Null Energy Condition Violation and Classical Stability in the Bianchi I Metric

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Abstract

The stability of isotropic cosmological solutions in the Bianchi I model is considered. We prove that the stability of isotropic solutions in the Bianchi I metric for a positive Hubble parameter follows from their stability in the Friedmann–Robertson–Walker metric. This result is applied to models inspired by string field theory, which violate the null energy condition. Examples of stable isotropic solutions are presented. We also consider the $k$-essence model and analyse the stability of solutions of the form $\Phi(t) = t$.

1 Introduction

Field theories which violate the null energy condition (NEC) are of interest for the solution of the cosmological singularity problem \cite{1,2,3} and for models of dark energy with the equation of state parameter $w < -1$ (see \cite{4}–\cite{14} and references therein). Generally speaking, models that violate the NEC have ghosts, and therefore are unstable and physically unacceptable.

However, the possibility of the existence of dark energy with $w < -1$ on the one hand\textsuperscript{1} and the cosmological singularity problem on the other hand encourage the investigation of

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\item\textsuperscript{1}This possibility is not excluded experimentally \cite{15}, see \cite{16,17} for reviews of dynamical dark energy models.
\end{itemize}

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models which violate the NEC. It is almost clear that such a possibility can be realized within an effective theory, while the fundamental theory should be stable and admit quantization. From this point of view the NEC violation might be a property of a model that approximates the fundamental theory and describes some particular features of the fundamental theory. With the lack of quantum gravity, we can just trust string theory or deal with an effective theory admitting the UV completion.

There have been several attempts to realize these scenarios [18, 19, 20]. The ghost condensation model [18, 21, 22, 23] proposed to describe a wide class of cosmological perturbations has a ghost in the perturbative vacuum and has no ghost in the ghost condensation phase within an effective theory. The new ekpyrotic scenario [20, 23, 25, 26] is a development of the ekpyrotic [27] and the cyclic scenarios [28], and it attempts to solve the singularity problem, among others, by involving violation of the NEC. Nonlocal cosmological models [19, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] inspired by the string field theory (SFT) admit a regime with $w < -1$.

All of these models possess higher derivatives terms, which produce well-known problems with quantum instability [42, 43]. Several attempts to solve these problems have been recently performed [44, 26]. A physical idea that could solve the problems is that the instabilities do not have enough time to fully develop. A mathematical one is that dangerous terms can be treated as corrections valued only at small energies below the physical cut-off. This approach implies the possibility to construct a UV completion of the theory, and this assumption requires detailed analysis.

The NEC plays an important role in classical general relativity, in particular, in the consideration of black holes and cosmological singularities [1, 3]. The NEC violating models can admit classically stable solutions in the Friedmann–Robertson–Walker (FRW) cosmology. In particular, there are classically stable solutions for self-interacting ghost models with minimal coupling to gravity. Moreover, there exists an attractor behavior (for details about attractor solutions for inhomogeneous cosmological models, see [45]) in a class of the phantom cosmological models [46, 47, 48]. One can study the stability of the FRW metric, specifying a form of fluctuations. It is interesting to know whether these solutions are stable under the deformation of the FRW metric to an anisotropic one, for example, to the Bianchi I metric. In comparison with general fluctuations we can get an explicit form of solutions in the Bianchi I metric, which can probably clarify some nontrivial issues of theories with NEC violation.

Stability of isotropic solutions in the Bianchi models [49, 50, 51] (see also [52]) has been considered in inflationary models (see [53, 54] and references therein for details of anisotropic slow-roll inflation). Assuming that the energy conditions are satisfied, it has been proved that all initially expanding Bianchi models except type IX approach the de Sitter space-time [55] (see also [56, 57, 58, 59]). The Wald theorem [55] shows that for space-time of Bianchi types I–VIII with a positive cosmological constant and matter satisfying the dominant and strong energy conditions, solutions which exist globally in the future have certain asymptotic properties at $t \to \infty$. It is interesting to consider a similar question in the case of phantom cosmology [47, 60, 48] and string inspired models [19, 30, 34, 36, 61, 62], as well as in the case of the ghost condensation models [18] or their modifications [26].

The Bianchi universe models [49, 50, 51] are spatially homogeneous anisotropic cosmo-
logical models. There are strong limits on anisotropic models from observations \cite{63, 64}. Anisotropic spatially homogeneous fluctuations have to be strongly suppressed, and models developing large anisotropy should be discarded as early or late cosmological models.

In this paper we consider the stability of isotropic solutions in the Bianchi I metric in the presence of phantom scalar fields. There are two classes of models whose stability we analyse in this paper. The first class includes the one phantom scalar field models of dark energy, which admit exact kink-type or lump-type solutions \cite{60, 30}. For this class of models we also analyse the stability with respect to small fluctuations of the initial value of the cold dark matter energy density (compare with \cite{46}). The second class includes models with a scalar field $\phi$, which have exact solutions $\phi \sim t$, for example, the $k$-essence models \cite{65, 66, 67, 68}, in particular, ghost condensate models \cite{18, 21, 26}.

For both classes of models we prove that the stability of the solutions in the Bianchi I metric is equivalent to the stability of the corresponding solutions in the FRW metric. The stability of a kink or lump solution in the FRW metric means the stability of the fixed point that the solution tends to. Using the Lyapunov theorem \cite{69, 70} we find conditions under which the fixed point and the corresponding kink (or lump) solution are stable. In these cases the necessary condition for the exact solution’s stability is boundedness of the first corrections for the positive time semiaxis. When we can not use this theorem we check the boundedness of the first corrections to the exact solutions explicitly.

The paper is organized as follows. In Section 2 we deal with an arbitrary $N$-component scalar potential model and a $k$-essence model in the Bianchi I metric. We also review the Lyapunov theorem and other important statement about stability. In Section 3 we consider the stability of solutions which tend to an isolated fixed point in one-field models with the cold dark matter (CDM). We find sufficient conditions for the stability of such solutions in the FRW and Bianchi I metrics. In Section 4 we find the connection between the first order corrections in the FRW and Bianchi I metrics for $N$-field models. The corresponding result for the $k$-essence model is presented in Section 5. In Section 6 we present examples of stable isotropic kink and lump solutions in SFT inspired cosmological models. We also analyse the first order corrections for solutions, which are proportional to time. In Section 7 we make a conclusion and propose directions for further investigations.

2 Setup

2.1 The Bianchi I cosmological model with scalar and phantom scalar fields and the CDM

Let us start with a cosmological model with $N$ scalar fields $\phi_1, \phi_2, \ldots, \phi_N$ in the Bianchi I metric

$$ ds^2 = -dt^2 + a_1^2(t) dx_1^2 + a_2^2(t) dx_2^2 + a_3^2(t) dx_3^2. $$

The action is

$$ S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G_N} - \sum_{k=1}^{N} \frac{C_k}{2} g^{\mu \nu} \partial_\mu \phi_k \partial_\nu \phi_k - V(\phi_1, \ldots, \phi_N) - \Lambda \right), $$

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where the potential $V$ is a twice continuously differentiable function, $G_N$ is the Newtonian gravitational constant, $\Lambda$ is a cosmological constant, and $C_k$ are nonzero real numbers. The sign of $C_k$ defines whether field $\phi_k$ is the phantom field ($C_k < 0$) or the ordinary scalar field ($C_k > 0$).

