On Some Stability Properties of Compactified
D=11 Supermembranes

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Abstract. We describe the minimal configurations of the bosonic membrane potential, when the membrane wraps up in an irreducible way over \( S^1 \times S^1 \). The membrane 2-dimensional spatial world volume is taken as a Riemann Surface of genus \( g \) with an arbitrary metric over it. All the minimal solutions are obtained and described in terms of 1-forms over an associated \( U(1) \) fiber bundle, extending previous results. It is shown that there are no infinite dimensional valleys at the minima.

1 Introduction

The Minkowski \( D = 11 \) Supermembrane, when the \( SU(N) \), \( N \to \infty \), regularization is used, was shown to have a continuous spectrum from zero to infinity (B. de Wit, M. Lüscher and H. Nicolai(1989)). The instability problem may be understood as a consequence of the existence of string like configuration with the same energy. The configurations which give the minimum of the potential consists of infinite dimensional functional subspaces. The potential as a functional may be described around those subspaces as having infinite dimensional valleys. This property appears already in the bosonic membrane, however because of the quantum zero point energy of the oscillators transversal to the valleys stability is attained. In the supermembrane, because of the property of the \( SUSY \) harmonic oscillators to have no zero point energy the theory is unstable. Because of duality arguments and the relation between the supermembrane and its dual D-brane, we are really more interested in studying the Hamiltonian of the compactified supermembranes where at least one dimension in the target space is compactified to \( S^1 \) (M.J.Duff, T. Inami, C.N. Pope, E. Sezgin and K.S.Stelle(1988)). The spectrum of the Hamiltonian of the compactified supermembrane has been recently studied by several authors without a conclusive result (J.G. Russo(1997))(B. de Wit, K. Peeters and J. Plefka(1997)).

In a recent paper (I. Martin, A. Restuccia and R. Torrealba(1998)) we showed that the Hamiltonian of the membrane wrapped up in an irreducible way over \( S^1 \times S^1 \) has no infinite dimensional valleys. Moreover we found all the minimal configurations, when the metric over \( \Sigma \) the 2-dimensional spatial world volume of the membrane was the canonical generalization of the induced metric over \( S^2 \) from \( R^3 \). We considered the general situation where \( \Sigma \) was a Riemann Surface
of genus $g$. The construction was performed in terms of intrinsic harmonic coordinates. The minimal configuration was found to be unique up to closed forms over $\Sigma$. We expect that this properties of the bosonic Hamiltonian will also be valid for the complete supermembrane Hamiltonian. That is, that there are no infinite dimensional valleys at the minimal configurations of the supermembrane wrapped up in an irreducible way over $S^1 \times S^1$. We will discuss this problem in a forthcoming paper. In the present work we would like to give the general solution for the minimal configurations when other metrics, not necessarily the canonical one, is assumed over $\Sigma$.

2 Minimal Configurations of the Membrane Potential

We will analyze in this section minimal configurations of the Hamiltonian of the bosonic membrane. To each configuration of the membrane, determined by $dx(\sigma)$ we will associate a connection 1-form on a $U(1)$ bundle over $\Sigma$, a compact Riemann surface of genus $g$. We will then show that the minimal configurations correspond exactly to the magnetic monopoles over Riemann surfaces found in (I. Marin and A. Restuccia(1997))( F. Ferrari(1993)). Given a set of maps $X_a$, $a = 1, \ldots, 9$, from $\Sigma$ to the target space $T$ we define

$$F_{ab} \equiv d(X_adX_b), \ a, b = 1, \ldots, 9. \quad (2.1)$$

as the $U(1)$ curvatures associated to the connection 1-forms

$$A_{ab} \equiv X_{[a}dX_{b]} \quad (2.2)$$

The potential of the bosonic membrane may then be rewritten as

$$< V >_{\sigma} \equiv \int_{\Sigma} d^2\sigma \sqrt{g} V(\sigma) = \int_{\Sigma} *F_{ab}F_{ab} \quad (2.3)$$

where $*F_{ab}$ is the Hodge dual of the curvature 2-form $F_{ab}$ and $g$ is the determinant of the metric over the Riemann surface $\Sigma$. We normalize the metric by the condition $\text{Vol} \Sigma = 2\pi$.

