Efimov Effect Revisited with Inclusion of Distortions

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Abstract: An elementary proof of the 3-body Efimov effect is provided in the case of a separable 2-body potential which binds at zero energy a light particle to a heavy one. The proof proceeds by two steps, namely i) a projection of the Hamiltonian in a subspace and the observation that the projected Hamiltonian generates an arbitrarily large number of bound states, and ii) a use of the Hylleraas-Undheim theorem to recover the unprojected Hamiltonian. The definition of the projectors we use can include mean field distortions.

1. Introduction

The Efimov effect\(^1\) consists in the claim that those 2-body forces which marginally bind pairs of particles in the 2-body problem induce an arbitrarily large number of bound states for the 3-body problem. This interesting phenomenon has been studied by many authors\(^2\)\(^-\)\(^5\), often in the framework of Faddeev equations, and, in particular, was validated by a formal proof\(^2\). An investigation of the same effect for the 4-body problem concluded\(^6\) that the effect does not exist in the case of identical bosons, but is present if three heavy particles meet with a fourth lighter particle\(^7,8\).

In this work, we follow the argument of Fonseca, Redish and Shanley\(^3,4\) and give again a proof of the effect, by means of a very elementary derivation. Symmetrization constraints between the three particles are strongly relaxed, two particles being chosen as heavy and the third particle as light. We do not use Faddeev equations, but rather
a Born-Oppenheimer (BO) approach. This is the subject of Section 2. Then Section 3 reinterprets the BO method as a generator coordinate method, thus as a projection into a trial subspace. We complete the proof by i) estimating the non adiabaticity corrections and ii) considering the residual coupling effects of the full Hamiltonian. Self consistent distortions of pair wave functions due to the presence of the third particle are discussed in Section 4. Finally we provide a discussion and conclusion in Section 5.

2. Elementary Proof

We consider the familiar Jacobi coordinates \( \vec{x} \) for pair (12) and \( \vec{y} \) for particle 3 with respect to the center of mass of (12). Then we consider the 3-body Hamiltonian

\[
H \equiv -\hbar^2 \Delta_{\vec{x}}/M + w(\vec{x}) - \hbar^2 \Delta_{\vec{y}}/(2\mu) + \lambda[v(\vec{y} - \vec{x}/2) + v(\vec{y} + \vec{x}/2)], \quad \mu = m/(1 + \varepsilon/2), \tag{2.1}
\]

where, for the sake of simplicity, the three particles are spinless, the masses \( M_1 \) and \( M_2 \) of particles 1 and 2 are equal, \( M_1 = M_2 = M \), and the third mass \( m \) is much lighter, \( \varepsilon \equiv m/M << 1 \).

Both potential functions \( v \) and \( w \) are assumed to have a short or a finite range, but their exact forms, local or even non local, are otherwise irrelevant. For the sake of simplicity, we assume that \( w \) is too weak to create bound states between the two heavy particles. Hence the continuum in the three-body problem could display two sorts of cuts at most, namely i) the three-body continuum at energy zero, naturally, and ii) two body-continuum(s) if the interaction \( v \) were able to bind a \((m + M)\) pair. Actually, the strength parameter \( \lambda \) has that critical value which makes semi-positive definite the 2-body Hamiltonian

\[
h_0 \equiv -\hbar^2 \Delta_{\vec{z}}/(2\mu) + \lambda v(\vec{z}). \tag{2.2}
\]

Here the degree of freedom \( \vec{z} \) may be either i) \( \vec{z} = \vec{y} \) or ii) \( \vec{z} = \vec{y} \pm \vec{x}/2 \), but it will be understood that \( h_0 \) acts only upon the degree of freedom \( \vec{y} \). The fact that \( h_0 \) generates a “zero energy ground state” does not depend on \( \vec{x} \), which plays for \( h_0 \) the rôle of a simple parameter, just defining the center of the critical potential \( \lambda v \).
It is noted that the critical value of $\lambda$ is linked here, from Eq.(2.2), to the value of the reduced mass $\mu$ rather than the slightly smaller reduced mass of a pair, $mM/(M+m) = m/(1+\varepsilon)$. With such a value of $\lambda$, critical for $\mu$, no bound state is induced for the 2-body problem with $m/(1+\varepsilon)$. The 2-body continuum in the 3-body problem starts at the same energy ($E = 0$, indeed) as the 3-body continuum. At most a resonance may occur in the 2-body channel at a slightly positive energy $E_r$.