The Einstein equations have the following form:

\begin{align}
H_1 H_2 + H_1 H_3 + H_2 H_3 &= 8\pi G_N \varrho, \\
\dot{H}_2 + H_2^2 + \dot{H}_3 + H_3^2 + H_2 H_3 &= -8\pi G_N p, \\
\dot{H}_1 + H_1^2 + \dot{H}_2 + H_2^2 + H_1 H_2 &= -8\pi G_N p, \\
\dot{H}_1 + H_1^2 + \dot{H}_3 + H_3^2 + H_1 H_3 &= -8\pi G_N p,
\end{align}

where

\begin{align}
\varrho &= \sum_{k=1}^{N} \frac{C_k}{2} \dot{\varphi}_k^2 + V(\phi_1, \ldots, \phi_N) + \Lambda + \rho_m, \\
p &= \sum_{k=1}^{N} \frac{C_k}{2} \dot{\varphi}_k^2 - V(\phi_1, \ldots, \phi_N) - \Lambda, \\
H_1 &= \frac{\dot{a}_1}{a_1}, \quad H_2 = \frac{\dot{a}_2}{a_2}, \quad H_3 = \frac{\dot{a}_3}{a_3}
\end{align}

and a dot denotes a time derivative.

Note that we couple, in a minimal way, pressureless matter (the CDM) with the energy density $\rho_m$ to our model. The equation for the CDM energy density is as follows:

\begin{equation}
\dot{\rho}_m = -(H_1 + H_2 + H_3)\rho_m.
\end{equation}

Introducing $\psi_k = \dot{\phi}_k$ we obtain from action (2) the following equations:

\begin{align}
\dot{\phi}_k &= \psi_k, \\
\dot{\psi}_k &= - (H_1 + H_2 + H_3)\psi_k - \frac{1}{C_k} V'_{\phi_k},
\end{align}

where $V'_{\phi_k} \equiv \frac{\partial V}{\partial \phi_k}, k = 1, 2, \ldots, N$. Thus we get the system of $2N + 4$ first order differential equations and one constraint (3).

It is convenient to express the initial variables $a_i$ in terms of new variables $a$ and $\beta_i$ (we use notations from [71]), subject to the following constraint:

\begin{equation}
\beta_1 + \beta_2 + \beta_3 = 0.
\end{equation}

One has the following relations

\begin{align}
a_i(t) &= a(t)e^{\beta_i(t)}, \quad \text{hence,} \quad a(t) = (a_1(t)a_2(t)a_3(t))^{1/3}, \\
H_i &\equiv H + \dot{\beta}_i, \quad \text{and} \quad H = \frac{1}{3}(H_1 + H_2 + H_3),
\end{align}

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where $H \equiv \dot{a}/a$. To obtain (14) we have used the following consequence of (12):

$$\dot{\beta}_1 + \dot{\beta}_2 + \dot{\beta}_3 = 0. \quad (15)$$

Note that $\beta_i$ are not components of a vector and, therefore, are not subjected to the Einstein summation rule. In the case of the FRW metric all $\beta_i$ are equal to zero and $H$ is the Hubble parameter. Following [50, 71] (see also [52]) we introduce the shear

$$\sigma^2 \equiv \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2. \quad (16)$$

It is useful to write equations (3)–(6), (10) and (11) in terms of new variables.

Using relation (15) we can write equation (3) as follows

$$3H^2 - \frac{1}{2}\sigma^2 = 8\pi G_N \varrho. \quad (17)$$

Summing equations (4)–(6) one can obtain

$$2\dot{H} + 3H^2 + \frac{1}{2}\sigma^2 = -8\pi G_N p. \quad (18)$$

Therefore

$$\dot{H} + 3H^2 = 4\pi G_N (\varrho - p). \quad (19)$$

Note that equations (10) and (11) in new variables,

$$\dot{\phi}_k = \psi_k, \quad \dot{\psi}_k = -3H\psi_k - \frac{1}{C_k}V'_k, \quad (20)$$

$$\dot{\rho}_m = -3H\rho_m, \quad (21)$$

as well as equation (19), look like the corresponding equations in the FRW metric.

Subtracting (4) from (5) we obtain

$$\dot{H}_1 + H_1^2 - \dot{H}_3 - H_3^2 + H_2(H_1 - H_3) = 0. \quad (22)$$

In terms of $H$ and $\beta_i$ equation (22) takes the form

$$\ddot{\beta}_1 + 3H\dot{\beta}_1 = \ddot{\beta}_3 + 3H\dot{\beta}_3. \quad (23)$$

Using (23) and (15) we obtain the following equations

$$\ddot{\beta}_i = -3H\dot{\beta}_i, \quad (24)$$

$$\frac{d}{dt}(\sigma^2) = -6H\sigma^2. \quad (25)$$

Functions $H(t)$ and $\sigma^2(t)$ together with $\phi_k(t)$ and $\rho_m(t)$ can be obtained from equations (18)–(21) and (25). If $H(t)$ is known, then $\beta_i$ can be trivially obtained from (24). We show in the next section that functions $H(t)$, $\dot{\beta}_i(t)$, and $\sigma^2(t)$ are very suitable to analyse the stability of isotropic solutions in the Bianchi I metric.
2.2 k-essence model in the Bianchi I metric

Let us consider the k-essence cosmological model, which is described by the action

\[ S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G_N} - \mathcal{P}(\Phi, X) - \Lambda\right), \]  

(26)

where

\[ X \equiv -g^{\mu \nu} \partial_\mu \Phi \partial_\nu \Phi. \]  

(27)

The pressure \( \mathcal{P}(\Phi, X) \) is of the form [18, 20]

\[ \mathcal{P}(\Phi, X) = \frac{1}{2} (p_q(\Phi) - \varrho_q(\Phi)) + \frac{1}{2} (p_q(\Phi) + \varrho_q(\Phi))X + \frac{1}{2} M^4(\Phi)(X - 1)^2. \]  

(28)

Here \( p_q(\Phi), \varrho_q(\Phi), \) and \( M^4(\Phi) \) are arbitrary functions of \( \Phi \). The energy density is

\[ \mathcal{E}(\Phi, X) = (p_q(\Phi) + \varrho_q(\Phi))X + 2M^4(\Phi)(X^2 - X) - \mathcal{P}(\Phi, X). \]  

(29)

In the Bianchi I metric for \( \Phi \), depending only on time, we have \( X = \dot{\Phi}^2 \). The Einstein equations are

\[
\begin{align*}
H_1H_2 + H_1H_3 + H_2H_3 &= 8\pi G_N (\mathcal{E} + \Lambda), \\
\dot{H}_2 + H_2^2 + H_3^2 + H_2H_3 &= -8\pi G_N (\mathcal{P} - \Lambda), \\
\dot{H}_1 + H_1^2 + H_2^2 + H_1H_2 &= -8\pi G_N (\mathcal{P} - \Lambda), \\
\dot{H}_1 + H_1^3 + H_3^2 + H_1H_3 &= -8\pi G_N (\mathcal{P} - \Lambda).
\end{align*}
\]  

(30–33)

From action (26) we also obtain the second order differential equation for the k-essence field \( \Phi \), which represents a consequence of system (30)-(33). Indeed, we differentiate (30) with respect to \( t \) and obtain

\[ (\dot{H}_2 + \dot{H}_3)H_1 + (\dot{H}_1 + \dot{H}_3)H_2 + (\dot{H}_1 + \dot{H}_2)H_3 = 8\pi G_N \ddot{\mathcal{E}}. \]  

