Adaptive Safety for Uncertain Nonlinear Systems
With Control Barrier Functions and Contraction Metrics

Brett T. Lopez\textsuperscript{1}, Jean-Jacques E. Slotine\textsuperscript{2}, and Jonathan P. How\textsuperscript{1}

Abstract—We present a framework for control of constrained nonlinear systems subject to structured parametric uncertainty. The approach extends the notion of adaptive safety to achieve forward invariance of a desired safe set through online parameter adaptation and data-driven model estimation. This adaptive safety paradigm is merged with a recent direct adaptive control algorithm for systems nominally contracting in closed-loop. Because the existence of a closed-loop contraction metric only requires stabilizability, the proposed unification is considerably more general than what currently exists, e.g., it can be applied to underactuated systems. Additionally, our framework is less expensive than nonlinear model predictive control as it does not require a full desired trajectory, but rather only a desired terminal state. The approach is illustrated on the pitch dynamics of an aircraft with uncertain aerodynamics.

Index Terms—Adaptive control, barrier functions, contraction analysis, contraction metrics, uncertain systems.

I. INTRODUCTION

State or actuator constraints are often encountered in real-world systems but systematic feedback controller design remains challenging. The main difficulty arises from needing to predict whether the system will remain in the feasible set with the current control input. Repeatedly solving a constrained finite-horizon optimal control problem (i.e., model predictive control) is one way to ensure feasibility, but solving a nonlinear optimization in real-time can be difficult. Alternatively, one can avoid trajectory optimization entirely by constructing safe invariant sets, i.e., a set of states that guarantee feasibility indefinitely. Through the development of control barrier functions (CBFs) \cite{1}, safe stabilizing controllers can be synthesized by simply solving a quadratic program (QP), and has recently been used in several applications \cite{2}. However, model error can significantly degrade the performance of these controllers to the extent that safety may no longer be guaranteed. We develop a general framework that guarantees safety through parameter adaptation and online model estimation for uncertain nonlinear systems.

Control barrier functions heavily rely on a model so it is critical to develop methodologies that maintain safety for uncertain systems. Robustness of CBFs was studied in \cite{3} which showed that the so-called zeroing CBFs are Input-to-State stable. This result was used to prove that a superset of the safe set is forward invariant; the size of the superset was characterized in \cite{4} by introducing Input-to-State safety. Stronger safety guarantees can be obtained via robust optimization as demonstrated in \cite{5}. Learning-based methods \cite{6}, \cite{7} have been developed to address the conservativeness of the robust strategies, but these can require extensive offline training to improve the model significantly. Adaptive CBFs (aCBFs) \cite{8} use ideas from adaptive control theory to ensure a safe set is forward invariant with online parameter adaptation. However, aCBFs have a much more stringent invariance condition that restricts the system to remain in sublevel sets of the aCBF, ultimately leading to conservative behavior. In \cite{8} parameter adaptation was limited to a region near the boundary of the safe set to address conservatism but at the expense of control chattering.

The contributions of this work are threefold. First, robust aCBFs (RaCBFs) are defined and shown to guarantee safety for uncertain nonlinear systems. Fundamentally, when combined with parameter adaptation, RaCBFs ensure forward invariance of a tightened set – the degree of tightening can be selected based on the desired conservativeness. RaCBFs are far less conservative than aCBFs and chatter is nonexistent. Second, RaCBFs are combined with the data-driven method set membership identification \cite{9} to safely reduce modeling error and expand the set of allowable states. This is the first work to utilize both parameter adaptation and data-driven model estimation simultaneously within the context of safety-critical control. And third, RaCBFs are merged with a recent direct adaptive controller \cite{10} based on the contraction metric framework \cite{11}, \cite{12}. Contraction expresses distance-like functions differentially rather than explicitly, and the existence of a metric is equivalent to stabilizability, which makes the proposed unification more general than what currently exists in the literature. The approach is demonstrated on the pitch dynamics of an aircraft with uncertain nonlinear aerodynamics. The system is non-invertible, not in strict-feedback form, and has non-polynomial dynamics, which highlights the generality of the framework.

II. PROBLEM FORMULATION & PRELIMINARIES

Consider the nonlinear system

\begin{equation}
\dot{x} = f(x) - \Delta(x)^\top \theta + B(x)u,
\end{equation}

with unknown parameters $\theta \in \mathbb{R}^p$ and known dynamics $\Delta(x) \in \mathbb{R}^{p \times n}$, state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, nominal dynamics $f(x)$, and control input matrix $B(x)$ with columns $b_i(x)$ for $i = 1, \ldots, m$. The nominal dynamics correspond to $\Delta(x) = 0$. The Lie bracket of two vector fields is $ad_x B = \frac{\partial f}{\partial x} B - \frac{\partial B}{\partial x} f$. Let $\mathcal{M}$ be a smooth manifold equipped with a Riemannian metric $M(x, t)$ that defines an inner product $\langle \cdot, \cdot \rangle_x$ on the tangent space $T_x \mathcal{M}$ at every point $x$. 

