Remarks on generalized Gauss-Bonnet dark energy

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Abstract
The modified gravity with $F(R,G)$ Lagrangian, $G$ is the Gauss-Bonnet invariant, is considered. It is shown that the phantom-divide-line crossing and the deceleration to acceleration transition generally occur in these models. Our results coincide with the known results of $f(R)$-gravity and $f(G)$-gravity models. The contribution of quantum effects to these transitions is calculated, and it is shown that in some special cases where there are no transitions in classical level, quantum contributions can induce transitions. The quantum effects are described via the account of conformal anomaly.

1 Introduction
Recent observational data indicate that our universe is currently in accelerated expansion phase. This is based on the redshift-distance relationship of type Ia supernovas [1], and many other observations [2]. This acceleration is explained in terms of the so-called dark energy which constitutes two third of the present universe. There are many candidates which have been introduced to explain the dark energy: the cosmological constant, the scalar fields(quintessence or phantom fields), scalar-tensor theory, k-essence models, etc. Two complete review articles which have been written in this subject are given in [3].

Among the interesting features of dark energy, the dynamical behavior of equation of state parameter $\omega = p/\rho$ is one which has been studied by many authors. As is well known, the accelerating universe demands $\omega < -1/3$. Also some astrophysical data seem to slightly favor an evolving dark energy and show a recent $\omega = -1$, the so-called phantom-divide-line, crossing [4]. These observations can not be explained by single scalar field theories. In quintessence model, which consists of a normal scalar field, $\omega$ is always $\omega > -1$, and in phantom model, a scalar field theory with unusual negative kinetic energy, $\omega$ satisfies $\omega < -1$. A possible theoretical solution to this problem is to consider the models known as hybrid models, the models in which there exist more than one scalar field. One of the famous hybrid model is quintom model which consists of one quintessence field and one phantom field. It has been shown that in quintom model with slowly-varying potentials, the $\omega = -1$ crossing always occurs [5].
An alternative approach for explaining the observational data is to postulate that the gravity is being nowadays modified by some terms which grow when curvature decrease. The first class of modified gravities are models known as $f(R)$-gravity, whose action is a general function $f(R)$ in terms of Ricci scalar $R$, i.e.

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} f(R) + \mathcal{L}_m \right]. \quad (1)$$

$\kappa^2 = 8\pi G$ and $\mathcal{L}_m$ is the Lagrangian density of dust-like matter. The possible cosmological applications of $f(R)$-gravity have been studied in [6]. Such theories and their extensions [7] do pass the local tests and successfully describe the (almost) $\Lambda$CDM epoch. For a review see [8].

Another modification of usual gravity is modified Gauss-Bonnet theory introduced in [9,10]. In this model, the Einstein action is modified by the function $f(G)$, $G$ being the Gauss-bonnet (GB) invariant:

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + f(G) \right], \quad (2)$$

in which

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}. \quad (3)$$

$G$ is topological invariant in four dimensions and may lead to some interesting cosmological effects in higher dimensional brane-world approach (for a review, see [11]). The Gauss-Bonnet coupling with scalar field and its contribution in creation of effective quintessence and phantom era have been studied in [12]. Other aspects of modified GB-gravity, such as its possibility to describe the inflationary era, transition from deceleration phase to acceleration phase, crossing the phantom-divide-line and passing the solar system tests have been studied in [11, 13, 14].

Recently, a generalization of modified GB-gravity has been introduced in [13]. The action of this generalized GB-gravity is:

$$S = \int d^4x \sqrt{-g} \left[ F(R, G) + \mathcal{L}_m \right]. \quad (4)$$

The modified $f(R)$-gravity and modified GB-gravity, i.e. the actions (1) and (2), are clearly the special examples of modified $F(R, G)$-gravity (4). The hierarchy problem of particle physics and the late time cosmology have been studied in $F(R, G)$ framework in [13]. See also [15].

The present paper is devoted to the study of some features of $F(R, G)$-gravity, i.e. the phantom-divide-line crossing and the possible transition between deceleration and acceleration phases. We seek the conditions under which these two important crossings occur, in both classical and quantum levels. It is seen that the quantum contributions can effectively change the transition conditions. The quantum effects are described via the the account of conformal anomaly, reminding about anomaly-driven inflation [16]. The contribution of conformal anomaly in energy conditions and Big Rip of phantom models has been discussed in [17], and its influence on $\omega = -1$ crossing of quintessence and phantom model has been studied in [18].

