Superconductor Meissner effects for gravito-electromagnetic fields in harmonic coordinates
due to non-relativistic gravitational sources

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Abstract

It is well known that a covariant Lagrangian for relativistic charged particles can lead to a van-
ishing Hamiltonian. Alternatively, it is shown that using a “space+time” Lagrangian leads to a new
canonical momentum and minimal coupling rule that describes the coupling of both electromagnetic
and gravitational fields to a relativistic charged particle. Discrepancies between Hamiltonians ob-
tained by various authors are resolved. The canonical momentum leads to a new form of the London
equations and London gauge. Using the linearized Einstein field equation in harmonic coordinates,
and a non-relativistic ideal fluid, leads to gravito-electromagnetic field equations. These are used
to obtain new penetration depths for both the magnetic and gravito-magnetic fields. A key result is
that the gravito-magnetic field is expelled from a superconductor only when a magnetic field is also
present. The flux quantum in the body of a superconductor, and the quantized supercurrent in a super-
conducting ring are derived. Lastly, the case of a superconducting ring in the presence of a charged
rotating mass cylinder is used as an example of applying the formalism developed.

Introduction

In 1949, the Gravity Research Foundation (GRF) was founded by Roger Babson with the goal of
finding practical applications of gravity such as partial insulators, reflectors, or absorbers of gravity.
“His views were reflected by the wording in the announcement of the first essay competition that said
the awards were to be given for suggestions for anti-gravity devices, for partial insulators, reflectors,
or absorbers of gravity, or for some substance that can be rearranged by gravity to throw off heat-
although not specifically mentioned in the announcement, he was thinking of absorbing or reflecting
gravity waves.”[1]

Yet by 1953, the winning GRF essay by Bryce DeWitt dispensed with such endeavors as unreal-
izable. DeWitt expressed doubt that a material which absorbs or reflects gravitational fields exists.[2]
He states, “...first fix our sights on those grossly practical things, such as ‘gravity reflectors’ or ‘in-
sulators’, or magic ‘alloys’ which can change ‘gravity’ into heat, which one might hope to find as
the usual by-products of new discoveries in the theory of gravitation... Of primary importance is
the extreme weakness of gravitation coupling between material bodies... The weakness of this cou-
pling has the consequence that schemes for achieving gravitational insulation, via methods involving
fanciful devices such as oscillation or conduction, would require masses of planetary magnitude.”

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However, in 1966 DeWitt’s viewpoint changed when he published an influential paper predicting the Lense-Thirring field (also known as frame-dragging or gravito-magnetic field) is expelled from superconductors in a Meissner-like effect. It is evident DeWitt considered quantum mechanical systems (such as superconductors) may possess characteristics that were not previously considered when the previous statement was made barring the possibility of gravitational reflectors or insulators.

In [3], DeWitt begins with the Lagrangian for a relativistic charged particle in an electromagnetic field in curved space-time and develops the associated Hamiltonian. He identifies a minimal coupling rule involving a gravitational vector potential and concludes this result implies the associated gravitational field must be expelled from a superconductor, just as the magnetic field is expelled from a superconductor in the Meissner effect.

DeWitt’s novelty and intuition is commendable, however, there are some technical shortcomings in his treatment. Also, his interpretation of the flux quantization condition, and his order of magnitude calculation for an induced electric current, are questioned here. In particular, the following items will be demonstrated.

1. The Hamiltonian formulated by DeWitt (shown below) contains errors.

\[ H_{DeWitt} = c \left( g^{jk} g_{0j} g_{0k} - g_{00} \right)^{1/2} \left[ m^2 c^2 + g^{jk} (P_j - eA_j) (P_k - eA_k) \right]^{1/2} - cg^{jk} g_{0k} (P_j - eA_j) - ceA_0 \]  

2. The weak field, low velocity limit of the Hamiltonian is missing a critical term. DeWitt reduced the Hamiltonian to

\[ H_{DeWitt} = \frac{1}{2m} \left( \vec{P} - e\vec{A} - m\vec{h}_0 \right)^2 + V \]  

where

\[ V = -eA_0 - \frac{1}{2} mh_{00} \quad \text{and} \quad \vec{h}_0 = c (h_{01}, h_{02}, h_{03}) \]  

Here \( e \) and \( m \) are the charge and mass of an electron, respectively. The Hamiltonian includes a term involving \( mh^2 \) which is inconsistent with a linear approximation in the metric perturbation. The Hamiltonian is also missing several terms that are of comparable magnitude as the ones retained.

3. The minimal coupling rule, \( \vec{P} \rightarrow \vec{P} - e\vec{A} - m\vec{h}_0 \) is missing several terms of comparable magnitude. The missing terms impact the associated London equations, London gauge, and penetration depths for the magnetic field and the gravito-magnetic field. A careful treatment will show that the gravito-magnetic field is only expelled while a magnetic field is also present. In the absence of a magnetic field, the superconductor exhibits a paramagnetic effect rather than a diamagnetic (Meissner) effect for the gravito-magnetic field.

4. The gravito-magnetic field used in DeWitt’s formulation is coordinate-dependent. With an appropriate coordinate transformation, the field can be made to vanish. Therefore, the corresponding gravitational Meissner-like effect is also coordinate-dependent and can be made to vanish as well.
5. DeWitt argues that the flux of the vector $\vec{G} = e\nabla \times \vec{A} + m\nabla \times \vec{h}_0$ must be quantized in units of $h/2$. In actuality, there is an additional term involving the flux of $\vec{E}_G \times \vec{A}$ that should be included.

6. Lastly, DeWitt claims that a magnetic field must arise if a superconducting ring is concentric with a rotating massive cylinder which is producing a gravito-magnetic flux in the ring. He predicts an electric current will be induced with an order-of-magnitude given by

$$I \sim \frac{GmMV}{ed}$$

where $V$ is the rim velocity, $d$ is the diameter, and $M$ is the mass of the rotating cylinder. However, it will be shown that the electric current DeWitt predicted is actually limited to $I < h/(2eL)$, where $L$ is the self-inductance. Furthermore, it will be argued that if there is no external magnetic field present, then the flux of the gravito-magnetic field will be quantized, but this will not induce the flux of a magnetic field.

In Section I, we begin with a covariant Lagrangian for charged relativistic test particles in an electromagnetic field in curved space-time. The Euler-Lagrange equation of motion leads to the geodesic equation of motion modified by the Lorentz four-force in curved space-time. Although the equation of motion correctly describes the dynamics of the particle, the associated Hamiltonian is identically zero and therefore cannot be used to describe a quantum mechanical system such as a superconductor.

Alternatively, a “space+time” Lagrangian is used to obtain a canonical three-momentum and Hamiltonian valid to all orders in the metric. The result is compared to DeWitt’s in (1). Some relevant metric relationships will be used to check the Hamiltonian with that of other authors for confirmation of its validity. The Hamiltonian is then expanded to first order in the metric to show the lowest order coupling of the momentum, electromagnetic fields, and gravitational fields. Again, the result is compared to DeWitt’s result in (2). The Hamiltonian is further simplified by introducing the trace-reversed metric perturbation and assuming non-relativistic gravitational sources.

In Section II, gravito-electric and gravito-magnetic fields are defined in terms of the metric perturbation. Using the stress tensor of a non-relativistic ideal fluid, and the linearized Einstein field equation in harmonic coordinates, leads to gravito-electromagnetic field equations. In addition, the canonical three-momentum is used to develop constitutive equations for the supercurrent. These lead to a new set of London equations describing the interaction of electromagnetic and gravito-electromagnetic fields with a superconductor. A modification to the London gauge condition is also identified.

Next, the constitutive equations are used in the field equations to identify a penetration depth associated with the magnetic field and the gravito-magnetic field. It is found that the usual London penetration depth for the magnetic field is modified by the presence of a gravito-magnetic field,
however, the modification is miniscule. It is also found that in the absence of a magnetic field, the superconductor demonstrates a paramagnetic effect rather than a diamagnetic (Meissner) effect for the gravito-magnetic field. In other words, the gravito-magnetic field is not expelled. However, when the magnetic field and the gravito-magnetic field are both present, then it is found that the gravito-magnetic field is expelled with a penetration depth on the same order as the London penetration depth. However, it is demonstrated that the gravito-magnetic field is a coordinate-dependent quantity and therefore effects associated with it can be made to vanish with an appropriate coordinate transformation.

In Section III, the new minimal coupling rule obtained in Section I will be used to write the Ginzburg-Landau supercurrent with coupling to electromagnetic and gravitational fields. Using the fact that all the fields vanish within the body of the superconductor, and the complex order parameter must be single-valued around a closed path, leads to a quantization condition for the flux of the magnetic and gravitational fields. The quantization condition is compared to DeWitt’s result which states the flux of the vector \( \vec{G} = e \nabla \times \vec{A} + m \nabla \times \vec{h}_0 \) must be quantized in units of \( \hbar/2 \).

