Construction of a Wilson action for the Wess-Zumino model

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We construct a Wilson action for the Wess-Zumino model by applying the exact renormalization group perturbatively. Using neither superfields nor auxiliary fields, we construct a supersymmetric action only with complex scalar and Majorana spinor fields. We adopt the BRST (antifield) formalism to show the consistency of the construction to all orders in loop expansions. The resulting action has a quadratically divergent scalar mass term which is absent in the superfield formalism.

\section{Introduction}

The purpose of this paper is to construct a Wilson action for the Wess-Zumino model by applying the exact renormalization group (ERG) perturbatively. ERG was originally introduced to define the continuum limit of a quantum field theory non-perturbatively.\textsuperscript{1)} A field theory is defined by an action with a momentum cutoff $\Lambda$. The renormalization group (RG) transformation is the change of the action as we lower $\Lambda$, while we keep the physics of the theory intact. This transformation generates an infinite number of interaction terms. The $\Lambda$ dependence of their coefficients is determined by a functional differential equation, which we call the ERG differential equation. (E for exact is added to RG as a reminder that the actions have an infinite number of terms.) The continuum limit corresponds to a solution of the differential equation that reaches a fixed point as we raise $\Lambda$ to infinity. In practice, non-perturbative application of ERG is difficult without any approximation; its main role has been to give us both an assurance to and an insight into the standard construction of a continuum limit by fine-tuning a small number of relevant parameters of a bare action.

In this paper we use ERG only perturbatively by introducing a cutoff through the propagators.\textsuperscript{2)} Within perturbation theory it becomes straightforward to construct the continuum limit (or renormalized theory) directly by solving the ERG differential equation; we obtain the continuum limit by finding a solution that reduces to an action with a finite number of interaction terms as we raise $\Lambda$ to infinity. The renormalized parameters of the theory characterize this asymptotic behavior of the solution.\textsuperscript{3)}

The presence of an ultraviolet cutoff often gives us a false impression that it is incompatible with local symmetry. The perturbative construction of gauge theories in the ERG formalism was pioneered by Becchi\textsuperscript{4)} and Ellwanger\textsuperscript{5)} among others. (See\textsuperscript{6)} for the background field method, and\textsuperscript{7)} for a manifestly gauge invariant method.) Gauge symmetry (or BRST invariance) is realized as the invariance of the action under an infinitesimal field transformation where the jacobian is properly taken into account. Hence, the theory has the full gauge invariance even though it must be
enforced order by order in loop expansions.

Here, we wish to construct the Wess-Zumino model,\(^8\) the simplest four-dimensional supersymmetric theory, by using ERG. On introducing supersymmetry, we choose to use no auxiliary fields for two reasons. First, auxiliary fields, necessary to close the supersymmetry algebra off-shell, are not available to all supersymmetric theories; we wish to build a formalism that does not rely on the existence of auxiliary fields. Second, the use of auxiliary fields (superfields to be more precise) does not bring any insight into the realization of supersymmetry in a Wilson action. With auxiliary fields, supersymmetry is a linear symmetry, and it is realized automatically once we adopt superfields.\(^9\) ERG then becomes merely a method of regularization. The proof of non-renormalization of the F-term, for example, depends on the familiar technicalities of supergraphs.\(^10\) We wish to construct the model using only its supersymmetry but no extra ingredient provided by the superfields.

In applying ERG to the Wess-Zumino model, we have an advantage that the model is defined strictly in four dimensions. Hence, there is no subtlety in defining \(\gamma_5\) or Majorana fermions, in contrast to when we use dimensional regularization. The method of dimensional reduction can handle \(\gamma_5\), but as far as we know its consistency has not been fully demonstrated. (For a recent review of the dimensional reduction, see\(^11\) for example.) We first introduce a most general renormalized theory with the same field contents as the Wess-Zumino model. This general theory has more number of renormalized parameters than the Wess-Zumino model. We then reduce the number of independent parameters by imposing supersymmetry. Using only component fields, the supersymmetry transformation is non-linear in fields, just like the gauge transformation of Yang-Mills theories. It is this non-linearity that calls for our attention and care.

The paper is organized as follows. In sect. 2, we give a brief summary of the ERG formalism, taking the four dimensional \(\phi^4\) theory as an example. In sect. 3, we apply the ERG formalism to construct the Wess-Zumino model, using only a complex scalar field and a Majorana spinor field (equivalently, a pair of right- and left-hand spinors). We choose two different cutoff functions for the scalar and spinor, since their equality is not required by supersymmetry. We also hope that this choice mimics some aspect of lattice realization of supersymmetry, if the realization is possible at all. We show that the theory has nine parameters if only renormalizability is imposed. We then derive an equation that gives the invariance of the action under the supersymmetry transformation. In sect. 4, we construct the action up to 1-loop. In sect. 5, we attempt to prove, to all orders in loop expansions, that we can realize supersymmetry by fine-tuning the parameters. The proof fails, however, and we explain why. To overcome this failure, we introduce antifields that generate the supersymmetry transformation in sect. 6. With the antifields, supersymmetry is reformulated as BRST invariance of the action. The antifields, which are classical external sources, transform under the supersymmetry transformation, and they play the role of auxiliary fields in making the BRST transformation nilpotent. (We explain this in some details for the classical action in Appendix B.) Outside the context of ERG, this procedure is well known. The antifield or BRST formalism has been
introduced to the Wess-Zumino model in\textsuperscript{12,13} and to super Yang-Mills theories with matter in\textsuperscript{14} the formalism has been extended also to include general global symmetries in\textsuperscript{15}. In sect. 7, we complete the perturbative proof that we can realize supersymmetry by fine-tuning the parameters. In sect. 8, we give brief comments on the quadratic divergences and holomorphy.\textsuperscript{16} We conclude the paper in sect. 9. Three appendices are given.

Before closing this introduction, we call the reader’s attention that we work in the four dimensional euclidean space throughout the paper. Contrary to the Minkowski space, the right- and left-hand spinors that constitute a Majorana spinor are not complex conjugate to each other. We summarize relevant properties of two-component spinors in Appendix A.

§2. Realization of symmetry in the ERG approach

To make the paper self-contained, we give a brief summary of the ERG formalism. For more details, we refer the reader to lecture notes such as\textsuperscript{4} and\textsuperscript{17} and references therein. In this section we only consider a real scalar field $\phi$ for simplicity.

Let $S(\Lambda)$ be a Wilson action with the momentum cutoff $\Lambda$. The action is given as the sum

$$ S(\Lambda) = S_{\text{free}}(\Lambda) + S_{\text{int}}(\Lambda) \quad (2.1) $$

where the free action is defined by

$$ S_{\text{free}}(\Lambda) \equiv - \int_p \frac{1}{K(p/\Lambda)} \frac{1}{2} \phi(-p)(p^2 + m^2)\phi(p) \quad \left( \int_p \equiv \int \frac{d^4p}{(2\pi)^4} \right) \quad (2.2) $$

The cutoff function $K(x)$ is a positive function of $x^2$, and has the following properties:

$$ K(x) \begin{cases} = 1 & (x^2 < 1) \\ \rightarrow 0 & (x^2 \rightarrow \infty) \end{cases} \quad (2.3) $$

The choice of $K(x)$ is arbitrary as long as it damps sufficiently fast ($1/x^6$ is more than enough) as $x^2 \rightarrow \infty$.

The cutoff dependence of the interaction action $S_{\text{int}}(\Lambda)$ is determined by the ERG differential equation\textsuperscript{2}

$$ -A \frac{\partial}{\partial A} S_{\text{int}} = \int_p \frac{\Delta(p/\Lambda)}{p^2 + m^2} \frac{1}{2} \left\{ \frac{\delta S_{\text{int}}}{\delta \phi(-p)} \frac{\delta S_{\text{int}}}{\delta \phi(p)} + \frac{\delta^2 S_{\text{int}}}{\delta \phi(-p) \delta \phi(p)} \right\} \quad (2.4) $$

where

$$ \Delta(p/\Lambda) \equiv A \frac{\partial}{\partial A} K(p/\Lambda) \quad (2.5) $$

is non-vanishing only for $p^2 > A^2$. Alternatively the ERG differential equation can be given for the full action as

$$ -A \frac{\partial}{\partial A} S = \int_p \frac{\Delta(p/\Lambda)}{p^2 + m^2} \left[ \frac{p^2 + m^2}{K(p/\Lambda)} \frac{\delta S}{\delta \phi(p)} \right. $$

$$ \left. + 1 \frac{1}{2} \left\{ \frac{\delta S}{\delta \phi(-p)} \frac{\delta S}{\delta \phi(p)} + \frac{\delta^2 S}{\delta \phi(p) \delta \phi(-p)} \right\} \right] \quad (2.6) $$
The ERG differential equation (2.4) or (2.6) implies the cutoff independence of the following connected correlation functions:

\[
\left\{ \begin{array}{l}
\langle \phi(p)\phi(-p) \rangle_\infty \equiv \frac{1 - K(p/\Lambda)}{p^2 + m^2} + \frac{1}{K(p/\Lambda)} \langle \phi(p)\phi(-p) \rangle_{S(\Lambda)} \\
\langle \phi(p_1)\cdots\phi(p_n) \rangle_\infty \equiv \prod_{i=1}^n \frac{1}{K(p_i/\Lambda)} \cdot \langle \phi(p_1)\cdots\phi(p_n) \rangle_{S(\Lambda)} 
\end{array} \right. \tag{2.7}
\]

Note that for (2.4) or (2.6) to have a unique solution, we must constrain the asymptotic behavior of \( S_{\text{int}}(\Lambda) \) for \( \Lambda \), large compared with \( m \) and the momenta carried by the fields. The asymptotic behavior of a renormalized theory has the form

\[
S_{\text{int}}(\Lambda) \xrightarrow{\Lambda \to \infty} \int d^4x \left[ (A^2a_2 + m^2b_2)\frac{1}{2}\phi^2 + c_2\frac{1}{2}(\partial_{\mu}\phi)^2 + a_4\frac{1}{4!}\phi^4 \right] \tag{2.8}
\]

where \( a_2, b_2, c_2, a_4 \) depend on \( \ln \Lambda/\mu \) (\( \mu \) is a renormalization scale). The values of \( b_2, c_2, a_4 \) at \( \Lambda = \mu \) can be chosen arbitrarily, and the simplest choice is

\[
b_2(0) = c_2(0) = 0, \quad a_4(0) = -\lambda \tag{2.9}
\]

where \( \lambda \) is a coupling constant.

Let \( \Phi(p) \) be a composite operator with momentum \( p \). A composite operator is a functional of \( \phi \) that satisfies the same ERG differential equation as an infinitesimal deformation of the interaction action. Hence,

\[
-A \frac{\partial}{\partial \Lambda} \Phi(p) = D \cdot \Phi(p) \tag{2.10}
\]

where

\[
D \equiv \int_p \frac{\Delta(p/\Lambda)}{p^2 + m^2} \left\{ \frac{1}{K(p_i/\Lambda)} \cdot \langle \phi(p_1)\cdots\phi(p_n) \rangle_{S(\Lambda)} \right\} \tag{2.11}
\]

The differential equation implies the cutoff independence of the correlation functions

\[
\langle \Phi(p)\phi(p_1)\cdots\phi(p_n) \rangle_\infty \equiv \prod_{i=1}^n \frac{1}{K(p_i/\Lambda)} \cdot \langle \phi(p_1)\cdots\phi(p_n) \rangle_{S(\Lambda)} \tag{2.12}
\]

for \( n \geq 1 \). For example, the elementary field \( \phi(p) \) is not a composite operator, but

\[
[\phi](p) \equiv \phi(p) + \frac{1 - K(p/\Lambda)}{p^2 + m^2} \frac{\delta S_{\text{int}}}{\delta \phi(-p)} \tag{2.13}
\]

is one with the correlation functions

\[
\langle \langle \phi \rangle(p)\phi(p_1)\cdots\phi(p_n) \rangle_\infty = \langle \phi(p)\phi(p_1)\cdots\phi(p_n) \rangle_\infty \quad (n \geq 1) \tag{2.14}
\]

The composite operator \( \frac{1}{2}\phi^2 \) is less easy to define. For non-vanishing momentum \( p \), it has the following asymptotic behavior:

\[
\left[ \frac{1}{2}\phi^2 \right](p) \xrightarrow{\Lambda \to \infty} z(\ln \Lambda/\mu) \frac{1}{2} \int_q \phi(p - q)\phi(q) \tag{2.15}
\]
The $\Lambda$ dependence of the coefficient is determined by the ERG, but the value $z(0)$ can be chosen arbitrarily; $z(0) = 1$ is the simplest choice.

Now, given an arbitrary composite operator $\Phi(p)$ of momentum $p$,

$$\Sigma \equiv \int_p K(p/\Lambda) \left( \frac{\delta S}{\delta \phi(p)} \Phi(p) + \frac{\delta \Phi(p)}{\delta \phi(p)} \right)$$

(2.16)

is a composite operator of zero momentum. Let us assume that $\Sigma$ vanishes for some $\Lambda$. Then

$$\Sigma(\Lambda) = 0$$

(2.17)

for any $\Lambda$, since $\Sigma$ satisfies the linear equation (2.10). (2.17) is equivalent to the Ward identity

$$\sum_{i=1}^{n} \langle \phi(p_1) \cdots \Phi(p_i) \cdots \phi(p_n) \rangle_\infty = 0$$

(2.18)

and it implies the invariance of the functional measure

$$[d\phi] e^{S(\Lambda)}$$

(2.19)

under the infinitesimal change of field

$$\phi(p) \rightarrow \phi(p) + \epsilon K(p/\Lambda) \Phi(p)$$

(2.20)

The second term of (2.16) takes into account the jacobian of this transformation. (2.17) is the generic expression of continuous symmetry in the ERG formalism, and in sect. 3 we introduce supersymmetry in this form. (See 18 for an application to QED.)

