Quantum conical designs

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Abstract

Complex projective \( t \)-designs, particularly SICs and full sets of mutually unbiased bases, play an important role in quantum information. We introduce a generalization which we call conical \( t \)-designs. They include arbitrary rank symmetric informationally complete measurements (SICs) and full sets of arbitrary rank mutually unbiased measurements (MUMs). They are deeply implicated in the description of entanglement (as we show in a subsequent paper). Viewed in one way a conical two-design is a symmetric decomposition of a separable Werner state (up to a normalization factor). Viewed in another way it is a certain kind of polytope in the Bloch body. In the Bloch body picture SICs and full sets of MUMs form highly symmetric polytopes (a single regular simplex in the one case; the convex hull of a set of orthogonal regular simplices in the other). We give the necessary and sufficient conditions for an arbitrary polytope to be what we call a homogeneous conical two-design. This suggests a way to search for new kinds of projective two-design.

Keywords: \( t \)-designs, quantum geometry, quantum information, SIC POVMs, MUBs

1. Introduction

Complex projective \( t \)-designs \cite{1–4} play an important role in quantum information (with the vast majority of current applications being for the case \( t = 2 \)). The best known examples are full sets of mutually unbiased bases (MUBs) \cite{5–8} and the families of projectors defining rank 1 symmetric informationally complete measurements (SICs) \cite{3, 9, 10}. However, these are not...
the only examples; in particular, it is known that such designs exist in every finite dimension [11, 12]. Among other things they have applications to quantum tomography [4, 7, 8, 12–16], cryptography [8, 17–21], dense-coding [8, 22], teleportation [8, 23], entanglement detection [24–27], quantum communication [28–31] and cloning [4, 8, 32, 33] (references given being representative only). They are important in quantum foundations, SCS being a mathematical cornerstone of QBism [34, 35]. They also have important applications in classical information, in particular to compressed sensing [36, 37].

A complex projective t-design consists of rank-1 projectors. In this paper we introduce a generalization which has many of the same properties, but in which the projectors are replaced by arbitrary positive semi-definite operators. The set of such operators is the cone generated by quantum state space (i.e. the set of all operators of the form $\lambda \rho$, with $\lambda \geq 0$ and $\rho$ a density matrix). We accordingly refer to the structures we introduce as conical t-designs. Although our definition is valid for arbitrary $t$ we will focus in this paper on the case $t = 2$, reserving the case $t > 2$ for later work.

One example of a conical design is the set of projectors forming a weighted projective two-design [4, 38–40], when appropriately re-scaled (here and in the sequel we use ‘projective two-design’ as shorthand for ‘complex projective two-design’). Other examples are arbitrary rank symmetric informationally complete measurements (SIMs) introduced in [41] (under the different acronym SI-POVMs) and full sets of arbitrary rank mutually unbiased measurements (MUMs) introduced in [42]. We thus provide an affirmative answer to the question posed by Dall’Arno [43], whether SIMs and MUMs are particular instances of a more general class of objects.

Conical two-designs are interesting in their own right. In particular, we will show in a subsequent publication [27] that they are deeply implicated in the description of entanglement. We show in section 6 that they provide simple decompositions of some separable Werner [44] and isotropic [45] states. Actually it is easy to see that conical two-designs provide simple expansions of all Werner and isotropic states (including the entangled ones), although, it remains to be seen whether such expansions can be put to any interesting use. Another reason for being interested in them is their potential relevance to projective two-designs. The problem of constructing projective two-designs is difficult. The MUB and SIC existence problems are still open, and in many dimensions the only known examples have a cardinality which grows extremely fast with dimension [11, 12, 46]. As we discuss in section 5, embedding the class of projective two-designs in a larger class of structures having nice mathematical properties casts some new and possibly fruitful light on the problem of constructing new projective two-designs.

The structure of the paper is as follows. We begin with two introductory sections. The generalized Bloch representation [41, 47–53] of quantum state space is central to our analysis. In section 2 we accordingly summarize the essential features of this representation. In section 3 we go on to describe SIMs and MUMs in terms of the Bloch picture. The existence of SIMs in every finite dimension was originally proved in [41] using a simple geometrical argument. It was subsequently re-proved in [54] using a more complicated algebraic argument. In [42] similar algebraic methods were used to prove the existence of MUMs in every finite dimension. Since our analysis of conical two-designs relies on the geometrical approach we begin our discussion by giving a unified geometrical description of SIMs and MUMs. The main part of the paper starts in section 4 where we define conical designs, and derive some basic properties. The class of all conical designs is large, and to make progress one needs to focus on some special cases. One important special case is the class of weighted projective two-designs; these, however, have already been extensively discussed in the literature [4, 38–40]. In section 5 we therefore examine another special case, which we call homogeneous.
conical two-designs. Projective two-designs are examples of homogeneous two-designs, as are SIMs and MUMs. In the Bloch body picture SIMs and full sets of MUMs form simple, highly symmetric polytopes (a single regular simplex in the case of SIMs; the convex hull of a set of orthogonal regular simplices in the case of MUMs). We fully characterize the polytope corresponding to an arbitrary homogeneous two-design. We further show that the problem of constructing a homogeneous two-design on a complex vector space reduces to the problem of constructing a one-design in a higher dimensional real vector space. We discuss how these geometrical results suggest a way to systematically search for new examples of projective two-designs. In section 6 we show that conical two-designs provide simple decompositions of some separable Werner and isotropic states. In this connection let us note that, although it is the entangled Werner states which are the most interesting, the problem of decomposing the separable states is not straightforward and has attracted some notice in the literature [55–57]. In section 7 we discuss some possible avenues of further research.

2. The Bloch body

Throughout \( \mathcal{H} \) will be a fixed \( d \)-dimensional complex Hilbert space. We define

- \( \mathcal{L} \) to be the complex vector space consisting of all linear operators on \( \mathcal{H} \),
- \( \mathcal{L}_{sa} \) to be the real vector space consisting of all self-adjoint operators on \( \mathcal{H} \),
- \( \mathcal{L}_{sa,0} \) to be the subspace of \( \mathcal{L}_{sa} \) consisting of all trace zero operators,
- \( \mathcal{L}_{sa,1} \) to be the hyperplane in \( \mathcal{L}_{sa} \) consisting of all trace one operators,
- \( \mathcal{C} \) to be the cone consisting of all positive semi-definite operators in \( \mathcal{L}_{sa} \),
- \( \mathcal{Q} \) to be quantum state space—i.e. the intersection \( \mathcal{C} \cap \mathcal{L}_{sa,1} \).

