From proximal point method to Nesterov’s acceleration*

Kwangjun Ahn
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology
kjahn@mit.edu

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Abstract

The proximal point method (PPM) is a fundamental method in optimization that is often used as a building block for fast optimization algorithms. In this work, building on a recent work by Defazio [Def19], we provide a complete understanding of Nesterov’s accelerated gradient method (AGM) by establishing quantitative and analytical connections between PPM and AGM. The main observation in this paper is that AGM is in fact equal to a simple approximation of PPM, which results in an elementary derivation of the mysterious updates of AGM as well as its step sizes. This connection also leads to a conceptually simple analysis of AGM based on the standard analysis of PPM. This view naturally extends to the strongly convex case and also motivates other accelerated methods for practically relevant settings.

1 Introduction

The proximal point method (PPM) [Mor65, Mar70, Roc76] is a fundamental method in optimization which solves the minimization of the cost function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ by iteratively solving the subproblem

$$x_{t+1} \leftarrow \arg\min_{x \in \mathbb{R}^d} \left\{ f(x) + \frac{1}{2\eta_{t+1}} \|x - x_t\|^2 \right\}$$  \hfill (1.1)

for a step size $\eta_{t+1} > 0$, where the norm is chosen as the $\ell_2$ norm. The motivation of the method is clear: we add a quadratic regularization to make the cost function well conditioned for faster optimization.

Nevertheless, solving (1.1) is in general as difficult as solving the original optimization problem, and PPM is largely regarded as a “conceptual” guiding principle for accelerating optimization algorithms [Dru17].

On the other hand, there is another prevalent accelerated method called Nesterov’s accelerated gradient method (AGM) [Nes83]. In contrast to PPM, AGM is implementable and has been applied to a myriad of applications, including sparse linear regression [BT09], compressed sensing [BBC11], the maximum flow problem [LRS13], and deep neural networks [SMDH13]. Nonetheless, in contrast with the clear motivation of PPM, AGM has an obscure driving principle. In particular, original construction of AGM relies on an ingenious yet abstruse technique called estimate sequence [Nes18, Section 2.2.1], which has motivated researchers to investigate numerous alternative explanations (see Section 6 for details).

Recently, Defazio [Def19] established an inspiring connection between PPM and AGM. The main observation is that for strongly convex costs, one can derive a version of AGM from the primal-dual form of PPM with a tweak of geometry. This observation constitutes an important step toward understanding AGM because PPM is not as difficult to understand. This inspiring result, nevertheless, leaves open many other

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1For instance, when $f$ is nonconvex, the regularization term can make each subproblem (1.1) convex, and even when $f$ is convex, the regularization term will serve to increase the strong convexity parameter, which results in faster optimization.
important questions. Most importantly, [Def19] lacks quantitative explanations as to why AGM achieves the accelerated convergence rates of $O(1/t^2)$ for smooth (Definition 1) costs and $O(\exp(-T/\sqrt{\kappa}))$ for smooth and strongly convex (Definition 2) costs, where $T$ is the number of iterations and $\kappa$ is the condition number of the problem. Moreover, it is not clear whether such connection can be made without assuming strong convexity and can be extended to other more general and practical versions of AGM.

In this work, we build a thorough understanding of Nesterov’s acceleration from the proximal point method by strengthening the connection made in [Def19]. The main observation in this paper is that the mysterious updates of AGM can be fully understood by viewing it as a simple approximation of PPM. In particular, this observation leads to a straightforward derivation of AGM that does not rely on duality unlike [Def19]. Moreover, the PPM view of AGM offers a simple analysis of AGM based on the standard analysis of PPM [Gül91]. We also demonstrate the generality of our view. More specifically, our view naturally extends to the strongly convex case and obtains a general\(^2\) version of AGM [Nes18, (2.2.19)], and our view also gives rise to the key idea of the method of similar triangles, a version of AGM shown to have simple extensions to practically relevant settings [Tse08, GN18, Nes18].

2 Baseline: analysis of the proximal point method

The baseline of our discussion is the following convergence rate of PPM for convex costs proved in a seminal paper by Güler [Gül91] (here $x_*$ denotes a global optimum point, i.e., $x_* \in \text{argmin}_x f(x)$):

$$f(x_T) - f(x_*) \leq O \left( \left( \sum_{t=1}^{T} \eta_t \right)^{-1} \right) \quad \text{for any } T \geq 1.$$ \hspace{1cm} (2.1)

In words, one can achieve an arbitrarily fast convergence rate by choosing step sizes $\eta_t$’s large. Below, we review a short Lyapunov function proof of (2.1), which will serve as a backbone to other analyses.

**Proof of (2.1).** It turns out that the following Lyapunov function is suitable:

$$\Phi_t := \left( \sum_{i=1}^{t} \eta_i \right) \cdot \left( f(x_t) - f(x_*) \right) + \frac{1}{2} \| x_* - x_t \|^2,$$ \hspace{1cm} (2.2)

where $\Phi_0 := \frac{1}{2} \| x_* - x_0 \|^2$ and here and below, $\| \cdot \|$ is the $\ell_2$ norm unless stated otherwise. Now, it suffices to show that $\Phi_t$ is decreasing, i.e., $\Phi_{t+1} \leq \Phi_t$ for all $t \geq 0$. Indeed, if $\Phi_t$ is decreasing, we have $\Phi_T \leq \Phi_0$ for any $T \geq 1$, which precisely recovers (2.1). To that end, we use a standard result:

**Proposition 1** (Proximal inequality (see e.g. [BC11, Proposition 12.26])). For a convex function $\phi : \mathbb{R}^d \to \mathbb{R}$, let $x_{t+1}$ be the unique minimizer of the following proximal step:

$$x_{t+1} \leftarrow \min_{x \in \mathbb{R}^d} \left\{ \phi(x) + \frac{1}{\eta} \| x - x_t \|^2 \right\}.$$ \hspace{1cm} (2.3)

Then, for any $u \in \mathbb{R}^d$, $\phi(x_{t+1}) - \phi(u) + \frac{1}{2} \| u - x_{t+1} \|^2 + \frac{1}{2} \| x_{t+1} - x_t \|^2 - \frac{1}{2} \| u - x_t \|^2 \leq 0$.

