Modelling of Contact Interaction with Allowance for Nonlinear Compliance in Unilateral Constraints

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Abstract. The finite element model and numerical solution of the contact problem in the presence of nonlinear deformable layer between interacting bodies are considered. To discretize the contact layer, frame-rod contact finite elements are used, which also model different contact states (separation, clutch, friction-slip, etc.). The use of frame-rod contact elements allows the physical nonlinearity of the contact layer to be reduced to the internal nonlinearity of the system of contact elements, while the nonlinear properties of the seam will be set through the nonlinear characteristics of the individual rods of the contact elements. The numerical algorithm for solving the contact problem is based on the method of step-by-step analysis with iterative refinement of the solution for the current load level. With its help, the numerical solutions of the problem of contact of the structure with the base under different conditions of the contact seam have been obtained and analyzed.

1. Introduction

At present, numerical methods, in particular the finite element method (FEM) [1–5], are used to solve contact problems with unilateral constraints and friction. In this case, the continuum contact problems of solid elastic bodies are reduced to finite-dimensional tasks with discrete unilateral constraints. In turn, special contact finite elements (CFE) can also be used to model contact interaction, which are introduced in the contact zone of elastic bodies, thereby discretizing a thin layer between the contacting surfaces [5–8]. This contact layer combines interacting bodies into a single system and, thanks to its special properties, can satisfy the necessary contact conditions [6]. The use of CFE allows to take the nonlinear compliance of the contact seam into account, reducing it to the internal nonlinearity of the contact elements discretizing the intermediate layer. Various iterative schemes of accounting for nonlinear effects on the contact of interacting bodies were considered in [8–11]. The nonlinear relationship between the deformation of the contact layer and the contact stresses is defined here by means of the corresponding nonlinear dependencies or diagrams.

The contact finite elements proposed by the author – in the form of a flat or space frame - are used for modeling of unilateral constraints [12, 13]. These elements interact with the usual planar or spatial elements and provide contact between nodes in the finite element mesh that are on the boundary surfaces of the contacting bodies. The algorithms for solving various contact problems, both with static and dynamic load action, have been developed on the basis of the frame-rod model of contact interaction and the method of step-by-step analysis. Among them there is the account of friction on the contact of interacting bodies, elastic compliance and strength of the contact joints, sequential erection and loading facilities and a number of other factors, the consideration of which approximates the design to the actual conditions of the structure [14–18].

The planar contact problem for linearly elastic bodies \(V^+\) and \(V^-\), between boundary surfaces of which \((S^+_c, S^-_c)\) there is the nonlinear deformable contact layer having thickness \(\zeta^0\) and compliance \(\rho_n, \rho_t\) (normal and tangential respectively) is discussed below. Modeling of contact in discrete model
of FEM is carried out by means of plane frame-rod contact elements. The CFE data interacts with the usual finite elements of a discrete calculation scheme and thus providing a connection between the grid nodes located on the boundary surfaces of the contacting bodies (Figure 1).

![Diagram of contact modeling by means of frame-rod CFE](image)

**Figure 1.** Modeling of contact by means of frame-rod CFE:

a) CFE in the contact area; b) displacement and effort in the CFE

Considering the $k$ contact element as the $k$ discrete unilateral compliance connection (normal and tangential) between interacting bodies, let us put down the conditions on the contact in terms of forces and displacements for the $k$ contact element element (at $S_c$, $S_c = S^+ _c \cup S^- _c$):

$$u_{nk} + u^c_{nk} \leq 0; \quad N_k \leq 0; \quad (u_{nk} + u^c_{nk}) N_k = 0.$$  \hspace{1cm} (1)

$$T_k \leq T_{pk}; \quad (u_{tk} + u^c_{tk}) \geq 0; \quad (|T_k| - T_{pk})(u_{tk} + u^c_{tk}) = 0.$$  

Here $u_{nk}$, $u_{tk}$ are the mutual displacements of the opposite nodes at $S^+_c$ and $S^-_c$ in normal and tangential direction; $u^c_{nk}$, $u^c_{tk}$ are the longitudinal and transverse strains in the $k$ CFE; $N_k, T_k$ are forces in the support rod in the $k$ CFE; $T_{pk} = -f_k N_k$ is the ultimate "Coulomb" friction force; $f_k \geq 0$ is the coefficient of friction at $k$ contact; $C_{nk}, C_{tk}$ are normal and tangential stiffness of the contact seam material. With the linear compliance: $C_{nk} = A_k / \rho_{nk}$, $C_{tk} = A_k / \rho_{tk}$, where $A_k$ is the contact area related to the $k$ CFE.