We equip \( \mathcal{L}_{sa} \) with the Hilbert–Schmidt inner product
\[
\langle A | A \rangle = \text{Tr}(A^* A)
\]
and the associated norm \( \|A\| = \sqrt{\langle A | A \rangle} \).

For a qubit \( \mathcal{Q} \) can be identified with the Bloch ball. For \( d > 2 \) the geometry of \( \mathcal{Q} \) is much more intricate than a simple ball [58]. Nevertheless, it is possible to construct a generalized Bloch representation, which preserves some essential features of the qubit case. We will follow the coordinate-free approach of [41] (also see [47–53]).

The key is to realize that any \( \rho \in \mathcal{Q} \) can be written in the form
\[
\rho = \frac{1}{d}(I + B)
\]
for some \( B \in \mathcal{L}_{sa,0} \). In the two-dimensional case it is customary to write \( B = n \cdot \sigma \) (where \( \sigma \) is the vector formed from the Pauli matrices) and to refer to \( n \) as the Bloch vector corresponding to \( \rho \). In the general case it is more convenient to adopt a coordinate-free point of view, and to regard \( B \) itself as the Bloch vector corresponding to \( \rho \). The Bloch body is then the set of all such vectors:
\[
B = \left\{ B \in \mathcal{L}_{sa,0} : \frac{1}{d}(I + B) \in \mathcal{Q} \right\}.
\]
We then define the in- and out-balls

\[ B_{\text{in}} = \{ B \in \mathcal{L}_{sa,0} : \|B\|_B \leq \frac{1}{d-1} \} , \]

\[ B_{\text{out}} = \{ B \in \mathcal{L}_{sa,0} : \|B\|_B \leq 1 \} . \]

We define \( S_{\text{in}} \) (respectively, \( S_{\text{out}} \)) to be the surface of \( B_{\text{in}} \) (respectively, \( B_{\text{out}} \)). One finds [41]

\[ B_{\text{in}} \subseteq B \subseteq B_{\text{out}} . \]

Moreover \( B_{\text{in}} \) (respectively, \( B_{\text{out}} \)) is the largest (respectively, smallest) ball centered on the origin and contained in \( B \) (respectively, containing \( B \)). The manifold of pure states is the intersection \( B \cap S_{\text{out}} \). If \( d = 2 \) then \( B_{\text{in}} = B_{\text{out}} = B \) and we recover the usual Bloch-ball description.

### 3. MUBs and SICs generalized

The Bloch body picture gives a particularly intuitive way of thinking about MUBs and SICs. A full set of MUBs is a family of \( d(d+1) \) rank-1 projectors \( \Pi_{b,j} \), where \( b \in \{1, \ldots, d+1\} \) and \( j \in \{1, \ldots, d\} \), such that

\[ \langle \Pi_{b,j} | \Pi_{b',j'} \rangle = \begin{cases} 
\delta_{j,j'} & \text{for } b = b', \\
\frac{1}{d} & \text{for } b \neq b'. 
\end{cases} \]

If \( B_{b,j} \) are the corresponding Bloch vectors we have

\[ \langle B_{b,j} | B_{b',j'} \rangle_B = \begin{cases} 
d\delta_{j,j'} - 1 & \text{for } b = b', \\
-1 & \text{for } b \neq b'. 
\end{cases} \]

The vectors thus form \( d+1 \) orthogonal \( d-1 \) dimensional regular simplices with vertices in \( B \cap S_{\text{out}} \). A SIC is a POVM consisting of \( d^2 \) effects \( E_j = (1/d) \Pi_j \), where the \( \Pi_j \) are rank-1 projectors satisfying

\[ \langle \Pi_j | \Pi_j \rangle = \frac{d\delta_{j,j'} + 1}{d^2 + 1}. \]

If \( B_j \) are the corresponding Bloch vectors we have

\[ \langle B_j | B_{j'} \rangle_B = \frac{d^2\delta_{j,j'} - 1}{d^2 - 1}. \]

The vectors thus form a single \( d^2 - 1 \) dimensional regular simplex with vertices in \( B \cap S_{\text{out}} \). The existence problems for MUBs and SICs are still open, notwithstanding the enormous amount of theoretical work which has been devoted to them. Full sets of MUBs have been shown to exist in every prime power dimension [5–7], but not in any other dimension. Moreover there is much evidence [59–61] supporting the conjecture [3] that a full set of MUBs...
does not exist for $d = 6$. Turning to SICs, these have been constructed numerically [9, 10, 62] for every $d \leq 121$ and exact solutions [3, 10, 59, 63–65] have been constructed for $d = 2–16, 19, 24, 28, 35, 48$. This encourages the conjecture that SICs exist in every finite dimension.

The Bloch body picture provides us with a simple geometrical explanation of why the MUB and SIC existence problems are so hard. It is easy to construct vectors satisfying (10) and (12) if one only requires that they lie in the $d^2 - 2$ dimensional manifold $S_{out}$. What is hard (if $d > 2$) is then to rotate the vectors so that they all lie in the measure zero, $2d - 2$ dimensional submanifold $B \cap S_{out}$.

The triviality of the problem of inscribing a regular simplex into a sphere motivated one of us [41] to introduce the concept of a SIM, or arbitrary-rank symmetric informationally complete measurement (what in [41] was called an SI-POVM). Suppose that in the definition of a SIC one drops the requirement that the effects are rank 1, so that one is only looking for a POVM which is

(i) Informationally complete.

(ii) Symmetric in the sense that the effects satisfy $\langle E_j|E_{j'} \rangle = \alpha \delta_{jj'} + \beta$ for some $\alpha, \beta$.

In terms of the Bloch body description this means [41] that, instead of looking for Bloch vectors satisfying (12), one is only demanding

$$\langle B_j|B_{j'} \rangle_B = \frac{\kappa^2 (d^2 \delta_{jj'} - 1)}{d^2 - 1}$$

(13)

for some $\kappa$ in the interval $(0, 1]$. In other words one is still looking for a regular simplex in $B$; however, one no longer insists that the vertices lie on the out-sphere. We refer to such a structure as a SIM, and to $\kappa$ as the contraction parameter. As $\kappa$ is reduced the area of the intersection of the sphere of radius $\kappa$ with $B$ becomes larger, and so the problem of finding a SIM becomes easier. Moreover, the fact that $B_m \subseteq B$ means that the problem becomes trivial once $\kappa \leq 1/(d - 1)$.

