Black hole magnetospheres in the Born-Infeld theory

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Abstract

We study the force-free electrodynamics on rotating black holes in the Born-Infeld (BI) effective theory. The stream equation describing a steady and axisymmetric magnetosphere is derived. From its near-horizon behavior, we obtain the modified Znajek regularity condition, with which we find that the horizon resistivity in the BI theory is generally not a constant. As expected, the outer boundary condition far away from the hole remains unchanged. In terms of the conditions at both boundaries, we derive the perturbative solution of split monopole in the slow rotation limit. It is interesting to realise that the correction to the solution relies not only on the parameter in the BI theory, but also on the radius (or the mass) of the hole. We also show that the quantum effects can undermine the energy extraction process of the magnetosphere in the non-linear theory and the extraction rate gets the maximum in the Maxwell theory.

1 Introduction

Analogous to neutron stars and other ordinary objects, magnetospheres can also form on astronomical black holes. In the black hole magnetosphere, plenty of electron-positron pairs can be created via the pair cascade processes \([1]\). With strong electromagnetic fields, this form the force-free magnetosphere in which the electric fields along the magnetic field lines are screened and the charges feel zero net force.

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The force-free magnetosphere can be used to extract the rotational energy of a rotating black hole. As a kind of Penrose process, the energy is extracted through the rotation of the magnetosphere dragged by the black hole spacetime. This process has become a promising mechanism nowadays that can explain the formation of powerful jets observed in many high energy objects, like AGN, GRB and microquasars.

It is known that the magnetic fields on neutron stars are very high, even exceeding the quantum electrodynamics (QED) critical value. In this case, the QED corrections should be included in the force-free magnetospheres and the Maxwell electrodynamics should be replaced by the non-linear theory. This has been discussed in magnetar magnetospheres (e.g., [2, 3, 4]), whose surface magnetic fields sometimes can be above $10^{15}$ G.

However, this is not the case for black hole magnetospheres. The astronomical black holes do not have their own magnetic fields. The magnetic fields on them come from accretion and are usually far weaker (e.g., $\lesssim 10^4$ G for a black hole with mass $M = 10^9$ $M_\odot$ [5]) than the QED critical value. But, the study of the black hole magnetosphere in non-linear electrodynamics is useful for that of the QED corrected magnetosphere near a magnetar where gravity is important. The latter can be obtained in a weak field limit of the former despite a difference of the inner boundary condition.

The study of black hole magnetospheres in non-linear electrodynamics is also of theoretical interest. The force-free magnetospheres on black holes are not well understood even in the Maxwell theory. We still do not know well the structure and geometry of the field lines in the force-free magnetosphere. The extension to the non-linear theory help find the analytical properties of black hole magnetospheres in a general sense.

Moreover, non-linear electrodynamics include quantum corrections to the Maxwell theory. As is known, strong quantum effects also happen in black holes. Thermal particles are excited and radiated in the near-horizon regions of black holes. It is interesting to examine the force-free non-linear electrodynamics in these regions.

In this work, we consider the black hole magnetospheres in non-linear electrodynamics, in particular in the Born-Infeld (BI) effective theory [6]. The BI theory has an explicit expression with well-regularized features. It also arises in string theory and so attracts much attention. The paper is organized as follows. In Section 2, we present the force-free theory in general non-linear electrodynamics. In this general framework, the stream equation describing steady and axisymmetric magnetospheres on rotating black holes is derived in Section 3. Based on the stream equation, the boundary conditions are discussed in Section 4 and the perturbative solution of split monopole is derived in Section 5. In the final section, we summarize and discuss the results.
2 Force-free non-linear electrodynamics

We start with the action of general electrodynamics

\[ S = \int \sqrt{-g} \left[ \frac{1}{4\pi} \mathcal{L}_{\text{EM}}(s, p) + A_\mu J^\mu \right] d^4x, \]  

(1)

where \( \mathcal{L}_{\text{EM}}(s, p) \) is general Lagrangian of the electromagnetic fields with

\[ s = \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad p = \frac{1}{4} \tilde{F}^{\mu\nu} F_{\mu\nu}. \]  

(2)

The dual field strength \( \tilde{F}^{\mu\nu} = (1/2) \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \). For the BI theory, the Lagrangian of electromagnetic fields takes the form

\[ \mathcal{L}_{\text{EM}}(s, p) = b^2 \left( 1 - \sqrt{1 + \frac{2s}{b^2} - \frac{p^2}{b^4}} \right), \]  

(3)

where \( b \) is an undetermined parameter, with the dimension of mass squared.

