ALP-Pions generalized

Triparno Bandyopadhyay, Subhajit Ghosh, and Tuhin S. Roy

Abstract: A light axion-like particle or an ALP not just gives rise to interesting and spectacular signals of new physics as final states in meson decays, it necessarily leaves tell-tale signatures in processes that involve standard model (SM) fields only (i.e., SM processes). These effects result in the violation of the Gell-Mann–Okubo mass relation, modified form factors, altered integrated and differential rates for various SM transitions etc. This suggests that in the presence of a low lying state, such as an ALP, extraction of masses, mixing angles, and form factors in an entirely data-driven way from meson-physics observables is a highly non-trivial exercise. However, once done correctly, these same observables may, in turn, provide important (indirect) bounds on ALP physics, which remain robust even in the limits where new physics effects conspire to weaken the bounds from direct searches. Starting with a generalized ALP-quark Lagrangian (where restrictions due to parity are removed) we demonstrate this approach by focussing on $K^+_{l3}$ decays, where we derive (indirect) bounds on ALP physics using NA48/2 data and lattice results. We also find sum rules which not just show deviations in the presence of an ALP, but also give hints towards the specific nature of the ALP physics itself.
1 Introduction

Studied primarily in the context of the strong-CP problem [1–5] and cold dark matter [6–8], sub-GeV pseudo-Nambu-Goldstone bosons (pNGB) are generic predictions of various new physics (NP) scenarios. These range from the dynamical generation of small neutrino masses (Majorons) [9], models attempting to solve the flavor problem (Flavons) [10], to models of universal extra dimensions [11] and string compactification [12–14]. Such pNGBs are also invoked in the context of the anomalous muon magnetic moment [15], the hierarchy problem [16], electroweak baryogenesis [17], and as a portal to dark matter [18]. These pNGBs are typically symmetric under a continuous shift of the field or—in less stringent cases—a periodic shift. For the prototypical axion [4, 5], QCD breaks the shift symmetry giving a mass to it at the corresponding scale ($\Lambda_{\text{QCD}}$). Yet, in general, one can decouple the mass of the pNGB from $\Lambda_{\text{QCD}}$ [19–23]. Such pNGBs of diverse origins and interactions—the masses of which are not strictly tied to $\Lambda_{\text{QCD}}$—are generally clubbed together in the literature under the hypernym of Axion-Like Particles (ALP). In this paper, we define an ALP, $a$, to be a pNGB with a periodic symmetry (compact, e.g., when the centre symmetry of a non-abelian group is preserved) that couples to QCD through dim-5 operators involving the quarks and the dim-5 $aG\tilde{G}$ coupling. We assume the presence of a shift-breaking spurion contributing to the mass of the ALP.

Owing to the diverse origins of ALPs, the detection strategies are varied, with different experiments—both dedicated and multi-purpose—probing different regions of the parameter space [24–37]. The standard sources of constraints on ALP couplings to QCD are Meson decay experiments [34–42]. The bulk of the analyses focuses on looking for ALPs in the decay products of heavier mesons. The SM interactions of the $\pi^0$ source these processes, with the ALP being emitted in place of the $\pi^0$—due to $a-$ $\pi^0$ mixing. Even though direct detection experiments are the most straightforward ways to look for a particle, these are somewhat impeded by assumptions about the properties of the hypothetical particle. These assumptions are related to the decay length (prompt/displaced/invisible), the decay channel, and the mass of the particle. The direct detection bounds are effective in constraining the Wilson coefficients corresponding to the ALP interactions with the mesons, insofar as allowed by these.
underlying assumptions. We can translate the bounds on the IR coefficients to those at the Electroweak (EW) scale, given a proper treatment of the corresponding effective field theories (EFT)—matched at (∼) \( \Lambda_{\text{QCD}} \). The pioneering work in this regard was presented in their 1986 paper by Georgi et al. [43]. Recently, over multiple papers, Bauer et al. expands [44] on the original work and points out a key omission [45]. In the latter, the authors present a meticulous matching of the chiral Lagrangian to the EW Lagrangian and then show the effects of the new operator on ALP-meson amplitudes.

We begin this work in a similar spirit and construct the chiral perturbation theory in the presence of a light ALP (A\( \chi \)PT). We call this setup ‘generalized’ in the sense that we derive the A\( \chi \)PT from a larger set of operators (up to \( d = 5 \)) involving the ALP and the quarks, where only the compactness criterion of the ALP is taken into account. Extending the framework to include operators where the ALP appears as a scalar and not a pseudo-scalar, allows us to include theories where the ALP may mix with CP even states. More importantly, it enables us to perform phenomenological studies of models where light CP-even scalars interact with the SM through a Higgs-portal like setup [46]. We construct the ALP-quark lagrangian in the EW basis where left-handed ‘u’ and ‘d’ quarks are treated on an equal footing. One advantage of such a basis is that it helps us to identify relationships among Wilson coefficients in the A\( \chi \)PT, which are hard to guess otherwise, and also to identify Wilson coefficients which have an additional suppression proportional to the EW breaking vev. Note that, in this work, we refrain from working with ‘all’ possible dim-5 contact terms between the ALP and the quarks.

Apart from working with a generalized version of the A\( \chi \)PT, this work differs from the bulk of the literature on ALPs in that we focus exclusively on processes that involve SM final states only. To be specific, we look at how the charged current (CC) interactions of the pions get modified in the presence of the low-lying ALP. The implications of the existence of a light ALP are many, starting with the modification of the meson mass spectra over the predictions of the SM chiral lagrangian (SM\( \chi \)PT). A particularly striking effect of this modification is the violation of the Gell-Mann–Okubo mass (GMO) mass relations [47, 48] at tree level. We also find important deviations of the form factors (FFs) in the A\( \chi \)PT, when compared to the SM\( \chi \)PT [49]. Take for example the FFs \( f_{K\pi}^{\pm}(0) \) defined as the matrix element \( \langle \pi | \bar{s}_L \gamma^\mu u_L | K \rangle \), where the ± refers to the components of the matrix element along \( p_K \pm p_\pi \) respectively. We find that

\[
\frac{f_{K^+\pi^0}(0)}{f_{K^0\pi^+}(0)} \bigg|_{\text{A}\chi\text{PT}} = \frac{f_{K^+\pi^0}(0)}{f_{K^0\pi^+}(0)} \bigg|_{\text{SM}\chi\text{PT}} + \frac{f_\pi^2}{f_\pi^2} K_1, \tag{1.1}
\]

where \( f_\pi \) and \( f_\pi \) are the characteristic scales associated with the pions and the ALP respectively, and \( K_1 \) is a function of various Wilson coefficients of the ALP-quark contact operators.

As expected, all these results simply suggest that even the rates of various processes in the SM meson sector will invariably deviate in the presence of an ALP. As a specific example, we consider in this work the CC decay of the charged Kaon to a neutral pion and two leptons, \( K^\pm \rightarrow \pi^0 \ell^+\nu (K^\pm_{\ell^3}) \). The decay amplitudes computed in the A\( \chi \)PT deviate from that in the SM primarily because of two reasons. Firstly, the matrix element of the SM operator changes as neutral pions mix with the ALP (this effect is captured in the altered FFs), and secondly, there exist new operators which contribute at the order \( (f_\pi/f_\pi)^2 \). The decay width and the corresponding differential distributions, therefore, differ from the SM expectations at \( \mathcal{O} \left( \frac{f_\pi^2}{f_\pi^2} \right) \). This leading order effect appears as interference between the SM piece and the NP part of the amplitude.

This presents us with a dilemma as well as an opportunity. Clearly, the extraction of SM parameters (masses, mixing angles, FFs etc.) in a data-driven way from pion data becomes a nontrivial exercise in the presence of a low-lying ALP. In fact, one requires a systematic study where these
quantities are either extracted from observables that remain relatively unaffected and/or calculated theoretically (such as lattice). Take for example, the CC interactions of mesons, which are standard sources for extracting CKM elements. The rate of \( K_{\ell 3} \), used as a standalone measurement of \( V_{us} \), and the differential distributions, used for data-driven determination of the \( f_{K^\mp}^\varphi \) FFs, are both sensitive to NP effects even at tree level. On the other hand, if one uses \( V_{us} \) extracted from channels unaffected by the ALP physics and FFs from lattice measurements, one can turn the argument around and use these precision measurements to constrain the ALP physics.

As a concrete demonstration, we use the observed \( K_{\ell 3}^+ \) 2D Dalitz distribution as reported by the NA48/2 collaboration [50] and the particle data group (PDG) average of the partial \( K^+\ell3 \) width [51] to constrain the parameters of the \( \Lambda\chiPT \). As SM inputs, we use \( V_{us} \) extracted from \( K^+ \rightarrow \mu^+ \nu \) [51] and the lattice computations of the \( f_{K^\mp}^\varphi \) as reported by the European Twisted Mass Collaboration [52]. We then go on to provide the first, to our knowledge, indirect constraints on the \( \Lambda\chiPT \) parameters using \( K_{\ell 3}^+ \). At first glimpse, the constraints appear much weaker than those from direct ALP searches. However, these constraints are somewhat model independent — in that these are largely independent of the details of the ALP mass, decay channels, and lifetime. More importantly, we show that these limits remain valid even in the corners of the theory space where the \( K^+ \) width to \( a\ell\nu \) is suppressed. Note that, the choice of the CC decay of the \( K^\pm \) over the much simpler \( \pi^\pm \) system (say, \( \pi^\beta \)) is motivated by the identification that there are multiple observable effects of the ALP which are exhibited only in \( K^\pm \) decays and not in those of the \( \pi^\pm \).

Finally, we construct sum rules out of the FFs corresponding to the CC semi-leptonic decays of the mesons. The sums can indicate the presence and the nature of an ALP in the chiral Lagrangian. For example, the sum corresponding to the \( K^+ \) FFs is identically equal to one in the SM, deviating from unity, at tree level, in the presence of the ALP. What is striking is that it can deviate on either side of unity, the sign of the deviation pointing to qualitatively different kinds of UV physics.

The paper is structured as follows. In the following section (section 2) we write down the dim-5 operators of the ALP in the EW symmetric phase of the SM and go on to match those onto the chiral Lagrangian. We go on to show the deviations to SM expectations in presence of the ALP. In section 3 we focus on \( K_{\ell 3}^+ \) decays. We first list all the different ALP sources modifying the \( K_{\ell 3}^+ \) amplitude and then go on to constrain these NP effects using data on \( K_{\ell 3}^+ \) decay spectra. In section 4, we discuss new contributions to the amplitudes with the ALP in it and discuss limits where the ALP amplitude is subdominant compared to the corresponding modification to SM amplitudes. In section 5, we go on to derive the sum rules discussed above. Finally, we conclude.

2 Formalism: Construction of the general ALP-Pion Lagrangian

Deriving the ALP-pion interactions in the IR is subtle, where subtleties arise from the matching of the ALP-quark Lagrangian above the QCD scale to the chiral Lagrangian. The most crucial piece in the calculation stems from the choice of the basis in which the ALP-quark Lagrangian is written. Before beginning the following subsection where we methodically derive the ALP-pion Lagrangian, here we initiate a brief discussion regarding the basis.

In order to substantiate the right choice for the basis, first note that non-trivial constraints exist in the IR Lagrangian, the origin of which lies in the demand that the ALP-quark Lagrangian must arise from a fully electroweak-symmetric theory at short distances. This observation gives the correct power counting (hence, the right suppression) for specific terms in the ALP-pion chiral Lagrangian.
which are harder to guess at low energy. Consider, for example, the two seemingly different operators,
\[ k_1 \frac{f_{\pi}}{2 f_a} \partial_{\mu} a \partial^\mu \pi_0 \quad \text{and} \quad i k_2 \frac{1}{2 f_a} \partial^\mu a \left( \pi^+ \partial_{\mu} \pi^- - \pi^- \partial_{\mu} \pi^+ \right). \tag{2.1} \]

In the above, \( k_1 \) and \( k_2 \) are dimensionless coupling constants, and the scales \( f_a \) and \( f_{\pi} \) are characteristic scales associated with the ALP and the pion physics respectively. The first operator generates the kinetic mixing between the ALP and \( \pi_0 \), while both these operators play important parts in pion decay. It turns out that in the EW limit one finds \( k_1 = -k_2 \). Consequently,
\[ k_1 + k_2 \simeq O(\nu/\Lambda_{UV}), \tag{2.2} \]
if we assume that even in the presence of NP, the Higgs vev, \( \nu \), remains the only source of electroweak symmetry breaking. Note, \( \Lambda_{UV} \) represents a high scale far above the EW scale, for example, corresponding to a higher dimensional Higgsed operator.

