CLOSING THE GAP OF SECURE QUANTUM KEY RATE WITH THE HERALDED PAIR-COHERENT STATES

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In this paper, we investigate the long-standing gap of quantum key rate between the Weak Coherent Pulse (WCP) and Heralded Single Photon Sources (HSPS) implementation of quantum cryptographical protocol. We prove that, by utilizing the Heralded Pair Coherent State (HPCS) photon sources, such a gap can be actually filled in both BB84 and SARG quantum key distribution. Thus, a universal photon source which achieves the up-to-date optimal key rate for each transmission distance is obtained.

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Quantum Key Distribution (QKD) is a powerful tool which allows two remote partners, Alice and Bob, to establish random and private keys with an unconditional security. However, the unavoidable quantum channel imperfections, such as the high loss of transmission line and the dark counts of single photon detectors, impose severe limitations on both key generation rate and transmission distance of the practical quantum cryptographical applications.

Recently, to give a solution to these problems, much effort has been focused on both cryptography protocol itself and experimental configuration to give a further improvement of the performance of the state-of-art quantum cryptography system. This includes: (1) decoy state method where several coherent state with different intensities are used; (2) searching for more efficient information reconciliation and privacy amplification protocol; (3) using spatial freedom of single photon[11] or by utilizing entanglement photon pairs to transfer more than 1 bit of information[12, 13]; (4) exploring more efficient photon number distribution sources to generate a higher secure key rate[14, 15, 16, 17], and more experimentally, (5) applying the high-speed signal modulation and single photon detection.

Of these efforts, the most efficient and prominent method is to bring (2) and (4) together, i.e., to use decoy state technology and explore its application in different kinds of photon source models. In Ref.[14], N. Lütkenhaus gave a proposal of the superiority of the HSPS photon sources in implementing QKD protocols. Later, following this seminar work, a lot of work has been done to bring such a topic into a much wider collection of variations[15, 20, 21]. It is shown by T. Horikiri et al. that the key rate can be improved with a HSPS sources and the secure distance can be prolonged from 140km to 170km for BB84 quantum cryptographical protocol[13]. Enlightened by such an observation, we gave a further consideration of the Scarani-Acin-Ribordy-Gisin(SARG)[22] protocol and proved that such an enhancement from 95km to 120 km for SARG in the transmission distance can also been obtained[21].

However, behind all these improvement in QKD practical implementation, there exist a long-standing problem that has not been addressed. That is the so-called gap of secure key rate between WCP and HSPS implementations of quantum cryptographical schemes. As is shown in Ref.[21], it is shown that there exists a threshold distance of 136km for BB84 (93km for SARG) beyond which the secret key rate is only enhanced and thus, the distance is greatly prolonged. However, within such a threshold distance, HSPS+BB84 (HSPS+SARG) has a less secure key generating rate than WCP+BB84(WCP+SARG). Put simply, to optimize the final performance, one should resort to WCP sources within the threshold distance but resort to HSPS sources if the distance is threshold-beyond.

In commercialization of QKD, it is important to find a universal photon-emitting sources which is the optimal in secure key generating rate for each transmission distance. Then, a question that naturally arises is that whether there exists a universal photon sources for QKD applications. In this paper we give a proof that the answer is actually affirmative.

Our derivation will be given by investigating the performance of Heralded Pair Coherent State (HPCS) and the corresponding Decoy State method. In this following, we will first start by introducing the famous GLLP formula for key generating rate and also the decoy state method. Then, we will move our attention to the decoy state implementation of BB84 and SARG protocol with HPCS sources. Finally, to obtain our conclusion, we give a numerical simulation with the same experiment parameters taken from[23] as in Ref.[6, 12, 21].

Different from any classical key distributing protocol, the QKD protocol, such as BB84 and SARG, bases on its unconditional security on the fundamentals of quantum physics. However, the security proof of these kind of protocol has not been obtained until many years after their inventions[24]. The first simple proof of the single photon key rate for BB84 is given by Shor and Preskill[25]. This proof is then further extended to explicitly accommodate the imperfections in practical devices, e.g., the
laser sources which occasionally emits multi-photon signals in each emitting. It is shown that the secure key rate for imperfect laser sources can be given by [26]:

$$R_{BB84} \geq -Q_{\mu} f(E_{\mu})H(E_{\mu}) + Q_{1}[1 - H(e_{1})]$$

(1)

where $Q_{\mu}$ is the Gain (the ratio of the number of Bob’s detector counts to the number of signals Alice emits when they are using the same basis) of Bob’s detector and $E_{\mu}$ is the bit error rate (QBER) of sift key when the average photon number of the laser source is $\mu$. $H(x) = -x \log_{2} x - (1 - x) \log_{2} (1 - x)$ is the Binary Shannon function. The function $f(E_{\mu})$ is the efficiency of key reconciliation and is often assumed to be an constant 1.22 for simplicity. $Q_{1}, e_{1}$ is the corresponding gain and error rate contributed by single-photon proportion of the laser source.

