An algebra of power series arising in the intersection theory of moduli spaces of curves and in the enumeration of ramified coverings of the sphere

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Abstract

A bracket is a function that assigns a number to each monomial in variables $\tau_0, \tau_1, \ldots$. We show that any bracket satisfying the string and the dilaton relations gives rise to a power series lying in the algebra $\mathcal{A}$ generated by the series $\sum n^{-1} q^n / n!$ and $\sum n^n q^n / n!$.

As a consequence, various series from $\mathcal{A}$ appear in the intersection theory of moduli spaces of curves.

A connection between the counting of ramified coverings of the sphere and the intersection theory on moduli spaces allows us to prove that some natural generating functions enumerating the ramified coverings lie, yet again, in $\mathcal{A}$. As an application, one can find the asymptotic of the number of such coverings as the number of sheets tends to $\infty$.

We believe that the leading terms of the asymptotics like that correspond to observables in 2-dimensional gravity.

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1
1 Introduction

Denote by $A$ the subalgebra of the algebra of power series in one variable, generated by the series

$$
\sum_{n \geq 1} \frac{n^{n-1}}{n!} q^n \quad \text{and} \quad \sum_{n \geq 1} \frac{n^n}{n!} q^n.
$$

We wish to show that this algebra plays an important role in the intersection theory of moduli spaces $\overline{M}_{g,n}$ of stable curves and in the problem of enumeration of ramified coverings of the sphere.

SECTION 2 contains a more explicit description of the algebra $A$. In this section we also prove some relations between $A$ and the combinatorics of Cayley trees (trees with numbered vertices).
SECTION 3 is devoted to the intersection theory on moduli spaces.

Notation 1.1 We denote by $\mathcal{M}_{g,n}$ the moduli space of smooth genus $g$ curves with $n$ marked and numbered distinct points.

Further, $\overline{\mathcal{M}}_{g,n}$ is the Deligne-Mumford compactification of this moduli space; in other words, $\overline{\mathcal{M}}_{g,n}$ is the space of stable genus $g$ curves with $n$ marked points.

We also use the standard notation $L_i$ for the following line bundle over $\overline{\mathcal{M}}_{g,n}$: consider a point $x \in \overline{\mathcal{M}}_{g,n}$ and the corresponding stable curve $C_x$; then the fiber of $L_i$ over $x$ is the cotangent line to $C_x$ at the $i$th marked point. These line bundles are called tautological.

We will need the first Chern classes $c_1(L_i)$ of the above line bundles, more precisely the expression

$$\frac{1}{1 - c_1(L_i)} = 1 + c_1(L_i) + c_1(L_i)^2 + \ldots \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Let $\beta \in H^*(\overline{\mathcal{M}}_{g,0}, \mathbb{Q})$ be any cohomology class of $\overline{\mathcal{M}}_{g,0}$. For any $n \geq 0$, there is a forgetful map from $\overline{\mathcal{M}}_{g,n}$ onto $\overline{\mathcal{M}}_{g,0}$, forgetting the marked points and contracting the components of the curve that have become unstable. By abuse of notation, the pull-back of $\beta$ to $\overline{\mathcal{M}}_{g,n}$ by this map will be again denoted by $\beta$.

**Theorem 1** For any $g \geq 2$ and $\beta \in H^*(\overline{\mathcal{M}}_{g,0}, \mathbb{Q})$, the power series

$$F_{g,\beta}(q) = \sum_{n \geq 0} \frac{q^n}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\beta}{(1 - c_1(L_1)) \ldots (1 - c_1(L_n))}$$

lies in the algebra $\mathcal{A}$.

In Section 3.2 we describe what happens for genus 0 and 1, when the moduli space $\overline{\mathcal{M}}_{g,0}$ does not exist.

Later we will give an important generalization of this theorem (Theorem 6).

To prove Theorem 1 we first establish the analogs of the “string equation” and the “dilaton equation” for the integrals involved in the right-hand part. Then we interpret these equations as operations on graphs with numbered vertices and use combinatorial results on these graphs to prove the theorem. It is important to note that the second part of the proof relies only on the string and dilaton relations.
SECTION 4 is devoted to the problem of enumerating the ramified coverings of the sphere with specified ramification types.

Consider a nonconstant holomorphic map \( f : C \to \mathbb{CP}^1 \) of degree \( n \) from a smooth complex curve \( C \) to the Riemann sphere. Such maps will be called ramified coverings with \( n \) sheets.

A ramification point of \( f \) is a point of the target Riemann sphere that has less than \( n \) distinct preimages.

For each ramification point \( y \) of a ramified covering, we are going to single out several simple preimages of \( y \).

Definition 1.2 A marked ramified covering is a ramified covering with a choice, for every ramification point \( y \), of a subset of the set of simple preimages of \( y \).

Consider a partition \( \mu = 1^{a_1}2^{a_2}\ldots \) of an integer \( m \leq n \). Here we use multiplicative notation for partitions: the partition \( \mu \) contains \( a_1 \) parts equal to 1, \( a_2 \) parts equal to 2, and so on, \( \sum i a_i = m \).

Suppose that a point \( y \in \mathbb{CP}^1 \) has \( a_1 \) marked simple preimages, \( a_2 \) double preimages, and so on. (Consequently, \( y \) also has \( n - m \) unmarked simple preimages.) We then say that \( y \) is a ramification point of \( f \) of multiplicity \( r = a_2 + 2a_3 + 3a_4 + \ldots \) and of ramification type \( \mu = 1^{a_1}2^{a_2}\ldots \). Sometimes the number \( r \) will also be called the degeneracy of the partition \( \mu \).

Definition 1.3 A Hurwitz number \( h_{n;\mu_1,\ldots,\mu_k} \) is the number of connected \( n \)-sheeted marked ramified coverings of \( \mathbb{CP}^1 \) with \( k \) ramification points, whose ramification types are \( \mu_1,\ldots,\mu_k \). Every such covering is counted with weight \( 1/|\text{Aut}| \), where \( |\text{Aut}| \) is the number of automorphisms of the covering.

Note that the genus \( g \) of the covering surface can be reconstituted from the data \( n;\mu_1,\ldots,\mu_k \) using the Riemann-Hurwitz formula: if the degeneracy of \( \mu_i \) equals \( r_i \), then

\[
2 - 2g = 2n - r_1 - \ldots - r_k.
\]

Fix \( k \) nonempty partitions \( \mu_1,\ldots,\mu_k \) with degeneracies \( r_1,\ldots,r_k \). Let \( r \) be the sum \( r = r_1 + \ldots + r_k \).

Notation 1.4 Denote by \( h_{g,n;\mu_1,\ldots,\mu_k} \) the number of \( n \)-sheeted marked ramified coverings of \( \mathbb{CP}^1 \) by a genus \( g \) surface, with \( k \) ramification points of
types $\mu_1, \ldots, \mu_k$ and, in addition, $2n + 2g - 2 - r$ simple (= of multiplicity 1) ramification points. Each covering is counted with weight $1/|\text{Aut}|$.

**Theorem 2** Fix any $g \geq 0$, $k \geq 0$. If $g = 1$, we suppose that $k \geq 1$. Then for any partitions $\mu_1, \ldots, \mu_k$, the series

$$H_{g,\mu_1,\ldots,\mu_k}(q) = \sum_{n \geq 1} h_{g,n;\mu_1,\ldots,\mu_k} \frac{n!}{q^n}$$

lies in the algebra $\mathcal{A}$.

This theorem is proved by induction on the number $k$ of nonsimple ramification points. Curiously, the base of induction (the case $k = 1$) is the most difficult part of the proof. We prove it using a generalization of Theorem 1 and the Ekedahl-Lando-Shapiro-Vainshtein (or “ELSV”) formula relating Hurwitz numbers to integrals over moduli spaces of stable curves.

Among other things, this theorem allows one to find the asymptotic of the coefficients of $H_{g,\mu_1,\ldots,\mu_k}$ as $n \to \infty$, knowing only the several first coefficients.

**SECTION 5** describes a relation between the asymptotic of Hurwitz numbers and 2-dimensional gravity.

We also compare the enumerative problems concerning ramified coverings of the sphere and of the torus. While the former are related to the intersection theory on $\overline{M}_{g,n}$ and give rise to the algebra $\mathcal{A}$, the latter are related with volumes of spaces of abelian differentials on Riemann surfaces and give rise to the algebra of quasi-modular forms.

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**List of notations.** Here we summarize some notation that we use consistently throughout the paper. We have tried to avoid using the same letter in different contexts unless it emphasizes a relation between the two contexts that appears in some of the proofs.

