DECOMPOSITIONS OF EHRHART $h^*$-POLYNOMIALS FOR RATIONAL POLYTOPES

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ABSTRACT. The Ehrhart quasipolynomial of a rational polytope $P$ encodes the number of integer lattice points in dilates of $P$, and the $h^*$-polynomial of $P$ is the numerator of the accompanying generating function. We provide two decomposition formulas for the $h^*$-polynomial of a rational polytope. The first decomposition generalizes a theorem of Betke and McMullen for lattice polytopes. We use our rational Betke–McMullen formula to provide a novel proof of Stanley’s Monotonicity Theorem for the $h^*$-polynomial of a rational polytope. The second decomposition generalizes a result of Stapledon, which we use to provide rational extensions of the Stanley and Hibi inequalities satisfied by the coefficients of the $h^*$-polynomial for lattice polytopes. Lastly, we apply our results to rational polytopes containing the origin whose duals are lattice polytopes.

1. INTRODUCTION

For a $d$-dimensional rational polytope $P \subset \mathbb{R}^d$ (i.e., the convex hull of finitely many points in $\mathbb{Q}^d$) and a positive integer $t$, let $L_P(t)$ denote the number of integer lattice points in $tP$. Ehrhart’s theorem [9] tells us that $L_P(t)$ is of the form $\text{vol}(P)t^d + k_{d-1}(t)t^{d-1} + \cdots + k_1(t)t + k_0(t)$, where $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ are periodic functions in $t$. We call $L_P(t)$ the Ehrhart quasipolynomial of $P$, and Ehrhart proved that each period of $k_0(t), k_1(t), \ldots, k_{d-1}(t)$ divides the denominator $q$ of $P$, which is the least common multiple of all its vertex coordinate denominators. The Ehrhart series is the rational generating function

$$Ehr(P;z) = \sum_{t \geq 0} L(P,t) z^t = \frac{h^*(P;z)}{(1-z^q)^{d+1}},$$

where $h^*(P;z)$ is a polynomial of degree less than $q(d+1)$, the $h^*$-polynomial of $P$.

Our first main contributions are generalizations of two well-known decomposition formulas of the $h^*$-polynomial for lattice polytopes due to Betke–McMullen [4] and Stapledon [33]. (All undefined terms are specified in the sections below.)

**Theorem 3.2.** For a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$,

$$Ehr(P;z) = \frac{\sum_{\Omega \in T} B(\Omega;z) h(\Omega;z^q)}{(1-z^q)^{d+1}}.$$

**Theorem 4.4.** Consider a rational $d$-polytope $P$ that contains an interior point $\frac{a}{\ell}$, where $a \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Fix a boundary triangulation $T$ of $P$ with denominator $q$. Then

$$h^*(P;z) = \frac{1-z^q}{1-z^d} \sum_{\Omega \in T} (B(\Omega;z) + B(\Omega';z)) h(\Omega;z^q).$$

Our second main result is a generalization of inequalities provided by Hibi [14] and Stanley [28] that are satisfied by the coefficients of the $h^*$-polynomial for lattice polytopes.

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1Note that the $h^*$-polynomial depends not only on $q$ (though that is implicitly determined by $P$), but also on our choice of representing the rational function $Ehr(P;z)$, which in our form will not be in lowest terms.*
Theorem 4.8. Let $P$ be a rational $d$-polytope with denominator $q$ and let $s := \deg h^*(P; z)$. The $h^*$-vector $(h^*_0, \ldots, h^*_q)_{q(d+1)-1}$ of $P$ satisfies the following inequalities:

(1) $h^*_0 + \cdots + h^*_{i+1} \geq h^*_{q(d+1)-1} + \cdots + h^*_{q(d+1)-1-i}$, $i = 0, \ldots, \left\lceil \frac{q(d+1)-1}{2} \right\rceil - 1$,

(2) $h^*_i + \cdots + h^*_{q-1} \geq h^*_0 + \cdots + h^*_i$, $i = 0, \ldots, q(d+1)-1$.

Inequality (1) is a generalization of a theorem by Hibi [14] for lattice polytopes, and (2) generalizes an inequality given by Stanley [28] for lattice polytopes, namely the case when $q = 1$. Both inequalities follow from the $a/b$-decomposition of the $h^*$-polynomial for rational polytopes given in Theorem 4.7 in Section 4, which in turn generalizes results (and uses rational analogues of techniques) by Stapledon [33]. Stapledon’s $a/b$-decomposition has been used by different authors to study connections to unimodality, dilated polytopes, open polytopes, order polytopes, and connections to chromatic polynomials [2, 19, 20, 23].

This paper is structured as follows. In Section 2 we provide notation and background. In Section 3 we prove Theorem 3.2 and use this to give a novel proof of Stanley’s Monotonicity Theorem. In Section 4 we prove Theorems 4.4 and 4.8. We conclude in Section 5 with some applications.

2. Set-Up and Notation

A pointed simplicial cone is a set of the form

$$K(W) = \left\{ \sum_{i=1}^{n} \lambda_i w_i : \lambda_i \geq 0 \right\},$$

where $W := \{w_1, \ldots, w_n\}$ is a set of $n$ linearly independent vectors in $\mathbb{R}^d$. If we can choose $w_i \in \mathbb{Z}^d$ then $K(W)$ is a rational cone and we assume this throughout this paper. Define the open parallelepiped associated with $K(W)$ as

$$\text{Box}(W) := \left\{ \sum_{i=1}^{n} \lambda_i w_i : 0 < \lambda_i < 1 \right\}.$$

Observe that we have the natural involution $1 : \text{Box}(W) \cap \mathbb{Z}^d \to \text{Box}(W) \cap \mathbb{Z}^d$ given by

$$1 \left( \sum_i \lambda_i w_i \right) := \sum_i (1 - \lambda_i) w_i.$$

We set $\text{Box}(\{0\}) := \{0\}$.

Let $u : \mathbb{R}^d \to \mathbb{R}$ denote the projection onto the last coordinate. We then define the box polynomial as

$$B(W; z) := \sum_{v \in \text{Box}(W) \cap \mathbb{Z}^d} z^{\mu(v)}.$$

If $\text{Box}(W) \cap \mathbb{Z}^d = \emptyset$, then we set $B(W; z) = 0$. We also define $B(\emptyset; z) = 1$.

Example 2.1. Let $W = \{(1,3), (2,3)\}$. Then

$$\text{Box}(W) = \{\lambda_1 (1,3) + \lambda_2 (2,3) : 0 < \lambda_1, \lambda_2 < 1\}.$$

Thus $\text{Box}(W) \cap \mathbb{Z}^2 = \{(1,2), (2,4)\}$ and its associated box polynomial is

$$B(W; z) = z^2 + z^4.$$

Lemma 2.2. $B(W; z) = \sum_{i} z^{\mu(w_i)} B(W; \frac{1}{z})$. 
We say a point is at height $k$ if we lift the vertices into $\mathbb{R}^{d+1}$ by appending a $1$ as the last coordinate. Then

$$z\sum_{v \in \text{Box}(W) \cap \mathbb{Z}^d} z^{u(v)} = \sum_{v \in \text{Box}(W) \cap \mathbb{Z}^d} z^{u(v)} = B(W; z).$$

Next, we define the fundamental parallelepiped $\Pi(W)$ to be a half-open variant of Box $(W)$, namely,

$$\Pi(W) := \left\{ \sum_{i=1}^{n} \lambda_i w_i : 0 \leq \lambda_i < 1 \right\}.$$

We also want to cone over a polytope $P$. If $P \subset \mathbb{R}^d$ is a rational polytope with vertices $v_1, \ldots, v_n \in \mathbb{Q}^d$, we lift the vertices into $\mathbb{R}^{d+1}$ by appending a $1$ as the last coordinate. Then

$$\text{cone}(P) = \left\{ \sum_{i=1}^{n} \lambda_i (v_i, 1) : \lambda_i \geq 0 \right\} \subset \mathbb{R}^{d+1}.$$

We say a point is at height $k$ in the cone if the point lies on $\text{cone}(P) \cap \{ x : x_{d+1} = k \}$. Note that $qP$ is embedded in $\text{cone}(P)$ as $\text{cone}(P) \cap \{ x : x_{d+1} = q \}$.

