Fundamental Strings as Black Bodies

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Abstract

We show that the decay spectrum of massive excitations in perturbative string theories is thermal when averaged over the (many) initial degenerate states. We first compute the inclusive photon spectrum for open strings at the tree level showing that a black body spectrum with the Hagedorn temperature emerges in the averaging. A similar calculation for a massive closed string state with winding and Kaluza-Klein charges shows that the emitted graviton spectrum is thermal with a “grey-body” factor, which approaches one near extremality. These results uncover a simple physical meaning of the Hagedorn temperature and provide an explicit microscopic derivation of the black body spectrum from a unitary $S$ matrix.
I. INTRODUCTION

High string excitations have a large degeneracy; the number of string states of a given mass increases exponentially in the excitation mass measured in string units (string tension or inverse string length). This degeneracy is consistent with the unitarity of the perturbative $S$ matrix. Unitarity guarantees that the analysis of final states that arise by the decay of a massive state must allow to retrieve the information on how the original massive state has been formed, i.e., on which is the coherent superposition of the many degenerate microstates that have been formed and subsequently decayed into the final states under analysis. A consistent and complete way of analysing final states is through semi-inclusive quantities, such as spectra, two particle correlations, three particle correlations, etc. The total set of these quantities is complete in the sense that it characterizes the initial state or, equivalently, the coherent superposition that created the final quantities in question. Semi-inclusive quantities are calculable in perturbative string theory (order by order in the string coupling) with a well-defined algorithm [1], which was extensively used in the past when string theory was being applied to strong interactions.

In this paper a remarkable property of the decay spectrum of a massive string state will be found: when averaging over all string excitations of mass $M$ ($M^2 \gg$ string tension), the decay spectrum exactly averages to a black body spectrum with Hagedorn temperature $T_H$. That the average procedure should wash out all the information stored in spectra and correlations leading to a total informationless thermal distribution was conjectured [2] in order to understand recently uncovered connections between string theory and black holes. For completeness we will briefly mention in section IV the possible relevance for black hole physics of this non-trivial property of a unitary microscopic quantum theory generating thermal properties in a decoherence procedure. The present results also unveil an important physical meaning of the Hagedorn temperature, as the radiation temperature of a macroscopic (averaged) string.
II. DECAY RATES IN OPEN STRING THEORY

The inclusive spectrum for the decay of a particular massive state $|\Phi\rangle$ of momentum $p_\mu = (M, \vec{0})$ into a particle with momentum $k_\mu$ ($k^2 = 0$) is represented by the modulus squared of the amplitude for the decay of $|\Phi\rangle$ into a state $|\Phi_X \otimes |k\rangle\rangle$, summed over all $|\Phi_X\rangle$. In string theory, the initial state $|\Phi\rangle$ will be described by a specific excitation of the string oscillators (for simplicity, we restrict the discussion to bosonic string theory). If $N_n$ is the occupation number of the $n$-th mode (with frequency $n$), a state $|\Phi\rangle$ of mass $M$ will be characterized by a partition $\{N_n\}$ of $N = \sum_{n=1}^{\infty} n N_n$, with

$$\hat{N} |\Phi_{(N_n)}\rangle = N |\Phi_{(N_n)}\rangle, \quad \hat{N} = \sum_{n=1}^{\infty} n a_n^\dagger a_n, \quad \alpha' M^2 = N - 1, \quad [a_n^\mu, a_m^\nu] = \delta_{nm}\eta^\mu\nu, \quad (2.1)$$

where $\mu, \nu = 1, ..., D$. For large $N$, there are $\{\mathcal{N}_N\}$ possible partitions, thus representing the degeneracy of the level $N$ associated with excitations of mass $M$. The total momentum $p'_\mu$ of $\Phi_X$ will be $p'_\mu = p_\mu - k_\mu$. At the tree level, $|\Phi_X\rangle$ will also be a superposition of string excitations, satisfying $\hat{N} |\Phi_X\rangle = N' |\Phi_X\rangle$,

$$N' = \alpha' p'^2 + 1 = N - 2k_0 \sqrt{\alpha'(N - 1)}, \quad (2.3)$$

Let us momentarily ignore twists or non-planar effects of open string theories $\{\mathcal{N}_N\}$. They will be incorporated later. The inclusive photon spectrum for the decay of a state $\Phi_{(N_n)}$ is then given by

$$d\Gamma_{\Phi_{(N_n)}}(k_0) = \sum_{\Phi_X|N'} |\langle \Phi_X|V(k)|\Phi_{(N_n)}\rangle|^2$$

