Finitely Correlated Generalized Spin Ladders

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We study two-leg $S = \frac{1}{2}$ ladders with general isotropic exchange interactions between spins on neighboring rungs, whose ground state can be found exactly in a form of finitely correlated (matrix product) wave function. Two families of models admitting an exact solution are found: one yields translationally invariant ground states and the other describes spontaneously dimerized models with twofold degenerate ground state. Several known models with exact ground states (Majumdar-Ghosh and Shastry-Sutherland spin-$\frac{1}{2}$ chains, Affleck-Kennedy-Lieb-Tasaki spin-1 chain, $\Delta$-chain, Bose-Gayen ladder model) can be obtained as particular cases from the general solution of the first family, which includes also a set of models with only bilinear interactions. Those two families of models have nonzero intersection, which enables us to determine exactly the phase boundary of the second-order transition into the dimerized phase and to study the properties of this transition. The structure of elementary excitations in the dimerized phase is discussed on the basis of a variational ansatz. For a particular class of models, we present exact wave functions of the elementary excitations becoming gapless at second-order transition lines. We also propose a generalization of the Bose-Gayen model which has a rich phase diagram with all phase boundaries being exact.

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1. INTRODUCTION

In recent years, Heisenberg spin ladders have attracted considerable attention, particularly motivated by their peculiar properties of being intermediate systems between dimensions one and two, as well as by the hope to get some insight into the physics of metal-oxide superconductors. It is now well established that “regular” (i.e., with only longitudinal “leg” and transverse “rung” exchange couplings) $S = \frac{1}{2}$ isotropic spin ladders with even number of legs have a spin liquid ground state with short-range correlations and an energy gap, while odd-legged ladders have a quasi-long-range ordered gapless ground state.

On the other hand, “generalized” ladders including other couplings beyond the simplest case of rung and leg exchange are interesting toy models which can interpolate between a variety of systems, exhibiting remarkably rich behavior. Diagonal interactions cause frustration and can change the structure of the ground state. Recently it has been also shown that biquadratic interactions are important since they tend to produce dimerization and may lead to phase transition into a “non-Haldane” spin liquid state with absence of magnon excitations. In real magnetic systems biquadratic terms can arise due to effective spin-spin interaction mediated by phonons.

In the present paper we study the class of generalized $S = \frac{1}{2}$ ladder models which admit an exact solution for the ground state in terms of the so-called finitely correlated, or matrix product (MP) states. The technique of MP states appears as a natural generalization of the well known valence bond solid (VBS) states which allows one to construct rather complicated wave functions with given quantum numbers and handle them easily. For standard Heisenberg spin models those MP states can serve as good trial wave functions while for certain “perturbations” of standard models those wave functions become exact ground states. MP states have also proved to be a convenient tool for variational study of elementary excitations.

We use two MP ansätze for the ground state wave function having different translational symmetry properties: one is invariant under translation for one ladder rung, and for the other ansatz the elementary cell consists of two rungs. Respectively, we obtain two families of Hamiltonians with exact ground states; one should emphasize that the Hamiltonian itself in both cases is translationally invariant, so that the second family describes models with spontaneous dimerization. (Those families were studied by us recently in a somewhat more restrictive formulation.) The two families have non-empty intersection, which makes possible to determine exactly the boundary of a second-order phase transition from translationally invariant to dimerized phase.

Thus we show that, in contrast to the common belief, finitely correlated states can describe critical models. However, it turns out that in the dimerized phase the dimer order parameter $D_i = S_{i,i+1} \cdot (S_{i+1,i} - S_{i,i-1})$ and spin correlation functions behave in a rather peculiar way at the transition point $\tau = \tau_c$ (where $\tau$ is certain model parameter): (i) the spin correlation function...
\[ \langle S_i^z S_{i+n}^z \rangle = A S e^{-n/\xi} \] always decays exponentially, and the correlation length \( \xi \) either does not exhibit any singularities, or diverges as \((\tau - \tau_c)^{-2}\), but in the latter case the prefactor \( A S \propto (\tau - \tau_c)^{-2} \) vanishes exactly at the transition, thus preventing the appearance of the long-range spin order; (ii) the dimer order correlation function \( \langle D_i D_{i+n} \rangle \propto (\tau - \tau_c)^{2} \) does not depend on the distance \( n \) and is just a constant smoothly vanishing at the transition.

For excitations in the spontaneously dimerized phase we propose a simple variational MP-type ansatz describing the elementary excitation as a pair of solitons in dimer order; we show that the variational estimate for the gap goes to zero at the phase boundary.

We show that for the certain class of translationally invariant models having effectively a spin-1 Affleck-Kennedy-Lieb-Tasaki (AKLT) VBS ground state (formed by only triplet degrees of freedom at each rung) it is possible to write an MP-type wave function for the singlet excitation which is an exact eigenstate of the Hamiltonian and whose gap closes at two different second-order phase transition boundaries. For another class of models whose ground state is a simple product of singlet bonds along the ladder rungs, we give the condition of existence of the exact triplet excitation.

We also consider in more detail a toy model being a matrix-product solvable “perturbation” of the “composite spin” model first introduced by Bose and Gayen and generalized later by Weihong et al., in the same sense as the AKLT model is a perturbation of the standard \( S = 1 \) Heisenberg spin chain. This model is remarkable because one can obtain exactly its full phase diagram, including several lines of phase transitions of the first and second order.

The paper is organized as follows: in Sect. II we explain the general procedure of finding a set of exact MP-type solutions, and sections III and IV describe the two families of solutions mentioned above; in Sect. V we also discuss the boundaries separating dimerized and non-dimerized phases and the structure of elementary excitations in the dimerized phase. Section V is devoted to the generalized Bose-Gayen model mentioned above and, finally, Sect. VI gives a brief summary.

II. FINITELY CORRELATED EXACT GROUND STATES: THE CONSTRUCTION ROUTINE

We start from a general form of the isotropic translationally invariant spin ladder Hamiltonian with exchange interaction only between spins on plaquettes formed by neighboring rungs, \( \hat{H} = \sum_i \hat{h}_{i,i+1} \), where the local Hamiltonian \( \hat{h} \) is defined as

\[
\hat{h}_{i,i+1} = \frac{1}{2}(J_R + \varepsilon_R)S_{1,i}S_{2,i} + \frac{1}{2}(J_R - \varepsilon_R)S_{1,i+1}S_{2,i+1} + J_L S_{1,i}S_{1,i+1} + J_D' S_{1,i}S_{2,i+1} + J_L' S_{2,i}S_{2,i+1} + J_D S_{1,i}S_{2,i+1} + J_D' S_{2,i}S_{1,i+1} + V_{LL}(S_{1,i}S_{1,i+1})(S_{2,i}S_{2,i+1}) + V_{RR}(S_{1,i}S_{2,i})(S_{1,i+1}S_{2,i+1}) - E_0,
\]

here the indices 1 and 2 distinguish lower and upper legs, and \( \varepsilon \) labels rungs. The model is schematically represented in Fig. 1. For periodic boundary conditions \( J_R \) is the coupling on the rungs, and Hamiltonians with different \( \varepsilon_R \) are physically indistinguishable; however, we will see later that it is necessary to introduce \( \varepsilon_R \) in order to include models with inequivalent legs. In addition, there are generally four different bilinear exchange couplings on the legs and diagonals and three biquadratic terms. Together with the constant term \(-E_0 \) we have a total of ten parameters. For later convenience, we will also use the following notation:

\[
J_{L,D}^{(\pm)} = \frac{1}{2}(J_{L,D} \pm J_{L,D}').
\]

We look for the ground state wave function \( \Psi_0 \) in a form of a so-called finitely correlated \( \Psi_0 \) or matrix product (MP) \( \Psi_0 \) state, and we use the following two ansätze for \( \Psi_0 \):

\[
\Psi_0^{\text{inv}} = \prod_i g_i(u), \quad \Psi_0^{\text{dim}} = \prod_i g_{2n-1}(u_1) \cdot g_{2n}(u_2),
\]

\[
g_i(u) = \left[ u|s_i\rangle + |t_0\rangle \frac{1}{\sqrt{2}}|t_{-1}\rangle, \quad u|s_i\rangle - |t_0\rangle \frac{1}{\sqrt{2}}|t_{-1}\rangle \right].
\]

