Abstract—We consider a unit memory channel, called Binary State Symmetric Channel (BSSC), in which the channel state is the modulo2 addition of the current channel input and the previous channel output. We derive closed form expressions for the capacity and corresponding channel input distribution, of this BSSC with and without feedback and transmission cost. We also show that the capacity of the BSSC is not increased by feedback, and it is achieved by a first order symmetric Markov process.

I. INTRODUCTION

Capacity of channels with feedback and associated coding theorems are often classified into Discrete Memoryless Channels (DMCs) and channels with memory. For DMCs with and without feedback, coding theorems, capacity expressions, and achieving distributions are derived by Shannon [1] and Dubushin [2]. Coding theorems for channels with memory and feedback, defined by Servers [7], the trapdoor can increase the capacity of channels with feedback, the first order moving average Gaussian noise channel [7], the trapdoor can increase the capacity of channels with feedback, the first [5], [6] where the authors proved that memory and feedback for stationary ergodic processes, directed information stable processes, and general nonstationary processes are given in [3, 4].

Although for several years great effort has been devoted to the study of channels with memory, with or without feedback, explicit or closed form expressions of capacity for channels with memory, are limited to few but ripe cases. Some of these are the additive Gaussian noisy channels with memory and feedback [5, 6] where the authors proved that memory can increase the capacity of channels with feedback, the first order moving average Gaussian noise channel [7], the trapdoor channel where it was shown that the feedback capacity is the log of the golden ratio [8], and the Ising channel [9].

The capacity of the Unit Memory Channel (UMC) with feedback, defined by \( \{P_{B_i|A_{i-1}B_{i-1}}(b_i|a_{i-1}b_{i-1}) : i = 0, 1, \ldots \} \), where \( b_i \) is the channel output and \( a_i \) the channel input, is investigated by Berger [10] and Chen and Berger [11]. Let \( a^n = \{a_0, a_1, \ldots, a_n\} \) and similarly for \( b^n \). It is conjectured in [11] that the capacity achieving distribution has the property

\[
P_{A_i|A_i^{-1}B_i}(a_i|a_i^{-1}, b_{i-1}) = P_{A_i|B_i}(a_i|b_{i-1}), i = 0, \ldots, (I.1)
\]

Recently, Asnani, Permuter and Weissman [13, 14] obtained an expression of the capacity of the so-called Previous Output S'Tate (POST) channel, which is a special case of (I.1). They have shown that feedback does not increase the capacity of the POST channel, among other results.

This paper is concerned with the Binary State Symmetric channel (BSSC) defined by (II.14) with and without feedback and transmission cost. Our interest in the BSSC is motivated by the desire to identify a Duality of Sources and channels, in the sense of Joint Source-Channel Coding (JSCC) design, in which the optimal transmission is nonanticipative and the nonanticipative Rate Distortion function of a source with memory [15, 12] is matched to the capacity of a channel with feedback. With respect to this motivation, the BSSC and Binary Symmetric Markov Source (BSMS) are a natural generalization of the JSCC design (uncoded transmission) of an Independent and Identically Distributed (IID) Bernoulli source over a Binary Symmetric Channel (BSC) [16]. The duality of the BSSC and BSMS source is discussed in the companion paper [17], and utilizes the results of this paper.

The main results derived in this paper are the explicit expressions of the capacity and corresponding achieving channel input distribution of the BSSC, with and without feedback and transmission cost.

Since the POST channel [13, 14] and the BSSC defined by (II.14) are within a transformation equivalent, our results for the case without transmission cost, complement the results in [13, 14], in the sense that, we give alternative direct derivations and we obtain the expression of the capacity achieving channel input distribution with feedback. Moreover, we show that a Markov channel input distribution achieves the capacity of the channel when there is no feedback, hence feedback does not increase capacity of the BSSC. Our capacity formulae highlights the optimal time sharing among two binary symmetric channels (states of the general unit memory channel). The case with transmission cost is necessary for the JSCC design found in [18, 12].

II. PROBLEM FORMULATION

In this section we present the optimization problems which correspond to the capacity of channels with memory with and without feedback and transmission cost, and discuss the special classes of UMCs and BSSCs.

