Model of the two level quantum dots ensemble interacting with coherent radiation

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ABSTRACT

We consider the model of quantum dots interacting with coherent radiation when the relaxation processes may be neglected. The system under investigation consists of two discrete energy levels of the quantum dots in the presence of strong electron-electron Coulomb interaction and the transitions between these levels in response to electromagnetic radiation. By using the suitable generalisation of the Hubbard model the system of equations describing the evolution of the state of quantum dot was derived.

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1 Introduction

During the past years, a considerable attention has been paid to the growth and characterisation of quantum dots [1, 2, 3, 4, 5, 6]. The strong interest in these systems is readily understood in view of their obvious potential for future high-speed electronic devices as well as for their intriguing physical properties. The (semiconductor) quantum dot (QD) is a system where electrons are confined in all the three dimensions. The famous Hubbard model [7] provides a description for such a system [3, 4, 5, 6].

The majority of the investigations are devoted to the transport properties of the coupled quantum dot system. At the same time the optical properties of low-dimension objects also attract the attention. For example, the possibility of the existence of the trapped states and coherent population transfer in semiconductor quantum walls has been demonstrated theoretically [8, 9, 10]. A solid-state implementation of stimulated Raman adiabatic passage in two coupled semiconductor quantum dots was proposed in [11]. The radiative effects of a QD array in the presence of a static magnetic field were investigated in [12]; with the peculiarities of the second harmonic generation in a two-dimensional array of QDs [13] dealt with.

In the theory of coherent optical transient phenomena the two-level atom model plays an important role [14]. A number of phenomena (as photon echo, optical nutation, free-induction decay and so on) can be described in the framework of this model. It is attracted to develop the simplest model to consider the interaction of coherent radiation with QD, which is a generalisation of the two-level atom model. In [15] the QD with only two one-particle energy states was proposed. There the interaction of electrons with scalar field of the ultra-short electromagnetic pulse (USP) was considered. If one takes both polarisation of the radiation and spin states of electrons into account, then the more complete model results. The goal of this paper is to derive the complete system of equations describing the two-level QD states evaluating under USP action. The term USP means that all relaxation processes in an ensemble of QD are neglected.

The rest of the paper is organised as follows. In Sec.2 the system of the equation describing the interaction of the QD with the USP is derived, Sec.3 demonstrates an example of the steady-state solution of this system and summary is given in Conclusion.
2 The constituent model

Let us consider the model of quantum dot, which has only two one-particle states $|a\rangle$ and $|b\rangle$ with corresponding energy levels $\varepsilon_a$ and $\varepsilon_b$. We suppose that state $|a\rangle$ lies in low energy band and state $|b\rangle$ belongs to the higher energy band of the bulk material.

The total Hubbard-type Hamiltonian of the model is written as

$$\hat{H} = \varepsilon_a \sum_{j\mu} \hat{n}_{j\mu} + U_a \sum_j \hat{n}_{j\uparrow} \hat{n}_{j\downarrow} + T_a \sum_{j\mu} \left( \hat{a}_{j+1\mu}^+ \hat{a}_{j\mu} + \hat{a}_{j\mu}^+ \hat{a}_{j+1\mu} \right) +$$

$$+ \varepsilon_b \sum_{j\mu} \hat{N}_{j\mu} + U_b \sum_j \hat{N}_{j\uparrow} \hat{N}_{j\downarrow} + T_b \sum_{j\mu} \left( \hat{b}_{j+1\mu}^+ \hat{b}_{j\mu} + \hat{b}_{j\mu}^+ \hat{b}_{j+1\mu} \right) +$$

$$+ \sum_j \left( V_1 \hat{a}_{j\uparrow}^+ \hat{b}_{j\downarrow} + V_2 \hat{a}_{j\downarrow}^+ \hat{b}_{j\uparrow} + V_1^* \hat{b}_{j\uparrow}^+ \hat{a}_{j\downarrow} + V_2^* \hat{b}_{j\downarrow}^+ \hat{a}_{j\uparrow} \right) + U_{ab} \sum_{j\mu\sigma} \hat{n}_{j\mu\upsigma} \hat{N}_{j\sigma}.$$

