AFFINE WALLED BRAUER ALGEBRAS

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ABSTRACT. A new class of associative algebras referred to as affine walled Brauer algebras are introduced. These algebras are free with infinite rank over a commutative ring containing 1. Then level two walled Brauer algebras over \( \mathbb{C} \) are defined, which are some cyclotomic quotients of affine walled Brauer algebras. We establish a super Schur-Weyl duality between affine walled Brauer algebras and general linear Lie superalgebras, and realize level two walled Brauer algebras as endomorphism algebras of tensor modules of Kac modules with mixed tensor products of the natural module and its dual over general linear Lie superalgebras, under some conditions. We also prove the weakly cellularity of level two walled Brauer algebras, and give a classification of their irreducible modules over \( \mathbb{C} \). This in turn enables us to classify the indecomposable direct summands of the said tensor modules.

1. Introduction

Schur-Weyl reciprocities set up close relationship between polynomial representations of general linear groups \( GL_n \) over \( \mathbb{C} \) and representations of symmetric groups \( S_r \) [13]. Such results have been generalized in various cases. Brauer [3] studied similar problems for symplectic groups and orthogonal groups. A class of associative algebras \( B_r(\delta) \), called Brauer algebras, came into the picture, which play the same important role as that of symmetric groups in Schur’s work.

Walled Brauer algebras or rational Brauer algebras \( B_{r,t}(\delta) \) (cf. Definition 2.2) are subalgebras of Brauer algebras \( B_{r+t}(\delta) \). They first appeared independently in Koike’s work [18] and Turaev’s work [25], which were partially motivated by Schur-Weyl dualities between walled Brauer algebras and general linear groups arising from mutually commuting actions on mixed tensor modules \( V^{\otimes r} \otimes (V^*)^{\otimes t} \) of the \( r \)-th power of the natural module \( V \) and the \( t \)-th power of the dual natural module \( V^* \) of \( GL_n \) for various \( r,t \in \mathbb{Z}_{\geq 0} \). Benkart, etc. [2] used walled Brauer algebras to study decompositions of mixed tensor modules of \( GL_n \). Since then, walled Brauer algebras have been intensively studied, e.g., [5, 7, 8, 9, 19, 20], etc. In particular, blocks and decomposition matrices of walled Brauer algebras over \( \mathbb{C} \) were determined in [7, 8]. Recently, Brundan and Stroppel [4] obtained \( \mathbb{Z} \)-gradings on \( B_{r,t}(\delta) \), proved the Koszulity of \( B_{r,t}(\delta) \) and established Morita equivalences between \( B_{r,t}(\delta) \) and idempotent truncations of certain infinite dimensional versions of Khovanov’s arc algebras.

In 2002, by studying mixed tensor modules of general linear Lie superalgebras \( \mathfrak{gl}_{m|n} \), Shader and Moon [20] set up super Schur-Weyl dualities between walled Brauer algebras and general linear Lie superalgebras. By studying tensor modules \( K_\lambda \otimes V^{\otimes r} \) of Kac modules \( K_\lambda \) with the \( r \)-th power \( V^{\otimes r} \) of the natural module \( V \) of \( \mathfrak{gl}_{m|n} \), Brundan and Stroppel [4] further established super Schur-Weyl dualities between level two Hecke algebras \( H^{p,q}_r \) (denoted as \( \mathcal{H}_{2,r} \) in the present paper) and general linear Lie superalgebras \( \mathfrak{gl}_{m|n} \), which provide powerful tools enabling them to obtain various results.
including a spectacular one on Morita equivalences between blocks of categories of finite dimensional $\mathfrak{gl}_{m|n}$-modules and categories of finite-dimensional left modules over some generalized Khovanov’s diagram algebras. A natural question is, what kind of algebras may come into the play if one replaces the tensor modules $K_\lambda \otimes V^r \otimes (V^*)^t$ of Kac modules $K_\lambda$ with the $r$-th power of the natural module $V$ and the $t$-th power of the dual natural module $V^*$ of $\mathfrak{gl}_{m|n}$. This is one of our motivations to introduce a new class of associative algebras $B_{r,t}^{\text{aff}}$ (cf. Definition 2.7), referred to as affine walled Brauer algebras, over a commutative ring containing 1.

At a first glance, the objects $B_{r,t}^{\text{aff}}$ we defined, which have as many as 26 defining relations, are artificial. However, a surprising thing is that not only level two walled Brauer algebras $B_{p,q}^{r,t}(m,n)$ (defined as cyclotomic quotients of affine walled Brauer algebras $B_{r,t}^{\text{aff}}$ with special defining parameters, cf. (5.35) and Definition 5.17) have weakly cellular structures (cf. Theorem 6.12), but also there is a super Schur-Weyl duality between affine walled algebras $B_{r,t}^{\text{aff}}$ and general linear Lie superalgebras $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$. In this case, level two walled Brauer algebras $B_{p,q}^{r,t}(m,n)$ can be realized as endomorphism algebras of tensor modules $V^r \otimes K_\lambda \otimes (V^*)^t$, under some conditions (cf. Theorem 5.16). Furthermore, using the weakly cellular structures on $B_{p,q}^{r,t}(m,n)$, we are able to give a classification of their irreducible modules (cf. Theorem 7.6). The result in turn enables us to classify the indecomposable direct summands of the said tensor modules of $\mathfrak{gl}_{m|n}$ (cf. Theorem 7.7). In contrast to level two Hecke algebras $H_{p,q}^{r,q}$ in \cite{IV}, which only depend on $p-q$ and $r$, level two walled Brauer algebras $B_{p,q}^{r,t}(m,n)$ heavily depend on parameters $p-q, r, t, m, n$ (cf. Remark 5.19). Nevertheless, affine walled algebras can be regarded as affinizations of walled Brauer algebras. In this sense, the appearing of affine walled algebras $B_{r,t}^{\text{aff}}$ is natural.

Another motivation comes from Nazarov’s works \cite{21} on the Juscy-Murphy elements of Brauer algebras and affine Wenzl algebras. We construct a family of Juscy-Murphy-like elements of walled Brauer algebras (cf. Definition 3.2), which have close relationship with their certain central elements. By studying properties of these elements in details, we are not only able to give the precise definition of affine walled Brauer algebras $B_{r,t}^{\text{aff}}$ (which can also be regarded as counterparts of Nazarov’s affine Wenzl algebras \cite{21} in this sense), but also able to set up a family of homomorphisms $\phi_k$ from affine walled Brauer algebras $B_{r,t}^{\text{aff}}$ to walled Brauer algebras $B_{k+r,k+r}(\omega_0)$ for any $k \in \mathbb{Z}^{\geq 1}$ (cf. Theorem 3.12). This then enables us to use the freeness of walled Brauer algebras to prove that affine walled Brauer algebras $B_{r,t}^{\text{aff}}$ are free with infinite rank over a commutative ring containing 1 (cf. Theorems 4.13 and 4.15).

It is very natural to ask whether level two walled Brauer algebras will play the role similar to that of level two Hecke algebras in Brundan-Stroppel’s work \cite{4}, etc. This will be persuaded in a sequel, where we will establish some relationship between decomposition matrices of level two walled Brauer algebras and structures of indecomposable direct summands of the above-mentioned tensor modules, which are tilting modules when Kac modules $K_\lambda$ are typical.

We organize the paper as follows. In Section 2, after recalling the notion of walled Brauer algebras, we introduce affine walled Brauer algebras over a commutative ring $R$ containing 1. In Section 3, we introduce a family of Juscy-Murphy-like elements of
walled Brauer algebras and establish a family of homomorphisms from affine walled Brauer algebras to walled Brauer algebras. Using these homomorphisms and the freeness of walled Brauer algebras, we prove the freeness of affine walled Brauer algebras in Section 4. In Section 5, we study the super Schur-Weyl duality between affine walled Brauer algebras (more precisely, level two walled Brauer algebras) with special parameters and general linear Lie superalgebras. In Section 6, we construct a weakly cellular basis of level two walled Brauer algebras. Finally in Section 7, we give a classification of their irreducible modules, and a classification of the indecomposable direct summands of the aforementioned tensor modules.

2. The walled Brauer algebra $B_{r,t}$ and its affinization

Throughout this section, let $R$ be a commutative ring containing 1. The walled Brauer algebra is an associative algebra over $R$ spanned by so called walled Brauer diagrams as follows.

Fix two positive integers $r$ and $t$. A walled $(r,t)$-Brauer diagram is a diagram with $(r+t)$ vertices on the top and bottom rows, and vertices on both rows are labeled from left to right by $r, \ldots, 2, 1, \bar{1}, \bar{2}, \ldots, \bar{t}$. Every vertex $i \in \{1, 2, \ldots, r\}$ (resp., $\bar{i} \in \{1, 2, \ldots, t\}$) on each row must be connected to a unique vertex $\bar{j}$ (resp., $j$) on the same row or a unique vertex $j$ (resp., $\bar{j}$) on the other row. In this way, we obtain 4 types of pairs $[i,j]$, $[i,\bar{j}]$, $[\bar{i},j]$ and $[\bar{i},\bar{j}]$. The pairs $[i,j]$ and $[\bar{i},\bar{j}]$ are called vertical edges, and the pairs $[i,\bar{j}]$ and $[\bar{i},j]$ are called horizontal edges. If we imagine that there is a wall which separates the vertices 1, $\bar{1}$ on both top and bottom rows, then a walled $(r,t)$-Brauer diagram is a diagram with $(r+t)$ vertices on both rows such that each vertical edge can not cross the wall and each horizontal edge has to cross the wall. For convenience, we call a walled $(r,t)$-Brauer diagram a walled Brauer diagram if there is no confusion.

Example 2.1. The following are $(r,t)$-Brauer diagrams:

Figure 1

\[
\begin{array}{c}
... \hline 3 & 2 & 1 & \bar{1} & 2 & 3 \\
... & \hline 3 & 2 & 1 & \bar{1} & 2 & 3 \\
\end{array}
\]

Figure 2

\[
\begin{array}{c}
... \hline i+2 & i+1 & i & i-1 & 1 & \bar{2} \\
... & \hline i+2 & i+1 & i & i-1 & 1 & \bar{2} \\
\end{array}
\]

Figure 3

\[
\begin{array}{c}
... \hline 2 & 1 & \bar{1} & \bar{2} & \bar{i-1} & \bar{i} & i+1 & i+2 \\
... & \hline 2 & 1 & \bar{1} & \bar{2} & \bar{i-1} & \bar{i} & i+1 & i+2 \\
\end{array}
\]
Throughout, we denote by $e_i, s_i, \bar{s}_i$ the diagrams in Figures 1–3, respectively.

In order to define the product of two walled Brauer diagrams, we consider the composition $D_1 \circ D_2$ of two walled Brauer diagrams $D_1$ and $D_2$, which is obtained by putting $D_1$ above $D_2$ and connecting each vertex on the bottom row of $D_1$ to the corresponding vertex on the top row of $D_2$. If we remove all circles of $D_1 \circ D_2$, we will get a walled Brauer diagram, say $D_3$. Let $n(D_1, D_2)$ be the number of circles appearing in $D_1 \circ D_2$. Then the product $D_1 D_2$ of $D_1$ and $D_2$ is defined to be $\delta^{n(D_1, D_2)} D_3$, where $\delta$ is a fixed element in $R$.

**Definition 2.2.** [18, 25] The walled Brauer algebra $B_{r,t}(\delta)$ with respect to the defining parameter $\delta$ is the associative algebra over $R$ spanned by all walled $(r,t)$-Brauer diagrams with product defined as above.

**Remark 2.3.** If we allow vertical edges can cross the wall and allow horizontal edges may not cross the wall (namely, a vertex can be connected to any other vertex), then we obtain $(r + t)$-Brauer diagrams. The Brauer algebra $B_{r+t}(\delta)$ is the free $R$-modules spanned by all $(r + t)$-Brauer diagrams with product defined as above. Thus a walled Brauer diagram is a Brauer diagram, and the walled Brauer algebra $B_{r,t}(\delta)$ is a subalgebra of the Brauer algebra $B_{r+t}(\delta)$.

The following result can be found in [14, Corollary 4.5] for a special case and [19, Theorem 4.1] in general.

**Theorem 2.4.** Let $R$ be a commutative ring containing 1 and $\delta$. Then $B_{r,t}(\delta)$ is an associative $R$-algebra generated by $e_i, s_i, \bar{s}_j$ with $1 \leq i \leq r - 1$ and $1 \leq j \leq t - 1$ subject to the following relations

1. $s_i^2 = 1, 1 \leq i < r,$
2. $s_i \bar{s}_j = s_j s_i, |i - j| > 1,$
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, 1 \leq i < r - 1,$
4. $e_1 s_i e_1 = e_i, 2 \leq i < r,$
5. $s_i e_1 e_1 = e_i,$
6. $e_i^2 = \delta e_i,$
7. $s_i \bar{s}_j = \bar{s}_j s_i,$
8. $\bar{s}_i^2 = 1, 1 \leq i < t,$
9. $\bar{s}_i \bar{s}_j = \bar{s}_j \bar{s}_i, |i - j| > 1,$
10. $s_i \bar{s}_{i+1} \bar{s}_i = \bar{s}_{i+1} \bar{s}_i \bar{s}_{i+1}, 1 \leq i < t - 1,$
11. $\bar{s}_i e_1 = e_1 \bar{s}_i, 2 \leq i < t,$
12. $e_1 \bar{s}_1 e_1 = e_1,$
13. $e_1 s_1 \bar{s}_1 e_1 s_1 = e_1 s_1 \bar{s}_1 e_1 s_1,$
14. $s_1 e_1 s_1 \bar{s}_1 e_1 = \bar{s}_1 e_1 s_1 \bar{s}_1 e_1.$

In particular, the rank of $B_{r,t}(\delta)$ is $(r + t)!$.

We remark that Jung and Kang gave a presentation of walled Brauer superalgebras in [15, Theorem 5.1], and the presentation of walled Brauer algebras in Theorem 2.4 can be obtained from those of walled Brauer superalgebras by removing the generators of Clifford algebras inside walled Brauer superalgebras.

The following two results can be deduced from Theorem 2.4 easily.

**Lemma 2.5.** There is an $R$-linear anti-involution $\sigma : B_{r,t}(\delta) \to B_{r,t}(\delta)$ fixing defining generators $s_i, \bar{s}_i$ and $e_1$ for all possible $i, j$’s.

**Proof.** The result follows from the symmetry of relations in Theorem 2.4 immediately. In particular, the image of a walled Brauer diagram $D$ under the map $\sigma$ is the diagram which is obtained from $D$ by reflecting along a horizontal line. □
Corollary 2.6. We have $\mathcal{B}_{r,t}(\delta) \cong \mathcal{B}_{t,r}(\delta)$. In particular, the corresponding isomorphism sends $s_i, e_1, \bar{s}_j$ of $\mathcal{B}_{r,t}(\delta)$ to $\bar{s}_i, e_1, s_j$ of $\mathcal{B}_{t,r}(\delta)$.

Proof. One can easily observe that the automorphism can be obtained by first rotating a diagram through $180^\circ$ and then reflecting along a horizontal line. \qed

In the present paper, we shall introduce a new class of associative algebras, which are natural generalizations of walled Brauer algebras, and thus can be regarded as affinizations of walled Brauer algebras. Such algebras can also be considered as the counterparts of Nazarov’s affine Wenzl algebras in \cite{21}. This is one of our motivations to introduce these algebras. Another motivation originates from super Schur-Weyl dualities in \cite{4, 20} and ours in Section 5.

Definition 2.7. Let $R$ be a commutative ring containing $1, \omega_0, \omega_1$. Fix $r, t \in \mathbb{Z}_{\geq 0}$. The affine walled Brauer algebra $\mathcal{B}_{r,t}^{\text{aff}}(\omega_0, \omega_1)$ is the associative $R$-algebra generated by $e_1, x_1, \bar{x}_1, s_i (1 \leq i \leq r-1), \bar{s}_j (1 \leq j \leq t-1)$, and two families of central elements $\omega_k (k \in \mathbb{Z}_{\geq 2}), \bar{\omega}_k (k \in \mathbb{Z}_{\geq 0})$, subject to the following relations:

1. $s_i^2 = 1$, $1 \leq i < r$,
2. $s_is_j = s_js_i, |i - j| > 1$,
3. $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, 1 \leq i < r - 1$,
4. $s_ie_1 = e_1s_i, 2 \leq i < r$,
5. $e_1s_1e_1 = e_1$,
6. $e_0^2 = \omega_0 e_1$,
7. $s_is_j = s_js_i$,
8. $e_1(x_1 + \bar{x}_1) = (x_1 + \bar{x}_1)e_1 = 0$,
9. $e_1s_1x_1s_1 = s_1x_1s_1e_1$,
10. $s_ix_1 = x_is_i, 2 \leq i < r$,
11. $s_i\bar{x}_1 = \bar{x}_1s_i, 1 \leq i < r$,
12. $e_1x_1^ek_1 = \omega_k e_1, \forall k \in \mathbb{Z}_{\geq 0}$,
13. $x_1(s_1x_1s_1 - s_1) = (s_1x_1s_1 - s_1)x_1$,
14. $s_is_j = s_js_i, |i - j| > 1$,
15. $\bar{s}_j\bar{s}_j = \bar{s}_j\bar{s}_j, |i - j| > 1$,
16. $\bar{s}_j\bar{s}_{i+1}\bar{s}_j = \bar{s}_{i+1}\bar{s}_j\bar{s}_i, 1 \leq i < t - 1$,
17. $\bar{s}_ie_1 = e_1\bar{s}_i, 2 \leq i < t$,
18. $\bar{x}_1\bar{x}_1 = \bar{x}_1\bar{x}_1$,
19. $e_1s_1\bar{s}_1e_1s_1 = e_1s_1\bar{s}_1e_1\bar{s}_1$,
20. $s_1e_1\bar{s}_1e_1s_1 = s_1e_1\bar{s}_1e_1s_1$,
21. $x_1(e_1 + \bar{x}_1) = (e_1 + \bar{x}_1)x_1$,
22. $\bar{x}_1\bar{x}_1\bar{x}_1 = \bar{x}_1\bar{x}_1\bar{x}_1$,
23. $\bar{s}_j\bar{x}_1 = \bar{x}_1\bar{s}_j, 2 \leq i < t$,
24. $\bar{s}_jx_1 = x_1\bar{s}_j, 1 \leq i < t$,
25. $e_1x_1^ek_1 = \bar{\omega}_k e_1, \forall k \in \mathbb{Z}_{\geq 0}$,
26. $x_1(\bar{s}_1\bar{x}_1\bar{s}_1 - \bar{s}_1) = (\bar{s}_1\bar{x}_1\bar{s}_1 - \bar{s}_1)x_1$.

For simplicity, we use $\mathcal{B}_{r,t}^{\text{aff}}$ instead of $\mathcal{B}_{r,t}^{\text{aff}}(\omega_0, \omega_1)$ later on. In other words, we always assume that $\mathcal{B}_{r,t}^{\text{aff}}$ is the affine walled Brauer algebra with respect to the defining parameters $\omega_0$ and $\omega_1$.

Remark 2.8. Later on we shall be mainly interested in the case when all central elements $\omega_a$ with $a \in \mathbb{Z}_{\geq 2}$ are specialized to some elements in $R$ (cf. Theorem 4.15). The reason we put $\omega_a$’s into generators is that in order to be able to prove the freeness of $\mathcal{B}_{r,t}^{\text{aff}}$ (cf. Theorem 4.13), we need to construct the homomorphism $\phi_k$ (cf. Theorem 3.12), which requires $\omega_a$’s to be generators.

In the next two sections, we shall prove that $\mathcal{B}_{r,t}^{\text{aff}}$ is a free $R$-algebra with infinite rank.

3. Homomorphisms from $\mathcal{B}_{r,t}^{\text{aff}}$ to $\mathcal{B}_{k+r,k+t}(\omega_0)$

The purpose of this section is to establish a family of algebraic homomorphisms $\phi_k$ from $\mathcal{B}_{r,t}^{\text{aff}}$ to $\mathcal{B}_{k+r,k+t}(\omega_0)$ for all $k \in \mathbb{Z}_{\geq 1}$. Then in the next section, we will use these homomorphisms and the freeness of walled Brauer algebras to prove the freeness of
\(B_{r,t}\). We remark that Nazarov \cite{21} used the freeness of Brauer algebras to prove the freeness of affine Wenzl algebras.