Now Proposition 1 completes the proof as follows: First, we apply Proposition 1 with $\phi = \eta_{t+1} f$ and $u = x_*$ and drop the term $\frac{1}{2} \| x_{t+1} - x_t \|^2$ to obtain:

$$\eta_{t+1} \left( f(x_{t+1}) - f(x_*) \right) + \frac{1}{2} \| x_* - x_{t+1} \|^2 - \frac{1}{2} \| x_* - x_t \|^2 \leq 0.$$ \hspace{1cm} (B1)

Next, from the optimality of $x_{t+1}$ for (2.3), it readily follows that

$$f(x_{t+1}) - f(x_t) \leq 0.$$ \hspace{1cm} (B2)

Now, computing (B1) + $\left( \sum_{i=1}^{t} \eta_i \right) \times$ (B2) yields $\Phi_{t+1} \leq \Phi_t$, which finishes the proof. \hfill \Box

\(^2\)It is general in the sense that it smoothly interpolates between the strongly convex case and the non-strongly convex case.
2.1 Our conceptual question

Although the convergence rate (2.1) seems powerful, it does not have any practical values as PPM is in general not implementable. Nevertheless, one can ask the following conceptual question:

Can we develop an implementable approximation of PPM for large step sizes η’s?

Perhaps, the most straightforward approximation would be to replace the cost function in (1.1) with its lower-order approximations. We implement this idea in the next section.

3 Two simple approximations of the proximal point method

To analyze approximation errors, let us assume that the cost function is smooth.

**Definition 1** (Smoothness). For L > 0, we say a differentiable function f : ℝ^d → ℝ is smooth if f(x) ≤ f(y) + ⟨∇f(y), x − y⟩ + 1/2L∥x − y∥^2 for any x, y ∈ ℝ^d.

From the convexity and the smoothness of f, we have the following lower and upper bounds:

\[ f(y) + ⟨∇f(y), x − y⟩ ≤ f(x) ≤ f(y) + ⟨∇f(y), x − y⟩ + \frac{L}{2}∥x − y∥^2 \]

for any x, y ∈ ℝ^d.

In this section, we use these bounds to approximate PPM.

3.1 First approach: using first-order approximation

Let us first replace f in the objective (1.1) with its lower approximation:

\[ x_{t+1} = \arg\min_x \left\{ f(x_t) + ⟨∇f(x_t), x − x_t⟩ + \frac{1}{2\eta_{t+1}}∥x − x_t∥^2 \right\}. \tag{3.1} \]

Writing the optimality condition, one quickly notices that (3.1) actually leads to gradient descent:

\[ x_{t+1} = x_t − \eta_{t+1}∇f(x_t). \tag{3.2} \]

Let us see how well (3.1) approximates PPM:

**Analysis of the first approach.** We first establish counterparts of (B1) and (B2). We begin with (B1).

We first apply Proposition 1 with \( φ(x) = \frac{L}{2}∥x − x_t∥^2 \) and \( u = x_t \):

\[ φ(x_{t+1}) − φ(x_t) + \frac{1}{2}∥x_t − x_{t+1}∥^2 + \frac{1}{2}∥x_{t+1} − x_t∥^2 − \frac{1}{2}∥x_t − x_t∥^2 ≤ 0. \]

Now using convexity and smoothness, we have \( φ(x) ≤ φ(x_{t+1}) \) and hence the above inequality implies the following approximate version of (B1):

\[ \eta_{t+1} [f(x_{t+1}) − f(x_t)] + \frac{1}{2}∥x_t − x_{t+1}∥^2 − \frac{1}{2}∥x_t − x_t∥^2 ≤ (E_1), \]

where \((E_1) := \frac{(Lη_{t+1} − \frac{1}{2})}{∥x_{t+1} − x_t∥^2} \). Next, we use the smoothness of f and the fact \( ∇f(x_t) = -\frac{L}{\eta_{t+1}}(x_t + x_{t+1}) \) (due to (3.2)), to obtain the following counterpart of (B2):

\[ f(x_{t+1}) − f(x_t) ≤ ⟨∇f(x_t), x_{t+1} − x_t⟩ + \frac{L}{2}∥x_{t+1} − x_t∥^2 = (E_2), \]

where \((E_2) := \frac{L}{∥x_{t+1} − x_t∥^2} \).

Now paralleling the proof of (2.1), to show that \( Φ_t \) (2.2) is a valid Lyapunov function, we need to find the step sizes \( η_t \)’s that satisfy the following relation: \((E_1) + (\sum_{i=1}^{t} \eta_i) × (E_2) ≤ 0 \). On the other hand, note that both \((E_1) \) and \((E_2) \) become positive numbers when \( η_{t+1} > 2/L \) (due to (3.2)). Hence, the admissible choices for \( η_t \) at each iteration are upper bounded by \( 2/L \), which together with the PPM convergence rate (2.1) implies that \( O(1/\sum_{i=1}^{t} \eta_i) = O(1/t) \) is the best convergence rate one can prove. Indeed, choosing \( η_t ≡ 1/L \), then we have \((E_1) = 0 \) and \((E_2) < 0 \), obtaining the well-known bound of \( f(x_T) − f(x_∗) ≤ \frac{L∥x_0 − x_∗∥^2}{2t} = O(1/t). \)
To summarize, the first approach only leads to a disappointing result: the approximation is valid only for the small step size regime of $\eta_t = O(1)$. We empirically verify this fact for a quadratic cost in Figure 1. As one can see from Figure 1, the lower approximation approach (3.1) overshoots for large step sizes like $\eta_t = \Theta(t)$ and quickly steers away from PPM iterates.

### 3.2 Second approach: using smoothness

After seeing the disappointing outcome of the first approach, our second approach is to replace $f$ with its upper approximation due to the $L$-smoothness:

$$x_{t+1} \leftarrow \arg\min_x \left\{ f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \frac{L}{2} \| x - x_t \|^2 + \frac{1}{2\eta_{t+1}} \| x - x_t \|^2 \right\}.$$  \hfill (3.3)

Writing the optimality condition, (3.3) actually leads to a conservative update of gradient descent:

$$x_{t+1} = x_t - \frac{1}{L + \eta_{t+1}^2} \nabla f(x_t).$$  \hfill (3.4)

Note that regardless of how large $\eta_{t+1}$ we choose, the actual update step size in (3.4) is always upper bounded by $1/L$. Although this conservative update prevents the overshooting phenomenon of the first approach, as we increase $\eta_t$, this conservative update becomes too tardy to be a good approximation of PPM; see Figure 1.

