Simulation of the spin-boson model with superconducting phase qubit coupled to a transmission line

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Based on the rapid experimental developments of circuit QED, we propose a feasible scheme to simulate a spin-boson model with the superconducting circuits, which can be used to detect quantum Kosterlitz-Thouless (KT) phase transition. We design the spin-boson model by using a superconducting phase qubit coupled with a semi-infinite transmission line, which is regarded as bosonic reservoir with a continuum spectrum. By tuning the bias current or the coupling capacitance, the quantum KT transition can be directly detected through tomography measurement on the states of the phase qubit. We also estimate the experimental parameters using numerical renormalization group method.

Quantum simulation is one of the original inspirations for quantum computing, proposed by Feynman[1] to solve the difficulties of simulating quantum systems on a classical computer. Simulating an arbitrary quantum system by the most powerful classical computer is very hard for large scale quantum systems, because of the exponential scaling of the Hilbert space with the size of the quantum system. Quantum phase transitions (QPT) [2] at zero temperature play a key role in the occurrence of important collective phenomena in quantum many-body systems, which occur as a result of competing ground state phases. Similarly, simulation of QPT with classical computer is also difficulty since it is usually relevant to a quantum many-body system.

On the other hand, the spin-boson model [3], a two-level system linearly coupled to a collective of harmonic oscillators, is a typical model to study decoherence effects and QPT. Those dissipative spin systems [4] are very interesting because they display both a localized (classical) and delocalized (quantum) phase for the spin. Spin-boson model has been primarily investigated by numerical renormalization group (NRG) [5,6] and find that different spectral functions of the bath may induce various kind of QPT. Many efforts have been made to observe such environment-induced QPT in various systems, such as mesoscopic metal ring [7], single-electron transistors [8,9] and cold atoms [10], etc.

In this paper, we propose a feasible scheme to simulate the spin-boson model with superconducting circuits, which can be used to observe the notable quantum KT phase transition. Here we focus on the ohmic case ($s = 1$), which can be mapped on the anisotropic Kondo model. The model shows a Kosterlitz-Thouless (KT) quantum phase transition [5], separating the localized phase at $\alpha \geq \alpha_c$ from the delocalized phase at $\alpha < \alpha_c$ [11], where $\alpha$ represents the strength of the dissipation and $\alpha_c$ is the critical value. Our idea is inspired by previous efforts to study the superconducting circuits as an artificial atom [12-13]. In the paper we design the spin-boson model using a superconducting phase qubit coupled with a semi-infinite transmission line, which is regarded as the qubit’s environment with gapless spectra. In this setup, the spectral function of the bosonic bath is treated as Ohmic. By tuning the bias current or the coupling capacitance, the coupling between the spin and the environment can be controlled. So the states of the qubit may transit from delocalized phase to localized state, which can be directly observed by measuring the phase qubit. We also estimate the experimental parameters of this transition with NRG method. Comparing with other candidates of spin-boson model [7,10], the proposed experimental setup based on superconducting system [12-14] may have some distinct advantages, such as the parameters in the model are tunable through experimentally controllable bias currents or driving microwaves, the bosonic reservoir with a continuum spectrum can be simulated easily, and the measurements are of high-fidelity, etc.

Our designed experimental setup is illustrated in Fig. 1, an approximately 1D transmission line (with the length as $L \rightarrow \infty$) is capacitively coupled to a current-biased phase qubit. The phase qubit is an artificial spin, while for relatively low frequencies the transmission line is well described by an infinite series of inductors with each node capacitively connected to ground and then can be considered as a boson

