Model of leptons from $SO(3) \to A_4$

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Abstract

The lepton sector masses and mixing angles can be explained in models based on $A_4$ symmetry. $A_4$ is a non-Abelian discrete group. Therefore, one issue in constructing models based on it is explaining the origin of $A_4$. A plausible mechanism is that $A_4$ is an unbroken subgroup of a continuous group that is broken spontaneously. We construct a model of leptons where the $A_4$ symmetry is obtained by spontaneous symmetry breaking of $SO(3)$.

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I. INTRODUCTION

Recent experiments [1, 2, 3, 4, 5] have given an increasingly accurate picture of the neutrino sector of the new Standard Model ($\nu$SM). Current best measurements are summarized in Table I. These pieces of evidence paint a picture radically different from that of the quark sector [6, 7, 8] that exhibits extremely small masses, small mass splittings and non-hierarchical mixing angles.

Many attempts were made to obtain the masses and mixing angles from a more fundamental theory. In this paper, we concentrate on the lepton sector and consider the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix, $U$ [9, 10]. Using the data presented in Table I, the current best fit for this matrix is

$$|U| = \begin{pmatrix} 0.823 & 0.554 & 0.126 \\ 0.480 & 0.558 & 0.677 \\ 0.305 & 0.618 & 0.725 \end{pmatrix}. \quad (1)$$

It has been pointed out that within $2\sigma$ this matrix is consistent with the Harrison-Perkins-Scott (HPS) mixing matrix [11]

$$U_{\text{HPS}} = \begin{pmatrix} \sqrt{2}/3 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \quad (2)$$

The HPS matrix has a definite pattern. This pattern has motivated explanations of the structure of $U$ using non-Abelian discrete flavor symmetries. Of particular interest is a class of models postulating an $A_4$ family symmetry [12, 13]. In these models, the left-handed lepton doublet and the right-handed neutrino singlet transform in three dimensional irreducible representations, while the right handed charged leptons transform under distinct one dimensional representations. For a review of such models, see Ref. [14]. Such models, however, are typically plagued by two issues.

The first is that of vacuum alignment. The $A_4$ symmetry is broken to $Z_3$ by a scalar $\phi$ that couples to the charged leptons and to $Z_2$ by a scalar $\phi'$ that couples to the neutrinos. There is no reason, a priori, for this particular vacuum structure. One approach to resolving this problem is to add scalars and symmetries, possibly with supersymmetry, to enforce that vacuum alignment [17, 18]. Placing the scalars $\phi$ and $\phi'$ on separate branes of an extra-dimensional model [12, 19] is another possibility.

The second problem of $A_4$ based models, which is the problem that we attempt to solve in this paper, is that of the origin of $A_4$. The symmetry group $A_4$ is chosen simply because it works, with no motivation from UV physics. This lack of motivation is exacerbated...
by the fact that gravity is believed to break global symmetries. Therefore, we look for possible motivations of the $A_4$ symmetry group. One possibility is that $A_4$ comes out as a subgroup of the modular group [20], which often arises in string theory. Another possibility is that $A_4$ arises in the low-energy effective theory obtained by orbifolds of a six dimensional theory [21]. In this paper we present a model where $A_4$ is obtained by spontaneously breaking a continuous symmetry which we take to be the minimal choice, $SO(3)$. The idea of embedding $A_4$ in $SO(3)$ has been discussed in [22]. Unlike in our case, where the $SO(3)$ is spontaneously broken, in [22] an explicit breaking of a global $SO(3)$ symmetry was introduced. The idea of spontaneously breaking a continuous symmetry to a discrete subgroup has been discussed in [23]. However, their procedure is different then ours.

In section 2, we briefly review the general structure of models of neutrino mixing using $A_4$ symmetry. In section 3, we review the vacuum structure of models with $SO(3)$ symmetry and how it can be broken to $A_4$. In section 4 we construct a model for the lepton sector based on spontaneously broken $SO(3) \rightarrow A_4$ symmetry. We conclude in section 5. Technical details are collected in the appendices. In appendix A, we summarize important properties of the group $A_4$ and introduce relevant group theory concepts. In appendix B, we describe one method for determining the vacua of a theory with $SO(3)$ symmetry and a scalar transforming in the 7 of the group.

II. MODELS WITH $A_4$ SYMMETRY

Implementing non-Abelian discrete flavor symmetries in a model generically leads to patterns in the mass matrices. These patterns yield patterns in the mixing matrices after changing to the mass basis. It is natural to try to obtain $U_{HPS}$ using such symmetries. In

| Observable | Value | Main Source |
|------------|-------|-------------|
| $\sin^2 \theta_{12}$ | $0.312^{+0.019}_{-0.018}$ | Solar neutrino experiments |
| $\Delta m^2_{21}$ | $7.67^{+0.16}_{-0.19} \times 10^{-5}$ eV$^2$ | |
| $\sin^2 \theta_{23}$ | $0.466^{+0.073}_{-0.058}$ | Atmospheric neutrino experiments |
| $|\Delta m^2_{32}|$ | $2.39^{+0.11}_{-0.08} \times 10^{-3}$ eV$^2$ | |
| $\sin^2 \theta_{13}$ | $0.016^{+0.010}_{-0.006}$ | Global fit with all current data |
| $\Delta m^2_{31}$ | $2.39^{+0.11}_{-0.08} \times 10^{-3}$ eV$^2$ | |

TABLE I: The current best fit values for neutrino mass splittings and mixing angles [15, 16]. All ranges are quoted at 1σ.
fact, several models \[18, 24\] did it using $A_4$ symmetry. These models have several common features which we describe in this section.