Unless otherwise indicated, all elements considered in this section are in the walled Brauer algebra \(B_{r,t}(\delta)\) for some \(r, t \in \mathbb{Z}^0\) with parameter \(\delta = \omega_0\).

Denote by \(\mathfrak{S}_r\) (resp., \(\tilde{\mathfrak{S}}_t\)) the symmetric group in \(r\) letters \(1, 2, \ldots, r\) (resp., \(t\) letters \(\bar{1}, \bar{2}, \ldots, \bar{t}\)). It is well-known that the subalgebra of \(B_{r,t}(\delta)\) generated by \(\{s_i \mid 1 \leq i < r\}\) (resp., \(\{\bar{s}_j \mid 1 \leq j < t\}\)) is isomorphic to the group algebra \(R\mathfrak{S}_r\) (resp., \(R\tilde{\mathfrak{S}}_t\)) of \(\mathfrak{S}_r\) (resp., \(\tilde{\mathfrak{S}}_t\)).

Let \((i, j) \in \mathfrak{S}_r\) (resp., \((\bar{i}, \bar{j}) \in \tilde{\mathfrak{S}}_t\)) be the transposition which switches \(i\) and \(j\) (resp., \(\bar{i}\) and \(\bar{j}\)) and fixes others. Then \(s_i\) and \(\bar{s}_j\) can be identified with 

\[
s_i = (i, i + 1) \quad \text{and} \quad \bar{s}_j = (j, \bar{j} + 1).
\]

Set \(L_1 = \bar{L}_1 = 0\) and

\[
L_i = \sum_{j=1}^{i-1} (j, i), \quad \bar{L}_i = \sum_{j=1}^{i-1} (\bar{j}, \bar{i}) \quad \text{for } i \geq 2.
\]

Then \(L_i\) for \(1 \leq i \leq r\) are known as the Jucys-Murphy elements of \(R\mathfrak{S}_r\), and \(\bar{L}_j\) for \(1 \leq j \leq t\) are the Jucys-Murphy elements of \(R\tilde{\mathfrak{S}}_t\). We will need the following well-known result.

**Lemma 3.1.** In \(R\mathfrak{S}_r\) and \(R\tilde{\mathfrak{S}}_t\), for all possible \(i, j\)'s, we have

1. \(L_is_j = s_js_i\), \quad \(\bar{L}_i\bar{s}_j = \bar{s}_j\bar{L}_i\) if \(i \neq j, j + 1\).
2. \(s_iL_i = L_{i+1}s_i - 1\), \quad \(\bar{s}_i\bar{L}_i = \bar{L}_{i+1}\bar{s}_i - 1\).
3. \((L_i + L_{i+1})s_i = s_i(L_i + L_{i+1})\), \((\bar{L}_i + \bar{L}_{i+1})\bar{s}_i = \bar{s}_i(\bar{L}_i + \bar{L}_{i+1})\).

For convenience, we define the following cycles in \(\mathfrak{S}_r\), where \(1 \leq i, j \leq r\),

\[
s_{i,j} = s_is_{i+1} \cdots s_{j-1} = (j, j - 1, \ldots, i) \quad \text{for } i < j,
\]

and \(s_{i,i} = 1\). If \(i > j\), we set \(s_{i,j} = s_{i,j}^{-1} = (j, j + 1, \ldots, i)\). Similarly, for \(1 \leq i, j \leq t\), we define \(e_{i,j}\) be the element whose corresponding diagram is the walled Brauer diagram such that any of its edge is of form \([k, k]\) or \([\bar{k}, \bar{k}]\) except two horizontal edges \([i, \bar{j}]\) on both top and bottom rows. Namely,

\[
e_{i,j} = s_{j,1}s_{i,1}e_{1}s_{i,1}s_{1,j} \quad \text{for } i, j \text{ with } 1 \leq i \leq r \text{ and } 1 \leq j \leq t.
\]

We also simply denote \(e_i = e_{i,i}\) for \(1 \leq i \leq \min\{r, t\}\).

It follows from \cite[Lemma 2.1]{5} and \cite[Proposition 2.5]{22} that

\[
c_{r,t} = \sum_{1 \leq i \leq r, 1 \leq j \leq t} e_{i,j} - \sum_{i=1}^{r} L_i - \sum_{j=1}^{t} \bar{L}_j,
\]

is a central element in \(B_{r,t}(\delta)\). Such a central element has already been used in \cite[Lemma 4.1]{8} to study blocks of \(B_{r,t}(\delta)\) over \(\mathbb{C}\). Motivated by \cite{34}, we define Jucys-Murphy-like elements \(y_i, \bar{y}_\ell\) below such that for any \(k \in \mathbb{Z}^\geq 1\), elements \(y_{k+1}, \bar{y}_{k+1}\) in the image of the homomorphism \(\phi_k\) (to be defined in Theorem 3.12) will play the same roles as that of \(x_1, \bar{x}_1\) in \(B_{r,t}^{\text{aff}}\).
Definition 3.2. Fix an element $\delta_1 \in R$. For $1 \leq i \leq r$ and $1 \leq \ell \leq t$, let
\[
y_i = \delta_1 + \sum_{j=1}^{i-1} e_{i,j} - L_i, \quad \text{and} \quad \bar{y}_\ell = -\delta_1 + \sum_{j=1}^{\ell-1} e_{j,\ell} - L_\ell.
\] (3.5)

Lemma 3.3. Let $i \in \mathbb{Z}$ with $1 \leq i \leq \min\{r, t\}$.

1. $e_i y_i = e_i (\delta_1 + L_i - L_i)$, \quad $e_i \bar{y}_i = e_i (-\delta_1 + L_i - L_i)$.
2. $e_i (y_i + \bar{y}_i) = 0$, \quad $(y_i + \bar{y}_i) e_i = 0$.
3. $e_i s_i y_i s_i = s_i y_i s_i e_i$, \quad $e_i \bar{s}_i \bar{y}_i \bar{s}_i = \bar{s}_i \bar{y}_i \bar{s}_i e_i$.
4. $y_i (e_i + \bar{y}_i) = (e_i + \bar{y}_i) y_i$.
5. $y_i (s_i y_i s_i - s_i) = (s_i y_i s_i - s_i) y_i$, \quad $\bar{y}_i (\bar{s}_i \bar{y}_i \bar{s}_i - \bar{s}_i) = (\bar{s}_i \bar{y}_i \bar{s}_i - \bar{s}_i) \bar{y}_i$.
6. $s_j y_i = y_i s_j$, \quad $\bar{s}_j \bar{y}_i = \bar{y}_i \bar{s}_j$ if $j \neq i - 1, i$.
7. $s_j \bar{y}_i = \bar{y}_i s_j$, \quad $s_j y_i = y_i s_j$ if $j \neq i - 1$.
8. $e_{i+1} y_i = y_i e_{i+1}$, \quad $e_{i+1} \bar{y}_i = \bar{y}_i e_{i+1}$ if $i < \min\{r, t\}$.
9. $y_i y_{i+1} = y_{i+1} y_i$, \quad $\bar{y}_i \bar{y}_{i+1} = \bar{y}_{i+1} \bar{y}_i$ if $i < \min\{r, t\}$.

Proof. We remark that the second assertion of (2) follows from the first assertion of (2) by applying the anti-involution $\sigma$ in Lemma 2.5. By Corollary 2.6, we need only check (4) and the first assertions of others.

Since $e_i e_{i,j} = e_i (j, i)$ and $e_i e_{j,i} = e_i (j, i)$ for $j \neq i$, we have (1) and (2). Further, (3) follows from the equalities $e_i e_{k,j} = e_{k,j} e_i$, $e_i (k,j) = (k,j) e_i$ if $i \not\in \{k,j\}$ together with (3.6) as follows:

\[
s_i y_i s_i = \sum_{j=1}^{i-1} e_{i+1,j} - \sum_{j=1}^{i-1} (j, i+1) + \delta_1 = y_{i+1} - s_i e_i s_i + s_i.
\] (3.6)

By Definition 3.2, we have
\[
y_i y_{i+1} - y_{i+1} y_i = \sum_{j=1}^{i-1} e_{i,j} + \sum_{k=1}^{i} e_{i+1,k} - \sum_{j=1}^{i-1} e_{i+1,j} - \sum_{j=1}^{i-1} e_{i,j} L_{i+1} + L_{i+1} \sum_{j=1}^{i-1} e_{i,j}
\]
\[
= \sum_{j=1}^{i-1} e_{i,j} s_i - s_i \sum_{j=1}^{i-1} e_{i,j} - i - 1 \sum_{j=1}^{i-1} e_{i,j} s_i + s_i \sum_{j=1}^{i-1} e_{i,j},
\]

which is equal to zero, proving (9).

Recall that $\sigma$ is the anti-involution on $\mathcal{B}_{r,t}(\delta)$ in Lemma 2.5. We have $\sigma(y_j) = y_j$ and $\sigma(s_j) = s_j$. Using (3.6) and $\sigma$, we have
\[
y_i (s_i y_i s_i - s_i) = y_i y_{i+1} - y_{i+1} y_i = s_i e_i s_i e_i - s_i e_i s_i.
\] (3.7)

By (3), we have $y_i s_i e_i s_i = s_i e_i s_i y_i$. So, (5) follows from (9).

We remark that (6) and (7) can be checked easily by using Theorem 2.4(2)\,(4). Since $e_{i+1} e_{i,j} = e_{i,j} e_{i+1}$ and $e_{i+1} (j, i) = (j, i) e_{i+1}$ for $1 \leq j \leq i-1$, we have (8).

Finally, we check (4). We have $(y_i + e_j) \bar{y}_{i+1} = \bar{y}_{i+1} (y_i + e_j)$ by $e_i \bar{y}_{i+1} = \bar{y}_{i+1} e_i$. By induction on $j$, we have $(y_j + e_j) \bar{y}_{i+1} = \bar{y}_{i+1} (y_j + e_j)$ and $e_j \bar{y}_{i+1} = \bar{y}_{i+1} e_j$ for all $j$ with $1 \leq j \leq i$. So,
\[
y_i \bar{y}_{i+1} = \bar{y}_{i+1} y_i.
\] (3.8)

By (3.6) and Corollary 2.6, $e_i + \bar{y}_i = \bar{s}_i \bar{y}_{i+1} \bar{s} + \bar{s}_i$. So, (4) follows from (3.8) and (7). \, $\square$
The following result is a special case of [8 Proposition 2.1].

**Proposition 3.4.** Let $\mathcal{B}_{r,t}(\delta)$ be defined over a field $F$. For $2 \leq k \leq \min\{r,t\}$, let $e = e_k$ if $\delta \neq 0$ or $e = e_k s_{k-1}$ otherwise. Let $\mathcal{B}_{k,k}(\delta)$ be the subalgebra of $\mathcal{B}_{r,t}(\delta)$ generated by $e_i, s_i, i, 1 \leq i \leq k$. Then $e \mathcal{B}_{k,k}(\delta)e = e \mathcal{B}_{k-1,k-1}(\delta)$, which is isomorphic to $\mathcal{B}_{k-1,k-1}(\delta)$ as an $F$-algebra.

We remark that we are assuming $\delta = \omega_0 \neq 0$. The following result immediately follows from Proposition 3.4 where elements $\omega_{a,k}, \bar{\omega}_{a,k}$ will be crucial in obtaining the homomorphisms $\phi_k$ in Theorem 3.12.

**Corollary 3.5.** For $a \in \mathbb{Z}^{>0}$, there exist unique $\omega_{a,k}, \bar{\omega}_{a,k} \in \mathcal{B}_{k-1,k-1}$ such that

$$
eq a_k e_k, \quad e_k \bar{a}_k e_k = \bar{\omega}_{a,k} e_k.$$

Furthermore, $\omega_{1,k} = -\bar{\omega}_{1,k} = \delta \delta_1$ and $\omega_{0,k} = \bar{\omega}_{0,k} = \delta$.

**Lemma 3.6.** For any $k \in \mathbb{Z}^{\geq 1}$, we have $e_i y_i^k = \sum_{j=0}^{k} a_{k,j}^{(i)} e_i y_j^i$ for some $a_{k,j}^{(i)} \in \mathcal{B}_{r,t}$ such that

1. $a_{k,k}^{(i)} = (-1)^k$,
2. $a_{k,j}^{(i)} = \omega_0 (a_{k-1,j}^{(i)} - a_{k-1,j-1}^{(i)})$, $1 \leq j \leq k - 1$,
3. $a_{k,0}^{(i)} = -\sum_{j=1}^{k-1} a_{k,j}^{(i)} \omega_{j,i}$.

In particular, $a_{k,j}^{(i)} \in R[\omega_{2,i}, \omega_{3,i}, \ldots, \omega_{k-1,i}]$ for any $j$ with $1 \leq j \leq k$ such that each monomial of $a_{k,j}^{(i)}$ is of form $\omega_{j_1,i} \cdots \omega_{j_{t-1,i}}$ with $\sum_{i=1}^{t} j_i = k - 1$.

**Proof.** By Lemma 3.3(2), the result holds for $k = 1$. In general, by Lemma 3.3(4),

$$
eq e_i y_i^k y_i = e_i e_i y_i^{k-1} - \omega_{j,i} e_i = \omega_0 e_i y_i^{k-1} - \omega_{j,i} e_i.$$

Now, the result follows from induction on $k$.

**Lemma 3.7.** For $k, a \in \mathbb{Z}^{>2}$, we have $\omega_{a,k} \in R[\omega_{2,k}, \omega_{3,k}, \ldots, \omega_{a,k}]$. Furthermore, both $\omega_{a,k}$ and $\bar{\omega}_{a,k}$ are central in $\mathcal{B}_{k-1,k-1}$.

**Proof.** The first assertion follows from Lemma 3.6. To prove the second, note that any $h \in \{e_i, s_i \mid 1 \leq i \leq k-2\}$ commutes with $e_k, y_k$. So, $e_k (h \omega_{a,k}) = e_k (\omega_{a,k} h)$. By Proposition 3.3 $h \omega_{a,k} = \omega_{a,k} h$. Finally, we need to check $e_k (h \omega_{a,k}) = e_k (\omega_{a,k} h)$ for any $h \in \{s_1, s_2, \ldots, s_{k-1}\}$. In this case, we use Lemma 3.6. More explicitly, we can use $y_k$ instead of $y_k$ in $e_k y_k e_k$. Therefore, $h \omega_{a,k} = \omega_{a,k} h$, as required.

The following result follows from (3.6) and induction on $a$.

**Lemma 3.8.** For $k, a \in \mathbb{Z}^{>1}$, we have

$$
eq s_k y_k^{a-1} = (y_k + e_k)^a s_k - \sum_{b=0}^{a-1} (y_k + e_k)^{a-1-b} y_k^b.$$

The elements $z_{j,k}, \bar{z}_{j,k}$ defined below will be crucial in the description of $\omega_{a,k}$ (cf. Lemma 3.10). For $1 \leq j \leq k - 1$, let

$$
eq z_{j,k} = s_{j,k-1}(y_{k-1} + e_{k-1}) \bar{s}_{k-1,j}, \quad \bar{z}_{j,k} = \bar{s}_{j,k-1}(\bar{y}_{k-1} + e_{k-1}) \bar{s}_{k-1,j}. \quad (3.9)$$

Then the following result can be verified, easily.
Lemma 3.9. For $1 \leq j \leq k-1$, we have
\begin{enumerate}
\item $z_{j,k} = \sum_{\ell=1}^{k-1} e_{j,\ell} - \sum_{1 \leq s \leq k-1, s \neq j} (s, j)$,
\item $\bar{z}_{j,k} = \sum_{\ell=1}^{k-1} e_{\ell,j} - \sum_{1 \leq s \leq k-1, s \neq j} (s, j)$.
\end{enumerate}

Note that $\omega_{0,k} = \delta$ and $\omega_{1,k} = \delta \delta_1$, and $e_k h = 0$ for $h \in B_{k-1,k-1}$ if and only if $h = 0$. We will use this fact freely in the proof of the following lemma, where we use the terminology that a monomial in $z_{j,k+1}$’s and $\bar{z}_{j,k+1}$’s is a leading term in an expression if it has the highest degree by defining $\deg z_{i,j} = \deg \bar{z}_{i,j} = 1$.

Lemma 3.10. Suppose $a \in \mathbb{Z}^{\geq 2}$. Then $\omega_{a,k+1}$ can be written as an $R$-linear combination of monomials in $z_{j,k+1}$’s and $\bar{z}_{j,k+1}$’s for $1 \leq j \leq k$ such that the leading terms of $\omega_{a,k+1}$ are $\sum_{j=1}^{k} (-z_{j,k+1}^{-1} + (-1)^{a-1} \bar{z}_{j,k+1}^{-1})$.

Proof. By Corollary 3.5 and Lemma 3.3(1), we have
\begin{equation}
\omega_{a,k+1} e_{k+1} = e_{k+1} a_{k+1} e_{k+1} = e_{k+1} (\bar{L}_{k+1} - L_{k+1}) g_{k+1}^{-a} e_{k+1}. \tag{3.10}
\end{equation}

Considering the right-hand side of (3.10) and expressing $L_{k+1}$ by (3.1), using $(j, k+1) = s_{j,k} k s_{k,j}$ (cf. (3.2)) and the fact that $s_{j,k}, s_{k,j}$ commute with $y_{k+1}, e_{k+1}$ (cf. (3.8) and Lemma 3.3(6)), we see that a term in the right-hand side of (3.10) becomes
\begin{equation}
-s_{j,k} e_{k+1} s_{j,k} k y_{k+1}^{-a} e_{k+1} s_{k,j} = s_{j,k} e_{k+1} \left( - (y_k + e_k)^a - \sum_{b=0}^{a-2} (y_k + e_k)^{a-2-b} \omega_{b,k+1} \right) s_{k,j}, \tag{3.11}
\end{equation}

where the equality follows from Lemma 3.8 and Corollary 3.5. By Lemmas 3.7, $\omega_{a,k+1}$ commutes with $s_{k,j}$. Now by induction assumption, the right-hand side of (3.11) can be written as an $R$-linear combination of monomials with the required form such that the leading term is $-z_{j,k+1}^{-1}$.

Now we consider terms in (3.10) concerning $\bar{L}_{k+1}$, namely we need to deal with $e_{k+1} (\bar{j}, k+1) y_{k+1}^{-a} e_{k+1}$. We remark that it is hard to compute it directly. However, by Lemma 3.6 and induction on $a$, we can use $(-1)^{a-1} y_{k+1}^{-a} e_{k+1}$ to replace $y_{k+1}^{-a} e_{k+1}$ in $e_{k+1} (\bar{j}, k+1) y_{k+1}^{-a} e_{k+1}$ (by forgetting lower terms). This enables us to consider $(-1)^{a-1} e_{k+1} (\bar{j}, k+1) y_{k+1}^{-a} e_{k+1}$ instead. As above, this term can be written as the required form with leading term $(-1)^{a-1} z_{j,k+1}$. The proof is completed. \hfill \Box

Lemma 3.11. For $a \in \mathbb{Z}^{\geq 0}, k \in \mathbb{Z}^{\geq 1}$, both $\omega_{a,k+1}$ and $\bar{\omega}_{a,k+1}$ commute with $y_{k+1}$ and $\bar{y}_{k+1}$.

Proof. By Corollary 2.6, Lemmas 3.7 and 3.10, it suffices to prove that both $z_{j,k+1}$ and $\bar{z}_{j,k+1}$ for $1 \leq j \leq k$, commute with $y_{k+1}$. By Lemma 3.3(9) and $y_{k+1} e_k = e_k y_{k+1}$, we have $y_{k+1} (e_k + y_k) = (e_k + y_k) y_{k+1}$. Note that $z_{k,k+1} = y_k + e_k$, we have $y_{k+1} z_{k,k+1} = z_{k,k+1} y_{k+1}$. In general, by Lemma 3.3(6), $y_{k+1} \bar{z}_{j,k+1} = \bar{z}_{j,k+1} y_{k+1}$. By (3.8) and Corollary 2.6, $y_{k+1} \bar{y}_k = \bar{y}_k y_{k+1}$. Since $\bar{y}_k + e_k = \bar{z}_{k,k+1}$ (cf. (3.9)), $y_{k+1} \bar{z}_{k,k+1} = \bar{z}_{k,k+1} y_{k+1}$. So, by Lemma 3.3(7), $y_{k+1} \bar{z}_{j,k+1} = \bar{z}_{j,k+1} y_{k+1}$. The result follows. \hfill \Box

The following is the main result of this section. It follows from Theorem 2.4, Lemmas 3.3, 3.11 and Corollary 3.5.

