Homogeneity of zero-divisors, units and colon ideals in a graded ring

Abolfazl Tarizadeh

Abstract. In this article, we first generalize Kaplansky’s zero-divisor conjecture of group-rings $K[G]$ (with $K$ a field) to the more general setting of $G$-graded rings $R = \bigoplus_{\alpha \in G} R_{\alpha}$ with $G$ a torsion-free group. Then we prove that if $I$ is an unfaithful left ideal of a $G$-graded ring $R$ with $G$ a totally ordered group, then there exists a (nonzero) homogeneous element $g \in R$ such that $gI = 0$. This theorem gives an affirmative answer to the new conjecture in the case that the group involved in the grading is a totally ordered group. Our result also generalizes McCoy’s famous theorem on polynomial rings to the more general setting of $G$-graded rings. Then we focus on Kaplansky’s unit conjecture. Although this conjecture was recently disproved by a counterexample in the general case, we discovered quite useful and general results that give an affirmative answer to the generalized version of the unit conjecture in the case that the group involved in the grading is a totally ordered group. Especially, we show that every invertible element of a $G$-graded domain with $G$ a totally ordered group is homogeneous. This key result enables us to provide a characterization of invertible elements in $G$-graded commutative rings. This theorem, in particular, tells us that the homogeneous components of an invertible element form a co-maximal ideal and all distinct double products are nilpotent. Next, we prove that if $I$ is a graded radical ideal of a $G$-graded commutative ring $R$ with $G$ a torsion-free Abelian group and $J$ an arbitrary ideal of $R$, then the colon ideal $I :_R J$ is a graded ideal. Our theorem vastly generalizes Armendariz’ result on reduced polynomial rings to the more general setting of graded rings.

1. Introduction

Kaplansky’s zero-divisor conjecture claims that if $G$ is a torsion-free group (not necessarily Abelian) and $K$ is a field, then the only zero-divisor element of the group-ring $K[G]$ is zero. Despite the proof of some special cases in the literature, this conjecture is still open in general. In order to achieve a unified framework and get deeper insight into the above conjecture, we first generalize it as follows:

2010 Mathematics Subject Classification. 13A02, 16S34, 13A15, 20K15, 20K20, 14A05.

Key words and phrases. Kaplansky’s zero-divisor conjecture; Kaplansky’s unit conjecture; McCoy’s theorem; Armendariz theorem; Homogeneity; Totally ordered group; Torsion-free group; Colon ideal.
Conjecture 1.1. For any ring $R$ and a torsion-free group $G$, if $f$ is a zero-divisor element of the group-ring $R[G]$ then there exists a nonzero $c \in R$ such that $cf = 0$.

The group-ring $R[G]$ is a typical example of $G$-graded rings. Hence, the above conjecture can be further generalized to the more general setting of graded rings and simultaneously to the setting of ideals rather than to that of elements:

Conjecture 1.2. (generalized zero-divisor conjecture) If $I$ is an unfaithful left ideal of a $G$-graded ring $R$ with $G$ a torsion-free group, then there exists a (nonzero) homogeneous element $g \in R$ such that $gI = 0$.

In Theorem 4.1, we completely resolve the above conjectures in the commutative case. In fact, we prove the above conjectures (including non-commutative rings) in the case that the group $G$ involved in the grading is a totally ordered group. More precisely, with the same proof of Theorem 4.1 (only taking into account that if $I$ is a left ideal of a ring $R$ then its annihilator $\text{Ann}_R(I) = \{r \in R : rI = 0\}$ is a two-sided ideal of $R$), we obtain the following general result (here $G$ is not necessarily an Abelian group and $R$ is not necessarily a commutative ring):

Theorem 1.3. If $I$ is an unfaithful left ideal of a $G$-graded ring $R$ with $G$ a totally ordered group, then there exists a (nonzero) homogeneous $g \in R$ such that $gI = 0$.

In the course of two classical articles [19, §3, Theorems 2-3] and [20, Theorem 1], McCoy proved a remarkable result on polynomial rings which asserts that every zero-divisor element (or more generally, every unfaithful ideal) of the polynomial ring $R[x_1, \ldots, x_d]$ is annihilated by a nonzero element of the ring $R$. Our theorem (Theorem 4.1) or more generally Theorem 1.3 also generalizes this result to the more general setting of graded rings. McCoy’s theorem is then deduced as a special case of our general result (see Corollaries 4.6 and 4.7). It is worth noting that in the proof of Theorem 4.1 we use an idea whose special case was already used by McCoy to establish his result [20]. McCoy’s theorem and McCoy rings have been extensively studied in the literature over the years (see e.g. [5, 8, 11, 12, 22, 27]).

Kaplansky’s unit conjecture claims that if $G$ is a torsion-free group (not necessarily Abelian) and $K$ is a field, then the units of the group-ring $K[G]$ are precisely homogeneous elements of the form $r e_x$ where $0 \neq r \in K$ and $x \in G$. Recently Gardam [9] proved that the group-ring $\mathbb{F}_2[F(6)]$ with $F(6)$ the 6th Fibonacci group, which is a torsion-free group and virtually abelian, is indeed a counterexample to the unit conjecture (in fact, he discovered a non-homogeneous invertible element in this group-ring with 21 homogeneous components). In light of this, the following theorem is the most general positive result one can expect about the generalized version of the unit conjecture. In fact, in Lemma 5.2 we first show that every invertible element of a $G$-graded domain with $G$ a totally ordered group is homogeneous. This
key result then enables us to give a complete characterization of invertible elements in graded commutative rings:

**Theorem 1.4.** Let \( f = \sum_{k \in G} r_k \) be an element of a \( G \)-graded ring \( R = \bigoplus_{k \in G} R_k \) with \( G \) a totally ordered group and \( r_k \in R_k \) for all \( k \). Then \( f \) is invertible in \( R \) if and only if the ideal \((r_k : k \in G)\) is the whole ring \( R \) and \( r_i r_k \) is nilpotent for all \( i \neq k \).

In [3, Lemma 1], Armendariz obtained an interesting result on reduced polynomial rings (which is closely related to McCoy’s theorem). His result asserts that in a reduced polynomial ring, if a polynomial \( f \) annihilates another polynomial \( g \) (i.e. \( fg = 0 \)) then each coefficient of \( f \) annihilates every coefficient of \( g \). In Theorem 6.1, we prove a more general result which asserts that if \( I \) is a graded radical ideal of a \( G \)-graded commutative ring \( R \) with \( G \) a torsion-free Abelian group and \( J \) an arbitrary ideal of \( R \), then the colon ideal \( I :_R J = \{ r \in R : rJ \subseteq I \} \) is a graded ideal. Our theorem vastly generalizes Armendariz’ result and several other related results on polynomial rings to the more general setting of graded rings and simultaneously to arbitrary (not necessarily reduced) rings. For more information on Armendariz’ result we refer the interested reader to the literature (see e.g. [1, 2, 3, 13, 14, 16, 25, 26]). Finally, in Theorem 6.9, we codify some important properties of monoid-rings. All of the results of this article are supplemented with (counter)examples showing that some conditions on the grading group \( G \) is required. In light of the (generalized) zero-divisor and unit conjectures, these examples of course heavily depend on torsion which highlight the necessity of the torsion-free assumption.

It is worth mentioning that in a subsequent work entitled “Homogeneity of idempotents and the Jacobson radical in a graded ring” [arXiv:2309.02880] we generalize Kaplansky’s third conjecture, the idempotent conjecture, to the more general setting of graded rings and then prove it for \( G \)-graded commutative rings with \( G \) an Abelian group. In spite of that the unit, zero-divisor and idempotent conjectures are already proved in the literature for some special cases (see e.g. [6], [7], [10], [15], [24, Lemma 1.9 on p. 589] and [21]) but the idempotent conjecture, like the zero-divisor conjecture, is still open in general.

2. Preliminaries

In this section, we recall some necessary background, for the convenience of the reader. Throughout the rest of the article, all monoids, groups and rings are assumed to be commutative.

