The Cluster Distribution as a Test of Dark Matter Models. IV: Topology and Geometry

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ABSTRACT
We study the geometry and topology of the large–scale structure traced by galaxy clusters in numerical simulations of a box of side $320 \ h^{-1} $ Mpc, and compare them with available data on real clusters. The simulations we use are generated by the Zel’dovich approximation, using the same methods as we have used in the first three papers in this series. We consider the following models to see if there are measurable differences in the topology and geometry of the superclustering they produce: (i) the standard CDM model (SCDM); (ii) a CDM model with $\Omega_0 = 0.2$ (OCDM); (iii) a CDM model with a ‘tilted’ power spectrum having $n = 0.7$ (TCDM); (iv) a CDM model with a very low Hubble constant, $h = 0.3$ (LOWH); (v) a model with mixed CDM and HDM (CHDM); (vi) a flat low–density CDM model with $\Omega_0 = 0.2$ and a non-zero cosmological $\Lambda$ term (ΛCDM). We analyse these models using a variety of statistical tests based on the analysis of: (i) the Euler–Poincaré characteristic; (ii) percolation properties; (iii) the Minimal Spanning Tree construction. Taking all these tests together we find that the best fitting model is ΛCDM and, indeed, the others do not appear to be consistent with the data. Our results demonstrate that despite their biased and extremely sparse sampling of the cosmological density field, it is possible to use clusters to probe subtle statistical diagnostics of models which go far beyond the low-order correlation functions usually applied to study superclustering.

Key words: Cosmology: theory – dark matter – galaxies: clustering – large–scale structure of Universe

1 INTRODUCTION
The study of the distribution of matter on the largest scales amenable to observation can provide important constraints on models of the formation of cosmological structures. In particular, it has now become well established that a very accurate and efficient way of describing very large scale structure in the galaxy distribution is obtained by not looking at galaxies themselves but at rich clusters of galaxies. If the ‘standard’ model of structure formation – the gravitational instability picture – is correct, the expected displacements of galaxy clusters from their primordial positions are much smaller than the typical separation of these objects. In principle, therefore, clusters of galaxies can yield clues about the primordial spectrum of perturbations that gave rise to them, without such clues being trampled on by the effects of non–linear evolution. Moreover, because clusters represent highly overdense regions in the cosmological density
field, these objects display an enhanced clustering signal relative to that of galaxies on the same scale, an effect usually known as biasing (Kaiser 1984).

This is the reason why so much effort has been devoted to compiling deep cluster surveys, starting with the pioneering work of Abell (1958), Zwicky et al. (1968) and Abell, Corwin & Olowin (1989), and leading up to extended redshift surveys both in the optical (e.g. Postman, Huchra & Geller 1992; Dalton et al. 1994; Collins et al. 1994, and references therein) and in the X-ray (e.g. Nichol, Briel & Henry 1994; Romer et al. 1994; Ebeling et al. 1996) regions of the spectrum.

The properties of galaxy clusters may help to resolve some of the issues that have led to the present relative stagnation in the theory of structure formation. Since the demise of the standard model of the 1980s – the standard Cold Dark Matter model (SCDM) – a number of contending theories have been proposed which are in better agreement with the observations than SCDM but between which it is difficult to discriminate using present observations of galaxy clustering and the cosmic microwave background; for a review, see Coles (1996). It is therefore important to try to find statistical diagnostics of clustering that may reveal differences between these models and the data to see if they do indeed explain the details of the observed clustering phenomenon, as well as between the models themselves so one can understand how the various extra ingredients involved in these models alter specific characteristics of the clustering pattern.

Simple two-point statistical descriptions of superclustering (i.e. the clustering of galaxy clusters) have already yielded important clues about the shape of the matter power spectrum on large scales (e.g. Peacock & Dodds 1994; Borgani et al. 1997) and, more recently, this has been extended to simple properties of the higher–order moments (e.g. Plionis & Valdarnini 1995; Plionis et al. 1995; Borgani et al. 1995; Gaztanaga, Croft & Dalton 1995). However, the complete statistical characterisation of the clustering requires knowledge of all the higher order moments or, equivalently, knowledge of the complete set of $n$–point correlation functions (Peebles 1980). Such a description is extremely laborious to construct, tends to be swamped by discreetness effects and sampling errors even at quite small $n$ and is in any case rather difficult to interpret geometrically.

For these reasons it is useful to seek a description of clustering which by-passes this more orthodox approach and looks for intrinsically geometrical or topological signatures. One can hope that such approaches might lead to robust quantitative descriptions of the void–filament network which is visually apparent in the distribution of galaxies, and to relate this visual appearance to the interaction of non–linear gravitational dynamics on an initial density field with some assumed power spectrum. The hope is therefore to pick out differences between these models and the data to see if they do indeed explain the details of the observed clustering phenomenon, as well as between the models themselves so one can understand how the various extra ingredients involved in these models alter specific characteristics of the clustering pattern.

In this paper, we aim to investigate a particular set of topological or geometrical descriptors of the pattern present in simulated cluster distributions and, where possible, to compare the results from simulations with the analogous results from the Abell/ACO cluster catalogue. We should stress at the outset that this is an exploratory work and there are reasons to suspect that the task of discriminating between these models and the data might be extremely difficult. First there is the problem of shot–noise we alluded to above. Secondly, the available cluster sample is quite small and may suffer from unknown selection effects. One can hope, however, that better controlled cluster samples may emerge fairly soon from ongoing galaxy redshift surveys. Third, it is extremely difficult to construct sufficiently large N-body simulations of galaxy clustering and select the appropriate clusters within them in the same way that clusters are selected observationally (e.g. Bahcall & Cen 1992; Croft & Efstathiou 1994; Eke et al. 1996). And finally, there is the ubiquitous problem of understanding how the objects one sees relate to the distribution of matter one calculates, a difficulty generically known by the name of biasing and which was first discussed in the context of rich clusters by Kaiser (1984).

In the spirit of exploration, therefore, we shall use simplified models of superclustering, generated by using a method based on the Zel'dovich approximation. This method has been used in a number of previous studies of the distribution of clusters in both position and velocity space (Borgani, Coles & Moscardini 1994; Plionis et al. 1995; Borgani et al. 1995; Tini Brunozzi et al. 1995; Moscardini et al. 1996; Borgani et al. 1997) and is known to be accurate in comparison with the full $N$–body approach, provided the degree of non–linear evolution at the scale of individual clusters is not too strong.
The outline of the paper is as follows. In Sections 2 and 3 we briefly describe our simulation method and the observed Abell/ACO cluster sample, respectively. We then go on to discuss the various clustering descriptors we use to analyse these data sets. First, in Section 4, we discuss the topological properties of the isodensity regions in the distribution traced by clusters, using a method described in detail by Coles, Davies & Pearson (1996) and which is similar (but not identical) to the well-known genus statistic (reviewed by Melott 1990) and which has recently been applied to cluster data by Rhoads, Gott & Postman (1994). We next, in Section 5, discuss an analysis based on percolation theory. The last of our three approaches, presented in Section 6, is based on properties of a graph-theoretical construction known as the minimal spanning tree, in conjunction with a set of mathematical quantities intended to describe the shapes of pieces of the trees obtained (Pearson & Coles 1995). Each of the three analyses we attempt is expected to perform better in some situations than others, so in Section 7 we present an analysis of the statistical power of these tests at discriminating between different models and between the models and the observed data. We also discuss the virtues of combining the various tests and show the statistical significance of the results we obtain by combining the different analyses into a composite test. We present our conclusions in Section 8.