For the sake of simplicity, the argument which follows assumes that $v$ is a scalar, nonlocal, rank one, separable potential

$$<\vec{p}'|v|\vec{p}> = -f(p')f(p),$$  \hspace{1cm} (2.3)$$

where $\vec{p}$ is the momentum conjugate to $\vec{z}$, and $p,p'$ are the lengths of $\vec{p},\vec{p}'$, respectively. The well-known binding condition for separable potentials then reads, at the critical value of $\lambda$,

$$1/\lambda = \int d\vec{p} \frac{|f(\vec{p})|^2}{p^2},$$  \hspace{1cm} (2.4)$$

Moreover an exactly soluble model is obtained if we choose the following form factor $f$,

$$f(p) = \frac{1}{\pi(p^2 + \gamma^2)},$$  \hspace{1cm} (2.5)$$

where $1/\gamma$ defines any suitable short range in coordinate space. With Eq.(2.5), the condition, Eq.(2.4), for a critical value of $\lambda$ becomes

$$\lambda = \gamma^3.$$  \hspace{1cm} (2.6)$$

In the following we set $\hbar = 1$ and $\hbar^2/(2\mu) = 1$, for a simpler system of units. We also choose $\vec{p}$ to be the momentum conjugate to $\vec{y}$, and introduce the translation operator $T = \exp(i\vec{p} \cdot \vec{x}/2)$ and its inverse $T^{-1} = \exp(-i\vec{p} \cdot \vec{x}/2)$. Then that part $h_f$ of $H$ which acts upon $\vec{y}$ reads,

$$h_f = p^2 + \lambda(TvT^{-1} + T^{-1}vT) = p^2 - \lambda(T|f><f'|T^{-1} + T^{-1}|f><f|T).$$  \hspace{1cm} (2.7)$$

Parametrized by $\vec{x}$, this Hamiltonian $h_f$ drives the BO dynamics of the fast, light particle. When compared with the semi-positive definite $h_0$, this Hamiltonian $h_f$ contains one additional attractive potential, and thus induces a truly bound state (square integrable) $\chi_{\vec{x}}(\vec{y})$
at a definitely negative energy \( \eta \). Since threshold singularities of a square root nature with respect to \( \eta \) are expected, we define such a square root \( \omega \) by
\[
\eta \vec{x} = -[\omega \vec{x}]^2.
\]
Naturally \( \omega \vec{x} \to 0 \) when \( x \to \infty \). The subscript \( \vec{x} \) will be omitted in the following, unless it is essential.

The wave function \( \chi \) is easily obtained in momentum representation, according to
\[
\chi(\vec{p}) = \lambda <f|T|\chi > 2 \cos(\vec{p} \cdot \vec{x}/2) f(\vec{p})/(p^2 + \omega^2),
\]
where we take advantage of the symmetry of the ground state, \(< f|T^{-1}|\chi >=< f|T|\chi >\). Projecting Eq.(2.8) against \( < f|T \), we obtain the equation which solves for the binding energy,
\[
1/\lambda = I(\omega, \vec{x}) + I(\omega, 0),
\]
with
\[
I(\omega, \vec{x}) =< f|(p^2 + \omega^2)^{-1}T^2|f >.
\]
We note incidentally that the condition for the value of \( \lambda \) to be critical reads
\[
1/\lambda = I(0, 0).
\]

With the choice of \( f(\vec{p}) \) as a scalar \( f(p) \), the integral \( I \), Eq.(2.10), does not depend on the orientation \( \hat{x} \) of \( \vec{x} \), but only on its length \( x \). More precisely, it becomes
\[
I(\omega, x) = 4\pi/x \int_0^\infty dp \ p \sin(px) [f(p)]^2/(p^2 + \omega^2).
\]
For the soluble model provided by the choice, Eq.(2.5), a straightforward contour integration gives
\[
I(\omega, x) = 2/(\pi x) \int_{-\infty}^{\infty} dp \ p \sin(px)/[(p^2 + \omega^2)(p^2 + \gamma^2)^2]
\]
\[
= [2\gamma \exp(-\omega x) + (\omega^2 x - \gamma^2 x - 2\gamma) \exp(-\gamma x)]/[\gamma x(\gamma^2 - \omega^2)^2].
\]
It is easy to obtain the value of \( I \) for \( x = 0 \),
\[
I_0(\omega) = I(\omega, 0) = 1/[(\gamma + \omega)^2].
\]
and then, for Eqs.(2.11,6), the number \( I_{00} = I(0, 0) = 1/\gamma^3 \).
If $\omega x$ has a finite limit $c$ when $x \to \infty$, it is also easy to find that $I(\omega, x)$ boils down to

$$I_\infty = \frac{2 \exp(-\omega x)}{x(\gamma^2 - \omega^2)^2}. \quad (2.15)$$

This simplification is also useful for the calculation of derivatives of $I$, because the contributions of the discarded term, proportional to $\exp(-\gamma x)$, are negligible for derivatives as well. Let us test this conjectured finite limit of $\omega x$, by means of an ansatz $\omega = c/x + d/x^2 + O(1/x^3)$ for Eq.(2.9). We obtain the condition

$$\frac{1}{\gamma^3} = \frac{2 \exp(-c - d/x)}{x [\gamma^2 - (c + d/x)^2/x^2]^2} + \frac{1}{\gamma [\gamma + (c + d/x)/x]^2} + O(1/x^3). \quad (2.16)$$

The ansatz is then consistent if $c$ and $d$ obey the conditions

$$\exp(-c) = c, \quad d = 3c^2/[2\gamma(1 + c)], \quad (2.17)$$

hence $c \simeq 0.5671$, $d \simeq 0.3079/\gamma$. We conclude that the “fast” BO Hamiltonian\(^4,5\) $h_f$ binds the light particle at energy $\eta(x) = -c^2/x^2 + O(1/x^3)$ when $x \to \infty$.