The modified Maxwell equations are given by

\[ \nabla_\mu \tilde{F}^{\mu\nu} = 0, \]  

(4)

\[ \nabla_\mu G^{\mu\nu} = 4\pi J^\nu, \]  

(5)

where \( J^\nu \) is the conserved current and

\[ G^{\mu\nu} = SF^{\mu\nu} + P \tilde{F}^{\mu\nu}, \]  

(6)

with

\[ S \equiv \partial_s \mathcal{L}_{\text{EM}}, \quad P \equiv \partial_p \mathcal{L}_{\text{EM}}. \]  

(7)

The energy-momentum tensor of the electromagnetic fields is obtained from the derivative of the Lagrangian with respect to the metric

\[ T^{\mu\nu}_{\text{EM}} = -\frac{1}{4\pi} [SF_\alpha^{\mu} F^{\nu\alpha} + P \tilde{F}_\alpha^{\mu} F^{\nu\alpha} - g^{\mu\nu} \mathcal{L}_{\text{EM}}], \]  

(8)

for which we have

\[ \nabla_\mu T^{\mu\nu}_{\text{EM}} = J_\mu F^{\mu\nu}. \]  

(9)

We are considering magnetospheres described by the BI theory in the force-free limit, i.e., the EM energy momentum tensor be conserved:

\[ J_\mu F^{\mu\nu} = 0. \]  

(10)

This implies \( p = 0 \) so that \( \mathcal{L}_{\text{EM}}(s, p) = \mathcal{L}_{\text{EM}}(s) \).
3 Steady magnetospheres on rotating black holes

Let us consider the steady and axisymmetric magnetosphere on a Kerr black hole, whose metric on the BL coordinates is

\[ ds^2 = -\frac{\rho^2\Delta}{A} dt^2 + \rho^2 dr^2 + \rho^2 d\theta^2 + \frac{A\sin^2\theta}{\rho^2} (d\phi - \omega dt)^2, \]

(11)

where \( \rho^2 = r^2 + a^2 \cos^2\theta, \Delta = r^2 - 2Mr + a^2, A = 2Mr(r^2 + a^2) + \rho^2\Delta \) and \( \omega = 2Mra/A. \)

3.1 The stream equation

On the Kerr black hole, the force-free condition (10) reads:

\[ \partial_r A_0 J^r + \partial_\theta A_0 J^\theta = 0, \]

(12)

\[ \partial_r A_0 J^0 + F_{r\theta} J^\theta + \partial_\theta A_\phi J^\phi = 0, \]

(13)

\[ \partial_\theta A_0 J^0 - F_{r\theta} J^r + \partial_r A_\phi J^\phi = 0, \]

(14)

\[ \partial_r A_\phi J^r + \partial_\theta A_\phi J^\theta = 0. \]

(15)

It is convenient to use the Poison bracket defined by

\[ [C, D] \equiv \partial_r C \partial_\theta D - \partial_\theta C \partial_r D. \]

(16)

When \( C \) is a function of \( D \), we must have \([C, D] = 0. \) From Eqs. (12) and (15), we get

\[ [A_0, A_\phi] = 0. \]

(17)

So \( A_0 \) should be a function of \( A_\phi \). We can define:

\[ dA_0 = -\Omega(A_\phi) dA_\phi, \]

(18)

where \( \Omega \) is the angular velocity of a magnetic field line, which is constant along any field line.