2 THE SIMULATIONS

2.1 The Zel’dovich Approach

The Zel’dovich approximation [ZA] (Zel’dovich 1970; Shandarin & Zel’dovich 1989) is based on the assumption of laminar flow for the motion of a self–gravitating non–relativistic collisionless fluid. Let \( q \) be the initial (Eulerian) position of a fluid element and \( r(q, t) = a(t) \mathbf{x}(q, t) \) the final position at the time \( t \), which is related to the comoving Lagrangian coordinate \( \mathbf{x}(q, t) \) through the cosmic expansion factor \( a(t) \). The ZA amounts to assume the expression

\[
r(q, t) = a(t) \left[ q + b(t) \nabla \psi(q) \right]
\]

for the Eulerian–to–Lagrangian coordinate mapping. In equation (1) \( b(t) \) is the growing mode for the evolution of linear density perturbations and \( \psi(q) \) is the gravitational potential, which is related to the initial density fluctuation field, \( \delta(q) \), through the Poisson equation

\[
\nabla^2 \psi(q) = -\frac{\delta(q)}{a(t)}.
\]

As a result of the factorization of the \( t \)– and \( q \)–dependence in the displacement term of equation (1), the fluid particles move under this approximation along straight lines, with comoving peculiar velocity

\[
v(q, t) = \dot{x}(q, t) = \dot{b}(t) \nabla \psi(q).
\]

Therefore, gravity determines the initial kick to the fluid particles through eqs. (2) and (3), and afterwards they do not feel any tidal interactions. Particles fall inside gravitational wells to form structures, which however quickly evaporate. In this sense, the ZA gives a good description of gravitational dynamics as far as particle trajectories do not intersect with each other, while its validity breaks down when shell–crossing occurs, and local gravity dominates.

Coles, Melott & Shandarin (1993) have shown that filtering out the small–scale wavelength modes in the linear power–spectrum reduces the amount of shell–crossing, thus improving the performance of the ZA. Melott, Pellman \& Shandarin (1993) claimed that an optimal filtering procedure is obtained by convolving the linear power–spectrum with the Gaussian filter

\[
W_G(kR_f) = e^{-\left(kR_f\right)^2/2}
\]

(cf. Sahni \& Coles 1995). The problem then arises of choosing the filtering radius \( R_f \) appropriately, in order to suppress shell–crossing as much as possible without preventing genuine clustering to build up.

Kofman et al. (1994) derived an analytical expression – their equation (7) – for the average number of streams at each Eulerian point, \( N_s \), as a function of the r.m.s. fluctuation level of the initial Gaussian density field. We decided to choose \( R_f \) for each model so that \( N_s = 1.1 \). We found this to be a reasonable compromise between smaller \( N_s \) values, giving rapidly increasing \( R_f \) and high suppression of clustering, and larger \( N_s \), at which the ZA progressively breaks down. The resulting r.m.s. fluctuation value corresponding to \( N_s = 1.1 \) is \( \sigma = 0.88 \).

By adopting this implementation of the ZA, the main steps of our cluster simulations are the following:
(a) Convolve the linear power-spectrum with the Gaussian window of equation (4) and $R_f$ chosen as previously described.

(b) Generate a random-phase realization of the density field on $128^3$ grid points for a cubic box of $L = 320 h^{-1}$Mpc aside.

(c) Move $128^3$ particles having initial Lagrangian position on the grid, according to the ZA. Each particle carries a mass of $4.4 \times 10^{12} h^{-1} \Omega_0 M_{\odot}$.

(d) Reassign the density and the velocity field on the grid through a TSC interpolation scheme (e.g. Hockney & Eastwood 1981) for the mass and the moment carried by each particle.

(e) Select clusters as local density maxima on the grid according to the following prescription. If $d_{cl}$ is the average cluster separation, then we select $N_d = (L/d_{cl})^3$ clusters as the $N_d$ highest density peaks. In the following, we assume $d_{cl} = 40 h^{-1}$Mpc, which is appropriate for the combined Abell/ACO cluster sample to which we will compare our simulation results (see Section 3). Therefore, we will analyze a distribution of 512 clusters in each simulation box, with periodic boundary conditions.

2.2 Dark Matter Models

We ran simulations for six different models of the initial fluctuation spectrum. For each model, we generate 5 random realizations, as a compromise between estimating the cosmic variance reliably and keeping the amount of data to be analysed within reasonable bounds. All the models, except OCDM, are normalized to be consistent with the COBE measured quadrupole of CMB temperature anisotropy (Bennett et al. 1994). Since we are primarily interested in these simply as tests of the method, we have not attempted to fine-tune the parameters of each scenario in order to maximise its performance: the models chosen are simply meant to represent the range of behaviours of contenders for a viable model of structure formation. The models we have considered are the following.

(1) The standard CDM model (SCDM), with $\Omega_0 = 1$, $h = 0.5$ and $\sigma_8 = 1$ for the r.m.s. fluctuation amplitude within a top-hat sphere of $8 h^{-1}$Mpc. This model has already been excluded by independent analyses but we include here for completeness and to see whether our pattern descriptors can also successfully reject it.

(2) An open CDM model (OCDM), with $\Omega_0 = 0.2$ and $n = 1$. We have chosen to normalise this model to $b = 1$, so that our results in this paper can be compared with Plionis et al. (1995) and Borgani et al. (1995); more detailed discussion of this model can be found in (Coles & Ellis 1994,1997; Ratra & Peebles 1994,1995; Liddle et al. 1996; Yamamoto & Bunn 1996).

(3) A tilted CDM model (TCDM), with $n = 0.7$ for the primordial spectral index. Tilting the primordial spectral shape from the scale-free one has been suggested in order to improve the CDM description of the large-scale structure (e.g. Cen et al. 1992; Tormen et al. 1993; Liddle & Lyth 1993; Adams et al. 1993; Moscardini et al. 1995).

(4) A low Hubble constant CDM model (LOWH), with $h = 0.3$. Decreasing the Hubble constant has the effect of increasing the horizon size at the equivalence epoch, thus pushing the turnover of the spectrum to its scale-free form out to larger scales (cf. Bartlett et al. 1994).

(5) A Cold + Hot DM model (CHDM), with $\Omega_{hot} = 0.3$ for the fractional density contributed by the hot particles. For a fixed large-scale normalization, adding a hot component has the effect of suppressing the power-spectrum amplitude at small wavelengths (e.g. Klypin et al. 1993). Although the small-scale peculiar velocities are lowered to an adequate level, the corresponding galaxy formation time is delayed so that such a model is strongly constrained by the detection of high-redshift objects (e.g. Ma & Bertschinger 1994; Klypin et al. 1995; Borgani et al. 1997).

