Generic Identifiability for REMIS: The Cointegrated Unit Root VAR

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Abstract

The “REtrieval from MIxed Sampling” (REMIS) approach based on blocking developed in Anderson et al. (2016a) is concerned with retrieving an underlying high frequency model from mixed frequency observations. In this paper we investigate parameter-identifiability in the Johansen (1995) vector error correction model for mixed frequency data. We prove that from the second moments of the blocked process after taking differences at lag $N$ (where $N$ is the slow sampling rate), the parameters of the high frequency system are generically identified. We treat the stock and the flow case as well as deterministic terms.

Index terms — Mixed Frequency, REMIS, VAR, Cointegration, Vector Error Correction Model, Identifiability

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1 Introduction

Econometric analysis is often encountered with multivariate time series data sampled at mixed frequencies. Examples for treating this are Zadrozny (1988), Ghysels et al. (2007) [MIDAS-regression], Schorfheide and Song (2015), Ghysels (2016), and Anderson et al. (2012), Anderson et al. (2016a) [REMIS] and for the cointegrated case Chambers (2020). Identifiability is a prerequisite for consistent estimation (see, e.g., Pötscher and Prucha, 1997) and often is necessary for economic interpretation of effects related to particular model parameters. This article investigates identifiability of the model parameters in a Johansen (1995) vector error correction model.

The general question is whether the internal characteristics, i.e. the model parameters $\theta$, can be retrieved from the external characteristics – in our case observable second moments. Identifiability means that the mapping from the parameters to these second moments is injective. Often injectivity of the mapping above can only be achieved for a certain subset of the parameterspace. Here, we prove that identifiability can be obtained for a generic subset (thus in a nontechnical sense “almost everywhere”) of the parameterspace (see Anderson et al., 2016a).

As opposed to MIDAS-regression, where the observations at high frequency are considered as additional information, we consider mixed frequency as either a “missing-values” or a “dis-aggregation”-problem, by which we mean the following: We commence from an underlying high frequency system (e.g., a VECM) parameterised by $\theta$ for a multivariate process

$$(y_t)_{t \in \mathbb{Z}} = \left( \left( y_t^f, y_t^s \right) \right)_{t \in \mathbb{Z}}$$

and our aim is to identify and estimate this system from the observed (mixed frequency) data. The observational scheme is as follows: While the fast variables $y_t^f$ are observed at $t \in \mathbb{Z}$, for the slow variables $y_t^s$ we consider:

1. **Stock-Case:** $y_t^s$ is observed only at $t \in N\mathbb{Z}$ for some sampling rate $N \geq 2$, hence we have a missing-value problem.

2. **Affine aggregation:** we observe an affine transformation $w_t = c_w + c_0 y_t^s + \cdots + c_p y_{t-p_c}$, where $c_i$ are known constants and $w_t$ is observed at $t \in N\mathbb{Z}$.

A special case of affine aggregation are flow variables: For example suppose $y_t^s = GDP_t$, the monthly gross domestic product of a country. The quarterly GDP, $w_t$ is the sum of three monthly GDPs. We call $y_t^s$ latent whenever it is not directly observed. Hence, our aim is to retrieve the underlying high
frequency parameters $\theta$ from data observed according to the observational schemes described above.

With the procedure described above, we are able to model all kinds of linear dynamic relationships between latent and observed variables, whereas the MIDAS (see, e.g., Ghysels, 2016) approach only covers relationships between observed variables. After identifying the parameters one may impute missing values or dis-aggregate observations in a model based way by using the retrieved parameters of the underlying high frequency system.

Estimation of continuous time models from mixed frequency data are investigated in Chambers (2003, 2016, 2020). In particular, Chambers (2003, 2020) consider co-integrating regressions and show that the scaled estimators proposed, converge in distribution to functionals of Brownian motion and to stochastic integrals. Hence, the estimators are consistent and by Deistler and Seifert (1978) the model parameters are identified.

For the stable vector auto-regressive model Anderson et al. (2012) and Anderson et al. (2016a) either used the blocking approach (see also Filler, 2010; Ghysels, 2016) or the extended Yule-Walker equations (see also Chen and Zadrozny, 1998; Anderson et al., 2016a) to show $g$-identifiability. For the same model class Gersing and Deistler (2021) present an alternative proof for identifiability using the so-called canonical projection form. This idea is also applied in this paper. On the other hand, Deistler et al. (2017) show that the parameters need not be identified in the auto-regressive-moving average (VARMA) case, if the order of the MA polynomial exceeds the order of the AR polynomial.

This article is organised as follows: Section 2 starts with a vector error correction model developed in Johansen (1995) as the underlying high frequency model. In Section 2.2 we describe the considered observational schemes considered in detail. In particular, we introduce a stationary blocked process containing all observed variables. Section 2.3 introduces conditions, which are later shown to be sufficient for identifiability. We prove that these conditions hold generically in the underlying high frequency parameters space. Section 3 extends the REMIS approach to the non-stationary case: Here, we use the result from Chambers (2020) that the cointegrating vectors can be identified from mixed frequency data. First, we derive a state-space representation of the blocked process that we call Canonical Projection Form (CPF) and where the high frequency parameters “shine out”. After that we start from the unique factor of the spectrum of the blocked process (see, e.g., Deistler and Scherrer, 2018, ch. 6.2 and 7.3) to get an arbitrary minimal realisation for this factor and relate this to the CPF. From there we can retrieve
the parameters of the underlying high frequency system using the structural properties of the CPF. Section 4 adds deterministic terms. Finally, Section 5 concludes.

2 Notation and Model Class

2.1 Representations and Parameterspace of the Underlying High Frequency System

In the first step, we introduce the class of underlying high frequency systems: We commence from a process which is integrated of order one and allow for cointegration. Suppose \((y_t)_{t \in \mathbb{Z}}\) is \(n \times 1\) and a solution on \(\mathbb{Z}\) of the vector error correction system:

\[
\Delta y_t = \Pi y_{t-1} + \sum_{j=1}^{p-1} \Phi_j \Delta y_{t-j} + \nu_t, \quad \nu_t \sim WN(\Sigma_\nu),
\]

where \((\nu_t)_{t \in \mathbb{Z}}\) is white noise and \(\Pi\) is of rank \(r > 0\) in the case of cointegrating relationships, but we also allow the case \(r = 0\). Such solutions always exist and can be constructed as described in detail in Bauer and Wagner (2012).

We obtain a unique factorisation of \(\Pi = \alpha \beta'\) with \(\alpha, \beta \in \mathbb{R}^{n \times r}\) applying the singular value decomposition to \(\Pi\) in the following way:

\[
\Pi = U \text{diag}(d_1, \ldots, d_r, 0, \ldots, 0) D' = U_1 \text{diag}(d_1, \ldots, d_r) V_1'
\]

\[
= U_1 \tilde{D} V_1' = U_1 Q^{-1} \beta' D V_1',
\]

where \(Q\) is a regular matrix of elementary row operations that transforms \(\tilde{D} V_1'\) into its reduced echelon form, s.t. \(Q \tilde{D} V_1' = (I_r \beta'_{n-r})\). We stack the parameters \(\alpha, \beta, \Phi_1, \ldots, \Phi_{p-1}\) to a vector \(\theta_{VECM} \in \mathbb{R}^d\), where \(d = nr + (n - r) r + (p - 1) n^2\).

We also have a VAR\((p)\) representation for \((y_t)\) of the form,

\[
y_t = A_1 y_{t-1} + \cdots + A_p y_{t-p} + \nu_t,
\]

which we obtain by the mapping \(\psi\):

\[
\psi : \theta_{VECM} \mapsto \theta_{AR}, \quad \text{defined as}
\]

\[
A_1 = I_n + \Pi + \Phi_1, \quad A_j = \Phi_j - \Phi_{j-1} \quad \text{for } 1 < j < p, \quad A_p = -\Phi_{p-1},
\]
with \( \theta_{AR} = \text{vec}(A_1 \cdots A_p) \). On the other hand for a \( \theta_{AR} \) which has a corresponding VECM representation, we compute \( \theta_{VECM} \) as follows:

\[
\psi^{-1}: \theta_{AR} \mapsto \theta_{VECM}
\]

\[
\Pi = -I_n + \sum_{j=1}^p A_j, \Phi_1 = -I_n + A_1 + \Pi, \Phi_2 = \Phi_1 + A_2, \ldots, \Phi_{p-1} = -A_p.
\]

Now, define the polynomial matrix 
\[
a(z) = I_n - A_1 z - \cdots - A_p z^p
\]

where \( z \) is a complex number or the lag operator on \( \mathbb{Z} \) depending on the context. For \( \tilde{c} = \begin{pmatrix} I_r \\ 0 \end{pmatrix} \in \mathbb{R}^{n \times r} \) and \( \tilde{c}_\perp = \begin{pmatrix} 0 \\ I_{n-r} \end{pmatrix} \in \mathbb{R}^{n \times (n-r)} \), \( \beta_\perp := (I_n - \tilde{c}(\beta')^{-1}\beta')\tilde{c}_\perp \) and \( \alpha_\perp \) analogously. We impose the following assumptions (Johansen, 1995, chapter 4):

**Assumption 1** (Cointegrated VAR-System)

**(C1)** \( \text{rk} \alpha \beta' = r < n \).

**(C2)** \( \det(\alpha_\perp' (I_n - \sum_{j=1}^{p-1} \Phi_j) \beta_\perp) \neq 0 \).

**(C3)** \( \det a(z) = 0 \Rightarrow z = 1 \) or \( |z| < 1 \).

**(C4)** \( \Sigma_\nu = \mathbb{E} \nu_t \nu_t' > 0 \).

We define the parameterspace as follows:\(^1\)

\[
\Theta_{VECM,1} := \psi^{-1} \left( \psi \left( \mathbb{R}^d \right|_{C_1,C_2} \right) \right|_{C_3} \right), \quad \Theta_1 := \psi (\Theta_{VECM,1})
\]

with \( \Theta_1 \leftrightarrow \Theta_{VECM,1} \)

Note that under these assumptions \( \psi \) is a homeomorphism. The set of vech \( \Sigma_\nu \) with \( \Sigma_\nu \in \mathbb{R}^{n \times n}, \Sigma_\nu = \Sigma_\nu' \) and \( \Sigma_\nu > 0 \) (condition (C4) in Assumption 1) is denoted by \( \Theta_2 \). The overall parameterspace for the VAR(\( p \)) representation is

\[
\Theta = \Theta_1 \times \Theta_2.
\]

\(^1\)We write \( \mathbb{R}^d \right|_{C_1,C_2} \) to denote the set of real vectors in \( \mathbb{R}^d \) for which \( C_1 \) and \( C_2 \) hold.
We will also need the state-space representation of \((y_t)_{t \in \mathbb{Z}}\), which follows from (2):

\[
\begin{pmatrix}
  y_t \\
  \vdots \\
  y_{t-p+1}
\end{pmatrix}
= \begin{pmatrix}
  \mathcal{A}_1 & \mathcal{A}_2 & \cdots & \mathcal{A}_p \\
  I_n & & & 0 \\
  & \ddots & & \vdots \\
  & & I_n & 0 \\
  & & & y_{t-p}
\end{pmatrix}
\begin{pmatrix}
  x_{t+1} \\
  \vdots \\
  x_r
\end{pmatrix}
+ \begin{pmatrix}
  I_n \\
  \vdots \\
  0
\end{pmatrix}
u_t
\tag{3}
\]

\[
y_t = \left(\mathcal{A}_1 \cdots \mathcal{A}_p\right) X_t + \nu_t.
\tag{4}
\]

Note that (3), (4) is always controllable as \(\Sigma_\nu\) and therefore \(\Gamma(t) := \mathbb{E}\left(X_{t+1}X_{t+1}'\right)\) are of full rank. The system (3), (4) is also observable whenever \(\mathcal{A}_p\) is of full rank. This follows since \(\mathcal{A}_p\) is nonsingular (and therefore \(\mathcal{A}\) is non-singular) from the BPH-test (see Kailath (1980) 2.4.3). Hence under Assumption 1 and if \(\mathcal{A}_p\) is nonsingular the system (3), (4) is minimal. For details on controllability and observability see Deistler and Scherrer (2018), ch. 7 and Hannan and Deistler (2012), ch. 2.

