SINGULAR YAMABE PROBLEM FOR SCALAR FLAT METRICS ON THE SPHERE

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Abstract. Let \( \Omega \subset \mathbb{S}^n \) be a domain of unit \( n \)-sphere and \( \hat{g} \) the standard metric of \( \mathbb{S}^n \), \( n \geq 3 \). We show that there exists conformal metric \( g \) with vanishing scalar curvature such that \((\Omega, g)\) is complete if and only if the Bessel capacity \( C_{\alpha,q}(\mathbb{S}^n \setminus \Omega) = 0 \), where \( \alpha = 1 + \frac{2}{n} \) and \( q = \frac{2}{n} \). Our analysis utilizes some well known properties of capacity and Wolff potentials, as well as ideas used to characterize the existence of negative scalar metric developed by Labutin in [La05].

1. Introduction

Let \( \Omega \) be an open subset of unit sphere \( \mathbb{S}^n \), \( n \geq 3 \) and \( \hat{g} \) the standard metric of \( \mathbb{S}^n \). We want to characterize the open sets \( \Omega \) with the following property: there exists a metric \( g \), conformal to \( \hat{g} \) such that \( (\Omega, g) \) is complete and \( g \) has vanishing scalar curvature. This question was studied by Schoen and Yau [Sch87], [SchY88]. If we are given a compact Riemannian manifold \( (M, g) \) then the question of existence of conformal deformation of metric into complete metric \( \bar{g} \) with constant scalar curvature is known as the Yamabe problem [Yam60]. Yamabe’s original approach was to formulate this as a variational problem. Later contributions of Trudinger [T67], Aubin [A76] and Schoen [Sch84] helped to complete Yamabe’s original approach.

If we impose structural assumptions on \( K := \mathbb{S}^n \setminus \Omega \), then it is known that there is a \( g \) with above properties. In particular, if \( K \) is a finite union of Lipschitz submanifolds of dimension \( k \leq (n - 2)/2 \) then this is indeed the case, see [De92], [KaN93], [M-McO92]. Some discussion on this and related open problems are contained in [McO98]. For a survey of related results see [G16], [La05] and references therein.

The aim of this work is to give a complete characterization of open set \( \Omega \) without any assumption on the structure of \( K = \mathbb{S}^n \subset \Omega \). Such characterization for negative scalar case was given by Labutin [La05]. In what follows \( \text{cap}(\cdot) := C_{1+\frac{2}{n},\frac{2}{n}}(\cdot) \) stands for Bessel’s capacity (see section 2 for precise definition). Our main result is

**Theorem 1.1.** Let \( \Omega \subset \mathbb{S}^n, n \geq 3 \), be an open set and \( K = \mathbb{S}^n \setminus \Omega \). Then the following properties are equivalent:

(i) In \( \Omega \) there exists a scalar flat complete metric conformal to \( \hat{g} \).

(ii) \( \text{cap}(K) = 0 \).

The proof is based on a characterization of Bessel’s capacities in terms of the Wolff potential [HWo83]. One of the main ingredients of the proof is the representation of positive harmonic functions in terms of Martin kernels [ArG], [He].
2. Background

In this section we recall some well known facts from conformal geometry which can be found in [SchY]. Let \((M, g)\) be a Riemannian manifold of dimension \(n \geq 3\). The operator

\[
L_g = -4\frac{n-1}{n-2} \Delta u_g + R_g
\]

is called conformal Laplacian. Here \(R_g\) is the scalar curvature of the metric \(g\) and \(\Delta_g\) is the Laplace-Beltrami operator. \(L_g\) has remarkable properties: under conformal change of metric \(\tilde{g} = \phi^{\frac{4}{n-2}} g, \phi \in C^\infty(M), \phi > 0\) we have

\[
R_{\tilde{g}} = \phi^{-\frac{n+2}{n-2}} L_g \phi, \quad L_{\tilde{g}} v = \phi^{-\frac{n+2}{n-2}} L_g(\phi v).
\]