Before we review the BRST formalism, let us briefly discuss how to introduce an external source to the action. Let $J(-p)$ be a classical external source coupled to a composite operator $\Phi(p)$. The action $\bar{S}$ in the presence of $J$ satisfies the same ERG differential equation (2.6) as $S$. If $\Phi$ is the composite operator (2.13), $\bar{S}$ satisfies the simple asymptotic condition

$$\bar{S}(\Lambda) - S(\Lambda) \xrightarrow{\Lambda \to \infty} \int_p J(-p) \phi(p)$$

(2.21)

and the solution of both the ERG differential equation and the above asymptotic condition is given by

$$\bar{S}(A) = S_{\text{tree}}(\Lambda) + \int_p J(-p) \phi(p) + \frac{1}{2} \int_p J(-p) \frac{1 - K(p/\Lambda)}{p^2 + m^2} J(p) + S_{\text{int}}(A)[\phi_{\text{sh}}]$$

(2.22)

where the last term is obtained from $S_{\text{int}}[\phi]$ by substituting the shifted field

$$\phi_{\text{sh}}(p) \equiv \phi(p) + \frac{1 - K(p/\Lambda)}{p^2 + m^2} J(p)$$

(2.23)

into the elementary field $\phi(p)$. We can easily check

$$\frac{\delta \bar{S}}{\delta J(-p)} \bigg|_{J=0} = [\phi](p)$$

(2.24)
Now, if \( \Phi \) is a more complicated composite operator such as \( \frac{1}{2} \phi^2 \), the asymptotic behavior has the form

\[
\bar{S}(A) - S(A) \xrightarrow{A \to \infty} w_1(\ln A/\mu) \int_{p,q} J(-p) \frac{1}{2} \phi(p - q) \phi(q) + \frac{1}{2} w_2(\ln A/\mu) \int_p J(p) J(-p)
\]

We can choose \( w_1(0) \) and \( w_2(0) \) at will, but we cannot give a closed formula like (2.22) in this case.

We now summarize the BRST formalism, an alternative way to realize symmetry using ERG. We introduce an antifield \( \phi^* \) which has the statistics (fermionic in this case) opposite to that of the conjugate field \( \phi \). We introduce \( \phi^*(-p) \) as a classical external source coupled to \( \Phi(p) \) of (2.16). We denote the resulting action as \( \bar{S}(A) \). Its interaction part

\[
\bar{S}_{\text{int}}(A) \equiv \bar{S}(A) - S_{\text{free}}(A)
\]

satisfies the same ERG differential equation as \( S_{\text{int}} \):

\[
-\Lambda \frac{\partial}{\partial \Lambda} \bar{S}_{\text{int}} = \int_p \frac{\Delta (p/\Lambda)}{p^2 + m^2} \left\{ \frac{\delta \bar{S}_{\text{int}}}{\delta \phi(-p)} \frac{\delta \bar{S}_{\text{int}}}{\delta \phi(p)} + \frac{\delta^2 \bar{S}_{\text{int}}}{\delta \phi(-p) \delta \phi(p)} \right\}
\]

Note \( \phi^* \) plays no role in ERG. Replacing \( \Phi(p) \) of (2.16) by the left-derivative of the action with respect to \( \phi^* \), we extend (2.17) to

\[
\bar{\Sigma}(A) = 0
\]

where \( \bar{\Sigma} \) is a composite operator defined by

\[
\bar{\Sigma}(A) \equiv \int_p K(p/\Lambda) \left( \frac{\delta \bar{S}}{\delta \phi(p)} \cdot \frac{\delta}{\delta \phi^*(-p)} \bar{S} + \frac{\delta}{\delta \phi(p)} \cdot \frac{\delta}{\delta \phi^*(-p)} \bar{S} \right)
\]

The above identity, or the vanishing of \( \bar{\Sigma} \), is the BRST invariance of the action in the ERG formalism.\(^\text{4,17}\) For any \( \bar{S} \) we can define BRST transformation by

\[
\delta_Q \equiv \int_p K(p/\Lambda) \left( \frac{\delta}{\delta \phi^*(-p)} \frac{\delta}{\delta \phi(p)} \right) \bar{S} \cdot \delta \phi(p) + \frac{\delta}{\delta \phi^*(-p)} \bar{S} \cdot \delta \phi(p) + \frac{\delta}{\delta \phi(p)} \cdot \frac{\delta}{\delta \phi^*(-p)} \bar{S}
\]

so that, given a composite operator \( \mathcal{O} \), \( \delta_Q \mathcal{O} \) gives another composite operator. We then obtain

\[
\delta_Q \bar{\Sigma} = 0
\]

for any \( \bar{\Sigma} \) given by (2.29) even if \( \Sigma \neq 0 \). This equation provides an essential algebraic structure that is missing in the formalism without antifields. We will take advantage of the algebraic structure to construct a supersymmetric Wilson action to all orders.
in loop expansions. Once we construct a BRST invariant action, satisfying \( \Sigma = 0 \), the BRST transformation automatically satisfies nilpotency

\[
\delta_Q \delta_Q = 0
\]  

(2.32)

but it plays no role in this paper.

§3. Construction without antifields

In this section we discuss how to construct a Wilson action of the Wess-Zumino model perturbatively without using antifields. The starting point is the classical action:

\[
S_{cl} = - \int d^4x \left[ \bar{\chi}_L \sigma \cdot \partial \chi_R + \frac{1}{2} (m \bar{\chi}_R \chi_R + \bar{m} \bar{\chi}_L \chi_L) + \partial_\mu \bar{\phi} \partial_\mu \phi + |m|^2 \bar{\phi} \phi 
+ g \phi \frac{1}{2} \bar{\chi}_R \chi_R + \bar{g} \bar{\phi} \frac{1}{2} \bar{\chi}_L \chi_L + m \phi \frac{\bar{g}}{2} \phi^2 + \bar{m} \bar{\phi} \frac{g}{2} \phi^2 + \frac{|g|^2}{4} |\phi|^4 \right]
\]  

(3.1)

The bar above a complex scalar \( \phi \) or the mass parameter \( m \) denotes complex conjugation, but the bar above a spinor denotes a transpose. (See Appendix A for our notation on two-component spinors.) We save \( * \) as a superscript to denote an antifield.

In order to classify all possible interaction terms, we introduce two types of charges. Though they are not conserved, we can use them to figure out which terms are allowed and which forbidden.\(^{16}\)

1. \( \alpha \) charge — We assign charge \( \alpha \) to \( \phi \), and \( -\alpha \) to \( \bar{\phi} \). The action is invariant under

\[
\phi \rightarrow e^{i\alpha} \phi, \quad \bar{\phi} \rightarrow e^{-i\alpha} \bar{\phi}
\]  

(3.2)

if we assign charge \( -\alpha \) to \( g \) and \( \alpha \) to \( \bar{g} \). In other words, the phase change of the coupling

\[
g \rightarrow e^{-i\alpha} g, \quad \bar{g} \rightarrow e^{i\alpha} \bar{g}
\]  

(3.3)

does not affect physics, since this can be compensated by [3.2].

2. \( \beta \) charge — We assign charge \( \beta \) to \( \chi_R \), and \( -\beta \) to \( \chi_L \). The action is invariant under

\[
\chi_R \rightarrow e^{i\beta} \chi_R, \quad \chi_L \rightarrow e^{-i\beta} \chi_L
\]  

(3.4)

if we assign charge \( -2\beta \) to \( g, m \), and \( 2\beta \) to \( \bar{g}, \bar{m} \). In other words, the phase change of the parameters

\[
g \rightarrow e^{-2i\beta} g, \quad m \rightarrow e^{-2i\beta} m, \quad \bar{g} \rightarrow e^{2i\beta} \bar{g}, \quad \bar{m} \rightarrow e^{2i\beta} \bar{m}
\]  

(3.5)

does not affect physics, since this can be compensated by [3.4].

We will impose the above properties on the Wilson action. Hence, the terms such as

\[
\bar{m}^2 g^2 \phi^2, \quad |m|^2 \bar{m} g \phi
\]  

(3.6)
can be generated, though they are absent in the classical action. The $\alpha$ and $\beta$ charges are obviously related to the standard R-symmetry; the R-charge is reproduced by choosing

$$\alpha + 2\beta = 0$$  

(3.7)

In the following we treat $\alpha$ and $\beta$ as independent. (One of the two U(1) charges of\(^{16}\) has $\alpha = \beta = 1$, and the other $\alpha = 1, \beta = -\frac{1}{2}$.) The classical action is invariant under the following supersymmetry transformation:

$$\begin{align*}
\delta_{cl}\phi & \equiv \bar{\xi}_{R}\chi_{R} + \eta_{\mu}\partial_{\mu}\phi \\
\delta_{cl}\bar{\phi} & \equiv \bar{\xi}_{L}\chi_{L} + \eta_{\mu}\partial_{\mu}\bar{\phi} \\
\delta_{cl}\chi_{R} & \equiv \bar{\sigma}_{\mu}\xi_{L}\partial_{\mu}\phi - \left(\bar{m}\bar{\phi} + \bar{\varphi}\frac{\varphi^{2}}{2}\right)\xi_{R} + \eta_{\mu}\partial_{\mu}\chi_{R} \\
\delta_{cl}\chi_{L} & \equiv \sigma_{\mu}\xi_{R}\partial_{\mu}\bar{\phi} - \left(m\phi + \varphi\frac{\phi^{2}}{2}\right)\xi_{L} + \eta_{\mu}\partial_{\mu}\chi_{L}
\end{align*}$$  

(3.8)

where $\xi_{R,L}$ are anticommuting constant spinors, and $\eta_{\mu}$ is an infinitesimal constant vector. Hence, we obtain

$$\delta_{cl}S_{cl} \equiv \int_{p} \left[ \frac{\delta S_{cl}}{\delta \phi(p)} \frac{\delta S_{cl}}{\delta \bar{\phi}(p)} \frac{\delta S_{cl}}{\delta \chi_{R}(p)} \frac{\delta S_{cl}}{\delta \chi_{L}(p)} \right] = 0$$  

(3.9)

We note two things here:

1. In (3.8), $\eta_{\mu}$ generates translation, and the invariance of the action under translation holds trivially. But it becomes crucial to include the translation as part of the transformation when we introduce a BRST invariant classical action $\bar{S}_{cl}$ in sect.\[^{6}\]. As is explained in Appendix B, its BRST invariance is equivalent to the nilpotency of (generalized) $\delta_{cl}$.

2. We can give mass dimensions and $\alpha, \beta$ charges to the parameters $\xi_{R,L}, \eta_{\mu}$ of the supersymmetry transformation (3.8) such that the field and its transformation have the same dimensions and charges. For later convenience, we summarize the mass dimensions and $\alpha, \beta$ charges of fields and constants (or parameters) in a table:

| fields | dimensions | $\alpha, \beta$ charges | constants | dimensions | $\alpha, \beta$ charges |
|--------|------------|-------------------------|-----------|------------|-------------------------|
| $\phi$ | 1          | $\alpha$               | $g$       | 0          | $-\alpha - 2\beta$     |
| $\bar{\phi}$ | 1          | $-\alpha$                        | $\bar{g}$ | 0          | $\alpha + 2\beta$     |
| $\chi_{R}$ | 3/2       | $\beta$                    | $m$       | 1          | $-2\beta$              |
| $\chi_{L}$ | 3/2       | $-\beta$                  | $\bar{m}$ | 1          | $2\beta$              |
| $\xi_{R}$ |           |                           | $\xi_{R}$ | $-1/2$     | $\alpha - \beta$      |
| $\xi_{L}$ |           |                           | $\xi_{L}$ | $-1/2$     | $-\alpha + \beta$     |
| $\eta_{\mu}$ |         |                           | $\eta_{\mu}$ | $-1$        | 0                       |

To construct a Wilson action of the Wess-Zumino model, we first split the action into the free and interaction parts:

$$S(A) = S_{\text{free}}(A) + S_{\text{int}}(A)$$
where the free part is given by
\[
S_{\text{free}}(A) = - \int p \left[ \frac{1}{K_b(p/A)} \bar{\phi}(-p)(p^2 + |m|^2)\phi(p) \right. \\
\left. + \frac{1}{K_f(p/A)} \left( \bar{\chi}_L(-p)i\sigma \chi_R(p) + \frac{m}{2} \bar{\chi}_R\chi_R + \frac{\bar{m}}{2}\chi_L\chi_L \right) \right]
\] (3.10)

We have chosen different cutoff functions for the scalars and spinors; as we shall see, supersymmetry does not require
\[
K_f = K_b
\] (3.11)

The propagators are given by
\[
\begin{align*}
\langle \phi(p)\bar{\phi}(-p) \rangle_{\text{free}} &= K_b(p/A)/(p^2 + |m|^2) \\
\langle \chi_R(p)\bar{\chi}_L(-p) \rangle_{\text{free}} &= (-ip\cdot\bar{\sigma})K_f(p/A)/(p^2 + |m|^2) \\
\langle \chi_R(p)\bar{\chi}_R(-p) \rangle_{\text{free}} &= \bar{m}K_f(p/A)/(p^2 + |m|^2) \\
\langle \chi_L(p)\bar{\chi}_L(-p) \rangle_{\text{free}} &= mK_f(p/A)/(p^2 + |m|^2)
\end{align*}
\] (3.12)

The interaction action \( S_{\text{int}} \) satisfies the ERG differential equation
\[
-A \frac{\partial}{\partial A} S_{\text{int}} = \int p \frac{\Delta_b(p/A)}{p^2 + |m|^2} \left\{ \frac{\delta S_{\text{int}}}{\delta \phi(p)} \cdot \frac{\delta S_{\text{int}}}{\delta \bar{\phi}(-p)} + \frac{\delta^2 S_{\text{int}}}{\delta \phi(p)\delta \bar{\phi}(-p)} \right\}
\]
\[
\times \left[ \frac{\bar{m}}{2} \left\{ S_{\text{int}} \frac{\delta}{\delta \chi_R(p)} \cdot \frac{\delta}{\delta \bar{\chi}_R(-p)} S_{\text{int}}^{(0)} - \text{Tr} \frac{\delta}{\delta \chi_R(-p)} S_{\text{int}}^{(0)} \frac{\delta}{\delta \bar{\chi}_R(p)} \right\} \\
+ \frac{m}{2} \left\{ S_{\text{int}}^{(0)} \frac{\delta}{\delta \chi_L(p)} \cdot \frac{\delta}{\delta \bar{\chi}_L(-p)} S_{\text{int}}^{(0)} - \text{Tr} \frac{\delta}{\delta \chi_L(-p)} S_{\text{int}}^{(0)} \frac{\delta}{\delta \bar{\chi}_L(p)} \right\} \\
+ S_{\text{int}} \frac{\delta}{\delta \chi_R(p)}(-ip\cdot\bar{\sigma}) \frac{\delta}{\delta \bar{\chi}_L(-p)} S_{\text{int}} - \text{Tr}(-ip\cdot\bar{\sigma}) \frac{\delta}{\delta \chi_L(-p)} S_{\text{int}} \right]
\] (3.13)

where
\[
\Delta_b(p/A) \equiv A \frac{\partial}{\partial A} K_b(p/A), \quad \Delta_f(p/A) \equiv A \frac{\partial}{\partial A} K_f(p/A)
\] (3.14)

The minus signs in front of the traces are due to the Fermi statistics. Note that the \( \alpha, \beta \) charges are preserved under the ERG transformation.

