Lévy stable distributions via associated integral transform

K. Górski$^{1,2,3,*}$ and K. A. Penson$^{3,†}$

$^1$H. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences,
ul. Eliaśa-Radzikowskiego 152, PL 31342 Kraków, Poland
$^2$Instituto de Física, Universidade de São Paulo,
P.O. Box 66318, B 05315-970 São Paulo, SP, Brasil
$^3$Laboratoire de Physique Théorique de la Matière Condensée (LPTMC),
Université Pierre et Marie Curie, CNRS UMR 7600
Tour 13 - 5ième ét., B.C. 121, 4 pl. Jussieu, F 75252 Paris Cedex 05, France

Abstract

We present a method of generation of exact and explicit forms of one-sided, heavy-tailed Lévy stable probability distributions $g_\alpha(x)$, $0 \leq x < \infty$, $0 < \alpha < 1$. We demonstrate that the knowledge of one such a distribution $g_\alpha(x)$ suffices to obtain exactly $g_{\alpha^p}(x)$, $p = 2, 3, \ldots$. Similarly, from known $g_\alpha(x)$ and $g_\beta(x)$, $0 < \alpha, \beta < 1$, we obtain $g_{\alpha\beta}(x)$. The method is based on the construction of the integral operator, called Lévy transform, which implements the above operations. For $\alpha$ rational, $\alpha = l/k$ with $l < k$, we reproduce in this manner many of the recently obtained exact results for $g_{l/k}(x)$. This approach can be also recast as an application of the Efros theorem for generalized Laplace convolutions. It relies solely on efficient definite integration.

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* kasia.gorska@o2.pl
† penson@lptljussieu.fr
I. INTRODUCTION

The one-dimensional Lévy stable one-sided distributions $g_\alpha(x)$, $0 < \alpha < 1$ are normalized probability density functions (PDF) defined on $0 \leq x < \infty$, with the peculiar property that their mean, variance and all higher moments diverge. The index $\alpha$ enters the principal characterization of $g_\alpha(x)$ via their Laplace transform

$$\mathcal{L}[g_\alpha(x); p] = \int_0^\infty e^{-px} g_\alpha(x) \, dx = e^{-p^\alpha}, \quad p > 0,$$

which is sometimes referred to as the Kohlrausch-Williams-Watts function [1]. The divergence of all the moments of $g_\alpha(x)$ of positive integer order is related to a slow decay of $g_\alpha(x)$, $(g_\alpha(x) \to x^{-(1+\alpha)}$, as $x \to \infty)$, usually termed as a "heavy-tail". Theoretical properties of $g_\alpha(x)$'s are discussed in [2], whereas the review of physical applications can be found in [3] and [4]. Many applications of $g_\alpha(x)$'s in financial mathematics can be found in [5], whereas their use in various simulations of jump processes is exposed in [6]. The quest for exact and explicit forms of $g_\alpha(x)$ was accomplished only recently [7–9]. The universal formulas of Eqs. (2), (3) and (4) in [7] give explicit forms of $g_\alpha(x)$ for fractional $\alpha = l/k$ with $l$ and $k$ arbitrary relatively prime integers, fulfilling the condition $l < k$. These results are satisfying as in the general case they furnish $g_{l/k}(x)$ as a sum of $k-1$ hypergeometric functions, which is a neat confirmation of the conjecture of Scher and Montroll [10]. As indicated in [7, 8], the implementation of Meijer’s G functions as well as of the hypergeometric functions in recent version of computer algebra systems [11] greatly facilitates the practical use of $g_{l/k}(x)$ in applications.

The purpose of this work is to present an alternative approach to generate $g_{l/k}(x)$ by an iterative procedure based on the self-reproducing property of $g_\alpha(x)$, directly resulting from Eq. (1). We shall proceed in an increasing order of complexity treating first in Sect. 2 the low-order values of $\alpha$, ($\alpha = 1/2, 1/3$ and $2/3$), for which exact forms were obtained in [7, 13, 14], see also [15]. In Sect. 2 we also formulate the formalism for general $\alpha$ and develop the notion of Lévy transform (initiated in [13]), which will allow one the generation of general distributions $g_{l/k}(x)$ by simple integration. In Sect. 3 we present applications of our method. Sect. 4 is devoted to discussion and conclusions. The definition and notations of Meijer’s G functions are given in the Appendix A. A compact formulation of our results using the Efros theorem is presented in Appendix B.
II. LOW-ORDER $\alpha = l/k$ AND REPRODUCING PROPERTIES OF $g_\alpha(x)$

a) The content of this subsection a) concerns the first genuine stable distribution $g_{1/2}(x)$ discovered by Lévy [12], and therefore all the results enumerated here are known and are included only for sake of completeness. This historically first Lévy stable distribution is:

$$g_{1/2}(x) = \frac{\exp\left(-\frac{1}{4x}\right)}{2\sqrt{\pi}x^{3/2}}, \quad x \geq 0,$$

satisfying $\mathcal{L}\left[g_{1/2}(x); p\right] = e^{-\sqrt{p}}$, with $p > 0$. We can formally introduce an additional variable $t$ in the problem by writing

$$\frac{1}{t^2} g_{1/2}\left(\frac{x}{t^2}\right) = \frac{t}{2\sqrt{\pi}x^{3/2}} e^{-t^2/(4x)}. \quad (3)$$

