Improving the Best-Worst Method Based on Optimal Completion of Incomplete Pairwise Comparison Matrix

SHIHUI WU, BO HE, AND XIAODONG LIU
Equipment Management and Unmanned Aerial Vehicle Engineering School, Air Force Engineering University, Xi’an 710051, China
Corresponding author: Shihui Wu (wu_s_h82@sina.com)

This work was supported by the National Natural Science Foundation of China under Grant 61601501.

ABSTRACT In this paper, the pairwise comparison matrix (PCM) of the best-worst method (BWM) is regarded as a special incomplete PCM (IPCM), and an improved method based on the optimal completion of the IPCM, called BWM-IPCM, is proposed. Firstly, as a special IPCM, BWM-IPCM has been proven to be effective/acceptable and ordinally consistent. Then, with the best consistency as the objective function, the optimization model of BWM-IPCM is established to achieve the optimal completion of incomplete elements, and the optimal weights and rankings are obtained from the completed PCM. In order to deal with possible inconsistencies in BWM, two methods are proposed: one is to invite experts to supplement a small amount of comparison information among inconsistent criteria found; the other is to fine-tune known elements to improve consistency without the participation of experts. Finally, illustrative examples show that compared with the original BWM, the proposed method can make more effective use of additional comparison information, has better consistency description and interpretation, and is easier to solve.

INDEX TERMS Decision analysis, incomplete pairwise comparison matrix (IPCM), best-worst method (BWM), optimization model, consistent ratio.

I. INTRODUCTION
Multi-criteria decision-making (MCDM) is a very important branch of decision theory, which has been widely used in many real-world fields such as economy [1], [2], [3], [4], healthcare [5], management [6], sports [7], [8], environment [9] and others. MCDM deals with three main types of decision problems: ranking, sorting, and choice. In the past decades, several MCDM methods have been proposed to help decision-makers (DMs) find efficient ranking decisions based on their preferences, such as TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) [10], ELECTRE (ELimination Et Choix Traduisant la REalité in French) [11], VIKOR (Vise Kriterijumska Optimizacija i Kompromisno Resenje) [5], [12], AHP (Analytic Hierarchy Process) [13], AQM (Alternative Queuing Method) [14], TODIM (an acronym in Portuguese for Interactive and Multi-Criteria Decision Making) [9], MULTIMOORA (Multi-Objective Optimization on the basis of Ratio Analysis) [4], EDAS (Evaluation based on Distance from Average Solution) [15], etc.

In recent years, one of the most developed MCDM methods, the best-worst method (BWM) [16], [17], has been proposed. It is a comparison-based method that conducts the comparisons in a particularly structured way. Compared with AHP, BWM has good performance in requiring less pairwise comparisons, i.e., the pairwise comparison matrix (PCM) is incomplete, and maintains better consistency between judgments, which has attracted many scholars’ attention [17], [18], [19], [20], [21], [22]. In the past few years, many methods combining BWM have been proposed, such as combining BWM with TOPSIS [23], VIKOR [24], ELECTRE [25], etc. The most popular application fields of BWM are supply chain [21], [26], supplier selection [23], [27], manufacturing [16],
performance evaluation [28], airline industry [24], energy [29], biology [30], transportation [31], education [32] and technology [33], etc.

From the perspective of theoretical research, BWM mainly focuses on using an optimization model to solve the optimal weight vector. Because the solution is not unique, Rezaei [17] also put forward the concept of optimal weight intervals, that is, the weights obtained are not crisp numbers, but possible intervals, so the weight intervals may have overlapping parts, which may lead to fuzzy criteria rankings. Another method is to linearize BWM to obtain a unique weight solution. However, the linearized BWM is essentially different from the original model [18], which may lead to different results.

From existing works of literature, BWM has the following shortcomings. First, how to determine an acceptable consistency ratio (CR) value and its threshold needs further studies. Although Rezaei [17] has defined a CR index, it does not have a threshold to measure acceptable consistency. To solve this problem, Liang et al. [22] proposed a cardinal consistency measurement called the input-based consistency ratio (CR^I) and its threshold, which is easier to obtain than the CR index in Rezaei [16] (called output-based consistency ratio (CR^O) in [22]). CR^I can be used as a substitute for CR^O, because the two measurements have high similarity, but it is very weird that the threshold of CR^1 varies with the value of the largest element of the PCM (a_{BW}). For example, when a_{BW} = 8 or 9, there may not be much difference for the DMs, but the consistency in these two cases may be completely different. Second, how to adjust the inconsistencies in BWM remains to be resolved. Although BWM maintains a higher consistency than AHP, there may still be inconsistencies in some cases. For high inconsistency cases, the results of BWM may be incredible, thus further research on consistency improvement is still needed.

In order to solve these problems, this paper proposes to regard the best-worst matrix as a special incomplete PCM (IPCM) [34], [35], [36], [37], [38] because it only gives the comparison information of the best and the worst criterion, and ignores other elements. When BWM is regarded as an IPCM, some properties and interesting findings are put forward, which can solve the shortcomings of the original BWM very well.

BWM was firstly introduced by Harker [39]. He proposed the geometric mean method based on the concept of connecting paths. Nowadays, decision-making based on IPCM has become very common in real life [7], [8]. For example, it has been used to rank large-scale data, such as historical Go players ranking [7], and historical top tennis players ranking [8]. Based on Harker’s method, many studies seek the optimal complement method for missing values of IPCM. For example, Shen et al. [40] used the connecting path method to estimate the missing judgments in IPCMs with a minimal geometric consistency index. Ergu et al. [6] extended the geometric mean induced bias matrix to estimate the missing values. Fedrizzi and Giove [36] completed the IPCMs by minimizing a measure of global inconsistency. Recently, Li, et al. [41] studied group decision-making (GDM) problems based on incomplete hesitant fuzzy linguistic preference relations (IHFLPR), and developed an optimization model to impute missing elements for an IHFLPR, in which the best additive consistency index (BCI) and the average additive consistency index (ACI) are utilized to measure the additive consistency level of an IHFLPR. These methods try to achieve the best consistency after optimal completion of the IPCM, which is what the DMs expect in decision-making.

Therefore, this paper proposes an improved BWM based on IPCM optimal completion, which we call the BWM-IPCM method. Based on the theory of IPCM, an optimization model is established to achieve the optimal completion of the IPCM, and afterward, the optimal weight and CR value of the completed PCM are calculated using AHP method, and the CR value is used as the consistency index of BWM-IPCM. In view of the possible inconsistencies in BWM, two improvement methods are proposed.

The rest of this paper is organized as follows. Section II introduces the original BWM. In Section III, the BWM-IPCM algorithm is proposed. In Section IV, numerical examples are given to illustrate the proposed method, and some comparisons with the original BWM are made. The last section is the conclusion and suggestions for future research.

II. BEST-WORST METHOD

Similar to AHP, BWM is also based on pairwise comparisons. However, compared with AHP, BWM reduces the number of pairwise comparisons between criteria, improves the inconsistency caused by experts’ subjective judgments, and thus reduces experts’ workload.

The basic steps of the original BWM can be summarized as follows [16], [17]:

**Step 1** Determine the best criterion c_B and the worst criterion c_W in the criteria set \{c_1, c_2, \ldots, c_n\}.

**Step 2** Use the scale of 1-9 to determine the preference degree of c_B compared with all other criteria, and obtain the comparison vector \(A_B = \{a_{B1}, a_{B2}, \ldots, a_{Bn}\}\), where \(a_{Bi}\) represents the preference degree of \(c_B\) over \(c_i\), \(i \in \{1, 2, \ldots, n\}\).

