Rayleigh–Faber–Krahn, Lyapunov and Hartmann–Wintner Inequalities for Fractional Elliptic Problems

Aidyn Kassymov, Michael Ruzhansky and Berikbol T. Torebek

Abstract. In this paper, in the cylindrical domain, we consider a fractional elliptic operator with Dirichlet conditions. We prove, that the first eigenvalue of the fractional elliptic operator is minimised in a circular cylinder among all cylindrical domains of the same Lebesgue measure. This inequality is called the Rayleigh–Faber–Krahn inequality. Also, we give Lyapunov and Hartmann–Wintner inequalities for the fractional elliptic boundary value problem.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N > 2$, be an open bounded domain with smooth boundary and let $(a, b), -\infty < a < b < +\infty$, be an interval. In the cylindrical domain $D = (a, b) \times \Omega$ we define the operator

\[
L_{\alpha,s}^\gamma u(x, y) \equiv D_{a+,x}^\alpha D_{b-,x}^\alpha u(x, y) + (-\Delta)_y^s u(x, y), \quad (x, y) \in D,
\]

with Dirichlet boundary conditions

\[
\begin{align*}
    u(a, y) &= u(b, y) = 0, \quad y \in \Omega, \\
    u(x, y) &= 0, \quad y \in \mathbb{R}^N \setminus \Omega,
\end{align*}
\]

where $1/2 < \alpha \leq 1$ and $s \in (0, 1)$. Here $D_{a+,x}^\alpha u(x, y) = \partial_x I_{a+,x}^{1-\alpha} u(x, y)$ is the left Riemann–Liouville, and $D_{b-,x}^\alpha u(x, y) = I_{b-,x}^{1-\alpha} u_x(x, y)$ is the right Caputo derivative.

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fractional derivatives of order $0 < \alpha \leq 1$ with the left and right Riemann—Liouville fractional integrals defined by

$$I_{a+}^\alpha u(x, y) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - s)^{\alpha - 1} u(s, y) \, ds$$

and

$$I_{b-}^\alpha u(x, y) = \frac{1}{\Gamma(\alpha)} \int_x^b (s - x)^{\alpha - 1} u(s, y) \, ds, \quad x \in (a, b),$$

respectively, and $(-\Delta)_y^s$ is the fractional Laplacian of order $s \in (0, 1)$ defined by

$$(-\Delta)_y^s u(x, y) = C_{N,s} \int_{\mathbb{R}^N} \frac{u(x, y) - u(x, \xi)}{|y - \xi|^{N+2s}} \, d\xi, \quad y \in \mathbb{R}^N,$$

where $C_{N,s}$ is some normalisation constant.

The classical Rayleigh–Faber–Krahn inequality asserts that the first eigenvalue of the Laplacian with the Dirichlet boundary condition in $\mathbb{R}^N$, $N \geq 2$, is minimised in a ball among all domains of the same measure. Recently, some Rayleigh–Faber–Krahn-type inequalities were obtained for the volume potentials [16–18], and for the fractional elliptic operators [1,2]. For the cylinder domains, Rayleigh–Faber–Krahn-type inequalities for $s$—numbers of the Cauchy–Diriclet and Cauchy–Robin heat operators were considered in [10] and [11], respectively. In this paper an analogue of the Rayleigh–Faber–Krahn inequality is proved for the fractional elliptic operator.

For the second-order differential equation with Dirichlet boundary condition Lyapunov [13] proved a necessary condition for existence of non-trivial solutions. In [5], Hartman and Wintner generalised the Lyapunov inequality. Recently, the study of Lyapunov-type inequalities were extended to the multidimensional elliptic problems by several authors [3,7–9,15]. The main goals of this paper are to obtain the Rayleigh–Faber–Krahn, Lyapunov and Hartmant–Wintner-type inequalities for the fractional elliptic problem (1.1), (1.2), (1.3) and, also combine these two inequalities.

For the convenience of the reader, let us briefly summarise the results of this paper:

- **Rayleigh–Faber–Krahn inequality for circular cylinder.** Suppose that $\frac{1}{2} < \alpha \leq 1$ and $s \in (0, 1)$. Then the first eigenvalue of the problem

$$\mathcal{L}^{\alpha,s} u(x, y) = \nu u(x, y), \quad \text{in } D = (a, b) \times \Omega,$$

with boundary conditions (1.2)–(1.3) is minimised in the circular cylinder $\mathcal{C}$ among all cylindric domains of a given measure, that is

$$\nu_1(D) \geq \nu_1(\mathcal{C}),$$

for all $D$ with $|D| = |\mathcal{C}|$.

