THE FREE GROUP OF RANK 2 IS A LIMIT
OF THOMPSON’S GROUP $F$

MATTHEW G. BRIN

Abstract. We show that the free group of rank 2 is a limit of 2-markings of
Thompson’s group $F$ in the space of all 2-marked groups. More specifically, we
find a sequence of generating pairs for $F$ so that as one goes out the sequence,
the length of the shortest relation satisfied by the generating pair goes to
infinity.

1. Introduction

From [10], a $k$-marked group is a pair $(G, S)$ where $S$ is an ordered $k$-tuple
of generators (the $k$-marking) of the group $G$. An isomorphism of marked groups
must preserve the markings. With $\mathcal{G}_k$ the set of all isomorphism classes of $k$-marked
groups, one says that two elements of $\mathcal{G}_k$ are no more than $e^{-R}$ apart if they have
satisfy the same relations of length no longer than $R$. This gives a metric on $\mathcal{G}_k$,
and a group $L$ is a $G$-limit if it is a limit in $\mathcal{G}_k$ of a sequence of marked groups each
of which is isomorphic as an unmarked group to $G$.

Marked groups and their limits are defined and studied in [6] where, among other
things, it is shown that $\mathcal{G}_k$ is compact. Limits of marked groups extend the notion
of limit groups of [9].

Groups that cannot be obtained as a limit of Thompson’s group $F$ are studied
in [10].

Recently Akhmedov, Stein and Taback [1] have announced that the free group
on $k$ generators is a limit of $k$-markings of Thompson’s group $F$ when $k \geq 3$. The
purpose of this paper is to prove the following.

Theorem 1. The free group on 2 generators is a limit of 2-markings of $F$.

See [5] for an introduction to Thompson’s group $F$.

There is a closely related notion of free-like. Essentially a group $G$ is $k$-free-like
if both (a) the free group on $k$ generators can be obtained as a $G$-limit, and (b)
the group $G$ is “uniformly non-amenable” with respect to the markings used in
the limit. See [8] for the full definition, examples and discussion of the free-like
property and related concepts.

The notion of a free group occurring as a $G$-limit can be easily restated. The free
group of rank $k$ is a $G$-limit if there is a sequence of $k$-tuples $S_n$ in $G$ so that (a)
each $S_n$ generates $G$, and (b) for each $n$, no reduced word of length at most $n$ in the
elements of $S_n$ and their inverses represents the trivial element of $G$. Alternatively
for (b), one can say that the homomorphism from the free group on $k$ generators

\[ \text{Date: April 20, 2009.} \]
\[ \text{AMS Classification (2000): primary 20F69, secondary 20F05, 20F38.} \]
taking the generators of the free group to the elements of \( S_n \) embeds the \( n \)-ball of the free group in \( G \).

Our proof combines two tendencies in \( F \). First, it is not hard to find elements in \( F \) that satisfy no short relations. Second it is rather “easy” to generate \( F \). The word “easy” is in quotes because while it might not be hard to pick out sets of elements that generate \( F \), the calculations that show that they generate might be complicated. This is discussed more in the body of the paper.

The technique that we use to embed large balls of the free group in \( F \) is taken from \([3]\). An alternative technique is found in \([7]\). However it is not clear that the alternative technique lends itself as well to building generators.

In Section 2 we give the outline. In Section 3 we give the details that show that the outline is correct. The outline is the more important part of the paper, and the details should be read only by those that wish to check for correctness.

Background for this paper would include \([5]\) for its general introduction to the group \( F \), for the normal form of an element in terms of the infinite generating set, and for the description in Section 1 of \([5]\) of the “rectangular diagrams” of Thurston which are aids to calculation. Rectangular diagrams will be used extensively in Section 3. We use the representation of \( F \) as a group of right acting homeomorphisms of the non-negative real numbers. A discussion close to this view is found in Section 2 of \([2]\) and to some extent in \([4]\).

We would like to thank Mark Sapir for supplying the question answered by Theorem \([4]\).

2. Outline

The model of \( F \) that we work with is the model on

\[
\mathbb{R}_{\geq 0} = \{ t \in \mathbb{R} \mid t \geq 0 \}
\]

in which the generators are the self homeomorphisms \( x_i, i \geq 0 \), of \( \mathbb{R}_{\geq 0} \) operating on the right defined by

\[
x_t x_i = \begin{cases} 
  t, & t \leq i, \\
  i + 2(t - i), & i \leq t \leq i + 1, \\
  t + 1, & t \geq i + 1.
\end{cases}
\]

The \( x_i \) satisfy the usual relations \( x_j x_i = x_i x_{j+1} \) whenever \( i < j \), and it is known that \( x_0 \) and \( x_1 \) generate \( F \).

Our method will be to modify \( x_0 \) and \( x_1 \) ”slightly” so that (I) they still generate, and (II) they don’t satisfy short relations. The slightness of the modification will mean that in a region large enough to be useful, the modifications agree with the originals.

2.1. Getting generators. Showing (I), that the modifications still generate, will need the more intricate argument. I learned the following use of the subgroups \( F_{[a,b]} \) from Collin Bleak and Bronlyn Wassink.

Let \([a,b]\) be a closed interval, and let \( F_{[a,b]} \) be all the elements in \( F \) whose support is contained in \([a,b]\). The support of an \( f \in F \) is the set \( \{ t \in \mathbb{R}_{\geq 0} \mid tf \neq t \} \).

