ON NONCOMMUTATIVE BASES OF THE FREE MODULE $W_n(\mathbb{K})$ OF ALL $\mathbb{K}$-DERIVATIONS OF THE POLYNOMIAL RING IN $n$ VARIABLES

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ABSTRACT. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $R = \mathbb{K}[x_1, x_2, ... x_n]$ the polynomial ring in $n$ variables over $\mathbb{K}$. We study bases of the free $R$-module $W_n(\mathbb{K})$ of all $\mathbb{K}$-derivations of the ring $R$, such that their linear span over $\mathbb{K}$ is a subalgebra of the Lie algebra $W_n(\mathbb{K})$. We proved that for any Lie algebra $L$ of dimension $n$ over $\mathbb{K}$ there exists a subalgebra $\mathcal{L}$ of $W_n(\mathbb{K})$ which is isomorphic to $L$ and such that every $\mathbb{K}$-basis of $\mathcal{L}$ is an $R$-basis of the $R$-module $W_n(\mathbb{K})$.

1. INTRODUCTION

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $W_n(\mathbb{K})$ the Lie algebra of all $\mathbb{K}$-derivations of the polynomial $R = \mathbb{K}[x_1, x_2, ... x_n]$ in $n$ variables over $\mathbb{K}$. The structure of the Lie algebra $W_n(\mathbb{K})$ and its subalgebras was studied by many authors (see, for example, [1, 3, 7]). One of the most important problem here is the question about structure of finite dimensional subalgebras of $W_n(\mathbb{K})$. The description of all such subalgebras in $W_1(\mathbb{K})$ and $W_2(\mathbb{K})$ in case of the field $\mathbb{K} = \mathbb{C}$ of complex numbers can be easily obtained from the works of S.Lie [4]. There is no such a description for the Lie algebra $W_n(\mathbb{K})$, $n \geq 3$, so it is of interest to study some classes of finite dimensional subalgebras of the algebra $W_n(\mathbb{K})$.

Define one of such classes in the following way: a subalgebra $L$ of $W_n(\mathbb{K})$ of dimension $n$ over $\mathbb{K}$ will be called basic, if every basis of the algebra $L$ over $\mathbb{K}$ is a basis of the $R$-module $W_n(\mathbb{K})$. It is obvious, that the Lie algebra $\mathbb{K}\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ... \frac{\partial}{\partial x_n} \rangle$ is abelian and basic. All abelian basic Lie subalgebras in $W_n(\mathbb{K})$ are described in [5] (see also [6]): let $f_1, f_2, ... f_n \in R$ such that $\det J(f_1, f_2, ... f_n) \in \mathbb{K}^*$, where $J(f_1, f_2, ... f_n)$ is the Jacoby matrix of polynomials $f_1, f_2, ... f_n \in R$. Then the derivations $D_1, \ldots, D_n \in W_n(\mathbb{K})$ defined by the conditions $D_i(h) = \det J(f_1, ..., f_{i-1}, h, f_{i+1}, ..., f_n)$ for any $h \in R$ pairwise commute and form a basis of the $R$-module $W_n(\mathbb{K})$. Conversely, every basis $\{D_1, D_2, ... D_n\}$ such that $[D_i, D_j] = 0$, $i, j = 1, \ldots, n$, can be obtained in such a way. But there exist also non-abelian basic subalgebras. For example, the two-dimensional subalgebra with a basis $\{\frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\}$ from $W_2(\mathbb{K})$ is basic and non-abelian. The main result of the paper, Theorem 2, shows that every Lie algebra of dimension $n$ over $\mathbb{K}$ is isomorphic to a basic subalgebra in $W_n(\mathbb{K})$. In Theorem 1 all nilpotent basic subalgebras of the Lie algebra $W_n(\mathbb{K})$ are characterized.

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Proof. Let \( \Delta \) be a divergence taken with the opposite sign, i.e.
\[
\Delta = \text{div}(ad) = \text{tr}(ad) = a_{ij} \frac{\partial}{\partial x^i} + \ldots + a_{in} \frac{\partial}{\partial x^n},
\]
i = 1, \ldots, n. If \( w_1, \ldots, w_n \in R = \mathbb{K}[x_1, \ldots, x_n] \) then by \( J(w_1, \ldots, w_n) \) will be denoted the Jacoby matrix of these polynomials.

2. ON NILPOTENT BASIC SUBALGEBRAS

**Proposition 1.** Let \( L \) be a basic Lie subalgebra in \( W_n(\mathbb{K}) \) and \( d \) an element of \( L \). Then the trace of \( d \) in \( L \) by the adjoint representation is equal to its divergence taken with the opposite sign, i.e. \( \text{tr} d = -\text{div} d \).

**Proof.** Let \( \{D_1, D_2, \ldots, D_n\} \) be an arbitrary basis of the Lie algebra \( L \) over \( \mathbb{K} \). Let \( D_i = a_{ij} \frac{\partial}{\partial x^j} + \ldots + a_{in} \frac{\partial}{\partial x^n} \) be a decomposition of \( D_i \) in standard basis, \( a_{ij} \in R \). Since \( \{D_1, D_2, \ldots, D_n\} \) is a basis of the free \( R \)-module \( W_n(\mathbb{K}) \), then
\[
\text{tr}(ad) = a_{ij} \frac{\partial}{\partial x^j} + \ldots + a_{in} \frac{\partial}{\partial x^n},
\]
i = 1, \ldots, n.

**Corollary 1.** If \( L \) is a semisimple or nilpotent basic subalgebra of \( W_n(\mathbb{K}) \) then \( L \in SW_n(\mathbb{K}) \), where \( SW_n(\mathbb{K}) \) is the Lie algebra of all \( D \in W_n(\mathbb{K}) \) such that \( \text{div} D = 0 \).

**Lemma 1.** Let \( \{D_1, D_2, \ldots, D_n\} \) be an arbitrary basis of the \( R \)-module \( W_n(\mathbb{K}) \) and \( [D, D_j] = \sum_{k=1}^n c_{ij}^k D_k \) for some \( c_{ij}^k \in R \). Write down
\[
\frac{\partial}{\partial x^i} = \sum_{j=1}^n b_{ij} D_j
\]
for all \( i = 1, \ldots, n \), where \( b_{ij} \in R \). Then
\[
\sum_{j=1}^n b_{pj} b_{qj} c_{ij}^k + \frac{\partial b_{jk}}{\partial x^p} - \frac{\partial b_{pk}}{\partial x^q} = 0
\]
for any \( p, q, k \).

Conversely, let \( \{D_1, D_2, \ldots, D_n\} \) be a basis of the \( R \)-module \( W_n(\mathbb{K}) \). Define the elements \( b_{ij} \) from the equations \( \frac{\partial}{\partial x^i} = \sum_{j=1}^n b_{ij} D_j \). If these elements satisfy relations (1) for some \( c_{ij}^k \in R \), then \( [D, D_j] = \sum_{k=1}^n c_{ij}^k D_k \).

