\textbf{A}^1\text{-CONNECTED VARIETIES OF RANK ONE OVER NON-CLOSED FIELDS}

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Abstract. In this paper, we proved two results regarding the arith-
metics of separably $A^1$-connected varieties of rank one. First we proved
over a large field, there is an $A^1$-curve through any rational point of the
boundary, if the boundary divisor is smooth and separably rationally
connected. Secondly, we generalize a theorem of Hassett-Tschinkel for
the Zariski density of integral points over function fields of curves.

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1. Introduction

Separably $A^1$-connected varieties has been introduced and studied in
[CZ13, CZ14]. They are the analogue of separably rationally connected
(SRC) varieties in the non-proper setting. When the non-proper variety ad-
mits a log smooth compactification, the recent developments on log stable
maps provide us a powerful tool to study $A^1$-connectedness. We refer to
[Kat89] for the basics of logarithmic geometry, and to [GS13, Che14, AC14,
ACMW14, Wis14] for the details of the theory of stable log maps.

In this paper, we study the arithmetics of simple separably $A^1$-connected
varieties of rank one with the SRC center over non-closed fields, or equiva-
lently, log pairs with the ambient variety smooth proper and the boundary
divisor smooth irreducible SRC. Our results consists of two parts: one is
over large fields and the other is over function fields of algebraic curves over
an algebraically closed field of characteristic zero.

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points, Zariski density.

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1.1. **Over large fields.** According to Iitaka’s philosophy, we expect the results for SRC varieties hold for separably $\mathbb{A}^1$-connected varieties in an appropriate form. Our first motivation here is to generalize Kollár’s theorem [Kol99, Theorem 1.4]: over a large field $K$, every rational point of a proper SRC variety is contained in a very free rational curve defined over $K$. In the logarithmic setting, we would like to find $\mathbb{A}^1$-curves on a proper separably $\mathbb{A}^1$-connected log variety defined over $K$. Since each $\mathbb{A}^1$-curve also gives a $K$-rational point on the boundary, a necessary condition for existence of $\mathbb{A}^1$-curves is $D(K) \neq \emptyset$. Conversely, we have the following:

**Theorem 1.1.** Let $K$ be a large field. Let $X = (\mathcal{X}, \mathcal{D})$ be a proper log smooth, simple, and separably $\mathbb{A}^1$-connected $K$-variety of rank one with the SRC center. Then there exists a very free $\mathbb{A}^1$-curve defined over $K$ through any $K$-rational point of $\mathcal{D}$.

1.2. **Over function fields.** Let $k$ be an algebraically closed field of characteristic zero. Let $B$ be a smooth projective algebraic $k$-curve, and let $F$ be its function field. Our second motivation is to study arithmetics of $\mathbb{A}^1$-connected varieties over $F$. Based on the work of [KMM92, GHS03, HT06a], Hassett-Tschinkel proposed the weak approximation conjecture:

**Conjecture 1.2.** [HT06a] Proper rationally connected varieties defined over $F$ satisfy the weak approximation.

Over number fields, number theorists are also interested in the approximation results for non-proper varieties, i.e. the strong approximation. Note that affine spaces satisfy the strong approximation. From our point of view, $\mathbb{A}^1$-connected varieties are generalizations of affine spaces. We propose the following question:

**Question 1.3.** Does strong approximation hold for $\mathbb{A}^1$-connected varieties over $F$?

A special case of Question 1.3 is the Zariski density of integral points studied by Hassett-Tschinkel [HT08]. Using the log deformation theory, we give another proof of Hassett-Tschinkel’s theorem in the $\mathbb{A}^1$-connectedness setting:

**Theorem 1.4.** Let $X = (\mathcal{X}, \mathcal{D})$ be a log smooth, proper, and $\mathbb{A}^1$-connected variety of rank one with the SRC center defined over $F$. Given a model $\pi : (\mathcal{X}, \mathcal{D}) \to B$ with the generic fiber $(\mathcal{X}, \mathcal{D})$, let $S$ be a non-empty finite set of places on $B$ containing the images of the singularities of $\mathcal{X}$ and $\mathcal{D}$. Then the set of $S$-integral points of the family are Zariski dense.