- \( n \): The number of vertices in a graph. The number of sheets of a covering. The power of the variable \( q \) in generating series. The number of marked points on a Riemann surface is sometimes \( n \) and sometimes \( n - r \).
- \( q \): The variable in generating series (to a sequence \( s_n \) we usually assign the series \( \sum s_n q^n / n! \)).
- \( g \): The genus of a Riemann surface.
- \( \mu \): A partition.
- \( p \): The number of parts of a partition \( \mu \).
- \( a_i \): The number of parts of a partition \( \mu \) that are equal to \( i \).
- \( b_i \): The parts of a partition \( \mu \) are denoted by \( b_1, \ldots, b_p \).
- \( r \): The degeneracy of a partition \( \mu \) defined by \( r = \sum (b_i - 1) \). The multiplicity of a ramification point.
- \( k \): The number of partitions. If \( k > 1 \), the partitions are denoted by \( \mu_1, \ldots, \mu_k \), their degeneracies by \( r_1, \ldots, r_k \), while \( r \) is the total degeneracy \( r = \sum r_i \).
- \( c(n) \): The number of simple ramification points in a ramified covering.
- \( \beta \): A cohomology class on the moduli space \( \overline{M}_{g,p} \).
- \( b \): We often take \( \beta \) to be a cohomology class of pure degree 2b.
- \( \psi_i \): The first Chern class \( c_1(L_i) \) of the line bundle \( L_i \).
- \( d_i \): The power of the class \( \psi_i \) in the intersection numbers we consider. We also consider graphs whose \( i \)th vertex has valency \( d_i + 1 \) for each \( i \).

**2 The algebra \( \mathcal{A} \) of power series**

The algebra of power series

\[
\mathcal{A} = \mathbb{Q} \left[ \sum_{n \geq 1} \frac{n^{n-1}}{n!} q^n, \sum_{n \geq 1} \frac{n^n}{n!} q^n \right]
\]

plays a central role in this paper. Here we give an explicit description of \( \mathcal{A} \) and show its relation with the combinatorics of Cayley trees. Many of
the results below are known, but have probably never been put together. As far as we know, the algebra $\mathcal{A}$ itself was first discovered by D. Zagier several years ago (unpublished), and then independently introduced in our paper [13], where most of the results of Section 2.1 are given.

## 2.1 How to make computations in $\mathcal{A}$

Denote by $Y$ and $Z$ the generators of $\mathcal{A}$

$$Y = \sum_{n \geq 1} \frac{n^{n-1}}{n!} q^n,$$

$$Z = \sum_{n \geq 1} \frac{n^n}{n!} q^n.$$

Denote by $D$ the differential operator $D = q \frac{\partial}{\partial q}$. Thus $Z = DY$.

Note that both $Y$ and $Z$ have a radius of convergence of $1/e$. Therefore the same is true of all series in $\mathcal{A}$. The function $Y(q)$, more precisely, $-Y(-q)$, was considered by J. H. Lambert in 1758 (we thank N. A’Campo for this reference).

**Proposition 2.1** We have

$$Y = q e^Y.$$

**Proof.** $Y$ is the exponential generating series for rooted Cayley trees (Definition 2.9). Therefore $e^Y$ is the exponential generating series for forests of rooted Cayley trees. Add a new vertex $*$ to such a forest and join $*$ to the root of each tree. We obtain a Cayley tree with root $*$. This operation is a one-to-one correspondence, hence $Y = q e^Y$.  

**Corollary 2.2** On the disc $|q| < 1/e$, the function $Y(q)$ is the inverse of the function $q(Y) = Y/e^Y$.

**Proposition 2.3** We have $(1 - Y)(1 + Z) = 1$.

**Proof.**

$$Z = DY = D(qe^Y) = qe^Y + qe^Y DY = qe^Y (1 + Z) = Y(1 + Z).$$

Hence $(1 - Y)(1 + Z) = 1$.  

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Corollary 2.4 As an abstract algebra, \( A \) is isomorphic to \( \mathbb{Q}[X, X^{-1}] \), where \( X = 1 - Y \).

Now we can show how to make computations in \( A \). For instance, let us prove that

\[
Y^2 = \sum_{n \geq 1} \frac{2(n - 1)n^{n-2}}{n!} q^n = 2 \sum_{n \geq 2} \frac{n^{n-3}}{(n-2)!} q^n.
\]

Indeed, \( D(Y^2) = 2YDY = 2YZ = 2(Z - Y) \), which determines \( Y^2 \) up to its free term, but the free term obviously vanishes.

More generally, the powers of \( Y \) are given by the following proposition.

**Proposition 2.5** We have

\[
Y^k = \sum_{n \geq 1} \frac{k(n - 1)\ldots(n - k + 1)n^{n-k}}{n!} q^n = k \sum_{n \geq k} \frac{n^{n-k-1}}{(n-k)!} q^n.
\]

**Proof.** Induction on \( k \). For \( k = 1 \) the assertion is true. To go from \( k \) to \( k + 1 \), one checks that the equality

\[
D \left( \frac{Y^{k+1}}{k+1} - \frac{Y^k}{k} \right) = (Y^k - Y^{k-1})DY = Y^{k-1}(Y - 1)Z = -Y^k
\]

is compatible with our expressions for \( Y^k \) and \( Y^{k+1} \). The constant term of \( Y^{k+1} \) vanishes, thus \( Y^{k+1} \) is uniquely determined by \( D(Y^{k+1}) \).

Now we study the powers of \( Z \).

**Definition 2.6** Denote by \( A_n \) the sequence of integers

\[
A_n = \sum_{p+q=n, p,q \geq 1} \frac{n!}{p! q!} p^p q^q.
\]

Its first terms are 0, 2, 24, 312, 4720, .... We have

\[
Z^2 = \sum_{n \geq 1} \frac{A_n}{n!} q^n.
\]
One can show that
\[ A_n = n! \sum_{k=0}^{n-2} \frac{n^k}{k!} \sim \sqrt{\pi/2} \, n^{n+\frac{1}{2}}. \]

As far as we know, there is no simple expression for the powers of \( Z \). However, we can prove that they are linear combinations of the series
\[ D^k Z = \sum_{n \geq 1} \frac{n^{n+k}}{n!} q^n \]
and
\[ D^k (Z^2) = \sum_{n \geq 1} \frac{n^k A_n}{n!} q^n. \]

**Proposition 2.7** For any integer \( k \geq 0 \), the power series \( D^k Z \) and \( D^k (Z^2) \) are polynomials in \( Z \) with positive integer coefficients, of degrees \( 2k + 1 \) and \( 2k + 2 \) respectively.

**Proof.** Applying \( D \) to both sides of the equality \( (1 - Y)(1 + Z) = 1 \) we get
\[ -Z(1 + Z) + (1 - Y) \cdot DZ = 0. \]
Thus
\[ DZ = \frac{Z(1 + Z)}{1 - Y} = Z(1 + Z)^2. \]
Hence
\[ D(Z^2) = 2Z^2(1 + Z)^2. \]
Now we proceed by induction on \( k \). \( \diamond \)

**Corollary 2.8** For any positive integer \( k \), the power series \( Z^k \) is a linear combination with integer coefficients of the first \( k \) series from the list \( Z, Z^2, DZ, D(Z^2), D^2 Z, D^2(Z^2), \ldots \)

From Proposition 2.7 and Corollary 2.8 we deduce the following theorem.

**Theorem 3** [15] The algebra \( A \) is spanned over \( \mathbb{Q} \) by the power series
\[ 1, \quad \sum_{n \geq 1} \frac{n^{n+k}}{n!} q^n, \quad k \in \mathbb{Z}, \quad \sum_{n \geq 1} \frac{n^k A_n}{n!} q^n, \quad k \in \mathbb{N}. \]
Note that the Stirling formula together with the asymptotic for the sequence $A_n$ allows one to determine the leading term of the asymptotic for the coefficients of any series in $A$. We have

$$\frac{n^n}{n!} \sim \frac{1}{\sqrt{2\pi n}} e^n, \quad \frac{A_n}{n!} \sim \frac{1}{2} e^n.$$

Note also that if, for some series $F \in A$, we know in advance its degree in $Y$ and in $Z$, then we can reconstitute the series $F$ using only a finite number of its initial terms.

Combining both remarks, we see that initial terms of the sequence of coefficients of $F$ determine the asymptotic of the sequence!

### 2.2 Dendrology

**Definition 2.9** A *Cayley tree* is a tree with numbered vertices.

It is well-known (Cayley theorem) that there are $n^{n-2}$ Cayley trees with $n$ vertices. Note that the corresponding exponential generating function

$$\sum_{n \geq 1} \frac{n^{n-2}}{n!} q^n$$

lies in the algebra $A$.

Consider a Cayley tree $T$ with two marked vertices $a$ and $b$. Denote by $l(T)$ the distance between these vertices, i.e., the number of edges in the shortest path joining them.

**Definition 2.10** Denote by $m_{n,k}$ and $p_{n,k}$ the sums

$$m_{n,k} = \sum_T l(T)^k, \quad p_{n,k} = \sum_T \frac{l(T)(l(T) - 1) \ldots (l(T) - k + 1)}{k!}$$

where the sum is taken over all Cayley trees $T$ with $n$ vertices, two of which are marked.

For instance, $m_{2,1} = p_{2,1} = 2$. Note that if we consider $l(T)$ as a random variable, then $m_{n,k}$ is its $k$th moment.
Theorem 4  For any \(k\), the power series

\[
\sum_{n \geq 1} \frac{m_{n,k}}{n!} q^n \quad \text{and} \quad \sum_{n \geq 1} \frac{p_{n,k}}{n!} q^n
\]

lie in \(A\).

Example 2.11  It follows from the proof below that \(p_{n,1} = m_{n,1} = A_n\). This number is called the total height of Cayley trees and was introduced in [12].

Proof of Theorem 4  It is sufficient to prove the theorem for \(p_{n,k}\).

