I. INTRODUCTION

General Relativity (GR) has been one of the most successful theories in explaining the nature of gravity on both astrophysical and cosmological scales. However, with recent developments of high precision cosmology able to probe physics at a very large redshifts, the large scale validity of GR has come under increasing scrutiny. This is due to the fact that in order to fit the standard model of cosmology one has to introduce two dark components, namely the Dark Matter and the Dark Energy, in order to achieve a consistent picture. The problems related to these dark components remains to be one of the greatest puzzles in contemporary physics. One of the theoretical proposals that has received a considerable amount of attention is that Dark Energy has a geometrical origin. This idea is driven by the fact that modifications to GR appear in the low energy limit of many fundamental theories and that these modifications lead naturally to cosmologies which admit a Dark Energy like era without the introduction of any additional cosmological fields. The most popular candidates among these ultraviolet modifications of GR are the fourth order gravity theories, where the standard Einstein-Hilbert action of GR is modified by adding terms that lead to field equations of order four in the metric tensor.

Although many of these modifications to GR has been somewhat successful in describing correctly the expansion history of the universe, on astrophysical scales there are considerable problems in modelling astrophysical objects like compact stars and black holes. This is partially due to the added mathematical complexity of these theories and also due to the fact that in many of these theories the astrophysical objects become unstable, contrary to our own experience. Hence, in order to reach a viable alternative to General Relativity, one must do a detailed investigation on both astrophysical and cosmological scales.

We know in GR, spherically symmetric vacuum spacetimes have an extra symmetry: they are either locally static or spatially homogeneous. This rigidity of spherically symmetric vacuum solutions is the essence of the Jebsen-Birkhoff theorem [1-3]. This theorem makes the Schwarzschild solution crucially important in astrophysics and underlies the way local astronomical systems decouple from the expansion of the universe [4]. The rigidity embodied in this property of the Einstein field equations is specific to vacuum GR solutions and is known not to hold for theories with extra degrees of freedom (e.g. $f(R)$ theories of gravity or other scalar-tensor theories [5]). It is, therefore, important to investigate the extra conditions required for a Jebsen-Birkhoff like theorem to hold modified theories of gravity.

It was recently shown [7-9], that in GR, the rigidity of spherical vacuum solutions of Einstein’s field equations continues even in the perturbed scenario: almost spherical symmetry and/or almost vacuum imply almost static or almost spatially homogeneous. This provides an important reason for the stability of the solar system and of black hole spacetimes and has interesting implications for the issue of how a universe made up of locally spherically symmetric objects imbedded in vacuum regions is able to expand, given that Birkhoff’s theorem tells us the local spacetime domains have to be static. A similar study of local stability is required for the spherically symmetric solutions in modified gravity theories to see if these theories are physically viable.

In this paper, we prove a Jebsen-Birkhoff like theorem for $f(R)$ theories of gravity, to find the necessary conditions required for the existence of a Schwarzschild solution in these theories. We discuss under what circumstances we can covariantly set up a scale in the problem and then perturb the vacuum spacetime with respect to this covariant scale to find the stability of the theorem. We do this in two steps: (a) First we maintain spherical symmetry and perturb the Ricci scalar $R$ around $R = 0$ to find the necessary conditions on the spatial and temporal derivatives of the Ricci scalar for the spacetime to be almost Schwarzschild. (b) We then define the notion of almost spherical symmetry with respect to the covariant scale and perturb the spherical symmetry to prove the stability of the theorem.

The important result that emerges covariantly from this investigation is that, there exists a non-zero measure in the parameter space of $f(R)$ theories for which
the Jebsen-Birkhoff like theorem remains stable under generic perturbations. Furthermore our result applies locally and hence does not depend on specific boundary conditions used for solving the perturbation equations. We prove the result by using the 1+1+2 covariant perturbation formalism \[11,13\], which developed from the 1+3 covariant perturbation formalism \[20\].

II. GENERAL EQUATIONS FOR FOURTH ORDER GRAVITY

The simplest generalisation of the Einstein-Hilbert action of GR is obtained by replacing the Ricci scalar \( R \) by a function of the Ricci scalar \( f(\mathcal{R}) \), resulting in the action

\[
\mathcal{A} = \frac{1}{2} \int d^4x \left[ \sqrt{-g} f(\mathcal{R}) + \mathcal{L}_m \right],
\]

where \( \mathcal{L}_m \) is the Lagrangian density of the standard matter fields. Varying the action with respect to the metric over a 4-volume gives the following field equations

\[
G_{ab} = \left( R_{ab} - \frac{1}{2} g_{ab} \mathcal{R} \right) = T_{ab} = \frac{T^M_{ab}}{f'} + T^R_{ab},
\]

where the right hand side is the “effective” energy momentum tensor \( T_{ab} \) comprising \( T^M_{ab} \), the standard matter energy momentum tensor and

\[
T^R_{ab} = \frac{1}{f'} \left[ \frac{1}{2} g_{ab} \left( f - R f' \right) + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f' \right],
\]

which we label the “curvature” energy momentum tensor.

III. 1+1+2 COVARIANT SPLITTING OF SPACETIME

A. Kinematics

In the 1+3 covariant formalism \[20\] a non-intersecting timelike family of worldlines (associated with fundamental observers comoving with the cosmological fluid) forms a congruence in spacetime \( (\mathcal{M}, g) \) representing the average motion of matter at each point. The congruence is defined by a timelike unit vector \( u^a \) (\( u^a u_a = -1 \)) splitting the spacetime in the form \( R \otimes V \) where \( R \) denotes the timeline along \( u^a \) and \( V \) is the tangent 3-space perpendicular to \( u^a \). The projection tensor

\[
h^a{}_b = g^a{}_b + u^a u_b , \quad h^a{}_a = 3 ,
\]

projects into the rest space orthogonal to \( u^a \) and the projected alternating Levi-Civita tensor \( \varepsilon_{abc} \) is the effective volume element for the 3-space.

Any spacetime 4-vector \( \psi_a \) may be covariantly split into a scalar, \( \psi \), which is the part of the vector parallel to \( u_a \), and a 3-vector, \( \psi_a \), lying orthogonal to \( u_a \):

\[
\psi_a = -\psi u_a + \psi^{(a)} , \quad \psi = \psi^b , \quad \psi^{(a)} = h^a{}_b \psi^b .
\]

and any projected rank-2 tensor \( \psi_{cd} \) can be split as

\[
\psi_{ab} = \psi_{(ab)} + \frac{1}{3} \psi h_{ab} + \psi_{[ab]} ,
\]

where \( \psi = h_{cd} \psi^{cd} \) is the spatial trace, \( \psi_{(ab)} \) is the orthogonally projected symmetric trace-free PSTF part of the tensor defined as

\[
\psi_{(ab)} = \left( h_{(a} \epsilon_{b)} \right) - \frac{1}{3} h_{ab} \psi^e ,
\]

and \( \psi_{[ab]} \) is the skew part of the rank-2 tensor which is spatially dual to the spatial vector \( \psi^{(a)} (\psi_{[ab]} = \epsilon_{abc} \psi^c) \). The angle brackets denote orthogonal projections of vectors and the orthogonally PSTF part of tensors.

