Posterior consistency and characteristic functional of normalized random measures with independent increments

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Abstract

In this paper, we obtain explicit form of the posterior moments and characteristic functional for normalized random measures with independent increments (NRMIs) in terms of their associated Lévy intensities, which is a class of Bayesian nonparametric priors that have been studied widely in the literature. These results are applied to solve the posterior consistency problem, the results of which are illustrated with examples.

Keywords: Dirichlet process; Normalized random measures with independent increments; characteristic functional; posterior consistency.

1 Introduction

Normalized random measures with independent increments (NRMIs) are introduced by (Regazzini et al., 2003) and represent a large class of Bayesian nonparametric priors, which include the very famous Dirichlet process (Ferguson, 1973), the σ-stable NRMIs (Kingman, 1975), the normalized inverse Gaussian process (Lijoi et al., 2005b), the normalized generalized gamma process (Lijoi et al., 2003, 2007), and generalized Dirichlet process (Lijoi et al., 2005a). We refer to (Hu and Zhang, 2021a), (Lijoi et al., 2010) for a review of these processes with their properties and applications. As ones of the most widely studied and applied Bayesian nonparametric models, their distributional properties and computational

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aspects have been extensively studied. However, the frequentist properties, such as posterior consistency, although previously discussed, considerations with respect to their Lévy intensities have been limited. Posterior consistency of Bayesian nonparametric models have always been the focus of a considerable amount of researches. Most contributions in the literature exploit the “frequentist” approach to Bayesian consistency, also termed the “what if” method according to (Freedman and Diaconis, 1983). This essentially considers what would happen to the posterior distribution if the sample were generated from a “true” distribution $P_0$: Will the posterior distribution concentrate in suitably defined neighborhoods of $P_0$? Let us mention the following inspiring works pioneered this paper. (James, 2008) obtains the posterior consistency and weak convergence of the two-parameter Poisson-Dirichlet process, which is not a NRMI, but closely related to NRMs. As a popular topic in Bayesian nonparametric models, stick-breaking representations of NRMs (see e.g. (Sethuraman, 1994; Perman et al., 1992; Favaro et al., 2016)) have been studied with various applications. (Hu and Zhang, 2021b) obtain the asymptotic theorems for the stick-breaking priors when the concentration parameter converges to infinity. Furthermore, the posterior consistency of species sampling priors and Gibbs-type priors are discussed by (Ho Jang et al., 2010) and (De Blasi et al., 2013).

Since NRMs are constructed by the normalization of completely random measures associated with their Lévy intensities (see e.g. section 2), it is quite natural to study their properties based on the corresponding Lévy intensities. The posterior consistent results in (Ho Jang et al., 2010) and (De Blasi et al., 2013) are given through the predictive distributions of the Bayesian nonparametric models and the NRMs they considered are constructed by homogeneous Lévy intensities. In this work, we discuss the posterior consistency of NRMs with non-homogeneous Lévy intensities (including the homogeneous case as a particular example). To do so, we establish the posterior moments and the characteristic functional of NRMs.

The outline of this paper is as follows. In Section 2 we recall the construction of NRMs and their posterior distributions. In Section 3 we obtain the moments and characteristic functional of the posterior of NRMs. As a consequence, we discuss the posterior consistency of NRMs based on some assumptions of the corresponding Lévy intensities. Examples are presented to see the posterior consistency results for some well-known Bayesian nonparametric priors. Finally, Section 4 provides a discussion of our results and some ideas that can be studied in the future. In order to ease the flow of the ideas, we delay the proofs in the Appendix.

2 Normalized random measures with independent increments

2.1 Constructions of NRMs

We start by recalling the notions of completely random measures, which play important roles in the construction of NRMs. See e.g. (Lijoi et al., 2010) for a more detailed discussion of constructing Bayesian nonparametric priors by using completely random measures.
The NRMIs can be defined via Poisson random measure. So we first recall this concept.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and let \(\mathcal{X}\) be a complete, separable metric space whose Borel \(\sigma\)-algebra is denoted by \(\mathcal{X}\). Denote \(\mathcal{S} = \mathbb{R}^+ \times \mathcal{X}\) and denote its Borel \(\sigma\)-algebra by \(\mathcal{S}\). A Poisson random measure \(\tilde{N}\) on \(\mathcal{S}\) with finite intensity measure \(\nu(ds,dx)\) is a random measure from \(\Omega \times \mathcal{S}\) to \(\mathbb{R}^+_+\) satisfying

(i) \(\tilde{N}(A) \sim \text{Poisson}(\nu(A))\) for any \(A\) in \(\mathcal{S}\);

(ii) for any pairwise disjoint sets \(A_1, \ldots, A_m\) in \(\mathcal{S}\), the random variables \(\tilde{N}(A_1), \ldots, \tilde{N}(A_m)\) are mutually independent.

The Poisson intensity measure \(\nu\) satisfies the condition (see \cite{DaleyVereJones2008} for details of Poisson random measures) that

\[
\int_0^\infty \int_{\mathcal{X}} \min(s,1) \nu(ds,dx) < \infty .
\]

Let \((\mathbb{M}_\mathcal{X}, \mathcal{M}_\mathcal{X})\) be the space of bounded finite measures on \((\mathcal{X}, \mathcal{X})\) endowed with the topology of weak convergence and let \(\tilde{\mu}\) be the random measure defined on \((\Omega, \mathcal{F}, \mathbb{P})\) that takes values in \((\mathbb{M}_\mathcal{X}, \mathcal{M}_\mathcal{X})\) defined as follows,

\[
\tilde{\mu}(A) := \int_0^\infty \int_A s \tilde{N}(ds,dx), \quad \forall A \in \mathcal{X}. \tag{2.1}
\]

It is trivial to verify that \(\tilde{\mu}\) is a completely random measure (see e.g. \cite{Kingman1967} and references therein for more details about this concept). It is also well-known that for any \(B \in \mathcal{X}\), \(\tilde{\mu}(B)\) is discrete and is uniquely characterized by its Laplace transform as follows:

\[
\mathbb{E} \left[ e^{-\lambda \tilde{\mu}(B)} \right] = \exp \left\{ -\int_0^\infty \int_B \left[ 1 - e^{-\lambda s} \right] \nu(ds,dx) \right\}. \tag{2.2}
\]

The measure \(\nu\) is called the Lévy intensity of \(\tilde{\mu}\) and we denote the Laplace exponent by

\[
\psi_B(\lambda) = \int_0^\infty \int_B \left[ 1 - e^{-\lambda s} \right] \nu(ds,dx). \tag{2.3}
\]

It is apparent that the completely random measure \(\tilde{\mu}\) is characterized completely by its Lévy intensity \(\nu\), which usually takes the following forms in the literature:

(a) \(\nu(ds,dx) = \rho(ds)\alpha(dx)\), where \(\rho : \mathcal{B}(\mathbb{R}^+) \to \mathbb{R}^+_+\) is some measure on \(\mathbb{R}^+_+\) and \(\alpha\) is a non-atomic measure on \((\mathcal{X}, \mathcal{X})\) so that \(\alpha(\mathcal{X}) = a < \infty\). The corresponding \(\tilde{\mu}\) is called homogeneous completely random measure.

(b) \(\nu(ds,dx) = \rho(ds|x)\alpha(dx)\), where \(\rho\) is defined on \(\mathcal{B}(\mathbb{R}^+) \times \mathcal{X}\) such that for any \(x \in \mathcal{X}\), \(\rho(\cdot|x)\) is a \(\sigma\)-finite measure on \(\mathcal{B}(\mathbb{R}^+)\) and for any \(A \in \mathcal{X}\), \(\rho(A|x)\) is \(\mathcal{B}(\mathbb{R}^+)\) measurable. The corresponding \(\tilde{\mu}\) is called non-homogeneous completely random measure.
It is obvious that the case (a) is a special case of case (b). Usually, we assume that $\alpha$ is a finite measure so we may write $\alpha(dx) = aH(dx)$ for some probability measure $H$ and some constant $a = \alpha(\mathcal{X}) \in (0, \infty)$.

To construct NRMIs, the completely random measure will be normalized, and thus one needs the total mass $\tilde{\mu}(\mathcal{X})$ to be finite and positive almost surely. This happens under the condition that $\rho(\mathbb{R}^+) = \infty$ in homogeneous case and that $\rho(\mathbb{R}^+|x) = \infty$ in non-homogeneous case (See e.g. [Regazzini et al., 2002] for a proof). Under the above conditions, an NRI $P$ on $(\mathcal{X}, \mathcal{X})$ is a random probability measure defined by

$$P(\cdot) = \frac{\tilde{\mu}(\cdot)}{\tilde{\mu}(\mathcal{X})}. \quad (2.4)$$

$P$ is discrete due to the discreteness of $\tilde{\mu}$. For notational simplicity, we denote $T = \tilde{\mu}(\mathcal{X})$ and let $f_T(t)$ be the density of $T$ throughout this paper. We point out that $P$ admits a stick-breaking representation (See [Sethuraman, 1994], [Pitman, 1996], [Pitman, 2003], [Favaro et al., 2010] for more discussions).