**Proof.** Using commutativity of the standard basis, we get:

\[
0 = \frac{\partial}{\partial x^p} \frac{\partial}{\partial x^q} - \frac{\partial}{\partial x^q} \frac{\partial}{\partial x^p} = \frac{\partial}{\partial x^p} \sum_{j=1}^n b_{qj} D_j - \frac{\partial}{\partial x^q} \sum_{j=1}^n b_{pj} D_j
\]
for any \( p, q \).

Observe that \( \frac{\partial}{\partial x^p} \sum_{j=1}^n b_{qj} D_j = \sum_{j=1}^n b_{qj} \frac{\partial}{\partial x^p} D_j + \sum_{j=1}^n \frac{\partial b_{qj}}{\partial x^p} D_j \), analogously, \( \frac{\partial}{\partial x^q} \sum_{j=1}^n b_{pj} D_j = \sum_{j=1}^n b_{pj} \frac{\partial}{\partial x^q} D_j + \sum_{j=1}^n \frac{\partial b_{pj}}{\partial x^q} D_j \). If we combine
this with (2), we have the relation
\[
0 = \sum_{j=1}^{n} b_{qj} \frac{\partial}{\partial x_p} D_j - \sum_{i=1}^{n} b_{pi} \frac{\partial}{\partial x_q} D_i + \sum_{k=1}^{n} \left( \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} \right) D_k.
\]
Substituting \( \sum_{j=1}^{n} b_{pj} D_j \) and \( \sum_{j=1}^{n} b_{qj} D_j \) instead of \( \frac{\partial}{\partial x_p} \) and respectively \( \frac{\partial}{\partial x_q} \), we get the equality:
\[
0 = \sum_{j=1}^{n} b_{qj} \left( \sum_{i=1}^{n} b_{pi} D_i \right) D_j - \sum_{i=1}^{n} b_{pi} \left( \sum_{j=1}^{n} b_{qj} D_j \right) D_i + \sum_{k=1}^{n} \left( \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} \right) D_k.
\]
It is easy to see that
\[
\sum_{j=1}^{n} b_{qj} \left( \sum_{i=1}^{n} b_{pi} D_i \right) D_j - \sum_{i=1}^{n} b_{pi} \left( \sum_{j=1}^{n} b_{qj} D_j \right) D_i = \sum_{i,j=1}^{n} b_{pi} b_{qj} [D_i, D_j].
\]
Combining this with the decomposition \([D_i, D_j] = \sum_{k=1}^{n} c^k_{ij} D_k\), we obtain the relation:
\[
\sum_{i,j,k=1}^{n} b_{pi} b_{qj} c^k_{ij} D_k + \sum_{k=1}^{n} \left( \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} \right) D_k = 0.
\]
The derivations \(\{D_1, D_2, \ldots, D_n\}\) are linearly independent over \(R\), therefore we have relation (1). This completes the proof of the first part of our statement.

To prove the second part of the Lemma we define elements \(b_{ij}\) by the next relations:
\[
\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} b_{ij} D_j, \quad i, j = 1, \ldots, n.
\]
Suppose the elements \(c^k_{ij}\) satisfy relations (1). We have \([D_i, D_j] = \sum_{k=1}^{n} \gamma^k_{ij} D_k\) for some \(\gamma^k_{ij} \in R\), because \(\{D_1, D_2, \ldots, D_n\}\) is a basis of the \(R\)-module \(W_n(K)\). By the first part of this Lemma the elements \(\gamma^k_{ij}\) satisfy the relations (1), that is, \(\sum_{i,j=1}^{n} b_{pi} b_{qj} c^k_{ij} + \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} = 0\). The system (1) can be regarded as a linear system in \(n^3\) variables \(c^k_{ij}\). This system can be decomposed into a direct sum of \(n\) linear systems:
\[
\sum_{i,j=1}^{n} b_{pi} b_{qj} c^k_{ij} + \frac{\partial b_{qk}}{\partial x_p} - \frac{\partial b_{pk}}{\partial x_q} = 0, \quad k \text{ is fixed, } k = 1 \ldots n.
\]
Let us prove that the system (3) has a unique solution. It is easy to see, that this system has the following matrix
\[
\begin{pmatrix}
    b_{11} & b_{11} & \ldots & b_{11} & b_{11} & b_{11} & \ldots & b_{11} & b_{11} \\
    b_{12} & b_{12} & \ldots & b_{12} & b_{12} & b_{12} & \ldots & b_{12} & b_{12} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    b_{n1} & b_{n1} & \ldots & b_{n1} & b_{n1} & b_{n1} & \ldots & b_{n1} & b_{n1} \\
    b_{n2} & b_{n2} & \ldots & b_{n2} & b_{n2} & b_{n2} & \ldots & b_{n2} & b_{n2} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    b_{nn} & b_{nn} & \ldots & b_{nn} & b_{nn} & b_{nn} & \ldots & b_{nn} & b_{nn}
\end{pmatrix}.
\]
Because this matrix is obviously the tensor square \((b_{ij}) \otimes (b_{ij})\) of the matrix \((b_{ij})\) and the determinant \(\det(b_{ij})\) is invertible (because \((b_{ij})\) is a transition matrix between two bases), it holds \(\det((b_{ij}) \otimes (b_{ij})) = (\det(b_{ij}))^{2n} \in \mathbb{K}^*\). Therefore, the system \((\text{3})\) has the unique solution \(\gamma_{ij}^k = c_{ij}^k\).

Remark 1. Let \(\bar{\text{Der}}(L)\) be a basic subalgebra of the Lie algebra \(L\) and denote by \(c_{ij}^k\) the structure constants of \(L\) in this basis, that is \([l_i, l_j] = \sum_{k=1}^n c_{ij}^k l_k\). It is well known that the tensor product \(R \otimes \mathbb{K} L\) of an associative and commutative \(\mathbb{K}\)-algebra \(R\) and the Lie algebra \(L\) is a Lie algebra over the field \(\mathbb{K}\). Further, we will always denote by \(\bar{\text{Der}}(L)\) the polynomial algebra \(\mathbb{K}[x_1, \ldots, x_n]\). Since the elements of the algebra \(R \otimes \mathbb{K} L\) are of the form \(\sum_{i=1}^n (f_i \otimes l_i), f_i \in R, i = 1, \ldots, n\), the tensor product \(R \otimes \mathbb{K} L\) is a free module of rank \(n\) over the ring \(R\). The elements \(\{1 \otimes l_1, \ldots, 1 \otimes l_n\}\) form obviously a basis of this module. Using the multiplication law in \(L\), we get the equality

\[
[\mathcal{J}, \mathcal{G}] = \sum_{k=1}^n \left( \sum_{i,j=1}^n c_{ij}^k f_i g_j \right) \otimes l_k.
\]

for any elements \(f = \sum_{i=1}^n f_i \otimes l_i, \ g = \sum_{i=1}^n g_i \otimes l_i \in R \otimes \mathbb{K} L\).

