Odd-\(J\) Pairing in Nuclei

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We point out a simplicity that arises when we use an interaction in which only an energy with odd \(J\) is non-zero. The emphasis is on \(J = J_{\text{max}}\) and in particular \(J = 9^+\) in the \(g_{9/2}\) shell. It is noted that high overlaps can be deceptive. In many cases a single set of unitary 9-j coefficients gives either an exact or a very good approximation to the wave function of a non-degenerate state. The many degeneracies that occur in these calculations are discussed and explained.

I. INTRODUCTION

The purpose of this work is to study the properties of a very simple interaction in a single-\(j\)-shell model space of neutrons and protons. A single proton–neutron (\(pn\)) pair in this space can have a total angular momentum from \(J = 0\) to \(J = J_{\text{max}} = 2J\). The even-\(J\) states have isospin \(T = 1\), i.e. they are members of an isoskillet—there are analog states of two neutrons and of two protons with these even angular momenta. The odd-\(J\) states have isospin \(T = 0\), i.e. isosinglet—in this model space, they only are present in the \(pn\) system.

There has been much study in journals and in textbooks of the \(J = 0\), \(T = 1\) “pairing interaction”, i.e. where only \(J = 0\) two-body matrix elements are non-zero (and attractive) \([1–5]\). Although the pairing interaction is not realistic, the conclusions of these studies has yielded results whose importance was well beyond expectation. Examples are seniority classifications, reduced isospin and indeed these studies can be regarded as precursors to the BCS theory in condensed matter theory. In turn the BCS theory is important in the nuclear context for explaining moments of inertia of deformed states. In the \(pn\) system, it is known that not only the \(J = 0\) two-body matrix element lies low but also \(J = 1\) and \(J = J_{\text{max}} = 2J\). As a counterpoint to the \(J = 0\) paring interaction, we will here consider \(J = J_{\text{max}}\) pairing interaction, for which \(V(2J)\) is non-zero and attractive and all other two-body matrix elements are zero. Such an interaction only acts between a neutron and a proton, not between two identical particles. This study will reveal some surprising results.

It should be noted that there has been a recent flurry of interest in \(J_{\text{odd}}\) pairing and in particular the case where \(J_{\text{odd}}\) is equal to \(J_{\text{max}}\). In a recent work, C. Qi et al. \([6]\) proposed a wave function for \(^{96}\text{Cd}\) in which each of two \(pn\) pairs couple to maximum angular momentum, which for the \(g_{9/2}\) shell is \(J = 9\). More generally, they consider the overlap of a wave function in which one \(pn\) pair couples to \(J_1\) and the other to \(J_2\), with basis states of two protons coupling to \(J_p\) and two neutrons to \(J_n\). In the latter basis, the effect of the Pauli principle is much clearer. Once one has antisymmetric wave functions of two protons, which is achieved by limiting the angular momenta to even \(J_n\), and likewise the two neutrons to even \(J_n\), one has a wave function which satisfies the Pauli principle. For the former wave function \(\langle pn\rangle J_1 \langle pn\rangle J_2\rangle\text{'}\), one does have to antisymmetrize and a priori things look complicated. The authors, however, make the statement “The overlap matrix automatically takes into account the Pauli principle”. Here the overlap matrix element is

\[
O^I(J_1J_2; J_pJ_n) = \langle \langle pn\rangle J_1 \langle pn\rangle J_2\rangle\text{'} | \langle pp\rangle J_p nn(J_n)\rangle\text{'} \rangle
\]

that is, the unitary 9-j symbol, which we will here call \(U9-j\).

