On initialized and ACM line bundles over a smooth sextic surface in $\mathbb{P}^3$

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ABSTRACT
Let $X \subset \mathbb{P}^3$ be a smooth hypersurface of degree 6 over complex numbers. In this article, we give a characterization of initialized and ACM bundles of rank 1 on $X$ with respect to the line bundle given by a smooth hyperplane section of $X$.

1. Introduction and the statement of the main result

Let $X \subset \mathbb{P}^n$ be a smooth projective variety and $H$ be the very ample line bundle given by a smooth hyperplane section $C$ of $X$. If the coordinate ring of $X$ is Cohen-Macaulay, then a vector bundle $F$ on $X$ is called arithmetically Cohen-Macaulay (ACM) if it has no intermediate cohomology, i.e. if:

$H^i(F \otimes H^t) = 0$, for $1 \leq i \leq \dim(X) - 1$ and for all $t \in \mathbb{Z}$.

It is always interesting to ask whether one can classify ACM bundles on a fixed variety. On a projective space $\mathbb{P}^n$, a higher rank vector bundle is ACM if and only if it's obtained as a direct sum of line bundles on $\mathbb{P}^n$(see [15]). This is not necessarily true for more general polarized smooth projective varieties (e.g. hypersurfaces of degree $d \geq 2$ in $\mathbb{P}^n$). In this direction, considerable efforts have been driven toward a classification of ACM bundles on hypersurfaces $X^{(d)} \subset \mathbb{P}^n$ of degree $d$. For $d = 2$, ACM bundles are classified by Knörrer (see [17]). For $d = 3$, Casanellas and Hartshorne (see [6]) have constructed a $n^2 + 1$-dimensional family of rank $n$ indecomposable ACM vector bundles on $X^{(d)}$ with Chern classes $c_1 = nH$ and $c_2 = \frac{1}{2}(3n^2 - n)$ for $n \geq 2$ and Faenzi (see [12]) gave a complete classification of rank 2 ACM bundles on $X^{(d)}$. For $d = 4, n = 3$, Coskun, Kulkarni and Mustopa have constructed a 14-dimensional family of simple Ulrich bundles on $X^{(d)}$ of rank 2 with $c_1 = H^\otimes 3$ and $c_2 = 14$, in the case where $X^{(d)}$ is a Pfaffian quartic surface (see [11]). For $d = 4, n = 3$, Casnati has classified indecomposable ACM bundles of rank
2 on $X^{(d)}$, where $X^{(d)}$ is general determinantal (see [7]). For $d = 5, n = 3$, rank 2 ACM bundles on general such $X^{(d)}$ are classified by Chiantini and Faenzi (see [8]). For $d = 6, n = 3$, rank 2 ACM bundles on general such $X^{(d)}$ are classified by M. Patnott (see [18]). Recently the present author has studied questions related to Brill-Noether locus on a very general sextic surface in $\mathbb{P}^3$ and their relation with rank 2 weakly Ulrich and Ulrich bundles on it (see [5]).

In this context, a characterization of ACM bundles of rank 1 on $X$ is useful for the construction of higher rank indecomposable ACM bundles on $X$. In this direction, Pons-Llopis and Tonini have classified ACM line bundles on a DelPezzo surface $X$ with respect to the anti-canonical line bundle on $X$ (see [19]). F. Chindea has classified ACM line bundles on a complex polarized elliptic ruled surface (see [9]). Recently K. Watanabe has classified ACM line bundles on smooth quartic hypersurfaces in $\mathbb{P}^3$ (see [22]), on polarized $K3$ surfaces (see [20]) and on smooth quintic hypersurfaces in $\mathbb{P}^3$ (see [21]). This motivates us to extend the study related to the classification of initialized (i.e. line bundles $L$ with $H^0(X, L) \neq 0, H^0(X, L \otimes H^*) = 0$) and ACM line bundles on smooth sextic hypersurfaces in $\mathbb{P}^3$. Our main result is as follows:

**Theorem 1.1.** Let $X \subset \mathbb{P}^3$ be a smooth sextic hypersurface. Let $H$ be the hyperplane class of $X$ and $C \in |H|$ be a smooth member. Let $D$ be a non-zero effective divisor on $X$ of arithmetic genus $P_a(D)$. Then $O_X(D)$ is initialized and ACM if and only if the following conditions are satisfied:

(i) either $2.C.D - 9 \leq P_a(D) \leq 2.C.D - 4$ or $P_a(D) = 2.C.D - 2$.

(ii) If $P_a(D) = 2.C.D - 2$, then $C.D = 1$.

(iii) If $P_a(D) = 2.C.D - 4$, then the following occurs:

(a) $C.D = 2$ or $5 \leq C.D \leq 6$ or $12 \leq C.D \leq 15$.

(b) If $C.D \in \{6, 12, 13, 14, 15\}$, then $h^0(O_C(D - C)) = 0$ and if $12 \leq C.D \leq 15$, then $h^0(O_C(D - 2C)) = 0$.

(c) If $C.D = 12$, then $h^0(O_C(2C - D)) = 0$ and if $C.D = 15$, then $h^0(O_C(D)) = 6$.

(iv) If $P_a(D) = 2.C.D - 5$, then the following occurs:

(a) $3 \leq C.D \leq 7$ or $10 \leq C.D \leq 14$.

(b) If $C.D \in \{6, 7, 10, 11, 12, 13, 14\}$, then $h^0(O_C(D - C)) = 0$ and for $C.D \in \{11, 12\}$, $h^0(O_C(2C - D)) = 0$.

(c) If $C.D = 6$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (3, 3)$ and $h^0(O_C(C - D)) = 0$.

(d) If $12 \leq C.D \leq 14$, then $h^0(O_C(D)) = 5$.

(v) If $P_a(D) = 2.C.D - 6$, then the following occurs:

(a) $3 \leq C.D \leq 13$.

(b) If $6 \leq C.D \leq 13$, then $h^0(O_C(D - C)) = 0$ and if $10 \leq C.D \leq 12$, then $h^0(O_C(2C - D)) = 0$.

(c) If $3 \leq C.D \leq 4$, then $(h^0(O_C(C - D)), h^0(O_C(2C - D))) \neq (1, 3)$ and $h^0(O_C(C - D)) = 0$ for $C.D = 6$.

(d) If $C.D = 7$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (3, 2)$.

(e) If $C.D = 6$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (i, j)$, where $i, j \in \{2, 3\}, i + j \in \{5, 6\}$.

(f) If $C.D = 5$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (2, 3)$ and if $11 \leq C.D \leq 13$, then $h^0(O_C(D)) = 4$.

(vi) If $P_a(D) = 2.C.D - 7$, then the following occurs:

(a) $4 \leq C.D \leq 12$.

(b) If $6 \leq C.D \leq 12$, then $h^0(O_C(D - C)) = 0$ and if $10 \leq C.D \leq 12$, then $h^0(O_C(2C - D)) = 0$.

(c) If $8 \leq C.D \leq 9$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (3, 1)$.

(d) If $C.D = 7$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (i, j)$, where $i \in \{2, 3\}, j \in \{1, 2\}, i + j \in \{4, 5\}$.

(e) If $C.D = 6$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (i, j)$, where $i, j \in \{1, 2, 3\}, i + j \in \{4, 5, 6\}$.

(f) If $C.D = 5$, then $(h^0(O_C(D)), h^0(O_C(2C - D))) \neq (i, j)$, where $i \in \{1, 2\}, j \in \{2, 3\}, i + j \in \{4, 5\}$.

(g) If $C.D = 4$, then $h^0(O_C(2C - D)) \neq 3$.

(h) If $C.D \in \{4, 6\}$, then $h^0(O_C(C - D)) = 0$ and if $10 \leq C.D \leq 12$, then $h^0(O_C(D)) = 3$. 
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1.1. The plan of the paper

Our plan of the paper is as follows:

Section 2 is dedicated to showcase the notions and results that will be useful in the next section. This section is divided into two subsections. The first subsection (cf. subsection 2.1) deals with the preliminary notions and results related to line bundles on curves. Here we begin by recalling the definitions and classical results concerning some important invariants of line bundles and smooth plane curves. We then also document results related to the existence of certain divisors on smooth plane curves. The second subsection (cf. subsection 2.2) deals with the results on line bundles over a smooth hypersurface in $\mathbb{P}^3$. To be precise, after mentioning basic facts on line bundles on smooth hypersurfaces, we note down a sufficient condition for the non-negativity of arithmetic genus of an effective divisor on surfaces of degree $d$. We then present a characterization of non-zero effective divisors of degree 1 and 2 on a smooth sextic surface. Next, we recall the definitions of initializedness and ACMness. Consequently, we document a result concerning $1$-connected divisors.

In Section 3, we study necessary and sufficient condition for a line bundle given by a non-zero effective divisor on a smooth sextic hypersurface to be initialized and ACM. In particular, toward the necessary condition we first study the possibilities in terms of their degree and arithmetic genus. This is one of the most important steps toward proving Theorem 1.1 and a major part of this article is devoted to obtain an optimal picture of possibilities of arithmetic genus and intersection numbers through a detailed case by case analysis (cf. Theorem 3.1). We then equip the already obtained necessary condition with some cohomological restrictions to obtain the only if part of the promised Theorem 1.1. Then initializedness is established for the divisors satisfying any of those conditions. Before proving ACMness, using the techniques related to $1$-connectedness, we prove a proposition regarding vanishing of $h^1(\mathcal{O}_X(-D))$ for some divisors (see Proposition 3.2). This proposition coupled with techniques from Section 2 enables us to prove ACMness for the divisors satisfying any of the conditions mentioned in the promised Theorem 1.1. We mention that the last part of Section 2 (cf. Proposition 2.18, Remark 2.22) plays the most crucial role in proving ACMness. We end this section by pointing out a concrete situation of the existence of a non-ACM line bundle on some smooth sextic surface.

Section 4, deals with the existence of smooth sextic hypersurfaces in $\mathbb{P}^3$ and initialized and ACM line bundles on them. Toward this, we first present a concrete example of lines on Fermat sextic. We then present a generalized version of [[21], Lemma 5.1], which gives us a sufficient condition for the existence of smooth hypersurfaces of degree $d > 4$ in $\mathbb{P}^3$ and initialized and ACM line bundles on them. This Lemma coupled with certain concrete constructions of quartic hypersurfaces in $\mathbb{P}^3$ and smooth curves on them as in [[21], Section 5], enables us to produce a series of examples of sextic surfaces and initialized and ACM line bundles on them.

(vii) If $P_a(D) = 2C.D - 8$, then the following occurs:
(a) $4 \leq C.D \leq 11$.
(b) If $5 \leq C.D \leq 9$, then $(h^0(\mathcal{O}_C(D)), h^0(\mathcal{O}_C(2C - D))) \neq (2, 1)$ and if $6 \leq C.D \leq 9$, then $h^0(\mathcal{O}_C(D)) \neq 3$.
(c) If $4 \leq C.D \leq 6$, then $h^0(\mathcal{O}_C(C - D)) = 0$ and if $5 \leq C.D \leq 7$, then $h^0(\mathcal{O}_C(2C - D)) \notin \{2, 3\}$.
(d) If $C.D = 4$, then $h^0(\mathcal{O}_C(2C - D)) = 1$ and if $10 \leq C.D \leq 11$, then $(h^0(\mathcal{O}_C(D)), h^0(\mathcal{O}_C(2C - D))) = (2, 0)$.

(viii) If $P_a(D) = 2C.D - 9$, then the following occurs:
(a) $5 \leq C.D \leq 10$, $h^0(\mathcal{O}_C(D)) = 1$ and $h^0(\mathcal{O}_C(2C - D)) = 0$.
(b) If $5 \leq C.D \leq 6$, then $h^0(\mathcal{O}_C(C - D)) = 0$. 

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In Section 3, we study necessary and sufficient condition for a line bundle given by a non-zero effective divisor on a smooth sextic hypersurface to be initialized and ACM. In particular, toward the necessary condition we first study the possibilities in terms of their degree and arithmetic genus. This is one of the most important steps toward proving Theorem 1.1 and a major part of this article is devoted to obtain an optimal picture of possibilities of arithmetic genus and intersection numbers through a detailed case by case analysis (cf. Theorem 3.1). We then equip the already obtained necessary condition with some cohomological restrictions to obtain the only if part of the promised Theorem 1.1. Then initializedness is established for the divisors satisfying any of those conditions. Before proving ACMness, using the techniques related to 1-connectedness, we prove a proposition regarding vanishing of $h^1(\mathcal{O}_X(-D))$ for some divisors (see Proposition 3.2). This proposition coupled with techniques from Section 2 enables us to prove ACMness for the divisors satisfying any of the conditions mentioned in the promised Theorem 1.1. We mention that the last part of Section 2 (cf. Proposition 2.18, Remark 2.22) plays the most crucial role in proving ACMness. We end this section by pointing out a concrete situation of the existence of a non-ACM line bundle on some smooth sextic surface.

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1.1.1. Notations and conventions

All curves and surfaces are always smooth and projective over $\mathbb{C}$. We denote the canonical bundle of a smooth variety $X$ to be $K_X$. For a divisor (or the corresponding line bundle) $\mathcal{L}$ on a variety $X$, we denote by $|\mathcal{L}|$ the linear system of $\mathcal{L}$ and denote the dual of a line bundle $\mathcal{L}$ by $\mathcal{L}^*$. In this context, $\deg(\mathcal{L})$ will always stand for the degree of the line bundle $\mathcal{L}$. For a divisor $D$ on a curve $C$, $r(D) = h^0(\mathcal{O}_C(D)) - 1$ is the projective dimension of the complete linear system $|D|$. A linear system on (a curve) $C$ having degree $d$ and projective dimension $r$ is denoted by $|D|^r$. We denote a smooth hypersurface of degree $d$ in $\mathbb{P}^3$ by $X(d)$. For a hyperplane section $C$ of $X(d)$, we denote the class of it in $\text{Pic}(X(d))$ by $H$. For an integer $t$, $H/t$ is denoted by $\mathcal{O}_X(t)$. For a vector bundle $E$ on $X(d)$, we will write $E/t = E(t)$.

For a real number $x$, we denote by $\lfloor x \rfloor$ the greatest integer less than or equal to $x$.

By S.E.S, we mean a short exact sequence.

2. Technical preliminaries

In this section, we will recall and establish a few preliminary results that we need in Section 3. This Section is divided into two subsections. In the first subsection, we document notions and results related to divisors on smooth curves and in the second subsection, we prepare several results on line bundles over surfaces that will be useful in Section 3.

2.1. Line bundles on curves

We begin by recalling the definitions of some important invariants of smooth projective curves of genus $g \geq 1$.

**Definition 2.1.** (gonality) Let $C$ be a curve of genus $g \geq 1$. The minimal degree of surjective morphisms from $C$ to $\mathbb{P}^1$ is called the gonality of $C$, and denoted by $\text{gon}(C)$:

$$\text{gon}(C) := \min \{ \deg(f) \mid f : C \to \mathbb{P}^1, \text{ a surjective morphism} \} = \min \{ d \mid \exists g_d^1 \text{ on } C \}$$

A curve of gonality $k$ is called a $k$-gonal curve.

The gonality is a classical invariant. We now mention another important invariant of smooth curves of genus $g \geq 1$, which is known as the Clifford index of a curve. Before recalling its definition, we note down a crucial motivating theorem:

**Theorem 2.2.** (Clifford’s Theorem) Let $\Delta$ be an effective divisor on a curve $C$. If $\Delta$ is special, i.e. $h^1(\mathcal{O}_C(\Delta)) \neq 0$, then $r(\Delta) \leq \frac{\deg(\Delta)}{2}$. Furthermore, equality occurs if and only if either $\Delta = 0$ or $\Delta = K_C$ or $C$ is hyperelliptic and $\Delta$ is a multiple of the unique $g_2^1$ on $C$.

**Proof.** See [13], Chapter-IV, Theorem 5.4.

In the context of Theorem 2.2, we recall the definition of the Clifford index:

**Definition 2.3.** (Clifford index)

(i) Let $\Delta$ be an effective divisor on $C$. The Clifford index of $\Delta$ is defined by $\text{Cliff}(\Delta) := \deg(\Delta) - 2r(\Delta)$. $\Delta$ is said to contribute to the Clifford index if both $h^0(\mathcal{O}_C(\Delta)) \geq 2$ and $h^1(\mathcal{O}_C(\Delta)) \geq 2$ holds.
We define the Clifford index of a curve of genus $g \geq 1$ as:

$$\text{Cliff}(C) := \min \{ \text{Cliff}(\Delta) \mid \Delta \text{ contributes to the Clifford index} \}.$$ 

The following classical result gives us the exact formula for the gonality and Clifford index of a smooth plane curve of degree $d \geq 5$ in terms of its degree $d$.

