Modelling of cyclic shells with complex geometry three-dimensional finite elements

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Abstract. At the present time, the manufacture of structures of complex shape becomes real. In the process of operation, the geometry of the structural elements can also change due to defects. However, the definition of strength is not simplified. A model of cyclic shells of complex geometry based on three-dimensional finite elements is described. Examples of calculation are considered

1. Introduction

In the process of creating constructions it is important to find rational geometry of construction’s elements based on its functional purpose, technological and economic capabilities [1, 2]. Modern technology makes it possible to manufacture various lightweight shell structures of various shapes. With the beginning of the development of 3D printing, the possibility of varying the shape is greatly expands. It becomes real to make various structural elements of complex geometry from various materials. However, the definition of strength is not simplified. To effectively use thin-walled structures of complex shape, it is important to know how to calculate the stress-strain state (SSS) of such objects. There is a need to develop methods for calculating shells of complex geometry or numerical models based on known complexes.

In [3-6], an effective method for calculating thin-walled shell structures having a complex geometry-the spline version of the finite element method is presented. The method, due to the combination of the idea of a preliminary specification of the middle surface by the parameters of a rectangular region and the finite element method (FEM) with cubic approximation of the unknown variables within each element, makes it possible to obtain high accuracy two-dimensional finite elements.

In the process of exploitation of constructions due to its contact with the environment, the structural elements of constructions are destroyed. With a relatively uniform deterioration of the surface of thin-walled structural elements to determine the change in the bearing capacity of elements, an experimental theoretical method can be used [7-9]. However, if there are local defects, for example, corrosive depressions, then to assess the load-bearing capacity it is necessary to know the level of stress concentration in this area. In this case, shell models for calculating the stress-strain state become ineffective. There is a need to use three-dimensional models, in particular [10].
2. The shell model in three-dimensional formulation

The problems of modeling cyclic shells of complex geometry by three-dimensional finite elements are considered. In a two-dimensional formulation it is impossible to estimate the level of stress concentration in the region of non-through local deepenings and surface scratches, in the region of local fixing of the contour etc.

The median surface of the shell under consideration is described by the cyclic surfaces [11], which are formed by the motion of a circle of variable radius along the elliptic curve (Fig. 1):

\[ \rho(\alpha) = \rho(\alpha) = a / \sqrt{b^2 \sin^2 \alpha + a^2 \cos^2 \alpha}, \]

where \( \tilde{r}(\alpha, \beta) \) – the radius-vector of a cyclic surface; \( \rho(\alpha) \) – radius-vector of the line of centers of forming surfaces; \( R(\alpha) \) – radius of forming circles; \( e_0(\alpha) \) – unit vectors of the rectangular coordinate system in the plane normal to the line of centers of the forming circles; \( \phi(\alpha) \) – the angle of inclination of the surface of the circle.

A special case of cyclic shells is a fragment of the toroidal shell.

We define the cyclic shell by the parameters of \( t^1, t^2, t^3 \) a rectangular parallelepiped:

\[
x = \{ \rho(t^1/2) + (R(t^1/2) + t^i) \cos \pi t^i \cos \varphi_a \} \cos(\pi t^i/2),
\]

\[
y = \{ \rho(t^1/2) + (R(t^1/2) + t^i) \cos \pi t^i \sin \varphi_a \} \sin(\pi t^i/2),
\]

\[
z = (R(t^1/2) + t^i) \sin(\varphi_a), \quad t^i \in [0; 1], \quad t^2 \in [-1; 1], \quad t^3 \in [0; 1]
\]

(2)

\[
\cos \varphi_a = \sqrt{1 - \cos^2 \varphi_a},
\]

where parameters of middle surface of shell on (1) through parameters \( t^1, t^2, t^3 \) expressed on formulas \( \alpha = \pi t^1/2, \ \beta = \pi t^3 \).

Knowing the relation (2), it is not difficult to determine the coordinate vectors: \( \tilde{F} = \partial \tilde{F} / \partial t^1, \ \tilde{F}_2 = \partial \tilde{F} / \partial t^2, \ \tilde{F}_3 = \partial \tilde{F} / \partial t^3 \); covariant components and the discriminant of the metric tensor:

\[ g_{11} = \tilde{F}_1 \cdot \tilde{F}_1, \quad g_{12} = \tilde{F}_1 \cdot \tilde{F}_2, \quad g_{22} = \tilde{F}_2 \cdot \tilde{F}_2, \quad g = g_{33} = (g_{11} g_{22} - g_{12}^2) - g_{32} (g_{11} g_{23} - g_{12} g_{13}) + g_{31} (g_{12} g_{23} - g_{13} g_{22}),
\]

Christoffel symbols: \( \Gamma^\gamma_{\beta \delta} = \partial \Gamma^\gamma_{\beta \delta} / \partial t^\beta + \partial \Gamma^\gamma_{\beta \delta} / \partial t^\delta + \partial \Gamma^\gamma_{\beta \delta} / \partial t^\gamma / 2 \).