This last function is of particular interest for us as it intervenes directly in the following property of standard Laplace transform (see formula 1.1.1.26, p. 4, vol. 5 of [16]): if $\mathcal{L}[f(x); p] = F(p)$, then if we define $\tilde{f}_{1/2}(x)$ such that

$$\tilde{f}_{1/2}(x) = \frac{1}{2\sqrt{\pi}x^{3/2}} \int_0^\infty t \exp\left(-\frac{t^2}{4x}\right) f(t) \, dt, \quad (4)$$

then

$$\mathcal{L}\left[\tilde{f}_{1/2}(x); p\right] = F(p^{1/2}). \quad (5)$$

The proof of Eq. (5) follows:

$$\mathcal{L}\left[\tilde{f}_{1/2}(x); p\right] = \int_0^\infty e^{-px} \left[ \frac{1}{2\sqrt{\pi}x^{3/2}} \int_0^\infty t e^{-t^2/(4x)} f(t) \, dt \right] \, dx \quad (6)$$

$$= \frac{1}{2\sqrt{\pi}} \int_0^\infty f(t) t \left[ \int_0^\infty e^{-px} e^{-t^2/(4x)} \, dx \right] \, dt. \quad (7)$$

To evaluate the internal integration in Eq. (7) we use the formula 2.2.2.5, p. 31, vol. 4 of [16], and Eq. (7) becomes

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty t \left[ \frac{2\sqrt{\pi}}{t} e^{-t\sqrt{p}} \right] f(t) \, dt = F(p^{1/2}), \quad (8)$$

and therefore, from Eq. (3)

$$\mathcal{L}\left[\frac{1}{t^2} g_{1/2}\left(\frac{x}{t^2}\right); p\right] = \mathcal{L}\left[\frac{t \exp\left(-\frac{t^2}{4x}\right)}{2\sqrt{\pi}x^{3/2}}; p\right] = e^{-t\sqrt{p}}. \quad (9)$$
We note that although all our PDF’s are one-dimensional, \( t^{-3} g_{\frac{1}{3}}(x/t^2) \) is formally equal to the three-dimensional heat kernel, through appropriate renaming of the variables.

b) As a next step in our development we consider the stable Lévy density for \( \alpha = 1/3 \) whose exact form is \[10, 15\]
\[
g_{\frac{1}{3}}(x) = \frac{1}{3\pi x^{\frac{2}{3}}} K_{\frac{1}{3}} \left( \frac{2}{3\sqrt{3} x} \right), \quad x \geq 0, \tag{10}
\]
where \( K_{\nu}(x) \) is the modified Bessel function of the second kind, which can also be expressed through the Airy function \( Ai(y) \) giving
\[
g_{\frac{1}{3}}(x) = \frac{Ai \left( (3x)^{-\frac{1}{4}} \right)}{(3x^{\frac{4}{3}})}. \tag{11}
\]
We shall evaluate the Laplace transform of \( g_{\frac{1}{3}}(x) \) using the formula 3.16.3.7, p. 355, vol. 4 of \[16\], that we quote here \[17\]:
\[
\mathcal{L} \left[ x^{-\frac{2}{3}} K_{\frac{1}{3}} \left( \frac{a}{\sqrt{x}} \right); p \right] = \frac{2\pi}{a^{\frac{1}{3}}} \exp \left[ -3 \left( \frac{a^2}{4} \right)^{\frac{1}{3}} p^{\frac{4}{3}} \right], \quad p > 0, \tag{12}
\]
which, upon choosing \( 3(a^2/4)^{\frac{1}{3}} = t \), transforms into
\[
\mathcal{L} \left[ \frac{1}{3\pi} \left( \frac{t}{x} \right)^{\frac{2}{3}} K_{\frac{1}{3}} \left( \frac{2t^{\frac{2}{3}}}{3\sqrt{3} x} \right); p \right] = e^{-tp^{\frac{4}{3}}}, \tag{13}
\]
or equivalently, can be rewritten as
\[
\mathcal{L} \left[ \frac{1}{t^{\frac{2}{3}}} g_{\frac{1}{3}} \left( \frac{x}{t^{\frac{2}{3}}} \right); p \right] = e^{-tp^{\frac{4}{3}}}. \tag{14}
\]
In full analogy with Eqs. (4), (5) and (6) we conclude that if \( \mathcal{L}[f(x); p] = F(p) \) then for \( \tilde{f}_{\frac{1}{3}}(x) \) defined through
\[
\tilde{f}_{\frac{1}{3}}(x) = \frac{1}{3\pi x^{\frac{2}{3}}} \int_0^{\infty} t^{\frac{2}{3}} K_{\frac{1}{3}} \left( \frac{2t^{\frac{2}{3}}}{3\sqrt{3} x} \right) f(t) \, dt, \quad x \geq 0, \tag{15}
\]
\[
\mathcal{L} [\tilde{f}_{\frac{1}{3}}(x); p] = F(p^{\frac{4}{3}}). \tag{16}
\]