Fix \(k\). There is a natural bijection between the following sets of objects.

\(E_n\) is the set of Cayley trees with \(n\) vertices, on which one has labeled two vertices by \(a\) and \(b\) and chosen \(k\) distinct edges on the shortest path from \(a\) to \(b\). The number of elements of \(E_n\) equals \(p_{n,k}\).

\(F_n\) is the set of ordered \((k+1)\)-tuples of trees with \(n\) vertices in whole; the vertices are numbered from 1 to \(n\) and, in addition, two vertices \(a_i\) and \(b_i\), \(1 \leq i \leq k+1\), are marked on each tree.

The bijection is established in the following way: taking a forest from the set \(F_n\) we draw new edges joining the vertices \(b_1\) to \(a_2\), then \(b_2\) to \(a_3\), \ldots, and finally \(b_k\) to \(a_{k+1}\). We obtain a tree with \(k\) marked edges lying on the path between \(a_1\) and \(b_{k+1}\), i.e., a tree from the set \(E_n\).

Now, the trees with two marked vertices are enumerated by the series \(Z\), therefore the exponential generating series for the sequence \(|F_n|\) is \(Z^{k+1}\). “

3  Intersection theory on \(\overline{M}_{g,n}\)

3.1  The bracket \(\langle \beta \tau_{d_1} \ldots \tau_{d_n} \rangle\)

Recall that \(\overline{M}_{g,n}\) is the moduli space of stable curves and \(\mathcal{L}_i\) the tautological line bundles (Notation 1.1). Further, \(\psi_i = c_1(\mathcal{L}_i)\). Let \(g \geq 2\) and denote by \(\beta \in H^*(\overline{M}_{g,0}, \mathbb{Q})\) a cohomology class of \(\overline{M}_{g,0}\) as well as its pull-backs to the spaces \(\overline{M}_{g,n}\), \(n \geq 0\), under the projections forgetting the marked points.

Notation 3.1  For \(n \geq 0\), we denote by \(\langle \beta \tau_{d_1} \ldots \tau_{d_n} \rangle\) the integral

\[
\langle \beta \tau_{d_1} \ldots \tau_{d_n} \rangle = \int_{\overline{M}_{g,n}} \beta \, \psi_1^{d_1} \ldots \psi_n^{d_n}.
\]
Remark 3.2 If the cohomology class $\beta$ has pure degree $2b$, then the bracket vanishes unless $b+\sum d_i = 3g-3+n$. If $\beta$ is odd, the bracket always vanishes.

In the case $\beta = 1$, our notation is compatible with the standard notation for intersection numbers of the first Chern classes of the bundles $L_i$ (see [14]).

The power series from Theorem 1, which is one of the main objects of our study, can be written in the form

$$F_{g,\beta}(q) = \sum_{n\geq 0} q^n \frac{\beta}{n!} \int_{\mathcal{M}_{g,n}} \frac{1}{(1-\psi_1)\cdots(1-\psi_n)} = \sum_{n\geq 0} q^n \sum_{d_1,...,d_n} \langle \beta \tau_{d_1} \cdots \tau_{d_n} \rangle .$$

3.2 Simplest cases: $g = 0$ and $g = 1$

Most of the theorems concerning the bracket $\langle \beta \ldots \rangle$ have three exceptional cases: $g = 0$, deg $\beta = 0$ and $g = 1$, deg $\beta = 0$ or 2. We will describe these cases here, which will allow us not to bother with them later and, at the same time, to give the simplest examples.

Strictly speaking, these examples are not totally compatible with our definitions, because there is no space $\mathcal{M}_{g,0}$ for $g = 0,1$. However, in some sense, the cohomology classes below could be seen as pull-backs from these non-existent spaces. At least, the bracket itself makes perfect sense.

The case $g = 0$, $\beta = 1$. In this case, we have

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g=0} = \frac{(n-3)!}{d_1! \cdots d_n!}$$

if $\sum d_i = n - 3$; otherwise the bracket vanishes. Using the explicit values of the bracket one gets

$$F_{g=0,\beta=1}(q) = \sum_{n\geq 3} q^n \frac{1}{n!} \sum_{d_1,...,d_n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g=0} = \sum_{n\geq 3} \frac{n^{n-3}}{n!} q^n .$$

The most logical thing to do is to add to this series two terms, $q$ and $q^2/4$, that would correspond to two non-existent moduli spaces $\mathcal{M}_{0,1}$ and $\mathcal{M}_{0,2}$. Thus we obtain the series

$$\sum_{n\geq 1} \frac{n^{n-3}}{n!} q^n \in A .$$
The case $g = 1$, $\deg \beta = 0$. In this case, we have the following formula for the values of the bracket. Denote by $\sigma_j$ the $j$th elementary symmetric function of $d_1, \ldots, d_n$. In other words,

$$(d - d_1) \cdots (d - d_n) = d^n - \sigma_1 d^{n-1} + \ldots + (-1)^n \sigma_n.$$

Then we have

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g = 1} = \frac{1}{24 d_1! \cdots d_n!} \left( n! - \sum_{j=2}^{n} (j - 2)! (n - j)! \sigma_j \right)$$

if $\sum d_i = n$; otherwise the bracket vanishes. From this formula (after a certain amount of calculations), one can deduce

$$\sum_{d_1, \ldots, d_n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g = 1} = \frac{1}{24} \left( \sum_{n \geq 1} A_n + n^{n-1} \right),$$

with the sequence $A_n$ from Definition 2.6. Thus

$$F_{g = 1, \beta = 1}(q) = \sum_{n \geq 1} \frac{q^n}{n!} \sum_{d_1, \ldots, d_n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g = 1} = \frac{1}{24} \sum_{n \geq 1} \left( \frac{A_n + n^{n-1}}{n!} \right) q^n.$$

The series $F_{g = 1, \beta = 1}$ does not lie in the algebra $A$, but the series

$$DF_{g = 1, \beta = 1}(q) = \frac{1}{24} \sum_{n \geq 1} \frac{A_n + n^n}{n!} q^n$$

does. A generalization of this fact is given in Theorem 6.

The case $g = 1$, $\deg \beta = 2$. In this paragraph we denote by $\beta$ the unique 2-cohomology class of $\overline{M}_{1,1}$ whose integral over the fundamental class of $\overline{M}_{1,1}$ is equal to 1. The other 2-cohomology classes are proportional to $\beta$, because the vector space $H^2(\overline{M}_{1,1}, \mathbb{Q})$ is 1-dimensional.

We have

$$\langle \beta \tau_{d_1} \cdots \tau_{d_n} \rangle = \frac{(n - 1)!}{d_1! \cdots d_n!}$$

if $\sum d_i = n - 1$; otherwise the bracket vanishes. The corresponding generating series equals

$$F_{g = 1, \beta}(q) = \sum_{n \geq 1} \frac{q^n}{n!} \sum_{d_1, \ldots, d_n} \langle \beta \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{n \geq 1} \frac{n^{n-1}}{n!} q^n = Y(q).$$

This series, of course, lies in $A$. 

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3.3 String and dilaton relations

The following proposition is well-known in the case $\beta = 1$ (see, for example, [14]). The proof in the general case is literally the same as for $\beta = 1$, but we still give it here for completeness.

In the right-hand side of the string relation below, we set, by convention, a bracket containing $\tau_{-1}$ to be 0.

**Proposition 3.3** For $n \geq 0$, we have

$$\langle \beta \tau_{d_1} \cdots \tau_{d_n} \tau_0 \rangle = \sum_{i=1}^{n} \langle \beta \tau_{d_1} \cdots \tau_{d_i-1} \cdots \tau_{d_n} \rangle \quad \text{(string relation)}$$

and

$$\langle \beta \tau_{d_1} \cdots \tau_{d_n} \tau_1 \rangle = (2g - 2 + n) \langle \beta \tau_{d_1} \cdots \tau_{d_n} \rangle \quad \text{(dilaton relation)}.$$

**Proof.** Let $D_{i,n+1} \subset \overline{M}_{g,n+1}$ be the divisor consisting of stable curves $C$ with the following property: one of the irreducible components of $C$ is a sphere containing the marked points $i$ and $n+1$ (and no other marked points) and exactly one node (Figure 1). We denote by $\Delta_i$ the 2-cohomology class Poincaré dual to $D_{i,n+1}$.

![Figure 1: A generic curve of the divisor $D_{i,n+1}$.

We will need to consider the line bundles $\mathcal{L}_i$ both on $\overline{M}_{g,n}$ and on $\overline{M}_{g,n+1}$. We momentarily denote the former by $\mathcal{L}'_i$, $1 \leq i \leq n$, the latter retaining the notation $\mathcal{L}_i$, $1 \leq i \leq n+1$. Further, denote by $\psi'_i$ the first Chern class of the line bundle $\mathcal{L}'_i$ on $\overline{M}_{g,n}$ and, by abuse of notation, its pull-back to $\overline{M}_{g,n+1}$ by the map forgetting the $(n+1)$st marked point. Let $\psi_i$ be the first Chern class of $\mathcal{L}_i$ on $\overline{M}_{g,n+1}$.

