INITIAL-BOUNDARY PROBLEM FOR DEGENERATE HIGH ORDER EQUATIONS WITH FRACTIONAL DERIVATIVE

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Annotation.. The mixed problem for a degenerate high order equation with a fractional derivative in a rectangular domain is considered in the article. The existence of a solution and its uniqueness are shown by the spectral method.

Keywords. Differential equation, high order, degeneracy, Riemann-Liouville fractional derivative, existence, uniqueness, series, uniform convergence.

1. Introduction

In the region $D = D_x \times D_y$, $D_x = \{x : 0 < x < 1\}$, $D_y = \{y : 0 < y < 1\}$, we consider the equation

$$(-1)^{k+1}D_{0x}^{\alpha}u(x, y) - y^m \frac{\partial^{2k}u(x, y)}{\partial y^{2k}} = 0, \quad 1 < \alpha < 2, \quad 0 \leq m < k, \quad m \notin N.$$  \hspace{1cm} (1)

where $k \in \mathbb{N}, D_{0x}^{\alpha} -$ is the Riemann-Liouville fractional differentiation operator of order $\alpha$

$$D_{0x}^{\alpha}u(x, y) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dx^2} \int_0^x \frac{u(\tau, y) d\tau}{(x - \tau)^{\alpha - 1}}.$$  \hspace{1cm} For equation (1), we consider the problem.

Problem A. Find a solution to equation (1) with the conditions:

$$D_{0x}^{\alpha}u(x, y) \in C\left(D_x \times \overline{D_y}\right), \quad x^{2-\alpha}u(x, y) \in C^1\left(\overline{D}\right), \quad \frac{\partial^{2k}u}{\partial y^{2k}} \in C\left(D_x \times \overline{D_y}\right),$$  \hspace{1cm} (2)

$$\frac{\partial^{2s}u(x, 0)}{\partial y^{2s}} = \frac{\partial^{2s}u(x, 1)}{\partial y^{2s}} = 0, \quad 0 < x \leq 1, \quad s = 0, 1, ..., k - 1,$$  \hspace{1cm} (3)

$$\lim_{x \to 0} x^{2-\alpha}u(x, y) = \psi(y),$$  \hspace{1cm} (4)

$$\lim_{x \to 0} \frac{d}{dx} \left(x^{2-\alpha}u(x, y)\right) = \varphi(y),$$  \hspace{1cm} (5)

here the functions $\varphi(y), \psi(y) -$ are quite smooth and the natural matching conditions are satisfied for them.

Fractional differential equations arise in mathematical modeling of various physical processes and phenomena [1]. Second-order equations, without degeneracy, with partial derivatives of fractional order, were studied in [1] - [5] and others. In these papers, the Cauchy problem, first, second and mixed boundary value problems were considered, a fundamental solution was found, a general representation of the solutions was constructed. Mixed equations and high-order equations with a fractional derivative were studied in [6] - [9]. Fractional order equations with degeneracy were studied in [1, 10]. The study will be carried out by the Fourier method. Earlier, by the Fourier method, boundary value problems for equations with a fractional derivative were studied in [5] - [6], [11].
2. Solution existence

We are looking for a solution in the form

\[ u(x, y) = X(x) Y(y). \]

Then with respect to the variable \( y \), taking into account condition (3), we obtain the following spectral problem:

\[
\begin{align*}
Y^{(2k)}(y) &= (-1)^k \lambda y^{-m} Y(y), \\
Y^{(s)}(0) &= Y^{(s)}(1) = 0, \quad s = 0, 1, \ldots, k - 1.
\end{align*}
\]  (6)

First, we show that \( \lambda = 0 \) is not an eigenvalue. Indeed, consider problem (6) with \( \lambda = 0 \)

\[
\begin{align*}
Y^{(2k)}(y) &= 0, \\
Y^{(s)}(0) &= Y^{(s)}(1) = 0, \quad s = 0, 1, \ldots, k - 1.
\end{align*}
\]

A solution to this problem satisfying the condition \( Y^{(s)}(0) = 0, s = 0, 1, \ldots, k - 1 \), has the form

\[ Y(y) = c_k x^k + c_{k+1} x^{k+1} + \ldots + c_{2k-1} x^{2k-1}, \]

to determine the unknowns \( c_j, j = k, k+1, \ldots, 2k-1 \), we obtain the system of equations

\[
\begin{align*}
c_k + c_{k+1} + \ldots + c_{2k-1} &= 0, \\
k c_k + (k+1) c_{k+1} + \ldots + (2k-1) c_{2k-1} &= 0, \\
\vdots \\
k (k-1) \ldots 2 \cdot c_k + (k+1) \ldots 3 \cdot c_{k+1} + \ldots + (2k-1) (2k-2) \ldots (k+1) c_{2k-1} &= 0,
\end{align*}
\]

the main determinant of this system \( \Delta \), has the form

\[
\Delta = \begin{vmatrix}
1 & 1 & \ldots & 1 \\
k & k+1 & \ldots & 2k-1 \\
k (k-1) \ldots 2 & (k+1) \ldots 3 & \ldots & (2k-1) (2k-2) \ldots (k+1)
\end{vmatrix}
= \prod_{j=1, j \neq i}^{k-1} (j-i) \neq 0.
\]

