ROBUST BPX PRECONDITIONER FOR FRACTIONAL LAPLACIANS
ON BOUNDED LIPSCHITZ DOMAINS

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Abstract. We propose and analyze a robust BPX preconditioner for the integral fractional Laplacian on bounded Lipschitz domains. For either quasi-uniform grids or graded bisection grids, we show that the condition numbers of the resulting systems remain uniformly bounded with respect to both the number of levels and the fractional power. The results apply also to the spectral and censored fractional Laplacians.

1. Introduction

Given \( s \in (0, 1) \), the fractional Laplacian of order \( s \) in \( \mathbb{R}^d \) is the pseudodifferential operator with symbol \( |\xi|^{2s} \). That is, denoting the Fourier transform by \( \mathcal{F} \), for every function \( v : \mathbb{R}^d \to \mathbb{R} \) in the Schwartz class \( \mathcal{S} \) it holds that
\[
\mathcal{F} ((-\Delta)^s v) (\xi) = |\xi|^{2s} \mathcal{F}(v)(\xi).
\]
Upon inverting the Fourier transform, one obtains the following equivalent expression:
\[
(-\Delta)^s v(x) = C(d, s) \text{ p.v.} \int_{\mathbb{R}^d} \frac{v(x) - v(y)}{|x - y|^{d+2s}} dy, \quad C(d, s) = \frac{2^{2s} s! (s + \frac{d}{2})}{\pi^{d/2} \Gamma(1 - s)}.
\]
The constant \( C(d, s) \simeq s(1 - s) \) compensates the singular behavior of the integrals for \( s \to 0 \) (as \( |y| \to \infty \)) and for \( s \to 1 \) (as \( y \to x \)), and yields \( \lim_{s \to 0} (-\Delta)^s v(x) = v(x), \quad \lim_{s \to 1} (-\Delta)^s v(x) = -\Delta v(x), \quad \forall v \in C^\infty_0(\mathbb{R}^d) \).

From a probabilistic point of view, the fractional Laplacian is related to a simple random walk with arbitrarily long jumps \([48]\), and is the infinitesimal generator of a \( 2s \)-stable process \([6]\). Thus, the fractional Laplacian has been widely utilized to model jump processes arising in social and physical environments, such as finance \([21]\), predator search patterns \([46]\), or ground-water solute transport \([5]\).

There exist several nonequivalent definitions of a fractional Laplace operator \( (-\Delta)^s \) on a bounded domain \( \Omega \subset \mathbb{R}^d \) (see \([8, 9]\)). Our emphasis is on the homogeneous Dirichlet problem for the integral Laplacian: given \( f : \Omega \to \mathbb{R} \), one seeks \( u : \mathbb{R}^d \to \mathbb{R} \) such that
\[
\begin{align*}
(-\Delta)^s u &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega^c,
\end{align*}
\]
where the pointwise definition of \( (-\Delta)^s u(x) \) is given by \([1, 2]\) for \( x \in \Omega \). Consequently, the integral fractional Laplacian on \( \Omega \) maintains the probabilistic interpretation and corresponds to a killed Lévy process \([6, 19]\). It is noteworthy that, as the underlying stochastic process admits jumps of arbitrary length, volume constraints for this operator need to be defined in the complement of the domain \( \Omega \).

Weak solutions to \( (1.4) \) are the minima of the functional \( v \mapsto \frac{1}{2} \| v \|^2_{H^s(\mathbb{R}^d)} - \int_{\Omega} f v \) on the zero-extension space \( H^s(\Omega) \) (see Section 2.1). In accordance with \( (1.3) \) restricted to any

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Consider a discretization of (1.4) using standard linear Lagrangian finite elements (see details in Section 2.4) on a mesh \( T \) whose elements have maximum and minimum size \( h_{\text{max}} \) and \( h_{\text{min}} \) respectively, and denote by \( A \) the corresponding stiffness matrix. Then, as shown in [4], the condition number of \( A \) obeys the relation

\[
\text{cond}(A) \approx (\dim \mathcal{V}(T))^{2s/d} \left( \frac{h_{\text{max}}}{h_{\text{min}}} \right)^{d-2s}
\]

for \( 0 < s < 1 \) with \( 2s \leq d \), and one can remove the factor involving \( \frac{h_{\text{max}}}{h_{\text{min}}} \) by preconditioning \( A \) by a diagonal scaling. On non quasi-uniform grids, the hidden constant in the critical case \( 2s = d \) is worse by a logarithmic factor.

In recent years, efficient finite element discretizations of (1.4) have been examined in several papers. Adaptive algorithms have been considered in [23, 22, 29], and an a posteriori error analysis has been addressed in [32, 27]. Standard finite element discretizations of the fractional Laplacian give rise to full stiffness matrices; matrix compression techniques have been proposed and studied in [57, 3, 35]. For the efficient resolution of the discrete problems, operator preconditioners have been considered in [30].

In this work, we propose a multilevel BPX preconditioner (cf. [49, 15]) \( B \) for the solution of (1.4), that yields \( \text{cond}(BA) \lesssim 1 \). In general, our result follows from the general theory for multigrid preconditioners (cf. [50, 31, 51, 55]). An important consequence of (1.5) and (1.6) is that, on any given grid, the stiffness matrices associated with integral fractional Laplacians of order \( s \) approach either the standard mass matrix (as \( s \to 0 \)) or the stiffness matrix corresponding to the Laplacian (as \( s \to 1 \); the latter because the canonical basis functions of \( \mathcal{V}(T) \) are Lipschitz and \( W_0^{1,\infty}(\Omega) \subset H^s(\Omega) \). This is consistent with (1.7): for example, on quasi-uniform grids of size \( h \), such a formula yields \( \text{cond}(A) \approx h^{-2s} \).

Based on the above observations, one of our main goals is to obtain a preconditioner that is uniform with respect to \( s \) as well as with respect to the number of levels \( J \). For such a purpose, we need to weight the contributions of the coarser levels differentially to the finest level. On a family of quasi-uniform grids \( \{ T_k \}_{k=0}^J \) with size \( h_k \), we shall consider a preconditioner in the operator form (cf. (1.4) below)

\[
\mathcal{B} = T_J h_{J}^{2s} Q_J + (1 - \tilde{\gamma}^s) \sum_{k=0}^{J-1} T_k h_k^{2s} Q_k,
\]

where the arbitrary parameter \( \tilde{\gamma} \in (0, 1) \). Above, \( Q_k \) and \( T_k \) are suitable \( L^2 \)-projection and inclusion operators, respectively. Clearly, if \( s \in (0, 1) \) is fixed, then the factor \( 1 - \tilde{\gamma}^s \) is equivalent to a constant. However, such a factor tends to 0 as \( s \to 0 \), and this correction is fundamental for the resulting condition number to be uniformly bounded with respect to \( s \).

We now present a simple numerical example to illustrate this point. Let \( \Omega = (-1, 1)^2 \), \( f = 1 \), \( s = 10^{-1}, 10^{-2} \), and choose either \( \tilde{\gamma} = 0 \) (i.e., no correction) and \( \tilde{\gamma} = \frac{1}{2} \) in the preconditioner above to compute finite element solutions to (1.4) on a sequence of nested grids. The left panel in Table 1 shows the number of iterations needed to solve the resulting linear system by using a Preconditioned Conjugate Gradient (PCG) method with a fixed tolerance. It is apparent that setting \( \tilde{\gamma} = \frac{1}{2} \) gives rise to a more robust behavior with respect to either \( s \) and the number of levels \( J \).

Another aspect to take into account in our problem is the low regularity of solutions. As we discuss in Section 2.4 by using uniform grids one can only expect convergence in the energy norm with order \( O \left( \dim \mathcal{V}(T)^{-1/(2d)} \right) \) (up to logarithmic factors) independently of
the smoothness of the data. The reason for such a low regularity of solutions is boundary behavior; exploiting the a priori knowledge of this behavior by means of suitably refined grids leads to convergence with order $O(d)$ if $d \geq 2$ and $O(\dim V(T)^{s-2})$ if $d = 1$. In spite of this advantage, graded grids give rise to worse-conditioned matrices, as described by (1.7). This work also addresses preconditioning on graded bisection grids, that can be employed to obtain the refinement as needed. Our algorithm on graded bisection grids builds on the subspace decomposition introduced in [18], that leads to optimal multilevel methods for classical ($s = 1$) problems. Our theory on graded bisection grids, however, differs from [18] to account for the uniformity with respect to $s$. As illustrated by the right panel in Table 1, including a correction factor on the coarser scales leads to a more robust preconditioner.

The integral (or restricted) Laplacian operator (1.2) turns out to be spectrally equivalent to the spectral Laplacian uniformly with respect to $s$ on bounded Lipschitz domains [19]. This property is also a consequence of our multilevel space decomposition. Such a spectral equivalence can be extended to the censored (or regional) Laplacian for $s \in (\frac{1}{2}, 1)$, and the uniformity of equivalence constant holds when $s \to 1$. However, the three operators have a strikingly different boundary behavior [8, 33, 44]. We present their definitions along with their properties in Section 4.3. Consequently, the BPX preconditioner (1.8) for the integral Laplacian on quasi-uniform grids and its counterpart on graded bisection grids apply as well to the spectral and censored Laplacians except for the censored one when $s \to \frac{1}{2}$. Further, the uniformity with respect to $s$ holds for both the integral or spectral Laplacians.

This paper is organized as follows. Section 2 collects preliminary material about problem (1.4), in particular regarding its variational formulation, regularity of solutions and its approximation by the finite element method. Next, in Section 3 we review some additional tools that we need to develop the theory of a robust BPX preconditioner; we discuss general aspects of the method of subspace corrections and introduce an $s$-uniform decomposition that plays a central role in our analysis. We introduce a BPX preconditioner for quasi-uniform grids in Section 4 and prove that it leads to condition numbers uniformly bounded with respect to the number of refinements $J$ and the fractional power $s$. Afterwards, we delve into the preconditioning of systems arising from graded bisection grids. For that purpose, Section 5 offers a review of the bisection method with novel twists, while Section 6 proposes and studies a BPX preconditioner on graded bisection grids. Section 7 presents some numerical experiments that illustrate the performance of the BPX preconditioners. The paper concludes with three Appendices that collect and prove a few technical results.

2. Preliminaries

In this section we set the notation used in the rest of the paper regarding Sobolev spaces and recall some preliminary results about their interpolation. We are particularly concerned with the zero-extension Sobolev space $\tilde{H}^s(\Omega)$ := $C_0^\infty(\Omega)^{1+\|\cdot\|_{H^s(\Omega)}}$, which is the set of functions in $H^s(\mathbb{R}^d)$ whose support is contained in $\Omega$. Given $u, v \in \tilde{H}^s(\Omega)$, we define

| Uniform grids | Graded bisection grids |
|---------------|------------------------|
| **DOFs** | **s = 10^{-1}** | **s = 10^{-2}** | **s = 10^{-1}** | **s = 10^{-2}** |
| $\gamma = 0$ | $\gamma = \frac{1}{2}$ | $\gamma = 0$ | $\gamma = \frac{1}{2}$ | $\gamma = 0$ | $\gamma = \frac{1}{2}$ |
| 225 | 14 | 10 | 10 | 16 | 10 | 11 | 9161 | 17 | 10 | 10 | 18 | 10 | 13 | 9397 | 19 | 10 | 21 | 10 | 2265 | 20 | 12 | 12 | 19 | 13 | 14 |

Table 1. Number of iterations needed when using a PCG method with BPX preconditioner without ($\tilde{\gamma} = 0$) and with ($\tilde{\gamma} = \frac{1}{2}$) a correction factor.
below the (scaled) inner product \((u, v)_\sigma = (u, v)_{\tilde{H}^\sigma(\Omega)}\) in \(\tilde{H}^\sigma(\Omega)\), the corresponding norm \(|u|_{\sigma} = |u|_{\tilde{H}^\sigma(\Omega)}\), and let \(\|u\|_0 := \|u\|_{L^2(\Omega)}\). Moreover, we discuss regularity of solutions to (1.4) and a priori error estimates for finite element approximations.

For convenience, we write \(X \lesssim Y\) (resp. \(X \gtrsim Y\)) to indicate \(X \leq CY\) (resp. \(CX \geq Y\)), where \(C\) denotes, if not specified, a generic positive constant that may stand for different values at its different occurrences but is independent of the number of levels or fractional power. The notation \(X \simeq Y\) means both \(X \lesssim Y\) and \(X \gtrsim Y\) hold.

2.1. Variational formulation. The natural setting to study the variational formulation of fractional diffusion problems such as (1.4) is in Sobolev spaces \(\tilde{H}^\sigma(\Omega)\) of non-integer order \(\sigma\). We refer to [1] for basic definitions and the notation we use here. We consider the symmetric bilinear form \((\cdot, \cdot)_\sigma: \tilde{H}^\sigma(\Omega) \times \tilde{H}^\sigma(\Omega) \to \mathbb{R},\)

\[
(u, v)_\sigma := \frac{C(d, \sigma)}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+2\sigma}} \, dx \, dy,
\]

where \(C(d, \sigma)\) is the constant from (1.2). We point out that, because functions in \(\tilde{H}^\sigma(\Omega)\) vanish in \(\Omega^c\), the integration takes place in \((\Omega \times \mathbb{R}^d) \cup (\mathbb{R}^d \times \Omega)\).

Since a Poincaré inequality is valid in \(\tilde{H}^s(\Omega)\) (cf. [1, Prop. 2.4], for example), the map \(u \mapsto (u, u)_s\) is an inner product \(H^s(\Omega)\). Given \(f \in H^{-s}(\Omega)\), the dual of \(H^s(\Omega)\), the weak formulation of the homogeneous Dirichlet problem (1.4) reads: find \(u \in \tilde{H}^s(\Omega)\) such that

\[
a(u, v)_s := (f, v)_{s, \Omega} \quad \forall v \in \tilde{H}^s(\Omega),
\]

where \((\cdot, \cdot)_{s, \Omega}\) stands for the duality pairing between \(H^{-s}(\Omega)\) and \(\tilde{H}^s(\Omega)\). Existence and uniqueness of solutions of (2.2) is a consequence of the Riesz representation theorem.

2.2. Interpolation and fractional Sobolev spaces. An important feature of the fractional Sobolev scale is that it can be equivalently defined by interpolation of integer-order spaces. This along with the observation that the norm equivalence constants are uniform with respect to \(s\) is fundamental for our work. In view of applications below, we now recall the abstract setting for two Hilbert spaces \(X^1 \subset X^0\) with \(X^1\) continuously embedded and dense in \(X^0\). Following [36, Section 2.1], the inner product in \(X^1\) can be represented by a self-adjoint and coercive operator \(S: D(S) \to X^0\) with domain \(D(S) \subset X^1\) dense in \(X_0\), i.e. \((v, w)_{X^1} = (Sv, w)_{X^0}\) for all \(v \in D(S), w \in X^1\). Invoking the spectral decomposition of self-adjoint operators [56], we let \(\Lambda: X^1 \to X^0\) be the square root of \(S\), which in turn is self-adjoint, coercive, and satisfies

\[
(v, w)_{X^1} = (\Lambda v, \Lambda w)_{X^0} \quad \forall v, w \in X^1.
\]

Suppose further that the spectrum \(\{\lambda_k\}_{k=1}^\infty\) of \(\Lambda\) is discrete and the corresponding eigenvectors \(\{\phi_k\}_{k=1}^\infty\) form a complete orthonormal basis for \(X^0\); hence \(\Lambda v = \sum_{k=1}^\infty \lambda_k \phi_k v \phi_k\) for all \(v = \sum_{k=1}^\infty \phi_k v \phi_k \in X^1\). Then, we can define a fractional power \(s \in (0, 1)\) of \(\Lambda\) as follows:

\[
\Lambda^s v := \sum_{k=1}^\infty \lambda_k^s \phi_k v \phi_k \quad \text{if} \quad \|\Lambda^s v\|_{X^0}^2 := \sum_{k=1}^\infty \lambda_k^{2s} \phi_k^2 < \infty.
\]

On the other hand, following [38, Appendix B], we consider a variant of the classical K-method which, for simplicity, we write for \(L^2\)-based interpolation. Decompose \(v = v^0 + v^1\) with \(v^0 \in X^0, v^1 \in X^1\), take \(t > 0\), and set

\[
K_2(t, v) := \inf_{v^0 \in X^0, v^1 \in X^1} \left( \|v^0\|_{X^0}^2 + t^2 \|v^1\|_{X^1}^2 \right)^{\frac{1}{2}}.
\]

It is immediate to verify that \(K_2\) is equivalent to the usual \(K\)-functional:

\[
K_2(t, v) \leq K(t, v) \leq \sqrt{2} K_2(t, v) \quad \forall v, t.
\]
Given $s \in (0, 1)$, we consider the interpolation space $(X^0, X^1)_{s,2}$ with norm
\begin{equation}
\|v\|_{(X^0, X^1)_{s,2}} := \left( \frac{2 \sin(\pi s)}{\pi} \int_0^\infty t^{-1-2s} K_2(v, t)^2 dt \right)^\frac{1}{2}.
\end{equation}

The following theorem gives an intrinsic spectral equivalence between the interpolation by $K$-method and spectral theory; see [36] Theorem 15.1 for a more general statement.

