STABILITY CRITERION FOR CONVOLUTION-DOMINATED INFINITE MATRICES

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Abstract. Let $\ell^p$ be the space of all $p$-summable sequences on $\mathbb{Z}$. An infinite matrix is said to have $\ell^p$-stability if it is bounded and has bounded inverse on $\ell^p$. In this paper, a practical criterion is established for the $\ell^p$-stability of convolution-dominated infinite matrices.

1. Introduction

Let $C$ be the set of all infinite matrices $A := (a(j,j'))_{j,j' \in \mathbb{Z}}$ with

$$\|A\|_C = \sum_{k \in \mathbb{Z}} \sup_{j-j' = k} |a(j,j')| < \infty.$$ 

Let $\ell^p := \ell^p(\mathbb{Z})$ be the set of all $p$-summable sequences on $\mathbb{Z}$ with the standard norm $\|\cdot\|_p$. An infinite matrix $A := (a(j,j'))_{j,j' \in \mathbb{Z}} \in C$ defines a bounded linear operator on $\ell^p$, $1 \leq p \leq \infty$, in the sense that

$$Ac = \left( \sum_{j' \in \mathbb{Z}} a(j,j')c(j') \right)_{j \in \mathbb{Z}}$$

where $c = (c(j))_{j \in \mathbb{Z}} \in \ell^p$. Given a summable sequence $h = (h(j))_{j \in \mathbb{Z}} \in \ell^1$, define the convolution operator $C_h$ on $\ell^p$, $1 \leq p \leq \infty$, by

$$C_h : \ell^p \ni (b(j))_{j \in \mathbb{Z}} \mapsto \left( \sum_{k \in \mathbb{Z}} h(j-k)b(k) \right)_{j \in \mathbb{Z}} \in \ell^p.$$ 

Observe that the linear operator associated with an infinite matrix $A \in C$ is dominated by a convolution operator in the sense that

$$|(Ac)(j)| \leq (C_h|c|)(j) := \sum_{j' \in \mathbb{Z}} h(j - j')|c(j')|, \quad j \in \mathbb{Z}$$

for any sequence $c = (c(j))_{j \in \mathbb{Z}} \in \ell^p$, $1 \leq p \leq \infty$, where $|c| = (|c(j)|)_{j \in \mathbb{Z}}$ and the sequence $(\sup_{j-j' = k}|a(j,j')|)_{k \in \mathbb{Z}}$ can be chosen to be the sequence $h = (h(j))_{j \in \mathbb{Z}}$ in (1.3). So infinite matrices in the set $C$ are said to be convolution-dominated.

Convolution-dominated infinite matrices were introduced by Gohberg, Kaashoek, and Woerdeman [12] as a generalization of Toeplitz matrices. They showed that the class $C$ equipped with the standard matrix multiplication and the above norm $\|\cdot\|_C$ is an inverse-closed Banach subalgebra of $B(\ell^p)$ for $p = 2$. Here $B(\ell^p)$, $1 \leq p \leq \infty$, is the space of all bounded linear operators on $\ell^p$ with the standard operator norm, and a subalgebra $A$ of a Banach algebra $\mathcal{B}$ is said to be inverse-closed if an operator $T \in \mathcal{A}$ has an inverse $T^{-1}$ in $\mathcal{B}$ then $T^{-1} \in \mathcal{A}$ ([7, 11, 21]). The inverse-closed property for convolution-dominated infinite matrices was rediscovered by Sjöstrand [25].

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with a completely different proof and an application to a deep theorem about pseudodifferential operators. Recently Shin and Sun [23] generalized Gohberg, Kaashoek and Woerdeman’s result and proved that the class $C$ is an inverse-closed Banach subalgebra of $B(ℓ^p)$ for any $1 \leq p \leq ∞$. The readers may refer to [5, 10, 20, 23, 25, 27] and the references therein for related results and various generalizations on the inverse-closed property for convolution-dominated infinite matrices.

Convolution-dominated infinite matrices arise and have been used in the study of spline approximation ([8, 9]), wavelets and affine frames ([6, 18]), Gabor frames and non-uniform sampling ([3, 14, 15, 26]), and pseudo-differential operators ([13, 16, 24, 25] and the references therein). Examples of convolution-dominated infinite matrices include the infinite matrix $\left( a(j - j') \right)_{j,j' \in \mathbb{Z}}$ associated with convolution operators, and the infinite matrix $\left( a(j - j') e^{-\frac{2\pi}{\sqrt{-1}} \theta j' (j - j')} \right)_{j,j' \in \mathbb{Z}}$ associated with twisted convolution operators, where $\theta \in \mathbb{R}$ and the sequence $a = (a(j))_{j \in \mathbb{Z}}$ satisfies $\sum_{j \in \mathbb{Z}} |a(j)| < ∞$ ([1, 14, 19, 27, 29]).