- **Rayleigh–Faber–Krahn inequality for polygonal cylinder.** Suppose that $\frac{1}{2} < \alpha \leq 1$ and $s \in (0, 1)$. Then the first eigenvalue of the problem

$$\mathcal{L}^{\alpha,s} u(x, y) = \nu u(x, y), \quad \text{in } D = (a, b) \times \Omega,$$

...
with boundary conditions (1.2)–(1.3) is minimised in the equilateral triangular (or square) cylinder $D^\star = (a, b) \times \Omega^\star$ among all triangular (or quadrilateral) cylindric domains of a given measure, that is
\[ \nu_1(D) \geq \nu_1(D^\star), \] (1.5)
for all $D$ with $|D| = |D^\star|$.

- **Lyapunov inequality.** Assume that $\frac{1}{2} < \alpha \leq 1$, $s \in (0, 1)$ and $q \in C([a, b])$. If the fractional elliptic equation
  \[ \mathcal{L}^{\alpha,s} u(x, y) = q(x)u(x, y), \text{ in } D = (a, b) \times \Omega, \]
with boundary conditions (1.2)–(1.3) admits a solution, then
  \[ \int_a^b |q(x) - \lambda_1(\Omega)|dx \geq \left( \sup_{a < x < b} G(x, x) \right)^{-1}, \]
where $\lambda_1(\Omega)$ is the first eigenvalue of the fractional Dirichlet–Laplacian (3.2) and
  \[ G(x, t) = K(x, t) - \frac{K(a, t)K(x, a)}{K(a, a)} \]
and
  \[ K(x, t) = \frac{1}{\Gamma^2(\alpha)} \int_{\max\{x, t\}}^b (s - x)^{\alpha-1}(s - t)^{\alpha-1}ds. \]

- **Hartmann–Wintner inequality.** Assume that $\frac{1}{2} < \alpha \leq 1$, $s \in (0, 1)$ and $q \in C([a, b])$. If the fractional elliptic equation
  \[ \mathcal{L}^{\alpha,s} u(x, y) = q(x)u(x, y), \text{ in } D = (a, b) \times \Omega, \]
with boundary conditions (1.2)–(1.3) admits a solution, then
  \[ \int_a^b \left( K(a, a)K(s, s) - K^2(a, s) \right) [q(x) - \lambda_1(\Omega)]^+ds > \frac{(b - a)^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha - 1)}, \]
where $[q(x) - \lambda_1(\Omega)]^+ = \max\{q(x) - \lambda_1(\Omega), 0\}$.

2. One-Dimensional Fractional Boundary Value Problem

Let us consider the following problem:
\[
\begin{aligned}
D_{a+}^\alpha D_{b-}^\alpha u(x) - q(x)u(x) &= 0, \quad x \in (a, b), \quad \alpha \in \left(\frac{1}{2}, 1\right], \\
u(a) = u(b) &= 0.
\end{aligned}
\] (2.1)

**Theorem 2.1.** Suppose that $\alpha \in \left(\frac{1}{2}, 1\right]$ and let $u$ be the solution of (2.1), such that $D_{a+}^\alpha D_{b-}^\alpha u \in C([a, b])$. Then, the solution of (2.1) is equivalent to the solution of the following integral equation
\[ u(x) = \int_a^b G(x, t)q(t)u(t)dt, \] (2.2)
where
\[ G(x, t) = K(x, t) - \frac{K(a, t)K(x, a)}{K(a, a)} \]

and
\[ K(x, t) = \frac{1}{\Gamma^2(\alpha)} \int_{\max\{x, t\}}^{b} (s - x)^{\alpha - 1}(s - t)^{\alpha - 1} ds. \]

Proof. By combining property of the Riemann–Liouville fractional integral for (2.1) with [12, Lemma 2.5], we have
\[ I_\alpha^a + D_\alpha^a u(x) = u(x) - \frac{I_{\alpha+}^a u(a)}{\Gamma(\alpha)} (x - a)^{\alpha - 1}, \quad 0 < \alpha \leq 1, \]
we establish
\[ 0 = I_\alpha^a + D_\alpha^a + D_\alpha^b u(x) - I_\alpha^a (q(x)u(x)) \]
\[ = D_\alpha^b u(x) - \frac{I_{\alpha+}^a D_\alpha^b u(a)}{\Gamma(\alpha)} (x - a)^{\alpha - 1} - I_\alpha^a (q(x)u(x)). \]