In the rest of our paper, we will give a thorough analysis of both BB84 and SARG protocol. So, it is now convenient to introduce the secure consideration relevant to SARG protocol. For SARG protocol, it is known that the security key rate $R_{BB84}$ contains the contribution of both single-photon and two-photon pulses [6, 27]

$$R_{SARG} \geq -Q_{\mu} f(E_{\mu})H(E_{\mu}) + Q_{1}[1 - H(g(e_{1}))]$$

$$+ Q_{2}[1 - H(h(e_{p,2}))]$$

(2)

where $e_{p,2} = e_{2}$ is the phase shift error rate contributed by double-photon pulses and $g(\cdot), h(\cdot)$ are functions that can be obtained in the Appendix of Ref. [6].

In effect, the average gain $Q_{\mu}$ of laser and the error rate $E_{\mu}$ can be directly observed in experiments. To derive the unconditional secure key rate in Eq. (1) and Eq. (2), what one is required to do is only to give an accurate estimation of $Q_{1}, e_{1}, Q_{2}, e_{2}$. However, due to strong channel loss and all kinds of potential and powerful attacks by eavesdroppers, to obtained the required estimation is not a trivial. Fortunately, one can apply the decoy state method and continue our derivation.

The main core of decoy state [6, 10, 11, 12, 13] is to replace the single intensity laser sources with signal states and decoys states which are generated respectively by lasers with different intensities. If the signal state and the decoy state have the same wavelength, timing, and many other physical characters, no eavesdropper will be able to distinguish a decoy state from a signal state successfully. Thus the condition probability that Bob’s detector clicks when a $n$-photon pulse is emitted from Alice will be the same for signal state and decoy state. If we use $Y_{n}$ to denote such a probability, we have $Y_{n}(signal) = Y_{n}(decoy) = Y_{n}$ and the error rate of $n$-photon pulse reads $e_{p}(signal) = e_{p}(decoy) = e_{p}$. Typically, if we denote the intensity of signal state by $\mu$, the decoy state by $\nu_{1}, \nu_{2}, \ldots, \nu_{k}$ (Here we assume, $k$ different decoy state are involved.), by averaging over all the possible $n$-photon pulses, one can obtain that the overall gain $Q_{\mu}, Q_{\nu}$ and over error $EQ_{\mu}, EQ_{\nu}$ follows

$$Q_{\mu} = \sum_{n} Y_{n} P(\mu, n), \quad Q_{\nu} = \sum_{n} Y_{n} P(\nu, n)$$

(3)

$$EQ_{\mu} = \sum_{n} e_{n} Y_{n} P(\mu, n), \quad EQ_{\nu} = \sum_{n} e_{n} Y_{n} P(\nu, n)$$

where $P(\mu, n)$ is the probability that the pulse contains $n$ photon. $E_{\mu}, E_{\nu}$ can be obtained by $E_{\mu} = EQ_{\mu}/Q_{\mu}, E_{\nu} = EQ_{\nu}/Q_{\nu}$. The laser sources will be utilized in encoding of BB84 or SARG quantum signal is Heralded Pair Coherent State (HPCS). In fact, the Pair Coherent State (PCS), which is originally proposed by G. S. Agarwal [28], is in essence another kind of photo-number correlated state. But we will see that its correlation is quite powerful and could induce a fundamental improvement compared with the HSPS state. According to Ref. [16, 28], the PCS is a two-mode correlated coherent and can be written in the Fock basis

$$|\mu\rangle = \frac{1}{\sqrt{I_{0}(2\mu)}} \sum_{n=0}^{\infty} |n\rangle_{1} |n\rangle_{2},$$