\[c\) We continue our review of Lévy stable densities expressible with standard special functions with \( g_{\frac{4}{3}}(x) \) which reads \[7, 14, 15, 18\]:
\[
g_{\frac{4}{3}}(x) = \frac{2\sqrt{3}}{27\pi x^{3}} e^{-\frac{x^2}{27}} \left[ K_{\frac{4}{3}} \left( \frac{2}{27 x^2} \right) + K_{\frac{4}{3}} \left( \frac{2}{27 x^2} \right) \right], \quad x \geq 0 \tag{17}
\]
\[
= \frac{\Gamma \left( \frac{4}{3} \right)}{\sqrt{3}\pi} x^{-\frac{5}{3}} \text{F}1 \left( \frac{5}{6}, \frac{-2^2}{33 x^2} \right) + \frac{2}{9} \Gamma \left( \frac{4}{3} \right) x^{-\frac{5}{3}} \text{F}1 \left( \frac{7}{6}, \frac{-2^2}{33 x^2} \right), \tag{18}
\]
where in Eq. (18) $\Gamma_1(\alpha | z) = _1F_1(a; b; z)$ is Kummer’s confluent hypergeometric function. In analogy with Eqs. (3) and (14) we form $t^{-\frac{a}{c}}g_\frac{2}{3}(x/t^\frac{2}{3})$ in the version of Eq. (18):

$$
\frac{1}{t^\frac{2}{3}}g_\frac{2}{3}(\frac{x}{t^\frac{2}{3}}) = \frac{\Gamma(\frac{5}{6})}{\sqrt{3\pi}} \frac{t}{x^\frac{1}{2}}_1F_1\left(\frac{5}{6}; \frac{2}{3}; -\frac{2t^3}{3^4x^2}\right) + \frac{2}{9}\frac{t^2}{\Gamma(\frac{5}{6})} x^\frac{2}{7} _1F_1\left(\frac{7}{6}; \frac{4}{3}; -\frac{2t^3}{3^4x^2}\right).
$$

To calculate the Laplace transform of Eq. (19) we apply the formula 3.35.1.16, p. 511, vol. 4 of [16] and, after simplifications, we arrive at the identity

$$
\mathcal{L}\left[\frac{1}{t^\frac{2}{3}}g_\frac{2}{3}(\frac{x}{t^\frac{2}{3}}); p\right] = e^{-\frac{t}{p^\frac{2}{3}}}. 
$$

This yields in turn, for $\mathcal{L}[f(x); p] = F(p)$, the pair of equations

$$
\tilde{f}_\frac{2}{3}(x) = \int_0^\infty \frac{1}{t^\frac{2}{3}}g_\frac{2}{3}(\frac{x}{t^\frac{2}{3}}) f(t) \, dt, \quad x \geq 0
$$

and

$$
\mathcal{L}[\tilde{f}_\frac{2}{3}(x); p] = F(p^\frac{2}{3}),
$$

in complete analogy to the pairs of Eqs. (4), (9) and Eqs. (15), (16).

**d)** The above pattern suggests the validity of the following general integral identities for $\mathcal{L}[f(x); p] = F(p)$: if

$$
\tilde{f}_\alpha(x) = \int_0^\infty \frac{1}{t^\alpha}g_\alpha(\frac{x}{t^\alpha}) f(t) \, dt, \quad 0 < \alpha < 1, \quad x \geq 0,
$$

then

$$
\mathcal{L}[\tilde{f}_\alpha(x); p] = F(p^\alpha).
$$

We are going to prove Eq. (24) using Eq. (1). To this end we calculate explicitly Eq. (24):

$$
\mathcal{L}[\tilde{f}_\alpha(x); p] = \int_0^\infty e^{-px} \left[ \int_0^\infty \frac{1}{t^\alpha}g_\alpha(\frac{x}{t^\alpha}) f(t) \, dt \right] \, dx
$$

$$
= \int_0^\infty f(t) \left[ \int_0^\infty e^{-p\frac{t}{x^\alpha}} \frac{1}{t^\alpha}g_\alpha(\frac{x}{t^\alpha}) \, dx \right] \, dt
$$

$$
= \int_0^\infty f(t) \left[ \int_0^\infty e^{-p\frac{t}{y^\alpha}} g_\alpha(y) \, dy \right] \, dt
$$

$$
= \int_0^\infty f(t) e^{-p^\alpha t} \, dt = F(p^\alpha).
$$
In Eq. (26) we have used a simple change of variable, whereas in Eq. (27) we have applied Eq. (1).

