Entanglement invariant for the double Jaynes-Cummings model

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We study entanglement dynamics between four qubits interacting through two isolated Jaynes-Cummings Hamiltonians, via an entanglement measure based on the wedge product. We compare the results with similar results obtained using bipartite concurrence resulting in what is referred to as “entanglement sudden death”. We find a natural entanglement invariant under evolution demonstrating that entanglement spreads out over all of the system’s degrees of freedom that become entangled through the interaction. We also provide an analysis why certain initial states lose all their entanglement in a finite time although their excitation and coherence only vanishes asymptotically with time.

I. INTRODUCTION

Entanglement plays a key role in quantum information processes and therefore it is important to study entanglement dynamics in different scenarios. The simplest situation, two-qubit entanglement dynamics, has been extensively studied in different contexts. A recent experimental demonstration was presented in and an open systems analysis was done in . Nevertheless, if we include the “reservoirs” in the studied system and consider the full entanglement between two non-interacting partitions of the system, one would expect the entanglement to be preserved, and therefore an associated entanglement invariant should exist. In addition, considering the reservoir as part of closed system until the moment it is ignored (and in mathematical terms “traced over”), one can reap insights into both the qualitative and quantitative transfer of excitation (and associated entanglement) from the atoms to the reservoir.

An important bipartite interaction is described by the Jaynes-Cummings (JC) model that describes, in a concise and elegant way, the near-resonant interaction between a single two-level atom and a single-mode quantized field. If the initial atom-field system contains a single excitation, the system is a model of a two-qubit system. Since the JC model is excitation number preserving, the system will always stay within the two qubit Hilbert space (but of course such a JC model spans only the one-excitation subspace of the full two-qubit space). However, since this model is one of the few exactly solvable models in quantum physics it has been exploited for studying the dynamics of entanglement.

Recently, a double JC model has been proposed in this context. The model consists of two separate JC-model systems (atom A interacting only with the cavity field a and similar for the atom B and the field b), where it has been assumed that the systems are identical. (Note that this model is applicable to any one-excitation, two-qubit system that is linearly coupled, e.g., to the experiment in where fields couple pairwise to each other rather than atoms to fields.) A major reason this particular interaction has been chosen is because it is local to subsystems and . If the atoms, or the fields, couple to each other, the coupling will alter the entanglement between the system partitions and in general, and subsequently, if the fields are traced over, between and . The whole point with these studies, however, is to study the entanglement dynamics between and in absence of any coupling, direct or indirect, between them.

The focus of interest has been the pairwise entanglement dynamics in terms of the concurrence between the initially entangled atoms. Through the JC interaction they may become unentangled through the excitation transfer to the initially unexcited fields which are traced over after the interaction. In Ref. , in particular, the authors study entanglement between the two atoms and they find that for the initial state \( |\psi(0)\rangle = \cos \alpha |\uparrow\rangle + \sin \alpha |\downarrow\rangle \), where we have used the notation \( |\uparrow\rangle \otimes |\downarrow\rangle_B = |\uparrow\uparrow\rangle \), etc. and \( |\downarrow\rangle (|\uparrow\rangle) \) denotes the atom’s excited (ground) state, the concurrence \( C_{AB} \) behave in a harmonic oscillatory manner. Translating this into a dissipation language, the entanglement vanishes asymptotically with increasing coupling to the reservoir. In contrast, the concurrence \( C_{AB} \) of the state \( |\phi(0)\rangle = \cos \alpha |\uparrow\rangle + \sin \alpha |\downarrow\rangle \) for certain values of \( \alpha \), specifically \( \tan \alpha < 1 \), falls rapidly and non-sinusoidally to zero in a finite time and remains zero for some time. In a dissipation language this means that the entanglement will vanish in a finite time although both the atomic excitation and the atomic coherence decay asymptotically.

In order to study the transfer of entanglement between the atom and the reservoir the authors extend the work in and study all the 6 concurrences, \( C_{AB}, C_{Aa}, C_{Bb}, C_{ab}, C_{Ab}, \) and \( C_{Ba} \), for the four-qubit system in . They find that
Therefore, there cannot exist any coherence between the states. An atom transfers its excitation to the initially empty field, it leaves a signature in terms of the excitation in the field. This is not the case, by definition, for the "X"-states, and therefore they will retain their "X" form under the assumed evolution whose form was motivated in the Introduction.

