Instantons and the Ground State of the Massive Schwinger Model

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Abstract

We study the massive Schwinger model, quantum electrodynamics of massive, Dirac fermions, in 1+1 dimensions; with space compactified to a circle. In the limit that transitions to fermion–anti-fermion pairs can be neglected, we study the full ground state. We focus on the effect of instantons which mediate tunnelling transitions in the induced potential for the dynamical degree of freedom in the gauge field.

I. Introduction and Summary

The massive Schwinger model is the direct analog of the quantum electrodynamics of electrons and positrons in 1 + 1 dimensions. It was studied1 in a non-perturbative analysis of the effects of the mass term on the phenomena which manifest themselves in its massless relative2, the usual Schwinger model, quark trapping (confinement) and spontaneous symmetry breaking without massless scalars (Higg’s phenomena). The usual Schwinger model, is exactly solvable and has led to much insight concerning the actual structure of quantum field theories. It has afforded the first reconciliation of gauge invariance and the absence of massless states2. It also provided a scenario of how confinement could manifest itself at long distances while asymptotic freedom was valid at short distances3. The massive model, further modified with a current-current interaction was rigorously proven to exist by Fröhlich and Seiler4. Recently, it was shown that its fermionic determinant is directly relevant to the fermionic determinant in four dimensional Q.E.D., in the presence of non-constant, though uni-directional electromagnetic fields5. Thus the two dimensional model has direct bearing on a physical four dimensional theory.

The massive Schwinger model however, is not exactly solvable. Bosonization yields a scalar field with a $m \cos(\phi)$ self-interaction term and an electromagnetic interaction which is no easier to solve. The mass term prohibits the possibility of performing chiral gauge
transformations, even at the classical level, hence the gauge field cannot be removed from
the Lagrangean. Imposition of Gauss’ law however, allows for the elimination of all non-
zero momentum modes of the gauge field at the expense of introducing the Coulomb
interaction into the Hamiltonian. On the circle, we are left with one dynamical degree of
freedom, the Wilson loop of the gauge field. For the infinite line the Wilson loop degree
of freedom is still there, however it becomes infinitesimal.

Yang-Mills theories defined on a circle have been studied by Manton (Schwinger model),
by Rajeev (pure Yang-Mills) and more recently by Langmann and Semenoff including
massless fermions. Some of the results in pre-empt some of our results, however there
are no dynamical calculations done there, and the normal ordering ground state energy
that has been left out there is very important in the massive case. Topologically non-
trivial gauge transformations on the circle, render the Wilson loop variable compact, in
fact, also a circle. The resulting Hamiltonian contains a kinetic term for the loop variable,
its interaction with the fermions and the properly normal ordered fermionic kinetic term.
The normal ordering introduces an induced potential for the loop degree of freedom. This
potential tends to localize the loop variable at its minimum. Instantons, however, mediate
tunnelling transitions around the circle on which the loop variable is defined, which tend
to delocalize the loop variable.

We focus on the effect of these instantons. We calculate the semi-classical correction
to the ground state energy. When these corrections become appreciable, the spectrum
must change from that of a localized particle to that of an essentially free particle which
is constrained to be on a circle. This signals the breakdown of the semi-classical limit,
however, the corresponding energy gives an estimate of the energy of the transition regime.
We therefore find two regimes in the low energy spectrum of the massive Schwinger model.
At low temperature, the spectrum is essentially a harmonic oscillator spectrum with a
frequency \( \omega \). This gives a specific heat which is constant for high temperature, \( \beta \omega \to 0 \),
but vanishing exponentially for low temperature, \( \beta \omega \to \infty \). Heating the system will
eventually move it into a new regime, where the loop variable delocalizes. Here the energy
spectrum behaves like \( \omega' n^2 \), yielding a specific heat which behaves as \( \sqrt{\frac{\omega'}{\beta}} \) as \( \beta \omega' \to 0 \).

II. Hamiltonian

The massive Schwinger model is governed by the Lagrangean density

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}(x,t) F^{\mu\nu}(x,t) + \Psi^\dagger(x,t) (i\partial_t - h(x,t)) \Psi(x,t),
\]

where \( F_{\mu\nu}(x,t) \) is the electromagnetic field strength and \( h(x,t) \) is the Hamiltonian of the
massive Dirac fermion with a minimal electromagnetic interaction. We fix the gauge by
taking \( A_0(x,t) = 0 \). This leaves one gauge field \( A_1(x,t) \), and hence

\[
F_{01} = \partial_t A_1(x,t) = \dot{v}(x,t) = -F_{10}.
\]
The fermionic Hamiltonian is

\[ h(x,t) = -i\gamma_5(\partial_x + iev(x,t)) + m\gamma^0, \]  

(3)

where \( \gamma_5 = i\gamma^0\gamma^1 \), and we take the representation \( \gamma_5 = \sigma^3, \gamma^0 = \sigma^1 \) in terms of the Pauli matrices.