The forgetful map $\overline{M}_{g,n+1} \to \overline{M}_{g,n}$ identifies the line bundles $\mathcal{L}_i$ and $\mathcal{L}'_i$ everywhere except over the divisor $D_{i,n+1}$. Using this one can check that

$$\psi_i = \psi'_i + \Delta_i. \quad (1)$$
Moreover, we have
\[
\psi_i \cdot \Delta_i = \psi_{n+1} \cdot \Delta_i = 0 \tag{2}
\]
(because the line bundles \(\mathcal{L}_i\) and \(\mathcal{L}_{n+1}\) are trivial over \(D_{i,n+1}\)) and
\[
\Delta_i \cdot \Delta_j = 0 \quad \text{for} \quad i \neq j \tag{3}
\]
(because the divisors have an empty geometric intersection).

Let us first prove the string relation. We have
\[
\psi_i^d - (\psi'_i)^d = \Delta_i \left(\psi_i^{d-1} + \ldots + (\psi'_{i})^{d-1}\right) = \Delta_i (\psi'_{i})^{d-1},
\]
where we set by convention \((\psi'_{i})^{-1} = 0\). Thus
\[
\psi_i^d = (\psi'_{i})^d + \Delta_i (\psi'_{i})^{d-1}.
\]

It follows that
\[
\int_{\overline{\mathcal{M}}_{g,n+1}} \beta \psi_1^d \ldots \psi_n^d = \int_{\overline{\mathcal{M}}_{g,n+1}} \beta \left[ (\psi'_1)^d_1 + \Delta_1 (\psi'_1)^{d_1-1} \right] \ldots \left[ (\psi'_n)^d_n + \Delta_n (\psi'_n)^{d_n-1} \right] \tag{3}
\]
\[
= \int_{\overline{\mathcal{M}}_{g,n+1}} \beta (\psi'_1)^{d_1} \ldots (\psi'_n)^{d_n} + \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n+1}} (\psi'_i)^{d_i} \ldots \Delta_i (\psi'_i)^{d_i-1} \ldots (\psi'_n)^{d_n}.
\]
The first integral is equal to 0, because the integrand is a pull-back from \(\overline{\mathcal{M}}_{g,n}\). As for the integrals composing the sum, we integrate the class \(\Delta_i\) over the fibers of the projection \(\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}\). This is equivalent to restricting the integral to the divisor \(D_{i,n+1}\), which is naturally isomorphic to \(\overline{\mathcal{M}}_{g,n}\). Finally, we obtain
\[
\int_{\overline{\mathcal{M}}_{g,n+1}} \beta \psi_1^d \ldots \psi_n^d = \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n}} \beta (\psi'_1)^{d_1} \ldots (\psi'_i)^{d_i-1} \ldots (\psi'_n)^{d_n}.
\]
This proves the string relation.

Now let us prove the dilaton relation. We have
\[
\int_{\overline{\mathcal{M}}_{g,n+1}} \beta \psi_1^d \ldots \psi_n^d (\psi_{n+1}^d) \tag{1}
\]
\[
\int_{\mathcal{M}_{g,n+1}} \beta \left( \psi_1' + \Delta_1 \right)^{d_1} \cdots \left( \psi_n' + \Delta_n \right)^{d_n} \psi_{n+1} \overset{(2)}{=} \\
\int_{\mathcal{M}_{g,n+1}} \beta \left( \psi_1' \right)^{d_1} \cdots \left( \psi_n' \right)^{d_n} \psi_{n+1} = \\
(2g - 2 + n) \int_{\mathcal{M}_{g,n}} \beta \left( \psi_1' \right)^{d_1} \cdots \left( \psi_n' \right)^{d_n}.
\]

The last equality is obtained by integrating the factor \(\psi_{n+1}\) over the fibers of the projection \(\mathcal{M}_{g,n+1} \to \mathcal{M}_{g,n}\).

This proves the dilaton relation.

\[\Diamond\]

### 3.4 Graphs with numbered vertices

We are now going to interpret the string and the dilaton relations as operations on graphs with numbered vertices. Each variable \(\tau_d\) will correspond to a vertex of valency \(d + 1\). The string relation will allow us to erase a vertex of valency 1, simultaneously decreasing by 1 the valency of its neighboring vertex. The dilaton relation will allow us to erase a vertex of valency 2, merging together the two edges that were adjacent to it. We will arrange for our graphs to have \(2g - 2 + n\) edges, which will account for the factor \(2g - 2 + n\) in the dilaton relation. We will show that using such graphs one can define all possible brackets \(\langle \beta \tau_{d_1} \cdots \tau_{d_n} \rangle\) satisfying the string and dilaton relations. Now let us give precise formulations.

The valency of a vertex in a graph is the number of edges issuing from this vertex (a loop being counted twice).

We introduce two operations (S) and (D) on finite graphs.

**Definition 3.4** The operation (S) consists in erasing a vertex of valency 1 and the edge adjacent to it. The operation (D) consists in erasing a vertex of valency 2 and merging its two adjacent edges into one edge. These operations are shown in Figure 2.

To define new brackets we will need the following type of graphs.

**Definition 3.5** We call an \(n\)-decorated graph a graph \(G\) with \(n + 1\) vertices, one of which is labeled by \(*\) and the remaining ones are numbered from 1 to \(n\). Moreover, we suppose that each connected component of \(G\) either contains
the vertex ∗ or has at least two independent cycles. Loops and multiple edges are allowed.

It is helpful to imagine that the factors β, τ_{d_1}, ..., τ_{d_n} from the bracket correspond, respectively, to the vertices ∗, 1, ..., n. The vertex ∗ will soon be required to have valency g − 1 + \frac{1}{2} \deg β. The vertex number i, for each i, will soon be required to have valency d_i + 1.

Now we are going to apply the operations (S) and (D) to decorated graphs. The vertex ∗ is special: we will erase neither ∗ itself nor the edges adjacent to it.

Let G be an n-decorated graph.

**Definition 3.6** The simplification of G is the graph H obtained from G by (i) applying as many times as possible the operations (S) and (D) without erasing the vertex ∗ or changing its valency; (ii) forgetting the numbers of the vertices.

Conversely, the graph G is called an extension of H.

A simple graph is any finite graph with one vertex labeled by ∗ and several other vertices, each of which has a valency ≥ 3.

It is easy to see that the simplification of G is well-defined and unique, in other words, it does not depend on the order in which one applies the operations (S) and (D).

Let g ≥ 2 be an integer and H a simple graph with Euler characteristic χ(H) = |vertices| − |edges| equal to 3 − 2g. For n ≥ 0, we define a bracket \langle τ_{d_1} ... τ_{d_n} \rangle_H by the following procedure.

**Definition 3.7** The value of the H-bracket \langle τ_{d_1} ... τ_{d_n} \rangle_H is equal to the weighted number of n-decorated graphs G such that: (i) the simplification of G is H; (ii) for 1 ≤ i ≤ n, the valency of the ith vertex of G equals d_i + 1. The weight of a graph G equals 1/(number of its symmetries).
Remark 3.8 A \textit{symmetry} of a decorated graph is a permutation of its half-edges that preserves the vertex of each half-edge and does not split the edges.

**Proposition 3.9** The bracket $\langle \cdot \rangle^H$ satisfies the string and dilaton relations of Proposition 3.3.

\textbf{Proof.}

String relation. Suppose $d_{n+1} = 0$. Definition 3.7 assigns to the list $d_1, \ldots, d_n, 0$ a set of $(n+1)$-decorated graphs. Consider a graph $G$ from this set. Its $(n+1)$st vertex has valency 1. First note that it cannot be joined to the vertex *. Indeed, otherwise the vertex number $n + 1$ can never be erased by the operations (S) and (D), and therefore will remain as a vertex of $H$. But the vertices of $H$ have valency at least 3. Thus the vertex number $n + 1$ is joined to some vertex number $i$. Then, applying the operation (S) to the $(n+1)$st vertex, we obtain an $n$-decorated graph $G'$ from the set assigned to the list $d_1, \ldots, d_{i-1}, d_i, \ldots, d_n$. This operation is obviously a one-to-one correspondence between the set of graphs $G$ corresponding to the list $d_1, \ldots, d_n, 0$ and the disjoint union of the sets of graphs $G'$ corresponding to the lists $d_1, \ldots, d_{i-1}, d_i, \ldots, d_n$ for $1 \leq i \leq n$. This leads immediately to the string relation for the bracket $\langle \cdot \rangle^H$.

Dilaton relation. Suppose $d_{n+1} = 1$. As above, consider an $(n+1)$-decorated graph $G$ assigned to the list $d_1, \ldots, d_n, 1$. Its $(n+1)$st vertex has valency 2. Applying operation (D) to this vertex, we obtain a graph $G'$ from the set assigned to the list $d_1, \ldots, d_n$. This time, the operation is not a bijection, but becomes one if we mark the edge of $G'$ on which the $(n+1)$st vertex of $G$ has disappeared. More precisely, we have a one-to-one correspondence between the set of graphs $G$ assigned to the list $d_1, \ldots, d_n, 1$ and the set of pairs $(G', e)$, where $G'$ is graph assigned to $d_1, \ldots, d_n$ and $e$ is its edge. Now, recall that the simple graph $H$, and therefore the graphs $G$ and $G'$ as well, have Euler characteristic $3 - 2g$. Thus each graph $G'$ has $2g - 2 + n$ edges. This leads to the dilaton relation for the bracket $\langle \cdot \rangle^H$. 