**Step 3** Similarly, determine the preference degree of all other criteria compared with \(c_W\), and obtain a comparison vector \(A_W = \{a_{W1}, a_{W2}, \ldots, a_{Wn}\}\)\(^T\), where \(a_{Wj}\) represents the preference degree of \(c_j\) over \(c_W\), \(j \in \{1, 2, \ldots, n\}\).

**Step 4** Use the optimization model to find the optimal weights, \(\{w_1, w_2, \ldots, w_n\}\). The goal is to minimize the maximum absolute differences \(|w_B/w_W - a_{Bj}|\) and \(|w_j/w_W - a_{Wj}|\) for all \(j\), that is, to solve the following minmax model:

**Model 1:**

\[
\min_{j} \max \left\{ \left| \frac{w_B}{w_W} - a_{Bj} \right|, \left| \frac{w_j}{w_W} - a_{Wj} \right| \right\}
\]

s.t. \( \sum_{j} w_j = 1 \)

\( w_j > 0, \quad \forall j \)
Generally, the equivalent form of Model 1 - Model 2 is used to solve the above problem.

**Model 2:**

\[
\begin{align*}
\text{min } \xi \\
\text{s.t. } \frac{w_B}{w_j} - a_{Bj} \leq \xi, \quad \forall j \\
\frac{w_j}{w_W} - a_{jW} \leq \xi, \quad \forall j \\
\sum_j w_j = 1 \\
w_j > 0, \quad \forall j
\end{align*}
\]

By solving Model 2, the optimal weights \([w_1^*, w_2^*, \ldots, w_n^*]\) and \(\xi^*\) are obtained. Note that the optimal solution of Model 2 is not unique, in other words, for the same optimal objective value \(\xi^*\), the optimal weights are interval-valued, which can be determined by Model 3. The upper and lower bounds of the weight interval are determined by the two optimal models in Model 3 respectively.

**Model 3:**

\[
\begin{align*}
\text{min } w_j \\
\text{s.t. } \frac{w_B}{w_j} - a_{Bj} \leq \xi^*, \quad \forall j \\
\frac{w_j}{w_W} - a_{jW} \leq \xi^*, \quad \forall j \\
w_j > 0, \quad \forall j
\end{align*}
\]

The center of the weight intervals obtained from Model 3 can be used to rank the criteria [17].

In [16] and [17], \(\xi^*\) is used to define the consistency ratio:

\[
\text{CR}_{BWM} = \frac{\xi^*}{\text{CI}}.
\]

where consistency index (CI) depends on the preference degree between the best criterion and the worst criterion \(a_{BW}\), as shown in Table 1. As mentioned in [16] and [17], \(\text{CR}_{BWM}\) is within the range of \([0, 1]\). The smaller the \(\text{CR}_{BWM}\) value, the better the consistency. On the contrary, the greater the \(\text{CR}_{BWM}\) value, the worse the consistency.

| \(a_{BW}\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| CI | 0 | 0.44 | 1.00 | 1.63 | 2.30 | 3.00 | 3.73 | 4.47 | 5.23 |

When \(\xi^* = 0\), we have \(\text{CR}_{BWM} = 0\) by (4), which means perfect/full consistency, and the optimal weights obtained by Models 2 and 3 are unique [17]; When \(\xi^* > 0\), we have \(\text{CR}_{BWM} > 0\) by (4), which means that it is not perfectly consistent. Note that the larger \(\xi^*\) is, the more inconsistent the comparison information is. This may lead to larger weight intervals [17], and more likely the intervals are to overlap. In other words, ranking the overlapped criteria becomes more difficult.

## III. BWM-IPCM ALGORITHM

### A. DEFINITIONS AND THEORIES

Since the best-worst matrix is a special IPCM in essence, it has some special properties besides the common characteristics of IPCM.

**Definition 1 ((IPCM) [39]):** An incomplete matrix \(A = (a_{ij})_{n \times n}\) is called an IPCM, if it contains both incomplete elements (unknown elements or missing elements) and known elements that satisfy \(a_{ij} = 1, a_{ii} = 1, a_{jB} > 0, \forall i, j \in [1, 2, \ldots, n]\).

**Definition 2 (Equivalent Decision Vector):** For an IPCM \(A = (a_{ij})_{n \times n}\), the equivalent decision vector \(\ddot{a}\) is formed by the upper triangle elements of \(A\), arranged from top to bottom, and from left to right, i.e., \(\ddot{a} = \{a_{12}, \ldots, a_{n1}, a_{23}, \ldots, a_{n(n-1)}, a_{34}, \ldots, a_{n(n-2)}, \ldots, a_{n-1n}\}\), where \(m\) unknown elements are denoted as variables, \(x_k \in [1/9, 9]\), where \(k = 1, 2, \ldots, m\).

**Definition 3 (BWM-IPCM):** An IPCM \(A = (a_{ij})_{n \times n}\) is called a BWM-IPCM, if the known elements are from two comparison vectors, i.e., one is the best criterion compared with other criteria, denoted as \(A_B = (a_{B1}, a_{B2}, \ldots, a_{Bn})\), and the other one is all other criteria compared with the worst criterion, denoted as \(A_W = (a_{1W}, a_{2W}, \ldots, a_{nW})^T\).

For example, suppose the best criterion is \(a_{1}, a_{2}, \ldots, a_{4}\), arranged from top to bottom, and from left to right. Then the equivalent decision vector \(\ddot{a}\) can be rewritten as (4).

\[
A_1 = \begin{bmatrix} 1 & x & x & x & 4 \\ 2 & 1 & 4 & 3 & 8 \\ x & x & 1 & 2 \\ x & x & 3 \\ x & x & x & 1 \end{bmatrix}.
\]

According to Definition 2, the equivalent decision vector \(\ddot{a} = \{0.5, x_1, x_2, 4, 4, 3, 8, x_3, 2, 3\}\), which has three unknown elements, thus \(m = 3\).

For an \(n\)-order BWM-IPCM, the number of elements in the upper triangle is \(n(n-1)/2\), since \(a_{BB} = 1, a_{BW} = 1\) and \(a_{BW} = 1\) exists both in \(A_B\) and \(A_W\), the number of known elements is \(2n-3\), thus the number of unknown elements will be \(m = n(n-1)/2-2(n-3) = (n-2)(n-3)/2\). In particular, for a 3rd-order matrix, we have \(m = 0\), which means that there is no unknown element, i.e., it is a complete PCM.
Definition 4 (Three-Way Cycle [42]): For a PCM \( A = (a_{ij})_{n \times n} \), a three-way cycle \( i-j-k-i \) is formed by three elements \( (a_{ij}, a_{jk}, a_{ki}) \) in \( A \), if \( c_i > c_j > c_k > c_i \) is satisfied (“\( > \)” means better than).

Definition 5 (Ordinal Consistency [43], [44]): A PCM \( A = (a_{ij})_{n \times n} \) is said to be ordinally consistent, if there is no three-way cycle in \( A \). Otherwise, \( A \) is ordinally inconsistent.

The AHP allows PCM to have a certain level of inconsistency, which is measured by the consistency ratio (CR) of Saaty [13]. The consistency ratio of the complete PCM \( C \) is:

\[
CR(C) = \frac{\lambda_{\text{max}}(C) - n}{(n - 1)RI}.
\]

where \( \lambda_{\text{max}} \) is the principal eigenvalue of \( C \). If \( C \) is perfectly consistent, then \( \lambda_{\text{max}}(C) = n \), \( CR(C) = 0 \); if \( C \) is inconsistent, then \( \lambda_{\text{max}}(C) > n \), \( CR(C) > 0 \), the greater the \( CR(C) \), the worse the consistency. \( RI \) is a random consistency index, which depends on order \( n \) and can be found in [13].