A convolution-dominated infinite matrix $A$ is said to have $ℓ^p$-stability if there are two positive constants $C_1$ and $C_2$ such that

$$C_1 \|c\|_p \leq \|Ac\|_p \leq C_2 \|c\|_p \quad \text{for all } c \in ℓ^p.$$  

The $ℓ^p$-stability is one of basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators, see [1, 3, 6, 8, 9, 10, 14, 15, 16, 18, 19, 23, 24, 25, 26, 27, 29] and the references therein. Practical criteria for the $ℓ^p$-stability of a convolution-dominated infinite matrix will play important roles in the further study of those topics.

However, up to the knowledge of the author, little is known about practical criteria for the $ℓ^p$-stability of an infinite matrix. For an infinite matrix $A = \left( a(j - j') \right)_{j,j' \in \mathbb{Z}}$ associated with convolution operators, there is a very useful criterion for its $ℓ^p$-stability. It states that $A$ has $ℓ^p$-stability if and only if the Fourier series $\hat{a}(ξ) := \sum_{j \in \mathbb{Z}} a(j) e^{-ijξ}$ of the generating sequence $a = (a(j))_{j \in \mathbb{Z}} \in ℓ^1$ does not vanish on the real line, i.e.,

$$\hat{a}(ξ) \neq 0 \quad \text{for all } ξ \in \mathbb{R}. \quad (1.5)$$

Applying this criterion for the $ℓ^p$-stability, one concludes that the spectrum $σ_p(C_a)$ of the convolution operator $C_a$ as an operator on $ℓ^p$ is independent of $1 \leq p \leq ∞$, i.e.,

$$σ_p(C_a) = σ_q(C_a) \quad \text{for all } 1 \leq p, q \leq ∞ \quad \text{see [4, 17, 22, 23]}$$

and the references therein for the discussion on spectrum of various convolution operators. Applying the above criterion again, together with the classical Wiener’s lemma ([29]), it follows that the inverse of an $ℓ^p$-stable convolution operator $C_a$ is a convolution operator $C_b$ associated with another summable sequence $b$.

For a convolution-dominated infinite matrix $A = \left( a(j,j') \right)_{j,j' \in \mathbb{Z}}$, a popular sufficient condition for its $ℓ^1$-stability and $ℓ^∞$-stability is that $A$ is diagonal-dominated,
i.e.,

\[ (1.7) \quad \inf_{j \in \mathbb{Z}} \left( \left| a(j, j) \right| - \max \left( \sum_{j' \neq j} \left| a(j, j') \right|, \sum_{j' \neq j} \left| a(j', j) \right| \right) \right) > 0. \]

In this paper, we provide a practical criterion for the \( \ell^p \)-stability of convolution-dominated infinite matrices. We show that a convolution-dominated infinite matrix \( A \) has \( \ell^p \)-stability if and only if it has certain “diagonal-blocks-dominated” property (see Theorem 2.1 for the precise statement).

### 2. Main Theorem

To state our criterion for the \( \ell^p \)-stability of convolution-dominated infinite matrices, we introduce two concepts. Given an infinite matrix \( A \), define the truncation matrices \( A_s, s \geq 0 \), by

\[ A_s = (a(i, j) \chi(-s, s)(i - j))_{i, j \in \mathbb{Z}} \]

where \( \chi_E \) is the characteristic function on a set \( E \). Given \( y \in \mathbb{R} \) and \( 1 \leq N \in \mathbb{Z} \), define the operator \( \chi_y^N \) on \( \ell^p \) by

\[ \chi_y^N : \ell^p \ni (c(j))_{j \in \mathbb{Z}} \mapsto (c(j) \chi(-N, N)(j - y))_{j \in \mathbb{Z}} \in \ell^p. \]

The operator \( \chi_y^N \) is a diagonal matrix \( \text{diag}(\chi(-N, N)(j - y))_{j \in \mathbb{Z}} \).

**Theorem 2.1.** Let \( 1 \leq p \leq \infty \), and \( A \) be a convolution-dominated infinite matrix in the class \( C \). Then the following statements are equivalent.

(i) The infinite matrix \( A \) has \( \ell^p \)-stability.

