The Cauchy Problem for Stochastic Generalized Benjamin-Ono Equation

Wei YAN†

†School of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, P. R. China
Email: yanwei19821115@sina.cn

Jianhua Huang

* College of Science, National University of Defense and Technology, Changsha, P. R. China 410073
Email: jhhuang32@nudt.edu.cn

and

Boling Guo

‡Institute of Applied Physics and Computational Mathematics, Beijing, 100088
Email: gbl@iapcm.ac.cn

Abstract. The current paper is devoted to the Cauchy problem for the stochastic generalized Benjamin-Ono equation. By establishing the bilinear estimate, trilinear estimates in some Bourgain spaces, we prove that the Cauchy problem for the stochastic generalized Benjamin-Ono equation is locally well-posed for the initial data $u_0(x, \omega) \in L^2(\Omega; H^s(\mathbb{R}))$ which is $\mathcal{F}_0$ measurable with $s \geq \frac{1}{2} - \frac{\alpha}{4}$ and $\Phi \in L^{0,s}_2$. In particular, when $\alpha = 1$, we prove that it is globally well-posed for the initial data $u_0(x, \omega) \in L^2(\Omega; H^1(\mathbb{R}))$ which is $\mathcal{F}_0$ measurable and $\Phi \in L^{0,1}_2$. The key ingredients that we use in this paper are trilinear estimates, Itô formula and the BDG inequality as well as the stopping time technique.

Keywords: Cauchy problem; Stochastic generalized Benjamin-Ono equation

Short Title: Stochastic Generalized Benjamin-Ono equation equation

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1. Introduction

In this paper, we consider the following stochastic fractional Benjamin-Ono type equation

\[
\begin{aligned}
\begin{cases}
du(t) &= [|\partial_x|^{\alpha+1} \partial_x u(t) - \frac{1}{k} \partial_x (u^k)]dt + \Phi dW(t), \\
u(0) &= u_0,
\end{cases}
\end{aligned}
\]

(1.1)

where \( W(t) = \frac{\partial B}{\partial x} = \sum_{j=1}^{\infty} \beta_j e_j, \) \( e_j \) is an orthonormal basis of \( L^2(\mathbb{R}) \) and \( (\beta_j)_{j \in \mathbb{N}} \) is a sequence of mutually independent real Brownian motions in a fixed probability space and is a Wiener process on \( L^2(\mathbb{R}) \). In fact, (1.1) is equivalent to the following equations:

\[
\begin{aligned}
\begin{cases}
du(t) &= [|\partial_x|^{\alpha+1} \partial_x u(t) - \frac{1}{k} \partial_x (u^k)]dt + \Phi \frac{dW(t)}{dt}, \\
u(0) &= u_0,
\end{cases}
\end{aligned}
\]

(1.2)

(1.2) is considered as the Benjamin-Ono type equation

\[
\begin{aligned}
\begin{cases}
du(t) &= [|\partial_x|^{\alpha+1} \partial_x u(t) - \frac{1}{k} \partial_x (u^k)], \\
u(0) &= u_0.
\end{cases}
\end{aligned}
\]

forced by a random term \( \Phi \frac{du(t)}{dt} \).

When \( \alpha = 1 \) and \( k = 2 \), (1.3) reduces to the KdV equation which has been investigated by many authors, we refer the readers to [4, 6–9, 12, 14, 20–24, 28]. The result of [23] and [24] implies that \( s = -\frac{3}{4} \) is the critical well-posedness indices of the Cauchy problem for the KdV equation. Guo [14] and Kishimoto [28] almost proved that the KdV equation is globally well-posed in \( H^{-3/4} \) with the aid of \( I \)-method and the dyadic bilinear estimates at the same time. When \( \alpha = 1 \) and \( k = 2 \), (1.2) reduces to the stochastic KdV equation which has been studied by some people, we refer the readers to [1–3]. Recently, motivated by [2], Chen et al. [5] studied the Cauchy problem for the stochastic Camassa-Holm equation.

When \( \alpha = 0 \) and \( k = 2 \), (1.3) reduces to the Benjamin-Ono equation which has been studied by many people, we refer the readers to [29, 30, 32–36, 44]. By using the gauge transformation introduced by [44] and a new bilinear estimate, Ionescu and Kenig [25] proved that the Benjamin-Ono equation is globally well-posed in \( H^s(\mathbb{R}) \) with \( s \geq 0 \).

When \( 0 < \alpha < 1 \) and \( k = 2 \), (1.3) has been investigated by some people, we refer the readers to [10, 11, 16, 17, 20]. In [17], the author proved that (1.3) is locally well-posed in \( H^{(s,a)} \), \( a = \frac{1}{\alpha+1} - \frac{1}{2}, s > -\frac{3a}{4} \) and globally well-posed in \( H^{(0,a)} \), \( a = \frac{1}{\alpha+1} - \frac{1}{2} \). Recently, by using a frequency dependent renormalization method, Herr et al. [18] proved that (1.3) is globally well-posed in \( L^2 \) if \( 0 < \alpha < 1 \) and \( k = 2 \). Very recently, Guo [15] proved that (1.3) is locally well-posed in \( H^s \) with \( s \geq 1 - \alpha \) if \( 0 \leq \alpha \leq 1 \) with \( k = 2 \) and in \( H^s \) with \( s \geq \frac{1}{2} - \frac{\alpha}{4} \), \( k = 3 \).

When \( \alpha = 1 \) and \( k = 3 \), (1.3) reduces to the mKdV equation which has been investigated by many authors, for instance, see [9, 13, 14, 21, 22, 24, 28, 37, 38, 45] and the references therein. In [29], by using the inverse scattering method, Koch and Tzvetkov proved that the Cauchy problem for the mKdV equation is locally well-posed on \( T \) in \( H^s \) with \( s \geq 0 \). In [45], Takaoka and Tsutsumi proved that the Cauchy problem for the mKdV possesses a unique solution on \( T \) in \( H^s \) with \( \frac{3}{8} < s < \frac{1}{2} \). By using the modified Fourier restriction norm method, Nakanishi et al. [38] proved that the Cauchy problem for the mKdV on \( T \) in \( H^s \) with \( s > \frac{1}{3} \) is locally well-posed and is locally well-posed in \( H^s \) with \( s > \frac{1}{4} \) with the help of the additional assumption on initial data. Recently, Molinet [31] proved that
the solution-maps associated with the mKdV equation is discontinuous for the $H^s$ topology for $s < 0$. Soonsik and Oh [43] studied the unconditional well-posedness of mKV equation. By using the Itô formula, BDG inequality and the conserved laws of the KdV equation, de Bouard and Debussche [1] studied the existence of and uniqueness of solutions to the Cauchy problem for the Stochastic KdV in $H^1(R)$ in the case of additive noise and existence of martingale solutions in $L^2(R)$ in the case of multiplicative noise with the aid of Strichartz estimates and Itô formula as well as BDG inequality. de Bouard et al. [2] obtained the existence of the solution to the stochastic KdV in $L^2$ with the aid of the modified Bourgain spaces.

In this paper, inspired by [1, 2], we focus on the case $0 < \alpha \leq 1$ and $k = 3$ of (1.1). By using the Sobolev spaces and the Bourgain spaces, we proved that (1.1) is locally well-posed for the initial data $u_0(x, w) \in L^2(\Omega; H^s(R))$ with $s \geq \frac{1}{2} - \frac{\alpha}{4}$, where $0 < \alpha \leq 1$. In particular, when $\alpha = 1$, we prove that it is globally well-posed for the initial data $u_0(x, w) \in L^2(\Omega; H^1(R))$. Compared to the deterministic KdV and Benjamin-Ono equation, the structure of stochastic Benjamin-Ono equation is more complicated. The perturbation of the noise destroyed the structure of original structure of Benjamin-Ono. More precisely, Lemma 2.6 requires $0 < b < \frac{1}{2}$. By using the idea of [46], we firstly establish the bilinear estimate, then, apply the bilinear estimate which is just Theorem 3.1 to establish the trilinear estimate which are Lemmas 4.1-4.2, thus, we need to use Lemmas 2.2, 2.3 which are not used in the deterministic KdV and Benjamin-Ono to establish bilinear and trilinear estimates. Then, the trilinear estimate in combination with the fixed point argument yields Theorem 1.1. For the Theorem 1.2, we use the frequency truncated technique rather than the method of [2].

