Do liquid drops roll or slide on inclined surfaces?

Sumesh P. Thampi,1 Ronoy Adhikari,2 and Rama Govindarajan1

1Engineering Mechanics Unit, Jawaharlal Nehru Centre for Advanced Scientific Research, Bangalore 560064, India
2The Institute of Mathematical Sciences, CIT Campus, Chennai 600113, India

A solid sphere is likely to roll, while a rectangular box is likely to slide, on an inclined surface. In contrast, a liquid drop moving on an inclined surface can exhibit a variety of shapes and hence complex but interesting dynamics. Combining lattice Boltzmann method for hydrodynamics with a diffuse interface model, a hybrid numerical scheme is used to study the dynamics of binary fluids on an inclined plate under the action of gravity. Using a triple decomposition of the velocity gradient tensor inside a drop, the vorticity associated with the rolling motion is distinguished from that of shearing motion. The average angular velocity based on this residual vorticity becomes significant when the external fluid viscosity is reduced and the shape of the drop approaches a circle. For a given slip length and viscosity ratio with the outer fluid, irrespective of the equilibrium contact angle, plate inclination or Bond number, a universal curve is observed for the amount of rotation as a function of the drop shape characterized by the isoperimetric quotient. Results from a large number of simulations over a wide range of parameters are shown to lie on this curve. The rolling motion is also found to be strongly dependent on the slip length at the contact line.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

A solid sphere on an inclined surface rolls down under the action of gravity. A rectangular object however is more likely to slide down. The choice of sliding versus rolling motion is determined by the shape and weight of the body and the frictional forces at the supporting surface. On the other hand, a liquid drop of a given volume on an inclined surface can attain a variety of shapes. The static shape depends on the equilibrium contact angle, solid surface characteristics including the pinning location, interfacial tension between the substrates, gravity and plate inclination, while a moving drop has to contend in addition with viscous and inertial stresses [1]. Moreover, the reaction forces and moments provided by the supporting surface are distributed and depend strongly on the shape, see e.g. [2–4]. These differences between a solid and a liquid make the study of the latter complex, but pose an interesting question, viz. whether a liquid drop sitting on an inclined solid surface will roll, slide, or do both.

The motion of the liquid drops on solid surfaces is much studied, especially in the two limiting cases of perfect wetting and zero wetting. When the contact angle is small, the lubrication approximation of the Navier-Stokes equation is a good model [5], to describe the dynamics, with a prescribed parabolic velocity profile. The sliding motion of the drops and the associated instabilities of the receding front has received much attention theoretically and experimentally, see e.g. [5–7]. At the other limit, when the contact angle approaches 180°, a complete rolling of the drop is observed under certain situations [8]. An anomalous increase in the speed of smaller drops has been observed on super hydrophobic surfaces [8] and explained based on the scaling arguments in the model of [9]. The shape of these droplets as they roll down has been studied by [10]. While these two limits are well studied, intermediate contact angles are studied much less. They are harder to analyze since they do not lend themselves to simplifying approximations. While analytical solutions are practically impossible, numerical solutions pose considerable challenges due to the multiple length scales present and the coupling between the evolving field and the interface shape [11]. Here we analyze the entire spectrum of shapes for two-dimensional drops, for a range of the relevant non-dimensional parameters. A splitting of the motion into sliding, shear and rolling leads to a better understanding of the dynamics.

The competition between rolling and sliding motion of droplets on hydrophobic surfaces has been investigated experimentally [12,13]. In these studies the effect of surface coating and roughness on the internal fluidity of the droplet was of primary concern. The velocity inside the droplet was obtained by particle image velocimetry, and the contribution to slip versus to roll was evaluated for an accelerating drop on an inclined surface. Rolling motion is clearly observed in molecular dynamics [14] and lattice Boltzmann, simulations [15] on cylindrical drops. However, such studies have concentrated on the total velocity of the droplet and its dependence on the driving force and the contact angle. With a somewhat different goal, we investigate here the motion of cylindrical drops of various contact angles on smooth surfaces, where the effect of fluid properties like density and viscosity, and also of slip are evaluated.

The idea of splitting the motion into shear and roll is standard in fluid mechanics. However, the standard splitting does not distinguish global rotation from local rotation of the fluid element. Such distinction is necessary to understand the global dynamics of a drop, so we distinguish between these, in the manner introduced by [16] in a different context. A splitting of the velocity field in a drop into slip and roll was done recently by [17], although in a different manner than done here. The linear
part of the velocity profile on a line passing through the center of the drop was attributed to rotation whereas we will see in section II.D that this would overestimate the rotation inside the drop. In addition, the study of 17 is valid for drop sizes smaller than the capillary length, so large deformations from the circular shape were not under consideration, while the present approach has no such restriction. We show that the shape, and hence the size is very important in determining the amount of rolling inside the drop.

Movement of the contact line, by slipping or otherwise, is imperative for a moving drop unless the contact angle is exactly 180° 9. Though the exact mechanism of contact line movement is not understood 18, it is generally believed that the macroscopic behavior of the drop is independent of the assumptions at the contact line. However, it is clear that even a small amount of global rotation can manifest itself as tank-treading 19 near the contact line, which means that contact line behavior is not local in nature, and must be consistent with the macroscopic motion. We demonstrate this using the diffuse interface (DI) model. Most earlier studies have concentrated on the rolling motion near the contact line 20, 21 while we look at the rolling motion in the bulk of the drop. The association between the two, if any, implies the nonlocal hydrodynamic effects of the contact line movement 11.

Understanding the kinematics is not just a curiosity but has practical relevance too. For example, rolling droplets play an important role in self-cleaning devices. As they roll, they pick up and remove dirt as observed on hydrophobic surfaces 22. Hence it is desirable to know under what situations one can maximize the rolling motion inside the drop.

II. THEORY: DIFFUSE INTERFACE MODEL AND HYDRODYNAMICS

We use a coupled system of equations describing the hydrodynamics of a conserved order parameter $\psi$ and the conserved momentum density $\rho u$, where $\rho$ and $u$ are the total density and the local fluid velocity.