(6) A spatially flat, low-density CDM model ($\Lambda$CDM), with $\Omega_0 = 0.2$, $\Omega_\Lambda = 0.8$ for the cosmological constant term (e.g. Efstathiou, Sutherland & Maddox 1990; Bahcall & Cen 1992; Baugh & Efstathiou 1993; Kofman, Gnedin & Bahcall 1993; Peacock & Dodds 1994) and $\sigma_8 = 1.3$, so as to be consistent with the two-year COBE results.

The transfer functions for the above models have been taken from Holtzman (1989), except that of LOWH, which is taken from Bond & Efstathiou (1984), with suitably chosen shape parameter $\Gamma = \Omega_0 h = 0.3$. We note that the latter transfer function assumes the baryonic component to be negligible, which is probably not accurate if nucleosynthesis is correct, but this would only affect the shape of the transfer function on small scales, below those we are interested in here. All the model parameters are listed in Table 1.

It is worth making a specific point about the $\Lambda$CDM model we use here. Strictly speaking, the amplitude of matter fluctuations required for this model to be compatible with COBE is larger than can be treated with great accuracy.
by our simulation method (Borgani et al. 1995). However, we found in the course of this analysis that this feature of \( \Lambda \)CDM reveals some interesting properties of the topological descriptors we use so, rather than using an alternative model with a lower normalisation (c.f. Borgani et al. 1995), we will keep the model with a higher normalisation in this analysis. In any case, as we shall show, changing the normalisation from \( \sigma_8 = 1.3 \) to, say, \( \sigma_8 = 0.8 \) does not significantly change the large–scale topological properties of selected clusters.

### 3 THE CLUSTER SAMPLE

We use the combined Abell/ACO \( R \geq 0 \) cluster sample, as defined in Plionis & Valdarnini (1995) and in Borgani et al. (1995). The declination limit between the northern (Abell) and southern (ACO; Abell, Corwin & Olowin 1989) sample is \( \text{dec} \geq -17^\circ \) while both samples are limited in Galactic latitude by \( |b| \geq 30^\circ \).

To take into account the effect of Galactic absorption, we assume the usual cosecant law:

\[
P(|b|) = \text{dex} \left[ \alpha (1 - \csc |b|) \right]
\]

with \( \alpha \approx 0.3 \) for the Abell sample (Bahcall & Soneira 1983; Postman et al. 1989) and \( \alpha \approx 0.2 \) for the ACO sample (Batuski et al. 1989). The cluster–redshift selection function, \( P(z) \), is determined in the usual way (cf. Postman et al. 1989), by fitting the cluster density, as a function of \( z \). Cluster distances are estimated using the standard relation:

\[
R = \frac{c}{H_o q_o^2 (1 + z)} \left[ q_o z + (1 - q_o) (1 - \sqrt{2q_o z + 1}) \right],
\]

with \( H_o = 100 \ h \ \text{km sec}^{-1} \ \text{Mpc}^{-1} \) and \( q_o = \Omega_o/2 \). Strictly speaking, equation (5) holds only for the case of a vanishing cosmological constant. Therefore, for a consistent comparison with the simulation models, we should use different \( R \sim z \) relations for the Abell/ACO analysis. However, we verified that final results are essentially independent of the choice of the \( (\Lambda, \Omega_o) \) parameters used in the simulations. For this reason, in the following we will present results for real data only based on assuming equation (5) with \( q_o = 0.2 \).

Note that due to the size of our simulations \( L = 320 \ h^{-1} \ \text{Mpc} \) we will restrict our analysis within a sphere of radius \( 160 \ h^{-1} \ \text{Mpc} \). Within this volume our Abell/ACO cluster sample is complete \( P(z) \approx 1 \) containing clusters all of which have measured redshifts. The Abell and ACO cluster number densities, corrected for galactic absorption according to equation (5) and within the present sample limits, are \( \sim 1.7 \times 10^{-5} \ h^3 \ \text{Mpc}^{-3} \) and \( \sim 2.3 \times 10^{-5} \ h^3 \ \text{Mpc}^{-3} \) respectively. The higher space–density of ACO clusters is partly due to the unique Shapley concentration (Shapley 1930; Scaramella et al. 1989), but a part is also due to systematic density differences between the Abell and ACO cluster samples which has been noted in a number of studies (cf. Plionis & Valdarnini 1991 and references therein) and which could be attributed to the high sensitivity of the IIIa–J emulsion plates. In fact, excluding the small \( b > 30^\circ \) region of the ACO sample where the Shapley concentration lies (corresponding to a solid angle of \( \delta \Omega \approx 0.08\pi \) smaller than the ACO cluster density \( \sim 1.9 \times 10^{-5} \ h^3 \ \text{Mpc}^{-3} \). Therefore the mean Abell/ACO cluster separation of our sample is \( d_{sv} \approx 38 \sim 40 \ h^{-1} \ \text{Mpc} \).

In the following, we compare results based on the Abell/ACO sample with those derived from our simulated cluster populations, selected to have a similar number-density, sample volume and sky coverage. We have verified...
that variations in $d_{41}$ of the order of the Abell–ACO difference, do not significantly affect the resulting statistical properties.

It is important to stress that there is a possibility that these catalogues may be contaminated by selection effects, perhaps due to the line-of-sight projection effects (e.g. Sutherland 1988; but see Jing, Plionis & Valdarnini 1992). The effect of such errors may be less pronounced on the ‘morphological’ measures of clustering we employ in this paper than on the quantities such as the two–point correlation function used in previous work. Nevertheless, the uncertain reliability of the catalogue, together with its relatively small size, requires us to be circumspect when presenting our conclusions.

4 TOPOLOGY

In this section we explore the behaviour of a topological characteristic of the large-scale distribution of clusters, called the Euler–Poincaré characteristic, for the different simulated data sets. For general background material on topology, see Adler (1981) and Nash & Sen (1983).

4.1 Theory

One of the commonly–used quantitative measures of clustering pattern used in cosmology is the so-called genus statistic, described in detail in Melott (1990) who gives the genus $g$ of a solid object as

$$g \equiv \text{(no. holes)} - \text{(no. isolated regions)} + 1.$$  \hspace{1cm} (7)

This characteristic is generally applied to the observational data by first smoothing them to form a continuous density field, $\delta$, and then locating the regions where the smoothed field exceeds a given threshold density. Isodensity surfaces thus define solid three-dimensional objects whose topology can be defined in terms of the genus. One typically labels the threshold density in the dimensionless form, $\nu$, defined as the number of standard deviations of $\delta$ above the mean: $\delta = \nu \sigma$ (the mean value of $\delta$ is zero by construction). One of the great advantages of the characteristic $g$ is that, for a Gaussian density field in three dimensions, its mean value per unit volume, $g_S$, as a function of $\nu$ can be obtained in a simple closed form:

$$g_S = A(1 - \nu^2) \exp(-\nu^2/2)$$  \hspace{1cm} (8)

(Doroshkevich 1970; Adler 1981; Bardeen et al. 1986; Hamilton et al. 1986); the constant $A$ depends only on the first and second moments of the power spectrum of $\delta$ and can be expressed in terms of the coherence length of the random field, $\lambda_c$:

$$A = \frac{1}{4\pi^2 \lambda_c^2}$$  \hspace{1cm} (9)

where

$$\frac{1}{\lambda_c^2} = \frac{<k^2>}{3},$$  \hspace{1cm} (10)

and

$$<k^2> \equiv \int_0^\infty P(k)k^2 dk / \int_0^\infty P(k)k^4 dk.$$  \hspace{1cm} (11)

The dependence [8] means that all Gaussian fields produce the same shape curve for $g_S(\nu)$ and that the amplitude can, in principle at least, be used to determine properties of the power spectrum, $P(k)$, relatively directly from the data. Note that the coherence length will be determined both by the shape of the transfer function of the model in question and by the scale of smoothing adopted to produce the continuous density field; see below.

