STABILIZATION OF SOME ELASTODYNAMIC SYSTEMS WITH LOCALIZED KELVIN-VOIGT DAMPING

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Abstract. We consider the dynamic elasticity equations with a locally distributed damping of Kelvin-Voigt type in a bounded domain. The damping is localized in a suitable open subset, of the domain under consideration, which satisfies the piecewise multipliers condition of Liu. Using multiplier techniques combined with the frequency domain method, we show that: i) the energy of this system decays polynomially when the damping coefficient is only bounded measurable, ii) the energy of this system decays exponentially when the damping coefficient as well as its gradient are bounded measurable, and the damping coefficient further satisfies a structural condition. These results generalize and improve, at the same time, on an earlier result of Liu and Rao involving the wave equation with Kelvin-Voigt damping; those authors proved the exponential decay of the energy provided that the damping region is a neighborhood of the whole boundary, and further restrictions are imposed on the damping coefficient.

1. Introduction and statement of main results. The stabilization of the wave equation with localized damping has received a particular attention since the seventies e.g. [7, 14, 15, 16, 17, 21, 22, 23, 35, 41, 44, 46, 49, 50, 53, 56, 57, 58, 59, 60, 63, 64, 72, 73]. The stabilization of locally damped elastodynamic systems is poorly covered; there are only a few papers including [5, 20, 61, 62] in the literature. However it should be noted that the situation is completely different for the boundary stabilization of elastodynamic systems; beginning with the work of Lagnese [31], numerous other papers discuss this topic e.g.[1, 33, 26, 32, 47] just to name a few. The purpose of this work is to study the stabilization of a material composed of two parts: one that is elastic and the other one that is a Kelvin-Voigt type viscoelastic material. This type of material is encountered in real life when one uses patches to suppress vibrations, the modeling aspect of which may be found in [6]. This type of question was examined in the one-dimensional setting in [42] where it was shown that the longitudinal motion of an Euler-Bernoulli beam modeled by a locally damped wave equation with Kelvin-Voigt damping is not exponentially stable when the junction between the elastic part and the viscoelastic part of the beam is not smooth enough. Later on, the wave equation with Kelvin-Voigt damping in the multidimensional setting was examined in [44]; in particular, those authors

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showed the exponential decay of the energy by assuming that \( \omega \) is a neighborhood of the whole boundary, and the damping coefficient \( a \) satisfies \([43, 44]\): 
\[
a \in C^{1,1}(\Omega), \quad \Delta a \in L^\infty(\Omega), \quad \text{and } |\nabla a(x)|^2 \leq M_0 a(x), \quad \text{for almost every } x \in \Omega, \text{ for some positive constant } M_0.
\]
The author of this paper showed in an earlier note that the exponential decay of the energy could be obtained without imposing \( \Delta a \in L^\infty(\Omega) \), and for a larger class of feedback regions \( \omega \) \([64]\). The main purpose of the present work is to investigate what happens for elastodynamic systems; in particular, extra technical difficulties arise due to the fact that we are dealing with non-homogeneous and non-isotropic materials.

For the sequel we need some notations. Let \( \Omega \) be a bounded nonempty subset of \( \mathbb{R}^N \) with boundary \( \Gamma \) of class \( C^2 \), where \( \nu \) denotes the unit normal vector pointing into the exterior of \( \Omega \). Let \( \omega \) be a nonvoid open subset contained in \( \Omega \). Throughout the paper subscripts following a comma stand for differentiation, and we use the Einstein summation convention on repeated indices.

Consider the damped elasticity system
\[
\begin{align*}
y_{i,tt} - \sigma_{ij,j}(y) - (a(x)\hat{\sigma}_{ij}(y,t)), \, j &= 0 \text{ in } \Omega \times (0, \infty) \\
y_i &= 0 \text{ on } \Gamma \times (0, \infty) \\
y_i(0) &= y_i^0, \quad y_i(t) = y_i^1, \quad i = 1, 2, \ldots, N,
\end{align*}
\]
where \( a : \Omega \rightarrow \mathbb{R} \) is a nonnegative measurable function.

In (1) the elasticity stress tensor \((\sigma_{ij})\) is given by
\[
\sigma_{ij}(y) = a_{ijkl}\varepsilon_{kl}(y),
\]
where \((\varepsilon_{kl})\) defined by
\[
\varepsilon_{kl}(y) = \frac{1}{2}(y_{k,l} + y_{l,k})
\]
is the strain tensor. The \( a_{ijkl} \) are the elasticity coefficients. They satisfy the symmetry properties
\[
a_{ijkl} = a_{jikl} = a_{klij}, \quad \forall i, j, k, l.
\]
Throughout the paper we assume that the \( a_{ijkl} \) depend on the space variable \( x \) but not on time, and that they are continuously differentiable, and satisfy the ellipticity condition
\[
\exists m_0 > 0 : a_{ijkl}u_{ij}u_{kl} \geq m_0 u_{ij}u_{kl} \quad \text{for all second order symmetric tensors } (u_{ij}).
\]

The tensor \((\hat{\sigma}_{ij})\) is given by
\[
\hat{\sigma}_{ij} = \hat{\sigma}_{ij}(y_t) = c_{ijkl}\varepsilon_{kl}(y_t),
\]
where the coefficients \( c_{ijkl} \) satisfy the same properties as the elasticity coefficients \( a_{ijkl} \).

We note that when the space dimension \( N = 2, 3 \), and the damping coefficient \( a \equiv 1 \), system (1) models the small vibrations of an elastic material with short memory (cf. Chap. 3, Sect. 6 in \([19]\)). System (1) where the damping coefficient is allowed to vanish outside \( \omega \) may be viewed as a model for the interaction between an elastic material (portion of the domain where the damping coefficient \( a \equiv 0 \), and a viscoelastic material of Kelvin-Voigt type, portion of the domain where \( a(x) > 0 \).

Under the above assumptions on the coefficients, if for all \( i \), \( (y_i^0, y_i^1) \in H_0^1(\Omega) \times L^2(\Omega) \), then it is well-known that System (1) has a unique weak solution
\[
y \in C([0, \infty); [H_0^1(\Omega)]^N) \cap C^1([0, \infty); [L^2(\Omega)]^N).
\]

(3)
Similarly if for all \(i\), \((y_0^i, y_1^i) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)\) then it can be shown that the unique solution of System (1) satisfies

\[
y \in C([0, \infty); [H^1_0(\Omega)]^N) \cap C^1([0, \infty); [H^1_0(\Omega)]^N). \tag{4}
\]

If one pays attention to (4), one notices that there is a discrepancy on the regularity of the initial data and that of the solutions; this is due to the structure of the Kelvin-Voigt damping. This makes the stabilization problem much more difficult to solve than in the case of a frictional damping \(a(x)y_t\). As we shall see in the proof of the exponential stabilization result later on, we need to introduce a new variable and a cohort of suitable auxiliary elliptic systems to cope with this loss of derivative. This loss of derivative seems intuitively unbelievable since strong damping would usually make the solution smoother than the initial data as the dynamical system evolves with time, but in the present framework where the strong dissipation is localized, the smoothing effect is also localized; in other words, there is no smoothing on the whole domain under consideration.

Introduce the energy

\[
E(t) = \frac{1}{2} \int_\Omega \{|y_t(x, t)|^2 + (\sigma_{ij}(y)\varepsilon_{ij}(y))(x, t)\} \, dx, \quad \forall t \geq 0. \tag{5}
\]

The energy \(E\) is a nonincreasing function of the time variable \(t\) and its derivative satisfies

\[
E'(t) = -\int_\Omega a(x)\hat{\sigma}_{ij}(y, t)\varepsilon_{ij}(y, t) \, dx, \quad \forall t \geq 0. \tag{6}
\]

The questions that we would like to address in the rest of this note are:

1. Does the energy \(E(t)\) go to zero as the time variable \(t\) tends to infinity?
2. If so, then how fast does \(E(t)\) decay to zero?

