NORM CONVERGENCE OF PARTIAL SUMS OF $H^1$ FUNCTIONS

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Abstract. A classical observation of Riesz says that truncations of a general $\sum_{n=0}^{\infty} a_n z^n$ in the Hardy space $H^1$ do not converge in $H^1$. A substitute positive result is proved: these partial sums always converge in the Bergman norm $A^1$. The result is extended to complete Reinhardt domains in $\mathbb{C}^n$. A new proof of the failure of $H^1$ convergence is also given.

1. Introduction

Let $U \subset \mathbb{C}$ be the unit disc and $\mathcal{O}(U)$ denote the set of holomorphic functions on $U$. If $f \in \mathcal{O}(U)$, with power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$S_N f(z) := \sum_{k=0}^{N} a_k z^k \to f(z)$$

uniformly on compact subsets of $U$. If $(X, \| \cdot \|_X) \subset \mathcal{O}(U)$ is a Banach space of functions, it is natural to ask whether $S_N f$ also converges to $f$ in the norm $\| \cdot \|_X$.

Two classically studied spaces, the Bergman and Hardy spaces, will be considered here. For $p > 0$, the Bergman space $A^p(U)$ is the set of $f \in \mathcal{O}(U)$ such that

$$\| f \|^p_{A^p(U)} = \int_U |f|^p dV < \infty,$$

d$s$ denoting Lebesgue measure on $\mathbb{C}$. The Hardy space $H^p(U)$ is the set of $f \in \mathcal{O}(U)$ such that

$$\| f \|^p_{H^p(U)} = \sup_{0 < r < 1} \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p d\theta < \infty, \quad (1.1)$$

$d\theta$ denoting Lebesgue measure on $[0, 2\pi]$. For $1 \leq p < \infty$, $A^p(U)$ and $H^p(U)$ are Banach spaces.

When $p = 2$, norm convergence of $S_N f$ in either the Bergman or Hardy norm is elementary. For $f \in H^2(U)$, orthogonality of $\{ e^{ik\theta} \}$ on $\partial U$ shows $\| f \|^2_{H^2(U)} = \sum_{k=0}^{\infty} |a_k|^2$. Orthogonality also shows $\| S_N f - f \|^2_{H^2(U)} = \sum_{k=N+1}^{\infty} |a_k|^2$, which tends to 0 as $N \to \infty$. Minor modifications of the argument hold when $A^2(U)$ replaces $H^2(U)$. When $1 < p < \infty$ and $p \neq 2$, the result is the same as in the Hilbert space case but proving this is no longer elementary. Norm convergence of $S_N f$ in $H^p(U)$ for $1 < p < \infty$ is considered classical; a proof is contained in [10] on pages 104–110. Convergence of $S_N f$ in $A^p(U)$ for the same range of $p$ is established by Zhu [19], utilizing the result on $H^p(U)$.

The focus in this paper is $p = 1$. Our interest in this case stems from the widespread occurrence of $L^1$ holomorphic functions, not as an endpoint consideration. For $A^1(U)$ and $H^1(U)$, it is known that partial sum approximation fails; this is also addressed in [10] and [19]. For $A^1(U)$, [19] gives an explicit family of functions $g_\alpha \in A^1(U)$, $\alpha \in U$, such that $\| S_N g_\alpha \|_{A^1}$ is not bounded uniformly in $\alpha$ and $N$. The fact that $\| S_N f - f \|_{A^1} \neq 0$ for all $f \in A^1(U)$ then follows from the uniform boundedness principle. For $H^1(U)$, the proofs in print are somewhat oblique. In [10], it is first shown that $S_N f$ approximating $f$ in

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$H^1$ is equivalent to $h^1$ boundedness of the harmonic conjugation operator, where $h^1(U)$
denotes the harmonic functions with the norm $\| \cdot \|_{H^1}$. Both properties are also shown to
be equivalent to the $L^1$ boundedness of the Szegő projection on $U$. The fact that harmonic
conjugation is not bounded on $h^1(U)$ -- evidenced, e.g., by the Poisson kernel, as discussed in
[10] and [3] -- then implies that partial sum approximation on $H^1(U)$ fails.

The role of harmonic conjugation in this argument does not readily generalize to domains
in $\mathbb{C}^n$ or to non-simply connected domains in the plane. Consequently a new proof of the
failure of $H^1$-approximation of partial sums, in the spirit of [19], is given in the Section 2.1.