A. Landau-Ginzburg theory

For a binary fluid system consisting of species I and II with local densities $n_I$ and $n_{II}$, the order parameter is defined as the normalized density difference, $\psi = \frac{n_{II} - n_I}{n_{II} + n_I}$ which quantifies the local composition. The equilibrium thermodynamics of the fluid is described by the Landau free-energy functional 23, 24

$$F(\psi) = \int \left[f(\psi) + \frac{K}{2} |\nabla \psi|^2 \right] dr,$$

where $r$ stands for the spatial dimensions. The first term represents the local free energy density of the bulk fluid, and is approximated as $f(\psi) = \frac{A}{2} \psi^2 + \frac{B}{4} \psi^4$ with $A < 0$ and $B > 0$. The three parameters $A$, $B$, and $K$ control the interfacial thickness and interfacial energy of the mixture. The second term of Eq. 1 involving the square gradient gives a free energy cost to any variation in the order parameter, and is related to the interfacial tension between the two fluid phases 25. Two uniform solutions $\psi = \pm \sqrt{A/B}$ can coexist across a fluid interface. For a planar interface; the concentration profile between the two bulk phases is given by

$$\psi(z) = \sqrt{\frac{A}{B}} \tanh \frac{z}{\xi},$$

where $z$ is the coordinate normal to the interface while $\xi = \sqrt{\frac{2K}{A}}$ determines the interfacial thickness. The energy associated with this profile in excess of the energy in the bulk, defined per unit area, provides the interfacial tension $\gamma = \frac{A}{2B} \sqrt{\frac{2K}{A}}$. The corresponding chemical potential is given by the variational derivative of the free energy with respect to the order parameter $\mu = \frac{\partial F}{\partial \psi} = A\psi + B\psi^3 - K\nabla^2 \psi$. Gradients in the order parameter produce additional stresses, which follow from the relation $\psi \nabla \mu = \nabla \cdot \sigma$ 26, including Laplace and Marangoni stresses due to a fluid-fluid interface.

B. Governing equations

The order parameter is described by a Cahn-Hilliard equation (CHE), which includes advection by fluid flow and relaxation due to chemical potential gradients,

$$\partial_t \psi + \nabla \cdot (u \psi) = \nabla \cdot (M \nabla \mu).$$

The mobility $M$ is the constant of proportionality in the linear phenomenological law relating the thermodynamic flux of $\psi$ to the thermodynamic force $\nabla \mu$. The order parameter dynamics is coupled to a Navier-Stokes equation (NSE) 27 with additional stress densities arising from the order parameter. For an incompressible fluid, the dynamics is governed by

$$\partial_t (\rho u) + \nabla \cdot (\rho uu) = -\nabla p + \eta \nabla^2 u + \psi \nabla \mu + G$$

together with the continuity equation for the density. In the above, $p$ stands for the isotropic contribution of the pressure, $\eta$ is the shear viscosity and $G$ is the gravitational force density.

C. Numerical Algorithm

We now briefly review the numerical algorithm used to solve these coupled equations. We use a hybrid algorithm by combining the lattice Boltzmann method for hydrodynamics and method of lines for the order parameter dynamics. The interested reader is referred to 28 for a detailed description.
The Lattice Boltzmann (LB) method for solving the Navier-Stokes equations is modified to include force densities such as the divergences of order parameter stresses and gravity \[29\]. In a standard \(DdQn\) LB model where the velocity space is discretized into \(n\) components in \(d\) dimensional space, the discrete form of the Boltzmann equation reads

\[
\partial_t f_i + c_i \cdot \nabla f_i + [\mathbf{F} \cdot \nabla_c f]_i = -\sum_j L_{ij}(f_j - f^0_j), \tag{5}
\]

where \(\mathbf{F}(\mathbf{x}, t)\) is an effective force density. The moments of the single particle distribution function \(f_i\) defined at lattice node \(x\) with velocity \(c_i\) at time \(t\), give the fluid mass, momentum and stress densities:

\[
\rho = \sum_{i=0}^{n} f_i, \quad \rho \mathbf{v} = \sum_{i=0}^{n} f_i c_i, \quad S_{\alpha\beta} = \sum_{i=0}^{n} f_i Q_{\alpha\beta}, \tag{6}
\]

where \(Q_{\alpha\beta} = c_{\alpha} c_{\beta} - c^2 \delta_{\alpha\beta}\). The collision operator \(L_{ij}\) is the discrete form of the collision integral. It controls the relaxation of \(f_j\) to equilibrium, \(f^0_j\), and may be modeled in terms of a single relaxation time \(\tau\) as \(L_{ij} = \delta_{ij}/\tau\). The viscosity is then obtained as \(\eta = \tau c^2\) where \(c = 1/\sqrt{3}\) is the sound speed in LB units.

The spatial discretization of the Cahn-Hilliard equation is based on a finite-volume formulation \[30\]. For a given node, the divergence is written as a sum of fluxes defined on the midpoint of the link connecting the node to its neighbors. The resulting equation is temporally integrated using a Runge-Kutta algorithm \[28\].

A liquid drop will typically be of higher viscosity and density than its surroundings. Secondly, the contact line must be treated with care. In the present work, these features are incorporated into the numerical approach as discussed below.

1. Viscosity Ratio

The differences in properties between the two fluids could be introduced at a molecular level, as done by \[31, 32\]. Our approach is macroscopic, and the simplest way to introduce a viscosity difference across the fluid interface is to prescribe the relaxation time as a function of the order parameter. The underlying assumption is that the molecular structure of two fluids is the same, and is analogous to the introduction of interfacial tension using Cahn-Hilliard theory. To test this, we first design a model problem, of the laminar pressure-driven flow of two fluids in a two-dimensional channel. Fluid I is of higher viscosity \(\eta_I\), and occupies the lower portion of the channel, while fluid II of lower viscosity \(\eta_{II}\) occupies the upper portion.

In the literature we find two different expressions for the relationship between the concentration and the relaxation time.

\[
\tau = 0.5[\tau_I(1 - \psi)] + [\tau_{II}(1 + \psi)], \tag{7}
\]

(i) Effective relaxation time as a polynomial function of order parameter, the simplest being linear \[32, 34\],

\[
\tau = 0.5[\tau_I(1 - \psi)] + [\tau_{II}(1 + \psi)]. \tag{7}
\]

(ii) As Arrhenius suggested, prescribe an effective relaxation time as a product of relaxation times, but raised to a power proportional to concentration \[35\],

\[
\tau = \tau_I^{(\frac{1}{\tau_{II}})} \tau_{II}^{(\frac{1}{\tau_{II}})}. \tag{8}
\]

We test both relationships by simulating the two-fluid flow described above, first defining the viscosity ratio simply by \(\eta_r = \eta_{II}/\eta_I = \tau_I/\tau_{II}\). The velocity profile from simulations using the two expressions above are compared to the analytical solution for two immiscible fluids separated by a sharp interface in Fig. 1. When the viscosity contrast is small, both match well with the analytical solution. However, when the ratio is large \(\eta_r = 10\), Eq. 8 is closer to the immiscible result than Eq. 7. This is because the effective mixed layer where the viscosity varies between \(\eta_I\) and \(\eta_{II}\) is smaller by the former relationship. This is evidenced by the fact that a corresponding analytical solution for the laminar velocity profile for the parallel flow of two miscible fluids with a thin mixed region between them agrees in each case with the computed result. We have thus shown that this approach is a good one for incorporating viscosity contrasts in LB simulations. Note that no ad-hoc fixes are needed. We have used Eq. 8 in our calculations.
The hybrid algorithm written using a single particle distribution function is modified to include gravitational effects as follows. Following [36], we may write the gravitational force acting on the fluids as,
\[
G = \frac{\rho}{2} [g_I(1 - \psi) + g_{II}(1 + \psi)] \mathbf{1}_g,
\]
where \( \mathbf{1}_g \) is the direction in which gravity is acting. Therefore, when \( \psi = 1 \), \( G = \rho g_I \mathbf{1}_g \) and when \( \psi = -1 \), \( G = \rho g_{II} \mathbf{1}_g \). At the interface when \( \psi = 0 \), \( G = \frac{\rho g}{2}(g_I + g_{II}) \mathbf{1}_g \). Thus \( g_I / g_{II} \) determines the density ratio between two fluids. In order to ensure that we are in the incompressible limit, we must have \( |G_z| << \rho c_s^2 \) where \( z \) is the vertical extent of the simulation domain, which means that thermodynamic pressure is large compared to the hydrostatic pressure difference.