(34)

Using (30)-(33) to exclude \( \dot{H}_i \), we transform this equation into the following form:

\[ \ddot{\mathcal{E}} = -(H_1 + H_2 + H_3)(\mathcal{E} + \mathcal{P}). \]  

(35)

Substituting explicit forms of \( \mathcal{E} \) and \( \mathcal{P} \), we obtain

\[
\left(2T_q + M^4(3\dot{\Phi}^2 - 1)\right)\ddot{\Phi} = -T_q'\dot{\Phi}^2 - V_q' - 2M^3M' \left(3\dot{\Phi}^4 - 2\dot{\Phi}^2 - 1\right) - 2(H_1 + H_2 + H_3)\dot{\Phi} \left(T_q + 2M^4(\ddot{\Phi}^2 - 1)\right),
\]  

(36)

where a prime denotes a derivative with respect to \( \Phi \),

\[ V_q(\Phi) \equiv \frac{1}{2} (\varrho_q(\Phi) - p_q(\Phi)), \quad T_q(\Phi) \equiv \frac{1}{2} (p_q(\Phi) + \varrho_q(\Phi)). \]  

(37)
The $k$-essence model has one important property. For any real differentiable function $H_0(t)$, there exist such real differentiable functions $\rho_q(\Phi)$ and $\pi_q(\Phi)$ that the functions $H_i(t) = H_0(t)$ and $\Phi(t) = t$ solve system (30)–(33) and, therefore, equation (36). Indeed at $\Phi(t) = t$,

$$\mathcal{E} = \rho_q(\Phi) = \rho_q(t), \quad \mathcal{P} = \pi_q(\Phi) = \pi_q(t).$$

(38)

So, one can obtain from (30)–(33)

$$\rho_q(t) = \frac{3}{8\pi G_N} H_0^2(t) - \Lambda, \quad \pi_q(t) = -\rho_q(t) - \frac{1}{4\pi G_N} \dot{H}(t).$$

(39)

Substituting the obtained $\rho_q(\Phi)$ and $\pi_q(\Phi)$ in (28), we see that the system (30)–(33) has a particular solution $H_i(t) = H_0(t)$ and $\Phi(t) = t$.

Bianchi–type models I–VIII coupled to $k$-essence matter representing dark energy and other matter which satisfies the strong and dominant energy conditions have been considered in [72]. A general criterion for isotropization of these models has been derived [72]. In this paper we do not assume that the energy conditions are satisfied when considering $k$-essence models in the Bianchi I metric.

### 2.3 A few known facts about stability

Let us remember a few facts about the stability [69, 70, 73] of solutions for a general system of the first order autonomic equations

$$\dot{y}_k = F_k(y), \quad k = 1, 2, \ldots, N. \quad (40)$$

By definition a solution (a trajectory) $y_0(t)$ is attractive (stable) if

$$\|\tilde{y}(t) - y_0(t)\| \to 0 \quad \text{at} \quad t \to \infty$$

(41)

for all solutions $\tilde{y}(t)$ that start close enough to $y_0(t)$.

If all solutions of the dynamical system that start out near a fixed (equilibrium) point $y_f$,

$$F_k(y_f) = 0, \quad k = 1, 2, \ldots, N \quad (42)$$

stay near $y_f$ forever, then $y_f$ is a Lyapunov stable point. If all solutions that start out near the equilibrium point $y_f$ converge to $y_f$, then the fixed point $y_f$ is an asymptotically stable one. Asymptotic stability of fixed point means that solutions that start close enough to the equilibrium not only remain close enough but also eventually converge to the equilibrium. A solution $y_0(t)$ of (41), which tends to the fixed point $y_f$, is attractive if and only if the point $y_f$ is asymptotically stable.

The Lyapunov theorem [69, 70] states that to prove the stability of fixed point $y_f$ of non-linear system (41) it is sufficient to prove the stability of this fixed point for the corresponding linearized system

$$\dot{\hat{y}} = Ay, \quad A_{ik} = \left. \frac{\partial F_i(y)}{\partial y_k} \right|_{y=y_f}. \quad (43)$$
The stability of the linear system means that real parts of all solutions of the characteristic equation

\[ \det \left( \frac{\partial F}{\partial y} - \lambda I \right) |_{y=y_f} = 0 \]

are negative.

In the case of a hyperbolic fixed point, i.e. the case when the Jacobian matrix of \( F \) at the fixed point does not have eigenvalues with zero real parts, one can use the Hartman–Grobman theorem \([74, 75, 76]\). This theorem reduces the study of the system of the first order nonlinear equations near the hyperbolic fixed point to the study of the behavior of its linearization near the origin.

The case with pure imaginary eigenvalues of the Jacobian matrix of \( F \) at the fixed point requires a more specific treatment \([76]\).

### 3 Stability of isolated fixed points and kink-type solutions in one-field models with the CDM

Let us consider the gravitational model with one scalar field \( \phi \) and an arbitrary potential \( V(\phi) \), described by action \((2)\) at \( N = 1 \). Equations \((19)\) and \((20)\) for one-field models are as follows

\[
\begin{align*}
\dot{H} &= -3H^2 + 8\pi G_N (V(\phi) + \Lambda), \\
\dot{\phi} &= \psi, \\
\dot{\psi} &= -3H\psi - \frac{1}{C} V'_\phi.
\end{align*}
\]

This system of three first order equations is valid in the Bianchi I metric as well as in the FRW one. Different initial values of \( \sigma^2 \) in \((17)\) specify these different cases.

Let us define

\[
I = \frac{3}{8\pi G_N} H^2 - \frac{C}{2} \psi^2 - V(\phi) - \Lambda.
\]

From system \((44)\) it is follows that the function \( I \) should be a solution of the following equation:

\[
\dot{I} = -6HI.
\]

If the case \( H(t) \equiv 0 \) is excluded, then \( I \) is an integral of motion of \((44)\) if and only if \( I = 0 \). From \((17)\) we see that

\[
I = \frac{1}{16\pi G_N} \sigma^2,
\]

so, \( I \) is an integral of motion only at \( \sigma^2 = 0 \), i.e. in the FRW metric. From \((16)\) and \((47)\) it follows that equation \((25)\) is a consequence of \((44)\).