We specify a renormalized theory by the asymptotic behavior of \( S_{\text{int}}(A) \) for \( A \) large compared with the momenta of fields and the mass parameters. The asymptotic behavior is parametrized as follows:
\[
S_{\text{int}}(A) \xrightarrow{A \to \infty} \int d^4x \left[ z_1(\ln A/\mu) \bar{\chi}_L\sigma_\mu \partial_\mu \chi_R \\
+ z_2(\ln A/\mu) \left( \frac{m}{2} \bar{\chi}_R\chi_R + \frac{\bar{m}}{2}\chi_L\chi_L \right) \right]
\]
\[+ z_3 (\ln \Lambda/\mu) \partial_\mu \bar{\phi} \partial_\mu \phi + \{ A^2 a_4 (\ln \Lambda/\mu) + |m|^2 z_4 (\ln \Lambda/\mu) \} |\phi|^2 + \{ -1 + z_5 (\ln \Lambda/\mu) \} \left( g \frac{\phi}{2} \bar{\chi}_R \chi_R + \bar{g} \frac{\phi}{2} \bar{\chi}_L \chi_L \right) + \{ -1 + z_6 (\ln \Lambda/\mu) \} \left( m \phi \frac{g}{2} \bar{\phi}^2 + \bar{m} \phi \frac{g}{2} \phi^2 \right) + \{ -1 + z_7 (\ln \Lambda/\mu) \} \frac{|g|^2}{4} |\phi|^4 + z_8 (\ln \Lambda/\mu) \left( g^2 m^2 \frac{1}{2} \phi^2 + \bar{g}^2 m^2 \frac{1}{2} \bar{\phi}^2 \right) + \{ A^2 a_9 (\ln \Lambda/\mu) + |m|^2 z_9 (\ln \Lambda/\mu) \} \left( g \bar{m} \phi + \bar{g} m \bar{\phi} \right) \] (3.15)

where we have adopted coordinate space notation. The real coefficients \(a_4, a_9\) and \(z_1, \ldots, z_9\) all depend on \(\ln \Lambda/\mu\), where \(\mu\) is an arbitrary renormalization scale. The values of \(z\)'s at \(\Lambda = \mu\) can be chosen at will, and they constitute part of the parameters of the theory:

\[z_1(0), \ldots, z_9(0)\] (3.16)

(A brief remark on why the coefficients are real as opposed to complex: \(a_4, z_{1,3,4,7}\) are real because the action is real, and \(a_9, z_{2,5,6,8,9}\) are real additionally because the theory is CP invariant. In euclidean space CP transformation is given by

\[
\begin{align*}
\phi(\vec{x}, x_4) & \to \bar{\phi}(\vec{x}, x_4), \quad \bar{\phi}(\vec{x}, x_4) \to \phi(\vec{x}, x_4) \\
\chi_R(\vec{x}, x_4) & \to \chi_L(\vec{x}, x_4), \quad \chi_L(\vec{x}, x_4) \to \chi_R(\vec{x}, x_4)
\end{align*}
\] (3.17)

We can choose both \(m\) and \(g\) real by choosing the phases of the fields appropriately. We can then demand the theory be invariant under the above CP.)

We now introduce a supersymmetry transformation:

\[
\begin{align*}
\delta \phi(p) & \equiv \xi_R(\chi_R)(p) + \eta_\mu \bar{\phi}_\mu [\phi](p) \\
\delta \bar{\phi}(p) & \equiv \xi_L(\chi_L)(p) + \eta_\mu \bar{\phi}_\mu [\bar{\phi}](p) \\
\delta \chi_R(p) & \equiv \bar{\sigma}_\mu \xi_L [\bar{\phi}](p) - \left( \bar{m} \bar{\phi}(p) + \bar{g} \cdot \frac{\bar{\phi}^2}{2} \right) \xi_R(p) \\
\delta \chi_L(p) & \equiv \sigma_\mu \xi_R [\phi](p) - \left( m \phi(p) + g \cdot \frac{\phi^2}{2} \right) \xi_L(p)
\end{align*}
\] (3.18)

Here, the fields in square brackets are composite operators:

\[
\begin{align*}
[\phi](p) & \equiv \phi(p) + \frac{1 - K_b (p/\Lambda)}{p^2 + |m|^2} \frac{\delta S_{\text{int}}}{\delta \phi(-p)} \\
[\bar{\phi}](p) & \equiv \bar{\phi}(p) + \frac{1 - K_b (p/\Lambda)}{p^2 + |m|^2} \frac{\delta S_{\text{int}}}{\delta \bar{\phi}(-p)} \\
[\chi_R](p) & \equiv \chi_R(p) + \frac{1 - K_f (p/\Lambda)}{p^2 + |m|^2} \left( -i p \cdot \bar{\sigma} \frac{\bar{\delta}}{\delta \chi_L(-p)} + m \frac{\bar{\delta}}{\delta \chi_R(-p)} \right) S_{\text{int}} \\
[\chi_L](p) & \equiv \chi_L(p)
\end{align*}
\] (3.19)
Wilson action for the Wess-Zumino model

\[ 1 - K_f \left( \frac{p}{\Lambda} \right) \left( -i p \cdot \sigma \frac{\delta}{\delta \chi_R(-p)} + m \frac{\delta}{\delta \chi_L(-p)} \right) S_{\text{int}} \]  

(3.22)

These behave trivially as \( \Lambda \to \infty \):

\[
\begin{align*}
[\phi](p) & \to \phi(p), \\
[\chi_R](p) & \to \chi_R(p), \\
[\chi_L](p) & \to \chi_L(p)
\end{align*}
\]

(3.23)

The composite operators \([\phi^2](p), [\bar{\phi}^2](p)\) are defined by ERG differential equations (the same as those satisfied by infinitesimal deformations of \(S_{\text{int}}\)) and their asymptotic behaviors:

\[
\begin{align*}
\left[ \frac{\phi^2}{2} \right](p) & \xrightarrow{\Lambda \to \infty} \left( 1 + z_{10} \right) \frac{\phi^2}{2}(p) + z_{11} \bar{g} m \phi(p) + z_{12} \bar{g}^2 m^2 \cdot (2\pi)^4 \delta^{(4)}(p) \\
\left[ \frac{\bar{\phi}^2}{2} \right](p) & \xrightarrow{\Lambda \to \infty} \left( 1 + z_{10} \right) \frac{\bar{\phi}^2}{2}(p) + z_{11} g \bar{m} \bar{\phi}(p) + z_{12} g^2 \bar{m}^2 \cdot (2\pi)^4 \delta^{(4)}(p)
\end{align*}
\]

(3.24, 3.25)

where \(z_{10,11,12}\) are real coefficients, dependent on \(\ln \Lambda/\mu\). The values of \(z_{10,11,12}\) at \(\Lambda = \mu\)

\[
z_{10}(0), z_{11}(0), z_{12}(0)
\]

(3.26)

should be considered as additional parameters of the theory. Hence, altogether, the theory has twelve real parameters

\[
z_1(0), \ldots, z_{12}(0)
\]

(3.27)

We now define a bosonic composite operator \(\Sigma\) by

\[
\Sigma \equiv \int_p K_b \left( \frac{p}{\Lambda} \right) \left[ \delta\phi(p) \frac{\delta S}{\delta \phi(p)} + \frac{\delta}{\delta \phi(p)} \delta\phi(p) \right] + \int_p K_f \left( \frac{p}{\Lambda} \right) \left[ S \frac{\delta}{\delta \chi_R(p)} \delta\chi_R(p) - \text{Tr} \frac{\delta}{\delta \chi_R(p)} \delta\chi_R(p) \right]
\]

(3.28)

where \(\delta\phi\), etc., are defined by (3.18). Supersymmetry of the Wilson action is the vanishing of this composite operator:

\[
\Sigma(\Lambda) = 0
\]

(3.29)

Note that if \(\Sigma\) vanishes asymptotically for large \(\Lambda\), it vanishes identically, since \(\Sigma\) satisfies a linear ERG differential equation.

Let us check (3.29) at tree level. At tree level, the interaction action, \(S_{\text{int}}^{(0)}\), satisfies a simpler ERG differential equation

\[
- \Lambda \frac{\partial}{\partial \Lambda} S_{\text{int}}^{(0)} = \int_p \frac{\Delta_b(p/\Lambda)}{p^2 + |m|^2} \frac{\delta S_{\text{int}}^{(0)}}{\delta \phi(p)} \cdot \frac{\delta S_{\text{int}}^{(0)}}{\delta \phi(-p)}
\]

(3.29)
\[ H. \text{ Sonoda and K. } \ddot{\text{U}}l\ddot{\text{e}}\text{ker} \]

\[ + \int \frac{\Delta_f(p/A)}{p^2 + |m|^2} \left[ S_{\text{int}}^{(0)} \frac{\bar{\sigma}}{\delta \chi_R(p)} (-ip \cdot \bar{\sigma}) \frac{\bar{\sigma}}{\delta \chi_L(-p)} S_{\text{int}}^{(0)} \right] \]

\[ + \frac{\bar{m}}{2} S_{\text{int}} \frac{\bar{\sigma}}{\delta \chi_R(p)} \cdot \frac{\bar{\sigma}}{\delta \chi_R(-p)} S_{\text{int}}^{(0)} + \frac{m}{2} S_{\text{int}} \frac{\bar{\sigma}}{\delta \chi_L(p)} \cdot \frac{\bar{\sigma}}{\delta \chi_L(-p)} S_{\text{int}}^{(0)} \]  \hspace{1cm} (3.30)

We impose the asymptotic condition that \( S_{\text{int}}^{(0)} \) reduces to the interaction part of the classical action:

\[ S_{\text{int}}^{(0)} \xrightarrow{A \to \infty} - \int d^4 x \left[ \frac{g}{2} \chi_R \chi_R + \bar{g} \frac{1}{2} \chi_L \chi_L + m \phi \bar{\phi} \phi^2 + \bar{m} \phi \bar{g} \bar{\phi}^2 + \frac{|g|^2}{4} |\phi|^4 \right] \]  \hspace{1cm} (3.31)

Then \( S_{\text{int}}^{(0)} \) is given by the tree level Feynman diagrams where the elementary vertices from \( S_{\text{cl}} \) are connected by high momentum propagators proportional to either \( 1 - K_b \) or \( 1 - K_f \). At tree level, (3.28) simplifies to

\[ \Sigma^{(0)} \equiv S_{\text{free}} + S_{\text{int}}^{(0)} \]  \hspace{1cm} (3.33)

and \( \delta^{(0)} \phi, \) etc., are tree-level supersymmetry transformations. For \( A \) large compared with \( m \) and the momenta carried by the fields, (3.32) reduces to

\[ \Sigma^{(0)}(A) \xrightarrow{A \to \infty} \delta_{cl} \]  \hspace{1cm} (3.34)

The last equality is due to the classical supersymmetry (3.9). Hence, \( \Sigma^{(0)}(A) \) vanishes identically for arbitrary \( A \).

We wish to construct \( S(A) \) that satisfies (3.29) to all orders in the number of loops. The question is twofold:

1. Can we satisfy (3.29) by fine-tuning the twelve parameters?
2. How many parameters are left arbitrary?

In the next section we will answer these questions at 1-loop level.

\section*{§4. 1-loop results}

Let \( S_{\text{int}}^{(1)} \) be the 1-loop correction to \( S_{\text{int}} \). This is obtained by solving (3.13) approximately. Here, we only summarize the 1-loop corrections to \( a_{4,9}(\ln A/\mu) \) and \( z_i(\ln A/\mu) \) \((i = 1, \cdots, 12)\):

\[ a_4^{(1)}(\ln A/\mu) = -\frac{|g|^2}{2} \int_p \left\{ -\Delta_b(p) + 2\Delta_f(p) (1 - K_f(p)) \right\} / p^2 \]

\[ a_9^{(1)}(\ln A/\mu) = -\frac{1}{2} \int_p \left\{ \Delta_f(p) - \Delta_b(p) \right\} / p^2 \]  \hspace{1cm} (4.1)
and

\[
\begin{align*}
n_1^{(1)}(\ln \Lambda/\mu) &= \frac{|g|^2}{(4\pi)^2} \ln \Lambda/\mu + c_1 \\
n_2^{(1)}(\ln \Lambda/\mu) &= c_2 \\
n_3^{(1)}(\ln \Lambda/\mu) &= \frac{|g|^2}{(4\pi)^2} \ln \Lambda/\mu + c_3 \\
n_4^{(1)}(\ln \Lambda/\mu) &= -\frac{|g|^2}{(4\pi)^2} \ln \Lambda/\mu + c_4 \\
n_5^{(1)}(\ln \Lambda/\mu) &= c_5 \\
n_6^{(1)}(\ln \Lambda/\mu) &= \frac{|g|^2}{(4\pi)^2} \ln \Lambda/\mu + c_6 \\
n_7^{(1)}(\ln \Lambda/\mu) &= -\frac{|g|^2}{(4\pi)^2} \ln \Lambda/\mu + c_7 \\
n_8^{(1)}(\ln \Lambda/\mu) &= c_8 \\
n_9^{(1)}(\ln \Lambda/\mu) &= c_9 \\
n_{10}^{(1)}(\ln \Lambda/\mu) &= \frac{|g|^2}{(4\pi)^2} \ln \Lambda/\mu + c_{10} \\
n_{11}^{(1)}(\ln \Lambda/\mu) &= \frac{|g|^2}{(4\pi)^2} \ln \Lambda/\mu + c_{11} |g|^2 \\
n_{12}^{(1)}(\ln \Lambda/\mu) &= c_{12} |g|^2 
\end{align*}
\]