Lastly, in Section IV, the canonical momentum is used to develop an expression for the Ginzburg-Landau phase around the perimeter of a superconducting ring. Once again, using the fact that the wave function is single-valued leads to an extension of the Byers-Yang theorem to the case of electromagnetic and gravitational fields. The result is a quantization condition involving the flux of the magnetic and gravitational fields, and the supercurrent around the perimeter of a superconducting ring. A charged, rotating mass cylinder is introduced as a source for electromagnetic and gravitational fields. The effect of placing the rotating cylinder concentric with the superconducting ring is analyzed. It is argued that the electric current predicted by DeWitt in \( (4) \) does not occur. Rather, any pre-existing supercurrent in the ring is quantized along with the flux of electromagnetic and gravitational fields through the ring.

I. A Hamiltonian for Cooper pairs coupled to electromagnetism in curved space-time

Using an action of the form \( S = \int_{\tau_1}^{\tau_2} L d\tau \), where \( \tau \) is proper time, leads to a Lagrangian for charged relativistic test particles in an electromagnetic field in curved space-time that can be written as\(^2\)

\[
L = -mc \sqrt{-g_{\mu\nu} u^\mu u^\nu} - q g_{\mu\nu} u^\mu A^\nu \tag{5}
\]

where \( u^\mu = dx^\mu / d\tau \) is the four-velocity, \( A^\mu = (\phi/c, A^i) \) is the electromagnetic four-potential, and \( m \) and \( q \) are the rest mass and charge of the test particles, respectively. It is well known that using \( (5) \) in the Euler-Lagrange equation of motion

\[
\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial (dx^\mu / d\tau)} \right) = 0 \tag{6}
\]

\(^2\)The signature of the Minkowski metric used here is diag\((-1,+1,+1,+1\)). Greek space-time indices \( \mu, \nu, ... \) run from 0 to 3. Latin spatial indices \( i, j, ... \) run from 1 to 3.
leads to the geodesic equation of motion modified by the Lorentz four-force in curved space-time

\[ \frac{dp^\mu}{d\tau} + m\Gamma^\mu_{\sigma \rho} u^\sigma u^\rho = q g_{\nu \alpha} u^\alpha F^{\mu \nu} \]  

(7)

where \( \Gamma^\mu_{\sigma \rho} \) are the metric connections (Christoffel symbols)\(^3\). This demonstrates that the Lagrangian in \( (5) \) correctly characterizes the dynamics in a covariant form. However, evaluating a canonical four-momentum, \( P_\mu = \partial L / \partial u^\mu \), leads to \( mu^\mu = P^\mu + qA^\mu \), where \( q = -e \) for electrons. This implies a minimal coupling rule given by \( P^\mu \to P^\mu - eA^\mu \). It is well known that applying this minimal coupling rule, and utilizing a covariant Legendre transformation, \( H = P_\mu u^\mu - L \), leads to a Hamiltonian that is identically zero.\(^4\) Therefore, it is evident that a different approach must be taken for obtaining a Hamiltonian as will be shown next.

Note that the four-velocity can be written as \( u^\mu = \gamma v^\mu \), where the Lorentz factor is \( \gamma \equiv dt / d\tau \), the coordinate velocity is \( v^\mu = dx^\mu / dt = (c, v^i) \), and \( t \) is the coordinate time. The action can also be written as \( S = \int_{t_1}^{t_2} Ld\tau = \int_{t_1}^{t_2} L\gamma^{-1} dt \) which is now reparametrized in terms of coordinate time rather than proper time. Therefore, the “space+time” Lagrangian becomes\(^3\)

\[ L = -mc \sqrt{-g_{\mu \nu} v^\mu v^\nu} - q g_{\mu \nu} v^\mu A^\nu \]  

(8)

Using \( g_{\mu \nu} u^\mu u^\nu = -c^2 \), the Lorentz factor in curved space-time can be evaluated as

\[ \gamma = \sqrt{-g_{\mu \nu} v^\mu v^\nu} = (-g_{00} - 2g_{0j}v^j/c - g_{ij}v^jv^j/c^2)^{-1/2} \]  

(9)

Then the canonical three-momentum, \( P_i = \partial L / \partial \dot{v}^i \), can be found from \( (8) \) to be\(^5\)

\[ P_i = \gamma m \left( c g_{0i} + g_{ij}v^j \right) - q \left( g_{0i}A^0 + g_{ij}A^j \right) \]  

(10)

In the absence of electromagnetic and gravitational fields, this reduces to \( P_i = \gamma mv_i \). Using \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \), where \( \eta_{\mu \nu} \) is the Minkowski metric of flat space-time, and \( h_{\mu \nu} \) is a perturbation, leads to

\[ \gamma mv_i = P_i - \gamma m \left( c h_{0i} + h_{ij}v^j \right) + q \left( A_i + h_{0i}A^0 + h_{ij}A^j \right) \]  

(11)

\(^3\)See Box 13.3 in MTW \(^4\) or 3.3 of Wald \(^5\).

\(^4\)This is mentioned in Chapter 12 of Jackson \(^6\) for flat space-time, and discussed in more detail in \( (7) \), \( (8) \), and \( (9) \) for curved space-time. It is also shown in \( (10) \) that a covariant Lagrangian of the form \( L_2 = \frac{1}{2m} g_{\mu \nu} p^\mu p^\nu + eA_\mu u^\mu \) will lead to a non-vanishing Hamiltonian, \( H = \frac{1}{2m} (p^\mu - eA^\mu)^2 \). Using a “space+time” approach leads to the Hamiltonian shown in \( (16) \).

\(^5\)This is essentially the Lagrangian used by DeWitt \(^3\) except he uses the notation \( v^\mu = x^\mu \) and sets \( c = 1 \). Most authors such as DeWitt \(^3\), Cognola, et al. \(^9\) and Bertschinger \(^10\) leave the electromagnetic field in the Lagrangian \( (8) \) as \( A_\mu x^\mu \) instead of \( g_{\mu \nu} A^\mu x^\nu \). However, this neglects the coupling of gravity to the electromagnetic field. As a result, the term involving \( g_{0i}A^0 \) would not appear in the canonical momentum \( (10) \) or the Hamiltonian \( (16) \).

\(^6\)In \( (12) \), a canonical momentum is also proposed in the form of \( P_i = mg_{ik}x^k - eA_i \). However, since it was not formally derived from a Lagrangian, it is missing all the other terms in \( (10) \).
This implies that in the presence of electromagnetic and gravitational fields, there is a modified minimal coupling rule given by

\[ P_i \rightarrow P_i - \gamma m \left( c h_{0i} + h_{ij} v^j \right) + q \left( A_i + h_{0i} A^0 + h_{ij} A^j \right) \]  

(12)

For an electron \((q = -e)\) in flat space-time \((h_{\mu \nu} = 0)\), the minimal coupling rule reduces to the usual \(P_i \rightarrow P_i - e A_i\).

The first term in \([10]\) can be defined as a relativistic kinetic momentum in curved space-time

\[ \pi_i \equiv \gamma m \left( c g_{0i} + g_{ij} v^j \right) \]  

(13)

Notice that in flat space-time, this becomes the usual relativistic kinetic momentum, \(\pi_i = \gamma_d m v_i\) where \(\gamma_d = \left(1 - v^2/c^2\right)^{-1/2}\). Using a Legendre transformation, \(H = P_i v^i - L\), requires solving \([13]\) for \(v^i\) which necessitates constructing the inverse of \(g_{ik}\). This is shown in \([76]\) of Appendix A to be \(g^{ik} \equiv g^{ik} - g^{0i} g^{0k}/g^{00}\) so that \(g^{ik} g_{ik} = \delta^i_i\). Then the velocity and the Lorentz factor can be expressed, respectively, as

\[ v^i = \frac{\tilde{g}^{ik} \pi_k}{\gamma m} - c \tilde{g}^{ik} g_{0k} \quad \text{and} \quad \gamma = \frac{1}{mc} \sqrt{\frac{m^2 c^2 + \tilde{g}^{ik} \pi_j \pi_k}{\tilde{g}^{ik} g_{0j} g_{0k} - g_{00}}} \]  

(14)

Using \([10]\) and \([13]\), as well as \(q = -e\), makes Hamiltonian become

\[ H = c \left( \tilde{g}^{ik} g_{0j} g_{0k} - g_{00} \right)^{1/2} \left[ m^2 c^2 + \tilde{g}^{ik} \left( P_j - e g_{0j} A^0 - e g_{ij} A^i \right) \left( P_k - e g_{0k} A^0 - e g_{ik} A^i \right) \right]^{1/2} \]

\[ - c \tilde{g}^{ik} g_{0k} \left( P_j - e g_{0j} A^0 - e g_{ij} A^i \right) - e c \left( g_{00} A^0 + g_{0i} A^i \right) \]  

