EQUIVARIANT SATAKE CATEGORY AND KOSTANT-WHITTAKER REDUCTION

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To Victor Ginzburg on his birthday

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Abstract. We explain (following V. Drinfeld) how the $G(\mathbb{C}[[t]])$ equivariant derived category of the affine Grassmannian can be described in terms of coherent sheaves on the Langlands dual Lie algebra equivariant with respect to the adjoint action, due to some old results of V. Ginzburg. The global cohomology functor corresponds under this identification to restriction to the Kostant slice. We extend this description to loop rotation equivariant derived category, linking it to Harish-Chandra bimodules for the Langlands dual Lie algebra, so that the global cohomology functor corresponds to the quantum Kostant-Whittaker reduction of a Harish-Chandra bimodule. We derive a conjecture of [11], which identifies the loop-rotation equivariant homology of the affine Grassmannian with quantized Toda lattice.

1. Introduction

Let $G$ be a semi-simple algebraic group over an algebraically closed characteristic zero field $k$. The fundamental object of the geometric Langlands duality theory is the so-called Satake category $\text{Perv}_{G_0}(\text{Gr})$. The latter is defined as the category of perverse sheaves on the affine (loop) Grassmannian $\text{Gr}$ equivariant with respect to the group of regular loops $G_0$.

It turns out that convolution provides $\text{Perv}_{G_0}(\text{Gr})$ with a tensor structure, and the celebrated geometric Satake isomorphism theorem establishes an equivalence between $\text{Perv}_{G_0}(\text{Gr})$ and the category of representations of the Langlands dual group $\check{G}$.

By its very definition $\text{Perv}_{G_0}(\text{Gr})$ arises as the heart of the t-structure on a monoidal triangulated category – the equivariant derived category $D_{G_0}(\text{Gr})$. It is a natural question (raised, in particular, by V. Drinfeld) to describe $D_{G_0}(\text{Gr})$ in terms of the dual group. Drinfeld has also noticed that at least some form of the answer follows from the results of V. Ginzburg’s preprint [18]. In the present paper we reproduce this description and extend it to a description of the loop rotation equivariant derived Satake category $D_{G_0 \times G_m}(\text{Gr})$.

The description of $D_{G_0}(\text{Gr})$ links it to conjugation equivariant coherent sheaves on the Langlands dual Lie algebra. The additional $S^1$ (or $G_m$) equivariance is connected to quantization of these to Harish-Chandra bimodules (see Theorem 1 for a precise formulation).

The argument follows the strategy of [18]; it is based on another result of Ginzburg [17], which reduces the question to computation of the global equivariant

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1We do not address Drinfeld’s problem to find a more natural derivation of the description, making compatibility with finer structures transparent. We understand that D. Gaitsgory and J. Lurie have made a significant progress in this direction.
cohomology of IC sheaves as modules over the global equivariant cohomology algebra \( H^\bullet_{GO \times G_m}(Gr) \). By an explicit calculation we show that \( H^\bullet_{GO \times G_m}(Gr) \) is related to the tensor square of the center of the enveloping of the dual Lie algebra \( \hat{g} \), while the global cohomology modules correspond to the bimodules, which describe twisting with a finite dimensional \( \hat{G} \)-modules on the category of Whittaker modules. This allows us to relate \( D_{GO \times G_m}(Gr) \) to Harish-Chandra bimodules, so that the global cohomology is identified with the Kostant-Whittaker reduction.

As an application we prove a conjecture of [11] which identifies the algebra of global equivariant homology of \( Gr \) equipped with the convolution algebra structure with the quantized Toda lattice (Theorem 3). Note that the quantized Toda lattice also appears in the apparently related computations by Givental, Kim and others of quantum \( D \)-module (quantum cohomology) of the flag variety of \( G \), see e.g. [20].

D. Ben-Zvi and D. Nadler have informed us that they have a more conceptual proof of some of our results, see [8].

Acknowledgements. As should be clear from the above, this modest token of gratitude to Victor Ginzburg is largely an outgrowth of his own works. Throughout the mathematical biography of the authors his ingenuity, generosity and enthusiasm have been an important source of support and inspiration. We wish Vitya many happy returns of the day and a lot of exciting mathematics ahead!

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2. Notations and statements of the results

2.1. Notations. \( k \) is the algebraically closed characteristic zero coefficient field. Let \( G \) be a semisimple complex algebraic group, \( G_O = G(\mathbb{C}[[t]]) \), \( G_F = G(\mathbb{C}((t))) \). The affine Grassmannian \( Gr = Gr_G = G_F/G_O \) carries the category \( \text{Perv}_{G_O}(Gr) \) of \( G_O \)-equivariant perverse constructible sheaves. It is equipped with the convolution monoidal structure, and is tensor equivalent to the tensor category \( Rep(\hat{G}) \) of representations of the Langlands dual group \( \hat{G} \) over the field \( k \) (see [23], [6], [18]). We denote by \( \hat{S} : Rep(\hat{G}) \to \text{Perv}_{G_O}(Gr) \) the geometric Satake isomorphism functor, and by \( S : Rep(\hat{G}) \to \text{Perv}_{G_O \times G_m}(Gr) \) its extension to the monoidal category of \( G_O \times G_m \)-equivariant perverse constructible sheaves. The Lie algebra of \( \hat{G} \) is denoted by \( \hat{g} \). We choose a Cartan subalgebra \( \mathfrak{t} \subset \hat{g} \); the corresponding Cartan torus in \( \hat{G} \) is denoted \( \hat{T} \). We choose the opposite Borel subalgebras \( \mathfrak{b}_\pm \supset \mathfrak{t} \) with nilpotent radicals \( \mathfrak{n}_\pm \) and corresponding unipotent subgroups \( \mathcal{N}_\pm \subset \mathcal{G} \). The Weyl group of \( \hat{G} \) is denoted by \( W \).
Let \( e, h, f \in \mathfrak{g} \) be a principal \( \mathfrak{sl}_2 \) triple such that \( f \in \mathfrak{n}_- \), \( e \in \mathfrak{n}_+ \). We have the Kostant slice \( e + \mathfrak{z}(f) \) to the principal nilpotent orbit. It is known that \( e + \mathfrak{z}(f) \sim \mathfrak{g}/Ad(G) \), and also \( e + \mathfrak{z}(f) \sim (e + \mathfrak{b}_-)/\mathcal{N}_- \); moreover the \( \mathcal{N}_- \) action on \( e + \mathfrak{b}_- \) is free. Let \( \Sigma \), \( \Upsilon \) be the images of \( e + \mathfrak{z}(f) \), \( e + \mathfrak{b}_- \) under a \( G \)-invariant isomorphism \( \mathfrak{g} \cong \mathfrak{g}^* \). Thus we have \( \Sigma \sim \Upsilon/\mathcal{N}_- = \mathfrak{i}^*/W \) canonically.

The total space of the tangent bundle of \( \mathfrak{i}^*/W \) is denoted \( T(\mathfrak{i}^*/W) \).

### 2.2. Asymptotic \( \mathfrak{g} \)-modules

Let \( U = U(\mathfrak{g}) \) be the enveloping algebra, and let \( U_h \) be the “graded enveloping” algebra, i.e. the graded \( k[\hbar] \)-algebra generated by \( \mathfrak{g} \) with relations \( xy - yx = \hbar[x, y] \) for \( x, y \in \mathfrak{g} \) (thus \( U_h \) is obtained from \( U \) by the standard Rees construction which produces a graded algebra from a filtered one); the adjoint action extends to the action of \( \tilde{G} \) on \( U_h \). We define the category \( \mathcal{HC}_h \) of “\( h \)-Harish-Chandra bimodules” as follows: an object \( M \) of this category is a graded \( U_h^\mathbb{Z} := U_h \otimes_{k[\hbar]} U_h \simeq U_h \otimes_k \mathcal{U} \)-module equipped with an algebraic action \( \rho \) of \( \mathcal{G} \) such that: (1) the action map \( U_h \otimes_{k[\hbar]} U_h \otimes M \to M \) is \( \mathcal{G} \) equivariant, and (2) for \( x \in \mathfrak{g} \) the action of \( (x \otimes 1 + 1 \otimes x) \in U_h \otimes_{k[\hbar]} U_h \) coincides with \( \hbar \cdot \rho(x) \). The functor of restriction from \( U_h \otimes U_h \) to \( U_h \otimes 1 \) is an equivalence between \( \mathcal{HC}_h \) and the category of \( \mathcal{G} \)-modules equipped with an equivariant \( U_h \)-action; the same is true for the restriction to \( 1 \otimes U_h \). We let \( \mathcal{HC}_h \subset \mathcal{HC} \) denote the full subcategory of objects which are finitely generated as \( U_h \otimes 1 \) modules (equivalently, as \( 1 \otimes U_h \) modules).

Notice that the full subcategory of \( \mathcal{HC}_h \) consisting of objects where \( h \) acts by zero is identified with the category \( \mathcal{Coh}^{\tilde{G}}(\mathfrak{g}^*) \) of coherent sheaves on \( \mathfrak{g}^* \) equivariant under the coadjoint action; while for \( s \in k \), \( s \neq 0 \) the subcategory where \( h \) acts by \( s \) is identified with the category of Harish-Chandra bimodules.

### 2.3. Kostant functor \( \kappa_h \)

We now proceed to define a functor \( \kappa_h : \mathcal{HC}_h \to Q \mathcal{Coh}^{G_m}((\mathfrak{i}^*/W)^2 \times \mathbb{A}^1) \).

Let \( \psi : U_h(\mathfrak{n}_-) \to k[\hbar] \) be a homomorphism such that \( \psi(f_\alpha) = 1 \) for any simple root \( \alpha \), and a root generator \( f_\alpha \in \mathfrak{n}_- \subset U_h(\mathfrak{n}_-) \).

Define \( U_h^2(\mathfrak{n}_-) \subset U_h^2 \) by \( U_h^2(\mathfrak{n}_-) = U_h(\mathfrak{n}_-) \otimes U(\mathfrak{n}_-) \). We extend \( \psi \) to a character \( \psi(2) : U_h^2(\mathfrak{n}_-) = U_h(\mathfrak{n}_-) \otimes U(\mathfrak{n}_-) \to k[\hbar] \) trivial on the second multiple. Note that its restriction to the first copy of \( U_h \) is \( \psi \), and its restriction to the second copy is \( -\psi \).

We set \( \kappa_h(M) = (M \otimes_{U_h(\mathfrak{n}_-)} (-\psi))^N \) — where the action of the second copy of \( U_h \) is used (though using the first one we get a canonically isomorphic functor). Clearly, \( \kappa_h(M) \) is equipped with the action of the Harish-Chandra center \( Z(U_h) \otimes_{k[\hbar]} Z(U_h) = \mathcal{O}(U(\mathfrak{i}^*/W) \times (\mathfrak{i}^*/W) \times \mathbb{A}^1) \), and with the grading (coming from the action of the Cartan element \( h \) of the principal \( \mathfrak{sl}_2 \)), so we may view \( \kappa_h(M) \) as a \( G_m \)-equivariant quasicoherent sheaf on \( (\mathfrak{i}^*/W)^2 \times \mathbb{A}^1 \).

If \( X \) is a scheme, and \( Z \subset X \) is a closed subscheme let \( N_XZ \) be the deformation to the normal cone to \( Z \), see [15]. It is equipped with a morphism \( N_XZ \to X \times \mathbb{A}^1 \) (with coordinate \( h \) on \( \mathbb{A}^1 \)), and is defined as the relative spectrum of the sheaf of subalgebras in \( \mathcal{O}_X[h^{\pm 1}] \) generated by the elements \( f \ h^{-1}, \ f \in \mathcal{O}_X : \ f|_Z = 0 \).
If $M$ is a free $k[h]$-module, the action of $\mathcal{O}((t^*/W)^2 \times A^1)$ on $\kappa_h(M)$ extends uniquely to the action of $\mathcal{O}(N(t^*/W)^2 \Delta)$ where $\Delta \subset (t^*/W)^2$ is the diagonal. So we can and will view $\kappa_h(M)$ as a $\mathbb{G}_m$-equivariant coherent sheaf on $N(t^*/W)^2 \Delta$.

For $V \in \text{Rep}(\hat{G})$ we define the $h$-Harish-Chandra bimodule $Fr(V)$ by $Fr(V) = U_h \otimes V$ with its natural $\hat{G}$-module structure $g(y \otimes v) = Ad(g)(y) \otimes g(v)$, and with the $U_h \otimes U_h$ action specified by $x \otimes u(y \otimes v) = xyu \otimes v + h \cdot xy \otimes u(v)$ for $x, u \in \mathfrak{g} \subset U_h$. In other words, $Fr(V)$ is obtained by applying the induction (left adjoint to the restriction functor) $\text{Rep}(\hat{G}) \to \mathcal{H} \mathcal{E}_h$ to $V$. We set $\phi(V) := \kappa_h(Fr(V))$.

Clearly $Fr(V)$ is a projective object of $\mathcal{H} \mathcal{E}_h$ for any $V \in \text{Rep}(\hat{G})$; we call an object of the form $Fr(V)$ a free $h$-Harish-Chandra bimodule. We define the full subcategory $\mathcal{H} \mathcal{E}_h^{fr} \subset \mathcal{H} \mathcal{E}_h$ to consist of all free objects.

2.4. Equivariant cohomology of $\text{Gr}_G$. Note that $H_G^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr})) = H_{\mathcal{G}_m}^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}))$, whence two morphisms $pr_1^*, pr_2^* : \mathcal{O}(t/W) = H_G^{\bullet}(\text{pt}) \to H_{\mathcal{G}_m}^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}))$. We also have a morphism $pr^* : k[h] = H_{\mathcal{G}_m}^{\bullet}(\text{pt}) \to H_{\mathcal{G}_m}^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}))$.

Theorem 1. a) Assume $G$ is simply connected. We have a canonical isomorphism $H_G^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr})) \cong \mathcal{O}(N(t^*/W \times t^*/W \Delta))$ where $\Delta \subset (t^*/W)^2$ is the diagonal. Here the projection $N(t^*/W \Delta) \to \mathbb{A}^1$ corresponds to the homomorphism $H_G^{\bullet}(\text{pt}) \to H_{\mathcal{G}_m}^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}))$; and the two projections $N(t^*/W \Delta) \to t^*/W = t^*/W \Delta$ correspond to the two homomorphisms $H_G^{\bullet}(\text{pt})\to H_{\mathcal{G}_m}^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}))$. The isomorphism is specified uniquely by these requirements.

b) For arbitrary $G$ we have a canonical isomorphism $H_G^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr})) \cong \bigoplus \pi_1(G)^* \mathcal{O}(N(t^*/W \times t^*/W \Delta))$.

Remark 1. To simplify the exposition we assume from now on that $G$ is simply connected.

Cohomology of any complex of sheaves on a topological space carries an action of the cohomology algebra of the space; thus we have the functor of equivariant cohomology $H_{\mathcal{G}_m}^{\bullet} : D_{\mathcal{G}_m} \to H_{\mathcal{G}_m}^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}))_{\text{mod}^{fr}} = Coh_{\mathcal{G}_m}(N(t^*/W \Delta))$

where $D_{\mathcal{G}_m}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}))$ denotes the bounded constructible equivariant derived category, and the grading on $H_{\mathcal{G}_m}^{\bullet}(\mathcal{O}_G \ltimes \mathcal{G}_m(\text{Gr}), ?)$ is the one by the cohomology degree.

Theorem 2. The functor $S : \text{Rep}(\hat{G}) \to \text{Perv}_{\mathcal{G}_m}(\text{Gr})$ extends to a full imbedding $S_h : \mathcal{H} \mathcal{E}_h^{fr} \to D_{\mathcal{G}_m}(\text{Gr})$, such that

\begin{equation}
\kappa_h \cong H_{\mathcal{G}_m}^{\bullet} \circ S_h.
\end{equation}

Such an extension $S_h$ (for a fixed isomorphism (1)) is unique.

2.5. Equivariant homology and quantum Toda lattice. Let $\mathcal{D}_h(\hat{G})$ stand for the sheaf of $h$-differential operators on $\hat{G}$; its global sections is the smash product of $U_h$ and $\mathcal{O}(\hat{G})$. The action of $\mathfrak{h}_-$ by the left-invariant (resp. right-invariant) vector fields on $\hat{G}$ gives rise to the homomorphism $l$ (resp. $l_r$) $U_h(\mathfrak{h}_-) \to \mathcal{D}_h(\hat{G})$. Let $I_\psi \subset \mathcal{D}_h(\hat{G})$
be the left ideal generated by the $h$-differential operators of the sort $l(u_1) - \psi(u_1) + r(u_2) + \psi(u_2)$; $u_1, u_2 \in U_h(\hat{n}_-)$. We consider the quantum hamiltonian reduction

$$(\mathcal{D}_h(\hat{G})/I_\psi)^{\hat{N}_- \times \hat{N}_-}$$

where the first (resp. second) copy of $\hat{N}_-$ acts on $\hat{G}$ (and hence on $\mathcal{D}_h(\hat{G})$) by the left (resp. right) translations: $(n_1, n_2) \circ g := n_1 gn_2^{-1}$. It is an algebra containing a commutative subalgebra $Z(U_h)$ (via the embedding $Z(U_h) \hookrightarrow \mathcal{D}_h(\hat{G})$ as both left- and right-invariant $h$-differential operators). Note that the action of $\hat{N}_- \times \hat{N}_-$ on the “big Bruhat cell” $C_{w_0} := N_- \cdot T \cdot w_0 \cdot N_- \subset \hat{G}$ is free, and hence the quantum hamiltonian reduction of $\mathcal{D}_h(C_{w_0})$ is isomorphic to $\mathcal{D}_h(\hat{T})$. This is the classical Kazhdan-Kostant construction of the quantum Toda lattice, see [22]. Thus, the quantum Toda lattice is a certain localization of $(\mathcal{D}_h(\hat{G})/I_\psi)^{\hat{N}_- \times \hat{N}_-}$ “the quantized Toda lattice”, somewhat abusing the language. The following result was conjectured in [11].

**Theorem 3.** The convolution algebra of equivariant homology $H^*_{GO\times G_m}(Gr)$ is naturally isomorphic to the quantized Toda lattice $(\mathcal{D}_h(\hat{G})/I_\psi)^{\hat{N}_- \times \hat{N}_-}$. The embedding $Z(U_h) \simeq H_{GO\times G_m}(pt) \hookrightarrow H^*_{GO\times G_m}(Gr)$ corresponds to the embedding $Z(U_h) \hookrightarrow (\mathcal{D}_h(\hat{G})/I_\psi)^{\hat{N}_- \times \hat{N}_-}$.