The equations of motion resulting from the Lagrangean are Ampère’s Law,

\[ \ddot{v}(x,t) = e\Psi^\dagger(x,t)\gamma_5\Psi(x,t), \]  

(4)

and the Dirac equation,

\[ i\partial_t\Psi(x,t) = h(x,t)\Psi(x,t) = (-i\gamma_5(\partial_x + iev(x,t)) + m\gamma^0)\Psi(x,t), \]  

(5)

Gauss’ Law, however, is absent

\[ \partial_x\dot{v}(x,t) = e\Psi^\dagger(x,t)\Psi(x,t). \]  

(6)

Invariance of the Lagrangean under static, local gauge transformations

\[ v(x,t) \rightarrow v(x,t) - \partial_x\Lambda(x) \]  
\[ \Psi(x,t) \rightarrow e^{ie\Lambda(x)}\Psi(x,t) \]  

(7)

yields a local conserved charge \( G(x) \),

\[ G(x) = \partial_x\dot{v}(x,t) - e\Psi^\dagger(x,t)\Psi(x,t) \]  

(8)

that is, the time derivative of Gauss’ Law is zero,

\[ \partial_t G(x) = 0. \]  

(9)

The canonical formalism is straightforward, yeilding the Hamiltonian density

\[ \mathcal{H} = \frac{1}{2}(\dot{v}(x))^2 + \Psi^\dagger(x,t)h(x,t)\Psi(x,t), \]  

(10)

with canonically conjugate variable pairs

\[ v(x,t), \quad \Pi_v(x,t)(x,t) = \dot{v}(x,t) \]  
\[ \Psi(x,t), \quad \Pi_{\Psi}(x,t)(x,t) = i\Psi^\dagger(x,t) \]  

(11)

and Poisson brackets

\[ \{v(x,t_x), \dot{v}(y,t_y)\}_{t_x = t_y} = \delta(x - y) \]  
\[ \{\Psi(x,t_x), i\Psi^\dagger(y,t_y)\}_{t_x = t_y} = \delta(x - y). \]  

(12)
For the quantum theory we work in the Schrödinger picture with time independent operators but time dependent states. The classical Poisson brackets are replaced with commutators or anti-commutators,

\[ [v(x, t_x), \dot{v}(y, t_y)]_{t_x = t_y} = i \delta(x - y), \]
\[ \{\Psi(x, t_x), \Psi^\dagger(y, t_y)\}_{t_x = t_y} = \delta(x - y). \]

(13)

The quantization proceeds essentially without subtlety, other than operator ordering ambiguities. Once the normal ordering infinities are subtracted, this yields a finite well defined Hamiltonian and Gauss operator, requiring no regularization. The normal ordering constants are fixed by gauge invariance and locality.\(^9\)

The fermions can be quantized in the Hilbert space of free, massive Dirac fermions, while the gauge fields can be quantized in their corresponding free Hilbert space. The free Hamiltonian is

\[ \hat{H}^0 = \int dx \left( \frac{1}{2} (\dot{v}(x))^2 + : \Psi^\dagger(x) h^0(x) \Psi(x) : \right) \]

(14)

where

\[
\Psi(x) = \sum_{p \in \mathbb{Z}} \psi^0_+(x, p) a_p + \psi^0_-(x, p) b_p^\dagger
\]
\[ \dot{v}(x) = \sum_{p \in \mathbb{Z}} v(p) e^{i \frac{p x}{2\pi L}} \]
\[ \dot{v}(x) = \sum_{p \in \mathbb{Z}} \frac{e^{-i \frac{p x}{2\pi L}} - id}{2\pi L} \frac{dv(p)}{dp} \]
\[ h^0(x) = -i \gamma^5 \partial_x + m \gamma^0 \]
\[ \psi^0_{\pm}(x, p) = \frac{e^{i \frac{p x}{2\pi L}}}{\sqrt{2\pi L}} \psi^0_{\pm}(p) \]
\[ \psi^0_{\pm}(p) = \frac{1}{\sqrt{2(\pm \sqrt{p^2 + (mL)^2})(\pm \sqrt{p^2 + (mL)^2 - p})}} \begin{pmatrix} (mL) \\ \pm \sqrt{p^2 + (mL)^2 - p} \end{pmatrix} \]

(15–20)

and

\[ \{a_p, a_q^\dagger\} = \{b_p, b_q^\dagger\} = \delta_{p, q}, \text{ all others zero.} \]

(21)

The normal ordering in (14) is with respect to these operators, the vacuum state defined by

\[ a_p |0 >= b_p |0 >= 0, \]

(22)

for all \(p\). Explicitly the free fermionic Hamiltonian is

\[ \hat{H}^0_F = \frac{1}{L} \sum_{p \in \mathbb{Z}} \sqrt{p^2 + (mL)^2} (a_p^\dagger a_p + b_p^\dagger b_p). \]

(23)

4
The gauge field Hilbert space is presumably a "wave functional" of the variables \( v(p) \). This is rather formal here, since the ground state wave functional would be the infinite fold product of normalized ground state wave functions for each independent variable, which does not exist. The problem arises simply because the infinite fold tensor product is not isomorphic to the space of normalizable complex valued wave functionals of an infinite number of variables, with the momentum operator represented by the functional derivative. This is in contra-distinction to the analogous case for a finite number of variables. Since we will be able to eliminate all but a single gauge degree of freedom on imposing Gauss’ law, we will not overly concern ourselves with the precise definition of the gauge field Hamiltonian and Hilbert space.