\textsuperscript{1}Aerospace Control Laboratory, Massachusetts Institute of Technology, Cambridge MA, \{btlopez, jhow\}@mit.edu
\textsuperscript{2}Nonlinear Systems Laboratory, Massachusetts Institute of Technology, Cambridge MA, jjs@mit.edu
The metric $M(x, t)$ defines local geometric notions such as angles, length, and orthogonality. The directional derivative of a metric $M(x, t)$ along vector $v$ is $\partial_v M = \sum_i \frac{\partial M}{\partial x_i} v_i$. Let $c : [0, 1] \to M$ be a regular (i.e., $\frac{dc}{dt} = c_a \neq 0 \forall s \in [0, 1]$) parameterized differentiable curve. The length $L$ and energy $E$ of curve $c$ are

$$L(c, t) = \int_0^1 \sqrt{c^\top \dot{c}(c, t)c}ds, \quad E(c, t) = \int_0^1 c^\top \dot{c}(c, t)c}ds.$$  

By the Hopf-Rinow theorem, a minimizing curve known as a minimal geodesic $\gamma : [0, 1] \to M$ is guaranteed to exist with the unique property $E(\gamma, t) = L(\gamma, t)^2 \leq L(c, t)^2 \leq E(c, t)$. The first variation of energy with respect to time is [13]

$$\frac{1}{2} \dot{E}(c, t) = \frac{\partial E}{\partial t} + \langle c_s(s), \dot{c}(s) \rangle|_{s=0} - \int_0^1 \left\langle \frac{D}{ds}c_s, \dot{c} \right\rangle ds,$$

where $D x(ds)$ is the covariant derivative. For a geodesic $\gamma(s)$, $\frac{Dc_s}{ds} = 0$ so the integral term vanishes. The time argument in $M(c, t)$ and $E(c, t)$ is dropped in the sequel for clarity.

III. ADAPTIVE SAFETY

A. Background

First consider the nominal dynamics of (1), i.e., $\Delta(x) = 0$. Let a closed convex set $C \subset \mathbb{R}^n$ be a superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ where

$$\mathcal{C} = \{x \in \mathbb{R}^n : h(x) \geq 0\},
\partial \mathcal{C} = \{x \in \mathbb{R}^n : h(x) = 0\},
\text{Int}(\mathcal{C}) = \{x \in \mathbb{R}^n : h(x) > 0\}.$$

If the nominal dynamics are locally Lipschitz, then given an initial condition $x_0$, there exists a maximum time interval $I(x_0) = [t_0, T)$ such that $x(t)$ is a unique solution on $I(x_0)$.

Definition 1. The set $\mathcal{C}$ is forward invariant if for every $x_0 \in \mathcal{C}, x(t) \in \mathcal{C}$ for all $t \in I(x_0)$.

Definition 2. The nominal system is safe with respect to set $\mathcal{C}$ if the set $\mathcal{C}$ is forward invariant.

Definition 3 ([11]). A continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ is an extended class $K_{\infty}$ function if it is strictly increasing, $\alpha(0) = 0$, and is defined on the entire real line.

Definition 4 ([11]). Let $C$ be a superlevel set for a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, then $h$ is a control barrier function if there exists an extended class $K_{\infty}$ function $\alpha$ such that

$$\sup_{u \in U} \left[ \frac{\partial h}{\partial x}(x) (f(x) + B(x)u) \right] \geq -\alpha(h(x)).$$  

Theorem 1 ([11]). Let $C \subset \mathbb{R}^n$ be a superlevel set of a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$, if $h$ is a CBF on $C$, then any Lipschitz continuous controller satisfying (2) renders the nominal system safe with respect to set $C$.

B. Adaptive CBFs

Adaptive CBFs (aCBFs) developed in [8] provide a general framework to achieve safety through parameter adaptation when a system is subject to structured uncertainty. By introducing model uncertainty, the notion of safety must be extended to a family of safe sets $\mathcal{C}_\theta$ parameterized by $\theta$. More precisely, the family of safe sets are superlevel sets of a continuously differentiable function $h_\theta : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$. If the uncertain dynamics in (1) are locally Lipschitz then the definitions of forward invariance and safety can be directly extended to $\mathcal{C}_\theta$.

Definition 5 ([8]). Let $\mathcal{C}_\theta$ be a family of superlevel sets parameterized by $\theta$ for a continuously differentiable function $h_\theta : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, then $h_\theta$ is an adaptive control barrier function if

$$\sup_{u \in U} \left[ \frac{\partial h_\theta}{\partial x}(x, \theta) (f(x) - \Delta(x)^\top \Lambda(x, \theta) + B(x)u) \right] \geq 0,$$

where $\Delta(x, \theta) := \theta - \Gamma (\frac{\partial h_\theta}{\partial \theta_p}(x, \theta))^\top$ and $\Gamma \in \mathbb{R}^{p \times p}$ is symmetric and positive definite.