$$\times V(S^{D-2}) k_0^{D-3} dk_0, \quad V(S^{D-2}) = \frac{2\pi^{D-1}}{\Gamma(D-1)}, \quad (2.4)$$

where $\sum_{\Phi_X|N'}$ means the sum over all states $\Phi_X$ satisfying the mass condition (2.3), and $V(k)$ is the photon vertex operator.
\( V(k) = \xi_\mu \partial_\tau X^\mu(0)e^{ik.X(0)}, \quad \xi_\mu k^\mu = 0, \quad k^2 = 0, \quad (2.5) \)

\[
X^\mu(0) = \hat{x}^\mu + i \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (a_n^\mu - a_n^{\dagger \mu}),
\]

\[
\partial_\tau X^\mu(0) = \hat{p}^\mu + \sum_{n=1}^{\infty} \sqrt{n} (a_n^{\dagger \mu} + a_n^\mu), \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta_\mu^\nu,
\]

where \( \xi_\mu \) is the photon polarization vector, and we have set \( \alpha' = 1/2 \). The \( \delta \)-function of energy-momentum conservation, coming from the zero-mode part of the amplitude, has been factored out from (2.4). Let us define a projector \( \hat{P}_N \) over states of level \( N \) as

\[
\hat{P}_N = \oint_C \frac{dz}{z} z^{-N}, \quad \hat{P}_N|\Phi_{N'}\rangle = \delta_{NN'}|\Phi_{N'}\rangle, \quad (2.6)
\]

where \( C \) is a small contour around the origin. Introducing \( \hat{P}_{N'} \) in (2.4) we can convert the sum into a sum over all physical states \( |\Phi_X\rangle \), and use completeness to write

\[
F_{\Phi_{\{Nn\}}} = \oint_C \frac{dz}{z} z^{-N} \oint_C \frac{dz'}{z'} z'^{-N'} \langle \Phi_{\{Nn\}}|V(-k)z'\hat{N}V(k)|\Phi_{\{Nn\}}\rangle, \quad (2.7)
\]

\[
F_{\Phi_{\{Nn\}}}(k) \equiv \frac{k_0^{3-D}}{V(S^{D-2})} \frac{d\Gamma_{\Phi_{\{Nn\}}}}{dk_0}. \quad (2.8)
\]

Let us now introduce a coherent set basis,

\[
|\{\lambda_n\}\rangle = \prod_{n=1}^{\infty} \exp \{\lambda_n \cdot a_n^\dagger\}|0; p_\mu\rangle, \quad (2.9)
\]

where \( \lambda_n, \ n = 1, ..\infty, \) is a set of arbitrary (complex) parameters, and define

\[
F_{\lambda^*\lambda} = \oint_C \frac{dz}{z} z^{-N} \oint_{C'} \frac{dz'}{z'} z'^{-N'} \\
\times \langle \{\lambda_n\}|W(-k)z'\hat{N}W(k)z\hat{N}|\{\lambda_n\}\rangle, \quad (2.10)
\]

where \( W(k) \equiv \exp[\xi \cdot \partial_\tau X^\mu]\exp[ik.X] \). From standard properties of coherent states \[3,4\], we find

\[
F_{\lambda^*\lambda} = \oint_C \frac{dz}{z} z^{-N} \oint_{C'} \frac{dz'}{z'} z'^{-N'} e^{p_\mu(\xi_1+\xi_2)} \prod_{n=1}^{\infty} \exp[K_n \lambda_n \cdot \lambda_n^* z^n z'^n + \lambda_n \cdot J_n + \lambda_n \cdot J_n^*], \quad (2.11)
\]
\[ J_{\mu} = \sqrt{n}(\xi_2^\mu + \xi_1^\mu z^n) - \frac{1}{\sqrt{n}} k^\mu (1 - z^n), \]
\[ J_{\mu}^* = \sqrt{n}(\xi_2^\mu z^n z + \xi_1^\mu z^n) - \frac{1}{\sqrt{n}} k^\mu z^n (1 - z^n), \]
\[ K_n = \xi_1 \cdot \xi_2 n z^n. \] (2.12)

The spectrum \( F_{\Phi(N_n)}(k) \) may be directly computed from (2.7) using standard operator techniques or from the explicit expression (2.11), (2.12), and (we omit Lorentz indices)
\[ F_{\Phi(N_n)}(k) = \prod_{n=1}^{\infty} \left( \frac{\partial}{\partial \lambda_n} \right)^{N_n} \left( \frac{\partial}{\partial \lambda_n^*} \right)^{N_n} \left. F_{\lambda}, \lambda^* \right|_{\lambda_n = \lambda_n^* = 0} \text{linear in } \xi_1 \xi_2, \] (2.13)
which follows after inverting (2.9) and noticing that \( V(k) \) is represented by the linear term in \( \xi \) of \( W(k) \). For every \( |\Phi(N_n)\rangle \), the \( F_{\Phi(N_n)}(k) \) are easily obtained and are polynomials in \( k_\mu \).