It is clear from (2.3) that the infinite dimensional configuration

$$X_a(\sigma) = \lambda_a X(\sigma), \ a = 1, \ldots, 9. \quad (2.4)$$

where $\lambda_a$ are arbitrary parameters, has zero potential.

In fact $F_{ab} = 0$ for all $a$ and $b$. The configuration (2.4) is infinite dimensional since $X(\sigma)$ is an arbitrary map, with the only restriction to have a well defined potential (2.3). The space of functions over $\Sigma$ satisfying that requirement is infinite dimensional. These are the valleys which give rise to the continuous spectrum from zero to infinite for the non-compactified supermembrane.

The existence of the valleys is, of course, not restricted to the minima, as emphasized in (B. de Wit, K. Peeters and J. Plefka(1997)).
We are interested in the analysis of the supermembrane in the case in which the target space $T$ is compactified. Let $X(\sigma)$ denote a compactified coordinate over $T$. Let us assume $X$ is a map from

$$\Sigma \to S^1$$  \hspace{1cm} (2.5)

It is then straightforward to see that $dx$ satisfies the following conditions for a 1-form $L$

$$dL = 0$$  \hspace{1cm} (2.6)
$$\oint_{C_i} L = 2\pi n^i$$

where $C_i$ denotes a basis of the integral homology of dimension one over $\Sigma$. It is interesting to note that the converse to (2.7) is also valid. That is, given a globally defined 1-form $L$ over $\Sigma$ satisfying (2.7) there exists a map $X$ from

$$\Sigma \to S^1$$  \hspace{1cm} (2.7)

for which $L = dX$.

We will say the supermembrane wraps up in a non-trivial way when at least one of the $n^i$ is different from zero.

We also say the supermembrane wraps up in an irreducible way over $S^1 \times S^1$ when

$$\int_{\Sigma} dX \wedge dY \neq 0$$  \hspace{1cm} (2.8)

where $X$ and $Y$ are two maps. That is, the irreducibility condition requires a compactification of at least two coordinates on the target $T$. If the membrane wraps up in an irreducible way over $S^1 \times S^1$ it does it in a nontrivial way over each of the $S^1$. Moreover

$$\frac{1}{2\pi} \int_{\Sigma} dX \wedge dY = 2\pi N \neq 0,$$  \hspace{1cm} (2.9)

$N$ being an integer.

(2.8) may be interpreted in terms of the $U(1)$ bundle associated to the membrane configuration, using (2.1) and (2.2). It tell us that the corresponding Chern class is non-trivial. The integral number $N$ in (2.9) determines the $U(1)$ bundle over which the connection (2.2) is defined. We notice that the existence of infinite dimensional valleys still persist when the target space $T$ is of the form

$$T = M_{10} \times S^1.$$  \hspace{1cm} (2.10)

In fact the coordinates mapping $\Sigma \to M_{10}$, 10-dimensional Minkowski space, are single valued over $\Sigma$.

We may always take the compactified coordinate, say $X_1$ to be

$$X_1 = \phi$$  \hspace{1cm} (2.11)
where $\phi$ is the angular coordinate of $S^1$ in (2.10). An admissible configuration is then given by

$$X_2 = X_3 = \ldots = X_9 = \frac{dX(\phi)}{d\phi}$$

$$\phi = \phi(\sigma),$$

(2.12)

where $X(\phi)$ is a differentiable single valued function on $\phi$, that is a regular periodic function, and $\phi(\sigma)$ is a map from $\Sigma$ to $S^1$. It then follows that the curvature (2.1) is zero for all $a$ and $b$. The subspace (2.12) is still infinite dimensional.