In [41], in addition to the above simple, geometrical argument to show SIMs exist for all $d$ and all $\kappa \leq 1/(d - 1)$, it was also shown that for odd $d$ there exist SIMs with $\kappa = 1/\sqrt{d + 1}$ (termed Wigner POVMs on account of their intimate relation to the Wigner function). Gour and Kalev [54] subsequently constructed SIMs using a more complicated algebraic method, involving the generalized Gell–Mann matrices. In odd dimension greater than 3 their SIMs have a contraction parameter significantly less than $1/\sqrt{d + 1}$ and are therefore not an improvement on the Wigner POVM. In even dimension, on the other hand, their SIMs are an improvement on the ones constructed in [41]. For further discussion of SIMs and their applications see [26, 66, 67].

For the sake of completeness let us note that if $B_j$ is a set of Bloch vectors satisfying (13) then the corresponding SIM consists of the $d^c$ effects

$$E_j = \frac{1}{d^2} (I + B_j),$$

(14)

satisfying

$$\langle E_j|E_{j'} \rangle = \frac{d^2 \kappa^2 \delta_{jj'} + d + 1 - \kappa^2}{d^3 (d + 1)}.$$ 

(15)

A similar approach can be taken with MUBs. If we relax the requirement that the Bloch vectors lie on the out-sphere then (10) becomes
\[ \langle B_{b,j}|B_{b',j'}\rangle = \begin{cases} \frac{\kappa^2(d\delta_{j,j'} - 1)}{d - 1} & \text{for } b = b', \\ 0 & \text{for } b \neq b'. \end{cases} \] (16)

with \( \kappa \in (0, 1] \). Given a solution to these equations the operators

\[ E_{b,j} = \frac{1}{d}(I + B_{b,j}) \] (17)

form a full set of MUMs [42]. If \( \kappa < 1 \) the \( E_{b,j} \) are not rank-1 projectors. However, they still form a POVM for each fixed \( b \). Moreover, the POVMs are unbiased in the sense that, just as for a full set of MUBs, \( \langle E_{b,j}|E_{b',j'}\rangle = 1/d \) for \( b \neq b' \):

\[ \langle E_{b,j}|E_{b',j'}\rangle = \begin{cases} \kappa^2\delta_{j,j'} + \frac{1 - \kappa^2}{d} & \text{for } b = b', \\ \frac{1}{d} & \text{for } b \neq b'. \end{cases} \] (18)

As with SIMs one immediately sees, from the basic geometrical properties of the Bloch body, that full sets of MUMs exist for all \( \kappa \leq 1/(d - 1) \). In [42], where these structures were first introduced, it was shown that full sets of MUMs can in fact be constructed with

\[ \kappa = \sqrt{\frac{2}{d(d - 1)}} \] (19)

(note that the definition of \( \kappa \) in [42] is different from the one adopted here). For the application of MUMs to entanglement detection see [68].

4. Enter conical two-designs

A full set of MUBs, and the family of projectors defining a SIC, are examples of projective designs (recall that we are using ‘projective design’ as a shorthand for ‘complex projective design’). In this section we will introduce a more general kind of design having MUMs and SIMs as special cases.

A projective two-design is a non-empty family of rank-1 projectors \( \Pi_j \) such that \( \sum_j \Pi_j \otimes \Pi_j \) commutes with \( U \otimes U \) for every unitary \( U \). It is natural to ask what can be said of an arbitrary family of operators \( A_j \in \mathcal{L} \) having this property. Theorem 1 answers that question.

Before stating the theorem it will be convenient to introduce some notation. We define

- \( \Pi_{\text{sym}} \) and \( \Pi_{\text{asym}} \) to be, respectively, the projectors onto the symmetric and antisymmetric subspaces of \( \mathcal{H} \otimes \mathcal{H} \),
- \( \mathcal{W} \) to be the unitary swap operator on \( \mathcal{H} \otimes \mathcal{H} \) which takes \( |\psi\rangle \otimes |\phi\rangle \) to \( |\phi\rangle \otimes |\psi\rangle \),
- \( \mathcal{I} \) to be the identity superoperator on \( \mathcal{L} \).

Relative to some fixed ONB \( \{e_j\} \) we also define

- \( |\Phi_+\rangle \) to be the maximally entangled state \((1/\sqrt{d}) \sum_j |e_j\rangle \otimes |e_j\rangle \),
- \( A^* \) to be, for given \( A \in \mathcal{L} \), the operator \( \sum_j (e_j|A|e_j)^*|e_j\rangle \langle e_j| \),
- \( \mathcal{T} \) to be the transpose superoperator which acts on \( \mathcal{L} \) according to \( \mathcal{T}(|e_j\rangle\langle e_k|) = |e_k\rangle\langle e_j| \).
Theorem 1. Let \{A_1, \ldots, A_m\} be a family of operators in \(\mathcal{L}\). Then the following statements are equivalent

(i) \[ \sum_{j=1}^{m} A_j \otimes A_j \text{ commutes with } U \otimes U \text{ for every unitary } U. \]

(ii) For some \(k_+ \geq k_- \geq 0\)

\[ \sum_{j=1}^{m} A_j \otimes A_j = k_+ \Pi_{\text{sym}} + k_- \Pi_{\text{asym}}. \] (20)

(iii) For some \(k_+ \geq k_- \geq 0\)

\[ \sum_{j=1}^{m} A_j \otimes A_j^\dagger = k_+ I + d \kappa_+ |\Phi_+\rangle \langle \Phi_+|. \] (21)

(iv) For some \(k_+ \geq k_- \geq 0\)

\[ \sum_{j=1}^{m} |A_j\rangle \langle A_j| = k_+ |I\rangle \langle I| + k_- T. \] (22)

(v) For some \(k_+ \geq k_- \geq 0\)

\[ \sum_{j=1}^{m} |A_j\rangle \langle A_j| = k_+ |I\rangle \langle I| + k_+ I. \] (23)

If these equivalent conditions are satisfied, then the quantities \(k_\pm\) in conditions (iii)–(v) are the same, and are related to the quantities \(k_\nu, k_\sigma\) in condition (ii) by \(k_\pm = (k_\nu \pm k_\sigma)/2\). The \(A_j\) span \(\mathcal{L}_{sa}\) if and only if \(k_+ > k_-\) (equivalently, \(k_- > 0\)).