We are interested in the stability of kink and lump solutions, namely, we consider such solutions in which the Hubble parameter tends to a finite value at \( t \to +\infty \). In this case \( \phi(t) \) tends to a finite value as well. Thus, there exists a fixed point \( y_f \equiv (H_f, \phi_f, \psi_f) \), which
corresponds to \( t = +\infty \). We consider the stability of isotropic solutions only, so \( \sigma_f^2 = 0 \) and \( \dot{\beta}_i = 0 \). It is easy to see that

\[
\psi_f = 0, \quad V''_\phi(\phi_f) = 0, \quad H_f^2 = \frac{8}{3} \pi G_N (\Lambda + V(\phi_f)).
\] (48)

To analyse the stability of \( y_f \), we present solutions as follows:

\[
\begin{align*}
H &= H_f + \varepsilon h(t) + \mathcal{O}(\varepsilon^2) \quad (49a) \\
\phi &= \phi_f + \varepsilon \varphi(t) + \mathcal{O}(\varepsilon^2) \quad (49b) \\
\psi &= \varepsilon \chi(t) + \mathcal{O}(\varepsilon^2), \quad (49c) \\
\dot{\beta}_i &= \varepsilon \zeta_i(t) + \mathcal{O}(\varepsilon^2), \quad (49d)
\end{align*}
\]

where \( \varepsilon \) is a small parameter. To first order in \( \varepsilon \) we obtain the following system of equations:

\[
\begin{align*}
\dot{h}(t) &= -6H_fh(t), \quad (50a) \\
\dot{\varphi}(t) &= \chi(t), \quad (50b) \\
\dot{\chi}(t) &= -3H_f\chi(t) - \frac{1}{C} V''(\phi_f)\varphi. \quad (50c)
\end{align*}
\]

Equation (50a) has the solution

\[
h(t) = b_0 e^{-6H_ft}, \quad (51)
\]

where \( b_0 \) is a constant.

From (50b)–(50c) we obtain the following solutions:

- at \( V''(\phi_f) \neq 0 \) and \( V''(\phi_f) \neq \frac{9C}{4} H_f^2 \),

\[
\varphi(t) = D_1 e^{-3(H_f + \sqrt{H_f^2 - \frac{9C}{4} V''(\phi_f)})t/2} + D_2 e^{-3(H_f - \sqrt{H_f^2 - \frac{9C}{4} V''(\phi_f)})t/2}, \quad (52)
\]

- at \( V''(\phi_f) = \frac{9C}{4} H_f^2 \),

\[
\varphi(t) = e^{-3H_ft/2} (D_1 + D_2 t), \quad (53)
\]

- at \( V''(\phi_f) = 0 \),

\[
\varphi(t) = \tilde{D}_1 - \frac{1}{3H_f} D_2 e^{-3H_ft}, \quad (54)
\]

where \( \tilde{D}_1, D_1, \) and \( D_2 \) are arbitrary constants.

Using the Lyapunov theorem we state that fixed point \( y_f \) is asymptotically stable and, therefore, the exact kink-type or lump-type solution \( y_0(t) \) is stable if

\[
\frac{V''(\phi_f)}{C} > 0 \quad \text{and} \quad H_f > 0. \quad (55)
\]

Namely, \( y_f \) is
• a stable focus at $\frac{V''(\phi_f)}{C} > \frac{9}{4}H_f^2$,

• a stable node at $\frac{9}{4}H_f^2 > \frac{V''(\phi_f)}{C} > 0$,

• a stable improper node at $V''(\phi_f) = \frac{9}{4}H_f^2$.

From (24) we obtain

$$\zeta_i(t) = C_i e^{-3H_f t},$$

(56)

where $C_i$ are constants. If $H_f > 0$, then $\zeta_i$ and $\sigma^2$ tend to zero at $t \to \infty$. Thus the obtained conditions (55) are sufficient to prove the stability of isotropic fixed points both in the Bianchi I and in FRW metrics.

At $V''(\phi_f) = 0$ or $H_f = 0$ we need an additional analysis of stability, because the Lyapunov theorem does not state the correspondence of the behavior of solutions to the initial system (44) and the obtained linear system (50).

At $H_f < 0$ the fixed point $y_f$ is unstable. Note that both $h(t)$ and $\zeta_i(t)$, as well as $\varphi(t)$, tend to infinity at $H_f < 0$.

Let us introduce the CDM into our model with a scalar field. Adding to system (44) the CDM energy density $\rho_m$ and the corresponding equation (21), we get

$$\dot{H} = -3H^2 + 4\pi G_N (2V(\phi) + 2\Lambda + \rho_m),$$

$$\dot{\phi} = \psi,$n

$$\dot{\psi} = -3H\psi - \frac{1}{C}V'\phi,$n

$$\dot{\rho}_m = -3H\rho_m.$$n

(57)

Let us consider the possible fixed points of system (57). From the last equation of this system, it follows that at the fixed point we have either $H_f = 0$ or $\rho_{mf} = 0$. Substituting (49) and

$$\rho_m(t) = \rho_{mf} + \varepsilon\tilde{\rho}_m(t) + \mathcal{O}(\varepsilon^2),$$

(58)

into the system (57), we obtain the following system in first order to $\varepsilon$:

$$\dot{\tilde{\rho}}_m(t) = -3H_f \tilde{\rho}_m(t) - 3\rho_{mf}h(t),$$

(59a)

$$\dot{h}(t) = -6H_f h(t) + 8\pi G_N \tilde{\rho}_m(t),$$

(59b)

$$\dot{\varphi}(t) = \chi(t),$$

(59c)

$$\dot{\chi}(t) = -3H_f \chi(t) - \frac{1}{C}V''(\phi_f)\varphi.$$n

(59d)

It is easy to see that the third and fourth equations of (59) coincide with the corresponding equations of system (44). Therefore, the case $H_f = 0$ can not be analysed by the Lyapunov theorem. Let us prove that condition (55) is sufficient for the stability of fixed points for models with the CDM. First of all, from $H_f \neq 0$ it follows that $\rho_{mf} = 0$. Solving the first and second equations of (59), we obtain

$$\tilde{\rho}_m(t) = b_1 e^{-3H_f t}, \quad h(t) = b_0 e^{-6H_f t} + \frac{b_1}{3H_f} e^{-3H_f t},$$

(60)
where \( b_1 \) is an arbitrary constant.

We come to the conclusion that if conditions (55) are satisfied, then the solution, which is stable in the model without the CDM, is stable with respect to the CDM energy density fluctuations as well.

4 Connections between the first order corrections to isotropic solutions in the FRW and Bianchi I metrics

In the previous section we studied one-field models and the first corrections near a fixed point. In this section we consider the first corrections of an arbitrary isotropic solution.

We consider an \( N \)-field cosmological model, which is described by action (2) and the Einstein equations (1)–(11). In this section we do not assume that the isotropic solution tends to a fixed point. We do not prove the stability of solutions, we only analyse the first corrections in the FRW and Bianchi I metrics. To apply the Lyapunov theorem it was convenient to consider functions \( H \) and \( \dot{\beta}_i \) instead of \( H_i \). In this section we return to functions \( H_i \).