Eq. (3) restricts level sets of $h_\theta$ to be forward invariant making it a much stricter condition than (2). The effects of this are displayed in Fig. 1a where the distance between the system’s trajectory and the boundary of the safe set must monotonically increase, i.e., $\dot{h}_\theta(x, \theta) \geq 0$ for all time. This can lead to extremely conservative behavior as the system is only allowed to operate in a set that is continuously shrinking. A possible solution is to define a modified aCBF

$$\tilde{h}_\theta(x, \theta) = \begin{cases} 
\sigma^2 & \text{if } h_\theta(x, \theta) \geq \sigma \\
\sigma^2 - (h_\theta(x, \theta) - \sigma)^2 & \text{otherwise,}
\end{cases}$$

which can be shown to satisfy (3) if $h_\theta$ is a valid aCBF [8]. While this modification expands the set of allowable states without compromising safety, in practice high-frequency control inputs arise since the barrier condition switches between being trivially satisfied, i.e., $\tilde{h}_\theta = 0 \geq 0$ for all $u$ when $h_\theta(x, \theta) \geq \sigma$, to satisfied only for a particular input, i.e., a $u$ such that $\tilde{h}_\theta \geq 0$ when $h_\theta(x, \theta) \leq \sigma$. The approach taken in this work not only addresses the conservativeness of aCBFs but does so without high-frequency control switching.

C. Robust aCBFs

First consider a continuously differentiable function $h_\eta : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ that defines a family of superlevel sets $\mathcal{C}_\theta$ parameterized by $\theta$. This section will show that a tightened set can be made forward invariant if the unknown model parameters are bounded and the parameter adaptation rate is sufficiently fast (to be defined more precisely in Theorem 2).

Assumption 1. Parameters $\theta$ belong to a known closed convex set $\Theta$. Furthermore, the parameter estimation error $\bar{\theta} := \hat{\theta} - \theta$ also belongs to a known closed convex set $\Theta$ and the maximum possible parameter error is $\bar{\theta}$.
Let \( C^r_\theta \) be a family of superlevel sets parameterized by \( \theta \) for a continuously differentiable function \( \bar{h}_r : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R} \)

\[
C^r_\theta = \left\{ x \in \mathbb{R}^n : \bar{h}_r(x, \theta) \geq 0 \right\},
\]

\( \partial C^r_\theta = \left\{ x \in \mathbb{R}^n : \bar{h}_r(x, \theta) = 0 \right\}, \)

\[
\text{Int}(C^r_\theta) = \left\{ x \in \mathbb{R}^n : \bar{h}_r(x, \theta) > 0 \right\},
\]

where \( \bar{h}_r(x, \theta) := h_r(x, \theta) - \frac{1}{2} \bar{\theta}^T \Gamma^{-1} \bar{\theta}. \) The set \( C^r_\theta \) can be interpreted as tightened set with respect to \( C_\theta \), i.e., \( C^r_\theta \subset C_\theta \), which is illustrated in Fig. 1b. The degree of tightening can be selected \textit{a priori} by choosing \( x_r \) and \( \theta_r \) such that \( h_r(x_r, \theta_r) = 0 \). More precisely, one can select a desired subset of \( C_\theta \) to be made forward invariant. In the limit of infinitely fast parameter adaptation, \( C^r_\theta \) can be chosen to \( C_\theta \).

In practice however this is not possible as one is limited to a maximum allowable adaptation rate. As before it is assumed (1) is locally Lipschitz so the forward invariance and safety definitions hold for \( C_\theta \) and \( C^r_\theta \).

\textbf{Definition 6.} Let \( C^r_\theta \) be a family of superlevel sets parameterized by \( \theta \) for a continuously differentiable function \( h_r : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R} \), then \( h_r \) is a \textit{robust adaptive control barrier function} if there exists an extended class \( \kappa_\infty \) function \( \alpha \) such that

\[
\sup_{u \in U} \left[ \frac{\partial h_r}{\partial x}(x, \theta) \left( f(x) - \Delta(x)^T \Lambda(x, \theta) + B(x)u \right) \right] \geq -\alpha \left( h_r(x, \theta) - \frac{1}{2} \bar{\theta}^T \Gamma^{-1} \bar{\theta} \right),
\]

for all \( \theta \in \Theta \) where \( \Lambda(x, \theta) := \theta - \Gamma \left( \frac{\partial h_r}{\partial \theta}(x, \theta) \right)^T \), \( \bar{\theta} \) is the maximum possible parameter error, and \( \Gamma \in \mathbb{R}^{p \times p} \) is symmetric, positive definite, and is sufficiently large.

The invariance condition (5) is reminiscent of that in (2) and is less conservative than that in (3) because the system is allowed to approach the boundary of \( C_\theta \). Theorem 2 shows the existence of a RaCBF, coupled with an adaptation law, renders the set \( C^r_\theta \) forward invariant.

\textbf{Theorem 2.} Let \( C^r_\theta \subset \mathbb{R}^n \) be a superlevel set of a continuously differentiable function \( h_r : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R} \), if \( h_r \) is a RaCBF on \( C^r_\theta \) defined by \( x_r \) and \( \theta_r \), then any Lipschitz continuous controller satisfying (5) with adaptation law and adaptation gain

\[
\dot{\hat{\theta}} = \Gamma \Delta(x) \left( \frac{\partial h_r}{\partial x}(x, \hat{\theta}) \right)^T, \quad \lambda_{\min}(\Gamma) \geq \frac{\|\bar{\theta}\|^2}{2h_r(x_r, \theta_r)},
\]

renders the uncertain system safe with respect to the set \( C^r_\theta \).

