Charged spherically symmetric Taub-NUT black hole solutions in $f(R)$ gravity

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$f(R)$ theory is a modification of Einstein general relativity which has many interesting results in cosmology and astrophysics. To derive black hole solution in this theory is difficult due to the fact that it is fourth order differential equations. In this study, we use the first reliable deviation from general relativity which is given by the quadratic form of $f(R) = R + \beta R^2$, where $\beta$ is a dimensional parameter. We calculate the energy conditions of the charged black holes and show that all of them are satisfied for the Taub-NUT spacetime. Finally, we study some thermodynamic quantities such as entropy, temperature, specific heat and Gibbs free energy. The calculations of heat capacity and free energy show that the charged Taub-NUT black hole have positive values which means that it has thermal stability.

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I. INTRODUCTION

The Hilbert-Einstein Lagrangian which involves higher order corrections of Ricci scalar has been used a long time ago as a result of the quantum correction to the gravitational field of the matter source [1]. Due to this feature, it was thought that such terms are successful to describe the early epoch of the growth of our universe. The successful model which describes this period is the Starobinsky one which explains the inflation epoch successfully [2]. Psaltis et al. [3] have obtained important results to test general relativity (GR) in the strong-field regime using astrophysical black holes.

Recently, observations confirm that our universe suffers from acceleration. Since that, the correspondence between the accelerated epoch and the inflationary mechanism help scientists to assume that dark energy may have a geometric origin [4–6]. Later, scientists discovered that the terms which are related to the quantum higher order corrections are responsible for the accelerating expansion rate of our universe at large structures. This phenomenon has been investigated at the fourth order of \( f(R) \) [7–15] (for more references on modified gravity theories as well as dark energy problem, see, e.g., [16, 17]). In this study, we are going to consider \( f(R) = R + \beta R^2 \) which we consider as a physical model due to the following: For this theory, in the early universe, the \( R^2 \) term is dominant and it can lead to the so-called Starobinsky inflation (the \( R^2 \) inflation) [2] (for review on inflation in modified gravity theories, see, e.g., [15]). It is known that Starobinsky inflation can be consistent with the recent Planck result [18]. Moreover, in the late-time universe, the cosmological constant becomes dominant and can play a role of dark energy, so that the accelerated expansion of the universe can occur at the present time [19]. Thus, in the model of \( f(R) = R + \beta R^2 \), both inflation in the early universe and the late-time cosmic acceleration can be realized in a unified manner. This is why we consider the \( R^2 \) model including a cosmological constant in the Lagrangian.

The mechanism which determines the difference between \( f(R) \) gravitational theories and the Einstein GR, that is ensured to be detected at the scales of astrophysics and strong gravitational field, is to derive black holes that are different from those of GR (vacuum or electro-vacuum) [20–25]. An identification to modified gravity theories that can intrinsically share the same solutions with GR has been investigated in [26]. There are many applications carried out on \( f(R) \), among them are: The one-loop effective action of \( f(R) \) theories on the de-Sitter background that has been explained in [27]. One of the most interesting phenomena which can occur in charged black hole solutions is the anti-evaporation process which explains the primordial black hole. In order that the anti-evaporation may occur in the Einstein theory, one needs to involve the quantum correction terms from the matter field. Anti-evaporation could occur at the classical level on modified gravity theories like \( f(R) \) gravity [28, 29]. A subject to acquire solutions for static spherically symmetric black holes in \( f(R) \) gravity has been addressed in [30]. In \( f(R) \) gravity, solutions for spherically symmetric black holes have been obtained by assuming that the scalar curvature is constant [31–33]. Also, spherically symmetric non-charged and charged black hole solutions have been discussed with no constraints on the scalar curvature and the Ricci tensor in [34–36]. One of the merits of \( f(R) \) theory is the fact that it is able to investigate the epoch from inflation to the accelerated expansion [11]. Moreover, \( f(R) \) theory has been tested using many cosmological and astrophysical applications [7, 13, 37] as well as many local tests to constrain it [38–41]. It has been shown that the successful modified gravitational theory is the one that could describe the evolution of our universe, from the big bang to the present time, and to be consistent with the astrophysical prediction given by GR [42, 43].

In spite of the fact that the gravitational field equations of \( f(R) \) gravity are of the fourth order and therefore the non-linear terms become too complicated, many great successful efforts have been achieved. Here, we give a brief summary of these successful efforts: Through the method with the Lagrange multiplier, in \( f(R) \) gravity, a Lagrangian for the gravitational field equations has been derived in spherically symmetric spacetimes [44]. The Friedmann universe filled with a perfect fluid has been investigated for \( f(R) = R^{1+\gamma} \) with \( \gamma \) a constant [45]. This study has demonstrated that the derivatives in the gravitational field equations are at most the first order and it has obtained novel analytic solutions [44]. In addition, static interior solutions for spherically symmetric spacetime have been found in [46]. With the Noether symmetries, the solutions for static spherically symmetric black holes have been analyzed [47]. It has been pointed out that if the universe is filled with a barotropic fluid, the expression of \( f(R) \) cannot be determined by the time evolution of the scale factor [48]. With the Weyl’s canonical coordinates, the static solutions for axially symmetric black holes in vacuum have been acquired [49]. Furthermore, \( d \)-dimensional static spherically symmetric black holes have been obtained using the generator method [50]. For more references on the static spherically symmetric black holes, we refer to [28, 51–75] and references therein. Also, Psaltis et al. and Motohashi and Minamitsuji [3, 26] who are succeeded to put constraints on the modified gravitational theories so the one can easily derive the solutions of GR. However, until now, in \( f(R) \) gravity, there is no study on the solutions for Taub-NUT black hole. Therefore, the main purpose of this work is to use the non-charged and charged gravitational field equations for \( f(R) \) gravity theory with its quadratic form in Taub-NUT spacetime and try to find new solutions.