(ii) There exist a positive constant \( C_0 \) and a positive integer \( N_0 \) such that

\[ (2.1) \quad \| \chi_y^{2N} A \chi_y^N c \|_p \geq C_0 \| \chi_y^N c \|_p, \quad c \in \ell^p, \]

hold for all integers \( N \geq N_0 \) and \( n \in N \mathbb{Z} \).

(iii) There exist a positive integer \( N_0 \) and a positive constant \( \alpha \) satisfying

\[ (2.2) \quad \alpha > 2(5 + 2^{1-p})^{1/p} \inf_{0 \leq s \leq N_0} \left( \| A - A_s \| c + \frac{s}{N_0} \| A \| c \right) \]

such that

\[ (2.3) \quad \| \chi_y^{2N_0} A \chi_y^{N_0} c \|_p \geq \alpha \| \chi_y^{N_0} c \|_p, \quad c \in \ell^p, \]

hold for all \( n \in N_0 \mathbb{Z} \).

Taking \( N_0 = 1 \) in (2.2) and (2.3), we obtain a sufficient condition (2.4), which is a strong version of the diagonal-domination condition (1.7), for the \( \ell^\infty \)-stability of a convolution-dominated infinite matrix.

**Corollary 2.2.** Let \( A = (a(j, j'))_{j, j' \in \mathbb{Z}} \) be a convolution-dominated infinite matrix in the class \( C \). If

\[ (2.4) \quad \inf_{j \in \mathbb{Z}} \left| a(j, j) \right| - 2 \sum_{0 \neq k \in \mathbb{Z}, j - j' = k} \sup_{j' = k} \left| a(j, j') \right| > 0, \]

then \( A \) has \( \ell^\infty \)-stability.
We say that an infinite matrix \( A = (a(i,j))_{i,j \in \mathbb{Z}} \) is a \textit{band matrix} if \( a(i,j) = 0 \) for all \( i, j \in \mathbb{Z} \) satisfying \( j > i + k \) or \( j < i - k \). The quantity \( 2k + 1 \) is the \textit{bandwidth} of the matrix \( A \). For a band matrix \( A \) with bandwidth \( 2k + 1 \), \( A - A_s \) is the zero matrix if \( s > k \). Therefore for \( N > k \),

\[
\inf_{0 \leq s \leq N} \left( \|A - A_s\|_c + \frac{s}{N} \|A\|_c \right) \leq \frac{k}{N} \|A\|_c.
\]

This, together with Theorem 2.1, gives the following sufficient condition for a band matrix to have \( \ell^p \)-stability.

\textbf{Corollary 2.3.} Let \( 1 \leq p \leq \infty \) and \( A \) be a convolution-dominated band matrix in the class \( C \) with bandwidth \( 2k + 1 \). If there exists an integer \( N_0 > k \) such that

\[
\|A \chi_{N_0}^p c\|_p \geq \alpha \|\chi_{N_0}^p c\|_p, \quad c \in \ell^p,
\]

holds for some constant \( \alpha \) strictly larger than \( 2(5 + 2^{1-p})^{1/p} k \|A\|_c / N_0 \), then \( A \) has \( \ell^p \)-stability.

If we further assume that the infinite matrix \( A \) in Corollary 2.3 has the form \( A = (a(j - j'))_{j,j' \in \mathbb{Z}} \) for some finite sequence \( a = (a(j))_{j \in \mathbb{Z}} \) satisfying \( a(j) = 0 \) for \( |j| > k \), then \( \|A\|_c = \sum_{|j| \leq k} |a(j)| \) and the condition (2.5) can reformulated as follows:

\[
\|\tilde{A}_{N_0} c\|_p \geq \frac{\gamma k}{N_0} \left( \sum_{|j| \leq k} |a(j)| \right) \|c\|_p, \quad c \in \mathbb{R}^{2N_0 + 1},
\]

holds for some \( \gamma > 2(5 + 2^{1-p})^{1/p} \), where

\[
\tilde{A}_{N_0} = (a(j - j'))_{-N_0 \leq j \leq N_0, -k \leq j' \leq N_0}
\]

and

\[
\|c\|_p = \begin{cases} \left( \sum_{j=-k}^{k} |c(j)|^p \right)^{1/p} & \text{if } 1 \leq p < \infty \\ \sup_{-k \leq j \leq k} |c(j)| & \text{if } p = \infty, \end{cases}
\]

for \( c = (c(-k_1), \ldots, c(0), \ldots, c(k_2))^T \in \mathbb{R}^{k_1 + k_2 + 1} \). As a conclusion from (2.6) and (2.7), we see that if \( A = (a(j - j'))_{j,j' \in \mathbb{Z}} \) does not have \( \ell^p \)-stability, then for any large integer \( N \),

\[
\inf_{0 \neq c \in \mathbb{R}^{2N + 1}} \frac{\|\tilde{A}_{N} c\|_p}{\|c\|_p} \leq \frac{2(5 + 2^{1-p})^{1/p} k}{N} \left( \sum_{|j| \leq k} |a(j)| \right).
\]