For an arbitrary element \(\mathcal{J} = \sum_{i=1}^n f_i \otimes l_i \in R \otimes \mathbb{K} L\) and an arbitrary index \(p = 1, \ldots, n\) define \(\frac{\partial \mathcal{J}}{\partial x_p} = \sum_{i=1}^n \frac{\partial f_i}{\partial x_p} \otimes l_i\). It is easy to see that the map \(\mathcal{J} \mapsto \frac{\partial \mathcal{J}}{\partial x_p}\) is a derivation of the Lie algebra \(R \otimes \mathbb{K} L\) (we will denote this map also by \(\frac{\partial}{\partial x_p}\)). Since the derivation \(\frac{\partial}{\partial x_p}\) acts on the coordinates \(f_i\) of the element \(\mathcal{J} = \sum_{i=1}^n f_i \otimes l_i\), it holds \(\frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q} - \frac{\partial}{\partial x_q} \frac{\partial}{\partial x_p} = 0\) for arbitrary \(p, q = 1, \ldots, n\). Denote by \(\mathcal{A}\) the abelian Lie subalgebra of \(\text{Der}(R \otimes \mathbb{K} L)\) with the basis \(\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\}\) and by \(\mathcal{L}\) the subalgebra \(\mathcal{L} = \mathcal{A} + R \otimes \mathbb{K} L\) of the semidirect product of Lie algebras \(\text{Der}(R \otimes \mathbb{K} L) \rtimes R \otimes \mathbb{K} L\).

Remark 1. Let \(L\) be a basic subalgebra of the Lie algebra \(W_n(\mathbb{K})\). Then the equations \((\text{4})\) are equivalent to the following relations in the Lie algebra \(\mathcal{L}\):

\[
[b_p, b_q] + \left[ \frac{\partial}{\partial x_p}, b_q \right] - \left[ \frac{\partial}{\partial x_q}, b_p \right] = 0.
\]

Since \(\left[ \frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q} \right] = 0\), we can rewrite relations \((\text{5})\) as the following relations in the Lie algebra \(\mathcal{L}\)

\[
\left[ \frac{\partial}{\partial x_p} + b_p, \frac{\partial}{\partial x_q} + b_q \right] = 0.
\]

Let \(L\) be a nilpotent Lie algebra over the field \(\mathbb{K}\) with \(\dim L_{\mathbb{K}} = n\). By Engel’s theorem, \(L\) has a flag of ideals \(L = L_0 \supseteq L_1 \supseteq \ldots \supseteq L_{n-1} \supseteq L_n = \{0\}\). Take any elements \(l_i \in L_{i-1}\backslash L_i, i = 1, \ldots, n\) and consider the Lie algebra \(\mathcal{L}\) constructed in such a way as it was mentioned above. The structure constants of \(L\) in the basis \(\{l_1, \ldots, l_n\}\) satisfy the relations

\[
c_{ij}^k = 0, \text{ if } k \leq \max(i, j).
\]
Since the Lie algebra \( L \) is nilpotent, the tensor product \( R \otimes_K L \) is also nilpotent. Then for any element \( \overline{w} \in R \otimes_K L \) the inner derivation \( \text{ad} \overline{w} \) of the Lie algebra \( \hat{L} \) is nilpotent. We collect some properties of the Lie algebras \( R \otimes_K L \) and \( \hat{L} \) in the following Lemma.

**Lemma 2.** Let \( L \) be a nilpotent Lie algebra of dimension \( n \) over the field \( K \). Let \( \{l_1, \ldots, l_n\} \) be a basis of \( L \) and \( \overline{w} = \sum_{i=1}^{n} w_i \otimes l_i \) be an arbitrary element of \( R \otimes_K L \). Then it holds

1. \( \text{ad} \overline{w} \) is a nilpotent endomorphism of the \( R \)-module \( R \otimes_K L \), that is \( \text{ad} \overline{w}(f \overline{m}) = f \text{ad} \overline{w}(\overline{m}) \) for any \( f \in R \) and \( \overline{m} = \sum_{i=1}^{n} u_i \otimes l_i \in R \otimes_K L \);  
2. \( \varphi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad} \overline{w})^{i-1} \) is an automorphism of the \( R \)-module \( R \otimes_K L \);  
3. if \( \overline{w} = \sum_{i=1}^{n} w_i \otimes l_i \) is an element of \( R \otimes_K L \) with the property \( \det J(w_1, \ldots, w_n) = c \in K^* \), then the set of elements \( \overline{b}_p = \sum_{i=1}^{n} b_{pi} \otimes l_i \) we have \( \det(b_{ij})_{i,j=1}^{n} = c \in K^* \).

**Proof.** (1) Obvious.

(2) As \( \text{ad} \overline{w} \) is a nilpotent endomorphism of \( R \)-module \( R \otimes_K L \), the map \( \varphi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad} \overline{w})^{i-1} \) is well defined and is an endomorphism of the \( R \)-module \( R \otimes_K L \). Since \( \varphi = E + \sum_{i=2}^{\infty} \frac{1}{i!} (\text{ad} \overline{w})^{i-1} \) is the sum of the identity and a nilpotent endomorphisms, the map \( \varphi \) is an automorphism of the free \( R \)-module \( R \otimes_K L \).

(3) Let \( \overline{w} = \sum_{i=1}^{n} w_i \otimes l_i \) be an element of \( R \otimes_K L \) such that \( \det J(w_1, \ldots, w_n) = c \in K^* \). It is easy to see that the set of elements \( \{\frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n}\} \) is a basis of the \( R \)-module \( R \otimes_K L \). Since the map \( \varphi \) defined above is an automorphism of the \( R \)-module \( R \otimes_K L \), the set of elements \( \overline{b}_p = \varphi(\frac{\partial w}{\partial x_p}), \ p = 1, \ldots, n \) is the basis of this module. Therefore \( \det(b_{ij})_{i,j=1}^{n} = c \in K^* \). \( \square \)

**Theorem 1.** (1) Let \( L \) be an arbitrary nilpotent Lie algebra over any field \( K \) of characteristic 0. Then there exists a basic subalgebra \( \overline{L} \) of \( W_n(K) \), such that \( \overline{L} \) is isomorphic to \( L \) (by that every basis of \( \overline{L} \) over \( K \) is a basis of \( R \)-module \( W_n(K) \)).

(2) Let \( \overline{L} \) be a basic Lie subalgebra of \( W_n(K) \), such that \( \overline{L} \) is isomorphic to a nilpotent Lie algebra \( L \) with a basis \( \{l_1, l_2, \ldots, l_n\} \), satisfying the relations \( \{ \overline{m} \} \), let \( D_i \) be the images of the elements \( l_i \) by this isomorphism. Write down \( \frac{\partial \overline{w}}{\partial x_1} = \sum_{j=1}^{n} b_{ij} D_j, \overline{b}_p = \sum_{i=1}^{n} b_{pi} \otimes l_i, \ p = 1, \ldots, n \). Then there exists an element \( \overline{w} = \sum_{i=1}^{n} w_i \otimes l_i \in R \otimes_K L \) such that \( \det J(w_1, w_2, \ldots, w_n) \in K^* \) and the following equalities hold:

\[
\overline{b}_p = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad} \overline{w})^{i-1} \left( \frac{\partial w}{\partial x_p} \right), \ p = 1, \ldots, n
\]

(the number of nonzero summands in this series is finite because the Lie algebra \( R \otimes_K L \) is nilpotent).

