Radially symmetric minimizers for a $p$-Ginzburg Landau type energy in $\mathbb{R}^2$

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Abstract  We consider the minimization of a $p$-Ginzburg-Landau energy functional over the class of radially symmetric functions of degree one. We prove the existence of a unique minimizer in this class, and show that its modulus is monotone increasing and concave. We also study the asymptotic limit of the minimizers as $p \to \infty$. Finally, we prove that the radially symmetric solution is locally stable for $2 < p \leq 4$.

Keywords  Ginzburg-Landau energy · $p$-Laplacian · Radial symmetry · Stability

Mathematics Subject Classification (2000)  35Q56

1 Introduction

Given $p > 2$ consider the minimization problem of the energy functional

$$E_p(u) = \int_{\mathbb{R}^2} |\nabla u|^p + \frac{1}{2} (1 - |u|^2)^2$$

(1.1)

over the class of maps $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ that satisfy $E_p(u) < \infty$ and have a degree $d$ “at infinity”. In our previous work [1] it was shown that the notion of degree at infinity is...
well-defined. Hence, minimization over the homotopy class of maps with degree \( d \) is a sensible task. Moreover, in the case of degree \( d = 1 \) we proved that a minimizer does exist. An important open question is whether any minimizer \( u \) is necessarily radially symmetric, i.e., \( u = f(r)e^{i\theta} \) for some function \( f(r) \) satisfying \( f(0) = 0 \) (thanks to invariance with respect to translations we may assume that \( u(0) = 0 \)). We show in the sequel that a (unique) minimizer within the radially symmetric class \( u_p = f_p(r)e^{i\theta} \) exists. We were, however, unable to determine whether \( u_p \) is a minimizer or not. As a preliminary step towards establishing the minimality properties of \( u_p \), we study in the present paper its stability properties. One of our main results (see Theorem 2 below) establishes that \( u_p \) is indeed stable if \( p \in (2, 4) \). We conjecture that this result remains valid for any \( p > 2 \). It should be mentioned that the analogous stability problem for \( p = 2 \) on the disc \( B_1(0) \) with the boundary condition 

\[
u(z) = \frac{z}{|z|} \quad \text{on} \quad \partial B_1(0)
\]

was solved by Mironescu [9] and in a weaker form, by Lieb and Loss [8]. Going back to the problem on \( \mathbb{R}^2 \), but again for \( p = 2 \), we recall that the \( L^2 \)-stability of the radially symmetric solution was proved by Ovchinnikov and Sigal [11] and in a more natural energy space by del Pino et al. [5]. However, Mironescu [10] showed a stronger result, namely, that the radially symmetric solution is the unique (up to rotations and translations) local minimizer on \( \mathbb{R}^2 \), that is, on every disc \( B_R(0) \) it is minimizing for its boundary values on \( \partial B_R(0) \). Note that for \( p = 2 \) (in contrast with \( p > 2 \)) only the notion of local minimizer makes sense since the admissible maps have infinite energy.

The manuscript is organized as follows. In Sect. 2 we establish existence and uniqueness of the minimizer \( u_p = f_p(r)e^{i\theta} \) in the radially symmetric class, as well as its regularity. We also show that \( f_p \) is increasing and concave and obtain some precise estimates for \( f_p(r) \) for large values of \( r \). In Sect. 3 we study the limit of \( f_p \) as \( p \) tends to infinity. We show that 

\[
\lim_{p \to \infty} f_p = f_{\infty}
\]

is the piecewise linear function given by \( \frac{x}{\sqrt{2}} \) for \( r < \sqrt{2} \) and is identically equal to 1 for \( r \geq \sqrt{2} \). Finally, Sect. 4 is devoted to the study of the stability of the radially symmetric solution.

## 2 Radially symmetric solutions

In this section we consider some of the properties of the minimizer of

\[
I_p(f) = \int_0^\infty \left[ (f')^2 + \frac{f^2}{r^2} \right]^{p/2} + \frac{1}{2} (1 - f^2)^2 \right] r dr \quad \text{(2.1)}
\]

for any \( p > 2 \). Note that \( I_p(f) = \frac{1}{2\pi} E(u) \) where \( u = f(r)e^{i\theta} \).

### 2.1 Existence

For each \( p > 2 \) we define the space

\[
X_p = \left\{ f \in W^{1,p}_{\text{loc}}(0, \infty) : \int_0^\infty \left[ (f')^2 + \frac{f^2}{r^2} \right]^{p/2} r dr < \infty \right\} \quad \text{(2.2)}
\]

Existence of a solution will be established by minimization of \( I_p(f) \) over \( X_p \). Note that \( X_p \subseteq C^{\alpha}_{\text{loc}}[0, \infty) \), with \( \alpha = 1 - 2/p \), since whenever \( f \in X_p \), the function 

\[
F(x_1, x_2) = f \left( \sqrt{x_1^2 + x_2^2} \right)
\]

belongs to \( W^{1,p}_{\text{loc}}(\mathbb{R}^2) \), and then we can apply Morrey’s theorem. Furthermore,
for every $f \in X_p$ we must have $f(0) = 0$. This follows from the continuity of $f$ and the fact that
\[ \int_0^1 \frac{|f|^p}{r^{p-1}} < \infty. \]

**Proposition 2.1** The minimum of $I_p(f)$ over $X_p$ is attained by a function $f_p \in X_p$ satisfying $0 \leq f_p(r) \leq 1$, $\forall r \in [0, \infty)$.

**Proof** Put
\[ m_p = \inf_{f \in X_p} I_p(f). \]
We first note that $m_p < \infty$ since the function $g^* \in X_p$ defined by
\[ g^*(r) = \begin{cases} r & r \leq 1, \\ 1 & r > 1, \end{cases} \]
verifies $I_p(g^*) < \infty$. Consider a minimizing sequence $\{g_m\}$ for (1.1), i.e.,
\[ \lim_{m \to \infty} I_p(g_m) = m_p. \]
By passing to a diagonal sequence we may assume that for any compact interval $[a, b] \subset (0, \infty)$ we have
\[ g_m \rightharpoonup g \text{ weakly in } W^{1,p}(a, b). \]
Since the convexity of the Lagrangian
\[ L(P, Z, r) = \left\{ \left( \frac{p^2 + Z^2}{r^2} \right)^{p/2} + \frac{1}{2}(1 - Z^2)^2 \right\} r \]
in the variable $P$ implies weak lower-semi-continuity of the functional $I_p^{(a,b)}(f) := \int_a^b L(f', f, r)dr$ (see [7, Theorem 1, Sect. 8.2]), we deduce from (2.4) that
\[ I_p^{(a,b)}(g) \leq m_p. \]
Since the interval $[a, b]$ is arbitrary, we conclude from (2.5) that $g \in X_p$, $I_p(g) \leq m_p$, so that necessarily $I_p(g) = m_p$, and $g$ is a minimizer in (1.1). Since replacing $g$ by
\[ \tilde{g}(r) = \min(1, |g(r)|), \]
gives a map $\tilde{g} \in X_p$ such that $I_p(\tilde{g}) \leq I_p(g)$ (with strict inequality, unless $|g| \leq 1$) we conclude that we may assume $0 \leq g(r) \leq 1$ for all $r$, and the result follows for $f_p = g$. \qed

The next lemma shows that $f$ is positive on $(0, \infty)$.

**Lemma 2.1** $f_p > 0$ for all $r > 0$.

**Proof** We first claim that there is no interval of the form $[0, a]$, with $a > 0$ such that
\[ f \equiv 0 \text{ on } [0, a]. \]
Indeed, suppose that (2.6) holds for some $a$. Fix any function $g \in C^\infty[0, a]$ satisfying $g(0) = g(a) = 0$ and $g(r) > 0$ for $r \in (0, a)$. Then, for any small $\varepsilon > 0$ consider the function $h_\varepsilon$ defined by

$$h_\varepsilon(r) = \begin{cases} 
\varepsilon g(r) & 0 \leq r \leq a, \\
f_p(r) & r > a.
\end{cases}$$

A simple computation gives

$$I_p(h_\varepsilon) = I_p^{(a, \infty)}(f_p) + \varepsilon^p \int_0^a \left(|g'|^2 + \left(\frac{g}{r}\right)^2\right)^{p/2} r \, dr + \int_0^a (1 - \varepsilon^2 g^2) r \, dr < I_p(f_p),$$

provided $\varepsilon$ is chosen small enough.

Next, we turn to the proof itself and assume by negation that $f_p(r_0) = 0$ for some $r_0 > 0$. Put

$$\delta_0 = \max_{r \in [0, r_0]} f_p(r).$$

By the above claim $\delta_0 > 0$. Let $\delta \in (0, \delta_0)$ and consider the set $S_\delta = \{r > 0 : f_p(r) < \delta\}$. Denote by $J = (\alpha, \beta)$ the component of $S_\delta$ containing $r_0$. Since $\delta < \delta_0$ we have $\alpha > 0$. There is a $\delta_1 > 0$ such that the function

$$H_r(t) = \left(\frac{t}{r}\right)^p + \frac{1}{2}(1 - t^2)^2$$

is decreasing on $[0, \delta_1]$ for every $r \geq \alpha$. We may now replace $\delta$ by $\min(\delta, \delta_1)$ and set

$$\tilde{f}(r) = \begin{cases} 
f_p & r \notin J \\
\delta & r \in J.
\end{cases}$$

From the monotonicity of $H_r$ it follows that $I_p(\tilde{f}) < I_p(f_p)$. A contradiction. \qed

2.2 Uniqueness

**Proposition 2.2** The non-negative minimizer for $I_p(f)$ is unique.

**Proof** We use a convexity argument due to Benguria (see [4]) for the case of the Laplacian (see [4]) and by Diaz and Saá [6] and Anane [2] for the case of the $p$-Laplacian. More specifically, we follow the presentation of Belloni and Kawohl [3]. Assume $f$ and $g$ are both minimizers in (2.1). By an argument from the proof of Proposition 2.1 it follows that necessarily $f(r) \leq 1$ and $g(r) \leq 1$ for each $r$. Set

$$\eta = \frac{f^p + g^p}{2} \quad \text{and} \quad w = \eta^{\frac{1}{p}}.$$ 

Denote also

$$s(r) = \frac{f^p}{f^p + g^p}.$$ 

Note that

$$w' = \frac{1}{2} \eta^{\frac{1}{p}-1} \left(f^{p-1}f' + g^{p-1}g'\right).$$

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Next we compute
\[
\left( w'^2 + \frac{w^2}{r^2} \right)^\frac{p}{2} = \left| \eta \left( \frac{(s(r)f')^2}{f} + \frac{(1-s(r))g'}{g} \right)^2 + \frac{1}{r^2} \right|^{\frac{p}{2}} \\
\leq \eta \left( s(r) \left( \frac{|f'|^2}{f^2} + \frac{1}{r^2} \right) + (1-s(r)) \left( \frac{|g'|^2}{g^2} + \frac{1}{r^2} \right) \right) \\
= \frac{\eta}{f^p + g^p} \left( f'^2 + \frac{f^2}{r^2} \right)^\frac{p}{2} + \frac{\eta}{f^p + g^p} \left( g'^2 + \frac{g^2}{r^2} \right)^\frac{p}{2} \\
= \frac{1}{2} \left( \left( f'^2 + \frac{f^2}{r^2} \right)^\frac{p}{2} + \left( g'^2 + \frac{g^2}{r^2} \right)^\frac{p}{2} \right)
\]