In our simulations, wall boundary conditions are applied on two sides of the domain. Thus a computation of a single component fluid, with gravity prescribed along the flow direction, develops a channel flow between the walls. For droplet simulations, we implement the body force only on one fluid, which is equivalent to solving the NSE with the Boussinesq approximation as described below. For small density variations, i.e., \( \Delta \rho \equiv g_I - g_{II} << g_I \), the Boussinesq approximation provides that we may neglect the density variation everywhere in the NSE except in the buoyancy term, \( G \) of Eq. 4. For simplicity we may further absorb the body force \( \rho g_{II} \) into the pressure term by redefining pressure. Density in the interface region is prescribed as a linear function of the order parameter. Note that the validity of our simulations is thus limited to situations where the Boussinesq approximation holds.

3. Wetting Boundary Conditions

The algorithm implemented to get the correct contact angle on the wall is based on [37, 38]. The solid-fluid surface tensions are introduced by defining the Landau free energy functional
\[
F = F_{\text{bulk}}(\psi) + \int f(\psi_x)ds,
\]
where \( \psi_x \) is the value of order parameter at the wall. Minimization of this energy functional near the wall gives a relation between energy gradient and the gradient of the order parameter
\[
\frac{df_s}{d\psi_s} = k\nabla \psi \cdot \mathbf{n},
\]
where \( \mathbf{n} \) is normal to the wall. The form \( f_s = C\psi_x^2 + H\psi_x \) is known to be sufficient to produce various wetting behavior. By tuning the parameters \( C \) and \( H \) we can modify the properties of the surface. If \( H = 0 \) we have neutral wetting. Nonzero values of \( H \) therefore allow an asymmetry in the surface value of the order parameter and a contact angle different from 90\(^\circ\). Therefore we have,
\[
\frac{df_s}{d\psi_s} = C\psi_s + H = K \nabla \psi \cdot \mathbf{n}.
\]
It is found sufficient to retain only the linear term of the surface energy functional, i.e., to set \( C = 0 \). We use a second order central difference formula to calculate the normal derivative of order parameter at the wall. Thus, \( H = K \frac{\psi_x - \psi_0}{\Delta x} \), where the subscripts 0 and 1 represents the 0\(^{th}\) and first node respectively, which may be used to obtain the order parameter \( \psi_0 \) at the boundary node. The wall is placed at the \( \frac{x}{L} \) location, as is usual in the bounce back schemes used to represent wall in LB procedures [38]. Defining a parameter \( h \equiv H \sqrt{\frac{\Delta x}{kB}} \), the contact angles may be calculated as
\[
\cos \theta = \frac{1}{2} \left[ (1 + h)^{3/2} - (1 - h)^{3/2} \right].
\]
of Eq. 13 where $h$ is assumed to be small and Taylor series expansions are used. In spite of this, we restrict our simulations as far as possible to the wide range of intermediate contact angles where we are sure the simulations capture the relevant physics in this respect.

## D. Measure of Rolling Motion

Before discussing the simulations and results, we take a typical simulated drop and discuss how the slide, shear and roll may be estimated. The droplet is illustrated in Fig. 3(a). The rolling motion inside the drop is evident in the corresponding velocity field, plotted in the center of mass reference frame of the moving drop as illustrated in 3(b). Our objective is to quantify this rotation. The first quantity to look at would be the vorticity field, shown in 3(c). The velocity gradient tensor is usually split into a symmetric part $S$ and an antisymmetric part $\Omega$ as

$$\nabla \mathbf{u} = S + \Omega. \quad (14)$$

$S(x, y)$ for shear tensor is a measure of the deformation of a small fluid element located at $(x, y)$, while the vorticity tensor $\Omega(x, y)$ identifies the angular velocity of the fluid element, with $(x, y)$ as the center of rotation. There are different measures available in the literature to identify regions of high vorticity relative to shear \[10\], and most of them have been derived in the context of turbulence. A commonly used measure is $W = \frac{1}{2}||\Omega||^2 - ||S||^2$ of the Weiss criterion. This is illustrated in Fig. 3(d) for the drop under consideration. Another measure is the kinematic vorticity number, defined as $m = ||\Omega||/||S||$ where $||\cdot|| = \text{trace}([\cdot] \cdot [\cdot]^T)^{1/2}$. This quantity is plotted in Fig. 3(e). Different criteria have been developed later on to define a vortex exactly and many of them reduce in two dimensions to the Weiss criterion \[41\]. However, the main drawback of these measures is that they all estimate vorticity, which does not in general give a direct measure of rolling motion. This is because the vorticity $\omega = \nabla \times \mathbf{u}$ is a local quantity which includes both solid body rotation and shearing motion of a fluid element, and thus cannot distinguish between them. The residual vorticity which we describe below is shown in Fig. 3(f). Although all three measures broadly describe the region of high rotationality in a similar fashion, only the last is good for obtaining a quantitative estimate of solid body rotation.

To demonstrate that these conventional measures of vorticity are inadequate in describing the global rolling motion inside a drop, let us discuss a simple flow configuration, namely, a shear flow with $u = \hat{\gamma} y$ where $x$ is the flow direction and $y$ is the gradient direction. The velocity gradient tensor $\nabla \mathbf{u}$ splits into its symmetric and antisymmetric parts as follows:

$$\nabla \mathbf{u} = \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \gamma/2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\gamma/2 \\ 0 & 0 \end{pmatrix} = S + \Omega. \quad (15)$$

Although $\Omega$ is non-zero, there is no rolling motion, which tells us that we need a modified vorticity to characterize rolling motion.