We will only refer to \( F_{[a,b]} \) when \( 0 \leq a < b \) and both \( a \) and \( b \) are dyadic (that is, of the form \( m/2^n \) with \( m \) and \( n \) in \( \mathbb{Z} \)). It is standard that under these restrictions \( F_{[a,b]} \) is isomorphic to \( F \).
When \(a\) and \(b\) are two consecutive integers, it is very easy to write down generators for \(F_{[a,b]}\). For \(a = 0\) and \(b = 1\), we have the usual model of \(F\) on \([0,1]\) and typical generators for \(F_{[0,1]}\) are \(y_0\) and \(z_0\) defined by

\[
ty_0 = \begin{cases} 
2t, & 0 \leq t \leq \frac{1}{4}, \\
t + \frac{1}{4}, & \frac{1}{4} \leq t \leq \frac{3}{4}, \\
1 - \frac{1}{2}(1 - t), & \frac{3}{4} \leq t \leq 1,
\end{cases}
\]

\[	z_0 = \begin{cases} 
2t, & 0 \leq t \leq \frac{1}{8}, \\
t + \frac{1}{8}, & \frac{1}{8} \leq t \leq \frac{1}{4}, \\
\frac{1}{2} - \frac{1}{2}(2 - t), & \frac{3}{8} \leq t \leq \frac{1}{2}, \\
t, & \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

It is standard and an easy exercise that

\[y_0 = x_0^2x_1^{-1}x_0^{-1},\]

\[z_0 = x_0^3x_1^{-1}x_0^{-2}.
\]

To generate \(F_{[i,i+1]}\) for a positive integer \(i\), we use \(y_i\) and \(z_i\) where

\[
y_i = x_i^2x_{i+1}^{-1}x_i^{-1},
\]

\[
z_i = x_i^3x_{i+1}^{-1}x_i^{-2}.
\]

We will also use the elements

\[w_i = x_ix_{i+1}^{-1}\]

and it is also standard and easy exercise that \(w_i\) and \(y_i\) generate \(F_{[i,i+2]}\) in a manner "identical" to the manner in which \(y_i\) and \(z_i\) generate \(F_{[i,i+1]}\).

**Lemma 2.1.** Let \(X_0\) and \(X_1\) generate a subgroup \(G\) of \(F\) and assume the following hold.

(a) The support of \(X_0x_0^{-1}\) is contained in a compact subset of \((0,\infty)\).

(b) The support of \(X_1x_1^{-1}\) is contained in a compact subset of \((0,\infty)\).

(c) The group \(G\) generated by \(X_0\) and \(X_1\) contains \(w_1\) and \(y_1\).

(d) The translates of the open interval \((1,3)\) under \(G\) cover all of \((0,\infty)\).

Then \(G = F\).

**Proof.** From (c), we know that \(G\) contains \(F_{[1,3]}\), from (d) we know that \(G\) contains all elements of \(F\) whose support is contained in a compact subset of \((0,\infty)\), and from (a) and (b) we know that \(G\) contains \(X_0x_0^{-1}\) and \(X_1x_1^{-1}\) and thus \(x_0\) and \(x_1\). Thus \(G\) contains all of \(F\). \(\square\)

2.2. **Avoiding relations.** Showing (II), that our chosen generators don’t satisfy short relations, will be done as in [3]. We will take homeomorphisms on the circle that generate a free group and lift these to the real line. The lifts will not be elements of \(F\), but approximations with supports on compact subsets will be elements of \(F\). If the compact subsets are long enough, then as shown in [3] the only relations that the approximations satisfy must be long. Let us refer to these approximations as "almost free" elements.

We will need to know more about these "almost free elements" than just the fact that they only satisfy long relations. We will also need to know some specific relations that they do satisfy. As is typically the case with \(F\), these relations will
be commutators of certain words in the almost free elements. These relations will be used to build our generating set.

2.3. Getting generators, revisited. We can now give better descriptions of our generators $X_0$ and $X_1$.

We will work with two intervals. On one interval $[0, b - 5)$ for some $b > 5$ that we will choose, $X_i$ will agree with $x_i$ for $i = 0, 1$. (The strange way of giving the upper limit of the interval is to make things convenient later.) On a second interval $(b - 5, d)$ for a $d > b$ that will depend on $n$, $X_i$ will agree with an element $g_i$ for $i = 0, 1$, (also depending on $n$) for which it is known that the $g_i$ satisfy no relations shorter than $n$ but which do satisfy certain specific relations that are longer than $n$.

We then consider words in the $X_i$. We will use capital letters to denote such words. For example

$$C = X_0^2 X_1^2 X_0^{-2} X_1^{-2}$$

will be one word. The corresponding lower case letter will denote the corresponding word in the $x_i$. Thus

$$c = x_0^2 x_1^2 x_0^{-2} x_1^{-2}$$

denotes the word corresponding to $C$.

For certain words (let $W$ denote such a word for this discussion) in the $X_i$, the corresponding word ($w$ in this example) in the $x_i$ will satisfy $W = w$ in $F$ because

1. $W$ will be trivial on $(b - 5, d)$,
2. the $X_i$ agree with the $x_i$ on $(0, b - 5)$, and
3. the word $w$ in the $x_i$ has support in $[0, b - 5)$.

2.4. Avoiding relations, revisited. We now describe the $X_i$. Our first basic building block will be the piece of function below on the left and its inverse below on the right.