II. THE MODEL

Consider a model consisting of two two-level atoms A, B, each interacting with a single-mode near-resonant cavity field denoted a and b, respectively. Following [8,9], we will assume that each atom-cavity system is isolated and that the cavities are initially in the unexcited state while the atoms are initially in an entangled state.

The dynamics of this model is given by the double JC Hamiltonian

\[ \hat{H}_\text{tot} = \hat{H}_A + \hat{H}_B, \]

where the Hamiltonians (under the rotating wave approximation and setting \( \hbar = 1 \)) are [13],

\[ \hat{H}_k = \nu_k (\hat{a}_k^\dagger \hat{a}_k + 1/2) + \frac{\omega_k}{2} \hat{\sigma}_z^k + g_k (\hat{a}_k^\dagger \hat{\sigma}_+^k + \hat{a}_k \hat{\sigma}_-^k), \]

where \( k = A, B \) (where the letter case is to be interpreted as appropriate), \( \nu_k \) is the field frequency, \( \omega_k \) is the transition frequency between the atomic excited and ground states, and \( g_k \) is the coupling constant between the cavity field and the atom. The field annihilation operators are \( \hat{a}_k \), and \( \hat{\sigma}_z^k \) are the spin-flip operators defined by \( \hat{\sigma}_z^k |\uparrow\rangle_k = |\downarrow\rangle_k \), \( \hat{\sigma}_z^k |\downarrow\rangle_k = |\uparrow\rangle_k \), and \( \hat{\sigma}_z^k \) is the atomic inversion operator, viz., \( \hat{\sigma}_z^k |\uparrow\rangle_k = |\downarrow\rangle_k \) and \( \hat{\sigma}_z^k |\downarrow\rangle_k = |\uparrow\rangle_k \).

As mentioned in the Introduction, if we have at most one excitation in each atom-cavity system, each such system will stay within a two-qubit space. Hence, since the two atom-cavity systems don’t interact, the double JC model will result in a four qubit state (but again, spanning only a subspace of the whole four-qubit Hilbert space).

The corresponding evolution operator for the Hamiltonian (2) is

\[ \hat{U}_k = e^{-i \hat{H}_k^0 / \hbar} \left\{ \cos(\hat{\Omega}_k t) \right. \]

\[ \left. -it \sin(\hat{\Omega}_k t) \left[ \frac{\Delta_k}{2} \hat{\sigma}_z^k + g_k (\hat{a}_k^\dagger \hat{\sigma}_+^k + \hat{a}_k \hat{\sigma}_-^k) \right] \right\}, \]

where \( \hat{H}_k^0 = \nu_k [\hat{a}_k^\dagger \hat{a}_k + (1 + \hat{\sigma}_z^k)/2] \) is a constant of motion, proportional to the total number of excitations of system \( k \), \( \Delta_k = \omega_k - \nu_k \) is the detuning between the atom and the cavity for each system, \( \sin(x) = x - x^3/3 + \cdots \), and

\[ \hat{\Omega}_k = \left\{ g_k^2 [\hat{a}_k^\dagger \hat{a}_k + (1 + \hat{\sigma}_z^k)/2] + \Delta_k^2/4 \right\}^{1/2}. \]

We will consider that the atoms are initially in an “X”-state, characterized by a (reduced) atom density operator whose non-zero elements are found only in the main diagonal and antidiagonal in the basis \( |\uparrow\rangle_k, |\downarrow\rangle_k, |\downarrow\rangle_k \), and \( |\downarrow\rangle_k \). This class of atom states has the property that the corresponding two-qubit density matrix preserves the “X” form when evolving under the action of certain system dynamics [8,9]. In this case, the reason is simple. When an atom transfers its excitation to the initially empty field, it leaves a signature in terms of the excitation in the field. Therefore, there cannot exist any coherence between the states \( |\uparrow\rangle_k \rangle \) and the states \( |\downarrow\rangle_k \rangle \rangle \rangle \rangle unless such coherence existed initially. This is not the case, by definition, for the “X”-states, and therefore they will retain their “X” form under the assumed evolution whose form was motivated in the Introduction.
As pointed out in [3], the “X”-class of states include the Bell states and the Werner states. Following [8, 9] we will focus on the Bell-like pure states,

\[ |\phi(0)\rangle = \cos \alpha |↑↑\rangle + \sin \alpha e^{i\beta} |↓↓\rangle, \]
\[ |\psi(0)\rangle = \cos \alpha |↓↓\rangle + \sin \alpha e^{i\beta} |↑↑\rangle, \]

where \(0 \leq \alpha \leq \pi/2, 0 \leq \beta \leq \pi\). (In Sec. V we will consider more general states.) As motivated above, we will assume that the initial state for the four qubit model is

\[ |\Phi(0)\rangle = |\phi(0)\rangle \otimes |00\rangle, \quad |\Psi(0)\rangle = |\psi(0)\rangle \otimes |00\rangle, \]  