### 2.6. Quasiclassical limit.

Recall that the fiber of $N_X Z$ over $0 \in \mathbb{A}^1$ is the normal cone to $Z$ in $X$. In particular, the fiber of $N_{(1^* / W)\Delta}$ over $0 \in \mathbb{A}^1$ is the total space of the tangent bundle $T(\hat{t}^*/W)$. Thus, Theorem 1 implies the canonical isomorphism $H^*_{GO}(Gr) \simeq \mathcal{O}(T(\hat{t}^*/W))$. On the other hand, $H^*_{GO}(Gr)$ was computed by V. Ginzburg in [18] in terms of the universal centralizer bundle of $\hat{g}$. The two computations are related as follows.

The variety $(\hat{g}^*)^\text{reg}$ of regular elements in $\hat{g}^*$ carries a sheaf of commutative Lie algebras $\mathfrak{g} \subset \hat{g} \otimes \mathcal{O}$ whose fiber at a point $\xi \in (\hat{g}^*)^\text{reg}$ is the stabilizer of $\xi$. We claim a canonical isomorphism $\mathfrak{g} \cong pr^*(T^*)$ where $pr : (\hat{g}^*)^\text{reg} \to \hat{t}^*/W$ is the projection to the spectrum of invariant polynomials, and $T^*$ stands for the cotangent sheaf. Indeed, the fiber of $pr^*(T^*)$ at a point $\xi \in \hat{g}^*$ is dual to the cokernel of the map $\hat{g} \to \hat{g}^*$, $x \mapsto \text{coad}(x)(\xi)$; thus it is canonically isomorphic to the kernel of the dual map (which happens to coincide with the original map), which is exactly the fiber of $\mathfrak{g}$ at $\xi$.

In view of this identification, one should compare Lemma 9 in subsection 4.7 below with Ginzburg’s description in [18] of $G_O$-equivariant Intersection Cohomology of a $G_O$-orbit in $\text{Gr}$ as a $\mathfrak{g}$-module.

We now proceed to define the Kostant functor $\kappa : \text{Coh}_{\hat{G} \times G_m}(\hat{g}^*) \to \text{Coh}^{G_m}(\mathcal{T}(\hat{t}^*/W))$, $G_m$-equivariant coherent sheaves on the tangent bundle to $\hat{t}^*/W$.

If $\mathcal{F} \in \text{Coh}_{\hat{G}}(\hat{g}^*)$ is equipped with an equivariant structure, then $\mathcal{F}|_{(\hat{g}^*)^\text{reg}}$ carries an action of $\mathfrak{g}$; thus by the previous paragraph it defines a coherent sheaf on the total space of the pull-back of the tangent bundle under $pr$. Restricting this sheaf to the preimage of $\Sigma$ we get a coherent sheaf on the tangent bundle to $\Sigma = \hat{t}^*/W$ which we denote by $\pi(\mathcal{F})$. Notice that $\pi(\mathcal{F}) = (\mathcal{F}|_{\Sigma})^{\hat{N}_-}$ (where we do not distinguish between a coherent sheaf on an affine variety and the module of its global sections). An obvious
modification of this definition yields a functor \( \kappa : \text{Coh}^{\hat{G} \times \hat{G}_m}(\hat{\mathfrak{g}}^*) \to \text{Coh}^{\hat{G}_m}(T(\hat{\mathfrak{g}}^*/W)) \)
(where the action of \( \hat{G}_m \) on \( \hat{\mathfrak{g}}^*/W \) is the natural one).

Define the full subcategory \( \text{Coh}^{\hat{G} \times \hat{G}_m}(\hat{\mathfrak{g}}^*) \subset \text{Coh}^{\hat{G} \times \hat{G}_m}(\hat{\mathfrak{g}}^*) \) to consist of all objects of the form \( V \otimes \mathcal{O}_{\hat{g}^*} \), for \( V \in \text{Rep}(\hat{G} \times \mathbb{G}_m) \).

Recall that \( \tilde{S} : \text{Rep}(\hat{G}) \to \text{Perv}_{\mathcal{G}_O}(\text{Gr}) \) is the composition of \( S \) with the forgetful functor \( \text{Perv}_{\mathcal{G}_O \times \mathbb{G}_m}(\text{Gr}) \to \text{Perv}_{\mathcal{G}_O}(\text{Gr}) \). We have the functor of \( \mathcal{G}_O \)-equivariant cohomology

\[
H^*_\mathcal{G}_O : D_{\mathcal{G}_O}(\text{Gr}) \to H^*_\mathcal{G}_O(\text{Gr}) - \text{mod}^{\mathbb{Z}} = \text{Coh}^{\hat{G}_m}(T(\hat{\mathfrak{g}}^*/W))
\]

**Theorem 4.** The functor \( \tilde{S} \) extends to a full imbedding \( \tilde{S}_{qc} : \text{Coh}^{\hat{G} \times \hat{G}_m}(\hat{\mathfrak{g}}^*) \to D_{\mathcal{G}_O}(\text{Gr}) \), such that there exists an isomorphism

\[
(2) \quad \kappa \cong H^*_\mathcal{G}_O \circ \tilde{S}_{qc}.
\]

Such an extension \( \tilde{S}_{qc} \) (for a fixed isomorphism (2)) is unique.

### 2.7. Equivalences

To a differential graded algebra \( A \) one can associate the triangulated category \( D(A) \) of differential graded modules localized by quasi-isomorphism; and a full triangulated subcategory \( D_{\text{perf}}(A) \subset D(A) \) of perfect complexes. Thus \( D_{\text{perf}}(A) \) is the full subcategory in the latter category consisting of perfect complexes (i.e. generated by the free module under cones and direct summands). Given an algebraic group \( H \) acting on a dg-algebra \( A \), we can consider equivariant dg-modules and localize them by quasi-isomorphisms, arriving at the equivariant version \( D^H_{\text{perf}}(A) \).

We now consider the “differential-graded versions” \( \text{Sym}^\bullet(\hat{\mathfrak{g}}), U^\bullet_h \) of the graded algebras \( \text{Sym}(\mathfrak{g}), U_h(\mathfrak{g}) \). By definition \( \text{Sym}^\bullet(\hat{\mathfrak{g}}), U^\bullet_h \) are differential graded algebras with zero differential, which as algebras are isomorphic to \( \text{Sym}(\mathfrak{g}), U_h(\mathfrak{g}) \) respectively. The cohomological grading is defined so that elements of \( \hat{\mathfrak{g}} \) and \( \mathfrak{h} \) have degree two. Recall from section 2.2 that an asymptotic Harish-Chandra bimodule \( M \in \text{H}C(\mathfrak{h}) \) is nothing but a \( \hat{G} \)-equivariant \( U^\bullet_h \)-module. Using this identification we can transfer tensor product of asymptotic Harish-Chandra bimodules to a monoidal structure on the category of \( \hat{G} \)-equivariant \( U^\bullet_h \)-modules. It gives rise to a monoidal structure on \( D^G_{\text{perf}}(U^\bullet_h) \). Similarly, we define a monoidal structure on \( D^G_{\text{perf}}(\text{Sym}^\bullet(\hat{\mathfrak{g}})) \).

**Theorem 5.** There exist canonical equivalences of monoidal triangulated categories

\[
D^G_{\text{perf}}(U^\bullet_h) \cong D_{\mathcal{G}_O \times \mathbb{G}_m}(\text{Gr}), \quad D^G_{\text{perf}}(\text{Sym}^\bullet(\hat{\mathfrak{g}})) \cong D_{\mathcal{G}_O}(\text{Gr}).
\]

The following statement is an immediate consequence of the Theorem, which has the (psychological) advantage of bypassing the notion of a dg-algebra.

**Corollary 1.** a) The derived graded Harish-Chandra bimodule category is Koszul dual to a graded version of the loop-rotation equivariant Satake category, i.e. (cf. [7], [2], [10]) there exists a functor \( \Phi : D^h(\text{H}C(\mathfrak{h})) \to D_{\mathcal{G}_O \times \mathbb{G}_m}(\text{Gr}) \), such that

i) \( \Phi(M(1)) \cong \Phi(M)[1] \) in a natural way.

ii) For \( M_1, M_2 \in D^h(\text{H}C(\mathfrak{h})) \), \( \Phi \) induces an isomorphism

\[
\sum_{n,m} \text{Hom}(M_1, M_2(n)[m]) \cong \sum_k \text{Hom}(\Phi(M_1), \Phi(M_2)[k])
\]
3.1. **Proof of Theorem 1.**

a) First we construct the morphism $α : \mathcal{O}(N_{t/W × t/W} \Delta) → H^*_G \times G_m(Gr)$. Recall that $H^*_G \times G_m(Gr) = H^*_{G_m} (G_G \setminus G_F / G_G)$, whence two morphisms $pr_{1}^*, pr_{2}^* : \mathcal{O}(t/W) = H^!_{G_G} (pt) → H^*_G \times G_m(Gr)$. We also have a morphism $pr^* : \{h\} = H^*_{G_m}(pt) → H^*_G \times G_m(Gr)$. Since $H^*_G \times G_m(Gr)|_{h=0} = H^*_G (Gr)$, it follows that $pr^*|_{h=0} = pr_{2}^*|_{h=0}$. Hence the morphism $(pr_{1}^*, pr_{2}^*, pr^*) : \mathcal{O}(t/W × t/W × A^1) → H^*_G \times G_m(Gr)$ factors through the desired morphism $\mathcal{O}(t/W × t/W × A^1) → \mathcal{O}(N_{t/W × t/W} \Delta) α H^*_G \times G_m(Gr)$.

Next we prove that $α$ is an embedding. It suffices to prove that the localized morphism

$$α_{t/loc} : \mathcal{O}(N_{t/W × t/W} \Delta) ⊕ \mathcal{O}(t/W × A^1) \text{Frac}(\mathcal{O}(t × A^1)) → H^*_G \times G_m(Gr) ⊕ \mathcal{O}(t/W × A^1) \text{Frac}(\mathcal{O}(t × A^1))$$

is an embedding. Note that the RHS is the localized equivariant cohomology $H^*_T \times G_m(Gr)_{t/loc}$, which embeds into the inverse limit of the localized equivariant cohomology $H^*_T \times G_m(Gr)_{loc}$ of the $G_G$-orbit closures $Gr_\lambda \subset Gr$. By the Localization Theorem for torus-equivariant cohomology, the latter is $\prod_\lambda \text{Frac}(\mathcal{O}(t × A^1))$ (cf. 3.2), and $α_{t/loc}$ is an embedding.

Finally, it remains to check that the graded dimensions of $H^*_G \times G_m(Gr)$, and of $\mathcal{O}(N_{t/W × t/W} \Delta)$ coincide. Here the grading of $\mathcal{O}(N_{t/W × t/W} \Delta)$ comes from the natural $G_m$-actions on $t$ and $A^1$. To this end note that the graded dimension of $H^*_G \times G_m(Gr)$ coincides with that of $H^*_G (pt) ⊕ H^*_{G_m} (pt) ⊕ H^* (Gr)$, that is $k[x_1, \ldots , x_r, y_1, \ldots , y_r , h]$. Here $r$ is the rank of $G$; the degree of $h$ is 2; the degrees of $x_i$’s are twice the exponents of $\mathfrak{g}$; the degrees of $y_i$’s are twice the exponents of $\mathfrak{g}$ minus 2.

Now to compute the graded dimension of $\mathcal{O}(N_{t/W × t/W} \Delta)$ we use that $t/W$ is isomorphic to the vector space $\Sigma$. More generally, for vector spaces $V, V'$ we have an isomorphism $β : V × V' × A^1 \sim N_{V × V'} V$. In effect, the map $γ : V × V' × A^1 → V × V' × A^1, (v, v', a) → (v, av', a)$, factors through the desired isomorphism: $V × V' × A^1 β N_{V × V'} V → V × V' × A^1$. We derive an isomorphism $N_{t/W × t/W} \Delta β t/W × t/W × A^1$ which lowers the $G_m$-weights in the second copy of $t/W$ by 1, whence the desired formula for the graded dimension of $\mathcal{O}(N_{t/W × t/W} \Delta)$.

This completes the proof of the part a) of the theorem.

b) Let $\tilde{G}$ stand for the simply connected cover of $G$. Then the Grassmannian $Gr_G$ is a union of connected components numbered by the characters $χ ∈ π_1(\tilde{G})$, and each connected component is isomorphic to $Gr_G$. Moreover, the isomorphisms of various connected components are $T × G_m$-equivariant. Hence $H^*_T \times G_m (Gr)$ is equal
Clearly, the canonical filtration with the associated graded of this filtration is 
Levi-equivariant cohomology.

Canonical filtration on $H^\bullet_{T\times G_m}(Gr_G)$. For a $G_O \times G_m$-equivariant perverse sheaf $\mathcal{F}$ on $Gr_G$, we will define a canonical filtration on $H^\bullet_{T\times G_m}(\mathcal{F}) = H^\bullet_{G_O \times G_m}(\mathcal{F}) \otimes_{O(t/W)} \mathcal{O}(t)$. Note that if $\pi$ is the projection $t \to t/W$, and $(\pi, \text{Id}, \text{Id})$ is the projection $t \times (t/W) \times A^1 \to t/W \times t/W \times A^1$, then $H^\bullet_{G_O \times G_m}(\mathcal{F}) \otimes_{O(t/W)} \mathcal{O}(t) = (\pi, \text{Id}, \text{Id})^* H^\bullet_{G_O \times G_m}(\mathcal{F})$. For $\lambda \in X_*(T)$ we denote by $\lambda$ the corresponding $T$-fixed point of $Gr_G$. We denote by $\Xi_{\lambda}$ the closure of $\Xi_{\mu}$ through $\lambda$. We denote by $\Xi_{\lambda}$ the union of $\Xi_{\mu}$ over $\mu \geq \lambda$. We filter $H^\bullet_{T\times G_m}(\mathcal{F})$ by the images of $r_\lambda : H^\bullet_{\Xi_{\lambda}, T\times G_m}(Gr_G, \mathcal{F}) \to H^\bullet_{T\times G_m}(Gr_G, \mathcal{F})$ (cohomology with supports). The associated graded of this filtration is $\bigoplus_{\lambda} H^\bullet_{\Xi_{\lambda}, T\times G_m}(Gr_G, \mathcal{F})$. Since $\lambda$ is the only $T \times G_m$-fixed point of $\Xi_{\lambda}$, we have $H^\bullet_{T\times G_m}(Gr_G, \mathcal{F}) = H^\bullet_{T\times G_m}(\lambda) \otimes j^\lambda_{\Xi_{\lambda}} \mathcal{F}$ where $j_{\lambda}$ is the locally closed embedding of $\Xi_{\lambda}$ into $Gr_G$, and $j_{\lambda}$ is the embedding of $\lambda$ into $\Xi_{\lambda}$.

Now recall that $\mathcal{F} \mapsto j_{\Xi_{\lambda}}^\lambda \mathcal{F}$ is the $\lambda$-weight component of the Mirković-Vilonen fiber functor on the category of $G_O \times G_m$-equivariant perverse sheaves on $Gr_G$. In other words, if $V$ is a $G$-module, and $\mathcal{F} = S(V)$, then $\mathcal{F} \mapsto j_{\Xi_{\lambda}}^\lambda \mathcal{F} = _V \mathcal{F}$ where $_V \mathcal{F}$ is the $\lambda$-weight component of $V$.

Furthermore, we claim that the $\mathcal{O}(t \times (t/W) \times A^1)$-module $H^\bullet_{T\times G_m}(\lambda)$ is canonically isomorphic to $(\text{Id}, \pi, \text{Id})_* \mathcal{O}(\Gamma_{\lambda})$ where $\Gamma_{\lambda} \subset t \times t \times A^1$ is given by the equation $\Gamma_{\lambda} = \{(x_1, x_2, a) : x_2 = x_1 + a \lambda\}$. Since $p$ is the projection from the affine flag variety $Fl_G$ to the affine Grassmannian $Gr_G$. Let $\lambda$ be a $T \times G_m$-fixed point of $Fl_G$ such that $p$ projects $\lambda$ isomorphically onto $\lambda$. We stand for the Iwahori subgroup of $G_O$. We have $H^\bullet_{T\times G_m}(Fl_G) = H^\bullet_{2m}(I, G_F/I)$, and so $H^\bullet_{T\times G_m}(\lambda)$ is a module over $\mathcal{O}(t \times (t \times A^1)$.

Clearly, the $\mathcal{O}(t \times (t/W) \times A^1)$-module $H^\bullet_{T\times G_m}(\lambda)$ is isomorphic to the direct image of the $\mathcal{O}(t \times t \times A^1)$-module $H^\bullet_{T\times G_m}(\lambda)$ under the projection $t \times t \times A^1 \to t \times (t/W) \times A^1$. So it suffices to check that the $\mathcal{O}(t \times t \times A^1)$-module $H^\bullet_{T\times G_m}(\lambda)$ is isomorphic to $\mathcal{O}(\Gamma_{\lambda})$ after localization along $t \times A^1$.

The set of $T$-fixed points in $Fl_G$ is canonically identified with the extended affine Weyl group $W_{aff}$ of $G$, and we choose $\lambda$ so that it coincides with $\lambda \in W_{aff}$. Then the preimage $T_{\lambda}$ of $\lambda \in Fl_G$ in $G_F$ is homotopically equivalent to $T$, and the action of $T \times T \times G_m$ on $T_{\lambda}$ is homotopically equivalent to $(t_1, t_2, z)(t) = t_1 t_2^{-1} \lambda(z)$. We conclude that the $\mathcal{O}(t \times t \times A^1)$-module $H^\bullet_{T\times G_m}(\lambda) = \mathcal{O}(T \setminus T_{\lambda}/T)$ is isomorphic to $\mathcal{O}(\Gamma_{\lambda})$.