More generally, let \(\tilde{M}\) be another manifold with the metric \(\tilde{g}\), and let \(f: M \to \tilde{M}\) be a diffeomorphism changing the metrics conformally. If \(f^* \tilde{g} = \tilde{g} \circ f\) is the pull-back then

\[
f^*(R_{\tilde{g}}) = \phi^{-\frac{n+2}{n-2}} L_g \phi, \quad f^*(L_{\tilde{g}} v) = \phi^{-\frac{n+2}{n-2}} L_g(\phi f^* v).
\]

Next we define the stereographic projection to be \(\sigma: \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n\) where \(N\) is the north pole. \(\sigma\) is a diffeomorphism between \((\mathbb{S}^n \setminus \{N\}, \tilde{g})\) and \((\mathbb{R}^n, g_E)\) because

\[
(\sigma^{-1})^* \tilde{g} = \left(\frac{2}{1 + |x|^2}\right)^2 g_E = U^{\frac{4}{n-2}},
\]

where

\[
U(x) := \left(\frac{2}{1 + |x|^2}\right)^{\frac{n-2}{2}} x \in \mathbb{R}^n.
\]

Since we consider the scalar flat case, i.e. \(R_{\tilde{g}} = 0\), then (2.5) yields

\[
L_{\tilde{g}} v = 0, \quad v > 0 \text{ in } \Omega.
\]

Introduce the function

\[
u(x) = U(x)(\sigma^{-1})^* v(x) = U(x)v(\sigma^{-1} x), \quad x \in \mathbb{R}^n.
\]

Then from (2.3) we obtain

\[
L_{g_E} v = 0, \quad v > 0 \text{ in } \sigma(\Omega) \subset \mathbb{R}^n.
\]

Since \(R_{g_E} = 0\) then we get

\[
\Delta_{g_E} v = 0, \quad v > 0 \text{ in } \Omega.
\]
2.1. Characterization of capacity. Let $\alpha > 0, 1 < q \leq \frac{n}{2}$ and $C_{\alpha,q}(E)$ be the Bessel capacity of $E \subset \mathbb{R}^n$ [HWo83]. For given Radon measure $\mu$ we can consider the Wolff potentials defined as

\[
W_{\alpha,q}(\mu, x) = \int_0^1 \left( \frac{\mu(B(\delta, x))}{\delta^{n-\alpha q}} \right)^{p-1} \frac{d\delta}{\delta}
\]

where $p + q = pq$.

Then we say that a set $E$ is $(\alpha, q)$-thin at $x_0 \in E$ if and only if there is a Radon measure $\mu$ such that

\[
W_{\alpha,q}(\mu, x_0) < \liminf_{x \in E \setminus \{x_0\}, x \to x_0} W_{\alpha,q}(\mu, x).
\]

In what follows we take $\alpha = 1 + \frac{2}{n}, q = \frac{n}{2}$ and denote $\text{cap}(\cdot) = C_{1+\frac{2}{n}, \frac{n}{2}}(\cdot)$. Then for this choice of parameters the Wolff potential takes the form

\[
W(\mu, x) = \int_0^1 \left( \frac{\mu(B(x, r))}{r^{\frac{n-2}{2}}} \right)^{\frac{2}{n-2}} \frac{dr}{r}.
\]

In view of theorem 4 [HWo83] we have

**Proposition 2.1.** Let $K \subset \mathbb{R}^n$ be a compact set such that $\text{cap}(K) = 0$. Then there exists a Radon measure $\mu, \|\mu\| = 1$, such that supp $\mu \subset K$ and

\[
W(\mu, x) = +\infty \quad \text{for all} \quad x \in K.
\]

3. Proof of $(i) \Rightarrow (ii)$: existence of $u$ implies $\text{cap}(K) = 0$

We can use the stereographic projection and without loss of generality assume that the north pole $N \in \Omega$ such that $\sigma(K) \subset B(0, 1/2)$ and there is $u : \mathbb{R}^n \setminus \sigma(K)$ such that $g = u^{\frac{4}{n-2}} \tilde{g}$ is complete where

\[
\Delta u = 0, \quad \text{in} \quad \Omega = \{u > 0\}.
\]