We consider only the lepton sector. The basic required matter content are the $\nu$SM fermions (including the RH singlet neutrinos), the SM Higgs and two more scalars that are denoted by $\phi$ and $\phi'$. The fermion field content is

$$\begin{align*}
\psi_\ell (2,3)_{1/2}, & \quad \psi_\nu (1,1)_{-1}, & \quad \psi_\mu (1,1')_{-1}, & \quad \psi_\tau (1,1'')_{-1}, & \quad \psi_n (1,3)_0,
\end{align*}$$

and the scalars are

$$\begin{align*}
H(2,1)_{1/2}, & \quad \phi(1,3)_0, & \quad \phi'(1,3)_0.
\end{align*}$$

We use standard notation, $(S, A)_Y$, where $S$ [$A$] is the representation under $SU(2)_L$ [$A_4$] and $Y$ is the hypercharge. In specific models more fields are added in order to satisfy vacuum alignment conditions. In addition, further symmetries are usually required to forbid unwanted terms in the Lagrangian, as well as to obtain the correct vacuum alignment. The purpose of the two scalars $\phi$ and $\phi'$ is to break the $A_4$ symmetry down to its $Z_3$ and $Z_2$ subgroups respectively. For the standard basis described in Appendix A this breaking is achieved by the VEVs:

$$\langle \phi \rangle = (v, v, v), \quad \langle \phi' \rangle = (v', 0, 0).$$

The two scalars are then made (by symmetries, for example) to couple to different sectors of the model. The $\phi$ couples to the charged leptons, giving a $Z_3$ symmetric mass matrix, while the $\phi'$ couples to the neutrinos, giving a $Z_2$ symmetric mass matrix.

The Lagrangian for the fermions with the properties and fields described above is:

$$\begin{align*}
\mathcal{L} = -\frac{y_\ell}{\Lambda} \bar{\psi}_\ell \phi H \psi_E - \frac{y_\mu}{\Lambda} \bar{\psi}_\mu \phi' H \psi_\mu - \frac{y_\tau}{\Lambda} (\bar{\psi}_\tau \phi')'' H \psi_\tau - M \bar{\psi}_n \psi_n - x_\nu \bar{\psi}_\nu \psi_n \phi' - y_\nu \bar{\psi}_\nu H \psi_n,
\end{align*}$$

where $(\bar{\psi}_\ell \phi)' [(\bar{\psi}_\ell \phi)'']$ denotes that the product is taken such that the result transforms in the $1'$ [$1''$]. This Lagrangian provides an effective description up until a cutoff $\Lambda$. We assume that $M$ is much larger than the weak scale. Notice that charged lepton masses would not be allowed without including non-renormalizable operators. We did not include terms that are suppressed by $1/\Lambda^2$.

We emphasize that the Lagrangian (6) is not the most general one. It is missing several terms allowed by the symmetries listed so far. Any of the terms coupling to $\phi$ is allowed with $\phi \rightarrow \phi'$ and vice-versa. For example, $\bar{\psi}_n \psi_\nu \phi$ is allowed. This issue is generally solved by including additional discrete or continuous Abelian symmetries. For example, ref. [18] describes a supersymmetric model with an additional $Z_4$ and $U(1)_R$ symmetry under which $\phi$ and $\phi'$ transform differently.

The heavy neutrino states present due to the see-saw mechanism can be integrated out.
The resulting low-energy Majorana mass matrix for the neutrinos has the form

\[
m_\nu = \begin{pmatrix}
a & 0 & 0 \\
0 & b & d \\
0 & d & b
\end{pmatrix},
\]

(7)

where \(a\), \(b\), and \(d\) depend on the specifics of the model. The off-diagonal \(d\) entries are a reflection of the \(A_4 \rightarrow Z_2\) breaking. It is made possible by the fact that a singlet can be formed out of the product of three triplets. The mass matrix for the charged leptons has the form

\[
m_\ell = \begin{pmatrix}
y_e & y_\mu & y_\tau \\
y_e & y_\mu \omega & y_\tau \omega^2 \\
y_e & y_\mu \omega^2 & y_\tau \omega
\end{pmatrix},
\]

(8)

where \(\omega \equiv e^{2\pi i/3}\) (see Appendix A for more details). This mass matrix is diagonalized by multiplying on the left by

\[
V = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega
\end{pmatrix}.
\]

(9)

The rotation matrix \(V\) in (9) does not depend on any of the parameters of the theory. This fact helps ensure that no hierarchy will appear in the neutrino mixing matrix. No change of basis is required for the right-handed leptons. Performing the full diagonalization procedure, the physical PMNS matrix, \(U\), is then given by \(U_{HPS}\).