Before stating our main results we need some additional notations. Setting \(Au = -\operatorname{div}(\sigma(u))\), and \(Z = \begin{pmatrix} y \\ y_t \end{pmatrix}\) (1) may be recast as:

\[
\begin{cases}
Z' - AZ = 0 \text{ in } (0, \infty), \\
Z(0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},
\end{cases} \tag{7}
\]

where the unbounded operator \(A\) is given by

\[
A = \begin{pmatrix} 0 & I \\ -A & \operatorname{div}(a\hat{\sigma}) \end{pmatrix} \tag{8}
\]

with \(D(A) = \{ (u, v) \in [H^1_0(\Omega)]^N \times [H^1_0(\Omega)]^N; Au - \operatorname{div}(a\hat{\sigma}(v)) \in [L^2(\Omega)]^N \}\).

Introduce the Hilbert space over the field \(\mathbb{C}\) of complex numbers \(\mathcal{H} = [H^1_0(\Omega)]^N \times [L^2(\Omega)]^N\), equipped with the norm (a norm indeed thanks to Korn’s inequality)

\[
||Z||_\mathcal{H}^2 = \int_\Omega \{|v|^2 + \sigma_{ij}(u)\varepsilon_{ij}(u)\} \, dx, \quad \forall Z = (u, v) \in \mathcal{H}. \tag{9}
\]

We now introduce a geometric constraint (GC) on the subset \(\omega\) where the dissipation is effective; we proceed as in [41], Theorem 4.1, and [46], pp. 93-95 (GC). (See Remark 1.1 below for more details.) There exist open sets \(\Omega_j \subset \Omega\) with piecewise smooth boundary \(\partial\Omega_j\), and points \(x_0^j \in \mathbb{R}^N, \ j = 1, 2, \ldots, J\), such
that $\Omega_i \cap \Omega_j = \emptyset$, for any $1 \leq i < j \leq J$, and:

$$\Omega \cap \mathcal{N}_\delta \left[ \left( \bigcup_{j=1}^J \Gamma_j \right) \bigcup \left( \Omega \setminus \bigcup_{j=1}^J \Omega_j \right) \right] \subset \omega,$$

for some $\delta > 0$, where

$$\mathcal{N}_\delta(S) = \bigcup_{x \in S} \{ y \in \mathbb{R}^N ; |x - y| < \delta \}, \text{ for } S \subset \mathbb{R}^N,$$

$$\Gamma_j = \{ x \in \partial \Omega_j ; m^j(x) \cdot \nu^j(x) > 0 \},$$

$\nu^j$ being the unit normal vector pointing into the exterior of $\Omega_j$, and $m^j(x) = x - x^j_0$.

**Remark 1.** We note that the geometric condition just stated is a generalization of the geometric condition found in Lions’ book on controllability [40], where the control region is a neighborhood of the portion of the boundary $\Gamma(x_0) = \{ x \in \partial \Omega ; (x - x_0) \cdot \nu(x) \geq 0 \}$, for some $x_0 \in \mathbb{R}^N$. The point $x_0$ may be viewed as the position of an observer, and $\Gamma(x_0)$ is the portion of the boundary that is invisible to the observer. That control region naturally arises when one uses a multiplier built using the radial vector field $m(x) = x - x_0$. The idea of Liu in [41] was to build a larger class of control regions by using, instead of one observer, finitely many observers. So the geometric constraint (GC) provides a much larger class of admissible control regions. One cannot obtain in the above figure $\omega$ as a neighborhood of a diagonal by using a single observer; instead, two observers are needed. The use of a single observer would require $\omega$ to be a neighborhood of two adjacent sides. We note that, when $J = 1$, (GC) reduces to the geometric constraint found in Lions’ book. We emphasize that the constraint (GC) is a sufficient condition, not a necessary one.
In the sequel, $|u|_q$ denotes the $L^q(\Omega)$-norm of $u$ when $q \geq 1$. We are now in the position to state our main results:

**Theorem 1.1.** Suppose that $\omega$ is an arbitrary nonempty open set in $\Omega$. Let the damping coefficient $\alpha$ be bounded measurable, and positive in $\omega$. The operator $A$ generates a $C_0$ semigroup of contractions $(S(t))_{t \geq 0}$ on $H$. In addition, the following unique continuation result holds: Given any nonzero $b \in \mathbb{R}$, 

$$
\begin{align*}
\left\{ \begin{array}{l l}
b^2u + \text{div}(\sigma(u)) = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega \\
u \equiv 0 & \text{in } \omega
\end{array} \right. \\
\Rightarrow u \equiv 0 \text{ in } \Omega,
\end{align*}
$$

then the semigroup contractions $(S(t))_{t \geq 0}$ is strongly stable:

$$
\lim_{t \to \infty} ||S(t)Z^0||_{H} = 0, \ \forall Z^0 \in H.
$$

Furthermore, if $\alpha$ satisfies

$$
a \in L^\infty(\Omega) \text{ and } \exists \alpha_0 > 0 : \alpha(x) \geq \alpha_0, \ a.e. \ x \in \omega, \ (12)
$$

$\omega$ satisfies the geometric constraint (GC) above, and the elasticity coefficients $a_{ijkl}$ satisfy for all symmetric tensors $\xi_{pq}$, (keeping in mind that $\tr(x) = x - x_0^1$, for each $j = 1, 2, ..., J$)

$$
(3a_{pqr}(x) - 2m_{x}^{2}(x)\partial_{p}a_{pqs}(x))\xi_{r}\xi_{spq} \geq 0, \ \forall x \in \Omega, \ \forall j.
$$

Then, we have the polynomial decay estimate

$$
\exists C_0 > 0 : ||S(t)Z^0||_H \leq \frac{C_0||Z^0||_{D(A)}}{(1 + t)^{\frac{3}{2}}}, \ \forall t \geq 0, \ \forall Z^0 \in D(A).\ (14)
$$

**Theorem 1.2.** Suppose that $\omega$ satisfies the geometric constraint (GC) above. Assume that the elasticity coefficients $a_{pqr}$ satisfy (13) as well as the proportionality condition $c_{pqr} = \gamma a_{pqr}$ for all $p, q, r, s$ and for some positive constant $\gamma$. As for the damping coefficient $\alpha$, suppose that it satisfies the coerciveness condition of (12) in some open $\omega_1$ of $\omega$, with $\omega_1 \subset \subset \omega$ satisfying the constraint (GC) as well. Moreover, assume that $a \in W^{1,\infty}(\Omega)$ with $|\nabla a(x)|^2 \leq M_0 a(x)$, for almost every $x$ in $\Omega$, for some positive constant $M_0$. The semigroup $(S(t))_{t \geq 0}$ is exponentially stable, viz., there exist positive constants $M$ and $\lambda$ with

$$
||S(t)Z^0||_H \leq M \exp(-\lambda t)||Z^0||_H, \ \forall Z^0 \in H. \ (15)
$$

**Remark 2.** The unique continuation (10) is valid in many instances including the case where the elastic medium is analytic, and special cases with less smoothness assumptions on the medium:

- in [48], $\Omega$ is just a bounded domain in $\mathbb{R}^2$, the elasticity coefficients satisfy $a_{ijkl} \in W^{1,\infty}(\Omega)$, and are further required to meet a special spectral condition (see (2.17) in [48])
- in [2], $\Omega$ is a bounded domain in $\mathbb{R}^N$, $(N \geq 2)$, and the material is isotropic: $a_{ijkl}(x) = 2\mu(x)\delta_{ik}\delta_{jh} + \lambda(x)\delta_{ij}\delta_{kh}$, where $\lambda, \mu \in C^{1,1}(\Omega)$ with $\mu(x) \geq \alpha_0$ and $2\mu(x) + \lambda(x) \geq \beta_0$, for all $x \in \Omega$, for some positive constants $\alpha_0$ and $\beta_0$.
- in [70], $\Omega$ is a bounded domain in $\mathbb{R}^3$, $(N = 3)$, and the material is isotropic as in the second point above, with $\lambda$ and $\mu$ being Lipschitz continuous, and further satisfying the coerciveness conditions above.
Other works about the unique continuation for the Lamé system of elasticity include e.g. \([3, 18, 39]\). We want to draw the reader’s attention to the fact that the unique continuation property(10) is needed in the proof of strong stability only; it is not needed in the proof of polynomial or exponential decay of the semigroup, as we are able to get precise decay estimates by weakening the hypotheses on the feedback control support \(\omega\), and on the elasticity coefficients.