**Theorem 2.4.** ([16], Theorem 1.1) Let $C$ be a smooth plane curve of degree $d \geq 5$. Then the gonality and the Clifford index of $C$ are determined by the degree of $C$:

$$\text{gon}(C) = d - 1, \text{Cliff}(C) = d - 4.$$ 

The following general remark helps estimate an upper bound of the dimension of global sections of line bundles on a smooth hyperplane section of $X^{(d)}$.

**Remark 2.5.** Let $C$ be a smooth hyperplane section of $X^{(d)}$. Then $|\mathcal{O}_C(1)|$ gives an embedding $C \hookrightarrow \mathbb{P}^2$. Since $C$ is a plane curve of degree $d$, by Theorem 2.4, $C$ is a $(d - 1)$-gonal curve. Therefore, it can be established by induction that, if $\mathcal{L}$ is a line bundle on $C$ satisfying $h^0(\mathcal{L}) \geq 2$, then $\deg(\mathcal{L}) \geq h^0(\mathcal{L}) + (d - 1) - 2$, i.e. $\deg(\mathcal{L}) \geq h^0(\mathcal{L}) + d - 3$. In particular, for $d = 6$ one has, if $\mathcal{L}$ is a line bundle on $C$ satisfying $h^0(\mathcal{L}) \geq 2$, then $\deg(\mathcal{L}) \geq h^0(\mathcal{L}) + 3$. Note that, for $d = 5$, we recover [[21], Remark 2.1].

We now mention an interesting result which deals with the existence (nonexistence) of certain line bundles on smooth plane curves of degree $d$. We will extensively use this result in the next Section. For some specific incidents of nonexistence of some special divisors (of certain degree and global sections) on a smooth plane sextic curve, see [2, 3].

**Proposition 2.6.** Let $g_n^r$ be a linear system on $C$ (not necessarily very special). Write $g(C) = \frac{(d-1)(d-2)}{2}$.

(i) If $n > d(d - 3)$, then $r = n - g$ (the non-special case).

(ii) If $n \leq d(d - 3)$, then write $n = kd - e$ with $0 \leq k \leq d - 3, 0 \leq e < d$, one has

$$\begin{cases} r \leq \frac{(k-1)(k+2)}{2}, & \text{if } e > k + 1 \\ r \leq \frac{k(k+3)}{2} - e, & \text{if } e \leq k + 1 \end{cases}$$

**Proof.** See [14] Theorem 2.1 or [10] Remark 1.3. $\square$

**Corollary 2.7.** On a smooth plane sextic curve $C$, let $g_n^r$ be a linear system on $C$, then

(i) If $n = 15$, then $r \leq 6$.

(ii) If $12 \leq n \leq 14$, then $r \leq 5$.

(iii) If $n = 11$, then $r \leq 4$.

(iv) If $n = 10$, then $r \leq 3$.

(v) If $7 \leq n \leq 9$, then $r \leq 2$.

We end the discussion on curves by mentioning the Riemann-Roch Theorem for line bundles on curves. Let $\mathcal{L}$ be a line bundle on a curve $C$ of genus $g$. Then the Riemann-Roch Theorem states that (see [13], Chapter-IV, Theorem 1.3):

$$h^0(C, \mathcal{L}) - h^0(C, \mathcal{L}^\ast \otimes K_C) = \deg(\mathcal{L}) - g + 1. \quad (2.1)$$
2.2. Line bundles on surfaces

Next, we will discuss several useful results related to line bundles on $X^{(d)}$.

From the piece of the following long exact cohomology sequence:

$$H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \rightarrow H^1(X^{(d)}, \mathcal{O}_{X^{(d)}}(m)) \rightarrow H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m - d))$$

and using the facts from the Cohomology of projective space (cf. [13], Chapter-3, Theorem 5.1(b)), we see that $H^1((X^{(d)}, \mathcal{O}_{X^{(d)}}(m)) = 0$, for $m \in \mathbb{Z}$. More precisely, using Serre duality and the above observation, we have the following:

- $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \cong H^0(X^{(d)}, \mathcal{O}_{X^{(d)}}(m))$, for $m \leq d - 1$.
- $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \cong H^1(X^{(d)}, \mathcal{O}_{X^{(d)}}(m)) = 0$, for $m \in \mathbb{Z}$.
- $H^2(X^{(d)}, \mathcal{O}_{X^{(d)}}(m)) \cong H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d - 4 - m))$, for $m \geq -3$.

Note that, on $X^{(d)}$, one has by the adjunction formula $K_{X^{(d)}} \cong \mathcal{O}_{X^{(d)}}(d - 4)$. One can compute its Euler characteristic as $\chi(\mathcal{O}_{X^{(d)}}) = 1 + \left(\frac{d - 1}{3}\right)$. Let $C$ be a smooth hyperplane section of $X^{(d)}$. If $D$ is a non-zero effective divisor on $X^{(d)}$, then

- the arithmetic genus is given by: $P_a(D) = \frac{1}{2}D.(D + (d - 4)C) + 1$.
- the Riemann-Roch theorem for $\mathcal{O}_{X^{(d)}}(D)$ is given by: $\chi(\mathcal{O}_{X^{(d)}}(D)) = \frac{1}{2}D(D - (d - 4)C) + 1 + \left(\frac{d - 1}{3}\right)$

Next, we make a remark on sextic surfaces, which expresses the Euler characteristic of a divisor in terms of its degree and arithmetic genus. This remark is exploited significantly in the next section.

**Remark 2.8.** Consider, $X = X^{(6)}$. Let $D$ be a non-zero divisor on $X$. Then one has the following Riemann-Roch relation:

$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(2C - D)) = \chi(\mathcal{O}_X(D)) = P_a(D) - 2C.D + 10. \ (2.2)$$

If $D$ is reduced and irreducible, then $P_a(D) \geq 0$. The following Proposition gives another sufficient condition for $P_a(D) \geq 0$, which can be considered as a generalized version of [[21], Proposition 2.2].

**Proposition 2.9.** Let $d \geq 4$. Let $D$ be a non-zero effective divisor on $X^{(d)}$. If $h^1(\mathcal{O}_{X^{(d)}}(-D)) = 0$, then $P_a(D) \geq 0$.

**Proof.** Applying Riemann-Roch Theorem for line bundle $\mathcal{O}_{X^{(d)}}(-D)$ on the surface $X^{(d)}$, we have the following:

$$\chi(\mathcal{O}_{X^{(d)}}(-D)) = \frac{1}{2}(-D)(-D - (d - 4)C) + 1 + \left(\frac{d - 1}{3}\right)$$

$$= \frac{1}{2}D(D + (d - 4)C) + 1 + \left(\frac{d - 1}{3}\right)$$

$$= P_a(D) + \left(\frac{d - 1}{3}\right)$$
From the ampleness of $C$, we have $C.D \geq 1$, which implies $C.(-D) \leq -1$ and hence $h^0(\mathcal{O}_{X^{(d)}}(-D)) = 0$. This observation along with the Serre duality and first cohomology vanishing hypothesis give us:

$$h^0(\mathcal{O}_{X^{(d)}}(D + (d - 4)C)) = p_a(D) + \binom{d - 1}{3}.$$ 

Now in order to obtain a lower bound of L.H.S, we look at the following:

$$D \hookrightarrow X^{(d)} \quad \text{induces} \quad \mathcal{O}_{X^{(d)}}(-D) \hookrightarrow \mathcal{O}_{X^{(d)}},$$

Twisting with $\mathcal{O}_{X^{(d)}}(D + (d - 4)C)$ yields the following inclusion: $\mathcal{O}_{X^{(d)}}((d - 4)C) \hookrightarrow \mathcal{O}_{X^{(d)}}(D + (d - 4)C)$. This means, $h^0(\mathcal{O}_{X^{(d)}}((d - 4)C)) \leq h^0(\mathcal{O}_{X^{(d)}}(D + (d - 4)C))$. Since $d - 4 \leq d - 1$, we have $h^0(\mathcal{O}_{X^{(d)}}((d - 4)C)) \cong h^0(\mathcal{O}_{X^{(d)}}(d - 4))$ and therefore, one has $h^0(\mathcal{O}_{X^{(d)}}(D + (d - 4)C)) \geq \binom{d - 1}{3}$, which in turn implies $p_a(D) \geq 0$. 

Let $D$ be a divisor on $X^{(d)}$ such that $C.D > 0$. Consider the following S.E.S:

$$0 \to \mathcal{O}_{X^{(d)}}(-D) \to \mathcal{O}_{X^{(d)}}(C - D) \to \mathcal{O}_C(C - D) \to 0 \quad (2.3)$$

In the context of S.E.S 2.3, we note down an easy remark regarding an upper bound of $h^0(\mathcal{O}_{X^{(d)}}(C - D))$.

**Remark 2.10.** Applying long exact cohomology sequence to the S.E.S 2.3 and noting $h^0(\mathcal{O}_{X^{(d)}}(-D)) = 0$, one obtains $h^0(\mathcal{O}_{X^{(d)}}(C - D)) \leq h^0(\mathcal{O}_C(C - D))$. By Remark 2.5, we have $h^0(\mathcal{O}_{X^{(d)}}(C - D)) \leq 2$.

Let $X = X^{(6)}$. The following lemmas are useful in showing the emptiness of certain linear systems in Section 3. These two lemmas are adaptations of [[21], Lemma 2.1, Lemma 2.2] to the case of sextic surface and can be thought of as a characterization of non-zero effective divisors of degrees 1 and 2 on $X$.

**Lemma 2.11.** Let $D$ be a divisor on $X$ satisfying $C.D = 1$. Then (a) $\Rightarrow$ (b), where

(a) $h^0(\mathcal{O}_X(D)) > 0$

(b) $h^0(\mathcal{O}_X(D)) = 1, h^0(\mathcal{O}_X(2C - D)) \geq 7$ and $D^2 \geq -4$.

Moreover, under the assumption $C.D = 1$ and $D^2 \geq -4$, if $h^0(\mathcal{O}_X(2C - D)) \leq 7$, then $h^0(\mathcal{O}_X(D)) > 0$.

**Proof.** Assume (a). By the hypothesis, we may assume that $D$ is effective. Since $C$ is ample, $D$ is reduced and irreducible. This forces $p_a(D) \geq 0$ and hence we have $D^2 \geq -4$.

Consider the S.E.S:

$$0 \to \mathcal{O}_X(D - C) \to \mathcal{O}_X(D) \to \mathcal{O}_C(D) \to 0 \quad (2.4)$$

Since $C.(D - C) < 0$, by the ampleness of $C$, we have $h^0(\mathcal{O}_X(D - C)) = 0$. By Remark 2.5, one sees that $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_X(D)) = 1$. From Riemann-Roch Theorem we have:

$$h^0(\mathcal{O}_X(D)) + h^0(\mathcal{O}_X(2C - D)) \geq 2g(\mathcal{O}_X(D)) \geq 8 \quad (2.5)$$

Since $h^0(\mathcal{O}_X(D)) = 1$, we have $h^0(\mathcal{O}_X(2C - D)) \geq 7$. Conversely, under the assumption $C.D = 1$ and $D^2 \geq -4$, we still have the inequality 2.5, whence it follows that if $h^0(\mathcal{O}_X(2C - D)) \leq 7$, then $h^0(\mathcal{O}_X(D)) > 0$. 

**Remark 2.12.** Note that, under the assumption $C.D = 1$ and $D^2 \geq -4$, if $h^0(\mathcal{O}_X(2C - D)) \geq 8$, then we might end up having no non-trivial global section of $\mathcal{O}_X(D)$. 

Lemma 2.13. Let $D$ be a non-zero effective divisor on $X$ with $C.D = 2$. If $D^2 \leq -8$, then one of the following cases occurs:

(i) There exists a curve $D_1$ on $X$ with $D = 2D_1$, $D_1^2 \geq -4$ and $C.D_1 = 1$.
(ii) There exist curves $D_1$ and $D_2$ with $D = D_1 + D_2$, $D_1.D_2 = 0$, $D_i^2 = -4$ and $C.D_i = 1$ (for $i = 1, 2$).

Conversely, Let $D$ be a non-zero effective divisor on $X$ with $C.D = 2$. If one of the following cases occurs:

(i) There exists a curve $D_1$ on $X$ with $D = 2D_1$, $D_1^2 \geq -4$, $C.D_1 = 1$ and $P_a(D_1) \leq 1$.
(ii) There exist curves $D_1$ and $D_2$ with $D = D_1 + D_2$, $D_1.D_2 = 0$, $D_i^2 = -4$ and $C.D_i = 1$ (for $i = 1, 2$).

Then, $D^2 \leq -8$.

Proof. If $D^2 \leq -8$, then $D$ can’t be both reduced and irreducible. Therefore, there exists a non-trivial effective decomposition $D = D_1 + D_2$. As $C.D = 2$, we must have $C.D_1 = C.D_2 = 1$. Hence, by Lemma 2.11, $D_i^2 \geq -4 (i = 1, 2)$. If $D_1 = D_2$, then $D = 2D_1$ and $D^2 = 4D_1^2 \geq -16$. If $D_1 \neq D_2$, then $D_1.D_2 \geq 0$. This forces, $D^2 = -8$ and hence $D_1.D_2 = 0$ and $D_i^2 = 4 (i = 1, 2)$. The converse assertion is clear.

Next, we note down a well-known fact about ACM bundles on $X = X^{(6)}$. Then we move on to preparing a proposition which will be crucial in proving the if part of our main theorem. We start by recalling the definitions of the main objects concerned.

Definition 2.14. (ACMness) We call a vector bundle $E$ on $X$ an arithmetically Cohen-Macaulay (ACM) bundle if $H^1(E(t)) = 0$ for all integers $t \in \mathbb{Z}$.

Definition 2.15. (Initialization) For a sheaf $F$ on $X$, we define the initial twist as the integer $t$ such that $h^0(F(t)) \neq 0$ and $h^0(F(t-1)) = 0$. $F$ is called initialized if its initial twist is 0, i.e. if $h^0(F) > h^0(F(-1)) = 0$.

For an ACM bundle $E$ on $X$, we consider the graded module $H^0_t(E) := \bigoplus_{t \in \mathbb{Z}} H^0(E(t))$ over the homogeneous coordinate ring of $X$. The following result gives an upper bound of the minimal number of generators of it. It also deals with an upper bound of global sections of an initialized and ACM line bundle on $X$.

Proposition 2.16. ([6], Theorem 3.1 and Corollary 3.5). Let $E$ be an ACM bundle of rank $r$ on $X$, and let $\mu(E)$ be the minimal number of generators of $H^0_t(E)$. Then we get $\mu(E) \leq 6r$. Moreover, if $E$ is initialized, then $h^0(E) \leq 6r$.

The following property of ACM line bundles will be useful in proving Proposition 2.18.

Remark 2.17. A line bundle on $X^{(d)}$ is ACM if and only if the dual of it is ACM. Indeed, if we assume that a line bundle $L$ is ACM on $X^{(d)}$, then by Serre duality and the isomorphism $K_{X^{(d)}} \cong O_{X^{(d)}}(d - 4)$, one obtains $H^1(L^*(t)) \cong (H^1(L(d - t - 4)))^* = 0$, for $t \in \mathbb{Z}$. This shows $L^*$ is ACM. By interchanging the roles of $L$ and $L^*$, we get the other direction.

Next, we generalize [[21], Proposition 3.2] for $X^{(d)}$. This plays a crucial role in dealing with ACMness in the if part of Theorem 1.1.