We have proved the following

**Lemma 1.** For $V \in \text{Rep}(G)$, the $\mathcal{O}(t \times (t/W) \times A^1)$-module $H^\bullet_{T\times G_m}(Gr_G, S(V))$ has a canonical filtration with the associated graded $\bigoplus_{\lambda} (\text{Id}, \pi, \text{Id})_* \mathcal{O}(\Gamma_{\lambda}) \otimes _V \mathcal{F}$. In particular, $H^\bullet_{T\times G_m}(Gr_G, S(V))$ is flat as an $\mathcal{O}(t \times A^1)$-module.

\[\square\]

3.3. Levi-equivariant cohomology. Let $T \subset L \subset G$ be a Levi subgroup. We denote by $P_L$ (resp. $P^L_-$) the parabolic subgroup generated by $L$ and the positive (resp. negative) Borel subgroup $B$ (resp. $B_-$. We denote by $W_L \subset W$ the Weyl group of
transitivity for a pair of Levi subgroups. We have a natural projection from $X_L^+$ to the lattice $X^* (Z (L))$ of characters of the center of $Z (L)$ of $L$. The set of $P_L^- (F)$-orbits in $Gr_G$ is numbered by $X^* (Z (L))$. For $\lambda \in X^* (Z (L))$ we will denote the corresponding orbit by $L_{\lambda}$, and its closure by $L_{\bar{\lambda}}$. The locally closed embedding of $\Sigma_{\lambda}$ into $Gr_G$ is denoted by $\iota_{\lambda}$. For an $L$-module $V$ we denote by $S_L (V)$ the corresponding $L (O) \rtimes \mathbb{G}_m$-equivariant perverse sheaf on $Gr_L$.

Lemma 2. For $V \in \text{Rep}(G)$, the $\mathcal{O} (t/W_L \times t/W \times \mathbb{A}^1)$-module $(\pi_L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V))$ carries a canonical filtration $F^*_L$ such that the associated graded is equipped with a canonical isomorphism

$$\top \Xi_L : gr (\pi_L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V)) \sim (\text{Id}, \pi_L, \text{Id})_* H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_L, S_L (V|_L)).$$

Proof: We have $(\pi_L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V)) = H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_G, S (V)) = H^*_{L \times X \rtimes \mathbb{G}_m} (Gr_G, S (V)).$ The canonical filtration in question is filtration by the images of $\tau_{\lambda} : H^*_{L \times \mathbb{X} \rtimes \mathbb{G}_m} (Gr_G, S (V)) \to H^*_{L \times \mathbb{X} \rtimes \mathbb{G}_m} (Gr_G, S (V))$ (cohomology with supports; here $\lambda \in X^* (Z (L))$). The associated graded of this filtration is $\bigoplus_{\lambda \in X^* (Z (L))} H^*_{L \times \mathbb{X} \rtimes \mathbb{G}_m} (Gr_G, S (V)).$ Let $p^\lambda : L_{\lambda} \to \text{Gr}_L$ denote the natural $L (O) \rtimes \mathbb{G}_m$-equivariant projection. Then we have $H^*_{L \times \mathbb{X} \rtimes \mathbb{G}_m} (Gr_G, S (V)) = H^*_{L \times \mathbb{X} \rtimes \mathbb{G}_m} (Gr_L, p^\lambda_! S (V)).$ However, according to [6], we have a canonical isomorphism $\bigoplus_{\lambda \in X^* (Z (L))} p^\lambda_! S (V) = S_L (V|_L).$ The lemma is proved. □

3.4. Transitivity for a pair of Levi subgroups. We have a canonical isomorphism $\top \Xi_L : gr (\pi_L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V)) \sim (\text{Id}, \pi_L, \text{Id})_* H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_L, S_L (V|_L)).$ In the RHS we have the restriction of $\mathcal{O} (t/W_L \times t/W \times \mathbb{A}^1)$-module $H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_L, S_L (V|_L))$ to $\mathcal{O} (t/W_L \times t/W \times \mathbb{A}^1)$. To save a bit of notation in what follows we will write simply

$$\top \Xi_L : gr (\pi_L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V)) \sim (\text{Id}, \pi_L, \text{Id})_* H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_L, S_L (V|_L)).$$

If $\bar{T} \subset \bar{L} \subset \bar{L}$ is another Levi subgroup, then we denote by $\pi_{\bar{L}}^L$ the projection from $t/W_L$ to $t/W_{\bar{L}}$. Note that the filtration $F_{\bar{L} L}^*$ on $(\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V)) = (\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V))$ is a refinement of the filtration $(\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* F_{\bar{L} L}$, and hence induces a canonical filtration $F_{\bar{L} L}^*$ on $(\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* F_{\bar{L} L}^* (\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V))$. The isomorphism $(\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* \top \Xi_L$ carries the filtration $F_{\bar{L} L}^*$ to the filtration $F_{\bar{L} L}^*$ on $(\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_L, S_L (V|_L)).$ We have a canonical isomorphism $\top \Xi_{\bar{L} L} : gr F_{\bar{L} L}^* (\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_L, S_L (V|_L)) \sim H^*_{L (O) \rtimes \mathbb{G}_m} (Gr_{\bar{L}}, S_{\bar{L} L} (V|_{\bar{L}})).$

We consider the composition

$$(3) \quad gr F_{\bar{L} L}^* (\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* H^*_{G_O \rtimes \mathbb{G}_m} (Gr_G, S (V)) \xrightarrow{\text{gr} F_{\bar{L} L}^* (\pi_{\bar{L}}^L, \text{Id}, \text{Id})^* \top \Xi_L}$$
In the RHS we have the restriction of $O$ and after tensoring with $V$ we have a canonical isomorphism $H^*_{O}(G) \cong H^*_{\text{Gr}}(\mathcal{O}_{V})$. Then we have $gr_{F^*_L}(\pi^L, \text{Id, Id})^*\Xi^L_{V} \rightarrow H^*_{L^*\text{Gr}}(\mathcal{O}_{V^L})$. Then we have

**Lemma 3.** $\Xi^L_{V} \circ gr_{F^*_L}(\pi^L, \text{Id, Id})^* \Xi^L_{L} = \Xi^L_{L'}$.

\[\square\]

3.5. **Tensor structure on equivariant cohomology.** A $G_{m} \times \mathfrak{g}_{m}$-equivariant sheaf $\mathcal{F}$ on $\text{Gr}_{G}$ will be viewed as a sheaf on the stack $G_{O} \times \mathfrak{g}_{m}/G_{F} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}$, given two such sheaves $\mathcal{F}_{1}, \mathcal{F}_{2}$ we define $\mathcal{F}_{1} \boxtimes \mathcal{F}_{2}$ as the descent of $\mathcal{F}_{1} \boxtimes \mathcal{F}_{2}$ from

\[(G_{m} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}) \times (G_{m} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}) \times (G_{m} \times \mathfrak{g}_{m}/G_{F} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}) \times (G_{m} \times \mathfrak{g}_{m}/G_{F} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}) \times (G_{m} \times \mathfrak{g}_{m}/G_{F} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m})
\]

Clearly,

\[H^*(G \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}/G_{F} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}) = H^*(G \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}/G_{F} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}, \mathcal{F}_{1} \boxtimes \mathcal{F}_{2}) = H^*(G \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}/G_{F} \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}, \mathcal{F}_{1} \boxtimes \mathcal{F}_{2})
\]

The multiplication in $G \times \mathfrak{g}_{m}$ gives rise to the map $m : G \times \mathfrak{g}_{m} \rightarrow G \times \mathfrak{g}_{m}$.

The convolution $\mathcal{F}_{1} \ast \mathcal{F}_{2}$ is defined as $m_{*}(\mathcal{F}_{1} \boxtimes \mathcal{F}_{2})$. Hence

\[H^*_{O \times \mathfrak{g}_{m}}(G, \mathcal{F}_{1} \ast \mathcal{F}_{2}) = H^*(G \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}, \mathcal{F}_{1} \boxtimes \mathcal{F}_{2}) = H^*(G \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}, \mathcal{F}_{1} \boxtimes \mathcal{F}_{2}) = H^*(G \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}, \mathcal{F}_{1} \boxtimes \mathcal{F}_{2}) = H^*(G \times \mathfrak{g}_{m}/G_{O} \times \mathfrak{g}_{m}, \mathcal{F}_{1} \boxtimes \mathcal{F}_{2})
\]

Now for $V_{1}, V_{2} \in \text{Rep}(G)$, and $\mathcal{F}_{1} = S(V_{1}), \mathcal{F}_{2} = S(V_{2})$ we have a canonical isomorphism $S(V_{1} \otimes V_{2}) \sim S(V_{1}) \ast S(V_{2})$, and thus

\[\Xi^L_{V_{1} \otimes V_{2}} : H^*_{O \times \mathfrak{g}_{m}}(G, S(V_{1} \otimes V_{2})) \sim H^*_{O \times \mathfrak{g}_{m}}(G, S(V_{1})) \ast H^*_{O \times \mathfrak{g}_{m}}(G, S(V_{2}))
\]

According to Lemma 1 (cf. also Lemma 2), we have a canonical isomorphism

\[\Xi^L_{V} = \Xi^L_{V, T} \circ gr(\pi, \text{Id, Id})^* H^*_{O \times \mathfrak{g}_{m}}(G, S(V)) \sim (\text{Id, Id})_{*} H^*_{O \times \mathfrak{g}_{m}}(G, S_{T}(V_{T}))
\]

In the RHS we have the restriction of $O(t \ltimes t \ltimes \mathfrak{A}^1)$-module $H^*_{T \ltimes \mathfrak{g}_{m}}(G_{R_{T}}(V_{T}))$ to $O(t \ltimes (t/W) \ltimes \mathfrak{A}^1)$. To save a bit of notation in what follows we will write simply $\Xi^L_{V} : gr(\pi, \text{Id, Id})^* H^*_{O \times \mathfrak{g}_{m}}(G, S(V)) \sim H^*_{T \ltimes \mathfrak{g}_{m}}(G_{R_{T}}(V_{T}))$. It follows that after tensoring with $k(t \ltimes \mathfrak{K}^1)$ (over the first and third factors in $O(t/W \ltimes t/W \ltimes \mathfrak{A}^1)$) we have a canonical isomorphism

\[\Xi^L_{V, T} \ast \Xi^L_{V} : H^*_{O \times \mathfrak{g}_{m}}(G, S(V_{1})) \ast H^*_{O \times \mathfrak{g}_{m}}(G, S(V_{2})) \otimes O(t/W \ltimes \mathfrak{A}^1) \sim k(t \ltimes \mathfrak{K}^1)
\]

Now we have a canonical isomorphism $O(\Gamma_{\mu}) \ast O(\Gamma_{\nu}) = O(\Gamma_{\mu} \otimes O(t \ltimes \mathfrak{A}^1) \otimes O(\Gamma_{\nu}) = O(\Gamma_{\mu} \ast \nu)$. Hence we get a canonical isomorphism

\[\Xi^L_{V_{1} \ast \Xi^L_{V_{2}} : (H^*_{O \times \mathfrak{g}_{m}}(G, S(V_{1})) \ast H^*_{O \times \mathfrak{g}_{m}}(G, S(V_{2}))) \otimes O(t/W \ltimes \mathfrak{A}^1) \sim k(t \ltimes \mathfrak{K}^1)
\]
Proposition 1.

\[
\sim \bigoplus_{\mu + \nu = \lambda} \left( \mathcal{O}(\Gamma_\lambda) \otimes \mathcal{O}(t \times A^1) k(t \times A^1) \right) \otimes \mu V_1 \otimes \nu V_2 = \\
= \bigoplus_{\lambda} \left( \mathcal{O}(\Gamma_\lambda) \otimes \mathcal{O}(t \times A^1) k(t \times A^1) \right) \otimes \lambda (V_1 \otimes V_2)
\]

We want to compare it with \( \text{top} \varepsilon^\text{gen}_{V_1 \otimes V_2} \):

\[
H^*_{G \times G_m} (\text{Gr}_G, S(V_1 \otimes V_2)) \otimes \mathcal{O}(I/W^{1,1}) k(t \times A^1) \sim \bigoplus_{\lambda} \left( \mathcal{O}(\Gamma_\lambda) \otimes \mathcal{O}(t \times A^1) k(t \times A^1) \right) \otimes \lambda (V_1 \otimes V_2)
\]

**Proposition 1.** \( \text{top} \varepsilon^\text{gen}_{V_1 \otimes V_2} = ( \text{top} \varepsilon^\text{gen}_{V_1} \ast \text{top} \varepsilon^\text{gen}_{V_2}) \circ \omega_{V_1, V_2} \).

**Proof:** The equality readily reduces to the following compatibility. Let \( \Phi_{MV} = \bigoplus_{\lambda} \Phi^\lambda_{MV} : \mathcal{F} \mapsto \bigoplus_{\lambda} j^\lambda_{\lambda} \mathcal{F} \) be the Mirković-Vilonen fiber functor on the Satake category \( \text{Perv}_{G \times G_m} (\text{Gr}) \) (notations of 3.2, see [24]). The (proof of) Lemma 1 provides a canonical isomorphism

\[
\text{gr} (H^*_{T \times G_m} (\text{Gr}_G, \mathcal{F})) \cong \bigoplus_{\lambda} \Phi^\lambda_{MV} \otimes \mathcal{O}(\Gamma_\lambda).
\]

We have to check that this isomorphism is compatible with the tensor structure, i.e. for \( \mathcal{F}, \mathcal{G} \in \text{Perv}_{G \times G_m} (\text{Gr}) \) we have to check coincidence of the two embeddings from \( \Phi_{MV}(\mathcal{F}) \otimes \Phi_{MV}(\mathcal{G}) \) in \( \text{gr} (H^*_{T \times G_m} (\text{Gr}_G, \mathcal{F} \ast \mathcal{G})) \), where the first one comes from the isomorphisms (4) for \( \mathcal{F}, \mathcal{G} \) and tensor structure on the functor \( \text{gr} (H^*_{T \times G_m} (\mathcal{F} \ast \mathcal{G})) \), and the second one comes from the tensor structure on the functor \( \Phi_{MV} \) and isomorphism (4) for \( \mathcal{F} \ast \mathcal{G} \).

To check the equality, we recall a “filtration” in the \( I \)-equivariant derived category on a \( G \) equivariant perverse sheaf \( \mathcal{F} \), which induces the above filtration on \( H^*_{T \times G_m} (\text{Gr}_G, \mathcal{F}) \). Here and below by a “filtration” on an object \( X \) of a triangulated category we mean a collection of object \( X_0 = 0, \ldots, X_n = X \) and distinguished triangles \( X_i \to X_{i+1} \to Y_i \); the objects \( Y_i \) will be called the “subquotients” of the filtration.

Let \( C, D \) be the equivariant constructible derived category with respect to the natural action of \( I \times G_m \) on \( \text{Gr}_G \) and on \( \text{Fl}_G \) respectively. Thus convolution \( *_I \) provides \( D \) with a monoidal structure, and \( C \) with an action of the monoidal category \( D \).

Recall the **Wakimoto sheaves** \( J_\lambda \in D \), characterized by the following properties: \( J_{\lambda + \mu} \cong J_\lambda *_I J_\mu \), while for a dominant weight \( \lambda \in \Lambda^+ \) the sheaf \( J_\lambda \) is the \( * \)-extension of the constant perverse sheaf from the Iwahori orbit corresponding to \( \lambda \), see, e.g. [1].

Recall that \( p \) stands for the projection \( \text{Fl}_G \to \text{Gr}_G \). We set \( J^G_\lambda = p_*(J_\lambda) \). It is not hard to show that for \( \mathcal{F} \in \text{D}_{G \times G_m} (\text{Gr}), J^G_\lambda \ast \mathcal{F} \cong p_*(J_\lambda \ast \mathcal{F}) \) canonically. Also \( J^G_\lambda \) can be characterized by \( j^G_\mu \circ j^G_\lambda = k^{\delta_{\lambda, \mu}} \) (cf. [1]).

Fix \( \mathcal{F} \in \text{Perv}_{G \times G_m} (\text{Gr}) \), and choose a coweight \( \lambda \) deep inside the dominant chamber. Then one shows that \( j^\nu_\mu(J_\lambda \ast \mathcal{F}) \cong k^{[\dim \text{Gr}_\nu]} \otimes j^\nu_\lambda j^\nu_\mu \mathcal{F} \) for all \( \nu \). Thus one can consider the Cousin “filtration” on \( J_\lambda \ast \mathcal{F} \) with subquotients \( j^\nu_\mu(J_\lambda \ast \mathcal{F}) \) and apply the functor \( J_{-\lambda *_I} \) to it, thereby obtaining a “filtration” on \( \mathcal{F} \) with “subquotients” \( \Phi^\mu_{MV}(\mathcal{F}) \otimes j^G_\mu \). It is clear that this “filtration” induces the above filtration on \( H^*_{T \times G_m} (\mathcal{F}) \).
Let now \( \mathcal{F}, \mathcal{G} \) be a pair of objects of \( \text{Perv}_{G_{0} \times G_{m}}(\text{Gr}) \). The above “filtration” on \( \mathcal{F} \) induces a ”filtration” on \( \mathcal{F} \ast \mathcal{G} \) with “subquotients”

\[
\Phi_{MV}^{\mu}(\mathcal{F}) \otimes J_{\mu}^{Gr} \ast \mathcal{G} = \Phi_{MV}^{\mu}(\mathcal{F}) \otimes J_{\mu} \ast \mathcal{G}.
\]

Using the “filtration” on \( \mathcal{G} \) with “subquotients” \( \Phi_{MV}^{\nu}(\mathcal{G}) \otimes J_{\nu}^{Gr} \) we get a “filtration” on \( \mathcal{F} \ast \mathcal{G} \) with “subquotients”

\[
\Phi_{MV}^{\mu}(\mathcal{F}) \otimes \Phi_{MV}^{\nu}(\mathcal{G}) \otimes J_{\mu} \ast J_{\nu}^{Gr} = \Phi_{MV}^{\mu}(\mathcal{F}) \otimes \Phi_{MV}^{\nu}(\mathcal{G}) \otimes J_{\mu+\nu}^{Gr}.
\]

Comparing it with the “filtration” on \( \mathcal{F} \ast \mathcal{G} \) with “subquotients” \( \Phi_{MV}^{\eta}(\mathcal{F} \ast \mathcal{G}) \) we get an isomorphism \( \Phi_{MV}^{\eta}(\mathcal{F} \ast \mathcal{G}) \cong \Phi_{MV}^{\mu}(\mathcal{F}) \otimes \Phi_{MV}^{\nu}(\mathcal{G}) \). It is not hard to see that this isomorphism coincides with any of the standard definitions of tensor structure on \( \Phi_{MV} \); in fact, a close description of the tensor structure appears in [12], Theorem 3.2.8.