Let \( \mathbb{N}^n = \{0, 1, 2, \ldots, n\} \), \( n \in \mathbb{N} \), \( A, B \) denote the channel input and output alphabets, respectively, and \( A^n = \times_{i=0}^{n} A \) and \( B^n = \times_{i=0}^{n} B \), their product spaces, respectively. Moreover, let \( a^n = \{a_0, a_1, \ldots, a_n\} \in A^n \) denote the channel input sequence of length \( n + 1 \), and similarly for \( b^n \). We associate the above product spaces by their measurable spaces \((A^n, \mathcal{B}(A^n)), (B^n, \mathcal{B}(B^n))\).

Definition 1. (Channels with memory with and without feedback)
A channel with memory is a sequence of conditional distributions \( \{ P_{B_i|B_{i-1},A_i}(db_i|b^{i-1},a^i) : i \in \mathbb{N} \} \) defined by
\[
\tilde{P}_{B^n|A^n}(db^n|a^n) \triangleq \otimes_{i=0}^n P_{B_i|B_{i-1},A_i}(db_i|b^{i-1},a^i).
\]

The channel input distribution of a channel with feedback is a sequence of conditional distributions \( \{ P_{A_{i-1},B_{i-1}}(da_i|a^{i-1},b^{i-1}) : i \in \mathbb{N} \} \) defined by
\[
\tilde{P}_{A^n|B_{n-1}}(da^n|b^{n-1}) \triangleq \otimes_{i=0}^n P_{A_{i-1},B_{i-1}}(da_i|a^{i-1},b^{i-1}).
\] (II.2)

The channel input distribution of a channel without feedback is a sequence of conditional distributions \( \{ P_{A_i}(da_i|a^{i-1}) : i \in \mathbb{N} \} \) defined by
\[
P_{A^n}(a^n) = \otimes_{i=0}^n P_{A_i}(da_i|a^{i-1}).
\] (II.3)

### Definition 2. (Transmission cost)
The cost of transmitting symbols over a channel with memory is a measurable function \( \gamma_{0,n} : \mathcal{A}^n \times \mathcal{B}^{n-1} \mapsto [0, \infty) \) defined by
\[
\gamma_{0,n}(a^n, b^{n-1}) \triangleq \sum_{i=0}^n c_{0,i}(a^i, b^i).
\] (II.4)

The transmission cost constraint of a channel with feedback is defined by
\[
P_{0,n}^{fb}(\kappa) \triangleq \left\{ P_{A^n} : \frac{1}{n+1} \mathbb{E} \left[ \gamma_{0,n}(a^n, b^{n-1}) \right] \leq \kappa \right\},
\] (II.5)
where \( \kappa \in [0, \infty) \).

The transmission cost constraint of a channel without feedback is defined by
\[
P_{0,n}^{fb}(\kappa) \triangleq \left\{ P_{A^n} : \frac{1}{n+1} \mathbb{E} \left[ \gamma_{0,n}(a^n, b^{n-1}) \right] \leq \kappa \right\}.
\] (II.6)

Define the following quantities.
\[
C_{0,n}^{fb}(\kappa) \triangleq \sup_{P_{A^n|B_{n-1}} \in \mathcal{P}_{0,n}^{fb}(\kappa)} \frac{1}{n+1} I(A^n \rightarrow B^n),
\] (II.7)
\[
C_{0,n}^{nf,b}(\kappa) \triangleq \sup_{P_{A^n} \in \mathcal{P}_{0,n}^{fb}(\kappa)} \frac{1}{n+1} I(A^n; B^n),
\] (II.8)
where \( I(A^n \rightarrow B^n) \triangleq \sum_{i=0}^n I(A_i; B_i|B_{i-1}) \). If there is no transmission cost the above expressions are denoted by \( C_{0,n}^{fb}, C_{0,n}^{nf,b} \).

Note that for a fixed channel distribution, \( \tilde{P}_{B^n|A^n}(db^n|a^n) \), the set of causal conditional distributions \( \tilde{P}_{A^n|B_{n-1}} \) is convex, which implies that \( I(A^n \rightarrow B^n) \) is a convex functional of \( \tilde{P}_{A^n|B_{n-1}} \), and that the transmission cost constraint \( \mathcal{P}_{0,n}^{fb}(\kappa) \) is a convex set. Hence, \( C_{0,n}^{fb}(\kappa) \) is a convex optimization problem. The fact that \( C_{0,n}^{fb}(\kappa) \) is a convex optimization problem is well known.

