On the rigidity of harmonic-Ricci solitons

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Abstract
In this paper we introduce the notion of rigidity for harmonic-Ricci solitons and we provide some characterizations of rigidity, generalizing some known results for Ricci solitons. In the compact case we are able to deal with not necessarily gradient solitons while, in the complete non-compact case, we restrict our attention to steady and shrinking gradient solitons. We show that the rigidity can be traced back to the vanishing of certain modified curvature tensors that take into account the geometry a Riemannian manifold equipped with a smooth map \( \varphi \), called \( \varphi \)-curvatures, which are a natural generalization of the standard curvature tensors in the setting of harmonic-Ricci solitons.

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1 Introduction
Ricci solitons, i.e., self-similar solutions of the Ricci flow, had been intensively studied since the pioneering work of R. S. Hamilton [H]. They have been subject of the studies of an incredible number of scientists: we recommend, for instance, the survey paper [C] or Chapter 8 of [AMR] and all the references therein. More recently, B. List and R. Müller, see [L] and [M], respectively, combined the Ricci flow for a metric with the heat flow for a map obtaining the so called harmonic-Ricci flow. Self-similar solutions of the harmonic-Ricci flow are called harmonic-Ricci solitons and quite recently started attracting some attention; see, for instance, [W], [YS] and [YZ].

Recall that a Riemannian manifold \((M, g)\) is called harmonic-Ricci soliton (with respect to the positive constant \(\alpha\), the smooth map \(\varphi : M \to N\), where the target \((N, \langle \cdot, \cdot \rangle_N)\) is a Riemannian manifold, the

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vector field $X$ and $\lambda \in \mathbb{R}$) if the following hold

$$
\begin{cases}
\text{Ric} - \alpha \varphi^* \langle \cdot, \cdot \rangle_N + \frac{1}{2} \mathcal{L}_X g = \lambda g \\
\tau(\varphi) = d\varphi(X),
\end{cases}
$$

(1.1)

where $\tau(\varphi)$ denotes the tension field of $\varphi$ and $\mathcal{L}_X g$ the Lie derivative of the metric $g$ in the direction of $X$. The harmonic-Ricci soliton is called shrinking, steady or expanding if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, and it is called gradient if the vector field can be replaced with the gradient of a smooth function $f$ on $M$, called potential function. For gradient harmonic-Ricci solitons, (1.1) reads

$$
\begin{cases}
\text{Ric} - \alpha \varphi^* \langle \cdot, \cdot \rangle_N + \text{Hess}(f) = \lambda g \\
\tau(\varphi) = d\varphi(\nabla f).
\end{cases}
$$

While not much is yet known about harmonic-Ricci solitons, many interesting classification results are available for Ricci solitons. A key point in the study of Ricci solitons is the understanding of the central role of Einstein metrics. Indeed, any Einstein manifold endowed with a Killing vector field give rise to a trivial Ricci soliton. Starting from Einstein manifolds it is possible to build more general and less trivial examples of gradient Ricci solitons, the so-called rigid gradient Ricci solitons introduced by P. Petersen and W. Wylie in [PW]. A gradient Ricci soliton is said to be rigid if it is isometric to a quotient of the Riemannian product $L \times \mathbb{R}^k$, where $L$ is an Einstein manifold and $f(x) = \frac{1}{2}|x|^2$ on the Euclidean factor.

Actually, in the compact case nothing changes: a compact gradient Ricci soliton is rigid if and only if it is Einstein and the potential function is constant and, moreover, rigidity is equivalent to the constancy of the scalar curvature, see for instance the work [ELM] of M. Eminenti, G. La Nave and C. Mantegazza. In the complete non-compact case we have an analogous situation for steady gradient Ricci solitons, since they are rigid precisely when the scalar curvature is constant, see Proposition 3.2 of [PW].

The key role played by Einstein metrics for Ricci solitons, in the setting of harmonic-Ricci solitons, is fundamental, starting from the harmonic-Einstein manifolds. Recall that a Riemannian manifold that take in account the geometry of a smooth manifold equipped with a smooth map $\varphi : M \to N$, where the target $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold and $\lambda \in \mathbb{R}$, is called harmonic-Einstein manifold with respect to the positive constant $\alpha$, the smooth map $\varphi : M \to N$, where the target $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold and $\lambda \in \mathbb{R}$, if the following hold

$$
\begin{cases}
\text{Ric} - \alpha \varphi^* \langle \cdot, \cdot \rangle_N = \lambda g \\
\tau(\varphi) = 0.
\end{cases}
$$

The results of [FG] and [MS] reveals fundamental in the work [CC] of H. D. Cao and Q. Chen, where the authors studied complete Bach-flat gradient Ricci soliton obtaining, among other things, their rigidity (see Theorem 1.1 and Theorem 1.2). By an accurate analysis of the geometry of the regular level sets of the potential function they obtained that the vanishing of the Bach tensor implies that the Weyl tensor is harmonic, hence they reduce the proof to the results of [FG] and [MS].

The first aim of this article is to introduce the notion of rigidity for gradient harmonic-Ricci soliton. The key role played by Einstein manifolds for Ricci solitons, in the setting of harmonic-Ricci solitons is occupied by harmonic-Einstein manifolds. Recall that a Riemannian manifold $(M, g)$ of dimension $m \geq 2$ is called harmonic-Einstein (with respect to the positive constant $\alpha$), the smooth map $\varphi : M \to N$, where the target $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold and $\lambda \in \mathbb{R}$, if the following hold

$$
\begin{cases}
\text{Ric} - \alpha \varphi^* \langle \cdot, \cdot \rangle_N = \lambda g \\
\tau(\varphi) = 0.
\end{cases}
$$

The author, in [ACR] and [A], studied the geometric properties of some curvature tensors on a Riemannian manifold that take in account the geometry of a smooth manifold equipped with a smooth map $\varphi$, the so-called $\varphi$-curvatures. It is important to observe that for a harmonic-Einstein manifold the $\varphi$-scalar curvature $S^\varphi := S - \alpha |d\varphi|^2$, where $|d\varphi|^2$ is the Hilbert-Schmidt norm of the differential of $\varphi$, being the trace of the $\varphi$-Ricci tensor $\text{Ric}^\varphi := \text{Ric} - \alpha \varphi^* \langle \cdot, \cdot \rangle_N$, is constant on the whole $M$. Starting from the $\varphi$-Ricci tensor one may define, among the other $\varphi$-curvatures, the $\varphi$-Cotton, the $\varphi$-Weyl and the $\varphi$-Bach tensors, see Section 2.1 for the details.

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We say that a gradient harmonic-Ricci soliton with respect to $\alpha, \varphi : M \to N$ smooth and $\lambda$ and with potential function $f$ is rigid if it is isometric to a quotient of the Riemannian product $L \times \mathbb{R}^k$, where $L$ is a harmonic-Einstein manifold with respect to $\alpha$, $\psi : M \to N$ smooth and $\lambda$, the potential $f = \frac{1}{2} |x|^2$ on the Euclidean factor and the map $\varphi = \psi$ on the harmonic-Einstein factor, see Definition 4.1 below for a more precise formulation.

A compact harmonic-Ricci soliton (not necessarily gradient) is called rigid if it is harmonic-Einstein. We will see that for compact harmonic-Ricci soliton rigidity is equivalent to having constant $\varphi$-scalar curvature. The main result for compact solitons is Theorem 5.20 that is a generalization, at least for dimension $m \geq 3$, of Theorem 2.1 of [FG]. At first one may think that the natural generalization of the hypothesis of having harmonic-Weyl tensor is to have harmonic $\varphi$-Weyl tensor, but this is not true. For manifolds of dimension $m \geq 4$ having harmonic Weyl tensor is equivalent to the vanishing of the Cotton tensor. What is true then is that, as stated in Theorem 5.20, $\varphi$-Cotton flat compact harmonic-Ricci soliton of dimension $m \geq 3$ are rigid.

Moving on to the complete case, in Proposition 4.38 we show that the constancy of the $\varphi$-scalar curvature is necessary and sufficient for the rigidity of complete gradient steady Ricci-harmonic soliton, extending Proposition 3.2 of [PW] in our setting. For complete non-steady gradient harmonic-Ricci solitons we characterize rigidity via the condition of having parallel $\varphi$-Ricci tensor, see Theorem 4.31. This result is similar to Theorem 1.2 of [PW], although our hypothesis is slightly stronger, see Remark 4.37. Nevertheless, relying on Theorem 4.31 we are able to extend the results [FG] and [MS] regarding complete non-compact shrinking soliton. Indeed, in Corollary 6.25 we characterize the rigidity of complete non-compact gradient shrinking harmonic-Ricci soliton via the vanishing of the $\varphi$-Cotton tensor.

Our final aim is the generalization of the results of [CC] mentioned above. What we obtained is that the vanishing of the $\varphi$-Bach tensor characterize rigidity for complete gradient shrinking harmonic-Ricci solitons (actually, in dimension $m \neq 4$, we require the vanishing of the totally traceless part of the $\varphi$-Bach tensor, see Remark 7.12 for further comments on this assumption). Recall that Bach flat metrics for compact four dimensional Riemannian manifold are critical points of the conformal invariant functional
\[
g \mapsto \int_M |W_g|^2 \mu_g,
\]
where $g$ is a Riemannian metric, $W_g$ is the Weyl tensor and $\mu_g$ is the Riemannian volume element. The study of Bach flat metrics is interesting in view of their role in General Relativity, see [B]. In [A20] the author studied critical points of the conformal invariant functional $S_2$, that is a more general version of the functional (1.2) that does not depend only on a Riemannian metric $g$ but also on a smooth map $\varphi$, on a compact four dimensional manifold. Its critical points $(g, \varphi)$ are characterized by the validity of
\[
B^\varphi = 0, \quad J = 0,
\]
where $B^\varphi$ is the $\varphi$-Bach tensor and $J$ is the section of the pullback bundle $\varphi^{-1}TN$ defined in (2.20) below. The main interest of (1.3) resides in their applications in General relativity, as pointed out in [A20]. See Remark 7.15 for more details and the definition of $S_2$.

The study of $\varphi$-Bach flat harmonic-Ricci solitons is a first attempt to find non-trivial examples (i.e, not harmonic-Einstein) of critical points of $S_2$ and it has been the main motivation of our study. As mentioned above our main result says, essentially, that $\varphi$-Bach flat complete gradient shrinking harmonic-Ricci soliton are rigid: as usual in the compact case we are able to deal with harmonic-Ricci solitons that are not necessarily gradient, see Corollary 7.13 while in the complete non-compact case we restricted our attention to gradient shrinking harmonic-Ricci solitons, see Corollary 7.14. It is interesting that the validity of the equation $J = 0$ follows from $B^\varphi = 0$ for the harmonic-Ricci solitons in consideration, as pointed out at the end of Remark 7.15.

The techniques used to prove our results are very close to one used by the authors of the articles we mentioned, with the difference that one have to take care to the geometry of the smooth map $\varphi$ during all the process. This is not particularly difficult once one understands the relevant role played by the $\varphi$-curvatures. Although we were able to generalize in the context of harmonic-Ricci solitons some of the results of [PW], [FG], [MS] and [CC] we are far from the more satisfactory description of rigidity of Ricci solitons. What is missing, up to now, is a more detailed study in the lower dimensional case $m = 2, 3$. 

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(that will be object a future study) and the detection of a condition comparable notion to the one of locally conformally flatness for Ricci solitons.

The paper is organized as follows. In Section 2 we fix the notations and we collect the necessary preliminaries on $\varphi$-curvatures, harmonic-Einstein manifolds and harmonic-Ricci solitons, providing the adequate references where one may find the proofs. In Section 3 we obtain some general formulas that will be used in the next Sections. Section 4 marks the beginning of the core of the paper: it is dedicated to the definition of rigidity for harmonic-Ricci solitons and the study of their property. The characterization of rigidity via the vanishing of the $\varphi$-Cotton tensor is the aim of Section 5 and Section 6. In Section 5 we deal with not necessarily gradient compact solitons while in Section 6 we deal with complete non-compact shrinking solitons. In the final Section, Section 7, we characterize rigidity via vanishing condition on the $\varphi$-Bach tensor.

2 Notations and preliminaries

All the manifolds in this paper are assumed to be smooth, connected and without boundary.

Let $M$ be a smooth manifold and $g$ be a Riemannian metric on $M$, we denote by $\nabla$ the Levi-Civita connection of $(M,g)$. For the Riemann tensor $\text{Riem}$ of $(M,g)$ we use the sign conventions

$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for every $X,Y,Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of vector fields on $M$ and $[\ ,\ ]$ denotes the Lie bracket, and

$$\text{Riem}(W,Z,X,Y) = g(R(X,Y)Z,W)$$

for every $X,Y,Z,W \in \mathfrak{X}(M)$.

Let $(N,(\ ,\ )_N)$ be a Riemannian manifold of dimension $n$ and $\varphi : M \to N$ a smooth map. All the computation will be carry on using the moving frame formalism introduced by E. Cartan and we refer to Chapter 1 of [AMR]. We fix the indexes ranges

$$1 \leq i,j,k,t,\ldots \leq m, \quad 1 \leq a,b,c,d,\ldots \leq n$$

and from now on we adopt the Einstein summation convention over repeated indexes. In a neighborhood of each point of $M$ we can write

$$g = \delta_{ij} \theta^i \otimes \theta^j = \theta^i \otimes \theta^i$$

where $\delta_{ij}$ is the Kronecker delta and $\{\theta^i\}$ is a local orthonormal coframe. The dual frame will be denoted $\{e_i\}$ and is an orthonormal frame with respect to $g$.

The Levi-Civita connection forms $\{\theta^i\}$ are characterized by the skew-symmetry $\theta^i_j + \theta^j_i = 0$ and the validity of the first structure equations $d\theta^i + \theta^i_j \wedge \theta^j = 0$ and the curvature forms $\{\Theta^i_j\}$ are given by

$$\Theta^i_j = \frac{1}{2} R^i_{jkt} \theta^k \wedge \theta^t, \quad \text{where } R^i_{jkt} \text{ are the components of the } (0,4)\text{-version of the Riemann tensor } \text{Riem} \text{ of } (M,g),$$

$$\text{Riem} = R^i_{jkt} \theta^i \otimes \theta^j \otimes \theta^k \otimes \theta^t,$$

and they satisfy the second structure equations $\Theta^i_j = d\theta^i_j + \theta^i_k \wedge \theta^k_j$.

The Riemann tensor present the symmetries $R_{ijkt} + R_{ijkl} = 0$, $R_{ijkt} + R_{jikt} = 0$ and $R_{ijkt} = R_{kijt}$. Moreover it satisfy the two Bianchi identities

$$R_{ijkt} + R_{kijt} + R_{itjk} = 0$$

and

$$R_{ijkt,l} + R_{ijtl,k} + R_{ijkl,t} = 0, \quad (2.1)$$

where, for an arbitrary tensor field $T$ of type $(r,s)$

$$T = T^{i_1 \ldots i_r} j_1 \ldots j_s \theta^i_1 \otimes \ldots \otimes \theta^i_r \otimes e_{i_1} \otimes \ldots \otimes e_{i_r},$$

its covariant derivative is defined as the tensor field of type $(r,s+1)$

$$\nabla T = T^{i_1 \ldots i_r} j_1 \ldots j_s k \theta^k \otimes \theta^{i_1} \otimes \ldots \otimes \theta^{i_r} \otimes e_{i_1} \otimes \ldots \otimes e_{i_r},$$

and is an orthonormal frame with respect to $\mathfrak{X}(M)$.
where its components satisfies
\[ T^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} \theta^k = dT^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} - \sum_{t=1}^s T^{i_1 \ldots i_r}_{j_1 \ldots j_{s-t} h_{j_{s-t+1}} \ldots j_s} \theta^h \theta^k + \sum_{t=1}^r T^{i_1 \ldots i_r}_{j_1 \ldots j_{s-1} h_{j_{s}} \ldots h_{s+t}} \theta^h \theta^k. \]

Later on we will make use of the following commutation relation
\[ T^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} = T^{i_1 \ldots i_r}_{j_1 \ldots j_s, tk} + \sum_{t=1}^s R^{i_1 \ldots i_r}_{j_1 \ldots j_{s-t} h_{j_{s-t+1}} \ldots j_s} + \sum_{t=1}^r R^{i_1 \ldots i_r}_{h_{j_1} j_{s-1} h_{j_{s}} \ldots h_{s+t}} \]
that when \( T \) is given by the gradient of a smooth function \( f \) yields
\[ f_{ijk} = f_{hkl} + R^{l}_{ijk} f_t, \quad (2.2) \]
and when \( T \) is a two times covariant tensor field reads
\[ T_{ij,kt} = T_{ij,tk} + R^{l}_{hkl} T_{ij} + R^{l}_{ijkl} T_{kl}. \quad (2.3) \]

The trace of the Riemann tensor is called Ricci tensor \( \text{Ric} \) of \((M,g)\) and, in a local orthonormal coframe \( \{\theta^i\} \), is given by
\[ \text{Ric} = R_{ij} \theta^i \otimes \theta^j, \quad R_{ij} = R_{hijk}. \quad (2.4) \]
The scalar curvature \( S \) of \((M,g)\) is defined as the trace of the Ricci tensor and it is locally given by \( S = \theta^i R_{ij} \). Finally, the Riemannian volume element of \((M,g)\) is locally given by \( \mu = \theta^1 \wedge \ldots \wedge \theta^n \).