Hence \( c_j = 0, j = k, k+1, \ldots, 2k-1 \), from here

\[ Y(y) = c_k x^k + c_{k+1} x^{k+1} + \ldots + c_{2k-1} x^{2k-1} \equiv 0. \]

We proceed to find the general solution of equation (6) for \( \lambda \neq 0 \). We make a change of variables

\[ t = y^a, \]

\[
D_y^{2k} Y(y) = D_y^{2k-1} \left( D_t^1 Y(y) D_t^1 t \right) = \sum_{k_1=0}^{2k-1} \left( C_{2k-1}^{k_1} D_y^{2k-1-k_1} (D_y^1 t) D_y^{k_1} (D_t^1 Y(y)) \right) =
\]

\[ = D_y^{2k-1} \left( D_y^1 t \right) D_t^1 Y + \sum_{k_1=1}^{2k-1} C_{2k-1}^{k_1} D_y^{2k-1-k_1} (D_y^1 t) D_y^{k_1-1} (D_t^1 Y D_y^1 t). \]

We introduce the following notation: \( (a)_{j+1} = a (a-1) (a-2) \ldots (a-j) \), and we assume that \( (a)_0 = 1 \), then \( D_y^1 (D_y^1 t) = (a)_{j+1} t \). From here we have

\[
D_y^{2k} Y(y) = \frac{(a)_{2k+1} t}{y^{2k}} D_t^1 Y(y) + \sum_{k_1=1}^{2k-1} C_{2k-1}^{k_1} \frac{(a)_{2k-1-k_1} t}{y^{2k-1-k_1}} \sum_{k_2=0}^{k_1-1} C_{k_1-1}^{k_2} D_y^{k_1-k_2-1} (D_y^1 t) D_y^{k_2} (D_t^1 Y(y)) =
\]
We substitute (7) into equation (6) then

\[ \sum_{k_i=1}^{2k-1} C_{k_1}^{k_1} \frac{(a)_{2k-1} (a)_{k_1-k_1} t D_y^{k_1} (y) t D_y^2 Y (y)}{y^{2k-k_1}} + \sum_{k_1=1}^{2k-1} C_{k_2}^{k_2} \frac{(a)_{k_1-k_2} t^2 D_y^{k_2} (y) D_y^2 Y (y)}{y^{k_1-k_2}}. \]

Continuing this process, we obtain the following formula:

\[ D_y^{2k} Y (y) = y^{-2k} \sum_{j=1}^{2k} \left( A_j^{2k} (a) t^j D_y^j Y (y) \right), \tag{7} \]

where

\[ A_j^{2k} (a) = \sum_{k_1=1}^{2k-1} C_{k_1}^{k_1} \left( \sum_{k_2=1}^{k_1-j} \cdots \sum_{k_j=j}^{k_1-1} (a)_{k_j} \prod_{s=1}^{j} C_{k_{s-1}}^{k_s} (a)_{k_{s-1}-k_s} \right), \]

moreover \( k_0 = 2k \), \( j = 1, 2, \ldots, k_1 > k_2 > \ldots > k_j \geq 1 \). Further, we assume that \( A_j^i (a) = 0 \), for \( i \geq j \). We note some properties of the coefficients \( A_j^i (a) \), that were established in [11].

**Lemma.**

1. \( A_{i+1}^i = a_i^i + 1; \)
2. \( A_j^i = \sum_{k=i}^{j-1} C_{j-1}^k (a)_{j-k} A_{i-1}^k; \)
3. \( A_j^i = a (i+1) A_{j+1}^i + A_{j-1}^{i+1} \) - \( (j-1) A_i^{j-1}; \)
4. \( A_0^j = (a)_j; \)
5. \( \sum_{j=1}^{a} (x)_j A_{j-1}^a (a) = (ax)_s. \)

We substitute (7) into equation (6)

\[ y^{-2k} \sum_{s=1}^{2k} A_{s-1}^{2k} (a) t^s Y (s) (t) = (-1)^k \lambda y^{-m} Y (t), \]

or

\[ \sum_{s=1}^{2k} A_{s-1}^{2k} (a) t^s Y (s) (t) = (-1)^k \lambda y^{2k-m} Y (t). \]

Let be

\[ a = 2k - m, \]

then

\[ \sum_{s=1}^{2k} A_{s-1}^{2k} (a) t^s Y (s) (t) = (-1)^k \lambda t Y (t). \]

The solution to this equation will be sought in the form of the following series

\[ Y (t) = \sum_{j=0}^{\infty} c_j t^{a+j}, \]

here \( \alpha \) – unknown parameter.