We look now for the stationary points of (2.3) over the space of maps defining supermembranes with irreducible wrapping over $S^1 \times S^1$. It is straightforward to see in this case that the minimal configurations occur when all but $X, Y$ maps are zero. Associated to this space we may introduce an $U(1)$ principle bundle. We proceed by noting that

$$F = \frac{1}{2\pi} dX \wedge dY$$

(2.13)

is a closed 2-form globally defined over $\Sigma$ satisfying (2.9). By Weil’s Theorem (A. Weil(1957)), (M. Caicedo, I. Martin and A. Restuccia (1997), there exists a $U(1)$ principle bundle and a connection over it such that its pull back by sections over $\Sigma$ are 1-form connections with curvatures given by (2.13).

The stationary points of the potential satisfy

$$\delta X dY \wedge d^* F = 0$$

$$\delta Y dX \wedge d^* F = 0$$

(2.14)

which imply

$$d^* F = 0.$$  

(2.15)

Now, since $*F$ is a 0-form we get

$$*F = \text{constant}.$$  

over $\Sigma$, and using (2.9) we finally obtain

$$*F = \frac{2\pi N}{\text{Vol} \Sigma} = N.$$  

(2.16)

We will now show that the configurations with $X$ and $Y$ satisfying (2.16) and all other $X_a$ maps to zero are minima of the potential (2.3) within the space of configurations with irreducible winding (2.9).

For any connection on the $U(1)$ principle bundle over $\Sigma$ determined by $N$ the associated curvature 2-form satisfies

$$dF = 0$$

$$\int_\Sigma F = 2\pi N$$

(2.17)
Let $A_0$ be a connection 1-form satisfying (2.16), and $A_1$ any other connection 1-form on the sample principle bundle. Then, using (2.18) for $F(A_1)$, we obtain

$$\int_{\Sigma} *F(A_1 - A_0)F(A_1 - A_0) = \int_{\Sigma} [*F(A_1)F(A_1) - *F(A_0)F(A_0)].$$

(2.18)

The left hand member of (2.18) is greater or equal to zero, we then have

$$\int_{\Sigma} *F(A_1)F(A_1) \geq \int_{\Sigma} *F(A_0)F(A_0)$$

(2.19)

The equality in (2.19) is obtained when the left hand member of (2.18) is zero. This implies

$$*F(A_1 - A_0) = *F(A_1) - *F(A_0) = 0.$$  

(2.20)

That is, the equality in (2.19) is obtained if and only if

$$A_1 = A_0 + dA$$  

(2.21)

where $dA$ is a closed 1-form globally defined over $\Sigma$.

The space of regular closed 1-forms, modulo exact 1-forms, over a compact (closed) Riemann Surface is finite dimensional. The exact 1-forms correspond to gauge transformations on the $U(1)$ bundle. In (I. Martin, A. Restuccia ant R. Torrealba(1998)) it was shown that they are generated by the area preserving transformations on the membrane maps. This implies the non-existence of infinite dimensional valleys at the minima for the membranes wrapping up in an irreducible way onto $S^1 \times S^1$.

3 Minimal Connections: Magnetic Monopoles over Riemann Surfaces of genus $g$

We will show in this section how to construct all the minimal connections over $S^2$ and over all topologically non-trivial Riemann Surfaces. To do so we will construct one minimal connection for each $N$. All others are obtained from (2.21). The space of closed 1-forms modulo exact forms is the space of harmonic 1-forms over $\Sigma$. It has been extensively studied in the literature, so it is not necessary to discuss it here. Our problem reduces then to find one minimal connection for each $N$. That is for each principle bundle over $\Sigma$. We will describe now that construction.