The properties of gravitomagnetic of the Taub-NUT spacetime are characterized by the non-diagonal metric whose source comes from the NUT parameter \( \chi \). This non-diagonal expression yields a singularity on the half-axis \( \theta = \pi \), which is called Misner string, that is different from ordinary coordinate singularity which is related to the use of spherical coordinates. Misner [76] supposed to elude such singularity is to include a periodic time coordinate and two coordinate patches. The first one covers the northern hemisphere and the singularity is located along the axis \( \theta = \pi \), while the other patch covers the southern hemisphere with the singularity extending along the axis \( \theta = 0 \). The price paid for a Taub-NUT spacetime to be free from these axial singularities is the use of a periodic time coordinate [77].
The use of periodic identification of the time coordinate makes the Taub-NUT solution has a problem for physical applications because of causality violations. Bonnor [78] suggested another explanation of the Taub-NUT space-time, to avoid the unphyscial property of the Misner explanation. He preserved the singularity at \( \theta = \pi \), and endowed it with a semi-infinite massless rotating rod. In this investigation, the NUT space-time is being created by a spherically symmetric body of mass \( M \) at the origin and by a source of pure angular momentum which is uniformly distributed along the \( \theta = \pi \) axis.

The organization of the present work is as the following. In Section II, the ABC of \( f(R) \) gravity is presented. The gravitational field equations for \( f(R) \) gravity with its quadratic form are applied to the Taub-NUT spacetime and derived their non-linear differential equations. We solve this system of differential equations and derive an exact solution. In Section III, the charged gravitational field equations in \( f(R) \) gravity is given and applied to the Taub-NUT spacetime considered in Section II. We also solve the charged field equations analytically and derive exact solutions. In Section IV, the relevant physics of black holes derived in Section II and III are analyzed by calculating their singularities and energy conditions. In Section V, we investigate thermodynamics of black holes derived in Sections II and III and demonstrate that locally, the solution of charged black holes is stable. In the final section, we discuss the results of the present study.

II. ABC OF \( f(R) \) GRAVITY

The Lagrangian of \( f(R) \) gravity has the form

\[
L_g := \frac{1}{2\kappa} \int d^4 x \sqrt{-g} (f(R) - \Lambda),
\]

where \( \kappa = 8G\pi \) is the Einstein gravitational constant and \( G \) is the Newtonian gravitational constant. In Eq. (1), \( R \) is the Ricci scalar, \( g \) is the determinant for the metric tensor \( g_{\mu\nu} \) and \( f(R) \) is a function which is analytic and differentiable. The variations of the Lagrangian (1) in terms of \( g_{\mu\nu} \) lead to the gravitational field equations in vacuum [27, 79]

\[
S_{\mu\nu} = R_{\mu\nu} f_R - \frac{1}{2} g_{\mu\nu} f(R) - 2 g_{\mu\nu} \Lambda + g_{\mu\nu} \Box f_R - \nabla_\mu \nabla_\nu f_R = 0,
\]

where \( R_{\mu\nu} \) is the Ricci tensor. The operator \( \Box \) is the D’Alembertian operator that is defined as \( \Box = \nabla_\alpha \nabla^\alpha \) where \( \nabla_\alpha W^\beta \) is the covariant derivatives of the vector \( W^\beta \) and \( f_R = \frac{dR(\kappa)}{d\kappa} \). The trace of equation (2) leads to:

\[
R f_R - 2 f(R) - 8 \Lambda + 3 \Box f_R = 0.
\]

The solution of Eq. (3) for constant Ricci scalar has the form [80, 81]

\[
R = -8 \Lambda.
\]

Finally, since the power-law of \( f(R) \) is the one with best agreement with cosmological data [82, 83], therefore, in the following sections we are going to focus our attention to the choice

\[
f(R) = R + \beta R^2,
\]

with \( \beta \) being the dimensional model parameter.

A. Taub-NUT spacetime

We consider spacetimes whose metric can be written locally in the form

\[
d\Sigma^2 = -s(r)dt^2 + \frac{1}{N_1(r)}dr^2 + k(r)d\Sigma^2 + 2\chi s(r)(\theta) [dt - 2\chi k_1(\theta) d\phi] d\phi,
\]

where \( s(r), k(r) \) and \( N_1(r) \) are arbitrary functions and \( \chi \) is the Taub-NUT parameter. Here \( d\Sigma^2 \) is a 2-dimensional Einstein-Kähler manifold, which can be taken to be the unit sphere \( S^2 \), torus \( T^2 \) or the hyperboloid \( H^2 \) which respectively have the form [84]:

\[
k_1(\theta) = \begin{cases} \cos \theta, & \text{for } \delta = 1 \text{ sphere}, \\ \theta, & \text{for } \delta = 0 \text{ torus}, \\ \cosh \theta, & \text{for } \delta = -1 \text{ hyperboloid}. \end{cases}
\]
Here in this study we are interested in the case of unit sphere $S^2$. Therefore the above metric takes the form

$$ds^2 = -s(r)dt^2 + \frac{1}{N_1(r)}dr^2 + k(r)\left(d\theta^2 + \sin^2 \theta d\phi^2\right) + 2\chi s(r)\cos \theta (dt - 2\chi \cos \theta d\phi) d\phi,$$

which is the Taub-NUT spacetime. By substituting Eq. (2) into Eq. (7), we acquire

$$S_i' = \frac{\mathcal{N}(k^2 N_1 s' - k^2 s' N_1') - 2k^2 s N_1 s'' - 2ks N_1 k' - 8s^3 \chi^2 + 8k^2 s^2 \Lambda}{4k^2 s^2} = 0,$$

$$S_r' = \frac{\mathcal{N}(k^2 N_1 s' - 2k^2 s N_1 s'') - sk^2 s' N_1' + 2s^2 N_1 k' - 4ks^2 N_1 k' - 2ks^2 k' N_1' + 8k^2 s^2 \Lambda}{4s^2 k^2} = 0,$$