Proposition 2.18. Let $D$ be a non-zero effective divisor on $X^{(d)}$. Let $C$ be a smooth hyperplane section of $X^{(d)}$ and $k$ be a positive integer satisfying $C.D + d^2 < (k + 4)d$. If $h^1(O_{X^{(d)}}(tC - D)) = 0$ for $0 \leq t \leq k$, then $O_{X^{(d)}}(D)$ is ACM.
Proof. Note that, If \( n \geq k \), then by hypothesis, \( \deg(\mathcal{O}_C((d - n - 4)C)) = C.D + d(d - n - 4) < 0 \) and therefore, using Serre duality one has \( h^1(\mathcal{O}_C((n + 1)C - D)) = h^0(\mathcal{O}_C(D + (d - n - 4)C)) = 0 \). Since \( h^1(\mathcal{O}_X(tC - D)) = 0 \), for \( 0 \leq t \leq k \), therefore, using the following S.E.S:

\[
0 \to \mathcal{O}_X(nC - D) \to \mathcal{O}_X((n + 1)C - D) \to \mathcal{O}_C((n + 1)C - D) \to 0,
\]

it can be seen that we have by induction \( h^1(\mathcal{O}_X(\mathcal{X} - D)) = 0 \), for \( n \geq 1 \). Next, note that, for \( m \geq 0 \), we have \( h^0(\mathcal{O}_C(-mC - D)) = 0 \) and from the assumption, we have \( h^1(\mathcal{O}_X(\mathcal{D} - D)) = 0 \). This means by the following S.E.S:

\[
0 \to \mathcal{O}_X(-(m + 1)C - D) \to \mathcal{O}_X(-mC - D) \to \mathcal{O}_C(-mC - D) \to 0,
\]

for any integer \( m \geq 0 \), we have \( h^1(\mathcal{O}_X(-mC - D)) = 0 \) by induction. Therefore, \( \mathcal{O}_X(D) \) is ACM, whence \( \mathcal{O}_X(D) \) is ACM by Remark 2.17.

Corollary 2.19. For \( X = X^{(6)} \), we obtain the following precise statement: Let \( D \) be a non-zero effective divisor on \( X \). Let \( C \) be a smooth hyperplane section of \( X \) and \( k \) be a positive integer satisfying \( C.D + 12 < 6k \). If \( h^1(\mathcal{O}_X(tC - D)) = 0 \) for \( 0 \leq t \leq k \), then \( \mathcal{O}_X(D) \) is ACM.

We end this section by mentioning the concept of \( m \)-connected divisors, which will play an essential role in the proof of the vanishing of \( h^1(\mathcal{O}_X(-D)) \) for certain non-zero effective divisors in Proposition 3.2. We first recall the definition of \( m \)-connected divisors.

Definition 2.20. \( m \)-connected divisor) Let \( m \) be a positive integer. Then a non-zero effective divisor \( D \) on a surface is called \( m \)-connected, if \( D_1.D_2 \geq m \), for each effective decomposition \( D = D_1 + D_2 \).

The following Lemma gives us a necessary condition for a non-zero effective divisor \( D \) to be \( 1 \)-connected.

Lemma 2.21. If \( D \) is \( 1 \)-connected effective divisor, then \( h^0(\mathcal{O}_D) = 1 \).

Proof. See [4], Corollary 12.3.

Remark 2.22. If a non-zero effective divisor \( D \) on \( X = X^{(6)} \) is \( 1 \)-connected, then by Lemma 2.21, we have, \( h^0(\mathcal{O}_D) = 1 \). Therefore, by applying long exact cohomology sequence to the following S.E.S:

\[
0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0
\]

one has, \( h^1(\mathcal{O}_X(-D)) = 0 \). This means, if \( h^1(\mathcal{O}_X(-D)) \neq 0 \), then \( D \) is not \( 1 \)-connected.

3. Initialized and ACM line bundles on sextic surfaces

In this section, we give a proof of Theorem 1.1, which is divided into two subsections. In the first subsection, we investigate the necessary condition, and in the second subsection, we show that if a non-zero effective divisor \( D \) on \( X = X^{(6)} \) satisfies the necessary condition (i.e. condition (i) and any one of the remaining conditions from (ii) to (viii) of the promised Theorem 1.1), then the line bundle \( \mathcal{O}_X(D) \) given by it is initialized and ACM.
3.1. Necessary condition

Let $C$ be a smooth hyperplane section of $X = X^{(6)}$ as before. Let’s assume that the line bundle $O_X(D)$ given by a non-zero effective divisor $D$ on $X$ is initialized and ACM. Then consider the following S.E.S:

$$0 \to O_X(C - D) \to O_X(2C - D) \to O_C(2C - D) \to 0. \quad (3.1)$$

Applying long exact cohomology sequence to the S.E.S 3.1 and using Remark 2.17, one obtains the following equality:

$$h^0(O_X(2C - D)) = h^0(O_X(C - D)) + h^0(O_C(2C - D)). \quad (3.2)$$

From Remark 2.10, we already know that $h^0(O_X(C - D)) \leq 2$. Note that, by Remark 2.5, we have $h^0(O_C(2C - D)) \leq 8$. This means $h^0(O_X(2C - D)) \leq 10$. In what follows, we obtain all possibilities of arithmetic genus and the corresponding range of degrees of $D$ for each such values of $h^0(O_X(2C - D)) \in \{0, 1, 2, ..., 9, 10\}$. In this pursuit, our strategy is as follows: we divide the numbers 0 to 10 into five distinct groups and use a unique technique for each one of them. To be more precise, the case $h^0(O_X(2C - D)) = 0$ will be discussed as CASE-(i), the cases $h^0(O_X(2C - D)) = 1, 2, 3$ will be discussed as CASE-(ii), the cases $h^0(O_X(2C - D)) = 4, 5, 6$ will be discussed as CASE-(iii), the cases $h^0(O_X(2C - D)) = 7, 8$ will be discussed as CASE-(iv) and finally it will be established that the remaining two cases can’t occur.

**CASE-(i) $h^0(O_X(2C - D)) = 0$.**

Here, we first analyze what happens when for a degree $d \geq 5$ surface $X^{(d)}$, one has $h^0(O_{X^{(d)}}(d - 4)C - D)) = 0$. We have from [[6], Theorem 3.1 and Corollary 3.5], $1 \leq h^0(O_{X^{(d)}}(D)) \leq d$. This can be rewritten as $h^0(O_{X^{(d)}}(D)) = d - k$, where $0 \leq k \leq d - 1$. By the Riemann-Roch Theorem, one has:

$$\frac{1}{2}D^2 = d - k + \frac{(d - 4)}{2}C.D - 1 - \binom{d - 1}{3}.$$

Therefore, $P_a(D) = d - k + (d - 4)C.D - \binom{d - 1}{3}$.

From Proposition 2.9, we have $P_a(D) \geq 0$, hence we have:

$$\begin{cases} 
C.D \geq \frac{1}{d - 4}\left(\binom{d - 1}{3} - 1\right) + 1, & \text{if } \frac{1}{d - 4}\left(\binom{d - 1}{3} - 1\right) \text{ is not an integer} \\
C.D \geq \frac{1}{d - 4}\left(\binom{d - 1}{3} - 1\right), & \text{otherwise}
\end{cases}$$

We also have, $\chi(O_{X^{(d)}}((d - 3)C - D)) = h^0(O_{X^{(d)}}(D)) + \frac{1}{2}d(d - 3) - C.D$. By Remark 2.17 and the assumptions of ACMness and initializedness, we have $C.D \leq h^0(O_{X^{(d)}}(D)) + \frac{1}{2}d(d - 3)$.

This means, for $h^0(O_{X^{(d)}}(D)) = 1$, we have:

$$P_a(D) = (d - 4)C.D + 1 - \binom{d - 1}{3}$$

either $\frac{1}{d - 4}\left(\binom{d - 1}{3} - 1\right) + 1 \leq C.D \leq 1 + \frac{1}{2}d(d - 3)$ or $\frac{1}{d - 4}\left(\binom{d - 1}{3} - 1\right) \leq C.D \leq 1 + \frac{1}{2}d(d - 3)$.

Consider $h^0(O_{X^{(d)}}(D)) \geq 2$. Then by S.E.S 2.4 and assumptions of initializedness and ACMness, one has $h^0(O_{X^{(d)}}(D)) = h^0(O_C(D)) = d - k \geq 2$. By Remark 2.5, we have $C.D \geq 2d - k - 3$. This means, for $h^0(O_{X^{(d)}}(D)) \geq 2$, we have:

$$P_a(D) = (d - 4)C.D + d - k - \binom{d - 1}{3} \text{ and } 2d - k - 3 \leq C.D \leq \frac{1}{2}d^2 - \frac{1}{2}d - k.$$
Note that, for \( d = 5 \), we recover \([21],\) Lemma 4.3. In particular, when \( d = 6 \), we obtain the following:

for \( h^0(O_X(D)) = 1 \): \( P_a(D) = 2C.D - 9, 5 \leq C.D \leq 10 \) and
for \( h^0(O_X(D)) \geq 2 \): \( P_a(D) = 2C.D - 4 - k, 9 - k \leq C.D \leq 15 - k \)

We complete the analysis of this case (for \( d = 6 \)) by mentioning that some possibilities can’t occur.

Note that, for each \( k \), one has from the assumption of ACMness and initializedness, \( h^0(O_C(D)) = h^0(O_X(D)) = 6 - k \). In view of that, it’s not very difficult to see the following:

- The cases \( k = 0, 9 \leq C.D \leq 10 \) and \( k = 1, C.D = 8 \) are not possible by **Theorem 2.2**.
- The cases \( k = 0, C.D = 11, k = 1, C.D = 9 \) and \( k = 2, C.D = 7 \) are not possible. In all of these cases one obtains Cliff\((O_C(D)) = 1 \), a contradiction to **Theorem 2.4**.
- The cases \( k = 1, C.D = 10 \) and \( k = 2, 8 \leq C.D \leq 9 \) are not possible by **Corollary 2.7**.

We represent the remaining possibilities in **Table 1**.

**Table 1.** CASE-(i). This table, together with all the other tables appearing in this article, apply to the case \( d = 6 \)

| \( P_a(D) \) | \( C.D \) |
|-------------|---------|
| 2C.D - 9    | \( 5 \leq C.D \leq 10 \) |
| 2C.D - 8    | \( 5 \leq C.D \leq 11 \) |
| 2C.D - 7    | \( 6 \leq C.D \leq 12 \) |
| 2C.D - 6    | \( 10 \leq C.D \leq 13 \) |
| 2C.D - 5    | \( 11 \leq C.D \leq 14 \) |
| 2C.D - 4    | \( 12 \leq C.D \leq 15 \) |

Note that, since in each of the remaining cases, we have \( |2C - D| \neq 0 \), from 3.2, one sees that in each such cases at least one of \( h^0(O_C(2C - D)) \) or \( h^0(O_X(C - D)) \) is non-zero.

**CASE-(ii) \( h^0(O_X(2C - D)) = 3 - m \), where \( m \in \{0, 1, 2\} \).**

Let’s assume \( h^0(O_C(2C - D)) \neq 0 \), then one must have \( C.(2C - D) \geq 0 \). This means we have \( C.D \leq 12 \). If \( C.D = 12 \), then \( O_X(D) \cong O_X(2C) \), a contradiction to the assumption that \( O_X(D) \) is initialized. This forces \( C.D \leq 11 \). From the assumption of ACMness, initializedness and **Remark 2.5**, we have \( h^0(O_C(D)) = h^0(O_X(D)) \leq 8 \). Therefore, one can write \( h^0(O_C(D)) = h^0(O_X(D)) = 8 - r \), where \( r \in \{0, 1, 2, 3, 4, 5, 6\} \). For such a fixed \( r \), one can write \( C.D = 11 - r + s \) (by **Remark 2.5**), where \( s \in \{0, \ldots r\} \). For fixed \( m, r \) and \( s \) in the admissible range, one obtains by assumption and relation 2.2,

\[
P_a(D) = 23 - m - 3r + 2s.
\]

(3.3)

Note that, for any \( m \in \{0, 1, 2\} \), the following possibilities can’t occur:

- The cases \( r = 0, 1 \leq r \leq 2 \) with \( 0 \leq s \leq 1 \) and \( r = 3 \) with \( s = 0 \) are not possible by **Theorem 2.2**.
- The cases \( r = 2 \) with \( s = 2 \), \( r = 3 \) with \( s = 1 \) and \( r = 4 \) with \( s = 0 \) are not possible. In all three cases one obtains \( \text{Cliff}(O_C(D)) = 1 \), a contradiction to **Theorem 2.4**.
- The cases \( r = 3 \) with \( s = 2 \) and \( r = 4 \) with \( 1 \leq s \leq 2 \) are not possible by **Corollary 2.7**.

Next we consider the case, \( h^0(O_C(D)) = h^0(O_X(D)) = 1 \). Since \( C.D \leq 11 \), from relation 2.2, one obtains
\[ P_a(D) = 2C.D - 6 - m \leq 16 - m. \] (3.4)

Note that, for any \( m \in \{0, 1, 2\} \), the following possibilities can’t occur:

- The case \( P_a(D) = 16 - m \) is not possible. Because in that case \( C.(2C - D) = 1 \) implies by Lemma 2.11, one must have \((2C - D)^2 \geq -4\), a contradiction.
- The case \( P_a(D) = 14 - m \) is not possible. Note that, as \( C.(2C - D) = 2 \), if we take \( \Gamma \in [2C - D] \), then by Lemma 2.13, in each case one must obtain that there exists a curve \( \Gamma \) on \( X \) with \( C.\Gamma = 1, \Gamma^2 \geq -4 \) such that \( \Gamma = 2.\Gamma_1 \). This means for \( m = 2 \), \( 4\Gamma_1^2 = -14 \), a contradiction. Finally for \( m = 1 \), this gives us \( \Gamma_1^2 = -3 \) and hence \( P_a(\Gamma_1) \) is a fraction, a contradiction. Finally for \( m = 0 \), this forces \( 4.\Gamma^2 = -10 \), a contradiction.

Note that, since \( \mathcal{O}_X(D) \) is ACM, by Proposition 2.9, we have \( P_a(D) \geq 0 \). In this situation (i.e. when \( h^0(\mathcal{O}_C(D)) = 1 \)), we are left with the following possibilities:

- For \( m = 0 \), \( P_a(D) = 2i \) and \( C.D = 3 + i \), where \( i \in \{0, 1, 2, 3, 4, 5, 6\} \)
- For \( m = 1 \), \( P_a(D) = 2i - 1 \) and \( C.D = 3 + i \), where \( i \in \{1, 2, 3, 4, 5, 6\} \)
- For \( m = 2 \), \( P_a(D) = 2i \) and \( C.D = 4 + i \), where \( i \in \{0, 1, 2, 3, 4, 5\} \)

Next, let’s consider the case \( |C - D| \neq \emptyset \). From the S.E.S 2.3 and the assumption of ACMness one has \( h^0(\mathcal{O}_C(C - D)) \neq 0 \). This forces \( C.D \leq 6 \). If \( C.D = 6 \), then \( \mathcal{O}_X(D) \cong \mathcal{O}_X(C) \), a contradiction to the assumption that \( \mathcal{O}_X(D) \) is initialized. This forces \( C.D \leq 5 \). From the assumption of ACMness, initializedness and Remark 2.5, we have \( h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_X(D)) \leq 2 \). Note that, the case \( h^0(\mathcal{O}_C(D)) = 2 \), \( C.D = 5 \) is not possible. This is because, in that case \( C.(C - D) = 1 \), which implies by Lemma 2.11, \( (C - D)^2 \geq -4 \), a contradiction (for each values of \( m \)). Therefore, we are left with the case \( h^0(\mathcal{O}_C(D)) = 1 \). Note that, as before in this situation one finds using \( C.D \leq 5 \) and relation 2.2,

\[ P_a(D) = 2C.D - 6 - m \leq 4 - m. \] (3.5)

Note that, for any \( m \in \{0, 1, 2\} \), the case \( P_a(D) = 4 - m \) is not possible. Because in that case \( C.(C - D) = 1 \), which implies by Lemma 2.11, one must have \((C - D)^2 \geq -4\), a contradiction.