**Theorem 2.1.** (intrinsic spectral equivalence) Let $X^1 \subset X^0$ be two Hilbert spaces with $X^1$ continuously embedded and dense in $X^0$. Let the self-adjoint and coercive operator $\Lambda : X^1 \to X^0$ satisfy \((2.3)\) and have a discrete, complete and orthonormal set of eigenpairs $(\lambda_k, \varphi_k)_{k=1}^\infty$ in $X^0$. Given $s \in (0, 1)$, for any $v \in X^0$ with $\|\Lambda^s v\|_{X^0} < \infty$ we have
\begin{equation}
\|\Lambda^s v\|_{X^0} = \|v\|_{(X^0, X^1)_{s,2}}.
\end{equation}

**Proof.** Given $v = \sum_{k=1}^\infty v_k \varphi_k \in X^0$ we split it as $v = v^0 + v^1$, with
\begin{align*}
v^0 &= \sum_{k=1}^\infty (1-a_k) v_k \varphi_k, \\
v^1 &= \sum_{k=0}^\infty a_k v_k \varphi_k,
\end{align*}
and $\{a_k\}_{k=1}^\infty$ to be determined. Combining \((2.3)\) with the definition \((2.5)\) of the $K_2$-functional and the orthonormality of $\{\varphi_k\}_{k=1}^\infty$ in $X^0$ yields
\begin{align*}
K_2(t, v)^2 &= \inf_{\{a_k\}_{k=1}^\infty} \left\| \sum_{k=1}^\infty (1-a_k) v_k \varphi_k \right\|_{X^0}^2 + t^2 \left\| \sum_{k=1}^\infty a_k v_k \varphi_k \right\|_{X^1}^2 \\
&= \inf_{\{a_k\}_{k=1}^\infty} \left\| \sum_{k=1}^\infty (1-a_k) v_k \varphi_k \right\|_{X^0}^2 + t^2 \left\| \sum_{k=1}^\infty a_k \lambda_k v_k \varphi_k \right\|_{X^0}^2 \\
&= \inf_{\{a_k\}_{k=1}^\infty} \sum_{k=1}^\infty \left( (1-a_k)^2 + t^2 a_k^2 \lambda_k^2 \right) v_k^2.
\end{align*}

We choose $a_k = (1 + \lambda_k^2 t^2)^{-1}$, which minimizes the terms in parenthesis above, to obtain
\begin{equation*}
K_2(t, v)^2 = \sum_{k=1}^\infty \frac{\lambda_k^2 t^2 v_k^2}{1 + \lambda_k^2 t^2}.
\end{equation*}
Recalling \((2.6)\) and applying the change of variables $\theta = \lambda_k t$, we end up with
\begin{align*}
\|v\|_{(X^0, X^1)_{s,2}}^2 &= \sum_{k=1}^\infty \frac{2 \sin(\pi s)}{\pi} \int_0^\infty t^{-1-2s} \frac{\lambda_k^2 t^2}{1 + \lambda_k^2 t^2} v_k^2 dt \\
&= \frac{\pi}{2 \sin(\pi s)} \left( \int_0^\infty \frac{\theta^{1-2s}}{1 + \theta^2} d\theta \right) \lambda_k^2 v_k^2 = \sum_{k=1}^\infty \lambda_k^2 v_k^2 = \|\Lambda^s v\|_{X^0}^2,
\end{align*}
because $\int_0^\infty \theta^{1-2s} \frac{1}{1+\theta^2} d\theta = \frac{\pi}{2 \sin(\pi s)}$ (see [38] Exercise B.5). This concludes the proof. \(\square\)

We now apply Theorem 2.1 (intrinsic spectral equivalence) to $L^2$-based Sobolev spaces. Let $X^0 = \tilde{L}^2(\Omega)$ and $X^1 = \tilde{H}^1_0(\Omega)$ denote the spaces of functions in $L^2(\Omega)$ and $H^1_0(\Omega)$ extended by zero to $\Omega$, respectively, and let the inner product in $X^1$ be given by $(v, w)_{X^1} = \int_{\Omega} \nabla v \cdot \nabla w = \int_{\Omega} \nabla v \cdot \nabla w$. The corresponding operator $S$ equals the Laplacian $-\Delta$ with zero Dirichlet condition and $\Lambda = (-\Delta)^\frac{1}{2}$. Therefore, the $k$-th eigenvalues $\lambda_k$ of $-\Delta$ and $\lambda_k$ of $(-\Delta)^\frac{1}{2}$ satisfy $\tilde{\lambda}_k = \lambda_k^2$ whereas the $k$-th eigenfunctions are the same, whence
\begin{equation*}
\|\Lambda^s v\|_{X^0}^2 = \|(-\Delta)^\frac{s}{2} v\|_{X_0}^2 = \sum_{k=1}^\infty \lambda_k^2 v_k^2 = \sum_{k=1}^\infty \tilde{\lambda}_k v_k^2
\end{equation*}
is the norm square of the interpolation space
\begin{equation*}
\tilde{H}^s(\Omega) = \left( \tilde{L}^2(\Omega), \tilde{H}^1_0(\Omega) \right)_{s,2}.
\end{equation*}
Since this norm is equivalent to the Gagliardo norm $| \cdot |_s$ induced by (2.1) for $\sigma = s$ with a constant independent of $s$ (cf. [38] Theorem B.8, Theorem B.9 and [17]), we deduce

$$
|v|^2 \simeq \|\Lambda^s v\|^2_0 = \sum_{k=1}^{\infty} \lambda_k v_k^2 \quad \forall v \in \tilde{H}^s(\Omega).
$$

(2.8)

2.3. Regularity of solutions. We next discuss the regularity of solutions to (1.4) in either standard or suitably weighted Sobolev spaces. Grubb’s [33] accurate elliptic regularity estimates, expressed in terms of Hörmander $\mu$-spaces, can be interpreted in the Sobolev scale but they require the domain to be smooth. Regularity estimates valid for arbitrary bounded Lipschitz domains and a right-hand side function $f \in L^2(\Omega)$ are derived in [12].

Reference [14] studies Hölder regularity of solutions by using a boundary Harnack method and establishes that, if $f \in L^\infty(\Omega)$, then the solution to (1.4) satisfies $u \in C^\sigma(\bar{\Omega})$. This is consistent with the boundary behavior [8] [33] [14]

$$
u \sim d(\cdot, \partial \Omega)^s,
$$

(2.9)

where $d(\cdot, \partial \Omega)$ denotes the distance to $\partial \Omega$. The sharp characterization of boundary behavior in [14] serves as a guide to derive Sobolev regularity estimates in [1].

Proposition 2.1 (regularity on Lipschitz domains). Let $s \in (0, 1)$ and $\Omega$ be a bounded Lipschitz domain satisfying the exterior ball condition. If $s \in (0, \frac{1}{2})$, let $f \in C^{\frac{1}{2} - s}(\bar{\Omega})$; if $s = \frac{1}{2}$, let $f \in L^\infty(\Omega)$; and if $s \in (\frac{1}{2}, 1)$, let $f \in C^\beta(\Omega)$ for some $\beta > 0$. Then, for every $\varepsilon > 0$, the solution $u$ to (1.4) satisfies $u \in \tilde{H}^{s+\frac{1}{2}-\varepsilon}(\Omega)$, with

$$
|u|_{s+\frac{1}{2}-\varepsilon} \lesssim \frac{1}{\varepsilon} \|f\|_s.
$$

Above, $\| \cdot \|_s$ denotes the $C^{\frac{1}{2} - s}(\bar{\Omega})$, $L^\infty(\Omega)$ or $C^\beta(\Omega)$, for $s < \frac{1}{2}$, $s = \frac{1}{2}$ or $s > \frac{1}{2}$, respectively, and the hidden constant depends on $\Omega$, $d$ and $s$.

Remark 1 (sharpness). Proposition 2.1 is sharp according to the following example [28]. Let $\Omega = B(0, 1) \subset \mathbb{R}^d$ and $f = 1$. Then, the solution to (1.4) is

$$
u(x) = \frac{\Gamma\left(\frac{d}{2}\right)}{2^s \Gamma\left(\frac{d+2s}{2}\right) \Gamma(1+s)} (1 - |x|^2)^s_+.
$$

In view of Proposition 2.1, we expect that conforming finite element approximations over quasi-uniform grids would converge with order $\frac{1}{2}$ in the energy norm. To mitigate such a low convergence rate we could increase the mesh grading towards $\partial \Omega$ and compensate for (2.9). This idea was exploited in [1] (see also [9, 13]), where the regularity of the solution is characterized in weighted Sobolev spaces, with the weight being a power of $d(\cdot, \partial \Omega)$. We refer to either of these references for a definition of the spaces $\tilde{H}^s_\alpha(\Omega)$.

Proposition 2.2 (regularity in weighted spaces). Let $\Omega$ be a bounded, Lipschitz domain satisfying the exterior ball condition. Let $f \in C^\beta(\bar{\Omega})$ for some $\beta \in (0, 2 - 2s)$, $\alpha \geq 0$, $t < \min\{\beta + 2s, \alpha + s + \frac{1}{2}\}$ and $u$ be the solution of (1.4). Then, we have $u \in \tilde{H}^{\alpha}(\Omega)$, with

$$
\|u\|_{\tilde{H}^\alpha(\Omega)} \leq \frac{C(\Omega, d, s)}{\sqrt{(\beta + 2s - t)(1 + 2(\alpha + s - t))}} \|f\|_{C^\beta(\partial \Omega)}.
$$

Remark 2 (optimal parameters). The optimal choice of parameters $t$ and $\alpha$ for finite element applications depends on either the smoothness of $f$ and the dimension of the space. For instance, in dimension $d = 2$ and a sufficiently smooth right-hand side $f$, one can set $t = 1 + s - 2\varepsilon$ and $\alpha = \frac{1}{2} - \varepsilon$ for an arbitrary $\varepsilon \in (0, \frac{1}{2})$. The resulting constant scales as $\varepsilon^{-\frac{1}{2}}$ and one obtains a linear convergence rate (up to logarithmic terms) with respect to the dimension of the finite element spaces, which is optimal for approximations on shape-regular meshes. We refer to [11] for a thorough discussion on this aspect.
2.4. Finite element discretization. Given a conforming and shape-regular triangulation \( T \) of \( \Omega \), we consider discrete spaces consisting of continuous piecewise linear functions that vanish on \( \partial \Omega \),
\[
\mathcal{V}(T) = \{ v_h \in C(\overline{\Omega}) : v_h|_T \in P_1(T) \ \forall T \in T, \ v_h|_{\partial \Omega} = 0 \}.
\]
It is clear that \( \mathcal{V}(T) \subset H^s(\Omega) \), independently of the value of \( s \). Therefore, we can pose a conforming discretization of \( (2.2) \); we seek \( u_h \in \mathcal{V}(T) \) such that
\[
a(u_h, v_h) = (f, v_h)_{s,\Omega} \ \forall v_h \in \mathcal{V}(T).
\]
Thus, the finite element solution is the elliptic projection of the solution \( u \) to \( (1.4) \) onto the discrete space \( \mathcal{V}(T) \),
\[
|u - u_h|_s = \inf_{v_h \in \mathcal{V}(T)} |u - v_h|_s \ \forall v_h \in \mathcal{V}(T).
\]
Convergence rates in the energy norm are derived by combining the estimate above with suitable interpolation estimates [20, 1] and the regularity described in Proposition 2.1 (regularity on Lipschitz domains); cf. [9, Theorem 3.7].

**Proposition 2.3** (convergence rates in uniform meshes). Assume \( s \in (0, 1) \) and \( \Omega \) is a bounded Lipschitz domain. Let \( u \) denote the solution to \( (2.2) \) and denote by \( u_h \in \mathcal{V}(T) \) the solution of the discrete problem \( (2.10) \), computed over a mesh \( T \) consisting of elements with maximum diameter \( h \). Under the hypotheses of Proposition 2.1, we have
\[
|u - u_h|_s \lesssim h^\frac{d}{2} \log h |h|^{1+s} \| f \|_*.
\]
Above, \( \| \cdot \|_* \) denotes the \( C^{\frac{d}{2}-s}(\overline{\Omega}) \), \( L^\infty(\Omega) \) or \( C^3(\overline{\Omega}) \), depending on whether \( s < \frac{1}{2} \), \( s = \frac{1}{2} \) or \( s > \frac{1}{2} \), and \( \kappa = 1 \) if \( s = \frac{1}{2} \) and zero otherwise.

We point out that, on quasi-uniform grids, this approximation rate is optimal due to (2.9). As shown in [1], when solving \( (1.4) \) it is possible to increase the a priori convergence rates by utilizing suitably graded grids and making use of the regularity estimate from Proposition 2.2 (regularity in weighted spaces). More precisely, given a grading parameter \( \mu \geq 1 \) and a mesh size parameter \( h \), assume that the element size \( h_T \) satisfies
\[
h_T \simeq \begin{cases} 
h^\mu & \text{if } S_T \cap \partial \Omega \neq \emptyset, \\
h \min(d, \mu^{-1})^{\mu} & \text{otherwise}.
\end{cases}
\]

We wish the mesh size parameter \( h \) to be such that \( \dim \mathcal{V}(T) \simeq h^{-d} \). Shape regularity limits the range for \( \mu \) when \( d \geq 2 \): we have
\[
\dim \mathcal{V}(T) \simeq \begin{cases} 
h^{-d} & \text{if } \mu \in [1, \frac{d}{d-1}], \\
h^{-d} \log h & \text{if } \mu = \frac{d}{d-1}, \\
h(1-d)\mu & \text{if } \mu > \frac{d}{d-1}.
\end{cases}
\]
Thus, for a sufficiently smooth right hand side \( f \), the optimal choice for \( \mu \) is \( \frac{d}{d-1} \), and one obtains the following convergence rates with this strategy.

**Proposition 2.4** (convergence rates in graded meshes). Let \( s \in (0, 1) \) and \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain satisfying the exterior ball condition, \( u \) be the solution to \( (2.2) \) and denote by \( u_h \in \mathcal{V}(T) \) the solution of the discrete problem \( (2.10) \). Let \( \beta > 0 \) be such that
\[
\beta \geq \begin{cases} 
2 - 2s & \text{if } d = 1, \\
\frac{d}{2(d-1)} - s & \text{if } d \geq 2,
\end{cases}
\]
and \( \mu = \begin{cases} 
2 - s & \text{if } d = 1, \\
\frac{d}{d-1} & \text{if } d \geq 2.
\end{cases} \)

Then, if \( f \in C^3(\overline{\Omega}) \), and the mesh \( T \) is graded according to \( (2.11) \), we have
\[
|u - u_h|_s \lesssim \begin{cases} 
h^{2-s} \log h |f|_{C^3(\overline{\Omega})} & \text{if } d = 1, \\
h \min(d, \mu^{-1}) \log h |h|^{1+s} \| f \|_{C^3(\overline{\Omega})} & \text{if } d \geq 2.
\end{cases}
\]
where the hidden constant depends on \( \Omega \), \( s \) and the shape regularity of \( T \) and \( \kappa = 1 \) if \( s = \frac{1}{2} \) and zero otherwise.
3. Robust Additive Multilevel Preconditioning

Let \((\cdot, \cdot)\) be the \(L^2\)-inner product in \(\Omega\) and \(V := \mathbb{V}(T)\) denote the discrete space. Let \(A: V \to V\) be the symmetric positive definite (SPD) operator defined by \((Au, v) := a(u, v)\) for any \(u, v \in V\), and let \(f \in V\) be given by \((f, v) = \langle f, v \rangle_{s,T}\) for any \(v \in V\). With this notation at hand, the discretization \((2.10)\) leads to the following linear equation in \(V\)

\[
Au = f.
\]

In this section, we give some general and basic results that will be used to construct the additive multilevel preconditioners for \((3.1)\).

3.1. Space decomposition. We now invoke the method of subspace corrections \([39, 31, 50]\). We first decompose the space \(V\) as the sum of subspaces \(V_j \subset V\)

\[
V = \sum_{j=0}^{J} V_j.
\]

For \(j = 0, 1, \ldots, J\), we consider the following operators:

- \(Q_j: V \to V_j\) is the \(L^2\)-projection operator defined by \((Q_j v, v_j) = (v, v_j)\) for all \(v \in V, v_j \in V_j\);
- \(I_j: V_j \to V\) is the natural inclusion operator given by \(I_j v_j = v_j\) for all \(v_j \in V_j\);
- \(\tilde{R}_j: V_j \to V\) is an approximate inverse of the restriction of \(A\) to \(V_j\) (often known as smoother); we set \(\|v_j\|_{R_j^{-1}}^2 := (R_j^{-1} v_j, v_j)\) for all \(v_j \in V_j\) provided that \(R_j\) is SPD on \(V_j\).

A straightforward calculation shows that \(Q_j = I_j^1\) because \((Q_j v, v_j) = (v, I_j v_j) = (I_j^1 v, v_j)\) for all \(v \in V, v_j \in V_j\). Let the fictitious space be \(\tilde{V} = V_0 \times V_1 \times \ldots \times V_J\). Then, the Parallel Subspace Correction (PSC) preconditioner \(B: V \to V\) is defined by

\[
B := \sum_{j=0}^{J} I_j R_j Q_j = \sum_{j=0}^{J} I_j R_j I_j^1.
\]

The next two lemmas follow from the general theory of preconditioning techniques based on fictitious or auxiliary spaces \([39, 31, 50, 51, 52, 55]\). For completeness, we give their proofs in Appendix A.

**Lemma 3.1** (identity for PSC). If \(R_j\) is SPD on \(V_j\) for \(j = 0, 1, \ldots, J\), then \(B\) defined in \((3.2)\) is also SPD under the inner product \((\cdot, \cdot)\). Furthermore,

\[
(B^{-1} v, v) = \inf_{\sum j=0}^{J} v_j = v \sum_{j=0}^{J} (R_j^{-1} v_j, v_j) \quad \forall v \in V.
\]

**Lemma 3.2** (estimate on \(\text{cond}(BA)\)). If the operator \(B\) in \((3.2)\) satisfies

(A1) Stable decomposition: for every \(v \in V\), there exists \((v_j)_{j=0}^{J} \in \tilde{V}\) such that \(\sum_{j=0}^{J} v_j = v\) and

\[
\sum_{j=0}^{J} \|v_j\|_{R_j^{-1}}^2 \leq c_0 \|v\|_A^2,
\]

where \(\|v\|_A^2 = (Av, v)\), then \(\lambda_{\min}(BA) \geq c_0^{-1}\);

(A2) Boundedness: For every \((v_j)_{j=0}^{J} \in \tilde{V}\) there holds

\[
\left\| \sum_{j=0}^{J} v_j \right\|_A^2 \leq c_1 \sum_{j=0}^{J} \|v_j\|_{R_j^{-1}}^2,
\]

then \(\lambda_{\max}(BA) \leq c_1\).

