Abstract. The notion of $f$-derivations was introduced by Beidar and Fong to unify several kinds of linear maps including derivations, Lie derivations and Jordan derivations. In this paper we introduce the notion of $f$-biderivations as a natural "biderivation" counterpart of the notion of "$f$-derivations". We first show, under some conditions, that any $f$-biderivation is a Jordan biderivation. Then, we turn to study $f$-biderivations of a unital algebra with an idempotent. Our second main result shows, under some conditions, that every Jordan biderivation can be written as a sum of a biderivation, an antibiderivation and an extremal biderivation. As a consequence we show that every Jordan biderivation on a triangular algebra is a biderivation.

2010 Mathematics Subject Classification. 16W25, 47B47.

Key Words. $f$-derivation; $f$-biderivation; Jordan biderivation.

1 Introduction

Throughout the paper $\mathcal{R}$ will denote a commutative ring with unity, $\mathcal{A}$ will be a unital $\mathcal{R}$-algebra with center $Z(\mathcal{A})$ and $\mathcal{M}$ will be a unital $\mathcal{A}$-bimodule.

Recall that an $\mathcal{R}$-linear map $\mathcal{D}$ from $\mathcal{A}$ into $\mathcal{M}$ is said to be a derivation (resp., an antiderivation) if, for all $a, b \in \mathcal{A}$, $\mathcal{D}(ab) = \mathcal{D}(a)b + a\mathcal{D}(b)$ (resp., $\mathcal{D}(ab) = \mathcal{D}(b)a + b\mathcal{D}(a)$). The inner derivations are classical examples of derivations. Recall that an $\mathcal{R}$-linear map $\mathcal{D}$ is said to be inner if it is of the form $\mathcal{D}(a) = [m_0, a]$ for some $m_0 \in \mathcal{M}$, where $[-,-]$ stands for the Lie bracket.
In [4], Beidar and Fong introduced the notion of \(f\)-derivations which unifies several particular kinds of linear maps including the classical derivations as follows:

Consider a fixed nonzero multilinear polynomial \(f\) in noncommuting indeterminates \(x_i\) over \(R\):

\[
f(x_1, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_{\pi} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}, \quad \alpha_{\pi} \in R,
\]

where \(S_n\) denotes the symmetric group of order an integer \(n \geq 2\). An \(R\)-linear map \(D : A \rightarrow M\) is called an \(f\)-derivation if it satisfies

\[
D(f(x_1, \ldots, x_n)) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, D(x_i), x_{i+1}, \ldots, x_n)
\]

for all \(x_1, \ldots, x_n \in A\).

Thus,

- a derivation is an \(f\)-derivation for the polynomial \(f(x_1, x_2) = x_1 x_2\),
- a Jordan derivation is an \(f\)-derivation for the polynomial \(f(x_1, x_2) = x_1 \circ x_2 := x_1 x_2 + x_2 x_1\),
- a Jordan triple derivation is an \(f\)-derivation for the polynomial \(f(x_1, x_2, x_3) = x_1 x_2 x_3 + x_3 x_2 x_1\),
- a Lie derivation is an \(f\)-derivation for the polynomial \(f(x_1, x_2) = [x_1, x_2] := x_1 x_2 - x_2 x_1\), and
- a Lie triple derivation is an \(f\)-derivation for the polynomial \(f(x_1, x_2, x_3) = [[x_1, x_2], x_3]\).

In [5, Theorem 1.3], Benkovič proved (under some conditions) that every \(f\)-derivation is a Jordan derivation. This means that in some situations studying \(f\)-derivation is based on the study of Jordan derivations. In [6], Benkovič and Širovnik investigated Jordan derivations on algebras with an idempotent. They proved that under certain “nice” conditions every Jordan derivation is a sum of a derivation and an antiderviation.

Our aim in this paper is to investigate the “biderivation” counterpart of the above results.
Naturally one can define a “biderivation” counterpart of the \( f \)-derivations as follows:

In what follows, we consider a fixed nonzero multilinear polynomial \( f \) as defined in (1.1). An \( \mathcal{R} \)-linear map \( F : A \times A \to M \) is called an \( f \)-biderivation, if

\[
F(f(x_1, \ldots, x_n), z) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, F(x_i, z), x_{i+1}, \ldots, x_n)
\]

and

\[
F(z, f(x_1, \ldots, x_n)) = \sum_{i=1}^{n} f(x_1, \ldots, x_{i-1}, F(z, x_i), x_{i+1}, \ldots, x_n)
\]

for all \( x_1, \ldots, x_n, z \in A \).