Using polar coordinates, it is easy to see $H^p(U) \subset A^p(U)$. The main purpose of this
paper is a substitute positive result for the failure of $H^1$ partial sum approximation: partial
sums of $f \in H^1(U)$ are norm convergent, but in the weaker norm $A^1(U)$.

**Theorem 1.2.** If $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^1(U)$ and $S_N f(z) = \sum_{k=0}^{N} a_k z^k$, then

$$\| S_N f - f \|_{A^1(U)} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$ 

In the Section[3] Theorem[1.2] is extended to give an analogous result on complete bounded
Reinhardt domains $\mathcal{R} \subset \mathbb{C}^n$.

There are other substitutes for the failure of $H^1(U)$ partial sum approximation in the
literature: [16] [17] [2] on the $H^1(U)$ boundedness of the Cesàro operator, [18] [14] on certain
Toeplitz and Hankel operators, [9] [12] on boundedness of the Hausdorff operator with
particular choices of Borel measure, and [13] on boundedness of the Libera operator from
$H^1(U)$ to $H^p(U)$ with $0 < p < 1$. Unlike Theorem 1.2 these results involve modifications of $S_N f$.

As notational shorthand, $|a| \lesssim |b|$ will mean there exists a constant $C > 0$ such that
$|a| \leq C |b|$, with $C$ independent of specified parameters. Let $|a| \approx |b|$ mean both $|a| \lesssim |b|
and $|b| \lesssim |a|$ hold.

2. The result on $U$

For $h(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{O}(U)$ and $N \in \mathbb{Z}^+$, let $S_N h(z) = \sum_{k=0}^{N} a_k z^k$ denote the $N$-th
partial sum of the power series of $h$. Since power series are unique, call these polynomials
partial sums of $h$ for short.

2.1. Failure of norm convergence in $H^1(U)$. A family of integral estimates is used in the
proof of Theorem 2.2 below.

**Lemma 2.1.** For $z \in U$ and $c$ real, define

$$I_c(z) = \int_{0}^{2\pi} \frac{1}{|1 - z e^{-i\theta}|^{1+c}} d\theta.$$ 

If $c < 0$ then $I_c \in L^\infty(U)$. Furthermore,

$$I_c(z) \approx \frac{1}{(1 - |z|^2)^c} \text{ if } c > 0 \text{ and } I_0(z) \approx \log \frac{1}{1 - |z|^2},$$

for constants independent of $z \in U$.

**Proof.** See [15] Proposition 1.4.10, [8], or [20] for the standard proof involving asymptotics
of the Gamma function. See [4] [5] [6] for alternate, elementary proofs that extend to other
singular integrands.

**Theorem 2.2.** There exists $g \in H^1(U)$ such that $S_N g$ does not converge in $H^1(U)$. 

Proof. For \( a \in U \), define \( f_a(z) = \frac{1-|a|^2}{1-\bar{a}z} \). By Lemma 2.1, \( \|f_a\|_{H^1(U)} \lesssim 1 \) with constant independent of \( a \). The power series of \( f_a \) is

\[
f_a(z) = \sum_{k=0}^{\infty} (1 - |a|^2) (k+1)(\bar{a}z)^k.
\]

Consider the partial sum

\[
S_N f_a(z) = \sum_{k=0}^{N} (1 - |a|^2) (k+1)(\bar{a}z)^k = (1 - |a|^2) \sum_{k=0}^{N} \frac{d}{dt} \left( t^{k+1} \right) \bigg|_{t=\bar{a}z} \\
= (1 - |a|^2) \left[ \frac{1 - (\bar{a}z)^{N+2}}{(1-\bar{a}z)^2} - \frac{(N+2)(\bar{a}z)^{N+1}}{1-\bar{a}z} \right] \\
= T_1 + T_2.
\]

