Graph centrality is a question of scale

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Classic measures of graph centrality capture distinct aspects of node importance, from the local (e.g., degree) to the global (e.g., closeness). Here we exploit the connection between diffusion and geometry to introduce a multiscale centrality measure. A node is defined to be central if it breaks the metricity of the diffusion as a consequence of the effective boundaries and inhomogeneities in the graph. Our measure is naturally multiscale, as it is computed relative to graph neighbourhoods within the varying time horizon of the diffusion. We find that the centrality of nodes can differ widely at different scales. In particular, our measure correlates with degree (i.e., hubs) at small scales and with closeness (i.e., bridges) at large scales, and also reveals the existence of multi-centric structures in complex networks. By examining centrality across scales, our measure thus provides an evaluation of node importance relative to local and global processes on the network.

Identifying the centrality nodes in a network is a topic of wide interest across fields, from finding critical junctions in a road network or important stations in a power grid to establishing which people are best poised to spread (or stop) gossip in a social network [1,2]. Starting in the 1950s, and motivated by heuristics from different application areas [3-5], various notions of node importance have been proposed as a measure of influence. These ideas were formalised and extended by several authors into classical graph-theoretical measures of node centrality [6]; some of a combinatorial nature (e.g., degree, closeness, or betweenness centrality [7]), others with a spectral character (e.g., eigenvector centrality [8]). Spurred by the recent interest in networks, such measures have been reformulated in different contexts, from extended notions of neighbourhood (e.g., k-core [9]) to information propagation (e.g., current-flow closeness [10]). Measures based on random walks (e.g., random walk accessibility [11], extensions of betweenness [12], non-backtracking centrality [13], embeddability in graph cycles [14], or subgraph centrality [15]) have also been proposed. For a survey of centralities, see the recent monograph [16].

These different measures capture relevant, yet distinct aspects of node importance. Some centralities are based on a local notion such as the number of connections of the node (i.e., the degree), whereas others stem from global network properties, such as closeness centrality, which considers the sum of shortest paths with all other nodes. However, the implicit scale of each centrality measure is rarely obvious. For instance, whilst betweenness centrality might be expected to be as global a measure as closeness, its value is computed only from a potentially small fraction of nodes where geodesic paths concentrate. Hence the effective scale of betweenness depends on the varying time horizon of the diffusion. We find that the centrality of nodes can differ widely at different scales. In particular, our measure correlates with degree (i.e., hubs) at small scales and with closeness (i.e., bridges) at large scales, and also reveals the existence of multi-centric structures in complex networks. By examining centrality across scales, our measure thus provides an evaluation of node importance relative to local and global processes on the network.

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Identifying the centrality nodes in a network is a topic of wide interest across fields, from finding critical junctions in a road network or important stations in a power grid to establishing which people are best poised to spread (or stop) gossip in a social network [1,2]. Starting in the 1950s, and motivated by heuristics from different application areas [3-5], various notions of node importance have been proposed as a measure of influence. These ideas were formalised and extended by several authors into classical graph-theoretical measures of node centrality [6]; some of a combinatorial nature (e.g., degree, closeness, or betweenness centrality [7]), others with a spectral character (e.g., eigenvector centrality [8]). Spurred by the recent interest in networks, such measures have been reformulated in different contexts, from extended notions of neighbourhood (e.g., k-core [9]) to information propagation (e.g., current-flow closeness [10]). Measures based on random walks (e.g., random walk accessibility [11], extensions of betweenness [12], non-backtracking centrality [13], embeddability in graph cycles [14], or subgraph centrality [15]) have also been proposed. For a survey of centralities, see the recent monograph [16].

These different measures capture relevant, yet distinct aspects of node importance. Some centralities are based on a local notion such as the number of connections of the node (i.e., the degree), whereas others stem from global network properties, such as closeness centrality, which considers the sum of shortest paths with all other nodes. However, the implicit scale of each centrality measure is rarely obvious. For instance, whilst betweenness centrality might be expected to be as global a measure as closeness, its value is computed only from a potentially small fraction of nodes where geodesic paths concentrate. Hence the effective scale of betweenness depends on the graph structure and varies for each node. This link between scale and centrality is recognisable in the classic Katz centrality [11] as well as recent measures [17] [18], where a free parameter is tuned to weigh the relative importance of local vs. global properties.