It is assumed that there is a local (corrosive) deepening on the surface of the considered shell. There are two vary of specifying a deepening. In the first variant - in the defect area, we exclude the
needed number of elements from the grid of the partition (Figure 2a). In the second variant, in the defective element (Figure 2b), we specify a deepening of elliptical shape in the plan with the semi-axes a and b in the form:

\[
x = \{ \rho (\pi t^3/2) + [R(\pi t^3/2) + (t^1 - t_{s_{wop}}(t^2, t^3)(t^1 - t_{p}^1))] \cos \pi t^2 \cos \varphi_a \} \cos (\pi t^3/2),
\]

\[
y = \{ \rho (\pi t^3/2) + [R(\pi t^3/2) + (t^1 - t_{s_{wop}}(t^2, t^3)(t^1 - t_{p}^1))] \cos \pi t^2 \sin \varphi_a \} \sin (\pi t^3/2),
\]

\[
z = [R(\pi t^3/2) + (t^1 - t_{s_{wop}}(t^2, t^3)(t^1 - t_{p}^1))] \sin \varphi_a, \quad h_{wop}(t^2, t^3) = h_p e^{-\chi},
\]

\[
\chi = \gamma \left( (t^3 - t_{p}^3)^2/a^2 + (t^2 - t_{p}^2)^2/b^2 \right), \quad t^1 \in [0; 1], \quad t^2 \in [-1; 1], \quad t^3 \in [0; 1],
\]

(3)

where \( \gamma \) – degree of compression-extension of the ellipse with respect to the coordinates \( t^2 \) and \( t^3 \); \( h_p \) – the minimum shell thickness in the defective region at the point \( t^2 = t_{p}^2, t^3 = t_{p}^3 \); \( t^1, t^2, t^3 \) – coordinates of the center of the defective area.

Differentiating (3) by the parameters \( t^1, t^2, t^3 \), we determine the coordinate vectors \( \vec{r}_1, \vec{r}_2, \vec{r}_3 \), the parameters of the metric tensor \( g_{ij} \), and the Christoffel symbols \( \Gamma^i_{jk} \) for the defect area.

The resolving relations are derived from the Lagrange variational equation:

\[
\delta \int_0^1 \int_0^1 \int_0^1 W \sqrt{g} dt^1 dt^2 dt^3 = \int_0^1 \int_0^1 \int_0^1 \rho f^{ji} \delta u_j \sqrt{g} dt^1 dt^2 dt^3 + \int_S p^i \delta u_i dS, \quad (4)
\]

where \( W \) – the specific potential energy of deformation of the three-dimensional body; \( f^i, p^i \) – components of the mass and surface forces vector; \( \rho \) – is the mass density; \( u_i \) are components of the vector of the required variables; \( S \) – is the surface of the lateral faces of the body.

In deriving the resolving equations, the connection of components of the strain tensor \( e_{ik} \) are through the components \( u_i \) is used: \( e_{ik} = (\partial u_i / \partial t^j + \partial u_j / \partial t^i)/2 - \Gamma^i_{jk} u_k \) and the connection of the components of the stress tensor \( \sigma^{ij} \) through \( e_{ik} \) is used: \( \{ \sigma^{ij} \} = (B_0) \{ e^{ik} \} + (B_{ik}) \{ e^{ijk+1} \} \), \( \{ \sigma^{ij} \} = (B_{i, j, k}) \{ e^{ik} \} + (B_{i, j, k+1}) \{ e^{ijk+1} \} \), \( \sigma^j = \sigma^{ij} \), \( i, j, k = 1, 2, 3; B_{ij} \) – values that depend on the properties of the material and the metric.