Theorem 3.12. Let $F$ be a field containing $\omega_0, \omega_1$ with $\omega_0 \neq 0$. For any $k \in \mathbb{Z}^{\geq 0}$, let $B_{r+k,t+k}(\omega_0)$ be the walled Brauer algebra over $F$. Then there is an $F$-algebraic
homomorphism \( \phi_k : B_{r,t}^{\text{aff}} \to B_{r+k,t+k}(\omega_0) \) sending
\[
s_i, s_j, e_1, x_1, \bar{x}_1, \omega_a, \bar{\omega}_a \mapsto s_{i+k}, \bar{s}_{j+k}, e_{k+1}, y_{k+1}, \bar{y}_{k+1}, \omega_{a,k+1}, \bar{\omega}_{a,k+1},
\]
respectively such that \( \delta_1 = \omega_0^{-1} \omega_1 \).

4. A basis of an affine walled Brauer algebra

Throughout this section, we assume that \( R \) is a commutative ring containing 1, \( \omega_0 \) and \( \omega_1 \). The main purpose of this section is to prove that \( B_{r,t}^{\text{aff}} \) is free over \( R \) with infinite rank.

**Lemma 4.1.** There is an \( R \)-linear anti-involution \( \sigma : B_{r,t}^{\text{aff}} \to B_{r,t}^{\text{aff}} \) fixing defining generators \( s_i, s_j, e_1, x_1, \bar{x}_1, \omega_a \) and \( \bar{\omega}_b \) for all possible \( a, b, i, j \)'s.

**Proof.** This follows from the symmetry of the defining relations in Definition 2.7 (cf. Lemma 2.5). \( \square \)

The following can be proven by arguments similar to those for Lemma 3.6.

**Lemma 4.2.** For any \( k \in \mathbb{Z}_{\geq 1} \), we have \( e_1 x_1^k = \sum_{i=0}^{k} a_{k,i} e_1 x_1^i \) for some \( a_{k,i} \in B_{r,t}^{\text{aff}} \) such that
\[
\begin{align*}
(1) & \quad a_{k,k} = (-1)^k, \\
(2) & \quad a_{k,i} = \omega_0 a_{k-1,i} - a_{k-1,i-1}, 1 \leq i \leq k-1, \\
(3) & \quad a_{k,0} = -\sum_{i=1}^{k-1} a_{k-1,i} \omega_i.
\end{align*}
\]

In particular, \( a_{k,i} \in R[\omega_2, \omega_3, \ldots, \omega_k, 0, \bar{0}, \bar{1}, \bar{2}] \) for all \( i \) with \( 1 \leq i \leq k \) such that each monomial of \( a_{k,i} \) is of form \( \omega_{j_1} \cdots \omega_{j_t} \) with \( \sum_{i=1}^{t} j_i \leq a - 1 \).

**Corollary 4.3.** Assume \( e_1 \) is \( R[\omega_2, \omega_3, \ldots, 0, \bar{0}, \bar{1}, \bar{2}] \)-torsion-free. Then \( \bar{\omega}_0 = \omega_0, \bar{\omega}_1 = -\omega_1 \) and \( \bar{\omega}_k \in R[\omega_2, \omega_3, \ldots, \omega_k] \) for \( k \geq 2 \).

**Proof.** Applying \( e_1 \) on the right hand side of \( e_1 x_1^k \) and using Lemma 4.2 yield the result as required. \( \square \)

**Remark 4.4.** By Corollary 4.3, \( B_{r,t}^{\text{aff}} \) can be generated by \( s_i, s_j, e_1, x_1, \bar{x}_1, \omega_a \) for all possible \( i, j, a \) if \( e_1 \) is \( R[\omega_2, \omega_3, \ldots, 0, \bar{0}, \bar{1}, \bar{2}] \)-torsion-free. In fact, when we prove the freeness of \( B_{r,t}^{\text{aff}} \), we do not need to assume that \( e_1 \) is \( R[\omega_2, \omega_3, \ldots, 0, \bar{0}, \bar{1}, \bar{2}] \)-torsion-free. What we need is that \( \bar{\omega}_k \)'s are determined by \( \omega_2, \ldots, \omega_k \) and \( a_{k,i} \) in Lemma 4.2.

If so, \( B_{r,t}^{\text{aff}} \) is free over \( R \), forcing \( e_1 \) to be \( R[\omega_2, \omega_3, \ldots, \bar{0}, \bar{1}, \bar{2}] \)-torsion-free, automatically.

In the remaining part of this paper, we always keep this reasonable assumption on \( e_1 \). So, \( B_{r,t}^{\text{aff}} \) is generated by \( s_i, s_j, e_1, x_1, \bar{x}_1, \omega_a \) for all possible \( i, j, a \)'s.

The elements defined below will play similar roles to that of \( x_1 \) and \( \bar{x}_1 \):
\[
x_i = s_{i-1} x_{i-1} s_{i-1} - s_{i-1}, \quad \bar{x}_j = \bar{s}_{j-1} \bar{x}_{j-1} \bar{s}_{j-1} - \bar{s}_{j-1},
\]
for \( 2 \leq i \leq r, 2 \leq j \leq t \). The following result can be checked easily.

**Lemma 4.5.** We have
\[
\begin{align*}
(1) & \quad s_i x_i = x_{i+1} s_i + 1, \quad x_i x_j = x_j x_i \quad \text{for} \quad 1 \leq i < j \leq r, \\
(2) & \quad \bar{s}_i \bar{x}_i = \bar{x}_{i+1} \bar{s}_i + 1, \quad \bar{x}_i \bar{x}_j = \bar{x}_j \bar{x}_i \quad \text{for} \quad 1 \leq i < j \leq t.
\end{align*}
\]
(3) Let $\phi_k : B_{r,t}^{\text{aff}} \rightarrow B_{r+k,t+k}(\omega_0)$ be the homomorphism in Theorem 3.12. Then (recall notation $e_{i,j}$ in (3.3))

(i) $\phi_k(x_{i,j}) = \sum_{j=1}^k e_{k+\ell,j} - L_{k+\ell} + \omega^{-1}_0 \omega_1$,

(ii) $\phi_k(\bar{x}_{i,j}) = \sum_{j=1}^k e_{j+k+\ell} - \bar{L}_{k+\ell} + \omega^{-1}_0 \omega_1$.

Lemma 4.6. For $1 \leq i \leq r$ and $1 \leq j \leq t$, we have

1. $x_i(x_j + e_{i,j}) = (x_j + e_{i,j})x_i$, $\bar{x}_j(x_i + e_{i,j}) = (x_i + e_{i,j})\bar{x}_j$.
2. $e_{i,j}(x_i + \bar{x}_j) = -e_{i,j}(\bar{L}_j + L_i)$, $(x_i + \bar{x}_j)e_{i,j} = -(\bar{L}_j + L_i)e_{i,j}$.

Proof. By symmetry and Lemma 4.1, we need only check the first assertions of (1)–(2). In fact, if $i = 1$, then (1) follows from Definition 2.7(21), (24). In general, it follows from induction on $i$. By Definition 2.7(8), $e_{1,2}(x_1 + \bar{x}_2) = -e_{1,2}L_2$. Using Definition 2.7(24), and induction on $j$ yields $e_{1,j}(x_1 + \bar{x}_j) = -e_{1,j}\bar{L}_j$. This is (2) for $i = 1$. The general case follows from induction on $i$.

□

Lemma 4.7. Suppose $1 \leq i, j \leq r$ and $1 \leq k, \ell \leq t$.

1. If $i \neq j$, then $e_{i,k}(x_j + L_j) = (x_j + L_j)e_{i,k}$.
2. If $k \neq \ell$, then $e_{i,k}(\bar{x}_\ell + L_\ell) = (\bar{x}_\ell + L_\ell)e_{i,k}$.

Proof. By Corollary 2.6, we need only to check (1). By Definition 2.7(9), we have $e_1(x_2 + L_2) = (L_2 + x_2)e_1$. Using induction on $j$ yields $e_1(x_j + L_j) = (x_j + L_j)e_1$ for $j \geq 3$. This is (1) for $i = k = 1$. By induction on $k$, $e_{1,k}(x_j + L_j) = (x_j + L_j)e_{1,k}$. If $i = j$, by Lemmas 3.1 and 4.5, we have $e_{i,k}(x_j + L_j) = (x_j + L_j)e_{i,k}$.

In order to prove (1) for $j > 1$, we need $e_{2,1}x_1 = x_1e_{2,1}$, which follows from Definition 2.7(9). By Definition 2.7(4), (24), we have $e_{i,k}x_1 = x_1e_{i,k}$. By induction on $j$, we have $e_{i,k}(x_j + L_j) = (x_j + L_j)e_{i,k}$ for all $j$ with $j < i$, as required.

□

Lemma 4.8. Suppose $1 \leq i \leq r - 1$ and $1 \leq j \leq t - 1$.

1. $s_i(x_i + L_i) = (x_i + L_i)s_i$, $\bar{s}_j(x_j + \bar{L}_j) = (x_j + \bar{L}_j)s_i$.
2. $e_{i,j}(x_i + L_i)^a e_{i,j} = \omega_0^{a}e_{i,j}$, $\bar{e}_{i,j}(\bar{x}_j + \bar{L}_j)^a e_{i,j} = \bar{\omega}_0^{a}e_{i,j}$ for $a \in \mathbb{Z}^\geq 0$.

Proof. By Corollary 2.6, it suffices to check the first assertions of (1) and (2). We remark that (1) follows from Lemmas 3.1 and 4.5, and (2) follows from (1) together with induction on $i$.

□

We consider $B_{r,t}^{\text{aff}}$ as a filtrated algebra defined as follows. Set

$$\deg s_i = \deg \bar{s}_j = \deg e_1 = \deg \omega_0 = 0 \quad \text{and} \quad \deg x_k = \deg \bar{x}_\ell = 1,$$

for all possible $a, i, j, k, \ell$'s. Let $(B_{r,t}^{\text{aff}})^{(k)}$ be the $R$-submodule spanned by monomials with degrees less than or equal to $k$ in $\mathbb{Z}^\geq 0$. Then we have the following filtration

$$B_{r,t}^{\text{aff}} \supset \cdots \supset (B_{r,t}^{\text{aff}})^{(1)} \supset (B_{r,t}^{\text{aff}})^{(0)} \supset (B_{r,t}^{\text{aff}})^{(-1)} = 0.$$  \hspace{1cm} (4.2)

Let $\text{gr}(B_{r,t}^{\text{aff}}) = \oplus_{i \geq 0}(B_{r,t}^{\text{aff}})^{(i)}$, where $(B_{r,t}^{\text{aff}})^{(i)} = (B_{r,t}^{\text{aff}})^{(i)}/(B_{r,t}^{\text{aff}})^{(i-1)}$. Then $\text{gr}(B_{r,t}^{\text{aff}})$ is a $\mathbb{Z}$-graded algebra associated to $B_{r,t}^{\text{aff}}$. We use the same symbols to denote elements in $\text{gr}(B_{r,t}^{\text{aff}})$. We remark that we will work with $\text{gr}(B_{r,t}^{\text{aff}})$ when we prove the freeness of $B_{r,t}^{\text{aff}}$. 

Fix \( r, t, f \in \mathbb{Z}_{}^+ \) with \( f \leq \min\{r, t\} \). We define the following subgroups of \( \mathfrak{S}_r \), \( \mathfrak{S}_r \times \mathfrak{S}_t \) and \( \mathfrak{S}_t \) respectively,

\[
\mathfrak{S}_{r-f} = \langle s_j \mid f+1 \leq j < r \rangle,
\]

\[
\mathfrak{S}_f = \langle s_i s_i \mid 1 \leq i < f \rangle,
\]

\[
\mathfrak{S}_{t-f} = \langle s_j \mid f+1 \leq j < t \rangle.
\]

(4.3)

Observe that \( \mathfrak{S}_f \) is isomorphic to the symmetric group in \( f \) letters. The following result has been given in [22], without a detailed proof. We remark that \( \mathcal{D}_{r,t}^f \) in (4.4) was defined in [10] Proposition 6.1 via certain row-standard tableaux.

**Lemma 4.9.** [22, Lemma 2.6] The following (recall notation \( s_{i,j} \) in (3.2))

\[
\mathcal{D}_{r,t}^f = \{ s_{f,i,j} \mid 1 \leq i_1 < \cdots < i_f \leq r, k \leq j \},
\]

(4.4)

is a complete set of right coset representatives for \( \mathfrak{S}_{r-f} \times \mathfrak{S}_f \times \mathfrak{S}_{t-f} \) in \( \mathfrak{S}_r \times \mathfrak{S}_t \).

**Proof.** We denote by \( \tilde{\mathcal{D}}_{r,t}^f \) the right-hand side of (4.4), and by \( \mathcal{D}_{r,t}^f \) a complete set of right coset representatives. Then obviously \( \tilde{\mathcal{D}}_{r,t}^f \subset \mathcal{D}_{r,t}^f \). In order to verify the inverse inclusion, it suffices to prove that \( |\tilde{\mathcal{D}}_{r,t}^f| \), the cardinality of \( \tilde{\mathcal{D}}_{r,t}^f \), is

\[
\frac{r!}{(r-f)!} (r-f)! f! = C_r^f C_t^f f!,
\]

which is clearly the cardinality of \( \mathcal{D}_{r,t}^f \), where \( C_r^f \) is the binomial number. This will be done by induction on \( f \) as follows.

If \( f = 0 \), there is nothing to be proven. Assume \( f \geq 1 \). For any element in (4.4), we have \( i_f \geq f \). For each fixed \( i := i_f \), there are \( t - f + 1 \) choices of \( j_f \) with \( j_f \geq f \), and further, conditions for other indices are simply conditions for \( \mathcal{D}_{r,t}^{f-1} \). So,

\[
|\tilde{\mathcal{D}}_{r,t}^f| = (t-f+1) \sum_{i=f}^{r} |\mathcal{D}_{r,t}^{f-1}|
\]

\[
= (t-f+1) \sum_{i=f}^{r} C_{r-1}^{f-1} C_t^{f-1} (f-1)! = \sum_{i=f}^{r} C_{r-1}^{f-1} C_t^{f} f! = C_r^f C_t^f f!,
\]

where the second equality follows from induction assumption on \( f \), and the last follows from the well-known combinatorics formula \( C_r^f = C_r^0 + C_r^{f-1} \). \( \square \)

We denote

\[
e^f = e_1 e_2 \cdots e_f \quad \text{for any } f \text{ with } 1 \leq f \leq \min\{r, t\}.
\]

(4.5)

If \( f = 0 \), we set \( e^0 = 1 \). In [10] Theorem 6.13, Enyang constructed a cellular basis for \( q \)-walled Brauer algebras. The following result follows from this result immediately.

**Theorem 4.10.** [10] The following is an \( R \)-basis of \( \mathcal{B}_{r,t}(\omega_0) \),

\[
\mathcal{M} = \{ c^{-1} e^f w d \mid 1 \leq f \leq \min\{r, t\}, w \in \mathfrak{S}_{r-f} \times \mathfrak{S}_{t-f}, c, d \in \mathcal{D}_{r,t}^f \}.
\]

**Definition 4.11.** We say that

\[
m := \prod_{i=1}^{r} x_i^{\alpha_i} c^{-1} e^f w d \prod_{j=1}^{t} x_j^{\beta_j} \prod_{k \in \mathbb{Z}_{}^+} \omega_k^{a_k}
\]

(4.6)

is a regular monomial if \( c, d \in \mathcal{D}_{r,t}^f \), \( \alpha_i, \beta_j \in \mathbb{Z}_{}^+ \), and \( a_k \in \mathbb{Z}_{}^+ \) for \( k \geq 2 \) such that \( a_k = 0 \) for all but a finite many \( k \)'s.

**Proposition 4.12.** Suppose \( R \) is a commutative ring which contains 1, \( \omega_0 \), \( \omega_1 \). As an \( R \)-module, \( \mathcal{B}_{r,t}^{\text{aff}} \) is spanned by all regular monomials.
Proof. Let \( M \) be the \( R \)-submodule of \( \B_{r,t}^{\text{aff}} \) spanned by all regular monomials \( m \in \B_{r,t}^{\text{aff}} \) given in (4.6). We want to prove

\[
hm = h \prod_{i=1}^{r} x_i^{\alpha_i} c^{-1} e^f wd \prod_{i=1}^{t} \bar{x}_i^{\beta_i} \prod_{i \in \mathbb{Z}_{\geq 2}} \omega_i^{\alpha_i} \in M \quad \text{for any generator } h \text{ of } \B_{r,t}^{\text{aff}}. \quad (4.7)
\]

If so, then \( M \) is a left \( \B_{r,t}^{\text{aff}} \)-module, and thus \( M = \B_{r,t}^{\text{aff}} \) by the fact that \( 1 \in M \).

We prove (4.7) by induction on \( |\alpha| := \sum_{i=1}^{r} \alpha_i \). If \( |\alpha| = 0 \), i.e., \( \alpha_i = 0 \) for all possible \( i \)'s, then by Theorem 4.10, we have (4.7) unless \( h = \bar{x}_1 \).

If \( h = \bar{x}_1 \), by Lemma 4.5 we need to compute \( \bar{x}_k e^f \) for all \( k \) with \( 1 \leq k \leq t \). If \( k \in \{1, 2, \ldots, f\} \), by Lemma 4.6(3), we can use \( -x_k \) instead of \( \bar{x}_k \). So, \( hm \in M \).

Otherwise, by Lemma 4.7(2), we can use \( e^f \bar{x}_k \) instead of \( \bar{x}_k e^f \). So, (4.7) follows from Lemma 4.5 and Theorem 4.10.

Suppose \( |\alpha| > 0 \). By Lemma 4.5 and Theorem 4.10 we see that (4.7) holds for \( h \in \{s_1, \ldots, s_{r-1}, \bar{s}_1, \ldots, \bar{s}_{t-1}, x_1\} \). If \( h = \bar{x}_1 \), then (4.7) follows from Lemma 4.6(1), and induction assumption.

Finally, we assume \( h = e_1 \). If \( \alpha_i \neq 0 \) for some \( i \) with \( 2 \leq i \leq r \), then (4.7) follows from Lemma 4.7(1) and induction assumption. Suppose \( x_\alpha = x_i^{\alpha_i} \) with \( \alpha_1 > 0 \). We need to verify

\[
e_1 x_i^{\alpha_i} c^{-1} e^f wd \prod_{i=1}^{t} x_i^{\beta_i} \in M \quad \text{for } \alpha_1 > 0. \quad (4.8)
\]

Note that \( ce_1 c^{-1} = e_{i,j} \) for some \( i, j \). By Lemma 4.5 and induction assumption on \( |\alpha| \), we can use \( c^{-1} x_i^{\alpha_i} \) to replace \( x_1^{\alpha_1} c^{-1} \) in (4.8). So, we need to verify

\[
e_{i,j} x_i^{\alpha_i} e^f wd \prod_{i=1}^{t} \bar{x}_i^{\beta_i} \in M. \quad (4.9)
\]

In fact, by Lemma 4.6(2) and induction assumption, it is equivalent to verifying

\[
e_{i,j} \bar{x}_i^{\alpha_i} e^f wd \prod_{i=1}^{t} \bar{x}_i^{\beta_i} \in M. \quad (4.10)
\]

If \( j \geq f + 1 \), (4.10) follows from Lemma 4.7(2) and Theorem 4.10. Otherwise, \( j \leq f \).

If \( i = j \), by induction assumption, we use \( (x_i + L_i)^{\alpha_i} \) instead of \( x_i^{\alpha_i} \) in \( e_{i,j} x_i^{\alpha_i} e_j \). So, (4.10) follows from Lemma 4.8(2). If \( i \neq j \), we have

\[
e_{i,j} x_i^{\alpha_i} e_j = e_{i,j} e_j x_i^{\alpha_i} = (i, j) x_i^{\alpha_i} e_j = x_j^{\alpha_i} (i, j) e_j,
\]

which holds in \( \text{gr}(\B_{r,t}^{\text{aff}}) \). By induction assumption and our previous result on \( h \in \{s_1, \ldots, s_{r-1}, x_1\} \), we have (4.9) and hence (4.8). This completes the proof. \( \square \)

Now we are able to prove the main result of this section. We remark that the idea of the proof is motivated by Nazarov’s work on affine Wenzl algebras in [21].