Even though the statement in eq. (2.2) appear to be highly nontrivial, it can be easily understood when the low energy chiral Lagrangian is derived from the manifestly EW symmetric quark Lagrangian. In order to demonstrate it, we begin with QCD with the number of flavors \( N_f = 2 \). The theory contains an approximate \( SU(2)_L \times SU(2)_R \), realized as
\[ \begin{align*}
q &\equiv \left( \begin{array}{c}
u \\ d \\
\end{array} \right), \\
q_L &\equiv P_L q \rightarrow L q_L, \quad \text{and} \quad q_R \equiv P_R q \rightarrow R q_R, \\
\end{align*} \tag{2.3} \]
where \( P_L \) and \( P_R \) are the projection operators that projects out the left- and right-handed spinors of the Dirac fermions \( u \) and \( d \). The contact operators relevant for our discussion are given as
\[ \sum_{i=0}^3 k^{L/R}_i \frac{f_a}{f_{\pi}} \partial_{\mu} a \gamma^\mu q_L, \quad \text{and} \quad \sum_{i=0}^3 k^{L/R}_i \frac{f_a}{f_{\pi}} \gamma^\mu q_R, \tag{2.4} \]
where \( t^i \) are the generators of \( SU(2) \) and \( k^{L/R}_i \) play the role of Wilson coefficients. Note that all such terms are not allowed by the symmetries of the SM. The operators proportional to \( t^1, t^2 \), or rather \( t^3 \), break \( U(1) \) electromagnetism and are therefore not allowed. More constraints follow once we recognize that the \( SU(2)_L \) in this particular case can be identified with the electroweak \( SU(2)_W \), and therefore \( q_L \) transforms as a doublet under \( SU(2)_W \). Therefore, in the EW limit one obtains the result that all \( k^{L/R}_i \rightarrow 0 \), except for \( i = 0 \). Summarizing:
\[ \begin{align*}
k^{L/R}_{1,2} &= 0 \quad : \quad \text{from Electromagnetic invariance}, \\
k^{L/R}_{1,2,3} &\rightarrow 0 \quad : \quad \text{in the EW symmetric limit}. \tag{2.5} \end{align*} \]

We now show that eq. (2.2) follows from the fact that \( k^{L/R}_3 \rightarrow 0 \) in the EW limit.

In order to derive the IR effective operators, which match to the operators in eq. (2.4) at the leading order, we use the symmetry properties of the pion fields and current matching. To be specific, note the symmetry transformation of the pion field and the corresponding currents under \( SU(2)_L \times SU(2)_R \):
\[ \begin{align*}
U_\pi &\equiv \exp \left( \frac{2 i \pi^a t^a}{f_{\pi}} \right) \rightarrow L U_\pi R^t \\
\Rightarrow J_{\mu \rho L}^a &= -i \frac{f_{\pi}^2}{2} \text{Tr} \left[ U_\pi t^a \partial^\rho U_\pi^t \right] + \cdots \quad \text{whereas} \quad J_{\mu \rho R}^a &= -i \frac{f_{\pi}^2}{2} \text{Tr} \left[ U_\pi t^a \partial^\rho U_\pi \right] + \cdots. \tag{2.6} \end{align*} \]
Therefore, at leading order, the interactions between the axion and the pions can be derived by simply matching currents. In particular, consider the operators with Wilson coefficient $k_L^3$ and $k_R^3$ in eq. (2.4):

$$k_L^3 \frac{\partial \mu_a \gamma^\mu q_L}{f_a} \rightarrow k_L^3 \frac{\partial \mu_a f_3^L}{f_a} \gamma^\mu q_L + \cdots \rightarrow -ik_L^3 \frac{f_2^L}{2 f_a} \partial_\mu a \text{Tr} [U^{\dagger}_a t^3 \partial^\mu U_a] + \cdots ,$$  \hspace{1cm} (2.7)

$$k_R^3 \frac{\partial \mu_a \gamma^\mu q_R}{f_a} \rightarrow k_R^3 \frac{\partial \mu_a f_3^R}{f_a} \gamma^\mu q_R + \cdots \rightarrow -ik_R^3 \frac{f_2^R}{2 f_a} \partial_\mu a \text{Tr} [U_\pi t^3 \partial^\mu U_\pi^\dagger] + \cdots .$$  \hspace{1cm} (2.8)

Expanding the exponential, one finds the interactions listed in eq. (2.4), with the identification

$$k_1 = k_L^3 - k_R^3 \quad \text{and} \quad k_2 = k_L^3 + k_R^3 .$$  \hspace{1cm} (2.9)

The conditions referred to in eq. (2.2) therefore follows from noting that $k_1 = \mathcal{O}(v/\Lambda_{\text{UV}})$.

The discussion above should make it clear that it is important to choose the right basis in order to write the general ALP-quark Lagrangian before matching, where the right basis is where the EW symmetry (as a global symmetry) in the ALP-quark interactions is manifest and is only broken by the EW-breaking parameters of the SM. However, this consideration does not uniquely define the basis, and additional redefinitions ($a$-dependent) of the quark fields (both chiral and vector in nature) are allowed that keep the EW symmetry manifest. A convenient use of the chiral redefinition can be invoked to get rid of the operator $aG\tilde{G}$. In this particular case, the chiral rotation must be flavor universal. Therefore, we pick the right basis to be the basis with manifest EW symmetry and without the $aG\tilde{G}$ operator. Further, any new flavor universal vectorial redefinition of the quark fields does not introduce any new independent operator.

![Figure 1](image.png)

**Figure 1.** Diagrams contributing to $\pi^+ \rightarrow a\ell^+\nu$ in a generic basis.

To expand, it is instructive to compare the fate of eq. (2.2), in the ‘wrong’ basis. It should be clear from the discussion above that a basis where the components of the electroweak doublet (namely, $q_L$) are treated differently can be obtained from the ‘right’ basis after a flavor dependent redefinition of the quark fields is performed. However, since electromagnetism remains a good symmetry in the IR, we are left with only redefinitions generated by $t^3$. Further, the redefinition must be vectorial to ensure that the $aG\tilde{G}$ operator remains absent in the new basis. It is easy to check that such a redefinition keeps $k_1$ the same but changes $k_2$ and consequently eq. (2.2). Therefore, eq. (2.2) is true only in the manifest EW basis. Of course, there is no need to assert that observable are basis independent. To confirm, consider the decay $\pi^+ \rightarrow a\mu^+\nu$. Starting with the manifest EW basis, we perform a flavor dependent vectorial redefinition (parameterized by, say, $\zeta$) of the quark fields:

$$q \rightarrow \exp \left( i \zeta \frac{a}{f_a} t^3 \right) q ;$$

$$\mathcal{L} \rightarrow \mathcal{L} + \zeta \frac{\partial^\mu a}{f_a} \left( \pi^- \partial_\mu \pi^+ - \pi^+ \partial_\mu \pi^- \right) + 2G_F \zeta \frac{f}{f_a} \left( \partial_\mu a\pi^+ + a\partial_\mu \pi^+ \right) \bar{\nu} \gamma_\mu P_L \nu + \cdots .$$  \hspace{1cm} (2.10)
In the above, \( \cdots \) refer to other deviations which do not impact the decay \( \pi^+ \rightarrow a \mu^+ \nu \) at the leading order. The new contributions to the amplitude proportional to \( \zeta \) is shown in Figure 1. It is easy to check that the contributions to the amplitude get neatly canceled out between the two diagrams. As expected, there is no effect in the amplitude for observables even though the redefinition in eq. (2.10) gives rise to a large number of operators (proportional to \( \zeta \)). In the next subsection, when we work with three flavors, we will see that the three-scalar interaction is relevant even if we work in the manifestly EW symmetric basis. Indeed, for the three flavor case, the left-handed interactions corresponding to \( t^8 \) of \( SU(3) \) are EW symmetric while being flavor dependent. Therefore, there will be a three-scalar interaction present even in the manifestly EW basis.

2.1 The chiral Lagrangian with the ALP: A\( \chi \)PT

In this subsection, we begin with and categorize the general ALP-quark Lagrangian in the manifestly EW basis, where EW symmetry is realized as a global symmetry and is only broken by the Higgs vev. As described before, this basis is also characterized by ‘no’ \( a \bar{G} \bar{G} \) operator. Before proceeding, however, note that we discuss now an \( N_f = 3 \) scenario, which is characterized by an approximate global \( U(3)_L \times U(3)_R (\equiv SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R) \). The EW factor \( SU(2)_W \) is identified as an \( SU(2) \) subgroup of \( U(3)_L, SU(2)_L \), which suggests that EW invariant but flavor dependent redefinitions of the quark triplets exist. In particular, by choosing the quark basis as in the equation below, the \( SU(2)_L \) can be understood to have been embedded in the top-left 2 \( \times \) 2 block of the \( SU(3)_L \). Under the action of \( SU(3)_L \times SU(3)_R \), we have,

\[
\text{using } q \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad q_L \equiv P_L q \rightarrow L_3 q_L \quad \text{and} \quad q_R \equiv P_R q \rightarrow R_3 q_R \ , \quad (2.11)
\]

where \( L_3 \) and \( R_3 \) represent transformation matrices for the \( SU(3)_L \) and \( SU(3)_R \) respectively. In this basis, the structure of the EM charge matrix is

\[
Q = \begin{pmatrix}
  2/3 & 0 & 0 \\
  0 & -1/3 & 0 \\
  0 & 0 & -1/3
\end{pmatrix} = t^3 + \frac{1}{\sqrt{3}} t^8 . \quad (2.12)
\]

Finally, note that as far as the ALP is considered, we only impose the compactness condition,

\[
a \rightarrow a + 2n \pi f_a , \quad n \in \mathbb{Z} . \quad (2.13)
\]

Operators involving derivatives of the ALP automatically satisfy the compactness criterion. In fact, these are additionally symmetric under any shift of the ALP \( (n \in \mathbb{R}) \)—a property of the NGBs. In our generalized ALP-Quark framework we stick to the compactness condition, and do not extend it to the full shift symmetry of an NGB. This allows for the presence of sinusoidal functions of \( a \). Note that, the polynomial \( \sin(na/f_a) \) is especially important since it can provide a term linear in \( a/f_a \).

With the symmetry properties of the ALP defined, we are ready to categorize the interactions of the ALP with the SM quarks at the leading order (linear in \( a/f_a \)). In particular, we find that all such interactions can be broadly categorized in five classes of operators involving the ALP and various quark bilinears. These operators are:

\[
\sum_{i=0}^{8} (C^i_L \mathcal{O}^i_L + C^i_R \mathcal{O}^i_R + C^i_{LR} \mathcal{O}^i_{LR}) + C_W \mathcal{O}_W + C_Z \mathcal{O}_Z , \quad (2.14)
\]
\[ \mathcal{O}_L^i = \frac{1}{f_a} \partial_\mu a \bar{q}_L t^i \gamma^\mu q_L \]

\[ i = 0, 8 : \text{Allowed} \]
\[ i = 1, 2, 4, 5 : \text{break EM} \]
\[ i = 1(1)7 : \text{break } SU(2)_W \]
\[ i = 6, 7 : \text{tree FCNC} \]

\[ \mathcal{O}_R^i = \frac{1}{f_a} \partial_\mu a \bar{q}_R t^i \gamma^\mu q_R \]

\[ i = 0, 3, 8 : \text{Allowed} \]

\[ \mathcal{O}_{LR}^i = \frac{a}{f_a} \bar{q}_L t^i M q_R \]

\[ i = 1, 2, 4, 5 : \text{break EM} \]
\[ i = 6, 7 : \text{tree FCNC} \]

\[ \mathcal{O}_W = -\frac{a}{f_a} \bar{q}_L Q^W j_{zL} q_L \]

\[ \mathcal{O}_Z = -\frac{a}{f_a} (\bar{q}_L Q^Z_{LR} j_Z q_L + \bar{q}_R Q^Z_{R} j_Z q_R) \]

**Table 1.** We tabulate the dimension-5 operators generated by the presence of an ALP in the EW basis. We indicate the generators allowed by symmetry for each of the operators and indicate the ones we have dropped. We tabulate the operators in each of these categories in Table 1, where \( t^0 \) is the identity matrix and \( t^{(1)8} \) are generators of \( SU(3) \). The \( j^\mu_\pm \) current consists of quark and lepton bilinears, which arise as we replace the EW field \( g_W W^\mu_{\pm} \rightarrow 4G_F j^\mu_\pm \) using the equation of motion in order the match to the EFT (Fermi theory) at the EW scale. Similarly, \( j^\mu_\pm \) is the replacement for the EW gauge boson \( Z^\mu \).

The matrices \( Q^W \) and \( Q^Z_{L/R} \) are the three-dimensional representations of the strengths, as in the SM, corresponding to the \( j^\mu_{zL} \) and \( j^\mu_{zR} \) currents respectively. We match the operators at the EW scale at the tree level and do not take into consideration effects due to renormalization below the EW scale. The strength of these interactions is of course given by the Fermi constant, defined as \( G_F = \sqrt{2}g^2_W/(8M_W^2) \).

Before proceeding, first note the important points regarding Table 1:

- Operators of the type \( \mathcal{O}_{LR}^i, \mathcal{O}_L^i \), and \( \mathcal{O}_R^i \) have been studied exhaustively [34, 43, 45, 53]. The first type ends up giving rise to the mass mixing between the ALP and pions, whereas operators of the second and the third kind generate kinetic mixings as discussed in the last subsection.

- Additional comments are due regarding the operators \( \mathcal{O}_L^0 \) and \( \mathcal{O}_R^0 \), which correspond to ALP derivative couplings to the \( U(1)_L \) and \( U(1)_R \) currents respectively, out of which the axial \( U(1)_{L-R} \) is anomalous and broken by QCD. Not surprisingly, one finds that a nonzero \( (C^0_L - C^0_R) \) generates kinetic mixing between the ALP and the \( \eta' \) meson. Since we are primarily interested in the phenomenology of the pion octet in the presence of the ALP, we set \( C^0_L = C^0_R = 0 \) for subsequent discussions and leave the exercise of including the \( \eta' \) meson for future endeavours.

\( \mathcal{O}_L^i \) \( \mathcal{O}_R^i \) \( \mathcal{O}_{LR}^i \) \( \mathcal{O}_W \) \( \mathcal{O}_Z \)

where \( C_a \)s are real Wilson coefficients for the operators belonging to the class \( \mathcal{O}_a \).

