Local thermodynamics of KS black hole

Myungseok Eune,1, Bogeun Gwak,1,2, and Wontae Kim1,2,3

1Research Institute for Basic Science, Sogang University, Seoul, 121-742, Korea
2Center for Quantum Spacetime, Sogang University, Seoul 121-742, Korea
3Department of Physics, Sogang University, Seoul 121-742, Korea

(Dated: December 10, 2012)

Abstract

We study the thermodynamics of the KS black hole which is an asymptotically flat solution of the HL gravity. In particular, we introduce a cavity to describe the thermodynamics at a finite isothermal surface on general ground and to get a well-defined thermodynamics. We show that there exists a locally stable small black hole which tunnels to the hot flat space below the critical temperature and to the large black hole above the critical temperature. Moreover, it turns out that the remnant decays into the vacuum through a quantum tunneling.

Keywords: Hořava-Lifshitz Gravity, Thermodynamics of Black Holes
I. INTRODUCTION

It has been claimed that the intrinsic entropy of a black hole is proportional to the area of the event horizon by Bekenstein [1]. Subsequently, Hawking has shown that the black hole has thermal radiation through a quantum field theoretical analysis [2] and studied the thermodynamic phase transition [3]. In particular, the thermodynamics of black holes has been also studied in a cavity to get well-defined canonical ensembles [4–12]. In the quasilocal thermodynamics of a black hole, the thermodynamical quantities such as the energy, the temperature, and so on, are related to the size of cavity, whereas the entropy is not since the entropy can be regarded as a conserved Noether charge [13].

On the other hand, Hořava has recently proposed the Hořava-Lifshitz (HL) gravity [14, 15], inspired by condensed matter models of dynamical critical systems [16]. The HL gravity is a power-counting renormalizable theory of gravity with anisotropic scaling of space and time. The scale transformations of time and space are given by $t \rightarrow b^z t$ and $x^i \rightarrow bx^i$, respectively, where $i = 1, \cdots, D$ is the spatial index, $D$ is the dimension of space, and the Lifshitz index $z$ is the “critical exponent” in the Lifshitz scalar field theory. One can expect that it recovers the general relativity in the IR region whereas it becomes a nonrelativistic gravity in the UV region. The static and spherically symmetric black hole solutions in the HL gravity have been investigated [17, 18]. While some of solutions give the asymptotically anti-de Sitter spacetime [17], the black hole solution called the KS solution [18] behaves like the Schwarzschild metric at infinity. Then, there have been many studies for the entropy of these black holes [19–27]. Especially, the ADM mass for the KS black hole was identified with a parameter from the asymptotic expansion of the metric. Moreover, it has been shown that the entropy is modified by the logarithmic term from the first law of thermodynamics [23]. This logarithmic correction to the entropy can be also obtained from the quantum tunneling method [27, 28].

In this paper, we would like to study the thermodynamics of the KS black hole using the logarithmic corrected entropy in the cavity. It should give the expected Hawking-Page type phase transition in the large black hole like the ordinary thermodynamics of the Schwarzschild black hole since the KS metric is asymptotically same with the Schwarzschild metric [3]. However, for the small black hole, its behavior will be different from that of the Schwarzschild black hole. We find that there exists a locally stable small black hole which
decays into the thermal state without the black hole below a critical temperature while it can decay into the large stable black hole above a critical temperature. Moreover, it will be shown that the remnant of the lowest mass of the black hole tunnels to the vacuum.

The paper is organized as follows. In section II we recapitulate the HL gravity and introduce the KS solution. In section III we get the thermodynamic quantities of Tolman temperature, heat capacity including free energies. In section IV the thermodynamics and phase transitions are studied for a given range of the coupling $\omega$. Finally, the summary and discussion are given in section V.