### 4 Nesterov’s acceleration via alternating two approaches

In the previous section, we have seen that the two simple approximations of PPM both have limitations. Nonetheless, observe that their limitations are opposite to each other: while the first approach is too “reckless,” the second approach is too “conservative.” This observation motivates us to consider a combination of the two approaches which could mitigate each other’s limitation.

Let us implement this idea by alternating between the two approximations (3.1) and (3.3) of PPM. The key modification is that for both approximations, we introduce an additional sequence of points $\{y_t\}$ for cost function approximation; i.e., we use the following approximations for the $t$-th iteration:

$$f(y_t) + \langle \nabla f(y_t), x - y_t \rangle \leq f(x) \leq f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{L}{2} \| x - y_t \|^2.$$  

Indeed, this modification is crucial: if we just use approximations at $x_t$, the resulting alternation merely concatenates (3.1) and (3.3) during each iteration, and the two limitations we discussed in Section 3 will remain in the combined approach.

Having introduced a separate sequence $\{y_t\}$ for cost approximations, we consider the following alternation where during each iteration, we update $x_t$ with (3.1) and $y_t$ with (3.3):
Approximate PPM with alternating two approaches. Given \( x_0 \in \mathbb{R}^d \), let \( y_0 = x_0 \) and run:

\[
\begin{align*}
x_{t+1} & \leftarrow \text{argmin}_x \left\{ f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{1}{2\eta_{t+1}} \| x - x_t \|^2 \right\}, \\
y_{t+1} & \leftarrow \text{argmin}_x \left\{ f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{\lambda}{2} \| x - y_t \|^2 + \frac{1}{2\eta_{t+1}} \| x - x_{t+1} \|^2 \right\}.
\end{align*}
\]

(4.1a)

In Figure 1, we empirically verify that (4.1) indeed gets the best of both worlds: this combined approach successfully approximates PPM even for the regime \( \eta_t = \Theta(t) \). More remarkably, (4.1) exactly recovers Nesterov’s AGM. More specifically, turning (4.1) into the equational form by writing the optimality conditions, and introducing an auxiliary iterate \( z_{t+1} := y_t - \frac{1}{t} \nabla f(y_t) \), we obtain the following (\( z_0 := x_0 = y_0 \)):

An equivalent representation of (4.1):

\[
\begin{align*}
y_t & = \frac{1}{L + \eta} x_t + \frac{\eta}{L + \eta} z_t, \\
x_{t+1} & = x_t - \eta_{t+1} \nabla f(y_t), \\
z_{t+1} & = y_t - \frac{1}{t} \nabla f(y_t).
\end{align*}
\]

(4.2a) (4.2b) (4.2c)

Figure 2: The iterates of (4.2).

Hence, we arrive at AGM without relying on any non-trivial derivations in the literature such as estimate sequence [Nes18] or linear coupling [AZO17]. To summarize, we have demonstrated:

*Nesterov’s accelerated method is a simple approximation of the proximal point method!*

**Remark 1.** Our derivation is inspired by the one in the recent work by Defazio [Def19, Sections 5 and 6]. However, unlike the approach in [Def19], our derivation does not rely on duality, which could be advantageous in the settings where duality fails.

**Remark 2** (Understanding mysterious parameters of AGM). It is often the case in the literature that the interpolation step (4.2a) is written as an abstract form \( y_t = \tau_t x_t + (1 - \tau_t) z_t \) with a weight parameter \( \tau_t > 0 \) to be chosen [AZO17, LR16, WOJ16, BG19, AS20]. That said, in the previous works, \( \tau_t \) is carefully chosen according to the analysis without conveying much intuition. One important aspect of our PPM view is that it reveals a close relation between the weight parameter \( \tau_t \) and the step size \( \eta_t \). More specifically, \( \tau_t \) is chosen so that the ratio of the distances \( \| y_t - x_t \| : \| y_t - z_t \| \) is equal to \( \eta_t : 1/L \) (see Figure 2).

### 4.1 Understanding the accelerated convergence rate

In order to determine \( \eta_t \)’s in (4.2), we revisit the analysis of PPM from Section 3. In turns out that following Section 3.1, one can derive from first principles the following inequalities using Proposition 1 (we defer the derivations to Appendix A):

\[
\begin{align*}
\text{Counterpart of (B1)}: \quad & \eta_{t+1} (f(z_{t+1}) - f(x_*) ) + \frac{1}{2} \| x_* - x_{t+1} \|^2 - \frac{1}{2} \| x_* - x_t \|^2 \leq (F_1) \quad \text{and} \\
\text{Counterpart of (B2)}: \quad & f(z_{t+1}) - f(z_t) \leq (F_2),
\end{align*}
\]

where \((F_1) : = \left( \frac{\eta_{t+1}^2}{2} - \frac{\eta_t^2}{2} \right) \| \nabla f(y_t) \|^2 + L \eta_{t+1} \| \nabla f(y_t) \|^2 - \| \nabla f(y_t) \|^2 \| z_t - y_t \|^2 - \| \nabla f(y_t) \|^2 \| z_t - y_t \|^2 \).