Using \( I_\alpha^b, u(b) = 0, \) and [12, Lemma 2.22], \( I_\alpha^a D_\alpha^b u(x) = u(x) - u(b), \) \( 0 < \alpha \leq 1, \)
we get
\[ 0 = I_\alpha^a D_\alpha^b u(x) - \frac{I_{\alpha+}^a D_\alpha^b u(a)}{\Gamma(\alpha)} I_\alpha^b (x - a)^{\alpha - 1} - I_\alpha^a (q(x)u(x)) \]
\[ = u(x) - u(b) - \frac{I_{\alpha+}^a D_\alpha^b u(a)}{\Gamma(\alpha)} I_\alpha^b (x - a)^{\alpha - 1} - I_\alpha^a (q(x)u(x)) \]
\[ = u(x) - \frac{I_{\alpha+}^a D_\alpha^b u(a)}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1}(t - a)^{\alpha - 1} dt \]
\[ - \frac{1}{\Gamma^2(\alpha)} \int_x^b (s - x)^{\alpha - 1} \left( \int_a^s (s - t)^{\alpha - 1} q(t)u(t) dt \right) ds \]
\[ = u(x) - \frac{I_{\alpha+}^a D_\alpha^b u(a)}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1}(t - a)^{\alpha - 1} dt \]
\[ - \frac{1}{\Gamma^2(\alpha)} \int_a^b q(t)u(t) \left( \int_{\max\{x, t\}}^b (s - x)^{\alpha - 1}(s - t)^{\alpha - 1} ds \right) dt \]
\[ = u(x) - I_{\alpha+}^a D_\alpha^b u(a)K(x, a) - \int_a^b K(x, t)q(t)u(t) dt. \]

From \( u(a) = 0, \) we have
\[ 0 = u(a) - I_{\alpha+}^a D_\alpha^b u(a)K(a, a) - \int_a^b K(a, t)q(t)u(t) dt \]
\[ = -I_{\alpha+}^a D_\alpha^b u(a)K(a, a) - \int_a^b K(a, t)q(t)u(t) dt, \]
then, we get
\[ I_{\alpha+}^a D_\alpha^b u(a) = -\frac{1}{K(a, a)} \int_a^b K(a, t)q(t)u(t) dt. \]
Finally, we establish

\[ u(x) = \int_a^b \left[ K(x, t) - \frac{K(a, t)K(x, a)}{K(a, a)} \right] q(t)u(t)dt \]

\[ = \int_a^b G(x, t)q(t)u(t)dt, \]

completing the proof. \( \square \)

Let us prove one of the main lemmas to show the Lyapunov inequality.

**Lemma 2.2.** Suppose that \( \alpha \in \left( \frac{1}{2}, 1 \right) \). Then, we have

\[ \sup_{a < t < x} G(x, t) = G(x, x) > 0, \quad a < t < x < b, \]

where \( G(x, t) \) is defined in Theorem 2.1.

**Proof.** From Theorem 2.1, we have

\[ G(x, a) = K(x, a) - \frac{K(a, a)K(x, a)}{K(a, a)} = 0. \]

Hence, we get

\[ \frac{\partial G(x, t)}{\partial t} = \frac{\partial K(x, t)}{\partial t} - \frac{K(x, a) \partial K(a, t)}{K(a, a)} \]

Let us compute the first term of the right-hand side of (2.5). Then, we get

\[ \frac{\partial K(x, t)}{\partial t} = \frac{1}{\Gamma^2(\alpha)} \frac{\partial}{\partial t} \int_{\max\{x, t\}}^b (s - x)^{\alpha - 1}(s - t)^{\alpha - 1}ds \]

\[ = \frac{1}{\Gamma^2(\alpha)} \frac{\partial}{\partial t} \int_x^b (s - x)^{\alpha - 1}(s - t)^{\alpha - 1}ds \]

\[ = 1 - \frac{\alpha}{\Gamma^2(\alpha)} \int_x^b (s - x)^{\alpha - 1}(s - t)^{\alpha - 2}ds > 0, \text{ for } a < t < x < b. \]

