I. INTRODUCTION

The damped random-frequency harmonic oscillator is a fundamental tool in statistical physics that has been extensively used to describe a myriad of physical systems in different research fields [1]. The statistical properties of dye lasers [2], the propagation of electromagnetic waves in random media [3], the population dynamics of living organisms [4], and the distribution of stock market price changes in economics [5] are examples of systems described by a noisy harmonic oscillator model.

Recently, it has been shown that networks of interacting noisy classical oscillators can feature certain physical process originally found in the study of open quantum systems [6, 7], such as the phenomenon called environment-assisted quantum transport [8] or dephasing-assisted energy transport [9]. In this scenario, energy transport from a specific oscillator, connected to a trapping center by a network of randomly fluctuating oscillators, can show a remarkable higher efficiency than the corresponding network with fixed frequencies [10].

Two experimental schemes have been proposed to observe the noise-assisted energy transfer process, both based on quantum-mechanical systems, namely superconducting qubits [11], and coupled quantum-optical cavities [12]. Although these experiments can be done with presently available technology, they represent a great challenge and require a big effort in their implementation.

Based on the fact that highly efficient noise-assisted energy transport can also take place in networks of purely classical random harmonic oscillators, one can think of observing the effect in these systems, which may reduce the complexity of the experimental schemes, thus making them more appealing for their implementation. Although the simulation of the coherent (noise-free) evolution of classical oscillator systems is straightforward [14], the question of how to introduce and control dephasing effects due to a surrounding environment remains open.

In this paper, we introduce an experimental setup that realizes a tunable environment for classical electrical oscillators. We test our scheme by implementing the case of a damped random-frequency harmonic oscillator. The tunability of our system is then demonstrated by gradually modifying the statistics of frequency fluctuations, which is managed by properly controlling the mean and variance of the oscillator’s frequency distribution.

This is particularly relevant, because it implies that the system introduces directly fluctuations in the frequency of the signal, which contrasts with previous experimental studies of electrical oscillators subjected to noise, where fluctuations in the amplitude, rather than frequency, are introduced in the system [13].

Because of its high degree of tunability and control, this system can readily be used to design various types of noise bearing different frequency probability distributions, which makes it an important tool for experimentally studying the effect of multiplicative and additive noise on instabilities of harmonic oscillators [15, 16], and for investigating the so-called noise-assisted energy transport in coupled oscillator networks.

II. MODEL

We consider a damped random-frequency harmonic oscillator whose temporal evolution reads as [17]

$$\frac{d^2 x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 [1 + \phi(t)] x = 0,$$

where $x$ is the oscillator coordinate, $\omega_0$ is the undamped frequency, $\Gamma$ is the damping parameter, and $\phi(t)$ is a random process that allows one to introduce dephasing effects into the system.
where $\Gamma$ is the damping coefficient, $\omega_0$ is the average frequency of the oscillator and $\phi(t)$ describes a stochastic Gaussian process with zero average $\langle \phi(t) \rangle = 0$, and a specific autocorrelation function defined by $\langle \phi(t) \phi(t') \rangle = \kappa(t - t')$, where the function $\kappa(t - t')$ defines the type of noise that is considered. For instance, in the case of ideal white noise, the autocorrelation function is defined as $\langle \phi(t) \phi(t') \rangle = 2D\delta(t - t')$, where $D$ denotes the intensity of the noise. A more realistic example is colored noise, where the autocorrelation function writes $\langle \phi(t) \phi(t') \rangle = (D/\tau_c) \exp(-|t - t'|/\tau_c)$, with $\tau_c$ being the correlation time of the stochastic process \cite{18}.

Using the cumulant expansion described by Van Kampen \cite{19}, one can show that the equation for $\langle x \rangle$ has the form (see Appendix for details)

$$
\frac{d^2}{dt^2} \langle x \rangle + \left( \Gamma + \frac{\omega_0^4}{2\nu c_2} \right) \frac{d}{dt} \langle x \rangle + \omega_0^2 \left[ 1 - \frac{\omega_0^2}{2\nu} \left( c_1 - \frac{\Gamma}{2\nu c_2} \right) \right] \langle x \rangle = 0, \quad (2)
$$

where $\nu = (\omega_0^2 - \Gamma^2/4)^{1/2}$, and the coefficients $c_1$ and $c_2$ are defined by

$$
c_1 = \int_0^{\infty} \langle \phi(t) \phi(t - \xi) \rangle \sin(2\omega_0\xi) \, d\xi, \quad (3)
$$

$$
c_2 = \int_0^{\infty} \langle \phi(t) \phi(t - \xi) \rangle [1 - \cos(2\omega_0\xi)] \, d\xi. \quad (4)
$$

Notice that the existence of frequency fluctuations in Eq. \cite{1} introduces a noise-induced additional damping and a noise-induced frequency shift to the average signal of the oscillator.