The considered area of the parallelepiped is divided into finite elements and the solution \( u_1 = u, u_2 = v \) and \( u_3 = w \) in each of them is represented as an interpolation Hermitian cubic spline of three variables. The relations for the unknown variables \( u, v \) and \( w \) are:

\[
u = \left[ \kappa_1 \left( t^1 - t^1_l \right) \times \kappa_2 \left( t^2 - t^2_l \right) \times \kappa_3 \left( t^3 - t^3_l \right) \right] \otimes F_v, \quad v = \left[ \kappa_1 \left( t^1 - t^1_l \right) \times \kappa_2 \left( t^2 - t^2_l \right) \times \kappa_3 \left( t^3 - t^3_l \right) \right] \otimes F_v,
\]

\[
w = \left[ \kappa_1 \left( t^1 - t^1_l \right) \times \kappa_2 \left( t^2 - t^2_l \right) \times \kappa_3 \left( t^3 - t^3_l \right) \right] \otimes F_w, \quad (5)
\]
where $\kappa_1$, $\kappa_2$ и $\kappa_3$ are vectors of coordinate functions in three corresponding coordinate lines, $F_u$, $F_v$, $F_w$ are three-dimensional matrices of the unknown unknowns $u$, $v$ and $w$ and its derivatives:

$u^{000}_{i,j,k}, u^{100}_{i,j,k}, u^{010}_{i,j,k}, u^{110}_{i,j,k}, u^{101}_{i,j,k}, u^{011}_{i,j,k}, v^{000}_{i,j,k}, v^{100}_{i,j,k}, v^{010}_{i,j,k}, v^{110}_{i,j,k}, v^{101}_{i,j,k}, v^{011}_{i,j,k}$ and

$w^{000}_{i,j,k}, w^{100}_{i,j,k}, w^{010}_{i,j,k}, w^{110}_{i,j,k}, w^{101}_{i,j,k}, w^{011}_{i,j,k}$. 

The problem reduces to a system of 24$A$ algebraic equations of the form $[A] \{U\} = \{R\}$, where $[A]$ is the symmetric stiffness matrix of the ribbon structure system, $\{U\}$ is a vector of unknowns, $\{R\}$ is the load vector, the total number of nodes in considered 3D object $A = MNL$, where $M$, $N$ and $L$ are the number of nodes in three directions, respectively.

Calculations are performed with double precision. The above described approach is implemented in the form of a software package on the basis of which test tasks are solved. In particular, the fragment of the toroidal shell (a special case of cyclic shells) is calculated. The analysis of the results showed the reliability of the obtained results. Specific examples of calculation of cyclic shells "with" and "without" local deepening are also considered, the analysis of the results is made, and conclusions are drawn.

**Example 1**

The smooth cyclic shell (Fig. 3) with parameters (according to Fig. 1) is considered: $R_0 = 50$ cm, $R_a = 30$ cm, $a = 170$ cm, $b = 150$ cm, thickness $h = 2$ cm, shell, fixed at the end ($R_b = 50$ cm) along the inner contour. The shell is under internal pressure $p = 10$ МПа. The elastic modulus is $E = 210000$ MPa, the Poisson's ratio is $\mu = 0.3$. The shell is divided into 64 elements (total 144 nodes) along the thickness $t_1 = \{0; 2\}$; with respect to the angle $\alpha \in [0, \pi/2]$: $t_2 = \pi/2 \{0.0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1.0\}$ and in the angle $\beta \in [-\pi, +\pi]$: $t_3 = \pi \{-1.0, -0.75, -0.5, -0.25, 0.0, 0.25, 0.5, 0.75, 1.0\}$.

Figure 4 shows the distribution of radial displacement $u$ (in cm) at a depth of 0.1 cm from the outer surface.
As can be seen from Figures 5-6, the maximum stresses $\sigma_{22}$ are observed on the inner and outer surfaces near $R_a = 30$ cm. Maximum stresses $\sigma_{33}$ are observed on the inner and outer surfaces near $R_a = 50$ cm.

Example 2

The cyclic shell with the local deepening on the outer surface is considered. Shell parameters as in Example 1. Coordinates of the center $t_p = 0.5$, $t_t = 0$ of the defective area. Deepening depth of the defective area $h_p = 1.2$ cm.

Figures 9 and 10 show stress distributions $\sigma_{22}$ in the defect area near the inner and outer surfaces of the shell, respectively. As can be seen from Figures 9 and 10 in the defective region the stress
concentration of $\sigma_{22}$ is observed. The stresses near the outer surface are higher than on the inner surface.

**Figure 9** - Intensity of stress $\sigma_{22}$ (kg/cm$^2$) near the inner surface

**Figure 10** - Intensity of stress $\sigma_{22}$ (kg/cm$^2$) near the outer surface

**Conclusion**

The combination of the idea of parameterization of the entire region by the parameters of a parallelepiped and the approximation of the unknown variables within each element by Hermitian cubic polynomials in all three directions makes it possible to obtain highly accurate matched three-dimensional finite elements and to calculate the stress-strain state of cyclic shells of complex geometry. The considered variants of with local deepening make it possible to estimate the level of stress concentration in the defective area.

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