Here, we introduce a multiscale centrality measure with a notion of scale that is intrinsic to the graph. The centrality of each node is computed over the set of nodes that are reachable by a dynamical process on the graph over a given time horizon $\tau \in \mathbb{R}^+$. A simple such process, which can be derived as an approximation to more realistic types of dynamics, is a diffusion dynamics governed by the Laplacian of the graph. The time horizon of the diffusion plays the role of a natural scale factor: as the diffusion probes the surroundings of a node, it becomes sensitive to the presence of effective boundaries, via the shape of its neighborhood. Exploiting its links with geometry, as in the classic "hearing the shape of a drum" [19], diffusion allows us to establish the position of the node relative to its scale-dependent neighborhood. Our measure thus provides a geometric interpretation of centrality: central nodes are those that are away from the effective boundaries found via diffusive dynamics.

To gain some intuition, we start by considering a compact Euclidean subspace, and recall how diffusive dynamics can be used to compute the center of the domain relative to its boundaries. Consider the one-dimensional diffusion equation with constant coefficient $D$ on a finite domain $x \in [0,1]$

$$\partial_t p(x,t) = D \partial_x^2 p(x,t),$$

with Neumann boundary conditions. If the initial condition is localised at $x_0$, $p(x,0|x_0) = \delta(x-x_0)$, the solution can be computed by the method of images:

$$p(x,t|x_0) = \sum_{k=-\infty}^{k=\infty} G_t(x + 2k + x_0) + G_t(x + 2k - x_0),$$

where the Green’s function is

$$G_t(x) = (4Dt)^{-1/2} \exp \left( -x^2 / 4Dt \right).$$

Figure 1(a) shows the solution $p(x,t|x_0)$ for $x_0 = 0.3$, and the insets highlight the temporal responses at three observation points, denoted $x_0, x_1, x_2$. At the input point $x_0$, the impulse decays monotonically, whereas $p(x,t|x_0)$ grows towards the stationary value $p_\infty(x) = 1, \forall x \neq x_0$. 

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However, the approach to the stationary value is qualitatively different depending on the distance to the input point \(x_0\). Specifically, the temporal response exhibits an overshooting peak if the observation point is close enough to \(x_0\) (e.g., \(x_1\)), whereas there is no overshooting if the observation point is further away from \(x_0\) (e.g., \(x_2\)).

We can thus define

\[
t^\star(x_0, x) = \arg\max_{t \in [0, \infty)} p(x, t|x_0),
\]

the peak time at which the maximum overshooting appears as a function of the input point \(x_0\) and observation point \(x\). When no overshooting peak is observed at \(x\), we adopt the convention \(t^\star(x_0, x) = \infty\) and denote these points as non-reachable. Conversely, points with a finite peak time are reachable. A key aspect of this dynamical perspective is the intrinsic scale afforded by the time horizon \(\tau\) over which we observe the diffusion. To establish the reachability within time \(\tau\), we modify the peak time function:

\[
t^\star_\tau(x_0, x) = \begin{cases} \infty & \text{if } t^\star(x_0, x) \geq \tau \\ t^\star(x_0, x) & \text{otherwise} \end{cases}
\]

such that \(t^\star_\tau(x_0, x) = t^\star(x_0, x)\). Figure 1(b) shows the peak times \((4)\), with the non-reachable regions in white.