Furthermore, when $S$ is nonempty containing all places of bad reductions [HT08, Definition 4], there exists an $S$-integral point through any finite collection of integral points lying in the strongly $\mathbb{A}^1$-uniruled locus of the fiber $(\mathcal{X}_t, \mathcal{D}_t)$ for $t \in B \setminus S$. 
For a log variety $X$ given by a pair $(X, D)$ over an algebraically closed field, we defined the strongly $\mathbb{A}^1$-uniruled locus of $X$ to be the open subset of $X \setminus D$ consisting of points contained in the image of free $\mathbb{A}^1$-curves.

By [CZ13, Corollary 1.10], the above theorem generalizes the previous work of Hassett and Tschinkel [HT08, Theorem 1]. It also includes the pairs $(\mathbb{P}^1, \infty)$ and Hirzebruch surface $H_n$ with the $(-n)$-curve as the boundary, where the original argument of Hassett and Tschinkel does not apply. We wish to further study Zariski density and strong approximation in our subsequent work for $\mathbb{A}^1$-connected varieties with more general boundaries.

1.3. Notations. In this paper, all log structures are fine and saturated [Kat89, Section 2]. Capital letters such as $C, S, X, Y$ are reserved for log schemes. Their associated underlying schemes are denoted by $\mathcal{C}, \mathcal{S}, \mathcal{X}, \mathcal{Y}$ respectively.

A log scheme $X$ is called of rank one, if geometric fiber of the characteristic monoid $\mathcal{M}_{X,x} := \mathcal{M}_{X,x}/\mathcal{O}_{X,x}^*$ is either $\mathbb{N}$ or $\{0\}$ for any geometric point $x \in X$. Given a pair $(X, D)$ with $D \subset X$ a cartier divisor, denote by $X$ the canonical log scheme associated to the pair $(X, D)$, see [Kat89, Complement 1]. Such log scheme $X$ is of rank one. For simplicity, we may write $X = (X, D)$ to denote the corresponding log scheme and the underlying pair. We say that a log smooth proper variety $X = (X, D)$ of rank one is simple if the boundary divisor is irreducible and smooth. We will keep using the terminology in [AC14], and call $D$ the center.

Let $K$ be a field, and $X$ be a proper, log smooth $K$-variety defined by a log smooth pair $(X, D)$ such that $D \subset X$ is a smooth divisor. Given a stable log map $f : C/S \rightarrow X$ over $S$, a marking $\Sigma \subset C$ is called a contact marking if the corresponding contact order is non-trivial. See [AC14, Section 3.8] and [ACGM10] for more details of contact orders.

Recall that a stable log map $f : C/S \rightarrow X$ is non-degenerate if the log structure $\mathcal{M}_S$ is trivial over every geometric point on $S$. A stable log map is called an $\mathbb{A}^1$-curve if it is a non-constant, non-degenerate genus zero stable log map with only one contact marking. Otherwise, we called it a stable $\mathbb{A}^1$-map.

We use $\mathcal{M}_{\mathbb{A}^1,n}(X, \beta)$ to denote the log algebraic $K$-stack of stable $\mathbb{A}^1$-maps with target $X$, $n$ non-contact markings, and curve class $\beta \in H_2(X)$. Denoted by $\mathcal{M}_{\mathbb{A}^1,0}(X, \beta)$ its underlying algebraic stack. When $n = 0$, we write $\mathcal{M}_{\mathbb{A}^1}(X, \beta)$ instead of $\mathcal{M}_{\mathbb{A}^1,0}(X, \beta)$.