Moreover, two derivatives can be defined: the vector \( u^a \) is used to define the covariant time derivative (denoted with a dot \( \cdot \)) along the observers’ worldlines, where for any tensor \( S^{a..b}_{c..d} \)

\[
\dot{S}^{a..b}_{c..d} = u^e \nabla_e S^{a..b}_{c..d} ,
\]

and the spatial projection tensor \( h_{ab} \) is used to define the fully orthogonally projected covariant spatial derivative ‘D’, such that,

\[
D_c S^{a..b}_{c..d} = h^{r.e}_c h^{f..g} f \ldots h^p_c h^q_d \nabla_r S^{f..g..p..q} ,
\]

with the projection on all the free indices.

In the 1+1+2 approach, we further split the 3-space \( V \), by introducing the spacelike unit vector \( e^a \) orthogonal to \( u^a \) so that

\[
e_a u^a = 0 , \quad e_a e^a = 1 .
\]

Then the projection tensor

\[
N^a{}_b \equiv h^a{}_b - e_a e^b \quad N^a{}_a = 2 ,
\]

projects vectors onto the tangent 2-surfaces orthogonal to \( e^a \) and \( u^a \), which, following \[10\], we will refer to as ‘sheets’. The sheet carries a natural 2-volume element:

\[
\varepsilon_{ab} \equiv \varepsilon_{abc} e^c .
\]

In 1+1+2 slicing, any 3-vector \( \psi_{(a)} \) as defined in \[10\], can be irreducibly split into a component along \( e^a \) and a sheet component \( \Psi^a \), orthogonal to \( e^a \) i.e.

\[
\psi^{(a)} = \Psi^a + \Psi^a , \quad \Psi \equiv \psi_{(a)} e^a , \quad \Psi^a \equiv N^{ab} \psi_{(b)} .
\]

and a similar decomposition can be done for a PSTF 3-tensor \( \psi_{(ab)} \) as defined \[17\], which can be split into scalar along \( e^a \), a 2-vector and a 2-tensor part as follows:

\[
\psi_{ab} = \psi_{(ab)} = \Psi \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2 \Psi_{(a} e_{b)} + \Psi_{ab} .
\]
where
\[ \Psi \equiv e \psi^a e_i^a \Psi_{ab} = -N^a \psi_{ab} , \]
\[ \Psi_a \equiv N_a e^i e_i^a \psi_{bc} , \]
\[ \Psi_{ab} \equiv \psi_{(ab)} = \left( N^e (a N_b)^d - \frac{1}{2} N_{ab} N^{cd} \right) \psi_{cd} , \quad (15) \]
and the curly brackets denote the PSTF part of a tensor with respect to \( e^a \).

Apart from the ‘time’ (dot) derivative of an object (scalar, vector or tensor), in the 1+1+2 formalism we introduce two new derivatives, which \( e^a \) defines, for any object \( \psi_{ab} \) as:
\[ \dot{\psi}_{ab}^{a...b...c...d} = e^f D_f \psi_{ab}^{a...b...c...d} , \quad (16) \]
\[ \delta_f \psi_{ab}^{a...b...c...d} = N^f \psi_{ab}^{a...b...c...d} , \quad (17) \]
The hat-derivative ‘\( \hat{\cdot} \)’ is the derivative along the \( e^a \) vector-field in the surfaces orthogonal to \( u^a \). The \( \delta \)-derivative is the projected derivative onto the orthogonal 2-sheet, with the projection on every free index.

The fundamental geometrical quantities in the space-time in the 1+1+2 formalism for \( f(R) \) gravity are (see [12] for a detailed physical description of these variables),
\[ \left\{ R, \Theta, \Lambda, \Sigma, \xi, \alpha, \Omega^a, \Omega^b, \right\} , \]
\[ \alpha^a, \alpha^b, \xi^a, \alpha_i, \alpha_{ab}, \xi_{ab}, \Lambda_{a}, \zeta_{ab} , \quad (18) \]
and their dynamics give us information about the space-time geometry.

In terms of these variables, the expression for the full covariant derivative of \( e^a \) in its irreducible form is
\[ \nabla_a e_b = -A u_a u_b + u_a \alpha_b + \left( \frac{1}{3} \Theta + \Sigma \right) e_a u_b + \xi e_{ab} + \zeta_{ab} + \left( \Sigma_a - \xi_{ac} \Omega^c \right) u_b + e_a a_b + \frac{1}{2} \phi N_{ab} , \quad (19) \]
from which we can obtain the spatial derivative of \( e^a \) as
\[ D_a e_b = e_a a_b + \frac{1}{2} \phi N_{ab} + \xi e_{ab} + \zeta_{ab} . \quad (20) \]
The other derivative of \( e^a \) is its change along \( u^a \),
\[ \dot{e}_a = A u_a + \alpha_a \quad \text{where} \quad A \equiv e^i u_i \quad \text{and} \quad \alpha_a \equiv N^a \dot{e}_b . \quad (21) \]
From equation (20) we see that along the spatial direction \( e^a \), \( \phi = \delta_a e^a \) represents the expansion of the sheet, \( \zeta_{ab} = \delta_a e_b \) is the shear of \( e^a \) (i.e., the distortion of the sheet) and \( a_a = e^c D_c e_a = \dot{e}_a \) its acceleration, while \( \xi = \frac{1}{2} e^b \delta^c_{ab} e_c \) represents the vorticity associated with \( e^a \) (‘twisting’ of the sheet).

We include here the expression for the 1+1+2 split of the full covariant derivative of \( u^a \)
\[ \nabla_a u_b = -u_a (A e_b + A_b) + e_a e_b \left( \frac{1}{3} \Theta + \Sigma \right) + e_a (\Sigma_b + \xi_{bc} \Omega^c) + (\Sigma_a - \xi_{ac} \Omega^c) e_b + N_{ab} \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) + \Omega e_{ab} + \Sigma_{ab} , \quad (22) \]

In general the three derivatives defined so far, dot - ‘\( \cdot \)’, hat - ‘\( \hat{\cdot} \)’ and delta - ‘\( \delta_a \)’, do not commute. The commutations relations for these derivatives of any scalar \( \psi \) are
\[ \dot{\psi} - \dot{\hat{\psi}} = -A \psi + \left( \frac{1}{3} \Theta + \Sigma \right) \psi + (\Sigma_a + \xi_{ab} \Omega^b - \alpha_a) \delta^a \psi , \quad (23) \]
\[ \delta_a \dot{\psi} - (\delta_a \psi) \dot{\hat{\psi}} = -A_a \dot{\psi} + (\alpha_a + \xi_{ab} \Omega^b) \hat{\psi} + \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) \delta_a \psi + (\Sigma_{ab} + \Omega \xi_{ab}) \delta^b \psi , \quad (24) \]
\[ \delta_a (\delta_b \psi) = \xi_{ab} (\Omega \dot{\psi} - \xi \hat{\psi}) + a_{ab} \delta_b \psi . \quad (25) \]
From the above relations it is clear that the 2-sheet is a genuine 2-surface (instead of just a collection of tangent planes) if and only if the commutator of the time and hat derivative do not depend on any sheet component, i.e., when \( \Sigma_a + \xi_{ab} \Omega^b - \alpha_a = 0 \) and the sheet derivatives commute, i.e., when \( \Omega = \alpha_a = 0. \)