II. SCHWARZSCHILD-LIKE BLACK HOLE IN THE HL GRAVITY

In this section, we would like to recapitulate the HL gravity for our purposes and introduce a static and spherically symmetric black hole. Let us start with the four-dimensional line element parametrized as

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt),$$

where $N(t, x^i)$, $N^i(t, x^j)$, and $g_{ij}(t, x^k)$ are the lapse function, the shift functions, and the spatial metric, respectively. Along with the ADM decomposition, the Einstein-Hilbert action is written as

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} N \left( K_{ij} K^{ij} - K^2 + R - 2\Lambda \right),$$

where $G$ and $\Lambda$ are Newton’s constant and cosmological constant, respectively. The extrinsic curvature $K_{ij}$ is defined by

$$K_{ij} = \frac{1}{2N}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i).$$

Here, the dot denotes a derivative with respect to $t$ and $\nabla_i$ is a covariant derivative with the spatial metric $g_{ij}$. The action in the HL gravity is given by

$$S_{HL} = \int dt d^2x \sqrt{g} N \left[ \frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) + \frac{\kappa^2 \mu^2}{8(1 - 3\lambda)} \left( \frac{1}{4} - \frac{4\lambda}{3}R^2 + \Lambda_W R - 3\Lambda_W^2 \right) \right] - \frac{\kappa^2}{2w^4} \left( C_{ij} - \frac{\mu w^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu w^2}{2} R^{ij} \right),$$

where $\kappa$, $\mu$, $\lambda$, $w$, and $\Lambda_W$ are constant parameters, and the Cotton tensor $C_{ij}$ is defined by

$$C_{ij} = (\epsilon^{ikl}/\sqrt{g}) \nabla_k \left( R^{jl} - \frac{1}{4}\delta^j_l R \right).$$

In the IR limit, comparing the action (2) to the Einstein-Hilbert action (1), the speed of light $c$, Newton’s constant $G$, and the cosmological constant $\Lambda$ are identified with

$$c = \frac{1}{4} \kappa^2 \mu \sqrt{\Lambda_W / (1 - 3\lambda)}, \quad G = \kappa^2 / (32\pi c), \quad \Lambda = \frac{3}{2} \Lambda_W.$$

Although the kinetic term in Eq. (2) agrees with that of the Einstein-Hilbert action (1) when $\lambda = 1$, the action (2) without $\Lambda_W$ does not fully recover Eq. (1) in the IR limit. In order to obtain it in the IR limit, one can deform the HL gravity by introducing a soft
violation term as $\mathcal{L}_V \rightarrow \mathcal{L}_V + \sqrt{g} N \mu^2 R$ and taking the limit of $\Lambda_W$ as $\Lambda_W \rightarrow 0$, which is called the “deformed HL gravity”. In the deformed HL gravity, the UV properties are unchanged, whereas there exists a Minkowski vacuum in the IR limit. In this case, the constants are given by $c = \kappa \mu^2 / \sqrt{2}$, $G = \kappa^2 / (32 \pi c)$, and $\lambda = 1$. In the deformed HL gravity, a static and spherically symmetric black hole is described by the line element [18]

$$ds^2 = - f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

(3)

with the metric function $f(r)$ given by

$$f(r) = 1 + \omega r^2 \left( 1 - \sqrt{1 + \frac{4M}{\omega r^3}} \right),$$

(4)

where $\omega = 16\mu^2 / \kappa^2$ and $M$ is a positive constant. The black hole described by the metric function (4) is called the KS black hole. In the limit of $\omega \rightarrow \infty$ or $r \rightarrow \infty$, it can be reduced to the form of $f(r) = 1 - 2M/r + 2M^2/(\omega r^4) + \cdots$, which gives the Schwarzschild metric function and has the Minkowski vacuum at infinity. The third term in the expansion shows that the IR properties of the KS black hole are different from the Reissner-Nordström (RN) black hole whose metric function is given by $f(r) = 1 - 2M/r + Q^2/r^2$, where $Q$ is the electric charge of the RN black hole. From the metric function (4), the outer $r_+$ and the inner $r_-$ horizons are given by

$$r_{\pm} = M \left( 1 \pm \sqrt{1 - \frac{1}{2\omega M^2}} \right),$$

(5)

where the mass is restricted to $M \geq 1 / \sqrt{2\omega}$. There exists the extremal black hole with the horizon $r_e$ and the mass $M_e$ given by $r_e = M_e = 1 / \sqrt{2\omega}$.