Introducing the interaction at the first quantized level yields the Hamiltonian

\[
h(x) = -i\gamma^5(\partial_x + iev(x)) + m\gamma^0
\]  

with putative second quantized version

\[
\hat{H}_F = \hat{H}^0_F + \int dx v(x)(\Psi^\dagger(x)\gamma^5\Psi(x)).
\]

We must define the last term in the R.H.S., the interaction current density

\[
j^1(x) = \Psi^\dagger(x)\gamma^5\Psi(x) = \bar{\Psi}(x)\gamma^1\Psi(x)
\]

and we will also need the charge density

\[
\rho(x) = \Psi^\dagger(x)\Psi(x) = \bar{\Psi}(x)\gamma^0\Psi(x).
\]

Precisely, we will define, their corresponding momentum components, for \( p \neq 0 \)

\[
j^\mu(p) = \sum_{q \in \mathbb{Z}} \left( \langle \psi^0_+(q)|\gamma^0\gamma^\mu|\psi^0_+(p + q)\rangle a_q^\dagger a_{p+q} + \langle \psi^0_+(q)|\gamma^0\gamma^\mu|\psi^0_-(p + q)\rangle a_q^\dagger b_{p+q}^\dagger 
+ \langle \psi^0_-(q)|\gamma^0\gamma^\mu|\psi^0_+(p + q)\rangle b_q a_{p+q} - \langle \psi^0_-(q)|\gamma^0\gamma^\mu|\psi^0_-(p + q)\rangle b_{p+q}^\dagger b_q \right),
\]

where the bracket \( \langle \cdot|\cdot \rangle \) is between the spinors in equation (20). These are well defined operators with their domain consisting of states corresponding to finitely many excitations above the free vacuum state. For \( p = 0 \) we must actually normal order by subtracting infinite constants, we take

\[
Q = \int dx j^0(x) = \sum_{q \in \mathbb{Z}} (a_q^\dagger a_q - b_q^\dagger b_q)
\]

\[
Q_5 = \int dx j^1(x) = \sum_{q \in \mathbb{Z}} \left( \frac{q}{\sqrt{q^2 + (mL)^2}} a_q^\dagger a_q + \frac{(mL)}{\sqrt{q^2 + (mL)^2}} a_q b_q 
+ \frac{(mL)}{\sqrt{q^2 + (mL)^2}} b_q^\dagger a_q^\dagger - \frac{q}{\sqrt{q^2 + (mL)^2}} b_q^\dagger b_q \right).
\]
These charges clearly have arbitrary definitions up to c-numbers, as far as the fermionic variables are concerned. We can fix these c-numbers by imposing gauge invariance and locality. These c-numbers will only affect the fermionic Hamiltonian by c-numbers if we take equations (28) as the basic building blocks for constructing the Hamiltonian (25).

Gauge invariance manifests itself with the condition that the spectrum of (25) is independent of $v(x)$, we can gauge away $v(x)$ with the gauge transformation

$$e^{ie \int_0^x dy v(y)}.$$ \hspace{1cm} (30)

Actually we may not remove all of the constant part of $v(x)$, the gauge transformation (27) must be single valued on the circle. Thus we must modify (30) to

$$e^{ie \int_0^x dy (v(y) - v)}$$ \hspace{1cm} (31)

with

$$v = \frac{1}{2\pi L} \int_0^{2\pi L} dv(x).$$ \hspace{1cm} (32)

As explained in reference 5, this gives the gauge covariant Hamiltonian

$$\hat{H}_F = \hat{H}_F^0 + e \sum_{p \in \mathbb{Z}} v(-p) j^1(p) + Le^2 \sum_{p \in \mathbb{Z}} v(-p)v(p) + evQ_5.$$ \hspace{1cm} (33)

The gauge covariance is manifest if we remove $v(-p)$ dependence by a unitary transformation

$$\hat{H}_F = \mathcal{U}^\dagger (\hat{H}_F^0 + evQ_5 + Le^2v^2) \mathcal{U} = \mathcal{U}^\dagger \hat{H}_F \mathcal{U}$$ \hspace{1cm} (34)

where

$$\mathcal{U} = e^{-e \sum_{p \neq 0} \frac{1}{L}v(-p)j^0(p)}.$$ \hspace{1cm} (35)

This follows from the commutation relations, which can be rigorously established,

$$[j^0(p), j^1(q)] = 2p \delta p, -q$$

$$[\hat{H}_F^0, j^0(p)] = -\frac{p}{L} j^1(p)$$

$$[Q, Q_5] = [Q, j^\mu(p)] = [Q_5, j^\mu(p)] = [j^0(p), j^0(q)] = [j^1(p), j^1(q)] = 0.$$ \hspace{1cm} (36)

The full Hamiltonian then is

$$\hat{H} = \sum_{p \in \mathbb{Z}} \frac{1}{2\pi L} \frac{d^2}{2 dv(p) dv(-p)} + \hat{H}_F$$

$$= \mathcal{U}^\dagger \left( -\frac{1}{2\pi L} \sum_{p \in \mathbb{Z}} 2 \left( \frac{d}{dv(-p)} + \frac{eLj^0(p)}{p} \right) \left( \frac{d}{dv(p)} - \frac{eLj^0(-p)}{p} \right) + \frac{-1}{4\pi L} \frac{d^2}{dv^2} + \hat{H}_F \right) \mathcal{U}$$

$$= \mathcal{U}^\dagger \hat{H} \mathcal{U}$$ \hspace{1cm} (37)
and we are looking for the ground state of $\hat{H}$ or equivalently $\bar{H}$.