We here note that work we previously did shows that certain \(U9-j\) coefficients form components of the \(J = 0^+\) ground state wave function of a \(J_{\text{odd}}\) pairing interaction. We here expand on this work. We feel our method is very simple to understand. We refer to a previous work by E. Moya de Guerra et al. \([7]\) and explicitly to Eqs. (70) and (74). In that work we describe the wave functions in a \(\langle pp\rangle J_p \langle mn\rangle J_n\) basis. The Pauli principle is easily satisfied by constraining \(J_p\) and \(J_n\) to be even. Our previous example was \(^{44}\text{Ti}\), but the same mathematics holds for \(^{96}\text{Cd}\). We previously considered various schematic interactions as well and the more realistic MBZ interaction taken from experiment \([8]\), for which detailed wave functions were subsequently published in the archives \([9]\) (with some modification of the two-body matrix elements). In this work we use the symbol \(I\) for the total angular momentum of the state and \(J\) otherwise.

II. CALCULATIONS

We now proceed to the calculations. We will be extensively using these two relations for the \(U9-j\)’s:

\[
\sum_{J_{13}J_{24}} \langle (jj)J_{13}(jj)J_{24}\rangle x \times \delta_{a,c}\delta_{b,d} = \delta_{a,c}\delta_{b,d}
\]

(2)

\[
\sum_{J_{13}J_{24}} (-1)^s \langle (jj)J_{12}(jj)J_{24}\rangle x \times \langle (jj)J_{13}(jj)J_{24}\rangle = \langle (jj)J_{12}(jj)J_{24}\rangle
\]

(3)

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where $s = J_{24} + J_{23} - J_{34} - 1$.

Let us first consider the $I = 0^+$ states in $^{96}$Cd (or $^{44}$Ti). For most interactions, the diagonalization is a fairly complicated procedure. However for certain interactions it is much easier. For example, the interaction used in Ref. [7] was one in which all two-body matrix elements were set equal to zero except for the $J = 1$, $T = 0$ two-body matrix element. In such a case, the matrix element of the secular four-particle Hamiltonian factorizes (the same result holds for any odd-$J$ interaction). This is the key point. The basis states are $|J_p J_p' J_{odd} J_{odd}'\rangle$. We have

$$H_{J_p J_p'} = E(J_{odd})f(J_p)f(J_p')$$

where $f(J_p)$ is twice the $U9-j$ symbol:

$$f(J_p) = 2(2J_p + 1)(2J_{odd} + 1)\frac{j j j J_p J_p}{J_{odd} J_{odd} 0}$$

If we write the wave function as $\sum X_{J_p J_p'} X_{J_p}^{J_{odd}}$ (as in Ref. [3]), then it was shown in Ref. [7] that $X_{J_p}^{J_{odd}}$ is proportional to $f(J_p)$. The other eigenstates are degenerate and, if $E(J_{odd})$ is negative, they are at higher energies. In other words, what we have shown in Ref. [7] is that the wave-function components $X_{J_p J_p'}$ of the lowest $I = 0^+$ state are proportional to the overlap factor of Ref. [3]; alternately, they are equal within a normalization to the $U9-j$ coefficients.

The eigenvalue is given by

$$E(I = 0^+) = E(J_{odd})\sum f(J_p)X_{J_p J_p}^2$$

Note that our very simple interactions are charge independent. This means that the lowest (non-degenerate) $I = 0^+$ state has good isospin, presumably $T = 0$. It is amusing that we can assign the isospin quantum number to a wave function with $U9-j$ coefficients.

In Table I we present the wave functions for the following interactions:

| $I^+$ | CCGI | E(0) | E(9) | E(0,9) | E(1) |
|-------|------|------|------|--------|------|
| $X_{00}$ | 0.7725 | 0.8563 | 0.6164 | 0.8103 | 0.2903 |
| $X_{22}$ | 0.5280 | 0.1741 | 0.7518 | 0.4814 | 0.5704 |
| $X_{34}$ | 0.2915 | 0.2335 | 0.2385 | 0.2514 | 0.5190 |
| $X_{66}$ | 0.1704 | 0.2807 | 0.0233 | 0.1718 | 0.1586 |
| $X_{88}$ | 0.1020 | 0.3210 | 0.0005 | 0.1831 | -0.5540 |