Here \( |s_i\rangle \) and \( |t_0\rangle \) are the singlet and triplet states of the \( i \)-th rung, and \( u_1, u_2 \) are free parameters. Strictly speaking, \( \Psi_0 \) in (2) is a matrix whose trace should be taken to get the wave function of a ladder with periodic boundary conditions. Three other linear independent wave functions \( \text{tr}(\sigma^\mu \Psi_0), \mu = 0, \pm 1 \) describe states of a ladder with open ends having different behavior at the edges, which is similar to the edge states in the effective \( S = 1 \) chain. The ansatz (3) obeys rotational symmetry, i.e., \( \text{tr}(\Psi_0) \) is a global singlet, and the states \( \text{tr}(\sigma^\mu \Psi_0) \) form a spin-1 triplet. The state \( \Psi_0^{\text{inv}} \) is translationally invariant under the translation for one ladder rung, and \( \Psi_0^{\text{dim}} \) describes dimerized states with spontaneously broken translational invariance (unless \( u_1 = u_2 \) when it obviously becomes identical to \( \Psi_0^{\text{inv}} \)). Translation of \( \Psi_0^{\text{dim}} \) for one rung leads to a different state with the same energy. The construction (4) was originally proposed as a variational wave function later the translationally invariant form \( \Psi_0^{\text{inv}} \) was used by us to construct a class of exact ground states for the model (1) under somewhat restrictive assumptions \( \varepsilon_R = 0 \), \( J_L = J_L' \), \( V_{LL} = V_{DD} \), \( V_{RR} = 0 \), and we have reported several particular solutions of the type \( \Psi_0^{\text{dim}} \) in Ref. [6].

One can observe that ansätze (3) describe several known examples of VBS-type states, e.g., \( \Psi_0^{\text{inv}} \) at \( u = 0 \) yields the ground state of the effective AKLT chain whose \( S = 1 \) spins are composed from pairs of \( S = \frac{1}{2} \) spins of the ladder rungs,
\[ \hat{H} = \sum_n S_n S_{n+1} + \frac{1}{3} (S_n \cdot S_{n+1})^2, \]  

while for \( u = 1 \) and \( u = \infty \) one obtains two degenerate dipole ground states of the Majumdar-Ghosh (MG) model with singlets residing on the diagonals and on the rungs, respectively \([\text{the MG model results from}]\) at \( V_{RR,LL,DD} = 0, J'_R = 0, J_L = J'_L, J_R = J_D = 2J_L \). The wave function \( \Psi_0^{\text{dim}} \) at \( u_1 = -u_2 = \pm 1 \) describes a state with checkerboard-type ordered singlet bonds along the ladder legs. Thus one may think of the construction \([\text{as of}}\) interpolating between several types of VBS states mentioned above.

The construction of exact ground states can be performed in the way outlined in Refs. \([\text{10,17}]\) one has to require \( \Psi_0 \) to be a zero-energy ground state of the local Hamiltonian \( \hat{h}_{i,i+1} \), which ensures that it is a ground state of the global Hamiltonian \( \hat{H} \). This yields the following conditions:

(i) for \( \Psi_0^{\text{inv}} \), the local Hamiltonian \( \hat{h}_{12} \) should annihilate all states being matrix elements of the product \( g_1(u) \cdot g_2(u) \):

\[ \hat{h}_{12}\{g_1(u) \cdot g_2(u)\} = 0, \]  

while for \( \Psi_0^{\text{dim}} \) it is necessary that \( \hat{h}_{12} \) annihilates all states contained in the two products \( g_1(u_1) \cdot g_2(u_2) \) and \( g_1(u_2) \cdot g_2(u_1) \),

\[ \hat{h}_{12}\{g_1(u_1) \cdot g_2(u_2)\} = 0, \]  

\[ \hat{h}_{12}\{g_1(u_2) \cdot g_2(u_1)\} = 0; \]  

(ii) all the other eigenstates of \( \hat{h}_{12} \) have positive energy. Then \( \Psi_0 \) is the zero-energy ground state of \( \hat{H} \); if one drops the constant term \( -E_0 \) in \( \[\text{[8]}\] \), the remaining Hamiltonian has the energy density \( E_0 \) per rung.

It is convenient to write the local Hamiltonian \( \hat{h}_{i,i+1} \) in terms of projectors on the states with fixed angular momentum \( |\psi_{\mu}^{(k)}\rangle \) of the two-rung plaquette \((i,i + 1)\). The complete set of the plaquette states contains one quintuplet \((j = 2)\), three triplets \((j = 1)\) and two singlets \((j = 0)\), therefore a general form of \( \hat{h} \) reads as:

\[ \hat{h}_{i,i+1} = \lambda_2 \sum_{\mu} |\psi_{2\mu}\rangle\langle\psi_{2\mu}| + \sum_{k,l=1}^{3} \lambda_1^{(k,l)} \sum_{\mu} |\psi_{1\mu}^{(k)}\rangle\langle\psi_{1\mu}^{(l)}| 
+ \sum_{k,l=1,2} \lambda_0^{(k,l)} |\psi_{00}^{(k)}\rangle\langle\psi_{00}^{(l)}|. \]  

III. MODELS WITH UNBROKEN TRANSLATIONAL SYMMETRY

For \( \Psi_0^{\text{inv}} \), it is easy to check that generally the matrix product \( g_1(u) \cdot g_2(u) \) contains only the following two multiplets which should become local ground states:

\[ (3 + u^2)^{-1/2} \left( u^2 |ss\rangle - \sqrt{3} |tt\rangle \right) \equiv |\psi_{00}^{(g)}\rangle, \]  

\[ (1 + u^2)^{-1/2} \left( \frac{u}{\sqrt{2}} (|t\rangle |s\rangle + |s\rangle |t\rangle) - |tt\rangle \right) \equiv |\psi_{1\mu}^{(g)}\rangle; \]  

\( \text{(in fact, the second multiplet above enters only in the combination} \) \( (3 + u^2)^{-1} |\psi_{1\mu}^{(g)}\rangle \), so that \( |\psi_{1\mu}^{(g)}\rangle \) disappears from \( g_1(u) \cdot g_2(u) \) in the limit \( u \to \infty \), see below). The remaining multiplets becoming local eigenstates can be chosen as...
\[ |\psi_{00}^{(e)}\rangle = (3 + u^4)^{-1/2} \left( \sqrt{3} |ss\rangle + u^2 |tt\rangle \right) \]
\[ |\psi_{11}^{(e,1)}\rangle = \frac{1}{\sqrt{2}} (|tt_s\rangle - |st\rangle) \]
\[ |\psi_{11}^{(e,2)}\rangle = (1 + u^2)^{-1/2} \left\{ \frac{1}{\sqrt{2}} (|tt_s\rangle + |st\rangle) + u |tt_1\rangle \right\} \]
\[ |\psi_{22}^{(e)}\rangle = |tt_2\rangle \, . \]

Here \(|tt_1\rangle\) and \(|tt_0\rangle\) denote respectively the triplet \((j = 1)\) and the singlet \((j = 0)\) constructed from two triplet states on rungs \(i\) and \(i + 1\).

