Formation of Spherical D2-brane from Multiple D0-branes

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Abstract

We study D-branes in $SU(2)$ WZW model by means of the boundary state techniques. We realize the “fuzzy sphere” configuration of multiple D0-branes as the boundary state with the insertion of suitable Wilson line. By making use of the path-integral representation we show that this boundary state preserves the appropriate boundary conditions and leads to the Cardy state describing a spherical D2-brane under the semi-classical approximation. This result directly implies that the spherical D2-brane in $SU(2)$ WZW model can be well described as the bound state of D0-branes.

After presenting the supersymmetric extension, we also investigate the BPS and the non-BPS configurations of D-branes in the NS5 background. We demonstrate that the non-BPS configurations are actually unstable, since they always possess the open string tachyons. We further notice that the stable BPS bound state constructed by the tachyon condensation is naturally interpreted as the brane configuration of fuzzy sphere.
1 Introduction

Toward thorough understanding of D-brane dynamics, the studies on the flat D-branes with constant $B$-field have recently received a great deal of interests [1, 2, 3]. One of the important precepts of them is the existence of two equivalent descriptions of such backgrounds; one is the “commutative description”, of which low energy effective action is well-known DBI action, and the other is the “non-commutative description” based on the geometry induced by the open string vertices. A transparent viewpoint for the understanding of appearance of non-commutativity is the interpretation of D-branes with constant $B$-field as the bound states of infinitely many lower dimensional D-objects (D0-branes or D-instantons, typically). This has been a well-known aspect in the context of Matrix theory [4, 5]. Roughly speaking, the classical solution of Matrix theory corresponding to the “non-commutative configuration”

$$[X^i, X^j] = i\theta^{ij},$$  \hspace{1cm} (1.1)

leads to such a bound state. This aspect was also discussed from the viewpoints of perturbative string theory in the papers [3, 7]. In those works the matrix coordinates $X^i$ are naturally identified with the CP factors and are incorporated appropriately as the (T-dualized) Wilson line such as $\text{Tr} \left( P \exp \left( i \int d\sigma P_i(\sigma)X^i \right) \right)$.

A simple extension of these non-commutative descriptions of D-branes to the curved background is realized by considering the linear $B$-field (namely, the constant field strength $H = dB = \text{const} \neq 0$) rather than the constant $B$-field;

$$[X^i, X^j] = i\epsilon^{ijk}X^k, \hspace{1cm} (i, j, k = 1, 2, 3),$$  \hspace{1cm} (1.2)

which is the configuration called the “fuzzy sphere” [8].

It is known that the fuzzy sphere configuration of D0-branes (1.2) is a classical solution under the suitable RR flux and describes the spherical brane [8, 10], often called “giant graviton” [11]. On the other hand, there have been many studies on the various aspects of D-branes in $SU(2)$ WZW model, which has a constant NS field strength $H \neq 0$ [12, 13, 14, 15]. Among other things, it was shown in [13] that the spherical D2-branes in $SU(2)$ WZW model are stabilized by the flux of D0-brane charges [3]. It was also claimed [13, 14] that the spherical D2-branes are no other than the bound states of D0-branes. This claim is based on the

\footnote{Related works about the quantization of flux on the spherical D2-brane are also given in [10].}
low energy effective field theory in the “non-commutative description” of D-branes in WZW model, which is given as an analog of non-commutative Yang-Mills theory with Chern-Simons term \[14\]. Related works treating such a non-commutative gauge theory have been given in the recent literatures \[17, 18\].

The main purpose of this paper is to clarify such a proposed aspect with respect to the “formation of spherical branes” from the picture of boundary states. Starting from the boundary state description of D0-brane in the background of \(SU(2)\) WZW model, we realize the several configurations of multiple D0-branes by inserting the Wilson lines following the works \[6, 7\]. Especially, we focus on the fuzzy sphere configuration of D0-branes in the similar manner as the configuration of non-commutative \(R^2\) (or non-commutative torus) \([1.1]\) in the case of flat string background. We prove that such a boundary state with the Wilson line of fuzzy sphere satisfies the appropriate boundary condition using the path-integral representation. We further demonstrate how it reduces to the boundary state describing the stable spherical D2-brane under the large \(k\) limit.

One of the most important applications of these results to superstring theory is surely the study of D-branes in the NS5 background, since the near horizon geometry of this background is known to be described by the CHS \(\sigma\)-model \[19\]. We consider the type II string theory in this background and investigate various configurations of multiple D0-branes with the world-sheet superconformal symmetry preserved. We show that, among these on-shell configurations of D0-branes, only the configuration of fuzzy sphere is BPS, and the other non-BPS configurations always include tachyonic excitations in the open string spectrum. This means that the system of multiple D0-branes is unstable and decays into a single BPS D2-brane described by the CP factor of fuzzy sphere after the open string tachyons condensate. This aspect is reminiscent of the tachyon condensation in the various systems of non-BPS brane configurations under the flat string background \[20\], and seems to fit with the observation by \[13, 14\] at the level of low energy effective field theory. The other studies of D-branes in the NS5 background from different viewpoints are given in \[21, 22, 23, 24\].

This paper is organized as follows. In section 2 we study D-branes in bosonic \(SU(2)\) WZW model. We examine various configurations of multiple D0-branes by inserting the Wilson line operators and explain how the fuzzy sphere configuration of D0-branes leads to a spherical D2-brane wrapped on a conjugacy class. In section 3, after providing the extension to the
superstring case, we especially study whether or not the brane configuration preserves the space-time SUSY and investigate the stability of the system, namely, the absence/existence of the tachyons in the open string spectrum. We also discuss the formation of the spherical brane from the viewpoints of tachyon condensation. Section 4 is devoted to a summary and some discussions.

2 Spherical D2-brane from D0-branes in Bosonic $SU(2)$ WZW model

2.1 D-branes in Bosonic $SU(2)$ WZW model

We shall begin with a brief review about D-branes in $SU(2)$ WZW model, which are established in many works [12]. In WZW model there are two currents; one is the left mover and the other is the right mover,

$$
J^a(\tau, \sigma) = \sum_n J^a_n e^{-in(\tau+\sigma)}, \quad \tilde{J}^a(\tau, \sigma) = \sum_n \tilde{J}^a_n e^{in(-\tau+\sigma)},
$$

(2.1)

where $a$ takes the values 1, 2, 3 and $n$ is an integer. D-branes should be described by the boundary of the string world-sheet. We have to impose an appropriate boundary condition.

In maximally symmetric case, this boundary condition reads as

$$
(J^a_n + \Lambda^a_b \tilde{J}^b_{-n})|B\rangle = 0,
$$

(2.2)

where $\Lambda$ is an automorphism of $SU(2)$. If this is an inner automorphism, we can readily reduce this condition to the simplest one

$$
(J^a_n + \tilde{J}^a_{-n})|B\rangle = 0,
$$

(2.3)

by considering the rotation of the type $|B\rangle \rightarrow e^{iu_a \tilde{J}^a_0}|B\rangle$. The case when $\Lambda$ is an outer automorphism is quite interesting and has been attracting much attentions in the recent works [25, 26]. However, it is beyond the scope of this paper and we shall start with the simplest “gluing condition” (2.3). We must also impose the conformal invariance at the boundary;

$$
(L_n - \tilde{L}_{-n})|B\rangle = 0.
$$

(2.4)

Throughout this paper we shall write down the boundary conditions in the closed string channel.
However, since the energy-momentum tensor is given as the quadratic form of currents, this condition is automatically satisfied because of the gluing condition for the currents (2.2) or (2.3).

For convenience we define the following currents at the boundary with setting $\tau = 0$, 

$$ J^a_\pm(\sigma) = \sum_n (J^a_n \pm \tilde{J}^a_{-n}) e^{-in\sigma}. \quad (2.5) $$

These currents satisfy the commutation relations

$$ [J^a_\pm(\sigma_1), J^b_\pm(\sigma_2)] = 2\pi i \epsilon^{abc} J^c_\pm(\sigma_1) \delta(\sigma_1 - \sigma_2), \quad (2.6) $$

$$ [J^a_\pm(\sigma_1), J^b_\mp(\sigma_2)] = 2\pi i \epsilon^{abc} J^c_-\mp(\sigma_1) \delta(\sigma_1 - \sigma_2) + 2\pi i k \delta^{ab} \delta'(\sigma_1 - \sigma_2). \quad (2.7) $$

Here we use $'$ as the derivative with respect to $\sigma_1$. By using these currents and the "boundary energy-momentum tensor"

$$ T_(\sigma) = \sum_n (L_n - \tilde{L}_{-n}) e^{-in\sigma}, \quad (2.8) $$

we can rewrite the gluing conditions (2.3), (2.4) as

$$ J^a_\pm(\sigma)|B\rangle = 0, \quad (2.9) $$

$$ T_-(\sigma)|B\rangle = 0. \quad (2.10) $$

Since the currents $J^a_\pm(\sigma)$ generate the adjoint action of $SU(2)$ at the boundary, the gluing condition (2.9) roughly means that the directions along the conjugacy class should obey the Neumann boundary condition, which is an origin of the geometrical interpretations of the boundary states of interest.

A complete system of solutions of the gluing conditions (2.3), (2.4) is given in [27] and called “Ishibashi states”. The Ishibashi states $|\ell\rangle_I (\ell = 0, \ldots, k)$ are characterized by the relation

$$ I\langle \ell | q^{H(c)} | \ell' \rangle I = \delta_{\ell\ell'} \chi^{(k)}_{\ell}(\bar{q}), \quad (2.11) $$

where $H^{(c)} = \frac{1}{2}(L_0 + \tilde{L}_0 - \frac{c}{24})$ denotes the Hamiltonian in the closed string channel. Here $c$ is the central charge of the system and $\bar{q} = \exp(-2\pi i/\tau)$ is the modulus in the closed string channel. Later we will use $q = \exp(2\pi i\tau)$ as the open string one. Furthermore $\chi^{(k)}_{\ell}(\bar{q})$ denotes the $SU(2)$ character and its modular transformation is given by

$$ \chi^{(k)}_{\ell}(\bar{q}) = \sum_{\ell'} S_{\ell\ell'} \chi^{(k)}_{\ell'}(q), \quad (2.12) $$
where we set
\[ S_{\ell\ell'} = \sqrt{\frac{2}{k+2}} \sin \left( \frac{\pi (\ell + 1)(\ell' + 1)}{k+2} \right). \] (2.13)

Because of the linearity of the gluing conditions, arbitrary linear combinations of Ishibashi states also satisfy them. Among them it is convenient in some physical reasons to take the “Cardy states” [28] defined by
\[ |L\rangle_C = \sum_{\ell} \frac{S_{\ell\ell}}{\sqrt{S_{0\ell}}} |\ell\rangle_I, \] (2.14)
where the label $L$ takes also the values $L = 0, 1, \cdots, k$. It is well-known [12] that the each Cardy state corresponds to the D-brane wrapped on the conjugacy class located at the quantized azimuthal angle $\theta = \frac{\pi L}{k}$, which has the spherical topology and is centered at the origin $e$ (the identity element of $SU(2)$). The special example $L = 0$ (and also $L = k$) corresponds to the point-like conjugacy class, and thus it is naturally identified with the boundary state of D0-brane;
\[ |D0\rangle_C = \sum_{\ell} \frac{S_{0\ell}}{\sqrt{S_{0\ell}}} |\ell\rangle_I = \sum_{\ell} \sqrt{S_{0\ell}} |\ell\rangle_I. \] (2.15)

The other Cardy states $L \neq 0, k$ correspond to spherical D2-branes stabilized by the $U(1)$ fluxes $\int F = L + 1$ [13]:
\[ |D2; L\rangle_C = \sum_{\ell} \frac{S_{\ell\ell}}{\sqrt{S_{0\ell}}} |\ell\rangle_I. \] (2.16)

### 2.2 Multiple D0-branes on $SU(2)$ WZW Model and Wilson Line

Let us consider a configuration of $(L + 1)$ D0-branes. As is well-known, multiple D-branes are described by the degrees of freedom of the CP indices. Following the discussion in the flat background [3, 4], we shall start with the boundary state of D0-branes with the Wilson line inserted in order to incorporate the CP degrees of freedom;
\[ |D0; \{M^a\}\rangle_C = \text{Tr} \left( P \exp \left( -\frac{i}{k} \int_0^{2\pi} d\sigma J^a_-(\sigma)M^a(\sigma) \right) \right) |D0\rangle_C, \] (2.17)
where $P$ indicates the path-order and the coefficients of matrices are chosen for convenience.