Consequently, if \(B\) satisfies (A1) and (A2), then \(\text{cond}(BA) \leq c_0 c_1\).
3.2. Instrumental tools for s-uniform preconditioner. We assume that the spaces \( \{V_j\}_{j=0}^J \) are nested, i.e.
\[
V_{j-1} \subset V_j \quad \forall 1 \leq j \leq J.
\]
With the convention that \( Q_{-1} = 0 \), we consider the \( L^2 \)-slicing operators
\[
\tilde{Q}_j : V \rightarrow V_j : \quad \tilde{Q}_j := Q_j - Q_{j-1} \quad (j = 0, 1, \ldots, J).
\]
Clearly, the \( L^2 \)-orthogonality implies that \( Q_k Q_j = Q_{k \wedge j} \), where \( k \wedge j := \min\{k, j\} \). Hence,
\[
(3.6) \quad \tilde{Q}_j Q_k = Q_k \tilde{Q}_j = \begin{cases} \tilde{Q}_j & j \leq k, \\ 0 & j > k, \end{cases} \quad \tilde{Q}_k \tilde{Q}_j = \delta_{kj} \tilde{Q}_j.
\]

The following lemma plays a key role in the analysis of an s-uniform preconditioner, which is obtained by using the identity of PSC \[\text{[3.3]}\] and reordering the BPX preconditioner \cite{49,15}.

Lemma 3.3 (s-uniform decomposition). Given \( \gamma \in (0, 1) \), \( 0 < s \leq 1 \), it holds that, for every \( v \in V \),
\[
\sum_{j=0}^{J} \gamma^{-2sj} \|Q_j - Q_{j-1}\|_0^2 = \inf_{v_j \in V_j} \left[ \sum_{j=0}^{J} \gamma^{-2sj} \|v_j\|_0^2 + \sum_{j=0}^{J-1} \gamma^{-2sj} \|v_j\|_0^2 \right].
\]

Proof. This proof is an application of Lemma 3.1 (identity for PSC). Taking
\[
B = \sum_{j=0}^{J} \gamma^{2sj}(Q_j - Q_{j-1}),
\]
by the \( L^2 \)-orthogonality \[\text{(3.6)}\], we easily see that \( B^{-1} = \sum_{j=0}^{J} \gamma^{-2sj}(Q_j - Q_{j-1}) \) and
\[
(B^{-1} v, v) = \sum_{j=0}^{J} \gamma^{-2sj} \|Q_j - Q_{j-1}\|_0^2.
\]
On the other hand, to identify \( R_j \) we reorder the sum in the definition of \( B \)
\[
B = \sum_{j=0}^{J} \gamma^{2sj}(Q_j - Q_{j-1}) = \gamma^{2s}Q_j + \sum_{j=0}^{J-1} (1 - \gamma^{2s}) \gamma^{2sj}Q_j = \sum_{j=0}^{J} I_j R_j Q_j,
\]
where
\[
R_j v_j := \begin{cases} (1 - \gamma^{2s}) \gamma^{2sj} v_j & j = 0, \ldots, J - 1, \\ \gamma^{2sj} v_j & j = J, \end{cases}
\]
for all \( v_j \in V_j \). Finally, the identity \[\text{(3.3)}\] of PSC gives the desired result. \( \square \)

The next lemma is an application of space interpolation theory and is crucial to obtain stable decompositions in fractional-order norms. We postpone its proof to Appendix B.

Lemma 3.4 (s-uniform interpolation). Assume that the spaces \( \{V_j\}_{j=0}^J \) are nested, and
\[
(3.7) \quad \sum_{j=0}^{J} \gamma^{-2sj} \|Q_j - Q_{j-1}\|_0^2 \lesssim |v|^2 \quad \forall v \in V.
\]
Then, the following inequality holds, with the hidden constant independent of \( s \) and \( J \),
\[
(3.8) \quad \sum_{j=0}^{J} \gamma^{-2sj} \|Q_j - Q_{j-1}\|_0^2 \lesssim |v|^2 \quad \forall v \in V.
\]

We conclude this section with a standard local inverse estimate valid on graded grids \( \mathcal{T} \).

The proof is elementary, and we give it in Appendix B. Given \( \tau \in \mathcal{T} \), we define
\[
S_\tau := \bigcup \{ \tau' \in \mathcal{T} : \overline{\tau'} \cap \tau \neq \emptyset \}.
\]
Lemma 3.5 (local inverse inequality). Let $\sigma \in [0, 3/2]$ and $\mu \in [0, \sigma]$. Then,

$$|v|_\sigma \lesssim \left( \sum_{\tau \in \mathcal{T}} h_{\tau}^{2(\mu-\sigma)} |v|_{H^\mu(\mathcal{T})}^2 \right)^{1/2}, \quad \forall v \in \mathcal{V}(\mathcal{T}),$$

where the hidden constant only blows up as $\sigma \to 3/2$.

4. ROBUST BPX PRECONDITIONER FOR QUASI-UNIFORM GRIDS

In this section, we propose and study a BPX preconditioner \cite{49, 15, 26} for the solution of the systems arising from the finite element discretizations \cite{2.10} on quasi-uniform grids. We emphasize that, in contrast to \cite{25}, the proposed preconditioner is uniform with respect to both the number of levels and the order $s$. To this end, we introduce a new scaling for coarse spaces which differs from the original BPX preconditioners given in \cite{15, 26}. Our theory applies as well to the spectral and censored fractional Laplacians except for the censored one when $s \to \frac{1}{2}$; see Section 4.3.

Consider a family of uniformly refined grids \{${\mathcal{T}_k}$\}$_{k=0}^J$ on $\Omega$, where $\mathcal{T}_0 = \mathcal{T}_0$ is a quasi-uniform initial triangulation. On each of these grids we define the space $\mathcal{V}_k := \mathcal{V}(\mathcal{T}_k)$. Let $\mathcal{V} = \mathcal{V}_J$ and $\mathcal{A}$ be the SPD operator on $\mathcal{V}$ associated with $a(\cdot, \cdot) : (\mathcal{V}, w) = a(v, w)$ for all $v, w \in \mathcal{V}$. Let the grid size be $\bar{h}_k \simeq \gamma^k$, where $\gamma \in (0, 1)$ is a fixed constant. For instance, we have $\gamma = \frac{1}{2}$ for uniform refinement, in which each simplex is refined into $2^d$ children, and $\gamma = (\frac{1}{2})^{1/d}$ for uniform bisection, in which each simplex is refined into 2 children.

Let $\bar{Q}_k : \mathcal{V} \to \mathcal{V}_k$ and $\bar{I}_k : \mathcal{V}_k \to \mathcal{V}$ be the $L^2$-projection and inclusion operators defined in Section 3.1 and let $\bar{Q}_{-1} := 0$. The standard BPX preconditioner reads \cite{15, 26}

$$\bar{B} = \sum_{k=0}^J \bar{I}_k \bar{h}_k^{2s} \bar{Q}_k : \mathcal{V} \to \mathcal{V}.$$ \hfill (4.1)

A rough analysis of (4.1) proceeds as follows. Let $\bar{S}_k : \mathcal{V} \to \mathcal{V}_k$ be the Scott-Zhang interpolation operator \cite{45} and let $\bar{S}_k - \bar{S}_{k-1} : \mathcal{V} \to \mathcal{V}_k$ be the slicing operator for all $k = 0, 1, \ldots, J$ with $\bar{S}_{-1} := 0$. We thus have the decomposition of any $v \in \mathcal{V}$:

$$v = \sum_{k=0}^J v_k, \quad v_k := (\bar{S}_k - \bar{S}_{k-1})v.$$

If $\bar{R}_k v_k := \tilde{h}_k^{2s} v_k$, then the stable decomposition \cite{3.4} is a consequence of

$$\sum_{k=0}^J \|v_k\|_{\bar{R}_k}^2 \lesssim \sum_{k=0}^J \tilde{h}_k^{-2s} \|(\bar{S}_k - \bar{S}_{k-1})v\|_{\mathcal{V}_k}^2 \lesssim \sum_{k=0}^J \tilde{h}_k^{-2s} \|v - \bar{S}_k v\|_{\mathcal{V}_k}^2 \lesssim \sum_{k=0}^J \|v\|_{\mathcal{V}_k}^2 \lesssim \bar{J} \|v\|_{\mathcal{V}_k}^2,$$

whence $c_0 \lesssim \bar{J}$. On the other hand, the boundedness \cite{3.5} follows from an inverse estimate

$$\sum_{k=0}^J \|v_k\|_{\bar{R}_k}^2 \lesssim \bar{J} \sum_{k=0}^J \|v_k\|_{\mathcal{V}_k}^2 \lesssim \bar{J} \sum_{k=0}^J \tilde{h}_k^{-2s} \|v_k\|_{\mathcal{V}_k}^2 = \bar{J} \sum_{k=0}^J \|v_k\|_{\bar{R}_k}^2,$$

whence $c_1 \lesssim \bar{J}$. Therefore, Lemma 3.2 (estimate of cond$(BA)$) yields the condition number estimate cond$(\bar{B} \mathcal{A}) \lesssim \bar{J}^2$ but independent of $s$. To remove the dependence on $\bar{J}$ we deal below with the slicing $L^2$-projectors $\bar{Q}_k - \bar{Q}_{k-1}$. However, a naive replacement of $\bar{S}_k - \bar{S}_{k-1}$ by $\bar{Q}_k - \bar{Q}_{k-1}$ would make cond$(\bar{B} \mathcal{A})$ independent of $\bar{J}$ but blow-up as $s \to 0$. This is an unnatural dependence on $s$ because $(-\Delta)^s$ tends to the identity as $s \to 0$. We circumvent this issue by a suitable rescaling of coarse levels and redefinition of the smoothers $\bar{R}_k$.

Let $\bar{\gamma} \in (0, 1)$ be a fixed constant; it can be taken equal to $\gamma$ but this is not needed. For every $v_k \in \mathcal{V}_k, k = 0, \ldots, \bar{J}$, we define $\bar{R}_k : \mathcal{V}_k \to \mathcal{V}_k$ to be

$$\bar{R}_k v_k := \begin{cases} (1 - \bar{\gamma}^k) \tilde{h}_k^{2s} v_k & k = 0, \ldots, \bar{J} - 1, \\ \tilde{h}_k^{2s} v_k & k = \bar{J}. \end{cases}$$ \hfill (4.3)
Theorem 4.2 given in [52, Theorem 10.5].

Consider the operator \( \Lambda = \sum_{k=0}^{J} T_k \tilde{R}_k T_k^T = T_J \tilde{h}^{2s}_J \mathcal{Q}_J + (1 - \tilde{\gamma}^s) \sum_{k=0}^{J-1} T_k \tilde{h}_k^{2s} \mathcal{Q}_k. \)

Our next goal is to prove the following theorem, namely that \( \mathcal{B} \) satisfies the two necessary conditions (4.4) and (3.5) of Lemma 3.2 (estimate of \( \text{cond}(BA) \)) uniformly in \( J \) and \( s \) over quasi-uniform grids. We observe that the scaling \( (1 - \tilde{\gamma}^s)^{-1} \) makes it easier to prove (3.5) but complicates (3.4). We prove (4.4) in Section 4.1 and (3.5) in Section 4.2.

Theorem 4.1 (uniform preconditioning on quasi-uniform grids). Let \( \Omega \) be a bounded Lipschitz domain and \( s \in (0, 1) \). Consider discretizations to (1.4) using piecewise linear Lagrangian finite elements on quasi-uniform grids. Then, the preconditioner (4.4) satisfies \( \text{cond}((BA) \lesssim 1 \), where the hidden constant is uniform with respect to both \( J \) and \( s \).

4.1. Stable decomposition: Proof of (3.4) for quasi-uniform grids. We start with a norm equivalence for discrete functions. We rely on operator interpolation and the decomposition for \( s = 1 \) [50, 43, 10], which was proposed earlier in [49, 15] with a removable logarithmic factor. A similar result, for the interpolation norm of \( (L^2(\Omega), H^s_0(\Omega)) \), was given in [52, Theorem 10.5].

Theorem 4.2 (norm equivalence). Let \( \Omega \) be a bounded Lipschitz domain and \( s \in (0, 1) \). If \( \overline{Q}_k : \overline{V} \to \mathcal{Q}_k \) denotes the \( L^2 \)-projection operators onto discrete spaces \( \mathcal{Q}_k \), and \( \overline{Q}_-1 := 0 \), then for any \( v \in \overline{V} \) the decomposition \( v = \sum_{k=0}^{J} (\overline{Q}_k - \overline{Q}_{k-1})v \) satisfies

\[
|v|^2_s \simeq \sum_{k=0}^{J} \tilde{h}_k^{-2s} ||(\overline{Q}_k - \overline{Q}_{k-1})v||^2_0.
\]

The equivalence hidden constant is independent of \( s \) and \( \tilde{\gamma} \).

Proof. Consider the operator \( \Lambda = \sum_{k=0}^{J} \tilde{h}_k^{-1} (\overline{Q}_k - \overline{Q}_{k-1}) : \overline{H}^1_0(\Omega) \to \overline{L}^2(\Omega) \), which happens to be self-adjoint and coercive in \( \overline{L}^2(\Omega) \). Combining the Poincaré inequality in \( \overline{H}^1_0(\Omega) \) with the \( H^1 \)-norm equivalence [50, 43, 10] yields

\[
|v|^2_1 \simeq ||v||^2_1 \simeq \sum_{k=0}^{J} \tilde{h}_k^{-2} ||(\overline{Q}_k - \overline{Q}_{k-1})v||^2_0 = ||\Lambda v||^2_0 \quad \forall v \in \overline{H}^1_0(\Omega),
\]

whence \( |v|_s \simeq ||\Lambda^s v||_0 \) according to (2.8). It remains to characterize \( ||\Lambda^s v||_0 \). To this end, notice that \( \mathcal{V}_k := (\overline{Q}_k - \overline{Q}_{k-1})^1 \mathcal{V} \) is an eigenspace of \( \Lambda \) with eigenvalue \( \tilde{\lambda}_k = \tilde{h}_k^{-1} \), namely \( \Lambda |\tilde{v}_k = \tilde{h}_k^{-1} I \). Moreover, \( \mathcal{V} = \oplus_{k=0}^{J} \mathcal{V}_k \) is an \( L^2 \)-orthogonal decomposition of \( \mathcal{V} \) and \( \Lambda^s |\tilde{v}_k = \tilde{h}_k^{-s} I \) according to (2.4). This implies \( ||\Lambda^s v||^2_0 = \sum_{k=0}^{J} \tilde{h}_k^{-2s} ||(\overline{Q}_k - \overline{Q}_{k-1})v||^2_0 \) and thus (4.5), as asserted.

Corollary 4.1 (stable decomposition). For every \( v \in \overline{V} = \overline{V}_J \), there exists a decomposition \( (v_0, \ldots, v_J) \in \overline{V}_0 \times \ldots \times \overline{V}_J \), such that \( \sum_{k=0}^{J} v_k = v \) and

\[
\tilde{h}_J^{-2s} ||v_J||^2_0 + \left( \frac{1}{1 - \tilde{\gamma}^s} \sum_{k=0}^{J-1} \tilde{h}_k^{-2s} ||v_k||^2_0 \right) \simeq ||v||^2_s.
\]

Proof. This is a direct consequence of Lemma 3.3 (s-uniform decomposition) and Theorem 4.2 (norm equivalence) because \( \tilde{h}_k \simeq \gamma^k \).

4.2. Boundedness: Proof of (3.5) for quasi-uniform grids. We now prove the boundedness estimate in Lemma 3.2 (estimate on \( \text{cond}(BA) \)) with a constant independent of both \( J \) and \( s \).
Proposition 4.1 (boundedness). The preconditioner $\mathcal{B}$ in (4.4) satisfies (3.5), namely
\begin{equation}
\|\sum_{k=0}^{\tilde{J}} v_k\|_s^2 \leq c_1 \left( \tilde{h}_j^{-2s} \|v_j\|_0^2 + \frac{1}{1 - \gamma^s} \sum_{k=0}^{\tilde{J}-1} \tilde{h}_k^{-2s} \|v_k\|_0^2 \right),
\end{equation}
where $\gamma \in (0,1)$ can be taken arbitrarily and the constant $c_1$ is independent of $J$ and $s$.