Then,

- every \( f \)-biderivation \( F \) is a biderivation when \( f(x, y) = xy \) (see [9]),
- every \( f \)-biderivation \( F \) is a Jordan biderivation when \( f(x, y) = x \circ y \) (see for instance [1]), and
- every \( f \)-biderivation \( F \) is a Jordan triple biderivation when \( f(x, y, z) = xyz + zyx \) (see for example [8]).

We start our paper with the first main result, Theorem 2.1, which is the “biderivation” counterpart of Benković’s result [5, Theorem 1.3]. It shows, under some conditions, that any \( f \)-biderivation is a Jordan biderivation. Then, in the remainder of the paper we focus on the study of Jordan biderivations. Namely, we aim to establish the “biderivation” counterpart of Benković and Širovnik’s main result [6, Theorem 4.1]. As a main result (Theorem 2.7), we show, under some conditions, that every Jordan biderivation can be written as a sum of a biderivation, an antibiderivation and an extremal biderivation. Recall that a bilinear map \( D : A \times A \to A \) is called an antibiderivation if it is an antiderivation with respect to both components. A bilinear map \( D : A \times A \to A \) is called an extremal biderivation if it is of the form \( D(x, y) = [x, [y, a]] \) for all \( x, y \in A \), where \( a \notin Z(A) \) and \( [[A, A], a] = 0 \).

As a consequence of our second main result, we show that every Jordan biderivation of a triangular algebra is a biderivation (Corollary 2.10). Recall, for two \( \mathcal{R} \)-algebras...
A and B and an \((A, B)\)-bimodule \(M\), the set

\[
\text{Tri}(A; M; B) := \{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} | a \in A, m \in M, b \in B \}
\]

equipped with the usual matrix operations is an \(\mathcal{R}\)-algebra called a (generalized) triangular \(\mathcal{R}\)-algebra (see [10] for more details about this construction). In this paper we assume that \(M\) is also a faithful \((A, B)\)-bimodule. As interesting examples of triangular matrix algebras one can cite the (classical) upper triangular matrix algebras, the block upper triangular matrix algebras and the nest algebras. It is important to recall that an algebra \(A\) is isomorphic to a triangular matrix algebra if there exists a non trivial idempotent \(e \in A\) such that \((1-e)Ae = 0\) (see, for instance, [10, Theorem 5.1.4]). Namely, in this case, \(A\) is isomorphic to \(\text{Tri}(eAe; eA(1-e); (1-e)A(1-e))\).

2 Main results

Let us start with the first main result which investigates \(f\)-biderivations under some conditions.

We say that an element \(r \in \mathcal{R}\) is \(\mathcal{M}\)-regular if for every \(m \in \mathcal{M}\), \(rm = 0\) implies \(m = 0\). Let

\[
\alpha = \sum_{\pi \in S_n} \alpha_\pi \in \mathcal{R}
\]

be the sum of coefficients of the polynomial \(f\) from (1.1).

The following result is the “biderivation” counterpart of [9, Theorem 1.3].

Theorem 2.1 Let \(A\) be a unital algebra and \(\mathcal{M}\) a unital \(A\)-bimodule. Let \(F : A \times A \to \mathcal{M}\) be an \(f\)-biderivation, with \(\alpha \neq 0\). If \(\mathcal{M}\) is \((n - 1)\)-torsion free and \(\alpha\) is \(\mathcal{M}\)-regular, then \(F\) is a Jordan biderivation.

Proof. First we prove that \(F(1, y) = 0\) for all \(y \in A\). Let \(x_i = 1\) for \(i = 1, \ldots, n\). Then, by the definition of \(f\)-biderivation,

\[
\alpha F(1, y) = n \alpha F(1, y).
\]

Then, \((n - 1)\alpha F(1, y) = 0\), and consequently \(F(1, y) = 0\).
Now, we decompose the sum $f(x_1, x_2, \ldots, x_n) = \sum_{\pi \in S_n} \alpha_{\pi} x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(n)}$ according to the order of $x_1$ and $x_2$ in the products $x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(n)}$. So let us decompose $S_n$ into the following two disjoints subsets:

$$S_n^\leq = \{ \pi \in S_n; \pi^{-1}(1) < \pi^{-1}(2) \} \text{ and } S_n^\geq = \{ \pi \in S_n; \pi^{-1}(1) > \pi^{-1}(2) \}.$$ 

Then, $f$ can be decomposed as a sum of $f^<$ and $f^>$, where

$$f^<(x_1, x_2, \ldots, x_n) = \sum_{\pi \in S_n^\leq} \alpha_{\pi} x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(n)}$$

and

$$f^>(x_1, x_2, \ldots, x_n) = \sum_{\pi \in S_n^\geq} \alpha_{\pi} x_{\pi(1)}x_{\pi(2)}\ldots x_{\pi(n)}.$$ 

