Abstract. We discuss an Abel-Jacobi invariant for algebraic cobordism cycles whose image in topological cobordism vanishes. In particular, we show that this invariant can be defined via integrals over forms similar to the classical case.

1. Introduction

The Abel-Jacobi map is an important invariant in algebraic geometry. We will briefly recall its construction.

Let $S$ be a connected smooth projective curve of genus $g$ over $\mathbb{C}$. Let $\text{Div}^0(S)/\sim$ be the group of divisors of degree 0 on $S$ modulo linear equivalence, $H_1(S;\mathbb{Z})$ be the first homology group of $S$, and let $H^0(S;\Omega^1_{\text{hol}})^*$ denote dual of the global sections of the sheaf of holomorphic 1-forms on $S$. The Abel-Jacobi map is the group homomorphism

$$\text{Div}^0(S)/\sim \to \text{Jac}(S) := H^0(S;\Omega^1_{\text{hol}})^*/H_1(S;\mathbb{Z}), \sum_i (p_i - q_i) \mapsto \left( \omega \mapsto \sum_i \int_{q_i}^{p_i} \omega \right).$$

The theorem of Abel-Jacobi states that this map is an isomorphism. The quotient $\text{Jac}(S)$ is called the Jacobian of $S$.

In his seminal work [9], Griffiths generalized the Abel-Jacobi map to algebraic varieties of higher dimensions. Let $X$ be a smooth projective complex variety of dimension $n$. Instead of looking at divisors, we would like to study irreducible closed subvarieties $Y \subset X$, or, more generally, cycles on $X$, i.e., elements in the free abelian group generated by irreducible closed subsets of $X$. Let $p$ be the codimension of $Y$ in $X$. The condition of being of degree zero is then replaced by assuming that $Y$ is homologous to zero, i.e., there is a singular chain $\Gamma$ of (real) dimension $2n - 2p + 1$ on $X$ whose boundary equals $Y$. we can consider the assignment

$$\omega \mapsto \int_{\Gamma} \omega$$

which sends a form $\omega$ to its integral over $\Gamma$. The value of this integral depends on the choice of $\Gamma$. After taking the quotient by the appropriate homology of $X$, we obtain a well-defined homomorphism

$$\Phi': Z^p_{\text{hom}}(X) \to F^{n-p+1}H^{2n-2p+1}(X;\mathbb{C})^*/H_{2n-2p+1}(X;\mathbb{Z})^*, \left( \omega \mapsto \int_{\Gamma} \omega \right).$$

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where $Z^p_{\text{hom}}(X)$ denotes the group of cycles of codimension $p$ in $X$ which are homologous to zero, and $F^{n-p+1}H^{2n-2p+1}(X; \mathbb{C})$ the $(n-p+1)$th step in the Hodge filtration on the cohomology of $X$ with complex coefficients. By Poincaré duality and Hodge theory, we know that the group $F^{n-p+1}H^{2n-2p+1}(X; \mathbb{C})^*/H_{2n-2p+1}(X; \mathbb{Z})$ is isomorphic to the $p$th intermediate Jacobian of Griffiths defined by

$$J^{2p-1}(X) := H^{2p-1}(X; \mathbb{C})/(F^p H^{2p-1}(X; \mathbb{C}) + H^{2p-1}(X; \mathbb{Z})).$$

The homomorphism $\Phi'$ in (2) is Griffiths’ generalization of the Abel-Jacobi map. Deligne showed that we can consider this map in the following larger picture. Let $H^{2p}_D(X; \mathbb{Z}(p))$ denote the $2p$th Deligne cohomology of $X$ with coefficients in $\mathbb{Z}(p)$, $\text{Hdg}^{2p}(X)$ the group of integral Hodge classes in $H^{2p}(X; \mathbb{Z})$, and let $CH^{p}(X)$ be the $p$th Chow group, i.e., the free abelian group of cycles of codimension $p$ modulo rational equivalence. Moreover, let $CH^p_{\text{hom}}(X)$ be the subgroup of $CH^p(X)$ of cycles which are homologous to zero. Then there is a commutative diagram

\[
\begin{array}{ccc}
CH^p_{\text{hom}}(X) & \longrightarrow & CH^p(X) \\
\Phi \downarrow & & \downarrow \text{cl} \\
0 & \longrightarrow & J^{2p-1}(X) & \longrightarrow & H^D^{2p}(X; \mathbb{Z}(p)) & \longrightarrow & \text{Hdg}^{2p}(X) & \longrightarrow & 0.
\end{array}
\]