\textbf{Remark 3.10} The string and dilaton relations are \textit{linear}, i.e., if several different brackets satisfy these relations then so does their arbitrary linear combination. Therefore Proposition 3.9 immediately implies that the two relations are satisfied by any linear combination

$$s_1 \langle \cdot \rangle_{H_1} + \cdots + s_l \langle \cdot \rangle_{H_l}$$
for simple graphs \( H_1, \ldots, H_l \).

Now we are going to prove that linear combinations of brackets \( \langle \cdot \rangle_H \) for different simple graphs \( H \) represent all possible brackets of nonnegative degree (see Definition 3.12 below) that satisfy the string and the dilaton relations. In particular, for any cohomology class \( \beta \), the bracket \( \langle \beta \ldots \rangle \) can be represented in that way.

**Example 3.11** Consider the case \( g = 2, \beta = 1 \). The corresponding bracket is entirely determined by the string and dilaton relations together with the “initial conditions” (we do not explain here how these are found):

\[
\langle \tau_4 \rangle = \frac{1}{1152}, \quad \langle \tau_2 \tau_3 \rangle = \frac{29}{5760}, \quad \langle \tau_2 \tau_2 \tau_2 \rangle = \frac{7}{240}.
\]

For each of the three brackets above, we choose a simple graph that will represent it, for instance those shown in Figure 3. Now we have

\[
\langle \cdot \rangle_{g=2} = 8 \cdot \frac{1}{1152} \langle \cdot \rangle_{H_4} + 4 \cdot \frac{29}{5760} \langle \cdot \rangle_{H_{2,3}} + 4 \cdot \frac{7}{240} \langle \cdot \rangle_{H_{2,2,2}}.
\]

The factors 8, 4, and 4 are the numbers of symmetries of the graphs.

![Figure 3](image)

**Figure 3**: The 3 graphs used to represent the bracket \( \langle \cdot \rangle \) for \( g = 2, \beta = 1 \).

Note that although from the point of view of the intersection theory the case \( \beta = 1 \) is the simplest and the most natural, the graph representation of the corresponding bracket is in no way simpler than for any other bracket. Note also that the three simple graphs we have chosen can be replaced by any other graphs with the same valencies of vertices.
**Definition 3.12** Let \( \langle \cdot \rangle \) be a bracket satisfying the string and dilaton relations.

The *genus* of the bracket is the number \( g \) that appears in the dilaton relations.

We say that the bracket is *of pure degree* \( b \) if it vanishes unless \( b + \sum d_i = 3g - 3 + n = \dim \mathcal{N}_{g,n} \). The bracket is *of nonnegative degree* if it is a finite linear combination of brackets of pure nonnegative degrees, in other words, if it vanishes unless

\[
3g - 3 + n \geq \sum d_i.
\]

The bracket \( \langle \beta \ldots \rangle \) is always of nonnegative degree and is of pure degree \( b \) if \( \beta \) is a \( 2b \)-cohomology class.

**Remark 3.13** In some cases, Theorem 5 below can be generalized to brackets whose degree may be negative, but bounded from below. To do that, one must consider other types of marked graphs then the ones we introduced in Definition 3.5. However, brackets with components of negative degree do not naturally appear either in the study of intersection theory of moduli spaces or in the enumeration of ramified coverings. Therefore we do not investigate them further.

**Theorem 5** For any rational-valued bracket \( \langle \cdot \rangle \) of nonnegative degree, satisfying the string and dilaton relations there exists a set of simple graphs \( H_1, \ldots, H_l \) and rational numbers \( q_1, \ldots, q_l \) such that

\[
\langle \cdot \rangle = q_1 \langle \cdot \rangle_{H_1} + \cdots + q_l \langle \cdot \rangle_{H_l}.
\]

**Proof.** Let \( \langle \cdot \rangle \) be a genus \( g \) bracket of nonnegative degree. It follows from the inequality \( 3g - 3 + n - \sum d_i \geq 0 \) that if \( d_i \geq 2 \) for all \( i \), then \( n \leq 3g - 3 \). In other words, there is only a finite number of brackets that cannot be simplified using the string or the dilaton relation. We will call the values of these brackets the *initial values*. The initial values and the string and dilaton relations completely determine all values of the bracket.

Consider the set \( S \) of all initial values of the bracket. An element \( v \) of \( S \) is a monomial \( \tau_{d_1} \ldots \tau_{d_m} \) (with \( d_i \geq 2 \) for all \( i \)) and a rational number \( q_v = \langle \tau_{d_1} \ldots \tau_{d_m} \rangle \). For a given \( v \), we denote by \( b \) the number \( b = 3g - 3 + m - \sum d_i \) (the degree of the initial value).

To each initial value \( v \in S \) we assign an arbitrary simple graph \( H_v \) with \( m \) vertices of valencies \( d_i + 1, 1 \leq i \leq m \), and a vertex \( * \) of valency \( b + g - 1 \).
The condition of nonnegative degree $b \geq 0$, together with $g \geq 2$ insures that the valency of the vertex * is positive. It is easy to see that a simple graph with these conditions can always be constructed. Moreover, it can be chosen to be connected. A simple computation shows that a simple graph like that has $2g - 2 + m$ edges, in other words, it has Euler characteristic $3 - 2g$.

We claim that the bracket $\langle \cdot \rangle$ can be represented as

$$\langle \cdot \rangle = \sum_{v \in S} q_v |\text{Aut}(H_v)| \langle \cdot \rangle_{H_v}.$$ 

Indeed, the left-hand side bracket and the right-hand side bracket have the same initial values and satisfy the string and dilaton relations. Therefore they coincide.

Thus we have represented any given bracket $\langle \cdot \rangle$ as a linear combination of brackets assigned to simples graphs. ♦

**Proposition 3.14** Consider a simple graph $H$. Let $e$ be its number of edges and $v$ its number of vertices without counting the vertex *. Denote by $|\text{Aut}(H)|$ the number of automorphisms of the graph $H$. Then the series

$$F_H(q) = \sum_{n \geq 0} \frac{q^n}{n!} \sum_{d_1, \ldots, d_n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_H$$

equals $\frac{1}{|\text{Aut}(H)|} Y^v (1 + Z)^e$ and, as a consequence, lies in the algebra $A$.

Various facts similar to this proposition were independently discovered by D. Zagier (unpublished).

**Proof.** The series $F_H$ is the exponential generating series that enumerates the $n$-decorated graphs $G$ whose simplification equals $H$. Every graph $G$ like that can be obtained by the following procedure shown in Figure [41]. First to every vertex of $H$ we assign a rooted tree. Second, to every edge of $H$ we assign either a tree with two marked vertices (possibly twice the same vertex) or the empty tree (with 0 vertices). Now join the marked vertices of these trees as shown in the figure to obtain a graph $G$ homotopic to $H$. Finally, number the vertices of $G$.

Since the exponential generating function for rooted trees (respectively, for trees with two marked vertices) is $Y$ (respectively, $Z$), we obtain

$$F_H = \frac{1}{|\text{Aut}(H)|} Y^v (1 + Z)^e,$$
Figure 4: Constructing the extensions of a simple graph \( H \).

where the factor \( 1/|\text{Aut}(H)| \) comes from the fact that we have counted as different the graphs \( G \) obtained from each other by symmetries of \( H \). Recalling that \( Y \) and \( Z \) are generators of \( A \), we see that \( F_H \) lies in \( A \). This proves the proposition.

Now we have all the necessary elements to prove Theorem 1, that we restate here.

\textbf{Theorem 1} For any \( g \geq 2 \) and \( \beta \in H^*(\overline{\mathcal{M}}_{g,0}, \mathbb{Q}) \), the power series

\[
F_{g,\beta}(q) = \sum_{n \geq 0} \frac{q^n}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\beta}{(1 - \psi_1) \cdots (1 - \psi_n)}
\]

lies in the algebra \( A \).

\textbf{Proof.} We have

\[
F_{g,\beta}(q) = \sum_n q^n \frac{1}{n!} \sum_{d_1 \ldots d_n} \langle \beta \tau_{d_1} \cdots \tau_{d_n} \rangle.
\]

According to Theorem 5, the bracket \( \langle \beta \ldots \rangle \) is a linear combination (with rational coefficients) of brackets \( \langle \cdot \rangle_H \) for a finite number of simple graphs \( H \). The assertion of Theorem 1 now follows from Proposition 3.14. \( \diamond \)
3.5 A generalization

Here we formulate and prove a generalization of Theorem 1. The class $\beta$ will now be a cohomology class of $H^*(\overline{M}_{g,p}, \mathbb{Q})$ for some fixed nonnegative integer $p$. Moreover, the generalization can be directly applied to the Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula from [4], see Section 4, Theorem 7.

Let $p \geq 0$ be an integer and $\beta \in H^*(\overline{M}_{g,p}, \mathbb{Q})$ a cohomology class of $\overline{M}_{g,p}$. Fix $p$ positive integers $b_1, \ldots, b_p$ and denote by $r$ the number $r = \sum (b_i - 1)$.