Now we see that the isomorphism

\[
gr H_{T \times G_{m}}^{\mu}(\mathcal{F} \ast \mathcal{G}) \cong gr H_{T \times G_{m}}^{\mu}(\mathcal{F}) \ast gr H_{T \times G_{m}}^{\nu}(\mathcal{G})
\]

breaks as a direct sum of maps

\[
(\Phi_{MV}^{\mu}(\mathcal{F}) \otimes H_{T \times G_{m}}^{\mu}(J_{\mu}^{Gr})) \ast (\Phi_{MV}^{\nu}(\mathcal{G}) \otimes H_{T \times G_{m}}^{\nu}(J_{\nu}^{Gr})) \rightarrow \Phi_{MV}^{\mu+\nu}(\mathcal{F} \ast \mathcal{G}) \otimes H_{T \times G_{m}}^{\mu+\nu}(J_{\mu+\nu}^{Gr}),
\]

coming from the map \( \Phi_{MV}^{\mu}(\mathcal{F}) \otimes \Phi_{MV}^{\nu}(\mathcal{G}) \rightarrow \Phi_{MV}^{\mu+\nu}(\mathcal{F} \ast \mathcal{G}) \) induced by the tensor structure on \( \Phi_{MV} \), and the natural isomorphism

\[
\mathcal{O}(\Gamma_{\mu}) \ast \mathcal{O}(\Gamma_{\nu}) = H_{T \times G_{m}}^{\mu}(J_{\mu}^{Gr}) \ast H_{T \times G_{m}}^{\nu}(J_{\nu}^{Gr}) \cong H_{T \times G_{m}}^{\mu+\nu}(J_{\mu+\nu}^{Gr}) = \mathcal{O}(\Gamma_{\mu+\nu}).
\]

The claim follows. \( \square \)

4. Algebra

4.1. Some properties of the Kostant functor \( \kappa_{h} \). The following properties of the Kostant functor will play an important role in the proof of the main results.

**Lemma 4.** a) The functors \( \kappa, \kappa_{h} \) are exact.

b) The functors \( \kappa|_{\text{Coh}^{\hat{G}_{0} \times G_{m}}(\hat{\mathfrak{g}}^*)} \), \( \kappa_{h}|_{\text{Coh}^{L}} \) are full embeddings.

**Proof** is essentially due to B. Kostant, cf. [21].

a) For the exactness of \( \kappa \), note that the functor of restriction \( \mathcal{F} \rightarrow \mathcal{F}|_{\mathcal{Y}} \) is exact on \( \text{Coh}^{\hat{G}}(\tilde{\mathfrak{g}}^*) \), and then \( \mathcal{F}|_{\mathcal{Y}} \) is an \( \hat{N} \)-equivariant coherent sheaf on \( \mathcal{Y} \). Recall that \( \hat{N} \) acts on \( \mathcal{Y} \) freely, and \( \mathcal{Y}/\hat{N} \simeq \Sigma \). Hence the functor of invariants \( \mathcal{G} \mapsto \mathcal{G}^{\hat{N}} \) is exact on \( \text{Coh}^{\hat{N}}(\mathcal{Y}) \). The exactness of \( \kappa \) follows.

For the exactness of \( \kappa_{h} \), we will prove that both the functors of \( -\psi \)-coinvariants, and \( \hat{N} \)-invariants are exact, and hence \( \kappa_{h} \) is exact as their composition. It is enough to check it on the positively graded \( h \)-Harish-Chandra bimodules. Then it is enough to check the exactness on the subcategory of \( h \)-Harish-Chandra bimodules with grading degrees between 0 and \( n \) for a fixed \( n \gg 0 \). Thus it suffices to check the exactness on the subcategory of \( h \)-Harish-Chandra bimodules with nilpotent action of \( h \), and then it suffices to consider the subcategory of \( h \)-Harish-Chandra bimodules with the *trivial* action of \( h \). However, an \( h \)-Harish-Chandra bimodule \( M \) with the trivial action of \( h \) is nothing else than a \( \hat{G} \)-equivariant coherent sheaf on \( \tilde{\mathfrak{g}}^* \), and \( M \otimes_{U_{h}(\hat{a}^{-})}(-\psi) = M|_{\mathcal{Y}} \). In
and for the same reason, \( M \mapsto (M \otimes_{U_h(\tilde{n}_-)} (-\psi))^{\tilde{N}^*} = (M|_Y)^{\tilde{N}^*} \) is exact. This completes the proof of a).

b) \( \kappa|_{Coh_{\mathfrak{g}^*}^{\mathfrak{g}}} \) is fully faithful since the complement to \((\mathfrak{g}^*)^{reg}\) in \( \mathfrak{g}^* \) has codimension 2, and the centralizer of a generic regular element is connected.

To prove that \( \kappa_h|_{\mathfrak{g}^*} \) is fully faithful, we consider free \( \hbar \)-Harish-Chandra bimodules \( M_1 = U_h \otimes V_1, \ M_2 = \hat{U}_h \otimes V_2, \) and the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}(M_1, M_2) & \xrightarrow{\epsilon} & \text{Hom}(\kappa_h M_1, \kappa_h M_2) \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\text{Hom}(M_1/h, M_2/h) & \xrightarrow{\alpha} & \text{Hom}(\kappa(M_1/h), \kappa(M_2/h)) \\

\end{array}
\]

We have just proved that \( \alpha \) is an isomorphism. Moreover, \( \beta \) is surjective since \( \text{Hom}(U_h \otimes V_1, U_h \otimes V_2) = \text{Hom}_G(V_1 \otimes V_2^*, U_h) \), and all the \( G \)-modules in question are semisimple. It follows that \( \gamma \) is surjective. On the other hand, \( \gamma \) is injective since \( \kappa_h M_1, \kappa_h M_2 \) are free over \( k[h] \). Now that \( \gamma \) is proved to be an isomorphism, the composition \( \delta \circ \epsilon \) must be surjective. Hence \( \epsilon \) is surjective by Nakayama Lemma. It remains to prove that \( \epsilon \) is injective. Since \( \kappa_h \) is exact, it is enough to prove that \( \kappa_h M \neq 0 \) for a nonzero subobject \( M \) of a free \( \hbar \)-Harish-Chandra bimodule \( M_2 \). We consider a nonzero subobject \( M/h \subset M_2/h \) of a free \( \mathcal{O}(\mathfrak{g}^*) \)-module \( M_2/h \). It suffices to prove that \( \kappa M \neq 0 \). However, the support of any nonzero section of a free \( \mathcal{O}(\mathfrak{g}^*) \)-module is the whole of \( \mathfrak{g}^* \), hence its restriction to \( Y \) is nonzero.

The lemma is proved. \( \Box \)

4.2. De-symmetrized Kostant functor \( \zeta_h \). We denote by \( \pi \) the projection \( \mathfrak{t}^*/W \), and we denote by \( (\pi, \text{Id}, \text{Id}) \) the projection \( \mathfrak{t}^* \times (\mathfrak{t}^*/W) \times A^1 \rightarrow \mathfrak{t}^*/W \times \mathfrak{t}^*/W \times A^1 \).

For \( V \in \text{Rep}(\hat{G}) \) we are going to describe \( (\pi, \text{Id}, \text{Id})^* \phi(V) \in Coh(\mathfrak{t}^* \times (\mathfrak{t}^*/W) \times A^1) \).

To this end we consider the universal Verma module \( M_h(-\rho) = U_h \otimes_{U_h(\mathfrak{b})} k[h][\tilde{t}](-\rho) \), and \( k[h][\tilde{t}](-\rho) \) is a \( U_h(\mathfrak{b}) \)-module which factors through the \( U_h(\tilde{t}) \) as the centralizer of a generic regular element is connected.

For an \( h \)-Harish-Chandra bimodule \( M \) we set \( \zeta_h(M) := M_h(-\rho) \otimes_{U_h} M \otimes_{U_h(\tilde{n}_-)} \psi = k[h][\tilde{t}](-\rho) \otimes_{U_h(\mathfrak{b})} M \otimes_{U_h(\tilde{n}_-)} \psi \). This is an \( \mathcal{O}(\mathfrak{t}^*/W) \times A^1 \)-module: the action of \( \mathcal{O}(\mathfrak{t}^*/W) \times A^1 \) on \( U_h(\tilde{t}) \) comes from the fact that \( U_h(\tilde{t}) \) normalizes \( U_h(\mathfrak{b}) \), and the action of \( \mathcal{O}(\mathfrak{t}^*/W) \times A^1 \) is the action of the center \( Z(U_h) \) of the second copy of \( U_h \), as before.

For \( V \in \text{Rep}(\hat{G}) \) we set \( \varphi(V) := \zeta_h(Fr(V)) \).

**Lemma 5.** For \( V \in \text{Rep}(\hat{G}) \) we have a canonical isomorphism \( \varphi(V) \simeq (\pi, \text{Id}, \text{Id})^* \phi(V) \).

**Proof:** We denote by \( W^- : U_h \otimes_{U_h(\tilde{n}_-)} \psi \) the Whittaker \( U_h \)-module. By a theorem of Kostant, \( \text{End}_{U_h}(W^-) = Z(U_h) \), and the category \( \mathcal{A} \) of \( (U_h(\tilde{n}_-), \psi) \)-integrable \( U_h \)-modules is equivalent to the category of \( Z(U_h) \)-modules (here a \( U_h(\tilde{n}_-) \)-module
Lemma 6. Canonical filtration on $\varphi(V)$. For $V \in \operatorname{Rep}(G)$ we have $\varphi(V) = (M_h(-\rho) \otimes V) \otimes_{U_h(n_-)} \psi$. Note that $M_h(-\rho) \otimes V$ has a canonical filtration with associated graded $\bigoplus_{\lambda} M_h(\lambda - \rho) \otimes \lambda V$, where $\lambda$ is a weight of $\mathfrak{t}$, and $\lambda V$ is the corresponding weight space of $V$; furthermore, $M_h(\lambda - \rho) = U_h \otimes_{U_h(b)} k[\mathfrak{t}] \otimes_{U_h(b)} k[\mathfrak{t}] \otimes (\lambda - \rho)$ is a $U_h(b)$-module which factors through the $U_h(\mathfrak{t}) = k[\mathfrak{t}] \otimes (\mathfrak{t} - \mathfrak{t})$-module where $t \in \mathfrak{t}$ acts by multiplication by $t + h(\lambda(t) - h(\rho(t))$.

It follows that $\varphi(V)$ has a canonical filtration with associated graded $\bigoplus_{\lambda} (M_h(\lambda - \rho) \otimes_{U_h(n_-)} \psi) \otimes \lambda V$. Note that $M_h(\lambda - \rho) \otimes_{U_h(n_-)} \psi$ is a $\mathcal{O}(\mathfrak{t}^* \times (\mathfrak{t}^*/W) \times \mathbb{A}^1)$-module since $M_h(\lambda - \rho)$ is a $U_h(\mathfrak{t})$-module. To describe $M_h(\lambda - \rho) \otimes_{U_h(n_-)} \psi$ as a coherent sheaf on $\mathfrak{t}^* \times (\mathfrak{t}^*/W) \times \mathbb{A}^1$, we denote by $(\operatorname{Id}, \pi, \operatorname{Id})$ the projection from $\mathfrak{t}^* \times \mathfrak{t}^* \times \mathbb{A}^1$ to $\mathfrak{t}^* \times (\mathfrak{t}^*/W) \times \mathbb{A}^1$, and we denote by $\Gamma_\lambda \subset \mathfrak{t}^* \times \mathfrak{t}^* \times \mathbb{A}^1$ the subscheme defined by the equations $\Gamma_\lambda = \{(t_1, t_2, a) : t_2 = t_1 + a\lambda\}$. Then $M_h(\lambda - \rho) \otimes_{U_h(n_-)} \psi = (\operatorname{Id}, \pi, \operatorname{Id})_s \mathcal{O}(\Gamma_\lambda)$.

We have proved the following

**Lemma 6.** For $V \in \operatorname{Rep}(G)$, the $\mathcal{O}(\mathfrak{t}^* \times (\mathfrak{t}^*/W) \times \mathbb{A}^1)$-module $\varphi(V)$ has a canonical filtration with associated graded $\bigoplus_{\lambda} (\operatorname{Id}, \pi, \operatorname{Id})_s \mathcal{O}(\Gamma_\lambda) \otimes \lambda V$. In particular, $\varphi(V)$ is flat as an $\mathcal{O}(\mathfrak{t}^* \times \mathbb{A}^1)$-module.

4.4. Whittaker modules for Levi subalgebras. Let $\tilde{T} \subset \tilde{L} \subset \tilde{G}$ be a Levi subgroup with the Lie algebra $\mathfrak{t} \subset \mathfrak{t} \subset \mathfrak{g}$. We denote by $\mathfrak{p}_L$ (resp. $\mathfrak{p}_L^-$) the parabolic subalgebra generated by $\mathfrak{t}$ and the positive (resp. negative) Borel subalgebra $\mathfrak{b}$ (resp. $\mathfrak{b}_-$). We denote by $\pi_L$ the projection from $\mathfrak{t}^*/W_L$ to $\mathfrak{t}^*/W$. 

$L$ is called a $(U_h(n_-), \psi)$-integrable if the action of $\mathfrak{n}_-$ on $L \otimes (-\psi)$ is locally nilpotent. Namely, a $(U_h(n_-), \psi)$-integrable $U_h$-module $L$ goes to the $Z(U_h)$-module $\operatorname{Hom}_{U_h}(W^-_h, L)$. Conversely, a $Z(U_h)$-module $R$ goes to $W^-_h \otimes_{Z(U_h)} R$. In particular, $W^-_h$ goes to the free module $Z(U_h)$.

For an $h$-Harish-Chandra bimodule $M$ we will construct a canonical isomorphism $\kappa_h(M) \simeq \operatorname{Hom}_{U_h}(W^-_h, M \otimes_{U_h} W^-_h)$. In effect, $L \mapsto M \otimes_{U_h} L$ is a right-exact endofunctor of the category of $(U_h(n_-), \psi)$-integrable $U_h$-modules. Under Kostant’s equivalence, this endofunctor goes to the convolution with the $Z(U_h)$-bimodule $X$ which corresponds by Kostant to our endofunctor applied to $W^-_h$. In other words, $X = \operatorname{Hom}_{U_h}(W^-_h, M \otimes_{U_h} W^-_h)$. We have a tautological isomorphism $X \otimes_{Z(U_h)} W^-_h \simeq M \otimes_{U_h} W^-_h$. This yields the desired isomorphism $X \simeq \kappa_h(M)$. In particular, for a free $h$-Harish-Chandra bimodule $M = Fr(V)$, we obtain $\phi(V) \otimes_{Z(U_h)} W^-_h \simeq V \otimes_{W^-_h} W^-_h$.

Now let us compute $(\mathcal{M}_h(-\rho) \otimes V \otimes W^-_h) \otimes_{U_h} k[h] = (\mathcal{M}_h(-\rho) \otimes (V \otimes W^-_h)) \otimes_{U_h} k[h] \xrightarrow{\sim} (\mathcal{M}_h(-\rho) \otimes (W^-_h \otimes_{Z(U_h)} \phi(V))) \otimes_{U_h} k[h] \xrightarrow{\sim} (\pi, \operatorname{Id})_* \phi(V)$. The last isomorphism arises from $(\mathcal{M}_h(-\rho) \otimes W^-_h) \otimes_{U_h} k[h] \xrightarrow{\sim} U_h(\mathfrak{t}) = \mathcal{O}(\mathfrak{t}^*)$, since $M_h(-\rho) = U_h \otimes_{U_h(b)} k[h][\mathfrak{t}](-\rho)$, and $W^-_h = U_h \otimes_{U_h(n_-)} \psi$.

On the other hand, $((\mathcal{M}_h(-\rho) \otimes V) \otimes W^-_h) \otimes_{U_h} k[h] = ((\mathcal{M}_h(-\rho) \otimes V) \otimes W^-_h) \otimes_{U_h} k[h] \xrightarrow{\sim} \mathcal{M}_h(-\rho) \otimes (V \otimes U_h(\mathfrak{t})) \otimes_{U_h(b)} (U_h \otimes V) \otimes_{U_h(n_-)} \psi = \phi(V)$.

This completes the proof of the lemma.
**Lemma 7.** For $V \in \text{Rep}(\tilde{G})$, the $\mathcal{O}(\tilde{t}^* / W_L \times \tilde{t}' / W \times \mathbb{A}^1)$-module $(\pi_L, \text{Id}, \text{Id})^* \phi(V)$ carries a canonical filtration $F_L^*$ such that the associated graded is equipped with a canonical isomorphism $\text{alg} \Xi_L : \text{gr}(\pi_L, \text{Id}, \text{Id})^* \phi(V) \sim \sim \text{Id, } \pi_L, \text{Id})_* \phi(L(V|_{\tilde{L}}))$.

**Proof:** We have $\tilde{I} = [\tilde{I}, \tilde{I}] \oplus \tilde{3}_{\tilde{I}}$ where $\tilde{3}_{\tilde{I}}$ stands for the center of $\tilde{I}$. We consider the nilpotent subalgebra $\tilde{n}_{\tilde{L}} = \tilde{I} \cap \tilde{n}_L$, and a nondegenerate homomorphism $\psi_L : U_h(\tilde{n}_{\tilde{L}}) \to k[h]$ such that $\psi(f_\alpha) = 1$ for any simple root $\alpha$ of $I$. We define the Whittaker $U_h([\tilde{I}, \tilde{I}])$-module $\mathcal{W}_{\tilde{L}}^{-}$ as $U_h([\tilde{I}, \tilde{I}]) \otimes_{U_h(\tilde{n}_{\tilde{L}})} \psi_L$. We define a free $U_h(\tilde{3}_{\tilde{I}}) = k[h][\tilde{3}_{\tilde{I}}]$-module $\tilde{3}_{\tilde{I}}(-\rho + \rho_L)$ as $k[h][\tilde{3}_{\tilde{I}}]$ where $t \in \tilde{3}_{\tilde{I}}$ acts by multiplication by $t - h(\rho - \rho_L)(t)$ (here $\rho_L$ is the halfsum of positive roots of $I$). We define a $U_h(\tilde{I})$-module $\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L)$ as $\mathcal{W}_{\tilde{L}}^{-} \otimes k[h][\tilde{3}_{\tilde{I}}](-\rho + \rho_L)$. The projection $\tilde{p}_L \to \tilde{I}$ gives rise to the homomorphism $U_h(\tilde{p}_L) \to U_h(\tilde{I})$, and thus we can consider $\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L)$ as a $U_h(\tilde{p}_L)$-module. Finally, we define the Verma-Whittaker $U_h$-module $\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L)$ as $U_h \otimes_{U_h(\tilde{p}_L)} \mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L)$. Note that the center $Z(U_h(\tilde{I})) = \mathcal{O}(\tilde{I}^* / W_L \times \mathbb{A}^1)$ acts by endomorphisms of $\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L)$, and hence of $\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L)$.