Moreover, the systems is safe with respect to the set \( C^r_\theta \).

\textbf{Proof.} See Appendix. \( \Box \)

\textbf{Remark 1.} If a RaCBF \( h_r \) exists and is independent of the model parameters \( \theta \), then the modified adaptation law

\[
\dot{\hat{\theta}} = \text{Proj}_{\Theta} \left\{ \Gamma \Delta(x) \left( \frac{\partial h_r}{\partial x}(x) \right)^T \right\}
\]

where \( \text{Proj}_{\Theta} \{ \} \) projects the parameter estimates onto \( \Theta \), renders the uncertain system safe with respect to set \( C^r \). The proof is omitted for brevity but one can show that when the projection operation temporarily stops adaptation, then a positive semi-definite term appears in the same composite candidate CBF used in Theorem 2. Projection is helpful to ensure the parameters remain bounded and can simplify RaCBF construction by eliminating parameter dependency.

Several remarks can be made about Theorem 2. First, safety is \textit{guaranteed} for all possible parameter realizations through adaptation with minimal conservatism. Furthermore, RaCBFs expands upon and improves the notion \textit{adaptive safety}. Second, the condition for sufficiently fast adaptation is now more precise as there is a requirement on the minimum eigenvalue of the adaptation gain. Intuitively, the adaptation gain must be large enough so the parameter estimates can change fast enough to ensure forward invariance as the system approaches \( \partial C^r_\theta \). Third, the RaCBF condition in (5) can be used as a safety filter for an existing tracking controller or as a constraint within an optimization. Section V will show the latter but with a contraction-based min-norm controller. Lastly, while RaCBFs are strictly better than aCBFs in terms of conservativeness, if the maximum parameter error is large (or the adaptation gain must be small) then RaCBFs can also exhibit conservative behavior, albeit not to the same extent as aCBFs. Combining RaCBFs with a model estimation technique that accurately and robustly identifies model parameters can greatly improve performance.

\textbf{D. RaCBFs with Set Membership Identification}

Set membership identification (SMID) is a model estimation technique that constructs an unfalsified set of model parameters. SMID was originally developed to identify transfer functions for uncertain linear systems [14], but has been more recently applied to linear [9], [15] and nonlinear adaptive MPC [16]. First assume that the true parameters \( \theta^* \) belong to an initial set of possible parameter values \( \Theta^0 \), i.e., \( \theta^* \in \Theta^0 \). Given \( k \) state, input, and rate measurements (denoted as \( x_{1:k} \) and so forth), a set \( \Xi \) can be constructed such that

\[
\Xi = \left\{ \theta : \| \hat{x}_{1:k} - f_{1:k} + \Delta_{1:k}^T \theta - B_{1:k} u_{1:k} \| \leq D \right\},
\]

where \( D \) can be treated as a tuning parameter that dictates the conservativeness of SMID. It can also represent a disturbance or noise bound (see [9], [16]). The set of possible parameter
Model uncertainty monotonically decreases via set membership identification, i.e., \( \tilde{\Theta} \). Lemma 1. Model uncertainty monotonically decreases via set membership identification, i.e., \( \Theta^{j+1} \subseteq \Theta^j \) for all \( j \geq 0 \). Proof. Since \( \theta^* \in \Theta^{j+1} \) then \( \theta^* \in \Theta^j \cap \Xi \) which is true if \( \theta^* \in \Theta^j \) so \( \Theta^{j+1} \subseteq \Theta^j \) and \( \Theta^{j+1} \subseteq \tilde{\Theta} \).

The motivation to combine SMID with RaCBFs is to enlarge the tightened set \( C^p_\theta \). To do so, one must ensure \( C^p_\theta \) remains forward invariant as the set of model parameters is updated. In general this can be non-trivial to prove since the maximum possible parameter error \( \hat{\theta} \) is now time varying. Theorem 3 shows that safety is maintained if the model uncertainty monotonically decreases.

Theorem 3. Let \( C^p_\theta \) be a superlevel set of a continuously differentiable function \( h_\theta : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R} \). If the system is safe on \( C^p_\theta \) then it remains safe if the maximum allowable model parameter error \( \hat{\theta} \) monotonically decreases. Moreover, the tightened set \( C^p_\theta \) converges to \( C_\theta \) monotonically.

Proof. See Appendix.

Combining RaCBFs and SMID provides a mechanism to 1) modify model parameters via adaptation to achieve safety and 2) update the model to reduce uncertainty and conservatism. Safety is guaranteed even as the model parameters are modified online, and the system’s performance will only improve as more data is collected. The proposed combination can be merged with a stabilizing adaptive controller for safe reference tracking. To maximize the generality of the proposed unification, the adaptive controller must have minimal restrictions on the type of systems it can be applied to.