To demarcate a coherent vortex correctly from high shear regions, Kolar \[10\] proposed a scheme to remove the ‘shear’ vorticity from the total vorticity. He proposed a triple decomposition of the relative motion of a fluid element, where the velocity gradient tensor is split into a straining part, a rigid body rotation and a simple shear flow part. For clarity, we illustrate these pictorially in Fig. 4. Given a velocity field in two dimensions, $\mathbf{u}$, the velocity gradient tensor (Fig. 4(a)) is a $2 \times 2$ matrix

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & \frac{u_x+v_y}{2} \\ v_x & v_y \end{pmatrix} + \begin{pmatrix} 0 & \frac{u_x-v_y}{2} \\ 0 & 0 \end{pmatrix}. \quad (15)$$

The symmetric part can be diagonalized to give $\begin{pmatrix} s/2 & 0 \\ 0 & -s/2 \end{pmatrix}$ where $s = \sqrt{4u_x^2 + (u_y + v_y)^2}$. This is the strain rate tensor in the principal coordinates, see Fig. 4(b), which represents the total straining of the fluid element. The rotation tensor being antisymmetric will not change with a rotation of the coordinate system, and remains as $\begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}$ where $\omega = \dot{v}_x - \dot{u}_y$, the vorticity.

Therefore, in the principal axis coordinates, the velocity gradient tensor is $\begin{pmatrix} s/2 - \omega/2 \\ \omega/2 - s/2 \end{pmatrix}$. We now rotate the coordinate system further by $\pi/4$ (Fig. 4(b)). This frame is called a basic frame of reference (BFR), and the velocity gradient tensor in this frame is

$$\begin{pmatrix} 0 & (s - \omega)/2 \\ (s + \omega)/2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & s/2 \\ s/2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\omega/2 \\ \omega/2 & 0 \end{pmatrix}. \quad (15)$$

In this reference frame, the contribution due to shear is maximized in a triple decomposition of $\nabla \mathbf{u}$. One may write \[16\]

$$\nabla \mathbf{u} = \nabla u_{\text{shear}} + \nabla u_{\text{residual}} = S_{\text{shear}} + \Omega_{\text{shear}} + S_{\text{residual}} + \Omega_{\text{residual}}.$$
FIG. 4: (a) Standard velocity gradient decomposition into symmetric and antisymmetric parts for a flow with $u_x = 0.1$, $u_y = 0.3$, $v_x = -0.2$, $v_y = -0.1$. (b) Strain rate tensor in the principal coordinate system is rotated by 45° and added to the antisymmetric tensor to generate the same flow in BFR. The residual tensor will combine to produce a simple shear flow. The residual straining, shown by the third matrix, is zero. The portion of the velocity gradient tensor contributing to solid body rotation is given by the fourth matrix as illustrated in Fig. 4(c). On the other hand, for a flow which is strain dominated, i.e., $|s| > |\omega|$, an example of which is demonstrated in Fig. 4(d). We may write

\[
\nabla u_{BFR} = \begin{pmatrix}
0 & s/2 & 0 \\
(s/2) & 0 & -sgn(\omega)|\omega|/2 \\
0 & sgn(\omega)|\omega|/2 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -sgn(\omega)|\omega|/2 \\
sgn(\omega)|\omega|/2 & 0 & 0
\end{pmatrix},
\]

where the first two matrices will combine to produce a simple shear flow. The third matrix, which is the local velocity along the streamline $n$, is illustrated in figure 4(c). For this case we may write

\[
\nabla u_{BFR} = \begin{pmatrix}
0 & sgn(s)|\omega|/2 & 0 \\
sgn(s)|\omega|/2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -sgn(\omega)|\omega|/2 \\
sgn(\omega)|\omega|/2 & 0 & 0
\end{pmatrix}.
\]

Again the first two matrices on the right hand side together produce a simple shear flow. The remainder is seen in the third matrix to be purely straining, as illustrated in Fig. 4(d). We may now see that for the simple shear flow discussed above, $|s| = |\omega|$ and $\omega_{res} = 0$, while for a solid body rotation, $\omega = \omega_{res}$. If $|s| < |\omega|$, the flow is vorticity dominated, and the residual tensor will consist of only rotation, and vice versa. Therefore residual vorticity can characterize the rolling motion inside a drop.

This separation of the shear vorticity from residual is different from the shear and curvature vorticity used by the atmospheric science community. There the shear vorticity is defined as $-\frac{\partial u}{\partial y}$, where $u$ is the local velocity along the streamline and $n$ is normal to the stream line. The remainder is defined as the ‘curvature vorticity’, which is associated with the curvature of streamlines. Note that these values will not be Galilean invariant. This procedure, when applied to a solid body rotation, predicts equal values for both shear and curvature vorticity, though there is no shear component present in solid body rotation. Hence, we will not benefit here from this procedure. The residual vorticity that we use does not suffer from this drawback.

In contrast, in an irrotational vortex where $u_r = 0$ and $u_\theta = 1/r$ the residual vorticity is zero because vorticity associated with shear and rotation are equal and of opposite signs. However the curvature vorticity is non-zero showing the swirling motion of fluid elements. We may neglect such a contribution as we do not expect a point vortex like motion inside the drop. In other words, in the strict limit of Stokes flow, the velocity field inside a drop is constituted by growing harmonics alone. Therefore it is reasonable to consider that the residual vorticity gives the correct quantitative measure of rolling motion inside

Only one of the residual terms $S_{residual}$ or $\Omega_{residual}$ will be non-zero. The residual matrices above are constructed as explained below. The residual vorticity and strain may be written respectively as

\[
\omega_{res} = \begin{cases} 
0 & \text{if } |s| \geq |\omega| \\
sgn(\omega)|\omega| - |s| & \text{if } |s| < |\omega|.
\end{cases}
\]

\[
s_{res} = \begin{cases} 
sgn(s)|s| & \text{if } |s| \geq |\omega| \\
sgn(s)|s| - |s| & \text{if } |s| < |\omega|.
\end{cases}
\]

An example flow which is vorticity dominated, where $|s| < |\omega|$ is illustrated in figure 4(c). For this case we may write

\[
\nabla u_{BFR} = \begin{pmatrix}
0 & s/2 & 0 \\
(s/2) & 0 & -sgn(\omega)|\omega|/2 \\
0 & sgn(\omega)|\omega|/2 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -sgn(\omega)|\omega|/2 \\
sgn(\omega)|\omega|/2 & 0 & 0
\end{pmatrix},
\]

where the first two matrices will combine to produce a simple shear flow. The residual straining, shown by the third matrix, is zero. The portion of the velocity gradient tensor contributing to solid body rotation is given by the fourth matrix as illustrated in Fig. 4(c). On the other hand, for a flow which is strain dominated, i.e., $|s| > |\omega|$, an example of which is demonstrated in Fig. 4(d). We may write

\[
\nabla u_{BFR} = \begin{pmatrix}
0 & sgn(s)|\omega|/2 & 0 \\
sgn(s)|\omega|/2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -sgn(\omega)|\omega|/2 \\
sgn(\omega)|\omega|/2 & 0 & 0
\end{pmatrix}.
\]

Again the first two matrices on the right hand side together produce a simple shear flow. The remainder is seen in the third matrix to be purely straining, as illustrated in Fig. 4(d). We may now see that for the simple shear flow discussed above, $|s| = |\omega|$ and $\omega_{res} = 0$, while for a solid body rotation, $\omega = \omega_{res}$. If $|s| < |\omega|$, the flow is vorticity dominated, and the residual tensor will consist of only rotation, and vice versa. Therefore residual vorticity can characterize the rolling motion inside a drop.