We give some notations before giving the main result. We denote $X \sim Y$ by $A_1|X| \leq |Y| \leq A_2|X|$, where $A_j > 0$ ($j = 1, 2$) and denote $X \gg Y$ by $|X| > C|Y|$, where $C$ is some positive number which is larger than 2. $\langle \xi \rangle^s = (1 + \xi^2)^{\frac{s}{2}}$ for any $\xi \in \mathbb{R}$, and $\mathcal{F}u$ denotes the Fourier transformation of $u$ with respect to its all variables. $\mathcal{F}^{-1}u$ denotes the Fourier inverse transformation of $u$ with respect to its all variables. $\mathcal{F}_xu$ denotes the Fourier transformation of $u$ with respect to its space variable. $\mathcal{F}_x^{-1}u$ denotes the Fourier inverse transformation of $u$ with respect to its space variable. $H^s(\mathbb{R})$ is the Sobolev space with norm $\|f\|_{H^s(\mathbb{R})} = \|\langle \xi \rangle^s \mathcal{F}_x f\|_{L^2(\mathbb{R})}$. For any $s, b \in \mathbb{R}$, $X_{s, b}(\mathbb{R}^2)$ is the Bourgain space with phase function $\phi(\xi) = \xi|\xi|^{1+\alpha}$. That is, a function $u(x, t)$ belongs to $X_{s, b}(\mathbb{R}^2)$ iff

$$\|u\|_{X_{s, b}(\mathbb{R}^2)} = \|\langle \xi \rangle^s (\tau - \xi)|\xi|^{\alpha+1}b \mathcal{F}u(\xi, \tau)\|_{L^2(\mathbb{R})L^2(\mathbb{R})} < \infty.$$  

For any given interval $L$, $X_{s, b}(\mathbb{R} \times L)$ is the space of the restriction of all functions in $X_{s, b}(\mathbb{R}^2)$ on $\mathbb{R} \times L$, and for $u \in X_{s, b}(\mathbb{R} \times L)$ its norm is

$$\|u\|_{X_{s, b}(\mathbb{R} \times L)} = \inf\{\|U\|_{X_{s, b}(\mathbb{R}^2)}; U|_{\mathbb{R} \times L} = u\}.$$  

When $L = [0, T]$, $X_{s, b}(\mathbb{R} \times L)$ is abbreviated as $X_{s, b}^T$. Throughout this paper, we always assume that $w(\xi) = \xi|\xi|^{\alpha+1}$, $\psi$ is a smooth function, $\psi_b(t) = \psi(\frac{t}{b})$, satisfying $0 \leq \psi \leq 1$, $\psi = 1$ when $t \in [0, 1]$, supp$\psi \subset [-1, 2]$ and $\sigma = \tau - \xi|\xi|^{\alpha+1}$, $\sigma_k = \tau_k - \xi_k|\xi_k|^{\alpha+1}$ ($k = 1, 2)$,

$$U(t)u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i(x\xi - \xi|\xi|^{\alpha+1})} \mathcal{F}_x u_0(\xi) d\xi,$$

$$\|f\|_{L^q_tL^p_x} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}$$

$$\|f\|_{L^q_tL^p_x} = \|f\|_{L^q_tL^p_x}.$$
We assume that $B(x,t)$, $t \geq 0, x \in \mathbb{R}$, is a zero mean gaussian process whose covariance function is given by

$$E(B(t,x)B(s,y)) = (t \wedge s)(x \wedge y)$$

for $t, s \geq 0, x, y \in \mathbb{R}$. $(.,.)$ denotes the $L^2$ space duality product, i.e., $(f,g) = \int_{\mathbb{R}} f(x)g(x)dx$. $(\Omega, \mathcal{F}, P)$ is a probability space endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$. $\hat{E}f = \int_{\Omega} fdP$. $W(t)$ is a cylindrical Wiener process $(W(t))_{t \geq 0}$ on $L^2(\mathbb{R})$ associated with the filtration $(\mathcal{F})_{t \geq 0}$. For any orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $L^2(\mathbb{R})$, $W = \sum_{k=0}^{\infty} \beta_k e_k$ for a sequence $(\beta_k)_{k \in \mathbb{N}}$ of real, mutually independent brownian motions on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})$. Let $H$ be a Hilbert space, $L^0_2(L^2(\mathbb{R}), H)$ the space of Hilbert-Schmidt operators from $L^2(\mathbb{R})$ into $H$. Its norm is given by $\|\Phi\|_{L^0_2(L^2(\mathbb{R}), H)}^2 = \sum_{j \in \mathbb{N}} |\Phi e_j|^2_H$. When $H = H^s(\mathbb{R})$, $L^0_2(L^2(\mathbb{R}), H^s(\mathbb{R})) = L^0_{2,s}$.

The main results of this paper are as follows:

**Theorem 1.1.** Let $u_0(x,\omega) \in L^2(\Omega; H^s(\mathbb{R}))$ with $s \geq \frac{1}{2} - \frac{\alpha}{4}$ and $\Phi \in L^0_{2,s}$ and $u_0$ be $\mathcal{F}_0$ measurable. Then, for a.e. $\omega \in \Omega$, there exists a $T_\omega > 0$ and a unique solution of the Cauchy problem for (1.1) on $[0, T_\omega]$ satisfying

$$u \in C([0, T_\omega]; H^s(\mathbb{R})) \cap X^{T_\omega}_{s,b}.$$  

**Theorem 1.2.** Let $\alpha = 1$, $u_0(x,\omega) \in L^2(\Omega; H^1(\mathbb{R}))$ and $\Phi \in L^0_{2,1}$ and $u_0$ and $\mathcal{F}_0$ be measurable. Then the solution to the Cauchy problem for (1.1) global and belongs to

$$L^2(\Omega; C([0, T_0]; H^1(\mathbb{R}))$$

for any $T_0 > 0$.

The rest of the paper is organized as follows. In Section 2, some key interpolation inequalities and preliminary estimates are established. In the Section 3, we establish bilinear estimate with the aid of Fourier restriction norm method. In Section 4, we will show the trilinear estimate. In section 5, we prove Theorem 1.1. In section 6, we prove Theorem 1.2.

### 2. Preliminaries

In this section, we give some preliminaries which plays the crucial role in establishing the main theorems.

**Lemma 2.1.** Let $\theta \in [0, 1]$, $\gamma > 0$ and $U_\gamma(t)u_0(x) = \int_{\mathbb{R}} e^{i(t\phi(x) + x\xi)} |\phi''(\xi)|^{\frac{2}{3}} \mathcal{F}_x u_0(\xi) d\xi$. Then

$$\|U_\gamma^2(t)u_0\|_{L^p_t L^q_x} \leq C\|u_0\|_{L^2_x},$$

where $(p, q) = \left(\frac{2}{1-\gamma}, \frac{4}{\gamma}\right)$.

For the proof of Lemma 2.1, we refer the readers to Theorem 2.1 of [21].

**Lemma 2.2.** Let $b = \frac{1}{2} + \epsilon$, $0 < \epsilon \ll 1$, then

$$\|u\|_{L^4_t} \leq C\|u\|_{X_{0, \frac{\alpha+3}{2(\alpha+2)+\frac{1}{2}+\epsilon}}}$$

and

$$\left\|D^\alpha_x u\right\|_{L^6_x} \leq C\|u\|_{X_{0, \frac{\alpha}{4}b}}.$$  




Proof. Let $\theta = \frac{2}{3}$, it follows from Lemma 2.1 that
\[
\left\| \int_{-\infty}^{\infty} e^{i\theta(x+ix)} |\phi''(x)|\mathcal{F}_x u_0(x) d\xi \right\|_{L^2_{x,t}} \leq C \left\| u_0 \right\|_{L^2_t},
\]
where $|\phi| = |\xi|^\alpha + 1$, $|\phi''| = c|\xi|^\alpha$, then
\[
\left\| \int_{-\infty}^{\infty} e^{i\theta(x+ix)} |\xi|^\alpha \mathcal{F}_x u_0(x) d\xi \right\|_{L^6_{x,t}} \leq C \left\| u_0 \right\|_{L^2_t}.
\]
Due to $\|f\|_{L^2_{x,t}} \leq C \|D_2^\gamma D_t^\theta f\|_{L^6_{x,t}}$, where $\gamma = \frac{\alpha}{6(\alpha+3)}$. Then
\[
\|U(t)u_0(x)\|_{L^{2\alpha+6}_{x,t}} = C \left\| \int_{-\infty}^{\infty} e^{i(t\phi+x\xi)} \mathcal{F}_x u_0(x) d\xi \right\|_{L^{2\alpha+6}_{x,t}}
\leq C \left\| D_x^\gamma D_t^\theta \int_{-\infty}^{\infty} e^{i(t\phi+x\xi)} \mathcal{F}_x u_0(x) d\xi \right\|_{L^6_{x,t}}
\leq C \left\| \int_{-\infty}^{\infty} e^{i(t\phi+x\xi)} |\xi|^\alpha \mathcal{F}_x u_0(x) d\xi \right\|_{L^6_{x,t}} \leq C \left\| u_0 \right\|_{L^2_t}.
\] (2.3)
Combining $\|U(t)u_0(x)\|_{L^{2\alpha+6}_{x,t}} \leq C \left\| u_0 \right\|_{L^2}$ with a standard argument, we have
\[
\|u(x)\|_{L^{2\alpha+6}_{x,t}} \leq C \|u\|_{X_0, \frac{1}{2} + \epsilon}. \quad (4.4)
\]
By using the Plancherel identity, we have that
\[
\|u\|_{L^2_{x,t}} = C \|u\|_{X_0,0}. \quad (5.5)
\]
Interpolating (4.4) with (5.5) yields
\[
\|u\|_{L^4_{x,t}} \leq C \|u\|_{X_0, \frac{\alpha+3}{2(\alpha+2)}(\frac{1}{2} + \epsilon)}. \quad (6.6)
\]
From (5.5), by using a standard proof, we have that
\[
\|D_x^\alpha u\|_{L^6_{x,t}} \leq C \|u\|_{X_0,6}. \quad (7.7)
\]
Interpolating (7.7) with (5.5) yields
\[
\|D_x^\alpha u\|_{L^4_{x,t}} \leq C \|u\|_{X_0, \frac{1}{4}^6}. \quad (8.8)
\]
We have completed the proof of Lemma 2.2. \qed