Above we used the convexity of the function \( t \mapsto \left( t^2 + \frac{1}{r^2} \right)^\frac{p}{2} \). Note that equality holds in the above only if \( \frac{f'}{f} = \frac{g'}{g} \). If such an equality holds for all \( r \), we conclude easily that \( g = cf \) for some constant \( c \), which then must be equal to 1. Therefore, the uniqueness claim follows from the above inequality and the convexity of the second term \((1 - f^2)^2 \) as a function of \( f^p \) for \( p \geq 2 \) and \( 0 \leq f \leq 1 \). \( \square \)

**Remark 2.1** As a matter of fact, the only minimizers of \( I_p \) are \( f_p \) and \(-f_p \). In view of Lemma 2.1 a non-negative minimizer must be strictly positive. Since \( I_p(|f|) = I_p(f) \), it follows that a minimizer may not change sign, and our assertion follows from the uniqueness for non-negative minimizers.

### 2.3 Regularity

This subsection is devoted to the study of the regularity properties of the minimizer \( f_p \).

**Proposition 2.3** We have \( f_p \in C^\infty(0, \infty) \).

**Proof** The Euler–Lagrange equation associated with (2.1) is
\[
\frac{1}{r} \left( r|\nabla u_p|^{p-2} f_p \right)' = |\nabla u_p|^{p-2} f_p - \eta \left( f_p \left( 1 - f_p^2 \right) \right) \tag{2.7}
\]
where
\[
u_p = f_p(r)e^{i\theta}.
\]

A direct consequence of (2.7) is that \( |\nabla u_p|^{p-2} f_p \in W^{1,p-2}_\text{loc}(0, \infty) \subset C(0, \infty) \) and we immediately obtain that \( f_p \in C^1(0, \infty) \) (using that \( f_p > 0 \) by Lemma 2.1). Inserting this new information into (2.7) we deduce that \( f_p \in C^2(0, \infty) \). Bootstrapping gives \( f_p \in C^k(0, \infty) \) for all \( k \), as claimed. \( \square \)

Our next objective is to prove the differentiability of \( f \) at 0.
Proposition 2.4 $\lim_{r \to 0^+} \frac{f_p(r)}{r}$ exists and is a positive number.

Proof We denote for convenience $f$ for $f_p$ and get from (2.7),

$$0 = f'' \left( 1 + \frac{p - 2}{|\nabla u_p|^2} |f'|^2 \right) + \frac{f''}{r} \left( 1 - \frac{p - 2}{|\nabla u_p|^2} f^2 \right) - \frac{f}{r^2} \left( 1 - \frac{p - 2}{|\nabla u_p|^2} |f'|^2 \right)$$

+ $\frac{2}{p}|\nabla u_p|^{2-p} f(1 - f^2)$,

or equivalently,

$$\frac{rf''}{f'} = -\left( |f'|^2 - (p - 3) \frac{f^2}{r^2} \right) + \frac{f^2}{rf'} \left( \frac{f^2}{r^2} - (p - 3)|f'|^2 \right)$$

+ $\frac{2}{p}|\nabla u_p|^{2-p} f f'' r^2 (1 - f^2) \cdot \frac{1}{1 + (p - 2)^2 |\nabla u_p|^2}$. (2.9)

Put

$$h = \frac{rf'}{f}. \quad (2.10)$$

We divide the rest of the proof into several steps.

Step 1: $-\frac{1}{p - 1} < h(r) < 1$ for all $r > 0$.

We can rewrite (2.9) as

$$rf'' = \frac{-h^2 + (p - 3) + h - (p - 3)h}{1 + (p - 1)h^2} - \frac{2}{p}|\nabla u_p|^{2-p} r^2 (1 - f^2) \frac{1 + h^2}{1 + (p - 1)h^2}. \quad (2.11)$$

Since

$$h' = \frac{f''h}{f'} + \frac{h}{r}(1 - h), \quad (2.12)$$

substituting (2.11) into (2.12) yields

$$h' = \left( \frac{1 - h}{r} \right) \cdot \left( \frac{1 + (p - 2)h + h^2}{1 + (p - 1)h^2} + h \right) - \frac{2}{p}|\nabla u_p|^{2-p} r(1 - f^2) \frac{1 + h^2}{1 + (p - 1)h^2}$$

+ $\frac{1 + h^2}{1 + (p - 1)h^2} \left[ \frac{(1 - h)(1 + (p - 1)h)}{r} - \frac{2}{p}|\nabla u_p|^{2-p} r(1 - f^2) \right]. \quad (2.13)$

By (2.13) we have

$$h' \leq \frac{1}{r} F_p(h), \quad (2.14)$$

where

$$F_p(h) = \frac{(1 + h^2)(1 - h)(1 + (p - 1)h)}{1 + (p - 1)h^2}. \quad (2.15)$$

We now prove that $h(r) < 1$ for all $r > 0$. Suppose to the contrary that there exists $r_0 > 0$ for which $h(r_0) \geq 1$. Then, (2.14) yields $h'(r) < 0$ and $h(r) > 1$ for all $r < r_0$. Therefore, by (2.15) also $F_p(h) < 0$ for $r < r_0$. Integrating (2.14) gives
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\[ \int_{h(r_0)}^{h(r)} \frac{dh}{F_p(h)} \geq \ln \frac{r_0}{r}, \quad \forall r < r_0. \]

(2.16)

Since $\int_{h(r_0)}^{\infty} \frac{dh}{F_p(h)} < \infty$, (2.16) leads to a contradiction for $r > 0$ small enough.

Finally, we show that $h(r) > -\frac{1}{p-1}$ on $(0, \infty)$. Suppose to the contrary that $h(r_0) \leq -\frac{1}{p-1}$ for some $r_0$. Then, from (2.14) and (2.15) it follows that

\[ h(r) \leq -\frac{1}{p-1} \quad \text{and} \quad h'(r) < 0, \quad \forall r \geq r_0. \]

Therefore, also $f'_p(r) < 0$ for all $r \geq r_0$, violating $I_p(f_p) < \infty$. Step 1 is established.

Step 2: $f'_p(r)$ is strictly decreasing on $(0, \infty)$.

From Step 1 we get that

\[ \left( \frac{f}{r} \right)' = \frac{f}{r^2} (h - 1) < 0, \quad \forall r > 0, \]

(2.17)

and the conclusion follows.

Step 3: $\lim_{r \to 0^+} h(r) = 1$.

Fix any $r_0 > 0$. By Step 2 we have,

\[ |\nabla u_p|(r) \geq \frac{f(r)}{r} > \frac{f(r_0)}{r_0}, \quad \forall r < r_0. \]

Consequently, we have by (2.13),

\[ h' \geq \frac{F_p(h)}{r} - C_0 r, \quad \forall r \in (0, r_0). \]

(2.18)

for some positive $C_0$, which is independent of $r$. For a contradiction, we assume that $\liminf_{r \to 0^+} h(r) = a < 1$. Then, using (2.18) we can find $r_1 \in (0, r_0)$ small enough so that $h'(r_1) > 0$. Bootstrapping we obtain that $h'(r) > 0$ for all $r < r_1$. In particular, the full limit $\lim_{r \to 0^+} h(r) = a$ exists. Integration of (2.18) then yields

\[ \int_{h(r_1)}^{h(r)} \frac{dh}{F_p(h)} \geq \ln \frac{r_1}{r} - C, \quad \forall r < r_1. \]

(2.19)

Here we used the fact that $F_p(h) > 0$ by Step 1. Passing to the limit $r \to 0^+$ in (2.19) gives $\int_a^{h(r_1)} \frac{dh}{F_p(h)} = \infty$. In view of (2.15) we must have

\[ a = \lim_{r \to 0^+} h(r) = -\frac{1}{p-1}. \]

In particular, for $r$ sufficiently small we have $\frac{r}{f'} \leq -\frac{1}{2(p-1)}$, implying

\[ f(r) \geq Cr^{-\frac{1}{2(p-1)}}. \]

A contradiction.

Step 4: $f'(0)$ exists and it is a positive number.
By Step 2, the (possibly generalized) limit \( \lim_{r \to 0^+} \frac{f(r)}{r} \) exists, so we only need to exclude the possibility that the limit equals \(+\infty\). From Step 3 and (2.18) we get that
\[
h(r) \geq 1 - cr^2, \quad \forall r < r_0,
\]
i.e.,
\[
\frac{f'}{f} \geq \frac{1}{r} - cr.
\]
Therefore, \( f(r) \leq Cr \) for some positive constant \( C \), independently of \( r \), and the differentiability of \( f \) at 0 follows. Finally, \( f'(0) > 0 \) since \( \frac{f(r)}{r} \) is decreasing. \( \square \)

### 2.4 Monotonicity

**Proposition 2.5** \( f' > 0 \) in \((0, \infty)\).

**Proof** First we show that \( f_p \) is non-decreasing on \((0, \infty)\). Recall that \( f'_p(0) > 0 \) and define
\[
r_1 = \sup\{r : f'_p(s) \geq 0 \text{ on } [0, r]\}.
\]
If \( r_1 = \infty \) then clearly \( f_p \) is non-decreasing on \((0, \infty)\). Assume then that \( r_1 < \infty \), and then obviously
\[
f'_p(r_1) = 0.
\]

By the definition of \( r_1 \) we have also
\[
f''_p(r_1) \leq 0. \tag{2.20}
\]

Next we distinguish between two cases:

(i) There exists a right-neighborhood of \( r_1, [r_1, R] \), in which \( f'_p \leq 0 \).

(ii) There exists no neighborhood as in (i).

