STABILITY CONDITIONS ON FIBRED THREEFOLDS

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ABSTRACT. We give a conjectural construction of Bridgeland stability conditions on the derived category of fibred threefolds. The construction depends on a conjectural Bogomolov-Gieseker type inequality for certain stable complexes. It can be considered as a relative version of the construction of Bayer, Macrì and Toda. We prove the conjectural Bogomolov-Gieseker type inequality in the case of relative projective planes over curves. This gives the existence of Bridgeland stability conditions on such threefolds.

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1. INTRODUCTION

Stability conditions for triangulated categories were introduced by Bridgeland in [6]. Since then, they have drawn a lot of attentions, and have been investigated intensively. The existence of stability conditions on three-dimensional varieties is often considered the biggest open problem in the theory of Bridgeland stability conditions. In [4, 3, 5], the authors introduced a conjectural construction of Bridgeland stability conditions for any projective threefold. Here the problem was reduced to proving a Bogomolov-Gieseker type inequality for the third Chern character of tilt-stable objects. It has been shown to hold for Fano 3-folds [5, 15], abelian 3-folds [3], some product type threefolds [11], quintic threefolds [13], threefolds with vanishing Chern classes [17], etc. Recently, Yuchen Liu [14] showed the existence of stability conditions on product type varieties by a different method.

In this paper, we give a conjectural construction of Bridgeland stability conditions on fibred threefolds. The construction depends on a conjectural Bogomolov-Gieseker type inequality for mixed tilt-stable complexes (Conjecture 5.2). We show that this conjecture gives the existence of stability conditions on fibred threefolds.
Our construction can be considered as a relative version of that of Bayer, Macrì and Toda [4]. We prove the conjectural Bogomolov-Gieseker type inequalities in the case of relative projective planes over curves. This gives the the existence of stability conditions on such threefolds:

**Theorem 1.1** (= Corollary 6.3). Let $E$ be a rank three vector bundle on a complex smooth projective curve $C$. Then there exist locally finite stability conditions on $\mathbb{P}(E)$.

Throughout this paper, we let $f : \mathcal{X} \to C$ be a smooth projective morphism from a complex smooth projective variety of dimension $n \geq 2$ to a complex smooth projective curve. We denote by $F$ the general fiber of $f$, and fix a nef and relative ample divisor $H$ on $\mathcal{X}$.

We now give a sketch of our construction; the details will be given in Section 5. We use the $\mu_C$-stability introduced in [1] to construct a torsion pair in $\text{Coh}(\mathcal{X})$. Let $T_{\beta H} \subset \text{Coh}(\mathcal{X})$ be the category generated by $\mu_C$-stable sheaves of slope $\mu_C > \beta$ via extension. Similarly, let $F_{\beta H}$ be the subcategory generated by $\mu_C$-stable sheaves of slope $\mu_C \leq \beta$. We define $\text{Coh}_{\beta H}^c(\mathcal{X}) \subset \mathbb{D}(\mathcal{X})$ as a tilt with respect to the torsion pair $(T_{\beta H}, F_{\beta H})$:

$$\text{Coh}_{\beta H}^c(\mathcal{X}) = \langle T_{\beta H}, F_{\beta H}[1] \rangle.$$  

Different from the torsion pair defined via the classical slope-stability, there are torsion sheaves in $F_{\beta H}$. We prove that there is a double-dual operation on $\text{Coh}_{\beta H}^c(\mathcal{X})$ as well as for coherent sheaves (Lemma 4.6). For any $(\alpha, \beta, t) \in \sqrt{\mathbb{Q}} \times \mathbb{Q} \times \mathbb{Q} \geq 0$, we then define the following function on $\text{Coh}_{\beta H}^c(\mathcal{X})$:

$$\nu_{\alpha, \beta, t}(\mathcal{E}) = \frac{(H^{n-2} + tFH^{n-3}) \; \text{ch}_2(\mathcal{E}) - t + 1}{FH^{n-2} \; \text{ch}_1(\mathcal{E})}. $$

We show that it is a slope-function associated to a very weak stability condition, which we call mixed tilt-stability. Like tilt-stability, we prove that mixed tilt-stable objects also satisfy a Bogomolov type inequality (Theorem 4.3). Using mixed tilt-stability, we can define a torsion pair in $\text{Coh}_{\alpha, \beta}^c(\mathcal{X})$ exactly as in the case of $\mu_C$-stability for $\text{Coh}(\mathcal{X})$ above. Tilting at this torsion pair produces a heart $\mathcal{A}_{\alpha, \beta}^c(\mathcal{X})$ of a t-structure. We prove that $\mathcal{A}_{\alpha, \beta}^c(\mathcal{X})$ is noetherian by the double-dual operation and the Bogomolov type inequality. Finally, Conjecture 5.2 guarantees the positivity property for some central charge on $\mathcal{A}_{\alpha, \beta}^c(\mathcal{X})$ when $n = 3$.

**Organization of the paper.** Our paper is organized as follows. In Section 2 we review some basic notions and results of stability for coherent sheaves on a fibred variety in [1]. Then in Section 3 we recall the definition of very weak stability and give a relative version of the tilt-stability constructed in [4]. We will introduce the mixed tilt-stability and give its basic properties in Section 4. In Section 5 we give the conjectural construction of Bridgeland stability conditions on fibred threefolds and propose Conjecture 5.2. We prove this conjectural for relative projective planes over curves in Section 6.

**Notation.** Let $X$ be a smooth projective variety. We denote by $T_X$ and $\Omega_X^1$ the tangent bundle and cotangent bundle of $X$, respectively. $K_X$ and $\omega_X$ denote the canonical divisor and canonical sheaf of $X$, respectively. We write $c_i(X) := c_i(T_X)$ for the $i$-th Chern class of $X$. We write $\text{NS}(X)$ for the Néron-Severi group of divisors.
up to numerical equivalence. We also write $\text{NS}(X)_{\mathbb{Q}}$, $\text{NS}(X)_{\mathbb{R}}$, etc. for $\text{NS}(X) \otimes \mathbb{Q}$, etc. For a triangulated category $D$, we write $K(D)$ for its Grothendieck group.

Let $\pi : \mathcal{X} \rightarrow S$ be a flat morphism of Noetherian schemes and $W \subset S$ be a subscheme. We denote by $\mathcal{X}_W = \mathcal{X} \times_S W$ the fiber of $\pi$ over $W$, and by $i_W : \mathcal{X}_W \hookrightarrow \mathcal{X}$ the embedding of the fiber. In the case that $S$ is integral, we write $K(S)$ for its fraction field, and $\mathcal{X}_K(S)$ for the generic fiber of $\pi$. We denote by $D^b(\mathcal{X})$ the bounded derived category of coherent sheaves on $\mathcal{X}$. Given $E \in D^b(\mathcal{X})$, we write $E_W$ (resp., $E_K(S)$) for the pullback to $\mathcal{X}_W$ (resp., $\mathcal{X}_K(S)$).

Let $F$ be a coherent sheaf on $X$. We write $H^j(F)$ ($j \in \mathbb{Z}_{\geq 0}$) for the cohomology groups of $F$ and write $\dim F$ for the dimension of its support. We write $\text{Coh}_{\leq d}(X) \subset \text{Coh}(X)$ for the subcategory of sheaves supported in dimension $\leq d$. Given a bounded t-structure on $D^b(X)$ with heart $\mathcal{A}$ and an object $E \in D^b(X)$, we write $\mathcal{H}^j_\mathcal{A}(E)$ ($j \in \mathbb{Z}$) for the cohomology objects with respect to $\mathcal{A}$. When $\mathcal{A} = \text{Coh}(X)$, we simply write $\mathcal{H}^j(E)$. Given a complex number $z \in \mathbb{C}$, we denote its real and imaginary part by $\Re z$ and $\Im z$, respectively. We write $\sqrt{\mathbb{Q}_{> 0}}$ for the set $\{\sqrt{x} : x \in \mathbb{Q}_{> 0}\}$.

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2. Relative slope-stability

We will review some results in [1] and some basic notions of stability for coherent sheaves.

For any $\mathbb{R}$-divisor $D$ on $\mathcal{X}$, we define the twisted Chern character $\text{ch}^D = e^{-D} \text{ch}$. More explicitly, we have

\[
\begin{align*}
\text{ch}_0^D &= \text{ch}_0 = \text{rk} \\
\text{ch}_2^D &= \text{ch}_2 - D \text{ch}_1 + \frac{D^2}{2} \text{ch}_0 \\
\text{ch}_1^D &= \text{ch}_1 - D \text{ch}_0 \\
\text{ch}_3^D &= \text{ch}_3 - D \text{ch}_2 + \frac{D^2}{2} \text{ch}_1 - \frac{D^3}{6} \text{ch}_0.
\end{align*}
\]

The first important notion of stability for a sheaf is the relative slope-stability. We define the relative slope $\mu_{H,F}$ of a coherent sheaf $\mathcal{E} \in \text{Coh}(\mathcal{X})$ by

\[
\mu_{H,F}(\mathcal{E}) = \begin{cases} 
+\infty, & \text{if } \text{ch}_0(\mathcal{E}) = 0, \\
\frac{\text{EH}^{n-2}_{n-1} \text{ch}_1(\mathcal{E})}{\text{EH}^{n-1}_{n-2} \text{ch}_0(\mathcal{E})}, & \text{otherwise}.
\end{cases}
\]

Definition 2.1. A coherent sheaf $\mathcal{E}$ on $\mathcal{X}$ is $\mu_{H,F}$-(semi)stable (or relative slope-(semi)stable) if, for all non-zero subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}$, we have

\[
\mu_{H,F}(\mathcal{F}) < (\leq) \mu_{H,F}(\mathcal{E}/\mathcal{F}).
\]

Similarly, for any point $s \in C$, we can define $\mu_{H_s}$-stability (or slope-stability) of a coherent sheaf $\mathcal{G}$ on the fiber $\mathcal{X}_s$ over $s$ for the slope $\mu_{H_s}$:

\[
\mu_{H_s}(\mathcal{G}) = \begin{cases} 
+\infty, & \text{if } \text{ch}_0(\mathcal{G}) = 0, \\
\frac{\text{EH}^{n-2}_{n-1} \text{ch}_1(\mathcal{G})}{\text{EH}^{n-1}_{n-2} \text{ch}_0(\mathcal{G})}, & \text{otherwise}.
\end{cases}
\]

One sees that $\mu_{H,F}(\mathcal{E}) = \mu_{H_s}(\mathcal{E}_s)$. 

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Definition 2.2. Let $\mathcal{A}_C$ be the heart of a $C$-local t-structure on $\mathcal{D}^b(X)$ (see [1] Definition 4.10), and let $\mathcal{E} \in \mathcal{A}_C$.

1. We say $\mathcal{E}$ is $C$-flat if $\mathcal{E}_c \in \mathcal{A}_c$ for every point $c \in C$, where $\mathcal{A}_c$ is the heart of the t-structure given by [1] Theorem 5.3 applied to the embedding $c \hookrightarrow C$.

2. An object $\mathcal{F} \in \mathcal{D}^b(X)$ is called $C$-torsion if it is the pushforward of an object in $\mathcal{D}^b(\mathcal{X}_W)$ for some proper closed subscheme $W \subset C$.

3. $\mathcal{E}$ is called $C$-torsion free if it contains no nonzero $C$-torsion subobject.

We denote by $\mathcal{A}_C$-tor the subcategory of $C$-torsion objects in $\mathcal{A}_C$, and by $\mathcal{A}_C$-tf the subcategory of $C$-torsion free objects. We say $\mathcal{A}_C$ has a $C$-torsion theory if the pair of subcategories $(\mathcal{A}_C$-tor, $\mathcal{A}_C$-tf) forms a torsion pair in the sense of [1, Definition 4.6].

Lemma 2.3. Let $\mathcal{E} \in \mathcal{A}_C$ be as in Definition 2.2. Then

1. $\mathcal{E}$ is $C$-flat if and only if $\mathcal{E}$ is $C$-torsion free;
2. $\mathcal{E}$ is $C$-torsion if and only if $\mathcal{E}_{K(C)} = 0$.

Proof. See [1] Lemma 6.12 and Lemma 6.4.

Since Coh($X$) is the heart of the natural $C$-local t-structure on $\mathcal{D}^b(X)$, one can applies the above definition and lemma to coherent sheaves. The following lemma shows the relation of the relative slope-stability to the slope-stability.

Lemma 2.4. Let $\mathcal{E}$ be a $C$-torsion free sheaf on $X$. Then $\mathcal{E}$ is $\mu_{H,F}$-(semi)stable if and only if there exists an open subset $U \subset C$ such that $\mathcal{E}_s$ is $\mu_{H,F}$-(semi)stable for any point $s \in U$.

Proof. By Lemma 2.3 one deduces any subsheaf $\mathcal{F}$ of $\mathcal{E}$ is flat over $C$. Thus $\mathcal{F}_{K(C)} \in \text{Coh}(\mathcal{X}_{K(C)})$. Since $
\mu_{H,F}(\mathcal{F}) = \mu_{H_{K(C)}}(\mathcal{F}_{K(C)}),
$ from [1] Ex. II.5.15, one sees that $\mathcal{E}$ is $\mu_{H,F}$-(semi)stable if and only if $\mathcal{E}_{K(C)}$ is $\mu_{H_{K(C)}}$-(semi)stable. Hence the desired conclusion follows from the openness of slope-stability.