The reader can verify that the function on the left is $w_0^2$, a fact that will be convenient for notation but that is not terribly important otherwise. The most important property that we need is that $\frac{1}{2}$ is taken to $\frac{3}{2}$. That is, all points in $[0, 2]$ not within $\frac{1}{2}$ of the left fixed point are carried to within $\frac{1}{2}$ of the right fixed point. Also important is that no point moves more than one.
Our second basic building block will be $w_0^{-2}w_2^2$ that is pictured below.

\[ (4) \]

We consider a composition of $m$ copies of the block from (4) translated by multiples of 4 as follows:

$$(w_0^{-2}w_2^2)(w_4^{-2}w_6^2)\cdots(w_{4(m-1)}^{-2}w_{4(m-1)+2}^2).$$

The graph looks something like the picture below, but the small scale prevents a truly accurate picture.

If we conjugate this function by a translation by $b > 0$ (so that it is the product $(w_b^{-2}w_{b+2}^2)(w_{b+4}^{-2}w_{b+6}^2)\cdots$ etc.), then we get a function with graph similar to that above and whose support is on $[b, b+4m]$. Call this function $g_0$. To make the next discussion easier, we take $b$ to be a multiple of 4.

The function $g_0$ has the property that positive powers of $g_0$ have the multiples of 4 in $[b, b+4m]$ as attracting fixed points and the odd multiples of 2 in $[b, b+4m]$ as repelling fixed points.

If we conjugate $g_0$ by translation by 1, then we get a function $g_1$ whose support is on $J = [b+1, b+4m+1]$, which has the elements from $\mathbb{Z}$ that are equal to 1 modulo 4 in $J$ as attracting fixed points, and which has the elements from $\mathbb{Z}$ that are equal to 3 modulo 4 in $J$ as repelling fixed points.

Let $n = 2m-3$. It is standard that given any reduced word in $g_0$ and $g_1$ of length less than $n$, then the function corresponding to that word cannot be the identity, and thus the pair $g_0$ and $g_1$ cannot satisfy any relation shorter than $n$. One sees this as follows. Associate to each of the four elements $g_0^{\pm1}$ and $g_1^{\pm1}$ its set of attracting fixed points in $[b, b+4m]$. Thus $g_0$ is associated to the integers in $[b, b+4m]$ that are equal to 0 modulo 4, $g_0^{-1}$ is associated to the integers in $[b, b+4m]$ that are equal to 2 modulo 4, and so forth. Let $w$ be a reduced word in $g_i^{\pm1}$, $i = 0, 1$. Let $\zeta$
be an integer within two of \( b + 2m \) in a set associated with neither the last letter in \( w \), nor the inverse of the first letter. One then shows inductively on the length of \( w \) (based on the fact that \( w \) is reduced), that the image of \( \zeta \) under \( w \) is within \( \frac{1}{2} \) of a point in the set associated to the last letter. Since the image of \( \zeta \) never moves more than one under the action of each successive letter in \( w \), and since the induction persists as long as the image of \( \zeta \) stays within \([b + 1, b + 4m]\), the induction will survive as long as the length of \( w \) is not more than \( n \). (We start the interval in the previous sentence at \( b + 1 \) since the behavior of \( g_1 \) is under the control of the blocks in (3) only starting at \( b + 1 \).)

2.5. The generators themselves. We wish to combine the function \( g_0 \) with \( x_0 \). This cannot be done directly since their behaviors at \( b \) do not match. At \( b \), the first is fixed and the second translates by 1. Thus we will alter \( g_0 \) in the interval \([b - 4, b]\) so as to agree with the picture below on the right instead of its originally defined behavior as pictured below on the left.

The exact values involved in the new \( g_0 \) will be made clear in Section 3. What we need to know now is that the function on the right acts as translation by 1 at \( b - 2 \).

We must similarly change the behavior above \( b + 4m \) so that its behavior is as shown below. Once again, exact values will be given in Section 3. For now we only need to know that the function acts as translation by 1 on \([b + 4m + 3, \infty)\).

Now we can define \( X_0 \) to agree with \( x_0 \) on \([0, b - 2]\), with the modified \( g_0 \) on \([b - 2, b + 4m + 3]\) and with \( x_0 \) again on \((b + 4m + 3, \infty)\).
We make similar modifications to $g_1$ and end up with an $X_1$ that is $X_0$ conjugated by a translation by 1. In particular $X_1$ agrees with $x_1$ on $[0, b - 1]$.

We take $m$ to be a positive integer. We let $i = m + 2$.

In the following definitions, we set $X_i = X_0^{-1}x_1X_0^{-1}$ for each $i > 1$ to parallel the relation $x_i = x_0^{-1}x_1x_0^{-1}$ that holds in $F$. We now define:

\[
\begin{align*}
C &= X_0^2X_1^2X_0^{-2}X_1^{-2}, \\
S &= X_0X_2X_1^{-2}, \\
T &= X_0^2X_2X_4X_3^{-2}X_1^{-1}X_0^{-1}, \\
\Sigma &= C^{-1}SC^i, \\
\Theta &= C^{-i}TC^i, \\
Z &= [S, \Sigma] = S\Sigma S^{-1}\Sigma^{-1}, \\
W &= [T, \Theta] = T\Theta T^{-1}\Theta^{-1}, \\
P &= Z^{-1}W, \\
Q &= X_1^{-1}PX_1P^{-1}, \\
H &= X_1^{-2}QX_1^2, \\
K &= X_1HX_1^{-1}.
\end{align*}
\]

As mentioned above, we define the corresponding lower case symbols, $c$ for $C$, $\theta$ for $\Theta$, etc., as the corresponding words in the $x_i$.

It is clear that $X_0$ and $X_1$ satisfy (a) and (b) of Lemma 2.1.