Consider first case (i). Since \( f_p \xrightarrow{r \to \infty} 1 \), there must exist a maximal right-neighborhood, where \( f'_p \leq 0 \) which we denote by \([r_1, r_2] \). Clearly, we must have \( f'_p(r_2) = 0 \). From (2.8) we get that
\[
r^2 f''_p = 1 - \frac{2}{p} \left( \frac{f_p}{r} \right)^{-(p-2)} r^2(1 - f_p^2), \quad \text{for } r = r_i, \ i = 1, 2. \tag{2.21}
\]

By Step 2 of the proof of Proposition 2.4 we have
\[
\left( \frac{f_p(r_2)}{r_2} \right)^{-(p-2)} > \left( \frac{f_p(r_1)}{r_1} \right)^{-(p-2)}. \tag{2.22}
\]

Furthermore, since \( f'_p \leq 0 \) in \([r_1, r_2] \) we have
\[
\left( 1 - f_p^2 \right)(r_2) \geq \left( 1 - f_p^2 \right)(r_1). \tag{2.23}
\]

Substituting (2.22), (2.23) into (2.21) and using (2.20) yields
\[
\left. \frac{r^2 f''_p}{f_p} \right|_{r=r_2} < \left. \frac{r^2 f''_p}{f_p} \right|_{r=r_1} \leq 0,
\]
i.e., \( f''_p(r_2) < 0 \), which clearly contradicts the definition of \( r_2 \).
Next we turn to case (ii). In this case we have \( f''(r_1) = 0 \). Differentiating the Eq. (2.7) at \( r = r_1 \) yields

\[
f_p^{(3)}(r_1) = -p \frac{f_p}{r_1^2} < 0.
\]

(2.24)

This implies that \( r_1 \) is a maximum point for \( f_p' \) which is obviously impossible.

Finally, we prove that \( f_p' > 0 \) on \([0, \infty)\) (we know already that \( f_p'(0) > 0 \)). Suppose, for a contradiction, that there exists \( r_0 > 0 \) such that

\[
f_p'(r_0) = f_p''(r_0) = 0.
\]

We then obtain the same identity as in (2.24), but this time at \( r = r_0 \). Again we get that \( f_p' \)

has a maximum at \( r_0 \), a contradiction. \[\Box\]

To prove monotonicity of \( f_p' \) we need the following result

**Lemma 2.2** We have

\[
h' \leq 0, \quad \forall r > 0.
\]

(2.25)

Furthermore,

\[
\lim_{r \to \infty} h(r) = 0.
\]

Proof Suppose, for a contradiction that (2.25) does not hold. Since \( \lim_{r \downarrow 0} h(r) = 1 \) and \( h < 1 \) on \((0, \infty)\) (see Steps 1 and 4 in the proof of Proposition 2.4) \( h \) must have a minimum point at some \( r = r_0 \). By (2.13) we have

\[
h''(r_0) = -\frac{1}{r_0^2} F_p(h) - \frac{2}{p} \frac{1 + h^2}{1 + (p - 1)h^2} |\nabla u_p|^{2-p} \left[ -\frac{(|\nabla u_p|^{2-p})'}{2} (1 - f_p^2) - \frac{2}{2} r_0 (1 - f_p^2) \right] + (1 - f_p^2) - 2r_0 f_p f_p'.
\]

(2.26)

Furthermore, as \( h'(r_0) = 0 \) we also have

\[
\frac{1}{r_0^2} F_p(h) = \frac{2}{p} \frac{1 + h^2}{1 + (p - 1)h^2} |\nabla u_p|^{2-p} \left( 1 - f_p^2 \right)
\]

\[
|\nabla u_p|^{2-p} \bigg|_{r=r_0} = \left( \frac{f_p^2}{r_0^2} (1 + h^2) \right) \bigg|_{r=r_0} = -2(1 + h^2) \frac{f_p}{r_0^2} \left( f_p - f_p' \right).
\]

Substituting the above into (2.26) we obtain

\[
\text{sign } h''(r_0) = \text{sign } g(r_0),
\]

(2.27)

where

\[
g(r) := 2hf_p^2 - ((p - 2)(1 - h) + 2)(1 - f_p^2).
\]

(2.28)

Since \( r_0 \) is a minimum point of \( h \), we must have \( g(r_0) \geq 0 \). Put

\[
r_1 = \sup\{r \in (r_0, \infty) : h' \geq 0 \text{ on } (r_0, r)\}.
\]
If \( r_1 = \infty \) then, since \( h < 1 \), \( h \xrightarrow{r \to \infty} h_\infty \) where \( 0 < h_\infty \leq 1 \). But this leads to a contradiction since then also
\[
rf'_p \xrightarrow{r \to \infty} h_\infty,
\]
which is inconsistent with \( \lim_{r \to \infty} f(r) = 1 \). If \( r_1 < \infty \) then necessarily \( h'(r_1) = 0 \) and \( h''(r_1) \leq 0 \), implying that \( g(r_1) \leq 0 \) too. But since \( h \) is non-decreasing on \((r_0, r_1)\) while \( f \) is strictly increasing on \((r_0, r_1)\) (by Proposition 2.5), it follows from (2.28) that \( g \) is strictly increasing on \((r_0, r_1)\). Therefore, \( g(r_1) > g(r_0) \geq 0 \), implying as in (2.27) that \( h''(r_1) > 0 \).

This contradiction completes the proof of (2.25).

Finally, as \( h \) is both positive and decreasing it must converge to a limit \( h_\infty \geq 0 \). From the above argument we obtain that \( h_\infty = 0 \).

\[\square\]

**Corollary 2.1** \( f'_p \) is monotone decreasing in \( \mathbb{R}_+ \).

The corollary follows immediately from the fact that \( f'_p \) is a product of the positive functions \( h \) and \( f_p/r \), the first of which is non-increasing, and the second is strictly decreasing.

2.5 Asymptotic behavior

In the following we derive the behavior of \( 1 - f_p^2 \) as \( r \to \infty \). The first lemma is a well-established result in asymptotic analysis. We include the proof for the convenience of the reader.

**Lemma 2.3** Let \( g(x) \) be monotone decreasing on \((0, \infty)\). Let further
\[
\int_r^\infty g(t)\,dt = \frac{1}{r^\alpha} [1 + o(1)] \quad \text{as } r \to \infty.
\]
for some positive \( \alpha \). Then,
\[
g(r) = \frac{\alpha}{r^{\alpha + 1}} [1 + o(1)] \quad \text{as } r \to \infty.
\]

**Proof** Put \( G(r) = \int_r^\infty g(t)\,dt \). Then, for any \( h > 0 \),
\[
hg(r) \geq \int_r^{r+h} g(t)\,dt = G(r) - G(r + h) = \frac{1}{r^\alpha} - \frac{1 + \eta(r)}{(r + h)^\alpha},
\]
where \( \lim_{r \to \infty} \eta(r) = 0 \). By (2.29),
\[
hg(r) \geq (1 + \eta(r)) \left( \frac{1}{r^\alpha} - \frac{1}{(r + h)^\alpha} \right) + \frac{\eta(r) - \eta(r + h)}{r^\alpha}
\geq \frac{1}{r^\alpha} \left( (1 - \eta_m) \left\{ 1 - \left( 1 + \frac{h}{r} \right)^{-\alpha} \right\} - 2\eta_m \right),
\]
where \( \eta_m(r, h) = \max(|\eta(r)|, |\eta(r + h)|) \). Let \( \epsilon = \frac{h}{r} \). Since for some \( C > 0 \) we have
\[
1 - (1 + \epsilon)^{-\alpha} \geq 1 + \alpha \epsilon - C \epsilon^2, \quad \epsilon \in \left[ 0, \frac{1}{2} \right],
\]
\[\square\] Springer
it follows that
\[ hg(r) \geq \frac{1}{r^\alpha} \left( (1 - \eta_m)(\alpha \epsilon - C \epsilon^2) - 2\eta_m \right). \]

Therefore,
\[ g(r) \geq \frac{1}{r^{\alpha+1}} \left( (1 - \eta_m)(\alpha - C \epsilon) - 2\eta_m \right). \tag{2.30} \]
Choosing \( \epsilon = \eta_m^{1/2} \) we get from (2.30) (since \( \lim_{r \to \infty} \sup_{h > 0} \eta_m(r, h) = 0 \)),
\[ g(r) \geq \frac{\alpha}{r^{\alpha+1}} (1 + o(1)), \text{ as } r \to \infty. \]
The second direction is proved in a similar manner. \( \square \)

We use the above lemma to prove the following result

**Lemma 2.4**

\[ 1 - f_p^2 \sim \frac{p}{2} \frac{1}{r^p} \text{ as } r \to \infty, \tag{2.31a} \]
\[ f_p' \sim \frac{p^2}{4} \frac{1}{r^{p+1}}. \tag{2.31b} \]

**Proof** Integrating by parts (2.7) between \( r \) and infinity yields
\[ \int_{r}^{\infty} f_p(1 - f_p^2)\,dt = \frac{p}{2} \int_{r}^{\infty} |\nabla u_p|^{p-2} \left[ \frac{f_p}{t} - f_p' \right] \frac{dt}{t} + \frac{p}{2} |\nabla u_p|^{p-2} f_p', \]
or equivalently that
\[ \int_{r}^{\infty} f_p(1 - f_p^2)\,dt = \frac{p}{2} \int_{r}^{\infty} [1 + h^2]^{(p-2)/2} (1 - h) \left[ \frac{f_p}{t} \right]^{p-1} \frac{dt}{t} + \frac{p}{2} |1 + h^2|^{(p-2)/2} h \left[ \frac{f_p}{r} \right]^{p-1}. \tag{2.32} \]
Applying the integral mean value theorem yields the existence of \( r^* \in [r, \infty) \) such that
\[ \int_{r}^{\infty} [1 + h^2]^{(p-2)/2} (1 - h) \left[ \frac{f_p}{t} \right]^{p-1} \frac{dt}{t} = \frac{1 + h^2(r^*)}{}^{(p-2)/2} (1 - h(r^*)) f_p^{p-1}(r^*) r^{-(p-1)} \frac{r^{-(p-1)}}{p-1}. \]
Hence, in view of Lemma 2.2 and the fact that \( f_p \underset{r \to \infty}{\longrightarrow} 1 \) we obtain
\[ \int_{r}^{\infty} |1 + h^2|^{(p-2)/2} (1 - h) \left[ \frac{f_p}{t} \right]^{p-1} \frac{dt}{t} = \frac{r^{-(p-1)}}{p-1} [1 + o(1)] \text{ as } r \to \infty. \tag{2.33} \]
Further, in view of Lemma 2.2 we have
\[ |1 + h^2|^{(p-2)/2} h \left[ \frac{f_p}{r} \right]^{p-1} = o(r^{-(p-1)}). \tag{2.34} \]
Substituting (2.33)–(2.34) into (2.32) yields
\[
\int_{r}^{\infty} f_p(1 - f_p^2)dt = \frac{p}{2} \frac{r^{-p+1}}{p-1} [1 + o(1)] \quad \text{as } r \to \infty.
\]
As \( f_p \to 1 \) we have
\[
\int_{r}^{\infty} f_p(1 - f_p^2)dt = [1 + o(1)] \int_{r}^{\infty} (1 - f_p^2)dt \quad \text{as } r \to \infty,
\]
and hence
\[
\int_{r}^{\infty} (1 - f_p^2)dt = \frac{p}{2} \frac{r^{-p+1}}{p-1} [1 + o(1)] \quad \text{as } r \to \infty.
\]
The proof of (2.31a) follows immediately from Lemma 2.3 and the monotonicity of \( f_p \).