For a better understanding of this $1/x^2$ binding, a thorough examination of Eq.(2.9) is in order. We rephrase it as

$$1/\lambda - I(\omega, 0) = I(\omega, x), \quad (2.18)$$

and notice on one hand that, because of the Fourier transform introduced explicitly by the translation $T^2$ in the definition of $I$, Eq.(2.10), we can take advantage of Lebesgue’s theorem to predict a $1/x$ factor in the right-hand side of Eq.(2.18) when $x \to \infty$. On the other hand, the left-hand side of Eq.(2.18), once it is written as a function of $\omega$ rather than $\eta$, is a regular function near $\omega = 0$. Since $1/\lambda = I(0, 0)$, the difference $I(0, 0) - I(\omega, 0)$ is proportional to first order with respect to $\omega$. Hence the possibility of a connection $\omega \propto 1/x$. Had $\lambda$ been overcritical, inducing an actual negative eigenvalue $-\omega_0^2$, $\omega_0 \neq 0$, then $1/\lambda = I(\omega_0, 0)$, and one might have tried a connection $\omega - \omega_0 \propto 1/x$, but failed because of the resulting decay of $\exp(-\omega_0 x)$ in the right-hand side. This shows the strict importance, for the Efimov argument, of the condition that the value $\lambda$ be exactly critical.
It is now trivial to consider the “slow” BO Hamiltonian, which drives the heavy degree of freedom \( \vec{x} \),

\[
h_s = -\Delta_{\vec{x}}/M + w(\vec{x}) + \eta(x). \tag{2.19}
\]

The effective potential \( \eta(x) \) induced by the binding of the light particle has the expected long range behavior, \( \eta \propto -1/x^2 \), that justifies the Efimov prediction of an infinite number of bound states for the 3-body system. It is interesting to point out, incidentally, that the mass ratio \( \varepsilon \) appears strictly nowhere in this derivation of \( \eta \). Hence, if the decoupling of the full Schroedinger equation into the two separate steps described by \( h_f, h_s \) can be justified by any other argument than the Born-Oppenheimer one, it can be claimed that the \(-1/x^2\) nature of the induced potential is valid also for particles with similar masses.

We can also offer a qualitative argument for a generalisation of this \(-1/x^2\) nature in the case of potentials \( v \) other than separable. Let us assume that \( v \) is semi-negative definite and let \( \rho(\omega) \) be the highest eigenvalue of the operator \( R(\omega) \equiv (-v)^{1/2}(p^2 + \omega^2)^{-1/2}(v)^{1/2} \). In the same way, let \( \sigma(\omega, x) \) be the highest eigenvalue of \( S(\omega, x) \equiv (-v - T^2 v T^{-2})^{1/2}(p^2 + \omega^2)^{-1/2}(-v - T^2 v T^{-2})^{1/2} \). The fact that \( \lambda \) is critical is expressed by the condition \( 1/\lambda = \rho(0) \).

Then the binding energy in presence of the two potentials \( TvT^{-1} \) and \( T^{-1}vT \) is obtained when solving for \( \omega \) the “spectral” condition \( \rho(0) = \sigma(\omega, x) \). If, when \( x \to \infty \), a property of the form \( \sigma(\omega, x) = \rho(\omega) + \tau(\omega x)/x \) occurs, where \( \tau \) is a suitable function depending on the product \( \omega x \) only, then the “spectral” condition which provides \( \omega \) amounts to \( \rho(0) - \rho(\omega) = \tau/x \). It is also reasonable to assume that \( \rho(0) - \rho(\omega) = O(\omega) \) when \( \omega \to 0 \). Then the whole scheme discussed for Eq.(2.18) is recovered, and thus \( \omega \propto 1/x \).

In summary for this section, elementary arguments indicate that if the distance \( x \) between two particles is frozen, and if a third particle is “bound at energy zero” to, e.g., the first particle, the presence of the second particle will induce true binding of the third particle, at an energy \( \eta \propto -1/x^2 \) when \( x \to \infty \). This \( \eta \) will in turn behave as a long range potential, efficient for creating many bound states of the pair made by the first and second particles.
3. Born-Oppenheimer Approximation and Generator Coordinate Method

Let \( \varphi \) be a bound eigenstate of \( h_s \), and \( E \) the corresponding eigenvalue. For the sake of simplicity, we assume in the following that \( w \) is a scalar, like \( \eta \) and \( v \), and we then omit vector notations whenever possible. The function \( \varphi \) is a scalar (s state). Although \( \chi_{\vec{x}} \) is not a scalar, it transforms very simply under any rotation \( \mathcal{R} \), namely \( \mathcal{R}(\chi_{\vec{x}}) = \chi_{\mathcal{R}(\vec{x})} \). As stated already, the vector nature of \( \vec{x} \) as a label can often be understood, and/or shortened into a simple scalar label \( x \).