Lemma 2.3. Let $b = \frac{1}{2} + \epsilon$. Then, for $0 \leq s \leq \frac{1}{2}$, we have that
\[
\|I^s(u_1,u_2)\|_{L^2_{x,t}} \leq C \prod_{j=1}^{2} \|u_j\|_{X_0, \frac{\alpha+3+2(\alpha+1)\epsilon}{2(\alpha+2)\epsilon}}, \quad (9.9)
\]
where
\[
\mathcal{F} I^s(u_1,u_2)(\xi,\tau) = \int_{\xi = \xi_1 + \xi_2} \tau = \tau_1 + \tau_2 \left| \frac{\alpha+1}{\alpha+1} - |\xi_2|^\alpha \right| \mathcal{F} u_1(\xi_1,\tau_1) \mathcal{F} u_2(\xi_2,\tau_2) d\xi_1 d\tau_1.
\]
Proof. Let \( F_j(\xi, \tau_j) = \langle \sigma_j \rangle \frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} \mathcal{F} u_j(\xi, \tau_j) (j = 1, 2) \). To prove Lemma 2.3, by the Plancherel identity, it suffices to prove that

\[
\left\| \oint_{\xi = \xi_1 + \xi_2 \atop \tau = \tau_1 + \tau_2} |\xi_1|^{\alpha + 1} - |\xi_2|^{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{\alpha + 2 + 2(\alpha + 1)s} \frac{F_1}{\langle \sigma_1 \rangle^{2\alpha + 2}} \left\langle \sigma_2 \right\rangle^{\alpha + 2 + 2(\alpha + 1)s} \frac{F_2}{\langle \sigma_2 \rangle^{2\alpha + 2}} d\xi_1 d\tau_1 \right\|_{L_{\xi_1}^2} \leq C \prod_{j=1}^2 \left\| F_j \right\|_{L_{\xi_1}^2}.
\]

(2.10)

Assume that \( b_1 = \frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b \). By using the Young inequality, since \( 0 < s < \frac{1}{2} \), we have that

\[
\left\| \xi_1 \right\|_{\alpha + 1} - \left\| \xi_2 \right\|_{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{-b_1} \left\langle \sigma_2 \right\rangle^{-b_1} = \left\| \xi_1 \right\|_{\alpha + 1} - \left\| \xi_2 \right\|_{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{-2bs} \left\langle \sigma_2 \right\rangle^{-2bs} \left\langle \sigma_1 \right\rangle^{-b_1} \left\langle \sigma_2 \right\rangle^{-b_1 - 2bs} \leq 2s \left\| \xi_1 \right\|_{\alpha + 1} - \left\| \xi_2 \right\|_{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{-b_1} \left\langle \sigma_2 \right\rangle^{-b_1 - 2bs} + \left( 1 - 2s \right) \left\langle \sigma_1 \right\rangle^{-\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b} \left\langle \sigma_2 \right\rangle^{-\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b} \leq \left\| \xi_1 \right\|_{\alpha + 1} - \left\| \xi_2 \right\|_{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{-b_1} \left\langle \sigma_2 \right\rangle^{-b_1 - 2bs} + \left( 1 - 2s \right) \left\langle \sigma_1 \right\rangle^{-\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b} \left\langle \sigma_2 \right\rangle^{-\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b} \right. \right. \leq \left \| \xi_1 \right\|_{\alpha + 1} - \left\| \xi_2 \right\|_{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{-b_1} \left\langle \sigma_2 \right\rangle^{-b_1 - 2bs} + \left( 1 - 2s \right) \left\langle \sigma_1 \right\rangle^{-\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b} \left\langle \sigma_2 \right\rangle^{-\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b} \right. \right. \right.
\]

(2.11)

By using (2.11), Plancherel identity, Lemma 3.1 in [17], we have that

\[
\left\| \oint_{\xi = \xi_1 + \xi_2 \atop \tau = \tau_1 + \tau_2} |\xi_1|^{\alpha + 1} - |\xi_2|^{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{\alpha + 2 + 2(\alpha + 1)s} \frac{F_1}{\langle \sigma_1 \rangle^{2\alpha + 2}} \left\langle \sigma_2 \right\rangle^{\alpha + 2 + 2(\alpha + 1)s} \frac{F_2}{\langle \sigma_2 \rangle^{2\alpha + 2}} d\xi_1 d\tau_1 \right\|_{L_{\xi_1}^2} \leq \left\| \oint_{\xi = \xi_1 + \xi_2 \atop \tau = \tau_1 + \tau_2} |\xi_1|^{\alpha + 1} - |\xi_2|^{\alpha + 1}|^s \left\langle \sigma_1 \right\rangle^{\alpha + 2 + 2(\alpha + 1)s} \frac{F_1}{\langle \sigma_1 \rangle^{2\alpha + 2}} \left\langle \sigma_2 \right\rangle^{\alpha + 2 + 2(\alpha + 1)s} \frac{F_2}{\langle \sigma_2 \rangle^{2\alpha + 2}} d\xi_1 d\tau_1 \right\|_{L_{\xi_1}^2} \]

\[
+ \left\| \oint_{\xi = \xi_1 + \xi_2 \atop \tau = \tau_1 + \tau_2} \prod_{j=1}^2 \left\langle \sigma_j \right\rangle^{\alpha + 3 + 2(\alpha + 1)s} \frac{F_j}{\langle \sigma_j \rangle^{2\alpha + 2}} d\xi_1 d\tau_1 \right\|_{L_{\xi_1}^2} \leq C \prod_{j=1}^2 \left\| \mathcal{F}^{-1} \left( \frac{F_j}{\langle \sigma_j \rangle^{\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b}} \right) \right\|_{X_0, b} + C \prod_{j=1}^2 \left\| \mathcal{F}^{-1} \left( \frac{F_j}{\langle \sigma_j \rangle^{\frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b}} \right) \right\|_{X_0, \frac{\alpha + 3 + 2(\alpha + 1)s}{2\alpha + 4} b} \leq C \prod_{j=1}^2 \left\| F_j \right\|_{L_{\xi_1}^2}.
\]

(2.12)

We have completed the proof of Lemma 2.3.

Lemma 2.4. Let \( u_0 \in H^s(\mathbb{R}) \), \( c > 1/2 \), \( 0 < b < 1/2 \). Then for \( t \in [0, T] \), \( U(t)u_0 \in X_{s,c}^T \) and there is a constant \( k_2 > 0 \) such that

\[
\|U(t)u_0\|_{X_{s,c}^T} \leq k_2\|u_0\|_{H^s}.
\]

(2.13)

There is a constant \( c > 0 \) such that for \( t \in [0, 1] \) and \( f \in X_{s,b}^T \),

\[
\left\| \int_0^T U(t - s)f(s)ds \right\|_{X_{s,b}^T} \leq CT^{1-2b}\|f\|_{X_{s-b}^T}.
\]

(2.14)

For the proof of Lemma 2.4, we refer the readers to Lemma 3.1 of [2].

Lemma 2.5. Let

\[
\bar{u} = \int_0^t U(t - s)\Phi dW(s)
\]
and $\Phi \in L^0_{2,s}$, for $t \in [0,T]$, we have

$$E(\sup_{t \in [0,T]} \|\overline{\pi}\|^2_{H^s}) \leq 38T\|\Phi\|^2_{L^0_{2,s}}. \quad (2.15)$$

Lemma 2.5 can be proved similarly to Proposition 2.1 of [2].

**Lemma 2.6.** Let

$$\tilde{u} = \int_0^t U(t-s)\Phi dW(s),$$

$s, b, b_1, b_2 \in \mathbb{R}$ with $b < \frac{1}{2}$ and $\Phi \in L^0_{2,s}$. Then, we have that

$$E\left(\|\psi \tilde{u}\|^2_{X_{s,b}}\right) \leq C\|\Phi\|^2_{L^0_{2,s}}. \quad (2.16)$$

For the proof of Lemma 2.6, we refer the readers to Proposition 2.1 of [2].

### 3. Bilinear estimate

In this section, we give an important bilinear estimate which can be used to establish two important trilinear estimates.