**Remark 3.** Hypothesis (13), or its analogue found in Lagnese work \([31]\), Theorem 1.1 (hypothesis (1.12)), is required in general when dealing with variable coefficients; we use this inequality in estimate (44) later on. We shall emphasize here that the hypothesis (13) is automatically satisfied when the elasticity coefficients are constant, which is the case for homogeneous elastic materials. We also note that (13) holds when each elasticity coefficient is much larger in magnitude than its gradient:

\[
\frac{3}{2} m_0 - R||\nabla A||_\infty \geq 0,
\]

where \(m_0\) is the positive constant appearing in the ellipticity hypothesis (2), \(R = \max\{|x - x_0^i|; x \in \Omega \text{ and } 1 \leq j \leq J\}\), and \(||\nabla A||_\infty = \max\{||\nabla a_{ijkh}(x)||; x \in \Omega \text{ and } 1 \leq i, j, k, h \leq N\}\). However, when the domain \(\Omega\) and the coefficients of the principal operator are smooth enough, microlocal analysis or Riemann geometry approach \([7, 12, 36, 68, 69]\) makes that hypothesis unnecessary in the case of the wave equation; in particular hypothesis (13) is not needed in the proofs of Theorems 1.1 & 1.2 when \(N = 1\). To the best of the author’s knowledge, getting rid of (13) for elastodynamic systems remains an open problem.

**Remark 4.** When \(N = 3\), and

\[
a_{ijkl} = 2c^2 \delta_{ik} \delta_{jl} - c^2 \delta_{ij} \delta_{kl},
\]

where \(c\) is a positive constant related to Lamé constants, and \(\delta_{ij}\) denotes the Kronecker symbol, our system reduces to a system of uncoupled wave equations; so our results encompass earlier results involving the wave equation.

The rest of the paper is organized as follows: Section 2 is devoted to some preliminary technical lemmas while in Section 3 we provide the proofs of Theorems 1.1 and 1.2. Section 4 deals with some further comments and open problems.

2. Some technical lemmas.

**Lemma 2.1.** ([24, 52]) Let \(A\) be the generator of a bounded \(C_0\) semigroup \((S(t))_{t \geq 0}\) on a Hilbert space \(\mathcal{H}\). Then \((S(t))_{t \geq 0}\) is exponentially stable if and only if:

1) \(i\mathbb{R} \subset \rho(A)\), and

2) \(\sup\{||((ib - A)^{-1})||; b \in \mathbb{R}\} < \infty\), where \(\rho(A)\) denotes the resolvent of \(A\).

**Lemma 2.2.** ([10]) Let \(A\) be the generator of a bounded \(C_0\) semigroup \((S(t))_{t \geq 0}\) on a Hilbert space \(\mathcal{H}\) such that \(i\mathbb{R} \subset \rho(A)\), where \(\rho(A)\) denotes the resolvent of \(A\). Then \((S(t))_{t \geq 0}\) is polynomially stable, viz., there are positive constants \(M\) and \(\alpha\) that are independent of the initial data such

\[
||S(t)Z^0||_\mathcal{H} \leq \frac{M||Z^0||_{D(A)}}{(1 + t)^{\frac{\alpha}{2}}}, \quad \forall t \geq 0, \quad Z^0 \in D(A)
\]

if and only if

\[
\exists C_0 > 0 : ||(ib - A)^{-1}||_{\mathcal{L}(\mathcal{H})} \leq C_0|b|^\alpha, \forall b \in \mathbb{R} \text{ with } |b| \geq 1.
\]
Weaker versions of Lemma 2.2 may be found in [8, 9, 45].

3. Proofs of Theorems 1.1 and 1.2. The energy decay estimates will be derived from resolvent estimates. For that derivation, we will rely on Lemma 2.2 for the case of Theorem 1.1, and Lemma 2.1 for the case of Theorem 1.2.

Proof of Theorem 1.1. This proof will be established in two steps. Step 1. Well-posedness and strong stability. First, we shall prove that the unbounded operator \( A \) generates a \( C_0 \) semigroup of contractions \( (S(t))_{t \geq 0} \), then we shall show that \( i \mathbb{R} \subset \rho(A) \).

We have:
- the operator \( A \) is dissipative as:
  \[
  \Re(AZ, Z) = - \int_{\Omega} a(x)\sigma_{ij}(v)\bar{\sigma}_{ij}(v) \, dx \leq 0, \quad \forall Z = (u, v) \in D(A).
  \]
- \( \mathcal{I} - A \) is onto, by Lax-Milgram Lemma, (\( \mathcal{I} \) denotes the identity operator).

Consequently, the operator \( A \) generates a \( C_0 \) semigroup of contractions on \( \mathcal{H} \) by Lumer-Phillips Theorem [51]; note that \( D(A) = \mathcal{H} \), by [51], Theorem 4.6, p. 16.

We now observe that the operator \( A \) does not have a compact resolvent; this is due to the fact that the localized viscoelastic damping has the same order as the principal operator \( A \), thereby precluding the embedding of \( D(A) \) into \( \mathcal{H} \) to be compact. Next, we note that \( 0 \in \rho(A) \). Now, let \( b \in \mathbb{R} \) with \( b \neq 0 \), the assertion about the resolvent will be established once we prove: i) \( \text{Ker}(ib - A) = \{0\} \) and ii) \( R(ib - A) = \mathcal{H} \), where \( \text{Ker}(B) \) stands for the kernel of the operator \( B \) and \( R(B) \) stands for the range of \( B \).

Proof of i). Let \( b \) be a nonzero real number and let \( Z = (u, v) \in D(A) \) with \( AZ = ibZ \), we shall prove that \( Z = 0 \). The equation \( AZ = ibZ \) easily yields
\[
\Re(AZ, Z) = - \int_{\Omega} a(x)\sigma_{ij}(v)\bar{\sigma}_{ij}(v) \, dx = 0; \quad \text{from which one derives } \varepsilon(v) = 0 \text{ in } \omega \text{ thanks to the fact that the damping coefficient } a \text{ is positive in } \omega.
\]
Since \( v = ibv = \text{div}(\sigma(u)) \); it then follows that \( v = 0 \) and \( u = 0 \) in \( \omega \). Invoking the unique continuation property (10), one derives \( Z = 0 \).