2.1. **Grothendieck group.** Recall that the Grothendieck group of a commutative monoid \( M \) is constructed in the following way. We first define a relation over the set \( M \times M \) as \((a, b) \sim (c, d)\) if there exists some \( m \in M \) such that \((a + d) + m = (b + c) + m\). It can be seen that it is an equivalence relation. Here we denote the equivalence class containing of an ordered pair
(a, b) simply by [a, b], and we denote by $G = \{[a, b] : a, b \in M\}$ the set of all equivalence classes obtained by this relation. Then the set $G$ by the operation $[a, b] + [c, d] = [a + c, b + d]$ is an Abelian group. Indeed, $[0, 0]$ is the identity element of $G$ where $0$ is the identity of $M$, and for each $[a, b] \in G$ its inverse is $[b, a]$. This group $G$ is called the Grothendieck group of $M$. Note that in the above construction, the commutativity of the monoid $M$ plays a vital role. We have the canonical morphism of monoids $M \to G$ which is given by $m \mapsto [m, 0]$. This map is injective if and only if $M$ has the cancellation property. The Grothendieck group of the additive monoid of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ is called the additive group of integers and denoted by $\mathbb{Z}$.

2.2. Totally ordered groups. By a totally (linearly) ordered monoid we mean a monoid $M$ equipped with a total ordering $<$ such that its operation is compatible with its ordering, i.e. if $a < b$ for some $a, b \in M$, then $a + x \leq b + x$ for all $x \in M$. If moreover, $M$ has the cancellation property then from $a < b$ we get that $a + x < b + x$. If $d \geq 1$ is a natural number, then the additive monoid $\mathbb{N}^d$ (and more generally, for an index set $I$, the direct sum additive monoid $\bigoplus_{k \in I} \mathbb{N}$) with the lexicographical ordering is a typical example of a totally ordered monoid.

If $M$ is a totally ordered monoid (by an ordering $<$) with the cancellation property, then it can be seen that its Grothendieck group is a totally ordered group whose order is defined by $[a, b] < [c, d]$ if $a + d < b + c$ in $M$.

It is well-known that an Abelian group is a totally ordered group if and only if it is a torsion-free group (every non-identity element is of infinite order). Its proof can be found in [17, Theorem 6.31] and [18, §3]. Hence, all of the results (of this article) remain true if one replaces “totally ordered group” by “torsion-free Abelian group”.

In the same vein, it can be shown that a monoid $M$ with the cancellation property is a totally ordered monoid if and only if $M$ is strongly torsion-free (i.e. whenever $na = nb$ for some $n \geq 2$ and some $a, b \in M$, then $a = b$), or equivalently, the Grothendieck group of $M$ is a torsion-free group. For the proof of the first equivalence see [23, §2.12, Theorem 22], the second equivalence is clear. In this regard, see also Example 3.7.

If $d \geq 1$, then the additive group $\mathbb{Z}^d$ (and more generally the additive group $\bigoplus_{k \in I} \mathbb{Z}$) with the lexicographical ordering is a typical example of a totally ordered group. If $n \geq 2$ then the additive group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is not a totally ordered group.

2.3. Graded rings. Let $R = \bigoplus_{m \in M} R_m$ be an $M$-graded ring with $M$ a monoid. Every nonzero element $0 \neq r \in R_m$ is called a homogeneous element of $R$ of degree $m \in M$ and written $\deg(r) = m$. If $r, r' \in R$ are homogeneous elements with $rr' \neq 0$, then $rr'$ is homogeneous and $\deg(rr') = \deg(r) + \deg(r')$. If $r \in R_m$ and $r' \in R_n$, then $rr' \in R_{m+n}$.
deg(r'). Whenever we write f = \( \sum_{m \in M} r_m \in R \), if it is not stated, then it should be understood that \( r_m \in R_m \) for all \( m \in M \). If \( M \) has the cancellation property, then \( 1 \in R_0 \) where 0 is the identity element of \( M \) (in other words, \( R_0 \) is a subring of \( R \)).

Let \( I \) be an ideal of an \( M \)-graded ring \( R \). Recall that \( I \) is called a graded ideal if whenever \( f \in I \) then all homogeneous components of \( f \) are members of \( I \). The intersection of every family of graded ideals is a graded ideal. Clearly, \( I \) is a graded ideal of \( R \) if and only if \( I = \bigoplus_{m \in M} I \cap R_m \), or equivalently, \( R/I = \bigoplus_{m \in M} (R/I)_m \) is an \( M \)-graded ring with the homogeneous components 

\[ (R/I)_m = (R_m + I)/I. \]

All of these criteria are well-known, except the latter which seems to be new. For the sake of completeness, we provide a proof. Assume \( R/I \) is naturally an \( M \)-graded ring. Take \( f = \sum r_m \in I \). Then

\[ \sum_{m \in M} (r_m + I) = 0. \]

Now the direct sum assumption in \( R/I \) yields that \( r_m \in I \) for all \( m \in M \). Hence, \( I \) is a graded ideal.

Let \( R = \bigoplus_{n \in G} R_n \) be a \( G \)-graded ring with \( G \) an Abelian group. If \( f = \sum_{n \in G} r_n \in R \) with \( r_n \in R_n \) for all \( n \in G \), then we define the support of \( f \) as the finite set \( \text{Supp}(f) := \{ n \in G : r_n \neq 0 \} \). If \( G \) is a totally ordered group and \( f \) is a nonzero element, then \( n_*(f) \) denotes the smallest element, and \( n^*(f) \) denotes the largest element, of \( \text{Supp}(f) \) with respect to the ordering on \( G \). We have \( n_*(f) \leq n^*(f) \), and the equality holds if and only if \( f \) is homogeneous. It is also clear that \( n_*(f + g) \leq n_*(fg) \) and \( n^*(fg) \leq n^*(f) + n^*(g) \) for all \( f, g \in R \) with \( fg \neq 0 \).

We also recall changing the grading on a given graded ring, as we shall need it later on. Let \( R = \bigoplus_{m \in M} R_m \) be an \( M \)-graded ring and \( \varphi : M \to M' \) a monoid morphism. Then \( R = \bigoplus_{d \in M'} S_d \) can be viewed as an \( M' \)-graded ring with the homogeneous components \( S_d = \sum_{m \in M, \varphi(m) = d} R_m \) (it is clear that if \( d \in M' \) is not in the image of \( \varphi \), then \( S_d = 0 \)).

2.4. **Monoid-rings.** Let \( R \) be a ring and \( M \) a monoid. By \( R[M] \) we mean the set of all elements \((r_m)_{m \in M}\) in the direct product set \( \prod_{m \in M} R \) such that \( r_m = 0 \) for all but finitely many indices \( m \). It can be seen that the set \( R[M] \) forms a ring with the componentwise addition \((a_x) + (b_x) = (a_x + b_x)\) and Cauchy product \((a_x) \cdot (b_y) = (r_m)\) where \( r_m := \sum_{(x,y) \in M^2, x+y=m} a_x b_y \) for all \( m \in M \).

Note that in this sum, the index set \( \{(x,y) \in M^2 : x + y = m\} \) is not necessarily a finite set. But it can be replaced with \( \{(x,y) \in M^2 : x + y = m\} \).
$m, a_x b_y \neq 0 \}$ which is always a finite set, because it is a subset of the finite set 
\( \{ x \in M : a_x \neq 0 \} \times \{ y \in M : b_y \neq 0 \} \). Also note that in the Cauchy product, 
\( r_m = 0 \) for all but finitely many \( m \). Indeed, \( \{ m \in M : r_m \neq 0 \} \) is contained in 
the image of the function \( f : \{ x \in M : a_x \neq 0 \} \times \{ y \in M : b_y \neq 0 \} \to M \) 
given by \( (x, y) \mapsto x + y \) which is a finite set. In order to show that \( R[M] \)
is a ring, the only nontrivial assertion that needs to be checked is that the 
Cauchy product is associative (its proof can be found in [28, Lemma 2.1]).

In mathematics, the ring \( R[M] \) is of particular interest and it is called the 
monoid-ring of \( R \) over \( R \). For each \( m \in M \), we call \( \epsilon_m := (\delta_{a,m})_{a \in M} \) the 
monomial of \( R[M] \) of degree \( m \) where \( \delta_{a,m} \) is the Kronecker delta. Clearly 
\( \epsilon_m \cdot \epsilon_n = \epsilon_{m+n} \) for all \( m, n \in M \). The element \( e_0 \) is the multiplicative identity 
of the ring \( R[M] \) where \( 0 \) is the identity element of \( M \). If \( r \in R \) then \( r \cdot \epsilon_m \)
or simply \( r \epsilon_m \) denotes the element in \( R[M] \) which has \( r \) in the \( m \) component 
and zero in all other components. Then each \( (r_m) \in R[M] \) can be written 
(uniquely) as \( (r_m) = \sum_{m \in M} r_m \epsilon_m \). We often denote the element \( r e_0 \) simply 
by \( r \) if there is no confusion. If \( M \) has the cancellation property, then \( \epsilon_m \) is 
a non-zero-divisor element of \( R[M] \) for all \( m \in M \). In the literature, \( r \cdot \epsilon_m \)
is often denoted by \( r \cdot m \), but we must add that our innovation of using \( \epsilon_m \) 
instead of \( m \) has many advantages. For instance, it considerably simplifies 
computations in monoid-rings (and surprisingly, this usage led us to discover 
several results in this article).