In the proposition below, the elements $y_1$ and $w_1$ are as defined in (1) and (2), and $w_1$ is not related to the $w$ that corresponds to $W$ defined in (7).

**Proposition 2.2.** The following hold for all sufficiently large values of $b$.

(i) The symbols defined in the right hand column of (7) represent the same elements of $F$ as the corresponding lower case symbols.

(ii) We have the equalities $H = w_1^{-1}$ and $K = y_1^{-1}$.

(iii) The elements $X_1$ and $X_2$ satisfy no relation of length less than $2m - 3$.

(iv) Hypothesis (d) of Lemma 2.1 is satisfied.

Since (ii) in the above proposition gives (c) of Lemma 2.1, we have that $X_0$ and $X_1$ generate $F$. This and (iii) of the proposition give Theorem 1.

We add a bit more to the outline before surrendering this paper to the details. Items (i) and (ii) are the most technical.

For (i), we will divide $R_{\geq 0}$ into two regions, $[0, b - 5)$ and $(b - 5, \infty)$. With $b$ sufficiently large, the lower case symbols corresponding to the words defined in (7) will have supports in $[0, b - 5)$. This is the only requirement on $b$ and contains the meaning of “sufficiently large.”

We will show that $C$ has two parts to its support. There will be a part in $[0, b - 5)$ where $C$ and $c$ agree and a part in a closed interval $I_C$ in $(b - 5, \infty)$ for which $\zeta C > \zeta$ for all $\zeta$ in the interior of $I_C$.

The functions defined as $S$ and $T$ will also have two parts to their support. There will be a part in $[0, b - 5)$ where $S$ and $T$ will agree with $s$ and $t$, respectively, and a part with closure in the interior of $I_C$.

We will show that the the conjugates of $S$ and $T$ by $C^i$ will have the parts of their supports in $(b - 5, \infty)$ disjoint from the supports of $S$ and $T$. Thus the commutators $Z$ and $W$ will be trivial on $(b - 5, \infty)$. This will verify (i) for $Z$ and $W$, and the truth of (i) for the rest will follow easily.

The truth of (ii) will follow from a long algebraic calculation.

The argument for (iii) has already been given for the original functions $g_0$ and $g_1$. The argument applies to $X_0$ and $X_1$ since these agree with the original $g_0$ and $g_1$, respectively, on the interval $[b + 1, b + 4m]$ needed for the argument.
The truth of (iv) will follow from the information gathered in the arguments for (ii) and from the definitions of \(X_0\) and \(X_1\).

The next section gives the details needed to verify the truth of (i), (ii) and (iv) of Proposition 2.2. This will complete the proof of Theorem 1.

3. Details

The elements in (7) were found using a computer program for doing calculations in \(F\) that was written by the author almost 20 years ago. It was written partly as an exercise in learning how to use a compiler compiler (sic). The elements in (7) were found by “experiment guided by experience.” Once elements were found with the required properties, the problem of how to write a proof of Theorem 1 arose. Listing the definitions in (7) and then instructing the reader to use the program to check the claimed behaviors was not an option since the program is very large and its inner workings are (after almost 20 years) opaque even to the author. It was then discovered that the algebra behind (ii) of Proposition 2.2 was not that bad, and that the “rectangular diagrams” of Thurston as described in [5] made the verification of all that is needed for (i) of Proposition 2.2 very visual. The result, while manageable, is still not attractive.

A failed attempt was made to find more attractive examples. There may very well be less complicated generators, or ones whose properties are easier to prove. However if such examples exist, they seem hard to find.

3.1. Proposition 2.2 part (ii). We start with the more algebraic calculations. We will not end with a proof of (ii), but a proof that (ii) follows from (i). If (i) holds, then \(H = h\) and \(K = k\) and proving that \(h = w_1^{-1}\) and \(k = y_1^{-1}\) will give (ii). Thus we analyze the lower case symbols.

We put the lower case symbols in normal form. Some, such as \(s\) and \(t\) are already given in normal form.

First

\[ c = x_0^2(x_1^2x_3^{-2})x_0^{-2}. \]

Next

\[ c^i = x_0^2(x_1^2x_3^{-2})^i x_0^{-2} = x_0^2(x_1^2 x_3^{-2} x_1^2 x_3^{-2} \cdots x_1^2 x_3^{-2}) x_0^{-2} = x_0^2(x_1 x_2 i + 1 x_2 i - 1 \cdots x_5 x_3^{-2}) x_0^{-2}. \]

We skip \(\sigma\) and \(\theta\) since they are absorbed into \(z\) and \(w\). We start with \(z\).

We have

\[ z = s(c^{-1} s c^{-1}) s^{-1} (c^{-1} s^{-1} c^{-1}) \\ = (x_0 x_2 x_1^{-2})(x_0^2 x_3^2 x_5 \cdots x_2 i x_3^2 x_1^{-2 i} x_0^{-2}) \\ (x_0 x_2 x_1^{-2})(x_0^2 x_3^2 x_5 \cdots x_2 i x_3^2 x_1^{-2 i} x_0^{-2}) \\ (x_1 x_2 i x_0^{-1})(x_0^2 x_3^2 x_5 \cdots x_2 i - 1 x_3^2 x_1^{-2 i} x_0^{-2}) \\ (x_1 x_2 i x_0^{-1})(x_0^2 x_3^2 x_5 \cdots x_2 i - 1 x_3^2 x_1^{-2 i} x_0^{-2}). \]
Now we move the appearances of $x_0^{±2}$ from the long parenthesized expressions on the right of each line.