Under the assumption that \( \{ (A_i, B_i) : i = 1, 2, \ldots \} \) is jointly ergodic or \( \frac{1}{n} \log \frac{P_{A^n|B^n}(db^n|a^n)}{P_{A^n}(db^n|a^n)} \) is information stable \([2, 3]\), then the channel capacity with feedback encoding and without feedback encoding are given by
\[
C_{fb} \triangleq \lim_{n \to \infty} C_{0,n}^{fb}, \quad C_{nf,b} \triangleq \lim_{n \to \infty} C_{0,n}^{fb},
\] (II.9)
Under appropriate assumptions then \( C_{fb}(\kappa) \) and \( C_{nf,b}(\kappa) \) corresponds to the channel capacity as well.

### A. Unit Memory Channel with Feedback

A Unit Memory Channel (UMC) is defined by
\[
\tilde{P}_{B^n|A^n}(db^n|a^n) \triangleq \otimes_{i=0}^n P_{B_i|B_{i-1},A_i}(db_i|b_{i-1}, a_{i-1}).
\] (II.10)

For a UMC the following results are found in \([11]\). If the channel is indecomposable then
\[
C_{fb} = \lim_{n \to \infty} \sup_{P_{A^n|B_{n-1}}} \frac{1}{n+1} I(A^n \rightarrow B^n)
\]
\[
= \lim_{n \to \infty} \sup_{P_{A^n|B_{n-1}}} \frac{1}{n+1} \sum_{i=0}^n I(A_i; B_i|B_{i-1})
\] (II.11)
\[
= \sup_{P_{A^n|B_{n-1}}} \left\{ H(B_i|B_{i-1}) - H(B_i|A_i, B_{i-1}) \right\}, \forall i (II.12)
\]
and the following hold.

1) \( \{ (A_i, B_i) : i = 0, 1, 2, \ldots \} \) is a first order stationary Markov process.
2) \( \{ B_i : i = 0, 1, 2, \ldots \} \) is a first order stationary Markov process.

Suppose the cost of transmitting symbols is letter-by-letter and time invariant, restricted to \( \gamma_{0,n}(a^n, b^{n-1}) = \sum_{i=0}^n c(a_i, b_{i-1}) \). Then by repeating the derivation in \([11]\), if necessary, by introducing the Lagrangian functional associated with the average transmission cost constraint (and assuming existence of an interior point of the constraint), then
\[
C_{fb}(\kappa) = \lim_{n \to \infty} \sup_{P_{A_i|B_{i-1}}} \frac{1}{n+1} \sum_{i=0}^n \mathbb{E} \{ c(A_i, B_{i-1}) \} \leq \kappa
\]
\[
= \sup_{P_{A_i|B_{i-1}}} \left\{ H(B_i|B_{i-1}) - H(B_i|A_i, B_{i-1}) \right\}, \forall \kappa \leq \kappa_m
\] (II.13)
where for \( \kappa \in [0, \kappa_m] \) and \( \kappa_m \) the maximum value of the cost constraint, both 1) and 2) remain valid. Moreover, \( C_{fb}(\kappa) \) is a concave, nondecreasing function of \( \kappa \in [0, \kappa_m] \).

### B. The Binary State Symmetric Channel

In this section, we consider a special class of the UMC channel, the BSSC, and discuss its physical meaning, and that of the imposed transmission cost constraint.
The BSSC($\alpha, \beta$) is a unit memory channel defined by

$$P_{B_i|A_i,B_{i-1}}(b_i|a_i,b_{i-1}) = \begin{bmatrix} 0 & 0.1 & 1.0 & 1.1 \\ \alpha & \beta & 1-\beta & 1-\alpha \\ 1 & 1-\alpha & 1-\beta & \beta \end{bmatrix}. \quad (II.14)$$

Introduce the following change of variables called state of the channel, $s_i \triangleq a_i \oplus b_{i-1}$, $i \in \mathbb{N}^n$, where $\oplus$ denotes the modulo2 addition. This transformation is one to one and onto, in the sense that, for a fixed channel input symbol value $a_i$ (respectively channel output symbol value $b_{i-1}$) then $s_i$ is uniquely determined by the value of $b_{i-1}$ (respectively $a_i$) and vice-versa. Then the following equivalent representation of the BSSC is obtained.

$$P_{B_i|A_i,S_i}(b_i|a_i,s_i=0) = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{bmatrix}. \quad (II.15)$$

$$P_{B_i|A_i,S_i}(b_i|a_i,s_i=1) = \begin{bmatrix} \beta & 1-\beta \\ 1-\beta & \beta \end{bmatrix}. \quad (II.16)$$

The above transformation highlights the symmetric form of the BSSC for a fixed state $s_i \in \{0,1\}$, which decomposes (II.14) into binary symmetric channels with crossover probabilities $(1-\alpha)$ and $(1-\beta)$, and motivates the name state symmetric channel.

