Combined topological and Landau order from strong correlations in Chern bands

Stefanos Kourtis and Maria Daghofer

Institute for Theoretical Solid State Physics, IFW Dresden, 01171 Dresden, Germany

(Dated: May 31, 2013)

We present a class of states with both topological and conventional Landau order that arise out of strongly interacting spinless fermions in fractionally filled and topologically non-trivial bands with Chern number $C = \pm 1$. These quantum states show the features of fractional Chern insulators, such as fractional Hall conductivity and interchange of ground-state levels upon insertion of a magnetic flux. In addition, they exhibit charge order and a related additional trivial ground-state degeneracy. Band mixing and geometric frustration of the charge pattern place these lattice states markedly beyond a single-band description.

PACS numbers: 71.27.+a, 03.65.Vf, 71.45.Lr, 73.43.-f

Introduction and Model.– The fractional quantum Hall (FQH) effect is a paradigmatic example for a correlation-driven state with topological order [1], and the proposal [2–4] that the concept might be extended from Landau levels arising through a magnetic field to topologically nontrivial bands on a lattice has thus raised considerable interest. That fractionally filled and topologically non-trivial bands with non-zero Chern number $C$ can host corresponding states, dubbed fractional Chern insulators (FCI), has since been well established numerically [2–4, 5–8] for various models. Analytical considerations have established connections to FQH states by proposing wave functions [5–8] and pseudopotentials [9], as well as by noting the close mathematical relation between density operators in partially filled Chern bands and those of partially filled Landau levels [12–14]. As paths to realizing topologically nontrivial and nearly flat bands cold quantum gases [15–16], oxide interfaces [17] and layered oxides with an orbital degree of freedom [17–18], strained graphene [18], and organometalic systems [19] have been proposed. A review of this rapidly evolving field can be found in Ref. 20.

Apart from the intrinsic interest in such an effect, one motivation for the search for FCI’s is their energy and thus temperature scale: as it is given by the scale of the interaction, it is expected to be considerably higher than the sub-Kelvin range of the FQH effect, especially in the case of oxide-based proposals [15–16, 17]. As one would moreover not need strong magnetic fields, both realization of such states and their potential application to qubits [21] then appear more feasible. It has been established that FCI states can persist the influence of several aspects that make bands on lattices different from perfect flat Landau levels with uniform Berry curvature, e.g., finite dispersion, a moderate staggered chemical potential [2], and a related additional trivial ground-state degeneracy. Band mixing and geometric frustration of the charge pattern place these lattice states markedly beyond a single-band description.

In this Letter, we go beyond the limit of weak interactions, into a regime where Chern bands with $C = +1$ and $C = -1$ mix. We find states that show the features of both a CDW (revealed by the charge structure factor) and of an FCI (fractional Hall conductivity and spectral flow). The states are related to the ‘pinball liquid’ of the triangular lattice [28] that combines charge-ordered with metallic [29, 30] or superconducting [31] electrons. In the latter case, the system combines charge order with ‘off-diagonal long-range order’. Simultaneous existence of two such different order parameters has been extensively investigated, especially in its bosonic counterpart [32–35], the supersolid [36]. The present case, however, differs fundamentally from supersolids or superconducting pinball liquids, as the second type of order in addition to charge order is topological, i.e., non-local and without an order parameter. We thus arrive at an exotic state of matter that is characterized by both Landau-type and topological order and is understood in terms of both band topology and lattice geometry.

We discuss here spinless fermions on a triangular lattice. The topologically nontrivial kinetic energy can conveniently be expressed in momentum space as

$$\mathcal{H}^0 = \sum_{k,\mu,\nu} c_{k,\mu}^\dagger H_{\mu,\nu}^0(k)c_{k,\nu}$$

(1)

where the indices $\mu, \nu$ refer to a two-site unit cell and $c_{k,\mu}^\dagger$ ($c_{k,\mu}$) are fermion creation (annihilation) operators. The momentum dependence is contained in the $2 \times 2$ matrix $H^0(k)$ expressed in terms of the vector of isospin Pauli matrices $\tau$ and the unity matrix $\hat{1}$ as

$$H^0(k) = g(k) \cdot \tau + g_0(k) \hat{1},$$

with

$$g_0(k) = 2\tau \cos(k \cdot a_i), \quad i = 1, 2, 3$$

understood by focusing exclusively on the fractionally filled Chern band. FCI states considered so far are weakly interacting in the sense that the interactions stabilizing them are too weak to mix in the other band(s) with different Chern number. Those can even be projected out of the Hamiltonian, which keeps the band topology intact but obscures the impact of other aspects like the lattice geometry, which again reflects the similarity to Landau levels with their weak lattice potential.