Proof. Let $v := \sum_{k=0}^{\tilde{J}} v_k$. Then, we use Theorem 4.2 (norm equivalence), the fact that $\tilde{h}_k \approx \gamma^k$ and Lemma 3.3 (s-uniform decomposition) to write
\begin{equation}
\|\sum_{k=0}^{\tilde{J}} v_k\|_s^2 \leqslant \sum_{k=0}^{\tilde{J}} \tilde{h}_k^{-2s} \|(\overline{Q}_k - \overline{Q}_{k-1})v\|_0^2 \leqslant \sum_{k=0}^{\tilde{J}} \gamma^{-2sk} \|(\overline{Q}_k - \overline{Q}_{k-1})v\|_0^2 = \inf_{w_k \in \mathcal{V}_k} \left[ \sum_{k=0}^{\tilde{J}} \gamma^{-2sk} \|w_j\|_0^2 + \frac{1}{1 - \gamma^s} \sum_{k=0}^{\tilde{J}-1} \tilde{h}_k^{-2s} \|v_k\|_0^2 \right].
\end{equation}
Therefore, upon setting $w_k = v_k$ for $j = 0, \ldots, \tilde{J}$ above, we deduce that
\begin{equation}
\|\sum_{k=0}^{\tilde{J}} v_k\|_s^2 \leq \sum_{k=0}^{\tilde{J}} \gamma^{-2sk} \|v_j\|_0^2 + \sum_{k=0}^{\tilde{J}-1} \gamma^{-2sk} \|v_k\|_0^2 \leq c_1 \left( \tilde{h}_j^{-2s} \|v_j\|_0^2 + \frac{1}{1 - \gamma^s} \sum_{k=0}^{\tilde{J}-1} \tilde{h}_k^{-2s} \|v_k\|_0^2 \right).
\end{equation}
The proof is thus complete. $\blacksquare$

An alternative derivation of boundedness could be carried with the aid of a strengthened Cauchy-Schwarz inequality, which plays an important role in the analysis of multigrid methods (cf. [49, 50, 52]). We provide a proof of such an inequality together with a second proof of Proposition 4.1 (boundedness) in Appendix C. This tool also allows us to illustrate the need of the correction factor $1 - \gamma^s$ in the coarser scales in (4.4).

Remark 3 (preconditioner (4.1)). We wonder how the boundedness (3.5) changes if we consider the standard preconditioner (4.1) instead of (4.4). Since $\|v_k\|_{\overline{Q}_k} = \tilde{h}_k^{-2s} \|v_k\|_0$ in this case, Lemma C.1 (generalized strengthened Cauchy-Schwarz inequality) and (C.2) yield
\begin{equation}
\|\sum_{k=0}^{\tilde{J}} v_k\|_s^2 = \inf_{\sum_{k=0}^{\tilde{J}} w_k = v} \sum_{k=0}^{\tilde{J}} \gamma^{-2sk} \|w_j\|_0^2 + \frac{1}{1 - \gamma^s} \sum_{k=0}^{\tilde{J}-1} \tilde{h}_k^{-2s} \|v_k\|_0^2 \leq \sum_{k=0}^{\tilde{J}} \gamma^{-2sk} \|v_j\|_0^2 + \sum_{k=0}^{\tilde{J}-1} \gamma^{-2sk} \|v_k\|_0^2 \leq c_1 \left( \tilde{h}_j^{-2s} \|v_j\|_0^2 + \frac{1}{1 - \gamma^s} \sum_{k=0}^{\tilde{J}-1} \tilde{h}_k^{-2s} \|v_k\|_0^2 \right).
\end{equation}
This together with (4.2) implies that the constant $c_1 \lesssim \min\{(1 - \gamma^s)^{-1}, \tilde{J}\}$ of (3.5) blows up as $s \to 0$ and $\tilde{J} \to \infty$; this is observed in the experimental results reported in Table 1

4.3. Spectral equivalence: Spectral and censored Laplacians. We now exploit the fact that Theorem 4.2 (norm equivalence) is insensitive to the number of levels $\tilde{J}$, whence letting $\tilde{J} \to \infty$ we obtain the multilevel decomposition of any $v \in \tilde{H}^s(\Omega)$
\begin{equation}
|v|_s^2 \simeq \sum_{k=0}^{\infty} \tilde{h}_k^{-2s} \| (\overline{Q}_k - \overline{Q}_{k-1})v \|_0^2.
\end{equation}
An alternative definition of $|\cdot|_s$, but equivalent to (2.1), relies on the spectral decomposition of the Laplacian $-\Delta$ in a bounded Lipschitz domain $\Omega$. Recall from Section 2.2 that if $(\lambda_k, \varphi_k)_{k=1}^{\infty}$ is the sequence of eigenpairs of $-\Delta$ with zero Dirichlet boundary condition and normalized in $L^2(\Omega)$, then (2.8) implies that the space
\begin{equation}
\tilde{H}^s(\Omega) := \left\{ v = \sum_{k=1}^{\infty} v_k \varphi_k \in L^2(\Omega) : \|v\|_s^2 = \sum_{k=1}^{\infty} \lambda_k^{s} v_k^2 < \infty \right\}
\end{equation}
coincides with $\tilde{H}^s(\Omega)$ and has equivalent norms. However, these norms induce different fractional operators. Minima of the functional $v \mapsto \frac{1}{2}|v|_{\tilde{H}^s(\Omega)}^2 - \int_\Omega fv$ are weak solutions of the spectral fractional Laplacian in $\Omega$ with homogeneous Dirichlet condition for $0 < s < 1,$
whose eigenpairs are \((\tilde{X}_k^s, \varphi_k^s)\) \(_{k=1}^{\infty}\). If \(\{\mu_k^s\}_{k=1}^{\infty}\) are the eigenvalues of the integral Laplacian \((\ref{eq:laplacian})\), the following equivalence is derived in \([19]\)

\[
C(\Omega)\hat{\lambda}_k^s \leq \mu_k^s \leq \tilde{\lambda}_k^s, \quad k \in \mathbb{N}.
\]

There is yet a third family of fractional Sobolev spaces, namely \(H^s_0(\Omega)\), which are the completion of \(C_0^\infty(\Omega)\) with the \(L^2\)-norm plus the usual \(H^s\)-seminorm

\[
|v|_{H^s(\Omega)}^2 = C(d, s) \int \int \Omega \frac{|u(x) - u(y)|^2}{|x - y|^{d + 2s}} \, dx \, dy.
\]

If \(\Omega\) is Lipschitz, it turns out that \(H^s_0(\Omega) = \tilde{H}^s(\Omega)\) for all \(0 < s < 1\) such that \(s \neq \frac{1}{2}\); in the latter case \(\tilde{H}^s(\Omega) = H^s_0(\Omega)\) is the so-called Lions-Magenes space. The seminorm \((\ref{eq:seminorm})\) is a norm equivalent to \(|v|_s\) for \(s \in (\frac{1}{2}, 1)\) but not for \(s \in (0, \frac{1}{2}]\); note that \(1 \in H^0_0(\Omega)\) and \(|1|_{H^s(\Omega)} = 0\) for \(s \in (0, \frac{1}{2}]\). Functions in \(H^0_0(\Omega)\) for \(s \in (\frac{1}{2}, 1)\) admit a trace on \(\partial \Omega\) and minima of the functional \(v \mapsto \frac{1}{2}|v|_{H^s(\Omega)}^2 - \int_\Omega f v\) are weak solutions of the censored fractional Laplacian. In view of the norm equivalence \(|v|_{H^s_0(\Omega)} \simeq |v|_s\) for \(s \in (\frac{1}{2}, 1)\), the multilevel decomposition \((\ref{eq:decomposition})\) applies to \(H^s_0(\Omega)\) uniformly in \(s\) as \(s \to 1\) but not as \(s \to \frac{1}{2}\). This is in agreement with the fact that in the inequality

\[
|v|_{H^s_0(\Omega)}^2 \leq C|v|_{H^s_0(\Omega)}^2, \quad v \in \tilde{H}^s(\Omega) = H^s_0(\Omega), \quad s > \frac{1}{2},
\]

the constant \(C\) scales as \((s - 1/2)^{-1}\). Indeed, splitting the integration to compute \(\tilde{H}^s(\Omega)\) above, one readily finds that

\[
|v|_{H^s_0(\Omega)}^2 = |v|_{H^s_0(\Omega)}^2 + 2C(d, s) \int \int \Omega \frac{|u(x)|^2}{|x - y|^{d+2s}} \, dy \, dx \simeq |v|_{H^s_0(\Omega)}^2 + \frac{C(d, s)}{s} \int \int \Omega \frac{|u(x)|^2}{d(x, \partial \Omega)^{2s}} \, dx,
\]

and it is therefore necessary to bound the last integral in the right hand side in terms of the \(H^s(\Omega)\)-seminorm. Such is the purpose of the Hardy inequality (cf. \([32]\) Theorem 1.4.4.4), for which the optimal constant is of order \((s - 1/2)^{-1}\) \([7]\).

In spite of their spectral equivalence, the inner products that give rise to the integral, spectral and censored fractional Laplacians are different and yield a strikingly different boundary behavior \([8]\). In contrast to \((\ref{eq:laplacian})\) for the integral Laplacian, for a generic right-hand side function \(f \in L^\infty(\Omega)\) the boundary behavior of solutions \(u\) of the spectral Laplacian is roughly like

\[
u \simeq d(\cdot, \partial \Omega)^{\min\{2s, 1\}},
\]

except for \(s = \frac{1}{2}\) that requires an additional factor \(|\log d(\cdot, \partial \Omega)|\), whereas solutions of the censored Laplacian are quite singular at the boundary \([8]\)

\[
u \simeq d(\cdot, \partial \Omega)^{s - \frac{1}{2}}.
\]

Nevertheless, the above norm equivalences and Theorem \([41]\) imply that the preconditioner \(B\) in \((\ref{eq:bp preconditioner})\) leads to \(\text{cond}(BA)\) being bounded independently of either \(s\) and \(\bar{J}\) if \(A\) is associated to the spectral Laplacian operator. For the censored Laplacian, the \(\text{cond}(BA)\) is uniform with respect to \(\bar{J}\) for \(s \in (\frac{1}{2}, 1)\) but blows up as \(s \to \frac{1}{2}\).

5. Graded Bisection grids

This section briefly reviews the bisection method with emphasis on graded grids, following \([18]\), and presents new notions. We also refer to \([40, 41, 54]\) for additional details.

5.1. Bisection rules. For each simplex \(\tau \in T\) and a refinement edge \(e\), the pair \((\tau, e)\) is called labeled simplex, and \((T, \mathcal{L}) := \{(\tau, e) : \tau \in T\}\) is called a labeled triangulation. For a labeled triangulation \((T, \mathcal{L})\), and \(\tau \in T\), a bisection \(b_\tau : \{(\tau, e)\} \to \{(\tau_1, e_1), (\tau_2, e_2)\}\) is a map that encodes the refinement procedure. The formal addition is defined as follows:

\[
T + b_\tau := (T, \mathcal{L}) \setminus \{(\tau, e)\} \cup \{(\tau_1, e_1), (\tau_2, e_2)\}.
\]

For an ordered sequence of bisections \(\mathcal{B} = (b_{\tau_1}, b_{\tau_2}, \ldots, b_{\tau_N})\), we set

\[
T + \mathcal{B} := (T + b_{\tau_1} + b_{\tau_2} + \cdots + b_{\tau_N}).
\]
Given an initial grid $\mathcal{T}_0$, the set of conforming grids obtained from $\mathcal{T}_0$ using the bisection method is defined as

$$\mathcal{T}(\mathcal{T}_0) := \{ \mathcal{T} = \mathcal{T}_0 + \mathcal{B} : \mathcal{B} \text{ is a bisection sequence and } \mathcal{T} \text{ is conforming} \}.$$  

The bisection method considered in this paper satisfies the following two assumptions:

(A1) Shape regularity: $\mathcal{T}(\mathcal{T}_0)$ is shape regular.

(A2) Conformity of uniform refinement: $\mathcal{T}_k := \mathcal{T}_{k-1} + \{ b_\tau : \tau \in \mathcal{T}_{k-1} \} \in \mathcal{T}(\mathcal{T}_0) \forall k \geq 1$.

5.2. Compatible bisections. We denote by $\mathcal{N}(\mathcal{T})$ the set of vertices of the mesh $\mathcal{T}$, and define the first ring of either a vertex $p \in \mathcal{N}(\mathcal{T})$ or an edge $e \in \mathcal{E}(\mathcal{T})$ as

$$\mathcal{R}_p = \{ \tau \in \mathcal{T} : p \in \tau \}, \quad \mathcal{R}_e = \{ \tau \in \mathcal{T} : e \subset \tau \},$$

and the local patch of either $p$ or $e$ as $\omega_p = \cup_{\tau \in \mathcal{R}_p} \tau$, and $\omega_e = \cup_{\tau \in \mathcal{R}_e} \tau$. An edge $e$ is called compatible if $e$ is the refinement edge of $\tau$ for all $\tau \in \mathcal{R}_e$. Let $p$ be the midpoint of a compatible edge $e$ and $\mathcal{R}_p$ be the ring of $p$ in $\mathcal{T} + \{ b_\tau : \tau \in \mathcal{R}_e \}$. Given a compatible edge $e$, a compatible bisection is a mapping $b_e : \mathcal{R}_e \rightarrow \mathcal{R}_p$. The addition is thus defined by

$$\mathcal{T} + b_e := \mathcal{T} + \{ b_\tau : \tau \in \mathcal{R}_e \} = \mathcal{T} \setminus \mathcal{R}_e \cup \mathcal{R}_p,$$

which preserves the conformity of triangulations. Figure 5.1 depicts the two possible configurations of a compatible bisection $b_{e_j}$ in 2D.

![Figure 5.1](image_url)

Figure 5.1. Two possible configurations of a compatible bisection $b_{e_j}$ in 2D. The edge with boldface is the compatible refinement edge, and the dash-line represents the bisection.

We now introduce the concepts of generation and level. The generation $g(\tau)$ of any element $\tau \in \mathcal{T}_0$ is set to be 0, and the generation of any subsequent element $\tau$ is 1 plus the generation of its father. For any vertex $p$, the generation $g(p)$ of $p$ is defined as the minimal integer $k$ such that $p \in \mathcal{N}(\mathcal{T}_k)$. Therefore, $g(\tau)$ and $g(p)$ are the minimal number of compatible bisections required to create $\tau$ and $p$ from $\mathcal{T}_0$. Once $p$ belongs to a bisection mesh, it will belong to all successive refinements; hence $g(p)$ is a static quantity insensitive to the level of resolution around $p$. To account for this issue, we define the level $\ell(p)$ of a vertex $p$ to be the maximal generation of elements in the first ring $\mathcal{R}_p$; this is then a dynamic quantity that characterizes the level of resolution around $p$.

We then have the decomposition of bisection grids in terms of compatible bisections; see [18, Theorem 3.1].

Theorem 5.1 (decomposition of bisection grids). Let $\mathcal{T}_0$ be a conforming mesh with initial labeling that enforces the bisection method to satisfy assumption (A2), i.e. for all $k \geq 0$ all uniform refinements $\mathcal{T}_k$ of $\mathcal{T}_0$ are conforming. Then for every $\mathcal{T} \in \mathcal{T}(\mathcal{T}_0)$, there exists a compatible bisection sequence $\mathcal{B} = (b_1, b_2, \ldots, b_J)$ with $J = \# \mathcal{N}(\mathcal{T}) - \# \mathcal{N}(\mathcal{T}_0)$ such that

$$\mathcal{T} = \mathcal{T}_0 + \mathcal{B}.$$  

For a compatible bisection $b_j$ with refinement edge $e_j$, we introduce the bisection triplet

$$T_j := \{ p_j, p_j^+, p_j^- \},$$
where \( p_j^- \) and \( p_j^+ \) are the end points of \( e_j \) and \( p_j \) is its middle point; see Figure 5.2. A vertex can be a middle point of a bisection solely once, when it is created, but instead it can be an end point of a refinement edge repeatedly; in fact this is the mechanism for the level to increment by 1. In addition, since \( p_j^- \) and \( p_j^+ \) already exist when \( p_j \) is created, it follows that

\[ g_j := g(p_j) \geq g(p_j^-), \]

The notion of generation of the bisection is well-defined due to the following lemma, see [18, Lemma 3.3].

**Lemma 5.1** (compatibility and generation). If \( b_j \in B \) is a compatible bisection, then all elements in \( R_j := R_{p_j} \) have the same generation \( g_j \).

In light of the previous lemma, we say that \( g_j \) is the generation of the compatible bisection \( b_j \) : \( R_{e_j} \rightarrow R_{p_j} \). Because by assumption \( h(\tau) \simeq 1 \) for \( \tau \in T_0 \), we have the following important relation between generation and mesh size:

\[ h_j \simeq \gamma^{g_j}, \quad \text{with} \quad \gamma = \left( \frac{1}{2} \right)^{1/d} \in (0,1). \]

Moreover, there exists a constant \( k_* \) depending on the shape regularity of \( T(T_0) \) such that for every vertex \( p \in \mathcal{N}(T_j) \)

\[ \max_{\tau \in R_p} g(\tau) - \min_{\tau \in R_p} g(\tau) \leq k_*, \quad \#R_p \leq k_* \]

Combining this geometric property with Lemma 5.1 (compatibility and generation), we deduce that

\[ g_j - k_* \leq g(\tau) \leq g_j + k_* \quad \forall \tau \in \bar{R}_j := R_{p_j} \cup R_{p_j^-} \cup R_{p_j^+}. \]

Another ingredient for our analysis is the relation between the generation of compatible bisections and their local or enlarged patches [18, Lemmas 3.4 and 3.5].

**Lemma 5.2** (generation and patches). Let \( T_j = T_0 + B \in T(T_0) \) with compatible bisection sequence \( B = (b_1, \ldots, b_J) \). Then the following properties are valid:

- **Nonoverlapping patches:** For any \( j \neq k \) and \( g_j = g_k \), we have

\[ \hat{\omega}_j \cap \hat{\omega}_k = \emptyset. \]

- **Quasi-monotonicity:** For any \( j > i \) and \( \hat{\omega}_j \cap \hat{\omega}_i \neq \emptyset \), we have

\[ g_j \geq g_i - 2k_*. \]

where \( k_* \) is the integer defined in (5.3).

We now investigate the evolution of the level \( \ell(p) \) of a generic vertex \( p \) of \( T \).
Figure 5.3. Two cases of bisection in $\mathcal{R}_q$: the bisection edge $e$ is on the boundary of the patch and $q$ does not belong to the bisection triplet (middle); the node $q$ is an endpoint of $e$ and belongs to the bisection triplet (right). The former can happen a fixed number $k_*$ of times before the second takes place, where $k_*$ depends on the shape regularity of $T(T_0)$.