(5)

where $\mu \in \mathbb{C}$ and $I_{0}(\cdot)$ is the modified Bessel’s function of the first kind. Interestingly, by tracing over arbitrary one of the two modes, the photon number distribution follows $P_{1} = Tr_{2}[|\mu\rangle\langle\mu|] = \frac{1}{I_{0}(2\mu)} \sum_{n} \mu^{2n} |n\rangle_{1} \langle n|_{1}$, which demonstrates a sub-Poissonian statistics. To decrease the potential multi-photon pulses, we consider the photon heralding technique. After the generation, one mode of the PCS state is sent to a trigger detector (only distinguish click and non-click), and the other one is sent to the Alice’s encoding module. Only when the trigger detector clicks, will the laser signal be deemed to have been sent to Bob. Therefore, when we consider the the probability that $n$-photon pulse is emitted form Alice enclave, the quantum detection efficiency $Q_{A}$ and also the dark count rate $d_{A}$ should be included [13]:

$$P(\mu, n) = \frac{1}{I_{0}(2\mu)} \frac{\mu^{2n}}{(n!)^{2}} [1 - (1 - Q_{A})^{n} + d_{A}]$$

(6)

With the photon distribution in Eq. (1), we will be able to give a general theory of decoy state method for HPCS-based quantum key distribution schemes. The decoy state method we present here is practical in the sense that we can fulfill such a task by only a finite different number of signal state and decoy state. In fact, it will be shown that only one signal state $\mu$ and two decoy state $\nu_{1}, \nu_{2} (1 > \mu > \nu_{1} > \nu_{2}$ and $\nu_{1}^{2} + \nu_{2}^{2} \leq \mu^{2}$) will be enough to give an estimation. First of all, it is sufficient to use only $\nu_{1}, \nu_{2}$ and obtain the estimation of $Y_{1}$ and $e_{1}$.
Form $I_0(2
u_1) \times \nu_1 - I_0(2
u_2) \times \nu_2$, we have

$$I_0(2
u_1)Q_{\nu_1} - I_0(2
u_2)Q_{\nu_2} = Y_1(\eta_A + d_A) + \sum_{n \geq 2} \frac{\nu_1^{2n} - \nu_2^{2n}}{n!^2} [1 - (1 - \eta_A)^n + d_A]$$

$$\leq Y_1(\eta_A + d_A) + \frac{\nu_1^2 - \nu_2^2}{\mu^2} \sum_{n \geq 2} Y_n \nu_2^{2n} [1 - (1 - \eta_A)^n + d_A],$$

where in last line we have applied the relation $\nu_1^{2n} - \nu_2^{2n} \leq (\nu_1^2 - \nu_2^2)\mu^{2n-4}$ for any $n \geq 2$. The lower bound for single photon gain $Y_1$ can be obtained:

$$Y_1 \geq Y_1^{U;\nu_1,\nu_2} = \frac{I_0(2
u_1)Q_{\nu_1} - I_0(2
u_2)Q_{\nu_2}}{(\eta_A + d_A) (\nu_1^2 - \nu_2^2)}.\quad (9)$$

Using a similar way, from $I_0(2
u_1) \times EQ_{\nu_1} - I_0(2
u_2) \times EQ_{\nu_2}$, we have

$$e_1 \leq e_1^{U;\nu_1,\nu_2} = \frac{I_0(2
u_1)EQ_{\nu_1} - I_0(2
u_2)EQ_{\nu_2}}{(\eta_A + d_A) Y_1^2 (\nu_1^2 - \nu_2^2)}.\quad (10)$$

Furthermore, to derive the two-photon gain $Y_2$ and bit error rate $e_2$, one need the help of the gain rate for signal state $Q_{\mu}$:

$$Q_{\mu} = \sum_{n=0}^{\infty} Y_n \frac{1}{I_0(2\mu)} \frac{\mu^{2n}}{n!^2} [1 - (1 - \eta_A)^n + d_A]$$

$$= \sum_{n=0}^{2} Y_n \frac{1}{I_0(2\mu)} \frac{\mu^{2n}}{n!^2} [1 - (1 - \eta_A)^n + d_A]$$

$$+ \sum_{n=3}^{\infty} Y_n \frac{1}{I_0(2\mu)} \frac{\mu^{2n}}{n!^2} [1 - (1 - \eta_A)^n + d_A].\quad (11)$$

With some frustrating algebra manipulation, the two-photon gain and two-photon error rate can be given by

$$Y_2 \geq Y_2^{Est}$$

$$= \frac{2}{[1 - (1 - \eta_A)^2 + d_A] [(a^2 - b^2) - \frac{a^3 - b^3}{c^3}]} [I_0(2
u_1)Q_{\nu_1} - I_0(2
u_2)Q_{\nu_2} - \frac{a^3 - b^3}{c^3} I_0(2\mu)Q_{\mu} +]$$

$$+ \nu_1^2 d_A \frac{a^3 - b^3}{c^3} - Y_1 \eta_A [(a - b) - \frac{a^3 - b^3}{c^3}].\quad (12)$$

$$e_2 < e_2^{Est} = \frac{\nu_1^2 I_0(2
u_1)EQ_{\nu_1} - \nu_2^2 I_0(2
u_2)EQ_{\nu_2}}{Y_2 [1 - (1 - \eta_A)^2 + d_A] \nu_1^2 \nu_2^2 (\nu_1^2 - \nu_2^2)}.$$

