Perturbation of Rayleigh waves in anisotropic media

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Abstract. Herein we consider Rayleigh waves propagating along the traction-free surface of a homogeneous anisotropic elastic half-space. We investigate the perturbation of the phase velocity and of the polarization of Rayleigh waves, i.e., the shift of phase velocity and the shift of polarization of Rayleigh waves from their respective value pertaining to a comparative isotropic state of the elastic medium, as caused by the anisotropic part of the elasticity tensor. With the help of the Stroh formalism, we derive first-order perturbation formulas for these shifts. We also discuss the problem of determining elastic constants of a weakly anisotropic elastic medium by making measurements of Rayleigh waves propagating in all the directions on the traction-free surface.

1. Introduction
Rayleigh waves are elastic surface waves which propagate along the traction-free surface with the phase velocity in the subsonic range, and whose amplitude decays exponentially with depth below that surface. Such waves serve as a useful tool in nondestructive characterization of materials. The problem there is what material information we obtain if we could measure accurately Rayleigh waves propagating in any direction on the traction-free surface.

We consider Rayleigh waves propagating along the traction-free surface of a homogeneous elastic half-space. For isotropic elasticity, such waves are well known: Their phase velocity $v_{R}^{\text{iso}}$ is determined from the secular equation, which is a bi-cubic equation written in terms of the Lamé constants $\lambda$ and $\mu$ (see (14)), and their polarization at the surface is given explicitly in terms of $\lambda, \mu$ and $v_{R}^{\text{iso}}$ (see (68)).

An elasticity tensor can be decomposed uniquely as the sum of its isotropic and its anisotropic part. In what follows we consider elastic media for which the anisotropic part of the elasticity tensor is sufficiently small as compared with the isotropic part. The isotropic part of a given elasticity tensor is itself also an elasticity tensor, which we interpret as a comparative ‘unperturbed’ isotropic state. The anisotropic part then gives the deviation of the elasticity tensor from the comparative isotropic state and represents the anisotropy that the elastic material carries. Here we do not put any restriction on the material symmetry of the anisotropic part so that it has 21 independent components.

In this article we investigate the perturbation of the phase velocity and of the polarization of Rayleigh waves, i.e., the shift of phase velocity and the shift of polarization of Rayleigh waves from their respective value pertaining to the comparative isotropic state in question, as caused by the anisotropic part. Two physical quantities are determined from the polarization of Rayleigh
waves on the traction-free surface. One is the polarization ratio and the other is the phase shift. The polarization ratio is the ratio of the maximum of the longitudinal component of the displacements to the maximum of the normal component (i.e., the transversal component that is normal to the traction-free surface) of the displacements of the Rayleigh waves at the surface. The phase shift is the shift in phase measured from that of the longitudinal component to that of the normal component of the displacements at the surface. Hence we present perturbation formulas for the phase velocity, the polarization ratio, and the phase shift, which are correct to first order in the components of the anisotropic part. These formulas show explicitly how the anisotropic part, to first order of itself, affects the phase velocity, the polarization ratio, and the phase shift of Rayleigh waves. We obtain these formulas by a consistent method on the basis of the Stroh formalism.

For definiteness, we choose a Cartesian coordinate system such that the material half-space occupies the region \( x_3 \leq 0 \), whereas the 1- and 2-axis are arbitrarily chosen. The following consequence immediately follows from the perturbation formulas: Only four components \( a_{2222}, a_{2233}, a_{3333} \) and \( a_{2323} \) of the anisotropic part can affect the first-order perturbations of the phase velocity and the polarization ratio of Rayleigh waves propagating in the direction of the 2-axis on the surface \( x_3 = 0 \), whereas only two components \( a_{2223} \) and \( a_{3323} \) of the anisotropic part can affect the first-order perturbation of the phase shift of such Rayleigh waves.

We also discuss the problem of determining the anisotropic part in the elasticity tensor by making measurements of perturbation of Rayleigh waves which propagate in any direction on the free surface. We show that, to first order of the material anisotropy, the totality of phase velocities of Rayleigh waves propagating in all the directions on the free surface carries information only on 13 components of the anisotropic part. The totality of the polarization ratios of such Rayleigh waves also carries information only on the same 13 components of the anisotropic part. On the other hand, it can be proved that the totality of the phase shifts of those Rayleigh waves carries information only on the remaining 8 components of the anisotropic part.

The present article is a survey paper which summarizes our approach and findings on first-order perturbation formulas for Rayleigh waves and which outlines what information on the elastic constants of a weakly anisotropic medium those measurements of Rayleigh waves can theoretically deliver. The objective of this article is to introduce the reader to the full papers [1, 2], where complete proofs are given and the effects of initial stress are also taken into consideration.

2. Preliminaries

In the Cartesian coordinates \((x_1, x_2, x_3)\), the elastic wave equations for dynamic deformations of the homogeneous elastic medium can be written as

\[
\sum_{j,k,l=1}^{3} C_{ijkl} \frac{\partial^2 u_k}{\partial x_j \partial x_l} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1, 2, 3.
\]  

Here \( C = (C_{ijkl})_{i,j,k,l=1,2,3} \) is the elasticity tensor, which has the physically natural symmetries

\[
C_{ijkl} = C_{jikl} = C_{klij}, \quad i,j,k,l = 1,2,3,
\]

and satisfies the strong convexity condition,

\[
\sum_{i,j,k,l=1}^{3} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} > 0 \quad \text{for any non-zero } 3 \times 3 \text{ real symmetric matrix } (\varepsilon_{ij})_{i,j=1,2,3},
\]
\( \rho \) is the uniform mass density, \( t \) is the time, and \( \mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3) \) is the displacement at the place \( \mathbf{x} = (x_1, x_2, x_3) \) at time \( t \).

The elasticity tensor \( \mathbf{C} = (C_{ijkl}) \) can be decomposed \([3]\) uniquely in the form

\[
\mathbf{C} = \mathbf{C}^{\text{iso}} + \mathbf{A},
\]

where \( \mathbf{C}^{\text{iso}} \) is the isotropic part of \( \mathbf{C} \),

\[
\mathbf{C}^{\text{iso}} = (C_{ijkl}^{\text{iso}})_{i,j,k,l=1,2,3}, \quad C_{ijkl}^{\text{iso}} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj})
\]

with Lamé constants \( \lambda \) and \( \mu \), and \( \mathbf{A} \) is the anisotropic part.

From the symmetry conditions (2) of \( \mathbf{C} \) it follows that

\[
a_{ijkl} = a_{iklj} = a_{klij}, \quad i, j, k, l = 1, 2, 3,
\]

but we do not assume any other symmetry for \( \mathbf{A} \). Hence the anisotropic part \( \mathbf{A} \) has 21 independent components, which can be written in the matrix form as

\[
\mathbf{A} = (a_{ijkl}) = \begin{bmatrix}
    a_{1111} & a_{1122} & a_{1133} & a_{1223} & a_{1213} & a_{1112} \\
    a_{2222} & a_{2233} & a_{2323} & a_{2323} & a_{2213} & a_{2222} \\
    a_{3333} & a_{3323} & a_{3233} & a_{2313} & a_{3313} & a_{3312} \\
    a_{2323} & a_{3233} & a_{3323} & a_{3213} & a_{3312} & a_{2312} \\
    a_{1313} & a_{1312} & a_{1212} & a_{1212} & a_{1313} & a_{1312} \\
    \text{Sym.} & \text{Sym.} & \text{Sym.} & \text{Sym.} & \text{Sym.} & \text{Sym.}
\end{bmatrix}.
\]

Assuming that the anisotropic part \( \mathbf{A} \) is sufficiently small as compared with the isotropic part \( \mathbf{C}^{\text{iso}} \), we consider Rayleigh waves in the half-space \( x_3 \leq 0 \) which propagate along the surface \( x_3 = 0 \) in the direction of the 2-axis with the phase velocity \( v_R \).