Another important notion of stability for a sheaf on a fibration is the stability introduced in [1] Example 15.3]. We define the slope $\mu_C$ of a coherent sheaf $\mathcal{E} \in \text{Coh}(X)$ by

$$\mu_C(\mathcal{E}) = \begin{cases} \frac{FH^{n-2}ch_1(\mathcal{E})}{H^{n-1}ch_0(\mathcal{E})}, & \text{if } ch_0(\mathcal{E}) \neq 0, \\ \frac{H^{n-2}ch_1(\mathcal{E})}{H^{n-1}ch_0(\mathcal{E})}, & \text{if } \mathcal{E}_{K(C)} = 0 \text{ and } H^{n-1}ch_1(\mathcal{E}) \neq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

We can define $\mu_C$-stability as in Definition 2.1.

Definition 2.5. A coherent sheaf $\mathcal{E}$ on $X$ is $\mu_C$-(semi)stable if, for all non-zero subsheaves $\mathcal{F} \hookrightarrow \mathcal{E}$, we have

$$\mu_C(\mathcal{F}) \leq (\leq) \mu_C(\mathcal{E}/\mathcal{F}).$$

For $\mathcal{E} \in \mathcal{D}^b(X)$, we define

$$Z_{K(C)}(\mathcal{E}) := -FH^{n-2}ch_1(\mathcal{E}) + iFH^{n-1}ch_0(\mathcal{E})$$
Proof. (1) Since one sees Lemma 2.7.

It turns out that \( D \) deduces that \((Z_{K(C)}(\mathcal{E}))\) over \( X \).

Let \( \mu \) be the following weak see-saw property and the Harder-Narasimhan property.

By the definition of the weak Harder-Narasimhan structure, one sees that like the slope-stability and the relative slope-stability, the \( \mu \)-stability also satisfies the following weak see-saw property and the Harder-Narasimhan property.

**Proposition 2.6.** Let \( \mathcal{E} \in \text{Coh}(\mathcal{X}) \) be a non-zero sheaf.

(1) For any short exact sequence

\[
0 \to F \to \mathcal{E} \to \mathcal{G} \to 0
\]

in \( \text{Coh}(\mathcal{X}) \), we have

\[
\mu_C(F) \leq \mu_C(\mathcal{E}) \leq \mu_C(\mathcal{G}) \quad \text{or} \quad \mu_C(F) \geq \mu_C(\mathcal{E}) \geq \mu_C(\mathcal{G}).
\]

(2) There is a filtration (called Harder-Narasimhan filtration)

\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}
\]

such that: \( \mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1} \) is \( \mu \)-semistable, and \( \mu_C(\mathcal{G}_1) > \cdots > \mu_C(\mathcal{G}_m) \).

We write \( \mu^+_C(\mathcal{E}) := \mu_C(\mathcal{G}_1) \) and \( \mu^-_C(\mathcal{E}) := \mu_C(\mathcal{G}_m) \).

The below lemma gives the relation between the Chern characters of objects on fibers and their pushforwards.

**Lemma 2.7.** Let \( W \) be a closed subscheme of \( C \), and \( j \) be a positive integer.

(1) For any \( \mathcal{E} \in D^b(\mathcal{X}) \) and \( \mathbb{Q} \)-divisor \( D \) on \( \mathcal{X} \), we have

\[
\text{ch}^D_j(i_{W*}\mathcal{E}_W) = \mathcal{X}_W \text{ch}^D_{j-1}(\mathcal{E}).
\]

(2) Assume that \( W \) is a closed point of \( C \). Then for any \( \mathcal{Q} \in D^b(\mathcal{X}_W) \) we have

\[
\text{ch}_j(i_{W*}\mathcal{Q}) = i_{W*}\text{ch}_{j-1}(\mathcal{Q}).
\]

**Proof.** (1) Since \( \mathcal{X}_W \) is a divisor of \( \mathcal{X} \), from the standard exact triangle

\[
\mathcal{E} \otimes O_X(-\mathcal{X}_W) \to \mathcal{E} \to i_{W*}\mathcal{E}_W,
\]

one sees

\[
\text{ch}^D_j(i_{W*}\mathcal{E}_W) = \text{ch}^D_j(\mathcal{E}) - \text{ch}^D_j(\mathcal{E} \otimes O_X(-\mathcal{X}_W))
\]

\[
= \text{ch}^D_j(\mathcal{E}) - \left( \text{ch}^D_j(\mathcal{E}) - \mathcal{X}_W \text{ch}^D_{j-1}(\mathcal{E}) + \frac{1}{2} \mathcal{X}_W^2 \text{ch}^D_{j-2}(\mathcal{E}) + \cdots \right)
\]

\[
= \mathcal{X}_W \text{ch}^D_{j-1}(\mathcal{E}).
\]
(2) Applying the Grothendieck-Riemann-Roch theorem for the embedding $i_W : X_W \hookrightarrow X$, we conclude that 
$$
\text{ch}(i_W^* Q) = i_W^* \left( \text{ch}(Q)(\text{td}(O_{X_W}))^{-1} \right) = i_W^* \text{ch}(Q).
$$
This implies the desired equalities. \hfill \Box

Lemma 2.4 says that the usual notion of relative slope-stability for a torsion free sheaf is equivalent to the slope-stability of the general fiber of the sheaf. In contrast, $\mu_C$-stability requires stability for all fibers:

**Proposition 2.8.** Let $\mathcal{E}$ be a $C$-torsion free sheaf on $X$. Then $\mathcal{E}$ is $\mu_C$-semistable if and only if $\mathcal{E}$ is $\mu_{H,F}$-semistable and for any closed point $p \in C$ and any quotient $\mathcal{E}_p \twoheadrightarrow \mathcal{Q}$ in $\text{Coh}(X_p)$ we have $\mu_{H_p}(\mathcal{E}_p) \leq \mu_{H_p}(\mathcal{Q})$.

**Proof.** See [1, Lemma 15.7]. \hfill \Box

3. Relative tilt-stability

In this section, we recall the definition of very weak stability conditions on $\mathcal{D}$ introduced in [3, Appendix 2], [16, Section 2.1] and [18, Section 2] and give a relative version of the tilt-stability constructed in [4]. We keep the same notations as that in the previous sections.

### 3.1. Very weak stability condition

Let $\mathcal{D}$ be a triangulated category, for which we fix a finitely generated free abelian group $\Lambda$ and a group homomorphism $v : K(\mathcal{D}) \to \Lambda$.

**Definition 3.1.** A very weak stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, A)$, where $A$ is the heart of a bounded t-structure on $\mathcal{D}$, and $Z : \Lambda \to \mathbb{C}$ is a group homomorphism (called central charge) such that

1. $Z$ satisfies the following positivity property for any $E \in A$:
   $$
   Z(v(E)) \in \{ re^{i\pi \phi} : r \geq 0, 0 < \phi \leq 1 \}.
   $$

2. $(Z, A)$ satisfies the Harder-Narasimhan property: every object of $A$ has a Harder-Narasimhan filtration in $A$ with respect to $\nu_\sigma$-stability, here the slope $\nu_\sigma$ of an object $E \in A$ is defined by
   $$
   \nu_\sigma(E) = \begin{cases} 
   +\infty, & \text{if } \Im Z(v(E)) = 0, \\
   -\frac{\Re Z(v(E))}{\Im Z(v(E))}, & \text{otherwise}. 
   \end{cases}
   $$

A very weak stability condition $\sigma = (Z, A)$ is called a stability condition if for any $0 \neq E \in A$ we have $Z(v(E)) \neq 0$. This notion coincides with the notion of Bridgeland stability conditions [19].

We say $\mathcal{E} \in A$ is $\nu_\sigma$-(semi)stable if for any non-zero subobject $F \subset \mathcal{E}$ in $A$, we have

$$
\nu_\sigma(F) < (\leq) \nu_\sigma(\mathcal{E}/F).
$$

The Harder-Narasimhan filtration of an object $\mathcal{E} \in A$ is a chain of subobjects

$$
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}
$$

in $A$ such that $\mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is $\nu_\sigma$-semistable and $\nu_\sigma(\mathcal{G}_1) > \cdots > \nu_\sigma(\mathcal{G}_m)$. We set $\nu_\sigma^+(\mathcal{E}) := \nu_\sigma(\mathcal{G}_1)$ and $\nu_\sigma^-(\mathcal{E}) := \nu_\sigma(\mathcal{G}_m)$.
Definition 3.2. For a very weak stability condition \((Z, A)\) on \(D\) and for \(0 < \phi \leq 1\), we define the subcategory \(P(\phi) \subset D\) to be the category of \(\nu_\sigma\)-semistable objects \(E \in A\) satisfying \(\tan(\pi \phi) = -1/\nu_\sigma(E)\). For other \(\phi \in \mathbb{R}\) the subcategory \(P(\phi)\) is defined by the rule:

\[
P(\phi + 1) = P(\phi)[1].
\]

The objects in \(P(\phi)\) is still called \(\nu_\sigma\)-semistable objects.

For an interval \(I = (a, b) \subset \mathbb{R}\), we denote by \(P(I)\) the extension-closure of \(\bigcup_{\phi \in I} P(\phi) \subset D\).

\(P(I)\) is a quasi-abelian category when \(b - a < 1\) (cf. [6, Definition 4.1]). If we have a distinguished triangle

\[
A_1 \xrightarrow{h} A_2 \xrightarrow{g} A_3 \rightarrow A_1[1]
\]

with \(A_1, A_2, A_3 \in P(I)\), we say \(h\) is a strict monomorphism and \(g\) is a strict epimorphism. Then we say that \(P(I)\) is of finite length if \(P(I)\) is Noetherian and Artinian with respect to strict epimorphisms and strict monomorphisms, respectively.

Definition 3.3. A very weak stability condition \(\sigma = (Z, A)\) is called locally finite if there exists \(\varepsilon > 0\) such that for any \(\phi \in \mathbb{R}\), the quasi-abelian category \(P((\phi - \varepsilon, \phi + \varepsilon))\) is of finite length.

In Definition 3.1 we let \(\Lambda_0\) be the saturation of the subgroup of \(\Lambda\) generated by \(\{v(E) : E \in A, Z(v(E)) = 0\}\).

Note that \(Z\) descends to the group homomorphism \(Z' : \Lambda/\Lambda_0 \rightarrow \mathbb{C}\). For \(w \in \Lambda\), we denote by \(w'\) its image in \(\Lambda/\Lambda_0\), and let \(\|\cdot\|\) be a fixed norm on \((\Lambda/\Lambda_0) \otimes_{\mathbb{Z}} \mathbb{R}\).

Definition 3.4. We say a very weak stability condition \(\sigma = (Z, A)\) satisfies the support property if there is a quadratic form \(Q\) on \(\Lambda/\Lambda_0\) satisfying \(Q(v(E)) \geq 0\) for any \(\nu_\sigma\)-semistable object \(E \in A\), and \(Q|_{\ker Z}\) is negative definite.

Remark 3.5. The local finiteness condition automatically follows if the support property is satisfied (cf. [10, Section 1.2] and [7, Lemma 4.5]).

3.2. Relative tilt-stability. Let \(\beta\) be a rational number and \(\alpha\) be a positive real number such that \(\alpha^2 \in \mathbb{Q}\). We will construct a family of very weak stability conditions on \(D^b(\mathcal{X})\) that depends on these two parameters. For brevity, we write \(\text{ch}\beta\) for the twisted Chern character \(\text{ch}\beta\).

There exists a torsion pair \((T_{\beta H}, F_{\beta H})\) in \(\text{Coh}(\mathcal{X})\) defined as follows:

\[
T_{\beta H} = \{ E \in \text{Coh}(\mathcal{X}) : \mu^-_C(E) > \beta \}
\]

\[
F_{\beta H} = \{ E \in \text{Coh}(\mathcal{X}) : \mu^+_C(E) \leq \beta \}.
\]

Equivalently, \(T_{\beta H}\) and \(F_{\beta H}\) are the extension-closed subcategories of \(\text{Coh}(\mathcal{X})\) generated by \(\mu_C\)-stable sheaves with \(\mu_C\)-slope \(> \beta\) and \(\leq \beta\), respectively.

Definition 3.6. We let \(\text{Coh}^\beta_\mathcal{H}(\mathcal{X}) \subset D^b(\mathcal{X})\) be the extension-closure

\[
\text{Coh}^\beta_\mathcal{H}(\mathcal{X}) = \langle T_{\beta H}, F_{\beta H}[1] \rangle.
\]
By the general theory of torsion pairs and tilting \cite{[8]}, \( \text{Coh}^{\beta H}(\mathcal{X}) \) is the heart of a bounded t-structure on \( \text{D}^b(\mathcal{X}) \); in particular, it is an abelian category. For any point \( s \in C \), similar as Definition \ref{def:3.6} one can define the subcategory

\[
\text{Coh}^{\beta H_s}(\mathcal{X}_s) = \langle \mathcal{T}_{\beta H_s}, \mathcal{F}_{\beta H_s}[1] \rangle \subset \text{D}^b(\mathcal{X}_s)
\]

via the \( \mu_{H_s} \)-stability (see \cite{[1]} Section 14.2).