Eq. (5) can be expressed as

\[ J^0 = \frac{1}{4\pi} \nabla \cdot [Sg^{00}(\omega - \Omega) \nabla A_\phi], \]

(19)

\[ J^r = -\frac{1}{4\pi \sqrt{-g}} \partial_\theta (SB_T), \]

(20)

\[ J^\theta = \frac{1}{4\pi \sqrt{-g}} \partial_r (SB_T), \]

(21)
\[ J^\phi = \frac{1}{4\pi} \nabla \cdot \left[ S \left( g^{\phi\phi} - g^{0\phi} \Omega \right) \nabla A_\phi \right], \]  
\[ (22) \]
where the operator \( \nabla_i = (\nabla_r, \nabla_\theta) \) is associated with the full Kerr metric. The toroidal field \( B_T = (\Delta \sin \theta/\rho^2) F_{r\theta} \).

From Eqs. (12), (15), (20) and (21), we find that
\[ [A_\phi, SB_T] = 0. \]  
\[ (23) \]
So \( \sin \theta SF_{r\theta} \) is also a function of \( A_\phi \). Let us denote
\[ \psi \equiv 2\pi A_\phi, \quad I(\psi) \equiv -2\pi SB_T. \]  
\[ (24) \]
From Eq. (13) or (14), we can have
\[ J^\phi = \Omega J^0 - \frac{II'}{8\pi^2 S \Delta \sin^2 \theta}, \]  
\[ (25) \]
where the prime denotes the derivative with respect to \( \psi \).

By comparing Eqs. (22) and (25) after insertion of Eq. (19), we derive the stream equation of the black hole magnetosphere in the non-linear theory:
\[ S \nabla \cdot \left\{ \frac{\rho^2 S}{A \sin^2 \theta} \left[ 1 - \frac{A^2 \sin^2 \theta (\Omega - \omega)^2}{\rho^4 \Delta} \right] \nabla \psi \right\} + \frac{AS^2 (\Omega - \omega)}{\rho^2 \Delta} \Omega' (\nabla \psi)^2 + \frac{II'}{\Delta \sin^2 \theta} = 0. \]  
\[ (26) \]
The equation is just modified with the factor \( S \). When \( S \rightarrow -1 \), the equation recovers the case in the Maxwell theory. As shown by the equation, the positions of the lightsurfaces are not changed.

With the above equations, we get
\[ s = \frac{1}{8\pi^2 A \sin^2 \theta} \left\{ \frac{AI^2}{\Delta S^2} + \left[ 1 - \frac{A^2 \sin^2 \theta (\Omega - \omega)^2}{\rho^4 \Delta} \right] [\Delta (\partial_r \psi)^2 + (\partial_\theta \psi)^2] \right\}. \]  
\[ (27) \]
From this, the expression of \( S \) for the BI theory is obtained:
\[ S^2 = \frac{A(f \Delta - I^2)}{Af \Delta + \left[ \Delta - \frac{A^2 \sin^2 \theta (\Omega - \omega)^2}{\rho^4} \right] [\Delta (\partial_r \psi)^2 + (\partial_\theta \psi)^2]}, \]  
\[ (28) \]
where \( f(\theta) = 4\pi^2 b^2 \sin^2 \theta \).

### 3.2 The energy and momentum extraction rates

The field components observed by the Zero Angular Momentum Observers (ZAMOs) in the unit basis vectors of the absolute space [7] are
\[ E = -\frac{D}{S} = -\frac{\Omega - \omega}{2\pi \Lambda \sqrt{\rho^2}} \left( \sqrt{\Delta \partial_r \psi e_r} + \partial_\theta \psi e_\theta \right), \]  
\[ (29) \]
\[ \mathbf{B} = -\frac{\mathbf{H}}{S} = \frac{1}{2\pi\sqrt{A}\sin\theta} \left( \partial_\theta \psi \mathbf{e}_r - \sqrt{\triangle} \partial_r \psi \mathbf{e}_\theta - \frac{I}{SA} \rho^2 \mathbf{e}_\phi \right), \] (30)

where \( \Lambda^2 = \rho^2 \triangle / A \).