Eqs. (23) and (24) acquire additional importance when for \( f(t) \) we choose another Lévy stable density, say \( g_\beta(x) \) with arbitrary \( \beta \), such that \( 0 < \beta < 1 \). Then

\[
\int_0^\infty \frac{1}{t^{1/\alpha}} g_\alpha \left( \frac{x}{t^{1/\alpha}} \right) g_\beta(t) \, dt = \int_0^\infty \frac{1}{t^{1/\beta}} g_\beta \left( \frac{x}{t^{1/\beta}} \right) g_\alpha(t) \, dt, \quad x \geq 0,
\]

which defines the following transitive property of Lévy laws:

\[
g_{\alpha \beta}(x) = \int_0^\infty \frac{1}{t^{1/\alpha}} g_\alpha \left( \frac{x}{t^{1/\alpha}} \right) g_\beta(t) \, dt = \int_0^\infty \frac{1}{t^{1/\beta}} g_\beta \left( \frac{x}{t^{1/\beta}} \right) g_\alpha(t) \, dt,
\]

for \( 0 < \alpha, \beta < 1 \). Eqs. (24) and (30) constitute the key results of the present investigation.

The following remarks are in order.

i) The formulas Eqs. (4) and (5) deserve to be better known. They appear (along with their several extensions) in various works of Russian school [16, 19, 20], but are conspicuously absent from other monographs and handbooks [21, 22].

ii) the formulas Eqs. (16), (20) and more generally, Eqs. (23) and (24) should complement the formula of Eq. (5) in the lists of properties of Laplace transform. In subsequent Sections we obtain many other formulas of this type, by specifying other values of \( \alpha \) and \( \beta \).

iii) Eq. (30) should be viewed as a tool to generate the Lévy stable PDF’s, starting with those with arbitrary \( \alpha \) and \( \beta \) and yielding one with \( \alpha \beta \), via integration.

The content of Eqs. (23) and (24) can be seen as an integral transform with the positive kernel \( \kappa_\alpha(t, x) = t^{-1/\alpha} g_\alpha(x/t^{1/\alpha}) \) with \( x, t > 0 \), which we define as the Lévy2 transform of index \( \alpha \), and is denoted by \( L_\alpha^{(2)} \):

\[
L_\alpha^{(2)}[f(t); x] = \int_0^\infty \kappa_\alpha(t, x) f(t) \, dt = \tilde{f}_\alpha(x).
\]

According to Eq. (30) the action of \( L_\alpha^{(2)} \) on Lévy’s PDFs satisfy

\[
L_\alpha^{(2)}[g_\beta(t); x] = L_\beta^{(2)}[g_\alpha(t); x] = g_{\alpha \beta}(x),
\]

which well illustrates the reproducing property of Lévy densities under the Lévy2 transforms. It can be seen from Eqs. (31) and (32) that for \( 0 < \alpha < 1 \) the inverse of Lévy2 transform
cannot be meaningfully defined. As a consequence, Eq. (32) reflects the semigroup property of $L_\alpha^{(2)}$ in accordance with the general theory [26].

Alternative way to derive the results of this section is to use the Efros theorem of theory of Laplace transform, see Appendix B.

III. SOME APPLICATIONS OF THE METHOD

a) We shall exemplify now how the method works by choosing in Eq. (32) $\alpha = \beta = 1/2$ which accordingly should furnish $g_4(x)$. We use Eqs. (2) and (3) and obtain

$$g_4(x) = \frac{1}{4\pi x^{3/2}} \int_0^\infty t^{-1/2} \exp \left[-\left(\frac{t^2}{4x} + \frac{1}{4t}\right)\right] dt.$$  

(33)

The integral in Eq. (33) is not elementary, but with a new variable $y = t^2$ it becomes a particular case of the formula 2.3.2.14, p. 322, vol. 1 of [23] (the alternative way is to employ the formula 2.2.2.7, p. 32, vol. 4 of [16]), which involves a finite sum of hypergeometric functions. The final answer, after the simplifications in hypergeometric functions

$$g_4(x) = \frac{\Gamma(\frac{3}{4})}{2^{\frac{3}{4}}\pi^{\frac{3}{4}} x^{\frac{3}{4}}} 0F_2 \left(\begin{array}{c} -1 \\ \frac{5}{4}, \frac{3}{2} \end{array}; -\frac{1}{2^8 x}\right)$$

$$- \frac{1}{4\sqrt{\pi} x^{\frac{3}{2}}} 0F_2 \left(\begin{array}{c} -1 \\ \frac{3}{4}, \frac{5}{4} \end{array}; -\frac{1}{2^8 x}\right)$$

$$+ \frac{1}{4\Gamma(\frac{3}{4}) x^{\frac{3}{4}}} 0F_2 \left(\begin{array}{c} -1 \\ \frac{7}{4}, \frac{3}{2} \end{array}; -\frac{1}{2^8 x}\right),$$