### III. THERMODYNAMIC QUANTITIES

We now introduce a cavity with a finite size $r$ in order to obtain the well-defined thermodynamics for the asymptotically flat black hole [6]. In the line element (3), there is the maximum of a mass for a given coupling and a given size $r$, $M_{\text{max}} = (1 + 2\omega r^2)/(4\omega r)$ because of $r \geq r_+$. By the way, from the reality condition of Eq. (5), there exists the minimum of mass $M_{\text{min}} = 1 / \sqrt{2\omega}$, which is just the mass of the extremal black hole. Thus, the black hole mass should be bounded.
Next, the Tolman temperature $T$ on the boundary of the cavity is given by \[ T = \frac{T_H}{\sqrt{f(r)}}, \] (6)

where the Hawking temperature is $T_H = (-1 + 2\omega r^2_+) / [8\pi r_+(1 + \omega r^2_+)]$. In particular, it is interesting to note that the entropy $S$ of the KS black hole is written as the Bekenstein-Hawking entropy with the logarithmic term \[ S = \frac{A_H}{4} + \frac{\pi}{\omega} \ln \left( \frac{A_H}{4} \right) + S_0 \]

\[ = \pi r^2_+ + \frac{\pi}{\omega} \ln(\pi r^2_+) - \frac{\pi}{2\omega} - \frac{\pi}{2\omega} \ln \frac{\pi}{2\omega}. \] (7)

where $A_H = 4\pi r^2_+$ is the area of the horizon. The integration constant $S_0$ was determined in order that the black hole entropy is non-negative at the minimum mass, which has been eventually chosen as $S_0 = -\frac{\pi}{2\omega} - \frac{\pi}{2\omega} \ln \frac{\pi}{2\omega}$. The negative entropy (or zero) has been discussed in the higher-derivative models \[30\].

The first law of thermodynamics $dE = T dS$ gives the thermodynamic internal energy

\[ E = r + \frac{2}{3\omega r} - \frac{2}{3\omega r} \left[ 1 + \omega r^2 \left( 1 + \frac{1}{2} \sqrt{1 + \frac{4M}{\omega r^3}} \right) \right] \left[ 1 + \omega r^2 \left( 1 - \sqrt{1 + \frac{4M}{\omega r^3}} \right) \right]^{1/2}, \] (8)

where the integration constant is fixed so that it becomes zero for $M = 0$. Then, it recovers the mass of the black hole at the infinity. Next, for the critical phenomena, one can calculate the heat capacity $C$ defined by

\[ C = \frac{\partial E}{\partial T} = T \frac{\partial S}{\partial T}. \] (9)

where the energy and the entropy are given by Eqs. (8) and (7), respectively. The critical phenomena appear at the extrema of the local temperature because the energy and the entropy are monotonic functions with respect to the mass. The logarithmic correction sometimes gives the negative contribution to the entropy \[31–34\]. Then, the critical phenomena may appear in the extrema of the energy or the entropy; however, this is not the case in the present model.

In connection with phase transitions, one can define the free energy,

\[ F = E - TS, \] (10)

using the local temperature \[6\]. Note that the extrema of the on-shell free energy are coincident with those of the local temperature. On the other hand, the off-shell free energy
will be useful to analyze the stable states of black holes; however, the meaningful equilibrium occurs at the extrema of the off-shell free energy.

IV. CRITICAL PHENOMENA AND PHASE TRANSITIONS

The thermodynamic behaviors depend on the coupling constant $\omega$, which means that the KS metric has several black hole states for $\omega > \omega_{\text{cr}}$ and only a single black hole state for $\omega \leq \omega_{\text{cr}}$ at a particular temperature. Note that the critical value of $\omega_{\text{cr}}$ can be found from the condition that the extrema of the local temperature yield an equal root.

Let us first study for $\omega > \omega_{\text{cr}}$. The black hole temperature can start with $T = 0$ since the black hole may become an extremal black hole at $M = M_{\text{min}}$, and it can have two extrema of $M_1$ and $M_2$ corresponding to the temperatures $T_1$ and $T_2$ as seen from Fig. 1. Around $M_1$, let us define the large black hole for $M > M_1$ and the small black hole for $M < M_1$ for convenience. We find that the black hole has a single state for $T > T_1$ or $T < T_2$, three states for $T_2 < T < T_1$, and two states at $T = T_1$ or $T = T_2$.