Let \( \{\Omega_a\}, \{\omega^a\}, \{\Omega^a_b\} \) be an orthonormal frame, coframe, the respectively Levi-Civita connection forms and curvature forms on an open subset \( V \) on \( N \) such that \( \varphi^{-1}(V) \subseteq U \). We set
\[ \varphi^* \omega^a = \varphi_t^a \theta^i \]
so that the differential \( d\varphi \) of \( \varphi \), a 1-form on \( M \) with values in the pullback bundle \( \varphi^{-1}(TN) \), can be written as
\[ d\varphi = \varphi_t^a \theta^i \otimes E_a. \]
The generalized second fundamental tensor of the map \( \varphi \) is given by \( \nabla d\varphi \), locally
\[ \nabla d\varphi = \varphi^b_{ia} \theta^j \otimes \theta^i \otimes E_a, \]
where its coefficient are defined according to the rule
\[ \varphi^b_{ia} \theta^j = d\varphi_t^b - \varphi^b_k \theta^k + \varphi^b_i \omega^a_k. \]
The tension field \( \tau(\varphi) \) of the map \( \varphi \) is the section of \( \varphi^{-1}TN \) given by
\[ \tau(\varphi) = \text{tr}(\nabla d\varphi) = \varphi^b_{ia} E_a. \quad (2.5) \]
Finally the bi-tension field \( \tau_2(\varphi) \) of the map \( \varphi \) is the section of \( \varphi^{-1}TN \) with components
\[ \tau_2(\varphi)^a = \varphi^a_{ijj} - \varphi^a_{ijj} - N R^{b}_{iade} \varphi^e_{i} \varphi^d_{j}, \quad (2.6) \]
where \( N R^{a}_{iade} \) are the components of the Riemann tensor of \((N, (\cdot, \cdot)_N)\). Recall that \( \varphi \) is said to harmonic if \( \tau(\varphi) = 0 \) and bi-harmonic if \( \tau_2(\varphi) = 0 \). Clearly harmonic maps are bi-harmonic.

We denote by \( \Delta \) the Laplace-Beltrami operator \( \Delta u = \text{tr}(\text{Hess}(u)) \) acting on functions \( u : M \to \mathbb{R} \), where \( \text{Hess}(u) = u_{ij} \theta^i \otimes \theta^j \), that is, \( \Delta u = u_{ij} \). Moreover, for every \( u \in C^\infty(M) \) the \( f \)-Laplacian of \( u \) is given by
\[ \Delta_f u := c f \text{div}(e^{-f} \nabla u). \]
It is easy to see that
\[ \Delta_f u = \Delta u - \langle \nabla f, \nabla u \rangle = u_{kk} - f_k u_k. \]
More generally, for every \( X \in \mathfrak{X}(M) \) we set
\[ \Delta_X u = \Delta u - \langle X, \nabla u \rangle. \]
If $A$ is a symmetric two times covariant tensor on $(M,g)$ the totally traceless part of $A$ is given by

$$\hat{A} := A - \frac{\text{tr}(A)}{m} g$$

and the symmetric two times covariant tensors $A^2$, $\Delta A$ and $\Delta f A$ have components, in a local orthonormal coframe, respectively

$$A^2_{ij} = A_{ik}A_{kj}, \quad \Delta A_{ij} = A_{ij, kk}$$

and

$$\Delta f A_{ij} = A_{ij, kk} - f_k A_{ij,k}.$$  

### 2.1 $\varphi$-Curvatures, harmonic-Einstein manifolds and harmonic-Ricci solitons

We recall the definition of $\varphi$-curvatures that we shall need later on, for their proof and other details we refer to [ACR] or Section 1.2 of [A].

Let $(M, g)$ be a Riemannian manifold of dimension $m \geq 2$, $(N, \langle , \rangle_N)$ a target Riemannian manifold and $\alpha$ a positive real constant. The $\varphi$-Ricci tensor is defined as

$$\text{Ric}^\varphi := \text{Ric} - \alpha \varphi^* \langle , \rangle_N$$

and its trace is denoted $S^\varphi$ and is called $\varphi$-scalar curvature. The $\varphi$-Schouten tensor is given by

$$A^\varphi := \text{Ric}^\varphi - \frac{S^\varphi}{2(m-1)} g$$

and the $\varphi$-Cotton tensor, that measures the failure of the commutation of the covariant derivatives of the $\varphi$-Schouten tensor, in global notation is given by

$$C^\varphi(X,Y,Z) := \nabla_Z A^\varphi(X,Y) - \nabla_Y A^\varphi(X,Z)$$

for every $X, Y, Z \in X(M)$

while, in moving frame notation, its components in a local orthonormal coframe are given by

$$C^\varphi_{ijk} = A^\varphi_{ij,k} - A^\varphi_{ik,j},$$

where we denoted by $A^\varphi_{ij}$ the components of the $\varphi$-Schouten tensor in a local orthonormal coframe.

For manifolds of dimension $m \geq 3$ we are able to define the $\varphi$-Weyl tensor by

$$W^\varphi := \text{Riem} - \frac{1}{m-2} A^\varphi \otimes g,$$

where $\otimes$ denotes the Kulkarni-Nomizu product of two times covariant symmetric tensors, that is,

$$(T \otimes V)(X, Y, Z, W) = T(X, Z)V(Y, W) - T(X, W)V(Y, Z) + T(Y, W)V(X, Z) - T(Y, Z)V(X, W).$$

In a local orthonormal coframe

$$W^\varphi_{ijk} = R_{ij,k} - \frac{1}{m-2}(A^\varphi_{ij}\delta_{ik} - A^\varphi_{ik}\delta_{ij} + A^\varphi_{ik}\delta_{ij} - A^\varphi_{ij}\delta_{ik}),$$

We point out that the divergence of the $\varphi$-Weyl is not, in general, a multiple of $\varphi$-Cotton tensor. Indeed the following holds

$$W^\varphi_{ijk, l} = \frac{m-3}{m-2} C^\varphi_{ikj} + \alpha (\varphi^a_i \varphi^a_j - \varphi^a_i \varphi^a_j) + \frac{\alpha}{m-2} \varphi^a_i (\varphi^a_j \delta_{ik} - \varphi^a_k \delta_{ij}).$$

The traces of $\varphi$-Cotton and of $\varphi$-Weyl are given by, respectively,

$$C^\varphi_{jji} = \alpha \varphi^a_j \varphi^a_i$$

and

$$W^\varphi_{kikj} = \alpha \varphi^a_i \varphi^a_j.$$
Following P. Baird and J. Eells, see [BaE], we define the stress-energy tensor of $\varphi$ (with a different sign convention) by

$$T := \varphi^* \langle \cdot, \cdot \rangle_N - \frac{|d\varphi|^2}{2} g,$$  \hspace{1cm} (2.15)

where $|d\varphi|^2 = \text{tr}(\varphi^* \langle \cdot, \cdot \rangle_N)$ is the square of the Hilbert-Schmidt norm of $d\varphi$. It is easy to see that, in a local orthonormal coframe,

$$\text{div}(T)_j = \varphi^a_{ij} \varphi^j_i.$$  \hspace{1cm} (2.16)

A map $\varphi$ is called conservative if the energy-stress tensor $T$ is divergence free. From the formula above harmonic maps are conservative.

The generalized Schur’s identity is given by

$$\text{div}(\text{Ric}^\varphi) = \frac{1}{2} dS^\varphi - \alpha \text{div}(T),$$

locally

$$R^\varphi_{ij,j} = \frac{1}{2} S^\varphi_i - \alpha \varphi^a_{ij} \varphi^a_i,$$  \hspace{1cm} (2.17)

where $R^\varphi_{ij}$ are the components of the $\varphi$-Ricci tensor in a local orthonormal coframe.

Finally, the $\varphi$-Bach tensor $B^\varphi$ has components, in a local orthonormal coframe and for manifolds of dimension $m \geq 3$,

$$(m - 2)B^\varphi_{ij} = C^\varphi_{ijk,k} + R^\varphi_{ik} (W^\varphi_{ikj} - \alpha \varphi^a_i \varphi^a_j \delta_{ijk}) + \alpha \left( \varphi^a_{ij} \varphi^a_{kk} - \varphi^a_{kkj} \varphi^a_i - \frac{1}{m - 2} |\tau(\varphi)|^2 \delta_{ij} \right).$$  \hspace{1cm} (2.18)

It is not immediate to see but the $\varphi$-Bach tensor is symmetric and its trace is given by

$$(m - 2)\text{tr}(B^\varphi) = \alpha \frac{m - 4}{m - 2} |\tau(\varphi)|^2.$$  \hspace{1cm} (2.19)

It remains only to define the tensor field $J$: its components are given by

$$J^a := \frac{m S^\varphi}{(m - 1)(m - 2)} \varphi^a_{ii} - \frac{m - 2}{2(m - 1)} S^\varphi_i \varphi^a_i - 2 R^\varphi_{ij} \varphi^a_{ij} + 2 \tau(\varphi) \varphi^a_i \varphi^a_i - \tau_2(\varphi)^a.$$  \hspace{1cm} (2.20)

For the motivation that led to its definition we refer to [A20].

**Definition 2.21.** Let $(M, g)$ be Riemannian manifold of dimension $m \geq 2$, $\alpha$ a positive constant, $\varphi : M \to N$ a smooth map, where the target $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold, and $\lambda \in \mathbb{R}$. Then $(M, g)$ is called harmonic-Einstein (with respect to $\alpha, \varphi$ and $\lambda$) if

$$\begin{cases}
\text{Ric}^\varphi = \lambda g \\
\tau(\varphi) = 0.
\end{cases}$$  \hspace{1cm} (2.22)

In case $\lambda = 0$ we say that $(M, g)$ is $\varphi$-Ricci flat with respect to $\alpha$.

**Remark 2.23.** Notice that harmonic-Einstein manifolds have parallel $\varphi$-Ricci tensor and thus they are $\varphi$-Cotton flat. Moreover it is possible to see that they satisfy $B^\varphi = 0$ and $J = 0$.

**Definition 2.24.** Let $(M, g)$ be Riemannian manifold of dimension $m \geq 2$, $\alpha$ a positive constant, $\varphi : M \to N$ a smooth map, where the target $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold, $X \in \mathfrak{X}(M)$ and $\lambda \in \mathbb{R}$. Then $(M, g)$ is called harmonic-Ricci soliton (with respect to $\alpha, \varphi, X$ and $\lambda$) if

$$\begin{cases}
\text{Ric}^\varphi + \frac{1}{2} \mathcal{L}_X g = \lambda g \\
\tau(\varphi) = d\varphi(X).
\end{cases}$$  \hspace{1cm} (2.25)

The harmonic-Ricci soliton is called shrinking, steady or expanding if, respectively, $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$. 7
Every harmonic-Einstein endowed with a vertical Killing vector field $Y \in \mathfrak{X}(M)$, i.e., a solution of
\[
\begin{align*}
\mathcal{L}_Y g &= 0 \\
\delta \varphi(Y) &= 0,
\end{align*}
\]
is trivially a harmonic-Ricci soliton. Furthermore, in (2.25), the vector field $X$ can be replaced with $X + Y$, for a vertical Killing vector field $Y$. In case $X = \nabla f + Y$ for some vertical Killing vector field $Y \in \mathfrak{X}(M)$, the equations (2.25) can be written as
\[
\begin{align*}
\text{Ric}^\varphi + \text{Hess}(f) &= \lambda g \\
\tau(\varphi) &= d\varphi(\nabla f)
\end{align*}
\]
and we say that $(M, g)$ is a gradient harmonic-Ricci soliton. The function $f$ is called potential function and it is defined up to an additive constant.

2.2 Preliminaries

We list the statement of some results that shall be useful in the rest of the article.

We start with Theorem 5.1.1 of [A].

**Theorem 2.28.** Let $(M, g)$ be a compact Riemannian manifold of dimension $m \geq 2$ with an Einstein-type structure of the form
\[
\begin{align*}
\text{Ric}^\varphi + \frac{1}{2} \mathcal{L}_X g &= \lambda g \\
\tau(\varphi) &= d\varphi(X),
\end{align*}
\]
for some $X \in \mathfrak{X}(M)$, $\lambda \in C^\infty(M)$, $\alpha > 0$ and $\varphi : M \to N$ a smooth map with target a Riemannian manifold $(N, \langle \cdot, \cdot \rangle_N)$. If $S^\varphi$ is constant then the structure (2.29) reduces to a harmonic-Einstein structure, that is,
\[
\begin{align*}
\text{Ric}^\varphi &= S^\varphi g \\
\tau(\varphi) &= 0
\end{align*}
\]
with $S^\varphi$ constant.

The fundamental step in the proof of the above Theorem is to show that integrating the equation
\[
\frac{1}{2} \Delta_X S^\varphi = -\alpha |\tau(\varphi)|^2 - |\text{Ric}^\varphi|^2 - (S^\varphi - m\lambda) \frac{S^\varphi}{m} + (m - 1) \Delta \lambda,
\]
one gets the validity of
\[
\frac{m-2}{2m} \int_M \langle X, \nabla S^\varphi \rangle = \int_M (|\text{Ric}^\varphi|^2 + \alpha |\tau(\varphi)|^2).
\]
The following is part of Theorem 7.3.3 of [A]. Its proof follows closely the one of Theorem 8.6 of [AMR] and, once again, relies on the validity of (2.30) and a clever use of the maximum principle.

**Theorem 2.31.** Let $(M, g)$ be a complete gradient harmonic-Ricci soliton of dimension $m \geq 2$ with respect to $\varphi : M \to N$ smooth map, where $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold, $f \in C^\infty(M)$, $\alpha > 0$ and $\lambda \geq 0$. Denoting $S^\varphi := \text{inf}_M S^\varphi$ we have $S^\varphi > -\infty$. Moreover

i) If $\lambda = 0$ then

\[ S^\varphi = 0. \]

Then either $S^\varphi > 0$ on $M$ or, if $f$ is non constant, $(M, g)$ splits as the Riemannian product of $\mathbb{R}$ with a totally geodesic $\psi$-Ricci flat hypersurface $\Sigma$, where $\psi := \varphi|_\Sigma$. Moreover $\varphi = \psi \circ \pi_\Sigma$ on $\mathbb{R} \times \Sigma$, where $\pi_\Sigma : \mathbb{R} \times \Sigma \to \Sigma$ is the canonical projection and the function $f$ can be expressed on $\mathbb{R} \times \Sigma$ as
\[
f(t, x) = at + b \quad \text{for every } t \in \mathbb{R} \text{ and } x \in \Sigma,
\]
for some $a > 0$ and $b \in \mathbb{R}$ such that $\Sigma = f^{-1}(\{b\})$.  

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ii) If $\lambda > 0$ then

$$0 \leq S_\varphi^2 \leq m\lambda. \quad (2.33)$$

If there exists $x_0 \in M$ such that $S_\varphi(x_0) = 0$ then $(M, g)$ is isometric to the euclidean space $\mathbb{R}^m$ and $\varphi$ is a constant map. Moreover, the potential $f$ can be expressed on $\mathbb{R}^m$ as $f(x) = \frac{\lambda}{2} |x|^2 + \langle b, x \rangle + c$ for some $b \in \mathbb{R}^m$ and $c \in \mathbb{R}$, for every $x \in \mathbb{R}^m$.

If $S_\varphi^2 = m\lambda$ either $S_\varphi > m\lambda$ or $M$ is compact and $f$ is constant.

We combine Theorem 1.1 and Proposition 4.1 of [YS] in a single statement.

**Theorem 2.34.** Let $(M, g)$ be a complete non-compact gradient shrinking harmonic-Ricci soliton. Then for every point $p \in M$ there exist positive constants $C$ and $c$ independent from $R$ and $x$, respectively, such that, for $R$ sufficiently large

$$\text{vol}(B_p(R)) \leq CR^m,$$

and, for every $x \in M$,

$$\frac{\lambda}{2}(r(x) - c)^2 \leq f(x) \leq \frac{\lambda}{2}(r(x) + c)^2, \quad (2.35)$$

where $B_p(R)$ is the geodesic ball of centre $p$ and radius $r$ and $r(x)$ is the geodesic distance from $x \in M$ to $p$.

