Resonance Form-Factors: \( L_8 \) determination at Next-to-Leading Order in \( 1/N_C \)

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Abstract. This talk presents a systematic procedure for the computation of the SS-PP correlator beyond the large–\( N_C \) limit. The present calculation is carried on within a perturbative \( 1/N_C \) framework. By constraining the meson form-factors at leading order in \( 1/N_C \), one obtains a one-loop spectral function well behaved at short distances. The Weinberg sum-rules get modified, gaining an extra contribution suppressed by \( 1/N_C \). This leads to a prediction for the low energy chiral perturbation theory coupling \( L_8(\mu) \) at the one-loop level, i.e., up to next-to-leading order in \( 1/N_C \).

1 The large–\( N_C \) limit

Quantum Chromodynamics (QCD) has provided a wide understanding of hadronic processes. At long distances, the theory becomes non-perturbative. An approach that has been found to be useful is to consider QCD in the limit of an infinite number of colors \( N_C \rightarrow \infty \), keeping \( N_C \alpha_s \) finite \([1]\). Assuming confinement, large–\( N_C \) QCD results equivalent to a theory with an infinite number of flavors. The symmetry group gets enlarged into \( SU(N) \). The large–\( N_C \) limit of a chiral theory of resonances (meson resonance) must be given by a chiral theory of resonances [2]. Symmetry imposes stringent constraints on the structure of the low energy interaction, so it is an essential ingredient to recover the proper low-energy QCD behavior. Hence, the interactions between mesons (Goldstones and resonance) must be given by a chiral theory of resonances \( \chi PT \) \([3,4]\).

Following Refs. \([4,5,6,7,8,9,10]\), this talk proposes a systematic program to abord the analysis of hadronic amplitudes beyond the leading order in \( 1/N_C \) (LO). The example of the isovector SS-PP correlator is studied:

\[
    \Pi(t) = i \int d^4x e^{iqx} \langle T\{J(x)J(0)^\dagger - J_5(x)J_5(0)^\dagger\} \rangle , \tag{1}
\]

with \( J = \bar{u}d \) and \( J_5 = i\bar{u}\gamma_5u \), and \( t = q^2 \). The calculation is carried on within the chiral limit.

2 A program for calculations at NLO in \( 1/N_C \)

2.1 \( R_\chi T \) lagrangian

Although the large–\( N_C \) spectrum contains an infinite number of hadrons, the Green-functions that are chiral order parameters are mainly governed the lightest states. In general, one truncates the tower of states and works under a minimal hadronical approximation (MHA) \([11]\), keeping the minimal number of resonance multiplets enough to fulfill the short-distance constraints. Though the truncation of the spectrum induces uncertainties \([12,13]\), these can be estimated through the last absorptive channel included the calculation \([7]\).

The particles included in our meson lagrangian are the chiral Goldstones and the lightest \( 1^{--}, 1^{++}, 0^{++}, 0^{--} \) resonance multiplets. Since we work within the large \( N_C \) framework, the hadrons are classified into \( U(n_f) \) multiplets.

The hadronic lagrangian contains all the available operators consistent with chiral symmetry. Our building blocks are the resonance fields and the chiral tensors containing the Goldstones \([10,11]\). For the spin–1 fields we use the antisymmetric tensor formalism \([5,14]\). In addition, in order to avoid a wrong growing behavior of the Green-functions at high energies, the operators only contain tensors up to \( O(p^2) \). The operators of the lagrangian can be organized on the number of resonance fields:

\[
    \mathcal{L}_{R_\chi T} = \mathcal{L}_{\chi PT}^{(2)} + \sum_{R_1} \mathcal{L}_{R_1} + \sum_{R_1,R_2} \mathcal{L}_{R_1,R_2} + \sum_{R_1,R_2,R_3} \mathcal{L}_{R_1,R_2,R_3} + \ldots \tag{2}
\]

where the first term on the r.h.s. is the \( O(p^2) \) \( \chi PT \) lagrangian \([2]\) and the second term, linear in the resonance fields, was long ago constructed in Ref. \([5]\). For the form-factors we are interested on (those with two mesons in the final state), only the operators with three or less res-
2.2 Form-factors at large–\(N_C\)

In the large \(N_C\) limit the correlator is given by the tree-level exchange of Goldstones and resonances. At the one loop level one may find the two-meson absorptive diagrams shown in Fig. 1. The calculation is carried on within perturbation theory and no Dyson-Schwinger resummation is performed. Hence, all the lines in Fig. 1 stand for tree-level meson propagators.