We claim that there exists a Radon measure $\mu$ with supp $\mu \subset K$,

\[
u(x) = \int_{\mathbb{R}^n} k(x, y) d\mu(y) \quad \forall x \in B(0, 3) \setminus K,
\]

where $k(x, y)$ is the Martin kernel (see [ArG] Theorem 8.4.1 or Chapter 12 [He] p 251) and $k$ is locally integrable in $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, there two universal constants $c_1, c_2$ such that

\[
\frac{c_1}{|x-y|^{n-2}} \leq k(x, y) \leq \frac{c_2}{|x-y|^{n-2}}.
\]

Suppose $\text{cap}(K) > 0$. By Proposition 2.1 the Wolff potential of $\mu$ must be finite at some point $x_0 \in K$. Without loss of generality we assume that $x_0 = 0$ and $\mu$ is a probability measure such that

\[
W(\mu, 0) < +\infty, \quad 0 \in K.
\]

We first establish a technical

**Lemma 3.1.** Let $u > 0$ be as above and supp $\mu \subset B(0, 1/2)$. Then there is a constant $C > 0$ such that

\[
\int_{B(0,1)} \frac{(u(x))^{\frac{n-2}{2}}}{|x|^{n-1}} \, dx \leq CW(\mu, 0).
\]
Proof. Denote $D_m = B(0, \rho_m) \setminus B(0, \rho_{m+1}), \rho_m = 2^{-m}$. We have
\[
\int_{B(0,2)} \frac{u(x)^{\frac{2}{n-2}}}{|x|^{n-1}} dx = \sum_{m=0}^{\infty} \int_{D_m} \frac{u(x)^{\frac{2}{n-2}}}{|x|^{n-1}} dx + \int_{B(0,2) \setminus B(0,1)} \frac{u(x)^{\frac{2}{n-2}}}{|x|^{n-1}} dx
\]
(3.6)
Since $u = k * \mu$ then from (3.4) we see that
\[
u \leq C \text{ in } B(0,2) \setminus B(0,1).
(3.7)
Hence
As for $I_1$ in (3.6) we have
\[
\int_{D_m} \frac{1}{|x|^{n-1}} \left( \int_{B(0,2)} \frac{d\mu(y)}{|x - y|^{n-2}} \right)^{\frac{2}{n-2}} dx =
\int_{D_m} \frac{1}{|x|^{n-1}} \left( \int_{B(0,\rho_{m+2})} \frac{d\mu(y)}{|x - y|^{n-2}} + \int_{B(0,\rho_{m+2}) \setminus B(0,\rho_{m+1})} \frac{d\mu(y)}{|x - y|^{n-2}} + \int_{B(0,1) \setminus B(0,\rho_{m-2})} \frac{d\mu(y)}{|x - y|^{n-2}} \right)^{\frac{2}{n-2}} dx.
\]
For $x \in D_m$ we have
\[
\int_{B(0,\rho_{m+2})} \frac{d\mu(y)}{|x - y|^{n-2}} \leq \frac{1}{\rho_{m+2}^{n-2}} \mu(B(0,\rho_{m+1}))
\]
and
\[
\int_{B(1) \setminus B(0,\rho_{m+2})} \frac{d\mu(y)}{|x - y|^{n-2}} = \sum_{k=0}^{m-3} \int_{D_k} \frac{d\mu(y)}{|x - y|^{n-2}}.
\]
Noting that
\[
\frac{1}{2k+1} - \frac{1}{2m} = \frac{1}{2k+1} \left( 1 - \frac{1}{2m-k-1} \right) \geq 3 \frac{1}{4} \frac{1}{2k+1}
\]
we get
\[
\sum_{k=0}^{m-3} \int_{D_k} \frac{d\mu(y)}{|x - y|^{n-2}} \leq \left( \frac{8}{3} \right)^{-2} \sum_{k=0}^{m-3} \frac{\mu(B(0,\rho_k))}{\rho_k^{n-2}}.
\]
Combining
\[
I_1 \leq C(n) \sum_{m=0}^{\infty} \left\{ \int_{D_m} \left( \sum_{k=0}^{m-3} \frac{\mu(B(0,\rho_k))}{\rho_k^{n-2}} \right)^{\frac{2}{n-2}} + \left( \frac{\mu(B(0,\rho_{m+1}))}{\rho_{m+2}^{n-2}} \right)^{\frac{2}{n-2}} + \left( \int_{B(0,\rho_{m+2}) \setminus B(0,\rho_{m-2})} \frac{d\mu(y)}{|x - y|^{n-2}} \right)^{\frac{2}{n-2}} \right\}
\leq C(n) \sum_{m=0}^{\infty} \left\{ \rho_m \left( \sum_{k=0}^{m} \frac{\mu(B(0,\rho_k))}{\rho_k^{n-2}} \right)^{\frac{2}{n-2}} + \frac{1}{\rho_{m+1}^{n-1}} \int_{D_m} \left( \int_{B(0,\rho_{m+2}) \setminus B(0,\rho_{m+2})} \frac{d\mu(y)}{|x - y|^{n-2}} \right)^{\frac{2}{n-2}} \right\}
= C(I_3 + I_4).
\]
For $n = 3$ we take a sequence of smooth functions $f_i$ weakly converging to $\mu$ in $\tilde{D}_m := B(0, \rho_{m-2}) \setminus B(0, \rho_{m+2})$ (see Lemma 0.2 [Lan]) then applying lemma 7.12 from [GT] to
\[
V_s f_i(x) = \int_{\tilde{D}_m} |x - y|^{n(s-1)} f_i(y) dy
\]
with \( q = 2, p = 1, \delta = 1 - \frac{1}{q} = \frac{1}{2} \) and \( s - \delta = \frac{1}{6} \) we get