**Theorem 3.1.** For all $u, v$ on $\mathbb{R} \times \mathbb{R}$, $0 \ll \epsilon \leq 1$ and $b = \frac{1}{2} - \epsilon$, we have

$$\|u_1 u_2\|_{L^2} \leq C\|u_1\|_{X_{\frac{1}{2},b}}\|u_2\|_{X_{\frac{1}{2},b}}. \quad (3.1)$$

**Proof.** Define

$$F_1(\xi_1, \tau_1) = \langle \xi_1 \rangle^{-\frac{1}{2}}(\sigma_1)^b F u_1(\xi_1, \tau_1) \quad F_2(\xi_2, \tau_2) = \langle \xi_2 \rangle^{-\frac{1}{2}}(\sigma_2)^b F u(\xi_2, \tau_2)$$

$$\sigma_j = \tau_j - |\xi_j|^{a+1}\xi_j, \quad j = 1, 2.$$  

To obtain (3.1), it suffices to prove that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K_1(\xi_1, \tau_1, \xi, \tau)|F_1|^{\frac{2}{2}}|F_2|^{\frac{2}{2}}d\xi_1 d\tau_1 d\xi d\tau \leq C\|F\|_{L^2_{\epsilon}} \prod_{j=1}^{2} \|F_j\|_{L^2_{\epsilon}}, \quad (3.2)$$

where

$$K_1(\xi_1, \tau_1, \xi, \tau) = \frac{\langle \xi_2 \rangle^{\frac{2}{2}}(\sigma_2)^b}{\langle \sigma_1 \rangle^b(\sigma_2)^b}.$$  

Without loss of generality, we assume that $F \geq 0, F_j \geq 0 (j = 1, 2)$.

$$\Omega_1 = \left\{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \leq |\xi_2| \leq 6\right\},$$

$$\Omega_2 = \left\{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_2| \geq 6, |\xi_2| \gg |\xi_1|\right\},$$

$$\Omega_3 = \left\{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_2| \geq 6, |\xi_2| \sim |\xi_1|\right\},$$

$$\Omega_4 = \left\{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_2| \leq |\xi_1| \leq 6\right\},$$

$$\Omega_5 = \left\{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \geq 6, |\xi_1| \gg |\xi_2|\right\},$$

$$\Omega_6 = \left\{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \geq 6, |\xi_1| \geq |\xi_2|, |\xi_1| \sim |\xi_2|\right\},$$
We define

\[ f_j = \mathcal{F}^{-1} \frac{F_j}{(\sigma_j)^b}, j = 1, 2. \]

(1) \( \Omega_1 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^2 \xi_j, \tau = \sum_{j=1}^2 \tau_j, |\xi_1| \leq |\xi_2| \leq 6 \} \). In this subregion, we have that

\[ K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{1}{\prod_{j=1}^2 \langle \sigma_j \rangle^b}. \]

By using the Plancherel identity and the Hölder inequality and \( \frac{\alpha+3}{2(\alpha+2)}(\frac{1}{2} + \epsilon) < \frac{1}{2} - \epsilon \), we have that

\[ J_1 \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathcal{F}^{-1}(F) f_1 f_2 d\xi d\tau \]

\[ \leq C \int_{\mathbb{R}^2} \mathcal{F}^{-1}(F) f_1 f_2 dx dt \leq C \|\mathcal{F}^{-1}(F)\|_{L^2} \prod_{j=1}^2 \|f_j\|_{L^2_t} \]

\[ \leq C\|F\|_{L^2_t} \prod_{j=1}^2 \|f_j\|_{L^2_t} = \prod_{j=1}^2 \|f_j\|_{L^2_t}. \] (3.3)

(2) \( \Omega_2 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^2 \xi_j, \tau = \sum_{j=1}^2 \tau_j, |\xi_2| \geq 6, |\xi_2| \gg |\xi_1| \} \).

If \(|\xi_1| \leq 1\), we have that

\[ K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{1}{\prod_{j=1}^2 \langle \sigma_j \rangle^b}. \]

This case can be proved similarly to \( \Omega_1 \).

If \(|\xi_1| \geq 1\), we have

\[ K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_2|^\frac{\alpha}{4}}{\prod_{j=1}^2 \langle \sigma_j \rangle^b} \leq C \frac{|\xi_2|^{\alpha+1} - |\xi_2|^{\frac{\alpha}{4} + \frac{\alpha}{4(\alpha+1)}}}{\prod_{j=1}^2 \langle \sigma_j \rangle^b}. \]

By using Lemma 2.3, we have

\[ J_2 \leq C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\xi_2|^{\alpha+1} - |\xi_2|^{\alpha + 1}}{\prod_{j=1}^2 \langle \sigma_j \rangle^b} F \prod_{j=1}^2 F_j d\xi d\tau \]

\[ \leq C\|F\|_{L^2_t} \prod_{j=1}^2 \|f_j\|_{L^2_t} = \prod_{j=1}^2 \|f_j\|_{L^2_t}. \]

(3) \( \Omega_3 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^2 \xi_j, \tau = \sum_{j=1}^2 \tau_j, |\xi_2| \geq 6, |\xi_2| \sim |\xi_1| \} \).

\[ K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi_2|^\frac{\alpha}{4}}{\prod_{j=1}^2 \langle \sigma_j \rangle^b} \leq C \frac{\prod_{j=1}^2 |\xi_j|^\frac{\alpha}{4}}{\prod_{j=1}^2 \langle \sigma_j \rangle^b}. \]
By using the Plancherel identity and the Cauchy-Schwartz inequality, we have

\[
J_3 \leq C \int_{\mathbb{R}^2} \int_{\xi = \xi_1 + \xi_2 \atop \tau = \tau_1 + \tau_2} \frac{\prod_{j=1}^{2} \gamma_j^{\alpha} F_j^{\frac{1}{2}} \prod_{j=1}^{2} F_j}{\prod_{j=1}^{2} (\sigma_j)^b} d\xi_1 d\tau_1 d\xi d\tau \\
\leq C \|F\|_{L^2_{x,\tau}} \prod_{j=1}^{2} \left\| \frac{D_x^{\frac{1}{2}} \mathcal{F}^{-1} \left( \frac{F_j}{\langle \sigma_j \rangle^b} \right)}{\prod_{j=1}^{2} (\sigma_j)^b} \right\|_{L^1_{x,\tau}} \\
\leq C \|F\|_{L^2_{x,\tau}} \prod_{j=1}^{2} \|F_j\|_{L^2_{x,\tau}}.
\tag{3.4}
\]

(4). \( \Omega_4 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_2| \leq |\xi_1| \leq 6 \}. \) In this subregion, we have that

\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq \frac{C}{\prod_{j=1}^{2} (\sigma_j)^b}.
\]

Thus subregion can be proved similarly to \( \Omega_1 \).

(5). \( \Omega_5 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \geq 6, |\xi_1| \gg |\xi_2| \}. \) In this subregion, we have

\[
K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{\|\xi_1\|^{\alpha+1} - |\xi_2|^{\alpha+1}}{\prod_{j=1}^{2} (\sigma_j)^b}.
\]

By using Lemma 2.3, we have that

\[
J_5 \leq C \int_{\mathbb{R}^2} \int_{\xi = \xi_1 + \xi_2 \atop \tau = \tau_1 + \tau_2} \frac{\|\xi_2\|^{\alpha+1} - |\xi_1|^{\alpha+1}}{\prod_{j=1}^{2} (\sigma_j)^b} F_j^{\frac{1}{2}} \prod_{j=1}^{2} F_j d\xi_1 d\tau_1 d\xi d\tau \\
\leq C \|F\|_{L^2_{x,\tau}} \left\| \int_{\xi = \xi_1 + \xi_2 \atop \tau = \tau_1 + \tau_2} \frac{\|\xi_2\|^{\alpha+1} - |\xi_1|^{\alpha+1}}{\prod_{j=1}^{2} (\sigma_j)^b} F_j^{\frac{1}{2}} \prod_{j=1}^{2} F_j d\xi_1 d\tau_1 \right\|_{L^2_{x,\tau}} \\
\leq C \|F\|_{L^2_{x,\tau}} \prod_{j=1}^{2} \|F_j\|_{L^2_{x,\tau}}.
\]

(6). \( \Omega_6 = \{(\xi_1, \tau_1, \xi, \tau) \in \mathbb{R}^4, \xi = \sum_{j=1}^{2} \xi_j, \tau = \sum_{j=1}^{2} \tau_j, |\xi_1| \geq 6, |\xi_1| \geq |\xi_2|, |\xi_1| \sim |\xi_2| \}. \)

This subregion can be proved similarly to \( \Omega_3 \).

We have completed the proof of Lemma 3.1.

4. Trilinear estimates

In this section, we will establish two new trilinear estimates which play a crucial role in establishing the local well-posedness of solution.