Definition 6 (Cardinal consistency [13]): A PCM \( A = (a_{ij})_{n \times n} \) is said to be cardinally consistent, if \( CR(A) < 0.1 \); Otherwise, \( A \) is cardinally inconsistent.

Definition 7 (Effective IPCM): An IPCM \( A \) is said to be effective (also called acceptable in [45] and [46]), if all missing elements can be determined by known elements.

Proposition 1: All BWM-IPCMs are effective IPCMs.

Proof: According to Harker [47], IPCM is an irreducible matrix if and only if the directed graph is strongly connected. A strongly connected directed graph means that all missing elements can be determined by known elements. Therefore, if the IPCM is irreducible, it is effective. Obviously, a complete row or column can be found in a BWM-IPCM, such as the row corresponding to the best criterion and the column corresponding to the worst criterion. Since IPCM with a complete row or column is a special case of irreducible matrix, BWM-IPCM is effective. \( \square \)

Proposition 2: All known elements in BWM-IPCM (except those available for \( a_{ij} = 1/a_{ji} \)) are greater than or equal to 1.

Proof: Obviously, for the best criterion, it is better than all other criteria, then we have \( a_{ij} \geq 1 \), \( \forall j \). Meanwhile, all other criteria are better than the worst criterion, so we have \( a_{ij} \geq 1 \), \( \forall j \). Therefore, for a BWM-IPCM, all known elements are greater than or equal to 1. \( \square \)

Definition 8 (Standard BWM-IPCM): A BWM-IPCM \( A_s = (a_{ij})_{n \times n} \) is called standard BWM-IPCM, when \( c_1 \) is the best criterion and \( c_n \) is the worst criterion. The process of transforming BWM-IPCM to standard BWM-IPCM is called BWM-IPCM standardization.

If one exchanges the best criterion with the first criterion \( c_1 \), and the worst criterion with the last criterion \( c_n \), the equivalent standard BWM-IPCM can be obtained. For the standard BWM-IPCM, according to Proposition 2, all known elements in the upper triangle are greater than or equal to 1, and \( a_{1n} (= a_{n1}) \) is the largest element among all known elements. For example, the equivalent standard BWM-IPCM of \( A_1 \) mentioned earlier is:

\[
A_3 = \begin{bmatrix}
1 & 2 & 4 & 3 & 8 \\
\times & 1 & \times & \times & 4 \\
\times & \times & 1 & \times & 2 \\
\times & \times & \times & 1 & 3 \\
\times & \times & \times & \times & 1
\end{bmatrix}.
\]

Obviously, for \( A_3 \), only the first row and the last column are known, and all known elements of the upper triangle are greater than or equal to 1. The optimal weights and the consistency ratio obtained by \( A_3 \) should be the same as \( A_1 \), because the difference between the two matrices is only the positions exchange of some criteria. For example, \( A_3 \) is the equivalent standard BWM-IPCM of \( A_1 \).

By solving Model 2, we have \( w(A_1) = \{0.222, 0.453, 0.110, 0.158, 0.055\} \), CRBWM (\( A_1 \)) = 0.032, and \( w(A_3) = \{0.453, 0.222, 0.110, 0.158, 0.055\} \), CRBWM (\( A_3 \)) = 0.032. It can be seen that the weight vector and CR value are completely the same, indicating that the standardized BWM-IPCM \( A_3 \) is the equivalent transformation of the original matrix \( A_1 \).

The purpose of the standard BWM-IPCM is that it has good characteristics, that is, all known elements in the standard BWM-IPCM’s equivalent decision vector are greater than or equal to 1, which is conducive to the subsequent optimization process, that is, when we need to fine-tune some known elements (see Step 7 in Section III(C)), the changing amount can have the same meaning. For example, for the unstandardized BWM-IPCM \( A_2 \), when \( a_{12} \) changes from 0.5 to 0.4 and \( a_{15} \) changes from 4 to 4.1 (see \( A_4 \)), this is equivalent to \( a_{21} \) changing from 2 to 2.5, thus the changing amount in the equivalent decision vector reaches 0.5, which is far greater than the change in \( a_{15} \), i.e., 0.1. However, for the standard BWM-IPCM \( A_3 \), the changing amounts of \( a_{12} \) and \( a_{25} \) are the same, both are 0.1 (see \( A_3 \)).

\[
A_4 = \begin{bmatrix}
1 & 0.4 & \times & \times & 4.1 \\
\times & 1 & 4 & 3 & 8 \\
\times & \times & 1 & \times & 2 \\
\times & \times & \times & 1 & 3 \\
\times & \times & \times & \times & 4.1
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
1 & 2.1 & 4 & 3 & 8 \\
\times & 1 & \times & \times & 4.1 \\
\times & \times & 1 & \times & 2 \\
\times & \times & \times & 1 & 3 \\
\times & \times & \times & \times & 1
\end{bmatrix}
\]

Proposition 3: BWM-IPCM has ordinal consistency.

Proof: See the Appendix section. \( \square \)

Proposition 3 can also be proven by the directed graph corresponding to BWM-IPCM. Suppose \( A_4 \) is a standard BWM-IPCM, as shown below, according to Proposition 2 and Definition 8, all known elements in the upper triangle of \( A_4 \) are greater than 1. In Fig. 1, \( G \) represents the directed graph corresponding to \( A_4 \), where the direction of the arrow indicates greater than, e.g., \( a_{12} > 1 \), so the arrow points from \( c_1 \) to \( c_2 \). Obviously, it is seen that there is no three-way cycle...
in \( \tilde{G} \). Therefore, according to Definition 5, \( A_4 \) has ordinal consistency. Since all BWM-IPCMs can be standardized to the form of \( A_4 \), Proposition 3 is true for any BWM-IPCM.

\[
A_4 = \begin{bmatrix}
1 & a_{12} & a_{13} & \cdots & a_{1n} \\
\times & 1 & \times & \cdots & \times a_{2n} \\
\times & \times & 1 & \cdots & \times \\
\times & \times & \times & \cdots & 1
\end{bmatrix}
\]

### B. OPTIMIZATION MODEL

The main idea of IPCM completion methods is to try to make the completed PCM as consistent as possible (see [6], [36], [39], [40]). Similarly, we have developed an optimization model to achieve the optimal completion of BWM-IPCM. For the missing element \( x_i \), the optimization goal is to find the best value of \( x_i \) so that the CR value of the completed BWM-IPCM is as small as possible. The optimization model is shown below.

**Model 4:**

\[
\min_{x_1, x_2, \ldots, x_m} \{ CR(A(x)) \} \\
\text{s.t. } x_i \in [1/9, 9], \quad \forall i \in \{1, 2, \ldots, m\} \quad (6)
\]

where \( x = \{x_1, x_2, \ldots, x_m\} \) are the incomplete elements to be optimized within the range \([1/9, 9]\), and CR is the consistency ratio of the completed PCM after optimal completion, which is calculated by (5).

**Definition 9 (Consistency Ratio of BWM-IPCM):** Let \( A = (a_{ij})_{n \times n} \) be a BWM-IPCM, the minimum CR value obtained from Model 4 is called the consistency ratio of \( A \), denoted as \( CR_{\min}(A) \). In order to keep consistent with Saaty [13], if \( CR_{\min}(A) < 0.1 \), \( A \) is said to have acceptable consistency; Otherwise, \( A \) is said to have unacceptable consistency.

Note that according to Proposition 3, all BWM-IPCMs are ordinably consistent. Therefore, in Definition 9, we use cardinal consistency to represent whether BWM-IPCMs have acceptable consistency.