(4.2)

where

\[ c_i \equiv n^{(1)}(0) \]  

(4.3)

are arbitrary constants. In calculating these, we have used the formula

\[
\int_p \frac{\Delta_b(p) (1 - K_b(p))^n}{p^4} = \int_p \frac{\Delta_f(p) (1 - K_f(p))^n}{p^4} = \frac{1}{(4\pi)^2} \frac{2}{n + 1} 
\]

(4.4)

Next we consider \( \Sigma^{(1)} \), the 1-loop correction to \( \Sigma \). It is given by

\[
\begin{align*}
\Sigma^{(1)} &= \int_p K_b(p/\Lambda) \left[ \frac{\delta^{(0)} \phi(p)}{\delta \phi(p)} + \frac{\delta^{(1)} \phi(p)}{\delta \phi(p)} \right. \\
&\quad \left. + \frac{\delta}{\delta \phi(p)} \delta^{(0)} \phi(p) + (\phi \to \bar{\phi}) \right] \\
&\quad + \int_p K_f(p/\Lambda) \left[ S^{(1)} \frac{\delta}{\delta \chi_R(p)} \delta^{(0)} \chi_R(p) + S^{(0)} \frac{\delta}{\delta \chi_R(p)} \delta^{(1)} \chi_R(p) \\
&\quad - \text{Tr} \delta^{(0)} \chi_R(p) \frac{\delta}{\delta \chi_R(p)} + (R \to L) \right] 
\end{align*}
\]

(4.5)

where \( \delta^{(1)} \phi \) is the 1-loop correction to \( \delta \phi \), etc. We only need the asymptotic behavior of \( \Sigma^{(1)} \), which is calculated as

\[
\begin{align*}
\Sigma^{(1)} &\xrightarrow{\Lambda \to \infty} \int d^4 x \bar{\xi}_R \chi_R \left[ t_1 \bar{\phi}^2 + t_2 g |m|^2 \bar{\phi} + t_3 |m|^2 \phi + t_4 g^2 m^2 \phi \\
&\quad + t_5 m \bar{\phi}^2 + t_6 \bar{m} g|\phi|^2 + t_7 |g|^2 \phi \bar{\phi}^2 \right] \\
&\quad + \int d^4 x \bar{\xi}_R \bar{\sigma}_\mu \chi_L \cdot \partial_\mu \bar{\phi} (t_8 g \bar{\phi} + t_9 \bar{m}) + (R \leftrightarrow L, \phi \leftrightarrow \bar{\phi}) 
\end{align*}
\]

(4.6)

where \( t \)'s are constants with two parts:

\[ t_i = u_i + v_i \quad (i = 1, \ldots, 9) \]  

(4.7)
We have defined and

\[ \Sigma \]

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Note three points about the above results:

1. All the integrands of (4.9) vanish for \( p^2 < 1 \). (We have redefined \( p/\Lambda \) as \( p \).)
   This makes the integrals infrared finite.

2. The values of the integrals depend on the choice of \( K_{b,f} \).

3. The coefficients satisfy one algebraic constraint:

   \[ t_5 - t_6 + t_8 - t_9 = 0 \]  (4.11)

For \( \Sigma^{(1)} \) to vanish, the nine constants \( t_1, \ldots, 9 \) must vanish. (4.11) leaves eight independent conditions on the twelve constants \( c_1, \ldots, 12 \). Taking

\[ c_1, c_2, c_5, c_9 \]  (4.12)

as arbitrary, we obtain the rest as

\[
\begin{align*}
c_3 &= c_1 - v_1 \\
c_4 &= -c_1 + 2c_2 + v_3 + v_5 - v_6 + v_8 \\
c_6 &= -c_1 + c_2 + z_5 + v_5 + v_8 \\
c_7 &= -c_1 + 2c_5 + v_7 + v_8 \\
c_8 &= c_9 - v_2 + v_4 \\
c_{10} &= c_1 - c_5 - v_8 \\
|g|^2c_{11} &= c_1 - c_2 - v_5 + v_6 - v_8 \\
|g|^2c_{12} &= -c_9 + v_2 \\
c_{13} &= -c_1 + v_5 - v_6 + v_8 - v_{10}
\end{align*}
\]
Thus, by fine-tuning we have obtained

\[ \Sigma^{(1)} = 0 \]  

(4.14)

We have chosen \( c_1, c_2, c_5, c_9 \) as arbitrary constants because they have clear physical meanings: \( c_{1,2,5} \) correspond to normalization of fields, \( m \), and \( g \); \( c_9 \) corresponds to a finite shift of \( \phi \) proportional to \( \bar{g}m \) and that of \( \bar{\phi} \) proportional to \( g \bar{m} \).

\section{5. Attempt at an all order construction without antifields}

We wish to prove that fine-tuning makes \( \Sigma \) vanish. We outline the inductive proof and show where it fails. Let us first introduce a loop expansion of the action:

\[ S(A) = \sum_{l=0}^{\infty} S^{(l)}(A) \]  

(5.1)

Correspondingly, we expand \( \Sigma \) defined by (3.28):

\[ \Sigma(A) = \sum_{l=0}^{\infty} \Sigma^{(l)}(A) \]  

(5.2)

We have already shown

\[ \Sigma^{(0)} = \Sigma^{(1)} = 0 \]  

(5.3)

We also expand the constants \( z_i(0) (i = 1, \ldots, 12) \):

\[ z_i(0) = \sum_{l=1}^{\infty} c_i^{(l)} \]  

(5.4)

where \( c_i^{(l)} \) determine the asymptotic behavior of \( S^{(l)} \). In the previous section we have constructed \( c_i^{(1)} \) (called \( c_i \) there). We also adopt the same notation for the supersymmetry transformation of the fields:

\[ \delta \phi(p) = \sum_{l=0}^{\infty} \delta^{(l)} \phi(p), \ldots \]  

(5.5)

Suppose \( S^{(1)}, \ldots, S^{(l-1)} \) have been constructed by fine-tuning \( c_i^{(1)}, \ldots, c_i^{(l-1)} \) such that

\[ \Sigma^{(0)} = \Sigma^{(1)} = \ldots = \Sigma^{(l-1)} = 0 \]  

(5.6)

We wish to determine \( c_i^{(l)} \) so that

\[ \Sigma^{(l)} = 0 \]  

(5.7)

First, we show that the asymptotic behavior of \( \Sigma^{(l)} \) has no \( \Lambda \) dependence. We show this using ERG. Since \( \Sigma \) is a composite operator, \( \Sigma^{(l)} \) satisfies

\[ -\Lambda \frac{\partial}{\partial \Lambda} \Sigma^{(l)} = \int \frac{p}{p^2 + m^2} \left[ \frac{\delta S_{\text{int}}^{(l)}}{\delta \phi(p)} \frac{\delta}{\delta \phi(p)} + \frac{\delta S_{\text{int}}^{(l)}}{\delta \bar{\phi}(p)} \frac{\delta}{\delta \bar{\phi}(-p)} \right] \Sigma^{(l)} \]
+ \int_p \frac{\Delta(p/\Lambda)}{p^2 + |m|^2} \left[ S^{(0)}_{\text{int}} \frac{\delta}{\delta \chi_R(p)} \left\{ -ip \cdot \sigma \frac{\delta}{\delta \chi_L(-p)} + \bar{m} \frac{\delta}{\delta \chi_R(-p)} \right\} \right]
+ S^{(0)}_{\text{int}} \frac{\delta}{\delta \chi_L(p)} \left\{ -ip \cdot \sigma \frac{\delta}{\delta \chi_L(-p)} + m \frac{\delta}{\delta \chi_R(-p)} \right\} \right] \Sigma^{(l)} \right) \tag{5.8}

thanks to the induction hypothesis (5.6). We now take \( \Lambda \) large compared with \( m \) and the momenta carried by the fields. Since \( p \) in the argument of \( \Delta \) is of the same order as the momenta carried by the fields, \( \Delta \) vanishes asymptotically. Hence, the asymptotic behavior of \( \Sigma^{(l)} \) is independent of \( \Lambda \). We can then write

\[
\Sigma^{(l)} \xrightarrow{\Lambda \to \infty} \int d^4 x \xi_R \chi_R \left[ t_1^{(l)} \bar{\phi}^2 \phi + t_2^{(l)} g |m|^2 \bar{m} \phi + t_3^{(l)} |m|^2 \phi^2 + t_4^{(l)} g^2 \bar{m} \phi \right]
+ t_5^{(l)} m \bar{g} \bar{\phi}^2 + t_6^{(l)} \bar{m} g |\phi|^2 + t_7^{(l)} |g|^2 \bar{\phi}^2 \right]
+ \int d^4 x \xi_R \sigma_{\mu \nu} \chi_L \cdot \partial_{\mu} \phi \left( t_8^{(l)} \bar{g} \phi + t_9^{(l)} \bar{m} \phi \right) + (R \leftrightarrow L, \phi \leftrightarrow \bar{\phi}) \tag{5.9}
\]

where \( t \)'s are real constants independent of \( \Lambda \). (The \( t \)'s are real because the action is real and CP invariant.) The nine terms given above exhaust the possibilities allowed by the following:

1. \( \Sigma \) has mass dimension 0.
2. \( \Sigma \) is a bosonic scalar.
3. \( \Sigma \) has zero \( \alpha, \beta \) charges.
4. \( \Sigma \) is linear in either \( \xi_R \) or \( \xi_L \).

There is no term proportional to \( \eta_{\mu} \) due to the translation invariance.

Second, we consider the dependence of \( \Sigma^{(l)} \) on the \( l \)-loop constants \( c^{(l)} \). By expanding (3.28) in the number of loops, we obtain the following expression for \( \Sigma^{(l)} \):

\[
\Sigma^{(l)} = \Sigma^{(l),1} + \Sigma^{(l),2} \tag{5.10}
\]

where

\[
\Sigma^{(l),1} \equiv \int_p K_b(p/\Lambda) \left[ \frac{\delta^{(l)} \phi(p)}{\delta \phi(p)} \delta S^{(l)} + \delta^{(l)} \phi(p) \delta S^{(0)} \frac{\delta}{\delta \phi(p)} + (\phi \to \bar{\phi}) \right] \tag{5.11}
+ \int_p K_f(p/\Lambda) \left[ S^{(0)} \frac{\delta}{\delta \chi_R(p)} \delta^{(l)} \chi_R(p) + S^{(0)} \frac{\delta}{\delta \chi_R(p)} \delta^{(l)} \chi_R(p) + (R \to L) \right]
\]

and

\[
\Sigma^{(l),2} \equiv \int_p K_b(p/\Lambda) \left[ \sum_{k=1}^{l-1} \delta^{(k)} \phi(p) \delta S^{(l-k)} + \frac{\delta}{\delta \phi(p)} \delta^{(l-1)} \phi(p) + (\phi \to \bar{\phi}) \right]
+ \int_p K_f(p/\Lambda) \left[ \sum_{k=1}^{l-1} S^{(l-k)} \frac{\delta}{\delta \chi_R(p)} \delta^{(k)} \chi_R(p) - \text{Tr} \delta^{(l-1)} \chi_R(p) \frac{\delta}{\delta \chi_R(p)} \right]
\]
Only $\Sigma^{(l),1}$ depends on the $l$-loop constants $c_1^{(l)}$. We extract only the part of its asymptotic behavior that depends on $c^{(l)}$'s:

$$
\Sigma^{(l),1} \xrightarrow{\Lambda \rightarrow \infty} \int d^4 x \bar{\xi}_R \chi \left[ u_1^{(l)} \partial^2 \bar{\phi} + u_2^{(l)} g |m|^2 \bar{m} + u_3^{(l)} m^2 \bar{\phi} + u_4^{(l)} g^2 m^2 \phi \\
+ u_5^{(l)} m \bar{\phi} \frac{\partial^2 \phi}{2} + u_6^{(l)} \bar{m} g |\phi|^2 + u_7^{(l)} |g|^2 \bar{\phi} \frac{\partial^2 \phi}{2} \right] \\
+ \int d^4 x \bar{\xi}_R \bar{\sigma}_{\mu \chi L} \cdot \partial_{\mu} \bar{\phi} \left( u_8^{(l)} \bar{g} \bar{\phi} + u_9^{(l)} \bar{m} \right) + (R \leftrightarrow L, \phi \leftrightarrow \bar{\phi})
$$

(5.13)

where

$$
\begin{align*}
u_1^{(l)} &= c_1^{(l)} - c_2^{(l)} \\
u_2^{(l)} &= c_9^{(l)} + c_{12}^{(l)} |g|^2 \\
u_3^{(l)} &= -c_2^{(l)} + c_4^{(l)} + c_{11}^{(l)} |g|^2 \\
u_4^{(l)} &= c_8^{(l)} + c_{12}^{(l)} |g|^2 \\
u_5^{(l)} &= -c_2^{(l)} + c_6^{(l)} + c_{10}^{(l)} \\
u_6^{(l)} &= c_1^{(l)} + c_6^{(l)} + c_{11}^{(l)} |g|^2 \\
u_7^{(l)} &= -c_5^{(l)} + c_6^{(l)} + c_{10}^{(l)} \\
u_8^{(l)} &= c_1^{(l)} - c_5^{(l)} - c_{11} |g|^2 \\
u_9^{(l)} &= c_1^{(l)} - c_2^{(l)} - c_{11} |g|^2
\end{align*}
$$

(The calculation is a repetition of what we have already done at 1-loop.) The $u^{(l)}$ coefficients satisfy one algebraic constraint

$$
v_5^{(l)} - u_6^{(l)} + u_8^{(l)} - u_9^{(l)} = 0
$$

(5.15)

We note that $u_i^{(l)}$ does not give the whole $t_i^{(l)}$, but an additional contribution, $v_i^{(l)}$, comes from $\Sigma^{(l),2}$ so that

$$
t_i^{(l)} = u_i^{(l)} + v_i^{(l)}
$$

(5.16)

The coefficients $v_i^{(l)}$ are independent of $c^{(l)}$'s; we can tune only $u_i^{(l)}$ through $c^{(l)}$'s. But the algebraic constraint (5.15) implies that we cannot satisfy

$$
t_i^{(l)} = 0 \quad (i = 1, \ldots, 9)
$$

(5.17)

unless $t^{(l)}$s also satisfy the same constraint:

$$
t_5^{(l)} - t_6^{(l)} + t_8^{(l)} - t_9^{(l)} = 0
$$

(5.18)

Equivalently, $v^{(l)}$s must satisfy

$$
v_5^{(l)} - v_6^{(l)} + v_8^{(l)} - v_9^{(l)} = 0
$$

(5.19)

But we have no control over $v^{(l)}$s; they are fixed by the choice of the lower loop constants $c^{(1)}, \ldots, c^{(l-1)}$. We do not know how to derive (5.18) or (5.19). This is where our proof fails. We resort to the BRST formalism for help.
§6. BRST formalism

In this section we introduce antifields that generate supersymmetry transformation. We do this for the sole purpose of obtaining an algebraic structure that enables us to derive the constraint \( (5.18) \) (or equivalently \( (5.19) \)). Once we know the construction works, we can discard antifields entirely and go ahead with constructing a Wilson action in terms of fields alone. From now on we take

\[
K_b = K_f = K
\]

for simplicity, even though this equality is not required by supersymmetry.