$$S_\theta = S_\phi = \frac{\mathcal{N}(2sk N_1 k'' + kk' (s N)' - 8s^2 \chi^2 - 4ks[1 + 2k\Lambda])}{4sk^2} = 0,$$

$$S_\phi = \frac{\chi N \cos \theta (2ks^2 N_1 k'' - 2k^2 s N_1 s'' + k^2 N_1 s' - sk' (s' N_1 - s N')) - k^2 s^2 s' N_1' - 4s^2 (2\chi^2 s + k)}{2k^2 s^2} = 0,$$

where $\mathcal{N} = (1 - 16\beta \Lambda)$ and we have used Eq. (4). The solution for Eq. (8) when $\mathcal{N} \neq 0$ become

$$k(r) = \varrho^2, \quad s(r) = s_1(r) N_1(r), \quad \text{where} \quad s_1(r) = \frac{r^2}{c_1 \varrho^2 - \chi^2},$$

$$N_1(r) = \frac{3c_2 c^3 \sqrt{c_1 \varrho^2 - \chi^2} + c_1^2 \varrho^2 [2\Lambda \varrho^2 + 3] + 2c_1 \chi^2 [4\Lambda \varrho^2 - 3] - 16\chi^4 \Lambda [c_1 \varrho^2 - \chi^2]}{3c_1^3 r^2 \varrho^2},$$

where

$$\varrho = \sqrt{r^2 + \chi^2}.$$

It is of interest to note that when the constant $c_1 = 1$ the function $s_1(r) = 1$ and the functions $s(r)$ and $N_1(r)$ will be identical. The horizon of Eq. (9) is shown in figure 1(a), for $r > 0$, that corresponds to a black hole horizon. The metric of solution (9) takes the form

$$ds^2 = \left\{\frac{3c_2 c^3 \sqrt{c_1 \varrho^2 - \chi^2} + c_1^2 \varrho^2 [2\Lambda \varrho^2 + 3] + 2c_1 \chi^2 [4\Lambda \varrho^2 - 3] - 16\chi^4 \Lambda}{3c_1^3 \varrho^2}\right\} dt^2 + \varrho^2 d\theta^2$$

$$+ \frac{3c_1^3 \varrho^2}{(c_1 \varrho^2 - \chi^2) [3c_2 c^3 \sqrt{c_1 \varrho^2 - \chi^2} + c_1^2 \varrho^2 [2\Lambda \varrho^2 + 3] + 2c_1 \chi^2 [4\Lambda \varrho^2 - 3] - 16\chi^4 \Lambda]} d\theta^2$$

$$- \frac{3c_1^3 \varrho^2 \sin^2 \theta - 12\chi^2 c_2 c^3 \cos^2 \theta \sqrt{c_1 \varrho^2 - \chi^2} - 4\chi^2 \cos \theta c_1 \varrho^2 [2\Lambda \varrho^2 + 3] - 2c_1 \chi^2 [4\Lambda \varrho^2 - 3] - 16\chi^4 \Lambda}{3c_1^3 \varrho^2} d\phi^2$$

$$+ 4\chi \cos \theta \frac{3c_2 c^3 \sqrt{c_1 \varrho^2 - \chi^2} + c_1^2 \varrho^2 [2\Lambda \varrho^2 + 3] + 2c_1 \chi^2 [2\Lambda \varrho^2 - 3] - 16\chi^4 \Lambda}{3c_1^3 \varrho^2} dtd\phi.$$

Equation (10) shows that solution (9) behaves asymptotically as AdS/dS. When $c_1 = 1$, the metric spacetime (10) is singular at $\varrho = 0$ and $r = 0$. Moreover, the metric (10) has another singularity at

$$3c_2 r + \varrho^2 [2\Lambda \varrho^2 + 3] + 2\chi^2 [4\Lambda \varrho^2 - 3] - 16\chi^4 \Lambda = 0.$$

Equation (11) is a fourth algebraic equation that has two real positive solutions.

Calculating all the invariants of solution (9) we get

$$R^\alpha_\beta R_\alpha^\beta = \frac{\mathcal{F}_1(r)}{3\varrho^2}, \quad R^\alpha_\beta R_\alpha^\beta = 16\Lambda^2,$$

with $\mathcal{F}_1(r)$ being a lengthy polynomial function and we have put $c_1 = 1$ to make the line element (10) has a well-known asymptote behavior. Equation (12) shows that a true singularity exists when $\varrho = 0 \Rightarrow r = 0$. It is interesting to note that the singularity that arise from Eq. (11) and makes the metric (10) singular does not make the Kretschmann invariant and squared Ricci of Eq. (12) divergent.
III. CHARGED TAUB-NUT BLACK HOLE SOLUTION

The Lagrangian of $f(R)$ gravity with a coupling between geometry and matter takes the form

$$ \mathcal{L} := \mathcal{L}_g + \mathcal{L}_{em}, \quad (13) $$

with $\mathcal{L}_g$ being the Lagrangian of the gravitation that is given by Eq. (1). The Lagrangian of matter is given by

$$ \mathcal{L}_{em} := -\frac{1}{2} F \wedge *F, \quad (14) $$

where $F = dA$ with $A = A_\mu dx^\mu$ the 1-form of the gauge potential [23] and $*F$ is the dual of $F$. The variations of the action (14) in terms of $g_{\mu\nu}$ and the vector potential $A_\mu$ give the field equation in the Maxwell-$f(R)$ gravity as [27, 79]:

$$ R_{\mu\nu}f_{R} - \frac{1}{2} g_{\mu\nu}f(R) - 2g_{\mu\nu}\Lambda + g_{\mu\nu}\square f_{R} - \nabla_\mu \nabla_\nu f_{R} = 2kT_{\mu\nu}, \quad \partial_\nu \left( \sqrt{-g} F^{\nu\rho} \right) = 0, \quad (15) $$

with $T_{\mu\nu}$ being the energy-momentum tensor for the Maxwell field, defined as

$$ T_{\mu}^{\nu} := g_{\nu\rho}F^{\rho\mu} - \frac{1}{4} \delta_\nu^{\rho} g_{\mu\rho} F_{\alpha\beta} F^{\alpha\beta}. \quad (16) $$

The trace of the field equations (15) gives Eq. (3).