**Theorem 1:** Let \( A = (a_{ij})_{n \times n} \) be a BWM-IPCM, the optimal solution for Model 4, \( CR_{\min}(A) \), exists and is unique, and the optimal weights are also unique.

**Proof:** See the Appendix section. \( \square \)

Xu and Wei [48] indicated that when the elements in PCM are modified in a specified way, it can guarantee a monotonic continuous decline of \( \lambda_{\max} \), which also indirectly proved that when \( A \) is not perfectly consistent, \( CR_{\min}(A) \) has only one global extreme and no local extreme.

According to Theorem 1, Model 4 is an unconstrained nonlinear minimization problem with only one optimal solution. We use the function \texttt{fmincon} in MATLAB v.7.0 to solve this minimization problem. First, establish the objective function, which inputs \( m \) variables \( x = \{x_1, x_2, \ldots, x_m\} \), and outputs \( CR(A(x)) \). Then, we initialize all \( m \) variables to 1, i.e., \( x_0 = \{1, 1, \ldots, 1\} \). For each \( x_i (i = 1, 2, \ldots, m) \), set the lower bound to 1/9, and the upper bound to 9. Input the objective function name, initial solution \( x_0 \), lower bounds and upper bounds into the \texttt{fmincon} function to obtain the optimal solution. Please note that Model 4 has a unique minimum value. Any gradient based method can reach the minimum value after some iterations. Therefore, the \texttt{fmincon} function can solve this problem quickly and accurately.

For the \( n \)th-order problem, 10 000 standard BWM-IPCMs are randomly generated. According to Proposition 2, since all known elements in the standard BWM-IPCM are greater than or equal to 1, each standard BWM-IPCM is generated by \((n-2)(n-3)/2\) random integers from 1 to 9, and the element \( a_{BW} = a_{11} \) is specified as the maximum value. Then, by solving Model 4, the \( CR_{\min} \) value of each BWM-IPCM is obtained, from which the percentage of cardinal inconsistency (\( CR_{\min} > 0.1 \)), and the maximum value of \( CR_{\min} \) in 10 000 cases are calculated. See Table 2 for the results.

For the 3rd-order cases, the probability of cardinal inconsistency is very high, reaching 42.83%. As mentioned earlier, the 3rd-order BWM-IPCM is actually a complete PCM, that is, there are no incomplete elements, and thus Model 4 is not required. The cardinal inconsistency probability of the 4th order is 21.61%, the 5th order is 1.8%, and the 6th order and above is almost 0. This is because the higher the order, the more elements are missing in the BWM-IPCM, which means that by solving Model 4 to complete the missing elements, more suitable elements can be found within the range of \([1/9, 9]\) so that BWM-IPCM can achieve better cardinal consistency. It can also be seen from the maximum \( CR_{\min} \) value in Table 2 that the higher the order, the higher the cardinal consistency that BWM-IPCM can achieve.

| Order | Number of missing elements | Percentage of \( CR_{\min} > 0.1 \) in 10 000 cases | Maximum \( CR_{\min} \) in 10 000 cases |
|-------|-----------------------------|---------------------------------|----------------------------------|
| 3     | 0                           | 42.83%                          | 0.539                            |
| 4     | 1                           | 21.61%                          | 0.227                            |
| 5     | 3                           | 1.8%                            | 0.158                            |
| 6     | 6                           | 0.02%                           | 0.101                            |
| 7     | 10                          | 0%                              | 0.093                            |
| 8     | 15                          | 0%                              | 0.069                            |
| 9     | 21                          | 0%                              | 0.059                            |
| 10    | 28                          | 0%                              | 0.047                            |

Some may doubt that according to Table 2, \( CR_{\min} > 0.1 \) holds true for almost all high-order BWM-IPCMs. Therefore,
it may not make sense to calculate $CR_{\min}$ of high-order BWM-IPCMs. However, the truth is that $CR_{\min}$ is equal to 0 only when the BWM-IPCM is perfectly consistent according to (5). Generally, $CR_{\min}$ is greater than 0, that is, the optimal completion of BWM-IPCM cannot be perfectly consistent, since there already exists cardinal inconsistency among known elements. As a result, $CR_{\min}$ actually reflects the consistency level of all known elements in BWM-IPCM, in other words, it reflects the consistency of the judgments given by the DM.

From Table 2, one can see that $CR_{\min}$ may be much greater than 0.1 for orders of 3, 4, 5, and 6. In this case, the consistency of BWM-IPCM can be improved by fine-tuning some known elements, such as changing from 6 to 5.5. For details on the fine-tuning method, please refer to Section 2.4 of Wu et al. [37]. However, it is almost impossible that $CR_{\min}$ is greater than 0.1 for orders of 7 and above.

For example, $A_3$ is the BWM-IPCM mentioned earlier. According to Definition 2, its equivalent decision vector is $\tilde{a} = [2, 4, 3, 8, x_1, x_2, 4, x_3, 2, 3]$. Since there are three unknown variables in $A_3$, in order to show how $CR_{\min}$ changes with the variables in 3D graph, we assume $x_1 = 2$ to get an IPCMA$_5$. Let $x_2$ and $x_3$ change from 1/9 to 9, and observe how $CR_{\min}$ changes with $x_2$ and $x_3$, as shown in Fig. 2.

$$A_5 = \begin{bmatrix} 1 & 2 & 4 & 3 & 8 \\ \times & 1 & \times & 4 \\ \times & \times & 1 & \times & 2 \\ \times & \times & \times & 1 & 3 \\ \times & \times & \times & \times & 1 \end{bmatrix}.$$  

It can be seen from Fig. 2 that the optimal solution $CR_{\min}(A_5)$ is unique, and the corresponding completed elements, that is, the optimal solution ($x^*_2, x^*_3$) is also unique. This example shows that the CR function (variables here are $x_2$ and $x_3$) can be regarded as a convex function, and the optimal solution can be easily obtained using common optimization algorithms.

From Table 3, it can be seen that the number of variables and constraints in Model 2 [17] (we call it the original BWM) is far greater than those in Model 4, which makes it more difficult for Model 2 to find the optimal solution. In Model 2, there are $5n-8$ variables, and $4n-5$ inequality constraints [20]. In Model 4, there are $n(n-2)(n-3)/2$ variables, and no inequality constraints. When $n > 3$, the number of variables in Model 2 is greater than the number of constraints, which will lead to multiple optimal solutions. Model 4 is essentially an unconstrained multivariable function optimization problem. Its objective function is to minimize $CR(x_1, x_2, \ldots, x_m)$, and there is no constraint except the upper and lower bounds of variables.

According to Theorem 1, Model 4 has at most one global minimum and no local minimum, and the optimal solution is unique. Meanwhile, the number of variables and constraints is relatively small, so it is much easier to solve than Model 2 and Model 3.

### C. PROCEDURE OF BWM-IPCM ALGORITHM

As shown in Fig. 3, the procedure of the proposed BWM-IPCM algorithm is summarized as follows.

**Step 1.** Experts select the best and worst criteria from the criteria set.

**Step 2.** Determine the best and worst comparison vector. According to BWM, under the scale of 1-9, the preference degree of the best criterion compared with other criteria, and that of all other criteria compared with the worst criterion are determined, and two comparison vectors are obtained:

$$A_B = \{a_{B1}, a_{B2}, \ldots, a_{Bn}\}$$

$$A_W = \{a_{W1}, a_{W2}, \ldots, a_{Wn}\}^T$$

**Step 3.** Construct standard BWM-IPCM. According to $A_B$ and $A_W$, BWM-IPCM $A$ is constructed by Definition 3, and then the equivalent standardized form $A_s$ is obtained by Definition 8.