respectively 0.9020, 0.9467, 0.9944, and 0.6484. We find that E(9) gives higher overlap than the much studied E(0) pairing interaction and a much higher overlap than E(1). This might lead one to believe that the idea of $J = 9^+$ pairing is a valid concept. But overlaps can be deceiving. We also present E(0,9), where the only non-vanishing matrix elements are for $J = 9^+$ and $0^+$, both set to $-2.0000$ MeV. Now the overlap is even higher—0.9944. This might not be startlingly different than 0.9467, but let us now look at the energies of the lowest even-$I$ states in Table II. They are given respectively for interactions CCGI, E(9), and E(0,9). The results of the second column (CCGI) were previously given [11] and the point was made that the $I = 16^+$ state is isomeric since it lies below the lowest $14^+$ and $15^+$ states. This is in agreement with experiment [12].

| $I^+$ | CCGI | E(0) | E(9) | E(0,9) |
|-------|------|------|------|--------|
| $0^+$ | 0.0000 | 1.0587 | 0.0000 |
| $2^+$ | 1.0812 | 1.0589 | 1.2740 |
| $4^+$ | 2.1096 | 1.0588 | 1.8584 |
| $6^+$ | 2.8883 | 1.0588 | 2.3929 |
| $8^+$ | 3.2302 | 1.0571 | 2.5125 |
| $10^+$ | 4.8815 | 1.0464 | 3.2142 |
| $12^+$ | 5.3394 | 0.9670 | 3.1348 |
| $14^+$ | 5.4031 | 0.8563 | 2.8247 |
| $16^+$ | 5.2247 | 0.8563 | 2.1678 |

We see that despite the 0.9467 overlap, the even-$I$ spectrum for Int1 in which only the $J = 9^+$ matrix element is non-zero is drastically different than CCGI. First of all, the ground state does not have $I = 0$, rather it has $I = J_{max} = 16$ and indeed the two spectra seem to have nothing to do with each other.

Let us briefly digress and look at the spectrum for Int1 for its own sake. It is quite remarkable. The energies of the $I = 0, 2, 4, 6,$ and $8$ states are very close to each other, differing at most by 0.002 MeV and the $I = 10^+$ state is 0.012 lower. All six states are essentially degenerate. Then there is a drop in energy with $I = 16^+$ becoming the ground state. Such a strange spectrum and this for an interaction that gives a 0.9467 overlap with a realistic
interaction for the $I = 0^+$ state.

In the last column of Table III we improve things by also lowering the $J = 0^+$ matrix element to the same value as for $J = 9^+$, $-2.0000$ MeV. The spectrum is better, with $I = 0^+$ now the lowest state, but it is far from satisfactory. Even an overlap exceeding 0.99 does not guarantee overall good results. Clearly all two-body matrix elements come into play.

As noted above, the eigenfunction of the lowest $I = 0^+$ state for the $E(9)$ interaction is $N|(j^2)^I_J(j^2)^h_Jn⟩|(j^2)^9_J⟩$. It can be shown that the normalization factor is $\sqrt{2}$. For the $I = 1^+$ states with the $E(9)$ interaction, the secular matrix is also separable. This is not true for other values of $J_{odd}$. If we were to replace $I = 0$ by $I = 1$ in the above expression, all the $U9_j$ coefficients would vanish. We must make a different choice. The eigenfunction of the lowest $I = 1^+$ state is then given by a single set of $U9_j$ coefficients: $2|(j^2)^I_J(j^2)^h_Jn⟩|(j^2)^9_J⟩$. This state has isospin $T = 1$. Indeed all $I = 1^+$ states in this model space have isospin $T = 1$. The other four $I = 1^+$ states are degenerate at a higher energy.

