Integers with a large smooth divisor

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Abstract

We study the function \( \Theta(x, y, z) \) that counts the number of positive integers \( n \leq x \) which have a divisor \( d > z \) with the property that \( p \leq y \) for every prime \( p \) dividing \( d \). We also indicate some cryptographic applications of our results.

1 Introduction

For every integer \( n \geq 2 \), let \( P^+(n) \) and \( P^-(n) \) denote the largest and the smallest prime factor of \( n \), respectively, and put \( P^+(1) = 1, P^-(1) = \infty \). For real numbers \( x, y \geq 1 \), let \( \Psi(x, y) \) and \( \Phi(x, y) \) denote the counting functions of the sets of \( y \)-smooth numbers and \( y \)-rough numbers, respectively; that is,

\[
\Psi(x, y) = \# \{ n \leq x : P^+(n) \leq y \},
\Phi(x, y) = \# \{ n \leq x : P^-(n) > y \}.
\]
For a very wide range in the $xy$-plane, it is known that

$$\Psi(x, y) \sim \varrho(u) x \quad \text{and} \quad \Phi(x, y) \sim \omega(u) \frac{x}{\log y},$$

where $u$ denotes the ratio $(\log x)/\log y$, $\varrho(u)$ is the Dickman function, and $\omega(u)$ is the Buchstab function; the definitions and certain analytic properties of $\varrho(u)$ and $\omega(u)$ are reviewed in Sections 2 and 3 below.

In this paper, our principal object of study is the function $\Theta(x, y, z)$ that counts positive integers $n \leq x$ for which there exists a divisor $d \mid n$ with $d > z$ and $P^+(d) \leq y$; in other words,

$$\Theta(x, y, z) = \# \{n \leq x : n_y > z\},$$

where $n_y$ denotes the largest $y$-smooth divisor of $n$. The function $\Theta(x, y, z)$ has been previously studied in the literature; see [1, 6, 7, 8].

For $x, y, z$ varying over a wide domain, we derive the first two terms of the asymptotic expansion of $\Theta(x, y, z)$. We show that the main term can be naturally defined in terms of the partial convolution $C_{\omega, \varrho}(u, v)$ of $\varrho$ with $\omega$, which is defined by

$$C_{\omega, \varrho}(u, v) = \int_v^\infty \omega(u-s)\varrho(s) \, ds.$$

Using precise estimates for $\Psi(x, y)$ and $\Phi(x, y)$, we also identify the second term of the asymptotic expansion of $\Theta(x, y, z)$, which is naturally expressed in terms of the partial convolution $C_{\omega, \varrho'}(u, v)$ of $\varrho'$ with $\omega$:

$$C_{\omega, \varrho'}(u, v) = \int_v^\infty \omega(u-s)\varrho'(s) \, ds.$$

We use this formula to give a heuristic prediction for the density of certain integers of cryptographic interest which appear in [3]. An alternative approach, which establishes a two term asymptotic formula for $\Theta(x, y, z)$ over a wider range, has been developed recently by Tenenbaum [8].

**Theorem 1.** For fixed $\varepsilon > 0$ and uniformly in the domain

$$x \geq 3, \quad y \geq \exp\{(\log \log x)^{5/3+\varepsilon}\}, \quad y \log y \leq z \leq x/y,$$

we have

$$\Theta(x, y, z) = (\varrho(u) + C_{\omega, \varrho}(u, v))x - \gamma C_{\omega, \varrho'}(u, v) \frac{x}{\log y} + O(E(x, y, z)),$$
where \( u = (\log x) / \log y \), \( v = (\log z) / \log y \), \( \gamma \) is the Euler-Mascheroni constant, and

\[
\mathcal{E}(x, y, z) = \frac{x}{\log y} \left\{ \varrho(u - 1) + \frac{\varrho(v) \log(v + 1)}{\log y} + \frac{\varrho(v)}{\log(v + 1)} \right\}.
\]

The proof of Theorem 1 is given below in Section 4; our principal tools are the estimates of Lemma 4 (Section 2) and Lemma 6 (Section 3). In Section 5 we outline some cryptographic applications of our results.

