Extended Pairing Model for Heavy Nuclei

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We study binding energies in three isotopic chains (\( ^{100-130} \text{Sn} \), \( ^{152-181} \text{Yb} \), and \( ^{181-202} \text{Pb} \)) using the extended pairing model with Nilsson single-particle energies. The exactly solvable nature of the model means that the pairing strength \( G(A) \) required to reproduce the experimental binding energies can be determined uniquely. The valence space consists of the neutron single-particle levels between two closed shells corresponding to the magic numbers 50-82 and 82-126. In all three isotopic chains, \( \log(G(A)) \) has a smooth quadratic behavior for even as well as odd nucleon numbers \( A \); \( \log(G(A)) \) for even and odd \( A \) are very similar. Remarkably, \( G(A) \) for all the Pb isotopes can be also described by a two parameter expression that is inversely proportional to the dimensionality of the model space.

To set the stage and establish the notation we begin with a brief review of the underpinnings of the extended pairing model. The standard Nilsson plus pairing Hamiltonian is given by

\[
\hat{H} = \sum_{j=1}^{p} \epsilon_j n_j - G \sum_{i,j=1}^{p} a_i^+ a_j^\dagger,
\]

where \( p \) is the total number of single-particle levels considered, \( \epsilon_j \) are single-particle energies, \( G \) is the overall pairing strength \( (G > 0) \), \( n_j = c_{j \uparrow}^\dagger c_{j \downarrow} + c_{j \downarrow}^\dagger c_{j \uparrow} \) is the number operator for the \( j \)-th single-particle level, \( a_i^+ = c_{i \uparrow}^\dagger c_{i \downarrow} \) (\( a_i^+ = (a_i^\dagger)^\dagger = c_{i \downarrow}^\dagger c_{i \uparrow} \)) are pair creation (annihilation) operators where \( c_{j \uparrow} (c_{j \downarrow}^\dagger) \) creates a fermion in the \( j \)-th single-particle level. The up and down arrows refer to time-reversed states. Since each Nilsson level can only be occupied by one pair due to the Pauli Exclusion Principle, the operators \( a_i^\dagger, a_i \) and \( n_i \) form a hard-core boson algebra:

\[
[a_i, a_j^\dagger] = \delta_{ij} (1 - n_i), \quad [a_i^\dagger, a_j^\dagger] = 0 = (a_i^\dagger)^2.
\]

As a generalization of \cite{11}, we consider the following extended pairing Hamiltonian \cite{12}:

In a previous paper we introduced the extended pairing model, explored its properties, and applied it to some well-deformed nuclei \cite{11}. Metallic clusters of nano-scale size may provide another physical system where one can test and study the applicability of the extended pairing interaction \cite{12}. In this paper we exploit the exactly solvable nature of the model to explore its applicability in reproducing experimental nuclear binding energy of well-deformed nuclei \cite{13}. In particular, for the Pb isotopes the even and odd \( A \) are very similar. The results suggest that the model is useful for its simplicity, its exact solvability, and its ability to track and predict experimental binding energies of long sequences of nuclei.

For each isotopic chain we take the binding energy of a closed neutron shell to be our zero-energy reference point. Each such closed neutron shell nucleus (\( ^{100-130} \text{Sn} \), \( ^{152-181} \text{Yb} \), and \( ^{181-202} \text{Pb} \)) is assumed to be well described by the independent particle model with Nilsson single-particle energies; thus no extended pairing interaction terms are needed for these nuclei. The energy scale of the Nilsson model space is reflected in the values of \( G(A) \). In particular, in all the cases studied \( \log(G(A)) \) has a smooth quadratic behavior for even and odd \( A \) with a minimum in the middle of the model space where the size of the space is a maximal; \( \log(G(A)) \) for even \( A \) and odd \( A \) are very similar which suggests that more detailed shell-model analyses may result in the same \( G(A) \) functional form for even \( A \) and odd \( A \) isotopes. In particular, for the Pb isotopes the even and odd \( A \) nuclei are described by a single two-parameter expression for the pairing strength \( G(A) \) that is inversely proportional to the dimensionality of the valence model space.
Besides the usual single-particle terms and the standard pairing interaction $H_0$, this interaction includes many-pair interaction terms which connect configurations that differ by more than one pair.

