp} \zeta \subset B_2 \). Multiplying (20) by \( \zeta_R \partial_t \phi_n \), we have that

\[
\frac{1}{2} \zeta_R \frac{\partial}{\partial t} |\partial_t \phi_n|^2 - \Delta_x \phi_n \cdot \partial_t \phi_n \cdot \zeta_R = -\mu_n \cdot \zeta_R \cdot \partial_t \phi_n.
\]

3. Proof of the main result. In this section, we shall finish the proof of Theorem 1.2 by utilizing Proposition 1.

Proof of Theorem 1.2. We first show that under the assumptions of Theorem 1.2, we have for almost all \( t \in [0, T] \),

\[
\frac{1}{2} (|\partial_t \phi(t)|^2_{L^2} + |\nabla_x \phi(t)|^2_{L^2}) + \int_0^t \int_{\mathbb{R}^3} \partial_t \phi(s, x) \cdot \mu(s, x) ds dx
\]

\[
= \frac{1}{2} (|\phi_1(t)|^2_{L^2} + |\nabla_x \phi_0||L^2_{L^2}).
\]

Actually, let \( \delta \) be the three dimensional mollifier, i.e., \( 0 \leq \delta \in C_c^\infty (\mathbb{R}^3), \int_{\mathbb{R}^3} \delta dx = 1 \) and \( \delta \) be even, and set \( \delta_n = n^3 \delta (nx) \). We consider the regularized equation of (2):

\[
\partial_t^2 \phi_n - \Delta_x \phi_n = -\mu_n,
\]

where \( \mu_n = \mu * \delta_n, \phi_n = \phi * \delta_n \). Then the scalar field \( \phi_n \) with respect to \( x \) is smooth and \( \phi_n(t, x) \in H^3_{loc}([0, T[ \times \mathbb{R}^3) \). By the chain rule in Sobolev spaces, we know that for almost all \( t \in ]0, T[ \),

\[
\frac{1}{2} \frac{\partial}{\partial t} (|\partial_t \phi_n(t)|^2 + |\nabla_x \phi_n(t)|^2) = \partial_t \phi_n \cdot \partial_t^2 \phi_n + \nabla_x \phi_n \cdot \partial_t \nabla_x \phi_n.
\]

In order to complete the proof, we take a family of smooth cut-off functions \( \zeta_R = \zeta (\frac{x}{R}) \), where \( R \geq 1 \), \( \zeta \in C_c^\infty (\mathbb{R}^3), \zeta \geq 0 \) and \( \zeta = 1 \) on \( B_1 \) such that \( \text{supp} \zeta \subset B_2 \). Multiplying (20) by \( \zeta_R \partial_t \phi_n \), we have that

\[
\frac{1}{2} \zeta_R \frac{\partial}{\partial t} |\partial_t \phi_n|^2 - \Delta_x \phi_n \cdot \partial_t \phi_n \cdot \zeta_R = -\mu_n \cdot \zeta_R \cdot \partial_t \phi_n.
\]
Integrating the above equality with respect to $t$ and $x$, using the divergence theorem we have that

$$\frac{1}{2} \int_{R^3} \left( |\partial_t \phi_n(t, x)|^2 + |\nabla_x \phi_n(t, x)|^2 \right) \zeta_R dx \quad - \frac{1}{2} \int_{R^3} \left( |\partial_t \phi_n(0, x)|^2 + |\nabla_x \phi_n(0, x)|^2 \right) \zeta_R dx \quad = \quad - \int_0^t \int_{R^3} \nabla_x \phi_n(s, x) \cdot \partial_x \zeta_R(x, t) \cdot \partial_s \phi_n(s, x) dx ds \quad - \int_0^t \int_{R^3} \mu_n(s, x) \partial_s \phi_n(s, x) \zeta_R dx ds.$$

For the left side above, it follows by the theorem of smooth approximations [10] that as $n \to \infty$,

$$\frac{1}{2} \int_{R^3} \left( |\partial_t \phi_n(t, x)|^2 + |\nabla_x \phi_n(t, x)|^2 \right) \zeta_R dx \quad - \frac{1}{2} \int_{R^3} \left( |\partial_t \phi_n(0, x)|^2 + |\nabla_x \phi_n(0, x)|^2 \right) \zeta_R dx \quad \to \quad \frac{1}{2} \int_{R^3} \left( |\partial_t \phi(t, x)|^2 + |\nabla_x \phi(t, x)|^2 \right) \zeta_R dx \quad - \frac{1}{2} \int_{R^3} \left( |\partial_1(x)|^2 + |\nabla_x \phi_0(x)|^2 \right) \zeta_R dx.$$

