Blow-up profiles of solutions for the exponential reaction-diffusion equation

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We consider the blow-up of solutions for a semilinear reaction-diffusion equation with exponential reaction term. It is known that certain solutions that can be continued beyond the blow-up time possess a non-constant self-similar blow-up profile. Our aim is to find the final time blow-up profile for such solutions. The proof is based on general ideas using semigroup estimates. The same approach works also for the power nonlinearity. Copyright © 2011 John Wiley & Sons, Ltd.

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1. Introduction

We consider the following problem

\[
\begin{align*}
  u_t &= \Delta u + f(u), & x \in \Omega, & t > 0, \\
  u &= 0, & x \in \partial \Omega, & t > 0, \\
  u(x, 0) &= u_0(x) \geq 0, & x \in \Omega,
\end{align*}
\]

where \( \Omega = B_R(0) = \{ x \in \mathbb{R}^N : |x| < R \} \) and \( N \) is supercritical, that is, \( N \in [3, 9) \) and the initial condition \( u_0 \) is non-negative and in \( C^1(\overline{\Omega}) \). We are mainly interested in the case \( f(u) = e^{du} \), but some results work with more general nonlinearities. Before stating our results, see Theorems 2, 3 and 5, we give a brief introduction to the subject.

We are interested in solutions that blow up in finite time, which means that there exists \( T \in (0, \infty) \) such that \( \|u(\cdot, t)\|_{\infty} < \infty \) for \( t < T \) and

\[
\limsup_{t \to T} \|u(\cdot, t)\|_{\infty} = \infty.
\]

By standard theory of parabolic regularity, this implies that \( u \) is a classical solution for every \( t \in (0, T) \). Blow-up is said to be of type I, if the blow-up rate is the same as that of the ordinary differential equation \( u' = f(u) \). For \( f(u) = e^{du} \) this means that

\[
-C_1 \leq \log(T - t) + \|u(\cdot, t)\|_{\infty} \leq C_2.
\]

for some constants \( C_1 \) and \( C_2 \). If blow-up is not of type I then it is said to be of type II. A point \( x_0 \in \Omega \) is a blow-up point if there exists a sequence \( \{ (x_n, t_n) \}_n \subset \Omega \times (0, T) \) such that \( (x_n, t_n) \to (x_0, T) \) and \( u(x_n, t_n) \to \infty \) as \( n \to \infty \).

A solution can exist beyond the blow-up time \( t = T \) as a weak solution. To be more precise, let us give the following definition.

**Definition 1.1**

By an \( L^1 \)-solution of (1) on \( [0, T] \), we mean a function \( u \in C([0, T]; L^1(\Omega)) \) such that \( f(u) \in L^1(Q_T) \), \( Q_T := \Omega \times (0, T) \) and the equality

\[
\int_{\Omega} [u\psi]_{t_1}^{t_2} dx - \int_{t_1}^{t_2} \int_{\Omega} u\psi_t dx dt = \int_{t_1}^{t_2} \int_{\Omega} (u\Delta \psi + f(u)\psi) dx dt
\]

holds for any \( 0 \leq t_1 < t_2 \leq T \) and \( \psi \in C^2(\overline{Q_T}), \psi = 0 \) on \( \partial \Omega \times [0, T] \).
Blow-up is said to be complete if the minimal continuation of the solution is infinite everywhere for \((x, t) \in \Omega \times (T, \infty)\).

In this paper, we want to focus on solutions that blow up at \(x = 0\) and have a nontrivial self-similar blow-up profile, that is, the convergence (2)–(4) below holds. The following theorem (proved in [1]) states that every radially symmetric solution that blows up and continues to exist as a weak solution has this property.

**Theorem 1**

Let \(f(u) = e^u, N \in [3, 9]\) and let \(u\) be a radially symmetric and radially nonincreasing \(L^1\)-solution of (1) on \([0, T]\) that blows up at \(t = T < \infty\). Then

\[
\lim_{t \uparrow T} \left[ \log(T - t) + u(y, \sqrt{T - t}, t) \right] = \varphi(y)
\]

(2)

uniformly for \(y\) in compact sets of \(\mathbb{R}^N\), where \(\varphi\) satisfies

\[
\begin{cases}
\Delta \varphi - \frac{N}{2} \varphi + e^\varphi - 1 = 0, & |y| > 0, \\
\varphi(0) = \alpha, & \nabla \varphi(0) = 0,
\end{cases}
\]

(3)

and

\[
\lim_{|y| \to \infty} \left( \varphi(y) + 2 \log |y| \right) = C_\alpha
\]

(4)

for some \(\alpha > 0\) and \(C_\alpha \in \mathbb{R}\).

If the above convergence (2) holds for some function \(\varphi\), we will refer to \(\varphi\) as the self-similar blow-up profile of \(u\).

In the last section of this paper, we will slightly improve Theorem 1 by showing that the assumption on \(u\) being radially nonincreasing is redundant. We will thus obtain the following.

**Theorem 2**

Let \(u\) be a radially symmetric \(L^1\)-solution of (1) with \(f(u) = e^u\) on \([0, T]\) that blows up with type I rate at \((x, t) = (0, T)\), where \(T < \infty\). Then the convergence (2)–(4) holds.

In this theorem, we assume that blow-up is of type I that holds if \(N < 2\) and the maximum of \(u\) is attained at the origin [1].

The existence of global \(L^1\)-solutions of (1) with \(f(u) = u^p\) that blow-up in finite time is proved in [2] and [3]. The previous two theorems then give the asymptotic behavior for such solutions as the blow-up time is approached.

For subcritical dimensions \(N < 2\), the only solution \(\varphi\) of (3)–(4) is \(\varphi = 0\) [4] and so the convergence (2) should always hold with \(\varphi = 0\). This is proved in [5] under the extra assumption that \(u\) is nondecreasing in time and radially decreasing. For general solutions, the problem is how to obtain the blow-up rate.

For the power nonlinearity \(f(u) = u|u|^{p-1}\) blow-up is always complete and \(u\) has a constant self-similar blow-up profile, that is, the convergence (7) holds for \(\varphi\) equal to a constant, whenever \(p < p_s\), where

\[
p_s = \begin{cases}
\infty, & \text{if } N < 2, \\
\frac{N+2}{N-2}, & \text{if } N > 2,
\end{cases}
\]

[6].

For \(f(u) = e^u\) and supercritical dimensions \(N \in [3, 9]\), however, there exists a sequence \(\{\alpha_j\}\) of initial values tending to infinity such that the solutions \(\psi_j\) satisfy (3)–(4) with \(\alpha = \alpha_j\) [7]. Similar result is also true for the power nonlinearity, [8].

The idea of the proof of Theorem 1 is to assume that the convergence (2) holds with \(\varphi = 0\) and then prove that the final time blow-up profile of \(u\) is given by

\[
u(x, T) + 2 \log |x| + \log \log |x| \to C,
\]

(5)

as \(|x| \to 0\), which implies complete blow-up by the results in [9]. Convergence to a nontrivial \(\varphi\) is then obtained by an energy argument. In [10] the existence of solutions with the final time profile as in (5) is proved.

In [11–13], the case of \(f(u) = u^p\) is discussed and a variety of final time blow-up profiles is obtained by assuming constant self-similar blow-up profile. See also [14, 15] for other works in that direction. Later, these methods were used in [16] to prove Theorem 4, which corresponds to Theorem 1, but for the power type nonlinearity. The exponential nonlinearity in dimension one is discussed in [17] and [18], and final time blow-up profiles are found, provided that the solution has a constant self-similar profile.

So there are many results concerning final time blow-up profiles of solutions provided that the self-similar blow-up profile is a constant one. The behavior of solutions as in Theorem 1 at the blow-up moment is, however, not directly evident from the asymptotics (2)–(4). Our main theorem of this paper is the following that in fact does give the final time blow-up profile for solutions satisfying (2)–(4).

**Theorem 3**

Assume that \(u\) is a solution of (1) with \(f(u) = e^u\) that blows up with type I rate at \((x, t) = (0, T)\) for some \(T < \infty\) and verifies (2)–(4). Then the final time blow up profile of \(u\) is given by

\[
u(x, T) + 2 \log |x| - C_\alpha \to 0, \quad \text{as } |x| \to 0,
\]

(6)

where \(C_\alpha\) is the constant from (4).
The existence of the limit $\lim_{t \to T} u(x, t)$ for $x \neq 0$ is a consequence of the parabolic estimates as will be seen in the proof of the above theorem.

In this theorem, we merely assume that $u$ is a continuous solution of (1) that blows up with type I rate at $(x, t) = (0, T)$ and has a nontrivial self-similar blow-up profile, that is, convergence as in (2) with (3)–(4) holds. We do not need to assume that the solution is decreasing or even radially symmetric. It is of course a different matter whether there exist radially nonsymmetric solutions of (1) verifying (2)–(4) with radially symmetric $\psi$. It is also not known if there exist any radially nonsymmetric self-similar solutions.

Even though Theorem 3 is stated with $f(u) = e^u$, my analysis works for a larger class of nonlinearities, including $f(u) = u^p$. For the algebraic nonlinearity, Theorem 2 corresponds to the following result [16].

**Theorem 4**

Let $p > 1$, $f(u) = u^p$ and $u$ be a radially symmetric $L^1$-solution of (1) on $(0, T)$ that blows up with type I rate at $(x, t) = (0, T)$, where $T < \infty$. Then

$$
\lim_{t \to T} [(T - t)^{(p-1)/2} u(y \sqrt{T-t}, t)] = \psi(y)
$$

(7)

uniformly for $y$ in compact sets of $\mathbb{R}^N$, where $\psi$ satisfies

$$
\begin{align*}
\Delta \psi - \frac{\lambda}{2} \nabla \psi - \frac{1}{p-1} \psi + \psi^p &= 0, \\
\psi(0) &= \kappa + \alpha, \quad \nabla \psi(0) = 0,
\end{align*}
$$

(8)

with

$$
\kappa = \left( \frac{1}{p-1} \right)^{1/(p-1)}
$$

(9)

and

$$
\lim_{|y| \to \infty} |y|^{2/(p-1)} \psi(y) = C_\alpha
$$

(10)

for some $\alpha > 0$ and $C_\alpha > 0$.

Theorem 3 stated for $f(u) = u^p$ will then be the following, which is already known and proved in [19]. Our method, however, gives a new proof, which we do not present in this paper, because it proceeds very much in the same way as the proof of Theorem 3.

**Theorem 5**

Assume that $u$ is a solution of (1) with $f(u) = u^p$ for some $p > 1$ that blows up with type I rate at $(x, t) = (0, T)$ for some $T < \infty$ and verifies (7)–(10). Then the final time blow up profile of $u$ is given by

$$
\lim_{x \to 0} |x|^{2/(p-1)} u(x, T) = C_\alpha
$$

(11)

where $C_\alpha$ is the same as in (10).

In a forthcoming paper of the author, it will be shown that if $u$ is a so-called minimal limit $L^1$-solution on $(0, T)$ that blows up at $t = T < \infty$ and if the assumptions of Theorem 2 hold, then $u$ becomes regular immediately after blow-up. Moreover, under some additional assumptions, the regularization is asymptotically self-similar, that is, $u$ approaches a forward self-similar solution as $t \to T$ from above. This improves somewhat the results in [20].

The question about the behavior of the final time blow-up profiles $u(x, T)$ near the blow-up point has been studied in many papers, but usually in the case where the self-similar blow-up profile is the constant one [11], [18], [13], [16]. In these cases, the final time profile $u(x, T)$ is greater than those in (6) and (11) near the blow-up point and it holds that

$$
\lim_{x \to 0} |u(x, T) + 2 \log |x|| = \infty,
$$

if $f(u) = e^u$, and

$$
\lim_{x \to 0} |x|^{2/(p-1)} u(x, T) = \infty,
$$

if $f(u) = u^p$. An example of this type of profiles for $f(u) = e^u$ is the one in (5).