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2 Integers free of large prime factors

In this section, we collect various estimates for the counting function \( \Psi(x, y) \) of \( y \)-smooth numbers:

\[
\Psi(x, y) = \#\{n \leq x : P^+(n) \leq y\}.
\]

As usual, we denote by \( \varrho(u) \) the Dickman function; it is continuous at \( u = 1 \), differentiable for \( u > 1 \), and it satisfies the difference-differential equation

\[
u \varrho'(u) + \varrho(u - 1) = 0 \quad (u > 1)
\]

along with the initial condition

\[
\varrho(u) = 1 \quad (0 \leq u \leq 1).
\]

It is convenient to define \( \varrho(u) = 0 \) for all \( u < 0 \) so that (1) is satisfied for \( u \in \mathbb{R} \setminus \{0, 1\} \), and we also define \( \varrho'(u) \) by right-continuity at \( u = 0 \) and \( u = 1 \). For a discussion of the analytic properties of \( \varrho(u) \), we refer the reader to [6, Chapter III.5].

We need the following well known estimate for \( \Psi(x, y) \), which is due to Hildebrand [2] (see also [6, Corollary 9.3, Chapter III.5]):
Lemma 1. For fixed $\varepsilon > 0$ and uniformly in the domain
\[ x \geq 3, \quad x \geq y \geq \exp\{(\log \log x)^{5/3+\varepsilon}\}, \]
we have
\[ \Psi(x, y) = \varrho(u) x \left\{ 1 + O\left(\frac{\log(u + 1)}{\log y}\right) \right\}, \]
where $u = (\log x)/\log y$.

We also need the following extension of Lemma 1, which is a special case of the results of Saias [5]:

Lemma 2. For fixed $\varepsilon > 0$ and uniformly in the domain
\[ x \geq 3, \quad y \geq \exp\{(\log \log x)^{5/3+\varepsilon}\}, \quad x \geq y \log y, \]
the following estimate holds:
\[ \Psi(x, y) = \varrho(u) x + (\gamma - 1) \varrho'(u) \frac{x}{\log y} + O\left(\varrho''(u) \frac{x}{\log^2 y}\right), \]
where $u = (\log x)/\log y$.

The following lemma provides a precise estimate for the sum
\[ S(y, z) = \sum_{d > z, P^+(d) \leq y} \frac{1}{d} \]
over a wide range, which is used in the proofs of Lemmas 4 and 6 below. The sum $S(y, z)$ has been previously studied; see, for example, [7].

Lemma 3. For fixed $\varepsilon > 0$ and uniformly in the domain
\[ y \geq 3, \quad 1 \leq z \leq \exp\{(\log y)^{3/5-\varepsilon}\}, \]
we have
\[ S(y, z) = \tau(v) \log y - \gamma \varrho(v) + O\left(E(y, z)\right), \]
where $v = (\log z)/\log y$,
\[ \tau(v) = \int_v^\infty \varrho(s) \, ds, \]
and
\[
E(y, z) = \begin{cases} 
\frac{\varrho(v) \log(v+1)}{\log y} & \text{if } z \geq y \log y; \\
z^{-1} + \frac{\log \log y}{\log y} & \text{if } z < y \log y.
\end{cases}
\]

Proof. Let \( Y = y \log y \). First, suppose that \( z > Y \), and put
\[
T = \exp\left\{ (\log y)^{3/5-\epsilon/2} \right\}.
\]
By partial summation, it follows that
\[
S(y, z) = \sum_{\substack{z < d \leq y^T \\
P^+(d) \leq y}} \frac{1}{d} + S(y, y^T)
= \frac{\Psi(y^T, y)}{y^T} - \frac{\Psi(z, y)}{z} + \log y \int_v^T \frac{\Psi(y^s, y)}{y^s} ds + S(y, y^T).
\]
By Lemma 1, we have the estimate
\[
\frac{\Psi(z, y)}{z} = \varrho(v) + O \left( \frac{\varrho(v) \log(v+1)}{\log y} \right).
\]
Also, by our choice of \( T \) we have
\[
\frac{\Psi(y^T, y)}{y^T} \ll \varrho(T) \ll \frac{\varrho(v) \log(v+1)}{\log y}.
\]
The following bound is given in the proof of [7, Corollaire 2]:
\[
S(y, y^T) = \sum_{\substack{d > y^T \\
P^+(d) \leq y}} \frac{1}{d} \ll \varrho(T) e^{\epsilon T} + y^{(1-\epsilon)T},
\]
from which we deduce that
\[
S(y, y^T) \ll \frac{\varrho(v) \log(v+1)}{\log y}.
\]
To estimate the integral in (2), we apply Lemma 2 and write
\[
\int_v^T \frac{\Psi(y^s, y)}{y^s} ds = I_1 + I_2 + O(I_3),
\]
where