2.2 Mixed Frequency Data: Stock and Flow Variables

A main challenge of the identifiability proof in the integrated case - as opposed to the stationary case (Anderson et al., 2016a) - is that the second moments of an integrated process are time dependent and cannot be estimated directly. Instead, for the sake of practical relevance of identifiability considerations, we identify from observable second moments of stationary transformations of the level process (see details below). The second moments for these transformations can be estimated consistently by the sample (auto-) covariances.

Suppose for the moment, that the matrix of cointegration vectors \(\beta\) is known. Our proof commences from what we call the “blocked process”, where we distinguish between the Stock- and the Flow-case:

1. **Stock Variables**: In this case for \(t \in N\mathbb{Z}\), we get the co-stationary vector of observed (after identification of \(\beta\)) random variables. We will use \(\tilde{n} := r + n + (N - 1)n_f\) for the dimension of \(\tilde{y}_t\) henceforth. Let \(u^S_t := \beta'y_t\) and \(\Delta_N y_t := y_t - y_{t-N} = \sum_{j=0}^{N-1} \Delta y_{t-j}\).
\[ \tilde{y}_t = \begin{pmatrix} \beta'y_t \\ y_t - y_{t-N} \\ y_{t-1} - y_{t-N-1} \\ \vdots \\ y_{t-N+1} - y_{t-2N+1} \end{pmatrix} = \begin{pmatrix} u_t^S \\ \Delta_N y_t \\ \Delta_N y_{t-1}^f \\ \vdots \\ \Delta_N y_{t-N+1}^f \end{pmatrix}. \] (5)

2. Flow Variables: In a similar way, we may consider the case where all slow variables are flow variables, in which case we are able to observe the temporal aggregate \( w_t := \sum_{j=0}^{N-1} y_{t-j}^s \) at \( t \in N\mathbb{Z} \). So

\[ \Delta_N^\Sigma y_t := \sum_{j=0}^{N-1} y_{t-j} - \sum_{j=0}^{N-1} y_{t-N-j} = \Delta_N \sum_{j=0}^{N-1} \left( y_t^f - y_t^s \right). \]

If all slow variables are flow variables, we can observe \( \sum_{j=0}^{N-1} y_{t-j} = \left( w_t, \sum_{j=0}^{N-1} y_{t-j}^f \right) \), \( t \in N\mathbb{Z} \). Since \( \beta'y_t \) is stationary, we have that \( (\beta'y_t)_{t \in N\mathbb{Z}} \) and \( u_t^f := \beta' \sum_{j=0}^{N-1} y_{t-j} \in \mathbb{R}^r \) are integrated of order zero. For the flow case we define the co-stationary vector process

\[ \tilde{y}_t = \begin{pmatrix} u_t^f \\ \Delta_N^\Sigma y_t \\ \Delta_N y_{t-1}^f \\ \vdots \\ \Delta_N y_{t-N+1}^f \end{pmatrix}. \] (6)

We call the autocovariance function of the (stationary) blocked process

\[ \tilde{\gamma} : h \mapsto \mathbb{E} \tilde{y}_{t+h} \tilde{y}_t^\prime \quad \text{where} \ h \in N\mathbb{Z} \] (7)

obscured second moments, which can be consistently estimated from the data (if \( \beta \) is known) under standard assumptions.

The motivation to consider this blocked process for identifiability is the following:

1. We take differences at lag \( N \) (as opposed to lag one) because these differences can be directly computed from the mixed frequency data and are stationary.
2. Note that the set of observable autocovariances given mixed frequency data is

\[ \gamma_{ff}(h) := \mathbb{E} \Delta N y_{t+h}^f \Delta N y_t^f, \quad h \in \mathbb{Z} \]
\[ \gamma_{fs}(h) := \mathbb{E} \Delta N y_{t+h}^f \Delta N y_t^s, \quad h \in \mathbb{Z} \]
\[ \gamma_{ss}(h) := \mathbb{E} \Delta N y_{t+h}^s \Delta N y_t^s, \quad h \in \mathbb{N} \mathbb{Z} \]
\[ \gamma_{\beta}(h) := \mathbb{E} u_{t+h} u_t', \quad h \in \mathbb{N} \mathbb{Z} . \]

Note that these are exactly the second moments of the autocovariance function \( \tilde{\gamma} \) of the blocked process defined in equation (7). So the blocked process “contains the whole second moment information available” from which we can identify. The same idea is also applied for the stationary case in Anderson et al. (2016a).

3. Our interest in the particular blocked process (5), (6) having \( u_t \) in the first coordinates, originates in the fact that we can obtain a minimal representation for this process (see Section 3), where the parameters are fairly simple functions of the parameters of the underlying high frequency system. This will finally help us to retrieve the high frequency model parameters. Next, we define the concept of generic identifiability. Here, identifiability is concerned with the problem whether the parameters of the underlying high frequency system (3), (4) or (1) are uniquely determined from the observable second moments (defined below in this section). To be more precise, a subset \( \Theta_I \subset \Theta \) is called identifiable, if the mapping attaching the observable second moments to the parameters \( \theta \in \Theta_I \) is injective. In our setting identifiability for the whole set \( \Theta \) cannot be obtained. However, in this paper we prove that identifiability holds for a so called generic subset of \( \Theta \). Note that a set \( \Theta_I \subset \Theta \) is called generic in \( \Theta \), if it contains a subset that is open and dense in \( \Theta \).

Let \( \Theta_I := (G \cap \Theta_1) \times \Theta_2 \). In this paper we show firstly that \( \Theta_I \) is generic in \( \Theta \) (see Section 2.3) and secondly that the set of high frequency systems corresponding to \( \Theta_I \) is identifiable from the observable second moments (see Section 3). Or formally, we show that

\[ \pi : \theta \mapsto \tilde{\gamma} \]

is injective on \( \Theta_I \subset \Theta \).

Finally, in terms of identifiability, we may suppose without loss of generality

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\(^2\)The example of a non-identifiable AR(1)-system considered in Anderson et al. (2016a) [Section 3] holds analogously for the integrated case.
that $\beta$ is known: Chambers (2020) developed a spectral regression estimator for mixed frequency observations in the cointegrated case, accounting for stock and flow variables, respectively. Chambers (2020)[Theorem 2] obtained weak convergence of the estimator scaled by $T$ to a product of the inverse of a functional of Brownian motions and a stochastic integral. Hence, by Deistler and Seifert (1978) the matrix of cointegrating vectors $\beta \in \mathbb{R}^{n \times r}$ is identified from mixed frequency observations given the assumptions imposed in Chambers (2020). These assumptions are only posed on the stochastic properties of the high frequency innovations $(\nu_t)_{t \in \mathbb{Z}}$ and therefore do not restrict our results on the genericity of the identifiability conditions from section 2.3.

2.3 Generic Identifiability and Topological Properties of the Parameterspace

In this section we define the conditions that we need for identifiability and prove that these conditions result in a generic subset of the parameterspace. Define a set $G \subset \mathbb{R}^{n^2 \times p}$ by the following assumptions:

**Assumption 2 (g-Identifiability Assumptions)**

(I1) $\text{rk} A_p = n$.

(I2) $\text{rk} \Gamma(t) = np$ where $\Gamma(t) = \mathbb{E}(X_{t+1}X'_{t+1})$.

(I3) The eigenvalues of $A$ are of the form: $(1, \ldots, 1, \lambda_{n-r+1}, \ldots, \lambda_{np})$ where $|\lambda_j| < 1$ and $\lambda_i \neq \lambda_j$ for $i \neq j$ with $i, j = n - r + 1, \ldots, np$.

(I4) For non-unit eigenvalues $\lambda_i \neq \lambda_j$ it follows that $\lambda_i^N \neq \lambda_j^N$.

(I5) For all eigenvalues $\lambda$ of $A$ smaller than one, it holds that $1 + \lambda + \cdots + \lambda^N = 0$ or $v_1$ consisting of the first $n$ elements of the eigenvector $v$ of $A$ corresponding to $\lambda$, it holds that $\beta'v_1 \neq 0$.

(I6) The pair $(S^{(1)}_{nt}, A)$ is observable, where $S^{(1)}_{nt}$ is defined in equations (13), (14).

Recall that $\Theta_I = (G \cap \Theta_1) \times \Theta_2$. In the stationary case, considered in Anderson et al. (2016a), Anderson et al. (2016b) we had the following
setting: The parameterspace $\Theta' \subset \mathbb{R}^{n^2p}$ is open and the set defined by the identifiability conditions $G' \subset \mathbb{R}^{n^2p}$ is generic in $\mathbb{R}^{n^2p}$. It easily follows that the intersection $\Theta' \cap G'$ is generic in $\Theta'$. However, in the integrated case, where unit roots occur, the situation is more intricate since neither $\Theta_1$ nor $G$ is open in $\mathbb{R}^{n^2p}$. This follows from the fact that for a process with $n - r$ common trends, the $n - r$ eigenvalues of $A$ in (3) are equal to one [note that the eigenvalues of $A$ are the reciprocals of the zeros of $a(z)$]. The following Theorem 1 implies that the identifiability conditions are generically fulfilled in $\Theta$:

**Theorem 1**

*Let $\Theta_1$ be endowed with the Euclidean norm $d$. The set $\Theta_1 \cap G$ is open and dense in $\Theta_1$.*

Since genericity is a topological property, it also holds for the homeomorphic parameterspace corresponding the vector error correction representation in (1) defined by Assumption 1.

## 3 Generic Identifiability

In this section, we first define a canonical state-space representation for the blocked process running on $t \in \mathbb{N} \mathbb{Z}$. We prove that this representation is minimal under our identifiability conditions. Then under an additional assumption on the lag order $p$, we show that the high frequency parameters are generically identifiable. The proofs of minimality and identifiability make use of the canonical representation.

We follow Hansen and Johansen (1999) and obtain from (1) the following state-space system for $\beta' y_t$ and first differences of $y_t$, that is $\Delta y_t = y_t - y_{t-1}$. 