Proof. To see that (ii) \iff (iii) observe that (20) can be written

\[ \sum_{j=1}^{m} A_j \otimes A_j = k_+ I + k_- W \] (24)

with \(k_\pm = (k_\nu \pm k_\sigma)/2\). Taking the partial transpose on both sides we obtain (21).

Now let \(J\) be the familiar Choi–Jamiolkowski isomorphism \([69, 70]\) which takes a superoperator \(\Lambda\) to

\[ J(\Lambda) = \frac{1}{d} \sum_{j,k} \Lambda(\langle e_j| \langle e_k|) \otimes |e_j\rangle \langle e_k|. \] (25)

One easily verifies that

\[ J^{-1}(I) = d |I\rangle \langle I|, \] (26)

\[ J^{-1}(W) = d T, \] (27)

\[ J^{-1}(|\Phi_+\rangle \langle \Phi_+|) = I \] (28)

and \(\forall A, B \in \mathcal{L}_{sa},\)

\[ J^{-1}(A \otimes B) = d |A\rangle \langle B^*|. \] (29)

Consequently, applying \(J^{-1}\) to both sides of (24) gives (22) while applying it to both sides of (21) gives (23).
We have shown that statements (ii)–(v) are equivalent. The implication (ii) \(\Rightarrow\) (i) is immediate. So it only remains to show that (i) \(\Rightarrow\) (ii). To see this observe that (i) implies \[ A_k \Pi_{\text{sym}} + k_s \Pi_{\text{asym}}. \] (30)

To see that \(k_s\) and \(k_a\) satisfy the stated inequalities, let \(|\Psi\rangle\) be an arbitrary normalized element of the antisymmetric subspace of \(\mathcal{H} \otimes \mathcal{H}\). Then

\[ k_s = \sum_{j=1}^{m} \langle \Psi | A_j \otimes A_j | \Psi \rangle \geq 0. \] (31)

Moreover partially transposing and applying \(J^{-1}\) to (30) gives (23) with \(k_s = (k_s \pm k_a)/2\). So \((k_s - k_a)/2 = \sum \|\langle A_j | B \rangle\|^2\) for all normalized \(B \in \mathcal{L}_{\text{as},0}\). It follows that \(k_s \geq k_s, k_a\). It also follows that if \(k_s = k_a\) then the \(A_j\) are not a spanning set for \(\mathcal{L}_{\text{as}}\). If, on the other hand, \(k_s > k_a\) then it follows from (23) that \(\sum \|\langle A_j | B \rangle\|^2 \geq k_s \|B\|^2 > 0\) for all non-zero \(B \in \mathcal{L}_{\text{as}}\), implying that the \(A_j\) are a spanning set.

It is easily seen that the theorem generalizes to arbitrary families of operators in \(\mathcal{L}\) (with the appropriate modification of the conditions on \(k_s\) and \(k_a\)). Lemma 1 in [72] is a partial version of this more general result, in which the positivity requirement is relaxed, but not the requirement of self-adjointness, and in which only two of the five equivalent conditions are stated.

It will be seen that, up to normalization, the right-hand sides of (20) and (21) are, respectively, separable Werner states [44], and separable isotropic states [45]. We return to this point in section 6. It will also be seen that if \(k_s = k_a\) then, up to normalization, the right-hand side of (22) is the structural approximation to the transpose superoperator

\[ T = \frac{1}{d + 1} |I\rangle \langle I| + \frac{1}{d + 1} T \] (32)

introduced by Horodecki [73]. As Kalev and Bae [74] have noted this means that, if \(|\psi\rangle \ldots |\psi\rangle\) are the normalized vectors defining a sac, then \(T\) has the minimum cardinality Kraus decomposition in terms of \(B_j = (1/\sqrt{d})|\psi\rangle \langle \psi|\)

\[ T(A) = \sum_j B_j A B^\dagger_j. \] (33)

Theorem 1 suggests the following definition:

**Definition 1.** A conical two-design is a family of non-zero operators \(A_1 \ldots A_m \in \mathcal{C}\) satisfying the five equivalent conditions (i)–(v) in theorem 1 with \(k_s > k_a\) (equivalently, \(k_+ > 0\)).

The requirement that the \(A_j\) all be non-zero is not essential, and is made for convenience only. We require that \(k_s > k_a\) so as to ensure that the \(A_j\) are a spanning set. Note that if it were a projective two-design which was in question, so that the \(A_j\) were rank-1 projectors, it would be enough to require that the set was non-empty in order to ensure that it was a spanning set. But in the more general case that is no longer so.

In the same way one can define a conical \(t\)-design for \(t > 2\) to be a set of non-zero operators \(A_j \in \mathcal{C}\) such that \(\sum A_j^\otimes t\) commutes with every unitary of the form \(U^\otimes t\), and then use Schur–Weyl duality [75] to derive an analogue of (20). However, in this paper we will
confine ourselves to the case $t = 2$, deferring a consideration of the general case to a later publication.

The fact that the $A_j$ are a spanning set means that $m \geq d^2$. Using (23) one easily derives the following expansion formula for an arbitrary operator $L$

$$
L = \frac{1}{k_-} \sum_j \left( \text{Tr}(A_j L) - \frac{k_+ \text{Tr}(A_j) \text{Tr}(L)}{dk_+ + k_-} \right) A_j.
$$

(34)

If $m = d^2$ the expansion is unique; otherwise not.

Obviously every projective two-design is a conical two-design. More generally a conical two-design $A_j$ has $k_+ = 0$ if and only if the $A_j$ are all rank 1, so that when suitably re-scaled they form a weighted projective two-design. In fact, taking the trace on both sides of (20) and (23) gives

$$
\sum_j ((\text{Tr}(A_j))^2 - \text{Tr}(A_j^2)) = d(d - 1)k_+.
$$

(35)

So $k_+ = 0$ if and only if $(\text{Tr}(A_j))^2 = \text{Tr}(A_j^2)$ for all $j$, which in turn is true if and only if the $A_j$ are all rank 1.

It is easily seen that a full set of MUMs is a conical two-design. In fact, let $E_{b_j}$ be such a set. If we define $N = \sum_{b_j} |E_{b_j}\rangle \langle E_{b_j}|$ it follows from (18) that

$$
N|E_{b_j}\rangle = \frac{1}{d} (d + 1 - \kappa^2 |I\rangle + \kappa^2 |E_{b_j}\rangle).
$$

(36)

Since the $E_{b_j}$ are a spanning set this means

$$
N = \frac{1}{d} (d + 1 - \kappa^2 |I\rangle + \kappa^2 |I\rangle).
$$

(37)

The claim now follows from theorem 1.