We will establish the Lemma 4.1 with the aid of the idea in [44]. Let \( Z = \mathbb{R} \) and \( \Gamma_k(Z) \) denote the hyperplane in \( \mathbb{R}^k \)

\[
\Gamma_k(Z) := \{(\xi_1, \cdots, \xi_k) \in Z^k, \xi_1 + \cdots + \xi_k = 0\}
\]

endowed with the induced measure

\[
\int_{\Gamma_k(Z)} f := \int_{Z^{k-1}} f(\xi_1, \cdots, \xi_{k-1}, -\xi_1 - \cdots - \xi_{k-1}) d\xi_1 \cdots d\xi_k.
\]
A function $m : \Gamma_k(Z) \to C$ is said to be a $[k; Z]$-multiplier, and we define the norm $\|m\|_{[k; Z]}$ to be the best constant such that the inequality
\[
\left| \int_{\Gamma_k(Z)} m(\xi) \prod_{j=1}^{k} f_j(\xi_j) \right| \leq \|m\|_{[k; Z]} \prod_{j=1}^{k} \|f_j\|_{L^2}.
\]
holds for all test function $f_j$ on $Z$.

**Lemma 4.1.** Let $s_0 = \frac{1}{2} - \frac{\alpha}{4}$, $b = \frac{1}{2} - \epsilon$. Then
\[
\|\partial_x(u_1 u_2 u_3)\|_{X_{s_0-b}} \leq C \prod_{j=1}^{3} \|u_j\|_{X_{s_0,b}}. \tag{4.1}
\]

**Proof.** By duality, Plancherel identity and the definition, to obtain (4.1), it suffices to prove that
\[
\left| \int_{\Gamma_k(Z)} (\sum_{j=1}^{3} \xi_j)(\xi_4)^{\frac{1}{2} - \frac{\alpha}{4}} \prod_{j=1}^{4} (\tau_j - w(\xi_j))^{\frac{1}{2} - \epsilon} \prod_{j=1}^{3} (\xi_j)^{\frac{1}{2} - \frac{\alpha}{4}} \right|_{[4; R \times R]} \leq C. \tag{4.2}
\]
By using the symmetry and
\[
\langle \xi_4 \rangle^{\frac{1}{2} - \frac{\alpha}{4}} \leq C \langle \xi_4 \rangle^{\frac{1}{2}} \left[ \sum_{j=1}^{3} \langle \xi_j \rangle^{1 - \frac{\alpha}{4}} \right]
\]
resulting from
\[
|\xi_1 + \xi_2 + \xi_3| \leq \langle \xi_4 \rangle,
\]
to obtain (4.2), it suffices to prove
\[
\left| \int_{\Gamma_k(Z)} \langle \xi_4 \rangle^{1/2} \langle \xi_3 \rangle^{1/2} \prod_{j=1}^{4} (\tau_j - w(\xi_j))^{\frac{1}{2} - \epsilon} \right|_{[4; R \times R]} \leq C. \tag{4.3}
\]
(4.3) follows from $TT^*$ identity in Lemma 3.7 of [44] and Lemma 3.1.

We have completed the proof of Lemma 4.1.

**Lemma 4.2.** Let $s \geq s_0 = \frac{1}{2} - \frac{\alpha}{4}$, $b = \frac{1}{2} - \epsilon$. Then
\[
\|\partial_x(u_1 u_2 u_3)\|_{X_{s-b}} \leq C \prod_{j=1}^{3} \|u_j\|_{X_{s,b}}. \tag{4.4}
\]

**Proof.** (4.4) is equivalent to the following inequality
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\xi| \langle \xi \rangle^s F \prod_{j=1}^{3} F_j}{\langle \sigma \rangle^b \prod_{j=1}^{3} \langle \sigma_j \rangle^b} d\xi_1 d\tau_1 d\xi_2 d\tau_2 d\xi d\tau \leq C \|F\|_{L^2_{\xi \tau}} \prod_{j=1}^{3} \|F_j\|_{L^2_{\xi \tau}} \tag{4.5}
\]
Since
\[
\langle \xi \rangle^{s-s_0} \leq C \prod_{j=1}^{3} \langle \xi_j \rangle^{s-s_0}, \tag{4.6}
\]
(4.5) is equivalent to the following inequality
\[
\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\xi| \langle \xi \rangle^{s_0} F \prod_{j=1}^{3} F_j}{\langle \sigma \rangle^b \prod_{j=1}^{3} \langle \sigma_j \rangle^{s_0}} d\xi_1 d\tau_1 d\xi_2 d\tau_2 d\xi d\tau \leq C \|F\|_{L^2_{\xi \tau}} \prod_{j=1}^{3} \|F_j\|_{L^2_{\xi \tau}} \tag{4.7}
\]
which is just the Lemma 4.1.

We have completed the proof of Lemma 4.2.
Lemma 4.3. Let $s \geq s_0 = \frac{1}{2} - \frac{\alpha}{4}$, $b = \frac{1}{2} - \epsilon$. Then

$$\|\partial_x(u_1u_2u_3)\|_{X_{s,b}^T} \leq C \prod_{j=1}^{3} \|u_j\|_{X_{s,b}^T}. \quad (4.8)$$

Combining Lemma 4.2 with a standard proof, we can obtain Lemma 4.3.

5. Local well-posedness

In this section, we prove Theorem 1.1. Let $z(t) = U(t)u_0$ and $\bar{u} = \int_0^t U(t - s)\Phi ds$. The solution to (1.1) is equivalent to the following integral equation

$$u(t) = U(t)u_0 + \frac{1}{3} \int_0^t U(t - s)\partial_x(u^3)ds + \int_0^t U(t - s)\Phi dsW. \quad (5.1)$$

and $v(t) = u(t) - z(t) - \bar{u}$. Then, we have that

$$v(t) = u(t) - z(t) - \bar{u} = \frac{1}{3} \int_0^t U(t - s)\partial_x(v(t) + \bar{u})^3ds. \quad (5.2)$$

We define

$$G(v) = \frac{1}{3} \int_0^t U(t - s)\partial_x(v(t) + \bar{u})^3ds. \quad (5.3)$$

By using Lemma 4.4, Lemmas 2.4, 2.5, 2.7, we have that

$$\|G(v)\|_{X_{s,b}^T} \leq \left| \frac{1}{3} \int_0^t U(t - s)\partial_x(v(t) + \bar{u})^3ds \right|_{X_{s,b}^T}$$

$$\leq CT^{1-2b} \left( \|v\|_{X_{s,b}^T}^3 + \|z(t)\|_{X_{s,b}^T}^3 + \|\psi \left( \frac{t}{T} \right) \bar{u}\|_{X_{s,b}^T}^3 \right)$$

$$\leq CT^{1-2b} \left( \|v\|_{X_{s,b}^T}^3 + \|u_0\|_{X_{s,b}^T}^3 + \|\psi \left( \frac{t}{T} \right) \bar{u}\|_{X_{s,b}^T}^3 \right), \quad (5.4)$$

similarly, we have that

$$\|G(v_1) - G(v_2)\|_{X_{s,b}^T} \leq \left| \frac{1}{3} \int_0^t U(t - s)\partial_x(v(t) + \bar{u})^3ds \right|_{X_{s,b}^T}$$

$$\leq CT^{1-2b} \|v_1 - v_2\|_{X_{s,b}^T} \left( \|v_1\|_{X_{s,b}^T}^2 + \|v_2\|_{X_{s,b}^T}^2 + \|z(t)\|_{X_{s,b}^T}^2 + \|\psi \left( \frac{t}{T} \right) \bar{u}\|_{X_{s,b}^T}^2 \right)$$

$$\leq CT^{1-2b} \|v_1 - v_2\|_{X_{s,b}^T} \left( \|v_1\|_{X_{s,b}^T}^2 + \|v_2\|_{X_{s,b}^T}^2 + \|u_0\|_{H^s}^2 + \|\psi \left( \frac{t}{T} \right) \bar{u}\|_{X_{s,b}^T}^2 \right). \quad (5.5)$$

Let

$$R_\omega = \left[ \|\psi \left( \frac{t}{T} \right) \bar{u}\|_{X_{s,b}} + \|u_0\|_{H^s} + 2 \right]^{\frac{3}{2}}. \quad (5.6)$$

and define

$$T_\omega = \inf \left\{ T > 0, CT^{1-2b}R_\omega^3 \geq \frac{1}{4} \right\}. \quad (5.7)$$
From Lemma 2.6, for any $0 < T < 1$, we have that
\[ \|\chi_{[0,T]}u\|_{X_{s,b}} \leq C\|u\|_{X_{s,b}^T} \leq C(\omega) \]
a.s. Moreover, since $b = \frac{1}{2} - \epsilon$, $\|\chi_{[0,T]}u\|_{X_{s,b}}$ is a.s. continuous with respect to $T$. From (5.6), we know that $T_\omega > 0$ a.s. Combining (5.6) with the fact that $\|\chi_{[0,T]}u\|_{X_{s,b}}$ is $\mathcal{F}_T$-measurable, we know that $T_\omega$ is a stopping time. Combining (5.4), (5.8) with (5.6), (5.7), we have that $G$ maps the ball of radius 1 in $X_{s,b}^T$ into itself and
\[ \|G(v_1) - G(v_2)\|_{X_{s,b}^T} \leq \frac{1}{2}\|v_1 - v_2\|_{X_{s,b}^T}, \] (5.8)
consequently, $G$ has a unique fixed point, which is the unique process $u$ satisfying (1.1) on $[0, T_\omega]$. Now we prove that $u \in C([0, T]; H^s(\mathbb{R}))$. Since $0 < b < \frac{1}{2}$, thus we obtain $\|z(t)\|_{C([0,T]; H^s)} \leq \|z(t)\|_{X_{s,1-b}}$. From Proposition 4.7 of [42] and Theorem 6.10 of [40], we know that $\bar{u} \in C([0, T]; H^s(\mathbb{R}))$. Obviously, we have that
\[ \|v\|_{C([0,T]; H^s)} \leq \left\|\frac{1}{3} \int_0^t U(t-s)\partial_x u^3ds\right\|_{X_{s,1-b}^T} \leq C\|u\|^3_{X_{s,b}^T} \leq C(1 + \|u_0\|_{H^s} + C(\omega))^3 < \infty. \]
Thus, $v \in C([0, T]; H^s)$. In conclusion, we have that $u = z(t) + \bar{u} + v \in C([0, T]; H^s)$.

