PROCESS, POPULATION, AND SAMPLE:
THE RESEARCHER’S INTEREST

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Abstract. A case is made that researchers are interested in studying processes. Often the inferences they are interested in making are about the process and its associated population. On other occasions, a researcher may be interested in making an inference about the collection of individuals the process has generated. We will call the statistical methods employed by the researcher to make such inferences about the process/population “estimation methods.” The statistical methods used in making an inference about the collection of individuals generated we call “prediction methods.” Methods for obtaining interval estimates of a parameter and prediction intervals for a statistic are given. The analytical and enumerative methods discussed in Deming (1953) are simply estimation and prediction methods, respectively.

1. Introduction

Researchers draw inferences from data. Data is obtained from taking measurements on individuals. Immediately, one is faced with the fundamental question, “Where do these individuals come from?” The individuals are created by a repetative process—natural, human made, or some combination thereof. Many authors refer to the set of all individuals that were created by a process as the “population of interest,” but we regard this view as incorrect, or at least incomplete. To begin to make our case, we return to the thinking of R. A. Fisher.

Fisher believed statistics to be a branch of applied mathematics. He stated in Fisher (1958, p. 1): “[t]he science of statistics is essentially a branch of Applied Mathematics, and may be regarded as mathematics applied to observational data. As in other mathematical studies, the same formula is equally relevant to widely different groups of subject-matter. Consequently the unity of the different applications had usually been overlooked, the more naturally because the development of the underlying mathematical theory had been much neglected.” It was not clear what he meant by “applied” mathematics, and indeed there is no universally agreed upon definition of “applied mathematics.” Nonetheless, “statistical thinking” would then be just a special case of “mathematical thinking” according to him. Fisher (1958) further regarded statistics as “(i) the study of populations, (ii) as the study of variation,” and “(iii) as the study of methods of the reduction of data.”

A variety of researchers and practitioners make use of statistical methods to analyze their data. These applications include, among others, environmental, pharmaceutical, medical, legal, biological, political, and industrial data. The techniques used to analyze these data range from simple graphical representations of data to rather involved formal statistical methods. At the heart of

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these applications, is a \textit{process} and associated \textit{population} of interest to the researcher. The complete information about the process and population will never be available to the researcher. The information the researcher will have available is always partial information about the process and population in the form of a finite subcollection of the population called a \textit{sample}. In this article, we will discuss the meaning of a population and sample. Further, we will examine and explicitly differentiate two statistical methods of inference, estimation and prediction. We will argue that the two areas of statistical inference, called \textit{analytical} and \textit{enumerative} in Deming (1953) are, respectively, estimation and predication as discussed in this article.

\section*{2. Population and Sample}

An individual to be measured by a researcher is one that has been produced by a repetitive process. This process may be natural, human made, or a combination of the two. The collection of individuals the process has generated (actual individuals) or could have generated (conceptual individuals) is the researcher’s \textit{population} of interest. We assert that the collection of conceptual individuals must be not only an infinite collection (as stated by Fisher, quoted below), but is inherently an \textit{uncountably infinite} collection. Accordingly, the population (as it contains both hypothetical individuals and a finite number of actual individuals produced by the process) is also an uncountably infinite collection of individuals. To see this, consider the following example:

\textbf{Example 1}. Consider the simple experiment of tossing a coin once, and observing whether the coin lands heads up or tails up. The population consists of all possible coin tosses. For simplicity, suppose that we have one person designated as the coin tosser. Now the coin tosser can stand (or sit) in any of a range of positions. Considering the floor to be a two-dimensional surface, there are already uncountably infinitely many standing/sitting locations even within a small area in the room. The height above the floor from which the coin is tossed can also be an interval of values, and again, any interval of positive length in the real numbers contains uncountably many points. There are also slight variants in the angle at which the coin is launched, the amount of force applied in the launch of the coin, and even variants in the air currents through which the coin will travel once released. All of these reinforce the idea of an uncountably infinite number of hypothetical individuals (coin tosses) contained in the population.