We express the CP factors as $(L + 1) \times (L + 1)$ hermitian matrices $M^a$ $(a = 1, 2, 3)$. For example, in the simplest case $M^a = 0_{(L+1) \times (L+1)}$, we have
\[ |D0; \{M^a\}\rangle_C = (L + 1)|D0\rangle_C, \] (2.18)
which merely stands for a stack of $(L + 1)$ D0-branes at the origin $e$. 


Slightly more non-trivial example is the case that $M^a$ are constant matrices which mutually commute with each other. In that case we can eliminate the path-order symbol and the boundary state reduces to

$$\langle D_0; \{M^a\}\rangle_C = \text{Tr} \left( \exp \left( -\frac{2\pi i}{k} J_{-0}^a M^a \right) \right) \langle D_0 \rangle_C. \quad (2.19)$$

This is nothing but a linear combination of $(L + 1)$ D0 boundary states, each of which is characterized by the gluing condition of the general type (2.2). The simultaneous eigenvalues of $M^a$ roughly express the positions of D0-branes on $S^3$. Since the Wilson line factor only includes the zero-modes of $SU(2)$ currents, it is obvious that this boundary state preserves the conformal invariance at the boundary.

It is more interesting to consider the example with the constant matrices $M^a$ satisfying the commutation relation

$$[M^a, M^b] = i \epsilon^{ab}_{\ c} M^c, \quad (2.20)$$

which defines the fuzzy sphere configuration of D0-branes [8]. In view of the conjecture given in [13, 14] one will expect that this configuration leads to the spherical D2-brane as the bound state of $(L + 1)$ D0-branes in the analogous way as the discussion [6] in the flat background.

From now on, we shall focus on this configuration and will show that this expectation is indeed the case at least under the large $k$ limit.

Since our CP matrices (2.20) are non-commutative, the path-ordering in the Wilson line essentially contributes and makes it difficult to confirm whether or not the boundary state (2.17) satisfies the appropriate gluing condition. At this point it is more convenient to rewrite the path-ordered trace by means of the path-integral representation [29, 6, 7, 30]. The formula of path integral on a group manifold is given in [31] (see also [32]) and we apply it to our case. The Wilson line hence can be represented (up to normalization) as

$$\text{Tr} \left( P \exp \left( -\frac{i}{k} \int_0^{2\pi} d\sigma J^a_{-}(\sigma) M^a \right) \right) = \int \mathcal{D}g \exp \left[ -A \int_0^{2\pi} d\sigma \langle g(\sigma)|D_\sigma^{-}|g(\sigma)\rangle \right], \quad (2.21)$$

where we set

$$D_\sigma^{-} = \frac{d}{d\sigma} + \frac{i}{k} J^a_{-}(\sigma) M^a. \quad (2.22)$$

The integral variable $g(\sigma)$ is the map of $S^1$ to $SU(2)$ and we define the “coherent state” as

$$|g(\sigma)\rangle = R_L(g(\sigma))|0\rangle, \quad \langle g(\sigma)| = \langle 0|R_L(g(\sigma)^{-1})\rangle, \quad (2.23)$$

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where $R_L$ denotes the spin $L/2$ representation of $SU(2)$ and $|0\rangle, \langle 0|$ denote the highest weight vector and its dual. We can also suppose that $M^a = R_L(T^a)$ without loss of generality, where $T^a$ denote the generators of $SU(2)$ algebra. The factor $A$ is nothing but a normalization constant and we can prove that the path-integral does not depend on this constant (up to the overall normalization), which will be explained just below.

Precisely speaking, we have to perform the gauge fixing about the gauge symmetry $g(\sigma) \rightarrow g(\sigma)h(\sigma)\ (\forall h(\sigma) \in U(1))$ in order to define the path-integral properly. The reduced phase space, which has the topology $S^2 \cong SU(2)/U(1)$, is identified with the co-adjoint orbit of $SU(2)$ and thus canonically equipped with the Kirillov-Kostant symplectic structure. This can be also identified with the conjugacy class located at $\theta = \frac{\pi L}{k}$. This aspect is quite expected for our later discussion and is similar to that for the flat background [6], in which the phase space where the path-integral is carried out is naturally identified with the world-volume of the brane created as a bound state.

The canonical quantization on the reduced phase space (so-called the “geometric quantization“ [33]) provides the finite dimensional quantum Hilbert space naturally identified with the representation space of $SU(2)$ with spin $L/2$. The quantum mechanical operators, which are necessarily finite dimensional matrices, are important objects in the non-commutative description of spherical brane. The finite dimensionality of them is the origin of the fuzziness of spherical brane as was pointed out in many literatures. To be more specific, the function $m^a(g) \equiv \langle g|M^a|g\rangle$ defines an observable on the reduced phase space, since this is $U(1)$ gauge invariant, $m^a(gh) = m^a(g), (\forall h \in U(1))$. According to the argument of [31], we can show that

$$\{m^a, m^b\}_{\text{PB}} = \frac{1}{A}\langle g|[M^a, M^b]|g\rangle = \frac{1}{A}i\epsilon^{abc}\langle g|M^c|g\rangle = \frac{1}{A}i\epsilon^{abc}m^c. \quad (2.24)$$

This implies that the quantum mechanical operator $\hat{m}^a$ corresponding to $m^a(g)$ should be identified with the matrix $\frac{1}{A}M^a$. We here point out that this factor $1/A$ cancels the overall factor $A$ in the path-integral formula (2.21), which proves that the path-integral does not actually depend on the choice of $A$. From now on, we shall set $A = 1$, which is the convention taken in [31].

We should also remark that it is not manifest whether this path-integral is well-defined or not, since the non-commutative operators $J^a_\sigma(\sigma)$ appear in the integrand. Nevertheless we can
safely use this representation because the terms $J_a^\sigma (g)(\sigma)|M^a|g(\sigma))$ and $J_a^\sigma (g')(\sigma')|M^a|g(\sigma'))$ commute with each other for an arbitrary choice of $\sigma, \sigma'$ and thus we can deal with it just like a c-number. In this evaluation of commutator it is crucial that currents $J_a^\sigma$ have no Schwinger term. (See (2.6).)

Now we show that the boundary state (2.17) with the CP factor of fuzzy sphere satisfies the gluing conditions (2.3), (2.4). To this end it is enough to prove that the Wilson line (2.21) commutes with all the currents $J_a^\sigma(\sigma)$ and the boundary stress tensor $T_-(\sigma)$.

Because of the commutation relations

\[
\left[ J_a^\sigma(\sigma_1), J_b^\sigma(\sigma_2) \right] = 2\pi i\epsilon^{abc} J_c^\sigma(\sigma_1) \delta(\sigma_1 - \sigma_2) + 2\pi i k \delta^{ab} \delta'(\sigma_1 - \sigma_2) ,
\]
\[
\left[ M^a, M^b \right] = i\epsilon^{abc} M^c ,
\]

we can easily derive the equality

\[
\hat{U} J_-(\sigma) \hat{U}^{-1} = U(\sigma) J_-(\sigma) U(\sigma)^{-1} - ikU(\sigma) \frac{d}{d\sigma} U(\sigma)^{-1} ,
\]

or equivalently,

\[
\hat{U} D_{\sigma}^J \hat{U}^{-1} = U(\sigma) D_{\sigma}^J U(\sigma)^{-1} ,
\]

where we write $J_-(\sigma) = J_a^\sigma(\sigma)M^a$ and set

\[
\hat{U} = \exp \left( i \int_0^{2\pi} u^a(\sigma) J_a^\sigma(\sigma) d\sigma \right) , \quad U(\sigma) = \exp(-2\pi i u^a(\sigma)M^a) .
\]

The action of currents to the CP factors is then written as

\[
\hat{U} \text{Tr} \left( P \exp \left( -\frac{i}{k} \int_0^{2\pi} d\sigma J_+^\sigma(\sigma)M^a \right) \right) \hat{U}^{-1}
\]
\[
= \hat{U} \int \mathcal{D}g \exp \left[ -\int_0^{2\pi} d\sigma \langle g(\sigma)|D_{\sigma}^J|g(\sigma)\rangle \right] \hat{U}^{-1}
\]
\[
= \int \mathcal{D}g \exp \left[ -\int_0^{2\pi} d\sigma \langle g(\sigma)|U(\sigma)D_{\sigma}^J U(\sigma)^{-1}|g(\sigma)\rangle \right] .
\]

Consider the transformation of integral variable $g(\sigma) \rightarrow g'(\sigma) = U(\sigma)^{-1}g(\sigma)$. Since the path integral measure should be invariant under this transformation, i.e. $\mathcal{D}g' = \mathcal{D}g$, we can conclude that the Wilson line (2.21) is invariant under the arbitrary transformations generated by the $SU(2)$ currents $J_a^\sigma(\sigma)$, which proves our claim.

One may be afraid that the Wilson line operator (2.21) was treated in a formal way. Especially it would include a potential subtlety due to the UV divergence. However, we can
define it by a suitable regularization and the above statement is confirmed in a more rigorous way in appendix A.

As we already mentioned, the conformal invariance at the boundary (2.4) is also satisfied. In this way we have proved that the boundary state with the CP factor of fuzzy sphere obeys the same gluing conditions (2.3), (2.4). This fact contrasts with the cases of commutative CP factors, in which the gluing condition (2.3) suffers the “twist” into the general one (2.2).

2.3 Formation of Spherical D2-brane

Now we again focus on the constant CP matrices $M^a$ of the fuzzy sphere (2.20). In the previous subsection we proved the boundary state (2.17) with these CP matrices satisfies the same gluing condition (2.3). Since the complete system of the solutions of the gluing condition (2.3) is given by the Ishibashi states, the boundary state of interest should be represented as a linear combination of Ishibashi states. Moreover, it is obvious that the action of Wilson line operator (2.21) closes in each of the integrable module of $\hat{SU}(2)$. Therefore we can obtain the next simple relation for each of the Ishibashi states;

$$\text{Tr} \left( P \exp \left( -\frac{i}{k} \int_0^{2\pi} d\sigma J_+^a(\sigma) M^a \right) \right) |\ell\rangle_I = c(L, \ell, k) |\ell\rangle_I ,$$

(2.30)

where $c(L, \ell, k)$ is nothing but a c-number depending on $L, \ell, k$. To evaluate this coefficient $c(L, \ell, k)$ we only have to consider the components of primary states. However, it is still rather difficult to calculate precisely the coefficient $c(L, \ell, k)$. The best we can do now is to evaluate it under the semi-classical limit $k \to +\infty$. Naively one might suppose that we merely have

$$\text{Tr} \left( P \exp \left( -\frac{i}{k} \int_0^{2\pi} d\sigma J_+^a(\sigma) M^a \right) \right) \approx 1 \text{ in this limit. But this is not correct. We point out that in this limit the components of Ishibashi states with } \ell \sim k \gg 1 \text{ are dominant in the D0-brane Cardy state } |D0\rangle_C \text{ (2.15)}.$$

Hence we can assume the order estimation $J_+^a \sim \ell \sim k$ and obtain the following evaluation;

$$\left[ P \left( -\frac{i}{k} \int_0^{2\pi} d\sigma J_+^a(\sigma) M^a \right)^n |D0\rangle \right]_{\text{primary state}} = \left( -\frac{2\pi i}{k} J^a M^a \right)^n |D0\rangle_{\text{primary state}} + O \left( \frac{1}{k} \right).$$

(2.31)