Problems of this type have been studied by Matano and Merle in the paper [19] in more detail. The authors consider $f(u) = |u|^{p-1} u$, sign changing solutions, and $\Omega$ being either a ball or $\mathbb{R}^N$. Their result characterizes the size of the final time blow-up profile in terms of the blow-up rate and the behavior of the self-similar profile, but their technique does not seem to directly apply to the case with exponential nonlinearity and possibly increasing solutions.
The result of Matano and Merle gives that for $p>p_*$ one has
\[
\lim_{x \to 0} |x|^{2/(p-1)} u(x,T) = \begin{cases} 
\infty \text{ or } -\infty & \leftrightarrow \text{ type I with } \varphi = \kappa \text{ or } -\kappa, \\
\text{finite but } \neq \pm 1, 0 & \leftrightarrow \text{ type I with nonconstant } \varphi, \\
1 \text{ or } -1 & \leftrightarrow \text{ type II,} \\
0 & \leftrightarrow \text{ no blow up at } x = 0,
\end{cases}
\]
where
\[
L^{p-1} = \frac{2}{p-1} \left( N - 2 - \frac{2}{p-1} \right)
\]
and $\kappa$ is defined in (9). Our Theorem 5 corresponds to the second equivalence.

Their techniques for obtaining the results are very different from ours. They obtain a priori bounds for the solutions and their derivatives by using some energy estimates and super solutions. They also work with radial solutions in order to be able to use parabolic estimates for one dimensional equations. These estimates then allow them to obtain the final time blow-up profile both in the case where $\varphi$ is regular and in the case where $\varphi$ is singular, and they prove immediate regularization with self-similar rate also for non-minimal $L^1$-solutions.

Non-uniqueness of $L^1$-continuations of $u$ for $f(u) = u^p$ was proved in [21].

Our technique of proving Theorem 3 is based on the variation of constants formula and certain semigroup estimates. The assumptions, here, are not very strong, and the ideas could be used also for different types of equations, but we cannot attack the situation where $\varphi$ is singular.

The next section is devoted to a discussion on some properties of certain semigroups. We prove that the semigroup generated by the operator $A = \Delta - \frac{\kappa^2}{2} \nabla + \Phi$ has specific regularization properties, similar to those of the semigroup generated by the Hermite operator $A = \Delta - \frac{\kappa^2}{2} \nabla$, in case the function $\Phi = \Phi(y)$ decays to 0 as $|y| \to \infty$.

In the third section, we prove Theorem 3 by using the variation of constants formula, the semigroup estimates from Section 2, and some properties of the solution $u$ of (1), specifically, the blow-up rate and the fact that
\[
|\nabla u(x,t)| \leq \sqrt{2} e^{\max_{x \in \Omega} u(x,t)/2}
\]  
for every $x \in B_R(0)$ and $t \in (0, T]$; see the paper [22].

In the last section, we demonstrate that the results in [1] can be proved also without the assumption that $u$ is radially decreasing and thereby prove Theorem 2.

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2. Semigroup estimates

To study the convergence (2) in more detail, define the similarity variables
\[
s = -\log(T-t) \quad \text{and} \quad y = \frac{x}{\sqrt{T-t}},
\]
and let
\[
w(y,s) = \log(T-t) + u(x,t)
\]
for $|y| \leq e^{s/2} R$ and $s \in [-\log(T), \infty)$. Here, $u$ solves (1) with $f(u) = e^u$. Then $w$ satisfies
\[
w_s = \Delta w - \frac{y}{2} \nabla w + G(w), \quad \text{for } |y| \leq e^{s/2} R, \quad s > -\log(T),
\]
with $w(y,s) = 0$ for $|y| = e^{s/2} R$, and
\[
G(w) = e^w - 1.
\]
The convergence in (2) is equivalent to $w \to \varphi$ uniformly on compact sets as $s \to \infty$. Even though, we take the equation with exponential nonlinearity to be the model problem, all the analysis goes through for a large class of nonlinearities.

Denoting $W = w - \varphi$ one notices that $W$ verifies
\[
W_s = \Delta W - \frac{y}{2} \nabla W + e^\varphi W + e^\varphi (e^W - 1 - W) = \Lambda W + e^\varphi (e^W - 1 - W)
\]
for $|y| \leq R e^{s/2}$ and $s > -\log(T)$, where we have defined the operator
\[
\Lambda = \Delta - \frac{y}{2} \nabla + \Phi,
\]
with $\Phi = e^\varphi$. 

The idea of the proof of Theorem 3 is simple. We prove that the convergence in (2) also holds for $|y| \leq C(T-t)^{-1/2} = C\varepsilon^{1/2}$. The claim is then achieved by using the asymptotics (4) of $\varphi$.

To obtain the convergence for $|y| \leq C\varepsilon^{1/2}$, we take a look at the shifted function $\mathcal{W}(y + e^{t/2}\xi, s)$ as $s$ tends to infinity. Thus, it is convenient to define the shifted and weighted $L^q$-norms as follows

$$
\mathcal{N}_q^\xi(\varphi) = \sup_{|\xi| \leq r} \left( \int_{\mathbb{R}^N} |\varphi(y)|^q e^{-(y-\xi)^2/4} dy \right)^{1/q} \text{ for } r \geq 0,
$$

and

$$
\mathcal{L}_q^\xi(\varphi) = \left( \int_{\mathbb{R}^N} |\varphi(y)|^q e^{-(y-\xi)^2/4} dy \right)^{1/q} \text{ for } \xi \in \mathbb{R}^N.
$$

In what follows, we consider the semigroup generated by $\Lambda$ and assume only that $\Phi > 0$ is bounded and verifies

$$
\Gamma = \max_{y \in \mathbb{R}^N} \Phi(y) < \infty \text{ and } \Phi(y) \leq \frac{C}{|y|^2}
$$

for $y \in \mathbb{R}^N \setminus \{0\}$ and some constant $C > 0$. We want to prove the necessary estimates that characterize the regularizing properties of the semigroup $\{e^{\Lambda t}\}$. Defining

$$
A = \Delta - \frac{y^2}{2}
$$

to be the standard Hermite operator, we know that $A$ and $\Lambda$ are self-adjoint operators with domain $H_0^2(\mathbb{R}^N)$, which denotes the weighted Sobolev space with weight $\rho = \rho(y) = e^{-|y|^2/4}$. They generate strongly continuous semigroups in $L^2_\rho(\mathbb{R}^N)$, which we denote by $\{e^{At}\}_{t \geq 0}$ and $\{e^{\Lambda t}\}_{t \geq 0}$ respectively. We use the notation $\| \cdot \|_{L^2_\rho}$ for the norm in $L^2_\rho(\mathbb{R}^N)$.

One has the following formula for the action of the semigroup $e^{At}$ on functions $\psi \in L^2_\rho(\mathbb{R}^N)$,

$$
e^{At}\psi(y) = \frac{1}{(4\pi(1-e^{-t}))^{N/2}} \int_{\mathbb{R}^N} \exp \left( -\frac{(ye^{-t/2} - \lambda)^2}{4(1-e^{-t})} \right) \psi(\lambda) d\lambda, \quad \text{for } t > 0.
$$

Because the spectrum of $A$ consists of non-positive real numbers, one also knows that there exists a constant $C > 0$ such that

$$
\|e^{At}\psi\|_{L^2_\rho} \leq C \|\psi\|_{L^2_\rho}
$$

for every $\psi \in L^2_\rho(\mathbb{R}^N)$ and $t \geq 0$. Because we assumed (16), it follows that

$$
\|e^{\Lambda t}\psi\|_{L^2_\rho} \leq C \Gamma^{1/2} \|\psi\|_{L^2_\rho}
$$

for some constant $C > 0$ and every $t \geq 0$.

The following regularizing property for the semigroup generated by $A$ can be found in [13]. The semigroup $\{e^{At}\}$ regularizes in the sense that it maps functions from $L^2_\rho(\mathbb{R}^N)$ to $L^q_\rho(\mathbb{R}^N)$ for any $q > \beta$ if $t$ is large enough. The following proposition states this property in terms of the shifted norms.

**Proposition 2.1**

Assume $1 < q, \beta < \infty$ and $r, \tau > 0$. Set $\beta' = \frac{\beta}{\beta-1}$. Then for any $t > 0$ and any $\psi$ with $\mathcal{N}_{\beta'}^q(\psi) < \infty$, one has

$$
\mathcal{N}_q^\xi(e^{At}\psi) \leq \frac{1}{(4\pi(1-e^{-t}))^{N/2}} \left( \frac{4\pi}{\beta'(\beta - 1 - e^{-t})} \right)^{N/2} \left( \frac{4\pi}{\beta - 1 - (q-1)e^{-t}} \right)^{N/2} \exp \left( \frac{e^{-t}(r - \tau e^{t/2})^2}{4(\beta - 1 - (q-1)e^{-t})} \right) \mathcal{N}_{\beta'}^r(\psi)
$$

for $t \geq 0$ such that $\beta - 1 - (q-1)e^{-t} > 0$.

We would like to obtain an analogous regularizing property for the semigroup generated by $\Lambda$.

This is done in different cases in Propositions 2.3 and 2.4 and Corollary 2.5 below. We will estimate the norm $\mathcal{L}_{\beta/2}^\xi(e^{\Lambda t}\psi)$ first when $t$ is strictly less than $s$ in Proposition 2.3, then with $t = s$ in Proposition 2.4 and finally for $s$ large and $t$ close to $s$ in Corollary 2.5. Because

$$
\frac{d}{dt} e^{At} = \Lambda e^{At} = (A + \Phi)e^{At},
$$

we know that

$$
e^{At}\psi = e^{At}\psi + \int_0^t e^{A(t-s)}\Phi e^{At}\psi dt.
$$

The next proposition restates Proposition 2.1 using the $L^q$-norms instead of $\mathcal{N}_{\beta'}^q$-norms.
Proposition 2.2
Assume that $1 < q, \beta < \infty$ and $|\mu| \geq 0$. Then for any $\hat{\tau}$ such that $qe^{-\hat{\tau}((\beta - 1 + e^{-\hat{\tau}})^{-1}} < 1$ there exists a constant $C$ such that

$$L^q_{e^{\hat{\tau}/2 \hat{\mu}}} (e^{\hat{\tau} \psi}) \leq C(1 - e^{-\hat{\tau}})^{-\epsilon} L^\beta_{\lambda}(\psi)$$

for $\epsilon = N/2\beta$ and any $t > \hat{\tau}$. The constant $C$ is independent of $\mu$ and depends only on $\beta$, $q$, $N$ and $\hat{\tau}$.

Proof
From the proof of Proposition 2.1. in [13], one obtains that

$$\mathbf{L}_t^p e^{\hat{\tau}/2 \hat{\mu}} (e^{\hat{\tau} \psi})^q \leq \mathbf{C}(\beta) (1 - e^{-\hat{\tau}})^{\mathbf{N}_t^\beta}$$

By taking $\xi = e^{\hat{\tau}/2 \mu}$ and by a change of variables, we immediately have that

$$L^q_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi})^q \leq C(\beta) \left(1 - e^{-\hat{\tau}}\right)^{\mathbf{N}_t^\beta} L^\beta_{\lambda}(\psi)$$

which gives the claim provided that $t > \hat{\tau}(q, \beta)$ because the function $qe^{-\hat{\tau}((\beta - 1 + e^{-\hat{\tau}})^{-1}}$ is decreasing for $t > 0$. \hfill \Box

Using this result, we can prove regularizing properties also for the semigroup generated by the operator $\Lambda$. The first is the following.