**Step 4.** If experts are involved, considering only one of the comparison vectors each time, two weight vectors will be obtained:

$$w_B = \{w_{B1}, w_{B2}, \ldots, w_{Bn}\}$$

$$w_W = \{w_{W1}, w_{W2}, \ldots, w_{Wn}\}$$

where $w_{Bi} = 1/a_{Bi} / \sum_{i=1}^{n} 1/a_{Bi}$, and $w_{Wi} = a_{Wi} / \sum_{i=1}^{n} a_{Wi}$.

If no experts are involved, go to Step 6.

**Step 5.** Judge whether the two weight vectors $w_B$ and $w_W$ are in the same rankings. If the rankings are different (See Example 3), experts are suggested to supplement a small

| Order | Number | Constraints | Variables |
|-------|--------|-------------|-----------|
| 3     |        | 0           | 0         |
| 4     |        | 0           | 0         |
| 5     |        | 0           | 0         |
| 6     |        | 0           | 0         |
| 7     |        | 0           | 0         |
| 8     |        | 0           | 0         |
| 9     |        | 0           | 0         |
amount of comparison information on the inconsistent elements. For example, suppose $n = 8$, if there is a conflict in ranking $c_3$ and $c_7$, experts can supplement a comparison between $c_3$ and $c_7$, that is, give the value of $a_{37}$ in the IPCM, and then go to the next step. If $\mathbf{w}_B$ and $\mathbf{w}_W$ are in the same rankings, go to Step 6.

**Step 6.** Solve the optimization model. Put the standard BWM-IPCM $\mathbf{A}_s$, supplemented IPCM (from Step 5) or fine-tuned BWM-IPCM (from Step 7) into Model 4, and use the MATLAB function `fmincon` to obtain the optimal weight vectors and $\text{CR}_{\text{min}}$. If $\text{CR}_{\text{min}} < 0.1$ holds, the IPCM is considered as having acceptable cardinal consistency, and go to Step 8; Otherwise, go to Step 7.

**Step 7.** Consistency fine-tuning of the BWM-IPCM. According to Wu et al. [37], fine-tune some known elements to make the fine-tuned BWM-IPCM meet the requirements of cardinal consistency, i.e., $\text{CR}_{\text{min}} < 0.1$, and return to Step 6.

**Step 8.** Output results, and end the algorithm. Output the optimal solution, including the optimal weight vector $\omega$ and consistency ratio $\text{CR}_{\text{min}}$.

### IV. NUMERICAL ILLUSTRATIONS

**Example 1 [17]:** $A$ is a 5th-order BWM-IPCM.

$$
A = \begin{bmatrix}
1 & x & x & 4 \\
2 & 1 & 4 & 2 & 8 \\
x & x & 1 & 2 \\
x & x & 1 & 4 \\
x & x & x & 1
\end{bmatrix}, \quad A_s = \begin{bmatrix}
1 & 2 & 4 & 2 & 8 \\
1 & x & x & 4 \\
x & 1 & x & 2 \\
x & x & 1 & 4 \\
x & x & x & 1
\end{bmatrix}
$$

First, transform $A$ by exchanging the best criterion $c_2$ with the first criterion $c_1$, and the equivalent standard BWM-IPCM $A_s$ is obtained. We will discuss $A_s$ instead of $A$.

**Step 1 – Step 2.** Obviously, for the standard BWM-IPCM $A_s$, we have $\mathbf{A}_B = \{1, 2, 4, 2, 8\}, \mathbf{A}_W = \{8, 4, 2, 4, 1\}$.

**Step 3 – Step 4.** It can be computed that $\mathbf{w}_B = [0.4210, 0.2105, 0.1053, 0.2105, 0.0526]$. For comparison, Table 4 also shows the optimal weights obtained in Rezaei [17] and the consistency index calculated by (4). It can be seen that the consistency information is perfectly consistent since $\text{CR} = 0$.

The PCM in Example 1 is perfectly consistent, as shown in Table 4. The comparative analysis between our method - BWM-IPCM and the original BWM [17] is as follows.

1. In the case of perfect consistency, the BWM-IPCM and the original BWM can obtain the same weight vectors and rankings, as shown in Table 4.
2. The consistency index for BWM-IPCM - $\text{CR}_{\text{min}}$ is more suitable to describe consistency than that of the original BWM - $\text{CR}_{\text{BWM}}$. The reasons are as follows: On the one hand, $\text{CR}_{\text{min}}$ has the same meaning as the traditional CR in Saaty [13], which can be applied to explain whether a BWM-IPCM has acceptable consistency, i.e., when $\text{CR}_{\text{min}} < 0.1$, the PCM is of acceptable consistency. However, the consistency index for the original BWM - $\text{CR}_{\text{BWM}}$ has no threshold or widely acknowledged threshold that measures acceptable consistency, which has been discussed in Section I. On the other hand, compared with $\text{CR}_{\text{min}}, \text{CR}_{\text{BWM}}$ is very sensitive to the expert’s judgment. Fig. 4 shows how the two CR indexes change when the known element $a_{13}$ changes from 2 to 8.

It can be seen that both CR indexes get the minimum value 0 when $a_{13} = 4$, and both indexes show similar increase and decrease trends. However, we find that $\text{CR}_{\text{BWM}}$ is very sensitive to the expert’s judgment, i.e., minor changes in expert’s judgment scores will lead to great changes in $\text{CR}_{\text{BWM}}$ value. For example, when $a_{13}$ changes from 4 to 3,

### Table 4. Results of Example 1.

| Methods          | Optimal weight vectors and rankings | Consistency indicators |
|------------------|------------------------------------|------------------------|
| Our method       | $\{0.4210, 0.2105, 0.1053, 0.2105, 0.0526\}$ | $\text{CR}_{\text{min}} = 0$, $\text{CR}_{\text{BWM}} = 0$ |
| Rezaei (2016)    | $\{0.4210, 0.2105, 0.1053, 0.2105, 0.0526\}$ | $\text{CR}_{\text{min}} = 0$, $\text{CR}_{\text{BWM}} = 0$ |

\[ \mathbf{w}_W = [0.4210, 0.2105, 0.1053, 0.2105, 0.0526] \].

**Step 5.** Since the two weight vectors $\mathbf{w}_B$ and $\mathbf{w}_W$ are in the same rankings, i.e., $1>2\sim4>3\sim5$, we can solve the optimization model - Model 4 according to the procedure of BWM-IPCM algorithm.

**Step 6 - Step 8.** The equivalent decision vector of $A_s$ is, $\bar{a} = \{2, 4, 2, 8, x_1, x_2, 4, x_3, 2, 4\}$. Substitute $\bar{a}$ into Model 4 referring to (6), and use the MATLAB function `fmincon` to find the optimal solution to obtain the completed PCM $A'$. The optimization results are shown in Table 4.

For comparison, Table 4 also shows the optimal weights obtained in Rezaei [17] and the consistency index calculated by (4). It can be seen that the comparison information is perfectly consistent since CR = 0.
the CR\textsubscript{BWM} increases from 0 to 0.070 8, while when a\textsubscript{13} decreases to 2, CR\textsubscript{BWM} reaches 0.156 9. It can be said that the degree of inconsistency is very high. For the BWM-IPCM algorithm, when a\textsubscript{13} changes from 4 to 2, CR\textsubscript{min} increases from 0 to 0.008 7, which still maintains a good consistency level. According to the expert scoring principle [13], a score of 4 means between "moderately important" and "strongly important", and a score of 2 means between "moderately important" and "equally important". Therefore, from the subjective judgment of experts, the difference between a score of 2 and a score of 4 is not significant, so the influence on the consistency index should not be too large. However, as shown in Fig. 4, when a\textsubscript{13} changes from 4 to 2, CR\textsubscript{BWM} changes greatly, that is, from 0 to 0.156 9, while CR\textsubscript{min} changes from 0 to 0.0087, which means that CR\textsubscript{min} is more consistent with the subjective judgment of experts.