Having named the collection of all actual and hypothetical individuals associated with a given process as the population, we need a different name for the (finite) collection of actual individuals the process has generated. This collection of actual individuals is a sample representative of what the process can generate, so let us call it the \textbf{representative sample}. This representative sample is a finite collection of some cardinality, say $N$. Again, many authors refer to the representative sample as “the population,” but we believe this to be incorrect for a number of reasons. This collection generated by the process is just \textit{one} of an \textit{uncountable collection of samples of size $N$ that the process could have generated}. One might refer to this latter collection as the \textbf{population of representative samples}. The uncertainty as to \textbf{which} representative sample the process has generated \textit{must be taken into account}, and will inevitably be ignored if we view the representative sample as the \textbf{“(finite) population” of interest}. Note that the size $N$ of the representative sample is in fact a random variable. A subcollection of the representative sample of size $n$, where $1 \leq n \leq N$, selected by the researcher is what we shall call the \textbf{researcher’s sample}. The sampling methods
available to the researcher at best gives the researcher a good chance of obtaining a sample that is representative of the representative sample. The researcher’s sample is a census if \( n = N \).

It is not difficult to argue via examples that a researcher is interested in making inferences about a process and the associated population or about the representative sample the process has generated. Most often researchers are interested in making inferences about a process or population. Further, it is usually of interest to study the (joint) distribution of one (or more) variable(s) and/or measurement(s) to be taken on individual members of the population. This distribution often characterizes the process and population in a way that is of interest to the researcher. The measurement(s) taken on an individual is sometimes referred to as an observation on the individual. Although the collection of observations on the individuals of a population are also often referred to as the population (c.f. Fisher (1958)), for clarity, we shall carefully draw a distinction between the individuals and their associated measurement(s).

To emphasize again the importance of regarding both hypothetical and actual individuals as the population of interest, we present an example where the population consists entirely of conceptual individuals:

**Example 2.** Suppose a researcher is interested in comparing treatments A and B. The population of interest is adult women. Presently, no adult woman is receiving treatment A or treatment B. Associated with treatment A is the hypothetical population of adult women who would be receiving treatment A. There is also a similar hypothetical population of adult women who would be receiving treatment B. At the beginning of the study, both of these populations contain only conceptual individuals. The representative sample is empty. So how does the researcher obtain samples from these two populations? Simply take a sample from the representative sample of adult women. Split this sample into two subsamples. The subsample that receives treatment A is a sample from the population of adult women who would be receiving treatment A. Similarly, for treatment B. In both cases, the researcher’s sample is the representative sample. As strange as it may seem, each of the researcher’s samples constitute a census.

Let us consider another example.

**Example 3.** A pollster is interested in the proportion of individuals who will say they will vote for a given presidential candidate one month before the actual election. At that point in time there will be a finite number of voters. So how does one view this collection as a representative sample of what some process has produced and not the population of interest? What if the election to be studied by the pollster is to occur four years from now? The collection of voters that will exist one month before the election date has yet to be (completely) generated. Consequently, this collection of voters can only be viewed as a representative sample of what the process can generate one month before the election as well as at any time. It then follows that a census occurs when the measurement(s) of interest is taken on each individual in the representative sample.

Fisher (1958, p. 33) states: “[w]hen a large number of individuals are measured in respect of physical dimensions, weight, colour, density, etc., it is possible to describe with some accuracy the population of which our experience may be regarded as a sample. By this means it may be possible to distinguish it from other populations differing in their genetic origin, or in environmental circumstances.” Further, Fisher (1958, p. 41) states the following about population and frequency
distribution. “The idea of an infinite population distributed in a frequency distribution in respect of one or more characters is fundamental to all statistical work. From a limited experience, for example, of individuals of a species, or of the weather of a locality, we may obtain some idea of the infinite hypothetical population from which our sample is drawn, and so of the probable nature of future samples to which our conclusions are to be applied.” Here, he alludes to a process (genetic origin or environmental circumstances), describes a population as an infinite collection, and does not make a distinction between the population as a collection of individuals or as the collection of the measurements on these individuals.

Gentle et al. (2012) state:

Statistical analysis involves use of observational data together with domain knowledge to develop a model to study and understand a data-generating process. The data analysis is used to refine the model or possibly to select a different model, to determine appropriate values for terms in the model, and to use the model to make inferences concerning the process. This has been the paradigm followed by statisticians for centuries. The advances in statistical theory over the past two centuries have not changed the paradigm, but they have improved the specific methods. Not only has the exponentially-increasing computational power allowed use of more detailed and better models, however, it has shifted the paradigm slightly. Many alternative views of the data can be examined. Many different models can be explored. Massive amounts of simulated data can be used to study the model/data possibilities.

The process must first generate an individual on which one or more measurements are to be taken resulting in the generation of data. We reiterate our definition of population as the collection of individuals the process has or could have generated.