Equations \((12)\) [in this case \(n_0^q = n_5^q = 1\), \(n_2^q = 2\), \(n_5^q = 1\), \(n_2 = 0\)], which impose the demand that the local Hamiltonian annihilates all the states \((13)\), can be rewritten in a form of relations between the Hamiltonian parameters as follows:

\[ J_R = \frac{5}{2} \lambda_2 + \frac{A B}{6} \lambda_0 - \frac{1}{2} (\lambda_1^{(1,1)} - C \lambda_1^{(2,2)}), \]
\[ J_L^{(+)} = \frac{5}{12} \lambda_2 - \frac{A (A + B)}{12} \lambda_0 - \frac{1}{4} (\lambda_1^{(1,1)} + C \lambda_1^{(2,2)}), \]
\[ J_D^{(+)} = \frac{5}{12} \lambda_2 - \frac{B (A + B)}{12} \lambda_0 - \frac{1}{4} (\lambda_1^{(1,1)} - \lambda_1^{(2,2)}), \]
\[ V_{RR} = \frac{1}{3} \lambda_2 - \frac{A B}{3} \lambda_0 - \lambda_1^{(1,1)} + C \lambda_1^{(2,2)}, \]
\[ V_{LL} = \frac{1}{3} \lambda_2 + \frac{A (A + B)}{3} \lambda_0 - \lambda_1^{(1,1)} - C \lambda_1^{(2,2)}, \]
\[ V_{DD} = \frac{1}{3} \lambda_2 + \frac{B (A + B)}{3} \lambda_0 + \lambda_1^{(1,1)} - \lambda_1^{(2,2)}, \]
\[ \varepsilon_R = \frac{2 \lambda_1^{(1,2)}}{\sqrt{1 + u^2}}, \quad J_L^{(-)} = -\frac{u \lambda_1^{(1,2)}}{\sqrt{1 + u^2}}, \quad J_D^{(-)} = -\frac{u \lambda_1^{(2,2)}}{1 + u^2}, \]

the energy density per rung \(E_0\) being
\[ E_0 = -\frac{5}{16} \lambda_2 - \frac{1}{16} \lambda_0 - \frac{3}{16} (\lambda_1^{(1,1)} + \lambda_1^{(2,2)}). \]  

Here we have used the following shorthand notation:
\[ A = \frac{u^2 + 3}{(3 + u^4)^{1/2}}, \quad B = \frac{u^2 - 3}{(3 + u^4)^{1/2}}, \quad C = \frac{u^2 - 1}{u^2 + 1}, \]

It is sufficient to consider only positive values of \(u\), because changing the sign \(u \rightarrow -u\) amounts just to interchanging the ladder legs and thus does not bring in any new physics. One can see that the “unphysical” parameter \(\varepsilon_R = -2 J_L^{(-)} / u\) is just needed to allow the ladder legs to be inequivalent. The conditions \((14)\) now take the form
\[ \lambda_0 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_1^{(1,1)} + \lambda_1^{(2,2)} \geq 0, \quad \left[ \lambda_1^{(1,2)} \right]^2 \leq \lambda_1^{(1,1)} \lambda_1^{(2,2)}. \]

Equations \((12)\) describe a five-parametric family of spin ladder models (five \(\lambda_i\'s\) and \(u\), but one of the \(\lambda_i\'s\) is irrelevant since it just determines the energy scale) whose translationally invariant ground state can be written exactly in the form of a matrix product \(\Psi_0^{\text{inv}}\).

Though the limit \(u \rightarrow \infty\) of Eqs. \((12)\) is formally well defined, the family \((12)\) does not contain all models having a product of singlet bonds on the rungs as their exact ground states, e.g., the model of Bose and Gayen is missing there. For \(u = \infty\) obviously only the state \(|ss\rangle\) enters the matrix product \(g_1 \cdot g_2\), and this case should be considered separately. The resulting general solution for the \(u = \infty\) case is
\[ J_L^{(+)} - J_D^{(+)} = (V_{LL} - V_{DD}) / 4, \]
the energy density per rung being
\[ E_0 = -3J_R / 4 + 3(V_{LL} + V_{DD} + 3V_{RR}) / 16. \]  

The conditions of positivity \((13)\) result in the constraints
\[ \mu_0 = J_R - 2J_L^{(+)} + (V_{DD} - V_{RR}) / 2 \geq 0, \]
\[ \mu_2 = J_R + J_D^{(+)} - (V_{DD} + 2V_{RR}) / 4 \geq 0, \]
\[ \bar{\mu}_1^{(i)} \geq 0, \quad i = 1, 2, 3, \]  

where \(\bar{\mu}_1^{(i)}\) are the eigenvalues of a symmetric 3×3 matrix \([\mu_1^{(i)}]\), with the matrix elements defined as
\[ \mu_1^{(1)} = \frac{J_R}{2} - V_{LL} - V_{DD} / 4, \quad \mu_1^{(2)} = \frac{J_R}{2} - \frac{V_{DD}}{2} - \frac{3V_{RR}}{4}, \]
\[ \mu_1^{(3)} = \frac{J_R - J_D^{(+)} - (2V_{DD} + 2V_{RR} + V_{LL}) / 4. \]  

One can check that the families \((12)\) and \((13)\) match each other if \(\lambda_0 = 0\), so that it is natural to assume that vanishing \(\lambda_0\) signals the first-order phase transition into the rung-dimer phase. Indeed, imagine we have a MP-solvable model with certain ground state and gradually decrease \(\lambda_0\) to 0, then exactly at \(\lambda_0 = 0\) the rung-dimer state is degenerate with the “old” ground state, so that we have level crossing. The same reasoning can be applied to the transition into the completely polarized ferromagnetic phase which is driven by vanishing \(\lambda_2\); note that one cannot do the same for \(\bar{\lambda}_1^{(i)}\) because no uniform global wave function can be constructed from the local Hamiltonian eigenstates corresponding to those eigenvalues. However, later we will see that simultaneous vanishing of \(\lambda_1^{(1,1)}\) and \(\lambda_1^{(2,2)}\) can be associated with another phase transition, the second order phase transition into a spontaneously dimerized phase. In special cases, other eigenvalues may become relevant for determining the phase transition points (see Sect. III C and Sect. V below).

In the rest of this section we consider several particular solutions belonging to the general families derived above.

A. Models with only bilinear exchange coupling

If one demands that all biquadratic interactions in \((1)\) are absent, the general family of models \((12)\) reduces to the following two “branches:”
1. Models with completely dimerized bonds, whose ground state is just a product of singlet dimers along the ladder diagonals, are obtained from (12) by setting $u = 1$, $\lambda_1^{(2,2)} = 1$, $\lambda_1^{(1,1)} = y$, $\lambda_1^{(1,2)} = x\sqrt{2}$, $\lambda_2 = 2 - y$:

$$J_R = 2(1 - y), \quad J_L = 1 - y - x, \quad J'_L = 1 - y + x, \quad J_D = 1, \quad J_B = 0, \quad \varepsilon_R = 2x,$$

the conditions (4) require that $1/2 \leq y \leq 2$, $x^2 \leq y^2/2$. It is easy to see that at $x = 0$ this model describes an $S = 1/2$ zigzag chain with alternated nearest-neighbor (NN) exchange and frustrating next-nearest neighbor (NNN) interaction, with the alternation proportional to $2y - 1$ and the ratio of NNN exchange constant to the smaller NN one equal to $1/2$. For $1/2 \leq y \leq 1$ the ground state of this model was found by Shastry and Sutherland, and $y = 1$ corresponds to the well-known Majumdar-Ghosh (MG) model. At the MG point there is another, degenerate ground state with singlet bonds on the rungs which is contained in the family (13). At $x = y = 1$, (13) describes the so-called $\Delta$-chain, or see-saw chain, which, like the MG model, also has two degenerate ground states at $y = 1$. The interval $1 \leq y \leq 2$ corresponds to partially ferromagnetic interactions, and at $y = 2$ the model (19) exhibits a first-order phase transition into the completely polarized ferromagnetic state.

One can consider a more general model of a zigzag chain with two unequal NN interactions $\alpha(1 \pm \gamma)$ and alternating NN couplings 1 and $\beta$, $\beta < 1$, as shown in Fig. 2a. This model interpolates between the usual frustrated chain ($\gamma = 0$) and the $\Delta$-chain ($\gamma = 1$). For fixed $\gamma$, the solution (13) determines the line

$$\beta = 2\alpha, \quad \max(-1, \zeta_-) \leq \alpha \leq \min\left(\frac{1}{2}, \zeta_+\right), \quad (20)$$

in the $(\alpha, \beta)$ plane where the ground state is a product of NN dimers on $J = 1$ bonds, see Fig. 2b. For $|\gamma| \leq 1$ the minimal and maximal values of $\alpha$ in (20) are always $-1$ and $1/2$, the point $\alpha = -1$, $\beta = -2$ lies on the boundary of the transition into the ferromagnetic (FM) phase, and $\alpha = 1/2$, $\beta = 1$ is the (generalized) Majumdar-Ghosh point. As we will see below, the situation for $|\gamma| > 1$ is more complicated.