Consider the following central charge

\[
z_{H,F}^{\alpha,\beta}(\mathcal{E}) = \frac{\alpha^2}{2} F H^{n-1} ch_0^\beta(\mathcal{E}) - F H^{n-3} ch_2^\beta(\mathcal{E}) + i F H^{n-2} ch_1^\beta(\mathcal{E}),
\]

here we set \( FH^{n-3} = 1 \) if \( n = 2 \). We think of it as the composition

\[
v(\mathcal{E}) = (F H^{n-1} ch_0(\mathcal{E}), F H^{n-2} ch_1(\mathcal{E}), F H^{n-3} ch_2(\mathcal{E})),
\]

and the second map is defined by

\[
Z_{H,F}^{\alpha,\beta}(e_0, e_1, e_2) = \frac{1}{2}(\alpha - \beta^2) e_0 + \beta e_1 - e_2 + i(e_1 - \beta e_0).
\]

We recall the classical Bogomolov inequality:

**Theorem 3.7.** Assume that \( \mathcal{E} \) is a \( \mu_{H,F} \)-semistable torsion free sheaf on \( \mathcal{X} \). Then we have

\[
F H^{n-3} \Delta(\mathcal{E}) := F H^{n-3}(ch_1^\beta(\mathcal{E}) - 2ch_0(\mathcal{E})ch_2(\mathcal{E})) \geq 0;
\]

\[
H^{n-2} \Delta(\mathcal{E}) := H^{n-2}(ch_2^\beta(\mathcal{E}) - 2ch_0(\mathcal{E})ch_2(\mathcal{E})) \geq 0.
\]

**Proof.** See \cite{[12]} Theorem 3.2. \( \square \)

A short calculation shows

\[
\Delta(\mathcal{E}) := (ch_1(\mathcal{E}))^2 - 2ch_0(\mathcal{E})ch_2(\mathcal{E})
= (ch_1^\beta(\mathcal{E}))^2 - 2ch_0^\beta(\mathcal{E})ch_2^\beta(\mathcal{E}).
\]

**Definition 3.8.** We define the generalized relative discriminants

\[
\Delta_{H,F}^{\beta H} := (F H^{n-2}ch_1^\beta)^2 - 2F H^{n-1}ch_0^\beta \cdot (F H^{n-3}ch_2^\beta)
\]

and

\[
\overline{\Delta}_{H,F}^{\beta H} := (F H^{n-2}ch_1^\beta)(H^{n-1}ch_1^\beta) - F H^{n-1}ch_0^\beta \cdot (H^{n-2}ch_2^\beta).
\]

A short calculation shows

\[
\overline{\Delta}_{H,F}^{\beta H} = (F H^{n-2}ch_1)^2 - 2F H^{n-1}ch_0 \cdot (F H^{n-3}ch_2) = \overline{\Delta}_{H,F}.
\]

when \( n \geq 3 \). Hence the first generalized relative discriminant \( \overline{\Delta}_{H,F}^{\beta H} \) is independent of \( \beta \) when \( n \geq 3 \). In general \( \overline{\Delta}_{H,F}^{\beta H} \) is not independent of \( \beta \), but we have

\[
\overline{\Delta}_{H,F}^{\beta H}(\mathcal{E} \otimes \mathcal{O}_X(mF)) = \overline{\Delta}_{H,F}^{\beta H}(\mathcal{E}),
\]

for any \( \mathcal{E} \in \text{D}^b(\mathcal{X}) \) and \( m \in \mathbb{Z} \).

**Lemma 3.9.** Let \( D \) be a \( \mathbb{Q} \)-divisor on \( \mathcal{X} \). Then we have

1. \( (H^{n-1}F)(F H^{n-3}D) \leq (H^{n-2}FD)^2 \) if \( n \geq 3 \);
2. \( (H^{n-1}F)(H^{n-2}D) \leq 2(H^{n-1}D)(H^{n-2}FD) \) if \( n \geq 2 \).
Proof. Since

\[ FH^{n-2}((FH^{n-1})D - (FDH^{n-2})H) = 0, \]

the Hodge index theorem gives

\[ FH^{n-2}((FH^{n-1})D - (FDH^{n-2})H)^2 \leq 0. \]

An easy computation shows the inequality (1) holds.

For the inequality (2), one notices that

\[ H^{n-1}((H^{n-1}F)D - (H^{n-1}D)F) = 0. \]

From the Hodge index theorem, it follows that

\[ H^{n-2}((H^{n-1}F)D - (H^{n-1}D)F)^2 \leq 0. \]

Expanding the left hand side of the above inequality, one obtains the desired conclusion.

By Lemma 3.9 and Theorem 3.7 we have:

Theorem 3.10. Assume that \( E \) is a \( \mu_{H,F} \)-semistable torsion free sheaf on \( X \). Then we have \( \Delta^H_{\mu_H} \geq 0 \) when \( n \geq 3 \) and \( \Delta^H_{\mu_H} \geq 0 \) when \( n \geq 2 \).

The following theorem gives a relative version of the tilt-stability in [4].

Theorem 3.11. For any \( (\alpha, \beta) \in \sqrt{Q_{>0}} \times Q \), \( \sigma_{H,F}^{\alpha,\beta} = (Z_H^{\alpha,\beta}, \text{Coh}_C^{\beta H}(X)) \) is a very weak stability condition.

Proof. Step 1. The pair \( \sigma_{H,F}^{\alpha,\beta} = (Z_H^{\alpha,\beta}, \text{Coh}_C^{\beta H}(X)) \) satisfies the positivity property for any \( 0 \neq E \in \text{Coh}_C^{\beta H}(X) \).

By the construction of \( \text{Coh}_C^{\beta H}(X) \), one sees that

\[ \Delta Z_{H,F}^{\alpha,\beta}(v(E)) = FH^{n-2} \text{ch}_1^\beta(E) - \beta FH^{n-1} \text{ch}_0(E) \geq 0, \]

for any \( E \in \text{Coh}_C^{\beta H}(X) \). Now we assume that \( FH^{n-2} \text{ch}_1^\beta(E) = 0 \). One obtains

\[ (3.1) \quad \text{ch}_0(\mathcal{H}^0(E)) = FH^{n-2} \text{ch}_1(\mathcal{H}^0(E)) = 0, \quad \mu_C(\mathcal{H}^0(E)) > \beta \]

and one of the following cases occurs:

1. \( \text{ch}_0(\mathcal{H}^{-1}(E)) > 0 \) and \( \mathcal{H}^{-1}(E) \) is \( \mu_C \)-semistable with \( \mu_C(\mathcal{H}^{-1}(E)) = \beta \);
2. \( \mathcal{H}^{-1}(E) \) is \( C \)-torsion and \( \mu_C^C(\mathcal{H}^{-1}(E)) \leq \beta \).

One sets \( G \) be the maximal subsheaf of \( \mathcal{H}^0(E) \) whose support has codimension \( \geq 2 \). Then \( \mathcal{H}^0(E)/G \) is pure and \( C \)-torsion. The condition 3.11 implies that

\[ H^{n-1} \text{ch}_1(\mathcal{H}^0(E)) = H^{n-1} \text{ch}_1(\mathcal{H}^0(E)/G) \geq 0 \]

and

\[ \mu_C(\mathcal{H}^0(E)/G) = H^{n-2} \text{ch}_2(\mathcal{H}^0(E)/G)/H^{n-1} \text{ch}_1(\mathcal{H}^0(E)/G) = H^{n-2} \text{ch}_2^\beta(\mathcal{H}^0(E)/G)/H^{n-1} \text{ch}_1(\mathcal{H}^0(E)/G) + \beta > \beta. \]

Thus we get \( H^{n-2} \text{ch}_2^\beta(\mathcal{H}^0(E)) = H^{n-2} \text{ch}_2^\beta(\mathcal{H}^0(E)/G) + H^{n-2} \text{ch}_2^\beta(G) \geq 0. \) When \( n \geq 3 \), one has \( FH^{n-3} \text{ch}_2^\beta(\mathcal{H}^0(E)/G) = 0 \) and \( FH^{n-3} \text{ch}_2^\beta(G) \geq 0 \). Hence we obtain

\[ FH^{n-3} \text{ch}_2^\beta(\mathcal{H}^0(E)) \geq 0 \]
when \( n \geq 2 \).

On the other hand, in the case (2), since \( \mu_C^+(H^{-1}(E)) \leq \beta \), one sees that \( H^{n-1} \text{ch}_1(H^{-1}(E)) > 0 \) and

\[
\mu_C(H^{-1}(E)) = \frac{H^{n-2} \text{ch}_2(H^{-1}(E))}{H^{n-1} \text{ch}_1(H^{-1}(E))} = \frac{H^{n-2} \text{ch}_2(H^{-1}(E))}{H^{n-1} \text{ch}_1(H^{-1}(E))} + \beta \leq \beta.
\]

These imply that \( H^{n-2} \text{ch}_2(H^{-1}(E)) \leq 0 \) when \( n \geq 2 \) and \( FH^{n-3} \text{ch}_2(H^{-1}(E)) = 0 \) when \( n \geq 3 \). Therefore in the case (2) we obtain

\[
\Re Z_{H,F}^{\alpha,\beta}(v(E)) = \frac{\alpha^2}{2} FH^{n-1} \text{ch}_0(E) - FH^{n-3} \text{ch}_2(E) \leq 0.
\]

For the case (1), we let \( T \) be the torsion part of \( H^{-1}(E) \). One sees that \( T \) is \( C \)-torsion and

\[
FH^{n-2} \text{ch}_1(T) = FH^{n-2} \text{ch}_1(H^{-1}(E)/T) = 0 \text{ and } \mu_C^+(T) \leq \beta.
\]

By the same way as the proof in the case (2), one obtains \( H^{n-2} \text{ch}_2(H^{-1}(E)) \leq 0 \) when \( n \geq 2 \) and \( FH^{n-3} \text{ch}_2(H^{-1}(E)) = 0 \) when \( n \geq 3 \). Thus \( FH^{n-3} \text{ch}_2(T) \leq 0 \). Since \( FH^{n-2} \text{ch}_1(H^{-1}(E)/T) = 0 \), we infer that

\[
\mu_{H,F}(H^{-1}(E)) = \mu_{H,F}(H^{-1}(E)/T) = \beta.
\]

Thus \( H^{-1}(E)/T \) is a \( \mu_{H,F} \)-semistable sheaf by Proposition 2.8. By Theorem 3.10 we have \( H^{n-2} \text{ch}_2(H^{-1}(E)/T) \leq 0 \) if \( n \geq 2 \) and \( FH^{n-3} \text{ch}_2(H^{-1}(E)/T) \leq 0 \) if \( n \geq 3 \). Hence \( FH^{n-3} \text{ch}_2(H^{-1}(E)) \leq 0 \), and in the case (1) we conclude \( \Re Z_{H,F}^{\alpha,\beta}(v(E)) < 0 \).

**Step 2.** The category \( \text{Coh}_C^{\beta H}(X) \) is noetherian, and \( \sigma_{H,F}^{\alpha,\beta} \) satisfies the Harder-Narasimhan property.

For every \( s \in C \) we let

\[
\sigma_s^{\alpha,\beta} := (Z_s = \text{ch}_1^{\beta}(X_s,0) + i \text{ch}_2^{\beta}(X_s))
\]

By Proposition 25.1, one sees that the collection \( \sigma^{\alpha,\beta} := (\sigma_s^{\alpha,\beta}) \) is a flat family of fiberwise weak stability condition on \( D^b(\mathcal{O}) \) over \( C \) (See Definition 20.5). Hence from Corollary 20.10 and Proposition 15.14, it follows that \( \text{Coh}_C^{\beta H}(X) \) has a \( C \)-torsion theory and is noetherian. Since \( Z_{H,F}^{\alpha,\beta} \) has discrete image, we conclude that the Harder-Narasimhan filtrations exist for objects in \( \text{Coh}_C^{\beta H}(X) \) with respect to \( Z_{H,F}^{\alpha,\beta} \) (cf. Lemma 2.18). \( \square \)

**Remark 3.12.** Let \( E \) be an object in \( \text{Coh}_C^{\beta H}(X) \) with \( Z_{H,F}^{\alpha,\beta}(v(E)) = 0 \). Since \( \text{Coh}_C^{\beta H}(X) \) has a \( C \)-torsion theory, we denote by \( \mathcal{E}_{C,\text{tor}} \) and \( \mathcal{E}_{C,\text{tf}} \) the \( C \)-torsion part and \( C \)-torsion free part of \( \mathcal{E} \), respectively. Then one sees that

\[
Z_{H,F}^{\alpha,\beta}(v(\mathcal{E}_{C,\text{tor}})) = Z_{H,F}^{\alpha,\beta}(v(\mathcal{E}_{C,\text{tf}})) = 0.
\]

By the proof of Theorem 3.11 we can deduce that \( H^{-1}(\mathcal{E}_{C,\text{tf}}) = 0 \) and \( H^0(\mathcal{E}_{C,\text{tf}}) \in \text{Coh}_{C_{n-3}}(X) \). In particular, \( \mathcal{E}_{C,\text{tf}} = 0 \) when \( n \leq 3 \). On the other hand, one has \( Z_{H,F}^{\alpha,\beta}(v(\mathcal{F})) = 0 \) for any \( C \)-torsion object \( \mathcal{F} \in \text{Coh}_C^{\beta H}(X) \) when \( n \geq 3 \). Hence we conclude that if \( n = 3 \), then \( \mathcal{F} \in \text{Coh}_C^{\beta H}(X) \) is \( C \)-torsion is equivalent to \( Z_{H,F}^{\alpha,\beta}(v(\mathcal{F})) = 0 \).
Remark 3.13. The pair $\sigma^{n,\beta}_\alpha = (Z^{n,\beta}_\alpha, \text{Coh}^{\beta}_C(\mathcal{X}))$ in Theorem 3.1 is not a stability condition, since

$$Z^{n,\beta}_\alpha(\mathcal{O}_F) = Z^{n,\beta}_\alpha(\mathcal{O}_F[1]) = 0$$

and one of $\mathcal{O}_F$ and $\mathcal{O}_F[1]$ is in $\text{Coh}^{\beta}_C(\mathcal{X})$.