Therefore, let us compute the second term of (2.5). Using the integration by parts, we obtain

\[ K(a, t) = \frac{1}{\Gamma^2(\alpha)} \int_{\max\{a, t\}}^b (s - a)^{\alpha - 1}(s - t)^{\alpha - 1}ds \]

\[ = \frac{1}{\Gamma^2(\alpha)} \frac{1}{\Gamma^2(\alpha)} \int_t^b (s - a)^{\alpha - 1}(s - t)^{\alpha - 1}ds \]

\[ = \frac{1}{\Gamma^2(\alpha)}(b - a)^{\alpha - 1}(b - t)^\alpha - \frac{\alpha - 1}{\Gamma^2(\alpha)} \int_t^b (s - a)^{\alpha - 2}(s - t)^\alpha ds \]

\[ = \frac{1}{\Gamma^2(\alpha)}(b - a)^{\alpha - 1}(b - t)^\alpha + \frac{1 - \alpha}{\Gamma^2(\alpha)} \int_t^b (s - a)^{\alpha - 2}(s - t)^\alpha ds. \]
Thus, we get
\[
\frac{\partial K(a,t)}{\partial t} = -\frac{1}{\Gamma^2(\alpha)} (b-a)^{\alpha-1}(b-t)^{\alpha-1} - \frac{1-\alpha}{\Gamma^2(\alpha)} \int_t^b (s-a)^{\alpha-2}(s-t)^{\alpha-1} ds < 0.
\] (2.8)

Obviously, we have that the \( K(x,a) > 0 \) for all \( x \in [a,b] \). By combining (2.6) and (2.8), we establish
\[
\frac{\partial G(x,t)}{\partial t} = \frac{\partial K(x,t)}{\partial t} - \frac{K(x,a) \partial K(a,t)}{K(a,a)} > 0.
\]
That is, \( G(x,t) \) is an increasing function in the variable \( t \). From this fact with (2.4), we get
\[
\sup_{a < t < x} G(x,t) = G(x,x) > 0, \quad a < t < x < b,
\]
completing the proof. \( \square \)

3. Rayleigh–Faber–Krahn-Type Inequality on Cylindrical Domains

Let us consider the following eigenvalue problem in cylindrical domain:
\[
L^{\alpha,s}u(x,y) = \nu u(x,y), \quad \text{in } D = (a,b) \times \Omega,
\] (3.1)
with Dirichlet boundary conditions (1.2)–(1.3), where \( \Omega \) is a bounded domain.

First, we give the following auxiliary statement for the application of further research.

**Lemma 3.1.** [19] Let \( \frac{1}{2} < \alpha \leq 1 \). Then, the operator \( D^{\alpha}_{a+}D^{\alpha}_{b-} \) with the domain \( \mathcal{D}(L^{\alpha}) := \{ v \in L^2([a,b]), D^{\alpha}_{a+}D^{\alpha}_{b-} v \in L^2([a,b]), v(a) = v(b) = 0 \} \):
- is self-adjoint and positive in \( L^2([a,b]) \);
- the spectrum is discrete, positive and increasing.

**Theorem 3.2.** Let \( \frac{1}{2} < \alpha \leq 1 \) and \( s \in (0,1) \). Then, the operator (1.1)–(1.3) is self-adjoint and positive in \( L^2(D) \), and all its eigenvalues are discrete, positive and increasing.

The theorem is proved by the method of separation of variables (so-called the Fourier method), by reducing problem (1.1)–(1.3) to two self-adjoint operators: fractional Dirichlet–Laplacian and \( D^{\alpha}_{a+}D^{\alpha}_{b-} \) with Dirichlet boundary conditions.

From [1], we can choose the first eigenfunction of
\[
\begin{cases}
(-\Delta)^s y \varphi_1(y) = \lambda_1(\Omega)\varphi_1(y), & y \in \Omega, \\
\varphi_1(y) = 0, & y \in \mathbb{R}^N\setminus\Omega,
\end{cases}
\] (3.2)
to be positive, corresponding to the simple and positive first eigenvalue \( \lambda_1(\Omega) > 0 \). It is known that the first eigenvalue \( \lambda(\Omega) \) of the fractional Dirichlet–Laplacian (3.2) is minimised in a ball \( B \) among all domains of the same Lebesgue measure (see [1, Theorem 3.5]), i.e.
\[
\lambda_1(\Omega) \geq \lambda_1(B), \quad |\Omega| = |B|.
\] (3.3)
Let us denote by $|\cdot|$ the Lebesgue measure, and let us introduce the Rayleigh quotient for the problem (3.1), (1.2)–(1.3) in the following form:

$$
\nu_1(D) = \inf_{u \neq 0} \frac{\langle L^{\alpha,s}u, u \rangle}{\|u\|_{L^2(D)}^2},
$$

where $\langle \cdot, \cdot \rangle$ is an inner product in $L^2(D)$.