### 2.2 Posterior of NRMIs

The posterior analysis of NRMIs is a key topic in Bayesian nonparametric analysis. Let us recall an important result that this work is based on. Let $P$ be an NRI on $\mathcal{X}$. A sample of size $n$ from $P$ is an exchangeable sequence of random variables $X = (X_i)_{i=1}^n$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in $\mathcal{X}$, such that given $P$, $(X_i)_{i \geq 1}$ are i.i.d. with distribution $P$, i.e.

$$P[X_1 \in A_1, \ldots, X_n \in A_n | P] = \prod_{i=1}^n P(A_i). \quad (2.5)$$

Let $Y = (Y_j)_{j=1}^{n(\pi)}$ be the distinct observations of the sample $X$ and let $n(\pi)$ be the number of unique values of $X$. This means, $\pi = (i_1, \ldots, i_{n_1}, \ldots, i_{n_{\pi(n)-1}}, \ldots, i_{n_{\pi(n)}})$ is the partition of $\{1, \ldots, n\}$ of size $n(\pi)$. The number of the $j$th set of the partition is $n_j$, so that $\sum_{j=1}^{n(\pi)} n_j = n$, and $Y_1 := X_{i_1} = \cdots = X_{i_{n_1}}, \ldots, Y_{n(\pi)} := X_{n_{\pi(n)-1}+1} = \cdots = X_{n_{\pi(n)}}$. Let

$$\tau_k(u, Y) = \int_0^\infty s^k e^{-us} \rho(ds | Y) \quad \text{for any positive integer } k \text{ and } Y \in \mathcal{X}. \quad (2.6)$$

With these notations, the posterior distribution of $P$ conditional on the observations of the sample $X_1, \ldots, X_n$ is given by the following theorem.

**Theorem 1** ([James et al., 2009]). Let $P$ be an NRI with intensity $\nu(ds, dx) = \rho(ds | x) \alpha(dx)$. The posterior distribution of $P$, given a latent random variable $U_n$, is an NRI that coincides in distribution with the random measure

$$\frac{\tilde{\mu}(U_n)}{T(U_n)} + (1 - \kappa) \sum_{j=1}^{n(\pi)} \frac{J_j \delta_{Y_j}}{\sum_{j=1}^{n(\pi)} J_j}, \quad (2.7)$$

where
(i) The random variable $U_n$ has density
\[ f_{U_n}(u) = \frac{u^{n-1}}{\Gamma(n)} \int_0^\infty t^n e^{-ut} f_T(t) dt. \]  

(ii) Given $U_n$, $\tilde{\mu}_{(U_n)}$ is the conditional completely random measure of $\tilde{\mu}$ with the Lévy intensity $\nu_{(U_n)} = e^{-U_n s} \alpha(dx) \rho(ds|x)$.

(iii) $\{J_1, \ldots, J_{n(\pi)}\}$ are random variables depending on $U_n$ and $Y_j$ and having density
\[ f_{J_j}(s|U_n = u, X) = \frac{s^{n_j} e^{-us} \rho(s|Y_j)}{\int_0^\infty s^{n_j} e^{-us} \rho(ds|Y_j)}. \]  

(iv) The random elements $\tilde{\mu}_{(U_n)}$ and $J_j$, $j \in \{1, \ldots, n(\pi)\}$ are independent.

(v) $T_{(U_n)} = \tilde{\mu}_{(U_n)}(X)$ and $\kappa = \frac{T_{(U_n)}}{T_{(U_n)} + \sum_{j=1}^{n(\pi)} J_j}$.

(vi) The conditional density of $U_n$ given $X$ is given by
\[ f_{U_n|X}(u|X) \propto u^{n-1} e^{-\psi(u)} \prod_{j=1}^{n(\pi)} \tau_{n_j}(u, Y_j). \]  

The above theorem shows that, given some latent variable $U_n$, the posterior of $P$ is a weighted sum of another NRMIs $\tilde{\mu}_{(U_n)}$ and the normalization of Delta function of distinct observations $Y_j$ multiplied by its corresponding jumps $J_j$. This gives a rather complete description of the posterior distribution of NRMIs. More details of the posterior analysis of $\tilde{\mu}$ and $P$ is discussed in [James et al., 2009].

3 Main results

In this section, we aim at deriving the posterior moments and the characteristic functional of NRMIs $P$, which will be used to obtain the posterior consistency of $P$ assuming some conditions on its associated Lévy intensity. We will illustrate the posterior consistency results and check the assumptions for several examples.

3.1 Posterior moments of NRMIs

Recall $\psi_A$ defined by (2.3). For any $A \in \mathcal{X}$, let
\[ V^{(k)}_{\alpha(A)}(y) = (-1)^k \frac{d^k}{dy^k} e^{-\psi_A(y)}. \]  

The posterior moments of NRMIs are given as follows.
Theorem 2. Let $X = (X_i)_{i=1}^n$ be a random sample from a normalized random measure with independent increments $P$. The moments and the mixed moments of the posterior moments of $P$ given $X$ are given as follows (we use the notation of Theorem 1).

(i) For any $A \in \mathcal{X}$ and $m \in \mathbb{N}$, the posterior moments of $P$ are given by

$$
\mathbb{E}[(P(A))^m|X] = \frac{\Gamma(n)}{\Gamma(m+n)} \sum_{0 \leq l_1 + \cdots + l_{\alpha} \leq m} \left( \prod_{i=1}^{\alpha} \frac{\tau_{l_i}}{\tau_{l_i}(u,Y_j)} \right) \left( \int_0^\infty u^m f_{U_n|X}(u|X) \right) \left( \prod_{j=1}^{\alpha} \frac{\tau_{l_j}}{\tau_{l_j}(u,Y_j)} \right) du. \tag{3.2}
$$

(ii) For any family of pairwise disjoint subsets $\{A_1, \cdots, A_q\}$ of $\mathcal{X}$ and any integers $\{m_1, \cdots, m_q\}$, we have

$$
\mathbb{E} [P(A_1)^{m_1} \cdots P(A_q)^{m_q}|X] = \frac{\Gamma(n)}{\Gamma(m+n)} \int_0^\infty u^m f_{U_n|X}(u|X) \prod_{i=1}^{q+1} \sum_{0 \leq l_1 + \cdots + l_{\max(\lambda_i)} \leq m_i} \left( \prod_{j=1}^{\max(\lambda_i)} \frac{\tau_{l_j}}{\tau_{l_j}(u,Y_j)} \right) \left( \prod_{j=1}^{\max(\lambda_i)} \frac{\tau_{l_j}}{\tau_{l_j}(u,Y_j)} \right) du, \tag{3.3}
$$

where $m = \sum_{i=1}^{q} m_i$, $A_{q+1} = (\cup_{i=1}^{q} A_i)^c$, $m_{q+1} = 0$, $\lambda_i = \{ j : Y_j \in A_i \}$ is the set of the index of $Y_j$’s that are in $A_i$, and $\max(\lambda_i)$ is the maximal value of $\lambda_i$’s.

The proof of this theorem is given in the Appendix. To apply the above theorem, one needs to deal with the term $V_{\alpha(A)}^{(k)}(y)$ defined by (3.1). We give the following recursion formula

$$
V_{\alpha(A)}^{(k)}(y) = \sum_{i=0}^{k-1} \binom{k - 1}{i} \xi_i(y) V_{\alpha(A)}^{(i)}(y),
$$

where $\xi_i(y) = \int_A \tau_i(y,x) \alpha(dx)$. The above theorem provides the posterior moments of NRMIs without including the latent random variable $U_n$. Such results can be reduced to the computation of the moments of NRMIs by letting the sample size $n = 0$. The above formula can be used to compute the Bayesian nonparametric estimators related to the moments such as variance, covariance, skewness. Using the Taylor’s expansion, one can obtain the characteristic function of posterior distribution of NRMIs.