It is worthwhile to make an attempt to estimate the boundaries of stability of the FM phase by using a standard spin wave theory. Introducing two kinds of Holstein-Primakoff bosons for representing the spin operators on two different legs, and passing to the momentum space, one obtains in the lowest order the following spin wave Hamiltonian:

$$\hat{H} = \sum_k \left\{ \varepsilon_{1k} a_{1k}^\dagger a_{1k} + \varepsilon_{2k} a_{2k}^\dagger a_{2k} + (\Phi_k a_{1k}^\dagger a_{2k} + \text{h.c.}) \right\},$$

$$\varepsilon_{1k,2k} = -\frac{1 + \beta}{2} - \alpha(1 + \gamma)(1 - \cos k), \quad \Phi_k = \beta + e^{-i k}.$$

This Hamiltonian is trivially diagonalized, yielding the following expressions for the dispersion of two magnon branches:

$$\varepsilon_{A,B}(k) = \varepsilon_{1k,2k} + \frac{(\varepsilon_{1k} - \varepsilon_{2k}) + 2|\Phi_k|^2}{(1 + \varepsilon_k^2)}, \quad (21)$$

$$\varepsilon_k = (\varepsilon_{1k} - \varepsilon_{2k})/2|\Phi_k| + \sigma(1 + (\varepsilon_{1k} - \varepsilon_{2k})^2/4|\Phi_k|^2)^{1/2}, \quad \text{here } \sigma = \text{sgn}(\varepsilon_{1k} - \varepsilon_{2k}).$$

It is easy to check that $\varepsilon_{A}(k) \simeq Ck^2$ at $k \to 0$, $C = -\frac{1}{8}\{2\alpha + (1 + \beta)\}$. Thus $\varepsilon_A$ represents the usual branch of ferromagnons, and the condition $C = 0$ determines the point where this branch becomes unstable. The other branch $\varepsilon_B$ represents “optical” ferromagnons, $\varepsilon_B(k = 0) = -1 - \beta$. For $\gamma = 0$ the branch $\varepsilon_B$ is always gapped, but for large enough $|\gamma|$ it closes at $k = \pi$. Thus the region of stability of the FM phase is determined by the following two conditions:

$$\beta \leq -\frac{2\alpha}{1 + 2\alpha^2}, \quad |\gamma| \leq \gamma_c(\alpha, \beta), \quad (22)$$

where the critical point $\gamma_c$ is defined as a real positive root of the following fourth-order polynomial equation:

$$64\alpha^4 \gamma_c^4 - [32\alpha(2\alpha + 1) + 12\beta^2 - 8\beta + 12] \alpha^2 \gamma_c^2 - (\beta - 1)^2(1 + 2\alpha)(\beta + 2\alpha) = 0. \quad (23)$$

It is interesting to note that the curve in the phase space of the model determined by the first condition in (22) at $\gamma = 0$ gives precisely the set of models with exact singlet ground states degenerate with the FM state which were studied in Ref. [20]. Thus at least the first condition in (22), obtained from a simple spin wave calculation, remarkably coincides with the exact result. It should be mentioned that the exact singlet wave functions presented in Ref. [20] exhibit double spiral order, and are very different from our MP wave functions, which indicates high degeneracy on the line of transition from the FM to singlet phase.

The second condition in (22) becomes relevant only at $|\gamma| > 1$, i.e., when the leg couplings have different signs; at $|\gamma| < 1$ the boundary of FM phase is determined solely by the first condition, see Fig. 3b.

One can also observe that the critical value of $\gamma$ determined by (23) is different from that which can be obtained from our MP calculation: at $\alpha = -1$, $\beta = -2$ it follows from (20) that $|\gamma| \leq 1$, while from (23) one obtains $|\gamma| < \gamma_c \simeq 1.7$. As we discussed before, vanishing of any $\lambda$‘s except $\lambda_0$ and $\lambda_2$ cannot be associated with a phase transition, and generally the conditions (4), being sufficient but not necessary, tend to underestimate the stability region.

2. Bilinear models with nontrivial ground states result from (12) when

$$\lambda_2 = u^2(9 - u^2), \quad \lambda_0 = 2(3 + u^4), \quad \lambda_1^{(1,1)} = 5u^2 + 3, \quad \lambda_1^{(2,2)} = (3 + u^2)(1 + u^2), \quad \lambda_1^{(1,2)} = x(1 + u^2)^{1/2}, \quad (24)$$

which gives the Hamiltonian parameters as
with the energy density per rung \( E_0 = -\frac{3}{5}(1+3u^2) \). From (23) it follows that \( x^2 \leq (3+u^2)(5u^2+3), |u| \leq 3 \). At \( u = 3 \) the eigenvalue \( \lambda_2 \) vanishes, i.e., the system undergoes a phase transition into the completely polarized state, and at the same time one of the diagonal interactions vanishes, making the ladder equivalent to a frustrated one. One can check that this solution belongs to the form (2) with \( 1 = 0 \), which defines a one-parametric family of models of the form (24) with arbitrary parameter \( u \) is a ground state. One can easily calculate the overlap between two such wave functions having different values of \( u \):

\[
\langle \Psi_0(u_1) | \Psi_0(u_2) \rangle = q^N, \quad q = \frac{(1 + u_1 u_2)^2}{(1 + u_1^2)(1 + u_2^2)} \leq 1,
\]

i.e., the two g.s. wave functions with different values of \( u \) are asymptotically orthogonal in the thermodynamic limit \( N \to \infty \) with the overlap vanishing exponentially with the increase of \( N \). This means that the dimension of the basis of this subspace \( \{ \Psi_0(u) \} \), i.e., the number of mutually orthogonal ground states, is exponentially large in the thermodynamic limit. For \( x = 0 \) this statement becomes trivial, because then the local Hamiltonian projects only onto the states with the total spin of two neighboring rungs equal to 2, and any sequence of singlets and triplets on the rungs which does not contain neighboring triplets is a ground state; in this case the SU(2) symmetry of the original Hamiltonian is spontaneously broken because the ground states are not necessarily eigenstates of the total spin. Later we show that at \( x = \frac{1}{2} \) the model (24) is gapless, by constructing the exact excitation wave function (see Sect. V).

### C. AKLT-type models

One can put \( u = 0 \) in (12) and arrive at a large family of models having the AKLT ground state formed by effective \( S = 1 \) spins made up of the ladder rungs. An example of such solution was presented in our earlier paper. The simplest model within this class can be obtained by requiring that the diagonal interactions vanish together with the rung-rung biquadratic coupling, which is achieved by setting \( \lambda_2 = 6/5, \lambda_1(1,1) = \frac{1}{5} \lambda_0 - \frac{4}{5}, \lambda_1(2,2) = \frac{1}{5} \lambda_0 + \frac{4}{5}, \lambda_1(1,2) = 0 \). The model then describes a ladder with ferromagnetic rungs and two additional biquadratic interactions:

\[
J_R \leq -4/5, \quad \varepsilon_R = 0, \quad J_L = J_L' = 1, \quad J_D = J_D' = 0, \quad V_{RR} = 0, \quad V_{LL} = 12/5, \quad V_{DD} = -8/5.
\]

It is worthwhile to mention that there exists a one-parametric family of models connecting smoothly the Majumdar-Ghosh chain to the effective AKLT chain, thus proving that the Haldane chain and the dimerized chain are “in the same phase” in the sense that there is a continuous path in the phase space which links the two states and does not cross any singularities. However, as we will see below, this does not generally exclude the possibility to have a phase boundary between the Haldane (AKLT) and dimer states. A model exhibiting such a boundary will be considered later in Sect. V.