The well-known BO ansatz for the 3-body state \( \psi(x, y) \) corresponding to \( E \) reads,

\[
\psi(x, y) = \varphi(x) \chi_x(y).
\] (3.1)

This may be written as well under the form,

\[
\psi(x, y) = \int d\xi \, \varphi(\xi) \, \delta(x - \xi) \chi_\xi(y),
\] (3.2)

which is nothing but a generator coordinate\(^{10-12}\) expansion on a basis

\[
\phi_\xi(x, y) = \delta(x - \xi) \chi_\xi(y),
\] (3.3)

of states \( \phi_\xi \) parametrized by the continuous label \( \xi \). The “slow” wave function \( \varphi(x) \) is thus reinterpreted as a mixture amplitude \( \varphi(\xi) \) with respect to the generator coordinate \( \xi \).

If this reinterpretation can be shown to be consistent, we will thus obtain that any BO eigenvalue \( E \) is an eigenvalue of the projection \( PHP \) of \( H \) in that subspace spanned by the wave functions \( \phi_\xi \). This will reinforce the BO result, namely that there are an infinity of such negative eigenvalues \( E \). Indeed, we can then take advantage of the Hylleraas-Undheim (HU) theorem\(^9\), which we summarize by the short statement “the unprojected Hamiltonian has more (or at least as many) bound states than (as) its projection into any subspace ”.

That is, any exact wave function \( \Psi \) may be written as

\[
\Psi = P\Psi + Q\Psi,
\] (3.4)

7
where $Q$ is a projector orthogonal to $P$. Then, the effect of the added $Q$-term modifies the spectrum of $PHP$ in such a way that the correct number $N$ of bound states satisfies the inequality

$$N_P \leq N_t \leq N.$$  \hspace{1cm} (3.5)

Here $N$ is the exact number of bound states of the original $H$, but we have also inserted an intermediate number $N_t$ of bound states in Eq.(3.5) to include the situation where $Q$ is not the full complement $1 - P$ of $P$, but only part of it, for a variational theory with trial states $\Psi_t$ in an enlarged subspace. In many of the earlier treatments of the Efimov effect in terms of models and simple ansatz on the wave functions, the effect of the $Q$-component of the wave function was not estimated, which made the proof incomplete. The HU theorem for the bound states may be re-stated as a deepening influence of the effective $Q$-space potential $PV G^Q V P < 0$ where $G^Q$ is the Green’s function in the $Q$-space. Therefore, all the bound states produced by $PHP$ will be uniformly pushed down in energy, and possibly additional bound states are created by the $Q$-component.

We now proceed by proving that the slow Hamiltonian, Eq.(2.19), is formally equivalent to a Griffin-Hill-Wheeler (GHW) kernel $H(\xi', \xi) \equiv \langle \phi_{\xi'} | H | \phi_{\xi} \rangle$, hence a representation of the projection $PHP$ of $H$ in the generator subspace spanned by the states $\phi_{\xi}$. For this, we first notice that such states $\phi_{\xi}$ are not orthogonal to one another if one measures their scalar product by $\langle \chi_{\xi'} | \chi_{\xi} \rangle$, namely an integration upon $y$ only, but become trivially orthogonal if an integration upon $x$ is also included. Indeed such states are strictly localized in $x$-space and can be normalized according to $\langle \phi_{\xi'} | \phi_{\xi'} \rangle = \delta(\xi' - \xi)$. Nothing prevents us from normalizing these states to unity in the $y$-space, since they are strictly bound eigenstates of the fast Hamiltonian $h_f$, Eq.(2.7),

$$\int dy \ [\chi_{\xi}(y)]^2 = 1.$$ \hspace{1cm} (3.6)

We find incidentally, as a consequence of Eq.(3.6), that

$$\int dy \ \chi_{\xi}(y) \ [\nabla_{\xi} \chi_{\xi}(y)] = \frac{1}{2} \nabla_{\xi} \left( \int dy \ [\chi_{\xi}(y)]^2 \right) = 0,$$ \hspace{1cm} (3.7)
and furthermore, as a consequence of Eq.(3.7), that

$$u(\xi) \equiv \int dy \, \chi(\xi) \, [\Delta \chi(y)] = \int dy \, [\nabla \chi(y)]^2 > 0. \quad (3.8)$$

The GHW kernel thus reads

$$\langle \phi_{\xi'}|H|\phi_\xi \rangle = \langle \phi_{\xi'}|[-\Delta_x/M + w(x) + h_f(x,y)]\phi_\xi \rangle = \delta(\xi' - \xi) \left( \frac{2\varepsilon}{1 + \varepsilon/2} \left[ -\Delta_{\xi} + u(\xi) \right] + w(\xi) + \eta(\xi) \right). \quad (3.9)$$

Except for a correction proportional to $u(\xi)$, this is nothing but $h_x$ in coordinate representation. The GHW equation,

$$\int d\xi \, \langle \phi_{\xi'}|(H - E)|\phi_\xi \rangle \, \varphi(\xi) = 0, \quad (3.10)$$

and the slow BO equation,

$$\left( -\frac{2\varepsilon}{1 + \varepsilon/2} \Delta_x + w(x) + \eta(x) - E \right) \varphi(x) = 0, \quad (3.11)$$

are thus equivalent if this correction brought by $u$, a repulsive effective potential, can be shown to be negligible. For that purpose we restate Eq.(2.8) in the form

$$|\chi> = (\langle \zeta|\zeta \rangle)^{-1/2} |\zeta>, \quad (3.12)$$

with

$$|\zeta> = (p^2 + \omega^2)^{-1} (T + T^{-1}) |f>, \quad <\zeta|\zeta> = 2 <f|(p^2 + \omega^2)^{-2} (1 + T^2) |f>. \quad (3.13)$$