**Proof.** (1) Let \( L \) be a nilpotent Lie algebra over the field \( K \) of dimension \( n \) and \( \{l_1, l_2, \ldots, l_n\} \) be its basis such that the structure constants \( c_{ij}^k \) satisfy the
relations (7). We will construct $b_{ij} \in R, i, j = 1, \ldots, n$, which satisfy relations (1) and have property $\det(b_{ij})^{n}_{i,j=1} \in K^*$. If we consider the Lie algebra $\hat{L}$, then by Remark 1 the conditions (1) are equivalent to the conditions (6) in the terms of $\hat{L}$ for $\overline{b}_p = \sum_{i=1}^{n} b_{pi} \otimes l_i$, $p = 1, \ldots, n$:

$$\left[ \partial_{x_p} + \overline{b}_p, \partial_{x_q} + \overline{b}_q \right] = 0.$$ 

Take an arbitrary element $\overline{w} = \sum_{i=1}^{n} w_i \otimes l_i \in \hat{L}$. Put $\overline{b}_p = \sum_{i=1}^{\infty} \frac{1}{i!} (ad \overline{w})^{i-1} \left( - \frac{\partial \overline{w}}{\partial x_p} \right)$, $p = 1, \ldots, n$. We took into account that $\overline{w}$ is an automorphism of the Lie algebra $\hat{L}$ and $\left[ \partial_{x_p}, \partial_{x_q} \right] = 0$, $p, q = 1, \ldots, n$.

Thus, we have elements $\overline{b}_1 = \sum_{i=1}^{n} b_{1i} \otimes l_i$, $\overline{b}_2 = \sum_{i=1}^{n} b_{2i} \otimes l_i, \ldots, \overline{b}_n = \sum_{i=1}^{n} b_{ni} \otimes l_i$ satisfying equations (6). We will build an element $\overline{w}$ of $R \otimes K L$ satisfying the relations (6). We will construct $\overline{w}$ in the form $\sum_{i=1}^{n} \overline{w}_i \otimes l_i$ such that

$$e^{ad \overline{w}} \left( \partial_{x_p} \right) = \partial_{x_p} + \overline{b}_p.$$ 

We have shown that $K(D_1, D_2, \ldots D_n)$ is a basic subalgebra of $W_n(K)$, which is isomorphic to $L$. This completes the proof of part (1).

(2) Take any elements

$$\overline{b}_1 = \sum_{i=1}^{n} b_{1i} \otimes l_i, \overline{b}_2 = \sum_{i=1}^{n} b_{2i} \otimes l_i, \ldots, \overline{b}_n = \sum_{i=1}^{n} b_{ni} \otimes l_i$$

of $R \otimes K L$ satisfying the relations (6). We will build an element $\overline{w} \in R \otimes K L$ such that $\overline{b}_p = \sum_{i=1}^{\infty} \frac{1}{i!} (ad \overline{w})^{i-1} \left( - \frac{\partial \overline{w}}{\partial x_p} \right)$, $p = 1, \ldots, n$. This equality is equivalent to the relation $e^{ad \overline{w}} \left( \partial_{x_p} \right) = \partial_{x_p} + \overline{b}_p$. Applying the automorphism $e^{ad \overline{w}}$ to the both sides of this relation, we obtain

$$\frac{\partial}{\partial x_p} = e^{ad - \overline{w}} \left( \frac{\partial}{\partial x_p} + \overline{b}_p \right).$$
Thus, we must prove that this equality holds for \( p = 1, \ldots, n \). Consider the matrix
\[
B = \begin{pmatrix}
b_{11} & \cdots & b_{1n} \\
\vdots & \ddots & \vdots \\
b_{m1} & \cdots & b_{mn}
\end{pmatrix}
\]
with entries that are coordinates of the elements \( b_1, \ldots, b_n \). Let \( m_1 \) be the number of the first nonzero column of \( B \) (for \( B = 0 \) take \( w = 0 \), then (8) obviously holds). Write the equations (12) for \( k = m_1 \):

\[
\sum_{i,j=1}^{n} b_{pi} b_{qj} c_{ij}^{m_1} + \partial b_{pm_1} \frac{\partial b_{qn_1}}{\partial x_p} - \partial b_{pm_1} = 0, \ p, q = 1, \ldots, n.
\]

Since \( m_1 \) is the number of the first nonzero column of \( B \), we have \( b_{pi} = 0 \) for \( i < m_1, \ p = 1, \ldots, n \). On the other hand, if \( i \geq m_1 \), then \( c_{ij}^{m_1} = 0 \) by (7) (because \( L \) is nilpotent). Hence (9) is equivalent to the equations

\[
\partial b_{pm_1} - \partial b_{pm_1} = 0, \ p, q = 1, \ldots, n.
\]

Then there exists a polynomial \( h_1 \) such that \( b_{pm_1} = \partial b_{pm_1}, \ p = 1, \ldots, n \) (see, for example [6]). Denote \( \overline{h}_1 = h_1 \otimes \lambda_{m_1} \) and consider the elements

\[
e^{\text{ad} \overline{h}_1} \left( \frac{\partial}{\partial x_p} + \overline{b}_p \right) \ p = 1, \ldots, n.
\]

Note that

\[
e^{\text{ad} \overline{h}_1} \left( \frac{\partial}{\partial x_p} + \overline{b}_p \right) = e^{\text{ad} \overline{h}_1} \left( \frac{\partial}{\partial x_p} \right) + e^{\text{ad} \overline{h}_1} (\overline{b}_p) = 0.
\]

Therefore, it holds

\[
\frac{\partial}{\partial x_p} - \frac{\partial h_1}{\partial x_p} \otimes \lambda_{m_1} + e^{\text{ad} \overline{h}_1} \left( \frac{\partial}{\partial x_p} \otimes \lambda_{m_1} + \sum_{i=m_1+1}^{n} b_{pi} \otimes \lambda_i \right), \ p = 1, \ldots, n.
\]

Since \( \left[ \overline{h}_1, \frac{\partial}{\partial x_p} \otimes \lambda_{m_1} \right] = 0 \), we get

\[
e^{\text{ad} \overline{h}_1} \left( \frac{\partial}{\partial x_p} \otimes \lambda_{m_1} \right) = \frac{\partial h_1}{\partial x_p} \otimes \lambda_{m_1}, \ p = 1, \ldots, n.
\]

It is easy to see that \( e^{\text{ad} \overline{h}_1} \left( \sum_{i=m_1+1}^{n} b_{pi} \otimes \lambda_i \right) \in R \otimes \langle \lambda_{m_1+1}, \ldots, \lambda_n \rangle \), because

\[
R \otimes \langle \lambda_{m_1+1}, \ldots, \lambda_n \rangle \text{ is an ideal of the algebra } R \otimes L.
\]

Then we have

\[
e^{\text{ad} \overline{h}_1} \left( \frac{\partial}{\partial x_p} + \overline{b}_p \right) = \frac{\partial}{\partial x_p} - \frac{\partial h_1}{\partial x_p} \otimes \lambda_{m_1} + \frac{\partial h_1}{\partial x_p} \otimes \lambda_{m_1} + \overline{d}_p,
\]

for \( \overline{d}_p = e^{\text{ad} \overline{h}_1} \left( \sum_{i=m_1+1}^{n} b_{pi} \otimes \lambda_i \right) \in R \otimes \langle \lambda_{m_1+1}, \ldots, \lambda_n \rangle \). Therefore,

\[
e^{\text{ad} \overline{h}_1} \left( \frac{\partial}{\partial x_p} + \overline{b}_p \right) = \frac{\partial}{\partial x_p} + \overline{d}_p, \ p = 1, \ldots, n.
\]