IV. ADAPTIVE CONTROL WITH CONTRACTION METRICS

Several techniques have been proposed for adaptive control of nonlinear systems, including methods based on feedback linearization, variable structure, and backstepping. Fundamentally, these strategies require constructing a distance-like function explicitly to prove stability, and it is this dependency that limits the type of systems these approaches can be applied to. This work will instead utilize the contraction metric framework to synthesize stable adaptive controllers since only stabilizability is required – a much less restrictive requirement compared to other approaches.

A. Contraction Metrics

For the nominal dynamics of (1), the corresponding differential dynamics are \( \dot{x}_r = A(x) \dot{x} + B(x) u \), where \( A(x) = \frac{\partial f}{\partial x} + \sum_{i=1}^m \frac{\partial h_i}{\partial x} u_i \). Contraction analysis searches for a metric \( M(x) \) such that the Riemannian energy satisfies \( E \leq -2\lambda E \), i.e., the squared distance between two points shrinks exponentially. Theorem 4 provides necessary and sufficient conditions for \( M(x) \) to be a contraction metric.

Theorem 4 ([12]). If there exists a uniformly bounded metric \( M(x) \) such that the following implication is true

\[
\delta_x^T M B = 0 \implies \delta_x^T \left( A^T M + M A + \dot{M} + 2\lambda M \right) \delta_x \leq 0,
\]

for all \( \delta_x \neq 0 \). \( x, u \) then the nominal system is universally exponentially stabilizable via continuous feedback defined almost everywhere, and everywhere in a neighborhood of the target trajectory.

Remark 2. Theorem 4 essentially states that the orthogonal dynamics of a system must be naturally contracting which can be interpreted as a stabilizability condition. The implication in Theorem 4 can be transformed into a convex constructive condition for the metric \( M(x) \) by introducing the dual metric \( W(x) = M(x)^{-1} \) and a change of variables (see [12] for more details). The existence of a contraction metric \( M(x) \) is necessary and sufficient for stabilizability via Theorem 4 so there exists a stabilizing feedback control law. Section V will show that one such controller can be found by solving a QP similar to that in [17] but without constructing a CLF explicitly.

B. Adaptive Control & Contraction

The control contraction metric framework provides a systematic way to synthesize feedback controllers for stabilizable nonlinear systems by representing CLFs differentially (see Fig. 2). However, model uncertainty or constraints are not directly considered in the framework. Recent work [10] developed a direct adaptive control approach for closed-loop contracting systems when the uncertainty satisfies the matched or extended matched condition, i.e., \( \Delta(x)^T \theta \in \text{span}\{B, \text{adj} B\} \). For systems with extended matched uncertainty the metric must also depend on the unknown parameters. Definition 7 formalizes this type of metric.

Definition 7 ([10]). A parameter-dependent (dual) contraction metric is a uniformly bounded metric, positive definite in both arguments, that satisfies the contraction condition for every possible value of the estimated unknown parameters \( \hat{\theta} \).

Theorem 5 ([10]). For the uncertain nonlinear system (1) satisfying the matched or extended matched condition, if a parameter-dependent metric exists then the adaptation law

\[
\dot{\hat{\theta}} = -\Gamma \Delta(x) M \left( x, \hat{\theta} \right) \gamma_s(1)
\]
renders the closed-loop system asymptotically stable where \( \gamma_s(s) \) is the speed of geodesic \( \gamma(s) \) and \( \Gamma \in \mathbb{R}^{p \times p} \) is symmetric and positive definite.

**Remark 3.** Several modifications can be made to (6) that improve transients or robustness including the projection operator in Section III-C (see [10] for more details).

**C. Offline/Online Computation**

The adaptive controller has an offline and online optimization component. The contraction metric is synthesized offline via sum-of-squares [12] (for polynomial systems) or via gridding. The generality of the approach leads to increased online computation in the form of a nonlinear optimization under reasonable conditions, a solution is guaranteed to exist. The Chebychev Pseudospectral method and Clenshaw-Curtis quadrature scheme have been shown to be effective techniques for finding geodesics with low computation time [18]. A stabilizing controller can be computed via a QP with \( E \leq -2\lambda E \) as a constraint.

**V. ADAPTIVE SAFETY-CRITICAL CONTROL**

RaCBFs, SMID, and adaptive control with contraction metrics can be unified into a single optimization for a safe and stabilizing controller. The steps are outlined below with their corresponding computational complexity.

