INDEPENDENT SETS IN EDGE-CLIQUE GRAPHS

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Abstract. We show that the edge-clique graphs of cocktail party graphs have unbounded rankwidth. This, and other observations lead us to conjecture that the edge-clique cover problem is NP-complete for cographs. We show that the independent set problem on edge-clique graphs of cographs and of distance-hereditary graphs can be solved in $O(n^4)$ time. We show that the independent set problem on edge-clique graphs of graphs without odd wheels remains NP-complete.

1 Introduction

Let $G = (V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. A clique is a complete subgraph of $G$.

Definition 1. An edge-clique covering of $G$ is a family of complete subgraphs such that each edge of $G$ is in at least one member of the family.

The minimal cardinality of such a family is the edge-clique covering number, and we denote it by $\theta_e(G)$.

The problem of deciding if $\theta_e(G) \leq k$, for a given natural number $k$, is NP-complete [38, 51, 33]. The problem remains NP-complete when restricted to graphs with maximum degree at most six [34]. Hoover [34] gives a polynomial time algorithm for graphs with maximum degree at most five. For graphs with maximum degree less than five, this was already done by Pullman [54]. Also for linegraphs the problem can be solved in polynomial time [51, 54].

In [38] it is shown that approximating the clique covering number within a constant factor smaller than two remains NP-complete.

Gyárfás [31] showed the following interesting lowerbound. Two vertices $x$ and $y$ are equivalent if they are adjacent and have the same closed neighborhood.

Theorem 1. If a graph $G$ has $n$ vertices and contains neither isolated nor equivalent vertices then $\theta_e(G) \geq \log_2(n + 1)$. 
Gyárfás result implies that the edge-clique cover problem is fixed-parameter tractable (see also [28]). Cygan et al showed that, under the assumption of the exponential time hypothesis, there is no polynomial-time algorithm which reduces the parameterized problem \((\theta_e(G), k)\) to a kernel of size bounded by \(2^{o(k)}\). In their proof the authors make use of the fact that \(\theta_e(\text{cp}(2^\ell))\) is a [sic] “hard instance for the edge-clique cover problem, at least from a point of view of the currently known algorithms.” Note that, in contrast, the parameterized edge-clique partition problem can be reduced to a kernel with at most \(k^2\) vertices [48]. (Mujuni and Rosamond also mention that the edge-clique cover problem probably has no polynomial kernel.)

2 Rankwidth of edge-clique graphs of cocktail parties

Definition 2. The cocktail party graph \(\text{cp}(n)\) is the complement of a matching with \(2n\) vertices.

Notice that a cocktail party graph has no equivalent vertices. Thus, by Theorem 1,

\[
\theta_e(\text{cp}(n)) \geq \log_2(2n + 1).
\]

For the cocktail party graph an exact formula for \(\theta_e(\text{cp}(n))\) appears in [29]. In that paper Gregory and Pullman prove that

\[
\lim_{n \to \infty} \frac{\theta_e(\text{cp}(n))}{\log_2(n)} = 1.
\]

Definition 3. Let \(G = (V, E)\) be a graph. The edge-clique graph \(K_e(G)\) has as its vertices the edges of \(G\) and two vertices of \(K_e(G)\) are adjacent when the corresponding edges in \(G\) are contained in a clique.

Albertson and Collins prove that there is a 1-1 correspondence between the maximal cliques in \(G\) and \(K_e(G)\) [1]. The same holds true for the intersections of maximal cliques in \(G\) and in \(K_e(G)\).