Since the FQH effect can be discussed in tight-binding models instead of Landau levels [24, 25], the most intriguing features of FCI states are those that go beyond their FQH counterpart. FCI states in bands with higher Chern numbers were proposed [26], which may be non-Abelian and thus suitable for quantum computation. It has also been noted that FCI do not share the particle-hole symmetry of partially filled Landau levels [22, 27]. All these extensions can, however, be

expressed in terms of both band topology and lattice geometry.

We discuss here spinless fermions on a triangular lattice. The topologically nontrivial kinetic energy can conveniently be expressed in momentum space as

$$\mathcal{H}^0 = \sum_{k,\mu,\nu} c_{k,\mu}^\dagger H_{\mu,\nu}^0(k)c_{k,\nu}$$

(1)

where the indices $\mu, \nu$ refer to a two-site unit cell and $c_{k,\mu}^\dagger$ ($c_{k,\mu}$) are fermion creation (annihilation) operators. The momentum dependence is contained in the $2 \times 2$ matrix $H^0(k)$ expressed in terms of the vector of isospin Pauli matrices $\tau$ and the unity matrix $\hat{1}$ as

$$H^0(k) = g(k) \cdot \tau + g_0(k) \hat{1},$$

with

$$g_0(k) = 2\tau \cos(k \cdot a_i), \quad i = 1, 2, 3$$

understood by focusing exclusively on the fractionally filled Chern band. FCI states considered so far are weakly interacting in the sense that the interactions stabilizing them are too weak to mix in the other band(s) with different Chern number. Those can even be projected out of the Hamiltonian, which keeps the band topology intact but obscures the impact of other aspects like the lattice geometry, which again reflects the similarity to Landau levels with their weak lattice potential.

Since the FQH effect can be discussed in tight-binding models instead of Landau levels [24, 25], the most intriguing features of FCI states are those that go beyond their FQH counterpart. FCI states in bands with higher Chern numbers were proposed [26], which may be non-Abelian and thus suitable for quantum computation. It has also been noted that FCI do not share the particle-hole symmetry of partially filled Landau levels [22, 27]. All these extensions can, however, be

expressed in terms of both band topology and lattice geometry.
FIG. 1. (color online) CDW and FCI on the triangular lattice at $v = 2/3$, resp. $n = 1/3$. (a) Phase diagram of model Eqs. (1) and (5) in the $V_1 - V'$ plane for a $3 \times 3$ unit-cell ($6 \times 3$ lattice-site) system with 6 fermions. In contrast to the system of Ref. 23, the CDW is here commensurate; the location of the two phases is consistent. Arrowed black lines represent complex hoppings with phases $0$, circles) arising in the CDW and the hoppings. Solid, dashed and a

The unit cell and the phases of the hoppings can be seen in Fig. 1(b). $a_1 = (1/2, -\sqrt{3}/2)^T$, $a_2 = (1/2, \sqrt{3}/2)^T$ and $a_3 = -(a_1 + a_2)$ are the triangular-lattice unit vectors. The nearest-neighbor (NN) and third-nearest-neighbor hopping matrix elements are $t$ and $t'$, the latter is tuned to change the dispersion of the Chern band $[23]$, with the flattest bands achieved for $t'/t \approx 0.2$. We keep $t'/t = 0.2$ unless mentioned otherwise, but we have verified that the main results remain valid for other values. The interaction

$$ \mathcal{H}^1 = V_1 \sum_{\langle i,j \rangle} \hat{n}_i \hat{n}_j + V_2 \sum_{\langle\langle i,j \rangle\rangle} \hat{n}_i \hat{n}_j + V_3 \sum_{\langle i,j \rangle} \hat{n}_i \hat{n}_j, \quad (5) $$

is a repulsion of strength $V_1$ between NN sites, denoted by $\langle i,j \rangle$ and longer-range repulsion $V_2$ and $V_3$ between second and third neighbors, respectively. We present here results for $V_2 = V_3$, but have verified that unequal values do not qualitatively change the results unless $V_2$ becomes strong enough to destabilize the CDW supported by $V_1$. Operator $\hat{n}_i$ measures particle density at site $i$. While the model was originally introduced to describe topologically nontrivial phases in Kondo-lattice $[37]$ and $t_{2g}$ $[8, 23]$ systems, we use it here to study the more generic question of CDW states in the limit of strong interactions. We treat the model Eqs. (1) and (5) with Lanczos exact diagonalization on small clusters with periodic boundaries, here clusters with $3 \times 6$ and 30 sites, which corresponds to 9 resp. 15 unit cells. This is done directly in real space, without projection onto the single-particle bands - both bands with Chern numbers $C = \pm 1$ are kept.