Now we turn to the convergence of the second term on the right side above. Due to (5), Hölder inequality and Lebesgue dominate convergence theorem, we have

$$\int_0^t \int_{R^3} \mu_n(s, x) \partial_s \phi_n(s, x) dx ds \to \int_0^t \int_{R^3} \mu(s, x) \partial_s \phi(s, x) dx ds, \quad n \to \infty,$$

and the same method also implies that as $n \to \infty$,

$$\int_0^t \int_{R^3} \nabla_x \phi_n(s, x) \cdot \partial_x \zeta_R(x, t) \cdot \partial_s \phi_n(s, x) dx ds \quad \to \quad \int_0^t \int_{R^3} \nabla_x \phi(s, x) \cdot \partial_x \zeta_R(x, t) \cdot \partial_s \phi(s, x) dx ds.$$

Then for almost all $t \in [0, T]$, we have

$$\frac{1}{2} \int_{R^3} \left( |\partial_t \phi(t, x)|^2 + |\nabla_x \phi(t, x)|^2 \right) \zeta_R dx \quad - \frac{1}{2} \int_{R^3} \left( |\partial_t \phi(0, x)|^2 + |\nabla_x \phi(0, x)|^2 \right) \zeta_R dx \quad = \quad - \int_0^t \int_{R^3} \nabla_x \phi(s, x) \cdot \partial_x \zeta_R(x, t) \cdot \partial_s \phi(s, x) dx ds \quad - \int_0^t \int_{R^3} \mu(s, x) \partial_s \phi(s, x) \cdot \zeta_R dx ds.$$

We estimate the first term on right side and find that

$$\left| \int_0^t \int_{R^3} \nabla_x \phi(s, x) \cdot \partial_s \phi(s, x) \cdot \partial_x \zeta_R(x) dx ds \right| \leq \frac{C_T}{R} \|\partial_x \phi\|_{L^\infty((0,T), L^2(R^3))} \|\partial_t \phi\|_{L^\infty((0,T), L^2(R^3))}.$$
Let $R \to \infty$, we obtain that for almost all $t \in [0, T[$

$$
\frac{1}{2} \int_{\mathbb{R}^3} (|\partial_t \phi(t, x)|^2 + |\nabla_x \phi(t, x)|^2) dx - \frac{1}{2} \int_{\mathbb{R}^3} (|\partial_t \phi(0, x)|^2 + |\nabla_x \phi(0, x)|^2) dx
$$

$$
= - \int_0^t \int_{\mathbb{R}^3} \mu(s, x) \partial_s \phi(s, x) dx ds,
$$

which completes the proof of (19). Now, the desired result follows from Proposition 1 and (19).

4. Concluding remarks. Here we stress a few points about our assumption made in Theorem 1.2. (i) Since $W^{1,1}(U) \subset BV(U)$ for any open set $U \subseteq \mathbb{R}^3$, we point out that if the derivatives of the scalar field $\phi$ is in $L^\infty_{loc}([0, T]; W^{1,1}_{loc}(\mathbb{R}^3))^3$, our main result also holds true. Notice that the proof of the estimates for the scalar field $\phi$ in [7], under the condition

$$
\int_{\mathbb{R}^3} |p|^k f(t, x, p) dp \in L^\infty_{loc}([0, T]; L^p_{\mathbb{R}^3}), \quad k \geq 2,
$$

the scalar field $\phi$ belongs to $L^\infty_{loc}([0, \infty); W^{1,\infty}(\mathbb{R}^3))$. Maybe the regularity on $\phi$ would be improved for weak solution by using the method in [7] and we also refer the reader to the reference [3], in which a priori regularity result on the electromagnetic field for weak solution to the Relativistic Vlasov-Maxwell system have been obtained.

(ii) If initial data has moments in velocity and the propagation for the velocity moment can be obtained, then the assumption made in Proposition 1 hold true. In the following, we recall [17, Lemma 5.1] as follows:

**Lemma 4.1.** For $0 \leq k \leq k' < \infty, 1 \leq p, p' < \infty$ with $\frac{1}{p} + \frac{1}{p'} = 1$, if $f \in L^\infty([0, T]; L^p(\mathbb{R}^6))$ with $M_k(f)(t) := \int_{\mathbb{R}^3} |p|^k f(t, x, p) dp < \infty$, then the $k$-th order moment density $m_k(f)(t, x) = \int_{\mathbb{R}^3} |p|^k f(t, x, p) dp \in L^\infty([0, T]; L^r(\mathbb{R}^3))$ and satisfies the following inequality:

$$
\|m_k(f)\|_{L^r_t(L^r_x)} \leq C \|f\|_{L^p_t(L^{p'}_x)} \|M_k(f)(t)\|_{L^p_t},
$$

where $r := \frac{k' + \frac{1}{p} + \frac{1}{p'}}{k + \frac{1}{p} + \frac{1}{p'}}$ and $C = C(k, k', p)$.

Notice that the assumption made in Proposition 1 is the special case in Lemma 4.1. Hence, we predict that if initial data has moments in velocity higher than 1, the weak solution of the (NV) system also has moments in velocity higher 1. But we failed in proving it. In addition, we also refer the readers to the reference [12] and the relevant reference, in which the propagation for moment has been obtained for Vlasov-Poisson system.

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