Proof of ii). For this proof, we borrow some ideas from [44]. Let \( b \) be a nonzero real number, and let \( U = (f, g) \in \mathcal{H} \). We shall show that there exists \( Z = (u, v) \in D(A) \) such that \( ibZ - AZ = U \), which may be recast as:
\[
\begin{align*}
ibu - v &= f \\
ibv + Au + Bv &= g, \quad \text{with } Bv = -\text{div}(a(x)\sigma(v)).
\end{align*}
\]
We may use the first equation in (17) to eliminate \( v \) in the second one, thereby getting, (denoting \( V = [H^1_0(\Omega)]^N \) and \( H = [L^2(\Omega)]^N \))
\[
-b^2u + Au + ibBu = g + ibf + Bf \in V' = [H^{-1}(\Omega)]^N.
\]
If we set \( A_b = A + ibB : V \rightarrow V' \), then Korn’s inequality and Lax-Milgram theorem show that \( A_b \) is an isomorphism. Further, one checks that \( A_b^{-1} \) is compact as \( A_b^{-1}(V') = V \), and the embedding \( V \subset H \) is compact. We may rewrite (18) as
\[
u - b^2 A_b^{-1}u = A_b^{-1}(g + ibf + Bf).
\]
Thanks to the Fredholm alternative e.g. [11], Theorem VI.6, p. 92, solving (19) in \( H \) amounts to showing that the equation \( u - b^2 A_b^{-1}u = 0 \) has the unique solution \( u = 0 \), or equivalently that \( u = 0 \) is the unique solution of the equation \( -b^2u + Au + ibBu = 0 \). Taking the duality product between \( V' \) and \( V \) of \( u \) on both sides of the last equation, we get: \(-b^2|u|^2 + |A^2 u|^2 + ib\langle Bu, u \rangle = 0\), so that taking the imaginary
we immediately derive \( Z \) to be recast as parts, and keeping in mind that \( b \neq 0 \), one finds: \( \langle Bu, u \rangle = 0 \); from which one derives \( u = 0 \) in \( \omega \), as in the proof of i). Invoking the unique continuation property (10) once more, one derives \( u = 0 \) in \( \Omega \). Hence ii) holds. Therefore, combining i), ii) and the closed graph theorem, one derives \( i\mathbb{R} \subset \rho(A) \). One may now invoke the stability theorem in [4] to conclude that the semigroup \((S(t))_{t \geq 0}\) is strongly stable.

In the sequel, we will quantify that strong stability property by establishing polynomial and exponential decay estimates.

**Step 2.** Polynomial decay estimate. Thanks to a recent result [10], Theorem 2.4, the polynomial decay estimate will follow from the resolvent estimate \( ||(ibI - A)^{-1}||_{\mathcal{L}(\mathcal{H})} = O(|b|^2) \) as \( |b| \nearrow +\infty \). To this end, let \( U \in \mathcal{H} \), and let \( b \) be a real number with \( |b| \geq 1 \). Since the range of \( ibI - A \) is \( \mathcal{H} \), there exists \( Z \in D(A) \) such that
\[
i bZ - AZ = U. \tag{20}
\]
We shall prove
\[
||Z||_{\mathcal{H}} \leq K_0|b|^2||U||_{\mathcal{H}}, \tag{21}
\]
where, here and in the sequel, \( K_0 \) is a generic positive constant that may eventually depend on \( \Omega, \omega \), and \( a \), and the other parameters of the system, but not on \( b \). From now on, we also use the notation \( |u|_2 \) for \( ||u||_{\mathcal{L}_2(\Omega)} \).

To establish (21), first, we note that if \( Z = (u, v) \), and \( U = (f, g) \), then (20) may be recast as
\[

\begin{align*}
ibu - v &= f \\
ibv - \text{div}(\sigma(u)) - \text{div}(a\hat{\sigma}(v)) &= g. \tag{22}
\end{align*}

\]
Taking the inner product with \( Z \) on both sides of (20), then taking the real parts, we immediately derive
\[
\int_{\Omega} a\hat{\sigma}_{pq}(v)\varepsilon_{pq}(v) \, dx \leq ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}. \tag{23}
\]
It now follows from the first equation in (22), and (23):
\[

\begin{align*}
b^2 \int_{\Omega} a\hat{\sigma}_{pq}(u)\varepsilon_{pq}(u) \, dx &\leq 2 \int_{\Omega} a(\hat{\sigma}_{pq}(v)\varepsilon_{pq}(v) + \hat{\sigma}_{pq}(f)\varepsilon_{pq}(f)) \, dx \\
&\leq 2 ||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + K_0||U||_{\mathcal{H}}^2. \tag{24}
\end{align*}

\]
In the remaining portion of the proof, we will be using a first order multiplier. Now, the function \( u \) in (22) lies in \( [H^1_0(\Omega)]^N \) only, thereby not suited for the ensuing operations. Consequently, we are going to introduce a change of variable in order to increase smoothness; set \( u_1 = u + w \), where \( \text{div}(\sigma(w)) = \text{div}(a\hat{\sigma}(v)) \), with \( w \in [H^1_0(\Omega)]^N \). Since \((u, v) \) lies in \( D(A) \), elliptic regularity shows that \( u_1 \in [H^2(\Omega) \cap H^1_0(\Omega)]^N \). Thanks to (23) and Korn’s inequality, we note that
\[
||u||_{[H^1_0(\Omega)]^N}^2 \leq K_0||U||_{\mathcal{H}}||Z||_{\mathcal{H}}, \quad ||u_1||_{[H^1_0(\Omega)]^N} \leq ||Z||_{\mathcal{H}} + K_0\sqrt{||U||_{\mathcal{H}}||Z||_{\mathcal{H}}}. \tag{25}
\]
On the other hand, the second equation in (22) becomes
\[
ibv - \text{div}(\sigma(u_1)) = g. \tag{26}
\]
It immediately follows from (26):
\[
|b||v||_{[H^{-1}(\Omega)]^N} \leq K_0||u_1||_{[H^1_0(\Omega)]^N} + K_0|g|_2. \tag{27}
\]
Let $\alpha > 0$ and $\beta$ be real constants with $\alpha(N - 2) < \beta < \alpha N$. Multiply (26) by $\beta \bar{u}_1$, integrate on $\Omega$, and take real parts to find

$$\beta \Re \int_{\Omega} g \bar{u}_1 \, dx = \beta \Re \int_{\Omega} (\bar{b}v - \text{div}(\sigma(u_1))) \bar{u}_1 \, dx = \beta \|u_1\|^2_{H^1_0(\Omega)} + \beta \Re \int_{\Omega} v(\bar{b}u + i\bar{b}\bar{w}) \, dx.$$  

(28)

Using (22), it follows:

$$\beta \Re \int_{\Omega} v(\bar{b}u + i\bar{b}\bar{w}) \, dx = \beta \Re \int_{\Omega} v(-\bar{v} - \bar{f} + i\bar{b}\bar{w}) \, dx. \quad (29)$$

Hence

$$\beta \Re \int_{\Omega} g \bar{u}_1 \, dx = \beta \|u_1\|^2_{H^1_0(\Omega)} - \beta \|v\|^2_2 - \beta \Re \int_{\Omega} v(\bar{f} - i\bar{b}\bar{w}) \, dx. \quad (30)$$

It follows from (25) and (27):

$$\left| \beta \Re \int_{\Omega} \{g \bar{u}_1 + v(\bar{f} - i\bar{b}\bar{w})\} \, dx \right| \leq K_0 \left( \|U\|_{H} \|Z\|_{H} + \|U\|^\frac{1}{2}_{H} \|Z\|^\frac{3}{2}_{H} + \|U\|^\frac{1}{2}_{H} \|Z\|^\frac{1}{2}_{H} \right).$$

(31)

Whence

$$K_0 \left( \|U\|_{H} \|Z\|_{H} + \|U\|^\frac{1}{2}_{H} \|Z\|^\frac{3}{2}_{H} + \|U\|^\frac{1}{2}_{H} \|Z\|^\frac{1}{2}_{H} \right) \geq \beta \|u_1\|^2_{H^1_0(\Omega)} - \beta \|v\|^2_2. \quad (32)$$

For the sequel, we need some additional notations.