**Theorem 4.13.** Suppose \( R \) is a commutative ring which contains \( 1, \omega_0, \omega_1 \). If \( e_1 \) is \( R[\omega_2, \omega_3, \ldots, \omega_0, \omega_1, \ldots]\)-torsion free, then \( \B_{r,t}^{\text{aff}} \) is free over \( R \) spanned by all regular monomials in (4.6). In particular, \( \B_{r,t}^{\text{aff}} \) is of infinite rank.

**Proof.** Let \( M \) be the set of all regular monomials of \( \B_{r,t}^{\text{aff}} \). First, we prove that \( M \) is \( F \)-linear independent where \( F \) is the quotient field of \( \mathbb{Z}[\omega_0, \omega_1] \) with \( \omega_0, \omega_1 \) being indeterminates.
Suppose conversely there is a finite subset $\mathcal{S}$ of $M$ such that $\sum_{m \in \mathcal{S}} r_m m = 0$ with $r_m \neq 0$ for all $m \in \mathcal{S}$. Recall from Definition 4.11 that each regular monomial is of the form in (4.10). For each $m \in \mathcal{S}$ as in (4.6), we set

$$k_m = \max \{ |\alpha| + \sum_j j a_j, |\beta| + \sum_j j a_j \}, \quad k = \max \{ k_m | m \in \mathcal{S} \},$$

(4.11)

where $|\alpha| = \sum_{i=1}^r \alpha_i$, $|\beta| = \sum_{i=1}^t \beta_i$. Consider the homomorphism $\phi_k : \mathcal{R}_{r,t}^{\text{aff}} \rightarrow \mathcal{B}_{r+k,t+k}(\omega_0)$ in Theorem 5.12. Then $\phi_k(m)$ can be written as a linear combinations of $(r+k, t+k)$-walled Brauer diagrams.

Using Lemma 4.5(3) to express $\phi_k(\bar{x}_e)$ and $\phi_k(\bar{x}_t)$, and using Lemma 3.10 to express $\omega_{a,k+1}$ for $a \in \mathbb{Z}^2$, we see that some terms of $\phi_k(m)$ are of forms (we will see in the next paragraph that other terms of $\phi_k(m)$ will not contribute to our computations)

$$\prod_{i=1}^r (k+i, i_1) \cdots (k+i, i_{\alpha_i}) \phi_k(c^{-1}e^j w d) \prod_{j=1}^t (\bar{k}+i, j_1) \cdots (\bar{k}+i, j_{\beta_j}) \prod_{i \geq 2} c_i,$$

(4.12)

where $c_i$ ranges over products of some disjoint cycles in $\mathfrak{S}_k$ (or $\hat{\mathfrak{S}}_k$) with total length $i a_i$. We remark that such $c_i$’s come from $\omega_{i,k+1}$. Further, the walled Brauer diagram corresponding to $\phi_k(c^{-1}e^j w d)$ have vertical edges $[i, i]$ and $[\bar{j}, \bar{j}]$ for all $i, j$ with $1 \leq i, j \leq k$. We call the terms of the form (4.12) the leading terms if

(i) either $k = |\alpha| + \sum_j j a_j$ or $k = |\beta| + \sum_j j a_j$ (cf. (4.11)), and
(ii) the corresponding $f$ in (4.12) is minimal among all terms satisfying (i), and
(iii) in the first case of (i), the juxtaposition of the sequences $i_1, i_2, \ldots, i_{\alpha_i}$ for $1 \leq i \leq r$ and $c_i$, $i \geq 2$ run through all permutations of the sequences in $1, 2, \ldots, k$; while in the second case of (i), the juxtaposition of the sequences $j_1, j_2, \ldots, j_{\beta_j}$ for $1 \leq j \leq r$ and $c_i$, $i \geq 2$ run through all permutations of the sequences in $1, 2, \ldots, k$.

If we identify the factor $\phi_k(c^{-1}e^j w d)$ in the leading terms with the corresponding walled Brauer diagrams, we have

(1) there are exactly $f$ horizontal edges in both top and bottom rows,
(2) no vertical edge of form $[i, i]$, $1 \leq i \leq k$ in the first case,
(3) no vertical edge of form $[\bar{i}, \bar{i}]$, $1 \leq i \leq k$ in the second case,
(4) no horizontal edge of form $[i, j]$, $1 \leq i \leq k$, $\bar{i} \leq \bar{j} \leq \bar{k}$ in both rows.

These leading terms exactly appear in $\phi_k(m)$ when conditions (i)–(iii) are satisfied.

Other terms in $\phi_k(\sum_{m \in \mathcal{S}} r_m m)$ are non-leading terms, which are terms obtained by (4.12) by using some $e_{k+i+j}$’s (resp., $e_{j,k+i}$’s) or scalars instead of some $(k+i, i_j)$’s (resp., $(\bar{k}+i, \bar{j}_i)$’s) or using certain product of $e_{i,j}$’s, $1 \leq i, j \leq k$ instead of some factors of some cycles $c_i$’s. Thus such terms can not be proportional to any leading terms. Therefore $\mathcal{S}$ is $F$-linear independent. By Proposition 4.12 $M$ is a $\mathbb{Z}[\omega_0, \omega_1]$-basis.

Now, for an arbitrary commutative ring $R$ containing $1, \omega_0, \omega_1$, we can regard $R$ as a left $\mathbb{Z}[\omega_0, \omega_1]$-module such that the indeterminates $\omega_0, \omega_1 \in \mathbb{Z}[\omega_0, \omega_1]$ act on $R$ as the scalars $\omega_0, \omega_1 \in R$ respectively. By standard arguments on specialization, $\mathcal{R}_{r,t}^{\text{aff}}$, which is defined over $R$, is isomorphic to $A \otimes \mathbb{Z}[\omega_0, \omega_1]$, where $A$ is the algebra $\mathcal{B}_{r,t}^{\text{aff}}$ defined over $\mathbb{Z}[\omega_0, \omega_1]$. So, $\mathcal{R}_{r,t}^{\text{aff}}$ is free over $R$ with infinite rank. □
Let $R$ be a commutative ring containing 1, $\omega_i$ for $i \in \mathbb{Z}_{\geq 0}$. Let $I$ be the two-sided ideal of $\mathcal{B}_{r,t}^{\text{aff}}$ generated by $\omega_n - \omega_a$, $a \in \mathbb{Z}_{\geq 2}$. Then there is an epimorphism $\psi : \mathcal{B}_{r,t}^{\text{aff}} \to \mathcal{B}_{r,t}^{\text{aff}}/I$. Let $\mathcal{B}_{r,t} = \mathcal{B}_{r,t}^{\text{aff}}/I$, namely $\mathcal{B}_{r,t}$ is the specialization of $\mathcal{B}_{r,t}$ with $\omega_a$ being specialized to $\omega_a$ for $a \in \mathbb{Z}_{\geq 2}$. Without confusion, we will simply denote elements $\omega_a$ of $R$ as $\omega_a$.

**Definition 4.14.** We say that the image of a regular monomial $m$ of $\mathcal{B}_{r,t}^{\text{aff}}$ (cf. (4.6)) is a *regular monomial* of $\mathcal{B}_{r,t}$ if $m$ does not contain factors $\omega_i$’s for $i \geq 2$.

The following result follows from Theorem 4.13, immediately.

**Theorem 4.15.** Suppose $R$ is a commutative ring which contains 1, $\omega_i$ with $i \in \mathbb{Z}_{\geq 0}$. Then $\mathcal{B}_{r,t}$ is free over $R$ spanned by all regular monomials. In particular, $\mathcal{B}_{r,t}$ is of infinite rank.

We close this section by giving some relationship between affine walled Brauer algebras and degenerate affine Hecke algebras [17], walled Brauer algebras, etc.

**Definition 4.16.** The degenerate affine Hecke algebra $\mathcal{H}_n^{\text{aff}}$ is the unital $R$-algebra generated by $S_1, \ldots, S_{n-1}, Y_1, \ldots, Y_n$ and relations

\[
S_i S_j = S_j S_i, \quad Y_i Y_k = Y_k Y_i, \\
S_i Y_j - Y_{i+1} S_i = -1, \quad Y_i S_j - S_i Y_{i+1} = -1, \\
S_j S_{j+1} S_j = S_{j+1} S_j S_{j+1}, \quad S_k^2 = 1,
\]

for $1 \leq i < n$, $1 \leq j < n - 1$ with $|i - j| > 1$, and $1 \leq k \leq n$.

**Proposition 4.17.** Let $R$ be a commutative ring containing 1, $\omega_i$ with $i \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{B}_{r,t}$ be the affine walled Brauer algebra over $R$. Let $I$ (resp., $J$) be the two-sided ideal of $\mathcal{B}_{r,t}$ generated by $x_1$ and $\bar{x}_1$ (resp., $e_1$).

1. $\mathcal{B}_{r,t}/I \cong \mathcal{B}_{r,t}$.
2. $\mathcal{B}_{r,t}/J \cong \mathcal{H}_r^{\text{aff}} \otimes \mathcal{H}_t^{\text{aff}}$.
3. The subalgebra of $\mathcal{B}_{r,t}$ generated by $e_1, s_1, \ldots, s_{r-1}, \bar{s}_1, \ldots, \bar{s}_{t-1}$ is isomorphic to the walled Brauer algebra $\mathcal{B}_{r,t}$ over $R$.
4. The subalgebra of $\mathcal{B}_{r,t}$ generated by $s_1, \ldots, s_{r-1}$ and $x_1$ (resp., $\bar{s}_1, \ldots, \bar{s}_{t-1}$ and $\bar{x}_1$) is isomorphic to the degenerate affine Hecke algebra $\mathcal{H}_r^{\text{aff}}$ (resp., $\mathcal{H}_t^{\text{aff}}$).

Note that the isomorphism in (4) sends $x_1$ (resp., $\bar{x}_1$) to $-Y_1$.

In the rest part of the paper, we will be interested in the specialized algebra $\mathcal{B}_{r,t}$. Without confusion, we will use $\mathcal{B}_{r,t}^{\text{aff}}$ to denote it.

5. **Super Schur-Weyl duality**

The main purpose of this section is to set up the relationship between affine walled Brauer algebras $\mathcal{B}_{r,t}^{\text{aff}}$ with special parameters and general linear Lie superalgebras $\mathfrak{gl}_{m|n}$. Throughout the section, we assume the ground field is $\mathbb{C}$.

Denote $\mathfrak{g} = \mathfrak{gl}_{m|n}$. Let $V = \mathbb{C}^{m|n}$ be the natural $\mathfrak{g}$-module. As a $\mathbb{C}$-vector superspace $V = V_0 \oplus V_1$ with $\dim V_0 = m$ and $\dim V_1 = n$. Take a natural basis $\{v_i \mid i \in I\}$ of $V$, where $I = \{1, 2, \ldots, m + n\}$. For convenience we define the map $[\cdot] : I \to \mathbb{Z}_2$ by $[i] = 0$ if
where \( \delta_{jk} = 1 \) if \( j = k \) and 0, otherwise. Let \( V^* \) be the dual space of \( V \) with dual basis \( \{ \bar{v}_i \mid i \in I \} \). Then \( V^* \) is a left \( g \)-module with action

\[
E_{ab} \bar{v}_i = -(-1)^{\|a\|+\|b\|} \delta_{ia} \bar{v}_b. \tag{5.2}
\]

Let \( \mathfrak{h} = \text{span}\{E_{ii} \mid i \in I\} \) be a Cartan subalgebra of \( g \), and \( \mathfrak{h}^* \) the dual space of \( \mathfrak{h} \) with \( \{ \varepsilon_i \mid i \in I \} \) being the dual basis of \( \{E_{ii} \mid i \in I\} \). Then an element \( \lambda \in \mathfrak{h}^* \) (called a weight) can be written as

\[
\lambda = \sum_{i \in I} \lambda_i \varepsilon_i = (\lambda_1, ..., \lambda_m \mid \lambda_{m+1}, ..., \lambda_{m+n}) \text{ with } \lambda_i \in \mathbb{C}. \tag{5.3}
\]

Take

\[
\rho = \sum_{i=1}^{m}(1-i)\varepsilon_i + \sum_{j=1}^{n}(m-j)\varepsilon_{m+j} = (0, -1, ..., 1-m \mid m-1, m-2, ..., m-n),
\]

and denote

\[
|\lambda| := \sum_{i \in I} \lambda_i \text{ (called the size of } \lambda), \tag{5.4}
\]

\[
\lambda^\rho = \lambda + \rho = (\lambda_1^\rho, ..., \lambda_m^\rho \mid \lambda_{m+1}^\rho, ..., \lambda_{m+n}^\rho), \text{ where,} \tag{5.5}
\]

\[
\lambda_i^\rho = \lambda_i + 1 - i \text{ if } i \leq m, \text{ and } \lambda_i^\rho = \lambda_i + 2m - i \text{ if } i > m.
\]

A weight \( \lambda \) is called integral dominant if \( \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \) for \( i \in \{m, m+n\} \). It is called typical if \( \lambda_i^\rho + \lambda_j^\rho \neq 0 \) for any \( i \leq m < j \) (otherwise it is called atypical) [16].

**Example 5.1.** For any \( p, q \in \mathbb{C} \),

\[
\lambda_{pq} = (p, ..., p \mid -q, ..., -q), \tag{5.6}
\]

is a typical integral dominant weight if and only if

\[
p - q \notin \mathbb{Z} \text{ or } p - q \leq -m \text{ or } p - q \geq n. \tag{5.7}
\]

(Note that the \( \lambda_{pq} \) defined in [14, IV] is the \( \lambda_{p,q+m} \) defined here.) In this case, the finite-dimensional irreducible \( g \)-module \( L_{\lambda_{pq}} \) with highest weight \( \lambda_{pq} \) coincides with the Kac-module \( K_{\lambda_{pq}} \) [14, IV], [16].

Let \( M \) be any \( g \)-module. For any \( r, t \in \mathbb{Z}_{\geq 0} \), set \( M^{rt} = V^{\otimes r} \otimes M \otimes (V^*)^{\otimes t} \). For convenience we denote the ordered set

\[
J = J_1 \cup \{0\} \cup J_2, \text{ where } J_1 = \{r, ..., 1\}, J_2 = \{\tilde{1}, ..., \tilde{t}\}, \tag{5.8}
\]

and \( r < \cdots < 1 < 0 < \tilde{1} < \cdots < \tilde{t} \). We write \( M^{rt} \) as

\[
M^{rt} = \bigotimes_{i \in J} V_i, \text{ where } V_i = V \text{ if } i < 0, V_0 = M \text{ and } V_i = V^* \text{ if } i > 0, \tag{5.9}
\]

(hereafter all tensor products will be taken according to the order in \( J \)), which is a left \( U(g)^{\otimes (r+t+1)} \)-module (where \( U(g) \) is the universal enveloping algebra of \( g \)), with the action given by

\[
\left( \bigotimes_{i \in J} g_i \right) \left( \bigotimes_{i \in J} x_i \right) = (-1)^{\sum_{i \in J} |g_i|} \sum_{j \in \mathbb{Z}_+} \left| x_j \right| \bigotimes_{i \in J} (g_i x_i) \text{ for } g_i \in U(g), \quad x_i \in V_i.
\]
In particular, if we delete the tensor $M$ (or take $M = \mathbb{C}$), then $M^{rt}$ is the mixed tensor product studied in [20], and if $t = 0$ and $M = K_{pq}$, then $M^{rt}$ is the tensor module studied in [4, IV].

We denote

$$\Omega = \sum_{i,j \in I} (-1)^{|j|} E_{ij} \otimes E_{ji} \in g^{\otimes 2}. \quad (5.10)$$

Because of the following well-known property of $\Omega$, it is called a Casimir element.

**Lemma 5.2.** For any $g \in g$, we denote $\Delta(g) = g \otimes 1 + 1 \otimes g \in g^{\otimes 2}$ (i.e., $\Delta$ is the comultiplication of the quantum group $U(g)$). Then

$$[\Delta(E_{ab}), \Omega] = 0 \quad \text{for all} \quad a, b \in I. \quad (5.11)$$

**Proof.** By definition, the left-hand side of (5.11) is equal to

$$\sum_{j \in I} (-1)^{|j|} \left( E_{ab} \otimes E_{ij} + (-1)^{(|a| + |b|)(|i| + |j|)} E_{ij} \otimes [E_{ab}, E_{ji}] \right)$$

$$= \sum_{j \in I} (-1)^{|j|} E_{aj} \otimes E_{jb} - \sum_{j \in I} (-1)^{|a| + (|a| + |b|)(|i| + |j|)} E_{ib} \otimes E_{ai}$$

$$+ \sum_{i \in I} (-1)^{|b| + (|a| + |b|)(|i| + |j|)} E_{ib} \otimes E_{ai} - \sum_{j \in I} (-1)^{|j|} E_{aj} \otimes E_{jb},$$

which is equal to zero by noting that

$$|a| + (|a| + |b|)(|i| + |j|) = (|a| + |b|)|i| + |a||b| = |b| + (|a| + |b|)(|i| + |j|). \quad \square$$

For $a, b \in J$ with $a < b$, we define $\pi_{ab} : U(g)^{\otimes 2} \to U(g)^{\otimes (r+t+1)}$ by

$$\pi_{ab}(x \otimes y) = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1, \quad (5.12)$$

where $x$ and $y$ are in the $a$-th and $b$-th tensors respectively.

**Notation 5.3.** From now on, we always suppose $M = K_\lambda$ is the Kac-module with highest weight $\lambda = \lambda_{pq}$ in (5.6) for $p, q \in \mathbb{C}$ (at this moment, we do not impose any condition on $p, q$) and a highest weight vector $v_\lambda$ defined to have parity 0.

Note that

$$E_{ij}v_\lambda = \begin{cases} \phantom{-}pv_\lambda & \text{if } 1 \leq i = j \leq m, \\
-qv_\lambda & \text{if } m < i = j \leq m + n, \\
\phantom{-}0 & \text{if } 1 \leq i \neq j \leq m \text{ or } m \leq i \neq j \leq m + n, \end{cases} \quad (5.13)$$

and $K_\lambda$ is $2^{mn}$-dimensional with a basis

$$B = \left\{ b^\sigma := \prod_{i=1}^n \prod_{j=1}^m E_{\sigma_{ij}} \varepsilon_{ij} v_\lambda \bigg| \sigma = (\sigma_{ij})_{1 \leq i,j \leq m} \in \{0,1\}^{n \times m} \right\}, \quad (5.14)$$

where the products are taken in any fixed order (changing the order only changes the vectors by $\pm 1$). Then $M^{rt}$ is $2^{mn}(m + n)^{r+t}$-dimensional with a basis

$$B_M = \left\{ b_M = \bigotimes_{i \in J_1} v_{k_i} \otimes b \otimes \bigotimes_{i \in J_2} \bar{v}_{k_i} \bigg| b \in B, k_i \in I \right\}. \quad (5.15)$$

Due to Lemma 5.2, the elements defined below are in the endomorphism algebra $\text{End}_g(M^{rt})$ of the $g$-module $M^{rt}$, which will be used throughout the section.
Definition 5.4. By (5.11), we can use (5.12) to define the following elements of the endomorphism algebra \( \text{End}_g(M^r) \),
\[

t_i = \pi_{i+1,i}(\Omega)|_{M^r} \quad (1 \leq i < r), \quad s_j = \pi_{j+1,j}(\Omega)|_{M^r} \quad (1 \leq j < t),
\]
\[
x_1 = \pi_{10}(\Omega)|_{M^r}, \quad \bar{x}_1 = \pi_{01}(\Omega)|_{M^r}, \quad e_1 = \pi_{11}(\Omega)|_{M^r}. \quad (5.16)
\]

For convenience, we will write elements of \( \text{End}_g(M^r) \) as right actions on \( M^r \). However, one shall always keep in mind that all endomorphisms are defined by left multiplication of the Casimir element \( \Omega \) such that the first and second tensors in \( \Omega \) act on some appropriate tensor positions in \( M^r \), and also one shall always do bookkeeping on the sign change whenever the order of two elements (factors) in a term are exchanged.