**Theorem 6** For any $g, p$ such that $2 - 2g - p < 0$ and for any positive integers $b_1, \ldots, b_p$, the power series

$$F_{g; \beta; b_1, \ldots, b_p}(q) = \sum_{n \geq p+r} \frac{q^n}{(n - p - r)!} \int_{\overline{M}_{g,n-r}} \frac{\beta}{(1 - b_1 \psi_1) \ldots (1 - b_p \psi_p)(1 - \psi_{p+1}) \ldots (1 - \psi_{n-r})}$$

lies in the algebra $\mathcal{A}$.

The proof goes along the lines of that of Theorem 1, but differs in some details.

First, we slightly modify the definition of an $n$-decorated graph.

**Definition 3.15** An $n$-decorated graph $G$ is a graph with $n + 1$ vertices one of which is labeled by * and the others are numbered from 1 to $n$. Moreover, we suppose that every connected component of $G$ either contains one of the vertices *, 1, $\ldots$, $p$ or has at least two independent cycles.

**Definition 3.16** Let $p$ be a nonnegative integer and $G$ an $n$-decorated graph. The $p$-simplification of $G$ is the graph $H$ obtained from $G$ by (i) applying as many times as possible the operations (S) and (D) without erasing the vertex * or changing its valency, and also without erasing the vertices with numbers from 1 to $p$; (ii) forgetting the numbers of the vertices, except those from 1 to $p$.

Conversely, the graph $G$ is called an extension of $H$.

A $p$-simple graph is any finite graph with $p$ vertices numbered from 1 to $p$, one more vertex labeled by *, and possibly several non-numbered vertices, each of which has a valency $\geq 3$.  

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It is easy to see that the $p$-simplification of a graph $G$ is unique, in other words, it does not depend on the order in which one applies the operations (S) and (D).

To any $p$-simple graph $H$ we assign a bracket $\langle \cdot \rangle_H$ using the same rule as in Definition 3.7 replacing the word “simplification” by “$p$-simplification”.

**Proposition 3.17** The bracket $\langle \cdot \rangle_H$ for a $p$-simple graph $H$ with Euler characteristic $3 - 2g$ satisfies the string and dilaton relations for $n \geq p$.

**Proof.** Same as that of Proposition 3.9. ♦

**Proposition 3.18** Consider any rational-valued bracket either of nonnegative degree and genus $g \geq 1$ or of positive degree and genus $g = 0$. Suppose that the bracket satisfies the string and the dilaton relations for $n \geq p$ ($p$ being a nonnegative integer). Then there exists a set of $p$-simple graphs $H_1, \ldots, H_l$ and of rational numbers $s_1, \ldots, s_l$ such that the bracket is equal to

$$s_1 \langle \cdot \rangle_{H_1} + \ldots + s_l \langle \cdot \rangle_{H_l}.$$

**Proof.** The proof repeats that of Theorem 5. The restriction that the genus $g$ and the degree $b$ of the bracket cannot vanish simultaneously comes from the fact that the vertex $\ast$ must have the valency $b + g - 1$. ♦

**Proof of Theorem 6.** The generating series $F_{g,\beta:b_1,\ldots,b_p}(q)$ from Theorem 6 can be rewritten as

$$F_{g,\beta:b_1,\ldots,b_p}(q) = \sum_{n \geq p+r} \frac{q^n}{(n - p - r)!} \prod_{d_1,\ldots,d_n} b_1^{d_1} \ldots b_p^{d_p} \langle \beta \tau_{d_1} \ldots \tau_{d_{n-r}} \rangle_H.$$

The exceptional case $g = 0, \deg \beta = 0$ can be easily treated using the explicit values of the bracket given in Section 3.2. We leave this as an exercise to the reader.

In all the other cases, by Proposition 3.18 the bracket $\langle \beta \ldots \rangle$ can be decomposed as a linear combination over $\mathbb{Q}$ of brackets $\langle \cdot \rangle_H$ for some $p$-simple graphs $H$. Therefore it is enough to prove that the series

$$F_{g,H:b_1,\ldots,b_p}(q) = \sum_{n \geq p+r} \frac{q^n}{(n - p - r)!} \prod_{d_1,\ldots,d_n} b_1^{d_1} \ldots b_p^{d_p} \langle \tau_{d_1} \ldots \tau_{d_{n-r}} \rangle_H.$$

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lies in $\mathcal{A}$ for every $p$-simple graph $H$. This series can be rewritten as follows

$$F_{g,H:b_1,...,b_p}(q) = \sum_{n \geq p+r} \frac{q^n}{(n-p-r)!} \sum_G b_1^{v_1-1} \ldots b_p^{v_p-1},$$

where the second sum is taken over all $(n-r)$-decorated graphs $G$ that simplify into $H$, while $v_1, \ldots, v_p$ are the valencies of the first $p$ vertices of $G$.

We are going to give a way of enumerating such graphs that shows that the corresponding generating series lies in $\mathcal{A}$.

Consider a $p$-simple graph $H$. Denote by $V_n$ the set $\{1, \ldots, n\}$. Consider the set $C_n$ of the following objects. (i) An ordered list of $p$ subsets $U_1, \ldots, U_p \subset V_n$ satisfying $|U_i| = b_i$. (Here $|U_i|$ is the number of elements in $U_i$.) (ii) A graph $\hat{G}$ whose set of vertices is $V_n \cup \{\ast\}$. Moreover, we impose the condition that if, in the graph $\hat{G}$, we glue together the vertices of $U_i$ for each $i$ and attribute to the obtained vertex the label $i$, we obtain an extension $G$ of the $p$-simple graph $H$. Note that the vertices of $G$ are numbered incorrectly (i.e., not form 1 to $n-r$).

Denote by $|C_n|$ the number of objects like that. We claim that

$$\frac{|C_n|}{n!} = \frac{1}{(b_1 - 1)! \ldots (b_p - 1)!} \cdot \frac{1}{(n-p-r)!} \sum_G b_1^{v_1-1} \ldots b_p^{v_p-1},$$

where, as before, the sum is taken over the $(n-r)$-decorated graphs that simplify into $H$. Indeed, the vertex number $i$, $1 \leq i \leq p$, in the graph $G$ is separated into $b_i$ different vertices in the graph $\hat{G}$. Thus every graph $G$ will be counted with weight $b_1^{v_1} \ldots b_p^{v_p}$. On the other hand, each graph $\hat{G}$ possesses $n!/b_1! \ldots b_p!$ different numberings of the unmarked vertices, while the graph $G$ has only $(n-p-r)!$ of them.

Now it remains to prove that the series

$$\sum_{n \geq p+r} \frac{|C_n|}{n!} \frac{q^n}{n!}$$

lies in $\mathcal{A}$. To do that, we propose the following way to enumerate the objects from $C_n$. (i) Construct $b_1 + \ldots + b_p$ disjoint rooted trees with vertices chosen in the set $V_n$. Regroup these trees into $p$ forests with $b_1, \ldots, b_p$ trees. The roots of these trees will form the sets $U_1, \ldots, U_p$. (ii) To each non-numbered vertex of $H$ assign yet another rooted tree with vertices in $V_n$. To each edge of $H$ assign either an empty tree (with 0 vertices) or a tree with two
marked vertices. (iii) Join the marked points of the trees assigned to edges to the roots of the rooted trees assigned to the non-numbered vertices (see Figure 4). (iv) Now consider an edge of \( H \) adjacent to a numbered vertex, say to the vertex number \( i \). There is a whole forest of \( b_i \) rooted trees assigned to this vertex. We choose one of these trees \( T \) and join one of the marked points of the “edge tree” to the root of \( T \) (see Figure 5). Here it is crucial to note that since the valency of each vertex of \( H \) is finite and fixed, there is only a finite and fixed (i.e., independent of \( n \)) number of choices.

![Figure 5: There are \( b_i \) choices for each edge of \( H \) adjacent to the vertex number \( i \).](image)

Our new description of the objects of \( C_n \) shows that the generating function

\[
\sum_{n \geq p+r} |C_n| \frac{q^n}{n!}
\]

is a finite sum (over a finite number of choices) of finite products of series \( Y \) (for rooted trees) and \( 1 + Z \) (for trees with two marked points). Thus the generating function lies in \( A \).

\[\diamondsuit\]

4 Counting ramified coverings of the sphere

This section is devoted to the enumeration of ramified coverings of the sphere by surfaces of a fixed genus \( g \) and to a proof of Theorem \( \Box \)
4.1 The ELSV formula

Curiously, the most difficult part of the proof of Theorem 2 is the case with only one complicated ramification point, $k = 1$. We know no other way to prove it than to use the intersection theory on moduli spaces and the results of the previous section. Apart from these results, the main ingredient of the proof is a theorem by T. Ekedahl, S. K. Lando, M. Shapiro, and A. Vainshtein that we formulate below after introducing some notation.

Let $\mu = 1^{a_1}2^{a_2} \ldots$ be a partition with degeneracy $r$. We define $|\text{Aut}(\mu)|$ to be $|\text{Aut}(\mu)| = a_1!a_2! \ldots$. For the formulation of the theorem it is more convenient to switch to using the additive notation for the partition $\mu$, $\mu = (b_1, \ldots, b_p)$, the $b_i$ being the parts of $\mu$.