(4)

respectively, where the abbreviated notation \(|0\rangle_a \otimes |0\rangle_b = |00\rangle\) has been used. Notice that Bell states can be recovered by setting \(\alpha = \pi/4\) and \(\beta = 0, \pi/2\). The initial states \([3]\) under the action of the operator \(\hat{U}_A \otimes \hat{U}_B\) evolve as

\[ |\Phi(t)\rangle = x_1 |↑↑00\rangle + x_2 |↓↑01\rangle + x_3 |↓↓10\rangle + x_4 |↑↓00\rangle, \]
\[ |\Psi(t)\rangle = y_1 |↑↑00\rangle + y_2 |↓↓00\rangle + y_3 |↓↑00\rangle + y_4 |↑↓01\rangle, \]  

(5)

(6)

where the coefficients for the state \([5]\) are given by

\[ x_1 = f_A(t)f_B(t) \cos \alpha, \]
\[ x_2 = f_A(t)g_B(t) \cos \alpha, \]
\[ x_3 = g_A(t)f_B(t) \cos \alpha, \]
\[ x_4 = g_A(t)g_B(t) \cos \alpha, \]
\[ x_5 = h_A(t)h_B(t) e^{i\beta} \sin \alpha. \]  

(7)

(8)

(9)

(10)

(11)

The functions \(f_k(t), g_k(t),\) and \(h_k(t)\) are given by

\[ f_k(t) = e^{-i\nu_k t} \left[ \cos(\Omega_k t) - i \frac{\Delta_k}{2\Omega_k} \sin(\Omega_k t) \right], \]
\[ g_k(t) = -i \frac{g_k}{\Omega_k} e^{-i\nu_k t} \sin(\Omega_k t), \]
\[ h_k(t) = e^{i\Delta_k t/2}, \]  

(12)

(13)

(14)

where the Rabi frequencies are \(\Omega_k = (g_k^2 + \Delta_k^2/4)^{1/2}\).

Similarly, the state \([6]\) will have the the coefficients

\[ y_1 = f_A(t)h_B(t) \cos \alpha, \]
\[ y_2 = h_A(t)f_B(t) e^{i\beta} \sin \alpha, \]
\[ y_3 = g_A(t)h_B(t) \cos \alpha, \]
\[ y_4 = h_A(t)g_B(t) e^{i\beta} \sin \alpha. \]  

(15)

(16)

(17)

(18)

### III. ENTANGLEMENT DYNAMICS

In this section we will analyze the entanglement evolution in the double JC model. As an entanglement measure we will use a wedge-product based measure introduced in [17], which, for the two-qubit case, coincides with the well-known concurrence [15], and in the multiqubit case with the entanglement monotones [18]. This measure is defined for any number of subsystems, each having an arbitrary, but finite, dimension.

By partitioning the total system into two we can compute the entanglement between these partitions. Consider some partition composed by \(P_1\) with dimension \(M\) and \(P_2\) with dimension \(N\) (note that each partition could contain more than one physical subsystem). Assume a pure system defined by

\[ |\psi\rangle = \sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_{mn} |m\rangle \otimes |n\rangle, \]  

(19)
vanish, the entanglement between these two vectors. The square of the measure introduced in [17] can hence be written as the determinant 

\[ A^2(m, l) = \left| \begin{array}{cc} \langle \psi_m | \psi'_m \rangle & \langle \psi_m | \psi_l \rangle \\ \langle \psi_l | \psi_m \rangle & \langle \psi_l | \psi_l \rangle \end{array} \right| . \]

Summing all contributions and using symmetry and the fact that the wedge product between a vector and itself vanish, the entanglement between \( P_1 \) and \( P_2 \) can finally be defined

\[ E_{P_1-P_2} = \frac{1}{2} \sum_{m=1}^{M} \sum_{l=1}^{M} A^2(m, l). \]  

(20)

In the case of the double-JC model, the possible partitions are: (a) one qubit - three qubit partitions, \( A - Bab, \) \( B - Aab, \) \( a - ABb, \) and \( b - ABa; \) (b) two qubit - two qubit partitions, \( Aa - Bb, Ab - Ba, \) and \( AB - ab. \)