We claim that for $V \in \text{Rep}(\tilde{G})$, we have a canonical isomorphism

$$(\pi_L, \text{Id}, \text{Id})^* \phi(V)[n_L] \cong (\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes V \otimes \mathcal{W}_h^{-}) \otimes_{U_h} k[h]$$

(the LHS is homologically shifted to the degree $-n_L$, that is negative dimension of $\tilde{n}_{\tilde{L}}$). In effect, arguing like in the proof of Lemma 5, we only have to check that

$$(\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes \mathcal{W}_h^{-}) \otimes_{U_h} k[h] \sim Z(U_h(\tilde{I}))[n_L] = \mathcal{O}(\tilde{I}^* / W_L \times \mathbb{A}^1)[n_L].$$

To this end we note that $$(\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes \mathcal{W}_h^{-}) \otimes_{U_h} k[h] \sim \mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes_{U_h(\tilde{p}_L)} (\mathcal{W}_h^{-} \otimes_{U_h(\tilde{p}_L)} \psi_L) \sim \mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes_{U_h(\tilde{p}_L)} \psi_L \sim Z(U_h(\tilde{I}))[n_L].$$

Moreover, it follows that for an $\tilde{L}$-module $W$ we have a canonical isomorphism of $\mathcal{O}(\tilde{t}_L^* / W_L \times \tilde{t}' / W \times \mathbb{A}^1)$-modules $[\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes W] \otimes_{U_h} k[h] \sim \sim \text{Id, } \pi_L, \text{Id})_* \phi(L(W)).$ Now it remains to notice that for a $\tilde{G}$-module $V$ the $U_h$-module $\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes V$ has a canonical filtration with associated graded $U_h \otimes_{U_h(\tilde{p}_L)} (\mathcal{W}_{\tilde{L}}^{-}(-\rho + \rho_L) \otimes V|_{\tilde{L}})$. This completes the proof of the lemma. 

**4.5. Transitivity for a pair of Levi subgroups.** We have a canonical isomorphism $\text{alg} \Xi_L : \text{gr}(\pi_L, \text{Id}, \text{Id})^* \phi(V) \sim \sim \text{Id, } \pi_L, \text{Id})_* \phi(L(V|_{\tilde{L}})).$ In the RHS we have the restriction of $\mathcal{O}(\tilde{t}^* / W_L \times \tilde{t}' / W \times \mathbb{A}^1)$-module $\phi_L(V|_{\tilde{L}})$ to $\mathcal{O}(\tilde{t}^* / W_L \times \tilde{t}' / W \times \mathbb{A}^1)$. To save a bit of notation in what follows we will write simply $\text{alg} \Xi_L : \text{gr}(\pi_L, \text{Id}, \text{Id})^* \phi(V) \sim \sim \phi(L(V|_{\tilde{L}})).$

If $\tilde{L} < \tilde{L}' < \tilde{L}$ is another Levi subgroup, then we denote by $\pi_{\tilde{L}'}$ the projection from $\tilde{t}_L^*$ to $\tilde{t}' / W_{\tilde{L}}$. Note that the filtration $F_{\tilde{L}'*}$ on $(\pi_{\tilde{L}'}, \text{Id}, \text{Id})^* \phi(V)$ is a refinement of the filtration $(\pi_L, \text{Id}, \text{Id})^* \phi(V)$ and hence induces a canonical filtration $F_{\tilde{L}'*}$ on $(\pi_{\tilde{L}'}^L, \text{Id}, \text{Id})^* \phi(V)$. The isomorphism $(\pi_L^L, \text{Id}, \text{Id})^* \text{alg} \Xi_{\tilde{L}'} : \text{gr}_{F_{\tilde{L}'*}}(\pi_{\tilde{L}'}, \text{Id}, \text{Id})^* \phi(L(V|_{\tilde{L}'})).$ We have a canonical isomorphism $\text{alg} \Xi_{\tilde{L}'} : \text{gr}_{F_{\tilde{L}'*}}(\pi_{\tilde{L}'}, \text{Id}, \text{Id})^* \phi(L(V|_{\tilde{L}'}) \sim \sim \phi(L'(V|_{\tilde{L}'})$.

We consider the
composition

\[ \xrightarrow{\text{Lemma } 5 \text{ for } \mathbf{Tensor structure on Kostant functor.}} \]

\[ \phi(V) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} W_h \sim V \otimes W_h. \]

Thus, for \( V_1, V_2, W_1 \in \text{Rep}(\hat{G}) \), we have

\[ \phi(V_1) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_2) = \phi(V_1 \circ \phi(V_2)) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_2) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_1 \circ \phi(V_2)) \]

and thus

\[ \phi(V_1 \circ \phi(V_2)) = \phi(V_1) \circ \phi(V_2). \]

According to Lemma 6 (cf. also Lemma 7), we have a canonical isomorphism \( \mathcal{E}_V = \mathcal{E}_{T, V} : \)

\[ \xrightarrow{\text{Lemma } 5 \text{ for } \mathbf{Tensor structure on Kostant functor.}} \]

\[ \phi(V) \otimes_{\mathcal{O}(\pi/\mathbb{Z} \times \mathbb{A})} \phi(V) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_2) = \phi(V_1) \circ \phi(V_2). \]

Now we have a canonical isomorphism \( \phi(V_1) \otimes \phi(V_2) \sim (\phi(V_1) \circ \phi(V_2)) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_2) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_1 \circ \phi(V_2)) \)

and thus

\[ \phi(V_1 \circ \phi(V_2)) = \phi(V_1) \circ \phi(V_2). \]

We want to compare it with

\[ \mathcal{E}_{V_1 \otimes V_2} : \phi(V_1 \otimes V_2) = \phi(V_1) \circ \phi(V_2) \]

\[ \mathcal{E}_{V_1 \otimes V_2} : \phi(V_1 \otimes V_2) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_2) \otimes_{\mathcal{O}(\pi^*/\mathbb{Z} \times \mathbb{A})} \phi(V_1 \circ \phi(V_2)) \]

Proposition 2.

\[ \mathcal{E}_{V_1 \otimes V_2} = (\mathcal{E}_{V_1} \ast \mathcal{E}_{V_2}) \circ \mathcal{E}_{V_1 \circ V_2}. \]
Proof: We consider the generic universal Verma module $M_h^{\text{gen}}(-\rho) = \mathbb{U}_h \otimes_{\mathbb{U}_h(\mathfrak{i})} k(\mathfrak{t}^* \times \mathbb{A}^1)(-\rho)$, and $k(\mathfrak{t}^* \times \mathbb{A}^1)(-\rho)$ is a $U_h(\mathfrak{g})$-module which factors through the $U_h(\mathfrak{i}) = \mathbb{O}(\mathfrak{t}^* \times \mathbb{A}^1)$-module where $t \in \mathfrak{t}$ acts by multiplication by $t - h \rho(t)$. It is well known that $\text{End}_{U_h}(M_h^{\text{gen}}(-\rho)) = k(\mathfrak{t}^* \times \mathbb{A}^1)$, and the category $\mathcal{B}$ of $U_h(\mathfrak{n})$-integrable $U_h \otimes k(\mathfrak{t}^* \times \mathbb{A}^1)$-modules is semisimple, and any simple object is isomorphic to $B$. In particular, $\mathcal{B}$ is equivalent to the category of $\mathfrak{b}$-modules isomorphism from $\mathbb{O}(\Sigma)$.

For $V \in \text{Rep}(\hat{G})$ we put $\varphi^{\text{gen}}(V) := \varphi(V) \otimes_{\mathbb{O}(\mathfrak{t}^* \times \mathbb{A}^1)} k(\mathfrak{t}^* \times \mathbb{A}^1) = \text{gr} \varphi(V) \otimes_{\mathbb{O}(\mathfrak{t}^* \times \mathbb{A}^1)} k(\mathfrak{t}^* \times \mathbb{A}^1)$. This is the restriction of a $k(\mathfrak{t}^* \times \mathbb{A}^1) \otimes \mathbb{O}(\mathfrak{t}^*)$-module to $k(\mathfrak{t}^* \times \mathbb{A}^1) \otimes \mathbb{O}(\mathfrak{t}^*/W)$, but we will view it as a $k(\mathfrak{t}^* \times \mathbb{A}^1) \otimes \mathbb{O}(\mathfrak{t}^*)$-module. Arguing like in the proof of Lemma 5, we obtain a canonical isomorphism $\varphi^{\text{gen}}(V) \otimes_{k(\mathfrak{t}^* \times \mathbb{A}^1)} M_h^{\text{gen}}(-\rho) \cong V \otimes M_h^{\text{gen}}(-\rho)$. This gives rise to the tensor structure on the functor $\varphi^{\text{gen}}$:

$$
\varphi^{\text{gen}}(V_1 \otimes V_2) \cong \varphi^{\text{gen}}(V_1) \otimes \varphi^{\text{gen}}(V_2).
$$

Clearly, the identification $\text{alg} \Sigma V^\text{gen} \colon \varphi^{\text{gen}}(V) \cong \bigoplus \mathbb{O}(\Gamma_i) \otimes k(\mathfrak{t}^* \times \mathbb{A}^1) \otimes \lambda V$ commutes with the obvious tensor structure in the RHS.

On the other hand, arguing like in the proof of Lemma 5, we obtain a canonical isomorphism $\varphi^{\text{gen}}(V) \cong (M_h^{\text{gen}}(-\rho) \otimes V \otimes W_h^\lambda) \otimes U_h k[\mathfrak{h}]$ which implies that the tensor structures on $\varphi$ and $\varphi^{\text{gen}}$ are compatible as well. This completes the proof of the proposition. □

4.7 Quasiclassical limit of $\varphi(V)$. For $V \in \text{Rep}(\hat{G})$, Lemma 6 implies that the $\mathbb{O}(\mathfrak{t}^*/W)$-bimodule $\varphi(V)$ is supported at the diagonal $\Delta \subset \mathfrak{t}^*/W \times \mathfrak{t}^*/W$. It follows that the action of $\mathbb{O}(\mathfrak{t}^*/W \times \mathfrak{t}^*/W \times \mathbb{A}^1)$ on $\varphi(V)$ actually extends to the action of $\mathbb{O}(\mathfrak{N}_W \times \mathfrak{t}^*/W \times \mathfrak{t}^*/W \Delta)$. As we know from 2.6, $\mathbb{O}(\mathfrak{N}_W \times \mathfrak{t}^*/W \times \mathfrak{t}^*/W \Delta)/h \cong \mathbb{O}(\mathfrak{T}(\mathfrak{t}^*/W))$ is the universal centralizer.

Lemma 9. For $V \in \text{Rep}(\hat{G})$, the $\mathbb{O}(\mathfrak{T}(\mathfrak{t}^*/W))$-module $\varphi(V)|_{h=0}$ is canonically isomorphic to the module $\mathbb{O}(\Sigma) \otimes V$ over the universal centralizer.

Proof: Consider the $\mathbb{O}(\mathfrak{g}^*)^\mathbb{G}$-module $\text{Pol}(\mathfrak{g}^*, \mathfrak{g})^\mathbb{G}$ of $\mathbb{G}$-invariant polynomial maps from $\mathfrak{g}^*$ to $\mathfrak{g}$; this is a vector bundle over $\text{Spec} \mathbb{O}(\mathfrak{g}^*)^\mathbb{G} = \Sigma$. Given a polynomial $P \in \mathbb{O}(\mathfrak{g}^*)^\mathbb{G}$, its differential $dP$ defines a section of this vector bundle. For a central element $z \in Z(U(\mathfrak{g})) = \mathbb{O}(\mathfrak{g}^*)^\mathbb{G}$ we denote the corresponding section by $\sigma_z$. If $z$ runs through a set of generators of $Z(U(\mathfrak{g}))$, the corresponding sections $\sigma_z$ form a basis of the universal centralizer bundle, and identify it with the cotangent bundle $\mathfrak{T}^*(\Sigma)$.

Thus it suffices to check the following statement about the free $h$-Harish-Chandra bimodule $U_h \otimes V$. Let $z^{(1)}$ (resp. $z^{(2)}$) stand for the left (resp. right) action of $z$ in $U_h \otimes V$. Then the action of $\frac{z^{(1)} - z^{(2)}}{h}$ on $(U_h \otimes V)|_{h=0} = \mathbb{O}(\mathfrak{g}^*) \otimes V$ coincides with the action of $\sigma_z \in \mathbb{O}(\mathfrak{g}^*)^\mathbb{G} \otimes \mathfrak{g}$.

In effect, if $v \in V$, and $z = \sum z_i x_{i_1} \ldots x_{i_k}$ where $x_{i_j} \in \mathfrak{g}$, and $\tilde{y} \in U_h$ is a lift of $y \in \mathbb{O}(\mathfrak{g}^*) = U_h|_{h=0}$, then $\left(\frac{z^{(1)} - z^{(2)}}{h}\right)|_{h=0} = \sum x_{i_j} a_x x_{i_1} \ldots x_{i_{k-1}} y \otimes x_{i_k}(v) = \sigma_z(y \otimes v)$.

The lemma is proved. □
5. Rank 1

5.1. Equivariant cohomology for $G = PGL(2)$. Let us describe $H^*_G \times \mathbb{G}_m(Gr,G(S(V)))$ in the case $G = PGL(2)$. Then $G$ is isomorphic to $SL(2)$. For $n > 1$ let $V_n$ be the $n$-dimensional irreducible $SL(2)$-module. Let $Gr_n$ be the closure of the $n$-dimensional $G_0$-orbit in $Gr$. It is known that $Gr_n$ is rationally smooth, so we are interested in $H^*_G \times \mathbb{G}_m(Gr_n)$ as a module over $H^*_G \times \mathbb{G}_m(Gr)$. The non-equivariant cohomology $H^*(Gr_n)$ is a cyclic $H^*(Gr)$-module (see e.g. [19]). Hence, by graded Nakayama lemma, $H^*_G \times \mathbb{G}_m(Gr_n)$ is a cyclic module over $H^*_G \times \mathbb{G}_m(Gr)$. Recall that $H^*_G \times \mathbb{G}_m(Gr)$ is the structure sheaf of a subscheme $A_n \subset N_{U/W \times U/W}$ of a copy of $N_{U/W \times U/W}$ specified by the parity of $n$.

Now we describe the subscheme $A_n$. Let $P_n = \{n\omega, (n-2)\omega, \ldots, (2-n)\omega, -n\omega\}$ be the set of weights of $G$-module $V_n$. We have $V_n \subset t^i$. For $i = -n, -n+2, \ldots, n-2, n$, let $\Gamma_i \subset t \times t \times \mathbb{A}^1$ be a subscheme defined by the equations $\Gamma_i = \{t_1, t_2, a : t_2 = t_1 + i\omega\}$. Let $\Gamma(n)$ stand for the subscheme defined by the product of the above equations (over $i = -n, -n+2, \ldots, n-2, n$). Recall that $\pi$ stands for the projection $t \to t/W$, and consider the subscheme $(\pi,\pi,\pi)(\Gamma(n)) \subset t/W \times t/W \times \mathbb{A}^1$. Finally, we can formulate

**Lemma 10.** $A_n$ is the proper preimage of $(\pi,\pi,\pi)(\Gamma(n))$ in $N_{U/W \times U/W}$.

**Proof:** Since $H^*_G \times \mathbb{G}_m(Gr_n)$ is a flat $O(t/W \times \mathbb{A}^1)$-module, it suffices to identify $A_n$ with $(\pi,\pi,\pi)(\Gamma(n))$ generically over $t/W \times \mathbb{A}^1$, or else to identify $(\pi,\pi,\pi)^{-1}(A_n)$ with $(\pi,\pi,\pi)(\Gamma(n))$ generically over $t \times \mathbb{A}^1$. This was done in Lemma 1. $\Box$

5.2. Generic splitting of the canonical filtration on equivariant cohomology for $G = PGL(2)$. Recall the canonical filtration on $H^*_T \times \mathbb{G}_m(Gr_n)$ (see 3.2). We will compare the identification (see Lemma 1) of the associated graded with $\bigoplus_{i=-n}^n (\pi,\pi,\pi)(\Gamma_i) \otimes V_n$ with the identification (see Lemma 10) $H^*_T \times \mathbb{G}_m(Gr_n) \cong O((\pi,\pi,\pi)^{-1}(A_n))$. To this end we recall some basic facts about the cohomology of $Gr_n$. For $i = -n, -n+2, \ldots, n-2, n$, let $v_i \in H^{i+n}(Gr_n)$ stand for the (Poincaré dual of the) fundamental class of $Gr_n \cap \Sigma_i$ (an irreducible subvariety of $Gr_n$ of dimension $\frac{n+n}{2}$). The action of $e, h, f \in sl_2$ on $H^*(Gr_n)$ in this basis is given by

$$hv_i = iv_i, \quad ev_{i-2} = \frac{n+i}{2}v_i, \quad fv_{i+2} = \frac{n-i}{2}v_i$$

(recall that $e$ is defined as the multiplication by the first Chern class of the determinant line bundle).