![FIG. 1: The solid line shows the behavior of the local temperature (6) for $\omega > \omega_{\text{cr}}$. We set $r = 10$ and $\omega = 1$. Then, the numerical values become $\omega_{\text{cr}} = 0.215$, $M_1 = 1.01$, $M_2 = 3.29$, $T_1 = 0.0321$, and $T_2 = 0.0201$.](image)

The stabilities of these black holes can be obtained from the heat capacity (9), which is shown in Fig. 2. For $M < M_1$ or $M > M_2$, the black hole is stable whereas it is unstable for $M_1 < M < M_2$. It means that a large black hole is stable and the end state after evaporation of the black hole is also stable. And there exist two stable black holes and an unstable small black hole simultaneously between $T_2 < T < T_1$. 

6
FIG. 2: The solid line shows the heat capacity (9) for $\omega > \omega_{cr}$. This is plotted at $r = 10$ and $\omega = 1$.

FIG. 3: The thick dashed and solid curves are the on-shell free energies of the black hole and the hot flat space, respectively. The right black dot is a point of the phase transition and a left black dot is a remnant. The five thin solid curves are the off-shell free energies depending on temperature of heat reservoir. These are plotted at $r = 10$ and $\omega = 1$ for $\omega > \omega_{cr}$.

As for the phase transition from the hot flat space to the stable large KS black hole, the Hawking-Page type phase transition appears only above a temperature $T^*$ as seen from Fig. 3. The free energy of the hot flat space is $F_{hfs} = -(4/135)\pi^3 r^3 T^4$ which is actually higher than that of the black hole. On the other hand, in the ordinary Schwarzschild black hole, the small unstable black hole decays either into the large stable black hole or to the thermal state; however, it can decay into the small stable black hole in this KS solution. Note that a stable black hole exists at $0 < T < T_1$. It is locally stable, so that it should undergo a quantum tunneling and decay to the hot flat space below the critical temperature $T^*$ as seen from Fig. 1 and Fig. 3. And, the remnant whose mass is $M_{min}$ should decay
into the vacuum through the quantum tunneling. At $T = 0$, it disappears and becomes an extremal black hole, which is no longer a thermodynamic object. The reason why the on-shell and the off-shell free energies at $M = M_{\text{min}}$ are all the same $F_{\text{on}} = F_{\text{off}} = E$ is that the entropy is zero for any temperature. This extremal configuration is similar to the case of the noncommutative black holes from the profile of the temperature [35, 36].

Let us now consider the case of $\omega \leq \omega_{\text{cr}}$. The local temperature (6) monotonically increases with respect to the mass as shown in Fig. 4. Any black hole state is stable since the heat capacity is always positive. Specifically, the maximum of the heat capacity appears at $M_1$ for $\omega < \omega_{\text{cr}}$, and it is positively divergent at that point for $\omega = \omega_{\text{cr}}$. Especially, the heat capacities vanish at $M = M_{\text{min}}$ and $M = M_{\text{max}}$. In this case also, there appears the Hawking-Page type phase transition since the on-shell free energy of the hot flat space is higher than that of the black hole, so that the stable black hole is more preferable above the critical temperature $T^*$ as seen from Fig. 5. For usual phase transitions, it seems to be necessary that the black hole mass should be getting smaller and smaller and then eventually its mass disappears to decay completely to the hot flat space without the black hole. On the contrary to the ordinary Schwarzschild black hole which has a continuous mass spectrum, the KS black hole has a lower bound. At first sight, the black hole does not decay into the hot flat space below $T^*$; however, this is not the case since the on-shell free energy of the black hole is higher than the free energy of the hot flat space below the critical mass $M^*$ as seen from Fig. 5, so that it should decay into the hot flat space quantum-mechanically.
As a result, the Hawking-Page phase transition from the black hole to the hot flat space can occur at a critical temperature for any coupling constant of $\omega$. For $\omega > \omega_c$, even though there exists a locally stable small black hole below the critical temperature, it should tunnel to the large black hole which eventually decays into the hot flat space. For a very high temperature, it directly decays into the large black hole thermodynamically. For $\omega \leq \omega_c$, all black hole states are stable and they tunnel to the hot flat space quantum-mechanically below the critical temperature. Finally, the remnant should also tunnel to the vacuum.