**CDW vs. FCI.**—The model has been shown to yield FCI states at several filling fractions $[8, 23]$. As we will later discuss a phase with both charge order and topological order, let us first review the two phases and their signatures. For a filling of $n = 1/3$ (one fermion per three lattice sites) and small to moderate $V_1$, the electrons occupy $2/3$ of the lower band and an FCI with $v = 2/3$ arises. CDW fluctuations destroy it at stronger $V_1$, see the phase diagram Fig. 1(a) and Ref. 23. A schematic picture of the CDW is given in Fig. 1(b).

In both a CDW with $\bar{n} = 1/3$ and an FCI with $v = 2/3$, we expect a ‘ground-state’ manifold of three nearly degenerate states that are separated from the remaining spectrum by a gap. In the CDW, the three states always have different center-of-mass momentum, while they can have the same in the FCI. If their momenta are different, the FCI ground states are expected to exhibit spectral flow upon insertion of one flux quantum through one of the handles of the torus $[38, 39]$, as has been demonstrated for the $v = 2/3$ state $[8, 23]$.

In addition to eigenvalue spectra, groundstate properties are crucial to diagnose the two phases. Charge order shows up in the static charge-structure factor

$$ N(k) = \left| \sum_i e^{ikr} \langle \hat{n}_i - \bar{n} \rangle \right|^2, \quad (6) $$

where $\hat{n}_i$ is a vector of the ground-state manifold and $r_i$ denotes the location of site $i$. The CDW with $\bar{n} = 1/3$ induces sharp peaks at momenta $k = \pm K$ that grow with interaction strength $V_1$. In the FCI, their weight should remain comparable to that of other momenta $[23]$.

FCI states are identified by a fractional Hall conductivity $\sigma_{HI}$, which is obtained by integrating the many-body Berry curvature in the $\varphi = (\varphi_{x_1}, \varphi_{x_2})$-plane of magnetic fluxes through the handles of the torus, where the fluxes are introduced as phase factors in the hoppings along $a_2$ and $a_3$ respectively. It is evaluated with the Kubo formula $[23, 40, 41]$

$$ \sigma_{HI} = \frac{N_c}{\pi q} \sum_{n,m} \int_0^{2\pi} \frac{d\varphi_{x_1} d\varphi_{x_2}}{\varepsilon_{n'} - \varepsilon_{n}} \sum_{n' \neq \bar{n}} \frac{\langle \partial / \partial \varphi_{x_1} |n'\rangle \langle \partial / \partial \varphi_{x_2} |n\rangle}{\varepsilon_n - \varepsilon_{n'}}, \quad (7) $$

where $N_c$ is the number of unit cells and $|n'|$ are higher-energy eigenstates with eigenenergies $\varepsilon_{n'}$. $\varepsilon_n$ are the energies of the ground states. All values of $\sigma_{HI}$ are given in units of $e^2/h$. As this approach does not involve projections onto the lower (flat) band, it remains valid for arbitrary interactions and band mixing. In the FCI regime, we find a very precisely quantized $\sigma_{HI} = 2/3$. For the $3 \times 6$-sites cluster, the ground states have the same momentum and the lowest-energy state contributes 2, while the other two states do not conduct, averaging in total to the expected value. Disorder, which mixes states at different momentum, produces the same effect both in FQH $[42, 43]$ and in FCI states $[23, 44]$.

Using all eigenvalue and ground-state properties discussed, we trace the approximate phase diagram for $V_2 = V_3 = 0$ in Fig. 1(a). The phase boundaries have been determined by requiring the eigenvalue spectrum to be gapped for any values of the fluxes threading the handles of the torus. Comparison to earlier results $[23]$ for a lattice that is not commensurate with the CDW and hence suppresses it, reveals substantial finite-size effects, but the presence and approximate location of both
The Hall conductivity establishes this similarity to $\nu = p/q$, with $n \neq q$. The Hall conductivity is thus not given by the usual heuristic $\sigma_H = V \times C$. This is a strong indication, related to earlier results for the FQH effect [47], that the ‘topological’ degeneracy in both cases is $n = 5$ and not $q = 15$.