For the special case \( p = 2 \), the above inequality (2.8) can be interpreted as the minimal eigenvalue of \( (\tilde{A}_N)^T \tilde{A}_N \) is less than or equal to \( \frac{2k}{N^2} \left( \sum_{|j| \leq k} |a(j)| \right)^2 \), and it can also be rewritten as

\[
\inf_{0 \neq P_N \in \Pi_N} \left( \int_{-\pi}^{\pi} |\tilde{a}(\xi)|^2 |P_N(\xi)|^2 d\xi \right)^{1/2} \leq \frac{\sqrt{2k}}{N} \left( \sum_{|j| \leq k} |a(j)| \right),
\]

where \( \tilde{a}(\xi) = \sum_{j \in \mathbb{Z}} a(j) e^{-ij\xi} \) and \( \Pi_N \) is the set of all trigonometrical polynomial of degree at most \( N \).

If the sequence \( a = (a(j))_{j \in \mathbb{Z}} \) satisfies \( a(0) = 1, a(-1) = -1, \) and \( a(j) = 0 \) otherwise, then the bandwidth of the infinite matrix \( A = (a(j - j'))_{j,j' \in \mathbb{Z}} \) is equal
to 3, the norm \( \|A\|_C \) of the associated infinite matrix \( A \) is equal to 2,

\[
\tilde{A}_N = \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 & 0 \\
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
& & & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 & -1 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},
\]

(2.10)

and

\[
\inf_{0 \neq c \in \mathbb{R}^{2N+1}} \frac{\|\tilde{A}_N c\|_p}{\|c\|_p} \geq \frac{1}{N+1},
\]

where the last inequality holds since the matrix

\[
\tilde{B}_N := \begin{pmatrix}
-1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & -1 & -1 & \cdots & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

is a left inverse of the matrix \( \tilde{A}_N \). Therefore the order \( N^{-1} \) in (2.8) can not be improved in general, but the author believes that the bound constant \( 2(5+2^{1-p})^{1/p} \) in (2.2) and (2.8) is not optimal and could be improved.

3. Proof

We say that a discrete subset \( \Lambda \) of \( \mathbb{R}^d \) is relatively-separated if

\[
R(\Lambda) := \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} \chi_{\lambda+[-1/2,1/2]^d}(x) < \infty
\]

([1, 23, 27]). Clearly, the set \( \mathbb{Z} \) of all integers is a relatively-separated subset of \( \mathbb{R} \) with

\[
R(\mathbb{Z}) = 1.
\]

Given a discrete set \( \Lambda \), let \( \ell^p(\Lambda) \) be the set of all \( p \)-summable sequences on the set \( \Lambda \) with standard norm \( \| \cdot \|_{\ell^p(\Lambda)} \) or \( \| \cdot \|_p \) for brevity.

Given two relatively-separated subsets \( \Lambda \) and \( \Lambda' \) of \( \mathbb{R}^d \), define

\[
\mathcal{C}(\Lambda, \Lambda') = \left\{ A := (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'} \left| \|A\|_{\mathcal{C}(\Lambda, \Lambda')} < \infty \right. \right\},
\]

where

\[
\|A\|_{\mathcal{C}(\Lambda, \Lambda')} = \sum_{k \in \mathbb{Z}^d} \sup_{\lambda \in \Lambda, \lambda' \in \Lambda'} |a(\lambda, \lambda')| \chi_{k+[-1/2,1/2]^d}(\lambda - \lambda').
\]
It is obvious that
\[(3.3) \quad C(\mathbb{Z}, \mathbb{Z}) = \mathcal{C}.
\]

Given an infinite matrix \( A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'} \) define its truncation matrices
\[
A_s, s \geq 0, \text{ by } A_s = \left( a(\lambda, \lambda') \chi_{\{s, s\}}(\lambda - \lambda') \right)_{\lambda \in \Lambda, \lambda' \in \Lambda'}.
\]

For any \( y \in \mathbb{R}^d \) and a positive integer \( N \), define the operator \( \chi_N^y \) on \( \ell^p(\Lambda) \) by
\[(3.4) \quad \chi_N^y : \ell^p(\Lambda) \ni (c(\lambda))_{\lambda \in \Lambda} \mapsto (c(\lambda)\chi_{(-N, N)}(\lambda - y))_{\lambda \in \Lambda} \in \ell^p(\Lambda).
\]

In this section, we establish the following criterion for the \( \ell^p \)-stability of infinite matrices in the class \( \mathcal{C}(\Lambda, \Lambda') \), which is a slight generalization of Theorem 2.1 by (3.2) and (3.3).