![FIG. 1: A phase qubit capacitively coupled to a semi-infinite transmission line simulates the spin-boson model.](image-url)
The Hamiltonian of the whole system can be written as
\[ H = H_q + H_T + H_{int}, \tag{1} \]
where
\[ H_q = \frac{C_J}{2} \left( \Phi_0 \phi_J \right)^2 - \Phi_0 I_b \phi_J - E_J \cos \phi_J \tag{2} \]
is the Hamiltonian of the current-biased phase qubit,
\[ H_T = \int_0^L \left\{ \frac{1}{2c} \left[ \frac{\partial \theta(x,t)}{\partial x} \right]^2 + \frac{l}{2} \frac{\partial^2 \theta(x,t)}{\partial t^2} \right\} dx \tag{3} \]
is the Hamiltonian of the transmission line, and
\[ H_{int} = \int_0^L \frac{C_0}{2} \left( \frac{1}{c} \frac{\partial \theta(0,t)}{\partial x} - \Phi_0 \phi_J \right)^2 dx \tag{4} \]
is the interaction between the phase qubit and the transmission line. In these equations \( E_J = \frac{\Phi_0 I_b}{2\pi} \) is the magnitude of maximum Josephson coupling energy, \( I_b \) is the critical current of the junction, \( \Phi_0 = \frac{\hbar}{2e} \) is the superconducting flux quantum, \( I_0 \) is the bias current, \( \phi_J \) is the phase difference of the junction, \( C_J \) and \( C_0 \) are the junction and coupling capacitance, and \( l \) and \( C \) are the capacitance and inductance per unit length, respectively; \( \theta(x,t) = \int_0^x q(x',t) \, dx' \) is the collective charge variable on the transmission line.

For the current-biased Josephson junction, the charge operator \( \hat{Q} = C' \Phi_0 \phi_J \) and phase difference operator \( \hat{\phi}_J \) have the commutation relationship \([\hat{\phi}_J, \hat{Q}] = 2ci\). Quantum mechanical behavior can be observed for large area junctions in which \( E_J \gg E_C = \epsilon^2/2C \) when the bias current is slightly smaller than the critical current \( I_b \lesssim I_0 \). In this regime, the last two terms of the Hamiltonian of the phase qubit (as Eq. (2)) can be accurately approximated by a cubic potential \( U(\phi_J) \) parameterized by a barrier height \( \Delta U(I_b) = (2\sqrt{2}I_b\Phi_0/3\pi)[1 - (I_b/I_0)]^{3/2} \) and a quadratic curvature at the bottom of the well that gives a classical oscillation frequency \( \omega_p(I_b) = 2^{1/4}(2\pi I_0/\Phi_0 C)^{1/2}[1 - (I_b/I_0)]^{1/4} \) with \( \omega_{10} \approx 0.95 \omega_p \) as the energy difference between the ground state and the first excited state of the phase qubit. The states can be fully manipulated with low- and microwave frequency control currents, which can be chosen as \( I_b = I_{dc} + I_{muw}(t) \), so the phase qubit can be expressed as
\[ H_q = \frac{c}{2} \sigma_x - \frac{\Delta}{2} \sigma_z, \tag{5} \]
where \( \sigma_x, \sigma_z \) are Pauli matrices, \( \epsilon = \frac{\hbar}{2\omega_{10}c} I_{muw}(t) \) and \( \Delta = \hbar \omega_{10} \). In the system, the charge operator can be described as \( \hat{Q} = \sqrt{\frac{C_0}{2}} \hat{\sigma}_y \).

The corresponding Euler-Lagrange equation for the transmission line is a wave equation
\[ \frac{1}{c} \frac{\partial^2 \theta(x,t)}{\partial x^2} - \frac{\partial \theta(x,t)}{\partial x} \delta(x) \left[ \frac{C_0}{2} \right] + \frac{l}{2} \frac{\partial^2 \theta(x,t)}{\partial t^2} = 0 \]
with the mode speed \( v = 1/\sqrt{cL} \). This is analogous to the problem of the Schrödinger equation with a delta function potential. By separation of variables, the spatial part of the solution of the modes is of the form
\[ \hat{\phi}_m(t) + \omega_m \phi_m(t) = 0. \]
In the limit \( C_0 \ll c \), the spatial mode can be expressed as \( k \approx \frac{\pi m}{L} \), where \( m \) is odd for symmetric modes and is even for antisymmetric modes. The eigenfrequencies of the modes are \( \omega = kv \). The time dependent part for transmission line still has the form of Euler-Lagrange equation
\[ \hat{\phi}_m(t) + \omega_m \phi_m(t) = 0. \]