To study the stability of this solution, we present solutions whose initial conditions are close to the isotropic one, in the following form:

\[
H_i(t) = H_0(t) + \varepsilon h_i(t) + \mathcal{O}(\varepsilon^2),
\]
\[
\phi_k(t) = \phi_{0k}(t) + \varepsilon \phi_k(t) + \mathcal{O}(\varepsilon^2),
\]
\[
\psi_k(t) = \psi_{0k}(t) + \varepsilon \psi_k(t) + \mathcal{O}(\varepsilon^2),
\]
\[
\rho_m(t) = \rho_{m0}(t) + \varepsilon \rho_m(t) + \mathcal{O}(\varepsilon^2),
\]

where \( i = 1, 2, 3 \) and \( k = 1, \ldots, N \). From (3)–(11) we obtain to zero order in \( \varepsilon \) the system of Einstein equations and equations of motion in the FRW metric. To first order in \( \varepsilon \) we have the following system:

\[
\dot{\phi}_k = \chi_k,
\]
\[
\dot{\chi}_k = -(h_1 + h_2 + h_3)\psi_{0k} - 3H_0\chi_k - \frac{1}{C_k} \sum_{m=1}^{N} V''_{\phi_k\phi_m}(\phi_0) \psi_m,
\]
\[
\ddot{\rho}_m = -(h_1 + h_2 + h_3)\rho_{m0} - 3H_0\ddot{\rho}_m,
\]
\[
\dot{h}_1 + \dot{h}_2 = -3H_0(h_1 + h_2) + 8\pi G_N \sum_{k=1}^{N} \left( V'_{\phi_k}(\phi_0) \varphi_k - C_k \phi_{0k} \chi_k \right),
\]
\[
\dot{h}_1 + \dot{h}_3 = -3H_0(h_1 + h_3) + 8\pi G_N \sum_{k=1}^{N} \left( V'_{\phi_k}(\phi_0) \varphi_k - C_k \phi_{0k} \chi_k \right),
\]
\[
\dot{h}_2 + \dot{h}_3 = -3H_0(h_2 + h_3) + 8\pi G_N \sum_{k=1}^{N} \left( V'_{\phi_k}(\phi_0) \varphi_k - C_k \phi_{0k} \chi_k \right).
\]
From equations (68)–(70) we get
\[
\dot{h}_1(t) - \dot{h}_2(t) + 3H_0(t)(h_1(t) - h_2(t)) = 0, \tag{71}
\]
\[
\dot{h}_1(t) - \dot{h}_3(t) + 3H_0(t)(h_1(t) - h_3(t)) = 0, \tag{72}
\]
and we also have
\[
H_0(h_1 + h_2 + h_3) = 4\pi G_N \sum_{k=1}^{N} \left( C_k \dot{\phi}_0 \dot{\varphi}_k + V'_{\phi k}(\phi_0) \varphi_k \right). \tag{73}
\]

**Theorem 1**

Let \(H_0(t)\) be a smooth function bounded at all finite values of time and \(\int_0^\infty H_0(\tau) d\tau\) be bounded from below, in other words, this integral is equal to either a finite number or plus infinity. Functions \(h_1(t), h_2(t), h_3(t), \tilde{\rho}_m(t),\) and \(\varphi_k(t),\) which are solutions of (65)–(70), are bounded if and only if isotropic solutions, namely, solutions, which satisfy the condition \(h_1(t) = h_2(t) = h_3(t),\) are bounded.

**Proof.** It is trivial that if the full set of solutions includes only boundary functions, then any subset which satisfies an additional condition includes only boundary functions. Let us prove that the boundedness of isotropic solutions is not only a necessary condition, but also a sufficient one.

From equations (71) and (72) we obtain:
\[
h_1(t) - h_2(t) = (h_1(0) - h_2(0)) e^{-3 \int_0^t H_0(\tau) d\tau}, \quad h_1(t) - h_3(t) = (h_1(0) - h_3(0)) e^{-3 \int_0^t H_0(\tau) d\tau}. \tag{74}
\]

So we obtain that if the integral \(\int_0^t H_0(\tau) d\tau\) is uniformly bounded from below, then anisotropy is bounded at all \(t.\) Note that in the most of cosmological models \(H_0(t) > 0\) for all \(t > 0\) and the anisotropy tends to zero at \(t \to \infty.\)

Using (74), one can express \(h_2(t)\) and \(h_3(t)\) via \(h_1(t)\) and reduce system (68)–(70) to one equation. System (65)–(70) takes the following form:
\[
2H_0 \left(3h_1 - C_0 e^{-3 \int_0^t H_0(\tau) d\tau}\right) = 8\pi G_N \left( \sum_{k=1}^{N} C_k \dot{\phi}_0 \dot{\varphi}_k + \sum_{k=1}^{N} V'_{\phi k}(\phi_0) \varphi_k \right), \tag{75}
\]
\[
2\dot{h}_1 + 6H_0 h_1 = 8\pi G_N \left( \sum_{k=1}^{N} V'_{\phi k}(\phi_0) \varphi_k - \sum_{k=1}^{N} C_k \dot{\phi}_0 \dot{\varphi}_k \right), \tag{76}
\]
where \(C_0 = 2h_1(0) - h_2(0) - h_3(0).\)

Let us introduce a new function,
\[
h_0(t) \equiv h_1(t) - \frac{C_0}{3} e^{-3 \int_0^t H_0(\tau) d\tau}. \tag{77}
\]
It is easy to check that
\[ 3h_0(t) = h_1(t) + h_2(t) + h_3(t). \] (78)

System (75)–(76) in terms of \( h_0 \) and \( \varphi_k \) coincides with the system of equations (65)–(70) with \( h_1(t) = h_2(t) = h_3(t) = h_0(t) \). In other words, we obtain that the functions \( \varphi_k(t) \) in the Bianchi I and FRW metrics are the same. Functions \( h_1(t) \), \( h_2(t) \), and \( h_3(t) \) differ from the correction for the Hubble parameter \( h_0(t) \) on a finite value. Thus the theorem is proven.

Note that Theorem 1 connects the stability properties of the FRW and Bianchi I metrics not only for solutions which tend to a fixed point, but also for solutions which tend to infinity at \( t \to \infty \). Examples of such solutions in the cosmological models are presented in Sections 5 and 6.

5 Stability of solutions in the \( k \)-essence model in the Bianchi I metric

5.1 First order corrections

Let us consider the first order corrections in the \( k \)-essence model. Substituting
\[ E = E_0 + \varepsilon E_1 + \mathcal{O}(\varepsilon^2), \quad P = P_0 + \varepsilon P_1 + \mathcal{O}(\varepsilon^2), \] (79)
in (30)–(33) and expanding (61) to first order in \( \varepsilon \), we obtain the following system:
\[ 2H_0(h_1 + h_2 + h_3) = 8\pi G_N E_1, \] (80)
\[ \dot{h}_1 + \dot{h}_2 = -3H_0(h_1 + h_2) - 8\pi G_N P_1, \] (81)
\[ \dot{h}_1 + \dot{h}_3 = -3H_0(h_1 + h_3) - 8\pi G_N P_1, \] (82)
\[ \dot{h}_2 + \dot{h}_3 = -3H_0(h_2 + h_3) - 8\pi G_N P_1. \] (83)

It is easy to see that equations (71) and (72) can be obtained from (81)–(83), therefore, formula (74) is valid for solutions of system (80)–(83). So, it is useful to introduce \( h_0 \) by formula (77). Because of (78) we obtain that system (80)–(83) in terms of \( h_0, E_1 \), and \( P_1 \) coincides with the corresponding equations in the FRW metric, so if \( H_0 \) satisfies the conditions of Theorem 1, then solutions of (80)–(83) are bounded if and only if isotropic solutions, namely, solutions which satisfy the condition \( h_1(t) = h_2(t) = h_3(t) \), are bounded. This means that it is sufficient to calculate the first order corrections for the given background solutions in the FRW metric to describe their behavior in the Bianchi I metric.