**Theorem 3.1.** Let \( 1 \leq p \leq \infty \), the subsets \( \Lambda, \Lambda' \) of \( \mathbb{R}^d \) be relatively-separated, and the infinite matrix \( A \) belong to \( \mathcal{C}(\Lambda, \Lambda') \). Then the following statements are equivalent to each other:

(i) The infinite matrix \( A \) has \( \ell^p \)-stability, i.e., there exist positive constants \( C_1 \) and \( C_2 \) such that
\[(3.5) \quad C_1 \|c\|_{\ell^p(\Lambda)} \leq \|Ac\|_{\ell^p(\Lambda')} \leq C_2 \|c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda').
\]

(ii) There exist a positive constant \( C_0 \) and a positive integer \( N_0 \) such that
\[(3.6) \quad \|\chi_n^{2N}A_n^\Lambda c\|_{\ell^p(\Lambda)} \geq C_0 \|\chi_n^N c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'),
\]
where \( N_0 \leq N \in \mathbb{Z} \) and \( n \in N\mathbb{Z}^d \).

(iii) There exist a positive integer \( N_0 \) and a positive constant \( \alpha \) satisfying
\[(3.7) \quad \alpha > 2(5 + 2^{1-p})^d R(\Lambda)^{-1/p} R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N_0} \left( \|A - A_s\|_{\ell^p(\Lambda, \Lambda')} + \frac{d_0}{N_0} \|A\|_{\ell^p(\Lambda, \Lambda')} \right)
\]
such that
\[(3.8) \quad \|\chi_n^{2N_0}A_n^\Lambda c\|_{\ell^p(\Lambda)} \geq \alpha \|\chi_n^{N_0} c\|_{\ell^p(\Lambda')}
\]
hold for all \( c \in \ell^p(\Lambda') \) and \( n \in N_0\mathbb{Z} \).

Using the above theorem, we obtain the following equivalence of \( \ell^p \)-stability for infinite matrices having certain off-diagonal decay, which is established in [2, 28, 23] for \( \gamma > d(d + 1), \gamma > 0 \), and \( \gamma \geq 0 \) respectively.

**Corollary 3.2.** Let \( \Lambda, \Lambda' \) be relatively-separated subsets of \( \mathbb{R}^d \), and \( A = (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'} \) satisfy
\[
\|A\|_{\ell^p(\Lambda, \Lambda')} = \sum_{k \in \mathbb{Z}^d} (1 + |k|)^\gamma \sup_{\lambda \in \Lambda, \lambda' \in \Lambda'} |a(\lambda, \lambda')| \chi_{|k+|1/2,1/2|^d}(\lambda - \lambda') < \infty
\]
where \( \gamma > 0 \). Then the \( \ell^p \)-stability of the infinite matrix \( A \) are equivalent to each other for different \( 1 \leq p \leq \infty \).