From the energy-momentum tensor (8), we can obtain the poloidal components of the energy and angular momentum flux densities:

\[ \mathcal{E}^r = \Omega L^r = -\frac{\Omega I}{16\pi^2 \rho^2 \sin\theta} \partial_\theta \psi, \] (31)

\[ \mathcal{E}^\theta = \Omega L^\theta = \frac{\Omega I}{16\pi^2 \rho^2 \sin\theta} \partial_r \psi. \] (32)

The total rate of angular momentum and energy extraction from the hole is given by the integration of the radial densities over all the accessible spacetime:

\[ L = -\frac{1}{8\pi^2} \int Id\psi, \quad E = \frac{1}{4\pi} \int (\mathbf{E} \times \mathbf{H}) \cdot ds = -\frac{1}{8\pi^2} \int \Omega I d\psi. \] (33)

So they take the same form in appearance as in the Maxwell theory, containing no \( S \). But, it should be noticed that the functions actually have been corrected by the nonlinear factor \( S \).

## 4 Boundary behaviors

As done in the Maxwell theory in the previous work [8], the conditions of the differential equation (26) at the event horizon and spatial infinity can be determined. In what follows, we examine the conditions in the BI theory.

### 4.1 The condition at the event horizon

#### 4.1.1 The Znajek condition

The equation (26) at the event horizon \( r \to r_+ = M + \sqrt{M^2 - a^2} \) is simplified to

\[ II' = \frac{S \sqrt{A} \sin \theta (\Omega - \omega)}{\rho^2} \partial_\theta \frac{S \sqrt{A} \sin \theta (\Omega - \omega) \partial_\theta \psi}{\rho^2}. \] (34)

This exactly gives the Znajek regularity condition [9] at the horizon:

\[ I_+ = -\frac{2Mr_+S_+ \sin \theta (\Omega_+ - \omega_+)}{\rho^2_+} \partial_\theta \psi_+, \] (35)

where the quantities with the subscript + indices denote values at the \( r = r_+ \). Note that \( S \) is negative here. It is interesting to find that the expression of \( S_+ \) obtained from Eq.
(28) with \( r \to r_+ \) gives the same condition. As it is seen, the Znajek condition is modified in the non-linear theory compared to the Maxwell theory case.

This relation is actually the result that the electromagnetic fields satisfy the radiation condition \[10\] at the horizon:

\[ E_\theta = \pm B_\phi. \]  

As determined in the ingoing frame \[9\], the field components (29) and (30) at the horizon generally satisfy the following conditions \[7\]:

\[ E_r, B_r, E_H, B_H \sim \mathcal{O}(1), \]

\[ B_H = E_H \times e_r, \]

where the horizon fields are defined by

\[ E_H = \Lambda E_\theta e_\theta|_{r \to r_+}, \quad B_H = \Lambda B_\phi e_\phi|_{r \to r_+}. \]

So the condition (38) corresponds to the negative sign case of the radiation condition (36), which also leads to the Znajek condition (35). Similarly, we can have the field components \( D_r, H_r, D_H \) and \( H_H \) according to the relations given in Eqs. (29) and (30).

### 4.1.2 The horizon resistivity

The boundary conditions of the electrodynamics on the horizon give rise to the notions of surface charge and current \[5, 11\]. In the non-linear theory, their definitions are changed to

\[ \sigma_H = \frac{D_r}{4\pi}, \]  

\[ j_H = -\frac{1}{4\pi} H_H \times e_r. \]

When \( \psi = \psi(\theta) \) at the horizon, \( D_r = 0 \) and so the surface charge is zero. It can be checked that the surface charge and current satisfy the charge conservation equation \[7\]. Combined with equation (38), we have

\[ j_H = \frac{E_H}{R_H}, \]

where the resistivity of the horizon is now

\[ R_H = -\frac{4\pi}{S_+}. \]

So the resistivity is not constant any more on the horizon. As indicated by the monopole solution that will be derived in the next section, \(-S_+ > 1\) in the BI theory. So the
resistivity here should be larger than in the Maxwell theory, for which \( S_+ = -1 \) and the resistivity gets the minimum value \( R_H = 4\pi \simeq 377 \) ohms.