\[ I_1 = \int_v^T \varrho(s) \, ds = \tau(v) - \tau(T), \]

\[ I_2 = \frac{(\gamma - 1)}{\log y} \int_v^T \varrho'(s) \, ds = \frac{(\gamma - 1)(\varrho(T) - \varrho(v))}{\log y}, \]

\[ I_3 = \frac{1}{\log^2 y} \int_v^T \varrho''(s) \, ds = \frac{\varrho'(T) - \varrho'(v)}{\log^2 y}. \]

Since \(|\varrho'(v)| \asymp \varrho(v) \log(v + 1)|\), and

\[ \tau(T) \ll \varrho(T) \ll \frac{\varrho(v) \log(v + 1)}{\log^2 y}, \]

it follows that

\[ \int_v^T \frac{\Psi(y^*, y)}{y^*} \, ds = \tau(v) - \frac{(\gamma - 1)\varrho(v)}{\log y} + O \left( \frac{\varrho(v) \log(v + 1)}{\log^2 y} \right). \] (5)

Inserting the estimates (3), (4) and (5) into (2), we obtain the desired estimate in the case \( z > Y \).

Next, suppose that \( y \leq z \leq Y \), and put

\[ V = \frac{\log Y}{\log y} = 1 + \frac{\log \log y}{\log y}. \]

Since \( \varrho(s) = 1 - \log s \) for \( 1 \leq s \leq 2 \), we have

\[ 1 \geq \varrho(v) \geq \varrho(V) = 1 + O \left( \frac{\log \log y}{\log y} \right); \]

therefore,

\[ \varrho(v) - \varrho(V) \ll \frac{\log \log y}{\log y}. \] (6)

By partial summation, it follows that

\[ S(y, z) = \sum_{z < d \leq Y \atop P^+(d) \leq y} \frac{1}{d} + S(y, Y) \]

\[ = \frac{\Psi(Y, y)}{Y} - \frac{\Psi(z, y)}{z} + \log y \int_v^V \frac{\Psi(y^*, y)}{y^*} \, ds + S(y, Y). \] (7)
Using Lemma 1 together with (3), it follows that
\[ \frac{\Psi(Y, y)}{Y} - \frac{\Psi(z, y)}{z} = \varrho(V) - \varrho(v) + O\left(\frac{1}{\log y}\right) \ll \frac{\log \log y}{\log y}. \] (8)

Applying the estimate from the previous case, we also have
\[ S(y, Y) = \tau(V) \log Y - \gamma \varrho(V) + O\left(\frac{1}{\log y}\right). \] (9)

To estimate the integral in (7), we use Lemma 1 again and write
\[ \int_v^V \frac{\Psi(y^s, y)}{y^s} \, ds = I_4 + O(I_5), \]
where
\[ I_4 = \int_v^V \varrho(s) \, ds = \tau(v) - \tau(V), \]
\[ I_5 = \frac{1}{\log y} \int_v^V ds = \frac{\log(Y/z)}{\log^2 y} \ll \frac{\log \log y}{\log^2 y}. \]

Therefore,
\[ \int_v^V \frac{\Psi(y^s, y)}{y^s} \, ds = \tau(v) - \tau(V) + O\left(\frac{\log \log y}{\log^2 y}\right). \] (10)

Inserting the estimates (8), (9) and (10) into (7), and taking into account (6), we obtain the stated estimate for \( y \leq z \leq Y \).