Denote by \( d_{pi} \) the coordinates of the element \( \overline{d}_p \) in basis \( \{ 1 \otimes \lambda_1, \ldots, 1 \otimes \lambda_n \} \), \( p = 1, \ldots, n \), i.e.

\[
\overline{d}_p = \sum_{i=1}^{n} d_{pi} \otimes \lambda_i.
\]

Consider the matrix
\[
D = \begin{pmatrix}
d_{11} & \cdots & d_{1n} \\
\vdots & \ddots & \vdots \\
d_{m1} & \cdots & d_{mn}
\end{pmatrix}
\]

We have just proved that the first nonzero column of the matrix \( D \) has the number \( m_2 > m_1 \). Analogously, applying the
autonomorphism $e^\text{ad}T_1$ to the elements $\frac{\partial}{\partial x_p} + f_p$, $p = 1, \ldots n$, we get the elements $\frac{\partial}{\partial x_p} + f_p$. Define the elements $f_{ij}$, $i, j = 1, \ldots n$ from the equalities $f_p = \sum_{i=1}^{n} f_{pi} \otimes l_i$. The first nonzero column in the matrix $F = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}$ has the number $m_3 > m_2$. It is obvious that after $s$ steps ($s \leq n$) we get the elements $\frac{\partial}{\partial x_p}$, $p = 1, \ldots n$ with the corresponding zero matrix. Therefore $e^\text{ad}T_1 \cdots e^\text{ad}T_1 \left( \frac{\partial}{\partial x_p} + f_p \right) = \frac{\partial}{\partial x_p} + f_p$, $p = 1, \ldots n$. Since the Lie algebra $R \otimes L$ is nilpotent, there exists (by Campbell-Baker-Hausdorff formula) an element $w \in R \otimes L$ such that $e^\text{ad}T_1 \cdots e^\text{ad}T_1 = e^{\text{ad}w}$. Finally, let $\mathcal{T}$ be a basic subalgebra which is isomorphic to $L$, $\{D_1, \ldots, D_n\}$ be a basis of $L$. Write $\frac{\partial}{\partial x_p} = \sum_{i=1}^{n} b_{pi} D_i$, the isomorphism is defined by the map $l_i \mapsto D_i$.

Then $\overline{b}_p = \sum_{i=1}^{n} b_{pi} \otimes l_i$, satisfies (1), therefore $\overline{b}_p = \sum_{i=1}^{n} \frac{1}{n!} (\text{ad} \overline{w})^{i-1} \left( \frac{\partial}{\partial x_p} \right)$ for an element $\overline{w} \in R \otimes \mathcal{K}$ such that $J(\overline{w})$ is invertible. This completes the proof of the theorem. \hfill \Box

Example 1. Let $L = H_n$ be the $2n+1$-dimensional Heisenberg Lie algebra, $\{l_1, \ldots, l_{2n+1}\}$ be its standard basis with multiplication rule $[l_i, l_{n+i}] = l_{2n+1}$ for $1 \leq i \leq n$, (other products are zero). Then, in this basis $c_{i,n+i}^{2n+1} = 1$, $c_{n+i,i}^{2n+1} = -1$, $1 \leq i \leq n$, other structure constants are zero. Take $\overline{w} = \sum_{i=1}^{n} -x_i \otimes l_i$. It is clear that $\frac{\partial}{\partial x_p} = -1 \otimes l_p$ and

$$\overline{b}_p = 1 \otimes l_p - \frac{1}{2} \left[ \sum_{i=1}^{n} x_i \otimes l_i, 1 \otimes l_p \right], 1 \leq p \leq n$$

in the Lie algebra $\overline{L}$. Easy calculation shows that $\overline{b}_p = 1 \otimes l_p + \frac{1}{2} x_{p+n} \otimes l_{2n+1}$, $p \leq n$, $\overline{b}_p = 1 \otimes l_p + \frac{1}{2} x_{p-n} \otimes l_{2n+1}$, $n < p \leq 2n$ and $\overline{b}_{2n+1} = 1 \otimes l_{2n+1}$. It can be easily shown that $\det(b_{ij}) = 1$. Passing to the inverse matrix $B^{-1}$ to the matrix $B = (b_{pi})_{p,i=1}^{n}$ one can easily show that the linear span over $\mathbb{K}$ of the following derivations is a basic subalgebra of $W_{2n+1}(\mathbb{K})$ which is isomorphic to $H_n$:

$$D_i = \frac{\partial}{\partial x_i} - \frac{1}{2} x_{n+i} \frac{\partial}{\partial x_{2n+1}}, 1 \leq i \leq n,$$

$$D_1 = \frac{\partial}{\partial x_i} + \frac{1}{2} x_{n-i} \frac{\partial}{\partial x_{2n+1}}, n < i \leq 2n,$$

$$D_{2n+1} = \frac{\partial}{\partial x_{2n+1}}.$$

3. ON THE SOLVABLE BASIC LIE SUBALGEBRAS

Some known properties of finite dimensional Lie algebras and modules over them are collected in the next Lemma.

Lemma 3. Let $L$ be a finite dimensional Lie algebra over an algebraically closed field of zero characteristic and let $H$ be any its Cartan subalgebra. Then

(1) if the algebra $L$ is solvable, then $L = H + [L, L]$;
(2) if $L$ is semisimple and $L = N_- \oplus H \oplus N_+$ is its triangular decomposition, then the subalgebras $N_+$ and $N_-$ act nilpotently on every finite dimensional module $M$ over the Lie algebra $L$.

**Proposition 2.** Let $L$ be an arbitrary $n$-dimensional ($n \geq 1$) solvable Lie algebra over an algebraically closed field $\mathbb{K}$ of zero characteristic, then there is a basic subalgebra $\mathcal{T}$ of $W_n(\mathbb{K})$, such that $\mathcal{T}$ is isomorphic to $L$.

**Proof.** Let $H$ be any Cartan subalgebra of $L$. Take a basis $\{l_1, \ldots, l_n\}$ of $L$ with the following property: $\{l_1, \ldots, l_m\}$ is a basis of $H$, and if $H \cap [L, L] \neq 0$, then $\{l_{k+1}, \ldots, l_n\}$ is a basis of $[L, L]$. Note that $[L, L]$ is a nilpotent ideal of $L$ because $L$ is solvable and the field $\mathbb{K}$ has zero characteristic; the subalgebra $H$ is nilpotent as a Cartan subalgebra of $L$.

Now, put $\varpi = \sum_{i=1}^k x_i \otimes l_i$. Consider the linear map $\phi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad} \varpi)^{-1}$ from the $R$-module $R \otimes \mathbb{K} H$ to itself (since $H$ is nilpotent, the sum is finite).