For any ring \( R \), the monoid-ring \( R[N] \) is denoted by \( R[x] \) and it is called the 
ring of polynomials over \( R \) with the variable \( x := \epsilon_1 \) where \( N \) is the 
additive monoid of natural numbers. It is clear that \( x^n = \epsilon_n \) for all \( n \geq 0 \). 
Similarly, the monoid-ring \( R[N^2] \) is denoted by \( R[x, y] \) and it is called the 
ring of polynomials over \( R \) with the variables \( x := \epsilon_{(1,0)} \) and \( y := \epsilon_{(0,1)} \). 
Then \( x^m y^n = \epsilon_{(m,n)} \) for all \( (m, n) \in N^2 \). In this way, we can define the ring 
of polynomials with any (finite or infinite) number of variables in a quite 
formal and systematic way.

The monoid-ring \( R[M] = \bigoplus_{m \in M} S_m \) is an \( M \)-graded ring with the homogeneous 
components \( S_m = R e_m = \{ r \epsilon_m : r \in R \} \). If \( d \geq 1 \) is a natural number, 
then the monoid-ring \( S := R[M^d] = \bigoplus_{m \in M} S_m \) is also an \( M \)-graded ring with 
homogeneous components \( S_m = \sum_{(c_1, \ldots, c_d) \in M^d, c_1 + \cdots + c_d = m} R \epsilon_{(c_1, \ldots, c_d)} \). More generally, let 
\( I \) be an index set, then the monoid-ring \( S := R[I M] = \bigoplus_{m \in M} S_m \) is an \( M \)- 
graded ring with homogeneous components \( S_m = \sum_{z \in M} R \epsilon_z \) where the sum is 
taken over the set of all \( z = (a_k) \in \bigoplus M \) such that \( \sum_{k \in I} a_k = m \). In particular, the polynomial ring 
\( R[x_k : k \in I] = R[I N] = \bigoplus_{n \geq 0} S_n \) with the variables 
\( x_k := \epsilon_{s_k} \) is an \( \mathbb{N} \)-graded ring with homogeneous components \( S_0 = R \) and
\[ S_n = \sum_{(i_1, \ldots, i_n) \in I^n} R x_{i_1} \ldots x_{i_n} \text{ for all } n \geq 1 \text{ where } s_k = (\delta_{i,k})_{i \in I} \in \bigoplus_{i \in I} \mathbb{N} \text{ and } \delta_{i,k} \text{ is the Kronecker delta.} \]

3. Grading by a totally ordered group

We begin by stating an important fact about gradings by totally ordered groups:

**Lemma 3.1.** Let \( G \) be a totally ordered group and \( I \) a graded proper ideal of a \( G \)-graded ring \( R = \bigoplus_{x \in G} R_x \) with the property that whenever \( rr' \in I \) for some homogeneous elements \( r, r' \in R \) then \( r \in I \) or \( r' \in I \). Then \( I \) is a prime ideal of \( R \).

**Proof.** It is an interesting exercise (and well-known). \( \square \)

**Corollary 3.2.** Let \( M \) be a totally ordered cancellative monoid and \( I \) a graded proper ideal of an \( M \)-graded ring \( R \) with the property that whenever \( rr' \in I \) for some homogeneous elements \( r, r' \in R \) then \( r \in I \) or \( r' \in I \). Then \( I \) is a prime ideal of \( R \).

**Proof.** Let \( G \) be the Grothendieck group of \( M \) which is a totally ordered group. By changing the grading, using the canonical morphism of monoids \( M \to G \), then \( R \) can be viewed as a \( G \)-graded ring. Since \( M \) is cancellative, the above map is injective and so the homogeneous components remain the same with this new grading. Now the assertion follows from Lemma 3.1. \( \square \)

Recall that if \( I \) is an ideal of an \( M \)-graded ring \( R \) (with \( M \) a monoid), then by \( I^* \) we mean the ideal of \( R \) generated by all homogeneous elements of \( R \) contained in \( I \). In fact, \( I^* \) is the largest graded ideal of \( R \) which is contained in \( I \). If \( p \) is a prime ideal of an \( M \)-graded ring \( R \), then \( p^* \) is a graded proper ideal of \( R \) and has the above property, i.e. if whenever \( ab \in p^* \) for some homogeneous elements \( a, b \in R \) then \( a \in p^* \) or \( b \in p^* \). But note that in general, \( p^* \) is not necessarily a prime ideal of \( R \) (see Example 3.6).

**Corollary 3.3.** If \( p \) is a prime ideal of a \( G \)-graded ring \( R \) with \( G \) a totally ordered group, then \( p^* \) is a prime ideal of \( R \).

**Proof.** It follows from Lemma 3.1. \( \square \)

**Corollary 3.4.** Every minimal prime ideal of a \( G \)-graded ring with \( G \) a totally ordered group is a graded ideal.

**Proof.** It follows from the above corollary. \( \square \)

**Remark 3.5.** (a) By the above corollary, the nilradical of every \( G \)-graded ring with \( G \) a totally ordered group is a graded ideal.

(b) The above corollaries can easily be generalized to the setting of \( M \)-graded rings where \( M \) is a totally ordered monoid with the cancellation property.
Example 3.6. It is important to notice that the above conclusions need not hold for gradings by Abelian groups $G$ that are not totally ordered groups. For example, let $R$ be an integral domain of characteristic $p > 0$. By changing the grading, we can naturally consider the polynomial ring $R[x] = \bigoplus_{d=0}^{n-1} S_d$ as a $\mathbb{Z}_p$-graded ring with the homogeneous components $S_d = \sum_{n \geq 0} R x^{np+d}$ (note that the additive group $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is not a totally ordered group). We know that $\mathfrak{p} = (x-1)$ is a prime ideal of $R[x]$, because $R[x]/(x-1) \simeq R$. But with this grading, $\mathfrak{p}^*$ is not a prime ideal of $R[x]$, because the Frobenius endomorphism gives us $(x-1)^p = x^p - 1 \in S_0 \cap \mathfrak{p}$ and so $(x-1)^p \notin \mathfrak{p}^*$ but $x-1 \notin \mathfrak{p}^*$. Also note that $R[x]$ modulo the graded ideal $I = (x^p - 1)$ is a $\mathbb{Z}_p$-graded ring. The element $(x-1) + I$ of this ring is nilpotent, since $(x-1)^p \in I$. But $1 \notin I$ (and also $x + I$ is not nilpotent). Thus the nilradical of the $\mathbb{Z}_p$-graded ring $R[x]/I$ is not a graded ideal, and hence it has a minimal prime ideal which is not a graded ideal. In fact, $\mathfrak{p}/I$ is the only minimal prime ideal of this ring which is not a graded ideal.

Example 3.7. Even if $M$ is a torsion-free monoid (i.e. every non-identity element is of infinite order) with the cancellation property, then the Grothendieck group of $M$ is not necessarily a torsion-free group (and hence is not necessarily a totally ordered group). For example, let $R$ be the submonoid of the additive group $\mathbb{Z} \times \mathbb{Z}$ (with $n \geq 2$) which is generated by $x = (1,0)$ and $y = (1,1)$. Then $M = \mathbb{N}x + \mathbb{N}y$ has the cancellation property, because every submonoid of a group has this property. It can be easily seen that $M$ is torsion-free. Let $G = \{[r,s] : r,s \in M\}$ be the Grothendieck group of $M$. Then we obtain an injective morphism of groups $f : G \to \mathbb{Z} \times \mathbb{Z}_n$ which is given by $[r,s] \mapsto r-s$. Then we show that it is also surjective. Take $(a,b) \in \mathbb{Z} \times \mathbb{Z}_n$ then $(a,b) = (a,0) + (0,b)$. If $a \geq 0$ then $[ax,0] \in G$ is mapped into $(a,0)$. But if $a < 0$ then $[0,-ax] \in G$ is mapped into $(a,0)$. Clearly $[y,x] \in G$ is mapped into $(y-x,0,1)$. Thus if $b \geq 0$ then $b[y,x] \in G$ is mapped into $(0,b)$. If $b < 0$ then $-b[x,y] \in G$ is mapped into $-b(0,-1) = (0,b)$. But $G \simeq \mathbb{Z} \times \mathbb{Z}_n$ is not a torsion-free group.