Specific implementations of the ideas described above have several obstacles to overcome. First, in general, \(A_4\) based models only explain the mixing parameters and not the mass hierarchies. (Both mixing and masses can be obtained in an RS-type model [19].) Another issue, as we already discussed, is the fact that extra symmetries are needed in order to forbid problematic terms. There is also an issue of vacuum alignment, which has been discussed in the introduction. Finally, there is the issue of the origin for the \(A_4\) symmetry group, which is the issue we discuss in this paper.

III. SPONTANEOUS BREAKING OF \(SO(3) \rightarrow A_4\)

In order to motivate the use of \(A_4\), we use a model where the group \(A_4\) arises from spontaneous breaking of a continuous symmetry. The simplest choice of gauge group is \(SO(3)\) [25, 26]. We discuss the representation necessary for a scalar to break \(SO(3)\) to \(A_4\) and write down a potential for this scalar to demonstrate how spontaneous symmetry breaking (SSB) is achieved.
| Representation | Decomposition |
|----------------|--------------|
| 1              | 1            |
| 3              | 3            |
| 5              | 3 + 1' + 1'' |
| 7              | 3 + 3 + 1    |

**Table II**: Decomposition of the four smallest representations of $SO(3)$ into irreducible representations of $A_4$.

Let $T$ be a scalar that transforms under an irreducible representation of $SO(3)$. This irreducible representation of $SO(3)$ induces a representation of $A_4$ since $A_4$ is a subgroup of $SO(3)$. In Appendix A, we write down a general method for decomposing an irreducible representation of $SO(3)$ into irreducible representations of $A_4$. The decomposition of the four smallest representations is given in Table I. For now, it is important to note that the smallest non-trivial representation of $SO(3)$ that contains a singlet of $A_4$ is the 7. This is the smallest representation that could in principle result in an $A_4$ invariant vacuum. Thus, it is natural to start our attempt to construct a model using a scalar in the 7.

A model with a scalar transforming in the 7 of $SO(3)$ has been described in [25, 26]. We summarize the results of [25]. The 7 of $SO(3)$ can be described by symmetric, traceless rank 3 tensors in 3D, denoted as $T^{abc}$. The most general renormalizable potential that can be written is

$$V = -\frac{\mu^2}{2}T^{abc}T^{abc} + \frac{\lambda}{4}(T^{abc}T^{abc})^2 + cT^{abc}T^{bcd}T^{def}T^{efa}. \quad (10)$$

Naively, there are other quartic terms that can be written down, but they are linear combinations of the two quartic terms in (10). Also note that cubic terms vanish since the cubic singlet is formed by an antisymmetric product of identical fields. A technique for minimizing the potential is presented in Appendix B. The results of the minimization are as follows. In order to have a stable potential we need $\lambda > 0$. In order to have a VEV at all we require $\mu^2 > 0$. Then, the residual symmetry depends on the relation between $c$ and $\lambda$. For $c < -\lambda/2$, the potential becomes unstable. For $c > 0$, the residual symmetry is $D_3$. For $-\lambda/2 < c < 0$, the residual symmetry is $A_4$. We learn that there is a large area in parameter space where $SO(3)$ is broken to $A_4$. In our model we choose the parameters such that this is the case.
TABLE III: Left: Matter content for the lepton and scalar sectors of the model. The blocks contain the left-handed fermions, right-handed fermions, and scalars respectively. Right: Vacuum expectation values for the scalars and the subgroup of $SO(3)_F$ under which they are invariant. The $H$ gets the usual SM-like VEV and the $T$ gets a VEV as described in Section III.

IV. MODEL OF LEPTON BASED ON $SO(3) \rightarrow A_4$

We move to describe the model. The symmetry of the model is

$$SU(2)_L \times U(1)_Y \times SO(3)_F \times Z_2.$$  (11)

At this stage we do not care if the $SO(3)_F$ is gauged or not. For the fermions, we consider only the leptons. The full matter content of the scalar and lepton sectors of the model are summarized in Table III. We also describe the symmetry breaking induced by each of the scalars.

We start with the scalar sector of the model. There are five scalar fields in the model. Three of them, $H$, $\phi$ and $\phi'$ are needed in the $A_4$ model. When extending the model to an $SO(3)_F$ symmetry, we add two scalars, $T$ and $\phi_5$. We need $T$ as it is responsible for the $SO(3)_F \rightarrow A_4$ breaking. As we discuss later, $\phi_5$ is needed because without it the tau and the muon would be degenerate. In term of scales, things are simpler if we decouple the $SO(3)_F \rightarrow A_4$ breaking (triggered by $v_T$) and the $A_4$ breaking (which is done by $v$, $v'$ and $v_5$). That is, we assume the following hierarchies of scales

$$\Lambda \gg v_T \gg v \sim v' \sim v_5 \gg v_H.$$  (12)
We do not try to explain these hierarchies.