Numerical implementation of the contact conditions (1) is performed with the help of step-by-step analysis of the contact state changes in the process of sequential application of the given load. Here, we are able to better meet the friction conditions, since the solution of the friction problem depends on the history of the structure loading. The transition from one state to the other is, respectively, an event of switching off or switching on of unilateral constraint, slippage or clutching. The method of step-by-step analysis is the most effective for the considered class of contact problems, in addition, it is possible to monitor the current state of the contact seam during the structure loading.

The algorithms of step-by-step analysis, including the sequence of actions at each step, both under static and dynamic loading, are described in details in [6, 13–16]. In the base of these algorithms is the representation of constructively nonlinear contact problem as a sequence of a finite number of linear problems with subsequent changes in the working schemes of the structure. The time of the next event on the contact is determined by the analysis of the behavior of the working scheme. As a result of the next step, the working scheme is changed and the new state of the contact is established, while the method of compensating loads is used to fulfill the contact conditions for friction sliding.

Below, for the purpose of the subsequent comparison, some expressions of step-by-step algorithm (at a linearly deformable contact layer) defining the moments of occurrence of the next event on contact are given, in particular, the moment of slippage for the constraint which has been before in epy state of clutch and the moment of shutdown of constraint from work (separation):
\[ \Delta x_{k}^{s+1} = \Delta x^{s+1} \left( \frac{T_{p,k}^{s} - T_{k}^{s}}{\Delta T_{p,k}^{s+1} - \Delta T_{k}^{s+1}} \right), \quad k \in S_{1c}; \quad \Delta x_{k}^{s+1} = \Delta x^{s+1} \left( \frac{-N_{k}^{s}}{\Delta N_{k}^{s+1}} \right), \quad k \in S_{1c}, S_{2c}. \]  

Here the sign "tilde" denotes the values corresponding to the trial step of loading \( \Delta x^{s+1} \), values of effort \( T_{p,k}^{s}, T_{k}^{s} \) and \( N_{k}^{s} \) correspond to the "s" level of loading. \( k \in S_{1c} \) are the constraints in the state of pre-ultimate friction (clutching); \( k \in S_{2c} \) – in conditions of ultimate friction (sliding).

At nonlinear law deformations of the contact layer rigidity in the normal and tangent direction will be functions of the variables, respectively, compression and shear contact of the seam: \( C_{n} = C_{n}(u_{c}^{i}) \); \( C_{c} = C_{c}(u_{c}^{i}) \). Using a frame-rod CFE allows to reduce the physical nonlinearity of the contact layer to the internal nonlinearity of the system, only the contact elements, while the nonlinear properties of the seam are determined via non-linear characteristics of the single rods of the CFE.

In step-by-step solution of the contact problem (based on the discrete model of FEM) for each \((s+1)\)-th level of loading, the following matrix equilibrium equation is valid:

\[ \left[ K_{lin} + K_{nel}(u^{s+1}) \right] \Delta u^{s+1} = P^{s+1} - \left[ K_{lin} + K_{nel}(u^{s+1}) \right] u^{s}. \]  

Here \( u^{s+1} \) and \( P^{s+1} \) are correspondingly, the vectors of nodal displacements and external load at the end \((s+1)\) of the step; \( \Delta u^{s+1} \) is the value of increments of displacements at \((s+1)\)-th step; \( K_{lin}, K_{nel}(u^{s+1}) \) are linear and non-linear components of the stiffness matrix of the system of finite elements.

By moving the nonlinear component of the equation \( K_{nel}(u^{s+1}) \Delta u^{s+1} \) to the right side, you can write the following recurrent equation to determine the increments of displacements \( \Delta u^{s+1} \):

\[ K_{lin} \Delta u_{i}^{s+1} = P_{lin}^{s+1} - P_{lin}^{s+1}(u_{i}^{s+1}), \]  

where \( \Delta u_{i}^{s+1} \) are the values of increments of displacements on the current \( i \) iteration; \( P_{lin}^{s+1} - K_{lin} u^{s} \) is the constant (linear) part of the vector of right parts; \( P_{lin}^{s+1}(u_{i}^{s+1}) = K_{nel}(u_{i}^{s+1}) \Delta u_{i}^{s+1} \) is the changeable part of the vector of right parts.