The following notation will be used in the rest of the paper.

1) BSSC($\alpha, \beta$) denotes the BSSC with transition probabilities defined by (II.14);
2) BSC(1-$\alpha$) denotes the “state zero” channel defined by (II.15);
3) BSC(1-$\beta$) denotes the “state one” channel defined by (II.16).

Next, we discuss the physical interpretation of the cost constraint. Consider $\alpha > \beta \geq 0.5$. Then the capacity of the state zero channel, $(1-H(\alpha))$, is greater than the capacity of the state one channel, $(1-H(\beta))$. With “abuse” of terminology, we interpret the state zero channel as the “good channel” and the state one channel, as the “bad channel”. It is then reasonable to consider a higher cost when employing the “good channel” and a lower cost when employing the “bad channel”. We quantify this policy by assigning the following binary pay-off to each of the channels.

$$c_i(a_i,b_{i-1}) = \begin{cases} 1 & \text{if } a_i=b_{i-1}, (s_i=0) \\ 0 & \text{if } a_i\neq b_{i-1}, (s_i=1) \end{cases} \quad (II.17)$$

The letter-by-letter average transmission cost is given by

$$\mathbb{E}\{c(A_i,B_{i-1})\}=P_{A_i,B_{i-1}}(0,0)+P_{A_i,B_{i-1}}(1,1)=P_{S_i}(0). \quad (II.18)$$

**Remark 1.** The binary form of the constraint does not downgrade the problem, since it can be easily upgraded to more complex forms, without affecting the proposed methodology (i.e., $(1-\delta)$, $\delta$, where $\delta$=constant). Moreover, for $\beta > \alpha \geq 0.5$ we reverse the cost, while for $\alpha$ and/or $\beta$ are less than 0.5 we flip the respective channel input.

**III. Explicit Expressions of Capacity of BSSC with Feedback with & without Transmission Cost**

In this section we provide explicit (or closed form) expressions for the capacity of the BSSC with feedback, with and without transmission cost.

**A. Capacity with Feedback and Transmission Cost**

Consider the case when there is feedback and transmission cost. Without loss of generality in the optimization problem (II.13) we replace the inequality by an equality, because the optimization problem is convex, and hence the optimal channel input distributions occurs on the boundary of the constraint, provided $\kappa \in [0, \kappa_m]$, where $\kappa_m \in [0,1]$.

Hence, we discuss the problem with an equality transmission cost constraint defined by

$$C_{fb}(\kappa) = \sup_{P_{A_i,B_{i-1}}} I(A_i;B_i|B_{i-1}), \quad \kappa \in [0,1]. \quad (III.19)$$

where $\kappa \in [0,1]$. In section III-B (Remark 2), we discuss the case when inequality is considered.

The constraint rate of the BSSC with feedback is illustrated in Figure III.1. The projection on the distribution plane, denoted by the black dotted line, shows all possible pairs of input distributions that satisfy the transmission cost $\mathbb{E}\{c(A_i,B_{i-1})\}=\kappa$.

Next, we state the main theorem from which all other results (no feedback, inequality transmission cost, no transmission cost) will be derived.

**Theorem 1.** (Capacity of BSSC($\alpha, \beta$) with feedback & transmission cost)

The capacity of BSSC($\alpha, \beta$) with feedback and transmission cost $\mathbb{E}\{c(A_i,B_{i-1})\}=\kappa$, $\kappa \in [0,1]$ is given by

$$C_{fb}(\kappa) = H(\lambda) - \kappa H(\alpha) - (1-\kappa) H(\beta). \quad (III.20)$$

where $\lambda=\alpha \kappa + (1-\kappa)(1-\beta)$. The optimal input and output distributions are given by...
The conditional distribution of the output is given by

$$P^*_B|B_{i-1}(b_i|b_{i-1}) = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}.$$  \hfill (III.22)

**Proof:** We outline the proof. The second term of the RHS of (III.19) is fixed by the cost constraint, and it is given by

$$H(B_i|B_{i-1},A_i)=\kappa H(\alpha)+(1-\kappa)H(\beta).$$  \hfill (III.23)