A triangulation $T$ of a $d$-polytope $P$ is a subdivision of $P$ into simplices (of all dimensions). If all the vertices of $T$ are rational points, define the denominator of $T$ to be the least common multiple of all the vertex coordinate denominators of the faces of $T$. For each $\Delta \in T$, we define the $h$-polynomial of $\Delta$ with respect to $T$ as

$$h_T(\Delta; z) := (1-z)^{d-\dim(\Delta)} \sum_{\Delta \subseteq \Phi \in T} \left( \frac{z}{1-z} \right)^{\dim(\Phi) - \dim(\Delta)},$$

where the sum is over all simplices $\Phi \in T$ containing $\Delta$. When $T$ is clear from context, we omit the subscript. Note that when $T$ is a boundary triangulation of $P$, the definition of the $h$-vector will be adjusted according to dimension, that is, $d$ should be replaced by $d-1$ in (7).

For a $d$-simplex $\Delta$ with denominator $q$, let $W$ be the set of ray generators of $\text{cone}(\Delta)$ at height $p$, which are all integral. We then define the $h^*$-polynomial of $\Delta$ as the generating function of the last coordinate of integer points in $\Pi(W) := \Pi(\Delta)$, that is,

$$h^*(\Delta; z) = \sum_{v \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} z^{\mu(v)}.$$

With this consideration, the Ehrhart series of $\Delta$ can be expressed as

$$\text{Ehr}(\Delta; z) = \frac{h^*(\Delta; z)}{(1-z^p)^{d+1}}.$$  

We use a modified convention when $\Delta$ is a rational $m$-simplex of a triangulation $T$, where $T$ has denominator $q$. In this case, it is possible that the denominator of $\Delta$ as an individual simplex might be different from $q$, but for coherence among all simplices in $T$ we use $q$ to select the height of the ray generators in $\Delta$. Namely, we let $W = \{ (r_1, q), \ldots, (r_{m+1}, q) \}$, where the $(r_i, q)$ are integral ray generators of $\text{cone}(\Delta)$ at height $q$. The corresponding $h^*$-polynomial of $\Delta$ is a function of $q$ and the Ehrhart series of $\Delta$ can be expressed as

$$\text{Ehr}(\Delta; z) = \frac{h^*(\Delta; z)}{(1-z^q)^{m+1}}.$$  

We may think of $h^*(\Delta; z)$ as computed via $\sum_{v \in \Pi(W) \cap \mathbb{Z}^{d+1}} z^{\mu(v)}$. 

Proof. Using the involution $t$,

$$z\sum_{v \in \text{Box}(W) \cap \mathbb{Z}^d} z^{u(v)} = \sum_{v \in \text{Box}(W) \cap \mathbb{Z}^d} z^{u(v)} = \sum_{v \in \text{Box}(W) \cap \mathbb{Z}^d} z^{\mu(v)} = B(W; z).$$
3. RATIONAL BETKE–MCMULLEN DECOMPOSITION

3.1. Decomposition à la Betke–Mcmullen. Let $P$ be a rational $d$-polytope and $T$ a triangulation of $P$ with denominator $q$. For an $m$-simplex $\Delta \in T$, let $W = \{ (r_1, q), \ldots, (r_m, q) \}$, where the $(r_i, q)$ are the integral ray generators of cone $(\Delta)$ at height $q$ as above. Further, set $B(W; z) = B(\Delta; z)$ and similarly $\text{Box}(W) = \text{Box}(\Delta)$. We emphasize that the $h^*$-polynomial, fundamental parallelepiped, and box polynomial of $\Delta$ depend on the denominator $q$ of $T$.

A point $v \in \text{cone}(\Delta)$ can be uniquely expressed as $v = \sum_{i=1}^{m+1} \lambda_i (r_i, q)$ for $\lambda_i \geq 0$. Define

$$I(v) := \{ i \in [m+1] : \lambda_i \in \mathbb{Z} \} \quad \text{and} \quad \overline{I}(v) := [m+1] \setminus I,$$

where $[m+1] := \{1, \ldots, m+1\}$.

**Lemma 3.1.** Fix a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$ and let $\Delta \in T$. Then $h^*(\Delta; z) = \sum_{\Omega \subseteq \Delta} B(\Omega; z)$.

**Proof.** First we show that $\Pi(\Delta) = \bigcup_{\Omega \subseteq \Delta} \text{Box}(\Omega)$. The reverse containment follows from the fact that any element in $\text{Box}(\Omega)$ is a linear combination of the ray generators of cone $(\Omega)$.

For the forward containment, if $v \in \Pi(\Delta)$, then

$$v = \sum_{i=1}^{m+1} \lambda_i (r_i, q) = \sum_{i \in \overline{I}(v)} \lambda_i (r_i, q) \in \text{Box}(\Omega),$$

for $\Omega := \text{conv} \left\{ \frac{r_i}{q} : i \in \overline{I}(v) \right\} \subseteq \Delta$. Note that $v$ will always lie in a unique $\text{Box}(\Omega)$ because every $\Omega$ corresponds to a different subset of $[m+1]$, which also tells us that the union we desire is disjoint.

Thus $\Pi(\Delta) = \bigcup_{\Omega \subseteq \Delta} \text{Box}(\Omega)$, and so

$$h^*(\Delta; z) = \sum_{v \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}} z^{h(v)} = \sum_{\Omega \subseteq \Delta} \sum_{v \in \text{Box}(\Omega) \cap \mathbb{Z}^{d+1}} z^{h(v)} = \sum_{\Omega \subseteq \Delta} B(\Omega; z).$$

**Theorem 3.2.** For a triangulation $T$ with denominator $q$ of a rational $d$-polytope $P$,

$$\text{Ehr}(P; z) = \sum_{\Delta \in T} \frac{B(\Omega; z) h(\Omega; z^q)}{(1 - z^q)^{d+1}}.$$

**Proof.** We write $P$ as the disjoint union of all open nonempty simplices in $T$ and use Ehrhart–Macdonald reciprocity [9, 24]:

$$\text{Ehr}(P; z) = 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} \text{Ehr}(\Delta^*; z) = 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} (-1)^{\dim(\Delta)+1} \text{Ehr} \left( \frac{\Delta}{z} \right)$$

$$= 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} (-1)^{\dim(\Delta)+1} \frac{h^* \left( \frac{\Delta}{z} \right)}{(1 - \frac{1}{z})^{\dim(\Delta)+1}} = 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} \frac{(z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)} h^* \left( \frac{\Delta}{z} \right)}{(1 - z^q)^{d+1}}.$$