The canonical filtration $0 = F^{n+2} \subset F^n \subset F^{n-2} \subset \ldots \subset F^{2-n} \subset F^{-n} = H^*_T \times \mathbb{G}_m(Gr_n)$ is given by $F^i = Im \left( r_i : H^*_T \times \mathbb{G}_m(Gr_n) \to H^*_T \times \mathbb{G}_m(Gr_n) \right)$. On the other hand, the proof of Lemma 10 shows that $H^*_T \times \mathbb{G}_m(Gr_n)$ is generated by the (Poincaré dual of the) fundamental class $\bar{v}_{-n}$ of $Gr_n = Gr_n \cap \overline{\Sigma}_{-n}$. Recall that $j_i$ stands for the embedding of the $T$-fixed point $i$ into $\Sigma_i$, while $i_i$ stands for the locally closed embedding of $\Sigma_i$ into $Gr$. Let us denote by $i_i$ the closed embedding of $\Sigma_i$ into $Gr$. The image of $i_i^* \bar{v}_{-n}$ in the nonequivariant cohomology $H^*(Gr_n \cap \overline{\Sigma}_i)$ is the fundamental
class $v_i$ of $\text{Gr}_n \cap \mathfrak{t}_i$. We can further restrict it to $\text{Gr}_n \cap \mathfrak{s}_i$, and then to $i$ to get $i^*_i \tilde{e}_{-n}$. To compare it with $j^*_i i^*_i \tilde{e}_{-n}$ we consider a transversal slice $\text{Gr}_n \cap \mathfrak{t}_i$ to $\text{Gr}_n \cap \mathfrak{s}_i$ in $\text{Gr}_n$ where $\mathfrak{s}_i$ is the $N(F)$-orbit through the point $i$.

It is known that $\text{Gr}_n \cap \mathfrak{s}_i$ is isomorphic to a vector space $A^{n+1}$ with the origin at $i$, and the action of $T \times \mathfrak{g}_m$ is linear with weights $x + (i-1)h, x + (i-2)h, \ldots, x + i^*_n h$. It follows that $i^*_i \tilde{e}_{-n} = (x + (i-1)h)(x + (i-2)h) \ldots (x + i^*_n h) i^*_i \tilde{e}_{-n}$. We conclude that the generator $\tilde{v}_i$ of $F^1$ whose class in the nonequivariant cohomology $H^*(\text{Gr}_n)$ is equal to $v_i$ is given by

$$v_i = (x + i^*_1 h)(x + i^*_2 h) \ldots (x + i^*_n h)$$

$5.3. \textbf{Kostant functor for } \bar{G} = SL(2).$ Now we consider the group $\bar{G} = SL(2)$ with the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{s}_2$ and Cartan subalgebra $\mathfrak{t} \subset \tilde{\mathfrak{g}}$.

\textbf{Lemma 11.} The $\mathcal{O}(\mathfrak{t}^*/W \times \mathfrak{t}^*/W \times \mathbb{A}^1)$-module $\phi(V_n) = \kappa_h(U_h \otimes V_n)$ is isomorphic to $\mathcal{O}(A_n)$.

\textit{Proof:} According to Lemma 9, the restriction of $\phi(V_n)$ to $h = 0$ is isomorphic to the $\mathcal{O}(\mathcal{F}(\mathfrak{t}^*/W))$-module $V_n \otimes \mathcal{O}(\mathfrak{t}^*/W)$, that is $V_n$ viewed as a module over the universal centralizer. Further restricting it to $0 \in \mathfrak{t}^*/W$ we obtain $V_n$ viewed as a module over the centralizer of the regular nilpotent $e \in \mathfrak{s}_2$. Clearly, $V_n$ is a cyclic $k[e]$-module. By the graded Nakayama Lemma, $\phi(V_n)$ is a cyclic $\mathcal{O}(\mathcal{F}(\mathfrak{t}^*/W \times \mathfrak{t}^*/W))$-module as well, hence $\phi(V_n)$ is isomorphic to the structure sheaf of a subscheme $B_n \subset \mathcal{F}(\mathfrak{t}^*/W \times \mathfrak{t}^*/W)$. We have to check $B_n = A_n$.

According to Lemmas 5, 6, $\phi(V_n)$ is a flat $\mathcal{O}(\mathfrak{t}^*/W \times \mathbb{A}^1)$-module, so it suffices to identify $B_n$ and $A_n$ generically over $\mathfrak{t}^*/W \times \mathbb{A}^1$. Moreover, it suffices to identify $(\pi, \pi, \text{Id})^{-1}(B_n)$ with $\Gamma(n) \subset \mathfrak{t}^* \times \mathfrak{t}^* \times \mathbb{A}^1$. This was done in Lemma 6. This completes the proof of the lemma. \qed

$5.4. \textbf{Generic splitting of the canonical filtration on Kostant functor for } \bar{G} = SL(2).$ Recall the canonical filtration on $\varphi(V_n)$ (see 4.3). We will compare the identification (see Lemma 6) of the associated graded with $\bigoplus_{n \geq 0} (\text{Id, } \pi, \text{Id})\mathcal{O}(\mathfrak{t}^*_n) \otimes \mathcal{O}(\mathfrak{t}^*_n)$ with the identification (see Lemma 11) $\varphi(V_n) \cong \mathcal{O}((\pi, \text{Id, Id})^{-1}(A_n))$. To this end we recall some basic facts about $U_h(\mathfrak{s}_2)$-modules. First, $V_n$ is a free $k[h]$-module with a basis $\{v_n, v_{n-2}, \ldots, v_{2-n}, v_{-n}\}$. The action of $e, h, f \in \mathfrak{s}_2$ is given by

$$hv_i = ihv_i, \quad ev_{i-2} = \frac{n+i}{2} hv_i, \quad f v_{i+2} = \frac{n-i}{2} hv_i$$

Second, $\mathcal{M}_h(1)$ is a free $k[h, x]$-module with a basis $\{m_{-1}, m_{-2}, m_{-3}, \ldots\}$. The action of $e, h, f$ is given by

$$hm_i = (x + ih)m_i, \quad em_i = -i - 1 \frac{h(x - i + 1)h}{2} m_{i+2}, \quad fm_i = m_{i-2}$$

We have a canonical filtration $0 = F^{n+2} \subset F^n \subset F^{n-2} \subset \ldots \subset F^{2-n} \subset F^{-n} = \mathcal{M}_h(-1) \otimes_{k[h]} V_n$ by $U_h$-submodules such that $F^i/F^{i+2} = \mathcal{M}_h(i-1) \otimes \mathcal{O}(\mathfrak{t}^*_n)$ (notations of 4.3). Recall that $\mathcal{O}(\mathfrak{t}^*_n)$ is spanned by $v_i$. There is a unique vector $s_i \in (\mathcal{M}_h(-1) \otimes V_n) \otimes_{k[h, x]} k(h, x)$ such that $es_i = 0$, and $s_i \equiv m_{-1} \otimes v_i$ modulo $U_h(s_{i+2}, s_{i+4}, \ldots)$.
Then \( F^i = U_h(s_i, s_{i+2}, \ldots, s_n) \cap (M_h(-1) \otimes V_n) \). The image \( \bar{s}_i \) of this vector in the \( \psi \)-coinvariants \( U_h(s_i) \otimes U_h(n) \psi = M_h(-1) \otimes U_h(n) \psi \) is the generator of \( (\text{Id}, \pi, \text{Id})_s \Omega(\Gamma) \otimes iV_n \). Note that the space of \( \psi \)-coinvariants is just the quotient modulo the image of \( f - 1 \).

On the other hand, the proof of Lemma 11 shows that \( \varphi(V_n) = (M_h(-1) \otimes V_n) \otimes \kappa[h,x] \) \( \psi \) is generated by the image \( m_{-1} \otimes v_n \) of \( m_{-1} \otimes v_n \) in the space of \( \psi \)-coinvariants. Thus we have to express \( m_{-1} \otimes v_n \) in terms of \( \bar{s}_{-n}, \ldots, \bar{s}_n \).

**Lemma 12.**

\[
m_{-1} \otimes v_n = \sum_{i=-n}^{n} (x + (i-1)h)^{-1}(x + (i-2)h)^{-1} \cdots (x + \frac{i-n}{2}h)^{-1} \bar{s}_i
\]

**Proof:** Recall that the space of \( \psi \)-coinvariants is just the quotient modulo the image of \( f - 1 \). This means we have to prove the following equality in \( (M_h(-1) \otimes V_n) \otimes \kappa[h,x] \):

\[
m_{-1} \otimes v_n = \sum_{i=-n}^{n} (x + (i-1)h)^{-1}(x + (i-2)h)^{-1} \cdots (x + \frac{i-n}{2}h)^{-1} f^{\frac{n+1}{2}} s_i
\]

For \( l = 0, 1, \ldots, n \) we introduce a new vector \( s^l_i \) such that \( cs^l_i = 0 \), and \( s^l_i \equiv e^l(m_{-1} \otimes v_{i-2l}) \) modulo \( U_h(s_{i+2}, s_{i+4}, \ldots, s_n) \) (evidently, \( s^l_i \) is proportional to \( s_i = s^0_i \) when \( i-2l \geq -n \); otherwise, \( s^l_i \) is not defined). Consider the following collection of equalities:

\[
e^l(m_{-1} \otimes v_n) = \sum_{i=-n+2l}^{n} h^l(x+(i-1)h)^{-1}(x+(i-2)h)^{-1} \cdots (x + \frac{i-n+2l}{2}h)^{-1} f^{\frac{n-2l+1}{2}} s^l_i
\]

Then the \( n \)-th equality is obvious, while the \( l+1 \)-st equality is equivalent to the \( l \)-th one by applying \( e \) to both sides. Thus the desired equality (for \( l = 0 \)) follows by descending induction in \( l \). \( \Box \)

6. **Topology vs Algebra**

6.1. **Comparison of Kostant functor with equivariant cohomology.** Recall that \( t \) is identified with \( i^* \).

**Theorem 6.** a) For \( V \in \text{Rep}(\hat{G}) \) there is a unique isomorphism of \( \mathcal{O}(t/W \times t/W \times \mathbb{A}^1) \) modules \( \eta_V : H^*_{G \times \mathbb{G}_m}(\text{Gr}_G, S(V)) \simeq \phi(V) \) such that \( (\pi, \text{Id, Id})^* \eta_V \) preserves the canonical filtrations and induces the identity isomorphism of the associated graded \( gr(\pi, \text{Id, Id})^* H^*_{G \times \mathbb{G}_m}(\text{Gr}_G, S(V)) = (\text{Id, \pi, Id})_s \bigoplus_{\lambda} \mathcal{O}(\Gamma_\lambda) \otimes \lambda V = gr(\pi, \text{Id, Id})^* \phi(V) \).

b) For \( V_1, V_2 \in \text{Rep}(\hat{G}) \) the composition

\[
H^*_{G \times \mathbb{G}_m}(\text{Gr}_G, S(V_1)) \times H^*_{G \times \mathbb{G}_m}(\text{Gr}_G, S(V_2)) \xrightarrow{\text{top}_{\omega V_1, V_2}} H^*_{G \times \mathbb{G}_m}(\text{Gr}_G, S(V_1 \otimes V_2)) \xrightarrow{\eta_{V_1} \otimes \eta_{V_2}} \phi(V_1 \otimes V_2) \xrightarrow{\text{alg}_{\omega V_1, V_2}} \phi(V_1) \otimes \phi(V_2)
\]

equals \( \eta_{V_1} \ast \eta_{V_2} \) (notations of 3.5 and 4.6).
Proof: a) We have

\[ (\pi, \text{Id}, \text{Id})^*H^*_G \times \mathbb{G}_m (\text{Gr}_G, S(V)) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1) = \]

\[ gr(\pi, \text{Id}, \text{Id})^*H^*_G \times \mathbb{G}_m (\text{Gr}_G, S(V)) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1) = \]

\[ \left( \bigoplus_{\lambda} (\text{Id}, \pi, \text{Id}), \mathcal{O}(\Gamma_\lambda) \otimes \lambda V \right) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1) = \]

\[ gr(\pi, \text{Id}, \text{Id})^*\phi(V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1) = (\pi, \text{Id}, \text{Id})^*\phi(V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1). \]

Thus we have to identify \((\pi, \text{Id}, \text{Id})^*H^*_G \times \mathbb{G}_m (\text{Gr}_G, S(V))\) with the natural \(W\)-action on it, and \((\pi, \text{Id}, \text{Id})^*\phi(V)\) with the natural \(W\)-action on it, as two \(\mathcal{O}(t \times t/W \times A^1)\)-submodules of \((\bigoplus_{\lambda} (\text{Id}, \pi, \text{Id}), \mathcal{O}(\Gamma_\lambda) \otimes \lambda V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\). First we will show that the \(W\)-action on \((\bigoplus_{\lambda} (\text{Id}, \pi, \text{Id}), \mathcal{O}(\Gamma_\lambda) \otimes \lambda V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\) arising from its identification with \((\pi, \text{Id}, \text{Id})^*H^*_G \times \mathbb{G}_m (\text{Gr}_G, S(V)) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\) coincides with the \(W\)-action arising from the identification with \((\pi, \text{Id}, \text{Id})^*\phi(V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\).

Let \(\alpha\) be a simple root, \(s_\alpha \in W\) the corresponding simple reflection, and \((e_\alpha, h_\alpha, f_\alpha)\) the corresponding \(\mathfrak{sl}_2\)-triple in \(\hat{\mathfrak{g}}\). Let \(v \in \lambda V\) be a vector such that \(f_\alpha v = 0\). Then \(h_\alpha v = \lambda(h_\alpha)v, \lambda(h_\alpha)\) is a nonpositive integer. We consider the following vector in \((\bigoplus_{\lambda} (\text{Id}, \pi, \text{Id}), \mathcal{O}(\Gamma_\lambda) \otimes \lambda V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\) \(\ni p_\alpha(v) := \)

\[ \sum_{k=0}^{\lambda(h_\alpha)} \left( h_\alpha + (2k-1+\lambda(h_\alpha))h^{-1}(h_\alpha + (2k-2+\lambda(h_\alpha))h^{-1} \ldots (h_\alpha + (k+\lambda(h_\alpha))h^{-1}1_{A^1} \otimes e_\alpha^k v \right) \]

where \(1_{A^1}\) is the constant function 1 on \((\text{Id}, \pi, \text{Id}) \Gamma_\mu\).

It follows from the computation in rank 1 (Lemma 12), and the transitivity equation (8) of subsection 4.5 (applied to the case where \(L\) is a subminimal Levi containing just one positive root \(\alpha\), and \(\bar{L} = \bar{T}\) that \(p_\alpha(v)\) is \(s_\alpha\)-invariant (with respect to the action arising from the identification with \((\pi, \text{Id}, \text{Id})^*H^*_G \times \mathbb{G}_m (\text{Gr}_G, S(V)) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\), and \(M_\alpha := \sum_{\lambda} (\bigoplus_{\lambda} (\text{Id}, \pi, \text{Id}), \mathcal{O}(\Gamma_\lambda) \otimes \lambda V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\) contains \((\pi, \text{Id}, \text{Id})^*\phi(V)\).

On the other hand, it follows from the computation in rank 1 (equation (7) of subsection 5.2), and the transitivity equation (3) of subsection 3.4 (applied to the case where \(L\) is a subminimal Levi containing just one positive root \(\alpha\), and \(\bar{L} = \bar{T}\) that \(p_\alpha(v)\) is \(s_\alpha\)-invariant (with respect to the action arising from the identification with \((\pi, \text{Id}, \text{Id})^*H^*_G \times \mathbb{G}_m (\text{Gr}_G, S(V)) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\), and \(M_\alpha := \sum_{\lambda} (\bigoplus_{\lambda} (\text{Id}, \pi, \text{Id}), \mathcal{O}(\Gamma_\lambda) \otimes \lambda V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\) coincide. In particular, we have defined unambiguously a \(W\)-action on \((\bigoplus_{\lambda} (\text{Id}, \pi, \text{Id}), \mathcal{O}(\Gamma_\lambda) \otimes \lambda V) \otimes_{\mathcal{O}(t \times A^1)} k(t \times A^1)\).

Now we claim that \((\pi, \text{Id}, \text{Id})^*\phi(V) = \bigcap_{w \in W} w(M_\alpha) = (\pi, \text{Id}, \text{Id})^*H^*_G \times \mathbb{G}_m (\text{Gr}_G, S(V))\). In effect, note that if \((\text{Id}, \pi, \text{Id}) \Gamma_\nu \cap (\text{Id}, \pi, \text{Id}) \Gamma_\mu\) has codimension 1 in \((\text{Id}, \pi, \text{Id}) \Gamma_\nu\) (and in \((\text{Id}, \pi, \text{Id}) \Gamma_\mu\) as well), then necessarily \(\mu = \nu + k\beta\) for certain integer \(k\), and certain positive root \(\beta\) (not necessarily a simple root). Let us choose \(w \in W\) such that \(\alpha := w^{-1}(\beta)\) is a simple root. Then \(w^{-1}(\mu) = w^{-1}(\nu) + k\alpha\). Since we know that any section in \(M_\alpha\) extends through the generic point of \((\text{Id}, \pi, \text{Id}) \Gamma_{w^{-1}(\mu)} \cap (\text{Id}, \pi, \text{Id}) \Gamma_{w^{-1}(\mu) + k\alpha}\), we conclude that any section in \(w(M_\alpha)\)
extends through the generic point of \((\text{Id}, \pi, \text{Id})_{n} \cap (\text{Id}, \pi, \text{Id})_{p}\). It follows that any section in \(\bigcap_{w \in \mathcal{W}} w(M_{a})\) is regular off a codimension 2 subvariety. Since both \((\pi, \text{Id}, \text{Id})^{\alpha}_{\text{simple}}(V)\) and \((\pi, \text{Id}, \text{Id})^{\alpha}_{\text{simple}}(\text{Gr}_{G}, S(V))\) are flat \(\mathcal{O}(t \times A^{1})\)-modules coinciding with \(\bigcap_{w \in \mathcal{W}} w(M_{a})\) generically, we conclude that \((\pi, \text{Id}, \text{Id})^{\alpha}_{\text{simple}}(V) = \bigcap_{w \in \mathcal{W}} w(M_{a}) = (\pi, \text{Id}, \text{Id})^{\alpha}_{\text{simple}}(\text{Gr}_{G}, S(V)).\) This completes the proof of the a).

b) follows by comparing Propositions 1 and 2. □

6.2. Cohomology is fully faithful. The following property of the cohomology functor will play an important role in the proof of the main results. Let \(\mathcal{C}\) (resp. \(\mathcal{C}\)) denote the full subcategory of semisimple complexes in \(D_{G_{G} \times G_{m}}(\text{Gr})\) (resp. \(D_{G_{G}}(\text{Gr})\)).