It is clear that

$$f^<(x_1, x_2, 1, \ldots, 1) = \beta x_1 x_2 \text{ where } \beta = \sum_{\pi \in S_n^\leq} \alpha_{\pi}$$

and

$$f^>(x_1, x_2, 1, \ldots, 1) = \gamma x_2 x_1 \text{ where } \gamma = \sum_{\pi \in S_n^\geq} \alpha_{\pi}.$$ 

Then, $f(x_1, x_2, 1, \ldots, 1) = \beta x_1 x_2 + \gamma x_2 x_1$. Since $F(1, y) = 0$,

$$F(f(x_1, x_2, 1, \ldots, 1), y) = f(F(x_1, y), x_2, 1, \ldots, 1) + f(x_1, F(x_2, y), 1, \ldots, 1).$$ 

Then, for all $x_1, x_2, y \in A$,

$$F(\beta x_1 x_2 + \gamma x_2 x_1, y) = \beta F(x_1, y)x_2 + \beta x_1 F(x_2, y) + \gamma F(x_2, y)x_1 + \gamma x_2 F(x_1, y). \quad (2.1)$$ 

Now we exchange the roles of $x_1$ and $x_2$ in (2.1) so that we get, for all $x_1, x_2, y \in A$,

$$F(\beta x_2 x_1 + \gamma x_1 x_2, y) = \beta F(x_2, y)x_1 + \beta x_2 F(x_1, y) + \gamma F(x_1, y)x_2 + \gamma x_1 F(x_2, y). \quad (2.2)$$ 

The sum of (2.1) and (2.2) is equal to

$$F(\alpha x_1 x_2 + \alpha x_2 x_1, y) = \alpha F(x_1, y)x_2 + \alpha x_1 F(x_2, y) + \alpha F(x_2, y)x_1 + \alpha x_2 F(x_1, y)$$

for all $x_1, x_2, y \in A$. Since $\alpha$ is $\mathcal{M}$-regular, we have, for all $x_1, x_2, y \in A$,

$$F(x_1 x_2 + x_2 x_1, y) = F(x_1, y)x_2 + x_1 F(x_2, y) + F(x_2, y)x_1 + x_2 F(x_1, y).$$ 

Similarly we prove that

$$F(y, x_1 x_2 + x_2 x_1) = F(y, x_1) x_2 + x_1 F(y, x_2) + F(y, x_2) x_1 + x_2 F(y, x_1)$$

for all $x_1, x_2, y \in A$. Therefore, $F$ is a Jordan biderivation. ■
Now we turn to our second aim of this paper. We study Jordan biderivations of unital algebras with idempotents.

Throughout the remainder of this section, we will fix the following condition and notation:

**Setup and notation.** We assume that the algebra $\mathcal{A}$ admits a nontrivial idempotent $e$. Then,
\[
\mathcal{A} = e\mathcal{A}e + e'\mathcal{A}e + e\mathcal{A}e',
\]
where $e' = 1 - e$. To simplify notation we will use the following convention:

- $a = eae \in e\mathcal{A}e = \mathcal{A}_{11}$,
- $m = eme' \in e\mathcal{A}e' = \mathcal{A}_{12}$,
- $n = e'ne \in e'\mathcal{A}e = \mathcal{A}_{21}$,
- $b = e'be' \in e'\mathcal{A}e' = \mathcal{A}_{22}$.

Then each element $x = exe + exe' + e'xe + e'xe' \in \mathcal{A}$ can be represented in the form
\[
x = a + m + n + b,
\]
where $a \in \mathcal{A}_{11}$, $m \in \mathcal{A}_{12}$, $n \in \mathcal{A}_{21}$ and $b \in \mathcal{A}_{22}$. Hence, every bilinear mapping $J : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ can be represented in the form
\[
J(x, y) = J(a, a') + J(a, b') + J(a, m') + J(a, n') + J(m, a') + J(m, b') + J(m, m')
\]
\[
+ J(m, n') + J(n, a') + J(n, b') + J(n, m') + J(b, a') + J(b, b')
\]
\[
+ J(b, m') + J(b, n').
\]
(2.3)

for all $x = a + m + n + b$, $y = a' + m' + n' + b' \in \mathcal{A}$.

Also, in the rest of this paper we assume that any algebra, in particular $\mathcal{A}$, is 2-torsion free (i.e., for every $x \in \mathcal{A}, 2x = 0$ implies $x = 0$). Notice that in this case a bilinear map $J : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a Jordan biderivation if and only if, for all $x, y \in \mathcal{A}$,
\[
J(x^2, y) = xJ(x, y) + J(x, y)x\]
and
\[
J(x, y^2) = yJ(x, y) + J(x, y)y.
\]

The second main result uses the following lemmas.