Note that the Ehrhart series of each $\Delta$ is being written as a rational function with denominator $(1 - z^q)^{d+1}$. Using Lemma 3.1,

$$\text{Ehr}(P; z) = 1 + \sum_{\Delta \in T \setminus \{\emptyset\}} \frac{(z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)} \sum_{\Omega \subseteq \Delta} B(\Omega; z)}{(1 - z^q)^{d+1}}$$

$$= \frac{\sum_{\Delta \in T} \left[ (z^q)^{\dim(\Delta)+1} (1 - z^q)^{d - \dim(\Delta)} \sum_{\Omega \subseteq \Delta} B(\Omega; z) \right]}{(1 - z^q)^{d+1}}.$$
By Lemma 2.2,
\[ h^*(P;z) = \sum_{\Delta \in T} \left[ (z^d)^{\dim(\Delta)} \cdot 1 \right] \sum_{\Omega \subseteq \Delta} B(\Omega; z) \frac{1}{1 - z^d} \]
\[ = \sum_{\Delta \in T} \left[ (z^d)^{\dim(\Delta)} \cdot 1 \right] \sum_{\Omega \subseteq \Delta} B(\Omega; z) \frac{1}{1 - z^d} \]
\[ = \sum_{\Omega \subseteq \Delta} B(\Omega; z) \frac{1}{1 - z^d} \sum_{\Delta \in T} \sum_{\Omega \subseteq \Delta} (z^d)^{\dim(\Delta) - \dim(\Omega)} B(\Omega; z) \]
\[ = \sum_{\Omega \subseteq T} B(\Omega; z) (1 - z^d)^{\dim(\Omega)} \sum_{\Omega \subseteq \Delta} \left( \frac{z^d}{1 - z^d} \right)^{\dim(\Delta) - \dim(\Omega)} \]

Using the definition of the $h$-polynomial, the theorem follows. \qed

3.2. **Rational $h^*$-Monotonicity.** We now show how the following theorem follows from our rational Betke–McMullen formula.

**Theorem 3.3** (Stanley Monotonicity [30]). Suppose that $P \subseteq Q$ are rational polytopes with $qP$ and $qQ$ integral (for minimal possible $q \in \mathbb{Z}_{>0}$). Define the $h^*$-polynomials via

\[ \text{Ehr}(P;z) = \frac{h^*(P;z)}{(1 - z^d)^{\dim(P)+1}} \quad \text{and} \quad \text{Ehr}(Q;z) = \frac{h^*(Q;z)}{(1 - z^d)^{\dim(Q)+1}}. \]

Then $h^*_i(P;z) \leq h^*_i(Q;z)$ coefficient-wise.

In addition to Stanley’s original proof, Beck and Sottile [3] provide a proof of Theorem 3.3 using irrational decompositions of rational polyhedra. In the case of lattice polytopes, Jochemko and Sanyal [21] prove Theorem 3.3 using combinatorial positivity of translation-invariant valuations and Stapledon [32] gives a geometric interpretation of Theorem 3.3 by considering the $h^*$-polynomials of lattice polytopes in terms of orbifold Chow rings. The following lemma assumes familiarity with Cohen–Macaulay complexes and related theory; see [31] for definitions and further reading.

**Lemma 3.4.** Suppose $P$ is a polytope and $T$ a triangulation of $P$. Let $P \subseteq Q$ be a polytope and $T'$ a triangulation of $Q$ such that $T'$ restricted to $P$ is $T$. Further, if $\dim(P) < \dim(Q)$, assume that there exists a set of affinely independent vertices $v_1, \ldots, v_n$ of $Q$ outside the affine span of $P$ such that (1) the join $T \ast \text{conv} \{v_1, \ldots, v_n\}$ is a subcomplex of $T'$ and (2) $\dim(P \ast \text{conv} \{v_1, \ldots, v_n\}) = \dim(Q)$. For every face $\Omega \in T$, the coefficient-wise inequality $h_T(\Omega;z) \leq h_{T'}(\Omega;z)$ holds.

**Proof.** Suppose first that $\dim(P) = \dim(Q)$. Let $T$ be a triangulation of $P$ and $T'$ a triangulation of $Q$ such that $T'$ restricted to $P$ is $T$. Note that $T$ and $T'$ are geometric simplicial complexes covering $P$ and $Q$, respectively. Let $\Omega \in T$. Then $\text{link}_T(\Omega)$ and $\text{link}_{T'}(\Omega)$ are either balls or spheres, hence Cohen–Macaulay. Now, consider $\mathcal{R} := \text{link}_{T'}(\Omega) - \text{link}_T(\Omega)$, which is a relative simplicial complex. By [31, Corollary 7.3(iv)] $\mathcal{R}$ is also Cohen–Macaulay. From [31, Proposition 7.1] it follows that

\[ h_{\mathcal{R}}(\emptyset;z) = h_T(\Omega;z) - h_{\Omega}(\emptyset;z) \quad \text{and} \quad h_{\mathcal{R}}(\emptyset;z), h_T(\Omega;z), h_{T'}(\Omega;z) \geq 0. \]

Rearranging, we obtain that $h_{T'}(\Omega;z) = h_{\mathcal{R}}(\emptyset;z) + h_T(\Omega;z)$, which implies that $h_T(\Omega;z) \leq h_{T'}(\Omega;z)$ Hence, for each face in $T$, the result follows.

Now, consider the case when $\dim(P) < \dim(Q)$. Again, let $T$ be a triangulation of $P$ and $T'$ a triangulation of $Q$ such that $T'$ restricted to $P$ is $T$, where we further assume that there exists a set of affinely independent
vertices \( v_1, \ldots, v_n \) of \( Q \) outside the affine span of \( P \) such that (1) the join \( T + \text{conv} \{ v_1, \ldots, v_n \} \) is a subcomplex of \( T' \) and (2) \( \dim(P + \text{conv} \{ v_1, \ldots, v_n \}) = \dim(Q) \). Note that the affine independence of the \( v_i \)'s implies that
\[
\dim(\text{conv} \{ P \cup v_1 \cup \cdots \cup v_k \}) = \dim(\text{conv} \{ P \cup v_1 \cup \cdots \cup v_{k-1} \}) + 1.
\]
Let \( T_k \) denote the join of \( T \) with the simplex \( \text{conv} \{ v_1, \ldots, v_k \} \). Let \( \Omega \in T_k \). Since \( \Omega \subseteq \partial T_{k+1} \) and \( \text{link}_{T_k}(\Omega) \) and \( \text{link}_{T_{k+1}}(\Omega) \) are both balls, \( \mathcal{R} := \text{link}_{T_{k+1}}(\Omega) - \text{link}_{T_k}(\Omega) \) is Cohen–Macaulay by [31, Proposition 7.3(iii)]. Thus, by a similar argument as given in the paragraph above,
\[
h_{T_k}(\Omega; z) \leq h_{T_{k+1}}(\Omega; z).
\]
Combining this with the fact that \( \dim(P + \text{conv} \{ v_1, \ldots, v_n \}) = \dim(Q) \), it follows by induction (for the first inequality) and our previous case (for the second inequality) that for \( \Omega \in T \)
\[
h_{T}(\Omega; z) \leq h_{T_{k}}(\Omega; z) \leq h_{T'}(\Omega; z).
\]