Then,

\[
\begin{pmatrix}
\beta' y_t \\
\Delta y_t \\
\vdots \\
\Delta y_{t-p+2}
\end{pmatrix}
\begin{pmatrix}
\beta' + I_r \\
\Phi_1 \\
\vdots \\
\Phi_{p-1}
\end{pmatrix}
\begin{pmatrix}
\alpha \\
I_n \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
\beta' y_{t-1} \\
\Delta y_{t-1} \\
\vdots \\
\Delta y_{t-p+1}
\end{pmatrix}
+ \begin{pmatrix}
\beta' \\
I_n \\
\vdots \\
0
\end{pmatrix}
\nu_t
\]

By iterating the system (9), (10), we get the non-miniphase system (in the sense that the transfer-function is not causally invertible as the input

1. **Case: Stock Variables:** We define a new state vector \( x_{t+1} \) in the following way, with the condition that \( p \geq N + 2 \):

\[
\begin{pmatrix}
u_t^S \\
\Delta y_t \\
\vdots \\
\Delta y_{t-p+1}
\end{pmatrix}
= \begin{pmatrix}
I_r \\
0 \\
\vdots \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
I_n \\
\vdots \\
I_n
\end{pmatrix}
\begin{pmatrix}
0 \\
I_n \\
\vdots \\
I_n
\end{pmatrix}
\begin{pmatrix}
\Delta y_{t-N} \\
\Delta y_{t-N+1} \\
\vdots \\
\Delta y_{t-p+2}
\end{pmatrix}
\]

By iterating the system (9), (10), we get the non-miniphase system (in the sense that the transfer-function is not causally invertible as the input.
dimension exceeds the output dimension, noting that $\Sigma_{\nu} > 0$):

\[ x_{t+1} = c A^N c^{-1} x_{t-N+1} + \frac{c B_b}{b_{b,c}} \nu_t^b \]

\[ : = A_b \]

\[ y_t = S_c A^N c^{-1} x_{t-N+1} + D_b \nu_t^b \]

\[ : = C_b \]

where

\[ S_{\xi} := \begin{pmatrix} \sum_{\xi} (r \times m) \{ \begin{array}{ccc} (I_r & 0 & \cdots & 0) A^N \\ S_{n_1} & A^N \\ S_{n_2} & A^N \\ S_{n_3} & A^N \\ \vdots \\ S_{n_N} & A^N \end{array} \end{pmatrix} \end{pmatrix} , \]

\[ C_b = \begin{pmatrix} (I_r & 0 & \cdots & 0) A^N \\ S_{n_1} & A^N \\ S_{n_2} & A^N \\ S_{n_3} & A^N \\ \vdots \\ S_{n_N} & A^N \end{pmatrix} = \begin{pmatrix} (I_r & 0 & \cdots & 0) A^N \\ S_{n_1} & A^N \\ S_{n_2} & A^N \\ S_{n_3} & A^N \\ \vdots \\ S_{n_N} & A^N \end{pmatrix} , \]

\[ \nu_t^b := \begin{pmatrix} \nu_t \\ \vdots \\ \nu_{t-N+1} \end{pmatrix} \in \mathbb{R}^{Nn} . \]

The matrices $B_{b,c} \in \mathbb{R}^{r+n(p-1) \times Nn}$ and $D_b \in \mathbb{R}^{r+n \times Nn}$ are obtained from $B$ and $A$.

**2. Case: Flow Variables:** Next, we obtain the state vector $x_{t+1}$ for the flow case. Note that $y_{t-j} = y_t - \sum_{\ell=1}^{j} \Delta y_{t-\ell}$, such that $\sum_{j=0}^{N-1} y_{t-j} = \sum_{j=0}^{N-1} \left( y_t - \sum_{\ell=1}^{j} \Delta y_{t-\ell} \right) = N y_t - (N-1) \Delta y_{t-1} - \cdots - \Delta y_{t-N+1}$. Analogously
to equation (11), this yields for \( p \geq 2N + 1 \) that

\[
\begin{pmatrix}
\beta' \sum_{t=0}^{N-1} y_{t-j} \\
\Delta_N y_t \\
\Delta_N y_{t-N+1} \\
\vdots \\
\Delta_N y_{t-p+2}
\end{pmatrix}_{t+1 \in \mathbb{R}^{r+n(p-1)}} =
\begin{pmatrix}
NI_r & -(N-1)\beta' & -(N-2)\beta' & \cdots & -\beta' & 0 & \cdots \\
0 & I_n & \cdots & I_n & -I_n & \cdots & -I_n & 0 \\
0 & 0 & \cdots & I_n & -I_n & \cdots & -I_n & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & I_n & 0 & \cdots & \cdots \\
\end{pmatrix}_{c \in \mathbb{R}^{r+n(p-1) \times r+n(p-1)}}
\begin{pmatrix}
\eta_t^s \\
\Delta y_t \\
\vdots \\
\Delta y_{t-p+2}
\end{pmatrix}.
\tag{15}
\]

We use the same notation for \( \tilde{y}_t, x_t, c \) for both cases. With this notation, we obtain the following state-space representation for blocked process in the flow case:

\[
x_{t+1} = c A_{b,c}^{-1} x_{t-N+1} + c B_{b,c} y_t^b 
\tag{16}
\]

\[
\tilde{y}_t = S_c A_{c}^{-1} x_{t-N+1} + D_{b,c} y_t^b,
\tag{17}
\]

where

\[
S_c =
\begin{pmatrix}
NI_r & -(N-1)\beta' & -(N-2)\beta' & \cdots & -\beta' & 0 & \cdots \\
0 & I_n & \cdots & I_n & -I_n & \cdots & -I_n & 0 \\
0 & 0 & \cdots & (I_n,0) & (I_n,0) & \cdots & (I_n,0) & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & (I_n,0) & (I_n,0) & \cdots & (I_n,0) & 0 \\
\end{pmatrix}.
\]

The matrix \( D_{b,c} \in \mathbb{R}^{\tilde{n} \times Nn} \) follows from \( D_b \), the matrix \( c \) and the selection of the corresponding rows resulting in \( \tilde{y}_t \).

3. Case: Mixed Case: Consider the case where we have slow stock as well as slow flow variables:

\[
w_t = \begin{pmatrix}
   c_1 y_t^s \\
   \vdots \\
   c_N y_{t-N+1}^s 
\end{pmatrix}.
\]

Now the problem is that \( \beta' w_t \) does in general not need to be stationary: For example, if \( (y_t) \) is a two-dimensional process and \( c_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), then \( \beta' c_2 y_{t-1}^s \) is not stationary. However, in special cases, such as separate cointegrating relationships among the slow flow variables only, or among the slow stock...
and fast variables only, etc. we can proceed similarly to the flow case. In the following we only consider the stock or the flow case.

The problem with the systems considered above is that the inputs $\nu^b_t$ are not the innovations of $\tilde{y}_t$. However, from the stable miniphase spectral factorisation, we only obtain transfer functions corresponding to systems in innovation form. The following Theorem 2 is the first step for obtaining a canonical state-space representation for the blocked process. A minimal state-space representation is called “canonical” if its parameters are uniquely determined from the transfer function. We introduce the following notation for specific subspaces of $L^2(\Omega,\mathcal{A},P)$, the space of square integrable random variables on the underlying probability space $(\Omega,\mathcal{A},P)$:

\[
\mathbb{H}(y) := \overline{\text{sp}}(y_{it} | t \in \mathbb{Z}, i = 1, ..., n)
\]

\[
\mathbb{H}_t(y) := \overline{\text{sp}}(y_{it} | s \leq t, i = 1, ..., n)
\]

\[
\mathbb{H}_N(y) := \overline{\text{sp}}(y_{it} | t \in N\mathbb{Z}, i = 1, ..., n)
\]

where $\overline{\text{sp}}(\cdot)$ denotes the closed span and $\text{proj}(v | U)$ the projection of $v$ on a closed subspace $U$ of $L^2$.

**Theorem 2**

Suppose that Assumption 1 and [Assumptions 1,2] from Chambers (2020) hold. Consider the blocked process $(\tilde{y}_t)_{t \in \mathbb{N}\mathbb{Z}}$ and set

\[
s_{t-N+1} := \text{proj}(x_{t-N+1} | N\mathbb{H}_{t-N}(\tilde{y}))
\]

\[
\tilde{\nu}_t := \tilde{y}_t - \text{proj}(\tilde{y}_t | N\mathbb{H}_{t-N}(\tilde{y})).
\]

Then there exists $\tilde{B}_c \in \mathbb{R}^{np \times n}$ s.t.

\[
s_{t+1} = A_{b,c}s_{t-N+1} + \tilde{B}_c\tilde{\nu}_t \tag{18}
\]

\[
\tilde{y}_t = C_{b,c}s_{t-N+1} + \tilde{\nu}_t \tag{19}
\]

is a miniphase and stable state-space representation of $(\tilde{y}_t)_{t \in \mathbb{N}\mathbb{Z}}$, i.e. it is in innovation form.

We call the representation in (18), (19) canonical projection form (CPF) of $\tilde{y}_t$. Note that the CPF provides an algorithm for computing the transfer function $\tilde{k}(\tilde{z})$ of $(\tilde{y}_t)_{t \in \mathbb{N}\mathbb{Z}}$ which corresponds to the Wold representation, where $\tilde{z} := z^N$.

Next we show that the system (18) and (19) is observable and controllable and therefore minimal (see, e.g., Hannan and Deistler, 2012, Theorem 2.3.3) for all $\theta \in \Theta_I$. 

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Theorem 3
For \( \theta \in \Theta_I \), the system (18) and (19) is minimal.

By Theorem 3, we know that the McMillan degree of the transferfunction of the blocked process \((\tilde{y}_t)_{t \in \mathbb{N}_Z}\) corresponding to an underlying high-frequency VECM is \( m = r + n(p - 1) \). This will be used in the proof of the subsequent Theorem 4, where we can relate an arbitrary minimal realisation \((\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})\) of the transfer function \( \bar{k}(\tilde{z}) = (\bar{C}_{b,c} (I_m \tilde{z}^{-1} - \bar{A}_{b,c}) \bar{B}_{b,c} + I_\tilde{b}) \) (where \( \tilde{z} := z^N \)) to the CPF \((A_{b,c}, \tilde{B}_c, C_{b,c})\). The minimal realisation \((\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})\) can be either obtained by the spectral factorisation and e.g. the eche-lon realisation from the Hankel matrix of the transfer function (see e.g. Hannan and Deistler, 2012, Theorem 2.6.2) or directly from the Hankel matrix of the observed second moments (see, e.g. Anderson et al., 2016a, Proof of Theorem 8). In the next step we relate the CPF to the underlying VECM/VAR - exploiting the fact that the parameters \( \theta \) of the underlying VECM reappear in the CPF.

Finally, we show that the parameters of the high frequency system are generically identifiable from the observed second moments, i.e. from \( \tilde{\gamma} \).