### B. Energy momentum tensor

In terms of the 1+1+2 variables, fluid description of the energy momentum tensor in (2) is given by
\[ T_{ab} = \mu u_a u_b + p \left[ N_{ab} + e e^b \right] + 2 u_a \left[ Q e_b + Q a \right] \]
\[ + \Pi \left[ e_a e_b - \frac{1}{2} N_{ab} \right] + 2 \Pi (e_a e_b) + \Pi_{ab} . \quad (27) \]
We recall here that the “effective” thermodynamic quantities as presented in (24) are representative of the total combination of the standard matter and curvature quantities as follows:
\[ \mu = \mu^M + \frac{1}{f} \left[ \frac{1}{2} (R f' - f) - \theta f'' R + f''' X^2 + f'' \dot{X} \right] + \frac{1}{f} \left[ \frac{1}{2} (R f' - f) - \theta f'' R + f''' X^2 + f'' \dot{X} \right] \]
\[ - A f'' R - A \xi a f'' \delta a R - \frac{2}{3} \left( \phi f'' X + f''' \delta a R \right) \]
\[ + f'' \delta a R + f''' X^2 + f'' \dot{X} - a a f'' \delta a R \] \quad , \quad (28)
\[ p = p^M + \frac{1}{f} \left[ \frac{1}{2} (f - R f') + \frac{2}{3} \theta f'' R + \frac{1}{2} f'' R \right] - A f'' X - A \xi a f'' \delta a R - \frac{2}{3} \left( \phi f'' X + f''' \delta a R \right) \]
\[ + f'' \delta a R + f''' X^2 + f'' \dot{X} - a a f'' \delta a R \] \quad , \quad (29)
\[ Q = Q^M - \frac{1}{f^t} \left[ f'' f X + f'' f X - \alpha^a f'' \delta_a R \right], \]  
(30)

\[ Q_a = Q_a^M + \frac{1}{f^t} \left[ (\Sigma_a - \varepsilon_a^b \Omega_b) f'' X - f'' \delta_a R \right. \]
\[ \left. + (\varepsilon_a^b + \varepsilon_a^b \Omega^b) f'' \delta_a R - f f'' \delta_a R \right] + \left( \frac{1}{3} \theta - \frac{1}{2} \Sigma \right) f'' \delta_a R \right], \]  
(31)

\[ \Pi = \Pi^M + \frac{1}{f^t} \left[ \frac{1}{3} \left( 2 f'' X^2 + 2 f'' X - 2 A_a f'' \delta_a R \right. \right. \]
\[ \left. - \phi f'' X - f'' \delta_a R \delta_a R + \left( 1 + 2 \right) f'' \delta_a R \right] - \Omega \left. f'' \delta_a R \right] - \Sigma f'' \delta_a R \right], \]  
(32)

\[ \Pi_a = \Pi_a^M + \frac{1}{f^t} \left[ - \Sigma_a X f'' \delta_a R + f'' \delta_a X \right. \]
\[ - \frac{1}{2} \phi f'' \delta_a R + \left( \xi \varepsilon_a^b - \zeta_a^b \right) f'' \delta_b R \]
\[ - \frac{1}{2} \left( \Sigma_a + \varepsilon_a^b \Omega_b \right) f'' \delta_a R \right], \]  
(33)

\[ \Pi_{ab} = \Pi_{ab}^M + \frac{1}{f^t} \left[ - \Sigma_{ab} f'' \delta_b R + \zeta_{ab} f'' X \right. \]
\[ \left. + f'' \delta_b \delta_{a b} R + f'' \delta_{a b} \delta_a R \right] \]  
(34)

where \( \mu^M \) is the energy density relative to \( u^a \), \( \Pi \) is the isotropic pressure, \( Q \) and \( Q_a \) are the components of the \( u^a \) energy flux parallel and orthogonal to \( e^a \) respectively, \( \Pi, \Pi^a \) and \( \Pi_{ab} \) are the PSTF ( w.r.t \( e^a \)) parts of the anisotropic pressure. We define also define \( \bar{R} = X \).

The set of thermodynamic matter variables,

\[ \{ \mu^M, p^M, Q^M, \Pi^M, Q_a^M, \Pi_a^M, \Pi_{ab}^M \}, \]  
(35)

for a given equation of state, together with \( \mathbf{I} \) form an irreducible set that completely describes the vacuum spacetime in \( f(R) \) gravity. For the complete set of evolution equations, propagation equations, mixed equations and constraints for the above irreducible set of variables please see equations (48-81) of \[12\].

IV. 1+1+2 EQUATIONS FOR VACUUM LRS-II SPACETIMES

A spacetime is said to be locally rotationally symmetric (LRS) if there exists a continuous isotropy group at each point and hence is characterised by the existence of a multi-transitive isometry group acting on the spacetime manifold \[14\][15]. These spacetimes exhibit locally (at each point) a unique preferred spatial direction that constitutes a local axis symmetry such that the geometry is invariant under rotations about it. We choose the vector field \( e^a \) as the preferred spatial direction in the LRS spacetime, namely the ‘radial’ vector. Now since LRS spacetimes are defined to be isotropic, this allows for the vanishing of all \( 1+1+2 \) vectors and tensors, such that there are no preferred directions in the sheet. Thus, in vacuum \( (\mu^M = p^M = Q^M = \Pi^M = 0) \), all the non-zero \( 1+1+2 \) variables are the covariantly defined scalars

\[ \mathbf{LRS} : \{ R, A, \Theta, \phi, \zeta, \Sigma, \Omega, E, H \}, \]  
(36)

A detailed discussion of the covariant approach to LRS perfect fluid space-times can be seen in \[10\].

One subsets of LRS spacetimes is the LRS class II, which contains all the LRS spacetimes that have no vorticity or spatial rotation. As a consequence, the vorticity components \( \Omega \) and \( \zeta \) associated with \( u^a \) and \( e^a \), respectively, and \( H \) which is a component of the magnetic Weyl curvature (all these quantities are in the surfaces orthogonal to \( u^a \)), are identically zero in the LRS-II spacetimes. The set of remaining variables are

\[ \mathbf{LRS} \text{ class II} : \{ R, A, \Theta, \phi, \zeta, \Sigma \}, \]  
(37)

where \( \Theta \) the 3-volume rate of expansion, \( \Sigma \) is the component of shear parallel to \( e^a \) and \( \zeta \) is the component of the electric Weyl tensor, also parallel to \( e^a \). These quantities fully characterise the kinematics and dynamics of the LRS II spacetime and their dynamics, based on the Ricci and Bianchi identities, is governed by the following equations:

\[ \dot{\phi} = - \frac{1}{2} \phi^2 + \left( \frac{1}{3} \Theta + \Sigma \right) \left( \frac{2}{3} \Theta - \Sigma \right), \]  
(38)

\[ \dot{\Sigma} - \frac{2}{3} \dot{\Theta} = - \frac{3}{2} \phi \dot{\Sigma} - Q, \]  
(39)

\[ \dot{\zeta} - \frac{1}{3} \dot{\mu} + \frac{1}{2} \dot{\Pi} = - \frac{3}{2} \phi \left( E + \frac{1}{2} \Pi \right) + \left( \frac{1}{2} \Sigma - \frac{1}{3} \Theta \right) \]  
(40)