W. E. Deming developed sampling techniques that are still used today by the United States Department of the Census and the Bureau of Labor Statistics. He is perhaps best known and recognized by the Japanese for his post-World War II work in helping the Japanese capture world markets in several key industries such as automotive, steel, and electronics. Their success was due in large part to producing high quality products. Deming (1953) stated the following:

Statistical data are supposedly collected to provide a rational basis for action. The action may call for the enumerative interpretation of the data, or it may call for the analytic interpretation. The aim here is to exhibit some of the consequences of failing to distinguish between the enumerative and the analytic uses of data. This distinction is necessary in the statement of the aims of a survey, census, or experiment, in order that the plans for the collection of the data and for the tabulations may most economically meet the needs of the consumer, and it is equally important in the interpretation of data. Thus, to draw on a result from a later paragraph, information obtained in a complete census concerning every person in an area (e.g., on occupation, income, or education) still possesses for analytic purposes a sampling error that is actually about a quarter as great as the sampling error of a 6 per cent sample. The consequences are far-reaching. In using a census-table for analytic purposes, even though the figures come from a perfect complete count, it
is therefore necessary to bear in mind that small numbers in a cell are unreliable in the sense that they have a standard error, just as if they had arisen in sampling, as indeed they did. Moreover, in the planning of a complete census, it is therefore imperative to use sampling for every bit of information that is not necessary as an aid to complete coverage, or required to give detail for small areas (such as the block statistics). Name, relationship to the head, age, sex, marital status, color are probably all necessary for the sake of completeness of coverage. These things, plus a few questions on rent, tenure, year built, will provide the information required for the block statistics.

Thus Deming pointed out that even a census is a sample. Recall that a census occurs when the measurement(s) of interest is taken on each individual in the representative sample. Although Deming does not explicitly point to an infinite population, the implication here is that the population is a collection of individuals that contains more than the individuals that have been generated by a process.

Throughout most of their textbook, Moore et al. (2018) refer to what we call the representative sample as the population. This can be summarized from the statement they make on page 376: “The condition that the population is large relative to the size of the sample will be satisfied if the population is, say, at least 20 times as large.” However, in Chapter 31 titled “Statistical Process Control,” they state, “[w]e can accommodate processes in our sample-versus-population framework: think of the population as containing all the outputs that would be produced by the process if it ran forever in its present state. The outputs produced today or this week are a sample from this population. Because the population doesn’t actually exist now, it is simpler to speak of a process and of recent output as a sample from the process in its present state.” Note, however, that we maintain that this viewpoint is applicable and appropriate to all processes and their corresponding populations, not just to the context of statistical process control in industrial settings.

3. Describing the Data Model

Montgomery (2009, p. 292, Ex. 7.1) gives an example of a process packaging frozen orange juice concentrate in six ounce cardboard containers. One measurement of interest is whether the container leaks or not. A number that completely characterizes the distribution of this measurement is the rate, $p$, at which the process is producing containers that leak. Another equivalent view of the meaning of $p$ is that $p$ is the probability that a given container will leak. Since the number $p$ characterizes the population of interest, it is generally referred to as a parameter. If $N$ of these containers of frozen orange juice concentrate produced by this process are to be purchased by a store, the $N$ items only constitute a representative sample from the process. There is no reason for the store to view this collection of $N$ items as a population. The uncertainty as to which collection of $N$ cans the process has generated must be taken into account. The store may be interested in the number $Y_N$ or proportion $\hat{p}_N$ of these $N$ containers that leak, but these values will depend on the sample received. Such numbers that characterize a sample (whether a representative sample or researcher’s sample) are referred to as statistics. As statistics, they have distributions often referred to as sampling distributions. Their distributions take into account the uncertainty as to which representative sample the process has generated with respect to the measurement of interest on the individuals.
It is useful to classify variables and measurements of interest. Virtually all of the variables we will study can be classified as either discrete or continuous. A discrete variable is a variable in which its set of possible values is a countable (i.e., either a finite or an uncountably infinite) set. Measurements such as length, weight, time, area, volume, and rates are variables in which the set of possible outcomes is a continuum of the real numbers, typically, some interval of the real numbers. These are examples of continuous variables. In the orange juice concentrate example, the measurement of interest has only two possible outcome—leaks or does not leak, and is therefore a discrete measurement.