Theorem 4 (Generic-Identifiability: Flow or Stock Case)
Let \( p \geq N + 2 \) for stock case or \( p \geq 2N + 1 \) for the flow case. Then,

1. The mapping, \( \pi \) in equation (8) which attaches the second moments of \((\tilde{y}_t)_{t \in \mathbb{N}_Z}\) to the high frequency parameters \( \theta \) is injective on \( \Theta_I \).

2. Its inverse, \( \pi^{-1} \), is continuous on \( \pi(\Theta_I) \).

Since by Theorem 1, \( \Theta_I \) is a generic subset of \( \Theta \), we say that \( \theta \) is generically identifiable from the observed autocovariance function \( \tilde{\gamma} \). Theorems 3 and 4 imply that the representation (18), (19) is indeed canonical on \( \pi(\Theta_I) \). Since the second moments of \((\tilde{y}_t)\) can be consistently estimated from the data under mild conditions, by the continuity of \( \pi^{-1} \) it follows that we have a consistent estimator for \( \theta \). The mapping \( \pi^{-1} \) is also called realisation procedure, since we realise the system parameters from the external characteristics of the data, i.e. the second moments, the spectrum or the transfer function respectively.

Finally, we consider the question whether there exists \( \pi^{-1}(\pi(\Theta_I)) = \Theta_I \):

Theorem 5
For all \( \theta_{-I} \in \Theta \setminus \Theta_I \) there exists no \( \theta \in \Theta_I \) s.t. \( \pi(\theta_{-I}) = \pi(\theta) = \tilde{\gamma} \).
4 Deterministic Terms

Consider the VECM with deterministic terms

$$\Delta y_t = \mu_0 + \mu_1 t + \Pi y_{t-1} + \sum_{j=1}^{p-1} \Phi_j \Delta y_{t-j} + \nu_t , \quad \nu_t \sim WN(\Sigma_{\nu}) .$$

For this model, we are able to obtain $\beta$ from mixed frequency data (see Chambers, 2020). For $(\Delta y_t)_{t \in \mathbb{Z}}$ and $(\beta' y_t)_{t \in \mathbb{Z}}$ we can compute the expectations: By considering the Granger-Representation-Theorem for the solution on $\mathbb{Z}$ in Bauer and Wagner (2012) [equation (26)], (see also Johansen, 1995, Theorem 4.3), we obtain (see also Hansen and Johansen, 1998, Exercise 4.5)

$$E \Delta y_t = C(\mu_0 + \mu_1 t) + M_c \mu_1$$  \hspace{1cm} (20)

with

$$C := \beta_1 \left( \alpha' \left( I_n - \sum_{j=1}^{p-1} \Phi_j \right) \beta_{1} \right)^{-1} \alpha'$$  \hspace{1cm} (21)

and $M_c$ is the limit of the stable part of the impulse responses in the particular solution defined in Bauer and Wagner (2012) [equation (8)] (this is $k_\bullet(1)$ in Bauer and Wagner (2012) equation (26)).

Let $\bar{\alpha}' = (\alpha' \alpha)^{-1} \alpha'$, then

$$E \beta' y_{t-1} = \bar{\alpha}' \left[ C(\mu_0 + \mu_1 t) + M_c \mu_1 - \mu_0 - \mu_1 t \right. \left. - \sum_{j=1}^{p-1} \Phi_j \left( C(\mu_0 + \mu_1 (t-j)) + M_c \mu_1 \right) \right] .$$  \hspace{1cm} (22)

Now, both moments in equations (20) and (22) are observable (or consistently estimable from observed data) for the stock as well as for the flow case. For $N > 1$ this is also straightforward. With these equations we can remove the deterministic terms from the VECM and reduce the case of deterministic terms to the case without deterministic terms discussed above. Note that by means of $E \beta' y_{t-1}$ and $E \Delta y_t$ we can also identify the following five cases of deterministic terms which are described in Johansen (1995)[page 81], see also Johansen (2004):

Case $H_2(r)$: $(\mu_0 = \mu_1 = 0)$ No deterministic terms. This case was shown above.
Case \( H_1(r) \): (\( \mu_1 = 0 \) and \( \mu_0 \neq 0 \)) We have a linear trend in \( \mathbb{E}y_t \), and a constant in \( \mathbb{E}\Delta y_t \), and constant in \( \mathbb{E}\beta'y_{t-1} \). Let the matrix \( S_C \) select \( n - r \) basis rows of \( C \) (e.g., \( S_C = \alpha'_\perp \) can be used). In this case it holds that

\[
\mathbb{E} \begin{pmatrix} \beta'y_{t-1} \\ S_C \Delta y_t \end{pmatrix} = \begin{pmatrix} \alpha'((I_n - \sum_{j=1}^{p-1} \Phi_j)C - I_n) \\ S_C C \end{pmatrix} \mu_0 ,
\]

where the matrix on the LHS before \( \mu_0 \) is of rank \( n \) since the first rowblock has rank \( r \) and \( C \) has rank \( n-r \) and both are mutually orthogonal as \( \alpha'_\perp \alpha = 0 \) and therefore (with \( \alpha'_\perp \) the last term of \( C \))

\[
\alpha'_\perp((I_n - \sum_{j=1}^{p-1} \Phi_j)C - I_n)' \alpha = 0 .
\]

From this we can compute \( \mu_0 \).

Case \( H^*_r(r) \): (\( \mu_1 = 0, \mu_0 \neq 0 \) and \( \alpha'_\perp \mu_0 = 0 \)) no linear trend but constant in \( \mathbb{E}y_t \), constant in \( \mathbb{E}\beta'y_{t-1} \), and \( \mathbb{E}\Delta y_t = 0 \). This results in \( r \) parameters in the cointegration equation. In this case we write \( \mu_t = \alpha \rho_0 \), where \( \rho_0 \in \mathbb{R}^r \). Note that \( C \mu_t = C \alpha \rho_0 = 0 \), which results in

\[
\mathbb{E} \begin{pmatrix} \beta'y_{t-1} \\ \Delta y_t \end{pmatrix} = \begin{pmatrix} \alpha'((I_n - \sum_{j=1}^{p-1} \Phi_j)C - I_n) \\ C \end{pmatrix} \alpha \rho_0
\]

\[
= \begin{pmatrix} -\alpha' \rho_0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\rho_0 \\ 0 \end{pmatrix} .
\]

This allows to uniquely retrieve the unknown parameter \( \rho_0 \) from \( \mathbb{E} \beta'y_{t-1} \).

Case \( H(r) \): (\( \mu_0 \neq 0 \) and \( \mu_1 \neq 0 \)) quadratic trend in \( \mathbb{E}y_t \), linear trend in \( \mathbb{E}\beta'y_{t-1} \), linear trend in \( \mathbb{E}\Delta y_t \). For this case we get the system of linear equations

\[
\mathbb{E} \begin{pmatrix} \beta'y_{t-1} \\ S_C \Delta y_t \\ \beta'y_{t-2} \\ S_C \Delta y_{t-1} \end{pmatrix} = M_\mu \begin{pmatrix} \mu_0 \\ \mu_1 \end{pmatrix} ,
\]

where

\[
M_\mu := \begin{pmatrix} \alpha'(C - I_n - \sum_{j=1}^{p-1} \Phi_j) \\ S_C C \\ \alpha'[(Ct + M_t - tI_n - \sum_{j=1}^{p-1} \Phi_j(C(t-j) - M_t))] \\ S_C (tC + M_t) \\ \alpha'(C - I_n - \sum_{j=1}^{p-1} \Phi_j) \\ S_C C \\ \alpha'[(C(t-1) + M_t - (t-1)I_n - \sum_{j=1}^{p-1} \Phi_j(C(t-1-j) - M_t))] \\ S_C (C(t-1)C + M_t) \end{pmatrix}
\]

\[
= \begin{pmatrix} M_{\mu 11} & M_{\mu 12} \\ M_{\mu 21} & M_{\mu 22} \end{pmatrix} .
\]
The matrix $S_C$ selects $n - r$ basis rows of $C$ (e.g., $S_C = \alpha_i'$ can be used) and $M_{\mu_{11}} = M_{\mu_{21}}$. The matrix $M_{\mu_{11}}$ is of full column rank $n$ by the above calculations where we set $\mu_1 = 0$. The determinant of a blocked matrix is given by

$$
\det M_\mu = \det M_{\mu_{11}} \det \left( M_{\mu_{22}} - M_{\mu_{21}} M_{\mu_{11}}^{-1} M_{\mu_{12}} \right) = \det M_{\mu_{11}} \det (M_{\mu_{22}} - M_{\mu_{12}}).
$$

Note that $M_{\mu_{22}} - M_{\mu_{12}} = \left( \tilde{\alpha}' (C - I_n - \sum_{j=1}^{p-1} C \Phi_j) \right) = M_{\mu_{11}}$. Therefore, $\det M_\mu = (\det M_{\mu_{11}})^2 \neq 0$ since we have already shown that $M_{\mu_{11}}$ is of full rank $n$. This allows to uniquely solve for $\mu_0$ and $\mu_1$.

**Case $H^*(r)$:** $(\mu_0 \neq 0, \mu_1 \neq 0, \alpha_i' \mu_1 = 0)$ linear trend in $\mathbb{E} y_t$, linear trend in $\mathbb{E} \beta' y_t$, and constant in $\mathbb{E} \Delta y_t$. $\mu_0$ contains $n$ free parameters, while $\mu_t = \mu_0 + \alpha \rho_1 t$, where $\rho_1 \in \mathbb{R}$. Hence, $C \mu_t = C \mu_0 + C \alpha \rho_1 t = C \mu_0$. This results in

$$
\mathbb{E} \begin{pmatrix} \beta' y_{t-1} \\ S_C \Delta y_t \\ \beta' y_{t-2} \end{pmatrix} = [M_\mu]_{1:r+n,1:2(r+m)} \begin{pmatrix} \mu_0 \\ \alpha \rho_1 \end{pmatrix}
= [M_\mu]_{1:r+n,1:2(r+n)} \begin{pmatrix} I_n & 0_{n \times r} \\ 0_{n \times n} & \alpha \end{pmatrix} \begin{pmatrix} \mu_0 \\ \rho_1 \end{pmatrix}.
$$

Since $M_\mu$ has full rank $2n$, the matrix $M_{\mu H^*_1(r)}$ has rank $r + n$. This allows to uniquely obtain $\mu_0$ and $\rho_1$.

## 5 Conclusion

In this paper, we generalise the results on identifiability from mixed frequency data in Anderson et al. (2016a,b) obtained for stationary VAR-systems to the case of unit-roots and cointegrating relationships. As is well known these systems have also a vector error correction representation. The corresponding parameterspaces are homeomorphic.

We commence from a solution of the (unstable) VAR system on the integers $\mathbb{Z}$ (see Bauer and Wagner, 2012, for the existence of such a solution). Then we take differences at lag $N$ (which is the sampling rate of the slow/aggregated
process) and stack these to what we call the “blocked process”. In addition, the blocked process also contains the stationary process $\beta' y_t$, where $\beta$ is the matrix of cointegrating relationships. This matrix is identified from mixed frequency data as already shown in Chambers (2020). This blocked process is stationary and contains all relevant differences of the observations.