This separation of the shear vorticity from residual is different from the shear and curvature vorticity used by the atmospheric science community. There the shear vorticity is defined as $-\frac{\partial u}{\partial y}$, where $u$ is the local velocity along the streamline and $n$ is normal to the stream line. The remainder is defined as the ‘curvature vorticity’, which is associated with the curvature of streamlines. Note that these values will not be Galilean invariant. This procedure, when applied to a solid body rotation, predicts equal values for both shear and curvature vorticity, though there is no shear component present in solid body rotation. Hence, we will not benefit here from this procedure. The residual vorticity that we use does not suffer from this drawback.

In contrast, in an irrotational vortex where $u_r = 0$ and $u_\theta = 1/r$ the residual vorticity is zero because vorticity associated with shear and rotation are equal and of opposite signs. However the curvature vorticity is non-zero showing the swirling motion of fluid elements. We may neglect such a contribution as we do not expect a point vortex like motion inside the drop. In other words, in the strict limit of Stokes flow, the velocity field inside a drop is constituted by growing harmonics alone. Therefore it is reasonable to consider that the residual vorticity gives the correct quantitative measure of rolling motion inside
a drop. Note that neither residual vorticity nor curvature vorticity are complete measures of rolling motion. One may use residual vorticity in low Re flows and curvature vorticity in high Re flows.

By splitting the vorticity into two parts, one can identify the regions of shear. This could have also been identified by looking at the shear rate or viscous dissipation. However, such an approach will miss a very important factor to the motion which is solid body rotation which will not produce shear, but is important in determining the dynamics. Hence looking at the actual and residual vorticity plots gives an idea about overall motion, straining regions and rotating regions. As we will see that solid body rotation is indeed an important ingredient in the dynamics.

III. RESULTS AND DISCUSSION

Simulations have been performed in a box of dimensions $512 \times 256 \times 1$ LB units for a cylindrical drop. Wall boundary conditions are applied on two sides and periodic boundary conditions are applied on the other two sides. The simulation is initiated with a semicircular drop of radius $L = 60$ sitting on one wall, which is inclined at an angle $\alpha$ to the horizontal. In response to gravity and surface forces, the drop starts moving on the solid surface. We impose a smooth surface which thus does not display any hysteresis. The simulation is continued till the drop reaches a steady state velocity $V$. Before compiling all the results into a unified framework, we first examine the effect of varying one physical quantity at a time. We define the Bond number as $Bo \equiv L^2|G|/\sigma$, the Reynolds number as $Re \equiv LV\rho/\eta$ and the Capillary number as $Ca \equiv \eta V/\sigma$.

The effect of increasing gravity on the steady state drop shape and streamline patterns, total and residual vorticity and angular velocity are illustrated in Fig. 4. A larger driving force means larger deformations, so the drop deviates further from its equilibrium shape at zero plate inclination. The streamlines are plotted in the center of mass frame. Fixing the center as the innermost streamline, vorticity and residual vorticity along different streamlines are plotted as functions of azimuthal angle, thus mapping the entire vorticity field inside the drop. The thick interface and the contact line region are excluded from quantitative consideration as there may be spurious velocities generated due to the LB-DI model [12]. In Fig. 5(a) the total vorticity is seen to lie within a small range almost everywhere in the drop, but as discussed above, we may not use this to determine whether there is a solid body rotation. In this case, the residual vorticity indicates that the bulk of the drop is indeed in solid body rotation. The total and residual vorticity values are comparable, showing that there is hardly any shear vorticity. However, in a region near the rear contact line, shear vorticity dominates. Thus the up-down symmetry is broken. Finally the angular velocity based on residual vorticity, $\omega_{res} = r\omega_{res}$ where $r$ is taken as the radial distance from the center of the innermost streamline, is plotted as a function of azimuthal angle. A perfect solid body rotation, such as that of a solid wheel would have appeared as concentric circles in this plot. At small Bond number, we do have something like this, except for a slight loss of symmetry in that the outer streamlines move faster at the top and slower at the bottom. In the case of large $Bo$, the drop is elongated normal to gravity, with a clear breakdown in left-right symmetry in its shape, as illustrated in Fig. 5(b). Except for the very center of the drop there is no resemblance to solid body rotation or even to tank-treading. This is reflected in the angular velocity plot as well. The residual vorticity is higher in the direction of elongation. Interestingly the residual vorticity is now higher near the rear of the drop, exactly where it was lower at low $Bo$. This is because at higher gravity the rear of the drop has a tendency to lift off the surface. A given fluid element accelerates and decelerates significantly as it moves on a streamline.

If a solid body is rolling on an inclined surface with an angular velocity of $N$, then the corresponding vorticity is $2N$. Therefore, we can find the average residual vorticity inside a drop and calculate a corresponding forward velocity of the drop corresponding to the roll as

$$V_{rolling} = \frac{\text{Average}(\omega_{res})}{2} \times \text{Height of the drop} \times \frac{2}{2}.$$  \hspace{1cm} (16)

Here we take the radius of the drop to be half of the maximum height. Then a quantity called percentage rotation, denoted by $%R$, is calculated based on the total translational velocity $V$ of the drop, as

$$%R = \frac{V_{rolling}}{V} \times 100.$$  \hspace{1cm} (17)

This is different from the roll versus slip velocity as defined in [7], where the velocity profile at a single location is considered and the definition does not distinguish the shear vorticity from residual vorticity. In Fig. 5 one may see that increasing gravity increases the translational velocity by an order of magnitude, as seen in the increase in Reynolds number, but the associated deformation reduces the percentage rotation $%R$ by 10%.

Therefore one may infer that an important property that determines the motion of a drop is its geometrical characteristics. Needless to say, the geometry is in turn determined by the volume, the contact angle, the gravitational force and the plate inclination, apart from the viscosity and density ratios. The present study is thus valid over a wide range of parameters in contrast to [9] wherein a spherical drop deformed by incremental gravity was studied. The crucial assumption in [9] was that the deviation of the shape from a sphere is very small. Relevant length scales of the deformation as a response to gravity were thence derived. These scaling arguments break down when $\theta_c \neq 180^\circ$ due to a finite contact area as we have in our simulations. Also, since we do not restrict our analysis to small $Bo$, the changes in the surface
FIG. 5: Effect of gravity on the drop shape, streamline patterns, vorticity ($\omega$), residual vorticity ($\omega_{\text{res}}$) and residual angular velocity ($v_{\text{res}}$) are illustrated in a coordinate frame moving with the center of mass of the drop. A streamline of a given color in the left-most figure is shown in the same color in the three polar plots of $\omega$, $\omega_{\text{res}}$ and $v_{\text{res}}$. The azimuthal angle, measured from a line parallel to the solid plate, corresponds to that of the streamline, and the radial location at a given polar angle indicates the magnitude of the respective quantities. The magenta lines represent $\psi = 0.9$, $\psi = 0$ and $\psi = -0.9$, showing the thick interface. The drop is moving on a surface inclined to the horizontal, and the black dashed line indicates the direction of gravity. The red dashed line is normal to it.

energy need not scale with that of gravitational potential energy unlike in [9].