**Remark 2.36.** F. Yang and J. Shen in [YS] deals with the normalized case where $\lambda = \frac{1}{2}$, but rescaling the metric clearly one can obtain the result for any $\lambda > 0$.

**Remark 2.37.** The estimate (2.35) shows that the potential function of a complete non-compact gradient shrinking harmonic-Ricci soliton is proper.

**Remark 2.38.** For a complete gradient shrinking harmonic-Ricci soliton of dimension $m \geq 2$ we have the following Hamilton-type identity

$$S_\varphi + |\nabla f|^2 - 2\lambda f$$

is constant on $M$, see for instance equation (7.1.7) of Proposition 7.1.5 in [A]. Hence, by adding a suitable constant to the potential function, we may assume

$$S_\varphi + |\nabla f|^2 = 2\lambda f. \quad (2.40)$$

Then, for every $x \in M$,

$$0 \leq S_\varphi(x) \leq \lambda^2(r(x) + c)^2, \quad (2.41)$$

indeed, combining (2.33), (2.40) and (2.35), one has the following chain of inequalities

$$0 \leq S_\varphi^2 \leq S_\varphi = 2\lambda f - |\nabla f|^2 \leq 2\lambda f \leq \lambda^2(r + c)^2.$$

Now, since from (2.41) and (2.35) both the potential function and the $\varphi$-scalar curvature have polynomial growth and the volume of $M$ is at most Euclidean, it is clear that for every $\mu, \gamma \in \mathbb{R}$

$$\int_M (S_\varphi)^\gamma e^{-\mu f} < +\infty, \quad \int_M |\nabla f|^\gamma e^{-\mu f} < +\infty. \quad (2.42)$$

The validity of (2.42) will be crucial in Section 6.

### 3 Fundamental calculations

We denote by $F_\varphi$ the three-times covariant tensor representing the obstruction to the commutation of the covariant derivative of $\text{Ric}_\varphi$, more precisely we give

**Definition 3.1.** Let $(M, g)$ be a Riemannian manifold of dimension $m \geq 2$, $\alpha$ a positive constant and $\varphi : M \to N$ a smooth map, where $(N, \langle \cdot, \cdot \rangle_N)$ is a target Riemannian manifold. The components of the three times covariant tensor field $F_\varphi$ are given, in a local orthonormal coframe, by

$$F_{ijk} = R^\alpha_{ij,k} - R^\alpha_{ik,j}. \quad (3.2)$$
Recalling the definitions of the $\varphi$-Schouten tensor (2.8) and of the $\varphi$-Cotton tensor (2.9), the following Proposition is easy to prove.

**Proposition 3.3.** Let $(M, g)$ be a Riemannian manifold of dimension $m \geq 2$, $\alpha$ a positive constant and $\varphi : M \to N$ a smooth map, where $(N, \langle , \rangle_N)$ is a target Riemannian manifold. Then

$$|F^\varphi|^2 = |C^\varphi|^2 + \frac{2\alpha}{m-1} \text{div}(T)(\nabla S^\varphi) + \frac{1}{2(m-1)} |\nabla S^\varphi|^2,$$

(3.4)

where $T$ is the energy-stress tensor of the smooth map $\varphi$.

**Proof.** By the definition (3.2) of $F^\varphi$

$$|F^\varphi|^2 = 2F^\varphi_{ijk} A^\varphi_{ijk}.$$  

(3.5)

Using (2.9) and (2.8) we easily get

$$F^\varphi_{ijk} = C^\varphi_{ijk} + \frac{1}{2(m-1)} (S^\varphi_k \delta_{ij} - S^\varphi_j \delta_{ik}),$$

and by plugging it into the above we get

$$|F^\varphi|^2 = 2R^\varphi_{ijk} C^\varphi_{ijk} + \frac{1}{m-1} (S^\varphi_k R^\varphi_{ii,k} - S^\varphi_j R^\varphi_{ij,i}).$$

Using once again the definition of $\varphi$-Schouten (2.8) combined with (2.17) from the above we infer

$$|F^\varphi|^2 = 2A^\varphi_{ijk} C^\varphi_{ijk} + \frac{1}{m-1} |\nabla S^\varphi|^2 - \frac{1}{m-1} S^\varphi_j \left( \frac{1}{2} S^\varphi_j - \alpha \varphi^a_i \varphi^a_j \right).$$

Now, using (2.13) and

$$|C^\varphi|^2 = 2C^\varphi_{ijk} A^\varphi_{ijk},$$

from the above we easily get

$$|F^\varphi|^2 = |C^\varphi|^2 + \frac{2\alpha}{m-1} S^\varphi_j \varphi^a_i \varphi^a_j + \frac{1}{2(m-1)} |\nabla S^\varphi|^2.$$ 

Then (3.4) follows, recalling the validity of (2.16).

**Remark 3.6.** From (3.3) we deduce

i) If $A^\varphi$ is a Codazzi tensor, i.e., $C^\varphi = 0$, we have that $\varphi$ is conservative (since the trace of $\varphi$-Cotton is a constant multiple of the divergence of $T$) and thus from (3.3)

$$|F^\varphi|^2 = \frac{1}{2(m-1)} |\nabla S^\varphi|^2.$$  

(3.7)

ii) If Ric$^\varphi$ is a Codazzi tensor, i.e., if $F^\varphi = 0$, we have from (3.4) that $S^\varphi$ is constant and $A^\varphi$ is Codazzi.

**Remark 3.8.** Assume that $\varphi$ is a constant map. We have that

$$F^\varphi = -\text{div}(...),$$

indeed the second Bianchi identity (2.1) implies

$$R_{ik,j,t} = R_{ij,k} - R_{ik,j}.$$  

Moreover, it is well known that

$$\text{div}(W) = \frac{m-3}{m-2} C,$$

see for instance (1.87) of [AMR], hence when $m \geq 4$

$$C^\varphi = -\frac{m-2}{m-3} \text{div}(W).$$

(3.10)

Relations (3.9) and (3.10) shows how $F^\varphi = 0$ and $C^\varphi = 0$, in case $\varphi : M \to N$ is a non-constant smooth map, generalize the notions of harmonic curvature and harmonic Weyl tensor, respectively.
**Definition 3.11.** Let \((M, g)\) be a Riemannian manifold of dimension \(m \geq 2\), \(\alpha\) a positive constant and \(\varphi : M \to N\) a smooth map, where \((N, \langle , \rangle_N)\) is a target Riemannian manifold. We say that \((M, g)\) has **parallel \(\varphi\)-Ricci tensor** if \(\nabla \text{Ric}^\varphi = 0\).

**Remark 3.12.** Assume \((M, g)\) has parallel \(\varphi\)-Ricci tensor, for some smooth map \(\varphi : M \to N\), where \((N, \langle , \rangle_N)\) is a Riemannian manifold, and some real constant \(\alpha \neq 0\). Then the \(\varphi\)-scalar curvature is constant and \(\varphi\) is conservative. Indeed, since the \(\varphi\)-scalar curvature is the trace of the \(\varphi\)-Riemann tensor, one has

\[
S^\varphi = R^\varphi_{ij,i} = 0,
\]

and since \(M\) is connected then \(S^\varphi\) is constant on \(M\). Then, using (2.17) and the constancy of the \(\varphi\)-scalar curvature,

\[
0 = R^\varphi_{ij,j} = \frac{1}{2} S^\varphi_i - \alpha \varphi^a_{jj} \varphi^a_i = - \alpha \varphi^a_{jj} \varphi^a_i,
\]

and thus \(\varphi\) is conservative.

**Remark 3.13.** Let \(\varphi : (M, g) \to (N, \langle , \rangle_N)\) be a conservative map between two Riemannian manifolds. Assume

\[
\tau(\varphi) = d\varphi(X)
\]

for some \(X \in \mathfrak{X}(M)\). Then, using (2.10), we get

\[
|\tau(\varphi)|^2 = \text{div}(T)(X) = 0,
\]

that is, \(\varphi\) is harmonic.

**Remark 3.14.** Assume that a harmonic-Ricci soliton of dimension \(m \geq 3\) has parallel \(\varphi\)-Ricci tensor. Then, as seen in [Remark 3.12] the \(\varphi\)-scalar curvature is constant and \(\varphi\) is conservative and thus, from [Remark 3.13] we have that \(\varphi\) is harmonic. Then the components of \(\varphi\)-Bach are given by

\[
(m-2)B^\varphi_{ij} = W_{tikj}^\varphi R^\varphi_{tk} - \alpha R^\varphi_{kj} \varphi^a_{ik} \varphi^a_i .
\]

(3.15)

Indeed, \(S^\varphi\) is constant \(\nabla A^\varphi = \nabla \text{Ric}^\varphi = 0\) and thus \(C^\varphi = 0\). Using also that \(\varphi\) is harmonic from (2.18) we immediately get (3.15). Furthermore, since \(S^\varphi\) is constant and \(\varphi\) is harmonic (recall that a harmonic map is also bi-harmonic) in (2.20), we also get

\[
J^a = -2 R^\varphi_{jk} \varphi^a_{jk} .
\]

(3.16)

In the next Proposition we collect a list of useful formulas for gradient harmonic-Ricci solitons. Some of them are not new (see for instance [A]), but we provide their proof for the reader convenience. This is not the case for (3.23) (or equivalently, (3.24)), the most difficult to obtain, in terms of computation.

**Proposition 3.17.** Let \((M, g)\) be a gradient harmonic-Ricci soliton with respect to \(\varphi : M \to N\) smooth, where \((N, \langle , \rangle_N)\) is a target Riemannian manifold, \(\alpha > 0\) and \(\lambda \in \mathbb{R}\). Then, the following formulas hold.

\[
F^\varphi_{ijk} = R^\varphi_{ijk} - R^\varphi_{ik,j} = R_{tikj} f_i ;
\]

(3.18)

\[
\frac{1}{2} S^\varphi = R^\varphi_{ij,i} ;
\]

(3.19)

\[
\text{div}(\text{Riem})_{tik} = R_{tik,j,i} = \alpha (\varphi_{ik} \varphi^a_j - \varphi^a_{ij} \varphi_k) ;
\]

(3.20)

\[
(R_{tikj} e^{-f})_{,i} = (\text{div}(\text{Riem})_{tik} - f_t R_{tikj}) e^{-f} = \alpha (\varphi_{ik} \varphi^a_j - \varphi^a_{ij} \varphi_k) e^{-f} ;
\]

(3.21)

\[
f_t R_{tikj} R^\varphi_{ij,k} = \frac{1}{2} |F^\varphi|^2 ;
\]

(3.22)

\[
\frac{1}{2} \Delta_f R^\varphi_{ij} = \lambda R^\varphi_{ij} + R_{tij,k} R^\varphi_{ik} + \frac{\alpha}{2} \varphi^a_{ik} (R^\varphi_{kj} \varphi^a_j + \varphi^a_{jk} R^\varphi_{ik}) - \alpha \varphi^a_{ik} \varphi^a_j ;
\]

(3.23)

\[
\frac{1}{2} \Delta_f R^\varphi_{ij} = - R_{tij,k} f_k - \frac{\alpha}{2} \varphi^a_{ik} (f_{kj} \varphi^a_j + \varphi^a_{jk} f_i) - \alpha \varphi^a_{ik} \varphi^a_j ;
\]

(3.24)

\[
\frac{1}{2} \Delta_f S^\varphi = \lambda S^\varphi - |\text{Ric}^\varphi|^2 - \alpha |\tau(\varphi)|^2 .
\]

(3.25)
Proof. In a local orthonormal coframe the harmonic-Ricci soliton equations are given by

\[
\begin{cases}
R^\varphi_{ij} + f_{ij} = \lambda \delta_{ij} \\
\varphi^a_{ii} = \varphi^a_i f_i
\end{cases}
\]  \hspace{1cm} (3.26)

Taking the covariant derivative of the first equation of (3.26) we get

\[
R^\varphi_{ij,k} + f_{ij,k} = 0.
\]

Skew-symmetrizing the above with respect to the indexes \(j\) and \(k\), using the commutation rule (2.2) and recalling the definition (3.2) of \(F\), we infer the validity of (3.18).

Summing on \(i = j\) the relation (3.18) we obtain

\[
S^\varphi_i - \frac{1}{2} S^\varphi_i + \alpha \varphi^a_i \varphi^a_k = R_{ik} f_t,
\]

that is, (3.19), using the second equation of (3.26) and recalling that, by definition, \(R^\varphi_{ij} = R_{ij} - \alpha \varphi^a_i \varphi^a_j\).

The second Bianchi identity (2.1) with \(l = t\) gives

\[
0 = R_{tikj,t} + R_{ti,jk} + R_{tij,k} = \text{div(Riem)}_{ikj} - R_{ij,k} + R_{ik,j},
\]

hence

\[
\text{div(Riem)}_{ikj} = R_{ij,k} - R_{ik,j}.
\]

Using the definition of \(\varphi\)-Ricci the above gives

\[
\text{div(Riem)}_{ikj} = R^\varphi_{ij,k} - R^\varphi_{ik,j} + \alpha (\varphi_{ik} \varphi^a_j - \varphi^a_i \varphi^a_k),
\]

and thus (3.20) follows with the aid of (3.18).

From (3.20) it is immediate to obtain (3.21).

Using (3.18) we get \(R_{tikj} f_t = F^\varphi_{ijk}\), hence recalling the validity of (3.5) we have

\[
f_t R_{tikj} R^\varphi_{ij,k} = F^\varphi_{ijk} R^\varphi_{ij,k} = \frac{1}{2} |F^\varphi|^2,
\]

that is, (3.22).

Now we prove the validity of (3.23). First of all, the following hold

\[
R^\varphi_{ij,k} f_k = R_{tikj} f_t f_k + \frac{1}{2} S^\varphi_{ij} + R^\varphi_{ik} R^\varphi_{kj} - \lambda R^\varphi_{ij}.
\]  \hspace{1cm} (3.27)

Indeed

\[
R^\varphi_{ij,k} f_k = (R^\varphi_{ij,k} - R^\varphi_{ik,j}) f_k + R^\varphi_{ik,j} f_k = (R^\varphi_{ij,k} - R^\varphi_{ik,j}) f_k + (R^\varphi_{ik} f_t)_{ik} - R^\varphi_{ik} f_t,
\]

and then (3.27) follows immediately using (3.18), (3.19) and the first equation of (3.26) as follows

\[
R^\varphi_{ij,k} f_k = (R^\varphi_{ij,k} - R^\varphi_{ik,j}) f_k + (R^\varphi_{ik} f_t)_{ik} - R^\varphi_{ik} f_t
\]

\[
= R_{tikj} f_t f_k + \left( \frac{1}{2} S^\varphi_{ij} \right)_{j} - R^\varphi_{ik} f_t - \lambda \delta_{kj}
\]

\[
= R_{tikj} f_t f_k + \frac{1}{2} S^\varphi_{ij} + R^\varphi_{ik} R^\varphi_{kj} - \lambda R^\varphi_{ij}.
\]

Next, we claim the validity of

\[
R^\varphi_{ij,kk} = R_{tik,jk} f_k f_t - 2 R_{tikj} R^\varphi_{ik} + \lambda R^\varphi_{ij} + \frac{1}{2} S^\varphi_{ij} + R^\varphi_{ik} R^\varphi_{kj} + \alpha \varphi^a_i \varphi^a_j R^\varphi_{ik} + \varphi^a_i \varphi^a_j R^\varphi_{ik} - 2 \alpha \varphi^a_{ik} \varphi^a_{kj}. \hspace{1cm} (3.28)
\]

To prove the claim notice that, using (3.18), the commutation rule (2.3) and Schur’s lemma (2.17),

\[
R^\varphi_{ij, kk} = (R^\varphi_{ij, kk} - R^\varphi_{ik, jk}) + R^\varphi_{ik, jk}
\]

\[
= (R_{tikj, f_t})_k + R^\varphi_{ik, jk} + R^\varphi_{ij, tk} + R^\varphi_{ik, jk} R^\varphi_{it}
\]

\[
= R_{tikj, f_t} + R_{tikj, f_t} + \left( \frac{1}{2} S^\varphi_{ij} - \alpha \varphi^a_{ik} \varphi^a_{jk} \right)_{j} + R_{tikj, f_t} + R_{tikj, f_t}.
\]
Then, using the definition of \( \varphi \)-Ricci and the symmetries of the Riemann tensor from the above we get

\[
R^\varphi_{ij,kk} = \text{div}(\text{Riem})_{jki} f_k + R_{tikj} f_{tk} + \frac{1}{2} S^\varphi_{ij} - \alpha \varphi^0_{ikj} \varphi_1^0 + \varphi^0_{kk} \varphi^0_{ij} + R_{tijk} R^\varphi_{tk} + \frac{1}{2} R^\varphi_{ikj} R^\varphi_{tk} + \alpha R^\varphi_{ikj} \varphi^0_1 \varphi^0_j.
\]

Now we plug (3.24) into the above obtaining

\[
R^\varphi_{ij,kk} = R_{tikj} f_{tk} + \alpha (\varphi_{jk} f_k + \varphi^0_k f_k) + R_{tikj} f_{tk} + \frac{1}{2} S^\varphi_{ij} + \alpha (R^\varphi_{ik} \varphi^0_j + \varphi^0_k f_{kj} \varphi^0_k) + R_{tijk} R^\varphi_{tk} + R^\varphi_{ikj} R^\varphi_{tk}.
\]

Notice that the from the second equation of (3.26), we have

\[
\varphi^0_{kkj} = (\varphi^0_{k} f_{kj}) = \varphi^0_{ikj} f_k + \varphi^0_{k} f_{kj}.
\]

Using the above and both the equations of (3.26) from (3.24), after a few simplifications, we infer

\[
R^\varphi_{ij,kk} = R_{tikj} f_{tk} - 2 \alpha \varphi^0_{ij} \varphi^0_k - R_{tikj} R^\varphi_{tk} + \lambda R_{tij} + \frac{1}{2} S^\varphi_{ij} + \alpha (R^\varphi_{ik} \varphi^0_j + \varphi^0_k f_{kj} \varphi^0_k) + R_{tijk} R^\varphi_{tk} + R^\varphi_{ikj} R^\varphi_{tk},
\]

so that, using once again the symmetries of the Riemann tensor, the first equation of (3.26) and the definition of \( \varphi \)-Ricci we obtain the claimed equality (3.28).