Propagation equations
Evolution equations

\[ \dot{\phi} = -\left( \Sigma - \frac{2}{3} \Theta \right) \left( A - \frac{1}{2} \phi \right) + Q, \quad (41) \]
\[ \dot{\Sigma} - \frac{2}{3} \dot{\Theta} = -A \phi + 2 \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right)^2 + \frac{1}{3} (\mu + 3p) - \mathcal{E} + \frac{1}{2} \Pi, \quad (42) \]
\[ \dot{\epsilon} - \frac{1}{2} \dot{\mu} + \frac{1}{2} \dot{\Pi} = \left( \frac{3}{2} \Sigma - \Theta \right) \mathcal{E} + \frac{1}{4} \left( \Sigma - \frac{2}{3} \Theta \right) \Pi + \frac{1}{2} \phi Q - \frac{1}{2} (\mu + p) \left( \Sigma - \frac{2}{3} \Theta \right). \quad (43) \]

Propagation/Evolution Equations

\[ \mu + \dot{Q} = -\Theta (\mu + p) - (\phi + 2A) Q - \frac{3}{2} \Sigma \Pi, \quad (44) \]
\[ \dot{Q} + \dot{p} + \dot{\Pi} = -\left( \frac{3}{2} \phi + A \right) \Pi - \left( \frac{4}{3} \Theta + \Sigma \right) Q - (\mu + p) A, \quad (45) \]
\[ \dot{A} - \dot{\Theta} = -(A + \phi) A + \frac{1}{3} \Theta^2 + \frac{3}{2} \Sigma^2 + \frac{1}{2} (\mu + 3p). \quad (46) \]

where

\[
\begin{align*}
\mu &= \frac{1}{f^2} \left[ \frac{1}{2} (Rf' - f - \theta f'' \hat{R} + f'''X^2 + f'' \hat{\Phi} + \phi f'' X) \right], \\
p &= \frac{1}{f^2} \left[ \frac{1}{2} (f - Rf') + f''' \hat{R} + f'' \hat{\Phi} - A f'' X \\
&\quad + \frac{2}{3} \left( \theta f'' \hat{R} - \phi f'' X - f''' X^2 - f'' \hat{\Phi} \right) \right], \\
Q &= -\frac{1}{f^2} \left[ f''' \hat{R} X + f'' \left( \hat{\Phi} - A \hat{R} \right) \right], \\
\Pi &= \frac{1}{f^2} \left[ \frac{1}{3} \left( 2f''' X^2 + 2f'' \hat{\Phi} X - \phi f'' X \right) - \Sigma f''' \hat{R} \right].
\end{align*}
\]

Commutation relation

\[ \hat{\psi} - \hat{\phi} = -A \hat{\phi} + \left( \frac{1}{3} \Theta + \Sigma \right) \hat{\psi}. \quad (47) \]

Due to the additional degrees of freedom, equations (48) are not closed and we have to add an additional equation that we label the trace equation:

\[ Rf' - 2f = 3 \left( f''' \hat{R} - f'' \hat{\Phi} + f''' \hat{R} - (\phi + A) f'' X - f''' X^2 + f'' \hat{R}^2 \right) \quad (48) \]

Since the vorticity vanishes, the Gauss equation for \( e^a \) together with the 3-Ricci identities determine the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to \( u^a \) to be [11]:

\[ 3R_{ab} = - \left[ \hat{\phi} + \frac{1}{2} \phi^2 \right] e_a e_b - \left[ \frac{1}{2} \phi + \frac{1}{2} \phi^2 - K \right] N_{ab}. \quad (49) \]

This gives the 3-Ricci-scalar as

\[ 3R = - 2 \left[ \frac{1}{2} \phi + \frac{3}{4} \phi^2 - K \right], \quad (50) \]

where \( K \) is the Gaussian curvature of the 2-sheet and is related to the two dimensional Riemann curvature tensor and two dimensional Ricci tensor as

\[ (2) R^a_{bca} = K (N^a c N_{bd} - N^a d N_{bc}), \quad \Rightarrow 2R_{ab} = K N_{ab}. \quad (51) \]

From (50) and (58) an expression for \( K \) is obtained in the form

\[ K = \frac{1}{3} \mu - \frac{1}{2} E - \frac{1}{4} \phi^2 - \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right)^2. \quad (52) \]

From (58) and (63), we obtain the evolution and propagation equations of \( K \) as

\[ \dot{K} = - \left[ \frac{2}{3} \Theta - \Sigma \right] K, \quad (53) \]
\[ \ddot{K} = - \phi K. \quad (54) \]

From equation (63), it follows that whenever the Gaussian curvature of the sheet is non-zero and constant in time, then the shear is always proportional to the expansion as

\[ K \neq 0 \quad \text{and} \quad \dot{K} = 0 \quad \Rightarrow \quad \Sigma = \frac{2}{3} \Theta. \quad (55) \]

V. SYMMETRIES

We know geometrically LRS-II space times have some inherent symmetries that lie on the 2-sheets. To investigate the extra symmetry of vacuum LRS-II space times for the modified theories, we follow [7], by trying to solve the Killing equation for a Killing vector of the form \( \xi_a = \Psi u_a + \Phi e_a \), where \( \Psi \) and \( \Phi \) are scalars. The Killing equation gives

\[ \nabla_a (\Psi u_b + \Phi e_b) + \nabla_b (\Psi u_a + \Phi e_a) = 0. \quad (56) \]

Using equations (129) and (132), and multiplying the Killing equation by \( u^a u^b, u^a e^b, e^a e^b \) and \( N^{ab} \) results in the following differential equations and constraints:

\[ \hat{\Psi} + A \hat{\Phi} = 0, \quad (57) \]
\[ \hat{\Psi} - \Psi \hat{A} + \Phi \left( \Sigma + \frac{1}{3} \Theta \right) = 0, \quad (58) \]
\[ \hat{\Phi} + \Psi \left( \frac{1}{3} \Theta + \Sigma \right) = 0, \quad (59) \]
\[ \Psi \left( \frac{2}{3} \Theta - \Sigma \right) + \Phi \phi = 0. \quad (60) \]
Considering that $\xi^a \xi_a = -\Psi^2 + \Phi^2$, if $\xi^a$ is timelike (that is, $\xi^a \xi_a < 0$), then because of the arbitrariness in choosing the vector field $\xi^a$, we can always make $\Phi = 0$, while if $\xi^a$ is spacelike (that is $\xi^a \xi_a > 0$), then we can make $\Psi = 0$.

Let us first assume that $\xi^a$ is timelike and $\Phi = 0$. Looking at equations (57) and (58), we know that their solutions always exist. For a non trivial $\Psi$, the constraints (59) and (60) together imply, that in general $\Theta = \Sigma = 0$, that is, the expansion and shear of a unit vector field along the timelike Killing vector vanishes. We also see that the time derivatives of all the quantities in the field equations (38)-(48) vanish and hence the spacetime is static. If the Killing vector is time-like, with $\xi^a \xi_a = 0$, then because of the arbitrariness in choosing the vector field $\xi^a$, we can always make $\Phi = 0$, while if $\xi^a$ is spacelike (that is $\xi^a \xi_a > 0$), then we can make $\Psi = 0$.

The local Gaussian curvature of the 2-sheets in this case (38)-(48) vanish and hence the spacetime is static.