We must not forget Gauss’ law, which actually simplifies matters. At the quantum level we impose Gauss’ law as a constraint on physical states. Physical states are those which are annihilated by the Gauss operator:

$$\hat{G}(x)|\text{physical} >= 0. \quad (38)$$

We can write $\hat{G}(x)$ by Fourier decomposition as

$$\hat{G}(x) = \sum_{p \in \mathbb{Z}} \frac{e^{ipx}}{2\pi L} \hat{G}(p) \quad (39)$$

where

$$\hat{G}(p) = \frac{p}{L} \frac{d}{dv(-p)} - ej^0(p)$$

with $Q$ the charge operator.

Thus we look for eigenstates

$$\bar{H}|\mathcal{E} >= \mathcal{E}|\mathcal{E} > \quad (41)$$

subject to the simpler conditions

$$\begin{cases}
\frac{p}{L} \frac{d}{dv(-p)}|\mathcal{E} >= 0 & p \neq 0 \\
Q|\mathcal{E} >= 0 & p = 0.
\end{cases} \quad (42)$$

Clearly the conditions are trivial to satisfy, $|\mathcal{E} >$ is a charge zero state, that is independent of $v(p)\forall p$. Then the eigenvalue problem for $|\mathcal{E} >$ reduces to

$$\left( -\frac{1}{2\pi L} \sum_{p \in \mathbb{Z}} \frac{1}{2} e^2 L^2 \frac{j^0(p) j^0(-p)}{p^2} - \frac{1}{4\pi L} \frac{d^2}{dv^2} + \bar{H}_F \right) |\mathcal{E} >= \mathcal{E}|\mathcal{E} >. \quad (43)$$

The first term is just the Coulomb energy of the fermions while the $v$ is the only physical degree of freedom in the gauge field.

We must still deal with topologically non-trivial gauge transformations

$$g_k(x) = e^{-i \frac{2\pi k}{L}} \quad k \in \mathbb{Z}. \quad (44)$$

The effect on $v$ is to shift it by a constant,

$$v \rightarrow v + \frac{k}{eL}. \quad (45)$$
Hence $v$ is really a circular variable with circumference $\frac{1}{eL}$. The fermionic part of the Hamiltonian transforms covariantly under these gauge transformations, the unitary operator $V$ which implements these transformations for $k = 1$, effects the mapping on the annihilation and creation operators which diagonalize $\bar{H}_F$,

$$a_p(v) = V^\dagger a_{p+1}(v - \frac{1}{eL})V$$

$$\bar{b}_p(v) = V^\dagger \bar{b}_{p+1}(v - \frac{1}{eL})V.$$  \hspace{1cm} (46)

The expression for $V$ is simple,

$$V = \prod_{p \in \mathbb{Z}} e^{-\frac{i}{2}(a_{p+1}(v)\bar{a}_p(v) - \bar{a}_{p+1}(v)\bar{a}_p(v))} e^{-\frac{i}{2}(\bar{b}_{p+1}(v)\bar{b}_p(v) - \bar{b}_{p+1}(v)\bar{b}_p(v))}.$$  \hspace{1cm} (47)

$V$ is in fact independent of $v$, as can be seen by reexpressing $\bar{a}_p(v)$ in terms of the free annihilation and creation operators,

$$\langle \psi_+(p,v) | (|\psi_0^0(p)\rangle a_p + |\psi_0^0(p)\rangle \bar{b}_p^\dagger) = V^\dagger \langle \psi_+(p+1,v - \frac{1}{eL}) | (|\psi_0^0(p+1)\rangle a_{p+1} + |\psi_0^0(p+1)\rangle \bar{b}_{p+1}^\dagger) V.$$  \hspace{1cm} (48)

Therefore, if

$$a_p = V^\dagger \langle \psi_0^0(p) | (|\psi_0^0(p+1)\rangle a_{p+1} + |\psi_0^0(p+1)\rangle \bar{b}_{p+1}^\dagger) V$$

$$\bar{b}_p^\dagger = V^\dagger \langle \psi_0^0(p) | (|\psi_0^0(p+1)\rangle a_{p+1} + |\psi_0^0(p+1)\rangle \bar{b}_{p+1}^\dagger) V$$  \hspace{1cm} (49)

equation (46) will be satisfied. This is a (unitarily implementable) Bogoliubov transformation that is completely independent of $v$. The Hamiltonian satisfies

$$V^\dagger \hat{H}_F(v)V = \hat{H}_F^0 + e(v + \frac{1}{eL})Q_5 + e^2L(v + \frac{1}{eL})^2 = \hat{H}_F(v + \frac{1}{eL}).$$  \hspace{1cm} (50)

Thus the spectrum of the fermionic Hamiltonian is invariant after going around the circle in $v$, however the eigenstates form sections of a bundle over $v$, the fermionic states satisfy

$$V |v + \frac{1}{eL} > = |v >.$$  \hspace{1cm} (51)

The physical state condition, equation (38), which corresponds to implementing Gauss’ law, makes the states invariant under infinitesimal gauge transformations. The integrated version of equation (38), corresponding to finite gauge transformations that are still continuously connected to the identity (small), simply implies that the states are invariant under these small gauge transformations

$$e^{i \int dx \Lambda(x) G(x)} |\text{physical} >= |\text{physical} >.$$  \hspace{1cm} (52)
In $\hat{G}(x)$, the part corresponding to the divergence of the electric field, effects the transformation on the gauge field, while the fermionic charge density effects the appropriate transformation on the fermions. For the topologically non-trivial gauge transformations (large), we can only work with the finite form of these transformations, evidently an infinitesimal generator does not exist. We should impose invariance of the states under these transformations also. The transformation on the gauge field is a translation operator, $\mathcal{K}$,

$$\mathcal{K}(v) = v + \frac{1}{eL}$$

(53)

while the fermions are transformed by $V$, equation (46). Thus invariance of the states under large gauge transformations implies

$$\mathcal{K}V|\text{physical} = |\text{physical} > .$$

(54)