Often the process and population of interest is studied by examining the (joint) distribution of the measurement(s) of interest. A useful way to view the distribution of a continuous variable is with a probability density curve (function). For many continuous variables \( X \) of interest, the density (function) curve describing its distribution is simply a (function) curve \( y = f(x) \) such that the curve is on or above the measurement axis and the total area under the curve is one. Further, area under the curve and over an interval is the proportion of the population having their measurements \( X \) falling in this interval. One can view the probability density function as a continuous analog of the population proportion. We note for completeness that there exist continuous measurements that do not have a probability density function that describes its distribution, e.g., the Cantor distribution (cf. Gut, 2005, pp. 40 ff.). However, this is not the case for the vast majority of measurements taken by researchers.

It is of importance to keep in mind that the researcher is typically interested in making an inference about the process and/or the associated population. The population is an uncountable collection and for this reason alone it is impossible for the researcher view the entire population. Since one cannot view the whole population, then only partial information about the process and population can be obtained by looking at a finite subcollection (sample) of what the process has produced. Ideally, we would like for this sample to be representative of the population. What the process has produced is a representative sample of what the process can produce. Often this sample is either too large or does not initially exist, so that it would be impossible (or at least impractical) for the researcher to view the full representative sample. One solution is to take a “representative” sample from this representative sample. Unfortunately, there is no sampling method that guarantees a representative sample. The best for which one can hope is a method that will “most likely” produce a representative sample. Also, it should be kept in mind that even after a sample is produced by the process, or selected by the researcher (as in the case of a designed experiment), and before the sample is examined, the uncertainty about which representative sample has been obtained is the same as that which existed before the sample was brought into existence. It is this uncertainty that must be accounted for by the researcher in making an inference. The often used argument that this sample generated by the process is now “fixed” which is to imply that somehow this uncertainty has been removed. But consider the following scenario: a penny is to be flipped and a “head” or “tail” is to be observed. Assume the model that the probability of a “head” is 50%. If the penny has been flipped and the face-up side has not been observed, the outcome is “fixed” but the uncertainty as to whether the outcome is a “head” still remains to be 50%.
4. Further Examples

Example 4. A coin is to be flipped. A measurement $X$ to be taken on the coin has the value 1 if the face-up side of the coin is a “head” and a value of 0 if the face-up side of the coin is a “tail.” The process is the flipping of the coin. The population is the collection of all possible flips of the coin. This is an uncountable collection of possibilities. For the American penny, it is common to assume that the outcomes of a “head” ($X = 1$) and a “tail” ($X = 0$) are equally likely (fifty-fifty). Assume this model. Suppose that the coin has been flipped but the outcome of “head” or “tail” has not been revealed. Does the knowledge that the coin has been flipped provide information that would cause one to now assume a different model than the fifty-fifty model? No, since we do not have a reason to assume our level of uncertainty has changed.

Example 5. A penny is to be placed on a flat surface, spun, and its resting side that is up is to be observed. This is a process of generating “heads” and “tails.” The collection of all possible ways of spinning this penny is an uncountable collection. Let $p$ be the rate at which the process is generating spins that result in a resting side up of “heads.” Define the measurement $X$ to have the value of 1 if the spin of the penny results in the coin resting heads-up and $X = 0$ if tails-up. The coin is now spun two hundred times and it is observed that the number of times the coin was “heads” in its resting position was 83. Thus, the proportion of the 200 spins that resulted in “heads” is $83/200 = 0.415$. Is $p = 0.415$ or is $0.415$ an estimate of the parameter $p$?

Example 6. A process has generated a collection of healthy adults. The population of interest is the collection of healthy adults the process has or could have generated. At three hour time intervals, the body temperatures $X_1, X_2, \ldots, X_8$ of each adult are to be measured over a 24-hour period and the average $\overline{X} = \frac{X_1 + X_2 + \ldots + X_8}{8}$ is recorded. The measurement $\overline{X}$ (daily average body temperature) on each health adult is a continuous variable. Hence, there is a density curve that describes how $\overline{X}$ is distributed over the population of healthy adults. It is generally assumed that average body temperature $\mu$, in degrees Fahrenheit, is 98.6°F. This is the mean of the distribution of body temperature measurements $X$ over all healthy adults. The assumption that $\mu = 98.6^\circ$ F is a model. It can be shown that under the model $\mu = 98.6^\circ$ F, the average $\mu\overline{X}$ over all healthy adults is also 98.6°F.