The result is different from that in [12], where the obtained resistivity is the same as in the Maxwell theory. The reason is that the authors have chosen a special frame in which the BI theory looks like the Maxwell theory. This should be not true for a general observer’s frame.

4.2 The condition at spatial infinity

Similar to the Maxwell theory case [13, 14], the finiteness of both the energy and the momentum fluxes in Eqs. (31) and (32) requires \( \Omega \) be independent of \( r \) at infinity:

\[
\Omega(r, \theta) \to \Omega_0(\theta) \quad \text{as} \quad r \to \infty,
\]  

(44)

where \( \Omega_0(\theta) \) is the value at the infinite boundary. Since \( I(\Omega) \) and \( \psi(\Omega) \) are functions of \( \Omega \), then the values \( I_0 \) and \( \psi_0 \) at infinity should be independent of \( r \) as well. In combination with the expression (28) of \( S \), we have

\[
r \to \infty : \quad \psi(\Omega) \to \psi_0(\Omega_0(\theta)), \quad I(\Omega) \to I_0(\Omega_0(\theta)), \quad S^2 \to 1.
\]  

(45)

At spatial infinity, the BI theory with weak fields approaches the Maxwell theory.

In this case, the stream equation (26) at infinity reduces to

\[
I_0 \frac{\partial I_0}{\partial \psi_0} = \sin \theta \Omega_0 \partial \theta (\sin \theta \Omega_0 \partial \theta \psi_0).
\]  

(46)

Similarly, the equation gives rise to the relation:

\[
I_0 = -\sin \theta \Omega_0 \partial \theta \psi_0.
\]  

(47)

Here, the negative sign is chosen for \( \Omega_+ \leq \omega_+ \). It corresponds to the positive sign case of the radiation condition (36), which guarantees outflow of energy from the hole.

4.3 Matching the boundary conditions

It is seen that the two boundaries of any field line in the black hole magnetosphere are in two different regimes: one is in the Maxwell theory and the other is in the BI theory.

As in [8], we first consider the case that the functions at the horizon and at infinity are matched to be identical:

\[
\psi_0 = \psi_+, \quad \Omega_0 = \Omega_+, \quad I_0 = I_+.
\]  

(48)
Then, from Eqs. (35) and (47), we obtain

$$S_+ = -\frac{\Omega_+(1 - a \sin^2 \theta \omega_+)}{\omega_+ - \Omega_+}. \quad (49)$$

Thus, for a given black hole, the quantum correction to the magnetosphere near the horizon is relying on the angular velocity $\Omega_+$ of the field lines. The correction factor $-S_+$ gets larger when $\Omega_+$ increases.

Instead, we may also make the identifications:

$$\Omega_0 = \Omega_+, \quad \frac{\partial_\theta \psi_0}{I_0} = S_+ \frac{\partial_\theta \psi_+}{I_+} \quad (50)$$

By comparing Eqs. (35) and (47), we then have the solution

$$\Omega_+ = \Omega_0 = \frac{a}{2Mr_+ + \rho_+^2}. \quad (51)$$

If we choose positive sign in Eq. (47), the resulting solution is

$$\Omega_+ = \Omega_0 = \frac{1}{a \sin^2 \theta}. \quad (52)$$

In this case, the angular velocity is larger than the one of the black hole. These two solutions at the boundaries are exactly the asymptotical solutions found in [13, 14]. It is easy to check that the latter solution with $S = -1$ is still an exact solution to the stream equation in all regions in the BI theory. But this trivial solution is unphysical since it admits null current.