Proposition 2.3
Let $\Phi$ satisfy (16). For every $\beta > N/2$, $0 < \delta < \beta - N/2$, and $r > 0$ there exist constants $C = C(\beta, \delta, r, \Gamma, \epsilon) = \epsilon(\beta > 0)$ and $M = M(\delta, \beta, r, \Gamma) > 0$, such that

$$L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) \leq C(1 - e^{-\hat{\tau}})^{-\epsilon} L^\beta_{e^{\hat{\tau}/2 \lambda}}(\psi)$$

which holds for every positive $t$ and $0 < t + M < s$, for every $p \in (1, \beta - \delta)$, every $|\xi| \geq r$, and non-negative $\psi \in \mathbf{L}^p(\mathbf{R}^N)$.

Proof
Consider $\beta, \delta, r, \Phi, \xi, \psi$ as in the claim. By the variation of constants formula,

$$L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) \leq L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) + \int_0^t L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) d\tau.$$

The previous Proposition 2.2 implies that

$$\sup_{\rho \in (1, \beta - \delta/3)} L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) \leq C_1(\delta, \beta)(1 - e^{-\hat{\tau}})^{-\epsilon} L^\beta_{e^{\hat{\tau}/2 \lambda}}(\psi)$$

for some $C_1(\delta, \beta) > 1$ and $\epsilon = \epsilon(\beta) = N/2\beta < 1$ for $t$ close to 0.

Now, we want to write the estimates for the integral part in (23). To that end, by using first the previous proposition and then Hölder’s inequality, we notice that there exists a constant $C_2(\beta, \beta)$ such that

$$L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) \leq C_2(\beta, \beta) (1 - e^{-\hat{\tau}})^{-\epsilon} L^\beta_{e^{\hat{\tau}/2 \lambda}}(\psi)$$

for any $p \leq \beta - \delta$ and $\beta' = \beta - 2/3$. Because $|\Phi| \leq \beta'$ and verifies (16), and because $|\xi| \geq r$, one can estimate, for any $\alpha > 1$ and $t \geq 0$,

$$L^\alpha_{e^{\hat{\tau}/2 \lambda}}(\Phi) = \int_{|y| < e^{\hat{\tau}/2} |\xi|/2} \Phi(y) e^{-\gamma - (\gamma - e^{\hat{\tau}/2})^2/4} dy + \int_{|y| > e^{\hat{\tau}/2} |\xi|/2} \Phi(y) e^{-\gamma - (\gamma - e^{\hat{\tau}/2})^2/4} dy$$

$$\leq C_3 \Gamma^{\alpha} e^{N/2 |\xi|^2/16} + C_4 e^{-\alpha \tau} \Phi(\xi) \int_{|y| > e^{\hat{\tau}/2} |\xi|/2} e^{-\gamma - (\gamma - e^{\hat{\tau}/2})^2/4} dy \leq C_4 (t, \Gamma, \alpha) e^{-\alpha \tau}.$$

By the boundedness of $\Phi$, we can take $K > 0$ such that, using Proposition 2.2, we have

$$L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) \leq L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} + \Gamma(\tau-K)/2 \psi}) \leq C_5(\Gamma, K) L^\beta_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} + \Gamma(\tau-K)/2 \psi})$$

for $p < \beta - \delta$ and $\beta' = \beta - 2/3$. Therefore, by using the previous inequalities and Hölder’s inequality, we have

$$L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} \psi}) \leq L^p_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} + \Gamma(\tau-K)/2 \psi}) \leq C_5(\Gamma, K) L^\beta_{e^{\hat{\tau}/2 \lambda}} (e^{\hat{\tau} + \Gamma(\tau-K)/2 \psi})$$

for $p < \beta - \delta$ and $\beta' = \beta - 2/3$. Therefore, by using the previous inequalities and Hölder’s inequality, we have
for any \( s > \tau \geq K/2 \). By Proposition 2.2, we may also assume that the constant \( C_5(\Gamma, K) > 1 \) is such that

\[
\mathcal{L}^{\beta}_{\varepsilon/2 \xi}(e^{\Lambda_\tau} \psi) \leq \mathcal{L}^{\alpha}_{\varepsilon/2 \xi}(e^{(\beta+\Gamma)\tau} \psi) \leq C_5(\Gamma, K)(1 - e^{-\tau})^{-\varepsilon} \mathcal{L}^{\beta}_{\varepsilon(1-\tau)/2 \xi}(\psi) \tag{28}
\]

for all \( \tau < K/2 \) and \( s > \tau \). Using (26) and assuming that \( M \) is large, we obtain

\[
\int_0^t \mathcal{L}^{\alpha}_{\varepsilon(t+\tau)/2 \xi}(\Phi)(1 - e^{-(t+\tau)})^{-\varepsilon} (1 - e^{-\tau})^{-\varepsilon} d\tau \\
\leq \int_0^\theta \ldots d\tau + \int_\theta^t \ldots d\tau + C_6(\theta, \tau, \Gamma) \int_\theta^t e^{-(t+\tau)} d\tau (1 - e^{-\theta})^{-\varepsilon} < \frac{1}{2C_2(\delta, \beta)C_5(\Gamma, K)} \tag{29}
\]

for every \( s \geq M/2 \) and \( t \in (0, s - M/2) \). To show (29), first take \( \theta \) small and then \( M \) large. Similarly, let \( M \) be large enough so that also

\[
\int_{K/2}^s \mathcal{L}^{\alpha}_{\varepsilon(s+\tau)/2 \xi}(\Phi)(1 - e^{-(s+\tau)})^{-\varepsilon} (1 - e^{-(s-K/2)})^{-\varepsilon} d\tau < \frac{1}{8C_2(\delta, \beta)C_5(\Gamma, K)} \tag{30}
\]

for every \( s > t + M > K/2 + M \).

For \( s > M \), let \( t_\beta(s) \) be the supremum of such \( t \in (0, s - M) \) for which

\[
\sup_{\rho \in (1, \beta - \delta)} \mathcal{L}^{\rho}_{\varepsilon/2 \xi}(e^{\rho \tau} \psi) = 2C_1(\delta, \beta)(1 - e^{-\tau})^{-\varepsilon} \mathcal{L}^{\beta}_{\varepsilon(1-\tau)/2 \xi}(\psi).
\]

By inequality (24) and because the integral part in (23) tends to zero as \( t \to 0 \), we know that \( t_\beta(s) \) is positive. Let

\[
t(s) = \sup_{s' \in (M, s)} (s' - t_\beta(s')).
\]

We want to show that \( t(s) \leq M \) for every \( s > M \), which implies \( t_\beta(s) \geq s - M \) for every \( s > M \). This will give the claim by the definition of \( t_\beta(s) \).

Let us first show that \( t_\beta(s) \geq K/2 \). Assume, to obtain a contradiction, that \( t_\beta(s) < K/2 \) for some \( s > M \). Then \( t_\beta(s) < M/2 < s - M/2 \), because we may assume \( M > K \). We may estimate the integral part of (23) through

\[
\int_0^{t_\beta(s)} \mathcal{L}^{\rho}_{\varepsilon/2 \xi}(e^{\rho t_\beta(s)} \Phi e^{\Lambda_\tau} \psi) d\tau \leq C_2 \int_0^{t_\beta(s)} (1 - e^{-(t_\beta(s) - \tau)})^{-\varepsilon} \mathcal{L}^{\alpha}_{\varepsilon(t_\beta(s) + \tau)/2 \xi}(\Phi) \mathcal{L}^{\beta}_{\varepsilon(t_\beta(s) + \tau)/2 \xi}(e^{\Lambda_\tau} \psi) d\tau \\
\quad < C_2 \int_0^{t_\beta(s)} (1 - e^{-(t_\beta(s) - \tau)})^{-\varepsilon} \mathcal{L}^{\alpha}_{\varepsilon(t_\beta(s) + \tau)/2 \xi}(\Phi) C_5(1 - e^{-\tau})^{-\varepsilon} \mathcal{L}^{\beta}_{\varepsilon(t_\beta(s) + \tau)/2 \xi}(\psi) d\tau \\
\quad < \frac{1}{2} \mathcal{L}^{\beta}_{\varepsilon(t_\beta(s)/2 \xi)}(\psi), \tag{31}
\]

by inequalities (25), (28), and (29). Here, the constants \( C_2 \) and \( C_5 \) are as above, even though — for the sake of notation — their dependence on the parameters is not written out explicitly. Using this, together with (23) and Proposition 2.2, we have

\[
\sup_{\rho \in (1, \beta - \delta)} \mathcal{L}^{\rho}_{\varepsilon/2 \xi}(e^{\rho t(s)} \psi) \leq C_1(\delta, \beta)(1 - e^{-t(s)})^{-\varepsilon} \mathcal{L}^{\beta}_{\varepsilon(t(s)/2 \xi)}(\psi) + \frac{1}{2} \mathcal{L}^{\beta}_{\varepsilon(t(s)/2 \xi)}(\psi) d\tau \\
\quad \leq \frac{3}{2} C_1(\delta, \beta)(1 - e^{-t(s)})^{-\varepsilon} \mathcal{L}^{\beta}_{\varepsilon(t(s)/2 \xi)}(\psi)
\]

by assuming \( C_1(\delta, \beta) \geq 1 \). This contradicts the definition of \( t_\beta(s) \). Therefore, \( t_\beta(s) \geq K/2 \) for every \( s > M \) and so \( t(s) \leq s - K/2 \).

Let us then show that \( t(s) \leq M \). We proceed again by contradiction and assume that \( t(s) > M \) for some \( s > M \), which implies \( t(s) \in (M, s - K/2) \). Without loss of generality, we may assume that \( t(s) = s - t_\beta(s) \), which gives \( t_\beta(s) \in [K/2, s - M) \). For \( t \in [K/2, t_\beta(s)] \), we have that \( \tilde{t} = t(s) + s - K/2 \), then \( s - \tilde{t} = s - t - s + t - K/2 \) for \( -t_\beta(s), -K/2 \) and so \( s - \tilde{t} < s \). Therefore, \( t(s) \leq t(\tilde{t}) \) and defining \( \tilde{s} = \tau - K/2 \), we have \( \tilde{s} = s - t(\tilde{s}) \leq t_\beta(\tilde{s}) \). Thus

\[
\mathcal{L}^{\beta}_{\varepsilon(t(s) + \tau - K/2)/2 \xi}(e^{\Lambda_\tau} \psi) \leq \sup_{\rho \in (1, \beta - \delta)} \mathcal{L}^{\rho}_{\varepsilon/2 \xi}(e^{\rho \tau} \psi) \leq 2C_1(\delta, \beta)(1 - e^{-(s-K/2)})^{-\varepsilon} \mathcal{L}^{\beta}_{\varepsilon(t(s)/2 \xi)}(\psi) \tag{32}
\]

by the definition of \( t_\beta(\tilde{s}) \).

Precisely as in (31), one obtains

\[
\int_0^{K/2} \mathcal{L}^{\beta}_{\varepsilon/2 \xi}(e^{\rho \tau} \psi) d\tau < \frac{1}{2} \mathcal{L}^{\beta}_{\varepsilon(t(s)/2 \xi)}(\psi),
\]
and, by inequalities (25), (27), (30), and (32), we can estimate
\[
\int_{K/2}^{t(s)} L^p_{e^{t/2}} \left( e^{A(t(s))} \Phi e^{At} \right) d\tau \\
\leq C_2 \int_{K/2}^{t(s)} L^p_{e^{(t-K)(\xi/2)}} \left( \Phi \left( 1 - e^{-\Phi t(\xi/2)} \right) - e' L^p_{e^{(t-K)(\xi/2)}} \right) d\tau \\
\leq C_2 C_5 \int_{K/2}^{t(s)} L^p_{e^{(t-K)(\xi/2)}} \left( \Phi \left( 1 - e^{-\Phi t(\xi/2)} \right) - e' \left( 1 - e^{-(t-K)/2} \right) - e \right) d\tau C^p_{e^{(t-K)(\xi/2)}}(\Psi),
\]
where the dependence of C_1, C_2, and C_5 on the parameters is not explicitly written out.
This implies that
\[
\sup_{p \in (1, \beta, \delta)} L^p_{e^{t/2}} \left( e^{A(t(\xi))} \Psi \right) \leq \left( 1 + \frac{1}{2} + \frac{1}{4} \right) C_1(\beta, \delta) \left( 1 - e^{-\Phi t(\xi/2)} \right) - e L^p_{e^{(t-K)(\xi/2)}}(\Psi),
\]
which contradicts the definition of t_B(s). Therefore, the only possibility is that t(s) \leq M and so the claim is proved.