We shall see in Section 6 that in the comparative isotropic medium defined by \( \mathbf{C} = \mathbf{C}^{\text{iso}} \) and \( \mathbf{A} = \mathbf{0} \) (i.e., the zero fourth-order tensor), the phase velocity \( v_R^{\text{iso}} \) of Rayleigh waves is the unique solution to the cubic equation (14) with \( V = \rho (v_R^{\text{iso}})^2 \) in the range \( 0 < V < \mu \). Thus, we first investigate the perturbation of the phase velocity \( v_R \) of Rayleigh waves, i.e., the shift in \( v_R \) from its comparative isotropic value \( v_R^{\text{iso}} \), as caused by the anisotropic part \( \mathbf{A} \). We present a velocity formula which is correct to within terms linear in the components of \( \mathbf{A} = (a_{ijkl}) \).

Next we investigate the perturbation of the polarization of Rayleigh waves. A real form of the displacements on the surface \( x_3 = 0 \) due to such Rayleigh waves are written by using two linearly independent real vectors

\[
\mathbf{a}^+ = \begin{bmatrix} a_1^+ \\ a_2^+ \\ a_3^+ \end{bmatrix} \quad \text{and} \quad \mathbf{a}^- = \begin{bmatrix} a_1^- \\ a_2^- \\ a_3^- \end{bmatrix}
\]

as

\[
\begin{bmatrix}
    a_1^+ \\
    a_2^+ \\
    a_3^+
\end{bmatrix} \cos k(x_2 - v_R t) + \begin{bmatrix}
    a_1^- \\
    a_2^- \\
    a_3^-
\end{bmatrix} \sin k(x_2 - v_R t),
\]

where \( k \) is the wave number and \( v_R \) is the phase velocity of Rayleigh waves (see (33)). Since the 2-axis is the propagation direction of Rayleigh waves, the longitudinal component of (7) is

\[
a_2^+ \cos k(x_2 - v_R t) + a_2^- \sin k(x_2 - v_R t) = \sqrt{(a_2^+)^2 + (a_2^-)^2} \sin (k(x_2 - v_R t) + \alpha).
\]
Since Rayleigh waves propagate along the surface $x_3 = 0$, the normal component of (7) is

$$a_3^+ \cos k(x_2 - v_R t) + a_3^- \sin k(x_2 - v_R t) = \sqrt{(a_3^+)^2 + (a_3^-)^2} \sin (k(x_2 - v_R t) + \beta),$$

where $\alpha$ and $\beta$ are the angles determined by

$$\tan \alpha = \frac{a_3^+}{a_2}, \quad \tan \beta = \frac{a_3^+}{a_3}, \quad (-\frac{\pi}{2} \leq \alpha, \beta \leq \frac{\pi}{2}).$$

The polarization ratio $r_R$ of Rayleigh waves on the surface $x_3 = 0$ is defined by the ratio of the maximum longitudinal component to the maximum normal component of the displacements on $x_3 = 0$

$$r_R = \frac{\sqrt{(a_2^+)^2 + (a_2^-)^2}}{\sqrt{(a_3^+)^2 + (a_3^-)^2}}. \quad (8)$$

The phase shift $\Delta_R$ of Rayleigh waves on the surface $x_3 = 0$ is defined by the shift in phase measured from that of the longitudinal component to that of the normal component of the displacements on $x_3 = 0$

$$\Delta_R = \beta - \alpha. \quad (9)$$

We shall see in Section 7 that in the comparative isotropic medium defined by $C = C^{\text{Iso}}$ and $A = 0$, the polarization ratio and the phase shift are given by

$$r_R^{\text{Iso}} = \frac{2\sqrt{\mu (\mu - V_R^{\text{Iso}})}}{2\mu - V_R^{\text{Iso}}}, \quad (10)$$

and

$$\Delta_R^{\text{Iso}} = \frac{\pi}{2}, \quad (11)$$

respectively.

Thus, we investigate the perturbation of the polarization ratio $r_R$ and of the phase shift $\Delta_R$ of Rayleigh waves, i.e., the shifts in $r_R$ and $\Delta_R$ from their comparative isotropic values $r_R^{\text{Iso}}$ and $\Delta_R^{\text{Iso}}$, respectively, as caused by the anisotropic part $A$. We present formulas for $r_R$ and $\Delta_R$ which are correct to first order in the components of $A = (a_{ijkl})$.

Finally, we apply the perturbation formulas obtained here to the problem of determining the anisotropy of an elastic material by making measurements of perturbation of Rayleigh waves which propagate in any direction on the free surface. Namely, we discuss what information on the anisotropic part $A = (a_{ijkl})$ can be obtained from knowledge of the perturbations of $v_R$, $r_R$ and $\Delta_R$ which pertain to Rayleigh waves propagating in all the directions on the surface $x_3 = 0$.

3. Perturbation formulas
Throughout this section we restrict our attention to an anisotropic elastic medium whose elasticity tensor $C$ is given by (4), (5) and (6), and we consider Rayleigh waves which propagate along the surface of the half-space $x_3 \leq 0$ in the direction of the 2-axis.

**Theorem 3.1** ([1]) The phase velocity of Rayleigh waves can be written, to within terms linear in the anisotropic part $A = (a_{ijkl})$, as

$$v_R = v_R^{\text{Iso}} - \frac{1}{2\rho v_R^{\text{Iso}}} \left[ \gamma_1 (v_R^{\text{Iso}} a_{2222} + \gamma_2 (v_R^{\text{Iso}} a_{2323} + \gamma_3 (v_R^{\text{Iso}} a_{2233} + \gamma_4 (v_R^{\text{Iso}} a_{3333}) \right], \quad (12)$$
where

\[ \gamma_1(v) = \frac{(\lambda + 2\mu) [-8\mu^2(\lambda + \mu) + 2\mu(5\lambda + 6\mu)V - (2\lambda + 3\mu)V^2]}{D(v)}, \]

\[ \gamma_2(v) = -\frac{8\mu(\lambda + 2\mu - V)[2\mu(\lambda + \mu) - (\lambda + 2\mu)V]}{D(v)}, \]

\[ \gamma_3(v) = \frac{4\lambda(\mu - V)[4\mu(\lambda + \mu) - (\lambda + 2\mu)V]}{D(v)}, \]

\[ \gamma_4(v) = \frac{(\lambda + 2\mu - V)[-8\mu^2(\lambda + \mu) + 2\mu(5\lambda + 6\mu)V - (2\lambda + 3\mu)V^2]}{D(v)}, \]

\[ D(v) = (\lambda + \mu) [8\mu^2(3\lambda + 4\mu) - 16\mu(\lambda + 2\mu)V + 3(\lambda + 2\mu)V^2], \]

\[ V = \rho v^2 \]

and \( v_{R}^{\text{iso}} \) is the velocity of Rayleigh waves in the comparative isotropic medium defined by \( C = C_{\text{iso}} \) and \( A = O \), i.e., \( v_{R}^{\text{iso}} = \rho (v_{R}^{\text{iso}})^2 \) is the unique solution to the cubic equation

\[ V^3 - 8\mu V^2 + \frac{8\mu^2(3\lambda + 4\mu)}{\lambda + 2\mu} V - \frac{16\mu^3(\lambda + \mu)}{\lambda + 2\mu} = 0 \]

in the range \( 0 < V < \mu \).