For a graph \(G\) we denote the vertex-clique cover number of \(G\) by \(\kappa(G)\). Thus

\[
\kappa(G) = \chi(\bar{G}).
\]

Notice that, for a graph \(G\),

\[
\theta_e(G) = \kappa(K_e(G)).
\]

Albertson and Collins mention the following result (due to Shearer) [11] for the graphs \(K_e^{r}(\text{cp}(n))\), defined inductively by \(K_e^{r}(\text{cp}(n)) = K_e(K_e^{r-1}(\text{cp}(n)))\).

\[
\alpha(K_e^{r}(\text{cp}(n))) \leq 3 \cdot (2^r)!.
\]

Thus, for \(r = 1\), \(\alpha(K_e(\text{cp}(n))) \leq 6\). However, the following is easily checked.
Lemma 1. For $n \geq 2$
\[ \alpha(K_e(cp(n))) = 4. \]

Proof. Let $G$ be the complement of a matching $\{x_i, y_i\}$, for $i \in \{1, \ldots, n\}$. Let $K = K_e(G)$. Obviously, every pair of edges in the matching induces an independent set with four vertices in $K$.

Consider an edge $e = \{x_i, x_j\}$ in $G$. The only edges in $G$ that are not adjacent to $e$ in $K$ must have endpoints in $y_i$ or in $y_j$. Consider an edge $f = \{y_i, y_k\}$ for some $k \notin \{i, j\}$. The only other edge incident with $y_i$, which is not adjacent in $K$ to $f$ nor to $e$ is $\{y_i, x_k\}$.

The only edge incident with $y_j$ which is not adjacent to $e$ nor $f$ is $\{y_j, x_i\}$. This proves the lemma. \hfill \Box

Definition 4. A class of graphs $\mathcal{G}$ is $\chi$-bounded if there exists a function $f$ such that for every graph $G \in \mathcal{G}$,
\[ \chi(G) \leq f(\omega(G)). \]

Dvořák and Král proved that the class of graphs with rankwidth at most $k$ is $\chi$-bounded [23].

We now easily obtain our result.

Theorem 2. The class of edge-clique graphs of cocktail parties has unbounded rankwidth.

Proof. It is easy to see that the rankwidth of any graph is at most one more than the rankwidth of its complement [52]. Assume that there is a constant $k$ such that the rankwidth of $K_e(G)$ is at most $k$ whenever $G$ is a cocktail party graph. Let \[ \mathcal{K} = \{ K_e(G) : G \simeq cp(n), n \in \mathbb{N} \}. \]

Then the rankwidth of graphs in $\mathcal{K}$ is uniformly bounded by $k + 1$. By the result of Dvořák and Král, there exists a function $f$ such that
\[ \kappa(K_e(G)) \leq f(\alpha(K_e(G))) \]
for every cocktail party graph $G$. This contradicts Lemma[1] and Theorem[1]. \hfill \Box

Remark 1. It is easy to see that for cographs $G$, $K_e(G)$ is not necessarily perfect. For example, when $G$ is the join of $P_3$ and $C_4$ then $K_e(G)$ contains $C_5$ as an induced subgraph.
3 Independent set in edge-clique graphs of cographs

Notice that, for any graph $G$, the clique number of its edge-clique graph satisfies

$$\omega(K_e(G)) = \binom{\omega(G)}{2}.$$  

For the independent set number $\alpha(K_e(G))$ there is no such relation. For example, when $G$ has no triangles then $K_e(G)$ is an independent set and the independent set problem in triangle-free graphs is NP-complete. We write

$$\alpha'(G) = \alpha(K_e(G)).$$

We say that a subset of edges in a graph $G$ is independent if it induces an independent set in $K_e(G)$. In other words, a set $A$ of edges in $G$ is independent if no two edges of $A$ are contained in a clique of $G$.

A graph is trivially perfect if it does not contain $C_4$ nor $P_4$ as an induced subgraph.

**Theorem 3.** If a graph $G$ is connected and trivially perfect then $\alpha'(G) = \theta_e(G)$.

**Proof.** When a graph $G$ is trivially perfect then the independence number is equal to the number of maximal cliques in $G$. Therefore, $\alpha(G) = \theta_e(G)$ and since $G$ is connected $\alpha'(G) \geq \alpha(G)$.