The existence of these two qualitatively distinct temporal profiles (reachable and non-reachable) follows from the presence of boundaries in the underlying space. Indeed, for the infinite domain, the solution is \(p(x, t|x_0) = G_\tau(x - x_0)\), and the peak time is given by \(t^\star(x_0, x) = (x - x_0)^2/2D\), \(\forall x_0\), i.e., all points are reachable and the peak time is simply the square of the (Euclidean) distance. For a finite segment, however, this relation holds only approximately for points away from the boundaries over small horizons, and breaks down as the horizon grows and the solution aggregates information from the boundaries. Due to these boundary effects, the peak time function \(t^\star_\tau(x_0, x)\) becomes non-unique and non-convex in...
its arguments, resulting in violations of the triangular inequality. Specifically, for a given \( x_0 \), there can be pairs of points \( (x_1, x_2) \) for which we have

\[
\Delta \tau(x_1, x_2| x_0) := t^*_\tau (x_0, x_1) + t^*_\tau (x_0, x_2) - t^*_\tau (x_1, x_2) \leq 0.
\]

(6)

If a point \( x_0 \) participates in a high proportion of such violations of the triangular inequality, it can be considered to be highly central: a point near the center of the interval reaches a large fraction of it, whereas points at both extremes do not reach each other, yielding violations of the triangle inequality. This leads us to our definition of the multiscale centrality for \( x_0 \) at scale \( \tau \) as the fraction of all pairs of points \( (x_1, x_2) \in [0, 1]^2 \) that violate the triangle inequality \( (6) \), relative to \( x_0 \) over time horizon \( \tau \):

\[
\mathcal{M}_\tau(x_0) = \left\langle \mathbf{1}_{\Delta \tau(x_1, x_2| x_0) \leq 0} \right\rangle,
\]

(7)

where \( \mathbf{1}_A \) is the indicator function for event \( A \) and \( \langle \cdot \rangle \) denotes the average over \([0, 1]^2\).

Figure 1(c) shows the multiscale centrality \( \mathcal{M}_\tau(x_0) \) on the segment \( x_0 \in [0, 1] \) for different values of the horizon \( \tau \). Overall, the centrality is large in the center and decreases toward the boundaries. Yet the multiscale centrality also captures subtle effects due to the boundaries, including local peaks near each boundary at short scales.

As the scale \( \tau \) grows, the centre of the segment is identified clearly. Note that in the absence of boundaries, we would find no center. For the infinite line, we have a constant centrality: \( \mathcal{M}_\tau(x_0) = m(\tau), \forall x_0 \), as there are no violations of the triangle inequality, similar to what is observed for small \( \tau \) in Fig. 1(c). The multiscale centrality thus allows us to establish the center of the domain, relative to boundaries of the underlying space, from the observation of the diffusion dynamics.

The notion of geometry that follows from the diffusive dynamics not only captures the effect of boundaries but is also sensitive to inhomogeneities in the domain. Indeed, although we started by considering a constant diffusion coefficient \( D \), the same procedure applies for a non-homogeneous diffusion. In that case, the centrality measure incorporates information about the mass distribution, i.e., it is analogous to finding scale-dependent centers of mass. An illustration of this notion is discussed in the SI (Figs. 4E), where we compare the centrality of a uniform Delaunay mesh on the plane with irregular and noisy meshes with inhomogeneous densities; at small scales, the centrality measure \( \mathcal{M}_\tau(x_0) \) captures local centres with high density, whereas at large scales it recovers the centre of mass of the mesh.

The crucial ingredient in our definition of centrality is the use of the diffusion dynamics to infer the centre of a (possibly inhomogeneous) compact space relative to its boundaries. In spaces such as graphs, the topological notion of boundary is not easy to establish. Yet it is straightforward to generalize our approach to such spaces if a diffusive dynamics exists. Consider an undirected, connected graph with \( N \) nodes, (weighted) adjacency matrix \( A \), and degree matrix \( D = \text{diag}(A) \). (For simplicity, we concentrate on undirected graphs initially and present the case of directed graphs in the SI.) The definition of the multiscale centrality translates directly to graphs by choosing a diffusion process

\[
\partial_t \mathbf{p}(t) = -L \mathbf{p},
\]

(8)

where \( L = D - A \) is the graph Laplacian, and we rescale time by the spectral gap of the Laplacian (i.e., \( t = \tau/\lambda_2 \)) to have a comparable time scale across different graphs. (Other choices of linear dynamics on graphs could be considered, such as the normalised Laplacian, or other biased diffusions.[20][21].) The function \( p(x, t) \) in (1) is replaced by a \( N \times 1 \) time-dependent node vector \( \mathbf{p}(t) \).