The remaining ‘trivial’ three-fold degeneracy stems from Landau-type charge order. The static charge-structure factor, see Fig. 3(a) and 3(b) clearly shows that the static charge-structure factor is not given by the usual $\nu = 2/5$ or $\nu = 3/5$, totaling to the 15 ground states. Five FCI states per CDW state exhibit spectral flow, as seen in Fig. 3 and return to the original point after insertion of 5 fluxes. A further check is to drive the system through a topological phase transition to a trivial state by setting $g_3 = m$, where $m = 2t$ is a constant [48]. Despite competing ordering tendencies due to the mass term, the charge order remains, see Fig. 3(c) but we find $\sigma_H = 0$.

The TPL, characterized by both topological and Landau order, can be intuitively understood as comprising of particles that play two roles simultaneously. Most of them form the CDW occupying $1/3$ of the lattice sites, see Fig. 1(b). As has been discussed in the context of the pinball liquid [29, 30, 49], additional particles can then move on the remaining sites, up to a total density of $\bar{n} = 2/3$. One can view the remaining sites as an effective honeycomb lattice with a 4-site unit cell [see Fig. 1(b)], and the remaining fermions give exactly the filling fractions $\nu = 2/5$ (for a total filling 12/15) resp. $\nu = 3/5$ (for 13/15) per unit cell. Due to the nontrivial topology of the underlying kinetic energy and due to residual as well as longer-range Coulomb interactions, these particles then form the corresponding FCI out of the pinball liquid’s metal. It has

FIG. 2. (Color online) Spectral flow upon flux insertion for the model of Eqs. 1 and 5 for a 15 unit-cell system at (a) $\nu = 12/15$ and (b) $\nu = 13/15$. In (c-e) and (f-h), the 15 low-energy states of (a) resp. (b) are divided into three groups of five; each group shows spectral flow consistent with a denominator-5 FCI. Fluxes are inserted as additional phases in the hoppings, totaling to $\Phi$ for a loop around the cluster along direction $a_3$. Parameters are $t'/t = 0.2$, $V_1/t = 10$ and $V_2 = V_3 = 2t$.

FIG. 3. (Color online) Static charge-structure factor $N(k)$. (a) For $\bar{n} = 12/30$ and various values of the Coulomb repulsion. (b) For $\bar{n} = 13/30$, $V_2 = V_1 = 2t$, and $V_1 = 8t$ (+), $V_1 = 10t$ (x), and $V_1 = 12t$ (+). (c) $N(k)$ for $V_1/t = 8$, $V_2 = V_3 = 3t$ (+), where dominant charge fluctuations are driven by $V_2$ rather than $V_1$ and for a topologically trivial kinetic energy ($g_3 = m = 2t$, see text) with $V_1/t = 10$, $V_2 = V_3 = 2t$ (+); $\sigma_H = 0$ in this latter case. $t' = 0.2t$ in all cases.

states are considerably more exotic as $\sigma_H = m/n$ at $\nu = p/q$, with $n \neq q$. The Hall conductivity is thus not given by the usual heuristic $\sigma_H = V \times C$. This is a strong indication, related to earlier results for the FQH effect [47], that the ‘topological’ degeneracy in both cases is $n = 5$ and not $q = 15$.

The remaining ‘trivial’ three-fold degeneracy stems from Landau-type charge order. The static charge-structure factor, see Fig. 3(a) and 3(b) clearly shows that the peaks at $\pm K$ grow with $V_1$, as expected for the $\bar{n} = 1/3$ CDW. We thus conclude that we have for each ground state of the CDW five FCI states (corresponding to $\nu = 2/5$ or $\nu = 3/5$), totaling to the 15 ground states. Five FCI states per CDW state exhibit spectral flow, as seen in Fig. 3 and return to the original point after insertion of 5 fluxes. A further check is to drive the system through a topological phase transition to a trivial state by setting $g_3 = m$, where $m = 2t$ is a constant [48]. Despite competing ordering tendencies due to the mass term, the charge order remains, see Fig. 3(c) but we find $\sigma_H = 0$.