In Figs. 1 and 2 we plot the evolution of the concurrence \( C_{AB} \) between the two atoms \( A \) and \( B, \) and the evolution of \( 4E_{Aa-Bb} \) (since \( 0 \leq C_{AB} \leq 1, \) we use \( 4E_{Aa-Bb} \) to scale it to essentially the same range of values as \( C \) may obtain) in the case of exact resonance \( (\Delta_A = \Delta_B = 0). \) Note that while \( C_{AB} \) represent only the remaining entanglement between the two atoms after the field states have been traced out, \( E_{Aa-Bb} \) represent the entire bipartite entanglement between the atom-field system \( Aa \) and \( Bb. \)

First we consider different values of \( \alpha. \) In Fig. 1(a), corresponding to \( g_A = g_B, \) we can observe that concurrence for the initial state \( |\Psi(0)\rangle \) evolve in a typical oscillating way between 0 and 1 [8][12], meanwhile \( E_{Aa-Bb} = \sin^2 \alpha \cos^2 \alpha \) is invariant and equals 1/4. In Fig. 1(b) we show the time evolution of the concurrence between the atoms, and that of \( 4E_{Aa-Bb}, \) for \( g_B/g_A = 2. \) When \( g_A \neq g_B \) the concurrence is not evolving in a typical oscillatory manner as was pointed out in [12]. Meanwhile, \( E_{Aa-Bb} \) depends only on the initial state, namely on \( \alpha, \) and not on the ratio \( g_A/g_B. \)

Fig. 2 shows the evolution of the concurrence and \( 4E_{Aa-Bb} \) for the state \( |\Phi(0)\rangle \) when both cavity-atom systems are in exact resonance. In Fig. 2(a) we can observe the so-called entanglement sudden death for different values of \( \alpha. \)

FIG. 1: (a) The concurrence \( C_{AB} \) between the atoms in state \( |\Psi(0)\rangle \) as a function of the coupling parameter \( \Omega_{Aa} \) when \( g_A/g_B = 1 \) for \( \alpha = \pi/4 \) (dashed line), \( \alpha = \pi/6 \) (dash-dotted line); and \( 4E_{Aa-Bb} \) for \( \alpha = \pi/4 \) (solid line) and \( \alpha = \pi/6 \) (dotted line). (b) The concurrence when \( g_B/g_A = 2 \) for the parameter values \( \alpha = \pi/4 \) (dashed line) and \( \alpha = \pi/6 \) (dash-dotted line); and \( 4E_{Aa-Bb} \) for \( \alpha = \pi/4 \) (solid line) and \( \alpha = \pi/6 \) (dotted line).
the states \( |\uparrow\uparrow\rangle \) established. When the excitation of these intermediate states is large compared to the remaining coherence between \( \alpha < \alpha < \text{as pointed out in } [6, 10] \). If \( \cos \alpha < \alpha \), different “signatures” in the reservoirs (the states \( |\downarrow\downarrow \rangle \)), has decayed. The state \( |\downarrow\uparrow\rangle \) and the ground state cannot be written as a convex sum of any separable states no matter to what extent the state \( |\downarrow\downarrow \rangle \) and \( |\downarrow\uparrow\rangle \) are traced out) become zero in a time \( \tau < \pi/2 \) are transferred to the fields in a monotonic fashion during the time interval where \( 0 < \Omega_k t \leq \pi/2 \). In the resonant case (\( \Delta_A = \Delta_B = 0 \Rightarrow \Omega_k = g_k \)), all the excitation will be transferred from the atoms to the cavity fields. Formally, during this time interval, one can then see the fields as a dissipative channel for the atoms’ excitation. On can subsequently map a dissipative evolution, e.g., spontaneous emission of the atom’s excitation obeying an exponential decay \( \propto \exp(-\gamma_k t') \), onto the JC dynamics by the identification between the times \( t \) and \( t' \): \( \exp(-\gamma_k t') = \cos^2(\Omega_k t) \). Quite obviously, \( \Omega_k t \rightarrow \pi/2 \) correspond to \( t' \rightarrow \infty \). Hence, if the entanglement between the atoms (after the fields are traced out) become zero in a time \( \tau < \pi/(2\Omega_k) \), then in the dissipative picture it vanishes in a finite time \( t' = \gamma^{-1} \ln(\cos^2(\Omega_k \tau)) \). This is indeed what happens for the state \( |\phi(0)\rangle \) as it evolves.