**Lemma 5.3** (levels of a vertex). If $p \in T_j \cap T_k$, where $T_j$ is a bisection triplet and $T_k$ is the next one to contain $q$ after $T_j$, and $\ell_j(p)$ and $\ell_k(p)$ are the corresponding levels, then

$$\ell_k(p) - \ell_j(p) \leq k_*$$

where $k_*$ is the integer given in (5.3).

**Proof.** Every time a bisection changes the ring $\mathcal{R}_q$, the level of $q$ may increase at most by 1. If the refinement edge $e$ of the bisection is on the boundary of the patch $\omega_q$, then $q$ does not belong to the bisection triplet; see Figure 5.3 (middle). The number of such edges is smaller than a fixed integer $k_*$ that only depends on the shape regularity of $T(T_0)$. Therefore, after at most $k_*$ bisections the vertex $q$ is an endpoint of a bisection triplet $T_k$; see Figure 5.3 (right). This implies $\ell_k(p) \leq \ell_j(p) + k_*$ as asserted. □

We conclude this section with the following sequence of auxiliary meshes

(5.5) $\hat{T}_j := \hat{T}_{j-1} + \{b_i \in \mathcal{B} : g_i = j\}$, $j \geq 1$, $\hat{T}_0 := T_0$,

where $\mathcal{B}$ is the set of compatible bisections (5.1). Note that each bisection $b_i$ in (5.1) does not require additional refinement beyond the refinement patch $\omega_i$ when incorporated in the order of the subscript $i$ according to (5.1). This is not obvious in (5.5) because the bisections are now ordered by generation. The mesh $\hat{T}_j$ contains all elements $\tau$ of generation $g(\tau) \leq j$ leading to the finest graded mesh $T = T_j$. The sequence $\{\hat{T}_j\}_{j=1}^J$ is never constructed but is useful for theoretical purposes in Section 6.

**Lemma 5.4** (conformity of $\hat{T}_j$). The meshes $\hat{T}_j$ are conforming for all $j \geq 0$.

**Proof.** We argue by induction. The starting mesh $\hat{T}_0$ is conforming by construction. Suppose that $\hat{T}_{j-1}$ is conforming. We observe that the bisections $b_i$ with $g_i = j$ are disjoint according to Lemma 5.2 (generation and patches). Suppose that adding $b_i$ does lead to further refinement beyond the refinement patch $\omega_i$. If this were the case, then recursive bisection refinement would end up adding compatible bisections of generation strictly less than $j$ that belong to the refinement chains emanating from $\omega_i$. But such bisections are all included in $\hat{T}_{j-1}$ by virtue of (5.5). This shows that all bisections $b_i$ with $g_i = j$ are compatible with $\hat{T}_{j-1}$ and yield local refinements that keep mesh conformity. □

6. Robust BPX preconditioner for graded bisection grids

In this section, we design and analyze a BPX preconditioner for the integral fractional Laplacian [1,2] on graded bisection grids that it is uniform with respect to both number
of levels $J$ and fractional order $s$. We combine the BPX preconditioner on quasi-uniform grids of Section 4 with the theory for graded bisection grids from [10, 18, 41], summarized in Section 3, that bridges the gap between graded and quasi-uniform grids. Building on Section 4.3, the results in this section—especially Theorem 5.2—apply to the spectral and censored fractional Laplacians, the latter if $s > \frac{1}{2}$ (with a blow up as $s \rightarrow \frac{1}{2}$), because of their spectral equivalence.

6.1. Space decomposition and BPX preconditioner. Let $T_j = T_0 + \{b_1, \ldots, b_J\} \in T(\mathcal{T}_0)$ be a conforming bisection grid obtained from $\mathcal{T}_0$ after $j \leq J$ compatible bisections $\{b_i\}_{i=1}^J$ and let $\mathcal{N}_j = \mathcal{N}(T_j)$ denote the set of interior vertices of $T_j$. Let $\mathcal{V}(T_j)$ be the finite element space of $C^0$ piecewise linear functions over $T_j$ that vanish on $\partial \Omega$ and its nodal basis functions be $\phi_{j,p}$, namely $\mathcal{V}(T_j) = \text{span}\{\phi_{j,p} : p \in \mathcal{N}_j\}$. We define the local spaces

\begin{equation}
V_j = \text{span}\{\phi_{j,q} : q \in T_j \cap \mathcal{N}_j\}, \quad j = 1, \ldots, J
\end{equation}

associated with each bisection triplet $T_j$. We observe that $\dim V_j \leq 3$ and $\text{supp} \phi \subseteq \bar{\mathcal{N}}_j$ for $\phi \in V_j$ and $1 \leq j \leq J$; see Figure 5.2. We indicate by $V := \mathcal{V}(T_j)$ the finite element space over the finest graded grid $T_J$, with interior nodes $\mathcal{P} = \mathcal{N}_J$ and nodal basis functions $\phi_p$

\begin{equation}
V = \text{span}\{\phi_p : p \in \mathcal{P}\}, \quad V_p = \text{span}\{\phi_p\};
\end{equation}

hence $\dim V_p = 1$. Adding the spaces $V_p$ and $V_j$ yields the space decomposition of $V$

\begin{equation}
V = \sum_{p \in \mathcal{P}} V_p + \sum_{j=0}^J V_j.
\end{equation}

We stress that the spaces $V_j$ appear in the order of creation and not of generation, as is typical of adaptive procedures. Remarkably, the functions $\phi_{j,q}$ with $q = \frac{p}{2}$ depend on the order of creation of $V_j$ (see Figure 5.2). Consequently, reordering of $V_j$ by generation, which is convenient for analysis, must be performed with caution; see Sections 6.2 and 6.3.

Let $Q_p$ (resp. $Q_j$) and $I_p$ (resp. $I_j$) be the $L^2$-projection and inclusion operators to and from the discrete spaces $V_p$ (resp. $V_j$), defined in Section 3.1. Inspired by the definition (4.3), we now define the subspace smoothers to be

\begin{align*}
R_j v_j &:= (1 - \eta^s)h_j^2 v_j \quad \forall v_j \in V_j, \\
R_p v_p &:= h_p^2 v_p \quad \forall v_p \in V_p,
\end{align*}

where $R_p$ plays the role of the finest scale whereas $R_j$ represents the intermediate scales. This in turn induces the following BPX preconditioner on graded bisection grids

\begin{equation}
B = \sum_{p \in \mathcal{P}} I_p R_p I_p' + \sum_{j=0}^J I_j R_j I_j' = \sum_{p \in \mathcal{P}} I_p h_p^2 Q_p + (1 - \eta^s) \sum_{j=0}^J I_j h_j^2 Q_j.
\end{equation}

6.2. Boundedness: Proof of (3.5) for graded bisection grids. Let $\bar{J} = \max_{\tau \in T_j} g_{\tau}$ denote the maximal generation of elements in $T_J$. This quantity is useful next to reorder the spaces $V_j$ by generation because $g_j \leq \bar{J}$.

**Proposition 6.1** (boundedness). Let $v = \sum_{p \in \mathcal{P}} v_p + \sum_{j=0}^J v_j$ be a decomposition of $v \in V$ according to (6.3). Then, there exists a constant $c_1 > 0$ independent of $J$ and $s$ such that

\begin{equation}
|v|^2_s \leq c_1 \left( \sum_{p \in \mathcal{P}} h_p^{-2s} \|v_p\|_0^2 + \frac{1}{1 - \eta^s} \sum_{j=0}^J h_j^{-2s} |v_j|_0^2 \right),
\end{equation}

whence the preconditioner $B$ in (6.4) satisfies $\lambda_{\text{max}}(BA) \leq c_1$.

**Proof.** We resort to Lemma 3.5 (local inverse inequality) with $\sigma = s$ and $\mu = 0$, which is valid on the graded grid $T_J$, to write

\begin{equation}
|v|^2_s = \left| \sum_{p \in \mathcal{P}} v_p + \sum_{j=0}^J v_j \right|^2_s \leq \left| \sum_{p \in \mathcal{P}} v_p \right|^2_s + \sum_{j=0}^J |v_j|^2_s \leq \sum_{p \in \mathcal{P}} h_p^{-2s} \|v_p\|_0^2 + \sum_{j=0}^J |v_j|^2_s.
\end{equation}
In order to deal with the last term, we reorder the functions \( v_j \) by generation and observe that \( \text{supp} \, v_j \subset \tilde{\omega}_j \). We thus define \( w_k = \sum_{g_j=k} v_j \) and use (6.4) to infer that \( w_k \in V_{k+k_s} = \mathbb{V}(T_{k+k_s}) \). Similar to the proof of Proposition 4.1 using Theorem 4.2 (norm equivalence), the fact that \( h_k \simeq \gamma^k \) and Lemma 3.3 (s-uniform decomposition), we have

\[
\left| \sum_{j=0}^J v_j \right|_s^2 = \left| \sum_{k=0}^{J} \sum_{g_j=k} v_j \right|_s^2 = \left| \sum_{k=0}^{J} w_k \right|_s^2 \\
\simeq \sum_{\ell=0}^{J+k_s} \gamma^{-2sf} \left| Q_\ell - Q_{\ell-1} \right| \sum_{k=0}^{J} w_k \\
= \inf_{z_\ell \in V_\ell} \sum_{\ell=0}^{J+k_s} \left[ \gamma^{-2s(J+k_s)} \left\| z_{J+k_s} \right\|^2_0 + \sum_{\ell=0}^{J+k_s} \gamma^{-2s} \left\| z_\ell \right\|^2_0 \right].
\]

Choosing \( z_\ell = 0 \) for \( \ell \leq k_s - 1 \) and \( z_\ell = w_{\ell-k_s} \in V_\ell \) for \( \ell \geq k_s \) we get

\[
\left| \sum_{j=0}^J v_j \right|_s^2 \lesssim \gamma^{-2s} \sum_{k=0}^{J} \gamma^{-2sk} \left\| w_k \right\|^2_0.
\]

In view of Lemma 3.2 we see that the enlarged patches \( \tilde{\omega}_j \) and \( \tilde{\omega}_i \) have finite overlap depending only on shape regularity of \( \mathcal{T}(T_0) \) provided \( g_j = g_i \), whence

\[
\left\| w_k \right\|^2_0 \lesssim \sum_{g_j=k} \left\| v_j \right\|^2_0.
\]

This in conjunction with \( \frac{1-\gamma}{1-\gamma^s} \simeq 1 \) and the fact that \( k_s \) is uniformly bounded yields

(6.7) \[
\left| \sum_{j=0}^J v_j \right|_s^2 \lesssim \frac{1}{1-\gamma} \sum_{k=0}^{J} \gamma^{-2sk} \sum_{g_j=k} \left\| v_j \right\|^2_0 \simeq \frac{1}{1-\gamma} \sum_{j=0}^{J} h_j^{-2s} \left\| v_j \right\|^2_0.
\]

Combining (6.6) and (6.7) leads to (6.5) as asserted. Finally, the estimate \( \lambda_{\text{max}}(BA) \leq c_1 \) follows directly from Lemma 3.2 (estimate on \( \text{cond}(BA) \)).

6.3. Stable decomposition: Proof of (3.4) for graded bisection grids. We start with a review of the case of quasi-uniform grids in Corollary 4.1 (stable decomposition) and a roadmap of our approach. We point out that robustness with respect to both \( J \) and \( s \), most notably the handling of factor \( (1-\gamma^s)^{-1} \) on coarse levels, is due to the combination of Lemma 3.3 (s-uniform decomposition) and Theorem 4.2 (norm equivalence), which in turn relies on Lemma 3.4 (s-uniform interpolation). Since Lemma 3.4 fails on graded bisection grids, applying Lemma 3.3 to such grids faces two main difficulties: (a) Theorem 4.2 does not hold even for \( s = 1 \); (b) the spaces \( V_j \) in (6.1) and \( V_p \) in (6.2) are locally supported, while the s-uniform interpolation requires nested spaces (see in Lemma 3.4). To overcome these difficulties, we create a family of nested spaces \( \{W_k\}_{k=0}^J \) with \( W_j = V \) upon grouping indices according to generation and level around \( k \): if

(6.8) \[
J_k := \{ 0 \leq j \leq J : g_j \leq k \}, \quad \mathcal{P}_k := \{ p \in \mathcal{P} : \ell(p) \leq k \},
\]

then we define \( W_k \) to be

(6.9) \[
W_k := \sum_{j \in J_k} V_j + \sum_{p \in \mathcal{P}_k} V_p.
\]

Our approach consists of three steps. The first step, developed in Section 6.3.1 is to derive a global decomposition based on \( W_k \). Since the levels within \( W_k \) are only bounded above, to account for coarse levels we invoke a localization argument based on a slicing Scott-Zhang operator as in [18], which gives the stability result (3.7) on \( \{W_k\}_{k=0}^J \) via Lemma 3.3 (s-uniform decomposition) for \( s = 1 \); we bridge the gap to \( 0 < s < 1 \) via Lemma 3.4 (s-uniform interpolation). The space \( W_k \) is created for theoretical convenience, but never constructed in
practice, because there is no obvious underlying graded bisection grid on which the functions of $W_k$ are piecewise linear. This complicates the stable decomposition of $W_k$ into local spaces and requires a characterization of $W_k$ in terms of the space $\hat{V}_k = \mathcal{V}(\bar{T}_k)$ of piecewise linear functions over $\bar{T}_k$. The second step in Section 6.3.2 consists of proving
\[
\hat{V}_k \subset W_k \subset \hat{V}_{k+k_*},
\]
where $k_*$ is constant. Therefore, the space $W_k$ of unordered bisections of generation and level $\leq k$ is equivalent, up to level $k_*$, to the space $\hat{V}_k$ of ordered bisections of generation $\leq k$; note that the individual spaces $V_j$ might not coincide though. In the last step, performed in Section 6.3.3 we construct a stable decomposition for graded bisection grids and associated BPX preconditioner $\hat{B}$. We also show that $\hat{B}$ is equivalent to $B$ in (6.4).

6.3.1. Global $L^2$-orthogonal decomposition of $W_k$. We recall that the Scott-Zhang quasi-interpolation operator $S_j : V \to \mathcal{V}(T_j)$ can be defined at a node $p \in P$ through the dual basis function on arbitrary elements $\tau \subset \mathcal{R}_p$ [15,18]. We exploit this flexibility to define a suitable quasi-interpolation operator $S_j$ as follows provided $S_j^{-1} : V \to \mathcal{V}(T_j^{-1})$ is already known. Since $T_j = T_j^{-1} + b_j$ and the compatible bisection $b_j$ changes $T_j^{-1}$ locally in the bisection patch $\omega_{p_j}$ associated with the new vertex $p_j$, we set $S_j v(p) := S_j^{-1} v(p)$ for all $p \in \mathcal{N}_j \setminus T_j$, where $T_j$ is the bisection triplet (5.2). We next define $S_j v(p_j)$ using a simplex $\tau \in \mathcal{R}_j$ newly created by the bisection $b_j$. If $p = p_j^\pm \in T_j$ and $\tau \in T_j^{-1}$ is the simplex used to define $S_j^{-1} v(p)$, then we define $S_j v(p)$ according to the following rules:

1. if $\tau \subset \omega_p(T_j)$ we keep the nodal value of $S_j^{-1} v$, i.e. $S_j v(p) = S_j^{-1} v(p)$;
2. otherwise we choose a new $\tau \subset \omega_p(T_j') \cap \omega_p(T_j^{-1})$ to define $S_j v(p)$;

note that $\tau \in \mathcal{R}_j$ in case (2). Once $\tau \in T_j$ has been chosen, then definition of $S_j v(p)$ for $p \in T_j$ is the same as in [15,10]. This construction guarantees the local stability bound [15]
\[
(6.10) \quad h_p^{d/2} |S_j v(p)| \lesssim \|v\|_{\omega_p}, \quad \forall p \in \mathcal{N}_j,
\]
and that the slicing operator $S_j - S_j^{-1}$ is supported in the enlarged patch $\bar{\omega}_j$, namely
\[
(6.11) \quad (S_j - S_j^{-1}) v \in V_j \quad \forall 1 \leq j \leq J.
\]

**Lemma 6.1** (stable $L^2$-orthogonal decomposition). Let $\hat{Q}_k : V \to W_k$ be the $L^2$-orthogonal projection operator onto $W_k$ and $\hat{Q}_k = 0$. For any $v \in V$, the global $L^2$-orthogonal decomposition $v = \sum_{k=0}^J (\hat{Q}_k - \hat{Q}_{k-1}) v$ satisfies
\[
(6.12) \quad \sum_{k=0}^J \gamma^{-2k} \|(\hat{Q}_k - \hat{Q}_{k-1}) v\|_0^2 \lesssim |v|_T^2,
\]
where the hidden constant is independent of $0 \leq s \leq 1$ and $J$.