The genus curve for Gaussian fields is symmetric about the mean and positive for $|\nu| < 1$, indicating that threshold values around the mean give rise to contour surfaces which are multiply connected. This is characteristic ‘sponge’ topology in which high density and low density regions interlock. Non–Gaussian alternatives would be a ‘meatball’ topology in which isolated high density regions sit in a low-density background and the mirror–image of this, a ‘swiss–cheese’ topology.
The quantity $g_s$ is usually measured in practice by invoking the Gauss–Bonne theorem to relate it to the integrated curvature of the contour surfaces; the algorithm CONTOUR3D is the standard tool for performing this calculation on smoothed observational or simulated data sets (Gott, Melott & Dickinson 1986; Hamilton, Gott & Weinberg 1986; Melott, Weinberg & Gott 1988; Gott et al. 1989; Melott 1990; Moore et al. 1992; Vogeley et al. 1994). In a more recent paper, however, Coles et al. (1996) have shown that a much more simple and efficient topology-measuring algorithm can be developed from ideas presented by Adler (1981). This algorithm basically computes an approximation to the Euler–Poincaré Characteristic (EPC) $\chi$ of data defined on a grid or lattice in three dimensions. If the genus is defined according to equation (7) then $g = -\chi/2$ so that the curves of $\chi(\nu)$ and $g(\nu)$ are of the same shape except for a sign. We shall concentrate on $\chi$ from now on. The algorithm we use is explained in more detail in Coles et al. (1996), but basically one constructs a three-dimensional framework of points, lines, squares and cubes linking neighbouring points above the threshold density contrast. If there are $P$ points, $L$ lines, $S$ squares and $C$ cubes then

$$\chi \simeq P - [L + C] + S. \quad (12)$$

Points are counted whether or not they belong to lines, squares or cubes; lines are counted whether or not they belong to squares or cubes; squares are counted whether or not they form part of cubes. This calculation is a simple generalisation of the 2D equivalent which counts only points, lines and squares: the 2D version has been explored in detail in (Coles 1988; Coles & Plionis 1991; Plionis, Valdarnini & Coles 1992; Davies & Coles 1993; Coles et al. 1993). Alternative algorithms are also discussed in Coles et al. (1996).

In order to define the excursion sets appropriately one needs to smooth the initial point set with some kind of local averaging procedure. Clearly the smoothing radius adopted must be greater than, or of the order of the mean distance between points otherwise a continuous field is not created. To implement our algorithm as described above we also need to grid the data on a regular cubic lattice. The choices of grid resolution and smoothing scale are user-defined quantities and must be chosen in a pragmatic fashion. For example, the coherence of the density field should not be too large compared with the sample volume, otherwise edge effects dominate. A correction for edge effects is straightforward for periodic boundaries, such as in our simulations but is less reliable if there is a complicated boundary. One also wants the gridding to be fine enough that each piece of the excursion set is sampled by a sufficient number of cell points and that the ratio of the total number of points in the sample volume to the number on the edges is large. The smoothing scale adopted also depends on the number density of points selected: for richer clusters we need a longer smoothing length, and this may also affect the optimal choice of gridding. We discuss our final choices below.

We finally remark that we prefer to plot the behaviour $\chi(\nu)$ as a function of $\nu$ as defined above in terms of the standard deviation of the density fluctuations. Other authors (e.g. Melott 1990) prefer to plot a different version of these curves which uses the volume fraction above the threshold to calculate the effective value $\nu$ would have for the same volume fraction of a Gaussian random field (using the error function). Any dependence of the results on the one-point distribution of the fluctuations is transformed away in this latter definition, so it has the advantage of removing any effect of a monotonic local bias (e.g. Kaiser 1984; Coles 1993) on initially Gaussian fluctuations: the volume fraction remains the same in such a transformation, since excursion sets in the unbiased field are mapped into the same sets in the biased field. The justification for this is that one might be able to recover the topology of the initial density field from that of a set of locally-biased mass tracers by exploiting this property. On the other hand, this definition may conceal information about the form of the bias if it is non-local or non-monotonic. In the case we are interested in, the clusters are defined as peaks of a non-linear and therefore non-Gaussian density field and it is not clear what the effect of mapping back onto a Gaussian distribution will have. We therefore feel that it is better not to attempt to remove one-point information, as this may yield important clues about the biasing of clusters of different density relative to the mass distribution. One expects, for example, that clusters with higher density would be more biased than those of lower density and would therefore have a curve with a stronger apparent meatball shift.

For reference, we should point out that related approaches to the analysis of superclustering have been implemented recently. Rhoads, Gott & Postman (1994) have used the more standard ‘genus’ algorithm to study the topological properties of contour surfaces constructed from the cluster distribution, including the different choice of $\nu$ we described above. On the other hand, Kerscher et al. (1997) have used a different mathematical approach, based on the so-called Minkowski functionals, which incorporates as one of the descriptors a quantity analogous to the genus;
for a further development of these ideas, see Schmalzing & Buchert (1997). These analyses are to some extent similar in spirit to that which we present here, but there are significant differences in both philosophy and implementation.

### 4.2 Analysis

In this section, we discuss only the comparison of our models with each other and leave the detailed analysis of errors, confidence intervals and comparisons with the data until Section 6.

We performed a series of calculations of the EPC for the simulations with the standard correction for periodic boundary conditions (Coles et al. 1996). To get a feel for the effect of cluster selection on the strength of the meatball distortion produced we have considered the distribution of all the density peaks identified on the grid, as well as the distribution of the highest peaks, selected so as to produce clusters with a mean spacing of $40h^{-1}$ Mpc, comparable to the Abell/ACO catalogue.