$$
z = (x_0^3 x_4 x_3^{-2}) (x_5^2 x_6^2 \cdots x_{2i-1}^2 x_{2i+1}^2 x_1^{-2}) \\
(x_0 x_4 x_3^{-2}) (x_1^{2i} x_{2i+1} x_{2i-1}^2 \cdots x_5^2 x_3^{-2}) \\
(x_0^2 x_4^2) (x_1^2 x_5^2 x_6^2 \cdots x_{2i-1}^2 x_{2i+1}^2 x_1^{-2}) \\
(x_0^2 x_4^{-1}) (x_1^{2i} x_{2i+1} x_{2i-1}^2 \cdots x_5^2 x_3^{-2} x_0^{-2}).
$$

We cancel adjacent inverse items.

$$
z = (x_0^3 x_4) (x_5^2 \cdots x_{2i-1}^2 x_{2i+1} x_1^{-2}) \\
(x_0 x_4 x_3^{-2}) (x_1^{2i} x_{2i+1} x_{2i-1}^2 \cdots x_5^2) \\
(x_0^2 x_4^{-1}) (x_1^2 x_5^2 \cdots x_{2i-1}^2 x_{2i+1} x_1^{-2}) \\
(x_0^2 x_4^{-1}) (x_1^{2i} x_{2i+1} x_{2i-1}^2 \cdots x_5^2 x_3^{-2} x_0^{-2}).
$$

Now we move the remaining appearances of $x_0^{±1}$ from the middle of the expression.

$$
z = (x_0^4 x_5) (x_6^2 \cdots x_{2i}^2 x_{2i+2} x_2^{-2i}) \\
(x_4 x_5^{-2}) (x_1^{2i} x_{2i+1} x_{2i-1}^2 \cdots x_5^{-2}) \\
(x_4^2 x_6^2 \cdots x_{2i}^2 x_{2i+2} x_2^{-2i}) \\
(x_4^2 x_5^{-2}) (x_3^2 x_{2i+3} x_{2i+1} x_{2i-1}^2 \cdots x_7^{-2} x_5^{-2} x_0^{-4}).
$$

We cancel one adjacent pair.

$$
z = (x_0^4 x_5) (x_6^2 \cdots x_{2i}^2 x_{2i+2} x_2^{-2i}) \\
(x_4 x_5^{-2}) (x_1^{2i} x_{2i+1} x_{2i-1}^2 \cdots x_5^{-2}) \\
(x_4^2 x_6^2 \cdots x_{2i}^2 x_{2i+2} x_2^{-2i}) \\
(x_0^2 x_5^{-2}) (x_3^2 x_{2i+3} x_{2i+1} x_{2i-1}^2 \cdots x_7^{-2} x_5^{-2} x_0^{-4}).
$$

We move the $x_1^{2i}$ from the second line and the $x_2^{-2i}$ from the third line.

$$
z = (x_0^4 x_1^{2i} x_{2i+5}) (x_2^{2i+6} \cdots x_4^2 x_{2i+2} x_{2i+4} x_2^{-2i}) \\
(x_2^{2i+4} x_{2i+3}) (x_2^{-2} x_{2i+1} x_{2i-1}^2 \cdots x_5^{-2}) \\
(x_4^2 x_6^2 \cdots x_{2i}^2 x_{2i+2}) \\
(x_0^2 x_5^{-2}) (x_3^2 x_{2i+3} x_{2i+1} x_{2i-1}^2 \cdots x_7^{-2} x_5^{-2} x_0^{-4}).
$$

We move the $x_4$ from the beginning of the third line.

$$
z = (x_0^4 x_1^{2i} x_{2i+6}) (x_2^{2i+7} \cdots x_4^2 x_{2i+1} x_{2i+3} x_{2i+4} x_2^{-2i}) \\
(x_2^{2i+5} x_{2i+4}) (x_2^{-2} x_{2i+2} x_{2i}^2 \cdots x_6^{-2}) \\
(x_2^2 \cdots x_{2i}^2 x_{2i+2}^2) \\
(x_0^2 x_5^{-2}) (x_3^2 x_{2i+3} x_{2i+1} x_{2i-1}^2 \cdots x_7^{-2} x_5^{-2} x_0^{-4}).
$$

We cancel inverse pairs.

$$
z = (x_0^4 x_1^{2i} x_{2i+6}) (x_2^{-2} x_{2i+5} x_2^{-2i} x_0^{-4}).
$$
Now we work on \( w \).

\[
w = (c^it) t^{-1} (c^{-i} t c^i) = (x_0^2 x_{2} x_4 x_3^{-2} x_1^{-1} x_0^{-1}) (x_0^2 x_{2} x_4 x_3^{-2} x_1^{-1} x_0^{-1}) (x_0^2 x_{2} x_4 x_3^{-2} x_1^{-1} x_0^{-1}) (x_0^2 x_{2} x_4 x_3^{-2} x_1^{-1} x_0^{-1}) \]

Moving the internal \( x_0^{+1} \) gives the following.

\[
w = (x_0^2 x_{4} x_5 x_6 x_7 x_8^{-2} x_9^{-1} (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) \]

We move the \( x_7^2 \) from the second line and the \( x_2^{-2i} \) from the third line.

\[
w = (x_0^2 x_{4} x_5 x_6 x_7 x_8^{-2} x_9^{-1} (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) \]

We move \( x_2 \) and \( x_3^{-1} \) from the third line and cancel some adjacent pairs.

\[
w = (x_0^2 x_{4} x_5 x_6 x_7 x_8^{-2} x_9^{-1} (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) \]