Next, we discuss the fermions. The fields $\psi_\ell$, $\psi_e$, and $\psi_n$ have the same representations under $SO(3)_F$ as under $A_4$. They correspond directly to fields in the $A_4$ model. Complications arise when considering the right handed muon and tau fields that transform as $1'$ and $1''$ respectively. The issue is that the $1'$ and $1''$ do not correspond to irreducible representations of $SO(3)_F$. Thus, they must be obtained as parts of $SO(3)_F$ representations that include extra singlets or triplets of $A_4$. Further complications arise from the fact that irreducible representations of $SO(3)$ are real and, therefore, $1'$ and $1''$ must be part of the same $SO(3)$ representation in the scenario with minimal matter content. The simplest choice of representation that contains both $1'$ and $1''$ is the $5$. This explains why we introduce $\psi_m$, which is the field that after $SO(3)_F$ breaking gives us the right handed muon and tau fields.

A fermion that transforms in the $5$ of $SO(3)_F$ decomposes into pieces that transform under the $1'$, $1''$, and $3$ representations of $A_4$. A field transforming in the $5$ can be written as a traceless, symmetric matrix. In this form, the decomposition is

$$\psi_m = \begin{pmatrix} \psi_\mu + \psi_\tau & \psi_3^3 & \psi_3^2 \\ \psi_3^3 & \omega \psi_\mu + \omega^2 \psi_\tau & \psi_3^1 \\ \psi_3^2 & \psi_3^1 & \omega^2 \psi_\mu + \omega \psi_\tau \end{pmatrix},$$

(13)

where $\psi_\mu$ transforms as a $1'$, $\psi_\tau$ transforms as a $1''$, and $\psi_h$ transforms as a $3$. The use of a fermion in the $5$ implies that further matter content is required. Without it, we end up with extra right-handed fields. These extra field can be “removed” by adding a triplet left-handed fermion giving them a large Dirac mass. This is the reason we add the left-handed triplet, $\psi_f$.

The most general Lagrangian, including $1/\Lambda$ terms, that is responsible for charged lepton masses is given by

$$\mathcal{L} = -y_e \bar{\psi}_e \frac{H}{\Lambda} \phi^a \psi_e - \frac{y_m}{\Lambda} \bar{\psi}_e \frac{H}{\Lambda} \phi^b \psi_m - y_m \bar{\psi}_e \frac{H}{\Lambda} T^{abc} \psi_m - y_e \bar{\psi}_f \phi^a \psi_e$$

$$-y' \bar{\psi}_f \phi^b \psi_m - y_m \bar{\psi}_f T^{abc} \psi_m - y_m e^{abc} \bar{\psi}_f H \phi^d \psi_m - y' \bar{\psi}_f \phi^d \psi_m.$$  

(14)

The scalars get VEVs as indicated in Table III. Consider the masses of the charged fermions. There are six left-handed and six right-handed fields that can mix. Working in the basis where the right handed fields are $(\psi_e, \psi_\mu, \psi_\tau, \psi_h^1, \psi_h^2, \psi_h^3)$ and the left-handed ones are $(\psi_\ell, \psi_f)$ the mass matrix is roughly

$$m_\ell \sim \begin{pmatrix} v_H v/\Lambda & v_H v_T/\Lambda \\ v_H v_T/\Lambda & v_T \end{pmatrix},$$

(15)

where each block describes a $3 \times 3$ matrix. We see that there are three heavy states (of order $v_T$), three light states (of order $v_H v/\Lambda$), and that there is very small mixing between
these two sets of states. We identify the light states as the three charged leptons, and we neglect the mixing between them and the heavy states. This procedure leaves a charged lepton Dirac mass matrix of the form \( \mathbf{S} \), which is given by

\[
m_\ell = \begin{pmatrix}
  y_e \frac{v}{\Lambda} & y_m \frac{v}{\Lambda} + y_5^5 \frac{(\omega^2 - \omega) v_5}{\Lambda} & y_m \frac{v}{\Lambda} + y_5^5 \frac{(\omega - \omega^2) v_5}{\Lambda} \\
y_e \frac{v}{\Lambda} & \omega [y_m \frac{v}{\Lambda} + y_5^5 \frac{(\omega^2 - \omega) v_5}{\Lambda}] & \omega^2 [y_m \frac{v}{\Lambda} + y_5^5 \frac{(\omega - \omega^2) v_5}{\Lambda}] \\
y_e \frac{v}{\Lambda} & \omega^2 [y_m \frac{v}{\Lambda} + y_5^5 \frac{(\omega^2 - \omega) v_5}{\Lambda}] & \omega [y_m \frac{v}{\Lambda} + y_5^5 \frac{(\omega - \omega^2) v_5}{\Lambda}]
\end{pmatrix}, \quad (16)
\]

In order to diagonalize this matrix, we multiply on the left by \( V \) introduced in \( \text{(9)} \). The resulting diagonal mass matrix for the charged leptons is

\[
m_\ell^{\text{diag}} = \begin{pmatrix}
  |y_e \frac{v}{\Lambda}| & 0 & 0 \\
  0 & |y_m \frac{v}{\Lambda} - y_5^5 i \sqrt{3} \frac{v}{\Lambda} | & 0 \\
  0 & 0 & |y_m \frac{v}{\Lambda} + y_5^5 i \sqrt{3} \frac{v}{\Lambda} |
\end{pmatrix}. \quad (17)
\]

Two remarks are in order regarding the mass matrix for the charged leptons. First, note that the charged lepton scale is smaller than the electroweak scale by a factor of \( v/\Lambda \). This implies that \( v \) and \( \Lambda \) are at most a factor of 10\(^2\) apart. Recalling that we assume that \( \Lambda \gg v_T \gg v \), we conclude that the different scales cannot be widely separated. That is, the ratio of scales is of order ten. Since this ratio is not very large, the fact that we neglected \( 1/\Lambda^2 \) terms may not be justified.