Again as before, since \( \mathcal{O}_X(D) \) is ACM, by Proposition 2.9, we have \( P_a(D) \geq 0 \). In this situation (i.e. when \( h^0(\mathcal{O}_C(D)) = 1 \)), we are left with the following possibilities:

- For \( m = 0 \), \( P_a(D) = 2i \) and \( C.D = 3 + i \), where \( i \in \{0, 1\} \)
- For \( m = 1 \), \( P_a(D) = 2i - 1 \) and \( C.D = 3 + i \), where \( i = 1 \)
- For \( m = 2 \), \( P_a(D) = 2i \) and \( C.D = 4 + i \), where \( i = 0 \)

We represent the possibilities that are obtained from the above analysis in the following tables: Table 2 (for \( m = 2 \)), Table 3 (for \( m = 1 \)), and Table 4 (for \( m = 0 \)).

| Table 2. CASE-(ii), \( m = 2 \). |
|---------------------------------|-----------------|
| \( P_a(D) \)                  | \( C.D \)      |
| \( 2C.D - 8 \)                | \( 4 \leq C.D \leq 9 \) |
| \( 2C.D - 7 \)                | \( 5 \leq C.D \leq 11 \) |
| \( 2C.D - 6 \)                | \( 6 \leq C.D \leq 11 \) |
| \( 2C.D - 5 \)                | \( 10 \leq C.D \leq 11 \) |
| \( 2C.D - 4 \)                | \( C.D = 11 \)  |
Note that, in this situation we have from Remark 2.10 and relation 3.2 that \( h^0(O_C(2C - D)) \geq 4 - m \geq 2 \). From Remark 2.5, we see that \( C.(2C - D) \geq h^0(O_C(2C - D)) + 3 \geq 7 - m \). This means, in this case, we have \( 1 \leq C.D \leq 5 + m \). From the assumption of ACMness, initializedness and Remark 2.5, we have \( h^0(O_C(D)) = h^0(O_X(D)) \leq 2 + m \). Therefore, one can write \( h^0(O_C(D)) = h^0(O_X(D)) = 2 + m - r \), where \( r \in \{0, ..., m\} \). For fixed \( m, r \), one can write \( C.D = 5 + m - r + s \) (by Remark 2.5), where \( s \in \{0, ... r\} \). For fixed \( m, r \) and \( s \) in the admissible range, one obtains by assumption and the relation 2.2,

\[
P_a(D) = 8 - 3r + 2m + 2s.
\]

Note that, for \( m = 2 \), the possibility \( r = 0 \) can’t occur. Indeed, in this case, one obtains \( \text{Cliff}(O_C(D)) = 1 \), a contradiction to Theorem 2.4.

Next, we consider the case \( h^0(O_C(D)) = h^0(O_X(D)) = 1 \). Since \( C.D \leq 5 + m \), from the relation 2.2, one obtains

\[
P_a(D) = 2C.D - 3 - m \leq 7 + m.
\]

Note that, for \( m \in \{0, 1, 2\} \), \( P_a(D) \neq 7 + m, 5 + m \) and for \( m \in \{1, 2\} \), \( P_a(D) \neq 3 + m \). These can be realized by the following arguments.

- Note that, for \( P_a(D) = 7 + m \), one can show that for \( m \in \{1, 2\} \), \( |C - D| = \emptyset \). Indeed, this can be seen for \( m = 1 \) from initializedness, for \( m = 2 \) from the observation \( C.(C - D) < 0 \) and the S.E.S 2.3 (and using the assumptions of ACMness and initializedness). This forces \( h^0(O_C(2C - D)) = 6 - m \) (by 3.2), a contradiction by Remark 2.5. For \( m = 0 \), by Remark 2.5 and the S.E.S 2.3, we must have \( h^0(O_X(C - D)) \leq 1 \). This forces \( h^0(O_C(2C - D)) \geq 5 \) (by 3.2), a contradiction by Remark 2.5.

- Note that, for \( P_a(D) = 5 + m \), we see that for \( m = 2 \), one has by initializedness \( |C - D| = \emptyset \), which implies \( h^0(O_C(2C - D)) = 4 \) (by 3.2), a contradiction by Remark 2.5. For \( m = 1 \), we have by Remark 2.5 and the S.E.S 2.3, \( h^0(O_X(C - D)) \leq 1 \) and hence \( h^0(O_C(2C - D)) \geq 4 \) (by 3.2). This forces \( \text{Cliff}(O_C(2C - D)) \leq 1 \), a contradiction by Theorem 2.4. For \( m = 0 \), again by Remark 2.5 and the S.E.S 2.3, we have \( h^0(O_X(C - D)) \leq 1 \). This implies \( h^0(O_C(2C - D)) \geq 5 \) (by 3.2), a contradiction by Theorem 2.2.
• Note that, for $P_a(D) = 3 + m$, we observe that for $m = 2$, by Lemma 2.11, one obtains $|C - D| = 0$ and hence $h^0(O_C(2C - D)) = 4$ (by 3.2). This forces $\text{Cliff}(O_C(2C - D)) = 1$, a contradiction by Theorem 2.4. For $m = 1$, one has by Remark 2.5 and the S.E.S 2.3, $h^0(O_X(C - D)) \leq 1$ and hence $h^0(O_C(2C - D)) \geq 4$ (by 3.2), a contradiction by Corollary 2.7.

Observe that, since $O_X(D)$ is ACM, by Proposition 2.9, we have $P_a(D) \geq 0$. In this situation (i.e. when $h^0(O_C(D)) = 1$), we are left with the following possibilities:

for $m = 0$, $P_a(D) = 2i - 1$ and $C.D = 1 + i$, where $i \in \{1,2\}$
for $m = 1$, $P_a(D) = 2i$ and $C.D = 2 + i$, where $i \in \{0,1\}$
for $m = 2$, $P_a(D) = 2i - 1$ and $C.D = 2 + i$, where $i \in \{1,2\}$

We can compactly rewrite the possibilities that arise from above calculations in tabular form (Table 5 for $m = 2$, Table 6 for $m = 1$, Table 7 for $m = 0$).

**Table 5.** CASE-(iii), $m = 2$.

| $P_a(D)$ | $C.D$ |
|-----------|-------|
| $2C.D - 5$ | $3 \leq C.D \leq 4$ |
| $2C.D - 4$ | $5 \leq C.D \leq 7$ |
| $2C.D - 3$ | $6 \leq C.D \leq 7$ |

**Table 6.** CASE-(iii), $m = 1$.

| $P_a(D)$ | $C.D$ |
|-----------|-------|
| $2C.D - 4$ | $2 \leq C.D \leq 3$ |
| $2C.D - 3$ | $5 \leq C.D \leq 6$ |
| $2C.D - 2$ | $C.D = 6$ |

**Table 7.** CASE-(iii), $m = 0$.

| $P_a(D)$ | $C.D$ |
|-----------|-------|
| $2C.D - 3$ | $2 \leq C.D \leq 3$ |
| $2C.D - 2$ | $C.D = 5$ |

**CASE-(iv) $h^0(O_X(2C - D)) = 8 - m$, where $m \in \{0,1\}$.**

Consider the situation $h^0(O_X(C - D)) = 2$. In this case, by the assumption of initializedness, Remark 2.5 and the S.E.S 2.3, we have, $h^0(O_C(C - D)) = 2$. This means by Remark 2.5, $C.D = 1^2$ and hence by relation 2.2,

$$P_a(D) = 1 - m$$

(3.8)

Next, let’s consider the situation $h^0(O_X(C - D)) = 2 - j$, where $j \in \{1,2\}$. In this case, by relation 3.2, we obtain $h^0(O_C(2C - D)) = 6 - m + j \geq 2$. By Remark 2.5, this implies, $C.D \leq 3 + m - j$. Therefore, for a fixed $m$ and $j$ in the admissible range we can write $C.D = 3 + m - j - l$, where $l \in \{0,1,2\}$. Also note that, in this situation, one must have by Remark 2.5 and the assumptions of ACMness and initializedness, $h^0(O_X(D)) = h^0(O_C(D)) = 1$. Thus from relation 2.2, one obtains:

$$P_a(D) = 5 + m - 2j - 2l.$$
Therefore, the possibilities that arise out of the above analysis are as follows:

for $m = 1$, $P_a(D) = 2C.D - 2$ and $1 \leq C.D \leq 3$.
for $m = 0$, $P_a(D) = 2C.D - 1$ and $1 \leq C.D \leq 2$.

Next, we will show that the cases $h^0(\mathcal{O}_X(2C-D)) = 10 - m$, where $m \in \{0, 1\}$, are not possible. Note that, in this situation, by ampleness of $C$, 3.2 and Remark 2.5, we must have $h^0(\mathcal{O}_X(C-D)) \neq 0, 1 - m$. Now consider the situation $h^0(\mathcal{O}_X(C-D)) = 2$. In this case, by the assumption of initializedness, Remark 2.5 and the S.E.S 2.3, we have, $h^0(\mathcal{O}_C(C-D)) = 2$. This means, by Remark 2.5, $C.D = 1$ and hence by relation 2.2,

$$P_a(D) = 3 - m.$$ (3.9)

Next, let’s consider the case $h^0(\mathcal{O}_X(C-D)) = 1$ (observe that, this can only happen when $m = 1$). Then from 3.2 and Remark 2.5, one obtains $C.D = 1$ (and hence $h^0(\mathcal{O}_C(D)) = 1$). Therefore, from relation 2.2, one obtains $P_a(D) = 2$. Thus, the possibilities that arise out of the above analysis are as follows:

for $m = 1$, $P_a(D) = 2C.D$ and $C.D = 1$.
for $m = 0$, $P_a(D) = 2C.D + 1$ and $C.D = 1$.

These, in particular, tell us that in each case, we have $D^2 = 2(1 - m)$. Therefore, by Riemann-Roch Theorem, we obtain:

$$h^0(\mathcal{O}_X(C + D)) + h^0(\mathcal{O}_X(C - D)) = \chi(\mathcal{O}_X(C + D)) = 9 - m$$

This means by Remark 2.10, $h^0(\mathcal{O}_X(D + C)) \geq 7 - m$. Applying the long exact cohomology sequence to the following S.E.S:

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + C) \to \mathcal{O}_C(D + C) \to 0$$ (3.10)

and using the assumption of ACMness and initializedness, one obtains $h^0(\mathcal{O}_C(C + D)) \geq 6 - m \geq 2$. This forces, by Remark 2.5, $m \geq 2$, a contradiction.

Now that we have analyzed all the cases and obtained all the possibilities of $P_a(D)$ and $C.D$, we further move on to improving the scenario of all possibilities. To be more precise, in what follows next, we show that certain cases that were obtained as possibilities in the previous analysis can’t occur.

We start by showing that the case $P_a(D) = 2C.D - 1$ is not possible. Note that, in this case, we had $C.D = 1 + i$, where $i \in \{0, 1\}$. In this situation, it is easy to see that $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(C - D)) \leq 2 - i$. Therefore, by relations 3.2 and 2.2, we end up having $h^0(\mathcal{O}_C(2C - D)) \geq 6 + i$. This amounts to Cliff$(\mathcal{O}_C(2C - D)) \leq 1$, a contradiction to Theorem 2.4.

Next, we will establish that in case $P_a(D) = 2C.D - 2$, except $C.D = 1$, the other possibilities can’t occur.

Consider the possibility $C.D = 11$. Note that, in this situation one has, $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) \leq 6$. Otherwise from 2.1, we have $h^0(\mathcal{O}_C(3C - D)) \geq 5$, a contradiction by Remark 2.5. Then from relation 2.2, one obtains $h^0(\mathcal{O}_X(2C - D)) \geq 2$, a contradiction by Lemma 2.11.

Consider the possibilities $C.D = 5 + i$, where $i \in \{0, 1\}$. By Remark 2.5, we have $h^0(\mathcal{O}_C(D)) \leq 2 + i$. Also it’s easy to see that $h^0(\mathcal{O}_X(C - D)) \leq 1 - i$. Then from relations 2.2 and 3.2, we end up having $h^0(\mathcal{O}_C(2C - D)) \geq 5$, a contradiction by Remark 2.5.
Consider the possibilities $C.D = 2 + i$, where $i \in \{0, 1\}$. By Remark 2.5, we have $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_X(D)) = 1$, $h^0(\mathcal{O}_X(C - D)) \leq 1$. Therefore, by relations 2.2 and 3.2, we obtain $h^0(\mathcal{O}_C(2C - D)) \geq 6$, a contradiction by Theorem 2.2.

Next, we will show that the case $P_a(D) = 2C.D - 3$ can’t occur.

Consider the possibility $C.D = 11$. Note that, in this situation we have $h^0(\mathcal{O}_C(D)) \geq 7$ (observe that, if $|2C - D| \neq 0$, then this can be realized from Lemma 2.11 and the S.E.S 2.4 and if $|2C - D| = 0$, then it follows from relation 2.2 and the S.E.S 2.4), a contradiction by Theorem 2.2.

Consider the possibility $C.D = 10$. Note that, in this situation, one has, $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) \leq 6$. Otherwise from 2.1, we have $h^0(\mathcal{O}_C(3C - D)) \geq 6$, a contradiction by Remark 2.5. Also by Theorem 2.2, $h^0(\mathcal{O}_C(D)) \neq 6$. Then from relations 2.2 and 3.2, we have $h^0(\mathcal{O}_C(2C - D)) \geq 2$, a contradiction by Remark 2.5.

Next, let’s consider the possibilities $C.D = 6 + i$, where $i \in \{0, 1\}$. In this situation, by Remark 2.5, we have $h^0(\mathcal{O}_C(D)) \leq 3 + i$. This coupled with relation 2.2 gives us $h^0(\mathcal{O}_X(2C - D)) \geq 4 - i$. It’s not difficult to see that in both cases we have $h^0(\mathcal{O}_X(C - D)) = 0$. This forces by 3.2, $h^0(\mathcal{O}_C(2C - D)) \geq 4 - i$, a contradiction by Remark 2.5.

Now consider the possibilities $C.D = 2 + i$, where $i \in \{0, 1, 3\}$. By Remark 2.5, we have the following: for $i = 3$, $h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) \leq 2$, for $i \in \{0, 1\}$, $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_X(D)) = 1$ and for all values of $i$, $h^0(\mathcal{O}_X(C - D)) \leq 1$. Therefore, from relations 2.2 and 3.2, we have for $i = 3$, $h^0(\mathcal{O}_C(2C - D)) \geq 4$ and for $i \in \{0, 1\}$, $h^0(\mathcal{O}_C(2C - D)) \geq 5$. For $i \in \{1, 3\}$, this means $\text{Cliff}(\mathcal{O}_C(2C - D)) \leq 1$, a contradiction to Theorem 2.4. By the same argument for $i = 0$, $h^0(\mathcal{O}_C(2C - D))$ can’t be $\geq 6$, whence we have $h^0(\mathcal{O}_C(2C - D)) = 5$, a contradiction to Corollary 2.7.

Next, we show that for $P_a(D) = 2C.D - 4$, the possibilities $C.D = 11, 10, 9, 8, 7, 3$ can’t occur.

For $C.D = 11$, one can argue as in the previous case (i.e. $P_a(D) = 2C.D - 3$ and $C.D = 11$) to obtain in any case $h^0(\mathcal{O}_C(D)) \geq 6$. This forces $\text{Cliff}(\mathcal{O}_C(D)) \leq 1$, a contradiction to Theorem 2.4.

Consider the possibilities $C.D = 9 + i$, where $i \in \{0, 1\}$. Then as before, by 2.1 and Remark 2.5, one must obtain $h^0(\mathcal{O}_C(D)) \leq 6$. Note that, by Theorem 2.2, $h^0(\mathcal{O}_C(D)) \neq 6$. For $i = 1$, by Corollary 2.7, $h^0(\mathcal{O}_C(D)) \neq 5$ and for $i = 0$, by Theorem 2.4, $h^0(\mathcal{O}_C(D)) \neq 5$. This yields by relations 2.2 and 3.2, $h^0(\mathcal{O}_C(2C - D)) \geq 2$, a contradiction by Remark 2.5.

Consider the possibilities $C.D = 7 + i$, where $i \in \{0, 1\}$. Then by Remark 2.5, one has $h^0(\mathcal{O}_C(D)) \leq 4 + i$. This implies by relations 2.2, 3.2 and Remark 2.5, $h^0(\mathcal{O}_C(2C - D)) = h^0(\mathcal{O}_X(2C - D)) = 2 - i$. This forces $h^0(\mathcal{O}_C(D)) = 4 + i$, a contradiction by Corollary 2.7.

For $C.D = 3$, by Remark 2.5, we have $h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_X(D)) = 1, h^0(\mathcal{O}_X(C - D)) \leq 1$. Thus, by relations 2.2 and 3.2, we end up having $h^0(\mathcal{O}_C(2C - D)) \geq 4$, a contradiction by Corollary 2.7.

Finally we show that for $P_a(D) = 2C.D - 5$, the cases $C.D = 8, 9$ can’t occur.

Here in both the cases using relations 2.2, 3.2 and Remark 2.5, one can show that the situation $h^0(\mathcal{O}_X(D)) \leq 3$ is not possible. This forces $h^0(\mathcal{O}_C(D)) \geq 4$, a contradiction by Corollary 2.7.

We summarize the content of the above discussion in the following Theorem.

**Theorem 3.1.** With the notations as before, If $\mathcal{O}_X(D)$ is ACM and initialized, then the following conditions are satisfied:

(a) either $2C.D - 9 \leq P_a(D) \leq 2C.D - 4$ or $P_a(D) = 2C.D - 2$.

(b) If $P_a(D) = 2C.D - 9$, then $5 \leq C.D \leq 10$.

(c) If $P_a(D) = 2C.D - 8$, then $4 \leq C.D \leq 11$. 