We tabulate the operators in each of these categories in Table 1, where \( t^0 \) is the identity matrix and \( t^{(1)8} \) are generators of \( SU(3) \). The \( j^\mu_\pm \) current consists of quark and lepton bilinears, which arise as we replace the EW field \( g_W W^\mu_{\pm} \rightarrow 4G_F j^\mu_\pm \) using the equation of motion in order the match to the EFT (Fermi theory) at the EW scale. Similarly, \( j^\mu_\pm \) is the replacement for the EW gauge boson \( Z^\mu \).
One finds scalars in pion decays even for the cases where the light scalar is not in an exact parity eigenstate. This allows us to include the phenomenology of light EFT. In such models, the hidden sector talks to the SM through the only available super-renormalizable operators (after appropriate tuning) that there in no FCNC in the ALP-quark EFT in the mass basis. For a detailed discussion of flavor violation generated by the ALP-quark Lagrangian check Ref. [35].

As mentioned before, operators of the type $O^\ell_{LR}$, $O^\ell_L$, and $O^\ell_R$ are well studied. On the other hand, operators $O_W$ and $O_{ZL/R}$ are not. In fact, we are not aware of any previous literature which includes these operators in the derivation of the ALP-meson EFT. There are many ways via which these operators can be generated. Here we give one possible UV scenario which results in an EFT with $O_W$ and $O_Z$. Begin with a fully EW (gauged) symmetric theory and consider the following potential:

$$V(H, a) = -\mu^2(a) H^\dagger H + \frac{1}{2} \lambda(a) (H^\dagger H)^2.$$  \hspace{1cm} (2.15)

In the above, we have simply extended the SM parameters in the Higgs potential to include polynomials of sinusoidal functions of the ALP, making the compactness condition manifest. The simplest way to analyze this theory is to expand it around the $a$-dependent minimum, bringing in $a$-dependent masses for the EW gauge bosons. Keeping terms linear in $a$, we find that a replacement of the EW vev (and therefore of the Fermi constant) by its $a$-dependent value allows us to derive the low energy EFT:

$$v^2 \rightarrow v^2(a) = \frac{\mu^2(a)}{\lambda(a)} \equiv v^2 \left(1 - C_v \frac{a}{f_a} + \ldots \right) \hspace{1cm} (2.16)$$

One finds $O_W$ and $O_Z$ with the constraint that $C_W = C_Z = C_v$.

Coupling the ALP to the scalar potential allows us to incorporate Higgs-portal scenarios into our EFT. In such models, the hidden sector talks to the SM through the only available super-renormalizable term in the SM, the Higgs quadratic term [46]. This allows us to include the phenomenology of light scalars in pion decays even for the cases where the light scalar is not in an exact parity eigenstate.

Note that, even though we talked about the $aG\bar{G}$ operator (and subsequently chose a basis where it is eliminated), we do not discuss other contact operators between the ALP and gauge field bilinears. We note, the operators $aWW$ and $aBB$—$W$ and $B$ being the field strengths corresponding to $SU(2)_W$ and $U(1)_Y$ respectively—can be eliminated by redefining the lepton doublets and singlets. Of the other operators, namely, $aG^2$, $aW^2$, and $aB^2$, the effects of the latter two are already captured by $O_{W/Z}$. This is because, one can incorporate the effects of these operators by simply substituting the EW gauge couplings $g_{W/Z} \rightarrow g_{W/Z}(1 + Ca/f_a)$ which in turn generate $O_{W/Z}$. As for the $aG^2$ operator, we ignore it here, and will follow up on it in a future work. However, do note that the $aG^2$ operator can only appear at one loop order and therefore is small.

In order to derive the ALP-pion Lagrangian, we write the quark Lagrangian in a convenient form,

$$\mathcal{L} \supset \bar{q}_L \gamma^\mu (i\partial_\mu + L_\mu) q_L + \bar{q}_R \gamma^\mu (i\partial_\mu + R_\mu) q_R + \bar{q}_L M q_R + \cdots,$$  

where $\mathcal{L} = Q A^\mu + \left(1 + C_Z \frac{a}{f_a}\right) Q L\bar{J}_Z^\mu + \left(1 + C_W \frac{a}{f_a}\right) Q W^\mu j_z^\mu + \frac{\partial^\mu a}{f_a} C_L t_8^s$, 

- 8 -
\[ R^\mu = QA^\mu + \left(1 + C_Z \frac{a}{f_\pi} \right) Q_{R/\pi}^2 \mu + \frac{\partial_{\mu a}}{f_\pi} \sum_{i=3,8} C_{R}^{i} t^{i}, \]

\[ \overline{M} = \sum_{i} \left(1 + i C_{L/R}^{i} \frac{a}{f_\pi} t^{i} + \cdots \right) M. \quad (2.17) \]

Here, we lightly gauge \( SU(3)_L \times SU(3)_R \) with \( L_\mu, R_\mu \) compensating for the corresponding gauge transformations of the quark fields. With the \( \overline{M} \) as a spurion, even the mass term acts gauge invariant.

With the quark Lagrangian defined in the presence of the ALP, the pion Lagrangian is obtained by current matching. Consider the fundamental construct in the pion Lagrangian, \( U_\pi \) (the exponential representation of the pions \([54]\)), which transforms as a bi-fundamental under \( SU(3)_L \times SU(3)_R \):

\[ U_\pi \equiv \exp \left( \frac{2i\pi i \mu a}{f_\pi} \right) \xrightarrow{L \times R} L_3 U_\pi R_3^\dagger, \quad (2.18) \]

so that, the ‘gauge-invariant’ kinetic piece of \( U_\pi \) and the mass term nicely match to eq. (2.17):

\[ \mathcal{L} \supset \frac{f_\pi^2}{4} \text{Tr} \left[ |\partial_\mu U_\pi - i(L_\mu U_\pi - U_\pi R_\mu)|^2 \right] + \frac{\Lambda f_\pi^2}{2} \text{Tr} \left[ \overline{M} U_\pi^4 \right] + \text{h.c.} + \cdots. \quad (2.19) \]

Here, \( \Lambda \) is the UV cut off of the EFT, and \( \cdots \) represent higher order terms in the chiral Lagrangian. The Lagrangian in eq. (2.19), with \( L_\mu, R_\mu, \) and \( M \) given by eq. (2.17), gives the leading order terms in the \( \Delta \chi \)PT. This completely reduces to the familiar chiral Lagrangian in the limit \( f_\pi \to \infty \).

While the introduction of the ALP fields by symmetry matching is straightforward, a subtle point needs to be addressed with regards to power counting. Before the addition of external currents, there are two power counting parameters in the chiral Lagrangian. The first of these is the momentum \( p^2/\Lambda^2 \), where \( \Lambda \) is the cutoff and is taken to be of the order of the \( \rho \) meson mass, the second is the quark mass(es), \( m_q/\Lambda \) (see, e.g., \([55]\)). The external currents come with their own power counting parameters, \( \alpha_{EM} \) for EM and \( p^2 G_F \) for the electroweak currents. The ALP field, however, introduces additional derivatives through the external currents. The introduction of these ‘new’ derivatives do not spoil the momentum power counting of the chiral Lagrangian. This is because the ALP derivatives come suppressed by a characteristic scale, \( f_\pi \), of its own. In essence, there are two new power counting parameters that the ALP brings in, \( f_\pi/f_\pi \) and \( p_\mu/f_\pi \). Therefore, corresponding to each derivative of the ‘pure’ chiral Lagrangian, the ALP derivatives are \( f_\pi/f_\pi \) suppressed.

We break down the \( \Delta \chi \)PT Lagrangian in eq. (2.18), to find leading terms in the ALP-pion interactions, starting with the so-called ‘mixing’-terms between the ALP and the neutral pions:

\[ -\frac{if_\pi^2}{4} \text{Tr} \left[ \partial_\mu U_\pi^\dagger L_\mu U_\pi \right] + \text{h.c.} = -\frac{f_\pi}{2} \frac{c_{L/R}^8}{f_\pi} \partial_\mu a \partial_\mu \eta + \cdots, \quad (2.20a) \]

\[ \frac{if_\pi^2}{4} \text{Tr} \left[ \partial_\mu U_\pi^\dagger U_\pi R_\mu \right] + \text{h.c.} = \frac{f_\pi}{2} \frac{c_{L/R}^8}{f_\pi} \partial_\mu a \partial_\mu \pi^0 \left( \frac{1}{2} \frac{f_\pi}{f_\pi} c_{L/R}^8 \partial_\mu a \partial_\mu \eta \right) + \cdots, \quad (2.20b) \]

\[ \frac{\Lambda f_\pi^2}{2} \text{Tr} \left[ \overline{M} U_\pi^4 \right] + \text{h.c.} = \frac{B_0 f_\pi}{f_\pi} \left[ C_{L/R}^0 \frac{m_\Delta}{\sqrt{3}} + C_{L/R}^3 \hat{m} \right] a \pi^0 \right. \]

\[ -\frac{1}{\sqrt{3}} \left( C_{L/R}^0 m_\Delta + C_{L/R}^8 \frac{m_\hat{m}}{\sqrt{3}} + \frac{2}{\sqrt{3}} C_{L/R}^8 m_s \right) a \eta + \cdots, \quad (2.20c) \]

where \( m_\Delta = \frac{m_u - m_d}{2}, \quad \hat{m} = \frac{m_u + m_d}{2}, \quad \text{and} \quad B_0 = \Lambda. \quad (2.20d) \]
Note the absence of ALP mixing with $K^0$ and $\bar{K}^0$. Even though electric charges allow for it, the ALP interactions in the quark mass basis effectively preserve strangeness, since, $a$-$K^0$ or $a$-$\bar{K}^0$ mixing terms are absent. Further, both kinetic and mass mixing terms are suppressed by the factor $\xi \equiv f_a / f_a$, which we use as the universal power counting parameter relevant to the physics of the ALP. This suggests that one way to calculate observables would be to treat the mixing terms as interactions. However, in this work we take the more straightforward approach of redefining the fields such that the Lagrangian is written in terms of canonically normalized mass eigenstates. The redefinition first brings the kinetic terms in the canonically normalized form followed by another rotation (orthogonal transformation) which diagonalizes the squared mass matrix. From this point, we would refer to the basis in which eqs. (2.18) to (2.20) are given as the original-basis, whereas the basis in which the kinetic terms are orthonormal and masses diagonal would be referred to as the eigenbasis.

We begin with the kinetic mixing terms. The following field redefinitions for $a, \pi^0$, and $\eta$ get rid of the kinetic mixing terms at the order quadratic in $\xi$:

$$\begin{align*}
a &\rightarrow a \left[ 1 - \frac{\xi^2}{8} \left( (C_R^a)^2 + (C_A^a)^2 \right) \right] - \pi^0 \frac{\xi}{2} C_R^a - \eta \frac{\xi}{2} C_A^a; \\
\pi^0 &\rightarrow \pi^0 \left[ 1 + \frac{\xi^2}{8} (C_R^\pi)^2 \right] + \eta \frac{\xi^2}{4} C_R^\pi C_A^\pi; \\
\eta &\rightarrow \eta \left[ 1 + \frac{\xi^2}{8} (C_A^\eta)^2 \right], \text{ where } C_A^8 = C_R^8 + C_L^8.
\end{align*}$$

(2.21)

In addition to the kinetic mixing with the $\pi^0, \eta$, the diagonal ALP kinetic term is also rescaled in the original basis. The rescaling is sourced by the Tr$[L_\mu L^\mu + R_\mu R^\mu]$ term given by the gauged kinetic term corresponding to $U_\pi$ in eq. (2.19):

$$\text{Tr}[L_\mu L^\mu + R_\mu R^\mu] \supset \frac{1}{8} \left[ (C_R^a)^2 + (C_A^a)^2 \right].$$

(2.22)

The redefinitions of the fields, as given in eq. (2.21) reflect this scaling of the ALP kinetic term in the original basis. Said redefinitions change the squared mass matrix for the $a, \pi^0$, and $\eta$ fields to:

$$\begin{align*}
\frac{1}{2} \begin{pmatrix} a & \pi^0 & \eta \end{pmatrix} \begin{pmatrix} m_a^2 & m_{a\pi}^2 & m_{a\eta}^2 \\ m_{a\pi}^2 & m_{\pi^0}^2 & m_{\pi\eta}^2 \\ m_{a\eta}^2 & m_{\pi\eta}^2 & m_\eta^2 \end{pmatrix} \begin{pmatrix} a \\ \pi^0 \\ \eta \end{pmatrix}, \text{ where } \\
m_a^2 &= 2B_0 \hat{m} \left[ 1 - \xi^2 \frac{C_R^3}{4} \left( 2C_R^3 - C_A^3 + \frac{2}{\sqrt{3}} C_L^8 \frac{m_\Delta}{\hat{m}} \right) \right] + \cdots; \\
m_{\pi^0}^2 &= \frac{2B_0}{3} (\hat{m} + 2m_\pi) \left[ 1 + \frac{\xi^2}{4} C_A^8 \left( C_A^3 - 2C_L^8 \right) \right] + \cdots; \\
m_{\pi\eta}^2 &= 2B_0 \sqrt{\frac{2}{3}} m_\Delta \left[ 1 - \frac{m_\pi}{m_\Delta} \xi^2 \frac{C_R^3 C_L^3}{2} \right] + \cdots; \\
m_{a\pi}^2 &= \frac{B_0}{\sqrt{3}} \left[ \frac{C_L^8}{\sqrt{3}} m_\Delta + C_L^8 \hat{m} \right] + \cdots; \\
m_{a\eta}^2 &= \frac{B_0}{\sqrt{3}} \left[ C_L^8 m_\Delta + \frac{C_L^8}{\sqrt{3}} (\hat{m} + 2m_\pi) \right] + \cdots.
\end{align*}$$

(2.23)

Here $\cdots$ represents higher order terms suppressed by additional powers of $\xi, m_u/d/m_s, m_a/B_0 m_\pi$ etc.