1) **Compute geodesic (NLP)**

\[
\gamma(s) = \arg\min_{c(s) \in \mathcal{Y}(x,x_d)} E(\Theta(s), \theta_C) \]

2) **Compute controller (QP & Quadrature)**

\[
\kappa = \arg\min_{u \in \mathcal{U}} \frac{1}{2} u^T u + re^2 \\
\text{s.t. } \tilde{E} \left( x, x_d, u, \hat{\theta}_C \right) \leq -2LE \left( \gamma(s), \hat{\theta}_C \right) + \epsilon \\
\tilde{h}_r \left( x, u, \hat{\theta}_B \right) \geq -\alpha \left( \hat{h}_r \left( x, \hat{\theta}_B \right) - \hat{\theta}_r^T \Gamma_B^{-1} \hat{\theta} \right) \]

3) **Update parameters (Quadrature)**

\[
\dot{\theta}_C = \text{Proj}_{\Theta} \left\{ -\Gamma_C \Delta(x) M \left( x, \hat{\theta}_C \right) \gamma_s(1) \right\} \\
\dot{\theta}_B = \Gamma_B \Delta(x) \frac{\partial h_r}{\partial x} \left( x, \hat{\theta}_B \right) \]

4) **Update parameter error bounds (LP)**

\[
\Xi = \{ \dot{x}_{1:k} - f_{1:k} + \Delta \gamma, |B_{1:k} - k_{1:k}| \leq D \} \\
\Theta^{j+1} = \Theta^j \cap \Xi, \quad \tilde{\theta} = \sup_{\dot{\Theta} \in \Xi} \Theta^{j+1} - \inf_{\dot{\Theta} \in \Xi} \Theta^{j+1} \]

**Remark 4.** Updating parameter bounds via SMID (Step 4) can be done in parallel to Steps 1-3 at its own rate since stability and safety are guaranteed for all \( \theta \in \Theta \) so real-time parameter bounds are not necessary but desirable.

**Remark 5.** SMID requires \( \dot{x} \) which can be difficult to obtain. However, based on Remark 4, non-causal filtering can be used to accurately estimate \( \dot{x} \). Estimation bounds can be added to \( D \) to prevent overly-confident parameter estimates.

**Remark 6.** Parameter adaptation for the controller should be temporarily stopped when the safety constraint is active to prevent undesirable transients in the parameter estimates.

**VI. ILLUSTRATIVE EXAMPLE**

Consider the simplified pitch dynamics of an aircraft [19]

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\alpha} \\
\dot{q} \\
\dot{q}'
\end{bmatrix} = 
\begin{bmatrix}
q \\
q - L(\alpha) \\
-kq + M(\alpha) \\
-kq + M(\alpha)
\end{bmatrix} + 
\begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix} u, \quad (7)
\]

where \( \theta \) is the pitch angle, \( \alpha \) is the angle of attack, \( q \) is the pitch rate, and \( L(\alpha) \) and \( M(\alpha) \) are the aerodynamic lift and moment respectively. System (7) is not feedback linearizable since the controllability matrix drops rank when \( L'(\alpha) = 0 \) and is not in strict-feedback form. The aerodynamics of a high-performance aircraft can be approximated via flat plate theory [19] so \( L(\alpha) = 0.8 \sin(2\alpha) \) and \( M(\alpha) = -L(\alpha) \). The model parameters \( k \) and \( \ell_\alpha \) are unknown but bounded, i.e., \( k \in [0.1, 0.8] \) and \( \ell_\alpha \in [-3, 1] \). A contraction metric was synthesized via gridding for \( \alpha \in [-60^\circ, 60^\circ] \) and \( q \in [-20^\circ/s, 20^\circ/s] \). Note that \( \alpha_c \) so \( L'(\alpha_c) = 0 \) is in the
chosen grid range. A constant metric was unable to satisfy the contraction condition, so a quadratic CLF cannot stabilize (7); a metric quadratic in $\alpha$ was found to be valid over the grid range. The function $h_r(q) = 1 - (q/q_m)^2$ where $q_m = 20^\circ/s$ is a valid RaCBF independent of $k$ and $\ell_\alpha$ so the approach in Remark 1 is used.

Fig. 3 compares the modified aCBFs from (4), RaCBFs, and RaCBFs with SMID for a full desired trajectory described by $\theta_d = -20^\circ\cos(t)$. Fig. 3a shows aCBFs and RaCBFs exhibit similar behavior in terms of conservatism, but high-frequency oscillations with aCBFs are present. The cause is seen in Fig. 3b where several instances of chatter are observed with aCBFs; RaCBFs generate continuous control inputs. High-frequency oscillations are also seen in the barrier function, shown in Fig. 3c, but is completely absent from RaCBFs. Fig. 3a–Fig. 3c show that augmenting RaCBFs with SMID can improve performance significantly. The tracking is much better in Fig. 3a due to the reduced model error and the norm of the control input is smaller compared to its counterparts in Fig. 3b. The set of allowable states is also considerably larger as is evident by the small proximity to $\partial C$. By selecting $D = 0.1$, the estimation error did not converge to zero but the max parameter error was reduced by 16.5% and 77.3% for $k$ and $\ell_\alpha$, respectively. The small reduction in error for $k$ is due to the trajectory not sufficiently exciting $q$ relative to $\dot{L}(\alpha)$. Nonetheless, the total model error was reduced by 73.4%, resulting in considerably less conservative behavior. Note that RaCBFs with SMID is less conservative after $t = 3s$ corresponding to when the largest error reduction occurred.