The bottom row of this diagram is exact. The homomorphism $\Phi$ is induced by the Deligne-cycle map $\text{cl}_D$ and the fact that $CH^p_{\text{hom}}(X)$ is the kernel of the classical cycle map $\text{cl}$. Using the above mentioned isomorphism between $J^{2p-1}(X)$ and the quotient $F^{n-p+1}H^{2n-2p+1}(X; \mathbb{C})^*/H_{2n-2p+1}(X; \mathbb{Z})$, we can identify $\Phi$ in (3) with the Abel-Jacobi map $\Phi'$ in (2). Hence we may view $\Phi'$ as a concrete description of the abstract map $\Phi$ in diagram (3).

Even though the Abel-Jacobi map in higher dimensions is, in general, not an isomorphism, it is an important invariant for algebraic cycles. For example, one can use $\Phi$ to detect cycles that are homologous to zero. This is particularly interesting when one can show that the cycle in question is not algebraically equivalent to zero. The quotient of $CH^p(X)_{\text{hom}}$ modulo algebraic equivalence is called the Griffiths group and is denoted by $\text{Griff}^p(X)$. In [9], Griffiths used the Abel-Jacobi map to show that $\text{Griff}^p(X)$ can be an infinite group.

The purpose of this paper is to discuss an analog of the Abel-Jacobi map when we replace the role of Chow groups and cohomology by algebraic and complex cobordism respectively. In [17], Levine and Morel constructed algebraic cobordism as the universal oriented cohomology theory on the category $\text{Sm}_k$ of smooth quasi-projective schemes over a field $k$ of characteristic zero. For $X \in \text{Sm}_k$, the algebraic cobordism ring of $X$ is denoted by $\Omega^*(X)$. For a given $p \geq 0$, $\Omega^p(X)$ is generated by prime cycle of the form $f : Y \to X$ where $Y$ is a smooth projective scheme over $k$ and $f$ is a projective $k$-morphism. In the case $k = \mathbb{C}$, taking complex points induces a natural homomorphism of rings

$$\varphi_{MU} : \Omega^*(X) \to MU^{2*}(X) := MU^{2*}(X(\mathbb{C})).$$

More recently, in [12] Hopkins and the author constructed natural generalizations of Deligne-Beilinson cohomology on $\text{Sm}_C$ for any topological spectrum $E$. For $E = MU$, we obtain logarithmic Hodge filtered complex bordism groups. For $n, p \in \mathbb{Z}$ and $X \in \text{Sm}_C$, they are denoted by $MU_{\log}^n(p)(X)$. Taking the sum over all
For given $n$ and $p$, $MU_{\log}$ is equipped with a ring structure, and, for every $X \in \text{Sm}_\mathbb{C}$, there is a natural ring homomorphism

$$\varphi_{MU_{\log}}: \Omega^*(X) \to MU_{\log}^{2p}(\ast)(X).$$

Furthermore, there is a generalization of diagram (3) in the following sense. For given $p$, let $\Omega^p_{\text{top}}(X)$ be the kernel of the map $\varphi_{MU}$. Then for every smooth projective algebraic variety $X$ over $\mathbb{C}$, there is a natural commutative diagram

(4)

$$\begin{array}{ccc}
\Omega^p_{\text{top}}(X) & \longrightarrow & \Omega^p(X) \\
\Phi_{MU} \downarrow & & \varphi_{MU_{\log}} \downarrow \\
0 & \longrightarrow & J^{2p-1}_{MU}(X) & \longrightarrow & MU_{\log}^{2p}(p)(X) & \longrightarrow & \text{Hdg}^{2p}_{\text{MU}}(X) & \longrightarrow & 0.
\end{array}$$

The bottom row of this diagram is again exact (see [12, Theorem 4.13]). The group $\text{Hdg}^{2p}_{\text{MU}}(X)$ is the subgroup of elements in $MU^{2p}(X)$ which are mapped to Hodge classes under the canonical map $MU^{2p}(X) \to H^{2p}(X; \mathbb{Z})$. We consider the group $J^{2p-1}_{MU}(X)$ as a generalized Jacobian of $X$. It is a complex torus which is determined by the Hodge structure of the cohomology and the complex cobordism of $X$. The map $\Phi_{MU}$ is induced by $\varphi_{MU_{\log}}$ and can be considered as an analog of the Abel-Jacobi map.