By Lemma 2, $\phi$ is an automorphism of the $R$-module $R \otimes \mathbb{K} H$. As $R \otimes \mathbb{K} (H \cap [L, L])$ is an ideal of the algebra $R \otimes \mathbb{K} H$, the map $\text{ad} \varpi$ saves the submodule $R \otimes \mathbb{K} (H \cap [L, L])$. The set of elements $\{-1 \otimes l_i\}, i = 1, \ldots, m$ is a basis of the $R$-module $R \otimes \mathbb{K} H$, so it is obvious that $\phi(-1 \otimes l_i), i = 1, \ldots, m$ is also a basis of $R \otimes \mathbb{K} H$. Further, the set $\{\phi(-1 \otimes l_i), i = k+1, \ldots, m\}$ is a basis of the submodule $R \otimes \mathbb{K} (H \cap [L, L])$. Therefore, it is clear that the elements $\phi(-1 \otimes l_i), i = 1, \ldots, m$ together form a basis of the $R$-module $R \otimes \mathbb{K} H$.

Put $d_p = \phi(-1 \otimes l_i), i = 1, \ldots, k$ and consider the automorphism $\theta = \exp \text{ad} w$ of the algebra $\hat{H}$. Then we get for $p = 1, \ldots, k$:

$$\theta \left( \frac{\partial}{\partial x_p} \right) = \frac{\partial}{\partial x_p} + \left[ w, \frac{\partial}{\partial x_p} \right] + \frac{1}{2!} \left[ w, \left[ w, \frac{\partial}{\partial x_p} \right] \right] + \ldots =$$

$$= \frac{\partial}{\partial x_p} - \frac{\partial w}{\partial x_p} - \frac{1}{2!} \left[ w, \frac{\partial w}{\partial x_p} + \ldots \right] = \frac{\partial}{\partial x_p} + \phi(-1 \otimes l_p) = \frac{\partial}{\partial x_p} + d_p.
$$

It follows from this that

$$\left[ \frac{\partial}{\partial x_p} + d_p, \frac{\partial}{\partial x_q} + d_q \right] = \theta \left( \frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q} \right) = \theta \left( \frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q} \right) = 0, \; p, q = 1, \ldots, k.$$ 

Consider the elements $d_p, p = 1, \ldots, k$ as elements of the algebra $R \otimes L$. Put $d_p = 0, p = k+1, \ldots, n$. It is clear, taking into account the choice of $\varpi$, that $d_p = 0, q = k+1, \ldots, n$. Therefore, $\left[ \frac{\partial}{\partial x_p} + d_p, \frac{\partial}{\partial x_q} + d_q \right] = 0, \; p, q = 1, \ldots, n$.

Now put $\varpi = \sum_{i=k+1}^n x_i \otimes l_i$. Consider the linear map $\psi$ from the $R$-module $R \otimes L$ to $R \otimes L$: $\psi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad} \varpi)^{-1}$. (The sum is finite, because $R \otimes [L, L]$ acts nilpotently on $R \otimes L$ and $u \in R \otimes [L, L]$.) By Lemma 2, $\psi$ is an automorphism of the $R$-module $R \otimes L$. It is obvious that the elements $d_p, p = 1, \ldots, k$ and $\varpi, i = k+1, \ldots, n$ form together a basis of the $R$-module $R \otimes L$. Note that the set $\{-1 \otimes l_i\}, i = k+1, \ldots, n$ is a basis of the $R$-module $R \otimes [L, L]$. Therefore the elements $\psi(-1 \otimes l_i), i = k+1, \ldots, n$ form a basis of the $R$-module $R \otimes L$ and the set $\psi(-1 \otimes l_p), p = k+1, \ldots, n$ is a basis.
of the $R$-submodule $R \otimes [L, L]$. Note, that $\psi(-1 \otimes l_p), p = k + 1, \ldots n$ is also a basis of the $R$-submodule $R \otimes [L, L]$. Then, the elements $\eta(d_p), p = 1, \ldots k$ and $\psi(-1 \otimes l_p), p = k + 1, \ldots n$ form together a basis of the $R$-module $R \otimes L$. Put $b_p = \eta(d_p), p = 1, \ldots k$ and $b_p = \psi(-1 \otimes l_p), p = k + 1, \ldots n$. Writing down $b_p = \sum_{i=1}^n (b_{pi} \otimes l_i)$, we have $\det(b_{pi})_{p,i=1}^n \in K^*$.

By construction of the element $\overline{u}$, it holds $\overline{u} \overline{\frac{\partial}{\partial x_p}} = 0$ for $p = 1, \ldots k$, so we have $e^{ad \overline{u}} \left( \frac{\partial}{\partial x_p} \right) = \frac{\partial}{\partial x_p}, p = 1, \ldots k$. Note that

\[
\frac{\partial}{\partial x_p} + b_p = \frac{\partial}{\partial x_p} + \eta(d_p) = e^{ad \overline{u}} \left( \frac{\partial}{\partial x_p} \right) + e^{ad \overline{u}}(d_p) = e^{ad \overline{u}} \left( \frac{\partial}{\partial x_p} + d_p \right) p = 1, \ldots k.
\]

It is easy to see, that for $p = k + 1, \ldots n$ it holds

\[-1 \otimes l_p = - \overline{\frac{\partial}{\partial x_p}} = \left[ \overline{u}, \frac{\partial}{\partial x_p} \right].\]

Hence, the relations

\[
\frac{\partial}{\partial x_p} + b_p = \frac{\partial}{\partial x_p} + \psi(-1 \otimes l_p) = \frac{\partial}{\partial x_p} + \left( E + \frac{1}{2!} (ad \overline{u}) + \frac{1}{3!} (ad \overline{u})^2 + \ldots \right) (-1 \otimes l_p) = \\
\frac{\partial}{\partial x_p} + (-1 \otimes l_p) + \frac{1}{2!} [\overline{u}, -1 \otimes l_p] + \frac{1}{3!} [\overline{u}, [\overline{u}, -1 \otimes l_p]] + \ldots = \\
\frac{\partial}{\partial x_p} + \left[ \overline{u}, \frac{\partial}{\partial x_p} \right] + \frac{1}{2!} \left[ \overline{u}, \overline{\frac{\partial}{\partial x_p}} \right] + \ldots = e^{ad \overline{u}} \left( \frac{\partial}{\partial x_p} \right).
\]

hold for $p = k + 1, \ldots n$. Since $d_p = 0$ for $p = k + 1, \ldots n$, we get

\[
\frac{\partial}{\partial x_p} + b_p = e^{ad \overline{u}} \left( \frac{\partial}{\partial x_p} + d_p \right).
\]

Thus,

\[
\left[ \frac{\partial}{\partial x_p} + b_p, \frac{\partial}{\partial x_q} + b_q \right] = \left[ e^{ad \overline{u}} \left( \frac{\partial}{\partial x_p} + d_p \right), e^{ad \overline{u}} \left( \frac{\partial}{\partial x_q} + d_q \right) \right] = \\
e^{ad \overline{u}} \left[ \frac{\partial}{\partial x_p} + d_p, \frac{\partial}{\partial x_q} + d_q \right] = 0.
\]

Therefore, by Lemma 4 there exists a basic subalgebra of $W_n(K)$ which is isomorphic to $L$.