In the BRST formalism we regard \( \xi_{R,L} \) as constant bosonic ghost spinors, and \( \eta_\mu \) as a constant fermionic ghost vector. We introduce antifields (sources) such that we reproduce the original supersymmetry transformation at the vanishing sources. The antifields have the opposite statistics to the corresponding fields: \( \phi^*, \bar{\phi}^* \) are fermionic, and \( \chi_R^*, \chi_L^* \) are bosonic. Let us summarize the properties of antifields and ghosts in a table:

| antifields/ghosts | statistics | dimensions | \( \alpha, \beta \) charges |
|------------------|------------|------------|--------------------------|
| \( \phi^* \)     | f          | 3          | \( -\alpha \)            |
| \( \bar{\phi}^* \) | f          | 3          | \( \alpha \)             |
| \( \chi_R^* \)   | b          | 5/2        | \( -\beta \)            |
| \( \chi_L^* \)   | b          | 5/2        | \( \beta \)             |
| \( \xi_R \)      | b          | \(-1/2\)   | \( \alpha - \beta \)    |
| \( \xi_L \)      | b          | \(-1/2\)   | \(-\alpha + \beta \)    |
| \( \eta_\mu \)   | f          | \(-1\)     | 0                        |

The ghost number is assigned as

\[
\begin{array}{c|c|c}
\text{ghost number } 1 & \text{ghost number } -1 \\
\hline
\xi_R, \xi_L, \eta_\mu & \phi^*, \bar{\phi}^*, \chi_R^*, \chi_L^* \\
\end{array}
\]

so that the action has zero ghost number.

At the classical level, we obtain the following action:

\[
\bar{S}_{ct} \equiv S_{ct} + \int d^4x \left[ \phi^* \left( \xi_R \chi_R + \eta_\mu \partial_\mu \phi \right) + \bar{\phi}^* \left( \xi_L \chi_L + \eta_\mu \partial_\mu \bar{\phi} \right) \right. \\
\left. + \tilde{\chi}_R^* \left( \sigma_\mu \xi_L \partial_\mu \phi - \left( \bar{m} \phi + \bar{g} \frac{\bar{\phi}^2}{2} \right) \xi_R + \eta_\mu \partial_\mu \chi_R \right) \right. \\
\left. + \tilde{\chi}_L^* \left( \sigma_\mu \xi_R \partial_\mu \phi - \left( m \phi + g \frac{\phi^2}{2} \right) \xi_L + \eta_\mu \partial_\mu \chi_L \right) - \tilde{\chi}_R^* \xi_R \cdot \tilde{\chi}_L^* \xi_L \right] \tag{6.2}
\]

The last term, quadratic in antifields, is necessary so that \( \bar{S}_{ct} \) satisfies the Zinn-Justin equation

\[
\Sigma_{ct} \equiv \int_p \left[ \frac{\delta \bar{S}_{ct}}{\delta \phi(p)} \frac{\partial}{\partial \phi^*(-p)} \bar{S}_{ct} + \frac{\delta \bar{S}_{ct}}{\delta \bar{\phi}(p)} \frac{\partial}{\partial \bar{\phi}^*(-p)} \bar{S}_{ct} \right. \\
\left. + \bar{S}_{ct} \frac{\partial}{\partial \chi_R(p)} \frac{\partial}{\partial \chi_R^*(-p)} \bar{S}_{ct} + \bar{S}_{ct} \frac{\partial}{\partial \chi_L(p)} \frac{\partial}{\partial \chi_L^*(-p)} \bar{S}_{ct} \right]
\]
This is the classical limit of the BRST invariance that we will introduce at the end of this section.

Let $\bar{S}$ be the total action in the presence of antifields. It is split as

$$\bar{S}(A) = S_{\text{free}}(A) + \bar{S}_{\text{int}}(A)$$

where $\bar{S}_{\text{int}}$ satisfies the same ERG differential equation as $S_{\text{int}}$, but their asymptotic behaviors differ due to antifields:

$$\bar{S}_{\text{int}}(A) - S_{\text{int}}(A) \xrightarrow{A \to \infty} \int d^4x \left[ \phi^* (\tilde{\xi}_R \chi_R + \eta_\mu \partial_\mu \phi) + \tilde{\phi}^* (\tilde{\xi}_L \chi_L + \eta_\mu \partial_\mu \tilde{\phi}) + \bar{\chi}_R^* \left( (1 + z_{10}) g \frac{\tilde{\phi}^2}{2} + (1 + z_{11}) \tilde{m} \tilde{\phi} + z_{12} \tilde{g} m^2 \right) ight.$$ \begin{align*} & - j_R \left( (1 + z_{10}) \tilde{g} \frac{\phi^2}{2} + (1 + z_{11}) \tilde{m} \phi + z_{12} \tilde{g} m^2 \right) \\ & - j_L \left( (1 + z_{10}) g \frac{\phi^2}{2} + (1 + z_{11}) m \phi + z_{12} g m^2 \right) \\ & + \{ -1 + z_{13} (\ln A / \mu) \} j_R j_L \right] \end{align*}

$$= \int_p \left[ \phi^* (-p) (\tilde{\xi}_R \chi_R (p) + \eta_\mu i p_\mu \phi(p)) \\ + \tilde{\phi}^* (-p) (\tilde{\xi}_L \chi_L (p) + \eta_\mu i p_\mu \tilde{\phi}(p)) + \bar{\chi}_R^* (-p) (\tilde{\sigma}_R i p_\mu \phi(p) - \xi_R \tilde{m} \phi(p) + \eta_\mu i p_\mu \chi_R(p)) \\ + \bar{\chi}_L^* (-p) (\sigma_R i p_\mu \tilde{\phi}(p) - \xi_L m \phi(p) + \eta_\mu i p_\mu \chi_L(p)) \\ - j_R (-p) j_L (p) \right]$$

The $A$ dependent coefficients $z_{10,11,12}$ are the same as those in sect. 3. The real coefficient $z_{13}$ is generated from the product of $[\phi^2](p)$ and $[\tilde{\phi}^2](-p)$. We note three points here:

1. Three parameters $z_{10}(0), z_{11}(0), z_{12}(0)$ that define supersymmetry transformation is now incorporated into the action $\bar{S}_{\text{int}}$.
2. $\bar{S}_{\text{int}}$ depends on one extra parameter $z_{13}(0)$, which is absent in the definition of $S_{\text{int}}$ and in the supersymmetry transformation at the vanishing antifields.
3. The antifields $j_R, j_L$ are similar to the auxiliary fields that close the off-shell supersymmetry algebra. Unlike the auxiliary fields, $j_R$ and $j_L$ are external sources not to be integrated over.

Most antifield dependence is introduced through linear couplings to the elementary fields, and can be given explicitly:
its asymptotic behavior is determined from (6.5) as
\[ S \]
where we have suppressed the $\ln \frac{2}{\Lambda}$
where the shifted fields are defined as follows:

\[
\begin{align*}
\phi_{sh}(p) & \equiv \phi(p) + \frac{1-K(p/A)}{p^2+|m|^2} \left\{ -\bar{\phi}^*(p)i\rho_{\mu\nu} - (\bar{\chi}_{L}^*(p)i\rho + \sigma + \bar{m}\bar{\chi}_{R}(p)) \right\} \\
\bar{\phi}_{sh}(p) & \equiv \bar{\phi}(p) + \frac{1-K(p/A)}{p^2+|m|^2} \left\{ -\phi^*(p)i\rho_{\mu\nu} - (\chi_{R}^*(p)i\rho + \bar{\sigma} + m\chi_{L}(p)) \right\} \\
\chi_{R,sh}(p) & \equiv \chi_{L}(p) + \frac{1-K(p/p)}{p^2+|m|^2} \left\{ -m\chi_{R}(p) + ip\cdot \bar{\sigma} \right\} + \frac{1-K(\bar{p}/p)}{p^2+|m|^2} \left\{ -m\chi_{R}^*(p) + ip\cdot \sigma \right\} \\
\chi_{L,sh}(p) & \equiv \chi_{L}(p) + \frac{1-K(p/p)}{p^2+|m|^2} \left\{ -m\chi_{R}(p) + ip\cdot \bar{\sigma} \right\} + \frac{1-K(\bar{p}/p)}{p^2+|m|^2} \left\{ -m\chi_{R}^*(p) + ip\cdot \sigma \right\}
\end{align*}
\]

\( \hat{S}_{int} \) is obtained from \( S_{int} \) by coupling an external source \( j_R \) to the composite operator \( [\phi^2/2] \), and \( j_L \) to \( [\bar{\phi}^2/2] \). \( \hat{S}_{int} \) satisfies the same ERG differential equation as \( \hat{S}_{int} \); its asymptotic behavior is determined from (6.5) as

\[
\hat{S}_{int}(A) - S_{int}(A) \rightarrow \int d^4x \left\{ (1 + z_{10})g \frac{\bar{\phi}^2}{2} + z_{11}m\bar{\phi} + z_{12}m^2 \right\}
\]

\[ -j_L \left\{ (1 + z_{10})g \frac{\phi^2}{2} + z_{11}m\phi + z_{12}m^2 \right\} + z_{13}j_Rj_L \]

where we have suppressed the $\ln A/\mu$ dependence of $z$'s.

We now define a fermionic composite operator by

\[
\tilde{\Sigma} \equiv \int p K(p/A) \left[ \frac{\delta}{\delta \phi^*(p)} - \delta \frac{\delta \tilde{S}}{\delta \phi(p)} + \delta \frac{\delta \tilde{S}}{\delta \phi*(p)} - \tilde{S} \right] + \frac{\delta}{\delta \phi^*(p)} - \delta \frac{\delta \tilde{S}}{\delta \phi(p)} + \delta \frac{\delta \tilde{S}}{\delta \phi*(p)} - \tilde{S} \left| \frac{\partial}{\partial \xi^*_{\rho \mu}} \tilde{S} \right| \]

We wish to fine-tune the thirteen parameters

\[ z_1(0), \cdots, z_{13}(0) \]
for the BRST invariance
\[ \Sigma = 0 \] (6.12)
At the vanishing antifields this reduces to (3.29), the supersymmetry of \( S \). The classical BRST invariance (6.3) implies that (6.12) holds at tree level. \( \Sigma \) defined by (6.10) satisfies the following algebraic constraint:
\[
\delta Q \bar{\Sigma} \equiv \int \bar{K} (p/\Lambda) \left[ \frac{\delta}{\delta \phi^*(-p)} \bar{\Sigma} \cdot \delta \bar{S} + \frac{\delta}{\delta \phi(p)} \delta \bar{\Sigma} + \frac{\delta}{\delta \phi^*(-p)} \delta \phi(p) \right] + \delta \bar{S} \cdot \delta \Sigma + \frac{\delta}{\delta \phi^*(-p)} \delta \phi(p) \nonumber
\]
\[
\quad + \frac{\delta}{\delta \phi^*(-p)} \left[ \frac{\delta}{\delta \chi_R^*(p)} \frac{\delta \bar{\Sigma}}{\delta \chi_R(p)} + \frac{\delta}{\delta \chi_L^*(p)} \frac{\delta \bar{\Sigma}}{\delta \chi_L(p)} \right] + \delta \bar{S} \cdot \delta \Sigma + \frac{\delta}{\delta \phi^*(-p)} \delta \phi(p) \nonumber
\]
\[
\quad + \frac{\delta}{\delta \phi^*(-p)} \left[ \frac{\delta}{\delta \chi_R^*(p)} \frac{\delta \bar{\Sigma}}{\delta \chi_R(p)} + \frac{\delta}{\delta \chi_L^*(p)} \frac{\delta \bar{\Sigma}}{\delta \chi_L(p)} \right] \right] - \xi_L \sigma \mu \frac{\delta}{\partial \eta^\mu} \bar{\Sigma} = 0 \] (6.13)
We shall see, in the next section, that this provides the algebraic constraint (5.18) that we could not derive without antifields.