The scheme of the paper is as follows. In section 2 we discusses the Friedmann equations of generalized GB-gravity in a spatially flat Friedman-Robertson-Walker background. Section 3 is devoted to obtaining the conditions of $\omega = -1$
crossing, and in section 4 the condition for $\dot{a} < 0$ to $\ddot{a} > 0$ transition is derived. It is seen that for special cases where the generalized GB-gravity is reduced to $f(R)$-gravity or GB-gravity, our results lead to the known ones for these cases. Finally in section 5, the contribution of quantum effects on these conditions is obtained and it is seen that under special initial conditions, these transitions occur as a result of only quantum effects.

2 The $F(R, G)$ gravity

Consider the generalized GB-gravity action \[^4\], with $G$ defined in eq. \[^3\]. Varying the action \[^4\] with respect to metric $g_{\mu \nu}$ results in \[^13\]:

$$
\frac{1}{2} T^{\mu \nu} + \frac{1}{2} g^{\mu \nu} F(R, G) - 2 F_G(R, G) R R^{\mu \nu} + 4 F_G(R, G) R^\rho_\mu R^{\nu \rho}
- 2 F_G(R, G) R^{\rho \sigma \tau} R^\nu_{\rho \sigma \tau} - 4 G_R(G, R) R^{\rho \sigma \nu} R^\rho_{\sigma \nu} + 2 (\nabla^\mu \nabla^\nu F_G(R, G)) R
- 2 g^\nu_{\mu} (\nabla^2 F_G(R, G)) R - 4 (\nabla_\rho \nabla^\mu F_G(R, G)) R^\rho - 4 (\nabla_\rho \nabla^\nu F_G(R, G)) R^\rho
+ 4 (\nabla^2 F_G(R, G)) R^{\mu \nu} + 4 g^\nu_{\mu} (\nabla_\rho \nabla_\sigma F_G(R, G)) R^{\rho \sigma} - 4 (\nabla_\rho \nabla_\sigma F_G(R, G)) R^{\mu \rho \sigma}
- F_R(R, G) R^{\mu \nu} + \nabla^\mu \nabla^\nu F_R(R, G) - g^{\mu \nu} \nabla^2 F_R(R, G) = 0.
$$

(5)

In the above evolution equation, $T_{\mu \nu}$ is energy-momentum tensor of matter field:

$$
T_{\mu \nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu \nu}},
$$

(6)

and $F_R$ and $F_G$ are:

$$
F_G(R, G) = \frac{\partial F(R, G)}{\partial G}, \quad F_R(R, G) = \frac{\partial F(R, G)}{\partial R}.
$$

(7)

A spatially flat Friedman-Robertson-Walker (FRW) space-time in co-moving coordinates $(t, x, y, z)$ is defined through

$$
ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),
$$

(8)

where $a(t)$ is the scale factor. For the metric \[^5\], the $(t, t)$-component of \[^5\] has the following form:

$$
-6 H^2 F_R(R, G) = F(R, G) - R F_R(R, G) + 6 H \dot{F}_R(R, G)
+ 24 H^3 \ddot{F}_G(R, G) - G F_G(R, G) - \rho_m.
$$

(9)

$H = \dot{a}(t)/a(t)$ is the Hubble parameter, and $\rho_m$ is matter energy density with evolution equation

$$
\dot{\rho}_m + 3 H (\rho_m + p_m) = 0.
$$

(10)

Here $R$ and $G$ have the following forms:

$$
R = 6 (\dot{H} + 2 H^2), \quad G = 24 (H^2 \dot{H} + H^4).
$$

(11)

The sum of $(i, i)$ components of eq. \[^5\] for FRW-metric becomes

$$
(2 \dot{H} + 3 H^2) F_R(R, G) = \frac{1}{2} [R F_R(R, G) - F(R, G)] - 2 H \dot{F}_R(R, G) - \ddot{F}_R(R, G)
+ \frac{1}{2} G F_G(R, G) - \frac{G}{3 H} \ddot{F}_G(R, G) - 4 H^2 \ddot{F}_G(R, G) - \frac{1}{2} \rho_m.
$$

(12)
Eqs. (9) and (12) are the Friedman equations of \( F(R, G) \) gravity. Note that the eqs. (9-12) are not independent, i.e. the time derivative of eq. (9), using eqs. (10) and (11), leads to a linear combination of eqs. (9) and (12).