For the proof of the rest of Theorem 1.1, we refer the readers to Theorem 1.1 of [2, 41].

We have completed the proof of Theorem 1.1.

6. Proof of Theorem 1.2

In this section, inspired by [41, 42], we prove Theorem 1.2.

Firstly, we consider the following the frequency truncated stochastic PDE
\[ \begin{aligned}
\left\{ 
&\begin{aligned}
&du^m(t) = [\partial_x^2 u^m - \frac{1}{3}\partial_x((u^m)^3)]dt + \Phi_m dW(t), \\
u^m(x, 0) = u_0^m(x) = P_m u_0(x),
\end{aligned}
\end{aligned} \tag{6.1} \]
where $\mathcal{F}_x P_m u_0(x) = \psi \left( \frac{x}{m} \right) \mathcal{F}_x u_0(\xi)$. Obviously, (6.1) can be rewritten as follows:
\[ u^m = U(t)u_0^m - \frac{1}{3} \int_0^t S(t-\tau)[(u^m)^3]d\tau + \int_0^t U(t-\tau)\Phi^m dW(\tau). \tag{6.2} \]

Firstly, we establish the following Lemmas.

**Lemma 6.1.** Let $u_0(x, \omega) \in L^2(\Omega; H^s(\mathbb{R}))$ with $s \geq \frac{1}{4}$ and $u_0$ be $\mathcal{F}_0$ measurable and $\Phi \in L_{-2}^0$. Suppose that $\Omega \subset \Omega$ is such that, for $\omega \in \Omega$, there exists $u^m(t)$ which is a solution to (6.2) for $t \in [0, T]$ with $T \leq T_{\omega, m}$, where
\[ T_{\omega, m} := \inf \left\{ T > 0, 2CT^{1-2b}\left(\|u_0^m\|_{H^s} + 2\left\|\psi \left( \frac{t}{T} \right) \bar{u}^m \right\|_{X_{s,b}} \right)^3 \geq 1 \right\}. \tag{6.3} \]
Then for all $t \in [0, T]$ and any $p \in N$, we have that
\[ \mathbb{E}\left( \sup_{t \in [0, T]} \|u^m\|_{H^s_{-1,2}}^2 \right) \leq C(p, m), \tag{6.4} \]
where $C(p, m) = C \left( p, T, \|u_0^m\|_{H^s_{-1,2}}; \|\Phi^m\|_{L_{-2}^0} \right)$. 

Combining (6.6) with (6.10), we have that
\[
E \left( \sup_{t \in [0,T]} \| u^m(t \wedge T_{\omega,m}) \|_{H^1_2}^2 \right) \leq E \left( \sup_{t \in [0,T]} \| u^m(t \wedge T_{\omega,m}) \|_{H^1_2}^{2p} \right). \tag{6.5}
\]

Since \((a+b)^p \leq 2^{p-1}(a^p + b^p)\) with \(a \geq 0, b \geq 0, p \geq 1\), we have that
\[
E \left( \sup_{t \in [0,T]} \| u^m(t \wedge T_{\omega,m}) \|_{H^1_2}^{2p} \right) \leq \sum_{j=1}^{2^p} I_j, \tag{6.6}
\]
where
\[
I_1 = 2^{p-1}E \left( \sup_{t \in [0,T]} \| u^m(t \wedge T_{\omega,m}) \|_{L^2_4}^{2p} \right),
\]
\[
I_2 = 2^{p-1}E \left( \sup_{t \in [0,T]} \| u^m_x(t \wedge T_{\omega,m}) \|_{L^4_2}^{2p} \right).
\]

Obviously,
\[
I_2 = 2^{p-1}E \left( \sup_{t \in [0,T]} \left( \| u^m_x(t \wedge T_{\omega,m}) \|_{L^2_4}^2 - \frac{1}{6} \| u(t \wedge T_{\omega,m}) \|_{L^4_4}^4 + \frac{1}{6} \| u^m(t \wedge T_{\omega,m}) \|_{L^4_4}^4 \right)^p \right)
\leq I_{21} + I_{22}. \tag{6.7}
\]

where
\[
I_{21} = 4^{p-1}E \left( \sup_{t \in [0,T]} \left( \| u^m_x(t \wedge T_{\omega,m}) \|_{L^2_4}^2 - \frac{1}{6} \| u(t \wedge T_{\omega,m}) \|_{L^4_4}^4 \right)^p \right),
\]
\[
I_{22} = \frac{1}{4} \left( \frac{2}{3} \right)^p E \left( \sup_{t \in [0,T]} \| u(t \wedge T_{\omega,m}) \|_{L^4_4}^{4p} \right).
\]

By using the interpolation Theorem, we have that
\[
I_{22} \leq \frac{2^{p-1}}{4} E \left( \sup_{t \in [0,T]} \| u^m_x(t \wedge T_{\omega,m}) \|_{L^2_4}^{2p} \right) + C(p) E \left( \sup_{t \in [0,T]} \| u(t \wedge T_{\omega,m}) \|_{L^2_4}^{6p} \right). \tag{6.8}
\]

Combining (6.8) with (6.9), we have that
\[
\frac{3}{4} I_2 \leq I_{21} + C(p) E \left( \sup_{t \in [0,T]} \| u(t \wedge T_{\omega,m}) \|_{L^2_4}^{6p} \right). \tag{6.9}
\]

From (6.9), we have that
\[
I_2 \leq \frac{4}{3} I_{21} + C(p) E \left( \sup_{t \in [0,T]} \| u(t \wedge T_{\omega,m}) \|_{L^2_4}^{6p} \right). \tag{6.10}
\]

Combining (6.6) with (6.10), we have that
\[
E \left( \sup_{t \in [0,T]} \| u^m(t \wedge T_{\omega,m}) \|_{H^1_2}^{2p} \right) \leq 2^{p-1} E \left( \sup_{t \in [0,T]} \| u(t \wedge T_{\omega,m}) \|_{L^2_4}^{2p} \right) + \frac{4}{3} I_{21} + C(p) E \left( \sup_{t \in [0,T]} \| u(t \wedge T_{\omega,m}) \|_{L^2_4}^{6p} \right), \tag{6.11}
\]
Combining (6.11) with a proof similar to (5.3.10) of Lemma 5.17 of [42], we have Lemma 6.1.

We have completed the proof of Lemma 6.1.

**Lemma 6.2.** Let \( \alpha = 1 \) and \( u_0(x, \omega) \in L^2(\Omega; H^s(\mathbb{R})) \) with \( s \geq \frac{1}{4} \) and \( \Phi \in L^0_{2,s} \) and \( u_0 \) be \( \mathcal{F}_0 \) measurable. For any \( m \) and any \( T_0 > 0 \), there exists an almost surely unique solution \( u^m \) to (6.2) for all \( t \in [0, T_0] \).

**Proof.** Combining Lemma 6.1 with a proof similar to Proposition 4.8 of [42], we have that Lemma 6.2 is valid.

We have completed the proof of Lemma 6.2.