In this paper, we show that the new Abel-Jacobi map $\Phi_{MU}$ has an interpretation via integrals which is similar to the description of the classical Abel-Jacobi map in [2]. Diagram (4) implies that $\Phi_{MU}$ detects algebraic cobordism cycles $Y \to X$ which are topologically trivial, i.e., such that $Y(\mathbb{C})$ is the boundary of a smooth manifold over $X(\mathbb{C})$. Our result shows that one way to check this is by calculating integrals over forms just as for ordinary cycles.

Let us add a few more words on the relationship of diagrams (3) and (4). By [17] and [12] there is a natural commutative diagram

(5)

$$\begin{array}{ccc}
\Omega^p(X) & \longrightarrow & MU_{\log}^{2p}(p)(X) & \longrightarrow & MU^p(X) \\
\varphi \downarrow & & \varphi_{\text{hom}} \downarrow & & \varphi \\
CH^p(X) & \longrightarrow & H^{2p}_p(X; \mathbb{Z}(p)) & \longrightarrow & H^{2p}(X; \mathbb{Z})
\end{array}$$

where the composite $CH^p(X) \to H^{2p}(X; \mathbb{Z})$ is the cycle map we mentioned above. The composite $\Omega^p(X) \to H^{2p}(X; \mathbb{Z})$ is the canonical homomorphism induced by the transformation of oriented cohomology theories. The classical Abel-Jacobi map is defined on cycles which vanish under the cycle map. We could also consider the subgroup $\Omega^p_{\text{hom}}(X)$ of elements in $\Omega^p(X)$ which are mapped to zero under the composite $\Omega^p(X) \to H^{2p}(X; \mathbb{Z}(p))$. It is clear from diagram (5) that we have a natural inclusion

$$\Omega^p_{\text{top}}(X) \subset \Omega^p_{\text{hom}}(X).$$

On $\Omega^p_{\text{hom}}(X)$ we have an induced map $\vartheta_{\text{hom}}: \Omega^p_{\text{hom}}(X) \to CH^p_{\text{hom}}(X)$. An Abel-Jacobi map for $\Omega^p(X)$ which would correspond to the canonical homomorphism $\Omega^p(X) \to H^{2p}(X; \mathbb{Z})$ would factor through $\vartheta_{\text{hom}}$. But the homomorphism $\vartheta_{\text{hom}}$ has a huge kernel to which such an Abel-Jacobi map would be insensitive. The map $\Phi_{MU}$ is a finer Abel-Jacobi invariant for algebraic cobordism and does detect the kernel of $\vartheta$. 

\[\varphi_{MU_{\log}}: \Omega^*(X) \to MU_{\log}^{2p}(\ast)(X).\]
In [12], we considered simple examples of algebraic cobordism classes in \( \Omega^\ast_{\text{top}}(X) \) and which lie in the kernel of \( \theta \). These classes can be obtained for example from the cycles which Griffiths constructed in [9] to show that \( \text{Griff}^2(X) \) can be infinite. We hope the the paper will help finding other kinds of examples.

Before we briefly describe the organization of the paper, we would like to remark that there is another interesting subgroup of \( \Omega^p(X) \) which is given by algebraic cobordism cycles which are algebraically equivalent to zero in the sense of the work of Krishna and Park in [16]. This group should sit in between \( \Omega^p_{\text{top}}(X) \) and \( \Omega^p_{\text{hom}}(X) \). As for ordinary cycles, one can consider the equivalence relation generated by the (rational) double point relation which can be used to define \( \Omega^*_{\ast}(X) \) by [18]. This is finer than the equivalence relation generated by the algebraic relation studied in [16]. The topological triviality condition we consider in this paper should be strictly coarser than the one in [16]. It would be very interesting to understand the difference between these relations in more detail.