Corollary 3. The characteristic function of $P(A)$ and $P(A)|X$ are given as

$$
\phi_{P(A)}(t) = \mathbb{E}[e^{itP(A)}] = \sum_{k=0}^{\infty} \frac{\Gamma(k)}{k!} \int_0^\infty e^{-s} \phi_{P(A)}^{(k)}(u) du \tag{3.4}
$$

$$
\phi_{P(A)|X}(t) = \mathbb{E}[e^{itP(A)|X}] = \sum_{k=0}^{\infty} \frac{\Gamma(n/k)}{\Gamma(n/k+1)/\Gamma(k)} \sum_{0 \leq l_1 + \cdots + l_{\alpha} \leq k} \left( \prod_{i=1}^{\alpha} \frac{\tau_{l_i}}{\tau_{l_i}(u,Y_j)} \right) \left( \prod_{j=1}^{\max(\lambda_i)} \frac{\tau_{l_j}}{\tau_{l_j}(u,Y_j)} \right) \left( \int_0^\infty u^m f_{U_n|X}(u|X) \right) \left( \prod_{j=1}^{\max(\lambda_i)} \frac{\tau_{l_j}}{\tau_{l_j}(u,Y_j)} \right) du.
$$
\[
\int_{0}^{\infty} u^{k} f_{U_{n}}(u) V_{\alpha(A)}^{(k - (l_{1} + \cdots + l_{n}(x)))}(u) \frac{\prod_{j=1}^{n} \tau_{n_{j}}(u, y_{j})}{\tau_{n_{j}}(u, y_{j})} \delta_{Y_{j}}(A) \, du. \tag{3.5}
\]

### 3.2 Consistency of posterior NRMI

Assume that \( X = \{X_{1}, \ldots, X_{n}\} \) is a sample from the “true” distribution \( P_{0} \) in \( \mathcal{M}_{\mathcal{X}} \). Namely, \( X = \{X_{1}, \ldots, X_{n}\} \) is i.i.d. \( P_{0} \) distributed. Let \( Q_{n} \) denote the posterior distribution \( Q(\cdot|X) \) of the random probability measure \( P_{n} \), conditional on the sample. The posterior distribution is said to be weakly consistent if \( Q_{n} \) concentrates on a weak neighbourhood of \( P_{0} \) almost surely. More precisely, for a weak neighbourhood \( O_{\epsilon} \) of \( P_{0} \) with arbitrary radius \( \epsilon > 0 \),

\[
Q_{n}(O_{\epsilon}) \rightarrow 1 \quad \text{a.s.} - P_{0}^{\infty},
\]
as \( n \rightarrow \infty \).

As a special NRMI, the weak consistency of the Dirichlet process is quite obvious and is verified by [James, 2008]. We shall study the weak consistency for more general NRMI\( s. \) To do so, we need the following assumption. To simplify the presentation, we use \( f(u) \overset{\mathcal{U}}{=} g(u) \) as the notation of \( \lim_{u \rightarrow \infty} \frac{f(u)}{g(u)} = C \) for some finite nonzero constant \( C \) in the remaining part of the paper.

**Assumption 4.** Let \( \tau_{k}(u, x) \) be defined by (2.6) and let \( \rho_{x}(s) \) be a function such that \( u \frac{\tau_{k+1}(u, x)}{\tau_{k}(u, x)} \) is nondecreasing in \( u \) and bounded from above by \( k - C_{k}(x) \) uniformly for all \( k \in \mathbb{Z}^{+} \) and \( x \in \mathcal{X} \), where \( \{C_{k}(x)\} \) is a sequence of functions from \( \mathcal{X} \) to \( [0, 1) \). Namely, there is an increasing positive function \( \phi(u) \) with \( \lim_{u \rightarrow \infty} \phi(u) = 1 \) such that

\[
\sup_{k \in \mathbb{Z}^{+}, x \in \mathcal{X}} \frac{u \frac{\tau_{k+1}(u, x)}{\tau_{k}(u, x)}}{k - C_{k}(x)} \leq \phi(u), \quad \forall \ u \in \mathbb{R}_{+}.
\]

**Theorem 5.** Let \( P \) be an NRMI with Lévy intensity \( \nu(ds, dx) = \rho_{x}(s)ds \alpha(dx) \), where \( \rho_{x}(s) \) satisfies Assumption 4. Then

1. If \( P_{0} \) is continuous, then the posterior of \( P \) converges weakly to a point mass at \( \bar{C}_{1}\alpha(\cdot) + (1 - \bar{C}_{1})P_{0}(\cdot) \) a.s. - \( P_{0}^{\infty} \), where \( \bar{C}_{1} \) is the population mean of \( \{C_{1}(X_{i})\}_{i=1}^{\infty} \).

2. If \( P_{0} \) is discrete with \( \lim_{n \rightarrow \infty} \frac{n_{[\pi]}}{n} = 0 \), then \( P \) is weakly consistent, i.e. the posterior of \( P \) converges weakly to a point mass at \( \bar{C}_{1}\alpha(\cdot) + (1 - \bar{C}_{1})P_{0}(\cdot) \) a.s. - \( P_{0}^{\infty} \).

We shall give the proof of this theorem in the appendix.

**Remark 6.** When \( P_{0} \) is discrete and \( \lim_{n \rightarrow \infty} \frac{n_{[\pi]}}{n} \neq 0 \), the posterior of \( P \) converges weakly to a point mass at \( \bar{C}_{1}\alpha(\cdot) + (1 - \bar{C}_{1})P_{0}(\cdot) \) a.s. - \( P_{0}^{\infty} \).

The assumption 4 is equivalent to the following assumption.

**Assumption 7.** \( \rho_{x}(s) \) is a function such that \( u \frac{d}{du} \ln (\tau_{k}(u, x)) \) is nonincreasing in \( u \) and bounded from below by \( C_{k}(x) - k \) for all \( k \in \mathbb{Z}^{+} \) and \( x \in \mathcal{X} \).
Remark 8. The above theorem can be extended to more general construction of NRMIs. For example, James (2002) introduced the h-biased random measures $\tilde{\mu}$ by $\int_{Y \times X} g(s) \tilde{N}(ds, dx)$, where $g : Y \to \mathbb{R}^+$ is an integrable function on any complete and separable metric space $Y$.

One interesting quantity to be considered is $n(\pi)$, the number of distinct observations of the sample $\{X_i\}_{i=1}^n$. In Bayesian nonparametric mixture models, $n(\pi)$ is the number of clusters in the sample observations and thus is studied in a number of works concerning the clustering and so on. Among the literature let us mention that the distribution of $n(\pi)$ is obtained in (Korwar and Hollander, 1973) for Dirichlet process; in (Antoniak, 1974) for the mixture of Dirichlet process; in (Pitman, 2003) for the two-parameter Poisson-Dirichlet process. For the general NRMIs we have by a result of (James et al., 2009):

Proposition 9. For any positive integer $n$, the distribution of $n(\pi)$ is

$$\mathbb{P}(n(\pi) = k) = \int_0^\infty nu^{n-1} - e^{-\int_0^\infty (1-e^{-y^\sigma})\rho(ds|x)\alpha(dx)} \prod_{j=1}^k \int_{n_j} \tau_n \tau_j(u, x)\alpha(dx) du,$$

(3.6)

where $k = 1, \ldots, n$, and the summation is over all vectors of positive integers $(n_1, \ldots, n_k)$ such that $\sum_{j=1}^k n_j = n$.

The assumption 4 is in fact quite easy to verify. We provide in the following examples to see the applicability of Theorem 5.

Example 10. The normalized generalized gamma process (Lijoi et al., 2003, 2007) is an NRMI with the following homogeneous Lévy intensity

$$\nu(ds, dx) = \frac{1}{\Gamma(1-\sigma)} s^{1-\sigma} e^{-\theta s} ds \alpha(dx),$$

(3.7)

where the parameter $\sigma \in (0, 1)$ and $\theta > 0$. It is easy to see that the Laplace transform for $\tilde{\mu}(A)$ is

$$\mathbb{E}[e^{-\lambda \tilde{\mu}(A)}] = \exp \left\{ -\frac{\alpha(A)}{\sigma} [(\lambda + \theta)^\sigma - \theta^\sigma] \right\}.$$

When $\theta \to 0$, this NRMI yields the homogeneous $\sigma$-stable NRMI introduced by (Kingman, 1973). Letting $\sigma \to 0$, this NRMI becomes the Dirichlet process (Ferguson, 1973). If we let $\sigma = \theta = \frac{1}{2}$ then this NRMI becomes the normalized inverse-Gaussian process (Lijoi et al., 2005b).

It is easy to check that for any nonnegative integer $k$,

$$\tau_k(u, x) = \tau_k(u) = \frac{1}{\Gamma(1-\sigma)} \int_0^\infty s^{k-\sigma-1} e^{-(u+\theta)s} ds = \frac{\Gamma(k-\sigma)}{\Gamma(1-\sigma)(u + \theta)^{k-\sigma}}.$$

It is obvious that $u \tau_{k+1}(u, x) / \tau_k(u, x) = u^{k-\sigma-1} - e^{-\theta s}$ is increasing in $u$ with the upper bound $k-\sigma$. Thus, the assumption 4 is verified and Theorem 2 yields that the normalized generalized gamma process is posterior consistent when $\sigma \to 0$ (i.e. the Dirichlet process), or when $P_0$ is discrete with $\lim_{n \to \infty} \frac{n(\pi)}{n} = 0$. 

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Example 11. The generalized Dirichlet process (Lijoi et al., 2005a) is an NRMI with the following homogeneous Lévy intensity

\[ \nu(ds, dx) = \sum_{j=1}^{\gamma} \frac{e^{-js}}{s} ds \alpha(dx), \]  

(3.8)

where \( \gamma \) is a positive integer. The corresponding Laplace transform of \( \tilde{\mu}(A) \) is

\[ E\left[e^{-\lambda \tilde{\mu}(A)}\right] = \left(\frac{(\gamma)!}{(\lambda + 1)\gamma}\right)^\alpha(A), \]

where for \( c > 0, c_k = \frac{\Gamma(c+k)}{\Gamma(c)} \) is the ascending factorial of \( c \) for any positive integer \( k \). When \( \gamma = 1 \), the generalized Dirichlet process is reduced to the Dirichlet process.