The main advantage of the extended pairing model is that it is exactly solvable \([1]\). It is easy to see that any term of the form $a_i^+ \cdots a_j^+$ that forms eigenstates of (3) are given by (any term of the form $a_i^+ \cdots a_j^+$)

$$ \sum_{i_1 \neq i_2 \neq \cdots \neq i_{2\mu}} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_{\mu}}^+ a_{i_{\mu+1}} a_{i_{\mu+2}} \cdots a_{i_{2\mu}} \right) .$$

(3)

where $y(\zeta)$ is a number that is to be determined. To solve (4) using (3) and (4), one directly applies Hamiltonian (5) on the $k$-pair state (4). Using the hard-core boson algebraic relations given by (2), one can determine the action of the mean-field part of the Hamiltonian (6):

$$ \sum_{\mu=1}^{\infty} \frac{1}{(\mu!)^2} \sum_{i_1 \neq i_2 \neq \cdots \neq i_{2\mu}} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_{\mu}}^+ a_{i_{\mu+1}} a_{i_{\mu+2}} \cdots a_{i_{2\mu}} |k; \zeta; 0\rangle = \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} C^{(\zeta)}_{i_1 i_2 \cdots i_k} \right) \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} a_{i_1}^+ a_{i_2}^+ \cdots a_{i_k}^+ |0\rangle + (k - 1) |k; \zeta; 0\rangle .$$

(7)

By combining (6) and (7), the $k$-pair excitation energies of (6) are given by

$$ E_k^{(\zeta)} = \frac{1}{y(\zeta)} - G(k - 1), $$

and the undetermined variable $y(\zeta)$ is given by

$$ \frac{1}{y(\zeta)} + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq p} \frac{G}{(1 - y(\zeta) E_{i_1 \cdots i_k})} = 0. $$

(8)

(9)

The additional quantum number $\zeta$ can now be understood as the $\zeta$-th solution of (9). Similar results for many broken-pair systems can be derived by using this approach except that the indexes $j_s$ of the level occupied by the single nucleons should be excluded from the summation in (1) and the single-particle energy term $\epsilon_{j_s}$ from the first part of (3) should be added to the total eigenenergy.

By comparing (8) and (9) to the exact solutions of the Heisenberg algebraic Hamiltonian with a one-body interaction \([9]\), one can regard the operator product $a_{i_1}^+ a_{i_2}^+ \cdots a_{i_k}^+$ in (3) as a ‘grand’ boson. The corresponding ‘single-particle energy’ of the ‘grand’ boson
is \( E_{i_1 i_2 \ldots i_k} = \sum_{\mu=1}^{k} 2\epsilon_{i\mu} \). The eigenstates \( |i\rangle \) are not normalized, but can be normalized once the coefficients \( C^{(C)}_{i_1 i_2 \ldots i_k} \) are known. The eigenstates \( |i\rangle \) with different roots given by \( \zeta \) are mutually orthogonal since they correspond to eigenstates with different eigenvalues.

The \( k \) coupled non-linear equations of the standard pairing model \( \xi \) are difficult to solve numerically, especially when the number of pairs \( k \) and number of levels \( p \) are large. Specifically, there should be \( \langle i \rangle = \frac{1}{\Gamma(p-k)\xi} \) distinct roots, which can be a very large number for an entire deformed major shell. While there have been major advances in methods for solving the associated Richardson equations, for example \( \xi \), the theory is limited because of the coupled non-linear nature of the equations. In contrast to this, for the extended pairing model there is but one variable \( y^{(C)} \) and one equation \( \zeta \), so a relatively simple Mathematica code, for example, can be used to solve for the roots.