**Figure 1**

Results are shown in Figure 1 for the EPC characteristic $\chi(\nu)$. We have chosen a Gaussian smoothing radius of $10h^{-1}$ Mpc, and have binned the smoothed field onto a $32^3$ grid. The curves shown are an average over 5 realizations of the model concerned, with error bars representing the standard deviation over this ensemble. Notice the slight asymmetry compared to the expected behaviour for a Gaussian field and the apparently anomalous shape of the $\Lambda$CDM, which appears to be very different to the other models. One should be suspicious that this difference might be due to the fact that this model is more highly evolved (i.e. it has a higher value of $\sigma_8$) than the others and the strange topological behaviour is simply due to the fact that our Zel’dovich simulation method is behaving badly for this model. In fact, this is not the case. We applied the same test to a less evolved $\Lambda$CDM model ($\sigma_8 = 0.8$; cf. Borgani et al. 1995), which was demonstrated to be evolved quite accurately by our Zel’dovich technique, but found a graph of the EPC which matched the more evolved model closely in shape. It seems therefore that the more pronounced behaviour of the $\Lambda$CDM is actually connected with intrinsic properties of the model, rather than with any limitation of our simulation method.

**Figure 2**

Figure 2 shows analogous results to Figure 1, but for clusters selected to resemble Abell/ACO clusters; we have chosen a Gaussian smoothing radius of $30h^{-1}$ Mpc, and have binned the smoothed field onto a $32^3$ grid. Notice the slight asymmetry compared to the expected behaviour for a Gaussian field. The curves shown are again an average over 5 realizations. Note the reduction in amplitude, due to the increase in smoothing length resulting in an increased coherence length, and a more drastic meatball effect as a consequence of the bias: the point of minimum $\chi$ is moved further to the left than in Figure 1. Notice also the substantially greater noise, even after averaging over 5 simulations.

It is important also to note the similarity of the curves for different models in Figure 2 compared to the clear systematic variations of the curves with model in Figure 1. This shows that the dominant effect on the behaviour of the EPC in the cluster–selected samples is that of thresholding rather than in difference in the amount of evolution or in the shape of the initial power spectrum. In particular, notice that the shape of the EPC curve for $\Lambda$CDM, although anomalous in Figure 1 is consistent with the other models when only selected clusters are used. There are nevertheless residual differences due to these other factors and, as we shall show later, they do allow discrimination between the models with some degree of statistical confidence.

### 5 PERCOLATION

#### 5.1 Theory

The use of percolation methods (e.g. Stauffer & Aharony 1992; Isichenko 1992), which have been borrowed by cosmologists from condensed matter physics, to study aspects of galaxy clustering dates back to Zel’dovich (1982) and Shandarin (1983), and for quantifying observed properties of galaxy clustering to Einasto et al. (1984); see also Zel’dovich, Einasto & Shandarin (1982). Initially, the method used was based on the idea of “decorating” each point (galaxy or cluster) in a point set with a sphere of some radius and determining the point at which these spheres overlap to ‘percolate’ the entire set. Suppose we have $N$ objects in a cubic sample of side $L$. The mean object–object
separation is defined to be $l = LN^{-1/3}$. One (notionally) draws a sphere of diameter $d = bl$, where $b$ is dimensionless, around each point and determines $L_p(b)$, the maximum distance that can be traversed while still remaining within such spheres. The spheres around neighbouring points may, of course, overlap with each other. When $L_p(b) = L$ then the set is said to percolate and the critical value of $b = b_*$ is called the percolation parameter. For a uniform set of points on a Cartesian lattice, $b_* = 1$, while if the points are distributed along lines or sheets, $b_* < 1$. This simple indicator of clustering, using only $b_*$, to quantify the connectivity of structures has not been altogether successful (Bhavsar & Barrow 1983; Dekel & West 1985; Dominik & Shandarin 1992), partly due to sensitivity to sample selection parameters but mainly due to the inadequacy of encoding in one numerical quantity (i.e. $b_*$) all the properties of the transition from the unpercolated distribution to a percolated one.

To remedy this shortcoming, and to keep the analysis as comparable as possible to that of the preceding Section on the EPC, we adopt a more sophisticated approach, as described by Klypin & Shandarin (1993); see also Mo & Börner (1990), de Lapparent et al. (1991), Yess & Shandarin (1996), Sathyaprakash, Sahni & Shandarin (1996) and Sahni, Sathyaprakash and Shandarin (1997). In this approach, we consider the application of percolation techniques to a cubic lattice on which, according to some density threshold criterion, cells are labelled as either ‘filled’ or ‘empty’. Once a cell is so labelled, clusters of cells are identified. A cluster can be defined as a connected neighbourhood of cells, where cells are connected if they either share a common side (not an edge or corner) or have a neighbour which is connected according to the previous criterion (i.e. ‘friends–of–friends’). From this definition of clusters (actually, in this context, it would be more accurate to call them ‘superclusters’), we define the size of a cluster to be the number of cells in the cluster. An infinite cluster is so called if it connects antipodal sides of the cubic lattice and the critical threshold level $\rho_c$ is the threshold level at which the first infinite cluster is formed. At $\rho_c$ the system can be thought of as undergoing a kind of phase transition, from an unconnected to a connected state. If the cubic lattice has side $L$ then the filling factor of the lattice is defined to be simply the fraction of cells labelled as “filled”, this yields the probability $p$ of a randomly–selected cell being filled. We then define the multiplicity function, $n(\eta)$, to be the average number density of clusters of size $\eta$. From this a cell can be thought of as being in one of three states: empty (with probability $1 - p$); member of a finite (non–spanning) cluster with probability $p_+ = \sum_\eta \eta n(\eta)$; or part of an infinite cluster with probability $p_\infty = \eta_{\text{max}}/L^3$. We then employ two sample statistics derived from these considerations. First is the fraction of cells belonging to infinite clusters in a given sample, which is an estimator of $p_\infty$ and which we shall call $\mu_\infty$. We define the second parameter $\mu^2$ to be the (weighted) mean square size of all clusters excluding the largest one:

$$\mu^2 = \frac{\sum_\eta \eta^2 n(\eta)}{L^{2/3} \sum_\eta n(\eta)} ;$$

the factor of $L^{2/3}$ is simply to scale results for different $L$ onto each other more simply (Klypin & Shandarin 1993).

5.2 Analysis

For the percolation analysis of the cluster simulations we have followed a similar approach to that of the EPC. That is to say, we have analysed samples containing both “all” clusters and those selected according to the Abell/ACO number–density criteria. We have analysed the simulations according to a variety of different grid resolutions but, for conciseness and to show only the most pertinent trends, we describe only those results for the same smoothing and gridding onto a cubic mesh as in Section 4; as before we define the density threshold $\rho_c$ in terms of the number of standard deviations from the mean density.

Figures 3 & 4.

Figure 3 shows the behaviour of $\mu_\infty$ for all the clusters in each model, while Figure 4 shows the same plot for the Abell/ACO selected samples. Notice that the fraction of points in the infinite cluster is generally unity for low thresholds and zero for high thresholds independent of the model, as expected. The transition from one limit to the other, however, takes place at different threshold levels for the different models: it begins at around $\nu \simeq -2$ for the
all data’ samples and at around \( \nu = -1 \) for the selected clusters; the transition is more rapid in the latter case. This must again be mainly influenced by the shape of the one–point distribution of the density fields in the two cases rather than differences in the amount of evolution or in the shape of the primordial fluctuation spectrum. As was the case for the EPC curves, it is not obvious to the uneducated eye whether the curves which are averages over 5 simulations, reveal significant departures from one model to another: we shall discuss this in more detail later.