5. Some Distributional Results

Let the $X$ measurements on the $N$ individuals in the representative sample be denoted by $X_1, \ldots, X_N$. From these $N$ individuals, the researcher will take a subcollection of size $n$. Without loss of generality let us re-number the measurements $X_1, \ldots, X_N$ so that the $X$ measurements on the $n$ individuals that appear in the researcher’s sample are $X_1, \ldots, X_n$ and the $X$ measurements on the $N-n$ individuals not selected by the researcher are $X_{n+1}, \ldots, X_N$. We will make the assumption that the $X$ measurements $X_1, \ldots, X_N$ are independent and identically distributed. It follows that the $X$ measurements of any subcollection of these $N$ individual that are selected at random is a random sample. Hence, the researcher’s sample is a random sample provided the representative
sample is a random sample. Assume the common distribution is a Bernoulli distribution with parameter p such that 0 < p < 1.

We define the following statistics:

\[ Y_N = X_1 + \ldots + X_N, \hat{p}_N = \frac{1}{N} Y_N, \]
\[ Y_n = X_1 + \ldots + X_n, \hat{p}_n = \frac{1}{n} Y_n, \]
\[ Y_{N-n} = Y_N - Y_n, \text{ and } \hat{p}_{N-n} = \frac{1}{N-n} Y_{N-n}. \]

Under our model, the statistic \( Y_N \) has a Binomial distribution with distributional parameters \( N \) and \( p \). The conditional distribution of \( Y_n \) given \( X \) is a hypergeometric distribution with distributional parameters \( Y_N \) and \( N \), where \( X = [X_1, \ldots, X_N]^T \). It is now easy to show that

\[ E(Y_n | X) = n \frac{Y_N}{N} = n \hat{p}_N, \]
\[ V(Y_n | X) = \left(1 - n \frac{1}{N - 1}\right) n \hat{p}_N (1 - \hat{p}_N), \]
\[ E(\hat{p}_n | X) = n \frac{Y_N}{N} = \hat{p}_N, \text{ and } V(\hat{p}_n | X) = \left(1 - n \frac{1}{N - 1}\right) \frac{\hat{p}_N (1 - \hat{p}_N)}{n}. \]

Conditioned on \( X \), the statistic \( \hat{p}_n \) is an unbiased predictor of \( \hat{p}_N \), and as \( n \) approaches \( N \), it becomes a more precise predictor of \( \hat{p}_N \). Further, one can show that the unconditional distribution of \( Y_n \) has a Binomial distribution with distributional parameters \( n \) and \( p \) with the unconditional distribution of \( \hat{p}_n \) given by

\[ P(\hat{p}_n = \frac{y}{n}) = P(Y_n = y). \]

Hence,

\[ E(\hat{p}_n) = p \] and \( V(\hat{p}_n) = \frac{p(1-p)}{n} \).

Thus, the researcher’s sample proportion \( \hat{p}_n \) is an unbiased estimator of the population proportion \( p \), and as \( n \) increases it is a more precise estimator of \( p \).

An estimator for \( V(\hat{p}_n) \) is

\[ \tilde{V}(\hat{p}_n) = \frac{\hat{p}_n (1 - \hat{p}_n)}{n}. \]

However,

\[ E\left(\frac{\hat{p}_n(1 - \hat{p}_n)}{n}\right) = \frac{E(\hat{p}_n) - E(\hat{p}_n^2)}{n} = \frac{E(\hat{p}_n) - V(\hat{p}_n) - [E(\hat{p}_n)]^2}{n} \]
\[ = \frac{p - \frac{p(1-p)}{n} - p^2}{n} = \frac{n - 1}{n} \frac{p(1-p)}{n}. \]

Hence, \( \tilde{V}(\hat{p}_n) \) is a biased estimator of \( V(\hat{p}_n) \). We can now see that

\[ \hat{V}(\hat{p}_n) = \frac{\hat{p}_n (1 - \hat{p}_n)}{n} = \frac{\hat{p}_n (1 - \hat{p}_n)}{n - 1} \]

is an unbiased estimator of \( V(\hat{p}_n) \). This results is given in Scheaffer et al. (2012).