(15)

Comparing this Hamiltonian to DeWitt’s result in \([11]\), it is evident DeWitt uses \(g^{ik}\) rather than \(\tilde{g}^{ik}\). However, it is shown in \([77]\) that \(\tilde{g}^{ik} \approx g^{ik}\) is only true to first order in the metric perturbation. Also note that using the metric relations developed in \([78]\) and \([80]\) leads to \([15]\) taking a form that matches \([9]\) and \([10]\).

\[ H = c \left[ \frac{m^2 c^2 + \tilde{g}^{ik} \pi_j \pi_k}{-g^{00}} \right]^{1/2} + c \tilde{g}^{0j} \pi_j - e c \left( g_{00} A^0 + g_{0i} A^i \right) \]  

(16)

Staying to first order in the metric perturbation and assuming non-relativistic velocities makes the Hamiltonian become\(^7\)

\[ H = mc^2 + \frac{\pi^2}{2m} - \frac{1}{2} h_{00} m c^2 - h_{00} \frac{\pi^2}{4m} - ch_{0i} \pi^i + \frac{h_{ij} \pi^i \pi^j}{2m} + e c A^0 - e c h_{00} A^0 - e c h_{0i} A^i \]  

(17)

Using \([10]\) and \([13]\) gives

\[ \pi_i = P_i - e h_{0i} A^0 - e A_i - e h_{ik} A^i \]  

(18)

\(^7\)Note that when working to first order in the metric, spatial indices can be freely raised and lowered.
Inserting (18) into (17) makes the term involving $h_{0i}A^i$ cancel. The Hamiltonian is then expressed in terms of the canonical momentum (to first order in the metric perturbation). Using $P^2 \equiv P^i P^i$, we have

$$H = mc^2 + \frac{1}{2m} (P^i - eA^i)^2 + e\phi$$

$$- \frac{1}{2} h_{00} mc^2 - h_{00} \frac{P^2}{4m} - c h_{0i} P^i + \frac{1}{2m} h_{ij} P^i P^j$$

$$- e h_{00} \phi + \frac{e^2}{2m} \left( 3 h_{ij} A^i A^j - \frac{1}{2} h_{00} A^2 \right) + \frac{e}{2m} \left( h_{00} A^i + \frac{1}{c} \phi h_{0i} - 4 h_{ij} A^i \right) P^i$$

(19)

The Hamiltonian consists of the following terms.

- The first line contains the standard terms for a charged particle coupled to electromagnetic fields. (No coupling to gravity.)

- The second line describes the coupling of the gravitational potentials to the mass and canonical momentum of the test particles. (No coupling to electromagnetism.)

- The third line describes the coupling of the electromagnetic and gravitational potentials together to the charge and canonical momentum of the test particles.

Note that the Hamiltonian contains the scalar, vector, and tensor parts of the metric perturbation which are, respectively, $h_{00}$, $h_{0i}$, and $h_{ij}$. In particular, the tensor part is still first order in the metric perturbation but is missing in DeWitt’s Hamiltonian (2). To eliminate the tensor part, we can use the trace-reversed metric perturbation defined as $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h$, where $h = \eta^{\mu\nu} h_{\mu\nu}$. It will be shown in the following section that using the harmonic coordinate condition, $\partial^\nu \bar{h}_{\mu\nu} = 0$, leads to $\bar{h}_{ij} = 0$ for non-relativistic gravitational sources. Then using $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \bar{h}$, leads to

$$h_{00} = \frac{1}{2} \bar{h}_{00}, \quad h_{0i} = \bar{h}_{0i}, \quad h_{ij} = \frac{1}{2} \bar{h}_{00} \delta_{ij}$$

(20)

Using $A^2 \equiv A^i A^i$, it is found that (19) simplifies to

$$H = mc^2 + \frac{1}{2m} (P^i - eA^i)^2 + e\phi - \frac{1}{4} \bar{h}_{00} mc^2 + \frac{1}{8m} \bar{h}_{00} P^2 - c\bar{h}_{0i} P^i$$

$$+ \frac{e}{2} \bar{h}_{00} \phi + \frac{5e^2}{8m} \bar{h}_{00} A^2 + \frac{e}{2m} \left( \frac{1}{c} \phi \bar{h}_{0i} - \frac{3}{2} \bar{h}_{00} A^i \right) P^i$$

(21)

It is evident that even in this approximation of non-relativistic gravitational sources, there are still additional terms in the Hamiltonian that are missing in (2). Furthermore, notice that DeWitt combines the terms involving $P^i, A^i$, and $h_{0i}$ into a single term expressed as $(P^i - eA^i - cmh_{0i})^2$. This leads to the following errors.
• It implies a minimal coupling rule given by \( \vec{P} \rightarrow \vec{P} - e\vec{A} - m\vec{h} \) rather than the full minimal coupling rule found in [12].

• It predicts a coupling term of the form \( \frac{1}{2m}qA^i h_{0i} \) in the Hamiltonian which is absent when the electromagnetic field in the Lagrangian is properly expressed as \( g_{\mu\nu}A^\mu v^\nu \) instead of \( A_{\mu} v^\mu \).

• It implies a term involving \( m\vec{h}^2 \) which is not consistent with a linear approximation.

II. London equations, Meissner effects, and penetration depths

In this section, gravito-electromagnetic field equations are introduced in harmonic coordinates, then constitutive equations are developed from the canonical momentum and combined with the field equations to obtain new London equations. Using the trace-reversed metric perturbation, \( \bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \), and the harmonic coordinate condition, \( \partial^\nu \bar{h}_{\mu\nu} = 0 \), makes the linearized Einstein equation, \( G_{\mu\nu} = \kappa T_{\mu\nu} \), become \( \Box \bar{h}_{\mu\nu} = -\frac{2}{\kappa} T_{\mu\nu} \), where \( \kappa = 8\pi G/c^4 \). For a non-relativistic ideal fluid, we have

\[
T^{00} = \rho c^2, \quad T^{0i} = \rho cV^i, \quad T^{ij} = 0
\]

which means \( \bar{h}^{ij} = 0 \). A gravito-scalar potential and gravito-vector potential can be defined respectively as

\[
\phi_G \equiv -\frac{c^2}{4} \bar{h}_{00} = -\frac{c^2}{4} \bar{h}^{00} \quad \text{and} \quad h^i \equiv \frac{c}{4} \bar{h}_{0i} = -\frac{c}{4} \bar{h}^{0i}
\]

A gravito-electric field (the Newtonian gravitational field) and a gravito-magnetic field (Lense-Thirring field) can also be defined respectively as

\[
\vec{E}_G \equiv -\nabla \phi_G \quad \text{and} \quad \vec{B}_G \equiv \nabla \times \vec{h}
\]

Defining the mass current density as \( J^i_m = T^{0i}/c = \rho_m V_i \), where \( \rho_m \) is mass density, leads to non-homogeneous field equations given by

\[
\nabla \cdot \vec{E}_G = -\rho_m/\rho_G \quad \text{and} \quad \nabla \times \vec{B}_G = -\mu_G J^i_m + \frac{1}{c^2} \partial_t \vec{E}_G
\]

where \( \rho_G \equiv (4\pi G)^{-1} \) and \( \mu_G \equiv 4\pi G/c^2 \). These can be described as a gravito-Gauss law (Newton’s law of gravitation), and a gravito-Ampere law, respectively.