In the above Hamiltonian operator $\hat{a}_{j\mu}^+ (\hat{b}_{j\mu}^+)$ create an electron with spin $\mu$ in the discrete level of the one-particle states $|a\rangle$ ($|b\rangle$) of the $j$th dot. The parameters $U_a$, $U_b$ and $U_{ab}$ correspond to the electronic repulsion on the two states. $T_a$ and $T_b$ are the tunnelling matrix element between the single particle states. The operators $\hat{n}_{j\mu} = \hat{a}_{j\mu}^+ \hat{a}_{j\mu}$ and $\hat{N}_{j\mu} = \hat{b}_{j\mu}^+ \hat{b}_{j\mu}$ are the particle number ones in the upper and lower states. Interaction of the electrons with electromagnetic field describes by the matrix elements $V_{1,2}$. The summation index $j$ runs over all QD and Greek index $\mu$ runs over $\downarrow$ and $\uparrow$.

The operators $\hat{a}_{j\mu}^+$ and $\hat{b}_{j\mu}^+$ obey the following anti-commutation relations

$$\hat{a}_{i\sigma} \hat{a}_{j\mu}^+ + \hat{a}_{j\mu}^+ \hat{a}_{i\sigma} = \delta_{\sigma\mu} \delta_{ij}, \quad \hat{b}_{i\sigma} \hat{b}_{j\mu}^+ + \hat{b}_{j\mu}^+ \hat{b}_{i\sigma} = \delta_{\sigma\mu} \delta_{ij},$$

$$\hat{a}_{i\sigma} \hat{a}_{j\mu} + \hat{a}_{j\mu} \hat{a}_{i\sigma} = 0, \quad \hat{b}_{i\sigma} \hat{b}_{j\mu} + \hat{b}_{j\mu} \hat{b}_{i\sigma} = 0,$$

$$\hat{a}_{i\sigma} \hat{b}_{j\mu} + \hat{b}_{j\mu} \hat{a}_{i\sigma} = 0, \quad \hat{a}_{i\sigma} \hat{b}_{j\mu}^+ + \hat{b}_{j\mu}^+ \hat{a}_{i\sigma} = 0.$$

In the Heisenberg pictures, operators $\hat{a}_{j\mu}$ and $\hat{b}_{j\mu}$ obey the usual Heisenberg equation of motion. By using the Hamiltonian (1) and the relations (2) we can write these equations as:

$$i\hbar \frac{\partial}{\partial t} \hat{a}_{j\sigma} = \varepsilon_a \hat{a}_{j\sigma} + T_a \left( \hat{a}_{j-1\sigma} + \hat{a}_{j+1\sigma} \right) + U_a \left( \delta_{\sigma\uparrow} \hat{a}_{j\uparrow} \hat{n}_{j\downarrow} + \delta_{\sigma\downarrow} \hat{a}_{j\downarrow} \hat{n}_{j\uparrow} \right) +$$

$$+ \sum_{j\mu\upsigma} U_{ab} \hat{n}_{j\mu\upsigma} \hat{N}_{j\upsigma}.$$
In order to consider the one-particle energy levels population dynamics it is instructive to write the equations for operators \( \hat{n}_{j\sigma} \) and \( \hat{N}_{j\sigma} \):