The contribution of this paper can be seen as an extension of the results in Chambers (2020), by proving that also the remaining parameters of the vector error correction model (i.e. besides $\beta$) are (generically) identified from mixed frequency observations.

The identifiability proof consists of two steps: In the first step, we derive a state-space representation of the blocked process (“the canonical projection form”) which is minimal, in innovation form (both, for the stock and the flow case) and unique. In the second step, we derive an algorithm, that retrieves the parameters of the underlying high frequency system from the parameters of the canonical projection form.

We show that the conditions (Assumption 2) which are sufficient for identifiability are generic in the parameterspace. This is more intricate than in the stationary case, since the parameterspace is not an open subspace of the Euclidean space, due to the fact that we allow for unit roots. Since the VECM and the VAR parameterspaces are homeomorphic, the genericity result holds for both.

Finally, we show that all common cases of deterministic terms in the VECM can be reduced to the case of non-deterministic terms.

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### A Proof of Theorem 1

**Proof.** 1. \( (G \cap \Theta_1 \text{ is dense.}) \)

Suppose that \( \theta_0 \in \Theta_1 \) does not satisfy at least one of the identifiability conditions. Let \( \epsilon > 0 \), we show that there exists \( \theta \in G \cap \Theta_1 \) s.t. \( \| \theta - \theta_0 \| < \epsilon \) by perturbing the eigenvalues / eigenvectors of the companion matrix \( A \) corresponding to \( \theta_0 \).
For this we define a mapping \( f_{\theta_0} \) that maps \( \mathcal{A} \) to a companion matrix \( \mathcal{A}^* \) with perturbed eigenvalues and eigenvectors such that \( \theta = \text{vec} \left( \mathcal{A}^*_1 \cdots \mathcal{A}^*_p \right) \) is in \( G \cap \Theta_1 \):

1. Compute the Jordan decomposition of \( \mathcal{A} = Q \Lambda Q^{-1} \).

2. Perturb the eigenvalues:
   \[
   \bar{\mathcal{A}}^* = Q \left( \Lambda + \text{diag} \left( 0, \ldots, 0, \xi_1, \ldots, \xi_{np-(n-r)} \right) \right) Q^{-1}.
   \]

3. We transform \( \bar{\mathcal{A}}^* \) to a similar matrix \( \mathcal{A}^* \) that has the companion structure by using the procedure from Anderson et al. (2016a):
   \[
   \mathcal{A}^* = T \bar{\mathcal{A}}^* T^{-1}, \quad \text{hence} \quad \mathcal{A}^* T = \begin{pmatrix} \mathcal{A}^*_1 & \cdots & \mathcal{A}^*_p \\ I_n & 0 & \vdots \\ \vdots & \ddots & \vdots \\ I_n & 0 & \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_p \end{pmatrix} = T \bar{\mathcal{A}}^*,
   \]

where \( T_j \) for \( j = 1, \ldots, p \) are the \( n \times np \) rowblocks of \( T \). Now we set \( T_1 = [I_n \ 0 \ \cdots \ 0] \) and solve the equation above:

\[
\mathcal{A}^*_1 T_1 + \cdots + \mathcal{A}^*_p T_p = T_1 \bar{\mathcal{A}}^*, \quad T_1 = T_2 \bar{\mathcal{A}}^*, \quad \ldots, \quad T_{p-1} = T_p \bar{\mathcal{A}}^*,
\]

which yields

\[
T_j = T_{j-1} \bar{\mathcal{A}}^{*-1} \quad \text{for} \quad j = 2, \ldots, p.
\]

Clearly, the mapping \( f_{\theta_0} : \xi \mapsto \mathcal{A}^* \mapsto \theta \) for \( \xi = (\xi_{n-r+1}, \ldots, \xi_{np})' \in \mathbb{R}^{np-(n-r)} \) is continuous at \( \theta_0 \) and \( f_{\theta_0}(0) = \theta_0 \) (as in this case \( T = I_{np} \)). So for the \( \varepsilon \)-neighborhood around \( \theta_0 \) denoted by \( U_\varepsilon(\theta_0) \) there exists a \( \delta > 0 \), s.t. for all \( \xi \in U_\delta(0) \) we have \( f_{\theta_0}(\xi) \in U_\varepsilon(\theta_0) \), where \( U_\delta(0) \) is the open \( \delta \)-neighborhood around 0 in \( \mathbb{R}^{np-(n-r)} \).

Now, \( \lambda^* := (1, \ldots, 1, \lambda_{n-r+1} + \xi_1, \ldots, \lambda_{np} + \xi_{np-(n-r)}) \) are the eigenvalues of \( \mathcal{A}^* \) because they are the zeros of the characteristic polynomial of \( \bar{\Lambda} \) in equation (23) from which we obtain \( \mathcal{A}^* \) by similarity transformation with \( T Q \). For any \( \delta > 0 \), we can find a \( \xi \in U_\delta(0) \) s.t. the corresponding eigenvalues \( \lambda^* \) of \( \mathcal{A}^* \) satisfy the conditions (I1), (I2) and (I3).
We have to ensure that the image \( f_{\theta_0}(\xi) \) is real valued. Since \( A \) is real valued, for any complex eigenvalue \( z = a + ib \in \mathbb{C} \setminus \mathbb{R} \), the conjugate \( \bar{z} = a - ib \) is also an eigenvalue of \( A \). If the algebraic multiplicity of \( z \) is larger than 1, \( z \) has to be perturbed. As is easily shown, if we add to \( z \) and \( \bar{z} \) the same small real number, the resulting \( \bar{A}^* \) (and therefore also \( A^* \)) is again real valued. Thus, we found \( \theta \in G \) close to \( \theta_0 \) and are left with checking whether \( \theta \) is also in \( \Theta \), (C3) is trivial.

For (C1), note that, still \( n - r \) eigenvalues of \( A^* \) equal unity which ensures that \( \text{rk} \Pi = r \). Applying the procedures described above, we obtain the vector error correction representation corresponding to \( f_{\theta_0}(\xi) = \theta \), say \( (\alpha(\xi), \beta(\xi), \Phi_1(\xi), ..., \Phi_p(\xi)) \), and see that

\[
g(\theta) = \det \alpha(\xi)'(I_n - \sum_{j=1}^{p-1} \Phi_j(\xi))\beta(\xi)
\]

is a continuous at \( \theta = \theta_0 \). We know that \( g(\theta_0) \neq 0 \) since \( \theta_0 \in \Theta_1 \). So there exists \( \varepsilon_3 > 0 \) s.t. the neighbourhood \( U_{\varepsilon_3}(g(\theta_0)) \) is bounded away from zero. By continuity there exists \( \varepsilon_2 > 0 \) s.t. for all \( \theta \in U_{\varepsilon_3}(\theta_0) \), we have \( g(\theta) \in U_{\varepsilon_2}(g(\theta_0)) \). For the same reasons as above we can find suitable \( \xi \) s.t. \( f_{\theta_0}(\xi) = \theta \in U_{\varepsilon_2}(\theta_0) \cap U_{\varepsilon_3}(\theta_0) \). Hence \( \Theta \cap G \) is dense in \( \Theta \).

2. \( G \cap \Theta_1 \) is open in \( (\Theta_1, d) \),

where \( d \) denotes the Euclidean metric. Suppose now for \( \theta^* \in G \cap \Theta_1 \), we have to show that there exists \( \varepsilon > 0 \) s.t. \( U_{\varepsilon}(\theta_0) \subset G \cap \Theta_1 \). The eigenvalues are the zeros of the characteristic polynomial of \( A \) and therefore continuous functions at \( \theta^* \) (since as is well known, the zeros of any polynomial are continuous function of its coefficients). So the mapping

\[
e(\theta) \mapsto A \mapsto (\lambda_{n-r+1} \cdots \lambda_n) = \lambda
\]

is continuous in \( \theta^* \). Clearly there is an open neighbourhood \( U \subset \mathbb{C}^{np-(n-r)} \) of \( \lambda^* = e(\theta^*) \) s.t. for all \( \lambda \in U \) the corresponding spectrum \( (1 \cdots 1 \lambda)' \) satisfies the identifiability conditions. The pre-image \( e^{-1}(U) \subset G \) is an open neighborhood of \( \theta_0 \). Analogously to the arguments applied above, we can establish (C2).

\[\text{XXX cite Bauer Wagner}\]
B Proof of Theorem 2

Proof. This follows from transforming a state-space system into prediction error form. See Hannan and Deistler (2012)[chapter 1] and Gersing and Deistler (2021). From Johansen (1995)[Proof of Theorem 4.2] it follows that the largest eigenvalue of \( A \) is in modulus smaller than one. Hence the system is stable. The linear expansion of the transfer function for a stable system is already the Wold representation as the inputs \( \nu_t \) are the innovations. Hence, the system is also miniphase (see, e.g., Deistler and Scherrer, 2018, Chapters 2 and 7.3).

C Proof of Theorem 3

Proof. By Johansen (1995)[Proof of Theorem 4.2] it follows that the eigenvalues of modulus smaller than 1 are the same, for \( A \) and \( A \).

1.1 Observability for the Stock Case: We use the PBH-Test (see, e.g., Kailath, 1980, Section 2.4.3) to prove that the pair \((A_b,C_b)\) is generically observable (note that the observability of \((A_b,C_b)\) also implies the observability of \((A_{b,c},C_{b,c})\) since \(c\) is regular). For this, note that the eigenvectors of \( A_b \) are the same as the eigenvectors of \( A \). Let \( \lambda \) be an eigenvalue of \( A \) and \( q = (q_\beta' \ q_1' \cdots \ q_{p-1}') \) the corresponding eigenvector. We write

\[
A q = \begin{bmatrix}
\beta' \alpha + I_r & \beta' \Phi_1 & \cdots & \beta' \Phi_{p-1} \\
\alpha & \Phi_1 & \cdots & \Phi_{p-1} \\
0 & I_n & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & I_n & 0
\end{bmatrix}
\begin{bmatrix}
q_\beta \\
q_1 \\
\vdots \\
q_{p-1}
\end{bmatrix} = \lambda
\begin{bmatrix}
q_\beta \\
q_1 \\
\vdots \\
q_{p-1}
\end{bmatrix},
\]

where \( q_\beta \) is \( r \times 1 \) and \( q_i \) is \( n \times 1 \) for \( i = 1, \ldots, p - 1 \). From this, we obtain the relations

\[
(\beta' \alpha + I_r)q_\beta + \sum_{i=1}^{p-1} \beta' \Phi_i q_i = \lambda q_\beta \tag{24}
\]

\[
\alpha q_\beta + \sum_{i=1}^{p-1} \Phi_i q_i = \lambda q_1 \tag{25}
\]

\[
q_i = \lambda q_{i+1}, \quad i = 1, \ldots, p - 2. \tag{26}
\]
Since $A$ is of full rank, $\lambda \neq 0$ and $q_1 = 0$ imply $q = 0$, which is a contradiction (noting that $\alpha$ has rank $r$). Now we look at

$$C_bq = \begin{pmatrix} I_r & 0 & \cdots & 0 \\ 0 & I_n & \cdots & 0 \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & I_n_f \end{pmatrix} \begin{pmatrix} \lambda^N q_\beta \\ \lambda^N q_1 \\ \vdots \\ \lambda^N q_{p-1} \end{pmatrix},$$

which is not equal to zero. If, for example,

$$\lambda^N q_1 + \cdots + \lambda^N q_N = \lambda^N q_1 + \lambda^{N-1} q_1 + \cdots + q_1 = (1 + \lambda + \cdots + \lambda^N)q_1 \neq 0 \iff (1 + \lambda + \cdots + \lambda^N) \neq 0,$$

which is generically the case (see Assumption 2). Recall that by $q$ we denote eigenvectors of $A$ and by $v$ eigenvectors of $A$, where both correspond to the same eigenvalue $|\lambda| < 1$. In Lemma 6, we show that

$$q_\beta = \frac{\lambda}{\lambda - 1} \beta' q_1 = \beta' v_1,$$

so if we suppose that $v_1$ is not in the right kernel of $\beta'$, we also get $C_b q \neq 0$.