5.2 Example

Let us consider the following example:
\[ \varrho_q(\Phi) = C_1 + B\Phi^2, \quad p_q(\Phi) = C_2 - B\Phi^2, \quad M(\Phi) = M_0, \] (84)
where \( B, C_1, C_2 \), and \( M_0 \) are constants. The Friedmann equations are
\[ 3H^2 = 8\pi G_N(E + \Lambda), \] (85)
\[ \dot{H} = -4\pi G_N(E + P). \] (86)
One can check that the following exact solution exists:

\[ \Phi_0(t) = t, \quad H_0(t) = -4\pi G_N (C_1 + C_2)t, \]  

if

\[ C_1 = -\Lambda, \quad B = 6\pi G_N (C_1 + C_2)^2. \]  

Let us analyse the stability of the exact solution,

\[ \Phi = \Phi_0(t) + \varepsilon \Psi(t), \quad H = H_0(t) + \varepsilon h(t). \]  

The equations for the first order fluctuations,

\[ \dot{h}(t) = -8\pi G_N (C_1 + C_2 + 2M^4) \dot{\Psi}(t), \]  
\[ \dot{\Psi}(t) = -12\pi G_N \frac{t(C_1 + C_2)}{(C_2 + C_1 + 4M^4)} \left[ (C_1 + C_2) \Psi(t) + 2m^2_p h(t) \right], \]

have the following general solution:

\[ h(t) = d_1 e^{6\pi G_N (C_1 + C_2)t^2} + d_2, \]  
\[ \Psi(t) = -\frac{1}{8\pi G_N} \left( \frac{d_1}{C_2 + C_1 + 2M^4} e^{6\pi G_N (C_1 + C_2)t^2} + \frac{2d_2}{C_1 + C_2} \right), \]

where \( d_1 \) and \( d_2 \) are arbitrary numbers.

If \( C_1 + C_2 < 0 \), then \( H_0 > 0 \) at \( t > 0 \) and the exact solution is stable in the sense that the first corrections are bounded functions. Similar solutions, obtained from the SFT inspired model, are considered in Subsection 6.5.

6 Examples of isotropic stable solutions in the SFT inspired models

6.1 String field theory inspired cosmological models

An interest in cosmological models coming from open string field theories [19] is caused by a possibility to get solutions rolling from a perturbative vacuum to the true one. When all other massive fields are integrated out by means of equations of motion, the open string tachyon acquires a nontrivial potential with a nonperturbative minimum. For the open fermionic NSR string with the GSO− sector [40] in a reasonable approximation, one gets the Mexican hat potential for the tachyon field (see [41] for a review). Rolling of the tachyon from the unstable perturbative extremum towards this minimum describes, according to the Sen conjecture [41], the transition of an unstable D-brane to a true vacuum. In fact one gets a nonlocal potential with a string scale as a parameter of nonlocality. After a suitable field redefinition the potential becomes local, meanwhile, the kinetic term becomes nonlocal. This nonstandard kinetic term has a so-called phantomlike behavior and can be approximated by a phantom kinetic term. Rolling solutions are particular examples of kink-type solutions.
It is also interesting to study lump-type solutions, which in particular have no time singularity. It was advocated in [30, 31] that such solutions are also available in the SFT inspired models.

In this section we consider the stability of the kink-type and lump-type solutions for the SFT inspired cosmological models [60, 30, 36] under perturbations in the Bianchi I metric.

In [60, 30] we have considered the SFT inspired phantom models with high degree polynomial potentials. We consider the stability of the obtained exact solutions in the next two subsections.

In [36] we have considered a nonlocal cosmological model with quadratic potential and obtained that exact solutions of this model are solutions of local models with quadratic or zero potential. In Subsections 6.4 and 6.5 we analyse the stability of these solutions in massless and massive cases correspondingly. The exact solutions in the massive case are similar to solutions (87) in the $k$-essence model.

In the examples we use a dimensionless parameter $m_p^2 \sim M_p^2 = 1/(8\pi G_N)$. The coefficient of proportionality arises when we construct effective cosmological models from the original SFT action (see [60, 30, 36] for details). For convenience, we write the Einstein equations for the SFT inspired cosmological models in the following form:

$$
\dot{H} = -\frac{3}{2}H^2 - \frac{1}{2m_p^2} \left( \frac{C\psi^2}{2} - V(\phi) - \Lambda \right),
$$

$$
\dot{\phi} = \psi,
$$

$$
\dot{\psi} = -3H\psi - \frac{1}{C} V'(\phi).
$$

We also have

$$
3m_p^2H^2 - \frac{C}{2}\phi^2 - V(\phi) = \Lambda. \quad \text{(95)}
$$

6.2 Model with a kink solution and the sixth degree potential

An exact solution to the Friedmann equations with a string inspired phantom scalar matter field has been constructed in [60] (see also [46]). The notable features of the model are a phantom sign of the kinetic term ($C = -1$) and a special polynomial form of the effective tachyon potential:

$$
V(\phi) = \frac{1}{2} (1 - \phi^2)^2 + \frac{1}{12m_p^2} \phi^2 (3 - \phi^2)^2. \quad \text{(96)}
$$

Note that this potential has been used in the string gas cosmology [8].

System (94) has the following exact kink-type solution [60]:

$$
\phi_0(t) = \tanh(t), \quad H_0(t) = \frac{1}{2m_p^2} \tanh(t) \left( 1 - \frac{1}{3} \tanh(t)^2 \right). \quad \text{(97)}
$$

Let us analyse the stability of this solution. At $t \to \infty$ solution (97) tends to a fixed point,

$$
H_f = \frac{1}{3m_p^2}, \quad \phi_f = 1. \quad \text{(98)}
$$
It is easy to see that
\[ V'_\phi(1) = 0, \quad V''_{\phi\phi}(1) = 2 \left( 2 - \frac{1}{m_p^2} \right). \]  

(99)

Using (55), we obtain that solution (97) is attractive in the Bianchi I metric at \( m_p^2 < 1/2 \). Note that this solution is stable with respect to small fluctuations of the initial value of the CDM energy density as well.

In [60] we have showed that the first corrections \( \varphi(t) \) and \( h(t) \) satisfy the following system:
\[
\begin{align*}
\dot{h} &= \frac{1}{m_p^2} \left( 1 - \tanh(t)^2 \right) \dot{\varphi}, \\
\dot{\varphi} &= \frac{(3 - 4m_p^2 + 4(m_p^2 - 1) \tanh(t)^2 + \tanh(t)^4) \tanh(t)}{2m_p^2 (1 - \tanh(t)^2)} \varphi - \\
&\quad - \frac{(3 - \tanh(t)^2) \tanh(t)}{1 - \tanh(t)^2} h,
\end{align*}
\]

(100)

and have the following explicit form:
\[
\begin{align*}
\varphi(t) &= 2m_p^2 C_1 \left( 1 - \tanh(t)^2 \right) + \\
&\quad + 2m_p^2 C_2 \frac{2J(t) + (\cosh(2t) - 1)(\cosh(t))^2 e^{1/m_p^2 (2m_p^2 \cosh(2t) + 1)}}{\cosh(2t) + 1}, \\
\end{align*}
\]

(101)

where \( C_1 \) and \( C_2 \) are arbitrary constants,
\[
J(t) = \int_0^t \sinh(\tau) (\cosh(\tau))^{1-1/m_p^2} \left( 2 (2m_p^2 - 1) \cosh(\tau)^2 - 1 \right) e^{1/m_p^2 \cosh(\tau)^2} d\tau.
\]

It is easy to see that if \( m_p^2 > 1/2 \) then at \( C_2 \neq 0 \) the function \( \varphi(t) \) tends to infinity as \( t \to \infty \) and, therefore, solution (97) is not stable. At \( m_p^2 = 1/2 \) we obtain from (101) that
\[
\begin{align*}
\varphi(t) &= - \left( \tanh(t)^2 - 1 \right) \left( C_1 - C_2 J_2 \right) - \frac{1}{2} C_2 e^{-\tanh(t)^2/2}, \\
\end{align*}
\]

where \( J_2 = \int_0^t e^{-\tanh(\tau)^2/2} \tanh(\tau) d\tau \). Thus, \( \varphi(t) \) and \( h(t) \) are bounded functions at \( m_p^2 = 1/2 \).

