Some algebraic notions related to analysis on metric spaces

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Abstract

There are versions of “calculus” in many settings, with various mixtures of algebra and analysis. In these informal notes we consider a few examples that suggest a lot of interesting questions.

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1 Functions on spaces

Let $X$ be a topological space, and let $\mathcal{C}(X)$ be the algebra of continuous real-valued functions on $X$. For each $p \in X$, the collection $I_p(X)$ of $f \in \mathcal{C}(X)$ such that $f(p) = 0$ is an ideal in $\mathcal{C}(X)$, since the product of two functions $f_1, f_2 \in \mathcal{C}(X)$ vanishes at $p$ if $f_1(p) = 0$ or $f_2(p) = 0$.

If $X$ is the $n$-dimensional Euclidean space $\mathbb{R}^n$ and $f$ is a continuously-differentiable function which vanishes at $p$, then

$$f(x) = O(|x - p|)$$

for $x$ near $p$. A similar statement holds for functions on smooth manifolds.
In general, a function \( f \in I_p(X) \) does not have to vanish at \( p \) at any particular rate. One might specify a rate at which a function vanishes with a condition like (1.1). On a metric space, one might work with functions in a Lipschitz class, for instance, which implies such a condition.

If \( f \) is a polynomial on \( \mathbb{R}^n \) such that \( f(p) = 0 \), then there are polynomials \( f_1, \ldots, f_n \) which satisfy

\[
f(x) = (x_1 - p_1) f_1(x) + \cdots + (x_n - p_n) f_n(x),
\]

where \( x_1, \ldots, x_n \) are the standard coordinates of \( x \in \mathbb{R}^n \). Of course, this implies (1.1).

This simple algebraic statement works just as well for polynomials with coefficients in any field. One can also consider rational functions where the denominators are nonzero.

If \( f \) is a continuously-differentiable function on \( \mathbb{R}^n \), then

\[
f(x) = o(|x - p|)
\]

if and only if \( f(p) = 0 \) and \( df_p = 0 \), where \( df_p \) denotes the differential of \( f \) at \( p \). If \( f \) is continuously-differentiable of order two, then it follows that

\[
f(x) = O(|x - p|^2).
\]

For any continuously-differentiable function \( f \) on \( \mathbb{R}^n \),

\[
f(x) - f(p) - df_p(x - p) = o(|x - p|),
\]

and

\[
f(x) - f(p) - df_p(x - p) = O(|x - p|^2)
\]

when \( f \) is continuously differentiable of order two. Thus the difference between vanishing to first order at \( p \) and vanishing more quickly is exactly described by the differential of \( f \) at \( p \).

For a polynomial \( f \) in \( n \) variables, the idea that \( f \) vanishes at \( p \) to second order can be expressed algebraically by

\[
f(x) = \sum_{j,l=1}^n (x_j - p_j) (x_l - p_l) f_{j,l}(x),
\]

where \( f_{j,l} \) are also polynomials.

On any metric space, it makes sense to talk about a function vanishing to first order, more quickly than first order, or second order at a point, as in (1.1), (1.3), and (1.4). Depending on the circumstances, there may be nice classes of regular functions with some kind of derivatives, suitable versions of Taylor’s theorem, and so on.

The real and complex numbers are equipped with metrics for which algebraic and analytic notions of vanishing to some order are compatible. There is a similar metric on the \( p \)-adic numbers.
There are translation-invariant metrics on $\mathbb{R}^n$ associated to nonisotropic dilations which lead to different measurements of vanishing, smoothness, and degrees of polynomials. This can be extended further to left-invariant metrics on nilpotent Lie groups. Just as a smooth manifold is approximately like a Euclidean space locally, and thus has a version of calculus, there are versions of calculus for sub-Riemannian spaces connected to those for nilpotent Lie groups.

2 Exponentiation

The classical exponential function on the real line can be defined by the power series

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \quad (2.1)$$

It can also be characterized by the differential equation $f' = f$ with the condition $f(0) = 1$.

This makes sense as a formal power series with coefficients in any field of characteristic 0 too. The radius of convergence on the $p$-adic numbers is positive and can be determined explicitly.

On an open set in $\mathbb{R}^n$, or any smooth manifold, one can consider a vector field $V$ with smooth coefficients. One can view this as a derivation on the algebra of smooth functions on the underlying open set or manifold.