Lemma 5.5. For \( w_1 \in V^{\otimes (r-1)} \), \( w_2 \in (V^*)^{\otimes t} \) and \( i \in I \), we have
\[
(w_1 \otimes v_i \otimes v_\lambda \otimes w_2)x_1 = \begin{cases} 
 pw_1 \otimes v_i \otimes v_\lambda \otimes w_2 & \text{if } i \leq m, \\
 qw_1 \otimes v_i \otimes v_\lambda \otimes w_2 + \sum_{a=1}^{m} w_1 \otimes v_a \otimes E_{ia}v_\lambda \otimes w_2 & \text{if } i > m.
\end{cases} \quad (5.17)
\]

Proof. Since the two tensors of \( \Omega \) in \( x_1 = \pi_{10}(\Omega)|_{M^r} \) act on \( v_i \) and \( v_\lambda \) respectively, the left-hand side of (5.17) is equal to
\[
w_1 \otimes \sum_{a,b \in I} (-1)^{|b|+|i|(|a|+|b|)} E_{ab}v_i \otimes E_{ba}v_\lambda \otimes w_2
\]
\[
= w_1 \otimes \sum_{a \in I, a \leq i} (-1)^{|a||b|} v_a \otimes E_{ia}v_\lambda \otimes w_2,
\]
which is equal to the right-hand side of (5.17) by (5.13).

Lemma 5.6. For \( w_1 \in V^{\otimes r} \), \( w_2 \in (V^*)^{\otimes (t-1)} \) and \( i \in I \), we have
\[
(w_1 \otimes v_\lambda \otimes \bar{v}_i \otimes w_2)x_1 = \begin{cases} 
 -pw_1 \otimes v_\lambda \otimes \bar{v}_i \otimes w_2 - \sum_{a=m+1}^{m+n} w_1 \otimes E_{ai}v_\lambda \otimes \bar{v}_a \otimes w_2 & \text{if } i \leq m, \\
 -qw_1 \otimes v_\lambda \otimes \bar{v}_i \otimes w_2 & \text{if } i > m.
\end{cases} \quad (5.18)
\]

Proof. By definition and (5.2), the left-hand side of (5.18) is equal to
\[
w_1 \otimes \sum_{a,b \in I} (-1)^{|b|} E_{ab}v_\lambda \otimes E_{ba}v_i \otimes w_2 = -w_1 \otimes \sum_{a \in I, a \geq i} (-1)^{|a||i|} E_{ai}v_\lambda \otimes \bar{v}_a \otimes w_2,
\]
which is equal to the right-hand side of (5.18) by (5.13).

Lemma 5.7. For \( w_1 \in V^{\otimes (r-1)} \), \( w_2 \in V^{\otimes (t-1)} \), \( i, j \in I \), we have
\[
(w_1 \otimes v_i \otimes v_\lambda \otimes \bar{v}_j \otimes w_2)e_1 = (-1)^{1+|i|} \delta_{ij} \sum_{a \in I} w_1 \otimes v_a \otimes v_\lambda \otimes \bar{v}_a \otimes w_2. \quad (5.19)
\]

Proof. By definition, the left-hand side of (5.19) is
\[
w_1 \otimes \sum_{a,b \in I} (-1)^{|b|+|i||a|+|b|} E_{ab}v_i \otimes v_\lambda \otimes E_{ba}v_j \otimes \bar{v}_a \otimes w_2,
\]
which equals the right-hand side of (5.19).

Similarly, one can easily verify the following.
Lemma 5.8. For \( a, b \in I \), and \( w_1 \in V^{\otimes (r-1-i)} \), \( w_2 \in V^{\otimes (i-1)} \otimes K_\lambda \otimes (V^*)^t \), \( w'_1 \in V^{\otimes r} \otimes K_\lambda \otimes (V^*)^{t(j-1)} \), \( w'_2 \in (V^*)^{(t-1-j)} \), we have

\[
(w_1 \otimes v_a \otimes v_b \otimes w_2) s_i = (-1)^{|a||b|} w_1 \otimes v_b \otimes v_a \otimes w_2,
\]
\[
(w'_1 \otimes v_a \otimes v_b \otimes w'_2) s_i = (-1)^{|a||b|} w'_1 \otimes v_b \otimes v_a \otimes w'_2.
\]

The following is obtained in [20, Proposition 3.2] when the tensor \( M = K_\lambda \) is omitted.

Proposition 5.9. There exists a \( \mathbb{C} \)-algebraic homomorphism \( \Phi : \mathcal{B}_{r,t}(m - n) \to \text{End}_\mathbb{C}(M^t) \), which sends generators \( e_1, s_i, \bar{s}_j, 1 \leq i \leq r-1, 1 \leq j \leq t-1 \) of \( \mathcal{B}_{r,t}(m - n) \) in Theorem 2.4 to elements with the same symbols.

We need the following results before stating a main result of this section.

Lemma 5.10. Let \( x_1, \bar{x}_1, e_1 \) be defined in (5.16). Then \( (x_1 + \bar{x}_1) e_1 = 0 = e_1 (x_1 + \bar{x}_1) \).

Proof. To prove the result, it suffices to consider the case \( r = t = 1 \). By (5.17), we have

\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) x_1 e_1 = \left( \sum_{a \in I} (-1)^{|a||j|} v_a \otimes E_{ia} v_\lambda \otimes \bar{v}_j \right) e_1
\]
\[
= (-1)^{|j||j|} (v_j \otimes E_{ij} v_\lambda \otimes \bar{v}_j) e_1
\]
\[
= (-1)^{|j||j|} \sum_{a, b \in I} (-1)^{|b|+|a|+|j|} E_{ab} v_j \otimes E_{ij} v_\lambda \otimes E_{ba} \bar{v}_j
\]
\[
= (-1)^{|j||j|} \sum_{a, b \in I} (-1)^{|j|+|a|+|j|+|j|+|a|} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a
\]

and by (5.18),

\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) x_1 e_1 = \left( \sum_{a \in I} (-1)^{|a|+|j|} v_i \otimes E_{aj} v_\lambda \otimes v_a \right) e_1
\]
\[
= (-1)^{|j||j|} (v_i \otimes E_{ij} v_\lambda \otimes \bar{v}_i) e_1
\]
\[
= (-1)^{|j||j|} \sum_{a, b \in I} (-1)^{|b|+|a|+|j|} E_{ab} v_i \otimes E_{ij} v_\lambda \otimes E_{ba} \bar{v}_i
\]
\[
= (-1)^{|j||j|} \sum_{a \in I} (-1)^{|i|+|a|+|j|+|i|+|a|} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a
\]

and by (5.19),

Thus \( (x_1 + \bar{x}_1) e_1 = 0 \). By (5.19), we have

\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) e_1 x_1 = (-1)^{|j|} \delta_{ij} \sum_{a \in I} (v_a \otimes v_\lambda \otimes \bar{v}_a) x_1
\]
\[
= \delta_{ij} \sum_{a, b \in I} (-1)^{|j|+|a|+|j|} E_{ba} v_a \otimes E_{ab} v_\lambda \otimes \bar{v}_a
\]
\[
= \delta_{ij} \sum_{a, b \in I} (-1)^{|j|+|a|} v_b \otimes E_{ab} v_\lambda \otimes \bar{v}_a,
\]

and similarly, we obtain \( e_1 \bar{x}_1 = -e_1 x_1 \). □

Lemma 5.11. Let \( x_1, \bar{x}_1, e_1 \) be defined in (5.16). Then \( x_1 (e_1 + \bar{x}_1) = (e_1 + \bar{x}_1) x_1 \).
Comparing this with (5.21) and (5.23), we obtain
\[
\delta_{ij} \sum_{a \in I} (-1)^{1+\bar{a}[a]+j} E_{ij} v_a \otimes E_{ab} v_b \otimes \bar{v}_a.
\]
and by (5.18),
\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) x_1 x_1 = \sum_{a \in I} (-1)^{1+\bar{a}[a]+j} (v_i \otimes E_{ab} v_\lambda \otimes \bar{v}_a) x_1
\]
Using $E_{ib} E_{aj} = (1)^{(a+b)(a+\bar{b})} E_{ij} E_{ab} + \delta_{ab} E_{ij} + (-1)^{1+(a+b)(a+\bar{b})} \delta_{ij} E_{ab}$ in (5.25), we obtain
\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) x_1 x_1 = (v_i \otimes v_\lambda \otimes \bar{v}_j) x_1 x_1 + \sum_{a \in I} (-1)^{1+\bar{a}[a]+j} v_a \otimes E_{ij} v_\lambda \otimes \bar{v}_a
\]
Comparing this with (5.21) and (5.23), we obtain $x_1 (e_1 + \bar{x}_1) = (e_1 + \bar{x}_1) x_1$.

Proof. As above, we can suppose $r = t = 1$. By (5.17),
\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) x_1 x_1 = \sum_{b \in I} (-1)^{b[i][b]} (v_b \otimes E_{ib} v_\lambda \otimes \bar{v}_j) \bar{x}_1,
\]
and by (5.18),
\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) x_1 x_1 = \sum_{a \in I} (-1)^{1+\bar{a}[a]j} (v_i \otimes E_{aj} v_\lambda \otimes \bar{v}_a) x_1.
\]
Lemma 5.12. Let $x_1, \bar{x}_1, e_1$ be defined in (5.16). For any $k \in \mathbb{Z}_{>0}$, we have $e_1 x_1^k e_1 = \omega_k e_1$ for some $\omega_k \in \mathbb{C}$ such that $\omega_0 = m - n, \omega_1 = nq - mp$.

Proof. As above, we can suppose $r = t = 1$. For any $k \in \mathbb{Z}_+$, we have
\[
(v_i \otimes v_\lambda \otimes \bar{v}_j) x_1^k e_1 = (-1)^{1+i} \sum_{\ell_0 \in I} (v_{\ell_0} \otimes v_\lambda \otimes \bar{v}_{\ell_0}) x_1^k e_1 = \omega_k (v_i \otimes v_\lambda \otimes \bar{v}_j) e_1
\]
where the last equality is obtained as follows: if $\ell_k \neq \ell_0$, the corresponding terms become zero after applying $e_1$ by (5.19); otherwise $E_{\ell_k-1, \ell_k} E_{\ell_k-2, \ell_k-1} \cdots E_{\ell_0, \ell_1} v_\lambda$ is a weight vector in $K_\lambda$ with weight $\lambda$, thus a multiple, denoted by $\omega_k$, of $v_\lambda$.

In particular, if $k = 0$, from the first equality of (5.26), we immediately obtain $\omega_0 = m - n$ by (5.19). If $k = 1$, from the second equality of (5.26) and the above arguments, there is only one factor $E_{\ell_1, \ell_0}$ with $\ell_1 = \ell_0$ we need to consider in the summand. Using (5.13), we obtain $\omega_1 = nq - mp$. 

□
Now we can prove the following.

**Theorem 5.13.** Let $M = K_{\lambda}$ be the Kac-module with highest weight $\lambda = \lambda_{pq}$ in (5.6) for $p, q \in \mathbb{C}$, and let $s_i, \bar{s}_j, x_1, e_1, \bar{x}_1 \in \text{End}_q(M^t)$ be defined as in (5.16)–(5.20). Then all relations in Definition 2.7 hold with $\omega_a$'s being specialized to the complex numbers

\[
\omega_0 = m - n, \quad \omega_1 = nq - mp, \\
\omega_a = (p + q - m)\omega_a - p(q - m)\omega_{a-1} \quad \text{for} \; a \geq 2. \tag{5.27}
\]

Furthermore, $x_1, \bar{x}_1$ satisfy

\[
(x_1 - p)(x_1 + m - q) = 0, \quad (\bar{x}_1 + p - n)(\bar{x}_1 + q) = 0. \tag{5.28}
\]

**Proof.** Note that those relations in Definition 2.7 which do not involve $x_1$ and $\bar{x}_1$ are relations of the walled Brauer algebra $B_{r,t}(m - n)$ in Definition 2.4 thus hold by Proposition 5.9.

Definition 2.7[9]–(11), (22)–(24) can be verified, easily. By Lemmas 5.10 and 5.11, we have Definition 2.7[8] and (21). Definition 2.7[12] and the first two equations of (5.27) follows from Lemma 5.12. Similarly by symmetry, one can prove Definition 2.7[25].

The last equation of (5.27) follows from (5.28) by induction on $a$. Note that the first equation of (5.28) is [4, IV, Corollary 3.2], which can also be obtained directly by (5.15) by noting that $x_1$ has two eigenvalues $p$ and $q - m$ as the summand in the second case of (5.15) is equal to $\sum_{\alpha=1}^{m} w_1 \otimes (E_{ia}(v_a \otimes v_\lambda) - v_i \otimes v_\lambda) \otimes w_2$. Similarly, we have the second equation of (5.28).

To prove Definition 2.7[13], let $x_i$ with $2 \leq i \leq r$ be defined as in (4.1). Then $x_i$'s are the exactly same as that in [4, IV, Lemma 3.1], in particular, they commute with each other, i.e., we have Definition 2.7[13] as $x_2$ coincides with $s_1x_1s_1 - s_1$. Similarly we have Definition 2.7[26].

Denote by $B_{q}(m, n)$ the subalgebra of $\text{End}_q(M^t)$ generated by $s_i, \bar{s}_j, x_1, \bar{x}_1, e_1$. Theorem 5.13 shows that $B_{r,t}(m, n)$ is a quotient algebra of the affine walled Brauer algebra $B_{r,t}^{\text{aff}}$ with specialized parameters (5.27).

We need to introduce the following notion in order to prove the next result.

**Definition 5.14.** For an element $b = b^a \in B$ as in (5.14), we denote $|b| = \sum_{i,j} \sigma_{i,j}$, called the degree of $b$. If $\sigma_{ij} \neq 0$, we say $E_{i+m,j}$ is a factor of $b$. For $b_M \in B_M$, we define its degree $|b_M|$ to be $|b|$, where $b_M \in B$ is a unique tensor factor of $b_M$.

For any $\alpha = (\alpha_1, ..., \alpha_r) \in \{0, 1\}^r$, $\beta = (\beta_1, ..., \beta_t) \in \{0, 1\}^t$, we define the following elements of $B_{r,t}^{p,q}(m, n)$:

\[
x^\alpha = \prod_{i=1}^{r} x_i^{\alpha_i}, \quad \bar{x}^\beta = \prod_{j=1}^{t} \bar{x}_j^{\beta_j}, \tag{5.29}
\]

where $x_i, \bar{x}_j$ are elements of $B_{r,t}^{p,q}(m, n)$ defined as in (4.1).

**Theorem 5.15.** We keep the assumption of Theorem 5.13, and assume $r + t \leq \min\{m, n\}$. Then the monomials

\[
m := c^{-1} x^\alpha c^f \bar{x}^\beta wd, \tag{5.30}
\]

with $\alpha \in \{0, 1\}^r$, $\beta \in \{0, 1\}^t$ and $c, c^f, w, d$ as in Theorem 4.10 and Definition 4.11 are $\mathbb{C}$-linearly independent endomorphisms of $M^t$. 

Proof. First we remark that for convenience we arrange factors of the monomial \( m \) in (5.30) in a different order from the corresponding monomial \( m \) in (4.6) (without factors \( \omega \)'s, cf. Definition 4.14). Note that changing the order only differs an element by some element with lower degree, where the degree of \( \omega \) in (5.30) in a different order from the corresponding monomial \( m \) is defined to be \( \text{deg } m := |\alpha| + |\beta| \), and \( |\alpha| = \sum_{i=1}^{r} \alpha_i, \beta = \sum_{i=1}^{t} \beta_i \).

Suppose there is a nonzero \( C \)-combination \( c := \sum_{m} r_{m} m \) of monomials (5.30) being zero. We fix a monomial \( m' := c^{-1} x^{\alpha'} x^{\beta'} w' d' \) in \( c \) with nonzero coefficient \( r_{m'} \neq 0 \) which satisfies the following conditions:

(i) \( |\alpha'| + |\beta'| \) is maximal;

(ii) \( f' \) is minimal among all monomials satisfying (i).

We take the basis element \( v = \otimes_{i \in J_1} v_{k_i} \otimes v_{\lambda} \otimes \otimes_{i \in J_2} \bar{v}_{k_i} \in B_M \) (cf. (5.15)) such that (note that here is the place where we require condition \( r + t \leq \min \{ m, n \} \))

1. \( k_i = i + \alpha_i m \) if \( i < 0 \);
2. \( k_i = i \) for \( 1 \leq i \leq f' \);
3. \( k_i = r + i + (1 - \beta_i)m \) if \( f' < i \leq t \).

We define \( p_v \) to be the maximal integer such that there exist \( p_v \) pairs \((i, j) \in J_1 \times J_2 \) satisfying \( k_i - k_j \in \{0, \pm m\} \). Then from the choice of \( v \), we have the following fact:

\[ p_v = f'. \] (5.31)

Now take

\[ u := (v)c' c d'^{-1} w'^{-1} \in M^{rt}, \quad \text{and} \quad b' = \prod_{i=1}^{r'} E_{t+m,i}^{\alpha'} \prod_{i=1}^{t} E_{r+i+m,i+r}^{\beta'} \in K_{\lambda}, \] (5.32)

such that \( b' \) is a basis element in \( B_M \) consisting of elements with \( b' \) being a tensor factor. We denote \( B_M' \) to be the subset of \( B_M \) consisting of elements with \( b' \) being a tensor factor. We define the projection \( \tilde{\pi}_v : M^{rt} \to \otimes_{i \in J_1 \setminus \{0\}} V_{i} \) (cf. (5.9)) by mapping a basis element \( b_M \in B_M \) to zero if \( b_M \notin B_M' \), or else to the element obtained from \( b_M \) by deleting the tensor factor \( b' \). Motivated by [IV, Corollary 3.3], we refer to \( \tilde{\pi}_v(u) \) as the \( b'-\text{component} \) of \( u \). We want to prove \( \tilde{\pi}_v(u) \neq 0 \).

Assume a monomial \( m \) in (5.30) appears in the expression of \( c \) with \( r_m \neq 0 \). Consider the following element of \( M^{rt} \) which contributes to \( u \) in (5.32),

\[ u_1 := (v)c' m d'^{-1} w'^{-1} = (v)c' c^{-1} x^\alpha x^\beta wdd^{-1} w'^{-1} \]
\[ = \left( \otimes_{i \in J_1} v_{k(i)'} \otimes v_{\lambda} \otimes \otimes_{i \in J_2} \bar{v}_{k(i)'} \right) \cdot x^\alpha x^\beta wdd^{-1} w'^{-1}, \] (5.33)

where the last equality follows by noting that elements in \( \mathfrak{S}_r \times \bar{\mathfrak{S}}_t \) have natural right actions on \( J_1 \cup J_2 \) by permutations. Write \( u_1 \) as a \( C \)-combination of basis \( B_M \), and for \( b_M \in B_M \), if \( b_M \) appears as a term with a nonzero coefficient in the combination, then we say that \( u_1 \) produces \( b_M \). By (5.17), (5.18), Definition 5.14 and condition (i), \( u_1 \) cannot produce a basis element with degree higher than \( |\alpha| + |\beta| \). Thus the \( b' \)-component of \( u_1 \) is zero if \( |\alpha| + |\beta| < |\alpha'| + |\beta'| \). So we can assume \( |\alpha| + |\beta| = |\alpha'| + |\beta'| \) by condition (i). Then \( f \geq f' \) by condition (ii).

Note from definitions (4.11) and (5.12) that

\[ x_i = \pi_i \Omega_{|M^{rt}|} + \text{some element of degree zero}, \]
and $e^J = e_1 \cdots e_J$ and $e_i = \pi_{i}(\Omega)|_{M^{rt}}$ (cf. (5.16)). By (5.19), we see that in order for $u_1$ in (5.33) to produce a basis element $b_M$ in $B^{rt}_M$ (note that $b_M \in B^{rt}_M$ has tensor factor $b'$ and all factors of $b'$ have the form $E_{i+m,i}$ by (5.32)), we need at least $f$ pairs $(i, j) \in J_1 \times J_2$ with $k_i - k_j \in \{0, \pm 1\}$ by (5.19). Thus we can suppose $f = f'$ by (5.31) and the fact that $f \geq f'$.