**Lemma 2.2** Let $J : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ be a Jordan biderivation. Then $[[x, y], J(x, y)] = 0$ for all $x, y \in \mathcal{A}$.

**Proof.** Since $J$ is a Jordan biderivation we have for every $x, y \in \mathcal{A}$,
\[
J(x^2, y^2) = xJ(x, y^2) + J(x, y^2)x.
\]

Then,
\[
J(x^2, y^2) = xyJ(x, y) + xJ(x, y)y + yJ(x, y)x + J(x, y)yx.
\]
By the same argument and using the fact that $J(x, y^2) = yJ(x, y) + J(x, y)y$ we get

$$J(x^2, y^2) = xyJ(x, y) + yJ(x, y)x + xJ(x, y)y + J(x, y)xy.$$ 

Comparing both relations leads to $[[x, y], J(x, y)] = 0$. ■

**Lemma 2.3** Let $J : A \times A \to A$ be a Jordan biderivation. Then $J = J_1 + J_2$, where $J_1(x, y) = [x, [y, J(e, e)]]$ is an extremal biderivation and $J_2$ is a Jordan biderivation such that $J_2(e, e) = 0$.

**Proof.** Let us first consider the identity $[[x, y], J(x, y)] = 0$ for all $x, y \in A$. Replacing $x$ by $x + e$, we get

$$[[x, y], J(e, y)] + [[e, y], J(x, y)] = 0.$$ 

This implies that, $[[x, e], J(e, e)] = 0$. Similarly, we obtain that

$$[[x, e], J(x, y)] + [[x, y], J(x, e)] = 0.$$ 

Next, replacing $x$ by $x + e$ and $y$ by $y + e$ in the relation $[[x, y], J(x, y)] = 0$ and summarizing the above conclusions, we see that

$$[[x, y], J(e, e)] + [[x, e], J(x, y)] + [[e, y], J(x, e)] = 0.$$ 

Hence, $[[exe, e ye], J(e, e)] = 0 = [[e' xe', e' ye'], J(e, e)]$.

Also, from the relations $[[x, e], J(e, e)] = 0$ and $e'J(e, e)e' = 0$, we get

$$J(e, e)exe' = [J(e, e), exe']e' = [[exe', e], J(e, e)]e' = 0$$

and

$$e' xeJ(e, e) = e'[exe, J(e, e)] = e'[[exe, e], J(e, e)] = 0.$$ 

In a similar manner, we can show that $exe'J(e, e) = 0 = J(e, e)e' xe$. Therefore, we conclude that $J_1$ is an extremal biderivation and $J_1(e, e) = J(e, e)$.

Indeed,

$$[[x, y], J(e, e)] = [e[x, y]e + e[x, y]e' + e'[x, y]e + e'[x, y]e', J(e, e)]$$

$$= [e[x, y]e + e'[x, y]e', J(e, e)]$$

$$= [[exe, eye] + [e' xe', e ye'], J(e, e)]$$

$$= 0.$$ 

It is easy to verify that $J_2 = J - J_1$ is a Jordan biderivation. ■
It is worthwhile mentioning that it was Herstein who initiated the study of Jordan derivations on associative rings. In [11], he proved that for every prime ring \( A \) of characteristic different from \( 2 \), a Jordan derivation of \( A \) is a derivation of \( A \). The following remark is the “biderivation” counterpart of his results [11, Lemmas 3.1. and 3.2.].

**Remark 1** If \( J : A \times A \rightarrow A \) is a Jordan biderivation, then the following assertions hold for all \( x, y, z, t \in A \):

1. \( J(xy, z) = J(x, z)yx + xJ(y, z)x + xyJ(x, z) \).
2. \( J(yz, x) = J(x, t)yz + xJ(y, t)z + xyJ(z, t) + J(z, t)yx + zJ(y, t)x + zyJ(x, t) \).

The following lemma is the key result for decomposing a Jordan biderivation as a sum of a biderivation and an antibiderivation.