We thus obtain in the large $k$ limit

$$|D0; \{M^a\}\rangle_C |_{\text{primary}} \approx \text{Tr}_{\text{RL}} \left( \exp \left( -\frac{2\pi i}{k} J_-^a M^a \right) \right) |D0\rangle_C |_{\text{primary}} \approx \text{Tr}_{\text{RL}} \left( \exp \left( -\frac{4\pi i}{k} J^a M^a \right) \right) |D0\rangle_C |_{\text{primary}}.$$

(2.32)
Moreover, since we can now suppose $\ell \gg 1$, we can make use of the semi-classical approximation for the “angular momentum” such as $J_0^a \sim \sqrt{J^2} n^a \sim \frac{\ell + 1}{2} n^a$, where we denote $n^a$ as a unit vector in some direction on $S^3$. In this approximation the right hand side of (2.32) can be rewritten as

$$\text{Tr}_{R_L} \left( \exp \left( -\frac{4\pi i}{k} J_0^a M^a \right) \right) |D0\rangle_C \rvert_{\text{primary}} \approx \sum_{\ell \gg 1} \text{Tr}_{R_L} \left( \exp \left( -\frac{4\pi i}{k} \frac{\ell + 1}{2} n^a M^a \right) \right) \frac{S_0\ell}{\sqrt{S_0\ell}} |\ell\rangle_I \rvert_{\text{primary}} .$$

(2.33)

$\chi_L$ denotes the $SU(2)$ character of the spin $L/2$ representation. Since this is a class function (i.e. $\chi_L(gh^{-1}) = \chi_L(g)$), we have

$$\chi_L \left( \exp \left( -\frac{4\pi i}{k} \frac{\ell + 1}{2} n^a M^a \right) \right) = \chi_L \left( \exp \left( -2i\pi \frac{\ell + 1}{k + 2} M^3 \right) \right) = \frac{\sin \left( \pi \frac{(L+1)(\ell+1)}{k+2} \right)}{\sin \left( \pi \frac{(\ell+1)}{k+2} \right)} = S_{L\ell} \frac{S_0\ell}{S_0\ell} .$$

(2.34)

In this way we can conclude that

$$c(L, \ell, k) \approx \frac{S_{L\ell}}{S_0\ell}, \quad (k \to +\infty) .$$

(2.35)

It leads to the remarkable result

$$|D0; \{M^a\}\rangle_C \approx |D2; L\rangle_C, \quad (k \to +\infty) ,$$

(2.36)

which means that the fuzzy sphere configuration of $(L+1)$ D0-branes forms the spherical D2-brane as their bound states and this is wrapped on the $(L+1)$-th conjugacy class of $SU(2)$.

More rigorous derivation of (2.35) is given as follows. Since $\text{Tr}_{R_L}$ is taken over the CP degrees of freedom belonging to the spin $L/2$ representation of $SU(2)$, we can also evaluate the (2.32) in the next way;

$$\text{Tr}_{R_L} \left( \exp \left( -i \frac{4\pi}{k + 2} J_0^a M^a \right) \right) |\ell\rangle_I \rvert_{\text{primary}} = \sum_{M = -\frac{L}{2}}^{M = \frac{L}{2}} \langle L, M | \exp \left( -i \frac{4\pi}{k + 2} J_0^a M^a \right) |\ell\rangle_I \rvert_{\text{primary}} \otimes |L, M\rangle .$$

(2.37)
Since the spin $\ell/2$ representation of the zero mode algebra $\{J^a_\ell\}$ appears in the Ishibashi state $|\ell\rangle_I$ primary, we can easily diagonalize the action of the operator $J^a_\ell M^a$. The eigen-value is evaluated as

$$J^a_\ell M^a = \frac{1}{2}((\vec{J}_0 + \vec{M})^2 - \vec{J}_0^2 - \vec{M}^2)$$

$$= \frac{1}{8}((\ell - L + 2m + 1)^2 - 1 - (\ell + 1)^2 + 1 - (L + 1)^2 + 1)$$

$$= -\frac{1}{4}(\ell + 1)(L - 2m) + \frac{1}{8}(2m + 1)^2 + \frac{1}{8} - \frac{1}{4}(L + 1)(2m + 1), \quad (2.38)$$

where $m$ runs over the range $m = 0, 1, \cdots, L$ because of the Clebsch-Gordan rule. We can now assume $\ell \gg L$ because we have $\ell \gg 1$, which implies that only the first term in (2.38) is dominant. We can thus continue the evaluation as follows;

$$c(L, \ell, k) \approx \sum_{m=0}^{L} \left( \exp \left( \frac{\pi i (\ell + 1)(L - 2m)}{k + 2} \right) \right)$$

$$= \frac{\sin \left( \frac{\pi (L+1)(\ell+1)}{k+2} \right)}{\sin \left( \frac{\pi (\ell+1)}{k+2} \right)} = \frac{S_{L\ell}}{S_{0\ell}}. \quad (2.39)$$

In this way we have got the same result as we previously obtained.

Lastly let us discuss a simple generalization of (2.36). Consider the boundary state

$$|L_1; \{M^a\}\rangle_C \equiv \text{Tr} \left( P \exp \left( -\frac{i}{k} \int_0^{2\pi} d\sigma J^a_\ell (\sigma) M^a \right) \right) |L_1\rangle_C, \quad (2.40)$$

where $\{M^a\}$ are the CP matrices of fuzzy sphere of the size $L_2 + 1$. We assume $L_1, L_2 \ll k \sim +\infty$. Then, the Ishibashi states with large $\ell$ are again dominant, and we can easily evaluate (2.40) thanks to (2.35);

$$|L_1; \{M^a\}\rangle_C = \sum_{\ell} \frac{S_{L_1 \ell}}{\sqrt{S_{0\ell}}} c(L_2, \ell, k)|\ell\rangle_I$$

$$\approx \sum_{\ell} \frac{S_{L_1 \ell}}{\sqrt{S_{0\ell}}} S_{L_2 \ell} |\ell\rangle_I$$

$$= \sum_{\ell, L_1, L_2} N^L_{L_1, L_2} \frac{S_{L_2 \ell}}{\sqrt{S_{0\ell}}} |\ell\rangle_I = \sum_{L} N^L_{L_1, L_2} |L\rangle_C, \quad (2.41)$$

where $N^L_{L_1, L_2}$ denotes the fusion matrix of $SU(2)_k$ and we used the Verlinde formula. This result (2.41) seems to be consistent with the formula given in [14, 23].

11
3 Spherical D2-brane from D0-branes in $SU(2)$ Super WZW model

3.1 Preliminary

In this section we extend the discussions in the previous section to the supersymmetric case. It is a familiar fact that the near horizon physics of NS5-branes in type II string theory is described by the CHS $\sigma$-model \[19\], $R \times S^3$ in which the $S^3$ sector is described by $SU(2)$ super WZW model. Our main purpose in this section is to study the similar aspects of the multiple D0-branes and spherical D2-brane under the NS5 background in superstring theory\[3\].

We start with the affine supercurrents of $\hat{SU}(2)_N$;

$$J^a(z, \theta) = \psi^a(z) + \theta J^a(z),$$

(3.1)

whose commutation relation is given by

$$J^a(z_1, \theta_1)J^b(z_2, \theta_2) \sim \frac{N}{2}\delta^{ab} \frac{\theta_{12}}{z_{12}} + \frac{\theta_{12}}{z_{12}} i\epsilon^{abc} \psi^c(z_2, \theta_2),$$

(3.2)

where $z_{12} = z_1 - z_2 - \theta_1 \theta_2$ and $\theta_{12} = \theta_1 - \theta_2$. In other words,

$$J^a(z_1)J^b(z_2) \sim \frac{N}{2}\delta^{ab} \frac{1}{(z_1 - z_2)^2} + \frac{i\epsilon^{abc}}{z_1 - z_2} \psi^c(z_2),$$

$$\psi^a(z_1)\psi^b(z_2) \sim \frac{N}{2}\delta^{ab} \frac{1}{z_1 - z_2}.$$

(3.3)

It is also convenient to rewrite the “total current” $J^a$ by the “bosonic current” $j^a$ and the “fermionic current” $j^a_f$;

$$J^a(z) = j^a(z) + j^a_f(z), \quad j^a_f(z) = -\frac{i}{N} \epsilon^{abc} \psi^b(z) \psi^c(z),$$

(3.4)

The terminologies “D0”, “D2” here are somewhat inaccurate. The reader should understand that we are now focusing only on the sector of $SU(2)$ super WZW model and the precise dimension of brane depends on the boundary conditions along the other sectors compatible with the GSO condition of the total system. For instance, it is known that $R_\phi$ direction must always obey the Neumann boundary condition to preserve the superconformal symmetry, as is mentioned by several authors (for example, see \[24\]).
which have the OPEs as follows

\begin{align}
  j^a_j(z_1) j^b_j(z_2) & \sim \frac{\delta^{ab}}{(z_1 - z_2)^2} + \frac{i\epsilon^{abc} j^c_j(z_2)}{z_1 - z_2}, \\
  j^a(z_1) j^b(z_2) & \sim \frac{N-2}{2} \delta^{ab} \frac{1}{(z_1 - z_2)^2} + \frac{i\epsilon^{abc} j^c(z_2)}{z_1 - z_2}, \\
  j^a(z_1) j^b_j(z_2) & \sim 0. 
\end{align}

(3.5)

In fact, \( j^a(z) \) is identified with the current defined by the bosonic sector of super WZW model.

The \( N = 1 \) superconformal algebra has a energy-momentum tensor and a superconformal current. In the superfield formalism, we can combine these currents as

\[
\mathcal{T}(z, \theta) = \frac{1}{2} G(z) + \theta T(z) = \frac{1}{N} : D J^a(z, \theta) J^a(z, \theta) : + \frac{2i}{3N^2} \epsilon_{abc} J^a(z, \theta) J^b(z, \theta) J^c(z, \theta),
\]

(3.6)

where we set \( D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \). We here note that the zero-modes of the total currents \( J^a \) commute with all the modes of superconformal currents;

\[
[G(z), J^a_0] = [T(z), J^a_0] = 0.
\]

(3.7)

The supersymmetric extensions of the gluing conditions (2.3), (2.4) are given by

\[
(J^a_n + \tilde{J}^a_n) |B; \epsilon\rangle = 0, \\
(\psi^a_r + i\epsilon \tilde{\psi}^a_r) |B; \epsilon\rangle = 0, \\
(L_n - \tilde{L}_n) |B; \epsilon\rangle = 0, \\
(G_r - i\epsilon \tilde{G}_r) |B; \epsilon\rangle = 0,
\]

(3.8)

where \( \epsilon = \pm 1 \) indicates the signature related to the choice of NS or R sectors in the open string channel.

For the later convenience we introduce some notations of the currents at the boundary just as in the bosonic case

\[
J^a_{\pm}(\sigma) = J^a(\sigma) \pm \tilde{J}^a(\sigma), \\
\psi^a_{\pm}(\sigma) = \psi^a(\sigma) \pm i\epsilon \tilde{\psi}^a(\sigma), \\
T_{-}(\sigma) = T(\sigma) - \tilde{T}(\sigma), \\
G_{-}(\sigma) = G(\sigma) - i\epsilon \tilde{G}(\sigma),
\]

(3.9)
where we set
\[
\psi_a^\pm (\sigma) = \sum_n \psi_{\pm, n} e^{-i n \sigma}, \quad \psi_{\pm, n} = \psi_n^a \pm i \epsilon \tilde{\psi}_n^a .
\] (3.10)

By using these notations the gluing conditions are expressed as
\[
J_+^a(\sigma)|B; \epsilon\rangle = 0 , \quad \psi_+^a(\sigma)|B; \epsilon\rangle = 0 , \quad T_-(\sigma)|B; \epsilon\rangle = 0 , \quad G_-(\sigma)|B; \epsilon\rangle = 0 .
\] (3.11)

However, as in the bosonic case, it is really enough to only impose the conditions
\[
J_+^a(\sigma)|B; \epsilon\rangle = 0 , \quad \psi_+^a(\sigma)|B; \epsilon\rangle = 0 ,
\] (3.12)
or equivalently,
\[
j_+^a(\sigma)|B; \epsilon\rangle = 0 , \quad \psi_+^a(\sigma)|B; \epsilon\rangle = 0 ,
\] (3.13)
since the superconformal invariance at the boundary can be derived from these conditions.