\( \|T_1\|_{H^1} \) is uniformly bounded in \( N \) and \( a \), since

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - |a|^2)}{|1 - \bar{a}e^{i\theta}|^2} d\theta \leq \frac{2}{2\pi} \int_0^{2\pi} \frac{(1 - |a|^2)}{|1 - \bar{a}e^{i\theta}|^2} d\theta < \infty
\]

by Lemma 2.1. \( \|T_2\|_{H^1} \) is estimated

\[
\frac{(1 - |a|^2)}{2\pi} \int_0^{2\pi} \frac{(N+2)(\bar{a}e^{i\theta})^{N+1}}{|1 - \bar{a}e^{i\theta}|} d\theta \gtrsim (1 - |a|^2)|a|^{N+1} (N+2) \log \frac{1}{1-|a|}
\]

for a constant independent of \( N \) and \( a \), by Lemma 2.1. Let \( a = \frac{N}{N+1} \). The lower bound on \( \|T_2\|_{H^1} \) goes to infinity as \( N \to +\infty \). Thus \( \|S_N f_a\|_{H^1} \) is unbounded as a function of \( N \) and \( a \). The uniform boundedness principle in contrapositive form gives the stated conclusion.

Remark 2.3. Holomorphic polynomials are dense in \( H^1(U) \) (see Proposition 3.6), but Theorem 2.2 says the sequence of natural polynomials \( \{S_N h\} \) does not approximate a general \( h \in H^1(U) \). Is there a best association of \( h \in H^1(U) \) to a sequence of holomorphic polynomials \( \{p_n\} \) such that \( \|p_n - h\|_{H^1} \to 0 \)? Several interpretations of “best” are possible; the authors are unaware of results in this direction.

2.2. Convergence in \( A^1(U) \). Convergence of \( S_N f \) can be reduced to a bound on the operator norm of \( S_N \). The following is a slight generalization of [19, Proposition 1].

Lemma 2.4. Let \( T_k \), \( k = 1, 2, \ldots \), be a sequence of bounded linear operators from a Banach space \( X \) to a Banach space \( Y \). Suppose that there is a dense subset \( D \) of \( X \) such that for each \( x \in D \), \( T_k x \to 0 \) in the norm of \( Y \) as \( k \to \infty \).

Then the following are equivalent

(i) \( \lim_{k \to \infty} \|T_k x\|_Y = 0 \) for each \( x \in X \).

(ii) there is a \( C > 0 \) such that for each \( k \), we have \( \|T_k\|_{op} \leq C \).

\( \|T_k\|_{op} \) is the operator norm of \( T_k : X \to Y \).

Proof. Assume (i). Then (ii) holds by the uniform boundedness principle.

Assume (ii). Fix \( x \in X \) and \( \epsilon > 0 \). Since \( D \) is dense in \( X \), there exists \( p \in D \) such that \( \|x - p\|_X < \frac{\epsilon}{2C} \). Therefore

\[
\|T_k x\|_Y \leq \|T_k x - T_k p\|_Y + \|T_k p\|_Y < \frac{\epsilon}{2} + \|T_k p\|_Y.
\]

Choosing \( k \) so large that \( \|T_k p\|_Y < \frac{\epsilon}{2} \) yields (i). \( \square \)
The key idea of the proof of Theorem 1.2 is to represent the coefficients in $S_N f$ as integrals, reducing the problem to estimates of geometric series.

Proof of Theorem 1.2. For $b \in \overline{U}$ and $\rho > 0$, let $U(b; \rho)$ denote the disc centered at $b$ of radius $\rho$. Let $U_r = U(0; r)$. For each $U_r$, choose $U_R$ such that $0 < r < R < 1$. If $f(z) = \sum_{j=0}^{\infty} a_j z^j \in H^1(U)$, the Cauchy integral formula gives $a_j = \frac{1}{2\pi i} \int_{\partial U_R} \frac{f(\xi)}{\xi^{j+1}} d\xi$.

Therefore

\[
\int_{U_r} \left| \sum_{j=0}^{N} a_j z^j \right| dV = \int_{U_r} \left| \sum_{j=0}^{N} \frac{1}{2\pi i} \int_{\partial U_R} \frac{f(\xi)}{\xi^{j+1}} d\xi \right| dV(z) \\
= \frac{1}{2\pi} \int_{U_r} \left| \int_{\partial U_R} f(\xi) \sum_{j=0}^{N} \frac{1}{\xi} \left( \frac{z}{\xi} \right)^j d\xi \right| dV(z) \\
= \frac{1}{2\pi} \int_{U_r} \left| \int_{\partial U_R} f(\xi) \frac{1 - \left( \frac{z}{\xi} \right)^{N+1}}{\xi - z} d\xi \right| dV(z) \\
\lesssim \int_{U_r} \int_{\partial U_R} |f(\xi)| \left| \frac{1 - \left( \frac{z}{\xi} \right)^{N+1}}{\xi - z} \right| d|\xi| dV(z) = I.
\]

Since $\xi \in \partial U_R$ and $z \in U_r$, $\left| \frac{z}{\xi} \right| < \frac{r}{R} < 1$. Thus Fubini’s theorem implies

\[
I \lesssim \int_{U_r} \int_{\partial U_R} |f(\xi)| \left| \frac{1}{\xi - z} \right| d|\xi| dV(z) = \int_{\partial U_R} |f(\xi)| \int_{U_r} \left| \frac{1}{\xi - z} \right| dV(z) d|\xi|, \tag{2.5}
\]

with constant independent of $N$.