The AKLT-type ladder models are especially remarkable because they admit constructing an exact eigenstate of the Hamiltonian being a singlet magnon which becomes gapless under certain conditions. Let us consider the state

\[
|n\rangle_S = \text{tr} \{ g_A^1 g_A^2 \cdots g_A^N g_A^{N+1} \cdots g_A^N \},
\]

where \( g_A^i = g_A(u = 0) \) is the ground state (AKLT) matrix and \( g_A^o = g_A(u = \infty) = 1 \cdot |s\rangle_n \) is the matrix describing a singlet bond on the \( n \)-th rung: a visual VBS-representation of this state is shown in Fig. 3. This state is orthogonal to the ground state wave function \( |\Psi_0^\text{g} \rangle = \text{tr} \{ g_A^1 \cdots g_A^N \} \). It can be straightforwardly checked that the action of the Hamiltonian on the state \( |n\rangle_S \) yields nothing but the states \( |n\rangle_S, |n \pm 1\rangle_S \):

\[
\hat{H} |n\rangle_S = \{ \lambda_1(1,1) + \lambda_1(2,2) \} |n\rangle_S
+ \frac{1}{2} \{ \lambda_1(2,2) - \lambda_1(1,1) \} \{ |n-1\rangle_S + |n+1\rangle_S \}.
\]
[Here the ground state energy is already subtracted, i.e., the term \(-E_0\) is included in the local Hamiltonian \(\hat{h}\) in (3)]. Trivially constructing the exact eigenstate \(|k\rangle_S = \sum_n e^{i kn}|n\rangle_S\) with total momentum \(k\), we obtain its energy as
\[
\varepsilon_S(k) = \lambda_1^{(1,1)} + \lambda_1^{(2,2)} + \{\lambda_1^{(2,2)} - \lambda_1^{(1,1)}\} \cos(k). \tag{31}
\]
The minimum of the dispersion is reached at \(k = \pi\) if \(\lambda_1^{(1,1)} < \lambda_1^{(2,2)}\) and at \(k = 0\) if \(\lambda_1^{(1,1)} > \lambda_1^{(2,2)}\), and the excitation becomes gapless when either \(\lambda_1^{(1,1)}\) or \(\lambda_1^{(2,2)}\) vanishes. Later we will see that the transition at \(\lambda_1^{(1,1)} = 0\) is into a spontaneously dimerized phase, and the transition at \(\lambda_1^{(2,2)} = 0\) is specific for the AKLT-type models.

Far from the phase boundaries one should expect that the singlet magnon branch will be high in energy, and the lowest excitation will be a usual triplet magnon being a soliton in the string order.

D. Rung-dimer models with exact triplet excitations

Inside the general family (3)1, (5), describing models with rung-dimer ground state, it is worthwhile to consider a special subclass of models satisfying the additional condition
\[
J_L^{(-)} + J_D^{(-)} = 0. \tag{32}
\]
Those models are remarkable because it is a straightforward exercise to check that the simplest excitation resulting from promoting one ladder rung from singlet to a triplet,
\[
|k\rangle_L = \sum_n e^{i kn}|n\rangle_L
\]
\[
|n\rangle_L = |s1\rangle|s2\rangle \cdots |s_{n-1}\rangle |t\rangle_n |s_{n+1}\rangle \cdots |s\rangle_N .
\]
is an exact eigenfunction with the energy
\[
\varepsilon_L(k) = J_R - \frac{1}{2}(V_{LL} + V_{DD} + 3V_{RR}) + 2(J_L^{(+)} - J_D^{(+)} \cos k). \tag{34}
\]
This exact excitation is of course not always the lowest one [e.g., for the model of \(\Delta\)-chain, \(\Delta > 1\) in (3)], it is dispersionless and has a high energy equal to 1, but in some cases the dispersion law (3) has gapless or almost gapless points and thus the triplet excitation defined above is relevant at least in some interval of momenta \(k\). One interesting realization is the ladder model with leg-leg biquadratic coupling, given by
\[
J_L = J'_L = \frac{1}{4}V_{LL}, \quad J_R \geq 4J_L \geq 0, \tag{35}
\]
with all the other couplings being zero. The dispersion (3) in this case reads as \(\varepsilon_L(k) = J_R - 2J_L (1 - \cos k)\), and becomes gapless at \(k = \pi\) for \(J_L \rightarrow \frac{1}{2}J_R\). Another interesting model within this class will be considered later in Sect. V.

IV. SPONTANEOUSLY DIMERIZED MODELS: PHASE BOUNDARIES AND ELEMENTARY EXCITATIONS

For the dimerized matrix product ansatz \(\Psi_0^{\text{dim}}\) the local ground states entering two matrix products \(g_1(u_1) \cdot g_2(u_2)\) and \(g_1(\bar{u}_2) \cdot g_2(\bar{u}_1)\) are
\[
|\psi_{00}^{(g)}\rangle = \left[3 + (u_1u_2)^2\right]^{-1/2} \{u_1u_2|ss\rangle - \sqrt{3}|tt\rangle_{00}\},
|\psi_{1\mu}^{(g)}\rangle = \left[1 + f^2/2\right]^{-1/2} \{f|st\rangle_{1\mu} + |ts\rangle_{1\mu}\},
|\psi_{1\mu}^{(g,2)}\rangle = \text{sgn}(u_1 - u_2) \frac{1}{\sqrt{2}}(|st\rangle_{1\mu} - |ts\rangle_{1\mu}),
\]
where \(f \equiv (u_1 + u_2)/\sqrt{2}\), and the remaining multiplets can be chosen as
\[
|\psi_{0\mu}^{(g)}\rangle = \left[3 + (u_1u_2)^2\right]^{-1/2} \{\sqrt{3}|ss\rangle + u_1u_2|tt\rangle_{00}\},
|\psi_{1\mu}^{(g)}\rangle = \left[2 + f^2\right]^{-1/2} \{f|tt\rangle_{1\mu} + |st\rangle_{1\mu} + |ts\rangle_{1\mu}\},
|\psi_{2\mu}^{(g)}\rangle = |tt\rangle_{1\mu}.
\]
Note that at \(u_1 = u_2\) the multiplet \(|\psi_{1\mu}^{(g,2)}\rangle\) disappears from the set of local ground states and becomes an eigenstate, in accordance with Eqs. (4), (5). Performing the same procedure as for undimerized case, we arrive at the following general family of solutions:
\[
J_R = \frac{1}{6}AB\lambda_0 + \frac{1}{6}C\lambda_1 + \frac{5}{6}\lambda_2, \quad \varepsilon_R = 0,
J_L = J'_L = -\frac{1}{12}A(\bar{A} + \bar{B})\lambda_0 - \frac{1}{4}C\lambda_1 + \frac{5}{12}\lambda_2,
J_D^{(+)} = -\frac{1}{12}\bar{B}(\bar{A} + \bar{B})\lambda_0 - \frac{1}{4}\lambda_1 + \frac{5}{12}\lambda_2,
J_D^{(-)} = -\frac{2}{12}(u_1 + u_2)^2 + \lambda_1,
V_{RR} = -\frac{1}{3}\bar{A}B\lambda_0 + \bar{C}\lambda_1 + \frac{1}{3}\lambda_2,
V_{LL} = \frac{1}{3}\bar{A}(\bar{A} + \bar{B})\lambda_0 - \bar{C}\lambda_1 + \frac{1}{3}\lambda_2,
V_{DD} = \frac{1}{3}\bar{B}(\bar{A} + \bar{B})\lambda_0 - \lambda_1 + \frac{1}{3}\lambda_2,
\]
with the energy density per rung
\[
E_0 = -\frac{1}{16}\lambda_0 - \frac{3}{16}\lambda_1 - \frac{5}{16}\lambda_2. \tag{39}
\]
Here the quantities \(\lambda_j \geq 0, j = 0,1,2\) denote the local eigenvalues corresponding to the multiplets \(|\psi_{0\mu}^{(g)}\rangle\) and the factors \(\bar{A}, \bar{B}, \bar{C}\) are defined as follows:
\[
\bar{A} = \frac{u_1u_2 + 3}{(u_1u_2 + 3)^{1/2}}, \quad \bar{B} = \frac{u_1u_2 - 3}{(u_1u_2 + 3)^{1/2}},
\bar{C} = \frac{(u_1 + u_2)^2 - 4}{(u_1 + u_2)^2 + 4}. \tag{40}
\]
Now we proceed to the study of general properties of the dimerized models derived above.
A. Phase boundaries