The explicit expression of the monopole connections is obtained in terms of the abelian differential $d\tilde{\Phi}$ of the third kind over the compact Riemann surface $\Sigma$ of genus $g$. $d\tilde{\Phi}$ is a meromorphic 1-form with poles of residue +1 and -1 at points $a$ and $b$, with real normalization. $\tilde{\Phi}$ is the abelian integral, its real part $G(z, \bar{z}, a, b, t)$ is a harmonic univalent function over $\Sigma$ with logarithmic behavior around $a$ and $b$.
\[
\ln\left(\frac{1}{|z + a|}\right) + \text{regular terms}, \\
\ln |z - b| + \text{regular terms}, 
\]
(3.1)

It is a conformally invariant geometrical object. \(z\) denotes the local coordinate over \(\Sigma\) and \(t\) the set of \(3g-3\) parameters describing the moduli space of Riemann surfaces. We are considering maps from \(\Sigma \mapsto S^1 \times S^1 \times M^7\) for a given \(\Sigma\), so the parameters \(t\) are kept fixed. They show however that the construction of minimal connections is a conformally invariant one.

Let \(a_i, \ i = 1, \ldots, m\) be \(m\) points over the compact Riemann surface. We associate to them integer weights \(\alpha_i, \ i = 1, \ldots, m\), satisfying
\[
\sum_{i=1}^{m} \alpha_i = 0 
\]
(3.2)

We define
\[
\phi = \sum_{i=1}^{m} \alpha_i G(z, \bar{z}, a_i, b, t). 
\]
(3.3)

and have
\[
\phi \to -\infty \text{ at } a_i \text{ with negative weights} \\
\phi \to +\infty \text{ at } a_i \text{ with positive weights.}
\]

\(\alpha_i\) are integers in order to have univalent transition functions over the nontrivial fiber bundle that we consider.

We denote \(\Phi\) the abelian integral with real part \(\phi\). Its imaginary part \(\varphi\) is also harmonic but multivalued over \(\Sigma\),
\[
\tilde{\Phi} = \phi + i\varphi. 
\]
(3.4)

Let us consider the curve \(C\) over \(\Sigma\) defined by
\[
\phi = \text{constant}. 
\]

It is a closed curve homologous to zero. It divides the Riemann surface into two regions \(U_+\) and \(U_-\), where \(U_+\) contains all the points \(a_i\) with negative weights and \(U_-\) the ones with positive weights.

We define over \(U_+\) and \(U_-\) the connection 1-forms
\[
A_+ = \frac{1}{2} (1 + \tanh(\phi)) d\varphi \\
A_- = \frac{1}{2} (-1 + \tanh(\phi)) d\varphi 
\]
(3.5)

respectively. \(A_+\) is regular in \(U_+\) and \(A_-\) in \(U_-\). In the overlapping \(U_+ \cap U_-\) we have
\[
A_+ = A_- + d\varphi 
\]
(3.6)
$g = \exp(i\varphi)$ defines the transition function on the overlapping $U_+ \cap U_-$, and because of the integer weights it is univalued over $U_+ \cap U_-$. The base manifold $\Sigma$, the transition function $g$ and the structure group $U(1)$ have a unique class of equivalent $U(1)$ principle bundles over $\Sigma$ associated to them. (3.6) defines a 1-form connection over $\Sigma$ with curvature

$$F = \frac{1}{2} \frac{1}{\cosh^2 \phi} d\phi \wedge d\varphi$$

(3.7)

The $U(1)$ principle bundles are classified by the sum of the positive integer weights $\alpha_i$

$$N = \sum_i \alpha_i^+ , \alpha_i^+ > 0.$$  

(3.8)

which is the only integer determining the number of times $\varphi$ wraps around $\mathcal{C}$. All the bundles with the same $N$ are equivalent. (3.7) satisfies (2.17), moreover it also satisfies (2.16). In fact, since $\varphi$ and $\phi$ are harmonic over $\Sigma$, the metric is

$$d^2s = \frac{1}{\cosh^2 \phi} ((d\varphi)^2 + (d\phi)^2),$$

(3.9)

and then (2.16) follows directly.