In the previous proposition, we assumed that t < s - M for some large M. The next proposition deals with the case t = s > M for some large M.

**Proposition 2.4**
Let \( \Phi \) satisfy (16). For every \( r \in (0, 1) \) there exist constants \( M = M(r, \Gamma) \) and \( C = C(r, \Gamma) \) such that
\[
L^2_{e^{t/2}}(e^{At}) \leq C(r, \Gamma, N) \| \Psi \|_{L^2_{e^{t}}},
\]
for every \( s > M, |\xi| \in (r, 1/r), \) and every non-negative \( \psi \in L^2_{e^{t}}(\mathbb{R}^N) \).

**Proof**
Fix \( r > 0 \) and take \( \theta = \log(2N) \). Assume that \( s \geq M + \theta \), where \( M > \theta \) will be defined later, and let \( |\xi| \in (r, 1/r) \).
Let us first note that for any \( 1 < p < q < \infty \) and \( p/q + 1/y = 1 \), we have
\[
L^p_{e^{t/2}}(\Psi) = \int_{\mathbb{R}^N} |\Psi|^p e^{-p|y|^2/4} e((1-y)\xi/2)|^{p/q} e^{-\frac{p}{q}|y^2/4} dy \\
= \left( \int_{\mathbb{R}^N} |\Psi|^p e^{-|y|^2/4} e^{-(y-\xi)\xi/2}|^{p/q} e^{-(y-\xi)^2/4} dy \right)^{1/y} \\
= \left( \int_{\mathbb{R}^N} |\Psi|^p e^{-|y|^2/4} e^{-(y-\xi)|y-\xi|^2/4} dy \right)^{1/y} \\
= \left( \int_{\mathbb{R}^N} |\Psi|^p e^{-|y|^2/4} e^{-(y-\xi)|y-\xi|^2/4} dy \right)^{1/y} = C(p, q, r) \| \Psi \|_{L^2_{e^{t}}}. \tag{33}
\]

If \( t \leq M \), then \( s - t \geq \theta \) and so by using first Proposition 2.2, then Hölder’s inequality, then the fact that \( L^{21/2}_{e^{t/2}}(\Psi) \leq C(r, \Gamma)e^{-t} \) by (26), and finally above inequality (33), we have
\[
L^2_{e^{t/2}}(e^{At}) \leq C_1(\theta) L^{3/2}_{e^{t/2}}(e^{At}) \leq C_1(\theta) L^{3/2}_{e^{t/2}}(e^{At}) \leq C_1(\theta) L^{7/4}_{e^{t/2}}(e^{At}) \leq C_2(\theta, r, \Gamma) e^{-t} \| \Psi \|_{L^2_{e^{t}}} \leq C_3(\theta, r, \Gamma, M) e^{-t} \| \Psi \|_{L^2_{e^{t}}} \leq C_4(\theta, r, \Gamma, M) e^{-t} \| \Psi \|_{L^2_{e^{t}}}.
\tag{34}
\]

If \( 0 < t < s - \theta \), then because of our choice of \( \theta \), we can use Proposition 2.2 with \( q = 2 \) and \( \beta = 3/2 \) for the first inequality below and Hölder’s inequality for the second, to obtain
\[
L^2_{e^{t/2}}(e^{At}) \leq C_1(\theta) L^{1/2}_{e^{t/2}}(e^{At}) \leq C_1(\theta) L^{1/2}_{e^{t/2}}(e^{At}) \leq C_4(\theta, r, \Gamma, M) e^{-t} L^2_{e^{t/2}}(e^{At}), \tag{35}
\]
where the last inequality is again due to (26).
If \( M \leq s - \theta \leq t \leq s \), then using first Proposition 2.2 with exponents \( q = 2 \) and \( \beta = 2 \), we obtain
\[
L^2_{e^{t/2}}(e^{At}) \leq C_2(s - (s-t))^{-1/2} L^2_{e^{t/2}}(e^{At}) \leq C_2(s - (s-t))^{-1/2} L^2_{e^{t/2}}(e^{At}) \leq C_6(\theta, r, \Gamma) e^{-t} \| \Psi \|_{L^2_{e^{t}}}.
\tag{36}
\]
Define $W(s) = L^2_{e^{t/2}}(e^{\Lambda t}\psi)$ and use the variation of constants formula (21) together with Proposition 2.1 and the above estimates (34) – (36) to observe that

$$W(s) \leq L^2_{e^{t/2}}(e^{\Lambda t}\psi) + \int_0^s L^2_{e^{t/2}}(e^{(s-t)} \Phi e^{\Lambda t}\psi)dt$$

$$\leq C_7 \|\psi\|_{L^2}^2 + \int_0^s C_3 e^{-t} \|\psi\|_{L^2}^2 dt + \int_0^s C_4 e^{-t} W(t) dt + \int_0^s C_6 e^{-t} (1 - e^{-(s-t)})^{-1/2} W(t - \theta) dt$$

(37)

for every $s \geq M + \theta$ and for some $C_7 > 0$ arising from estimate (20).

By Proposition 2.1, we have, for every $s \in [M, M + 2\theta]$, the estimate

$$W(s) \leq C_8 e^{\Gamma(M + \theta)} \|\psi\|_{L^2}^2$$

(38)

Take $M = M(\theta, r, \Gamma)$ large enough such that

$$C_4(\theta, r, \Gamma)e^{-M} + C_6(\theta, r, \Gamma)e^{-M} \int_0^\theta (1 - e^{-1})dt < \frac{1}{3}.$$  

(39)

Then let $\tilde{C}(\theta, r, \Gamma) = C_8 e^{\Gamma(M + \theta)}$ and take $K = K(\theta, r, \Gamma) > 1$ large enough such that

$$\frac{C_7}{K\tilde{C}(\theta, r, \Gamma)} + \frac{C_3(\theta, r, \Gamma, M)}{K\tilde{C}(\theta, r, \Gamma)} < \frac{1}{3}.$$  

(40)

Let $\tilde{s}$ be the supremum of such $s' > M$ for which $W(s) \leq \tilde{K}(\theta, r, \Gamma) \|\psi\|_{L^2}^2$ for every $s \in [M, s']$. By (38), we know that $\tilde{s} \geq M + 2\theta$.

Assuming that $\tilde{s} < \infty$, by using (37), (39), and (40), we have for every $s \in [M + 2\theta, \tilde{s}]$ that

$$W(s) \leq C_7 \|\psi\|_{L^2}^2 + C_3 \|\psi\|_{L^2}^2 + C_4(e^{-M} - e^{-(\theta - s)})K\tilde{C}(\theta, r, \Gamma) \|\psi\|_{L^2}^2 + C_6 e^{-M}K\tilde{C}(\theta, r, \Gamma) \int_0^\theta (1 - e^{-1})dt \|\psi\|_{L^2}^2$$

$$\leq K\tilde{C}(\theta, r, \Gamma) \|\psi\|_{L^2}^2 + \frac{C_7}{K\tilde{C}(\theta, r, \Gamma)} + \frac{C_3}{K\tilde{C}(\theta, r, \Gamma)} + C_4 e^{-M} + C_6 e^{-M} \int_0^\theta (1 - e^{-1})dt \leq \frac{2}{3} K\tilde{C}(\theta) \|\psi\|_{L^2}^2,$$

which contradicts the definition of $\tilde{s}$ and so $\tilde{s} = \infty$. Consequently, we have obtained that

$$W(s) = L^2_{e^{t/2}}(e^{\Lambda t}\psi) \leq K(\theta, r, \Gamma)\tilde{C}(\theta, r, \Gamma) \|\psi\|_{L^2}^2$$

for every $s > M(\theta, r, \Gamma)$. This gives the claim. \qed

The following corollary is almost a restatement of the previous proposition, but instead of considering $L^2_{e^{t/2}}(e^{\Lambda t}\psi)$ for $s = t$, we allow $s - t$ to be positive and bounded.

**Corollary 2.5**

Let $\Phi$ satisfy (16). For every $r \in (0, 1)$ and $M > 0$, there exist constants $K = K(M, r, \Gamma)$ and $C = C(M, r, \Gamma, \psi)$ such that

$$L^2_{e^{(s-t)/2}}(e^{\Lambda(s-t)}\psi) \leq C(M, r, \Gamma) \|\psi\|_{L^2}^2$$

for every $t \in [s_0, s + M]$, $s > s_0 + K(M, r, \Gamma)$, $s_0 \geq 0$, every $|\xi| \geq (r, 1/r)$, and every non-negative $\psi \in L^2_{\psi}(\mathbb{R}^N)$.

**Proof**

By the previous Proposition 2.4, there exist constants $M'(r, \Gamma)$ and $C(M, r, \Gamma)$ such that

$$L^2_{e^{-|\eta|/2}}(e^{\Lambda |\eta|/2}) \leq C(M, r, \Gamma) \|\psi\|_{L^2}^2$$

for every $\tau > M'(r, \Gamma)$ and $|\eta| \in (r, e^{M'/r})$.

Assume that $s, t$, and $\xi$ are as in the claim, and let $\tau = s - t$ and $\eta = e^{-s_0 - \xi}$. Then $\tau > K(M, r, \Gamma) - M > M'(r, \Gamma)$ by choosing $K$ large enough, and $|\eta| \in (r, e^{M'/r})$, which implies that

$$L^2_{e^{(s-t)/2}}(e^{\Lambda(s-t)}\psi) = L^2_{e^{-|\eta|/2}}(e^{\Lambda |\eta|/2}) \leq C(M, r, \Gamma) \|\psi\|_{L^2}^2,$$

and so the claim is proved. \qed

We have now the desired results that describe the regularizing properties of the semigroup generated by $\Lambda$. These results will be used in the next section to prove Proposition 3.4.
3. The final time blow-up profile

In this section, we prove Theorem 3. Let \( u \) and \( \psi \) be as in Theorem 3, define the usual similarity variables \( s \) and \( y \) as in (13), and let

\[
W(y, s) = \log(T - t) + u(x, t) - \psi(y).
\]

Our assumptions in Theorem 3 imply that \( W(y, s) \to 0 \) uniformly for \( y \) in compact sets as \( s \to \infty \).

Generally speaking, to prove Theorem 3, we want to show that the \( L^2_{t=\rho(s)}E^{s/2} \)-norm of \( W(t, s) \) can be estimated by the \( L^2_{\infty}E^{s/2} \)-norm of \( W(t, s) \). This is done in Proposition 3.4. In the proof of that proposition, we utilize the regularizing properties of the semigroup \( \{e^{At}\} \), obtained in the previous section. Using this results and the \( L^2 - L^\infty \) regularity of the semigroup generated by \( A \), one has that \( |W(e^{(s-\rho(s))2}E^{s/2}E, s)| \) tends to zero as \( s \to \infty \) to infinity. By the definition of \( W \) and by the asymptotics (4) of \( \psi \), we obtain the claim at the very end of this section.

Before stating and proving Proposition 3.4, we have to consider the properties of the function \( W \) in more detail. We also need some auxiliary results. In Proposition 3.1, we demonstrate how to move from \( L^2(E^{s/2}E) \)-norm to \( L^{\infty}(E^{s/2}E) \)-norm, in Proposition 3.2, we consider the \( L^2 - L^\infty \) regularization of \( e^{At} \), and in Proposition 3.3, we estimate the norm of the nonlinearity appearing in the equation for \( W \).