The validity of (3.23) follows immediately from the definition \( \Delta_f R^\varphi_{ij} = R^\varphi_{ij,kk} - R^\varphi_{ij,jk} f_k \), using (3.28) and (3.27).

To get (3.24) it is sufficient to use the first equation of (3.26) and the definition of \( \varphi \)-Ricci into (3.23).

Tracing (3.26) we get

\[
\frac{1}{2} \Delta_f S^\varphi = \lambda S^\varphi - R_{tk} R^\varphi_{tk} + \alpha \varphi^0_k R^\varphi_{ki} \varphi^0_t - \alpha |\tau(\varphi)|^2,
\]

that is, using the definition of the \( \varphi \)-Ricci tensor, (3.25). \( \square \)

## 4 Rigidity

Following what P. Petersen and W. Wylie did in [PW], we give the following

**Definition 4.1.** Let \((M, g)\) be a gradient harmonic-Ricci soliton with respect to a positive constant \(\alpha\), a smooth map \(\varphi : M \to N\), where \((N, (\cdot, \cdot)_N)\) is a Riemannian manifold, \(f \in C^\infty(M)\) and \(\lambda \in \mathbb{R}\). We say that \((M, g)\) is *rigid* if it is isometric to a quotient of the Riemannian product \(L \times \mathbb{R}^k\), where

i) \(0 \leq k \leq m\) is an integer and \(\mathbb{R}^k\) is endowed with the canonical metric \(g_{\text{can}}\),

ii) \((L, g_L)\) is a harmonic-Einstein manifold with respect to \(\alpha\), a smooth map \(\varphi_L : L \to N\) and \(\lambda \in \mathbb{R}\);

iii) Via the isometry \(\varphi\) is given the lifting of \(\varphi_L\) and \(f\) by the lifting of the map \(f_{R^k}(x) = \frac{1}{2} |x|^2 + \langle b, x \rangle + c\),

for some \(c \in \mathbb{R}\) and \(b \in \mathbb{R}^k\), i.e., via the isometry \(\varphi = \varphi_L \circ \pi_L\) and \(f = f_{R^k} \circ \pi_{R^k}\), where \(\pi_L : L \times \mathbb{R}^k \to L\) and \(\pi_{R^k} : L \times \mathbb{R}^k \to L\) are the canonical projections.

**Remark 4.2.** In the above:

i) When \(k = 0\) what we get is that \((M, g)\) is isometric to a quotient of the harmonic-Einstein manifold \((L, g_L)\) and, via the isometry, \(\varphi = \varphi_L \circ \pi_L\) and the potential is constant;

ii) When \(k = m\) what we get is that \((M, g)\) is isometric to a quotient of \((\mathbb{R}^m, g_{\text{can}})\) and, via the isometry, \(f = f_{R^k} \circ \pi_{R^k}\) and \(\varphi\) is constant.

Clearly a compact gradient harmonic-Ricci soliton is rigid if and only if it is harmonic-Einstein. Motivated by this we extend the notion of rigidity to any (not necessarily gradient) harmonic-Ricci soliton, in the following
Definition 4.3. Let \((M, g)\) be a compact harmonic-Ricci soliton with respect to a positive constant \(\alpha\), a smooth map \(\varphi : M \to N\), where \((N, \langle , \rangle_N)\) is a Riemannian manifold, \(X \in \mathfrak{X}(M)\) and \(\lambda \in \mathbb{R}\). We say that \((M, g)\) is rigid if it is harmonic-Einstein with respect to \(\alpha\), \(\varphi\) and \(\lambda\).

We will deal with the characterization of rigidity of compact harmonic-Ricci soliton in Section 5 and thus, for the rest of the Section, we will focus on complete non-compact gradient solitons.

The aim of the following Proposition is to justify Definition 4.1, showing that the Riemannian products \(L \times \mathbb{R}^k\) described in Definition 4.1 (for \(1 \leq k \leq m - 1\)) are actually harmonic-Ricci soliton. We will also study some of their geometric properties.

**Proposition 4.4.** Let \((M, g)\) a harmonic-Einstein manifold of dimension \(m \geq 2\) with respect to a positive constant \(\alpha\), a smooth map \(\varphi : M \to N\), where \((N, \langle , \rangle_N)\) is a target Riemannian manifold, and \(\lambda \in \mathbb{R}\). Let \(k \geq 1\) be an integer and consider on the Euclidean space \(\mathbb{R}^k\), endowed with its canonical metric \(g_{\text{can}}\), the function

\[
f(x) := \frac{\lambda}{2}|x|^2 + \langle b, x \rangle + c \quad \text{for every } x \in \mathbb{R}^k,
\]

where \(b \in \mathbb{R}^k\) and \(c \in \mathbb{R}\). Consider the Riemannian product \(\bar{M} := M \times \mathbb{R}^k\), with Riemannian metric

\[
\bar{g} = \pi_M^* g + \pi_{\mathbb{R}^k}^* g_{\text{can}} \equiv g + g_{\text{can}},
\]

where \(\pi_M : \bar{M} \to M\) and \(\pi_{\mathbb{R}^k} : \bar{M} \to \mathbb{R}^k\) are the canonical projections. Denote by \(\bar{f}\) and \(\bar{\varphi}\) the lifting of \(f\) and \(\varphi\), respectively, to \(\bar{M}\):

\[
\bar{f} := f \circ \pi_{\mathbb{R}^k} \in C^\infty(\bar{M}), \quad \bar{\varphi} := \varphi \circ \pi_M : \bar{M} \to N.
\]

Then \((\bar{M}, \bar{g})\) is a gradient harmonic-Ricci soliton of dimension \(\bar{m} = m + k \geq 3\) with respect to \(\alpha\), \(\bar{\varphi}\), \(\bar{f}\) and \(\lambda\), that is,

\[
\begin{cases}
\bar{Ric} - \alpha \bar{\varphi}^* \langle , \rangle_N + \bar{Hess}(\bar{f}) = \lambda \bar{g} \\
\bar{\tau}(\bar{\varphi}) = d\bar{\varphi}(\nabla \bar{f})
\end{cases}
\]

(4.6)

holds. Moreover

\[
\nabla^\bar{M} \bar{Ric} = 0,
\]

(4.7)

and, as a consequence, the \(\bar{\varphi}\)-scalar curvature is constant and \(\bar{\varphi}\) is harmonic. Furthermore,

\[
\bar{B}^\varphi = \frac{(k-1)\lambda^2}{(m+k-1)(m+k-2)^2}(k\pi_M^* g - m\pi_{\mathbb{R}^k}^* g_{\text{can}})
\]

(4.8)

and

\[
\bar{J} = 0.
\]

(4.9)

**Proof.** We will use the following indexes conventions

\[
1 \leq i, j, \ldots \leq m, \quad 1 \leq \alpha, \beta, \ldots \leq k, \quad 1 \leq A, B, \ldots \leq m + k.
\]

Let \(\{\theta^i\}\) be a local orthonormal coframe for \((M, g)\) on an open subset \(U\) of \(M\) and \(\{\psi^\alpha\}\) for \((\mathbb{R}^k, g_{\text{can}})\) on an open subset \(W\) of \(\mathbb{R}^k\). It is easy to see that, by setting

\[
\bar{\theta}^i := \pi_M^* \theta^i, \quad \bar{\theta}^{m+\alpha} := \pi_{\mathbb{R}^k}^* \psi^\alpha,
\]

then \(\{\bar{\theta}^A\}\) is a local orthonormal coframe for \(\bar{g}\) in \(\bar{U} := U \times W\). The same applies for the Levi-Civita connections forms, indeed let \(\{\theta^i_j\}\) and \(\{\psi^\alpha_\beta\}\) be, respectively, the Levi-Civita connection forms for \((M, g)\) on \(U\) and for \((\mathbb{R}^k, g_{\text{can}})\) on \(W\). Then the Levi-Civita connection forms for \((\bar{M}, \bar{g})\) on \(\bar{U}\) are given by

\[
\bar{\theta}^i_j = \pi_M^* \theta^i_j, \quad \bar{\theta}^{m+\alpha}_j = 0, \quad \bar{\theta}^{m+\alpha}_j = \pi_{\mathbb{R}^k}^* \psi^\alpha_\beta.
\]

From now on we will omit the pullback from our notation. We collect some well known fact needed in the rest of proof (for a proof see, for instance, Section 4 of [A20] where we deal with semi-Riemannian warped products), that relies on the fact that \(\mathbb{R}^k\) is flat and on the definitions (4.3) of \(\bar{f}\) and \(\bar{\varphi}\).
• The non-vanishing components of the Riemann tensor $\overline{\text{Riem}}$ of $(\tilde{M}, \tilde{g})$ in the coframe $\{\tilde{\theta}^A\}$ are determined by

$$\tilde{R}_{ijkt} = R_{ijkt}, \quad (4.10)$$

where $R_{ijkt}$ are the components of the Riemann tensors of $(M, g)$.

• The non-vanishing components of $\overline{\text{Ric}}$, the Ricci tensor of $(\tilde{M}, \tilde{g})$, in the coframe $\{\tilde{\theta}^A\}$ are given by

$$\tilde{R}_{ij} = R_{ij}, \quad (4.11)$$

• The scalar curvature $\tilde{S}$ of $(\tilde{M}, \tilde{g})$ is given by

$$\tilde{S} = S, \quad (4.12)$$

where $S$ is the scalar curvature of $(M, g)$.

• The components of $\nabla \tilde{f}$ and of $\text{Hess}(\tilde{f})$ in the coframe $\{\tilde{\theta}^A\}$ are given by

$$\tilde{f}_i = 0, \quad \tilde{f}_m = f_a, \quad \tilde{f}_{ij} = 0, \quad \tilde{f}_{m+\alpha} = f_a, \quad \tilde{f}_{m+\alpha+\beta} = f_a, \quad (4.13)$$

• The components of $\tilde{d} \tilde{\varphi}$ and of $\nabla \tilde{d} \tilde{\varphi}$ are given by

$$\tilde{\varphi}_a^a = \tilde{\varphi}_s^s, \quad \tilde{\varphi}_s^a = 0, \quad \tilde{\varphi}_i^a = \tilde{\varphi}_j^a, \quad \tilde{\varphi}_{i+\alpha}^a = 0, \quad \tilde{\varphi}_{i+\alpha+\beta}^a = 0. \quad (4.14)$$

The equations (4.16) in the coframe $\{\tilde{\theta}^A\}$ are given by

$$\begin{cases}
\tilde{R}_{AB} - \alpha \tilde{\varphi}_A^a \tilde{\varphi}_B^a + \tilde{f}_{AB} = \lambda \delta_{AB} \\
\tilde{\varphi}_A^a = \tilde{\varphi}_A^a \tilde{f}_A.
\end{cases} \quad (4.15)$$

Using the decomposition (4.11) of $\overline{\text{Ric}}$, (4.14) and (4.13) the first equation above is equivalent to

$$\begin{cases}
\tilde{R}_{ij} - \alpha \tilde{\varphi}_i^a \tilde{\varphi}_j^a = \lambda \delta_{ij} \\
\tilde{f}_{ij} = \lambda \delta_{ij}
\end{cases}$$

while the second is equivalent to

$$\tilde{\varphi}_a^a = 0,$$

and from our assumption we are able to conclude the validity of (4.15).

From now on we denote, as usual, by $\tilde{R}_{AB}^c$, the components of $\overline{\text{Ric}} = \overline{\text{Ric}} - \alpha \tilde{\varphi}^c(,)_{N}$ and by $\tilde{R}_{ij}^c$ the components of $\overline{\text{Ric}}^c = \overline{\text{Ric}} - \alpha \tilde{\varphi}^c(,)_{N}$. Combining (4.11) and (4.14), recalling that $(M, g)$ is harmonic-Einstein, the only non-trivial components of $\overline{\text{Ric}}^c$ are given by

$$\tilde{R}_{ij}^c = \tilde{R}_{ij}^c = \lambda \delta_{ij}. \quad (4.16)$$

Using only that $\overline{\text{Ric}}$ is flat we easily get

$$\nabla \overline{\text{Ric}}^c = \pi_M \nabla \overline{\text{Ric}}^c. \quad (4.17)$$

Indeed, by definition of covariant derivative

$$\tilde{R}_{AB,C}^c \tilde{\theta}^C = \tilde{d} \tilde{R}_{AB}^c - \tilde{R}_{CB}^c \tilde{\theta}_A^C - \tilde{R}_{AC}^c \tilde{\theta}_B^C,$$

that is

$$\tilde{R}_{AB,k}^c \theta^k + \tilde{R}_{AB}^c \theta_{m+\gamma}^\gamma = \tilde{d} \tilde{R}_{AB}^c - \tilde{R}_{KB}^c \theta^k_A - \tilde{R}_{m+\gamma}^c \theta_{A}^\gamma - \tilde{R}_{k}^c \theta_B^k - \tilde{R}_{A}^c \theta_{m+\gamma}^\gamma,$$

where we used the obvious notations for the Levi-Civita connection forms. Hence

$$\tilde{R}_{ij,k}^c \theta^k + \tilde{R}_{ij,m+\gamma}^c \theta^\gamma = \tilde{d} \tilde{R}_{ij}^c - \tilde{R}_{ij}^c \theta^k_i - \tilde{R}_{ik}^c \theta^k_j = \tilde{R}_{ij,k}^c \theta^k,$$
\[ R^\phi_{1m+\beta,k} \theta^k + R^\phi_{1m+\beta,m+\gamma} \psi^\gamma = \bar{d}R^\phi_{1m+\beta} - \bar{R}^\phi_{k m+\beta} \theta^k - R^\phi_{1m+\gamma} \psi^\gamma \]

and
\[ \bar{R}^\phi_{m+\alpha m+\beta,k} \theta^k + \bar{R}^\phi_{m+\alpha m+\beta,m+\gamma} \psi^\gamma = \bar{d}\bar{R}^\phi_{m+\alpha m+\beta} - \bar{R}^\phi_{m+\gamma m+\beta} \psi^\gamma - \bar{R}^\phi_{m+\alpha m+\gamma} \psi^\gamma = 0, \]

and thus \([4.17]\) holds.

Now, using also that \((M, g)\) is harmonic-Einstein the above yields \(\nabla\text{Ric}^\varphi = 0\) and thus \([4.17]\) holds.