We have then found for each $U(1)$ principle bundle over $\Sigma$, a connection 1-form (3.6) with curvature 2-form (3.7) satisfying (2.16) for the metric (3.9) on the Riemann Surface. We have obtained several expressions (3.6), (3.7) and (3.9), since we are allowed to consider different harmonic coordinates $\phi$ and $\varphi$. In fact, for any set of $\alpha_i$ with total positive weight $N$, we may define coordinates $\phi$ and $\varphi$ away from the points $a_1$ and $b$. By so doing we are only using different coordinates over $\Sigma$ to describe the same connection over the $U(1)$ principle bundle. It is, as if in the expressions of the 1-form connection describing the Dirac monopole we use different coordinates $(\theta, \varphi)$ with different North and South poles to describe the magnetic field of the monopole. Since $\ast F$ is scalar field over $\Sigma$ then it is always equal to $N$.

4 The General Solution

We have thus obtained all the solutions satisfying $\ast F = N$ for the metric (3.9). The question arises then, what happens when we consider, instead of (3.9), the metric

$$|\lambda(\phi, \varphi)|^2 \frac{1}{\cosh^2(\phi)}[(d\varphi)^2 + (d\phi)^2]$$

(4.1)

that is, an element of the conformal class of (3.9). The questions is relevant since (2.16) is not conformal invariant. It depends on the metric through the factor $(\sqrt{g})^{-1}$. We may then ask for the solutions of (2.16) when the new metric (4.1) is considered over $\Sigma$. We will answer the question starting with the case in
which $\Sigma$ is the sphere $S^2$, and giving afterwards the solutions for topologically non-trivial Riemann Surfaces.

We consider the Hopf fiber bundle over $S_2$. The three dimensional sphere $S_3$ may be defined by $z_0, z_1 \in C$, the complex numbers, satisfying
\[ z_0 \bar{z}_0 + z_1 \bar{z}_1 = 1. \tag{4.2} \]
The group $U(1)$ acts on $S_3$ by
\[ (z_0, z_1) \rightarrow (z_0 u, z_1 u) \tag{4.3} \]
where $u \bar{u} = 1$, $\bar{u}$ being the complex conjugate to $u \in C$. The projection $S_3 \rightarrow S_2$ is defined by the composition of
\[ (z_0, z_1) \rightarrow \begin{cases} \frac{z_1}{z_0} & z_0 \neq 0 \\ \frac{z_0}{z_1} & z_1 \neq 0 \end{cases} \tag{4.4} \]
with the stereographic projection
\[ C \rightarrow S_2 \]
defined by
\[ \rho = \frac{\sin(\theta)}{1 - \cos(\theta)} \tag{4.5} \]
where $z = \rho e^{i\phi} \in C$ and $(\theta, \phi)$ are the coordinates $S_2$.

There is a natural connection over the Hopf fiber bundle which may be obtained from the line element of $S_3$,
\[ ds^2 = 4 (d\bar{z}_0 dz_0 + d\bar{z}_1 dz_1) \tag{4.6} \]
where $z_0, z_1$ satisfy (4.2). We will use spherical coordinates $(\chi, \theta, \phi)$ over $S_3$, defined in the following way
\[ z_0 = \exp \left[ -\frac{i}{2} (\chi + \varphi) \right] C(\theta, \varphi) \tag{4.7} \]
\[ z_1 = \exp \left[ -\frac{i}{2} (\chi - \varphi) \right] S(\theta, \varphi) \]
where
\[ C^2 + S^2 = 1. \tag{4.8} \]

We then obtain the line element of $S_3$ as
\[ \frac{1}{4} ds^2 = (dc)^2 + (ds)^2 + \frac{1}{4} [(d\chi)^2 + (d\varphi)^2 + 2(c^2 - s^2)d\chi d\phi]. \]
We denote
\[ c^2 - s^2 \equiv g(u, \varphi) \]
\[ u \equiv \cos(\theta) \]
We then get

\[ ds^2 = (d\chi + g(u, \varphi)d\phi)^2 + \frac{(dg)^2}{1 - g^2} + (1 - g^2)(d\varphi)^2. \quad (4.9) \]