---

**III. EXPERIMENT**

**A. The setup**

The experimental setup that allows us to introduce random-frequency fluctuations into a harmonic oscillator model is the following. Firstly, note that one can construct a system governed by Eq. \cite{1} by making use of electrical $RLC$ oscillators (where $R$ stands for resistance, $L$ for inductance and $C$ for capacitance). In these systems, the charge in the capacitor satisfies the same equation as Eq. \cite{1}, where the coefficients $\Gamma$ and $\omega_0$ are defined by \cite{13}

$$
\Gamma = R/L, \quad (5)
$$

$$
\omega_0 = (LC_0)^{-1/2}, \quad (6)
$$

with $C_0$ denoting the average capacitance of the circuit. From Eq. \cite{1} one can see that fluctuations in the frequency of the $RLC$ oscillator can be introduced by randomly switching the values of the capacitance \cite{20}.

Random switching of capacitance is performed in the following way: An array of eight capacitors, each with equal capacitance $C_n$, is connected in parallel to a central capacitor $C$ of a $RLC$ circuit. To produce uncorrelated random switching, the individual capacitors are independently turned on/off by means of analog switches (NXP-74HC4066N quad bilateral switch), which are driven by independent digital signals provided by an arbitrary function generator (Signadyne digital I/O module SD-PXE-DIO-H0001), as shown in Fig. 1. Because we are interested in designing a Gaussian stochastic process, we program the arbitrary function generator, so each capacitor has the same probability to be on or off, in the same fashion as in a coin-tossing event. It is easy to show that the probability that $n$-capacitors in the array are on
satisfies a binomial distribution given by

$$P(n) = \binom{8}{n} \frac{1}{2^n}, \quad (7)$$

where \( n = \{0, 1, 2, \ldots, 8\} \). This distribution is defined by a mean value \( \langle n \rangle = 4 \) and a variance \( \sigma_n^2 = 2 \). Notice that the binomial distribution described in Eq. (7) is a discrete version of a Gaussian distribution with the same mean and variance, as depicted in Fig. 2. It is important to remark that due to the nonlinear relation between frequency and capacitance [Eq. (6)], when calculating the probability distribution of frequency, a Gaussian distribution is obtained provided that the condition \( C_a \ll C \) is satisfied [18].

B. Implementation and Results

To test the proposed scheme, we construct a RLC circuit where the central capacitance \( C \) is provided by a 1 nF ceramic capacitor, inductance \( L \) by a 1.5 mH ferrite core inductor, and resistance \( R \) represents parasitic losses within the system. For the random switching of capacitance, we have designed several arrays using different ceramic capacitors with capacitance value \( C_a = \{4.7, 10, 18, 33, 47, 68, 100\} \) pF. Notice from Fig. 2 that by changing the values of \( C_a \) one can modify the variance of the Gaussian distribution, which in turn modifies the statistics of the noise in the system [18].

Since the frequency fluctuations need to be faster than the characteristic time evolution of the system, the digital signals from the arbitrary function generator are set with a time rate \( \tau = 650 \) ns, which is longer than the response time of the analog switches (400 ns), and much shorter than the temporal evolution window of the measured signal (\( t = 100 \) µs).

Using the system described above, we have performed the simulation of Eq. (1). To this end, we keep the capacitor \( C \) fixed and measure the averaged signal of the oscillator connected to different capacitor arrays. Figure 3(a) shows the histograms of the measured frequency in each case. Histograms are obtained from 50000 different realizations, and they are normalized to the maximum number of events, where we define number of events as the number of realizations that have the same value of frequency. Notice that in all cases the probability distribution of the frequency follows a Gaussian distribution whose variance \( \sigma_n^2 \) depends strongly on the value of \( C_a \) used in the connected array. This implies that this scheme allows to control the statistics of the noise that is introduced in the system, which is important when simulating the dynamics of open systems [7, 8, 11]. To compare the results obtained in Fig. 3(a) with the theoretical model, we have measured the frequency shift that arises from the influence of frequency fluctuations, as predicted by Eq. (2). Figure 3(b) shows the frequency shift for each capacitor array. We have made use of Eq. (2), and the relation [18]: \( D = \sigma^2 \tau \), to find that the driving noise of our system can be described by a colored-noise-like autocorrelation function of the form

$$\langle \phi(t) \phi(t') \rangle = \frac{\sigma^2}{\omega_0^2} \exp \left( -\frac{|t-t'|}{\tau} \right), \quad (8)$$

where the mean value of the frequency is computed with \( C_0 = C + 4C_a \), and the variance of the driving noise is \( \sigma^2 = (\eta \sigma_n)^2 \), with \( \eta = 3.4 \). This relation between both variances can be understood as a consequence of the damping term in Eq. (11). The same effect can be found, for instance, in the Ornstein-Uhlenbeck process, where the resulting variance is proportional to the variance of the driving noise due to the presence of a damping term [19].