At LO, the only absorptive cuts are one-particle cuts. At NLO, one may also have intermediate two-meson states. For a particular two-particle cut \(M_1M_2\), its contribution to the spectral function is in general proportional to some form factor squared:

\[
\text{Im} \Pi(t)_{M_1M_2} \propto |\mathcal{F}_{M_1M_2}(t)|^2.
\]

By means of quark-counting rule arguments [15], it is usually accepted that the pion scalar form factor vanishes at infinite momentum. It accepts an unsubtracted dispersion relation which leads to the usual monopolar expression \(\mathcal{F}_{\pi\pi}(t) = \frac{M_\pi^2}{M_\pi^2 - t}\) and to a contribution \(\text{Im} \Pi(t)_{\pi\pi}\) to the spectral function which vanishes at \(t \rightarrow \infty\).

Demanding each separate absorptive contribution \(\text{Im} \Pi(t)_{M_1M_2}\) to vanish at least as fast as \(\text{Im} \Pi(t)_{\pi\pi}\) leads to a series of constraints for the form-factors at LO in \(1/N_C\). Furthermore, the \(\pi\pi\) and \(R\pi\) form factors become determined in terms of the resonance masses [7].

\[\text{Im} \Pi(t)_{M_1M_2} \propto |\mathcal{F}_{M_1M_2}(t)|^2.\]

\[\text{Im} \Pi(t) = \int \frac{dt' \text{Im} \Pi(t')}{t' - t} = \frac{1}{\pi} \int_0^\infty dt' \frac{\text{Im} \Pi(t')}{t' - t}.\]

At LO, the spectral function is given by a sum of delta functions centered on the meson masses. Within the MHA, this gives the large–\(N_C\) correlator

\[\Pi(t) = 2B_0^2 \left[ \frac{8c_m^2}{M_S^2 - t} - \frac{8d_m^2}{M_P^2 - t} + \frac{F^2}{t} \right]. \tag{5}\]

At NLO, the spectral function contains as well finite two-particle contributions \(\text{Im} \Pi(t)_{M_1M_2}\), related to the two-meson form-factors \(\mathcal{F}_{M_1M_2}(t)\). By means of Eq. (4), one finds that the two-meson cuts contribute to the correlator with a finite part, \(\Delta \Pi(t)_{M_1M_2}\), and a NLO renormalization of the scalar and pseudo-scalar masses and couplings. Hence, the whole correlator up to NLO shows the general structure:

\[\Pi(t) = 2B_0^2 \left[ \frac{8c_m^2}{M_S^2 - t} - \frac{8d_m^2}{M_P^2 - t} + \frac{F^2}{t} \right] + \sum_{M_1M_2} \Delta \Pi(t)_{M_1M_2}, \tag{6}\]

with the finite contribution from the \(M_1M_2\) cut,

\[\Delta \Pi(t)_{M_1M_2} = \lim_{\epsilon \to 0^+} \int \frac{dt \text{Im} \Pi(t')_{M_1M_2}}{t' - t} - \frac{2}{\pi} \lim_{t' \to M_R^2} \left( \frac{t_0}{t_0 + t'} \right) \left[ \left( M_R^2 - t' \right)^2 \frac{\text{Im} \Pi(t')_{M_1M_2}}{t' - t} \right], \tag{7}\]

where the NLO constants \(\delta^{(1)}_{\text{NLO}}, \delta^{(2)}_{\text{NLO}}, \delta^{(3)}_{\text{NLO}}\) depend on the decay constant \(F\) and the resonance masses \(M_S, M_P\).

2.4 Matching OPE up to NLO in \(1/N_C\)

In the high energy limit, the two-meson contribution is found to behave as

\[\Delta \Pi(t) = \frac{F^2}{t} \delta^{(1)}_{\text{NLO}} + \frac{F^2 M_S^2}{t^2} \left( \delta^{(2)}_{\text{NLO}} + \delta^{(2)}_{\text{NLO}} \ln \frac{-t}{M_S^2} \right), \tag{8}\]

where the NLO constants \(\delta^{(1)}_{\text{NLO}}, \delta^{(2)}_{\text{NLO}}\) and \(\delta^{(3)}_{\text{NLO}}\) depend on the decay constant \(F\) and the resonance masses \(M_S, M_P\).

The one-loop \(R\chi T\) correlator can be now matched to OPE in the deep euclidian region, finding similar expressions to the Weinberg sum-rules, but now containing extra terms, NLO in \(1/N_C\):

\[-8c_m^2 + 8d_m^2 + F^2 (1 + \delta^{(1)}_{\text{NLO}}) = 0, \tag{9}\]

\[-8c_m^2 M_S^2 - 8d_m M_P^2 + F^2 M_S^2 \delta^{(2)}_{\text{NLO}} 

\text{and it can be safely}}
neglected \cite{17}. The matching is fulfilled by demanding that the \( \frac{1}{4} \ln \frac{M_T^2}{M^2} \) term also vanishes, this is, \( \delta_{\text{NLO}}^{(2)} = 0 \).