Hence, we modify the Lyapunov function (2.2) by replacing the first \( x_t \) with \( z_t \):

\[
F_t := \left( \sum_{i=1}^t \eta_i \right) (f(z_t) - f(x_*)) + \frac{1}{2} \| x_* - x_t \|^2.
\]

Then as before, to prove the validity of the chosen Lyapunov function, it suffices to verify \((F_1) + (\sum_{i=1}^t \eta_i) \cdot (F_2) \leq 0\), which is equivalent to

\[
\frac{1}{2L} \left( \sum_{i=1}^{t+1} \eta_i \right) \| \nabla f(y_i) \|^2 + \left( \sum_{i=1}^t \eta_i \right) (f(z_t) - f(y_t)) \leq 0 \quad (4.4)
\]
From (4.4), it suffices to choose \( \{ \eta_t \} \) so that \( L \eta_t \eta_{t+1} = \sum_{i=1}^{t} \eta_i \). Indeed, with such a choice, the coefficient of the inner product term in (4.4) becomes zero and the coefficient of the squared norm term becomes \( \frac{1}{2L} (L \eta_t^2 + L \eta_{t+1} \eta_{t+2}) \leq 0 \) (if \( \{ \eta_t \} \) is increasing). Indeed, one can quickly notice that choosing \( \eta_t = \frac{1}{t} / \mu \) satisfies the desired relation. Therefore, we obtain the well known accelerated convergence rate of \( f(z_T) - f(x_*) \leq \frac{2t \| x_0 - x_* \|^2}{T(T+1)} = O(1/T^2) \) [Nes83].

### 4.2 Separating step sizes for flexibility

Since (4.1) is an approximation of PPM, it is helpful to give (4.1) more flexibility when we try to extend it to other settings. In particular, we relax (4.1) by separating the two step sizes:

**Approximate PPM with two separate step sizes \( \{ \eta_t \} \) and \( \{ \tilde{\eta}_t \} \).** Given \( x_0 = y_0 \in \mathbb{R}^d \),

\[
\begin{align*}
x_{t+1} &\leftarrow \text{argmin}_x \left\{ f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{1}{2 \tilde{\eta}_{t+1}} \| x - x_t \|^2 \right\}, \quad (4.5a) \\
y_{t+1} &\leftarrow \text{argmin}_x \left\{ f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{L}{2} \| x - y_t \|^2 + \frac{1}{2 \tilde{\eta}_{t+1}} \| x - x_{t+1} \|^2 \right\}. \quad (4.5b)
\end{align*}
\]

As we shall see in the next subsection, this simple relaxation allows us to recover a well known general version of AGM [Nes18, Section 2.2].

### 4.3 Acceleration for strongly convex costs

Let us apply our PPM view to the strongly convex cost case. We begin with the definition:

**Definition 2** (Strong convexity). For \( \mu > 0 \), we say a differentiable function \( f : \mathbb{R}^d \to \mathbb{R} \) is \( \mu \)-strongly convex if \( f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle + \frac{\mu}{2} \| x - y \|^2 \) for any \( x, y \in \mathbb{R}^d \).

Since \( f \) is additionally assumed to be strongly convex, one can strengthen the step (4.5a) by

\[
x_{t+1} \leftarrow \text{argmin}_{x \in \mathbb{R}^d} \left\{ f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{\mu}{2} \| x - y_t \|^2 + \frac{1}{2 \tilde{\eta}_{t+1}} \| x - x_t \|^2 \right\}. \quad (4.6)
\]

Writing the optimality condition of (4.6), it is straightforward to check that (4.5) is equivalent to the following form (again, we introduce another auxiliary iterate \( w_t \), and let \( z_0 := x_0 = y_0 \)):

**Approximate PPM for strongly convex costs:**

\[
\begin{align*}
y_t &= \frac{1/\mu}{\sqrt{L + \eta_t}} x_t + \frac{\eta_t}{\sqrt{L + \eta_t}} z_t, \quad (4.7a) \\
w_t &= \frac{1/\mu}{\sqrt{\mu + \eta_t + 1}} x_t + \frac{\eta_t}{\sqrt{\mu + \eta_t + 1}} y_t \quad (4.7b) \\
x_{t+1} &= w_t - \frac{\mu}{\sqrt{\mu + \eta_{t+1}} + \eta_{t+1}} \nabla f(y_t), \quad (4.7c) \\
z_{t+1} &= y_t - \frac{1}{\mu} \nabla f(y_t). \quad (4.7d)
\end{align*}
\]

Paralleling Remark 2, our derivation provides new insights into the choices of the AGM step sizes by expressing them in terms of the PPM step sizes \( \eta_t \)'s and \( \tilde{\eta}_t \)'s. Furthermore, our derivation actually demystifies the mysterious parameter choices made in the Nesterov’s book [Nes18, Section 2.2]. To see this, let us recall the well known convergence rate of PPM for strongly convex costs due to Rockafellar [Roc76, (1.14)]:

\[
f(x_T) - f(x_*) \leq O \left( \prod_{t=1}^{T} (1 + \mu \eta_t)^{-1} \right) \quad \text{for any } T \geq 1. \quad (4.8)
\]

In light of (4.8), it turns out that for the approximate PPM (4.7), choosing the following step sizes
\[ \eta_t \equiv \eta := \mu^{-1} \cdot (\sqrt{\kappa} - 1)^{-1} \quad \text{and} \quad \tilde{\eta}_t \equiv \tilde{\eta} := \mu^{-1} \cdot (\sqrt{\kappa})^{-1} \quad \text{(where} \quad \kappa := \ell/\mu) \]

actually recovers the well known parameters choice [Nes18, (2.2.22)] which leads to the convergence rate of \( O \left( (1 + \eta \mu)^{-T} \right) = O \left( (1 + 1/(\sqrt{\kappa} - 1))^{-T} \right) \). See [BG19, Section 5.5] for a simple Lyapunov function proof of this convergence rate.

**Remark 3 (Nesterov’s general method).** Remarkably, (4.7) also exactly recovers a general version of AGM [Nes18, (2.2.19)] which smoothly interpolates between the strongly convex case and the non-strongly convex case. To see this, we follow a simple equivalent representation of Nesterov’s general step sizes given in [AS20]. More specifically, given a sequence \( \{x_t\} \) of positive numbers defined as per the non-linear recursion \( \xi_t + (\kappa \xi_t + 1)^{-1} = \xi_t^2 \) with an initial value \( \xi_0 > 0 \), choosing \( \eta_t = \mu^{-1} \cdot (\kappa \xi_t - 1)^{-1} \) and \( \tilde{\eta}_t = \mu^{-1} \cdot (\kappa \xi_{t+1} - 1)^{-1} \cdot (1 - \xi_{t+1}) \) exactly recovers the choices in [AS20, Section 2] which are shown to be equivalent to Nesterov’s choices [Nes18, (2.2.19)].