\[
\begin{align*}
\frac{i\hbar}{\partial t} \hat{n}_{j\sigma} &= T_a \left( \hat{a}_{j_{\sigma}}^+ \hat{a}_{j-1\sigma} - \hat{a}_{j+1\sigma}^+ \hat{a}_{j\sigma} \right) + T_a \left( \hat{a}_{j\sigma}^+ \hat{a}_{j-1\sigma} - \hat{a}_{j-1\sigma}^+ \hat{a}_{j\sigma} \right) \\
&+ \delta_{\sigma\downarrow} \left( V_1 \hat{a}_{j\sigma}^+ \hat{b}_{j\uparrow} - V_1 \hat{b}_{j\uparrow}^+ \hat{a}_{j\sigma} \right) + \delta_{\sigma\uparrow} \left( V_2 \hat{a}_{j\sigma}^+ \hat{b}_{j\uparrow} - V_2 \hat{b}_{j\uparrow}^+ \hat{a}_{j\sigma} \right), \\
\frac{i\hbar}{\partial t} \hat{N}_{j\sigma} &= T_b \left( \hat{b}_{j\sigma}^+ \hat{b}_{j+1\sigma} - \hat{b}_{j+1\sigma}^+ \hat{b}_{j\sigma} \right) + T_b \left( \hat{b}_{j\sigma}^+ \hat{b}_{j-1\sigma} - \hat{b}_{j-1\sigma}^+ \hat{b}_{j\sigma} \right) \\
&- \delta_{\sigma\uparrow} \left( V_1 \hat{a}_{j\sigma}^+ \hat{b}_{j\uparrow} - V_1 \hat{b}_{j\uparrow}^+ \hat{a}_{j\sigma} \right) - \delta_{\sigma\uparrow} \left( V_2 \hat{a}_{j\sigma}^+ \hat{b}_{j\uparrow} - V_2 \hat{b}_{j\uparrow}^+ \hat{a}_{j\sigma} \right). 
\end{align*}
\]

One can see from these equations that population of the states \(|a\rangle\) and \(|b\rangle\) varies with time due to both tunnel processes and electromagnetic interaction determined by the transition operators \( \hat{P}_{ij} = \hat{b}_{j\uparrow}^+ \hat{a}_{j\downarrow} \) and \( \hat{P}_{2j} = \hat{b}_{j\downarrow}^+ \hat{a}_{j\uparrow} \). The equations (3) and (4) allow to write the system of equations for these operators:

\[
\begin{align*}
\frac{i\hbar}{\partial t} \hat{P}_{1j} &= (\varepsilon_a - \varepsilon_b) \hat{P}_{1j} - V_1 (\hat{n}_{j\downarrow} - \hat{N}_{j\uparrow}) + (U_a - U_{ab}) \hat{n}_{j\uparrow} \hat{P}_{1j} - \\
&- (U_b - U_{ab}) \hat{N}_{j\downarrow} \hat{P}_{1j} + T_a \hat{b}_{j\uparrow}^+ (\hat{a}_{j-1\downarrow} + \hat{a}_{j+1\downarrow}) - T_b (\hat{b}_{j-1\uparrow} + \hat{b}_{j+1\uparrow}) \hat{a}_{j\downarrow}, \\
\frac{i\hbar}{\partial t} \hat{P}_{2j} &= (\varepsilon_a - \varepsilon_b) \hat{P}_{2j} - V_2 (\hat{n}_{j\uparrow} - \hat{N}_{j\downarrow}) + (U_b - U_{ab}) \hat{n}_{j\downarrow} \hat{P}_{2j} - \\
&- (U_a - U_{ab}) \hat{N}_{j\uparrow} \hat{P}_{2j} + T_a \hat{b}_{j\downarrow}^+ (\hat{a}_{j-1\uparrow} + \hat{a}_{j+1\uparrow}) - T_b (\hat{b}_{j-1\downarrow} + \hat{b}_{j+1\downarrow}) \hat{a}_{j\uparrow}.
\end{align*}
\]
following relations result from the definition of \( \hat{\alpha} \). It should be pointed out that when the dots are not separated far enough, the strongly localised electrons on one dot produce a significant potential on adjacent dots. In this paper we suppose that the distances between QDs are so long that the tunnelling between neighbouring quantum dots can be neglected. It is one site approximation. The ensemble of such dots is similar to gas of the resonant atoms or impurities in glass. It is useful model being considered in the resonant optics. Henceforth the site index \( j \) of the operators can be omitted. Thus, we get the simpler model of the low-density ensemble of QDs, which is described by the following system of the equations

\[
\begin{align*}
\frac{i\hbar}{\partial t} \hat{P}_1 &= -\Delta \varepsilon \hat{P}_1 + \Delta U \hat{R}_2 \hat{P}_1 - V_1 \hat{R}_1, \\
\frac{i\hbar}{\partial t} \hat{P}_2 &= -\Delta \varepsilon \hat{P}_2 + \Delta U \hat{R}_1 \hat{P}_2 - V_2 \hat{R}_2, \\
\frac{i\hbar}{\partial t} \hat{R}_1 &= 2 \left( V_1 \hat{P}_1^+ - V_1^\ast \hat{P}_1 \right), \\
\frac{i\hbar}{\partial t} \hat{R}_2 &= 2 \left( V_2 \hat{P}_2^+ - V_2^\ast \hat{P}_2 \right),
\end{align*}
\]