Consider a different scenario where only a desired terminal state is specified rather than a full desired trajectory. For instance, let the desired terminal state correspond to the first portion of the aerobatic maneuver known as the Immelmann turn. The vehicle must execute a half loop, i.e., $\theta_d = 180^\circ$, before executing a half roll (not considered here) which results in level flight in the opposite direction. A metric quadratic in $\alpha$ was constructed for $\alpha \in [-5^\circ, 50^\circ]$ and $q \in [-10^\circ/s, 50^\circ/s]$ with $h_r(q) = q_m - q = 50^\circ/s - q$. This article presented a framework that guarantees safety for uncertain nonlinear systems through parameter adaptation and data-driven model estimation via RaCBFs and SMID. The unification with contraction-based adaptive controllers makes the framework more general than what currently exists as it only requires the system be stabilizable. Extending to systems with probabilistic model bounds, non-parametric uncertainty, and external disturbances is future work.

VII. CONCLUSION

This article presented a framework that guarantees safety for uncertain nonlinear systems through parameter adaptation and data-driven model estimation via RaCBFs and SMID. The unification with contraction-based adaptive controllers makes the framework more general than what currently exists as it only requires the system be stabilizable. Extending to systems with probabilistic model bounds, non-parametric uncertainty, and external disturbances is future work.

APPENDIX

Proof of Theorem 2. Consider the composite candidate CBF $h(x, \dot{\theta}) = h_r(x, \dot{\theta}) - \frac{1}{2} \dot{\theta}^\top \Gamma^{-1} \dot{\theta}$, where $\Gamma$ must be chosen so that $h(x_r, \theta_r) \geq 0$. Since $|\dot{\theta}| \leq \dot{\theta}_r$ then the minimum eigenvalue of $\Gamma$ must satisfy $\lambda_{\text{min}}(\Gamma) \geq \frac{||\dot{\theta}_r||^2}{2h_r(x_r, \theta_r)}$. Differentiating the composite candidate CBF with respect to (1),

$$
\dot{h} = \dot{h}_r(x, \dot{\theta}) - \dot{\theta}^\top \Gamma^{-1} \dot{\theta} \cdot 
= \frac{\partial h_r}{\partial x} \left( f(x) - \Delta(x)^\top \dot{\theta} + B(x)u \right) + \frac{\partial h_r}{\partial \dot{\theta}} \dot{\theta}^\top \Gamma^{-1} \dot{\theta} - \frac{\partial h_r}{\partial x} \Delta(x)^\top \left( \dot{\theta} \Gamma^{-1} \dot{\theta} \right) \left( \dot{\theta} - \Gamma \left( \frac{\partial h_r}{\partial \dot{\theta}} \right)^\top \right) + \frac{\partial h_r}{\partial x} \Delta(x)^\top \left( \dot{\theta} - \Gamma \left( \frac{\partial h_r}{\partial \dot{\theta}} \right)^\top \right).
$$

Fig. 4 compares the modified aCBFs from (4), RaCBFs, and RaCBFs with SMID for the Immelmann turn. High-frequency oscillations are again observed in the pitch rate, shown in Fig. 4a, with aCBFs. For RaCBFs with SMID, the aircraft utilizes the maximum allowable pitch rate resulting in faster maneuver execution. High-frequency chatter is again observed in Fig. 4b with aCBFs; RaCBFs generate continuous control inputs. The barrier function, shown in Fig. 4c, again confirms that RaCBFs with SMID increases the set of allowable states while guaranteeing safety. The parameter error bounds were reduced by 63.0% for $k$ and 90.5% for $\ell_\alpha$ – a total model error reduction of 88.7%. Fig. 4 shows that our approach is able to maintain safety even without a desired trajectory. The results in Fig. 3 and Fig. 4 demonstrate the benefits of RaCBFs and utilizing SMID for model estimation.
Using the definition of $\Delta(x, \hat{\theta})$,
\[
\dot{h} = \frac{\partial h_r}{\partial x} \left( f(x) - \Delta(x)^\top \Lambda(x, \hat{\theta}) + B(x)u \right) + \frac{\partial h_r}{\partial \hat{\theta}} \dot{\hat{\theta}}
- \dot{\hat{\theta}}^\top \Gamma^{-1} \dot{\hat{\theta}} + \frac{\partial h_r}{\partial x} \Delta(x)^\top \left( \dot{\hat{\theta}} - \Gamma \left( \frac{\partial h_r}{\partial \hat{\theta}} \right)^\top \right),
\]

Choosing $\dot{\hat{\theta}} = \Gamma \Delta(x)^\top \frac{\partial h_r}{\partial x}$, then
\[
\dot{h} = \frac{\partial h_r}{\partial x} \left( f(x) - \Delta(x)^\top \Lambda(x, \hat{\theta}) + B(x)u \right)
\geq -\alpha \left( h_r - \frac{1}{2} \dot{\hat{\theta}}^\top \Gamma^{-1} \dot{\hat{\theta}} \right) \geq -\alpha(h),
\]

where the first inequality is obtained via the definition of a RaCBF and the second by noting $|\dot{\hat{\theta}}| \leq \dot{\hat{\theta}}$ so $\dot{h} = h_r - \frac{1}{2} \dot{\hat{\theta}}^\top \Gamma^{-1} \dot{\hat{\theta}} \geq h_r - \frac{1}{2} \dot{\hat{\theta}}^\top \Gamma^{-1} \dot{\hat{\theta}}$. Since $h \geq 0$ and $h_r \geq h$ for all $t$, then $h_r \geq \frac{1}{2} \dot{\hat{\theta}}^\top \Gamma^{-1} \dot{\hat{\theta}} \geq 0$ and $C_\theta^r$ is forward invariant as desired.