As pointed out in Remark 3.14 we know that \(\bar{\varphi}\) is constant and \(\varphi\) is harmonic. Furthermore \([3.15]\) and \([3.16]\) read, respectively,
\[ (m + k - 2)\bar{B}_{AB}^\varphi = \bar{W}_{CADB}^\varphi \bar{R}_{CD} - \bar{\alpha}_A^\varphi \bar{\varphi}_C \bar{R}_{CB}^\varphi \]

and
\[ \bar{J}^a = -2\bar{R}_{AB}^\varphi \bar{\varphi}_{AB}. \]

Using \([4.14]\) and the fact that \((M, g)\) is harmonic-Einstein we infer
\[ \bar{J}^a = -2\bar{R}_{ij}^\varphi \bar{\varphi}_{ij} = -2\lambda \tau(\varphi)^a = 0, \]

hence \([4.9]\) holds.

Using \([4.14]\) and \([4.16]\) the components of the \(\bar{\varphi}\)-Bach tensor are given by
\[ (m + k - 2)\bar{B}_{AB}^\varphi = \lambda(\bar{W}_{kAB}^\varphi - \bar{\alpha}_k^\varphi \bar{\varphi}_A \delta_{AB} \delta_B) \tag{4.18} \]

To compute them we need the components of the \(\bar{\varphi}\)-Weyl tensor and to compute them we need the components of the \(\bar{\varphi}\)-Schouten tensor.

By definition the \(\bar{\varphi}\)-Schouten tensor has components
\[ \bar{A}_{AB}^\varphi = \bar{R}_{AB}^\varphi - \frac{\bar{S}^\varphi}{2(m + k - 1)} \delta_{AB}. \tag{4.19} \]

Since \(\mathbb{R}^k\) is flat, \(\bar{\varphi} = \varphi \circ \pi_M\) and \((M, g)\) is harmonic-Einstein, we have \(\bar{S}^\varphi = S^\varphi = m \lambda\) and thus the above can be written as
\[ \bar{A}_{AB}^\varphi = \bar{R}_{AB}^\varphi - \frac{m \lambda}{2(m + k - 1)} \delta_{AB}. \]

From \([4.19]\), using that \(R_{ij}^\varphi = \lambda \delta_{ij}\), the only non-trivial components of the \(\bar{\varphi}\)-Schouten tensor are given by
\[ \bar{A}_{ij}^\varphi = \frac{m - 2 + 2k}{2(m + k - 1)} \delta_{ij} \tag{4.20} \]

and
\[ \bar{A}_{m+\alpha m+\beta}^\varphi = -\frac{m}{2(m + k - 1)} \lambda \delta_{\alpha \beta} \tag{4.21} \]

By definition, see \([2.11]\), the components of the \(\bar{\varphi}\)-Weyl tensor are given by
\[ \bar{W}_{ABCD}^\varphi = \bar{R}_{ABCD} - \frac{1}{m + k - 2}(\bar{A}_{AC}^\varphi \delta_{BD} - \bar{A}_{AD}^\varphi \delta_{BC} + \bar{A}_{BD}^\varphi \delta_{AC} - \bar{A}_{BC}^\varphi \delta_{AD}). \]

Using \([4.10]\), \([4.20]\) and \([4.21]\) from the above we get
\[ \bar{W}_{\alpha \beta i m+\alpha k}^\varphi = \frac{k - 1}{(m + k - 1)(m + k - 2)} \lambda \delta_{\alpha \beta} \delta_{ik} \tag{4.24} \]

From \([4.18]\) with \(A = i\) and \(B = j\) we have
\[ (m + k - 2)\bar{B}_{ij}^\varphi = \lambda(\bar{W}_{kikj}^\varphi - \alpha_i^\varphi \varphi_j^\varphi) \tag{4.25} \]
Tracing (4.24) we infer
\[ W^\varphi_{tik} = R_{ik} = -\frac{(m - 2 + 2k)(m - 1)}{(m + k - 1)(m + k - 2)} \lambda \delta_{ik}, \]
hence, by definition of \( \varphi \)-Ricci tensor and since \((M, g)\) is harmonic-Einstein,
\[ W^\varphi_{kik} = \alpha \varphi \tau \xi_j = \frac{k(k - 1)}{(m + k - 1)(m + k - 2)} \lambda \delta_{ij}. \]
Plugging the above into (4.25) we infer
\[ \text{Theorem 4.31.} \]
Let \( f \) for further comments on the assumptions. We take care of the steady case in Proposition 4.38 below.

Indeed, in a local orthonormal coframe
\[ d\varphi = 0 \text{ in a local orthonormal coframe}. \]
Then we easily get
\[ \text{parallel, that is,} \]

Eventually after passing to the universal cover \((\tilde{M}, \tilde{g})\) of \((M, g)\) we can assume that \((M, g)\) is simply connected. Indeed, since the projection \( p : (M, \tilde{g}) \to (M, g) \) is a local isometry, \((M, \tilde{g})\) is itself a Ricci-harmonic soliton with respect to \( \alpha, \lambda \) and the lifting of \( f \) and \( \varphi \) to \( \tilde{M} \).

Since the \( \varphi \)-Ricci tensor is parallel, with the aid of Remark 3.14 we know that \( \varphi \) is harmonic, hence \( d\varphi(\nabla f) = 0 = \tau(\varphi) \), and the \( \varphi \)-scalar curvature is constant.

Recall that \( f \) satisfies the Hamilton-type identity (2.39). Hence, since \( S^\varphi \) is constant,
\[ \text{Hess}(|\nabla f|^2) = 2\lambda \text{Hess}(f). \]
Notice that since \( \text{Ric}^\varphi \) is parallel using the first equation of (4.22) we infer that also \( \text{Hess}(f) \) is parallel, that is, \( f_{ijk} = 0 \) in a local orthonormal coframe. Then we easily get
\[ \text{Hess}(|\nabla f|^2) = 2\lambda \text{Hess}(f)^2. \]
Indeed, in a local orthonormal coframe
\[ |\nabla f|^2 = 2(f_{ijk} f_k + f_{ik} f_j). \]
Then (4.33) reads
\[ \lambda \text{Hess}(f) = \text{Hess}(f)^2, \]

From the above equation the only possible eigenvalues of Hess$(f)$ are 0 and $\lambda$. Using the first equation of (4.32) the same holds also for Ric$^\phi$.

If $f$ is constant then $(M,g)$ is harmonic-Einstein, hence rigid. From now on we assume that $f$ is non-constant on $M$.

Since $S^2$ is constant and $f$ is non-constant, from (6.19) we see that the $\varphi$-Ricci tensor has always 0 as eigenvalue and thus the Hessian of $f$ has always $\lambda$ as eigenvalue, at least at a point $p \in M$.

We denote by $V_0$ and $V_\lambda$ the eigenspace bundles of Hess$(f)$. Since $M$ is connected and the eigenvalues of Hess$(f)$ are smooth we know that $TM$ splits orthogonally as $V_0 \oplus V_\lambda$. Furthermore, since we are assuming $(M,g)$ simply connected, $V_0$ and $V_\lambda$ are parallel and integrable distributions and $(M,g)$ splits as the Riemannian product of two complete totally geodesic submanifolds $(L, g_L)$ and $(E, g_E)$ such that $TE = V_\lambda|_E$ and $TL = V_0|_L$. Moreover, via the splitting,
\[ \text{Hess}(f) = \lambda \pi_E^* g_E \] (4.34)

and
\[ \text{Ric}^\phi = \lambda \pi_L^* g_L. \] (4.35)

This is essentially the de Rham splitting theorem [11], see for instance Section 3.2 of [1] (or also 2.3 at page 55 of [DMVVZ]). We denote by $k$ the dimension of $E$, we know that $k \geq 1$.

The validity of (4.34) implies that, via the splitting, $f = f_E \circ \pi_E$ for some $f_E \in C^\infty(E)$ such that
\[ E \text{Hess}(f_E) = \lambda g_E, \]
on the complete Riemannian manifold $(E, g_E)$. Then, via the classic theorem of Y. Tashiro [11] (see also Theorem 8.5 of [AMR]), since $\lambda \neq 0$ we deduce that $(E, g_E)$ is isometric to $\mathbb{R}^k$ with the Euclidean metric and, via the isometry, $f_E(x) = \frac{1}{2} |x|^2 + \langle b, x \rangle + c$ for some $b \in \mathbb{R}^k$ and $c \in \mathbb{R}$. Furthermore, (4.35) gives that $\pi_E^* \text{Ric}^\phi = \pi_E^* \text{Ric} - \alpha \pi_E^* \varphi^\phi(\cdot, \cdot)_N$ vanishes on $E$. Via the splitting $\pi_E^* \text{Ric} = \pi_L^* \text{Ric} = 0$, since $E$ is flat, hence $\pi_E^* \varphi^\phi(\cdot, \cdot)_N = (\varphi \circ \pi_L^*)^\phi(\cdot, \cdot)_N$ vanishes on $E$. Then $\varphi \circ \pi_E^* \circ \pi_L^* \circ \pi_L^* \circ \pi_L^* = \text{constant}$. Since the argument above applies for every leaf of the foliation $E \times L$ of $M$ we deduce that $\varphi = \varphi_L \circ \pi_L$, for some smooth map $\varphi_L : L \to N$. Since $\varphi$ is harmonic and $\varphi = \varphi_L \circ \pi_L$ we immediately get, via (4.14), that $\varphi_L$ is harmonic too. Now, using (4.35) we get
\[ L \text{Ric} - \alpha \varphi_L^\phi(\cdot, \cdot)_N = 0 \]
and since $\varphi_L$ is harmonic we conclude that $(L, g_L)$ is harmonic-Einstein with respect to $\varphi_L$, $\alpha$ and $\lambda$.

**Remark 4.36.** By the proof of the Theorem above we know that the universal cover $(\tilde{M}, \tilde{g})$ of $(M,g)$ is isometric to the Riemannian product of $L \times \mathbb{R}^k$ described in Proposition 4.3 for some $0 \leq k \leq m$.

When the soliton is shrinking, i.e., when $\lambda > 0$, we know that $L$ is compact. Indeed, using that $\alpha > 0$
\[ L \text{Ric} \geq L \text{Ric} - \alpha \varphi_L^\phi(\cdot, \cdot)_N = \lambda g_L, \]
hence we can apply Myers’s theorem. Furthermore, since
\[ \text{Ric}_f \geq \text{Ric} - \alpha \varphi^\phi(\cdot, \cdot)_N + \text{Hess}(f) = \lambda g, \]
using the weighted version of the Myers’s theorem (see Theorem 1 of [N]) we obtain that the fundamental group of $M$ is finite.

Then, for shrinking solitons, the theorem above can be strengthened, obtaining that $(M,g)$ is isometric to a finite quotient of $L \times \mathbb{R}^k$ for some $1 \leq k \leq m$ ($k = 0$ is excluded because $M$ is non-compact), where $(L, g_L)$ is a compact harmonic-Einstein manifold with respect to $\alpha$, $\varphi$ and $\lambda$ and, via the isometry, $\varphi = \varphi_L \circ \pi_L$ and $f = f_L \circ \pi_L$, where $f_L = \frac{1}{2} |x|^2 + \langle b, x \rangle + c$ for some $c \in \mathbb{R}$ and $b \in \mathbb{R}^k$ and $\pi_L : L \times \mathbb{R}^k \to L$ and where $\pi_{L^k} : L \times \mathbb{R}^k \to \mathbb{R}^k$ are the canonical projections.

**Remark 4.37.** In Theorem 1.2 of [PW] the authors characterized rigidity for complete non-compact non-steady Ricci soliton via the constancy of the scalar curvature and the radial flatness, that is,
\[ R(\cdot, \nabla f)\nabla f = 0. \]
We could have done the analogous here, i.e., characterize rigidity for complete non-compact non-steady harmonic-Ricci soliton via the constancy of the $\varphi$-scalar curvature and the radial flatness, but we preferred the assumption of parallel $\varphi$-Ricci tensor. The reason for this choice are two. The first is that, as seen in Proposition 4.4, rigid harmonic-Ricci soliton have actually parallel $\varphi$-Ricci tensor. The second is that the proof with this assumption is simpler, more similar to the one of Theorem 2 of [N] than the one of Theorem 1.2 of [PW], and for our purpose this characterization will be sufficient.

Notice that for a gradient harmonic-Ricci soliton having parallel $\varphi$-Ricci tensor implies the constancy of the $\varphi$-scalar curvature, as seen in Remark 3.14 and the radial flatness, since \(\text{Proposition 4.18}\) gives $f_t R_{tijk} = 0$, that is, $R(\cdot, \cdot) \nabla f = 0$.

To conclude this Section we deal with complete gradient steady harmonic-Ricci soliton.

**Proposition 4.38.** Let $(M, g)$ be a complete gradient steady harmonic-Ricci soliton of dimension $m \geq 2$ with respect to a positive constant $\alpha$, a smooth map $\varphi : M \rightarrow N$, where $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold, and $f \in C^\infty(M)$. If the $\varphi$-scalar curvature is constant and $f$ is non-constant then $(M, g)$ is isometric to the the Riemannian product of $\mathbb{R}$ with a totally geodesic $\psi$-Ricci flat (with respect to $\alpha$) hypersurface $\Sigma$, where $\psi := \varphi|_\Sigma$. Moreover $\varphi = \psi \circ \pi_\Sigma$ on $\mathbb{R} \times \Sigma$, where $\pi_\Sigma : \mathbb{R} \times \Sigma \rightarrow \Sigma$ is the canonical projection and the function $f$ can be expressed on $\mathbb{R} \times \Sigma$ as

$$f(t, x) = bt + c \quad \text{for every } t \in \mathbb{R} \text{ and } x \in \Sigma,$$

for some $b > 0$ and $c \in \mathbb{R}$ such that $\Sigma = f^{-1}(\{b\})$.

In particular, complete gradient steady harmonic-Ricci solitons with constant $\varphi$-scalar curvature are rigid.

**Proof.** It is an easy application of i) of [Theorem 2.31]. Indeed, since $\lambda = 0$ we have that $S^\varphi = 0$. Since we assumed the constancy of the $\varphi$-scalar curvature we have $S^\varphi = S^\varphi = 0$ on $M$, hence the thesis. \(\square\)

**Remark 4.39.** Proposition 4.38 is apparently in contrast with Proposition 4.4 where we showed that the Riemannian product $M$ of any complete $\varphi$-Ricci flat manifold with $\mathbb{R}^k$, for any non-negative integer $k$ (and not only for $k = 1$), produces a complete gradient steady harmonic-Ricci soliton with constant $\varphi$-scalar curvature.

To overcome this contrast notice that, when $\lambda = 0$, we can always reduce to the cases where $k = 0, 1$. Indeed, when $k \geq 2$, the affine function $f$ is given by

$$f : \mathbb{R}^k \rightarrow \mathbb{R}, \quad x \mapsto f(x) = \langle x, b \rangle + c,$$

for some $b \in \mathbb{R}^k$ and $c \in \mathbb{R}$. If $b = 0$ then $f$ is constant, hence we can reduce to the case $k = 0$ by replacing $M$ with $M \times \mathbb{R}^k$, that is $\varphi$-Ricci flat with respect to the trivial lifting of $\varphi$. Assume now that $b \neq 0$. Then we may choose a orthonormal basis $\{v_1, \ldots, v_k\}$ of $\mathbb{R}^k$ such that

$$v_k = \frac{b}{|b|}.$$

Then, letting $x = x^1 v_1 + \ldots + x^k v_k$, we see that $f(x) = |b| x^k + c$. Hence, via the following isometry

$$\psi : \mathbb{R}^k \rightarrow \mathbb{R}^{k-1} \times \mathbb{R}, \quad x \mapsto (x - x^k v_k, |b| x_k + c),$$

the map $f$ takes the form

$$f(y, r) = |b| r + c.$$

Then $\tilde{M} = M \times \mathbb{R}^k$ can be seen as the Riemannian product between the $\varphi$-Ricci flat manifold $\tilde{M} := M \times \mathbb{R}^{k-1}$ (with respect to the trivial lifting of $\varphi$ to $M \times \mathbb{R}^k$) and $\mathbb{R}$ while $\tilde{f}$ can be seen as the lifting of $f(x) = |b| x + c$, for $x \in \mathbb{R}$, to $\tilde{M} = M \times \mathbb{R}$.

## 5 Compact harmonic-Ricci solitons

In this section we consider compact harmonic-Ricci solitons of dimension $m \geq 2$

$$\begin{cases}
\text{Ric}^\varphi + \frac{1}{2} \mathcal{L} \varphi g = \lambda g \\
\tau(\varphi) = d \varphi(X)
\end{cases} \quad (5.1)$$
for $X \in \mathfrak{X}(M), \lambda \in \mathbb{R}$ and $\alpha > 0$. In Section 5.1 we show that if (5.1) is not rigid then it is gradient and shrinking. Using this important information, in Section 5.2 we prove that the soliton (5.1) is rigid if and only if it is $\varphi$-Cotton flat (see Theorem 5.20 below).