where \( \Delta \varepsilon = (\varepsilon_b - \varepsilon_a) \), \( \Delta U = (U_a - U_{ab}) = (U_b - U_{ab}) \). Hereafter, we suppose that \( U_a = U_b \neq U_{ab} \) and define the operators \( \hat{R}_1 = \hat{n}_\downarrow - \hat{N}_\uparrow \), \( \hat{R}_2 = \hat{n}_\uparrow - \hat{N}_\downarrow \). Furthermore, in the equations (9) and (10) the productions of the operators \( \hat{S}_1 = \hat{R}_2 \hat{P}_1 \) and \( \hat{S}_2 = \hat{R}_1 \hat{P}_2 \) are appeared. By using the equations (9 - 12) we can obtain

\[
\begin{align*}
\frac{i\hbar}{\partial t} \hat{S}_1 &= \left( \frac{i\hbar}{\partial t} \hat{R}_2 \right) \hat{P}_1 + \hat{R}_2 \left( \frac{i\hbar}{\partial t} \hat{P}_1 \right) = \\
&= -\Delta \varepsilon \hat{S}_1 + \Delta U \hat{R}_2 \hat{R}_2 \hat{P}_1 - V_1 \hat{R}_2 \hat{R}_1 + 2 \left( V_2 \hat{P}_2^+ - V_2^\ast \hat{P}_2 \right) \hat{P}_1, \\
\frac{i\hbar}{\partial t} \hat{S}_2 &= \left( \frac{i\hbar}{\partial t} \hat{R}_1 \right) \hat{P}_2 + \hat{R}_1 \left( \frac{i\hbar}{\partial t} \hat{P}_2 \right) = \\
&= -\Delta \varepsilon \hat{S}_2 + \Delta U \hat{R}_1 \hat{R}_1 \hat{P}_2 - V_2 \hat{R}_1 \hat{R}_2 + 2 \left( V_1 \hat{P}_1^+ - V_1^\ast \hat{P}_1 \right) \hat{P}_2.
\end{align*}
\]

Here we get new operator products. Rather than continue this process, the algebraic properties of the operators \( \hat{P}_1 \) and \( \hat{P}_2 \) will be considered. The following relations result from the definition of \( \hat{P}_1 \) and \( \hat{P}_2 \).

\[
\hat{P}_1 \hat{P}_2 = \hat{P}_2 \hat{P}_1, \quad \hat{P}_1^+ \hat{P}_2 = \hat{P}_2^+ \hat{P}_1^+, \quad \hat{R}_1 \hat{R}_2 = \hat{R}_2 \hat{R}_1, \quad \hat{P}_1 \hat{R}_2 = \hat{R}_2 \hat{P}_1, \quad \hat{R}_1 \hat{P}_2 = \hat{P}_2 \hat{R}_1,
\]
\[ \hat{R}_1 \hat{P}_1 = -\hat{P}_1, \quad \hat{N}_\downarrow \hat{P}_1 = \hat{P}_1, \quad \hat{P}_1 \hat{N}_\uparrow = 0, \quad \hat{P}_1 \hat{R}_1 = \hat{P}_1, \quad \hat{P}_1 \hat{n}_\downarrow = \hat{P}_1, \quad \hat{n}_\downarrow \hat{P}_1 = 0, \]
\[ \hat{R}_1 \hat{P}_1^+ = \hat{P}_1^+, \quad \hat{P}_1^+ \hat{N}_\uparrow = \hat{P}_1^+, \quad \hat{P}_1^+ \hat{n}_\downarrow = 0, \quad \hat{P}_1^+ \hat{R}_1 = -\hat{P}_1^+, \quad \hat{n}_\downarrow \hat{P}_1^+ = \hat{P}_1^+, \]
\[ \hat{R}_2 \hat{P}_2 = -\hat{P}_2, \quad \hat{N}_\downarrow \hat{P}_2 = \hat{P}_2, \quad \hat{P}_2 \hat{N}_\uparrow = 0, \quad \hat{P}_2 \hat{R}_2 = \hat{P}_2, \quad \hat{P}_2 \hat{n}_\uparrow = \hat{P}_2, \quad \hat{n}_\uparrow \hat{P}_2 = 0, \]
\[ \hat{R}_2 \hat{P}_2^+ = \hat{P}_2^+, \quad \hat{P}_2^+ \hat{N}_\uparrow = \hat{P}_2^+, \quad \hat{P}_2^+ \hat{n}_\downarrow = 0, \quad \hat{P}_2^+ \hat{R}_2 = -\hat{P}_2^+, \quad \hat{n}_\uparrow \hat{P}_2^+ = \hat{P}_2^+. \]