3 The \( \omega = -1 \) crossing

In the case of Einstein gravity with scalar fields responsible for dark energy, the equation of state parameter \( \omega = p/\rho \) satisfies \( \omega = -1 - 2 \frac{\dot{H}}{3H^2} \). For other theories, including the \( F(R, G) \)-gravity, the effective equation of state parameter \( \omega_{\text{eff}} \) is also defined through \([9, 13]\)

\[
\omega_{\text{eff}} = \frac{p}{\rho} = -1 - 2 \frac{\dot{H}}{3H^2}. \tag{13}
\]

So if \( H(t) \) has a relative extremum at \( t = t_0 \), the system crosses \( \omega = -1 \) line at time \( t = t_0 \).

Restricting ourselves to \( t - t_0 \ll h_0^{-1} \), where \( h_0 = H(t_0) \) and \( h_0^{-1} \) is of order of the age of universe, the Hubble parameter can be expanded as

\[
H(t) = h_0 + h_1(t - t_0)^\alpha + h_2(t - t_0)^{\alpha + 1} + O((t - t_0)^{\alpha + 2}), \tag{14}
\]

in which \( \alpha \geq 2 \) is the order of first non-vanishing derivative of \( H(t) \) at \( t = t_0 \) and \( h_1 = \frac{1}{\alpha!}H^{(\alpha)}(t_0) \). \( H^{(n)}(t_0) \) is the \( n \)-th derivative of \( H(t) \) at \( t = t_0 \). The transition from \( \omega > -1 \) to \( \omega < -1 \) regions occurs when \( \alpha \) is even positive integer and \( h_1 > 0 \). If \( h_1 < 0 \), the reverse transition occurs. Here we seek a solution for the Friedman equation (9), with \( \rho_m, R \) and \( G \) given by eqs. (10) and (11), when \( H(t) \) is expressed by eq. (14). The obtained value of \( \alpha \) will determine the possibility of \( \omega = -1 \) crossing of \( F(R, G) \)-gravity theories.

We first rewrite eq. (9) as following

\[
H^2 F_R(R, G) \equiv b(t), \tag{15}
\]

where

\[
b(t) = \frac{1}{6} \left[ F(R, G) - RF_R(R, G) + 6H \dot{R} F_{RR} + 6H(\dot{G} + 4H^2 \dot{R}) F_{RG}(R, G) \right. \\
\left. + 24H^3 \dot{G} F_{GG}(R, G) + GF_G(R, G) - \rho_m \right]. \tag{16}
\]

In eq. (16) we use:

\[
\frac{d}{dt} f(R, G) = \dot{R} f_R + \dot{G} f_G, \tag{17}
\]

and the subscripts of \( F(R, G) \) denote the partial derivatives, e.g.

\[
F_{RG}(R, G) = \frac{\partial^2 F(R, G)}{\partial R \partial G}. \tag{18}
\]

Expanding both sides of eq. (15) near \( t_0 \equiv 0 \), with \( H(t) \) given by eq. (14), result in the following relations:

\[
h_0^2 F_R(0) = b(0), \tag{19}
\]
\[ h_2^3(F_{RR} \dot{R} + F_{RG} \dot{G})_{t=0} = \dot{b}(0) , \quad (20) \]
and
\[ \frac{1}{2} h_0^3(F_{RRR} \ddot{R}^2 + 2F_{RRC} \dot{R} \dot{G} + F_{RR} \ddot{R} + F_{RGG} \ddot{G}^2 + F_{RG} \ddot{G})_{t=0} + 2h_0 h_1 F_{R}(0) \delta_{\alpha,2} = \frac{1}{2} \dot{b}(0) . \quad (21) \]