Set $J' = (J_1 \cup J_2) \cap \{ i \mid f' \leq i \leq f'' \}$ (cf. (5.8)). If $c \neq c'$, then by definition (4.4), we have

$$j' := (j)c'c^{-1} \notin J' \text{ for some } j \in J'.$$

Say $j' \in J_1$ (the proof is similar if $j' \in J_2$), then $f' < j' \leq r$. Condition (1) shows that either $f' < k_j = j' \leq r$ or else $f' + m < k_j = j' + m \leq r + m$. Then conditions (2) and (3) show that there is no $\ell \in J_2$ with $k_j - k_\ell \in \{0, \pm 1\}$. Since all factors of $b'$ have the form $E_{i+m,i}$, we see that $u_1$ cannot produce a basis element in $B^{rt}_M$. Thus we can suppose $c = c'$.

By conditions (1) and (2), we see that if $\alpha_i \neq \alpha'_i$ for some $i$ with $1 \leq i \leq f$, or $\alpha_i = 1 \neq \alpha'_i$ for some $i \in J_1$, then again $u_1$ cannot produce a basis element in $B^{rt}_M$. Thus we suppose: $\alpha_i = \alpha'_i$ if $1 \leq i \leq f$, and $\alpha'_i = 0$ implies $\alpha_i = 0$ for $i \in J$.

Consider the coefficient $\chi^{u_1}_{b_M}$ of the basis element $b_M := \otimes_{i \in J_1} v_i \otimes b' \otimes \otimes_{i \in J_2} v_{i+m}$ in $u_1$. If $\alpha'_i = 1$ but $\alpha_i = 0$ for some $i \in J$, then $u_1$ can only produce some basis elements which have at least a tensor factor, say $v_\ell$, with $\ell > m$, and thus $b_M$ cannot be produced. Thus we can suppose $\alpha = \alpha'$. Dually, we can suppose $\beta' = \beta$.

Now rewrite $wdd^{-1}w^{-1}$ as $wdd^{-1}w^{-1} = d\bar{d}^{-1}w''$, where $\bar{d} = wdw^{-1}$, $\bar{d}' = wd'w^{-1}$ and $w'' = w'w'^{-1}$. Note that $w'' \in \mathcal{S}_{r-f} \times \mathcal{S}_{t-f}$, which only permutes elements of $(J_1 \cup J_2) \setminus J'$. We see that if $\bar{d} \neq \bar{d}'$, then as in (5.34), there exists some $j \in J'$ with $j' := (j)d\bar{d}^{-1}w' \notin J'$, thus $b_M$ cannot be produced. So assume $\bar{d} = \bar{d}'$. Similarly we can suppose $w'' = 1$.

The above has in fact proved that if the coefficient $\chi^{u_1}_{b_M}$ is nonzero then $u_1$ in (5.33) must satisfy $(c, f, \beta, d, w) = (c', f', \beta', d', w')$, i.e., $u_1 = (v)x^{\alpha}e^J\bar{x}\beta'$. In this case, one can easily verify that $\chi^{u_1}_{b_M} = \pm 1$. This proves that $u$ defined in (5.32) is nonzero, a contradiction. The theorem is proven. \( \square \)

As in [1 IV], we shall be mainly interested in the case when the Kac module $K_\lambda$ is typical, namely, either $p - q \notin \mathbb{Z}$ or $p - q \leq -m$ or $p - q \geq n$. In this case, the tensor module $M^{rt}$ is a tilting module. Using the vector space isomorphism $\text{Hom}_g(M_1 \otimes V, M_2) \cong \text{Hom}_g(M_1, M_2 \otimes V^*)$ for any two $g$-modules $M_1, M_2$, one can easily obtain the vector space isomorphism: $\text{End}_g(M^{rt}) \cong \text{End}_g(K_\lambda \otimes V^\otimes(r+t))$. Thus $\dim \text{End}_g(M^{rt}) = 2^{r+t}(r+t)!$ by [3 IV], which is the same as the total number of all monomials of the form (5.30).

Now we can state the main result of this section.

**Theorem 5.16.** (Super Schur-Weyl duality) Assume $r + t \leq \min\{m, n\}$, and $p - q \notin \mathbb{Z}$ or $p - q \leq -m$ or $p - q \geq n$. We have $\text{End}_g(M^{rt}) = \mathcal{B}^{p,q}_{r,t}(m, n)$ and

$$\mathcal{B}^{p,q}_{r,t}(m, n) \cong \mathcal{B}^{\text{aff}}_{r,t}/\langle (x_1 - p)(x_1 + m - q), (x_1 + p - n)(x_1 + q) \rangle.$$  (5.35)
Proof. Recall that \( \mathcal{B}_{r,t}^{p,q}(m,n) \) is a subalgebra of \( \text{End}_q(M^{rt}) \) defined before Definition 5.14. By Theorem 5.15, \( \dim \mathcal{B}_{r,t}^{p,q}(m,n) \geq 2^{r+t}(r+t)! = \dim \text{End}_q(M^{rt}) \), forcing \( \text{End}_q(M^{rt}) = \mathcal{B}_{r,t}^{p,q}(m,n) \).

Denote the right-hand side of (5.35) by \( A \). By Theorem 4.15 and arguments on the degree, we see easily that the (image of) monomials in (5.30) span \( A \). Thus \( \dim \mathcal{B}_r A \leq 2^{r+t}(r+t)! \) (which is the number of monomials in (5.30)).

Using Theorem 5.13 yields an epimorphism from \( \mathcal{B}_{r,t}^{p,q} \) to \( \text{End}_q(M^{rt}) \) killing the two-sided ideal \( \langle (x_1-p)(x_1+m-q), (x_1+p-n)(x_1+q) \rangle \) of \( \mathcal{B}_{r,t}^{p,q} \), thus induces an epimorphism from \( A \) to \( \mathcal{B}_{r,t}^{p,q} \). Thus \( \dim \mathcal{B}_r A \geq \dim \mathcal{B}_{r,t}^{p,q} \). Comparing their dimensions, we have the isomorphism in (5.35), as required.

Because of Theorem 5.16, we refer to the right-hand side of (5.35) as a level two walled Brauer algebra, defined below.

Definition 5.17. Let \( \mathcal{B}_{r,t}^{p,q} \) be the affine walled Brauer algebra defined over \( \mathbb{C} \) with specialized parameters (5.27). The cyclotomic quotient associative \( \mathbb{C} \)-algebra \( \mathcal{B}_{r,t}^{p,q}/\langle (x_1-p)(x_1+m-q), (x_1+p-n)(x_1+q) \rangle \) is called a level two walled Brauer algebra. By abusing of notations, we denote this algebra by \( \mathcal{B}_{r,t}^{p,q}(m,n) \).

By arguments in the proof of Theorem 5.16, we obtain

Corollary 5.18. Let \( m, n, r, t \in \mathbb{Z} \), \( p, q \in \mathbb{C} \) such that \( r + t \leq \min\{m, n\} \). Then the level two walled Brauer algebra \( \mathcal{B}_{r,t}^{p,q}(m,n) \) is of dimension \( 2^{r+t}(r+t)! \) over \( \mathbb{C} \) with all monomials of the form (5.30) being a \( \mathbb{C} \)-basis of \( \mathcal{B}_{r,t}^{p,q}(m,n) \).

Remark 5.19. Note that level two walled Brauer algebras \( \mathcal{B}_{r,t}^{p,q}(m,n) \) heavily depend on parameters \( p-q, r, t, m, n \), in sharp contrast to level two Hecke algebras \( H_r^{p,q} \) in [14 IV] (denoted as \( \mathcal{H}_{2,r} \) in the present paper), which only depend on \( p-q, r \).

We are now going to determine when \( \mathcal{B}_{r,t}^{p,q}(m,n) \) is semisimple. For this purpose, we need the following result, which is a slight generalization of [23, Lemma 5.2] and [21, Lemma 3.6], where a \( g \)-highest weight of \( M^{rt} \) means a weight \( \mu \in \mathfrak{h}^* \) such that there exists a nonzero \( g \)-highest weight vector \( v \in M^{rt} \) with weight \( \mu \) (i.e., \( v \) is a vector satisfying \( E_{ii}v = \mu_iv, E_{ji}v = 0 \) for \( 1 \leq i < j \leq m + n \)).

Lemma 5.20. We keep the assumption of Theorem 5.15 and assume \( \mu \in \mathfrak{h}^* \) is a \( g \)-highest weight of \( M^{rt} \). Then \( |\mu| = |\lambda| + r - t \) and \( -t \leq \sum_{i \in S}(\mu_i - \lambda_i) \leq r \) for any subset \( S \subseteq I \), where \( |\lambda|, |\mu| \) are sizes of \( \lambda, \mu \) (cf. (5.4)).

Proof. Let \( w_\mu \) be a \( g \)-highest weight vector with weight \( \mu \), and write \( w_\mu \) in terms of basis \( B_m \) in (5.15). As in the proofs of [23, Lemma 5.2] and [21, Lemma 3.6], \( w_\mu \) must contain a basis element, say \( b_M \), with degree 0 (cf. Definition 5.14), i.e., \( b_M \) has the form \( w_1 \otimes v_\lambda \otimes w_2 \) for some \( w_1 \in V^{\otimes r} \) and \( w_2 \in (V^*)^{\otimes t} \) such that \( w_1 \) (resp., \( w_2 \)) is a weight vector with some weight \( \eta \) (resp., \( \zeta \)) of size \( r \) (resp., \( -t \)) satisfying \( \eta_i \in \mathbb{Z} \leq 0 \) (resp., \( \zeta_i \in \mathbb{Z} \leq 0 \)) for all \( i \in I \). The result follows.

Theorem 5.21. We keep the assumption of Theorem 5.16, then \( \mathcal{B}_{r,t}^{p,q}(m,n) \) is semisimple if and only if \( p-q \notin \mathbb{Z} \) or \( p-q \leq -m-r \) or \( p-q \geq n+t \).
Proof. First assume \( p - q \in \mathbb{Z} \) and \( p - q \leq -m - r \). Let \( \mu \in \mathfrak{h}^* \) be a \( \mathfrak{g} \)-highest weight of \( M^r \). For \( 1 \leq i \leq m < j \leq m + n \), by definition of \((5.5)\), we have (hereafter we define the partial order on \( \mathbb{C} \) such that \( a \leq b \) if and only if \( b - a \) is a nonnegative real number)

\[
\mu_i^\rho + \mu_j^\rho = \mu_i + \mu_j + 1 + 2m - i - j
\]

\[
\leq \lambda_i + \lambda_j + r + 1 + 2m - 1 - (m + 1)
\]

\[
= p + m + r - q - 1,
\]

which is strictly less than zero, i.e., \( \mu \) is a typical integral dominant weight, where the inequality follows from Lemma 5.20 and \( i \geq 1, j \geq m + 1 \). By \cite{[16]}, \( M^r \) is a completely reducible module which can be decomposed as a direct sum of typical finite dimensional irreducible modules: 
\[
M^{st} = \oplus_{\mu \in T} L_{\mu}^{k_{\mu}},
\]

where \( T \) is a finite set consisting of typical integral dominant weights, and \( k_{\mu} \in \mathbb{Z}^{\geq 1} \). Thus \( B^p_q(m, n) \cong \text{End}_\mathfrak{g}(M^r) \cong \oplus_{\mu \in T} M_{k_{\mu}} \) is a semisimple associative algebra, where \( M_{k_{\mu}} \) is the algebra of matrices of rank \( k_{\mu} \).

The case \( p - q \notin \mathbb{Z} \) or \( p - q \geq n + t \) can be proven similarly.

Now suppose \( p - q \in \mathbb{Z} \) and \( q - m - r < p < q + n + t \). This together with condition \((5.7)\) shows that either \( |p - q + m| \), the absolute value of difference of two eigenvalues of \( x_1 \), is an integer \( < r \), or, \( |p - q - n| \), the absolute value of difference of two eigenvalues of \( \bar{x}_1 \), is an integer \( < t \). Thus by \cite{[1]} Theorem 6.1, either the level two degenerate Hecke algebras \( \mathcal{H}_{2,r} := \mathcal{H}_r^{\text{aff}} / \langle (x_1 - p)(x_1 + m - q) \rangle \) (cf. Proposition \cite{[16]}(3)) is not semisimple or else \( \mathcal{H}_{2,t} := \mathcal{H}_t^{\text{aff}} / \langle (\bar{x}_1 + p - n)(\bar{x}_2 + q) \rangle \) is not semisimple. In any case, \( \mathcal{H}_{2,r} \otimes \mathcal{H}_{2,t} = B^{p,q}_r(m, n) / \langle e_1 \rangle \) is not semisimple. As a result, \( B^{p,q}_r(m, n) \), the preimage of \( \mathcal{H}_{2,r} \otimes \mathcal{H}_{2,t} \), cannot be semisimple. \( \square \)

In the next two sections, we shall study \( B^{p,q}_r(m, n) \) for given \( m, n \) (with the assumption \( r + t \leq \min\{m, n\} \)), thus we omit \( (m, n) \) from the notation, and simply denote it by \( B^{p,q}_r \). When there is no confusion, the notation is further simplified to \( B \).

6. Weakly cellular basis of level two walled Brauer algebras

In this section, we shall use Corollary 5.18 to construct a weakly cellular basis of \( B = B^{p,q}_r(m, n) \) over \( \mathbb{C} \) for \( m, n, r, t \in \mathbb{Z}^{\geq 1} \), \( p, q \in \mathbb{C} \) such that \( r + t \leq \min\{m, n\} \).

First recall that a partition \( \lambda \in \mathbb{Z}^{\geq 0} \) is a sequence of non-negative integers \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( \lambda_i \geq \lambda_{i+1} \) for all positive integers \( i \) and \( |\lambda| := \lambda_1 + \lambda_2 + \cdots = k \). Let \( \Lambda^+(k) \) be the set of all partitions of \( k \). A bipartition of \( k \) is an ordered 2-tuple of partitions \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \) such that \( |\lambda| := |\lambda^{(1)}| + |\lambda^{(2)}| = k \).

Let \( \Lambda^+_2(k) \) be the set of all bipartitions of \( k \). Then \( \Lambda^+_2(k) \) is a poset with the dominance order \( \triangleright \) as the partial order on it. More explicitly, we say \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \) is dominated by \( \mu = (\mu^{(1)}, \mu^{(2)}) \) and write \( \mu \triangleright \lambda \) if

\[
\sum_{j=1}^{i} \lambda_j^{(1)} \leq \sum_{j=1}^{i} \mu_j^{(1)} \quad \text{and} \quad |\lambda^{(1)}| + \sum_{j=1}^{\ell} \lambda_j^{(2)} \leq |\mu^{(1)}| + \sum_{j=1}^{\ell} \mu_j^{(2)},
\]

for all possible \( i, \ell \)'s. We write \( \mu \triangleright \lambda \) if \( \mu \triangleright \lambda \) and \( \lambda \neq \mu \).

For each partition \( \lambda \) of \( k \), the Young diagram \( [\lambda] \) is a collection of boxes arranged in left-justified rows with \( \lambda_i \) boxes in the \( i \)-th row of \( [\lambda] \). If \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda^+_2(k) \), then the corresponding Young diagram \( [\lambda] \) is \( ([\lambda^{(1)}], [\lambda^{(2)}]) \). In this case, a \( \lambda \)-tableau \( s = (s_1, s_2) \) is obtained by inserting \( i, 1 \leq i \leq k \) into \([\lambda]\) without repetition.
A $\lambda$-tableau $\mathfrak{s}$ is said to be standard if the entries in $\mathfrak{s}_1$ and $\mathfrak{s}_2$ increase from left to right in each row and from top to bottom in each column. Let $\mathcal{F}^{\text{std}}(\lambda)$ be the set of all standard $\lambda$-tableaux.

**Definition 6.1.** We define

- $t^\lambda$ to be the $\lambda$-tableau obtained from the Young diagram $[\lambda]$ by adding $1, 2, \cdots, k$ from left to right along the rows of $[\lambda^{(1)}]$ and then $[\lambda^{(2)}]$;
- $t_\lambda$ to be the $\lambda$-tableau obtained from $[\lambda]$ by adding $1, 2, \cdots, k$ from top to bottom along the columns of $[\lambda^{(2)}]$ and then $[\lambda^{(1)}]$.

For example, if $\lambda = ((3, 2, 1), (2, 1))$, then

$$t^\lambda = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & \end{pmatrix}, \quad t_\lambda = \begin{pmatrix} 4 & 7 & 9 \\ 5 & 8 & 9 \\ 6 & 1 & 3 \end{pmatrix} \quad (6.2)$$

The symmetric group $\mathfrak{S}_k$ acts on a $\lambda$-tableau $\mathfrak{s}$ by permuting its entries. For $w \in \mathfrak{S}_k$, if $t^\lambda w = \mathfrak{s}$, we write $d(\mathfrak{s}) = w$. Then $d(\mathfrak{s})$ is uniquely determined by $\mathfrak{s}$.

Given a $\lambda \in \Lambda^r_2(k)$, let $\mathfrak{S}_\lambda$ be the row stabilizer of $t^\lambda$. Then $\mathfrak{S}_\lambda$ (sometimes denoted as $\mathfrak{S}_\lambda$) is the Young subgroup of $\mathfrak{S}_k$ with respect to the composition $\bar{\lambda}$, which is obtained from $\lambda$ by concatenation. For example, $\bar{\lambda} = (3, 2, 1, 2, 1)$ if $\lambda = ((3, 2, 1), (2, 1))$.

Recall that $\mathcal{H}^{\text{aff}}(\lambda)$ is the degenerate affine Hecke algebra generated by $S_i$’s and $Y_j$’s (cf. Definition 6.16). Let $\mathcal{H}^{\text{aff}}(\lambda) = \mathcal{H}^{\text{aff}} / I$, where $I = \langle (Y_1 - u)(Y_1 - v) \rangle$ is the two-sided ideal of $\mathcal{H}^{\text{aff}}$ generated by $(Y_1 - u)(Y_1 - v)$ for $u, v \in \mathbb{C}$. Then $\mathcal{H}^{\text{aff}}(\lambda)$ is known as the level two degenerate Hecke algebra with defining parameters $u, v$. As mentioned in Proposition 4.17, our current elements $x_1, \bar{x}_1$, which are two generators of $\mathcal{H}^{\text{aff}}$, corresponds $-Y_1$ in Definition 4.16. Thus, when we use the construction of cellular basis for $\mathcal{H}^{\text{aff}}$ in [1], we need to use $-Y_1, -u, -v$ instead of $Y_1, u, v$, respectively. By abusing of notations, we will not distinguish between them. The following definition on $m^{\lambda}_{\mathfrak{st}}$ is a special case of that in [1] for degenerate Hecke algebra $\mathcal{H}^{\text{aff}}(\lambda)$ of type $G(r,1,m)$.

Suppose $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda^r_2$ with $a = |\lambda^{(1)}|$. We set $\pi_0 = 1$ if $a = 0$, and $\pi_a = (x_1 - v)(x_2 - v) \cdots (x_a - v)$ for $a \geq 1$. Let

$$m^{\lambda}_{\mathfrak{st}} = d(\mathfrak{s})^{-1} \pi_a m^{\lambda}_{\mathfrak{st}} d(t), \quad (6.3)$$

where $\mathfrak{s}, t \in \mathcal{F}^{\text{std}}(\lambda)$ and $m^{\lambda}_{\mathfrak{st}} = \sum_{w \in \mathfrak{S}_\lambda} w$. In the following, we shall always omit the $\lambda$ from notations $m^{\lambda}_{\mathfrak{st}}, C^{\lambda}_{\mathfrak{st}}$, etc., and simply denote them as $m_{\mathfrak{st}}, C_{\mathfrak{st}}$, etc.