Before proceeding, we need to comment about the sizes of these quantities and the order of precision
of our calculations. In the SMχPT, the only relevant mixing is between $\pi^0$ and $\eta$, which is rather trivial to take into consideration since the mixing (parameterized by $\epsilon$) is of the order of $1 \times 10^{-2}$. In this paper, we only consider tree level effects linear in $\epsilon$, which implies that we need to keep terms of the order of $\xi^2$ when scanning for $\xi \lesssim \sqrt{\epsilon}$. Additionally, we will also assume $m_a^2/B_0 m_\eta \lesssim m_\pi^2/m_\eta^2 \sim \epsilon$. This suggests, we keep terms of the order of $\xi^2$, $m_a^2/m_\pi^2$ but neglect terms of the order of $\xi \epsilon$, $m_a^2/B_0 m_\pi$ etc. To keep the text, somewhat, notationally simple, we represent all contributions to the ALP mass, i.e. those from the UV and those from QCD, by $m_a$.

An implication of these hierarchies is that the off-diagonal elements in the mass-squared matrix are smaller than the differences of the corresponding diagonal ones. Consequently, we can employ results from non-degenerate perturbation theory to find the eigenvectors and eigenvalues. Note that we are interested in determining the eigenvalues and eigenvectors at order $\xi^2$. Apart from the perturbative corrections, we also include the effects from redefinitions in eq. (2.21), which allows us to write the contributions to the ALP mass, i.e. those from the UV and those from QCD, by $m_a$.

The same procedure allows us to determine mass eigenvalues. Below, we summarize all the pion masses and include corrections due to electromagnetism (denoted here by the quantity $\Delta_a$). For completeness, we also include masses for charged pions and neutral kaons which remain the same as in the SM:

\[
\begin{align*}
M_{\pi^\pm}^2 &= 2B_0 m_\eta + \Delta_a, \\
M_{\pi^0}^2 &= 2B_0 m_\eta \left[ 1 + \frac{1}{6} \left( 3C_3^3 - 2\sqrt{3}C_3^8 C_\eta^3 \frac{m_\Delta}{m_\eta} \right) \xi^2 \right], \\
M_{K^\pm}^2 &= B_0 (m_\pi + m_\eta) + \Delta_a, \\
M_{K^0}^2 &= M_{K^0}^2 = B_0 (m_\pi + m_\eta), \\
M_{\eta}^2 &= \frac{4}{3} B_0 \left( m_\eta + \frac{1}{2} m_\eta \right) \left[ 1 + \frac{\xi^2}{4} C_8^2 \right].
\end{align*}
\]  

Hence, the spectrum of pions starts deviating from the SM trajectories. This deviation of the trajectories of the meson masses allows us to constrain $\xi^2$ (multiplied by the relevant function of Wilson coefficients). In order to do so, we construct the following sums out of the masses:

\[
\begin{align*}
\Delta_{M_s} &= \frac{M_{\pi^+}^2 - M_{\pi^0}^2 - \Delta_a}{M_{\pi^+}^2} = 0 + \frac{1}{6} \left( 2\sqrt{3}C_3^8 C_\eta^3 \frac{m_\Delta}{m_\eta} - 3C_3^2 \right) \xi^2 + \cdots, \\
\Delta_{GMO} &= \frac{4M_{K^0}^2 - 3M_\eta^2}{M_{\eta}^2 - M_{\pi^0}^2} = 0 - \frac{3}{4} \xi^2 C_8^2 + \cdots.
\end{align*}
\]  

Here, we use the isospin invariant definitions $M_K^2 = (M_{K^0}^2 + M_{K^\pm}^2) / 2$, and additionally use $M_a^2 = M_{\pi^0}^2$. As always, the ellipsis represents terms further suppressed. The first sum is just a measure of the charged and neutral pion mass difference, while the second one is the well known Gell-Mann–Okubo
mass relation. Both the sums contracted are zero in the SM (up to $\eta - \eta'$ mixing for GMO, which we discuss shortly). However, in the $A_\chi$PT, the sum relations get violated at the tree level itself.

We can use the measured values of the meson masses and the computed values of the EM contributions in an attempt to give bounds on $\xi^2$ from the mass differences. With $M_{\pi^+} = 139.57039(18)$, $M_{\pi^0} = 134.9768(5)$ [51], and $\sqrt{\Delta_e} = 4.538$ MeV [56], we have from eq. (2.26):

$$2\sqrt{3}C_{LR}^8 C_\eta^3 \frac{m_\Delta}{m} - 3C_3^2 < 0.38211(4). \quad (2.28)$$

In the isospin symmetric limit, $m_\Delta = 0$, the relationship is trivially satisfied. However, for $m_\Delta \neq 0$, the relationship above constrains $\xi^2$ times the Wilson coefficients. However, in this case, it’s contaminated by the presence of the quark masses, making the conclusion inconclusive.

The GMO relation, on the other hand, is constructed in a way that doesn’t involve the quark masses. Hence, it can be used to bound $C_8^2 \xi^2$. In order to do so, however, we need to carefully take into account $\eta - \eta'$ mixing\(^1\). For the mass-mixing, we have:

$$M_{\eta}^2 = M_{\eta'}^2 \cos^2 \vartheta_{\eta\eta'} + M_{\eta'}^2 \sin^2 \vartheta_{\eta\eta'} \quad (2.29)$$

Here, $M_{\eta}$ is the mass of the $\eta$ as given by $SU(3)_L \times SU(3)_R$ chiral perturbation theory, while, $M_{\eta'} = 547.862(17)$ MeV and $M_{\eta'} = 957.78(6)$ [51] are the physical masses of the $\eta$ and the $\eta'$ mesons, respectively. We use the lattice value $\vartheta_{\eta\eta'} = -14.1(2.8)^\circ$ [57, 58] for the $\eta - \eta'$ mixing angle. With the magnitudes of the different quantities in hand, we get for the LHS of eq. (2.27):

$$\Delta_{\text{GMO}} = -0.15 \pm 0.13. \quad (2.30)$$

This translates to a bound on $C_8^2 \xi^2$, at 95% CL. The, somewhat, weak bounds on the ALP parameters from the meson mass relationships have an interesting consequence. Quark masses are obtained on the lattice from the meson masses and the relationship between them (see, e.g., [59]), using the SM-only hypothesis. Hence, it is clear that these computed quark masses will deviate in the $A_\chi$PT hypothesis. It is beyond the scope of this work to predict the exact nature of these changes.

For example, we see that the Gell-Mann–Okubo formula gets violated at tree level (at $O(\xi^2)$) even after neglecting EM:

$$\Delta_{\text{GMO}} \equiv \frac{4M_{\eta}^2 - M_{\eta'}^2 - 3M_{\pi}^2}{M_{\eta}^2 - M_{\pi}^2} = 0 - \frac{3}{4} C_8^2 \xi^2 + \cdots, \quad (2.31)$$

where we use the isospin invariant definitions $M_{K}^2 = M_{K^0}^2 + M_{K^\pm}^2 / 2$, and additionally use $M_{\pi}^2 = M_{\pi^\pm}^2$. As always, the ellipsis represents terms further suppressed.

Another important deviation from SM expectations happen for the meson form factors (FF). Take, for example, the strangeness violating FFs defined via

$$\langle \pi^0(p_\pi) | \bar{s} \gamma_\mu u | K^+(p_K) \rangle \equiv \frac{1}{\sqrt{2}} \left[ f_{+_{\text{SM}}}^{K^{\pm}_{\text{SM}}}(q^2) Q_\mu + f_{-_{\text{SM}}}^{K^{\pm}_{\text{SM}}}(q^2) q_\mu \right], \quad (2.32a)$$

$$\langle \pi^+(p_\pi) | \bar{s} \gamma_\mu u | K^0(p_K) \rangle \equiv f_{+_{\text{SM}}}^{K^0_{\text{SM}}}(q^2) Q_\mu + f_{-_{\text{SM}}}^{K^0_{\text{SM}}}(q^2) q_\mu, \quad (2.32b)$$

where, $Q_\mu = p_{K^+}^\mu + p_{\pi^+}^\mu$; $q_\mu = p_{K^0}^\mu - p_{\pi^+}^\mu$. \(^{\scriptstyle \, (2.32c)}\)

\(^{\scriptstyle 1}\)We ignore the EM corrections to the charged meson masses and $\eta'$ mixing with the other neutral states as these are subdominant compared to $M_{\eta}$ and $\eta - \eta'$ mixing, respectively.
To find deviations in FFs in the AχPT, we match the operator $\bar{s}\gamma_{\mu}\gamma_{\nu}u$ to pions in the original basis:

$$\bar{s}\gamma_{\mu}u = -f_{\pi}^{2} \mathrm{Tr} \left( U_{\pi}^{\dagger} \left( t^{a} - it^{5} \right) \partial_{\mu} U_{\pi} \right)$$

$$\supset \left( K^{0} \partial_{\mu} \pi^{+} - \partial_{\mu} K^{0} \pi^{+} \right) + \frac{1}{\sqrt{2}} \left[ K^{+} \partial_{\mu} \left( \pi_{0} + \sqrt{3} \eta \right) - \partial_{\mu} K^{+} \left( \pi_{0} + \sqrt{3} \eta \right) \right],$$

(2.33)

then, noting that the observed states get created/annihilated from/to the vacuum by the fields (hatted) in the eigenbasis, we derive the form factors (at the tree level from the $O(p^2)$ Lagrangian) to find:

$$\frac{f_{K^{+}\pi^{0}}(0)}{f_{K^{0}\pi^{-}}(0)} = 1 - \sqrt{3} \epsilon - \xi^{2} \frac{C_{3}}{8} \left[ C_{A}^{3} + C_{LR}^{3} + 2\sqrt{3} C_{LR}^{8} \right].$$

(2.34)

Here, $f_{K^{+}\pi}(0)$ is the leading order term in the $q^{2}$ expansion of $f_{K^{+}\pi}(q^{2})$. Compare the ratios of the form factor as presented in eq. (2.34) to that of the SM (i.e., in the limit $\xi^{2} \to 0$). The deviation from the SM value of this ratio results due to the mixing of the ALP with the $\pi^{0}$. Similar deviations can be seen in the FFs corresponding to other light mesons as well, e.g., $f_{K^{+}\pi^{0}}(q^{2})$, $f_{K^{0}\pi^{-}}(q^{2})$ etc. As for the $f_{K^{+}\pi^{0}}(q^{2})$ FFs, these are zero at leading order in the SM. However, as we show in the next section, the ALP interactions source leading order contributions to these. The modification of the FFs, $f_{K^{+}\pi^{0}}(q^{2})$, are key elements in our discussion on $K^{+} \to \pi^{0} \ell^{+} \nu$ decays in the subsequent sections.

### 3 Phenomenology of $K_{L3}$ Decay

The CC mediated semi-leptonic decays of the light mesons have always been a testing ground for SM physics. The $\pi^{+} \to \pi^{0} \ell^{+} \nu$ ($\pi_{\beta}$) and the $K^{+} \to \pi^{0} \ell^{+} \nu$ ($K_{L3}^{+}$) decays are used to compute values of $V_{td}$ and $V_{us}$ respectively. These decays are driven exclusively by the $(\partial_{\mu} K^{+}(\pi^{0})\pi^{0} - K^{+}(\pi^{+})\partial_{\mu} \pi^{0})\ell^{\mu} \bar{\nu}_{\ell} \nu$ operators at leading order in SMχpt. However, in the AχPT, there are multiple terms that are relevant for these decays, even at leading order. To proceed further, we first list all the operators that generate contributions to $K_{L3}^{+}$ at $O(\xi^{2})$. Parametrizing the relevant Lagrangian in the original-basis as:

$$\mathcal{L}_{K_{L3}^{+}} = \sum_{i} C_{K_{L3}^{+}}^{i} \mathcal{O}_{K_{L3}^{+}}^{i},$$

(3.1)

we list the operators and the corresponding coefficients in Table 2 (with $J_{\mu\nu} = \bar{\nu}^{\mu} \frac{1}{2}(1 - \gamma_{5}) \gamma_{\nu} \ell$).