Because the function, \( W \) is defined on some \( s \) dependent subset of \( \mathbb{R}^N \), one needs to extend it to \( \mathbb{R}^N \). Because the blow-up set is a compact set of \( B(\mathbb{R}) \), we can take \( R \in (0, R) \) such that \( u(x, t) \) is bounded for \( (x, t) \in B(\mathbb{R}) \times (0, T) \). Then, let \( \zeta \) be a smooth function equal to 1 for \( |x| \leq R \) and equal to 0 for \( |x| > R \), and let

\[ \tilde{W}(y, s) = \zeta(e^{-s/2}y)W(y, s). \]

Now \( \tilde{W} \) is defined in the whole space \( \mathbb{R}^N \), and it satisfies the equation

\[ \tilde{W}_t = \Lambda \tilde{W} + \tilde{h}, \]

where

\[ \tilde{h} = -e^{-s} \Delta \zeta W - 2e^{-s/2} \nabla \zeta \cdot \nabla W + \frac{e^{-s/2}y}{2} \cdot \nabla \zeta W + \zeta e^s(e^W - 1) - e^s \tilde{W}. \]

for \( |y| \leq Re^{s/2} \) and \( \tilde{h} = 0 \), for \( |y| > Re^{s/2} \). Here, we use the notation \( \nabla \zeta \) for \( \nabla \zeta (e^{s/2}y) \), and the same applies to the Laplacian of \( \zeta \).

Because blow-up is assumed to be of type I, we obtain, by using (12),

\[ |\nabla \tilde{W}(y, s)| = |\nabla W(y, s)| = \sqrt{T - t} \nabla u(x, t) - \nabla \psi(y) | \leq C \]

for \( |y| > Re^{s/2} \). For such \( \zeta \), one also has \( \tilde{h} = e^s(e^{\tilde{W}} - 1 - \tilde{W}) \). Moreover, type I blow-up implies that \( W \) is bounded from above, which gives that the estimates \( |\tilde{h}| \leq C|\tilde{W}| \) and \( |\tilde{h}| \leq C|\tilde{W}|^2 \) are valid for some constant \( C > 0 \).

For \( |y| = |x|e^{s/2} \) and \( |x| \in (R_1, R) \), one has

\[ |\nabla \tilde{W}(y, s)| \leq e^{-s/2} |\nabla \zeta (e^{-s/2}y) | (u(x, t) + (s + \psi(y))) + |\zeta(e^{-s/2}y)| \sqrt{T - t} \nabla u(x, t) - \nabla \psi(y) | \leq C \]

by using (12) and the asymptotic behavior (4) of \( \psi \). Similarly, we have

\[ |\tilde{W}(y, s)| \leq |W(y, s)| \leq u(x, t) + |\psi(e^{s/2}x) + s| \leq C. \]

Therefore, we obtain the estimate

\[ |\tilde{h}| \leq A_0 \]

for some finite constant \( A_0 > 0 \).

For \( |y| > Re^{s/2} \), it holds that \( \tilde{W} = 0 \) and \( \tilde{h} = 0 \).

Defining

\[ Z(y, s) = |\tilde{W}(y, s)| = |\zeta(e^{-s/2}y) | - s + u(e^{-s/2}y, T - e^{-s}) - \psi(y)|, \]

we have obtained that \( Z \) satisfies

\[ Z_t \leq \Lambda Z + h + A_0 \chi, \]

where \( \chi(y, s) = \chi_{\{y > Re^{s/2}\}}(y) \) is the characteristic function of the set \( \{y \in \mathbb{R}^N : |y| > Re^{s/2}\} \). The function \( h \) verifies

\[ h \leq A_1 Z \quad \text{for} \quad y \in \mathbb{R}^N \]

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and, for some \( \theta(y, s) \in (0, \tilde{W}(y, s)) \),
\[
h = \frac{1}{Z} e^{\theta + \tilde{W}} Z^2 \leq A_2 Z^2 \quad \text{for } |y| \leq R_1 e^{t/2},
\]
(43)

where \( A_1 \) and \( A_2 \) are some positive constants. Because \( \psi \) is assumed to be as in Theorem 1, we have that
\[
\max_{y \in \mathbb{R}^n} \psi(y) = \psi(0) = \alpha.
\]

Therefore, \( Z \) satisfies also the inequality
\[
||Z(s)||_{L^p} \leq A_3 \quad \text{for } s \geq 0 \quad \text{and} \quad ||Z(s)||_{L^q} \rightarrow 0
\]
as \( s \rightarrow \infty \). Moreover,
\[
||\nabla Z(s)|| \leq A_4 \quad \text{for } (y, s) \in \mathbb{R}^N \times (0, \infty).
\]
(46)

In what follows, we will consider the parameters \( \{A_1, A_2, \ldots, A_7\} \) and \( \Gamma \) as given. The constants below may depend on these parameters, but it will not be stated explicitly.

Let us now derive some estimates for the \( L^q_{\xi} \) -norm of \( Z \). To that end, let \( t \in (s - \theta, s) \), \( |\lambda| > R_1 e^{t/2} \) and \( |y| < R_1 e^{t/2} / 2 \), which gives
\[
\frac{|ye^{-(s-t)/2} - \lambda|}{\sqrt{1 - e^{-s+t}}} \geq \frac{e^{t/2} R_1}{2} > \frac{e^{t/2} R_1}{2}
\]
and, by using the representation formula (17) for the semigroup, we have that, for such \( t \) and \( y \), it holds
\[
e^{\lambda(s-t)} \chi(y, t) = \frac{1}{(4\pi (1 - e^{-(s-t)})^{N/2}} \int_{\mathbb{R}^N} \exp \left( \frac{-(ye^{-(s-t)/2} - \lambda)^2}{4(1 - e^{-(s-t)})} \right) \chi(y, t) d\lambda \] 
(47)
\[
e^{\lambda(s-t)} \chi(y, t) = \frac{1}{(4\pi (1 - e^{-(s-t)})^{N/2}} \int_{|\lambda| > R_1 e^{t/2}} \exp \left( \frac{-(ye^{-(s-t)/2} - \lambda)^2}{4(1 - e^{-(s-t)})} \right) d\lambda
\]
(48)
\[
e^{\lambda(s-t)} \chi(y, t) \leq \frac{1}{(4\pi (1 - e^{-(s-t)})^{N/2}} \int_{|\lambda| > R_1 e^{t/2}} e^{-|\lambda|^2/4} d\lambda \leq C_1 e^{-t}.
\]
(49)

Above, to be more precise, we could have written \( e^{\lambda(s-t)} \chi(y, t) \) as \( [e^{\lambda(s-t)} \chi(t)(y)] \), but we obey the former option in what follows. If \( t \in (s - \theta, s) \) and \( |y| < R_1 e^{t/2} / 2 \), then
\[
e^{\lambda(s-t)} \chi(y, t) \leq \frac{1}{(4\pi (1 - e^{-(s-t)})^{N/2}} \int_{\mathbb{R}^N} \exp \left( \frac{-(ye^{-(s-t)/2} - \lambda)^2}{4(1 - e^{-(s-t)})} \right) d\lambda = \frac{1}{(4\pi (1 - e^{-(s-t)})^{N/2}} \int_{\mathbb{R}^N} e^{-|\lambda|^2/4} d\lambda = C_2.
\]

Therefore, for any \( \sigma > 0, |\xi| \leq \max(\sigma, e^{t/2} R_1 / 4) \), and \( t \in (s - \theta, s) \), we have
\[
L^q_{\xi}(e^{\lambda(s-t)} \chi(y, t)) \leq C_1 e^{-\sigma t} \int_{|y| < R_1 e^{t/2} / 2} e^{-(y-\xi)^2/4} dy + C_2 \int_{|y| > R_1 e^{t/2} / 2} e^{-(y-\xi)^2/4} dy \leq C_3(\sigma, q)e^{-\sigma t}.
\]

Next, we use the variation of constants formula and (44) to obtain
\[
L^q_{\xi}(Z(s, \cdot)) \leq L^q_{\xi}(e^{(A+\Gamma+A_1)\theta} Z(s, \cdot - \theta)) + \int_{s-\theta}^s L^q_{\xi}(e^{(A+\Gamma+A_1)\theta}(s-t) A_0 \chi(t, \cdot) dt
\]
\[
= L^q_{\xi}(e^{(A+\Gamma+A_1)\theta} Z(s, \cdot - \theta)) + A_0 C_3(\sigma, q) \int_{s-\theta}^s e^{(\Gamma+A_1)(s-t)} e^{-t} dt
\]
\[
= L^q_{\xi}(e^{(A+\Gamma+A_1)\theta} Z(s, \cdot - \theta)) + \frac{A_0 C_3(\sigma, q)}{\Gamma + A_1 + 1} (e^{(\Gamma+A_1+1)\theta} - 1)
\]
(50)

for every \( |\xi| \leq \max(\sigma, e^{t/2} R_1 / 4) \). By Proposition 2.2, one has
\[
L^q_{e^{t/2} \mu}(Z(s, \cdot)) \leq C_4(\sigma, \beta, \theta) \left( L^q_{\mu}(Z(s, \cdot - \theta)) + e^{-s} \right)
\]
(51)
for every $|e^{\beta/\mu}| \leq \max(\sigma, e^{1/2} R_1 / 4)$, $s > \theta$, and $\theta$ such that $qe^{-\theta}(\beta - 1 + e^{-\theta})^{-1} < 1$. By Proposition 2.1, it also holds that

$$N^2_{\gamma}(\zeta, s) \leq e^{(\Gamma + A_1)\gamma} N^2_{\gamma}(\zeta, s - \theta) + C_5^\theta(\theta, \alpha) e^{-s} \leq C_5(\theta, \alpha) \left( \|\zeta(s - \theta)\|_{L^2_p} + e^{-s} \right)$$

(52)

for every $\theta, \alpha > 0$.

We want to consider the solution $W$ by estimating the norm $L^2_{\xi=\eta=0/2}(\zeta(s), s)$ by the norm $\|\zeta(s, s)\|_{L^2_p}$ for $s_0$ large enough. This is done in Proposition 3.4. In this proof, we will need the constructed semigroup estimates from the previous section.

The next proposition is merely a simple change of variables, but it demonstrates how we are able to move from the $L^2_{\xi}$ norm to the $N^2_{\gamma}/|\xi|$ norm.

Proposition 3.1

Let $Z$ be as above. If $s_0 > 1, s' > 0$, and

$$\sup_{s \in (s_0, s_0 + s')} L^2_{\xi=\eta=0/2}(\zeta(s), s) \leq C_1$$

for every $|\xi| = 1$ and for every $\tau \geq 0$, then

$$\sup_{s \in (s_0, s_0 + s')} \sup_{|\xi| \leq 1} L^2_{\xi=\eta=0/2}(\zeta(s), s) < C(C_1).$$

Proof

For any $s \in (s_0, s_0 + s')$, let $\xi(s) \in \mathbb{R}^N$ be such that $|\xi(s)| \leq 1$ and

$$\sup_{|\xi| \leq 1} L^2_{\xi=\eta=0/2}(\zeta(s)) = L^2_{\xi=\eta=0/2}(\zeta(s)).$$

For $s \in (s_0, s_0 + s')$, define a function $\beta$ through $e^{(\beta(s)-\eta=0/2)} = e^{(s-\eta=0/2)}|\xi(s)|$. This gives that $\beta(s) = s + 2 \log(|\xi(s)|) \leq s$. Let $l = [s \in (s_0, s_0 + s') : \beta(s) > s_0]$. Then, for $s \in l$ and for $\xi(s) = \xi(s)/|\xi(s)|$, we have that

$$\sup_{|\xi| \leq 1} L^2_{\xi=\eta=0/2}(\zeta(s)) = L^2_{\xi=\eta=0/2}(\zeta(s)) \leq L^2_{\xi=\eta=0/2}(\zeta(s), s) \leq C_1$$

by assumption because $|\xi(s)| = 1$ and $s - \beta(s) \geq 0$.