A. Charged solution for Taub-NUT spacetime

By combining Eq. (15) with the spacetime (7), we get

$$ S' = \frac{N(k^2N_1s^2 - kss'[2k'N_1 + kN'_1] - 2k^2sN_1s'' - 8s^2\chi^2 + 8k^2s^2\Lambda) + 8sN_1k^2q^2 + 16s^2\chi^2q^2 + 8s \csc \theta_0 [\csc \theta_0 - 2q\chi]}{4k^2s^2} = 0, \quad (17) $$

where Eq. (4) is used and we assume the vector potential has the form

$$ A = q(r)d\tau + [2\chi q(r) \cos(\theta) + h(\theta)]d\phi. \quad (18) $$

Equation (18) shows that the vector potential consists of the electric field as well as the magnetic field. The solution of Eq. (17) takes the form

$$ k(r) = \theta^2, \quad s(r) = s_1(r)N_1(r), \quad \text{where} \quad s_1(r) = \frac{\theta^2}{c_3 \theta^2 - \chi^2}, $$

$$ N_1(r) = \frac{c_3 \theta^2 - \chi^2}{3c_3 r^2 \theta^2 N^2} \left( 3c_3^4 [c_3^2 + 4c_5^2 \chi^2] + 3c_3^3 [4c_5 c_5 \chi - c_7 N \sqrt{c_3 \theta^2 - \chi^2}] + c_3^2 \left[ 2\theta^2 \Lambda (\theta^2 - 24\theta + \chi^2) - 62\theta^4 \Lambda^2 + 3(c_3^2 + \theta^2) \right] + 4\chi^2 N [4\Lambda^2 - 3] - 16\chi^4 \Lambda N \right) b(\theta) = c_6 \cos \theta, \quad q(r) = \frac{1}{c_3 \theta^2} \left( c_3 c_5 \theta^2 - 2\chi^2 \right) + c_3 c_4 \sqrt{c_3 \theta^2 - \chi^2} - c_6 \chi. \quad (19) $$
The horizons of solution (19) are plotted in figure 1(b) which shows two horizons, one for the event and the second is the cosmological horizons. Using Eq. (7) then metric spacetime of solution (19) takes the form

\[
ds^2 = \left( -\frac{1}{3c_3^3q^2N} \left[ 3c_3^4[4c_4^2 + 4c_5^2\chi^2] + 3c_3^3[4c_6c_5\chi + c_7N\sqrt{c_5^2q - \chi^2}] + c_3^2[2q^2\Lambda(r^2 - 24\beta + \chi^2) - 32\beta q^4\Lambda^2] \\
+ 3(c_6^2 + r^2 + \chi^2) \right] + 2c_3\chi^2[4\Lambda q^2 - 3] - 16\chi^4\Lambda N \right) \frac{d\tau}{\Delta_1(r)} + \frac{d\phi}{d\tau} + \left\{ \frac{c_6^2 - r^2}{3c_3^3r^2q^6N} \left[ 3c_3^4[c_4^2 + 4c_5^2\chi^2] + 3c_3^3[4c_6c_5\chi + c_7N\sqrt{c_5^2q - \chi^2}] + c_3^2[2q^2\Lambda(r^2 - 24\beta + \chi^2) - 32\beta q^4\Lambda^2] + 3(c_6^2 + r^2 + \chi^2) \right] + 2c_3\chi^2[4\Lambda q^2 - 3] - 16\chi^4\Lambda N \right\}^{-\frac{1}{2}} \frac{d\phi}{d\tau}.
\]

From Eq. (20), we can see that solution (19) behaves asymptotically as AdS/dS and informs us that it is a new solution that depends on the dimension parameter \( \beta \) that must satisfies

\[ N \neq 0 \Rightarrow \beta \neq \frac{1}{16\Lambda}. \]

Equation (20) is singular at \( q = 0 \) and \( r = 0 \). Moreover, the metric (20) has another singularity at

\[ 3c_3^4[c_4^2 + 4c_5^2\chi^2] + 3c_3^3[4c_6c_5\chi + c_7N\sqrt{c_5^2q - \chi^2}] + c_3^2[2q^2\Lambda(r^2 - 24\beta + \chi^2) - 32\beta q^4\Lambda^2] + 3(c_6^2 + r^2 + \chi^2) \]

\[ + 2c_3\chi^2[4\Lambda q^2 - 3] - 16\chi^4\Lambda N \equiv 0. \]

Equation (21) is a fourth order algebraic equation that has at least two real positive roots.

The Kretschmann invariant, squared Ricci and Maxwell field of solution (19) take the form

\[
R^\mu\nu\rho\sigma R_{\mu\nu\rho\sigma} = \frac{F_2(r)}{N^2q^{12}}, \quad R^\mu\nu R_{\mu\nu} = \frac{F_3(r)}{N^2q^8}, \quad F^\mu\nu F_{\mu\nu} = \frac{F_4(r)}{q^8},
\]

where \( F_2(r) \), \( F_3(r) \) and \( F_4(r) \) are polynomial functions and we have put \( c_3 = 1 \). It is understood from Eq. (22) that a singularity exists at \( q = 0 \Rightarrow r = 0 \) when \( \chi = 0 \). It is important to mention that the singularity that arise from Eq. (21) and makes the metric (20) singular does not make the Kretschmann invariant and squared Ricci of Eq. (22) divergent.