To prove (2.31b) we first note that
\[
\lim_{r \to \infty} \frac{1 - f_p^2}{1 - f_p} = 2.
\]
Hence,
\[
\int_{r}^{\infty} f'_p dt = \frac{p}{4r^p} [1 + o(1)] \quad \text{as } r \to \infty.
\]
Lemma 2.3 provides, once again, the closing argument for the proof.

\section{Large \( p \)}

In this section we discuss the behavior of the radially symmetric solution in the large \( p \) limit. We prove the following result

\textbf{Theorem 1} \ Let
\[
f_\infty = \begin{cases} \frac{p}{\sqrt{2}} & r < \sqrt{2} \\ 1 & r \geq \sqrt{2}. \end{cases} \tag{3.1}
\]

There exists \( C > 0 \) such that for every \( p > 2 \) we have
\[
\| f_p - f_\infty \|_{L^\infty(\mathbb{R}_+)} = \| f_p - f_\infty \|_\infty \leq C \left( \frac{\ln p}{p} \right)^{1/2}. \tag{3.2}
\]

To prove the theorem we shall need to prove first a few auxiliary results. We first derive a simple upper bound.

\textbf{Lemma 3.1} \ We have
\[
I_p(f_p) \leq \left( \frac{1}{6} + C \frac{\ln p}{p} \right), \quad \forall p > 2. \tag{3.3}
\]

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Proof We use the test function
\[ \tilde{f} = \begin{cases} \frac{1}{\sqrt{2}} \left(1 - \frac{\ln p}{p}\right) r, & r < \frac{\sqrt{2}}{1 - \frac{\ln p}{p}} \\ 1, & r \geq \frac{\sqrt{2}}{1 - \frac{\ln p}{p}} \end{cases}. \]

It is easy to show that there exists \( C > 0 \), independent of \( p \) such that
\[ I_p(\tilde{f}) \leq \left(\frac{1}{6} + C \frac{\ln p}{p}\right), \quad \forall p > 2, \]
from which the lemma immediately follows. \( \square \)

We first deal with the interval \([0, \sqrt{2}]\).

**Proposition 3.1** We have
\[ \exists C > 0 : \| \nabla u_p \|_\infty \leq 1 + \frac{C}{p}, \forall p > 2. \] (3.4)

**Proof** We first note that by Lemma 2.2 and Step 2 of the proof of Proposition 2.4 both \( f'_p \) and \( f_p/r \) are decreasing. Therefore, the same holds for \( |\nabla u_p| \) and it follows that
\[ \| \nabla u_p \|_\infty = |\nabla u_p(0)|. \] (3.5)

Obviously, if we have \( |\nabla u_p| - 1 \gg 1/p \) over a sufficiently large right semi-neighborhood of \( r = 0 \), then \( I_p(f) \) would become larger than the upper bound (3.3). This, however, does not eliminate the possibility of a small neighborhood of \( r = 0 \) where \( p(|\nabla u_p| - 1) \) is large. Thus, the proof splits into two parts: at first, using regularity arguments, we bound from below the size of the above neighborhood as a function of \( |\nabla u_p(0)| \). Then, we use (3.3) to bound \( |\nabla u_p(0)| \) from above.

Suppose that \( |\nabla u_p(0)| = a > 1 \). Let
\[ s = \sup \left\{ r > 0 : |\nabla u_p(r)| > \frac{1+a}{2} \right\}. \] (3.6)

By (2.13) we have for all \( r < s \) that
\[ h' \geq -\frac{2}{p} \left(\frac{1+a}{2}\right)^{(p-2)} r, \] (3.7)

hence
\[ 1 - \frac{1}{p} \left(\frac{1+a}{2}\right)^{(p-2)} r^2 \leq h \leq 1, \quad \forall r \leq s. \] (3.8)

Assume first that
\[ s^2 \leq \frac{p}{2} \left(\frac{1+a}{2}\right)^{p-2}, \] (3.9)

implying by (3.8) that
\[ h \geq \frac{1}{2}, \quad \forall r \leq s. \] (3.10)
By (2.12) we have
\[ \frac{f_p''}{f_p'} = \frac{h'}{h} - \frac{1 - h}{r}. \] (3.11)

Therefore, using (3.10) and (3.7)–(3.8) we deduce that
\[ \left| \frac{f_p''}{f_p'} \right| = -\frac{f_p''}{f_p'} \leq C \left( \frac{1 + a}{2} \right)^{2-p} r, \quad \forall r \leq s, \]
implying
\[ \exp \left\{ -\frac{C}{2p} \left( \frac{1 + a}{2} \right)^{2-p} r^2 \right\} \leq \frac{f_p'(r)}{f_p'(0)}, \quad \forall r \leq s. \] (3.12)

Since \( h < 1 \),
\[ \frac{|\nabla u_p(r)|^2}{|\nabla u_p(0)|^2} = \frac{(1 + h^{-2})|f_p'(r)|^2}{2|f_p'(0)|^2} \geq \frac{|f_p'(r)|^2}{|f_p'(0)|^2}, \quad \forall r \leq s. \]

Consequently, by (3.12)
\[ \exp \left\{ -\frac{C}{p} \left( \frac{1 + a}{2} \right)^{2-p} r^2 \right\} \leq \frac{|\nabla u_p(r)|^2}{|\nabla u_p(0)|^2}, \quad \forall r \leq s. \] (3.13)

Setting \( r = s \) in (3.13) we obtain
\[ s^2 \geq Cp \left( \frac{1 + a}{2} \right)^{p-2} \ln \left( \frac{2a}{1 + a} \right) \geq Cp \left( \frac{1 + a}{2} \right)^{p-2} a - 1. \]

If (3.9) doesn’t hold, then clearly
\[ s^2 > \frac{p}{2} \left( \frac{1 + a}{2} \right)^{p-2}. \]

Therefore, in all cases we have
\[ s^2 \geq Cp \frac{a - 1}{a} \left( \frac{1 + a}{2} \right)^{p-2}. \] (3.14)

To conclude, we shall use the upper-bound for the energy from Lemma 3.1 in order to bound \( s \) from above. Combining (3.14) with (3.3) and (3.6) yields
\[ C \geq \int_0^s |\nabla u_p|^p r dr \geq \frac{s^2}{2} \left( \frac{1 + a}{2} \right)^p \geq Cp \frac{a - 1}{a} \left( \frac{1 + a}{2} \right)^{2(p-1)} \geq Cp(a - 1), \] (3.15)

From (3.15) we get
\[ a \leq 1 + \frac{C}{p}, \quad \forall p > 2, \]

and (3.4) follows from (3.5).

We can now obtain \( L^\infty \) convergence of \( f_p \) to \( f_\infty \) in every compact set in \([0, \sqrt{2}]\).
Proposition 3.2 For every $b \in (0, \sqrt{2})$ there exists $C = C(b) > 0$ such that,
\[
\left\| f_p - \frac{r}{\sqrt{2}} \right\|_{L^\infty(0,b)} \leq C \frac{\ln p}{p}, \quad p > 2,
\]
\[
\left\| f'_p - \frac{1}{\sqrt{2}} \right\|_{L^\infty(0,b)} \leq C \frac{\ln p}{p}, \quad p > 2.
\]

Proof First we note that by (3.4)
\[
|\nabla u_p(0)|^2 = 2 f'_p(0)^2 \leq 1 + \frac{C}{p} \implies f'_p(0) \leq \frac{1}{\sqrt{2}} + \frac{C}{p}.
\]
Since $f'_p$ is decreasing, we conclude that
\[
f'_p(r) \leq \frac{1}{\sqrt{2}} + \frac{C}{p}.
\]
Integrating (3.17), using $f_p(0) = 0$, yields the existence of $C > 0$ such that for every $p > 2$ we have, for all $r > 0$,
\[
f_p(r) \leq \frac{1}{\sqrt{2}} \left( 1 + \frac{C}{p} \right) r.
\]
Put
\[
w(r) = \frac{r}{\sqrt{2}} - f_p(r).
\]
By (3.17)–(3.18) we have
\[
w'(r) \geq -\frac{C}{p} \geq -C \frac{\ln p}{p} \quad \text{and} \quad w(r) \geq -\frac{C}{p} \geq -C \frac{\ln p}{p}, \quad \forall r > 0.
\]
In order to conclude, we need to prove that for each $b \in (0, \sqrt{2})$ there exists $C_b$ such that
\[
w'(r) \leq C_b \frac{\ln p}{p} \quad \text{and} \quad w(r) \leq C_b \frac{\ln p}{p}, \quad \forall r \in [0, b].
\]
For such $b$ we set $\bar{b} = \frac{b + \sqrt{2}}{2}$ and claim that
\[
\int_{\bar{b}}^{b} w(r) \, dr \leq C_b \frac{\ln p}{p}.
\]
To prove (3.20) we first note that
\[
\frac{1}{2} \int_0^\infty \left( 1 - f_p^2 \right)^2 r \, dr \geq \frac{1}{2} \int_0^{\sqrt{2}} \left( 1 - f_p^2 \right)^2 r \, dr = \frac{1}{2} \int_0^{\sqrt{2}} \left[ 1 - \frac{1}{2} r^2 \right]^2 r \, dr
\]
\[
+ \int_0^{\sqrt{2}} \left[ 1 - \frac{1}{2} r^2 \right] \left[ \frac{1}{2} r^2 - f_p^2 \right] r \, dr + \frac{1}{2} \int_0^{\sqrt{2}} \left[ \frac{1}{2} r^2 - f_p^2 \right]^2 r \, dr.
\]
Since
\[
\frac{1}{2} \int_0^{\sqrt{2}} \left[ 1 - \frac{1}{2} r^2 \right]^2 r \, dr = \frac{1}{6},
\]
we deduce, using (3.3), that
\[
\int_0^{\sqrt{2}} \left[ 1 - \frac{1}{2} r^2 \right] \left[ \frac{1}{2} r^2 - f_p^2 \right] r \, dr \leq C \frac{\ln p}{p}.
\]
Therefore
\[C \frac{\ln p}{p} \geq \left( 1 - \frac{\tilde{b}^2}{2} \right) \int_{\tilde{b}}^{b} w(r) \left( \frac{r}{\sqrt{2}} + f_p \right) \, dr,
\]
and (3.20) follows. Finally, using the convexity of \( w \) in conjunction with (3.20) gives
\[
w(b)(\tilde{b} - b) + \frac{w'(b)}{2}(\tilde{b} - b)^2 = \int_{b}^{\tilde{b}} (w(b) + (r - b)w'(b)) \, dr \leq \int_{b}^{\tilde{b}} w(r) \, dr \leq C \frac{\ln p}{p},
\]
implying, in particular, that
\[
w'(b) \leq C \frac{\ln p}{p}.
\]
(3.21)
Since \( w' \) is increasing we deduce the first inequality in (3.19). The second one follows by integration of the first one. \( \square \)

We now improve the estimates (3.16). We start by deriving a Pohozaev-type identity.