**Proof.** We rely on the auxiliary spaces $\bar{V}_k = \mathcal{V}(\bar{T}_k)$ defined over uniformly refined meshes $\bar{T}_k$ of $T_0$ for $0 \leq k \leq J$. Let $\hat{Q}_k : \bar{V}_j \to \bar{V}_k$ denote the $L^2$-orthogonal projection operator onto $\bar{V}_k$ and consider the global $L^2$-orthogonal decomposition $v = \sum_{k=0}^J \hat{v}_k$ of any $v \in V \subset \bar{V}_j$, where $\hat{v}_k := (\hat{Q}_k - \hat{Q}_{k-1}) v$. This decomposition is stable in $H^1$ [20,43,10]
\[
\sum_{k=0}^J \gamma^{-2k} \|\hat{v}_k\|_0^2 \lesssim |v|_T^2.
\]

If $g_j$ is the generation of bisection $b_j$ and $g_j > k$, then $\hat{v}_k$ is piecewise linear in $\omega_{e_j}$ (the patch of the refinement edge $e_j$), whence $(S_j - S_{j-1}) \hat{v}_k = 0$ and the slicing operator detects frequencies $k \geq g_j$. Consider now the decomposition $v = \sum_{k=0}^J \hat{v}_k$ of $v \in V$ where
\[
(6.13) \quad \hat{v}_k := \sum_{g_j = k}^{g_j = k} (S_j - S_{j-1}) v = \sum_{g_j = k}^{g_j = k} (S_j - S_{j-1}) \sum_{\ell = k}^J \hat{v}_\ell \in W_k.
\]
In view of Lemma 5.2 (generation and patches) and shape regularity of \(T(T_0)\), enlarged patches \(\tilde{\omega}_j\) with the same generation \(g_j = k\) have a finite overlapping property. This, in conjunction with (6.10) and (6.11) as well as the \(L^2\)-orthogonality of \(\{\tilde{v}_\ell\}_{\ell=1}^J\), yields

\[
\|v_k\|_0^2 \lesssim \sum_{g_j=k} \|S_j - S_{j-1}\|_{0,\tilde{\omega}_j}^2 \lesssim \sum_{g_j=k} \|S_j - S_{j-1}\|_{0,\tilde{\omega}_j}^2 \lesssim \sum_{\ell=k}^J \|S_j - S_{j-1}\|_{0,\tilde{\omega}_j}^2 \lesssim \sum_{\ell=k}^J \|\tilde{v}_\ell\|_0^2.
\]

We use Lemma 3.3 (s-uniform decomposition) with \(s = 1\), together with (6.13), to obtain

\[
\sum_{k=0}^J \gamma^{-2k}\|\tilde{Q}_k - \tilde{Q}_{k-1}\|_{0,\tilde{\omega}_j}^2 \lesssim \sum_{k=0}^J \gamma^{-2k}\|\tilde{v}_j\|_0^2 + \sum_{k=0}^{j-1} \frac{\gamma^{-2k}}{1 - \gamma^2} \|w_k\|_0^2.
\]

Employing the preceding estimate of \(\|v_k\|_0^2\) and reordering the sum implies

\[
\sum_{k=0}^J \gamma^{-2k}\|\tilde{Q}_k - \tilde{Q}_{k-1}\|_{0,\tilde{\omega}_j}^2 \lesssim \sum_{k=0}^J \gamma^{-2k}\|\tilde{v}_j\|_0^2 + \sum_{k=0}^{j-1} \frac{\gamma^{-2k}}{1 - \gamma^2} \|\tilde{v}_j\|_0^2
\]

\[
= \sum_{k=0}^J \gamma^{-2k}\|\tilde{v}_j\|_0^2 + \sum_{k=0}^{j-1} \frac{\gamma^{-2k}}{1 - \gamma^2} \|\tilde{v}_j\|_0^2
\]

\[
= \sum_{k=0}^J \gamma^{-2k}\|\tilde{v}_j\|_0^2 + \sum_{k=0}^{j-1} \frac{\gamma^{-2k}}{1 - \gamma^2} \|\tilde{v}_j\|_0^2 \lesssim \sum_{k=0}^J \gamma^{-2k}\|\tilde{v}_j\|_0^2 \lesssim |v|_s^2.
\]

Hence, we have shown that (6.12) holds for \(s = 1\). The desired estimate for arbitrary \(0 \leq s \leq 1\) follows by Lemma 3.4 (s-uniform interpolation).

As a consequence of Lemma 3.3 (s-uniform decomposition) and Lemma 6.1 (stable \(L^2\)-orthogonal decomposition), we deduce the following property.

**Corollary 6.1** (s-uniform decomposition on \(W_k\)). For every \(v \in V\), there exists a decomposition \(v = \sum_{k=0}^J w_k\) with \(w_k \in W_k\) for all \(k = 0, 1, \ldots, J\) and

\[
\gamma^{-2s}\|w_j\|_0^2 + \sum_{k=0}^{j-1} \frac{\gamma^{-2k}}{1 - \gamma^2} \|w_k\|_0^2 \lesssim \sum_{k=0}^J \gamma^{-2k}\|\tilde{v}_j\|_0^2 \lesssim |v|_s^2.
\]

**6.3.2. Characterization of \(W_k\).** We now study the geometric structure of the spaces \(W_k\), defined in (6.9), which is useful in the construction of a stable decomposition of \(V\). Recalling definition (5.5), our first goal is to compare \(W_k\) with the space

\[
\mathbb{V}_k := \mathbb{V}(\tilde{\mathcal{K}}_k)
\]

of \(C^0\) piecewise linear functions over \(\tilde{\mathcal{K}}_k\) that have vanishing trace. We will show below (6.14)

\[
\mathbb{V}_k \subset W_k;
\]

see Lemmas 6.3 and 6.4. We start with the set of interior vertices of \(W_k\)

\[
\mathcal{V}_k := \mathcal{B}_k \cup \mathcal{P}_k, \quad \mathcal{B}_k := \bigcup \{T_j : j \in \mathcal{J}_k\}, \quad \mathcal{P}_k = \{p \in \mathcal{P} : \ell(p) \leq k\}.
\]

**Lemma 6.2** (geometric structure of \(W_k\)). Functions in \(W_k\) are \(C^0\) piecewise linear on the auxiliary mesh \(\tilde{\mathcal{K}}_{k+s}\), where \(s\) is given in (5.3). Equivalently, \(W_k \subset \mathbb{V}_{k+s}\).

**Proof.** We examine separately each vertex \(q \in \mathcal{V}_k\). If \(q \in \mathcal{B}_k\), then \(\ell(q) \leq k\) and all elements \(\tau \in \mathcal{R}(q)\) have generation \(g(\tau) \leq k\) by definition of level; hence \(\tau \in \tilde{\mathcal{K}}_k\) for all \(\tau \in \mathcal{R}(q)\). If \(q \in \mathcal{B}_k \setminus \mathcal{P}_k\) instead, then the patch of \(q\) shares elements with that of the bisection node \(p_j\)

\[
\min_{\tau \in \mathcal{R}_j(q)} g(\tau) \leq g(p_j) = g_j \leq k,
\]
where $\mathcal{R}_j(q)$ is the ring of elements containing $q$ in the mesh $\mathcal{T}_j$. Property (5.3) yields
\[
\max_{\tau \in \mathcal{R}_j(q)} g(\tau) \leq \min_{\tau \in \mathcal{R}_j(q)} g(\tau) + k_* \leq k + k_*.
\]

It turns out that all elements $\tau \in \mathcal{R}_j$, the enlarged ring around $p_j$, have generation $g(\tau) \leq k + k_*$, whence $\tau \in \mathcal{T}_{k+k_*}$. It remains to realize that any function $w \in \mathcal{V}_j$ is thus piecewise linear over $\mathcal{T}_{k+k_*}$ and vanishes outside $\tilde{\omega}_j$. 

We next exploit the $L^2$-stability of the nodal basis $\{\hat{\phi}_q\}_{q \in \hat{\mathcal{V}}}$ of $\hat{\mathcal{V}}_{k+k_*}$, where $\hat{\mathcal{V}} = \hat{\mathcal{V}}_{k+k_*}$ is the set of interior vertices of $\hat{T} = \mathcal{T}_{k+k_*}$. In fact, if $w = \sum_{q \in \hat{\mathcal{V}}} w(q) \hat{\phi}_q$, then
\[
\|w\|_0^2 = \sum_{\tau \in \hat{T}} |\tau| \sum_{q \in \tau} w(q)^2 = \sum_{q \in \hat{\mathcal{V}}} w(q)^2 |\tau| \sum_{q \in \tau} w(q)^2 \leq \|\phi_q\|_0^2.
\]

Our goal now is to represent each function $\hat{\phi}_q \in \hat{\mathcal{V}}_{k+k_*}$ in terms of functions of $W_{k+k_*}$, which in turn shows $\hat{\mathcal{V}}_{k+k_*} \subset W_{k+k_*}$, and thus (6.14). We start with a partition of $\hat{\mathcal{V}}_{k+k_*}$,
\[
\hat{\mathcal{P}}_{k+k_*} := \{q \in \hat{\mathcal{V}}_{k+k_*} : \hat{\ell}(q) \leq k + k_* - 1\}, \quad \hat{\mathcal{P}}^c_{k+k_*} := \hat{\mathcal{V}}_{k+k_*} \setminus \hat{\mathcal{P}}_{k+k_*},
\]

where $\hat{\ell}(q) \leq k + k_*$ is the level of $q$ on $\hat{T}_{k+k_*}$. Consequently, $\hat{\ell}(q) = k + k_*$ for all $q \in \hat{\mathcal{P}}_{k+k_*}$ and the corresponding functions $\hat{\phi}_q$ have all the same scaling due to shape regularity of $\hat{T}(\mathcal{T}_0)$. In the next two lemmas we represent the functions $\hat{\phi}_q$ in terms of $W_{k+k_*}$.

**Lemma 6.3** (nodal basis $\hat{\phi}_q$ with $q \in \hat{\mathcal{P}}_{k+k_*}$). For any $q \in \hat{\mathcal{P}}_{k+k_*}$, there holds
\[
\hat{\phi}_q = \phi_q, \quad q \in \mathcal{P}_{k+k_*},
\]
where $\mathcal{P}_k$ is defined in (6.8); hence, $\hat{\phi}_q \in W_{k+k_*}$.

**Proof.** Since $\hat{\ell}(q) \leq k + k_* - 1$, all elements $\tau \in \mathcal{R}(q)$ have generation $g(\tau) \leq k + k_* - 1$. This implies that no further bisection is allowed in $\tau$ because all the bisections with generation lesser or equal than $k + k_*$ have been incorporated in $\hat{T}_{k+k_*}$ by definition. Therefore, $\mathcal{R}(q)$ belongs to the finest grid $T$ and $\ell(q) = \hat{\ell}(q) \leq k + k_* - 1$, whence $\hat{\phi}_q \in W_{k+k_*}$. 

Next, we consider a nodal basis function $\hat{\phi}_q$ corresponding to $q \in \hat{\mathcal{P}}^c_{k+k_*}$. There exists a bisection triplet $T_{y_j}$ that contains $q$ and $k \leq \ell_{y_j}(q) \leq k + k_*$, for otherwise $\ell_{y_j}(q) < k$ would violate Lemma (5.3) (levels of a vertex). We thus deduce
\[
k - k_* \leq \ell_{y_j}(q) - k_* \leq g_{y_j}(q) \leq \ell_{y_j}(q) \leq k + k_*.
\]

In accordance with (6.14), we denote by $\phi_{y_j,q}$ the nodal basis function of $V_{y_j}$ centered at $q$. We first show that $\phi_q$ can be obtained by a suitable modification of $\phi_{y_j,q}$ within $W_{k+k_*}$.

**Lemma 6.4** (nodal basis $\hat{\phi}_q$ with $q \in \hat{\mathcal{P}}^c_{k+k_*}$). For any $q \in \hat{\mathcal{P}}^c_{k+k_*}$, let
\[
\mathcal{S}_q := \{j \in \mathcal{J}_{k+k_*} : j > j_q, \omega_j \cap \text{supp} \phi_{j,q} \neq \emptyset\}
\]
be the set of bisection indices $j > j_q$ such that $g_j \leq k + k_*$, $\phi_{j,p_j}$ be the function of $V_j$ centered at the bisection vertex $p_j$ and $\omega_j = \text{supp} p_j$. Then there exist numbers $c_{j,q} \in (-1,0]$ for $j \in \mathcal{S}_q$ such that the nodal basis function $\phi_{y_j,q} \in V_{k+k_*}$ associated with $q$ can be written as
\[
\hat{\phi}_q = \phi_{y_j,q} + \sum_{j \in \mathcal{S}_q} c_{j,q} \phi_{j,p_j},
\]
and the representation is $L^2$-stable, i.e.,
\[
\|\phi_q\|_0^2 \simeq \|\phi_{y_j,q}\|_0^2 + \sum_{j \in \mathcal{S}_q} c_{j,q}^2 \|\phi_{j,p_j}\|_0^2.
\]
**Proof.** The discussion leading to (6.16) yields $k \leq \ell_{j_q}(q) \leq k + k_s$ which, combined with (5.3), implies that all elements $\tau \in \mathcal{R}_{j_q}(q)$ have generation between $k - k_s$ and $k + k_s$. The idea now is to start from the patch $\mathcal{R}_{j_q}(q)$, the local conforming mesh associated with $\phi_{j_q,q}$, and successively refine it with compatible bisections in the spirit of the construction of $\hat{T}_j$ in (5.5) until we reach the level $k + k_s$; see Figure 6.1. To this end, let $\hat{T}_{k-k_s}(q) := \mathcal{R}_{j_q}(q)$ and consider the sequence of local auxiliary meshes

$$
\hat{T}_j(q) := \hat{T}_{j-1}(q) + \{b_i \in B : i \in S_q, \ g_i = j \} \quad k - k_s + 1 \leq j \leq k + k_s,
$$

which are conforming according to Lemma 5.4 (conformity of $\hat{T}_j$).

**FIGURE 6.1.** Local auxiliary meshes $\hat{T}_{j,q}$ with $|j - k| \leq k_s = 2$. Index sets $S_{k-1,q} = \{i_1\}$, $S_{k,q} = \{i_2, i_3, i_4\}$, $S_{k+1,q} = \{i_5, i_6, i_7\}$, $S_{k+2,q} = \{i_8, i_9\}$ of compatible bisections to transition from $\hat{\phi}_{j-1,q}$ to $\hat{\phi}_{j,q}$. The support of $\hat{\phi}_{j,q}$ is monotone decreasing as $j$ increases and is plotted in grey.

We now consider the following recursive procedure: let $\hat{\phi}_{k-k_s,q} := \phi_{j,q}$ and

$$
(6.19) \quad \hat{\phi}_{j,q} := \hat{\phi}_{j-1,q} - \sum_{i \in S_{j,q}} \hat{\phi}_{j-1,q}(p_i) \phi_{i,p_i} \quad k - k_s + 1 \leq j \leq k + k_s,
$$

where $p_i$ is the bisection node of $b_i \in B$ and

$\quad S_{j,q} := \{i \in J_{k+2} : g_i = j, \ \omega_i \cap \text{supp} \hat{\phi}_{j-1,q} \neq \emptyset\}$.

Unless $p_i$ belongs to the boundary of supp $\hat{\phi}_{j-1,q}$, the construction (6.19) always modifies $\hat{\phi}_{j-1,q}$; compare Figure 6.1b with Figures 6.1c–6.1e. In view of Lemma 5.2 (generation and patches), the sets $\hat{\omega}_i$ for $i \in S_{j,q}$ are disjoint, whence $\hat{\phi}_{j,q}(p) = \delta_{pq}$ for all nodes $p$ of $\hat{T}_{j}(q)$ and $\hat{\phi}_{j,q}$ is the nodal basis function centered at $q$ on $\hat{T}_{j}(q)$. Morever,

$$
\hat{\phi}_{j,q} = \hat{\phi}_{j-1,q} + \sum_{i \in S_{j,q}} c_{i,q} \phi_{i,p_i},
$$

with coefficients $c_{i,q} \in (-1, 0)$. The scales of these functions being comparable yields

$$
\|\hat{\phi}_{j,q}\|^2_0 \simeq \|\hat{\phi}_{j-1,q}\|^2_0 + \sum_{i \in S_{j,q}} c_{i,q}^2 \|\phi_{i,p_i}\|^2_0.
$$

Since $k_s$ is uniformly bounded depending on shape regularity of $\overline{T}(T_0)$, iterating these two expressions at most $2k_s$ times leads to (6.17) and (6.18), and concludes the proof. \hfill \Box

We are now in a position to exploit the representation of nodal basis of $\hat{V}_{k+2s}$, given in Lemmas 6.3 and 6.4, to decompose functions in $W_k$. We do this next.

**Corollary 6.2** ($L^2$-stable decomposition of $W_k$). Given any $0 \leq k \leq \bar{j}$ consider the sets

$$
(6.20) \quad \mathcal{P}_{k+k_s} = \{q \in \mathcal{P} : \ell(q) \leq k + k_s\}, \quad \mathcal{I}_{k+k_s} = \{0 \leq i \leq J : k - k_s \leq g_i \leq k + k_s\}.
$$
Then, every function \( w \in W_k \) admits a \( L^2 \)-stable decomposition
\[
(6.21) \quad w = \sum_{q \in \tilde{P}_{k+k_*}} w_q + \sum_{j \in \mathcal{I}_{k+k_*}} w_j, \quad \|w\|_0^2 \simeq \sum_{q \in \tilde{P}_{k+k_*}} \|w_q\|_0^2 + \sum_{j \in \mathcal{I}_{k+k_*}} \|w_j\|_0^2,
\]
where \( w_q \in V_q \) for all \( q \in \mathcal{P}_{k+k_*} \) and \( w_j \in V_j \) for all \( j \in \mathcal{I}_{k+k_*} \).