Now if $\xi^a$ is spacelike and $\Psi = 0$, then we see in this case that solution of equations (58) and (59) always exists and the constraints (57) together imply that in general, (for a non trivial $\Phi$), $\phi = A = 0$. If we impose further the condition,

$$R = R_0 = \text{const.} \quad \text{and} \quad f_0' \neq 0,$$

which in turn implies

$$\mu = 0,$$
$$\Pi = \frac{1}{f_0} \left[ \frac{1}{2}R_0 f_0' - f_0 \right],$$

$$p = \frac{1}{f_0} \left[ \frac{1}{2}(f_0 - R_0 f_0') \right],$$

$$R_0 f_0' - 2 f_0 = 0,$$

where $f'(R_0) = f_0'$, then all the spatial derivatives of all the quantities in (38)-(48) vanish. From this we see that homogeneity is only achieved if $R = \text{constant}$, otherwise inhomogeneity is admitted for non-constant $R$. This result is unlike that of GR where the spacetime is spatially homogenous upon setting $\phi = A = 0$ in the list of LRS II equations.

We can now say that: **There always exists a Killing vector in the local $[u,e]$ plane for a vacuum LRS-II spacetime in $f(R)$ gravity.** If the Killing vector is timelike then the spacetime is locally static. If the Killing vector is spacelike, the spacetime is locally spatially homogeneous if and only if $R = R_0 = \text{const.} \quad \text{and} \quad f_0' \neq 0$.

### VI. JEBSEN-BIRKHOFF LIKE THEOREM IN $f(R)$ GRAVITY

If we apply the conditions (61), to the system of equations (35)-(48), then from (40), (43), (53) and (54) we get

$$\mathcal{E} = C K^{3/2}.$$

That is, the $1+1+2$ scalar of the electric part of the Weyl tensor is always proportional to the $(3/2)$th power of the Gaussian curvature of the 2-sheet. The proportionality constant $C$ sets up a scale in the problem in this particular case.

Furthermore, when $\Theta = \Sigma = 0$, we have $\dot{K} = 0$. We choose coordinates to make the Gaussian curvature $K$ of the spherical sheets proportional to the inverse square of the radius co-ordinate $r'$, (such that this coordinate becomes the *area radius* of the sheets), then this geometrically relates the ‘hat’ derivative with the radial co-ordinate $r$. Using (54), (16) and the definition of $\phi$ we can then define the hat derivative of any scalar $\mathcal{M}$ as

$$\hat{\mathcal{M}} = \frac{1}{2}r\phi \frac{d\mathcal{M}}{dr}$$

for a static spacetime.

If we take $R = R_0 = 0$, $f(0) = 0$ and $f_0' \neq 0$, the equations (38)-(48) reduce to

$$\dot{\phi} = -\frac{1}{2} \phi^2 - \mathcal{E},$$
$$\dot{\mathcal{E}} = -\frac{3}{2} \phi \mathcal{E},$$

together with the constraint:

$$\mathcal{E} + \mathcal{A} \phi = 0.$$

The local Gaussian curvature of the 2-sheets in this case becomes,

$$K = -\mathcal{E} + \frac{1}{4} \phi^2.$$

The parametric solutions for these variables (when $K > 0$) are

$$\phi = \frac{2}{r} \sqrt{1 - \frac{2m}{r}}, \quad \mathcal{A} = \frac{m}{r^2}, \quad \left[ 1 - \frac{2m}{r} \right]^{-\frac{3}{2}}$$
$$\mathcal{E} = \frac{2m}{r^2}, \quad K = \frac{1}{r^2},$$

where $m$ is the constant of integration.

Solving for the metric using the definition of these geometrical quantities we get [11]

$$ds^2 = -\left( 1 - \frac{2m}{r} \right) dt^2 + \frac{dr^2}{(1 - \frac{2m}{r})} + r^2 d\Omega^2,$$

which is the metric of a static Schwarzschild exterior. A similar derivation can be done for the case of space-like Killing vector and vanishing Ricci scalar, to get the Schwarzschild interior metric.

We can now give a generalisation of the Jebsen-Birkhoff-like theorem in $f(R)$ gravity:

For $f(R)$ gravity, where the function $f$ is of class $C^3$ at $R = 0$, with $f(0) = 0$ and $f'(0) \neq 0$, the only spherically symmetric solution with vanishing Ricci scalar in empty space in an open set $S$, is one that is locally equivalent to part of maximally extended Schwarzschild solution in $S$.

It is also interesting to note here that the covariant scale defined by equation (66) is equal to the Schwarzschild mass $m$. 
VII. SPHERICALLY SYMMETRIC SPACETIMES WITH AN ALMOST VANISHING RICCI SCALAR

From the previous section we know that for $f(R)$ gravity with $R = 0$, $f(0) = 0$ and $f''_0 \neq 0$, all spherically symmetric vacuum spacetimes are locally isomorphic to a part of Schwarzschild solution. In [5], the vacuum LRS II spacetime was perturbed by putting in a small amount of general matter that obeys weak and dominant energy conditions, to find out the amount of matter that can be introduced to the spacetime for the Jepsen-Birkhoff theorem to remain approximately true. Analogously, we investigate here the necessary conditions on the magnitude and spatial and temporal derivatives of the Ricci scalar, for the above theorem to remain approximately true. In this section we only deal with the static exterior background as it is astrophysically more interesting.

We have seen that the vacuum spherically symmetric spacetime with vanishing Ricci scalar has a covariant scale given by the Schwarzschild radius which sets up the scale for perturbation. Going by our description of the energy momentum tensor for vacuum LRS II spacetime in $f(R)$ gravity as consisting of curvature terms $\mu^R$, $p^R$, $\Pi^R$ and $Q^R$ and taking a static Schwarzschild background, then the set $\{R, \Theta, \Sigma\}$ describe the first-order quantities (and are gauge-invariant according to the Stewart and Walker lemma [17]). Performing a series expansion of $f(R)$ in the neighbourhood of $R = 0$ gives $f(R) = f'_0 R$ as a first-order term. Neglecting the higher order quantities in (28-34) we get the following equations

$$
\mu = \frac{f''}{f'_0} \left( \hat{X} + \phi X \right), \quad (75)
$$

$$
p = \frac{f'_0}{f'_0} \left( \hat{R} - AX - \frac{2}{3} \phi X - \frac{2}{3} \hat{X} \right), \quad (76)
$$

$$
Q = - \frac{f''}{f'_0} \left( \hat{X} - A \hat{R} \right), \quad (77)
$$

$$
\Pi = \frac{f'_0}{3f'_0} \left( 2\hat{X} - \phi X \right), \quad (78)
$$

and

$$
R f'_0 = 3f''_0 \left( \hat{X} + (A + \phi)X - \hat{R} \right) \quad (79)
$$

for the trace. Thus we see that by perturbing the Ricci scalar in the neighbourhood of $R = 0$ background, we are actually generating a ‘curvature fluid’ having the above thermodynamic quantities. Therefore the situation here is similar to introducing small amount of matter on a Schwarzschild background in GR. In [5] the sufficient conditions for the smallness of these matter perturbations in order for the spacetime to remain almost Schwarzschild are given. These conditions in our case become

$$
\begin{bmatrix}
\frac{|\hat{R}|}{K^{(3/2)}}, & \frac{f''_0}{f'_0 K^{(3/2)}}, & \frac{|\hat{X}|}{f'_0 K^{(3/2)}} \\
\frac{f''_0}{f'_0 K^{(3/2)}}, & \frac{|\hat{R}|}{K^{(3/2)}}, & \frac{f''_0}{f'_0 K^{(3/2)}} \\
\frac{|\hat{X}|}{f'_0 K^{(3/2)}}, & \frac{f''_0}{f'_0 K^{(3/2)}}, & \frac{|X|}{K^{(3/2)}}
\end{bmatrix} \ll C, \quad (80)
$$

and

$$
\begin{bmatrix}
\frac{f''_0^3/2}{K^{(3/2)}}, & \frac{f''_0^3/2}{K^{(3/2)}}, & \frac{|\hat{X}|}{f'_0 K^{(3/2)}} \\
\frac{f''_0^3/2}{K^{(3/2)}}, & \frac{f''_0^3/2}{K^{(3/2)}}, & \frac{|X|}{K^{(3/2)}}
\end{bmatrix} \ll C. \quad (81)
$$