**Lemma 6.3.** The sequence \( u^m \) is bounded in \( L^2(\Omega, L^\infty(0, T_0; H^1(\mathbb{R}))) \). More precisely, we have that

\[
E \left( \sup_{t \in [0, T_0]} \| u^m \|_{L^2}^2 \right) \leq C \left( E(\| u_0 \|_{H^1}^2), T_0, \| \Phi \|_{L^0_{2,1}} \right). \tag{6.12}
\]

**Proof.** Let \( \mathcal{E}(u^m) = \| u^m \|_{L^2}^6 \). Applying the Itô formula to \( \mathcal{E}(u^m) \) yields

\[
\| u^m \|_{L^2}^6 = \| u_0^m \|_{L^2}^6 + 6 \int_0^t \| u^m \|_{L^2}^4 (u^m, \Phi^m dW) + \frac{1}{2} \int_0^t Tr \mathcal{E}''(u^m) (\Phi^m) (\Phi^m)^* d\mathbb{E} \tag{6.13}
\]

with

\[
\mathcal{E}''(u^m) \phi = 24 \| u^m \|_{L^2}^2 (u^m, \phi) u^m + 6 \| u^m \|_{L^2}^2 \phi.
\]

By using a martingale inequality which can be seen in Theorem 3.14 of [40], we have

\[
E \left( \sup_{t \in [0, T_0]} \int_0^t \| u^m \|_{L^2}^4 (u^m, \Phi^m dW) \right) \leq 3E \left( \int_0^{T_0} \| u^m \|_{L^2}^8 \| (\Phi^m)^m u^m \|_{L^2} ds \right)^{1/2} \leq \frac{1}{16} E \left( \sup_{t \in [0, T_0]} \| u^m \|_{L^2}^6 \right) + CT_0^3 \| \Phi^m \|_{L^0_{2,0}}^6. \tag{6.14}
\]

By using the definition of trace operator and the Young inequality, we have

\[
Tr (\mathcal{E}''(u^m) \Phi^m \Phi^m_*) = \sum_{j \in N} \left[ 24 \| u^m \|_{L^2}^2 (u^m, \Phi^m e_j)^2 + 6 \| u^m \|_{L^2}^4 (\Phi^m e_j, \Phi^m e_j) \right] \leq 30 \| u^m \|_{L^2}^4 \| \Phi^m \|_{L^0_{2,0}}^2 \leq \frac{1}{12T_0} \| u^m \|_{L^2}^6 + CT_0^2 \| \Phi^m \|_{L^0_{2,0}}^6. \tag{6.15}
\]

Inserting (6.14), (6.15) into (6.13) yields

\[
\| u^m \|_{L^2}^6 \leq \| u_0^m \|_{L^2}^6 + \frac{1}{2} \| u^m \|_{L^2}^6 + CT_0^3 \| \Phi^m \|_{L^0_{2,0}}^6. \tag{6.16}
\]

From (6.16), we have

\[
E \left( \sup_{t \in [0, T]} \| u^m \|_{L^2}^6 \right) \leq 2E \left( \| u_0^m \|_{L^2}^6 \right) + CT_0^3 \| \Phi^m \|_{L^0_{2,0}}^6. \tag{6.17}
\]
Let $C(u^m) = \|u^m\|_{L^2}^8$. Applying the Itô formula to $C(u_m)$ yields

$$\|u^m\|_{L^2}^8 = \|u^m_0\|_{L^2}^8 + 8 \int_0^t \|u^m\|_{L^2}^6(u^m, \Phi^m dW) + \frac{1}{2} \int_0^t Tr C''(u^m) \Phi^m(\Phi^m)^* ds$$

(6.18)

with

$$C''(u^m)\phi = 48\|u^m\|_{L^2}^4(u^m, \phi) u^m + 8\|u^m\|_{L^2}^6 \phi.$$  

By using a martingale inequality which can be seen in Theorem 3.14 of [40], we have

$$E\left(\sup_{t \in [0,T_0]} \int_0^t \|u^m\|_{L^2}^6(u^m, \Phi^m dW)\right) \leq 3E\left(\int_0^{T_0} \|u^m\|_{L^2}^2(\Phi^m)^2 m(u^m) ds\right)^{1/2}$$

$$\leq \frac{1}{16} E\left(\sup_{t \in [0,T_0]} \|u^m\|_{L^2}^6\right) + CT_0^4 \|\Phi^m\|_{L^2}^8.$$  

(6.19)

By using the definition of trace operator and the Young inequality, we have

$$Tr(C''(u^m)\Phi^m(\Phi^m)^*)$$

$$= \sum_{j \in N} \left[48\|u^m\|_{L^2}^2(u^m, \Phi^m e_j)^2 + 8\|u^m\|_{L^2}^4 \Phi^m e_j \|_{L^2}^2\right]$$

$$\leq 56\|u^m\|_{L^2}^6 \Phi^m \|_{L^2}^2 + 8\|u^m\|_{L^2}^6 \Phi^m \|_{L^2}^2.$$  

(6.20)

Inserting (6.19), (6.20) into (6.18) yields

$$\|u^m\|_{L^2}^6 \leq \|u^m_0\|_{L^2}^6 + \frac{1}{2} \|u^m\|_{L^2}^6 + CT_0^3 \|\Phi^m\|_{L^2}^6.$$  

(6.21)

From (6.21), we have

$$E\left(\sup_{t \in [0,T_0]} \|u^m\|_{L^2}^6\right) \leq 2E\left(\|u^m_0\|_{L^2}^6\right) + CT_0^3 \|\Phi^m\|_{L^2}^6.$$  

(6.22)

Let

$$H_2(u^m) = \frac{1}{2} \int_R (u_{xx}^m)^2 dx - \frac{1}{4} \int_R (u^m)^4 dx.$$  

(6.23)

Applying the Itô formula to $I(u^m)$ yields

$$H_2(u^m) = H_2(u^m_0) - \int_0^t (u_{xx}^m + (u^m)^3, \Phi dW(s))$$

$$+ \frac{1}{2} \int_0^t Tr (H_2''(u^m)\Phi^m(\Phi^m)^*) ds.$$  

(6.24)

with

$$H_2''(u^m)\phi = -\phi_{xx} - 3(u^m)^2 \phi.$$
By using a martingale inequality which can be seen in Theorem 3.14 of [40], we have
\[
E \left( \sup_{t \in [0,T_0]} - \int_0^t \left( u_{xx}^m + (u^m)^3, \Phi^m dW(s) \right) \right) 
\leq 3E \left( \left( \int_0^{T_0} \left( (\Phi^m)^* \left( u_{xx}^m + (u^m)^3 \right) \right)^2 \right)^{1/2} \right).
\]

By using the Sobolev embedding \( H^1 \hookrightarrow L^\infty \), we have
\[
|((\Phi^m)^* (u_{xx}^m + (u^m)^3)|^2 = \sum_{j \in \mathbb{N}^+} \left( (u_{xx}^m, \Phi^m e_j) + ((u^m)^3, \Phi^m e_j) \right)^2 
\leq C \sum_{j \in \mathbb{N}^+} (\|u^m\|^2_{H^1} ||\Phi^m e_j\|_{H^1}^2 + \|u^m\|^4_{L^2} \|u^m\|^2_{L^\infty} \|\Phi^m e_j\|^2_{L^\infty}) 
\leq C \left( 1 + \|u^m\|^4_{L^2} \|u^m\|^2_{H^1} \|\Phi^m\|^2_{L^2_{0,1}} \right).
\]

Consequently, we have
\[
E \left( \sup_{t \in [0,T_0]} - \int_0^t \left( u_{xx}^m + (u^m)^3, \Phi^m dW(s) \right) \right) 
\leq 3E \left( \left( \int_0^{T_0} \left( (\Phi^m)^* \left( u_{xx}^m + (u^m)^3 \right) \right)^2 \right)^{1/2} \right) 
\leq CT_0^{1/2} \left( 1 + \|u^m\|^2_{L^2} \|u^m\|^2_{H^1} \|\Phi^m\|^2_{L^2_{0,1}} \right) 
\leq \frac{1}{4} \|u^m\|^2_{H^1} + CT_0 \|\Phi^m\|^2_{L^2_{0,1}} + CT_0^2 \|\Phi^m\|^4_{L^2_{0,1}} + C \|u^m\|_{L^2}^8.
\]

Thus, by using \( H^1 \hookrightarrow L^\infty \), we have
\[
Tr \left( H^m_2(u^m) \Phi^m(\Phi^m)^* \right) 
= - \sum_{j \in \mathbb{N}} \int_{\mathbb{R}} \left[ (\Phi^m e_j)_{xx} \Phi^m e_j + 3(u^m)^2(\Phi^m e_j)^2 \right] dx 
\leq \sum_{j \in \mathbb{N}} \left( \|\Phi^m e_j\|^2_{L^2} + 3 \|u^m\|^4_{L^\infty} \|\Phi^m e_j\|^2_{L^2} \right) \leq C \|\Phi^m\|^2_{L^2_{0,1}} \left[ \|u^m\|^2_{H^1} + 1 \right]
\]

By using the martingale inequality, we have
\[
E \left( \sup_{t \in [0,T_0]} - \int_0^t \left( u_{xx}^m + (u^m)^3, \Phi dW(s) \right) \right) 
\leq 3E \left( \left( \int_0^{T_0} \left( (\Phi^m)^* \left( u_{xx}^m + (u^m)^3 \right) \right)^2 ds \right)^{1/2} \right).
\]

Consequently, we have
\[
\frac{1}{2} \int_0^t Tr \left( H^m_2(u^m) \Phi^m(\Phi^m)^* \right) ds \leq \frac{1}{4} \|u^m\|^4_{L^2} + CT_0^2 \|\Phi^m\|^4_{L^2_{0,1}} + CT_0 \|\Phi^m\|^2_{L^2_{0,1}}.
\]