For an initial condition given by a delta function at node \( i \), the solution of (5) is

\[
p_j(t|i) = (e^{-t\Delta})_{ij},
\]

(9)

For a given input node \( i \) and horizon \( \tau \), some node functions \( p_j(t|i) \) have a peak at \( t^*_\tau j < \tau \), whereas others do not.[22]. In addition, as in (6) above, there are pairs \( (j, k) \) for which the triangle inequality \( t^*_\tau ij + t^*_\tau ik \geq t^*_\tau jk \) centered at node \( i \) will not be satisfied. Adapting the continuous definition (7) to the discrete setting, we compute the multiscale centrality of node \( i \), \( \mathcal{M}_\tau(i) \), as the proportion of violations of the triangle inequality in which \( i \) is involved up to scale \( \tau \). This gives us the normalized multiscale centrality node vector with elements

\[
(\mathcal{M}_\tau)_i = \mathcal{M}_\tau(i)/\max_j \mathcal{M}_\tau(j).
\]

(10)

We now illustrate the application of the centrality measure on graphs. Figure 1(d)-(f) shows the results of the Zachary Karate club,[23] a social network that has been widely studied as it underwent an acrimonious split into two factions led by the ‘Officer’ and ‘Mr Hi’. Figure 1(d) displays the propagation on the network of an impulse starting at Mr Hi: the impulse decreases to stationarity at the source node, whereas some nodes display a peak in the temporal response and others (e.g., the Officer) do not. In Fig. 1(e), we show the peak time \( t^*_\tau ij \) at which a peak appears at node \( j \) when an impulse is injected at node \( i \) (with the convention \( t^*_\tau ij = \infty \) if there is no peak). Note how the two known clusters of the Karate club are clearly separated by their reachability, yet there is a group of nodes that bridge across both clusters where two typically misclassified nodes (red, green) fall. Fig. 1(f) shows the multiscale centrality \( \mathcal{M}_\tau \) at three different scales \( \tau \) for all nodes of the network. The measure shows that both Mr. Hi and the Officer are central nodes at small scales (i.e., they are central to their local environments), whereas the nodes bridging across both clusters become central at larger scales (i.e.,
From this simple example, we see that nodes with high degrees ('hubs') are central at small scales, whereas 'inter-hub nodes' are central at larger scales relative to the global structure of the graph. This observation reflects the different heuristics that have been used in the literature to define centrality, from the local to the global. To examine the correspondence of our measure with other centralities, we have analysed in Figure 2 a second classic example, the interaction network of bottlenose dolphins, constructed between 1994 and 2001 in New Zealand [24]. (The same analysis for the Karate club is presented in Fig. 6 in the SI.) Figure 2(a) displays $\mathcal{M}_\tau$, the multiscale centrality for three values of the scale $\tau$. At small scales, the dolphin Grin (with the highest degree) has the highest centrality whereas at large scales another dolphin, called PL, becomes the most central, highlighting its role as a connector. Interestingly, both dolphins exhibit rare communication traits [24] (see SI for a more extended discussion). Figure 2(b) shows the Spearman correlation of the ranking based on $\mathcal{M}_\tau(i)$ compared to four classical notions of centrality. The multiscale centrality at low $\tau$ is highly correlated with the degree, whilst at large $\tau$ we observe a strong correlation with closeness. This result recovers our intuition: the degree is a local measure whereas closeness takes into account distances to all nodes in the network. Betweenness shows similar trends to degree centrality and eigenvector centrality shows similar behaviour to closeness, albeit they do not correlate as strongly with multiscale centrality and tend to correlate at various intermediate scales depending on the graph (see SI for additional examples).

In larger, complex networks, $\mathcal{M}_\tau$ can help reveal the multiscale structure of centrality. An example is given in Fig. 3 where we study the network of the European power grid [25] with $N = 2783$ nodes. At small scales, there are local centers distributed across Europe (e.g., France and Spain have high degree nodes perhaps due to differences in the design of the network in these countries). As the scale grows, high centralities emerge round a series of regional nuclei, which progressively coalesce into a strip extending across Central Europe (see SI for a more detailed explanation). Our analysis thus indicates a multi-centric structure in the European power grid, highly dependent on the scale of interest, which suggests that all scales should be considered when estab-
lishing the importance of nodes within this network.