One may then ask why one state’s entanglement vanishes in finite time while the others’ does not. The reason is the fundamentally different way the states decay. The state \( |\psi(0)\rangle \) decays directly into the ground state \( |\downarrow\downarrow\rangle \). Whatever excitation is left in the atoms will still be in a superposition state, and such a statistical mixture between a Bell state and the ground state cannot be written as a convex sum of any separable states no matter to what extent the state has decayed. The state \( |\phi(0)\rangle \) decays to the ground state via the intermediate states \( |\uparrow\downarrow\rangle \) and \( |\downarrow\downarrow\rangle \). As the decay leaves different “signatures” in the reservoirs (the states \( |01\rangle \) and \( |10\rangle \), respectively), no coherence between these states is established. When the excitation of these intermediate states is large compared to the remaining coherence between the states \( |\uparrow\downarrow\rangle \) and \( |\downarrow\downarrow\rangle \), the state can be written as a convex combination of separable states so the state is no longer entangled. This happens when

\[
\tan \alpha < \sin^2(\Omega_A t) = 1 - \exp(-\gamma_k t')
\]

as pointed out in [6, 10]. If \( \cos \alpha < \sin \alpha \), the atomic excitation is insufficient to excite the intermediate states \( |\uparrow\downarrow\rangle \) and \( |\downarrow\downarrow\rangle \) to the extent that the entanglement between the atoms vanishes in a finite time.

If the coupling constants \( g_A \) and \( g_B \) are different, say that \( g_A > g_B \) as in Fig. 1 (b) and 2 (b), the just presented dissipative picture is only valid as long as \( 0 \leq \Omega_A t \leq \pi/2 \), where \( \Omega_k = g_k \) when the atoms and cavities are resonant. For times longer than \( \pi/(2g_A) \) the excitation of atom \( A \) starts to revive again in the JC model, a phenomenon not compatible with dissipation. However, as seen from the figures, the behavior of the states for times \( \Omega_A t < \pi/2 \Leftrightarrow \Omega_B t < \pi/4 \approx 0.79 \) is qualitatively the same as in the symmetric \( g_A = g_B \) case.

IV. DISSIPATIVE DYNAMICS

While the model presented above is closed and does not involve any dissipation, the atoms’ evolution under dissipation can still be described. As can be seen from Eqs. (7)-(10), (12), (13), and (15)-(18) the excitation of the atoms is transferred to the fields in a monotonic fashion during the time interval where \( 0 \leq \Omega_k t \leq \pi/2 \). In the resonant case (\( \Delta_A = \Delta_B = 0 \Rightarrow \Omega_k = g_k \)), all the excitation will be transferred from the atoms to the cavity fields. Formally, during this time interval, one can then see the fields as a dissipative channel for the atoms’ excitation. On can subsequently map a dissipative evolution, e.g., spontaneous emission of the atom’s excitation obeying an exponential decay \( \propto \exp(-\gamma_k t') \), onto the JC dynamics by the identification between the times \( t \) and \( t' \): \( \exp(-\gamma_k t') = \cos^2(\Omega_k t) \). Quite obviously, \( \Omega_k t \rightarrow \pi/2 \) correspond to \( t' \rightarrow \infty \). Hence, if the entanglement between the atoms (after the fields are traced out) become zero in a time \( \tau < \pi/(2\Omega_k) \), then in the dissipative picture it vanishes in a finite time \( t' = \gamma^{-1} \ln(\cos^2(\Omega_k \tau)) \). This is indeed what happens for the state \( |\phi(0)\rangle \) as it evolves.

One may then ask why one state’s entanglement vanishes in finite time while the others’ does not. The reason is the fundamentally different way the states decay. The state \( |\psi(0)\rangle \) decays directly into the ground state \( |\downarrow\downarrow\rangle \). Whatever excitation is left in the atoms will still be in a superposition state, and such a statistical mixture between a Bell state and the ground state cannot be written as a convex sum of any separable states no matter to what extent the state has decayed. The state \( |\phi(0)\rangle \) decays to the ground state via the intermediate states \( |\uparrow\downarrow\rangle \) and \( |\downarrow\downarrow\rangle \). As the decay leaves different “signatures” in the reservoirs (the states \( |01\rangle \) and \( |10\rangle \), respectively), no coherence between these states is established. When the excitation of these intermediate states is large compared to the remaining coherence between the states \( |\uparrow\downarrow\rangle \) and \( |\downarrow\downarrow\rangle \), the state can be written as a convex combination of separable states so the state is no longer entangled. This happens when

\[
\tan \alpha < \sin^2(\Omega_A t) = 1 - \exp(-\gamma_k t')
\]
V. AN ENTANGLEMENT INVARIANT

The form of the chosen interaction Hamiltonian, which does not include any interaction between subsystems $Aa$ and $Bb$, ensures that no entanglement is formed between these subsystems that was not already present in the initial state. Therefore, it is reasonable to expect an entanglement invariant to exist that measures the net entanglement between these subsystems. As Fig.4 and Fig.2 suggest, we can find the invariant

$$\mathcal{E} = E_{Aa-Bb}$$

(22)

valid for all parameter values, that is, even for nonresonant coupling and different atom-cavity coupling ratios.