**Proposition 3.3** We have
\[
\int_0^{\infty} |\nabla u_p|^p r \, dr = \frac{2}{p} m_p,
\]
where \( m_p \) is defined in (2.3).

**Proof** Let \( f_p^{(\alpha)}(r) = f_p(\alpha r) \) and \( J = \int_0^{\infty} |\nabla u_p|^p r \, dr \). Clearly,
\[
M_\alpha = I_p(f_p^{(\alpha)}) = \alpha^{p-2} J + \frac{1}{\alpha^2} (m_p - J).
\]
Hence,
\[
\frac{dM_\alpha}{d\alpha} = (p - 2)\alpha^{p-3} J - \frac{2}{\alpha^3} (m_p - J).
\]
Since \( M_\alpha \) must have a global minimum at \( \alpha = 1 \), (3.22) follows. \( \square \)

**Corollary 3.1** We have
\[
\liminf_{p \to \infty} p |\nabla u_p(0)|^p \geq \frac{1}{3}.
\]
Proof Since \( f'_p < f_p / r < 1 / r \) we have

\[ |\nabla u_p| \leq \frac{\sqrt{2}}{r}. \]

Thus, for every \( l > 0 \),

\[
\int_{l}^{\infty} |\nabla u_p|^p r \, dr \leq \frac{2^{p/2}}{(p-2)l^{p-2}},
\]

from which we get

\[
\lim_{p \to \infty} \int_{l}^{\infty} |\nabla u_p|^p r \, dr = 0, \quad \forall l > \sqrt{2}.
\]

(3.24)

By (3.16) we have

\[
\liminf_{p \to \infty} m_p \geq \sup_{b \in (0, \sqrt{2})} \liminf_{p \to \infty} \frac{1}{2} \int_{0}^{b} (1 - f_p^2)^2 r \, dr = \frac{1}{6}.
\]

Thus, by (3.24) and (3.22) we have for all \( l > \sqrt{2} \),

\[
\liminf_{p \to \infty} p \int_{0}^{l} |\nabla u_p|^p r \, dr \geq \frac{1}{3}.
\]

As \( |\nabla u_p(r)| \leq |\nabla u_p(0)| \) we deduce that

\[
\liminf_{p \to \infty} p|\nabla u_p(0)|^2 \geq \frac{1}{3} \quad \forall l > \sqrt{2},
\]

from which (3.23) readily follows. \( \square \)

Lemma 3.2 Let \( g = |\nabla u_p|^p \), and

\[
g_0 = \frac{1}{p} \left(1 - \frac{1}{2}r^2\right)^2.
\]

Then,

\[
\lim_{p \to \infty} p\|g - g_0\|_{L^\infty(0,a)} = 0, \quad \forall a < \sqrt{2}.
\]

(3.25)

Proof Multiplying (2.7) by \( rf_p \) and integrating over \([0, r]\), we obtain

\[
\frac{p}{4}r^2g(1 - \alpha_p) - \frac{p}{2} \int_{0}^{r} g(t) t \, dt + h(r) = 0,
\]

(3.26a)

in which

\[
\alpha_p = 1 - \frac{2 f'_p}{f_p |\nabla u_p|_p^2} > 0,
\]

(3.26b)
and
\[
h(r) = \int_0^r f_p^2(1 - f_p^2) dt.
\]
(3.26c)

We may write
\[
h(r) = h_0(r)(1 + \beta_p) \quad \text{with} \quad h_0(r) = \int_0^r \frac{t^2}{2} \left(1 - \frac{t^2}{2}ight) t dt = \frac{1}{8} \left(r^4 - \frac{1}{3} r^6\right).
\]

Set
\[
\epsilon_p(a) = \max \left( \|\alpha_p\|_{L^\infty(0,a)}, \|\beta_p\|_{L^\infty(0,a)} \right).
\]
(3.27)

By (3.16) there exists \(C > 0\) such that
\[
\epsilon_p(a) \leq C \frac{\ln p}{p},
\]
(3.28)

for all fixed \(a < \sqrt{2}\).

Set
\[
G(r) = \int_0^r g(t) t dt,
\]
to obtain from (3.26) that
\[
G' - \gamma_p G + \frac{2}{p} \gamma_p h = 0,
\]
(3.29)

where
\[
\gamma_p = \frac{2}{r(1 - \alpha_p)}.
\]

Solving (3.29) and then evaluating \(G'\) once again from (3.29) yields the general solution of (3.26):
\[
g(r) = -\frac{2}{p} \frac{\gamma_p}{r} \left[ h + \int_0^r \exp \left\{ \int_t^r \gamma_p(s) ds \right\} \gamma_p(t)h(t) dt + C_0 \exp \left\{ -\int_r^a \gamma_p(t) dt \right\} \right],
\]
(3.30)

where \(C_0\) is arbitrary.

First we compute
\[
\frac{\gamma_p}{r} \exp \left\{ -\int_r^a \gamma_p(t) dt \right\} \leq \frac{\gamma_p}{r} \exp \left\{ -2 \int_r^a \frac{dt}{t} \right\} = \frac{\gamma_p}{a^2}.
\]

On the other hand, a similar computation gives
\[
\frac{\gamma_p}{r} \exp \left\{ -\int_r^a \gamma_p(t) dt \right\} \geq \frac{\gamma_p}{r} \left( \frac{r}{a} \right)^{\frac{2}{1-\epsilon_p}}.
\]
Therefore,
\[
\frac{2}{a^2} \left( \frac{r}{a} \right)^{2((1-\epsilon_p)^{-1} - 1)} \leq \gamma_p \exp \left\{ -\int_r^a \gamma_p(t) \, dt \right\} \leq \frac{2}{a^2} \left( 1 + C\epsilon_p \right).
\]

Similarly,
\[
\frac{\gamma_p}{r} \int_0^r \exp \left\{ \int_s^r \gamma_p(s) \, ds \right\} \gamma_p(t) h(t) \, dt \geq \frac{\gamma_p}{r} \int_0^r \left( \frac{r}{t} \right)^2 \gamma_p(t) h(t) \, dt \\
\geq 4 \int_0^r \frac{h(t)}{t^3} \, dt \geq \left( \frac{1}{4} r^2 - \frac{1}{24} r^4 \right) (1 - \epsilon_p),
\]
and
\[
\frac{\gamma_p}{r} \int_0^r \exp \left\{ \int_s^r \gamma_p(s) \, ds \right\} \gamma_p(t) h(t) \, dt \leq \left( \frac{1}{4} r^2 - \frac{1}{24} r^4 \right) (1 + C\epsilon_p).
\]

Combining the above with (3.30) we obtain that
\[
\frac{2}{p} \left[ \hat{C}_0 r^{2((1-\epsilon_p)^{-1} - 1)} - \frac{1}{2} r^2 + \frac{1}{8} r^4 - C\epsilon_p \right] \leq g \leq \frac{2}{p} \left[ \hat{C}_0 - \frac{1}{2} r^2 + \frac{1}{8} r^4 + C\epsilon_p \right]. \quad (3.31)
\]

Note that the above lower bound is unsatisfactory in some neighborhood of \( r = 0 \) where
\[
1 - r^{2((1-\epsilon_p)^{-1} - 1)} \sim O(\epsilon_p),
\]
which is valid for \( r \sim O(1) \) as \( p \to \infty \).

We defer the proof of convergence near \( r = 0 \) to a later stage and instead prove first the existence of \( \lim_{p \to \infty} \hat{C}_0(p) \), and then obtain its value. Clearly,
\[
\liminf_{p \to \infty} \hat{C}_0(p) \geq \frac{1}{2},
\]
otherwise \( g \) would become negative, for some sufficiently large \( p \) and a fixed \( r_0 < \sqrt{2} \)—a contradiction. Suppose now to the contrary, that a sequence \( \{ p_k \}_{k=1}^\infty \) exists such that \( \hat{C}_0(p_k) = C_k \to b \), where \( b \in (\frac{1}{2}, \infty) \). By (3.4) we have
\[
\| g(\cdot, p_k) \|_{L^\infty(\mathbb{R}^+)} \leq C,
\]
where \( C \) is independent of \( k \). Hence, by (3.31) we have
\[
C_k \leq C p_k. \quad (3.32)
\]

Set
\[
g_{0,k} = 2 \left[ C_k - \frac{1}{2} r^2 + \frac{1}{8} r^4 \right].
\]
Note that by our supposition \( \lim_{k \to 0} g_{0,k}(r) > 0 \) in \([0, \sqrt{2} + \delta]\) for some \( \delta > 0 \). It follows from (3.31) and (3.32) that

\[
\frac{\ln(g_{0,k} - \epsilon_k)}{p_k} - 2 \frac{\left| \ln(g_{0,k} - \epsilon_k) \right|^2}{p_k} \leq \left| \nabla u_p \right| - 1 + \frac{\ln p_k}{p_k} \leq \frac{\ln(g_{0,k} + \epsilon_k)}{p_k} + 2 \frac{\left| \ln(g_{0,k} + \epsilon_k) \right|^2}{p_k}
\]

(3.33)

where \( \epsilon_k(a) = \epsilon(p_k)(a) \).