(d) If $P_a(D) = 2C.D - 7$, then $4 \leq C.D \leq 12$.

(e) If $P_a(D) = 2C.D - 6$, then $3 \leq C.D \leq 13$.

(f) If $P_a(D) = 2C.D - 5$, then $3 \leq C.D \leq 7$ or $10 \leq C.D \leq 14$.

(g) If $P_a(D) = 2C.D - 4$, then $C.D = 2$ or $5 \leq C.D \leq 6$ or $12 \leq C.D \leq 15$.

(h) If $P_a(D) = 2C.D - 2$, then $C.D = 1$.

With Theorem 3.1 at hand, we are now ready to prove the only if direction of the promised Theorem 1.1.

Proof of the only if part of the Theorem 1.1

Proof. Note that, (i), (ii) and (iii)(a) follows from Theorem 3.1. To establish (iii)(b), (c), we argue as follows:

For (iii)(b) first part, consider $C.D = 6$ or $12 \leq C.D \leq 15$. Then applying long exact cohomology sequence to the following S.E.S:

$$
0 \rightarrow \mathcal{O}_X(D - 2C) \rightarrow \mathcal{O}_X(D - C) \rightarrow \mathcal{O}_C(D - C) \rightarrow 0
$$

(3.11)

and from the assumption of ACMness and initializedness, we have $h^0(\mathcal{O}_C(D - C)) = 0$.

For the second part let $12 \leq C.D \leq 15$. Then consider the following S.E.S:

$$
0 \rightarrow \mathcal{O}_X(D - 3C) \rightarrow \mathcal{O}_X(D - 2C) \rightarrow \mathcal{O}_C(D - 2C) \rightarrow 0
$$

(3.12)

Taking long exact cohomology sequence to the S.E.S (3.12) and using the assumption of ACMness and initializedness, one has $h^0(\mathcal{O}_C(D - 2C)) = 0$.

For (iii)(c) first part, note that, for $C.D = 12$, one must have $|2C - D| = \emptyset$ (otherwise $\mathcal{O}_X(D) \cong \mathcal{O}_X(2C)$, a contradiction). Then from relation 3.2, the desired result follows. For the second part, consider $C.D = 15$. Since $C.(2C - D) < 0$, we have $h^0(\mathcal{O}_C(2C - D)) = 0$ and hence $h^0(\mathcal{O}_X(2C - D)) = 0$ (by 3.2). Then from relation 2.2, one has $h^0(\mathcal{O}_X(D)) = 6$. This means $h^0(\mathcal{O}_C(D)) = 6$ by the S.E.S 2.4.

Next, consider case (iv). In this case, observe that (iv)(a) follows from Theorem 3.1 and (iv)(b) first part follows by the same argument as that of (iii)(b) first part.

For (iv)(b) second part, if $C.D = 11$, then by Lemma 2.11 and if $C.D = 12$, then by the same argument as in (iii)(c) first part, one obtains $|2C - D| = \emptyset$. This means by relation 3.2, $h^0(\mathcal{O}_C(2C - D)) = 0$.

For (iv)(c) first part, note that, by initializedness, one has $|C - D| = \emptyset$. Then from relations 2.2, 3.2 and S.E.S 2.4, one must have $h^0(\mathcal{O}_C(D)) + h^0(\mathcal{O}_C(2C - D)) = 5$, whence the claim follows. The second part follows from ACMness and the S.E.S 2.3.

For (iv)(d), let $12 \leq C.D \leq 14$. In this case, for $13 \leq C.D \leq 14$, from $C.(2C - D) < 0$ and relation 3.2, it’s easy to see that, we have $h^0(\mathcal{O}_C(2C - D)) = h^0(\mathcal{O}_X(2C - D)) = 0$. Note that, we already have this for $C.D = 12$. Then from relation 2.2 and the S.E.S 2.4, one obtains $h^0(\mathcal{O}_C(D)) = 5$.

Next, consider the case (v). In this case, observe that (v)(a) follows from Theorem 3.1, (v)(b) first part follows by the same argument as that of (iii)(b) first part and (v)(b) second part 11 $\leq C.D \leq 12$ follows by the same argument as (iv)(b) second part.

For (v)(b) second part, for $C.D = 10$, note that, by relation 3.2 and by $C.(C - D) < 0$, it’s enough to show that $|2C - D| = \emptyset$. This follows from Lemma 2.13.

Note that, (v)(c) first part follows from the fact $h^0(\mathcal{O}_X(D)) = 1$, relation 3.2, 2.2 and from S.E.S 2.3. The argument for (v)(c) second part is same as that of (iv)(c) second part.
It’s clear that \((v)(d), (v)(e)\) and \((v)(f)\) first part follows from S.E.S 2.4, relations 2.2, 3.2 and the observation that in each such case \(|C - D| = \emptyset\).^5

For \((v)(f)\) second part, let \(11 \leq C.D \leq 13\). Since \(C.(2C - D) < 0\) for \(C.D = 13\), we have \(h^0(O_C(2C - D)) = 0\). We have the same vanishing for \(C.D = 11,12\) from \((v)(b)\) second part. Then from relations 2.2, 3.2, one sees that in all three situations \(h^0(O_X(D)) = 4\). This means \(h^0(O_C(D)) = 4\) from the S.E.S 2.4.

Now consider the case \((vi)\). Note that, here \((vi)(a)\) follows from Theorem 3.1, \((vi)(b)\) follows by the same argument as that of \((v)(b)\). Note that, for \(C.D = 5, \ |C - D| = \emptyset\) follows from Lemma 2.11 and for \(C.D = 6, \ |C - D| = \emptyset\) follows from initializedness. Then \((vi)(c), (d), (e), (f)\) follows from the relations 2.2, 3.2 and the S.E.S 2.4.

For \((vi)(g)\), one obtains for \(C.D = 4, \ |C - D| = \emptyset\) by Lemma 2.13. Note that, here we have by Remark 2.5 and S.E.S 2.4, \(h^0(O_C(D)) = h^0(O_X(D)) = 1\). Then the desired result follows from relations 2.2 and 3.2.

Since for \(C.D = 4, 6\), we have \(|C - D| = \emptyset\), argument for \((vi)(h)\) first part is same as that of \((iv)(c)\) second part. For \((vi)(h)\) second part, observe that \((vi)(b)\) second part, relations 2.2, 3.2 and S.E.S 2.4 give us the desired condition.

Consider the case \((vii)\). As before \((vii)(a)\) follows from Theorem 3.1. It can be shown by Lemma 2.11 that for \(C.D = 5\), one has \(|C - D| = \emptyset\). Note that, we have the same result for \(C.D = 6\) by initializedness. Then \((vii)(b)\) first part follows by the same argument as that of \((vi)(c)\). \((vii)(b)\) second part is clear from initializedness, ACMness, S.E.S 2.4 and the relation 2.2.

For \(C.D = 4\), it is not difficult to see that \(|C - D| = \emptyset\) (by lemma 2.13). Then \((vii)(c)\) follows (all three cases) by the same argument as that of \((vi)(h)\) first part. For \((vii)(c)\) second part, note that, if \(h^0(O_C(2C - D)) \geq 2\), then by relations 2.2, 3.2, one obtains \(h^0(O_X(D)) = h^0(O_C(D)) = 0\), a contradiction.

\((vii)(d)\) first part follows by noting that \(h^0(O_X(D)) = 1\) and using relations 2.2, 3.2. To establish \((vii)(d)\) second part, we argue as before that in both the situations, one has \(h^0(O_C(2C - D)) = 0\). This forces by relations 2.2, 3.2, S.E.S 2.4, the desired condition.

Finally, let’s consider the case \((viii)\). Note that, \((viii)(a)\) first part follows from Theorem 3.1. For \(C.D = 5\), one can show by Lemma 2.11 that \(|C - D| = \emptyset\). The same observation can be made for \(C.D = 6\) by initializedness. Then it can be seen that \((viii)(a)\) second part follows by relation 2.2, 3.2 and the S.E.S 2.4. The last condition \((viii)(b)\) follows by the same argument as \((viii)(c)\) first part.

### 3.2. Sufficient condition

In this subsection, we give a proof of the “if” part of the Theorem 1.1. Let \(X \subset \mathbb{P}^3\) be a smooth sextic hypersurface. Let \(H\) be the hyperplane class of \(X\) and \(C \in |H|\) be a smooth member as before. Let \(D\) be a non-zero effective divisor on \(X\) of arithmetic genus \(P_a(D)\). Let \(D\) satisfy condition (i) and any one of the remaining conditions (i.e. from (ii) to (viii) with the restrictions mentioned under the condition) of the promised Theorem 1.1. Our first aim is to show that \(O_X(D)\) is initialized.

**Proof of the if part of the Theorem 1.1**

**Proof. Initializedness**

Note that, for any value of the arithmetic genus of \(D\), if \(C.D \leq 5\), then initializedness follows from the observation \(C.(D - C) < 0\).
Note that, irrespective of the value of the arithmetic genus of $D$, one has for $9 \leq C.D \leq 11$, $C.(D - 2C) < 0$ and hence $h^0(O_X(D - 2C)) = 0$. Therefore, in these cases it’s enough to have $h^0(O_C(D - C)) = 0$ (because in that case initializedness follows by applying long exact cohomology sequence to the S.E.S 3.11). Now for $9 \leq C.D \leq 11$, this is there by assumptions $(iv)(b), (v)(b), (vi)(b)$ and by the observation that in remaining cases $h^0(O_C(D)) < 3$. For $C.D = 6$, if $|D - C| \neq \emptyset$, then we have $O_X(D) \cong O_X(C)$, a contradiction (as then $P_a(D) = 10$).

Again for any value of the arithmetic genus of $D$, for $C.D = 7$, $|D - C| = \emptyset$ can be obtained by applying Lemma 2.11 on the divisor $D - C$.

Similarly, for the cases $(vi), (vii), (viii)$, if $C.D = 8$, $|D - C| = \emptyset$ can be obtained by applying Lemma 2.13 on the divisor $D - C$. For the case $(v)$, if $C.D = 8$, then initializedness can be realized by the same argument as that of the situations $9 \leq C.D \leq 11$.

Finally, consider the cases, $12 \leq C.D \leq 15$. Note that, in each such cases we have $C.(D - 3C) < 0$ and $h^0(O_C(D - 2C)) = 0$ (for $12 \leq C.D \leq 15$, this follows either by the assumption $(iii)(b)$ second part or by noting that $h^0(O_C(D)) < 6$). Then from the S.E.S 3.12, one has $h^0(O_X(D - 2C)) = 0$. Lastly, applying long exact cohomology sequence to the S.E.S 3.11 and using $h^0(O_C(D - C)) = 0$, we have $h^0(O_X(D - C)) = 0$ (Note that, $h^0(O_C(D - C)) = 0$ is either there by assumptions $(iii)(b), (iv)(b), (v)(b), (vi)(b)$ or can be seen by noting that in the remaining case one has $h^0(O_C(D)) < 3$).

To prove ACMness, we need a Proposition which deals with the vanishing of $h^1(O_X(-D))$ for effective divisors with low degrees. This Proposition uses a consequence of a non-zero effective divisor $D$ being not 1-connected.

**Proposition 3.2.** Let $D$ be a non-zero effective divisor on a smooth sextic surface $X \subset \mathbb{P}^3$ such that $h^1(O_X(D)) = 0$ and $h^0(O_X(D - C)) = 0$. Then we have, $h^1(O_X(-D)) = 0$, if the following conditions are satisfied:

(i) $2C.D - 7 \leq P_a(D) \leq 2C.D - 5$, $C.D = 5$.
(ii) $P_a(D) = 2C.D - 4$, $C.D = 6$.

**Proof.** We suitably adapt the techniques used in [[21], Lemma 2.3] according to our situation. In what follows, we note down an important consequence of such $D$ being not 1-connected. We work under the assumption that $D^2 = 2C.D - 2k$ for some positive integer $k$.

Assume that $D$ is not 1-connected. Let $D = D_1 + D_2$ be a non-trivial effective decomposition with $D_1, D_2 \leq 0$. Since $D^2 = D.D_1 + D.D_2$, we may assume that $D.D_1 = D^2 - D.D_2 \leq D_2^2$. Then we have, $\frac{D_1^2}{2} \leq D.D_2 = D_2^2 + D_1.D_2 \leq D_2^2$. Note that,

$$4\chi(O_X(D_2)) = 4\left(\frac{1}{2}D_2(D_2 - 2C) + 11\right)$$

$$= 2D_2^2 - 4C.D_2 + 44$$

and

$$4h^0(O_X(D)) = 4\chi(O_X(D)) - 4h^0(O_X(2C - D))$$

$$= 2D^2 - 4C.D + 44 - 4h^0(O_X(2C - D))$$
Therefore,
\[4\chi(O_X(D_2)) - 4h^0(O_X(D)) = (2D_2^2 - 4C.D_2) - (2D^2 - 4C.D) + 4h^0(O_X(2C - D))
\]
\[\geq D^2 - 4C.D + 2k + 4h^0(O_X(2C - D))
\]
\[= 2C.D - 2k - 4C.D_2 + 4k + 4h^0(O_X(2C - D))
\]
\[= 2C.D_2 - 2C.D_2 + 2k + 4h^0(O_X(2C - D))
\]

This in turn gives us:
\[\chi(O_X(D_2)) \geq h^0(O_X(D)) + h^0(O_X(2C - D)) + \frac{1}{2}(C.D_1 - C.D_2 + k) \quad (3.13)
\]

With inequality 3.13 at hand, we now analyze what happens in each case if we start with the assumption \(h^1(O_X(-D)) \neq 0\). Note that, by Remark 2.22, this assumption implies \(D\) is not 1-connected, which enables us to apply the inequality 3.13. Therefore, to conclude the Proposition, it is enough to obtain contradiction in both the cases.

(i) Let's consider \(P_\alpha(D) = 2C.D - r, C.D = 5\), where \(r \in \{5, 6, 7\}\)

Note that, in this situation, one has \(D^2 = 8 - 2r\) and hence \(D^2 = 2C.D - 2k\) for \(k = r + 1\). From 3.13, there exist divisors \(D_1, D_2\) on \(X\) with \(D = D_1 + D_2\) such that \(\chi(O_X(D_2)) \geq 10 - r + \frac{1}{2}(C.D_1 - C.D_2 + k)\). Assume that \(C.D_1 < C.D_2\). Since \(C.D_1 \geq 1\), we have \(C.D_2 \geq 2\). In this situation, the only possibilities are either \(C.D_1 = 1, C.D_2 = 4\) or \(C.D_1 = 2, C.D_2 = 3\). Consider the case \(C.D_1 = 1, C.D_2 = 4\). Since \(C.D_1 - C.D_2 + k = r - 2\), one obtains \(h^0(O_X(D_2)) + h^0(O_X(2C - D_2)) \geq \chi(O_X(D_2)) \geq 9 - \frac{r}{2}\). Applying long exact cohomology sequence to the S.E.S. 3.1 and noting \(h^0(O_X(D_2)) \leq h^0(O_X(C_2 - D_2)) \leq 1\), we have \(1 + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 9 - \frac{r}{2}\). Since \(C.D_2 = 4\), by Corollary 2.7, we have \(h^0(O_X(C_2 - D_2)) \leq 3\) and by Remark 2.5, we have \(h^0(O_X(C - D_2)) \leq 1\). These forces \(5 \geq 1 + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 9 - \frac{r}{2}\), a contradiction. Consider the case \(C.D_1 = 2, C.D_2 = 3\). Since \(C.D_1 - C.D_2 + k = r\), as before, one obtains \(h^0(O_X(D_2)) + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 10 - \frac{r}{2}\). Since \(C.D_2 = 3\), by Corollary 2.7, we have \(h^0(O_X(C_2 - D_2)) \leq 3\) and by Remark 2.5, we have \(h^0(O_X(C - D_2)) \leq 1\). This forces \(5 \geq 1 + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 10 - \frac{r}{2}\), a contradiction. Now consider the case \(C.D_2 \leq C.D_1\). In this situation, the only possibilities are either \(C.D_1 = 4, C.D_2 = 1\) or \(C.D_1 = 3, C.D_2 = 2\). Again as before, we have, \(1 + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 10 - r + \frac{1}{2}(C.D_1 - C.D_2 + k)\). Consider the case \(C.D_2 = 2, C.D_1 = 3\), then we have \(C.D_1 - C.D_2 + k = r + 2\) and therefore, one obtains \(1 + h^0(O_X(C - D_2)) + h^0(O_X(2C - D_2)) \geq 11 - \frac{r}{2}\). By Remark 2.5, \(h^0(O_X(C - D_2)) \leq 1\) and by Corollary 2.7, we have \(h^0(O_X(C_2 - D_2)) \leq 4\). But this then means \(6 \geq 1 + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 11 - \frac{r}{2}\), a contradiction. Finally, consider the case \(C.D_2 = 1, C.D_1 = 4\), then we have \(C.D_1 - C.D_2 + k = r + 4\) and therefore, one obtains \(1 + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 12 - \frac{r}{2}\). By Remark 2.5, \(h^0(O_X(C - D_2)) \leq 2\) and by Corollary 2.7, we have \(h^0(O_X(C_2 - D_2)) \leq 5\). This means \(8 \geq 1 + h^0(O_X(C_2 - D_2)) + h^0(O_X(C - D_2)) \geq 12 - \frac{r}{2}\), a contradiction.