Now let’s give some numerical simulation with the experimental parameters taken from Ref. [23]. To move on,
we use the $Q_\mu, EQ_\mu$ in the ideal scenario to replace the actual experiment and investigate the limitations of the key generation rate with HPCS. In precise, for a n-photon pulse which is not disturbed by eavesdropper, the expectation values of yields and bit error rate can be respectively evaluated\cite{[6]}.

For BB84:

$$Y_{BB84} = \frac{[\eta_n + (1-\eta_n)p_{dark}]}{2}, \quad e_{BB84} = \frac{\eta_n \frac{e_{det}}{2} + (1-\eta_n)p_{dark} \frac{1}{2}}{Y_{BB84}},$$  

and for SARG:

$$Y_{SARG} = \eta_n \left( \frac{e_{det}}{2} + \frac{1}{4} \right) + (1-\eta_n)p_{dark} \frac{1}{2}, \quad e_{SARG} = \frac{\eta_n \frac{e_{det}}{2} + (1-\eta_n)p_{dark} \frac{1}{2}}{Y_{SARG}},$$

where $p_{dark}$ is the dark count Bob’s detector and $e_{det}$ stands for the misalignment of the optical instrument. $\eta_n = 1 - (1-\eta)^n$ and $\eta$ denotes the total transmittance of quantum channel and Bob’s enclaves.

Substituting Eqs. (13)\cite{[13]} and Eq. (15) into Eqs. (14)\cite{[14]}, we get $Q_{\mu, BB84} = \left[ \frac{1}{4} - \left( 1 - p_{dark} \right) \xi \right] E_{\mu, BB84}$ \cite{[15]}:  

$$Q_{\mu, BB84} = \frac{e_{det}}{4} \xi - \left( \frac{e_{det}}{4} - \frac{e_{det}}{2} - \frac{e_{det}}{4} \right) \xi, E_{\mu, BB84} = EQ_{\mu, BB84}/Q_{\mu, BB84}.$$ 

Similarly, for SARG protocol, we obtain $Q_{\mu, SARG} = \left( \frac{e_{det}}{2} + \frac{1}{4} \right) \xi - \left( \frac{e_{det}}{2} - \frac{e_{det}}{4} - \frac{e_{det}}{4} \right) \xi, E_{\mu, SARG} = EQ_{\mu, SARG}/Q_{\mu, SARG}$ where 

$$\xi = 1 + d_A - \frac{I_0(2\mu \sqrt{1-\eta_A})}{I_0(\mu)} - \frac{I_0(2\mu \sqrt{1-\eta_A})}{I_0(\mu)} \cdot \frac{I_0(2\mu \sqrt{1-\eta_A})}{I_0(\mu)} \cdot \frac{I_0(2\mu \sqrt{1-\eta_A})}{I_0(\mu)}.$$

This ideal scenario is a good indication of the limitation of the performance of secure key rate. To see this, the single error $e_1$ and $Q_1$ is obtained directly from Eq.(15)\cite{[15]} and Eq.(17)\cite{[17]}. The parameters for trigger detector are $\eta_d = 0.6, d_A = 5 \times 10^{-8}$ as the ones chosen in Ref. [15]. The average photon number for signal state $\mu$ is optimized for each transmittance distance. As is shown in Fig 1, the original threshold distance for BB84 at 136km, doesn’t exist for HPCS state and an average gain of 1.5 in key generating rate can be observed. Fig. 2 the SARG protocol within different implementation of photon sources is depicted. Comparison with the BB84, the average gain is much more prominent, which is about 3.4 times than ever before.

In summary, by investigating the sub-possion photon sources, we show the gap of secrete key rate between the WCP and HSPS implementations can be actually filled. For the Heralded Pair Coherent State, the threshold distance does not exist and can be considered as a convenient and universal photon surces in QKD commercializations. However, we note that here, to look for a convenient and efficient photon sources is still an open an important question for the future robust and high speed QKD application.

After finishing our task we are aware that, more recently, the generation of pair coherent state has also been addressed in Ref \cite{[29]}.

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