IV. PHYSICAL PROPERTIES OF THE BLACK HOLE SOLUTIONS

We investigate the Taub–NUT non-charged case to the metric (10) which can be rewritten as\(^1\)

\[
ds^2 = -\Delta_1(r)dt^2 + g^2 \frac{d\theta^2}{\Delta_1(r)} + \frac{dr^2}{\Delta_1(r)} + \left[ g^2 \sin^2 \theta - 4\chi^2 \Delta_1(r) \cos^2 \theta \right] d\phi^2 + 4\chi \Delta_1(r) \cos \theta \frac{d\phi}{d\tau},
\]

where \( c_2 = -2m \) and

\[ \Delta_1(r) = 1 + \frac{2\Lambda q^2}{3} + \frac{2\chi^2[4\Lambda q^2 - 3]}{3q^2} - \frac{16\chi^4\Lambda}{3q^2} - \frac{2mr}{q^2} = 0. \]

\(^1\) The family of the Taub–NUT spacetimes is given by

\[
ds^2 = -\Delta(r)dt^2 + g^2 \frac{d\theta^2}{\Delta(r)} + \frac{dr^2}{\Delta(r)} + \left[ g^2 \sin^2 \theta - 4\chi^2 \Delta(r) \cos^2 \theta + C^2 \right] d\phi^2 - 4\chi \Delta(r) \cos \theta \frac{d\phi}{d\tau},
\]

where \( C \) is an additional parameter related to the large coordinate transformation \( t \rightarrow t + C\phi \). Note that \( C \) should be considered as physical rather than pure gauge parameter, since it changes the asymptotic behavior of the metric. Its introduction was often used to modify the position of the Minser string; for \( C = -1 \) it lies at the southern hemisphere, for \( C = 1 \) at the northern and for \( C = 0 \) at both of them. Equation (23) belong to \( C = 0 \) Taub-NUT family and in that case Minser string lies on both of the northern and southern hemispheres [85].
In the limiting case when $\chi \to 0$ Eq. (23) reduces to Schwarzschild AdS/dS spacetime [86]. The number of the horizons of Eq. (23) are the positive real roots of $\Delta_1(r) = 0$ which has four roots, two of them are real that represent the inner and outer horizons. The metric (23) is singular at $\mathcal{O} = 0 \to r = 0$ and $\chi = 0$ in addition to $\Delta_1(r) = 0$. If we want to overcome the problem encountered in the coordinate $(t, r, \theta, \phi)$ system, singularity at $\Delta_1(r) = 0$, an obvious place to start is the time-coordinate $t$. We will replace $t$ by a coordinate $(v, r, \theta, \phi)$, where $v = t + r$, and $r$ is defined by

$$dr_v = \Delta_1^{-1}(r)dr.$$

(24)

In terms of the new coordinates the line-element (23) becomes

$$ds^2 = -\Delta_1(r)dv^2 + 2dvdr + \mathcal{g}^2d\theta^2 + \left[\mathcal{g}^2 \sin^2 \theta - 4\chi^2 \Delta_1(r) \cos^2 \theta\right]d\phi^2 + 4\chi \cos \theta [\Delta_1(r)dv - dr]d\phi.$$

(25)

Equation (25) shows that there are no more factors of $\Delta_1(r)$ in the denominator, and the metric is regular at the inner and outer horizons. The only remaining singularity is the curvature singularity at $\mathcal{O} = 0$.

For the charged Taub-NUT the metric (20) can be rewritten as

$$ds^2 = -\Delta_2(r)dv^2 + \mathcal{g}^2d\theta^2 + \frac{dr^2}{\Delta_2(r)} + \left[\mathcal{g}^2 \sin^2 \theta - 4\chi^2 \Delta_2(r) \cos^2 \theta\right]d\phi^2 + 4\chi \cos \theta [\Delta_2(r)dv - dr]d\phi,$$

(26)

where we have used $c_1 = 1, c_4 = q/2, c_5 = q_1, c_6 = q/2$ and $c_7 = -2m$ to ensure a flat spacetime when $\chi \to 0, \beta \to 0$ and $\Lambda \to 0$. Here $\Delta_2$ is defined as

$$\Delta_2(r) = \frac{1}{3q^2\mathcal{N}} \left[3[q^2 + 4q_1^2 \chi^2] + 3[2q_1q_\chi - 2mr\mathcal{N}] + 2\mathcal{g}^2\Lambda (\mathcal{g}^2 - 24\beta) - 32\beta\mathcal{g}^4\Lambda^2 + 3\mathcal{g}^2 - 16\chi^4\Lambda\mathcal{N} + 2\chi^2\mathcal{N}[4\Lambda\mathcal{g}^2 - 3]\right].$$

In the limiting case $\chi \to 0$ and $\beta \to 0$, Eq. (26) reduces to Reissner-Nordström AdS/dS spacetime [86]. The most interesting thing is the fact that the charge $q_1$ is accompanied with the dimension parameter $\chi$. So, if $\chi = 0$ then the charge $q_1$ will disappear however, the inverse is not correct, i.e., when $q_1 \to 0$ then $\chi$ will not disappear. As usual the number of horizons of the spacetime (26) are the positive roots of

$$\Delta_2(r) = 3[q^2 + 4q_1^2 \chi^2] + 3[2q_1q_\chi - 2mr\mathcal{N}] + 2\mathcal{g}^2\Lambda (\mathcal{g}^2 - 24\beta) - 32\beta\mathcal{g}^4\Lambda^2 + 3\mathcal{g}^2 - 16\chi^4\Lambda\mathcal{N} + 2\chi^2\mathcal{N}[4\Lambda\mathcal{g}^2 - 3] = 0,$$

which has two real roots. Using the procedure applied in the neutral spacetime we can show that the line-element (26) can has the form

$$ds^2 = -\Delta_2(r)dv^2 + 2dvdr + \mathcal{g}^2d\theta^2 + \left[\mathcal{g}^2 \sin^2 \theta - 4\chi^2 \Delta_2(r) \cos^2 \theta\right]d\phi^2 + 4\chi \cos \theta [\Delta_2(r)dv - dr]d\phi,$$

(27)

where $v$ in this case has the same form given in the neutral case and $r_v$ is defined by

$$dr_v = \Delta_2^{-1}(r)dr.$$

(28)