1.2 *Observability for the Flow Case:* The first part of the proof is analogous to the stock case. It remains to show that there exists no eigenvector that is in the right kernel of $C_b$, where $C_b$ is now defined in (17). Now, analogously to the procedure in (27) we obtain, that an eigenvector of $A_b$ is not in the right kernel of $C_b$ if e.g.

$$\lambda^N q_1 + \cdots + \lambda^N q_N - \lambda^N q_{N+1} - \cdots - \lambda^N q_{2N} = \lambda^N q_1 + \lambda^{N-1} q_1 + \cdots + \lambda q_1 - q_1 - \cdots - \lambda^{-N+1} q_1 = \lambda^{N-1}(-1 - \lambda - \cdots - \lambda^{N-1} + \lambda^N + \cdots + \lambda^{2N-1})q_1 \neq 0 \iff (-1 - \lambda - \cdots - \lambda^{N-1} + \lambda^N + \cdots + \lambda^{2N-1}) \neq 0,$$

Also the second part is similar to the stock case: By Lemma 6, $q_\beta = \frac{\lambda}{\lambda - 1} \beta' q_1 = \beta' v_1$. Assume that $v_1$ is not in the right kernel of $\beta'$ (as already done in the
stock case). In addition, by considering the first $r$ rows of the matrix $c$ for the flow case, provided in (15), we get

$$NI_rq_\beta - (N - 1)\beta'q_1 - (N - 2)\beta'q_2 - \cdots - 2\beta'q_{N-2} - \beta'q_{N-1}$$

$$= NI_r\frac{\lambda}{\lambda - 1}\beta'q_1 - \frac{(N - 1)}{\lambda - 1}\beta'q_1 - \frac{(N - 2)}{\lambda - 1}\beta'q_1 - \cdots - \frac{2}{\lambda - 1}\beta'q_1 - \frac{1}{\lambda - 1}\beta'q_1$$

$$= \left( N\frac{\lambda}{\lambda - 1} - \frac{(N - 1)}{\lambda - 1} - \frac{(N - 2)}{\lambda - 1} - \cdots - \frac{2}{\lambda - 1} - \frac{1}{\lambda - 1} \right) \beta'q_1$$

$$= \frac{1}{\lambda - 1}\left( N\frac{\lambda}{\lambda - 1} - (N - 1)\lambda^{-1} - (N - 2)\lambda^{-2} - \cdots - 2\lambda - 1 \right) \beta'q_1 .$$

Note that $\lambda \neq 1$ and $\lambda \neq 0$ by the model assumptions (recall that by Johansen (1995)[Proof of Theorem 4.2] it follows that the eigenvalues of modulus smaller than 1 are the same, for $A$ and $A$). Hence, if $\nu_1$ is not in the right kernel of $\beta'$ and $N\frac{\lambda}{\lambda - 1} - (N - 1)\lambda^{-1} - (N - 2)\lambda^{-2} - \cdots - 2\lambda - 1 \neq 0$ we also get that $\tilde{C}_b\beta \neq 0$ for the flow case.

2. **Controllability:** It is enough to show that the matrix $E x_{t+1} \left( \tilde{y}_t' \ \tilde{y}_{t-N}' \cdots \right)'$ has full rank. For $k$ sufficiently large, we have

$$x_{t+N-1} = A_{b,c}^{k-1} x_{t-kN+1} + \sum_{j=0}^{k-1} A_{b,c}^j B_{b,c} \nu_{t-N-jN}$$

$$\Delta_Ny_{t-kN} = \left[ \begin{array}{cccc} I_n & 0 & \cdots & 0 \end{array} \right] x_{t-kN+1}$$

$$\mathbb{E} \Delta_Ny_{t-kN} x_{t-kN+1}' = \mathbb{E} \left\{ \Delta_Ny_{t-kN+1} x_{t-kN+1}' A_{b,c}^{k-1} + \Delta_Ny_{t-kN+1} \left( \sum_{j=0}^{k-2} A_{b,c}^j B_{b,c} \nu_{t-N-jN} \right)' \right\}$$

$$= \Delta_Ny_{t-kN} e^{G_{\Gamma_{rp,c}} A_{b,c}^{k-1}} .$$

Therefore

$$\mathbb{E} x_{t-N+1} \left( \Delta_Ny_{t-kN} \ \Delta_Ny_{t-(k+1)N} \ \cdots \ \Delta_Ny_{t-(k+p-1)N} \right)$$

$$= A_{b,c}^{k-1} \left[ \Gamma_{rp,c} S_{\Delta_Ny} A_{b,c} \Gamma_{rp,c} S_{\Delta_Ny} \cdots A_{b,c}^{p-1} \Gamma_{rp,c} S_{\Delta_Ny} \right] ,$$

which has full rank if $\Gamma_{rp} > 0$ as follows from the proof of Theorem 7 in Anderson et al. (2016a). By Hannan and Deistler (2012)[Theorem 2.3.3] controllability and observability imply that the system is minimal. ■
Lemma 6
Suppose the Assumption 1 and 2 hold. Then equation (28) holds.

Proof. Substracting $\beta'$ times (25) from (24), we obtain

$$q_\beta = \lambda q_\beta - \lambda \beta' q_1$$

such that

$$q_\beta = \frac{\lambda}{\lambda - 1} \beta' q_1.$$ 

Next, we consider the eigenvector $v = (v_1' \; \cdots \; v_p')'$ of $A$ corresponding to $\lambda$ (recall that eigenvalues in modulus smaller that one of $A$ and $\mathcal{A}$ are the same). By using the relations of the parameters between the VECM and VAR representation, we get

$$\lambda v_1 = (I_n + \alpha \beta') v_1 + \Phi_1 (v_1 - v_2) + \Phi_2 (v_2 - v_3) + \cdots + \Phi_{p-1} (v_{p-1} - v_1)$$

$$\alpha \beta' v_1 + \Phi_1 \frac{\lambda - 1}{\lambda} v_1 + \Phi_2 \frac{\lambda - 1}{\lambda^2} v_1 + \cdots \frac{\lambda - 1}{\lambda^{p-1}} v_1 = (\lambda - 1)v_1,$$

where the last relation follows from $v_i = \lambda v_{i+1}$ for $i = 1, \ldots, p-1$, which results by the companion structure of $A$. Now, we see that $q_1 = ((\lambda - 1)/\lambda)v_1$ solves (24) and (25). 

\[\boxright\]

D Proof of Theorem 4

Proof. Consider the stable, miniphase spectral factor $\tilde{k}(\bar{\tilde{z}})$, $\bar{\tilde{z}} := z^N$, corresponding to the Wold representation of $(\tilde{y}_t)_{t \in \mathbb{N}}$.

Step 1: We obtain an arbitrary minimal realisation $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$ of $\tilde{k}(\bar{\tilde{z}})$, e.g. by taking the echelon form, see Hannan and Deistler (2012)[Thm 2.5.2].

Step 2: (Obtain eigenvalues $\Lambda = diag(\lambda_1, \ldots, \lambda_{r+n(p-1)})$ and a linear combination of the eigenvectors of $A$, denoted $q_i$, from $A_{b,c}$).

By, e.g., Hannan and Deistler (2012)[Theorem 2.3.4] the parameter matrices of minimal systems relate via $\bar{A}_{b,c} = T^{-1}A_{b,c}T$, $\bar{C}_{b,c} = C_{b,c}T$ and $\bar{B}_{b,c} = T^{-1}B_{b,c}$, where $T$ is a regular matrix.

Since $\mathcal{A}$ is assumed to be diagonalizable (Assumption 2), $A$ can be expressed by means of $A = Q\Lambda Q^{-1}$, where $\Lambda = diag(\lambda_1, \ldots, \lambda_{r+n(p-1)})$ is the diagonal matrix of eigenvalues of $A$ and $Q = (q_1, \ldots, q_{r+n(p-1)})$ contains the eigenvectors. $A_b = A^N$, $C_{b,c} = C_b c^{-1}$ and $A_{b,c} = cA_b c^{-1}$, such that $\bar{A}_{b,c} = T^{-1}A_{b,c}T = T^{-1}cA_b c^{-1}T = (T^{-1}cQ) \Lambda^N (T^{-1}cQ)^{-1}$, $\bar{C}_{b,c} = C_{b,c}T = C_b c^{-1}T$.

By the eigen-decomposition of $\bar{A}_{b,c}$, we obtain $(T^{-1}cQ)$ and $\Lambda^N$. In addition,
\((0_{n \times r}, 0_{n \times n}, I_n 0 \ldots 0) A^2 = (0_{n \times r}, I_n 0 \ldots 0) A\) by the companion structure of \(A\). Hence by (14), we have

\[\bar{C}_{b,c} T^{-1} c Q = C_b c^{-1} T T^{-1} c Q = C_b Q = \begin{pmatrix} (I_r & 0 & \cdots & 0 & A^N \\ S_{n_f}^{(1)} A_N & S_{n_f}^{(1)} A_N & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ S_{n_f}^{(1)} A_N^{-1} & S_{n_f}^{(1)} A_N^{-2} & \cdots & \cdots & \cdots \\ S_{n_f}^{(1)}& S_{n_f}^{(1)}& \cdots & \cdots & I_n \end{pmatrix} Q. \tag{29}\]

Now we look at the last two rowblocks of \(C_b\) with the eigenvectors \(q_i, 1 \leq i \leq m\). From assumption 2 (14), it follows that the eigenvectors of \(A\) are the same as the eigenvectors of \(A^2\) (also as \(A^N\)) (see Felsenstein, 2014, Lemma 3.2.1), therefore we have

\[S_{n_f}^{(1)} A^2 q_i = S_{n_f}^{(1)} \lambda_i^2 q_i, \quad S_{n_f}^{(1)} A q_i = S_{n_f}^{(1)} \lambda_i q_i, \tag{30}\]

and we can compute all eigenvalues not equal to one since \(S_{n_f}^{(1)} q_i \neq 0\) by assumption 2 (16). The flow-case is analogous. Summing up, from \(\bar{A}_{b,c}\) we are able to obtain \(T^{-1} c Q, \Lambda^N = \text{diag}(\lambda_1^N, \ldots, \lambda_{r+n(p-1)}^N)\) and \(\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{r+n(p-1)})\).