We have introduced a scale-dependent graph centrality, which is based on a notion of reachable nodes from a localized diffusive source in terms of overshooting events. The key concept is to interpret the timing of the overshooting events as a proxy for a distance between nodes, such that the underlying geometry of the network is captured by the diffusive process. Within this framework, central nodes are those that are involved in breaking the metricity of the diffusion. The proposed measure also recovers the concept of a geometric center as a center of mass relative to the intrinsic boundaries of the graph probed by the diffusion. Because the diffusion has an inherent dynamical scale, our measure captures different notions of centrality, from the local (degree) to the global (closeness).

A Python code to compute the multiscale centrality [10] is available at https://github.com/barahona-research-group/MultiscaleCentrality.

In its current form, the method is applicable to relatively large graphs (thousands of nodes) but the evaluation of the matrix exponential [26] and the triangle inequalities can become expensive for larger graphs. The latter could be approximated, and other centrality measures could be extracted by, e.g., treating the matrix of peak times $t^*_ij$ as a distance matrix and computing a different version of the closeness centrality.

The dynamical foundation of the measure means that directed graphs can also be considered seamlessly within this approach. We provide an example of directed graphs in the SI, where we study the multiscale centrality of the neuronal network of C. Elegans, which captures differences between forward, backward and symmetrized graphs related to biological information flow from sensory to motor neurons (Fig. 8). Finally, we remark that although we have used diffusion here for conceptual clarity, similar notions can be implemented computationally with more complex dynamics, e.g., epidemic spreading, Kuramoto oscillators, or non-Markovian dynamics.

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[1] S. P. Borgatti and M. G. Everett, Social Networks 28, 466 (2006).
[2] M. Newman, Networks: An Introduction (2010).
[3] A. Bavelas, The Journal of the Acoustical Society of America 22, 725 (1950).
[4] L. Katz, Psychometrika 18, 39 (1953).
[5] M. A. Beauchamp, Behavioral science 10, 161 (1965).
[6] F. Harary, Graph Theory (1969).
[7] L. C. Freeman, Social Networks 1, 215 (1978).
[8] P. Bonacich, The Journal of Mathematical Sociology 2, 113 (1972).
[9] M. Kitsak, L. K. Gallos, S. Havlin, F. Liljeros, L. Muchnik, H. E. Stanley, and H. A. Makse, Nature Physics 6, 888 (2010).
[10] K. Stephenson and M. Zelen, Social Networks 11, 1 (1989).
[11] G. F. De Arruda, A. L. Barbieri, P. M. Rodrigues, F. A. Rodrigues, Y. Moreno, and L. D. F. Costa, Physical Review E 90, 032812 (2014).
[12] M. E. Newman, Social Networks 27, 39 (2005).
[13] T. Martin, X. Zhang, and M. E. Newman, Physical Review E 90, 052808 (2014).
[14] M. T. Schaub, J. Lehmann, S. N. Yaliraki, and M. Barahona, Network Science 2, 66 (2014).
[15] E. Estrada and J. A. Rodriguez-Velazquez, Physical Review E 71, 056103 (2005).
[16] F. Fouss, M. Saerens, and M. Shimbo, Algorithms and models for network data and link analysis (Cambridge University Press, 2016).
[17] E. Estrada, Journal of Theoretical Biology 263, 556 (2010).
[18] A. J. Gurfinkel and P. A. Rikvold, arXiv preprint arXiv:1904.05790 (2019).
[19] M. Kac, The american mathematical monthly 73, 1 (1966).
[20] R. Lambiotte, J. C. Delvenne, and M. Barahona, IEEE Transactions on Network Science and Engineering 1, 76 (2014) arXiv:0812.1770.
[21] M. T. Schaub, J.-C. Delvenne, R. Lambiotte, and M. Barahona, Phys. Rev. E 99, 062308 (2019) arXiv:1804.03733.
[22] K. A. Bacik, M. T. Schaub, M. Beguerisse-Díaz, Y. N. Billeh, and M. Barahona, PLoS Comp. Biology 12 (2016), 10.1371/journal.pcbi.1005055 arXiv:1511.00673.
[23] W. W. Zachary, Journal of Anthropological Research 33, 452 (1977).
[24] D. Lusseau, Evolutionary Ecology 21, 357 (2007).
[25] M. T. Schaub, J.-C. Delvenne, S. N. Yaliraki, and M. Barahona, PLoS ONE 7, e32210 (2012).
[26] A. H. Al-Mohy and N. J. Higham, SIAM Journal on Scientific Computing 33, 488 (2011).
Supplemental Material:  
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CENTER OF MASS AND CENTRALITY IN DELAUNAY MESHES