**Proof of Theorem 3.3.** Let \( P \) be a polytope contained in \( Q \). Let \( T \) be a triangulation of \( P \) and let \( T' \) be a triangulation of \( Q \) such that \( T' \) restricted to \( P \) is \( T \), where if \( \dim(P) < \dim(Q) \) the triangulation \( T' \) satisfies the conditions given in Lemma 3.4. (Note that such a triangulation \( T' \) can always be obtained from \( T \), e.g., by extending \( T \) using a placing triangulation.) By Theorem 3.2, \( h^*(P; z) = \sum_{\Omega \in T} B(\Omega; z) h_{T}(\Omega; z^\theta) \). Since \( P \) is contained in \( Q \),
\[
h^*(Q; z) = \sum_{\Omega \in T} B(\Omega; z) h_{T}(\Omega; z^\theta) + \sum_{\Omega \in T \setminus T} B(\Omega; z) h_{T}(\Omega; z^\theta).
\]
By Lemma 3.4, the coefficients of \( \sum_{\Omega \in T} B(\Omega; z) h_{T}(\Omega; z^\theta) \) dominate the coefficients of \( \sum_{\Omega \in T \setminus T} B(\Omega; z) h_{T}(\Omega; z^\theta) \). This further implies that the coefficients of \( h^*(Q; z) \) dominate the coefficients of \( h^*(P; z) \) since
\[
\sum_{\Omega \in T} B(\Omega; z) h_{T}(\Omega; z^\theta) \leq \sum_{\Omega \in T} B(\Omega; z) h_{T}(\Omega; z^\theta)
\]
\[
\leq \sum_{\Omega \in T} B(\Omega; z) h_{T}(\Omega; z^\theta) + \sum_{\Omega \in T \setminus T} B(\Omega; z) h_{T}(\Omega; z^\theta).
\]

4. \( h^* \)-**DECOMPOSITIONS FROM BOUNDARY TRIANGULATIONS**

4.1. **Set-up.** Throughout this section we will use the following set-up. Fix a boundary triangulation \( T \) with denominator \( q \) of a rational \( d \)-polytope \( P \). Take \( \ell \in \mathbb{Z}_{\geq 0} \), such that \( \ell P \) contains a lattice point \( a \) in its interior. Thus \( (a, \ell) \in \text{cone}(P)^{\circ} \cap \mathbb{Z}^{d+1} \) is a lattice point in the interior of the cone of \( P \) at height \( \ell \), and \( \text{cone}(a, \ell) \) is the ray through the point \( a, \ell \). We cone over each \( \Delta \in T \) and define \( W = \{(r_1, q), \ldots, (r_{m+1}, q)\} \) where the \( (r_i, q) \) are integral ray generators of cone \( \Delta \) at height \( q \). As before, we have the associated box polynomial \( B(W; z) = B(\Delta; z) \). Now, let \( W' = W \cup \{(a, \ell)\} \) be the set of generators from \( W \) together with \( (a, \ell) \) and we set \( \text{cone}(\Delta') \) to be the cone generated by \( W' \), with associated box polynomial \( B(W'; z) = B(\Delta'; z) \).

**Corollary 4.1.** For each face \( \Delta \) of \( T \),
\[
B(\Delta; z) = z^{q(\dim(\Delta)+1)} B\left(\Delta; \frac{1}{z}\right)
\]
and
\[
B(\Delta'; z) = z^{q(\dim(\Delta)+1)+\ell} B\left(\Delta'; \frac{1}{z}\right).
\]

**Proof.** The height of \( \sum_i (r_i, q) \) is \( q \) times the number of summands, which gives us \( q(\dim(\Delta)+1) \). The first equations now follow from the involution \( \iota \) and Lemma 2.2; note that we will have to use \( W \) in the first case and \( W' \) in the second.

Observe that when \( \Delta = \emptyset \) is the empty face, \( B(\emptyset; z) = 1 \), but \( B(\emptyset'; z) = B((a, \ell); z) \). This differs from the scenario in [33] where Stapledon’s set-up determined that \( B(\emptyset'; z) = 0 \).

For a real number \( x \), define \( \lfloor x \rfloor \) to be the greatest integer less than or equal to \( x \). Additionally, define the fractional part of \( x \) to be \( \{x\} = x - \lfloor x \rfloor \).
4.2. **Boundary Triangulations.** For each $v \in \text{cone}(P)$ we associate two faces $\Delta(v)$ and $\Omega(v)$ of $T$, as follows. The face $\Delta(v)$ is chosen to be the minimal face of $T$ such that $v \in \text{cone}(\Delta(v))$, and we define

$$\Omega(v) := \text{conv} \left\{ \frac{r_i}{q} : i \in I(v) \right\} \subseteq \Delta(v),$$

where $I(v)$ is defined as in (8) and the $(r_i, q)$ are ray generators of cone $(\Delta(v))$. In an effort to make our statements and proofs less notation heavy, for the rest of this section we write $\Delta(v) = \Delta$ and $\Omega(v) = \Omega$ with the understanding that both depend on $v$. Furthermore, for $v = \sum_{i=1}^{m+1} \lambda_i (r_i, q) + \lambda (a, \ell)$ where $\lambda, \lambda_i \geq 0$, define

$$\{v\} := \sum_{i \in I(v)} \{\lambda_i\} (r_i, q) + \{\lambda\} (a, \ell).$$

**Lemma 4.2.** Given $v \in \text{cone}(P)$, construct $\Delta = \Delta(v)$ as described above, with cone $(\Delta)$ generated by $(r_1, q), \ldots, (r_{m+1}, q)$. Then $v$ can be written uniquely as

$$\{v\} + \sum_{i \in I(v)} (r_i, q) + \sum_{i=1}^{m+1} \mu_i (r_i, q) + \mu (a, \ell),$$

where $\mu, \mu_i \in \mathbb{Z}_{\geq 0}$.

Below we will note the dependence of the unique coefficients $\mu_i$ and $\mu$ on $v$ by writing them as $\mu_i(v)$ and $\mu(v)$.

**Proof.** Since $v$ is in cone $(\Delta')$, it can be written as a linear combination of the generators of cone $(\Delta)$ and $(a, \ell)$. We further express $v$ as a sum of its integer and fractional parts.

$$v = \sum_{i=1}^{m+1} \lambda_i (r_i, q) + \bar{\lambda} (a, \ell),$$

where $\lambda_i > 0$ and $\bar{\lambda} \geq 0$

$$\begin{align*}
\text{for } \lambda_i > 0 \text{ and } \bar{\lambda} \geq 0 \\
= \sum_{i \in I(v)} \{\lambda_i\} (r_i, q) + \{\bar{\lambda}\} (a, \ell) + \sum_{i=1}^{m+1} \lambda_i (r_i, q) + \{\lambda_i\} (a, \ell) \\
= \{v\} + \sum_{i=1}^{m+1} \lambda_i (r_i, q) + \{\lambda_i\} (a, \ell).
\end{align*}$$

Note that each $\lambda_i > 0$ because of the minimality of $\Delta$. Recall that $\Omega = \text{conv} \left\{ \frac{r_i}{q} : i \in I(v) \right\} \subseteq \Delta$. Thus

- if $\lambda \notin \mathbb{Z}$, then $\{v\} \in \text{Box}(\Omega')$,  
- if $\lambda \in \mathbb{Z}$, then $\{v\} \in \text{Box}(\Omega)$.

Further observe that when $\lambda$ is an integer, $\{v\}$ is an element on the boundary of cone $(P)$.