To develop constitutive equations for a superconductor, we begin by promoting the canonical momentum in (10) to a quantum mechanical operator, \( \hat{P}_i = -i\hbar \partial_i \), and act on the complex order
parameter, \( \Psi (r) = \psi (r) e^{i \vec{p} \cdot \vec{r}} \), where \( \psi^2 = n_s \) is the number density of Cooper pairs. This gives

\[
\hat{P} \Psi = \left[ \gamma m \left( c g_{0i} + g_{ij} \hat{v}^i \right) - q \left( g_{0i} \hat{A}^0 + g_{ij} \hat{A}^j \right) \right] \Psi \tag{26}
\]

Since the bulk of the superconductor is in the zero-momentum eigenstate, then \( \hat{P} \Psi = p_0 \Psi = 0 \). Then taking the expectation value gives

\[
0 = \gamma m \left( c g_{0i} + g_{ij} \langle \hat{v}^i \rangle \right) - q \left( g_{0i} \langle \hat{A}^0 \rangle + g_{ij} \langle \hat{A}^j \rangle \right) \tag{27}
\]

Applying Ehrenfest’s theorem allows this equation to return to a classical equation of motion once again. To first order in the metric perturbation, and first order in test mass velocity, (9) becomes \( \gamma \approx 1 + h_{00}/2 + h_{0j} v^j / c \). Then (27) becomes

\[
(1 + h_{00}/2) mv_i = -m \left( c h_{0i} + h_{ij} v^j \right) + q \left( A_i + h_{0i} A^0 + h_{ij} A^j \right) \tag{28}
\]

Using (20) and (23) leads to

\[
v_i = \frac{q}{m} \vec{A} + \frac{q}{mc^2} \varphi_G \vec{A} - 4 \vec{h} + \frac{4q}{mc^2} \varphi \vec{h} \tag{29}
\]

The charge and mass supercurrent densities are, respectively\(^{11}\)

\[
\vec{J}_c = -n_s q \vec{v} \quad \text{and} \quad \vec{J}_m = n_s m \vec{v} \tag{30}
\]

where \( n_s \) is the number density of Cooper pairs. Inserting (29) into (30), and using \( q = -e \) and \( m = m_e \) for electrons, leads to

\[
\vec{J}_c = -\Lambda_L \left( \alpha \vec{A} - \beta \vec{h} \right) \tag{31}
\]

and

\[
\vec{J}_m = n_e e \left( \alpha \vec{A} - \beta \vec{h} \right) \tag{32}
\]

where \( \Lambda_L \equiv n_e e^2 / m_e \) is the London constant \(^{14}\), and\(^{12}\)

\[
\alpha \equiv 1 + \varphi_G / c^2 \quad \text{and} \quad \beta \equiv \frac{4 \left( m_e c^2 + e \varphi \right)}{ec^2} \tag{33}
\]

The expressions in (31) and (32) are the London constitutive equations for a non-relativistic supercurrent in the presence of electromagnetic fields and gravitational fields (from non-relativistic sources). Notice that if all gravitational fields are neglected, then \( \alpha = 1 \) and \( \vec{h} = 0 \), so (31) becomes the well-known London constitutive equation, \( \vec{J}_c = -\Lambda_L \vec{A} \). Also notice that for the case of a neutral superfluid, setting the charge to zero in (32) gives \( \vec{J}_m = -n_s m_e \vec{h} \) which is the constitutive equation for a neutral superfluid in the presence of a gravito-vector potential.

\(^{10}\)This can be considered a semiclassical approach where the gravitational field, \( h_{\mu \nu} \), is still a classical field while \( \hat{P}, \hat{v}, \) and \( \hat{A} \) are quantum operators that act on the Cooper pair state, \( \Psi \).

\(^{11}\)Note that a negative is used in \( \vec{J}_c = -n_s q \vec{v} \) so that when \( q = -e \) is used, then \( \vec{J}_c \) becomes positive and hence represents the conventional current.

\(^{12}\)Similar expressions to (31) and (32) can be found in \(^{15}\), however, with \( \alpha = 1 \) and \( \beta = m_e / e \).
Taking the time-derivative of (31) and (32), and using the fact that $\nabla \varphi = 0$ inside a superconductor and $\partial_t \vec{h} = 0$ in this approximation, leads to

$$\partial_t \vec{J}_c = \Lambda_L \left( \alpha \vec{E} - \frac{1}{c^2} \phi_G \vec{A} - \frac{4}{c^2} \vec{\phi} \right) \tag{34}$$

and

$$\partial_t \vec{J}_m = -en_s \left( \alpha \vec{E} - \frac{1}{c^2} \phi_G \vec{A} - \frac{4}{c^2} \vec{\phi} \right) \tag{35}$$

Notice that (34) is the usual electric London equation, $\partial_t \vec{J}_c = \Lambda_L \vec{E}$, but with correction terms due to gravity. However, (34) is a redundant equation since $\vec{J}_c = -\vec{J}_m \left( e / m_e \right)$.

Taking the curl of (31) and (32) leads to

$$\nabla \times \vec{J}_c = -\Lambda_L \left( \alpha \vec{B} - \beta \vec{B}_G - \frac{1}{c^2} \vec{E}_G \times \vec{A} \right) \tag{36}$$

and

$$\nabla \times \vec{J}_m = n_s e \left( \alpha \vec{B} - \beta \vec{B}_G - \frac{1}{c^2} \vec{E}_G \times \vec{A} \right) \tag{37}$$

Notice that (36) is the usual magnetic London equation, $\nabla \times \vec{J}_c = -\Lambda_L \vec{B}$, but with correction terms due to gravity. However, for the case of a neutral superfluid, setting the charge to zero makes (37) become $\nabla \times \vec{J}_m = -4n_s m_e \vec{B}_G$. This can be viewed as a gravito-magnetic London equation for a neutral superfluid.

Note that the usual London gauge is $\nabla \cdot \vec{A} \propto \nabla \cdot \vec{J}_c = 0$. By the continuity equation, this means $\partial_t \rho_c = 0$ which is consistent with a static number density of Cooper pairs since $\rho_c = 2n_s e$. Applying the same condition to the modified London constitutive equation in (31) leads to a new London condition given by $\nabla \cdot \left( \alpha \vec{A} - \beta \vec{\phi} \right) = 0$. Using (33) and $\nabla \varphi = 0$ leads to

$$\alpha \nabla \cdot \vec{A} - \frac{1}{c^2} \left( \vec{E}_G \cdot \vec{A} \right) - \beta \nabla \cdot \vec{\phi} = 0 \tag{38}$$

This is the modified London gauge condition associated with the London equations developed above. Since we are using harmonic coordinates, $\partial_{\nu} \vec{h}^{\mu \nu} = 0$, then the last term in (38) can also be expressed using $\nabla \cdot \vec{h} = -\phi_G / c^2$.

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13 In standard London theory, the requirement that $\nabla \varphi = 0$ inside a superconductor follows from inserting $\vec{E} = -\nabla \varphi - \partial_t \vec{A}$ into the electric London equation, $\partial_t \vec{J}_c = \Lambda_L \vec{E}$ which gives $\partial_t \left( \vec{J}_c + \Lambda_L \vec{A} \right) = -\Lambda_L \nabla \varphi$. Since $\vec{J}_c = -\Lambda_L \vec{A}$ is the London constitutive equation, then it follows that $\nabla \varphi = 0$. However, this assumption is not taken for granted and is widely discussed in the literature, as summarized in [16]. In the treatment used here, we can also insert $\vec{E} = -\nabla \varphi - \partial_t \vec{A}$ into (34) to obtain $\partial_t \left[ \vec{J}_c + \Lambda_L \left( \alpha \vec{A} - \beta \vec{\phi} \right) \right] = -\Lambda_L \alpha \nabla \varphi$. Then using (31) requires the bracket is zero and therefore $\nabla \varphi = 0$. 

10
For simplicity, consider a cylindrically symmetric superconductor with an axis normal to the surface of Earth so that \( \vec{A} \) is azimuthal and \( \vec{E}_G \) (due to Earth) is along the axis of the cylinder which means \( \vec{A} \times \vec{E}_G = 0 \). For a steady-state supercurrent, Ampère’s law is \( \nabla \times \vec{B} = \mu_0 \vec{J}_c \), and the gravito-Ampere law is \( \nabla \times \vec{B}_G = -\mu_G \vec{J}_m \). Taking the curl of both field equations, using the identity \( \nabla \times \nabla \times \vec{B} = \nabla \left( \nabla \cdot \vec{B} \right) - \nabla^2 \vec{B} \), where \( \nabla \cdot \vec{B} = 0 \), and using \( (36) \) and \( (37) \) in their respective field equations, leads to \( \nabla^2 \vec{B} = \mu_0 \Lambda_L \left( \alpha \vec{B} - \beta \vec{B}_G \right) \) \( (39) \)

and

\[ \nabla^2 \vec{B}_G = \mu_G n_s e \left( \alpha \vec{B} - \beta \vec{B}_G \right) \] \( (40) \)

Notice that neglecting gravity makes \( \alpha = 1 \) and \( \vec{B}_G = 0 \). Then \( (39) \) becomes a Yukawa-like equation, \( \nabla^2 \vec{B} = \mu_0 \Lambda_L \vec{B} \), where \( \Lambda_L = 1/\sqrt{\mu_0 \Lambda_L} \) is the London penetration depth. The only non-trivial physical solution is \( B(z) = B_0 e^{-|z|/\Lambda_L} \), where \( z \) is the distance from the surface to the interior of the superconductor, and \( B_0 \) is the magnitude of the magnetic field at the surface of the superconductor. This is the standard Meissner effect.