The eigenstates of the total Hamiltonian then must have the form

$$|E >= \sum_n \psi_n(v)|n,v >.$$

(55)

Here $|n,v >$ correspond to a complete set of fermionic states satisfying equation (51) and $\psi_n(v)$ is the bosonic wave function which is periodic under translation of $v$ by $\mathcal{K}$,

$$\mathcal{K}(\psi_n(v)) = \psi_n(v + \frac{1}{eL}) = \psi_n(v).$$

(56)

If we express $\bar{H}$ in terms of fermionic annihilation and creation operators which diagonalize $\bar{H}_F$ we get

$$\bar{H} = -\frac{1}{4\pi L} d^2v^2 + \sum_{p \in \mathbb{Z}} \sqrt{\frac{p^2}{L} + ev^2} + m^2 (\bar{a}_p(v)\bar{a}_p(v) + \bar{b}_p(v)b_p(v)) - g(v)$$

$$+ \frac{-1}{2\pi L} \sum_{p \in \mathbb{Z}, p \neq 0} \frac{1}{2} e^2L^2 \frac{j^0(p)j^0(-p)}{p^2}$$

(57)

and $g(v)$ is the induced potential from the fermions for the gauge degree of freedom $^9$,

$$g(v) = \frac{-1}{L} \frac{2mL}{\pi} \sum_{n=1}^{\infty} \frac{K_1(\pi nmL)}{n} (\cos(2\pi neLv) - 1).$$

(58)

III. Approximations

The basic approximation that we make, which is not exact $^{10}$, is that excitations to fermion–anti-fermion pairs (corresponding to $\bar{H}_F(v)$) are suppressed. Standard perturbation theory
shows that corrections to the wave function and energy levels coming from intermediate states are not only suppressed by explicit factors of some coupling constant, but also due to powers of the ratio of the characteristic energy scale of the interaction Hamiltonian with the energy difference between the unperturbed initial state and the intermediate state. If the Hamiltonian is a free part plus a perturbation,

$$H = H^0 + \lambda H'$$

and the full wave function admits an expansion of the form

$$\psi = \psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n \psi_n$$

with

$$H^0 \psi_n = E_n \psi_n \quad n = 0, 1, 2, \ldots,$$

we get

$$H \psi = (H^0 + \lambda H')(\psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n \psi_n)$$

$$= E_0 \psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n E_n \psi_n + \lambda H' \psi_0 + \lambda^2 \sum_{n=1}^{\infty} \alpha_n H' \psi_n$$

$$= E \psi = E(\psi_0 + \lambda \sum_{n=1}^{\infty} \alpha_n \psi_n).$$

This implies

$$E = E_0 + \lambda <\psi_0|H'|\psi_0> + o(\lambda^2)$$

and

$$\lambda \alpha_n = \lambda <\psi_n|H'|\psi_0> \frac{E_0 - E_n}{E_0 - E_{\text{pairs}}} + o(\lambda^2).$$

In our case we consider excitations from the fermionic ground state \(|0>>\) which is annihilated by \(a_p(v)\) and \(b_p(v)\). \(j^0(p)\) can be expressed as a bilinear in these operators, hence the Coulomb energy term mediates transitions to intermediate states with zero, one or two fermion–anti-fermion pairs, at zero total momentum. Then

$$\lambda H' = \frac{(eL)^2}{4\pi} \left( 1 \sum_{p \in \mathbb{Z}}^{p \neq 0} \frac{j^0(p)j^0(-p)}{p^2} \right),$$

and the coefficients of states involving fermion–anti-fermion pairs, first order in the perturbative expansion of the full ground state are

$$\frac{(eL)^2}{4\pi} \left( 1 \sum_{p \in \mathbb{Z}}^{p \neq 0} \frac{<\text{pairs}|j^0(p)j^0(-p)|0>>}{p^2} \right)$$

$$= \frac{(eL)^2}{4\pi 2mL} \left( 1 \sum_{p \in \mathbb{Z}}^{p \neq 0} \frac{<\text{pairs}|j^0(p)j^0(-p)|0>>}{p^2} \right).$$
This behaviour continues in each order. Thus the coupling constant emerges as \((eL)(\frac{e}{m})\), which we take to be arbitrarily small. Therefore we can neglect the contribution of pair states, arising because of the Coulomb term, to the full ground state. This makes intuitive sense in the following way: on a circle, we cannot separate charges to infinity, there is a maximum separation that we can separate charges. The Coulomb energy, which is linear in the separation, is bounded. It is a little more work to see that the corresponding operator is relatively bounded in comparison to the fermionic Hamiltonian about which we perturb. Hence in the limit that its coefficient goes to zero, we can rigorously neglect it.