The TPL, characterized by both topological and Landau order, can be intuitively understood as comprising of particles that play two roles simultaneously. Most of them form the CDW occupying 1/3 of the lattice sites, see Fig. 1(b). As has been discussed in the context of the pinball liquid [29, 30, 49], additional particles can then move on the remaining sites, up to a total density of $\bar{n} = 2/3$. One can view the remaining sites as an effective honeycomb lattice with a 4-site unit cell [see Fig. 1(b)], and the remaining fermions give exactly the filling fractions $\nu = 2/5$ (for a total filling 12/15) resp. $\nu = 3/5$ (for 13/15) per unit cell. Due to the nontrivial topology of the underlying kinetic energy and due to residual as well as longer-range Coulomb interactions, these particles then form the corresponding FCI out of the pinball liquid’s metal. It has
The system is not a Chern band, yet, charge fluctuations at finite order. The eigenvalue spectra of our interacting spinless-fermions exhibiting both Landau and topological features that go beyond the single-band picture usually sufficient to describe FCIIs and definitely goes beyond a Landau-level description. In analogy to the supersolid/pinball liquid connection on the triangular lattice, analogous states might be found on other lattices supporting supersolids or pinball liquids, possibly also in multi-orbital settings [50].

This work was supported by the Emmy-Noether program of the Deutsche Forschungsgemeinschaft (DFG). We thank A. Ralko, J. Venderbos, I. Vincon, T. Neupert and P. Kotetes for helpful discussions.

Figure 4 gives an approximate phase diagram for the $\nu = 2/5$ TPL. While it is stabilized by longer-range interactions $V_2, V_3 > 0$, it relies on the charge order driven by $V_1$ and disappears when the latter is destroyed by larger $V_2$, see Fig. 4(a). In contrast to the FCI at $\nu = 2/3$, the dispersion of the initial band (which is minimal at $t'/t \approx 0.19$) does not affect the TPL, see Fig. 4(b). This independence may be connected to the reduced dispersion recently reported for the metallic part of the pinball state [49] or may be due to the fact that the interactions stabilizing a CDW are already rather strong and can thus overcome substantial dispersion, as in the FCI for $\nu = 1/3$ [23].

**Discussion and Conclusions.**—In conclusion, we have presented numerical evidence for a class of composite states of spinless fermions exhibiting both Landau and topological order. The eigenvalue spectra of our interacting spinless-fermion model on the triangular lattice at filling fractions $\nu = 12/15 = 4/5$ and $13/15$ hint at ground states that are neither FCI nor CDW, but have features of both. The Landau order, commensurate charge modulation in the ground states, reveals itself in the interaction strength-dependent peaks in the static charge-structure factor. The topological order is established via the Hall conductivity, which is precisely quantized, but with a value $\sigma_H \neq \nu$. Instead, we find a quantization consistent with viewing the ground states as composites of a CDW state and a FCI state formed by additional particles in the part of the lattice that remains unoccupied by the CDW, in some sense similar to the superfluid coexisting with a CDW in a supersolid.

These states with coexisting Landau and topological order mix both bands of the model, with Chern numbers $C = \pm 1$, and are made possible by the geometric frustration of the triangular lattice. The TPL is thus a state that arises out of lattice features that go beyond the single-band picture usually sufficient to describe FCIIs and definitely goes beyond a Landau-level description. In analogy to the supersolid/pinball liquid connection on the triangular lattice, analogous states might be found on other lattices supporting supersolids or pinball liquids, possibly also in multi-orbital settings [50].

This work was supported by the Emmy-Noether program of the Deutsche Forschungsgemeinschaft (DFG). We thank A. Ralko, J. Venderbos, I. Vincon, T. Neupert and P. Kotetes for helpful discussions.