For states with $I = 2$ or higher, the secular matrix is no longer separable—rather it is a sum of separable terms. The eigenvalue equation is

$$4 \sum_{J_J} \frac{((j^2)^I_{J}(j^2)^h_{J}n)(j^2)^9_{J}I} \times \sum_{J'_J'} \frac{((j^2)^I_{J'}(j^2)^h_{J'})(j^2)^9_{J'}I}D(J_{J'}, J_{J'}) = \lambda D(J_{J}, J_{J}) \tag{7}$$

For $I = 2^+$ there are two terms corresponding to $J_x = 7$ and 9; for $I = 3^+$ the values are $J_x = 6$ and 8, etc.

Despite the complexity of the above equation, there are some surprising results. The eigenfunction components of the lowest $2^+$ state are numerically extraordinarily close to the single $U9_j$ symbols $\sqrt{2}\langle j^2 \rangle^I_J\langle j^2 \rangle^h_J\langle j^2 \rangle^9_JI = 2$. Furthermore, this $2^+$ state has also components exceedingly close to $2\langle j^2 \rangle^I_J\langle j^2 \rangle^h_J\langle j^2 \rangle^9_JI = 2$. This is by no means obvious because, as mentioned above, the interaction involves a sum of two separable terms corresponding to $J_x = 7$ and 9. We can explain this result by observing the sum over even $J_p$ and even $J_n$. We first note schematically

$$4 \sum_{J_p J_n} = \sum (1 + (-1)^{J_p})(1 + (-1)^{J_n}) = \sum (-1)^{J_p} + \sum (-1)^{J_n} + \sum (-1)^{J_p + J_n} \tag{8}$$

The first term vanishes because of Eq. (2). In the last term one of the $U9_j$’s has two rows that are the same, which means that the only non-vanishing terms in the sum have $(J_p + J_n)$ even. Thus, the last term is the same as the first term—zero. The two middle terms are the same, so we get

$$\sum_{even J_p J_n} = \frac{1}{2} \sum ((-1)^{J_p}(j^2)^I_J(j^2)^h_Jn)(j^2)^9_JI = 2 \times \times (j^2)^I_Jn|(j^2)^9_JI = 2 = -\frac{1}{2}(j^2)^9_JI = 2 \times (j^2)^9_JI = 2 \tag{9}$$

We obtain the above by using two orthogonality relations for $9_j$-symbols as shown e.g. in Nuclear Shell Theory (p. 516) by de-Shalit and Talmi [3]. We call the right-hand side of Eq. (9) the overlap.

Using similar arguments, one can show that the normalization for the $[9, 9], N$, is such that

$$N^{-2} = \frac{1}{2} - \frac{1}{2}(j^2)^9_JI = 2 = 0.49993950935 \tag{10}$$

For the $[9, 7]$ case, we obtain

$$N^{-2} = \frac{1}{4} - \frac{1}{2}(j^2)^9_JI = 2 = 0.25037626385 \tag{11}$$

To get this latter result, we use the following relationship

$$\sum_{(-1)^{J_p + J_n}}|(j^2)^9_JI = 2 = 0 \tag{12}$$

Therefore, we obtain that the overlap is exceedingly small for the $g9/2$ shell. From Eqs. (10) and (11), we find that the normalizations are 1.414222 and 1.998497, the latter slightly smaller than 2. The overlap in Eq. (11) is 0.0009113 and, if we include the normalization factors, we get 0.00025765.

In lower shells the deviations are larger. For example, in the $d5/2$ shell, we replace $[9, 9]$ by $[5, 5]$ and $[9, 7]$ by $[5, 3]$. The overlap is now 0.0107, small but not zero. The norms are no longer $\sqrt{2}$ and 2, but rather 1.4161 and 1.9204. With normalized states the overlap is 0.0290. The corresponding numbers in the $f7/2$ shell are 0.001037, 1.41434, 1.9875 and 0.002916. The overlap is here smaller than in the $d5/2$ shell, but it is not zero. As one can see above in the $g9/2$ shell, the overlap is an order of magnitude smaller than in the $f7/2$ shell. We can speculate that the overlap might vanish in the infinite-$j$ limit.