The operator $\mathcal{O}_{K_{L3}^{+}}^{0}$ is the familiar operator of the SM. However, as shown in the previous section, even this familiar operator gives rise to deviations from the SM, due to the redefinition of the physical pions. The contributions due to other operators can be calculated by converting fields in the original basis to the fields in the eigenbasis by using the expansion given in eq. (2.24):

$$C_{K_{L3}^{+}}^{1} \mathcal{O}_{K_{L3}^{+}}^{1} \supset \hat{C}_{K_{L3}^{+}}^{1} \hat{O}_{K_{L3}^{+}}^{1} = i G_{F} V_{su} \xi^{2} C_{3} \left[ C_{R}^{3} + \sqrt{3} \left( C_{8} - C_{L}^{8} \right) \right] \partial_{\mu} K^{+} \bar{\pi}^{0} - K^{+} \partial_{\mu} \bar{\pi}^{0} J_{\mu\ell},$$

(3.2a)

$$C_{K_{L3}^{+}}^{2} \mathcal{O}_{K_{L3}^{+}}^{2} \supset \hat{C}_{K_{L3}^{+}}^{2} \hat{O}_{K_{L3}^{+}}^{2} = i G_{F} V_{su} \xi^{2} C_{3} \left[ C_{R}^{3} + \sqrt{3} C_{L}^{8} \right] \partial_{\mu} K^{+} \bar{\pi}^{0} J_{\mu\ell},$$

(3.2b)

$$C_{K_{L3}^{+}}^{3} \mathcal{O}_{K_{L3}^{+}}^{3} \supset \hat{C}_{K_{L3}^{+}}^{3} \hat{O}_{K_{L3}^{+}}^{3} = \frac{i \xi^{2} C_{3}}{f_{\pi}^{2}} \left( C_{R}^{3} + \sqrt{3} C_{L}^{8} \right) \partial_{\mu} \bar{\pi}^{0} \left( \partial_{\mu} K^{+} K^{-} - K^{+} \partial_{\mu} K^{-} \right).$$

(3.2c)

It is straightforward to see that $\hat{O}_{K_{L3}^{+}}^{1}$ gives a contribution proportional to $Q^{\mu}$ to the matrix element, while $\hat{O}_{K_{L3}^{+}}^{2}$ gives a contribution proportional to $q^{\mu}$. What is, perhaps, not as clear from the operator
The definitions of the FFs that we provide in eq. (3.4) make their extraction from data a lot more straightforward. Note, the imaginary parts of \( \tilde{\alpha}_{K^\pm} \) leave any signatures, through interference with the leading SM contribution. The real parts of the FFs exhibit both contributions proportional to \( q^\mu \) as well. This has been alluded to in section 2 and we make it explicit here. The net contribution proportional to \( q^\mu \) is given by: \( \mathcal{O} \left( \frac{1}{\sqrt{s}} \right) \). This implies that at tree level and at \( \mathcal{O}(p^2, \ell, \xi^2) \), the hadronic part of the amplitude for \( K^\pm \rightarrow \pi^0 \ell \nu \) gets additional contributions which are not just limited to the modification of the FF \( \tilde{f}_K^{\pm \pi^0} \). Hence, we define the ‘effective’ form factors \( f_\pm(0) \) as:

\[
A = G_F V_{su} \left[ f_+^{K^+ \pi^0}(0) Q_{\mu} + f_-^{K^- \pi^0}(0) q_{\mu} \right] \bar{u}_\nu \gamma^\mu \frac{1}{2} (1 - \gamma_5) v_\ell.
\] (3.3)

After taking into account all operators listed in Table 2, we find that:

\[
\tilde{f}_+^{K^+ \pi^0}(0) = \alpha_{K^+}^{(0)} + \xi^2 \left( \alpha_{K^+}^{(2)} \right), \quad \tilde{f}_-^{K^- \pi^0}(0) = \beta_{K^-}^{(0)} + \xi^2 \left( \beta_{K^-}^{(2)} \right),
\]

with\[
\alpha_{K^+}^{(0)} = 1 - \sqrt{3} \xi, \quad \alpha_{K^+}^{(2)} = -\frac{C_3}{8} (C_{LR}^3 - C_{LR}^3 + 2 \sqrt{3} (C_{LR}^3 - C_{LR}^3)), \quad \beta_{K^+}^{(2)} = \frac{1}{2} C_3 C_W,
\]

and\[
\beta_{K^+}^{(0)} = 0, \quad \beta_{K^-}^{(0)} = -\sqrt{3} \xi C_3 C_W, \quad \beta_{K^-}^{(2)} = \frac{1}{2} C_3 C_W.
\] (3.4)

Note, the imaginary parts of \( \tilde{f}_K^{\pm \pi^0}(0) \) contribute to the decay amplitude only at \( \mathcal{O}(\xi^4) \). At \( \mathcal{O}(\xi^2) \), only the Re \( \tilde{f}_K^{\pm \pi^0}(0) \) leave any signatures, through interference with the leading SM contribution. The definitions of the FFs that we provide in eq. (3.4) make their extraction from data a lot more
straightforward. The only catch is that the FFs \( f_{K^+}^{\pi^0}(0) \) defined in eq. (2.32) as matrix elements of \( \delta_u L \gamma^\mu u_L \) is different from the effective \( \text{Re} \left( f_{K^+}^{\pi^0}(0) \right) \) by additional terms at \( \mathcal{O}(\xi^2) \).

There are additional contributions coming from higher orders in the chiral expansion, from electromagnetism, and from EW breaking operators [60]. One can simply absorb these by replacing \( f_{K^+}^{\pi^0}(0) \to f_{K^+}^{\pi^0}(t) \) where \( t \equiv q^2 \). Note, similar to the modifications to the leading order form factors \( f_{K^+}^{\pi^0}(0) \), as given in eq. (2.34), the strengths of these higher order contributions are also expected to be modified. Now, by the virtue of perturbativity of the Lagrangian wrt \( \xi \), we can intuit:

\[
\text{Re} \left( f_{K^+}^{\pi^0}(t) \right) = \left( \alpha_{K^+\pi^0}^{(0)} + \xi^2 \alpha_{K^+\pi^0}^{(2)} + \delta \alpha_{K^+\pi^0}^{(0)} + \xi^2 \delta \alpha_{K^+\pi^0}^{(2)} \right) \\
\times \left[ 1 + \left( \lambda_{K^+\pi^0}^{(0)} + \xi^2 \lambda_{K^+\pi^0}^{(2)} \right) \frac{t}{M^2} \right] + \cdots \\
\approx \left[ 1 + \xi^2 \frac{\alpha_{K^+\pi^0}^{(2)}}{\alpha_{K^+\pi^0}^{(0)}} \right] f_{K^+}^{\pi^0}(t),
\]

(3.5a)

\[
\text{Re} \left( f_{K^+}^{\pi^0}(t) \right) = \left( \delta \beta_{K^+\pi^0}^{(0)} + \xi^2 \delta \beta_{K^+\pi^0}^{(2)} + \xi^2 \beta_{K^+\pi^0}^{(2)} \right) \\
\times \left[ 1 + \left( \lambda_{K^+\pi^0}^{(0)} + \xi^2 \lambda_{K^+\pi^0}^{(2)} \right) \frac{t}{M^2} \right] + \cdots \\
\approx \left[ 1 + \xi^2 \frac{\beta_{K^+\pi^0}^{(2)}}{\beta_{K^+\pi^0}^{(0)}} \right] f_{K^+}^{\pi^0}(t),
\]

(3.5b)

where \( \delta \alpha_{K^+\pi^0}^{(0)} \) and \( \delta \alpha_{K^+\pi^0}^{(2)} \) represent higher-order corrections [60]. Out of these terms, we can neglect \( \xi^2 \delta \alpha_{K^+\pi^0}^{(2)} \) since we expect \( \delta \alpha_{K^+\pi^0}^{(2)} \lesssim \delta \alpha_{K^+\pi^0}^{(0)} \sim 10^{-2} \). The \( \lambda^{\pm} \) are ‘slope parameters’ that parametrize the effects of the higher order (in powers of \( q^2 \)) terms in the chiral expansion. Obviously, these slope parameters get \( \xi^2 \) dependent contributions as well. Similar arguments can be made for the parameters entering \( \text{Re} \left( f_{K^+}^{\pi^0}(t) \right) \). The only difference is that in the SM there is no leading order contribution to \( \text{Re} \left( f_{K^+}^{\pi^0}(0) \right) \), making \( \beta_{K^+\pi^0}^{(0)} = 0 \). Putting everything together, the spin-summed matrix element squared for \( K^+ \to \pi^0 \ell \nu \), at \( \mathcal{O}(\xi^2) \), is given by:

\[
\overline{|A|}^2_{K\ell_3} = 2G_F^2 |V_{us}|^2 C_{\text{cor}} \left[ 1 + 2 \xi^2 \frac{\alpha_{K^+\pi^0}^{(2)}}{\alpha_{K^+\pi^0}^{(0)}} \right] \left( 2H \cdot p_\ell H \cdot p_\nu - H^2 p_\ell \cdot p_\nu \right),
\]

(3.6)

where \( H_\mu = f_{K^+}^{\pi^0}(t) Q_\mu + \left[ 1 + \xi^2 \frac{\beta_{K^+\pi^0}^{(2)}}{\beta_{K^+\pi^0}^{(0)}} \right] f_{K^+}^{\pi^0}(t) q_\mu. \)

The factor \( C_{\text{cor}} \) encapsulates effects that are not captured by the lattice computations of the FFs that we use in our numerical analyses. It is defined as:

\[
C_{\text{cor}} = S_{\text{EW}} \left( 1 + \frac{\delta_{SU(2)}^{K^+}}{2} + \delta_{\text{EM}}^{K^+} \right)^2.
\]

(3.7a)

Here, \( S_{\text{EW}} = 1 + \frac{2\alpha_{\text{EM}}}{\pi} \left( 1 - \frac{\alpha_s}{4\pi} \right) \log \frac{M_Z}{M_\mu} + \mathcal{O} \left( \frac{\alpha_s}{\pi^2} \right) \approx \frac{0.0232 \pm 0.0003}{1 - \frac{\alpha_s}{4\pi} \log \frac{M_Z}{M_\mu} \mathcal{O} \left( \frac{\alpha_s}{\pi^2} \right)} \approx \frac{0.0232 \pm 0.0003}{1 - \frac{\alpha_s}{4\pi} \log \frac{M_Z}{M_\mu} \mathcal{O} \left( \frac{\alpha_s}{\pi^2} \right)}
\]

(3.7b)

encodes the short-distance contribution to the EM corrections [61, 62]. The other corrections, \( \delta_{SU(2)}^{K^+} = (2.45 \pm 0.19)\% \) [63] and \( \delta_{\text{EM}}^{K^+} = (0.016 \pm 0.25) \times 10^{-2} \) [62] are the isospin breaking and long distance electromagnetic corrections respectively. In our calculations, We drop the \( \delta_{\text{EM}}^{K^+} \) corrections as it is much smaller than the order we are working up to.
As evident from eq. (3.6), there are two distinct NP effects to the amplitude-squared. The first is an overall scaling that is entirely controlled by \( \alpha^{(2)}_{K^{+}\pi^0} \), and the second is a relative scaling between the two independent momentum directions. The relative scaling is proportional to the momentum transferred to the lepton system \( (q_{\mu}) \), and—with the leptons in the final state—it is essentially proportional to the mass of the charged lepton \( (m_{\nu} \approx 0) \). The net effect due to the relative scaling factor, being lepton-mass-suppressed, is then subdominant compared to that of the overall scaling term. Although, the relative scaling gets contributions from both \( \alpha^{(2)}_{K^{+}\pi^0} \) and \( \beta^{(2)}_{K^{+}\pi^0} \), the former is scaled by the leading order \( \alpha^{(0)}_{K^{+}\pi^0} \), as compared to the sub-leading \( \delta\beta^{(0)}_{K^{+}\pi^0} \) scaling of \( \beta^{(2)}_{K^{+}\pi^0} \), implying that \( \beta^{(2)}_{K^{+}\pi^0} \) is the primary driver of the distortion in the decay distribution. Therefore, the primary effect of \( \alpha^{(2)}_{K^{+}\pi^0} \) is an overall scaling of the decay rate, while those of \( \beta^{(2)}_{K^{+}\pi^0} \) are lepton-mass-suppressed and related to the distortion of the shape of the decay distribution. In Figure 2, we show the effects of varying \( \alpha^{(2)}_{K^{+}\pi^0} \) and \( \beta^{(2)}_{K^{+}\pi^0} \) on the \( K^{+}\mu^3 \) marginal distribution wrt \( E_\pi \), obtained by marginalising over \( E_\ell \) in:

\[
\frac{d^2\Gamma}{dE_\pi dE_\ell} = \frac{1}{2\pi^3} \frac{1}{8M_{K^+}} \Theta(1 - y^2) \Theta(M_{K^+} - E_\pi - E_\ell) |A|^2_{K^\ell\pi},
\]

with \( y = M_{K^+}^2 + M_{\mu}^2 + M_{\ell}^2 - 2M_{K^+}(E_\pi + E_\ell) + 2E_\pi E_\ell \),

where \( \Theta(x) \) is the Heaviside step function. From the bottom panel of Figure 2, where we show the

![Figure 2](image-url)
differential distributions of $K_{\mu 3}^+$, we can clearly see that compared to $\alpha_{K^{+}\pi^{0}}^{(2)}$, a large value of $\beta_{K^{+}\pi^{0}}^{(2)}$ is required to produce a similar change in the decay rate. From the top panel of the same figure, we can see the different momentum-dependence of the modifications sourced by $\alpha_{K^{+}\pi^{0}}^{(2)}$ and $\beta_{K^{+}\pi^{0}}^{(2)}$.

### 3.1 Results and constraints

In the previous subsection, we derived the amplitude-squared for the decay of $K^+ \to \pi^0 \ell^+\nu$ in the $\chi A$PT. In this subsection, we see how to constrain the effective parameters controlling the deviations from the SM, viz. $\xi^2 \alpha_{K^{+}\pi^{0}}^{(2)}$ and $\xi^2 \beta_{K^{+}\pi^{0}}^{(2)}$, using data corresponding to $K_{\mu 3}^+$ decays. To constrain these parameters, we use the following independent measurements:

- Measurement of the differential decay distributions of $K^\pm \to \pi^0 \mu^\pm \nu_\mu (K_{\mu 3}^\pm)$ and $K^\pm \to \pi^0 e^\pm \nu_e (K_{e 3}^\pm)$ by the NA48/2 collaboration at the CERN SPS [50].

- The total width measurements for $K^+_{\mu 3}$ and $K^+_{e 3}$ decays. We have used the experimental averages of the branching fractions from PDG [51] to calculate the rates.

As discussed in the last subsection, the effect of $\xi^2 \alpha_{K^{+}\pi^{0}}^{(2)}$ dominates over that of the lepton–mass–suppressed effect of $\xi^2 \beta_{K^{+}\pi^{0}}^{(2)}$ when it comes to the total decay rate. This suppression causes $\xi^2 \beta_{K^{+}\pi^{0}}^{(2)}$ to be essentially unbounded by observables related to $K_{e 3}^\pm$. However, as we show below, the marginal energy spectra of the decay rate of $K_{e 3}^\pm$ can be used effectively to constrain $\xi^2 \beta_{K^{+}\pi^{0}}^{(2)}$.