Let $s$ be a $K$-point of $D$. Denote by $\mathcal{M}_{\mathbb{A}^1}(X, \beta; s)$ the fiber of the contact evaluation morphism

$$\mathcal{M}_{\mathbb{A}^1}(X, \beta) \rightarrow D$$

over $s$.

2. A Gluing Technique

Definition 2.1. Let $K \subset L$ be any field extension. An $\mathbb{A}^1$-comb over $S = \text{Spec} L$ is a stable $\mathbb{A}^1$-map $f : C/S \rightarrow X$ in $\mathcal{M}_{\mathbb{A}^1}(X, \beta)(S)$ satisfying:
the underlying curve $\mathcal{C}$ of $C$ is given by a union of irreducible components $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_m$ over $S$ such that $\mathcal{C}$ is obtained by joining $\mathcal{C}_0$ and $\mathcal{C}_i$ along two $L$-points $q_i : S \to \mathcal{C}_0$ and $p_i : S \to \mathcal{C}_i$ for each $i \neq 0$. 

(2) the unique contact marking is given by an $L$-point $q_\infty : S \to \mathcal{C}_0$. 

(3) the general fiber of the restriction $f_i := f|_{\mathcal{C}_i}$ over $S$ defines a family of $\mathbb{A}^1$-curves on $X$ for $i \neq 0$. 

We call $f_i$ the $\mathbb{A}^1$-tooth of $f$, and $\mathcal{C}_0$ the handle of $f$. 

We introduce an $\mathbb{A}^1$-comb construction when the teeth are Galois conjugate to each other.

**Proposition 2.2.** Let $K \subset L$ be a Galois extension with $G = \text{Gal}(L/K)$. Given an $K$-rational point $s \in D(K)$, and $\mathbb{A}^1$-curves $[f_i : C_i \to X] \in \mathcal{M}_{\mathbb{A}^1}(X, \beta; s)(L)$ for $i = 1, \ldots, m$ with $m \geq 2$, such that they are contained in a $G$-orbit under the Galois action. Then there exists an $\mathbb{A}^1$-comb $[f : C/S \to X] \in \mathcal{M}_{\mathbb{A}^1}(X/K, \beta' ; s)(L)$ with $S$ the standard log point over $\text{Spec} L$ satisfying:

1. $\beta' = m \cdot \beta$;
2. $f_i$ is the tooth of $f$ for each $i$;
3. $C$ is obtained by gluing $C_1, \ldots, C_m$ along $m$ different $L$-rational points of $C_0 = \mathbb{P}^1$ contained in a $G$-orbit;
4. $f$ contracts the handle $C_0$ to the $K$-rational point $s$;
5. the log structure on $S$ is minimal in the sense of [Che14, AC14].

If furthermore the set of $\mathbb{A}^1$-curves $\{[f_i]\}$ forms a complete Galois orbit, then $[f]$ is $G$-invariant, and descents to a $K$-rational point in $\mathcal{M}_{\mathbb{A}^1}(X/K, \beta' ; s)(K)$.

**Proof.** Comparing with the case of usual stable maps, the major difficulty is to construct morphism on the level of log structures. We split the construction into several steps.

**Step 1. Construct the underlying map.**

Choose $C_0 = \mathbb{P}^1$ defined over $K$ with prescribed $m$ different $L$-rational points

$$q_1, \ldots, q_m,$$

and a $K$-rational point $q_\infty$, which will be the contact marking of the $\mathbb{A}^1$-comb. We may choose the $L$-rational points in (2.1) contained in a $G$-orbit compatible with the Galois action on $\{f_i\}$. Let $C$ be the nodal curve over $L$ obtained by gluing $C_0$ and $C_i$ by identifying $p_i$ with the contact marking $q_i \in C_i$. Then $f : C \to X$ is defined by gluing $f_i$ with the contraction map $f(C_0) = s$.