If $i \in I(v)$, then $\lambda_i \in \mathbb{Z}$ and $[\lambda_i] = \lambda_i \geq 1$ for $i \in I(v)$. This allows us to represent $v$ in the form

$$v = \{v\} + \sum_{i \in I(v)} (r_i, q) + \sum_{i=1}^{m+1} \mu_i (r_i, q) + \mu (a, \ell),$$

where $\mu, \mu_i \in \mathbb{Z}_{\geq 0}$. \hfill \Box

**Corollary 4.3.** Continuing the notation above,

$$u(v) = u(\{v\}) + q(\text{dim}(\Delta(v)) - \text{dim}(\Omega(v))) + \sum_{i=1}^{m+1} q \mu_i(v) + \mu(v) \ell.$$

**Proof.** This follows from considering the height contribution of each part in (9). \hfill \Box
The following theorem provides a decomposition of the $h^*$-polynomial of a rational polytope in terms of box and $h$-polynomials. It is important to note again that the $h^*$-polynomial depends on the denominator of the boundary triangulation.

**Theorem 4.4.** Consider a rational $d$-polytope $P$ that contains an interior point $\frac{a}{\ell}$, where $a \in \mathbb{Z}^d$ and $\ell \in \mathbb{Z}_{>0}$. Fix a boundary triangulation $T$ of $P$ with denominator $q$. Then

$$h^*(P; z) = \frac{1 - z^q}{1 - z^d} \sum_{v \in T} (B(\Omega; z) + B(\Omega'; z)) h(\Omega; z^q).$$

**Proof.** By Corollary 4.3,

$$h^*(P; z) = \sum_{v \in \text{cone}(P) \cap \mathbb{Z}^{d+1}} z^{|v|}.$$  

$$= \sum_{v \in \text{cone}(P) \cap \mathbb{Z}^{d+1}} z^{|v|} = \sum_{v \in \text{cone}(P) \cap \mathbb{Z}^{d+1}} \sum_{u \in \text{Box}(P) \cap \mathbb{Z}^{d+1}} z^{q(q)}$$  

$$= \sum_{\Delta \in T} \sum_{\Omega \subseteq \Delta} \frac{(B(\Omega; z) + B(\Omega'; z)) z^{q(q)}}{(1 - z^q)^{q}q^{q}q^{q}} = \frac{1}{1 - z^q} \sum_{\Delta \in T} \sum_{\Omega \subseteq \Delta} (B(\Omega; z) + B(\Omega'; z)) h(\Omega; z^q).$$

**Example 4.5.** Following the setup in Section 4.1, consider the line segment $P = \left[\frac{1}{3}, \frac{2}{3}\right]$ and so our boundary triangulation $T$ has denominator 3. In the cone over $P$, set $\{a, \ell\} = (2, 4)$. The simplices in $T$ are the empty face $\emptyset$ and the two vertices $\Delta_1 = 1$ and $\Delta_2 = 2$. The cones over the vertices have integral ray generators $W_1 = \{(1, 3)\}$ and $W_2 = \{(2, 3)\}$. We see that if $v \in \text{cone}(P)$ then the only options for $\Delta(v)$ to be chosen as a minimal face of $T$ such that $v \in \text{cone}(\Delta(v))$ are again to consider $\emptyset$, $\Delta_1$, and $\Delta_2$. In this example, $\Omega(v) = \Delta(v)$. Recall that since $T$ is a boundary triangulation of $P$, the definition of the $h$-vector (7) is adjusted according to dimension, that is, $d$ is replaced by $d - 1$.

From Figure 1 we determine the following:

| $\Omega \subseteq T$ | $\dim(\Omega)$ | $B(\Omega; z)$ | $B(\Omega'; z)$ | $h(\Omega, z^3)$ |
|---------------------|----------------|---------------|----------------|----------------|
| $\Delta_1$         | 0              | 0             | 0              | 1              |
| $\Delta_2$         | 0              | 0             | 0              | 1              |
| $\emptyset$        | -1             | 1             | $z^2$          | $1 + z^3$      |

Applying Theorem 4.4, we obtain

$$h^*(P; z) = \frac{1 - z^3}{1 - z^d} (1 + z^3 + z^2 + z^5) = 1 + z^2 + z^4,$$

which agrees with the computation obtained using Normaliz [7].
4.3. **Rational Stapledon Decomposition and Inequalities.** Using Theorem 4.4, we can rewrite the $h^*$-polynomial of a rational polytope $P$ as

$$h^*(P; z) = \frac{1+z+\cdots+z^{\ell-1}}{1+z+\cdots+z^{\ell-1}} \sum_{\Omega \in T} \left( B(\Omega; z) + B(\Omega'; z) \right) h(\Omega; z^\ell).$$

Next, we turn our attention to the polynomial

$$\overline{h^*}(P; z) := \left( 1 + z + \cdots + z^{\ell-1} \right) h^*(P; z).$$

We know that $h^*(P; z)$ is a polynomial of degree at most $q(d+1)-1$, thus $\overline{h^*}(P; z)$ has degree at most $q(d+1)+\ell-2$. We set $f$ to be the degree of $\overline{h^*}(P; z)$ and $s$ to be the degree of $h^*(P; z)$. We can recover $h^*(P; z)$ from $\overline{h^*}(P; z)$ for a chosen value of $\ell$; if we write

$$\overline{h^*}(P; z) = \overline{h^*_0} + \overline{h^*_1}z + \cdots + \overline{h^*_f}z^f,$$

then

$$\overline{h^*_i} = h^*_i + h^*_{i-1} + \cdots + h^*_{i-\ell+1} \quad i = 0, \ldots, f,$$

FIGURE 1. This figure shows cone $(P)$ (in orange), $P$, $3P$, $(a, \ell) = (2, 4)$, Box $(\Delta_1)$ (in yellow), Box $(\Delta_2')$ (in pink).
and we set \( h_i^* = 0 \) when \( i > s \) or \( i < 0 \).

**Proposition 4.6.** Let \( P \) be a rational \( d \)-polytope with denominator \( q \) and Ehrhart series

\[
\text{Ehr}(P;z) = \frac{h^*(P;z)}{(1 - z^q)^{d+1}}.
\]

Then \( \deg h^*(P;z) = s \) if and only if \((q(d + 1) - s)P\) is the smallest integer dilate of \( P \) that contains an interior lattice point.

**Proof.** Let \( L(P;t) \) and \( L(P^*;t) \) be the Ehrhart quasipolynomials of \( P \) and the interior of \( P \), respectively. Using Ehrhart–Macdonald reciprocity \([9,24]\) we obtain

\[
\text{Ehr}(P^*;z) = \sum_{t \geq 1} L(P^*;t)z^t = (-1)^{d+1} \frac{\sum_{j=0}^s h_j^* \left( \frac{1}{z} \right)^j}{(1 - \frac{1}{z})^{d+1}} = z^{q(d+1)} \frac{\sum_{j=0}^s h_j^* z^{-j}}{(1 - z^q)^{d+1}}
\]

then we compute that

\[
\left( \sum_{j=0}^s h_j^* z^{q(d+1) - j} \right) (1 + z^q + z^{2q} + \ldots)^{d+1}.
\]

Now, note that the minimum degree term of

\[
\left( \sum_{j=0}^s h_j^* z^{q(d+1) - j} \right) (1 + z^q + z^{2q} + \ldots)^{d+1}
\]

is \( h_s^* z^{q(d+1) - s} \), which implies that the term of \( \sum_{t \geq 1} L(P^*;t)z^t \) with minimum degree is \((q(d + 1) - s)P\). Hence, the degree of \( h^*(P;z) \) is \( s \) precisely if \((q(d + 1) - s)P\) is the smallest integer dilate of \( P \) that contains an interior lattice point. \( \square \)

The following result provides a decomposition of the \( \overline{h^*} \)-polynomial which we refer to as an \( a/b \)-decomposition. It generalizes \([33, \text{Theorem 2.14}]\) to the rational case.