The second remark is about the muon and tau masses. The matrix \( \text{(17)} \) leads degenerate muon and tau if the parameters of the theory are real or if \( v_5 = 0 \). Moreover, in order to reproduce the observed ratio of masses, \( m_\mu/m_\tau \sim 1/16 \), some amount of fine tuning is needed. Defining

\[
a \equiv y_m v, \quad b \equiv i \sqrt{3} y_5^5 v_5, \quad \alpha \equiv \arg(ab^*),
\]

we require

\[
\frac{|a|^2 + |b|^2 - 2 |a||b| \cos \alpha}{|a|^2 + |b|^2 + 2 |a||b| \cos \alpha} \approx \frac{1}{16^2}. \quad (19)
\]

That is, the phase between \( y_m \) and \( y_5^5 \) must be very close to \( \pi/2 \) and the values of \( a \) and \( b \) must be very close to each other. Given this fine-tuning, it is clear that this model does not try to explain the fermion mass hierarchy: the tuning of the scales of the charged lepton sector is exchanged for a tuning of the scales \( a \) and \( b \) to be very close to each other.

The neutrino sector works just as in the low-energy \( A_4 \) model described in section \( \text{[II]} \). Since the neutrinos are in triplet representation, the Lagrangian is almost the same as in Eq. \( \text{(6)} \). One issue is that the off-diagonal terms in the Majorana mass matrix require a coupling to \( T \). Coupling to \( \phi \) and \( \phi_5 \) are forbidden by the \( Z_2 \) symmetry used to forbid terms involving \( \phi' \) in \( \text{(14)} \). Then the terms relevant for neutrino masses are

\[
\mathcal{L} = - M \overline{\psi^a_n} \psi^a_n - \frac{X_\nu}{\Lambda} \overline{\psi^a_n} \psi^b_n \phi^c H T^{abc} - y_5 \overline{\psi^a_n} H \psi^a_n. \quad (20)
\]
Recalling that $\phi'$ gets a VEV $(v', 0, 0)^T$ and $T^{abc}$ gets a VEV $v_T x^{(a} y^{b} z^{c)}$, the neutrino Majorana mass matrix is given by

$$m^M_\nu = \begin{pmatrix} M & 0 & 0 \\ 0 & M & x_\nu v'^T \\ 0 & x_\nu v'^T & M \end{pmatrix},$$

while the Dirac mass matrix is given by

$$m^D_\nu = y_\nu v_H \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (22)$$

The low-energy effective Majorana matrix is then

$$\tilde{m}^M_\nu = -m^D_\nu (m^M_\nu)^{-1} (m^D_\nu)^{-1} = \begin{pmatrix} -\frac{y_\nu^2 v'^2}{M} & 0 & 0 \\ 0 & \frac{M v_H^2}{M^2} & \frac{v'^2 v_T}{M^2} \\ 0 & \frac{v'^2 v_T}{M^2} & \frac{M v_H^2}{M^2} \end{pmatrix}. \quad (23)$$

The matrix (23) has precisely the form (7). Taking into account the action of $V$ on the left-handed handed fields, it can then be diagonalized by rotating the left-handed neutrinos by $U_{\text{HPS}}$. The resulting diagonal mass matrix is

$$\tilde{m}^{\text{diag}}_\nu = y_\nu^2 v_H^2 \begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \frac{1}{M} & 0 \\ 0 & 0 & \Lambda \frac{v'^2 v_T}{M^2} \end{pmatrix}. \quad (24)$$

Two remarks are in order. First, we emphasize that the the result of the diagonalization is that the physical PMNS matrix is given by the HPS matrix, that is, $U = U_{\text{HPS}}$. The second remark is about the mass splittings in the neutrino sector. The form of the neutrino masses in (24) constraints the scales in the theory. If $x_\nu v'^T \ll M \Lambda$, then the splittings become very small, in contradiction to the $O(100)$ factor difference in the measured values of $\Delta m^2_{12}$ and $\Delta m^2_{23}$. We then conclude that $x_\nu v'^T \sim M \Lambda$. Since we require $\Lambda \gg v_T \gg v'$ and perturbative Yukawa couplings, we conclude that $v' \gg M$. This is not a problem, as both $v'$ and $M$ can be much above the weak scale.

V. DISCUSSION AND CONCLUSIONS

We have shown that, in principle, a model of lepton masses and mixings using an $A_4$ discrete symmetry can be obtained by spontaneously breaking a continuous symmetry. The
model, however, is not very elegant. We already mentioned the problem of the fine tuning required to get the correct muon and tau masses. We discuss a few other problems below.