We see easily that for any nonnegative integer \( k \),

\[ \tau_k(u, x) = \tau_k(u) = \sum_{j=1}^{\gamma} \frac{k}{(u+j)^k}. \]

This means \( \frac{\tau_{k+1}(u,x)}{\tau_k(u,x)} = k \frac{\sum_{j=1}^{\gamma} (u+j)^{-k-1}}{\sum_{j=1}^{\gamma} (u+j)^{-k}} \in (\frac{k}{u+\gamma}, \frac{k}{u+1}) \), which implies \( u^{\frac{\tau_{k+1}(u,x)}{\tau_k(u,x)}} = u^\frac{k}{u+c(\gamma)} \) with some constant \( c(\gamma) \in (1, \gamma) \). Thus, \( u^{\frac{\tau_{k+1}(u,x)}{\tau_k(u,x)}} \) is increasing in \( u \) with the upper bound \( k \). Theorem 5 can then be used to conclude that the generalized Dirichlet process is posterior consistent.

Example 12. As a non-homogeneous example, we consider the extended Gamma NRMI whose non-homogeneous Lévy intensity is given by

\[ \nu(ds, dx) = \frac{e^{-\beta(x)s}}{s} ds \alpha(dx), \]  

(3.9)

where \( \beta(x) : X \to \mathbb{R}^+ \) is an integrable function (with respect to \( \alpha(dx) \)). Such NRMI is constructed by the normalization of the extended Gamma process on \( \mathbb{R} \) introduced by (Dykstra and Laud, 1981). More generally, (Lo, 1982) studied the extended Gamma process, called weighted Gamma process on abstract spaces.

By a trivial computation, for any nonnegative integer \( k \), \( \tau_k(u, x) = \frac{k}{\Gamma(k)} (\frac{1}{u+\beta(x)})^k \) and thus \( u^{\frac{\tau_{k+1}(u,x)}{\tau_k(u,x)}} = u^\frac{k}{u+\beta(x)} \) and the assumption 4 is satisfied. Theorem 5 implies that the extended Gamma NRMI is posterior consistent when \( \beta(x) \) is finite.

Our theorem can also be applied to more general NRMI which are not been investigated earlier. For example, we may naturally consider the following generalized extended Gamma NRMI by letting the Lévy intensity be as follows:

\[ \nu(ds, dx) = \sum_{i=1}^{r} \frac{e^{-\beta_i(x)s}}{s} ds \alpha(dx), \]  

where \( \beta_i(x) : \mathbb{R} \to \mathbb{R}^+ \) are integrable functions.
where \( r \in \mathbb{Z}^+ \) and \( \beta_i(x) : \mathcal{X} \to \mathbb{R}^+ \) are integrable functions (with respect to \( \alpha(dx) \)). A similar argument to that of Examples 11 and 12 implies that the generalized extended Gamma NRMI is posterior consistent when \( \beta_i(x) \) are finite for all \( i \in \{1, \cdots, r\} \).

As we can see from assumption 4, the form \( \tau_{k+1}(u,x)/\tau_k(u,x) \) is mainly determined by \( \rho_x(s) \), so that it is natural to give conditions of \( \rho_x(s) \) such that the assumption 4 holds. The following remark is one of these possible conditions of \( \rho_x(s) \) and can be checked by simple computation.

**Remark 13.** If \( s^{-C(x)-1} \leq \rho_x(s) \leq M(x) \) for some \( C(x) \in [0,1) \) and integrable function \( M(x) \) (with respect to \( \alpha(x) \)), then assumption 4 is satisfied.

### 4 Discussion

To the best of our knowledge, the Lévy intensities of the well-studied NRMIs up-to-date are given in the form of the gamma density: \( s^{-\sigma-1}e^{-\beta s} \). It turns out that with the shape parameter \( \sigma = 0 \), the posterior consistency is always guaranteed for any “true” prior distribution \( P_0 \). Otherwise, the posterior consistency only holds for discrete prior \( P_0 \) but not for diffusive prior \( P_0 \). Such phenomenon does naturally make sense due to the discreteness of NRMIs (the completely random measures \((\text{Kingman, 1975})\)). As explained in the Bayesian literature, if \( P_0 \) is diffusive and the prior guess for the sample distribution \( \alpha \neq P_0 \), then the prior guess will always contribute to the posterior, no matter how large is the sample size. In such sense, the Bayesian nonparametric models never behave “better” than the empirical models asymptotically. However, this doesn’t mean the NRMIs are not useful. On the one hand, we are not able to know the “true” distribution of a given sample with any size \( n \), also the sample size \( n \) will never be \( \infty \). On the other hand, the NRMIs behave great for the data from discrete distributions. Furthermore, the mixture and hierarchical Bayesian nonparametric models based on NRMIs are showing great success in the applications and consistency behaviours \((\text{Lijoi et al., 2005})\). And the class of NRMIs is much larger than we expected, so that more study is necessary to develop more flexible subclasses of NRMIs or more general NRMIs like classes that are satisfying the consistency property.

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**Appendix**

In this section, we prove Theorems 2 and 5.
Proof of Theorem 2

Let \( I = \mathbb{E}[(P(A)|X)^m] \). Then, by Theorem 1, \( I \) can be computed as follows.

\[
I = \int_0^\infty \mathbb{E}[P(A)| U_n = u, X] f_{U_n|X}(u|X) \, du \\
= \int_0^\infty \left[ \sum_{j=1}^{n(\pi)} \frac{\mu(U_n)(A)}{T(U_n)} + \sum_{j=1}^{n(\pi)} \frac{J_j \delta Y_j(A)}{T(U_n) + \sum_{j=1}^{n(\pi)} J_j} \right] \, f_{U_n|X}(u|X) \, du \\
= \sum_{k=0}^m \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} \mathbb{E} \left[ e^{-y\mu(U_n)(A)} \mu(U_n)(A)^{m-k} \right] \mathbb{E} \left[ e^{-y\mu(U_n)(A^c)} \right] \\
\times \mathbb{E} \left[ e^{-y\sum_{j=1}^{n(\pi)} J_j \delta Y_j(A)} \right] f_{U_n|X}(u|X) \, dy \, du.
\]

Noticing that \( T(U_n) = \mu(U_n)(A) + \mu(U_n)(A^c) \), where \( \mu(U_n)(A) \) and \( \mu(U_n)(A^c) \) are independent, we can rewrite the expectation in (4.1) as

\[
I = \sum_{k=0}^m \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} \left( \sum_{j=1}^{n(\pi)} J_j \delta Y_j(A) \right)^k \, f_{U_n|X}(u|X) \, dy \, du \\
= \sum_{k=0}^m \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} (\sum_{j=1}^{n(\pi)} J_j \delta Y_j(A))^k \, f_{U_n|X}(u|X) \, dy \, du \\
= \sum_{k=0}^m \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} (\sum_{j=1}^{n(\pi)} J_j \delta Y_j(A))^k \, f_{U_n|X}(u|X) \, dy \, du.
\]

where the sum in front of \( \binom{n}{k} \) is over all the vector \((l_1, \ldots, l_{n(\pi)})\) such that \( \sum_{j=1}^{n(\pi)} l_j = k \). Taking the derivatives inside the expectation and using the Laplace transform of \( \mu(U_n)(A) \), we have

\[
I = \sum_{k=0}^m \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} (\sum_{j=1}^{n(\pi)} J_j \delta Y_j(A))^k \, f_{U_n|X}(u|X) \, dy \, du.
\]
we obtain

\[
\left( \sum \binom{k}{l_1, \ldots, l_{n(x)}} \prod_{j=1}^{n(x)} \frac{\tau_{n_j+l_j}(u+y, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) \right) f_{U_n|x}(u|X) dy du
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} V_{\alpha(A)}^{(m-k)}(u, y) e^{-\psi_x(u, y)} dy du \\
\left( \sum \binom{k}{l_1, \ldots, l_{n(x)}} \prod_{j=1}^{n(x)} \frac{\tau_{n_j+l_j}(u+y, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) \right) f_{U_n|x}(u|X) dy du , \quad (4.2)
\]

where \( \psi_x(u, y) = \int_x^\infty (1 - e^{-ys}) e^{-us} \rho(ds|x) \alpha(dx) \). By the fact that

\[
f_{U_n|x}(u|X) \propto u^{n-1} e^{-\psi_x(u)} \prod_{j=1}^{n(x)} \tau_{n_j}(u, Y_j)
\]

and \( e^{-\psi_x(u)} e^{-\psi_x(u+y)} = e^{-\psi_x(u+y)} \), we further simplify (4.2) to

\[
\mathcal{I} = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} u^{n-1} V_{\alpha(A)}^{(m-k)}(u+y) e^{-\psi_x(u+y)} dy du \\
\left( \sum \binom{k}{l_1, \ldots, l_{n(x)}} \prod_{j=1}^{n(x)} \frac{\tau_{n_j+l_j}(u+y, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) \right) dy du
\]

\[
= \sum_{k=0}^{m} \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} u^{n-1} V_{\alpha(A)}^{(m-k)}(u+y) e^{-\psi_x(u+y)} \prod_{j=1}^{n(x)} \tau_{n_j}(u+y, Y_j) \\
\left( \sum \binom{k}{l_1, \ldots, l_{n(x)}} \prod_{j=1}^{n(x)} \frac{\tau_{n_j+l_j}(u+y, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) \right) dy du .
\]