We notice that in the particular case

\[ C(\theta, \varphi) = \cos\left(\frac{\theta}{2}\right) \]
\[ S(\theta, \varphi = \sin\left(\frac{\theta}{2}\right)) \]

(4.10) yields

\[ ds^2 = (d\chi + \cos(\theta)d\phi)^2 + (d\theta)^2 + (\sin(\theta)d\phi)^2. \quad (4.11) \]

Coming back to the general case (4.9), the line element of \( S_3 \) decomposes into the line element of \( S_2 \)

\[ \frac{(dg)^2}{1 - g^2} + (1 - g^2)(d\varphi)^2 \quad (4.12) \]

and the tensorial square of the 1-form

\[ \omega = d\chi + g(u, \varphi)d\phi. \quad (4.13) \]

The above decomposition allows to determine a 1-form (4.13) over the fiber bundle \( S_3 \). Notice that from (4.3) and (4.7) the group acts on \( \chi \) as follows

\[ \chi \rightarrow \chi + \lambda \quad (4.14) \]

where \( u = \exp(i\lambda) \). We are then interested in considering

\[ \frac{1}{2}\omega \]

as a connection over the fiber bundle \( S_3 \).

To obtain the \( U(1) \) connection 1-form over \( S_2 \), one may consider the local section

\[ \hat{z}_0 = \exp(i\varphi)C(\theta, \varphi) \]
\[ \hat{z}_1 = S(\theta, \varphi) \]

over \( S_2 \) with the point \( \theta = 0 \) removed, which we denote \( U_+ \). The \( U(1) \) connection over \( U_+ \) is then

\[ A_+ = \frac{1}{2}(1 + g(u, \varphi))d\varphi. \quad (4.16) \]

To give a covering of \( S_2 \) we define \( U_- \), another local section, by

\[ \hat{z}_0 = C(\theta, \varphi) \]
\[ \hat{z}_1 = e^{-i\varphi}S(\theta, \varphi) \]

(4.17)
over $S_2$ with the point $\theta = \pi$ removed. We have assumed in (4.15) and (4.17) that
\begin{align*}
C|_{\theta = \pi} &= 0 \\
S|_{\theta = 0} &= 0.
\end{align*}
(4.18)

Over $U_-$ the connection 1-form is then given by
\begin{equation}
A_+ = \frac{1}{2}(g(u, \varphi) - 1)d\varphi.
\end{equation}
(4.19)

In the overlapping region $U_+ \cap U_-$ we have
\begin{equation}
A_+ - A_- = d\varphi.
\end{equation}
(4.20)

The curvature 2-form $F$ is then given by
\begin{equation}
F = \frac{1}{2} \partial_{\mu} g(u, \varphi) \sin(\theta) d\varphi \wedge d\theta.
\end{equation}
(4.21)

In the particular case (4.11), it reduces to
\begin{equation*}
F = \frac{1}{2} \sin(\theta) d\varphi \wedge d\theta.
\end{equation*}

In order to obtain the general solution satisfying (2.16) we may proceed in two ways. We may extend the Hopf fibering
\[ S_3 \to S_2 \]
to
\[ S_{2n+1} \to CP_n \]
as considered in (A. Trautman(1977)), and repeat the procedure. This approach yields the explicit expressions of the connection 1-form over $CP_n$ and then over $S_2$. Otherwise we may consider
\begin{equation}
F = N \frac{1}{2} \partial_{mu} g(u, \varphi) \sin(\theta) d\varphi \wedge d\theta
\end{equation}
(4.22)

and check (2.17). Weil’s theorem ensures the existence of the connection over a $U(1)$ fiber bundle over $\Sigma$, with a curvature 2-form given by (4.22). This second procedure, although more direct, does not provide the explicit expression of the connection 1-form as in (4.16) and (4.19).