The proposed notion of centrality can be related not only to the geometric center but also to the center of mass. We illustrate this fact through the analysis of Delaunay meshes on the plane.

In Fig. 4 (a)-(c), we present the analysis of a (non-random) grid of $10 \times 10$ nodes on the plane. First, we generate its Delaunay mesh using the standard algorithm in the Qhull library (http://www.qhull.org/). Note that the obtained mesh is not a regular graph but a certain discretization of the flat unit square $[0,1]^2$, where each node $i$ has a position $x_i$ on the plane, and each edge $ij$ has a length $l_{ij} = \|x_i - x_j\|$. We thus set the edge weights of the graph to $w_{ij} = 1/l_{ij}$. We compute the multiscale centrality $M_\tau$ of this weighted graph across scales $\tau$. As expected, the small scale centrality almost perfectly correlates with the (weighted) degree of the node, while the large scale centrality produces a function peaked at the center of the square, similar to Fig. 1(a)-(c). We also computed the center of mass of this network, where the mass of each node is its weighted degree. Since the mass distribution is close to uniform, the center of mass corresponds to the geometric center, and is well recovered by the centrality at large scales.

In Fig. 4(d)-(f), we studied an inhomogeneous Delaunay mesh obtained by adding 50 additional nodes on the

![Multiscale centrality of Delaunay mesh triangulations on the plane. (a)-(b) Analysis of a uniform grid. At small scales, $M_\tau$ is highly correlated with degree, whereas at large scale $M_\tau$ coincides with the center (geometric centre and centre of mass), thus correlated with closeness. The node with the highest $M_\tau$ at small scales is indicated with a blue line and the node with the highest $M_\tau$ at large scales is indicated with a red line. (c)-(d) A normally distributed patch of nodes was added to the grid around the point $(0.2, 0.2)$. This has the effect of a localised larger diffusion coefficient and an inhomogeneous mass distribution. At small scales, $M_\tau$ correlates with the degree, with high localization at the added patch of nodes. At large scales, the centrality $M_\tau$ is concentrated on nodes between the center of mass and the geometric center of the square.](image)
plane, drawn from a normal distribution centered at \([0.2, 0.2]\), with variance 0.05. This amounts to adding a localised mass in the bottom left corner of the square, thus making the mass distribution far from uniform. The computation of the multiscale centrality for the corresponding network shows that, at small scale, the centrality captures the higher degree (and mass) of the cluster of dense nodes, whereas for large scales, we find that the most central nodes are between the center of mass and the geometric center of the square. As expected, the center has thus shifted towards the cluster of nodes, but does not match the center of mass as the effect of boundaries remains important.

To test the robustness of these results under perturbations, we considered a noisy grid with random Gaussian shifts of increasing variance: 0.01, 0.02 and 0.03. Figure 5 shows the small scale and large scale multiscale centralities for three increasingly noisy realisations of these graphs. For small scales, the centrality reflects the randomness of the degree distribution, with no clear centers present. At large scales, however, the geometric center is well estimated for small amounts of noise, but less so as the noise grows due to the blurring of the boundaries.