### 3.1. Circular Cylinder Case

In this subsection, we give estimate of the first eigenvalue of (3.1) in the circular cylinder.

**Theorem 3.3.** Suppose that $\frac{1}{2} < \alpha \leq 1$ and $s \in (0,1)$. Then the first eigenvalue of (3.1) is minimised in the circular cylinder $C$ among all cylindrical domains of a given measure, that is

$$
\nu_1(D) \geq \nu_1(C),
$$

for all $D$ with $|D| = |C|$.

**Proof.** By $C = (a, b) \times \mathbb{B}$, we denote the circular cylinder which is the symmetric rearrangement of $D = (a, b) \times \Omega$ with the measure equal to the measure of $D$, where $B \subset \mathbb{R}^N$ is an open ball. Assume that $u$ is a nonnegative measurable function in $D$, such that all its positive level sets have finite measure. Hence, we can use the layer-cake decomposition [14] to give definition of the symmetric-decreasing rearrangement of $u$ in the following form:

$$
u(x, y) = \int_0^\infty \chi_{\{u(x, y) > z\}} \, dz, \quad \forall y \in \Omega,$$

where $\chi$ is the characteristic function of the domain. Then by

$$
u^*(x, y) = \int_0^\infty \chi_{\{u(x, y) > z\}} \cdot dz, \quad \forall y \in \Omega,$$

we define the (radially) symmetric-decreasing rearrangement of $u$.

If a domain $D$ is the cylindrical domain, we can use Fourier’s method, hence we have $u(x, y) = X(x)\varphi(y)$ and $u_1(x, y) = X_1(x)\varphi_1(y)$ is the first eigenfunction of the operator (1.1)–(1.3). Then we have,

$$
\varphi_1(y)D^\alpha_{a+,x}D^\alpha_{b-,x}X_1(x) + X_1(x)(-\Delta)^s_y\varphi_1(y) = \nu_1 X_1(x)\varphi_1(y).
$$

Let us denote $((-\Delta)^sg, g)_\Omega = \left( \int \int g(y)\left(\frac{y-x}{\rho}\right)^{-\frac{1}{2}} \, dx \, dy \right)^{\frac{1}{2}}$. By the variational principle for the self-adjoint positive operator $L^{\alpha,s}$, we get

$$
\nu_1(D) = \int_a^b X_1(x)D^\alpha_{a+,x}D^\alpha_{b-,x}X_1(x) \, dx \int_\Omega \varphi_1^2(y) \, dy + \left( \int_a^b X_1^2(x) \, dx \int_\Omega \varphi_1^2(y) \, dy \right)\nu_1
$$

$$
= \int_a^b X_1(x)D^\alpha_{a+,x}D^\alpha_{b-,x}X_1(x) \, dx \int_\Omega \varphi_1^2(y) \, dy + \lambda_1(\Omega) \int_a^b X_1^2(x) \, dx \int_\Omega \varphi_1^2(y) \, dy
$$

$$
= \int_a^b X_1(x)D^\alpha_{a+,x}D^\alpha_{b-,x}X_1(x) \, dx \int_\Omega \varphi_1^2(y) \, dy + \lambda_1(\Omega) \int_a^b X_1^2(x) \, dx \int_\Omega \varphi_1^2(y) \, dy,
$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the fractional Dirichlet–Laplacian operator (3.2).
For all nonnegative function \( v \in L^2(\Omega) \), we obtain
\[
\int_{\Omega} |v(y)|^2 \, dy = \int_B |v^*(y)|^2 \, dy.
\] (3.6)

Using Theorem A.1 in \([4]\) and (3.6), we establish
\[
\nu_1(D) = \int_B X_1(x) D_{\alpha,+x} D_{\alpha,-x} X_1(x) \, dx \int_\Omega \varphi_1^2(y) \, dy
+ \lambda_1(\Omega) \int_B X_1^2(x) \, dx \int_\Omega \varphi_1^2(y) \, dy
\]
\[
= \frac{\int_B X_1(x) D_{\alpha,+x} D_{\alpha,-x} X_1(x) \, dx \int_B (\varphi_1^*)^2(y) \, dy + \lambda_1(\Omega) \int_B X_1^2(x) \, dx \int_B (\varphi_1^*)^2(y) \, dy}{\int_B X_1^2(x) \, dx \int_B (\varphi_1^*)^2(y) \, dy}
\]
\[
\geq \inf_{r \in (x,y) \neq 0} \| L^{\alpha,s} u_1^, v_1^ \|_{L^2(\Omega)} = \nu_1(C).
\]
The proof is complete. \( \square \)

**Corollary 3.4.** Suppose that \( \frac{1}{2} < \alpha \leq 1 \) and \( s \in (0,1) \). Then first characteristic number \( \mu_1(D) = \frac{1}{\nu_1(D)} \) of (3.1) is maximised in the circular cylinder \( C \) among all cylindric domains of a given measure, that is
\[
\mu_1(D) \leq \mu_1(C),
\] (3.7)
for all \( D \) with \( |D| = |C| \).