**References**

[1] X.-G. Wen and Q. Niu, Phys. Rev. B 41, 9377 (1990).
[2] T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. 106, 236804 (2011).
[3] E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. 106, 236802 (2010).
[4] K. Sun, Z. Gu, H. Katsura, and S. D. Sarma, Phys. Rev. Lett. 106, 236803 (2010).
[5] D. N. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Nat. Comm. 2, 389 (2011).
[6] N. Regnault and B. A. Bernevig, Phys. Rev. X 1, 021014 (2011).
[7] J. W. F. Venderbos, M. Daghofer, and J. van den Brink, Phys. Rev. Lett. 107, 116401 (2011).
[8] J. W. F. Venderbos, S. Kourtis, J. van den Brink, and M. Daghofer, Phys. Rev. Lett. 108, 126405 (2011).
[9] X.-L. Qi, Phys. Rev. Lett. 107, 126803 (2011).
[10] Y.-L. Wu, N. Regnault, and B. A. Bernevig, Phys. Rev. B 86, 085129 (2012).
[11] C. H. Lee, R. Thomale, and X.-L. Qi, arXiv:1207.5587 (2012).
[12] B. A. Bernevig and N. Regnault, Phys. Rev. B 85, 075128 (2012).
[13] S. A. Parameswaran, R. Roy, and S. L. Sondhi, Phys. Rev. B 86, 241308 (2012).
[14] G. Murthy and R. Shankar, Phys. Rev. B 86, 195146 (2012).
[15] N. R. Cooper and J. Dalibard, Phys. Rev. Lett. 110, 185301 (2013).
[16] N. Y. Yao et al., Phys. Rev. Lett. 110, 185302 (2013).
[17] D. Xiao et al., Nat. Comm. 2, 596 (2011).
[18] P. Ghaemi, J. Cayssol, D. N. Sheng, and A. Vishwanath, Phys. Rev. Lett. 108, 266801 (2012).
[19] Z. Liu et al., Phys. Rev. Lett. 110, 106804 (2013).
[20] S. A. Parameswaran and R. Roy, arXiv:1302.6606v1 (2013).
[21] C. Nayak, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[22] S. Yang, K. Sun, and S. Das Sarma, Phys. Rev. B 85, 205124 (2012).
[23] S. Kourtis, J. W. F. Venderbos, and M. Daghofer, Phys. Rev. B 86, 235118 (2012).
[24] G. S. Khiros and N. d’Ambrumenil, J. Phys.: Condens. Matter 3, 4241 (1991).
[25] A. S. Sorensen, E. Demler, and M. D. Lukin, Phys. Rev. Lett. 94, 086803 (2005).
[26] M. Trescher and E. Bergholtz, Phys. Rev. B 86, 241111 (2012).
[27] A. Läuchli, Z. Liu, E. J. Bergholtz, and R. Moessner, arXiv:1207.6094v1 (2012).
[28] C. Hotta and N. Furukawa, Phys. Rev. B 74, 193107 (2006).
[29] S. Nishimoto, M. Shingai, and Y. Otta, Phys. Rev. B 78, 035113 (2008).
[30] L. Cano-Cortés et al., Phys. Rev. B 84, 155115 (2011).
[31] S. Morohoshi and Y. Fukumoto, J. Phys. Soc. Jpn. 77, 023708 (2008).
[32] M. Boninsegni and N. Prokof’ev, Phys. Rev. Lett. 95, 237204 (2005).
[33] S. Wessel and M. Troyer, Phys. Rev. Lett. 95, 127205 (2005).
[34] R. G. Melko et al., Phys. Rev. Lett. 95, 127207 (2005).
[35] D. Heidarian and K. Damle, Phys. Rev. Lett. 95, 127206 (2005).
[36] M. Boninsegni and N. V. Prokof’ev, Rev. Mod. Phys. 84, 759 (2012).
[37] I. Martin and C. D. Batista, Phys. Rev. Lett. 101, 156402 (2008).
[38] R. Tao and Y. S. Wu, Phys. Rev. B 30, 1097 (1984).
[39] D. J. Thouless, Phys. Rev. B 40, 12034 (1989).
[40] Q. Niu and D. J. Thouless, Phys. Rev. B 31, 3372 (1985).
[41] D. Xiao, M. C. Chang, and Q. Niu, Rev. Mod. Phys. 82, 1959 (2010).
[42] D. N. Sheng et al., Phys. Rev. Lett. 90, 256802 (2003).
[43] X. Wan et al., Phys. Rev. B 72, 1 (2005).
[44] Nevertheless, topologically conjugate ground states are expected to become equivalent in the thermodynamic limit.
[45] C. Chamon and C. Mudry, Phys. Rev. B 86, 195125 (2012).
[46] Changes of parameters that do not lead to a different ground state do not affect the quantization.
[47] A. Kol and N. Read, Phys. Rev. B 48, 8890 (1993).
[48] Additionally, the third-neighbor hopping along $a_3$ was set to 0, the lower band then has a similarly flat dispersion as in the topological case.
[49] J. Merino, A. Ralko, and S. Fratini, arXiv:1304.1700 (2013).
[50] F. Trousselet, A. Ralko, and A. M. Ole´ s, Phys. Rev. B 86, 014432 (2012).