**Proof.** Let \( 1 \leq p \leq \infty \) and \( A \) have \( \ell^p \)-stability. Then by Theorem 3.1 there exists a positive constant \( C_0 \) and a positive integer \( N_0 \) such that
\[(3.9) \quad \|\chi_n^{2N_0}A_n^\Lambda c\|_{\ell^p(\Lambda)} \geq C_0 \|\chi_n^N c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'),
\]
where \( N_0 \leq N \in \mathbb{Z} \) and \( n \in \mathbb{N}^d \). From the equivalence of different norms on a finite-dimensional space, we have that
\[
(2N)^d R(\Lambda)^{\min(1/q-1/p,0)} \|A_N^c\|_{p,(\Lambda)} \leq (2N)^d \|A_N^c\|_{p,(\Lambda)} \text{ for all } c \in \ell^p(\Lambda),
\]
where \( 1 \leq p, q \leq \infty, 1 \leq N \in \mathbb{Z} \) and \( n \in \mathbb{N}^d \) ([2, 23]). Therefore for \( 1 \leq q \leq \infty, \)
\[
\|A_N^c\|_{p,(\Lambda)} \leq C_0(2N)^{-d|1/p-1/q|} R(\Lambda)^{\min(1/p-1/q,0)}
\]
(3.10)
\[
\times R(\Lambda)^{-\max(1/p-1/q,0)} \|A_N^c\|_{p,(\Lambda)} \text{ for all } c \in \ell^q(\Lambda),
\]
where \( N_0 \leq N \in \mathbb{Z} \) and \( n \in \mathbb{N}^d \). We notice that
\[
\inf_{0 \leq s \leq N} \|A - A_s\|_{C(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{C(\Lambda,\Lambda')} \leq \|A\|_{C,\gamma(\Lambda,\Lambda')} \inf_{0 \leq s \leq N} s^\gamma + \frac{ds}{N}
\]
(3.11)
\[
\leq (d + 1) \|A\|_{C,\gamma(\Lambda,\Lambda')} N^{-\gamma/(1+\gamma)}.
\]
Thus for \( 1 \leq q \leq \infty \) with \( d|1/p-1/q| < \gamma/(1+\gamma) \), it follows from (3.10) and (3.11) that there exists a sufficiently large integer \( N_0 \) such that
\[
\|A_N^c\|_{p,(\Lambda)} \geq \alpha \|A_N^c\|_{p,(\Lambda)}
\]
hold for all \( c \in \ell^q(\Lambda), N \geq N_0 \) and \( n \in \mathbb{N}^d \), where \( \alpha \) is a positive constant larger than \( 2(5+2^{1-q})d/q R(\Lambda)^{1/q} R(\Lambda)^{1/q} \inf_{0 \leq s \leq N_0} (\|A - A_s\|_{C(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{C(\Lambda,\Lambda')} \).

Then by Theorem 3.1, the infinite matrix \( A \) has \( \ell^q \)-stability for all \( 1 \leq q \leq \infty \) with \( d|1/q-1/p| < \gamma/(1+\gamma) \). Applying the above trick repeatedly, we prove the \( \ell^q \)-stability of the infinite matrix \( A \) for any \( 1 \leq q \leq \infty \).

To prove Theorem 3.1, we first recall some basic properties for infinite matrices \( A \) in the class \( C(\Lambda,\Lambda') \) and its truncation matrices \( A_s, s \geq 0 \).

**Lemma 3.3.** ([23]) Let \( 1 \leq p \leq \infty \), the subsets \( \Lambda, \Lambda' \) of \( \mathbb{R}^d \) be relatively-separated, \( A \) be an infinite matrix in the class \( C(\Lambda,\Lambda') \), and \( A_s, s \geq 0 \), be the truncation matrices of \( A \). Then
\[
\|Ac\|_{p,(\Lambda')} \leq R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A\|_{C(\Lambda,\Lambda')} \|c\|_{p,(\Lambda')} \text{ for all } c \in \ell^p(\Lambda'),
\]
(3.13)
\[
\lim_{s \to +\infty} \|A - A_s\|_{C(\Lambda,\Lambda')} = 0,
\]
(3.14)
\[
\lim_{N \to +\infty} \inf_{0 \leq s \leq N} \|A - A_s\|_{C(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{C(\Lambda,\Lambda')} = 0,
\]
and
\[
\|A_s\|_c \leq \|A\|_c \text{ for all } s \geq 0.
\]

Let \( \psi_0(x_1, \ldots, x_d) = \prod_{i=1}^d \max(2 - 2|x_i|, 1, 0) \) be a cut-off function on \( \mathbb{R}^d \).

Then
\[
0 \leq \chi_{[-1/2,1/2]}(x) \leq \psi_0(x) \leq \chi_{[-1,1]}(x) \leq 1 \text{ for all } x \in \mathbb{R}^d,
\]
and
\[
|\psi_0(x) - \psi_0(y)| \leq 2d \|x - y\|_\infty \text{ for all } x, y \in \mathbb{R}.
\]
where \( \|x\|_\infty = \max_{1 \leq i \leq d} |x_i| \) for \( x = (x_1, \ldots, x_d) \). Define the multiplication operator \( \Psi^N_n \) on \( \ell^p(\Lambda) \) by
\[
(3.19) \quad \Psi^N_n : \ell^p(\Lambda) \ni (c(\lambda))_{\lambda \in \Lambda} \mapsto \left( \psi_0\left(\frac{\lambda - n}{N}\right) c(\lambda) \right)_{\lambda \in \Lambda} \in \ell^p(\Lambda).
\]

Applying (3.17) and (3.18) for the cut-off function \( \psi_0 \), we obtain the following properties for the multiplication operators \( \Psi^N_n, n \in N\mathbb{Z} \).