The spectrum obtained by averaging over string states with mass \( M \) is
\[ F_N(k_0) = \frac{1}{N_N} \sum_{\Phi|N} F_{\Phi(N_n)}(k_0). \] (2.14)

One can introduce again the projector (2.6) to convert the above sum into a sum over all states, so that
\[ F_N(k_0) = \frac{1}{N_N} \oint_{C_1} \oint_{C_2'} d\lambda d\lambda^* e^{-\lambda^* \cdot \lambda} \text{Tr} [z^N V(-k) z^{N'} V(k)] . \] (2.15)

The trace in (2.13) can then be computed from (2.11) by integrating over \( \lambda_n, \lambda_n^* \),
\[ F_N(k_0) = \frac{1}{N_N} \prod_{n, \mu} d\lambda_n^\mu d\lambda_n^{* \mu} e^{-\lambda_n^* \cdot \lambda_n} \left| \xi_1 \xi_2 \right. \right|_{\xi_1 \xi_2}, \] (2.16)
where \( |\xi_1 \xi_2 \rangle \) means the term linear in \( \xi_1 \mu, \xi_2 \nu \), and then setting \( \xi_1 \mu = \xi_2 \mu \). Using (2.11) we get
\[ F_N(k_0) = \frac{e^{p(\xi_1 + \xi_2)}}{N_N} \oint_{C_1} \oint_{C_2'} d\lambda d\lambda^* \left. e^{-\lambda^* \cdot \lambda} \right|_{\xi_1 \xi_2} , \]
\[ f(w) = \prod_{n=1}^{\infty} (1 - w^n), \quad w = z^{n} , \quad v = z^{n'}. \] (2.17)

\( C_1 \) and \( C_2 \) are two small contours around zero. The extra factor \( f^2(w) \) in the integrand arises as usual by incorporating the ghost contribution (the \( D - 2 \) power corresponds to the \( D - 2 \) transverse coordinates). Thus we obtain
\[ F_N(k_0) = \frac{1}{N_N^N} \xi_\mu \xi_\nu \oint_{C_1} dw \frac{w^{-N} f(w)^{2-D}}{w} \oint_{C_2} dv v^{N-N'} \times (p_\mu p_\nu + \eta_{\mu \nu} \Omega(v, w)) , \]  
\[ \Omega(v, w) = \sum_{n=1}^{\infty} \left( v^n + \frac{w^n(v^n + v^{-n})}{1 - w^n} \right) . \]  

The integral over \( v \) gives \((N' < N)\)

\[ F_N(k_0) = \frac{\xi^2 (N - N')}{N_N^N} \oint dw \frac{w^{-N'} f(w)^{2-D}}{1 - w^{N-N'}} . \]  

For large \( N' \), the integral in (2.20) can be computed by a saddle point approximation, with the main contribution coming from \( w \sim 1 \). This is similar to the familiar calculation of the number of states \( N_N \) of level \( N \). The behavior of \( f(w) \) at \( w \equiv 1 \) is well known and given by

\[ f(w) \approx \text{const.} \ (1 - w)^{-1/2} e^{-\frac{1}{\alpha'^2}} . \]  

Therefore the saddle point is at \( \ln w \approx -a/2\sqrt{N'} \). Using \( N' = N - 2k_0\sqrt{\alpha'^N} \) we obtain for large \( N' \)

\[ F_N(k_0) \approx \xi^2 2k_0\sqrt{\alpha'^N} e^a\sqrt{N-2k_0\sqrt{\alpha'^N} - a\sqrt{N}} \]  
\[ \frac{e^{-a\sqrt{N-2k_0\sqrt{\alpha'^N} - a\sqrt{N}}}}{1 - e^{-a\sqrt{N-2k_0\sqrt{\alpha'^N} - a\sqrt{N}}}} . \]  