We argue from here by bootstrapping. Let \( a \in (0, \sqrt{2} + \delta] \) be such that

\[
\limsup \epsilon_k(a) \leq \limsup \frac{g_{0,k}(a)}{2}.
\]

(3.34)

For sufficiently large \( k \) we have, in view of (3.33) and (3.32) and the fact that \( \epsilon_k(\sqrt{2} + \delta) \) is bounded, that

\[
2 \frac{f_k'}{r} f_k' \leq \left| \nabla u_p \right|^2 \leq 1 + \frac{C}{p_k} \quad \forall r \in [0, \sqrt{2} + \delta],
\]

where \( f_k = f_{p_k} \), from which we obtain that

\[
\frac{f_k}{r} \leq \frac{1}{\sqrt{2}} + \frac{C}{p_k} \quad \forall r \in [0, \sqrt{2} + \delta].
\]

(3.35)

Consequently, by (3.33) and (3.34), we have for sufficiently large \( k \) that

\[
f_k' \geq \frac{1}{\sqrt{2}} - C \frac{\ln p_k}{p_k} \quad \forall r \in [0, a],
\]

(3.36)

where \( C \) is independent of \( a \). Since \( f_k' \leq f_k/r \), we have by (3.26b), for sufficiently large \( k \), that

\[
\alpha_{p_k} \leq \frac{\ln p_k}{p_k} \quad \forall r \in [0, a],
\]

(3.37)

where \( C \) is independent of \( a \). Furthermore, by (3.26c, d), (3.35), (3.36), and the fact that \( f_k/r > f_k' \) there exists \( C > 0 \) which is independent of both \( k \) and \( a \) such that

\[
\beta_{p_k}(r) \leq C \frac{\ln p_k}{p_k}
\]

for all \( r \leq a \). Combining the above and (3.37) we obtain for sufficiently large \( k \)

\[
\limsup \epsilon_k(a) \leq \lim \frac{g_{0,k}(a)}{2} \Rightarrow \limsup \frac{p_k}{\ln p_k} \epsilon_k \leq C,
\]

(3.38)

where \( C \) is independent of \( a \). From (3.16) we thus have

\[
\limsup \frac{p_k \epsilon_k}{\ln p_k} \leq C
\]

for all \( a < \sqrt{2} \).

Let then \( a_0 \) be such that

\[
\limsup \epsilon_k(a_0) = \lim \frac{g_{0,k}(a_0)}{2}.
\]
Since by (3.38) we have \( \lim g_{0,k}(a_0) = 0 \), it follows that \( a_0 > \sqrt{2} + \delta \). Hence

\[
\lim \sup \frac{p_k}{\ln p_k} \epsilon_k(\sqrt{2} + \delta) \leq C.
\]

Substituting into (3.31) we obtain that

\[
\lim \frac{p_k}{\ln p_k} g(\sqrt{2} + \delta) > 0.
\]

Let \( l > \sqrt{2} \). Then, \( f'_p(l) < f_p(l)/l < 1/l \), and hence \( g(l) \leq (\sqrt{2}/l)^p \). Consequently, \( g(l) \) is exponentially small for all \( l > \sqrt{2} \) as \( p \to \infty \), and in particular at \( l = \sqrt{2} + \delta \)—a contradiction. Hence, we obtain that \( \lim_{p \to \infty} \tilde{C}_0(p) = 1/2 \).

To complete the proof of (3.25) we need to extend (3.31) to every neighborhood of \( r = 0 \). Since obtaining an \( O(\epsilon^p) \) accuracy in this neighborhood is a difficult task, we allow for an error of larger magnitude. Thus, requiring that

\[
\frac{2}{p} \left[ \tilde{C}_0 r^2(1-\epsilon^{p-1}) - \frac{1}{2} r^2 + \frac{1}{8} r^4 - C \epsilon^{\frac{1}{2}} p \right] \leq g \leq \frac{2}{p} \left[ \tilde{C}_0 - \frac{1}{2} r^2 + \frac{1}{8} r^4 + C \epsilon^{\frac{1}{2}} p \right].
\]

(3.39)

It is easy to show that the lower bound in (3.39) provides an estimate which is \( O(\epsilon^{1/2}) \)—accurate whenever \( r^2 > e^{-\epsilon^{p-1/2}} \). To complete the proof of (3.25), we just need to obtain an \( O\left(\epsilon^{1/2}\right) \)—accurate estimate for \( g \), valid for \( r^2 \leq e^{-\epsilon^{p-1/2}} \).

We argue again by bootstrapping. We may regroup the terms in (2.8) to get

\[
-\frac{2}{p} |\nabla u_p|^{2-p} f_p(1 - f_p^2) = f''_p \left( 1 + \frac{p - 2}{|\nabla u_p|^2} |f'_p|^2 \right) + \left( \frac{f'_p}{r} - \frac{f_p}{r^2} \right) \left( 1 + \frac{p - 2}{|\nabla u_p|^2} \right) |f_p| f'_p f''.
\]

(3.40)

By Step 1 in the proof of Proposition 2.4 we have \( \frac{f'_p}{r} - \frac{f_p}{r^2} > 0 \), and by Corollary 2.1, \( f''_p < 0 \). Hence,

\[
f''_p \geq -\frac{2}{p} |\nabla u_p|^{2-p} f_p(1 - f_p^2).
\]

It follows that as long as

\[
g \geq g(0) \left( 1 - \epsilon^{\frac{1}{2}} p \right),
\]

we must have, by (3.16) that

\[
f''_p \geq -\frac{2}{p} \left[ g(0) \left( 1 - \epsilon^{\frac{1}{2}} p \right) \right] (p-2)/p.
\]

Integrating the above yields, in view of (3.23),

\[
f'_p(r) \geq f'_p(0) - 4r^2 \left( 1 + 2 \epsilon^{\frac{1}{2}} p \right).
\]
Note that we can replace the constant 4 by any other constant greater than 3. Consequently,

$$|\nabla u_p| \geq \sqrt{2} |f_p' - 4r^2 (1 + 2\epsilon_p^2)| = \sqrt{2} |f_p'(0)| \left| 1 - \left( \frac{4\sqrt{2}r^2}{|\nabla u_p(0)|} \right) \left( 1 + 2\epsilon_p^2 \right) \right|.$$ 

Hence,

$$g(r) \geq g(0) \left( 1 - \epsilon_p^2 \right) \Rightarrow g(r) \geq g(0) \left| 1 - \frac{4\sqrt{2}r^2}{|\nabla u_p(0)|} \left( 1 + 2\epsilon_p^2 \right) \right|^p.$$ 

Applying again (3.23) we obtain that as long as

$$r^2 < \frac{1}{8p} \epsilon_p^2,$$

we have

$$g(r) \geq g(0) \left( 1 - \epsilon_p^2 \right). \quad (3.41)$$

On the other hand,

$$g(r) \leq g(0). \quad (3.42)$$

Since (3.41), (3.42), and (3.39) are simultaneously satisfied at $r^2 = 2e^{-\epsilon_p^2}$, we obtain

$$\begin{align*}
\frac{2}{p} \tilde{C}_0 \left( 1 + C \epsilon_p^2 \right) &\geq g(0) \left( 1 - \frac{1}{\epsilon_p} \right) \\
\frac{2}{p} \tilde{C}_0 \left( 1 - C \epsilon_p^2 \right) &\leq g(0)
\end{align*}.$$ 

Consequently,

$$\left| \tilde{C}_0 - \frac{p}{2} g(0) \right| \leq C \epsilon_p^2.$$ 

Furthermore, in view of (3.41) and (3.39) we can safely state that

$$\frac{2}{p} \left[ \tilde{C}_0 - \frac{1}{2} r^2 + \frac{1}{8} r^4 \tilde{C}_0 - C \epsilon_p^2 \right] \leq g$$

$$\leq \frac{2}{p} \left[ \tilde{C}_0 - \frac{1}{2} r^2 + \frac{1}{8} r^4 + C \epsilon_p^2 \right], \quad \text{in} \ [0, a], \ \forall a \in (0, \sqrt{2}). \quad (3.43)$$

Or

$$\| p(g - g_0) \| \leq |\tilde{C}_0(p) - 1| + C \epsilon_p^{1/2}.$$ 

\[ \square \]

Remark 3.1 From (3.25) we can obtain the next two terms in the asymptotic expansion of $f_p$ in the large $p$ limit

$$f_p = \frac{r}{\sqrt{2}} \left[ 1 - \frac{\ln p}{p} + \frac{\ln g_0(r)}{p} + o \left( \frac{1}{p} \right) \right]. \quad (3.44)$$

The above expansion is valid in $[0, a]$ for every $a < \sqrt{2}$. 

\[ \square \] Springer
Remark 3.2 Note that (3.43) is valid for all \( r > 0 \). It is only because of (3.28) that we have to confine the validity of (3.25) to closed intervals in \([0, \sqrt{2})\) whose edges do not depend on \( p \). Note further that by (3.27) we have that \( \epsilon_p \leq 2 \) for all \( r \leq \sqrt{2} \).

We can now extend the validity of the above estimate to \([0, \sqrt{2} - O(\sqrt{\ln p / p})] \).

Proposition 3.4 There exists \( C > 0 \), which is independent of \( p \), such that the estimate (3.25) holds for every \( r \in [0, \sqrt{2} - C(\ln p / p)^{1/2}] \).

Proof Let

\[
g_{0,C} = \tilde{C}_0 - \frac{1}{2} r^2 + \frac{1}{8} r^4.
\]

Suppose that \( \tilde{C}_0 \) is such that \( g_{0,C}(\sqrt{2} - \Delta p) = 0 \). From the previous lemma we have that \( \Delta_p \rightarrow 0 \) as \( p \rightarrow \infty \). It is easy to show that,

\[
g_{0,C} \left( \sqrt{2} - \frac{2}{3} \Delta_p \right) \leq -C \Delta_p^2,
\]

for all \( C < 1/6 \) and for sufficiently large \( p \).