The following lemma shows that the independent set problem in $K_e(G)$ can be reduced to the independent set problem in $G$.

**Lemma 2.** The computation of $\alpha'(G)$ for arbitrary graphs $G$ is NP-hard.

**Proof.** Let $G$ be an arbitrary graph. Construct a graph $H$ as follows. At every edge in $G$ add two simplicial vertices, both adjacent to the two endvertices of the edge. Add one extra vertex $x$ adjacent to all vertices of $G$. Let $H$ be the graph constructed in this way.

Notice that a maximum set of independent edges does not contain any edge of $G$ since it would be better to replace such an edge by two edges incident with the two simplicial vertices at this edge. Also notice that a set of independent edges incident with $x$ corresponds with an independent set of vertices in $G$. Hence

$$\alpha'(H) = 2m + \alpha(G),$$

where $m$ is the number of edges of $G$. 

$\Box$
A cograph is a graph without induced $P_4$. It is well-known that a graph is a cograph if and only if every induced subgraph with at least two vertices is either a join or a union of two smaller cographs. It follows that a cograph $G$ has a decomposition tree $(T, f)$ where $T$ is a rooted binary tree and $f$ is a bijection from the vertices of $G$ to the leaves of $T$. Each internal node of $T$, including the root, is labeled as $\otimes$ or $\oplus$. The $\otimes$-node joins the two subgraphs mapped to the left and right subtree. The $\oplus$ unites the two subgraphs. When $G$ is a cograph then a decomposition tree as described above can be obtained in linear time [20].

**Lemma 3.** Let $G$ be a cograph. Assume that $G$ is the join of two smaller cographs $G_1$ and $G_2$. Then any edge in $G_1$ is adjacent in $K_e(G)$ to any edge in $G_2$.

**Proof.** Let $e_1$ and $e_2$ be edges in $G_1$ and $G_2$, respectively. Then the four endpoints induce a clique in $G$. $\square$

For a vertex $x$, let $d'(x)$ be the independence number of the subgraph of $G$ induced by $N(x)$, that is,

$$d'(x) = \alpha(G[N(x)]). \quad (1)$$

**Theorem 4.** When $G$ is a cograph then

$$\alpha'(G) = \max \{ \sum_{x \in A} d'(x) | A \text{ is an independent set in } G \}. \quad (2)$$

There exists an $O(n^4)$ algorithm that computes the independence number of $K_e(G)$ for cographs $G$. Here $n$ is the number of vertices of $G$.

**Proof.** Let $G$ be a cograph. The algorithm first computes a decomposition tree $(T, f)$ for $G$ in linear time. For each node $p$ of $T$ let $G_p$ be the subgraph induced by the set of vertices that are mapped to leaves in the subtree rooted at $p$.

Notice that the independence number of each $G_p$ can be computed in linear time as follows.

Let $p$ be an internal node and let $c_1$ and $c_2$ be the two children of $p$. For $i \in \{1, 2\}$, write $G_i$ instead of $G_{c_i}$. Let $p$ be labeled with $\otimes$ and let $G_p$ be the join of $G_1$ and $G_2$. Then

$$\alpha(G_p) = \max \{ \alpha(G_1), \alpha(G_2) \}.$$  

Assume that $p$ is labeled with $\oplus$. Then let $G_p$ be the union of $G_1$ and $G_2$. In that case

$$\alpha(G_p) = \alpha(G_1) + \alpha(G_2).$$

Consider two disjoint, nonempty subsets of vertices, $A$ and $B$, such that the graph $G[A \cup B]$ is either a join or a union of $G[A]$ and $G[B]$. Let $\alpha'(A, B)$ be the
maximal cardinality of an independent set of edges in $G[A \cup B]$ such that no element has both endpoints in $B$. Assume that $G[A \cup B]$ is the union of $G[A]$ and $G[B]$. Then

$$\alpha'(A, B) = \alpha'(G[A]).$$

(3)