From the above equation we can obtain the time dependent Hamiltonian for multi-mode transmission line
\[ H_T(t) = \sum m \frac{l}{2} \phi_m^2(t) + \frac{1}{2c} \left( \frac{m\pi}{2L} \right)^2 \phi_m^2 \tag{7} \]
as a function of \( \phi_m(t) \) and its canonically conjugate momentum \( p_m = i\dot{\phi}_m(t) \). The Hamiltonian \( H_T(t) \) can be diagonalized by the bosonic creation \( a^\dagger \) and annihilation operators \( a \) \( \{ a_m, a^\dagger_m \} = \delta_{mm'}. \) Then the collective charge variable in the semi-infinite transmission line is \( \hat{\theta}(x,t) = \sum n=1 \sqrt{\frac{\hbar \omega_n c}{2L}} \left[ \phi_n(t) + \hat{a}_n(t) \right] \) \( \{ \cos \frac{m\pi}{2L} x \} \) (m odd), \( \{ \sin \frac{m\pi}{2L} x \} \) (m even).

The voltage at the first end of the transmission line \( x = 0 \) is
\[ \hat{V}(0,t) = \frac{1}{c} \frac{\partial \hat{\theta}(0,t)}{\partial x} = \sum_{n=1}^N \sqrt{\frac{\hbar \omega_n}{Lc}} \left[ \phi_n(t) + \hat{a}_n(t) \right], \tag{9} \]
where \( n = m/2 = 1, 2, \ldots, n_c. \) Here \( n \) is even for the maximal voltage. The voltage is zero for odd \( m \) and thus this part can be neglected. Substituting Eq. (9) into \( H_{int} \) in Eq. (4), we obtain the interaction Hamiltonian between the phase qubit and the transmission line given by
\[ H_{int} = C_0 \sum_{n=1}^{N_c} \sqrt{\frac{C_0}{2\lambda}} \sum_{n=1}^{N_c} \sqrt{\frac{\hbar \omega_n}{Lc}} (a_n + a_n^\dagger). \tag{10} \]
After rotating the frame of the phase qubit ($\sigma_x \rightarrow \sigma_x$, $\sigma_y \rightarrow \sigma_z$), the Hamiltonian of the circuit without microwave frequency control current ($\epsilon = 0$) can be written as the standard spin-boson model,

$$H = -\frac{\Delta}{2} \sigma_x + \hbar \omega_n a_n^\dagger a_n + \frac{\sigma_z}{2} \sum_n \lambda_n (a_n^\dagger + a_n), \quad (11)$$

where $\lambda_n = C_0 \sqrt{\frac{2 \hbar \omega_c}{C_L C_n}}$ are the coupling strengths between the spin and the modes of transmission line, and the frequencies of modes $\omega_n = \frac{n \pi}{L \sqrt{C}}$. It is notable that the coupling strengths can be controlled by the bias current $I_b$ (or the coupling capacitance $C_0$ if it can be modified in the experiments), so the parameters in the spin-boson model realized in this superconducting circuits are tunable and thus the system is suitable to be used to observe rich phenomena in the spin-boson model.

The spectra is gapless when the length of the transmission line is sufficient large. In this case, the bosonic bath can be characterized by its spectral function \[ J(\omega) = \pi \sum_n \lambda_n^2 \delta(\omega_n - \omega) = 2 \pi \alpha \omega \omega_c^{1-s}. \quad (12) \]

In the proposed experimental setup, the obtained spectral function is actually Ohmic case $s = 1$. In addition, the dimensionless parameter $\alpha = \frac{\Delta C_c^2}{C_L \pi \hbar \omega_c} \sqrt{\frac{C_c}{C_L}} \propto C_0 \frac{\hbar \omega_c}{L} \sqrt{t/c}$ measures the strength of the dissipation, which is determined by the coupling strengths $\lambda_n$ and may be modified with the bias current $I_b$ and the coupling capacitance $C_0$.

We now turn to address a method to observe one of the most interesting phenomena in the spin-boson model: the KT phase transition. In the dissipation case, the spin-boson model has a delocalized and a localized zero temperature phases separated by the KT transition \[ [3] \] at the critical value $\alpha_c \approx 1$ (for the unbiased case of $\epsilon = 0$). In the delocalized phase at small dissipation strength $\alpha$, the ground state is nondegenerate and represents a damped tunneling particle. For large $\alpha$, the dissipation leads to a localization of the particle in one of the two $\sigma_z$ eigenstates, thus the ground state is doubly degenerate.