Lemma 3.14. Let $\mathcal{E}$ be an object in $\text{Coh}^{\beta}_C(\mathcal{X})$.

1. We have $FH^{n-2}ch_1^\beta(\mathcal{E}) \geq 0$.
2. If $FH^{n-2}ch_1^\beta(\mathcal{E}) = 0$, then one has $H^{n-2}ch_2^\beta(\mathcal{E}) \geq 0$, $FH^{n-3}ch_3^\beta(\mathcal{E}) \geq 0$ and $ch_0(\mathcal{E}) \leq 0$.
3. If $FH^{n-2}ch_1^\beta(\mathcal{E}) = ch_0(\mathcal{E}) = H^{n-2}ch_2^\beta(\mathcal{E}) = 0$, then $H^0(\mathcal{E}) \in \text{Coh}^{\beta}_{\mathcal{X}}(\mathcal{X})$, $H^{-1}(\mathcal{E})$ is a $C$-torsion $\mu_C$-semistable sheaf with $\mu_C(H^{-1}(\mathcal{E})) = \beta$ and $H^{n-3}ch_3^\beta(\mathcal{E}) \geq 0$.

Proof. The first statement follows from the definition of $\text{Coh}^{\beta}_C(\mathcal{X})$. By Step 1 in the proof of Theorem 3.11 one obtains the second statement.

For the third statement, still by Step 1 in the proof of Theorem 3.11 one sees that if $FH^{n-2}ch_1^\beta(\mathcal{E}) = 0$, $ch_0(\mathcal{E}) = 0$ and $H^{n-2}ch_2^\beta(\mathcal{E}) = 0$ then the support of $H^0(\mathcal{E})$ has codimension $\geq 3$ and $H^{-1}(\mathcal{E})$ is a $C$-torsion $\mu_C$-semistable sheaf with $\mu_C(H^{-1}(\mathcal{E})) = \beta$. Hence under the assumptions of the third statement we have $H^{n-3}ch_3^\beta(H^0(\mathcal{E})) \geq 0$. Now we prove that $H^{n-3}ch_3^\beta(H^{-1}(\mathcal{E})) \leq 0$. Without loss of generality, we can assume that $H^{-1}(\mathcal{E})$ is a $C$-torsion sheaf set-theoretically supported over a closed point $s \in C$. Let $\pi$ be a local generator of $I_s$. By [11, Lemma 6.11], one obtains a filtration

$$0 = G_m \subset G_{m-1} \subset \cdots \subset G_1 \subset G_0 = H^{-1}(\mathcal{E})$$

where $G_j = \pi^j \cdot H^{-1}(\mathcal{E})$ and all filtration quotients $G_j/G_{j+1}$ are quotients of $G_0/G_1$ in $i_\ast(\text{Coh}(\mathcal{X}))$. Since $H^{-1}(\mathcal{E})$ is a $\mu_C$-semistable sheaf with $\mu_C(H^{-1}(\mathcal{E})) = \beta$, so are $G_j$ and $G_j/G_{j+1}$ for $j = 0, 1, \cdots, m - 1$. We write $G_j/G_{j+1} = i_\ast(\mathcal{F}_j)$, here $\mathcal{F}_j \in \text{Coh}(\mathcal{X}_s)$. Then from Bogomolov’s inequality for the semistable sheaves $\mathcal{F}_j$ on $\mathcal{X}_s$ and Lemma 2.27 it follows that

$$H^{n-3}ch_3^\beta(i_\ast(\mathcal{F}_j)) = H^{n-3}i_\ast(ch_3^\beta(\mathcal{F}_j)) = H^{n-3}ch_3^\beta(\mathcal{F}_j) \leq 0.$$ 

This implies

$$H^{n-3}ch_3^\beta(H^{-1}(\mathcal{E})) = \sum_{j=0}^{m-1} H^{n-3}ch_3^\beta(G_j/G_{j+1}) \leq 0.$$

Therefore one concludes that

$$H^{n-3}ch_3^\beta(\mathcal{E}) = H^{n-3}ch_3^\beta(H^0(\mathcal{E})) + H^{n-3}ch_3^\beta(H^{-1}(\mathcal{E})[1]) \geq 0.$$ 

This proves the third statement.

We write $\nu^{n,\beta}_H$ for the slope function on $\text{Coh}^{\beta}_C(\mathcal{X})$ induced by $Z^{n,\beta}_H$. Explicitly, for any $\mathcal{E} \in \text{Coh}^{\beta}_C(\mathcal{X})$, one has

$$\nu^{n,\beta}_H(\mathcal{E}) = \begin{cases} +\infty, & \text{if } FH^{n-2}ch_1^\beta(\mathcal{E}) = 0, \\ \frac{FH^{n-3}ch_3^\beta(\mathcal{E}) - \frac{\beta^2}{n^2}FH^{n-3}ch_3^\beta(\mathcal{E})}{FH^{n-2}ch_1^\beta(\mathcal{E})}, & \text{otherwise.} \end{cases}$$
Theorem 3.11 gives the notion of $\nu_{H,F}^{\alpha,\beta}$-stability. We can also consider the tilt-stability on the fibers of $f$. If $n \geq 3$, for any point $s \in C$, we define

$$
\sigma_s^{\alpha,\beta} := \left( Z_s^{\alpha,\beta} = iH^{n-2}ch_1^\beta s + \frac{\alpha^2}{2}H^{n-1}ch_0^\beta s - H_n^2ch_2^\beta s, \text{Coh}^{\beta H_s}(X_s) \right).
$$

This is the tilt-stability condition on $X_s$ defined in [3, Lemma 4.16.(2)]. We write $\nu_s^{\alpha,\beta}$ for the slope function on $\text{Coh}^{\beta H_s}(X_s)$ induced by $Z_s^{\alpha,\beta}$. One sees

$$
\nu_s^{\alpha,\beta}(E) = \nu_{H,F}^{\alpha,\beta}(E).
$$

We also call $\nu_{H,F}^{\alpha,\beta}$-stability relative tilt-stability.

**Lemma 3.15.** Let $E \in \text{Coh}^{\beta H}(X)$ be a $C$-torsion free object. Then the following conditions are equivalent:

1. $E$ is $\nu_{H,F}^{\alpha,\beta}$-(semi)stable;
2. $E_{K(C)}$ is $\nu_{K(C)}^{\alpha,\beta}$-(semi)stable;
3. there exists an open subset $U \subset C$ such that $E_s$ is $\nu_{s}^{\alpha,\beta}$-(semi)stable for any point $s \in U$.

**Proof.** By Lemma 2.3 one infers that any subobject $F$ of $E$ is $C$-flat. Thus $F_{K(C)} \in \text{Coh}^{\beta H_{K(C)}}(X_{K(C)})$. Since

$$
\nu_{K(C)}^{\alpha,\beta}(F_{K(C)}) = \nu_{H,F}^{\alpha,\beta}(F),
$$

from [3, Lemma 4.16.(2)], one deduces that $E$ is $\nu_{H,F}^{\alpha,\beta}$-(semi)stable if and only if $E_{K(C)}$ is $\nu_{K(C)}^{\alpha,\beta}$-(semi)stable. The implication “(3) \(\Rightarrow\) (2)” is obvious. For the other direction, by [3, Proposition 25.3], one sees that $(\sigma_s^{\alpha,\beta})_{s \in C}$ is a flat family of fiberwise weak stability condition on $D^b(X)$ over $C$. Hence [3, Definition 20.5.(2') and Lemma 20.4] gives the openness of tilt-stability and tilt-semistability. This completes the proof. \(\square\)

Similar to the slope $\mu_C$, we define the slope $\nu_C^{\alpha,\beta}$ of an object $E \in \text{Coh}^{\beta H}(X)$ by

$$
\nu_C^{\alpha,\beta}(E) = \begin{cases} 
\frac{FH^{n-3}ch_0^\beta(E) + \frac{\alpha^2}{2}FH^{n-1}ch_1^\beta(E)}{FH^{n-2}ch_1^\beta(E)}, & \text{if } FH^{n-2}ch_1^\beta(E) \neq 0, \\
\frac{H^{n-3}ch_0^\beta(E) + \frac{\alpha^2}{2}FH^{n-1}ch_1^\beta(E)}{H^{n-2}ch_2^\beta(E)}, & \text{if } E_{K(C)} = 0 \text{ and } H^{n-2}ch_2^\beta(E) \neq 0, \\
+\infty, & \text{otherwise.}
\end{cases}
$$

For $E \in D^b(X)$, we define

$$
Z_{K(C)}^{\alpha,\beta}(E) := \frac{\alpha^2}{2}FH^{n-1}ch_1^\beta(E) - FH^{n-3}ch_2^\beta(E) + iFH^{n-2}ch_1^\beta(E)
$$

and

$$
Z_{\text{C-tor}}^{\alpha,\beta}(E) := \frac{\alpha^2}{2}H^{n-1}ch_1^\beta(E) - H^{n-3}ch_3^\beta(E) + iH^{n-2}ch_2^\beta(E),
$$

respectively. Then by Lemma 2.7 one sees that

$$
Z_{K(C)}^{\alpha,\beta}(E) = Z_{\text{C-tor}}^{\alpha,\beta}(i_{W*}E_W),
$$

for all closed subscheme $W \subset C$, where $i_W : X_W \hookrightarrow X$ is the embedding of the fiber over $W$. From [3, Proposition 25.3], it follows that $(Z_{K(C)}^{\alpha,\beta}, Z_{\text{C-tor}}^{\alpha,\beta}, \text{Coh}^{\beta H}(X))$
is a weak Harder-Narasimhan structure on $D^b(\mathcal{X})$ over $C$. This gives the notion of $\nu_C^{\alpha,\beta}$-stability via the equality

$$
\nu_C^{\alpha,\beta}(\mathcal{E}) = \begin{cases} 
\frac{\mathbb{Z}^{\alpha,\beta}(\mathcal{E})}{\mathbb{Z}^{\alpha,\beta}(\mathcal{E})} & \text{if } \mathcal{E}_{\mathcal{K}(C)} \neq 0, \\
\frac{\mathbb{Z}^{\alpha,\beta}(\mathcal{E})}{\mathbb{Z}^{\alpha,\beta}(\mathcal{E})} & \text{otherwise}.
\end{cases}
$$

By [1] Lemma 15.7, one sees that $\nu_C^{\alpha,\beta}$-stability requires stability for all fibers:

**Proposition 3.16.** Let $\mathcal{E} \in \text{Coh}^{\beta H}_C(\mathcal{X})$ be a $C$-torsion free object. Then $\mathcal{E}$ is $\nu_C^{\alpha,\beta}$-semistable if and only if $\mathcal{E}$ is $\nu_{H,F}^{\alpha,\beta}$-semistable and for any closed point $p \in C$ and any quotient $\mathcal{E}_p \to \mathcal{Q}$ in $\text{Coh}^{\beta H_p}(X_p)$ we have $\nu_p^{\alpha,\beta}(\mathcal{E}_p) \leq \nu_p^{\alpha,\beta}(\mathcal{Q})$.

One can translate some basic properties of tilt-stability ([7] Proposition 14.2 and [1] Proposition 7.2.1) into relative tilt-stability.

**Lemma 3.17.** Let $\mathcal{E} \in \text{Coh}^{\beta H}_C(\mathcal{X})$ be a $C$-torsion free object.

1. If $\mathcal{E}$ is $\nu_{H,F}^{\alpha,\beta}$-semistable for $\alpha \gg 0$, then it satisfies one of the following conditions:
   (a) $\mathcal{H}^{-1}(\mathcal{E}) = 0$ and $\mathcal{H}^0(\mathcal{E})$ is a $\mu_{H,F}$-semistable torsion free sheaf.
   (b) $\mathcal{H}^{-1}(\mathcal{E}) = 0$ and $\mathcal{H}^0(\mathcal{E})$ is a torsion sheaf.
   (c) $\mathcal{H}^1(\mathcal{E})$ is a $\mu_{H,F}$-semistable torsion free sheaf and $\mathcal{H}^0(\mathcal{E})$ is a torsion sheaf with $FH^{n-2}ch_1(\mathcal{H}^0(\mathcal{E})) = 0$.

2. Let $\mathcal{F}$ be a $\mu_C$-stable locally free sheaf on $\mathcal{X}$ with

$$
\mathcal{X}_{H,F}(\mathcal{F}) = (FH^{n-2}ch_1(\mathcal{F}))^2 - 2FH^{n-1}ch_0(\mathcal{F}) \cdot (FH^{n-3}ch_2(\mathcal{F})) = 0.
$$

Then $\mathcal{F}$ or $\mathcal{F}[1]$ is a $\nu_{H,F}^{\alpha,\beta}$-stable object in $\text{Coh}^{\beta H}_C(\mathcal{X})$.

**Proof.** Assume that $\mathcal{E}$ is $\nu_{H,F}^{\alpha,\beta}$-semistable for $\alpha \gg 0$. One has the following exact sequence in $\text{Coh}^{\beta H}_C(\mathcal{X})$:

$$
0 \to \mathcal{H}^{-1}(\mathcal{E})[1] \to \mathcal{E} \to \mathcal{H}^0(\mathcal{E}) \to 0,
$$

where $\mathcal{H}^{-1}(\mathcal{E}) \in \mathcal{F}_{\beta H}$ and $\mathcal{H}^0(\mathcal{E}) \in \mathcal{T}_{\beta H}$. If $\mathcal{H}^{-1}(\mathcal{E}) = 0$ and $ch_0(\mathcal{H}^0(\mathcal{E})) \neq 0$, it is easy to see $\mathcal{H}^0(\mathcal{E})$ is a $\mu_{H,F}$-semistable torsion free sheaf by the definition $\nu_{H,F}^{\alpha,\beta}$.