The introduced measure (22) does not depend on time for the initial pure “X”-states (5) and (6). The value of $\mathcal{E}$ in both cases is

$$\mathcal{E} = \sin^2 \alpha \cos^2 \alpha.$$  

(23)

This value is proportional to the square of the two atoms’ initial concurrence $C$ [15]:

$$4\mathcal{E} = C^2.$$  

The result is expected, because as the Hamiltonian is chosen not to change the entanglement between $Aa - Bb$, it must remain equal to its initial value at all times. What is more significant is that for the initial generic state

$$|\xi_g⟩ = c_1 |↑⟩_0 + c_2 |↓⟩_0 + c_3 |↑↑⟩ + c_4 |↑↓⟩ + c_5 |↓↑⟩ + c_6 |↓↓⟩,$$

(24)

we find that (22) is still invariant during evolution under the action of $U_A ⊗ U_B$. Explicitly,

$$\mathcal{E}(|\xi_g⟩) = |c_1 c_4 - c_2 c_3|^2 + |c_1 d_3 - c_2 d_1|^2 + |c_3 d_3 - c_4 d_1|^2 + |c_1 d_4 - c_2 d_2|^2 + |c_2 d_4 - c_4 d_2|^2 + |c_1 c_5 - d_1 d_2|^2 + |c_2 c_5 - d_2 d_1|^2 + |c_3 c_5 - d_3 d_1|^2.$$  

(25)

It is worth noticing that each cavity-atom in the state $|\xi_g⟩$ will evolve in the subspace $\{|↑⟩_0, |↓⟩_0, |↑⟩_1, |↓⟩_1\}$ under the double JC Hamiltonian. Hence, the state (24) can be seen as a two-qutrit, pure state. In this case equation (25) is just one ninth of the square of the two-qutrit concurrence introduced in [19].

It should also be noted that in the dissipative picture, the state $|\xi_g⟩$ models coupling to excited reservoirs, and in general the states cannot, at any time, be written as a product state between the atoms and the fields. Hence, some entanglement between atom and fields is already there at the start of the evolution. The states $|Φ(0)⟩$ and $|Ψ(0)⟩$ are special cases of $|\xi_g⟩$ where the atoms couple to initially empty reservoirs, a relevant but special case.

When we study other partitions in (20), we have seen that for both states, any single term is zero only at discrete times. This means that at all times, except a discrete set of times of zero measure, all parts of the system become entangled in some degree through excitation transfer. This phenomenon is generic for entangled systems and has, e.g., been used to entangle subsystems that have never interacted through so-called entanglement swapping [20].

In a dissipative system, this entanglement spread is of course detrimental and may lead to complete elimination of entanglement.

VI. DISCUSSION

In this work we have discussed the entanglement dynamics for two excited atoms coupled to cavity field-modes through a Jaynes-Cummings Hamiltonian. We have discussed both the closed system dynamics and the dynamics if the cavities are viewed as reservoirs. We have discussed why different initial atomic states face different fates (asymptotic vs. sudden decay) with respect of their entanglement under dissipation. We have also shown that as expected, there exist an entanglement invariant valid for a large class of states, much larger than the class of so-called “X”-states in the closed system. When treating the fields as reservoirs (i.e., when tracing over the fields), some of the entanglement transferred to the cavities is ignored. The state $|φ(0)⟩$ transfers its excitation over a larger set of distinguishable field states ($|01⟩$, $|10⟩$, and $|11⟩$) than the state $|ψ(0)⟩$ (that excites only the field states $|01⟩$ and $|10⟩$), and therefore is is not so surprising that the former state may lose all of its entanglement through even a finite dissipation.

Since the proposed (bipartite) invariant is a measure of the entanglement between the two systems $Aa$ and $Bb$, and the measure is invariant to local unitary transformations, it will be a constant for any pure state, not only for the generic state (24), but also for states with higher excitation.
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