In particular, when $x \to \infty$,

$$\omega <\zeta|\zeta> = -\frac{\partial}{\partial \omega} [I_0(\omega) + I_\infty(\omega, x)] = 2 \frac{\gamma}{\gamma(\gamma + \omega)^3} + \frac{2 \exp(-\omega x)}{(\gamma^2 - \omega^2)^3} - \frac{8\omega \exp(-\omega x)}{x(\gamma^2 - \omega^2)^3}. \quad (3.14)$$

The right-hand side of Eq.(3.14) has a finite limit $\ell = 2[1 + \exp(-c)]/\gamma^4$ when $x \to \infty$, hence $<\zeta|\zeta> = 2(1 + c)x/(c\gamma^4) + \mathcal{O}(x^0)$ when $x \to \infty$.
Let us denote by $|\chi^r\rangle,|\zeta^r\rangle$ the full gradients of $|\chi\rangle,|\zeta\rangle$, respectively, with respect to $\vec{x}$, including the dependence of $\omega$ on $x$. It is easy to find that

$$
|\zeta^r\rangle = \frac{i\vec{p}}{2}(p^2 + \omega^2)^{-1}(T - T^{-1})f > -2\omega(p^2 + \omega^2)^{-1}(T + T^{-1})f > \nabla_{\vec{x}} \omega,
$$

where we know that $\nabla_{\vec{x}} \omega = -(c + 2d/x)\hat{x}/x^2 + O(1/x^4) = -(c + 2d/x)\vec{x}/x^3 + O(1/x^4)$. Here $\hat{x}$ is the unit vector defining the direction of $\vec{x}$, namely $\hat{x} = \vec{x}/x$. Then we have to investigate the behavior of

$$
u = \langle \vec{x}'|\chi'^r\rangle = (\langle \zeta|\zeta\rangle)^{-1} \langle \zeta^r|\left(1 - \frac{|\zeta\rangle\langle\zeta|}{\langle\zeta|\zeta\rangle}\right)|\zeta^r\rangle,$$

when $x \to \infty$.

From the linear asymptotic behaviour, $\langle \zeta|\zeta\rangle \propto x$, of the square norm of $\zeta$, we deduce that the length of its gradient, namely of $2\langle \zeta|\zeta^r\rangle$, has a finite limit when $x \to \infty$. Indeed, from the derivative of the right-hand side of Eq.(3.14), we find

$$
\langle \zeta|\zeta^r\rangle = \left[\frac{-\exp(-\omega x)}{(\gamma^2 - \omega^2)^2} + \frac{4\omega\exp(-\omega x)}{x(\gamma^2 - \omega^2)^3} + \frac{4\exp(-\omega x)}{x^2(\gamma^2 - \omega^2)^3}\right]\hat{x} - \left[\frac{1}{\omega^2\gamma(\gamma + \omega)} + \frac{3}{\omega\gamma(\gamma + \omega)^4}\right]\nabla_{\vec{x}} \omega

- \left[\frac{x\exp(-\omega x)}{\omega(\gamma^2 - \omega^2)^2} + \frac{\exp(-\omega x)}{\omega^2(\gamma^2 - \omega^2)^2} - \frac{8\exp(-\omega x)}{(\gamma^2 - \omega^2)^3} + \frac{24\omega\exp(-\omega x)}{x(\gamma^2 - \omega^2)^4}\right]\nabla_{\vec{x}} \omega,
$$

the limit of which is $(1 + c)\hat{x}/(c\gamma^4) = [\exp(c) + 1]\hat{x}/\gamma^4$. It can be concluded that the term $-(\langle \zeta^r|\zeta\rangle / \langle \zeta|\zeta\rangle)^2$ in the right hand side of Eq.(3.16) reinforces the potential $\eta = -c^2/x^2$ by a contribution $-\epsilon/[(2 + \epsilon)x^2]$.

We must now evaluate

$$
\langle \zeta^r|\zeta^r\rangle = \frac{1}{2} < f|p^2(p^2 + \omega^2)^{-2}(1 - T^2)|f > -4\omega(c + 2d/x) < f|\vec{p} \cdot \vec{x}(p^2 + \omega^2)^{-3}\sin(\vec{p} \cdot \vec{x})|f > /x^3

+ 8\omega^2(c + 2d/x)^2 < f|(p^2 + \omega^2)^{-4}(1 + T^2)|f > /x^4 + O(1/x^2).
$$

The matrix element of $\vec{p} \cdot \vec{x}\sin(\vec{p} \cdot \vec{x})$ can be obtained as

$$
\langle f|\vec{p} \cdot \vec{x}\sin(\vec{p} \cdot \vec{x})(p^2 + \omega^2)^{-3}|f > = \Im m < f|\vec{p} \cdot \vec{x}\exp(i\vec{p} \cdot \vec{x})(p^2 + \omega^2)^{-3}|f > =