Now for each $j = 1, \ldots, J$, where $J$ appears in the geometric constraint (GC) stated above, set $m^j(x) = x - x_0^j$ and $R_j = \sup\{|m^j(x)|, x \in \Omega\}$. Let $0 < \delta_0 < \delta_1 < \delta$, where $\delta$ is the one given in (GC). Set:

$$S = \left( \bigcup_{j=1}^{J} \Gamma_j \right) \bigcup \left( \Omega \setminus \bigcup_{j=1}^{J} \Omega_j \right), \quad Q_0 = N_{\delta_0}(S), \quad Q_1 = N_{\delta_1}(S), \quad \omega_1 = \Omega \cap Q_1,$$

and for each $j$, let $\varphi_j$ be a function satisfying

$$\varphi_j \in W^{1,\infty}(\Omega), \quad 0 \leq \varphi_j \leq 1, \quad \varphi_j = 1 \text{ in } \Omega_j \setminus Q_1, \quad \varphi_j = 0 \text{ in } \Omega \cap Q_0.$$

Now, multiply (26) by $2\alpha \varphi_j m^j \nabla \bar{u}_1$, integrate on $\Omega_j$, and take real parts to get

$$2\alpha \Re \int_{\Omega_j} g_p(\varphi_j m^j \cdot \nabla \bar{u}_1) \, dx = 2\alpha \Re \int_{\Omega_j} v_p \varphi_j m^j \cdot \nabla(-\bar{v}_p - f_p + i\bar{b}\bar{w}_p) \, dx$$

$$-2\alpha \Re \int_{\Omega_j} \text{div}(\sigma(u_1)) \cdot (\varphi_j m^j \nabla \bar{u}_1) \, dx. \quad (33)$$

The application of Green’s formula shows

$$-2\alpha \Re \int_{\Omega_j} v_p \varphi_j m^j \cdot \nabla \bar{v}_p \, dx = \alpha N \int_{\Omega_j} \varphi_j |v|^2 \, dx + \alpha \int_{\Omega_j} (m^j \cdot \nabla \varphi_j) |v|^2$$

$$-\alpha \int_{\partial \Omega_j} \varphi_j (m^j \cdot \nu) |v|^2 \, d\Gamma, \quad (34)$$
and

\[-2\alpha \Re \int_{\Omega_j} \text{div}(\sigma(u_1)) \cdot (\varphi_j m^j \nabla \bar{u}_1) \, dx = 2\alpha \int_{\Omega_j} \varphi_j \sigma_{pq}(u_1) \overline{\varepsilon_{pq}(u_1)} \, dx \]

\[+ 2\alpha \Re \int_{\Omega_j} \sigma_{pq}(u_1) \cdot (\partial_q \varphi_j) m^j \cdot \nabla \bar{u}_{1p} \, dx \]

\[+ 2\alpha \Re \int_{\Omega_j} \varphi_j \sigma_{pq}(u_1) m^j \cdot \varphi_j \sigma_{pq}(u_1) m^j \cdot \nabla \bar{u}_{1p} \, dx \]

\[= -2\alpha \Re \int_{\partial \Omega_j} \sigma_{pq}(u_1) \overline{\varepsilon_{pq}(u_1)} \varphi_j m^j \cdot \nabla \bar{u}_{1p} \, d\Gamma. \tag{35} \]

Using the symmetry properties of the elasticity coefficients, and applying Green’s formula once more, we find

\[2\alpha \Re \int_{\Omega_j} \varphi_j \sigma_{pq}(u_1) m^j \cdot \varphi_j \sigma_{pq}(u_1) m^j \cdot \nabla \bar{u}_{1p} \, dx = 2\alpha \Re \int_{\Omega_j} \varphi_j m^j \sigma_{pq}(u_1) \overline{\varepsilon_{pq}(u_1)} \, dx \]

\[= \alpha \Re \int_{\Omega_j} \varphi_j m^j \overline{\varepsilon_{pq}(u_1) \sigma_{pq}(u_1)} \, dx \]

\[= -\alpha \Re \int_{\Omega_j} (\partial_n a_{pqrs}) \varepsilon_{rs}(u_1) \varepsilon_{pq}(u_1) \varphi_j m^j \, dx \]

\[= -N \alpha \int_{\Omega_j} \varepsilon_{pq}(u_1) \varphi_j m^j \, dx \]

\[+ \alpha \int_{\partial \Omega_j} \sigma_{pq}(u_1) \overline{\varepsilon_{pq}(u_1)} \varphi_j m^j \cdot \nu^j \, d\Gamma. \tag{36} \]

If as in \([41]\), we set for each \(j, S_j = \Gamma_j \cup (\partial \Omega_j \cap \Omega)\), then one checks that \(\varphi_j = 0\) on \(S_j\). On the other hand, \(\partial \Omega_j \setminus S_j \subset \Gamma_j \cap \partial \Omega\), \((A^c)\) denotes the complement of \(A\); consequently, for each \(j\), one has

\[\int_{\partial \Omega_j} \varphi_j (m^j \cdot \nu^j) |v|^2 \, d\Gamma = 0 \]

\[-2\alpha \Re \int_{\partial \Omega_j} \sigma_{pq}(u_1) \nu^j \varphi_j m^j \cdot \nu^j \, d\Gamma + \alpha \int_{\partial \Omega_j} \sigma_{pq}(u_1) \overline{\varepsilon_{pq}(u_1)} \varphi_j m^j \cdot \nu^j \, d\Gamma \geq 0. \tag{37} \]

The last inequality follows from the fact that

\[-2\alpha \Re \int_{\partial \Omega_j \setminus S_j} \sigma_{pq}(u_1) \nu^j \varphi_j m^j \cdot \nabla \bar{u}_{1p} \, d\Gamma = -2\alpha \int_{\partial \Omega_j \setminus S_j} \sigma_{pq}(u_1) \overline{\varepsilon_{pq}(u_1)} \varphi_j m^j \cdot \nu^j \, d\Gamma. \]

Thus, reporting \((36)\) and \((37)\) in \((35)\), and combining \((34)\) and \((35)\), we find

\[-2\alpha \Re \int_{\Omega_j} v_p \varphi_j m^j \cdot \nabla \bar{v}_p \, dx - 2\alpha \Re \int_{\Omega_j} \text{div}(\sigma(u_1)) \cdot (\varphi_j m^j \nabla \bar{u}_1) \, dx \]

\[\geq \alpha N \int_{\Omega_j} |v|^2 \, dx + \alpha N \int_{\Omega_j} (\varphi_j - 1) |v|^2 \, dx \]

\[+ 2\alpha \Re \int_{\Omega_j} \sigma_{pq}(u_1) \cdot (\partial_q \varphi_j) m^j \cdot \nabla \bar{u}_{1p} \, dx - (N - 2) \alpha \int_{\Omega_j} \sigma_{pq}(u_1) \overline{\varepsilon_{pq}(u_1)} \, dx \]

\[+ 2\alpha \Re \int_{\Omega_j} \sigma_{pq}(u_1) \cdot (\partial_q \varphi_j) m^j \cdot \nabla \bar{u}_{1p} \, dx \]
The use of Poincaré and Korn inequalities and (23) lead to

\[-(N - 2)\alpha \int_{\Omega_j} (\varphi_j - 1)\sigma_{pq}(u_1)\varepsilon_{pq}(u_1) \, dx - \alpha \int_{\Omega_j} \sigma_{pq}(u_1)\varepsilon_{pq}(u_1)(m^j \cdot \nabla \varphi_j) \, dx\]

\[-\alpha \mathbb{R} \int_{\Omega_j} (\partial_n a_{pqrs})\varepsilon_{rs}(u_1)\varepsilon_{pq}(u_1)(\varphi_j - 1)m^l_p \, dx\]

\[-\alpha \mathbb{R} \int_{\Omega_j} (\partial_n a_{pqrs})\varepsilon_{rs}(u_1)\varepsilon_{pq}(u_1)m^l_n \, dx.\]

Adding the utmost right term in the first line of (33) in (38), then taking the sums over \( j \), we obtain

\[-2\alpha \mathbb{R} \sum_{j=1}^{J} \int_{\Omega_j} \{ \nu_p \varphi_j m^j \cdot \nabla \bar{u}_p + \text{div} (\sigma(u_1)) \cdot (\varphi_j m^j \nabla \bar{u}_k) - i\nu_p \varphi_j m^j \cdot \nabla \bar{w}_p \} \, dx\]

\[\geq \alpha N \sum_{j=1}^{J} \int_{\Omega_j} |v|^2 \, dx + \alpha N \sum_{j=1}^{J} \int_{\Omega_j} (\varphi_j - 1)|v|^2 \, dx + \alpha \int_{\Omega_j} (m^j \cdot \nabla \varphi_j)|v|^2 \, dx\]

\[+2\alpha \mathbb{R} \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1) \cdot (\partial_q \varphi_j) m^j \cdot \nabla \bar{u}_p \, dx - (N - 2)\alpha \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1)\varepsilon_{pq}(u_1) \, dx\]

\[-\alpha \mathbb{R} \sum_{j=1}^{J} \int_{\Omega_j} (\partial_n a_{pqrs})\varepsilon_{rs}(u_1)\varepsilon_{pq}(u_1)(\varphi_j - 1)m^l_p \, dx + 2\alpha \mathbb{R} \sum_{j=1}^{J} \int_{\Omega_j} \nu_p \varphi_j m^j \cdot \nabla \bar{w}_p \, dx\]