Similarly to [5], we also need to specify in what domain these equations will hold. This is important because eventually we will reach a radius $r$ where these inequalities may no longer hold. On the basis that in the real universe asymptotically flat regions are always of finite size, we will describe the local domain where our results will apply by [7],

- Defining finite infinity $\mathcal{F}$ as a 2-sphere of radius $R_F \gg C$ surrounding the star: this is infinity for all practical purposes [18, 19].
- Assuming the relations [80], [81] hold in the domain $D_F$ defined by $r_S < r < R_F$ where $r_S > C$ is the radius of the surface of the star.

We now linearise the field equations [38-48] by neglecting the higher order quantities and we obtain the following equations for the first-order quantities

$$
\Sigma - \frac{2}{3} \hat{\Theta} \approx - \frac{3}{2} \phi \Sigma + \frac{f''_0}{f'_0} \left( \hat{X} - A \hat{R} \right), \quad (82)
$$

$$
\hat{\Theta} \approx - \frac{f''_0}{2f'_0} \left( 3 \hat{R} - \hat{X} - (3A + \phi)X \right), \quad (83)
$$

$$
\Sigma - \frac{2}{3} \hat{\Theta} \approx \frac{f''_0}{f'_0} \left[ \hat{R} - X \left( A + \frac{1}{2} \phi \right) \right], \quad (84)
$$

$$
\phi \approx \left( \Sigma - \frac{2}{3} \hat{\Theta} \right) \left( A - \frac{1}{2} \phi \right) - \frac{f''_0}{f'_0} (\hat{X} - A \hat{R}), \quad (85)
$$

$$
\hat{\phi} \approx \left( \Sigma - \frac{2}{3} \hat{\Theta} \right) \left( A - \frac{1}{2} \phi \right) - \frac{f''_0}{f'_0} (\hat{X} - A \hat{R}), \quad (86)
$$

$$
\frac{1}{3} R f'_0 \approx \frac{f''_0}{f'_0} \hat{X} - f''_0 \hat{R} + (\phi + A) f''_0 X. \quad (87)
$$

From these equations we can see that if [80] and [81] are locally satisfied at any epoch, within the domain $D_F$, then the spatial and temporal variation of the expansion $\Theta$ and the shear $\Sigma$ are of same order of smallness as the perturbations and derivatives of the Ricci scalar. In that case a timelike vector will not exactly solve the Killing
we get
\[ \nicefrac{1}{3} - \frac{1}{2} \Xi \] approximately. To see this explicitly, let us set \( \Phi = 0 \) in the Killing equation \[ (88) \]
and we once again try to solve the equation for a Killing vector of the form \( \xi_a = \Psi u_a \) with an aim to see how close the \( \xi_a \) is to Killing vector in the perturbed scenario.

We consider the scalars constructed by multiplying the Killing equation by the vectors \( u^a, e^a \), the projection tensor \( N^{ab} \) and utilise equation \[ (110) \] and \[ (22) \] to facilitate the calculation. We know that multiplying the Killing equation by \( u^a u^b \) and \( e^a e^b \) results in equations for which the solution of the scalar \( \Psi \) always exists. The constraints obtained from multiplying the Killing equation by \( e^a e^b \) and \( N^{ab} \) only vanish if \( \Theta = \Sigma = 0 \), however, we are considering here the perturbed case which is characterised by non-zero \( \Theta \) and \( \Sigma \). As a result not all the equations are completely solved in general. If we set up \[ (88) \] as a symmetric tensor
\[ K_{ab} := \nabla_a (\Psi u_b) + \nabla_b (\Psi u_a) \] (89)
we can instead say that there always exists a non-trivial solution of the scalar \( \Psi \) for which \( |K_{ab} u^a u^b| \) and \( |K_{ab} e^a e^b| \) vanish and that \( |K_{ab} e^a e^b| \) and \( |K_{ab} N^{ab}| \) are non-zero since \( \Theta \) and \( \Sigma \) are non-zero. However, if the conditions
\[ \left| \frac{K_{ab} u^a u^b}{K^{3/2}} \right|, \left| \frac{K_{ab} e^a e^b}{K^{3/2}} \right|, \left| \frac{K_{ab} N^{ab}}{K^{3/2}} \right| < C \] (90)
are satisfied, then we can say that \( \xi_a = \Psi u_a \) is close to a Killing vector and that the spacetime is approximately static.

Subtracting the background equation \[ (77) \] from \[ (22) \], we get
\[ \left( \frac{1}{3} - \frac{1}{2} \Xi \right)^2 \approx \frac{f''}{2f_0} \phi \chi . \] (91)
Similarly subtracting \[ (68) \] from \[ (88) \] we get
\[ \left( \frac{1}{3} \Theta + \Sigma \right) \left( \frac{2}{3} \Theta - \Sigma \right) \approx \frac{f''}{2f_0} \left( 2 \chi + \phi \chi \right) . \] (92)
Using the above equations \[ (71) \] and \[ (72) \], we immediately see that if \[ (80) \] is locally satisfied, then the following conditions are satisfied
\[ |K_{ab} e^a e^b| = \Psi^2 \left( \frac{1}{3} \Theta + \Sigma \right)^2 \ll C K^{3/2} , \] (93)
\[ |K_{ab} N^{ab}| = \Psi^2 \left( \frac{2}{3} \Theta - \Sigma \right)^2 \ll C K^{3/2} . \] (94)
It follows that there always exists a timelike vector that satisfies \[ (81) \]. This vector almost solves the Killing equations in the open set \( S \) in the domain \( D_F \) and hence the spacetime is almost static in \( S \). Moreover, the resultant field equations are the zeroth-order equations \[ (68)-(70) \] with an addition of \( O(\epsilon) \) terms.

**VIII. ALMOST SPHERICALLY SYMMETRIC SPACETIME WITH VANISHING RICCI SCALAR**

In order to geometrically define an almost spherically symmetric spacetime, we begin by writing the geodesic deviation equation for a family of closely spaced geodesics on the 2-sheets with tangent vectors \( \psi^a(v) \) and separation vectors \( \eta^a(v) \) (where \( 'v' \) is the parameter which labels the different geodesics) as \[ (8) \],
\[ \psi^a \delta_{bc} (\psi^b \delta_f \eta^a) = K (\psi^a \psi_d \eta^d \eta^b - \eta^a \psi_c \psi^c) . \] (95)
We have used here the definition of the two dimensional Riemann curvature tensor equation \[ (51) \].

We now define a vector \( V^a \) by
\[ V^a = \psi^c \delta_{bc} (\psi^b \delta_f \eta^a) - K_0 (\psi^a \psi_d \eta^d \eta^b - \eta^a \psi_c \psi^c) , \] (96)
where \( K_0 \) is the Gaussian curvature for a spherical sheet at any point \( P \), which can be fixed by making the vector \( V^a = 0 \) at that point. This vector vanishes for exact spherical 2-sheets in any open neighbourhood of \( P \) but doesn’t for non-spherical sheets. As a result, from the magnitude of \( V^a (= \sqrt{V_a V_a}) \) we obtain a covariant measure of the deviation from the spherical symmetry.