### 3.2. Polygonal Cylindric Case

In this subsection, we show the Rayleigh–Faber–Krahn inequality on the triangular and quadrilateral cylinders. First, we recall the definition of the Steiner symmetrization (see \([6]\) and \([2]\)).

Let \( N \geq 2 \) and \( D = (a,b) \times \Omega \subset \mathbb{R}^{N+1} \) be measurable set. By \( D' \), we denote the projection in the \( y_N \)-direction:
\[
D' := \{(x, y_1, \ldots, y_{N-1}) \in \mathbb{R}^N : \text{there exists } y_N \text{ s.t. } (x, y_1, \ldots, y_{N-1}, y_N) \in D\}.
\] (3.8)

Also, for \((x, y_1, \ldots, y_{N-1}) \in \mathbb{R}^N\), we denote by \( D'(x, y_1, \ldots, y_{N-1}) \) the section of \( D \) by the variable \((x, y_1, \ldots, y_{N-1})\):
\[
D'(x, y_1, \ldots, y_{N-1}) := \{y_N \in \mathbb{R} : (x, y_1, \ldots, y_{N-1}, y_N) \in D\},
\]
\[
(x, y_1, \ldots, y_{N-1}) \in D'.
\]

**Definition 3.5.** Let \( D \subset \mathbb{R}^{N+1} \) be a bounded measurable set in \( \mathbb{R}^{N+1} \). The Steiner symmetrization \( D^* = (a,b) \times \Omega^* \) of \( D = (a,b) \times \Omega \) of a measurable set \( D = (a,b) \times \Omega \) with respect to the hyperplane \( y_{N} = 0 \) is defined in by
\[
D^* := \{(x, y_1, \ldots, y_N) : -\frac{1}{2}|D'(x, y_1, \ldots, y_{N-1})| \leq y_N
\]
\[
\leq \frac{1}{2}|D'(x, y_1, \ldots, y_{N-1})|,
\]
\[
(x, y_1, \ldots, y_{N-1}) \in D',
\] (3.9)
where \(|·|\) is the Lebesgue measure.

Let us give definition the Steiner symmetrization of the function \( u \).
Definition 3.6. A nonnegative, measurable function $u^*(x,y)$ on $D^* = (a,b) \times \Omega^*$ is called the Steiner symmetrization of $u(x,y)$ in $D = (a,b) \times \Omega$ which vanishes on $\partial D$ by

$$u^*(x,y) := \sup\{c : (y_1, \ldots, y_N) \in \{z \in \Omega : u(x,z) \geq c\}^*\}.$$  \hspace{1cm} (3.10)

Then, we have the Rayleigh–Faber–Krahn inequality on triangle and quadrilateral cylinders.

Theorem 3.7. [2, Theorem 1.1] The equilateral triangle has the least first eigenvalue for the fractional Dirichlet $p$-Laplacian among all triangles of given measure. The square has the least first eigenvalue for the fractional Dirichlet $p$-Laplacian among all quadrilaterals of given measure. Moreover, the equilateral triangle and the square are the unique minimizers in the above problems.

Let us consider the following eigenvalue problem in triangular (or quadrilateral) cylindrical domain:

$$L^{\alpha,s}u(x,y) = \nu u(x,y), \text{ in } D = (a,b) \times \Omega,$$  \hspace{1cm} (3.11)

with Dirichlet boundary conditions (1.2)–(1.3), where $\Omega$ is a triangle (or quadrilateral). Let us give the main result of this subsection.

Theorem 3.8. Suppose that $\frac{1}{2} < \alpha \leq 1$ and $s \in (0,1)$. Then, first eigenvalue of the (3.11) is minimised in the equilateral triangular (or square) cylinder $D^* = (a,b) \times \Omega^*$ among all triangular (or quadrilateral) cylindric domains of a given measure, that is

$$\nu_1(D) \geq \nu_1(D^*),$$  \hspace{1cm} (3.12)

for all $D$ with $|D| = |D^*|$.