Now we assume that $\mathcal{H}^{-1}(\mathcal{E}) \neq 0$. It turns out that

$$
\nu_{H,F}^{\alpha,\beta}(\mathcal{H}^{-1}(\mathcal{E})[1]) \leq \nu_{H,F}^{\alpha,\beta}(\mathcal{E}) \leq \nu_{H,F}^{\alpha,\beta}(\mathcal{H}^0(\mathcal{E}))
$$

for $\alpha \gg 0$. This implies

$$
F^{-1}ch_0^\beta(\mathcal{H}^{-1}(\mathcal{E})) \leq \frac{FH^{n-1}ch_0^\beta(\mathcal{E})}{FH^{n-2}ch_1^\beta(\mathcal{H}^{-1}(\mathcal{E}))} \leq \frac{FH^{n-1}ch_0^\beta(\mathcal{H}^0(\mathcal{E}))}{FH^{n-2}ch_1^\beta(\mathcal{H}^0(\mathcal{E}))}.
$$

Since $\mathcal{E}$ is $C$-torsion free, so is $\mathcal{H}^{-1}(\mathcal{E})$. Hence $\mathcal{H}^{-1}(\mathcal{E})$ is torsion free with $\mu_{H,F}^\beta(\mathcal{H}^{-1}(\mathcal{E})) \leq \beta$. From (3.2) and $\mu_C(\mathcal{H}^0(\mathcal{E})) > \beta$, one obtains

$$
FH^{n-2}ch_1^\beta(\mathcal{H}^0(\mathcal{E})) = ch_0(\mathcal{H}^0(\mathcal{E})) = 0.
$$

Thus one sees that

$$
\frac{FH^{n-1}ch_0^\beta(\mathcal{H}^{-1}(\mathcal{E}))}{FH^{n-2}ch_1^\beta(\mathcal{H}^{-1}(\mathcal{E}))} = \frac{FH^{n-1}ch_0^\beta(\mathcal{E})}{FH^{n-2}ch_1^\beta(\mathcal{E})}.
$$
For any subsheaf $K \subset \mathcal{H}^{-1}(\mathcal{E})$, we have $\nu_{H,F}^{\alpha,\beta}(K[1]) \leq \nu_{H,F}^{\alpha,\beta}(\mathcal{E})$ for $\alpha \gg 0$. This implies

$$\frac{F H^{-1} ch_{0}^{\beta}(\mathcal{H}^{-1}(\mathcal{E}))}{F H^{-2} ch_{1}^{\beta}(\mathcal{H}^{-1}(\mathcal{E}))} = \frac{F H^{-1} ch_{0}^{\beta}(\mathcal{E})}{F H^{-2} ch_{1}^{\beta}(\mathcal{E})} = \frac{F H^{-1} ch_{0}^{\beta}(K)}{F H^{-2} ch_{1}^{\beta}(K)}.$$  

Hence $\mathcal{H}^{-1}(\mathcal{E})$ is $\mu_{H,F}$-semistable. This concludes the first statement.

For the second statement, one notices that the $\mu_{C}$-stability of $\mathcal{F}$ implies that $\mathcal{F}$ or $\mathcal{F}[1]$ is an object in $\text{Coh}_{C}^{H}(\mathcal{X})$, and $\mathcal{F}_{s}$ is $\mu_{H,F}$-stable for a general $s \in \mathcal{C}$ by Lemma 2.4. By Proposition 7.4.1, one obtains the $\nu_{s}^{\alpha,\beta}$-stability of $\mathcal{F}_{s}$ or $\mathcal{F}_{s}[1]$. Hence Lemma 3.15 gives the second argument.

We now give the Bogomolov–Gieseker type inequality for relative tilt-stable complexes.

**Theorem 3.18.** If $\mathcal{E} \in \text{Coh}^{H}_{C}(\mathcal{X})$ is $\nu_{H,F}^{\alpha,\beta}$-semistable and $n \geq 3$, then

$$\Sigma_{H,F}(\mathcal{E}) \geq 0.$$  

**Proof.** If $F H^{-n} ch_{1}^{\beta}(\mathcal{E}) = 0$, by Lemma 3.14 one easily verifies that $\mathcal{E}$ satisfies the conclusion.

Now we assume that $F H^{-n} ch_{1}^{\beta}(\mathcal{E}) > 0$. Since the $\nu_{H,F}^{\alpha,\beta}$-slope of any $C$-torsion object in $\text{Coh}^{H}_{C}(\mathcal{X})$ is $+\infty$, by the $\nu_{H,F}^{\alpha,\beta}$-semistability of $\mathcal{E}$, one infers that $\mathcal{E}$ is $C$-torsion free. Hence Lemma 3.15 gives the $\nu_{s}^{\alpha,\beta}$-semistability of $\mathcal{E}_{s}$ for general $s \in \mathcal{C}$. From [1] Theorem 7.3.1 (see also [3] Theorem 3.5), it follows that

$$\Sigma_{H,F}(\mathcal{E}) = (H_{s}^{-n} ch_{1}(\mathcal{E}_{s}))^{2} - 2H_{s}^{-n} ch_{0}(\mathcal{E}_{s}) \cdot (H_{s}^{-n-3} ch_{2}(\mathcal{E}_{s})) \geq 0.$$  

\[Q := (F H^{-n} ch_{1}^{\beta})^{2} - 2(F H^{-1} ch_{0}^{\beta})(F H^{-n-3} ch_{2}^{\beta})\]

on $\Lambda := \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}$. Then from Theorem 3.18 one deduces that $Q$ satisfies the conditions in Definition 3.3. Hence $\sigma_{H,F}^{\alpha,\beta}$ satisfies the support property.

4. **Mixed tilt-stability**

In this section, we will introduce the mixed tilt-stability. We keep the same notations as that in the previous sections.

Let $t$ be a non-negative rational number. Consider the following central charge

$$z_{\alpha,\beta,t}(\mathcal{E}) = \frac{(t + 1)\alpha^{2}}{2} F H^{-1} ch_{0}(\mathcal{E}) - (H^{-2} + t F H^{-3}) ch_{1}^{\beta}(\mathcal{E}) + i F H^{-2} ch_{1}^{\beta}(\mathcal{E}).$$

We think of it as the composition

$$z_{\alpha,\beta,t} : K(D^{b}(\mathcal{X})) \rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{2}\mathbb{Z} Z_{\alpha,\beta,t} \rightarrow \mathbb{C},$$

where the first map is given by

$$v_{t}(\mathcal{E}) = (F H^{-1} ch_{0}(\mathcal{E}), F H^{-2} ch_{1}(\mathcal{E}), H^{-2} + t F H^{-3}) ch_{2}(\mathcal{E}),$$
and the second map is defined by
\[
Z_{\alpha, \beta, t}(e_0, e_1, e_1', e_2) = \frac{1}{2} (t + 1) \alpha^2 - \left( \frac{H^n}{FH^{n-1}} + t \right) \beta^2 \right) e_0 + \beta(e_1' + t e_1) - e_2 + \epsilon(e_1 - \beta e_0).
\]

**Theorem 4.1.** For any \((\alpha, \beta, t) \in \sqrt{Q_{>0}} \times Q \times Q_{\geq 0}\), \(\sigma_{\alpha, \beta, t} = (Z_{\alpha, \beta, t}, \text{Coh}_{\alpha}^{\beta}(\mathcal{X}))\) is a very weak stability condition.

**Proof.** The positivity property follows from Lemma 3.14. The proof of the Harder-Narasimhan property is the same as that of Theorem 3.11. \(\square\)

We write \(\nu_{\alpha, \beta, t}\) for the slope function on \(\text{Coh}_{\alpha}^{\beta}(\mathcal{X})\) induced by \(Z_{\alpha, \beta, t}\). Explicitly, for any \(\mathcal{E} \in \text{Coh}_{\alpha}^{\beta}(\mathcal{X})\), one has
\[
\nu_{\alpha, \beta, t}(\mathcal{E}) = \begin{cases} +\infty, & \text{if } FH^{-2} \text{ch}^2(\mathcal{E}) = 0, \\ \frac{(H^{-2} + t F H^{-3}) \text{ch}^2(\mathcal{E}) - t F H^{-1} \text{ch} \mathcal{E})}{FH^{-2} \text{ch}^2(\mathcal{E})}, & \text{otherwise.} \end{cases}
\]

Theorem 4.1 gives the notion of \(\nu_{\alpha, \beta, t}\)-stability. Since \(\nu_{\alpha, \beta, t} = \nu_{\alpha, \beta, 0} + t \nu_{H,F}\), we also call \(\nu_{\alpha, \beta, t}\)-stability mixed tilt-stability.

**Lemma 4.2.** Let \(\mathcal{E} \in \text{Coh}_{\alpha}^{\beta}(\mathcal{X})\) be a \(C\)-torsion free object. If \(\mathcal{E}\) is \(\nu_{\alpha, \beta, t}\)-semistable for \(\alpha \gg 0\), then it satisfies one of the following conditions:

1. \(H^{-1}(\mathcal{E}) = 0\) and \(H^0(\mathcal{E})\) is a \(\mu_{H,F}\)-semistable torsion free sheaf.
2. \(H^{-1}(\mathcal{E}) = 0\) and \(H^0(\mathcal{E})\) is a torsion sheaf.
3. \(H^{-1}(\mathcal{E})\) is a \(\mu_{H,F}\)-semistable torsion free sheaf and \(H^0(\mathcal{E})\) is a torsion sheaf with \(FH^{-2} \text{ch} \mathcal{E}(H^0(\mathcal{E})) = 0\).

**Proof.** The proof is the same as that of Lemma 3.17. \(\square\)

We now show the Bogomolov-Gieseker type inequality of mixed tilt-stable complexes.

**Theorem 4.3.** Let \(\mathcal{E} \in \text{Coh}_{\alpha}^{\beta}(\mathcal{X})\) be a \(\nu_{\alpha, \beta, t}\)-semistable object, and set \(H_t = H + t F\). Then we have
\[
\Delta_{H,F,t}(\mathcal{E}) := (FH^{-2} \text{ch}^2(\mathcal{E}))(H_t H^{-1} \text{ch}^2(\mathcal{E})) - FH^{-1} \text{ch}^2(\mathcal{E}) \cdot (H_t H^{-1} \text{ch}^2(\mathcal{E}))
\]
\[
= \Delta_{H,F}(\mathcal{E}) + \frac{t}{2} \Sigma_{H,F}(\mathcal{E}) + \frac{t}{2} (FH^{-2} \text{ch}^2(\mathcal{E}))^2 \geq 0.
\]

**Proof.** The proof is a mimic of that of [3] Theorem 3.5. We proceed by induction on \(FH^{-2} \text{ch}^2(\mathcal{E})\), which is a non-negative function with discrete values on objects of \(\text{Coh}_{\alpha}^{\beta}(\mathcal{X})\).

In the case of \(FH^{-2} \text{ch}^2(\mathcal{E}) = 0\), by Lemma 3.14 one infers \(\Delta_{H,F}(\mathcal{E}) \geq 0\). Now we assume that \(FH^{-2} \text{ch}^2(\mathcal{E}) > 0\). Thus \(\mathcal{E}\) is \(C\)-torsion free. We start increasing \(\alpha\). If \(\mathcal{E}\) remains stable as \(\alpha \rightarrow +\infty\), by Lemma 4.2 one sees that one of the following holds:
(1) $E$ is a $\mu_{H,F}$-semistable torsion free sheaf.
(2) $E$ is a torsion sheaf.
(3) $H^{-1}(E)$ is a $\mu_{H,F}$-semistable torsion free sheaf and $H^0(E)$ is a torsion sheaf with $FH^{n-2}\text{ch}_1(H^0(E)) = 0$.

One easily verifies that $E$ satisfies the conclusion in any of the possible cases by Theorem 3.10 and Lemma 3.14.

Otherwise, $E$ will get destabilized for some $\alpha_1 > \alpha$ with $\alpha^2 \in \mathbb{Q}$.

Consider the set
\[ W := \{ z_{\alpha,\beta,t}(\mathcal{K}) : 0 \neq \mathcal{K} \subset E \text{ and } \nu_{\alpha,\beta,t}(\mathcal{K}) > \nu_{\alpha,\beta,t}(E) \}. \]

Since $\nu_{\alpha,\beta,t}(\mathcal{K}) \leq \nu_{\alpha,\beta,t}^{-}(E)$, $FH^{n-2}\text{ch}_1(\mathcal{K}) \leq FH^{n-2}\text{ch}_1(E)$ and the image of $z_{\alpha,\beta,t}$ is discrete, one sees that $W$ is a finite subset of $\mathbb{C}$. For any element $w \in W$, we set
\[ M_w = \{ \mathcal{K} : \mathcal{K} \subset E \text{ and } z_{\alpha,\beta,t}(\mathcal{K}) = w \}. \]

By the discreteness of $z_{\alpha,\beta,t}$, one can find $\mathcal{K}_w \in M_w$ with
\[ \nu_{\alpha,\beta,t}(\mathcal{K}_w) = \max_{\mathcal{K} \in M_w} \nu_{\alpha,\beta,t}(\mathcal{K}). \]

Since $\nu_{\alpha,\beta,t}(\mathcal{K}_w) \leq \nu_{\alpha,\beta,t}(\mathcal{E})$ and $\nu_{\alpha,\beta,t}(\mathcal{K}_w) > \nu_{\alpha,\beta,t}(\mathcal{E})$, we can find $\alpha_w \in \sqrt{\mathbb{Q}}_{>0}$ such that $\alpha \leq \alpha_w < \alpha_1$ and $\nu_{\alpha_w,\beta,t}(\mathcal{K}_w) = \nu_{\alpha_w,\beta,t}(\mathcal{E})$.