The change of variable \((w, z) = (u+y, u)\) yields

\[
\mathcal{I} = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{\Gamma(m)} \int_0^\infty \int_0^w (w-z)^{m-1} z^{n-1} V_{\alpha(A)}^{(m-k)}(w) e^{-\psi_x(w)} \prod_{j=1}^{n(x)} \tau_{n_j}(w, Y_j) \\
\left( \sum \binom{k}{l_1, \ldots, l_{n(x)}} \prod_{j=1}^{n(x)} \frac{\tau_{n_j+l_j}(w, Y_j)}{\tau_{n_j}(w, Y_j)} \delta_{Y_j}(A) \right) dz dw .
\]

Using

\[
\int_0^w (w-z)^{m-1} z^{n-1} dz = w^{m+n-1} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} ,
\]

we obtain

\[
\mathcal{I} = \sum_{k=0}^{m} \binom{m}{k} \frac{1}{\Gamma(m)} \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} \int_0^\infty w^{n+m-1} V_{\alpha(A)}^{(m-k)}(w) e^{-\psi_x(w)} \prod_{j=1}^{n(x)} \tau_{n_j}(w, Y_j)
\]

\[14\]
A similar computation as that for $\mathcal{I}$ yields

$$\mathcal{L} = \frac{\Gamma(n)}{\Gamma(m+n)} \int_0^\infty w^{n+m-1} e^{-\psi_X(w)} \prod_{j=1}^{n(\pi)} \tau_{n_j}(w, Y_j) \prod_{i=1}^{q+1} \left\{ \sum_{0 \leq l_1 + \cdots + l_{\max(\lambda_i)} \leq m_i} \binom{m_i}{l_1, \ldots, l_{\max(\lambda_i)}} \right\} \left( \prod_{j \in \lambda_i} \tau_{\lambda_j+l_j}(w, Y_j) \right) dw$$

This is (3.2).

For any family of pairwise disjoint sets $\{A_1, \ldots, A_q\}$ in $\mathcal{X}$ and for any positive integers $\{m_1, \ldots, m_q\}$ we denote $A_{q+1} = (\bigcup_{i=1}^q A_i)^c$, $m_{q+1} = 0$, and $m = \sum_{i=1}^q m_i$. For any sample $\{X_i\}_{i=1}^n$ from $P$, let $\{Y_j\}_{j=1}^{n(\pi)}$ be the distinct values of $\{X_i\}_{i=1}^n$. Let $\lambda_i = \{j : Y_j \in A_i\}$ be the set of the index of $Y_j$'s that in $A_i$ and we denote by $\max(\lambda_i)$ the maximal value in $\lambda_i$. We can compute the following moments easily.

$$\mathcal{L} := \mathbb{E} \left[ P(A_1)^{m_1} \cdots P(A_q)^{m_q} | \mathbf{X} \right] = \int_0^\infty \mathbb{E} \left[ P(A_1)^{m_1} \cdots P(A_q)^{m_q} | U_n = u, \mathbf{X} \right] f_{U_n | \mathbf{X}}(u | \mathbf{X}) du$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} \mathbb{E} \left[ e^{-y(T_u + \sum_{j=1}^{n(\pi)} J_j)} \prod_{i=1}^{q+1} \left( \mu_{n_i}(A_i) + \sum_{j=1}^{n(\pi)} J_j \delta_{Y_j}(A_i) \right)^{m_i} \right] f_{U_n | \mathbf{X}}(u | \mathbf{X}) dy du$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty \int_0^\infty y^{m-1} \prod_{i=1}^{q+1} \left\{ \sum_{k=0}^{m_i} \mathbb{E} \left[ e^{-y\mu_{n_i}(A_i)\mu_{n_i}(A_i)^{m_i-k}} \right] \mathbb{E} \left[ \left( \sum_{j \in \lambda_i} J_j \right)^k \right] \right\} f_{U_n | \mathbf{X}}(u | \mathbf{X}) dy du.$$
Proof. Let

\[ \Gamma(n) = \frac{1}{\Gamma(m + n)} \int_0^\infty w^m f_{U_n|X}(w|X) \prod_{i=1}^{g+1} \left\{ \sum_{0 \leq l_1 + \cdots + l_{\text{max}(\lambda_i)} \leq m_i} m_i \left( l_1, \ldots, l_{\text{max}(\lambda_i)} \right) \right\} \, dw. \]

This is part (ii) of the theorem. Then the proof of Theorem 5 is completed.

**Proof of Theorem 5**

We need the following lemma to prove Theorem 5.

**Lemma 14.** Under the assumption 4, we have for any \( y \in \mathcal{Y} \) and \( k \in \mathbb{Z}^+ \),

\[
\lim_{n \to \infty} \int_0^\infty \tau_k(u, y) f_{U_n|X}(u|X) \, du = k - C_k(y). \tag{4.4}
\]

**Proof.** Let \( g_n(u) \) be a constant multiple of the density of \( f_{U_n|X}(u|X) \) given by (4.10).

Namely,

\[
g_n(u) = u^{n-1} e^{-\psi(u)} \prod_{j=1}^{\pi(n)} \tau_j(u, Y_j).
\]

\[
= u^{n-1} e^{-a} \int_0^\infty (1-e^{-u}) \rho(ds|x) \, dx \prod_{j=1}^{\pi(n)} s^j e^{-us} \rho(ds|Y_j). \tag{4.5}
\]

\[
f_{U_n|X}(u|X) = \frac{g_n(u)}{\int_0^\infty g_n(u) \, du}. \tag{4.6}
\]

The derivative of \( g_n(u) \) is computed as follows,

\[
g_n'(u) = u^{n-2} e^{-\psi(u)} \prod_{j=1}^{\pi(n)} \tau_j(u, Y_j) \left[ n - 1 - \left( u \int_\mathcal{X} \tau_1(u, y) \alpha(dy) + \sum_{j=1}^{\pi(n)} \frac{\tau_{j+1}(u, Y_j)}{\tau_j(u, Y_j)} \right) \right].
\]

Let \( h_n(u) = u \int_\mathcal{X} \tau_1(u, y) \alpha(dy) \), then \( h_n'(u) = \int_\mathcal{X} (\tau_1(u, y) - u \tau_2(u, y)) \alpha(dy) \). By the assumption 4 \( \frac{u \tau_2(u, y)}{\tau_1(u, y)} \leq 1 \). This means \( h_n(u) \geq 0 \) and then \( h_n(u) \) is nondecreasing in \( u \).

Similarly, from the assumption 4 it follows that \( \frac{\tau_{n+1}(u, Y_j)}{\tau_n(u, Y_j)} \) is also nondecreasing in \( u \) for all \( n_j \). Thus, we have

\[
\tilde{g}_n(u) := u \int_\mathcal{X} \tau_1(u, y) \alpha(dy) + \sum_{j=1}^{\pi(n)} \frac{\tau_{j+1}(u, Y_j)}{\tau_j(u, Y_j)}
\]

is nondecreasing in \( u \). Since \( g_n(u) \) is a continuously differentiable function such that \( \int_0^\infty g_n(u) \, du < \infty \), it is then bounded and attain its maximum point at some point \( u^2_{n, n(\pi)} \).
satisfying $g'_n(u_{n,n(\pi)}^2) = 0$ or $\tilde{g}_n(u_{n,n(\pi)}^2) = n - 1$. Note that $\tilde{g}_n$ is also a continuous function and is then bounded on bounded interval. We claim that $u_{n,n(\pi)}^2 \to \infty$ as $n \to \infty$. In fact, by assumption \( u \frac{\tau_{n+1}(u,y)}{\tau_n(u,y)} \leq \phi(u)(k - C_k(y)), \forall k \in \mathbb{Z}^+ \text{ and } y \in \mathcal{K}, \) for some function $\phi(u) \in (0,1)$ which is nondecreasing in $u$ and $\lim_{u \to \infty} \phi(u) = 1$. Assume that $u_{n,n(\pi)}^2 < \infty$ as $n \to \infty$. Then, $\phi(u_{n,n(\pi)}) = \alpha < 1$, which implies $\sum_{j=1}^{n(\pi)} u \frac{\tau_{n+1}(u,Y_j)}{\tau_n(u,Y_j)} < \alpha \left( n - \sum_{j=1}^{n(\pi)} C_j(Y_j) \right)$. Therefore,

$$n - 1 = \tilde{g}_n(u_{n,n(\pi)}^2) < u_{n,n(\pi)}^2 \int_\mathcal{K} \tau_1(u_{n,n(\pi)}^2,y)\alpha(dy) + \alpha \left( n - \sum_{j=1}^{n(\pi)} C_j(Y_j) \right),$$

which implies

$$n(1 - \alpha) + \alpha \sum_{j=1}^{n(\pi)} C_j(Y_j) - 1 < u_{n,n(\pi)}^2 \int_\mathcal{K} \tau_1(u_{n,n(\pi)}^2,y)\alpha(dy) < \infty,$$

which is a contradiction.