**Proof of Theorem 3.** Since the model uncertainty is changing via estimation, the maximum allowable parameter error is time varying, i.e., $\tilde{\theta}(t)$. From Lemma 1, $\Theta$ monotonically decreases so $\dot{\tilde{\theta}} \leq 0$. Let $\hat{h}_r$ be a candidate RaCBF, then $\dot{\hat{h}}_r = \dot{h}_r - \dot{\tilde{\theta}}^\top \Gamma^{-1} \dot{\hat{\theta}} \geq \hat{h}_r$ since $\dot{\tilde{\theta}} \leq 0$ for all $t$. Inequality (5) in Definition 6 is then still satisfied even for $\tilde{\theta}(t)$. Using identical steps from Theorem 2, the system can be shown to be safe on $C_\theta^r$. Since $\tilde{\theta} \leq 0$ then $\hat{h}_r \rightarrow h_r$ so $C_\theta^r$ converges to $C_\theta$ monotonically.

**Acknowledgements** We thank David Fan for stimulating discussions. This work was supported by the NSF Graduate Research Fellowship Grant No. 1122374.

**REFERENCES**

[1] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, “Control barrier function based quadratic programs for safety critical systems,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2016.

[2] A. D. Ames, S. Coogan, M. Egerstedt, G. Notomista, K. Sreenath, and P. Tabuada, “Control barrier functions: Theory and applications,” in *2019 18th European Control Conference (ECC)*, pp. 3420–3431, IEEE, 2019.

[3] X. Xu, P. Tabuada, J. W. Grizzle, and A. D. Ames, “Robustness of control barrier functions for safety critical control,” *IFAC-PapersOnLine*, vol. 48, no. 27, pp. 54–61, 2015.

[4] S. Kolathaya and A. D. Ames, “Input-to-state safety with control barrier functions,” *IEEE control systems letters*, vol. 3, no. 1, pp. 108–113, 2018.

[5] T. Gurriet, A. Singletary, J. Reher, L. Ciarletta, E. Feron, and A. Ames, “Towards a framework for realizable safety critical control through active set invariance,” in *2018 ACM/IEEE 9th International Conference on Cyber-Physical Systems (ICCPs)*, pp. 98–106, IEEE, 2018.

[6] D. D. Fan, J. Nguyen, R. Thakker, N. Alatur, A.-a. Agha-mohammadi, and E. A. Theodorou, “Bayesian learning-based adaptive control for safety critical systems,” *arXiv preprint arXiv:1910.02325*, 2019.

[7] A. Taylor, A. Singletary, Y. Yue, and A. Ames, “Learning for safety-critical control with control barrier functions,” *arXiv preprint arXiv:1912.10099*, 2019.

[8] A. J. Taylor and A. D. Ames, “Adaptive safety with control barrier functions,” *arXiv preprint arXiv:1910.00555*, 2019.

[9] M. Tanaskovic, L. Fagiano, R. Smith, and M. Morari, “Adaptive receding horizon control for constrained mimo systems,” *Automatica*, vol. 50, no. 12, pp. 3019–3029, 2014.

[10] B. T. Lopez and J.-J. E. Slotine, “Contraction metrics in adaptive nonlinear control,” *arXiv preprint arXiv:1912.13138*, 2019.

[11] W. Lohmiller and J.-J. E. Slotine, “On contraction analysis for nonlinear systems,” *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.

[12] I. R. Manchester and J.-J. E. Slotine, “Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design,” *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 3046–3053, 2017.

[13] M. P. Do Carmo, *Riemannian geometry*. Birkhäuser, 1992.

[14] R. L. Kosut, M. K. Lau, and S. P. Boyd, “Set-membership identification of systems with parametric and nonparametric uncertainty,” *IEEE Transactions on Automatic Control*, vol. 37, no. 7, p. 929, 1992.

[15] M. Lorenzen, F. Allgöwer, and M. Cannon, “Adaptive model predictive control with robust constraint satisfaction,” *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 3313–3318, 2017.

[16] T. B. Lopez, Adaptive robust model predictive control for nonlinear systems. PhD thesis, Massachusetts Institute of Technology, 2019.

[17] J. A. Primbs, V. Nevistic, and J. C. Doyle, “A receding horizon generalization of pointwise min-norm controllers,” *IEEE Transactions on Automatic Control*, vol. 45, no. 5, pp. 898–909, 2000.

[18] K. Leung and I. R. Manchester, “Nonlinear stabilization via control contraction metrics: A pseudospectral approach for computing geodesics,” in *2017 American Control Conference (ACC)*, pp. 1284–1289, IEEE, 2017.

[19] B. W. McCormick, *Aerodynamics, aeronautics, and flight mechanics*. John Wiley, 1999.