**Step 3:** (relate \(c^{-1} T\) to \(T\)) Let

\[A = \begin{pmatrix} A_\beta \\ A_1 \\ \vdots \\ A_{p-1} \end{pmatrix}, \quad T = \begin{pmatrix} T_\beta \\ T_1 \\ \vdots \\ T_{p-1} \end{pmatrix}, \quad \text{and} \quad R := c^{-1} T = \begin{pmatrix} R_\beta \\ R_1 \\ \vdots \\ R_{p-1} \end{pmatrix}. \tag{31}\]

Observe that for the stock case (the flowcase is treated analogously)

\[\bar{C}_{b,c} \bar{A}_{b,c}^{-1} = \left(\begin{pmatrix} I_r & c_{\beta 1} & \cdots & c_{\beta N-1} & 0 & \cdots & 0 \\ 0 & I_n & I_n & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & I_n & I_n & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} A^N \right) c^{-1} T \cdot T^{-1} c A^{-N} e^{-1} T \cdot c_{b,c}^{-1} \bar{A}_{b,c}^{-1} = \begin{pmatrix} I_r & c_{\beta 1} & \cdots & c_{\beta N-1} & 0 & \cdots & 0 \\ 0 & I_n & I_n & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 & I_n & I_n & \cdots \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} c^{-1} T = \begin{pmatrix} \lfloor c \rfloor_{(1+n+r,1:m)} \\ \cdots \end{pmatrix} c^{-1} T = \begin{pmatrix} T_\beta \\ T_1 \\ \vdots \end{pmatrix}. \tag{32}\]
Note that \( \bar{A}_{c} = T^{-1}cA_{c}c^{-1}T \). From Steps 1 and 2, we obtain \( \bar{A}_{c} := T^{-1}cA_{c}^{-1}T = T^{-1}c\Lambda Q^{-1}c^{-1}T \). \( c^{-1}T = c^{-1}T\bar{A}_{c} \) and

\[
AR = A \begin{pmatrix} R_{\beta} \\ R_{1} \\ \vdots \\ R_{p-1} \end{pmatrix} = \begin{pmatrix} (I_{r} + \beta'\alpha)R_{\beta} + \beta'\Phi_{1}R_{1} + \cdots + \beta'\Phi_{p-1}R_{p-1} \\ \alpha R_{\beta} + \Phi_{1}R_{1} + \cdots + \Phi_{p-1}R_{p-1} \\ \vdots \\ R_{1} R_{p-2} \end{pmatrix} = \begin{pmatrix} R_{\beta}\bar{A}_{c} \\ R_{1}\bar{A}_{c} \\ \vdots \\ R_{p-2}\bar{A}_{c} \end{pmatrix} = R\bar{A}_{c} \quad (33)
\]

\[
AR = R\bar{A}_{c} = c^{-1} \begin{pmatrix} T_{\beta}\bar{A}_{c} \\ T_{1}\bar{A}_{c} \\ \vdots \\ T_{p}\bar{A}_{c} \end{pmatrix} = c^{-1} \begin{pmatrix} T_{\beta}T^{-1}cA^{-1}T \\ T_{1}T^{-1}cA^{-1}T \\ \vdots \\ T_{p}T^{-1}cA^{-1}T \end{pmatrix} \quad (34)
\]

Now, \( R_{\beta} = c_{\beta}^{-1}T \) and \( R_{1} = c_{1}^{-1}T \), where \( c_{\beta}^{-1} := [c^{-1}]_{(1,r,1:m)} \) and \( c_{1}^{-1} := [c^{-1}]_{(r+1,r+n,1:m)} \). Therefore, we receive \( R_{i} \) for \( i = 2, \ldots, p-1 \), given \( R_{1} = c_{1}^{-1}T_{1} \) from the recursion \( R_{i+1} = R_{i}A_{c}^{-1} \), for \( i = 1, \ldots, p-2 \).

**Step 4**: (obtain \( R = c^{-1}T, T \) and \( \beta, \Phi_{1}, \ldots, \Phi_{p-1} \))

To jointly treat the stock and the flow case, we write

\[
c = \begin{pmatrix} c_{\beta} & c_{\beta_{1}} & c_{\beta_{2}} & \cdots & c_{\beta_{N-1}} & 0 & \cdots \\ 0 & c_{11} & c_{12} & \cdots & c_{1N} & c_{1,N+1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \end{pmatrix},
\]

where \( c_{\beta_{1}}, \ldots, c_{\beta_{N-1}} \) and \( c_{N-1+j}, \ j \geq 1 \) are zero for the stock case. For the flow case \( c_{11}, \ldots, c_{N-1,1} = I_{n} \) and \( c_{1N}, \ldots, c_{2N-1,1} = -I_{n} \). For the case of stock and flow variables the corresponding coordinates of \( c_{1N}, \ldots, c_{2N-1,1} \) are zero for stock variables.

To retrieve \( T \) and \( R \) we proceed as follows: By means of (32) and (34), and the assumption \( p \geq 2N \) we derive

\[
T = cR = \begin{pmatrix} c_{\beta}R_{\beta} + c_{\beta_{1}}R_{1} + c_{\beta_{2}}R_{2} + \cdots + c_{\beta_{N-1}}R_{N-1} \\ 0R_{\beta} + c_{11}R_{1} + c_{12}R_{2} + \cdots + c_{1N}R_{N} + c_{1,N+1}R_{N+1} + \cdots + c_{1,2N}R_{2N} \\ 0R_{\beta} + 0R_{1} + R_{2} + R_{3} + \cdots + R_{N+1} \\ \vdots \\ R_{N} + R_{N+1} + \cdots + R_{2N-1} \\ I_{n}R_{N+1} \\ \vdots \\ I_{n}R_{p-1} \end{pmatrix}.
\]

Recall that for stock case \( c_{1j} = I_{n}, \ j = 1, \ldots, N, \ c_{1j} = 0, \ j > N, \ c_{\beta j} = 0 \), for \( j \geq 1 \), while for the flow case \( c_{1j} = I_{n}, \ j = 1, \ldots, N, \ c_{1j} = -I_{n}, \ j = N+1, \ldots, 2N \), and \( c_{\beta j} = -(N-j)\beta' \), for \( j = 1, \ldots, N-1 \).
From the above considerations $T_1$ can be obtained from (32). Since $R_{i+1} = R_i A_c^{-1}$, equation (35) yields

$$T_1 = \begin{cases} R_1 + R_2 + \cdots + R_N, & \text{for the stock case,} \\ R_1 + R_2 + \cdots + R_N - R_{N+1} - \cdots - R_{2N}, & \text{for the flow case.} \end{cases} \quad (36)$$

In the above Step 3, we obtained $R_{i+1} = R_i A_c^{-1}$, which results in

$$T_1 = \begin{cases} R_1 + R_2 + \cdots + R_N, & \text{for the stock case,} \\ R_1 + R_2 + \cdots + R_N - (R_1 + \cdots + R_N) A_c^{-N}, & \text{for the flow case.} \end{cases} \quad (37)$$

such that $R_1 + \cdots + R_N = T_1$ for the stock and $R_1 + \cdots + R_N = T_1 (I_m - A_c^{-N})^{-1}$ for the flow case. As already obtained above, $R_{i+1} = R_i A_c^{-1}$. This yields $R_1 + \cdots + R_N = R_1 \sum_{j=1}^{N} A_c^{-j+1}$. Since $R_1 + \cdots + R_N$ follows from (37) we are also able to derive $R_1$ and therefore $R_{i+1}$ by the recursion $R_{i+1} = R_i A_c^{-1}$, $i = 2, \ldots, p - 1$. Finally, we observe

$$T_2 = R_2 + R_3 + \cdots + R_{N+1} = (R_1 + R_2 + \cdots + R_N) A_c^{-1}$$

$$\vdots$$

$$T_N = R_N + R_{N+1} + \cdots + R_{N+N-1} = \ldots$$

$$T_{N+1} = R_{N+1}$$

$$\vdots$$

$$T_{p-1} = R_{p-1} \cdot \quad (38)$$

Hence $T_i$, $i = 2, \ldots, p - 1$, are provided by (37). Recall that $T_\beta$ and $T_1$ follow from (31).

**Step 5:** (Obtain $\Sigma_\nu$)
Let
\[ \gamma_{\Delta_N y}(\kappa - \ell) := \mathbb{E} \Delta_N y_{t+\ell} \Delta_N y_{t-\kappa}, \]
\[ \gamma_{\beta}(\kappa - \ell) := \mathbb{E} \beta' y_{t+\ell} (\beta' y_{t-\kappa})', \]
\[ \gamma_{\beta,\Delta_N y}(\kappa - \ell) := \mathbb{E} \beta' y_{t+\ell} \Delta_N y_{t-\kappa} = (\mathbb{E} \Delta_N y_{t-\kappa} (\beta' y_{t-\ell})') = \gamma_{\Delta_N y,\beta}(\ell - \kappa)', \]
and
\[ \Gamma_{\tau} := \mathbb{E} \varphi_{\tau+1} \varphi_{\tau+1}^{T} \in \mathbb{R}^{m \times m} \]
\[
\left(\begin{array}{cccc}
\gamma_{\beta} (0) & \gamma_{\beta,\Delta y} (0) & \gamma_{\beta,\Delta y} (1) & \cdots & \gamma_{\beta,\Delta y} (p-2) \\
\gamma_{\Delta y,\beta} (0) & \gamma_{\Delta y} (0) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (p-2) \\
\gamma_{\Delta y,\beta} (-1) & \gamma_{\Delta y} (-1) & \gamma_{\Delta y} (0) & \cdots & \gamma_{\Delta y} (p-3) \\
\gamma_{\Delta y,\beta} (-p+2) & \gamma_{\Delta y} (-p+2) & \gamma_{\Delta y} (-p+3) & \cdots & \gamma_{\Delta y} (0) \\
\gamma_{\beta} (0) & \gamma_{\beta,\Delta y} (0) & \gamma_{\beta,\Delta y} (-1)' & \cdots & \gamma_{\beta,\Delta y} (p-2)' \\
\gamma_{\Delta y,\beta} (0) & \gamma_{\Delta y} (0) & \gamma_{\Delta y} (-1)' & \cdots & \gamma_{\Delta y} (-p+2)' \\
\gamma_{\Delta y,\beta} (-1) & \gamma_{\Delta y} (-1) & \gamma_{\Delta y} (0) & \cdots & \gamma_{\Delta y} (-p+3)' \\
\gamma_{\Delta y,\beta} (-p+2) & \gamma_{\Delta y} (-p+2) & \gamma_{\Delta y} (-p+3) & \cdots & \gamma_{\Delta y} (0)
\end{array}\right)
\]
where \( \varphi_t \) was defined in (9),(10). The last step follows from the fact that \( \varphi_t \) is stationary, such that \( \Gamma_{\tau} \) has to be symmetric.
Let \( S_{\beta} := (I_{nx}, 0_{n \times n}, \ldots, 0) \in \mathbb{R}^{r \times m} \), and \( S_{\Delta_N y} := (0_{nx}, I_n, 0, \ldots, 0) \in \mathbb{R}^{n \times m} \). Then (9) and (10) result in
\[
\left(\begin{array}{ccc}
\gamma_{u} (0) & \gamma_{u,\Delta_N y} (0) \\
\gamma_{u,\Delta_N y} (0) & \gamma_{\Delta_N y} (0) \\
\gamma_{u,\Delta_N y} (N) & \gamma_{\Delta_N y} (N) \\
\gamma_{u,\Delta_N y} ((np-2)N) & \gamma_{\Delta_N y} ((np-2)N)
\end{array}\right)
\]
\[ \Gamma_{\beta,\Delta_N y} \]
\[
\left(\begin{array}{c}
\mathcal{O}_{N} \quad \left(S_{\beta} \quad S_{\Delta_N y} \quad S_{\Delta_N y} A_{b,c}^{N} \quad \cdots \quad S_{\Delta_N y} A_{b,c}^{N(np-2)}\right)
\end{array}\right) \]
\[
c_{\tau} \Gamma_{\tau} \left(\begin{array}{c}
\mathcal{O}_{N} \quad \left(S_{\beta} \quad S_{\Delta_N y} \quad S_{\Delta_N y} A_{b,c}^{N} \quad \cdots \quad S_{\Delta_N y} A_{b,c}^{N(np-2)}\right)
\end{array}\right).
\]
Note that \( \mathcal{O}_{N} A_{b,c}^{-N} = \mathcal{O} \), where \( \mathcal{O} \) is defined in (43). The matrix \( \mathcal{O} \) has full column rank, as will be shown in Lemma 7, such that also \( \mathcal{O}_{N} \) has full rank. Thus we obtain the first two column blocks of \( \Gamma_{\tau} \). Now looking at the
specific structure of