Figure 5

Now we turn to the behaviour of \( \mu^2 \) displayed in Figure 5. For brevity we show only for the Abell/ACO type clusters and using a 32\(^3\) grid as before. Notice that the largest mean size of cluster is formed at thresholds between \( \nu = -1 \) and \( \nu = 0 \) which is where the topology also indicates the highest degree of multiple connectivity. There is an apparent ‘glitch’ in the case of the LOWH model displayed in the right–hand panel, but this is simply due to the fact that there are two effectively infinite clusters in this case: the pattern percolates in two directions at low thresholds and only one of the infinite clusters is removed in the averaging procedure. Other than this one feature (which occurs with a small but non–negligible probability) the distributions are visually similar for all models.

At this stage we simply remark that differences between the models appear to be less pronounced in terms of these two statistics than they do when we look at the topological EPC descriptor. This conclusion is not altered if one uses different gridding parameters; one simply probes the connectivity on a different scale. A more detailed analysis of the discrimination achieved between these models using these descriptors is deferred until Section 7.

6 MINIMAL SPANNING TREES AND STRUCTURE FUNCTIONS

In this section we describe the results of applying a test proposed by Pearson & Coles (1995) to these simulations. This test was originally suggested as a means to quantify the shape (i.e. filament or sheet–like geometry) of galaxy clustering, rather than its topology (i.e. connectivity) which was the case in the previous two descriptors. Rather than smoothing the data, one constructs the Minimal Spanning Tree (MST) of the point set and then quantifies the shape of pieces of the tree using a set of three shape parameters suggested by Babul & Starkman (1992); see also Luo & Vishniac (1995) and Davé et al. (1997). We should say at the outset that the shape of superclustering is expected to be much more poorly defined than that of galaxy clustering, both because the cluster distribution is dominated by small number statistics (which one ameliorates in the topology analysis by smoothing) and because the formation of structures of a well–defined dimensionality is not expected on the very large scales (which are evolving in a quasi–linear fashion) probed by clusters. One might expect the performance of any shape statistic therefore to be rather poorer than a topological descriptor. This will, in fact, be what we find.

6.1 Theory

Full details of our method are described in Pearson et al. (1995); we simply define our notation here. The MST (Ore 1962; Gower & Ross 1969; Zahn 1971) is derived from graph theory and is a construct that (uniquely) connects a set of \( N \) points (‘nodes’) with \( N – 1 \) straight lines (‘edges’) in such a way that the sum of the edge lengths is a minimum and there are no closed circuits in the graph thus formed. For applications of this construction to galaxy clustering problems, see (Barrow, Bhavsar & Sonoda 1984,1985; Bhavsar & Ling 1988a,b; Plionis, Valdarnini & Jing 1992; Krzewina & Saslaw 1996). For an interesting discussion of the relationship between the MST and the percolation approach used in the previous section, see Bhavsar & Splinter (1996).

Once the tree is constructed it can be separated by removing all the edges that exceed a specified cutoff length. As is usual, we define the cutoff length as a multiple of the mean edge length of the MST so as to delete chance linkages. After separation, the MST will fall into a number of disjoint trees whose properties can be further explored individually. Pearson & Coles (1995) showed how separation can be used to enhance structures relative to surrounding noise.

The MST is a construction rather than a statistic so in order to use it to describe galaxy clustering we have to quantify the shape of the tree(s). We have chosen to use the structure functions \( S_1, S_2 \) and \( S_3 \) defined originally by Babul & Starkman (1992). These are calculated by first defining the moment of inertia tensor around the centre of mass of each piece of the separated tree. The eigenvalues of this tensor are used to define quantities \( S_i \) such that \( 0 \leq S_i \leq 1 \) and \((S_1, S_2, S_3) = (0, 0, 1)\) for a spherical distribution, \((S_1, S_2, S_3) = (0, 1, 0)\) for a flat sheet, and
(S₁, S₂, S₃) = (1, 0, 0) for a straight filament. The functions are further designed to fall away rapidly from unity as a structure deviates from the shape specified by a particular value of Sᵢ. One can then look, for example, at the distribution of Sᵢ values over the pieces of the tree; see Pearson & Coles (1995) for further details.

6.2 Analysis

The analysis we carry out here follows that in Pearson & Coles (1995) which was applied to the simulations of Borgani et al. (1994) and which showed that this method could discern differences between the models presented in that paper. For each data set we first construct the MST, and then separate the tree as outlined above. The values of Sᵢ are then calculated for each separate piece of the tree. In Pearson & Coles (1995), it was found that the structure functions S₁ and S₂ were close to zero for the Borgani et al. (1994) simulations indicating the lack of any obvious filamentary or sheet–like pattern. We again found this for the newer simulations. The S₃ statistic, however, was shown to have an interesting behaviour, and we therefore look at it further here.

Figure 6 shows the distribution of S₃ values for all the simulated clusters and for a separation length equal to F\bar{x}, where \( \bar{x} \) is the mean edge length and the results are integrated over all values of F for simplicity of presentation; the trends with F are quite consistent in the different models. The TCDM and LOWH models have a more peaked distribution of S₃ values than the other models, while the ΛCDM model is much broader but has a lower average. Roughly speaking, this means that the distribution in the ΛCDM case is less spherical than in the others which is consistent with the greater amount of dynamical evolution on large scales in this model than in the other cases, and which is also evident in the analysis we performed in Section 4.

We have also looked at the differences in the number of trees formed at each model at different edge cutoff lengths (not shown). Again, the ΛCDM model stands out because it has the greatest distribution of values over which trees form: all curves peak at around F = 1.25 and fall off rapidly as F is increased.

Our most disappointing result, however, is that when we select clusters with a mean spacing of 40h⁻¹ Mpc, we find that the MST produces insufficient trees to proceed with the analysis on simulation volumes of this size. When the tree is constructed and separation performed with any reasonable value of F, one simply gets trees containing only one node. We are therefore unable to use this descriptor for more detailed tests of selected clusters. As we feared, small number effects prevents us using this method for samples as sparsely sampled and within such a small volume as our simulations.

The conclusion of this section is, therefore, that while the MST method proposed by Pearson & Coles (1995) can indeed discriminate between different underlying distributions, the shot–noise associated with clusters, and the lack of a pattern with a specific dimensionality, seems to pose insuperable problems for testing the cluster distribution using data sets of size comparable to those we have used in this study.