(34)

which is precisely the result obtained in [7] and [13]. We stress that here it was calculated with one definite integration using the tables [23].

b) Next example in this spirit will involve the indices $\alpha = 1/2$ and $\beta = 1/3$, i.e. we shall explicitly perform the Lévy2 transforms of type

$$L_\frac{1}{2}^{(2)}[g_4(t); x] = L_\frac{1}{3}^{(2)}[g_4(t); x] = g_6(x),$$

(35)

with Eqs. (3) and (10). The result reads ($b \equiv \frac{2}{3\sqrt{3}}$)

$$g_6(x) = \frac{1}{2\sqrt{\pi} x^{\frac{3}{4}}} \int_0^\infty t e^{-t^2/(4x)} \left[\frac{1}{3\pi t^{\frac{3}{2}}} K_\frac{1}{4} \left(\frac{b}{\sqrt{t}}\right)\right] dt,$$

(36)
which, with a change of variable, transforms into

\[ g_{\frac{1}{6}}(x) = \frac{1}{12(\pi x)^{\frac{3}{4}}} \int_0^\infty e^{-y/(4x)} \left[ y^{-\frac{3}{4}} K_{\frac{1}{3}} \left( \frac{b}{y^{\frac{1}{4}}} \right) \right] dy. \]  \hspace{1cm} (37)

The integral in Eq. (37) can be perceived as the Laplace transform of \( y^{-\frac{3}{4}} K_{\frac{1}{3}}(by^{-1/4}) \) for which an exact formula in terms of a specific Meijer’s G function exists, compare Eq. 3.16.3.9, p. 355, vol. 4 of [16], with the specializations: \( a = b = \frac{2}{3\sqrt{3}}, \ l = 1, \ k = 2, \ \mu = -3/4, \ \nu = 1/3 \) and \( p = 1/(4x) \). The final result is proportional to the Meijer’s G function of type \( G_{5,0}^{0,5} \left( 6^6 x \mid \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{13}{12} \right) \), see Appendix. By lumping together all the constants we obtain:

\[ g_{\frac{1}{6}}(x) = \frac{\sqrt{2}}{48\pi^{5/2} x^{5/4}} G_{5,0}^{0,5} \left( 6^6 x \mid \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{13}{12} \right) \]  \hspace{1cm} (38)

\[ = \frac{2^{-\frac{1}{3}} 3^{-\frac{3}{2}} \sqrt{\pi}}{\Gamma \left( \frac{3}{2} \right)^2} 0F_4 \left( -1 \mid 6^6 x \right) - \frac{1}{6\Gamma \left( \frac{2}{3} \right) x^{\frac{3}{2}}} 0F_4 \left( -1 \mid 6^6 x \right) \]

\[ + \frac{(12)^{-1}}{\sqrt{\pi} x^{\frac{3}{2}}} 0F_4 \left( \frac{5}{6}, \frac{7}{6}, \frac{4}{3} \mid 6^6 x \right) - \frac{\sqrt{3}\Gamma \left( \frac{2}{3} \right)}{72\pi x^{\frac{3}{2}}} 0F_4 \left( \frac{5}{6}, \frac{7}{6}, \frac{4}{3} \mid 6^6 x \right) \]

\[ + \frac{3^{-\frac{3}{2}} \Gamma \left( \frac{2}{3} \right)^2}{2^{15}\pi x^{1/2}} 0F_4 \left( \frac{7}{6}, \frac{4}{3}, \frac{5}{3} \mid 6^6 x \right). \]  \hspace{1cm} (39)

The Eq. (39) was obtained using the formulas 16.17.2 and 16.17.3 of [24]. The result of Eq. (39) is a special case of the solutions in [7]. The explicit form of \( g_{\frac{1}{6}}(x) \) in Eq. (39) appears to be written down for the first time here.