The conditional distribution of the output is given by

$$P_{B_i|B_{i-1}}=\sum_{a_i\in A} P_{B_i|A_i,B_{i-1}} P_{A_i|B_{i-1}}.$$  \hfill (III.24)

Then, by manipulating (II.18) and (III.24), we obtain

$$P_{B_i|B_{i-1}}(0|0) P_{B_i}(0)+P_{B_i|B_{i-1}}(1|1)(1-P_{B_i}(0)) = \lambda,$$  \hfill (III.25)

where $\lambda=\alpha\kappa+(1-\beta)(1-\kappa)$. Using (III.25) we obtain the following expressions for $P_{B_i|B_{i-1}}(0|0)$ and $P_{B_i}(0)$, as functions of $P_{B_i|B_{i-1}}(1|1)$, $\alpha, \beta, \kappa$.

$$P_{B_i}(0)=\frac{1+\lambda-2P_{B_i|B_{i-1}}(1|1)}{2(1-P_{B_i|B_{i-1}}(1|1))},$$  \hfill (III.26)

$$P_{B_i|B_{i-1}}(0|0)=\frac{2\lambda(1-\lambda)P_{B_i|B_{i-1}}(1|1)}{1+\lambda-2P_{B_i|B_{i-1}}(1|1)}.$$  \hfill (III.27)

To simplify the notation, we set $q_b=P_{B_i|B_{i-1}}(1|1)$, and then calculate $H(B_i|B_{i-1})$ as a function of $\lambda$ and $q_b$. Maximizing $H(B_i|B_{i-1})$ with respect to $q_b$, yields

$$1-\lambda \left( \frac{\log(2(\lambda-1+\lambda)q_b)}{2(q_b-1)^2} \right) = 0$$

$$\Rightarrow \frac{1-\lambda}{2(q_b-1)^2} \log \left( \frac{2\lambda(1+\lambda)q_b}{1+\lambda-2q_b} \right) = 0,$$  \hfill (III.28)

hence, $q_b=\lambda$ or $q_b=1$ (the trivial solution). By substituting the non-trivial solution $q_b=\lambda$ into the single letter expression of the constraint capacity we obtain (III.21), (III.22). Moreover, since the transition matrix (III.22) is doubly stochastic, the distribution of the output symbol, $B_i$, at each time instant $i\in\mathbb{N}$, is given by $P_{B_i}(0)=P_{B_i}(1)=0.5$, $i\in\mathbb{N}$.

**B. Capacity with Feedback without Transmission Cost**

When there is no transmission cost constraint any channel input distribution pair is permissible. An example for a possible rate of the BSSC with feedback without transmission cost is illustrated in Figure III.3.

Next, we derive the analogue of Theorem 1 when there is no transmission cost. For this case, the expression of the capacity highlights the optimal time sharing between the two states.

**Theorem 2. (Capacity of BSSC($\alpha, \beta$) with feedback without transmission cost)**

The capacity of the BSSC($\alpha, \beta$) with feedback without transmission cost is given by

$$C^{fb}=H(\lambda^*)-\kappa^* H(\alpha)-(1-\kappa^*) H(\beta),$$  \hfill (III.29)

where

$$\lambda^*=\frac{\alpha \kappa+(1-\kappa^*)(1-\beta)}{(\alpha+\beta-1)(1+2 \frac{H(\kappa; H(\alpha))}{\alpha+\beta-1})}.$$  \hfill (III.30)

$$\kappa^*=\frac{\beta(1+2 \frac{H(\kappa; H(\alpha))}{\alpha+\beta-1})-1}{(\alpha+\beta-1)(1+2 \frac{H(\kappa; H(\alpha))}{\alpha+\beta-1})}.$$  \hfill (III.31)

The optimal input and output distributions are given by
\begin{align}
P_{A_i|B_{i-1}}^*(a_i|b_{i-1}) &= \begin{bmatrix} \kappa^* & 1-\kappa^* \\ 1-\kappa^* & \kappa^* \end{bmatrix}. \quad (III.32) \\
P_{B_i|B_{i-1}}^*(b_i|b_{i-1}) &= \begin{bmatrix} \lambda^* & 1-\lambda^* \\ 1-\lambda^* & \lambda^* \end{bmatrix}. \quad (III.33) 
\end{align}