Comparing the general solutions (12) and (18), one can easily see that they match each other if

$$u_1 = u_2 = u, \quad \lambda_1^{(1,1)} = \lambda_1^{(1,2)} = 0,$$  \hspace{1cm} (41)

then \(\lambda_1^{(2,2)}, \lambda_0, \lambda_2\) in (12) should be identified respectively with \(\lambda_1, \lambda_0, \lambda_2\) in (18). The original Hamiltonian is translational invariant, thus the condition defines a critical line in the space of the model parameters where the second-order quantum phase transition into spontaneously dimerized phase occurs.

It should be mentioned that such a transition for ladders was first studied by Nersesyan and Tsvelik within a field-theoretical approach in the approximation of weak inter-leg coupling. They found that leg-leg biquadratic exchange is a relevant perturbation, so that at sufficiently strong \(V_{LL}\) the ladder can enter into a spontaneously dimerized non-Haldane spin liquid phase, with the elementary excitations being soliton-antisoliton pairs. Later we have proposed explicit examples of solvable models exhibiting such properties and from the general solution it follows that other interactions may be responsible for this transition as well. (For example, it is possible to get a spontaneously dimerized ladder with \(V_{LL} = 0\).) However, one can see that in MP-solvable ladder models described by (11) the dimerization transition has certain peculiarities. It is easy to calculate spin-spin and dimer-dimer correlation functions \(C_S(n) = \langle S^z_{1,i} S^z_{1,i+n} \rangle\) and \(C_D(n) = \langle D_i D_{i+n} \rangle, D_i = S_{1,i} \cdot (S_{1,i+1} - S_{1,i-1})\) being the dimerization order parameter:

$$C_S(n) = (u_2^2 + 3)^{-1}(Z_{12} Z_{21})^n (\delta_{n,2k} + Z_{21} \delta_{n,2k+1}),$$

$$C_D(n) = (u_2^2 + 3)^{-1} (u_2^2 + 3)^2 (u_2^2 + 3)^2, \quad Z_{ab} = \frac{(u_a - 1)(u_b + 1)}{3 + u_b^2},$$  \hspace{1cm} (42)

here \(a, b = 1, 2\). One can see that the dimer correlations exhibit long-range order vanishing for \(u_1 \rightarrow u_2\) or \(u_{1,2} \rightarrow \infty\), but remarkably there is no exponential tail present in the correlations. Spin-spin correlations always decay exponentially, and the correlation length \(\xi = -1/\ln(Z_{12} Z_{21})\) does not show any singular behavior at \(u_1 \rightarrow u_2\), but diverges at \(u_{1,2} \rightarrow \infty\): however, there is no long-range spin order at \(u_{1,2} \rightarrow \infty\) since the amplitude of correlations in this limit vanishes. From the field-theoretical point of view, such peculiarities correspond to the “fine tuning” of the theory parameters which makes certain pre-exponential factors zero.

It is easy to check that the general family of MP-solvable spontaneously dimerized models (35) includes one model of the multicritical type if \(u_1 = -u_2 = u\) and \(\lambda_0 = 0\), wave functions \(\Psi_{0,\text{sim}}(u)\) with any \(u\) have the same energy and are degenerate ground states. This solution can also be matched to the translational invariant family (12) at \(\lambda_1^{(1,1)} = \lambda_0 = 0, u = 0\).

B. Elementary excitations in dimerized phase

In order to discuss the structure of elementary excitations in the dimerized phase, let us consider a model defined by setting

$$u_1 = -u_2 = u, \quad \lambda_0(u^4 + 10u^2 + 5) = 16/3, \quad \lambda_1 = \lambda_0(3u^4 + 14u^2 + 15)/8, \quad \lambda_2 = \lambda_0(5u^4 + 18u^2 + 9)/8,$$

which yields the Hamiltonian (1) with the following couplings:

$$V_{DD} = -V_{RR} = J_R = \frac{8 (u^2 - 1)(u^2 + 3)}{3 u^4 + 10u^2 + 5},$$

$$V_{LL} = \frac{45u^4 + 2u^2 + 9}{3 u^4 + 10u^2 + 5}, \quad J_L = J'_L = 1, \quad J_D = J'_D = 0.$$

The ground state energy per rung given by

$$E_0 = -7u^4 + 22u^2 + 19 \frac{4}{3 u^4 + 10u^2 + 5}.\hspace{1cm}$$

A “generic” example from this family is the model at \(u = \pm 1\), when one gets purely biquadratic interchain interaction. It is remarkable first of all because of simplicity of its ground state, which is just a checkerboard-type product of singlet bonds along the ladder legs (see Fig. (a)); second, this model lies within the class of Hamiltonians considered by Nersesyan and Tsvelik.

Elementary excitations of the model (35) are pairs of solitons in the dimer order, and the scattering states of can be studied with the help of the following variational MP ansatz for a single soliton:

$$|p\rangle_t = \sum_n e^{ip(2n+1)} |n\rangle_t.$$

$$|n\rangle_{t,s} = \prod_{i=1}^n (g_{2i-1}(-u)g_{2i}(u)) g_{2i+1}^{s,t} \prod_{i=n+1}^N g_{2i}(-u)g_{2i+1}(u),$$

$$g_{2i}^s = g(u) - \zeta g(-u), \quad g_{2i+1}^s = \sigma^\mu g(u) + \zeta g(-u)\sigma^\mu.$$

Here we for the moment assume that the ladder has \(2N + 1\) rungs and periodic boundary conditions, so that the one-soliton wave function is well defined, \(\mu = 0, \pm 1\) denotes the \(z\)-projection of spin of the triplet excitation, and \(\zeta = \pm 1\) is the parity of a single soliton. The momenta are defined in terms of the Brillouin zone of non-dimerized ladder, so that \(p \in [0, \pi]\). In case of the “generic” model with \(u = \pm 1\) the states \(|n\rangle_{t,s}\) can be visualized as singlet or triplet diagonal bonds separating two “checkerboard-dimer” ground states, as shown in Fig. (b).

The lowest energy for the variational excitation (43) is always reached for the odd-parity state (\(\zeta = -1\)), and the variational gap for the Haldane triplet is always higher than the gap for a soliton-antisoliton pair. The energy of the elementary excitation (soliton pair) for the scattering states is given by
\[ \tilde{E}(k, q) = \varepsilon_{s,t} \left( \frac{(k + q)/2}{} + \varepsilon_{s,t} \left( \frac{(k - q)/2}{} , \right. \right) \]

where \( k \) and \( q \) are the total and relative momentum.

The following expressions for the variational gaps can be obtained:

\[ \Delta_s = \frac{-4u^4}{(u^2 + 3)^2}, \quad \Delta_t = \frac{4}{(u^2 + 3)^2}. \quad (45) \]

At \( u \to 0 \) the odd-singlet soliton gap goes to zero, indicating the second-order transition to the Haldane phase, and at \( u \to \infty \) the odd-triplet soliton gap vanishes, signaling another second-order transition into the rung-dimer phase. Thus, transition to the Haldane (AKLT) phase at \( u = 0 \) is governed by a vanishing singlet-singlet gap, and the other transition to the rung-dimer phase at \( u \to \infty \) is determined by the closing singlet-triplet gap; later in Sect. \[ \] we will see that the same property holds also for a different model, which suggests that it has a general character.