We also define the currents
\[
j_{f, \pm}^a(\sigma) = j_f^a(\sigma) \pm \tilde{j}_f^a(\sigma) ,
\] (3.14)
where the fermionic currents \( j_f^a, \tilde{j}_f^a \) are given by (3.4) and we can rewrite them as
\[
j_{f, +}^a(\sigma) = -\frac{i}{2 N} e^{abc} (\psi_+^b \psi_-^c + \psi_-^b \psi_+^c) ,
\]
\[
j_{f, -}^a(\sigma) = -\frac{i}{2 N} e^{abc} (\psi_+^b \psi_-^c + \psi_-^b \psi_+^c) .
\] (3.15)

### 3.2 Wilson Line in \( SU(2) \) Super WZW Model

We extend the discussion about the Wilson line in bosonic WZW model to the supersymmetric case. We shall start with the superfield representation of boundary currents;
\[
\mathcal{J}_\pm^a(\sigma, \theta) \equiv \psi_{\pm}^a(\sigma) + \theta J_\pm^a(\sigma), \quad \mathcal{T}_-(\sigma, \theta) \equiv \frac{1}{2} G_-(\sigma) + \theta T_-(\sigma) .
\] (3.16)

The gluing conditions (3.11) are now given by
\[
\mathcal{J}_\pm^a(\sigma, \theta)|B; \epsilon\rangle = 0 ,
\]
\[
\mathcal{T}_-(\sigma, \theta)|B; \epsilon\rangle = 0 .
\] (3.17)
A natural extension of the bosonic Wilson line (2.21) to the supersymmetric one is given by

\[ W(\{M^a\}) \equiv \int \mathcal{D}\mathcal{G} \exp \left[ i \int_0^{2\pi} d\sigma \int d\theta \langle \mathcal{G} | \left( D - \frac{1}{N} \mathcal{J}^a M^a \right) | \mathcal{G} \rangle \right] , \tag{3.18} \]

where the supercoordinate \( \mathcal{G}(\sigma, \theta) \) is defined as

\[ \mathcal{G}(\sigma, \theta) = \exp(i\theta \eta^a R_L(T^a)) g(\sigma) . \tag{3.19} \]

(\( \eta^a(\sigma) \) are Grassmann coordinates.) The path-integral measure is defined in the standard way, \( \mathcal{D}\mathcal{G} = \mathcal{D}g \mathcal{D}\eta \). Precisely speaking, we must define it so that it does not include the zero-modes of the fermionic coordinates \( \eta^a \) to obtain non-vanishing integrals. We also used the symbol of the superderivative defined by

\[ D = -\frac{\partial}{\partial \theta} + \theta i \frac{\partial}{\partial \sigma} . \tag{3.20} \]

We now focus on the two cases of CP matrices just as in the bosonic case; (1) the commutative CP matrices \([M^a, M^b] = 0\) and (2) the CP matrices of fuzzy sphere \([M^a, M^b] = i\epsilon^{abc} M^c\).

For the first case we assume for simplicity that \( M^1 = M^2 = 0 \) and \( M^3 \) is a diagonal matrix. By this assumption we can readily carry out the \( \eta \)-integral, since this is nothing but a Gaussian integral, and obtain

\[ W(\{M^a\}) = \text{Tr} \left( \exp \left( -\frac{2\pi i}{N} J^a_3 M^3 \right) \right) . \tag{3.21} \]

This Wilson line actually preserves the world-sheet superconformal invariance because of the commutation relations (3.7).

For the second case of fuzzy sphere we can again integrate out the fermionic coordinate \( \eta(\sigma) \) explicitly. However, we have to be more careful to evaluate the Gaussian integral. We again assume that \( M^a = R_L(T^a) \) and make use of the abbreviated notations \( \mathcal{J}_- \equiv \mathcal{J}^a M^a \), \( \psi_- \equiv \psi^a M^a \), etc. in the following discussion. We first note that

\[ \int \mathcal{D}\mathcal{G} \exp \left[ i \int_0^{2\pi} d\sigma \int d\theta \langle \mathcal{G} | \left( D - \frac{1}{N} \mathcal{J}_- \right) | \mathcal{G} \rangle \right] = \int \mathcal{D}g \mathcal{D}\eta \exp \left[ -\int_0^{2\pi} d\sigma \langle \mathcal{G} | \left( \frac{d}{d\sigma} + \frac{i}{N} \mathcal{J}_- \right) + i\eta + \frac{1}{N} (\eta \psi_- + \psi_- \eta) \right] | \mathcal{G} \rangle . \tag{3.22} \]

Thus we can apply the Gaussian integral

\[ \int \mathcal{D}\eta \exp \left[ -\int_0^{2\pi} d\sigma \langle \mathcal{G} | i\eta \rangle | \mathcal{G} \rangle \right] = C , \tag{3.23} \]
where $C$ is some constant which is independent of $g$ because the path-integral measure is invariant under the transformation $\eta \to U(\sigma) \eta U(\sigma)^{-1}$. Since the path integral (3.22) has the linear term of $\eta$, we obtain

$$
\int D\eta \exp \left[ -\int_0^{2\pi} d\sigma \langle g| \left(i\eta + \frac{1}{N}(\eta \psi_- + \psi_- \eta)\right) |g\rangle \right] = \int D\eta \exp \left[ -\int_0^{2\pi} d\sigma \left\{ \langle g|i(\eta - \frac{i}{N}\psi_-)^2|g\rangle + \langle g|\frac{i}{N^2}\psi_-^2|g\rangle \right\} \right], \tag{3.24}
$$

and can apply the Gaussian integral to the first term of the second line. Naively it seems that the integral again gives merely the same constant $C$ by means of the simple change of variable $\eta' = \eta - \frac{i}{N}\psi_-$. But this is not correct, since $\psi_-^a$ is not a c-number. We have to carefully evaluate the $g$ dependence of the path integral

$$
F(g) = \int D\eta \exp \left[ -\int_0^{2\pi} d\sigma \langle g|i \left( \eta - \frac{i}{N}\psi_- \right)^2 |g\rangle \right]. \tag{3.25}
$$

First we point out that $F(g)$ is obviously a c-number, which does not include the operator $\psi_-$. Replacing $g(\sigma)$ by $U(\sigma)g(\sigma)$, we obtain

$$
F(Ug) = \int D\eta \exp \left[ -\int_0^{2\pi} d\sigma \langle g|U^{-1}i(\eta - \frac{i}{N}\psi_-)^2U|g\rangle \right] = \int D\eta' \exp \left[ -\int_0^{2\pi} d\sigma \langle g|i \left( \eta' - \frac{i}{N}(\eta'U^{-1}\psi_- U + U^{-1}\psi_- U\eta') \right)^2 - \frac{1}{N^2}U^{-1}\psi_-^2 U |g\rangle \right], \tag{3.26}
$$

where we changed the integral variable $\eta \to \eta' \equiv U^{-1}\eta U$. We next make use of the following equalities

$$
U^{-1}(\sigma)\psi_- U(\sigma) = \hat{U}^{-1}\psi_- \hat{U}, \tag{3.27}
$$

$$
U^{-1}(\sigma)\frac{1}{N}\psi_-^2 U(\sigma) = \hat{U}^{-1} \frac{1}{N}\psi_-^2 \hat{U} - 2iU(\sigma)^{-1} \frac{d}{d\sigma} U(\sigma). \tag{3.28}
$$

where we set $U(\sigma) = \exp(-2\pi i u^a(\sigma) M^a)$ and $\hat{U} = \exp\left( i \int d\sigma u^a(\sigma) j^a_{f-}(\sigma) \right)$. The second term in the right hand side of the equality (3.28) is originating from the anomalous contribution when evaluating it on the boundary state. In fact, we point out that

$$
\frac{1}{N}\psi_-^2(\sigma)|B;\epsilon\rangle \equiv -j^a_{f-}(\sigma)M^a|B;\epsilon\rangle, \tag{3.29}
$$

and

$$
\hat{U}^{-1}j^a_{f-}(\sigma)\hat{U} = U(\sigma)^{-1}j^a_{f-}(\sigma)U(\sigma) - 2iU(\sigma)^{-1} \frac{d}{d\sigma} U(\sigma). \tag{3.30}
$$

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$F(Ug)$ is then evaluated as

$$F(Ug) = \hat{U}^{-1} \int D\eta' \exp \left[ -\int_0^{2\pi} d\sigma \langle g| i \left( \eta' - \frac{i}{N} \psi_- \right)^2 |g\rangle - \frac{2}{N} \langle g| U(\sigma)^{-1} \frac{d}{d\sigma} U(\sigma)|g\rangle \right] \hat{U} .$$

We can thus obtain

$$F(g) = C' \exp \left[ -\int_0^{2\pi} d\sigma \left\{ -\frac{2}{N} \langle g| \frac{d}{d\sigma} |g\rangle \right\} \right] ,$$

where $C'$ is some constant independent of $g$.

In this way we can obtain from (3.22)

$$\int Dg \exp \left[ -\int_0^{2\pi} d\sigma \langle g| \left( \left( 1 - \frac{2}{N} \right) \frac{d}{d\sigma} + \frac{i}{N} J_- \right) + \frac{i}{N^2} \psi_-^2 |g\rangle \right] = \int Dg \exp \left[ -\int_0^{2\pi} d\sigma \langle g| \left( \left( 1 - \frac{2}{N} \right) \frac{d}{d\sigma} + \frac{i}{N} j_- \right) |g\rangle \right] .$$

On the second line we used $J_- + \frac{1}{N} \psi_-^2 = j_- - \frac{1}{N} \psi_-^2$ and neglect the term $-\frac{1}{N} \psi_-^2$ because of the gluing condition $\psi_+^a(\sigma)|B; \epsilon\rangle = 0$. As we already pointed out, the factor $\frac{N - 2}{N}$ is irrelevant and we finally obtain for the CP matrices of fuzzy sphere

$$W(\{M^a\}) = \text{Tr} \left( P \exp \left( -\frac{i}{N - 2} \int_0^{2\pi} d\sigma j_+^a(\sigma) M^a \right) \right) .$$

This form is the same as the Wilson line in the bosonic case. It is quite important that the final expression (3.34) contains only the bosonic currents $j_+^a$ in contrast to the case of commutative CP matrices (3.21), in which the total currents $J_+^a$ appear instead of $j_+^a$. Because we already know that the Wilson line (3.34) commutes with all the currents $j_+^a(\sigma)$ (and also $J_+^a(\sigma)$), it is straightforward to show that this Wilson line preserves the superconformal invariance. (Recall that the bosonic currents $j_+^a(z)$ have the level $N - 2$.) It is also not difficult to show the same property directly from the expression (3.18) in the similar argument as in the bosonic case. In fact, we can prove the supersymmetric extension of the equality (2.27):

$$\hat{U} \left( D - \frac{1}{N} \mathcal{J}_-(\sigma, \theta) \right) \hat{U}^{-1} = U(\sigma, \theta) \left( D - \frac{1}{N} \mathcal{J}_-(\sigma, \theta) \right) U(\sigma, \theta)^{-1} ,$$

(3.35)
where we set
\[ \hat{U} \equiv \exp \left( i \int_0^{2\pi} d\sigma \int d\theta \, u^a(\sigma, \theta) J^a_+ (\sigma, \theta) \right), \quad U(\sigma, \theta) \equiv \exp (-2\pi i u^a(\sigma, \theta) M^a). \] \hspace{1cm} (3.36)

Hence we can likewise prove that the supercurrents \( J^a_+ (\sigma, \theta) \) commute with the Wilson line of fuzzy sphere.