**Remark 3.2** Only four components \( a_{2222}, a_{2233}, a_{3333} \) and \( a_{2323} \) of the anisotropic part \( A \) can affect the first-order perturbation of the phase velocity \( v_R \).

**Theorem 3.3 ([2])** The polarization ratio of Rayleigh waves can be written, to within terms linear in the anisotropic part \( A \), as

\[ r_R = v_{R}^{\text{iso}} + \eta_1(v_{R}^{\text{iso}}) a_{2222} + \eta_2(v_{R}^{\text{iso}}) a_{2233} + \eta_3(v_{R}^{\text{iso}}) a_{2323} + \eta_4(v_{R}^{\text{iso}}) a_{3333}, \]

where \( v_{R}^{\text{iso}} \) is given by \( (10) \),

\[ \eta_1(v) = \sqrt{\frac{\lambda + 2\mu - V}{\lambda + 2\mu} \left( \frac{\lambda + 2\mu}{(\lambda + 2\mu)^2(2\mu - V)(2\lambda + \mu - V)} \right)}, \]

\[ \eta_2(v) = \sqrt{\frac{\lambda + 2\mu - V}{\lambda + 2\mu} \left( \frac{8(\lambda + 2\mu)(\lambda + 2\mu - V)(2\mu(\lambda + \mu) - (\lambda + 2\mu)V)}{D_1(v)} \right)}, \]

\[ \eta_3(v) = \sqrt{\frac{\lambda + 2\mu - V}{\lambda + 2\mu} \left( \frac{4\lambda(\lambda + 2\mu)(2\mu(\lambda + \mu) - (\lambda + 2\mu)V)}{D_1(v)} \right)}, \]

\[ \eta_4(v) = -\sqrt{\frac{\lambda + 2\mu - V}{\lambda + 2\mu} \left( \frac{4\mu(\lambda + \mu)(\lambda^2 + 4\mu^2)}{\lambda^2 + 4\mu^2} - 2(\lambda^3 + 2\lambda^2\mu + 12\lambda\mu^2 + 16\mu^3)V - (\lambda - 6\mu)(\lambda + 2\mu)V^2 \right)}/D_1(v), \]

\[ D_1(v) = 4(2\mu - V) \left[ 8\mu(\lambda + \mu)(\lambda^2 + 2\lambda\mu + 2\mu^2) - (\lambda + 2\mu)^2(\lambda + 5\mu)V + 3\mu(\lambda + 2\mu)V^2 \right], \]

\[ V = \rho v^2 \]

and \( v_{R}^{\text{iso}} \) is the velocity of Rayleigh waves in the comparative isotropic medium.
Remark 3.4 Again only four components $a_{2222}, a_{2233}, a_{3333}$ and $a_{3323}$ of the anisotropic part $A$ can affect the first-order perturbation of the polarization ratio $r_R$.

Theorem 3.5 ([2]) The phase shift of Rayleigh waves can be written, to within terms linear in the anisotropic part $A$, as

$$\Delta_R = \Delta^{\text{Iso}}_R + \Theta_1(v^{\text{Iso}}_R) a_{2222} + \Theta_2(v^{\text{Iso}}_R) a_{3333},$$

where $\Delta^{\text{Iso}}_R$ is given by (11) and

$$\Theta_1(v) = \frac{N(v)}{D_2(v)}, \quad \Theta_2(v) = -\frac{\lambda}{\lambda + 2\mu} \frac{N(v)}{D_2(v)},$$

$$N(v) = 8\mu \left[ 2\mu^2(\lambda - 6\mu)(\lambda + \mu) + \mu(-3\lambda^2 + 12\lambda\mu + 22\mu^2) + \lambda^2 - 8\mu^2 \right],$$

$$D_2(v) = \sqrt{(\lambda + 2\mu)(\lambda + 2\mu - V)} (\lambda + \mu)(2\mu - V)^2 V^2,$$

$$V = \rho v^2.$$  \hspace{1cm} (17)

Remark 3.6 Only two components $a_{2233}$ and $a_{3333}$ of the anisotropic part $A$ can affect the first-order perturbation of the phase shift $\Delta_R$.

The preceding theorems are proved with the help of the Stroh formalism. We shall give a brief sketch of the proofs of these theorems in Sections 6 and 7.

4. Recovery of elastic constants

Consider Rayleigh waves propagating along the traction-free surface of the half-space $x_3 \leq 0$. There remains the problem what information on the material anisotropy we could obtain if we could measure accurately the phase velocity or the polarization of Rayleigh waves propagating in any direction on the free surface.

Consider Rayleigh waves whose propagation direction $\mathbf{m}$ on the surface $x_3 = 0$ makes an angle $\theta$ with the 2-axis. Let $v_R(\theta)$, $r_R(\theta)$ and $\Delta_R(\theta)$ be the phase velocity, the polarization ratio and the phase shift of such waves, respectively. Let $O'X'_1X'_2X'_3$ be the Cartesian coordinate system such that the $3'$-axis agrees with the 3-axis and the $2'$-axis has the same direction as $\mathbf{m}$. Let $a'_{ijkl}$ be the components of the anisotropic part $A$ under the primed coordinate system.

First, suppose that we could measure $v_R(\theta)$ and $r_R(\theta)$ for each $0 \leq \theta < 2\pi$. According to Theorem 3.1 and Theorem 3.3 only the four components $a'_{2222}, a'_{2233}, a'_{3333}$ and $a'_{2323}$ affect the first-order perturbations of the phase velocity $v_R(\theta)$ and the polarization $r_R(\theta)$. From the transformation formula for fourth-order tensors we have

$$a'_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}a_{pqls},$$

where the $Q_{ip}$’s are the entries of the rotation matrix

$$Q = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (19)

Hence the four components that affect the first-order perturbations of $v_R(\theta)$ and $r_R(\theta)$ are written in terms of the components $(a_{ijkl})$ of $A$ under the original coordinate system as

$$a'_{2222} = a_{1111} \sin^4 \theta - 4a_{1112} \sin^3 \theta \cos \theta + (2a_{1122} + 6a_{1212}) \sin^2 \theta \cos^2 \theta + 4a_{2212} \sin \theta \cos^3 \theta + a_{2222} \cos^4 \theta,$$

$$a'_{2233} = a_{1133} \sin^2 \theta - 2a_{3312} \sin \theta \cos \theta + a_{2233} \cos^2 \theta,$$

$$a'_{2323} = a_{1313} \sin^2 \theta - 2a_{2313} \sin \theta \cos \theta + a_{2323} \cos^2 \theta,$$

$$a'_{3333} = a_{3333}.$$
Theorem 4.1 The totality of phase velocities and polarization ratios of Rayleigh waves propagating in all the directions on the traction-free surface of the half-space \( x_3 \leq 0 \), to first order of the material anisotropy, carries information only on 13 components of the anisotropic part \( \mathbf{A} \), namely on \( a_{1111}, a_{1122}, a_{1133}, a_{1112}, a_{2222}, a_{2233}, a_{2212}, a_{3333}, a_{3312}, a_{2323}, a_{2313}, a_{1313}, \) and \( a_{1212} \).