Consider then $s \in (s_0, s_0 + s') \setminus l$. Because $\beta(s) \leq s_0$, we can write

$$\sup_{|\xi| \leq 1} L^2_{\xi=\eta=0/2}(\zeta(s), s) \leq L^2_{\xi=\eta=0/2}(\zeta(s), s) \leq N^2_{\gamma}(\zeta, s-1) \leq C_2 \left( \|\zeta(s-1)\|_{L^2_p} + e^{-s} \right) \leq C_3$$

by using (45) and (52), which finishes the proof.

In the next proposition, we consider another type of regularizing property of the semigroup generated by the operator $A$. It is an $L^2 - L^\infty$ regularization for solutions with bounded gradient.

Proposition 3.2

Let $Z$ be as above. Then

$$Z(\xi + \gamma, s) \leq C \left( L^2_{\xi=\eta=0/2}(\zeta(s-1)) + |\gamma| + e^{-s} \right)$$

for any $\gamma$ in $\mathbb{R}^N$, $|\gamma| \leq e^{1/2} R_1 / 2$ and $s \geq 1$.

Proof

Using the variation of constants formula together with inequality (44) and (47), it holds that, for every $|\xi| \leq e^{1/2} R_1 / 2$,

$$Z(\xi, s) \leq e^{(\alpha + \Gamma + A_1)\gamma} Z(\xi, s-1) + A_0 \int_{s-1}^s \sup_{|\xi| \leq e^{1/2} R_1 / 2} e^{(\alpha + \Gamma + A_1)\gamma} \chi(\xi, t) dt \leq e^{\Gamma + A_1 \gamma} Z(\xi, s-1) + A_0 C_1 e^{(\Gamma + A_1)s} \int_{s-1}^s e^{-(\Gamma + A_1)\gamma} e^{-t} dt.$$

By the representation formula (17), we estimate

$$(e^{A_1} Z)(\xi, s-1) = \frac{1}{(1 - e^{-1}) |\gamma|/2} \int_{\mathbb{R}^N} \exp \left( \frac{-e^{-1/2} \xi - \lambda \eta}{4 (1 - e^{-1})} \right) |Z(\lambda, s-1)| d\lambda \leq \frac{1}{(1 - e^{-1}) |\gamma|/2} \left\{ \int_{\mathbb{R}^N} e^{-(1 - e^{-1}) |\eta|^2/2} e^{-e^{-1/2} \xi \eta} d\eta \right\}^{1/2} \left\{ \int_{\mathbb{R}^N} |Z(e^{-1/2} \xi + \eta, s-1)|^2 e^{-(1 - e^{-1}) |\eta|^2} d\eta \right\}^{1/2} = C_2 e^{-1/2} Z(\xi, s-1).$$
Therefore, by using (46), one has for every $|\xi| \leq e^{1/2} R_1/2$ that
\[
Z(\xi + \gamma, s) \leq Z(\xi, s) + A_4 |\gamma| \leq C_3 \left( L^2_{e^{(t-s)/2}} (Z(t, s - 1)) + |\gamma| + e^{-\tilde{\gamma}} \right),
\]
which gives us the claim. \hfill \Box

In the next proposition, we estimate the shifted $L^\beta$-norm of $h(t)$ for some $\beta > 1$. What we want to obtain is that the norm is integrable with respect to $t$ if the shifted $L^2$-norm of $Z$ is bounded.

**Proposition 3.3**

Let $Z$ be as above and let $R_1 e^{s_0/2} > 2$ and $s_0 > 1$. If
\[
\sup_{s \in (s_0, s_0 + s')} L^2_{e^{(t-s)/2}} (Z(t, s + \tau)) \leq B
\]
for every $\xi$ such that $|\xi| = 1$ and for every $\tau \geq 0$, then for any $\beta > 1$ there exist constants $M$, $\theta$, and $C'$ such that
\[
L^\beta_{e^{(t-s)/2}} (h(t, t)) \leq C \left( e^{-(t-s)/2} + C' e^{2(t-s)/2} \right) L^2_{e^{(t-s)/2}} (Z(t, t)) + e^{-\tilde{\gamma}} L^2_{e^{(t-s)/2}} (Z(t, t - \theta)) + e^{-t}
\]
for every $|\xi| = 1$ and $t \in (s_0 + M, s_0 + s')$, provided that $s' > M$. Here, $M$ and $\theta$ depend only on the constants $\beta$, $\Gamma$, $R_1$, $N$, and $\{A_k\}_{k=0}^4$. The constants $C$ and $C'$ may, however, also depend on $\beta$.

**Proof**

By Proposition 3.1, we have that
\[
\sup_{s \in (s_0, s_0 + s')} \sup_{|\xi| \leq 1} L^2_{e^{(t-s)/2}} (Z(t, s)) \leq C_1 (B).
\]
Because our assumptions imply that $e^{(t-s)/2} \leq e^{s_0/2} R_1/2$ for every $|\xi| \leq 1$, Proposition 3.2 tells us that
\[
\sup_{s \in (s_0 + 1, s_0 + s')} \sup_{|\xi| \leq 1} L^2_{e^{(t-s)/2}} (Z(t, s + \tau)) \leq C_2 (B) (1 + |\gamma|).
\]
Assuming that $M$ is large enough such that $R_1 e^{-M} M^{1/2} < 1/2$, we have that
\[
e^{(t-s)/2} + (t-s_0) = R_1 e^{s_0/2} + R_1 e^{t-s_0} + R_1 e^{-t/2} (t-s_0) \leq R_1 e^{t/2}
\]
for every $t > s_0 + M$. For $|\gamma| > R_1 e^{t/2}$ and $|\xi| = 1$, we also have that $|\gamma| e^{(t-s)/2} \geq R_1 e^{t/2}$, because of the assumption $e^{-s_0/2} < R_1/2$. Then, using the assumptions (42) and (43), we can estimate, for every $|\xi| = 1$ and $1 < M < t-s_0 < s'$, to obtain
\[
L^\beta_{e^{(t-s)/2}} (h(t, t)) \leq 2^{-\tilde{\gamma}} \left( Z(\gamma, t) e^{2\tilde{\gamma}} e^{-|\gamma|/2} \int_{|\gamma| e^{t-s}/2} \right. dy + A_2 \left. \left( \frac{Z(\gamma, t)}{2} e^{-|\gamma|/4} \right) \right)
\]
\[
\leq 2^{-\tilde{\gamma}} \left( Z(\gamma, t) e^{2\tilde{\gamma}} e^{-|\gamma|/2} \int_{|\gamma| e^{t-s}/2} \right. dy + A_2 \left. \left( \frac{Z(\gamma, t)}{2} e^{-|\gamma|/4} \right) \right)
\]
Above $\theta(y, z) \in (0, \tilde{W}(y, z))$. Thus, by using (43) and (53) with $y = (t-s_0)/2 > M^{1/2} > 1$, we have that
\[
\theta(y, t) \leq \max_{0 \leq \theta \leq \tilde{W}(y, t)} \leq 2C_2 (B) (t-s_0)^{1/2}
\]
for every $|\gamma| < e^{(t-s)/2} + (t-s_0)/2$ and $t \in (s_0 + M, s_0 + s')$. This together with Hölder’s inequality gives us the following estimate for the above expression
\[
\leq 2^{-\tilde{\gamma}} e^{2\tilde{\gamma}} C_2 (B) (t-s_0)^{1/2} L^2_{e^{(t-s)/2}} (Z(t, t)) + e^{-|\gamma|/4} L^2_{e^{(t-s)/2}} (Z(t, t)) + e^{-|\gamma|/4} L^2_{e^{(t-s)/2}} (Z(t, t))
\]
Finally, by (26) and (51), we can estimate the above by
\[
\leq C_3 e^{2\tilde{\gamma}} C_2 (B) (t-s_0)^{1/2} e^{-|\gamma|/4} L^2_{e^{(t-s)/2}} (Z(t, t)) + C_4 e^{2\tilde{\gamma}} (t-s_0)^{1/4} L^2_{e^{(t-s)/2}} (Z(t, t)) + C_5 e^{-|\gamma|/4} L^2_{e^{(t-s)/2}} (Z(t, t))
\]
for $\theta > 0$ satisfying $4\tilde{\gamma} e^{-\theta} (1 + e^{-\theta})^{-1} < 1$. The claim follows after some simple estimations. \hfill \Box
Now, we are ready to state the proposition that is the cornerstone of the proof of Theorem 3. At the end of this section, we will use Proposition 3.4 to obtain Theorem 3 as a relatively simple corollary.

**Proposition 3.4**
Let $Z$ be as above. Then, there exist constants $Z_0, C, K > 0$, depending only on $\{A_i\}^4_{i=0}, R_1$ and $\Gamma$, such that

$$L^2_{e^{(s_0-\gamma)/2}}(Z_0) \leq C \left( \sup_{s \geq s_0-K} \|Z(s)\|_{L^2} + e^{-s_0} \right)$$

for every $|\gamma| = 1$ and $s > s_0 = Z_0$. 

**Proof**
Let $Z$ be as above, $|\gamma| = 1$ and define $Z_\tau(y, s) = Z(y, s + \tau)$ for $\tau \geq 0$. Then, because $\gamma(s + \tau) \leq \gamma(s)$, we have that $Z_\tau$ satisfies inequality (41) with $h$ replaced by $h_\tau(y, s) = h(y, s + \tau)$. Therefore, also (42)–(46) hold for $Z_\tau$ and $h_\tau$, respectively, with the same constants $\{A_i\}^4_{i=1}$. By the previous results, there exists a constant $M$ such that for every triplet $s, t, s_0$ for which $s \geq t > s_0 + M > M$, the following estimates (54)–(56) hold for $Z_\tau$ when $M(M) > M$ is large enough. Then, by (52) we have

$$L^2_{e^{(s_0-\gamma)/2}}(Z_\tau) \leq C_4(M) \left( \|Z_\tau(s, s - K(M))\|_{L^2} + e^{-s} \right)$$

for every $s \geq K(M)$ and $s \in (s_0, s_0 + M).$

Notice that, in all these estimates, the constants are independent of $\tau \geq 0$ and depend only on parameters such as $\{A_i\}^4_{i=0}, \Gamma, \beta, N,$ and $M$. Because all these parameters, apart from $M$, are fixed, we only point out the dependencies on $M$ below.

Let us estimate the $L^2_{e^{(s_0-\gamma)/2}}$-norm of $Z(s)$ for $s > s_0 + M > s_0 + M$ with $s_0 \geq K(M).$ The variation of constants formula and inequality (41) give that

$$L^2_{e^{(s_0-\gamma)/2}}(Z(s)) \leq \int_{s_0}^{s+M} L^2_{e^{(s_0-\gamma)/2}}(e^{A(s-t)}Z(t))dt + \int_{s_0}^{s} L^2_{e^{(s_0-\gamma)/2}}(e^{A(s-t)}\chi(t))dt = T_1 + T_2 + T_3.$$

First, we notice that (54) gives an estimate for the term $T_1.$ Then, we split the integral in $T_2$ in two parts and use (55), and (56) to obtain

$$T_2 = \int_{s_0}^{s+M} L^2_{e^{(s_0-\gamma)/2}}(e^{A(s-t)}h_t(t))dt + \int_{s_0}^{s} L^2_{e^{(s_0-\gamma)/2}}(e^{A(s-t)}h_\tau(t))dt \leq C_3(M) \sup_{s \geq s_0} \|Z_\tau(s, s)\|_{L^2} + \int_{s_0}^{s+M} C_2(1 - e^{-(s-t)}) - eL^2_{e^{(s_0-\gamma)/2}}(h_\tau(t))dt \leq C_3(M) \sup_{s \geq s_0} \|Z_\tau(s, s)\|_{L^2} + \int_{s_0}^{s+M} C_2(1 - e^{-(s-t)}) - eL^2_{e^{(s_0-\gamma)/2}}(h_\tau(t))dt.$$
which holds because we may assume that $|e^{t-t_0}/2| \leq e^{t/2} R_1/2$ and because one has that
\[ L^2_{\phi}(\chi(t)) \leq C e^{-t} \]
for every $q \geq 1$ and $|\xi| \leq e^{t/2} R_1/2$.