FIG. 5. Caption as in Fig. 4, but the Delaunay mesh is created by adding Gaussian random deviations to the positions of the nodes on the grid with increasing noise. At small scales, the centrality largely reflects the degree of the nodes. At large scales, the centrality measure localizes at the center of the square but becomes less reliable as the noise increases and the boundaries and topological features of the network become washed out.

EXTENDED ANALYSIS OF ZACHARY’S KARATE CLUB NETWORK

To illustrate multiscale centrality on graphs, we used in Figure 1 Zachary’s karate club network. This social network became a classic example after it was used to predict the groups that emerged from the splitting of the club into two factions. Here, we considered it as an example of a network with two communities, each represented by a clear leader (Mr Hi vs. Officer), which are expected to be central nodes. Graph centrality measures such as eigenvector, degree or betweenness find these two group leaders as the most central nodes within the network.

In addition to the Figure 1(d)-(f) in the main text, we show in Figure 6 the results of ranking the nodes across all scales \(\tau\), also highlighting the values of the centralities on the network at three different scales. We find that the
two leaders are the most central nodes for a large range of scales. Only at long scales do we find that a different node (that bridges across the two groups) becomes highly central. The Spearman correlation shows that the ranking induced by multiscale centrality is most correlated with betweenness at short scales (even more than to the degree). This higher correlation of betweenness is due to the particularities of the structure and size of this network. Closeness is the most correlated at large scales, and we observe a high correlation with eigenvector centrality at middle scales. However, we note that for all the changes across scales, the two leaders are the most central according to any of those measures across a large range of scales, suggesting that these two nodes do really play a central role in this network independent of scales.

FIG. 6. Multiscale centrality of the Zachary’s karate club network [23]. (a) Multiscale centrality as a function of $\tau$ for each node (with the values of $M_{\tau}$ mapped on the network for three instances at small, middling and large scales). The multiscale centrality of the Officer (blue), Mr Hi (green) and the bridge node (red) are highlighted. Mr Hi and the Officer are highly central across most scales. Only at large scales does the bridge node become central. (b) Spearman correlation between $M_{\tau}$ and four common centrality measures as a function of $\tau$.

EXTENDED DISCUSSION OF THE DOLPHIN NETWORK

In addition to the short description in the main text, we provide a more detailed description of the dolphin network analysis. A minority of the dolphins within the dataset displayed physical forms of communication to other dolphins; 7/63 dolphins display ‘lobtailing’ and 5/63 display ‘side flopping’, which are both forms of physical to auditory communication [24]. Figure 2(a) displays the normalized multiscale centrality of each node on the dolphin network at
three different scales. At short timescales, the dolphin Grin has the highest multiscale centrality. Grin is located at the centre of a dolphin cluster and is one of the few dolphins that displays lobtailing communication. At large scales, the dolphin PL becomes central to the whole network. Contrary to Glin, PL displayed ‘side flopping’ communication which suggests that, despite its low degree, PL communicates across the two main clusters of dolphins. Other nodes appear to be central at difference scales. In particular, for intermediate scales, there is more variation in the centralities, due to the complex topology of the graph. Nevertheless, the top scoring nodes are all in the center of the two clusters, but mostly in the larger one.

FIG. 7. Same as Figure 2 in the main text, but with the middle panel showing the multiscale centrality $M_\tau$ for all nodes as a function of the scale $\tau$.