**Lemma 3.4.** Let \( 1 \leq N \in \mathbb{Z}, \Lambda \) be a relatively-separated subset of \( \mathbb{R}^d \), and the multiplication operators \( \Psi^N_n, n \in N\mathbb{Z}^d \), be as in (3.19). Then
\[
(3.20) \quad \|\Psi^N_n c\|_{\ell^p(\Lambda)} \leq \|\chi^N_n c\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p(\Lambda)
\]
where \( 1 \leq p \leq \infty \),
\[
(3.21) \quad \|c\|_{\ell^p(\Lambda)} \leq \left( \sum_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^p(\Lambda)}^p \right)^{1/p} \leq 2^d/p\|c\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p(\Lambda)
\]
\[
(3.22) \quad A^{d/p}\|c\|_{\ell^p(\Lambda)} \leq \left( \sum_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^p(\Lambda)}^p \right)^{1/p} \leq (5 + 2^{1-p})d/p\|c\|_{\ell^p(\Lambda)} \quad \text{for all } c \in \ell^p(\Lambda),
\]
where \( 1 \leq p < \infty \), and
\[
(3.23) \quad \|c\|_{\ell^\infty(\Lambda)} = \sup_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^\infty(\Lambda)} = \sup_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^\infty(\Lambda)} \quad \text{for all } c \in \ell^\infty(\Lambda).
\]

To prove Theorem 2.1, we also need the following result.

**Lemma 3.5.** ([23]) Let \( N \geq 1 \), the subsets \( \Lambda, \Lambda' \) of \( \mathbb{R}^d \) be relatively-separated, \( A \) be an infinite matrix in the class \( C(\Lambda, \Lambda') \), \( A_N \) be the truncation matrix of \( A \), and \( \Psi^N_n, n \in N\mathbb{Z}^d \), be the multiplication operators in (3.19). Then
\[
(3.24) \quad \|\Psi^N_n A_N - A_N \Psi^N_n\|_{C(\Lambda, \Lambda')} \leq \inf_{0 \leq s \leq N} \left( \|A_N - A_s\|_{C(\Lambda, \Lambda')} + 2dsN \right).
\]

Now we start to prove Theorem 3.1.

**Proof of Theorem 3.1.** (i)\( \Rightarrow \) (ii): By the \( \ell^p \)-stability of the infinite matrix \( A \), there exists a positive constant \( C_0 \) (independent of \( n \in N\mathbb{Z}^d \) and \( 1 \leq N \in \mathbb{Z} \)) such that
\[
(3.25) \quad \|A\chi^N_n c\|_{\ell^p(\Lambda)} \geq C_0\|\chi^N_n c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'),
\]
where \( n \in N\mathbb{Z}^d \) and \( N \geq 1 \). Noting
\[
(3.26) \quad \chi^N_n A_N \psi^N_n = A_N \psi^N_n
\]
and applying (3.13) yield
\[
(3.27) \quad \|A\chi^N_n c - \chi^N_n A\chi^N_n c\|_{\ell^p(\Lambda)} \leq \|I - \chi^N_n A - A\chi^N_n c\|_{\ell^p(\Lambda)} \leq R(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\|A - A_N\|_{C(\Lambda, \Lambda')}\|\chi^N_n c\|_{\ell^p(\Lambda')}.
\]

where \( I \) is the identity operator. Combining the estimates in (3.25) and (3.27) proves that
\[
(3.28) \quad \|\chi^N_n A\chi^N_n c\|_{\ell^p(\Lambda)} \geq (C_0 - R(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\|A - A_N\|_{C(\Lambda, \Lambda')}\|\chi^N_n c\|_{\ell^p(\Lambda')}.
\]
hold for all \( c \in \ell^p(\Lambda') \), where \( n \in \mathbb{N}^d \) and \( N \geq 1 \). The conclusion (ii) then follows from (3.14) and (3.28).

(ii) \( \Rightarrow \) (iii): The implication follows from (3.15).