Therefore, we have proved that
\[ L^2_{\phi}(Z(\cdot, s)) \leq C_8(M) \left( \sup_{s \geq s_0} \|Z(\cdot, s)\|_{L^p}^2 + e^{-s_0} \right) + \int_{s_0+M}^s C_2(1-e^{-(s-t)})^{-1} e^{-(s-t)/2} L^2_{\phi}(h(t)) \, dt. \tag{58} \]

Let $\beta = \max\{C_4(M), C_6(M)\} \geq 1$ and define, for every $s_0 \geq K(M)$,
\[ s(s_0) = \sup \left\{ s' \geq s_0 : L^2_{\phi}(Z(\cdot, s')) \leq 2B \left( \sup_{s \geq K(M)} \|Z(\cdot, s)\|_{L^p} + e^{-s_0}/2 \right), \text{ for every } s \in (s_0, s') \text{ and } \beta > 0 \right\}. \]

Notice that, because of inequality (57), we have $s(s_0) \geq s_0 + K(M)$. We want to show that there exists $\delta_0$ such that $s(s_0) = \infty$ whenever $s_0 \geq \delta_0$. By the previous proposition, we may assume that $M$ is large enough, such that
\[ L^2_{\phi}(h(t)) \leq C_9(M) \left( e^{-(t-s_0)/4} + C_5(M) \sqrt{s_0} L^2_{\phi}(Z(\cdot, t-\theta)) + e^{-t} L^2_{\phi}(Z(\cdot, t-\theta)) + e^{-t} \right) \tag{59} \]
for every $s_0 + M < t < s(s_0)$ and $e^{b_0/2} R_1 > 2$ and for some $\theta < M$.

We may also choose $K(M)$ to be such that all the previous inequalities in this proof hold and in addition $2\beta e^{-K(M)}(1+e^{-K(M)})^{-1} < 1$. Then, one can use Hölder’s inequality and (51) to verify that, for $t \in (s(s_0), s(s_0) + \delta)$ and $\delta \in (0, 1)$, it holds
\[ L^2_{\phi}(Z(\cdot, t)) \leq C_{10}(M)^{\beta} \left( \sup_{s \geq s_0} \|Z(\cdot, s)\|_{L^p} + e^{-s_0} \right)^{\beta} \tag{60} \]
where the function $g$ is defined by the last equality and $s = s(s_0)$. Therefore,
\[ \int_\tau^{\tau+\delta} \left( 1 - e^{-(\tau+\delta-t)} \right) e^{-(t-s_0)/2} L^2_{\phi}(Z(\cdot, t)) \, dt \leq C_{10}(M)^{\beta} \delta^{1-\beta} g(s(s_0)) \left( \sup_{s \geq s_0} \|Z(\cdot, s)\|_{L^p} + e^{-s_0} \right). \]

Now, we use (59) to estimate the integral term in (58) with $s = s(s_0) + \delta$, to obtain
\[ I = \int_{s_0+M}^{s(s_0) + \delta} \left( 1 - e^{-(\tau+\delta-t)} \right) e^{-(t-s_0)/2} L^2_{\phi}(h(t)) \, dt \leq C_9 \int_{s_0+M}^{s(s_0) + \delta} \left( 1 - e^{-(\tau+\delta-t)} \right) e^{-(t-s_0)/2} L^2_{\phi}(Z(\cdot, t-\theta)) \, dt \]
\[ + C_9 \int_{s_0+M}^{s(s_0) + \delta} \left( 1 - e^{-(\tau+\delta-t)} \right) e^{-(t-s_0)/2} L^2_{\phi}(Z(\cdot, t-\theta)) \, dt \]
\[ + C_{10}(M)^{\beta} \delta^{1-\beta} g(s(s_0)) \left( \sup_{s \geq s_0} \|Z(\cdot, s)\|_{L^p} + e^{-s_0} \right). \]

Then use the definition of $s$ in the two first integrals above to estimate
\[ I \leq 2B \left( \sup_{s \geq s_0 - K(M)} \|Z(\cdot, s)\|_{L^p} + e^{-s_0}/2 \right) \]
\[ \times \left( 2C_9 \delta^{1-\beta} g(s(s_0)) + C_{10}(M)^{\beta} \delta^{1-\beta} \right) \]
\[ + C_9 \int_{s_0+M}^{s(s_0) + \delta} \left( 1 - e^{-(\tau+\delta-t)} \right) e^{-(t-s_0)/2} L^2_{\phi}(h(t)) \, dt \leq C_{10}(M)^{\beta} \delta^{1-\beta} g(s(s_0)) + C_9 \int_{s_0+M}^{s(s_0) + \delta} \left( 1 - e^{-(\tau+\delta-t)} \right) e^{-(t-s_0)/2} L^2_{\phi}(h(t)) \, dt. \]
Defining the constant $C_{11}$ through
\[
\int_{s_{0}+M}^{T} (1 - e^{-(\tau + \delta - t)}) e^{-(t-s_{0})/4\beta + C_{9} \sqrt{t-s_{0}}} dt \\
\leq C_{11}(M) \int_{T+\delta-1}^{T} (1 - e^{-(\tau + \delta - t)}) e^{-\epsilon t} dt + (1 - e^{-\epsilon}) \int_{s_{0}+M}^{T+\delta-1} e^{-(t-s_{0})/4\beta + C_{9} \sqrt{t-s_{0}}} dt \leq C_{11}(M)
\]
and similarly
\[
\int_{s_{0}+M}^{T} (1 - e^{-(\tau + \delta - t)}) e^{-\epsilon t} dt \leq C_{12} e^{-s_{0}},
\]
one has that
\[
l \leq 2B \left( \sup_{t \geq s_{0} - K(M)} \|Z(t,s)\|_{L_{p}^{2}} e^{-s_{0}/2} \right) \times \left\{ 2C_{9} BC_{11} \left( \sup_{t \geq s_{0} - K(M)} \|Z(t,s)\|_{L_{p}^{2}} e^{-s_{0}/2} \right) + C_{9} C_{12} e^{-s_{0}} \\
+ C_{10}(M) \delta^{1-\epsilon} g(\tau - s_{0}) \right\} \leq C_{2} C_{9} C_{12} e^{-s_{0}/2} \leq \frac{1}{16}
\]
for every $s_{0} > s_{0}$. Then, by (58) and (61), one has that
\[
\begin{align*}
L_{e^{(\tau + \delta - t)}}^{2} (Z(t,s) & ) \\
\leq 2B \left( \sup_{t \geq s_{0} - K(M)} \|Z(t,s)\|_{L_{p}^{2}} e^{-s_{0}/2} \right) \times \left\{ 2C_{9} BC_{11} \left( \sup_{t \geq s_{0} - K(M)} \|Z(t,s)\|_{L_{p}^{2}} e^{-s_{0}/2} \right) + C_{9} C_{12} e^{-s_{0}} + C_{2} C_{10} \delta^{1-\epsilon} g(\tau - s_{0}) \right\} \leq 2B \left( \sup_{t \geq s_{0} - K(M)} \|Z(t,s)\|_{L_{p}^{2}} e^{-s_{0}/2} \right) \frac{1}{2} + \frac{1}{16} + \frac{1}{16} + C_{2} C_{10} \delta^{1-\epsilon} g(\tau - s_{0}) \leq \frac{2C_{9} C_{12}}{2B} e^{-s_{0}/2}.
\end{align*}
\]
Assuming that $\bar{s}(s_{0}) < \infty$ for some $s_{0} > s_{0}$ and defining $\delta = \delta(\bar{s}(s_{0}))$ to be small enough such that
\[
C_{2} C_{10} \delta^{1-\epsilon} g(\tau - s_{0}) < \frac{1}{16},
\]
inequalities (62) and (63) yield that
\[
L_{e^{(\tau + \delta - t)}}^{2} (Z(t,s) ) \leq 2B \left( \sup_{t \geq s_{0} - K(M)} \|Z(t,s)\|_{L_{p}^{2}} e^{-s_{0}/2} \right) \frac{3}{4}
\]
for every $\gamma > 0$. This is in contradiction with the definition of $\bar{s} = \bar{s}(s_{0})$. We have thus proved the claim and reached the end of the proof of Proposition 3.4.

**Proof of Theorem 3.**

Proposition 3.2 with $\gamma = 0$ gives us that, for $e^{-s_{0}/2} \leq R_{1}/2$,
\[
Z(e^{(s-s_{0})/2} \xi, s) \leq C \left( L_{e^{(s-s_{0})/2} \xi}^{2} (Z(s-1)) e^{-s} \right)
\]
for every $s-1 > s_{0}$. Therefore, Proposition 3.4 states that
\[
Z(e^{(s-s_{0})/2} \xi, s) \leq C \left( \sup_{t \geq s_{0} - K(M)} \|Z(t,s)\|_{L_{p}^{2}} e^{-s_{0}} \right)
\]
whenever $s_{0} > s_{0}$ for some $s_{0}$ large enough.
So, for \( s \) large, \( s_0 > 30 \) and \( e^{-\delta_0/2}\mathbf{1} \leq \tilde{R}_1/2, \) we have
\[
Z(e^{(s-\delta_0)/2}x, s) = | -s + u(e^{-\delta_0/2}x, T - e^{-s}) - \varphi(e^{(s-\delta_0)/2}x) | \\
= | -s + u(e^{-\delta_0/2}x, T - e^{-s}) + 2\log(e^{(s-\delta_0)/2}x) - C_\alpha + o_{s \to \infty}(1) |\\
\leq C \left( \sup_{s \geq s_0 - K(M)} \| Z(\cdot, s) \|_{L^p} + e^{-\delta_0} \right),
\]
where \( C_\alpha \) is as in Theorem 1.

By the above inequality, the origin is the only blow-up point in the ball \( B_{e^{-\delta_0}}(0) \). Therefore, \( u(x, t) \) is bounded in every set \( B_{\gamma} \setminus B_{\delta} \) for \( 0 < \delta < \gamma < e^{-\delta_0} \), and parabolic estimates imply that also \( u_t(x, t) \) is bounded in every such set. This gives us the existence of the limit \( \lim_{t \to T} u(x, t) \) for \( x \in B_{e^{-\delta_0}}(0) \). By taking the limit as \( s \to \infty \), it holds
\[
\left| u(e^{-\delta_0/2}x, T) + 2\log(e^{-\delta_0/2}x) - C_\alpha \right| \leq C \left( \sup_{s \geq s_0 - K(M)} \| Z(\cdot, s) \|_{L^p} + e^{-\delta_0} \right).
\]

Using the assumption (45), we obtain
\[
\lim_{x \to 0} \left| u(x, T) + 2 \log |x| - C_\alpha \right| = \lim_{s \to \infty} \left| u(e^{-\delta_0/2}x, T) + 2\log(e^{-\delta_0/2}x) - C_\alpha \right| = 0,
\]
and we have found the blow-up profile. \( \square \)

4. Revisiting the case of constant self-similar profile

In this section, we will briefly go through some results of the paper [1] and notice that the conclusions of Theorem 1 hold by assuming only that \( u_0 \) is radially symmetric and blow-up takes place at the origin with type I rate; therefore, verifying Theorem 2.