Assume that $G[A \cup B]$ is the join of $G[A]$ and $G[B]$. Then consider the following three cases. First assume that $G[A]$ is the union of two smaller cographs $G[A_1]$ and $G[A_2]$. In that case

$$\alpha'(A, B) = \alpha'(A_1, B) + \alpha'(A_2, B).$$

(4)

Consider the case where $G[A]$ is the join of two smaller cographs $G[A_1]$ and $G[A_2]$. In that case

$$\alpha'(A, B) = \max \{ \alpha'(A_1, B \cup A_2), \alpha'(A_2, B \cup A_1) \}.$$  

(5)

Finally, assume that $|A| = 1$. In that case

$$\alpha'(A, B) = \alpha(G[B]).$$

(6)

Now, Equation (2) easily follows by induction from the recurrences (3), (4), (5) and (6). It is easy to see that this can be computed in $O(n^4)$ time.

Remark 2. Notice that Formula (2) confirms Lemma (1).

Remark 3. The independence number of $K_e(G)$ equals $\theta_e(G)$ for graphs $G$ that are chordal. For interval graphs the edge-clique cover number $\theta_e(G)$ equals the number of maximal cliques [59].

### 3.1 Distance-hereditary graphs

In this section we briefly look at the independence number of edge-clique graphs of distance-hereditary graphs.

A graph $G$ is distance hereditary if the distance between any two nonadjacent vertices, in any connected induced subgraph of $G$, is the same as their distance in the $G$ [35]. Bandelt and Mulder obtained the following characterization of distance-hereditary graphs.

Lemma 4 ([51]). A graph is distance hereditary if and only if every induced subgraph has an isolated vertex, a pendant vertex or a twin.

The papers [5] and [35] also contain a characterization of distance-hereditary graphs in terms of forbidden induced subgraphs.
Theorem 5. Let $G$ be distance hereditary. Then $\alpha'(G)$ satisfies Equation (2). This value can be computed in polynomial time.

Proof. Consider an isolated vertex $x$ in $G$. Then $A$ is a maximum independent set of edges in $G$ if and only if $A$ is a maximum independent set of edges in the graph $G - x$. By induction, Equation (2) is valid for $G - x$.

Let $x$ be a pendant vertex and let $y$ be the unique neighbor of $x$ in $G$. Since $\{x, y\}$ is not in any triangle, the edge $\{x, y\}$ is in any maximal independent set of edges in $G$. Therefore,

$$\alpha'(G) = 1 + \alpha'(G - x).$$

Let $Q$ be an independent set which maximizes Equation (2) for $G - x$. If $y \in Q$ then $d'(y)$ goes up by one when adding the vertex $x$. If $y \notin Q$, then $Q \cup \{x\}$ is an independent set in $G$ and $d'(x) = 1$.

Let $x$ be a false twin of a vertex $y$ in $G$. Let $A$ be a maximum independent set of edges in $G$. Let $A(x)$ and $A(y)$ be the sets of edges in $A$ that are incident with $x$ and $y$. Assume that $|A(x)| > |A(y)|$. Let $\Omega(x)$ be the set of endvertices in $N(x)$ of edges in $A(x)$. then we may replace the set $A(y)$ with the set

$$\{ \{y, z\} | z \in \Omega(x) \}.$$

The cardinality of the new set is at least as large as $|A|$. Notice that, for any maximal independent set $Q$ in $G$, either $\{x, y\} \subseteq Q$ or $\{x, y\} \cap Q = \emptyset$. By induction on the number of vertices in $G$, Equation (2) is valid.

Let $x$ be a true twin of a vertex $y$ in $G$. Let $A$ be a maximum independent set of edges in $G$ and let $A(x)$ and $A(y)$ be the sets of edges in $A$ that are incident with $x$ and $y$. If $\{x, y\} \in A$ then $A(x) = A(y) = \{x, y\}$.