Hence, for any \( \theta \) (\( 0 \leq \theta < 2\pi \)), the phase velocity \( v_R(\theta) \) and polarization ratio \( r_R(\theta) \), to first order of the material anisotropy, do not carry any information on the remaining 8 components \( a_{1123}, a_{1113}, a_{2223}, a_{2213}, a_{3323}, a_{3313}, a_{2312} \) and \( a_{1312} \) of \( \mathbf{A} \).

Secondly, suppose that we could measure the phase shift \( \Delta_R(\theta) \) for each \( 0 \leq \theta < 2\pi \). According to Theorem 3.5 only the two components \( a_{2223}', a_{3323}' \) affect the first-order perturbation of the phase shift \( \Delta_R(\theta) \). We see from (18) that these two components are written in terms of the components \( (a_{ijkl}) \) as

\[
a_{2223}' = -a_{1113} \sin^3 \theta + (2a_{1212} + a_{1123}) \sin^2 \theta \cos \theta - (a_{2212} + a_{2312}) \sin \theta \cos^2 \theta + a_{2223} \cos^3 \theta,
\]

\[
a_{3323}' = -a_{3313} \sin \theta + a_{3323} \cos \theta.
\]

Therefore, we obtain

Theorem 4.2 The totality of phase shifts of Rayleigh waves propagating in all the directions on the traction-free surface of the half-space \( x_3 \leq 0 \), to first order of the material anisotropy, carries information on 8 components of the anisotropic part \( \mathbf{A} \), namely on \( a_{1123}, a_{1113}, a_{2223}, a_{2213}, a_{3323}, a_{3313}, a_{2312}, a_{1312} \).

These components are exactly the same as those on which the phase velocity \( v_R(\theta) \) and polarization ratio \( r_R(\theta) \), to first order of the material anisotropy, do not carry any information.

5. Rayleigh waves in the Stroh formalism

The results in Section 4 follow almost immediately from the perturbation formulas in Section 3. Hence we shall briefly sketch a derivation of the perturbation formulas. The Stroh formalism plays an important role there. In this section we give a review of part of the Stroh formalism for Rayleigh waves in anisotropic elasticity.

Let \( \mathbb{R} \) denote the set of real numbers and \( \mathbb{C} \) the set of complex numbers. Let

\[
\mathbf{m} = (m_1, m_2, m_3) \quad \text{and} \quad \mathbf{n} = (n_1, n_2, n_3)
\]

be orthogonal unit vectors in \( \mathbb{R}^3 \). Then

\[
\mathbf{u} = (u_1, u_2, u_3) = \mathbf{a} e^{-\sqrt{-1}k(m \cdot x + p n \cdot x - v t)} \in \mathbb{C}^3
\]

is the surface-wave solution to (1) in the half-space \( \mathbf{n} \cdot \mathbf{x} \leq 0 \) which decays exponentially as \( \mathbf{n} \cdot \mathbf{x} \rightarrow -\infty \) and has direction of propagation \( \mathbf{m} \), phase velocity \( v \), wave number \( k \), and traction

\[
-\sqrt{-1}k \mathbf{l} e^{-\sqrt{-1}k(m \cdot x - v t)} \in \mathbb{C}^3
\]

on the surface \( \mathbf{n} \cdot \mathbf{x} = 0 \) if and only if \( p \in \mathbb{C} \) and \( \begin{bmatrix} \mathbf{a} \\ 1 \end{bmatrix} \in \mathbb{C}^6 \) satisfy the six-dimensional eigenrelation

\[
\mathbf{N} \begin{bmatrix} \mathbf{a} \\ 1 \end{bmatrix} = p \begin{bmatrix} \mathbf{a} \\ 1 \end{bmatrix}, \quad \text{Im} \, p > 0.
\]

Here \( \text{Im} \, p \) denotes the imaginary part of \( p \), and \( \mathbf{N} \) is the \( 6 \times 6 \) matrix defined by

\[
\mathbf{N} = \mathbf{N}(\mathbf{m}, \mathbf{n}) = \begin{bmatrix} -\mathbf{T}^{-1} \mathbf{R}^T & \mathbf{T}^{-1} \\ -\mathbf{Q} + \mathbf{RT}^{-1} \mathbf{R}^T & -\mathbf{RT}^{-1} \end{bmatrix},
\]
where \( Q, R, T \) are the \( 3 \times 3 \) matrices given by

\[
Q = \left( \sum_{j,l=1}^{3} C_{ijkl}^d m_j m_l \right)_{i|k=1,2,3}, \quad R = \left( \sum_{j,l=1}^{3} C_{ijkl}^d m_j n_l \right)_{i|k=1,2,3},
\]

\[
T = \left( \sum_{j,l=1}^{3} C_{ijkl}^d n_j m_l \right)_{i|k=1,2,3}
\]

and

\[
C_{ijkl}^d = C_{ijkl}^{lo} + a_{ijkl} - \rho v^2 m_j m_l \delta_{ik} \quad (1 \leq i, j, k, l \leq 3).
\]  

(25)

Since the matrix \( T \) is positive definite for small \( v \) (i.e., for \( 0 \leq v < v_L \), where \( v_L \) is the limiting velocity defined below), the eigenvalues \( p_\alpha \) \((1 \leq \alpha \leq 6)\) of \( N \) are not real, and they occur in complex conjugate pairs. Henceforth we take

\[
\text{Im} \ p_\alpha > 0, \quad \alpha = 1, 2, 3.
\]

If the eigenvalue problem (23) does not have six linearly independent eigenvectors, then the generalized eigenvector(s) must be introduced. Let \( \begin{bmatrix} a_\alpha \\ l_\alpha \end{bmatrix} \in \mathbb{C}^6 \) be linearly independent eigenvector(s) and generalized eigenvector(s) of \( N \) associated with the eigenvalues \( p_\alpha \) \((1 \leq \alpha \leq 3)\). In this case, the form of the surface-wave solution (21) must be modified according to the degeneracy of the eigenvalue problem (23). Nevertheless, the corresponding tractions on the surface \( n \cdot x = 0 \) always have the same form as (22), i.e.,

\[
-\sqrt{-1} k l_\alpha e^{-\sqrt{-1} k(m \cdot x - v t)}, \quad \alpha = 1, 2, 3,
\]

(26)

and the corresponding displacements on \( n \cdot x = 0 \) have the same form as (21) with \( n \cdot x = 0 \), i.e.,

\[
a_\alpha e^{-\sqrt{-1} k(m \cdot x - v t)}, \quad \alpha = 1, 2, 3.
\]

(27)

For the details, we refer to Section 3 of [4].