To establish (ii) let's consider \(P_\alpha(D) = 2C.D - r, C.D = 6\), where \(r = 4\). Note that, in this situation, one has \(D^2 = 10 - 2r\) and hence \(D^2 = 2C.D - 2k\) for \(k = r + 1\). From 3.13, there exist divisors \(D_1, D_2\) on \(X\) with \(D = D_1 + D_2\) such that \(\chi(O_X(D_2)) \geq 10 - r + \frac{1}{2}(C.D_1 - C.D_2 + k)\). Assume that \(C.D_1 < C.D_2\). Since \(C.D_1 \geq 1\), we have \(C.D_2 \geq 2\). In this situation, the only possibilities are either \(C.D_1 = 1, C.D_2 = 5\) or \(C.D_1 = 2, C.D_2 = 4\). Consider the case \(C.D_1 = 1, C.D_2 = 5\). Since \(C.D_1 - C.D_2 + k = r - 3\), one obtains \(h^0(O_X(D_2)) + h^0(O_X(C_2 - D_2)) \geq \chi(O_X(D_2)) \geq 8.5 - \frac{r}{2}\). Applying long exact cohomology sequence to the S.E.S. 3.1 and noting \(h^0(O_X(D_2)) \leq h^0(O_X(C_2 - D_2)) \leq 2\), we have \(2 + h^0(O_X(2C - D_2)) + h^0(O_X(C - D_2)) \geq 8.5 - \frac{r}{2}\). Since \(C.D_2 = 5\), by
Corollary 2.7, we have $h^0(O_C(2C - D_2)) \leq 3$ and by Remark 2.5, we have $h^0(O_X(C - D_2)) \leq 1$. This forces $6 \geq 2 + h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 8.5 - \frac{r}{2}$, a contradiction. Consider the case $C.D_1 = 2, C.D_2 = 4$. Since $C.D_1 - C.D_2 + k = r - 1$, as before, one obtains $h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 9.5 - \frac{r}{2}$. Since $C.D_2$ is 4, by Corollary 2.7, we have $h^0(O_C(2C - D_2)) \leq 3$ and by Remark 2.5, we have $h^0(O_X(C - D_2)) \leq 1$. This forces $5 \geq 1 + h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 9.5 - \frac{r}{2}$, a contradiction. Now consider the case $C.D_2 \leq C.D_1$. In this situation, we have $h^0(O_C(2C - D_2)) \leq 3$ and by Remark 2.5, we have $h^0(O_X(C - D_2)) \leq 1$ and by Corollary 2.7, we have $h^0(O_C(2C - D_2)) \leq 4$. But this then means $6 \geq 1 + h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 11.5 - \frac{r}{2}$, a contradiction. Next, consider the case $C.D_2 = 1, C.D_1 = 5$, then we have $C.D_1 - C.D_2 + k = r + 5$ and therefore, one obtains $1 + h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 12.5 - \frac{r}{2}$. By Remark 2.5, $h^0(O_X(C - D_2)) \leq 2$ and by Corollary 2.7, we have $h^0(O_C(2C - D_2)) \leq 5$. This means $8 \geq 1 + h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 12.5 - \frac{r}{2}$, a contradiction. Finally, consider the case $C.D_2 = 3, C.D_1 = 3$, then we have $C.D_1 - C.D_2 + k = r + 1$ and therefore, one obtains $1 + h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 10.5 - \frac{r}{2}$. By Remark 2.5, $h^0(O_X(C - D_2)) \leq 1$ and by Corollary 2.7, we have $h^0(O_C(2C - D_2)) \leq 3$. This means $5 \geq 1 + h^0(O_C(2C - D_2)) + h^0(O_X(C - D_2)) \geq 10.5 - \frac{r}{2}$, a contradiction.

Remark 3.3. From the strategy of the above proof, it can be seen that for case-(ii), if we consider $r=5$, then we get a contradiction in each case except the case $C.D_1 = 1, C.D_2 = 5$. In this situation, it can also be noticed that $D_i^2 = 0$ for $i \in \{1, 2\}$ and $D_1.D_2 = 0$. For case-(i), if we consider $r=8$, then we get contradiction except in two situations. The first such situation is when $C.D_1 = 1, C.D_2 = 4$. In this case, again one obtains $D_i^2 = -4$ for $i \in \{1, 2\}$ and $D_1.D_2 = 0$. In second situation, we have $C.D_1 = 4, C.D_2 = 1$. In this case, we have $D_i^2 = -4$ for $i \in \{1, 2\}$ and $D_1.D_2 = 0$. In this situation, it can also be noticed that $D_2$ is an ACM line bundle of type (ii) (i.e. satisfies the condition (ii) of the promised Theorem 1.1 and is ACM and initialized).

With Proposition 3.2 at hand, we are now ready to prove the ACMness part of the “ii” direction of the promised Theorem 1.1. Before proceeding further, we would like to briefly describe our strategy of proving ACMness (in the context of Corollary 2.19) in 6 steps.

Step-(I): We use Remark 2.5 (and occasionally Corollary 2.7) to obtain an upper bound of $h^0(O_C(D))$ and hence an upper bound of $h^0(O_X(D))$ by S.E.S 2.4 (Here in some situations for convenience we may start with an upper bound of $h^0(O_C(2C - D))$ and use the same technique as mentioned here to obtain precise cohomological values). By relation 2.2 this gives us a lower bound of $h^0(O_X(2C - D))$ and hence a lower bound of $h^0(O_C(2C - D))$ (by S.E.S 3.1). Then by assumption or Corollary 2.7 or by Remark 2.5, we obtain precise values of $h^0(O_X(D)) = h^0(O_C(D)), h^0(O_C(2C - D)), h^0(O_X(2C - D))$. This values when substituted to the relation 2.2 give us $h^1(O_X(D)) = h^1(O_X(2C - D)) = 0$. We here mention that this is the most important step toward proving ACMness.

Step-(II): We apply long exact cohomology sequence to the S.E.S 2.4. Then from the initializaedness and Step-(I), we have $h^1(O_X(3C - D)) = h^1(O_X(D - C)) = 0$ (Note that, from step-(I), we will always have $h^0(O_X(D)) = h^0(O_C(D))$).

Step-(III): We apply long exact cohomology sequence to the S.E.S 3.11. Then from Step-(II) and $h^0(O_C(D - C)) = 0$, one obtains $h^1(O_X(4C - D)) = h^1(O_X(D - 2C)) = 0$. 
Step-(IV): We apply long exact cohomology sequence to the S.E.S 3.12. Then from Step-(III) and \( h^0(O_C(D - 2C)) = 0 \), one obtains \( h^1(O_X(5C - D)) = h^1(O_X(D - 3C)) = 0 \).

Step-(V): We apply long exact cohomology sequence to the S.E.S 3.1. Then using the information obtained in Step-(I), we obtain \( h^1(O_X(C - D)) = 0 \).

Step-(VI): If we have \( h^0(O_C(C - D)) = h^0(O_X(C - D)) \), then we apply long exact cohomology sequence to the S.E.S 2.3 and use Step-(V) to see \( h^1(O_X(-D)) = 0 \) or else the vanishing of \( h^1(O_X(-D)) \) will follow from Proposition 3.2.

Also, note that, for establishing ACMness, by Corollary 2.19, irrespective of the value of the arithmetic genus of \( D \), we have the following:

- For \( C.D \leq 5 \), it is enough to carry out Steps-(I), (II), (V), (VI).
- For \( 6 \leq C.D \leq 11 \), it is enough to carry out all Steps except Step-(IV).
- For \( 12 \leq C.D \leq 15 \), we need to carry out all the 6 Steps.

**ACMness**

Consider \( P_2(D) = 2C.D - 2 \).

Let \( C.D = 1 \). By Remark 2.5, initializedness and S.E.S 2.4, 2.3, we have \( h^0(O_C(D)) = h^0(O_X(D)) = h^0(O_X(C - D)) \leq 2 \). By relation 2.2, one has \( h^0(O_X(2C - D)) \geq 7 \). From the S.E.S 3.1, we have \( h^0(O_X(2C - D)) \leq h^0(O_X(2C - D)) + h^0(O_X(C - D)) \). If \( h^0(O_X(2C - D)) \geq 6 \), then \( Clif(O_X(2C - D)) \leq 1 \), a contradiction to Theorem 2.4. This means \( h^0(O_X(2C - D)) = 5 \) and hence \( h^0(O_X(2C - D)) = 7 \), \( h^0(O_X(C - D)) = h^0(O_X(C - D)) = 2 \). Then from relation 2.2, we see that Step-(I) is completed and hence Step-(V) is also done. It’s clear that Step-(II) follows from Step-(I) and initializedness. Since from Step-(I), we have \( h^0(O_X(C - D)) = h^0(O_X(C - D)) \), Step-(VI) is done using Step-(V).

Consider \( P_2(D) = 2C.D - 4 \).

Let \( C.D = 2 \). Again as before by Remark 2.5, initializedness and S.E.S 2.4, 2.3, we have \( h^0(O_C(D)) = h^0(O_X(D)) = h^0(O_X(C - D)) \leq 1 \). By the same argument as in previous case Step-(I), one can obtain \( h^0(O_C(2C - D)) \geq 4 \). By Corollary 2.7, this means \( h^0(O_C(2C - D)) = 4 \). From S.E.S 3.1, one sees that, \( h^0(O_X(C - D)) = h^0(O_X(C - D)) = 1, h^0(O_X(2C - D)) = 5 \). Then from relation 2.2, Step-(I) is completed. Steps-(II), (V), (VI) are done by the same arguments as in the previous case. Since from Step-(I) we have \( h^0(O_X(C - D)) = h^0(O_X(C - D)) \), Step-(VI) is done using Step-(V).

Let \( C.D = 5 \). Note that, here we have \( h^0(O_X(C - D)) \leq 2, h^0(O_X(C - D)) \leq 1 \) (by Remark 2.5). By the same arguments as in previous case Step-(I), one can obtain \( h^0(O_C(2C - D)) \geq 3 \). By Corollary 2.7, this means \( h^0(O_C(2C - D)) = 3 \). From S.E.S 3.1, one sees that, \( h^0(O_X(C - D)) = h^0(O_X(C - D)) = 1, h^0(O_X(2C - D)) = 4 \). This forces by relation 2.2 and S.E.S 2.3, \( h^0(O_X(D)) = h^0(O_X(D)) = 2 \). Then from relation 2.2, Step-(I) is completed. The arguments for Steps-(II), (V), (VI) are same as that of previous case (i.e. the case \( C.D = 2 \)).

Let \( C.D = 6 \). From Remark 2.5, we have \( h^0(O_X(D)) \leq 3 \). Note that, in this case \( |C - D| = \emptyset \) (otherwise \( O_X(D) \cong O_X(C) \), a contradiction). Then by the same argument as in previous case Step-(I), one can obtain \( h^0(O_C(2C - D)) \geq 3 \). By Remark 2.5, this means \( h^0(O_C(2C - D)) = 3 \). From S.E.S 3.1, one sees that, \( h^0(O_X(2C - D)) = h^0(O_X(2C - D)) = 3 \). This forces by relation 2.2 and S.E.S 2.3, \( h^0(O_X(D)) = h^0(O_X(D)) = 3 \). Then from relation 2.2, Step-(I) is completed. The arguments for Steps-(II), (V) are the same as that of the previous case (i.e. the case \( C.D = 2 \)). Step-(III) follows from Step-(II) and assumption (iii)(b) first part. Step-(VI) follows from Proposition 3.2.
Let $12 \leq C.D \leq 15$. Note that, in this situation, we have $C.(2C - D) < 0$, for $13 \leq C.D \leq 15$ and for $C.D = 12$, we have $|2C - D| = \emptyset$ (from assumption (iii)(c) first part and S.E.S 3.1). This means $h^0((O_C(D)) \geq 6$ (by relation 2.2 and S.E.S 2.4). From Corollary 2.7 and from assumption (iii)(c) second part, we have $h^0(O_C(D)) = h^0(O_X(D)) = 6$. Then from relation 2.2, Step-(I) is completed. From Step-(I) one can see that Step-(II) follows. Step-(III) follows from Step-(II) and assumption (iii)(b) first part. Step-(IV) follows from Step-(III) and assumption (iii)(b) second part. Step-(V) follows from Step-(I) and from the facts $|C - D| = \emptyset, h^0(O_X(2C - D)) = h^0(O_C(2C - D)) = 0$. Finally, Step-(VI) follows from Step-(V) and the observation $C.(C - D) < 0$.

Consider $P_a(D) = 2C.D - 5$.

Let $3 \leq C.D \leq 4$. Here we have by Remark 2.5, S.E.S 2.4, S.E.S 2.3 and initializedness, $h^0(O_C(D)) = h^0(O_X(D)) = 1, h^0(O_C(C - D)) \leq 1$. Then using the relation 2.2 and applying long exact cohomology sequence to the S.E.S 3.1, we have $h^0(O_C(2C - D)) \geq 3$. Corollary 2.7 forces $h^0(O_C(2C - D)) = 3, h^0(O_X(C - D)) = h^0(O_C(C - D)) = 1, h^0(O_X(2C - D)) = 4$. Then from the relation 2.2, Step-(I) is done. Step-(II) follows from Step-(I) and the observation mentioned in the beginning of this paragraph. It’s easy to see that Step-(V) follows from Step-(I) (and the cohomological information obtained therein) and Step-(VI) follows from Step-(I) and Step-(V).

Let $C.D = 5$. By Remark 2.5, we have $h^0(O_C(D)) \leq 2$. Note that, by Lemma 2.11, one has $C.D = 5$. Then using the relation 2.2 and S.E.S 3.1, 2.4, we have $h^0(O_C(2C - D)) \geq 3$. Corollary 2.7 forces $h^0(O_C(2C - D)) = 3, h^0(O_X(C - D)) = h^0(O_C(C - D)) = 1, h^0(O_X(2C - D)) = 4$. Then from relation 2.2, Step-(I) is done. Steps-(II), (V) follow from Step-(I) and the cohomological data obtained there. Step-(VI) follows from Proposition 3.2.

Let $C.D = 6$. Again we have by Remark 2.5, $h^0(O_C(D)) \leq 3$. Note that, we have $C.D = 0$. Then using the relation 2.2 and S.E.S 3.1, 2.4, we have $h^0(O_C(2C - D)) \geq 2$. It can be easily deduced that $h^0(O_X(D)) \neq 1$. Then assumption (iv)(c) first part, the abovementioned S.E.S’s and relation 2.2 force that we have either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 3, h^0(O_X(D)) = h^0(O_C(D)) = 2$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 2, h^0(O_X(D)) = h^0(O_C(D)) = 3$. Then from relation 2.2, Step-(I) is done. Arguments for Steps-(II), (V) are the same as that of the case $C.D = 5$. Step-(III) follows from Step-(II) and assumption (iv)(b) first part. It can be seen that step-(VI) follows from Step-(V) and assumption (iv)(c) second part.

Let $C.D = 7$. In this case, we have by Corollary 2.7, $h^0(O_C(D)) \leq 3$. Then using the relation 2.2 and S.E.S 3.1, 2.4, one has $h^0(O_C(2C - D)) \geq 2$. Remark 2.5 forces $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 2$ and hence from relation 2.2 and S.E.S 2.4, one sees that $h^0(O_X(D)) = h^0(O_C(D)) = 3$. Then from relation 2.2, Step-(I) is done. Again the arguments for Steps-(II), (V) are the same as that of the case $C.D = 6$. Step-(III) follows from Step-(II) and assumption (iv)(b) first part. Finally, note that, Step-(VI) follows from Step-(V) and the observation that $C.(C - D) < 0$.