As is evident from the discussion around eq. (3.6), the NP effects will show up as deviations from the SM expectations of the FF parameters. This implies that we need to carefully estimate the SM inputs that enter eq. (3.5). For the slope parameters, $\lambda_\pm$, appearing in eq. (3.5), we use results obtained from lattice computations by the European twisted mass collaboration [52]. The collaboration reports the FFs, expressed as:

$$f_{+0}^{K^{+}\pi^{0}}(t) = f_{+0}^{K^{+}\pi^{0}}(0) \left[ 1 + \lambda_{K^{+}\pi^{0}}^{(0)}(t) \right],$$

$$f_{+0}^{K^{+}\pi^{0}}(t) = f_{+0}^{K^{+}\pi^{0}}(t) \left[ M_{K_{\pi}}^{2} - M_{\pi}^{2} \right].$$

We use the dispersive parametrization of the FFs, where the slope parameters are determined in terms of the slope of the vector, $f_{+0}^{K^{+}\pi^{0}}(0)$, FF at $t = 0$ and the slope of the scalar, $f_{+0}^{K^{+}\pi^{0}}(t)$, FF at the Callan-Treiman point ($t_{CT} = M_{K_{\pi}}^{2} - M_{\pi}^{2}$), $\Lambda_\pm$ and $C$ respectively [64]:

$$\lambda_{K^{+}\pi^{0}}^{(0)} = \Lambda_\pm;$$

$$\lambda_{K^{+}\pi^{0}}^{(0)} = \left( \lambda_{K^{+}\pi^{0}}^{(0)} \right)^2 + 5.79(97) \times 10^{-4} ;$$

$$\lambda_{K^{+}\pi^{0}}^{(0)} = M_{K_{\pi}}^{2} \left[ \log(C) - 0.0398(44) \right];$$

$$\lambda_{K^{+}\pi^{0}}^{(0)} = \left( \lambda_{K^{+}\pi^{0}}^{(0)} \right)^2 + 4.16(56) \times 10^{-4} .$$

We tabulate the lattice determination [52] of $\Lambda_\pm, C$, and $f_{+0}^{K^{+}\pi^{0}}(0)\equiv f_{+0}^{K^{+}\pi^{0}}(0)$ in Table 3. We also note the correlations among these parameters, which we include in our computations.

We have checked that the contributions from $\lambda_{K^{+}\pi^{0}}^{(0)} t^3/M_{\pi}^{2}$ and higher are at a sub percent level, much smaller than the experimental precision. The smallness of $\lambda_{K^{+}\pi^{0}}^{(0)} t^3/M_{\pi}^{2}$ is why we have truncated $f_{+0}^{K^{+}\pi^{0}}(t)$ after the $t/M_{\pi}^{2}$ term, even though $t/M_{\pi}^{2}$ can be as large as $\sim 10$ near the edge of the
Table 3. Lattice determined values (left) and correlations (right) of the FF parameters in the dispersive formalism [52]. The quantities $\rho[i, j]$ are the correlation coefficients between the $i$-th and the $j$-th parameters.

| Parameter | Correlation |
|-----------|-------------|
| $\Lambda_+ = 24.22(1.16) \times 10^{-3}$ | $\rho[\Lambda_+, \log(C)] = 0.376$ |
| $\log(C) = 0.1998(138)$ | $\rho[f^{K+\pi^0}_{+/0, SM}(0), \log(C)] = -0.719$ |
| $f^{K+\pi^0}_{+/0, SM}(0) = 0.9709(46)$ | $\rho[f^{K+\pi^0}_{+/0, SM}(0), \Lambda_+] = -0.228$ |

phase space. As is obvious from eq. (3.9), the coefficient of the $t^2$ term would be proportional to $(\lambda^{(0)}_{K^{+}π^{0}} - \lambda^{(0)}_{K^{+}π^{0}})$, which is below the experimental sensitivity.

Another subtlety to be considered in the analyses is the value of $V_{us}$. The $K^+_{\ell_3}$ total width measurements are often used to independently determine the CKM element $V_{us}$ [51]. We can’t use $V_{us}$, extracted from $K^+_{\ell_3}$ under the SM only hypothesis, as an independent parameter while fitting the NP hypothesis to the same $K^+_{\ell_3}$ data. To circumvent this issue, we need to use a determination of $V_{us}$ that does not use the $K^+_{\ell_3}$ data at all. Such a computation does exist, where the ratio of the $K^+ \rightarrow \mu^+\nu$ width to the $\pi^+ \rightarrow \mu^+\nu$ width is used to obtain $V_{us} = 0.2252 \pm 0.0005$ [51, 65]. This extraction is suitable for the analysis as none of these amplitudes are modified in the $\Lambda_\chi$PT at $O(\xi^2)$.

On the experiment side, we use the $K^+_{\ell_3}$ data obtained by the NA48/2 collaboration\(^2\) to fit the truth level distribution against the observed distribution. We combine this multi-variable fit with the constraints set by the independent measurements of the total decay rates to bound $\xi^2\alpha^{(2)}_{K^+π^0}$ and $\xi^2\beta^{(2)}_{K^+π^0}$. The data consist of bin-by-bin event distributions of the differential decay rate with respect to the pion and the lepton energies ($E_{\pi}, E_{\mu}$), for $4.4 \times 10^{6}$ and $2.3 \times 10^{6}$ reconstructed events corresponding to $K^+_{\ell_3}$ and $K^+_{\mu_3}$ respectively. Using the data for $K^+_{\ell_3}$, we draw the binned Dalitz distribution for the residual events, defined as the differences between accepted events and SM predictions. We show this in the bottom-left panel of Figure 3. The diagonal panels show the distributions of excess events with $1\sigma$ experimental error-bars with respect to $E_{\pi}$ (bottom-right) and $E_{\mu}$ (top-left), after marginalizing over the $E_{\mu}$ and $E_{\pi}$ bins respectively. On the top-right panel, we show the 2D distribution of excess NP events with $\xi^2\beta^{(2)}_{K^+π^0} = 0.06, \xi^2\alpha^{(2)}_{K^+π^0} = 0$. The binned and marginalized distributions of the excess BSM events are shown in the panels containing the corresponding marginal distributions for the data. Although the residual fluctuations show a slight systematic excess in Figure 3, the excess becomes consistent with the SM prediction when the theory error is taken into account.

The data corresponding to the differential rate for $K^+_{\mu_3}$ is the primary source of constraint for $\xi^2\beta^{(2)}_{K^+π^0}$. We note, both $\xi^2\beta^{(2)}_{K^{+}π^{0}}$ and $\xi^2\alpha^{(2)}_{K^{+}π^{0}}$ appear in the same footing as the relative factor between $q^{\mu}$ and $Q^{\mu}$ in the amplitude given in eq. (3.6). The exact form of this factor is:

$$
\left( \frac{\xi^2\beta^{(2)}_{K^{+}π^{0}}}{\delta^{(0)}_{K^{+}π^{0}}} - \frac{\xi^2\alpha^{(2)}_{K^{+}π^{0}}}{\alpha^{(0)}_{K^{+}π^{0}}} \right).
$$

However, the bound on $\xi^2\beta^{(2)}_{K^{+}π^{0}}$ from the differential distribution is stronger than that on $\xi^2\alpha^{(2)}_{K^{+}π^{0}}$. This is because $\xi^2\beta^{(2)}_{K^{+}π^{0}}$ and $\xi^2\alpha^{(2)}_{K^{+}π^{0}}$ are scaled by factors that are hierarchically separated, with $|\delta^{(0)}_{K^{+}π^{0}}| \sim 0.1 \times |\alpha^{(0)}_{K^{+}π^{0}}|$. Thus, decay rate and decay distribution measurements provide complementary constraints on these two parameters.

\(^2\)Publicly available at: [https://zenodo.org/record/3560600#.X-xCJulsKJuJ] [66] (CC BY 4.0).
Figure 3. Bottom Left: The binned Dalitz plot corresponding to the residuals, i.e., the difference between the collected data and the theoretical SM predictions. Sharing its x- and y-axes to the top and to the right are the corresponding marginals with respect to \( E_\mu \) and \( E_\pi \). The empty circles denote the residual signal and the bars denote the 1\( \sigma \) experimental error. Top Right: The binned Dalitz plot corresponding to the additional events generated by NP for \( \xi^2 \beta^{(2)}_{K^+\pi^0} = 0.06 \). The corresponding marginals are depicted in the panels containing the marginals of the residuals for comparison (Green filled circles). For effective comparison, we use the same normalisation of color-shading for the data and the theory distributions.

Before proceeding to a more sophisticated analysis to constrain \( \xi^2 \beta^{(2)}_{K^+\pi^0} \), it is instructive to get a heuristic estimate for the order of the constraint we can achieve. Note that, the NA48/2 collaboration fits the measured differential distribution assuming the SM \( \chi \)PT and determines the values of the slope parameters \[50\]. We can get an estimate of \( \xi^2 \beta^{(2)}_{K^+\pi^0} \) from the difference of the measured (fitted) and the SM computations of the slope parameters. Using eq. (3.9) and eq. (3.5), we can find an approximate expression for \( \xi^2 \beta^{(2)}_{K^+\pi^0} \):

\[
\xi^2 \beta^{(2)}_{K^+\pi^0} \approx f^{K^+\pi^0}_{-\text{Fit}}(0) - \delta \beta^{(0)}_{K^+\pi^0} \equiv f^{K^+\pi^0}_{-\text{Fit}}(0) - f^{K^+\pi^0}_{-\text{SM}}(0)
\approx \frac{M_K^2 - M_\pi^2}{M_\pi^2} \left[ \lambda^{K^+\pi^0,(0),\text{Fit}}(0) - \lambda^{K^+\pi^0,(0),\text{SM}}(0) \right] - f^{K^+\pi^0}_{+,\text{SM}}(0) \left[ \lambda^{K^+\pi^0,(0),\text{SM}}(0) - \lambda^{K^+\pi^0,(0),\text{SM}}(0) \right],
\]

(3.11)

if we take \( \xi^2 \alpha^{(2)}_{K^+\pi^0} \rightarrow 0 \). Using the SM values from Table 3 and fitted numbers from Table 4 of Ref.
Figure 4. The 95% C.L. allowed regions for the ALP parameters in the $\xi^2 \alpha^{(2)}_{K^+\pi^0} - \xi^2 \beta^{(2)}_{K^+\pi^0}$ plane. The yellow band indicates the region allowed by the combined $K^{+}_{e_3}$ data, i.e., the total rate and differential rates combined. The black patch shows the region allowed (at 95% C.L.) by the $K^{+}_{e_3}$ differential distribution. The hatched area is the corresponding region allowed by the $K^{+}_{\mu_3}$ total rate. The red patch is the 95% C.L. allowed region obtained by combining all the independent analyses. The cross marks the SMχPT point where the values of both the parameters are zero.

[50], eq. (3.11) roughly yields $\xi^2 \beta^{(2)}_{K^+\pi^0} \sim 0.01 \pm 0.04$. As we show now, a more involved computation using simultaneous fits yields a constraint of a similar size.

We compute $\chi^2$ distributions by comparing the truth-level signal against the differential distribution data and the total decay width measurement, after taking into account the experimental and theoretical errors and correlations. For the differential distributions, we normalize our histograms using the total number of events, as quoted in the last paragraph. In Figure 4, we show the 95% confidence limits (C.L.) obtained in our analysis in the ($\xi^2 \alpha^{(2)}_{K^+\pi^0} - \xi^2 \beta^{(2)}_{K^+\pi^0}$) plane. The yellow band shows the combined exclusion obtained from the differential and total decay rate measurements of $K^{+}_{e_3}$. Due to the smallness of the electron mass, the constraint on $\xi^2 \beta^{(2)}_{K^+\pi^0}$ is insensitive to the decay distribution measurement. For $K^{+}_{\mu_3}$, we show the individual exclusions obtained from the differential and the total decay rate data, the solid black and the hatched regions respectively. In analysing the figure, we clearly see that the $K^{+}_{\mu_3}$ total rate measurement (hatched) mostly constrains $\xi^2 \alpha^{(2)}_{K^+\pi^0}$. Whereas, the differential distribution (black) constraints primarily $\xi^2 \beta^{(2)}_{K^+\pi^0}$, as anticipated. The black patch is not centred around the SM point ($\xi^2 \alpha^{(2)}_{K^+\pi^0} = \xi^2 \beta^{(2)}_{K^+\pi^0} = 0$), but is compatible with it at 2σ. This is because there is a slight disagreement between the SM and the measured values of the FF parameters (see Figure 8 of Ref. [52]). The combined exclusion, from the four independent measurements is shown in red. In Table 4, we show the individual constraints by varying one parameter at a time. We can see from the table that the addition of the $K^{+}_{e_3}$ measurements only improves the constraint on $\alpha^{(2)}_{K^+\pi^0}$.