**Figure 4. Comparison of consistency index.**

**Example 2 [17]:** A is a 5th-order BWM-IPCM, and its standard form A\textsubscript{s} is as follows.

\[
A = \begin{bmatrix}
1 & x & x & x & 4 \\
2 & 1 & 4 & 3 & 8 \\
x & x & 1 & x & 2 \\
x & x & x & 1 & 3 \\
1 & 2 & 4 & 3 & 8 \\
x & 1 & x & 4 \\
x & x & 1 & x & 2 \\
x & x & x & 1 & 3 \\
x & x & x & x & 1 \\
\end{bmatrix}
\]

\[
A_s = \begin{bmatrix}
1 & x & x & x & 4 \\
2 & 1 & 4 & 3 & 8 \\
x & x & 1 & x & 2 \\
x & x & x & 1 & 3 \\
1 & 2 & 4 & 3 & 8 \\
x & 1 & x & 4 \\
x & x & 1 & x & 2 \\
x & x & x & 1 & 3 \\
x & x & x & x & 1 \\
\end{bmatrix}
\]

**Step 1 – Step 2.** For the standard BWM-IPCM A\textsubscript{s}, we have \(A_B = \{1, 2, 4, 3, 8\}, A_W = \{8, 4, 2, 3, 1\}\). **Step 3 – Step 4.** It can be computed that

\[
w_B = \begin{bmatrix} 0.452 8 & 0.226 4 & 0.113 2 & 0.150 9 & 0.056 6 \end{bmatrix}
\]

\[
w_W = \begin{bmatrix} 0.444 4 & 0.222 2 & 0.111 1 & 0.166 7 & 0.055 6 \end{bmatrix}
\]

**Step 5.** Since the two weight vectors w\textsubscript{B} and w\textsubscript{W} are in the same rankings, i.e., 1 \(\succ\) 2 \(\succ\) 4 \(\succ\) 3 \(\succ\) 5, we can solve the optimization model - Model 4 according to the procedure of BWM-IPCM algorithm.

**Table 5. Results of Example 2.**

| Methods     | Optimal weight vectors and rankings | Consistency indicators |
|-------------|------------------------------------|-----------------------|
| Our method  | \{0.451 9, 0.223 7, 0.111 7, 0.157 6, 0.055 2\} | CR\textsubscript{min} = 0.008 7 |
| Rezaee (2016) | \{0.453 4, 0.222 0, 0.110 6, 0.158 9, 0.055 5\} | CR\textsubscript{BWM} = 0.156 9 |

**Table 6. Weight interval calculated by Model 3.**

| Weight | Min       | Max       | Center  | Width     |
|--------|-----------|-----------|---------|-----------|
| w\textsubscript{1} | 0.446 1   | 0.457 1   | 0.451 6 | 0.005 5   |
| w\textsubscript{2} | 0.214 5   | 0.228 9   | 0.221 7 | 0.007 2   |
| w\textsubscript{3} | 0.108 5   | 0.117 6   | 0.113 1 | 0.004 6   |
| w\textsubscript{4} | 0.156 3   | 0.160 2   | 0.158 2 | 0.001 9   |
| w\textsubscript{5} | 0.054 8   | 0.056 1   | 0.055 4 | 0.000 7   |

**Step 6 - Step 8.** The equivalent decision vector of A\textsubscript{s} is, \(\vec{a} = \{2, 4, 3, 8, x_1, x_2, 4, x_3, 2, 3\}\), which is substituted into Model 4. The optimization results are shown in Table 5. It can be seen that CR\textsubscript{min} = 0.000 25 < 0.1, which indicates a high consistency, while the result of [17] is CR\textsubscript{BWM} = 0.032 6, which also indicates a good consistency of expert judgments. At the same time, according to Model 3, the optimal weight interval can be calculated, as shown in Table 6.

The PCM in Example 2 is not perfectly consistent, but it is highly consistent. The comparative analysis between our method – BWM-IPCM and the original BWM is as follows.

1. When the PCM has high consistency, our method can obtain the same rankings as the original BWM. In the case of high consistency, each weight interval obtained from the original BWM does not overlap, as shown in Table 6, and there is no fuzzy ranking, that is, the ranking is unique.

2. When the PCM has high consistency, our method can obtain a weight vector similar to the original BWM. The weight results are very close, both within the weight interval shown in Table 6, and almost the same as the center of the interval. However, the weight vector obtained from the original BWM is not unique. When DM wants to find the determined weight vector from the BWM, this may cause trouble. It can be seen from Table 5 and Table 6 that for the same \(\xi^*\) or CR\textsubscript{BWM}, there may be countless weight solutions, which may indicate that CR\textsubscript{BWM} is poor in measuring consistency, because CR\textsubscript{BWM} cannot correspond to a determined weight vector. Meanwhile, as suggested in [17], there is no theoretical basis for using the center of weight intervals as the optimal weights. For our method, there is only one optimal weight vector corresponding to CR\textsubscript{min}, which has been proven in Theorem 1.
Example 3 [17]: $A$ is a 5th-order BWM-IPCM, and its standard form $A_s$ is as follows.

$$
A = \begin{bmatrix}
1 & \times & \times & \times & 4 \\
2 & 1 & 4 & 3 & 8 \\
\times & \times & 1 & \times & 4 \\
\times & \times & \times & 1 & 2 \\
1 & 2 & 4 & 3 & 8
\end{bmatrix}
$$

$$
A_s = \begin{bmatrix}
1 & 2 & 4 & 3 & 8 \\
\times & \times & 1 & \times & 4 \\
\times & \times & \times & 1 & 2 \\
\times & \times & 1 & 4 \\
1 & 2 & 4 & 3 & 8
\end{bmatrix}
$$

Step 1 – Step 2. For the standard BWM-IPCM $A_s$, we have $A_B = \{1, 2, 4, 3, 8\}$, $A_W = \{8, 4, 4, 2, 1\}^T$.

Step 3 – Step 4. It can be computed that

$$w_B = \{0.452 8, 0.226 4, 0.113 2, 0.150 9, 0.056 6\}$$

$$w_W = \{0.421 1, 0.210 5, 0.210 5, 0.105 3, 0.052 6\}$$

Step 5. The two weight vectors $w_B$ and $w_W$ are in different rankings, i.e., according to $w_B$, we have $1 \succ 2 \succ 4 \succ 3 \succ 5$, according to $w_W$, we have $1 \approx 2 \approx 3 \approx 4 \approx 5$, where the symbol “$\approx$” means “the same as”, as shown in Table 7. From these two rankings, the main conflict occurs in the ranking of $w_3$ and $w_4$, which is also the reason why the weight interval obtained by Model 3 overlaps on $w_3$ and $w_4$, as shown in Table 8. Solve Model 2 and Model 4 for $A_s$ and the results are shown in Table 7. According to Model 4, the completed PCM is shown below.

$$A' = \begin{bmatrix}
1 & 2 & 4 & 3 & 8 \\
0.5 & 1 & 1.417 7 & 1.736 6 & 4 \\
0.25 & 0.705 4 & 1 & 1.223 7 & 4 \\
0.333 3 & 0.575 8 & 0.817 2 & 1 & 2 \\
0.125 & 0.25 & 0.25 & 0.5 & 1
\end{bmatrix}
$$

Step 6 - Step 8. According to the proposed algorithm, since the two weight vectors $w_B$ and $w_W$ are in different rankings, we go to Step 7 - consistency fine-tuning. It is suggested that experts add a comparison judgment, such as $a_{34}$, which will make the supplemented matrix no longer a BWM-IPCM, but a common PCM, which can still be solved by Model 4.