Let $C.D = 10$. In this case, we have by Remark 2.5, $h^0(O_C(2C - D)) \leq 1$. Then using the relation 2.2 and S.E.S 3.1, we have $h^0(O_X(D)) \geq 4$. Corollary 2.7 and S.E.S 2.4 force $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 1, h^0(O_X(D)) = h^0(O_C(D)) = 4$. Then from relation 2.2, Step-(I) is done. The arguments for the remaining cases are same as the case $C.D = 7$.

Let $C.D = 11$. By assumption (iv)(b) second part, 2.2 and S.E.S 3.1, we have $h^0(O_X(D)) \geq 5$. Corollary 2.7 and S.E.S 2.4 force $h^0(O_X(D)) = h^0(O_C(D)) = 5$. Then from relation 2.2, Step-(I) is done. The arguments for the remaining cases are same as the case $C.D = 10$.

Finally, let $12 \leq C.D \leq 14$. Note that, in each such situation one has $h^0(O_C(2C - D)) = 0$. Then by 2.2 and S.E.S 3.1, we have $h^0(O_X(D)) \geq 5$. Assumption (iv)(d) and S.E.S 2.4 force $h^0(O_X(D)) = h^0(O_C(D)) = 5$. Then from relation 2.2, Step-(I) is done. Step-(II) follows from
Step-(I) and the cohomological equalities obtained there. Step-(III) follows from Step-(II) and assumption (iv)(b) first part. Step-(IV) follows from Step-(III) and the observation $h^0(O_C(D)) < 6$. It is easy to see that Step-(V) follows from Step-(I) and assumption (iv)(b) second part. Finally, Step-(VI) follows by the same reason as $C.D = 10$.

Consider $P_a(D) = 2C.D - 6$.
Let $3 \leq C.D \leq 4$. As before by Remark 2.5, S.E.S 2.4, S.E.S 2.3 and initializedness, $h^0(O_C(D)) = h^0(O_X(D)) = 1, h^0(O_X(C - D)) \leq 1$. From relation 2.2 and S.E.S 3.1, we have $h^0(O_C(2C - D)) \geq 2$. Note that, by Corollary 2.7, $h^0(O_C(2C - D)) \leq 3$. Then one can deduce using the assumption $(v)(c)$ first part, relation 2.2 and S.E.S 3.1 that either we have $h^0(O_X(C - D)) = h^0(O_C(C - D)) = 1$ and $h^0(O_X(2C - D)) = 3, h^0(O_C(2C - D)) = 2$ or $h^0(O_X(C - D)) = 0$ and $h^0(O_X(2C - D)) = h^0(O_C(2C - D)) = 3$. Then from relation 2.2, Step-(I) is done. The remaining Steps follow by the same arguments as that of the case $P_a(D) = 2C.D - 5, 3 \leq C.D \leq 4$.

Let $C.D = 5$. By Remark 2.5, we have $h^0(O_C(D)) \leq 2$. Note that, by Lemma 2.11, one has $|C - D| = \emptyset$. Then using the relation 2.2 and S.E.S 3.1, 2.4, we have $h^0(O_C(2C - D)) \geq 2$. By Corollary 2.7, relation 2.2 and assumption $(v)(f)$ first part, one can have either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 3, h^0(O_X(D)) = h^0(O_C(D)) = 1$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 2, h^0(O_X(D)) = h^0(O_C(D)) = 2$. Then from relation 2.2, Step-(I) is done. The remaining Steps follow by the same arguments as that of the case $P_a(D) = 2C.D - 5, C.D = 5$.

Let $C.D = 6$. As before by Remark 2.5, one has $h^0(O_C(D)) \leq 3$. Note that, we have, $|C - D| = \emptyset$. Then using the relation 2.2 and S.E.S 3.1, 2.4, we have $h^0(O_C(2C - D)) \geq 1$. Assumption $(v)(e)$ and the abovementioned S.E.S’s force that we have either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 3, h^0(O_X(D)) = h^0(O_C(D)) = 1$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 2, h^0(O_X(D)) = h^0(O_C(D)) = 1. h^0(O_X(2C - D)) = h^0(O_C(2C - D)) = 3$. Then from relation 2.2, Step-(II) is completed. Steps-(II), (V) follow from Step-(I) and the cohomological observations obtained there. Step-(III) follows from Step-(II) and assumption $(v)(b)$ first part. Then Step-(VI) follows from Step-(V) and assumption $(v)(c)$ second part.

Let $C.D = 7$. In this case, we have by Corollary 2.7, $h^0(O_C(D)) \leq 3$. Then using relation 2.2 and S.E.S 3.1, 2.4, we have $h^0(O_C(2C - D)) \geq 1$. Then Remark 2.5, assumption $(v)(d)$, relation 2.2 force either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 2, h^0(O_X(D)) = h^0(O_C(D)) = 2$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 1, h^0(O_X(D)) = h^0(O_C(D)) = 3$. Then from relation 2.2, Step-(I) is done. Steps-(II), (V) follow from Step-(I) and the observations mentioned there. Step-(III) follows from Step-(II) and assumption $(v)(b)$ first part. Then Step-(VI) follows from Step-(V) and the observation that $C.(C - D) < 0$.

Let $8 \leq C.D \leq 9$. Then we have by Corollary 2.7, $h^0(O_C(D)) \leq 3$. Then using Remark 2.5, relation 2.2 and S.E.S 3.1, one can see that $h^0(O_X(D)) \leq 2$ is not possible. Then Remark 2.5, relation 2.2 yields $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 1, h^0(O_X(D)) = h^0(O_C(D)) = 3$. Then from relation 2.2, Step-(I) is done. The arguments for the remaining cases are the same as that of the previous case (i.e. $C.D = 7$).

Let $C.D = 10$. Then using relation 2.2, S.E.S 3.1 and assumption $(v)(b)$ second part, we have $h^0(O_X(D)) \geq 4$. Corollary 2.7 and S.E.S 2.4 force $h^0(O_X(D)) = h^0(O_C(D)) = 4$. Then from relation 2.2, Step-(I) is done. The arguments for the remaining cases are the same as that of the case $C.D = 7$.

Let $C.D = 11$. In this case, we have by assumption $(v)(b)$ second part, relation 2.2 and S.E.S 3.1, $h^0(O_X(D)) \geq 4$. Then assumption $(v)(f)$ second part and S.E.S 2.4 enable us to obtain
$h^0(O_X(D)) = h^0(O_C(D)) = 4$. Then from relation 2.2, Step-(I) is done. The arguments for the remaining cases are the same as that of the case $C.D = 7$.

Finally, let $12 \leq C.D \leq 13$. Note that, in each such situation one has $h^0(O_C(2C - D)) = 0$. Then by 2.2 and S.E.S 3.1, we have $h^0(O_X(D)) \geq 4$. Assumption $(vi)(f)$ second part and S.E.S 2.4 force $h^0(O_X(D)) = h^0(O_C(D)) = 4$. Then from relation 2.2, Step-(I) is done. Step-(II) follows from Step-(I) and the cohomological equality obtained there. Step-(III) follows from Step-(II) and assumption $(vi)(b)$ first part. Step-(IV) follows from Step-(III) and the observation $h^0(O_C(D)) < 6$. It is easy to see that Step-(V) follows from Step-(I), assumption $(vi)(b)$ second part (for $C.D = 12$) and the observation $C.(2C - D) < 0$ (for $C.D = 13$). Finally, Step-(VI) follows by the same reason as $C.D = 10$.

Consider $P_a(D) = 2C.D - 7$.

Let $C.D = 4$. By Remark 2.5, S.E.S 2.4 and initializedness, it follows that $h^0(O_C(D)) = h^0(O_X(D)) = 1$. That, one can use Lemma 2.13 to obtain $|C - D| = \emptyset$. From the relation 2.2 and S.E.S 3.1, we have $h^0(O_C(2C - D)) \geq 2$. Then one can deduce the assumption $(vi)(g)$, relation 2.2 and S.E.S 3.1, $h^0(O_X(2C - D)) = h^0(O_C(2C - D)) = 2$. Then from relation 2.2, Step-(I) is completed. Step-(II) follows from Step-(I) and the observation mentioned in the beginning of this paragraph. It is clear that Step-(V) follows from Step-(I) and Step-(VI) follows from assumption $(vi)(h)$ first part and Step-(V).

Let $C.D = 5$. In this case, we have by Remark 2.5, S.E.S 2.4 and initializedness, it follows that $h^0(O_C(D)) \leq 2$. Note that, by Lemma 2.11, one has $|C - D| = \emptyset$. Then using relation 2.2 and S.E.S 3.1, we have $h^0(O_C(2C - D)) \geq 1$. By corollary 2.7, relation 2.2 and assumption $(vi)(f)$, one can have either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 2$, $h^0(O_X(D)) = h^0(O_C(D)) = 1$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 1, h^0(O_X(D)) = h^0(O_C(D)) = 2$. Then from relation 2.2, Step-(I) is done. Steps-(II), (V) follow by the same arguments as that of the case $P_a(D) = 2C.D - 5, C.D = 5$. Step-(VI) follows from Step-(V) and Proposition 3.2.

Let $C.D = 6$. Then by Remark 2.5, one has $h^0(O_C(D)) \leq 3$. Note that, we have $|C - D| = \emptyset$. Then using assumption $(vi)(e)$, relation 2.2, S.E.S 3.1, S.E.S 2.4, we have either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 0, h^0(O_X(D)) = h^0(O_C(D)) = 3$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 2, h^0(O_X(D)) = h^0(O_C(D)) = 1$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 1, h^0(O_X(D)) = h^0(O_C(D)) = 2$. Therefore, from relation 2.2, Step-(I) is done. Steps-(III), (V) follow from Step-(I) and the observation obtained there. Step-(III) follows from Step-(II) and assumption $(vi)(b)$ first part. Then Step-(VI) follows from Step-(V) and assumption $(vi)(h)$ first part.

Let $C.D = 7$. In this case, we have by Remark 2.5, $h^0(O_C(2C - D)) \leq 2$. Then using relation 2.2 and S.E.S 3.1, we have $h^0(O_X(D)) \geq 1$. Then Remark 2.5, Corollary 2.7, assumption $(vi)(d)$ and relation 2.2 force either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 1, h^0(O_X(D)) = h^0(O_C(D)) = 2$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 0, h^0(O_X(D)) = h^0(O_C(D)) = 3$. Thus, from relation 2.2, Step-(I) is done. Steps-(II), (V) follow by the same arguments as in the previous case (i.e. $C.D = 6$). Step-(III) follows from Step-(II) and assumption $(vi)(b)$ first part. Then Step-(VI) follows from Step-(V) and the observation that $C.(C - D) < 0$.

Let $8 \leq C.D \leq 9$. Then we have by Corollary 2.7, $h^0(O_C(D)) \leq 3$. From Remark 2.5, relation 2.2 and S.E.S 3.1, one can see that $h^0(O_X(D)) = 1$ is not possible. Then Remark 2.5, relation 2.2, assumption $(vi)(c)$ force either $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 1, h^0(O_X(D)) = h^0(O_C(D)) = 2$ or $h^0(O_C(2C - D)) = h^0(O_X(2C - D)) = 0, h^0(O_X(D)) = h^0(O_C(D)) = 3$. Therefore, from relation 2.2, Step-(I) is done. The arguments for the remaining cases are the same as that of the previous case (i.e. $C.D = 7$).
Let $10 \leq C.D \leq 11$. In this case, we have by assumption (vi)(b) second part, relation 2.2 and S.E.S 3.1, $h^0(O_C(D)) \geq 3$. From assumption (vi)(h) second part and S.E.S 2.4, we have $h^0(O_C(D)) = h^0(O_C(D)) = 3$, whence Step-(I) is achieved from relation 2.2. The arguments for the remaining cases are the same as that of the case $C.D = 7$.

Finally, let $C.D = 12$. In this case, all the Steps except Step-(IV) follow by the same arguments as that of case $C.D = 10$. It is easy to see that Step-(IV) follows from Step-(III) and the observation $h^0(O_C(D)) < 6$.

Consider $P_a(D) = 2C.D - 8$.

Let $C.D = 4$. As before by Remark 2.5, S.E.S 2.4 and initializedness, one has $h^0(O_C(D)) = h^0(O_C(D)) = 1$ and by Lemma 2.13 one obtains $|C - D| = 0$. Then relation 2.2, S.E.S 3.1 and the assumption (vii)(d) first part enables us to deduce $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 1$, whence Step-(I) is completed by relation 2.2. Again as before, Step-(II) follows from initializedness, Step-(I) and the observation mentioned at the beginning of this paragraph. It is easy to see that Step-(V) follows from the cohomological information obtained in Step-(I) and Step-(VI) follows from the assumption (vii)(c) first part and Step-(V).

Let $C.D = 5$. Then by Remark 2.5, we have $h^0(O_C(D)) \leq 2$. Again as in the previous situations, by Lemma 2.11, one has $|C - D| = 0$. Then using relation 2.2, S.E.S 3.1, S.E.S 2.4, initializedness, assumptions (vii)(b) first part and (vii)(c) second part, we have either $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 1, h^0(O_C(D)) = h^0(O_C(D)) = 1$ or $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 0, h^0(O_C(D)) = h^0(O_C(D)) = 2$. Then from relation 2.2, Step-(I) is done. Steps-(II), (V) follow by the same arguments as that of the case $P_a(D) = 2C.D - 5, C.D = 5$. Step-(VI) follows from Step-(V) and the assumption (vii)(c) first part.

Let $C.D = 6$. By Remark 2.5 and assumption (vii)(b) second part, one has $h^0(O_C(D)) \leq 2$. Note that, by initializedness, we have $|C - D| = 0$. Then relation 2.2, S.E.S 3.1, S.E.S 2.4, initializedness and assumption (vii)(c) second part force either $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 1, h^0(O_C(D)) = h^0(O_C(D)) = 1$ or $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 0, h^0(O_C(D)) = h^0(O_C(D)) = 2$, whence Step-(I) is evident from relation 2.2. Step-(II) follows from Step-(I) and the cohomological observations obtained therein. Step-(III) follows from Step-(II) and the statement asserted in the second line of this paragraph. Again as in other cases Step-(V) follows from the cohomological values obtained in Step-(I) and Step-(VI) follows from Step-(V) and assumption (vii)(c) first part.

Let $C.D = 7$. In this situation, we have by Corollary 2.7 and assumption (vii)(b) second part, $h^0(O_C(D)) \leq 2$. Then relation 2.2 and S.E.S 3.1, S.E.S 2.4, initializedness, Remark 2.5 and assumptions (vii)(b) first part with (vii)(c) second part force either $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 1, h^0(O_C(D)) = h^0(O_C(D)) = 1$ or $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 0, h^0(O_C(D)) = h^0(O_C(D)) = 2$, whence Step-(I) is evident from relation 2.2. Step-(II) follows from Step-(I) and the cohomological observations obtained in the previous line. Step-(III) follows from Step-(II) and the upper bound mentioned in the second line of this paragraph. It is clear that Step-(V) follows from Step-(I) and two sets of cohomological possibilities seen therein. Then Step-(VI) follows from Step-(V) and the fact that $C.(C - D) < 0$.

Let $8 \leq C.D \leq 9$. Note that, by Remark 2.5, one has $h^0(O_C(2C - D)) \leq 1$. Then the rest of the arguments are the same as in the previous case (i.e. $C.D = 7$).

Let $10 \leq C.D \leq 11$. By (vii)(d) second part, S.E.S 2.4, initializedness, relation 2.2 and S.E.S 3.1, one obtains $h^0(O_C(2C - D)) = h^0(O_C(2C - D)) = 0, h^0(O_C(D)) = h^0(O_C(D)) = 2$. Therefore, Step-(I) is completed by relation 2.2. Arguments for Step-(II), (V), (VI) are the same as the case $C.D = 7$. Finally, Step-(III) follows from assumption (vii)(d) second part.
Consider $P_d(D) = 2C.D - 9$.

Note that, in each situation, by assumption (viii)(a), S.E.S 2.4, initializedness, relation 2.2 and S.E.S 3.1, one must have $h^0(\mathcal{O}_C(2C - D)) = h^0(\mathcal{O}_X(2C - D)) = 0, h^0(\mathcal{O}_X(D)) = h^0(\mathcal{O}_C(D)) = 1$. Then from relation 2.2, Step-(I) follows. Arguments for Step-(II), (V) are the same as before (i.e. case $P_a(D) = 2C.D - 8, C.D = 7$). For $5 \leq C.D \leq 6$, Step-(VI) follows from assumption (viii)(b) and Step-(V). As earlier, for $7 \leq C.D \leq 10$, Step-(VI) follows from Step-(V) and the observation that $C.(C - D) < 0$. Lastly, for $6 \leq C.D \leq 10$, Step-(III) follows from Step-(II) and assumption (viii)(a).