For any $\xi \in \partial U_R$ fixed, note $U \subset U(\xi; 2)$. Letting $z = \xi + se^{i\sigma}$,

\[
\int_{U_r} \left| \frac{1}{\xi - z} \right| dV(z) \lesssim \int_{U(\xi; 2)} \left| \frac{1}{\xi - z} \right| dV(z) = \int_{0}^{2\pi} \int_{0}^{2} \frac{1}{s} ds d\sigma \lesssim 1,
\]

with constant independent of $\xi$. Thus (2.5) implies

\[
\int_{U_r} \left| \sum_{j=0}^{N} a_j z^j \right| dV \lesssim \int_{\partial U_R} |f(\xi)| d|\xi| \lesssim \|f\|_{H^1(U)},
\]

with constant independent of $N$. Since $\lim_{r \to 1} \|S_N f\|_{A^1(U_r)} = \|S_N f\|_{A^1(U)}$,

\[
\|S_N f\|_{A^1(U)} \lesssim \|f\|_{H^1(U)}. \tag{2.6}
\]

Let $T_N = S_N - \text{id}$, $X = H^1(U)$, $Y = A^1(U)$, and $D = \{\text{holomorphic polynomials}\}$ in Lemma 2.4. Note that for any $p \in D$, $T_N p \equiv 0$ if $N \geq \deg(p)$. Lemma 2.4 says that (2.6) implies $\|S_N f - f\|_{A^1(U)} \to 0$ as $N \to \infty$. \qed

3. Several variable extension

A domain $\Omega \subset \mathbb{C}^n$ is a complete Reinhardt domain if $(z_1, \ldots, z_n) \in \Omega$ implies $(\lambda_1 z_1, \ldots, \lambda_n z_n) \in \Omega$ for all $\lambda_k \in \mathbb{C}$ with $|\lambda_k| < 1$, $k = 1, \ldots, n$.

Let $\mathcal{R}$ be a bounded complete Reinhardt domain in $\mathbb{C}^n$ and $\mathcal{O}(\mathcal{R})$ denote the set of holomorphic functions on $\mathcal{R}$. Each $f \in \mathcal{O}(\mathcal{R})$ has a power series expansion $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha} z^\alpha$, using standard multi-index notation, converging uniformly on compact subsets of $\mathcal{R}$. 


A choice of partial sum of \( f \in \mathcal{O}(\mathcal{R}) \) is required, since the index set is an \( n \)-dimensional lattice. Let \(|\alpha|_\infty = \max\{\alpha_j : \alpha = (\alpha_1, \cdots, \alpha_n)\} \) if \( \alpha \in \mathbb{N}^n \). For \( f(z) = \sum_\alpha b_\alpha z^\alpha \in \mathcal{O}(\mathcal{R}) \), define
\[
S_N f(z) =: \sum_{|\alpha|_\infty \leq N} b_\alpha z^\alpha. \tag{3.1}
\]
Call \( S_N f \) the square partial sum of \( f \).

Let \( \mathbb{T}^n = \{z \in \mathbb{C}^n : |z_j| = 1, j = 1, \cdots, n\} \) and \( \mathbb{U}^n = \{z \in \mathbb{C}^n : |z_j| < 1, j = 1, \cdots, n\} \) denote the unit torus and polydisc, respectively. In the sequel, quantities depending on several variables are sometimes written in bold typeface, scalar quantities in regular, to avoid ambiguity. For \( r = (r_1, \cdots, r_n) \in (\mathbb{R}^n)^+ \), \( \theta = (\theta_1, \cdots, \theta_n) \in \mathbb{R}^n \), and \( z = (z_1, \cdots z_n) \in \mathbb{C}^n \) let \( r \cdot e^{i\theta} = (r_1 e^{i\theta_1}, \cdots, r_n e^{i\theta_n}) \) and \( r \cdot z = (r_1 z_1, \cdots, r_n z_n) \). Dilations of \( \mathbb{T}^n \) and \( \mathbb{U}^n \) are denoted \( r \cdot \mathbb{T}^n = \{r \cdot e^{i\theta} : e^{i\theta} \in \mathbb{T}^n\} \) and \( r \cdot \mathbb{U}^n = \{r \cdot z : z \in \mathbb{U}^n\} \).