### 3.3 Boundary State Analysis for the D-branes in the NS5 Background and Space-time SUSY

Now let us study the D-branes in the NS5 background. We shall consider the near horizon geometry of \((N - 2)\) NS5-branes described by the CHS \(\sigma\)-model \[19\];
\[ R^5, R^1 \times R^\phi \times SU(2), \] \hspace{1cm} (3.37)
where \(R^\phi\) denotes the radial direction and is described by the Liouville theory. The background charge of Liouville field is given by \(Q = \sqrt{\frac{2}{N}}\). \(SU(2)\) stands for the \(SU(2)\) WZW model (the level of bosonic currents is equal to \(N - 2\)). There are also 8 (physical) free fermions, which make the \(SU(2)\) sector raised to the super WZW model with the shifted level \(N\). Since the bosonic coordinates along \(R^5, R^1\times R^\phi\) are not important for our discussion, we will neglect these parts. Nevertheless we need take account of the fermionic coordinates of these directions to impose the GSO condition properly. We shall hence deal with the system of \(SU(2)\) WZW model and 8 free fermions. We here only work on the structure of \(N = 1\) world-sheet SUSY, although it is known that the world-sheet SUSY of the CHS \(\sigma\)-model can be extended to \(N = 4\) \[\text{[4]}\]. We summarize several features of our system as superconformal theory with the extended SUSY in appendix B.

We begin our analysis by introducing the Ishibashi states for the system of interest;
\[ |\ell, s\rangle_I, \quad \ell = 0, 1, \ldots, N - 2, \quad s \in \mathbb{Z}_4, \] \hspace{1cm} (3.38)
where \(s\) labels the integral representations of \(SO(2n)\) \((n = 4)\) in our setup \((s = 0\) for basic, \(s = 2\) for vector, \(s = 1\) for spinor and \(s = 3\) for cospinor representations, respectively). These

\[4\text{We must incorporate the } R^\phi\text{-direction for the extended SUSY. (See appendix B.)} \]
states satisfy the following boundary conditions

\begin{align*}
(J^a_n + J^a_{-n})|\ell, s\rangle_I &= 0 , \\
\psi^a_r|\ell, s\rangle_I &= -i\bar{\psi}^a_{-r}|\ell, s + 2\rangle_I , \\
(L_n - \bar{L}_{-n})|\ell, s\rangle_I &= 0 , \\
G_r|\ell, s\rangle_I &= i\bar{G}_{-r}|\ell, s + 2\rangle_I .
\end{align*}

(3.39)

The “inner products” between Ishibashi states are given by

\begin{align*}
I\langle \ell, s|\bar{q}^H(c)|\ell', s'\rangle_I &= \delta_{\ell,\ell'}\delta_{s,s'}(-1)^s\chi^{(N-2)}_\ell(q)\chi^{SO(8)}_s(q) ,
\end{align*}

(3.40)

where \(\chi^{SO(8)}_s\) is the character of \(SO(8)_1\). In general, the character of \(SO(2n)_1\) is given by

\begin{align*}
\chi^{SO(2n)}_0 + \chi^{SO(2n)}_2 &= \left(\frac{\theta_3}{\eta}\right)^n , \\
\chi^{SO(2n)}_0 - \chi^{SO(2n)}_2 &= \left(\frac{\theta_1}{\eta}\right)^n , \\
\chi^{SO(2n)}_1 + \chi^{SO(2n)}_3 &= \left(\frac{\theta_2}{\eta}\right)^n , \\
\chi^{SO(2n)}_1 - \chi^{SO(2n)}_3 &= \left(-i\frac{\theta_1}{\eta}\right)^n .
\end{align*}

(3.41)

Then, we can construct the Cardy states

\begin{equation}
|L, S\rangle_C = \sum_{\ell=0}^{N-2} \sum_{s \in \mathbb{Z}_4} \frac{S_{L,\ell}}{\sqrt{S_{0,\ell}}} e^{-i\pi \frac{\ell^2}{2}} |\ell, s\rangle_I ,
\end{equation}

(3.42)

where \(S\) labels spin structure. The cylinder amplitude between these Cardy states is given by

\begin{equation}
C(L', S'|q^H(c)|L, S\rangle_C = \sum_{\ell, s} N_{L,\ell}^{S} \chi^{(N-2)}_\ell(q)\chi^{SO(8)}_s(q)\delta^{(4)}_{S-S'+s+2,0} ,
\end{equation}

(3.43)

where \(\delta^{(4)}_{a,0}\) defines \(a = 0 \mod 4\).

It is also convenient to introduce the projections of the Cardy states to the NSNS-sector \(|L; \epsilon\rangle_C^{(NS)}\) \((\epsilon = \pm 1)\) and to the RR-sector \(|L; \epsilon\rangle_C^{(R)}\);

\begin{align*}
|L; \epsilon\rangle_C^{(NS)} &= |L, S\rangle_C + |L, S + 2\rangle_C , \\
|L; \epsilon\rangle_C^{(R)} &= |L, S\rangle_C - |L, S + 2\rangle_C ,
\end{align*}

(3.44)

where \(\epsilon = +1\) corresponds to \(S = 0\) and \(\epsilon = -1\) corresponds to \(S = 1\). These boundary states satisfy the boundary condition (3.8) and especially preserve the \(N = 1\) world-sheet SUSY.

We must suitably take account of the GSO projection \(\frac{1-(-1)^F}{2}\) in the underlying string theory, where \(F\) denotes the world-sheet fermion number in the left mover. Since the action of operator \((-1)^F\) to the Cardy state is given by

\begin{equation}
(-1)^F |L, S\rangle_C = |L, S - 1\rangle_C ,
\end{equation}

(3.45)
the GSO invariant combinations in NSNS-sector and RR-sector are

\[ |B; L\rangle^{(NS)} = \frac{1}{\sqrt{2}} \left( |L; +1\rangle^{(NS)}_C - |L; -1\rangle^{(NS)}_C \right), \]

\[ |B; L\rangle^{(R)} = \frac{1}{\sqrt{2}} \left( |L; +1\rangle^{(R)}_C + |L; -1\rangle^{(R)}_C \right). \] (3.46)

Then the supersymmetric boundary state representing a D-brane is defined by

\[ |B; L\rangle = \frac{1}{\sqrt{2}} \left( |B; L\rangle^{(NS)} + |B; L\rangle^{(R)} \right). \] (3.47)

The boundary state describing the anti-brane is likewise given by

\[ |\bar{B}; L\rangle = \frac{1}{\sqrt{2}} \left( |B; L\rangle^{(NS)} - |B; L\rangle^{(R)} \right). \] (3.48)

We calculate the cylinder amplitude between the branes. If the brane configuration preserves the space-time SUSY, the amplitude should vanish. The amplitude between the branes labeled by \( L \) and \( L' \) becomes

\[ \langle B; L | \hat{q}^{(c)} \hat{q}^{(c)} | B; L' \rangle = \sum_{\ell} N_{L,L'}^{(N-2)} \left[ \left( \frac{\theta_3}{\eta} \right)^4 - \left( \frac{\theta_1}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right] (q) = 0, \] (3.49)

which implies the existence of space-time SUSY as pointed out. On the other hand, the amplitude between the brane and anti-brane becomes

\[ \langle \bar{B}; L | \hat{q}^{(c)} \hat{q}^{(c)} | B; L' \rangle = \sum_{\ell} N_{L,L'}^{(N-2)} \left[ \left( \frac{\theta_3}{\eta} \right)^4 + \left( \frac{\theta_1}{\eta} \right)^4 - \left( \frac{\theta_2}{\eta} \right)^4 \right] (q) \neq 0, \] (3.50)
as expected, since this configuration does not preserve any supersymmetry.

Now our main purpose is to study the aspects of boundary states with the insertion of Wilson line previously considered. The Cardy state \( |B; L = 0\rangle \) is the single “D0-brane state”, and the various configurations of multiple D0-branes are realized as

\[ |B; \{M^a\}\rangle = W(\{M^a\}) |B; L = 0\rangle, \] (3.51)

where \( W(\{M^a\}) \) is the Wilson line operator defined by (3.18).

Suppose \( M^a \) are the \((L + 1) \times (L + 1)\) matrices of the fuzzy sphere, then the boundary state \( |B; \{M^a\}\rangle \) satisfies the gluing conditions (3.11), as we already discussed. It is also clear that \( |B; \{M^a\}\rangle \) preserves the half of space-time SUSY, since the expression (3.34) does not
contain the fermionic degrees of freedom and thus it obviously commutes with any spin fields. Moreover we can obtain, under the large $N$ limit,

$$|B; \{M^a\} \approx |B; L\rangle ,$$

just as in the bosonic case. The right hand side is the Cardy state corresponding to the D2-brane wrapped on the $(L + 1)$-th conjugacy class of $SU(2)$.

Next we consider the commutative CP matrices and again we assume that $M^1 = M^2 = 0$ and $M^3 = \text{diag}(a_1, a_2, \ldots, a_{L+1})$. The boundary states $|B; \{M^a\} \rangle$ now becomes

$$|B; \{M^a\} \rangle = \text{Tr} \left( \exp \left( -\frac{2\pi i}{N} J^3_{-0} M^3 \right) \right) |B; 0\rangle = \sum_{i=1}^{L+1} |B; L = 0, a_i\rangle ,$$

where we set

$$|B; L, a\rangle \equiv e^{-\frac{4\pi i}{N} J^3_{-0} a} |B; L\rangle .$$

We now observe that the boundary states (3.53) cannot preserve any space-time SUSY except for the special case $a_1 = a_2 = \cdots = a_{L+1}$. In fact, let $Q_{\epsilon_1 \cdots \epsilon_4}, \tilde{Q}_{\bar{\epsilon}_1 \cdots \bar{\epsilon}_4}$ ($\epsilon_i, \bar{\epsilon}_i = \pm 1$) be the space-time SUSY charges defined by the four real bosons $H_1, \ldots, H_4$ bosonizing the transverse fermions. Especially, we take the convention such that

$$i\partial H_1 = i\psi_1 \psi^2 ,$$

and its counter part of the right mover. Assume that $|B; L = 0\rangle$ preserves the following SUSY with some coefficients $\Lambda_{\epsilon_1 \cdots \epsilon_4}$.

$$\left( Q_{\epsilon_1 \cdots} + \Lambda_{\epsilon_1 \cdots} \tilde{Q}_{\bar{\epsilon}_1 \cdots} \right) |B; L = 0\rangle = 0 .$$

Since the total current $J^3$ has a non-trivial commutation relations with $H_1$, we can find that

$$Q_{\epsilon_1 \cdots} e^{-\frac{4\pi i}{N} J^3_{0} a} = e^{-\frac{2\pi i}{N} a \epsilon_1 \cdots} e^{-\frac{4\pi i}{N} J^3_{0} a} Q_{\epsilon_1 \cdots} .$$

Hence the boundary state $|B; L = 0, a_i\rangle$ preserves the space-time SUSY

$$\left( Q_{\epsilon_1 \cdots} + \Lambda_{\epsilon_1 \cdots} e^{-\frac{2\pi i}{N} a_0} \tilde{Q}_{\bar{\epsilon}_1 \cdots} \right) |B; 0, a_i\rangle = 0 ,$$

but their summation (3.53) does not except for the case $a_1 = a_2 = \ldots = a_{L+1}$. 