These relations allow fixing the resonance couplings up to NLO:

\[
c_r^2 = \frac{F^2}{8} \frac{M_P^2}{M_P^2 - M_S^2} \left[ 1 + \delta_{(1)} - \frac{M_S^2}{M_P^2} \delta_{(2)} \right], \tag{11}
\]

\[
d_r^2 = \frac{F^2}{8} \frac{M_P^2}{M_P^2 - M_S^2} \left[ 1 + \delta_{(1)} - \delta_{(2)} \right]. \tag{12}
\]

When considering just \( \pi\pi \) and \( R\pi \) cuts, one finds that after imposing the QCD short distance conditions everything becomes determined in terms of the renormalized masses \( M_R^2 \).

At low energies, the contribution from higher and higher thresholds becomes more and more suppressed. The two-resonance cuts are neglected in the present work. The uncertainty from the truncation is estimated from the \( P\pi \) contribution, the higher threshold under consideration.

### 2.5 Recovery of \( \chi \)PT at low energies

One of the main advantages of working within a chiral invariant framework is the recovery of \( \chi \)PT at low energies even at the loop level. The one–loop \( R\chi \)T calculation exactly reproduces the one–loop \( \chi \)PT expression. This provides a prediction for the value of the renormalized low energy constant (LEC), \( L_8^\xi (\mu) \), in terms of \( R\chi \)T parameters. The two expressions match at any \( \mu \) and \( R\chi \)T generates the exact \( L_8^\xi (\mu) \) running found in \( \chi \)PT \cite{2}. There is not a specific saturation scale but a relation between renormalized LECs and renormalized \( R\chi \)T parameters.

In the low energy limit, the \( R\chi \)T expression can be expanded in powers of \( t \):

\[
\Pi(t) = B_0^\xi \left\{ \frac{2F^2}{t} + 32L_8^{U(3)} + \frac{3}{16\pi^2} \left( 1 - \ln \frac{-t}{M_S^2} \right) + O(t) \right\},
\]

with the constant

\[
L_8^{U(3)} = \frac{F^2}{16} \left[ \frac{1}{M_S^2} + \frac{1}{M_P^2} \right] \left( 1 + \frac{\delta_{\text{NLO}}^{(1)}}{M_S^2} - \frac{M_S^2}{M_P^2} \delta_{\text{NLO}}^{(2)} \right) - \frac{3 \Delta}{256\pi^2}.
\]

The \( O(1) \) constant \( \Delta \) is given in Ref. \cite{7}. It comes from the two-particle contribution \( \Delta \Pi(t) \) and is a function of \( F \) and \( M_R \).

Comparing this result with \( U(3) - \chi \)PT \cite{4}, one gets a prediction for the renormalized LEC \( L_8^\xi (\mu) \):

\[
L_8^\xi (\mu)_{U(3)} = L_8^{U(3)} - \frac{3}{512\pi^2} \ln \frac{\mu^2}{M_S^2}.
\]

The last step is to integrate out the chiral singlet \( \eta_0 \). In the large–\( N_C \) limit, the \( \eta_0 \) is the ninth Goldstone and it is massless \cite{4}. However, it gains mass due to higher order corrections. Naively, one would expect that the effect of the \( \eta_0 \) mass in the one-loop diagrams would be next-to-next-to-leading order, this is, suppressed by \( \frac{1}{M_\chi^2} \). Actually, since we study an energy limit below the \( \eta_0 \) threshold (\( t \ll M_\chi^2 \)), the first effect from the \( \eta_0 \) mass appears at order \( \frac{1}{N_C^2} \ln \frac{M_\chi^2}{N_C} \). Thus, the \( SU(3) - \chi \)PT constant is finally related to the \( U(3) \) prediction through \cite{17}

\[
L_8^\xi (\mu)_{SU(3)} = L_8^\xi (\mu)_{U(3)} - \frac{1}{384\pi^2} \ln \frac{M_\chi^2}{\mu^2}.
\]

### 3 Conclusions

One of the main advantages of a chirally invariant theory of resonances is that the symmetry properties ensures the right recovery of the QCD low energy limit, \( \chi \)PT, even at the loop level.

The absorptive \( \chi \)PT logarithms are exactly reproduced by our result at long distances. This removes the large–\( N_C \) ambiguity about the renormalization scale of saturation of \( L_8^\xi (\mu) \). The renormalized chiral coupling is given in terms of the renormalized resonance effective parameters \( c_r^2, d_r^2, M_R^2, M_P^2 \).

The systematic \( 1/N_C \) expansion within the \( R\chi \)T framework allows to derive Weinberg sum-rules beyond the leading order. This fixes the value of the renormalized scalar and pseudo-scalar couplings in terms of the renormalized resonance masses.

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