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### 5 Simple generalizations with similar triangles

In the previous section, we have demonstrated that Nesterov’s method is nothing but an approximation of PPM. This view point has not only provided simple derivations of versions of AGM, but also offered clear explanations of the step sizes. In this section, we demonstrate that these interpretations offered by PPM actually lead to a great simplification of Nesterov’s AGM in the form of the method of similar triangles [Nes18, GN18] which admits simple generalizations to practically relevant settings.

Our starting point is the observations made in the previous section: (i) from Remark 2, we have seen \( \|y_t - x_t\| : \|y_t - z_t\| = \eta_t : 1/L; \) (ii) from Section 4.1, we have seen that we need to choose \( \eta_t = \Theta(t) \), and hence, \( \eta_{t+1} \approx \eta_t \gg 1 \). From these observations, one can readily see that the triangle \( \Delta x_t x_{t+1} z_t \) is approximately similar to \( \Delta y_t z_{t+1} z_t \). Therefore, one can simplify AGM by further exploiting this fact: we modify the updates so that the two triangles are indeed **similar**:

**Similar triangle approximation of PPM:**

\[
\begin{align*}
\eta_t &= \frac{1}{L + \eta_t} x_t + \frac{\eta_t}{L + \eta_t} z_t, \quad (5.1a) \\
x_{t+1} &= x_t - \eta_{t+1} \nabla f(y_t), \quad (5.1b) \\
z_{t+1} &= \frac{1}{L + \eta_t} x_{t+1} + \frac{\eta_t}{L + \eta_t} z_t. \quad (5.1c)
\end{align*}
\]

We provide a PPM-based analysis of (5.1) for a more general setting in Section 5.1.

**Remark 4.** The updates akin to (5.1) can be found in [Tse08, Algorithm 1] (note that the step sizes are slightly different). Our derivation based on the PPM view indeed clarifies why such similar triangles are natural updates to consider. We also remark that the updates based on similar triangles is useful in developing universal methods for stochastic composite optimizations [GN18].

**Remark 5 (Momentum methods).** An alternative way to approximate PPM with similar triangles is:

**Alternative similar triangle approximation:**

\[
\begin{align*}
y_t &= \frac{1}{L + \eta_t} x_t + \frac{\eta_t}{L + \eta_t} z_t, \quad (5.2a) \\
z_{t+1} &= y_t - \frac{1}{L} \nabla f(y_t), \quad (5.2b) \\
x_{t+1} &= z_{t+1} + L \eta_t (z_{t+1} - z_t). \quad (5.2c)
\end{align*}
\]

In fact, (5.2) can be equivalently expressed without \( \{x_t\} \): at the \( t \)-iteration, we compute (5.2b) and then update \( y_{t+1} \) via \( y_{t+1} = z_{t+1} + \frac{L \eta_t}{L \eta_{t+1} + 1} (z_{t+1} - z_t) \) (these updates are illustrated with dots in Figure 5). Hence,
(5.2) is equal to the well-known momentum based AGM [Nes83, BT09]. Notably, it turns out that our PPM-based analysis suggests the choice of \( \{ \eta_t \} \) as per the recursive relation 
\[
(L\eta_t + 1 + \frac{a_t}{2})^2 = (L\eta_t + 1)^2 + \frac{a_t^2}{4},
\]
which after substitution \( L\eta + 1 \leftarrow a_t \) exactly recovers the popular recursive relation 
\[
a_{t+1} = \frac{1}{2}(1 + \sqrt{1 + 4a_t^2}) \]
in [Nes83, BT09]. See Appendix C for precise details.

The main advantage of this similar triangles approximation (5.1) becomes clearer in the constraint optimization case: when there is a constraint set, the steps (4.2b) and (4.2c) both become projections steps which could be costly when the constraint set does not admit simple projections. On the other hand, since (5.1) only requires a single projection in each iteration, it minimizes such costly computations.

### 5.1 Extension to composite costs and general norms

It turns out (5.1) admits a simple extension to the practically relevant setting of composite costs and general norms (see e.g. [Nes18, Section 6.1.3]). More specifically, for a closed convex set \( Q \subseteq \mathbb{R}^d \) and a closed convex function \( \Psi : Q \to \mathbb{R} \), consider

\[
\min_{x \in Q} f^\Psi(x) := f(x) + \Psi(x),
\]

where \( f : Q \to \mathbb{R} \) is a differentiable convex function which is \( L \)-smooth with respect to a norm \( \| \cdot \| \) that is not necessarily the \( \ell_2 \) norm (i.e., we regard the norm in Definition 1 to be our chosen norm). For the general norm case, we use the Bregman divergence for the regularizer:

**Definition 3.** Given a 1-strongly convex (w.r.t the chosen norm \( \| \cdot \| \)) function \( h : Q \to \mathbb{R} \cup \{ \infty \} \) that is differentiable on the interior of \( Q \), 

\[
D_h(u, v) := h(u) - h(v) - \langle \nabla h(v), u - v \rangle
\]

for all \( u, v \in Q \).

Under the above setting and assumption, (5.1) admits a simple generalization:

| Similar triangle approximations of PPM for composite costs and general norms: |
|---------------------------------------------------------------|
| \[
y_t = \frac{y_t}{1/L + \eta_t} x_t + \frac{\eta_t}{1/L + \eta_t} z_t,
\]
| \[
x_{t+1} = \arg\min_{x \in Q} \left\{ f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{1}{\eta_t} D_h(x, x_t) + \Psi(x) \right\},
\]
| \[
z_{t+1} = \frac{z_t}{1/L + \eta_t} x_{t+1} + \frac{1}{1/L + \eta_t} z_t.
\]

Again, the similar triangle approximation (5.3) is computationally advantageous in that it only requires a single projection in each iteration. Now we provide a simple PPM-based analysis of (5.3):

**PPM-based analysis of (5.3).** To obtain counterparts of (B1) and (B2), we now use a generalization of Proposition 1 to the Bregman divergence ([Teb18, Lemma 3.1]). With such a generalization, we obtain the following inequality for \( \phi^\Psi(x) := \eta_{t+1}[f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \Psi(x)] \):

\[
\phi^\Psi(x_{t+1}) - \phi^\Psi(x_*) + D_h(x_*, x_{t+1}) + D_h(x_{t+1}, x_t) - D_h(x_*, x_t) \leq 0,
\]

where \( x_* \in \arg\min_{x \in Q} f^\Psi(x) \). Now using (5.4), one can derive from first principles the following inequalities (we defer the derivations to Appendix B):

- **Counterpart of (B1):** \( \eta_{t+1}[f^\Psi(z_{t+1}) - f^\Psi(x_*)] + D_h(x_*, x_{t+1}) - D_h(x_*, x_t) \leq (G_1) \)
- **Counterpart of (B2):** \( f^\Psi(z_{t+1}) - f^\Psi(x_t) \leq (G_2) \).