Well-known results about systems of ordinary differential equations imply the local existence and uniqueness of integral curves for $V$. Under suitable conditions, one has global solutions.

One can also look at this in terms of the partial differential equation

$$\frac{\partial}{\partial t} f = V f, \quad (2.2)$$

where $f(x, t)$ is a smooth function of $x$ in the open set or manifold and $t \in \mathbb{R}$.

If there are global solutions, as in the case of a compact manifold without boundary, then one is basically exponentiating $V$ as a derivation on the algebra of smooth functions on the underlying space to get a one-parameter group of automorphisms of the algebra. These automorphisms can be described by composing smooth functions on the space with diffeomorphisms, where the diffeomorphisms correspond to the flow determined by the vector field.

A vector field on a sub-Riemannian space may be considered as a vector field on an ordinary underlying manifold. However, there are additional conditions for compatibility with the sub-Riemannian structure.

Integrating vector fields on other metric spaces can be a fascinating enterprise.

For any linear transformation $L$ on a vector space $A$, one can try to exponentiate $L$ or scalar multiples of $L$ to get invertible linear transformations on $A$. If $A$ is an algebra and $L$ is a derivation, then the exponential ought to be an automorphism of $A$. 

3
As in the case of the classical exponential function, one can try to approach exponentiation of linear transformations in terms of power series or differential equations. If $A$ is a vector space over a field of characteristic 0, then

$$\exp(t L) = \sum_{n=0}^{\infty} \frac{t^n L^n}{n!} \tag{2.3}$$

makes sense as a formal power series in $t$ with coefficients in linear transformations on $A$. If $A$ is a Banach space and $L$ is a bounded linear transformation on $A$, then this series converges absolutely for all $t$ in the Banach space of bounded linear transformations on $A$.

One might think of a vector field as a continuous linear transformation on a topological vector space, or as an unbounded linear transformation on a subspace of a Banach space. In this case, one normally approaches the exponential in terms of differential equations.

3 Buildings and symmetric spaces

Classical symmetric spaces are homogeneous spaces obtained from quotients of semi-simple Lie groups in a special way. They are quite interesting in particular for actions on them by discrete subgroups of semisimple Lie groups.

Buildings are generalizations of symmetric spaces with nice properties for actions by broader classes of discrete groups. Some of these discrete groups are related to $p$-adic matrix groups in much the same way as for real Lie groups in the classical case.

In the classical situation, the boundary of a rank 1 symmetric space of negative curvature is related to ordinary Euclidean geometry and the geometry of sub-Riemannian spaces in a natural way. In [11], it is shown that the boundaries of certain buildings are fractal spaces which enjoy Poincaré inequalities similar to those on Euclidean and sub-Riemannian spaces. These fractal boundary spaces have topological dimension 1 and are far from being manifolds even topologically.

Of course, one can look at versions of calculus on discrete structures too, on discrete groups in particular.

4 Locally compact groups

Locally compact topological groups are very attractive as general objects on which to consider basic notions of harmonic analysis. In particular, one always has Haar measure and convolutions. However, the solution of Hilbert’s fifth problem shows that such a group is a Lie group under some conditions related to connectivity and finite-dimensionality. Thus, there is so much structure that something relatively close to the classical situation may very well already be in the classical situation.
5 Holomorphic functions

A basic advantage of holomorphic functions as compared to harmonic functions is that the product of two holomorphic functions is holomorphic. This leads to algebras of holomorphic functions, modules over such algebras, and so on. Local versions of this are closely connected to formal power series.

One can also look at holomorphic functions on suitable classes of complex analytic metric spaces. Let us suppose that there is some local regularity for holomorphic functions. It seems natural to ask about analogues of power series expansions or related properties. Some information may follow from appropriate elliptic analysis.

6 Quasiconformal mappings

In the theory of quasiconformal mappings in the plane, there are remarkable existence results for mappings with prescribed dilatation. These results imply that perturbations of the classical Cauchy–Riemann equations are in a sense equivalent to the original ones. Specifically, holomorphic functions with respect to perturbed equations can be expressed as compositions of ordinary holomorphic functions with quasiconformal mappings under suitable conditions. This probably does not work very well in general for complex analytic metric spaces, even when the complex dimension is equal to 1.

7 Uniform algebras

It seems natural to take a fresh look at deformation theory of uniform algebras, as in [37, 38, 39, 40, 41, 58, 59, 60, 61], in connection with complex analytic metric spaces.

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