The functions \( h_i \) have the form
\[
\begin{align*}
h_i(t) &= h(t) + \tilde{C}_i e^{-\tanh(t)} \left( 1 - \tanh^2(t) \right)^{1/(2m_p^2)}, \\
\end{align*}
\]

(102)

where \( \tilde{C}_i \) are real constants, \( i = 1, 2, 3 \), which satisfy the following relation:
\[
\tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 = 0.
\]

(103)

We conclude that exact solutions obtained in [60] are stable in the Bianchi I metric at \( m_p^2 < 1/2 \) and unstable at \( m_p^2 > 1/2 \). The case of \( m_p^2 = 1/2 \) needs a more detailed analysis. The first corrections are bounded.
6.3 Model with a lump solution

In the previous subsection kink solutions were considered. In this subsection we consider the stability of a lump solution in the model [30] which is motivated by a description of D-brane decay within the string field theory framework. We take the one-field cosmological model with the potential

\[ V(\phi) = 2(1 - \phi)\phi^2 - \frac{4(\phi - 1)^3(2 + 3\phi)^2}{75m_p^2} \]  

and \( C = -1 \). The Friedmann equations (94) have the following exact solution [30]:

\[ \phi_0 = \text{sech}^2(t), \quad H_0 = \frac{2(3 + 2\cosh(t)^2)\tanh^3(t)}{15m_p^2\cosh(t)^2}. \]  

At \( t \rightarrow \infty \) solution (105) tends to a fixed point:

\[ H_f = \frac{4}{15m_p^2}, \quad \phi_f = 0. \]  

It is easy to see that

\[ V'_\phi(0) = 0, \quad V''_{\phi\phi}(0) = 4 \left(1 - \frac{2}{5m_p^2}\right). \]  

Using (55), we obtain that solution (105) is attractive in the Bianchi I metric at \( m_p^2 < 2/5 \). In [30] the authors consider a model without the CDM, at the same time, the results of Section 2 show that solution (105) is stable with respect to small fluctuations of the initial value of the CDM energy density as well.

Let us perturb the Friedmann equations in the standard way,

\[ H = H_0(t) + \epsilon h(t), \quad \phi = \phi_0(t) + \epsilon \varphi(t). \]  

To first order in \( \epsilon \) we have the following system of equations:

\[ \dot{h} + \frac{2}{m_p^2} \text{sech}^2(t) \tanh^2(t)\dot{\varphi} = 0, \]

\[ \frac{1}{m_p^2} \left( \frac{4}{5}(4 + \cosh(2t)) \text{sech}^2(t) \tanh^3(t)h + (6 \text{sech}^4(t) - 4 \text{sech}^2(t))\varphi \right) - \frac{4(2 + 3 \text{sech}^2(t))^2}{25m_p^2} (\tanh^4(t) - 2 \tanh^6(t))\varphi + 2 \text{sech}^2(t) \tanh(t)\dot{\varphi} = 0. \]  

System (109) has the following solutions:

\[ \varphi = \frac{1}{2\sinh(t)\cosh^3(t)} \left( 5C_2m_p^2\cosh(t) - \frac{4 + 30m_p^2}{5m_p^2} \right) e^{\left( \frac{2\cosh^2(t) - 3}{10m_p^2\cosh^4(t)} \right)} - 2C_1 \cosh^2(t) - 2C_2 \int \frac{1}{\sinh^3(t)} (-15m_p^2\cosh^4(t) + 10m_p^2\cosh^6(t) + 8 \cosh^2(t) - 6 - 4 \cosh^6(t) + 2 \cosh^4(t)) \times \]
\[ \times \cosh(t) \left( \frac{e^{4+5m_p^2}}{5m_p^2} \right) \left( \frac{2 \cosh^2(t) - 3}{10m_p^2 \cosh^4(t)} \right) \left( \cosh^2(t) - m_p^2 \right) dt + 2C_1 \),
\]

\[ h = \frac{16(\cosh(2t) - 1)}{\cosh(6t) + 6\cosh(4t) + 15 \cosh(2t) + 10} \left( C_1 + C_2 \int \cosh(t) \left( \frac{e^{4+5m_p^2}}{5m_p^2} \right) \left( \frac{2 \cosh^2(t) - 3}{10m_p^2 \cosh^4(t)} \right) \right. \]
\[ \times \left. \left[ -15m_p^2 \cosh^4(t) + 10m_p^2 \cosh^6(t) + 8 \cosh^2(t) - 6 - 4 \cosh^6(t) + 2 \cosh^4(t) \right] dt \right) . \]

Using (105b) we get
\[ h_i = h + \tilde{C}_i \cosh(t) \left( \frac{e^{4+5m_p^2}}{5m_p^2} \right) \left( \frac{2 \cosh^2(t) - 3}{10m_p^2 \cosh^4(t)} \right), \] (110)

where \( \tilde{C}_i \) are arbitrary real constants which satisfy (103).

It is easy to verify that \( h(t) \) and \( h_i(t) \) are bounded functions for any values of the parameters. Taking into account that
\[ \lim_{t \to -\infty} \exp \left( \frac{2 \cosh^2(t) - 3}{10m_p^2 \cosh^4(t)} \right) = 1, \] (111)

we obtain that \( \varphi \) is bounded at \( m_p^2 \leq 2/5 \) and unbounded at \( m_p^2 > 2/5 \). The stability in the case of \( m_p^2 = 2/5 \) cannot be analysed without using high order corrections.

In the examples considered in this and the previous subsections, the potentials depend on \( m_p^2 \), which results in the fact that extrema of potentials are either minima, maxima, or inflection points. In the next two examples we consider the opposite case, when potentials do not depend on \( m_p^2 \).

### 6.4 Model with a massless phantom field

At present nonlocal cosmological models are being studied very actively [19], [30]–[38]. In this and the next subsections we consider the stability of solutions of local models which correspond to the nonlocal model with a quadratic potential. These solutions have been presented in [36], where the method of localizing the nonlocal model with a quadratic potential has been proposed.

Let us consider the one-field model with zero potential, \( V(\phi) = 0 \). From the Friedmann equations (94) we obtain
\[ 3H^2 = \frac{C}{2m_p^2} \frac{\dot{\phi}^2}{\phi^2} + \frac{\Lambda}{m_p^2} ; \] (112)
\[ \dot{H} = - \frac{C}{2m_p^2} \frac{\dot{\phi}^2}{\phi^2} . \] (113)

At \( \Lambda > 0 \) and \( C < 0 \) there are the following real solutions:
\[ \phi_0(t) = \pm \sqrt{- \frac{2m_p^2}{3C}} \arctan \left( \frac{3\Lambda}{m_p^2} (t - t_0) \right) + C_1, \]
\[ H_0(t) = \sqrt{\frac{\Lambda}{3m_p^2}} \tanh \left( \sqrt{\frac{3\Lambda}{m_p^2}} (t - t_0) \right) , \] (114)
where $t_0$ and $C_1$ are arbitrary real constants.