These equalities lead to commutation relations
\[ \hat{P}_1 \hat{R}_1 - \hat{R}_1 \hat{P}_1 = 2\hat{P}_1, \quad \hat{P}_1 \hat{R}_1 + \hat{R}_1 \hat{P}_1 = 0, \]
\[ \hat{P}_2 \hat{R}_2 - \hat{R}_2 \hat{P}_2 = 2\hat{P}_2, \quad \hat{P}_2 \hat{R}_2 + \hat{R}_2 \hat{P}_2 = 0, \]
\[ \hat{P}_1^+ \hat{R}_1 - \hat{R}_1 \hat{P}_1^+ = -2\hat{P}_1^+, \quad \hat{P}_1^+ \hat{R}_1 + \hat{R}_1 \hat{P}_1^+ = 0, \]
\[ \hat{P}_2^+ \hat{R}_2 - \hat{R}_2 \hat{P}_2^+ = -2\hat{P}_2^+, \quad \hat{P}_2^+ \hat{R}_2 + \hat{R}_2 \hat{P}_2^+ = 0. \]

There are more complete expressions
\[ \hat{P}_1^+ \hat{P}_1 = \hat{n}_\downarrow (1 - \hat{N}_\uparrow), \quad \hat{P}_1 \hat{P}_1^+ = \hat{N}_\uparrow (1 - \hat{n}_\downarrow), \]
\[ \hat{P}_2^+ \hat{P}_2 = \hat{n}_\uparrow (1 - \hat{N}_\uparrow), \quad \hat{P}_2 \hat{P}_2^+ = \hat{N}_\downarrow (1 - \hat{n}_\uparrow), \]

which cause the following commutation relations
\[ \hat{P}_1^+ \hat{P}_1 - \hat{P}_1 \hat{P}_1^+ = (\hat{n}_\downarrow - \hat{N}_\uparrow) = \hat{R}_1, \quad \hat{P}_1^+ \hat{P}_1 + \hat{P}_1 \hat{P}_1^+ = \hat{R}_1^2, \]
\[ \hat{P}_2^+ \hat{P}_2 - \hat{P}_2 \hat{P}_2^+ = (\hat{n}_\uparrow - \hat{N}_\downarrow) = \hat{R}_2, \quad \hat{P}_2^+ \hat{P}_2 + \hat{P}_2 \hat{P}_2^+ = \hat{R}_2^2, \]

where the properties \( \hat{n}_\downarrow = n_\sigma, \hat{N}_\uparrow = \hat{N}_\sigma \) have been used.

Let us denote the energy states of the QD by vectors \( |n_\sigma; n_\downarrow; n_\uparrow\rangle \), where \( n_\sigma \) (\( n_\downarrow \)) is number of electrons with spin in state \( |a\rangle \) (\( |b\rangle \)). It should be noted that the state of the quantum dot in one site approximation evaluates in the Hilbert space \( \mathfrak{H} \) generated by four basis vectors
\[ |1, 1; 0, 0, \rangle = |1\rangle, \quad |1, 0; 0, 1, \rangle = |2\rangle, \quad |0, 1; 1, 0, \rangle = |3\rangle, \quad |0, 0; 1, 1, \rangle = |4\rangle. \]