A similar argument shows that every SIM is a conical two-design. However, for SIMs we have the stronger statement, that a POVM of cardinality $d^2$ is a conical two-design if and only if it is a SIM. To show that the condition is necessary as well as sufficient let $E_1, \ldots, E_{d^2}$ be a conical two-design which is also a POVM. (23) implies

$$
\sum_{j=1}^{d^2} \text{Tr}(E_j)E_j = (dk_+ + k_- |I\rangle = \sum_{j=1}^{d^2} (dk_+ + k_-)E_j.
$$

(38)

Since the $E_j$ are a basis this means we must also have $\text{Tr}(E_j) = dk_+ + k_-$. The fact that $\text{Tr}(E_j)$ is a constant means we must also have $\text{Tr}(E_j) = 1/d$. Taking account of the inequalities $k_+ \geq k_- > 0$ we deduce

$$
k_+ = \frac{d + 1 - \kappa^2}{d^2(d + 1)},
$$

(39)

$$
k_- = \frac{\kappa^2}{d(d + 1)}
$$

(40)

for some $\kappa \in (0, 1]$. By another application of (23)

$$
\sum_{k=1}^{d^2} \text{Tr}(E_kE_j)E_k = \sum_{k=1}^{d^2} \left( \frac{k_+}{d} + k_+ \delta_{j,k} \right) E_k.
$$

(41)
Since the $E_j$ are a basis this means
\[ \text{Tr}(E_j E_k) = \frac{d^2 \kappa^2 \delta_{jk} + d + 1 - \kappa^2}{d^3 (d + 1)}. \] (42)

Comparing with (15) we see that the $E_j$ are a SIM.

It is not possible to prove an equally strong statement for MUMs. A full set of MUMs, scaled by a factor of $1/(d + 1)$, is a POVM and a conical two-design. However, there are other POVMs of cardinality $d(d + 1)$ which are conical two-designs. For instance, if $E_j$ is a SIM then the POVM with effects
\[ E'_j = \begin{cases} \frac{1}{2} E_j & \text{for } 1 \leq j \leq d^2, \\ \frac{1}{2d} I & \text{for } d^2 < j \leq d(d + 1) \end{cases} \] (43)
is a conical two-design of cardinality $d(d + 1)$.

At this stage it will be helpful to introduce some new notation. Let $A_1, \ldots, A_m$ be an arbitrary conical two-design, and let $t_j = \text{Tr}(A_j)$. Then $A_j$ has the Bloch representation
\[ A_j = \frac{t_j}{d} (I + B_j). \] (44)

Define $\kappa_j = \|B_j\|_B$ and
\[ t = \sqrt{\frac{1}{m} \sum_j t_j^2}, \] (45)
\[ \kappa = \sqrt{\frac{1}{mt} \sum_j t_j^2 \kappa_j^2}. \] (46)

So $t$ is the rms trace, and $\kappa$ is the weighted rms Bloch vector norm. Note that $\kappa \in (0, 1]$ and that $\kappa = 1$ if and only if the $A_j$ are all rank 1. As with SIMs and MUMs we will refer to $\kappa$ as the contraction parameter. Taking the trace on both sides of (20) and (23) we find
\[ \frac{1}{2} d(d + 1)k_x + \frac{1}{2} d(d - 1)k_x = \sum_j (\text{Tr}(A_j))^2, \] (47)
\[ \frac{1}{2} d(d + 1)k_x - \frac{1}{2} d(d - 1)k_x = \sum_j \text{Tr}(A_j^2), \] (48)

from which it follows
\[ k_x = \frac{mt^2}{d^2} \left( 1 + \frac{(d - 1) \kappa^2}{d} \right), \] (49)
\[ k_a = \frac{mt^2 (1 - \kappa^2)}{d^2}. \] (50)

Taking a partial trace on both sides of (20) we find
\[ \sum_j t_j A_j = \frac{mt^2}{d} I. \] (51)
It follows that the operators
\[ E_j = \frac{dt_j}{mt^2} A_j \] (52)
constitute a POVM. In the case when the \( A_j \) have constant trace (but not in general) this POVM is also a conical two-design.

Lastly, the Bloch vectors satisfy
\[ \sum_j c_j^2 B_j = 0, \] (53)
\[ \sum_j c_j^2 |B_j\rangle\langle B_j| = \frac{m d t^2 \kappa^2}{d + 1} \Pi_B, \] (54)

where
\[ \Pi_B = I - \frac{1}{d} |I\rangle\langle I| \] (55)
is the Bloch projector, i.e. the projector onto the subspace \( \mathcal{L}_{\text{sa},0} \).

5. Bloch geometry in the homogeneous case

The class of all conical two-designs is large, and to make progress one needs to focus on special cases. One important special case is the class of weighted projective two-designs, concerning which much is known \[4, 38–40\]. In this section we consider another special case. Specifically, we consider conical two-designs which are homogeneous in the sense that \( \text{Tr}(A_j) \) and \( \text{Tr}(A_j^2) \) are constant, so that \( t_j = t \), \( \kappa_j = \kappa \) for all \( j \). This class of two-designs includes SIMs and full sets of MUMs. It also includes all projective two-designs. Specifically the projective two-designs are precisely the homogeneous conical two-designs for which
\[ t = \kappa = 1 \] (56)
or, equivalently
\[ k_s = \frac{2m}{d(d + 1)} \quad \text{and} \quad k_a = 0. \] (57)

In the remainder of this section we study the Bloch geometry of a homogeneous conical two-design. We know that the Bloch vectors of SIMs and full sets of MUMs form polytopes having a simple geometrical description. We would like to describe the polytope corresponding to an arbitrary homogeneous conical two-design. The geometry of the polytope is fully specified by the Gram matrix \( G \) with matrix elements
\[ G_{jk} = \langle B_j|B_k \rangle. \] (58)

We will therefore focus on the problem of characterizing this matrix.

**Theorem 2.** Let \( B_1, \ldots, B_m \) be a set of vectors in \( \mathcal{B} \). Then the following statements are equivalent

(i) The \( B_j \) are the Bloch vectors of a homogeneous conical two-design.