V. GENERALIZED BOSE-GAYEN MODEL

In this section we introduce a toy model which allows one to view simultaneously nearly all possibilities offered by the MP approach. Let us consider the model described by the Hamiltonian \[ ] with

\[ J_L = J_L' = 1, \quad J_R = y_1, \quad J_D = J_D' = y_2, \quad V_{RR} = 0. \quad (46) \]

At \( y_2 = 1 \) and \( V_{LL} = V_{DD} = 0 \) this model was first introduced by Bose and Gayen[2] and its generalization to the case of arbitrary \( y_2 \) was recently considered by WeiHong, Kotov and Oitmaa[3].

The line \( y_2 = 1 \) is quite peculiar because, as was shown by XiAn[4] in this case the operator of the rung interaction commutes with the rest of the Hamiltonian, which enables one to classify the eigenstates by the total spin of each rung. At \( y_1 > \varepsilon_0 \approx 1.4 \) the exact ground state is just a product of singlet bonds along the rungs, and at \( y_1 < \varepsilon_0 \) the ground state coincides with that of the effective \( S = 1 \) Haldane chain whose \( S = 1 \) spins are formed by the pairs of \( S = \frac{3}{2} \) spins on the ladder rungs; \( \varepsilon_0 \) is exactly the ground state energy per spin of the \( S = 1 \) Haldane chain, and at \( y_1 = \varepsilon_0 \) a first-order phase transition from the Haldane phase to the rung-dimer phase occurs. When \( y_2 \neq 1 \), this point of transition develops into a line as shown numerically by WeiHong et al. It is also worthwhile to mention that when the rung exchange \( y_1 \) is allowed to alternate, the system exhibits a nice sequence of first-order phase transitions.

It is natural to try to “deform” the model by including biquadratic couplings \( V_{LL} \) and \( V_{DD} \) in such a way that the Haldane phase becomes simply the AKLT phase, and then its ground state can be expressed through a matrix product wave function. It is a straightforward exercise to check that the general family \[ ] of MP-solvable models with translational invariant ground state reduces to the form \[ ] under the following choice of parameters:

\[ \begin{align*}
\lambda_1^{(1,1)} &= -\frac{1}{2}y_1 + \frac{2}{5}(4y_2 - 1) \geq 0, \quad \lambda_1^{(1,2)} = 0, \\
\lambda_1^{(2,2)} &= -\frac{1}{2}y_1 + \frac{2}{5}(4 - y_2) \geq 0, \quad u = 0, \\
\lambda_0 &= -y_1 + \frac{4}{5}(1 + y_2) \geq 0, \quad \lambda_2 = \frac{6}{5}(1 + y_2) \geq 0.
\end{align*} \quad (47) \]

Then the biquadratic couplings are expressed through \( y_2 \) as

\[ V_{LL} = \frac{4}{5}(3 - 2y_2), \quad V_{DD} = \frac{4}{5}(3y_2 - 2). \quad (48) \]

This model has the AKLT-type ground state with \( u = 0 \) under the conditions imposed by the inequalities \[ ], with the ground state energy per rung

\[ E_0^{AKLT} = \frac{1}{4}y_1 - \frac{13}{20}(1 + y_2). \quad (49) \]

From the general analysis presented in previous sections we know that vanishing \( \lambda_0 \) corresponds to the phase transition into the rung-dimer phase, \( \lambda_2 = 0 \) determines the transition into the completely polarized ferromagnetic phase, and \( \lambda_1^{(1,1)} = 0 \) gives the phase boundary of a transition into a spontaneously dimerized state, only the last transition being of the second order. The Hamiltonian \[ ] has a special symmetry (cf. Ref. \[ ]): exchanging the two spins of every second rung is equivalent to exchanging the leg coupling \( 1 \) with the diagonal coupling \( y_2 \) with simultaneous exchanging \( V_{LL} \) and \( V_{DD} \), so that there is a one-to-one correspondence between the eigenstates of the Hamiltonian at \( y_2 > 1 \) and \( 0 < y_2 < 1 \), defined as

\[ y_1 \mapsto y_1/y_2, \quad y_2 \mapsto 1/y_2, \quad y_2 \geq 0. \quad (50) \]

Exploiting this symmetry, one can obtain the “mirror” of the second-order transition line \( \lambda_1^{(1,1)} = 0 \) in the other half-plane: this happens to be exactly the line defined by the equation \( \lambda_1^{(2,2)} = 0 \). The line \( \lambda_0 = 0 \) is invariant under the transformation \[ ]. At the intersection of lines \( \lambda_0 = 0 \) and \( \lambda_1^{(2,2)} = 0 \) we have the multicritical point \( (y_1 = 2, y_2 = \frac{2}{3}) \) of the type \( g(u) g(u) \ldots \) corresponding to the model \[ ] with \( x = \frac{1}{2} \) (see Sect. \[ ]). And its “mirror” is obviously another multicritical point \( (y_1 = \frac{2}{5}, y_2 = 2) \) of the type \( g(u) g(-u) g(u) \ldots \) (see Ref. \[ ] and Sect. \[ ]), lying at the intersection of the lines \( \lambda_1^{(1,1)} = 0 \) and \( \lambda_0 = 0 \).

Since the model \[ ] belongs to the class of AKLT-type models considered in Sect. \[ ] it is possible to write the exact wave function of the singlet magnon in the AKLT phase whose softening drives the second-order phase transitions at \( y_2 = \frac{1}{1 + \frac{1}{10}y_1 y_2} \) and \( y_2 = 4 - \frac{3}{5}y_1 \).

According to \[ ], its dispersion law reads as

\[ \varepsilon(k) = -y_1 + \frac{6}{5}(1 + y_2) + 2(1 - y_2) \cos(k). \quad (51) \]
so that the excitation becomes dispersionless at $y_2 = 1$. 
Near the lines $\lambda^{(1,1)} = 0$, $\lambda^{(2,2)} = 0$ those singlet
excitation should be the lowest ones, and their condensation
determines the physics of second-order phase transition. On the line of transition into the dimerized phase, $y_2 = \frac{k}{y_1}$, the gap closes at $k = \pi$, and on the other
line $y_2 = 4 - \frac{2}{5}y_1$ the gap closes at $k = 0$.

The Hamiltonian (46), (48) satisfies also the special
$u = \infty$ solution (15), which means that it has a rung-
dimer ground state within the region determined by the
conditions (17), (18); those conditions take the form

$$
\mu_0 = \tilde{\mu}_1^{(3)} = y_1 - \frac{4}{5}(1 + y_2) \geq 0,
$$

$$
\tilde{\mu}_1^{(1)} = \frac{1}{2}y_1 + \frac{2}{5}(2y_2 - 3) \geq 0,
$$

$$
\tilde{\mu}_1^{(2)} = \frac{1}{2}y_1 + \frac{2}{5}(2 - 3y_2) \geq 0,
$$

$$
\mu_2 = y_1 + \frac{2}{5}(1 + y_2) \geq 0,
$$

and the ground state energy per rung for the rung-dimer
phase is

$$
E_0^{RD} = \frac{3}{4}y_1 + \frac{3}{20}(1 + y_2). 
$$

It should be remarked that the equations (12)
indicate only the region of guaranteed stability of the rung-dimer
phase, and generally they do not necessarily correspond
to the real phase boundaries (recall that the conditions of
stability imposed by the inequalities (1) are sufficient
but not necessary); for example, a naive application of the Eqs. (22)
to the original Bose-Gayen model would give $y_{1c} = 2$
as the boundary of the rung-dimer phase, while the correct region of stability, according to Xian (2),
is wider, $y_{1c} = 1.4$. However, in the present case the lines
$\mu_0 = 0$, $\tilde{\mu}_1^{(1)} = 0$, $\tilde{\mu}_1^{(2)} = 0$
are true phase boundaries. The line $\mu_0 = 0$ obviously determines the first-order phase transition because it precisely coincides with the $\lambda_0 = 0$
line, so that we know exact ground states on both sides.