---

\(^3\)This means that the epigraph of the function is closed. See [Nes18, Definition 3.1.2].

\(^4\)The reason why we need the Bregman divergence in place of the norm squared regularization is because for norms other than the \( \ell_2 \) norm, \( \frac{1}{2} \| u - v \|^2 \) is not strongly convex with respect to the chosen norm.
where $(\mathcal{G}_1) := -\frac{1}{2}||x_{t+1} - x_t||^2 + \eta_t\left(\frac{1}{2}||z_{t+1} - y_t||^2 + \langle \nabla f(y_t), z_{t+1} - x_{t+1} \rangle + \Psi(z_{t+1}) - \Psi(x_{t+1}) \right)$ and $(\mathcal{G}_2) := \frac{1}{2}||z_{t+1} - y_t||^2 + \langle \nabla f(y_t), z_{t+1} - z_t \rangle + \Psi(z_{t+1}) - \Psi(z_t)$. Similarly to Section 4.1, yet replacing the norm squared term with the Bregman divergence, we choose
\[
\Phi_t := \left( \sum_{i=1}^t \eta_i \right) \cdot \left( f^\Psi(z_t) - f^\Psi(x_*) \right) + D_h(x_*, x_t).
\]
Then, it suffices to show $(\mathcal{G}_1) + \left( \sum_{i=1}^t \eta_i \right) \cdot (\mathcal{G}_2) \leq 0$. Using the facts (i) $z_{t+1} - x_{t+1} = L\eta_t(z_t - z_{t+1})$ and (ii) $||x_{t+1} - x_t|| = (L\eta_t + 1)||z_{t+1} - y_t||$ (both are immediate consequences of the similar triangles) and rearranging, one can easily check that $(\mathcal{G}_1) + \left( \sum_{i=1}^t \eta_i \right) \cdot (\mathcal{G}_2)$ is equal to
\[
\frac{1}{2} \left(- (L\eta_t + 1)^2 + L\eta_{t+1} + L \sum_{i=1}^t \eta_i \right) \||z_{t+1} - y_t||^2 + \left(L\eta_{t+1} - \sum_{i=1}^t \eta_i \right) \langle \nabla f(y_t), z_t - z_{t+1} \rangle + \eta_{t+1} [\Psi(z_{t+1}) - \Psi(x_{t+1})] + \left( \sum_{i=1}^t \eta_i \right) \cdot [\Psi(z_{t+1}) - \Psi(z_t)]. \tag{5.5}
\]
Now choosing $\eta_t = t/2L$ analogously to Section 4.1, one can easily verify (5.5) + (5.6) + (5.7) $\leq 0$. Indeed, for (5.5), since $L\eta_{t+1} = \sum_{i=1}^t \eta_i$, the coefficient becomes $1/2(L\eta_{t+1})/(L\eta_t - 1)$ which is a negative number since $L\eta_{t+1} - L\eta_t - 1 = -1/2$; for (5.6), the coefficient becomes zero due to the relation $L\eta_t\eta_{t+1} = \sum_{i=1}^t \eta_i$; lastly, for (5.7), we have
\[
(5.7) = \eta_{t+1} \left[(1 + L\eta_t)\Psi(z_{t+1}) - \Psi(x_{t+1}) - L\eta_t \Psi(z_t) \right] \leq 0, \tag{5.8}
\]
where the equality is due to the relation $L\eta_t\eta_{t+1} = \sum_{i=1}^t \eta_i$, and the inequality is due to the update (5.3c) (which can be equivalently written as $\gamma(z_{t+1}) = x_{t+1} + L\eta_tz_t$ and the convexity of $\Psi$. Hence, we obtain the accelerated rate of $f^\Psi(z_T) - f^\Psi(x_*) \leq \frac{4L\eta_t(z_T - x_*)}{1/L + 1} = O(1/t^2).
\]

6 Related work

Motivated by the obscure scope of Nesterov’s estimate sequence technique, there have been a flurry of works on developing alternative approaches to Nesterov’s acceleration. The most contributions are made based on understanding the continuous limit dynamics of Nesterov’s AGM [SBC16, KBB15, WWJ16]. These continuous dynamics approaches have brought about new intuitions about Nesterov’s acceleration, and follow-up works have developed analytical techniques for such dynamics [WRJ16, DO19]. However, these approaches share a limitation that when applying discretization techniques to obtain optimization algorithms, some auxiliary modifications are required to recover/obtain analyzable algorithms. In contrast, our PPM approach directly yields accelerated methods and does not require additional adjustments.

Another notable contribution is made based on the linear coupling framework [AZO17]. The main observation is that the two most popular first-order methods, namely gradient descent and mirror descent, have complementary performances, and hence, one can come up with a faster method by linearly coupling the two methods. This view indeed offers a general framework of developing fast optimization algorithms; however, for understanding Nesterov’s acceleration, this view has less expressive power compared to our PPM view. More specifically, it is a priori not clear why one needs to couple two methods linearly. Moreover, with the linear coupling view, one cannot interpret the interpolation weight as we did in Remark 2.