(ii) Their Gram matrix is of the form
\[ G = \lambda P, \] (59)
where $\lambda$ is a positive constant and $P$ is a rank $d^2 - 1$ projector which is constant on the diagonal and such that $\sum_k P_{jk} = 0$ for all $j$. If these equivalent conditions are satisfied then $\lambda \leq md/(d+1)$ and $P_{jk} \leq (d^2 - 1)/m$ for all $j, k$ with equality when $j = k$. The associated conical two-designs have contraction parameter

$$\kappa = \sqrt{\frac{\lambda(d+1)}{md}}. \quad (60)$$

**Remark.** Notice that if the conditions of the theorem are satisfied then there are infinitely many conical two-designs with Bloch vectors $B_j$ since the trace $t$ in (61) can take any positive value.

**Proof.** To show that (i) $\implies$ (ii), suppose

$$A_j = \frac{1}{d} (I + B_j) \quad (61)$$

is a homogeneous conical two-design with contraction parameter $\kappa$. It follows from (54) that

$$\sum_j B_j = 0, \quad (62)$$

$$\sum_j |B_j\rangle\langle B_j| = \lambda I_G, \quad (63)$$

where $\lambda = md\kappa^2/(d+1)$. So

$$G^2 = \lambda G, \quad (64)$$

implying that $P = (1/\lambda)G$ is a projection operator. Taking the trace on both sides of (63) we find

$$\text{Tr}(P) = \frac{1}{\lambda} \sum_j G_{jj} = d^2 - 1. \quad (65)$$

So $P$ is rank $d^2 - 1$. Moreover

$$P_{jj} = \frac{1}{\lambda} \langle B_j | B_j \rangle = \frac{d^2 - 1}{m}, \quad (66)$$

for all $j$. So $P$ is constant on the diagonal. Finally, it follows from (53) that $\sum_k B_k = 0$, implying that $\sum_k P_{kj} = 0$ for all $j$.

To show that (ii) $\implies$ (i), suppose the Gram matrix has the stated form. Observe that the fact that the rank of the Gram matrix is $d^2 - 1$ means that the $B_j$ are a spanning set for $L_{sa,0}$. Let $N = \sum_j |B_j\rangle\langle B_j|$. Then

$$\langle B_j | N | B_k \rangle = \lambda \langle B_j | B_k \rangle \quad (67)$$

for all $j, k$. Since the $B_j$ are a spanning set for $L_{sa,0}$, and since $N|I\rangle = 0$, this implies

$$\sum_j |B_j\rangle\langle B_j| = \lambda I_G. \quad (68)$$

The fact that $\sum_k P_{jk} = 0$ means $\sum_k B_k = 0$. So if we define $A_j = (t/d)(I + B_j)$ for any fixed positive $t$ we will have
\[ \sum_j |A_j\rangle \langle A_j| = \frac{\lambda^2 (md - \lambda)}{d^3} |I\rangle \langle I| + \frac{\lambda^2}{d^2}. \]  

(69)

If we can show that \( \lambda \leq md/(d+1) \) it will follow that the \( A_j \) are a conical two-design. To see that this is the case observe that the fact that \( P \) is constant on the diagonal means

\[ mP_{jj} = \text{Tr}(P) = d^2 - 1 \]  

for all \( j \). Consequently

\[ 1 \geq \|B_j\|^2 = \frac{\lambda P_{jj}}{d(d-1)} = \frac{\lambda(d+1)}{md} \]  

from which the claim follows. We have incidentally shown that the two-design is homogeneous, with contraction parameter

\[ \kappa = \sqrt{\frac{\lambda(d+1)}{md}}. \]  

(72)

To prove the last part of theorem observe that the only statement not proved in the course of establishing the implication (ii) \( \Rightarrow \) (i) is the bound on the matrix elements of \( P \). This is an immediate consequence of the fact that \( \|B_j\|^2 = \lambda(d+1)/(md) \).

Let \( \mathcal{P}_m \) be the set of all \( m \times m \) rank \( d^2 - 1 \) projectors \( P \) with the properties:

1. \( \forall j : \sum_k P_{jk} = 0 \),

2. \( \forall j, k : |P_{jk}| \leq \frac{d^2 - 1}{m} \), with equality when \( j = k \).

We have shown that some projectors of this type are associated to homogeneous conical two-designs via (59). It remains to show that all of them are, for every \( m \geq d^2 \).

For given \( m \geq d^2 \) and \( P \in \mathcal{P}_m \) let \( S(P) \) be the set of all \( m \)-tuples of associated vectors in \( \mathcal{B} \). Thus \( B = (B_1, \ldots, B_m) \in S(P) \) if and only if

\[ \langle B_j | B_k \rangle = \lambda P_{jk} \]  

(75)

for some positive \( \lambda \). Also define, for each \( B \in S(P) \)

\[ \kappa_B = \|B_1\|_\mathcal{B} = \cdots = \|B_m\|_\mathcal{B} \]  

(76)

and let \( K(P) = \{ \kappa_B : B \in S(P) \} \). The convexity of the Bloch body means that if \( B \in S(P) \) then so does \( \eta B \) for all \( \eta \in (0, 1] \) (so \( S(P) \) is either empty or infinite). It follows that if \( \kappa \in K(P) \) then \((0, \kappa] \subseteq K(P) \). So if

\[ c_P = \begin{cases} 
\sup(K(P)) & K(P) \text{ non-empty} \\
0 & K(P) \text{ empty} 
\end{cases} \]  

(77)

then \((0, c_P] \subseteq K(P) \subseteq (0, c_P] \). We claim that in fact \( K(P) = (0, c_P] \). The claim is trivial if \( c_P = 0 \), so we may assume without loss of generality that \( c_P > 0 \). Choose a sequence \( B_n \in S(P) \) such that \( \kappa_B \uparrow c_P \). Since \( \mathcal{B}^m \) is a closed, bounded subset of a finite dimensional, real inner-product space it is compact [76]. We can therefore choose a convergent subsequence \( B_{n_k} \rightarrow B \in \mathcal{B}^m \). We have
\[
\langle B_j | B_k \rangle = \lim_{a \to \infty} \left( \frac{mdc_P^2}{d+1} \right) P_{jk} = \frac{mdc_P^2}{d+1} P_{jk}.
\]  
(78)

So \( B \in S(P) \) and \( c_P = \kappa_B \in K(P) \).

It is known \([11, 12]\) that projective two-designs exist in every dimension (although it should be noted \([46]\) that the cardinality of the projective two-designs constructed in these papers grows extremely fast with dimension). Since \( c_P = 1 \) for the projector corresponding to a projective two-design this means that homogeneous conical two-designs exist in every dimension and for every \( \kappa \in (0, 1] \).