For \( n < N \) and \( \lambda = N/n \), if \( n \to \infty \), then \( N - n \to \infty \). We observe that

\[ \frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \to D N(0, 1) \] and \( \frac{\hat{p}_{N-n} - p}{\sqrt{\frac{p(1-p)}{N-n}}} \to D N(0, 1) \)
according to the central limit theorem as \( n \to \infty \). Another limiting distribution result states that
\[
\frac{\hat{p}_n - p}{\sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}} \xrightarrow{d} N(0,1),
\]
as \( n \to \infty \). See Bain and Engelhardt (1992, p. 249, Example 7.7.2). We now examine the limiting distributions of
\[
\frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \frac{p(1-p)}{n}}}, \quad \frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \frac{\hat{p}_n(1-\hat{p}_n)}{n}}}, \quad \text{and} \quad \frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \frac{\hat{p}_N(1-\hat{p}_N)}{N}}},
\]
Observe that we can write
\[
\frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \frac{p(1-p)}{n}}} = \frac{\frac{N-n}{N} (\hat{p}_n - \hat{p}_{N-n})}{\sqrt{\frac{N-n}{N} \frac{p(1-p)}{n}}} = \sqrt{\frac{N-n}{N}} \frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} - \sqrt{\frac{n}{N}} \frac{\hat{p}_{N-n} - p}{\sqrt{\frac{p(1-p)}{N-n}}}
= \sqrt{\frac{N-n}{N}} Z_n - \sqrt{\frac{n}{N}} Z_{N-n},
\]
where
\[
Z_n = \frac{\hat{p}_n - p}{\sqrt{\frac{p(1-p)}{n}}} \quad \text{and} \quad Z_{N-n} = \frac{\hat{p}_{N-n} - p}{\sqrt{\frac{p(1-p)}{N-n}}}.
\]
We see that
\[
\sqrt{\frac{N-n}{N}} = \sqrt{\frac{\lambda n-n}{\lambda n}} = \sqrt{\frac{\lambda-1}{\lambda}} \quad \text{and} \quad \sqrt{\frac{n}{N}} = \sqrt{\frac{n}{\lambda n}} = \sqrt{\frac{1}{\lambda}}.
\]
Using Slutsky’s theorem, we have
\[
U_n = \sqrt{\frac{\lambda-1}{\lambda}} Z_n \xrightarrow{d} N \left(0, \frac{\lambda-1}{\lambda}\right) \quad \text{and} \quad V_n = \sqrt{\frac{1}{\lambda}} Z_{(\lambda-1)n} \xrightarrow{d} N \left(0, \frac{1}{\lambda}\right)
\]
as \( n \to \infty \). Making the transformation
\[
T_n = U_n - V_n \quad \text{and} \quad Q_n = V_n
\]
with inverse transformation \( U_n = T_n + Q_n \) and \( V_n = Q_n \) with Jacobian \( J = 1 \), we have
\[
f_{T_n,Q_n}(t,q) = f_{U_n}(t+q) f_{V_n}(q).
\]
It follows that
\[
\lim_{n \to \infty} f_{T_n,Q_n}(t,q) = \lim_{n \to \infty} f_{U_n}(t+q) f_{V_n}(q) = \lim_{n \to \infty} f_{U_n}(t+q) \lim_{n \to \infty} f_{V_n}(q)
= f_N \left(0, \frac{1}{\lambda}\right) (t+q) f_N \left(0, \frac{1}{\lambda}\right) (q),
\]
where $f_{N(0,\sigma^2)}(y)$ is the probability density function of a Normal distribution with mean 0 and variance $\sigma^2$. We then have

$$
\lim_{n \to \infty} f_{T_n}(t) = \int_{-\infty}^{\infty} f_{N(0,\frac{1}{n})}(t + q) f_{N(0,\frac{1}{n})}(q) \, dq = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.
$$

Hence,

$$
\frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \frac{p(1-p)}{n}}} \xrightarrow{d} N(0,1).
$$

Again using Slutsky’s theorem, we have

$$
\frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}}} \xrightarrow{d} N(0,1) \quad \text{and} \quad \frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \sqrt{\frac{\hat{p}_N(1-\hat{p}_N)}{n}}}} \xrightarrow{d} N(0,1)
$$

as $n \to \infty$. Using similar arguments, one can show that

$$
\frac{\overline{X}_n - \overline{X}_N}{\sqrt{\frac{N-n}{N} \frac{S_n}{\sqrt{n}}}} \xrightarrow{d} N(0,1),
$$

where $\overline{X}_n$ and $\overline{X}_N$ are the means of the researcher’s sample and the representative sample and $S_n$ is the standard deviation of the researcher’s sample.