**Step 2. Expansion along $D$**
Denote by \( N := N_{D/X} \) the normal bundle of \( D \) in \( X \), and form \( \mathbb{P} = \mathbb{P}(N \otimes \mathcal{O}_D) \). Thus, we have a \( \mathbb{P}^1 \)-fibration:
\[
\phi : \mathbb{P} \to D
\]
with two disjoint sections \( D_0 \cong D_\infty \cong D \) such that
\[
N_{D_0/P} \cong N_{D_\infty/P} \cong N^\vee.
\]
Consider \( W = \mathbb{P} \cup_{D_0 \cong D} X \) obtain by gluing \( \mathbb{P} \) and \( X \) using the canonical identification \( D_0 \cong D \). By [Ols03], there is a canonical log smooth family
\[
(2.2) \quad \psi : W \to B
\]
over \( \psi : W \to B := \text{Spec } K \). The underlying family \( \psi \) is called the expansion along \( D \). We call \( \psi \) the logarithmic expansion along \( D \). Note that we have a natural morphism of log schemes
\[
(2.3) \quad \pi : W \to X
\]
whose underlying morphism \( \pi \) is the contraction of the \( \mathbb{P}^1 \)-fibration \( \phi \). This can be shown by a similar argument as in for example [GS13, Proposition 6.1].

**Step 3. Lift \( f \) to underlying stable map to the expansion**

Denote by \( c = \beta \cap D \in \mathbb{Z}_{>0} \). The integer \( c \) is the contact order of \( f_i \) at the contact marking \( p_i \) for each \( i \). Since both \( X \) and \( D \) are defined over \( K \), the fiber of the restriction \( \phi|_W \) is a \( \mathbb{P}^1_K \) defined over \( K \). We then construct the underlying stable map
\[
f'_0 : C_0 \cong \mathbb{P}^1 \to \mathbb{P}^1_K
\]
such that

1. \( f'_0 \) factors through \( \mathbb{P}^1_B \);
2. \( f'_0 \) tangent to \( D_0 \) at \( q_i \) with contact order \( c \) for \( i = 1, \ldots, m \);
3. \( f'_0 \) tangent to \( D_\infty \) at \( q_\infty \) with contact order \( m \cdot c \).

Such \( f'_0 \) can be defined by choosing a non-zero \( L \)-rational function in
\[
(2.4) \quad H^0(\mathcal{O}_{\mathbb{P}^1}(m \cdot c \cdot q_\infty - \sum_i c \cdot q_i)).
\]

Gluing \( f'_0 \) and \( f_i \) by identifying the \( L \)-rational points \( p_i \) and \( q_i \), we obtain the underlying stable map \( f' : C \to W \).

**Step 4. Lift \( f' \) to a stable log map to \( W/B \).**

We next construct a stable log maps \( f' \) over \( f'_0 \) as in the following commutative diagram
\[
(2.5) \quad \begin{array}{ccc}
C & \xrightarrow{f'} & W \\
S & \xrightarrow{h} & B
\end{array}
\]
where $S \cong B$.

Denote by $C^\sharp := (\mathcal{C}, \mathcal{M}^\sharp) \to B^\sharp := (\mathcal{B}, \mathcal{M}_{B^\sharp})$ the log curve with the canonical log structure over the underlying curve $C$. Let $\sigma_i \in C$ be the node obtained by gluing $q_i$ and $p_i$. By [Ols03], there is a canonical log structure $\mathcal{N}_i$ over $B$ associated to the node $\sigma_i$ of the underlying curve $C$. Furthermore, we have

$$\mathcal{M}^\sharp_B \cong \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_m.$$  

Since the log structure $\mathcal{M}_B$ and $\mathcal{N}_i$ are canonically associated to the underlying structure of the fibers, by the same argument as in [Kim10, Section 5.2.3], for each $i \neq \infty$ the underlying map $f'$ induces a morphism of log structures defined over $L$:

$$h_i : \mathcal{M}_B \to \mathcal{N}_i. \tag{2.6}$$

To construct the stable log map as in (2.5), it suffices to construct a log scheme $S = (\mathcal{S}, \mathcal{M}_S)$ with isomorphisms

$$\mathcal{M}_S \cong \mathcal{N}_i, \quad \text{for each } i. \tag{2.7}$$

Since by our construction of the morphism, the Galois action provides a canonical set of such isomorphisms by permuting the nodes and the underlying maps. This provides the log map as needed.