Also notice that for a neutral superfluid, \( (40) \) becomes \( \nabla^2 \vec{B}_G = -4\mu_G n_s m_e \vec{B}_G \) which is a Helmholtz-like differential equation (rather than a Yukawa-like differential equation), and therefore only allows sinusoidal solutions, not exponential solutions. Since there is no exponential decay of the field then there is no penetration depth and no associated Meissner effect. The reason can be traced back to the difference in the sign appearing in the magnetic field equation, \( \nabla \times \vec{B} = \mu_0 \vec{J}_c \), and the gravito-magnetic field equation, \( \nabla \times \vec{B}_G = -\mu_G \vec{J}_m \). The negative sign in the gravito-Ampere law eliminates a gravitational Meissner effect for a neutral superfluid. Physically speaking, this implies a paramagnetic effect instead of a diamagnetic (Meissner) effect. This is in agreement with \([20], [21]\) but in disagreement with \([18]\).

In fact, for a maximum gravito-magnetic field at the surface of the superconductor, the solution to the gravito-Ampere field equation would have the form \( \vec{B}_G = B_{G,0} \cos(kz) \), where \( k^2 = 4\mu_G n_s m_e \). The associated spatial periodicity in the field would be given by \( 2\pi/k \) which gives

\[ \lambda_{(\text{periodicity of } \vec{B}_G)} = \frac{2\pi}{\sqrt{4\mu_G n_s m_e}} \] \( (41) \)

To obtain a numerical estimate for this quantity, consider if each atom contributes two conduction electrons, and only \( 10^{-3} \) of the conduction electrons are in a superconducting state, then \( n_s \approx 2n \left( 10^{-3} \right) \), where \( n = \rho_m/m \) is the number density of atoms. For Niobium, the mass density is \( \rho_m \approx 8.6 \times 10^3 \text{kg/m}^3 \) and the mass per atom is \( m \approx 1.5 \times 10^{-25} \text{kg/atom} \). Then the number density of atoms is \( n \approx 5.7 \times 10^{28} \text{m}^{-3} \) and therefore the number density of Cooper pairs is \( n_s \approx 2n \left( 10^{-3} \right) \approx 1.1 \times 10^{26}\text{m}^{-3} \). Then \( (41) \) gives approximately \( 3.3 \times 10^{15}\text{m} \) which is clearly not measurable on a terrestrial scale.

\(^{14}\)Coupled differential equations similar to \( (39) \) and \( (40) \) can also be found in \([17]\) and \([18]\).
For a charged supercurrent in the presence of both magnetic and gravito-magnetic fields, the differential equations (39) and (40) need to be decoupled to obtain solutions. Solving (39) for $\vec{B}_G$, substituting the result into (40), and canceling common terms gives

$$\nabla^2 \left[ \nabla^2 \vec{B} - \lambda_{L(\text{modified})}^{-2} \vec{B} \right] = 0 \tag{42}$$

where a “modified” London penetration due to the presence of gravity is defined as

$$\lambda_{L(\text{modified})}^{-2} \equiv \alpha \lambda_{L}^{-2} - \beta \mu_G n_s e \tag{43}$$

Assuming $\nabla^2 \vec{B}$ and $\vec{B}$ both go to zero as $r \to \infty$, then the solution to (42) requires that the bracket is zero which leads to $\nabla^2 \vec{B} = \lambda_{L(\text{mod})}^{-2} \vec{B}$. Notice that the first term in (43) encodes a correction due to the gravitational scalar potential since $\alpha \equiv 1 + \varphi_G / e^2$. The second term in (43) encodes a correction due to the gravito-magnetic potential since it can be traced back to the terms involving $\beta \vec{B}_G$ in (39) and (40).

An order of magnitude can be calculated for each term in (43). For example, the London penetration depth for niobium is known to be $\lambda_L \sim 10^{-9}$ m. At the surface of earth, $\varphi_G / c^2 \sim 10^{-10}$ which means the correction to the London penetration due to the earth’s gravitational scalar potential is on the order of $10^{-19}$ m and therefore not observable. For the second term in (43), we can note from (33) that if $m_e c^2 >> e \varphi$, then $\beta \approx 4 m_e / e \sim 10^{-11}$ kg/C. Also using $n_s \sim 10^{26}$ m$^{-3}$ the second term in (43) is $\sim 10^{-30}$ m$^{-2}$. Since the first term is $\alpha \lambda_{L}^{-2} \sim 10^{18}$ m$^{-2}$, and the second term is completely negligible. Hence we find that the presence of a Newtonian and/or Lense-Thirring gravitational field cannot have a measurable effect on the penetration depth of the magnetic field.

Next, solving (40) for $\vec{B}$, substituting the result into (39), and canceling common terms gives

$$\nabla^2 \left[ \nabla^2 \vec{B}_G + k_{\text{(modified)}}^2 \vec{B}_G \right] = 0 \tag{44}$$

where

$$k_{\text{(modified)}}^2 \equiv \mu_G n_s e \beta - \alpha \mu_0 \Lambda_L \tag{45}$$

Assuming $\nabla^2 \vec{B}$ and $\vec{B}$ both go to zero as $r \to \infty$, then the solution to (42) requires that the bracket is zero which leads to $\nabla^2 \vec{B}_G = k_{\text{(mod)}}^2 \vec{B}$. Notice that for a neutral superfluid (or in the absence of a magnetic field), the expression in (45) reduces to $k^2 = 4 \mu_G n_s m_e$ which leads to a paramagnetic effect, as stated before.

However, for a charged supercurrent, the first term in (45) encodes a correction due to the electric scalar potential, while the second term encodes a correction due to the gravitational scalar potential and the electric charge. An order of magnitude can be calculated for the first term using $\beta \approx 4 m_e / e \sim 10^{-11}$ kg/C and $n_s \sim 10^{26}$ m$^{-3}$ which gives $\sim 10^{-30}$ m$^{-2}$. Since $\mu_0 \Lambda_L = \lambda_{L}^{-2}$, then the second term can be expressed in terms of the London penetration depth which gives $\alpha \lambda_{L}^{-2} \sim 10^{18}$ m$^{-2}$. In that case, the second term in (45) far exceeds the first term, and therefore (45) is negative. This leads to a diamagnetic (Meissner) effect. In fact, the gravito-magnetic field is expelled with effectively the same
London penetration depth as the magnetic field. The possibility of a gravito-magnetic Meissner effect is in agreement with [22], [23], [24].

It is helpful to compare the physics contained in (43) and (45). They are analogous in the sense that (43) predicts a diamagnetic (Meissner) effect for the magnetic field, but it is altered by the presence of the gravito-magnetic field, while (45) predicts a paramagnetic effect for the gravito-magnetic field but it is altered by the presence of the magnetic field. In the case of (43), the alteration due to the presence of the gravito-magnetic field is extremely small, so that the diamagnetic (Meissner) effect remains but with a slightly larger penetration depth. However, for the case of (45), the alteration due to the presence of the magnetic field is so substantial that it switches a paramagnetic effect into a diamagnetic (Meissner) effect for the gravito-magnetic field.

Hence the findings above are summarized as follows:

- Supercurrent in the presence of only $\vec{B}$: magnetic Meissner effect.
- Supercurrent in the presence of only $\vec{B}_G$: no gravito-magnetic Meissner effect.
- Supercurrent in the presence of both $\vec{B}$ and $\vec{B}_G$: both magnetic and gravito-magnetic Meissner effects.
- Neutral superfluid in the presence of $\vec{B}_G$: no gravito-magnetic Meissner effect

These results demonstrate an important interaction between electromagnetism, gravitation, and a quantum mechanical system that only occurs when all three are present. The superconductor provides the quantum mechanical system which is necessary to have any kind of Meissner effect. The gravito-magnetic field is required to create a novel gravitational effect. Lastly, the magnetic field is necessary to mediate the interaction. In the absence of a magnetic field, the novel gravitational effect would not take place.