Substituting into the equation, we have

\[ \sum_{s=1}^{2k} A_{s-1}^{2k} (a) t^s \sum_{j=0}^{\infty} c_j (\alpha + j) s^{\alpha+j-s} = (-1)^k \lambda t \sum_{j=0}^{\infty} c_j t^{\alpha+j}, \]
\[ \sum_{j=0}^{\infty} c_j t^j \sum_{s=1}^{2k} A_{s-1}^{2k} (2k-m)(\alpha+j)_s = (-1)^k \lambda \sum_{j=0}^{\infty} c_j t^{j+1}, \]

property 5 of the lemma implies

\[ \sum_{s=1}^{2k} A_{s-1}^{2k} (2k-m)(\alpha+j)_s = ((2k-m)(\alpha+j))_{2k}, \]

from here

\[ \sum_{j=0}^{\infty} c_j t^j ((2k-m)(\alpha+j))_{2k} = (-1)^k \lambda \sum_{j=0}^{\infty} c_j t^{j+1}, \]

now if

\[ \alpha = \frac{s}{2k-m}, \ s = 0, 1, ..., 2k-1, \]

then

\[ c_0 \neq 0, \ c_1 = \frac{(-1)^k \lambda}{(2k-m)^{2k} \prod_{s=0}^{2k-1} \left( \alpha - \frac{s}{2k-m} + 1 \right)}, \]

because

\[ 2k-m \notin N, \]

then

\[ \prod_{s=0}^{2k-1} \left( \alpha - \frac{s}{2k-m} + 1 \right) \neq 0, \]

further

\[ c_2 = \frac{(-1)^k \lambda c_1}{(2k-m)^{2k} \prod_{s=0}^{2k-1} \left( \alpha - \frac{s}{2k-m} + 2 \right)} = \frac{(-1)^k \lambda c_0}{(2k-m)^{2k} \prod_{s=0}^{2k-1} \left( \alpha - \frac{s}{2k-m} + 1 \right)} , \]

\[ c_j = \frac{c_0}{\prod_{s=0}^{2k-1} \left( \alpha - \frac{s}{2k-m} + 1 \right)} \left( \frac{(-1)^k \lambda}{(2k-m)^{2k}} \right)^j, \]

\[ Y(t) = t^\alpha \sum_{j=0}^{\infty} c_0 \prod_{s=0}^{2k-1} \left( \alpha - \frac{s}{2k-m} + 1 \right)^j \frac{(-1)^k \lambda}{(2k-m)^{2k}}^j t^j \]

\[ Y_i(y) = y^i \sum_{j=0}^{\infty} \frac{(-1)^k \lambda y^{2k-m}}{(2k-m)^{2k}}^j \prod_{s=0}^{2k-1} \left( \frac{1-s}{2k-m} + 1 \right)^j, \ i = 0, 1, ..., 2k-1. \]

So in terms of special functions, we got 2k pieces of linearly independent solutions

\[ Y_i(y) = y^i \text{ } _0F_{2k-1} \left[ \frac{i}{2k-m} + 1, ..., \frac{i-(i-1)}{2k-m} + 1, \frac{i-(i+1)}{2k-m} + 1, ..., \frac{i-(2k-1)}{2k-m} + 1, \frac{(-1)^k \lambda y^{2k-m}}{(2k-m)^{2k}} \right], \]

\[ i = 0, 1, ..., 2k-1, \]

where

\[ _pF_q \left[ \begin{array}{c} a_1, ..., a_p, x \\ b_1, ..., b_q \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_p)_k}{(b_1)_k ... (b_q)_k} \frac{x^k}{k!} \]
- is the generalized hypergeometric function, here

\[(a)_k = a(a+1)\ldots(a+k-1)\]

- Pochhammer symbol.

In particular, for \(k = 1\) we have \((c_0, \ldots, c_3 - \text{const})\)

\[Y_0(t) = c_0 \left(\sqrt{\lambda y \frac{2-m}{2}}\right) \frac{1}{2-m} \sum_{j=0}^{\infty} \frac{(-1)^j \left(2 \sqrt{\lambda y \frac{2-m}{2}}\right)^{2j+1}}{j! \Gamma\left(j - \frac{1}{2-m} + 1\right)} = c_1 \sqrt{y} J_{\frac{1}{2-m}}\left(\frac{2 \sqrt{\lambda y \frac{2-m}{2}}}{2 - m}\right),\]

\[Y_1(y) = c_2 y \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{\lambda y \frac{2-m}{(2-m)}\right)^{2j+1}}{j! \Gamma\left(j + \frac{1}{2-m} + 1\right)} = c_3 \sqrt{y} J_{\frac{1}{2-m}}\left(\frac{2 \sqrt{\lambda y \frac{2-m}{2}}}{2 - m}\right),\]

where

\[J_{\nu}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j \left(\frac{z}{2}\right)^{2j+\nu}}{j! \Gamma\left(j + \nu + 1\right)}\]

- Bessel functions \([12]\).