It is also important to note that PPM has been given new attention as a building block for designing and analyzing fast optimization methods [Dru17]. To list few instances, PPM has given rise to methods for weakly convex problems [DG19], the prox-linear methods for composite optimizations [BF95, Nes07, LW16], accelerated methods for stochastic optimizations [LH15], and methods for saddle-point problems [MOP19]. Moreover, using Proposition 1 (and its generalization to the Bregman divergence) as a unified tool for analyzing first-order methods has been discussed in [Teb18].
7 Conclusion

This work provides a complete understanding of Nesterov’s acceleration by making analytical and quantitative connections to the proximal point method. The key observation is that an alternation of two simple approximations of the PPM exactly recovers Nesterov’s AGM. Through this connection, we are able to explain all the step sizes of AGM in terms of the PPM step sizes, demystifying the mysterious choices made in the literature. This view naturally extends to the strongly convex case and recovers Nesterov’s general accelerated method from his book. Moreover, our PPM view motivates a simplification of AGM using similar triangles, which admits a simple PPM-based analysis as well as a simple extension to the general norm and composite optimization case. For future directions, it would be interesting to connect our PPM view to accelerated stochastic methods \cite{LMH15, LZ18} and other accelerated methods, including geometric descent \cite{BLS15}.

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11
A Deferred derivations from Section 4.1

Counterpart of (B1). Apply Proposition 1 with \( \phi(x) = \eta_{t+1}[f(y_t) + \langle \nabla f(y_t), x - y_t \rangle] \) to (4.1a):

\[
\phi(x_{t+1}) - \phi(x_*) + \frac{1}{2} \|x_* - x_{t+1}\|^2 + \frac{1}{2} \|x_{t+1} - x_t\|^2 - \frac{1}{2} \|x_* - x_t\|^2 \leq 0 .
\]

(A.1)

Now from the convexity, we have \( \phi(x_*) \leq \eta_{t+1} f(x_*) \), and from the \( L \)-smoothness, we have

\[
\phi(x_{t+1}) = \eta_{t+1}[f(y_t) + \langle \nabla f(y_t), z_{t+1} - y_t \rangle + \langle \nabla f(y_t), x_{t+1} - z_{t+1} \rangle] \\
\geq \eta_{t+1} \left[ f(z_{t+1}) - \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), x_{t+1} - z_{t+1} \rangle \right].
\]

Plugging these inequalities back to (A.1) and rearranging, we obtain the following inequality:

\[
\eta_{t+1}[f(z_{t+1}) - f(x_*)] + \frac{1}{2} \|x_* - x_{t+1}\|^2 - \frac{1}{2} \|x_* - x_t\|^2 \\
\leq - \frac{1}{2} \|x_{t+1} - x_t\|^2 + \eta_{t+1} \left[ \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), z_{t+1} - x_{t+1} \rangle \right].
\]

(A.2)

Now decomposing the inner product term in (A.2) into

\[
\eta_{t+1} \langle \nabla f(y_t), z_{t+1} - y_t \rangle + \eta_{t+1} \langle \nabla f(y_t), y_t - x_t \rangle + \eta_{t+1} \langle \nabla f(y_t), x_t - x_{t+1} \rangle,
\]

and using \( x_{t+1} - x_t = -\eta_{t+1} \nabla f(y_t) \) and \( z_{t+1} - y_t = -\frac{1}{L} \nabla f(y_t) \) (which are (4.2b) and (4.2c), respectively), (A.2) becomes

\[
\left( \frac{\eta_{t+1}^2}{2} - \frac{\eta_{t+1}}{2L} \right) \|\nabla f(y_t)\|^2 + \eta_{t+1} \langle \nabla f(y_t), y_t - x_t \rangle.
\]

Now, using the relation \( y_t - x_t = L\eta_t (z_t - y_t) \) (which is (4.2a)), we obtain \( (F_1) \).

Counterpart of (B2). It readily follows from the \( L \)-smoothness and the convexity of \( f \):

\[
f(z_{t+1}) - f(z_t) = f(z_{t+1}) - f(y_t) + f(y_t) - f(z_t) \\
\leq \langle \nabla f(y_t), z_{t+1} - y_t \rangle + \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), y_t - z_t \rangle \\
\overset{(a)}{=} - \frac{1}{2L} \|\nabla f(y_t)\|^2 + \langle \nabla f(y_t), y_t - z_t \rangle = (F_2),
\]

where (a) is due to \( z_{t+1} - y_t = -\frac{1}{L} \nabla f(y_t) \).
B  Deferred derivations from Section 5

Counterpart of (B1). From convexity, we have $\phi^\Psi(x_*) \leq \eta_{t+1} f^\Psi(x_*)$, and from the $L$-smoothness, we have the following lower bound:

$$
\phi^\Psi(x_{t+1}) = \eta_{t+1}[f(y_t) + \langle \nabla f(y_t), z_{t+1} - y_t \rangle + \langle \nabla f(y_t), x_{t+1} - z_{t+1} \rangle + \Psi(x_{t+1})] \\
\geq \eta_{t+1} \left[ f^\Psi(z_{t+1}) - \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), x_{t+1} - z_{t+1} \rangle + \Psi(x_{t+1}) - \Psi(z_{t+1}) \right].
$$

Plugging these back to (5.4), and using the bound $-D_h(x_{t+1}, x_t) \leq -\frac{1}{2} \|x_{t+1} - x_t\|^2$, we obtain:

$$
\eta_{t+1} (f^\Psi(z_{t+1}) - f^\Psi(x_*)) + D_h(x_*, x_{t+1}) - D_h(x_*, x_t) \\
\leq -\frac{1}{2} \|x_{t+1} - x_t\|^2 + \eta_{t+1} \left[ \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), z_{t+1} - x_{t+1} \rangle + \Psi(z_{t+1}) - \Psi(x_{t+1}) \right],
$$

which is precisely equal to ($G_1$).

Counterpart of (B2). We use $L$-smoothness and the convexity of $f$ to obtain the following:

$$
\begin{align*}
&f^\Psi(z_{t+1}) - f^\Psi(z_t) \leq f(z_{t+1}) - f(y_t) + f(y_t) - f(z_t) + \Psi(z_{t+1}) - \Psi(z_t) \\
&\leq \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), z_{t+1} - x_{t+1} \rangle + \Psi(z_{t+1}) - \Psi(z_t),
\end{align*}
$$

which is precisely equal to ($G_2$).