Now assume that $\{x, y\} \notin A$. Endvertices in $N(x)$ of edges in $A(x)$ and $A(y)$ are not adjacent nor do they coincide. Replace $A(x)$ with

$$\{ \{x, z\} | \{x, z\} \in A(x) \text{ or } \{y, z\} \in A(y) \}$$

and set $A(y) = \emptyset$. Then the new set of edges is independent and has the same cardinality as $A$.

Let $Q$ be an independent set in $G$. At most one of $x$ and $y$ is in $Q$. The validity of Equation (2) is easily checked. \qed

4 Graphs without odd wheels

A wheel $W_n$ is a graph consisting of a cycle $C_n$ and one additional vertex adjacent to all vertices in the cycle. The universal vertex of $W_n$ is called the hub. It is unique unless $W_n = K_4$. The edges incident with the hub are called the spokes of the wheel. The cycle is called the rim of the wheel. A wheel is odd if the number of vertices in the cycle is odd.
Lakshmanan, Bujtás and Tuza investigate the class of graphs without odd wheels in [43]. They prove that Tuza’s conjecture holds true for this class of graphs.

Notice that a graph $G$ has no odd wheel if and only if every neighborhood in $G$ induces a bipartite graph. It follows that $\omega(G) \leq 3$. Obviously, the class of graphs without odd wheels is closed under taking subgraphs. Notice that, when $G$ has no odd wheel then every neighborhood in $K_e(G)$ is either empty or a matching. Furthermore, it is easy to see that $K_e(G)$ contains no diamond (every edge is in exactly one triangle), no $C_5$ and no odd antihole.

For graphs $G$ without odd wheels $K_e(G)$ coincides with the anti-Gallai graphs introduced by Le [46]. For general anti-Gallai graphs the computation of the clique number and chromatic number are NP-complete.

Let us mention that the recognition of anti-Gallai graphs of general graphs is NP-complete [4]. The recognition of edge-clique graphs of graphs without odd wheels is, as far as we know, open. Let us also mention that the edge-clique graphs of graphs without odd wheels are clique graphs [14]. The recognition of clique graphs of general graphs is NP-complete [2].

**Theorem 6.** The computation of $\alpha'(G)$ is NP-complete for graphs $G$ without odd wheels.

**Proof.** We reduce 3-SAT to the vertex cover problem in edge-clique graphs of graphs without odd wheels.

Let $H \cong L(K_4)$, i.e., the complement of $3K_2$. Let $S$ be a 3-sun. The graph $H$ is obtained from $S$ by adding three edges between pairs of vertices of degree two in $S$. Call the three vertices of degree four in $S$, the ‘inner triangle’ of $H$ and call the remaining three vertices of $H$ the ‘outer triangle.’

Notice that $H$ has 3 maximum independent sets of edges. Each maximum independent set of edges is an induced $C_4$ consisting of one edge from the inner triangle, one edge from the outer triangle, and two edges between the two triangles. The three independent sets partition the edges of $H$. The six edges of $H$ between the inner and outer triangle form a 6-cycle in $K_e(H)$. Let $F$ denote this set of edges in $H$.

For each clause $(x_i \lor x_j \lor x_k)$ take one copy of $H$. Take an independent set of three edges contained in $F$ and label these with $x_i$, $x_j$ and $x_k$.

For each variable $x$ take a triangle. Label one edge of the triangle with the literal $x$ and one other edge of the triangle with its negation $\bar{x}$.

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3 In [44, Theorem 14] the authors prove that every maximal clique in $K_e(G)$ contains a simplicial vertex if and only if $G$ does not contain, as an induced subgraph, $K_4$ nor a 3-sun with 0, 1, 2 or 3 edges connecting the vertices of degree two.
Construct links between variable gadgets and clause gadgets as follows. Let \((x_i \lor x_j \lor x_k)\) be a clause. Identify one endpoint of the edge \(x_i\) in the clause gadget with an endpoint of the edge labeled \(x_i\) in the variable gadget. Add an edge between the other two endpoints. Construct links for the other two literals in the clause in the same manner.