Denote

$$\tilde{\tau}_k(u,y) = u \frac{\tau_{k+1}(u,y)}{\tau_k(u,y)}.$$

And let $u_{n,n(\pi)}$ be the positive square root of $u_{n,n(\pi)}^2$. Then, we have the following inequalities,

$$k - C_k(y) \geq \int_0^\infty u \frac{\tau_{k+1}(u,y)}{\tau_k(u,y)} f_{U_n}[X(u)|X]du = \frac{\int_0^\infty \tilde{\tau}_k(u,y)g_n(u)du}{\int_0^\infty g_n(u)du} = \tilde{\tau}_k(u_{n,n(\pi)},y) \left( 1 + \frac{\int_{u_{n,n(\pi)}}^{U_n} g_n(u)du}{\int_{u_{n,n(\pi)}}^{U_n} g_n(u)du} \right)^{-1} \geq \tilde{\tau}_k(u_{n,n(\pi)},y) \left( 1 + \frac{\int_{u_{n,n(\pi)}}^{U_n} g_n(u)du}{\int_{u_{n,n(\pi)}}^{U_n} g_n(u)du} \right)^{-1} \geq \tilde{\tau}_k(u_{n,n(\pi)},y) \left( 1 + \frac{\int_{u_{n,n(\pi)}}^{U_n} g_n(u_n,n(\pi))du}{\int_{u_{n,n(\pi)}}^{U_n} g_n(u_n,n(\pi))du} \right)^{-1} = \tilde{\tau}_k(u_{n,n(\pi)},y) \left( 1 + \frac{u_n,n(\pi)g_n(u_n,n(\pi))}{u^2_n,n(\pi) - u_n,n(\pi)g_n(u_n,n(\pi))} \right)^{-1} = (u_{n,n(\pi)} - 1) \frac{\tau_{k+1}(u_{n,n(\pi)},y)}{\tau_k(u_{n,n(\pi)},y)} \xrightarrow{n \to \infty} k - C_k(y).$$

This completes the proof of the lemma. \(\square\)
Now we are ready to give the proof of Theorem 5. To show the finiteness of \( \alpha \), we use the notation that \( \alpha = aH \), where \( \alpha = \alpha(\mathcal{X}) \) is finite and \( H \) is some probability measure.

We first follow the similar idea to that in (Freedman and Diaconis, 1983) to define a class of semi-norms on \( \mathcal{M}_\mathcal{X} \) such that convergence under such norms implies weak convergence. Let \( \mathcal{A} = \{ A_i \}_{i=1}^\infty \) be a measurable partition of \( \mathcal{X} \). The semi-norm between two probability measures \( P_1 \) and \( P_2 \) in \( \mathcal{M}_\mathcal{X} \) with respect to the partition \( \mathcal{A} \) is defined by

\[
|P_1 - P_2|_A = \sqrt{\sum_{i=1}^\infty [P_1(A_i) - P_2(A_i)]^2}.
\]

(4.7)

In order to show the posterior distribution of NRMI concentrates around its posterior mean, we need to evaluate the following quantity

\[
\mathbb{E} \left[ |P - \mathbb{E}[P\mid \mathbf{X}]|_A^2 \mid \mathbf{X} \right] = \sum_{i=1}^\infty \text{Var}[P(A_i)\mid \mathbf{X}].
\]

(4.8)

We claim that for any given measurable partition \( \mathcal{A} \), the above expectation \( (4.8) \) converges to 0 a.s. \( P_0^\infty \) as \( n \to \infty \). To prove this claim, we shall evaluate the first and second posterior moments of \( P \) for any \( A \in \mathcal{X} \). For the first moment we have

\[
\mathbb{E}[P(A)\mid \mathbf{X}] = \frac{1}{n} \int_0^\infty u f_{U_n}(u) V_{\alpha(A)}^{(1)}(u) du + \frac{1}{n} \sum_{j=1}^{n(\pi)} \int_0^\infty u f_{U_n}\mid \mathbf{X}(u\mid \mathbf{X}) \frac{\tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) du
\]

\[
= \frac{a}{n} \int_0^\infty u f_{U_n}(u) \int_A \tau_1(u, x) H(dx) du + \frac{1}{n} \sum_{j=1}^{n(\pi)} \int_0^\infty u f_{U_n}\mid \mathbf{X}(u\mid \mathbf{X}) \frac{\tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) du.
\]

For the second moment we have

\[
\mathbb{E}[P(A)^2\mid \mathbf{X}] = \frac{a}{n(n+1)} \int_0^\infty u^2 f_{U_n}\mid \mathbf{X}(u\mid \mathbf{X}) \int_A \tau_2(u, x) H(dx) du
\]

\[
+ \frac{a^2}{n(n+1)} \int_0^\infty u^2 f_{U_n}\mid \mathbf{X}(u\mid \mathbf{X}) \left( \int_A \tau_1(u, x) H(dx) \right)^2 du
\]

\[
+ 2 \frac{a}{n(n+1)} \sum_{j=1}^{n(\pi)} \int_0^\infty u^2 f_{U_n}\mid \mathbf{X}(u\mid \mathbf{X}) \frac{\tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) \int_A \tau_1(u, x) H(dx) du
\]

\[
+ \frac{1}{n(n+1)} \sum_{j=1}^{n(\pi)} \int_0^\infty u^2 f_{U_n}\mid \mathbf{X}(u\mid \mathbf{X}) \frac{\tau_{n_j+2}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) du
\]

\[
+ 2 \frac{1}{n(n+1)} \sum_{j\neq k}^{n(\pi)} \int_0^\infty u^2 f_{U_n}\mid \mathbf{X}(u\mid \mathbf{X}) \frac{\tau_{n_k+1}(u, Y_k) \tau_{n_j+1}(u, Y_j)}{\tau_{n_k}(u, Y_k) \tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) \delta_{Y_j}(A) du.
\]

Then, we can write

\[
\sum_{i=1}^\infty \text{Var}[P(A_i)\mid \mathbf{X}] = \sum_{i=1}^\infty \left( \mathbb{E}[P(A)^2\mid \mathbf{X}] - \mathbb{E}[P(A)\mid \mathbf{X}]^2 \right) = \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4,
\]

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where the terms $J_1$, $J_2$, $J_3$, $J_4$ are defined as follows.

$$J_1 = \frac{a}{n(n+1)} \int_0^\infty u^2 f_{U_n|X}(u|X) \left( \int_{A_i} \tau_1(u,x) H(dx)du \right)$$

$$+ \frac{a^2}{n(n+1)} \sum_{i=1}^{\infty} \int_0^\infty u^2 f_{U_n|X}(u|X) \left( \int_{A_i} \tau_1(u,x) H(dx)du \right)^2 du$$

$$- \frac{a^2}{n^2} \sum_{i=1}^{\infty} \left( \int_0^\infty u f_{U_n}(u) \int_{A_i} \tau_1(u,x) H(dx)du \right)^2; \quad (4.9)$$

$$J_2 = 2 \frac{a}{n(n+1)} \sum_{i=1}^{\infty} \sum_{j=1}^{n(\pi)} \int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \delta_{Y_j}(A_i)$$

$$\times \int_{A_i} \tau_1(u,x) H(dx)du - 2 \frac{a^2}{n^2} \sum_{i=1}^{\infty} \sum_{j=1}^{n(\pi)} \int_0^\infty u f_{U_n}(u) \int_{A_i} \tau_1(u,x) H(dx)du$$

$$\times \int_0^\infty u f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \delta_{Y_j}(A_i)du; \quad (4.10)$$

$$J_3 = \frac{1}{n(n+1)} \sum_{i=1}^{\infty} \sum_{j=1}^{n(\pi)} \int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \delta_{Y_j}(A_i)du$$

$$- \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{j=1}^{n(\pi)} \left( \int_0^\infty u f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \delta_{Y_j}(A_i)du \right)^2 \quad (4.11)$$

$$= \frac{1}{n(n+1)} \sum_{j=1}^{n(\pi)} \int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} du$$

$$- \frac{1}{n^2} \sum_{j=1}^{n(\pi)} \left( \int_0^\infty u f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} du \right)^2; \quad (4.12)$$

and

$$J_4 = 2 \frac{1}{n(n+1)} \sum_{i=1}^{\infty} \sum_{j \neq k}^{n(\pi)} \int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_k)}{\tau_{n_k}(u,Y_k)} \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \delta_{Y_k}(A_i) \delta_{Y_j}(A_i)$$

$$- 2 \frac{1}{n^2} \sum_{i=1}^{\infty} \sum_{j \neq k}^{n(\pi)} \int_0^\infty u f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_k)}{\tau_{n_k}(u,Y_k)} \delta_{Y_k}(A_i)du$$

$$\times \int_0^\infty u f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \delta_{Y_j}(A_i)du. \quad (4.13)$$
We will first consider the terms \( \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4 \) and then \( \mathcal{J}_1 \). But before dealing with them, we need some prior preparations. By the identity \( E[\mathcal{P}(Y)|X] = 1 \) we have

\[
\frac{a}{n} \int_0^\infty u f_{\nu_n}(u) \int_\mathbb{X} \tau_1(u, x) H(dx) du + \frac{1}{n} \sum_{j=1}^{n(\pi)} \frac{u f_{\nu_n}|X(u|X) \tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} du = 1. 
\]