\[
\int_{\mathcal{D}_m} |V_{2/3} f_i| \leq C \left( \text{Vol}_{g_E}(D_m) \right)^{\frac{1}{2}} \left( \int_{\mathcal{D}_m} f_i^2 \right)^{\frac{1}{2}}.
\]

After letting \( i \to \infty \) this yields

\[
I_4 \leq \sum_{m=0}^{\infty} \frac{1}{\rho_m^3} \left( \mu(B(0, \rho_m)) \right)^2 \leq C \int_0^1 \frac{\mu(B(0, t))}{t^2} \, dt.
\]

Moreover, denoting \( m(t) = \mu(B(0, t)) \) and using integration by parts together with Cauchy-Schwarz inequality we get

\[
\int_0^1 \left( \int_0^1 \frac{m(\tau)}{\tau^2} \, d\tau \right)^2 \, dt = 2 \int_0^1 \frac{m(t)}{t} \left( \int_0^1 \frac{m(\tau)}{\tau^2} \, d\tau \right) \, dt \leq 2 \left[ \int_0^1 \left( \frac{m(t)}{t} \right)^2 \, dt \int_0^1 \left( \int_0^1 \frac{m(\tau)}{\tau^2} \, d\tau \right)^2 \, dt \right]^{\frac{1}{2}}
\]

implying that

\[
\int_0^1 \left( \int_0^1 \frac{m(\tau)}{\tau^2} \, d\tau \right)^2 \, dt \leq 4 \int_0^1 \frac{m(t)}{t} \, dt.
\]

Hence

\[
I_3 \leq C \int_0^1 \left( \int_0^1 \frac{m(\tau)}{\tau^2} \, d\tau \right)^2 \, dt \leq 4C \int_0^1 \frac{m(t)}{t} \, dt = 4CW(\mu, 0).
\]

If \( n = 4 \) we have

\[
\int_{\mathcal{D}_m} d\mu(y) \int_{\mathcal{D}_m} \frac{dx}{|x-y|^{n-2}} \leq \mu(B(0, \rho_m)) \rho_m^2
\]

and then from Fubini’s theorem we get as above

\[
I_4 \leq \sum_{m=0}^{\infty} \frac{1}{\rho_m^3} \mu(B(0, \rho_m)) \leq C \int_0^1 \frac{\mu(B(0, t))}{t^2} \, dt.
\]

The estimate for \( I_3 \) follows from integration by parts.