In general, when simulating open systems, one is interested in keeping the mean frequency of each oscillator fixed while increasing the strength of the noise [9, 11]. This can be achieved in our system by controlling the values of the central capacitance \( C \) and the time duration \( \tau \) of the digital signals. Figure 4 shows the frequency

![Graph](image-url)
FIG. 4. Frequency histograms for different capacitor arrays centered in the same mean frequency $f_0 \simeq 123$ kHz. Dotted line: Experimental data, Solid line: Gaussian fitting.

| $C_a$ (pF) | 4.7 | 10 | 18 | 33 | 47 | 68 | 100 |
|------------|-----|----|----|----|----|----|-----|
| $C$ (nF)   | 1.120 | 1.090 | 1.053 | 0.978 | 0.933 | 0.840 | 0.355 |
| $\tau$ (ns) | 650 | 650 | 750 | 780 | 800 | 720 | 650 |

TABLE I. Experimental parameters used to obtain the histograms shown in Fig. 4.

histograms measured with different capacitor arrays. Notice that by properly controlling the parameters of the system, we are able to center all the probability distributions in the same value of frequency $f_0 \simeq 123$ kHz. This demonstrates the flexibility of our system when modifying the statistical properties of the environmental noise that interacts with the oscillator. Parameters of the system used in each case are summarized in Table 1.

IV. CONCLUSIONS AND OUTLOOK

In this paper, we have demonstrated a system that performs as a tunable environment for classical electrical oscillators. We have shown its operation by simulating the case of a damped random-frequency oscillator, where a perfect agreement with the theoretical model has been obtained. Finally, we have demonstrated the degree of control that one can achieve with this system by gradually modifying the variance of the frequency fluctuations, while maintaining a fixed central frequency of oscillation, which is of critical importance when simulating energy transfer mechanisms in different scenarios, such as the case, for instance, in molecular aggregates. This high degree of tunability and control can be further used to design various types of noise with different probability distributions. Moreover, it might allow us to study the transition from Markovian to non-Markovian dynamics of open systems. The results reported here represent an important step towards the simulation of certain types of exciton transport effects in classical systems, such as the above mentioned noise-assisted energy transport in coupled networks.

ACKNOWLEDGMENTS

We thank Adam Vallés, Luis José Salazar Serrano, Yannick de Icaza Astiz, Daniel Mitrani and José Carlos Cifuentes for valuable discussions. This work was supported by the projects funded by the Government of Spain FIS2010-14831 and Severo Ochoa. This work has also been partially supported by Fundacio Privada Cellex Barcelona. J. Svozilík acknowledges projects CZ.1.07/2.3.00/30.0004 of MSMT of CR and PrF-2013-006 of IGA UP Olomouc.

Appendix: Derivation of the averaged amplitude equation of a damped random-frequency harmonic oscillator

Let us consider the equation for a damped random-frequency harmonic oscillator

$$\frac{d^2 x}{dt^2} + \Gamma \frac{dx}{dt} + \omega_0^2 [1 + \alpha \phi(t)] x = 0,$$

(A.1)

where $\Gamma$ is the damping coefficient, $\omega_0$ is the average frequency of the oscillator, and $\phi(t)$ is a dimensionless stochastic variable with zero average $\langle \phi(t) \rangle = 0$, and an autocorrelation function satisfying $\langle \phi(t_1) \phi(t_2) \rangle \to 0$ for any two time points $t_1$, $t_2$ such that $|t_2 - t_1| > \tau_c$, where $\tau_c$ is the correlation time of the stochastic process. To simplify the derivation of Eq. (2), and for consistency with Ref. [19], we have included in Eq. (A.1) the parameter $\alpha$, which represents the strength of the stochastic fluctuations.

In order to solve Eq. (A.1), we first transform it into a set of first-order differential equations

$$\frac{d}{dt} X = (M_d + M_s) X,$$

(A.2)

where

$$X = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix},$$

(A.3)

$$M_d = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\Gamma \end{bmatrix},$$

(A.4)

$$M_s = \begin{bmatrix} 0 & \alpha \omega_0^2 \phi(t) \\ -\alpha \omega_0^2 \phi(t) & 0 \end{bmatrix}. $$

(A.5)

Here, the matrices $M_d$ and $M_s$ represent the deterministic and stochastic evolution of the oscillator, respectively.
and \( \dot{x}(t) \) stands for the time derivative of the oscillator’s amplitude \( x \).