4. Zero-divisors in a Graded Ring

The following theorem gives an affirmative answer to Conjecture 1.1 and Conjecture 1.2 in the case that the group involved in the grading is a totally ordered group. This theorem also generalizes McCoy’s celebrated theorem (see Corollary 4.7) to the more general setting of graded rings. First recall that if the annihilator of a module over a ring is a nonzero ideal of the ring, then the module is called unfaithful.

Theorem 4.1. If $I$ is an unfaithful ideal of a $G$-graded ring $R = \bigoplus_{n \in G} R_n$ with $G$ a totally ordered group, then there exists a (nonzero) homogeneous element $g \in R$ such that $gI = 0$. 
Proof. Amongst all nonzero elements of $\text{Ann}_R(I)$, by the well-ordering principle of the natural numbers, we may choose some $g$ such that the number \( \ell := |\text{Supp}(g)| \) is minimal. We show that \( \ell = 1 \), i.e. $g$ is homogeneous. We have \( \ell \geq 1 \), because $g \neq 0$. Suppose that \( \ell \geq 2 \). Assume $g = \sum_{n \in G} g_n$ with $g_n \in R_n$ for all $n$. Put $s := n^*(g) \in G$. Then $g_s \notin \text{Ann}_R(I)$, because $|\text{Supp}(g_s)| = \{|s|\} = 1$. Thus there exists some $f = \sum_{n \in G} f_n \in I$ (with $f_n \in R_n$ for all $n$) such that $g_sf \neq 0$ and so $g_sf_k \neq 0$ for some $k \in G$. It follows that $gf_k \neq 0$. Choose $t \in \text{Supp}(f) \subseteq G$ to be the largest element (with respect to the ordering) such that $gf_t \neq 0$. Thus $gf_n = 0$ for all $n > t$. But $gf = 0$ and so $(gf)_{s+t} = g_sf_t + \sum_{x+y=s+t, (x,y) \neq (s,t)} g_x f_y = 0$. Take $x,y \in G$ with $x + y = s + t$ and $(x,y) \neq (s,t)$. If $x < s$ and $y < t$ then, since $G$ is a totally ordered group, we will have $s + t = x + y < s + t$ which is impossible. Hence, $x > s$ or $y > t$. If $x > s$ then $g_x = 0$ and so $g_xf_y = 0$. If $y > t$ then $gf_y = 0$ and so $g_xf_y = 0$. Thus $\sum_{x+y=s+t, (x,y) \neq (s,t)} g_x f_y = 0$ and so $g_sf_t = 0$.

Note that $0 \neq gf_t \in \text{Ann}_R(I)$ and $|\text{Supp}(gf_t)| \leq \ell - 1$, because $g_sf_t = 0$. This contradicts the minimality of $\ell = |\text{Supp}(g)|$. Thus, $\ell = 1$. Hence, $g$ is homogeneous. \( \square \)

Note that in the above theorem, $I$ is an arbitrary (not necessarily graded) ideal. It is also important to notice that, as mentioned in the Introduction, the above theorem also holds in the non-commutative case (see Theorem 1.3).

Example 4.2. Note that Theorem 4.1 fails if $G$ is not a totally ordered group. For example, $f = (1/2)(\epsilon_0 + \epsilon_1)$ is an idempotent element of the group-ring $R = \mathbb{Q}[\mathbb{Z}_2] = \mathbb{Q}e_0 + \mathbb{Q}e_1$ which is a $\mathbb{Z}_2$-graded ring. Hence, $f$ is a zero-divisor of $R$. In fact, $\text{Ann}_R(f) = Rg$ where $g = (1/2)(\epsilon_0 - \epsilon_1)$. But there is no homogeneous element in $R$ which could vanish $f$.

Corollary 4.3. If $I$ is an unfaithful ideal of an $M$-graded ring $R$ with $M$ a totally ordered cancellative monoid, then there exists a (nonzero) homogeneous $g \in R$ such that $gI = 0$.

Proof. It follows from Theorem 4.1 by passing to the $G$-grading where $G$ is the Grothendieck group of $M$ which is a totally ordered group. \( \square \)

Corollary 4.4. Let $R$ be an $M$-graded ring with $M$ a totally ordered cancellative monoid. If $f$ is a zero-divisor element of $R$, then there exists a (nonzero) homogeneous $g \in R$ such that $fg = 0$.

Proof. Clearly, $I = Rf$ is an unfaithful ideal of $R$, since $\text{Ann}_R(I) = \text{Ann}_R(f) \neq 0$. The desired conclusion now follows from Corollary 4.3. \( \square \)

Corollary 4.5. Let $R$ be a ring and $M$ a totally ordered cancellative monoid. If $I$ is an unfaithful ideal of the monoid-ring $R[M]$, then $aI = 0$ for some nonzero $a \in R$. 


Proof. We know that the monoid-ring $R[M]$ is an $M$-graded ring with homogeneous components $R_m$. Then by Corollary 4.3 there exists a (nonzero) homogeneous element $ae_m \in R[M]$ such that $(ae_m)I = 0$ where $0 \neq a \in R$ and $m \in M$. But $e_m$ is a non-zero-divisor of $R[M]$, because $M$ has the cancellation property. Hence, $aI = 0$. □

The above result makes the multi-variable versions of McCoy’s theorem quite easy, with no induction on the number of variables required:

**Corollary 4.6.** If $I$ is an unfaithful ideal of the polynomial ring $R[x_k : k \in S]$, then $aI = 0$ for some nonzero $a \in R$.

Proof. We have $R[x_k : k \in S] = R[\bigoplus_{k \in S} \mathbb{N}]$ and each variable $x_k := \epsilon_{sk}$ is a non-zero-divisor where $s_k = (\delta_{i,k})_{i \in I} \in \bigoplus_{k \in S} \mathbb{N}$ and $\delta_{i,k}$ is the Kronecker delta.

Hence, the assertion follows from Corollary 4.5 □

**Corollary 4.7.** If $f$ is a zero-divisor element of the polynomial ring $R[x_k : k \in S]$, then $af = 0$ for some nonzero $a \in R$.

Proof. It follows from the above corollary. □

The next result follows immediately from Corollary 4.4.

**Corollary 4.8.** If $M$ is a totally ordered cancellative monoid, then every homogeneous component of a zero-divisor element of an $M$-graded ring is a zero-divisor.

**Example 4.9.** The converse of Corollary 4.8 does not hold. To see this, consider the $\mathbb{N}$-graded ring $\mathbb{Z}_6[x]$ in which $2$ and $3x$ are zero-divisors. By Corollary 4.7, however, $2 + 3x$ is not a zero-divisor.

By $Z(R)$ we mean the set of zero-divisors of a ring $R$.

If $f$ is an element of a graded ring $R$, then $C(f)$ denotes the ideal of $R$ generated by the homogeneous components of $f$. It is the smallest graded ideal containing $Rf$.

**Corollary 4.10.** If $R = \bigoplus_{m \in M} R_m$ is an $M$-graded ring with $M$ a totally ordered cancellative monoid, then $Z(R) \subseteq \bigcup_{p \in \text{Spec}(R)} p^*$. 

Proof. If $f = \sum_{m \in M} r_m \in Z(R)$ with $r_m \in R_m$ for all $m$, then by Corollary 4.4 there exists a (nonzero) homogeneous $g \in R$ such that $fg = 0$ and so $r_m g = 0$ for all $m \in G$. Since $g \neq 0$, we get that $C(f)$ is a proper ideal of $R$. Thus, there exists a prime ideal $p$ of $R$ such that $f \in C(f) \subseteq p$. But $C(f) \subseteq p^*$, because $C(f)$ is a graded ideal. □

**Remark 4.11.** Note that Theorem 4.1 cannot be generalized to modules. More precisely, the statement “if $M$ is an unfaithful module over a $G$-graded
ring $R$ with $G$ a totally ordered group, then there exists a nonzero homogeneous $g \in R$ such that $gM = 0$ is false. As a counterexample, $M = \mathbb{Z}_2[x]/(1+x)$ is an unfaithful module over the \(\mathbb{N}\)-graded ring $R = \mathbb{Z}_2[x]$. Suppose that there is a nonzero homogeneous $g \in R$ such that $gM = 0$. Then $g \in \text{Ann}_R(M) = (1+x)$. Clearly, $g = x^d$ for some $d \geq 1$. Hence $x^d = (1+x)f(x)$. But $R$ is a UFD and the elements $x$ and $1+x$ are irreducible. This yields that $x = 1 + x$ and so $1 = 0$ which is a contradiction.