### 5 The perturbative monopole solution in the BI theory

It is easy to find that the monopole solution $\psi = -\cos \theta$ still exists to the stream equation (26) on Schwarzschild back holes with $a = 0$ (and so $\Omega = I = 0$). Based on the solution, the perturbative monopole solution can be derived in slowly rotating black holes, as done in [1]. In this section, we shall explore the corresponding monopole solution in the BI theory.

Let us define the dimensionless parameters:

$$x = \frac{r}{r_0}, \quad \alpha = \frac{a}{r_0} \quad (53)$$
where \( r_0 = 2M \) is the radius of the horizon in the Schwarzschild case. The functions can be expanded in powers of \( \alpha \):

\[
\psi = \psi_0 + \alpha^2 \psi_2 + \cdots, \quad (54)
\]

\[
\tilde{\Omega} = r_0 \Omega = \alpha \tilde{\Omega}_1 + \alpha^3 \tilde{\Omega}_3 + \cdots, \quad (55)
\]

\[
\tilde{I} = r_0 I = \alpha \tilde{I}_1 + \alpha^3 \tilde{I}_3 + \cdots, \quad (56)
\]

where \( \psi_0 \) is the monopole solution of the zero-th order equation:

\[
\psi_0 = -\cos \theta. \quad (57)
\]

With them, we get the expanded form of \( S^2 \):

\[
S^2 = \frac{k x^4}{1 + k x^4} + \alpha^2 \frac{g}{(x - 1)(1 + k x^4)^2} + \cdots, \quad (58)
\]

where

\[
k = 4\pi^2 b^2 r_0^4, \quad (59)
\]

\[
g = k x (x - 1) [2x - (x - 1) \sin^2 \theta] + k x \sin^2 \theta (1 - \tilde{\Omega}_1 x^3)^2 \]

\[
-2k x^4 (x - 1) \frac{\partial_\theta \psi_2}{\sin \theta} - x^3 (1 + k x^4) \frac{\tilde{I}_1^2}{\sin^2 \theta}. \quad (60)
\]

The expanded equation (26) with (58) at the order \( O(\alpha^2) \) gives the equation:

\[
L^2 \psi_2 = \frac{1}{1 - \frac{x}{1}} \left[ \sin^2 \theta \left( \tilde{\Omega}_1 - \frac{1}{x^3} \right) \partial_\theta \tilde{\Omega}_1 + \sin 2\theta \left( \tilde{\Omega}_1 - \frac{1}{x^3} \right)^2 - \frac{\tilde{I}_1 \tilde{I}_1'}{\sin \theta} \right] + \frac{\sin 2\theta}{x^5} + \delta, \quad (61)
\]

where

\[
L^2 = \frac{1}{\sin \theta} \partial_x \left( 1 - \frac{1}{x} \right) \partial_x + \frac{1}{x^2} \partial_\theta \left( \frac{1}{\sin \theta} \partial_\theta \right), \quad (62)
\]

and the correction terms

\[
\delta = -\frac{1}{k x^4} \left[ \frac{1}{1 - \frac{x}{1}} \left( \frac{\tilde{I}_1 \tilde{I}_1'}{\sin \theta} + \frac{\partial_\theta g}{2x^3(1 + k x^4)} \right) + \frac{2k x^3(1 - \frac{1}{x})}{\sin \theta(1 + k x^4)} \partial_x \psi_2 \right]. \quad (63)
\]

Towards the horizon with \( x \to 1 \), the equation diverges as \( O(1/(1 - 1/x)) \). So, to avoid divergence, the relevant terms must cancel out, i.e.,

\[
\tilde{I}_1(x = 1, \theta) = \sqrt{\frac{k}{1 + k} (\tilde{\Omega}_1 - 1) \sin \theta \partial_\theta \psi_0}. \quad (64)
\]

This is also the boundary condition (35) at the horizon.
Since the magnetosphere in asymptotical regions is in the regime of the Maxwell theory, we can still take the outer boundary condition as the Michel monopole solution obtained in the flat spacetime. Inserting it into the condition (47) at infinity leads to

\[ \tilde{I}_1(x \to \infty, \theta) = -\tilde{\Omega}_1 \sin^2 \theta. \]  