The first issue is that of vacuum alignment in the full scalar potential. In previous incarnations of the $A_4$ model, additional symmetries and, often, scalars are needed in order to ensure the correct vacuum alignment. The question is even trickier in our case. All four scalars in the model need very specific alignments. Without additional symmetries, there are many couplings in the potential between these scalars which affect the vacuum structure. In particular, a possibility is the case where the additional scalars force the scalar $T$ away from the $A_4$ invariant vacuum. With the many additional degrees of freedom in this model, it is difficult to verify the vacuum alignment or to correct the alignment if it does not follow from the current iteration of the model.

The second is that of anomalies. The most natural way to implement the model would be to gauge the $SO(3)_F$ symmetry. This avoids possible issues with breaking due to gravity, as well as eating any massless Goldstone bosons. If $SO(3)_F$ were a global symmetry, there would be Goldstone bosons that would have to be extremely weakly coupled to the standard model fields in order to not have been detected. Even though they are not directly coupled in this model, it is unclear that, at loop level, the couplings remain small enough to evade bounds. If the symmetry were gauged, however, it would induce a $U(1)_Y$ anomaly via the triangle diagram in Figure 1. Using the Casimir square operator for the 5 dimensional representation, $C(5) = 10$, the anomaly is given by

$$A^{ab} = \sum_\ell Y_\ell \text{Tr} \left( \{ t^a_\ell, t^b_\ell \} \right) - \sum_r Y_r \text{Tr} \left( \{ t^a_r, t^b_r \} \right) = 12\delta^{ab}, \quad (25)$$

where $\ell$ are left-handed fermions and $r$ are right-handed fermions. Such anomalies can be eliminated by introducing new fermions. Note the need to introduce new fermions in the full model once the quarks are included. The additional fermions may lead to new light states. It is beyond the scope of this work to resolve this issue and to present an anomaly-free model.
for an $SO(3)$ flavor gauge theory.

Our last remark is about possible variation of our model. Our model is minimal in many ways, like the choice of the gauge group, the scalars that breaks $SO(3)_F$, and the fields we choose. It is likely that in order to achieve the desired vacuum alignment further structure would be necessary, including addition symmetries and matter content. Furthermore, in our model, the origin of the $Z_2$ symmetry is unexplained. However, Abelian discrete symmetries are easier to produce naturally in the context of orbifolds or spontaneous symmetry breaking. Finally, no attempt has been made to incorporate solutions to the hierarchy problem or other extensions of the Standard Model. In particular, the model has not been made supersymmetric and is four dimensional, while many current models using $A_4$ symmetry work in supersymmetric theories \[17, 18\] or theories with extra dimensions \[12, 19\]. It should be possible to extend our model to fit within the structure of these theories.

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APPENDIX A: MATHEMATICS OF $A_4$

The non-Abelian discrete group $A_4$ arises in the context of neutrinos as described in section II. For the purposes of this paper, we would like to be able to determine the irreducible representations of this group, to determine the result of products of representations, and to decompose reducible representations into irreducible representations. Accomplishing these goals requires some mathematical background.

The group $A_4$ is defined to be the group of even permutations of 4 objects. It is isomorphic to the group of rotational symmetries of the tetrahedron. The latter description will be used throughout this work. The group is of order 12 with the elements given as follows:

- The identity 1;
- Rotations by $180^\circ$ about three orthogonal axes (edge-to-edge);
- Rotations by $120^\circ$ and $240^\circ$ about 4 different axes (vertex-to-face).

This description gives the defining representation, which clearly has dimension 3 and indicates that $A_4$ is a subgroup of rotations in 3 dimensions $SO(3)$. Typically, a basis is chosen
where the two generators $S$ and $T$ are given by:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (A1)$$

This basis is chosen such that the three 180° rotation axes are the Cartesian coordinate axes.

Two irreducible representations are immediately seen at this point: the defining dimension 3 representation described above and the trivial representation 1. There are two more irreducible representations of $A_4$. The 1′ and the 1″ are dimension 1 representations that map the 120° rotations onto $\omega = e^{2\pi i/3}$ and $\omega^* = e^{4\pi i/3}$ respectively. The number $\omega$ is a cube root of 1 and satisfies

$$1 + \omega + \omega^2 = 0. \quad (A2)$$

Notice that these representations are not real. The combination 1′ $\oplus$ 1″, however, is a real representation isomorphic to the group generated by a 120° rotations in 2 dimensions. Thus, any real representation of $A_4$ must contain 1′ and 1″ in equal multiplicities.