We move the \( x_4 \) from the beginning of the third line.

\[
w = (x_0^2 x_{4} x_5 x_6 x_7 x_8^{-2} x_9^{-1} (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) \]

We cancel inverse pairs.

\[
w = (x_0^2 x_{4} x_5 x_6 x_7 x_8^{-2} x_9^{-1} (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) \]

\[
w = (x_0^2 x_{4} x_5 x_6 x_7 x_8^{-2} x_9^{-1} (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) (x_4^2 x_5 x_6 x_7 x_8^{-2} x_9^{-1}) \]

RANK 2 FREE LIMITS 10
We continue with \( p, q, h, \) and \( k \).

\[
p = z^{-1}w = (x_0 z^{-1})^2 (x_0 z^{-1})^{-1} = (x_0 z^{-1})^2 \frac{x_1^2 x_2 + 6 x_2 + 3 x_2^2 - 2 x_2^1 - x_2^0}{x_1^2 x_2 + 6 x_2 + 3 x_2^2 - 2 x_2^1 - x_2^0}
\]

Now,

\[
x_1^{-1} p x_1 = x_1^{-1} (x_0 z^{-1})^2 (x_0 z^{-1})^{-1} = \frac{x_0^2 x_1^2 x_2 + 6 x_2 + 3 x_2^2 - 2 x_2^1 - x_2^0}{x_1^2 x_2 + 6 x_2 + 3 x_2^2 - 2 x_2^1 - x_2^0}
\]

So,

\[
q = x_1^{-1} p x_1 p^{-1} = (x_0 z^{-1})^2 (x_0 z^{-1})^{-1} = \frac{x_0^2 x_1^2 x_2 + 6 x_2 + 3 x_2^2 - 2 x_2^1 - x_2^0}{x_1^2 x_2 + 6 x_2 + 3 x_2^2 - 2 x_2^1 - x_2^0}
\]

Finally,

\[
h = x_1^{-2} q x_1^2 = x_1^{-2} (x_0 x_1^2) x_1^{-2} = x_2 x_1^{-1} = w_1^{-1},
\]

\[
k = x_1 h x_1^{-1} = x_1 x_2 x_1^{-2} = y_1^{-1}.
\]

This completes the proof of (ii) from (i). The calculations above might give the impression that generating \( F \) (or more to the point, generating some \( F_{[a,b]} \)) is a rare phenomenon. This is not so. While certain obvious obstructions make the generation of some \( F_{[a,b]} \) a “probability zero” event, it is still surprisingly easy to arrange that it happens.
3.2. Proposition 2.2 Part (i). Almost all of the effort here will go to understanding the supports of the elements in (7). What we do here will also supply the missing details about the defined behavior of \(g_0\) and \(g_1\) that were only vaguely described in (5) and (6).

3.2.1. First look at the supports. We start with some preliminary estimates that relate to \(b\). These will be sharpened later.

Each of the elements defined in (7) acts on \(\mathbb{R}_{\geq 0}\) as the identity on certain intervals. It will be necessary to know something about what these intervals are. We will look at both the upper and lower case symbols.

We note that the symbols in (7) occur in three groups. The symbols \(C, S\) and \(T\) are defined in terms of the \(X_i\), the symbols \(\Sigma, \Theta, Z, W\) and \(P\) are defined in terms of \(C, S\) and \(T\), and the last three symbols are defined in terms of \(P\) and \(X_1\).

We consider \(C, S\) and \(T\) first. When written in terms of \(X_0\) and \(X_1\) using \(X_i = X_0^{1-i}X_1^{i-1}\), we see that

\[
T = X_0^3X_2X_4X_3^{-2}X_1X_0^{-1}X_0X_1^{-1}X_1^{-1}X_0X_1^{-1}X_0^{-1}
\]

is the longest at 12 letters. We also note that the total exponent sum over all the generators in each of \(C, S\) and \(T\) is zero. Thus the sum of the positive exponents is never more than 6. Since \(X_0^{\pm 1}\) and \(X_1^{\pm 1}\) are translations by \(\pm 1\) on at least \([3, b-2]\), it follows that if the length of \([3, b-2]\) is 12 or more, then each of \(C, S\) and \(T\) has a fixed point at 9. From this point we can take \(b\) to be at least 17. We will see later that this is overly cautious.

With \(b\) as above, it follows similarly from the fact that each of \(X_0^{\pm 1}\) and \(X_1^{\pm 1}\) is translation by \(\pm 1\) on at least \([3, \infty)\) that each of \(C, S\) and \(T\) has support in \([0, 9]\), and that each of \(C, S\) and \(T\) agrees with \(c, s\) and \(t\), respectively, on \([0, 9]\).

It now follows that each of the symbols in the second group \(\Sigma, \Theta, Z, W, P\) has a fixed point at 9 and that it agrees with the corresponding lower case symbol on \([0, 9]\).

Discussion of the last group can wait until it is shown that \(W = w\) and \(Z = z\).

3.2.2. Describing the elements. We make use of the rectangular diagrams of (5). In (8) below are the digrams for the basic building blocks shown in (5).

The numbers at the top give coordinates in the domain, and the numbers at the bottom give coordinates for the range. The function is viewed as going from the top of the rectangle to the bottom. The numbers in the middle give the slopes. The slopes will be useful in calculating the effects of some compositions.

The two figures in (8) are mutual inverses. Note that, with the exceptions of the numbers across the middle, the two figures in (8) are reflections of each
other across a horizontal line through the center. The slopes in one figure are the
reciprocals of the slopes in the other.