Further, one can show that the lines $\tilde{\mu}_1^{(1)} = 0$, $\tilde{\mu}_1^{(2)} = 0$
determine second-order transitions. Indeed, the model
(46), (48) satisfies the condition (32), and thus the wave
function (33) is an exact excited eigenstate, with the dispersion
given by

$$
\varepsilon_t(k) = y_1 - \frac{2}{5}(1 + y_2) + 2(1 - y_2) \cos k.
$$

It is easy to see that the $k = \pi$ gap closes at the $\tilde{\mu}_1^{(1)} = 0$
line, and the $k = 0$ gap closes at the $\tilde{\mu}_1^{(2)} = 0$ line. Note that the symmetry transformation (50) leaves the
dispersions (51) and (54) invariant, except for the change
$k \rightarrow \pi - k$.

Boundaries of the ferromagnetic phase can be obtained
by requiring that $\lambda_3$ is the lowest eigenvalue of the local
Hamiltonian (in this case only relative sign of $\lambda$’s matters), which gives

$$
\frac{5}{4}y_1 + 4 < y_2 < -1.
$$

It is interesting to remark that the same conditions of stability of the FM phase can be obtained from the standard
lowest order spin-wave theory: one has two branches of ferromagnons with the dispersion laws

$$
\varepsilon^{(+)}(k) = -\frac{6}{5}(1 + y_2)(1 - \cos k),
$$

$$
\varepsilon^{(-)}(k) = -\frac{6}{5}(1 + y_2) - y_1 + 2(1 - y_2) \cos k,
$$

and demanding the excitation energies to be positive one arrives at the same result. The transition at the line $y_2 = -1$ is of the first order (the entire branch $\varepsilon^{(+)}(k)$
of “usual” ferromagnons collapses), while for the line $y_2 = \frac{5}{2}y_1 + 4$ the situation is different: the $k = \pi$ gap of the “optical” ferromagnons branch $\varepsilon^{(-)}(k)$
closes, which suggests a second-order transition.

Gathering up all that, one arrives at the complete
phase diagram presented in Fig. 1. It is interesting that the model (46), (48), despite its simplicity, has a rich
phase diagram with five phases and three multicritical
points. All transition lines are exact, and only inside the phases marked D1 and D2 the ground state is not known exactly. Existence of second-order transition boundaries to the AKLT and RD phases means that the symmetry of D1, D2 is lower than that of the AKLT phase; this spontaneously broken symmetry can be only a discrete symmetry
connected with the spatial parity of the system, which
means that the ground state in D1, D2 is twofold degenerate. The AKLT-D1 boundary matches our condition (41)
for the phase transition into the dimerized state with spontaneously broken translational symmetry, which
indicates that D1 is dimerized. The properties of D2 are equivalent to those of D1 in the sense that the two phases are connected by the symmetry transformation (51).

VI. SUMMARY

We have studied a class of generalized $S = \frac{1}{2}$
ladder models admitting exact solution for the ground state
in terms of finitely correlated, or matrix product (MP)
states. We use two different MP ansiztze for the ground state
wave function and respectively obtain two families of Hamiltonians with exact ground states, one being translationally invariant and the other one spontaneously dimerized.

The two families have non-empty intersection, which
enables us to determine the boundary of a second-order
phase transition from translationally invariant to dimer-
ized phase. We show that the behavior of the dimer order
parameter and spin correlation functions at the transition
is peculiar: The spin-spin correlation length $\xi$ can be ei-
ergher finite or diverge when approaching the transition
point, but in the latter case there is no long-range spin
order since the preexponential factor vanishes exactly at the transition; the dimer order correlation function is just a distance-independent constant term vanishing at the transition.

We show that for a certain class of translationally invariant models (having effectively a spin-1 AKLT ground state formed by the triplet degrees of freedom at each rung) it is possible to write an exact wave function for the singlet branch of elementary excitations whose gap closes exactly at two second-order phase transition boundaries; one of those boundaries corresponds to the transition into dimerized phase, and the origin of the other one is different.

For excitations in the spontaneously dimerized phase we propose a simple variational MP-type ansatz describing the elementary excitation as a pair of solitons in dimer order; we show that variational estimate for the gap goes to zero at the phase boundary.

As an illustration of the technique and the ideas of the present approach, we also propose a toy model being an MP-solvable deformation of the “composite spin” model recently considered by Bose and Gayen and later by Weihong et al. One can obtain exactly its complete phase diagram, including several lines of phase transitions of the first and second order.

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FIG. 1. Generalized $S = \frac{1}{2}$ spin ladder described by the Hamiltonian (1), $V$'s denote the biquadratic couplings.

FIG. 2. (a) The generalized zigzag chain with two unequal next-nearest neighbor couplings; (b) its phase diagram. Thick dashed line $\beta = F(\alpha) \equiv -2\alpha/(1 + 2\alpha)$ and thin dashed lines $\beta = f_\gamma(\alpha)$ denote the boundaries of stability of the ferromagnetic (FM) phase resulting from the first and the second conditions in (22), respectively; the actual boundary between the FM phase and the singlet phase is determined by the two conditions $\beta < F(\alpha)$, $\beta < f_\gamma(\alpha)$. The line $\beta = 1$, $\alpha \leq \alpha_c \simeq 0.2411$ is gapless, the rest of the singlet phase is gapped. For $|\gamma| \leq 1$ the FM-singlet boundary is determined solely by the thick dashed line, and for $|\gamma| > 1$ the FM phase starts to shrink with increasing $|\gamma|$ as indicated by the thin dashed lines. On the thick solid line ($\beta = 2\alpha$) the ground state is a product of dimers along the $J = 1$ bonds. For $|\gamma| \leq 1$ it is stable for $-1 \leq \alpha \leq \frac{1}{2}$, and $(\frac{1}{2}, 1)$ is the generalized Majumdar-Ghosh point; for $|\gamma| > 1$, according to the criterion (20) obtained from the matrix product calculation, the region of guaranteed stability of the dimer-product ground state on the line $\beta = 2\alpha$ shrinks with increasing $|\gamma|$ as shown with the corresponding symbols (however, the MP criterion generally gives the conditions which are only sufficient but not necessary, so that the actual region of stability of the dimer-product ground state may be wider, see the discussion in the text). The point $(\frac{1}{2}, -\frac{3}{2})$ at the thick dashed line (shown with an open circle) has a matrix-product state $\Psi^{uv}_0$ of the type (2) with $u = 3$, which is stable at least for $|\gamma| < 1$. 
FIG. 3. Schematic representation of the exact excitation wave function in the “AKLT-type” ladder model. Thick solid lines denote valence bonds connecting effective $S = 1$ spins (indicated by ovals) which are formed by the triplet ($t$) degrees of freedom on the ladder rungs, and $s$ denotes a singlet bond connecting two spins of the $n$-th rung.

FIG. 4. (a) Two degenerate ground states of the “generic” dimerized model with $u = 1$; (b) the soliton states $|n\rangle$ used in Eq. (44), in the same special case $u = 1$. Thick solid lines indicate singlet bonds, and thick dashed lines can be either singlets or triplets. Arrows indicate the “direction” of the singlet bonds [i.e., $|s_{1\rightarrow 2}\rangle = 2^{-1/2}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$].

FIG. 5. Phase diagram of the generalized Bose-Gayen model. AKLT and FM denote the effective Affleck-Kennedy-Lieb-Tasaki ($u = 0$) and ferromagnetic phase, respectively, RD stays for the rung-dimer phase, and D1, D2 indicate phases with spontaneously broken symmetry where exact ground state is not known. The transitions AKLT-RD, AKLT-FM (shown with thick dashed lines) are of the first order, while all the other ones (shown with thick solid lines) are second-order transitions. Points M1 ($4/3, 2/3$) and M2 (2, 3/2) are multicritical with exponential degeneracy of the ground state, and M3 ($-4, 1$) is also a multicritical point but of different nature. The indicated eigenvalues of the local Hamiltonian vanish on the corresponding transition lines (on the FM-D1 line $\lambda_2 = \lambda^{(1,1)}_1$).