### 3.1. Hardy spaces of Reinhardt domains

There is not a canonical definition of Hardy spaces on a general domain, especially in several variables. See [7, 11, 1] for a few of the definitions used. On a Reinhardt domain, the following is reasonable.

#### Definition 3.2

Let \( 0 < p < \infty \). Say \( f \in H^p(\mathcal{R}) \) if \( f \in \mathcal{O}(\mathcal{R}) \) and
\[
\|f\|_{H^p(\mathcal{R})}^p =: \sup_{r \in \mathcal{F}} \int_{\mathbb{T}^n} |f(r \cdot e^{i\theta})|^p \, d\theta_1 \cdots d\theta_n < \infty,
\]
where \( \mathcal{F} = \{r : r \cdot \mathbb{T}^n \subset \mathcal{R}\} \).

An alternate form of the integrals in Definition 3.2 is used in section 3.2. As shorthand, let \( d\theta = d\theta_1 \cdots d\theta_n \). Then
\[
\int_{\mathbb{T}^n} |f(r \cdot e^{i\theta})|^p \, d\theta \approx \int_{r \cdot \mathbb{T}^n} |f(\xi)| |d| |\xi_1| \cdots |d| |\xi_n|,
\]
where \( |d| |\xi_k| \) denotes arc length measure.

#### 3.1.1. Density of polynomials in \( H^p(\mathcal{R}) \)

A fundamental fact about holomorphic functions on \( U \) is
\[
\int_0^{2\pi} |h(Re^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |h(Re^{i\theta})|^p \, d\theta, \quad h \in \mathcal{O}(U), \tag{3.3}
\]
if \( r \leq R < 1 \). See [3, Theorem 1.5].

A version of this monotonicity holds on \( \mathcal{R} \). For \( r = (r_1, \ldots, r_n) \) and \( R = (R_1, \ldots, R_n) \), write \( r \prec R \) to denote \( r_k \leq R_k \) for all \( k = 1, \ldots, n \).

#### Lemma 3.4

If \( r, R \in \mathcal{F}, r \prec R \), and \( f \in \mathcal{O}(\mathcal{R}) \),
\[
\int_{\mathbb{T}^n} |f(r \cdot e^{i\theta})|^p \, d\theta \leq \int_{\mathbb{T}^n} |f(R \cdot e^{i\theta})|^p \, d\theta. \tag{3.5}
\]

**Proof.**
\[
\int_{\mathbb{T}^n} |f(r \cdot e^{i\theta})|^p \, d\theta = \int_{\mathbb{T}^{n-1}} \left( \int_0^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, \ldots, r_n e^{i\theta_n})|^p \, d\theta_1 \right) \, d\theta_2 \cdots d\theta_n 
\leq \int_{\mathbb{T}^{n-1}} \left( \int_0^{2\pi} |f(R_1 e^{i\theta_1}, R_2 e^{i\theta_2}, \ldots, R_n e^{i\theta_n})|^p \, d\theta_1 \right) \, d\theta_2 \cdots d\theta_n
\]
by (3.3). Iteratively applying this to the integrals \( d\theta_2 \cdots d\theta_n \) gives (3.5). \( \square \)

The density of holomorphic polynomials in \( H^p(U) \) is well-known, see [3, Theorem 3.3]. This fact also holds on complete Reinhardt domains in \( \mathbb{C}^n \):
Proposition 3.6. If $\mathcal{R} \subset \mathbb{C}^n$ is a bounded complete Reinhardt domain and $0 < p < \infty$, the set of holomorphic polynomials is dense in $H^p(\mathcal{R})$.

Proof. Let $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \in H^p(\mathcal{R})$. For $0 < s < 1$, define $f_s(z) = f(sz)$.