By \( F_{R}(0) \) we mean \( F_{R}(R, G)|_{t=0} \). Using eqs.\((10)\) and \((11)\) and expansion \((14)\), two relations \((19)\) and \((20)\) lead to :

\[
- h_0^2 F_{R}(0) + \frac{1}{6} F(0) - 4h_0^4 F_{G}(0) - \frac{1}{6} \rho_m(0) \\
+ 12h_0 h_1 (F_{RR} + 8h_0^2 F_{RG} + 16h_0^4 F_{GG})_{t=0} \delta_{\alpha,2} = 0 , \quad (22)
\]

and

\[
36h_0 \left[(F_{RR} + 8h_0^2 F_{RG} + 16h_0^4 F_{GG})h_2 \\
+ 4(F_{RRR} + 12h_0^2 F_{RRC} + 48h_0^4 F_{RGG} + 64h_0^6 F_{GGG})h_1^2\right]_{t=0} \delta_{\alpha,2} \\
+ 36h_1 (F_{RR} + 8h_0^2 F_{RG} + 16h_0^4 F_{GG})_{t=0} (h_0^2 \delta_{\alpha,2} + h_0 \delta_{\alpha,3}) + \frac{1}{2} \gamma_m h_0 \rho_m(0) = 0 . \quad (23)
\]

In above equation, \( \gamma_m \) is defined by \( \gamma_m = 1 + \omega_m \) where \( \omega_m = \rho_m / \rho_m \).

As is clear from eq.\((23)\), for \( \alpha \geq 4 \), the only solution is

\[ h_0 = 0 \quad \text{or} \quad \rho_m(0) = 0 , \quad (24) \]

which both are unphysical. In the case \( \alpha = 3 \), eq.\((22)\) does not depend on \( h_1 \) and can be used to determine \( h_0 \) in terms of \( \rho_m(0) \). Of course its explicit expression can not be found before the function \( F(R, G) \) is known. Note that at \( t = 0 \), \( R(0) = 12h_0^2 \) and \( G(0) = 24h_0^3 \). On the other hand, the value of \( h_0 \) is also determined from eq.\((10)\) in term of \( \rho_m \) and \( \rho_m \) at \( t = 0 \):

\[ h_0 = - \frac{\dot{\rho}_m(0)}{3 \gamma_m \rho_m(0)} . \quad (25) \]

The \( \alpha = 3 \) solution exists only if these two different expressions are equal, which is a very special choice of initial values. Under these conditions there is no \( \omega = -1 \) transition. Except these fine-tuned initial values, the only remaining solution is \( \alpha = 2 \) which we now discuss it.

For \( \alpha = 2 \), eqs.\((22)\) and \((23)\) result in \( h_1 \) and \( h_2 \), respectively. The parameter \( h_1 \) becomes

\[ h_1 = \left. \frac{6h_0^2 F_R + 24h_0^4 F_G - F + \rho_m(0)}{72h_0(F_{RR} + 8h_0^2 F_{RG} + 16h_0^4 F_{GG})} \right|_{R=12h_0^2, G=24h_0^3} , \quad (26) \]

which is, in general, a non-zero quantity. So \( F(R, G) \)-gravity can explain the phantom-divide-line crossing. Depending on the explicit form of \( F(R, G) \), \( h_1 \) can be positive or negative. In the case of \( f(R) \)-gravity

\[ F(R, G) = \frac{1}{2R^2} f(R) , \quad (27) \]
eq.(26) leads to
\[ h_1 = \frac{6h_0^2 f'(12h_0^2) - f(12h_0^2) + 2\kappa^2 \rho_m(0)}{72h_0^2 f''(12h_0^2)}, \tag{28} \]
which is consistent with the result of [8]. In Gauss-Bonnet gravity
\[ F(R,G) = \frac{1}{2\kappa^2} R + f(G), \tag{29} \]
eq.(26) results in
\[ h_1 = -\frac{3}{15} h_0^2 - f(24h_0^2) + 24h_0^2 f'(24h_0^2) + \rho_m(0) \]
\[ \frac{1152h_0^7 f''(24h_0^2)}{72h_0^2 f''(12h_0^2)}, \tag{30} \]
which is the same as one obtained in [9].