EXTENDED DISCUSSION OF THE GRAPH OF THE EUROPEAN POWER GRID NETWORK

We have computed the multiscale centrality for the large European power grid network (data from Union for the Coordination of Transmission Energy, see [25] for more details on this dataset). This network contains 2783 nodes and 3762 unweighted edges, with community structure at several scales, see [25]. In Figure 3 we displayed the Spearman correlation of multiscale centrality with four classic centrality measures along with four instances of the multiscale centrality $M_\tau$ computed at different scales. At $\log_{10}(\tau) = -4.3$, where $M_\tau$ correlates strongly with degree centrality, we find that the most central nodes tend to be highly local: the most central node is in the south west of France, followed by various cities located predominantly in France and Spain. Interestingly, this would suggest that countries such as France and Spain contain some nodes with a higher degree relative to the rest of Europe, maybe
due to their absolute size or some systematic difference in the design of the electrical network in these countries. At slightly longer timescales, \( \log_{10}(\tau) = -3.02 \), we find the most central nodes located predominantly in the north of Spain and near Paris in France. At yet longer timescales, \( \log_{10}(\tau) = -1.16 \), the nodes in Spain become less central, reflecting their more peripheral location relative to the global network, and instead we observe three main regions of high \( \mathcal{M}_\tau \): a region around the Pyrenees, on the border between Spain and France; a region along the border between France and Germany; and a region on the western part of Eastern Europe. At the largest timescales, \( \log_{10}(\tau) = 0.6 \), we find a single predominant region of highly central nodes stretching from the north-east of France to the north-west of Italy, where the east and west centers present at shorter scales collapse. Our analysis indicates a multi-centric structure in the European electrical network, which is highly dependent on the scale of interest and suggests that all scales be considered when trying to understand the importance of nodes within a network.

### Multiscale Centrality of a Directed Graph: The Neuronal Network of C. elegans

In this section, we showcase the application of the multiscale centrality to directed networks using the directed and weighted graph representing the connectome of the nematode *C. elegans* [22].

For directed graphs \( A \neq A^T \), and we need the following modification of the definition of the multiscale centrality. Let us consider the standard directed combinatorial Laplacian with teleportation [20, 21]:

\[
L_{\text{dir}} = \Phi - \frac{1}{2} (\Phi P + P^T \Phi),
\]

where \( \Phi \) is the Perron vector of the transition matrix \( P \) with teleportation:

\[
P = \alpha D^{-1} A + \left((1 - \alpha) + \alpha \text{diag}(a)\right) \frac{11^T}{n}.
\]

Here \( \alpha = 0.85 \) is the (Google) teleportation parameter and \( a \) is an indicator function for nodes with vanishing out-degrees. The directionality of the graph implies that in general \( t_{ij}^* \neq t_{ji}^* \), thus we consider the triangle inequality

\[
t_{ij}^* + t_{ik}^* \geq \frac{t_{jk}^* + t_{kj}^*}{2}, \quad (11)
\]

and count the fraction of violations of this inequality for any node \( i \). This directed version of the multiscale centrality is therefore sensitive to the directionality of the graph.

To see this, we have computed and compared the multiscale centrality for the directed *C. elegans* network, its reverse, and the undirected network (where directionality is ignored). The results are presented in Fig. 8. We find that in the directed, reverse directed and undirected connectomes the AVAR and AVAL neurons are consistently the most central nodes at short timescales. This is a consequence of AVAR and AVAL having both the highest in and out degrees of any nodes in the network. At larger scales, however, there are differences between the three networks: in the directed network, FLPR (a sensory neuron) is the most central, whereas in the reverse directed network, RIMR/RIML (two motor neurons) become the most central. In the undirected network, we find PVT (inter neuron) and AQR (sensory neuron) as the most central nodes at large scales.

In general, we find that the most central nodes in the directed network are sensory neurons or interneurons, whereas the most central nodes in the reverse directed connectome tend to be motor neurons or interneurons. In the undirected connectome, interneurons are the most central and both sensory and motor neurons have low centrality. This is shown in Fig. 8(b), where we compute the average centrality for each of the three classes of neurons.
FIG. 8. Multiscale centrality of the directed weighted neuronal network of C. Elegans. (a) We compare: (i) the directed network (left column); (ii) the reverse directed network (central column); and (iii) the undirected network (right column). The most central nodes at small scales $\tau$ for all three graphs (blue line) are the AVAR/L neurons. The most central nodes at large scales for each of the networks are: (i) FLPR (a sensory neuron); (ii) RIMR/RIML (two motor neurons); (iii) PVT (interneuron) and AQR (sensory neuron). (b) The averaged multiscale centrality for the three types of neurons: sensory (S), motor (M), and inter-neurons (I). Interneurons are most central for all three networks at all scales. Sensory neurons only become central at large scales for the directed network, whereas motor neurons only become central at large scales for the reverse directed network. In the undirected network, neither motor nor sensory neurons are central at any scale.