![Graph](image_url)

FIG. 5: The thick dashed and solid curves are the on-shell free energies for the black hole and for the hot flat space, respectively for $\omega < \omega_{cr}$. The phase transition occurs at $T^*$. The plot for $\omega = \omega_{cr}$ is very similar to this case. This is plotted for $r = 10$ and $\omega = 0.1$, which results in $\omega_{cr} = 0.215$, $M^* = 4.65$, and $T^* = 0.0218$.

V. DISCUSSION

We have studied the thermodynamics and the phase transitions of the KS black hole at the finite isothermal surface. Note that in this model the mass of the black hole should be bounded. Especially, the lower bound of the mass is related to the remnant and the upper bound comes from the cavity effect. The KS black hole metric depends on a coupling constant $\omega$ which gives inner and outer horizons. When two horizons are coincident, it gives the minimum mass. By the way, the black hole is surrounded by the cavity, that is, the radius of the outer horizon should be smaller than the size of cavity. When the size of cavity equals to the outer horizon, the black hole has a maximum mass. So, the temperature of the black hole with the minimum mass is zero from the extremality while the temperature
at the maximum mass is infinity. Between them, the extrema have something to do with the critical phenomena. For the range of $\omega \leq \omega_{cr}$, the temperature has no extremum and monotonically increases as seen from Fig. 4. For $\omega > \omega_{cr}$, there are two extrema which are related to the stability of the black hole states.

On general ground, a lower on-shell free energy state is more preferable so that Hawking-Page type phase transition from the hot flat space to the black hole naturally appears for the large black hole. As for the small black hole, a locally unstable small black hole can decay into the large black hole or into the locally stable small black hole thermodynamically below the critical temperature. Subsequently, the stable small black hole tunnels to the hot flat space quantum-mechanically as seen from Fig. 3 and Fig. 5.

Finally, let us discuss the evolution of a small unstable black hole especially for $\omega > \omega_{cr}$. Suppose that it loses its mass through Hawking radiation, then the negative heat capacity shows that the black hole temperature should be increased while the temperature of the heat reservoir is still maintained. Eventually, it can decay into much smaller stable black hole which lies in the lower free energy state compared to that of the unstable black hole. In other words, it arrives at the on-shell state of the same temperature with that of the heat reservoir since there are two black hole states for a given temperature. If this black hole loses its mass further, it subsequently acquires some energy from the heat reservoir because of the positive heat capacity. Even though it can arrive at the minimum mass throughout the non-trivial mechanism, it can not evolve any more. Since the extremal black hole with the non-vanishing temperature of the heat reservoir is not in thermal equilibrium, so that it should return to the stable black hole eventually. Then, the stable small black hole should decay into the thermal vacuum via a quantum tunneling.

**Acknowledgments**

We would like to thank S.-H. Yi for exciting discussion. M. Eune was supported by National Research Foundation of Korea Grant funded by the Korean Government (Ministry of Education, Science and Technology) (NRF-2010-359-C00007). B. Gwak and W. Kim are supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) through the Center for Quantum Spacetime(CQUeST) of Sogang University with grant number 2005-0049409, and W. Kim was also supported by the Basic
Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-0002880).