The products of these representations are as follows:

$$1′ \times 1′ = 1″, \quad 1′ \times 1″ = 1, \quad 1″ \times 1″ = 1′, \quad 1′ \times 3 = 3,$$

$$1″ \times 3 = 3, \quad 3 \times 3 = 3_1 + 3_2 + 1 + 1′ + 1″. \quad (A3)$$

Given two triplets $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$, the results of the multiplication of $3 \times 3$ gives

$$1 = x_1y_1 + x_2y_2 + x_3y_3,$$

$$1′ = x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3,$$

$$1″ = x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3,$$

$$3_1 = (x_2y_3, x_3y_1, x_1y_2),$$

$$3_2 = (x_3y_2, x_1y_3, x_2y_1). \quad (A4)$$

Furthermore, for a 1′ (denoted by $u$) and an 1″ (denoted by $v$), the multiplications $3 \times 1′$ and $3 \times 1″$ give respectively

$$3 = u(x_1, \omega x_2, \omega^2 x_3), \quad 3 = v(x_1, \omega^2 x_2, \omega x_3). \quad (A5)$$

Next we need a way to decompose reducible representations of $A_4$ into a direct sum of irreducible representations. In order to do this decomposition, we use a theorem about the characters of an element of a representation. Given an arbitrary group $G$, an element $g \in G$, and a representation $\rho$ of $G$, the character is defined as

$$\chi_\rho(g) = \text{Tr}\rho(g). \quad (A6)$$
Since the trace is invariant under similarity transformation, every element of a given conjugacy class will have the same character. There are four conjugacy classes for $A_4$ given by each of the four possible angles of rotation: $0^\circ$, $180^\circ$, $120^\circ$, and $240^\circ$. The number of conjugacy classes is the same as the number of irreducible representations. This is a general result that holds for any finite group. It allows the construction of a character table listing the characters by irreducible representation and conjugacy class. For $A_4$, the character table is given in Table IV.

Given a representation $\rho$ which is not necessarily irreducible, irreducible representations $\rho_i$ and an element $g \in G$, the following relation holds:

$$\chi_{\rho}(g) = \sum_i n_i \chi_{\rho_i}(g), \quad (A7)$$

where $n_i$ is the multiplicity of $\rho_i$ in the decomposition of $\rho$ into irreducible representations. In the case of $A_4$, $i = 1, 1', 1'', 3$. Notice that the number of multiplicities $n_i$ is given by the number of irreducible representations of $G$. Such an equation can be written down for each conjugacy class of $G$. Thus, if we wish to determine the multiplicities $n_i$, we have the same number of variables as equations given by (A7). Given the characters in the representation under study and the irreducible representations, it is then possible to determine the decomposition of the representation $\rho$ into irreducible representations. The characters of the irreducible representations are given by the character table. The characters of the representation under study can be computed directly. In our case, we are interested in studying the representations of $A_4$ induced by irreducible representations of $SO(3)$. In this case, computing the characters is even simpler as a general formula for the characters in $SO(3)$ has been determined [26]:

$$\chi_j(\theta) = \frac{\sin [(2j + 1)\theta/2]}{\sin (\theta/2)}, \quad (A8)$$

where $j$ is the spin of the representation and $\theta$ is the angle of rotation.

|   | $0^\circ$ | $120^\circ$ | $240^\circ$ | $180^\circ$ |
|---|-----------|--------------|--------------|-------------|
| $\chi_1$ | 1         | 1            | 1            | 1           |
| $\chi_2$ | 1         | $\omega$    | $\omega^2$  | 1           |
| $\chi_3$ | 1         | $\omega^2$  | $\omega$    | 1           |
| $\chi_4$ | 3         | 0            | 0            | $-1$        |

TABLE IV: The character table for $A_4$, listing the conjugacy classes on the horizontal and the representations on the vertical. Here $\omega$ satisfies the equation $\omega^2 + \omega + 1 = 0$. The table is taken from [27].
TABLE V: Decomposition of the six smallest representations of $SO(3)$ into irreducible representations of $A_4$. The 4 rightmost columns indicate the multiplicity of the four irreducible representations of $A_4$.

| $j$ | $n_1$ | $n_1'$ | $n_1''$ | $n_3$ |
|-----|-------|--------|---------|-------|
| 0   | 1     | 0      | 0       | 0     |
| 1   | 0     | 0      | 0       | 1     |
| 2   | 0     | 1      | 1       | 1     |
| 3   | 1     | 0      | 0       | 2     |
| 4   | 1     | 1      | 1       | 2     |
| 5   | 0     | 1      | 1       | 3     |

For a spin $j$ representation of $SO(3)$, the decomposition under $A_4$ proceeds as follows. There are four conjugacy classes of $A_4$, corresponding to rotations by $0^\circ$, $180^\circ$, $120^\circ$, and $240^\circ$. The characters of these rotations under the representation of $SO(3)$ are given by (A8). The multiplicities of $1'$ and $1''$ must be equal since the group $SO(3)$ is real. Then, using (A7), the following set of equations can be written:

$$2j + 1 = n_1 + 2n_1' + 3n_3,$$

$$(-1)^j = n_1 + 2n_1' - n_3,$$

$$\frac{2}{\sqrt{3}} \sin \left(\frac{(2j + 1)\pi}{3}\right) = n_1 - n_1' + \omega^2 n_3$$

(A9)

Note that the last two equations are cyclic in $j$ with period 6. This results in a pattern with that period. The decomposition for the first six representations is given in Table V. The pattern for a higher representation $j$ can be determined as follows. Let

$$q = \lfloor j/6 \rfloor, \quad r = j \mod 6.$$  

(A10)

Then for $i = 1, 1', 1''$ we have

$$n_i(j) = n_i(r) + q.$$  

(A11)

For $i = 3$ we have

$$n_3(j) = n_3(r) + 3q.$$  

(A12)

For example, the spin $j = 23$ representation has $q = 3$ and $r = 5$, and thus $n_1(23) = 3$, $n_1'(23) = n_1''(23) = 4$ and $n_3(23) = 12$. 