7 STATISTICAL TESTS

The results we have discussed so far have been displayed for visual interpretation only, and without a detailed study of the errors and resulting confidence limits. We have shown results only for the theoretical simulations, and not for the real cluster catalogue. Now there are two main tasks one might be interested in setting for clustering descriptors of the kind we have discussed so far in this paper. One is to indicate differences between the pattern displayed by the real cluster catalogue. Now there are two main tasks one might be interested in setting for clustering descriptors (i) between the models and (ii) between the models and the data. We constructed distributions for each of
Table 2. Power of EPC as a discriminator. The diagonal (in boldface) shows the comparison with the real Abell/ACO data, above right of the diagonal shows discrimination between models using the distribution of all the density peaks, below left shows discrimination between models using selected clusters only.

|        | CHDM | OCDM | SCDM | ΛCDM | LOWH | TCDM |
|--------|------|------|------|------|------|------|
| CHDM   | 0.92 | 0.58 | 0.17 | 1.00 | 0.58 | 0.75 |
| OCDM   | 0.33 | 0.92 | 0.25 | 0.91 | 0.75 | 0.91 |
| SCDM   | 0.50 | 0.83 | 0.92 | 1.00 | 0.50 | 0.91 |
| ΛCDM   | 0.66 | 0.58 | 0.91 | 0.42 | 1.00 | 1.00 |
| LOWH   | 0.50 | 0.91 | 0.91 | 0.91 | 0.23 | 1.00 |
| TCDM   | 0.50 | 0.66 | 0.09 | 0.75 | 0.00 | 1.00 |

Table 3. Power of $\mu_\infty$ as a discriminator. The diagonal (in boldface) shows the comparison with the real Abell/ACO data, above right of the diagonal shows discrimination between models using the distribution of all the density peaks, below left shows discrimination between models using selected clusters only.

|        | CHDM | OCDM | SCDM | ΛCDM | LOWH | TCDM |
|--------|------|------|------|------|------|------|
| CHDM   | 0.92 | 0.33 | 0.25 | 0.66 | 0.33 | 0.91 |
| OCDM   | 0.33 | 0.92 | 0.91 | 0.75 | 0.66 | 0.83 |
| SCDM   | 0.09 | 0.50 | 1.00 | 0.66 | 0.50 | 0.83 |
| ΛCDM   | 0.66 | 0.25 | 0.75 | 0.50 | 1.00 | 1.00 |
| LOWH   | 0.17 | 0.50 | 0.17 | 0.83 | 1.00 | 0.33 |
| TCDM   | 0.25 | 0.41 | 0.17 | 1.00 | 0.00 | 0.92 |

The quantities involved in the analyses we have described, and then used a Monte-Carlo test using the Kolmogorov-Smirnov statistic to find the fraction of times that the distributions were found to be different. This then yields a robust estimate of the statistical significance of differences between the models under each of these descriptors.

We now display the results of this procedure in a series of tables which all have the same format: along the downward diagonal we see the significance level of departures of the simulated Abell/ACO clusters (i.e. clusters selected according to the same criteria as the Abell/ACO sample) against the real data; above the diagonal shows the significance level of departures of models from each other based on the properties of all clusters in the simulation; below the diagonal we have discrimination between models based on the properties of clusters selected with a mean spacing of $40h^{-1}$ Mpc (but still in a cubic volume). To give an example, in Table 2, we see that the ΛCDM model disagrees with the Abell/ACO data at a 42% confidence level; while it is different from OCDM at the 91% level if all data are included and different to CHDM at the 66% level if only selected clusters are used.

Table 2 shows the results for the EPC only. Looking first at the diagonal reveals that all but the LOWH and TCDM models are consistent with the Abell/ACO data within 95% confidence. The best fit is ΛCDM. The ability of the method to discriminate between models is variable and can be very poor if only the selected clusters are used; for example, LOWH and TCDM appear identical in this case. Table 3 shows the results for $\mu_\infty$ only. This appears to rule out both SCDM and LOWH at the 95% level, while it is different from OCDM at the 91% level if all data are included and different to CHDM at the 66% level if only selected clusters are used.

The need for such a detailed statistical study is demonstrated by the form of Figure 7 which shows the results obtained for the EPC smoothed and gridded as in Section 4 for the Abell/ACO sample and for samples extracted from the simulations according to the same selection criteria (i.e. radial distribution and sky coverage). It is by no means obvious whether there are any systematic departures, although some of the model curves appear to be discrepant. Merely plotting error bars on this curve would not help much as differences in the shape are more important than differences in the amplitude.

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Table 4. Power of $\mu^2$ as a discriminator. The diagonal (in boldface) shows the comparison with the real Abell/ACO data, above right of the diagonal shows discrimination between models using the distribution of all the peaks, below left shows discrimination between models using selected clusters only.

|       | CHDM | OCDM | SCDM | ACDM | LOWH | TCDM |
|-------|------|------|------|------|------|------|
| CHDM  | 1.00 | 0.66 | 0.66 | 0.91 | 0.91 | 1.00 |
| OCDM  | 0.58 | 1.00 | 0.50 | 1.00 | 1.00 | 1.00 |
| SCDM  | 0.58 | 0.75 | 1.00 | 0.75 | 0.91 | 0.91 |
| ACDM  | 0.66 | 0.83 | 0.50 | 1.00 | 1.00 | 1.00 |
| LOWH  | 0.73 | 0.83 | 0.83 | 0.66 | 1.00 | 0.91 |
| TCDM  | 0.83 | 0.66 | 0.75 | 0.66 | 0.50 | 1.00 |

Table 5. Power of all tests combined into a single discriminator. The diagonal (in boldface) shows the comparison with the real Abell/ACO data, above right of the diagonal shows discrimination between models using the distribution of all the density peaks, below left shows discrimination between models using selected clusters only.

|       | CHDM | OCDM | SCDM | ACDM | LOWH | TCDM |
|-------|------|------|------|------|------|------|
| CHDM  | 0.94 | 0.53 | 0.36 | 0.86 | 0.61 | 0.89 |
| OCDM  | 0.39 | 0.94 | 0.28 | 0.88 | 0.80 | 0.91 |
| SCDM  | 0.39 | 0.69 | 0.97 | 0.81 | 0.75 | 0.89 |
| ACDM  | 0.67 | 0.56 | 0.72 | 0.64 | 1.00 | 1.00 |
| LOWH  | 0.47 | 0.75 | 0.36 | 0.81 | 1.00 | 0.50 |
| TCDM  | 0.53 | 0.58 | 0.33 | 0.83 | 0.17 | 1.00 |

different way in the percolation analysis than in the EPC analysis. In any case, there certainly seems to be stronger discrimination between models with $\mu^2$ than with $\mu_\infty$ alone.

Since the tests are not overwhelmingly powerful on an individual basis, we look at the results of combining a battery of these three into one ‘supertest’ which would make use of any complementarity that exists in these descriptors. Table 5 shows the effectiveness of combining all the tests into one. Notice that LOWH, SCDM and TCDM are all excluded with at least 95% confidence, but that the best discrimination that can be achieved between the models, though better than in the previous tables, is generally less than 95%.

8 DISCUSSION AND CONCLUSIONS

In the Introduction to this paper, we stressed that this analysis was to be treated as exploratory because there were reasonable grounds to doubt the quality of present clustering data and that looking for geometrical signatures of the pattern of superclustering was in any case difficult because of the extreme rareness of rich clusters and the consequent sparse sampling and shot-noise this implies.