§7. All order proof

The all order proof in the BRST formalism proceeds analogously to the attempted proof in sect. [5]. Before we begin, we make some preparation. As can be seen from (6.17), most of the antifield dependence of \( \bar{S} \) is given by the shift (6.8) of the fields. It is then natural to express \( \bar{\Sigma} \) in terms of (6.8) by replacing each field in \( \bar{\Sigma} \) by the corresponding shifted field. We obtain
\[
\bar{\Sigma}[\phi, \tilde{\phi}, \chi_R, \chi_L, \phi^*, \tilde{\phi}^*, \chi_R^*, \chi_L^*] = \bar{\Sigma} \left[ \phi_{sh}, \tilde{\phi}_{sh}, \chi_{R,sh}, \chi_{L,sh}, \phi^*, \tilde{\phi}^*, \chi_{R}^*, \chi_{L}^* \right] \] (7.1)
where \( \bar{\Sigma} \) is defined by
\[
\bar{\Sigma} \left[ \phi, \tilde{\phi}, \chi_R, \chi_L, \phi^*, \tilde{\phi}^*, \chi_R^*, \chi_L^* \right] \equiv \int \bar{K} (p/\Lambda) \left[ \frac{\delta}{\delta \phi^*(-p)} \bar{\Sigma} \cdot \delta \bar{S} + \frac{\delta}{\delta \phi(p)} \delta \bar{\Sigma} + \frac{\delta}{\delta \phi^*(-p)} \delta \phi(p) \right] - \frac{\delta}{\delta \chi_R(p)} \frac{\delta \bar{\Sigma}}{\delta \chi_R(p)} + \frac{\delta}{\delta \chi_L(p)} \frac{\delta \bar{\Sigma}}{\delta \chi_L(p)} \right] \]

Wilson action for the Wess-Zumino model
\[ -\xi_L \sigma_{\mu} \xi_R \cdot \frac{\partial}{\partial \eta_{\mu}} \tilde{S} \]  
and \( \tilde{S} \) is defined by 
\[ \tilde{S} \left[ \phi, \bar{\phi}, \chi_R, \chi_L, \phi^*, \bar{\phi}^*, \chi_R^*, \chi_L^* \right] \]
\[ \equiv S_{\text{free}}[\phi, \bar{\phi}, \chi_R, \chi_L] + \tilde{S}_{\text{int}}[\phi, \bar{\phi}, \chi_R, \chi_L; j_R, j_L] \]
\[ + \int \frac{1}{p} K(p/\Lambda) \left[ \phi^*(-p) \left\{ i p_{\mu} \eta_{\mu} \bar{\phi}(p) + \xi_R \chi_R(p) \right\} \right] \]
\[ + \xi^*(p) \left\{ i p_{\mu} \eta_{\mu} \bar{\phi}(p) + \bar{\xi}_L \chi_L(p) \right\} \]
\[ + \chi_R^*(-p) \left\{ i p_{\mu} \eta_{\mu} \chi_R(p) + i p \cdot \bar{\sigma} \xi_L \bar{\phi}(p) - m \xi_R \bar{\phi}(p) \right\} \]
\[ + \chi_L^*(-p) \left\{ i p_{\mu} \eta_{\mu} \chi_L(p) + i p \cdot \sigma \xi_R \bar{\phi}(p) - m \xi_L \bar{\phi}(p) \right\} \]
\[ - j_R(-p) j_L(p) \]  
(7.3)

Thus, the identity (6.12) is equivalent to the identity 
\[ \tilde{\Sigma} \left[ \phi, \bar{\phi}, \chi_R, \chi_L, \phi^*, \bar{\phi}^*, \chi_R^*, \chi_L^* \right] = 0 \]  
(7.4)
for the unshifted fields. This identity is simpler to consider than (6.12), since unshifting makes the antifield dependence of \( \tilde{\Sigma} \) simpler than that of \( \Sigma \). Note, however, that for \( \Lambda \) large compared with \( m \) and the momenta of the fields, \( \Sigma \) and \( \tilde{\Sigma} \) are the same: 
\[ \Sigma(\Lambda) - \tilde{\Sigma}(\Lambda) \xrightarrow{\Lambda \to \infty} 0 \]  
(7.5)
since the shifts in (6.8) vanish in this limit.

By definition, \( \Sigma \) satisfies its own algebraic identity:
\[ \tilde{\delta}_Q \Sigma \equiv \int \frac{1}{p} K(p/\Lambda) \left[ \right. \]
\[ \left. \frac{\delta}{\delta \phi^*(-p)} \tilde{S} \cdot \frac{\delta \tilde{\Sigma}}{\delta p_{\mu} \eta_{\mu}} + \frac{\delta}{\delta \phi^*(-p)} \tilde{S} \cdot \frac{\delta \Sigma}{\delta \phi(p)} + \frac{\delta}{\delta \phi^*(-p)} \frac{\delta}{\delta \phi(p)} \tilde{\Sigma} \right] \]
\[ + \text{Tr} \left\{ \right. \]
\[ \left. \frac{\delta}{\delta \chi_R^*(-p)} \tilde{S} \cdot \frac{\delta \tilde{\Sigma}}{\delta \chi_R(p)} + \frac{\delta}{\delta \chi_R^*(-p)} \tilde{S} \cdot \frac{\delta \Sigma}{\delta \chi_R(p)} + \frac{\delta}{\delta \chi_R^*(-p)} \frac{\delta}{\delta \chi_R(p)} \tilde{\Sigma} \right\} \]
\[ + \text{Tr} \left\{ \right. \]
\[ \left. \frac{\delta}{\delta \chi_L^*(-p)} \tilde{S} \cdot \frac{\delta \tilde{\Sigma}}{\delta \chi_L(p)} + \frac{\delta}{\delta \chi_L^*(-p)} \tilde{S} \cdot \frac{\delta \Sigma}{\delta \chi_L(p)} + \frac{\delta}{\delta \chi_L^*(-p)} \frac{\delta}{\delta \chi_L(p)} \tilde{\Sigma} \right\} \]
\[ - \xi_L \sigma_{\mu} \xi_R \cdot \frac{\partial}{\partial \eta_{\mu}} \tilde{\Sigma} = 0 \]  
(7.6)

In fact we can show 
\[ \delta_Q \Sigma \left[ \phi, \bar{\phi}, \chi_R, \chi_L, \phi^*, \bar{\phi}^*, \chi_R^*, \chi_L^* \right] = \delta_Q \Sigma \left[ \phi_{\text{sh}}, \bar{\phi}_{\text{sh}}, \chi_R, \chi_L, \phi^*, \bar{\phi}^*, \chi_R^*, \chi_L^* \right] \]  
(7.7)
Hence, the two identities (7.13), (7.10) are really the same.

We now begin our inductive proof. We will show that we can satisfy (6.12) by fine-tuning the thirteen parameters \( z_1(0), \ldots, z_{13}(0) \). The proof proceeds analogously to the attempted proof in sect. 5. We use the same notation for loop expansions:

\[
S = \sum_{l=0}^{\infty} S^{(l)}, \quad \Sigma = \sum_{l=1}^{\infty} \Sigma^{(l)}, \quad z_i(0) = \sum_{l=1}^{\infty} c_i(0) \quad (i = 1, \ldots, 13)
\]

(7.8)

As the induction hypothesis, we assume that we have chosen \( c^{(1)}, \ldots, c^{(l-1)} \) so that

\[
\Sigma^{(0)} = \ldots = \Sigma^{(l-1)} = 0
\]

(7.9)

We need to show that fine-tuning of \( c^{(l)} \)'s makes

\[
\Sigma^{(l)} = 0
\]

(7.10)

First, we show that the asymptotic behavior of \( \Sigma^{(l)}(A) \) is independent of \( A \), using ERG and (7.9). We omit the proof since it is exactly the same as in sect. 5.

Second, we enumerate all possible terms in the asymptotic behavior of \( \Sigma^{(l)} \), which is the same as that of \( \Sigma^{(l)} \). The asymptotic form of \( \Sigma^{(l)} \) is parametrized by twelve real constants \( t_1^{(l)}, \ldots, t_{12}^{(l)} \) as follows:

\[
\Sigma^{(l)} \xrightarrow{A \to \infty} \int \bar{\xi}_{R,L} \left[ t_1^{(l)} \partial^2 \bar{\phi} + t_2^{(l)} g |m|^2 \bar{m} + t_3^{(l)} |m|^2 \bar{\phi} + t_4^{(l)} g^2 \bar{m}^2 \phi + t_5^{(l)} m \bar{g} \bar{\phi}^2 \right. \\
+ t_6^{(l)} mg |\phi|^2 + t_7^{(l)} |g|^2 \phi \bar{\phi}^2 + t_8^{(l)} g \bar{g} \bar{\phi} + t_9^{(l)} |\bar{m}|^2 \phi + t_{10}^{(l)} m + t_{11}^{(l)} g \phi \right] \\
+ \int \chi_L \sigma \mu \xi_R \cdot \partial_{\mu} \bar{\phi} \left( t_8^{(l)} g \bar{\phi} + t_9^{(l)} |\bar{m}|^2 \phi \right) + t_{12}^{(l)} \int \chi_L \sigma \mu \xi_R \cdot \partial_{\mu} \bar{J}_L \\
+ (R \leftrightarrow L, \phi \leftrightarrow \bar{\phi}, m \leftrightarrow \bar{m}, g \leftrightarrow \bar{g})
\]

(7.11)

This is the most general form allowed by the following constraints:

1. \( \Sigma \) has mass dimension 0.
2. \( \Sigma \) is a fermionic scalar.
3. \( \Sigma \) has no \( \alpha, \beta \) charges.
4. \( \Sigma \) has ghost number 1.
5. \( \Sigma^{(l)} \) is independent of \( \eta_{L,R}, \phi^*, \bar{\phi}^* \).
6. \( \Sigma^{(l)} \) depends on \( \chi^*_R, \chi^*_L \) only through \( j_R, j_L \).

We derive the last two properties in Appendix C. As in sect. 5 we now divide each \( t_i^{(l)} \) into two parts:

\[
t_i^{(l)} = u_i^{(l)} + v_i^{(l)}
\]

(7.12)

where \( u_i^{(l)} \)'s are linear combinations of \( c^{(l)} \)'s, and \( v_i^{(l)} \)'s are determined by the action up to \((l - 1)\)-loop level. \( u_i^{(l)} \)'s are obtained from the asymptotic behavior of

\[
\hat{\Sigma}^{(l),1} \equiv \int_p K(p/A) \left[ \frac{-\delta}{\delta \phi^* (-p)} \hat{S}^{(0)} \right. \\
\left. \delta \hat{S}^{(l)} / \delta \phi(p) + (\phi \rightarrow \bar{\phi}, \phi^* \rightarrow \bar{\phi}^*) \right]
\]

(7.13)
\[ -\frac{\delta \bar{S}^{(l)}}{\delta \chi_R(p)} \cdot \frac{\delta}{\delta \bar{x}_R(-p)} \bar{S}^{(0)} - \frac{\gamma}{\delta \chi_R(p)} \cdot \frac{\delta}{\delta \bar{x}_R(-p)} S^{(l)} + (R \to L) \]

as follows:

\[
\begin{align*}
 u_1^{(l)} &= c_1^{(l)} - c_3^{(l)} \\
u_1^{(l)} &= -c_2^{(l)} + c_4^{(l)} + c_{11}^{(l)} |g|^2 \\
u_2^{(l)} &= c_5^{(l)} + c_7^{(l)} + c_{12}^{(l)} |g|^2 \\
u_5^{(l)} &= -c_2^{(l)} + c_6^{(l)} + c_{10}^{(l)} \\
u_7^{(l)} &= c_5^{(l)} + c_7^{(l)} + c_{12}^{(l)} |g|^2 \\
u_9^{(l)} &= c_1^{(l)} - c_2^{(l)} - c_{11}^{(l)} |g|^2 \\
u_{10}^{(l)} &= -c_1^{(l)} - c_2^{(l)} + c_{12}^{(l)} |g|^2 - c_{13}^{(l)} \\
u_{11}^{(l)} &= -c_6^{(l)} - c_{10}^{(l)} - c_{13}^{(l)} \\
u_{12}^{(l)} &= c_1^{(l)} + c_{13}^{(l)}
\end{align*}
\]

The calculation is mostly the same as that done in sect. 5. The \(u^{(l)}\)'s are not linearly independent, but they satisfy the following three linear relations:

\[
\begin{align*}
u_5^{(l)} - u_6^{(l)} + u_8^{(l)} - u_9^{(l)} &= 0 \\
u_8^{(l)} - u_{11}^{(l)} - u_{12}^{(l)} &= 0 \\
u_9^{(l)} - u_{10}^{(l)} - u_{12}^{(l)} &= 0
\end{align*}
\]

The first relation is the same as that found in sect. 5.

Third, we derive algebraic constraints on the \(t^{(l)}\) constants. For fine-tuning to work, \(t^{(l)}\)'s must satisfy the same linear relations as (7.15). These come from the algebraic identity (7.46). Since \(\Sigma\) vanishes up to \((l - 1)\)-loop, we obtain

\[
\begin{align*}
&\left( \delta_Q \bar{S} \right)^{(l)} = \int_p K(p/\Lambda) \left[ \right. \\
&\quad \frac{\delta}{\delta \phi^*(-p)} \bar{S}^{(l)} + \frac{\delta}{\delta \phi^*(p)} S^{(0)} + \frac{\delta}{\delta \phi(p)} \delta \bar{S}^{(l)} + \delta \bar{S}^{(0)} \cdot \frac{\delta}{\delta \chi_R(p)} \left. \right] (R \to L) \]
\end{align*}
\]

Taking the asymptotic part, we obtain

\[
\begin{align*}
&\left( \delta_Q \bar{S} \right)^{(l)} \xrightarrow{\Lambda \to \infty} \\
&\quad (t_5^{(l)} - t_6^{(l)} - t_{10}^{(l)} + t_{11}^{(l)}) \int d^4 x \xi_R \sigma \xi \chi_L \left( \bar{m} \phi - \bar{m} \phi \right) \\
&\quad + \int d^4 x \xi_R \sigma \xi \partial \mu \chi_R \left[ (-t_9^{(l)} + t_{10}^{(l)} + t_{12}^{(l)}) m + (-t_8^{(l)} + t_{11}^{(l)} + t_{12}^{(l)}) g \phi \right] \\
&\quad + \xi_R \sigma \xi \chi_L \int d^4 x \left[ (t_5^{(l)} - t_6^{(l)} + t_8^{(l)}) m \phi \phi - \phi^2 \right] \\
&\quad + \partial_\mu \phi \cdot j_L \left\{ (t_{10}^{(l)} - t_9^{(l)} + t_{12}^{(l)}) m + (t_{11}^{(l)} - t_8^{(l)} + t_{12}^{(l)}) g \phi \right\} + \{\phi, \chi_R, \} \leftrightarrow \{ \bar{\phi}, \chi_L \} \right. \]
\end{align*}
\]
This must vanish. Hence, \( i^{(l)} \)'s satisfy the same linear relations as (7.15):

\[
\begin{align*}
\begin{cases}
\quad t_5^{(l)} - t_6^{(l)} + t_8^{(l)} - t_9^{(l)} = 0 \\
\quad t_8^{(l)} - t_{11}^{(l)} - t_{12}^{(l)} = 0 \\
\quad t_9^{(l)} - t_{10}^{(l)} - t_{12}^{(l)} = 0
\end{cases}
\end{align*}
\tag{7.18}
\]

This also implies

\[
\begin{align*}
\begin{cases}
\quad v_5^{(l)} - v_6^{(l)} + v_8^{(l)} - v_9^{(l)} = 0 \\
\quad v_8^{(l)} - v_{11}^{(l)} - v_{12}^{(l)} = 0 \\
\quad v_9^{(l)} - v_{10}^{(l)} - v_{12}^{(l)} = 0
\end{cases}
\end{align*}
\tag{7.19}
\]

Last, we fine-tune \( c^{(l)} \)'s to make \( i^{(l)} \)'s vanish. Because of (7.18), we have nine independent conditions to satisfy using thirteen constants. Taking

\[
c_1^{(l)}, c_2^{(l)}, c_5^{(l)}, c_9^{(l)}
\tag{7.20}
\]
as arbitrary, we obtain the rest as follows:

\[
\begin{align*}
\quad c_3^{(l)} &= c_1^{(l)} - v_1^{(l)} \\
\quad c_4^{(l)} &= -c_1^{(l)} + 2c_2^{(l)} + v_3^{(l)} + v_5^{(l)} - v_6^{(l)} + v_8^{(l)} \\
\quad c_6^{(l)} &= -c_1^{(l)} + c_2^{(l)} + c_5^{(l)} + v_5^{(l)} + v_8^{(l)} \\
\quad c_7^{(l)} &= -c_1^{(l)} + 2c_5^{(l)} + v_7^{(l)} + v_8^{(l)} \\
\quad c_8^{(l)} &= c_9^{(l)} - v_2^{(l)} + v_4^{(l)} \\
\quad c_{10}^{(l)} &= c_1^{(l)} - c_5^{(l)} - v_8^{(l)} \\
\quad |g|^2 c_{11}^{(l)} &= c_1^{(l)} - c_2^{(l)} - v_5^{(l)} - v_6^{(l)} - v_8^{(l)} \\
\quad |g|^2 c_{12}^{(l)} &= -c_9^{(l)} + v_2^{(l)} \\
\quad c_{13}^{(l)} &= -c_1^{(l)} + v_5^{(l)} - v_6^{(l)} + v_8^{(l)} - v_{10}^{(l)}
\end{align*}
\tag{7.21}
\]

This concludes the proof by induction.