A final important consideration is the issue of coordinate-freedom in linearized General Relativity. The gravito-magnetic field, $\vec{B}_G = \nabla \times \vec{h}$, is a coordinate-dependent quantity which can be made to vanish by a linear coordinate transformation, $x'^\mu = x^{\mu} - \xi^{\mu}$. Since the linearized metric perturbation transforms as

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

then $\vec{B}_G$ transforms as

$$\vec{B}_G' = \vec{B}_G - \frac{1}{4} \nabla \times \vec{\xi}$$

Therefore, the effects associated with $\vec{B}_G$ can be made to vanish. Alternatively, a coordinate-invariant approach can be used which also applies to gravitational waves. This is discussed in [25], [26], [27].

\[15\] A neutral superfluid in the presence of $\vec{B}$ is not expected to have any interaction simply because there is no charge to couple to the magnetic field.
III. Flux quantum in the body of a superconductor

In Ginzburg-Landau theory, the minimal coupling rule, $\hat{P}_i \rightarrow \hat{P}_i - q\hat{A}_i$, makes the supercurrent become \[ \vec{J} = \frac{e}{2m} \left[ \Psi^* (-i\hbar \nabla) \Psi - \Psi (-i\hbar \nabla) \Psi^* - 2e\vec{A} |\Psi|^2 \right] \] (48)

where $\Psi(r)$ is the complex order parameter. Using (20) and (23) in (12), and promoting the canonical momentum to a quantum mechanical operator makes the minimal coupling rule become

$$\hat{P}_i \rightarrow \hat{P}_i - \gamma m \left( 4\vec{h} - \frac{2}{c^2} \varphi_G \vec{v} \right) + q \left( A_i + \frac{4}{c^2} \varphi \vec{h} - \frac{2}{c^2} \varphi_G A_i \right) \quad (49)$$

To first order in the metric perturbation, and first order in test mass velocity, we can use $\gamma \approx 1$ in (49). For convenience, we can also define the entire coupling vector as \[ \vec{C} \equiv -m \left( 4\vec{h} - \frac{2}{c^2} \varphi_G \vec{v} \right) + q \left( \vec{A} + \frac{4}{c^2} \varphi \vec{h} - \frac{2}{c^2} \varphi_G \vec{A} \right) \] (50)

Then the corresponding supercurrent becomes

$$\vec{J} = \frac{e}{2m} \left[ \Psi^* (-i\hbar \nabla) \Psi - \Psi (-i\hbar \nabla) \Psi^* + \vec{C} |\Psi|^2 \right] \quad (51)$$

which reduces to (48) in the absence of gravitation. Using $\Psi(r) = \sqrt{n_s(r)} \, e^{i\theta(r)}$, where $\theta$ is the phase, leads to

$$\vec{J} = \frac{e}{2m} \left( 2\hbar \nabla \theta + \vec{C} \right) n_s \quad (52)$$

In the previous section, it was shown that inside the body of the superconductor (beyond the London penetration depth), all the fields in (29) vanish and therefore the supercurrent velocity is zero. Therefore, using $J_i = 0$ in (52) and $v_i = 0$ in (50) makes (52) become

$$\hbar \nabla \theta = 4\hbar \vec{A} - q\vec{A} - \frac{4q}{c^2} \varphi \vec{h} + \frac{2q}{c^2} \varphi_G \vec{A} \quad (53)$$

Integrating around a closed loop gives

$$\int_C \left( 4m\vec{h} - q\vec{A} - \frac{4q}{c^2} \varphi \vec{h} + \frac{2q}{c^2} \varphi_G \vec{A} \right) \cdot d\vec{l} \quad (54)$$

Since the order parameter is single-valued, then it must return to the same value when the line integral returns to the same point. Therefore, the left side must be $2\pi n$, where $n$ is an integer. Applying Stokes’ theorem on the right side gives

$$\hbar 2\pi n = \int_S \nabla \times \left( 4m\vec{h} - q\vec{A} - \frac{4q}{c^2} \varphi \vec{h} + \frac{2q}{c^2} \varphi_G \vec{A} \right) \cdot d\vec{S} \quad (55)$$

\[16\] An expression similar to (52) is found in [17], however, the terms involving $\varphi$ and $\varphi_G$ in (50) are missing.
where $S$ is the surface bounded by $C$. Using $\nabla \varphi = 0$ within the superconductor, and $q = -2e$ and $m = 2m_e$ for Cooper pairs, gives

$$
e \left( 1 - \frac{2\varphi_G}{c^2} \right) \Phi_B + 4m_e \left( 1 - \frac{q}{2m_e} \frac{\varphi}{c^2} \right) \Phi_{BG} + \frac{2e}{c^2} \int_S \left( \vec{E}_G \times \vec{A} \right) \cdot d\vec{S} = \frac{\hbar}{2}$$  \hspace{1cm} (56)

where $\Phi_B$ and $\Phi_{BG}$ are the flux of $\vec{B}$ and $\vec{B}_G$, respectively. Since $\varphi_G \ll c^2$ and $e \varphi \ll m_e c^2$, then the result can be approximated to

$$e\Phi_B + 4m_e \Phi_{BG} + \frac{2e}{c^2} \int_S \left( \vec{E}_G \times \vec{A} \right) \cdot d\vec{S} = \frac{\hbar}{2}$$  \hspace{1cm} (57)

This result is consistent with [18], [19], and DeWitt’s statement in [3] that “the total flux of $\vec{G}$ linking a superconducting circuit must be quantized in units of $\frac{1}{2}\hbar$,” where $\vec{G} = e \nabla \times \vec{A} + m \nabla \times \vec{h}$. However, none of these authors have the additional term in (57) involving the flux of $\vec{E}_G \times \vec{A}$.

IV. Quantized supercurrent for a superconducting ring in the presence of a “mass solenoid”

To obtain the quantized supercurrent, we begin by promoting the canonical momentum in (10) to a quantum mechanical operator and act on the complex order parameter.

$$\hat{P}_i \Psi = \gamma m \left( g_{0i} + g_{ij} \hat{\psi}^j \right) - q \left( g_{0i} \hat{A}_i^0 + g_{ij} \hat{A}_j^i \right) \Psi$$  \hspace{1cm} (58)

Again we use $\Psi = \psi e^{i\theta}$, but now let $\psi = \sqrt{n_s}$ be a uniform number density around a ring. Therefore, $\hat{P}_i \Psi = \hbar \Psi \partial^i \theta$. Then using (58) and taking the expectation value gives

$$\hbar \left\langle \partial^i \theta \right\rangle = \gamma m \left( g_{0i} + g_{ij} \left\langle \hat{\psi}^j \right\rangle \right) - q \left( g_{0i} \left\langle \hat{A}_i^0 \right\rangle + g_{ij} \left\langle \hat{A}_j^i \right\rangle \right)$$  \hspace{1cm} (59)

Applying Ehrenfest’s theorem allows this equation to return to a classical equation once again. To first order in the metric perturbation, and first order in test mass velocity, we can use $\gamma \approx 1 + \hbar_0/2$. Also using (20) and (23) gives

$$\hbar \nabla \theta = \left( 1 - \frac{3\varphi_G}{c^2} \right) m\vec{v} - \left( 1 - \frac{2\varphi_G}{c^2} \right) q\vec{A} + 4 \left( 1 - \frac{q\varphi}{mc^2} \right) \vec{m}\vec{h}$$  \hspace{1cm} (60)

We can write this expression in terms of the supercurrent density which is $J^i = -qn_s \hat{v}^i$ and integrate around a circular superconducting ring.

$$\hbar \oint_C \left( \nabla \theta \right) \cdot d\vec{l} = \oint_C \left[ -\frac{m}{q} \left( 1 - \frac{3\varphi_G}{c^2} \right) - \left( 1 - \frac{2\varphi_G}{c^2} \right) q\vec{A} + 4 \left( 1 - \frac{q\varphi}{mc^2} \right) \vec{m}\vec{h} \right] \cdot d\vec{l}$$  \hspace{1cm} (61)

Since the order parameter is single-valued, then it must return to the same value when the line integral returns to the same point. Therefore, the left side must be $2\pi n$, where $n$ is an integer.\footnote{This can also be identified as an extended application of the Byers-Yang theorem [29] which ordinarily applies only to a wave function in the presence of a magnetic vector potential.} Applying
Stokes’ theorem on the right side, using \( q = -2e \) and \( m = 2m_e \) for Cooper pairs, and assuming \( \varphi_G \ll c^2 \) and \( q\varphi \ll mc^2 \), gives
\[
\frac{m_e}{2en_s} \oint_C \mathbf{J} \cdot d\mathbf{l} + e\Phi_B + 4m_e\Phi_{BG} + \frac{2e}{c^2} \int_S \left( \mathbf{E}_G \times \mathbf{A} + 2\nabla \varphi \times \mathbf{h} \right) \cdot d\mathbf{S} = \frac{n\hbar}{2} \tag{62}
\]
where \( S \) is the surface bounded by \( C \). The result is similar to (57) which applies to the bulk of the superconductor. However, there is an additional term involving the supercurrent density, and a term involving \( \nabla \varphi \times \mathbf{h} \) since \( \mathbf{J} \) and \( \nabla \varphi \) can be non-zero on the surface of the ring.

As a practical example, we consider a superconducting ring in the presence of a “mass solenoid” as described in Appendix B. This is effectively the same system that was considered by DeWitt [3]. He states, “Now consider an experiment in which the superconductor is a uniform circular ring surrounding a concentric, axially symmetric, quasirigid mass. Suppose the mass, initially at rest, is set in motion until a constant final angular velocity is reached. This produces a Lense-Thirring field
\[
\mathbf{\tilde{V}} \times \mathbf{\tilde{h}}_0 = 16\pi G\mathbf{V}^{-2} \mathbf{\tilde{V}} \times \left( \rho \mathbf{\tilde{V}} \right) \tag{63}
\]
where \( \rho \) and \( \mathbf{\tilde{V}} \) are, respectively, the mass density and velocity field of the rotating mass.\(^{18}\) For a steady-state current, the gravito-Ampere field equation in (25) can indeed be written as \( \nabla^2 \mathbf{\tilde{h}} = \left( 4\pi G/c^2 \right) \rho_m \mathbf{\tilde{V}} \), which matches DeWitt’s equation up to a factor of 4.