In special case where
\[ F_{RR} = F_{RG} = F_{GG}|_{R=12h_0^2, G=24h_0^4} = 0, \tag{31} \]
eq.(26) is not the solution. In this case, eq.(22) determines \( h_0 \), which as has been discussed before, is possible only for specific choice of initial values. Under these conditions, \( h_1 \) is determined from eq.(23) as following:
\[ h_1 = \left[ -\frac{1}{288} F_{RRR} + 12h_0^2 F_{RRG} + 48h_0^3 F_{RRG} + 64h_0^5 F_{GGG} \right]_{R=12h_0^2, G=24h_0^4}^{1/2}. \tag{32} \]
It is clear that this solution exist only when the matter energy density \( \rho_m(0) \) is not zero.

4 Deceleration to acceleration transition

To study the \( \ddot{a} < 0 \) to \( \ddot{a} > 0 \) transition, it must be noted that \( G = 24H^2(\dot{H} + H^2) = 24H^2 \dot{a}/a \). So at the point of transition \( \ddot{a} = 0 \), one has \( G(t_0) = 0 \).

We expand \( H(t) \) and \( G(t) \) around \( t_0 = 0 \) as following:
\[ H(t) = H_0 + H_1 t + H_2 t^2 + \cdots, \tag{33} \]
\[ G(t) = \dot{G}(0)t + \frac{1}{2} \ddot{G}(0)t^2 + \cdots = G_1 t + G_2 t^2 + \cdots. \tag{34} \]

Since \( G(0) = 0 \), one finds
\[ H_1 = -H_0^2. \tag{35} \]

Using \( G = 24H^2(\dot{H} + H^2) \), eqs.(33) and (35) determine \( G_1 \) and \( G_2 \) as following:
\[ G_1 = 48H_0^2(H_2 - H_0^3), \]
\[ G_2 = 24H_0^4(5H_0^4 - 2H_0^2H_2 + 3H_4). \tag{36} \]

We seek any consistent solution of Friedman equation (11), along with eqs. (10) and (11), when \( H(t) \) and \( G(t) \) are given by eqs.(33) and (34), respectively. Instead of relations (19) and (20), here we obtain:
\[ H_0^2 F_R(0) = b(0), \tag{37} \]
and
\[ 2H_0 H_1 F_R(0) + H_0^2 (F_{RR} \dot{R} + F_{RG} \dot{G})_{t=0} = \dot{b}(0), \] (38)
respectively, from them the parameters \( H_2 \) and \( H_3 \) can be found in terms of \( H_0 \). (Note that \( H_1 \) is determined by eq.(35)). The result for \( H_2 \) is:

\[ H_2 = \frac{-F + 144H_0^4 F_{RR} + 864H_0^4 F_{RG} + 1152H_0^4 F_{GG} + \rho_m(0)}{72H_0(F_{RR} + 8H_0^2 F_{RG} + 16H_0^2 F_{GG})} \bigg|_{R=6H_0^2, G=0}. \] (39)

\( H_3 \), and other parameters of Hubble rate, can be also found consistently, which suggests the existence of the transition between deceleration and acceleration phases of the universe.

In the case of Gauss-Bonnet gravity, i.e. eq.(29), eq(39) is reduced to

\[ H_2 = H_0^3 - \frac{1}{24f''(0)H_0^2} \left( \frac{1}{16\kappa^2} + \frac{f(0) + f''(0)}{48H_0^2} \right), \] (40)
which is the same as in [9]. (Note that there is a little mistake in eq.(12) of [9] and the factor 48 of that equation must be replaced by 24).

For the cases in which eq.(31) satisfies, the solution (39) does not exist. In this case, eq.(37) results in

\[ F(R, G) \bigg|_{R=6H_0^2, G=0} = \rho_m(0), \] (41)
which determines \( H_0 \) in terms of \( \rho_m(0) \). This is again held only for special initial values (by initial, we mean at transition time \( t' = 0 \)). \( H_2 \) is thus found by solving eq.(38), which results in two lengthy solutions. Note that if \( F \big|_{R=6H_0^2, G=0} = 0 \), this solution exists only when there is no matter energy density at \( t' = 0 \), i.e. \( \rho_m(0) = 0 \).