**Lemma 2.4** Let \( J : A \times A \rightarrow A \) be a Jordan biderivation such that \( J(e, e) = 0 \). Then, the following assertions hold for all \( a, a' \in A_{11}, m, m' \in A_{12}, n, n' \in A_{21} \) and \( b, b' \in A_{22} \):

1. \( J(a, a') = eJ(a, a')e \) and \( J(b, b') = e'J(b, b')e' \).
2. \( J(a, m) = aJ(e, m) + J(e, m)a \) and \( J(m, a) = aJ(m, e) + J(m, e)a \).
3. \( J(b, m) = bJ(e', m) + J(e', m)b \) and \( J(m, b) = bJ(m, e') + J(m, e')b \).
4. \( J(a, n) = aJ(e, n) + J(e, n)a \) and \( J(n, a) = aJ(n, e) + J(n, e)a \).
5. \( J(b, n) = bJ(e', n) + J(e', n)b \) and \( J(n, b) = bJ(n, e') + J(n, e')b \).
6. \( J(m, n) = eJ(m, n)e' + e'J(m, n)e + [J(e, n), m] = eJ(m, n)e' + e'J(m, n)e + [n, J(m, e)] \).
7. \( J(n, m) = eJ(n, m)e' + e'J(n, m)e + [J(n, e), m] = eJ(n, m)e' + e'J(n, m)e + [n, J(e, m)] \).
8. \( J(n, n') = eJ(n, n')e' + e'J(n, n')e + [n', J(n, e)] = eJ(n, n')e' + e'J(n, n')e + [n', J(e, n')] \).
9. \( J(m, m') = eJ(m, m')e' + e'J(m, m')e + [J(e, m'), m] = eJ(m, m')e' + e'J(m, m')e + [J(m, e), m'] \).
On the other hand, (8) and (9) also hold. Indeed, we show that
\[ J(a, a') = J(e, a') = J(e', a') = 0. \]

We have from \( J(x^2, y) = xJ(x, y) + J(x, y)x \), then, taking \( x = e \) and \( y = a \), we get \( eJ(e, a)e = 0 \). Similarly, from \( J(x, y^2) = yJ(x, y) + J(x, y)y \), we get \( eJ(a, e)e = 0 \). Now, according to Remark 1 and identity \( eJ(e, a)e = eJ(e, a)e = 0 \), we have
\[ J(a, a') = J(ea, ea') = eaJ(ea') + J(e, ea')e + J(e, ea')ae \]
\[ = e(aea'J(e, e) + eJ(e, a')e + J(e, e)a' e) \]
\[ + e(2)J(a, e) + eJ(a, ea')e + J(a, e)a' e) \]
\[ + (ea' J(e, e) + eJ(e, a')e + J(e, e)a' e)ea \]
\[ = eJ(a, a').e. \]

Similarly, we can show that \( J(b, b') = e'J(b, b')e' \) and that the relation (10) is true. Now to prove the assertion (2), we use Remark 1. Namely, we obtain
\[ J(a, m) = eaJ(e, em') + eJ(a, em')e + J(e, em')ae \]
\[ = ea(2)J(e, m)e' + eJ(e, m)eae \]
\[ = aJ(e, m) + J(e, m)a. \]

In a similar manner, we can prove that the conditions (3), (4) and (5) hold. Next, we show that \( J(m, n) = eJ(e, m, n)e + eJ(m, n)e + [J(e, n), m] = eJ(m, n)e' + e'J(m, n)e + [J(m, e), n] \), and one can prove analogously that the conditions (7), (8) and (9) also hold. Indeed,
\[ J(m, n) = J(em', e'm, n) = emJ(e', n) + eJ(m, n)e' + J(e, n)me' \]
\[ + e'mJ(e, n) + e'J(m, n)e + J(e', n)me \]
\[ = eJ(m, n)e' + e'J(m, n)e + [J(e, n), m]. \]

On the other hand, \( J(m, n) = J(m, en'e + e'n) = eJ(m, n)e' + e'J(m, n)e' + [n, J(m, e)]. \)

Consider the decomposition of a bilinear mapping \( J : A \times A \rightarrow A \) given in (23). When \( J \) is a Jordan biderivation, we could continue this decomposition using all the assertions of Lemma 2, so we get a new larger decomposition. We will show, in the following two lemmas, that one part \( \Delta \) of this new decomposition is an antibiderivation and another part \( D \) is a biderivation. So we get
\[ J = \Delta + D + eJ(m, n)e' + e'J(m, n)e + eJ(n, m)e' + e'J(n, m)e. \]
Under the condition given in Theorem 2.7, we will show that the part \( eJ(m,n)e' + e'J(m,n)e + eJ(n,m)e' + e'J(n,m)e \) is zero.