The shift in the ground state energy has the leading contribution

\[
< 0 | \lambda H' | 0 > > = \left(\frac{eL}{2}\right)^2 \left(\frac{1}{L} \sum_{p \in \mathbb{Z}} \frac{< 0 | j^0(p) j^0(-p) | 0 > >}{p^2}\right)
\]

\[
< 0 | j^0(p) j^0(-p) | 0 > > = \left(\frac{eL}{2}\right)^2 \left(\frac{1}{L} \sum_{p \in \mathbb{Z}} \frac{\left|\langle \psi_-(q,v) | \psi_+(p+q,v) \rangle \right|^2}{p^2}\right)
\]

\[
< 0 | j^0(p) j^0(-p) | 0 > > = \left(\frac{eL}{2}\right)^2 \left(\frac{1}{L} \sum_{p,q \in \mathbb{Z}} \frac{1}{p^2} \left(\frac{1}{2\sqrt{(q+eLv)^2+(mL)^2} - (p+q+eLv)}\right) \times \right.
\]

\[
\left(\frac{1}{2\sqrt{(q+eLv)^2+(mL)^2} - (p+q+eLv)}\right) \right.
\]

This has as coefficient \((eL)^2\) multiplying a function of \(eLv\), which is of order 1. This shift in the ground state energy is suppressed relative to \(-g(v)\), the normal ordering contribution to the ground state energy, by again a factor of \((eL)(\frac{e}{m})\).

This yields the truncated Hamiltonian

\[
H = \frac{-1}{4\pi L} \frac{d^2}{dv^2} + \frac{1}{L} \sum_{p \in \mathbb{Z}} \sqrt{(p+ev)^2 + (mL)^2} (\bar{a}^+_p(v) \bar{a}_p(v) + \bar{b}^+_p(v) \bar{b}_p(v)) - g(v)
\]

and the ground state will be of the form

\[
|E_0 > = \psi(v) | 0 > >
\]

The state |0 > > depends on \(v\), hence the derivative \(\frac{d}{dv}\) will give transitions to other fermionic states. We have

\[
|0 > > = \mathcal{W}^\dagger | 0 >
\]
with $|0>$ the free fermion vacuum, and
\[ \mathcal{W} = \prod_{p \in \mathbb{Z}} e^{\theta(p,v)}(a_p^\dagger b_p^\dagger - b_p a_p). \] (71)
\[ W \] is the unitary operator implementing the Bogoliubov transformation
\[ \bar{a}_p(v) = \cos(\theta(p,v))a_p + \sin(\theta(p,v))b_p^\dagger = \mathcal{W}_p a_p \mathcal{W}^\dagger, \]
\[ \bar{b}_p(v) = -\sin(\theta(p,v))a_p + \cos(\theta(p,v))b_p^\dagger = \mathcal{W}_p b_p^\dagger \mathcal{W}, \] (72)
with
\[ \cos(\theta(p,v)) = e^{\psi_+(p,v)\psi_0^0(p)}, \]
\[ \sin(\theta(p,v)) = e^{\psi_+(p,v)\psi_0^-(p)}. \] (73)
Then,
\[ \frac{d}{dv}|0>> = -\sum_{p \in \mathbb{Z}} \frac{d\langle\psi_+(p,v)|\psi_0^+(p)\rangle}{dv} \bar{a}_p(v)\bar{b}_p(v)|0>>. \] (74)
The derivative can be expressed as
\[ \frac{d\theta(p,v)}{dv} = \frac{1}{\cos(\theta(p,v))} \frac{d\sin(\theta(p,v))}{dv} = \frac{1}{\langle\psi_+(p,v)|\psi_0^+(p)\rangle} \frac{d\langle\psi_+(p,v)|\psi_0^-(p)\rangle}{dv}. \] (75)
The denominator is a smooth function of order 1 in $eLv$, and also in $mL$. Note that $eLv$ is always in $[0, 1]$. The numerator is
\[ \langle\psi_+(p,v)|\psi_0^+(p)\rangle = \frac{(mL)^2 - (\sqrt{p + eLv})^2 + (mL)^2 - (p + eLv)(\sqrt{p^2 + (mL)^2} + p)}{2\sqrt{(p + eLv)^2 + (mL)^2}} \times \frac{1}{\sqrt{\sqrt{p^2 + (mL)^2} + mL}^2 + p}. \] (76)
It is easy to see that $\langle\psi_+(p,v)|\psi_0^-(p)\rangle$ behaves like $\frac{eLv}{mL}$ for $mL >> p$. For $mL << p$ it behaves like $\frac{mLeLv}{p^2}$, which is then much smaller than $\frac{eLv}{mL}$ and for $mL = \alpha p$, with $\alpha \approx 1$, we can directly factor the powers of $mL$ out of the expression leaving a function of $\frac{eLv}{mL}$ and $\alpha$. Thus differentiating with respect to $v$ in each case gives a factor of $\frac{eLv}{mL}$. Thus when this parameter is small we can neglect the action of the derivative $\frac{d}{dv}$ on the fermionic state, yielding the equation for $\psi(v)$
\[ \left(\frac{-1}{4\pi} \frac{d^2}{dv^2} - g(v)\right)\psi(v) = \mathcal{E}_0^0\psi(v), \] (77)
a simple, one-dimensional quantum mechanics problem on a circle with a periodic potential. For $m \neq 0$, $g(v)$ is a smooth periodic potential with a single, symmetric well at $v = \frac{1}{2\pi L}$.
It is clear what the excitation spectrum will be. For low energies, the variable will be localized in the bottom of the approximately harmonic well. The energy will be

$$\mathcal{E}_n = \hbar \omega (n + \frac{1}{2}) \quad n = 0, 1, 2, \ldots,$$

(78)

where $\omega$ is the curvature at the bottom of the well. Then for high energies, the variable $v$ will hardly notice the small potential $-g(v)$, but will be essentially constrained by the size of the circle upon which it must sit. The circumference is $\frac{1}{eL}$, giving rise to an energy spectrum

$$\mathcal{E}_n = \hbar \omega'(n')^2 \quad n' = 1, 2, \ldots,$$

(79)