DeWitt goes on to state, “Suppose the rotating mass is kept electromagnetically neutral (which means compensating for any Schiff-Barnhill polarization which may be induced in it).” He also writes the “Schiff-Barnhill” field as \( \mathbf{\tilde{F}} = e\mathbf{\tilde{E}} + \frac{1}{2}m\nabla \varphi_0 \). Therefore, by eliminating this field, we would be effectively eliminate \( \mathbf{\tilde{E}}_G \times \mathbf{\tilde{A}} \) and \( \nabla \varphi \times \mathbf{\tilde{h}} \) found in (62) and obtain
\[
\frac{m_e}{2en_s} \oint_C \mathbf{J} \cdot d\mathbf{l} + e\Phi_B + 4m_e\Phi_{BG} = \frac{n\hbar}{2} \tag{64}
\]
DeWitt also states, “Because of the flux quantization condition, the flux of \( G \) through the superconducting ring must remain zero. But since \( \mathbf{\tilde{V}} \times \mathbf{\tilde{h}}_0 \) is nonvanishing in the final state, a magnetic field must be induced.” This implies that DeWitt is effectively setting \( n = 0 \) in (64). Next, DeWitt states, “Then the magnetic field must arise from a current induced in the ring.” It is at this point that we must disagree with DeWitt. It appears that he has also set the first term in (64) independently to zero, then solved for \( \Phi_B \) to obtain \( \Phi_B = -4m_e\Phi_{BG}/e \). This is evident since using the definition of self-inductance, \( L = \Phi_B/I \), and the condition \( \Phi_B = -4m_e\Phi_{BG}/e \), leads to
\[
I = -\frac{4m_e}{eL} \int_S \left( \mathbf{\tilde{V}} \times \mathbf{\tilde{h}} \right) \cdot d\mathbf{S} = -\frac{16\pi Gm_e}{eLc^2} \int_S \left( \nabla^{-2} \rho \mathbf{\tilde{V}} \right) \cdot d\mathbf{\tilde{r}} \tag{65}
\]
which matches DeWitt’s equation (11). However, there is a contradiction since using the condition \( \Phi_B = -4m_e\Phi_{BG}/e \) required setting the first term in (64) to zero and therefore excluded any current in the superconducting ring.

\(^{18}\)Note that DeWitt’s \( \mathbf{\tilde{h}}_0 \) is related to \( \mathbf{\tilde{h}} \) used in this paper by \( \mathbf{\tilde{h}} = 4\mathbf{\tilde{h}}_0 \).
Therefore, we would suggest that a correct interpretation of (64) requires recognizing that $\Phi_B$ and $\Phi_{BG}$ are the flux of external fields that are introduced to the superconductor independently. There is no inductive relationship between them. In fact, if there is initially no external magnetic field, then starting back at the canonical momentum in (58), we must set $A^i = 0$ and hence $\Phi_B$ would be absent in (64) and the expression simply becomes

$$\frac{m_e}{2en_s} \oint_C \vec{J} \cdot d\vec{l} + 4m_e\Phi_{BG} = \frac{h}{2}$$

(66)

If there is no persistent current initially set up in the superconducting ring, then the first term is zero and (66) simply predicts that the flux of any external gravito-magnetic field through the ring must be quantized in units of $h/2$.

Furthermore, DeWitt obtains an order-of-magnitude estimate for his predicted electric current\(^{19}\) by applying Stokes’ theorem to (65), integrating around the perimeter with diameter $d = 2R$, and using $\rho = M/ (\pi R^2 \ell)$, where $\ell$ and $M$ are the length and mass of the cylinder, respectively, to obtain

$I = -64\pi Gm_e MV^{-2}\tilde{V}/ (eLc^2 \ell d)$. DeWitt sets $c = 1$ and evidently assumes $64\pi \tilde{V}^{-2} \sim 1$ to obtain a result of

$$I \sim \frac{Gm_e MV}{ed}$$

(67)

in his equation (12). However, again we must keep in mind that this result was obtained by setting $n = 0$ in (64). Preserving this state requires that the left side remains less than $h/2$. This means requiring $m_e\Phi_{BG} < h/2$ and therefore (65) would give $I = -\frac{4m_e}{eL}\Phi_{BG} < \frac{2h}{eL}$ which is an exceedingly small electric current contrary to DeWitt’s order of magnitude in (67).

Lastly, we return to (62) and develop an expression that correctly describes the quantization condition for a superconducting ring in the presence of a charged “mass solenoid.” Rather than applying $\nabla^{-2}$ to both sides of the gravito-Ampere law as DeWitt did in (63), we can evaluate the gravito-magnetic flux directly as is done in (93) of Appendix B which gives $\Phi_{BG} = \frac{1}{2}\mu_G \pi R^4 \rho_m \omega$. Similarly, the magnetic flux can be found using $\mu_G \rightarrow -\mu_0$ and $\rho_m \rightarrow \rho_c$ which gives $\Phi_B = -\frac{1}{2}\mu_0 \pi R^4 \rho_c \omega$. Also using (82), (94), and (95) in (62) gives

$$\frac{m_e}{en_s} \oint_C \vec{J} \cdot d\vec{l} + F = nh$$

(68)

where

$$F \equiv \pi R^4 \omega \left(4m_e\mu_G\rho_m - e\mu_0\rho_c\right) - \frac{\pi eR^6 \omega \rho_m \rho_c}{2\varepsilon_G \varepsilon_0 c^4}$$

(69)

This result demonstrates that the persistent current is quantized in units of $nh$ with an offset given by $F$ due to the presence of a charged mass solenoid.

\(^{19}\)Papini [30] calculates a similar result using the gravito-magnetic field of the earth which leads to an electric current given by $I = \frac{2\pi MG}{5c^4 R} m_o a$, where $M$, $R$ and $\omega$ are, respectively the mass, radius and angular velocity of the earth, and $L$ and $a$ are the self-inductance and radius of the loop.
Appendix A: The spatial inverse metric

The “spatial inverse metric” can be defined as \( \tilde{g}^{ik} \) where

\[
\tilde{g}^{ik} g_{jk} = \delta^i_j \tag{70}
\]

To find an expression for \( \tilde{g}^{ik} \), we can develop relations between the metric and inverse metric components. Since \( g^{\mu \nu} g_{\lambda \nu} = \delta^\mu_\lambda \), then summing over \( \nu \) gives

\[
g^{\mu 0} g_{\lambda 0} + g^{\mu k} g_{\lambda k} = \delta^\mu_\lambda \tag{71}
\]

We can consider the various combinations of choosing space and time components for \( \mu \) and \( \lambda \). Using (71) and recognizing that \( \delta^i_i = \delta^0_0 = 1 \) and \( \delta^i_0 = 0 \), gives the following.

For \( (\mu, \lambda) = (i, j) \) \( \implies \) \( g^{0i} g_{j0} + g^{ik} g_{jk} = \delta^i_j \tag{72} \)

For \( (\mu, \lambda) = (0, j) \) \( \implies \) \( g^{00} g_{j0} + g^{0k} g_{jk} = \delta^0_j \implies g^{0k} g_{jk} = -g^{00} g_{j0} \tag{73} \)

For \( (\mu, \lambda) = (j, 0) \) \( \implies \) \( g^{j0} g_{00} + g^{jk} g_{0k} = \delta^j_0 \implies g^{jk} g_{0k} = -g^{j0} g_{00} \tag{74} \)

For \( (\mu, \lambda) = (0, 0) \) \( \implies \) \( g^{00} g_{00} + g^{0k} g_{0k} = \delta^0_0 \implies g^{0k} g_{0k} = 1 - g^{00} g_{00} \tag{75} \)

Inserting (72) into (70) and dividing by \( g_{jk} \) gives \( \tilde{g}^{ik} = g^{ik} + g^{0i} g_{j0} / g^{0k} \). From (73) we also have \( g_{jk} = -g^{00} g_{j0} / g^{0k} \) which leads to

\[
\tilde{g}^{jk} = g^{jk} - g^{0j} g_{0k} / g^{00} \tag{76}
\]

This expression is valid to all orders in the metric. If we use \( g_{\mu \nu} = \eta_{\mu \nu} + h_{\mu \nu} \) and \( g_{\mu \sigma} g^{\sigma \nu} = \delta^\nu_\mu \), then the inverse metric (to first order in the metric perturbation) is \( g^{\mu \nu} = \eta^{\mu \nu} - h^{\mu \nu} \). Also using \( (1 + h^{00})^{-1} \approx 1 - h^{00} \) leads to

\[
\tilde{g}^{jk} \approx g^{jk} + h^{0i} h^{0k} (1 - h^{00}) \tag{77}
\]

Therefore, \( \tilde{g}^{jk} \approx g^{jk} \) is only true to first order in the metric perturbation.