Let \( m' \) and \( n' \) be orthogonal unit vectors which are obtained by rotating the orthogonal unit vectors \( m \) and \( n \) around \( m \times n \) by an angle \( \phi \) so that

\[
m'(\phi) = (m'_1, m'_2, m'_3) = m \cos \phi + n \sin \phi, \\
n'(\phi) = (n'_1, n'_2, n'_3) = -m \sin \phi + n \cos \phi.
\]

(28)

Let \( Q(\phi), R(\phi), T(\phi) \) be the \( 3 \times 3 \) matrices given by

\[
Q(\phi) = \left( \sum_{j,l=1}^{3} C_{ijkl}^d m'_j m'_l \right), \quad R(\phi) = \left( \sum_{j,l=1}^{3} C_{ijkl}^d m'_j n'_l \right),
\]

(29)

\[
T(\phi) = \left( \sum_{j,l=1}^{3} C_{ijkl}^d n'_j m'_l \right).
\]

The limiting velocity \( v_L = v_L(m, n) \) is defined as the lowest velocity for which the matrix \( T(\phi) \) becomes singular for some angle \( \phi \). Then for \( 0 \leq v < v_L \), \( T(\phi) \) is positive definite for all real \( \phi \). It can be easily seen that \( v_L = \sqrt{\mu/\rho} \) for isotropic elasticity.
Let $N(\phi)$ be the $6 \times 6$ matrix defined by

$$N(\phi) = N(\phi, v) = \begin{bmatrix} -T^{-1}(\phi)R^T(\phi) & T^{-1}(\phi) \\ -Q(\phi) + R(\phi)T^{-1}(\phi)R^T(\phi) & -R(\phi)T^{-1}(\phi) \end{bmatrix}.$$ 

Then $N(0) = N(0, v)$ is equal to $N$ in (24). We define the angular average of $N(\phi)$ over $[0, 2\pi]$ as follows:

$$S(v) = S(m, n) = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_3^T \end{bmatrix} = \frac{1}{2\pi} \int_0^{2\pi} N(\phi) d\phi, \quad (30)$$

where $S_1 = S_1(v), S_2 = S_2(v)$ and $S_3 = S_3(v)$ are $3 \times 3$ real matrices defined by

$$S_1 = \frac{1}{2\pi} \int_0^{2\pi} -T^{-1}(\phi)R^T(\phi) d\phi, \quad S_2 = \frac{1}{2\pi} \int_0^{2\pi} T^{-1}(\phi) d\phi,$$

$$S_3 = \frac{1}{2\pi} \int_0^{2\pi} (-Q(\phi) + R(\phi)T^{-1}(\phi)R^T(\phi)) d\phi. \quad (31)$$

**Lemma 5.1** ([5, 6, 7, 8]) For $0 \leq v < v_L$, take $C^3$-vectors $a = a(v)$ and $l = l(v)$ so that the $C^6$-vector $\begin{bmatrix} a \\ l \end{bmatrix}$ is an eigenvector or a generalized eigenvector of $N(0) = N(0, v)$ corresponding to an eigenvalue with positive imaginary part. Then for $0 \leq v < v_L$,

$$S(v) \begin{bmatrix} a \\ l \end{bmatrix} = \sqrt{-1} \begin{bmatrix} a \\ l \end{bmatrix}.$$

Rayleigh waves are elastic surface waves which propagate along the surface $\mathbf{n} \cdot \mathbf{x} = 0$ in the direction of $\mathbf{m}$ with the phase velocity $v_R$ in the subsonic range $0 < v < v_L$, whose amplitude decays exponentially as $\mathbf{n} \cdot \mathbf{x} \longrightarrow -\infty$, and which produce no tractions on $\mathbf{n} \cdot \mathbf{x} = 0$. We take $C^3$-vectors $a_\alpha = a_\alpha(v)$ and $l_\alpha = l_\alpha(v)$ ($\alpha = 1, 2, 3$) so that $\begin{bmatrix} a_\alpha \\ l_\alpha \end{bmatrix}$ ($\alpha = 1, 2, 3$) are linearly independent eigenvector(s) and generalized eigenvector(s) of $N(0) = N(0, v)$ corresponding to the eigenvalues $p_\alpha$ ($\alpha = 1, 2, 3, \text{Im} p_\alpha > 0$). Then the existence of Rayleigh waves implies that a linear combination of the tractions of the form (26) vanishes for $v = v_R$. This means that there exists a set of complex numbers $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that

$$\sum_{\alpha=1}^{3} c_\alpha l_\alpha = 0 \quad \text{at } v = v_R. \quad (32)$$

Then by (27),

$$\sum_{\alpha=1}^{3} c_\alpha a_\alpha e^{-\sqrt{-1}k(m \cdot x - v_R t)}$$

at $v = v_R$ is the displacement field of the Rayleigh waves at the surface $\mathbf{n} \cdot \mathbf{x} = 0$, and

$$a_{\text{pol}} = \sum_{\alpha=1}^{3} c_\alpha a_\alpha \in \mathbb{C}^3$$

at $v = v_R$ is the polarization vector of the Rayleigh waves at $\mathbf{n} \cdot \mathbf{x} = 0$. The real part $a^+ \in \mathbb{R}^3$ and the imaginary part $a^- \in \mathbb{R}^3$ of $a_{\text{pol}}$ define the plane to which the paths of surface particles are
confined, so that their displacements are expressed by taking the real part of \( a_{\text{pol}} e^{-\sqrt{-1} k(m \cdot x - v_R t)} \) or
\[
a^+ \cos k(m \cdot x - v_R t) + a^- \sin k(m \cdot x - v_R t).
\] (33)

On the other hand, from Lemma 5.1 and (30) it follows that
\[
S_1 a_\alpha + S_2 l_\alpha = \sqrt{-1} a_\alpha, \quad \alpha = 1, 2, 3
\] (34)
and
\[
S_3 a_\alpha + S_1^T l_\alpha = \sqrt{-1} l_\alpha, \quad \alpha = 1, 2, 3.
\] (35)

Then (34) and (35), combined with (32), imply that for \( c_\alpha \) in (32)
\[
S_1 a_{\text{pol}} = \sqrt{-1} a_{\text{pol}} \quad \text{at} \quad v = v_R
\] (36)
and
\[
S_3 a_{\text{pol}} = 0 \quad \text{at} \quad v = v_R.
\] (37)

It is easily seen that the two vectors \( a^+ \) and \( a^- \) are linearly independent. In fact, suppose that there exists a real number \( k \) such that \( a^+ = k a^- \) holds. Then, since (36) implies that
\[
S_1 a^+ = -a^- \quad \text{and} \quad S_3 a^- = a^+,
\] (38)
it follows that \( k^2 = -1 \), which is a contradiction. Therefore, the null space of \( S_3(v_R) \) is two-dimensional, which leads to

**Theorem 5.2 ([7, 9])** Let \( v_R \) be the phase velocity of Rayleigh waves which propagate along the surface of the half-space \( n \cdot x \leq 0 \) in the direction of \( m \), and let \( S_3 = S_3(v) \) be the \( 3 \times 3 \) matrix defined by (31). Then
\[
\text{rank } S_3 = 1 \quad \text{at} \quad v = v_R.
\] (39)

We shall use this theorem in our investigation of the perturbation of \( v_R \) caused by the anisotropic part \( A \).