We can now define an almost spherically symmetric spacetime in following way \[ (95) \]:

Any \( C^3 \) spacetime with positive Gaussian curvature everywhere, which admits a local \( 1+1+2 \) splitting at every point is called an almost spherically symmetric spacetime, if and only if the following quantities are either zero or much smaller than the scale defined by the modulus of the proportionality constant in equation \[ (67) \] :

- The magnitude of all the 2-vectors (defined by \( \sqrt{\psi_a \psi^{ab}} \)) and PSTF 2-tensors (defined by \( \sqrt{\psi_{ab} \psi^{ab}} \))
- The magnitude of vector \( V^a \) defined above in \[ (70) \].

We have seen that subject to the conditions \[ (80) \] and \[ (81) \] on any spherically symmetric local domain \( D_F \), the spacetime remains “almost” Schwarzschild for all the \( f(R) \)-theories that admit a Schwarzschild background, (that is, a background characterised by a vanishing Ricci scalar with \( f(0) = 0 \) and \( f_0' \neq 0 \)). We now wish to see to what extent the conditions hold when we perturb this geometry.

As previously stated, the sheet will be a genuine two surface if and only if the commutator of the time and hat derivative do not depend on any sheet component and the sheet derivatives commute in \[ (23) \] and \[ (20) \]. Following from the definition of almost spherical symmetry, in the perturbed scenario we will require the sheet to be an almost genuine 2-surface such that the commutator of the time and hat derivative almost do not depend on any sheet component and the sheet derivatives almost commute. In that case we see from \[ (23) \] and \[ (20) \] that the scalars \( \Omega \) and \( \xi \) will be of the same order of smallness.
as the other vectors and PSTF 2-tensors on the sheet. Furthermore, from the constraint equation
\[ \delta_a \Omega^a + \varepsilon_{abc} \Sigma^b = (2A - \phi) \Omega - 3\xi \Sigma + \varepsilon_{abc} \Sigma^b c + \mathcal{H}, \]
we see that the scalar \( \mathcal{H} \) is of the same order of smallness. Dealing once again with the static exterior background, we now have it that the set of 1+1+2 variables
\[ \begin{align*}
R, \Theta, \Sigma, \Omega, \mathcal{H}, \xi, A^a, \Omega^a, \Sigma^a, \alpha^a, \\
a^a, E^a, \Omega^c, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}, \zeta_{ab}, \end{align*} \]
are all of \( \mathcal{O}(\varepsilon) \) with respect to the invariant scale. We shall treat these variables along with their derivatives and the dot - ‘’ and delta - ‘’ derivatives of \{\( A, E, \phi \)\} as first-order relative to the background terms.

Performing a series expansion of \( f(R) \) in the neighbourhood of \( R = 0 \) and linearising by neglecting all products of first-order quantities in \([28]-[34]\), we obtain
\[ \mu \approx \frac{f''}{f_0} \left( \dot{X} + \phi \dot{X} + \dot{\delta}^2 R \right), \]
\[ p \approx \frac{f''}{f_0} \left[ \dot{R} - A X - \frac{2}{3} \left( \phi X + \dot{X} + \dot{\delta}^2 R \right) \right], \]
\[ Q \approx -\frac{f''}{f_0} \left( \dot{X} - A \dot{R} \right), \]
\[ Q_a \approx -\frac{f''}{f_0} \delta_a \dot{R}, \]
\[ \Pi \approx \frac{f''}{f_0} \left( 2 \dot{X} - \phi X - \dot{\delta}^2 R \right), \]
\[ \Pi_a \approx \frac{f''}{f_0} \left( \delta_a X - \frac{1}{2} \phi \delta_a \right), \]
\[ \Pi_{ab} \approx \frac{f''}{f_0} \delta_{(a} \delta_{b)} R. \]

Linearising the field equations (48-81) in \([12]\) and substituting in equations \([49]-[105]\) we obtain:

**Evolution equations**

The evolution equations for \( \xi \) and \( \zeta_{(ab)} \) are:
\[ \dot{\xi} = \left( A - \frac{1}{2} \phi \right) \Omega + \frac{1}{2} \varepsilon_{abc} \alpha^b + \frac{1}{2} \mathcal{H}, \]
\[ \dot{\zeta}_{(ab)} = \left( A - \frac{1}{2} \phi \right) \Sigma_{ab} + \delta_{(a} \mathcal{H}_{b)} c + \varepsilon_{c(a} \delta_{b)} \]

Vorticity evolution equation:
\[ \dot{\Omega} = \frac{1}{2} \varepsilon_{abc} A^b + A \xi, \]
\[ \dot{\Omega}_a + \frac{1}{2} \varepsilon_{abc} \dot{A}^b = \frac{1}{2} \varepsilon_{abc} \left( \delta^b A - A^b - \frac{1}{2} \phi A^b \right); \]

**Shear evolution:**
\[ \hat{\Sigma}_a = \frac{1}{2} \hat{A}_a = \frac{1}{2} \delta_a A + \left( A - \frac{1}{4} \phi \right) A_a + \frac{1}{2} \mathcal{A}_a - \mathcal{E}_a \]
\[ + \frac{f''}{f_0} \left( \delta_a X - \frac{1}{2} \phi \delta_a R \right), \]
\[ \hat{\Sigma}_{(ab)} = \delta_{(a} A_{b)} + A \delta_{ab} - \mathcal{E}_{ab} + \frac{f''}{f_0} \delta_{(a} \delta_{b)} R; \]

**Magnetic Weyl evolution:**
\[ \dot{\mathcal{H}} = -\varepsilon_{abc} \mathcal{E}^b \delta \xi, \]
\[ \dot{\mathcal{H}}_a = -\frac{3}{4} \xi \varepsilon_{abc} A^b - \frac{1}{2} \varepsilon_{abc} \mathcal{E} - \frac{1}{2} (\phi - 2A) \varepsilon_{abc} \mathcal{E}^b \]
\[ + \varepsilon_{c(a} \delta_{b)} \mathcal{E} \mathcal{E} - \mathcal{E} \frac{f''}{f_0} \varepsilon_{abc} \mathcal{E} \mathcal{E} \]
\[ \dot{\mathcal{H}}_{(ab)} + \varepsilon_{c(a} \delta_{b)} \mathcal{E} c + \frac{3}{2} \xi \varepsilon_{c(a} \delta_{b)} \mathcal{E} c \]
\[ - \left( \frac{1}{2} \phi + \frac{1}{2} A \right) \varepsilon_{c(a} \delta_{b)} \mathcal{E} c; \]

**Electric Weyl evolution:**
\[ \dot{\mathcal{E}}_a + \frac{1}{2} \varepsilon_{abc} \hat{H}^b = \frac{3}{4} \xi \varepsilon_{abc} \mathcal{E} - \frac{1}{4} \phi - 2A \]
\[ + \frac{3}{4} \varepsilon_{abc} \mathcal{E} \]
\[ - \left( \frac{1}{2} \phi + \frac{1}{2} A \right) \varepsilon_{c(a} \delta_{b)} \mathcal{E} c, \]
\[ \dot{\mathcal{E}}_{(ab)} - \varepsilon_{c(a} \delta_{b)} \mathcal{E} \]
\[ - \varepsilon_{c(a} \delta_{b)} \mathcal{E} - \frac{3}{2} \xi \varepsilon_{ab} \]
\[ + \left( \frac{1}{2} \phi + \frac{1}{2} A \right) \varepsilon_{c(a} \delta_{b)} \mathcal{E} c; \]