The physical meaning of each arbitrary parameters is clear:
1. \( z_1(0) \) — overall normalization of scalar and spinor fields; this is unphysical.
2. \( z_2(0) \) — normalization of \( m \)
3. \( z_5(0) \) — normalization of \( g \)
4. \( z_9(0) \) — constant shift of \( \phi \) proportional to \( \bar{g} m \), and \( \bar{\phi} \) proportional to \( g \bar{m} \); this is also unphysical.

### §8. Quadratic divergences and holomorphy

Before concluding the paper, we make remarks on the quadratic divergences and holomorphy of the Wilson action \( S(\Lambda) \).

First on the quadratic divergences, by which we mean the two leading terms in the asymptotic behavior of the action:

\[
S(\Lambda) \xrightarrow{\Lambda \to \infty} \int d^4x \ [\Lambda^2 a_4 (\ln \Lambda / \mu)|\phi|^2 + \Lambda^2 a_9 (\ln \Lambda / \mu) (\bar{m} g \phi + m \bar{g} \bar{\phi})] \tag{8.1}
\]
In sect. 4 we have found the following 1-loop results:

\[
\begin{align*}
  a_4^{(1)} &= -\frac{|g|^2}{2} \int \{ -\Delta_b(p) + 2\Delta_f(p) (1 - K_f(p)) \} / p^2 \\
  a_9^{(1)} &= -\frac{1}{2} \int \{ \Delta_f(p) - \Delta_b(p) \} / p^2
\end{align*}
\]  

(8.2)

We observe two points and make a comment:

1. The values of the constants \(a_4^{(1)}, a_9^{(1)}\) depend on the choice of the cutoff functions \(K_{b,f}\), and therefore non-universal. For a general choice of \(K_{b,f}\), both the constants are non-vanishing.

2. Though supersymmetry does not require \(K_b = K_f\), this choice would make \(a_9^{(1)}\) zero, but not \(a_4^{(1)}\).

3. If we adopt the superfields, we must choose \(K_b = K_f\). The F-term in the superfield formalism does not receive any quantum correction.\(^{10}\) This suggests a possibility of choosing the parameters \(z_{1,2,5,9}\) so that

\[
a_9 = 0
\]

(8.3)

to all orders in loop expansions.

The Wilson action in terms of superfields does not have any quadratic divergence, but it is generated by integration over the auxiliary fields. At 1-loop, it is easy to see this explicitly. Let us give an outline here. The tree level action has a vertex

\[
-|g|^2 \bar{\phi}(p_3) F(p_4) \frac{1 - K((p_1 + p_2)/\Lambda)}{(p_1 + p_2)^2 + |m|^2} \phi(p_1) F(p_2)
\]

(8.4)

where \(F, \bar{F}\) are auxiliary fields. Contracting \(F\) and \(\bar{F}\) using the propagator

\[
K(p/\Lambda) \frac{p^2}{p^2 + |m|^2}
\]

(8.5)

we obtain the scalar mass term (at zero momentum) as follows:

\[
-|g|^2 \int K(p/\Lambda) (1 - K(p/\Lambda)) \frac{p^2}{(p^2 + |m|^2)^2}
\]

\[
= -|g|^2 \Lambda^2 \int \frac{K(p)(1-K(p))}{p^2} + O(|g|^2|m|^2)
\]

\[
= -|g|^2 \Lambda^2 \frac{1}{2} \int \frac{\Delta(p) - 2\Delta(p)(1 - K(p))}{p^2} + O(|g|^2|m|^2)
\]

reproducing \(a_4^{(1)}\) of (8.2).

We must conclude that quadratic divergences generally exist in the Wilson action of the Wess-Zumino model. This is analogous to the presence of a quadratically divergent gauge boson mass term in the Wilson actions of gauge theories. (For an explicit calculation in QED, see.\(^{18}\)) We expect that any realization of supersymmetry on a lattice (if it exists) is analogous to the choice \(K_b \neq K_f\), and that fine-tuning of order \(\mu^2/\Lambda^2\) will be inevitable.
Second on holomorphy. In\(^{(16)}\) the non-renormalization of the F-term of the Wilson action in superfields has been explained as a consequence of holomorphy, i.e., the F-term depends either only on \(\phi, \chi_R, \bar{F}, g, m\), or only on \(\phi, \chi_L, \bar{F}, \bar{g}, \bar{m}\). This concept of holomorphy has been introduced with help of the \(\alpha, \beta\) charges and of the idea of local couplings that is inspired by string theory. The non-renormalization theorem has been also derived in\(^{(19)}\) based on the observation that the chiral vertices can be written as multiple supersymmetry variations. The approach of\(^{(19)}\) has been further generalized to the component field formalism with on-shell supersymmetry.\(^{(20)}\) Thus, it is possible to formulate holomorphy in terms of component fields. We have not, however, studied how to incorporate holomorphy in the ERG formalism. Nevertheless we can expect that holomorphy corresponds to some constraints on the free parameters \(z_{1,2,5,9}(0)\) of the theory. Examining the 1-loop results of sect. 4, we notice \(z_{2,5,9}(A/\mu)\) have no dependence on \(A/\mu\). We then speculate that holomorphy is equivalent to

\[
z_2(\ln A/\mu) = z_5(\ln A/\mu) = z_9(\ln A/\mu) = 0 \quad (8.7)
\]

under the following choice of the parameters

\[
z_2(0) = z_5(0) = z_9(0) = 0 \quad (8.8)
\]

The verification is left for a future study.

§9. Conclusions

In this paper we have constructed a Wilson action of the Wess-Zumino model using ERG perturbatively: solving the exact renormalization group differential equation and imposing supersymmetry. The action is built out of scalar and spinor fields without auxiliary fields, and satisfies the invariance (3.29) under the supersymmetry transformation (3.18). In order to prove the consistency of the construction, we have resorted to the BRST formalism, by introducing antifields that generate the transformation. But the antifields are necessary only for the proof, and once we know the construction works, we can discard the antifields entirely. Hence, the Wilson action is a functional only of scalar and spinor fields.

The Wilson action, thus constructed, has four arbitrary parameters:
1. the mass parameter \(m\)
2. the coupling parameter \(g\)
3. the common normalization of scalar and spinor fields — the relative normalization is fixed by our choice of supersymmetry transformation
4. the constant shift of \(\phi, \bar{\phi}\) proportional to \(\bar{g}m, g\bar{m}\) — physics does not change, but the action changes its appearance

In the formulation without auxiliary fields, each parameter gets its own beta function that gives the dependence of the parameter on the renormalization scale \(\mu\). How the beta functions arise in the context of ERG has been discussed in ref.\(^{(21)}\) The derivation of the beta functions for the Wess-Zumino model is left for a future study. Since our construction of the supersymmetric Wilson action depends only on its supersymmetry, we expect that the ERG method can be applied straightforwardly
to supersymmetric Yang-Mills theories and theories with extended supersymmetry; this is also left for future studies.

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Appendix A

Notation on spinors

Throughout this paper we work on the four dimensional euclidean space. In this appendix we summarize the basic properties of spinors, emphasizing what is relevant to this paper.

Denoting the orthogonal coordinates by $x_\mu$ ($\mu = 1, 2, 3, 4$), we define

$$x \equiv i x_\mu \sigma_\mu$$

where $\sigma_\mu$ are defined by the Pauli matrices and the 2-by-2 unit matrix as

$$\sigma_\mu \equiv (\vec{\sigma}, -i 1_2)$$

We then find

$$\det x = x_\mu x_\mu = x^2$$

An arbitrary rotation can be given as

$$x \rightarrow x' = L x R^{-1}$$

where $L, R$ are arbitrary SU(2) matrices, and hence $\det x$ is invariant under any rotation.

Under the rotation (A.4) a right-hand two-component spinor field $\chi_R(x)$ transforms as

$$\chi_R(x) \rightarrow \chi'_R(x') = R \chi_R(x)$$

Similarly, a left-hand two-component spinor field $\chi_L(x)$ transforms as

$$\chi_L(x) \rightarrow \chi'_L(x') = L \chi_L(x)$$

Denoting the transpose of a spinor (either right or left) by

$$\bar{\chi} \equiv \chi^T \sigma_y = (i \chi_2, -i \chi_1)$$

we can construct scalars:

$$\bar{\chi}_R \chi'_R, \quad \bar{\chi}_L \chi'_L$$
We obtain
\[ \bar{\chi}_R' \chi_R = \pm \bar{\chi}'_R \chi_R, \quad \bar{\chi}_L' \chi_L = \pm \bar{\chi}'_L \chi_L \] (A.9)
depending on whether the two spinors are mutually anticommuting (plus) or commuting (minus).

We call the hermitian conjugate of \( \sigma_\mu \) by
\[ \bar{\sigma}_\mu \equiv \sigma_\mu^\dagger = -\sigma_\mu^T \sigma_y \] (A.10)
Using \( \sigma_\mu \) or \( \bar{\sigma}_\mu \), we can construct a vector:
\[ \bar{\chi}_L \sigma_\mu \chi_R = \pm \bar{\chi}_R \bar{\sigma}_\mu \chi_L \] (A.11)
Here, the sign is minus for mutually anticommuting spinors, and plus for mutually commuting spinors.

For mutually commuting spinors (either R or L), the following identity holds:
\[ \chi(\bar{\chi}' \chi'') + \chi'(\bar{\chi}'' \chi) + \chi''(\bar{\chi} \chi') = 0 \] (A.12)
We have used this often in our calculations.

Appendix B

Classical BRST invariance

In this appendix we verify the classical BRST invariance of the classical action, defined by
\[ S_{cl} \equiv - \int d^4x \left[ \bar{\chi}_L \sigma_\mu \partial_\mu \chi_R + \partial_\mu \bar{\phi} \partial_\mu \phi + \bar{\phi} F(\phi) + F'(\bar{\phi}) \frac{1}{2} \bar{\chi}_R \chi_R + \frac{1}{2} \bar{\chi}_L \chi_L \right] \] (B.1)
where \( F(\phi) \) is an arbitrary function of \( \phi \), and \( \bar{F} \) its complex conjugate. \( F' \) is the derivative of \( F \) with respect to \( \phi \). We introduce a supersymmetry transformation as the following BRST transformation:
\[
\begin{align*}
\delta \phi &= \bar{\xi}_R \chi_R + \eta_\mu \partial_\mu \phi \\
\delta \bar{\phi} &= \bar{\xi}_L \chi_L + \eta_\mu \partial_\mu \bar{\phi} \\
\delta \chi_R &= \bar{\sigma}_\mu \xi_L \partial_\mu \phi - \bar{F}(\bar{\phi}) \xi_R + \eta_\mu \partial_\mu \chi_R \\
\delta \chi_L &= \sigma_\mu \xi_R \partial_\mu \bar{\phi} - F'(\phi) \xi_L + \eta_\mu \partial_\mu \chi_L
\end{align*}
\] (B.2)
where \( \xi_{R,L} \) are constant spinors, and \( \eta_\mu \) a constant vector. We take \( \xi_{R,L} \) commuting, and \( \eta_\mu \) anticommuting. It is straightforward to check the BRST invariance of \( S_{cl} \):
\[
\delta S_{cl} \equiv \int d^4x \left[ \frac{\delta S_{cl}}{\delta \phi} \delta \phi + \frac{\delta S_{cl}}{\delta \bar{\phi}} \delta \bar{\phi} + S_{cl} \frac{\delta}{\delta \chi_R} \delta \chi_R + S_{cl} \frac{\delta}{\delta \chi_L} \delta \chi_L \right] = 0
\] (B.3)
The BRST transformation (B.2) is not nilpotent. Defining the transformation of \( \xi_{R,L} \) and \( \eta_\mu \) by
\[
\delta \xi_{R,L} = 0, \quad \delta \eta_\mu = -\bar{\xi}_R \sigma_\mu \xi_L = -\bar{\xi}_L \sigma_\mu \xi_R
\] (B.4)
we obtain

\[
\begin{align*}
\delta^2 \phi &= 0 \\
\bar{\delta}^2 \bar{\phi} &= 0 \\
\delta^2 \chi_R &= \xi_R \cdot \bar{\xi}_L \frac{\partial}{\partial \delta \chi_L} S_{cl} \\
\delta^2 \chi_L &= \xi_L \cdot \bar{\xi}_R \frac{\partial}{\partial \delta \chi_R} S_{cl}
\end{align*}
\]  

(B.5)

Hence, to make \( \delta \) nilpotent, we must invoke the equations of motion:

\[
\frac{\delta}{\delta \chi_L} S_{cl} = 0, \quad \frac{\delta}{\delta \chi_R} S_{cl} = 0
\]

(B.6)

In other words, the algebra of the classical supersymmetry transformation is closed only on the mass shell.