This implies $\nu_{\alpha_w,\beta,t}(\mathcal{K}) \leq \nu_{\alpha_w,\beta,t}(\mathcal{E})$ for any $\mathcal{K} \in M_w$. Taking $\alpha_0 = \min_{w \in W} \alpha_w$, since $\nu_{\alpha,\beta,t}$ is a linear function of $\alpha$, one can easily check that $E$ is strictly $\nu_{\alpha_0,\beta,t}$-semistable. Let
\[ 0 \rightarrow \mathcal{E}_1 \rightarrow E \rightarrow \mathcal{E}_2 \rightarrow 0 \]
be a short exact sequence where both $\mathcal{E}_1$ and $\mathcal{E}_2$ have the same $\nu_{\alpha_0,\beta,t}$ slope. Since both $\mathcal{E}_1$ and $\mathcal{E}_2$ have strictly smaller $FH^{n-2}\text{ch}_1(\mathcal{E})$, by the induction assumption we have $\Delta_{H,F,t}^\beta(\mathcal{E}_1) \geq 0$ and $\Delta_{H,F,t}^\beta(\mathcal{E}_2) \geq 0$.

On the other hand, we think of $\Delta_{H,F,t}^\beta$ as a composition
\[ K(D^{\beta}(\mathcal{X})) \xrightarrow{\nu^\beta} \mathbb{R} \oplus \text{NS}(\mathcal{X})_{\mathbb{R}} \oplus \mathbb{R} \xrightarrow{q^\beta} \mathbb{R}, \]
where $\nu^\beta$ is given by
\[ \nu^\beta_t(E) = (FH^{n-1}\text{ch}_0(E), \text{ch}_1^\beta(E), H_tH^{n-3}\text{ch}_2^\beta(E)) \]
and $q^\beta$ is the quadratic form
\[ q_t^\beta(r,c,d) = (H_tH^{n-2}c)(FH^{n-2}c) - rd. \]

Let $Z : \mathbb{R} \oplus \text{NS}(\mathcal{X})_{\mathbb{R}} \oplus \mathbb{R} \rightarrow \mathbb{C}$ be the linear map defined by
\[ Z(r,c,d) = \frac{(t+1)}{2}a^2r - d + iFH^{n-2}c. \]

It obvious that the kernel of $Z$ is semi-negative definite with respect to $q_t^\beta$. Since
\[ \nu^\beta_t(E) = \nu^\beta_t(\mathcal{E}_1) + \nu^\beta_t(\mathcal{E}_2), \]
we deduce $\Delta_{H,F,t}^\beta(\mathcal{E}) \geq 0$ by [3] Lemma 11.7].

**Remark 4.4.** I have no examples of quadratic forms satisfies the conditions in Definition 3.3 for $\sigma_{\alpha,\beta,t}$. I do not think they exist.

The following lemma gives a relation between relative tilt-stability and mixed tilt-stability.
Lemma 4.5. Assume that $n \geq 3$. Let $E \in \text{Coh}_{\beta}^{H}(\mathcal{X})$ be a $\nu_{H,F}^{H,F}$-stable object. Then there exists a non-negative rational number $t_0$ only depending on $\alpha, \beta$ and $E$, such that $E$ is $\nu_{H,F}$-stable for any $t \geq t_0$.

**Proof.** Since $\nu_{H,F}^{H,F} = \nu_{H,F}^{\alpha,\beta} + tv_{H,F}^{\alpha,\beta}$, one sees $E$ is $\nu_{H,F}$-stable for any $t \geq 0$ if $E$ is $\nu_{H,F}^{\alpha,\beta}$-stable. Now we assume that $E$ is not $\nu_{H,F}$-stable. Let $\nu_1$ be the maximal $\nu_{H,F}$-slope of $E$. Then $\nu_1 \geq \nu_{H,F}(E)$. Let

$$\nu_2 = \max \left\{ \nu_{H,F}(F) : \text{F is a subobject of } E \text{ with } FH^{n-2}ch_{1}^{\beta}(F) < FH^{n-2}ch_{1}^{\beta}(F) \right\}.$$ 

The $\nu_{H,F}$-stability of $E$ implies that $\nu_2 < \nu_{H,F}(E)$. Therefore, when

$$t > \frac{\nu_1 - \nu_{H,F}(E)}{\nu_{H,F}(E) - \nu_2}$$

one sees that

$$\nu_{H,F}(E) = \nu_{H,F}(E) + tv_{H,F}(E)$$

$$\geq \nu_{H,F}(F) + tv_{H,F}(F)$$

$$= \nu_{H,F}(F)$$

for any subobject $F \subset E$ with

$$FH^{n-2}ch_{1}^{\beta}(F) < FH^{n-2}ch_{1}^{\beta}(E).$$

This completes the proof. \qed

Let $\mathbb{D}(-) := \mathbb{R}\text{Hom}(-, \mathcal{O}_{\mathcal{X}})[1]$ denote the duality functor. We will show that there is a double-dual operation on $\text{Coh}_{\beta}^{H}(\mathcal{X})$ as well as for coherent sheaves. This is a relative analogy of [2] Lemma 2.19.

Lemma 4.6. Let $E$ be an object in $\text{Coh}_{\beta}^{H}(\mathcal{X})$ with $\nu_{\alpha,\beta}^{+}(E) < +\infty$ and $E^{\ast}$ the cohomology object $\mathcal{H}_{\text{Coh}_{\beta}^{H}(\mathcal{X})}^{0}(\mathbb{D}(E))$.

1. There exists an exact triangle

$$E^{\ast} \rightarrow \mathbb{D}(E) \rightarrow Q$$

with $\mathcal{H}^{j}(Q) = 0$ for $j \leq 0$ and $\mathcal{H}^{j}(Q)$ a torsion sheaf supported in codimension at least $j + 2$ for $j \geq 1$.

2. There exists an exact sequence in $\text{Coh}_{\beta}^{H}(\mathcal{X})$

$$0 \rightarrow E \rightarrow E^{\ast} \rightarrow E^{\ast}/E \rightarrow 0,$$

with $E^{\ast}/E \in \text{Coh}_{\beta}^{H}(\mathcal{X})$, and $E^{\ast}/E$ is quasi-isomorphic to a two term complex $B^{-1} \rightarrow B^{0}$ with $B^{-1}$ locally-free and $B^{0}$ reflexive.

**Proof.** The proof is similar to that of [2] Lemma 2.19. We sketch it here for reader’s convenience. Let

$$D_{\alpha,\beta}^{0} = \{ E \in \text{D}^{b}(\mathcal{X}) : \mathcal{H}^{i}(E) \in T_{\pm H}, \mathcal{H}^{i}(E) = 0 \text{ for } i > 0 \}$$

$$D_{\pm,\beta}^{0} = \{ E \in \text{D}^{b}(\mathcal{X}) : \mathcal{H}^{i}(E) \in F_{\pm H}, \mathcal{H}^{i}(E) = 0 \text{ for } i < 0 \}.$$
By the general theory of torsion pairs and tilting [3], $\mathbb{D}_{\frac{\mu_+}{\beta}, \frac{\mu_-}{\beta}}(\mathbb{D}^b_{\mathcal{A}})$ is a bounded $t$-structure on $\mathbb{D}^b_{\mathcal{A}}$. We denote by $(\tau_{\frac{\mu_-}{\beta}, \frac{\mu_+}{\beta}})$ the associated truncation functors. We also write $(\mathbb{D}^f_{\mu_-}, \mathbb{D}^g_{\mu_+})$ for the standard $t$-structure on $\mathbb{D}^b_{\mathcal{A}}$ and $(\tau_{\frac{\mu_-}{\beta}, \frac{\mu_+}{\beta}})$ for the associated truncation functors.

We first notice that for a coherent sheaf $G \in \text{Coh}(\mathcal{X})$, the complex $\mathbb{D}(G)$ satisfies

$$\mathcal{H}^j(\mathbb{D}(G)) = \begin{cases} 0, & \text{if } j < -1, \\ \text{Hom}(G, \mathcal{O}_\mathcal{X}), & \text{if } j = -1, \\ \text{Ext}^{j+1}(G, \mathcal{O}_\mathcal{X}), & \text{if } j \geq 0, \end{cases}$$

where $\text{Ext}^{j+1}(G, \mathcal{O}_\mathcal{X})$ is a sheaf supported in codimension $\geq j + 1$. In particular, if $G$ is supported in codimension $k$, then $\mathcal{H}^{k-1}(\mathbb{D}(G))$ is the smallest degree with a nonvanishing cohomology sheaf.

(1) Dualizing the triangle $\mathcal{H}^{-1}(\mathcal{E})[1] \rightarrow \mathcal{E} \rightarrow \mathcal{H}^0(\mathcal{E})$, one gets an exact triangle $\mathbb{D}(\mathcal{H}^0(\mathcal{E})) \rightarrow \mathbb{D}(\mathcal{E}) \rightarrow \mathbb{D}(\mathcal{H}^{-1}(\mathcal{E})[1])$.

Taking the long exact cohomology sequence, we first see that $\mathcal{H}^j(\mathbb{D}(\mathcal{E})) = 0$ for $j < -1$ and

$$\text{Hom}(\mathbb{D}(\mathcal{E}), \mathcal{O}_\mathcal{X}) \cong \mathcal{H}^{-1}(\mathbb{D}(\mathcal{E})).$$

Since $\mu_C(\mathcal{H}^0(\mathcal{E})) > \beta$, we infer that $\mathcal{H}^{-1}(\mathbb{D}(\mathcal{E}))$ is either zero or a torsion free sheaf with $\mu_C(\mathcal{H}^{-1}(\mathbb{D}(\mathcal{E}))) < -\beta$, i.e., $\mathcal{H}^{-1}(\mathbb{D}(\mathcal{E})) \in \mathcal{F}_{-\beta}$.

We next obtain a long exact sequence

$$0 \rightarrow \text{Ext}^1(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_\mathcal{X}) \rightarrow \mathcal{H}^0(\mathbb{D}(\mathcal{E})) \rightarrow \text{Hom}(\mathcal{H}^{-1}(\mathcal{E}), \mathcal{O}_\mathcal{X}) \rightarrow \text{Ext}^2(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_\mathcal{X}) \rightarrow \text{Ext}^1(\mathcal{H}^{-1}(\mathcal{E}), \mathcal{O}_\mathcal{X}) \rightarrow \cdots$$

As $\nu_{\alpha, \beta, \delta}(\mathcal{E}) < +\infty$, one sees that any subobject of $\mathcal{E}$ has non-zero $H^{n-2}F \text{ch}_1^\delta$.

So are the subobjects of $\mathcal{H}^{-1}(\mathcal{E})$. From the definition of $\mathcal{F}_{-\beta}$ and $\mu_C$, it follows that $\mathcal{H}^{-1}(\mathcal{E})$ is a torsion free sheaf with $\mu_C(\mathcal{H}^{-1}(\mathcal{E})) < -\beta$. This implies $\mu_{\mathcal{H}, \mathcal{F}}(\text{Hom}(\mathcal{H}^{-1}(\mathcal{E}), \mathcal{O}_\mathcal{X})) > -\beta$. Since the support of $\text{Ext}^1(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_\mathcal{X})$ is of codimension at least two, we have $\mu_{\mathcal{H}, \mathcal{F}}(\ker \delta) > -\beta$. As $\text{Ext}^1(\mathcal{H}^0(\mathcal{E}), \mathcal{O}_\mathcal{X})$ is a torsion sheaf, one deduces that $\mathcal{H}^0(\mathbb{D}(\mathcal{E})) \subseteq K_{-\beta}$.

Let $T := \tau_{-\alpha}(\mathcal{H}^0(\mathbb{D}(\mathcal{E})))$ be the torsion part of $\mathcal{H}^0(\mathbb{D}(\mathcal{E}))$ with respect to the torsion pair $(\mathcal{F}_{-\beta}, \mathcal{F}_{-\beta})$. Since the Harder-Narasimhan filtration of $\mathcal{H}^0(\mathbb{D}(\mathcal{E}))$ with respect to the slope $\mu_C$ induces the Harder-Narasimhan filtration of $\mathcal{H}^0(\mathbb{D}(\mathcal{E}))$, one infers that $T_{K(C)} = \mathcal{H}^0(\mathbb{D}(\mathcal{E}))_{K(C)}$, and thus $\mathcal{H}^0(\mathbb{D}(\mathcal{E}))/T$ is a $C$-torsion sheaf in $\mathcal{F}_{-\beta}$.