We conclude this Section by pointing out a concrete example where we have $D$ is a non-zero effective divisor on some smooth sextic surface in $\mathbb{P}^3$ such that $P_a(D) = 2C.D - 9, C.D = 5$ and $h^0(\mathcal{O}_X(D)) = 1$ but $h^0(\mathcal{O}_X(2C - D)) \neq 0$ (i.e. $\mathcal{O}_X(D)$ is not an ACM line bundle). This also gives us a naive understanding of why we are forced to impose the condition $h^0(\mathcal{O}_C(2C - D)) = 0$ in case (viii)(a).

Example 3.4. (See [1]) Let $A$ be a plane conic and $B$ be a plane cubic curve in $\mathbb{P}^3$ (lying in different planes) which meets transversally at 1 point. Then it can be shown that there exists a smooth sextic surface in $\mathbb{P}^3$ containing $A$ and $B$ (say $X$). Denote $D := A + B$. If $C$ is a hyperplane section of $X$, then we have $C.D = 5$. Note that, $\deg(K_A) = K_X.A + A^2$ gives us $A^2 = -6$. Similarly, one obtains $B^2 = -6$. This in turn, by construction, means $D^2 = -10$ and hence $P_a(D) = 2C.D - 9$ for $C.D = 5$. The existence of non-trivial sections of $\mathcal{O}_X(2C - D)$ follows from construction.

4. Examples

In this section, we deal with the existence of smooth sextic hypersurfaces $X \subset \mathbb{P}^3$ and a non-zero effective divisor $D$ on it such that $\mathcal{O}_X(D)$ becomes initialized and ACM.

We begin by demonstrating an explicit example.

Example 4.1. Let $X \subset \mathbb{P}^3$ be the Fermat sextic hypersurface i.e. $X$ is given by the equation $x^6 + y^6 + z^6 + w^6 = 0$ for a suitable homogeneous coordinate $(x : y : z : w)$ on $\mathbb{P}^3$. Let $C$ be the hyperplane section of $X$ given by the equation $x = 0$. Consider two lines $D_1, D_2$ on $X$ given by the equations:

- $D_1$ is given by $x - \eta y = z - \eta w = 0$.  
- $D_2$ is given by $x - \bar{\eta} y = z - \bar{\eta} w = 0$.

where $\eta := \frac{\sqrt{5}}{2} + \frac{i}{2}$ and $\bar{\eta}$ stands for the complex conjugate of $\eta$.\footnote{Since $P_a(D_i) = 0, C.D_i = 1$ for $i \in \{1, 2\}$, $D_i$ ($i = 1, 2$) satisfies condition (ii) of Theorem 1.1 and hence each line bundle $\mathcal{O}_X(D_i)$ ($i = 1, 2$) is initialized and ACM.

Next, let us consider $\tilde{D}_i = C - D_i$ for $i \in \{1, 2\}$. Then one has $P_a(\tilde{D}_i) = 6, C.\tilde{D}_i = 5$ ($i = 1, 2$). Therefore, $\tilde{D}_i$ ($i = 1, 2$) satisfies condition (iii)(a) (corresponding to the intersection number 5) of Theorem 1.1 and hence each line bundle $\mathcal{O}_X(\tilde{D}_i)$ ($i = 1, 2$) is initialized and ACM.

The rest of this section is devoted to establish the existence of certain smooth sextic hypersurfaces in $\mathbb{P}^3$ and initialized and ACM line bundles on them using certain concrete constructions of smooth quartic hypersurfaces in $\mathbb{P}^3$ and some specific smooth curves on them as demonstrated in [[21], Section 5]. Toward this end, we present a generalized version of [[21], Lemma 5.1].

Lemma 4.2. Let $X^{(4)} \subset \mathbb{P}^3$ be a smooth quartic hypersurface and $D$ be a smooth curve on $X^{(4)}$ such that $\mathcal{O}_{X^{(4)}}(D)$ is ACM. If $d > 4$ is an integer such that $h^0(\mathcal{O}_{X^{(4)}}(D)(3 - d)) = 0$, then there exists a
smooth hypersurface $X^{(d)} \subset \mathbb{P}^3$ containing $D$ such that $\mathcal{O}_{X^{(d)}}(D)$ is an initialized and ACM line bundle on $X^{(d)}$.

**Proof.** Let $\mathcal{I}_D$ be the ideal sheaf of $D$ in $\mathbb{P}^3$. Note that, under the hypothesis of the Lemma, if we are able to show that $\mathcal{I}_D(d)$ is globally generated, then by [21], Theorem 5.1, there exists a smooth hypersurface $X^{(d)}$ of degree $d$ in $\mathbb{P}^3$ containing $D$. Therefore, we focus our attention on proving that $\mathcal{I}_D(d)$ is globally generated. By [21], Theorem 5.1, it is enough to show the following:

- $h^1(\mathcal{I}_D(d-1)) = 0$.
- $h^2(\mathcal{I}_D(d-2)) = 0$.
- $h^3(\mathcal{I}_D(d-3)) = 0$.

Since $D$ is on $X^{(4)}$, we can consider the following S.E.S:\n
$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \mathcal{I}_D \rightarrow \mathcal{O}_{X^{(4)}}(-D) \rightarrow 0$$  \hspace{1cm} (4.1)

Since $h^i(\mathcal{O}_{\mathbb{P}^3}(d-5)) = 0$, for $i \in \{1, 2\}$ and $\mathcal{O}_{X^{(4)}}(D)$ is ACM, tensoring S.E.S 4.1 by $\mathcal{O}_{\mathbb{P}^3}(d-1)$ and taking long exact cohomology sequence, one obtains $h^1(\mathcal{I}_D(d-1)) = h^1(\mathcal{O}_{X^{(4)}}(-D)(d-1)) = 0$.

Since $h^i(\mathcal{O}_{\mathbb{P}^3}(d-6)) = 0$, for $i \in \{2, 3\}$, tensoring S.E.S 4.1 by $\mathcal{O}_{\mathbb{P}^3}(d-2)$ and taking long exact cohomology sequence, one obtains $h^2(\mathcal{I}_D(d-2)) = h^0(\mathcal{O}_{X^{(4)}}(D)(2-d)) = 0$.

Since $h^3(\mathcal{O}_{\mathbb{P}^3}(d-7)) = 0$, tensoring S.E.S 4.1 by $\mathcal{O}_{\mathbb{P}^3}(d-3)$ and taking long exact cohomology sequence, one obtains $h^3(\mathcal{I}_D(d-3)) = 0$.

Note that, the ACMness of the line bundle $\mathcal{O}_{X^{(d)}}(D)$ on $X^{(d)}$ follows from [21], Proposition 5.1. Therefore, we are now left to establish the initializedness of the line bundle $\mathcal{O}_{X^{(d)}}(D)$. Observe that, By Serre duality, it’s enough to show that $h^2(\mathcal{O}_{X^{(d)}}(-D)(d-3)) = 0$. Consider the following S.E.S:\n
$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-d) \rightarrow \mathcal{I}_D \rightarrow \mathcal{O}_{X^{(d)}}(-D) \rightarrow 0$$  \hspace{1cm} (4.2)

Since $h^i(\mathcal{O}_{\mathbb{P}^3}(d-3)) = 0$, for $i \in \{2, 3\}$, twisting the S.E.S 4.2 by $\mathcal{O}_{\mathbb{P}^3}(d-3)$ and taking long exact cohomology sequence, we have $h^2(\mathcal{O}_{X^{(d)}}(-D)(d-3)) = h^2(\mathcal{I}_D(d-3))$. Therefore, we need to show that $h^2(\mathcal{I}_D(d-3)) = 0$. Since $h^i(\mathcal{O}_{\mathbb{P}^3}(d-7)) = 0$, for $i \in \{2, 3\}$, twisting the S.E.S 4.1 by $\mathcal{O}_{\mathbb{P}^3}(d-3)$ and taking long exact cohomology sequence, we have $h^2(\mathcal{I}_D(d-3)) = h^0(\mathcal{O}_{X^{(4)}}(D)(3-d)) = 0$.

**Remark 4.3.** Although we expect that Lemma 4.2 can be used to guarantee the existence of several smooth hypersurfaces $X^{(d)} \subset \mathbb{P}^3$ for any $d > 4$ and some initialized and ACM line bundles on them, in this article, we limit our applications to the case $d = 6$. Also, note that, the case $d = 5$ is dealt in [21], Section 5.

With Lemma 4.2 at hand, we proceed to give several examples:

**Example 4.4.** Let $Y, C, D_1, D_2, D_3$ be as in [21], Example 5.2. Note that, one can choose a smooth curve $D_3 \in |3C - D_1|$. By [21], Proposition 5.2, we have the line bundles $\mathcal{O}_Y(D_i)$ ($i = 2, 3, 4$) are ACM on $Y$. Since $h^0(\mathcal{O}_Y(D_i)(-3)) = 0$, for $i \in \{2, 3, 4\}$, Lemma 4.2 applies and we
have the existence of smooth sextic hypersurfaces \( X_i \subset \mathbb{P}^3 \) containing \( D_i(i = 2, 3, 4) \) such that the line bundles \( \mathcal{O}_{X_i}(D_i) \) are initialized and ACM over \( X_i \). Finally, observe that:

- \( \mathcal{O}_{X_1}(D_2) \) satisfies the condition as in Theorem 1.1(iv)(a) (corresponding to the intersection number 3).
- \( \mathcal{O}_{X_1}(D_3) \) satisfies the conditions as in Theorem 1.1(vii) (corresponding to the intersection number 7).
- \( \mathcal{O}_{X_1}(D_4) \) satisfies the conditions as in Theorem 1.1(vi) (corresponding to the intersection number 11).

**Example 4.5.** Let \( Y, C, D_1, D_2 \) be as in [[21], Example 5.3]. Note that, one can choose a smooth curve \( D_3 \in [2C + D_1] \). By [[21], Proposition 5.2], we have the line bundles \( \mathcal{O}_Y(D_i) \) (\( i = 1, 2, 3 \)) are ACM on \( Y \). Since \( \mathcal{O}_Y(D_1) \) is also initialized, we have \( h^0(\mathcal{O}_Y(D_i)(-3)) = 0 \), for \( i \in \{1, 2, 3\} \). By Lemma 4.2, there exist smooth sextic hypersurfaces \( X_i \subset \mathbb{P}^3 \) containing \( D_i(i = 1, 2, 3) \) such that the line bundles \( \mathcal{O}_{X_i}(D_i) \) are initialized and ACM over \( X_i \). Finally, note the following:

- \( \mathcal{O}_{X_1}(D_1) \) satisfies the conditions as in Theorem 1.1(vi) (corresponding to the intersection number 4).
- \( \mathcal{O}_{X_1}(D_2) \) satisfies the conditions as in Theorem 1.1(viii) (corresponding to the intersection number 8).
- \( \mathcal{O}_{X_1}(D_3) \) satisfies the conditions as in Theorem 1.1(vi) (corresponding to the intersection number 12).

**Example 4.6.** Let \( Y, C, D_1, D_2 \) be as in [[21], Example 5.4]. Note that, one can choose a smooth curve \( D_3 \in [2C + D_1] \). By [[21], Proposition 5.2], we have the line bundles \( \mathcal{O}_Y(D_i) \) (\( i = 1, 2, 3 \)) are ACM on \( Y \). Since \( \mathcal{O}_Y(D_1) \) is initialized, we have \( h^0(\mathcal{O}_Y(D_i)(-3)) = 0 \), for \( i \in \{1, 2, 3\} \). By Lemma 4.2, there exist smooth sextic hypersurfaces \( X_i \subset \mathbb{P}^3 \) containing \( D_i(i = 1, 2, 3) \) such that the line bundles \( \mathcal{O}_{X_i}(D_i) \) are initialized and ACM over \( X_i \). Finally, we have the following:

- \( \mathcal{O}_{X_1}(D_1) \) satisfies the conditions as in Theorem 1.1(viii) (corresponding to the intersection number 6).
- \( \mathcal{O}_{X_1}(D_2) \) satisfies the conditions as in Theorem 1.1(vii) (corresponding to the intersection number 10).
- \( \mathcal{O}_{X_1}(D_3) \) satisfies the conditions as in Theorem 1.1(iv) (corresponding to the intersection number 14).

**Example 4.7.** For each \( 1 \leq i \leq 3 \), let \( Y_i, C_i, D_i, \tilde{D}_i \) be as in [[21], Example 5.5]. Note that, one can choose smooth curves \( E_i \in [2C_i + D_i] \), for \( i \in \{1, 2, 3\} \). By [[21], Proposition 5.2], we have the line bundles \( \mathcal{O}_{Y_i}(D_i), \mathcal{O}_{Y_i}(\tilde{D}_i), \mathcal{O}_{Y_i}(E_i) \) are ACM on \( Y_i(i = 1, 2, 3) \). Since \( \mathcal{O}_{Y_i}(D_i) \) (\( i = 1, 2, 3 \)) are initialized, we have \( h^0(\mathcal{O}_{Y_i}(D_i)(-3)) = 0, h^0(\mathcal{O}_{Y_i}(\tilde{D}_i)(-3)) = 0 \) and \( h^0(\mathcal{O}_{Y_i}(E_i)(-3)) = 0 \), for \( i \in \{1, 2, 3\} \). By Lemma 4.2, there exist smooth sextic hypersurfaces \( X_i, \tilde{X}_i, \tilde{X}'_i \subset \mathbb{P}^3 \) such that \( D_i \subset X_i, \tilde{D}_i \subset \tilde{X}_i, E_i \subset \tilde{X}'_i \) (\( i = 1, 2, 3 \)) and the line bundles \( \mathcal{O}_{X_i}(D_i), \mathcal{O}_{\tilde{X}_i}(\tilde{D}_i), \mathcal{O}_{\tilde{X}'_i}(E_i) \) are initialized and ACM over \( X_i, \tilde{X}_i \) and \( \tilde{X}'_i \) (\( i = 1, 2, 3 \)) respectively. Finally, one has the following:

- \( \mathcal{O}_{X_1}(D_1) \) satisfies the condition as in Theorem 1.1(iii)(a) (corresponding to the intersection number 2).
- \( \mathcal{O}_{\tilde{X}_1}(\tilde{D}_1) \) satisfies the conditions as in Theorem 1.1(vii) (corresponding to the intersection number 6).
\[ \mathcal{O}_{X'_1}(E_1) \text{ satisfies the conditions as in Theorem 1.1(vii)} \] (corresponding to the intersection number 10).

\[ \mathcal{O}_{X_2}(D_2) \text{ satisfies the conditions as in Theorem 1.1(v)(a)} \] (corresponding to the intersection number 3).

\[ \mathcal{O}_{X'_2}(D'_2) \text{ satisfies the conditions as in Theorem 1.1(viii)} \] (corresponding to the intersection number 7).

\[ \mathcal{O}_{X'_3}(D'_3) \text{ satisfies the conditions as in Theorem 1.1(vii)} \] (corresponding to the intersection number 11).

\[ \mathcal{O}_{X_3}(E_3) \text{ satisfies the conditions as in Theorem 1.1(vii)} \] (corresponding to the intersection number 5).

\[ \mathcal{O}_{X'_3}(D'_3) \text{ satisfies the conditions as in Theorem 1.1(viii)} \] (corresponding to the intersection number 9).

\[ \mathcal{O}_{X'_3}(E'_3) \text{ satisfies the conditions as in Theorem 1.1(v)} \] (corresponding to the intersection number 13).

**Notes**

1. Observe that, \( K_C \cong \mathcal{O}_C(d-3) \).
2. In this case, it is easy to see that we have, \( h^0(\mathcal{O}_C(D)) = h^0(\mathcal{O}_X(D)) = 1 \).
3. In this representation, we must have \( C.D \geq 1 \). Therefore, under this notation the following situations don’t occur:
   - For \( m = 0, j = 1, \ l \) can’t be 2 and for \( m = 0, j = 2, \ l \) can’t be 1 or 2.
   - For \( m = 1, j = 2, \ l \) can’t be 2.
4. For \( i = 1 \), this follows from initializedness and for \( i = 0 \) this follows by Remark 2.5.
5. For \( C.D = 7 \), this follows from \( C.(C - D) < 0 \), for \( C.D = 6 \), this follows from initializedness and for \( C.D = 5 \), this follows from Lemma 2.11.
6. Observe that, when \( P_x(D) = 2C.D - r \), then we always have this expression for some integer \( k \).
7. Note that, we have \( \eta^6 = \bar{\eta}^6 = -1 \).

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