Satisfying the boundary conditions, we obtain the condition for the existence of eigenvalues

\[J_{\frac{1}{2-m}}\left(\frac{2 \sqrt{\lambda y \frac{2-m}{2}}}{2 - m}\right) = 0.\]

Let’s get back to the general case. Because \((2k - m) \notin N\), then the system of functions \(\{Y_i(y)\}_{i=0}^{2k-1}\) - forms a fundamental system of solutions. Hence the general solution of equation (6) has the form

\[Y(y) = c_0 Y_0(y) + c_1 Y_1(y) + \ldots + c_{2k-1} Y_{2k-1}(y),\]

from the first boundary condition we obtain

\[Y(y) = c_k Y_k(y) + c_{k+1} Y_{k+1}(y) + \ldots + c_{2k-1} Y_{2k-1}(y),\]

mean

\[Y(y) = O\left(y^k\right), y \to +0,\]

from the second boundary condition we have the system

\[
\begin{cases}
c_k Y_k(1) + c_{k+1} Y_{k+1}(1) + \ldots + c_{2k-2} Y_{2k-2}(1) + c_{2k-1} Y_{2k-1}(1) = 0, \\
\ldots \\
(c_k Y_k(y) + c_{k+1} Y_{k+1}(y) + \ldots + c_{2k-2} Y_{2k-2}(y) + c_{2k-1} Y_{2k-1}(y))^{(k-1)}_{y=1} = 0,
\end{cases}
\]
equating to zero the main determinant of the system, one can find the eigenvalues of problem (6). But in view of the complexity of this process, we will proceed in a different way, namely: we reduce problem (6) to the integral equation using the Green function and obtain the necessary estimates for the eigenfunctions. But first, we show that \(\lambda > 0\). Indeed, we have

\[
\int_0^1 Y(y) Y^{(2k)}(y) dy = (-1)^k \lambda \int_0^1 y^{-m} Y^2(y) dy,
\]
\[ \int_0^1 \left( Y^{(k)} \right)^2 dy = \lambda \int_0^1 y^{-m} Y^2(y) dy, \]

because \( \lambda = 0 \) is not an eigenvalue, it follows that \( \lambda > 0 \). It remains to show the existence of eigenvalues and eigenfunctions of problem (6). The integral equation equivalent to problem (6) has the form

\[ Y(y) = (-1)^k \lambda \int_0^1 \xi^{-m} G(y, \xi) Y(\xi) d\xi, \quad (8) \]

where

\[ G(y, \xi) = -\frac{1}{(2k-1)!} \begin{cases} 
G_1(y, \xi), & 0 \leq y \leq \xi, \\
G_2(y, \xi), & \xi \leq y \leq 1,
\end{cases} \]

- the Green function of problem (6) (see [13]), here

\[ G_1(y, \xi) = (1 - \xi)^k \xi^k \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i C_{2k-1}^i C_{k-1+j}^i y^{k-i-1} \xi^{j+i}, \]
\[ G_2(y, \xi) = (1 - y)^k \xi^k \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i C_{2k-1}^i C_{k-1+j}^i y^{k-i-1} \xi^{j+i}, \]
\[ C_k^i = \frac{n!}{k! (n-k)!}. \]

Rewrite (8) as

\[ y^{-\frac{m}{2}} Y(y) = \lambda \int_0^1 \xi^{-\frac{m}{2}} \left[ (-1)^k G(y, \xi) \right] y^{-\frac{m}{2}} \left( \xi^{-\frac{m}{2}} Y(\xi) \right) d\xi, \]

we introduce the notation

\[ \overline{Y}(y) = y^{-\frac{m}{2}} Y(y), \]
\[ \overline{G}(y, \xi) = \xi^{-\frac{m}{2}} \left[ (-1)^k G(y, \xi) \right] y^{-\frac{m}{2}}, \]

then we have

\[ \overline{Y}(y) = \lambda \int_0^1 \overline{G}(y, \xi) \overline{Y}(\xi) d\xi, \quad (9) \]

(9) - there is an integral equation with a continuous, in both variables, and a symmetric kernel. According to the theory of equations with symmetric kernels, equation (9) has no more than a countable number of eigenvalues and eigenfunctions. So, problem (6) has eigenvalues \( \lambda_n > 0, \ n = 1, 2, ..., \) and the corresponding eigenfunctions are \( Y_n(y) \). Further, we assume that

\[ \|Y_n(y)\|^2 = \int_0^1 y^{-m} Y_n^2(y) dy = 1, \]

then, taking into account (9), we have the Bessel inequality

\[ \sum_{n=0}^{\infty} \left( \frac{Y_n(y)}{\lambda_n} \right)^2 \leq \int_0^1 y^{-m} G^2(y, \xi) dy < \infty. \quad (10) \]
Now we find the conditions under which the given function $\varphi(y)$ is expanded in a series according to the eigenfunctions $Y_n(y)$. For this we use the Hilbert-Schmidt theorem.