**Lemma 13.** a) The functor \(H_{G_{G} \times G_{m}}^{\bullet} : \mathcal{C} \rightarrow \text{Coh}^{G_{m}}(\mathcal{N}(t^{\ast}/W))\) is a full embedding.

b) The functor \(H_{G_{G}}^{\bullet} : \mathcal{C} \rightarrow \text{Coh}^{G_{m}}(\mathcal{T}(t^{\ast}/W))\) is a full embedding.

**Proof:** is due to V. Ginzburg, see [17]. We prove a), and the proof of b) is identical. For \(V_{1}, V_{2} \in \text{Rep}(G)\) we have \(\text{Ext}^{\bullet}_{G_{G} \times G_{m}}(S(V_{1}), S(V_{2})) = \text{Ext}^{\bullet}_{G_{m}}(S(V_{1}), S(V_{2})).\) Let us denote by \(\text{Res}_{G}^{T}\) the forgetting functor from the \(G \times G_{m}\)-equivariant derived category to the \(T \times G_{m}\)-equivariant derived category. Then the Weyl group \(W\) acts naturally on \(\text{Ext}^{\bullet}_{T \times G_{m}}(\text{Res}_{G}^{T}S(V_{1}), \text{Res}_{G}^{T}(V_{2}))\), and \(\text{Ext}^{\bullet}_{T \times G_{m}}(S(V_{1}), S(V_{2})) = \text{Ext}^{\bullet}_{T \times G_{m}}(\text{Res}_{G}^{T}S(V_{1}), \text{Res}_{G}^{T}(V_{2}))^{W}.\) On the other hand, we know by Theorem 6 b) that \(\text{H}^{\bullet}_{T \times G_{m}}(\text{Gr}, (V_{1}, 2)) \cong \varphi(V_{1}, 2)\), and clearly, \(\text{Hom}(\varphi(V_{1}), \varphi(V_{2})) = \text{Hom}(\varphi(V_{1}), \varphi(V_{2}))^{W}.\) Thus it suffices to prove that

\[
\text{Ext}^{\bullet}_{T \times G_{m}}(\text{Res}_{G}^{T}S(V_{1}), \text{Res}_{G}^{T}(V_{2})) \simeq \text{Hom}_{\mathcal{O}(t \times (t^{\ast}/W)) \times A^{1}}(\text{H}^{\bullet}_{T \times G_{m}}(\text{Gr}, (V_{1})), \text{H}^{\bullet}_{T \times G_{m}}(\text{Gr}, (V_{2})))
\]

\[
\text{Hom}_{\text{H}^{\bullet}_{T \times G_{m}}(\text{Gr})}(\text{H}^{\bullet}_{T \times G_{m}}(\text{Gr}, (V_{1})), \text{H}^{\bullet}_{T \times G_{m}}(\text{Gr}, (V_{2}))).
\]

Following [9], recall the definition of the LHS: we choose finite dimensional approximations \(P_{i}\) to the classifying space of \(T \times G_{m}\), and we have ind-varieties \(P_{i}\text{Gr}\) fibered over \(P_{i}\) with fibers isomorphic to \(\text{Gr}\). We also have semisimple perverse sheaves \(P_{i}S(V_{1,2})\) on \(P_{i}\text{Gr}\), and finally \(\text{Ext}^{\bullet}_{T \times G_{m}}(\text{Res}_{G}^{T}S(V_{1}), \text{Res}_{G}^{T}(V_{2}))\) is defined as \(\text{lim Ext}^{\bullet}_{P_{i}\text{Gr}}(P_{i}S(V_{1}), P_{i}S(V_{2})).\) Since \(T \times G_{m}\) is a torus, we can choose \(P_{i}\) to be the products of projective spaces (of increasing dimension). We can choose a generic action of \(G_{m}\) on \(P_{i}\text{Gr}\) (linear along \(P_{i}\), and via a one-parametric subgroup of \(T \times G_{m}\) along \(\text{Gr}\)) such that the corresponding Bialynicki-Birula decomposition of \(P_{i}\text{Gr}\) is cellular. Then we can apply the Theorem of [17] to conclude that \(\text{Ext}^{\bullet}_{P_{i}\text{Gr}}(P_{i}S(V_{1}), P_{i}S(V_{2})) \simeq \text{Hom}_{\text{H}^{\bullet}(P_{i}\text{Gr})}(\text{H}^{\bullet}(P_{i}\text{Gr}, S(V_{1})), \text{H}^{\bullet}(P_{i}\text{Gr}, S(V_{2}))).\) But the limit of the RHS as \(i\) grows is \(\text{Hom}_{\text{H}^{\bullet}_{T \times G_{m}}(\text{Gr})}(\text{H}^{\bullet}_{T \times G_{m}}(\text{Gr}, (V_{1})), \text{H}^{\bullet}_{T \times G_{m}}(\text{Gr}, (V_{2}))).\)

The lemma is proved. □

6.3. **Proof of Theorems 2, 4.** The Theorems follow from Theorem 6 in view of Lemmas 4(b), 13.
6.4. **Homology.** The goal of this subsection is to express equivariant cohomology of arbitrary (not necessarily semisimple) equivariant complexes in terms of Harish-Chandra bimodules, and to prove Theorem 3. By Theorem 2 we have a monoidal equivalence $\mathcal{S}$ between the category $\mathcal{HC}_h^f$ of free asymptotic Harish-Chandra bimodules and the category $\mathcal{IC}$ of semisimple complexes in $D_{GO \times G_m}(Gr)$.

A standard argument shows any functor on free Harish-Chandra bimodules is representable by a unique up to a unique isomorphism (not necessarily free) Harish-Chandra bimodule. Thus we have a functor from $\mathcal{F} : D_{GO \times G_m}(Gr) \to \mathcal{HC}_h$ equipped with a functorial isomorphism

$$\text{Hom}(\mathcal{S}(F), \mathcal{F}) \cong \text{Hom}(F, \mathcal{F} \circ \mathcal{S});$$

$\mathcal{F}$ is defined uniquely up to a unique isomorphism.

Since $\mathcal{S}$ is a full embedding, $\mathcal{F} \circ \mathcal{S} \cong \text{Id}$ canonically, i.e. $\mathcal{F}$ restricted to the category of semi-simple complexes is the inverse equivalence to $\mathcal{S}$. It is easy to see from the definition that $\mathcal{F}$ is a homological functor; thus it actually lands in the category $\mathcal{HC}_h$ of finitely generated Harish-Chandra bimodules, $\mathcal{F} : D_{GO \times G_m}(Gr) \to \mathcal{HC}_h$.

We will say that a functor $F$ between two monoidal categories is quasi-monoidal if a functorial map $F(X) \ast F(Y) \to F(X \ast Y)$ is fixed for any objects $X, Y$ of the target category, compatible with the associativity isomorphisms in the two categories in the natural sense.

**Proposition 3.** a) $\mathcal{F}$ carries a unique quasi-monoidal structure, whose restriction to $\mathcal{IC}$ induces the natural monoidal structure on $\text{Id}_{\mathcal{HC}_h^f} \cong \mathcal{F} \circ \mathcal{S}$.

b) We have a natural isomorphism $H^*_\mathcal{F} \cong \kappa_h \circ \mathcal{F}$ compatible with the (quasi)monoidal structure.

**Proof** a) It is easy to see that

$$\mathcal{F}(\mathcal{F}) = R \text{Hom}(IC_0, O(\hat{G}) \ast \mathcal{F}) \cong R \text{Hom}(IC_0, \mathcal{F} \ast O(\hat{G}))$$

canonically (where the last isomorphism comes from the fact that both sides can be identified with $\bigoplus_\Lambda R \text{Hom}(V^*_\Lambda \otimes IC_\Lambda, \mathcal{F})$). Then the quasi-monoidal structure comes from the coalgebra structure on $O(\hat{G})$. Uniqueness follows from the fact that for every $\mathcal{F} \in D_{GO \times G_m}$ we can find a free asymptotic Harish-Chandra bimodule $V$ and a surjection $V \to \mathcal{F} \mathcal{F}$; by definition of $\mathcal{F}$ it comes from a map $L = S(V) \to \mathcal{F}$. Then, in view of functoriality, for $\mathcal{F}_1, \mathcal{F}_2 \in D_{GO \times G_m}$ the map $\mathcal{F}(\mathcal{F}_1) \ast \mathcal{F}(\mathcal{F}_2) \to \mathcal{F}(\mathcal{F}_1 \ast \mathcal{F}_2)$ is uniquely determined by the isomorphism $\mathcal{F}(L_1) \ast \mathcal{F}(L_2) \to \mathcal{F}(L_1 \ast L_2)$ for $L_i \to \mathcal{F}$ ($i = 1, 2$) as above.

b) For $\mathcal{F} \in D_{GO \times G_m}(Gr)$ we can find an exact sequence $V_1 \to V_2 \to \mathcal{F} \to 0$, where $V_i$ are free asymptotic Harish-Chandra bimodules. To this sequence there corresponds a sequence of maps in $D_{GO \times G_m}(Gr)$ with zero composition: $L_1 \to L_2 \to \mathcal{F}$, where $L_i = S(V_i)$. We have $\kappa_h \circ \mathcal{F} \mathcal{F} = \text{CoKer}(\kappa_h(V_1) \to \kappa_h(V_2)) = \text{CoKer}(H^*_\mathcal{F} \to H^*_\mathcal{F}(\mathcal{F}_1 \ast \mathcal{F}_2))$. The latter module maps canonically to $H^*_\mathcal{F}(\mathcal{F})$. Thus we defined a map $\kappa_h \circ \mathcal{F} \mathcal{F} \to H^*_\mathcal{F}(\mathcal{F})$. A standard argument shows that this map does not depend on the choice of $V_1, V_2$. We have obtained a natural transformation between the two functors. This transformation is an isomorphism on semi-simple complexes.
Since both functors are homological (where we use exactness of \( \kappa_h \), Lemma 4(a)) and semi-simple objects generate the triangulated category \( D_{\mathcal{G}_G \times \mathfrak{g}_m}^*(\text{Gr}) \), we see that the transformation is an isomorphism. \( \square \)

We can clearly extend all of the above to Ind-objects. We will be particularly interested in the Ind-object \( \mathcal{D} = \lim \mathcal{D}_\lambda \), where \( \mathcal{D}_\lambda \) is the dualizing sheaf of the closure of the \( \mathcal{G}_O \)-orbit \( \text{Gr}_\lambda \). Notice that \( H_{\mathcal{G}_G \times \mathfrak{g}_m}^*(\text{Gr}) = H_{\mathcal{G}_G \times \mathfrak{g}_m}(\mathcal{D}) \), and the convolution algebra structure on homology comes from the structure of an algebra in the monoidal category \( D_{\mathcal{G}_G \times \mathfrak{g}_m} \) on the object \( \mathcal{D} \) and the monoidal structure on the cohomology functor.

We will now describe the corresponding object in the category of Harish-Chandra bimodules. Recall first the duality Proposition 4. We have a canonical isomorphism of algebra Ind-objects: \( \mathcal{F}(\mathcal{D}) \cong \mathcal{K} \).

To prove Proposition we need another auxiliary Lemma 14. We have a canonical isomorphism \( \mathcal{D} \circ \mathcal{S} \cong \mathcal{S} \circ \mathcal{C}_G \circ \mathcal{D} \), where \( \mathcal{D} \), \( \mathcal{C}_G \) denote, respectively, Verdier duality and the functor induced by the canonical outer automorphism of \( \check{G} \) interchanging conjugacy classes of \( g \) and \( g^{-1} \), \( g \in \check{G} \) (the Chevalley involution).

Proof : Recall that a monoidal category \( \mathcal{C} \) is rigid if for any object \( V \in \mathcal{C} \) there exists another object \( V^* \in \mathcal{C} \) and morphisms \( \iota : 1 \rightarrow V \otimes V^* \) and \( \tau : V^* \otimes V \rightarrow 1 \) satisfying a certain compatibility constraint, see [14]. Given \( V \), an object \( V^* \) together with morphisms \( \iota, \tau \) is unique up to a unique isomorphism if it exists. Thus for a rigid category \( \mathcal{C} \) we have a canonical (up to a canonical isomorphism) functor \( \mathcal{C} \rightarrow \mathcal{C}^{op} \) sending \( V \) to \( V^* \). It is immediate to check that monoidal categories \( \text{Perv}_{\mathcal{G}_G \times \mathfrak{g}_m}((\text{Gr})) \) and \( \mathcal{H} \mathcal{C}_h^G \) are rigid categories with duality functors given by \( \mathcal{C}_G \circ \mathcal{D} \) and \( \mathcal{D} \) respectively, where the functor \( \mathcal{C}_G \) is induced by the Chevalley involution of \( G \). The equivalence between the two categories intertwines the canonically defined dualities. Also, it is well-known that \( \mathcal{S} \circ \mathcal{C}_G \cong \mathcal{C}_G \circ \mathcal{S} \) canonically. The Lemma follows. \( \square \)

Proof of Proposition 4. The Ind-object \( \mathcal{D} \) represents the functor \( \mathcal{F} \mapsto H_{\mathcal{G}_G \times \mathfrak{g}_m}^*(\mathcal{D}(\mathcal{F})) \). In view of Lemma 14 and Proposition 3(c) we see that the Ind-object \( \mathcal{F}(\mathcal{D}) \) represents the functor \( M \mapsto \kappa_h(\mathcal{D}(M)) \) on the category of free asymptotic Harish-Chandra modules. It is straightforward to see from the definitions that the Ind-object \( \mathcal{K} \) represents the same functor. The isomorphism of functors yields an isomorphism of Harish-Chandra bimodules. Since the isomorphism of functors is compatible with the monoidal structure, the isomorphism of Harish-Chandra bimodules is compatible with the algebra structure. \( \square \)
Proof of Theorem 3. By Propositions 3 and 4 we have an isomorphism of algebras
\[ H^*_G \times G_m(\text{Gr}) = H^*_G \times G_m(\mathfrak{D}) \cong \kappa_h(K). \]
The latter is by definition the algebra of the quantum Toda lattice. □

6.5. Formality from purity. In this section we combine the above Ext computation with a standard argument which allows one to derive formality of the RHom algebra from purity of the Ext spaces. The result is a description of the derived Satake category, including the version equivariant with respect to the loop rotation.

Except for some technical details, this section does not contain original contributions of the authors. We have learned the geometric (respectively, algebraic) ideas exposed here from V. Ginzburg (respectively, L. Positselski) around 1998.

In order to be able to use Frobenius weights we extend the basic setting, and consider $G_0$, Gr etc. over $\overline{\mathbb{F}}_q$, and the categories of equivariant l-adic sheaves on Gr.

Consider the following general situation. Let $R$ be a finitely localized ring of integers of a number field $E$, and let $\mathbb{F}_q$ be a finite field quotient of $R$. Let $X_R$ be a flat scheme over $R$ acted upon by a smooth affine group scheme $G_R$, such that the set of orbits is finite. We denote by $(X_{\overline{\mathbb{F}}_q}, G_{\overline{\mathbb{F}}_q})$ (resp. $(X_E, G_E)$) the base change of $(X_R, G_R)$ to a geometric point of $R$ over $\mathbb{F}_q$ (resp. over the generic point). We choose a prime $l$ invertible in $R$. Let $D_{G_{\overline{\mathbb{F}}_q}}(X_{\overline{\mathbb{F}}_q})$ (resp. $D_{G_E}(X_E)$) stand for the bounded equivariant constructible derived category of étale $\mathbb{Q}_l$-sheaves on $X_{\overline{\mathbb{F}}_q}$ (resp. $X_E$) (see e.g. [6], 7.4).

We choose an isomorphism $\mathbb{Q}_l \simeq k$ (under a technical assumption that $k$ has the same cardinality as $\mathbb{Q}_l$), and an embedding $E \hookrightarrow \mathbb{C}$, and we denote by $(X_C, G_C)$ the base change of $(X_E, G_E)$ to $\mathbb{C}$. Let $D_{G_C}(X_C)$ (resp. $D_{G_C}^\text{top}(X_C, \mathbb{Q}_l)$, $D_{G_C}^\text{top}(X_C)$) stand for the bounded equivariant constructible derived category of étale $\mathbb{Q}_l$-sheaves on $X_C$ (resp. bounded equivariant constructible derived category of sheaves with $\mathbb{Q}_l$-coefficients, resp. $k$-coefficients, in the classical topology of $X_C$).

**Proposition 5.** There exists a localization $R_{(r)}$ of $R$ such that for any point $R_{(r)} \to \mathbb{F}_q$, we have the following chain of natural equivalences:

\[ D_{G_{\overline{\mathbb{F}}_q}}(X_{\overline{\mathbb{F}}_q}) \xrightarrow{\alpha} D_{G_E}(X_E) \xrightarrow{\beta} D_{G_C}(X_C) \xrightarrow{\gamma} D_{G_C}^\text{top}(X_C, \mathbb{Q}_l) \xrightarrow{\delta} D_{G_C}^\text{top}(X_C) \]

**Sketch of proof.** The argument is taken from [5], 6.1. The first equivalence $\alpha$ is constructed in [5], 6.1.9 (existence of good models). To justify the finiteness assumptions of *loc. cit.* we note that the set of isomorphism classes of $G$-equivariant irreducible perverse sheaves on $X$ is finite. Since the equivariant derived categories are not considered in *loc. cit.* we note that according to [6], 7.4, to compare the equivariant Exts between equivariant irreducible perverse sheaves on $X$ it suffices to compare the usual Exts between the lifts of these sheaves to $X \times G^n$ (see the canonical spectral sequence (312) of *loc. cit.*). These are calculated by the Künneth formula.

The second equivalence $\beta$ is just the base change from $E$ to $\mathbb{C}$. The third equivalence $\gamma$ is the classical M. Artin’s comparison theorem of étale and classical cohomology, see [5], 6.1.2(B’’). Finally, $\delta$ is induced by our isomorphism $\mathbb{Q}_l \simeq k$. □
Proposition 6. There exists a canonical functor 
\[ \Phi_X : D_{perf}(\mathcal{E}) \to D(X_{\bar{q}}) \]
sending the free module to \( i \). Let \( \mathcal{F} \) be a pure weight zero object of the \( l \)-adic derived category of \( X_{\bar{q}} \). Let \( \mathcal{F} \) be a pure weight zero object of the \( l \)-adic derived category of \( X \) of geometric origin. The space \( \text{Ext}^i(\mathcal{F}_{\bar{q}}, \mathcal{F}_{\bar{q}}) \) carries a canonical grading by Frobenius weights; here the subindex denotes base change to \( \bar{q} \), and \( \text{Ext}^i_{\bar{q}} \) is the component of weight \( j \). Recall that by Deligne Theorem \([13]\) \( \text{Ext}^j_{\bar{q}} = 0 \) for \( j < i \).