Let us present a nonlinear matrix, \( K_{nel}(u) \) corresponding to a discrete layer, as the sum of the nonlinear components of the CFE stiffness matrices:

\[ K_{nel}(u) = \sum_{k} \left[ K_{n,nel}^{(k)}(u_{n}^{i}) + K_{n,nel}^{(k)}(u_{c}^{i}) \right]. \]  

In turn, the nonlinear component of the stiffness matrix of \( k \) CFE, for example, \( K_{n,nel}^{(k)}(u_{n}^{i}) \), can be written as follows:

\[ K_{n,nel}^{(k)}(u_{n}^{i}) = C_{n}(u_{n}^{i}) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \]  

Here, the contribution of the individual CFE to the changeable “nonlinear” part of the right-hand side vector \( K_{nel}(u_{i+1}^{s+1}) \Delta u_{i}^{s+1} \) will be as follows:

\[ C_{n}(u_{n}^{i}) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (u_{n}^{i})^{s} + (\Delta u_{n}^{s+1})_{i}^{s+1} \\ (u_{n}^{i})^{s} + (\Delta u_{n}^{s+1})_{i}^{s+1} \end{bmatrix} = C_{n}(u_{n}^{i}) \begin{bmatrix} -(u_{n}^{i})^{s} - (\Delta u_{n}^{s+1})_{i}^{s+1} \\ (u_{n}^{i})^{s} + (\Delta u_{n}^{s+1})_{i}^{s+1} \end{bmatrix}, \]  

where \((u_{n}^{i})^{s}, (u_{c}^{i})^{s}\) are moving opposite points to \( S_{c}^{+}, S_{c}^{-} \) at the end of the “s” step; \((\Delta u_{n}^{s+1})_{i}^{s+1} = (u_{n}^{i})_{i+1}^{s+1} - (u_{n}^{i})^{s}\) are respectively the increments of displacements of opposite points in

\[ \]
Taking into account the nonlinear deformation of the contact layer, both in the normal and tangential direction, according to (5), the iterative expression (4) takes the following form

\[ K_{li}^n \Delta u_{i}^{s+1} = P_{lin}^{s+1} - \left( \sum_{k \in S_{1c}, S_{2c}} F_{nk}^+ \right) + \sum_{k \in S_{1c}} \left( F_{nk}^- \right) \right]_{i=1}^{s+1}. \]  

(8)

Here \( F_{nk}^+, F_{nk}^-, F_{tk}^+, F_{tk}^- \) are the forces applied on the \( i \)-iteration in the process of iterative refinement \( \Delta u_i^{s+1} \) for \((s+1)\) step of loading to the opposite nodes of contact surfaces and, respectively, normal and tangential;

The left part of equations (8) for the given working scheme of contact here does not change that makes it able to hold the factorization of the stiffness matrix once, and then adjusting the value \( \Delta u_i^{s+1} \) until the difference between two subsequent iterations does not satisfy the given accuracy of calculation.

Therewith the values of the correcting forces \( F_{nk}^+, F_{nk}^-, F_{tk}^+, F_{tk}^- \) are calculated based on the results of the previous \((i-1)\)-th iteration:

\[ \begin{align*}
F_{nk}^+ & = C_n (u_i^c) \left[ -(u_i^c)^s - (\Delta u_i^c)_{i=1}^{s+1} \right], \\
F_{nk}^- & = C_n (u_i^s + (\Delta u_i^c)_{i=1}^{s+1}), \\
F_{tk}^+ & = C_t (u_i^c) \left[ -(u_i^c)^s - (\Delta u_i^c)_{i=1}^{s+1} \right], \\
F_{tk}^- & = C_t (u_i^c)^s + (\Delta u_i^c)_{i=1}^{s+1}. 
\end{align*} \]  

(9)

Thus, the corresponding expressions for determining the moment of occurrence of the next event on the contact (switching off, switching of constraints, slipping or clenching) will now be put down in the form of iterative formulas. So, the expression (2) will take the following form:

\[ (\Delta \alpha_k^{s+1})_{i=1}^{s+1} = (\Delta \alpha_k^{s+1})_{i=1}^{s+1} \left( \frac{T_k^s - T_k^s}{(\Delta T_k^{s+1})_{i=1}^{s+1}} \right)_{k \in S_{1c}} \]  