Finally, let us consider the case \( n \geq 5 \).

\[
\int_{\mathcal{D}_m} \left( \int_{\mathcal{D}_m} \frac{d\mu(y)}{|x-y|^{n-2}} \right) \frac{1}{\rho_m^2} \leq \left( \int_{\mathcal{D}_m} \int_{\mathcal{D}_m} \frac{d\mu}{|x-y|^{n-2}} \right)^{\frac{1}{2}} \left( \text{Vol}_{g_E}(D_m) \right)^{\frac{1}{2} - \frac{2}{n-2}} \leq C \left( \rho_m^2 (B(0, \rho_m)) \right)^{\frac{1}{2}} \rho_m^{n-\frac{2n}{n-2}} \rho_m \rho_m \rho_m \rho_m \rho_m \rho_m \rho_m \rho_m.
\]

Thus

\[
I_4 \leq \sum_{m=0}^{\infty} \frac{1}{\rho_m^3} \mu(B(0, \rho_m)) \frac{1}{\rho_m^2} \leq C \int_0^1 \mu(B(0, t))^\frac{n-2}{n-2} \, dt.
\]

As for \( I_3 \) one can easily see that

\[
\left( \sum_{k=0}^{m} \frac{\mu(B(0, \rho_k))}{\rho_k^2} \right)^{\frac{1}{2}} \leq C \sum_{k=0}^{m} \left( \frac{\mu(B(0, \rho_k))}{\rho_k^2} \right)^{\frac{1}{2}}
\]

and consequently after integration by parts we get

\[
I_3 \leq C \int_0^1 \int_0^1 \frac{m(\tau)}{\tau^{n-2}} \, d\tau \, dt \leq C \int_0^1 \frac{m(t)}{t^{n-2}} \, dt.
\]
and the proof of lemma is complete. □

Proof of (i)⇒(ii) in Theorem 1.1.

We claim that there exists a smooth curve \( c \)

\[
L_g(c) = \int_0^1 (u(c(t)))^{\frac{n-2}{2}} |c'(t)| \, dt < +\infty
\]

for a smooth curve \( \gamma : [0,1] \to B(0,2) \setminus K \),
such that \( \gamma(t) \to 0 \) as \( t \to 1 \).

Observe that (3.8) is impossible if \( u^{4/(n-2)}g_E \) is complete, Hence to finish the proof we have to establish (3.8). But from (3.5) the existence of such curve can be deduced as in [La05] p 23 and therefore the result follows.

4. Proof of (ii)⇒(i): \( \text{cap}(K) = 0 \) implies existence of metric

Proof of (ii)⇒(i) in Theorem 1.1. In what follows we assume, without loss of generality, that the north pole \( N \in \Omega \). Since \( \sigma(K) \subset \mathbb{R}^n \) is the image of \( K \) under stereographic projection then it is compact such that \( \text{cap}(\sigma(K)) = 0 \).

From Proposition 2.1 it follows that there is a probability measure \( \mu \), such that \( \text{supp} \mu \subset \sigma(K) \) and

\[
\mathcal{W}(\mu, x) = +\infty \quad \text{for all} \quad x \in \sigma(K).
\]

The convolution

\[
u(x) = \int \frac{d\mu(y)}{|x-y|^{n-2}}
\]
solves

\[
\Delta u = 0, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n \setminus \sigma(K).
\]

To finish the proof we have to show that

\[
\sigma^*(u^{\frac{n-2}{2}}g_E) \text{ is a complete metric in } \Omega.
\]

To see this we use a version of Hopf-Rinow theorem formulated in terms of divergent paths [PV16]. A continuous path \( c : [0, \infty) \to \Omega \) is said to be a divergent path if, for every compact set \( E \subset \Omega \), there exists \( t_0 \geq 0 \) such that \( c(t) \not\in E \) for every \( t > t_0 \). \((\Omega, g)\) is called "divergent paths complete" (or complete with respect to divergent paths) if every locally Lipschitz divergent path has infinite length. Using this version of Hopf-Rinow theorem one can see that \((\Omega, g)\) is complete if and only if every smooth (or even Lipschitz) divergent path has infinite length.