\[\square\]

4. The main theorem

Lemma 4. Let $L$ be $n$-dimensional Lie algebra over an algebraically closed field $K$ of zero characteristic. Let $L = L_1 + L_2$, where $L_1, L_2$ are subalgebras of $L$ such that $L_1 \cap L_2 = \{0\}$, dim $L_1 = m < n$. Assume that the subalgebra $L_2$ acts nilpotently (by means of multiplication) on $L$, that is $(ad L_2)^k(L) = 0$ for some $k$. If there exists a basic subalgebra $\overline{L}_1$ of $W_n(K)$ such that $\overline{L}_1$ is isomorphic to $L_1$, then there exists a basic subalgebra of $W_n(K)$ such that $\overline{L}$ is isomorphic to the Lie algebra $L$. 
Proof. Take a basis \( \{l_1, \ldots, l_m\} \) of the subalgebra \( L_1 \) and a basis \( \{l_{m+1}, \ldots, l_n\} \) of the subalgebra \( L_2 \). Then the elements \( l_1, \ldots, l_n \) form a basis of \( L \). Denote by \( R_1 \) the subring \( \mathbb{K}[x_1, \ldots, x_m] \) of the polynomial ring \( R = \mathbb{K}[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n] \). Since there exists a basic subalgebra \( L_1 \) of \( W_m(\mathbb{K}) \) such that \( L_1 \) is isomorphic to \( L_1 \), by Lemma \( \beta \) there exists elements \( d_p = \sum_{i=1}^m d_{pi} \otimes l_i \in R_1 \otimes L_1 \), \( p = 1, \ldots, m \) such that for the polynomials \( d_{pi} \) the relation \( \beta \) holds with the structure constants of the algebra \( L_1 \) and \( \det(d_{pi})_{p,i=1}^n \in \mathbb{K}^* \). As \( L_1 \) is a subalgebra of \( L \), we have a natural embedding of the Lie algebra \( R_1 \otimes L_1 \) into \( R \otimes L \) (an element \( \overline{d}_p = \sum_{i=1}^m d_{pi} \otimes l_i \in R_1 \otimes K L_1 \) maps to the element \( d_p = \sum_{i=1}^m d_{pi} \otimes l_i \in R \otimes K L \), \( d_{pi} = 0 \) for \( i = m+1, \ldots, n \)). Put \( d_p = 0 \) for \( p = m+1, \ldots, n \). Using Remark \( \beta \) it is easy to see that the following relations hold in \( L \):

\[
[\frac{\partial}{\partial x_p} + \overline{d}_p, \frac{\partial}{\partial x_q} + d_q] = 0, \quad p, q = 1, \ldots, n.
\]

Put \( \overline{b}_p = \sum_{i=m+1}^n x_i \otimes l_i \in R \otimes L_2 \). By the assumption for the subalgebra \( L_2 \), the derivation \( \text{ad} \overline{b} \) of the Lie algebra \( R \otimes L \) is nilpotent and therefore the automorphism \( \theta = \exp(\text{ad} \overline{b}) \) of the Lie algebra \( R \otimes L \) (and \( L \)) is well defined. Denote \( \overline{b}_p = -\frac{\partial}{\partial x_p} + \theta \left( \frac{\partial}{\partial x_p} + \overline{d}_p \right) \), \( p = 1 \ldots n \). Then \( \frac{\partial}{\partial x_p} + \overline{b}_p = \theta \left( \frac{\partial}{\partial x_p} + \overline{d}_p \right) \), and the following equalities hold:

\[
[\frac{\partial}{\partial x_p} + \overline{b}_p, \frac{\partial}{\partial x_q} + \overline{d}_q] = \theta \left( \frac{\partial}{\partial x_p} + \overline{d}_p \right), \theta \left( \frac{\partial}{\partial x_q} + \overline{d}_q \right) = 0, \quad p, q = 1 \ldots n.
\]

Let us show that the set of the elements \( \overline{b}_p, p = 1, \ldots n \) is a basis of the free \( R \)-module \( R \otimes L \). It holds

\[
\overline{b}_p = -\frac{\partial}{\partial x_p} + \theta \left( \frac{\partial}{\partial x_p} + \overline{d}_p \right) = -\frac{\partial}{\partial x_p} + \theta \left( \frac{\partial}{\partial x_p} \right) + \theta (\overline{d}_p), \quad p = 1, \ldots, m
\]

and then

\[
\theta \left( \frac{\partial}{\partial x_p} \right) = \left( E + \text{ad} \overline{b} + \frac{1}{2!} (\text{ad} \overline{b})^2 + \ldots \right) \left( \frac{\partial}{\partial x_p} \right) = \frac{\partial}{\partial x_p} + \left[ \overline{b}, \frac{\partial}{\partial x_p} \right] + \left[ \overline{b}, \left[ \overline{b}, \frac{\partial}{\partial x_p} \right] \right] + \ldots = -\frac{\partial}{\partial x_p},
\]

since \( \left[ \overline{b}, \frac{\partial}{\partial x_p} \right] = -\frac{\partial}{\partial x_p} = 0 \) for \( p = 1, \ldots, m \). Now consider the elements \( \overline{b}_p \) for \( p = m+1, \ldots, n \). In this case \( \overline{d}_p = 0 \). Therefore

\[
\overline{b}_p = -\frac{\partial}{\partial x_p} + \theta \left( \frac{\partial}{\partial x_p} + \overline{d}_p \right) = -\frac{\partial}{\partial x_p} + \left( E + \text{ad} \overline{b} + \frac{1}{2!} (\text{ad} \overline{b})^2 + \ldots \right) \left( \frac{\partial}{\partial x_p} \right) = -\frac{\partial}{\partial x_p} + \frac{\partial}{\partial x_p} + \left[ \overline{b}, \frac{\partial}{\partial x_p} \right] + \left[ \overline{b}, \left[ \overline{b}, \frac{\partial}{\partial x_p} \right] \right] + \ldots = -\frac{\partial}{\partial x_p} - \frac{1}{2!} \left[ \overline{b}, \frac{\partial}{\partial x_p} \right] - \frac{1}{3!} \left[ \overline{b}, \left[ \overline{b}, \frac{\partial}{\partial x_p} \right] \right] = \sum_{i=1}^\infty \frac{1}{i!} (\text{ad} \overline{b})^{i-1} \left( -\frac{\partial}{\partial x_p} \right).
\]
(Because of nilpotency of \( \text{ad} \overline{b} \), the number of nonzero summands in this series is finite).

Denote \( \phi = \sum_{i=1}^{\infty} \frac{1}{i!} (\text{ad} \overline{b})^{i-1} \). It is easy to see that \( \phi \) is an automorphism of the free \( R \)-module \( R \otimes L \) (see Lemma \( 2 \)). Since \( \overline{b} = \sum_{i=m+1}^{n} x_i \otimes l_i \in R \otimes L_2 \), the \( R \)-module \( R \otimes L_2 \) is invariant under action of \( \phi \).