Let \( \mathcal{E} \) be a graded algebra and \( \phi : \mathcal{E} \to \text{Ext}^\bullet(\mathcal{F}_{\bar{q}}, \mathcal{F}_{\bar{q}})^{op} \) be a homomorphism sending a graded component \( \mathcal{E}_i \) to \( \text{Ext}^i(\mathcal{F}_{\bar{q}}, \mathcal{F}_{\bar{q}}) \).

We will consider the graded algebra \( \mathcal{E} \) as a dg-algebra with zero differential.

**Proposition 6.** There exists a canonical functor \( \Phi_X : D_{perf}(\mathcal{E}) \to D(X_{\bar{q}}) \) sending the free module to \( \mathcal{F}_{\bar{q}} \) and inducing the map \( \phi \) on \( \text{Ext} \) groups.

**Sketch of proof.** The complex \( \mathcal{F}_{\bar{q}} \) is semi-simple, i.e. is isomorphic to \( \oplus \text{IC}_i[d_i] \), where \( \text{IC}_i \) is an irreducible perverse sheaf and \( d_i \in \mathbb{Z} \). By Beilinson’s Theorem \([4]\) the \( l \)-adic derived category contains the derived category of perverse sheaves as a full subcategory, thus \( \text{Ext}^\bullet(\mathcal{F}_{\bar{q}}, \mathcal{F}_{\bar{q}}) \) coincides with \( \text{Ext} \) in the category of perverse sheaves.

We can assume without loss of generality that \( \mathcal{E} \) is finitely presented; thus the map \( \phi \) factors through a map \( \phi_{\text{fin}} : \mathcal{E} \to \text{Ext}^\bullet(\mathcal{F}_{\bar{q}}, \mathcal{F}_{\bar{q}}) \), where \( \mathcal{A} \) is the Serre subcategory in the category of perverse sheaves on \( X_{\bar{q}} \) generated by a finite set of irreducible objects, including \( \text{IC}_i \). Moreover, the argument of \([4]\) (cf. Examples in loc. cit., 1.2, p.28) shows that all irreducible objects of \( \mathcal{A} \) can be assumed to be of geometric origin.

We can identify the abelian category \( \mathcal{A} \) with the category of finite length \( \mathcal{A} \)-modules, where the pro-finite dimensional algebra (algebra in the tensor category of pro-finite dimensional vector spaces) \( \mathcal{A} \) is defined by \( \mathcal{A} = \text{End}(\oplus \mathcal{P}_s) \); here \( s \) runs over the (finite) set of isomorphism classes of irreducible objects in \( \mathcal{A} \), and \( \mathcal{P}_s \) is a pro-object in \( \mathcal{A} \) which is a projective cover of the corresponding irreducible object \( L_s \), cf. \([3]\). We fix an isomorphism \( \text{Fr}_q^*(L_s) \cong L_s \), which induces a pure weight zero Weil structure on \( L_s \) (this is possible because \( L_s \) has geometric origin, see \([5]\)). Since projective cover of an irreducible object is unique up to a non-unique isomorphism, we can (and will) fix an isomorphism \( \text{Fr}_q^*(\mathcal{P}_s) \cong \mathcal{P}_s \). Then conjugation with Frobenius is an automorphism of \( \mathcal{A} \) (which we will also call Frobenius).

By a result of \([5]\), Frobenius acts on \( \text{Ext}^1(L_s, L_{s'}) \) with positive weights. It follows that Frobenius finite elements are dense in \( \mathcal{A} \), and they form a graded subalgebra \( \mathcal{A}^{gr} \) with finite dimensional graded components, where the grading comes from Frobenius weights. Moreover, components of negative degree in \( \mathcal{A}^{gr} \) vanish, while \( \mathcal{A}^{gr}_0 \) is semisimple. Obviously, \( \mathcal{A} \) is identified with the category of finite length \( \mathcal{A}^{gr} \) modules, on which \( \mathcal{A}^{gr}_N \) acts by zero for \( N \gg 0 \).

We now consider the object \( L = \oplus L_i[d_i] \in D^b(\mathcal{A}) \) (where \( L_i \) corresponds to \( \text{IC}_i \)) and a dg-algebra \( D = \text{RHom}_A(L, L) \) (well defined as an object of the category of dg-algebras with inverted quasi-isomorphisms). Recall that we have an equivalence \( D^b(\mathcal{A}) \cong D_{perf}(D^{op}), M \mapsto \text{RHom}(L, M) \). We lift \( L \) to an object \( \tilde{L} = \oplus \tilde{L}_i[d_i](d_i) \)
of the derived category of graded $A^{gr}$-modules, where $\tilde{L}_i$ is the irreducible $A$-module concentrated in degree zero, and $(d)$ stands for shift of grading by $d$. Then the algebra $\text{Ext}^\bullet(L, L)$ acquires an additional grading, and $D$ can be chosen to carry also an additional grading compatible with the grading on $\text{Ext}$'s. We have a homomorphism $\mathcal{E} \to \bigoplus_i \text{Ext}^i_\mathcal{F}(L, L)$, where the lower index denotes the additional grading, and the fact that $A^{gr}$ is positively graded implies that $\text{Ext}^j_\mathcal{F} = 0$ for $j < 0$. Thus existence of a canonically defined functor $D_{\text{perf}}(\mathcal{E}) \to D^b(A)$ follows from the standard Lemma 15a) below.

We leave it as an exercise to the reader to show that the composed functor $\Phi_X : D_{\text{perf}}(\mathcal{E}) \to D^b(A) \to D(X_{\mathbb{F}_q})$ does not depend on the choice of $A$ up to a canonical isomorphism. $\blacksquare$

We will also have to use functoriality properties of the above construction. We spell these out now.

**Proposition 7.** a) Let $X$, $Y$ be algebraic varieties over a finite field $\mathbb{F}_q$, and $F : D(X_{\mathbb{F}_q}) \to D(Y_{\mathbb{F}_q})$ be a functor satisfying the following conditions

1. $F$ commutes with the pull back under Frobenius functor, i.e. an isomorphism $F \circ F_{X Y} \cong F_{Y X} \circ F$ is fixed.
2. $F$ sends pure weight zero Weil complexes to pure weight zero Weil complexes.
3. Let $\mathcal{A}$ be a finitely generated Serre subcategory in $\text{Perv}(X_{\mathbb{F}_q})$ invariant under the Frobenius pull-back functor. Then there exists a natural exact functor $F_{\mathcal{A}} : \text{Com}(\mathcal{A}) \to \text{Com}(\text{Perv}(Y_{\mathbb{F}_q}))$, compatible with Frobenius and equipped with a natural isomorphism $\beta_Y \circ F_{\mathcal{A}} \cong F \circ \beta_X$. Here $\beta_X : \text{Com}(\mathcal{A}) \to D(X_{\mathbb{F}_q})$, $\beta_Y : \text{Com}(\text{Perv}(Y_{\mathbb{F}_q})) \to D(Y_{\mathbb{F}_q})$ are the natural functors.

Then the construction of Proposition 6 is compatible with $F$, that is, there is a natural isomorphism $\psi_{X \to Y} : \Phi_Y \cong F \circ \Phi_X$.

b) Assume furthermore that $F_1 : D(Y_{\mathbb{F}_q}) \to D(Z_{\mathbb{F}_q})$ is a functor satisfying the above conditions. Then $F_1 \circ F$ also satisfies these conditions, and the two isomorphisms $\psi_{X \to Z}$, $F_1 \circ \psi_{X \to Y}$ between the two functors $\Phi_Z \cong F_1 \circ F \circ \Phi_X : D_{\text{perf}}(\mathcal{E}) \to D(Z_{\mathbb{F}_q})$ coincide.

**Proof.** As above, we find an abelian subcategory $\mathcal{A}' \subset \text{Perv}(Y)$ containing all subquotients of complexes $F_{\mathcal{A}}(\mathcal{G})$, $\mathcal{G} \in \mathcal{A}$; a complex $\mathcal{C}'$ of pro-objects in $\text{Perv}_{\text{mix}}(Y)$ quasiisomorphic to $F(\mathcal{C})$, whose terms with forgotten Frobenius action are projective pro-objects in $\mathcal{A}'$. We also have a dg-algebra $D'$ equipped with an additional grading, which acts on $\mathcal{C}'$ in a way compatible with the grading by Frobenius weights, so that the action induces a quasiisomorphism $D' \to \text{RHom}_{\mathcal{A}}(F(\mathcal{C}), F(\mathcal{C}))$.

Consider a dg-module $B_{X Y}$ of $D' \otimes D_{\mathbb{F}_q}$-module defined by $B_{X Y} := \text{Hom}(\mathcal{C}', F(\mathcal{C}))$. It is not hard to see that the composed functor $D_{\text{perf}}(D') \cong D^b(A') \to D^b(A') \cong D_{\text{perf}}((D')^{op})$ arises from this bimodule as described in Lemma 15(b). The action of Frobenius endows $B_{X Y}$ with an additional grading compatible with the gradings on $D, D'$. Thus Lemma 15(b) provides the sought for isomorphism between the two functors $D_{\text{perf}}(\mathcal{E}) \to D^b(A')$.

Finally part (b) of the Proposition can be deduced from Lemma 15(c). $\blacksquare$
Remark 2. We will apply Proposition 7 when \( F = f^* \) or \( F = g_* \) for a smooth map \( f : Y \to X \) or a proper map \( g : X \to Y \). Each of the functors \( F = f^* \), \( F = g_* \) satisfies the requirements of the Propositions: for \( f^* \) this is standard, and for \( g_* \) property 3 follows from a construction described in [4], page 41, and 2 follows from [5].

Thus Proposition 7 implies functoriality of the construction of Proposition 6 with respect to proper push-forward and smooth pull backs. Also, Proposition 7(b) implies compatibility of the isomorphism of Proposition 6 with the base change isomorphism for a proper map \( X \to Y \) and a smooth map \( Y' \to Y \).

Remark 3. A result similar to Proposition 7 holds, with a similar proof, for a functor \( F : D(\mathfrak{I}_q^\mathfrak{I}) \times \cdots \times D(\mathfrak{I}_n^\mathfrak{I}) \to D(Y_q^\mathfrak{I}) \). Examples of this situation arise when \( Y = X \times \cdots \times X \), \( F : (\mathfrak{I}_1, \ldots, \mathfrak{I}_n) \to \mathfrak{I}_1 \otimes \cdots \otimes \mathfrak{I}_n \); or in a twisted version of this situation (see below).

Lemma 15. a) Let \( D = \bigoplus D^i \) be a dg-algebra equipped with an additional “inner” grading (denoted by a subindex), which is compatible with the differential (thus we have \( d : D^j \to D^{j+1} \)). Assume that \( H^j(D) = 0 \) for \( j < i \). Then there is a canonical morphism in the category of dg-algebras with inverted quasiisomorphisms: \( H^i_{\text{pur}} = \bigoplus H^i_{\text{pur}}(D) \to D \).

In particular, we have a canonical push-forward functor \( D_{\text{perf}}(H^i_{\text{pur}}) \to D_{\text{perf}}(D) \).

b) Let \( D, D' \) be dg-algebras satisfying the assumptions of (a). Let \( B \in D(D' \otimes D'^{\text{op}}) \) be such that the forgetful functor \( D(D' \otimes D'^{\text{op}}) \to D(D') \) sends \( B \) to the free rank one module over \( D' \); thus \( B \) defines a homomorphism \( \phi_B : H^\bullet(D) \to H^\bullet(D') \). Let \( \mathcal{E} \), \( \phi : \mathcal{E} \to \bigoplus H^i_{\text{pur}}(D) \) be as above, and consider the functors \( \Phi : D_{\text{perf}}(\mathcal{E}) \to D_{\text{perf}}(D) \), \( \Phi' : D_{\text{perf}}(\mathcal{E}) \to D_{\text{perf}}(D') \) arising from \( \phi, \phi_B \circ \phi \) respectively by the construction of part (a).

Consider the functor \( \Phi_B : D_{\text{perf}}(D) \to D_{\text{perf}}(D') \) given by \( M \mapsto B \otimes_D M \). Assume that \( B \) carries an additional grading compatible with the gradings on \( D, D' \). Then we have a natural isomorphism \( \Phi' \cong \Phi_B \circ \Phi \).

c) Let \( D, D', D'' \) be three dg-algebras as above, and \( B, B'' \) be modules for \( D' \otimes D'^{\text{op}}, D'' \otimes (D')^{\text{op}} \) as above. For a homomorphism \( \mathcal{E} \to \bigoplus H^i_{\text{pur}}(D) \) the two isomorphisms between the two functors \( D_{\text{perf}}(\mathcal{E}) \to D_{\text{perf}}(D') \) arising from part (b) coincide.

Proof. We remind the idea of the construction in part (a), and leave (b,c) to the interested reader. Let \( D_i \subset D \) be the subcomplex of elements of inner degree \( i \). We have a sub dg-algebra \( D_{up} := \bigoplus_i \tau_{\leq i} D_i \), where we use the standard notation \( \tau \) for truncation of a complex. Furthermore, \( D_{up} \) has a quotient algebra with zero differential \( D_{\text{diag}} := \bigoplus \tau_{\geq i} \tau_{\leq i} D_i \). The conditions of part (a) guarantee that the projection homomorphism \( D_{up} \to D_{\text{diag}} \) is a quasi-isomorphism. The composition of the formal inverse to this quasi-isomorphism with the embedding \( D_{up} \hookrightarrow D \) is the desired morphism. □

6.6. **Proof of Theorem 5.** We construct the first equivalence, the second one is similar. We will construct a monoidal functor \( \Psi : D_{\text{perf}}^{\mathbb{G}_m}(U_h^\mathfrak{I}) \to D_{\mathbb{G}_m \times \mathbb{G}_m}(\mathfrak{I}) \), whose restriction to the full subcategory \( \mathfrak{I}_h^f \subset D_{\text{perf}}^{\mathbb{G}_m}(U_h^\mathfrak{I}) \) is identified with \( S \) (where the
full embedding sends a \( \hat{G} \)-equivariant graded \( U_h \)-module to the same module considered as a dg-module with zero differential). Then \( \Psi \) sends a set of generators of the source triangulated category to generators of the target category, and induces an isomorphism on \( \text{Hom} \)'s between the generators, hence it is an equivalence.

It suffices to construct a collection of functors \( \Psi_\lambda : D_{\text{perf}}^G(U_h^\parallel)_{\leq \lambda} \to D_{G_0 \times G_m}(\text{Gr}_{\leq \lambda}) \), where \( \lambda \) is a coweight of \( G \), \( \text{Gr}_{\leq \lambda} \) is the closure of the corresponding \( G_0 \) orbit on \( \text{Gr} \), and \( D_{\text{perf}}^G(U_h^\parallel)_{\leq \lambda} \) is the full subcategory in \( D_{\text{perf}}^G(U_h^\parallel) \) generated by the objects \( V \otimes U_h^\parallel \), where \( V \) is an irreducible representation of \( \hat{G} \) with a highest weight \( \mu \leq \lambda \). These functors will be compatible for comparable coweights (i.e. we have isomorphisms \( \Psi_\mu \cong \Psi_\lambda |_{D_{\text{perf}}^G(U_h^\parallel)_{\leq \mu}} \) for \( \mu \leq \lambda \), satisfying the obvious compatibility for a triple of coweights \( \nu \leq \mu \leq \lambda \).

The action of \( G_0 \times G_m \) on \( \text{Gr}_{\leq \lambda} \) factors through a finite dimensional algebraic group \( H_\lambda \), and \( D_{G_0 \times G_m}(\text{Gr}_{\leq \lambda}) \cong D_H(\text{Gr}_{\leq \lambda}) \) naturally. To describe \( \Psi_\lambda \) we need to provide the following data: for a smooth \( H_\lambda \)-equivariant map \( X \to \text{Gr}_{\leq \lambda} \), where \( H_\lambda \) acts on \( X \) freely we need to provide a functor \( D_{\text{perf}}^G(U_h^\parallel)_{\leq \lambda} \to D(X/H_\lambda) \), compatible with pull-backs (i.e. a Cartesian section of the category of resolutions of \( \text{Gr}_{\leq \lambda}/H_\lambda \), cf. [9], 2.4.3).

In view of Theorem 2 we have a map \( \text{End}^\bullet(V_{\leq \lambda} \otimes U_h^\parallel) \xrightarrow{\sim} \text{End}^\bullet(I_{\leq \lambda}) \to \text{End}(I_{\leq \lambda}^X) \); here \( V_{\leq \lambda} \) is the sum of all irreducible \( \hat{G} \)-modules with a highest weight less or equal than \( \lambda \), \( I_{\leq \lambda} \in D_{G_0 \times G_m}(\text{Gr}_{\leq \lambda}) \) is the sum of IC sheaves of all \( G_0 \) orbits in \( \text{Gr}_{\leq \lambda} \), and \( I_{\leq \lambda}^X \) is the pull-back of \( I_{\leq \lambda} \) to \( X \). Thus the required functor is given by Proposition 6 in view of purity of equivariant Ext’s between IC sheaves on the affine Grassmannian, see e.g. [16]. These functors do indeed form a Cartesian section in view of Proposition 7, cf. Remark 2. Compatibility between \( \Psi_\lambda \) and \( \Psi_\mu \) for \( \mu \leq \lambda \) is left as an exercise for the reader.

In view of Proposition 7 (cf. Remark 2), a monoidal structure for the constructed functor would follow if we show that the functors from \([D_{\text{perf}}^G(U_h^\parallel)]^2\), \([D_{\text{perf}}^G(U_h^\parallel)]^3\) to the derived category of sheaves on the convolution space, respectively, triple convolution space, given, respectively, by \( (M_1, M_2) \mapsto \Psi(M_1) \otimes \Psi(M_2) \) and \( (M_1, M_2, M_3) \mapsto \Psi(M_1) \otimes \Psi(M_2) \otimes \Psi(M_3) \) (where \( \otimes \) denotes the twisted external product on the convolution space), are compatible with the functors stemming from the construction of Proposition 6. This follows from Remark 3. □

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