To describe more complicated functions, we will put together smaller versions
of the pictures above, with less information about coordinates and no information
about slopes. For example, the function in (4) would be described by the following.

To describe the function in the right figure of (5) we will make use of the diagram
in the left of (9) below. The right figure in (9) is the inverse of the left figure.

The coordinates in (9) have been arbitrarily chosen to start at 0, and the left
edge is missing since 0 is not a fixed point. A diagram for the function in the right
part of (5) is as follows. We do not bother with the coordinates on the bottom.

To describe the function in (6), we will use the left figure in (10) below whose
inverse is in the right part of (10).

A diagram for the function in (6) is as follows where we show only a few coordinates.

As is seen, coordinates such as $b + 4m$ and $b + 4m + 6$ are cumbersome. From
now on $\xi$ will represent $b + 4m + 2$, the rightmost fixed point (which happens to be
repelling) of $X_0$.

The diagrams are too bulky to show all parts of a given element from (7). We
will restrict ourselves to diagrams in the neighborhood of $b$ and diagrams in the
neighborhood of $\xi = b + 4m + 2$. 
3.2.3. The analysis of supports near $\xi$. Let us first tackle diagrams at the right end, in the neighborhood of $\xi$. We start with the generators.

First we have $X_0$.

Next we have $X_1$.

Their inverses are obtained by reflecting across a central horizontal line. Compositions are shown by stacking the diagrams vertically. We start with the simpler of $S$ and $T$.

Since $S = X_1X_0X_1^{-1}X_1^{-1}$, we get the following diagram for $S$.

The picture above shows that $\xi + 1 \frac{1}{4}$ is an upper bound for the support of $S$. The actual right endpoint for the support needs more careful inspection. By tracing the slopes from top to bottom, using the figures in (8) and (10), it is seen that between $\xi + 1 \frac{1}{8}$ and $\xi + 1 \frac{1}{4}$ the slopes encountered are 4, 1, 1 and $\frac{1}{4}$ in that order from top to bottom. However, the slopes between $\xi$ and $\xi + 1 \frac{1}{8}$ are 4, 1, $\frac{1}{2}$ and $\frac{1}{4}$. Thus $\xi + 1 \frac{1}{8}$ is the right endpoint of the support of $S$. 
Our next task is to tackle $T = X_0X_1X_0^{-1}X_1X_0X_1^{-1}X_1^{-1}X_0X_0X_1^{-1}X_0^{-1}$.

In an analysis almost identical to that of $S$, we get that $\xi + 1\frac{1}{8}$ is the right endpoint of the support of $T$.

Now we look at $C = X_0X_0X_1X_1^{-1}X_0^{-1}X_1^{-1}X_1^{-1}$.

We get the following from the picture above. For an integer $j > 0$ for which $\xi - 4j$ is in the pattern above, we have $\xi - 4j + \frac{1}{4}$ is carried by $C$ to at least $\xi - 4j + 4\frac{3}{4}$. In particular $\xi - 4 + \frac{1}{4}$ is carried to greater than $\xi + \frac{3}{4}$. Using the information in
with the figure above, we get that $\xi + \frac{3}{4} \leq \xi + 1\frac{3}{16}$. Further, the interval from $\xi + 1$ to $\xi + 1\frac{1}{2}$ is carried affinely with slope $\frac{1}{2}$ to the interval from $\xi + 1\frac{1}{2}$ to $\xi + 1\frac{3}{4}$. Thus $\xi + 1\frac{3}{4}$ is the right endpoint of the support of $C$.

It follows that for an integer $j > 0$ for which $\xi - 4j$ is in the pattern above, we have that any $\eta$ in $(\xi - 4j + \frac{1}{4}, \xi + 1\frac{1}{2})$ has $\eta C > \eta$. Further, any such $\eta$ has

\[
\begin{align*}
\eta C^j & > \xi + \frac{3}{4}, \\
\eta C^{j+1} & > \xi + 1\frac{3}{16}.
\end{align*}
\]

The point is that $\xi + 1\frac{3}{16}$ is greater than $\xi + 1\frac{1}{8}$, the right endpoint of the support of $S$ and of $T$.

### 3.2.4. The analysis of supports near $b$.

We create pictures for the generators near $b$ in much the same way as we do near $\xi$. The reader can verify that the the following is an accurate combination of diagrams for $X^{\pm 1}_0$ and $X^{\pm 1}_1$ that gives the behavior of $C = X_0 X_0 X_1 X_1 X_0^{-1} X_0^{-1} X_1^{-1} X_1^{-1}$ near $b$.

From the diagram above, we get the following information. First, the left endpoint of the support of $C$ near $b$ is $b - 4\frac{1}{4}$. Second, a value of $j$ for which (11) is valid is that $j$ for which $\xi - 4j = b + 2$. This value of $j$ satisfies $b + 4m + 2 - 4j = b + 2$ or $j = m$. Third, we note that $(b - 3\frac{1}{4})C > b + 2\frac{3}{4}$. Combining this information with (11), we get that for any $\eta \geq (b - 3\frac{1}{4})$ we have

\[
\eta C^{m+2} > \xi + 1\frac{3}{16}.
\]

We next look at $S$ and $T$. 
The diagram for $S = X_1 X_0 X^{-1}_1 X^{-1}_0$ near $b$ follows.

```
| b-5 | b-3 | b-1 | b | b+2 | b+4 | b+6 | b+8 |
|-----|-----|-----|---|-----|-----|-----|-----|
```

The diagram for $T = X_0 X_1 X^{-1}_0 X^{-1}_0 X_1 X_0 X^{-1}_1 X^{-1}_0 X_0 X_1 X^{-1}_0 X^{-1}_0$ near $b$ is below.

```
| b-5 | b-3 | b-1 | b | b+2 | b+4 | b+6 | b+8 |
|-----|-----|-----|---|-----|-----|-----|-----|
```

A trace through the two diagrams above, using the information in §4 about the slopes, shows that the left endpoint of the supports of $S$ and $T$ near $b$ is $b - 2\frac{1}{2}$.