Now we consider the energy conditions that have the following constrains [34]:

- **SCEC**: $\rho + p_r \geq 0$, $\rho + p_t \geq 0$, $\rho + p_r + 2p_t \geq 0$,

- **WEC**: $\rho \geq 0$, $\rho + p_r \geq 0$, $\rho + p_t \geq 0$,

- **NEC**: $\rho + p_r \geq 0$, $\rho + p_t \geq 0$,

- **DEC**: $\rho \geq 0$, $\rho \geq 0$, $\rho \geq 0$,

(29)

where $T_0^0 = \rho$ is the density, $T_{11} = p_r$ is the radial pressure, and $T_{22} = T_3^3 = p_t$ is the tangential pressure. Straightforward calculations of charged Taub-NUT black hole solution (19) gives

- **Strong Energy Condition**: $\rho + p_r = \frac{2(qq_1\chi - 2q_1^2 \chi^2 - q^2)}{q^4} \geq 0$, $\rho + p_r + 2p_t = 0$,

- **Weak Energy Condition**: $\rho = \frac{2qq_1\chi - 2q_1^2 \chi^2 - q^2}{q^4} \geq 0$, $\rho + p_r = \frac{2qq_1\chi - 2q_1^2 \chi^2 - q^2}{q^4} \geq 0$, $\rho + p_t = 0$,

- **Null Energy Condition**: $\rho + p_r = \frac{2qq_1\chi - 2q_1^2 \chi^2 - q^2}{q^4} \geq 0$, $\rho + p_t = 0$,

- **Dominant Energy Condition**: $\rho = \frac{2qq_1\chi - 2q_1^2 \chi^2 - q^2}{q^4} \geq 0$, $\rho - p_r = 0$, $\rho + p_t = 0$.

(30)

Hence, all of the energy conditions are satisfied provided that $qq_1 > \frac{2q_1^2 \chi^2 + q^2}{q^4}$. We have shown that the charged Taub-NUT black hole configuration may fulfil the energy conditions.
(a) The Taub-NUT non-charged case

(b) The Taub-NUT charged case

FIG. 1. Schematic plots of $N(r)$ and $s(r)$ that characterize the horizons of the black holes by setting $N(r) = 0$ and $s(r) = 0$: (a) For the Taub-NUT non-charged case, the function $s(r)$ is given by (9); (b) For the Taub-NUT charged case, the function $s(r)$ is given by (19).

V. THERMODYNAMICS OF THE DERIVED BLACK HOLES

To study different thermodynamical properties [87–91] of the black hole solutions given by Eqs. (9) and (19), we start by obtaining roots for $N(r) = 0$ as well as $s(r) = 0$. These horizons can be seen in Figs. 1(a) and 1(b).

The Bekenstein-Hawking entropy is represented as

$$S(r_h) = \frac{1}{4} A = \pi r_h^2,$$

where $A$ is the area of the black hole horizon. The stability of the black hole thermodynamics of Eqs. (9) and (19) can be tested by exploring the heat capacity, $C_h$, which is defined as [92, 93]

$$C_h = \frac{\partial m}{\partial T} = \frac{\partial m}{\partial r_h} \left( \frac{\partial r_h}{\partial T} \right).$$

(32)

If $C_h > 0$, thermodynamics for black holes is stable. On the other hand, for $C_h < 0$, it is unstable. To understand this, we assume that at some point and due to thermal fluctuations, the black hole absorbs more radiation than it emits which makes the heat capacity positive. On the contrary, when the black hole emits more radiation than it absorbs, the heat capacity becomes negative. Thus, the black holes with negative heat capacities are thermally unstable.

In order to evaluate Eq. (32) we must calculate the black hole mass within the inner horizon $r_h$. To this end we set $N(r_h) = 0$, for Taub-NUT and charged Taub-NUT spacetimes, and $s(r_h) = 0$, for Taub-NUT and charged Taub-NUT spacetimes, then we get

$$m_h = \frac{1}{6r_h} \left[ 2\Lambda r_h^4 + 3r_h^2(1 + 4\chi^2\Lambda) - 3\chi^2(1 + 2\chi^2\Lambda) \right],$$

$$m_+ = \frac{1}{6r_h^2} \left[ 3(\varrho_h^2 + q^2) - 6\chi^2 + 6q_1(q + 2q_1) - 2\Lambda(3\chi^4 + 24\beta r_h^2 - 6\chi^2[r_h^2 + 4\beta] - r_h^4) - 32\beta \Lambda (r_h^4 + 6\chi^2r_h^2 - 3\chi^4) \right],$$

(33)

where $\varrho_h = \sqrt{r_h^2 + \chi^2}$. We plot the black hole mass within the radius of the horizon $r_h$ in Fig. 2. The black hole size varies between the inner $r_h$ and cosmological $r_c$ horizons [94, 95].

The black hole Hawking temperature is acquired by the request that there is no singularity at the Euclidean horizon. While, the temperature at the a black hole horizon $r = r_h$ is given by [96]

$$T = \frac{\kappa}{2\pi},$$

where $\kappa$ is the surface gravity, represented as $\kappa = \frac{N'(r_h)}{2} = \frac{s'(r_h)}{2}$.