Let \(G\) be the graph constructed in this manner. Notice that \(K_e(G)\) contains some simplicial vertices; namely the unlabeled edges in each variable gadget and the unlabeled edges in the links. Notice that these simplicial vertices can be removed without changing the complexity of the vertex cover problem. Let \(K\) be the graph obtained from \(K_e(G)\) by removing these simplicial vertices.

Let \(L\) be the number of variables and let \(M\) be the number of clauses in the 3-SAT formula. Assume that there is a satisfying assignment. Then choose the vertices in \(K\) corresponding to literals that are \(\text{true}\) in the vertex cover. The variable gadgets need \(L\) vertices in the vertex cover. Since this assignment is satisfying, we need at most \(8M\) vertices to cover the remaining edges in the clause structures, since the outgoing edge from each literal which is \(\text{true}\) is covered. Thus there is a vertex cover of \(K_e(G)\) with \(L + 8M\) vertices.

Assume that \(K_e(G)\) has a vertex cover with \(L + 8M\) vertices. At least \(L\) vertices in \(K\) are covering the edges in the variable gadgets. The other \(8M\) vertices of \(K\) are covering the edges in the clause gadgets. Take the literals of the variable gadgets that are in the vertex cover as an assignment for the formula. Each clause gadget must have one literal vertex of which the outgoing edge is covered. Therefore, the assignment is satisfying.

\[\square\]

Consider any graph \(W\). Assume that \(W\) has a simplicial vertex \(s\). Then the vertex cover number satisfies \(\kappa(W) = \kappa(W - s)\), unless the component that contains \(s\) is \(\{s\}\) or an edge. See eg [37, Theorem 2.64].
5 Concluding remark

As far as we know, the recognition of edge-clique graphs is an open problem.

Let $K_{m}^{n}$ denote the complete multipartite graph with $m$ partite sets each having $n$ vertices. Obviously, $K_{m}^{n}$ is a cograph with $n \cdot m$ vertices.

**Theorem 7 ([53]).** Assume that

$$3 \leq m \leq n + 1.$$  

Then $\theta_e(K_{m}^{n}) = n^2$ if and only if there exists a collection of at least $m - 2$ pairwise orthogonal Latin squares of order $n$.

Notice that, if there exists an edge-clique cover of $K_{m}^{n}$ with $n^2$ cliques, then these cliques are mutually edge-disjoint.

Finding the maximum number of pairwise orthogonal Latin squares of order $n$ is a renowned open problem. The problem has a wide field of applications, e.g. in combinatorics, designs of experiments, group theory and quantum informatics.

Unless $n$ is a prime power, the maximal number of MOLS is known for only a few orders. We briefly mention a few results. Let $f(n)$ denote the maximal number of MOLS of order $n$. The well-known ‘Euler-spoiler’ shows that $f(n) = 1$ only for $n = 2$ and $n = 6$. Also, $f(n) \leq n - 1$ for all $n > 1$, and Chowla, Erdős and Straus [19] show that

$$\lim_{n \to \infty} f(n) = \infty.$$  

Define

$$n_r = \max \{ n \mid f(n) < r \}.$$  

A lowerbound for the speed at which $f(n)$ grows was obtained by Wilson, who showed that $n_r < r^{17}$ when $r$ is sufficiently large [61]. Better bounds for $n_r$, for some specific values of $r$, were obtained by various authors (see e.g. [10]).

See e.g. [42] for some recent computational attempts to find orthogonal Latin squares. The problem seems extremely hard, both from a combinatorial and from a computational point of view. Despite many efforts, the existence of three pairwise orthogonal Latin squares of order 10 is, as far as we know, still unclear.

**Conjecture 1.** The edge-clique cover problem is NP-complete for cographs.

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