Proof: The derivation of the results can be done by following that of Theorem \[1\] with the maximization of directed information taken over all possible channel input distributions. Alternatively, by utilizing the statements of Theorem \[1\] the capacity without transmission cost corresponds to the double maximization over the channel input distributions that satisfy the average cost constraint, and over all possible values of \( \kappa \in [0,1] \), via

\[ C^{fb} = \max_{\kappa \in [0,1]} C^{fb}(\kappa) \quad (III.34) \]

Using \( (III.34) \), then \( (III.31) \) is obtained; the rest of the statements are easily shown. ■

The result of the unconstrained capacity with feedback is equivalent to \[13\]. However, since our derivations are different, the capacity formulae given here highlights the optimal time sharing, \( \kappa^* \), among the two binary symmetric channels.

**Remark 2.** For problem \[12\] with inequality constraint, in view of its convexity, and the fact that \( C^{fb}(\kappa) \) as a function of \( \kappa \) is concave and nondecreasing in \( \kappa \in [0, \kappa_m] \), then \( \kappa_m = \kappa^* \), and the solution occurs on the boundary of the cost constraint.

**IV. Explicit Expressions of Capacity of BSSC Without Feedback with & Without Transmission Cost**

In this section we show that for BSSC feedback does not increase capacity, and then we derive the analogue of Theorem \[1\] and Theorem \[2\].

**Theorem 3. (a):** For the BSSC(\( \alpha, \beta \)) with transmission cost, the first-order Markovian input distribution \( \{P_{A_i|A_{i-1}}^*: i=0,1,\ldots \} \) given by

\[ P_{A_i|A_{i-1}}^*(a_i|a_{i-1}) = \begin{bmatrix} 1-\kappa & \kappa \\ 1-2\gamma & \gamma \end{bmatrix}, \quad (IV.35) \]

where \( \gamma = \alpha \kappa + \beta (1-\kappa) \), induces the optimal channel input and channel output distributions \( P_{A_i|B_{i-1}}^* \) and \( P_{B_i|B_{i-1}}^* \), respectively, of the BSSC(\( \alpha, \beta \)) with feedback and transmission cost.

(b): For the BSSC(\( \alpha, \beta \)) without transmission cost \( (a) \) holds with \( \kappa = \kappa^* \) and \( \gamma = \gamma^* \).

(c): The capacity the BSSC without feedback and transmission cost is given by

\[ C^{fb} = \max_{P_{A_i|A_{i-1}}} I(A_i;B_i|A_{i-1})=C^{fb}, \quad (IV.36) \]

and similarly, if there is a transmission cost.

Proof: To prove the claims it suffices to show that a Markovian input distribution achieves the capacity achieving channel input distribution with feedback. Consider the following identities.

\[ P_{A_i|B_{i-1}}^* = \sum_{A_{i-1}} P_{A_i|A_{i-1},B_{i-1}} P_{A_{i-1}|B_{i-1}} = \sum_{A_{i-1}} P_{A_{i-1}} P_{A_{i-1}|B_{i-1}} \]

\[ = \sum_{A_{i-1}} P_{A_{i-1}} \sum_{B_{i-2}} P_{B_{i-1}|A_{i-1},B_{i-2}} P_{A_{i-1}|B_{i-2}} P_{B_{i-2}}, \quad (IV.37) \]

Thus, we search for an input distribution without feedback \( P_{A_i|A_{i-1},B_{i-1}} = P_{A_i|A_{i-1}} \) that satisfies \( (IV.37) \). Solving iteratively this system of equations yields the values of the optimal input distribution without feedback given by \( (IV.35) \). Since \( P_{A_i|A_{i-1}} \) given by \( (IV.35) \) induces \( P_{A_i|B_{i-1}}^* \), then the input distribution without feedback also induces the optimal output distribution \( P_{B_i|B_{i-1}}^* = \sum_{A_i} P_{A_i} P_{B_i|A_{i-1},B_{i-2}} P_{A_{i-1}|B_{i-2}} P_{B_{i-2}}, \) and joint processes \( P_{A_i,B_i|A_{i-1},B_{i-2}} \) and \( P_{A_i,A_{i-1}|B_{i-2}} \). This is sufficient to conclude (c). ■

**V. Conclusions**

In this paper we formulate the capacity of the UMC and the BSSC and provide the explicit expressions of the capacity and corresponding achieving channel input distribution of the BSSC, with and without feedback and with and without transmission cost.

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