**Theorem 1.** Let the function $\varphi(y)$ satisfy the conditions
1. $\varphi(y) \in C^{2k}[0, 1]$;
2. $\varphi^{(i)}(0) = \varphi^{(i)}(1) = 0$, $i = 0, 1, \ldots, k - 1$.

Then it can be expanded in a uniformly and absolutely converging series of the form

$$\varphi(y) = \sum_{n=1}^{\infty} \varphi_n Y_n(y),$$

where

$$\varphi_n = \int_0^1 y^{-m} \varphi(y) Y_n(y) dy.$$

**Proof.** We show the equality

$$y^{-\frac{m}{2}} \varphi(y) = \int_0^1 G(y, \xi) \left( (-1)^k \xi^{-\frac{m}{2k}} \frac{d^{2k} \varphi(\xi)}{d\xi^{2k}} \right) d\xi,$$

really

$$\int_0^1 \xi^{-\frac{m}{2}} \left( (-1)^k G(y, \xi) \right) y^{-\frac{m}{2}} \left( (-1)^k \xi^{-\frac{m}{2k}} \frac{d^{2k} \varphi(\xi)}{d\xi^{2k}} \right) d\xi =$$

$$= y^{-\frac{m}{2}} \int_0^1 G(y, \xi) \frac{d^{2k} \varphi(\xi)}{d\xi^{2k}} d\xi = y^{-\frac{m}{2}} \varphi(y).$$

Those for the function $y^{-\frac{m}{2}} \varphi(y)$ the conditions of the Hilbert-Schmidt theorem are satisfied and therefore

$$y^{-\frac{m}{2}} \varphi(y) = \sum_{n=1}^{\infty} y^{-\frac{m}{2}} \varphi_n Y_n(y),$$

dividing by $y^{-\frac{m}{2}}$, we have

$$\varphi(y) = \sum_{n=1}^{\infty} \varphi_n Y_n(y).$$

**Theorem 1 is proved.**

We proceed to solve the equation in the variable $x$. Taking into account conditions (4), (5), we obtain the following initial problem:

$$\begin{cases}
D_0^\alpha X_n(x) = -\lambda_n X_n(x), \\
\lim_{x \to 0} (x^{2-\alpha} X_n(x)) = \psi_n, \\
\lim_{x \to 0} \frac{d}{dx} (x^{2-\alpha} X_n(x)) = \varphi_n.
\end{cases} \quad (11)$$

where

$$\psi_n = \int_0^1 \psi(y) y^{-m} Y_n(y) dy,$$

$$\varphi_n = \int_0^1 \varphi(y) y^{-m} Y_n(y) dy.$$
We will search for solution (11) as a series

\[ X_n(x) = \sum_{j=0}^{\infty} c_j x^{\gamma j + \beta}, \]

where \( c_j, \beta, \gamma - \) are still unknown real numbers.

Formally, we have (the legality of rearranging the series and the integral will follow below)

\[
D_0^\alpha X_n(x, y) = \frac{1}{\Gamma (2 - \alpha)} \frac{d^2}{dx^2} \int_0^x \sum_{j=0}^{\infty} c_j \tau^{\gamma j + \beta} \frac{d\tau}{(x - \tau)^{\alpha - 1}} = \frac{1}{\Gamma (2 - \alpha)} \sum_{j=0}^{\infty} c_j \frac{d^2}{dx^2} \int_0^x \tau^{\gamma j + \beta} \frac{d\tau}{(x - \tau)^{\alpha - 1}} =
\]

\[
= \frac{1}{\Gamma (2 - \alpha)} \sum_{j=0}^{\infty} c_j \frac{d^2}{dx^2} \int_0^1 \frac{1}{x^{\alpha - 1}(1 - z)^{\alpha - 1}} \frac{z^{\gamma j + \beta - \alpha} \Gamma (\gamma j + \beta + 1)}{\Gamma (\gamma j + \beta + 3 - \alpha)} \frac{dz}{(1 - z)^{\alpha - 1}} =
\]