We have set \((N/N')^{D+1} \approx 1\), since in the regime of interest, \( k_0 \ll M \), \( N/N' \approx 1 \) (the emission of a photon with energy \( k_0 \) of order \( M \) and higher is suppressed by a factor of order \( e^{-\sqrt{N}} \)). Thus \( \sqrt{N'} - \sqrt{N} \approx -k_0\sqrt{\alpha'} \) and

\[ F_N(k_0) \approx \xi^2 2\alpha'k_0M \frac{e^{-a\sqrt{\alpha'}}}{1 - e^{-a\sqrt{\alpha'}}} , \]  

or

\[ d\Gamma_N(k_0) \approx \text{const.} \ \frac{e^{-\frac{k_0}{\sqrt{\alpha'}}}}{1 - e^{-\frac{k_0}{\sqrt{\alpha'}}}} k_0^{D-2} dk_0 \]  

where \( T_H \) is the Hagedorn temperature \( T_H = \frac{1}{a\sqrt{\alpha'}} \). Thus the radiation spectrum from a macroscopic (i.e. averaged over all degenerate microscopic states) string of mass \( M \) is exactly thermal (despite the spectrum for each microstate being absolutely non-thermal).
The way the exact Planck formula arises in the final result is striking, since the exponents in the numerator and denominator have different origins.

Let us now show that the non-planar contribution does not change the above result. It arises in open string theory due to the fact that the photon may be emitted by any of the two ends of the string. So, strictly speaking, we should have written

$$d\Gamma_{\Phi_{\{N_n\}}}(k_0) = \sum_{\Phi_{X}|_{N'}} |\langle \Phi_X|\frac{1}{\sqrt{2}}(V(k) + \Theta V(k)\Theta)|\Phi_{\{N_n\}}\rangle|^2$$

$$\times V(S^{D-2})k_0^{D-3}dk_0,$$

(2.25)

at the place of (2.4), where \(\Theta\) is the twist operator [3]. The modulus squared involving \(V(k)\) –and that with \(\Theta V(k)\Theta\)– leads each to half of the planar result (2.24) obtained before. The cross product gives rise to the non-planar contribution that may be computed analogously to the planar loop diagram, of which our spectrum represents the absorptive part. Instead of (2.15) for the non-planar contribution one has

$$F_{N}(k_0)_{n.p.} = \frac{1}{N_N} \oint_{C_1} \oint_{C'} dw \int_{1-w}^{w} \int_{0}^{\varepsilon} dz' \int_{1-z'}^{1-z} z'^{N-N'}$$

$$\times \text{Tr} [z^{N}\Theta V(-k)\Theta z^{N}V(k)].$$

(2.26)

Using the explicit expression for \(\Theta\) [3] and computing the trace [3] one finally obtains

$$F_{N}(k_0)_{n.p.} = \frac{1}{N_N} \xi_{\mu}\xi_{\nu} \oint_{C_1} \oint_{C'} dw \int_{1-w}^{w} \int_{0}^{\varepsilon} dz' \int_{1-z'}^{1-z} z'^{N-N'}$$

$$\times (p_{\mu}p_{\nu} + \eta_{\mu\nu}\Omega(v, w)),$$

(2.27)

where \(\Omega\) is given by eq. (2.19) with \(v = \frac{w-z'}{1-z'}\). The contour integral in \(z'\) now selects the pole at \(z' = w\). The remaining integral over \(w\) is dominated by the same saddle point that allowed the evaluation of (2.20). The final result is

$$F_{N}(k_0)_{n.p.} \simeq \frac{\xi^2a}{2\sqrt{N}} e^{-ak_0\sqrt{\varepsilon}}$$

(2.28)

which is smaller by a factor \(< 1/N\) with respect to the planar result (2.23) or (2.24) that dominates the sum (2.25).
III. SPECTRUM OF A CHARGED CLOSED STRING STATE

Let us now consider closed bosonic string theory with one dimension compactified on a circle of radius $R$. Let $m_0$ and $w_0$ be the integers representing the Kaluza-Klein momentum and winding number of the string state along the circle (we will assume $m_0, w_0 > 0$). The mass spectrum is given by

$$\alpha' M^2 = 4(\hat{N}_R - 1) + \alpha' Q^2_+ = 4(\hat{N}_L - 1) + \alpha' Q^2_- ,$$

where $Q_{\pm} = \frac{m_0}{R} \pm \frac{w_0 R}{\alpha'}$, Consider the decay rate of a state of level $N_L, N_R$ of given charges $m_0, w_0$, with $N_L - N_R = m_0 w_0$, by averaging over all physical states of charge $m_0, w_0$ and level $N_L, N_R$. The calculation factorizes in a left and right part, and it is similar as in the previous section. We assume that $N'_R$ and $N'_L$ are sufficiently large for the saddle-point approximation to apply. The final result is