**Evolution equation for \( \hat{\varepsilon}_a \):**
\[ \hat{\varepsilon}_a - \hat{\varepsilon}_a = \left( \frac{1}{2} \phi + \frac{1}{2} A \right) \alpha_a - \left( \frac{1}{2} \phi - A \right) \left( \Sigma_a + \varepsilon_{ab} \mathcal{E}^a \right) \]
\[ + \varepsilon_{abc} \mathcal{E} + \frac{f''}{f_0} \delta_{a} \delta_{b} R. \]

**Propagation equations**
\[ \dot{\zeta} = -\phi \xi + \frac{1}{2} \varepsilon_{abc} \mathcal{E} \]
\[ \dot{\zeta}_{(ab)} = -\phi \zeta + \delta_{(a} \delta_{b)} \mathcal{E}_{ab} - \frac{f''}{f_0} \delta_{(a} \delta_{b)} R; \]

**Shear divergence:**
\[ \hat{\Sigma}_a - \varepsilon_{abc} \hat{L}^b = \frac{1}{2} \delta_a \Sigma + \frac{2}{3} \delta_a \theta - \varepsilon_{abc} \mathcal{E} \]
\[ - \frac{3}{2} \phi \Sigma_a - \delta^a \Sigma_{ab} \]
\[ + \left( \frac{1}{2} \phi + 2A \right) \varepsilon_{abc} \mathcal{E} + \frac{f''}{f_0} \delta_{a} \delta_{b} R. \]
\[ \hat{\Sigma}_{(ab)} = \delta_{[a} \Sigma_{b]} - \varepsilon c_{[a} \delta^b \Omega_{b]} - \frac{1}{2} \phi \Sigma_{ab} - \varepsilon c_{(a} \mathcal{H}_{b)} \varepsilon \]  

(121)

Vorticity divergence equation:

\[ \hat{\Omega} = - \delta_a \Omega^a + (A - \phi) \Omega \]  

(122)

Electric Weyl Divergence:

\[ \hat{\mathcal{E}}_a = \frac{1}{2} \delta_a \mathcal{E} - \delta^b \mathcal{E}_{ab} - \frac{3}{2} \mathcal{E} a_a - \frac{3}{2} \phi \mathcal{E}_a + \mathcal{E} \frac{f_0''}{f_0} \delta_a R \]  

(123)

Magnetic Weyl divergence:

\[ \hat{\mathcal{H}} = - \delta_a \mathcal{H}^a - \frac{3}{2} \phi \mathcal{H} - 3 \mathcal{E} \Omega \]  

(124)

\[ \hat{\mathcal{H}}_a = \frac{1}{2} \mathcal{H} - \delta^b \mathcal{H}_{ab} + \frac{3}{2} \mathcal{E} (\Omega_a - \varepsilon_{ab} \Sigma^b) - \frac{3}{2} \phi \mathcal{H}_a \]  

(125)

Finally we have the linearised curvature trace equation

\[ \frac{1}{3} R = \frac{f''}{f_0} \left[ \tilde{X} - \tilde{R} + (\phi + \mathcal{A}) X + \delta^2 R \right] . \]  

(126)

From the evolution equations [106] - [117], it is evident that if the background is static with \( \Sigma = \Theta = 0 \) or “almost static” with \( \Sigma = \Theta = \mathcal{O}(\epsilon) \), the time derivatives of the first-order quantities at a given point are all of the same order of smallness as the variables themselves. Hence if at a given epoch these quantities are of order \( \mathcal{O}(\epsilon) \), then there exists an open set \( S \) in the domain \( D_F \) where these quantities continue to be of the same order.

This time if we project the Killing equation [55] for a Killing vector of the form \( \xi_a = \Psi u_a \), with \( N^a_c u^b \), \( N^c e^b \) and \( N^a N^b_d \), we obtain the following additional constraints on the 2-sheet:

\[ - \delta_c \Psi + \Psi \mathcal{A}_c = 0 , \]  

(127)

\[ \Psi \Sigma_c = 0 , \]  

(128)

\[ \Psi \Sigma_{cd} = 0 . \]  

(129)

The solution of [127] always exists and as we have just seen, the LHS of equations [128] and [129] remain \( \mathcal{O}(\epsilon) \) in \( S \). Hence a timelike vector almost solves the Killing equations, making the spacetime almost static.

 IX. LOCAL STABILITY OF JEBSEN-BIRKHOFF LIKE THEOREM

Let us now combine the results obtained in the previous two sections. Consider any \( f(R) \) theory of gravity which admits a Schwarzschild background and consider the following sets of scalars:

\[
\begin{bmatrix}
\frac{|R|}{K^{(3/2)}}, & \frac{f''(1/2)}{K^{(3/2)}}, & \frac{f'''}{K^{(3/2)}}, & \frac{f''(1/2)}{K^{(3/2)}}, & \frac{f''}{K^{(3/2)}}, & \frac{f'}{K^{(3/2)}}, & \frac{f''}{K^{(3/2)}}, & \frac{f''}{K^{(3/2)}}
\end{bmatrix}.
\]  

(130)

If these scalars locally satisfy [80], [81] and their sheet derivatives are of same order of smallness as themselves at any epoch within the domain \( D_F \), then there exists an open set \( S \) in \( D_F \) where the conditions continue to hold. Consequently then there will exist a timelike/spacelike vector that almost solves the Killing equations in \( S \) and hence the spacetime will be “almost” Schwarzschild. Hence we have demonstrated an important result: For \( f(R) \) gravity, where the function \( f \) is of class \( C^3 \) at \( R = 0 \), with \( f(0) = 0 \) and \( f'(0) \neq 0 \), any almost spherically symmetric solution with almost vanishing Ricci scalar in empty space in an open set \( S \), is locally almost equivalent to part of maximally extended Schwarzschild solution in \( S \).

We would like to emphasise here that the size of the open set \( S \) depends on the parameters of theory (namely the quantity \( f''(0) \)) and the covariant scale (which is the Schwarzschild mass of the star) and we can always tune the parameters of the theory such that the perturbations continue to remain small for a time period which is greater than the age of the universe. The above result shows that the local spacetime around almost spherical stars will be stable in the regime of linear perturbations in these modified gravity theories.

X. DISCUSSION

In this paper we used the 1+1+2 covariant perturbation formalism to prove a Jebsen-Birkhoff like theorem for \( f(R) \) theories of gravity in order to determine the conditions required for the existence of the Schwarzschild solution in these theories. We then discussed under what circumstances we can covariantly set up the fundamental scale in the problem and perturbed the vacuum spacetime with respect to this scale to find the stability of the theorem.

What emerges from this analysis is the important result that there exists a non-zero measure in the parameter space of \( f(R) \) theories of gravity for which the Jebsen-Birkhoff like theorem remains stable under generic perturbations. This result applies locally and therefore does not depend on specific boundary conditions used for solving the perturbation equations. A detailed analysis of generic linear perturbations of the Schwarzschild solution will be presented elsewhere, which supports the work presented in this paper.
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