By Lemma 14, we have the approximation

\[
\frac{a}{n} \int_0^\infty u f_{\nu_n}(u) \int_\mathbb{X} \tau_1(u, x) H(dx) du \approx \frac{1}{n} \sum_{j=1}^{n(\pi)} C_{n_j}(Y_j) 
\]

as \( n \) is large. On the other hand, let \( u_{n,n(\pi)} \) be the maximal point of \( g_n(u) \) as in Lemma 14. Under the assumption 4, we know that \( u \tau_1(u, x) \) is nondecreasing in \( u \) for all \( x \). We have

\[
a \int_0^{u_{n,n(\pi)}} u f_{\nu_n}(u) \int_\mathbb{X} \tau_1(u, x) H(dx) du = a \int_0^{u_{n,n(\pi)}} u g_n(u) \int_\mathbb{X} \tau_1(u, x) H(dx) du 
\]

\[
\leq a u_{n,n(\pi)} \int_\mathbb{X} \tau_1(u_{n,n(\pi)}, x) H(dx) \left( 1 + \frac{\int_0^{u_{n,n(\pi)}} g_n(u) du}{\int_0^{u_{n,n(\pi)}} g_n(u) du} \right)^{-1} 
\]

\[
= a \int_\mathbb{X} \tau_1(u_{n,n(\pi)}, x) H(dx) , 
\]

which goes to 0 as \( n \to \infty \), since \( \tau_1(u, x) \) is decreasing to 0 in \( u \) for all \( x \). Combining the above computation with the approximation (4.14), we have

\[
\lim_{n \to \infty} a \sum_{j=1}^{n(\pi)} C_{n_j}(Y_j) \int_{u_{n,n(\pi)}}^{\infty} u f_{\nu_n}|X(u|X) \int_\mathbb{X} \tau_1(u, x) H(dx) du = 1. 
\]

With this preparation we now deal with \( \mathcal{J}_2 \). Notice first that for any \( A_i \) and \( Y_j \), by the assumption 4, we will have

\[
I_1 := \int_0^\infty u^2 \frac{\tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A_i) \int_{A_i} \tau_1(u, x) H(dx) f_{\nu_n}|X(u|X) du 
\]

\[
\leq (n_j - C_{n_j}(Y_j)) \delta_{Y_j}(A_i) \int_0^\infty u f_{\nu_n}|X(u|X) \int_{A_i} \tau_1(u, x) H(dx) du. 
\]

On the other hand,

\[
I_1 \geq \int_{u_{n,n(\pi)}}^{\infty} u^2 \frac{\tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A_i) \int_{A_i} \tau_1(u, x) H(dx) f_{\nu_n}|X(u|X) du 
\]
By the above inequalities (4.18), (4.19) and the approximation (4.14), (4.17), we can see

Thus, for large $n$, we have

This combined with (4.14) yields

which has order $O\left(\frac{1}{n}\right)$.

For $J_3$, notice that under the assumption $\mathfrak{A}$, we have

is nondecreasing in $u$ and is bounded by $(n_j + 1 - C_{n_j+1}(Y_j))(n_j - C_{n_j}(Y_j))$. Using a similar approach as that in Lemma $\mathfrak{A}$, we have as $n$ is large,

Combining it with Lemma $\mathfrak{A}$, we have as $n$ becomes large

\begin{align*}
J_3 &\sim \frac{1}{n(n+1)} \sum_{j=1}^{n} (n_j + 1 - C_{n_j+1}(Y_j))(n_j - C_{n_j}(Y_j)) - \frac{1}{n^2} \sum_{j=1}^{n} (n_j - C_{n_j}(Y_j))^2 \\
&= \frac{1}{n^2(n+1)} \sum_{j=1}^{n} (n_j - C_{n_j}(Y_j))(n_j + (n+1)C_{n_j} - n - nC_{n_j+1})
\end{align*}
\[ n - \frac{\left( \sum_{j=1}^{n(\pi)} C_{n_j}(Y_j) \right)}{n(n+1)} \leq 2, \quad (4.22) \]

which has order at most \( O(\frac{1}{n}) \).

For \( J_4 \), we have that under the assumption \( u^2 \frac{\tau_{n_k+1}(u,Y_k)}{\tau_{n_k}(u,Y_k)} \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \) is nondecreasing in \( u \) and is bounded by \((n_k - C_{n_k}(Y_k))(n_j - C_{n_j}(Y_j))\). Using a similar argument to that in Lemma \( \square \) leads to

\[
\int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_k+1}(u,Y_k)}{\tau_{n_k}(u,Y_k)} \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} du \\
\approx \int_0^\infty u f_{U_n|X}(u|X) \frac{\tau_{n_k+1}(u,Y_k)}{\tau_{n_k}(u,Y_k)} du \int_0^\infty u f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} du \\
\approx (n_k - C_{n_k}(Y_k))(n_j - C_{n_j}(Y_j)).
\]

Thus

\[
J_4 \approx 2 \left( \frac{1}{n(n+1)} - \frac{1}{n^2} \right) \sum_{i=1}^{n(\pi)} \sum_{j \neq i} (n_k - C_{n_k}(Y_k))(n_j - C_{n_j}(Y_j)) \delta_{k}(A_i) \delta_{j}(A_i) \\
\approx 2 \sum_{j \neq k} (n_k - C_{n_k}(Y_k))(n_j - C_{n_j}(Y_j)) \frac{C_{n(\pi)}}{n^2(n+1)},
\]

which has an order at most \( O(\frac{1}{n}) \).

Finally, we deal with the term \( J_1 \). Notice that \( \mathbb{E}[P(X)^2|X] = 1 \). Using the computation we obtained for \( J_2, J_3, J_4 \), we have

\[
1 = \mathbb{E}[P(X)^2|X] = \frac{a}{n(n+1)} \int_0^\infty u^2 f_{U_n|X}(u|X) \int_x \tau_2(u,x) H(dx) du \\
+ \frac{a^2}{n(n+1)} \int_0^\infty u^2 f_{U_n|X}(u|X) \left( \int_x \tau_1(u,x) H(dx) \right)^2 du \\
+ 2 \frac{a}{n(n+1)} \sum_{j=1}^{n(\pi)} \int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_j+1}(u,Y_j)}{\tau_{n_j}(u,Y_j)} \int_x \tau_1(u,x) H(dx) du \\
+ \frac{1}{n(n+1)} \sum_{j=1}^{n(\pi)} \int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_j+2}(u,Y_j)}{\tau_{n_j}(u,Y_j)} du \\
+ 2 \frac{1}{n(n+1)} \sum_{j \neq k} \int_0^\infty u^2 f_{U_n|X}(u|X) \frac{\tau_{n_k+1}(u,Y_k) \tau_{n_j+1}(u,Y_j)}{\tau_{n_k}(u,Y_k) \tau_{n_j}(u,Y_j)} du \\
\approx \frac{a}{n(n+1)} \int_0^\infty u^2 f_{U_n|X}(u|X) \int_x \tau_2(u,x) H(dx) du \\
+ \frac{a^2}{n(n+1)} \int_0^\infty u^2 f_{U_n|X}(u|X) \left( \int_x \tau_1(u,x) H(dx) \right)^2 du
\]

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Combining the approximations (4.14) and (4.23), we have

\[
\sum_{j=1}^{n+1} C_{n_j}(Y_j) \left( n - \sum_{j=1}^{n+1} C_{n_j}(Y_j) \right) + 2 \sum_{j=1}^{n+1} C_{n_j}(Y_j) \left( n - \sum_{j=1}^{n+1} C_{n_j}(Y_j) \right)
\]

\[
+ \sum_{j=1}^{n+1} (n_j + 1 - C_{n_{j+1}}(Y_j) - \frac{n_j - C_{n_j}}{n})(n_j - C_{n_j}(Y_j))
\]

\[
+ 2 \sum_{j \neq k} (n_k - C_{n_k}(Y_k))(n_j - C_{n_j}(Y_j))
\]