$\omega$ is given by

$$\omega = \sqrt{\frac{1}{2\pi L} \left( \frac{d^2}{dv^2} g(v) \bigg|_{v = \frac{1}{eL}} \right)},$$

(80)

while

$$\omega' = \pi e^2 L.$$ 

(81)

**IV. Instantons**

We proceed along the lines pioneered by Langer$^{12}$, and popularized by Coleman$^{13}$, for using the Euclidean path integral to calculate the effects of tunnelling. The idea is simple, the matrix element of $e^{-TH}$ in the “position” eigenstate $|v = \frac{1}{2eL} \rangle$ has a representation in terms of a Euclidean path integral

$$\langle v = \frac{1}{2eL} | e^{-TH} | v = \frac{1}{2eL} \rangle = \int_{v(-\frac{T}{2})=v(\frac{T}{2})=\frac{1}{2eL}} \mathcal{D}v(\tau) e^{-\int_{\frac{T}{2}}^{\frac{3T}{2}} d\tau L^E(v(\tau))},$$

(82)

where $L^E(v(\tau))$ is the continuation to (dimensionless) Euclidean time $\tau$ of the usual Lagrangean. In this simple case it corresponds to

$$L^E(v(\tau)) = \frac{1}{2} (\dot{v}(\tau))^2 - 2\pi L g(v(\tau))$$

(83)

which can be thought of simply as the Lagrangean describing motion of a particle in minus the original potential, $-(-g(v))$. The matrix element equation (82) has the expansion

$$\langle v = \frac{1}{2eL} | e^{-TH} | v = \frac{1}{2eL} \rangle = e^{-\mathcal{E}_0} \langle v = \frac{1}{2eL} | \mathcal{E}_0 > | v = \frac{1}{2eL} \rangle + \cdots$$

(84)

thus in the limit that $T \to \infty$ we can extract $\mathcal{E}_0$, and the amplitudes $| \langle \mathcal{E}_0 | v = \frac{1}{2eL} \rangle |^2$; contributions from higher states will be exponentially suppressed.

The Euclidean functional integral can be evaluated in a saddle point approximation. The first step is to identify the saddle point, called an instanton here, and then perform the
functional integral in a Gaussian approximation about the saddle point. The Gaussian functional integral simply gives rise to $e^{-S^E_0}$ where $S^E_0$ is the Euclidean action for an instanton, multiplied by the inverse square root of a functional determinant. The determinant is the product of all the eigenvalues of the functional operator corresponding to the second variation of the Euclidean action about the instanton. This product has two main problems, it is of course infinite, but it is also zero!

The infinite product of a continuum of eigenvalues that become arbitrarily large is formally infinite, but in fact completely ill-defined. It is, however, only the ratio of this product relative to the corresponding (infinite) product for the free case that is important. This ratio is finite. The other problem comes from vanishing eigenvalues. These render the determinant zero. Such zero modes correspond to degeneracies of the original instanton. There usually exist a whole set of instantons with the same action. We should sum over the contribution from all saddle points (instantons) with the same, minimal action. When we perform this sum, we have already taken into account the direction in function space corresponding to the zero modes. Thus in the Gaussian integral, we should exclude the integration along the zero modes, the result being the determinant with the zero eigenvalues removed. There is a Jacobian factor which must be taken into account since the measure corresponding to summing over the contribution from degenerate instantons is different from that corresponding to integrating over the zero modes directions in the Gaussian functional integral.

Actually, for large but finite $T$, there are no exact zero modes, corresponding to invariance under translation of the instanton in Euclidean time. The corresponding eigenvalue however, is exponentially small in $T$, thus the infinite $T$ calculations will be exponentially close to those for finite, but large $T$. Furthermore, we must recognize that in this case, there are other approximate critical points, corresponding to $N$ widely separated instantons which must also be considered. The corresponding action is $N$ times the action of one instanton, implying naively that their contribution is suppressed by $N-1$ powers of $e^{-S^E_0}$ relative to the contribution for one instanton. The degeneracy factor of these approximate critical points is, however, $T^N N!$, corresponding to independent translation in Euclidean time of each instanton. This factor can be arbitrarily large compensating the suppression from the exponential factor, until $N$ surpasses $T$. $T$ must always of course be sufficiently large so that the space per instanton, $T^N$, is still much larger than the size of the instanton. The size of the instanton is determined by the parameters that appear in the Lagrangean, hence has nothing to do with $T$ and $N$. Thus it is always possible to satisfy this constraint. We should sum over $N$ until it is of the same order as $T$. However, once $N$ is of this size, the contribution of further terms in the expansion is exponentially small, due to the $1/N!$, thus we make only a negligible error to continue the sum up to $\infty$.