We can see in Table III that the results for matrix diagonalization for both $I = 2^+$ states yield wave function components which are very close to the normalized $U9_j$ coefficients. In fact, they are so close that one could wonder if they are exactly the same. But they are not. As seen in Eq. (9), the two $U9_j$ sets corresponding to $[9, 9]$ and $[9, 7]$ are very nearly orthogonal, but not quite.

The above $I = 2$ states have isospin $T = 0$. The $E(9)$ interaction also yields a $T = 1$ non-degenerate state with
components $2|(jj^9)^3(jj)^2(JJ^9)^2|$. This is a pure state—it does not mix with any other $T = 1$ state. This is because there is only one way of forming an $I = 2$, $T = 1$ state from $U9$-$j$ symbols, i.e. only one possible $J_x$.

For $I = 3$, $T = 0$ there is also a pure state $2|(jj^9)^3(jj)^2(JJ^9)^3|$. This wave function changes sign under the interchange of $J_p$ and $J_n$. It cannot admix with a state with $J_x = 8$ for which there is no change of sign when $J_p$ and $J_n$ are interchanged.

Table III: Comparison for the first two $I = 2^+$ states of the matrix diagonalization with the $E(9)$ interaction and with normalized $U9$-$j$ components. We give the energy in MeV in the second row.

| $[J_p, J_n]$ | $E(9)$  | $U9$-$j$   | $E(9)$  | $U9$-$j$   |
|---------------|----------|------------|----------|------------|
| $[0, 2]$      | 0.5334   | 0.5338     | 0.1349   | 0.1351     |
| $[2, 2]$      | $-0.4707$| $-0.4708$  | 0.5569   | 0.5567     |
| $[2, 4]$      | 0.3035   | 0.3035     | 0.3188   | 0.3189     |
| $[4, 4]$      | $-0.1388$| $-0.1390$  | 0.6300   | 0.6299     |
| $[4, 6]$      | 0.0531   | 0.0531     | 0.1320   | 0.1320     |
| $[6, 6]$      | $-0.0137$| $-0.0138$  | 0.1350   | 0.1350     |
| $[6, 8]$      | 0.0025   | 0.0025     | 0.0114   | 0.0114     |
| $[8, 8]$      | $-0.0003$| $-0.0003$  | 0.0052   | 0.0052     |

It turns out that all the other lowest even-$J$ states have eigenfunctions close although not exactly equal to $\sqrt{2}|(j^2)^2(JJ^2)^2(jj)^9|$. In Table IV, we compare, as an example, the wave function of the $J = 8^+$ state. In the second column, we give the single $U9$-$j$ symbols (normalized) and in the third column we give results of diagonalizing the $E(9)$ interaction. Since the coefficient $[J_p, J_n]$ is the same as $[J_n, J_p]$, we list only one of them. The overlap of the two wave functions, $(\psi_1, \psi_2)$, is 0.9944.

Table IV: Comparing the wave functions of a single $U9$-$j$ symbol with $J_x = 9$ with a full diagonalization of $E(9)$ for the lowest $I = 8^+$ state in $^{96}$Cd.

| $[J_p, J_n]$ | $U9$-$j$   | $E(9)$  |
|---------------|------------|----------|
| $[0, 8]$      | 0.0630     | 0.0644   |
| $[2, 6]$      | 0.4299     | 0.4271   |
| $[2, 8]$      | $-0.0522$  | $-0.0513$|
| $[4, 4]$      | 0.7444     | 0.7456   |
| $[4, 6]$      | $-0.1803$  | $-0.1729$|
| $[4, 8]$      | 0.0256     | 0.0280   |
| $[6, 6]$      | 0.0521     | 0.0657   |
| $[6, 8]$      | $-0.0076$  | $-0.0012$|
| $[8, 8]$      | 0.0011     | 0.0047   |