We now analyze the effect of contact angle alone as illustrated in Fig. 6 for drops of same volume. Here gravity is adjusted so that the drop attains the same terminal settling velocity in all cases and hence the same Reynolds number. We thus ensure that effects of inertia are nullified in this comparison. In Fig. 6 one may see that no contribution of rotation is present when $\theta_e = 42^\circ$. Here the entire vorticity of the fluid elements can be attributed to that associated with shear. As the equilibrium contact angle increases, the percentage rotation increases with the maximum in the case of an almost circular drop. The effect of equilibrium contact angle is thus an intuitive result. Compared to the case of Fig. 6(a) case 6(b) shows a marginal reduction in the percentage rotation despite an increase in the contact angle. This can be attributed to the deformation of the drop near the wall in the latter case. The total vorticity is a function of shape as seen in different cases with a large contribution coming from the shear vorticity.

It is of interest to study how the rolling behavior changes as a function of the tilt angle of the plate. This is because the ratio of the components of gravity normal and tangential to the plate changes. The application of the normal component alone does not produce any movement of the drop, but both components contribute to deciding the shape, and hence the dynamics. The effect of plate inclination on the shape, streamlines and vorticities and their angular dependencies are illustrated in Fig. 7. This illustration is for an equilibrium contact angle of $90^\circ$. As the plate inclination increases the height of the drop increases, and it tends to lift off from the plate at some inclination. In turn the percentage rotation increases, and is highest for a tilt angle of $176^\circ$. This is another indication that the drop shape is a very important parameter in determining the kinematics inside the drop. The presence of corners and deformed parts of the drop always increase the shear vorticity locally.

As the plate inclination changes, not only the ratio of
normal to tangential forces changes, but also their magnitudes. In order to study the effect of this ratio alone, the normal force component was artificially varied keeping the tangential force the same. This corresponds to a simultaneous variation in plate inclination and gravity to achieve the same settling velocity. The results are illustrated in Fig. 6. One can clearly see that as the normal component of gravity is slowly reduced, the shape becomes more and more elongated in the direction normal to the plate and this increases the amount of rotation considerably. This is yet another indication that the shape of the drop plays a big role in the rolling behavior of the drop.

In all these cases, we see that the deviation from a circular shape plays an important role in determining the dynamics. One can then suitably define a shape parameter to describe the closeness of the shape to a circle, for example, the isoperimetric quotient, $q$,

$$q = \frac{4\pi \times \text{Area}}{\text{Perimeter}^2}. \quad (18)$$

This ratio is unity for a circle and is less than this value for any other shape, since a circle has the least circumference for a given area. The percentage rotation in all the cases we have computed shows a direct dependence over 100 simulations, spanning a wide range of $\theta_e$, $\alpha$ and $\eta_r$. A larger reduction in $\omega_{res}$ as compared to $\omega$ may be observed as $\theta_e$ decreases.

FIG. 6: Effect of contact angle and hence the geometry on the rolling behavior. The $\%R$ is larger when the drop shape is closer to a circle. In all cases $Re$ and $Ca$ are kept constant by adjusting the $Bo$ and $\eta_r$ is kept as 10. A larger reduction in $\omega_{res}$ is observed as $\theta_e$ decreases.

FIG. 7: Effect of plate inclination on the shape and rotation behavior of drops. Equilibrium contact angle is $90^\circ$ and $\eta_r = 10$. A pendant drop is elongated to almost same size as the radius, producing more solid body rotation in the drop.

(a) $Bo = 0.027, \theta_e > 180^\circ (h = -0.7), \%R = 17.8, Re = 0.23, Ca = 0.002$

(b) $Bo = 0.028, \theta_e = 138^\circ, \%R = 18.5, Re = 0.23, Ca = 0.002$

(c) $Bo = 0.038, \theta_e = 90^\circ, \%R = 8.7, Re = 0.23, Ca = 0.002$

(d) $Bo = 0.068, \theta_e = 42^\circ, \%R = 1.9, Re = 0.23, Ca = 0.002$
variation in physical properties fall on this curve. The slip length and the viscosity are kept fixed. An exponential curve fitted through all data points is also shown. Note that moving drops for a wide gravity and surface tension. The slip length and the viscosity of the above.

FIG. 8: Effect of normal component of gravity on the shape and rolling behavior of drops. Equilibrium angle is 90° and \( \eta = 10 \). The tilt angle is chosen as 4°. To differentiate the effect of the tangential component of gravity, the normal component of gravity in (b) is artificially suppressed to 1/10^{th} of its value. However, the tangential component being maintained the same in both (a) and (b) yields a similar settling velocity. The percentage rotation can clearly be very different even at the same settling velocity, due to the change in shape.

FIG. 9: The variation of percentage rotation with the isoperimetric quotient is illustrated for different sets of simulations. Each color represents a fixed equilibrium contact angle. Within each set, plate inclinations vary from \( \alpha = 4° \) to 176°. Also, \( Bo \) ranges from \( 5 \times 10^{-3} \) to 1.5 by varying gravity and surface tension. The slip length and the viscosity ratio are kept fixed. An exponential curve fitted through all data points is also shown. Note that moving drops for a wide variation in physical properties fall on this curve.

FIG. 10: Isoperimetric quotient of static shapes compared with dynamic drops for the same Bond numbers. The equilibrium contact angle is 90° and three different plate inclinations, 30°, 90° and 135°, are chosen for comparison. Here ‘FP’ stands for ‘front pinned’, ‘RP’ stands for ‘rear pinned’ and ‘D’ stands for dynamic cases. The shape factor of a dynamic drop lies in between those corresponding to front pinned and back pinned static shapes. This behavior breaks down at large Bond numbers where inertia is higher. In that case, the moving drop is closer to circular than either static shape. Bo. However the viscosity, the mobility and the viscosity ratio are kept fixed in the simulations shown so far. As we will see below, percentage rotation curve gets shifted in response to changes in these parameters. We have defined the outline of the drop as a contour of \( \psi = -0.9 \) which can be thought of as the inner limit of the interface. Since we use a combination of LB and DI models, there can be spurious interface velocities, \( \psi = 0 \), and hence the data outside this line is not considered. We have also tried to calculate this shape parameter from \( \psi = 0 \) which is theoretically the interface. However this shape fails to capture the deformations correctly for large contact angles.