Nevertheless, as a guide to the results one might expect from larger and better controlled cluster samples the results we have obtained are extremely encouraging, at least for some of the tests we have used. Although this optimism is largely based on results from simulations which may be reasonably argued to be much ‘cleaner’ than real data are likely to be, our results show at least that there are perceptible differences between these models on large scales and that these do in principle allow one to discriminate between them using shape- and topology-based descriptors.

For our topological analysis, based on the EPC, clear differences emerge between the models. One has to be a little careful here, however, because the form of the statistic we use actually contains information about the one–point distribution function of the objects, because of the choice of threshold parameter $\nu$. Remember also that the amplitude of the EPC curve is related to the coherence length of the density field and that this is simply derived from the power spectrum. Comparing the trends we see in the EPC analysis with the trends of the one–point distribution found in an analysis of the same models by Borgani et al. (1995) together with the coherence lengths of the initial power spectra, shows that the behaviour of the EPC for different simulations can, roughly speaking, be ‘explained’ in terms of these other descriptions. Although differences therefore show up between the models, they are largely the same as the differences one finds in non–topological descriptors. One would be justified therefore in saying that this descriptor does not add very much: it just provides a different way of seeing differences in one and two–point information. Nevertheless, folding such information in with the topology (which is in any case very easy to measure)
does seem to provide a simple methodology for discriminating between models which does not require the computation of power-spectra and distribution functions and may in any case incorporate at least some extra information than these quantities do.

On the other hand, the topology of the Abell/ACO data does not display the same kind of EPC graph that one would expect by looking at the results of Plionis et al. (1995) and Borgani et al. (1995) and assuming it follows the same trends as our models. This may be telling us that the Abell/ACO is essentially different to all of the models we have looked at in this paper, which in turn may mean that either all the models are incorrect or that there is something suspicious about the catalogues or the way we have interpreted them. In particular, the effects of redshift selection, galactic extinction and the differences in number density between the Abell and ACO catalogues introduce some uncertainty into our conclusions.

The one model that does have a topological description in reasonable accord with the Abell/ACO data is the ΛCDM model, a result which agrees with the results of Kerscher et al. (1997) (although the model they used had a rather smaller value of Ω_Λ = 0.65 than the model we have used here). This model also survives the tests described in Borgani et al. (1995), but there was uncertainty attached to that analysis because of the possibility of that model being too strongly clustered to be adequately described by the Zel’dovich approximations. We have shown that this extra evolution does not influence the behaviour of the EPC to any significant extent and the claim that this model can reproduce the behaviour of Abell/ACO in terms of topology and low-order moments therefore stands up to scrutiny. This, of course, still admits the possibility that this is telling us more about problems with the catalogue than about the real distribution of overdensities.

The performance of our percolation test depends strongly on the kind of statistic one extracts from the percolated set. If one looks only at the statistic \( \mu_\infty \) then the power of discrimination is mediocre, but this rises strongly if one uses \( \mu^2 \) instead or together with \( \mu_\infty \).

The one disappointment of this analysis is the performance of the MST/shape functions we introduced in Pearson & Coles (1995). Although they do perform well for relatively well–sampled distributions, we were unable to get useful results for any of the simulated samples of clusters. The application of this statistic, at least in the form we have used it here, is not recommended for extremely sparsely-sampled distributions like those of Abell clusters.

Our final conclusion, however, is that topological and geometrical descriptors (of which we have studied only three) are at least in principle capable of diagnosing differences between very sparsely-sampled distributions in a fashion which is quite independent of the one- and two-point statistics which are more familiar in the cosmological community. With the arrival of larger and better controlled samples of galaxy redshifts and the cluster catalogues which will accompany them, clustering data will not only be more amenable to this type of analysis, they will also require such an approach if one is to extract as much information as possible.

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FIGURE CAPTIONS

Figure 1. Results for the EPC ($\chi$) as a function of density threshold, $\nu$, expressed in standard deviations from the mean, for all models described in the text using all peaks found in the simulations to define the structure. Panel (a) shows CHDM (solid lines), $\Lambda$CDM (dotted) and LOWH (dashed); Panel (b) shows OCDM (solid), SCDM (dotted) and TCDM (dashed). Error bars are shown in (a) for CHDM and in (b) for OCDM only to avoid crowding the plot. The vertical scaling is arbitrary, but is identical for all the models.

Figure 2. Results for the EPC ($\chi$) as a function of density threshold, expressed in standard deviations from the mean, $\nu$ for all models described in the text using clusters selected in the simulations so as to have a fiducial mean spacing of $40h^{-1}$ Mpc. Panel (a) shows CHDM (solid lines), $\Lambda$CDM (dotted) and LOWH (dashed); Panel (b) shows OCDM (solid), SCDM (dotted) and TCDM (dashed). Error bars are shown in (a) for CHDM and in (b) for OCDM only to avoid crowding the plot. The vertical scaling is arbitrary, but is identical for all the models.

Figure 3. Results for the percolation statistic $\mu_\infty$ for all models described in the text using all the peaks found in the simulation to define the structure. Panel (a) shows CHDM (solid lines), $\Lambda$CDM (dotted) and LOWH (dashed); Panel (b) shows OCDM (solid), SCDM (dotted) and TCDM (dashed). Error bars are shown in (a) for CHDM and in (b) for OCDM only to avoid crowding the plot.

Figure 4. Results for the percolation statistic $\mu_\infty$ for all models described in the text using selected clusters only, as in Figure 2. Panel (a) shows CHDM (solid lines), $\Lambda$CDM (dotted) and LOWH (dashed); Panel (b) shows OCDM (solid), SCDM (dotted) and TCDM (dashed). Error bars are shown in (a) for CHDM and in (b) for OCDM only to avoid crowding the plot.

Figure 5. Results for the percolation statistic $\mu^2$ for all models described in the text using selected clusters only, as in Figure 2. Panel (a) shows CHDM (solid lines), $\Lambda$CDM (dotted) and LOWH (dashed); Panel (b) shows OCDM (solid), SCDM (dotted) and TCDM (dashed). Error bars are shown in (a) for CHDM and in (b) for OCDM only to avoid crowding the plot.

Figure 6. Integrated distribution of the shape-space statistic $S_3$ for all clusters selected and for all models. The distributions are obtained by co-adding distributions for various values of $F$, as described in the text. Panel (a) shows CHDM (solid lines), $\Lambda$CDM (dotted) and LOWH (dashed); Panel (b) shows OCDM (solid), SCDM (dotted) and TCDM (dashed). Error bars are now shown, as the results come from co-adding all the simulation results.

Figure 7. Results for the EPC as a function of density threshold, expressed in standard deviations from the mean, for all models described in the text. Samples were extracted according to the same selection criteria as the Abell/ACO sample which is also shown for comparison. The figures show: (a) CHDM; (b) $\Lambda$CDM; (c) TCDM; (d) LOWH; (e) OCDM; (f) SCDM; appropriate error bars are drawn on these curves. The heavy solid line in each plot shows the corresponding results for the Abell/ACO catalogue. The noisiness of these curves demonstrates the need for careful statistical assessment of the discriminatory power.