Finally, the composition $f := \pi \circ f' : C/S \to X$ is a stable log map to $X$ lifting the underlying stable map $\tilde{f}$ as in STEP 1, which fulfills the conditions as in the statement. The minimality in (5) follows from a direct calculation of the minimal monoid.

When the set of $\mathbb{A}^1$-teeth forms a complete Galois orbit, we notice that the rational section of (2.4) can be choosing defined over $K$. Since the isomorphism (2.7) is given by the Galois conjugation, the $\mathbb{A}^1$-map is stable under the Galois action, hence descents to a $K$-rational point as in the statement.

Lemma 2.3. Let $\overline{K}$ be the algebraic closure of $K$. Let $X = (\mathcal{X}, \mathcal{D})$ be a log smooth, proper, simple, and separably $\mathbb{A}^1$-connected $\overline{K}$-variety of rank one with the SRC center. Then there exists a very free $\mathbb{A}^1$-curve over $\overline{K}$ through any $\overline{K}$-rational point of $\mathcal{D}$.

Proof. The proof is similar to that of [CZ13, Theorem 1.9]. We give a sketch as follows. Since $\mathcal{D}$ is SRC, by [Kol96, IV.3], given any point $p \in \mathcal{D}$, there exists a free rational curve $f : \mathbb{P}^1 \to \mathcal{D}$ connecting $p$ and a general point $q$. By separably $\mathbb{A}^1$-connectedness, we may choose a very free $\mathbb{A}^1$-curve $g : (\mathbb{P}^1, \{\infty\}) \to (\mathcal{X}, \mathcal{D})$ with the boundary marking $q$ such that $\deg(f^*\mathcal{O}_X(\mathcal{D}) + g^*\mathcal{O}_{\mathcal{X}}(\mathcal{D})) > 0$. By [CZ13, Lemma 3.6], we can glue $f$ and $g$ into a stable $\mathbb{A}^1$-map with the contact marking $p$. A general smoothing of the stable $\mathbb{A}^1$-map will do the job. 

Proof of Theorem 1.1. Given a $K$-point $p \in \mathcal{D}(K)$, by Lemma 2.3 we may choose a finite Galois extension $L$ over $K$ such that there exists a very free
$A^1$-curve $f_1$ passing through $p_L$. By Proposition 2.2, gluing the Galois orbit of $f_1$, we obtain an $A^1$-comb $f \in \mathfrak{M}_{A^1}(X/K, m\beta; \underline{s})(K)$. We may further assume that $f$ is automorphism-free. By construction, $f$ is unobstructed and the minimal log structure on $S$ has rank one. Thus, it gives a smooth point of the underlying scheme $\mathfrak{M}_{A^1}(X/K, m\beta; \underline{s})(K)$. Since the set of very free $A^1$-curves through $p$ forms a dense open subset of $\mathfrak{M}_{A^1}(X/K, m\beta; \underline{s})$, the theorem is proved when $K$ is large.

Remark 2.4. When $X$ is log Fano and the normal bundle of $D$ is nontrivial and effective, there is a simple proof using Kollár’s result and [CZ13, Lemma 3.5]. However, our condition is weaker. The normal bundle of $D$ could be negative, for example the Hirzebruch surface with the $(-n)$-curve as the boundary.

3. Zariski density

Proof of Theorem 1.4. By [HT08, Theorem 9], after passing to a good resolution, we may assume that both $X$ and $D$ are nonsingular.