Our Nilsson plus extended pairing model uses single-particle energies of each nucleus as calculated within the Nilsson deformed shell model with experimentally evaluated deformation parameters \( \xi \). Experimental binding energies are taken from reference \( \xi \). The theoretical binding energies are calculated relative to a particular core. We use \( ^{152}\text{Yb} \), \( ^{100}\text{Sn} \), and \( ^{208}\text{Pb} \) as cores in our calculations. While there are changes in the binding energy of the core since the corresponding Nilsson levels change as a function of the deformation, our results indicate that such core affects are an order of magnitude smaller than the overall even-odd staggering in the binding energy. From the binding energy of a nucleus next to the core, we calculate an overall energy scale for the Nilsson single-particle energies. For an even number of neutrons, we consider only pairs of particles (hard bosons). For an odd number of neutrons, we apply Pauli blocking of the Fermi level of the last unpaired fermion and consider the remaining fermions as if they were an even \( A \) fermion system. By using \( \xi \) and \( \zeta \), values of \( G \) are calculated so that the experimental and theoretical binding energy match exactly. With the help of \( \xi \) and \( \zeta \) and some algebra, one can see that for a given single particle energies there is an upper limit to the value of the binding energy for which a physically meaningful exact solution can be constructed. This upper value of the binding energy for each nucleus is given by the energy of the lowest “grand boson”, namely, \( E_{gb} = \sum_{\mu=1}^{k} 2\epsilon_{\mu} \).

We now turn to our results for the binding energies and \( \log(G) \) values within the Nilsson plus extended pairing model for three isotopic chains: \( ^{100\ldots130}\text{Sn} \), \( ^{152\ldots181}\text{Yb} \), and \( ^{181\ldots202}\text{Pb} \). Figure \( \zeta \) shows the results for the \( ^{100\ldots130}\text{Sn} \) isotopes. Calculations for the pairing strength \( G \) were carried out for the \( ^{102\ldots130}\text{Sn} \) isotopes within the 50-82 neutron shell. Once this was done, a quadratic polynomial fit of the \( \log(G) \) values for even and odd \( A \) was determined. By doing this we were able to fit two sets of 14 data points with two 3 parameter expressions. Overall the results are very good for the lower and middle part of the 50-82 neutron shell. Even though a particle-hole symmetry can be seen from the \( \log(G) \) inset, there is a discrepancy between the two as one moves towards the upper part of the shell. This discrepancy is due to the shell closure, the single-particle level structure, and the Pauli blocking for odd \( A \) nuclei.

![Figure 1](image1.png)

**FIG. 1:** The solid line gives the binding energies of the Sn isotopes relative to that of the \( ^{108}\text{Sn} \) core. The single-particle energy scale is set by the binding energy of \( ^{101}\text{Sn} \). The inset shows the fit to values of \( G \) that reproduces the experimental data exactly. The two fitting functions are: \( \log(G(A)) = 365.0584 - 6.4836A + 0.0284A^2 \) for even values of \( A \) and \( \log(G(A)) = 398.2277 - 7.0349A + 0.0307A^2 \) for odd values of \( A \). The Nilsson BE energy is the lowest energy of the non-interacting system.

![Figure 2](image2.png)

**FIG. 2:** The solid line gives the binding energies of Yb isotopes relative to that of the \( ^{152}\text{Yb} \) core. The single-particle energy scale is set from the binding energy of \( ^{153}\text{Yb} \). The inset shows the fit to values of \( G \) that reproduce exactly the experimental data. The two fitting functions are: \( \log(G(A)) = 662.2247 - 7.7912A + 0.0226A^2 \) for even values of \( A \) and \( \log(G(A)) = 716.3279 - 8.4049A + 0.0244A^2 \) for odd values of \( A \). The Nilsson BE energy is the lowest energy of the non-interacting system.

Figure \( \zeta \) shows our results for the \( ^{154\ldots161}\text{Yb} \) isotopes. We test the predictive power of the model on the \( ^{172\ldots177}\text{Yb} \) isotopes. In this case calculations of the pairing strength \( G \) are carried out only for the \( ^{154\ldots173,178\ldots181}\text{Yb} \) isotopes and not for the \( ^{172\ldots177}\text{Yb} \) iso-
topes that are in the middle of the model space and thus are more computationally involved. This is a fit of two sets of 11 data points with two 3 parameter expressions. Then from the obtained quadratic polynomial fit to the log($G$) values we calculate the theoretical values of the binding energy for these nuclei as shown in Figure 2. This prediction is very good when compared to the experiment. Thus, based on experimental data of the nuclei in the upper and lower part of the shell and a log($G$) fit to this data we can make a reasonable estimate for the mid-shell nuclei. Therefore, the model has a good predictive power.