c) The third example in this section will be of more academic value as it will concern three PDF’s known from the previous sections: \( g_{\frac{1}{2}}(x), \ g_{\frac{2}{3}}(x) \) and \( g_{\frac{1}{3}}(x) \). According to Eqs. (30) and (32) they are related through

\[ L_{\frac{1}{2}}^{(2)}[g_{\frac{2}{3}}(t); x] = L_{\frac{1}{2}}^{(2)}[g_{\frac{1}{3}}(t); x] = g_{\frac{1}{3}}(x). \]  \hspace{1cm} (40)
Indeed, the first integral transform of Eq. (40) is equal to
\[
\int_0^\infty \frac{1}{t^2} g_{\frac{1}{2}} \left( \frac{x}{t^2} \right) g_{\frac{1}{2}}(t) \, dt = \frac{\Gamma \left( \frac{3}{2} \right)}{2\sqrt{\pi} x^{3/2}} \int_0^\infty t^{-1/2} e^{-1/4t^3} \left[ 5/6 \right]_{-2^2t^3/3^3x^2}^1 F_1 \left( \frac{7/6}{4/3} \right) dt + \frac{1}{9\sqrt{\pi} \Gamma \left( \frac{3}{2} \right)} x^{7/3} \int_0^\infty t^{1/2} e^{-1/4t^3} \left[ 7/6 \right]_{-2^2t^3/3^3x^2}^1 F_1 \left( \frac{7/6}{4/3} \right) dt,
\]
and, with the help of the formula 3.38.1.30, p. 553, vol. 4 of \[16\], it can be written down as a special case of the Meijer’s G function, very much in the spirit of considerations leading to Eq. (38), however see \[25\]. (Alternatively the simplified formula 3.35.1.16, p. 511, vol. 4 of \[16\] can also be used.). We will skip further details of this evaluation and conclude that the result is equal to \( g_{\frac{1}{2}}(x) \) from Eq. (10), as it should be.

d) In the following examples we shall not use the reproducing property of Eq. (30). We shall instead concentrate on Eqs. (23) and (24) conceived as an operational property of the Laplace transform. To this end we, quite arbitrarily, choose the modified Bessel function \( K_0(x) \) as \( f(x) \) in Eq. (23) and we read off its Laplace transform from the Eq. 3.16.1.2, p. 349, vol. 4 of \[16\], which is
\[
\mathcal{L} [K_0(x); p] = \frac{\arccos(p)}{\sqrt{1 - p^2}}, \quad p > 0.
\]
If we set \( g_{\frac{1}{2}}(z) \) in Eq. (23) then the corresponding transformed function is
\[
\tilde{K}_{0,\frac{1}{2}}(x) = \frac{e^x \Gamma(0, x)}{2\sqrt{\pi x}}
\]
\[
= -\frac{e^x Ei(-x)}{2\sqrt{\pi x}}.
\]
In Eq. (43) we have used the formula 2.16.8.5, p. 352, vol. 2 of \[23\], with \( \Gamma(0, x) = -Ei(-x) \), see p. 726, vol. 2 of \[23\]. In Eqs. (43) and (44), \( \Gamma(\nu, z) \) and \( Ei(z) \) are incomplete gamma function and the exponential integral, respectively. The Laplace transform of the function in Eq. (44) can be evaluated to be
\[
\mathcal{L} \left[ \tilde{K}_{0,\frac{1}{2}}(x); p \right] = \frac{\arcsinh(\sqrt{p-1})}{\sqrt{p-1}}, \quad p > 0.
\]
Let us briefly sketch how Eq. (45) comes about.
We write out explicitly the Laplace transform in Eq. (4) as

\[ L \left[ \tilde{K}_0, \frac{1}{3} (x), p \right] = -\int_0^\infty e^{-(p-1)Ei(-x)} \frac{dx}{2\sqrt{\pi x}} \] (46)

\[ = \frac{1}{\sqrt{p}} {}_2F_1 \left( \frac{1}{2}, 1 \mid \frac{p-1}{p} \right) \] (47)

\[ = \frac{\text{arcsinh} (\sqrt{p-1})}{\sqrt{p-1}}, \quad p > 0. \] (48)

In Eq. (47) we have used the Eq. 3.4.1.3, p. 135, vol. 4 of [16], whereas Eq. (48) results from Eq. 7.3.2.83, p. 473, vol. 3 of [23]. Since for \( p > 0 \), \( \text{arccos}(\sqrt{p})/\sqrt{1-p} = \text{arcsinh}(\sqrt{p-1})/\sqrt{p-1} \), the above calculations evidently confirm the validity of Eq. (24).

We go now a step further: while still keeping \( f(x) = K_0(x) \) in Eq. (24), we use now \( g_{\frac{1}{3}}(z) \), see Eq. (10), to perform the integration in Eq. (23). The corresponding transformed function \( \tilde{K}_0, \frac{1}{3} (x) \) reads [11]

\[ \tilde{K}_0, \frac{1}{3} (x) = \frac{\sqrt{3/2}}{8\pi^2 x^{3/2}} G_{6,4}^{4,6} \left( \frac{x^2}{4}, \frac{\frac{5}{12}, \frac{7}{12}, \frac{11}{12}, \frac{13}{12}}{\frac{5}{12}, \frac{3}{12}, \frac{1}{12}, \frac{13}{12}} \right), \quad x > 0, \] (49)

whose Laplace transform can be calculated via 3.40.1.1, vol. 4 of [16]. This lengthy expression involves a combination of four different hypergeometric functions of argument \( p^2 \). We shall not reproduce it here. We emphasize however that we have verified numerically the relation

\[ L \left[ \tilde{K}_0, \frac{1}{3} (x); p \right] = \frac{\text{arccos} (p^{1/3})}{\sqrt{1 - p^{2/3}}}, \quad p > 0, \] (50)

thereby establishing an algebraic identity stating that the aforementioned combination of hypergeometric functions is equal to the r.h.s. of Eq. (50).