Since the geometric generic fiber of $D \to B$ is rationally connected, by [GHS03] there is a section $f_0 : B \to D$. We can use the comb smoothing argument as in [HT06b, Proposition 24], and assume that

1. $f_0$ is an immersion in $D$;
2. the morphism $df_0 : f_0^*\Omega_D \to \Omega_B$ is surjective, with the locally free kernel $N_{D/B}^\vee$;
3. $f_0$ is $S$-free, i.e., $H^1(N_{D/B}^\vee(-S)) = 0$, and $N_{D/B}^\vee(-S)$ is globally generated.

Notation 3.1. let $f : C \to X$ be a usual stable map over a geometric point such that

1. $C$ consists of irreducible components $C_0 \cup C_1 \cup \cdots \cup C_m$, where $C_0 \cong B$ is of genus $g$, and all other irreducible components are rational.
2. For each $i \neq 0$, the irreducible component $C_i$ is attached to $C_0$ at a general point $p_i \in C_0$ away from $S$, and there is no node on $C$ other than $p_i$ for $i = 1, \cdots, m$.
3. $f(C_0) \subset D$, and $q_i = f(p_i)$ is in general position of $D$ for all $i$.
4. $f|_{C_i}$ defines an $A^1$-curve for any $i \neq 0$.

Lemma 3.2. Notations as above, assume that $\deg_{C_0} f^*(D) = e$ for all $i$. Fix any point $\sigma \in C_0$. Assume that $c \geq 0$, and there is an isomorphism

$$N_{D/X|C_0} \cong \mathcal{O}_{C_0}(c \cdot \sigma - e(p_1 + \cdots + p_m)).$$

Then there is a log map $f : C/S \to X$ with a unique contact marking $\sigma$ of contact order $c$, where $X_D$ is the log scheme associated to the pair $(X, D)$. 
Proof. By assumption, we may choose a surjection
\[ O_{\mathbb{C}^0} \oplus N_D^{\vee} |_{\mathbb{C}^0} \rightarrow O_{\mathbb{C}^0} (e(p_1 + \cdots + p_m)) \]
where the restriction to the first factor is given by the divisor \( e \cdot \sum_{i \geq 1} p_i \), and to the second factor is given by \( c \cdot \sigma \). This induces a morphism
\[ \mathbb{C}^0 \rightarrow \mathbb{P}(O_{\mathbb{C}^0} \oplus N_D^{\vee} |_{\mathbb{C}^0}) \]
tangent to \( D_\infty \) at \( \sigma \) of order \( c \), and tangent to \( D_0 \) at \( p_i \) of order \( c_i \). By the same argument as in [CZ13, Lemma 3.6], the section \( s \) induces a map to the expansion. To further lift the underlying stable map to a stable log map, we will need the set of isomorphisms of the nodes as in (2.7). But since we are over algebraically closed fields, we could always make a choice of such isomorphisms. This provides the stable log map as needed.

Now consider the situation as in Notation 3.1. By the \( \mathbb{A}^1 \)-connectedness of the general fiber, we may choose \( m \) free \( \mathbb{A}^1 \)-curves as the teeth \( f|_{\mathbb{C}^0} : \mathbb{C}^0 \rightarrow X \). Then by [HT08, Lemma 21] and degree count, there is a log map \( f : C \rightarrow X \) as long as \( m \) is sufficiently large and all \( p_i \)'s are generic.

Furthermore, we may assume that \( \sigma \) be a point lying in \( S \) and \( f \) is a local immersion away from the special points. By [CZ14, (4.3.7) and Lemma 4.13], we conclude that \( H^1(N_f^{\vee}(-S)) = 0 \) and \( N_f^{\vee}(-S) \) is globally generated.

Theorem 1.4 follows from taking a general deformation of \( f \).

The approximation at a finite collection of integral points can be proved similarly as follows. By further gluing free \( \mathbb{A}^1 \)-teeth, we may assume \( f \) has \( \mathbb{A}^1 \)-teeth passing through those integral points. Then a general smoothing of \( f \) fixing those integral points will do the job.

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