5.1 Reduction to compact gradient shrinking harmonic-Ricci solitons

Remark 5.2. Using Theorem 2.28 it is easy to see that the harmonic-Ricci soliton is rigid (i.e., is harmonic-Einstein) if and only if $S^\varphi$ is constant.

Proposition 5.3. If the compact harmonic-Ricci soliton (5.1) is not rigid then it is shrinking.

Proof. Taking the trace of the first equation of (5.1) we get
$$S^\varphi + \text{div}(X) = m\lambda,$$

hence, using the divergence theorem
$$\int_M S^\varphi = m\lambda \text{vol}(M).$$

By setting
$$\bar{S}^\varphi := \frac{1}{\text{vol}(M)} \int_M S^\varphi, \quad S^\varphi_* := \min_M S^\varphi,$$

assuming that the soliton is not rigid, from Remark 5.2 cannot be constant and thus
$$S^\varphi_* < \bar{S}^\varphi = m\lambda.$$ \hspace{1cm} (5.4)

On the other hand equation (2.30) gives
$$\frac{1}{2} \Delta_X S^\varphi + |\text{Ric}^\varphi|^2 + \alpha |\tau(\varphi)|^2 + \frac{S^\varphi}{m} (S^\varphi - m\lambda) = 0,$$

hence
$$\frac{1}{2} \Delta_X S^\varphi + \frac{S^\varphi}{m} (S^\varphi - \bar{S}^\varphi) \leq 0.$$

Let $x_* \in M$ such that $S^\varphi(x_*) = S^\varphi_*$. Using that $\Delta_X S^\varphi(x_*) \leq 0$, evaluating the above at $x_*$ we get
$$S^\varphi_* (S^\varphi_* - \bar{S}^\varphi) \leq 0.$$

Since the harmonic-Ricci soliton is not rigid we have $S^\varphi_* - \bar{S}^\varphi < 0$, then from the above we infer $S^\varphi_* \geq 0$ and combining it with (5.4) we obtain
$$0 \leq S^\varphi_* < m\lambda,$$

and thus $\lambda > 0$. \hfill \Box

Combining Theorem 1.1 of [YZ] with Proposition 5.3 we get the following

Theorem 5.5. If a compact harmonic-Ricci soliton of dimension $m \geq 2$ is not rigid then it is gradient and shrinking.

In the rest of this Section, for completeness and for the reader convenience, we provide some details regarding the proof of Theorem 1.1 of [YZ].

Remark 5.6. The key point of Theorem 1.1 of [YZ] is to guarantee the existence of a smooth function $f$ on a compact shrinking harmonic-Ricci soliton $(M, g)$ such that
$$S^\varphi + 2\Delta f - |\nabla f|^2 + 2\lambda f \text{ is constant on } M.$$ \hspace{1cm} (5.7)

Notice that the existence of such function $f$ had been established by R. Müller in Section 7.3 of [M], where he proved the existence on any compact Riemannian manifold $(M, \tilde{g})$ of a unique $v \in C^\infty(M)$ positive normalized eigenvector of
$$L(v) = -4\tilde{\Delta} v + \left( \tilde{S}^\varphi - \frac{m}{2} \log(4\pi) - m \right) v - 2v \log v,$$

and thus $\lambda > 0$. \hfill \Box
where we denoted by $\tilde{\Delta}$ and $\tilde{S}_\varphi$, respectively, the Laplacian and the $\varphi$-scalar curvature evaluated with respect to the metric $\tilde{g}$. By setting $v = e^{-\varphi}$ the above yields the constancy on $M$ of

$$\tilde{S}_\varphi + 2\tilde{\Delta}f - |\tilde{\nabla}f|^2_{\tilde{g}} + f.$$ 

By setting $\tilde{g} := \beta^2 g$, for any positive constant $\beta$, the above gives the constancy of

$$S_\varphi + 2\Delta f - |\nabla f|^2 + \beta^2 f.$$ 

To obtain (5.7) it is sufficient to choose $\beta^2 = 2\lambda$, and this can be done since the harmonic-Ricci soliton is shrinking.

**Definition 5.8.** We call the unique (up to an additive constant) function $f$ such that (5.7) holds the *Müller-Perelman potential*.

**Remark 5.9.** Once we have the existence of the Müller-Perelman potential on a compact shrinking harmonic-Ricci soliton $(M, g)$ we can prove Theorem 1.1 of [YZ] relying on the validity of

$$\int_M \left( |h_1|^2 - \langle h_1, b_1 \rangle + \alpha|h_2|^2 - \alpha \langle b_2, b_2 \rangle \right) e^{-f} = \frac{1}{2} \int_M (S_\varphi + 2\Delta f - |\nabla f|^2 + 2\lambda f) \text{div}(e^{-f} Y),$$

where

$$Y := \nabla h - X,$$ (5.11) 

$$h_1 := \text{Ric}_\varphi + \text{Hess}(f) - \lambda g, \quad h_2 := \tau(\varphi) - d\varphi(\nabla f)$$ (5.12) 

and

$$b_1 := \text{Ric}_\varphi + \frac{1}{2} \mathcal{L}_X g - \lambda g, \quad h_2 := \tau(\varphi) - d\varphi(X),$$ (5.13)

whose proof (that is computational) is postponed to Proposition 5.15 below. Indeed, let $(M, g)$ be a compact shrinking harmonic-Ricci soliton with potential $X$ and let $f$ be the Müller-Perelman potential. Then $b_1 = 0 = b_2$ and thus (5.10) gives, using (5.7),

$$\int_M (|h_1|^2 + \alpha|h_2|^2) e^{-f} = 0.$$ 

Hence, since $\alpha > 0$ we have $h_1 = 0 = h_2$, that is,

$$\begin{cases} \text{Ric}_\varphi + \text{Hess}(f) = \lambda g \\ \tau(\varphi) = d\varphi(\nabla f). \end{cases}$$

The above shows the compact shrinking harmonic-Ricci soliton is gradient.

**Remark 5.14.** Recall that on any compact Riemannian manifold $(M, g)$ every vector field $X \in \mathfrak{X}(M)$ can be decomposed as

$$X = \nabla h + Y,$$

where $Y$ is a divergence free vector field and the smooth function $h \in C^\infty(M)$, defined up to an additive constant, is called Hodge-de Rham potential of $X$, see for instance [ABR].

As pointed out in Proposition 2.1 of [YZ], the Müller-Perelman potential coincide with the Hodge-de Rham potential, since $f - h$ is a harmonic function.

**Proposition 5.15.** For every compact Riemannian manifold $(M, g)$, $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, by setting $Y$ as in (5.11), $h_1$ and $h_2$ as in (5.12) and $b_1$ and $b_2$ as in (5.13), formula (5.10) holds.

**Proof.** We claim the validity of

$$\int_M \left( \frac{1}{2} \text{Ric}_\varphi + \text{Hess}(f) \right) e^{-f} = \int_M \left( \frac{1}{2} \text{Hess}(f) + 2\Delta f - |\nabla f|^2 \right) \text{div}(e^{-f} Y).$$

(5.16)
We have, in a local orthonormal coframe,
\[
\left(\text{Ric}^\varphi + \text{Hess}(f), \frac{1}{2} \nabla_Y g\right) e^{-f} = (R^\varphi_{ij} + f_{ij})Y^i e^{-f} = (R^\varphi_{ij} + f_{ij})Y^i e^{-f}.
\]
(5.17)

Using Schur’s lemma (2.17), the commutation relation (2.2) and the definition of the $\varphi$-Ricci tensor
\[
\langle R^\varphi_{ij} + f_{ij} \rangle Y^i e^{-f} = \left(\frac{1}{2} S^\varphi + \Delta f\right) e^{-f} Y^i - \alpha \varphi^a & \varphi^a + R^\varphi_{ij} f_j + \alpha \varphi^a & \varphi^a f_j,
\]
hence we easily get
\[
\langle R^\varphi_{ij} + f_{ij} \rangle Y^i e^{-f} = \left[\left(\frac{1}{2} S^\varphi + \Delta f\right) e^{-f} Y^i\right]_i + \left(\frac{1}{2} S^\varphi + \Delta f\right) (e^{-f} Y^i) + R^\varphi_{ij} f_j + \alpha \varphi^a & \varphi^a f_j.
\]
(5.18)

Furthermore
\[
f_{ij} f_j Y^i e^{-f} = \frac{1}{2} |\nabla f|^2 e^{-f} - \frac{1}{2} \left(|\nabla f|^2 e^{-f} Y^i\right) + \frac{1}{2} |\nabla f|^2 \text{div}(e^{-f} Y).
\]
(5.19)

The claim follows by plugging (5.18) and (5.19) into (5.17), using the divergence theorem and rearranging the terms.

Clearly have
\[
\langle h_1, h_1 - b_1 \rangle + \alpha \langle b_2, h_2 - b_2 \rangle = \langle h_1 \rangle^2 - \langle h_1, b_1 \rangle + \alpha |b_2|^2 - \alpha \langle h_2, b_2 \rangle
\]
and on the other hand, using the definitions (5.12), (5.13) and (5.11),
\[
\langle h_1, h_1 - b_1 \rangle + \alpha \langle b_2, h_2 - b_2 \rangle = \left\langle \text{Ric}^\varphi + \text{Hess}(f), \frac{1}{2} \nabla_Y g\right\rangle + \alpha \langle \varphi - \partial f(Y), \varphi (Y) \rangle - \lambda \text{div}(Y),
\]
then combining the above relations with (5.16) we get
\[
\int_M \left[|h_1|^2 - \langle h_1, b_1 \rangle + \alpha |b_2|^2 - \alpha \langle h_2, b_2 \rangle\right] e^{-f} = \int_M \left(\frac{1}{2} S^\varphi + \Delta f - \frac{1}{2} |\nabla f|^2\right) \text{div}(e^{-f} Y) + \lambda \int_M \text{div}(Y) e^{-f}.
\]

The validity of (5.10) follows from the above, since using that
\[
\text{div}(e^{-f} Y) = \text{div}(Y) e^{-f} - e^{-f} \langle \nabla f, Y \rangle, \quad \text{div}(f e^{-f} Y) = e^{-f} \langle \nabla f, Y \rangle
\]
we get
\[
\int_M \text{div}(Y) e^{-f} = \int_M \langle \nabla f, Y \rangle e^{-f} = - \int_M f \text{div}(e^{-f} Y).
\]

5.2 Rigidity of $\varphi$-Cotton flat compact harmonic-Ricci solitons

Using [Theorem 5.5], we able to prove the next

**Theorem 5.20.** Let $(M,g)$ be compact harmonic-Ricci soliton of dimension $m \geq 3$. If $(M,g)$ is $\varphi$-Cotton flat then $(M,g)$ is rigid.

**Proof.** Assume by contradiction that $(M,g)$ is not rigid. From [Theorem 5.5] we can assume that $(M,g)$ is a gradient (shrinking) harmonic-Ricci soliton, i.e.,
\[
\begin{align*}
\text{Ric}^\varphi + \text{Hess}(f) &= \lambda g, \\
\tau(\varphi) &= d\varphi(\nabla f)
\end{align*}
\]
(5.21)
holds for some \( f \in C^\infty(M), \lambda > \mathbb{R}, \varphi : M \to N \) smooth, where \((N, \langle \cdot, \cdot \rangle_N)\) is a Riemannian manifold and \(\alpha > 0\).

First of all, relying only on the validity of (5.21) (and without using that \(\lambda > 0\)), we show that

\[
\int_M |\nabla \text{Ric}^e|^2 e^{-f} = \int_M |F^e|^2 e^{-f} + 2\alpha \int_M \varphi^e_{kk} \varphi^e_{ij} R^e_{ij} e^{-f},
\]

(5.22)

where \(F^e\) is the tensor defined in (3.2). Afterwards we will see how the validity of above implies rigidity assuming that \((M, g)\) is Cotton flat.

The following Weitzenböck identity holds

\[
\frac{1}{2} \Delta_f |\text{Ric}^e|^2 = |\nabla \text{Ric}^e|^2 + \langle \Delta_f \text{Ric}^e, \text{Ric}^e \rangle.
\]

Multiplying the above by \(e^{-f}\), integrating it on \(M\) and using the divergence theorem we get

\[
\int_M |\nabla \text{Ric}^e|^2 e^{-f} = - \int_M \langle \Delta_f \text{Ric}^e, \text{Ric}^e \rangle e^{-f}.
\]

Contracting (3.21) against Ric we get

\[
\int_M \langle \Delta_f \text{Ric}^e, \text{Ric}^e \rangle e^{-f} = \int_M R^{e}_{ij} \Delta_f R^{e}_{ij} e^{-f}
\]

\[
= - \int_M R_{ijkl} f_{kt} R^{e}_{ij} e^{-f} - 2\alpha \int_M \varphi^e_{ik} \varphi^e_{kj} R^e_{ij} e^{-f} - 2\alpha \int_M \varphi^e_{ij} R^{e}_{ij} \varphi^e_{kk} e^{-f},
\]

hence the above relation gives

\[
\int_M |\nabla \text{Ric}^e|^2 e^{-f} = -2 \int_M R_{ijkl} f_{kt} R^{e}_{ij} e^{-f} + 2\alpha \int_M \varphi^e_{ik} \varphi^e_{kj} R^e_{ij} e^{-f} + 2\alpha \int_M \varphi^e_{ij} R^{e}_{ij} \varphi^e_{kk} e^{-f}.
\]

(5.23)

Using the divergence theorem

\[
- \int_M R_{ijkl} f_{kt} R^{e}_{ij} e^{-f} = \int_M (R_{ijkl} R^{e}_{ij}) f_k = \int_M (R_{ijkl} e^{-f})_t f_k + \int_M f_k R_{ijkl} R^{e}_{ij} e^{-f},
\]

that gives

\[
-2 \int_M R_{ijkl} f_{kt} R^{e}_{ij} e^{-f} = 2 \int_M (R_{ijkl} e^{-f})_t f_k + 2 \int_M f_k R_{ijkl} R^{e}_{ij} e^{-f}.
\]

(5.24)

Plugging (5.22) and (3.21) into (5.21) we get

\[
-2 \int_M R_{ijkl} f_{kt} R^{e}_{ij} e^{-f} = 2 \alpha \int M (\varphi^e_{ik} \varphi^e_{kj} - \varphi^e_{ij} \varphi^e_{kk}) R^e_{ij} f_k e^{-f} + \int M |F^e|^2 e^{-f}
\]

\[
= \int M |F^e|^2 e^{-f} + 2\alpha \int M \varphi^e_{ik} \varphi^e_{kj} R^e_{ij} f_k e^{-f} - 2\alpha \int M \varphi^e_{ij} \varphi^e_{kk} R^e_{ij} e^{-f}
\]

and thus, from (5.23) we obtain,

\[
\int_M |\nabla \text{Ric}^e|^2 e^{-f} = \int_M |F^e|^2 e^{-f} + 2\alpha \int M \varphi^e_{ik} \varphi^e_{kj} R^e_{ij} f_k e^{-f} + 2\alpha \int M \varphi^e_{ij} \varphi^e_{kk} R^e_{ij} e^{-f}.
\]

Taking the covariant derivative of the second equation of (5.21) we get \(\varphi^e_{kk} = \varphi^e_{ki} f_k + \varphi^e_{ki} f_{ki}\), hence from the above we conclude that (5.22) holds.

Now we assume \(C^e = 0\). Then \(\varphi\) is harmonic and thus (5.22) reduces to

\[
\int_M |\nabla \text{Ric}^e|^2 e^{-f} = \int_M |F^e|^2 e^{-f},
\]

so that, using (5.22),

\[
\int_M |\nabla \text{Ric}^e|^2 e^{-f} = \frac{1}{2(m-1)} \int_M |\nabla S^e|^2 e^{-f}.
\]
On the other hand via Cauchy-Schwarz inequality

$$|\nabla \text{Ric}^\varphi|^2 \geq \frac{1}{m} |\nabla S^\varphi|^2,$$

hence from the above we conclude that

$$\frac{m-2}{2m(m-1)} \int_M |\nabla S^\varphi|^2e^{-f} \leq 0.$$ 

Then, since $m \geq 3$, we conclude that $S^\varphi$ is constant on $M$. Then, in view of Remark 5.2, $(M, g)$ is rigid. Contradiction, hence the proof is concluded. 