We are now in a position to prove the second main result of this section (which can be regarded as a generalization of the existence proofs for SIMs and full sets of MUMs).

**Theorem 3.** For all \( m \geq d^2 \) and \( P \in \mathcal{P}_m \)

\[
c_P \geq \frac{1}{d - 1}.
\]  
(79)

In particular \( S(P) \) is non-empty.

**Proof.** The fact that \( P \) is a rank \( d^2 - 1 \) projector means we can choose \( d^2 - 1 \) orthonormal vectors \( \tilde{u}_a \in \mathbb{R}^m \) such that

\[
P_{jk} = \sum_a u_{a,j} \tilde{u}_{a,k}.
\]  
(80)

Let \( D_1, \ldots, D_{d^2-1} \) be an orthonormal basis for \( L_{sa,0} \) and define \( B = \langle B_1, \ldots, B_m \rangle \) by

\[
B_j = \frac{md}{(d+1)(d-1)^2} \sum_{a=1}^{d^2-1} u_{a,j} D_a.
\]  
(81)

Then

\[
\langle B_j | B_k \rangle = \frac{md}{d+1} (d-1)^2 P_{jk}.
\]  
(82)

In particular \( \| B_j \|_B = 1/(d - 1) \), implying that \( B_j \in B \subseteq S \). So \( B \in S(P) \) and \( 1/(d - 1) = \kappa_B \leq c_P \).

In this paper we are mainly focusing on conical two-designs. However, the result just established is potentially relevant to the problem of constructing projective two-designs. The projectors in \( \mathcal{P}_m \) which correspond to projective two-designs are precisely the ones for which \( c_P = 1 \). This suggests the following program:

(i) Classify the polytopes described by the projectors in \( \mathcal{P}_m \).
(ii) Identify those polytopes for which \( c_P = 1 \).

This program is, of course, extremely ambitious as success would carry with it, as a minor corollary, solutions to the MUB and SIC existence problems. However, even some partial results might be useful. It might, for instance, be useful if one could exclude some of the projectors in \( \mathcal{P}_m \), as definitely not having \( c_P = 1 \). One obvious way to do this is to exploit the fact \([52]\) that each vertex of the polytope corresponding to a projective two-design must be diametrically opposite a face which is tangential to \( S_m \). Having narrowed down the set of candidates, one might then investigate the remaining polytopes numerically, to see if any of them correspond to projective two-designs in low dimension. In essence, this procedure—
writing down a set of equations motivated by considerations of symmetry, and then looking for solutions in low dimension—was the way SICs were originally found \[3, 9\]. The same procedure might possibly be used to find other projective two-designs.

We conclude this section with a result which says that the problem of constructing a homogeneous two-design in a complex vector space reduces to the problem of constructing a one-design in a higher-dimensional real vector space.

**Theorem 4.** Let \(B_1, \ldots, B_m\) be a set of Bloch vectors. Then the following statements are equivalent

(i) The \(B_j\) are the Bloch vectors of a homogeneous conical two-design.

(ii) The \(B_j\) have the same norm and satisfy

\[
\sum_j B_j = 0, \tag{83}
\]

\[
\sum_j |B_j\rangle\langle B_j| = \lambda \Pi_B \tag{84}
\]

for some \(\lambda > 0\).

**Proof.** The implication (i) \(\Rightarrow\) (ii) is an immediate consequence of (53) and (54). To prove the converse let \(B_j\) be a set of Bloch vectors having the stated properties and define

\[
P_{jk} = \frac{1}{\lambda} \langle B_j | B_k \rangle. \tag{85}
\]

Then it follows from (84) that \(P^2 = P\), implying that \(P\) is a projector. Taking the trace on both sides of (84) we find

\[
\text{Tr}(P) = \frac{1}{\lambda} \sum_j \langle B_j | B_j \rangle = \text{Tr}(\Pi_B) = d^2 - 1. \tag{86}
\]

So \(P\) is rank \(d^2 - 1\). The fact that the \(B_j\) have the same norm means that \(P\) is constant on the diagonal, while (83) implies that \(\sum_j P_{jk} = 0\) for all \(j\). So it follows from theorem 2 that the \(B_j\) are the Bloch vectors of a conical two-design. \(\square\)

6. Werner and isotropic states

In section 4 we observed that, up to normalization, the right-hand sides of (20) and (21) are, respectively, separable Werner states \[44\], and separable isotropic states \[45\]. This merits a little discussion.

A Werner state is one of the form

\[
\rho_W = k_s \Pi_{\text{sym}} + k_a \Pi_{\text{asym}} \tag{87}
\]

with

\[
k_s = \frac{2(1 - p)}{d(d + 1)} \quad k_a = \frac{2p}{d(d - 1)} \tag{88}
\]

for some \(p \in [0, 1]\). The state is entangled if and only if \(p \in (1/2, 1]\). The entangled states are the ones of most interest since, in addition to Werner’s original motivation, it can be shown \[45\] that the existence of bound-entangled NPT states is equivalent to the existence of bound-entangled Werner states. The existence of the latter is still an open question, but there
are indications [77, 78] that the entanglement becomes bound as one approaches the cross-over point at $p = 1/2$. As we remarked in the introduction conical two-designs can be used to provide simple decompositions of all Werner states, both separable and entangled (although it remains to be seen how interesting they are). However, we will here confine ourselves to the point, which is already apparent from the definition, that they provide simple decompositions of some of the separable states. In this connection let us observe that, although less interesting, the problem of decomposing a separable Werner state is not straightforward, and has attracted some notice in the literature [55–57]. Conical two-designs cast additional light on the problem.

Let us define a symmetric decomposition of a separable Werner state to be one of the form

$$\rho_W = \sum_{j=1}^{m} \lambda_j \rho_j \otimes \rho_j,$$  \hspace{1cm} (89)

where $\rho_j \in \mathcal{Q}$, $\lambda_j \in (0, 1]$ and $\sum_j \lambda_j = 1$. We will say that the decomposition is homogeneous if $\lambda_j = 1/m$ for all $j$, and that it is pure if the $\rho_j$ are all pure. It follows from theorem 1 that $\rho_W$ does not have a symmetric decomposition if $k_s < k_a$ or, equivalently, if $p > (d - 1)/(2d)$. If $p = (d - 1)/(2d)$ then $\rho_W$ is the maximally mixed state, so the existence of a symmetric decomposition is trivial. If $p < (d - 1)/(2d)$ then (89) is equivalent to the statement that the operators $A_j = \sqrt[\lambda_j]{\rho_j}$ are a conical two-design.