Substituting the last expression in (11), we obtain

\[
\sum_{j=0}^{\infty} c_j (\gamma j + \beta + 2 - \alpha)^2 x^{\gamma j + \beta} \frac{\Gamma (\gamma j + \beta + 1)}{\Gamma (\gamma j + \beta + 3 - \alpha)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\gamma j + \beta} \Rightarrow
\]

\[
\sum_{j=0}^{\infty} c_j (\gamma j + \beta + 2 - \alpha)^2 x^{\gamma (j-1) + \gamma - \alpha} \frac{\Gamma (\gamma j + \beta + 1)}{\Gamma (\gamma j + \beta + 3 - \alpha)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\gamma j} \Rightarrow \gamma - \alpha = 0 \Rightarrow \gamma = \alpha \Rightarrow
\]

\[
\sum_{j=0}^{\infty} c_j (\alpha j + \beta + 2 - \alpha)^2 x^{\alpha (j-1)} \frac{\Gamma (\gamma j + \beta + 1)}{\Gamma (\gamma j + \beta + 3 - \alpha)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\alpha j} \Rightarrow \beta_1 = \alpha - 1, \beta_2 = \alpha - 2.
\]

Let be \( \beta_1 = \alpha - 1 \)

\[
\sum_{j=1}^{\infty} c_j (\alpha j + 1)^2 x^{\alpha (j-1)} \frac{\Gamma (\alpha j + \alpha)}{\Gamma (\alpha j + 2)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\alpha j} \Rightarrow
\]

\[c_j = -\lambda_n (\alpha j) (\alpha j + 1) \frac{\Gamma (\alpha j)}{\Gamma (\alpha (j+1))} c_{j-1} = \frac{(-\lambda_n)^j \Gamma (\alpha)}{\Gamma (\alpha (j+1))} c_0 \Rightarrow
\]

The first solution would be to

\[ X_{1n}(x) = x^{\alpha - 1} \sum_{j=0}^{\infty} \frac{(-\lambda_n x^{\alpha j})^j}{\Gamma (\alpha j + \alpha)}. \]
This series converges absolutely and uniformly for fixed values of \( \lambda_n \) and for limited values of \( x \). Indeed, on the basis of d’Alembert, we have

\[
\lim_{j \to +\infty} \left( \frac{(-\lambda_n x^\alpha)^{j+1}}{\Gamma(\alpha j + 2\alpha)} : \frac{(-\lambda_n x^\alpha)^j}{\Gamma(\alpha j + \alpha)} \right) = (-\lambda_n x^\alpha) \lim_{j \to +\infty} \frac{\Gamma(\alpha j + \alpha)}{\Gamma(\alpha j + 2\alpha)} = \]

\[
= (-\lambda_n x^\alpha) \lim_{j \to +\infty} O(\alpha j)^{\alpha-2} = 0,
\]
here we used the relation from [14]

\[
\frac{\Gamma(z + \alpha)}{\Gamma(z + \beta)} = z^{\alpha-\beta} \left[ 1 + \frac{(\alpha - \beta)(\alpha - \beta - 1)}{2z} + O(z^{-2}) \right].
\]

So the permutation of the series and the integral in the above was legal.

Now let \( \beta_2 = \alpha - 2 \), then

\[
\sum_{j=0}^{\infty} c_j (\alpha j) 2x^\alpha(j-1) \frac{\Gamma(\alpha j + \alpha - 1)}{\Gamma(\alpha j + 1)} = \]

\[
= -\lambda_n \sum_{j=0}^{\infty} c_j x^{\alpha j} \Rightarrow
\]

\[
\sum_{j=1}^{\infty} (\alpha j) 2x^\alpha(j-1) \frac{\Gamma(\alpha j + \alpha - 1)}{\Gamma(\alpha j + 1)} = -\lambda_n \sum_{j=0}^{\infty} c_j x^{\alpha j} \Rightarrow
\]

\[
c_j = \frac{-\lambda_n \Gamma(\alpha j + 1)}{(\alpha j) (\alpha j - 1) \Gamma(\alpha (j + 1) - 1)} c_{j-1} = \frac{-\lambda_n \Gamma(\alpha j + 1)}{\Gamma(\alpha j + 2)} c_{j-1} = \]

\[
= \ldots = \frac{(-\lambda_n)^j \Gamma(\alpha - 1)}{\Gamma(\alpha (j + 1) - 1)} c_0 \Rightarrow
\]

the second linearly independent solution will be

\[
X_{2n} (x) = x^{\alpha-2} \sum_{j=0}^{\infty} \frac{(-\lambda_n x^\alpha)^j}{\Gamma(\alpha j + \alpha - 1)}.
\]

So, the general solution of equation (11) has the form

\[
X_n (x) = d_1 x^{\alpha-1} \sum_{j=0}^{\infty} \frac{(-\lambda_n x^\alpha)^j}{\Gamma(\alpha j + \alpha)} + d_2 x^{\alpha-2} \sum_{j=0}^{\infty} \frac{(-\lambda_n x^\alpha)^j}{\Gamma(\alpha j + \alpha - 1)}, \quad d_1, d_2 - \text{const},
\]
or in terms of special functions

\[
X_n (x) = d_1 x^{\alpha-1} E_{1/\alpha} (-\lambda_n x^\alpha, \alpha) + d_2 x^{\alpha-2} E_{1/\alpha} (-\lambda_n x^\alpha, \alpha - 1),
\]

(12)

where

\[
E_{1/\alpha} (z, \mu) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j + \mu)}
\]

- Mittag-Leffler function [1].