It would be interesting to compare the isoperimetric quotient of static drops with that of dynamic cases and see whether any predictions can be made. This is illustrated in Fig. II. As explained in [2], we obtain minimum energy static shapes of drops with either the front end or the rear end pinned. As illustrated in Fig. II, the isoperimetric quotient of the dynamic drops resides between that of front pinned and back pinned cases. This is interesting because one can make predictions about kinematics inside the drop by the analysis of static drops, but only for small Bond numbers. As \( Bo \) increases, such monotonic variations in the shape factor is violated, necessitating the full calculations.

One of the main drawbacks of the DI models is that it imposes a finite thickness of the interface while for most macroscopic drops, the interface thickness would be negligible compared to any other length scale in the problem. In order to ensure that our results are indepen-
The contact line moves at a velocity to the entire drop, the contact line moves over a range of interface thicknesses were conducted. As shown in Fig. 11 both the Capillary number and the percentage rotation remain insensitive to interface width, to within numerical errors. Here $Cn = \xi/L$, the ratio of interfacial thickness to the macroscopic length, is the Cahn number.

Apart from the solid body rotation, which gives a forward velocity to the entire drop, the contact line moves due to the slip provided by the diffusion of the order parameter [43]. Balancing the advection and diffusion of order parameter across the interface provides a length scale for this diffusion as $\lambda = \sqrt{\eta M}$. We define a non-dimensional slip length as $S = \lambda/L$. This slip length is the same as that used in the slip-induced movement of contact line in sharp interface models [44], and is not an artificial parameter. They actually show that this slip length should not be dependent on the interfacial width. Hence we can use $\lambda$ as a measure of slip at the contact

The Cahn number.

In our simulations, the nondimensional slip length, $S$, is plotted as a function of the non-dimensionalised slip length, $S$. The simulation parameters are same as in Fig. 12. A strong dependence of the rolling behavior on the slip length may be observed.

line. This means that either mobility or viscosity can be independently or simultaneously varied to change the slip at the contact line. Slip length is here defined using the viscosity of the drop $\eta_I$. In principle, the external fluid plays a very important role and an effective viscosity to define the slip length may need to account for the viscosity ratio $\eta_I$. For example a geometric mean of the two is used in [44]. However, we refrain from using this relation as it lacks a physical significance.

Both viscosity and mobility are independently varied by at least one order of magnitude in Fig. 12 to obtain a range of $S$. As the slip length increases, the $Ca$ also increases as illustrated in Fig. 12. Intuitively, a slipping drop on an inclined surface will roll less. This is verified in our simulations as shown in Fig. 13 wherein the importance of slip length in determining the amount of rotation inside the drop may be inferred. And this dependence appears to be exponential. Larger percentage rotations than those shown, which would correspond to smaller slip lengths could not be obtained reliably with the present numerical simulations.

In our simulations, the nondimensional slip length, $S$, varies from $10^{-3}$ to $10^{-2}$. In the light of experimental evidence where slip length varies from $nm$ to $\mu m$ [15] we expect that our observations remain valid for a range of drop sizes. For macroscopic drops smaller than the capillary length, this ratio is very small and hence a larger fraction of rolling motion may be expected than those seen here. It is worth mentioning however that slip lengths of $10-100$ of micron have been reported on patterned surfaces [10] or when lubricating gas layers are present [47] or on super-hydrophobic surfaces [22]. Since we concentrate on the bulk motion of fluid elements, we expect that our simulations are relevant in several practical applications.

It is interesting to note that viscosity does not have
Navier's boundary condition \[18, 49\] and the above scaling relation. For this we use the generalized Navier's boundary condition \[18, 49\],

\[ \text{Ca} \sim \text{Bo} - \Delta \theta, \]  

(19)

where \( \Delta \theta = \cos \theta_r - \cos \theta_e \), which is small for most of our simulations. A plot of Ca vs Bo is shown in Fig. 14 which corresponds to the data described in Fig. 9. This scaling holds good for all our simulations. The small deviations seen are due to inertial effects not accounted for in Eq. 19 as well as by the intercept, as shown below. Having seen the effect of contact-line slip on the drop dynamics, we now make a connection between the slip velocity and the above scaling relation. For this we use the generalized Navier’s boundary condition \[18, 49\],

\[ \frac{u^{\text{slip}}}{\lambda} = \left[ \partial_n u \right] + \frac{\mathcal{L}(\psi)}{\eta} \partial_x \psi \]  

(20)

to make quantitative estimates of the sliding velocity. In the above equation \( n \) is a direction normal to the wall and \( \mathcal{L}(\psi) = K \partial_n \psi + \partial \sigma_{wf}/\partial \psi \), \( K \) is the coefficient in the free energy functional and \( \sigma_{wf} \) is the interfacial free energy per unit area at the fluid-solid boundary. The second term is called the uncompensated Young’s stress and one may see that

\[ \int_{\text{int}} dx [\mathcal{L}(\psi) \partial_x \psi] = \gamma (\cos \theta_d - \cos \theta_e), \]  

(21)

where int means ‘across the interface’ and \( \theta_d \) is the dy-
dynamic contact angle. We can calculate an order of magnitude estimate of $\Delta \theta$ from the above equation as $Ca \xi / \lambda$. Since we are at steady state, both the advancing and receding contact lines move with the same velocity as the center of mass of the drop. We assume that any variation in order parameter and hence the slip is felt over a region of interfacial thickness $\xi$ while the associated slip length $\lambda$ is the same slip length calculated in a DI model. Accounting for a possible pre-factor in the addition of scaling estimates in Eq. 19 we obtain,

$$Ca \left[1 + \frac{\xi}{\beta \lambda}\right] \sim Bo,$$

which explicitly includes the role of slip at the contact line. This relationship is verified in Fig. 10 where the change in only the slip length affects the linear relationship between $Ca$ and $Bo$. We define $Ca_M \equiv [1 + \xi / (\beta \lambda)]$. By choosing a suitable pre-factor $\beta$ one may see that the second term in the above equation explains in part the distribution in $Ca$ at a given $Bo$. In other words, the relative standard deviation, which is the ratio of standard deviation to the mean value expressed as a percentage comes down dramatically when Eq. 22 is used.

Finally we investigate the role of the viscosity ratio between the drop and the surrounding fluid, in Fig. 17 and 18. When the viscosity of the external fluid is reduced, the settling velocity and hence the $Ca$, as expected, increase. Also the percentage of rolling motion is larger. In line with this, one may expect significant amount of rolling in case of a water-air system where viscosity contrast is large. As the viscosity of the external fluid increases and goes beyond that of the drop, the entire dynamics shifts to the external fluid. The drops slides in that case. This too is consistent with intuition, since a ‘bubble’ will simply slide in a liquid rather than roll when moving on a surface. The changes in the vorticity and residual vorticity fields are illustrated in Fig. 19. One may observe that, despite the geometry remaining similar, the percentage rotation increases when the viscosity ratio increases. Therefore the universal curve obtained in Fig. 9 will be shifted appropriately by a change in the slip length and viscosity ratio.
IV. CONCLUSIONS

A hybrid simulation method implementing lattice Boltzmann algorithm with diffuse interface model is used to analyze the drop motion on inclined surfaces under gravity. Modification of the model to provide a viscosity contrast between the fluids, wetting boundary conditions and to introduce gravity are discussed. By removing shear vorticity from total vorticity, it is shown that residual vorticity is a good measure to characterize the rolling motion inside the drop. It is shown that the drop shape can be described by a geometrical quantity, isoperimetric quotient, that is primarily responsible for determining the fraction of forward motion accruing from solid body rotation. For a given slip length and viscosity ratio with the outer fluid, this dependence is universal, irrespective of the equilibrium contact angle, plate inclination and Bo. The importance of the slip mechanism of the contact line is discussed, not only in relation to the rolling motion inside the drop, but also in modifying the coefficient in the scaling relationship between Bo and Ca. The external fluid certainly affects the drop motion with larger rolling motion observed when its viscosity is small compared to the drop viscosity.