**Theorem 4.7.** Let \( P \) be a rational \( d \)-polytope with denominator \( q \), and let \( s := \deg h^*(P;z) \). Then \( \overline{h^*}(P;z) \) has a unique decomposition

\[
\overline{h^*}(P;z) = a(z) + z^\ell b(z),
\]

where \( \ell = q(d + 1) - s \) and \( a(z) \) and \( b(z) \) are polynomials with integer coefficients satisfying \( a(z) = z^{q(d+1) - 1} a \left( \frac{1}{z} \right) \) and \( b(z) = z^{q(d+1) - 1 - \ell} b \left( \frac{1}{z} \right) \). Moreover, the coefficients of \( a(z) \) and \( b(z) \) are nonnegative.

**Proof.** Let \( a_i \) and \( b_i \) denote the coefficients of \( z^i \) in \( a(z) \) and \( b(z) \), respectively. Set

\[
a_i + 1 = h_0^* + \cdots + h_{i+1}^* - h_{q(d+1)-1}^* - \cdots - h_{q(d+1)-1-i}^*,
\]

and

\[
b_i = -h_0^* - \cdots - h_i^* + h_{i+1}^* + \cdots + h_{n-i}^*.
\]

Using (12) and the fact that \( \ell = q(d + 1) - s \), we compute that
\[ a_i + b_{i-\ell} = h_0^* + \cdots + h_i^* - h_{q(d+1)-1}^* - \cdots - h_{q(d+1)-i}^* - h_0^* - \cdots - h_{i-\ell}^* + h_s^* + \cdots + h_{s-i+\ell}^* \\
= h_{i-\ell+1}^* + \cdots + h_{i+1}^*. \]
\[ a_i - a_{q(d+1)-1-i} = h_0^* + \cdots + h_i^* - h_{q(d+1)-1}^* - \cdots - h_{q(d+1)-i}^* - h_0^* - \cdots - h_{s-i-1}^* + h_0^* + \cdots + h_s^* - \cdots - h_{s-i-1}^* - h_s^* - \cdots - h_{i+1}^* \]
\[ = 0, \]
\[ b_i - b_{q(d+1)-1-\ell-i} = -h_0^* - \cdots - h_s^* + \cdots + h_{s-i}^* + h_0^* + \cdots + h_s^* - \cdots - h_{s-i-1}^* - h_s^* - \cdots - h_{i+1}^* \]
\[ = 0, \]
for \( i = 0, \ldots, q(d+1) - 1 \). Thus, we obtain the decomposition desired. The uniqueness property follows from (13) and (14).

Let \( T \) be a regular boundary triangulation of \( P \). By Theorem 4.4 and (11), we can set
\[ a(z) = (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega; z) h(\Omega; z^q), \]
and
\[ b(z) = z^{-\ell} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega'; z) h(\Omega; z^q), \]
so that \( \overline{h}(P; z) = a(z) + z^\ell b(z) \). By Proposition 4.6, the dilate \( kP \) contains no interior lattice points for \( k = 1, \ldots, \ell - 1 \), so if \( v \in \text{Box}(\Omega') \cap \mathbb{Z}^{d+1} \) for \( \Omega \in T \), then \( u(v) \geq \ell \). Hence, \( b(z) \) is a polynomial. We now need to verify that
\[ a(z) = z^{q(d+1)-1} a \left( \frac{1}{z} \right) \quad \text{and} \quad b(z) = z^{q(d+1)-1-\ell} b \left( \frac{1}{z} \right). \]
It is a well-known property of the \( h \)-vector in (7) that \( h(\Omega; z^q) = z^{q(d-\dim(\Omega)-1)} h(\Omega; z^{-q}) \) \([11, 25, 27]\).

Using the aforementioned and Corollary 4.1, we determine that
\[ z^{q(d+1)-1} a \left( \frac{1}{z} \right) = z^{q(d+1)-1} \left( 1 + \frac{1}{z} + \cdots + \frac{1}{z^{q-1}} \right) \sum_{\Omega \in T} B \left( \Omega; \frac{1}{z} \right) h \left( \Omega; \frac{1}{z^q} \right) \]
\[ = z^{q(d+1)-1} z^{-q} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B \left( \Omega; \frac{1}{z} \right) h \left( \Omega; \frac{1}{z^q} \right) \]
\[ = z^{qd} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} z^{-q(d-\dim(\Omega)+1)} B(\Omega, z) z^{-q(d-1-\dim(\Omega))} h(\Omega; z^q) \]
\[ = (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega, z) h(\Omega; z^q) = a(z) \]
and
\[ z^{q(d+1)-1-\ell} b \left( \frac{1}{z} \right) = z^{q(d+1)-1-\ell} z^{-\ell} \left( 1 + \frac{1}{z} + \cdots + \frac{1}{z^{q-1}} \right) \sum_{\Omega \in T} B \left( \Omega'; \frac{1}{z} \right) h \left( \Omega; \frac{1}{z^q} \right) \]
\[ = z^{q(d+1)-1} z^{-q} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B \left( \Omega'; \frac{1}{z} \right) h \left( \Omega; \frac{1}{z^q} \right) \]
\[ = z^{qd} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} z^{-q(d-\dim(\Omega)+1)-\ell} B(\Omega', z) z^{-q(d-1-\dim(\Omega))} h(\Omega; z^q) \]
\[ = z^{-\ell} (1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega', z) h(\Omega; z^q) = b(z). \]
Lastly, recall that the box polynomials and the $h$-polynomials have nonnegative coefficients [29], so a sum of products of box polynomials and $h$-polynomials will also have nonnegative coefficients. Thus, the result holds.

The next theorem follows as a corollary to Theorem 4.7 and gives inequalities satisfied by the coefficients of the $h^*$-polynomial for rational polytopes.

**Theorem 4.8.** Let $P$ be a rational $d$-polytope with denominator $q$ and let $s := \deg h^*(P; z)$. The $h^*$-vector $(h_0^*, \ldots, h_{q(d+1)-1}^*)$ of $P$ satisfies the following inequalities:

\[
\begin{align*}
& h_0^* + \cdots + h_{i+1}^* \geq h_{q(d+1)-1}^* + \cdots + h_{q(d+1)-1-i}^*, & i = 0, \ldots, \left\lfloor \frac{q(d+1) - 1}{2} \right\rfloor - 1, \\
& h_i^* + \cdots + h_{q(d+1)-i}^* \geq h_0^* + \cdots + h_i^*, & i = 0, \ldots, q(d+1) - 1.
\end{align*}
\]

**Proof.** By (13) and (14) if follows that (17) and (18) hold if and only if $a(z)$ and $b(z)$ have nonnegative coefficients, respectively, which in turn follows from Theorem 4.7.

5. Applications

5.1. Rational Reflexive Polytopes. A lattice polytope is reflexive if its dual is also a lattice polytope. Reflexive polytopes have enjoyed a wealth of recent research activity (see, e.g., [1, 5, 6, 12, 13, 16–18, 26]), and Hibi [15] proved that a lattice polytope $P$ is the translate of a reflexive polytope if and only if $\text{Ehr}(P, \frac{1}{b}) = (-1)^d z \text{Ehr}(P; z)$ as rational functions, that is, $h^*(z)$ is palindromic. More generally, Fiset and Kaspryzk [10, Corollary 2.2] proved that a rational polytope $P$ whose dual is a lattice polytope has a palindromic $h^*$-polynomial, complementing previous results by De Negri and Hibi [8]. The following proposition provides an alternate route to Fiset and Kaspryzk’s result.