Note that the representation in the form (12) coincides with the known results (see [1] and others) obtained using the properties of the Mittag-Leffler function.

Satisfying the initial conditions, we obtain a solution to problem (11) in the form

\[
X_n (x) = \Gamma(\alpha) \varphi_n x^{\alpha-1} E_{1/\alpha} (-\lambda_n x^\alpha, \alpha) + \Gamma(\alpha - 1) \psi_n x^{\alpha-2} E_{1/\alpha} (-\lambda_n x^\alpha, \alpha - 1),
\]
this representation implies the uniqueness of the solution to problem \((11)\).

Given the estimate (see [15], p.136)

\[
|E_{1/\alpha}(-\lambda_n x^{\alpha}, \alpha)| \leq \frac{M}{1 + \lambda_n x^{\alpha}}, \quad 0 < M - \text{const}, \quad \alpha < 2,
\]

we have

\[
|X_n(x)| \leq M_1 \left( |\varphi_n| + \frac{|\psi_n|}{x} \right), \quad 0 < M_1 - \text{const}.
\]

Thus, the formal solution of the problem \(A\) has the form

\[
u(x,y) = \sum_{n=0}^{\infty} X_n(x) Y_n(y),
\]

We show that \((13)\) is a classical solution of equation \((1)\), indeed

\[
|D_{n,x}^\alpha u(x,y)| \leq \sum_{n=0}^{\infty} |D_{n,x}^\alpha X_n(x)| |Y_n(y)| = \sum_{n=0}^{\infty} |\lambda_n X_n(x)| |Y_n(y)| \leq
\]

\[
\leq M_1 \left( \sum_{n=0}^{\infty} |\lambda_n \varphi_n| |Y_n(y)| + \frac{1}{x} \sum_{n=0}^{\infty} |\lambda_n \psi_n| |Y_n(y)| \right),
\]

We show the convergence of the first term \(\sum_{n=0}^{\infty} |\lambda_n \varphi_n| |Y_n(y)|\), convergence of the second: \(\sum_{n=0}^{\infty} |\lambda_n \psi_n| |Y_n(y)|\), is shown in the same way. So

\[
\sum_{n=0}^{\infty} |\lambda_n \varphi_n| |Y_n(y)| = \sum_{n=1}^{\infty} |\lambda_n^2 \varphi_n| \frac{|Y_n(y)|}{\lambda_n} \leq \sqrt{\sum_{n=1}^{\infty} |\lambda_n^4 \varphi_n^2|} \sqrt{\sum_{n=1}^{\infty} \frac{|Y_n^2(y)|}{\lambda_n^2}},
\]

we have

\[
\varphi_n = \int_0^1 y^{-m} \varphi(y) Y_n(y)dy = \frac{(-1)^k}{\lambda_n} \int_0^1 \varphi(y) Y_{n}^{(2k)}(y)dy =
\]

\[
= \frac{(-1)^k}{\lambda_n} \int_0^1 \varphi^{(2k)}(y) Y_{n}(y)dy =
\]

\[
= \frac{1}{\lambda_n^2} \int_0^1 y^m \varphi^{(2k)}(y) Y_{n}^{(2k)}(y)dy = \frac{1}{\lambda_n^2} \int_0^1 y^m \left( y^m \varphi^{(2k)}(y) \right)^{(2k)} Y_{n}(y)y^{-m}dy,
\]

Now we apply the Bessel inequality

\[
\sum_{n=1}^{\infty} |\lambda_n^4 \varphi_n^2| \leq \int_0^1 \left( \left( y^m \varphi^{(2k)}(y) \right)^{(2k)} \right)^2 y^{-m}dy < \infty.
\]

Now, in order for the calculations made above to be legal, we impose the following restrictions on the function \(\varphi(y)\):

\[
\varphi^{(s)}(0) = \varphi^{(s)}(1) = 0, \quad \varphi(y) \in C^{2k}[0,1],
\]

\[
\left( y^m \varphi^{(2k)}(y) \right)^{(s)}(0) = \left( y^m \varphi^{(2k)}(y) \right)^{(s)}(1) = 0,
\]

\[
y^m \varphi^{(2k)}(y) \in C^{2k}[0,1], \quad s = 0, 1, ..., k - 1.
\]
Taking into account (10) and (14), we obtain that the series

\[ D^\alpha_0 u (x, y) = \sum_{n=0}^{\infty} D^\alpha_0 X_n (x) Y_n (y) \]

converges uniformly. The uniform convergence of the series is proved in a similar way

\[ \frac{\partial^{2k} u (x, y)}{\partial y^{2k}} = \sum_{n=0}^{\infty} X_n (x) \frac{\partial^{2k} Y_n (y)}{\partial y^{2k}} = (-1)^k y^{-m} \sum_{n=0}^{\infty} \lambda_n X_n (x) Y_n (y). \]

So the following theorem holds.