Proof. Invoking Lemma 6.2 (geometric structure of \( W_k \)), we infer that \( w \in \tilde{V}_{k+k_*} \), which yields the \( L^2 \)-stable decomposition of \( w \) in terms of nodal basis of \( \tilde{V}_{k+k_*} \):
\[
w = \sum_{q \in \tilde{V}_{k+k_*}} w(q) \hat{\phi}_q = \sum_{q \in \tilde{P}_{k+k_*}} w(q) \hat{\phi}_q + \sum_{q \in \tilde{P}^c_{k+k_*}} w(q) \hat{\phi}_q.
\]
On the one hand, Lemma 6.3 (nodal basis \( \hat{\phi}_q \) with \( q \in \tilde{P}_{k+k_*} \)) implies that \( \hat{\phi}_q = \phi_q \) and \( \tilde{P}_{k+k_*} \subset \mathcal{P}_{k+k_*} \); hence we simply take \( w_q := w(q)\hat{\phi}_q \). On the other hand, using the representation (6.17) of \( \hat{\phi}_q \) from Lemma 6.4 (nodal basis \( \hat{\phi}_q \) with \( q \in \tilde{P}^c_{k+k_*} \)) and reordering, we arrive at
\[
\sum_{q \in \tilde{P}^c_{k+k_*}} w(q)\hat{\phi}_q = \sum_{q \in \tilde{P}^c_{k+k_*}} w(q) (\phi_{j_q,q} + \sum_{j \in S_q} c_{j,q} \phi_{j,p_j}) = \sum_{j \in \mathcal{I}_{k+k_*}} w_j,
\]
where
\[
w_j := \sum_{j_q = j} w(q) \phi_{j,q} + \sum_{S_q \ni j} w(q) c_{j,q} \phi_{j,p_j} \in V_j.
\]
This gives the decomposition (6.21). The \( L^2 \)-stability (6.15) of \( \{\hat{\phi}_q\}_{q \in \tilde{V}_{k+k_*}} \)
\[
\|w\|_0^2 \simeq \sum_{q \in \tilde{P}_{k+k_*}} w(q)^2 \|\hat{\phi}_q\|_0^2 + \sum_{q \in \tilde{P}^c_{k+k_*}} w(q)^2 \|\hat{\phi}_q\|_0^2,
\]
in conjunction with (6.18), gives
\[
\|w\|_0^2 \simeq \sum_{q \in \tilde{P}_{k+k_*}} \|w(q)\hat{\phi}_q\|_0^2 + \sum_{q \in \tilde{P}^c_{k+k_*}} w^2(q) \left( \|\phi_{j_q,q}\|_0^2 + \sum_{j \in S_q} c_{j,q}^2 \|\phi_{j,p_j}\|_0^2 \right)
\]
\[
= \sum_{q \in \tilde{P}_{k+k_*}} \|w_q\|_0^2 + \sum_{j \in \mathcal{I}_{k+k_*}} \left( \sum_{j_q = j} w^2(q) \|\phi_{j,q}\|_0^2 + \sum_{S_q \ni j} w^2(q) c_{j,q}^2 \|\phi_{j,p_j}\|_0^2 \right).
\]
To prove the \( L^2 \)-stability in (6.21), it remains to show that the term in parenthesis is equivalent to \( \|w_j\|_0^2 \) for any \( j \in \mathcal{I}_{k+k_*} \), which in turn is a consequence of the number of summands being bounded uniformly. We first observe that the cardinality of \( \{q : j_q = j\} \) is at most three because this corresponds to \( q \in T_j \), the \( j \)-th bisection triplet. Finally, the cardinality of the set \( \{q \in \tilde{P}^c_{k+k_*} : j \in S_q \cap \mathcal{I}_{k+k_*}\} \) is bounded uniformly by a constant that depends solely on shape regularity of \( T_0 \). To see this, note that \( \tilde{T}(q) = k + k_* \) yields \( k \leq g(\tau) \leq k + k_* \) for all elements \( \tau \) within \( \text{supp} \phi_{j,q} \) and \( k - k_* \leq g_j \leq k + k_* \), whence the number of vertices \( q \) such that \( \text{supp} \phi_{j,q} \cap \omega_j \neq \emptyset \) is uniformly bounded as asserted. Hence
\[
\|w_j\|_0^2 \simeq \sum_{j_q = j} w^2(q) \|\phi_{j,q}\|_0^2 + \sum_{S_q \ni j} w^2(q) c_{j,q}^2 \|\phi_{j,p_j}\|_0^2
\]
yields the norm equivalence in (6.21) and finishes the proof. \( \square \)

6.3.3. Construction of stable decomposition. We first construct a BPX preconditioner that hinges on the space decomposition of Section 6.3.1 and the nodal basis functions just discussed in Section 6.3.2. We next show that this preconditioner is equivalent to (6.4).
Theorem 6.1 (stable decomposition on graded bisection grids). For every \( v \in V \), there exist \( v_p \in V_p \) with \( p \in \mathcal{P} \), \( v_{p,k} \in V_{p,k} \) with \( p \in \mathcal{P}_{k+k_*} \), and \( v_{j,k} \in V_j \) with \( j \in \mathcal{I}_{k+k_*} \), such that

\[
(6.22) \quad v = \sum_{p \in \mathcal{P}} v_p + \sum_{k=0}^{j} \left( \sum_{q \in \mathcal{P}_{k+k_*}} v_{q,k} + \sum_{j \in \mathcal{I}_{k+k_*}} v_{j,k} \right),
\]

where \( \mathcal{P}_{k+k_*} \) and \( \mathcal{I}_{k+k_*} \) are given in (6.20), and there exists a constant \( c_0 \) independent of \( s \) and \( J \) such that

\[
(6.23) \quad \gamma^{-2s} \sum_{p \in \mathcal{P}} \| v_p \|^2_0 + \sum_{k=0}^{j} \gamma^{-2sk} \left( \sum_{p \in \mathcal{P}_{k+k_*}} \| v_{p,k} \|^2_0 + \sum_{j \in \mathcal{I}_{k+k_*}} \| v_{j,k} \|^2_0 \right) \leq c_0 |v|^2.
\]

Proof. We construct the decomposition (6.22) in three steps.

Step 1: Decomposition on \( W_k \). Applying Corollary 6.1 (\( s \)-uniform decomposition on \( W_k \)), we observe that there exist \( w_k \in W_k, k = 0, 1, \cdots, J \) such that

\[
(6.24) \quad \gamma^{-2s} \sum_{p \in \mathcal{P}} \| v_{p,k} \|^2_0 + \sum_{k=0}^{j-1} \gamma^{-2sk} \| w_k \|^2_0 \lesssim |v|^2.
\]

Step 2: Finest scale. We let \( \{ \phi_{p} \}_{p \in \mathcal{P}} \) be the nodal basis of \( V \) and set \( v_p := w_j(p)\phi_p \); hence \( w_j = \sum_{p \in \mathcal{P}} v_p \). Applying the \( L^2 \)-stability (6.15) to \( \{ \phi_p \}_{p \in \mathcal{P}} \) gives

\[
(6.25) \quad \| w_j \|^2_0 \lesssim \sum_{p \in \mathcal{P}} \| v_p \|^2_0.
\]

We also choose the finest scale of \( v_{p,k} \) and \( v_{j,k} \) to be \( v_{q,j} = 0 \) and \( v_{j,j} = 0 \).

Step 3: Intermediate scales. By Corollary 6.2 (\( L^2 \)-stable decomposition of \( W_k \)), we have the \( L^2 \)-stable decomposition (6.21) of \( w_k \in W_k \) for every \( k = 0, \ldots, J-1 \). Combining the stability bound (6.24) with (6.25) and (6.21), we deduce the stable decomposition (6.23). \( \square \)

In view of Theorem 6.1 above, we consider the BPX preconditioner

\[
(6.26) \quad \hat{B} := \gamma^{2s} \sum_{p \in \mathcal{P}} I_p Q_p + (1 - \gamma^{2s}) \sum_{k=0}^{J} \sum_{j \in \mathcal{I}_{k+k_*}} I_j Q_j.
\]

The following corollary is a direct consequence of (6.23) and (3.4).

Corollary 6.3 (uniform bound for \( \lambda_{\text{min}}(\hat{B}A) \)). The preconditioner \( \hat{B} \) in (6.26) satisfies

\[
\lambda_{\text{min}}(\hat{B}A) \geq c_0^{-1}
\]

We are now ready to prove the main result of this section, namely that \( B \) in (6.4) is a robust preconditioner for \( A \) on graded bisection grids. To this end, we need to show that \( \hat{B} \) in (6.26) is spectrally equivalent to \( B \).

Theorem 6.2 (uniform preconditioning on graded bisection grids). Let \( \Omega \) be a bounded Lipschitz domain and \( s \in (0,1) \). Let \( V \) be the space of continuous piecewise linear finite elements over a graded bisection grid \( \mathcal{T} \), and consider the space decomposition (6.3). The corresponding BPX preconditioner \( B \) in (6.4), namely

\[
B = \sum_{p \in \mathcal{P}} I_p h_p^{2s} Q_p + (1 - \gamma^s) \sum_{j=0}^{J} I_j h_j^{2s} Q_j,
\]

is spectrally equivalent to \( \hat{B} \) in (6.26), whence \( \lambda_{\text{min}}(BA) \geq c_0^{-1} \). Therefore, the condition number of \( BA \) satisfies

\[
\text{cond} (BA) \lesssim c_0 c_1,
\]

where the constants \( c_0 \) and \( c_1 \) are independent of \( s \) and \( J \) and given in (6.23) and (6.5).
Proof. We show that the ratio \( (Bv,v) \) is bounded below and above by constants independent of \( s \) and \( J \) for all \( v \in V \). We first observe that for \( p \in P \) with level \( \ell(p) \), we have

\[
h_p^{2s} \simeq \gamma^{2s\ell(p)} = \gamma^{2s(J+1)} + (1 - \gamma^{2s}) \sum_{k=\ell(p)}^{J} \gamma^{2sk},
\]

whence \( B_1 := \sum_{p \in P} I_p h_p^{2s} Q_p \) and \( v_p = Q_p v \) satisfy

\[
(B_1 v, v) \simeq \gamma^{2sJ} \sum_{p \in P} \|v_p\|_0^2 + (1 - \gamma^{2s}) \sum_{p \in P} \gamma^{2sk} \|v_p\|_0^2.
\]

The rightmost sum can be further decomposed as follows:

\[
\sum_{p \in P} \sum_{k=\ell(p)}^{J} \gamma^{2sk} \|v_p\|_0^2 = \sum_{j=0}^{J} \sum_{\ell(p) \leq k} \gamma^{2sk} \|v_p\|_0^2 \\
= \sum_{k=0}^{J} \gamma^{2sk} \sum_{\ell(p) \leq k} \|v_p\|_0^2 \leq \sum_{k=0}^{J} \gamma^{2sk} \sum_{\ell(p) \leq k+k_s} \|v_p\|_0^2 \\
= \gamma^{-2sk_s} \sum_{k=0}^{J} \gamma^{2sk} \sum_{\ell(p) \leq k+k_s} \|v_p\|_0^2 \leq \gamma^{-2sk_s} \sum_{k=0}^{J} \gamma^{2sk} \sum_{\ell(p) \leq k} \|v_p\|_0^2.
\]

Since \( \gamma^{-2sk_s} \simeq 1 \), there exist equivalence constants independent of \( s \) and \( J \) such that

\[
(6.27) \quad (B_1 v, v) \simeq \gamma^{2sJ} \sum_{p \in P} \|v_p\|_0^2 + (1 - \gamma^{2s}) \sum_{p \in P_{k+k_s}} \|v_p\|_0^2.
\]

We now consider the bisection triplets \( T_j \) and 3-dimensional spaces \( V_j \), for which \( h_j \simeq \gamma^{g_j} \).

We let \( \hat{B}_2 := \sum_{k=0}^{J} \gamma^{2sk} \sum_{j \in J_{k+k_s}} I_j Q_j \), \( B_2 := \sum_{j=0}^{J} I_j h^{2s}_j Q_j \) and \( v_j := Q_j v \), to write

\[
(\hat{B}_2 v, v) = \sum_{k=0}^{J} \gamma^{2sk} \sum_{k+k_s \leq g_j \leq k+k_s} \|v_j\|_0^2 \\
= \sum_{k=0}^{J} \gamma^{2sk} \sum_{k+k_s \leq g_j \leq k+k_s} \|v_j\|_0^2 \\
= \sum_{k=0}^{J} \gamma^{2sk} \sum_{g_j=k} \|v_j\|_0^2 \simeq \sum_{j=1}^{J} \gamma^{2sg_j} \|v_j\|_0^2 \simeq (B_2 v, v),
\]

because \( \sum_{i=k_s}^{k+k_s} \gamma^{2si} \simeq 1 \) due to the fact that \( k_s \) is a fixed integer depending solely on shape regularity of \( T(T_0) \). Combining (6.27) and (6.28) we obtain

\[
(B v, v) = (B_1 v, v) + (1 - \gamma^{g}) (B_2 v, v) \simeq (\hat{B} v, v) \quad \forall \ v \in V,
\]

whence the operators \( B \) and \( \hat{B} \) are spectrally equivalent. Invoking Corollary 6.3 (uniform bound for \( \lambda_{\min}(BA) \)), we readily deduce \( \lambda_{\min}(BA) \gtrsim c_0^{-1} \). We finally recall that \( \lambda_{\max}(BA) \lesssim c_1 \), according to Proposition 6.1 (boundedness), to infer the desired uniform bound \( \text{cond}(BA) = \lambda_{\max}(BA)\lambda_{\min}(BA)^{-1} \lesssim c_0 c_1 \). \( \square \)

7. Numerical Experiments

This section presents some experiments in both uniform and graded bisection grids. We provide some details about the implementation of the BPX preconditioners and their matrix representations in Appendix D.
In the sequel, we solve (1.4) with $\Omega = (-1, 1)^2$ and $f = 1$, and $s = 0.9$, $s = 0.5$ or $s = 0.1$. In all numerical experiments, the stopping criterion is $\|b - Ax\|_2 \|b\|_2 \leq 1 \times 10^{-6}$.

7.1. Uniform grids. In first place, we perform computations on a family of nested, uniformly refined meshes. Table 2 lists the number of iterations performed when solving the linear systems using either the Gauss-Seidel (GS), Conjugate Gradient (CG) and Preconditioned Conjugate Gradient (PCG) methods. Limited by computational capacity, the largest $\bar{J}$ we take in our computations is 6, which corresponds to 16129 degrees of freedom (DOFs). Even though this is a small-scale problem, the BPX preconditioner (D.1) performs well.

| $\bar{J}$ | $h_j$ | DOFs | $s = 0.9$ | $s = 0.5$ | $s = 0.1$ |
|---------|-------|------|--------|--------|--------|
|         |       | GS   | CG     | PCG    | GS     | CG     | PCG    |
| 1       | 2^{-1}| 9    | 4      | 4      | 8      | 4      | 4      |
| 2       | 2^{-2}| 49   | 12     | 12     | 16     | 8      | 8      |
| 3       | 2^{-3}| 225  | 25     | 16     | 33     | 11     | 10     |
| 4       | 2^{-4}| 961  | 46     | 19     | 68     | 17     | 11     |
| 5       | 2^{-5}| 3969 | 84     | 21     | 139    | 24     | 12     |
| 6       | 2^{-6}| 16129| 157    | 22     | 279    | 32     | 13     |

Table 2. Number of iterations: GS, CG, and PCG with BPX preconditioner (D.1), $\tilde{\gamma} = 0.5$.

7.2. Graded bisection grids. We next consider graded bisection grids. As described in Proposition 2.2 (regularity in weighted spaces) and Remark 2, the solution $u$ to (1.4) satisfies $u \in \cap_{\varepsilon > 0} H^{1+s-2\varepsilon}(\Omega)$ and this regularity can be optimally exploited by considering grids graded according to (2.11) with $\mu = 2$. In the energy norm, one obtains linear convergence rates with this strategy.

In order to obtain the graded refinement (2.11) when using bisection grids, we consider the following strategy. Given an element $\tau \in T$, let $x_\tau$ be its barycenter. Our strategy is based on choosing a number $\theta > 1$ and marking those elements $\tau$ such that

$$|\tau| > \theta N^{-1} \log N \cdot d(x_\tau, \partial \Omega)^2(\mu - 1)/\mu,$$

where $N = \dim V(T)$ is the number of degrees of freedom. We use the newest vertex marking strategy. Figure 7.1 shows some graded bisection grids by using the marking strategy with $\theta = 4$, $\mu = 2$.

![Figure 7.1](image)

**Figure 7.1.** Graded bisection grids on $(-1, 1)^2$, using strategy (7.1) with $\theta = 4$ and $\mu = 2$.

We document the number of iterations needed when solving the linear systems using GS, CG and PCG over graded bisection grids, for the same example as in the previous subsection,
and with the same stopping criterion. As shown in Table 3, the BPX preconditioner \( D.2 \) performed satisfactorily in the experiments we have carried out.

| \( J \) | DOFs(N) | \( s = 0.9 \) | \( s = 0.5 \) | \( s = 0.1 \) |
|---|---|---|---|---|
| 7 | 61 | 35 | 10 | 10 | 12 | 10 | 7 | 9 | 13 | 8 |
| 8 | 153 | 51 | 15 | 13 | 15 | 15 | 9 | 9 | 21 | 10 |
| 9 | 161 | 62 | 15 | 14 | 17 | 15 | 9 | 10 | 21 | 10 |
| 10 | 369 | 93 | 20 | 16 | 21 | 20 | 11 | 9 | 34 | 11 |
| 11 | 405 | 125 | 21 | 16 | 25 | 19 | 11 | 9 | 31 | 12 |
| 12 | 853 | 160 | 26 | 18 | 29 | 26 | 12 | 9 | 48 | 12 |
| 13 | 973 | 224 | 30 | 19 | 35 | 26 | 12 | 9 | 47 | 12 |
| 14 | 1921 | 282 | 34 | 20 | 41 | 33 | 13 | 9 | 72 | 12 |
| 15 | 2265 | 407 | 40 | 21 | 50 | 32 | 13 | 9 | 65 | 12 |
| 16 | 4209 | 532 | 46 | 22 | 58 | 39 | 14 | 9 | 97 | 13 |
| 17 | 5157 | 745 | 55 | 22 | 70 | 40 | 14 | 10 | 92 | 12 |
| 18 | 9397 | 997 | 64 | 24 | 83 | 48 | 14 | 9 | 135 | 13 |

Table 3. Number of iterations: GS, CG and PCG with BPX preconditioner \( D.2 \), \( \hat{\gamma} = \sqrt{2}/2 \).