**Definition 6.2.** [12] Let $A$ be an algebra over a commutative ring $R$ containing 1. Fix a partially ordered set $\Lambda = (\Lambda, \succeq)$, and for each $\lambda \in \Lambda$, let $T(\lambda)$ be a finite set. Further, fix $C_{\mathfrak{st}} \in A$ for all $\lambda \in \Lambda$ and $\mathfrak{s}, t \in T(\lambda)$. Then the triple $(\Lambda, T, C)$ is a cell datum for $A$ if:

1. $\mathcal{C} := \{ C_{\mathfrak{st}} \mid \lambda \in \Lambda, \mathfrak{s}, t \in T(\lambda) \}$ is an $R$-basis for $A$;
2. the $R$-linear map $*: A \to A$ determined by $(C_{\mathfrak{st}})^* = C_{\mathfrak{tu}}$ for all $\lambda \in \Lambda$ and all $\mathfrak{s}, t \in T(\lambda)$ is an anti-involution of $A$;
3. for all $\lambda \in \Lambda$, $\mathfrak{s} \in T(\lambda)$ and $a \in A$, there exist scalars $r_{\mathfrak{tu}}(a) \in R$ such that

$$C_{\mathfrak{st}} a = \sum_{u \in T(\lambda)} r_{\mathfrak{tu}}(a) C_{\mathfrak{su}} \quad (\text{mod } A^{\mathfrak{d}^{\lambda}}),$$
where \( A^{\lambda} = R\text{-span}\{C_{\mu}^u \mid \mu \triangleright \lambda, u, v \in T(\mu)\} \). Furthermore, each scalar \( r_u(a) \) is independent of \( s \).

An algebra \( A \) is a **cellular algebra** if it has a cell datum. We call \( C \) a **cellular basis** of \( A \).

The notion of **weakly cellular algebras** in [11] Definition 2.9 is obtained from Definition 6.2 with condition (2) replaced by: there exists an anti-involution \( * \) of \( A \) satisfying

\[
(C_{st})^* \equiv C_{is} \pmod{A^{\lambda}}.
\]

The results and proofs of [12] are equally valid for weakly cellular algebras, so in the remainder of the paper we will not distinguish between cellular algebras and weakly cellular algebras.

We remark that [1, Theorem 6.3] holds over any commutative ring containing 1. In this paper, we need its special case below.

**Theorem 6.3.** [1] The set \( \{m_{st} \mid s, t \in \mathcal{P}^{std}(\lambda), \lambda \in \Lambda^+_2(r)\} \) with \( m_{st} \) defined in (6.3) is a cellular basis of \( \mathcal{H}_{2,r} \) over \( \mathbb{C} \).

Now, we construct a weakly cellular basis of \( \mathcal{B} \) over \( \mathbb{C} \). Fix \( r, t, f \in \mathbb{Z}_{>0} \) with \( f \leq \min\{r, t\} \). We need to redefine some notations. In contrast to ([1,3]), we define the following subgroups of \( \mathcal{G}_r, \mathcal{G}_r \times \mathcal{G}_t \) and \( \mathcal{G}_t \), respectively,

\[
S_{r-f} = \langle s_j \mid 1 \leq j < r - f \rangle,
\]
\[
\mathcal{G}_f = \langle \bar{s}_{t-i} s_{r-i} \mid 1 \leq i < f \rangle,
\]
\[
\bar{S}_{t-f} = \langle \bar{s}_j \mid 1 \leq j < t - f \rangle.
\]

(6.5)

Let \( D_{r,t}^f \) be the set consisting of the following elements:

\[
c = s_{r-f+1,i_{r-f+1}} \bar{s}_{t-f+1,j_{t-f+1}} \cdots s_{r,i} \bar{s}_{t,j} \quad \text{with} \quad r \geq i_r > \cdots > i_{r-f+1} \quad \text{and} \quad j_k \leq t - k.
\]

(6.6)

Then by arguments similar to those for Lemma 4.9, \( D_{r,t}^f \) is a complete set of right coset representatives for \( S_{r-f} \times \mathcal{G}_f \times \bar{S}_{t-f} \) in \( \mathcal{G}_r \times \mathcal{G}_t \). Let

\[
\Lambda_{2,r,t} = \{(f, (\lambda, \mu)) \mid (\lambda, \mu) \in \Lambda^+_2(r-f) \times \Lambda^+_2(t-f), \ 0 \leq f \leq \min\{r, t\}\}.
\]

(6.7)

**Definition 6.4.** For \( (f, \lambda, \mu), (\ell, \alpha, \beta) \in \Lambda_{2,r,t} \), we define

\[
(f, (\lambda, \mu)) \triangleright (\ell, (\alpha, \beta)) \iff \text{either } f > \ell \text{ or } f = \ell \text{ and } \lambda \triangleright_1 \alpha, \mu \triangleright_2 \beta,
\]

where in case \( f = \ell \), the orders \( \triangleright_1 \) and \( \triangleright_2 \) are dominance orders on \( \Lambda^+_2(r-f) \) and \( \Lambda^+_2(t-f) \) respectively (cf. (6.1)). Then \( (\Lambda_{2,r,t}, \triangleright) \) is a poset.

For each \( c \in D_{r,t}^f \), as in (6.6), let \( \kappa_c \) be the \( r \)-tuple

\[
\kappa_c = (k_1, \ldots, k_r) \in \{0, 1\}^r \quad \text{such that} \quad k_i = 0 \quad \text{unless} \quad i = i_r, i_{r-1}, \ldots, i_{r-f+1}.
\]

(6.8)

Note that \( \kappa_c \) may have more than one choice for a fixed \( c \), and it may be equal to \( \kappa_d \) although \( c \neq d \) for \( c, d \in D_{r,t}^f \). We set \( x^{\kappa_c} = \prod_{i=1}^{k_r} x_i^{k_i} \). By Lemma 4.3

\[
x^{\kappa_c} = s_{r-f+1,i_{r-f+1}} x_{i_{r-f+1}}^{k_{i_{r-f+1}}} \cdots s_{r-1,i_{r-1}} x_{i_{r-1}}^{k_{i_{r-1}}} s_{r,i} x_r^{k_i} \bar{s}_{t-f+1,j_{t-f+1}} \cdots \bar{s}_{t,j}.
\]

(6.9)

For each \( (f, \lambda) \in \Lambda_{2,r,t} \) (thus \( \lambda \) is now a pair of bipartitions), let

\[
\delta(f, \lambda) = \{(t, c, \kappa_c) \mid t \in \mathcal{P}^{std}(\lambda), c \in D_{r,t}^f \text{ and } \kappa_c \in N_f\}.
\]

(6.10)
where \( N_f = \{ \kappa, c \in D^f_{r,t} \} \). We remark that in (6.10), \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \) with \( \lambda^{(1)} \in \Lambda^+_2 (r-f) \) and \( \lambda^{(2)} \in \Lambda^+_2 (t-f) \), and \( t = (t^{(1)}, t^{(2)}) \) with \( t^{(i)} \) being a \( i \)-th tableau for \( i = 1, 2 \). In contrast to (6.3), we define

\[
e^f = e_{r,f} e_{r-1,t-1} \cdots e_{r-f+1,t-f+1} \quad \text{if} \quad f \geq 1, \quad \text{and} \quad e^0 = 1.
\]  

(6.11)

**Definition 6.5.** For each \( (f, \lambda) \in \Lambda^+_2 r, t \) and \( (s, \kappa, d), (t, \kappa, c) \in \delta(f, \lambda) \), we define

\[
C_{(s, \kappa, d), (t, \kappa, c)} = x^\kappa d^{-1} e^f m_{st} x^\kappa c,
\]

where \( m_{st} \) is a product of cellular basis elements for \( \mathcal{H}_{2,r-f} \) and \( \mathcal{H}_{2,t-f} \) described in Theorem 6.3.

We remark that an element in \( \mathcal{H}_{2,r-f} \) (generated by \( s_1, \cdots, s_{r-f-1} \) and \( x_1 \) may not commute with an element of \( \mathcal{H}_{2,t-f} \) (generated by \( \bar{s}_1, \cdots, \bar{s}_{t-f-1} \) and \( \bar{x}_1 \)). So, we always fix \( m_{st} \) as the product \( ab \), such that \( a \) (resp., \( b \)) is obtained from the corresponding cellular basis element of \( \mathcal{H}_{2,r-f} \) (resp. \( \mathcal{H}_{2,t-f} \)) described in Theorem 6.3 by using \( -x_1, -p, m, q \) (resp. \( -\bar{x}_1, q, p - n \)) instead of \( Y_1, u, v \), respectively.

Let \( I \) be the two-sided ideal of \( \mathcal{B} \) generated by \( e_1 \). By Proposition 4.17(2), there is a \( \mathbb{C} \)-algebraic isomorphism \( \varepsilon_{r,t} : \mathcal{H}_{2,r} \times \mathcal{H}_{2,t} \cong \mathcal{B}/I \) such that

\[
\varepsilon_{r,t}(s_i) = s_i + I, \quad \varepsilon_{r,t}(\bar{s}_j) = \bar{s}_j + I, \quad \varepsilon_{r,t}(\bar{x}_k) = \bar{x}_k + I, \quad \varepsilon_{r,t}(\bar{x}_\ell) = \bar{x}_\ell + I,
\]

for all possible \( i, j, k, \ell \).

For each \( f \) with \( 0 \leq f \leq \min\{r, t\} \), let \( \mathcal{B}(f) \) be the two-sided ideal of \( \mathcal{B} \) generated by \( e^f \). Then there is a filtration of two-sided ideals of \( \mathcal{B} \) as follows:

\[
\mathcal{B} = \mathcal{B}(0) \supset \mathcal{B}(1) \supset \cdots \supset \mathcal{B}(k) \supset \mathcal{B}(k+1) = 0, \quad \text{where} \quad k = \min\{r, t\}.
\]

**Definition 6.6.** Suppose \( 0 \leq f \leq \min\{r, t\} \) and \( \lambda \in \Lambda^+_2 (r-f) \times \Lambda^+_2 (t-f) \). Define \( \mathcal{B}^\infty(f, \lambda) \) to be the two-sided ideal of \( \mathcal{B} \) generated by \( \mathcal{B}(f+1) \) and \( S \), where

\[
S = \{ e^f m_{st} \mid s, t \in \mathcal{T}^{std}(\mu) \} \quad \text{and} \quad \mu \in \Lambda^+_2 (r-f) \times \Lambda^+_2 (t-f) \quad \text{with} \quad \mu \geq \lambda.
\]

We also define \( \mathcal{B}^\infty(f, \lambda) = \sum_{\mu \geq \lambda} \mathcal{B}^\infty(f, \mu) \), where \( \mu \in \Lambda^+_2 (r-f) \times \Lambda^+_2 (t-f) \).

By Corollary 5.18, \( \mathcal{B}^p_q r, f, t, f \) can be embedded into \( \mathcal{B} \), thus we regard it as a subalgebra of \( \mathcal{B} \).

**Lemma 6.7.** Suppose \( d \in D^f_{r,t} \) with \( 0 \leq f \leq \min\{r, t\} \). Then \( e^f (e_1) \subset \mathcal{B}(f+1) \), where \( e_1 \) is the two-sided ideal of \( \mathcal{B}^p_q r, f, t, f \) generated by \( e_1 \).

**Proof.** By assumption, we have \( r-f \geq 1 \) and \( t-f \geq 1 \). It is easy to check that \( e^f \) commutes with any element in \( \mathcal{B}^p_q r, f, t, f \). Since \( e_{r-f,t-f} = s_{r-f,1} s_{r-f,1} e_1 s_{1,r-f} s_{1,t-f} \), we have \( \mathcal{B}^p_q r, f, t, f \), \( e_1 \subset \mathcal{B}(f+1) \), proving the result.

For \( 0 \leq f \leq \min\{r, t\} \), let \( \pi_{r,t} : \mathcal{B}(f) \rightarrow \mathcal{B}(f)/\mathcal{B}(f+1) \) be the canonical epimorphism. Since both \( \mathcal{B}(f) \) and \( \mathcal{B}(f+1) \) are \( \mathcal{B} \)-bimodules, \( \pi_{r,t} \) is a homomorphism as \( \mathcal{B} \)-bimodules. The following result follows from (6.13) and Lemma 6.7 immediately.

**Lemma 6.8.** For each \( f \in \mathbb{Z}^0 \) with \( f < \min\{r, t\} \), there is a well-defined \( \mathbb{C} \)-homomorphism \( \sigma_f : \mathcal{H}_{2,r-f} \times \mathcal{H}_{2,t-f} \rightarrow \mathcal{B}(f)/\mathcal{B}(f+1) \) such that

\[
\sigma_f(h) = e^f \varepsilon_{r-f,t-f} (h) + \mathcal{B}(f+1) \quad \text{for} \quad h \in \mathcal{H}_{2,r-f} \times \mathcal{H}_{2,t-f},
\]
where \( \varepsilon_{r-f,t-f}(h) \) is the preimage of the element \( \varepsilon_{r-f,t-f}(h) \in B_{r-f,t-f}^{p,q} / I \) in \( B_{r-f,t-f}^{p,q} \), where \( I \) is the two-sided ideal of \( B_{r-f,t-f}^{p,q} \) generated by \( e_1 \).

**Lemma 6.9.** Suppose \( \lambda \in \Lambda_{r,t} \) and \( 0 \leq f \leq \min\{r,t\} \). For any \( s, t \in \mathcal{F}^{\text{std}}(\lambda) \),

1. \( e^f \mathbf{m}_{st} = \mathbf{m}_{st} e^f \in \mathcal{B}(f) \).
2. \( \sigma_f(\mathbf{m}_{st}) = \pi_{f,r}(e^f \mathbf{m}_{st}) \).
3. \( \sigma(e^f \mathbf{m}_{st}) \equiv e^f \mathbf{m}_{st} \pmod{\mathcal{B}(f,\lambda)} \), where \( \sigma \) is the anti-involution on \( \mathcal{B} \) induced from that in Lemma 4.1.

**Proof.** By Lemma [4.7](1), \( e_{i,j}(x_k + L_k) = (x_k + L_k)e_{j,k} \) if \( i \neq k \). Furthermore, \( e_{i,j}(\ell,k) = (\ell,k)e_{i,j} \) if \( 1 \leq \ell < k < i \). So, \( e_{i,j}(L_k) = L_ke_{i,j} \), forcing \( e_{i,j}x_k = x_ke_{i,j} \).

Similarly, \( e_{i,j}(x_k) = x_k e_{i,j} \) for \( k < j \). So, \( e^f \mathbf{m}_{st} = \mathbf{m}_{st} e^f \). The second assertion is trivial. By Lemma [4.6](3), \( x_i x_j \equiv x_j x_i \pmod{J} \), where \( J \) is the two-sided ideal of \( B_{r,t}^{p,q} \) generated by \( e_1 \). Now, (3) follows from Lemma 6.7 and (1).

Recall that the degree of a monomial \( \mathbf{m} \in \mathcal{B} \) in (6.30) is \( |\alpha| + |\beta| = \sum_{i=1}^r \alpha_i + \sum_{j=1}^t \beta_j \). So, \( \mathcal{B} \) is a filtered algebra, which associates to a \( \mathbb{Z} \)-graded algebra \( \mathfrak{g}(\mathcal{B}) \) defined the same as in (4.2).

The following is motivated by Song and one of authors’ work on \( q \)-walled Brauer algebras [22, Proposition 2.9].

**Proposition 6.10.** Fix \( r,t,f \in \mathbb{Z}^>0 \) with \( f \leq \min\{r,t\} \). Let \( M_f \) be the left \( B_{r,t}^{p,q} \)-module generated

\[
V_{r,t}^{f} = \{ e^f dx^{\kappa_d} \mid (d, \kappa_d) \in \mathcal{D}_{r,t}^{f} \times \mathbb{N}_f \}. \tag{6.14}
\]

Then \( M_f \) is a right \( \mathcal{B} \)-module.

**Proof.** We prove the result by induction on the degree of \( e^f dx^{\kappa_d} \). If the degree is 0, then \( e^f dx^{\kappa_d} = e^f d \). By the result on the walled Brauer algebra (which is the special case of [22, Proposition 2.9]), we have \( e^f dh \in M_f \) for any \( h \in B_{r,t}(\omega_0) \). Note that \( B_{r,t}(\omega_0) \) is a subalgebra of \( \mathcal{B} \).

Now, we consider \( e^f dx_1 \), where \( d \) has the form in (6.6). If \( i_j = 1 \) for some \( j \geq r-f+1 \), then \( j = r-f+1 \) and \( e^f dx_1 \in V_{r,t}^{f} \). Otherwise, we have (1)\( d = 1 \), and \( dx_1 = x_1 d \). Note that \( r-f+1 \geq r-f+1 > 1 \), we have \( e^f x_1 d = x_1 e^f d \in M_f \).

We have \( e^f d \bar{x}_1 = e^f \bar{x}_k d + e^f w \) for some \( k, 1 \leq k \leq t \) and some \( w \in \mathbb{C} \mathcal{S}_r \times \mathbb{C} \mathcal{S}_t \). By corresponding result for walled Brauer algebras, we have \( e^f w \in M_f \). If \( k \leq r-f \), by Lemma 4.7, \( e^f \bar{x}_k d \) can be replaced by \( \bar{x}_k e^f d \in M_f \). If \( k \geq r-f+1 \), by Lemma 4.6(2), we can use \( x_1 \) instead of \( \bar{x}_k \) in \( e^f \bar{x}_k d \). So, the required result follows from our previous arguments on \( s_i, \bar{s}_j \) and \( x_1 \). This completes the proof when the degree of \( e^f dx^{\kappa} \) is 0.

Suppose the degree of \( e^f dx^{\kappa_d} \) is not 0. We want to prove \( e^f dx^{\kappa_d} h \in M_f \) for any generators \( h \) of \( \mathcal{B} \).

**Case 1:** \( h \in \mathcal{G}_t \). We have \( x^{\kappa_d} h = h x^{\kappa_d} \). By our pervious result on degree 0, we have \( e^f dh \in M_f \). Therefore, we need to check \( e^f (dh)x^{\kappa_d} \in M_f \). If \( x_j \) is a term of \( x^{\kappa_d} \), by induction on the degree, we have \( e^f dh x \in M_f \), where \( x \) is obtained from \( x^{\kappa_d} \) by removing the factor \( x_j \). So, \( e^f dh x^{\kappa_d} \in M_f \) by induction assumption on \( \deg(e^f dh x^{\kappa_d})-1 \).
Case 2: \( h \in S_r \). We have \( x^e dh = hx \in \text{gr}(B) \), where \( x \) is obtained from \( x^e \) by permuting some indices. By induction assumption, it suffices to verify \( \ell^i d x \in M_f \) with \( \deg(x) = \deg(x^e) \). This has already been verified in Case 1.

Case 3: \( h = x_1 \). If \( x_1 \) is a factor of \( x^e \), we have \( x_1^2 = (p + q - m)x_1 - p(q - m) \) (cf. (5.23)). So, \( \ell^i dx^e x_1 \in M_f \) by induction assumption on \( \deg(\ell^i dx^e x_1) = 1 \). If \( x_1 \) is not a factor of \( x^e \), and if \( i_{r-f+1} = 1 \), where \( d \) has the form in (6.6), then there is nothing to be proven. Otherwise, \( i_{r-f+1} > 1 \) and \( \ell^i dx^e x_1 = x_1 \ell^i dx^e \in M_f \).

Case 4: \( h = \bar{x}_1 \). By Lemma 4.6(1), \( x^e \bar{x}_1 = \bar{x}_1 x^e \in \text{gr}(B) \). So, the result follows from induction assumption on degree and our previous results in Cases 2–3.

Case 5: \( h = e_1 \). We can assume \( x^e = x_1 \). Otherwise, the result follows from Lemma 4.7 induction assumption and our previous results in Cases 1–3. In this case, \( i_{r-f+1} = 1 \) and \( e_{r-f+1,t-f+1}d = d e_{1,j} \) for some \( j, 1 \leq j \leq t \). So, \( \ell^i dx e_1 = \ell^{i-1} d e_{1,j} x_1 e_1 \).

If \( j = 1 \), the required result follows from the equality \( e_1 x e_1 = (nq - mp) e_1 \) (cf. (5.27)). Otherwise, by Lemmas 4.6 and 4.7

\[
e_{1,j} x e_1 = -\bar{x}_1 e_{1,j} e_1 = -e_{1,j} \bar{x}_1 (\bar{1}, \bar{j}).
\]

So, we need to verify \( \ell^{i-1} d e_{1,j} x_1 (\bar{1}, \bar{j}) = \ell^i d x_1 (\bar{1}, \bar{j}) \in M_f \), which follows from our previous results in Cases 1, 2 and 4. This completes the proof of Proposition 6.10. \( \square \)

Proposition 6.11. Suppose \( (f, \lambda) \in \Lambda_{2,t} \). Then \( \Delta^R f, \lambda \) (resp., \( \Delta^L f, \lambda \)) is a right (resp., left) \( B \)-module, where

- \( \Delta^R f, \lambda \) is \( \mathbb{C} \)-spanned by \( \{ \ell^i m_{\beta} dx^e | (s, d, \kappa_d) \in \delta(f, \lambda) \} \), and
- \( \Delta^L f, \lambda \) is \( \mathbb{C} \)-spanned by \( \{ d^{-1} \ell^i m_{\beta} + B^{(f, \lambda)} | (s, d, \kappa_d) \in \delta(f, \lambda) \} \).