5. **Units in a graded ring**

In this section, we discover positive results concerning the $G$-graded version of the unit conjecture. In particular, we give a complete characterization of invertible elements in graded rings.

**Lemma 5.1.** Let $f$ be an invertible homogeneous element of an $M$-graded ring $R = \bigoplus_{m \in M} R_m$ with $M$ a cancellative monoid. Then $f^{-1}$ is homogeneous with $\deg(f) + \deg(f^{-1}) = 0$.

**Proof.** Suppose $f^{-1} = \sum_{m \in M} r_m$ where $r_m \in R_m$ for all $m$. We have $\sum_{m \in M} fr_m = 1$. Since $M$ has the cancellation property, so on the left hand side of the above equality, $fr_m$ is the only homogeneous element of degree $\deg(f) + m$ for all $m \in \text{Supp}(f^{-1})$. Thus there exists a (unique) $d \in M$ such that $fr_d = 1$ and $fr_m = 0$ and so $r_m = 0$ for all $m \neq d$. This shows that $f^{-1} = r_d$ and $\deg(f) + \deg(f^{-1}) = 0$. \(\square\)

**Lemma 5.2.** Every invertible element of a $G$-graded domain with $G$ a totally ordered group is homogeneous.

**Proof.** Let $R$ be a $G$-graded domain. Using that $R$ is a domain, we have $n_*(fg) = n_*(f) + n_*(g)$ and $n^*(fg) = n^*(f) + n^*(g)$ for all nonzero $f, g \in R$. If $fg = 1$, then $n_*(fg) = n^*(fg) = 0$. If $n_*(f) < n^*(f)$, then $0 = n_*(f) + n_*(g) < n^*(f) + n_*(g) \leq n^*(f) + n^*(g) = 0$ which is impossible. Therefore, $n_*(f) = n^*(f)$, showing that $f$ is homogeneous. \(\square\)

The above lemma also holds for non-commutative $G$-graded domains with $G$ a totally ordered group (not necessarily Abelian).

**Corollary 5.3.** If $M$ is a totally ordered cancellative monoid, then every invertible element of an $M$-graded integral domain is homogeneous.

**Proof.** It follows from Lemma 5.2 by passing to the $G$-grading where $G$ is the Grothendieck group of $M$. \(\square\)

By $U(R)$ we denote the group of units (invertible elements) of a ring $R$.

**Remark 5.4.** Let $R$ be an integral domain and $M$ a totally ordered monoid with the cancellation property. Then by Corollary 5.2, $R[M]$ is an integral domain. If $f$ is an invertible element of $R[M]$ then by Corollary 5.3, $f = ax$ (and by Lemma 5.1, $f^{-1} = a^{-1}e_y$) where $a \in U(R)$ and $x + y = 0$ for some
Corollary 5.5. If \( R \) is a domain and \( G \) a totally ordered group, then the units of the group-ring \( R[G] \) are precisely of the form \( r\epsilon_x \) with \( r \in U(R) \) and \( x \in G \).

Proof. We know that the group-ring \( R[G] = \bigoplus_{x \in G} S_x \) is a \( G \)-graded ring with the homogeneous components \( S_x = R\epsilon_x \). Then by Lemma 5.1 the zero ideal of \( R[G] \) is a prime ideal. Hence, \( R[G] \) is a \( G \)-graded domain. Now the assertion easily follows from Lemmas 5.1 and 5.2. \( \Box \)

Remark 5.6. Note that Lemma 5.2 cannot be generalized to \( G \)-graded rings which are only reduced (cf. Corollary 5.14) or whose base subrings \( R_0 \) are integral domains. For the first case, consider for instance the reduced \( \mathbb{Z} \)-graded ring \( \mathbb{Z}[x, x^{-1}] \) where \( \mathbb{Z}_0 = \mathbb{Z}/6\mathbb{Z} \) is the ring of integers modulo 6. The element \( g = 2x + 3x^{-1} \) is clearly not homogeneous, but it is invertible with the inverse \( g^{-1} = 2x^{-1} + 3x \). For the second case, consider the associated graded ring \( \text{gr}_p(R) = \bigoplus_{n \geq 0} p^n/p^{n+1} = R/p \oplus p/p^2 \oplus \cdots \) where \( p \) is a prime ideal of a ring \( R \) with the property that there is some \( f \in p \setminus p^2 \) such that \( f^2 \in p^3 \). In this case, \( f + p^2 \) is nilpotent, since \((f + p^2)^2 = f^2 + p^3 = 0 \). Thus, the element \((f^n + p^{n+1})_{n \geq 0} \) of \( \mathbb{Z}[x, x^{-1}] \) is not homogeneous, but invertible in \( \text{gr}_p(R) \), because the sum of an invertible element and a nilpotent element is invertible. Finding such a prime ideal is not hard. For instance, in the ring \( \mathbb{Z}_4 \) we may take \( p = \{0, 2\} \) and \( f = 2 \).

Lemma 5.7. Let \( f = \sum_{k=1}^n r_k \) be an element of a ring \( R \). If \( (r_1, \ldots, r_n) = R \) and \( r_i r_k \) is nilpotent for all \( i \neq k \), then \( f \) is invertible in \( R \).

Proof. It suffices to show that \( Rf = R \). If \( Rf \neq R \), then \( Rf \subseteq p \) for some prime ideal \( p \) of \( R \). But there exists some \( k \) such that \( r_k \notin p \). By assumption, \( r_i r_k \in p \) and hence \( r_i \in p \) for all \( i \neq k \). Thus, \( r_k = f - \sum_{i \neq k} r_i \in p \) which is a contradiction. \( \Box \)

Theorem 5.8. Let \( f = \sum_{k \in G} r_k \) be an element of a \( G \)-graded ring \( R = \bigoplus_{k \in G} R_k \) with \( G \) a totally ordered group and \( r_k \in R_k \) for all \( k \). Then \( f \) is invertible in \( R \) if and only if \( (r_k : k \in G) = R \) and \( r_i r_k \) is nilpotent for all \( i \neq k \).

Proof. If \( f \) is invertible in \( R \) then \( R = Rf \subseteq (r_k : k \in G) \subseteq R \) and so \( (r_k : k \in G) = R \). To prove that \( r_i r_k \) is nilpotent for \( i \neq k \), it suffices to
show that $r_rk \in p$ for every minimal prime ideal $p$ of $R$. Since $G$ is a totally ordered group, thus by Corollary 5.9, every minimal prime ideal of $R$ is a graded ideal. Hence, $R/p$ is a $G$-graded integral domain. Clearly $f + p$ is invertible in $R/p$. Thus, by Lemma 5.12, there exists some $\ell \in G$ such that $r_n \in p$ for all $n \neq \ell$. Then $r_i$ or $r_k$ is always a member of $p$ and so $r_rk \in p$ for all $i \neq k$. The reverse implication follows from Lemma 5.7.

**Corollary 5.9.** Let $f = \sum r_x \in M$ be an invertible element of an $M$-graded ring $R = \bigoplus_{x \in M} R_x$ with $M$ a totally ordered cancellative monoid and $r_x \in R_x$ for all $x$. If $r_0$ is invertible in $R$, then $r_x$ is nilpotent for all $x \neq 0$.

**Proof.** Clearly $f$ is invertible in the $G$-graded ring $R$ where $G$ is the Grothendieck group of $M$ which is a totally ordered group. Thus by Theorem 5.8, $r_0r_x$ and so $r_x$ are nilpotent for all $x \neq 0$.

**Example 5.10.** The above results (Lemma 5.2, Theorem 5.8 and Corollary 5.9) fail to hold if $G$ is not a totally ordered group. For example, $f = (1/3)(\epsilon_0 + \epsilon_1 + \epsilon_2)$ is an idempotent element of the group-ring $R = \mathbb{Q}G$ which is a $\mathbb{Z}_3$-graded ring. Hence, $1 - 2f = (1/3)(\epsilon_0 - 2\epsilon_1 - 2\epsilon_2)$ is invertible in $R$ which is not homogeneous (recall that if $e$ is an idempotent of a ring then $1 - 2e$ is invertible, because $(1 - 2e)^2 = 1$). Also neither $-2/3$ nor $((-2/3)\epsilon_1)((-2/3)\epsilon_2) = (4/9)\epsilon_0 = 4/9$ is nilpotent.