[1] J. D. Bekenstein, Lett. Nuovo Cim. 4 (1972) 737.
[2] S. W. Hawking, Commun. Math. Phys. 43 (1975) 199 [Erratum-ibid. 46 (1976) 206].
[3] S. W. Hawking and D. N. Page, Commun. Math. Phys. 87 (1983) 577.
[4] B. Allen, Phys. Rev. D 30 (1984) 1153.
[5] J. W. York, Jr., Phys. Rev. D 33 (1986) 2092.
[6] B. F. Whiting and J. W. York, Jr., Phys. Rev. Lett. 61 (1988) 1336.
[7] J. D. Brown, J. Creighton and R. B. Mann, Phys. Rev. D 50 (1994) 6394 [gr-qc/9405007].
[8] J. D. E. Creighton and R. B. Mann, Phys. Rev. D 54 (1996) 7476.
[9] M. H. Dehghani and R. B. Mann, Phys. Rev. D 64 (2001) 044003 [hep-th/0102001].
[10] O. B. Zaslavskii, Phys. Rev. D 69 (2004) 044008 [hep-th/0310268].
[11] D. Grumiller and R. McNees, JHEP 0704 (2007) 074 [hep-th/0703230 [HEP-TH]].
[12] D. Astefanesei, R. B. Mann, M. J. Rodriguez and C. Stelea, Class. Quant. Grav. 27 (2010) 165004 [arXiv:0909.3852 [hep-th]].
[13] R. M. Wald, Phys. Rev. D 48 (1993) 3427 [gr-qc/9307038].
[14] P. Hořava, JHEP 0903 (2009) 020 [arXiv:0812.4287 [hep-th]].
[15] P. Hořava, Phys. Rev. D 79 (2009) 084008 [arXiv:0901.3775 [hep-th]].
[16] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. 11 (1941) 255; Zh. Eksp. Teor. Fiz. 11 (1941) 269.
[17] H. Lu, J. Mei and C. N. Pope, Phys. Rev. Lett. 103 (2009) 091301 [arXiv:0904.1595 [hep-th]].
[18] A. Kehagias and K. Sfetsos, Phys. Lett. B 678 (2009) 123 [arXiv:0905.0477 [hep-th]].
[19] R. G. Cai, L. M. Cao and N. Ohta, Phys. Lett. B 679 (2009) 504 [arXiv:0905.0751 [hep-th]].
[20] Y. S. Myung, Phys. Lett. B 690 (2010) 534 [arXiv:1002.4448 [hep-th]].
[21] M. Eune and W. Kim, Phys. Rev. D 82 (2010) 124048 [arXiv:1007.1824 [hep-th]].
[22] I. Radinschi, F. Rahaman and A. Banerjee, arXiv:1012.0986 [gr-qc].
[23] Y. S. Myung, Phys. Lett. B 684, 158 (2010) [arXiv:0908.4132 [hep-th]].
[24] M. Liu, J. Lu, Y. -L. Jia and J. Lu, Int. J. Theor. Phys. 50, 1978 (2011).
[25] R. Biswas and S. Chakraborty, Astrophys. Space Sci. 332, 193 (2011) [arXiv:1104.3719 [gr-qc]].
[26] P. B. Khatua, S. Chakraborty and U. Debnath, arXiv:1105.1533 [physics.gen-ph].
[27] M. Liu and J. Lu, Phys. Lett. B 699, 296 (2011) [arXiv:1107.5878 [gr-qc]].
[28] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. 85 (2000) 5042 [hep-th/9907001].
[29] R. C. Tolman, Phys. Rev. 35 (1930) 904.
[30] M. Cvetic, S. Nojiri and S. D. Odintsov, Nucl. Phys. B 628 (2002) 295 [hep-th/0112045].
[31] R. K. Kaul and P. Majumdar, Phys. Rev. Lett. 84 (2000) 5255 [gr-qc/0002040].
[32] T. R. Govindarajan, R. K. Kaul and V. Suneeta, Class. Quant. Grav. 18 (2001) 2877 [gr-qc/0104010].
[33] A. Chatterjee and P. Majumdar, Phys. Rev. Lett. 92 (2004) 141301 [gr-qc/0309026].
[34] R. -G. Cai, L. -M. Cao and N. Ohta, JHEP 1004 (2010) 082 [arXiv:0911.4379 [hep-th]].
[35] W. Kim, E. J. Son and M. Yoon, JHEP 0804 (2008) 042 [arXiv:0802.1757 [gr-qc]].
[36] P. Nicolini and G. Torrieri, JHEP 1108 (2011) 097 [arXiv:1105.0188 [gr-qc]].