On the basis of this theorem and its proof, we obtain another theorem which will be used in our investigation of the perturbations of \( r_R \) and \( \Delta_R \).

**Theorem 5.3 ([7])** Let \( S_1(v) \) and \( S_3(v) \) be the \( 3 \times 3 \) real matrices in (31), and let \( e_1 \) and \( e_2 \) be orthogonal unit vectors in \( \mathbb{R}^3 \) such that
\[
S_3(v_R) e_1 \neq 0, \quad (S_3(v_R) e_1) \times e_2 \neq 0,
\] (40)
where the symbol \( \times \) denotes the cross product of vectors. Then the Rayleigh waves described above have at the surface \( n \cdot x = 0 \) the polarization vector
\[
a_{\text{pol}} = (S_3(v_R) e_1) \times e_2 - \sqrt{-1} S_1(v_R) \left[ (S_3(v_R) e_1) \times e_2 \right].
\] (41)

**Proof.** Formula (39) implies that there is a non-zero vector \( s \in \mathbb{R}^3 \) such that
\[
S_3(v_R) = \pm s \otimes s,
\] (42)
where the symbol \( \otimes \) denotes the tensor product of two vectors. Then from (37) it follows that
\[
S_3(v_R) a_{\text{pol}} = \pm (s \otimes s) a_{\text{pol}} = \pm (a_{\text{pol}} \cdot s) s = 0.
\] (43)
Since \( s \neq 0 \), we have
\[
a_{\text{pol}} \cdot s = a^+ \cdot s + \sqrt{-1} a^- \cdot s = 0,
\]
and hence
\[
a^+ \cdot s = a^- \cdot s = 0.
\]
This implies that \( s \) is normal to the plane of polarization.

Let \( e_1 \) and \( e_2 \) be orthogonal unit vectors in \( \mathbb{R}^3 \) which satisfy (40). Then \( S_3(v_R)e_1 \), which is equal to \( (e_1 \cdot s)s \) or \(- (e_1 \cdot s)s\), is a non-zero scalar multiple of \( s \), and hence, is normal to the plane of polarization. Therefore,
\[
(S_3(v_R)e_1) \times e_2
\]
is a non-zero real vector lying in the plane of polarization. By taking this vector to be the real part \( a^+ \), from (38) the imaginary part \( a^- \) is given by
\[
-S_1(v_R) \left[ (S_3(v_R)e_1) \times e_2 \right].
\]

\[\square\]

6. Perturbation of phase velocity \( v_R \)

To use Theorem 5.2, we first give an expression for \( S_3(v) \) which is correct to first order in the components of the anisotropic part \( A \). Let \( I \) be the \( 3 \times 3 \) identity matrix. We see from (25), (29) and (31) that \( S_3(v) \) can be written as
\[
S_3(v) \approx S_3^{\text{iso}}(v) + S_3^{\text{Aniso}}(v).
\]

Here
\[
S_3^{\text{iso}}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( -Q_v^{\text{iso}}(\phi) + R_v^{\text{iso}}(\phi)T_v^{\text{iso}}(\phi)^{-1}R_v^{\text{iso}}(\phi)^T \right) d\phi
\]
is of zeroth order in \( A \), where
\[
Q_v^{\text{iso}}(\phi) = \left( \sum_{j,l=1}^{3} C_{ijkl} m'_j m'_l \right) - \rho v^2 \cos^2 \phi I,
\]
\[
R_v^{\text{iso}}(\phi) = \left( \sum_{j,l=1}^{3} C_{ijkl} m'_j m'_l \right) + \rho v^2 \cos \phi \sin \phi I,
\]
\[
T_v^{\text{iso}}(\phi) = \left( \sum_{j,l=1}^{3} C_{ijkl} m'_j m'_l \right) - \rho v^2 \sin^2 \phi I,
\]
and
\[
S_3^{\text{Aniso}}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( -Q_v^{\text{Aniso}}(\phi) + R_v^{\text{Aniso}}(\phi)T_v^{\text{iso}}(\phi)^{-1}R_v^{\text{iso}}(\phi)^T + R_v^{\text{iso}}(\phi)T_v^{\text{iso}}(\phi)^{-1}R_v^{\text{Aniso}}(\phi)^T \right. \\
- R_v^{\text{iso}}(\phi)T_v^{\text{iso}}(\phi)^{-1}T_v^{\text{Aniso}}(\phi)T_v^{\text{iso}}(\phi)^{-1}R_v^{\text{iso}}(\phi)^T d\phi
\]
is of first order in \( A \), where
\[
Q_v^{\text{Aniso}}(\phi) = \left( \sum_{j,l=1}^{3} a_{ijkl} m'_j m'_l \right),
\]

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\( \mathbf{R}^{\text{Aniso}}(\phi) = \left( \sum_{j,l=1}^{3} a_{ijkl} m'_{j} n'_l \right), \quad (51) \)

\( \mathbf{T}^{\text{Aniso}}(\phi) = \left( \sum_{j,l=1}^{3} a_{ijkl} n'_{j} m'_l \right), \quad (52) \)

and we have used the notation \( \approx \) in (44) to indicate that we are retaining terms up to those linear in the anisotropic part \( \mathbf{A} \) and are neglecting the higher order terms (see Section 4 of [4]).

From now on, we consider Rayleigh waves under the setting of Section 3, i.e., Rayleigh waves which propagate along the surface of the half-space \( x_3 \leq 0 \) in the direction of the 2-axis. Let \( v_R \) be the phase velocity of such Rayleigh waves. Then we may take

\[ \mathbf{m} = (0, 1, 0) \quad \text{and} \quad \mathbf{n} = (0, 0, 1) \]

and get from (28)

\[ \mathbf{m}' = (m'_1, m'_2, m'_3) = (0, \cos \phi, \sin \phi), \]
\[ \mathbf{n}' = (n'_1, n'_2, n'_3) = (0, -\sin \phi, \cos \phi). \quad (53) \]

The explicit form of \( \mathbf{S}_{3}^{\text{Iso}} \) is computed from (45)–(48) and (53) as (cf. (6.57) in [7] and Appendix 1 in [4])

\[ \mathbf{S}_{3}^{\text{Iso}}(v) = \begin{pmatrix} -\sqrt{\mu(\mu - V)} & 0 & 0 \\ 0 & R(v) & 0 \\ 0 & 0 & \sqrt{\frac{\lambda + 2\mu}{\lambda + 2\mu - V}} \sqrt{\frac{\mu - V}{\mu}} R(v) \end{pmatrix}, \quad (54) \]

where

\[ R(v) = \frac{1}{V} \left( \sqrt{\frac{\mu}{\mu - V}} (2\mu - V)^2 - \sqrt{\frac{\lambda + 2\mu - V}{\lambda + 2\mu}} 4\mu^2 \right), \quad (55) \]

and

\[ V = \rho v^2. \]

Since \( v_{R}^{\text{Iso}} \), the velocity of Rayleigh waves in the comparative isotropic medium, satisfies \( \rho (v_{R}^{\text{Iso}})^2 < \mu \), and since the strong convexity condition (3) for \( \mathbf{C}^{\text{Iso}} \) implies \( \mu < \lambda + 2\mu \), we obtain from Theorem 5.2 and (54) that

\[ R(v_{R}^{\text{Iso}}) = 0. \quad (56) \]

We can easily check that this is equivalent to (14).