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APPENDIX B: MINIMA OF THE POTENTIAL OF A 7 OF SO(3)

In this appendix, we present the determination of the minima of the potential (10) as done in [25]. In order to proceed, it is simplest to reparametrize $T^{abc}$ based on symmetries. Before we do that, however, we start with a simpler example: the case of a triplet. We can write the 3 as the product of a magnitude and a unit vector: $v^a = \alpha x^a$ such that the three parameters are the length of $v$, denote by $\alpha$, and the two angles that describe the orientation of $v^a$. The point to emphasize is that the potential for such a scalar is written as a function of only one of the parameters, the magnitude $\alpha$. It is given by

$$V = -\frac{\mu^2}{2} v^a v^a + \frac{\lambda}{4!} (v^a v^a)^2 = -\frac{\mu^2}{2} \alpha^2 + \frac{\lambda}{4!} \alpha^4. \quad (B1)$$

Furthermore, if $\mu^2 > 0$ and $\lambda > 0$, the resulting vacuum has the residual symmetry of the unit vector $x^a$, which is $SO(2)$.

For the 7, the parametrization and potential are both more complicated. There are three orthogonal terms with different symmetries. The first term is invariant under $SO(2)$ as it depends on a single unit vector. The second term is best described geometrically. Consider an arbitrary equilateral triangle in three dimensions. Define three vectors connecting the center of the triangle to each of the three vertices of the triangle. The object defined by these vectors is called a regular 3-point star. Mathematically, it can be written as the symmetric outer product of the three defining vectors. This construction is automatically traceless. The second term is then given by a regular 3-point star defined with unit vectors. Finally, the third term is given by the symmetric product of three orthonormal unit vectors. Explicitly, the parametrization is

$$T^{abc} = \alpha \left( x^a x^b x^c - \frac{3}{5} \delta^{(ab)} x^c \right) + \beta \chi_{(3)}^{abc} + \gamma x^a y^b z^c, \quad (B2)$$

where $\chi_{(3)}^{abc}$ describes an arbitrary 3 point regular star with unit length vectors, the vectors $x$, $y$, $z$ are orthonormal and $\chi$ is orthogonal to $x$. A general tensor written as in (B2) has 7 parameters as one would expect for a symmetric traceless tensor of rank 3: $\alpha$, $\beta$, $\gamma$, the two angles in $x^a$, the angle of $\chi_{(3)}$ about the $x$ axis, and the angle of $y$ about the $x$ axis. The angle of $z$ is determined by requiring orthogonality. There are two advantages to the parametrization (B2). The first is that since the terms are orthogonal and normalized, the potential can now be written in terms of the three parameters $\alpha$, $\beta$, and $\gamma$ rather than in terms of seven parameters. The second is that, once the vacua are determined, it is far easier to determine the symmetries in this parametrization. The three terms in the parametrization have well-defined symmetry groups. The first is invariant under $SO(2)$ (rotations orthogonal to $x$). The second is invariant under $D_3$ since a three point star has the symmetries of a triangle. The third is invariant under $A_4$, where the three vectors $x$, $y$, and $z$ are taken
to be the 180° rotation axes. If the basis is chosen such that \( x, y, \) and \( z \) are along the corresponding axes of the coordinate system, this term is invariant under both \( S \) and \( T \) given in (A1).

The potential of Eq. (10) written in terms of (B2) depend only of three out of the seven parameters, \( \alpha, \beta, \) and \( \gamma. \) It is given by

\[
V = -\frac{\mu^2}{2} \left( \frac{2}{5} \alpha^2 + \frac{1}{4} \beta^2 + \frac{1}{6} \gamma^2 \right) + \frac{\lambda}{4} \left( \frac{2}{5} \alpha^2 + \frac{1}{4} \beta^2 + \frac{1}{6} \gamma^2 \right)^2 + \\
\quad c \left( \frac{44}{25} \alpha^4 + \frac{1}{25} \alpha^2 \beta^2 + \frac{2}{25} \alpha^2 \gamma^2 + \frac{1}{24} \beta^2 \gamma^2 + \frac{3}{18} \gamma^4 \right). \tag{B3}
\]

In order for an \( A_4 \)-invariant vacuum to exist, there must be a minimum with \( \alpha = \beta = 0 \) and \( \gamma \neq 0. \) Indeed, there is such a minimum for a certain portion of parameter space. If \( c > 0, \) then there is a \( D_3 \) invariant vacuum (only \( \beta \neq 0). \) For \( -\lambda/2 < c < 0, \) there is an \( A_4 \) invariant vacuum. Finally, for \( c < -\lambda/2, \) the potential has a runaway direction. It is possible to spontaneously break \( SO(3) \) to \( A_4 \) using a single scalar in a spin 3 representation of \( SO(3) \) by picking the second case, \( -\lambda/2 < c < 0. \)

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