III. DEGENERACIES

With this $E(9)$ interaction, we get several degenerate states with an absolute energy zero. In some detail, for $I = 0$ there are five states, three with isospin $T = 0$ and two with $T = 2$. There is one non-degenerate state at an energy $2V(9)$ ($V(9)$ is negative). The other four $I = 0$ states have zero energy. For $I = 1$ all states have isospin $T = 1$. There is a single non-degenerate state at $V(9)$, the other three have zero energy. For $I = 2$ there are twelve states—six have $T = 0$, four have $T = 1$ and two have $T = 2$. There are two non-degenerate $T = 0$ states with approximate energies $2V(9)$ and $V(9)$ respectively, and one non-degenerate $T = 1$ state with energy $V(9)$. The other nine states have zero energy. To understand this, take a wave function

$$\Psi^\alpha = \sum C^\alpha (J_p, J_n)[J_pJ_n]^I$$

and the corresponding energies $E^\alpha = \langle \Psi^\alpha H \Psi^\alpha \rangle$. Consider the sum $\sum_\alpha E^\alpha$. We have

$$\sum_\alpha C^\alpha (J_p, J_n)C^\alpha (J'_p, J'_n) = \delta_{J_pJ'_p}\delta_{J_nJ'_n}$$

Thus

$$\sum_\alpha E^\alpha = \sum_{J_pJ_n} \langle [J_pJ_n]^I | H | [J_pJ_n]^I \rangle = 4V(9) \sum_{J_pJ_n} \langle [jj]^n(JJ)^n | [jj]^9(JJ)^9 \rangle$$

This expression does not depend on the detailed wave functions. Using the properties of $U9$-$j$’s, we can show that $\sum_\alpha E^\alpha = 2V(9)$ for $I = 0$, $V(9)$ for $I = 1$, and $4V(9)$ for $I = 2$. But we can alternately show, using the explicit wave functions, that for $I = 0$ the energy of the lowest state is $2V(9)$. Hence, all the other states must have zero energy. A similar story for $I = 1$. The $I = 2$ state is a bit more complicated because of the coupling between two states, however small it is. Still one can work it through and see that the $4V(0)$ energy is exhausted by the two $T = 0$ and the one $T = 1$ non-degenerate states.

IV. CLOSING REMARKS

In closing, we note that the subject of $J_{\text{max}}$ pairing is currently a very active field. Besides the work of Qi et al. [4], there are related works by Zerguine and Van Isacker [13], Cederwall et al. [14] and Xu et al. [15]. The topic of $J$-pairing interactions has also been addressed by Zhao and Arima [16]. In this work we expand on our 2003 work [2] by noting that the Hamiltonian matrix for a $2p-2n$ system for $I = 0^+$ states in a single $j$-shell is separable for a simple interaction which is non-zero only for a single odd angular momentum. This leads to an eigenfunction with components proportional to a
single set of unitary 9-\(j\) symbols. We apply this to the \(J = J_{\text{max}}\) interaction. The single set of \(U9-j\) components form the eigenfunction not only for the lowest \(I = 0^+\) state, but also of the lowest \(I = 1^+\) state and, to a surprisingly excellent approximation, for the lowest two \(I = 2^+\) states. A single set of \(U9-j\) coefficients yields a good approximation for the higher yrast even-\(I\) states. We use the \(J = J_{\text{max}}\) interaction to confirm the observation of Ref. [6] that the resulting wave function has a fairly high overlap with that of a realistic interaction, although we note the energies are not realistic. We also note that this wave function for the lowest \(I = 0^+, 1^+, \) and \(2^+\) states has good isospin. We have other examples too: we have found a quantum number \(J_x\) (see Eq. (7)) which can, either exactly or approximately, help classify some of the states.

Our cautionary remarks are for the topic of pairing in general, but are not intended specifically for any of the works mentioned in this paper.

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