5 The quantum corrections

In this section we study the contributions of quantum effects on transition conditions. For calculating the quantum corrections we use a standard method in which the interactions are considered between the quantum particles and classical gravitational field [16,19]. In this context, the renormalization of effective action leads to some extra terms in the trace of energy-momentum tensor, which is known as trace/conformal anomaly. Classically, this tensor is traceless. The extra terms are:

\[ T = b(F + \frac{2}{3} \Box R) + b'G + b''R, \] (42)
in which \( F \) is the square of 4d Weyl tensor

\[ F = \frac{1}{3} R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \] (43)
and \( G \) and \( R \) are Gauss-Bonnet and Ricci scalars, respectively. \( b, b' \) and \( b'' \) are given by

\[ b = \frac{N + 6N_{1/2} + 12N_1 + 611N_2 - 8N_{HD}}{120(4\pi)^2}, \]
\[ b' = -\frac{N + 11N_{1/2} + 62N_1 + 1411N_2 - 28N_{HD}}{360(4\pi)^2}, \] (44)
\[ b'' = 0, \] (45)
for situations which exist $N$ scalars, $N_{1/2}$ spinors, $N_1$ vector fields, $N_2 (= 0, \text{ or } 1)$ gravitons and $N_{HD}$ higher derivative conformal scalars (including phantoms).

It is worth noting that the calculation which is leaded to eq. (42) is independent of the gravitational part of the Lagrangian and therefore can contribute to both the standard and modified gravities. In fact, this method is based on the non-minimal coupling of scalar field to gravity, through the classical Lagrangian density

\[
\mathcal{L} = \frac{1}{2} \left\{ -g^\mu\nu \varphi,\mu(x) \varphi,\nu(x) - (m^2 + \xi R(x)) \varphi^2(x) \right\},
\]

which in one-loop level, results in the trace-anomaly [16].

For FRW metric [8], eq. (42) results in the following equations for the contribution of conformal anomaly to $\rho$ and $p$ [11, 20]:

\[
\rho_A = -\frac{1}{a^4} \left\{ b'(6a^4H^4 + a^2H^2) + \frac{2b + b''}{3} \right\}[a^4(-6H \ddot{H} - 18H^2 \dot{H} + 3 \dot{H}^2) + 6a^2H^2]
\]

\[
-2b + 6b' - 3b''
\]

and

\[
p_A = b'[6H^4 + 8H^2 \dot{H} + \frac{1}{a^2}(4H^2 + 8 \dot{H})]
\]

\[
+ \frac{2b + b''}{3} \left[ -2 \ddot{H} - 12H \dot{H} - 18H^2 \dot{H} - 9 \dot{H}^2 \right]
\]

\[
+ \frac{1}{a^2}(2H^2 + 4 \dot{H})] - \frac{-2b + 6b' - 3b''}{3a^4}.
\]

The natural way to consider the quantum correction is to add its contribution to Friedman equations, i.e. to change $\rho_m$ in eq. (16) to $\rho_m + \rho_A$. In this way we can study the effects of quantum phenomena on $\omega = -1$ and $\ddot{a} = 0$ crossings of the universe.

5.1 Quantum correction to $\omega = -1$ crossing

In $\omega = -1$ crossing, we take $\alpha = 2$ in eq. (14) and consider eq. (15) with $\rho_m$ replaced by $\rho_m + \rho_A$. Eq. (19) is then become

\[
h_0^2 F_R(0) = b(0) + \frac{\rho_A(0)}{6},
\]

from which $h_1$ is found as follows:

\[
h_1^{q.c.} = \frac{6h_0^2 F_R + 24h_1^2 F_G - F + \rho_m(0) - 6b' h_0^4 - (4b + 12b')(h_0/a_0)^2 + (2b - 6b')/a_0^4}{72h_0(F_{RR} + 8h_0^2 F_{RG} + 16h_1^2 F_{GG} - b/9)} \bigg|_{R=12h_0^2 G=24h_0^4}.
\]