Thus, we have
\[
E \left( \sup_{t \in [0,T_0]} H^m_2(u^m) \right) 
\leq E \left( H^m_2(u^m_0) \right) + \frac{1}{4} E \left( \sup_{t \in [0,T_0]} \|u^m_x\|^2_{L^2} \right) 
+ \frac{1}{4} E \left( \sup_{t \in [0,T_0]} \|u^m\|^4_{L^2} \right) + \frac{1}{2} E \left( \sup_{t \in [0,T_0]} \|u^m\|^8_{L^2} \right) + CT_0^2 \|\Phi^m\|^2_{L^2_{0,1}} + CT_0 \|\Phi^m\|^2_{L^2_{0,1}}.
\]
From the above inequality, by using the interpolation theorem
\[
\|u\|_{L^4}^4 \leq C\|u_x\|_{L^2}\|u\|_{L^3}^3 + \frac{1}{8}\|u_x\|_{L^2}^2 + C\|u\|_{L^2}^6
\]
and (6.24), we have
\[
\begin{align*}
E \left( \sup_{t \in [0,T_0]} \|u^m_x\|_{L^2}^2 \right) & \leq 4E \left( H_2(u^m_0) \right) + CE \left( \sup_{t \in [0,T_0]} \|u^m_0\|_{L^2}^2 \right) \\
+ CT_0^2 \|\Phi^m\|_{L^2_0,1}^4 + CT_0 \|\Phi^m\|_{L^2_0,1}^2 + CE \left( \sup_{t \in [0,T_0]} \|u^m\|_{L^2}^2 \right) \\
\leq CE \|u^m_0\|_{H^1_0}^2 + CE \left( \sup_{t \in [0,T_0]} \|u^m\|_{L^2}^2 \right)
\end{align*}
\]
and (6.25), we have
\[
\begin{align*}
&+ CE \left( \sup_{t \in [0,T_0]} \|u^m\|_{L^2}^4 \right) + CE \left( \sup_{t \in [0,T_0]} \|u^m\|_{L^2}^6 \right) + CE \left( \sup_{t \in [0,T_0]} \|u^m\|_{L^2}^8 \right) \\
+ CT_0 \|\Phi^m\|_{L^2_0,1}^2 + CT_0^2 \|\Phi^m\|_{L^2_0,1}^4 + CT_0^3 \|\Phi^m\|_{L^2_0,1}^6 + C^6 T_3 \|\Phi^m\|_{L^2_0,1}^6 + C T_0 \|\Phi^m\|_{L^2_0,0}^8 + C T_0 \|\Phi^m\|_{L^2_0,0}^8.
\end{align*}
\]
(6.25)

We define \( \mathcal{D}(u^m) = \left( \int (u^m)^2 dx \right)^4 \). Applying the Itô formula to \( \mathcal{D}(u) \) yields
\[
\mathcal{D}(u^m) = \mathcal{D}(u^m_0) + 4 \int_0^t \|u^m\|_{L^2}^2 (u^m, \Phi^m) dW + \frac{1}{2} \int_0^t \text{Tr} (\mathcal{D}''(u^m) \Phi^m \Phi^m) ds,
\]
where
\[
\mathcal{D}''(u^m) \phi = 8(u^m, \phi) u^m + 4 \|u^m\|_{L^2}^2 \phi.
\]
By using a computation similar to (6.25), we have
\[
E \left( \sup_{t \in [0,T_0]} \|u^m\|_{L^2}^4 \right) \leq 2E \left( \|u^m_0\|_{L^2}^4 \right) + C T_0^2 \|\Phi^m\|_{L^2_0,0}^4.
\]
(6.26)

In the same way, we have
\[
E \left( \sup_{t \in [0,T_0]} \|u^m\|_{L^2}^2 \right) \leq 2E \left( \|u^m_0\|_{L^2}^2 \right) + C T_0 \|\Phi^m\|_{L^2_0,0}^2.
\]
(6.27)

Inserting (6.26), (6.27) into (6.25) yields
\[
E \left( \sup_{t \in [0,T_0]} \|u^m_x\|_{L^2}^2 \right) \leq CE \left( \|u^m_0\|_{H^1} + 1 \right)^2 + C \left[ T_0 \|\Phi^m\|_{L^2_0,1} + 1 \right]^3.
\]
(6.28)

Combining (6.27) with (6.28), we have
\[
E \left( \sup_{t \in [0,T_0]} \|u^m\|_{H^1}^2 \right) \leq CE \left( \|u^m_0\|_{H^1} + 1 \right)^2 + C \left[ T_0 \|\Phi^m\|_{L^2_0,1} + 1 \right]^3.
\]
(6.29)
From (6.29), we have that
\[ E \left( \sup_{t \in [0,T_0]} \| u^m \|_{H^1}^2 \right) \leq C \left( \| u_0 \|_{H^1} + 1 \right)^2 + C \left[ T_0 \left\| \Phi \right\|_{L^2}^2 + 1 \right]^3. \]  
(6.30)

We have completed the proof of Lemma 6.3.
Now we are in a position to Theorem 1.2.
From Lemma 6.3, we know that after extraction of a subsequence, we can find a function \( \tilde{u} \in L^2(\Omega; L^\infty(0,T_0; H^1(\mathbb{R}))) \) such that
\[ u^m \rightharpoonup \tilde{u} \]  
(6.31)
in \( L^2(\Omega; L^\infty(0,T_0; H^1(\mathbb{R}))) \) weak star. Moreover, we have
\[ E \left( \sup_{t \in [0,T_0]} \| \tilde{u} \|_{H^1}^2 \right) \leq C. \]  
(6.32)

Let \( z^m(t) = U(t)u^0_m \) and \( \tilde{u}^m = \int_0^t U(t - \tau)\Phi^m d\tau \) and \( v^m = u^m - z^m - \tilde{u}^m \), then for each \( m \), \( v^m \) satisfies the truncated equation
\[ v^m = \frac{1}{3} \int_0^t U(t - \tau) \partial_x (v^m + z^m + \tilde{u}^m)^3 d\tau =: G_m(v^m). \]  
(6.33)

By repeating the proof of Theorem 1.1, it is easily checked that \( G_m \) is a.s. a contraction on a ball of radius 1 in \( X^T_{1,b} \) for any \( T > 0 \), satisfying
\[ 2CT^{-1-2b} \left( 2 + \| u^0_0 \|_{H^1} + \| \chi_{t \in [0,T]} \|_{X_{1,b}} \right)^3 \leq 1. \]  
(6.34)

Let
\[ D(\omega) = \sup_{0 \leq t \leq T_0} \| \tilde{u} \|_{H^1}^2. \]

Then
\[ E \left( \sup_{0 \leq t \leq T_0} \| \tilde{u} \|_{H^1}^2 \right) \leq C, \]
thus, we derive that \( D(\omega) < \infty \) a.s. We consider \( \tilde{T}_\omega > 0 \) satisfying
\[ 2CT^{-1-2b} \left( 2 + \| u^0_0 \|_{H^1} + D(\omega)^{1/2} + \| \chi_{t \in [0,T]} \|_{X_{1,b}} \right)^3 \leq 1. \]  
(6.35)

Then for any \( m \), we have that
\[ \| u^0_0 \|_{H^1} \leq \| u_0 \|_{H^1} \]
and
\[ \| \chi_{[0,T]} \|_{X_{1,b}} \leq \| \chi_{t \in [0,T]} \|_{X_{1,b}}. \]

It follows that (6.34) is valid a.s. for any \( m \) with \( T = \tilde{T}_\omega \). Furthermore, we have that \( \tilde{T}_\omega \leq T_\omega \), where \( T_\omega \) is the solution \( v \) from Theorem 1.1. Consequently, \( G \) and \( G_m \) are contractions in \( X^T_{1,b} \) for any \( m \), where \( \tilde{T}_\omega \) satisfies (6.35). Particularly, a unique solution \( v \in X^T_{1,b} \) to (5.2) a.s. exists. Moreover, for any \( m \), \( v^m \) and \( v \) are the unique fixed points of the contractions \( G_m \) and \( G \), respectively.
By using Lemmas 2.4, 2.5, 2.7 and Lemma 4.3, we have that \( u^m \to u \) in \( C([0, \tilde{T}_\omega]; H^1(\Omega)) \) and obtain that \( u = \bar{u} \) for \( t \in [0, \tilde{T}_\omega] \) a.s. with the aid of the idea of Section 4.3.2 of [42]. Consequently, we have that

\[
\|u(\tilde{T}_\omega)\|_{H^1}^2 \leq \sup_{t \in [0, T_0]} \|\bar{u}\|_{H^1}^2 = D_\omega. \tag{6.36}
\]

Combining (6.35) with (6.36), we can construct a solution on \([\tilde{T}_\omega, 2\tilde{T}_\omega]\) a.s. starting from \( u(2\tilde{T}_\omega) \), we obtain a solution on \([0, T_0]\) by reiterating this argument.

We have completed the proof of Theorem 1.2.

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