(iii) \( \Rightarrow \) (i): Let \( 1 \leq p < \infty \). Take any \( n \in \mathbb{N}_0^d \) and \( c \in \ell^p(\Lambda') \). By the assumption (iii) for the infinite matrix \( A \),

\[
\| A^{N_0} \Psi_n^{N_0} c \|_{p(\Lambda)} = \| A^{N_0} (A_n - A) \Psi_n^{N_0} c \|_{p(\Lambda)} \geq \alpha \| \Psi_n^{N_0} c \|_{p(\Lambda)}.
\]

This together with (3.13) and (3.26) implies that

\[
\| A^{N_0} \Psi_n^{N_0} c \|_{p(\Lambda)} = \| A^{N_0} (A_n - A) \Psi_n^{N_0} c \|_{p(\Lambda)} \geq \| A^{N_0} (A_n - A) \Psi_n^{N_0} c \|_{p(\Lambda)} - \| A^{N_0} (A_n - A) \Psi_n^{N_0} c \|_{p(\Lambda)} \geq (\alpha - R(\Lambda)1/p R(\Lambda')1^{-1/p} \| A - A_n \|_{\mathcal{C}(\Lambda, \Lambda')} \| \Psi_n^{N_0} c \|_{p(\Lambda)}).
\]

From (3.13) and (3.24) it follows that

\[
\| (\Psi_n^{N_0} A_n - A_n \Psi_n^{N_0}) c \|_{p(\Lambda)} \leq R(\Lambda)1/p R(\Lambda')1^{-1/p} \| A_n - \Psi_n^{N_0} \|_{\mathcal{C}(\Lambda, \Lambda')} \| \Psi_n^{N_0} c \|_{p(\Lambda)}.
\]

Combining (3.21), (3.22), (3.30) and (3.31), we get

\[
2^{d/p} \| A_n c \|_{p(\Lambda)} \geq \left( \sum_{n \in \mathbb{N}_0^d} \| \Psi_n^{N_0} A_n c \|_{p(\Lambda)}^p \right)^{1/p} \\
\geq \left( \alpha - R(\Lambda)1/p R(\Lambda')1^{-1/p} \| A - A_n \|_{\mathcal{C}(\Lambda, \Lambda')} \right) \left( \sum_{n \in \mathbb{N}_0^d} \| \Psi_n^{N_0} c \|_{p(\Lambda)}^p \right)^{1/p} \\
- R(\Lambda)1/p R(\Lambda')1^{-1/p} \inf_{0 \leq s \leq N_0} \left( \| A_n - A_s \|_{\mathcal{C}(\Lambda, \Lambda')} + \frac{2d}{N_0} \| A_n \|_{\mathcal{C}(\Lambda, \Lambda')} \right) \\
\times \left( \sum_{n \in \mathbb{N}_0^d} \| \Psi_n^{N_0} c \|_{p(\Lambda)}^p \right)^{1/p} \\
\geq \left( \alpha - R(\Lambda)1/p R(\Lambda')1^{-1/p} \| A - A_n \|_{\mathcal{C}(\Lambda, \Lambda')} - (5 + 2^{1-p})1/p R(\Lambda)1/p R(\Lambda')1^{-1/p} \\
\times \inf_{0 \leq s \leq N_0} \left( \| A_n - A_s \|_{\mathcal{C}(\Lambda, \Lambda')} + \frac{2d}{N_0} \| A_n \|_{\mathcal{C}(\Lambda, \Lambda')} \right) \right) \| c \|_{p(\Lambda)}.
\]
Therefore
\[
\|Ac\|_{\ell^p(\Lambda)} \geq \|A_{N_0}c\|_{\ell^p(\Lambda)} - \|(A - A_{N_0})c\|_{\ell^p(\Lambda)}
\geq 2^{-1/p} \left( \alpha - (1 + 2^{d/p}) R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A - A_{N_0}\|_{C(\Lambda, \Lambda')}
- (5 + 2^{1-p})^{d/p} R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \times \inf_{0 \leq s \leq N_0} \left( \|A_{N_0} - A_s\|_{C(\Lambda, \Lambda')} + \frac{2ds}{N_0} \|A_{N_0}\|_{C(\Lambda, \Lambda')} \right) \right) \|c\|_{\ell^p(\Lambda')}
\geq 2^{-d/p} \left( \alpha - 2(5 + 2^{1-p})^{1/p} R(\Lambda)^{1/p}
- \inf_{0 \leq s \leq N_0} \left( \|A - A_s\|_{C(\Lambda, \Lambda')} + \frac{ds}{N_0} \|A\|_{C(\Lambda, \Lambda')} \right) \right) \|c\|_{\ell^p(\Lambda')},
\]
and the conclusion (i) for $1 \leq p < \infty$ follows.

The conclusion (i) for $p = \infty$ can be proved by similar argument. We omit the details here.

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