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We also examine whether or not the brane configuration preserves the space-time SUSY by another way. After some straightforward calculations we find out the following cylinder amplitudes

$$\langle B; L', a' | \tilde{q}^{H^{(c)}} | B; L, a \rangle = \sum_{\ell,m} N_{\ell,L,L'}^{\ell} \left[ \left( \frac{\theta_3}{\eta} \right)^3 \text{Ch}_{\ell,m}^{(NS)} - \left( \frac{\theta_4}{\eta} \right)^3 \text{Ch}_{\ell,m}^{(NS)} - \left( \frac{\theta_2}{\eta} \right)^3 \text{Ch}_{\ell,m}^{(R)} \right] \frac{\theta_{m+2(a'-a),N}}{\eta} (q), \quad (3.59)$$

where Ch_{\ell,m}(q) is the character of N = 2 minimal model of level N - 2 (see appendix C). In this calculation the next “Gepner model like” reinterpretation of CHS σ-model is essential

$$\mathbf{R}_\phi \times \text{super } SU(2)_N \cong \left( \mathbf{R}_\phi \times S^1 \right) \times \left( SU(2)_N/U(1) \right) \text{Z}_N, \quad (3.60)$$

where $\mathbf{R}_\phi \times S^1$ denotes the N = 2 Liouville theory with $\hat{c} \equiv 1 + Q^2 = \frac{N + 2}{N}$ and SU(2)$_N$/U(1) denotes the Kazama-Suzuki model [37] for SU(2)/U(1), which is one of the concise realization of the N = 2 minimal model of level N - 2 ($\hat{c} = \frac{N - 2}{N}$). Note that $J_0^3$ commutes with all the generators of this Kazama-Suzuki model and the appearance of the minimal characters Ch_{\ell,m}(q) is due to this fact. Especially, we here only need the L = L' = 0 sector of (3.59);

$$\langle B; 0, a' | \tilde{q}^{H^{(c)}} | B; 0, a \rangle = \sum_{m} \left[ \left( \frac{\theta_3}{\eta} \right)^3 \text{Ch}_{0,m}^{(NS)} - \left( \frac{\theta_4}{\eta} \right)^3 \text{Ch}_{0,m}^{(NS)} - \left( \frac{\theta_2}{\eta} \right)^3 \text{Ch}_{0,m}^{(R)} \right] \frac{\theta_{m+2(a'-a),N}}{\eta} (q). \quad (3.61)$$

When $a = a'$ holds, the above amplitude (3.59) reduces to the previous one (3.49) because of the “branching relation” (3.5) and vanishes, which reflects the existence of space-time SUSY. On the other hand, when $a \neq a'$, the amplitude (3.59) (and, of course, (3.61)) does not vanish. This implies that the configuration of branes characterized by the different $a$’s does not preserve any space-time SUSY. It is also easy to show that the cylinder amplitude defined with respect to the boundary state (3.53) does not vanish except for the case of $a_1 = \ldots = a_{L+1}$, which confirms our above expectation.

Since $a_1, \ldots, a_{L+1}$ correspond to the positions of $(L + 1)$ D0-branes on $S^3$, the above result means that only the stack of D0-branes should be BPS among the commutative configurations. This fact contrasts to the aspects in the flat backgrounds, in which we can freely distribute multiple D0-branes without breaking the space-time SUSY. Arbitrary commutative configurations of D0-branes in the flat backgrounds are marginally stable. In our case
of $S^3$ we can expect that the non-BPS configuration \(3.53\) (with, say, \(a_1 \neq a_2\)) should be unstable. In fact, we can find out the tachyonic excitations by studying the spectrum in the open string channel. For example, observing the \(q\)-expansion of the cylinder amplitude \(3.61\) (when setting \(a = a_1, a' = a_2\)), we can show that the lowest mass in NS-sector is evaluated as

\[
(mass)^2 = \left(\frac{1}{2}m + \Delta a\right)^2 - \frac{1}{4}m^2 + \frac{Q^2}{8} = \frac{\Delta a(\Delta a + m)}{N} + \frac{1}{4N},
\]

where \(\Delta a = a_1 - a_2\) and \(|\Delta a| \leq N/2\). The term \(Q^2/8 \equiv 1/(4N)\) is the contribution from the mass gap in the Liouville sector \(R_\phi\), which we neglected in the above argument. This term is not important under the large \(N\) approximation. Therefore, if \(\Delta a \neq 0\) holds, there always exists a particular \(m \in \mathbb{Z}_{2N}\) which generates a negative mass squared. In other words, we always have an open string tachyon in the cases of \(\Delta a \neq 0\), which makes the system unstable. We will next discuss how this instability is related to the formation of spherical D2-brane.

### 3.4 Tachyon Condensation and Formation of Spherical Brane

As is discussed in the flat background in many literatures \([20]\), we can often expect that some stable BPS configurations of branes arise after the open string tachyons condensate. It is quite interesting to discuss the similar phenomenon in our case of the NS5 background. For the simplicity we shall take a simple example \(L = 1\), namely, the case of two D0-branes on \(S^3\). We start with the \(2 \times 2\) CP matrices \(M^1 = M^2 = 0, M^3 = \sigma^3/2\), which means that the Wilson line \(3.18\) contains the super affine currents as the form

\[
\begin{pmatrix}
J^3 - (\sigma) & 0 \\
0 & -J^3 - (\sigma)
\end{pmatrix}
\]

As we previously observed, this is a non-BPS configuration and we have the tachyonic modes. The tachyon fields should arise as the non-diagonal elements of CP matrices, just like the \(D - \bar{D}\) system in the flat background. Therefore it is reasonable to consider the following deformation of \(3.63\);

\[
\begin{pmatrix}
\mathcal{J}^a M^a = \frac{1}{2} \begin{pmatrix}
\mathcal{J}^3 - (\sigma) & 0 \\
0 & -\mathcal{J}^3 - (\sigma)
\end{pmatrix}
\end{pmatrix}
\]

(3.63)

where we introduced the “tachyon field” \(T, \bar{T}(\equiv T^\dagger)\). (We here only consider the constant tachyon fields.) At first glance this deformation \(3.64\) seems to be marginal, but our observation about the mass spectrum of open string implies that it should be precisely marginally...
Therefore the tachyon fields \( T, \bar{T} \) glow along the trajectory of RG flow starting from the point (3.63). Then, can we have the fixed point at which the tachyon fields condensate? It is a difficult problem to make a complete answer, since we need to solve the dynamics away from the conformal point. Nevertheless, we can expect the next natural solution for the fixed point

\[
T = e^{i\alpha}, \quad \bar{T} = e^{-i\alpha} \quad (\alpha \in \mathbb{R}).
\] (3.65)

This is no other than the fuzzy sphere configuration! As we already observed, the CP matrices of fuzzy sphere correspond to a BPS bound state, and at least under the large \( N \) approximation it is identified with the Cardy state (with \( L = 1 \)). Among other things, it is easy to show that the open string spectrum appearing in the cylinder amplitude like (3.49) has no tachyonic excitations. This means that the system is stable and resides at a fixed point of the boundary renormalization group.

Our claim here is summarized as follows;

\[
|B; \{M^a\}\rangle \equiv |B; L = 0, a = \frac{1}{2}\rangle + |B; L = 0, a = -\frac{1}{2}\rangle \xrightarrow{\text{tachyon condensation}} |B; L = 1\rangle. \quad (3.66)
\]

To close this subsection let us make several comments;

1. It may be natural to assume that the central charge, which is directly calculated from the RR-part of the boundary state \([38, 39, 12, 24]\), should not change through the tachyon condensation. In fact, we can easily check that the both sides of (3.66) have the equal central charges. More generically, the central charge of \( |B; \{M^a\}\rangle \) with \( M^1 = M^2 = 0 \), \( M^3 = \text{diag}(L/2, L/2-1, \ldots, -L/2) \) is computed, up to some factors of no interest, as follows\(^5\):

\[
\sin \left( \frac{\pi}{N} \right) \times \left( e^{i\pi \frac{L}{N}} + e^{i\pi \frac{L}{N}-\frac{2}{N}} + \cdots + e^{-i\pi \frac{L}{N}} \right) \equiv \sin \left( \frac{\pi L + 1}{N} \right), \quad (3.67)
\]

which is indeed equal to the central charge of the Cardy state \( |B; L\rangle \).

2. The brane mass (or tension) can be readily read off from the NSNS part of the boundary state \([39, 10]\). The left hand side in (3.66) has the brane mass

\[
\text{mass} \sim |\sin \left( \frac{\pi}{N} \right)| + |\sin \left( \frac{\pi}{N} \right)|, \quad (3.68)
\]

\(^5\)Strictly speaking, we must turn on the Liouville potential term in the \( N = 2 \) Liouville sector in order to obtain the non-zero central charges. See [24].
and the right hand side has

\[ \text{mass} \sim | \sin \left( \frac{2\pi}{N} \right) | . \]  

(3.69)

Clearly (3.68) is greater than (3.69) because of the triangular inequality. This feature reflects directly the fact that the left hand side expresses the non-BPS branes, while the right hand side corresponds to the BPS saturated configuration. It is also consistent with the \( g \)-theorem about the boundary RG flow [11].

3. It is easy to extend (3.66) to more general cases. Suppose that we start with the \((L + 1) \times (L + 1)\) CP matrices \(M^1 = M^2 = 0\), and \(M^3\) is a diagonal matrix. Let us further assume that there exists an \((L + 1)\)-dimensional (not necessarily irreducible) representation \(R\) of \(SU(2)\) such that \(R(T^3) = M^3\). In this situation, when we have the decomposition

\[ R \cong R_{L_1} \oplus \cdots \oplus R_{L_r} , \quad \left( \sum_{i=1}^{r} (L_i + 1) = L + 1 \right) , \]  

(3.70)

our claim should be addressed as follows;

\[ |B; \{ M^a \} \rangle \xrightarrow{\text{tachyon condensation}} \sum_{i=1}^{r} |B; L_i \rangle . \]  

(3.71)

It is not hard to show that the both sides of (3.71) have the equal central charges and the total mass of left hand side is greater than that of the right hand side, which is decomposed to the \(r\) pieces of Cardy states and describes (marginally stable) BPS bound states. One might feel that the D0-brane configurations we are treating are rather limited, since the eigen-values of \(M^3\) are now assumed to only take some discrete values. However, since our discussion here is based on the large \(N\) approximation, one can expect that the sufficiently dense distributions of D0-branes are realized in this argument. (Recall the definition (3.54).)

4. In the T-dualized framework, our NS5 background can be reinterpreted as the ALE space (with the vanishing \(B\)-field) of \(A_{N-1}\) type singularity [34, 42]. In the picture of ALE space the boundary states of “D0-branes” \(|L = 0, a = M/2\rangle \ (M = N, N - 2, \ldots)\) correspond to the “primitive vanishing cycles” which are in one to one correspondence with the simple roots of \(A_{N-1}\), and the “D2-brane states” \(|B; L\rangle \ (L \neq 0, N - 2)\) correspond to the supersymmetric cycles (special Lagrangian submanifolds) homologous to the non-trivial sums of the primitive vanishing cycles [22, 24]. It is an interesting point that the fuzzy sphere configurations in the
NS5 background is equivalent to the special Lagrangian configurations in the ALE side. The BPS saturation in the former seems to be due to a stringy effect, “fuzziness of the space-time coordinates”, while that of the latter is based on the classical geometry with no quantum corrections.

4 Summary and Discussions

In this paper we especially studied the BPS bound states of multiple D0-branes realized as the spherical D2-branes, which was proposed in [13, 14], from the view points of boundary states. We realized the configurations of D0-branes as the insertions of Wilson line and investigated the gluing condition by making use of the path integral techniques. We have further shown that the fuzzy sphere configuration of D0-branes directly leads to the Cardy states corresponding to several conjugacy classes of SU(2) group, which finely confirms the interpretation of spherical D2-branes as the stable bound state of D0-branes.

We also present a discussion about this subject from the view points of the tachyon condensation. In contrast to the flat background any commutative configurations of D0-branes are non-BPS (except for the case when all the D0-branes are stacked at one point) and always contain the tachyonic excitations in the open string channel. The existence of open string tachyons implies that the deformation of the system is marginally relevant, and we claimed that after the tachyon condensation, the system should flow into a fuzzy sphere configuration, which is manifestly BPS and has no tachyons in the open string spectrum. A similar observation was already given in [14]. However, there is a subtle point in relation to our discussion. In [14], being inspired by the argument of Kondo problem [11], the perturbation term such as $S_{\text{pert}} \sim \int J^a(\sigma)S^a$ is discussed ($S^a$ should be identified with the CP matrix $M^a$ in this paper.), and the combined currents $\hat{J}^a \equiv J^a + S^a$ are introduced, since they commute with the perturbation term $S_{\text{pert}}$. On the other hand, in our case we take the Wilson line operator defined by the path-ordered trace instead of $S_{\text{pert}}$, and we have no room to consider the combined currents like $\hat{J}^a$. In fact, the Wilson line of fuzzy sphere actually commutes with all the currents $J^a_+$ (not the currents like $\hat{J}^a$). It may be an important task to clarify the relation between these two approaches.