There are two other quantities that appear in the Hamiltonian in (15) that can be evaluated here. The first quantity is \( g^{ik} g_{0k} \). Writing (72) with \( \mu = 0 \), \( \lambda = k \) and using \( j \) for the repeated index gives \( g_{jk} = -g^{00} g_{k0} / g^{0j} \). Inserting this into (70) leads to

\[
\tilde{g}^{ik} g_{0k} = -g^{0i}_0 / g^{00} \tag{78}
\]
The second quantity is \( \tilde{g}^{ik}g_{0j}g_{0k} - g_{00} \) which can be evaluated by using (76). This gives
\[
\tilde{g}^{ik}g_{0j}g_{0k} - g_{00} = g^{ik}g_{0j}g_{0k} - \frac{g_{0j}g_{0k}}{g_{00}}g_{0j}g_{0k} - g_{00}
\]  
(79)

Using (74) and (75) leads to
\[
\tilde{g}^{ik}g_{0j}g_{0k} - g_{00} = -\frac{1}{g_{00}}
\]  
(80)

**Appendix B: Gravito-electromagnetic fields of a non-relativistic “mass solenoid”**

Here we consider a rotating cylindrical “mass solenoid” of length \( \ell \) and radius \( R \) (where \( \ell >> R \)) with the axis along the \( z \)-axis from \( z = -\ell/2 \) to \( z = \ell/2 \). We assume that the cylinder rotates at a constant non-relativistic angular velocity and hence has a non-time varying mass current.

Figure 1: A “mass solenoid” with a mass current, \( \vec{J}_m \), creating a gravito-vector potential, \( \vec{h} \), and corresponding gravito-magnetic field, \( \vec{B}_G \). Note that in the diagram, \( \vec{h} \) points in the *opposite* direction of \( \vec{J}_m \) as a result of the negative sign in \( \nabla \times \left( \nabla \times \vec{h} \right) = -\mu_G \vec{J}_m \).

We can use the gravito-Gauss law from (25) to obtain the gravito-electric field, \( \vec{E}_G \). Taking the volume integral of both sides and applying the Divergence theorem gives
\[
\oint_{\text{Surface of } V} \vec{E}_G \cdot d\vec{A} = -\frac{1}{\varepsilon_G} \int_V \rho_mdV
\]  
(81)
Assume a uniform mass distribution and use a cylindrical Gaussian surface with radius \( r \) surrounding the mass solenoid concentrically. This gives

\[
\vec{E}_G = -\frac{R^2 \rho_m}{2 \varepsilon_G r} \hat{r}
\]  

(82)

We can also use a line integral of \( \vec{E}_G \) to find the change in the gravito-scalar potential, \( \varphi_G \), when going from \( r' = r_0 \) to \( r' = r \), where \( r_0 \) is an arbitrary distance away from the mass solenoid such that \( \varphi_G (r_0) = 0 \).

\[
\Delta \varphi_G (r) = -\int_{r_0}^{r} \vec{E}_G \cdot d\vec{r}
\]  

(83)

Evaluating the integral gives

\[
\varphi_G (r) = \frac{R^2 \rho_0}{2 \varepsilon_G} \ln \left( \frac{r}{r_0} \right)
\]  

(84)

Next we can calculate the gravito-vector potential, \( \vec{h} \), outside the mass solenoid. Since the \( z \)-axis is the axis of symmetry as well as the axis of rotation, then \( \vec{h} = h_\phi (\vec{r}) \hat{\phi} \). To find an expression for \( \vec{h} \), we take a line integral of \( \vec{h} \) along a closed path around the solenoid, use \( \vec{B}_G = \nabla \times \vec{h} \) and apply Stokes’ theorem.

\[
\oint_{\text{Cross section of solenoid}} \vec{B}_G \cdot d\vec{S} = \oint_{\text{Cross section of solenoid}} (\nabla \times \vec{h}) \cdot d\vec{S} = \Phi_{\vec{B}_G}
\]  

(85)

where \( \Phi_{\vec{B}_G} \) is the total gravito-magnetic flux of \( \vec{B}_G \) through a cross-section of the solenoid. If we use a circular path along the \( \hat{\phi} \) direction (with \( z \) in the upward direction), then we also have

\[
\oint_{\text{Cross section of solenoid}} (\nabla \times \vec{h}) \cdot d\vec{S} = h_\phi 2 \pi r
\]  

(86)

So equating (85) and (86) gives

\[
\vec{h} = \frac{\Phi_{\vec{B}_G}}{2 \pi r} \hat{\phi}
\]  

(87)

We can also develop an expression for \( \Phi_{\vec{B}_G} \) (and hence for \( \vec{h} \)) by taking a surface integral of both sides of the gravito-Ampere law from (25) and applying Stokes’ theorem to change the surface integral into a line integral which gives

\[
\oint_{\text{Surrounding solenoid}} \vec{B}_G \cdot d\vec{l} = -\mu_G I_m
\]  

(88)

where \( I_m = \int J_m \cdot d\vec{S} \) is the mass-current. We can use a line integral along a rectangular loop with one edge inside the solenoid parallel to the axis (where \( \vec{B}_G \neq 0 \)) and the opposite edge outside the solenoid (where \( \vec{B}_G = 0 \)). If the length of the edge is \( L \), then we obtain

\[
\vec{B}_G L = -\mu_G I_m
\]  

(89)

The total current in a solenoid is \( I_m = Ni_m \) where \( i_m \) is the current in each loop. If the solenoid is a continuous mass shell, then it is effectively a “perfect” solenoid where the current is distributed
continuously over the surface. Then we can use \( J_m = \sigma_m \omega \) where \( \sigma_m \) is the effective surface mass density of the cylinder spinning with angular velocity \( \omega \). So the total current would be \( I_m = J_m A_\perp \) where \( A_\perp = RL \) is the area normal to the current, and \( R \) is the radius of the solenoid. Then we have \( B_G L = -\mu_G (\sigma_m \omega) (RL) \) which gives \( B_G = -\mu_G R \sigma_m \omega \) (90)

We can now determine the magnitude of the gravito-magnetic flux \( \Phi_{B_G} \) through a cross-sectional area of the solenoid, \( A_{cs} = \pi R^2 \). When we determined the gravito-magnetic field, we already treated the cylinder as a “perfect” solenoid which means it is effectively one “loop” so \( N = 1 \). Then we have \( \Phi_{B_G} = NB_G A_{cs} = (\mu_G R \sigma_m \omega) (\pi R^2) = \mu_G \pi R^3 \sigma_m \omega \) (91)

We can also express this in terms of a volume mass density since \( \sigma_m A = \rho_m V \) \( \implies \sigma_m (2\pi RL) = \rho_m (\pi R^2 L) \implies \sigma_m = R \rho_m / 2 \) (92)

Then the gravito-magnetic flux in (91) becomes \( \Phi_{B_G} = \frac{1}{2} \mu_G \pi R^4 \rho_m \omega \) (93)

We can substitute this back into (87) and (90) to express the gravito-vector potential and the gravito-magnetic field in terms of the physical parameters of the mass solenoid. This gives, respectively,

\[
\vec{\alpha} = -\frac{\mu_0 R^4 \rho_c \omega}{4r} \hat{\phi} \quad \text{and} \quad \vec{B}_G = -\frac{1}{2} \mu_G R^2 \rho_m \omega \hat{z}
\] (94)

If the mass solenoid is also charged, then there will also be a magnetic vector potential and magnetic field. These can be found by simply comparing \( \nabla \times \vec{B}_G = -\mu_0 \vec{J}_m \) and \( \nabla \times \vec{B} = \mu_0 \vec{J}_c \) which implies \( \mu_G \rightarrow -\mu_0 \) and \( \rho_m \rightarrow \rho_c \). Then (94) can be used to immediately obtain

\[
\vec{A} = -\frac{\mu_0 R^4 \rho_c \omega}{4r} \hat{\phi} \quad \text{and} \quad \vec{B} = \frac{1}{2} \mu_0 R^2 \rho_c \omega \hat{z}
\] (95)

Lastly, the electric field can also be obtained by using \( \epsilon_G \rightarrow -\epsilon_0 \) and \( \rho_m \rightarrow \rho_c \) in (82) to obtain

\[
\vec{E} = \frac{R^2 \rho_c}{2\epsilon_0 r} \hat{r}
\] (96)

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