6 Complete non-compact gradient shrinking harmonic-Ricci solitons

In this section $(M, g)$ is a complete non-compact harmonic-Ricci soliton, i.e., there exist $f \in C^\infty(M)$, $\alpha, \lambda > 0$, $\varphi : M \to N$ smooth, where $(N, \langle , \rangle_N)$ is a Riemannian manifold, such that

$$\begin{cases} \text{Ric}^\varphi + \text{Hess}(\varphi) = \lambda g \\ \tau(\varphi) = d\varphi(\nabla f) . \end{cases} \tag{6.1}$$

**Remark 6.2.** Recall that on any complete Riemannian manifold there exists, for every $R > 0$ and $p \in M$, a cutoff function $\rho_R$ such that $0 \leq \rho_R \leq 1$, $\rho_R = 1$ on $B_p(R)$, $\rho_R = 0$ on $M \setminus B_p(2R)$ and $|\nabla \rho_R| \leq \frac{C}{R}$, where $C$ is a positive constant independent on $R > 0$ and $p \in M$.

The following technical lemma shall be useful later on in the proof of Theorem 6.17.

**Lemma 6.3.** Let $(M, g)$ be a complete non-compact gradient harmonic-Ricci soliton, that is, $(6.1)$ holds for some $f \in C^\infty(M)$, $\alpha, \lambda > 0$, $\varphi : M \to N$ smooth, where $(N, \langle , \rangle_N)$ is a Riemannian manifold. For every positive constant $\mu$ we have

$$\int_M |\text{Ric}^\varphi|^2 e^{-\mu f}, \int_M |\tau(\varphi)|^2 e^{-\mu f} < +\infty . \tag{6.4}$$

In particular, for $\mu = 1$ we have the following relation

$$\int_M |\text{Ric}^\varphi|^2 e^{-f} + \alpha \int_M |\tau(\varphi)|^2 e^{-f} = \lambda \int_M S^\varphi e^{-f} < +\infty . \tag{6.5}$$

Furthermore

$$\int_M |\nabla S^\varphi|^2 e^{-f} < +\infty . \tag{6.6}$$

**Proof.** Using the first equation of $(6.1)$ and the divergence theorem, for a smooth cutoff $\rho$ we get

$$\int_M |\text{Ric}^\varphi|^2 \rho^2 e^{-\mu f} = \int_M \text{R}^\varphi \rho^2 e^{-\mu f} = \lambda \int_M S^\varphi \rho^2 e^{-f} + \int_M \text{R}^\varphi \rho^2 e^{-\mu f} ,$$

Using also $(2.17)$, the second equation of $(6.1)$ and $(6.16)$ the above gives

$$\int_M |\text{Ric}^\varphi|^2 \rho^2 e^{-\mu f} + \alpha \int_M |\tau(\varphi)|^2 \rho^2 e^{-\mu f} = \lambda \int_M S^\varphi \rho^2 e^{-f} + (1 - \mu) \int_M \text{R}^\varphi \rho^2 e^{-\mu f} + \int_M \text{R}^\varphi \rho \rho^2 e^{-\mu f} , \tag{6.7}$$

Notice that

$$|\text{ric}^\varphi(\nabla f)|^2 = \text{R}^\varphi f_j f_k R^\varphi f_k f_j = (\text{Ric}^\varphi)^2(\nabla f, \nabla f) , \tag{6.8}$$

where we are denoting by $\text{ric}^\varphi$ the $(1,1)$-version of $\text{Ric}^\varphi$. Clearly $(\text{Ric}^\varphi)^2 \geq 0$ and thus the following inequality in the sense of quadratic forms holds

$$(\text{Ric}^\varphi)^2 \leq \text{tr}[(\text{Ric}^\varphi)^2]g = |\text{Ric}^\varphi|^2 g ,$$

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then

\[(\text{Ric}^\varphi)^2(\nabla f, \nabla f) \leq |\text{Ric}^\varphi|^2|\nabla f|^2.\]  

(6.9)

Then (6.8) yields the validity of

\[|\text{ric}^\varphi(\nabla f)| \leq |\text{Ric}^\varphi||\nabla f|\]  

(6.10)

Now, from Cauchy-Schwarz inequality and (6.10) we deduce

\[|R^\varphi_{ij} f_i(\rho^2)_{ij}| \leq |\text{ric}^\varphi(\nabla f)||\nabla \rho^2| \leq |\text{Ric}^\varphi||\nabla f||\nabla \rho^2|,
\]

so that, since \(\nabla \rho = 2\rho \nabla \rho\), using the following Cauchy inequality, valid for every \(\varepsilon > 0\) and for every \(a, b \in \mathbb{R}\),

\[ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2,
\]

(6.11)

we get from the above

\[|R^\varphi_{ij} f_i(\rho^2)_{ij}| \leq 2\rho |\text{Ric}^\varphi||\nabla f||\nabla \rho| \leq 2\varepsilon |\text{Ric}^\varphi|^2 \rho^2 + \frac{1}{2\varepsilon} |\nabla f|^2 |\nabla \rho|^2,
\]

that, for \(\varepsilon = \frac{1}{3}\) reads

\[|R^\varphi_{ij} f_i(\rho^2)_{ij}| \leq \frac{1}{4} |\text{Ric}^\varphi|^2 \rho^2 + 4|\nabla f|^2 |\nabla \rho|^2,
\]

(6.12)

Furthermore, using Cauchy-Schwarz inequality and (6.11) we have

\[|R^\varphi_{ij} f_if_j| \leq |\text{Ric}^\varphi||\nabla f|^2 \leq \frac{\varepsilon}{2} |\text{Ric}^\varphi|^2 + \frac{1}{2\varepsilon} |\nabla f|^4,
\]

that for \(\varepsilon = \frac{2|1 - \mu|}{2|1 - \mu|} \) (when \(\mu \neq 1\), because when \(\mu = 1\) we do not have to deal with this term) gives

\[\int_M |R^\varphi_{ij} f_if_j| \leq \frac{1}{4} \int_M |\text{Ric}^\varphi|^2 \rho^2 e^{-\mu f} + \lambda \int_M S^\varphi \rho^2 e^{-\mu f} + (1 - \mu)^2 \int_M |\nabla f|^4 \rho^2 e^{-\mu f} + 2 \int_M |\nabla f|^2 |\nabla \rho|^2 e^{-\mu f}.
\]

(6.13)

Then, with the aid of (6.12) and (6.13), from (6.7) we get

\[\frac{1}{2} \int_M |\text{Ric}^\varphi|^2 \rho^2 e^{-\mu f} + \alpha \int_M |\nabla f|^2 |\nabla \rho|^2 e^{-\mu f} \leq \lambda \int_M S^\varphi \rho^2 e^{-\mu f} + (1 - \mu)^2 \int_M |\nabla f|^4 \rho^2 e^{-\mu f} + 2 \int_M |\nabla f|^2 |\nabla \rho|^2 e^{-\mu f}.
\]

(6.14)

Recalling the validity of (2.42), from (6.14) and choosing \(\rho = \rho_R\) as defined in Remark 6.2 we get

\[\frac{1}{2} \int_{B_{\rho}(R)} |\text{Ric}^\varphi|^2 e^{-\mu f} + \alpha \int_{B_{\rho}(R)} |\nabla f|^2 e^{-\mu f} \leq \lambda \int_{B_{\rho}(R)} S^\varphi e^{-\mu f} + (1 - \mu)^2 \int_{B_{\rho}(R)} |\nabla f|^4 e^{-\mu f} + \frac{1}{R} \int_{B_{\rho}(R)} |\nabla f|^2 e^{-\mu f} < +\infty,
\]

Letting \(R \to +\infty\) we get (6.14), since \(\alpha > 0\).

Once we know that \(\text{Ric}^\varphi \in L^2(M, e^{-f})\), using (6.7) with \(\rho = \rho_R\) and \(\mu = 1\) and passing to the limit for \(R \to +\infty\) we obtain (6.15). Indeed it is sufficient to show that

\[\lim_{R \to +\infty} \int_M R^\varphi_{ij} f_i(\rho^2)_{ij} e^{-f} = 0,
\]

(6.15)

and this follows easily from the inequality

\[|R^\varphi_{ij} f_i(\rho^2)_{ij}| \leq 2\rho |\text{Ric}^\varphi||\nabla f||\nabla \rho| \leq \frac{2C}{R} |\text{Ric}^\varphi||\nabla f| \leq \frac{C}{R} (|\text{Ric}^\varphi|^2 + |\nabla f|^2)
\]

and (2.42).

It remains to prove (6.6). Using (6.14) twice we have

\[|\nabla S^\varphi|^2 = 4(\text{Ric}^\varphi)^2(\nabla f, \nabla f),
\]

(6.16)
that with the aid of \((6.10)\) gives
\[
|\nabla S|^2 e^{-f} \leq |\text{Ric} \rho|^2 |\nabla f|^2 e^{-f}.
\]
Since \(f\) has polynomial growth we have
\[
|\nabla f|^2 e^{-f} \leq e^{-\mu f}
\]
for some \(0 < \mu < 1\). Then we deduce, using \((6.10)\) and the above,
\[
\int_M |\nabla S|^2 e^{-f} \leq 4 \int_M |\text{Ric} \rho|^2 e^{-\mu f},
\]
that is finite in view of \((6.4)\).

Now we are ready to prove the key result in order to extend the validity of Theorem 5.20 to complete non-compact gradient solitons.

**Theorem 6.17.** Let \((M, g)\) be a complete non-compact gradient harmonic-Ricci soliton of dimension \(m \geq 2\), that is, \((6.1)\) holds for some \(f \in C^\infty(M)\), \(\alpha, \lambda > 0\), \(\varphi : M \to N\) smooth, where \((N, \langle \cdot, \cdot \rangle_N)\) is a Riemannian manifold. If \(C^\varphi = 0\) then
\[
\int_M |\nabla \text{Ric} \rho|^2 e^{-f} = \frac{1}{2(m-1)} \int_M |\nabla S|^2 e^{-f} < +\infty. \tag{6.18}
\]
In particular, if \(m \geq 3\), \(\nabla \text{Ric} \rho = 0\).

**Proof.** Let \(\rho\) be a smooth function with compact support, then, in a local orthonormal coframe
\[
|\nabla \text{Ric} \rho|^2 \rho^2 e^{-f} = (R_{ij}^\varphi R_{ij,k} \rho^2 e^{-f})_k - R_{ij}^\varphi (R_{ij,k} \rho^2 e^{-f})_k.
\]
Integrating the above and using the divergence theorem we get
\[
\int_M |\nabla \text{Ric} \rho|^2 \rho^2 e^{-f} = - \int_M R_{ij}^\varphi (R_{ij,k} \rho^2 e^{-f})_k = - \int_M R_{ij}^\varphi \Delta_f R_{ij}^\varphi \rho^2 e^{-f} - \int_M R_{ij}^\varphi R_{ij,k} (\rho^2) e^{-f}. \tag{6.19}
\]
Contracting \((6.19)\) against Ric \(\rho\) we get
\[
R_{ij}^\varphi \Delta_f R_{ij}^\varphi = 2R_{tikj} f_{klt} R_{ij}^\varphi - 2\alpha \varphi_{ij}^a \varphi_{kjk}^a R_{ij}^\varphi - 2\alpha \varphi_{ij}^a \varphi_{kjk}^a R_{ij}^\varphi.
\]
Since \(C^\varphi = 0\) we know that \(\varphi\) is harmonic, hence the above becomes
\[
R_{ij}^\varphi \Delta_f R_{ij}^\varphi = 2R_{tikj} f_{klt} R_{ij}^\varphi - 2\alpha \varphi_{ij}^a \varphi_{kjk}^a f_{kjk} R_{ij}^\varphi,
\]
and thus \((6.19)\) can be rewritten as
\[
\int_M |\nabla \text{Ric} \rho|^2 \rho^2 e^{-f} = -2 \int_M R_{tikj} f_{klt} R_{ij}^\varphi \rho^2 e^{-f} + 2\alpha \int_M \varphi_{ij}^a \varphi_{kjk}^a R_{ij}^\varphi e^{-f} - \int_M R_{ij}^\varphi R_{ij,k} (\rho^2) e^{-f}. \tag{6.20}
\]
Integrating by parts
\[
- \int_M R_{tikj} f_{klt} R_{ij}^\varphi \rho^2 e^{-f} = \int_M (R_{tikj} R_{ij,k} \rho^2 e^{-f}) f_k
\]
\[
= \int_M (R_{tikj} \epsilon^f) f_k R_{ij}^\varphi f_k + \int_M f_k R_{tikj} R_{ij,k} \rho^2 e^{-f} + \int_M R_{tikj} f_{klt} R_{ij}^\varphi \rho^2 e^{-f}.
\]
By plugging \((6.22)\) and \((6.21)\) into the above we get
\[
- \int_M R_{tikj} f_{klt} R_{ij}^\varphi \rho^2 e^{-f} = \alpha \int_M (\varphi_{ij}^a \varphi_{kjk}^a - \varphi_{ij}^a \varphi_{kjk}^a) f_k R_{ij}^\varphi f_k + \frac{1}{2} \int_M |F|^2 \rho^2 e^{-f} + \int_M R_{tikj} f_{klt} R_{ij}^\varphi \rho^2 e^{-f},
\]
that gives, using \(\varphi_{ij}^a f_k = 0\) and \(\varphi_{ij}^a f_k = (\varphi_{ij}^a f_k) f_k - \varphi_{ij}^k f_k = -\varphi_{ij}^k f_k f_k,
\]
\[
- \int_M R_{tikj} f_{klt} R_{ij}^\varphi \rho^2 e^{-f} = \frac{1}{2} \int_M |F|^2 \rho^2 e^{-f} - \alpha \int_M \varphi_{ij}^a \varphi_{kjk}^a R_{ij}^\varphi f_k \rho^2 e^{-f} + \int_M R_{tikj} f_{klt} R_{ij}^\varphi \rho^2 e^{-f}.
\]
Inserting the above into (6.20) we conclude
\[
\int_M |\nabla \text{Ric}^\varphi|^2 \rho R^2 e^{-f} = \int_M |F\varphi|^2 \rho R^2 e^{-f} + 2 \int_M R_{i0j} f_k R_{ij}(\rho^2) e^{-f} - \int_M R_{ij}^\varphi R_{i,j,k}(\rho^2) e^{-f}.
\] (6.21)

Since \(C^\varphi = 0\) we know that (3.7) holds, hence
\[
|F\varphi|^2 = \frac{1}{2(m-1)} |\nabla S^\varphi|^2.
\]

In view of (6.6) we know that \(F^\varphi \in L^2(M, e^{-f})\).

To prove (6.18) it is sufficient to show the validity of
\[
\lim_{R \to +\infty} \int_M R_{i0j} f_k R_{ij}^\rho(\rho_R^2) e^{-f} = 0
\] (6.22)
and
\[
\lim_{R \to +\infty} \int_M R_{ij}^\varphi R_{i,j,k}(\rho^2_R) e^{-f} = 0,
\] (6.23)
where \(\rho_R\) is the cutoff function defined in Remark 6.2. Indeed, assuming the validity of (6.22) and (6.23), passing to the limit for \(R \to +\infty\) in (6.21) with \(\rho = \rho_R\) we obtain (6.18).

We start proving (6.22). Using (3.18) we have
\[
R_{i0j} f_k R_{ij}^\rho = R_{i0j} f_k R_{ij}^\rho = F_{it}^\rho R_{itij}
\]
and thus, using Cauchy-Schwarz inequality,
\[
|R_{i0j} f_k R_{ij}^\rho(\rho_R^2)| = |F_{it}^\rho R_{itij}(\rho_R^2)| \leq |F\varphi||\text{Ric}^\varphi||\nabla \rho_R^2|.
\]

Then, using the property of the cutoff \(\rho_R\), on \(B_p(2R) \setminus B_p(R)\)
\[
|R_{i0j} f_k R_{ij}^\rho(\rho_R^2)| \leq 2\rho_R |F\varphi||\text{Ric}^\varphi||\nabla \rho_R| \leq \frac{2C}{R} \left(\frac{1}{2} |F\varphi|^2 + \frac{1}{2} |\text{Ric}^\varphi|^2\right) = \frac{C}{R} (|F\varphi|^2 + |\text{Ric}^\varphi|^2),
\]
so that, using that \(\text{Ric}^\varphi, F^\varphi \in L^2(M, e^{-f})\), we guarantee the existence of a positive constant \(C_1\) independent from \(R\) such that
\[
\left|\int_M R_{i0j} f_k R_{ij}^\rho(\rho_R^2) e^{-f}\right| \leq \int_M |R_{i0j} f_k R_{ij}^\rho(\rho_R^2)| e^{-f} \leq \frac{C}{R} \left(\int_M |F\varphi|^2 e^{-f} + \int_M |\text{Ric}^\varphi|^2 e^{-f}\right) = \frac{C_1}{R},
\]
and thus (6.22) follows.