For the future directions it is an interesting subject to relate our analysis based on the
boundary conformal field theory with the approach of low energy effective field theory, especially, the analysis on some classical solutions of “unstable solitons” analogous to those given in the flat non-commutative spaces.

The analysis in the finite $k$ (or finite $N$) system is more challenging problem. However, if we intend to make the argument on the tachyon condensation for the D-branes in NS5 background as in section 3, there is a subtle point; the mass gap of Liouville sector in the evaluation of (3.62) is not necessarily small in the finite $N$ case. We will have to carefully treat the Liouville sector to work on this problem.

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Appendix A Some Remarks on Path Integral Representation

In section 2 the path integral representation of Wilson line (2.21) was used in a formal way. In order to remove the potential subtlety due to the UV divergence, we should first discretize the coordinate $\sigma$ and take the continuum limit after that. In this appendix we define the path integral in terms of the discretized coordinates and show that the formal analysis in section 2 can be confirmed with no subtlety.

We start with discretizing the coordinate $\sigma_n = na$, where $a = 2\pi/N$ and $n = 1, 2, \cdots, N$. The delta function and the integral should be replaced with
\[
\delta(\sigma_n - \sigma_m) \rightarrow \frac{1}{a} \delta_{n,m}, \quad \int_0^{2\pi} d\sigma \rightarrow a \sum_{n=1}^N ,
\]
and also we have
\[
\frac{d}{d\sigma} f(\sigma_n) \rightarrow \Delta f(\sigma_n) = \frac{f(\sigma_n + a) - f(\sigma_n - a)}{2a} .
\]
In order to construct the action in terms of the discretized coordinates we have to use the dimensionless variables such as
\[
\hat{\Delta} = a \Delta, \quad \hat{J}_a^+ (\sigma_n) = a J_a^+ (\sigma_n), \quad \hat{J}_a^- (\sigma_n) = a J_a^- (\sigma_n) .
\]

The discretized version of path integral representation should be defined as
\[
\int \prod_{m=1}^N dg(\sigma_m) \exp \left[ - \sum_{n=1}^N \langle g(\sigma_n) | (\hat{\Delta} + i \frac{\hat{J}_a^- (\sigma_n) M^a}{k} ) | g(\sigma_n) \rangle \right] ,
\]
where $dg(\sigma_m)$ denotes the Haar measure of $SU(2)$. The discretized version of commutation relations of the currents [2.6,2.7]
\[
[\hat{J}_a^\pm (\sigma_n), \hat{J}_b^\pm (\sigma_m)] = 2\pi i \epsilon^{abc} c J_c^+ (\sigma_n) \delta_{n,m} ,
\]
\[
[\hat{J}_a^\pm (\sigma_n), \hat{J}_b^- (\sigma_m)] = 2\pi i \epsilon^{abc} c J_c^- (\sigma_n) \delta_{n,m} + 2\pi i k \delta^{ab} \frac{1}{2} (\delta_{n+1,m} - \delta_{n-1,m}) .
\]
One might be afraid that the expression (A.4) is not well-defined due to the ordering problem of $\hat{J}_a^- (\sigma_n)$ in the exponential. However, this is not the case because of the next commutation relation
\[
[\hat{J}_a^- (\sigma_n) | g(\sigma_n) | M^a | g(\sigma_n) \rangle, \hat{J}_b^- (\sigma_m) | g(\sigma_m) | M^b | g(\sigma_m) \rangle]
\]
\[
= 2\pi i \epsilon^{abc} \hat{J}_c^+ (\sigma_n) | g(\sigma_n) | M^a | g(\sigma_n) \rangle | g(\sigma_n) | M^b | g(\sigma_n) \rangle \delta_{n,m}
\]
\[
= 0 .
\]
Since the coordinates are discrete, we can use Kronecker symbol rather than Dirac $\delta$-function, which removes the subtlety of the divergence of equal $\sigma \delta$-function. In this way we conclude that the path integral (A.4) is well-defined with no problem of the operator ordering.

We also remark that we need not here take account of the gauge fixing of $U(1)$-gauge symmetry: $g(\sigma_m) \to g(\sigma_m)a(\sigma_m)$, ($\forall a(\sigma_m) \in U(1)$) for the discretized framework. The gauge volume is finite as in the usual lattice gauge theory.

In our argument of section 2 it is quite important that the Wilson line of fuzzy sphere configuration (2.20) commutes with the currents $J^a_\pm(\sigma)$. We now show that this is indeed the case for the discretized version of Wilson line (A.4). For this purpose we only have to replace the unitary operators (2.28) with

$$
\hat{U} = \exp \left( i \sum_{n=1}^{N} u^a(\sigma_n) \hat{J}^a_+(\sigma_n) \right), \quad U(\sigma_n) = \exp(-2\pi i u^a(\sigma_n) M^a).
$$

Then the following identities

$$
\hat{U} \left[ \hat{\Delta} - \frac{i}{k} \hat{J}^a_-(\sigma_n) M^a \right] \hat{U}^{-1} = U(\sigma) \left[ \hat{\Delta} + \frac{i}{k} \hat{J}^a_+(\sigma_n) M^a \right] U(\sigma)^{-1},
$$

are also satisfied as in (2.27). Thus we can likewise show that the Wilson line commutes with the currents $J^a_\pm(\sigma)$ by using the identity;

$$
\hat{U} \int \prod_{m=1}^{N} dg(\sigma_m) \exp \left[ - \sum_{n=1}^{N} \langle g(\sigma_n) \vert \left( \hat{\Delta} + \frac{i}{k} \hat{J}^a_+(\sigma_n) M^a \right) \vert g(\sigma_n) \rangle \right] \hat{U}^{-1}
$$

$$
= \int \prod_{m=1}^{N} dg(\sigma_m) \exp \left[ - \sum_{n=1}^{N} \langle g(\sigma_n) \vert U(\sigma_n) \left( \hat{\Delta} + \frac{i}{k} \hat{J}^a_-(\sigma_n) M^a \right) U(\sigma_n)^{-1} \vert g(\sigma_n) \rangle \right],
$$

and the fact that the Haar measure $dg(\sigma_n)$ is invariant under the field redefinition $g(\sigma_n) \to g'(\sigma) = U(\sigma_n)^{-1} g(\sigma_n)$.

Up to now, we investigated the properties of the path integral representation when the coordinate $\sigma$ is discretized. We have to take the continuum limit $a \to 0$ to define the Wilson line operator (2.21) in section 2. Since it generically has divergent contributions in this limit, we will have to take account of the renormalization of coupling constants and would potentially suffer non-trivial radiative corrections. However, in our case (A.4), the story becomes quite simple as long as it is inserted at the boundary with the suitable gluing condition (2.3), of which discretized version is

$$
\hat{J}^a_+(\sigma_n) \vert B \rangle = 0.
$$
We showed in the above argument that the Wilson line operator (A.4) preserves the symmetry of discretized currents $\hat{J}_a^\pm(\sigma_n)$, and thus we obtain

$$\hat{J}_a^\pm(\sigma_n) W(\{M^a\};a)|B\rangle = 0,$$

(A.12)

where $W(\{M^a\};a)$ denotes the Wilson line operator defined in (A.4).

The discretized conformal invariance at boundary should be realized in terms of the discretized boundary stress tensor;

$$\hat{T}_-(\sigma_n) = \frac{1}{k} \hat{J}_-(\sigma_n) \hat{J}_+(\sigma_n) \equiv \sum_{m \in \mathbb{Z}_N} \hat{L}_{-,m} e^{-im\sigma_n},$$

(A.13)

which is the discretized counterpart of (2.8) as we will discuss below. Notice that the products of currents $\hat{J}_a^\pm(\sigma_n)$ at the equal points are now well-defined without any subtlety of UV divergence. We can directly check that the mode oscillators $\hat{L}_{-,m}$ generate a closed algebra together with $\hat{J}_{\pm,n}$ (defined by $\hat{J}_{\pm,m} = \sum_{n \in \mathbb{Z}_N} \hat{J}_{\pm,n} e^{-im\sigma_l}$);

$$[\hat{L}_{-,m}, \hat{J}_{\pm,n}] = -\frac{1}{a} \sin(an) \ \hat{J}_{\pm,n+m},$$

$$[\hat{L}_{-,m}, \hat{L}_{-,n}] = \frac{1}{a} \{ \sin(an) - \sin(\tilde{\sigma}n) \} \hat{L}_{-,m+n} + \sum_{l \in \mathbb{Z}_N} c(m, n, l; a) j_{-,m+n-l}^a \hat{J}_{+,l}^a,$$

(A.14)

where $c(m, n, l; a)$ are some constants depending on $m, n, l \in \mathbb{Z}_N$ and are of order $a$.

One might think our definition (A.13) to be peculiar, since we include the factor $1/k$ rather than the usual one $1/(k+2)$. By our construction of discretized currents we need not introduce the normal ordering, and the absence of the level shift $k \rightarrow k+2$ is originating from this fact. By this reason it seems subtle whether the continuum limit of $\hat{L}_{-,n}$ truly corresponds to the mode oscillator of boundary stress tensor $L_n - \tilde{L}_{-,n}$ (2.8) (defined with the usual normal ordering). However, the commutation relations (A.14) imply that the continuum limits of $\hat{L}_{-,n}$ satisfy the same commutation relations with the currents $J_{\pm,n}^a$ as those of $L_n - \tilde{L}_{-,n}$,

\footnote{In taking $a$ to zero limit, a subtle point is in the zero-mode part, since the normal ordering contribution would become important. However, we can show that $\langle 0 | \hat{L}_{-,0} | 0 \rangle = 0$ and $\langle 0 | [\hat{L}_{-,n}, \hat{L}_{-,n}] | 0 \rangle = 0$ hold for arbitrary $a$ without taking the normal ordering. The essential point is the cancellation of contributions from the central terms of left and right moving sectors and the second equality is derived from the property $c(n, -n, l; a) = c(n, -n, -l; a)$.}
and hence they are identified with each other on the states of the type;

\[ \sum_{a=\{a_1,a_2,\ldots\}, n=\{n_1,n_2,\ldots\}, r} N_{a,n,r} \prod_i J_{a_i}^{n_i} |B; r\rangle, \]

where $|B; r\rangle$ satisfies (A.11). All the states considered in relation to the Wilson lines in this paper are indeed of this type. Therefore we can regard (A.13) as the discretized version of (2.8) in our arguments.

Equation (A.12) and the definition (A.13) readily implies

\[ \hat{T}_- (\sigma_n) W(\{M^a\}; a) |B\rangle = 0. \]

(A.15)

In this way we can conclude that the boundary state $W(\{M^a\}; a) |B\rangle$ preserves the discretized conformal invariance for an arbitrary finite lattice spacing $a$. This fact means that $W(\{M^a\}; a) |B\rangle$ corresponds to the fixed point of boundary renormalization group flow\[7\], and thus we can take the continuum limit without suffering the renormalization and any radiative corrections. Therefore we can safely conclude that the Wilson line operator in the continuous theory (2.21) (with the CP matrices (2.20)) preserves the gluing condition (2.3). That is truly the statement we need for our argument in section 2.

For a general Wilson line operator such as $\text{Tr} \left( P \exp \left( i \lambda \oint J_a^a (\sigma) M^a \right) \right)$ (namely, with a general $\lambda$, and general matrices $M^a$), taking the continuum limit will be of course a non-trivial problem with complicated radiative corrections, since it is not a truly marginal operator. This fact makes the rigid analysis away from the conformal points difficult and it is beyond the scope of this paper.