**Theorem 5.1.** Let $P$ be a rational polytope containing the origin. The dual of $P$ is a lattice polytope if and only if $\overline{h^*}(P; z) = h^*(z) = a(z)$, that is, $b(z) = 0$ in the $a/b$-decomposition of $\overline{h^*}(P; z)$ from Theorem 4.4.

**Proof.** Let $P$ be a rational polytope containing the origin in its interior. Following Set-up 4.1, we let $T$ be a boundary triangulation of $P$ and we set $(a, \ell) = (0, 1)$. Recall that this implies

\[
b(z) = z^{-1}(1 + z + \cdots + z^{q-1}) \sum_{\Omega \in T} B(\Omega'; z)h(\Omega; z^q).
\]

Thus, $b(z) = 0$ if and only if $B(\Omega'; z) = 0$ for every $\Omega \in T$, which is true if and only if Box $(\Omega')$ contains no integer points for every $\Omega \in T$.

To establish the forward direction, assume that the dual of $P$ is a lattice polytope. We want to show that $b(z) = 0$ in the $a/b$-decomposition of $\overline{h^*}(P; z) = h^*(P; z)$. Each $\Omega \in T$ is contained in a facet $F$ of $P$. Since the dual of $P$ is a lattice polytope, the vector normal to cone $(F)$ is of the form $(p, 1)$, where $p$ is the vertex of the dual of $P$ corresponding to $F$. Let $(r_1, q), \ldots, (r_{m+1}, q)$ be the ray generators of Box $(\Omega)$. If $\sum_{i=1}^{m+1} \lambda_i(ri, q) \in \text{Box}(\Omega)$ for $0 < \lambda_i < 1$, then $(p, 1) \cdot (\sum_{i=1}^{m+1} \lambda_i(ri, q)) = 0$. Also, note that $(p, 1) \cdot (0, 1) = 1$, which tells us that $(0, 1)$ is at lattice distance 1 away from Box $(\Omega)$ with respect to $(p, 1)$. So, if

\[
\sum_{i=1}^{m+1} \lambda_i(ri, q) + \lambda(0, 1) \in \text{Box}(\Omega')
\]

then $(p, 1) \cdot \left[ \sum_{i=1}^{m+1} \lambda_i(ri, q) + \lambda(0, 1) \right] = \lambda$, where $0 < \lambda < 1$. This implies that $\sum_{i=1}^{m+1} \lambda_i(ri, q) + \lambda(0, 1)$ is not an integer point, from which it follows that Box $(\Omega')$ contains no lattice points. Thus $B(\Omega', z) = 0$ and so $b(z) = 0$ in the $a/b$-decomposition of $\overline{h^*}(P; z)$. Hence, $\overline{h^*}(P; z) = h^*(P; z) = a(z)$ is palindromic.

For the backward direction, assume that $b(z) = 0$, and thus for every $\Omega \in T$, the set Box $(\Omega')$ contains no integer points. Our goal is to use this fact to show that for every facet $F$ of $P$, the vertex of the dual
of $P$ corresponding to $F$ is a lattice point, i.e., to show that the primitive facet normal to cone $(F)$ is given by $(p, 1)$ for some lattice point $p$. Let $F$ be a facet of $P$, and let $\Omega = \text{conv}\{(r_1, q), \ldots, (r_{m+1}, q)\} \in T$ be a full-dimensional simplex contained in $F$. Since the origin lies in the interior of $P$, the dual of $P$ is a rational polytope containing the origin. Further, the vector normal to cone $(F)$ can be written in the form $(p, b)$ with $b > 0$, where $p$ is an integer vector that is primitive, i.e., the greatest common divisor of the entries in $(p, b)$ equals 1. Observe that $(p, b) \cdot (0, 1) = b$. If $b = 1$, then the vertex of the dual of $P$ corresponding to $F$ is a lattice point, and our proof is complete.

Otherwise, suppose that $b > 1$. Since $(p, b)$ is primitive, there exists an integer vector $v$ such that $(p, b) \cdot v = 1$. Since $b > 1 > 0$, $v$ is an element of the subset $S$ strictly contained between the hyperplane $H_0$ spanned by cone $(F)$ and the affine hyperplane $H_b = H_0 + (0, 1)$; we can precisely describe this subset as

$$S := \left\{ \sum_{i=1}^{m+1} \lambda_i (r_i, q) + \lambda (0, 1) : \lambda_i \in \mathbb{R} \text{ and } 0 < \lambda < 1 \right\}.$$

Since $b(z) = 0$, it follows that for each $\tau \subseteq \Omega$ the set $\text{Box}(\tau') = \text{Box}(\tau, (0, 1))$ contains no integer points. The key observation is that translates of $\bigcup_{\tau \subseteq \Omega} \text{Box}(\tau, (0, 1))$ by the integer ray generators of cone $(F)$ cover $S$, though this union is not disjoint, i.e.,

$$S = \bigcup_{\mu_1, \ldots, \mu_{m+1} \in \mathbb{Z}} \left( \sum_{i=1}^{m+1} \mu_i (r_i, q) \right) + \bigcup_{\tau \subseteq \Omega} \text{Box}(\tau, (0, 1)).$$

This cover property follows from taking an arbitrary $\sum_{i=1}^{m+1} \lambda_i (r_i, q) + \lambda (0, 1) \in S$ and expressing each coefficient as a sum of an integer and fractional part. It follows that $S$ contains no integer points, since $\bigcup_{\tau \subseteq \Omega} \text{Box}(\tau, (0, 1))$ contains no integer points. Hence, no such integer vector $v$ exists, implying that $b = 1$. Since $F$ was arbitrary, it follows that the dual of $P$ is a lattice polytope. \qed

### 5.2. Reflexive Polytopes of Higher Index

Kasprzyk and Nill [22] introduced the following class of polytopes.

**Definition 5.2.** A lattice polytope $P$ is a reflexive polytope of higher index $\mathcal{L}$ (also known as an $\mathcal{L}$-reflexive polytope), for some $\mathcal{L} \in \mathbb{Z}_{>0}$, if the following conditions hold:

- $P$ contains the origin in its interior;
- The vertices of $P$ are primitive, i.e., the line segment joining each vertex to 0 contains no other lattice points;
- For any facet $F$ of $P$ the local index $\mathcal{L}_F$ equals $\mathcal{L}$, i.e., the integral distance of 0 from the affine hyperplane spanned by $F$ equals $\mathcal{L}$.

The 1-reflexive polytopes are the reflexive polytopes mentioned earlier in the section. Kasprzyk and Nill proved that if $P$ is a lattice polytope with primitive vertices containing the origin in its interior then $P$ is $\mathcal{L}$-reflexive if and only if $\mathcal{L}P^*$ is a lattice polytope having only primitive vertices. In this case, $\mathcal{L}P^*$ is also $\mathcal{L}$-reflexive.