By direct calculations in space \( \mathfrak{H} \) we can obtain the following representations for the operators \( \hat{R}_1 \) and \( \hat{R}_2 \). They are \( \hat{R}_1 = \text{diag}(+1, +1, -1, -1) \) and \( \hat{R}_2 = \text{diag}(+1, -1, +1, -1) \). Thus we have \( \hat{R}_1^2 = \hat{R}_2^2 = 1 \). Furthermore, one can obtain the following expressions
\[ \hat{n}_\downarrow = \hat{P}_1^+ \hat{P}_1, \quad \hat{n}_\uparrow = \hat{P}_2^+ \hat{P}_2, \quad \hat{N}_\uparrow = \hat{P}_1 \hat{P}_1^+ , \quad \hat{N}_\downarrow = \hat{P}_2 \hat{P}_2^+. \]
By taking into account these results the reduced commutation relations can be written as

\[
\hat{P}_1^+ \hat{P}_1 - \hat{P}_1 \hat{P}_1^+ = \hat{R}_1, \quad \hat{P}_1^+ \hat{P}_1 + \hat{P}_1 \hat{P}_1^+ = 1,
\]
\[
\hat{P}_1 \hat{R}_1 - \hat{R}_1 \hat{P}_1 = 2\hat{P}_1, \quad \hat{P}_1 \hat{R}_1 + \hat{R}_1 \hat{P}_1 = 0,
\]
\[
\hat{P}_1^+ \hat{R}_2 - \hat{R}_2 \hat{P}_1^+ = -2\hat{P}_1^+, \quad \hat{P}_1^+ \hat{R}_1 + \hat{R}_1 \hat{P}_1^+ = 0,
\]
\[
\hat{P}_1^+ \hat{P}_m - \hat{P}_m \hat{P}_1^+ = 0, \quad \hat{P}_1 \hat{P}_m - \hat{P}_m \hat{P}_1 = 0,
\]
\[
\hat{P}_1 \hat{R}_m - \hat{R}_m \hat{P}_1 = 0, \quad \hat{R}_1 \hat{R}_m - \hat{R}_m \hat{R}_1 = 0.
\]

where \( l, m = 1, 2 \). Thus the constriction of the considered algebra of the operators \( \{ \hat{P}_l, \hat{R}_m \} \) on the space \( \mathfrak{G} \) generates the Lie-algebra \( su(2) \times su(2) \).

Taking the commutation relations (22) into account we can find \( \hat{R}_1^2 = \hat{R}_2^2 = 1 \). Let us introduce new operators

\[
\hat{N}_3 = \hat{R}_1 \hat{R}_2 = \hat{R}_2 \hat{R}_1, \quad \hat{W} = \hat{P}_1^+ \hat{P}_2 = \hat{P}_2 \hat{P}_1^+, \quad \hat{K} = \hat{P}_1 \hat{P}_2 = \hat{P}_2 \hat{P}_1. \tag{23}
\]

These operators are governed by the equations

\[
\frac{\hbar}{i} \frac{\partial}{\partial t} \hat{N}_3 = 2 \left( V_2 \hat{P}_2^+ - V_2^* \hat{P}_2 \right) \hat{R}_1 + 2 \hat{R}_2 \left( V_1 \hat{P}_1^+ - V_1^* \hat{P}_1 \right), \tag{24}
\]
\[
\frac{\hbar}{i} \frac{\partial}{\partial t} \hat{W} = \left( -\Delta \varepsilon \hat{P}_1^+ + \Delta U \hat{R}_2 \hat{P}_1^+ + V_1^* \hat{R}_1 \right) \hat{P}_2 + \hat{P}_1^+ \left( -\Delta \varepsilon \hat{P}_2 + \Delta U \hat{R}_1 \hat{P}_2 + V_2 \hat{R}_2 \right) = \Delta U \left( \hat{P}_1^+ \hat{R}_1 \hat{P}_2 - \hat{R}_2 \hat{P}_1^+ \hat{P}_2 \right) + V_1^* \hat{R}_1 \hat{P}_2 - V_2 \hat{P}_1^+ \hat{R}_2, \tag{25}
\]
\[
\frac{\hbar}{i} \frac{\partial}{\partial t} \hat{K} = -2\Delta \varepsilon \hat{P}_1 \hat{P}_2 + \Delta U \left( \hat{R}_2 \hat{P}_1 \hat{P}_2 + \hat{P}_1 \hat{R}_1 \hat{P}_2 \right) - V_1 \hat{R}_1 \hat{P}_2 - V_2 \hat{P}_1 \hat{R}_2. \tag{26}
\]