Proof. We remark that \( x_i \bar{x}_j = \bar{x}_j x_i \in \Delta^R f, \lambda \) (resp., \( \Delta^L f, \lambda \)) for all possible \( i, j \)’s. So, the result follows from Proposition 6.10 and Theorem 6.3 on the cellular basis of level two degenerate Hecke algebras \( \mathcal{H}_{2,r-f} \times \mathcal{H}_{2,t-f} \). \( \square \)

Theorem 6.12. Let \( m, n, r, t \in \mathbb{Z}^\geq 1 \), \( p, q \in \mathbb{C} \) such that \( r + t \leq \min\{m, n\} \). The set

\[
C = \{ C_{(s, \kappa, c)}((t, \kappa, d)) | (s, \kappa, c), (t, \kappa, d) \in \delta(f, \lambda), \forall (f, \lambda) \in \Lambda_{2,t} \},
\]

is a weakly cellular basis \( B = B^{p,q}_{r,f} (m, n) \) over \( \mathbb{C} \).

Proof. Suppose \( 0 \leq f \leq \min\{r, t\} \). By Proposition 6.11 \( B(f)/B(f+1) \) is spanned by \( C_{(s, c, \kappa)}((t, d, \kappa_d)) + B(f + 1) \) for all \( (s, c, \kappa), (t, d, \kappa_d) \in \delta(f, \lambda) \) and \( \lambda \in \Lambda_{2,r-f,t-f} \). So, \( B \) is spanned by \( C \). Counting the cardinality of \( C \), yields \( |C| = 2^{r+t}(r+t)! \), which is the dimension of \( B \), stated in Corollary 5.18. So, \( C \) is a \( \mathbb{C} \)-basis of \( B \). By Lemma 6.9(3) and Proposition 6.11 it is a weakly cellular basis in the sense of 6.4. \( \square \)

Remark 6.13. If we consider level two walled Brauer algebras over a commutative ring containing 1, and if we know its rank is equal to \( 2^{r+t}(r+t)! \), then all results in this section hold. We hope to prove this result elsewhere.

7. **Irreducible modules for \( B \)**

In this section, we classify irreducible \( B \)-modules over \( \mathbb{C} \) via Theorem 6.12. So, we assume \( r + t \leq \min\{m, n\} \).
First, we briefly recall the representation theory of cellular algebras \[12\]. At moment, we keep the notations in Definition \[6.2\]. So, \( R \) is a commutative ring \( R \) containing 1 and \( A \) is a (weakly) cellular algebra over \( R \) with a weakly cellular basis \( \{ C_s | s \in T(\lambda), \lambda \in \Lambda \} \). We consider the right \( A \)-module in this section.

Recall that each cell module \( C(\lambda) \) of \( A \) is the free \( R \)-module with basis \( \{ C_s | s \in T(\lambda) \} \), and every irreducible \( A \)-module arises in a unique way as the simple head of some cell module \[12\]. More explicitly, each \( C(\lambda) \) comes equipped with the invariant form \( \phi_\lambda \) which is determined by the equation

\[
C_s \psi \equiv \phi_\lambda(C_t, C_v) \cdot C_{ss} \pmod{A^{\Phi_\lambda}}.
\]

Consequently,

\[
\text{Rad} C(\lambda) = \{ x \in C(\lambda) | \phi_\lambda(x, y) = 0 \text{ for all } y \in C(\lambda) \},
\]

is an \( A \)-submodule of \( C(\lambda) \) and \( D^\lambda = C(\lambda)/\text{Rad} C(\lambda) \) is either zero or absolutely irreducible. Graham and Lehrer \[12\] proved the following result.

**Theorem 7.1.** \[12\] Let \((A, \Lambda)\) be a (weakly) cellular algebra over a field \( F \). The set \( \{ D^\lambda | D^\lambda \neq 0, \lambda \in \Lambda \} \) consists of a complete set of pairwise non-isomorphic irreducible \( A \)-modules.

By Theorem \[6.12\] we have cell modules \( C(f, \lambda) \) with \((f, \lambda) \in \Lambda_{2,r,f} \) for \( \mathcal{B} \). In fact, it is \( \Delta^R(f, \lambda) \) in Proposition \[6.11\] up to an isomorphism. Let \( \phi_{f,\lambda} \) be the corresponding invariant form on \( C(f, \lambda) \). We use Theorem \[7.1\] to classify the irreducible \( \mathcal{B} \)-module over \( \mathbb{C} \).

Let \( \mathcal{H}_{2,r,f} \) (resp., \( \mathcal{H}_{2,t,f} \)) be the level two Hecke algebra which is isomorphic to the subalgebra of \( \mathcal{H}_{2,r,f}^{p,q} \) generated by \( s_1, s_2, \ldots, s_{r-f-1} \) and \( x_1 \) (resp., \( s_1, \bar{s}_2, \ldots, \bar{s}_{t-f-1} \) and \( \bar{x}_1 \)). So, the eigenvalues of \( x_1 \) (resp., \( \bar{x}_1 \)) are given in \[5.28\]. By Theorem \[6.3\] we have

\[
\{ m_{st} | s, t \in \mathcal{T}^{std}(\lambda), \lambda \in \Lambda_2^+(r-f) \times \Lambda_2^+(t-f) \}
\]

is a cellular basis of \( \mathcal{H}_{2,r,f} \times \mathcal{H}_{2,t,f} \). We remark that \( m_{st} \) is a product of cellular basis elements of \( \mathcal{H}_{2,r,f} \) and \( \mathcal{H}_{2,t,f} \) described in Theorem \[6.3\].

Let \( C(\lambda) \) be the cell module with respect to \( \lambda \in \Lambda_{2,2,r,f-1} \) for \( \mathcal{H}_{2,r,f} \times \mathcal{H}_{2,t,f} \). Let \( \phi_\lambda \) be the invariant form on \( C(\lambda) \). For simplicity, we use \( \mathbf{H}(2, f) \) to denote \( \mathcal{H}_{2,r,f} \times \mathcal{H}_{2,t,f} \).

**Proposition 7.2.** Suppose \( r, t \in \mathbb{Z}^{>2} \). We have \( e_{r,t} \mathcal{B}^{p,q}_{r,t} e_{r,t} \subseteq e_{r,t} \mathcal{B}^{p,q}_{r-1,t-1} \).

**Proof.** Recall that \( \mathcal{B}^{p,q}_{r,t} \) is a (weakly) cellular algebra with cellular basis given in Theorem \[6.12\]. So, it suffices to verify

\[
e_{r,t} C(\mathbf{s}_{\mathbf{k},d})(t, \mathbf{k}, \mathbf{c}) e_{r,t} \in e_{r,t} \mathcal{B}^{p,q}_{r-1,t-1}, \quad (7.1)
\]

where \( C(\mathbf{s}_{\mathbf{k},d})(t, \mathbf{k}, \mathbf{c}) = x^{\mathbf{c}} d^{\mathbf{d}} e^{\mathbf{m}_{st}} c^{\mathbf{x}} \mathbf{c} (\text{cf. } (6.12)) \).

Let \( f = 0 \). Since \( m_{st} \) is a combination of monomials of form \( \prod_{i=1}^{r} x_i^{\alpha_i} w_1 \prod_{i=1}^{t} \bar{x}_i^{\beta_i} \), it suffices to verify

\[
e_{r,t} \prod_{i=1}^{r} x_i^{\alpha_i} w_1 \prod_{i=1}^{t} \bar{x}_i^{\beta_i} e_{r,t} \in e_{r,t} \mathcal{B}^{p,q}_{r-1,t-1} \quad (7.2)
\]

We prove \[7.2\] by induction on the degree of \( \prod_{i=1}^{r} x_i^{\alpha_i} w_1 \prod_{i=1}^{t} \bar{x}_i^{\beta_i} \). The case for degree 0 follows from \[6.2\] Proposition 2.1. In general, we assume that \( \alpha_i = 0 \) for
1 ≤ i ≤ r − 1 and β_j = 0 for 1 ≤ j ≤ t − 1. Otherwise, (7.2) follows from induction assumption and the equalities _e_ r,t x_i = x_i e_r,t and _e_ r,t x_j = x_j e_r,t for i ≠ r and j ≠ t.

By symmetry, we assume α_r = 1. Write w_1 = s_r,k w_2 for some k, 1 ≤ k ≤ r and some w_2 ∈ _S_ r−1. Since any element in _S_ r−1 commutes with _s j_ ≜ _S_ t, _x_ t and _e_ r,t, we can assume w_1 ∈ {1, s_r−1}.

If w_1 = 1, by Lemma 4.7(2), it suffices to verify _e_ r,t x_t w_1 x_t^β e_r,t e_r,t ∈ _e_ r,t _B_ r−1,t−1. In this case, we have _x_ t w_1 = _w_ 1 _x_ k + h for some h ∈ _C_ _S_ t and some k with ( _t_ ) _w_ 1 = _k_. By induction on degree, we need to verify

\[ _e_ r,t x_t w_1 x_k x_t^β e_r,t ∈ _e_ r,t _B_ r−1,t−1. \] (7.3)

By Lemma 4.7(2), and induction assumption on degree, we have (7.3) if k ≠ t. Otherwise, we have k = t and w_1 ∈ _S_ t−1. So, _e_ r,t w_1 = _w_ 1 _e_ r,t. By induction on degree, we use ( _x_ t + _L_ t)_1+β e_r,t in _e_ r,t _x_ t x_t^β e_r,t. So, the result follows from Lemma 4.8(2). This verifies (7.2) provided f = 0.

Suppose f > 0. By (6.12), we rewrite (7.1) as follows:

\[ _e_ r,t x^α d^{-1} _e f m_α c x^κ e_r,t ∈ _e_ r,t _B_ r−1,t−1. \] (7.4)

Applying our previous result for f = 0 on _e_ r,t x^α d^{-1} _e f and _e f m_α c x^κ e_r,t and noting that _e f = _e_ r,t… _e_ r−f+1,t−f+1, we have (7.4) as required.

By recalling the definitions of _ω_ 0 and _ω_ 1 in (5.27), we see that _ω_ 0 = _ω_ 1 = 0 if and only if m = n and p = q. For any λ ∈ _Λ_ 2 (r − f) × _Λ_ 2 (t − f) and t ∈ _T_ std(λ), we define _m_ t, _m_ t ≝ _H_ (2, f) ⊂ _f_ λ, where _H_ (2, f) is given above Proposition 7.2.

**Lemma 7.3.** Let _B_ be the level two walled Brauer algebra defined over _C_. Suppose (f, λ) ∈ _Λ_ 2 (r, t) and f ≠ r if t = r. Then _ρ_ f,λ ≠ 0 if and only if _ρ_ λ ≠ 0.

**Proof.** If r ≠ t or if r = t and f ≠ r, then either s_r,r−f or _s_ t,t−f is well-defined. We denote such an element by w. So, _e f w e f = _e f_.

If _ρ_ λ ≠ 0, then _ρ_ λ (_m_ s, _m_ t) ≠ 0 for some _s, t ∈ T_ std(λ). We have _ρ_ f,λ ≠ 0 since

\[ _m_ s _f e f w_0 _e f m_ t ≝ _ρ_ λ (_m_ s, _m_ t) _e f m_ t (mod _B_ (f, λ)). \]

We remark that _x_ i _x_ j = _x_ j _x_ i in _C_ (f, λ) for 1 ≤ i ≤ r − f and 1 ≤ j ≤ t − f (cf. Lemma 6.7).

Conversely, if _ρ_ f,λ ≠ 0, then

\[ _ρ_ f,λ (_e f m_ s c x^α, _e f m_ t c x^κ) ≠ 0, \]

for some ( _s, d, κ_ d), ( _t, c, κ_ c) ∈ δ(f, λ). We have _ρ_ λ ≠ 0. Otherwise,

\[ _m_ λ _ρ_ λ _h m_ t ≝ 0 (mod _H_ (2, f) ⊂ _f_ λ), \]

for all _h ∈ H_ (2, f). Using Proposition 7.2 repeatedly, we have

\[ _e f m_ s _f c d^{-1} _e f m_ t ≝ _m_ λ _h m_ t _e f (mod _B_ (f + 1)), \]

for some _h ∈ H_ (2, f), forcing _ρ_ f,λ (_e f m_ s c x^α, _e f m_ t c x^κ) = 0, a contradiction.

**Lemma 7.4.** Let _B_ be the level two walled Brauer algebra defined over _C_ with _r_ = _t_. Then _ρ_ r,0 ≠ 0 if at least one of _ω_ 0 and _ω_ 1 is non-zero. Otherwise, _ρ_ r,0 = 0.
Definition 7.5. For our purpose, we only consider bipartitions and only if $\lambda$ of $n$. If $\lambda$ is Kleshchev in the sense of [26, p273]. We recall the definition as follows.

Theorem 7.6. Let $\mathcal{B} = \mathcal{B}^{p,q}_{r,t}(m,n)$ be the level two walled Brauer algebra over $\mathbb{C}$ with condition $r + t \leq \min\{m,n\}$.

(1) Suppose either $r \neq t$ or $r = t$ and one of $\omega_0, \omega_1$ is non-zero. Then the set of pairwise non-isomorphic irreducible $\mathcal{B}$-modules are indexed by $\{(f, \lambda) \in \Lambda_2(t-f), 0 \leq f \leq \min\{r,t\}, \lambda$ being Kleshchev}. 

(2) If $r = t$ and $\omega_0 = \omega_1 = 0$, then the set of pairwise non-isomorphic irreducible $\mathcal{B}$-modules are indexed by $\{(f, \lambda) \in \Lambda_2, |0 \leq f < r, \lambda$ being Kleshchev}. 

Proof. Suppose $\omega_0 \neq 0$. We have $\phi_{r,0} \neq 0$ since $e^f e^f = \omega_0^f e^f$. Otherwise, $\omega_0 = m - n = 0$, forcing $m = n$ and $\omega_1 = n(q - p)$. We consider $e_re_{r-1} \cdots e_1 \prod_{i=1}^{r}(x_i + L_i)$ (where the product is in any order) and $e_re_{r-1} \cdots e_1$ in the cell module $C(r,0)$. By Lemma 4.8(2),

$$e_re_{r-1} \cdots e_1 \prod_{i=1}^{r}(x_i + L_i)e_re_{r-1} \cdots e_1 = \omega_1 e_re_{r-1} \cdots e_1.$$ 

We have $\phi_{r,0} \neq 0$ if $\omega_1 \neq 0$.

Finally, we assume $\omega_0 = \omega_1 = 0$ and $r = t$. In this case, we have $m = n$ and $p = q$. We claim that $\phi_{r,0} = 0$.

In fact, for any two basis elements $e^r c x^{k_r}$ and $e^d c x^{k_d}$ in $C(r,0)$ with $c, d \in D_{r,t}^r$, by using Proposition 7.2 repeatedly, we have

$$e^r c x^{k_r} x^{k_d} d^{-1} e^r \in e^{-1} e_1 \mathcal{B}^{p,q}_{1,1} e_1.$$ 

(7.5) However, since we are assuming that $\omega_0 = \omega_1 = 0$, it is routine to check $e_1 \mathcal{B}^{p,q}_{1,1} e_1 = 0$. So, $\phi_{r,0} = 0$, as required. □

In [17], Kleshchev classified the irreducible modules for degenerate cyclotomic Hecke algebra $\mathcal{H}_{r,n}$ over an arbitrary field via Kleshchev multipartitions of $n$. As mentioned in [11] page 130, one could not say that $\phi_\lambda \neq 0$ if and only if $\lambda$ is a Kleshchev multipartition. In our case, since level two walled Brauer algebras are only related to representations of cyclotomic degenerate Hecke algebras (more explicitly, level two Hecke algebras) over $\mathbb{C}$, we can use Vazirani’s result [26, Theorem 3.4]. In fact, it is not difficult to prove that there is an epimorphism from a standard module in [17, page 130], one could not say that $\phi_\lambda \neq 0$ if and only if $\lambda$ is Kleshchev in the sense of [26, p273]. We recall the definition as follows. For our purpose, we only consider bipartitions.

Definition 7.5. Fix $u_1, u_2 \in \mathbb{C}$ with $u_1 - u_2 \in \mathbb{Z}$. A bipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)}) \in \Lambda_2^+(n)$ of $n$ is called a Kleshchev bipartition with respect to $u_1, u_2$ if

$$\lambda^{(1)}_{u_1-u_2+i} \leq \lambda^{(2)}_i$$ 

for all possible $i$. If $u_1 - u_2 \not\in \mathbb{Z}$, then we say that all bipartitions of $n$ are Kleshchev bipartitions.

Since we consider a pair of bipartitions $(\lambda^{(1)}, \lambda^{(2)})$, where $\lambda^{(1)} \in \Lambda_2^+(r - f)$ and $\lambda^{(2)} \in \Lambda_2^+(t - f)$ for all $f, 0 \leq f \leq \min\{r,t\}$, we say that $\lambda$ is Kleshchev if both $\lambda^{(1)}$ and $\lambda^{(2)}$ are Kleshchev with respect to the parameters $u_1 = -p, u_2 = m - q$ and $u_1 = q, u_2 = p - n$ respectively. The following result follows from Lemmas 7.3, 7.4 and our previous arguments immediately.

Theorem 7.6. Let $\mathcal{B} = \mathcal{B}^{p,q}_{r,t}(m,n)$ be the level two walled Brauer algebra over $\mathbb{C}$ with condition $r + t \leq \min\{m,n\}$.

(1) Suppose either $r \neq t$ or $r = t$ and one of $\omega_0, \omega_1$ is non-zero. Then the set of pairwise non-isomorphic irreducible $\mathcal{B}$-modules are indexed by $\{(f, \lambda) \in \Lambda_2, 0 \leq f \leq \min\{r,t\}, \lambda$ being Kleshchev\}.

(2) If $r = t$ and $\omega_0 = \omega_1 = 0$, then the set of pairwise non-isomorphic irreducible $\mathcal{B}$-modules are indexed by $\{(f, \lambda) \in \Lambda_2, 0 \leq f < r, \lambda$ being Kleshchev\}.
We close the paper by giving a classification of non-isomorphic indecomposable direct summands of $\mathfrak{gl}_{m|n}$-modules $M^{rt}$ (cf. (5.9)) provided that $M = K_\lambda$ is typical. Such direct summands are called *indecomposable tilting modules* of $\mathfrak{gl}_{m|n}$.

**Theorem 7.7.** Assume $r+t \leq \min\{m,n\}$.

1. If $p-q \in \mathbb{Z}$ with either $p-q \leq -m$ or $p-q \geq n$, then $M^{rt}$ (cf. (5.9)) is a tilting module and the non-isomorphic indecomposable direct summands of $M^{rt}$ are indexed by $\{(f,\mu) \in \Lambda_{2,r,t} \mid 0 \leq f \leq \min\{r,t\}, \mu$ being Kleshchev$\}$.

2. If $p-q \notin \mathbb{Z}$, then the non-isomorphic indecomposable direct summands of $M^{t}$ are irreducible and indexed by $\Lambda_{2,r,t}$.

**Proof.** Under the assumptions in (1) and (2), the Kac module $K_{\lambda_{pq}}$ is typical and at least one of $\omega_0$ and $\omega_1$ is non-zero. In this case, $M^{rt}$ is a tilting module (see, e.g., [4, IV] for the case $t = 0$, from which one sees that it holds in general). By Theorem 5.16 there is a bijection between the set of non-isomorphic indecomposable direct summands of $M^{rt}$ and the irreducible modules of $\mathcal{B}$. So, (1)–(2) follows from Theorem 7.6(1). In particular, if $p-q \notin \mathbb{Z}$, all partitions $\lambda \in \Lambda_{2}^{+}(r-f) \times \Lambda_{2}^{+}(t-f)$ are Kleshchev. We remark that (2) also follows from Corollary 5.18 and Graham-Lehrer’s result in [12], which says that a cellular algebra is semisimple if and only if each cell module is equal to its simple head.

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