6. Confidence and Prediction Intervals

6.1. Confidence intervals for the population proportion $p$. Suppose that a process is producing individuals that possess a characteristic of interest at a rate $p$ and that a researcher will have a sample of $n$ individuals whose $X$ measurements are to be taken. We assume here that $0 < p < 1$. The random variable $X$ will assume the value 1 if the individual possesses the characteristic of interest and 0 if it does not. The random variable $X$ thus has a Bernoulli distribution with distributional parameter $p$. Virtually every elementary statistics textbook introduces the Wald approximate confidence interval for $p$, based on the observation that for sufficiently large $n$, $Y_n \sim N(n\hat{p}_n, n\hat{p}_N(1-\hat{p}_n))$, which leads to the approximate $100(1-\alpha)%$ confidence interval

$$
\left(\hat{p}_n - z_{\alpha/2} \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}, \hat{p}_n + z_{\alpha/2} \sqrt{\frac{\hat{p}_n(1-\hat{p}_n)}{n}}\right).
$$

for $p$. This is based on the random variable $(\hat{p}_n - p) / \sqrt{\hat{p}_N(1-\hat{p}_N)/n}$ having an approximate standard normal distribution. Another “large” sample confidence interval for $p$ is

$$
\left(\frac{2n\hat{p}_n + z_{\alpha/2}^2 - z_{\alpha/2}}{2 \left(n + z_{\alpha/2}^2\right)}, \frac{2n\hat{p}_n + z_{\alpha/2}^2 + z_{\alpha/2}}{2 \left(n + z_{\alpha/2}^2\right)}\right).
$$

This is based on the random variable $(\hat{p}_n - p) / \sqrt{p(1-p)/n}$ having an approximate standard Normal distribution. Several “exact” confidence intervals for $p$, based on $Y_n$ having a binomial distribution with distribution parameters $n$ and $p$, are known in the literature and implemented in statistical software. The most well known of these was given by Clopper and Pearson (1934). For a discussion of other “exact” confidence intervals for $p$, see Thulin (2014, p. 821).
6.2. **Prediction intervals for the representative sample proportion \( \hat{p}_N \).** A corresponding approximate 100(1 – \( \alpha \))% prediction interval for \( \hat{p}_N \) is given by

\[
\left( \hat{p}_n - z_{\alpha/2} \sqrt{1 - \frac{n}{N}} \hat{p}_n(1 - \hat{p}_n) \right) \leq \hat{p}_N \leq \left( \hat{p}_n + z_{\alpha/2} \sqrt{1 - \frac{n}{N}} \hat{p}_n(1 - \hat{p}_n) \right)
\]

for “large” \( n \). This is based on the limiting distribution of the statistic

\[
\frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \hat{p}_n(1-\hat{p}_n)}}
\]

being a standard Normal distribution. Since \( \sqrt{1 - n/N} < 1 \) and \( N \) is not known, an approximate at least 100(1 – \( \alpha \))% prediction interval for \( \hat{p}_N \) is

\[
\left( \hat{p}_n - z_{\alpha/2} \sqrt{\frac{\hat{p}_n(1 - \hat{p}_n)}{n}} \right) \leq \hat{p}_N \leq \left( \hat{p}_n + z_{\alpha/2} \sqrt{\frac{\hat{p}_n(1 - \hat{p}_n)}{n}} \right)
\]

for \( N \) “large” relative to \( n \). Also, the observed value of the random interval,

\[
\left( \frac{2\hat{p}_n + z_{\alpha/2}^2 \frac{N-n}{nN}}{2 \left( 1 + z_{\alpha/2}^2 \frac{N-n}{nN} \right)} - \frac{\sqrt{\left(2\hat{p}_n + z_{\alpha/2}^2 \frac{N-n}{nN}\right)^2 - 4 \left( 1 + z_{\alpha/2}^2 \frac{N-n}{nN} \right) \hat{p}_n^2}}{2 \left( 1 + z_{\alpha/2}^2 \frac{N-n}{nN} \right)} \right)
\]

is an approximate 100 (1 – \( \alpha \)) % prediction interval for \( \hat{p}_N \) which is based on the limiting distribution of the statistic

\[
\frac{\hat{p}_n - \hat{p}_N}{\sqrt{\frac{N-n}{N} \hat{p}_n(1-\hat{p}_n)}}
\]

being a standard Normal distribution. Note that

\[
\frac{N-n}{nN} = \frac{1}{n} \left( 1 - \frac{n}{N} \right) < \frac{1}{n}.
\]