**Lemma 6.1 ([1])** Let \( s_{ij}(v) \) \((i, j = 1, 2, 3)\) be the \((i, j)\) components of the matrix \( \mathbf{S}_{3}^{\text{Aniso}}(v) \). Then an approximate secular solution for \( v_R \), i.e., an equation which \( v_R \) satisfies to first order of the anisotropic part \( \mathbf{A} \), is

\[ \tilde{R}(v) = 0, \]

where

\[ \tilde{R}(v) = R(v) + s_{22}(v). \quad (57) \]
Proof. We see from (44) and (54) that

\[
S_3(v) \approx S_3^{\text{iso}}(v) + S_3^{\text{Aniso}}(v) = \begin{pmatrix}
-\sqrt{\mu(\mu - V)} + s_{11} & s_{12} & s_{13} \\
 s_{12} & R(v) + s_{22} & s_{23} \\
 s_{13} & s_{23} & \sqrt{\frac{\lambda + 2\mu}{\mu} \sqrt{\frac{\mu - V}{\mu}} R(v) + s_{33}}
\end{pmatrix}.
\]

By Theorem 5.2, the $(3, 3)$ minor of the preceding matrix, i.e., the determinant of the submatrix formed by striking out the third row and the third column, must vanish at \( v = v_R \). Since \( s_{ij}(v) \) are linear functions of the anisotropic part \( A \), the $(3, 3)$ minor becomes

\[
\det \begin{pmatrix}
-\sqrt{\mu(\mu - V)} + s_{11}(v) & s_{12}(v) \\
 s_{12}(v) & R(v) + s_{22}(v)
\end{pmatrix} = --\sqrt{\mu(\mu - V)}(R(v) + s_{22}(v)) - s_{12}(v)^2 \approx -\sqrt{\mu(\mu - V)}R(v) + s_{11}(v)R(v).
\]

Also, recalling that

\[
R(v_R^{\text{iso}}) = 0,
\]

we have

\[
s_{11}(v_R)R(v) \approx 0
\]

at \( v = v_R \). Since \( v_R^{\text{iso}} \) is less than \( \sqrt{\frac{\mu}{\rho}} \) and \( v_R \) is close to \( v_R^{\text{iso}} \), we get

\[
\sqrt{\mu(\mu - V)} \neq 0
\]

at \( v = v_R \). Hence (59) and Theorem 5.2 imply that

\[
R(v) + s_{22}(v) = \tilde{R}(v) = 0
\]

at \( v = v_R \). This proves the lemma. \(\square\)

Lemma 6.2 ([4])

\[
v_R \approx v_R^{\text{iso}} - \sum \frac{\partial s_{22}(v)}{\partial a_{ijkl}} \bigg|_{A=(a_{ijkl})=O, v=v_R^{\text{iso}}} \times a_{ijkl},
\]

where the summation on the right-hand side is taken for the indices of the 21 independent components in (6).

Proof. From the Taylor expansion of \( v_R \) around \( A = (a_{ijkl}) = O \), we get

\[
v_R \approx v_R^{\text{iso}} + \sum \frac{\partial v_R}{\partial a_{ijkl}} \bigg|_{A=O} a_{ijkl}.
\]

Since we have

\[
\tilde{R}(v_R) = 0,
\]

13
it follows from the implicit function theorem that
\[
\frac{\partial v_R}{\partial a_{ijkl}} \bigg|_{A=0} = - \frac{\partial \tilde{R}}{\partial a_{ijkl}} \bigg|_{A=0,v=v^1_{R}} \left( \frac{\partial \tilde{R}}{\partial v} \bigg|_{A=0,v=v^1_{R}} \right)^{-1}.
\] (62)

From (57) we get
\[
\frac{\partial \tilde{R}}{\partial a_{ijkl}} = \frac{\partial s_{22}(v)}{\partial a_{ijkl}} \quad \text{and} \quad \frac{\partial \tilde{R}}{\partial v} = \left( \frac{\partial R(v)}{\partial v} + \frac{\partial s_{22}(v)}{\partial v} \right). \quad (63)
\]

Since \( s_{22}(v) \) is a linear function of \( A \), it follows that
\[
\frac{\partial s_{22}(v)}{\partial v} \bigg|_{A=0} = 0.
\]

Hence we have
\[
\frac{\partial \tilde{R}}{\partial v} \bigg|_{A=0,v=v^1_{iso}, v^1_{R}} = \frac{\partial R(v)}{\partial v} \bigg|_{v^1_{R}}. \quad (64)
\]

Therefore, from (61)–(64) we obtain (60). \( \square \)

From Lemma (6.2) we see that only \( s_{22}(v) \), the (2,2) component of \( S^{Aniso}_3 \), needs to be computed in order to study the perturbation of \( v_R \). It is possible to compute \( s_{22}(v) \) directly by taking the angular average of the (2,2) component of the integrand of (49). However, we can prove that the effect of the anisotropic part \( A \) on \( s_{22}(v) \) comes only from \( a_{2222}, a_{2233}, a_{3333} \) and \( a_{2323} \). This observation allows us to reduce general anisotropy to the orthorhombic case and provides a much simpler derivation of the perturbation formula for \( v_R \). For the details, we refer to [1].

7. Perturbations of polarization ratio \( r_R \) and phase shift \( \Delta_R \)

To begin with, let us compute the polarization vector \( a_{pol}^{iso} \) of Rayleigh waves which propagate along the surface of the half-space \( x_3 \leq 0 \) of the comparative isotropic medium in the direction of the 2-axis. We use Theorem 5.3. The matrix \( S_1(v) \) for this medium is
\[
S_1^{iso}(v) = \frac{1}{2\pi} \int_{-\pi}^{\pi} -T_v^{iso}(\phi)^{-1} R_v^{iso}(\phi)^T d\phi.
\] (65)

Its explicit form is computed from (47), (48) and (53) as (cf. (6.55) in [7])
\[
S_1^{iso}(v) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -S(v) \\
0 & \sqrt{\frac{\mu}{\mu-V}} & \sqrt{\frac{\lambda+2\mu-V}{\lambda+2\mu}} S(v) \\
\end{pmatrix},
\] (66)

where
\[
S(v) = \frac{1}{V} \left( \sqrt{\frac{\lambda+2\mu}{\lambda+2\mu-V}} - \sqrt{\frac{\mu-V}{\mu}} \right), \quad V = \rho v^2.
\]