It is clear that these procedures can be carried out by choosing functions other than \( K_0(x) \) in Eq. (23). This circumstance paves the way for a scheme of generation of algebraic identities involving special functions using Eqs. (23) and (24). This will be a subject of future publication.
IV. DISCUSSION AND CONCLUSIONS

Our kernel $\kappa_\alpha(t, x)$ which defines the Lévy2 transform of index $\alpha$ through Eq. (31) above, can be related to the function $n(s, \tau)$ introduced by E. Barkai in [13] as follows

$$n(s, \tau) = \frac{1}{\alpha s} s^{\tau / \alpha} g_\alpha \left( \frac{\tau}{s^\alpha} \right) = \frac{1}{\alpha s} \kappa_\alpha(\tau, s),$$

(51)

where attention should be given to the order of variables in Eq. (51). This function serves as a kernel in another integral transform [13] which links a solution of ordinary Fokker-Planck equation $P_1(x, \tau)$, satisfying, for given functions $A(x)$ and $B(x)$,

$$\frac{\partial}{\partial \tau} P_1(x, \tau) = - \frac{\partial}{\partial x} [A(x) P_1(x, \tau)] + \frac{\partial^2}{\partial x^2} [B(x) P_1(x, \tau)],$$

(52)

with $P_\alpha(x, \tau)$ satisfying the fractional Fokker-Planck (FFP) equation,

$$\frac{\partial^\alpha}{\partial \tau^\alpha} P_\alpha(x, \tau) = - \frac{\partial}{\partial x} [A(x) P_\alpha(x, \tau)] + \frac{\partial^2}{\partial x^2} [B(x) P_\alpha(x, \tau)],$$

(53)

where $\frac{\partial^\alpha}{\partial \tau^\alpha}$ is appropriately defined fractional derivative. The relation is

$$P_\alpha(x, \tau) = \int_0^\infty n(s, \tau) P_1(x, s) \, ds,$$

(54)

and is called in [13] the inverse Lévy transform. The functional relation of this type appears also in Sokolov’s work [27], section III, Eqs. (15) and (16) of [27]. If we provisionally reserve the notation $\left( L_\alpha^{(1)} \right)^{-1}$ to the transform in Eq. (54), then it can be rewritten as

$$\left( L_\alpha^{(1)} \right)^{-1} [P_1(x, s); \tau] = P_\alpha(x, \tau).$$

(55)

The symbol $L_\alpha^{(1)}$ should be clearly distinguished from the Lévy2 transform appearing in Eqs. (31) and (35), denoted by $L_\alpha^{(2)}$. From Eq. (51) the relation between those two transforms is

$$L_\alpha^{(2)} \left[ \frac{1}{\alpha s} P_1(x, s); \tau \right] = \left( L_\alpha^{(1)} \right)^{-1} [P_1(x, s), \tau],$$

(56)

where it should be stressed that in both sides of Eq. (56) the integration variable is $s$. Here for the general $\alpha$ the direct Lévy transform $L_\alpha^{(1)}$ cannot be constructed, compare remarks after Eq. (32).

In both Refs. [13] and [27] the emphasis is on generation of solutions of FFP equations depending on the choice of $A(x)$, the drift function, and $B(x)$, the diffusion function, whereas our method here focuses on generation of explicit forms of $g_\alpha(x)$. In the separate work we
have undertaken the analysis of FFP equation using the explicit form of $g_L(x)$, see [28]. Further considerations on applications of relations of type Eq. (54) can be found in [29] and [30]. Extensions of the current framework to include two-sided Lévy distributions [8, 9] as well as log-Lévy distributions [31] are under study. Finally, note that newer versions of Mathematica® furnish numerical forms of $g_\alpha(x)$. Similarly, Matlab® can be also used for that purpose.