The next Figure 3 shows results for the $^{181-202}$Pb isotopes that were studied in the same way as the Sn and Yb isotopes. In the calculations for these Pb isotopes, however, the binding energy that was used is relative to that for $^{208}$Pb which was set to zero, but the core nucleus was chosen to be $^{164}$Pb. Note that for the Yb and Sn isotopes the core nucleus was also the binding energy reference nucleus ($^{100}$Sn and $^{152}$Yb). In contrast, the calculation for the Pb-isotopes was different because the core nucleus ($^{164}$Pb) and the binding energy reference nucleus ($^{208}$Pb) are different. We again have good quadratic fit to log($G$) as function of $A$.

The fact that there is a correlation between the pairing strength $G(A)$ and the size of the model space reflected in the minimum of $G$ that is at the maximal model space dimension prompted us to study $G(A)$ as function of the model space dimension dim($A$). In this respect, a remarkable results is shown in Figure 2. In this case the pairing strength $G(A)$ for all the 21 nuclei ($A=181-202$) was fit by a two parameter function $G(A) = \alpha / [\text{dim}(A)]^\beta$ with the values of the parameters taken to be $\alpha = 366.7702$ and $\beta = 0.9972$. This function is inversely proportional to the dimensionality of the model space dim($A$). Unfortunately, this is not the case for the other two isotopic chains. For example, in Figure 4 a log – log plot is shown for the Sn nuclei.

![Figure 3: The solid line gives the binding energy for the Pb isotopes relative to the $^{208}$Pb nucleus. The insets show the fit to the values of $G$ that reproduce exactly the experimental data using a $^{164}$Pb core. The lower inset shows the two fitting functions: log($G(A)$) = 382.3502 – 4.1375$A$ + 0.0111$A^2$ for even values of $A$ and log($G(A)$) = 391.6113 – 4.2374$A$ + 0.0114$A^2$ for odd values of $A$. The upper inset shows a fit to $G(A)$ that is inversely proportional to the size of the model space, dim($A$), that is valid for even as well as odd values of $A$: $G(A) = 366.7702 / \text{dim}(A)^{0.9972}$. The Nilsson BE energy is the lowest energy of the non-interacting system.](image1)

![Figure 4: Results for log($G$) versa log(dim) for the Sn nuclei.](image2)

It is unlikely that the linear relation between log($G$) and log(dim($A$)) for the Pb isotopes is due to the difference between the core and binding energy reference nucleus. What is more likely is that this is due to the fact that in these cases the model space dimensions are sufficiently large to result in a limiting form for the effective pairing strength. Another reason might be that the single-particle energies and the Pauli blocking mechanism are such that the even and odd parabolas of log($G(A)$) produce a linear log – log structure.

In the light of the above, it seems that the next step should be a study that tracks the results as a function of the increasing size of the model space to confirm or refute the log – log relation. Such a study could also address other questions such as the effect of the core binding energy as a function of the deformation that is used in the Nilsson model to derive the single-particle energies. Using a Woods-Saxon potential or other methods to generate more realistic single-particle energies is another opportunity for further studies.

In conclusion, we have studied binding energies of nuclei in three isotopic chains: $^{100-130}$Sn, $^{152-181}$Yb, and $^{181-202}$Pb within the recently proposed extended pairing model. By using Nilsson single-particle energies as the input mean-field energies. Overall, the results suggest that the model is applicable to well-deformed nuclei if the pairing strength is allowed to change as a (smooth) function of the nucleon number $A$. The remarkably similar behavior of log($G$) for even and odd $A$ values seems to suggest that there may be a single log($G$) function that bifurcates into an even-$A$ and an odd-$A$ branch when the fermion dynamics is restricted to hard boson pairs only and Pauli blocking is applied to exclude levels populated
by the unpaired fermion.

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