This implies

\[
\frac{a}{n(n+1)} \int_0^\infty u^2 f_{U_n|x}(u|X) \int_x \tau_2(u, x) H(dx)du
\]

\[
+ \frac{a^2}{n(n+1)} \int_0^\infty u^2 f_{U_n|x}(u|X) \left( \int_x \tau_1(u, x) H(dx) \right)^2 du
\]

\[
- \frac{a^2}{n^2} \left( \int_0^\infty u f_{U_n}(u) \int_x \tau_1(u, x) H(dx)du \right)^2
\]

\[
+ 2 \sum_{i \neq l} \int_0^\infty u f_{U_n}(u) \int_{A_i} \tau_1(u, x) H(dx)du \int_0^\infty u f_{U_n}(u) \int_{A_l} \tau_1(u, x) H(dx)du
\]

\[
- 2 \sum_{i \neq l} \int_0^\infty u^2 f_{U_n|x}(u|X) \int_{A_i} \tau_1(u, x) H(dx) \int_{A_i} \tau_1(u, x) H(dx)du
\]

\[
\sim \frac{n + \left( \sum_{j=1}^{n+1} C_{n_j}(Y_j) \right)^2 - \sum_{j=1}^{n+1} (n_j - C_{n_j}(Y_j))(1 + C_{n_j} - C_{n_{j+1}} - \frac{n_j - C_{n_j}}{n})}{n(n+1)}.
\] (4.23)

Combining the approximations (4.14) and (4.23), we have

\[
J_1 = \frac{a}{n(n+1)} \int_0^\infty u^2 f_{U_n|x}(u|X) \int_x \tau_2(u, x) H(dx)du
\]

\[
+ \frac{a^2}{n(n+1)} \int_0^\infty u^2 f_{U_n|x}(u|X) \left( \int_x \tau_1(u, x) H(dx) \right)^2 du
\]

\[
- \frac{a^2}{n^2} \left( \int_0^\infty u f_{U_n}(u) \int_x \tau_1(u, x) H(dx)du \right)^2
\]

\[
+ 2 \sum_{i \neq l} \int_0^\infty u f_{U_n}(u) \int_{A_i} \tau_1(u, x) H(dx)du \int_0^\infty u f_{U_n}(u) \int_{A_l} \tau_1(u, x) H(dx)du
\]

\[
- 2 \sum_{i \neq l} \int_0^\infty u^2 f_{U_n|x}(u|X) \int_{A_i} \tau_1(u, x) H(dx) \int_{A_i} \tau_1(u, x) H(dx)du
\]

\[
\sim \frac{n + \left( \sum_{j=1}^{n+1} C_{n_j}(Y_j) \right)^2 - \sum_{j=1}^{n+1} (n_j - C_{n_j}(Y_j))(1 + C_{n_j} - C_{n_{j+1}} - \frac{n_j - C_{n_j}}{n})}{n(n+1)}
\]

\[
- \frac{\left( \sum_{j=1}^{n+1} C_{n_j}(Y_j) \right)^2}{n^2}
\]

\[
+ 2 \frac{a^2}{n^2} \sum_{i \neq l} \int_0^\infty u f_{U_n}(u) \int_{A_i} \tau_1(u, x) H(dx)du \int_0^\infty u f_{U_n}(u) \int_{A_l} \tau_1(u, x) H(dx)du
\]

\[
- 2 \frac{a^2}{n(n+1)} \sum_{i \neq l} \int_0^\infty u^2 f_{U_n|x}(u|X) \int_{A_i} \tau_1(u, x) H(dx) \int_{A_i} \tau_1(u, x) H(dx)du.
\] (4.24)
We now treat the above last two summation terms. First, we see

\[ 2 \sum_{i \neq l}^{\infty} \int_{0}^{\infty} u f_{U_{n}}(u) \int_{A_{i}} \tau_{1}(u, x) H(dx) du \int_{0}^{\infty} u f_{U_{n}}(u) \int_{A_{l}} \tau_{1}(u, x) H(dx) du \]

\[ \sim n \left( \int_{0}^{\infty} u f_{U_{n}}(u) \int_{\mathcal{X}} \tau_{1}(u, x) H(dx) du \right)^{2} \]

and

\[ 2 \sum_{i \neq l}^{\infty} \int_{0}^{\infty} u^{2} f_{U_{n}|X}(u|X) \int_{A_{i}} \tau_{1}(u, x) H(dx) \int_{A_{l}} \tau_{1}(u, x) H(dx) du \]

\[ \sim \int_{0}^{\infty} u^{2} f_{U_{n}|X}(u|X) \left( \int_{\mathcal{X}} \tau_{1}(u, x) H(dx) \right)^{2} du. \]

Thus, we see

\[ J_{1} \sim \frac{n^{2} - n \sum_{j=1}^{n(\pi)} (n_{j} - C_{n_{j}}(Y_{j}))(1 + C_{n_{j}} - C_{n_{j} + 1}) - \left( \sum_{j=1}^{n(\pi)} C_{n_{j}}(Y_{j}) \right)^{2}}{n^{2}(n + 1)}. \]

It is easy to see

\[ \sum_{j=1}^{n(\pi)} (n_{j} - C_{n_{j}}(Y_{j}))(1 + C_{n_{j}} - C_{n_{j} + 1}) \leq 3n \sum_{j=1}^{n(\pi)} (n_{j} - C_{n_{j}}(Y_{j})) \leq 3n^{2}. \]

Thus \( J_{1} \) has an order \( O\left( \frac{1}{n} \right) \).

Summarizing the above discussion for \( J_{1}, J_{2}, J_{3}, J_{4} \), we have

\[ \sum_{i=1}^{\infty} \text{Var}[P(A_{i})|X] \sim O\left( \frac{1}{n} \right) \to 0 \quad \text{as } n \to \infty. \]

Now that the distribution of \( P(\cdot|X) \) converges weakly to the point mass at the distribution of \( \lim_{n \to \infty} \mathbb{E}[P(dx)|X] \). If the “true” distribution \( P_{0} \) of \( X \) is continuous, the posterior expectation has the following form for any \( A \in \mathcal{X} \).

\[
\mathbb{E}[P(A)|X] = \frac{a}{n} \int_{0}^{\infty} u f_{U_{n}}(u) \int_{A} \tau_{1}(u, x) H(dx) du \\
+ \frac{1}{n} \sum_{j=1}^{n} \int_{0}^{\infty} u f_{U_{n}|X}(u|X) \frac{\tau_{2}(u, X_{j})}{\tau_{1}(u, X_{j})} \delta_{X_{j}}(A) du.
\]

(4.25)

As \( n \to \infty \), the weight \( \int_{0}^{\infty} u f_{U_{n}|X}(u|X) \frac{\tau_{2}(u, X_{j})}{\tau_{1}(u, X_{j})} du \sim 1 - C_{1}(X_{j}) \), thus the second part of the summation in (4.25) has the form \( \sum_{j=1}^{n} \frac{1-C_{1}(X_{j})}{n} \delta_{X_{j}}(A) \) that converges uniformly over Glivenko-Cantelli classes to \( (1 - \bar{C}_{1})P_{0} \). Since the sum of the weights of \( \alpha(dx) = aH(dx) \) and \( \delta_{X_{j}}(dx) \) is equal to \( 1 \), we have \( \lim_{n \to \infty} \mathbb{E}[P(\cdot)|X] = \bar{C}_{1}\alpha(\cdot) + (1 - \bar{C}_{1})P_{0}(\cdot) \).
If the “true” distribution $P_0$ of $X$ is discrete with $\lim_{n \to \infty} \frac{n(\pi)}{n} = 0$, the posterior expectation has the following form:

$$
\mathbb{E}[P(A)|X] = \frac{a}{n} \int_{0}^{\infty} u f_{U_n}(u) \int_{A} \tau_1(u, x) H(dx) du \\
+ \frac{1}{n} \sum_{j=1}^{n(\pi)} \int_{0}^{\infty} u f_{U_n|x}(u|x) \frac{\tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \delta_{Y_j}(A) du.
$$

(4.26)

As $n \to \infty$, $\int_{0}^{\infty} u f_{U_n|x}(u|x) \frac{\tau_{n_j+1}(u, Y_j)}{\tau_{n_j}(u, Y_j)} \sim \frac{n_j - C_{n_j}(Y_j)}{n}$. Hence, the second part of the summation in (4.26) has the form $\sum_{j=1}^{n(\pi)} \frac{n_j - C_{n_j}(Y_j)}{n} \delta_{Y_j}(A)$. Notice that

$$
\sum_{j=1}^{n(\pi)} n_j - C_{n_j}(Y_j) \frac{\delta_{Y_j}(A)}{n} = \sum_{i=1}^{n} \frac{1}{n} \delta_{X_i}(A) - \sum_{j=1}^{n(\pi)} C_{n_j}(Y_j) \frac{\delta_{Y_j}(A)}{n},
$$

where the term $\sum_{j=1}^{n(\pi)} C_{n_j}(Y_j)$ converges to 0. Thus the weight of $\alpha(dx)$ is 0 and we have $\lim_{n \to \infty} \mathbb{E}[P(\cdot)|X] = P_0(dx)$. This completes the proof of Theorem 5.