Finally, suppose that \( 1 \leq z < y \). In this case,
\[ S(y, z) = \sum_{z < d \leq y} \frac{1}{d} + S(y, y). \] (11)

By partial summation, we have
\[ \sum_{z < d \leq y} \frac{1}{d} = \log y - \log z + O(z^{-1}) = (1 - v) \log y + O(z^{-1}) \]
\[ = \log y \int_v^1 \varrho(s) \, ds + O(z^{-1}) = (\tau(v) - \tau(1)) \log y + O(z^{-1}). \]
Applying the estimate from the previous case, we also have
\[
S(y, y) = \tau(1) \log y - \gamma \varrho(1) + O\left(\frac{\log\log y}{\log y}\right).
\]
Inserting these estimates into (11), and using the fact that \(\varrho(v) = \varrho(1) = 1\), we obtain the desired result. This completes the proof.

Lemma 4. For fixed \(\varepsilon > 0\) and uniformly in the domain
\[
x \geq 3, \quad y \geq \exp\{(\log\log x)^{5/3+\varepsilon}\}, \quad 1 \leq z \leq x/y,
\]
we have
\[
\sum_{z < d \leq x/y} \frac{\varrho(u - u_d)}{d} \ll C_{\varrho, \varrho}(u, v) \log(u + 1) + \varrho(u - v)\varrho(v) + \varrho(u - 1),
\]
where \(u = (\log x)/\log y\), \(v = (\log z)/\log y\), \(u_d = (\log d)/\log y\) for every integer \(d\) in the sum, and
\[
C_{\varrho, \varrho}(u, v) = \int_{v}^{\infty} \varrho(u - s)\varrho(s) \, ds.
\]
Proof. By partial summation, we have
\[
\sum_{z < d \leq x/y} \frac{\varrho(u - u_d)}{d} = S(y, x/y) - \varrho(u - v)S(y, z) + \int_{v}^{u-1} \varrho'(u - s)S(y, y^s) \, ds.
\]
Lemma 3 implies that
\[
S(y, x/y) = \tau(u - 1) \log y + O(\varrho(u - 1)),
\]
\[
S(y, z) = \tau(v) \log y + O(\varrho(v)),
\]
and
\[
\int_{v}^{u-1} \varrho'(u - s)S(y, y^s) \, ds = I_1 \log y + O(I_2),
\]
where
\[
I_1 = \int_{v}^{u-1} \varrho'(u - s)\tau(s) \, ds = \varrho(u - v)\tau(v) - \tau(u - 1) + C_{\varrho, \varrho}(u, v),
\]
\[
I_2 = \int_{v}^{u-1} |\varrho'(u - s)|\varrho(s) \, ds.
\]
Finally, using the bound
\[
|\varphi'(t)| \ll \varphi(t) \log(t + 1) \quad (t > 1),
\]
we see that
\[
I_2 \ll \log(u + 1) \int_v^{u-1} \varphi(u-s) \varphi(s) \, ds \leq C_{\varphi, \psi}(u,v) \log(u + 1).
\]
Putting everything together, the result follows. \qed

3 Integers free of small prime factors

In this section, we collect various estimates for the counting function \( \Phi(x,y) \)
of \( y \)-rough numbers:
\[
\Phi(x,y) = \#\{ n \leq x : P^-(n) > y \}.
\]
As usual, we denote by \( \omega(u) \) the Buchstab function; for \( u > 1 \), it is the unique
continuous solution to the difference-differential equation
\[
\left( u \omega(u) \right)' = \omega(u-1) \quad (u > 2) \tag{12}
\]
with initial condition
\[
u \omega(u) = 1 \quad (1 \leq u \leq 2).
\]
It is convenient to define \( \omega(1) = 0 \) for all \( u < 1 \) so that (12) is satisfied for
\( u \in \mathbb{R} \setminus \{1,2\} \), and we also define \( \omega'(u) \) by right-continuity at \( u = 1 \)
and \( u = 2 \). For a discussion of the analytic properties of \( \omega(u) \), we refer the reader
to [6, Chapter III.6]
The next result follows from [6, Corollary 7.5, Chapter III.6]:

Lemma 5. For fixed \( \varepsilon > 0 \) and uniformly in the domain
\[
x \geq 3, \quad x \geq y \geq \exp\{(\log \log x)^{5/3+\varepsilon}\},
\]
the following estimate holds:
\[
\Phi(x,y) = \left( x \omega(u) - y \right) \frac{e^y}{\zeta(1,y)} + O\left( \frac{x \varphi(u)}{\log^2 y} \right),
\]
where \( u = (\log x) / \log y \), and \( \zeta(1,y) = \prod_{p \leq y} (1 - p^{-1})^{-1} \).
Lemma 6. For fixed $\varepsilon > 0$ and uniformly in the domain
\[
x \geq 3, \quad y \geq \exp\{(\log \log x)^{5/3 + \varepsilon}\}, \quad 1 \leq z \leq x/y,
\]
we have
\[
\sum_{\substack{z < d \leq x/y \\ P^+(d) \leq y}} \frac{\omega(u - u_d)}{d} = C_{\omega, \varrho}(u, v) \log y - \gamma C_{\omega, \varrho'}(u, v) + O\left(E(y, z)\right),
\]
where $u = (\log x)/\log y$, $v = (\log z)/\log y$, $u_d = (\log d)/\log y$ for every integer $d$ in the sum, and $E(y, z)$ is the error term of Lemma 3.

Proof. By partial summation, it follows that
\[
\sum_{\substack{z < d \leq x/y \\ P^+(d) \leq y}} \frac{\omega(u - u_d)}{d} = S(y, x/y) - \omega(u - v)S(y, z) + \int_v^{u-1} \omega'(u - s)S(y, y^s) \, ds.
\]
By Lemma 3 we have the estimates
\[
S(y, x/y) = \tau(u - 1) \log y - \gamma \varrho(u - 1) + O\left(E(y, x/y)\right)
\]
and
\[
S(y, z) = \tau(v) \log y - \gamma \varrho(v) + O\left(E(y, z)\right).
\]
Also,
\[
\int_v^{u-1} \omega'(u - s)S(y, y^s) \, ds = I_1 \log y - \gamma I_2 + O(I_3),
\]
where
\[
I_1 = \int_v^{u-1} \omega'(u - s)\tau(s) \, ds = \omega(u - v)\tau(v) - \tau(u - 1) + C_{\omega, \varrho}(u, v),
\]
\[
I_2 = \int_v^{u-1} \omega'(u - s)\varrho(s) \, ds = \omega(u - v)\varrho(v) - \varrho(u - 1) + C_{\omega, \varrho'}(u, v),
\]
\[
I_3 = \frac{1}{\log y} \int_v^{u-1} \left|\omega'(u - s)\right|E(y, y^s) \, ds.
\]
Putting everything together, we see that the stated estimate follows from the bound
\[
E(y, x/y) + \omega(u - v)E(y, z) + I_3 \ll E(y, z). \quad (13)
\]
To prove this, observe that $E(y, z_1) \ll E(y, z_2)$ holds for all $z_1 \geq z_2 \geq 1$. Therefore, $E(y, x/y) \ll E(y, z)$, and

$$I_3 \ll \frac{E(y, z)}{\log y} \int_v^{u-1} |\omega'(u - s)| \, ds \ll \frac{E(y, z)}{\log y}.$$  

Taking into account the fact that $\omega(u - v) \approx 1$, we derive the bound (13), and this completes the proof. \( \square \)

### 4 Proof of Theorem 1

For fixed $y$, every positive integer $n$ can be uniquely decomposed as a product $n = de$, where $P^+(d) \leq y$ and $P^-(e) > y$. Therefore,

$$\Theta(x, y, z) = \sum_{z < d \leq x \atop P^+(d) \leq y} \sum_{e \leq x/d \atop P^-(e) > y} 1 = \sum_{z < d \leq x \atop P^+(d) \leq y} \Phi(x/d, y)$$

$$= \Psi(x, y) - \Psi(x/y, y) + \sum_{z < d \leq x/y \atop P^+(d) \leq y} \Phi(x/d, y).$$

Using Lemma 1 it follows that

$$\Psi(x, y) - \Psi(x/y, y) = \varrho(u) x + O \left( \frac{\varrho(u - 1) x}{\log y} \right).$$