Kasprzyk and Nill investigated $\mathcal{L}$-reflexive polygons. In particular, they show that there is no $\mathcal{L}$-reflexive polygon of even index. Furthermore, they provide a family of $\mathcal{L}$-reflexive polygons arising for each odd index:

$$P_{\mathcal{L}} = \text{conv}\{\pm (0, 1), \pm (2, \mathcal{L}, 2), \pm (\mathcal{L}, 1)\}.$$

We are interested in the dual of $P_{\mathcal{L}}$:

$$P_{\mathcal{L}}^* = \text{conv}\left\{ \pm \left( \frac{1}{\mathcal{L}}, 0 \right), \pm \left( \frac{2}{\mathcal{L}}, -1 \right), \pm \left( \frac{1}{\mathcal{L}}, -1 \right) \right\}.$$
Theorem

Let $\mathcal{L}$ be odd. Our goal in the remainder of this subsection is to compute the $h^*$-polynomial of $P_{\mathcal{L}}^*$ using Theorem 4.4, to illustrate how this theorem can be applied. Consider the boundary as its own triangulation $T$ (with denominator $\mathcal{L}$) of $P_{\mathcal{L}}^*$ and take the set of integral ray generators of cone $(P_{\mathcal{L}}^*)$ to be

$$\{\pm(1,0,\mathcal{L}), \pm(2,0,\mathcal{L}), \pm(1,\mathcal{L},\mathcal{L})\}.$$ 

Observe that $T$ contains six edges, six vertices, and the empty face $\emptyset$. It is not difficult to see that the box polynomials of the 0-simplices are 0. For example, in order for $B(\mathcal{L})$ to contain any lattice points, $2\lambda_1$ must be an integer between 0 and 2, implying that $\lambda_1 = \frac{1}{2}$. Also, $-\mathcal{L}\lambda_1$ and $\mathcal{L}\lambda_1$ must be integers, but since $\lambda_1 = \frac{1}{2}$ and $\mathcal{L}$ is odd, $-\mathcal{L}\lambda_1$ and $\mathcal{L}\lambda_1$ are never integers. Therefore, $\text{Box}((2,-\mathcal{L},\mathcal{L})) = \emptyset$.

Since $P_{\mathcal{L}}^*$ is a centrally symmetric hexagon, we can restrict our analysis to three of its facets: $F_1 := \text{conv} \{\pm(\frac{1}{2},-1), \pm(\frac{1}{2},1)\}$, $F_2 := \text{conv} \{\pm(\frac{1}{2},-1), \pm(\frac{1}{2},0)\}$, and $F_3 := \text{conv} \{\pm(\frac{1}{2},0), \pm(-\frac{1}{2},1)\}$. We consider each facet separately.

**Case:** $F_1$. Observe:

$$\text{Box}((F_1,\mathcal{L})) = \{\lambda_1(1,-\mathcal{L},\mathcal{L}) + \lambda_2(2,-\mathcal{L},\mathcal{L}) : 0 < \lambda_1, \lambda_2 < 1\}$$

$$= \{(\lambda_1 + 2\lambda_2, -\mathcal{L}\lambda_1 - \mathcal{L}\lambda_2, \mathcal{L}\lambda_1 + \mathcal{L}\lambda_2 : 0 < \lambda_1, \lambda_2 < 1\}.$$ 

Let $\mathcal{L} = 2k + 1$ for $k \in \mathbb{Z}_{\geq 0}$. We now want to determine when $(A,-B,B) \in \text{Box}((F_1,\mathcal{L}))$ is a lattice point. This reduces to solving a system of linear equations between $A$ and $B$. In order for $A$ to be an integer it must be 1 or 2. When $A = \lambda_1 + 2\lambda_2 = 1$, $B = \mathcal{L}\lambda_1 + \mathcal{L}\lambda_2$ equals $\mathcal{L} - k$, $\mathcal{L} - k + 1$, ..., $\mathcal{L} - 2$, or $\mathcal{L} - 1$ with the restriction that $0 < \lambda_1, \lambda_2 < 1$. When $A = \lambda_1 + 2\lambda_2 = 2$, $B = \mathcal{L}\lambda_1 + \mathcal{L}\lambda_2$ equals $\mathcal{L} + 1$, $\mathcal{L} + 2$, ..., $\mathcal{L} + k - 1$, or $\mathcal{L} + k$. Therefore, $\text{Box}((F_1,\mathcal{L})) \cap \mathbb{Z}^3$ contains the elements $\{(1,\mathcal{L} - k, \mathcal{L} - k), (1,k - \mathcal{L} - 1,\mathcal{L} - k + 1), (1,2,\mathcal{L},\mathcal{L} - 2), (1,\mathcal{L} - 1,\mathcal{L} - 1), (2,-\mathcal{L} - 2,\mathcal{L} + 2), (2,1-\mathcal{L} - \mathcal{L} + k + 1), (2,-\mathcal{L} - k,\mathcal{L} + k)\}$. Therefore, the box polynomial of $F_1$ is

$$B(F_1; z) = \sum_{i = \mathcal{L} - k}^{\mathcal{L} - 1} z^i + \sum_{i = \mathcal{L} + 1}^{\mathcal{L} + k} z^i.$$ 

![Figure 2. The rational hexagon $P_{\mathcal{L}}^*$.](image-url)
and conclude that for $L^1 \lambda A$ to solving a system of linear equations between $L$ reduces to 2 obtain $\lambda (Suppose $0 = L$, Combining the above analysis with the values in Table Case: $F$. Observe:

$$\text{Box}(F_2, L) = \{\lambda_1 (2, -L, L) + \lambda_2 (1, 0, L) : 0 < \lambda_1, \lambda_2 < 1\}$$

$$= \{(2\lambda_1 + \lambda_2, -L \lambda_1, L \lambda_1 + L \lambda_2) : 0 < \lambda_1, \lambda_2 < 1\}.$$ Suppose $(A, B, C)$ is an integer point in this set. Again, determining the integer points in the box reduces to solving a system of linear equations between $A$ and $C$ with the added condition coming from $B$ that $\lambda_1 = \frac{1}{2}, \ldots, \frac{L-1}{2}$. It is straightforward to verify that the resulting box polynomial of $F_2$ is the same as $F_1$.

**Case: $F_3$. Observe:**

$$\text{Box}(F_3, L) = \{\lambda_1 (-1, L, L) + \lambda_2 (1, 0, L) : 0 < \lambda_1, \lambda_2 < 1\}$$

$$= \{(-\lambda_1 + \lambda_2, L \lambda_1, L \lambda_1 + L \lambda_2) : 0 < \lambda_1, \lambda_2 < 1\}.$$ Suppose $(A, B, C)$ is an integer point in this set. For $A$ to be an integer it must be equal to zero, so we obtain $\lambda_1 = \lambda_2$. The expression for $B$ implies that $\lambda_2 = \frac{m}{L}$ for some integer $m \in [1, L - 1]$. Lastly, $C$ then reduces to $2L \lambda_1 = 2m$. Therefore, we conclude $\text{Box}((F_3, L))$ contains $L - 1$ lattice points of the form $(0, m, 2m)$, one for each integer $m \in [1, L - 1]$. This implies the box polynomial of $F_3$ is given by

$$B(F_3; z) = \sum_{i=1}^{L-1} z^{2i}.$$ Combining the above analysis with the values in Table 5.2, we apply Theorems 4.4 and 5.1 and conclude that for $L = 2k + 1$,

$$h^*(P^*_L; z) = (1 + z + \cdots + z^L) \left(1 + 4z^L + z^{2L} + 4 \left(\sum_{i=1}^{L-1} z^i + \sum_{i=L+1}^{L+k} z^i\right) + 2 \sum_{i=1}^{L-1} z^{2i}\right).$$
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