\Im m < f\left[-i\frac{d}{ds}\exp(is\vec{p} \cdot \vec{x})\right]_{s=1} (p^2 + \omega^2)^{-3} |f > = -\left(\frac{d}{2\omega d\omega}\right)^2 \left.\frac{d}{ds}\right|_{s=1} I(\omega, sx)/2.
$$

(3.19)
Then a somewhat tedious calculation, which cannot be reported here in detail, gives the contribution of $<\zeta' | \zeta'>$ to $u$,

$$
\frac{2\varepsilon <\zeta' | \zeta'>}{(1 + \varepsilon/2) <\zeta|\zeta>} = \left(\frac{\varepsilon}{2 + \varepsilon}\right) \times \left[ \frac{c \gamma}{(1 + c)x} + \frac{12 + 36c + 45c^2 + c^3 - 13c^4 - 2c^5}{6(1 + c)^3x^2} + O(x^{-3}) \right] \approx \left(\frac{\varepsilon}{2 + \varepsilon}\right) \left(\frac{0.36\gamma}{x} + \frac{1.98}{x^2}\right).
$$

(3.20)

It turns out, as a matter of fact, that $<\zeta' | \zeta'>$ has a finite limit $1/(2\gamma^3)$ when $x \to \infty$. Combined with the result already found, $<\zeta|\zeta> = 2(1+c)x/(c\gamma^4)+O(x^0)$, this asymptotic behavior of $<\zeta' | \zeta'>$ induces a repulsive potential of order $1/x$, which seems to contradict the leading term $-c^2/x^2$ derived from the straight BO approximation.

We notice however that this repulsive potential $\varepsilon c \gamma/[(2 + \varepsilon)(1 + c)x]$, and also the minute corrections brought by $u$ at order $1/x^2$, are weighted by $\varepsilon$, naturally. Hence the Efimov phenomenon is still present for $\varepsilon$ sufficiently small, because a generic potential of the form $\varepsilon/x - 1/x^2$ generates an arbitrarily large number of bound states.

More precisely, if we collect all relevant terms contributing to Eqs.(3.9,10), the radial equation in $s$-waves to be studied for binding is, for $\hat{\varphi} \equiv x \varphi$,

$$
\left( -\frac{d^2}{dx^2} + \frac{(1 + \varepsilon/2)}{2\varepsilon}w(x) + \frac{c \gamma}{4(1 + c)x} - \frac{\Lambda}{x^2} - \frac{(1 + \varepsilon/2)E}{2\varepsilon} \right) \hat{\varphi} = 0, \quad x \geq 0, \quad \hat{\varphi}(0) = 0,
$$

(3.21)

with

$$
\Lambda = -\frac{(1 + \varepsilon/2)c^2}{2\varepsilon} - \frac{1}{4} + \frac{12 + 36c + 45c^2 + c^3 - 13c^4 - 2c^5}{24(1 + c)^3}.
$$

(3.22)

Potentials of order $1/x^2$ when $x \to \infty$ do not modify our expected conclusion of a large number of bound states and can be simply neglected. We also keep in mind that the $1/x$ and $1/x^2$ forms contributing to Eq.(3.21) are not valid for small values of $x$, since the potentials they represent are actually finite.

If now we assume for the sake of simplicity that $w$ has a strictly finite range $x_0$, the radial equation Eq.(3.21) can be replaced by

$$
\left( -\frac{d^2}{dx^2} + \frac{c \gamma}{4(1 + c)x} - \frac{\Lambda}{x^2} - \frac{(1 + \varepsilon/2)E}{2\varepsilon} \right) \hat{\varphi} = 0, \quad x \geq x_0, \quad \hat{\varphi}(x_0) = 0.
$$

(3.23)
The hard core boundary condition $\psi(x_0) = 0$ removes all effects of the (finite) potentials at short distances. This introduction of a hard core is legitimate, for it can only underestimate the number of bound states.

A rescaling $x = 4(1 + c)z/(c\gamma)$ is then suitable to remove the coefficient of the $1/x$ term. The ratio between $x$ and $z$ is a fixed number, which depends only upon $c$ and $\gamma$, and we define a fixed, rescaled hard core radius, $z_0 = c\gamma x_0/[4(1 + c)]$, accordingly. Finally the energy scales according to $E' = 4(1 + c)^2(2 + \varepsilon)E/(c^2\gamma^2\varepsilon)$, and we find it strictly equivalent to study the number of bound states for the following problem,

$$\left(-\frac{d^2}{dz^2} + \frac{1}{z + z_0} - \frac{\Lambda}{(z + z_0)^2} - E'\right)\psi = 0, \quad z \geq 0, \quad \psi(0) = 0.$$ (3.24)

The positive coefficient $\Lambda$ is of order $1/\varepsilon$ and becomes infinite when $\varepsilon \to 0$. It is obvious that, as long as $z + z_0 \leq \Lambda/2$, then

$$\frac{1}{z + z_0} - \frac{\Lambda}{(z + z_0)^2} \leq -\frac{\Lambda}{2(z + z_0)^2}. \quad (3.25)$$