Proof. Since the Steiner symmetrization has the same property as the symmetric-decreasing rearrangement, proof of this theorem is similar to Theorem 3.3, but instead of the symmetric-decreasing rearrangement we use the Steiner symmetrization. \hfill \square

4. Lyapunov and Hartmann–Wintner Inequalities

4.1. Lyapunov Inequality

Let us consider the fractional elliptic equation:

$$L^{\alpha,s}u(x,y) = q(x)u(x,y), \text{ in } D = (a,b) \times \Omega,$$  \hspace{1cm} (4.1)

with boundary conditions (1.2)–(1.3), where $q(x)$ be a real-valued, continuous function.

In this section, we show a Lyapunov-type inequality for (4.1).

Theorem 4.1. Assume that $\frac{1}{2} < \alpha \leq 1$, $s \in (0,1)$ and $q \in C([a,b])$. If (4.1) with boundary conditions (1.2)–(1.3) admits a solution $u \in C(\bar{D})$, $L^{\alpha,s}u \in C(D)$, then we get

$$\int_a^b |q(x) - \lambda_1(\Omega)|dx \geq \left( \sup_{a < x < b} G(x,x) \right)^{-1},$$  \hspace{1cm} (4.2)
where $\lambda_1(\Omega)$ is the first eigenvalue of (3.2).

Proof. From (4.1), we get
\[
\int_\Omega D_{a+,x}^\alpha D_{b-,x}^\alpha u(x,y)\varphi_1(y)dy + \int_\Omega ((-\Delta_y)^{s}u(x,y))\varphi_1(y)dy - q(x)
\]
which yields
\[
\int_\Omega u(x,y)\varphi_1(y)dy = D_{a+,x}^\alpha D_{b-,x}^\alpha \int_\Omega u(x,y)\varphi_1(y)dy + \int_\Omega ((-\Delta_y)^{s}u(x,y))\varphi_1(y)dy - q(x)
\]
which in turn yields
\[
\int_\Omega u(x,y)\varphi_1(y)dy = D_{a+,x}^\alpha D_{b-,x}^\alpha v(x) - q_1(x)v(x) = 0,
\]
where $\varphi_1(y)$ is the first eigenfunction of (3.2), $v(x) = \int_\Omega u(x,y)\varphi_1(y)dy$, $q_1(x) = q(x) - \lambda_1(\Omega)$. Using (1.2), (1.3), we establish $v(a) = 0$, $v(b) = 0$, which yields
\[
D_{a+,x}^\alpha D_{b-,x}^\alpha v(x) - q_1(x)v(x) = 0, \quad x \in (a,b),
\]
$v(a) = 0$, $v(b) = 0$.

By combining Theorem 2.1 and Lemma 2.2, we get
\[
|u(x)| \leq \int_a^b |G(x,t)||q_1(t)||u(t)|dt \leq G(x,x) \left( \sup_{a<t<b} |u(t)| \right) \int_a^b |q_1(t)|dt.
\]
Hence, by taking supremum in both sides in $a < x < b$, we have
\[
\left( \sup_{a<x<b} G(x,x) \right)^{-1} \leq \int_a^b |q_1(t)|dt. \tag{4.3}
\]
Finally, using (4.3), we have
\[
\int_a^b |q_1(x)|dx = \int_a^b |q(x) - \lambda_1(\Omega)|dx \geq \left( \sup_{a<x<b} G(x,x) \right)^{-1}.
\]
The proof of Theorem 4.1 is complete. \qed

Corollary 4.2. By taking $\alpha = 1$, we obtain
\[
\int_a^b |q_1(x)|dx = \int_a^b |q(x) - \lambda_1(\Omega)|dx \geq \frac{4}{b-a}.
\]
Proof. It is easy to see that, $K(x, x) = K(x, a) = b - x$, and $K(a, a) = b - a$. Then using these facts, we have

$$G(x, x) = b - x - \frac{(b - x)^2}{b - a}.$$  

Then, supremum of the function $G(x, x)$ on $a < x < b$ equals to $\frac{b - a}{4}$. □

**Theorem 4.3.** Suppose that $\frac{1}{2} < \alpha < 1$ and $s \in (0, 1)$. If (4.1) with boundary conditions (1.2)-(1.3) admits a solution $u \in C(\bar{D})$, $\mathcal{L}^{\alpha,s}u \in C(D)$, then we get

$$\int_a^b |q(x)| dx + (b - a)\lambda_1(\Omega) \geq \int_a^b |q(x)| dx + (b - a)\lambda_1(B) \geq \left( \sup_{a < x < b} G(x, x) \right)^{-1},$$

where $\lambda_1(B)$ is the first eigenvalue of the eigenvalue problem (3.1) in a ball $B$ with $|\Omega| = |B|$.