The use of Poincaré and Korn inequalities and (23) lead to

\[(\alpha N - \beta) \sum_{j=1}^{J} \int_{\Omega_j} \alpha N \sum_{j=1}^{J} \int_{\Omega_j} (\varphi_j - 1)|v|^2 \, dx + \alpha \int_{\Omega_j} (m^j \cdot \nabla \varphi_j)|v|^2 \, dx\]

\[\geq (\alpha N - \beta)|v|^2 - K_0 \int_{\Omega_j} |v|^2 \, dx\]

\[\geq (\alpha N - \beta)|v|^2 - K_0 \int_{\Omega_j} |\nabla v|^2 \, dx\]

\[\geq (\alpha N - \beta)|v|^2 - K_0 \int_{\Omega_j} a\sigma_{pq}(v)\varepsilon_{pq}(v) \, dx\]

\[\geq (\alpha N - \beta)|v|^2 - K_0||U||_{H}||Z||_{H}.\]
On the other hand, choosing $\beta = (N - \frac{1}{2})\alpha$, and using (25), one immediately gets

$$-(N - 2)\alpha \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} \, dx$$

$$-(N - 2)\alpha \sum_{j=1}^{J} \int_{\Omega_j} (\varphi_j - 1)\sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} \, dx$$

$$-\alpha \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} (m^j \cdot \nabla \varphi_j) \, dx$$

$$-\alpha \sum_{j=1}^{J} \int_{\Omega_j} (\partial_n a_{pqrs}) \varepsilon_{rs}(u_1)\overline{\varepsilon_{pq}(u_1)} (\varphi_j - 1)m_n^j \, dx$$

$$+2\alpha Rb \sum_{j=1}^{J} \int_{\Omega_j} v_p \varphi_j m^j \cdot \nabla \tilde{w}_p \, dx$$

$$-\alpha R \sum_{j=1}^{J} \int_{\Omega_j} (\partial_n a_{pqrs}) \varepsilon_{rs}(u_1)\overline{\varepsilon_{pq}(u_1)m_n^j} \, dx.$$  \hfill (41)

First applying H"older inequality, then Korn inequality to the terms in the left hand side of (41), and using (25), one immediately gets

$$2\alpha R \sum_{j=1}^{J} \int_{\Omega_j} \{g_p (\varphi_j m^j \cdot \nabla \tilde{u}_1) + v_p \varphi_j m^j \cdot \nabla \tilde{f}_p\} \, dx$$

$$\leq K_0 (||U||_{H^1}||Z||_{H^1} + ||U||_{H^2}||Z||_{H^2}).$$  \hfill (42)

Now we are going to estimate the terms in the right hand side of (41).

Thanks to H"older inequality, Korn inequality, and (25), it easily follows

$$\left|2\alpha Rb \sum_{j=1}^{J} \int_{\Omega_j} v_p \varphi_j m^j \cdot \nabla \tilde{w}_p \, dx\right| \leq K_0 b ||U||_{H^2}||Z||_{H^2}.  \hfill (43)$$

On the other hand, choosing $\beta = (N - \frac{1}{2})\alpha$, and using the hypothesis (13), one derives

$$(\beta - (N - 2)\alpha) \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} \, dx$$

$$-\alpha R \sum_{j=1}^{J} \int_{\Omega_j} (\partial_n a_{pqrs}) \varepsilon_{rs}(u_1)\overline{\varepsilon_{pq}(u_1)}m_n^j \, dx$$

$$-\alpha \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} (m^j \cdot \nabla \varphi_j) \, dx$$

$$-(N - 2)\alpha \sum_{j=1}^{J} \int_{\Omega_j} (\varphi_j - 1)\sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} \, dx$$

$$= \frac{\alpha}{4} \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} \, dx + \alpha \sum_{j=1}^{J} \int_{\Omega_j} \sigma_{pq}(u_1)\overline{\varepsilon_{pq}(u_1)} (m^j \cdot \nabla \varphi_j) \, dx$$

$$+ \alpha \sum_{j=1}^{J} \int_{\Omega_j} \left(\frac{3}{2}a_{pqrs} - m_n^j (\partial_n a_{pqrs}) \varepsilon_{rs}(u_1)\overline{\varepsilon_{pq}(u_1)} \right).$$  \hfill (44)
\[-(N-2)\alpha \sum_{j=1}^{J} \int_{\Omega_j} (\varphi_j - 1) \sigma_{pq}(u_1) \varepsilon_{pq}(u_1) \, dx \]
\[\geq \frac{\alpha}{2} \int_{\Omega} \sigma_{pq}(u_1) \varepsilon_{pq}(u_1) \, dx - K_0 \int_{\Omega} \sigma_{pq}(u_1) \varepsilon_{pq}(u_1) \, dx.\]

Now, the definition of $u_1$, and (24)-(25) show (keeping in mind that $|b| \geq 1$)
\[\int_{\Omega} \sigma_{pq}(u_1) \varepsilon_{pq}(u_1) \, dx \]
\[= \int_{\Omega} \sigma_{pq}(u) \varepsilon_{pq}(u) \, dx + \Re \int_{\Omega} (2\sigma_{pq}(u) + \sigma_{pq}(w)) \varepsilon_{pq}(w) \, dx\]
\[\leq K_0(|U|_{H^1} |Z|_{H^1} + |U|_{H^1}^2), \quad \text{by Cauchy-Schwarz inequality.}\]

Gathering (32) and (41)-(45), we find
\[|v|^2 + \int_{\Omega} \sigma_{pq}(u) \varepsilon_{pq}(u) \, dx \leq K_0(|b||U|_{H^1}^2 |Z|_{H^1}^2 + |U|_{H^1} |Z|_{H^1} + |U|_{H^1}^2 |Z|_{H^1}^2 + |U|_{H^1}^4),\]

The definition of $u_1$ and (25), as in (45), yield
\[|v|^2 + \int_{\Omega} \sigma_{pq}(u) \varepsilon_{pq}(u) \, dx \leq K_0(|b||U|_{H^1}^2 |Z|_{H^1}^2 + |U|_{H^1} |Z|_{H^1} + |U|_{H^1}^2 |Z|_{H^1} + |U|_{H^1}^4),\]

or
\[|Z|_{H^1}^2 \leq K_0(|b||U|_{H^1}^2 |Z|_{H^1}^2 + |U|_{H^1} |Z|_{H^1} + |U|_{H^1}^2 |Z|_{H^1} + |U|_{H^1}^4).\]  

The use of Young inequality in (47) leads at once to (21). Applying [10], Theorem 2.4, one gets the claimed polynomial decay estimate, thereby completing the proof of Theorem 1.1.