**Lemma 2.5** For a Jordan biderivation \( J : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \) such that \( J(e,e) = 0 \), a mapping \( \Delta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \) is an antibiderivation if it satisfies the following conditions for all \( a, a' \in \mathcal{A}_{11}, m, m' \in \mathcal{A}_{12}, n, n' \in \mathcal{A}_{21} \) and \( b, b' \in \mathcal{A}_{22} \):

1. \( \Delta(m, m') = e'J(m, m')e \) and \( \Delta(n, n') = eJ(n, n')e' \).
2. \( \Delta(a, m) = J(e, m)a \) and \( \Delta(m, a) = J(m, e)a \).
3. \( \Delta(b, m) = bJ(e, m) \) and \( \Delta(m, b) = bJ(m, e) \).
4. \( \Delta(a, n) = aJ(e, n) \) and \( \Delta(n, a) = aJ(n, e) \).
5. \( \Delta(b, n) = J(e, n)b \) and \( \Delta(n, b) = J(n, e)b \).
6. \( \Delta(a, a') = \Delta(b, b') = \Delta(b, b') = \Delta(a, b) = \Delta(b, a) = \Delta(m, n) = \Delta(n, m) = 0 \).

**Proof.** Since \( J(m, e)[A, A] = 0 \) and \( J(a, m) = J(e, am) \), we conclude that

\[
\Delta(m, aa') - a'd\Delta(m, a) - \Delta(m, a')a = J(m, e)aa' - a'J(m, e)a - J(m, e)a'a = 0
\]

and

\[
\Delta(a'm, a) - \Delta(m, a)a' - m\Delta(a, a') = J(m, a')a - J(m, e)aa' = J(m, e)[a', a] = 0.
\]

Using the fact that \( J(e, mb) = J(e, mb + bm) = J(e, m)b + bJ(e, m) = -J(b, m) \) and the condition (3) in Lemma 2.4 we get

\[
\Delta(a, mb) - \Delta(a, b)m - b\Delta(a, m) = J(e, mb)a - J(mb, e)a - b(J(e, m)a + J(m, e)a) = 0.
\]

The other conditions for \( \Delta \) to be an antibiderivation can be proved with a similar calculation. \( \blacksquare \)

In what follows we will also use Benkovič and Širovnik’s conditions [4]; that is the algebra \( \mathcal{A} \) satisfies the following two implications which will be refereed as “the conditions (*)”:

- For all \( x \in \mathcal{A}, exe \cdot eAe' = \{ 0 \} = e'Ae \cdot exe \) implies \( exe = 0 \).
- For all \( x \in \mathcal{A}, eAe' \cdot e'xe' = \{ 0 \} = e'xe' \cdot e'Ae \) implies \( e'xe' = 0 \).
Some important examples of unital algebras with nontrivial idempotents having the conditions (*) are triangular algebras, matrix algebras and prime (and hence in particular simple) algebras with nontrivial idempotents.

**Lemma 2.6** Assume that \( A \) satisfies the conditions (*). Then, for a Jordan biderivation \( J : A \times A \to A \) such that \( J(e, e) = 0 \), a mapping \( D : A \times A \to A \) is a biderivation if it satisfies the following conditions for all \( a, a' \in A_{11}, m, m' \in A_{12}, n, n' \in A_{21} \) and \( b, b' \in A_{22} \):

1. \( D(a, a') = J(a, a') \) and \( D(b, b') = J(b, b') \).
2. \( D(a, m) = aJ(e, m) \) and \( D(m, a) = aJ(m, e) \).
3. \( D(b, m) = J(e', m)b \) and \( D(m, b) = J(m, e')b \).
4. \( D(a, n) = J(e, n)a \) and \( D(n, a) = J(n, e)a \).
5. \( D(b, n) = bJ(e', n) \) and \( D(n, b) = bJ(n, e') \).
6. \( D(m, m') = eJ(m, m')e' \) and \( D(n, n') = e'J(n, n')e \).
7. \( D(a, b) = D(b, a) = D(m, n) = D(n, m) = 0 \).

**Proof.** From the assumption \( J(e, e) = 0 \) we get \( J(e, e') = J(e', e) = J(e', e') = 0 \) and as in the first part of the proof of Lemma 2.4 \( J(x^2, y) = xJ(x, y) + J(x, y)x \) and \( J(x, y^2) = yJ(x, y) + J(x, y)y \), shows that \( eJ(e, a)e = eJ(a, e)e = 0 \). From Remark 1 and the identity \( eJ(e, a)e = eJ(a, e)e = 0 \), we have

\[
J(am, e) = J(am + ma, e) = J(eam + mae, e) = J(e, e)am + eJ(a, e)m + aJ(m, e) + J(m, e)a + mJ(a, e)e + maJ(e, e)e = J(m, e)J(m, e)a
\]