Now is the turn of (6.23). To prove if first of all we need to show that
\[
\int_M |\nabla \text{Ric}^\varphi|^2 e^{-f} < +\infty.
\] (6.24)

Using Cauchy-Schwarz inequality and the Cauchy inequality (6.11) we have, for every \(\varepsilon > 0\),
\[
|R_{ij}^\varphi R_{i,j,k}(\rho^2_R)| \leq |\nabla \text{Ric}^\varphi||\text{Ric}^\varphi||\nabla \rho_R^2| = 2\rho_R |\nabla \text{Ric}^\varphi||\text{Ric}^\varphi| \leq \left(2\varepsilon |\rho_R^2| |\nabla \text{Ric}^\varphi|^2 + \frac{1}{2\varepsilon} |\text{Ric}^\varphi|^2\right),
\]
so that, for \(\varepsilon = \frac{1}{4}\) from (6.21) with \(\rho = \rho_R\) we deduce
\[
\int_M |\nabla \text{Ric}^\varphi|^2 \rho_R^2 e^{-f} = \int_M |F\varphi|^2 \rho_R^2 e^{-f} + 2 \int_M R_{i0j} f_k R_{ij}(\rho^2_R) e^{-f} + \frac{1}{2} \int_M \rho_R^2 |\nabla \text{Ric}^\varphi|^2 e^{-f} + 2 \int_M |\text{Ric}^\varphi|^2 \rho_R^2 e^{-f},
\]
and using that \(\text{Ric}^\varphi, F^\varphi \in L^2(M, e^{-f})\) the above gives
\[
\frac{1}{2} \int_M |\nabla \text{Ric}^\varphi|^2 \rho_R^2 e^{-f} = 2 \int_M R_{i0j} f_k R_{ij}(\rho^2_R) e^{-f} + C_1.
\]
With the aid of (6.22) we conclude that (6.24) holds.

Now that we obtained the validity of (6.24), we are finally ready to prove (6.23). Using Cauchy-Schwarz inequality and the properties of the cutoff \( \rho_R \),
\[
|R^\varphi_{ij} R^\varphi_{ij,k} (\rho^2_R) k| \leq |\nabla \text{Ric}^\varphi| |\text{Ric}^\varphi| |\nabla \rho^2_R| \leq \frac{C}{R} (|\nabla \text{Ric}^\varphi|^2 + |\text{Ric}^\varphi|^2),
\]
hence, for some positive constant independent from \( R \),
\[
\int_M R^\varphi_{ij} R^\varphi_{ij,k} (\rho^2_R) k e^{-f} \leq \int_M |R^\varphi_{ij} R^\varphi_{ij,k} (\rho^2_R) k| e^{-f} \leq \frac{C}{R} \left( \int_M |\nabla \text{Ric}^\varphi|^2 e^{-f} + \int_M |\text{Ric}^\varphi|^2 e^{-f} \right) = \frac{C_1}{R}.
\]
Now (6.23) follows and thus (6.13) holds.

Using the inequality (6.2), the validity of (6.18) immediately gives that \( S^\varphi \) is constant on \( M \) and thus \( \text{Ric}^\varphi \) is parallel when \( m = 3 \), as seen in the proof of Theorem 5.20.

The above theorem, combined with Theorem 4.31 and Remark 4.36 gives the following

**Corollary 6.25.** Let \((M, g)\) be a complete non-compact harmonic-Ricci soliton of dimension \( m \geq 3 \), that is, (6.1) holds for some \( f \in C^\infty(M) \), \( \alpha, \lambda > 0 \), \( \varphi : M \to N \) smooth, where \((N, \langle \cdot, \cdot \rangle_N)\) is a Riemannian manifold. If \( C^\varphi = 0 \) then \((M, g)\) is isometric to a finite quotient of \( L \times \mathbb{R}^k \) for some \( 1 \leq k \leq m \), where \((L, g_L)\) is a compact harmonic-Einstein manifold (with respect to \( \alpha \), \( \varphi : L \to N \) and \( \lambda \)) and, via the isometry, \( \varphi = \varphi_L \circ \pi_L \) and \( f = f_{g_L} \circ \pi_{g_L} \), where \( f_{g_L} = \frac{1}{2} |x|^2 + (b, x) + c \) for some \( c \in \mathbb{R} \) and \( b \in \mathbb{R}^k \) and \( \pi_L : L \times \mathbb{R}^k \to L \) and \( \pi_{g_L} : L \times \mathbb{R}^k \to \mathbb{R}^k \) are the canonical projections.

### 7 Rigidity with assumptions on the \( \varphi \)-Bach tensor

In this section our aim is to extend the results of [CC] to the class of harmonic-Ricci solitons, where it is natural to replace the Bach tensor with the \( \varphi \)-Bach tensor. We begin by showing in the next theorem that we can reduce ourselves to the classification of the previous Sections.

**Theorem 7.1.** Let \((M, g)\) be a complete gradient shrinking harmonic-Ricci soliton of dimension \( m \geq 3 \). If the totally traceless part \( D^\varphi \) of the \( \varphi \)-Bach tensor vanishes on \( M \), then \((M, g)\) is \( \varphi \)-Cotton flat.

**Proof.** The theorem above has been essentially proved in Chapter 6 of [A] but, since here we need a little modification in one of the assumptions, we briefly recall how the proof works. Chapter 6 of [A] deals with complete gradient Einstein type structures of dimension \( m \geq 3 \), i.e., complete Riemannian manifolds \((M, g)\) such that
\[
\begin{aligned}
\text{Ric}^\varphi + \text{Hess}(f) - \mu df \otimes df &= \lambda g \\
\tau(\varphi) &= d\varphi(\nabla f),
\end{aligned}
\]

for some \( \alpha \in \mathbb{R} \setminus \{0\} \), \( \mu \in \mathbb{R} \), \( \lambda, f \in C^\infty(M) \) and \( \varphi : M \to N \), where \((N, \langle \cdot, \cdot \rangle_N)\) is a Riemannian manifold. We are interested to the more particular situation where \( \mu = 0 \), \( \alpha > 0 \) and \( \lambda \in \mathbb{R} \), that is, the situation where \((M, g)\) is a complete gradient harmonic-Ricci soliton of dimension \( m \geq 3 \).

Notice that, if \( f \) is constant then \((M, g)\) is harmonic-Einstein and thus \( C^\varphi = 0 \) is trivially satisfied. Otherwise, if \( f \) is non-constant, then it is proper. Indeed, in the compact case the statement is trivial, while in the complete non-compact case we rely on the estimates for the potential function, see Remark 2.37, to obtain its properness.

In Proposition 6.1.10 of [A] we proved
\[
C^\varphi_{ijk} + f_i W^\varphi_{ijk} = D^\varphi_{ijk},
\]
where, in a local orthonormal coframe, the components of \( D^\varphi \) are given by
\[
D^\varphi_{ijk} := \frac{1}{m-2} \left[ R^\varphi_{ijk} f_k - R^\varphi_{ijk} f_j + \frac{1}{m-1} f_i (R^\varphi_{ijk} \delta_{ij} - R^\varphi_{ij} \delta_{ik}) - \frac{S^\varphi}{m-1} (f_k \delta_{ij} - f_j \delta_{ik}) \right].
\]
The tensor \( D^\varphi \) has the following geometric meaning, as pointed out in Remark 6.1.4 of [A]
\[
\tilde{C}^\varphi = D^\varphi,
\]
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where $\tilde{C}^\varphi$ is the $\varphi$-Cotton tensor with respect to the conformal metric \( \tilde{g} := e^{\frac{-2\varphi}{m-2}} g \).

In Proposition 6.1.15 of \( \text{[A]} \), we proved

\[
(m - 2)B^\varphi_{ij} - \frac{m - 3}{m - 2}C^\varphi_{ijk} f_k = D^\varphi_{ijk,k} - \frac{\alpha}{m - 2} \varphi^a_{kk} \varphi^a_{ij} f_j. \tag{7.5}
\]

In Proposition 6.2.3 of \( \text{[A]} \), assuming

\[
B^\varphi(\nabla f, \cdot) = 0 \tag{7.6}
\]

we proved the validity of

\[
\frac{m - 2}{2} |D^\varphi|^2 + \frac{\alpha}{m - 2} \tau(\varphi)|\nabla f|^2 = \text{div}(Y), \tag{7.7}
\]

where the components of the vector field \( Y \) are given by, in a local orthonormal coframe,

\[
Y^k := D^\varphi_{ijk} f_i f_j. \tag{7.8}
\]

Formula (7.7) is the key point to obtain Theorem 2.9 of \( \text{[A]} \), that guarantees that under the assumption \( m \geq 3 \), if the potential is proper and non-constant then \( D^\varphi = 0 \) and \( \tau(\varphi) = 0 \). Indeed, to prove the theorem one integrates (7.7) on the sublevel of the potential function to deduce the validity of

\[
\frac{m - 2}{2} \int_M |D^\varphi|^2 + \frac{\alpha}{m - 2} \int_M \tau(\varphi)|\nabla f|^2 = 0.
\]

Finally, by studying the geometry of the level sets of the potential function in Proposition 6.3.29 we obtain that, if \( D^\varphi = 0 \) and \( \tau(\varphi) = 0 \), then \( C^\varphi = 0 \) on \{ \nabla f \neq 0 \}. Since we are assuming \( \lambda \in \mathbb{R} \) the potential function is real analytic in harmonic coordinates, see Remark 6.3.31 in \( \text{[A]} \). Then, from Remark 6.3.30, we have \( C^\varphi = 0 \) on the whole \( M \).

We show that instead of (7.6), one could assume

\[
B^\varphi \geq -\alpha \frac{1 - \varepsilon}{(m - 2)^2} |\tau(\varphi)|^2 g, \tag{7.9}
\]

for some \( \varepsilon > 0 \) to obtain that \( D^\varphi \) and \( \tau(\varphi) \) vanishes and then the vanishing of \( C^\varphi \) on the whole \( M \). Indeed, from the second integrability condition (7.5) we easily get

\[
(m - 2)B^\varphi(\nabla f, \nabla f) = D^\varphi_{ijk,k} f_i f_j - \frac{\alpha}{m - 2} |\tau(\varphi)|^2 |\nabla f|^2,
\]

and using (7.9) we obtain

\[
D^\varphi_{ijk,k} f_i f_j - \alpha \varepsilon |\tau(\varphi)|^2 |\nabla f|^2 \geq 0.
\]

The above, using the relation (6.2.8) of \( \text{[A]} \),

\[
|D^\varphi|^2 = \frac{2}{m - 2} D^\varphi_{ijk} R^\varphi_{ijk} f_k, \tag{7.10}
\]

and the definition (7.8) of \( Y \) gives

\[
\frac{m - 2}{2} |D^\varphi|^2 + \alpha \varepsilon |\tau(\varphi)|^2 |\nabla f|^2 \leq \text{div}(Y^\varphi).
\]

Integrating the above on the sublevel of the potential function we deduce

\[
\frac{m - 2}{2} \int_M |D^\varphi|^2 + \alpha \varepsilon \int_M |\tau(\varphi)|^2 |\nabla f|^2 \leq 0,
\]

Then we conclude exactly as in the previous case.

To conclude the proof it only remains to show that the assumption (7.9) is satisfied. To see that observe that the vanishing of the totally traceless part of $\varphi$-Bach implies (7.9), for some $\varepsilon > 0$. Notice that, using (2.19), the vanishing of $B^\varphi$ is equivalent to

\[
B^\varphi = \frac{\alpha}{m(m - 2)} |\tau(\varphi)|^2 g, \tag{7.11}
\]

that gives (7.9) with the equality sign when choosing $\varepsilon = \frac{2(m - 2)}{m} > 0$, since $m \geq 3$. □
Remark 7.12. Clearly in the Theorem above one could assume $B^\varphi = 0$ instead of $B^\varphi = 0$. When $m = 4$ the $\varphi$-Bach tensor is traceless, from (2.19), hence the two assumptions are actually the same. The reason why we preferred the latter assumption in the statement of the Theorem above is that, in dimension $m \neq 4$, the vanishing of $\varphi$-Bach implies automatically the harmonicity of $\varphi$ while the vanishing of $B^\varphi = 0$ does not. Hence the assumption $B^\varphi = 0$ does not require a priori that $\varphi$ is harmonic, exactly as assumption (7.6) of Chapter 6 of [A], and it is a geometric assumption on $M$ that does not involve the potential function $f$.

As a consequence of [Theorem 7.1] we get the following two Corollaries.

**Corollary 7.13.** Let $(M, g)$ be a compact harmonic-Ricci soliton of dimension $m \geq 3$. If the totally traceless part $B^\varphi$ of the $\varphi$-Bach tensor vanishes on $M$, then $(M, g)$ is rigid.

**Proof.** From [Theorem 7.1] we obtain that $(M, g)$ is $\varphi$-Cotton flat. Then the thesis follows from [Theorem 5.20]. □

**Corollary 7.14.** Let $(M, g)$ be a complete non-compact gradient shrinking harmonic-Ricci soliton of dimension $m \geq 3$ with respect to $f \in C^\infty(M)$, $\alpha, \lambda > 0$, $\varphi : M \to N$ smooth, where $(N, \langle \cdot, \cdot \rangle_N)$ is a Riemannian manifold. If the totally traceless part $B^\varphi$ of the $\varphi$-Bach tensor vanishes on $M$, then $(M, g)$ is isometric to a finite quotient of $L \times \mathbb{R}$, where $(L, g_L)$ is a compact harmonic-Einstein manifold with respect to $\alpha, \varphi_L : M \to N$ and $\lambda$ and, via the isometry, $\varphi = \varphi_L \circ \pi_L$ and $f = f_R \circ \pi_R$, where $f_R = \frac{1}{2}x^2 + bx + c$ for some $b, c \in \mathbb{R}$ and $\pi_L : L \times \mathbb{R} \to L$ and $\pi_R : L \times \mathbb{R} \to \mathbb{R}$ are the canonical projections. Furthermore we have $B^\varphi = 0$ and $J = 0$, where $J$ is defined by (2.20).

**Proof.** From [Theorem 7.1] we obtain that $(M, g)$ is $\varphi$-Cotton flat. From [Corollary 6.25] we deduce the isometry with $L \times \mathbb{R}$, for some $k$. Now, since $\varphi$ is harmonic, we have $B^\varphi = B^\varphi$. Notice that, from [Proposition 4.4] since $\lambda > 0$, the only chance to have $B^\varphi = 0$ is that $k = 1$, and moreover $J = 0$. Hence the proof is concluded. □

In this final Remark we motivate in which sense it is natural to replace the Bach tensor with the $\varphi$-Bach tensor.

**Remark 7.15.** In [A20] we motivated the study of the pair of equations

$$B^\varphi = 0, \quad J = 0$$

on the compact four dimensional smooth manifold $M$, for a Riemannian metric $g$ and the smooth map $\varphi : M \to N$, where $(N, \langle \cdot, \cdot \rangle_N)$ is a fixed target Riemannian manifold. The solutions of (7.16) are characterized as critical points of the functional

$$S_2(g, \varphi) := \int_M S_2(A^\varphi_g) \mu_g - \frac{\alpha}{2} \int_M |\tau_\varphi(\varphi)|^2 \mu_g,$$

where $\tau_\varphi(\varphi)$ denotes the tension field of $\varphi$ evaluated with respect to the metric $g$, $S_2(A^\varphi_g)$ denotes the second elementary symmetric polynomial in the eigenvalues of the $\varphi$-Schouten tensor $A^\varphi_g$ of $(M, g)$ and $\mu_g$ is the Riemannian volume element of $(M, g)$.

The functional (7.17) is the natural extension, in presence of the field $\varphi$, of the functional

$$g \mapsto \int_M |W_g|^2 \mu_g,$$

whose critical points in four dimension are characterized as Bach flat metrics.

[Corollary 7.13] shows that every $\varphi$-Bach flat compact four dimensional harmonic-Ricci soliton is harmonic-Einstein, i.e., critical point of the functional of normalized total $\varphi$-scalar curvature

$$(g, \varphi) \mapsto \left( \int_M \mu_g \right)^{-\frac{1}{2}} \int_M S^\varphi_3 \mu_g.$$

One may consider the equations (7.16) for a complete non-compact four dimensional manifolds.

[Corollary 7.14] shows that $\varphi$-Bach flat four dimensional complete non-compact gradient shrinking harmonic-Ricci solitons are isometric to a finite quotient of the Riemannian product of a three dimensional compact harmonic-Einstein manifold with the Gaussian shrinking soliton on $\mathbb{R}$. Furthermore, the validity of $B^\varphi = 0$ implies automatically that $J = 0$, exactly as in case $\varphi$ is a submersion a.e., see [A20].

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