By using the commutation relations one can find that

\[
\hat{P}_1^+ \hat{R}_1 \hat{P}_2 = -\hat{P}_1 \hat{P}_2, \quad \hat{R}_2 \hat{P}_1^+ \hat{P}_2 = \hat{P}_1^+ \hat{R}_2 \hat{P}_2 = -\hat{P}_1 \hat{P}_2, \quad \hat{R}_2 \hat{P}_1 \hat{P}_2 = -\hat{P}_2 \hat{P}_1, \quad \hat{P}_1 \hat{R}_1 \hat{P}_2 = \hat{P}_1 \hat{P}_2 = \hat{P}_2 \hat{P}_1.
\]

Hence, the first term in equation (25) and the second term in equation (24) vanishes. By combining the resulting equations we can represent completed system of the equations for relative operators:

\[
\frac{\hbar}{i} \frac{\partial}{\partial t} \hat{P}_1 = -\Delta \varepsilon \hat{P}_1 + \Delta U \hat{S}_1 - V_1 \hat{R}_1,
\]
\[ \hat{P}_2 = -\Delta \varepsilon \hat{P}_2 + \Delta U \hat{S}_2 - V_2 \hat{R}_2, \]
\[ \hat{R}_1 = 2 \left( V_1 \hat{P}_1^+ - V_1^* \hat{P}_1 \right), \]
\[ \hat{R}_2 = 2 \left( V_2 \hat{P}_2^+ - V_2^* \hat{P}_2 \right), \]
\[ \hat{S}_1 = -\Delta \varepsilon \hat{S}_1 + \Delta U \hat{P}_1 - V_1 \hat{N}_3 + 2V_2 \hat{W}^+ - 2V_2^* \hat{K}, \]
\[ \hat{S}_2 = -\Delta \varepsilon \hat{S}_2 + \Delta U \hat{P}_2 - V_2 \hat{N}_3 + 2V_1 \hat{W} - 2V_1^* \hat{K}, \]
\[ \hat{W} = V_1^* \hat{R}_1 \hat{P}_2 - V_2 \hat{P}_1^+ \hat{R}_2, \]
\[ \hat{K} = -2\Delta \varepsilon \hat{K} - \left( V_1 \hat{S}_2 + V_2 \hat{S}_1 \right). \] (27)

Instead of solving for the full system of operators’ equations (27) one may find the classical (c-number’s) equations describing the evolution of the QD variables. These equations play a similar role in the theory of coherent responses of the optically excited single QD much as the Bloch equations are used in the case of two-level atoms. To obtain the desired equations the operators in equations (27) should be substituted for them expectation values, i.e. \( \hat{A} \rightarrow \langle \hat{A} \rangle \), for any operator \( \hat{A} \). It is convenient to introduce the following variables and parameters:

\[ 2 \langle \hat{P}_{1,2} \rangle = p_{1,2}, \quad 2 \langle \hat{S}_{1,2} \rangle = s_{1,2}, \quad 2 \langle \hat{K} \rangle = u, \quad 2 \langle \hat{W} \rangle = w, \]
\[ \langle \hat{R}_{1,2} \rangle = n_{1,2}, \quad \langle \hat{N}_3 \rangle = n_3, \]
\[ V_{1,2} = \omega_R e_{1,2}, \quad \Omega = -\Delta \varepsilon / 2\hbar \omega_R, \quad \Delta = -\Delta U / 2\hbar \omega_R, \quad \tau = 2\omega_R t, \]

where \( \omega_R = d_{12} A_0 \) is picked Raby frequency, \( d_{12} \) is a matrix element of the dipole-moment operator, \( e_{1,2} = E_{1,2} / A_0 \) are the normalised electric fields of the electromagnetic waves with different polarisations. In terms of these variables we have following system of equations:

\[ i \frac{\partial p_1}{\partial \tau} = \Omega p_1 - \Delta s_1 - e_1 n_1, \]
\[ i \frac{\partial p_2}{\partial \tau} = \Omega p_2 - \Delta s_2 - e_2 n_2, \]
\[
\begin{align*}
  i \frac{\partial s_1}{\partial \tau} &= \Omega s_1 - \Delta p_1 - e_1 n_3 + 2e_2 w^* - 2e_2^* u, \\
  i \frac{\partial s_2}{\partial \tau} &= \Omega s_2 - \Delta p_2 - e_2 n_3 + 2e_1 w - 2e_1^* u, \\
  i \frac{\partial n_1}{\partial \tau} &= \frac{1}{2} (e_1 p_1^* - e_1^* p_1), \\
  i \frac{\partial n_2}{\partial \tau} &= \frac{1}{2} (e_2 p_2^* - e_2^* p_2), \\
  i \frac{\partial n_3}{\partial \tau} &= \frac{1}{2} (e_1 s_1^* - e_1^* s_1) + \frac{1}{2} (e_2 s_2^* - e_2^* s_2), \\
  i \frac{\partial w}{\partial \tau} &= \frac{1}{2} (e_1 s_2 - e_2 s_1^*), \\
  i \frac{\partial u}{\partial \tau} &= 2\Omega u - \frac{1}{2} (e_1 s_1 + e_2 s_2).
\end{align*}
\]

It is worthy of note that in these equations we take not the slowly varying envelope and phase approximation (SVEPA), hence \(e_{1,2} = e_{1,2}^*\). However, SVEPA is of frequent occurrence in resonant non-linear optics. To employ the equations (28) in framework of the SVEPA the variables \(e_{1,2}\) should be read as the slowly varying complex envelopes of the ultra-short electromagnetic pulses and parameter \(\Omega = -\Delta \varepsilon / 2\hbar \omega_R\) in (28) should be substituted for \(\Omega = -(\Delta \varepsilon - \hbar \omega_0) / 2\hbar \omega_R\). Here \(\omega_0\) is the carrier wave frequency.

3 Illustrative example

The ultra-short electromagnetic pulse propagation can be considered in framework of the equations (28) and reduced Maxwell equations

\[
\frac{\partial e_1}{\partial \zeta} = i \langle p_1 \rangle, \quad \frac{\partial e_2}{\partial \zeta} = i \langle p_2 \rangle,
\]

where \(\zeta = z / L_{ab}\) is normalised space co-ordinate, \(L_{ab}\) is absorption length. As a simple example of solving the system of equations (28) and (29), we consider the solution that describes the propagation of a steady state solitary wave, i.e., the USP. Let all variables depend only on \(\eta = \tau - \zeta / \alpha\), where \(\alpha\) is a parameter. It leads to system of ordinary differential equations. The solution
of the resulting system can be obtained if to consider the electromagnetic waves to be a circular polarised ones. In this case we found

\[ e_{1,2}(\tau, \zeta) = \left(\frac{2}{\tau_{1,2}}\right) \text{sech}\left[\frac{(\tau - \zeta/\alpha_{1,2})/\tau_{1,2}}{\tau_{1,2}}\right] \exp(-i\Omega\tau), \quad e_{2,1}(\tau, \zeta) = 0, \]

where \( \tau_{1,2} \) is re-normalised USP duration. These solutions represent the simplest \( 2\pi \)-pulse by McCall and Hahn [10] propagating in the low-density ensemble of quantum dots. The numerical simulation of the collision of two steady states with different circular polarisation shows an inelastic interaction of these USPs, i.e., the collision between pulses results in the generation of the radiation from these pulses. That reflects a strong electron-electron Coulomb interaction. If parameter \( \Delta \) in equations (28) set zero, then the different polarised steady state pulses (30) are interacted as solitons (or \( 2\pi \)-pulse).

4 Conclusion

We have introduced the simplest model of the quantum dots interacting with electromagnetic wave. That leads to new system of equations, which can be considered as generalisation of the famous Bloch equations of the theory of coherent optical processes [14]. This system in combination with reduced wave equations for electromagnetic waves provide the base for description of the coherent optical phenomena in the low-density ensemble of quantum dots. The simplest families of exact analytical solutions have been found for moving steady state pulses. Collisions between the pulses were simulated as the tentative, showing that they interact inelastic, resulting in generation of radiation and emerging a new weak pulse.

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