Then it follows from (56) that
\[
S(v_R^{iso}) = \frac{2\sqrt{\mu(\mu-V_R^{iso})}}{2\mu-V_R^{iso}} = \frac{1}{2\mu} \sqrt{\frac{\lambda+2\mu}{\lambda+2\mu-V_R^{iso}} (2\mu-V_R^{iso})} > 0, \quad (67)
\]
where
\[ V_{R}^{\text{Iso}} = \rho (v_{R}^{\text{Iso}})^2. \]

Now we apply Theorem 5.3 to isotropic elasticity. Take
\[ e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \]

Then from (54) we obtain
\[ S_3(v_{R}^{\text{Iso}}) e_1 = \begin{pmatrix} -\sqrt{\mu(\mu - V_{R}^{\text{Iso}})} \\ 0 \\ 0 \end{pmatrix}. \]

Since \( V_{R}^{\text{Iso}} = \rho (v_{R}^{\text{Iso}})^2 < \mu \), we get \( S_3(v_{R}^{\text{Iso}}) e_1 \neq 0 \). Next, taking
\[ e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \]
we have
\[ (S_3(v_{R}^{\text{Iso}}) e_1) \times e_2 = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{\mu(\mu - V_{R}^{\text{Iso}})} \end{pmatrix} \neq 0, \]
which is the real part of \( a_{\text{pol}}^{\text{Iso}} \). Then by (66), we get
\[ -S_1(v_{R}^{\text{Iso}}) [ (S_3(v_{R}^{\text{Iso}}) e_1) \times e_2 ] = \begin{pmatrix} 0 \\ 0 \\ -S(v_{R}^{\text{Iso}}) \sqrt{\mu(\mu - V_{R}^{\text{Iso}})} \end{pmatrix}, \]
which is the imaginary part of \( a_{\text{pol}}^{\text{Iso}} \). Thus, we obtain
\[ a_{\text{pol}}^{\text{Iso}} = -\sqrt{\mu(\mu - V_{R}^{\text{Iso}})} \begin{pmatrix} 0 \\ \sqrt{-1} S(v_{R}^{\text{Iso}}) \\ 1 \end{pmatrix}. \]

Therefore, from (33) and (68) the real displacement field of the Rayleigh waves at the surface \( n \cdot x = 0 \) can be written as
\[ -\sqrt{\mu(\mu - V_{R}^{\text{Iso}})} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cos k(\mathbf{m} \cdot \mathbf{x} - v_{R}^{\text{Iso}} t) + \begin{pmatrix} 0 \\ S(v_{R}^{\text{Iso}}) \\ 0 \end{pmatrix} \sin k(\mathbf{m} \cdot \mathbf{x} - v_{R}^{\text{Iso}} t) \right\}. \]

Since the 2-axis and the 3-axis are the propagation direction on the surface and the normal direction to the surface, respectively, the polarization ratio, i.e., the ratio of the maximum longitudinal component to the maximum normal component, is
\[ r_{R}^{\text{Iso}} = S(v_{R}^{\text{Iso}}) = \frac{2\sqrt{\mu(\mu - V_{R}^{\text{Iso}})}}{2\mu - V_{R}^{\text{Iso}}}. \]
Since \( S(v_{R}) \) > 0, the phase shift, i.e., the shift in phase measured from that of the longitudinal component to that of the normal component, is

\[
\Delta_{R}^{\text{iso}} = \frac{\pi}{2}.
\]

Now we compute the polarization vector \( \mathbf{a}_{\text{pol}} \) of Rayleigh waves in Section 3, to within terms linear in the components of the anisotropic part \( A \). We see from (25), (29) and (31) that \( S_{1}(v) \) can be written as

\[
S_{1}(v) \approx S_{1}^{\text{iso}}(v) + S_{1}^{\text{Aniso}}(v),
\]

where \( S_{1}^{\text{iso}}(v) \) is given by (65), which is of zeroth order in \( A \), and

\[
S_{1}^{\text{Aniso}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( -T_{v}^{\text{iso}}(\phi)^{-1}R_{\text{Aniso}}(\phi)^{T} + T_{v}^{\text{iso}}(\phi)^{-1}T_{\text{iso}}^{\text{Aniso}}(\phi)R_{v}^{\text{iso}}(\phi)^{T} \right) d\phi
\]

is of first order in \( A \).

Let \( t_{ij} = t_{ij}(v) \) be the \((i, j)\) components of the matrix \( S_{1}^{\text{Aniso}}(v) \). They are linear functions of \( A \). We see from (66) and (70) that

\[
S_{1}(v) \approx S_{1}^{\text{iso}}(v) + S_{1}^{\text{Aniso}}(v)
\]

\[
= \begin{pmatrix}
    t_{11} & t_{12} & t_{13} \\
    t_{21} & t_{22} & t_{23} \\
    t_{31} & \sqrt{\frac{\mu}{\mu - V}} \sqrt{\frac{\lambda + 2\mu - V}{\lambda + 2\mu}} S(v) + t_{32} & t_{33}
\end{pmatrix}.
\]

Taking \( e_{1} = (1, 0, 0) \) and \( e_{2} = (0, 1, 0) \), we use Theorem 5.3. By (58) and (72), we extract terms up to those linear in \( A \) from

\[
(S_{3}(v)e_{1}) \times e_{2} = \sqrt{-1} S_{1}(v) \left[ (S_{3}(v)e_{1}) \times e_{2} \right].
\]

The result is

\[
\begin{pmatrix}
    -s_{13} \\
    0 \\
    -\sqrt{\mu/\mu - V} + s_{11}
\end{pmatrix} + \sqrt{-1} \begin{pmatrix}
    \sqrt{\mu(\mu - V)} t_{13} \\
    \sqrt{\mu(\mu - V)} t_{13} \\
    \sqrt{\mu(\mu - V)} t_{33}
\end{pmatrix}.
\]

Hence a real form of the displacements on the surface \( x_{3} = 0 \) of the Rayleigh waves is given from (33), to first order in \( A \), as

\[
\begin{pmatrix}
    -s_{31}(v_{R}) \\
    0 \\
    -\sqrt{\mu(\mu - V_{R})} + s_{11}(v_{R})
\end{pmatrix} \cos k(x_{2} - v_{R}t) + \begin{pmatrix}
    \sqrt{\mu(\mu - V_{R})} t_{13}(v_{R}) \\
    \sqrt{\mu(\mu - V_{R})} t_{13}(v_{R}) \\
    \sqrt{\mu(\mu - V_{R})} t_{33}(v_{R})
\end{pmatrix} \sin k(x_{2} - v_{R}t),
\]

where

\[
V_{R} = \rho v_{R}^{2}
\]

16
and \(v_R\) is the phase velocity of Rayleigh waves.

From (7), (8), (9) and (74) we can compute \(r_R\) and \(\Delta_R\), to within terms linear in \(A\). In this process we observe that only \(S(v_R)\) and \(t_{23}(v_{Riso}^\text{iso})\) need to be computed in order to obtain the perturbation formula for \(r_R\) and that only \(t_{33}(v_{Riso}^\text{iso})\) needs to be computed in order to obtain the perturbation formula for \(\Delta_R\). For the computation of \(r_R\), we also need the result of Theorem 3.1. For these observations and the details of computations, we refer to [2].

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