Recall that in any ring, the sum of an invertible element and a nilpotent element is an invertible element.

**Corollary 5.11.** Let $f = \sum r_x \in M$ be an element of an $M$-graded ring $R = \bigoplus_{x \in M} R_x$ with $M$ a totally ordered cancellative monoid and $r_x \in R_x$ for all $x$. If the identity element of $M$ is the smallest member, then $f$ is invertible in $R$ if and only if $r_0$ is invertible in $R_0$ and $r_x$ is nilpotent for all $x \neq 0$.

**Proof.** If $f$ is invertible in $R$ then $fg = 1$ for some $g = \sum r_x' \in R$. It follows that $r_0r_0' + \sum_{x+y=0, (x,y) \neq (0,0)} r_xr_y' = 1$. But if $(x, y) \neq (0, 0)$ for some $x, y \in M$ then $0 < x + y$, because the identity element of $M$ is the smallest member. This yields that $\sum_{x+y=0, (x,y) \neq (0,0)} r_xr_y' = 0$ and so $r_0r_0' = 1$. Then by Corollary 5.9 $r_x$ is nilpotent for all $x \neq 0$. The reverse implication is clear.

**Corollary 5.12.** Let $f = \sum r_n \in N$ be an element of an $N$-graded ring $R = \bigoplus_{n \geq 0} R_n$ with $r_n \in R_n$ for all $n$. Then $f$ is invertible in $R$ if and only if $r_0$ is invertible in $R_0$ and $r_n$ is nilpotent for all $n \geq 1$.

**Proof.** It follows from Corollary 5.11.
The following two results are immediate consequences of Corollary 5.12.

**Corollary 5.13.** In an \( N \)-graded ring, every invertible homogeneous element is of degree zero.

**Corollary 5.14.** In a reduced \( N \)-graded ring, every invertible element is homogeneous of degree zero.

**Corollary 5.15.** An element of the polynomial ring \( R[x_k : k \in I] \) is invertible if and only if its constant term in invertible in \( R \) and all the remaining coefficients are nilpotent.

**Proof.** The identity element of the additive monoid \( M = \bigoplus_{k \in I} N \) is the smallest element with respect to the lexicographical ordering. We know that \( R[x_k : k \in I] = R[M] \). Hence the assertion follows from Corollary 5.11 with taking into account that each variable \( x_k \) is a non-zero-divisor. \( \square \)

**Corollary 5.16.** Let \( R \) be an \( N \)-graded ring. Then \( R \) is reduced if and only if \( R_0 \) is reduced and \( U(R_0) = U(R) \).

**Proof.** The “only if” statement follows from Corollary 5.12. Conversely, it suffices to show that every nilpotent homogeneous element \( f \in R \) of positive degree is zero. Clearly, \( 1 + f \) is invertible in \( R \). Thus, by assumption, \( f = 0 \). \( \square \)

6. Homogeneity of the colon ideal

Recall that if \( I \) and \( J \) are ideals of a ring \( R \), then the ideal quotient (or, colon ideal) of \( I \) by \( J \) is the ideal \( I : R J = \{ r \in R : rJ \subseteq I \} \) of \( R \).

Let \( (X, <) \) be a well-ordered set and let \( P \) be a property of elements of \( X \), i.e. \( P(x) \) is a mathematical statement for all \( x \in X \). Suppose that whenever \( P(y) \) is true for all \( y < x \), then \( P(x) \) is also true. Then it is easy to see that \( P(x) \) is true for all \( x \in X \). This statement is called the transfinite induction. Every finite totally ordered set is well-ordered, and hence the transfinite induction can be applied for such sets. Using this weak version of the transfinite induction, we will establish the following general result which vastly generalizes Armendariz’ result to the more general setting of graded rings.

**Theorem 6.1.** Let \( I \) be a graded radical ideal of a \( G \)-graded ring \( R = \bigoplus_{i \in G} R_i \) with \( G \) a totally ordered group and \( J \) an arbitrary ideal of \( R \). Then \( I : R J \) is a graded ideal.

**Proof.** Take \( f = \sum_{i \in G} r_i \in I : R J \) where \( r_i \in R_i \) for all \( i \). We must prove that each \( r_i \in I : R J \). Take \( g = \sum_{k \in G} r'_k \in J \) where \( r'_k \in R_k \) for all \( k \). It suffices to show that \( r_i r'_k \in I \) for all \( i,k \in G \). Since \( I \) is a radical ideal, it
will be enough to show that for each pair \((i, k) \in X = \text{Supp}(f) \times \text{Supp}(g)\) then \(r_ir_i' \in p\) where \(p\) is a prime ideal of \(R\) containing \(I\). We will prove the assertion by induction on the finite well-ordered set \(S = \{i + k : (i, k) \in X\} = \{d + s, \ldots, m + l\}\) where \(f = r_d + \cdots + r_m\) and \(g = r_s + \cdots + r_l\) with \(d = n_s(f) \leq m = n^*(f)\) and \(s = n_s(g) \leq l = n^*(g)\). We have \((fg)_{d+s} = \sum_{i+k=d+s} r_ir_i'k\). If \(i+k = d+s\) and \((i, k) \neq (d, s)\), then since \(G\) is a totally ordered group, we have \(i < d\) or \(k < s\) and so \(r_ir_i'k = 0\). Thus \((fg)_{d+s} = r_ir_i'\). But \(r_ir_i' \in I \subseteq p\), because \(I\) is a graded ideal and \(fg \in I\). We have thus established the base case of the induction \((n = d+s)\). Assume now \(n > d+s\) with \(n \in S\). By the induction hypothesis, if \(i+k < n\) for some \((i, k) \in X\) then \(r_ir_i'k \in p\). Seeking a contradiction, suppose that there exists \((a, b) \in X\) with \(a+b = n\) for which \(r_ar_b' \notin p\). Using that \(I\) is graded, we have \((fg)_n = r_ar_b' + \sum_{i+k=n, (i,k) \neq (a,b)} r_ir_i'k \notin I \subseteq p\). If \(i+k = n\) and \((i, k) \neq (a, b)\), then \(i < a\) or \(a < i\). If \(i < a\), then \(i+b < n\), because \(G\) is a totally ordered group. Hence by the induction, \(r_ir_i' \in p\) which yields \(r_i \in p\) (since \(r_i' \notin p\)). Similarly, if \(a < i\), then \(a+k < n\) and hence by the induction, \(r_ar_k' \in p\) which yields \(r_k \in p\) (since \(r_k' \notin p\)). Therefore for any such pair \((i, k)\) we have \(r_ir_i' \in p\) and so \(\sum_{i+k=n, (i,k) \neq (a,b)} r_ir_i' \in p\). It follows that \(r_ar_b' \in p\) which is a contradiction. \(\square\)

**Remark 6.2.** Note that in the proof of Theorem 6.1 we cannot do the transfinite induction on the set \(\{n \in G : d+s \leq n \leq m+l\}\), because this totally ordered set is not necessarily well-ordered (nor finite). For example, if \(G = \mathbb{Z}^2\) then \((0,1) < (1,n) < (2,0)\) for all \(n \in \mathbb{Z}\). Also note that although \(I :_R J = \text{Ann}_R ((I+J)/I)\) as \(R\)-modules, we cannot deduce Theorem 6.1 from Theorem 4.1 because Theorem 4.1 cannot be generalized to the setting of modules (see Remark 4.11).

In addition to its generalization to the setting of graded rings, the other main novelty and power of the above theorem is that \(J\) is an arbitrary (not necessarily graded) ideal. Some consequences of the above theorem are given below.

**Corollary 6.3.** Let \(I\) be a graded radical ideal and \(J\) an arbitrary ideal of an \(M\)-graded ring \(R\) with \(M\) a totally ordered cancellative monoid. Then \(I :_R J\) is a graded ideal.

**Proof.** It follows from Theorem 6.1 by passing to the \(G\)-grading where \(G\) is the Grothendieck group of \(M\) which is a totally ordered group. \(\square\)

**Corollary 6.4.** If \(I\) is an ideal of a \(G\)-graded ring \(R\) with \(G\) a totally ordered group, then \(\mathfrak{N} :_R I\) is a graded ideal.