The set of elements \( \overline{b}_p, p = 1, \ldots, m \) and \( -\frac{\partial}{\partial x_p} = -1 \otimes l_p, \ p = m + 1, \ldots, n \) is a basis of the free \( R \)-module \( R \otimes L \) (because this module is the direct sum of the \( R \)-modules \( R \otimes L_1 \) and \( R \otimes L_2 \)). Then, the elements \( \theta \left( -\frac{\partial}{\partial x_p} \right), p = m + 1, \ldots, n \) form a basis of \( R \otimes L \). Since \( \phi \) is an automorphism of the free \( R \)-module \( R \otimes L_2 \), the set of elements \( \phi \left( -\frac{\partial}{\partial x_p} \right), p = m + 1, \ldots, n \) is a basis of this submodule (note that the set of elements \( \theta \left( -\frac{\partial}{\partial x_p} \right), p = m + 1, \ldots, n \) is also a basis of \( R \otimes L_2 \)). It follows from this that the elements \( \theta \left( \overline{b}_p \right), p = 1, \ldots, m \) and \( \phi \left( -\frac{\partial}{\partial x_p} \right), p = m + 1, \ldots, n \) together form a basis of the free \( R \)-module \( R \otimes L \). Then, using the equalities \( \overline{b}_p = \theta \left( \overline{b}_p \right), p = 1, \ldots, m \) and \( \overline{b}_p = \phi \left( -\frac{\partial}{\partial x_p} \right), p = m + 1, \ldots, n \), we see that \( \{ \overline{b}_p \}, p = 1, \ldots, n \) is a basis of the free \( R \)-module \( R \otimes L \), and therefore \( \det(b_{pi})^{ij}_{p,i=1} \in \mathbb{K}^* \). Hence, by Lemma \( 4 \) there exists a basic subalgebra \( \widehat{L} \) of \( W_n(\mathbb{K}) \) such that \( \widehat{L} \) is isomorphic to the Lie algebra \( L \).

\[\Box\]

Now we can prove the main theorem of this paper.

**Theorem 2.** Let \( L \) be any Lie algebra \( L \) over \( \mathbb{K} \) of dimension \( n \geq 1 \). Then there exists a basic subalgebra \( \overline{L} \) of \( W_n(\mathbb{K}) \) such that \( \overline{L} \) is isomorphic to \( L \).

**Proof.** By Proposition \( 2 \) we may assume that \( L \) is not solvable. Let \( S = S(L) \) be the solvable radical of \( L \), and \( L = L_0 \ltimes S \) be the Levi decomposition of \( L \), where \( L_0 \) is a semisimple subalgebra of \( L \). Let \( H_0 \subseteq L_0 \) be a Cartan subalgebra of \( L_0 \) and let \( L = N_\cdot \oplus H_0 \oplus N_\cdot \) be the corresponding triangular decomposition for some choice of simple roots. Denote by \( B_0 \) the Borel subalgebra \( B_0 = H_0 + N_\cdot \) of \( L_0 \). Then the subalgebra \( L_1 = S + B_0 \) is solvable, therefore by Proposition \( 2 \) there exists a basic subalgebra of \( W_k(\mathbb{K}) \), which is isomorphic to \( \overline{L}_1 \), where \( k = \dim L_1 \). Since \( L = L_1 \oplus N_\cdot \), \( L_1 \cap N_\cdot = \{0\} \), and the subalgebra \( N_\cdot \) acts nilpotently (by multiplication) on the \( L_0 \)-module \( L \) (see Lemma \( 3 \)), there exists by Lemma \( 4 \) a basic subalgebra \( \overline{L} \) of \( W_n(\mathbb{K}) \) such that \( \overline{L} \) is isomorphic to the Lie algebra \( L \).

\[\Box\]

**Example 2.** Let \( L = sl_2(\mathbb{K}) \) and \( \{ E, H, F \} \) be its standard basis over \( \mathbb{K} \). We shall construct a basic subalgebra of \( W_3(\mathbb{K}) \), which is isomorphic to \( L \). As the Cartan subalgebra \( \langle H \rangle \) of \( L \) is one-dimensional, the basic subalgebra \( \langle -\frac{\partial}{\partial x_1} \rangle \) of \( W_1(\mathbb{K}) \) is isomorphic to \( \langle H \rangle \). To the element \( H \in L \) corresponds the element \( 1 \otimes H \) in the Lie algebra \( \widehat{L} \) (see the Remark \( 1 \)). The subalgebra \( N_\cdot \) has obviously generators \( E \) and \( [H, E] = 2E \). Put \( w = x_2 \otimes E \). Therefore, we have

\[
e^{\text{ad}(x_2 \otimes E)} \left( \frac{\partial}{\partial x_1} - 1 \otimes H \right) = \frac{\partial}{\partial x_1} + 2x_2 \otimes E - 1 \otimes H
\]
ON NONCOMMUTATIVE BASES OF THE FREE MODULE $W_n(\mathbb{K})$

$e^{ad_{x_2 \otimes E}} \left( \frac{\partial}{\partial x_2} \right) = \frac{\partial}{\partial x_2} - 1 \otimes E.$

Further, $N_-$ has generators $F$ and $[H, F] = -2F$, $[E, F] = H$. Analogously we obtain

$e^{ad_{(x_3 \otimes F)}} \left( \frac{\partial}{\partial x_1} + 2x_2 \otimes E - 1 \otimes H \right) = \frac{\partial}{\partial x_1} + 2x_2 \otimes E - (1 + 2x_2x_3) \otimes H - (2x_3 + 2x_2x_3^2) \otimes F$

$e^{ad_{x_3 \otimes H}} \left( \frac{\partial}{\partial x_2} - 1 \otimes E \right) = \frac{\partial}{\partial x_2} - 1 \otimes E + x_3 \otimes H + x_3^2 \otimes F$

$e^{ad_{x_3 \otimes F}} \left( \frac{\partial}{\partial x_3} \right) = \frac{\partial}{\partial x_3} - 1 \otimes F.$

Then, using the inverse matrix to the matrix of these elements, we get a basis of $W_3(\mathbb{K})$. The linear span of this basis over $\mathbb{K}$ is isomorphic to the Lie algebra $L = sl_2(\mathbb{K})$:

$E = -x_3 \frac{\partial}{\partial x_1} + (1 + 2x_2x_3) \frac{\partial}{\partial x_2} - x_3^2 \frac{\partial}{\partial x_3}$

$H = \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} + 2x_3 \frac{\partial}{\partial x_3}, \quad F = \frac{\partial}{\partial x_3}.$

Remark 2. Since the set of all linear homogeneous derivations of $W_n(\mathbb{K})$ form a Lie algebra, which is isomorphic to $gl_n(\mathbb{K})$, there are embeddings of any $n$-dimensional Lie algebra $L$ without center into $W_n(\mathbb{K})$. But for the image $\mathcal{L}$ of $L$ by such an embedding in $W_n(\mathbb{K})$ and a basis $\{D_1, \ldots, D_n\}$ of $\mathcal{L}$ such that $D_i = \sum_{j=1}^{n} f_{ij} \frac{\partial}{\partial x_j}$, the determinant $\det(f_{ij})$ is not a constant. Therefore $\mathcal{L}$ is not a basic Lie subalgebra of $W_n(\mathbb{K})$.

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