$$d\Gamma_{N_R N_L}(k_0) = \text{const.} \xi^2 M^2 \frac{e^{-\frac{k_0}{T_R}}}{1 - e^{-\frac{k_0}{T_R}}} \frac{e^{-\frac{k_0}{T_L}}}{1 - e^{-\frac{k_0}{T_L}}} \times V(S^{D-2}) k_0^{D-1} dk_0 ,$$

$$T_R = \frac{2\sqrt{M^2 - Q^2_+}}{a\sqrt{\alpha'^2} M}, \quad T_L = \frac{2\sqrt{M^2 - Q^2_-}}{a\sqrt{\alpha'^2} M} .$$

Equation (3.2) is remarkably similar to the analogous result derived for D-branes in \[\text{[3]}\] (see also \[\text{[3]}\] ), despite the two calculations being rather different, and describing different physical systems. Indeed, the D-brane calculation involves only a single oscillator $a, a^\dagger$ that excites modes with one unit of Kaluza-Klein momentum \[\text{[3]}\]; the present elementary string calculation involves all excitations $a_n, a^\dagger_n, \ n = 1, ..., \infty$, in a very specific way dictated by the vertex insertion.

By analogy with black holes \[\text{[7]}\], eq. (3.2) may be written as

$$d\Gamma_{N_R N_L}(k_0) = \text{const.} \sigma(k_0) \frac{e^{-\frac{k_0}{T}}}{1 - e^{-\frac{k_0}{T}}} k_0^{D-2} dk_0 ,$$

where $T^{-1} = T_R^{-1} + T_L^{-1}$ and $\sigma(k_0)$ is the “grey body” factor

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\[ \sigma(k_0) = k_0 \frac{1 - e^{-k_0}}{(1 - e^{-k_0})(1 - e^{-k_0/T_L})(1 - e^{-k_0/T_R})} \quad (3.5) \]

The radiation vanishes if the initial string state saturates the BPS condition \( M \geq Q_+ \), i.e. when \( M = Q_+ \). For a near BPS state, \( M \approx Q_+ \), \( N_L \gg N_R \), one gets

\[ T \approx T_R = \frac{2}{a\sqrt{\alpha'}} \frac{\sqrt{M^2 - Q_+^2}}{M} \quad (3.6) \]

In the superstring theory, owing to the supersymmetry of the BPS state, corrections to the "free string" picture are expected to be smaller for near-BPS configurations, which might allow to extrapolate eq. (3.6) from the weak to the strong coupling regime [8].

**IV. DISCUSSION**

The appearance of the Hagedorn temperature characterizing the radiation in the decay of a massive macrostate in string theory at zero temperature sheds light on an old problem concerning the physical interpretation of the Hagedorn temperature. The Hagedorn temperature was traditionally interpreted as the temperature at which the canonical thermal ensemble of a string gas breaks down. The technical reason is simple –at higher temperatures the integral defining the thermal partition function diverges– and it is related to the fast growing of the level density with mass. The question is what is the physical picture behind this instability. The present results lead to the following interpretation (this seems consistent with other suggestions [9,10]). When an open string in a macrostate with \( N \gg 1 \) is placed in a thermal bath of temperature \( T_{\text{bath}} < T_H \), the string will decay by emitting massless (as well as massive) particles, as a black body of temperature \( T_H \), until the level decreases to a point where the saddle-point prediction \( T_{\text{string}} = T_H \) ceases to apply, and the real temperature of the string is of order \( T_{\text{bath}} \). A situation of equilibrium occurs immediately if one places the string in a thermal bath with \( T_{\text{bath}} = T_H \). When \( T_{\text{bath}} > T_H \), the energy of the string increase endlessly, since it can only emit at a fixed rate determined by the temperature \( T_H \). The system becomes highly unstable, and there cannot be any
thermal equilibrium. The canonical thermal ensemble must therefore break down precisely at $T = T_H$.

It is interesting to witness how these statistical or classical concepts stem from microstate computations through average procedures (decoherence). The string scale is of course present in the spectrum and in the correlations of single microstates, but it is only after averaging that appears as the temperature characterizing the emission. The emergence of classical properties in a computable weak coupling string regime was indeed one of the motivation for this investigation. It substantiates the conjecture [2] that a similar mechanism is at work at large couplings in order to generate from string theory classical concepts such as space-time, causal relations and black hole physics.

Finally, it would be interesting to investigate possible cosmological implications related to the decay spectrum and lifetime of cosmic strings [3,4].

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