By Lemma 3 we also have

$$\sum_{z < d \leq x/y \atop P^+(d) \leq y} \Phi(x/d, y)$$

$$= \sum_{z < d \leq x/y \atop P^+(d) \leq y} \left\{ \left( \frac{x \omega(u - u_d)}{y} - y \right) \frac{e^\gamma}{\zeta(1, y)} + O \left( \frac{x \varrho(u - u_d)}{d \log^2 y} \right) \right\}$$

$$= \frac{e^\gamma x}{\zeta(1, y)} \sum_{z < d \leq x/y \atop P^+(d) \leq y} \frac{\omega(u - u_d)}{d} - \frac{e^\gamma y}{\zeta(1, y)} \left\{ \Psi(x/y, y) - \Psi(z, y) \right\} + O \left( \frac{x}{\log^2 y} \sum_{z < d \leq x/y \atop P^+(d) \leq y} \frac{\varrho(u - u_d)}{d} \right).$$

(14)
Applying Lemma 1 again, we have

\[-\frac{e^\gamma y}{\zeta(1,y)} \left\{ \Psi(x/y, y) - \Psi(z, y) \right\} \ll \frac{\varrho(u - 1) x}{\log y}.\]

Inserting the estimates of Lemmas 4 and 6 into (14), and making use of the trivial estimate

\[ C \varrho, \varrho(u, v) \log(u + 1) \ll \log y \int_v^\infty \varrho(s) \, ds \ll \frac{\varrho(v) \log y}{\log(v + 1)}.\]

it is easy to see that

\[ \Theta(x, y, z) = \left( \varrho(u) + C \varrho, \varrho(u, v) \frac{e^\gamma \log y}{\zeta(1,y)} \right) x - \gamma C \varrho, \varrho(u, v) \frac{e^\gamma x}{\zeta(1,y)} + O(\mathcal{E}(x, y, z)).\]

To complete the proof, we use the estimate (see [9]):

\[ \zeta(1,y) = e^\gamma \log y \left( 1 + \exp\{-c(\log y)^{3/5}\} \right),\]

which holds for some absolute constant \( c > 0 \), together with the trivial estimate

\[ \max\{C \varrho, \varrho(u, v), C \varrho, \varrho(u, v)\} \ll \int_v^\infty \varrho(s) \, ds \ll \frac{\varrho(v)}{\log(v + 1)}.\]

### 5 Cryptographic applications

Suppose that two primes \( p \) and \( q \) are selected for use in the Digital Signature Algorithm (see, for example, [4]) using the following standard method:

- Select a random \( m \)-bit prime \( q \);
- Randomly generate \( k \)-bit integers \( n \) until a prime \( p = 2nq + 1 \) is reached.

The large subgroup attack described in [3, Section 3.2.2] leads one naturally to consider the following question: What is the probability \( \eta(k, \ell, m) \) that \( n \) has a divisor \( s > q \) which is \( 2^\ell \)-smooth?

It is natural to expect that the proportion of those integers in the set \( \{2^{k-1} \leq n < 2^k\} \) having a large smooth divisor should be roughly the same as the proportion of integers in

\[ \{2^{k-1} \leq n < 2^k : n = (p - 1)/(2q) \text{ for some prime } p \equiv 1 \pmod{2q}\} \]
having a large smooth divisor. Accordingly, we expect that the probability \( \eta(k, \ell, m) \) is reasonably close to

\[
\frac{\Theta(2^k, 2^\ell, 2^m) - \Theta(2^{k-1}, 2^\ell, 2^m)}{2^{k-1}}.
\]

Theorem 1 then suggests that

\[
\eta(k, \ell, m) \approx 2 \varphi(k, \ell, m) - \varphi(k - 1, \ell, m),
\]

where

\[
\varphi(k, \ell, m) = \varphi(k/\ell) + \mathcal{C}_{\omega, \varphi}(k/\ell, m/\ell) - \frac{\gamma \mathcal{C}_{\omega, \varphi}(k/\ell, m/\ell)}{\ell \log 2}.
\]

In particular, the most interesting choice of parameters at the present time is \( k = 863, \ell = 80, \) and \( m = 160 \) (which produces a 1024-bit prime \( p \)), for which expect that \( \eta(863, 80, 160) \approx 0.09576. \)

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