If is straightforward to check that (4.22) satisfies (2.17). In fact
\begin{align*}
C^2 &= \frac{1 + g}{2} \\
S^2 &= \frac{1 - g}{2}
\end{align*}
(4.23)

and hence at $\theta = \pi$, $g = -1$ and at $\theta = 0$, $g = 1$. We may now evaluate $^* F$ for our general solution (4.22) and the metric (4.12).
If we use our normalization $\text{Vol}\Sigma = 2\pi$, we obtain
\[
\sqrt{g} = \frac{1}{2} \partial_{\mu} g \sin(\theta),
\]
and
\[
* F = \frac{2}{\sqrt{g}} F_{\varphi \theta} = N \tag{4.24}
\]
as required for the minima of the membrane potential. The above construction may be extended to obtain all the minimal connection 1-form over topologically non-trivial Riemann surfaces. Using the global coordinates introduced in section 3, the connection 1-form over $U_+$ and $U_-$ may be expressed as
\[
A_+ = \frac{N}{2} (1 + g(u, \varphi)) d\varphi, \tag{4.25}
\]
\[
A_- = \frac{N}{2} (-1 + g(u, \varphi)) d\varphi,
\]
respectively, where $u = \tanh(\phi)$ and $g(u, \varphi)$ is a single valued function over $\Sigma$ satisfying
\[
u \to +1 \quad g \to +1 \tag{4.26}
\]
\[
u \to -1 \quad g \to -1.
\]
The curvature 2-form has then the form
\[
F = N \frac{1}{2} \partial_{\mu} g(u, \varphi) \frac{1}{\cosh^2(\theta)} d\varphi \wedge d\varphi. \tag{4.27}
\]
Its Hodge dual, over the metric (4.12), is consequently
\[
* F = N. \tag{4.28}
\]
(4.25) gives then the general solution for the minimization problem of the membrane potential over topologically non-trivial Riemann Surface.

Having constructed all minimal configurations of the membrane potential, in terms of connection 1-forms over $U(1)$ fiber bundles, one has to determine the configurations maps in terms of them. If $\hat{A}$ is a minimal connection, then under an area preserving diffeomorphism
\[
\delta \hat{A}_r = \partial_r (-\epsilon^{rt} \partial_t \xi \hat{A}_s - \frac{1}{2} \xi^* \hat{F})
\]
where $\xi$ is the infinitesimal parameter of the transformation. It is then equivalent to a gauge transformation on the $U(1)$ fiber bundle. Using this property it was shown in (I. Martin, A. Restuccia ant R. Torrealba(1998)) that the space of all minimal connections may be generated from a particular minimal connection to which a representative of each real cohomology class of 1-form over $\Sigma$ has been added. The space of maps given rise to the particular minimal connection being finite dimensional. The argument in (I. Martin, A. Restuccia ant R. Torrealba(1998)) was performed using the canonical metric over $\Sigma$, its extension to the general case is straightforward.
References

B. de Wit, M. Lüscher and H. Nicolai(1989), Nucl. Phys. B320 135
M.J. Duff, T. Inami, C.N. Pope, E. Sezgin and K.S. Stelle(1988), Nucl. Phys. B297 515.
J.G. Russo(1997), Nucl. Phys. B492 205.
B. de Wit, K. Peeters and J. Plefka(1997), hep-th/9705225.
I. Martin, A. Restuccia and R. Torrealba(1998), Nucl. Phys. B521 117.
M. Caicedo, I. Martin and A. Restuccia (1997), hep-th/9701010; Proceedings of I SILAFAE, Yucatan, Mexico November 1996.
A. Weil(1957), *Varits Kaehliennes*, Hermann.
I. Martin and A. Restuccia(1997), Lett. Math. Phys. 39 (4).
F. Ferrari(1993), hep-th/9310024.
A. Trautman(1977), Internat. J. Theoret. Phys. 16, 561