Note that, fitting the NP signal against the residual fluctuations around the experimental best-fit
Table 4. Tabulated here are the 95% confidence limits on $\xi^2 \beta^{(2)}_{K^+\pi^0}$ and $\xi^2 \alpha^{(2)}_{K^+\pi^0}$ from the $K_{\mu3}^+$ decay analysis and the $K_{\mu3}^+ + K_{e3}^+$ combined analysis.

|                  | $\xi^2 \beta^{(2)}_{K^+\pi^0}$ ($\xi^2 \alpha^{(2)}_{K^+\pi^0} = 0$) | $\xi^2 \alpha^{(2)}_{K^+\pi^0}$ ($\xi^2 \beta^{(2)}_{K^+\pi^0} = 0$) |
|------------------|-------------------------------------------------|-------------------------------------------------|
| $K_{\mu3}^+$     | $-0.006 < \xi^2 \beta^{(2)}_{K^+\pi^0} < 0.026$ | $-0.021 < \xi^2 \alpha^{(2)}_{K^+\pi^0} < 0.007$ |
| $K_{\mu3}^+ + K_{e3}^+$ | $-0.006 < \xi^2 \beta^{(2)}_{K^+\pi^0} < 0.026$ | $-0.018 < \xi^2 \alpha^{(2)}_{K^+\pi^0} < 0.003$ |

would result in much stronger constraints on $\xi^2 \alpha^{(2)}_{K^+\pi^0}$ and $\xi^2 \beta^{(2)}_{K^+\pi^0}$. However, this approach assumes the true SM value to coincide with the experimental best-fit point, and we do not take this route. Instead, we estimate the theoretical SM spectrum by taking into account the full error in the FF computations, which dominates over the experimental precision. Naturally, these constraints would become much stronger with reduced error in the FF computation, assuming the SM prediction gets closer to the experimental number, with the other possibility pointing to a discovery. In this spirit, we perform a simplistic analysis to estimate the reach of future experiments by reducing the experimental error and assuming more precise theoretical predictions. We keep the central values of the experimental measurements and the theoretical predictions same as the current values, and then reduce both the experimental and theoretical uncertainties by a factor of 2. In Figure 5, we show the 95% C.L. allowed region for $\xi^2 \alpha^{(2)}_{K^+\pi^0}$ and $\xi^2 \beta^{(2)}_{K^+\pi^0}$ as obtained from that analysis.

Figure 5. A conservative future projection (red) for 95% C.L. exclusion limits on the ALP parameters, along with the current bound (hatched), in the $\xi^2 \alpha^{(2)}_{K^+\pi^0} - \xi^2 \beta^{(2)}_{K^+\pi^0}$ plane. We obtain this by a speculative reduction of the experimental and the theoretical uncertainties by 50%.

Before concluding this section, note that, we compute the MC event distribution, corresponding to
the NP signal, from our analytical calculations as follows. Using our analytical computation of the SM distribution and the SM MC Dalitz distribution (given by the NA48/2 collaboration) we can compute a bin-by-bin (as a function of \((E_\pi, E_\ell)\)) scaling factor, let’s call it \(\mathcal{R}(E_\pi, E_\mu)\), which represents the difference between the analytical and the simulated results. We then multiply this function with the analytical form of the NP distribution to get our proxy for the MC simulated NP distribution,

\[
\frac{d\Gamma}{dE_\pi dE_\mu}(E_\pi, E_\mu)_{\text{final}} \equiv \mathcal{R}(E_\pi, E_\mu) \otimes \frac{d\Gamma}{dE_\pi dE_\mu}(E_\pi, E_\mu),
\]

with the \(\otimes\) symbolically representing the bin-by-bin multiplication. Although this, somewhat pedestrian, approach is able to capture all the essential effects sourced by the ALP, a more sophisticated MC simulation might be of interest.

4 Direct vis á vis indirect detection

In the last section, we exclusively concentrated on the alterations of the \(K_L^+\) theoretical expectations in the presence of an ALP. This was motivated by our focus on indirect signatures of the ALP. It is, however, instructive to look at the decay amplitudes for processes involving the ALP itself (direct signatures) to fully appreciate the benefits of the indirect stratagem. This exercise also allows us to point out the contributions to the ALP CC amplitudes from the \(\mathcal{O}_W\) operator.

Following the same exercise that led us to the \(K_L^+\) Lagrangian in the AχPT (eq. (3.6)), we can write down the relevant interaction Lagrangian for \(K_L^+ \to a\ell^+\nu\) as:

\[
\mathcal{L}_{a\ell^+\nu} \supset i G_F V_{tsa} \xi \left[ \left( \alpha^{(1)}_{K^+a} + i \beta^{(1)}_{K^+a} \right) (K^+ \partial_\mu K^+ \partial_\mu \hat{a}) + \left( \beta^{(1)}_{K^+a} + i \beta^{(1)}_{K^+a} \right) \partial_\mu (K^+ \partial_\mu \hat{a}) \right] j_\mu^{\alpha, \ell},
\]

with

\[
\alpha^{(1)}_{K^+a} = - \frac{1}{2} \left( C_{LR}^a - C_{LR}^a + \sqrt{3} (C_{LR}^a - C_{R}^a) \right),
\]

\[
\beta^{(1)}_{K^+a} = - \frac{\sqrt{3}}{2} C_{L}^a, \quad \beta^{(1)}_{K^+a} = - \alpha^{(1)}_{K^+a} = C_W.
\]

In the limit that the final state lepton mass goes to zero (for simplicity), the corresponding amplitude-squared is proportional to:

\[
|A|_{K^+ \to a\ell^+\nu}^2 \propto \xi^2 \left[ \left| \alpha^{(1)}_{K^+a} \right|^2 + \left| \beta^{(1)}_{K^+a} \right|^2 \right] \propto \xi^2 \left[ \left( C_{LR}^a - C_{LR}^a + \sqrt{3} (C_{LR}^a - C_{R}^a) \right)^2 + 2C_W^2 \right].
\]

Comments are in order about the strength of the \(K_L^+ \to a\ell^+\nu\) amplitude-squared. The first noteworthy thing is that the amplitude with the ALP in the final state is of the order \(\xi^2\), the same as the deviation of the corresponding pion amplitudes from the SM expectations. The second thing to point out is that the amplitude gets a contribution from the new \(C_W\) coefficient, which—being purely imaginary—does not interfere with the other contributions. The consequence of this is that there exists no limit where the different Wilson coefficients conspire to set the amplitude to zero unless \(C_W\) is itself zero. This observation is relevant in the context of the so-called pion-phobia [43, 67] that is popular in the literature. The condition for said pion-phobia is generally presented in terms of the quark masses and the couplings of the ALP to the scalar quark-currents. This definition is, of course, model dependent. In a more model-independent way, pion-phobia is just the limit [43] where the \(K^+(\pi^+) \to a\ell^+\nu\) amplitude vanishes, signifying, in effect, a flat direction in the plane of Wilson coefficients. It has, correctly, been pointed out that said flat direction is not stable under renormalization [44]. However, eq. (4.2) shows that even at tree level this cancellation does not exist when \(\mathcal{O}_W\) is taken into account.
If we assume $C_W = 0$, there does exist a limit where the amplitude goes to zero, i.e., $\alpha_{K^{+}\ell a}^{(1)} = 0$. However, it is important to realize that even in this limit, the effects of the ALP do not decouple from the observables corresponding to the CC Lagrangian of the mesons. To see this, we recall the $K_{\ell 3}$ Lagrangian (eq. (3.6)) from the last section. As we can clearly see, by comparing eq. (3.4) and eq. (4.1), the effective coefficient governing the deviation of the $K_{\ell 3}$ amplitude from SM expectations (viz. $\alpha_{K^{+}\pi^0}^{(2)}$) expressed in terms of $\alpha_{K^{+}\ell a}^{(1)}$,

$$\alpha_{K^{+}\pi^0}^{(2)} = \frac{C_3}{2} \left( \alpha_{K^{+}\ell a}^{(1)} + \frac{C_3}{4} \right), \quad (4.3)$$

remains non-zero even in the limit where $\alpha_{K^{+}\ell a}^{(1)}$ is taken to be zero. Consequently, even in the limit the $K^{+} \to a\ell^+\nu$ amplitude is suppressed, the NP contribution to the $K_{\ell 3}$ amplitude continues to be sizeable. This implies, even in the pion-phobic limit the signatures of the ALP are not ‘invisible’, they are just buried in the distribution data of the processes with SM final states. Therefore, the condition for true pion-phobia is actually more non-trivial. For it to happen, both the $K^\pm \to a\ell\nu$ amplitude and the deviation to the $K_{\ell 3}$ amplitude simultaneously need to go to zero. Therefore, in terms of the Wilson coefficients, the condition for ‘pion-phobia’ is:

$$C_W = 0, \quad C_{LR}^3 = C_R^3, \quad \text{and} \quad C_{LR}^8 = C_R^8; \quad (4.4)$$

at leading order in $\xi^2$. This is the limit where the ALP coupling to the vectorial (RH) quark-currents identically cancels out the ALP couplings to the scalar quark-currents. It is easy to see that in this limit, even the $\pi^+ \to a\ell\nu$ amplitude and the corresponding NP effects to the $\pi^0\ell\nu$ amplitude also go to zero. To explicitly see this, note that the $\pi^+ \to a\ell\nu$ amplitude-squared is proportional to:

$$\left| \alpha_{\pi^+a}^{(1)} \right|^2 \propto \frac{(C_3)^2}{4}, \quad (4.5)$$

in the limit $C_W = 0$. The corresponding NP contribution to $\pi^+ \to \pi^0\ell\nu$ in terms of this $\alpha_{\pi^+a}^{(1)}$ is:

$$\left| \alpha_{\pi^+\pi^0}^{(2)} \right|^2 \propto \frac{(C_3)^2}{4} \left( \alpha_{\pi^+a}^{(1)} + \frac{C_3}{4} \right)^2. \quad (4.6)$$

Clearly, in the limit given in eq. (4.4), this factor goes to zero. However, as we can see, the $\pi^+$ case is slightly different from the $K^+$ decay case. We see that for the $\pi^+$, a vanishing of the $\pi^+ \to a\ell\nu$ amplitude-squared (i.e. $C_3 = 0, C_W = 0$) implies the vanishing of NP effects in the $\pi^+ \to \pi^0\ell\nu$ amplitude-squared as well. This is because unlike the $t^L_3$ current that contributes in $K^+$ decay, the $t^L_3$ current that contributes to the $\pi^+$ decay is EW vev suppressed and does not contribute at this order. Hence, the difference is a consequence of the global $SU(2)_L$ symmetry that we discussed in section 2.

5 Sum rules in meson decays

In this section, we discuss a way of identifying the presence and the nature of an ALP in the chiral Lagrangian. We do this by formulating sum rules involving the form factors corresponding to leptonic amplitudes of the light, viz. $\pi^0, \eta, \pi^+ \text{ and } K^+$, mesons. One of the sums we discuss reduces to unity in the SM limit, but deviates from unity in the presence of an ALP. We can use the sign of this deviation to distinguish between a meson Lagrangian where the ALPs enter through mixing alone from that where there are EW interactions of the ALP in the flavor basis itself. The sum rule, in principle, is a way to tackle the EFT ‘inverse problem’ and gives us a handle on the differentiation of UV scenarios.
In deriving the sum rules, we work in the limit \( m_\ell \to 0 \) where the operators corresponding to \( \beta^{(1,2)} \) vanish and it is only the operators corresponding to \( \alpha^{(1,2)} \) that contribute. Hence, we need only to concentrate on processes with the electron in the final state. As discussed in detail in the last section, this results in the vanishing of the momentum-dependent effect and the net modification is an overall scaling of the matrix element.

In the SM, owing to the completeness of the \( \pi^0 - \eta \) basis (neglecting mixing with \( \eta' \)), we have:

\[
\frac{1}{4} \left| f_{K^+\pi^0,SM}(0) \right|^2 + \frac{3}{4} \left| f_{K^+\eta,SM}(0) \right|^2 = 1. \tag{5.1}
\]

Here, the pre-factors of the FFs are the corresponding group theory factors. In the presence of the ALP, this relationship is obviously modified. The complete basis now includes the ALP, and this is reflected in the sum of the form factors. In the \( A\chi PT \), owing to the redefinition of the physical \( \pi^0 \) and the \( \eta \) mesons and due to the introduction of new operators, the FFs are modified, as shown in eqs. (3.3) and (3.4). These effective FFs, as we discussed, are the objects which are extracted by the experiments. Therefore, in the \( A\chi PT \), these effective FFs are the natural candidates for the construction of the sum. In terms of these FFs, we find after some algebra:

\[
\frac{1}{4} \left| f_{K^+\pi^0}(0) \right|^2 + \frac{3}{4} \left| f_{K^+\eta}(0) \right|^2 = 1 - \frac{\xi^2}{16} \left( C_{3LR}^3 - C_{3R}^3 + \sqrt{3}(C_{8LR}^8 - C_{8R}^8) \right)^2 + \xi^2 \frac{3}{16} C_L^8. \tag{5.2}
\]

We see that when \( C_L^8 = 0 \), the sum above is identically less than one, as expected from considerations of completeness. However, when there are \( t_{3L}^8 \) breaking interaction between the ALP and the mesons, i.e. \( C_L^8 \neq 0 \), the sum can be greater than one. Therefore, a positive deviation of the sum from unity is not only possible, it uniquely signals a \( t_{3L}^8 \) breaking interaction between the ALP and the quarks in the UV. That is, this sum can not only tell us about the presence of an ALP in the chiral Lagrangian, but it can also tell us about the corresponding UV model. The latter would not have been possible by just looking at the deviations from SM expectations of the individual decay widths.

Before concluding, it is instructive to look at the corresponding sum for a \( \pi^+ \) in place of a \( K^+ \):

\[
\left| f_{\pi^+\pi^0}(0) \right|^2 + \left| f_{\pi^+\eta}(0) \right|^2 = 1 - \frac{\xi^2}{4} C_3^8. \tag{5.3}
\]

As is obvious, unlike the previous case, this sum is always less than one. As seen from eq. (5.2), a value of the sum greater than one is possible only when \( C_L^8 \neq 0 \). That is, similar to the phenomenon discussed in the last section, this result is sourced by the \( t_{3L}^8 \) interaction of the ALP. The \( t_{3L}^8 \) counterpart for the \( \pi^+ \) sector is the \( t_{3L}^8 \). Now, as stressed in the first section, any \( t_{3L}^8 \) interaction of the ALP breaks the \( SU(2)_L \) subgroup of \( SU(3)_L \), hence, must be electroweak vev suppressed. This particular result, along with the other \( SU(3)_L \) breaking effects discussed throughout the paper, vindicates our choice of working with the \( K^\pm \) decays as opposed to \( \pi^\pm \) decays.