Using condition (2) of Lemma 2.4, we get \( J(am, e) = J(m, a) \). Then, from \( J(am, e) = aJ(m, e) + J(m, e)a \), we get \( eJ(m, e)e = 0 \). Indeed, we have \( J(m, e) = J(em, e) = eJ(m, e) + J(m, e)e \). Next, we claim that \( J(a, a'm) = [a', [a', J(m, e)] \). Since,

\[
a'aJ(m, e) + J(m, e)a'a' = a'J(m, a) + J(m, a)a'
= a'J(am, e) + J(am, e)a'
= J(am + ma, a')
= J(a, a')m + aJ(m, a') + J(m, a'a)
= J(a, a')m + a'a'J(m, e) + J(m, e)a'a.
\]
Next, we claim that \( J(a, a')m = [a', a]J(m, e) \) and \( J(m, e)[a, a'] = 0 \). Since,

\[
a'aj(m, e) + J(m, e)a'a = J(m, a'a) = J(a'am, e) = J(am, a') = J(am + ma, a') = a'aJ(m, e) + J(a, a')m + J(m, e)a'a.
\]

Now, using the conditions (*) , we get \( D(a_1a_2, a) = D(a_1, a)a_2 + a_1D(a_2, a) \) for all \( a_1, a_2, a \in A_{11} \). Indeed, for all \( m \in \mathcal{A}_{12} \), we have

\[
(D(a_1a_2, a) - a_1D(a_2, a) - D(a_1, a)a_2)m = (J(a_1a_2, a) - a_1J(a_2, a) - J(a_1, a)a_2)m
\]

\[
= [a, a_1a_2]J(m, e) - a_1[a, a_2]J(m, e) - [a, a_1]J(m, a_2)
\]

\[
= a_1a_2J(m, e) - a_1a_2J(m, e) - a_1a_2J(m, e) + a_1a_2J(m, e)
\]

\[
= a_1a_2J(m, e) - a_1a_2J(m, e) + a_1a_2J(m, e)a_2 - a_1a_2J(m, e)a_2
\]

\[
= J(a, a_1)ma_2
\]

\[
= 0.
\]

Similarly we obtain, for all \( n \in \mathcal{A}_{21} \), \( n(D(a_1a_2, a) - a_1D(a_2, a) - D(a_1, a)a_2) = 0 \). So the first implication of the conditions (*), gives \( D(a_1a_2, a) = D(a_1, a)a_2 + a_1D(a_2, a) \).

Moreover,

\[
D(am, a') - aD(m, a') - D(a, a')m = a'J(am, e) - aa'J(m, e) - J(a, a')m
\]

\[
= a'aJ(m, e) - aa'J(m, e) - J(a', a)m
\]

\[
= 0.
\]

Analogously, we can prove the other relations for \( D \) to be a biderivation. \( \blacksquare \)

We are now in a position to state and to prove the second main result.

**Theorem 2.7** Assume that \( A \) satisfies the conditions (*) and that the zero homomorphism is the only \( (e_Ae, e'Ae') \)-module morphism \( f : e_Ae' \rightarrow e_Ae' \) such that \( e[A, A]e \cdot f(e_Ae') = f(e_Ae') \cdot e[A, A]e' = 0 \), then every Jordan biderivation \( J : A \times A \rightarrow A \) can be written as a sum of a biderivation, an antibiderivation and an extremal biderivation.
Proof. Let \( J : A \times A \rightarrow A \) be a Jordan biderivation. Using Lemma 2.3, we get \( J = J_1 + J_2 \), where \( J_1(x, y) = [x, [y, J(e, e)]] \) is an extremal biderivation and \( J_2 \) is a Jordan biderivation with \( J_2(e, e) = 0 \). Following the discussion given before Lemma 2.5, we get the result if we prove that

\[
e J(m, n)e' = e' J(m, n)e = e J(n, m)e' = e' J(n, m)e = 0.
\]

Now fix \( n \in A_{21} \). Then, the map \( f : A_{12} \rightarrow A_{12} \) defined by \( f(m) = e J(m, n)e' \) (for all \( m \in A_{12} \)) is a module homomorphism. Moreover, according to Remark 1, we get \( [A_{11}, A_{11}]f(m) = f(m)[A_{22}, A_{22}] = 0 \). Indeed, for all \( a, a' \in A_{11} \) and \( b, b' \in A_{22} \), one can check easily that \( aa'f(m) = a'af(m) \) and \( f(m)bb' = f(m)bb' \). Hence, by hypothesis, \( f = 0 \). Similarly, we can obtain the other relations.

Let \( Id([A, A]) \) denotes the ideal generated by all commutators \([x, y]\) \((x, y \in A)\) of an algebra \( A \). The following corollary is an immediate consequence of Theorem 2.7.

Corollary 2.8 If \( A \) satisfies the conditions (*) and either \( Id([eAe, eAe]) = eAe \) or \( Id([e'Ae', e'Ae']) = e'Ae' \), then every Jordan biderivation \( J : A \times A \rightarrow A \) can be written as a sum of a biderivation and an antibiderivation.