(65)

Adopting the matching condition (48) given in the previous section for the above two boundary conditions (64) and (65), we derive

\[ \tilde{\Omega}_1 = \frac{\sqrt{k}}{\sqrt{1 + k + \sqrt{k}}} \]  

(66)

So the angular velocity grows with the value of \( k \) and reaches the maximum value \( 1/2 \) as \( k \to \infty \). With the angular velocity, we have

\[ \tilde{I}_1 = -\frac{\sqrt{k}}{\sqrt{1 + k + \sqrt{k}}} \sin^2 \theta. \]  

(67)

Inserting the above results into Eq. (61), the solution of \( \psi_2 \) can be derived basically. But it is hard to do so because the resulting equation is highly non-linear. Here, we only consider the solution in the asymptotical region. At large \( x \), the equation reduces to

\[ \frac{1}{\sin \theta} \partial_x^2 \psi_2 + \frac{1}{x^2} \partial_\theta \left( \frac{1}{\sin \theta} \partial_\theta \psi_2 \right) = -\frac{2\tilde{\Omega}_1 \sin 2\theta}{x^3}. \]  

(68)

The solution is

\[ \psi_2 = \frac{\sqrt{k} \sin^2 \theta \cos \theta}{(\sqrt{1 + k + \sqrt{k}})x}. \]  

(69)

When \( k \to \infty \), this recovers the result in [1].

6 Discussion and conclusion

Force-free non-linear electrodynamics on rotating black holes is discussed. Based on the derived stream equation, we analyze the boundary conditions at the horizon and at spatial infinity. Compared to the case in the Maxwell theory, the Znajek condition at the horizon is modified in the non-linear theory, while the one at infinity remains the same as in the Maxwell theory. We also show that the surface resistivity on the horizon is modified in the ZAMO frame.

In terms of the boundary conditions, we further obtain the perturbative solution of the split monopole in the slow rotation limit. With the solution, we can find that the
horizon resistivity given by Eq. (43) is larger than that in the Maxwell theory: \( R_H = 4\pi \sqrt{k}/\sqrt{1+k} \). Following the analysis in [5], this implies that the energy extraction rate should be lower in the BI theory. For given electric potential difference induced by the hole rotation, the energy output should be smaller with a larger impedance. Indeed, from Eq. (33) with the monopole solution, we can see that the energy extraction rate is smaller compared to that in the Maxwell theory:

\[
\frac{E^{(BI)}}{E^{(Maxwell)}} = \left( \frac{2\sqrt{k}}{\sqrt{1+k} + \sqrt{k}} \right)^3.
\]  

(70)

The ratio only relies on the parameter \( k \) defined in Eq. (59). It gets the Maximum value as \( k \to \infty \). It is interesting that the parameter \( k \) is related to the parameter \( b \) in the BI theory as well as the radius \( r_0 \) of the black hole horizon, which implies that we can recover the results in the Maxwell theory only with \( k \to \infty \), without need of a large \( b \). The non-linear correction in the BI theory become more important on a lighter black hole (with smaller \( r_0 \)). On the opposite, the BI theory with finite \( b \) can behave like the Maxwell theory on a massive black hole. This is different from the situation in the Minkowski spacetime.

The reason for this difference might be due to the converging effect of the horizon on the field lines. For given boundary conditions at infinity, the field lines are much denser across a horizon on a lighter black hole because the area of the horizon is smaller. This makes the fields to be stronger and easier to reach the QED regime. Similarly, for a massive black hole, the field field lines will be diluted on the horizon with a large area.

Finally, it should be pointed out that the result is also consistent with that from the quantum aspects of black holes. Thermal particles excited in the vacuum near the horizon should also contribute to the quantum corrections to the electrodynamics in this region. So the quantum corrections become more important for the electrodynamics on a smaller black hole who has a larger Hawking temperature. Of course, our discussion here does not include the quantum effects from the black hole spacetime. It is interesting to do further investigation in future study.

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