Therefore the problem,

$$\left(-\frac{d^2}{dz^2} - \frac{\Lambda}{2(z + z_0)^2} - E'\right)\psi = 0, \quad 0 \leq z \leq \Lambda/2, \quad \psi(0) = \psi(\Lambda/2) = 0,$$ (3.26)

generates a further lower bound for the number of bound states. From the potential present in Eq.(3.26), we finally use the trace formula\(^{13}\) for an estimate of a number of bound states,

$$N_B = \int_{0}^{\Lambda/2} dz \frac{\Lambda}{2(z + z_0)^2} \propto \Lambda \log(\Lambda). \quad (3.27)$$

This number $N_B$ does diverge when $\varepsilon \to 0$. Q E D

To summarize this section, the Born-Oppenheimer method is restated as a special case of the generator coordinate method. The latter is a projection of the Schroedinger equation onto a trial subspace. The proof that there is an arbitrarily large number of bound states in a subspace is extended to the full space of wave functions, by means of the Hylleraas-Undheim theorem. In the process of relating BO to GCM, however, a long range repulsive correction $\varepsilon u$ is found to perturb the static BO potential $\eta$ which creates
the Efimov effect. Small values of the mass ratio, the adiabaticity parameter $\varepsilon$, are then necessary to validate our proof of an arbitrarily large number of bound states for the three-body system.

4. Introduction of Distortions and Mean Field Approximation

In order to make less mandatory the restriction of $\varepsilon$ to small values, we suggest a more flexible formulation of the BO method. For this, we complete $H$, Eq.(2.1), by a constraint upon $<\vec{x}>$, via an auxiliary, external harmonic oscillator potential $(\vec{x} - \vec{\xi})^2$ with a strong spring constant, Lagrange multiplier $L$. This defines the constrained Hamiltonian

$$\mathcal{H} = \frac{P^2}{M} + L (\vec{x} - \vec{\xi})^2 + w + p^2 + \lambda (TvT^{-1} + T^{-1}vT).$$  \hspace{1cm} (4.1)

It is clear that the ground state $|\Phi_{\xi}>$ of $\mathcal{H}$ verifies $<\Phi_{\xi}|\vec{x}|\Phi_{\xi}> \to \vec{\xi}$ when $L \to \infty$. For finite values of $L$, however, the freezing of $\vec{x}$ is implemented by a wave packet, less stringent than a $\delta$–function $\delta(\vec{x} - \vec{\xi})$.

For simplicity of notations in the following, vectors will again be replaced by scalars. The simplest approximate description of the ground state of $\mathcal{H}$ is obtained by means of a Hartree ansatz $\Phi_{\xi}(x, y) = \Gamma_{\xi}(x) \Xi_{\xi}(y)$. It is clear that $\Gamma$ is not very different from a sharp Gaussian centered at $\xi$, and that, however, it incorporates distortions due to the various potentials present into the corresponding Hartree equation,

$$\left[\frac{P^2}{M} + L (x - \xi)^2 + w + V - \eta'\right] \Gamma = 0,$$  \hspace{1cm} (4.2)

where $V$ is the mean field potential induced by the convolution of $\lambda(TvT^{-1} + T^{-1}vT)$ with the density $[\Xi(y)]^2$. The corresponding Hartree eigenvalue $\eta'$ contains a spurious harmonic oscillator contribution, to be discarded if physical interpretations are needed. In turn, $\Xi$ is the bound state generated for the third particle according to the second Hartree equation,

$$(p^2 + U_{\xi} - \eta'') \Xi = 0,$$  \hspace{1cm} (4.3)
where $U_\xi$ is the mean field potential arising from the convolution of $\lambda(TvT^{-1} + T^{-1}vT)$ with the density $[\Gamma(x)]^2$. In so far as $\Gamma$ is strongly localized around $\xi$, there is not much difference between $U_\xi$ and $\lambda[v(y - \xi/2) + v(y + \xi/2)]$. Hence there is a strong similarity between this second Hartree equation, Eq.(4.3) and the fast BO equation driven by $h_f$, see Eq.(2.7). The same similarity holds for $\eta''(\xi)$ and the static BO potential $\eta(\xi)$. The former, however, contains those corrections arising from the differences between $\Gamma$ and a $\delta-$function. It is stressed that all these corrections and distortions represent self consistency between $\xi$ and $y$, and facilitate binding.

The generator coordinate ansatz,

$$|\Psi> = \int d\xi \ F(\xi) \ |\Phi_\xi>, \quad (4.4)$$

then leads to the usual GHW equation,

$$\int d\xi \ (\langle \Phi_{\xi'}|H|\Phi_\xi > - E < \Phi_{\xi'}|\Phi_\xi >) \ F(\xi) = 0, \quad (4.5)$$

where now the overlap kernel $N_{\xi',\xi} \equiv \langle \Phi_{\xi'}|\Phi_\xi >$ differs from $\delta(\xi - \xi')$. It is rather similar to a Gaussian, as an overlap of the quasi-Gaussian wave packets $\Gamma_\xi$ and $\Gamma_{\xi'}$. Its inverse matrix square root $N^{-1/2}$ will be necessary for the usual GCM deconvolution manipulation. The slow wave function will not be $F$, but rather $N_{\xi}/2 F$.