**Theorem 2.** Let the function \( \tau (y) \), where \( \tau (y) = \phi (y) \) or \( \tau (y) = \psi (y) \), satisfies the following conditions:

\[ \tau (y) \in C^{2k} [0, 1], \quad \tau^{(s)} (0) = \tau^{(s)} (1) = 0, \]

\[ \left( y^m \tau^{(2k)} (y) \right)^{(s)} (0) = \left( y^m \tau^{(2k)} (y) \right)^{(s)} (1) = 0, \]

\[ y^m \tau^{(2k)} (y) \in C^{2k} [0, 1], \quad s = 0, 1, \ldots, k-1. \]

Then a solution to Problem A exists.

### 3. Uniqueness of solution

Let the function \( u (x, y) \) be a solution to Problem A with zero initial and boundary conditions. We consider its Fourier coefficients with respect to the system of eigenfunctions of problem (6)

\[ u_n (x) = \int_0^1 y^{-m} u (x, y) Y_n (y) dy, \]

it is easy to show that \( u_n (x) \) is a solution to the problem

\[
\begin{cases}
D^\alpha_0 u_n (x) = -\lambda_n u_n (x), \\
\lim_{x \to 0} (x^{2-\alpha} u_n (x)) = 0, \\
\lim_{x \to 0} \frac{d}{dx} (x^{2-\alpha} u_n (x)) = 0.
\end{cases}
\]

This problem has only a zero solution, i.e.

\[ \int_0^1 y^{-m} u (x, y) Y_n (y) dy = 0, \quad \forall n. \]

Because

\[ G (y, \xi) \] - symmetrical, continuous

\[ \int_0^1 G^2 (y, \xi) d\xi < \infty, \quad \int_0^1 G^2 (y, \xi) dy < \infty, \quad \int_0^1 \int_0^1 G^2 (y, \xi) dyd\xi < \infty, \quad \lambda_n > 0, \quad \forall n, \]

then the conditions of the Mercer theorem are satisfied and

\[ G (y, \xi) = \sum_{n=0}^{\infty} \frac{Y_n (y) \bar{Y}_n (\xi)}{\lambda_n}. \]
From here we have

\[ y^{-\frac{m}{2}} u(x, y) = \int_0^1 \overline{G}(y, \xi) \left( (-1)^k \xi^m \frac{\partial^2 k u(x, \xi)}{\partial \xi^2 k} \right) d\xi = \]

\[ = (-1)^k \int_0^1 \sum_{n=0}^\infty \frac{Y_n(y) \overline{Y}_n(\xi)}{\lambda_n} \left( \xi^m \frac{\partial^2 k u(x, \xi)}{\partial \xi^2 k} \right) d\xi = \]

\[ = (-1)^k \sum_{n=0}^\infty \frac{y^{-\frac{m}{2}} Y_n(y)}{\lambda_n} \int_0^1 \xi^{-\frac{m}{2}} \overline{Y}_n(\xi) \frac{\partial^2 k u(x, \xi)}{\partial \xi^2 k} d\xi = \]

because since the series converges uniformly, then we can interchange the signs of integration and the sum

\[ = (-1)^k \sum_{n=0}^\infty \frac{y^{-\frac{m}{2}} Y_n(y)}{\lambda_n} \int_0^1 \overline{Y}_n(\xi) \frac{\partial^2 k u(x, \xi)}{\partial \xi^2 k} d\xi = \]

\[ = (-1)^k \sum_{n=0}^\infty \frac{y^{-\frac{m}{2}} Y_n(y)}{\lambda_n} \int_0^1 Y_n(2k)(\xi) u(x, \xi) d\xi = \]

\[ = (-1)^k \sum_{n=0}^\infty \frac{y^{-\frac{m}{2}} Y_n(y)}{\lambda_n} \int_0^1 \lambda_n (-1)^k \xi^{-m} Y_n(\xi) u(x, \xi) d\xi = \]

\[ = y^{-\frac{m}{2}} \sum_{n=0}^\infty Y_n(y) \int_0^1 \xi^{-m} Y_n(\xi) u(x, \xi) d\xi = 0 \Rightarrow \]

\[ u(x, y) \equiv 0. \]

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