(10)

Here \( T_k^s, N_k^s \) are the values of contact forces for the \("s\) level of loading; \((\Delta T_k^{s+1})_{i=1}^{s+1}, (\Delta N_k^{s+1})_{i=1}^{s+1}\) are the increment of effort on the \((i-1)\) iteration to the \((s+1)\) step; \((\Delta \alpha_k^{s+1})_{i=1}^{s+1}\) is the iterative refinement of the step value \((s+1)\). Step-by-step algorithm defining expressions also take the iterative form. The end of the iterative refinement for the loading step is to achieve the specified accuracy of calculations. In other respects, the sequence of computational solution of the contact problem with a nonlinearly deformable layer corresponds to the algorithms described in [6, 13–16].

In fact the nonlinear deformation law of the contact layer can be much more complicated than that recorded in the form of (5). First of all, this applies to tangential forces, which should take into account not only the shear deformation, but also the compression of the contact layer. In this case, using the method of step-by-step analysis allows us to establish the dependence of the forces on the deformation at each individual step of loading, then iterative refinement of the solution for the current level of loading.

Using the stated algorithm, numerical solutions for the test problem of the contact interaction of the structure with the base are obtained. Conditions of Coulomb friction-sliding (friction coefficient \( f = 0.2 \), as well as separation of boundary surfaces from each other, are possible at the contact areas.
The breakdown by finite element is shown in Figure 2, a (to the right of the axis of symmetry). Contact seam was simulated by nine frame-core CFE. The aim of the calculations was to assess the various conditions of contact interaction on the stress state of the soles of the structure.

The comparative analysis of the results of calculations obtained under the following conditions of the contact seam was conducted:

1) with zero compliance of the contact seam (hard unilateral contact taking into account the Coulomb friction);
2) with linear compliance of the contact seam in both directions (taking friction into account);
3) with nonlinear compliance of the contact seam in the direction of normal;

The results of calculations - mutual displacements and stresses on the contact are shown in Figure 2, b, c, d. Continuous line corresponds to the calculation of the first option, dashed-on the second one, dotted-on the third one.

![Figure 2. The problem of the interaction of a structure (1) with a hard base (2):](image)

- a) design scheme; b) contact stresses; c), d) mutual displacement

As it can be seen from the results of the calculation, the area of separation of contact surfaces has the largest dimensions in the first variant – with the hard contact of the structure with the base. When considering the compliance of the seam (thickness $\zeta^0 = 2$ mm, compliance $\rho_0 = 1.25\cdot10^{-7}$ m$^3$/MN, modulus of elasticity of material of structure is 3060 MPa) the area of separation is slightly reduced. At the same time, due to shear deformations of this layer, the mutual displacement of the contact surfaces horizontally increases by 20%. The intensity of contact stresses here is slightly less than in the case of hard contact. It should be noted that the calculation of step-by-step results for the first and second variants fully correspond to the solutions of the problem under consideration, obtained in a number of papers by iterative methods [19–21].

The nonlinear law of deformation of the contact seam in the direction of the normal to the boundary surfaces was conveniently set by the following function [8]:

\[
\sigma_n(x, y) = q \left(1 - \left(\frac{x}{a} + \frac{y}{b}\right)^2\right)
\]
\[ C_{nk}(u_{nk}^c) = A_k \sqrt{\frac{\rho_0}{1 - \frac{1}{n} \left( \frac{u_{nk}^c}{\mu^c} \right)^2}}. \] (11)

In the tangential direction the stiffness of the seam was taken as constant one: \( C_{tk} = A_k / \rho_0 \). Comparison of the calculation results shows that taking into account the nonlinear deformation of the contact seam leads to some redistribution of deformations when it is compressed.

The carried out calculations confirm that the account of the contact seam compliance, in particular, between the structure and the base, is important in assessing the stress-strain state and, therefore, for the normal operation of the structure. Modeling of initial strength and compliance of the unilateral constraints of the contact elements frame rod allows the constructive and the physical nonlinearity of the problem to reduce to the internal nonlinearity of only the discrete contact layer. In turn, the presence of the compliance factor in one-sided relationships can also be considered as a way of regularization, i.e. improving the properties of the contact problem with friction.

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