**Proof.** It follows from Theorem 6.1 because the nilradical of \(R\) is a graded radical ideal. \(\square\)
Corollary 6.5. If $I$ is an ideal of a reduced $G$-graded ring $R$ with $G$ a totally ordered group, then $\text{Ann}_R(I)$ is a graded ideal.

Proof. By Corollary [6.4] $\mathfrak{N}(R) :_R I = 0 :_R I = \text{Ann}_R(I)$ is a graded ideal. \hfill \Box

Corollary 6.6. Let $I$ be a graded radical ideal of a $G$-graded ring $R = \bigoplus_{i \in G} R_i$ and let $f = \sum_{i \in G} r_i$ and $g = \sum_{k \in G} r'_k$ be elements of $R$ with $G$ a totally ordered group and $r_i, r'_k \in R_i$ for all $i$. Then $fg \in I$ if and only if $r_i r'_k \in I$ for all $i, k \in G$.

Proof. If $fg \in I$, then $g \in I :_R Rf$. By Theorem [6.1] $I :_R Rf$ is a graded ideal. Thus, $fr'_k \in I$ for all $k \in G$. It follows that each $r_i r'_k \in I$, since $I$ is a graded ideal. The reverse implication is obvious. \hfill \Box

As an immediate consequence of Corollary 6.6, we get the following result.

Corollary 6.7. Let $f = \sum_{i \in G} r_i$ and $g = \sum_{k \in G} r'_k$ be elements of a $G$-graded ring $R = \bigoplus_{i \in G} R_i$ with $G$ a totally ordered group and $r_i, r'_k \in R_i$ for all $i$. Then $fg$ is nilpotent if and only if $r_i r'_k$ is nilpotent for all $i, k \in G$.

All of the above four results can be easily generalized to the setting of $M$-graded rings where $M$ is a totally ordered monoid with the cancellation property.

Example 6.8. (An ideal whose annihilator is not graded) In order to find an ideal in a $G$-graded ring (with $G$ a totally ordered group) whose annihilator is not a graded ideal, the ideal should not be a graded ideal and by Corollary 6.5 the ring must not be reduced. The question of whether such an ideal exists is highly interesting in the light of Theorem [6.1] and its consequences. Finding a concrete example of such an ideal is not an easy task, but fortunately Pierre Deligne has provided us with an example: Let $k$ be a field (or, an integral domain) and let $R$ be the polynomial ring $k[x_1, x_2, x_3, x_4]$ modulo the ideal $I = (x_1x_3, x_2x_4, x_1x_4 + x_2x_3)$. Then in the $\mathbb{N}$-graded polynomial ring $S := R[T]$ with $\deg(T) = 1$ we have $(a_1T + a_2)(a_3T + a_4) = 0$ where $a_i := x_i + I$. Thus, $a_1T + a_2 \in \text{Ann}_S(a_3T + a_4)$, but the annihilator does not contain $a_1T$ nor $a_2$, because $x_1x_4, x_2x_3 \notin I$. Hence, $\text{Ann}_S(a_3T + a_4)$ is not a graded ideal of $S$.

Recall that if $I$ is a nonempty subset of a ring $R$ and $M$ is a monoid, then by $I[M]$ we mean the set of all $\sum_{m \in M} r_m e_m \in R[M]$ such that $r_m \in I$ for all $m \in M$. If $I$ is an ideal of $R$ then $I[M]$ is a graded ideal of $R[M]$. In fact, $I[M]$ is the extension of $I$ under the canonical ring map $R \to R[M]$. It is also clear that the contraction of $I[M]$ under this map equals $I$. We have then the following result which codifies some important properties of monoid-rings.
Theorem 6.9. Let $R$ be a ring and $M$ a totally ordered cancellative monoid. Then the following assertions hold.

(i) A nonempty subset $p$ of $R$ is a prime ideal of $R$ if and only if $p[M]$ is a prime ideal of $R[M]$. In particular, $R$ is a domain if and only if $R[M]$ is a domain.

(ii) The minimal primes of $R[M]$ are precisely of the form $p[M]$ where $p$ is a minimal prime of $R$.

(iii) The nilradical of $R[M]$ is of the form $\mathfrak{N}[M]$ where $\mathfrak{N}$ is the nilradical of $R$.

(iv) $R$ is a reduced ring if and only if $R[M]$ is reduced.

(v) $Z(R[M]) \subseteq \bigcup_{p \in \text{Spec}(R)} p[M]$.

(vi) If $R$ is a zero-dimensional ring, then $Z(R[M]) = \bigcup_{p \in \text{Spec}(R)} p[M]$.

(vii) For given $f = \sum_{x \in M} r_x \epsilon_x$ and $g = \sum_{y \in M} r'_y \epsilon_y$ in $R[M]$, then $fg$ is nilpotent if and only if $r_x r'_y$ is nilpotent for all $x, y \in M$.

(viii) An ideal $I$ of $R$ is faithful if and only if $I[M]$ is a faithful ideal of $R[M]$.

Proof. (i): If $p$ is a prime ideal of $R$, then by Corollary 3.2, $p[M]$ is a prime ideal of $R$. Conversely, it can be easily seen that for a nonempty subset $p \subseteq R$, if $p[M]$ is a (prime) ideal of $R[M]$ then $p$ is a (prime) ideal of $R$.

(ii): If $p$ is a minimal prime ideal of $R$, then by (ii), $p[M]$ is a prime ideal of $R$. Let $Q$ be a minimal prime of $R[M]$ with $Q \subseteq p[M]$. Then $Q \cap R \subseteq p[M] \cap R = p$ and so $Q \cap R = p$. This yields that $Q = p[M]$. Now let $P$ be a minimal prime of $R[M]$. Then $p := P \cap R$ is a minimal prime of $R$ and it is easy to see that $p[M] \subseteq P$. By the $M$-graded version of Corollary 3.3, $P$ is a graded ideal of $R[M]$. Hence, to see the inclusion $P \subseteq p[M]$, it suffices to check it for homogeneous elements of $P$. Take $r \epsilon_m \in P$ where $r \in R$ and $m \in M$. We may write $r \epsilon_m = (r \epsilon_0) \epsilon_m$. But $\epsilon_m \notin P$, because $\epsilon_m$ is a non-zero-divisor of $R[M]$ while $P$ is a minimal prime of $R[M]$ and hence it is contained in the set of zero-divisors of $R[M]$. It follows that $r = r \epsilon_0 \in P \cap R = p$. Hence, $r \epsilon_m \in p[M]$.

(iii): It is clear that $\mathfrak{N}[M]$ is contained in the nilradical of $R[M]$. By the $M$-graded version of Corollary 3.3, the nilradical of $R[M]$ is a graded ideal. Hence, to prove the reverse inclusion, it suffices to check it for homogeneous nilpotents. If $r \epsilon_m = (r \epsilon_0) \epsilon_m$ is nilpotent with $r \in R$ and $m \in M$, then $r = r \epsilon_0$ is nilpotent because $\epsilon_m$ is a non-zero-divisor. Thus $r \epsilon_m \in \mathfrak{N}[M]$.

(iv): It is clear from (iii).

(v): If $f = \sum_{m \in M} r_m \epsilon_m$ is a zero-divisor of $R[M]$ then by Corollary 4.3 there exists a nonzero $a \in R$ such that $af = 0$. It follows that $ar_m = 0$ for all $m \in M$. Consider the ideal $I = (r_m : m \in M)$ of $R$. Then $I$ is a proper ideal of $R$, because $a \neq 0$. Thus, there exists a prime ideal $p$ of $R$ such that $I \subseteq p$ and so $f \in p[M]$.

(vi): If $p$ is a prime ideal of $R$ then by (ii), $p[M]$ is a minimal prime of $R[M]$.
But it is well-known that every minimal prime of a ring is contained in the set of zero-divisors. So \( \bigcup_{p \in \text{Spec}(R)} p[M] \subseteq Z(R[M]) \). The reverse inclusion follows from (v).

(vii): By the \( M \)-graded version of Corollary 6.7, \( fg \) is nilpotent if and only if every \( (r_x \epsilon_x)(r'_y \epsilon_y) = r_x r'_y \epsilon_{x+y} \) is nilpotent. But \( \epsilon_{x+y} \) is a non-zero-divisor. Hence, \( fg \) is nilpotent if and only if each \( r_x r'_y \) is nilpotent.