**Proof.** By the previous Theorem, let $B$ be an open ball, then using Theorem A.1 in [4], we have

$$\int_a^b |q(x)| dx + (b - a)\lambda_1(\Omega) \geq \int_a^b |q(x)| dx + (b - a)\lambda_1(B) \geq \int_a^b |q(x) - \lambda_1(B)| dx \geq \left( \sup_{a < x < b} G(x, x) \right)^{-1},$$

completing the proof. □

**4.2. Hartman–Wintner Inequality**

In this subsection, we show a Hartman–Wintner-type inequality.

**Theorem 4.4.** Let $\frac{1}{2} < \alpha \leq 1$ and $s \in (0, 1)$, and $q \in C([a, b])$. If (4.1) with boundary conditions (1.2)-(1.3) admits a solution $u \in C(\bar{D})$, $\mathcal{L}^{\alpha,s}u \in C(D)$, then we have

$$\int_a^b (K(a, a)K(s, s) - K^2(a, s)) [q(x) - \lambda_1(\Omega)]^+ ds \geq \frac{(b - a)^{2\alpha - 1}}{\Gamma^2(\alpha)(2\alpha - 1)}, (4.4)$$

where $[q(x) - \lambda_1(\Omega)]^+ = \max\{q(x) - \lambda_1(\Omega), 0\}$.

**Proof.** Similar to the previous theorem, we get:

$$\begin{cases} D^\alpha_{a+}D^\alpha_{b-}v(x) - q_1(x)v(x) = 0, & x \in (a, b), \\ v(a) = 0, v(b) = 0. \end{cases} (4.5)$$
Also, from Theorem 2.1, the problem (4.5) is equivalent to the integral equation

\[ v(x) = \int_a^b G(x, s) q_1(s) v(s) \, ds, \]

where

\[ G(x, t) = \frac{K(x, t)}{\Gamma^2(\alpha)} - \frac{K(a, t) K(x, a)}{\Gamma^2(\alpha) K(a, a)} \]

and

\[ K(x, t) = \frac{1}{\Gamma^2(\alpha)} \int_{\max\{x, t\}}^b (s - x)^{\alpha-1}(s - t)^{\alpha-1} \, ds, \]

and from Lemma 2.2, we have

\[ G(x, s) \leq G(s, s), \quad \text{for } a < x < s < b. \] (4.6)

From the last fact, by (4.6) for any \( a \leq x \leq b \), we establish

\[ |v(x)| \leq \int_a^b |G(x, s)||q_1(s)||v(s)| \, ds \leq \int_a^b G(s, s)|q_1(s)||v(s)| \, ds \]

\[ = \frac{\Gamma^2(\alpha)(2\alpha - 1)}{(b - a)^{2\alpha - 1}} \int_a^b (K(a, a)K(s, s) - K^2(a, s)) \, |q_1(s)||v(s)| \, ds, \]

thanks to \( K(a, a) = \frac{(b - a)^{2\alpha - 1}}{\Gamma^2(\alpha)(2\alpha - 1)} \). Theorem 4.4 is proved. \( \square \)

**Corollary 4.5.** By taking \( \alpha = 1 \) and \( s = 1 \) in (4.4), we get the classical Hartman-Wintner inequality

\[ \int_a^b (b - s)(s - a)q_1^+(s) \geq b - a. \] (4.7)

**Conclusion.** The paper was devoted to the study of a fractional elliptic problem with Dirichlet conditions in the cylindrical domain. For the considered problem the following results have been obtained:

- Rayleigh–Faber–Krahn inequality for circular cylinder;
- Rayleigh–Faber–Krahn inequality for polygonal cylinder;
- Lyapunov inequality;
- Hartmann–Wintner inequality.

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Aidyn Kassymov, Michael Ruzhansky and Berikbol T. Torebek
Ghent University
Ghent
Belgium
e-mail: berikbol.torebek@ugent.be

Aidyn Kassymov and Berikbol T. Torebek
Al-Farabi Kazakh National University
Almaty
Kazakhstan

and

Institute of Mathematics and Mathematical Modeling
Almaty
Kazakhstan

Michael Ruzhansky
Queen Mary University of London
London
UK

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