The standard way to elevate the on-shell algebra to the off-shell algebra is to introduce auxiliary fields. Here, we take a different route, however. We introduce antifields: fermionic scalars \( \phi^* \), \( \bar{\phi}^* \) and bosonic spinors \( \chi^*_R, \chi^*_L \). We first couple them linearly to the supersymmetry transformation:

\[
S_{cl,1} \equiv S_{cl} + \int d^4x \left[ \phi^* \delta \phi + \bar{\phi}^* \bar{\delta} \bar{\phi} + \bar{\chi}^*_R \delta \chi_R + \bar{\chi}^*_L \delta \chi_L \right]
\]

(B.7)

so that

\[
\begin{align*}
\frac{\delta}{\delta \phi} S_{cl,1} &= \delta \phi, & \frac{\delta}{\delta \phi^*} S_{cl,1} &= \delta \bar{\phi} \\
\frac{\delta}{\delta \chi_R} S_{cl,1} &= \delta \chi_R, & \frac{\delta}{\delta \chi_L} S_{cl,1} &= \delta \chi_L
\end{align*}
\]

(B.8)

We now transform \( S_{cl,1} \):

\[
\delta S_{cl,1} = \int d^4x \left[ \frac{\delta S_{cl,1}}{\delta \phi} \frac{\delta}{\delta \phi} S_{cl,1} + \frac{\delta S_{cl,1}}{\delta \phi^*} \frac{\bar{\delta}}{\bar{\delta} \phi^*} S_{cl,1} \\
- \frac{\delta S_{cl,1}}{\delta \chi_R} \frac{\delta}{\delta \chi_R} S_{cl,1} - \frac{\delta S_{cl,1}}{\delta \chi_L} \frac{\delta}{\delta \chi_L} S_{cl,1} \right] - \bar{\xi}_L \sigma_\mu \xi_R \frac{\partial}{\partial \eta^\mu} S_{cl,1}
\]

(B.9)

This can be calculated further as

\[
\delta S_{cl,1} = \int \left[ -\phi^* \delta^2 \phi - \bar{\phi}^* \bar{\delta}^2 \bar{\phi} + \bar{\chi}^*_R \delta^2 \chi_R + \bar{\chi}^*_L \delta^2 \chi_L \right]
\]

\[
= \int \left[ \bar{\chi}^*_R \xi_R \cdot \bar{\xi}_L \frac{\partial}{\partial \chi_L} S_{cl} + \bar{\chi}^*_L \xi_L \cdot \bar{\xi}_R \frac{\partial}{\partial \chi_R} S_{cl} \right]
\]

\[
= \int \left[ \bar{\chi}^*_R \xi_R \cdot \bar{\xi}_L \frac{\partial}{\partial \chi_L} S_{cl,1} + \bar{\chi}^*_L \xi_L \cdot \bar{\xi}_R \frac{\partial}{\partial \chi_R} S_{cl,1} \right]
\]

(B.10)

To cancel this, we modify the action by adding a term quadratic in antifields:

\[
\bar{S}_{cl} \equiv S_{cl,1} - \int d^4x \bar{\chi}^*_R \xi_R \cdot \bar{\chi}^*_L \xi_L
\]

(B.11)
such that

\[
\begin{align*}
\frac{\delta}{\delta x_R} \tilde{S}_{cl} &= \frac{\delta}{\delta x_R} S_{cl,1} - \xi_R \cdot \bar{\chi}_L \chi_L \\
\frac{\delta}{\delta x_L} \tilde{S}_{cl} &= \frac{\delta}{\delta x_L} S_{cl,1} - \xi_L \cdot \bar{\chi}_R \chi_R 
\end{align*}
\]  
(B.12)

This satisfies the classical BRST invariance (or Zinn-Justin equation):

\[
\int d^4x \left[ \frac{\delta \tilde{S}_{cl}}{\delta \phi} \frac{\delta}{\delta \phi^*} \tilde{S}_{cl} + \frac{\delta \tilde{S}_{cl}}{\delta \phi} \frac{\delta}{\delta \phi^*} \tilde{S}_{cl} \right. \\
\left. - \tilde{S}_{cl} \frac{\delta}{\delta \chi_R} \frac{\delta}{\delta \chi_R} \tilde{S}_{cl} - \tilde{S}_{cl} \frac{\delta}{\delta \chi_L} \frac{\delta}{\delta \chi_L} \tilde{S}_{cl} \right] - \xi_L \sigma \mu \xi_R \frac{\delta}{\partial \eta_{\mu}} \tilde{S}_{cl} = 0
\]  
(B.13)

**Appendix C**

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**Antifield dependence of \( \bar{\Sigma} \)**

---

In this appendix, we wish to show

1. \( \bar{\Sigma}^{(l>0)} \) is independent of \( \eta_{\mu}, \phi^*, \bar{\phi}^* \).
2. The dependence of \( \bar{\Sigma}^{(l>0)} \) on \( \chi_R, \chi_L \) comes only through \( j_R, j_L \).

We expand \( \tilde{S}, \bar{\Sigma} \) in the number of loops:

\[
\tilde{S} = \sum_{l=0}^{\infty} \bar{S}^{(l)}, \quad \bar{\Sigma} = \sum_{l=1}^{\infty} \bar{\Sigma}^{(l)}
\]  
(C.1)

Only \( \bar{S}^{(0)} \) has dependence on \( \eta_{\mu}, \phi^*, \bar{\phi}^* \); \( \bar{S}^{(l)} \) (\( l > 0 \)) depends only on \( j_R, j_L \) and regular fields \( \phi, \bar{\phi}, \chi_R, \chi_L \). Therefore, we obtain

\[
\bar{\Sigma} = \int_p K (p/\Lambda) \left[ \frac{\delta}{\delta \phi^*(-p)} \bar{S}^{(0)} \cdot \frac{\delta}{\delta \phi(p)} \bar{S}^{(0)} + \frac{\delta}{\delta \phi^*(-p)} \bar{S}^{(0)} \cdot \frac{\delta}{\delta \phi(p)} \right. \\
\left. - \bar{S} \frac{\delta}{\delta \chi_R(p)} \frac{\delta}{\delta \chi_R(p)} \bar{S} - \text{Tr} \left( \frac{\delta}{\delta \chi_R(p)} \frac{\delta}{\delta \chi_R(p)} \right) \bar{S} \right] \\
- \xi_L \sigma \mu \xi_R \frac{\delta}{\partial \eta_{\mu}} \bar{S}^{(0)}
\]  
(C.2)

Substituting the loop expansions of \( \tilde{S} \) into the above, we obtain

\[
\bar{\Sigma}^{(l>0)} = \int \left[ K (p/\Lambda) \left[ \frac{\delta}{\delta \phi^*(-p)} \bar{S}^{(0)} \cdot \frac{\delta}{\delta \phi(p)} \bar{S}^{(0)} + \frac{\delta}{\delta \phi^*(-p)} \bar{S}^{(0)} \cdot \frac{\delta}{\delta \phi(p)} \right. \\
\left. - \bar{S} \frac{\delta}{\delta \chi_R(p)} \frac{\delta}{\delta \chi_R(p)} \bar{S} - \text{Tr} \left( \frac{\delta}{\delta \chi_R(p)} \frac{\delta}{\delta \chi_R(p)} \right) \bar{S} \right] \\
- \sum_{l'=0}^{l} \bar{S}^{(l-l')} \frac{\delta}{\delta \chi_R(p)} \frac{\delta}{\delta \chi_R(p)} \bar{S}^{(l')} - \text{Tr} \left( \frac{\delta}{\delta \chi_R(p)} \frac{\delta}{\delta \chi_R(p)} \right) \bar{S}^{(l-1)} \right] + \text{similar terms}
\]  
(C.3)
Now, recall (3.3):

\[
\tilde{S} = S_{\text{free}}[\phi, \tilde{\phi}; \chi_R, \chi_L] + \tilde{S}_{\text{int}}[\phi, \tilde{\phi}; j_R, j_L]
\]

\[
+ \int_p \frac{1}{K(p/A)} \left[ \phi^*(-p) \left\{ i\mu \eta_{\mu} \phi(p) + \tilde{\xi}_R \chi_R(p) \right\} \\
+ \tilde{\phi}^*(-p) \left\{ i\mu \eta_{\mu} \tilde{\phi}(p) + \tilde{\xi}_L \chi_L(p) \right\} \\
+ \tilde{\chi}_R^*(-p) \left\{ i\mu \eta_{\mu} \chi_R(p) + ip \cdot \tilde{\sigma}_L \phi(p) - m \tilde{\xi}_R \tilde{\phi}(p) \right\} \\
+ \tilde{\chi}_L^*(-p) \left\{ i\mu \eta_{\mu} \chi_L(p) + ip \cdot \sigma_R \tilde{\phi}(p) - m \xi_L \phi(p) \right\} \\
- j_R(-p) j_L(p) \right]
\]

(C.4)

Since \( \tilde{S}^{(l)} \) for \( l > 0 \) is the \( l \)-loop part of \( \tilde{S}_{\text{int}} \), we obtain

\[
\frac{\delta}{\delta \tilde{\chi}_R} \tilde{S}^{(l)} = \xi_R \frac{\delta \tilde{S}^{(l)}}{\delta j_R}, \quad \frac{\delta}{\delta \tilde{\chi}_L} \tilde{S}^{(l)} = \xi_L \frac{\delta \tilde{S}^{(l)}}{\delta j_L}
\]

(C.5)

Therefore, we get

\[
\tilde{S}^{(l)} = \eta_{\mu} \int_p \left[ \phi^*(p) \frac{\delta \tilde{S}^{(l)}}{\delta \phi(p)} + \tilde{\phi}^*(p) \frac{\delta \tilde{S}^{(l)}}{\delta \tilde{\phi}(p)} \\
+ \tilde{\chi}_R^*(p) \frac{\delta \tilde{S}^{(l)}}{\delta \chi_R(p)} \chi_R(p) + \tilde{\chi}_L^*(p) \frac{\delta \tilde{S}^{(l)}}{\delta \chi_L(p)} \chi_L(p) \\
- \tilde{\chi}_R^*(-p) \frac{\delta \tilde{S}^{(l)}}{\delta \tilde{\chi}_R(-p)} - \tilde{\chi}_L^*(-p) \frac{\delta \tilde{S}^{(l)}}{\delta \tilde{\chi}_L(-p)} \right] \\
- \int_p \left[ \phi^*(-p) \tilde{\xi}_R \chi_R(-p) \frac{\delta \tilde{S}^{(l)}}{\delta j_R(-p)} + \tilde{\phi}^*(-p) \tilde{\xi}_L \chi_L(-p) \frac{\delta \tilde{S}^{(l)}}{\delta j_L(-p)} \right] \\
+ \int_p K(p/A) \left[ \frac{1}{K(p/A)} \left\{ \tilde{\xi}_R \chi_R(p) \frac{\delta \tilde{S}^{(l)}}{\delta \phi(p)} + \tilde{\xi}_L \chi_L(p) \frac{\delta \tilde{S}^{(l)}}{\delta \tilde{\phi}(p)} \right\} \\
- \sum_{l'=1}^{l-1} \tilde{S}^{(l-l')} \frac{\delta \tilde{S}^{(l')}}{\delta \chi_R(p)} \frac{\delta \tilde{S}^{(l')}}{\delta j_R(-p)} \right]
\]

\[
- \frac{1}{K(p/A)} \tilde{S}^{(l)} \frac{\delta}{\delta \chi_R(p)} \left( ip \cdot \tilde{\sigma}_L \phi(p) - m \tilde{\xi}_R \tilde{\phi}(p) + \xi_R \frac{\delta \tilde{S}^{(0)}}{\delta j_R(-p)} \right) \\
- \left( S_{\text{free}} + \tilde{S}^{(0)} \right) \frac{\delta}{\delta \chi_R(p)} \frac{\delta \tilde{S}^{(l)}}{\delta j_R(-p)} \frac{\delta \tilde{S}^{(l)}}{\delta j_R(-p)} \frac{\delta \tilde{S}^{(l)}}{\delta \chi_R(p)} \xi_R \\
- \sum_{l'=1}^{l-1} \tilde{S}^{(l-l')} \frac{\delta \tilde{S}^{(l')}}{\delta \chi_L(p)} \frac{\delta \tilde{S}^{(l')}}{\delta j_L(-p)} \\
- \frac{1}{K(p/A)} \tilde{S}^{(l)} \frac{\delta}{\delta \chi_L(p)} \left( ip \cdot \sigma_R \tilde{\phi}(p) - m \xi_L \phi(p) + \xi_L \frac{\delta \tilde{S}^{(0)}}{\delta j_L(-p)} \right)
\]
Wilson action for the Wess-Zumino model

\[-\left( S_{\text{free}} + \tilde{S}_{\text{int}}^{(0)} \right) \left( \frac{\delta}{\delta \chi_L(p)} \frac{\delta \tilde{S}^{(l)}}{\delta j_L(-p)} - \frac{\delta \tilde{S}^{(l)}}{\delta j_L(p)} \frac{\delta}{\delta \chi_L(p)} \right) \]  

(C.6)

The first part that is proportional to $\eta_\mu$ vanishes due to the translation invariance of $\tilde{S}^{(l)}$. Due to the Bose statistics of $\xi_R$, $\xi_L$, we obtain

\[\bar{\xi}_R \xi_R = \bar{\xi}_L \xi_L = 0 \]  

(C.7)

and this makes the second part zero. What is left is built out of $S_{\text{free}} + \tilde{S}_{\text{int}}^{(0)}$ and $\tilde{S}^{(1)}, \cdots, \tilde{S}^{(l)}$. Hence, $\tilde{\Sigma}^{(l)}$ satisfies the two properties stated at the beginning of this appendix.

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