It was shown in section 5 that homogeneous conical two-designs exist for all $d$ and all $\kappa \in (0, 1]$. We conclude that a separable Werner state has a symmetric decomposition if and only if $0 \leq p \leq (d - 1)/(2d)$. Furthermore, if $p$ is in this interval the decomposition can always be chosen to be homogeneous. Finally, it was shown in section 4 that a conical two-design is rank 1 if and only if $k_a = 0$ (in which case it is essentially the same thing as a weighted projective two-design [4, 38–40]). So $\rho_W$ has a pure symmetric decomposition if and only if $p = 0$.

We have thus shown that the interval $0 \leq p \leq 1/2$ splits into two sub-intervals separated by the maximally mixed state at $p = (d - 1)/(2d)$. States in the sub-interval $0 \leq p \leq (d - 1)/(2d)$ do have symmetric decompositions; states in the sub-interval $(d - 1)/(2d) \leq p \leq 1/2$ do not. One motivation for studying separable Werner states is the hope that, by looking at the states immediately below the cross-over at $p = 1/2$, one may get some insight into the bound-entangled states conjectured to exist just above it. From this point of view the most interesting feature of our discussion is the negative statement, that states immediately below the cross-over cannot be put into the simple form of (89).

In the case $p < (d - 1)/(2d)$ we define an ideal decomposition to be one which is symmetric, homogeneous and such that $m$ achieves its minimum value of $d^2$. A homogeneous conical two-design is a POVM up to re-scaling, so we can use one of the results proved in section 4 to conclude that an ideal decomposition must be of the form

$$\rho_W = \sum_{j=1}^{d^2} E_j \otimes E_j,$$ \hspace{1cm} (90)

where the $E_j$ constitute a SIM. In view of the discussion in section 5 this gives us the following reformulation of the SIC-existence problem: a SIC exists in dimension $d$ if and only if every Werner state with $0 \leq p < (d - 1)/2d$ has an ideal decomposition.

Conical two-designs can also be used to give simple decompositions of a subset of the isotropic states introduced in [45]. The states are defined by
\[ \rho_1 = \frac{1 - F}{d^2 - 1} I + \frac{d^2 F - 1}{d^2 - 1} |\Phi_+\rangle \langle \Phi_+ | \]  

(91)

with \( F \in [0, 1] \) and \(|\Phi_+\rangle\) the maximally entangled state defined at the beginning of section 4. They are separable for \( F \in [0, 1/d] \) and entangled for \( F \in (1/d, 1] \) (they are not, however, bound-entangled for any value of \( F \)). We define a symmetric decomposition of an isotropic state to be one of the form

\[ \rho_1 = \sum_{j=1}^{m} \lambda_j \rho_j \otimes \rho_j^\dagger, \]  

(92)

where \( \rho_j \in \mathcal{Q} \), \( \lambda_j \in (0, 1] \) and \( \sum_j \lambda_j = 1 \). Symmetric decompositions of isotropic states are in bijective correspondence with symmetric decompositions of Werner states. In fact let

\[ k_s \Pi_{\text{sym}} + k_a \Pi_{\text{asym}} = \sum_j \lambda_j \rho_j \otimes \rho_j \]  

(93)

be a symmetric decomposition of a Werner state with \( p \) in the interval \([0, (d - 1)/(2d)]\). Taking the partial transpose on both sides gives

\[ k_s I + d k_s |\Phi_+\rangle \langle \Phi_+ | = \sum_j \lambda_j \rho_j \otimes \rho_j^\dagger, \]  

(94)

where \( k_{\pm} = (k_s \pm k_a)/2 \). The fact that \( 0 \leq p \leq (d - 1)/(2d) \) means \( 1/(d(d + 1)) \leq k_+ \leq 1/d^2 \). So we obtain in this way a symmetric decomposition of every isotropic state with \( 1/d^2 \leq F \leq 1/d \). Reversing the argument it can be seen that, if one had a symmetric decomposition of an isotropic state with \( 0 \leq F < 1/d^2 \), then taking the partial transpose would give a symmetric decomposition of a Werner state with \( (d - 1)/(2d) < p \leq 1/2 \)—which we have shown to be impossible.

Similarly to the Werner case, we see that the interval \( 0 \leq F \leq 1/d \) corresponding to the separable states splits into two sub-intervals, situated either side of the maximally mixed state at \( F = 1/d^2 \). States in the sub-interval \( 1/d^2 \leq F \leq 1/d \) do have symmetric decompositions; states in the sub-interval \( 0 \leq F < 1/d^2 \) do not. The difference with the Werner case is that it is now the states with a symmetric decomposition which lie next to the set of entangled states.

7. Conclusion

We introduced a new class of geometric structures in quantum theory, conical designs, which are natural generalizations of projective designs. We showed that SIMs and MUMs are special cases, as are weighted projective two-designs (up to re-scaling). We began by establishing their basic properties. In particular we gave five equivalent conditions for a set of positive semi-definite operators to be a conical two-design (theorem 1). We then turned to the special case of homogenous conical two-designs, and analyzed their Bloch geometry. In the Bloch body picture SIMs and full sets of MUMs form simple, highly symmetric polytopes (a single regular simplex in the case of SIMs; the convex hull of a set of orthogonal regular simplices in the case of MUMs). We showed that the same is true of an arbitrary homogeneous conical two-design. Moreover, we derived necessary and sufficient conditions for a given polytope to be such a design (theorems 2 and 3). We also showed how the problem of constructing a homogeneous two-design in a complex vector space reduces to the problem of constructing a spherical one-design in a higher dimensional real vector space (theorem 4). Finally, we showed that conical two-designs provide simple decompositions of some separable Werner and isotropic states.
We show in a subsequent publication [27] that conical two-designs are deeply implicated in the description of entanglement. There are other questions which might be interesting to investigate. Firstly, there is our suggestion in section 5, that the results there proved could be used to search systematically for new projective two-designs. Secondly, all known examples of SCS and full sets of MUBs have important group covariance properties [79]. One would like to know how far this holds true in the more general setting of homogeneous conical two-designs. Thirdly, one would like to extend the analysis to conical $t$-designs with $t > 2$ via Schur–Weyl duality [75]. Fourthly, it is to be observed that the full class of conical two-designs is itself a convex set. It might be interesting to explore the geometry of that set. For instance, one might try to characterize the extreme points. Finally, it would be interesting to investigate conical designs in the larger context of general probabilistic theories [80].

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