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which gives a special decomposition that satisfies (3.3). For any other decomposition $v = \sum_{j=0}^{J} v_j$, we write $v_j = v_j^* + w_j$ with $\sum_{j=0}^{J} w_j = 0$ and observe that

$$\sum_{j=0}^{J} (R_j^{-1} v_j, v_j) = \sum_{j=0}^{J} (R_j^{-1} (v_j^* + w_j), v_j^* + w_j) = (B^{-1} v, v) + 2 \sum_{j=0}^{J} (R_j^{-1} v_j^*, w_j) + \sum_{j=0}^{J} (R_j^{-1} w_j, w_j).$$

Since

$$\sum_{j=0}^{J} (R_j^{-1} v_j^*, w_j) = \sum_{j=0}^{J} (B^{-1} v, w_j) = (B^{-1} v, \sum_{j=0}^{J} w_j) = 0,$$

we deduce

$$\sum_{j=0}^{J} (R_j^{-1} v_j, v_j) = (B^{-1} v, v) + \sum_{j=0}^{J} (R_j^{-1} w_j, w_j) \geq (B^{-1} v, v).$$

This gives (3.3) and concludes the proof. \qed

**Proof of Lemma 3.2** We note that $BA : V \to V$ is SPD with the inner product $(A \cdot, \cdot)$. If $(\lambda, v)$ is an eigenpair of $(BA)^{-1}$, then $B^{-1} v = \lambda A v$. The stable decomposition (3.4) thus yields

$$\lambda_{\text{max}}((BA)^{-1}) = \sup_{\|v\|_A = 1} (B^{-1} v, v) = \sup_{\|v\|_A = 1} \inf_{\sum_{j=0}^{J} v_j = v} \|v_j\|_{R_j^{-1}}^2 \leq c_0,$$
whence \( \lambda_{\min}(BA) \geq c_0^{-1} \). On the other hand, the boundedness \([3.5]\) implies
\[
\lambda_{\min}((BA)^{-1}) = \inf_{\|v\| = 1} (B^{-1}v, v) = \inf_{\|v\| = 1} \inf_{\sum_j v_j = w} \|v_j\|^2 \geq c_1^{-1},
\]
which gives \( \lambda_{\max}(BA) \leq c_1 \). Applying the definition \( \text{cond}(BA) = \lambda_{\max}(BA)\lambda_{\min}(BA)^{-1} \) concludes the proof.

□

APPENDIX B. TWO AUXILIARY RESULTS: PROOF OF LEMMAS 3.4 AND 3.5

The proof of Lemma 3.4 involves interpolation of weighted \( L^2 \) spaces \([47] \text{ Lemma 23.1} \).

**Lemma B.1** (interpolation of weighted \( L^2 \) spaces). Given \( w : \Omega \to (0, \infty) \) measurable, let
\[
E(w) := \left\{ v : \Omega \to \mathbb{R} : \int_\Omega |v|^2w < \infty \right\}, \quad \|v\|_{E(w)} := \left( \int_\Omega |v|^2w \right)^{\frac{1}{2}}.
\]
If \( w_0, w_1 \) are two functions as above, then for \( s \in (0, 1) \) one has \( E(w_0^{1-s}w_1^s) = E(w_s) \), where \( w_s = w_0^{1-s}w_1^s \). Moreover, the interpolation norm \( (2.6) \) is equivalent to the \( E(w_s) \) constant norm, with an equivalence constant independent of \( s \).

Upon invoking the modified \( K \)-functional \( (2.5) \), Lemma 3.4 (\( s \)-uniform interpolation) is a consequence of interpolation theory. See \([49] \) for a non-optimal version of this result.

**Proof of Lemma 3.4.** We consider the spaces
\[
X^0 = (V_J, \| \cdot \|_0), \quad Y^0 = (V_0 \times V_1 \times \ldots \times V_J, \| \cdot \|_{Y^0}),
\]
\[
X^1 = (V_J, \| \cdot \|_1), \quad Y^1 = (V_0 \times V_1 \times \ldots \times V_J, \| \cdot \|_{Y^1}),
\]
where, for \( v = (v_0, \ldots, v_J) \in V_0 \times V_1 \times \ldots \times V_J,
\[
\|v\|_{Y^0} := \left( \sum_{j=0}^{J} \|v_j\|^2 \right)^{\frac{1}{2}}, \quad \|v\|_{Y^1} := \left( \sum_{j=0}^{J} \gamma^{-2j}\|v_j\|^2 \right)^{\frac{1}{2}}.
\]
Furthermore, we shall denote, for \( i = 0, 1, j = 0, \ldots, J \), \((Y_i^0, \| \cdot \|_{Y_i^0}) = (V_j, \gamma^{-ij} \| \cdot \|_0) \).

We now consider the map \( TV = (\tilde{Q}_0v, \ldots, \tilde{Q}_Jv) \). By \( L^2 \)-orthogonality, this map satisfies
\[
\|TV\|_{Y^0} = \|v\|_{X^0} \quad \forall v \in X^0.
\]
The assumption guarantees that
\[
\|TV\|_{Y^1} \lesssim \|v\|_{X^1} \quad \forall v \in X^1.
\]

Therefore, by interpolation theory, the map \( T \) satisfies
\[
T : (X^0, X^1)_{s,2} \to (Y^0, Y^1)_{s,2} \quad \forall s \in (0, 1),
\]
with a continuity constant independent of \( s \). As discussed in Section 2.2, we have that \((X^0, X^1)_{s,2} = (V_J, \| \cdot \|_{X^s}) \), and that the interpolation norm is equivalent to the \( \| \cdot \|_s \) norm, with an equivalence constant independent of \( s \).

We need to verify that the interpolation norm in \((Y^0, Y^1)_{s,2}\) coincides with the left hand side in \((3.8)\). For that purpose, given \( w \in Y^0 + Y^1 \) with \( w = (w_0, \ldots, w_J) \), we have
\[
K_2(t, w)^2 = \inf_{w^0 \in X^0, w^1 \in X^1} \|w^0\|^2_{Y^0} + t^2\|w^1\|^2_{Y^1} = \sum_{j=0}^{J} K_2(t, w_j)^2.
\]

Therefore, we can write the interpolation norm as \( \|w\|^2_{(Y^0, Y^1)_{s,2}} = \sum_{j=0}^{J} \|w_j\|^2_{(Y^0, Y^1)_{s,2}} \). By Lemma B.1, we have \( \|w_j\|_{(Y^0, Y^1)_{s,2}} \approx \gamma^{-sj}\|w_j\|_0 \), with an equivalence constant independent of \( s \). Thus, we have proved the desired result \((3.8)\). □
The proof of Lemma 3.5 (local inverse inequality) exploits a localization property of fractional Sobolev spaces (cf. [49, Lemma 3.2] and [24, 25]) and standard local estimates.

**Proof of Lemma 3.5.** We distinguish between $\sigma \in [0, 1]$ and $\sigma \in (1, 3/2)$.

**Step 1:**

Fix and recalling that $\bar{\sigma}$

1. Transform by $F$

Combining this estimate with $|v|_{H^\sigma(S_\tau)} \lesssim h^{-\frac{1}{2}}\|v\|_{L^2(S_\tau)}$, the local inverse estimate

The case $\sigma = 1$ is similar and hinges on the local inverse estimate $|v|_{H^1(S_\tau)} \lesssim \bar{h}^{-1}\|v\|_{H^\sigma(S_\tau)}$, which in turn results from operator interpolation between the estimates $|v|_{H^1(S_\tau)} \lesssim \bar{h}^{-1}\|v\|_{L^2(S_\tau)}$ and $|v|_{H^1(S_\tau)} \lesssim \bar{h}^{-1}\|v\|_{H^\sigma(S_\tau)}$.

**Step 2:** Let $1 \leq \mu \leq \sigma$ and apply Step 1 to $\nabla v$

If $0 < \mu < 1$, instead, we concatenate the preceding estimate for $\mu = 1$ with the inverse estimate $|v|_{H^1(S_\tau)} \lesssim \bar{h}^{-1}\|v\|_{H^\mu(S_\tau)}$ of Step 1. We finally observe that $v \notin H^{3/2}(\Omega)$ because it is piecewise linear, which implies that the constant hidden in (3.9) blows up as $\sigma \to 3/2$.

This concludes the proof. □

**Appendix C. Generalized strengthened Cauchy-Schwarz inequality**

This appendix offers a proof of an inequality in the spirit of the well-known strengthened Cauchy-Schwarz inequality, that is amenable for applications in the analysis of fractional-order problems. The usual proof for second-order problems consists of an elementwise integration-by-parts argument, a local argument that quantifies the interaction between functions with different frequencies. This is not possible in the present context due to the nonlocal nature of the fractional norms. We resort instead to the well-known characterization of the fractional Sobolev space $\dot{H}^\sigma(\Omega)$ as a Bessel potential space; see also [33, 34].

**Lemma C.1** (generalized strengthened Cauchy-Schwarz inequality). Let $\sigma \in (0, 3/2)$ and $k \leq \ell$. Then, given $\beta > 0$ such that $\beta \leq \sigma$ and $\beta < \frac{3}{2} - \sigma$, there holds

$$(v_k, v_\ell) \lesssim \gamma^{\sigma - k - \beta} \bar{h}^{-\sigma} |v_k|\|v_\ell\|_0, \quad \forall v_k \in \mathcal{V}_k, v_\ell \in \mathcal{V}_\ell,$$

where the hidden constant only blows up as $\sigma \to 3/2$.

**Proof.** Fix $\beta$ as in the statement of the lemma, and recall that we denote the Fourier transform by $\mathcal{F}$. Applying Parseval’s identity, we deduce

$$(v_k, v_\ell) = \int_{\mathbb{R}^d} |\xi|^{\sigma} \mathcal{F}(v_k)|\xi|^{\sigma} \mathcal{F}(v_\ell) \, d\xi$$

$$= \int_{\mathbb{R}^d} |\xi|^{\sigma + \beta} \mathcal{F}(v_k)|\xi|^{-\beta} \mathcal{F}(v_\ell) \, d\xi \leq |v_k|_{\sigma + \beta} |v_\ell|_{\sigma - \beta}.$$  

The assertion follows upon applying the inverse inequality (3.9) on quasi-uniform meshes and recalling that $\bar{h}_k \simeq \gamma^k$:

$$(v_k, v_\ell) \lesssim \bar{h}^{-\beta} |v_k|_{\sigma} \bar{h}^{-\sigma + \beta} |v_\ell|_0 \simeq \gamma^{\beta |\ell - k|} \bar{h}^{-\sigma} |v_k|_{\sigma} |v_\ell|_0.$$
This completes the proof. □

Relying on the generalized strengthened Cauchy-Schwarz inequality, we next offer a second proof of the boundedness property in Lemma 3.2 (estimate on \( \text{cond}(BA) \)).

**Alternative proof of Proposition 4.1 (boundedness).** Combining the inverse inequality (3.9) with \( \sigma = \beta = s \) and \( \mu = 0 \), with Lemma C.1 (generalized strengthened Cauchy-Schwarz inequality), we obtain

\[
\sum_{k=0}^{j-1} |v_k|^2_s \lesssim |v_j|^2_s + \sum_{k=0}^{j-1} (v_k, v_k)_s \lesssim \tilde{h}_j^{-2s} \|v_j\|_0^2 + \sum_{k=0}^{j-1} \gamma_s^{k-\ell} \tilde{h}_k^{-s} \|v_k\|_0 \|v_\ell\|_0,
\]

provided \( \beta + \sigma = 2s < \frac{3}{2} \). We recall the elementary inequality for \( \theta < 1 \),

\[
\sum_{i,j=1}^{n} |\theta^{i-j}| x_i y_j \leq \frac{2}{1-\theta} \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{n} y_j^2 \right)^{\frac{1}{2}}, \quad \forall (x_i)_{i=1}^{n}, (y_i)_{i=1}^{n} \in \mathbb{R}^n.
\]

We next apply (C.1) with \( \theta = \gamma^s \) to obtain

\[
\sum_{k,\ell=0}^{j-1} \gamma_s^{k-\ell} \tilde{h}_k^{-s} \tilde{h}_\ell^{-s} \|v_k\|_0 \|v_\ell\|_0 \lesssim \frac{1}{1-\gamma^s} \sum_{k=0}^{j-1} \tilde{h}_k^{-2s} \|v_k\|_0^2 \lesssim \frac{1}{1-\gamma^s} \sum_{k=0}^{j-1} \tilde{h}_k^{-2s} \|v_k\|_0^2.
\]

Combining the two preceding estimates, we arrive at the desired bound (4.6), which is (3.5) for \( B \) in accordance with the definitions (4.3) and (4.4), provided \( s < \frac{3}{2} \). If \( \frac{3}{4} \leq s < 1 \), then our choices of \( \sigma, \beta \) in Lemma C.1 are restricted: we take \( \sigma = s \) and \( \beta = \frac{1}{2} - \frac{3}{2} - \sigma \). The previous argument still works but the prefactor in (4.6) becomes \( (1-\gamma^s)^{-1} \) instead. Since there is a constant \( C > 0 \), independent of \( s \), such that

\[
\frac{1-\gamma^s}{1-\gamma^s} \leq C
\]

the expression (4.6) is still valid in this case. This proof is thus complete. □

**APPENDIX D. MATRIX REPRESENTATION AND IMPLEMENTATION**

In this appendix we briefly discuss the implementation of BPX preconditioners. Denoting the nodal basis functions of \( V \) by \( \{ \phi_i \}_{i=1}^{N} \), we have the following matrices:

- **Stiffness matrix** \( K = (k_{ij})_{i,j=1}^{N} \in \mathbb{R}^{N \times N} \), where \( k_{ij} = a(\phi_j, \phi_i) \);
- **Mass matrix** \( M = (m_{ij})_{i,j=1}^{N} \in \mathbb{R}^{N \times N} \), where \( m_{ij} = (\phi_j, \phi_i) \);
- **Matrix representation** \( A = (a_{ij})_{i,j=1}^{N} \) of \( A \): \( A\phi_i = \sum_{j=1}^{N} a_{ij} \phi_j \) or equivalently

\[
A[\phi_1, \ldots, \phi_N] = [\phi_1, \ldots, \phi_N] A.
\]

Recalling the definition \( (A\phi_j, \phi_i) = a(\phi_j, \phi_i) \) of \( A \), we deduce \( K = MA \) or \( A = M^{-1}K \). Next we derive the matrix presentation of the BPX preconditioner (4.4) on quasi-uniform grids. We denote the nodal basis functions of \( \nabla V_k \) by \( \{ \phi^k_i \}_{i=1}^{N_k} \) and then have the following matrices:

- **Matrix representation** \( I_k \in \mathbb{R}^{N \times N_k} \) of the inclusion \( I_k \), often called prolongation matrix:

\[
I_k[\phi^k_1, \ldots, \phi^k_{N_k}] = [\phi_1, \ldots, \phi_N] I_k;
\]

- **Matrix representation** \( Q_k \in \mathbb{R}^{N_k \times N} \) of the \( L^2 \)-projector \( Q_k \):

\[
Q_k[\phi_1, \ldots, \phi_N] = [\phi^k_1, \ldots, \phi^k_{N_k}] Q_k;
\]

- **Matrix representation** \( T_k \in \mathbb{R}^{N \times N_k} \) of the prolongation matrix:

\[
T_k[\phi_1, \ldots, \phi_N] = [\phi_1, \ldots, \phi_N] T_k;
\]
If $M_k \in \mathbb{R}^{N_k \times N_k}$ denotes the mass matrix on $V_k$, the definition of $L^2$-projection yields

$$I^T_k M = \begin{bmatrix} I^T_k \phi_1, \ldots, \phi_{N_k} \end{bmatrix} = \begin{bmatrix} \phi_1^T, \ldots, \phi_{N_k}^T \end{bmatrix} = Q_k \begin{bmatrix} \phi_1, \ldots, \phi_{N_k} \end{bmatrix} = M_k Q_k,$$

Consequently, the matrix representation $\overline{B}$ of $\overline{B}$ in (4.4) reads

$$\overline{B} = I \bar{h}^{2s} + (1 - \bar{\gamma}^s) \sum_{k=0}^{j-1} \bar{I}_k \bar{h}_k^{2s} Q_k$$

where we have used the equivalence $M_k^{-1} \simeq \bar{h}_k^{-d} I_k$ to avoid inverting $M_k$ and

$$P := \bar{h}_j^{2s-d} I_j + (1 - \bar{\gamma}^s) \sum_{k=0}^{j-1} \bar{h}_k^{2s-d} I_k I^T_k.$$

This implies $\text{cond}(PK) = \text{cond}((PK)(K^{-1}K)) \simeq \text{cond}(BA) = \text{cond}(BA) \lesssim 1$, whence $P$ is a robust preconditioning matrix for the stiffness matrix $K$.

On graded bisection grids, our implementation of (6.4) hinges on the local scaling of $V_j$

$$h_{j,q} := \left( \frac{\omega_q}{\# R_q} \right)^{1/d} \quad q \in T_j \cap N_j;$$

note that $h_{j,q} \simeq h_j$ in view of shape regularity. The robust preconditioning matrix reads

$$P = \sum_{p \in P} h_{j,q}^{2s-d} I_p I^T_p + (1 - \bar{\gamma}^s) \sum_{j=0}^{j} \sum_{q \in T_j \cap N_j} h_{j,q}^{2s-d} I_{j,q} I^T_{j,q},$$

where $I_{j,q}$ is the prolongation matrix from $\text{span}\{\phi_{j,q}\}$ to $V$. 

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