$$A_{s4} = \begin{bmatrix}
1 & 2 & 4 & 3 & 8 \\
\times & \times & 1 & \times & 4 \\
\times & \times & \times & 1 & 2 \\
\times & \times & 1 & 2 \\
1 & 2 & 4 & 3 & 8
\end{bmatrix}
$$

$$A_{s4} = \begin{bmatrix}
1 & 2 & 4 & 3 & 8 \\
\times & \times & 1 & \times & 4 \\
\times & \times & 1 & 1/2 & 4 \\
\times & \times & 1 & 2 \\
\times & \times & \times & 1 & 1
\end{bmatrix}
$$

Assume that $a_{34} = 2$, $A_s$ becomes $A_{s4}$, and substitute $A_{s4}$ into Model 4. The results are shown in Table 7. We can see that $w_3 = 0.172 8$, $w_4 = 0.109 8$, which means that the gap between these two weights is significantly greater than the original, that is, the relationship of 3$\succ$4 is enhanced. Assume that $a_{34} = 1/2$, the IPCM $A_i$ becomes $A_{s2}$. Similarly, the weight vector is shown in Table 7. This time, $w_3 = 0.127 3$, $w_4 = 0.160 5$, and the weight ranking is reversed to 4$\succ$3. Therefore, by adding some comparison information to the conflict criteria, the problems of overlapping weight intervals and fuzzy ranking can be effectively solved. Note that $CR_{\min}$ increases regardless of whether $a_{34}$ is 2 or 1/2. This is because when $a_{34}$ is unknown, Model 4 can uniquely determine the optimal value of $a_{34}$, i.e. $1.223 7$ in $A'$, in order to achieve the highest possible level of consistency. If $a_{34} = 1.223 7$ is replaced by 2 or 1/2, its consistency will decrease, similar to that shown in Fig. 4, which can also be explained as improving consistency through less pairwise comparison.

The PCM in Example 3 is highly inconsistent. The comparative analysis between our method - BWM-IPCM and the original BWM is as follows.

(1) When the PCM is highly inconsistent, the rankings obtained by the original BWM may not be unique, which will make the DM confused about which alternative ranks first. As shown in Table 8, the weight intervals obtained by Model 3 overlap on $w_3$ and $w_4$. For example, $w_3 \in [0.142 9, 0.164 4]$, $w_4 \in [0.111 1, 0.157 9]$, so the ranking 3$\succ$4 or 4$\succ$3 may hold at the same time, that is, the rankings obtained by the original BWM may not be unique. For our method, there

| Methods          | Optimal weight vectors and rankings | Consistency indicators |
|------------------|------------------------------------|------------------------|
| Our method on $A_s$ | (0.453 7, 0.214 9, 0.154 5, 0.124 7, 0.052 2) | CR$_{\min}$ = 0.011 96 |
| Rezaei (2016)    | (0.429 1, 0.237 1, 0.143 0, 0.143 0, 0.047 7) | $\xi$ = 1, CR$_{\min}$ = 0.223 7 |
| Consider only the first row of $A_i$: $w_B$ | (0.452 8, 0.226 4, 0.113 2, 0.150 9, 0.056 6) | |
| Consider only the last column of $A_i$: $w_W$ | (0.421 1, 0.210 5, 0.210 5, 0.105 3, 0.052 6) | |

TABLE 7. Results of Example 3.

| Methods          | Optimal weight vectors and rankings | Consistency indicators |
|------------------|------------------------------------|------------------------|
| Our method on $A_{s1}$ | (0.454 3, 0.212 1, 0.172 8, 0.109 8, 0.051 0) | CR$_{\min}$ = 0.017 53 |
| Our method on $A_{s2}$ | (0.445 7, 0.213 1, 0.127 3, 0.160 5, 0.053 4) | CR$_{\min}$ = 0.030 82 |

TABLE 8. Weight interval calculated by Model 3.

| Weight | Min | Max | Center | Width |
|--------|-----|-----|--------|-------|
| $w_1$ | 0.428 6 | 0.493 2 | 0.460 9 | 0.032 3 |
| $w_2$ | 0.157 9 | 0.246 9 | 0.202 4 | 0.044 5 |
| $w_3$ | 0.142 9 | 0.164 4 | 0.153 6 | 0.010 8 |
| $w_4$ | 0.111 1 | 0.157 9 | 0.134 5 | 0.023 4 |
| $w_5$ | 0.047 6 | 0.054 8 | 0.051 2 | 0.003 6 |
is only one optimal weight vector corresponding to CRmin, which has been proven in Theorem 1.

(2) When the PCM is highly inconsistent, we extend the original BWM by asking experts to add some comparison information to the conflicting rankings that can be found by comparing \( w_B \) and \( w_W \). We believe this improvement can better reflect the real preferences of experts and reduce the inconsistency of decision-making.

**Example 4:** A is an 8th-order BWM-IPCM, which is modified from Example 3.2 of Xu and Wei [48], and only the best criterion \( c_8 \) and the worst criterion \( c_4 \) are selected, that is, only the 8th row and the 4th column are retained. The standard form \( A_8 \) is as follows.

\[
A = 
\begin{bmatrix}
1 & \times & 7 & \times & \times & \times & \times & \times \\
\times & 1 & 5 & \times & \times & \times & \times & \times \\
\times & \times & 1 & 6 & \times & \times & \times & \times \\
\times & \times & \times & 3 & 1 & \times & \times & \times \\
\times & \times & \times & 4 & \times & 1 & \times & \times \\
\times & \times & \times & \times & 7 & \times & 1 & \times \\
4 & 7 & 5 & 8 & 6 & 6 & 2 & 1
\end{bmatrix}
\]

This example is mainly used to compare the computational efficiency of our method and the original BWM on high-order BWM problems.

(1) When the PCM has high order, our method deals with much fewer variables and no constraints, which makes it much easier to solve the optimization problem. For this 8th-order problem, according to Table 3, the original BWM (Model 2) will have 27 nonlinear inequality constraints and one linear equality constraint, and the number of variables is as high as 32. It is difficult to find the optimal solution using conventional optimization methods such as MATLAB function \( \text{fmincon} \), but we still get the optimal solution of Model 2 through genetic algorithm (GA), see Table 9 for the results. But GA is not so efficient since it requires the selection of parameters to obtain the best results, especially when there are too many variables. In addition, the weight vector of the original BWM is not unique, but the weight intervals obtained by Model 3. In order to find the minimum and maximum values of the weight intervals shown in Table 10, we must solve 16 such nonlinear optimization problems, which is very time-consuming. However, according to Table 3, our method (Model 4) has only 15 variables and no constraints, which is easy to find the unique optimal weights and rankings. The results are shown in Table 9. Therefore, this example shows that our method can obtain weight vectors and rankings similar to that of the original BWM, and has high computational efficiency.

(2) When the PCM has high order, the weight vectors and rankings obtained by our method are similar to those of AHP, but our method requires less pairwise comparison and has higher consistency. Comparing with Xu and Wei [48], we noticed that Xu & Wei’s ranking results were given after the consistency of the original PCM was improved. We found that the differences between our method and Xu & Wei lies in the rankings among \( w_3 \), \( w_4 \), and \( w_7 \). The reasons are as follows: On the one hand, Xu & Wei provides a complete PCM, so under the comprehensive effect of more comparison information, the ranking obtained is different from BWM-ICPM, which only uses part of the comparison information. On the other hand, Xu & Wei’s ranking results are based on a PCM with ordinal inconsistency, that is, there is still a three-way cycle in the improved PCM, which leads to the wrong ranking between \( w_3 \) and \( w_7 \). In fact, \( a_{37} \) should be modified first, that is, to make \( a_{37} < 1 \) (Wu et al. [49]). In Wu et al., we corrected the ordinal inconsistency in Xu and Wei’s results, as shown in Table 9, which is consistent with our method.
example, we can see that a complete PCM with a higher order is more likely to be inconsistent than a BWM-IPCM, which requires less comparison information and has been proven to be ordinally consistent (Proposition 3).