It may then be convenient to write the Hamiltonian kernel $\hat{H}_{\xi',\xi} \equiv \langle \Phi_{\xi'}|H|\Phi_\xi >$ under the form

$$\hat{H}_{\xi',\xi} = \langle \Phi_{\xi'}|P^2/M + w + p^2 + U_\xi + \lambda(TvT^{-1} + T^{-1}vT) - U_\xi|\Phi_\xi > \quad (4.6)$$

$$= \langle \Phi_{\xi'}|(P^2/M + w)|\Phi_\xi > + \langle \Phi_{\xi'}|\lambda(TvT^{-1} + T^{-1}vT) - U_\xi|\Phi_\xi > + \eta''(\xi)N_{\xi',\xi},$$

for a hint that $\eta''$ will appear as a dynamical effective potential when the usual deconvolution $N^{-1/2}\hat{H}N^{-1/2}$ is performed. It will be noticed that, according to the Hartree definition of $U$, the diagonal matrix element $\langle \Phi_\xi|\lambda(TvT^{-1} + T^{-1}vT) - U_\xi|\Phi_\xi >$ vanishes identically. Finally it is known$^{12)}$ that the same deconvolution removes the zero-point kinetic energy which plagues $\langle \Phi_{\xi'}|P^2/M|\Phi_\xi >$.

All told, after deconvolution, the present GHW equation, which includes dynamical distortions, reads as a generalization of the slow BO equation. The self consistency inserted
by the Hartree method, via those distortions included in $\Gamma$ and $\Xi$, is expected to bring more binding. It should thus increase the number of bound states and allow larger values of $\epsilon$ to be compatible with a given, large number of bound states.

5. Discussion and Conclusion

In this paper, on one hand, we provide another rigorous proof of an Efimov effect. But the proven effect is weaker than the full expected effect. Namely, given the initial hypothesis that a pair potential is marginally able to bind, an arbitrarily large number of three-body bound states is obtained only if a suitable mass ratio is small enough. This may be the result of the BO ansatz, Eq.(3.1), and the particular coordinate system chosen.

On the other hand, several new results were found. For one, the scheme of our proof is based on arguments of moderate technicality only, such as the recognition that the BO method is but a special case of a projection of the three-body dynamics into a generator coordinate subspace. Hence, we were able to take advantage of the Hylleraas-Undheim theorem, which states that there are at least as many bound states in the full space of wave functions as there are in a subspace.

We were also able to relax the BO freezing of the heavy degree of freedom into more flexible Hartree calculations under harmonic oscillator constraints. This allows more precise descriptions of the wave functions of Efimov states, including mean field distortions.

Despite the slight technical difficulties we had to face when the effective potential, of the form $-1/x^2$, became plagued by corrections of the form $\epsilon/x$, we did not find it necessary here to introduce the now well known electron translation factor (ETF) correction of the BO wave functions for the calculation of the molecular potential$^{14-17}$. An introduction of this ETF correction is likely to be in order for future stages of the theory only. We conjecture that the $\epsilon/x$ perturbation may be eliminated when Eq.(3.1) is improved by this ETF asymptotically. The problem of ill-behaved boundary conditions raised by the conflict between the representation where the two heavy particles are considered close
by and the representation where these heavy particles are taken far apart is not new and smooth transitions between such representations have been proposed\textsuperscript{16}. We can point out, however, that each one of the two competing representations provides a complete basis of the Hilbert space of bound states. For our theory of Efimov bound states, the conflict between these representations is thus tempered.

Accordingly, we find it reasonable, while odd at first sight, that the reduced mass which we use to define the critical value of the potential strength is a three-body reduced mass $\mu = 2Mm/(2M + m)$ rather than a pair reduced mass $\mu' = Mm/(M + m)$. This choice is indeed imposed mathematically by the “fast” BO Hamiltonian, see Eq.\textsuperscript{(2.7)}. But it may also receive an intuitive, physical interpretation: that BO bound state $\chi_x(y)$, which reaches zero binding when $x \to \infty$, is even under the exchange of the two heavy particles and is defined with respect to a fictitious particle, namely the center of mass of these heavy particles. In that sense, the critical condition for marginal binding must refer to the relation between the light particle and that center of mass with mass $2M$. All told, for Efimov states, where long range effects are at work, a pair formed by the light particle and one of the heavy ones cannot really be isolated from the other heavy particle.

Finally, our approach can be extended\textsuperscript{18} by the consideration of coupling the three possible “channels” defined by the three possible pair partitions. A BO treatment, or the constrained Hartree(-Fock) generalization advocated in Section 4, can be undertaken for each such channel. There is no doubt that the resulting, projected Faddeev equations will be driven by long range potentials similar to those found in the one-channel, mathematically rigorous argument detailed in Sections 2 and 3.

Acknowledgements : One author (YH) thanks the Saclay theory group for their hospitality during three visits since 1990. The other (BG) is grateful to the University of Connecticut for a visit, where this work was started, and thanks J.Letourneux for pointing out an inconsistency in an earlier version of the manuscript.

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