We will also need several cohomology classes on the moduli space $\overline{\mathcal{M}}_{g,n-r}$. As before, $L_i$, $1 \leq i \leq n - r$ are the tautological line bundles (Notation 1.1), and $\psi_i = c_1(L_i)$ are their first Chern classes. We also consider the Hodge vector bundle $W$. The fiber of $W$ over a smooth curve is the set of holomorphic 1-forms on this curve. The fiber of $W$ over a general stable curve is the set of global sections of its dualizing sheaf. We do not give the details here (see the paper [4] itself). Suffice it to note that $W$ is a vector bundle of rank $g$.

One property of the vector bundle $W$ will be important to us. Recall that there is a canonical “forgetful” map from $\overline{\mathcal{M}}_{g,n-r}$ to $\overline{\mathcal{M}}_{g,0}$, forgetting the marked points and contracting the components of the curve that have become unstable. The Hodge bundle can be defined on both $\overline{\mathcal{M}}_{g,n-r}$ and $\overline{\mathcal{M}}_{g,0}$, and the former is the pull-back of the latter under the forgetful map. The same is true for the Chern classes of the Hodge bundle, in particular for the total Chern class $c(W^*)$ of its dual bundle.

The Hurwitz number $h_{g,n;\mu}$ is defined in Notation 1.4. It counts the marked ramified coverings with one ramification point of type $\mu$ and $c(n) = 2n + 2g - 2 - r$ simple ramifications.

Now we can write down the ELSV formula.

**Theorem 7 (The ELSV formula, [4])**  For any $g$, $n$, and $\mu$ such that $2 - 2g - (n - r) < 0$, we have

$$h_{g,n;\mu} = \frac{(2n + 2g - 2 - r)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{p} \frac{b_i^{b_i}}{b_i!} \times \ldots$$
× \frac{1}{(n-p-r)!} \int_{\text{M}_{g,n-r}} \frac{c(W^*)}{(1 - b_1 \psi_1) \ldots (1 - b_p \psi_p)(1 - \psi_{p+1}) \ldots (1 - \psi_{n-r})}.

Note that this formula seems to be waiting for someone to apply Theorem 6.

\subsection*{4.2 Proof of Theorem 2}

Recall that \( h_{g,n;\mu_1,\ldots,\mu_k} \) is the number of \( n \)-sheeted marked ramified coverings of \( \mathbb{C}P^1 \) by a genus \( g \) surface, with \( k \) ramification points of types \( \mu_1,\ldots,\mu_k \) and, in addition, \( c(n) = 2n + 2 - 2g - r \) simple ramification points, where \( r \) is the sum of degeneracies of the partitions (see Notation 1.4). We are going to prove that all generating series for the numbers \( h_{g,n;\mu_1,\ldots,\mu_k} \) with respect to the number of sheets \( n \) lie, once again, in the algebra \( \mathcal{A} \). In Section 5 we explain why these particular generating series are interesting to consider.

Let us restate the theorem we are going to prove.

**Theorem 2** Fix any \( g \geq 0, k \geq 0 \). If \( g = 1 \), we suppose that \( k \geq 1 \). Then for any partitions \( \mu_1,\ldots,\mu_k \), the series

\[ H_{g,\mu_1,\ldots,\mu_k}(q) = \sum_{n \geq 1} \frac{h_{g,n;\mu_1,\ldots,\mu_k}}{c(n)!} q^n \]

lies in the algebra \( \mathcal{A} \).

**Proof.** The theorem is proved by induction on the number \( k \) of partitions.

**Base of induction.** For \( k = 0,1 \), the result is obtained by a direct application of Theorem 6 to the ELSV formula. In the case \( k = 0 \), we must use the ELSV formula with an empty partition \( \mu \).

There are three exceptional cases in which Theorem 6 cannot be applied: \( g = 0, k = 0; g = 0, k = 1, p \leq 2; g = 1, k = 0 \). These cases are discussed in Remark 4.1 below. It turns out that the assertion of Theorem 2 fails only if \( g = 1, k = 0 \), as stated in the formulation.

**Step of induction.** The step of induction is an almost exact repetition of the proof of Theorem 2 from our previous work [15]. We only give a short summary of the argument here, referring to [15] for details.

The proof goes in the spirit of [6]. Recently, M. Kazaryan introduced a semi-group of colored permutations, which seems to be best fit for describing the proof (work in preparation).
It is easy to see that there is only a finite number of possible cycle structures for a permutation that can be obtained as a product of two permutations with given cycle structures \( \mu_1 \) and \( \mu_2 \).

Let \( \mu_1 \) and \( \mu_2 \) be two partitions from the list \( \mu_1, \ldots, \mu_k \). We can move the two corresponding ramification points on \( \mathbb{CP}^1 \) towards each other until they collapse. We obtain a new (not necessarily connected) ramified covering. Its monodromy at the new ramification point is the product of the monodromies of the two points that have collapsed.

Let us choose one of the possible cycle structures of the product monodromy and also one of the possible ways in which the covering can split into connected components. By the induction assumption, we obtain a series from the algebra \( \mathcal{A} \) assigned to each connected component of the covering. Indeed, each connected component is itself a ramified covering of the sphere as in Theorem 7 but with \( k - 1 \) fixed ramification types instead of \( k \). We obtain the generating series for the number of nonconnected ramified coverings by multiplying the series that correspond to the connected components. Since it is a finite product of series lying in \( \mathcal{A} \), we obtain again a series from \( \mathcal{A} \).

Finally, we must add the generating series described above for all choices of types of nonconnected coverings. Since the number of choices is finite, we obtain, once again, a series from \( \mathcal{A} \).

\[ h_{0,n;\mu} = \frac{(2n - 2 - r)!}{|\text{Aut}(\mu)|} \prod_{i=1}^{p} \frac{b_i^{h_i} b_i!}{b_i!} \cdot \frac{n^{n-r-3}}{(n-p-r)!}. \]

This formula is true for any \( n \geq p + r \) and for any partition \( \mu \) (including even the empty partition). We see that the corresponding generating series always lies in the algebra \( \mathcal{A} \).

The case \( g = 1, k = 0 \) is covered by the ELSV formula with an empty partition \( \mu \). Consider the moduli space \( \overline{\mathcal{M}}_{1,1} \). As in the last paragraph of Section 3.2, denote by \( \beta \) the 2-cohomology class of \( \overline{\mathcal{M}}_{1,1} \) whose integral over
the fundamental homology class equals 1. One can prove that the Hodge bundle over \( \mathcal{M}_{1,1} \) is a line bundle with first Chern class \( \beta/24 \). Therefore we obtain
\[
h_{1,n;\emptyset} = (2n)! \cdot \frac{1}{n!} \int_{\mathcal{M}_{1,n}} \frac{1 - \frac{1}{24}\beta}{(1 - \psi_1) \cdots (1 - \psi_n)}.
\]
Using the calculations of Section 3.2, we obtain
\[
\sum_{n \geq 1} h_{1,n;\emptyset} q^n = \frac{1}{24} \sum_{n \geq 1} \frac{A_n}{n!} \cdot q^n.
\]
This series does not lie in \( \mathcal{A} \) (and constitutes the only exception to the general rule). It suffices to consider the partition \( \mu = (1) \), which amounts to distinguishing one sheet in the ramified covering, to obtain the series
\[
\frac{1}{24} \sum_{n \geq 1} \frac{A_n}{n!} \cdot q^n \in \mathcal{A}.
\]

5 Random Riemannian metrics versus random abelian differentials

In this section we do not prove any theorems, but discuss the relation between the enumeration of ramified coverings of the sphere and the 2-dimensional gravity. We also draw a parallel between the study of spaces of Riemannian metrics using ramified coverings of the sphere and the study of spaces of abelian differentials using ramified coverings of the torus.

5.1 Two models of 2-dimensional gravity

In every problem of statistical physics one starts with introducing a space of states and by assigning an energy to every state.

In 2-dimensional gravity, a state is a 2-dimensional compact oriented real not necessarily connected surface endowed with a Riemannian metric. Two surfaces like that are equivalent, i.e., correspond to the same state, if they are isometric.

Consider a surface \( S \) with a Riemannian metric. Let \( \chi(S) \) be its Euler characteristic and \( A \) its total area. To such a surface one assigns an energy
\[
E = \lambda A + \mu \chi(S).
\]
Here $\lambda$ and $\mu$ are two constants called the cosmological constant and the gravitational constant, respectively. Note that $\chi(S)$ is actually the integral over $S$ of the curvature of the metric. The fact that this integral takes such a simple form is special to dimension 2.

Now the first thing to do is to compute the partition function

$$Z(\lambda, \mu) = \int_{\text{states}} e^{-E},$$

or, equivalently, the free energy

$$F(\lambda, \mu) = \ln Z(\lambda, \mu) = \sum_{g \geq 0} \int_{\text{metrics}} e^{-E}.$$

The free energy is the sum of contributions of connected surfaces, while the partition function is the sum of contributions of all surfaces.

Neither of the above integrals is well-defined mathematically, but we would still like to compute them. To do that, physicists introduced a discrete model of Riemannian metrics, replacing them by quadrangulations [2, 13] (see also [9], Chapter 3 for a mathematical description). In this model, instead of considering Riemannian metrics, one considers metrics obtained by gluings of squares of area $\varepsilon$. Our goal is to convince the reader that the ramified coverings of the sphere provide a new (maybe more natural) discrete model of Riemannian metrics.