Let

\[
|\nabla u_p| \left( \sqrt{2} - \frac{2}{3} \Delta_p \right) = 1 - \delta_p.
\]

Since \( f_p / r \leq 1 / \sqrt{2} \), we have \( f_p' \left( \sqrt{2} - 2 \Delta_p / 3 \right) \geq 1 - C \delta_p \). From here it is easy to show that \( \epsilon_p (\sqrt{2} - 2 \Delta_p / 3) \leq C \delta_p \). By (3.40) we have

\[
f_p'' \leq -\frac{C}{p^2} |1 - \delta_p|^2 - p \left( 1 - \frac{r}{\sqrt{2}} \right) + C p^{1/2} \quad \forall r \in [\sqrt{2} - 2 \Delta_p / 3, \sqrt{2} - \Delta_p / 3],
\]

where we have taken into account the fact that \( |\nabla u_p| \) is decreasing and that

\[
\frac{1 + \frac{p-2}{|\nabla u_p|^2} f_p f_p'}{r} \leq C p^{1/2}.
\]

Integrating (3.45) over \([\sqrt{2} - 2 \Delta_p / 3, \sqrt{2} - \Delta_p / 3] \) yields

\[
-\frac{1}{\sqrt{2}} \leq -C \frac{\Delta_p^2}{p^2} |1 - \delta_p|^{-p} + C p^{1/2} \Delta_p,
\]

from which we obtain

\[
(1 - \delta_p)^p \geq C \frac{\Delta_p^2}{p^{5/2}}.
\]

Consequently,

\[
\delta_p \leq \frac{5 \ln p}{2} - 2 \frac{\ln \Delta_p}{p} + \frac{C}{p}.
\]

We conclude from here that

\[
\epsilon_p (\sqrt{2} - 2 \Delta_p / 3) \leq C \delta_p \leq C \frac{\ln p}{p}.
\]
Since $g$ is positive we obtain by (3.31) that

$$\Delta p \leq C \left[ \frac{\ln p}{p} \right]^{1/2}.$$

Since $\epsilon_p(a)$ is an increasing function of $a$ (3.25) must be valid in $[0, \sqrt{2} - 2\Delta_p/3]$.

\[\Box\]

**Proof of Theorem 1** In view of proposition 3.4 there exists $C > 0$ such that (3.16a) and hence (3.44) hold for sufficiently large $p$ whenever $r < \sqrt{2} - C(\ln p/p)^{1/2}$. From the monotonicity of $f_p$ it follows that

$$f_p(\sqrt{2} - C/(\ln p/p)^{1/2}) \leq f_p(r) \leq 1.$$

\[\Box\]

### 4 Stability of the radial solution

In this section we prove our main stability result for $u_p = f_p(r)e^{i\theta}$, the degree one radially symmetric solution of

$$\frac{p}{2} \nabla \cdot (|\nabla u|^p - 2 \nabla u) + u(1 - |u|^2) = 0. \quad (4.1)$$

A simple computation gives the second variation of $E_p$ at $u_p$:

$$J_2(\phi) = \int_{\mathbb{R}^2} \left\{ \frac{p}{2} |\nabla u_p|^{p-2} \left[ |\nabla \phi|^2 + (p - 2) \frac{|\Re(\nabla u_p \cdot \nabla \phi)|^2}{|\nabla u_p|^2} \right] 
+ 2 |\Re(u_p \phi)|^2 - (1 - |u_p|^2)|\phi|^2 \right\}. \quad (4.2)$$

Because of (4.2) and analogously to [5], we consider perturbations in the “natural” Hilbert space $\mathcal{H}$ consisting of functions $\phi \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ for which

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left\{ \frac{p}{2} |\nabla u_p|^{p-2} \left[ |\nabla \phi|^2 + (p - 2) \frac{|\Re(\nabla u_p \cdot \nabla \phi)|^2}{|\nabla u_p|^2} \right] 
+ 2 |\Re(u_p \phi)|^2 + (1 - |u_p|^2)|\phi|^2 \right\} < \infty.$$

Note that $\mathcal{H}$ contains all “admissible perturbations” $\phi$, i.e., any $\phi$ for which $E_p(u_p + \phi) < \infty$. Note also that in contrast with the case $p = 2$, in our case $p > 2$, constant functions do belong to $\mathcal{H}$. Thanks to the invariance of the functional $E_p$ with respect to rotations and translations (see [11]) we have

$$J_2(\phi) = 0 \quad \text{for } \phi = \left\{ \begin{array}{l}
\frac{\partial u_p}{\partial \sigma} = if_p e^{i\theta}, \\
\frac{\partial u_p}{\partial x_1} = \frac{1}{2} \left( f_p' - \frac{f_p}{r} \right) e^{2i\theta} + \frac{1}{2} \left( f_p' + \frac{f_p}{r} \right), \\
\frac{\partial u_p}{\partial x_2} = -\frac{i}{2} \left( f_p' - \frac{f_p}{r} \right) e^{2i\theta} + \frac{i}{2} \left( f_p' + \frac{f_p}{r} \right). \end{array} \right. \quad (4.3)$$

Indeed, this leads to the equality cases in the next theorem.
Radially symmetric minimizers for a $p$-Ginzburg Landau type energy

**Theorem 2** For every $2 < p \leq 4$ the radially symmetric solution $u_p$ is stable in the sense that $J_2(\phi) \geq 0$ for all $\phi \in \mathcal{H}$. Moreover, we have $J_2(\phi) = 0$ if and only if

$$\phi = c_0 \frac{\partial u_p}{\partial \theta} + c_1 \frac{\partial u_p}{\partial x_1} + c_2 \frac{\partial u_p}{\partial x_2},$$

for some constants $c_0, c_1, c_2 \in \mathbb{R}$. \hfill (4.4)

Following [9] we represent each $\phi$ by its Fourier expansion

$$\phi = \sum_{n=-\infty}^{\infty} \phi_n(r)e^{in\theta}. \hfill (4.5)$$

Substituting into (4.2) we obtain

$$\frac{1}{2\pi}J_2(\phi) = E_1(\phi_1) + \sum_{n=2}^{\infty} E_n(\phi_n, \phi_{2-n}), \hfill (4.6)$$

in which

$$E_1(\phi_1) = \int_0^{\infty} \left\{ \frac{p}{2} |\nabla u_p|^{p-2} \left[ |\phi'_1|^2 + \frac{1}{r^2} |\phi_1|^2 + (p-2) \frac{|\Re(f_p' \phi'_1 + f_p \phi_1)|^2}{|\nabla u_p|^2} \right] 
+ 2 f_p^2 |\Re \phi_1|^2 - (1 - f_p^2) |\phi_1|^2 \right\} r \, dr, \hfill (4.7a)$$

and

$$E_n(\phi_n, \phi_{2-n}) = \int_0^{\infty} \left\{ \frac{p}{2} |\nabla u_p|^{p-2} \left[ |\phi'_n|^2 + |\phi'_{2-n}|^2 + \frac{n^2}{r^2} |\phi_n|^2 + \frac{(2-n)^2}{r^2} |\phi_{2-n}|^2 
+ \frac{1}{2} (p-2) \frac{f_p' (\phi'_n + \phi'_{2-n}) + f_p (n \phi_n + (2-n) \phi_{2-n})^2}{|\nabla u_p|^2} 
+ f_p^2 (\phi_n + \phi_{2-n})^2 - (1 - f_p^2) (|\phi_n|^2 + |\phi_{2-n}|^2) \right\} r \, dr. \hfill (4.7b)$$

A necessary and sufficient condition for the positive definiteness of $J_2$ is that the $E_n$’s are all positive definite. An appropriate Hilbert space for the study of the functionals $\{E_n\}$ is

$$S = \left\{ \phi \in H^1_{\text{loc}}(\mathbb{R}^+, \mathbb{C}) \cap L^2_r(\mathbb{R}^+, \mathbb{C}) : \int_0^{\infty} \frac{p}{2} |\nabla u_p|^{p-2} \left[ |\phi'|^2 + \frac{1}{r^2} |\phi|^2 \right] r \, dr < \infty \right\}.$$  

We also denote by $\tilde{S}$ the space of real-valued functions in $S$.

4.1 $n \neq 2$

We consider first the case $n = 1$.  

\[ \square \] Springer
Lemma 4.1

\[ \inf_{\phi \in \tilde{S}} E_1(\phi) = 0. \]  

(4.8)

Furthermore, the minimum in (4.8) is attained only for \( \phi = cf_p \), for any real constant c.

Proof Since \( E_1(i|\phi|) \leq E_1(\phi) \) for every \( \phi \) for which \( E_1(\phi) < \infty \), with strict inequality unless \( \phi \) takes only purely imaginary values, we may consider instead of \( E_1 \) the following functional

\[ \tilde{E}_1(\phi) = \int_{\tilde{S}} \frac{p}{2} |\nabla u_p|^2 - f_p^2 |\phi| \, dr. \]

over \( \tilde{S} \). Consider first \( \phi \in C^\infty(0, \infty) \) and set \( \phi = f_p w \). Integration by parts, with the aid of (2.7) yields

\[ F_1(w) = \tilde{E}_1(f_p w) = \int_{0}^{\infty} \frac{p}{2} |\nabla u_p|^2 - f_p^2 |w'|^2 r \, dr. \]  

(4.9)

A standard use of cut-off functions yields that (4.9) holds also for smooth \( \phi = f_p w \) with compact support in \([0, \infty)(i.e., the support may contain the origin). Finally, by density of smooth maps with compact support in \([0, \infty)\) in \( \tilde{S} \) it follows that (4.9) continues to hold for \( \phi = f_p w \in \tilde{S} \). Therefore, \( \tilde{E}_1(\phi) \geq 0 \) for all \( \phi \in \tilde{S} \) and \( F_1(w) = 0 \) if and only if \( w \equiv \text{const.} \). \( \square \)

We now consider the case \( n \geq 3 \).

Proposition 4.1 For each \( n \geq 3 \) we have

\[ E_n(u_1, u_2) > 0 \quad \text{for all } (u_1, u_2) \in \tilde{S} \times \tilde{S} \setminus \{(0, 0)\}. \]

Proof The result follows right away from the previous lemma and the inequality

\[ E_n(u_1, u_2) \geq \tilde{E}_1(|u_1|) + \tilde{E}_1(|u_2|), \]

with strict inequality, unless \( u_j \equiv 0, \ j = 1, 2. \) \( \square \)

4.2 \( n = 2 \)

It is easy to reduce the analysis of \( E_2 \) to that of a functional acting on real-valued functions. Indeed, writing a complex-valued function \( \phi \) as \( \phi = \phi^R + i\phi^I \), we have

\[ E_2(\phi_2, \phi_0) = E_2^R(\phi_2^R, \phi_0^R) + E_2^I(\phi_2^I, \phi_0^I), \]

where

\[ E_2^R(\phi_2^R, \phi_0^R) = E_2(\phi_2^R, \phi_0^R), \quad E_2^I(\phi_2^I, \phi_0^I) = E_2(i\phi_2^I, i\phi_0^I). \]

Clearly,

\[ E_2(i\phi_2^I, i\phi_0^I) = E_2^R(-\phi_2^I, \phi_0^I). \]
Hence,

\[ E_2(\phi_2, \phi_0) = E^I_2(-\phi^R_2, \phi^R_0) + E^I_2(\phi^I_2, \phi^I_0), \quad (4.10) \]

and it suffices to study the minimization to the functional \( E^I_2 \) over \( \bar{S} \times \bar{S} \).