For any $\sigma \in \mathcal{F}$, consider $I(\sigma) = \int_{\mathbb{T}^n} |f(\sigma e^{i\theta}) - f_s(\sigma e^{i\theta})|^p d\theta$. Lemma 3.4 implies

$$I(\sigma) = \int_{\mathbb{T}^n} |f(\sigma e^{i\theta}) - f_s(\sigma e^{i\theta})|^p d\theta \leq 2^p \left\{ \int_{\mathbb{T}^n} |f(\sigma e^{i\theta})|^p d\theta + \int_{\mathbb{T}^n} |f(s\sigma e^{i\theta})|^p d\theta \right\}$$

$$\leq 2^{p+1} \int_{\mathbb{T}^n} |f(\sigma e^{i\theta})|^p d\theta.$$ 

In particular $I(\sigma) \lesssim \|f\|^p_{H^p}$.

Let $\epsilon > 0$. Since $\lim_{s \to 1} (f(\sigma e^{i\theta}) - f(s\sigma e^{i\theta})) = 0$ for all $\theta$, the dominated convergence theorem gives $\rho$ such that for all $\rho \leq s < 1$

$$\int_{\mathbb{T}^n} |f(\sigma e^{i\theta}) - f_s(\sigma e^{i\theta})|^p d\theta < \epsilon.$$ 

(3.7)

Note $\rho$ depends on both $\sigma$ and $\epsilon$.

The function $f_\rho \in \mathcal{O}\left(\frac{1}{\rho} \mathcal{R}\right)$ and $\mathcal{R} \subset \frac{1}{\rho} \mathcal{R}$. Since the power series

$$f_\rho(z) = \sum_{\alpha \in \mathbb{N}^n} \left( a_\alpha \rho^{|\alpha|} \right) z^\alpha =: \sum_{\alpha \in \mathbb{N}^n} b_\alpha(\rho) z^\alpha$$

converges uniformly on $\bar{\mathcal{R}}$, there exists $M = M(\rho, \epsilon)$ such that

$$\sup_{z \in \mathcal{R}} \left| f_\rho(z) - \sum_{|\alpha| \leq M} b_\alpha(\rho) z^\alpha \right| < \epsilon.$$ 

(3.8)

Let $Q(z) = \sum_{|\alpha| \leq M} b_\alpha(\rho) z^\alpha$. Then

$$\int_{\mathbb{T}^n} |f(\sigma e^{i\theta}) - Q(\sigma e^{i\theta})|^p d\theta \leq 2^p \left\{ I(\sigma) + \int_{\mathbb{T}^n} |f_\rho(\sigma e^{i\theta}) - Q(\sigma e^{i\theta})|^p d\theta \right\}$$

$$< 2^p (\epsilon + \epsilon^p \cdot (2\pi)^n)$$

by (3.7) and (3.8). This holds for any $\sigma \in \mathcal{F}$ and $\epsilon > 0$ was arbitrary, so the claimed density holds.

\[ \square \]

3.2. Partial sums in $H^1(\mathcal{R})$ converge in $A^1(\mathcal{R})$. The definition of Bergman spaces is canonical: $f \in A^p(\mathcal{R})$ if

$$\|f\|^p_{A^p(\mathcal{R})} = \int_{\mathcal{R}} |f|^p d\mathcal{V} < \infty.$$ 

Theorem 1.2 generalizes to the pair $H^1(\mathcal{R}), A^1(\mathcal{R})$. As in one variable, the key fact is an estimate on the operator norm of $S_N$.

Proposition 3.9. There exists a constant independent of $N \in \mathbb{Z}^+$ and $f \in H^1(\mathcal{R})$ such that

$$\|S_N f\|_{A^1(\mathcal{R})} \lesssim \|f\|_{H^1(\mathcal{R})} \quad \forall f \in H^1(\mathcal{R}),$$

where $S_N f$ is the square partial sum $\sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \in H^1(\mathcal{R})$, the Cauchy integral formula implies