C  Details for Remark 5

We first develop counterparts of (B1) and (B2):

Counterpart of (B1). By the updates (5.2), we have $x_{t+1} = x_t - (\eta_t + \frac{1}{L}) \nabla f(y_t)$. Letting $\eta_{t+1} := \eta_t + \frac{1}{L}$, this relation can be equivalently written as:

$$
x_{t+1} \leftarrow \arg\min_x \left\{ f(y_t) + \langle \nabla f(y_t), x - y_t \rangle + \frac{1}{\eta_{t+1}} \|x - x_t\|^2 \right\} \quad (C.1)
$$

The rest is similar to Appendix A: we apply Proposition 1 with $\phi(x) = \eta_{t+1}[f(y_t) + \langle \nabla f(y_t), x - y_t \rangle]$:

$$
\phi(x_{t+1}) - \phi(x_*) + \frac{1}{2} \|x_* - x_{t+1}\|^2 + \frac{1}{2} \|x_{t+1} - x_t\|^2 - \frac{1}{2} \|x_* - x_t\|^2 \leq 0. \quad (C.2)
$$

Now from the convexity, we have $\phi(x_*) \leq \bar{\eta}_{t+1} f(x_*)$, and from the $L$-smoothness, we have

$$
\phi(x_{t+1}) = \bar{\eta}_{t+1} [f(y_t) + \langle \nabla f(y_t), z_{t+1} - y_t \rangle + \langle \nabla f(y_t), x_{t+1} - z_{t+1} \rangle] \\
\geq \bar{\eta}_{t+1} \left[ f(z_{t+1}) - \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), z_{t+1} - x_{t+1} \rangle \right].
$$

Plugging these inequalities back to (C.2) and rearranging, we obtain the following inequality:

$$
\begin{align*}
&\bar{\eta}_{t+1} [f(z_{t+1}) - f(x_*)] + \frac{1}{2} \|x_* - x_{t+1}\|^2 - \frac{1}{2} \|x_* - x_t\|^2 \\
&\leq -\frac{1}{2} \|x_{t+1} - x_t\|^2 + \bar{\eta}_{t+1} \left[ \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), z_{t+1} - x_{t+1} \rangle \right] \\
&= \left( - (L\eta_t + 1)^2 + L\bar{\eta}_{t+1} \right) \cdot \|z_{t+1} - y_t\|^2 + \bar{\eta}_{t+1} \cdot \langle \nabla f(y_t), z_{t+1} - x_{t+1} \rangle =: (H_1),
\end{align*}
$$

where the last line follows since $\|x_{t+1} - x_t\| = (L\eta_t + 1) \cdot \|z_{t+1} - z_t\|$ (see Figure 5).
Counterpart of (B2). It is identical to Appendix B: From the $L$-smoothness and the convexity of $f$:

$$
\begin{align*}
f(z_{t+1}) - f(z_t) &= f(z_{t+1}) - f(y_t) + f(y_t) - f(z_t) \\
&\leq \langle \nabla f(y_t), z_{t+1} - y_t \rangle + \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), y_t - z_t \rangle \\
&= \frac{L}{2} \|z_{t+1} - y_t\|^2 + \langle \nabla f(y_t), z_{t+1} - z_t \rangle =: (H_2).
\end{align*}
$$

Having established the counterparts of (B1) and (B2), following Section 4.1, we choose

$$
\Phi_t := (\sum_{i=1}^t \tilde{\eta}_i) \cdot (f(z_t) - f(x_*)) + \frac{1}{2} \|x_* - x_t\|^2 .
$$

(C.4)

To prove the validity of the chosen Lyapunov function, it suffices to verify

$$
(H_1) + (\sum_{i=1}^t \tilde{\eta}_i) \cdot (H_2) \leq 0
$$

(C.5)

which is equivalent to (since $z_{t+1} - x_{t+1} = -L\eta_t(z_{t+1} - z_t)$)

$$
\frac{1}{2} \left(- (L\eta_t + 1)^2 + \sum_{i=1}^{t+1} L\tilde{\eta}_i \right) \cdot \|z_{t+1} - y_t\|^2 + \left(L\eta_t \tilde{\eta}_{t+1} - \sum_{i=1}^{t+1} \tilde{\eta}_i \right) \langle \nabla f(y_t), z_{t+1} - z_t \rangle \leq 0 .
$$

(C.6)

From (C.6), it suffices to choose \{\eta_t\} so that $L\eta_t \tilde{\eta}_{t+1} = \sum_{i=1}^{t+1} \tilde{\eta}_i$. Indeed, with such a choice, the coefficient of the inner product term in (4.4) becomes zero and the coefficient of the squared norm term becomes

$$
\frac{1}{2} \left(- (L\eta_t + 1)^2 + \sum_{i=1}^{t+1} L\tilde{\eta}_i \right) = \frac{1}{2} \left(- (L\eta_t + 1)^2 + L\tilde{\eta}_{t+1} + L\tilde{\eta}_{t+1} \cdot L\eta_t \right) = \frac{1}{2} \left(- (L\eta_t + 1)^2 + L\tilde{\eta}_{t+1} + L\eta_t \cdot L\tilde{\eta}_{t+1} \right) = 0
$$

since $L\tilde{\eta}_{t+1} = L\eta_t + 1$. Indeed, one can actually simplify the relation $L\eta_t \tilde{\eta}_{t+1} = \sum_{i=1}^{t+1} \tilde{\eta}_i$:

$$
L\eta_{t+1} \cdot (L\eta_{t+1} + 1) = L\eta_{t+1} \cdot L\tilde{\eta}_{t+2} = \sum_{i=1}^{t+1} L\tilde{\eta}_i = L\tilde{\eta}_{t+1} + L\eta_t \cdot L\tilde{\eta}_{t+1} = (L\eta_t + 1)^2 .
$$

After rearranging, we obtain the recursive relation: $(L\eta_{t+1} + \frac{1}{2})^2 = (L\eta_t + 1)^2 + \frac{1}{4}$, which after the substitution $L\eta_t + 1 = a_t$ exactly recovers the popular recursive relation $a_{t+1} = \frac{1 + \sqrt{1 + 4a_t^2}}{2}$ in [Nes83, BT09].

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