Let us take a divergent path \( c \) in \( \Omega \), and denote \( \tilde{c} : [0, \infty) \to \mathbb{R}^n \setminus \sigma(K) \) its stereographic projection. Clearly \( \tilde{c} \) is divergent path in \( \mathbb{R}^n \setminus \sigma(K) \). Since by assumption \( N \in \Omega \) then \( \tilde{c} \) is contained in some ball in \( \mathbb{R}^n \). Recall the arc length formula

\[
L_g(c) = \int_0^\infty \sqrt{g(c'(t), c'(t))} \, dt = \int_0^\infty u(\tilde{c}(t))^{\frac{n-2}{2}} |\tilde{c}'(t)| \, dt.
\]

where \( g = u^{\frac{n-2}{2}}g_E \).

By assumption \( \tilde{c} \) is a divergent path, therefore there exists \( x_0 \in K \) such that

\[
\text{dist}_g(x_0, c(t_k)) \to 0 \quad \text{for a sequence} \quad \{t_k\}, \quad k \to +\infty.
\]

For \( m \in \mathbb{N} \), we let \( \gamma_m = D_m \cap \tilde{c} \) where

\[
D_m = \left\{ x \in \mathbb{R}^n : \frac{1}{2^m} < |x - \bar{x}_0| < \frac{1}{2^{m-1}} \right\}.
\]
If \( m \geq m_0 \) for sufficiently large \( m_0 \) it follows from the smoothness of \( c \) that \( \gamma_m \) is at most a countable union of open smooth curves. Moreover, from (4.4) we see that \( \gamma_m \neq \emptyset \), and

\[
 L_{ge}(\gamma_m) \geq \frac{1}{2m-1} - \frac{1}{2} = \frac{1}{2m-1} \quad \text{for all} \quad m \geq m_0.
\]

For \( y \in B(\tilde{x}_0, 2^{-(k+2)}) \) and \( x \in D_k \) we have \( |x - y| \leq \frac{1}{2^{k+2}} + \frac{1}{2^k} = \frac{1}{2^k} \). Therefore

\[
 u(x) \geq \int_{B(\tilde{x}_0, \rho_k+2)} \frac{d\mu(y)}{|x - y|^{n-2}} \\geq \frac{1}{5^{n-2}} \frac{\mu(B(\tilde{x}_0, \rho_k+2))}{\rho_k^{n-2}} \quad \text{for all} \quad x \in D_k,
\]

where we set \( \rho_i = 2^{-i} \). Let \( I_k \subset (0, +\infty) \) denote the open set such that

\[
 \tilde{c}: I_k \to \sigma(\Omega) \cap D_k.
\]

Then we derive that

\[
 L_g(c) = \int_0^\infty u(\tilde{c}(t)) \frac{\pi^{-2}}{\tilde{c}''(t)} \, dt
 = \sum_{k=0}^\infty \int_{I_k} u(\tilde{c}(t)) \frac{\pi^{-2}}{\tilde{c}''(t)} \, dt
 \geq \sum_{k=0}^\infty \left( \inf_{\partial D_k} u \right) \frac{\pi^{-2}}{L_{ge}(\gamma_k)}
 \geq \frac{2}{5^{n-2}} \sum_{k=0}^\infty \left( \inf_{\partial D_k} u \right) \frac{\pi^{-2}}{\frac{\mu(B(\tilde{x}_0, \rho_k+2))}{\rho_k^{n-2}}} \rho_k \quad \text{after using (4.5)}.
\]

Recalling the definition of \( W \) we see that

\[
 W(\mu, x) = \int_0^1 \left( \frac{\mu(B(x, r))}{r^{n-2}} \right)^{\frac{2}{n-2}} \, dr = \sum_{k=0}^\infty \int_{2^{-(k+1)}}^{2^{-k}} \left( \frac{\mu(B(x, r))}{r^{n-2}} \right)^{\frac{2}{n-2}} \, dr
 \leq 4 \sum_{k=0}^\infty \left( \frac{\mu(B(x, \rho_k))}{\rho_k^{n-2}} \right)^{\frac{2}{n-2}} \rho_k.
\]

Comparing the inequalities for \( L_g(c) \) and \( W \) and recalling (4.1)

\[
 W(\mu, \tilde{x}_0) = +\infty.
\]

we obtain (4.3). \( \square \)

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