From (4.3) and (4.6) it follows that the functions

\[ \Phi_0 = f'_p + \frac{f_p}{r} \quad \text{and} \quad \Phi_2 = -f'_p + \frac{f_p}{r}, \quad (4.11) \]

satisfy

\[ E^R_2(-\Phi_2, \Phi_0) = E^I_2(\Phi_2, \Phi_0) = 0. \]

We next claim:

**Proposition 4.2** For \( p \in (2, 4) \) we have \( E^I_2(\phi_2, \phi_0) \geq 0 \) for every \( (\phi_2, \phi_0) \in \bar{S} \times \bar{S} \) with equality if and only if \( (\phi_2, \phi_0) = c(\Phi_2, \Phi_0) \) for some \( c \in \mathbb{R} \) (see (4.11)).

For the proof of Proposition 4.2 we shall need some preliminary results. First, by (4.7b) we have

\[ E^I_2(\phi_2, \phi_0) = \int_0^\infty \left\{ \begin{array}{l}
\frac{p}{2} |\nabla u_p|^{p-2} \left[ (\phi'_2)^2 + (\phi'_0)^2 + \frac{4}{r^2} (\phi_2)^2 \\
+ \frac{1}{2} (p - 2) \frac{(f'_p(\phi'_0 - \phi'_2) - 2 f_p \phi_2)^2}{|\nabla u_p|^2} \right] \\
+ f_p^2 (\phi_0 - \phi_2)^2 - (1 - f_p^2) ((\phi_2)^2 + (\phi_0)^2) \end{array} \right\} r dr. \quad (4.12) \]

It is more convenient to consider an alternative form by applying the transformation

\[ A = \phi_0 + \phi_2, \quad B = \phi_0 - \phi_2 \]

to obtain

\[ E^I_2(\phi_0, \phi_2) = F_2(A, B) := \int_0^\infty \left\{ \begin{array}{l}
\frac{p}{4} |\nabla u_p|^{p-2} \left[ (A')^2 + (B')^2 + \frac{2}{r^2} (A - B)^2 \\
+ (p - 2) \frac{(f'_p B' - \frac{f_p}{r^2} (A - B))^2}{|\nabla u_p|^2} \right] \\
+ f_p^2 B^2 - \frac{1}{2} (1 - f_p^2) (A^2 + B^2) \end{array} \right\} r dr. \quad (4.13) \]

Clearly,

\[ F_2(f_p/r, f'_p) = 0. \quad (4.14) \]
The “problematic term” in (4.13) is the one involving the mixed product $AB'$. The difficulty in handling this term is the obstacle for determining the positivity of $F_2$ for every $p > 2$. We were able to overcome this difficulty only in the case $p \in (2, 4]$ thanks to the following lemma.

**Lemma 4.2** We have

$$F_2(A, B) = G_2(A, B) + \int_0^\infty \frac{p(p-2)}{4} |\nabla u_p|^{p-2} \frac{(hA' - \frac{1}{r}(h^2 A - B))^2}{1 + h^2} r dr, \quad (4.15)$$

with

$$G_2(A, B) = \int_0^\infty \left\{ \frac{p}{4} |\nabla u_p|^{p-2} \left[ (A')^2 + (B')^2 + \frac{2}{r^2} (A - B)^2 \\
+ (p-2) \frac{h^2((B')^2 - (A')^2) + \frac{1-h^4}{r^2} A^2 - \frac{2}{r^2} (1-h^2) AB}{1 + h^2} \right] \\
+ \frac{p(p-2)}{4r} \left[ H'(2AB - B^2) - (h^2 H)' A^2 \right] \\
+ f^2_p B - \frac{1}{2}(1-f^2_p)(A^2 + B^2) \right\} r dr, \quad (4.16)$$

where

$$H = \frac{h}{1+h^2} |\nabla u_p|^{p-2} \quad \text{and} \quad h = rf'_p/f_p \ (\text{as in } (2.10)).$$

Moreover, $G_2(f_p/r, f'_p) = 0$ and the pair $(f_p/r, f'_p)$ solves the Euler–Lagrange equations associated with $G_2$.

**Proof** First, a direct computation gives the identity

$$\frac{(f'_p B' - \frac{f_p}{r^2} (A - B))^2}{|\nabla u_p|^2} = \frac{h^2(|B'|^2 - |A'|^2) - \frac{2h}{r} [(AB)' - BB' - h^2 AA'] + \frac{1-h^4}{r^2} |A|^2 - \frac{2}{r^2} (1-h^2) AB}{1 + h^2}$$

$$+ \frac{(hA' - \frac{1}{(h^2 A - B))^2}{1 + h^2}. \quad (4.17)$$

Next, integration by parts yields

$$\int_0^\infty |\nabla u_p|^{p-2} \left\{ \frac{-2h}{r} [(AB)' - BB' - h^2 AA'] \right\} r dr$$

$$= \int_0^\infty \left\{ H'(2AB - B^2) - (h^2 H)' A^2 \right\} dr \quad (4.18)$$
Using (4.17)–(4.18) in conjunction with (4.13) leads to (4.15)–(4.16). Finally, a direct com-
putation shows that the integrand in the integral on the right-hand-side of (4.15) is identically
zero for $A = f_p/r$ and $B = f'_p$, and the last assertion of the lemma follows. □

Proof of Proposition 4.2 In view of Lemma 4.2 it suffices to show that
\[ G_2(u, v) \geq 0, \quad \forall (u, v) \in \tilde{S} \times \tilde{S}, \]
with equality iff:
\[ u = \phi := f_p/r \quad \text{and} \quad v = \psi := f'_p. \] (4.19)

We write $G_2$ in the form
\[ G(u, v) = \int_0^\infty \left( \alpha(r)u'^2 + \beta(r)v'^2 + a(r)u^2 + 2b(r)uv + c(r)v^2 \right) dr. \]

The properties of the coefficients which are important to us are
\[ \alpha(r), \beta(r) > 0 \quad \text{and} \quad b(r) < 0, \quad \text{for} \quad r > 0. \] (4.20)

Indeed, clearly $\beta(r) > 0$. Next,
\[ \alpha(r) = \frac{p}{4} |\nabla u_p|^{p-2}r \left( 1 - (p - 2) \frac{h^2}{1 + h^2} \right) > 0, \]
provided $p \leq 4$, since $0 < h < 1$ by Step 1 of Proposition 2.4 and Proposition 2.5. Finally,
\[ b = r \left\{ \frac{p}{4} |\nabla u_0|^{p-2} \left[ -\frac{2}{r^2} - \frac{(p - 2)}{r^2} \left( 1 - h^2 \right) \right] + \frac{p(p - 2)}{4r} H' \right\} < 0, \]
since $0 < h(r) < 1$, and
\[ H' = |\nabla u_p|^{p-2} \frac{(1 - h^2)h'}{(1 + h^2)^2} + (p - 2) \frac{h}{1 + h^2} \left( \left( \frac{f_p}{r} \right) \left( \frac{f'_p}{r} \right) + f''_p f'_p \right) < 0, \]
since $h' \leq 0$ by Lemma 2.2 and both $f'_p$ and $f_p/r$ are decreasing (as we noted already before,
by Lemma 2.2 and Step 2 of the proof of Proposition 2.4).

By Lemma 4.2 we know that $\phi$ and $\psi$ satisfy
\[ \begin{cases} 
-(\alpha \phi')' + a \phi + b \psi = 0, \\
-(\beta \psi')' + c \psi + b \phi = 0.
\end{cases} \] (4.21)

We consider first $u, v \in C_c^\infty(0, \infty)$. By Picone’s identity
\[ (u')^2 - \left( \frac{u^2}{\phi} \right)' \phi' = (u' - (u/\phi)\phi')^2 \geq 0 \] (4.22)
\[ (v')^2 - \left( \frac{v^2}{\psi} \right)' \psi' = (v' - (v/\psi)\psi')^2 \geq 0. \] (4.23)

Multiplying (4.22)–(4.23) by $\alpha$ and $\beta$ respectively, applying integration by parts and using
(4.21) we obtain
\[ 0 \leq \int_0^\infty \alpha(u')^2 - \alpha \left( \frac{u^2}{\phi} \right)' \phi' + \beta (v')^2 - \beta \left( \frac{v^2}{\psi} \right)' \]
\[ \begin{align*}
\int_0^\infty \alpha (u')^2 + \frac{u^2}{\phi} (a\phi + b\psi) + \beta (v')^2 + \frac{v^2}{\psi} (c\psi + b\phi) \\
= \int_0^\infty \alpha u^2 + \beta v^2 + au^2 + cv^2 + b \left( \frac{u^2}{\phi} \psi + \frac{v^2}{\psi} \phi \right) \\
= G(u, v) + \int_0^\infty b \left( u \left( \frac{\psi}{\phi} \right) \right)^{1/2} - v \left( \frac{\phi}{\psi} \right)^{1/2} \right)^2.
\end{align*} \tag{4.24} \]

From (4.24) and a density argument we conclude that

\[ G(u, v) \geq \int_0^\infty (-b) \left( u \left( \frac{\psi}{\phi} \right) \right)^{1/2} - v \left( \frac{\phi}{\psi} \right)^{1/2} \right)^2, \quad \forall u, v \in \tilde{S}, \]

and (4.19) follows.

Next we are ready to present the proof of our main stability theorem.

**Proof of Theorem 2** Representing each \( \phi \in \mathcal{H} \) by its Fourier expansion (4.5), we have by (4.6), Lemma 4.1, Proposition 4.1, (4.10) and Proposition 4.2 that \( J_2(\phi) \geq 0 \). Furthermore, by the equality cases in Lemma 4.1, Proposition 4.1 and Proposition 4.2 we have \( J_2(\phi) = 0 \) iff \( \phi = \phi_0 + \phi_1 e^{i\theta} + \phi_2 e^{2i\theta} \) where

\( \phi_1 = a_1 f_p, \quad (\phi_2^1, \phi_0^1) = a_2 (\Phi_2, \Phi_0) \) and \( (-\phi_2^R, \phi_0^R) = a_3 (\Phi_2, \Phi_0) \), with \( a_1, a_2, a_3 \in \mathbb{R} \).

It is easy to verify that these relations are equivalent to (4.4). \( \square \)

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