**Appendix B  Extended Supersymmetry**

In this appendix we show that our boundary state defined in the CHS background actually preserves the $N = 2$ and $N = 4$ superconformal symmetries. In addition to the $SU(2)$ supercurrents defined in section 3, the Liouville mode is expressed as $\phi$ and its superpartner

\[ ^{\text{7}}\text{This aspect seems to be consistent with the perturbative calculation of } \beta\text{-function presented in the works } [1]. \]
as $\psi$, which has the OPE:

$$\psi\psi(z)\psi(0) \sim 1/z.$$

We also define

$$\psi^\pm = \frac{1}{\sqrt{2}}(\psi^1 \pm i\psi^2), \quad \Psi^\pm = -\frac{1}{\sqrt{2}}(\psi^3 \pm i\psi^\phi), \quad j^\pm = j^1 \pm ij^2.$$  \hfill (B.1)

First, we investigate the system as an $N = 2$ superconformal field theory. The realizations of the $N = 2$ superconformal currents are given by

$$T = -\frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi + \frac{1}{N}(j^a j^a) - \frac{1}{2}(\psi^a \partial\psi^a) - \frac{1}{2}(\psi^\phi \partial\psi^\phi),$$

$$G^\pm = -\frac{1}{\sqrt{2}}\left(\sqrt{2}\frac{1}{N}j^3 \pm \partial\phi\right)\Psi^\pm + \frac{1}{\sqrt{N}}(j^\pm \psi^\mp) \mp \frac{Q}{\sqrt{2}}\partial\Psi^\pm,$$ 

$$J = \Psi^+\Psi^- - \psi^+\psi^-,$$  \hfill (B.2)

where $a = 1, 2, 3$ and we will use these indices below. In this theory two types of the boundary conditions preserving the $N = 2$ SUSY are possible and called the A-type and the B-type conditions 38:

**A-type**

$$\begin{align*}
(J(\sigma) - \tilde{J}(\sigma))|B; \epsilon) &= 0, \\
(G^\pm(\sigma) - i\epsilon\tilde{G}^\mp(\sigma))|B; \epsilon) &= 0. 
\end{align*}$$  \hfill (B.3)

**B-type**

$$\begin{align*}
(J(\sigma) + \tilde{J}(\sigma))|B; \epsilon) &= 0, \\
(G^\pm(\sigma) - i\epsilon\tilde{G}^\mp(\sigma))|B; \epsilon) &= 0. 
\end{align*}$$  \hfill (B.4)

The gluing conditions compatible with the B-type boundary condition (B.4) are given by

$$\begin{align*}
(j^\pm(\sigma) + e^{\pm\alpha}\tilde{j}^\pm(\sigma))|B; \epsilon) &= 0, \\
(j^3(\sigma) + \tilde{j}^3(\sigma))|B; \epsilon) &= 0, \\
(\partial\phi(\sigma) + \tilde{\partial}\phi(\sigma) - Q)|B; \epsilon) &= 0, \\
(\psi^\pm(\sigma) + i\epsilon e^{\pm\alpha}\tilde{\psi}^\pm(\sigma))|B; \epsilon) &= 0, \\
(\psi^{3,\phi}(\sigma) + i\epsilon\tilde{\psi}^{3,\phi}(\sigma))|B; \epsilon) &= 0. 
\end{align*}$$  \hfill (B.5)

---

8We here use the different normalization of fermions in $SU(2)$ sector and their OPEs are given by

$$\psi^a(z)\psi^b(0) \sim \frac{\delta^{ab}}{z}.$$
where $\alpha \in \mathbb{R}$ and there is the momentum shift $Q$ in the Liouville mode (see, for example, [24]). The case of $\alpha = 0$ is used in section 3 to define the Cardy states (3.44). It is obvious that the Wilson line of fuzzy sphere does not break the $N = 2$ SUSY. For the commutative CP matrices, $M^1 = M^2 = 0$ and $M^3 = \text{diag}(a_1, a_2, \ldots)$, let us consider the decomposition like (3.53). For each term we obtain the gluing conditions (B.5) with the various phase $\alpha$ depending on the value $a_i$. Hence the boundary state with such Wilson line obeys the B-type condition as well. In this way we have shown that the $N = 2$ superconformal symmetry is surely preserved on the boundary states we considered in section 3.

On the other hand, we have to change the gluing conditions to make it compatible with the A-type condition as follows;

\[
(j^{\pm}(\sigma) + e^{\pm i\alpha} \tilde{j}^{\mp}(\sigma))|B; \epsilon\rangle = 0 ,
\]
\[
(j^3(\sigma) - \tilde{j}^3(\sigma))|B; \epsilon\rangle = 0 ,
\]
\[
(\partial \phi(\sigma) + \tilde{\partial} \tilde{\phi}(\sigma) - Q)|B; \epsilon\rangle = 0 ,
\]
\[
(\psi^{\pm}(\sigma) + i e^{\pm i\alpha} \tilde{\psi}^{\mp}(\sigma))|B; \epsilon\rangle = 0 ,
\]
\[
(\psi^3(\sigma) - i e \tilde{\psi}^3(\sigma))|B; \epsilon\rangle = 0 ,
\]
\[
(\psi^\phi(\sigma) + i e \tilde{\psi}^\phi(\sigma))|B; \epsilon\rangle = 0 ,
\]

which is an example of general gluing condition (2.2).

Next, we focus on the (small) $N = 4$ superconformal structure. This theory has $SU(2)_R$ currents $A^a$ and two more superconformal currents other than $N = 2$ ones. We take their linear combinations so that one is a singlet under the $SU(2)$ transformation and the others are vectors. Their explicit forms in the CHS $\sigma$-model are obtained as follows [15, 19];

\[
T = -\frac{1}{2} (\partial \phi)^2 - \frac{Q}{2} \partial^2 \phi + \frac{1}{N} (j^a j^a) - \frac{1}{2} (\psi^a \partial \psi^a) - \frac{1}{2} (\psi^\phi \partial \psi^\phi) ,
\]
\[
G^0 = i \partial \phi \psi^\phi + Qi \partial \psi^\phi + \sqrt{\frac{2}{N}} (j^a \psi^a - i \psi^1 \psi^2 \psi^3) ,
\]
\[
G^a = i \partial \phi \psi^a + Qi \partial \psi^a + \sqrt{\frac{2}{N}} (-j^a \psi^\phi + e^{abc} j^b \psi^c + i e^{abc} \psi^\phi \psi^b \psi^c) ,
\]
\[
A^a = -\frac{i}{2} \psi^\phi \psi^a - \frac{i e^{abc}}{4} \psi^b \psi^c .
\]

In general the boundary condition preserving the $N = 4$ SUSY is given by [38];

\[
(A^a(\sigma) + \Lambda^a_b \tilde{A}^b(\sigma))|B; \epsilon\rangle = 0 ,
\]
\[(G^0(\sigma) - i\epsilon \tilde{G}^0(\sigma))|B; \epsilon\rangle = 0,\]
\[(G^a(\sigma) - i\epsilon \Lambda^a_b \tilde{G}^b(\sigma))|B; \epsilon\rangle = 0,\]  \hspace{1cm} (B.8)

where $\Lambda^a_b$ is an automorphism of $SU(2)$. This condition is compatible with the twisted gluing condition of the type (2.2);

\[(j^a(\sigma) + \Lambda^a_b \tilde{j}^b(\sigma))|B; \epsilon\rangle = 0,\]
\[(\partial \phi(\sigma) + \partial \tilde{\phi}(\sigma) - Q)|B; \epsilon\rangle = 0,\]
\[(\psi^a(\sigma) + i\epsilon \Lambda^a_b \tilde{\psi}^b(\sigma))|B; \epsilon\rangle = 0,\]
\[(\psi^a(\sigma) + i\epsilon \tilde{\psi}^a(\sigma))|B; \epsilon\rangle = 0,\]  \hspace{1cm} (B.9)

and thus we can easily construct the boundary state satisfying the $N = 4$ boundary condition (B.8). Notice that the gluing condition (B.5) with $\alpha = 0$ is the special case with $\Lambda^a_b = \delta^a_b$ and hence the boundary states (3.44) satisfy this condition. More general cases with the Wilson lines can be also discussed just as in the $N = 2$ argument.

Appendix C Convention of Conformal Field Theory

1. Theta functions

The Jacobi theta functions are defined by

\[\theta_1(q, z) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} z^{n-\frac{1}{2}}, \quad \theta_2(q, z) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} z^{n-\frac{1}{2}},\]

\[\theta_3(q, z) = \sum_{n=-\infty}^{\infty} q^\frac{n^2}{2} z^n, \quad \theta_4(q, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^\frac{n^2}{2} z^n,\]  \hspace{1cm} (C.1)

where we set $q = e^{2\pi i \tau}$ and $z = e^{2\pi i \nu}$. For an arbitrary positive integer $k$, the theta function of level $k$ is defined by

\[\theta_{m,k}(q, z) = \sum_{n=-\infty}^{\infty} q^{k \frac{m^2}{2} + \frac{m^2}{k}} z^{k(n+\frac{m}{k})},\]  \hspace{1cm} (C.2)

Therefore we can rewrite the Jacobi theta functions in terms of the theta function of level 2:

\[i\theta_1(q, z) = \theta_{1,2}(q, z) - \theta_{3,2}(q, z), \quad \theta_2(q, z) = \theta_{1,2}(q, z) + \theta_{3,2}(q, z),\]
\[\theta_3(q, z) = \theta_{0,2}(q, z) + \theta_{2,2}(q, z), \quad \theta_4(q, z) = \theta_{0,2}(q, z) - \theta_{2,2}(q, z).\]  \hspace{1cm} (C.3)
2. Characters of $N = 2$ minimal model

There is a discrete series of unitary representations of $N = 2$ superconformal algebra with $c < 3$, namely, with $c = \frac{3k}{k+2}$ ($k = 1, 2, 3, \ldots$). Based on these representations one can construct the family of rational conformal field theories known as the $N = 2$ minimal models. The discrete representations of the $N = 2$ algebra are related to the $\hat{SU}(2)_k$ representations.

The character of $\hat{SU}(2)_k$ with the spin $\ell$ ($0 \leq \ell \leq k$) representation is calculated as

$$\chi^{(k)}_\ell(q) = \theta_{\ell+1,k+2} - \theta_{\ell-1,k+2} \theta_{1,2} - \theta_{-1,2}$$

and the coefficient $c^\ell_m(q)$ is called the string function. The character of $N = 2$ representation labeled by $(\ell, m, s)$ is obtained through the “branching relation” \cite{36}:

$$\chi^{(k)}_\ell(q)\theta_{s,2}(q) = \sum_{m=-k-1}^{k+2} \chi^{\ell,s}_m(q)\theta_{m,k+2}(q) ,$$

where we set

$$\chi^{\ell,s}_m(q) = \sum_{r\in\mathbb{Z}_k} c^\ell_{m-s+4r}(q)\theta_{2m+(k+2)(-s+4r),2k(k+2)}(q) .$$

These “branching functions” $\chi^{\ell,s}_m(q)$ are defined in the range $\ell \in \{0, \ldots, k\}$, $m \in \mathbb{Z}_{2k+4}$, $s \in \mathbb{Z}_4$ and $\ell + m + s = 0 \text{ mod } 2$. The characters of $N = 2$ minimal model of level $k$ are then expressed as

$$\text{Ch}^{(NS)}_{\ell m}(q) = \chi^{\ell,0}_m(q) + \chi^{\ell,2}_m(q) , \quad \overline{\text{Ch}}^{(NS)}_{\ell m}(q) = \chi^{\ell,0}_m(q) - \chi^{\ell,2}_m(q) ,$$

$$\text{Ch}^{(R)}_{\ell m}(q) = \chi^{\ell,1}_m(q) + \chi^{\ell,3}_m(q) , \quad \overline{\text{Ch}}^{(R)}_{\ell m}(q) = \chi^{\ell,1}_m(q) - \chi^{\ell,3}_m(q) .$$

(C.7)
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