Proof. We have \( J(m, e)A_{11} = 0 \), then by hypothesis, we get \( J(m, e)A_{11} = 0 \). Since \( A_{11} \) has a unity element, it follows that \( J(m, e) = 0 \). So \( J(a, a')m = [a', a]J(m, e) = 0 \) for all \( m \in A_{12} \). Similarly we have \( nJ(a, a') = 0 \) for all \( n \in A_{21} \). Thus, using the conditions (*), \( J(e, e) = 0 \). Therefore, by Theorem 2.7, we get the result.

If \( A \) admits a nontrivial idempotent \( e \) such that \( eA'eA = \{0\} = e'Ae'Ae' \) and the bimodule \( eAe \) is faithful as both a left \( eAe \)-module and a right \( e'Ae' \)-module, then \( A \) satisfies the conditions (*). Then, we have the following corollary.

Corollary 2.9 Assume that \( eA'eA = \{0\} = e'Ae'Ae' \) and either \( Id([eAe, eAe]) = eAe \) or \( Id([e'Ae', e'Ae']) = e'Ae' \). If the bimodule \( eAe \) is faithful as both a left \( eAe \)-module and a right \( e'Ae' \)-module, then every Jordan biderivation \( J : A \times A \rightarrow A \) can be written as a sum of a biderivation and an antibiderivation.

Recall that an algebra \( A \) is isomorphic to a triangular matrix algebra if there exists a non trivial idempotent \( e \in A \) such that \((1 - e)Ae = 0\) (see, for instance, [10] Theorem 5.1.4). Thus, triangular algebras are examples of algebras that satisfies the conditions of Theorem 2.7. Namely, we get the following result which generalizes [2] Theorem 2.10.
Corollary 2.10  
Every Jordan biderivation on a triangular algebra is a biderivation.

At this stage we remark that, triangular algebras are special examples of trivial extension algebras on which the Jordan generalized and Lie generalized derivations are recently investigated in [3, 7]. We conclude this section with the following question, to the best of our knowledge, has not been studied/answered yet:

Question 1  
Under what conditions every Jordan biderivation on a trivial extension algebra is a biderivation?

Acknowledgement. The authors would like to thank the referee for the careful reading of the paper.

References

[1] C. Abdioğlua and T-K Lee, A basic functional identity with applications to Jordan σ-biderivations, Comm. Algebra 45 (2017), 1741–1756.

[2] D. Aiat Hadj Ahmed, On Jordan Biderivations of Triangular Matrix Rings, Journal of Mathematical Research with Applications 36 (2016), 162–170.

[3] M.A. Bahmani, D. Bennis, H.R. Ebrahimi Vishki, A. Erfanian Attar and B. Fahid Jordan generalized derivations on trivial extension algebras, Commun. Korean Math. Soc. 33 (2018), 721-739.

[4] K. I. Beidar, Y. Fong, On additive isomorphisms of prime rings preserving polynomials, J. Algebra 217 (1999), 650–667.

[5] D. Benkovič, A note on f-derivations of triangular algebras, Aequat. Math. 89 (2015), 1207–1211

[6] D. Benkovič and N. Sirovnik, Jordan derivations of unital algebras with idempotents, Linear Algebra Appl. 437 (2012), 2271–2284.

[7] D. Bennis, H.R. Ebrahimi Vishki, B. Fahid, A.A. Khadem-Maboudi and A.H. Mokhtari, Lie generalized derivations on trivial extension algebras, Boll. Unione Mat. Ital. 12 (2019), 441–452.

[8] M. Brešar, Jordan mappings of semiprime rings, J. Algebra 127 (1989), 1003–1006.
[9] M. Brešar, W.S. Martindale 3rd and C.R. Miers, *Centralizing maps in prime rings with involution*, J. Algebra 161 (1993), 342–357.

[10] G. F. Birkenmeier, J. K. Park and S. T. Rizvi, *Extensions of rings and Modules*, Birkhauser, (2013).

[11] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1104–1110.

1. Department of Pure Mathematics, Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, IRAN.
   mohamadali_bahmani@yahoo.com

2. Centre de Recherche de Mathématiques et Applications de Rabat (CeReMAR), Faculty of Sciences, Mohammed V University in Rabat, Morocco.
   driss.bennis@um5.ac.ma; driss_bennis@hotmail.com

3. Department of Pure Mathematics, Centre of Excellence in Analysis on Algebraic Structures (CEAAS), Ferdowsi University of Mashhad, P.O. Box 1159, Mashhad 91775, IRAN.
   vishki@um.ac.ir

4. Superior School of Technology, Ibn Tofail University, Kenitra, Morocco.
   fahid.brahim@yahoo.fr