In [1], we proved Theorem 1 by showing that if \( u \) is a radially symmetric and radially nonincreasing \( L^1 \)-solution of equation (1) with \( f(u) = e^u \) on \([0, T] \) that blows up at \( t = T < \infty \) and
\[
\log(T - t) + u(\sqrt{T - t}, t) \to 0
\]
uniformly on compact sets as \( t \to T \), then either
\[
\lim_{x \to 0} \left| u(x, T) + 2 \log |x| - \log \log |x| \right| = C \tag{64}
\]
or
\[
\lim_{x \to 0} \left| u(x, T) + m \log |x| \right| = C \tag{65}
\]
for some constant \( C \). This will then imply that blow-up is complete by Theorem 3.6 in [9], thereby contradicting the assumption on \( u \) being an \( L^1 \)-solution on \([0, T] \). For an \( L^1 \)-solution such as in Theorem 1, the only possibility is thus a nonconstant self-similar blow-up profile, see details in [1].

We will now demonstrate that it is not actually necessary to assume that \( u \) is radially nonincreasing in order for this analysis to go through.

Let
\[
\tilde{u}(x, t) = \zeta(x)u(x, t) - \left( \log(T - t) + 1 \right)(1 - \zeta(x))
\]
be a continuation of \( u \) to the whole space \( \mathbb{R}^N \), where \( \zeta \in C^\infty(\mathbb{R}^N) \) and \( \zeta(x) = 1 \) for \( |x| \leq R_1 < R_2 < R \) and \( \zeta(x) = 0 \) for \( |x| > R_2 \). It is proved in Propositions 3.4 and 3.6 in [1] that the following is true.

**Proposition 4.1**

Let \( u \) be a radially symmetric solution of (1) that blows up with type I rate at \((x, t) = (0, T)\), and assume that the convergence (2) holds with \( \varphi = 0 \). Then
\[
\lim_{t \to T} \left( \log(T - t) + u(\lambda(t)y, t) \right) = -\log \left( 1 + \sum_{|\alpha| = m} C_\alpha y^\alpha \right)
\]
uniformly for \( y \) in compact sets for some \( m \geq 2 \) and constants \( C_\alpha \), where \( \alpha \) is a multi-index and \( |\alpha| = \alpha_1 + \ldots + \alpha_N \). The function \( \lambda(t) \) is defined by \( \lambda(t) = |\log(T - t)|^{1/2} \) if \( m = 2 \), and by \( \lambda(t) = (T - t)^{1/m} \) if \( m > 2 \).
Notice that there is no assumption on υ being radially nonincreasing in the above proposition.

We show that the above proposition implies that either (64) or (65) holds. Therefore, the conclusion of Theorem 2 is obtained by using Theorem 3.4 in [9] and an energy argument as in [1]. We will first define an auxiliary function \( W_t \) and describe some of its properties below. Then we prove Proposition 4.2, which gives that the \( L^2 \)-norm of \( W_t(\cdot, s) \) is controlled by the \( L^2 \)-norm of \( W_t(\cdot, 0) \). Profiles (64) and (65) are obtained at the very end of this section.

For fixed \( \xi \) with \( |\xi| \leq 1 \), define

\[
W_t(y, s) = \log(T - \tau) + \tilde{u} \left( \lambda(\tau) \xi + \sqrt{T - \tau} \sqrt{1 - ty}, \tau + (T - \tau) t \right) + \log \left( 1 + \sum_{|a| = m} c_a \xi^a \right)
\]

for \( y \in \mathbb{R}^N \) and \( s \geq 0 \), where \( \tau \in (0, T) \) and \( s = -\log(1 - t) \). Then, by using Proposition 4.1, \( W_t \) satisfies

\[
\|W_t(\cdot, 0)\|_{L^2_0} \to 0
\]

as \( T \to T \), and, by (12), there exists a constant \( A_0 > 0 \) such that

\[
|\nabla W_t(y, s)| \leq A_0
\]

for every \( (y, s) \in \mathbb{R}^N \times (0, \infty) \). It can be also verified that \( W_t \) solves

\[
(W_t)_s = \Delta W_t - \frac{y}{2} \nabla W_t + e^{\phi} W_t + f_t = AW_t + e^{\phi} W_t + f_t,
\]

where \( A = \Delta - \frac{y}{2} \nabla \) and

\[
\tilde{\phi}(s) = -s - \log \left( e^{-s} + \sum_{|a| = m} c_a \xi^a \right).
\]

Above \( f_t \) is a certain function that we do not explicitly write out here, satisfying

\[
|f_t| \leq A_1 |W_t| \quad \text{and} \quad |f_t| \leq A_2 |W_t|^2
\]

for some constants \( A_1, A_2 > 0 \) and

\[
|f_t| \leq e^{\phi} \left( e^{W_t} - 1 - W_t \right)
\]

for \( |\lambda(\tau) \xi + \sqrt{T - \tau} \sqrt{1 - ty}| \leq R_1 \). We have the following result.

**Proposition 4.2**

Let \( W_t \) be as above. There exist constants \( C \) and \( \tau > 0 \) such that

\[
\|W_t(\cdot, s)\|_{L^2_0} \leq C\|W_t(\cdot, 0)\|_{L^2_0}
\]

for every \( s \geq 0 \) and \( T > \tau \).

**Proof**

Let \( Z_t = |W_t| \). Because \( e^{\phi} \) is bounded and because \( |f_t| \leq A_1 Z_t \), we have a constant \( C_1 > 0 \) such that for any \( s \geq s_0 \geq 0 \) it holds

\[
Z_t(\cdot, s) \leq e^{(A+C_1)(s-s_0)} Z_t(\cdot, s_0).
\]

By (66), one has that \( \|Z_t(\cdot, 0)\|_{L^2_0} \leq C \) for some \( C > 0 \) independent of \( \tau \), which implies

\[
\|Z_t(\cdot, s)\|_{L^2_0} \leq e^{C_1 s_0} \|Z_t(\cdot, 0)\|_{L^2_0} \leq C_2(s_0)
\]

for every \( s \leq s_0 \) and \( \tau \in (0, T) \) with \( C_2(s_0) \) independent of \( \tau \).

Now define,

\[
\mathcal{S}(s_0) = \sup \left\{ s' > 0 : \|Z_t(\cdot, s)\|_{L^2_0} \leq 4 e^{C_1 s_0} \|Z_t(\cdot, 0)\|_{L^2_0} \right\}
\]

for every \( s \in (0, s') \) and notice that (71) implies \( \mathcal{S}(s_0) > s_0 \).

Using the representation (17) and (70) together with the definition of \( \mathcal{S}(s_0) \), we obtain

\[
Z_t(0, s) \leq e^{C_1} e^{A_1} Z_t(0, s - 1) \leq C_3 \int_{\mathbb{R}^n} e^{-\frac{|\lambda|^2}{4(1-e^{-\tau})}} |Z_t(\lambda, s - 1)| d\lambda
\]

\[
\leq C_3 \left( \int_{\mathbb{R}^n} |Z_t(\lambda, s - 1)|^2 e^{-|\lambda|^2/4} d\lambda \right)^{1/2} \left( \int_{\mathbb{R}^n} e^{|\lambda|^2/4} e^{-|\lambda|^2/2} d\lambda \right)^{1/2} \leq C_2 e^{C_1 s_0} \|Z_t(\cdot, 0)\|_{L^2_0} \leq C_4(s_0)
\]
for every $s \in (1, 3(s_0))$. For $|y| \leq \sqrt{s}$, we have that $|\lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y| \leq R_1$ for $\tau$ large enough, and for such $y$ the inequalities (67), (69), and (72) thus imply

$$|f_{\tau}(y, s)| \leq \frac{1}{2} e^{-\gamma(s+1)k} s^{-1/2} Z_k(y, s)^2 \leq C_5 e^{-s/2} e^0 \sqrt{s} Z_0(y, s)^2 \leq C_6(s_0)e^{-s-\gamma_0} \sqrt{s} Z_0(y, s)^2.$$  \hspace{1cm} (73)

Now, we can estimate the $L^2$-norm of $f_{\tau}$, by using (68), (73), Hölder's inequality, Proposition 2.2, and the definition of $3(s_0)$, to obtain

$$\|f_{\tau}(\cdot, s)\|_{L^2}^2 \leq C_6(s_0)^2 \int_{|y| \leq \sqrt{s}} e^{-2s+2A_0 \sqrt{s} \lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y} \frac{|y|^2}{s} dy + \int_{|y| > \sqrt{s}} A_2^s Z_k(y, s)^4 e^{-|y|^2/4} dy$$

$$\leq C_6(s_0)^2 e^{-2s+2A_0 \sqrt{s} \lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y} \frac{|y|^2}{s} dy + A_2 \left( \int_{|y| > \sqrt{s}} e^{-|y|^2/4} dy \right)^{1/2} \|Z_k(\cdot, s)\|_{L^2}^4$$

$$\leq C_7(s_0)^2 e^{-2s+2A_0 \sqrt{s} \lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y} \frac{|y|^2}{s} dy + C_9 e^{-s/8} \|Z_k(\cdot, s)\|_{L^2}^4 \leq C_9(s_0)^2 e^{-s/8+2A_0 \sqrt{s} \lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y} \frac{|y|^2}{s} dy$$

for any $s \in (K, 3(s_0))$ and $s_0 > K$.

Now, let $C_{10}$ be such that $\lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y \leq C_{10} e^{-s}$ and take $s_0 > K$ large enough to satisfy

$$C_{10} e^{-s_0} < \frac{1}{4}$$

and define $\tau = \tau(s_0)$ to be such that

$$\frac{C_9(s_0)^2}{4 e^{s_0}} \int_{s_0}^{\infty} e^{-t/16+2A_0 \sqrt{s} \lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y} \frac{|y|^2}{s} dy < \frac{1}{4}$$

for every $\tau > \tau$. Then, by using the variation of constants formula together with the previous estimates, one obtains

$$\|Z_k(\cdot, s)\|_{L^2}^2 \leq \|\lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y\|_{L^2}^2 + \int_{s_0}^{\infty} \|e^{\lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y} \|_{L^2}^2 \int_{s_0}^{\tau} \|f_{\tau}(\cdot, t)\|_{L^2}^2 dt$$

$$\leq \|Z_k(\cdot, s_0)\|_{L^2}^2 + C_{10} e^{s_0} \int_{s_0}^{\tau} \|Z_k(\cdot, t)\|_{L^2}^2 \int_{s_0}^{\tau} \|f_{\tau}(\cdot, t)\|_{L^2}^2 dt$$

$$\leq 4 e^{s_0} \|Z_k(\cdot, 0)\|_{L^2}^2 \left( \frac{1}{4} + C_{10} e^{-s_0} + C_9(s_0)^2 \int_{s_0}^{\tau} e^{-t/16+2A_0 \sqrt{s} \lambda(\tau)\xi + \sqrt{T - \tau}e^{-1/2}y} \frac{|y|^2}{s} dy \right) \leq 3 e^{s_0} \|Z_k(\cdot, 0)\|_{L^2}^2$$

for every $s \in (s_0, 3(s_0))$ and $\tau > \tau$. This proves that $3(s_0) = \infty$ and the claim follows.

Using the previous proposition and (72), we have that

$$|W_{\tau}(0, s)| \leq C \|W_{\tau}(\cdot, s - 1)\|_{L^2} \leq C \|W_{\tau}(\cdot, 0)\|_{L^2}$$

for every $\tau > \tau$ and $s > 1$ when $\tau$ is close enough to $T$. Rewriting this gives

$$\log(T - \tau) + u \left( \lambda(\tau)\xi + \sqrt{T - \tau}(1 - e^{-1/2}y) \right) + \log \left( \sum_{|a| = m} c_{a} e^{\xi a} \right) \leq C \|W_{\tau}(\cdot, 0)\|_{L^2}^2,$$

and by taking the limit as $s \to \infty$, one notices that

$$\log(T - \tau) + u \left( \lambda(\tau)\xi, T \right) + \log \left( \sum_{|a| = m} c_{a} e^{\xi a} \right) \leq C \|W_{\tau}(\cdot, 0)\|_{L^2}^2 \to 0$$

as $\tau \to T$. The desired blow-up profiles (64) and (65) are then obtained by a change of variables as in [1].

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