**Proof of Theorem 1.2.** The proof of Theorem 1.2 is very similar to that of Theorem 1.1; only estimating the term just before the last term in the right-hand side of (39) is distinct in the present proof. Instead of the rough estimate (43), we must now get an estimate that is independent of $b$. So, following the proof of Theorem 1.1, we already have
\[\|Z\|^2_{H^1} \leq K_0(|U|^\frac{1}{2}_1 |Z|^\frac{3}{2}_1 + |U|_{H^1} |Z|_{H^1} + |U|^\frac{3}{2}_1 |Z|^\frac{1}{2}_1 + |U|_{H^1}^2)\]
\[+ K_0 \left| \Re b \sum_{j=1}^{J} \int_{\Omega_j} v_p \varphi_j m^j \cdot \nabla \tilde{w}_p \, dx \right|.\]

We shall now estimate the last term in (48) independently of $b$. To this end, introduce for each $j \in \{1, \ldots, J\}$, the function $z^j \in [H^1_0(\Omega)]^N$, solution of the system
\[\partial_t \sigma_{pq}(z^j) = \text{div}(1_{\Omega_j} v_p \varphi_j m^j) \quad \text{in} \quad \Omega, \quad p = 1, \ldots, N\]
where $1_{\Omega_j}$ stands for the characteristic function of $\Omega_j$.

Multiplying the system by $\tilde{w}_p$, applying Green’s formula over $\Omega$, and using an elasticity symmetry, we obtain
\[\int_{\Omega_j} v_p \varphi_j m^j \cdot \nabla \tilde{w}_p \, dx = \int_{\Omega_j} \sigma_{pq}(z^j) \varepsilon_{pq}(w) \, dx = \int_{\Omega} \sigma_{pq}(w) \varepsilon_{pq}(z^j) \, dx\]
\[= \int_{\Omega} \alpha \sigma_{pq}(w) \varepsilon_{pq}(z^j) \, dx,\]

where the last equality comes from the equation satisfied by $\tilde{w}$, and the variational method.
Now, if we multiply the system (49) by $\gamma a \v$, and apply Green’s formula once more, we find

$$\gamma \int_{\Omega} \sigma_{pq}(z^j)(\partial_q a) \bar{v}_p \, dx + \gamma \int_{\Omega} a \sigma_{pq}(z^j) \underline{\v}_{pq}(v) \, dx = \gamma \int_{\Omega} \varphi_j(m^j \cdot \nabla a) |v|^2 \, dx + \gamma \int_{\Omega} a \nu_p \varphi_j m^j \cdot \nabla \bar{v}_p \, dx,$$

Adding (50) and (51) side by side and using an elasticity symmetry, it follows

$$\int_{\Omega} v_p \varphi_j m^j \cdot \nabla \bar{w}_p \, dx = \gamma \int_{\Omega} \sigma_{pq}(z^j)(\partial_q a) \bar{v}_p \, dx + \gamma \int_{\Omega} \varphi_j(m^j \cdot \nabla a) |v|^2 \, dx + \gamma \int_{\Omega} a \nu_p \varphi_j m^j \cdot \nabla \bar{v}_p \, dx,$$

since

$$\int_{\Omega} a \sigma_{pq}(v) \underline{\v}_{pq}(z^j) \, dx = \gamma \int_{\Omega} a \sigma_{pq}(z^j) \underline{\v}_{pq}(v) \, dx = \gamma \int_{\Omega} a \sigma_{pq}(v) \underline{\v}_{pq}(z^j) \, dx,$$

so the two terms cancel out.

Consequently, by virtue of (52), one has

$$\Re ib \int_{\Omega} v_p \varphi_j m^j \cdot \nabla \bar{w}_p \, dx = -\Re ib \gamma \int_{\Omega} \sigma_{pq}(z^j)(\partial_q a) \bar{v}_p \, dx + \Re ib \gamma \int_{\Omega} a \nu_p \varphi_j m^j \cdot \nabla \bar{v}_p \, dx,$$

We shall now estimate the two terms in the right hand side of (53). Thanks to Cauchy-Schwarz inequality and the inequality constraint on the gradient of the damping coefficient $a$, estimating the left term yields

$$\left| \Re ib \gamma \int_{\Omega} \sigma_{pq}(z^j)(\partial_q a) \bar{v}_p \, dx \right| \leq K_0 |b| \sqrt{a} v |Z|_{H^1},$$

where we have utilized the easy to establish estimate $||z^j||_{\{H^1(\Omega)\}_{N}} \leq K_0 |v|_2$, for all $j$.

As for the other term, Cauchy-Schwarz inequality yields

$$\left| \Re ib \gamma \int_{\Omega} a \nu_p \varphi_j m^j \cdot \nabla \bar{v}_p \, dx \right| \leq K_0 |b| \sqrt{a} v \left( \int_{\Omega} a |\nabla v|^2 \, dx \right)^{\frac{1}{2}}.$$  \hspace{1cm} (55)

By virtue of Green’s formula, one checks that

$$\int_{\Omega} a |\nabla v|^2 \, dx \leq 4 \int_{\Omega} a |\v|^2 \, dx + K_0 |Z|_{H^1}^2 \leq K_0 \int_{\Omega} a \sigma_{pq}(v) \underline{\v}_{pq}(v) \, dx + K_0 |Z|_{H^1}^2 \leq K_0 (|U|_{H^1} |Z|_{H^1} + |Z|_{H^1}^2).$$  \hspace{1cm} (56)

Indeed, we have

$$\int_{\Omega} a |\v|^2 \, dx$$

$$= \frac{1}{4} \Re \int_{\Omega} a (v_{p,q} + v_{q,p}) (\bar{v}_{p,q} + \bar{v}_{q,p}) \, dx$$

$$= \frac{1}{4} \int_{\Omega} a \left( |\nabla v|^2 + v_{p,q} \bar{v}_{p,q} + v_{q,p} \bar{v}_{p,q} \right) \, dx.$$
\[ \frac{1}{2} \int_{\Omega} a|\nabla v|^2 \, dx - \frac{1}{2} \Re \int_{\Omega} a\bar{v}_q \partial_q (\text{div}v) \, dx - \frac{1}{2} \Re \int_{\Omega} \bar{v}_q (\partial_p a) v_{p\cdot q} \, dx \]
\[ = \frac{1}{2} \int_{\Omega} a|\nabla v|^2 \, dx + \frac{1}{2} \Re \int_{\Omega} \{a|\text{div}v|^2 + (\bar{v}_p \partial_p a)(\text{div}v)\} \, dx - \frac{1}{2} \Re \int_{\Omega} \bar{v}_q (\partial_p a) v_{p\cdot q} \, dx \]

(57)

Thanks to Young inequality, we have the estimates

\[ \frac{1}{2} \Re \int_{\Omega} (\bar{v} \cdot \nabla a)(\text{div}v) \, dx \geq -\frac{1}{4} \int_{\Omega} a|\text{div}v|^2 \, dx - K_0|v|^2 \]
\[ \geq -\frac{1}{4} \int_{\Omega} a|\text{div}v|^2 \, dx - K_0|Z|_H^2 - \frac{1}{2} \Re \int_{\Omega} \bar{v}_q (\partial_p a) v_{p\cdot q} \, dx \]
\[ \geq -\frac{1}{4} \int_{\Omega} a|\nabla v|^2 \, dx - K_0|v|^2 \geq -\frac{1}{4} \int_{\Omega} a|\nabla v|^2 \, dx - K_0|Z|_H^2, \]

whence the claimed inequality (56), by reporting (58) in (57).