In the proposed experimental setup, the strength of the dissipation is proportional to the controllable parameter as $\alpha \propto \Delta/2\pi$, which can be changed by tuning the bias current $I_b$. The parameter $\alpha$ can be experimentally modified in the regime $0.2 \sim 3.0$ if one takes the following typical data: the junction capacitance $C_j = 0.85 \text{pF}$, $C_0 = 5 C_j$, the impedance of transmission line $z = \sqrt{t/c} = 50 \Omega$, and $\Delta \simeq \hbar \omega_c / 2$. Therefore, the localized and delocalized states as well as the KT phase transition may be observed by tuning the parameter $\Delta/2$ through the biased current $I_b$.

The different phases and the KT phase transition can be demonstrated through measuring the populations of the qubit states ($\sigma_z$), which can be detected by a three-Josephson-junction superconducting quantum interference device (SQUID). The result of measurement can be defined as $\delta P = \frac{1}{2} |P_e - P_g|$, where $P_{e(g)}$ denotes the population of excited (ground) state of phase qubit. $\delta P = 0$ means that the system stands at the localized phase, while $\delta P = 0.5$ stands at the localized phase. The $\delta P$ as a function of the strength $\alpha$, calculated with numerical renormalization group method, is plotted in Fig. 2. The readout technique of phase qubit \[ [20] \] is achieved by applying a short bias current pulse $\delta I(t)$ (less than 5 ns) that adiabatically reduces the well depth $\Delta U / \hbar \omega_c$, so that the first excited state lies very near the top of the well when the current pulse is at its maximum value. It has been shown that only the expectation of $\sigma_z$ in the phase qubit can be measured; however, the other direction measurements can be achieved by applying a transformation that maps the detected eigenvector onto the eigenvector of $\sigma_z$ before detecting. The ratio of the tunneling rates for the two states $|1\rangle$ and $|0\rangle$ is about 200, so the fidelity of measurement of the phase qubit is about 96\% when properly biased.

Let us briefly introduce the critical behaviors of the model studied by NRG, which is an efficient way to treat the spin-boson model with a broad and continuous spectrum of energies \[ [5, 6] \]. The NRG method starts with a logarithmic discretization of the bosonic bath in intervals $[\Lambda^{-(n+1)} \omega_c, \Lambda^{-n} \omega_c] (n = 0, 1, 2, \ldots)$, where $\Lambda > 1$ is called as the NRG discretization parameter. We take the surface plasma frequency as the cutoff energy ($\omega_c = 10^{14} \text{Hz}$) and energy unit. After a sequence of transformations, the discretized model is mapped onto a semi-infinite chain with the spin representing the first site of the chain. The spin-boson model in the semi-infinite chain form \[ [6] \] is diagonalized iteratively, starting from the spin site and successively adding degrees of freedom to the chain. The exponentially growing Hilbert space in the iterative process is truncated by keeping a certain fraction of the lowest-lying many particle states. Due to the logarithmic discretization, the hopping parameters between neighboring sites fall off exponentially, going along the chain corresponds to accessing decreasing energy scales in the calculation. From the energy of the first excited state, the properties of the total system can be presented according to the parameters, i.e., $\Delta, \alpha, \epsilon$. It should be mentioned that, when the controlling parameters are close to the critical val-
The critical value separate two different phases, which can be observed directly by measuring the populations of the qubit states.

Before ending the paper, we make two additional remarks. If the coupling capacitance $C_0$ can be tuned, we can also detect the quantum KT phase transition through modifying $C_0$ but fixing $\Delta$. On the other hand, we just discussed the Ohmic spectra in the paper since the resistances of the transmission line are neglected; however, the sub-Ohmic spin-boson model can be designed if a RC-dominate transmission line is considered. In this case, the critical properties of the sub-Ohmic spin-boson model can be observed in a slightly expanded model.

In summary, we have presented a scheme to realize the spin-boson model by using a phase qubit coupled to a semi-infinite transmission line. By tuning the bias current or the coupling capacitance, the notable KT phase transition from the delocalized phase to the localized phase can be directly observed through measuring the states of the phase qubit. We have also estimated the required experimental parameters by using numerical renormalization group.

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