(34)
The Hawking temperatures for (9) and (19) are expressed as

\[ T_h^{(9)} = \frac{1}{4\pi r_h} \left[ 2\Lambda \rho_h^2 + 1 \right], \]
\[ T_h^{(19)} = \frac{1}{6r_hN} \left[ 3\rho_h^2 + q^2 \right] - 6\chi [\chi - 2\chi q_1^2 - qq] - 32\beta\Lambda^2 (r_h^4 + 6r_h^2\chi^2 - 3\chi^4) - \Lambda(6\chi^4 - 12\chi^2[r_h^2 - 4\beta] + 2r_h^2[24\beta - r_h^2]) \]

(35)

where \( T_h \) is the Hawking temperature for the cosmological horizon. We plot the temperatures \( T_h \) in Fig. 3. We show that \( r_{\text{min}} \) at which \( T_h \) vanishes for the NUT case, but that the ultra-cold black holes are considered for \( r_h < r_{\text{min}} \). With the effect of gravity for thermal radiation, in a very-high temperature \( T_{\text{max}} \), thermal radiation becomes unstable, so that it could collapse to black holes [97]. As a result, only for \( T < T_{\text{max}} \), the solution for the pure AdS is stable. Above \( T_{\text{max}} \), black holes with those very heavy masses could be stable [97].
We calculate the heat capacity, after substituting Eqs. (33) and (35) into Eq. (32) and get

\[ C_{h_{q}} = \frac{2\pi q_{1}^{2}(2\Lambda q_{1}^{2} + 1)}{2\Lambda(r_{h}^{2} - \chi^{2}) - 1}, \]

\[ C_{h_{q}} = \frac{2\pi[2\chi q_{1} - 2\chi^{2}N - \chi^{2}(N[1 + 4r_{h}^{2}\Lambda] + 4q_{1}^{2}) - 2N\Lambda r_{h}^{4} - q_{1}^{4}(Nr_{h}^{2} - q_{1}^{2})]}{2\Lambda r_{h}^{4} + \chi^{4}[N(2r_{h}^{2}\Lambda + 1) - 4q_{1}^{2}] - 2q_{1}\chi^{2} - \chi^{2}(2\Lambda r_{h}^{4}N - 2r_{h}^{2}[N - 6q_{1}^{2}] + q_{1}^{2}) - 6r_{h}^{2}q_{1} + (1 - 2\Lambda r_{h}^{2})r_{h}^{4}N - 3q_{1}^{2}r_{h}^{2}}. \]

(36)

To have information directly from Eq. (36) is not easy, so we plot them in Fig. 4 for particular values of the parameters of black holes. For the non-charged case, the heat capacity is negative when \( r_{h} < r_{\text{min}} \) at which the temperature has a vanishing value. For the case \( r_{h} > r_{\text{min}} \), the heat capacity become positive, so that the solution can locally be stable. The same conclusion is valid for the charged Taub-NUT black holes. Note that all the above heat capacity are characterized by a second-order phase transition [98, 99] as \( C_{h} \) diverges at some critical value \( r_{h} < r_{\text{min}} \). In conclusion, Eq. (33) shows that \( \partial m_{h}/\partial r_{h} > 0 \), while the sign for the heat capacity is equal to that for \( \partial T/\partial r_{h} \). Consequently, we find that \( C_{h} < 0 \) when \( r_{h} < r_{\text{min}} \) and \( C_{h} > 0 \) when \( r_{h} > r_{\text{min}} \). In this sense, there are two possible black hole solutions for a given temperature \( T > T(r_{\text{min}}) \), but only the bigger one is thermally stable.

The free energy in grand canonical ensemble also called Gibb's free energy is defined as

\[ G(r_{h}) = M(r_{h}) - T(r_{h})S(r_{h}). \]

(37)

Here, \( M(r_{h}) \) is the black hole mass, \( T(r_{h}) \) is the temperature for the a black hole horizon, and \( S(r_{h}) \) is the entropy. Using Eqs. (31), (33) and (35) in (37) we get

\[ G_{h_{q}} = \frac{18\Lambda\chi^{2}r_{h}^{2} + 3r_{h}^{2} - 2\Lambda r_{h}^{4} - 6\chi^{2} - 12\chi^{4}}{12r_{h}}, \]

\[ G_{h_{q}} = \frac{1}{12N\Omega_{h}^{2}r_{h}}\left(6q_{1}[2\chi^{2} + 3r_{h}^{2}] - 6\chi^{4}[2\Lambda\chi^{2}N - (N[1 + r_{h}^{2}\Lambda] + 4q_{1}^{2})] + \chi^{2}(16N\chi^{4} + 3r_{h}^{2}[N + 12q_{1}^{2}] + 6q_{1}^{2}) - r_{h}^{2}[2N\Lambda r_{h}^{6} + 3(Nr_{h}^{2} - 3q_{1}^{2})]\right). \]

(38)

If \( \chi \to 0 \), the Gibb's free energy in the non-charged case becomes equal to that in [100]. The Gibb's energy for black holes are depicted in Figs. 5(a) and 5(b). As 5(b) shows that the Gibb's energy is always positive which means that it is more globally stable than the other three spacetimes.
VI. SUMMARY AND DISCUSSION

In this study, we have addressed the Taub-NUT spacetimes in $f(R)$ gravitational theory. We describe the gravitational field equations for $f(R) = R + \beta R^2$ and apply them to Taub-NUT spacetime, using the fact that the solution of the trace of the gravitational field equations for $f(R)$ gravity gives a constant Ricci scalar, $R = -8\Lambda$. We have solved the resulting differential equations exactly and show that the solutions of the neutral Taub-NUT spacetimes did not depend on the dimensional parameter $\beta$. We have repeated the same calculations to the charged field equations of $f(R) = R + \beta R^2$ and used the same Taub-NUT spacetime. We have solved the resulting differential equations analytically and show that the output solution depends on the dimensional parameter $\beta$ and must satisfy the constraint $\beta/\text{nequal} \leq 1/16\Lambda$.

The physical quantities of these black hole solutions are studied. Among different things, we have studied the singularities and show that all the black hole solutions have a singularity at $\rho = 0 \Rightarrow r = 0$ when the parameter $\chi = 0$. Also, we have studied the horizons and show that there are two horizons corresponding to the event and the cosmological horizons. We have shown these horizons in Fig. 1 for the charged and the non-charged cases. Furthermore, the thermodynamics for black holes has been explored and the thermal phase transition based on the discontinuous sign changing of the specific heat has been investigated. We have calculated the mass in terms of the horizons and have shown the behavior of these quantities in Fig 2. Also, we have calculated the temperature in terms of the horizons and indicate their behavior in Fig. 3. Moreover, we have calculated the heat capacity of each black hole solution and have shown their behavior in Fig. 4. We have shown that the solution of Taub-NUT spacetime is unstable in the region $r < r_h$ and then has a phase transition at $r = r_h$, then has a stable value at $r > r_h$ as Fig. 4(a) shows. Same discussions can be applied for the other charged Taub-NUT, as Figs. 4(b) shows. In addition, the free energy for these solutions has been analyzed and the pattern of those has depicted in Fig. 5. It has been found from 5(b) that the charged Taub-NUT solution always has local stability [101].