One then derives from (53)-(56)

\[ \left| \Re b \sum_{j=1}^{N} \int_{\Omega} v_{p\cdot q} \partial_q v_{p\cdot j} \cdot \nabla \bar{w}_p \, dx \right| \leq K_0|b|\sqrt{\bar{v}}|_{\Omega}(||U||_H||Z||_H + ||Z||_H^2)^{\frac{1}{2}}. \]

(59)

To complete the proof of Theorem 1.2, we shall now estimate the term $|b|\sqrt{\bar{v}}|_{\Omega}$. For this purpose, multiplying the second equation in (22) by $iba\bar{v}$ and applying Green’s formula, one finds

\[ b^2 \int_{\Omega} a|v|^2 \, dx = \Re b \int_{\Omega} \{\sigma_{pq}(u)\partial_q(a\bar{v}_p) + a\sigma_{pq}(v)\partial_q(a\bar{v}_p)\} \, dx - \Re b \int_{\Omega} a g \cdot \bar{v} \, dx \]
\[ = \Re \int_{\Omega} \{\sigma_{pq}(v + f)\partial_q(a\bar{v}_p) + iba\sigma_{pq}(v)(\partial_q a)\bar{v}_p\} \, dx \]
\[ + \Re b \int_{\Omega} a^2\sigma_{pq}(v)\bar{e}_{pq}(v) \, dx - \Re b \int_{\Omega} a g \cdot \bar{v} \, dx \]
\[ = \Re \int_{\Omega} \{\sigma_{pq}(v + f)\partial_q(a\bar{v}_p) + iba\sigma_{pq}(v)(\partial_q a)\bar{v}_p - ibag \cdot \bar{v}\} \, dx. \]

(60)

Thanks to Cauchy-Schwarz inequality and (23), one gets the following estimate

\[ \left| \Re \int_{\Omega} \{\sigma_{pq}(v + f)(v_p \partial_q a + a \partial_q \bar{v}_p)\} \, dx \right| \]
\[ \leq K_0 \left[ \left( \int_{\Omega} a^2 \sigma_{pq}(v)\bar{e}_{pq}(v) \, dx \right)^{\frac{1}{2}} \right] \left[ \left( \int_{\Omega} a |\sigma_{pq}(v)|^2 \, dx \right)^{\frac{1}{2}} \right] \]
\[ \leq K_0(||U||_H^2||Z||_H^2 + ||U||_H(||Z||_H + ||Z||_H^2||Z||_H^2)) \]
\[ \leq K_0(||U||_H^2||Z||_H^2 + ||U||_H||Z||_H + ||U||_H^2||Z||_H^2). \]

(61)

Now, using Young inequality and (23) once more, one obtains

\[ \left| \Re \int_{\Omega} iba\sigma_{pq}(v)(\partial_q a)\bar{v}_p \, dx - \Re b \int_{\Omega} a g \cdot \bar{v} \, dx \right| \]
\[ \leq \frac{1}{4} \int_{\Omega} a|v|^2 \, dx + K_0 \int_{\Omega} a \sigma_{pq}(v)\bar{e}_{pq}(v) \, dx + \frac{b^2}{4} \int_{\Omega} a|v|^2 \, dx + K_0|g|^2 \]
\[ \leq \frac{b^2}{2} \int_{\Omega} a|v|^2 \, dx + K_0(||U||_H||Z||_H + ||U||_H^2). \]

(62)
Reporting (61) and (62) in (60), it follows
\[ b^2 \int_{\Omega} a|v|^2 \, dx \leq K_0(||U||_{\mathcal{H}}^\frac{3}{2}||Z||_{\mathcal{H}}^\frac{3}{2} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}^2 + ||U||_{\mathcal{H}}^2). \] (63)

The combination of (59) and (63) yields
\[
\begin{align*}
\left| \text{Re} \sum_{j=1}^{N} \int_{\Omega} v_j \varphi_j m^j \cdot \nabla \bar{w}_p \, dx \right| \\
\leq K_0(||U||_{\mathcal{H}}^\frac{3}{2}||Z||_{\mathcal{H}}^\frac{3}{2} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}^2 + ||U||_{\mathcal{H}}^2) \frac{1}{2} \\
\times (||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||Z||_{\mathcal{H}}) \frac{1}{2} \\
\leq K_0(||U||_{\mathcal{H}}^\frac{3}{2}||Z||_{\mathcal{H}}^\frac{3}{2} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}^2 + ||U||_{\mathcal{H}}^2) \frac{1}{2} \\
+ ||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}^2 + ||U||_{\mathcal{H}}^2). \end{align*}
\] (64)

Reporting (64) in (48), we get
\[
||Z||_{\mathcal{H}}^2 \leq K_0(||U||_{\mathcal{H}}^\frac{3}{2}||Z||_{\mathcal{H}}^\frac{3}{2} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}^2 + ||U||_{\mathcal{H}}^2) \frac{1}{2} \\
+ ||U||_{\mathcal{H}}||Z||_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}^2 + ||U||_{\mathcal{H}}^2). \] (65)

Using Young inequality, one derives the desired estimate from (65) for large enough $|b|$. By the continuity of the resolvent, one obtains the desired estimate for the remaining values of $b$, thereby completing the proof of Theorem 1.2. \hfill \Box

4. Further comments and open problems. The purpose of this section is to discuss some extensions of our results, and some open problems.

4.1. A modified damping mechanism. As the proof of Theorem 1.2 shows, the inequality constraint on the gradient of the damping coefficient $a$ is critical in carrying the estimates that lead to the exponential decay of the energy. However one may dispense with this constraint if instead of the damping $-\text{div}(a\sigma(y))$, one uses $-\text{div}(a^2\sigma(y))$; so by squaring the damping coefficient, one gets a slight improvement of Theorem 1.2. So far, it is not clear what the optimal hypotheses on the damping coefficient allowing to obtain the exponential decay of the energy of system (1) are, even in the case of the wave equation. In this regard, it is worth noting that the one-dimensional case is slightly better; in [71], the author shows that for $\Omega = (-1,1)$ and $\omega = (0,1)$, it is enough to choose the damping coefficient $a$ in $C^1([-1,1])$ with $a \equiv 0$ in $[-1,0]$, $a_x(0) = 0$, and $\int_0^1 \frac{a_x(x)^2}{a(x)} \, ds \leq c_0 |a_x(x)|$ for all $x \in [0,1]$, for some positive constant $c_0$. Still in the one-dimensional setting, an even better result was established earlier in [54]; indeed, with $\Omega = (-1,1)$ and $\omega = (0,1)$, and the damping coefficient satisfying: $a$ in $C^1([-1,1])$ with $a \equiv 0$ in $[-1,0]$, $a_x > 0$ on $(0,1]$, and $\lim_{x \to 0^+} a_x(x) = k > 0$ for some positive constant $r$, it was shown that for the underlying eigenvalue problem, if a sequence of eigenvalues goes to infinity, then the corresponding sequence of real parts tends toward negative infinity.

4.2. The Euler-Bernoulli with Kelvin-Voigt damping. The analogous problem for the plate equation
\[
\begin{align*}
y_{tt} + \Delta^2 y + \Delta(a\Delta y_t) &= 0 \text{ in } \Omega \times (0, \infty) \\
y &= 0 \& \partial_n y = 0 \text{ or } y = 0 \& \Delta y = 0 \text{ on } \Sigma \\
y(x, 0) &= y_0(x), \quad y_t(x, 0) = y_1(x) \text{ in } \Omega.
\end{align*}
\]
is still open. We note here that no smoothness on the damping coefficient is needed in the one-dimensional setting for the clamped boundary conditions [42].

4.3. More on the wave equation. After completing this work, I came across the work of Burq and Christianson [13] where the authors show, among other things, that the energy of the wave equation with Kelvin-Voigt damping decays exponentially provided:

- the feedback control support \( \omega = \{ x \in \Omega; a(x) > 0 \} \) satisfies the geometric control condition of Bardos-Lebeau-Rauch [7]: \( \text{There exists a time } T > 0 \text{ such that every ray of geometric optics enters } \omega \times (0,T) \text{ in a time less than } T \),
- the damping coefficient \( a \) is in \( C^\infty(\Omega) \) with
  \[
  |\partial^\alpha a(x)| \leq C_\alpha a^{k-|\alpha|}, \quad |\alpha| \leq 2,
  \]
  for some \( k > 2 \), and
- the initial displacement is identically zero in \( \Omega \).

The advantage of Theorem 3 in [13] over Theorem 1.2, in the present work, specialized to the wave equation is that their feedback control region is the best possible. However, to achieve this, the authors of [13] had to require more restrictions on the damping coefficient than in the present work, and to further impose null initial displacement. It would be very interesting to improve Theorem 3 in [13] by reducing the restrictions on the damping coefficient, and more importantly by allowing nonzero initial displacements; we note that the null initial displacement is inherent to the semi-classical analysis approach employed in [13].

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