We now consider $\tau_{-\beta}(\mathbb{D}(\mathcal{E}))$ and $Q := \tau_{-\beta}(\mathbb{D}(\mathcal{E}))$. By the definition of $\tau_{-\beta}$ and $\tau_{-1}$, one sees

$$T = \mathcal{H}^0(\tau_{-\beta}(\mathbb{D}(\mathcal{E}))) \subseteq \mathcal{F}_{-\beta} \text{ and } \mathcal{H}^0(\mathcal{Q}) = \mathcal{H}^0(\mathbb{D}(\mathcal{E}))/T \subseteq \mathcal{F}_{-\beta}.$$
$H^j(Q)$ is supported in codimension at least $\geq j + 2$ if and only if $D(H^j(Q)[−j]) \in D^{>1}$. Assume for contradiction that there is a largest possible $j_0 > 0$ such that

$$H^0(D(H^{j_0}(Q)[−j_0])) \neq 0.$$  

By induction on the number of non-zero cohomology objects, we see that

$$D(\tau^k Q), \ D(\tau^{\leq k} Q), \ D(Q) \in D^{>0}$$  

for all $k \in \mathbb{Z}$, $H^i(D(\tau^k Q))$ is supported in codimension at least two for any $i \geq 0$ and $l \geq 1$, $D(\tau^{j_0+1} Q) \in D^{>1}$ and $H^1(D(\tau^{j_0+1} Q))$ is supported in codimension at least $j_0 + 3$. Dualizing the exact triangle

$$H^{j_0}(Q)[-j_0] \to \tau^{j_0} Q \to \tau^{j_0+1} Q$$

and taking its long exact cohomology sequence, one gets the exact sequence

$$0 \to H^0(D(\tau^{j_0} Q)) \to H^0(D(H^{j_0}(Q)[−j_0])) \to H^1(D(\tau^{j_0+1} Q)).$$

Since the middle object is supported in codimension exactly $j_0 + 1$, and the right object is supported in codimension $j_0 + 3$, it follows that $H^0(D(\tau^{j_0} Q)) \neq 0$. Dualizing the exact triangle

$$\tau^{<j_0} Q \to Q \to \tau^{j_0} Q$$

gives an injection

$$0 \neq H^0(D(\tau^{j_0} Q)) \hookrightarrow H^0(D(Q)).$$

Similarly, dualizing the exact triangle

$$H^0(Q) \to Q \to \tau^{>1} Q$$

gives

$$(4.1) \quad 0 \to H^0(D(\tau^{>1} Q)) \to H^0(D(Q)) \to Ext^1(H^0(Q), O_X) \to H^1(D(\tau^{>1} Q)),$$  

thus $H^0(D(Q))$ is a torsion sheaf. Now consider the exact triangle

$$D(Q) \to D(D(E)) = E \to D(E^*).$$

The same arguments as before show $H^{-1}(D(E^*))$ is a torsion free sheaf in $F_{\beta H}$, and hence

$$\text{Hom}(H^0(D(Q)), D(E^*)[-1]) = \text{Hom}(H^0(D(Q)), H^{-1}(D(E^*))) = 0.$$  

Therefore, the composition $H^0(D(Q)) \to D(Q) \to E$ is non-zero. So is

$$H^0(D(\tau^{j_0} Q)) \hookrightarrow H^0(D(Q)) \to E.$$  

This is a contradiction to $\nu^+_{\alpha, \beta, t}(E) < +\infty$.

Now we show that $H^0(Q) = 0$. Assume for contradiction that $H^0(Q) \neq 0$. Then $Ext^1(H^0(Q), O_X)$ is a $C$-torsion sheaf supported in codimension one. Since $H^0(D(\tau^{>1} Q)) = 0$, from (4.1), it follows that $H^0(D(Q))$ is $C$-torsion. This is still a contradiction to $\nu^+_{\alpha, \beta, t}(E) < +\infty$.

From the above proof, we see that

$$E^* = \tau^{<0}(D(E)) \text{ and } Q = \tau^{>1}(D(E)).$$
Proof. It turns out that \( \text{Coh}^\beta C \) with \( \nu_{\alpha,\beta,t}(E) < +\infty \), there exists a short exact sequence \( E \hookrightarrow E^{**} \rightarrow E^0 \in \text{Coh}^{\leq_n - 3}(\mathcal{X}) \) and \( \text{Hom}(\text{Coh}^{\leq_n - 3}(\mathcal{X}), E^{**}[1]) = 0 \).

Proof. It turns out that \( C^0 \) is abelian. As we showed in the proof of Theorem 3.11, \( \text{Coh}^\beta C \) is noetherian. So is \( C^0 \). It follows that we can find a maximal subobject \( \tilde{E} \in C^0 \) of \( E \) satisfying property (1).

The proof of property (2) is the same as that of \([2\text{ Proposition 2.18}]. \)
5. Construction and Conjecture

In this section, we give the construction of the heart \( \mathcal{A}^\alpha,\beta_t(\mathcal{X}) \) of a bounded \( t \)-structure on \( \mathcal{D}^b(\mathcal{X}) \) as a tilt starting from \( \text{Coh}_{\mathbb{C}}^H(\mathcal{X}) \), and state our main conjectures. We always assume that \( n = \dim \mathcal{X} = 3 \) throughout this section.

We consider the torsion pair \((T'_t, F'_t)\) in \( \text{Coh}_{\mathbb{C}}^H(\mathcal{X}) \) as follows:

\[
T'_t = \{ E \in \text{Coh}_{\mathbb{C}}^H(\mathcal{X}) : \text{any quotient } E \rightarrow G \text{ satisfies } \nu_{\alpha,\beta,t}(G) > 0 \} \\
F'_t = \{ E \in \text{Coh}_{\mathbb{C}}^H(\mathcal{X}) : \text{any subobject } K \hookrightarrow E \text{ satisfies } \nu_{\alpha,\beta,t}(K) \leq 0 \}.
\]

**Definition 5.1.** We define the abelian category \( \mathcal{A}^\alpha,\beta_t(\mathcal{X}) \subset \mathcal{D}^b(\mathcal{X}) \) to be the extension-closure

\[
\mathcal{A}^\alpha,\beta_t(\mathcal{X}) = \langle T'_t, F'_t[1] \rangle.
\]

We propose the following conjecture which can be considered as a relative analogy of [4 Conjecture 1.3.1].

**Conjecture 5.2.** There exists a triple \((\alpha, \beta, t)\) in \( \sqrt{\mathbb{Q}_{\geq 0}} \times \mathbb{Q} \times \mathbb{Q}_{\geq 0} \) such that

\[
\text{ch}^\beta(E) \leq (a_1 H^2 + b_1 HF) \text{ch}_1^\beta(E) + (a_2 H + b_2 F) \text{ch}_2^\beta(E) + c \text{ch}_0^\beta(E)
\]

for any \( \nu_{\alpha,\beta,t}(E) \in \text{Coh}_{\mathbb{C}}^H(\mathcal{X}) \) with \( \nu_{\alpha,\beta,t}(E) = 0 \), where the constants \( a_1, b_1, a_2, b_2 \) and \( c \) are independent of \( E \) and \( a_1 > 0 \).

Consider the following central charge

\[
z_t = (a_1 H^2 + tHF) \text{ch}_1^\beta + (a_2 H + b_2 F) \text{ch}_2^\beta + c \text{ch}_0^\beta - \text{ch}_3^\beta + i \left( (H + tF) \text{ch}_3^\beta(E) - \frac{t+1}{2} a^2 F H^2 \text{ch}_0(E) \right).
\]

We think of it as the composition

\[
z_t : K(\mathcal{D}^b(\mathcal{X})) \xrightarrow{\tilde{\nu}} \mathbb{Z} \oplus \mathbb{Z} \oplus \frac{1}{2} \mathbb{Z} \oplus \frac{1}{2} \mathbb{Z} \oplus \frac{1}{6} \mathbb{Z} \xrightarrow{Z_t} \mathbb{C},
\]

where the first map is given by

\[
\tilde{\nu}(E) = (\text{ch}_0(E), F H \text{ch}_1(E), H^2 \text{ch}_1(E), F \text{ch}_2(E), H \text{ch}_2(E), \text{ch}_3(E)).
\]

**Conjecture 5.2** implies the existence of stability conditions on \( \mathcal{X} \).

**Theorem 5.3.** Assume Conjecture 5.2 holds for \((\alpha, \beta, t) \in \sqrt{\mathbb{Q}_{\geq 0}} \times \mathbb{Q} \times \mathbb{Q}_{\geq 0} \), then the pair \((Z_t, \mathcal{A}^{\alpha,\beta}_t(\mathcal{X}))\) is a locally finite stability condition on \( \mathcal{X} \) if \( l > \max\{b_1, 0\} \).

**Proof.** **Step 1.** The pair \((Z_t, \mathcal{A}^{\alpha,\beta}_t(\mathcal{X}))\) satisfies the positivity property for any \( 0 \neq E \in \mathcal{A}^{\alpha,\beta}_t(\mathcal{X}) \).

The construction of the heart \( \mathcal{A}^{\alpha,\beta}_t(\mathcal{X}) \) directly ensures that \( \Im Z_{s,t}(\tilde{\nu}(E)) \geq 0 \) for any \( E \in \mathcal{A}^{\alpha,\beta}_t(\mathcal{X}) \). Moreover, if \( \Im Z_{s,t}(\tilde{\nu}(E)) = 0 \), then \( E \) fits into an exact triangle

\[
\mathcal{K}[1] \rightarrow E \rightarrow G
\]

where

1. \( G \) is an object in \( \text{Coh}_{\mathbb{C}}^H(\mathcal{X}) \) with

\[
HF \text{ch}_1^\beta(G) = (H + tF) \text{ch}_2^\beta(G) - \frac{t+1}{2} a^2 F H^2 \text{ch}_0(G) = 0;
\]

2. \( K \in \text{Coh}_{\mathbb{C}}^H(\mathcal{X}) \) is \( \nu_{\alpha,\beta,t}(K) \text{-semistable with } \nu_{\alpha,\beta,t}(K) = 0 \) and \( HF \text{ch}_1^\beta(K) > 0 \).
By Lemma 3.14 one sees that
\[ HF \cdot \chi_1^\beta \mathcal{F} = H \cdot \chi_2^\beta \mathcal{G} = F \cdot \chi_2^\beta \mathcal{G} = \chi_0 \mathcal{G} = 0, \]
\[ \chi_2^\beta \mathcal{G} \geq 0, \mathcal{H}^{-1}\mathcal{G} \text{ is } C\text{-torsion and } \mathcal{H}^0\mathcal{G} \in \text{Coh}_{\leq 0}(\mathcal{X}). \] These imply that \( H^2 \cdot \chi_1(\mathcal{H}^{-1}\mathcal{G}) > 0 \) and \( \mathcal{R}_{\pi}(\mathcal{G}) < 0 \). On the other hand, Conjecture 3.2 implies that
\[
\chi^\beta(\mathcal{K}) \leq (a_1 H^2 + b_1 H F) \chi_1^\beta(\mathcal{K}) + (a_2 H + b_2 F) \chi_2^\beta(\mathcal{K}) + c \chi_0^\beta(\mathcal{K})
\]
\[
< (a_1 H^2 + l H F) \chi_1^\beta(\mathcal{K}) + (a_2 H + b_2 F) \chi_2^\beta(\mathcal{K}) + c \chi_0^\beta(\mathcal{K}),
\]
i.e., \( \mathcal{R}_{\pi}(\mathcal{K}) > 0 \). Therefore one concludes that
\[ \mathcal{R}_{\pi}(\mathcal{E}) = \mathcal{R}_{\pi}(\mathcal{G}) - \mathcal{R}_{\pi}(\mathcal{K}) < 0. \]

**Step 2.** The category \( \mathcal{A}^n_\alpha, \beta(\mathcal{X}) \) is noetherian.

We take a chain of surjections in \( \mathcal{A}^n_\alpha, \beta(\mathcal{X}) \):
\[ \mathcal{E}_0 \rightarrow \mathcal{E}_1 \rightarrow \cdots \rightarrow \mathcal{E}_m \rightarrow \cdots. \]
Since \( \mathcal{R}_{\pi}(\mathcal{E}_j) \geq 0 \) and the image of \( \mathcal{z}_i \) is discrete, we can assume that \( \mathcal{Z}_{\pi}(\mathcal{E}_j) \) is constant for all \( j \geq 0 \). Then one obtains short exact sequences in \( \mathcal{A}^n_\alpha, \beta(\mathcal{X}) \):
\[ 0 \rightarrow \mathcal{K}_j \rightarrow \mathcal{E}_0 \rightarrow \mathcal{E}_j \rightarrow 0 \]
with \( \mathcal{Z}_{\pi}(\mathcal{K}_j) = 0 \). By the Noetherianity of \( \text{Coh}_C^{\beta\mathcal{H}}(\mathcal{X}) \), we may assume that
\[ \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{E}_0) = \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{E}_j) \text{ and } \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_j) = \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_{j+1}) \]
for any \( j \geq 1 \). Setting \( \mathcal{U} = \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{E}_0)/\mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_j) \), one gets the short exact sequences
\[ 0 \rightarrow \mathcal{U} \rightarrow \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{E}_j) \rightarrow \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_j) \rightarrow 0. \]
Consider the exact sequence in \( \mathcal{A}^n_\alpha, \beta(\mathcal{X}) \):
\[ 0 \rightarrow \mathcal{K}_j \rightarrow \mathcal{K}_{j+1} \rightarrow \mathcal{Q}_{j+1} \rightarrow 0. \]
Since \( \mathcal{R}_{\pi}(\mathcal{K}_j) = 0 \), by the definition of \( \mathcal{A}^n_\alpha, \beta(\mathcal{X}) \), one sees that \( \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_j) \) has zero \( HF \cdot \chi_1, H \cdot \chi_2, F \cdot \chi_2 \) and \( \chi_0 \). Lemma 3.14 gives that \( \mathcal{H}^{-1}\left( \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_j) \right) \) is a \( C \)-torsion \( \mu_C \)-semistable sheaf with \( \mu_C \)-slope \( \beta \), and \( \mathcal{H}^0\left( \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_j) \right) \) is a sheaf supported in dimension zero. The same argument holds for \( \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{Q}_{j+1}) \).
Taking the long exact cohomology sequence of \( \mathcal{U} \), one sees that \( \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{Q}_{j+1}) \) is a subobject of \( \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_j) \), and thus
\[ \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{Q}_{j+1}) = 0 \]
by the definition of \( \mathcal{A}^n_\alpha, \beta(\mathcal{X}) \). Hence we have a chain of injections in \( \mathcal{T}_i^j \)
\[ \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_1) \subset \mathcal{H}^0_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{K}_2) \subset \cdots. \]
This gives a chain of injections in \( \mathcal{F}_i^j \):
\[ \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{E}_1) \subset \mathcal{H}^{-1}_{\text{Coh}_C^{\beta\mathcal{H}}}(\mathcal{E}_2) \subset \cdots. \]
with
\[ H_{\text{Coh}}^{-1}(\mathcal{E}_j) / H_{\text{Coh}}^{-1}(\mathcal{E}_j) \cong H_{\text{Coh}}^0(\mathcal{Q}_j) \cong Q_j \]
for \( j \geq 1 \).