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{\mathbb{R}^n} \frac{f(\xi)}{\xi_{a+1}^{a+1} \cdots \xi_{n+1}^{n+1}} d\xi_1 \cdots d\xi_n.$$
On $r \cdot U^n$, the $A^1$ norm of $S_N f$ is

$$
\int_{r \cdot U^n} \left| \sum_{|\alpha| \leq N} a_\alpha z^\alpha \right| dV = \int_{r \cdot U^n} \left| \sum_{|\alpha| \leq N} \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} f(\xi) \frac{1}{\xi_1^{\alpha_1+1} \cdots \xi_n^{\alpha_n+1}} d\xi_1 \cdots d\xi_n z^\alpha \right| dV(z)
$$

$$
\leq \int_{r \cdot U^n} \left| \int_{\mathbb{T}^n} f(\xi) \sum_{|\alpha| \leq N} \frac{z^\alpha}{\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}} d\xi_1 \cdots d\xi_n \right| dV(z)
$$

$$
= \int_{r \cdot U^n} \left| \int_{\mathbb{T}^n} f(\xi) \sum_{|\alpha| \leq N} \sum_{|\alpha| \leq N} \frac{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}{\xi_1^{\alpha_1+1} \cdots \xi_n^{\alpha_n+1}} d\xi_1 \cdots d\xi_n \right| dV(z)
$$

$$
= \int_{r \cdot U^n} \left| \int_{\mathbb{T}^n} f(\xi) \prod_{j=1}^n \frac{1 - (z_j/\xi_j)^{N+1}}{\xi_j - z_j} d\xi_1 \cdots d\xi_n \right| dV(z) = II.
$$

Since $\xi \in R \cdot T^n$, $z \in r \cdot U^n$, $\frac{z_j}{\xi_j} < 1$ for each $j$. Therefore

$$
II \leq \int_{r \cdot U^n} \left| \int_{\mathbb{T}^n} f(\xi) \prod_{j=1}^n \frac{1}{\xi_j - z_j} d\xi_1 \cdots d\xi_n \right| dV(z)
$$

$$
\leq \int_{r \cdot U^n} \left| \int_{\mathbb{T}^n} |f(\xi)| \prod_{j=1}^n \frac{1}{|\xi_j - z_j|} d|\xi_1| \cdots d|\xi_n| \right| dV(z)
$$

$$
= \int_{\mathbb{T}^n} |f(\xi)| \int_{r \cdot U^n} \prod_{j=1}^n \frac{1}{|\xi_j - z_j|} dV(z) d|\xi_1| \cdots d|\xi_n| \tag{3.10}
$$

by Fubini’s theorem.

Since $\mathcal{R}$ is bounded, for $r = (r_1, \ldots, r_n) \in \mathcal{F}$, the argument that gave (3.10) shows that for each $r_j$, $\int_{r_j U^n \frac{1}{|\xi_j - z_j|}} dV(z_j) \leq 1$ for a constant independent of $\xi \in R \cdot T^n$. Consequently

$$
\int_{r \cdot U^n} \prod_{j=1}^n \frac{1}{|\xi_j - z_j|} dV(z) = \prod_{j=1}^n \int_{r_j U} \frac{1}{|\xi_j - z_j|} dV(z_j) \leq 1.
$$

Inserting this into (3.10) yields

$$
\int_{r \cdot U^n} \left| \sum_{|\alpha| \leq N} a_\alpha z^\alpha \right| dV(z) \leq \int_{\mathbb{T}^n} |f(\xi)| d|\xi_1| \cdots d|\xi_n| \leq \|f\|_{H^1(\mathcal{R})},
$$

with all constants independent of $N$ and $f$.

Finally note that for any $f \in A^1(\mathcal{R})$ there are sequences $\{r_k\}_{k=1}^\infty \subseteq \mathcal{F}$ such that $\|f\|_{A^1(\mathcal{R})} = \lim_{k \to \infty} \int_{r_k \cdot U^n} |f| dV(z)$, by the dominated convergence theorem. Therefore

$$
\|S_N f\|_{A^1(\mathcal{R})} \lesssim \|f\|_{H^1(\mathcal{R})}
$$

as claimed. \qed

Convergence of $S_N f$ is $A^1(\mathcal{R})$ now follows as in the conclusion to the proof of Theorem 1.2. Let $T_N = S_N - \text{id}$, $X = H^1(\mathcal{R})$, $Y = A^1(\mathcal{R})$, and $D = \{\text{holomorphic polynomials}\}$. Note $T_N p \equiv 0$ for any $p \in D$ if $N$ is large enough. The hypotheses of Lemma 2.4 are thus satisfied. Proposition 3.9 and this lemma yield
Corollary 3.11. If \( f \in H^1(\mathcal{R}) \) and \( S_N f \) is given by (3.1), then
\[
\| S_N f - f \|_{A^1(\mathcal{R})} \to 0 \quad \text{as } N \to \infty.
\]

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