V. ACKNOWLEDGEMENTS

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Appendix A: Definition and notations of Meijer’s G functions

The Meijer’s G function is defined as an inverse Mellin transform denoted by $\mathcal{M}^{-1}$, see vol. 3 of [23]:

$$
G_{p,q}^{m,n}(z^{(\alpha_p)}_{(\beta_q)}) = \mathcal{M}^{-1}\left[\frac{\prod_{j=1}^{m}\Gamma(\beta_j + s)\prod_{j=1}^{n}\Gamma(1 - \alpha_j - s)}{\prod_{j=n+1}^{q}\Gamma(1 - \beta_j - s)\prod_{j=m+1}^{p}\Gamma(\alpha_j + s)}; z\right]
$$

$$
= G([\alpha_1, \ldots, \alpha_n], [\alpha_{n+1}, \ldots, \alpha_p], [\beta_1, \ldots, \beta_m], [\beta_{m+1}, \ldots, \beta_q], z),
$$

where in Eq. (A1) empty products are taken to be equal to one. In Eqs. (A1) and (A2) the parameters are subject of conditions:

$$
z \neq 0, \ 0 \leq m \leq q, \ 0 \leq n \leq p;
$$

$$
\alpha_j \in \mathbb{C}, \ j = 1, \ldots, p; \ \beta_j \in \mathbb{C}, \ j = 1, \ldots, q;
$$

$$
(\alpha_p) = \alpha_1, \alpha_2, \ldots, \alpha_p; \ (\beta_q) = \beta_1, \beta_2, \ldots, \beta_q.
$$
For a full description of integration contours in Eq. (A1), general properties and special cases of the $G$ functions, see vol. 3 of [23]. In Eq. (A2) we present a transparent notation inspired by computer algebra systems [11].

In this context we can give more details on how Eq. (38) comes about. The rewriting again of Eq. 3.16.3.9, p. 355, vol. 4 of [16] with the set of parameter $s$ given above, gives the following expression for $g_6^1(x)$:

$$g_6^1(x) \sim G_{5,0}^{0,5} \left( 6^6 x \left| \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{13}{12} \right. \right).$$

(A4)

According to (A1), in $G_{5,0}^{0,5}$ the parameters are: $m = 0$, $n = 5$, $p = 5$, and $q = 0$. If we apply this to the notation specified in the function $G_{5,0}^{0,5}$ in (A4), it becomes:

$$G_{5,0}^{0,5} \left( 6^6 x \left| \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{13}{12} \right. \right)$$

$$\equiv G \left( \left[ \left[ \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{13}{12} \right], \left[ \right] \right], \left[ \right], \left[ \right], 6^6 x \right),$$

(A5)

which is a required format of computer algebra systems [11].

All the Meijer’s G functions encountered in course of integrations via Lévy2 transform of Eq. (31) were converted to forms of type (A5) and consequently checked numerically.

**Appendix B: The Efros Theorem**

We quote here the A. M. Efros theorem (1935) which is the generalisation of the convolution theorem for the Laplace transform. See Refs. [19, 32, 33] for proof and many applications.

**Theorem:** If $G(p)$ and $q(p)$ are analytic functions, $L[f(x); p] = F(p)$ and

$$L[g(x,t); p] = \int_0^\infty g(x,t) e^{-px} dx = G(p) e^{-tq(p)},$$

(B1)

then

$$G(p) F(q(p)) = \int_0^\infty \left[ \int_0^\infty f(t) g(x,t) dt \right] e^{-px} dx.$$  

(B2)

We shall demonstrate now that Eq. (30) can be recast as a special case of the Efros theorem, Eq. (B2).
The use of the result (B2) in the context of Lévy stable laws entails the choice $f(t) = g_\beta(t)$, $0 < \beta < 1$, implying $F(p) = e^{-p^\beta}$ with Eq. [1]. Furthermore, we set $G(p) = 1$, $q(p) = p^\alpha$ and

$$g(x, t) = \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right), \quad 0 < \alpha < 1.$$  \hfill (B3)

From the Efros theorem (B2) we obtain

$$G(p) F(q(p)) = e^{-p^\alpha} = \int_0^\infty \left[ \int_0^\infty g_\beta(t) \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt \right] e^{-px} dx.$$ \hfill (B4)

On the other hand, Eq. [1] implies

$$e^{-p^\beta} = \int_0^\infty g_{\alpha\beta}(x) e^{-px} dx,$$ \hfill (B5)

which immediately yields

$$g_{\alpha\beta}(x) = \int_0^\infty g_\beta(t) \frac{1}{t^{1/\alpha}} g_\alpha\left(\frac{x}{t^{1/\alpha}}\right) dt,$$ \hfill (B6)

which is our Eq. (30).

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We have corrected a misprint in this formula: the correct prefactor in the numerator of r.h.s. of Eq. (12) is $2\pi$ and not $4\pi$ as indicated in [16].

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Two misprints in this last formula should be corrected: in the prefactor of the r.h.s. $k^\nu$ should be replaced by $k^\mu$ and in the parameter list of the G function the (meaningless) symbol $\Delta(1-(b_n))$ should be replaced by $\Delta(k,1-(b_n))$, where $\Delta(k,(a_p)) = \frac{(a_p)}{k}, \frac{(a_p)+1}{k}, \ldots, \frac{(a_p)+k-1}{k}$, see p. 798 of vol. 3 of [23].

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