\[
\Gamma_{rp,c} = \begin{pmatrix}
\gamma_u (0) & \gamma_{\Delta_N \gamma} (0) & \gamma_{\Delta y} (0) & \cdots & \gamma_{\Delta y} (N) \\
\gamma_{\Delta y} (0) & \gamma_{\Delta\gamma} (0) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\gamma_{\Delta N \gamma} (-1) & \gamma_{\Delta y} (-1) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\gamma_{\Delta N \gamma} (-2) & \gamma_{\Delta y} (-2) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{\Delta N \gamma} (-N) & \gamma_{\Delta y} (-N) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\gamma_{\Delta N \gamma} (-N-1) & \gamma_{\Delta y} (-N-1) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\gamma_{\Delta N \gamma} (-N-2) & \gamma_{\Delta y} (-N-2) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma_{\Delta N \gamma} (-N-3) & \gamma_{\Delta y} (-N-3) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\gamma_{\Delta N \gamma} (-N-2) & \gamma_{\Delta y} (-N-2) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\gamma_{\Delta N \gamma} (-N) & \gamma_{\Delta y} (-N) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\gamma_{\Delta N \gamma} (-N-1) & \gamma_{\Delta y} (-N-1) & \gamma_{\Delta y} (1) & \cdots & \gamma_{\Delta y} (N+1) \\
\end{pmatrix}
\]

we see the following relations

\[
\Gamma_{(2+h)}^{(2)} = \Gamma_{rp,c}^{(2)} (h) \quad \text{for} \quad h = 1, ..., m - 2
\]

\[
\Gamma_{rp,c} (h) = A_c \Gamma_{rp,c} \quad \text{for} \quad h = 1, 2, ...
\]

\[
\Gamma_{rp,c} (2 + h) = A_c \Gamma_{(2+h)}^{(2)}
\]

where by \(\Gamma_{rp,c}^{(j)} (h)\), we denote the \(j\)-th column block of \(\Gamma_{rp,c} (h)\). The first equation follows from the structure of the autocovariances of the states, i.e. \(\Delta y_t = E x_{t+h} x_t'\) for \(h \in \mathbb{N}_0\), the second equation follows from the Lyapunov equations. Hence, we receive all columns of \(\Gamma_{rp,c}\) and therefore of \(\Gamma_{rp} = c^{-1} \Gamma_{rp,c} c^{-1}\). Finally, again by using the Lyapunov equations we have all second moments of \((\Delta y_t)_{t \in \mathbb{Z}}\) and \((y_t \Delta y_t')_{t \in \mathbb{Z}}\).

Now \(\Sigma_{\nu}\) retained by using the “high frequency Yule-Walker type equations”, that is,

\[
\begin{align*}
\Delta y_t - \alpha \beta y_{t-1} - \Phi_1 \Delta y_{t-1} - \cdots - \Phi_{p-1} \Delta y_{t-p+1} = \nu_t \\
\Delta y_t \Delta y_t' - \alpha \beta y_{t-1} \Delta y_t' - \Phi_1 \Delta y_{t-1} \Delta y_t' - \cdots - \Phi_{p-1} \Delta y_{t-p+1} \Delta y_t' = \nu_t \Delta y_t' \\
E \Delta y_t \Delta y_t' - \alpha \beta y_{t-1} \Delta y_t' - \Phi_1 E \Delta y_{t-1} \Delta y_t' - \cdots - \Phi_{p-1} E \Delta y_{t-p+1} \Delta y_t' = \Sigma \nu_t \Delta y_t' \quad (42)
\end{align*}
\]

Hence, also generic identifiability of \(\Sigma_{\nu}\) is established.

Finally we prove continuity of \(\pi^{-1}\). This involves two steps: 1. The continuity of the mapping from the observed second moments to the parameters of a canonical minimal realisation \((\bar{A}_{h,c}, \bar{B}_{h,c}, \bar{C}_{h,c})\) (say the echelon form):

Recall that the set of transfer functions with McMillan-degree \(m\), call it \(\tilde{M}(m)\), can be decomposed in disjoint pieces corresponding to different Kronecker indices summing up to \(m\). The set of transfer functions where the
first $m$ rows of the Hankel matrix are a basis of the row space of the Hankel matrix is generic in $\tilde{M}(m)$ (w.r.t. the pointwise topology for $\tilde{M}(m)$ (see Hannan and Deistler, 2012, p. 65)). This set is also called the “generic neighbourhood”. As has been shown in Step 5 above, $\Gamma_{r,pc}$ from equation (41) has full rank $m$. We know that the linear dependencies in the Hankel matrix of the transfer function, say $\tilde{H}$, and the Hankel matrix of the second moments, say $\tilde{H}_\gamma$, are the same (for the definitions see Anderson et al., 2016a). Now since $\Gamma_{rp,c}$ is the upper left $m \times m$ block of $\tilde{H}_\gamma$, we know that the first $m$ rows of $\tilde{H}$ are a basis of the row space of $\tilde{H}$. Therefore $\Theta_I$ is a subset of the generic neighbourhood.

2. Note that from a given minimal realisation $(\bar{A}_{b,c}, \bar{B}_{b,c}, \bar{C}_{b,c})$ of $\theta \in \Theta_I$ all transformations involved in the retrieval algorithm described above are continuous. ■

### Lemma 7

Suppose that Assumptions 1 and 2 hold. The matrix

$$
O = \begin{pmatrix}
S_p A_{b,c} \\
S_{\Delta_y A_{b,c}} \\
S_{\Delta_y A_{b,c}^2} \\
\vdots \\
S_{\Delta_y A_{b,c}^{n(p-1)}}
\end{pmatrix}
$$

is of full column rank $m = r + n(p - 1)$.

**Proof.** The proof is very similar to the proof that the observability matrix is of full rank in Anderson et al. (2016a)[Proof of Theorem 7, page 823]. Since the matrix $c$ is of full rank $m$ we are allowed to consider $A^N$ and $A$. To see this, let $\tilde{q}_i$ now denote an eigenvector of $cA^{-1}$ with eigenvalue $\lambda_i$, then $(cA^N_{c^{-1}}) \tilde{q}_i = cA^{N-1}_{c^{-1}}cA^{-1}_{c^{-1}} \tilde{q}_i = \lambda_i cA^{N-1}_{c^{-1}} \tilde{q}_i = \lambda_i^N \tilde{q}_i$. In addition, if $q_i$ is an eigenvector of $A$, then $\tilde{q}_i = cq_i$ is an eigenvector of $A_{b,c}$. Moreover, $A^i_{b,c} \tilde{q}_i = cA^i_{c^{-1}}cq_i = \lambda_i^N \tilde{q}_i$. The eigenvalues of $A$ are such that $\lambda_i \neq \lambda_j$ implies $\lambda_i^2 \neq \lambda_j^2$, the eigenvectors of $A$ and $A^2$ coincide. To see this, let $q_i \in \mathbb{R}^m$ and $\lambda_i \in \mathbb{R}$ denote an eigenvector and an eigenvalue of the matrix $A$. Then, $Aq_i = \lambda_i q_i$ and $A^2 q_i = AAq_i = \lambda_i Aq_i = \lambda_i^2 q_i$; for $N > 2$ this works in the same way. Therefore it is sufficient to look at the eigenvectors and eigenvalues of the matrix $A$. Similar to Anderson et al. (2016a)[Lemma 2] we have shown in the proof of the above Theorem 3 that the first $r + n$ components of an eigenvector of $A$ or $cA^{-1}$ are not equal to a vector of zeros.
Therefore, by the Popov-Belevitch-Hautus (PBH)-eigenvector test (see, e.g., Kailath, 1980, page 135), the matrix $O$ has full column rank $r + n(p - 1)$. That is,

$$
\begin{pmatrix}
(A^N - \lambda_i^N I_m) \\
S_{\beta} \\
S_{\Delta y}
\end{pmatrix} q_i = \begin{pmatrix}
0_{m \times 1} \\
\lambda_i^N [q_i]_{1:n+r} \\ 0_{n+r \times 1}
\end{pmatrix}.
$$

(44)

E Proof of Theorem 5

Proof. The proof is constructed as follows: For each of the identifiability conditions in Assumption 2, we suppose that $(I_j)$ is violated for $j = 1, ..., 6$ and show that there exists no “observationally equivalent” $\theta \in \Theta_I$.

Suppose $(I_1)$ or $(I_2)$ are violated for $\theta \not\in \Theta_I$, then it follows that the McMillan degree of $\tilde{k}(\tilde{z})$ is less than $m$. Hence there exists no $\theta \in \Theta_I$ with the same auto-covariance function $\tilde{\gamma}$.

Suppose $(I_3)$ or $(I_4)$ are violated, then the minimal realisation of $\tilde{A}_{b,c}$, which is directly obtained from $\tilde{\gamma}$ has eigenvalues $\lambda_i^N = \lambda_j^N$ for some $i \neq j$, and thus $(I_4)$ is violated.

Suppose that neither of the conditions in $(I_5)$ hold, then by equations (27), we have $C_{b,q} = 0$ and the system is not observable (and therefore of McMillan degree smaller than $m$).

Suppose that condition $(I_6)$ is not satisfied, then after going through steps Steps 1 and 2 of the retrieval algorithm in the proof of Theorem 4, we obtain in equation (30) that $S_{n,l}^{(1)} q_i = 0$ for some $i$ and therefore we are outside of $\Theta_I$ already.