Let us consider the stability of the solution $(H_0, \phi_0)$. Substituting $H_0$ and $\phi_0$ into (61)–(62), to first order in $\varepsilon$ we obtain

$$
\varphi(t) = \pm \frac{2m_p^2 \sqrt{e^2 \sqrt{3m_p^2 \Lambda(t-t_0)/m_p^2} \Lambda(t-t_0)/m_p^2 + 1}}{\sqrt{-C\Lambda}} C_3 + C_2,
$$

(115)

$$
h(t) = \frac{2C_3}{\cosh \left(2 \sqrt{3m_p^2 \Lambda/m_p^2}(t-t_0)\right) + 1},
$$

where $C_2$ and $C_3$ are arbitrary real constants. It is obvious, that functions $h(t)$ and $\varphi(t)$ are bounded. In the Bianchi I metric we have

$$
h_i(t) = h(t) + \tilde{C}_i \sqrt{1 - \tanh^2 \left(\frac{\sqrt{3m_p^2 \Lambda/m_p^2}(t-t_0)}{m_p^2}\right)},
$$

(116)

where real constants $\tilde{C}_i$, $i = 1, 2, 3$, satisfy the following relation:

$$
\tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 = 0.
$$

(117)

Thus, we have obtained that the kink-type solutions (114) in the Bianchi I metric have the bounded first corrections.

### 6.5 Model with a quadratic potential and the cosmological constant

Let us consider the model of a scalar field with a quadratic potential and the cosmological constant. In this case the Friedmann equations are

$$
H^2 = \frac{8\pi G_N}{3} \left(\frac{C}{2} \dot{\phi}^2 + \frac{B}{2} \phi^2 + \Lambda\right),
$$

(118)

$$
\dot{H} = -4\pi G_N C \dot{\phi}^2,
$$

(119)

where $C$ and $B$ are arbitrary nonzero real numbers.

System (118)–(119) has the following particular solutions:

$$
H_0(t) = k_1 t, \quad \phi_0(t) = k_2 t,
$$

(120)

where

$$
k_1 = -\frac{B}{3C}, \quad k_2 = \frac{B}{12\pi G_N C^2}.
$$

(121)
From (121) it follows that the function $\phi$ is real if and only if $B > 0$. The above-mentioned solutions exist only if

$$\Lambda = -\frac{B}{24\pi G_N C}. \tag{122}$$

To analyse the stability of these exact solutions, we substitute

$$H(t) = k_1 t + \varepsilon h(t) \tag{123}$$

and

$$\phi(t) = k_2 t + \varepsilon \varphi(t). \tag{124}$$

in (118) and (119). To first order in $\varepsilon$ we obtain the following system of equations:

$$\dot{\varphi}(t) = -\frac{Bt}{C} \left( \varphi(t) + \frac{1}{4\pi G_N C k_2} h(t) \right), \tag{125}$$

$$\dot{h}(t) = -8\pi G_N C k_2 \dot{\varphi}(t). \tag{126}$$

Solutions of (125)–(126) are

$$h(t) = \tilde{D}_1 e^{\frac{B}{12\pi G_N C^2} t^2} + \tilde{D}_2, \tag{127}$$

$$\varphi(t) = -\frac{1}{8\pi C G_N k_2} \left( 2\tilde{D}_1 + \tilde{D}_2 e^{\frac{B}{12\pi G_N C^2} t^2} \right), \tag{128}$$

where $\tilde{D}_1$ and $\tilde{D}_2$ are arbitrary constants,

$$k_2 = \pm \sqrt{\frac{B}{12\pi G_N C^2}}. \tag{129}$$

Therefore, the functions $h(t)$ and $\varphi(t)$ are bounded at $C/B < 0$. Real solutions exist only if $B > 0$, and hence, $C < 0$. We come to the conclusion that solution (120) can be stable (the first corrections are bounded) only if $C < 0$, in other words, $\phi(t)$ is a phantom scalar field. In this case $H_0(t) > 0$ at $t > 0$, hence $h_i$ are bounded as well. Indeed,

$$h_i(t) = h(t) + \tilde{C}_i e^{-\frac{3}{6} \int_0^t H_0(\tau) d\tau} = (\tilde{D}_1 + \tilde{C}_i) e^{\frac{B}{12\pi G_N C^2} t^2} + \tilde{D}_2, \tag{130}$$

are bounded. Constants $\tilde{C}_i$ satisfy the relation (103).

### 7 Conclusion

We have analysed the stability of isotropic solutions for the models with NEC violation in the Bianchi I metric.

In our paper for the one-field model with the CDM we used the Lyapunov theorem and found sufficient conditions for stability of kink-type and lump-type solutions both in the FRW metric and in the Bianchi I metric. The obtained results allow us to prove that the exact solutions, found in string inspired phantom models [60, 30], are stable. A generalization
of this result to two-field models, for example, quintom models, requires further studies and will be considered in future investigations.

We found the explicit form of the connection between \( h_1(t) \), \( h_2(t) \), and \( h_3(t) \), which define metric perturbations in the Bianchi I metric, and \( h_0 \), which defines perturbations in the FRW metric. We have proved that fluctuations for the fields and the CDM energy density in both metric are the same. In particular, for \( H_0 \geq 0 \) the boundedness of \( h_0 \) is a sufficient and necessary condition for the boundedness of \( h_1(t) \), \( h_2(t) \), and \( h_3(t) \). This result is valid for both \( N \)-field and \( k \)-essence models.

Note that linear-in-time solutions for the simple models with a quadratic potential, which have been considered in Subsection 6.5, are stable with respect to the first corrections only in the phantom case (which corresponds to \( H > 0 \)). It means that NEC violation does not lead to the instability in this sense. It gives us an intuitive reason to expect that more complicated NEC violated models can also have stable isotropic solutions. This expectation has been confirmed in the models considered in Section 6.

To conclude, our results are for the NEC violated models. If energy conditions are satisfied, then results, similar to those obtained in Section 4, are consequences of the Wald theorem [55] and its generalizations [56, 57, 58, 59].

Our study of the stability of isotropic solutions for the models with NEC violation in the Bianchi I metric shows that the NEC is not a necessary condition for classical stability of isotropic solutions. Because of strong limits on anisotropic models from observations [63, 64], cosmological models developing large anisotropy should be discarded. In this paper we have shown that the models [30, 36, 60] have stable isotropic solutions and that large anisotropy does not appear in these models.

Acknowledgements

The authors are grateful to Alexei A. Starobinsky for drawing their attention to the stability problem of isotropic solutions in the Bianchi metrics, in particular, in the Bianchi I metric. L.J. would like to thank D. Mulryne and D. Wesley for useful discussions.

I.A., N.B. and S.V. are supported in part by a state contract from the Russian Federal Agency for Science and Innovations No. 02.740.11.5057. I.A. and S.V. are supported in part by RFBR grant No. 08-01-00798 and by the Russian Ministry of Education and Science though grant No. NSh-795.2008.1 (I.A.) and No. NSh-1456.2008.2 (S.V.). L.J. acknowledges the support of the Centre for Theoretical Cosmology, in Cambridge.

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