It is of interest to note that Lü et al. [73] have derived numerical spherically symmetric solution in higher order derivative gravity using the action

$$ I = \int d^4x \sqrt{-g}\left[\alpha R - \gamma C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2\right], \quad (39) $$

with $\alpha$, $\beta$ and $\gamma$ being constants and $C_{\mu\nu\rho\sigma}$ is the Weyl tensor\(^2\). It is of interest to note that if $\alpha = 1$ and $\gamma = 0$ the field equations resulting from Lü et al. [74] will be identical with the one studied here in the non-charged case. So, is it possible to derive a NUT or Taub-NUT solution in higher order derivative gravity? Moreover, the stability analysis using geodesic deviation [102] of the above black holes needs to be checked. These will be answered elsewhere.

Before we close this discussion we must stress on the fact that all the black holes derived in this study can easily transformed to Einstein frame using a constant scalar field, due to the fact that

$$ g_{\mu\nu} \rightarrow g_{\mu\nu}' = \Omega^2(x)g_{\mu\nu}^{\text{Jor}}, \quad \text{where} \quad \Omega^2 = f_R, \quad (40) $$

\(^2\) In this numerical spherically symmetric solution, $\gamma$-term is significant in deriving solutions.
with \( f_r \) being constant value and \( g_{\mu\nu,\text{Ein}} \) is the metric in Einstein frame and \( g_{\mu\nu,\text{Jor}} \) is the metric in Jordan frame.

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**Appendix**

In the appendix, we are going to list the necessary quantities of the spacetimes (7) that are used in the calculations of the field equations (2) and (15). The non-vanishing components of the Livi-Civita connection of spacetime (7) are:

\[
\begin{align*}
\left(\frac{t}{\ell}\right) & = \frac{N_1 t}{2} - \frac{s'}{2s}, & \left(\frac{t}{\ell}\right) & = \frac{2y^{s}\cos\theta}{k \sin\theta}, & \left(\frac{t}{\ell}\right) & = \frac{\chi s}{k \sin\theta}, & \left(\frac{t}{\ell}\right) & = \frac{\chi N_1 s' \cos\theta}, & \left(\frac{t}{\ell}\right) & = \frac{N_1^2}{2N_1}, \\
\left(\frac{\theta}{\ell}\right) & = -\frac{\chi \sin\theta}{k}, & \left(\frac{\phi}{\ell}\right) & = \frac{\chi \cos\theta}{k s} - \frac{s' k}{2k}, & \left(\frac{\theta}{\ell}\right) & = \frac{\chi \cos\theta}{k s}, & \left(\frac{\phi}{\ell}\right) & = \frac{\chi \cos\theta}{k s}, & \left(\frac{\phi}{\ell}\right) & = \frac{\chi \cos\theta}{k s}. \\
\end{align*}
\]

The non-zero components of the Riemann tensor for (7) are:

\[
\begin{align*}
R_{t\ell t\ell} & = \frac{\chi s}{k \sin\theta}, & R_{t\ell \phi\phi} & = \frac{\chi s}{k \sin\theta}, & R_{\ell\phi t\phi} & = \frac{\chi s}{k \sin\theta}, & R_{\ell\phi \phi t} & = \frac{\chi s}{k \sin\theta}, \\
R_{t\ell \phi \phi} & = -\frac{\chi s}{k \sin\theta}, & R_{\ell\phi t\ell} & = -\frac{\chi s}{k \sin\theta}, & R_{\ell\phi \phi \ell} & = -\frac{\chi s}{k \sin\theta}, & R_{\ell\ell \phi \phi} & = -\frac{\chi s}{k \sin\theta}. \\
\end{align*}
\]

The non-zero components of the Riemann tensor for (7) are:

\[
\begin{align*}
R_{t\ell} & = \frac{N_1(2k^2 s N_1 s' - N_1 k^2 s' - 2k^2 s' N_1' + 2k^2 s N_1 k' s' + 2k^2 s N_1 k' s' + 8\chi^2 N_1 s)}{4k}, \\
R_{t\phi} & = \frac{-\chi N_1 \cos\theta(3k^2 N_1 s' + 2k^2 s N_1 s' + 2k^2 s N_1 s' + 2k^2 s N_1 k' s' + 2k^2 s N_1 k' s' + 8\chi^2 N_1 s)}{4k}, \\
R_{\phi \ell} & = \frac{4s - 2k N_1 s' s' k' - 2k s N_1 k' + 8\chi^2 s N_1}{4k s^2}, \\
R_{\phi \phi} & = \frac{2k^2 N_1 k' - 2s N_1 k' - 2s N_1 k' - 2s N_1 s' + 3k^2 s N_1 s' + k^2 N_1 s' - 2k^2 s N_1 s'}{4s^2 N_1 k^2}, \\
R_{\ell\ell} & = \frac{-\chi N_1 \cos\theta(2k^2 s N_1 s' + 2k N_1 s' s' k' + 2k s' k N_1 k' + 8\chi^2 s N_1 - k^2 N_1 s^2 + 3k^2 s' N_1' + 2k^2 s' N_1')}{4s^2 N_1 k^2}, \\
R_{\phi \phi} & = \frac{-\sin\theta(2k^2 s N_1 s' + 2k^2 s' N_1 k' + 2k^2 s' N_1 k' - 8\chi^2 s N_1 k + 4k^2 s)}{4s^2 N_1 k^2}. \\
\end{align*}
\]

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