In the matrix representation, it is easy to show that the equation for the deterministic evolution of the oscillator,

\[
U_d(t) = \exp \left( -\frac{\Gamma}{2} t \right) \begin{bmatrix}
\cos(\nu t) + (\Gamma/2\nu) \sin(\nu t) \\
-\omega_0^2 \sin(\nu t) / \nu \\
\sin(\nu t) / \nu \\
\cos(\nu t) - (\Gamma/2\nu) \sin(\nu t)
\end{bmatrix}.
\]

Notice that the presence of damping in the harmonic oscillator produces a frequency shift that is given by \( \nu = \sqrt{\omega_0^2 - \Gamma^2/4} \).

Now, we make use of Eq. (A.7) to perform the transformation

\[
X(t) = U_d(t) \tilde{X}(t),
\]

which by substituting it into Eq. (A.2) allows us to write

\[
\frac{d}{dt} \tilde{X} = \alpha U_d(-t) M_s(t) U_d(t) \tilde{X}(t).
\]

Then, we iteratively solve Eq. (A.9) to find that the average of \( \tilde{X} \) writes

\[
\langle \tilde{X}(t) \rangle = \langle \tilde{X}(0) \rangle + \alpha^2 \int_0^t dt_1 \int_0^\infty dt_2 U_d(-t_1) M_s(t_1) \times \left[ U_d(t_1 - t_2) M_s(t_2) U_d(t_2) \langle \tilde{X}(0) \rangle \right].
\]

Notice that the linear term with \( \alpha \) disappears since \( \langle M_s(t) \rangle = 0 \). In writing Eq. (A.10), we have considered only the contributions up to \( \alpha^2 \), which is an approximation that is valid as long as the condition \( \alpha \tau_c \ll 1 \) is satisfied. In addition, we have assumed that the correlation time \( \tau_c \) is much shorter than the integration time, so we can take \( \langle \tilde{X}(t) \rangle \rightarrow \langle X(0) \rangle \), and integrate to infinity, in the second term of Eq. (A.10).

We now perform the time derivative of Eq. (A.10) to obtain

\[
\frac{d}{dt} \langle \tilde{X}(t) \rangle = \alpha^2 \int_0^\infty d\xi \langle U_d(-t) M_s(\xi) \rangle M_s(t - \xi) \times U_d(t - \xi) \langle \tilde{X}(0) \rangle,
\]

where the substitutions \( \xi = t_1 - t_2 \), and \( t = t_1 \) are used. Inverse transformation of Eq. (A.11), by means of Eq. (A.8), then gives

\[
\frac{d}{dt} \langle X(t) \rangle = \left[ M_d + \alpha^2 \int_0^\infty d\xi \langle M_s(\xi) U_d(\xi) \rangle \times M_s(t - \xi) U_d(-\xi) \right] \langle X(t) \rangle.
\]

Using Eqs. (A.5) and (A.7), we can readily find that the expression inside the integral of Eq. (A.12) writes

\[
\langle M_s(t) U_d(\xi) M_s(t - \xi) U_d(-\xi) \rangle = \omega_0^4 \langle \phi(t) \phi(t - \xi) \rangle \\
\times \left\{ \frac{\sin(\nu \xi)}{\nu} [\cos(\nu \xi) - (\Gamma/2\nu) \sin(\nu \xi)] - \frac{\sin^2(\nu \xi)}{\nu^2} \right\}
\]

Finally, by substituting Eq. (A.13) into Eq. (A.12), and transforming the set of two first-order equations to a single second-order differential equation, Eq. (2) is obtained.

[1] M. Gitterman, The Noisy Oscillator: The First Hundred Years, From Einstein Until Now (World Scientific, Singapore, 2005).
[2] R. Graham, M. Hömmerbach, and A. Schenzle, Phys. Rev. Lett. 48, 1396 (1982).
[3] A. Ishimaru, Wave Propagation and Scattering in Random Media (IEEE Press, Piscataway, NJ, 1997).
[4] M. Turelli, Theoretical Population Biology 12, 140 (1977).
[5] H. Takayasu, A.-H. Sato, and M. Takayasu, Phys. Rev. Lett. 79, 966 (1997).
[6] J. S. Briggs and A. Eisfeld, Phys. Rev. E 83, 051911.
For the sake of simplicity, we have selected a random switching of the capacitance. However, one can always choose to randomly change the values of the inductance. By doing this, one adds more complexity to the system since the damping coefficient would randomly fluctuate as well.

K. Jacobs, *Stochastic Processes for Physicists: Understanding Noisy Systems* (Cambridge University Press, UK, 2010)