For $mL$ sufficiently large we can keep only the first term in the series for $g(v)$,

$$-g(v) \to \frac{-4mL}{\pi L} K_1(2\pi mL) \sin^2(\pi eLv)$$

(85)
and
\[ \omega = 4\pi eL\sqrt{mLK_1(2\pi mL)}, \]  
(86)
is the relevant frequency for a (dimensionless) Euclidean time, \( \tau \). The corresponding instanton equation, obtained by varying the Euclidean Lagrangean is
\[ \frac{d^2}{d\tau^2} \vec{v}(\tau) = -8\pi eLmLK_1(2\pi mL)\sin(2\pi eL\vec{v}(\tau)). \]  
(87)
This equation is easily integrated to give
\[ \vec{v}(\tau) = \frac{2}{\pi eL}\tan^{-1}\left(e^{\pm\omega(\tau-\tau_0)}\right) - \frac{1}{2eL}, \]  
(88)
the \( \pm \) choosing an instanton or an anti-instanton. The action for either is given by
\[ S_0 = 8\sqrt{mLK_1(2\pi mL)} \]  
(89)
A method which we follow here, for calculating the ratio \( \frac{\text{det}'}{\text{det}_0} \), where \( \text{det}' \) is the determinant with the zero modes excluded and \( \text{det}_0 \) is the free determinant, is given in Coleman\textsuperscript{13}. Here it is shown that
\[ \frac{\text{det}'}{\text{det}_0} = \frac{\psi_0(T_{4\pi L})}{\lambda_0\psi_0'(T_{4\pi L})}, \]  
(90)
where \( \psi_0(T_{4\pi L}) \) is the eigenfunction (evaluated at \( T_{4\pi L} \)) with the smallest eigenvalue \( \lambda_0 \), for the differential equation corresponding to the Schrödinger operator on the interval \( [-T_{4\pi L}, T_{4\pi L}] \), with potential \(+g(\vec{v}(\tau))'\) satisfying the boundary conditions \( \psi_0(-\frac{T_{4\pi L}}{2}) = 0 \), \( \psi_0'(-\frac{T_{4\pi L}}{2}) = 1 \). \( \psi_0(\frac{T_{4\pi L}}{4\pi L}) \) is the analogous solution for the free problem. As \( T \to \infty \), these are easy to find. We find
\[ \frac{\text{det}'}{\text{det}_0} = \frac{S_0(\pi eL)^2}{8}\omega^3 = \frac{1}{4\omega^2}. \]  
(91)
The contribution from the instantons and the anti-instantons sums separately. There is no constraint on the order in which they must appear, either tunnels from the same initial and final state. This gives two times the same contribution. The factor\textsuperscript{13} \( K \), which takes into account the Jacobian factor and the ratio of the determinant in the presence of one instanton and the free determinant then is
\[ K = \left(\frac{S_0}{2\pi}\right)^\frac{1}{2} \left(\frac{\text{det}'}{\text{det}_0}\right)^{-\frac{1}{2}} = \left(\frac{\omega}{2\pi eL}\right)^\frac{1}{2} \frac{2\omega}{eL\pi}. \]  
(92)
Then we find the path integral, equation (82), is given by (in dimensionless Euclidean time)
\[ (\text{det}_0) \sum_{N_1,N_2=0}^{\infty} \frac{\left(T_{2\pi L}Ke^{S_0}\right)^{N_1}\left(T_{2\pi L}Ke^{S_0}\right)^{N_2}}{N_1!N_2!} = \left(\frac{\omega}{\pi}\right)^\frac{1}{2} e^{-\frac{T_{2\pi L}}{2\pi eL}} e^{\omega} e^{\frac{T_{2\pi L}}{2\pi eL}} Ke^{-S_0}. \]  
(93)
The sum over \( N_1 \) is for the instantons while that over \( N_2 \) is for the anti-instantons, the upshot is the factor of 2 in the exponent. The free determinant is calculated in Coleman\(^{13}\).

Finally we find,

\[
<v = \frac{1}{2eL} |e^{-TH}|v = \frac{1}{2eL} > = e^{-TE_0} < v = \frac{1}{2eL} |E_0 > < E_0 |v = \frac{1}{2eL} > + \cdots
\]

\[
= \left( \frac{\omega}{\pi} \right)^{1/2} e^{-\frac{T}{2\pi L}} (\frac{\omega}{4(\pi)})^{1/2} \frac{\omega}{\pi e L} e^{-S_0}).
\]

Hence

\[
E_0 = \frac{1}{2\pi L} \frac{1}{2} \omega \left( 1 - \frac{16}{\pi} \left( \frac{\sqrt{mLK_1(2\pi mL)}}{eL} \right)^{1/2} e^{-S_0} \right) - \frac{1}{2\pi L} 8mLK_1(2\pi mL)
\]

\[
= \frac{1}{2\pi L} \frac{1}{2} \omega \left( 1 - \frac{4\sqrt{2}}{\sqrt{\pi}} \sqrt{S_0} e^{-S_0} \right) - \frac{1}{2\pi L} 8mLK_1(2\pi mL),
\]

where we have included the offset due to the value of the minimum of the potential \(-g(v)\).

The correction that we have calculated is non-perturbative. Our approximations, \( eL \frac{e}{m} \to 0 \), \( e \to 0 \) and \( mL \to \infty \) still leave \( S_0 \) arbitrary (as can be seen from its expression). It can take values from 0 to \( \infty \). The function \( e^{-x} \frac{\sqrt{x}}{\sqrt{2e}} \) is maximum at \( x = \frac{1}{2} \), where it is equal to \( \frac{4\sqrt{2}}{\sqrt{\pi}} \frac{1}{\sqrt{2e}} \approx 1.3 \). Hence as \( S_0 \) approaches \( \frac{1}{2} \) from above or below, the effects of the instantons become non-negligible.

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