Fix a positive number $\varepsilon$. Consider a sphere with the standard (round) Riemannian metric and with total area $\varepsilon$. On this sphere, choose at random $2n + 2g - 2$ points. Now chose a random connected $n$-sheeted covering of the sphere with simple ramifications over the $2n + 2g - 2$ chosen points. The covering surface $S$ will automatically be of genus $g$. The metric on the sphere can be lifted to $S$, which will give us a metric with constant positive curvature except at the critical points, where it has conical singularities with angles $4\pi$. This metric is, of course, not Riemannian. However, one can argue that if $\varepsilon$ is very small and the number of sheets very large, a random metric obtained in this way looks similar to a random Riemannian metric (unless we look at them through a microscope to reveal the difference). We do not know any rigorous statement that would formalize this intuitive explanation, but the same argument is used by physicists to justify the usage of quadrangulations.

Using our discrete model of metrics, one can write the free energy for the
2-dimensional gravity in the following way:

\[ F(\lambda, \mu) = \sum_{g,n} \varepsilon^{2n+2g-2} (2n + 2g - 2)! \ h_{g,n;\emptyset} \ e^{-n\lambda \varepsilon - \mu(2-2g)}. \]

Here \( \varepsilon^{2n+2g-2}/(2n + 2g - 2)! \) is the volume of the space of choices of \( 2n + 2g - 2 \) unordered points on the sphere of area \( \varepsilon \).

Using Theorem 2 (and some additional considerations to find the degree in \( Y \) and \( Z \) of the series that are involved), one can show that the leading term of the asymptotic of \( h_{g,n;\emptyset}/(2n + 2g - 2)! \) as \( n \) tends to infinity (while \( g \) is fixed) looks like

\[ \frac{h_{g,n;\emptyset}}{(2n + 2g - 2)!} \sim e^n n^{\frac{g}{2}(g-1)-1} b_g, \]

with some constants \( b_g \). The first values of these constants are

\[
\begin{align*}
b_0 &= \frac{1}{\sqrt{2\pi}}, \quad b_1 = \frac{1}{2^4 \cdot 3}, \quad b_2 = \frac{1}{\sqrt{2\pi}} \cdot \frac{7}{2^5 \cdot 3^3 \cdot 5}, \\
b_3 &= \frac{5 \cdot 7^2}{2^16 \cdot 3^5}, \quad b_4 = \frac{1}{\sqrt{2\pi}} \cdot \frac{7 \cdot 5297}{2^{11} \cdot 3^8 \cdot 5^2 \cdot 11 \cdot 13}.
\end{align*}
\]

Now we make the final step by letting \( \varepsilon \) tend to 0 in the expression for the free energy \( F \). To obtain an interesting limit for the free energy, we must make \( \lambda \) and \( \mu \) depend on \( \varepsilon \). We want to use

\[
\sum_{n \geq 1} n^{a-1} e^{-\varepsilon n} \sim \frac{1}{\varepsilon^a} \Gamma(a) \quad \text{as} \quad \varepsilon \to 0.
\]

Therefore we let \( \lambda \varepsilon - 2 \ln \varepsilon - 1 \to 0 \), while

\[ y = \frac{(\varepsilon e^\mu)^{4/5}}{\lambda \varepsilon - 2 \ln \varepsilon - 1} \]

remains fixed. This gives us the final expression of the free energy, now depending on only one variable \( y \):

\[ F(y) = \Gamma(-5/2) b_0 y^{5/2} - b_1 \ln y + \sum_{g \geq 2} \Gamma\left(\frac{5(g - 1)}{2}\right) b_g \cdot y^{5(1-g)/2}. \]
The coefficients $\Gamma(5(g - 1)/2) b_g$ are rational for odd $g$ and rational multiples of $\sqrt{2}$ for even $g$.

Our above treatment is parallel to the treatment of the quadrangulation model in [14]. Denote by $Q_{g,n}$ the number of ways to divide a surface of genus $g$ into $n$ squares. Then the study of the quadrangulation model involves the asymptotic of $Q_{g,n}$, which is given by

$$Q_{g,n} \sim 12^n n^{5/2} (g - 1)^{-1} b'_g,$$

for another sequence of constants $b'_g$. This sequence was studied using matrix integrals, and it is known that a generating function for the sequence $b'_g$ satisfies the Painlevé I equation. Our numerical experiments show that the expressions for the free energy obtained in both models coincide up to a rescaling of $y$. More precisely, we formulate the following conjecture.

**Conjecture 5.1** We have

$$b'_g = 2 \frac{3}{2} (g - 1)^{1+1} b_g.$$

An equivalent statement: the function $u(y) = F''(y)$ satisfies the Painlevé I equation

$$\frac{1}{6} u''(y) + u(y)^2 = 2y.$$

It is also an open problem to find the Korteweg–de Vries hierarchy using the enumeration of ramified coverings, as it was done with quadrangulations in [14].

We believe that the connection between the two models can be obtained using A. Okounkov’s results on random Young diagrams: the identity between the distribution of highest eigenvalues of a random hermitian matrix and the distribution of lengths of longest columns in a random Young diagram [11]; and an appearance of integrable hierarchies in the study of random Young diagrams [10].

### 5.2 Ramified coverings of a torus and abelian differentials

Fix an integer $g \geq 1$ and a list of $p$ nonnegative integers $b_1, \ldots, b_p$ with the condition $\sum b_i = 2g - 2$. We consider the space $D_{g,b_1,\ldots,b_p}$ of abelian
differentials on Riemann surfaces of genus $g$, with zeroes of multiplicities $b_1, \ldots, b_p$. More precisely, $D_{g; b_1, \ldots, b_p}$ is the space of triples $(C, \{x_1, \ldots, x_p\}, \alpha)$, where $C$ is a smooth complex curve, $x_1, \ldots, x_p \in C$ are distinct marked points, and $\alpha$ is an abelian (= holomorphic) differential on $C$ whose zero divisor is precisely $b_1 x_1 + \ldots + b_p x_p$.

It turns out that the space $D_{g; b_1, \ldots, b_p}$ has a natural integer affine structure. This means that it can be covered by charts of local coordinates in such a way that the change of coordinates, as one goes from one chart to another, is an affine map with integer coefficients. Such local coordinates are introduced in the following way. Fix a basis $l_1, \ldots, l_{2g+p-1}$ of the relative homology group $H_1(C, \{x_1, \ldots, x_p\}, \mathbb{Z})$. Then the integrals of $\alpha$ over the cycles $l_i$ are the local coordinates we need. The area function

$$A: (C, \alpha) \mapsto \frac{i}{2} \int_C \alpha \wedge \bar{\alpha}$$

is a quadratic form with respect to the affine structure.

The integer affine structure allows one to define a volume measure on the space $D_{g; b_1, \ldots, b_p}$. It is then a natural question to find the total volume of the part of the space $D_{g; b_1, \ldots, b_p}$ defined by $A \leq 1$ (the volume of the whole space being infinite).

A. Eskin and A. Okounkov [5] obtained an effective way to calculate these volumes using the asymptotic for the number of ramified coverings of a torus. Consider the elliptic curve obtained by gluing the opposite sides of the square $(0, 1, i, 1+i)$ endowed with the abelian differential $dz$. Given a ramified covering of this elliptic curve with critical points of multiplicities $b_1, \ldots, b_p$, we can lift the abelian differential to the covering curve and obtain a point of $D_{g; b_1, \ldots, b_p}$. One can then easily show that such points are densely and uniformly distributed in $D_{g; b_1, \ldots, b_p}$ if one considers coverings with a big number of sheets. Moreover, R. Dijkgraaf [3] and S. Bloch and A. Okounkov [4] showed that the generating series for the ramified coverings of the torus that arise in this study are quasi-modular forms. In other words, they lie in the algebra

$$\mathbb{Q}[E_2, E_4, E_6],$$

where $E_{2k}$ are the Eisenstein series

$$E_{2k}(q) = \frac{1}{2} \zeta(1 - 2k) + \sum_{n \geq 1} \left( \sum_{d|n} d^{2k-1} \right) q^n.$$
We conclude with the following comparison between the counting of ramified coverings of a sphere and of a torus.

Sphere: The generating series enumerating the ramified coverings lie in the algebra $A$.

Torus: The generating series enumerating the ramified coverings lie in the algebra of quasi-modular forms.

Sphere: The coefficients of a generating series grow as $e^n \cdot n^\gamma \cdot c$. The exponent $\gamma$ is a half-integer. The number $1 - \gamma$ is called the string susceptibility. The constant $c$ is an observable in 2-dimensional gravity.

Torus: The sum of the first $n$ coefficients of a generating series grows as $n^d \cdot c$. The number $d$ is the complex dimension of the corresponding space of abelian differentials. The constant $c$ is its volume.

Sphere: It is known that some particular observables in 2-dimensional gravity can be arranged into a generating series (in an infinite number of variables), that turns out to be a $\tau$-function for the Korteweg–de Vries hierarchy. These observables have not been found yet in the model with coverings of the sphere.

Torus: As far as we know, nobody has tried to arrange the volumes of the spaces of abelian differentials into a unique generating series.

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