From the above examples, we believe that compared with AHP, BWM can reduce the number of pairwise comparisons and ensure good performance in terms of consistency, which has been proven by Proposition 3. However, it also has disadvantages. Firstly, there may be countless optimal weight solutions in BWM, forming weight intervals, which makes ranking difficult (see Example 2 and Example 3). Secondly, BWM may lack important comparison information, leading to deviation from actual preferences (see Example 3). Thirdly, in case of higher-order PCM, the optimization model of BWM is difficult to solve (see Example 4). The above shortcomings can be solved by the proposed BWM-IPCM. In the following, we have made some comparative discussions between the original BWM (Model 2 and Model 3) and BWM-IPCM (Model 4):

1. BWM-IPCM is a special case of IPCM. In order to obtain the best consistency, unknown elements in BWM can be uniquely determined by Model 4. For example, $\text{CR} = 0$ in Example 1 is the best consistency. By solving Model 4, the missing elements $a_{13}$, $a_{14}$, and $a_{34}$ can be uniquely determined from the known elements.

2. BWM-IPCM can make better use of important comparison information. The original BWM method is limited by its format, and only the best and worst criteria are compared with other criteria. This paper shows that the accuracy of BWM can be significantly improved by adding a small amount of key comparison information. As shown in Example 3, after the experts give a score of $a_{34}$, the weights and rankings become clearer and more accurate. In other words, the BWM may overlook important comparison information. We believe that a better way is to add a small amount of valuable and reliable comparison information in addition to $a_{ij}$ and $a_{ji}$, so as to improve the accuracy of the results.

3. BWM-IPCM method has advantages in describing consistency. The consistency index $\text{CR}_{\text{min}}$ of BWM-IPCM method has the same meaning as that of Satty’s AHP method, which is more explanatory, that is, $\text{CR}_{\text{min}} < 0.1$ indicates acceptable consistency. In contrast, the consistency index $\text{CR}_{\text{BWM}}$ of the original BWM is very sensitive to the judgments of experts, and the interpretability is not very good and clear. See Example 1 for details.

4. The optimization model of BWM-IPCM method is easier to solve. On the one hand, the original BWM method, Model 2, often has countless optimal weight solutions, which usually form overlapping weight intervals, so it is not conducive to accurate weights and ranking, as shown in Example 3. On the other hand, when the order is high, the number of constraints and variables of Model 2 becomes very large, which makes it difficult to solve, as shown in Example 4. In this paper, the BWM-IPCM method, Model 4, can be regarded as an unconstrained multivariable function optimization. According to Theorem 1, the minimum value exists and is unique, and the optimal weight vector is also unique, that is, the weights are determined numbers rather than weight intervals, and it is easier to solve.

V. CONCLUSION

This paper presents an improved BWM algorithm, called BWM-IPCM. As a special case of IPCM, BWM-IPCM has some special properties such as always being effective and ordinal consistent. This paper proposes the standardization of BWM-IPCM, that is, the best criterion comparison information is placed in the first row, and the worst criterion comparison information is placed in the last column, so that all known elements in the upper triangle are greater than or equal to 1, which is convenient for modeling and solving. By comparing BWM-IPCM with the original BWM method, we find that the former is superior to the latter in terms of the uniqueness of the solution and the computational efficiency of optimization. Meanwhile, in view of the possible inconsistency of BWM, two improvement methods are put forward: One is to use the two comparison vectors of the BWM to obtain weight vectors and rankings respectively, then find out the elements that are inconsistent in the two rankings, and let experts supplement a small amount of comparison information for the conflicting criteria (not to compare with the best or the worst criterion), and then use Model 4 to get the final results, refer to Example 3 for the detailed process. This method can solve the problem of overlapping weight intervals and improve the inconsistency of the original BWM. The other method is to improve the inconsistency by fine-tuning some known elements [37] when $\text{CR}_{\text{min}} > 0.1$ (mainly for the 3rd and 4th order BWM-ICPMs).

The future research will focus on whether the methods proposed in this paper can be extended to GDM issues, such as consensus issues in GDM [40] and large-group decision-making [51] based on BWM, as well as a further research on the acceptable threshold of $\text{CR}_{\text{min}}$ in BWM-IPCM.

APPENDIX

Proof of Proposition 3: Using reduction to absurdity. Because only the positions of the criteria are exchanged before and after the BWM-IPCM standardization, which will not change the ordinal consistency, we will analyze a standard BWM-IPCM here. Assume $A = (a_{ij})_{n \times n}$ is an $n$th-order standard BWM-IPCM with ordinal inconsistency. According to Definition 5, it has at least one three-way cycle. Since only the first row and the last column are known, let us assume that the three-way cycle is $1 \times n - 1$. For the standard BWM-IPCM $A$, all elements in the first row are greater than 1, that is $a_{1i} > 1$, $a_{i1} > 1$, so we have $c_1 > c_x$, $c_1 > c_n$. Meanwhile, all elements in the last row are also greater than 1, that is, $a_{ni} > 1$, so we have $c_x > c_n$. To sum up, we have $c_1 > c_x > c_n$ and $c_1 < c_n$. But according to the definition of three-way cycle (Definition 4), $c_n > c_1$ should be satisfied to form a cycle, obviously, there is a contradiction. Therefore, no known elements in BWM-IPCM can form a three-way...
cycle. According to Definition 5, the BWM-IPCM has ordinal consistency. □

**Proof of Theorem 1:** According to Theorem 2 in [34], the optimal solution of \(\min \lambda_{\text{max}}(A(x))\) is unique if and only if the graph \(\bar{G}\) corresponding to the IPCM is connected. Take the standard BWM-IPCM \(A_4\) as an example (see Section III(A)), Fig. 5 shows the undirected graph corresponding to \(A_4\), which is connected. Therefore, the optimal solution for \(\min \lambda_{\text{max}}(A(x))\) should be unique. According to (5), for a PCM, the minimum value of \(\lambda_{\text{max}}\) and CR is one-to-one, so the optimal solution for \(\min \text{CR}(A(x))\) will also be unique.

Suppose we have found the minimum \(\lambda_{\text{max}}\), according to the maximum eigenvalue method [13], since the eigenvector corresponding to \(\lambda_{\text{max}}\) is unique, thus the optimal weights are also unique. □

![FIGURE 5. Undirected graph of \(A_4\).](image)

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SHIHUI WU received the M.S. and Ph.D. degrees in management science and engineering from Air Force Engineering University, Xi’an, China, in 2007 and 2010, respectively. He is currently an Associate Professor with Air Force Engineering University. His research interests include multiple attribute decision making, game theory, and simulation optimization.

BO HE received the M.S. and Ph.D. degrees in military equipment from Air Force Engineering University. He is currently a Lecturer with Air Force Engineering University. His research interests include management of weapon equipment and weapon equipment economics.

XIAODONG LIU received the M.S. and Ph.D. degrees in aviation equipment management from Air Force Engineering University, in 1993 and 2001, respectively. He is currently a Professor and a Ph.D. Tutor with Air Force Engineering University. He has worked in aerial equipment management for more than 30 years. His current research interests include management of weapon equipment and weapon equipment economics.