In the aforementioned prediction interval for \( \hat{p}_N \), replacing \( (N-n)/(nN) \) with \( 1/n \) when \( N \) is “large” relative to \( n \), we obtain the prediction interval

\[
\left( \frac{2\hat{p}_n + z_{\alpha/2}^2/n}{2 \left( 1 + z_{\alpha/2}^2/n \right)} - \frac{\sqrt{\left(2\hat{p}_n + z_{\alpha/2}^2/n\right)^2 - 4 \left( 1 + z_{\alpha/2}^2/n \right) \hat{p}_n^2}}{2 \left( 1 + z_{\alpha/2}^2/n \right)} \right)
\]

\[
\left( \frac{2\hat{p}_n + z_{\alpha/2}^2/n}{2 \left( 1 + z_{\alpha/2}^2/n \right)} + \frac{\sqrt{\left(2\hat{p}_n + z_{\alpha/2}^2/n\right)^2 - 4 \left( 1 + z_{\alpha/2}^2/n \right) \hat{p}_n^2}}{2 \left( 1 + z_{\alpha/2}^2/n \right)} \right)
\]
is an approximate at least 100 \((1 - \alpha)\)% prediction interval for \(\hat{p}_N\) that does not depend on the knowing the value of \(N\).

Thus we see that the prediction interval for a statistic describing the representative sample is essentially the confidence interval for the corresponding population parameter with the extra factor \(\sqrt{(N - n)/N}\), which is sometimes called the “finite population correction” factor, as in Cochran (1977, p. 24). But under the view for which we are advocating, there are no finite populations, so what has been thought of as a “confidence interval for a parameter of a finite population” is really a prediction interval for a statistic of the representative sample in what Deming called an enumerative study.

7. Conclusion

We make the case that researchers are interested in drawing inferences about a process and population, and that a population is inherently an uncountably infinite collection. That which is often referred to as a “finite population” is better termed the “representative sample” because it is representative of what the process can produce, and is a sample in the sense that it possesses uncertainty. To distinguish the set of individuals actually produced by the process from the sample collected for study by the researcher, we call the latter the “researcher’s sample.” Finally, we draw a clear distinction between estimation and prediction, where the former refers to inference drawn about the population and the latter about the collection of all actual individuals produced by the process, herein called the “representative sample.” Various confidence and prediction intervals were derived and compared.

8. Disclosure

No potential competing interest was reported by the authors.

References

Bain, L.J. and Engelhardt, M. (1992), *Introduction to Probability and Mathematical Statistics*, 2nd ed., Pacific Grove, California: Duxbury.

Cochran, W. G. (1977), *Sampling Techniques*, 3rd ed., New York: Wiley.

Clopper, C. J. and E. S. Pearson (1934), “The Use of Confidence or Fiducial Limits Illustrated in the Case of the Binomial,” *Biometrika* 26, 404–413.

Deming, W. E. (1942), “On Probability as a Basis for Action,” *American Statistician* 29, 146-152.

Deming, W. E. (1942), “On a Classification of the Problems of Statistical Inference,” *Journal of the American Statistical Association* 37, 173-185.

Deming, W. E. (1953), “On the Distinction between Enumerative and Analytical Surveys,” *Journal of the American Statistical Association* 48, 244–255.

Fisher, R. A. (1958), *Statistical Methods for Research Workers*, 13th ed., New York: Hafner Publishing Company.

Gentle, J. E., W. K. H"ardle, and Y. Mori (2012), *Handbook of Computational Statistics*, 2nd ed., New York: Springer.
Gut, A. (2005), *Probability: a Graduate Course*, New York: Springer.
Montgomery, D. C. (2009), *Introduction to Statistical Quality Control*, 6th ed., New York: Wiley.
Moore, D. S., Notz, W. I., and Fligner, M. A. (2019) *The Basic Practice of Statistics*, 8th ed., New York: Wiley.
Scheaffer, R. L., W. Mendenhall III, R. L. Ott, and K. G. Gerow (2012), *Elementary Survey Sampling*, 7th ed., Brooks/Cole, Cengage Learning.
Thulin, Måns (2014), “The cost of using exact confidence intervals for a binomial proportion,” *Electronic Journal of Statistics* **8**, 817–840.

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