In above equation ”q.c.” stands for ”quantum-corrected”. As is obvious from eq. (19), in the limits $b \to 0$ and $b' \to 0$, one has $h_1^{q.c.} \to h_1^{cl}$, which shows the correct behavior of $h_1^{q.c.}$. By classical $h_1, h_1^{cl}$, we mean eq. (20). Also note that the quantum correction terms are much more smaller than the classical terms.
This can be seen by comparing $\rho_m(0)$ in numerator of (49) with terms rise from quantum corrections, if we note that:

$$h_0^4 \sim \frac{1}{a_0^2} << 1 . \quad (50)$$

Finally it is interesting to compare $h_{q,c}^1$ with $h_{cl}^1$ for the cases where eq.(31) holds. In classical level, $h_{cl}^1 = 0$ if $\rho_m(0) = 0$ (see eq.(32)), but here $h_{q,c}^1$ is not zero. Using (49), it becomes

$$h_{q,c}^1 = -6h_0^2F_R + 24h_0^4F_G - F - 6b'h_0^4 - (4b + 12b')(h_0/a_0)^2 + (2b - 6b')/a_0^4 \bigg|_{R=12h_0^2, G=24h_0^4} , \quad (51)$$

which is a purely quantum mechanical term. In other words, for $F(R,G)$s which satisfies (31), the classical transition exists only when a very specific condition is satisfied by initial values and also when $\rho_m \neq 0$. In fact, in the classical level, the $\omega = -1$ crossing is a matter-induced transition. But quantum mechanically, for the same $F(R,G)$, the transition occurs with or without matter; it is a purely quantum-induced transition.

### 5.2 quantum correction of $\ddot{a} = 0$ crossing

To study the quantum contributions in deceleration-acceleration transition, we must solve eq.(37), when $b(0)$ replaced by $b(0) + \rho_A(0)/6$. $H(t)$ and $G(t)$ are given by eqs.(33) and (34). The final expression of $H_2$ is as follows:

$$H_2^{q,c} = \left[ -F + 144H_0^4F_R + 864H_0^6F_G + 1152H_0^8F_{GG} + \rho_m(0) \\
+ (-6b' - 14b)H_0^4 + (-12b' + 4b)(H_0/a_0)^2 + (-6b' + 2b)/a_0^4 \right] \bigg|_{R=6H_0^2, G=0} \times (52)$$

The above equation has correct classical limit (comparing $b = b' = 0$ limit of eq.(52) with eq.(39)), and the correction terms are much smaller than the classical terms. In special case where

$$F = F_R = F_G = F_{GG} \bigg|_{R=6H_0^2, G=0} = 0 \quad (53)$$

hold, it was shown that the transition solution exists only when $\rho_m(0) = 0$ (see the discussion after eq.(41)). But here, under the conditions (52), $H_2$ in (51) is yet a solution, without any constraint on $\rho_m$. In other words, for the cases characterized by (53), the classical transition occurs only when the matter does not exist, but by considering the quantum effects, it is seen that this transition occurs in the presence of ordinary matters. This is a purely quantum effect.

### 6 Conclusion

To summarize, the generalized Gauss-Bonnet dark energy with Lagrangian (4) has been considered in Friedman-Robertson-Walker background metric. For times near the $\omega = -1$ crossing time, $H(t)$ can be expressed by (14). It has been shown that the only general solution is one with $\alpha = 2$. The Hubble parameter
$h_1$ has been obtained in [26], which proves the existence of phantom-divide-line crossing in $F(R,G)$-gravity models.

For obtaining an $\alpha = 0$ crossing solution, we take $H(t)$ as (33), with constraint (35), and the parameter $H_2$ of Hubble rate has been found in (39). This relation shows the existence of deceleration-acceleration transition in generalized GB models.

Finally the quantum corrections have been added by considering the conformal anomalies (46) and (47). It has been shown that these corrections modify $h_1^{cl}$ (in eq. (26)) to $h_1^{q.c.}$ (in eq. (49)). In some special cases, quantum phenomenon is the only reason for $\omega = -1$ crossing. The same corrections modify $H_2^{cl}$ (in eq. (39)) to $H_2^{q.c.}$ (in eq. (22)), and again for some special $F(R,G)$s, the transition from deceleration to acceleration phases is induced by quantum effects.

Acknowledgement: This work was partially supported by the "center of excellence in structure of matter" of the Department of Physics of University of Tehran, and also a research grant from University of Tehran

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