Acknowledgments

We gratefully acknowledge Ignacio Pagonabarraga for the fruitful discussions.

[1] P. de Gennes, F. Brochard-Wyart, and D. Quere, *Capillarity And Wetting Phenomena: Drops, Bubbles, Pearls, Waves* (Springer, Spring Street, New York, USA, 2004).
[2] S. P. Thampi and R. Govindarajan, Phys. Rev. E 84, 046304 (2011).
[3] P. T. Sumesh and R. Govindarajan, J. Chem. Phys. 133, 144707 (2010).
[4] E. B. Dussan and R. T. Chow, J. Fluid Mech. 137, 1 (1983).
[5] N. L. Grand, A. Daerr, and L. Limat, J. Fluid Mech. 541, 293 (2005).
[6] J. H. Snoeijer, E. Rio, N. L. Grand, and L. Limat, Phys. Fluids 17, 072101 (2005).
[7] D. Quere, Rep. Prog. Phys. 68, 2495 (2005).
[8] D. Richard and D. Quere, Europhys. Lett. 48, 286 (1999).
[9] L. Mahadevan and Y. Pomeau, Phys. Fluids 11, 2449 (1999).
[10] P. Aussillous and D. Quere, J. Fluid Mech. 512, 133 (2004).
[11] Y. D. Shikhmurzaev, *Capillary Flows with Forming Interfaces* (Chapman & Hall/CRC, 2008).
[12] M. Sakai, J. Song, N. Yoshida, S. Suzuki, Y. Kameshima, and A. Nakajima, Langmuir 22, 4906 (2006).
[13] S. Suzuki, A. Nakajima, M. Sakai, Y. Sakurada, N. Yoshida, A. Hashimoto, Y. Kameshima, and K. Okada, Chem. Lett. 37, 58 (2008).
[14] J. Servantie and M. Muller, J. Chem. Phys. 128, 014709 (2008).
[15] N. Moradi, F. Varnik, and I. Steinbach, Europhys. Lett. 95, 44003p1 (2011).
[16] V. Kolar, Int. J. Heat and Fluid Flow 28, 638 (2007).
[17] B. M. Mogenti, H. Kusumaatmaja, and J. M. Yeomans, Faraday Discuss. 146, 153 (2010).
[18] D. Bonn, J. Eggers, J. Indekeu, J. Meunier, and E. Rolley, Rev. Mod. Phys. 81, 739 (2009).
[19] E. B. Dussan and S. H. Davis, J. Fluid Mech. 65, 71 (1974).
[20] A. Clarke, Chem. Eng. Sci 50, 2397 (1995).
[21] Q. Chen, E. Rame, and S. Garoff, J. Fluid Mech. 337, 49 (1996).
[22] J. F. Rothstein, Annu. Rev. Fluid Mech. 42, 89 (2010).
[23] P. M. Chaikin and T. C. Lubensky, *Principles of Condensed Matter Physics* (Cambridge, 1995).
[24] J. S. Rowlinson and B. Widom, *Molecular Theory of Capillarity* (Dover Publications, 1982).
[25] V. M. Kendon, M. E. Cates, I. Pagonabarraga, J. C. Despalt, and P. Bladon, J. Fluid Mech. 440, 147 (2001).
[26] D. M. Anderson, G. B. McFadden, and A. A. Wheeler, Annu. Rev. Fluid. Mech. 30, 139 (1998).
[27] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, 1959).
[28] S. P. Thampi, I. Pagonabarraga, and R. Adhikari, Phys. Rev. E 84, 046709 (2011).
[29] R. W. Nash, R. Adhikari, and M. E. Cates, Phys. Rev. E 77, 026709 (2008).
[30] B. Rotenberg, I. Pagonabarraga, and D. Frenkel, Europhys. Lett. 83, 34004 (2008).
[31] L. Luo and S. S. Girimaji, Phys. Rev. E 67, 036302 (2003).
[32] P. Asinari, Phys. Fluids 17, 067102 (2005).
[33] X. He, S. Chen, and R. Zhang, J. Comput. Phys. 152, 642 (1999).
[34] D. Grunau, S. Chen, and K. Eggert, Phys. Fluids A 5, 2557 (1993).
[35] K. Langaas and J. M. Yeomans, Eur. Phys. J. B 15, 133 (2000).
[36] J. M. Buick and C. A. Great, Phys. Rev. E 61, 5307 (2000).
[37] J. C. Despalt, I. Pagonabarraga, and P. Bladon, Comput. Phys. Commun. 134, 2001 (2001).
[38] A. J. Briant, P. Papatzacos, and J. M. Yeomans, Phil. Trans. R. Soc. Lond. A 360, 485 (2002).
[39] S. Succi, *The Lattice Boltzmann Equation for Fluid Dynamics and Beyond* (Oxford University Press, 2001).
[40] M. Z. J. Wu, H. Ma, *Vorticity and Vortex Dynamics* (Springer, NewYourk, Berlin, Heidelberg, 2006).
[41] J. Jeong and F. Hussain, J. Fluid Mech. 285, 69 (1995).
[42] c. M. Pooley, H. Kusumaatmaja, and J. M. Yeomans, Phys. Rev. E 78, 056709 (2008).
[43] D. Jacqumin, J. Fluid Mech. 402, 57 (2000).
[44] P. Yue, C. Zhou, and J. J. Feng, J. Fluid Mech. 645, 279 (2010).
[45] D. C. Tretway and C. D. Meinhart, Phys. Fluids 14, L9 (2002).
[46] P. Tsai, A. M. Peters, C. Pirat, M. Wessling, R. G. H. Lammertink, and D. Lohse, Phys. Fluids 21, 112002
[47] F. Feuillebois, M. Z. Bazant, and O. I. Vinogradova, Phys. Rev. Lett 102, 026001 (2009).
[48] H. Kim, H. Lee, and B. H. Kang, J. Colloid. Interf. Sci. 247, 372 (2001).
[49] T. Qian, X. Wang, and P. Sheng, Phys. Rev. E 68, 016306 (2003).