6 Conclusion

A low-lying ALP leaves its signatures in amplitudes corresponding to SM processes, signatures that are manifest at the tree level itself. These signatures will be seen as variations from SM expectations in conventional observables of flavor physics, for example, form factors, differential distributions, decay rates etc. Therefore, indirect detection techniques, like the ones discussed in this work, open up novel avenues to look for ALPs and are complementary to standard direct detection searches. Furthermore, the tree level modification to SM physics behoves us to undertake a careful study of the ALP-meson
Lagrangian in the light of the existing flavor physics anomalies. The results presented in this work, whether the data-driven analysis corresponding to the $K^\pm$ decay or the sum rules constructed out of the form factors, are proof-of-concept examples that can be generalized and used in the context of other observables. Needless to stress that the efficacy of such indirect techniques will only increase with the inevitable improvements in lattice computations and with more precise measurements of SM observables. We expect that the methods discussed in this work will be further generalized and applied to constrain the ALP parameter space by focussing on data sets to be obtained from the plethora of ongoing and upcoming flavor physics experiments.

Acknowledgments

The authors thank Dmitry Madigozhin for providing them with the source of the NA48/2 dataset. The research of SG is supported by the NSF grant PHY-2014165.

References

[1] G. 't Hooft, *Symmetry Breaking Through Bell-Jackiw Anomalies*, Phys. Rev. Lett. 37 (1976) 8.
[2] R.D. Peccei and H.R. Quinn, *CP Conservation in the Presence of Instantons*, Phys. Rev. Lett. 38 (1977) 1440.
[3] R.D. Peccei and H.R. Quinn, *Constraints Imposed by CP Conservation in the Presence of Instantons*, Phys. Rev. D 16 (1977) 1791.
[4] S. Weinberg, *A New Light Boson?*, Phys. Rev. Lett. 40 (1978) 223.
[5] F. Wilczek, *Problem of Strong P and T Invariance in the Presence of Instantons*, Phys. Rev. Lett. 40 (1978) 279.
[6] J. Preskill, M.B. Wise and F. Wilczek, *Cosmology of the Invisible Axion*, Phys. Lett. B 120 (1983) 127.
[7] L.F. Abbott and P. Sikivie, *A Cosmological Bound on the Invisible Axion*, Phys. Lett. B 120 (1983) 133.
[8] M. Dine and W. Fischler, *The Not So Harmless Axion*, Phys. Lett. B 120 (1983) 137.
[9] Y. Chikashige, R.N. Mohapatra and R.D. Peccei, *Are There Real Goldstone Bosons Associated with Broken Lepton Number?*, Phys. Lett. B 98 (1981) 265.
[10] C.D. Froggatt and H.B. Nielsen, *Hierarchy of Quark Masses, Cabibbo Angles and CP Violation*, Nucl. Phys. B 147 (1979) 277.
[11] K.I. Izawa, T. Watari and T. Yanagida, *Higher dimensional QCD without the strong CP problem*, Phys. Lett. B 534 (2002) 93 [hep-ph/0202171].
[12] E. Witten, *Some Properties of O(32) Superstrings*, Phys. Lett. B 149 (1984) 351.
[13] P. Svrcek and E. Witten, *Axions In String Theory*, JHEP 06 (2006) 051 [hep-th/0605206].
[14] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper and J. March-Russell, *String Axiverse*, Phys. Rev. D 81 (2010) 123530 [0905.4720].
[15] D. Chang, W.-F. Chang, C.-H. Chou and W.-Y. Keung, *Large two loop contributions to g-2 from a generic pseudoscalar boson*, Phys. Rev. D 63 (2001) 091301 [hep-ph/0009292].
[16] P.W. Graham, D.E. Kaplan and S. Rajendran, *Cosmological Relaxation of the Electroweak Scale*, Phys. Rev. Lett. 115 (2015) 221801 [1504.07551].
[17] K.S. Jeong, T.H. Jung and C.S. Shin, \textit{Adiabatic electroweak baryogenesis driven by an axionlike particle}, \textit{Phys. Rev. D} \textbf{101} (2020) 035009 [1811.03294].

[18] Y. Nomura and J. Thaler, \textit{Dark Matter through the Axion Portal}, \textit{Phys. Rev. D} \textbf{79} (2009) 075008 [0810.5397].

[19] V.A. Rubakov, \textit{Grand unification and heavy axion}, \textit{JETP Lett.} \textbf{65} (1997) 621 [hep-ph/9703409].

[20] A. Hook, \textit{Anomalous solutions to the strong CP problem}, \textit{Phys. Rev. Lett.} \textbf{114} (2015) 141801.

[21] H. Fukuda, K. Harigaya, M. Ibe and T.T. Yanagida, \textit{Model of visible QCD axion}, \textit{Phys. Rev. D} \textbf{92} (2015) 015021 [1504.06084].

[22] G. Marques-Tavares and M. Teo, \textit{Light axions with large hadronic couplings}, \textit{JHEP} \textbf{05} (2018) 180 [1803.07675].

[23] T. Gherghetta, V.V. Khoze, A. Pomarol and Y. Shirman, \textit{The Axion Mass from 5D Small Instantons}, \textit{JHEP} \textbf{03} (2020) 063 [2001.05610].

[24] I.G. Irastorza and J. Redondo, \textit{New experimental approaches in the search for axion-like particles}, \textit{Prog. Part. Nucl. Phys.} \textbf{102} (2018) 89 [1801.08127].

[25] I.G. Irastorza, \textit{An introduction to axions and their detection}, in \textit{Les Houches summer school on Dark Matter}, 9, 2021 [2109.07376].

[26] D.S.M. Alves and N. Weiner, \textit{A viable QCD axion in the MeV mass range}, \textit{JHEP} \textbf{07} (2018) 092 [1710.03764].

[27] W.J. Marciano, A. Masiero, P. Paradisi and M. Passera, \textit{Contributions of axionlike particles to lepton dipole moments}, \textit{Phys. Rev. D} \textbf{94} (2016) 115033 [1607.01022].

[28] J. Jaeckel and M. Spannowsky, \textit{Probing MeV to 90 GeV axion-like particles with LEP and LHC}, \textit{Phys. Lett. B} \textbf{753} (2016) 482 [1509.00476].

[29] B. Döbrich, J. Jaeckel, F. Kahlhoefer, A. Ringwald and K. Schmidt-Hoberg, \textit{ALPtraum: ALP production in proton beam dump experiments}, \textit{JHEP} \textbf{02} (2016) 018 [1512.03069].

[30] S. Knapen, T. Lin, H.K. Lou and T. Melia, \textit{Searching for Axionlike Particles with Ultraperipheral Heavy-Ion Collisions}, \textit{Phys. Rev. Lett.} \textbf{118} (2017) 171801 [1607.06083].

[31] M. Bauer, M. Neubert and A. Thamm, \textit{Collider Probes of Axion-Like Particles}, \textit{JHEP} \textbf{12} (2017) 044 [1708.00443].

[32] M. Bauer, M. Heiles, M. Neubert and A. Thamm, \textit{Axion-Like Particles at Future Colliders}, \textit{Eur. Phys. J. C} \textbf{79} (2019) 74 [1808.10323].

[33] X. Cid Vidal, A. Mariotti, D. Redigolo, F. Sala and K. Tobioka, \textit{New Axion Searches at Flavor Factories}, \textit{JHEP} \textbf{01} (2019) 113 [1810.09452].

[34] D. Aloni, Y. Soreq and M. Williams, \textit{Coupling QCD-Scale Axionlike Particles to Gluons}, \textit{Phys. Rev. Lett.} \textbf{123} (2019) 031803 [1811.03474].

[35] M. Bauer, M. Neubert, S. Renner, M. Schnubel and A. Thamm, \textit{Flavor probes of axion-like particles}, \textit{2110.10698}.

[36] S. Chakraborty, M. Kraus, V. Loladze, T. Okui and K. Tobioka, \textit{Heavy QCD axion in b \to s transition: Enhanced limits and projections}, \textit{Phys. Rev. D} \textbf{104} (2021) 055036 [2102.04474].

[37] E. Bertholet, S. Chakraborty, V. Loladze, T. Okui, A. Soffer and K. Tobioka, \textit{Heavy QCD Axion at Belle II: Displaced and Prompt Signals}, \textit{2108.10331}. 
[38] M. Freytsis, Z. Ligeti and J. Thaler, *Constraining the Axion Portal with $B \to K^{\pm} l^\pm$, Phys. Rev. D* 81 (2010) 034001 [0911.5355].

[39] F. Björkeroth, E.J. Chun and S.F. King, *Flavourful Axion Phenomenology, JHEP* 08 (2018) 117.

[40] W. Altmannshofer, S. Gori and D.J. Robinson, *Constraining axionlike particles from rare pion decays, Phys. Rev. D* 101 (2020) 075002 [1909.00005].

[41] H. Ishida, S. Matsuzaki and Y. Shigekami, *New perspective in searching for axionlike particles from flavor physics, Phys. Rev. D* 103 (2021) 095022 [2006.02725].

[42] PIENU collaboration, *Search for three body pion decays $\pi^+ \to l^+ \nu X$, Phys. Rev. D* 103 (2021) 052006 [2101.07381].

[43] H. Georgi, D.B. Kaplan and L. Randall, *Manifesting the Invisible Axion at Low-energies, Phys. Lett.* 169B (1986) 73.

[44] M. Bauer, M. Neubert, S. Renner, M. Schnubel and A. Thamm, *The Low-Energy Effective Theory of Axions and ALPs, JHEP* 04 (2021) 063 [2012.12272].

[45] M. Bauer, M. Neubert, S. Renner, M. Schnubel and A. Thamm, *Consistent Treatment of Axions in the Weak Chiral Lagrangian, Phys. Rev. Lett.* 127 (2021) 081803 [2102.13112].

[46] B. Patt and F. Wilczek, *Higgs-field portal into hidden sectors, hep-ph/0605188.*

[47] M. Gell-Mann, *Symmetries of baryons and mesons, Phys. Rev.* 125 (1962) 1067.

[48] S. Okubo, *Note on Unitary Symmetry in Strong Interactions, Progress of Theoretical Physics* 27 (1962) 949.

[49] H. Leutwyler, *On the foundations of chiral perturbation theory, Annals Phys.* 235 (1994) 165 [hep-ph/9311274].

[50] NA48/2 collaboration, *Measurement of the form factors of charged kaon semileptonic decays, JHEP* 10 (2018) 150 [1808.09041].

[51] PARTICLE DATA GROUP collaboration, *Review of Particle Physics, PTEP* 2020 (2020) 083C01.

[52] N. Carrasco, P. Lami, V. Lubicz, L. Riggio, S. Simula and C. Tarantino, *K $\to \pi$ semileptonic form factors with $N_f = 2 + 1 + 1$ twisted mass fermions, Phys. Rev. D* 93 (2016) 114512 [1602.04113].

[53] H.-C. Cheng, L. Li and E. Salvioni, *A Theory of Dark Pions, 2110.10691.*

[54] J. Callan, Curtis G., S.R. Coleman, J. Wess and B. Zumino, *Structure of phenomenological Lagrangians. 2., Phys. Rev.* 177 (1969) 2247.

[55] D.B. Kaplan, *Five lectures on effective field theory, 10, 2005 [nucl-th/0510023].

[56] J.F. Donoghue and A.F. Perez, *The Electromagnetic mass differences of pions and kaons, Phys. Rev. D* 55 (1997) 7075 [hep-ph/9611331].

[57] N.H. Christ, C. Dawson, T. Izubuchi, C. Jung, Q. Liu, R.D. Mawhinney et al., *The $\eta$ and $\eta'$ mesons from Lattice QCD, Phys. Rev. Lett.* 105 (2010) 241601 [1002.2999].

[58] RQCD collaboration, *Properties of the $\eta$ and $\eta'$ mesons: Masses, decay constants and gluonic matrix elements, PoS LATTICE2021* (2021) 286 [2111.05656].

[59] S. Aoki et al., *Review of Lattice Results Concerning Low-Energy Particle Physics, Eur. Phys. J. C* 74 (2014) 2890 [1310.8555].

[60] V. Cirigliano, M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, *Radiative corrections to $K(l3)$ decays, Eur. Phys. J. C* 23 (2002) 121 [hep-ph/0110153].
[61] A. Sirlin, *Large m(W), m(Z) Behavior of the O(alpha) Corrections to Semileptonic Processes Mediated by W*, Nucl. Phys. B **196** (1982) 83.

[62] FlaviaNet Working Group on Kaon Decays collaboration, *Precision tests of the Standard Model with leptonic and semileptonic kaon decays*, in *5th International Workshop on e+ e- Collisions from Phi to Psi*, 1, 2008 [0801.1817].

[63] M. Moulson, *Experimental determination of V_{us} from kaon decays*, PoS CKM2016 (2017) 033 [1704.04104].

[64] V. Bernard, M. Oertel, E. Passemar and J. Stern, *Dispersive representation and shape of the K(l3) form factors: Robustness*, Phys. Rev. D **80** (2009) 034034 [0903.1654].

[65] W.J. Marciano, *Precise determination of —V(us)— from lattice calculations of pseudoscalar decay constants*, Phys. Rev. Lett. **93** (2004) 231803 [hep-ph/0402299].

[66] NA48/2 collaboration, M. Dmitry and S. Sergey, *NA48/2 program and data for calculation of charged kaon semileptonic form factors*, Dec., 2019. 10.5281/zenodo.3560600.

[67] L.M. Krauss and D.J. Nash, *A viable weak interaction axion?*, Phys. Lett. B **202** (1988) 560.