As we have shown that \( Q_j \in \text{Coh}^{-1}(\mathcal{X}) \), \( H_{\text{Coh}}^0(\mathcal{Q}_j) \) is supported in dimension zero and \( H_{\text{Coh}}^{-1}(\mathcal{Q}_j) \) is a \( C \)-torsion \( \mu_C \)-semistable sheaf with \( \mu_C(H_{\text{Coh}}^{-1}(\mathcal{Q}_j)) = \beta \), one sees that \( H^2 \operatorname{ch}_1^\beta(Q_j) < 0 \) if \( H_{\text{Coh}}^{-1}(\mathcal{Q}_j) \neq 0 \). Thus we have
\[ H^2 \operatorname{ch}_1^\beta \left( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_j) \right) > H^2 \operatorname{ch}_1^\beta \left( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_j) \right) \]
if \( H_{\text{Coh}}^{-1}(\mathcal{Q}_j) \neq 0 \). As the proof of \( \text{Lemma 2.15} \), by induction on the number of Harder-Narasimhan factors of \( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_1) \) with respect to \( \nu_{\alpha,\beta,t} \), one may assume that \( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_1) \) is \( \nu_{\alpha,\beta,t} \)-semistable. Hence one infers \( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_j) \) is \( \nu_{\alpha,\beta,t} \)-semistable with positive \( HF \operatorname{ch}_1^\beta \) for any \( j \geq 1 \) by \( \text{Sublemma 2.16} \).

Since \( HF \operatorname{ch}_1^\beta \), \( HF \operatorname{ch}_2^\beta \), \( F \operatorname{ch}_1^\beta \) and \( \operatorname{ch}_0 \) of \( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_j) \) are constant as \( j \) grows, by Theorem 4.3 one deduces that there is a rational number \( c_0 \) such that
\[ H^2 \operatorname{ch}_1^\beta \left( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_j) \right) \geq c_0 \]
for any \( j \geq 1 \). This implies that
\[ H^2 \operatorname{ch}_1^\beta \left( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_j) \right) = H^2 \operatorname{ch}_1^\beta \left( H_{\text{Coh}}^{-1}(\mathcal{X})(\mathcal{E}_j) \right) \]
when \( j > 0 \). Therefore, we conclude that \( H_{\text{Coh}}^{-1}(\mathcal{Q}_j) = 0 \) when \( j > 0 \). We may assume that \( Q_{j+1} \) is a sheaf supported in dimension zero for any \( j \geq 1 \).

Taking the long exact cohomology sequence of \( \mathcal{X} \) again, one obtains exact sequences of sheaves supported in dimension zero
\[ 0 \to H^0(\mathcal{E}_j) \to H^0(\mathcal{K}_j) \to Q_j \to 0 \]
and equalities
\[ H_{\text{Coh}}^{-1}(\mathcal{E}_j) = H_{\text{Coh}}^{-1}(\mathcal{K}_j) = \cdots . \]

Let \( \mathcal{G}_i \) be the kernel of the composition
\[ H_{\text{Coh}}^{-1}(\mathcal{E}_j) \to H^0(\mathcal{E}_j) \to H^0(\mathcal{K}_j). \]

Then we have the following commutative diagram
\[ \begin{array}{cccccc}
0 & \to \mathcal{G}_j & \to H_{\text{Coh}}^{-1}(\mathcal{E}_j) & \to H^0(\mathcal{E}_j) & \to 0 \\
\| & \| & \| & \| & \\
0 & \to \mathcal{G}_{j+1} & \to H_{\text{Coh}}^{-1}(\mathcal{E}_{j+1}) & \to H^0(\mathcal{E}_{j+1}) & \to 0 \\
\| & \| & \| & \| & \\
& Q_j & \to Q_{j+1} & & \\
\end{array} \]

The snake lemma gives \( \mathcal{G}_i = \mathcal{G}_j \) for any \( j \geq 1 \). By Proposition 4.7, one concludes that \( H^0(\mathcal{E}_j) \) is a subsheaf of the zero dimensional sheaf \( \mathcal{G}_1 \).
particular the degree of \( H^0 \left( \mathcal{H}^0_{\text{Coh}^H_{\beta C}(X)}(K_j) \right) \) is bounded. This shows that \( Q_j = 0 \) for large \( j \), and hence \( K_j = K_{j+1} \) for large \( j \). This implies that the chain (5.2) terminates, and thus \( A^{\alpha,\beta}_i(X) \) is Noetherian.

**Step 3.** The pair \((Z_l, A^{\alpha,\beta}_i(X))\) satisfies the Harder-Narasimhan property and the local finiteness property.

Since \( A^{\alpha,\beta}_i(X) \) is Noetherian, from the discreteness of \( Z_l \), we conclude that \((Z_l, A^{\alpha,\beta}_i(X))\) satisfies the Harder-Narasimhan property. The local finiteness follows immediately from [7, Lemma 4.4].

**Remark 5.4.** I do not know whether \((Z_l, A^{\alpha,\beta}_i(X))\) satisfies the support property.

6. **Stability conditions on projective bundles**

Throughout this section we let \( E \) be a locally free sheaf on \( C \) with \( \text{rk} E = 3 \) and \( X := \mathbb{P}(E) \) be the projective bundle associated to \( E \) with the projection \( f : X \to C \) and the associated relative ample invertible sheaf \( \mathcal{O}_X(1) \). Since \( \mathbb{P}(E) \cong \mathbb{P}(E \otimes L) \) for any line bundle \( L \) on \( C \), we can assume that \( H := c_1(\mathcal{O}_X(1)) \) is ample. One sees that \( H^3 = \deg E \) and \( H^2 F = 1 \). We freely use the notations in previous sections.

**Lemma 6.1.** Denote by \( g \) the genus of \( C \). Then we have
\[

c_1(T_X) = -f^*K_C - f^*c_1(E) + 3H \\
c_2(T_X) = 3H^2 - (6g - 6 + 2\deg E)HF.
\]

In particular, for any divisor \( D \) on \( X \) we have
\[
DFc_1(T_X) = 3DHF \\
DHc_1(T_X) = 3DH^2 - (2g - 2 + H^3)DFH \\
D \left( c_1^2(T_X) + c_2(T_X) \right) = 12DH^2 - (18g - 18 + 8H^3)DFH \\
\chi(\mathcal{O}_X) = \frac{1}{24}c_1(T_X)c_2(T_X) = 1 - g.
\]

**Proof.** The formulas follow from the relative Euler sequence
\[
0 \to \Omega_{X/C} \to f^*E \otimes \mathcal{O}_X(-H) \to \mathcal{O}_X \to 0
\]
and the standard exact sequence
\[
0 \to f^*\mathcal{O}_C \to \mathcal{O}_X \to \Omega_{X/C} \to 0.
\]

By Lemma 3.17, one sees that \( \mathcal{O}_X(H) \) and \( \mathcal{O}_X(K_X + H)[1] \) are \( \nu^{\alpha,\beta}_{H,F} \)-stable objects in \( \text{Coh}^H_{\beta C}(X) \) for any \( \alpha > 0 \) and \(-2 < \beta < 1 \). Hence from Lemma 4.4 it follows that there is a non-negative rational number \( t_0 \) such that \( \mathcal{O}_X(H) \) and \( \mathcal{O}_X(K_X + H)[1] \) are \( \nu^{\alpha,\beta,t}_{H,F} \)-stable for any \( t \geq t_0 \), \(-2 < \beta < 1 \) and \( \alpha > 0 \).

**Theorem 6.2.** Assume that the following inequalities hold:

1. \( \alpha - 2 < \beta < 1 - \alpha \);
2. \( t > \max\left\{ \frac{-\beta(\beta+2)H^3+4(\beta+2)(g-1)+\alpha^2}{(\beta+2)^2-\alpha^2}, t_0 \right\} \).

\( \square \)
Then there exist rational numbers $a_0$, $a_1$ and $a_2$ only depending on $\alpha$, $\beta$ and $X$, such that

\begin{equation}
\label{6.1}
\text{ch}^2_2(\mathcal{E}) \leq - \frac{\beta(\beta + 1)}{2} H^2 \text{ch}^2_1(\mathcal{E}) - (\beta + \frac{1}{2}) H \text{ch}^2_0(\mathcal{E}) + a_0 \text{ch}^0_0(\mathcal{E}) + a_1 HF \text{ch}^1_1(\mathcal{E}) + a_2 F \text{ch}^2_2(\mathcal{E})
\end{equation}

for any $\nu_{\alpha,\beta,t}$-semistable object $\mathcal{E}$ with $\nu_{\alpha,\beta,t}(\mathcal{E}) = 0$.

**Proof.** Under our assumptions on $\alpha$, $\beta$ and $t$, one sees that

\[
\nu_{\alpha,\beta,t}(\mathcal{O}_X(H)) = \nu_{\alpha,\beta,0}(\mathcal{O}_X(H)) + t \nu_{H,F}^{\alpha,\beta}(\mathcal{O}_X(H)) = \frac{H^3(1 - \beta)^2 - \alpha^2}{2(1 - \beta)} + t \frac{(1 - \beta)^2 - \alpha^2}{2(1 - \beta)} > 0
\]

and

\[
\nu_{\alpha,\beta,t}(\mathcal{O}_X(K_X + H)[1]) = \nu_{\alpha,\beta,0}(\mathcal{O}_X(-2H + (2g - 2 + H^3)F)) + t \nu_{H,F}^{\alpha,\beta}(\mathcal{O}_X(-2H)) = \frac{H^3(\beta + 2)^2 - 2(\beta + 2)(2g - 2 + H^3) - \alpha^2}{2(-2 - \beta)} + t \frac{(\beta + 2)^2 - \alpha^2}{2(-2 - \beta)} < 0.
\]

By the $\nu_{\alpha,\beta,t}$-semistability of $\mathcal{E}$, the inequalities above imply that

\[
\text{Hom}(\mathcal{O}_X(H), \mathcal{E}) \cong \text{Hom}(\mathcal{O}_X, \mathcal{E}(-H)) = 0
\]

and

\[
\text{Ext}^2(\mathcal{O}_X, \mathcal{E}(-H)) \cong \text{Hom}(\mathcal{E}, \mathcal{O}_X(K_X + H)[1]) = 0.
\]

Therefore the application of the Grothendieck-Riemann-Roch theorem leads to

\[
0 \geq \chi(\mathcal{E}(-H)) = \text{ch}^4_4(\mathcal{E}) + \frac{c_1}{2} \text{ch}^2_2(\mathcal{E}) + \frac{c_1^2 + c_2}{12} \text{ch}^2_2(\mathcal{E}) + \chi(\mathcal{O}_X) \text{ch}_0(\mathcal{E})
\]

\[
= \text{ch}^2_2(\mathcal{E}) + \left(\frac{c_1}{2} + \frac{c_2}{2} c_1 + \frac{1}{12} \text{ch}^2_2(\mathcal{E}) \right) + \left(\frac{(\beta - 1)^2}{2} H^2 + \frac{\beta - 1}{2} H c_1 + \frac{c_1^2 + c_2}{12} \right) \text{ch}^2_2(\mathcal{E})
\]

\[
+ \left(\frac{(\beta - 1)^3}{6} H^3 + \frac{(\beta - 1)^2}{4} H^2 c_1 + \frac{\beta - 1}{12} H (c_2^2 + c_2) + \chi(\mathcal{O}_X) \right) \text{ch}_0(\mathcal{E})
\]

\[
= \text{ch}^2_2(\mathcal{E}) + (\beta + \frac{1}{2}) H \text{ch}^2_2(\mathcal{E}) + \frac{1}{2} \beta(\beta + 1) H^2 \text{ch}^2_2(\mathcal{E})
\]

\[
- a_2 F \text{ch}^2_2(\mathcal{E}) - a_1 HF \text{ch}^2_2(\mathcal{E}) - a_0 \text{ch}_0(\mathcal{E}),
\]

where $c_i = c_i(T_X)$ for $i = 1, 2$ and the constants $a_0$, $a_1$ and $a_2$ only depend on $\alpha$, $\beta$ and $X$. This completes the proof of Theorem 6.2.

**Corollary 6.3.** There exist locally finite stability conditions on $\mathcal{X} := \mathbb{P}(\mathcal{E})$.

**Proof.** It turns out that the coefficient $-\frac{\beta(\beta + 1)}{2} H^2$ in (6.1) is positive when $-1 < \beta < 0$. Therefore the conclusion follows from Theorem 6.3 and Theorem 6.2. \qed
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