The fact that the left endpoint of the support of $C$ near $b$ is $(b - 4\frac{1}{2})$ and that the left endpoint of the supports of both $S$ and $T$ near $b$ is $(b - 2\frac{1}{2})$ explains why our claims in Section 2 refer to $[0, b - 5)$ and $(b - 5, \infty)$.

3.2.5. End of the proof of Part (i). From Sections 3.2.4 and 3.2.5 we have the following information. Using the fact that $\xi = b + 4m + 2$, we have that the supports of $C$, $S$ and $T$ in $(b - 5, \infty)$ are given by

- $C : (b - 4\frac{1}{2}, b + 4m + 3\frac{1}{2})$,
- $S, T : (b - 2\frac{1}{2}, b + 4m + 3\frac{1}{2})$. 
From (12) and the fact that \( i = m + 2 \), we know that for any \( \eta \geq (b - 3 \frac{1}{2}) \) we have
\[
\eta C^i > b + 4m + 3 \frac{3}{16}.
\]
From the facts above, we know that the supports of \( \Sigma = C^{-1} SC^i \) and \( \Theta = C^{-1} TC^i \) in \((b - 5, \infty)\) are both contained in
\[
(b + 4m + 3 \frac{3}{16}, b + 4m + 3 \frac{1}{2})
\]
which is disjoint from the supports of \( S \) and \( T \) in \((b - 5, \infty)\). Thus the restrictions of \( Z = [S, \Sigma] \) and \( W = [T, \Theta] \) to \((b - 5, \infty)\) are trivial.

From our discussion in Section 3.2.1, we see that \( W = w \) and \( Z = z \). From the definition \( P = Z^{-1} W \), we get \( P = p \). Lastly, with \( Q, H \) and \( K \) defined in terms of \( P \) and \( X_1 \), we get the rest of (i) of Proposition 2.2.

3.3. Proposition 2.2 Part (iv). We must show that the translates of the interval \((1, 3)\) under words in \( X_0 \) and \( X_1 \) cover all of \((0, \infty)\).

Since the orbit of \((b - 1)\) under \( X_0 \) includes all the integers below \( b - 1 \) as well as all fractions of the form \( 1/2^n \) for a positive integer \( n \), we get that the translates cover at least \((0, b - 1)\).

Since \((1, b - 1)\) is covered by finitely many translates of \((1, 3)\), we may work from now on with \((1, b - 1)\) instead of \((1, 3)\).

Since \( C \) has a fixed point at \((b - 5)\) which we take to be bigger than 1, and since \((b - 1)\) is carried by powers of \( C \) to at least \((b + 4m + 3 \frac{1}{2})\). We can now make another replacement and work from now on with the interval \((1, b + 4m + 3)\).

From the discussion above (6), we know that \( X_0 \) acts as translation by 1 on \([b + 4m + 3, \infty)\). Since \( X_0 \) has a fixed point at \((b + 4m + 2)\), we get all points in \((1, \infty)\) covered.

Combining the information in the four paragraphs above completes the argument for (iv).

References

1. Azer Akhmedov, Melanie Stein and Jennifer Taback, *Free limits of Thompson's group F*, announcement, conference on Geometric and Asymptotic Group Theory with Applications, March 9–12, 2009, Stevens Institute, Hoboken, N.J.
2. Matthew G. Brin and Fernando Guzmán, *Automorphisms of generalized Thompson groups*, J. Algebra 203 (1998), no. 1, 285–348.
3. Matthew G. Brin and Craig C. Squier, *Groups of piecewise linear homeomorphisms of the real line*, Invent. Math. 79 (1985), 485–498.
4. Kenneth S. Brown, *Finiteness properties of groups*, Journal of Pure and Applied Algebra 44 (1987), 45–75.
5. J. W. Cannon, W. J. Floyd, and W. R. Parry, *Introductory notes on Richard Thompson's groups*, Enseign. Math. (2) 42 (1996), no. 3-4, 215–256. MR 98g:20058
6. Christophe Champetier and Vincent Guirardel, *Limit groups as limits of free groups*, Israel J. Math. 146 (2005), 1–75. MR MR2151593 (2006d:20045)
7. Evgenii S. Esyp, *On identities in Thompson's group*, ArXiv preprint: http://front.math.ucdavis.edu/0902.0199 2009.
8. M. V. Sapir and A. Yu. Olshanskii, *On k-free-like groups*, ArXiv preprint: http://front.math.ucdavis.edu/0811.1607 2008.
9. Zlil Sela, *Diophantine geometry over groups. I. Makanin-Razborov diagrams*, Publ. Math. Inst. Hautes Études Sci. (2001), no. 93, 31–105. MR MR1863735 (2002h:20061)
10. Roland Zarzycki, *Limits of (Thompson's) group F*, ArXiv preprint: http://front.math.ucdavis.edu/0701.5601 2007.
Department of Mathematical Sciences
State University of New York at Binghamton
Binghamton, NY 13902-6000
USA
email: matt@math.binghamton.edu