(viii): Assume \( I \) is faithful. If \( I[M] \) is an unfaithful ideal of \( R[M] \), then by Corollary 4.3, \((r_m)I[M] = 0 \) for some \( 0 \neq r \in R \) and some \( m \in M \). But \( \epsilon_m \) is a non-zero-divisor, and hence \( r \in \text{Ann}(I) = 0 \) which is a contradiction. The reverse implication is clear. \( \square \)

The above theorem, in particular, can be applied to the ring of polynomials as well as to the ring of Laurent polynomials with any (finite or infinite) number of variables. Theorem 6.9(vii) generalizes Armendariz’ result \([3]\) Lemma 1 and also \([2]\) Proposition 2.1 to the more general setting of monoid-rings.

**Example 6.10.** Theorem 6.9 does not hold in general. For example, let \( G \) be an Abelian group which is not a totally ordered group. So \( G \) has a nonzero element \( g \) of finite order \( n \geq 2 \). Let \( p \) be a prime number which divides \( n \). Then \( x := (n/p)g \) is a nonzero element of order \( p \). Let \( R \) be an integral domain (or, a reduced ring) of characteristic \( p \). Then the group-ring \( R[G] \) is not reduced, because the Frobenius endomorphism yields that \( (\epsilon_0 - \epsilon_x)^p = (\epsilon_0)^p - (\epsilon_x)^p = \epsilon_0 - \epsilon_{px} = 0 \), but \( \epsilon_0 \neq \epsilon_x \). Furthermore, \( \epsilon_0 - \epsilon_x \) is a zero-divisor of \( R[G] \) but it is not contained in \( \bigcup_{p \in \text{Spec}(R)} p[G] \).

**Remark 6.11.** Here we describe an interesting connection between the general graded rings and monoid-rings (which is so useful in providing simple alternative proofs or discovering new observations in graded ring theory): Let \( R = \bigoplus_{m \in M} R_m \) be an \( M \)-graded ring with \( 1 \in R_0 \) where \( 0 \) is the identity element of \( M \) (e.g. \( M \) is a cancellative monoid). Then \( R \) can be naturally viewed as a graded subring of the monoid-ring \( R[M] \) via the embedding \( \theta : R \to R[M] \) which sends each \( r_m \in R_m \) into \( r_m \epsilon_m \) for all \( m \in M \). In particular, every \( \mathbb{N} \)-graded ring \( R = \bigoplus_{n \geq 0} R_n \) can be viewed as a graded subring of the polynomial ring \( R[x] \) via the embedding \( R \to R[x] \) which sends each \( r \in R_n \) into \( rx^n \) for all \( n \geq 0 \), and every \( \mathbb{Z} \)-graded ring \( R = \bigoplus_{n \in \mathbb{Z}} R_n \) can be viewed as a graded subring of the ring of Laurent polynomials \( R[x, x^{-1}] \) via the embedding \( R \to R[x, x^{-1}] \) which sends each \( r \in R_n \) into \( rx^n \) for all \( n \in \mathbb{Z} \). The ring \( R[x, x^{-1}] \) also can be viewed as a subring of the formal power series ring \( R[[x]] \). Indeed, using the universal property of polynomial rings, we obtain an injective morphism of \( R \)-algebras \( R[x] \to R[[x]] \) given by \( x \mapsto 1 - x \). But \( 1 - x \) is invertible in \( R[[x]] \) (remember that an element of \( R[[x]] \) is invertible if and only if its constant term is invertible in \( R \)). Thus
by the universal property of localizations, we get an injective morphism of $R$-algebras $\psi : R[x, x^{-1}] \to R[[x]]$ with $\psi(x) = 1 - x$ and $\psi(x^{-1}) = (1 - x)^{-1} = \sum_{k>0} x^k$. Finally, if $M$ is a totally ordered monoid with the cancellation property then an ideal $I$ of $R$ is faithful if and only if its extension under the above embedding $\theta : R \to R[M]$ is a faithful ideal of $R[M]$. Indeed, assume $I$ is faithful. If the extension ideal $I^e$ is unfaithful then by Corollary 4.5, $rI^e = 0$ for some $0 \neq r \in R$. If $f = \sum_{x \in M} r_x \in I$ then $\theta(f) = \sum_{x \in M} r_x \epsilon_x \in I^e$ and so $\sum_{x \in M} rr_x \epsilon_x = 0$. Since $M$ has the cancellation property, $rr_x \epsilon_x = 0$ and so $rr_x = 0$ for all $x \in M$. This shows that $rf = 0$. Hence, $r \in \text{Ann}_R(I) = 0$ which is a contradiction. The reverse implication is proved similarly, by applying Corollary 4.3 instead of Corollary 4.5.

References

[1] D. D. Anderson and V. Camillo, Armendariz rings and Gaussian rings, Comm. Algebra 26(7) (1998) 2265-2272.
[2] R. Antoine, Nilpotent elements and Armendariz rings, J. Algebra, 319(8) (2008) 3128-3140.
[3] E. P. Armendariz, A note on extensions of Baer and p.p. rings, J. Aust. Math. Soc. 18(4) (1974) 470-473.
[4] M. Atiyah and I. G. MacDonald, Introduction to commutative algebra, Addison-Wesley, (1969).
[5] V. Camillo and P. P. Nielsen, McCoy rings and zero-divisors, J. Pure Appl. Algebra 212(3) (2008) 599-615.
[6] G. H. Cliff, Zero divisors and idempotents in group rings, Canadian J. Math. 32(3) (1980) 596-602.
[7] D. A. Craven and P. Pappas, On the unit conjecture for supersoluble group algebras, J. Algebra, 394 (2013) 310-356.
[8] A. Forsythe, Divisors of zero in polynomial rings, Amer. Math. Monthly, 50(1) (1943) 7-8.
[9] G. Gardam, A counterexample to the unit conjecture for group rings, Ann. Math. 194(3) (2021) 967-979.
[10] G. Higman, The units of group-rings, Proc. London Math. Soc. s2-46(1) (1940) 231-248.
[11] C. Y. Hong, Y. C. Jeon, N. K. Kim and Y. Lee, The McCoy Condition on Noncommutative Rings, Comm. Algebra 39(5) (2011) 1809-1825.
[12] C. Y. Hong, N. K. Kim and Y. Lee, Extensions of McCoy’s theorem, Glasgow Math. J. 52 (2010) 155-159.
[13] Y. C. Jeon, N. K. Kim, Y. Lee and J. S. Yoon, On weak Armendariz ring, Bull. Korean Math. Soc. 46(1) (2009) 135-146.
[14] N. K. Kim and Y. Lee, Armendariz rings and reduced rings, J. Algebra, 223(2) (2000) 477-488.
[15] H. Kropholler, et al., Applications of a new $K$-theoretic theorem to soluble group rings, Proc. Amer. Math. Soc. 104(3) (1988) 675-684.
[16] T. K. Kwak, Y. Lee and S. J. Yun, The Armendariz property on ideals, J. Algebra, 354(1) (2012) 121-135.
[17] T. Y. Lam, A First Course in Noncommutative Rings, Springer-Verlag, (2001).
[18] F. W. Levi, Ordered groups, Proc. Indian Acad. Sci. A16 (4) (1942) 256-263.
[19] N. H. McCoy, Remarks on divisors of zero, Amer. Math. Monthly, 49(5) (1942) 286-295.
[20] N. H. McCoy, Annihilators in polynomial rings, Amer. Math. Monthly 64(1) (1957), 28-29.
[21] M. Mirowicz, Units in group rings of the infinite dihedral group, Canad. Math. Bull. 34(1) (1991) 83-89.
[22] P. P. Nielsen, Semi-commutativity and the McCoy condition, J. Algebra, 298(1) (2006) 134-141.
[23] D.G. Northcott, Lessons on Rings, Modules and Multiplicities, Cambridge University Press (1968).
[24] D. S. Passman, The Algebraic Structure of Group Rings, John Wiley and Sons, Inc. (1977).
[25] O. Prakash, S. Singh and K. P. Shum, On almost Armendariz ring, Algebra Colloquium, 27(2) (2020) 199-212.
[26] M. B. Rege and S. Chhawchharia, Armendariz rings, Proc. Japan Acad. Ser. A Math. Sci. 73 (1997) 14-17.
[27] W. R. Scott, Divisors of Zero in Polynomial Rings, Amer. Math. Monthly, 61(5) (1954) p. 336.
[28] A. Tarizadeh, A fresh look into monoid rings and formal power series rings, J. Algebra Appl. 19(1) (2020) 2050003.

Department of Mathematics, Faculty of Basic Sciences, University of Maragheh, Maragheh, East Azerbaijan Province, Iran.

Email address: ebulfaz1978@gmail.com