On the other hand, we could also consider the logarithmic Hodge filtered cohomology theory associated to a complex oriented cohomology theory \( E \) with maps \( \text{MU} \to E \to \text{HZ} \). This would induce an intermediate row in diagram (5). The natural map \( \Omega^p(X) \to E_{\log}^2(p)(X) \) would factor through a quotient of \( \Omega^p(X) \) which is determined by the formal group law of \( E \). One would get a subgroup \( \Omega^p_{E, \text{top}}(X) \) of elements in \( \Omega^p(X) \) which vanish under the natural map

\[
\Omega^p(X) \to E^{2p}(X) = E^{2p}(X(\mathbb{C})).
\]

This subgroup would be an intermediate step

\[
\Omega^p_{\text{top}}(X) \subseteq \Omega^p_{E, \text{top}}(X) \subseteq \Omega^p_{\text{hom}}(X).
\]

In order to obtain the alternative description of \( \Phi_{\text{MU}} \) we also provide a concrete representation of elements in logarithmic Hodge filtered cohomology theories which may be of independent interest. An element in \( \text{MU}^n_{\log}(p)(X) \) can be represented by pairs of elements consisting of holomorphic forms and cobordism elements which are connected by a homotopy. This resembles the way we can view elements in Deligne cohomology for complex manifolds (see e.g. [8] or [26]) and elements in differential cohomology theories for smooth manifolds as in [13]. In order to obtain this representation for elements in \( \text{MU}_{\log} \), we first discuss some facts about homotopy pullbacks for simplicial presheaves. Then we define logarithmic Hodge filtered spaces and study their global sections for smooth projective varieties. The above mentioned representation is an immediate consequence of the construction. We consider this as a manifestation that the constructions are natural. In the fourth section we use this representation to deduce the description of the Abel-Jacobi invariant for topologically trivial cobordism cycles.

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2. Homotopy pullbacks of simplicial presheaves

We briefly recall some basic facts about simplicial presheaves. In particular, we will discuss the construction of homotopy pullbacks of simplicial presheaves which will be needed in the next section.
2.1. Simplicial presheaves. Let $T$ be an essentially small site with enough points.

The following two examples of such sites will occur in this paper:

- The category $\text{Man}_C$ of complex manifolds and holomorphic maps which we consider as a site with the Grothendieck topology defined by open coverings.
- The category $\text{Sm}_{C,Nis} = \text{Sm}$ of smooth complex algebraic varieties (separated schemes of finite type over $\mathbb{C}$) with the Nisnevich topology. We recall that a distinguished square in $\text{Sm}_{C,Nis}$ is a cartesian square of the form

$$
\begin{array}{ccc}
U \times_X V & \to & V \\
\downarrow & & \downarrow p \\
U & \to & X
\end{array}
$$

such that $p$ is an étale morphism, $j$ is an open embedding and the induced morphism $p^{-1}(X-U) \to X-U$ is an isomorphism, where the closed subsets are equipped with the reduced induced structure. The Nisnevich topology is the Grothendieck topology generated by coverings of the form (6) (see [19, §3.1]).

We denote by $\text{sPre} = \text{sPre}(T)$ the category of simplicial presheaves on $T$, i.e. contravariant functors from $T$ to the category $\text{sS}$ of simplicial sets. Objects in $\text{sPre}$ will also be called spaces. There are several important model structures on the category $\text{sPre}$ (see [14], [2], [5]).

We start with the projective model structure on $\text{sPre}$. A map $F \to G$ in $\text{sPre}$ is called

- an objectwise weak equivalence if $F(X) \to G(X)$ is a weak equivalence in $\text{sS}$ (equipped with the standard model structure) for every $X \in T$;
- projective fibration if $F(X) \to G(X)$ is a fibration in $\text{sS}$ for every $X \in T$;
- projective cofibration if it has the left lifting property with respect to all acyclic fibrations.

In order to obtain a local model structure, i.e., one which respects the topology on the site $T$, we can localize the projective model structure at the hypercovers in $T$. More precisely, since the projective model structure on $\text{sPre}$ is cellular, proper and simplicial, it admits a left Bousfield localization with respect to all maps

$$\{\text{hocolim} \mathcal{U}_n \to X\}$$

where $X$ runs through all objects in $T$ and $\mathcal{U}$ runs through the hypercovers of $X$. The resulting model structure is the local projective model structure on $\text{sPre}$ (see [2] and [3]). The weak equivalences, fibrations and cofibrations in the local projective model structure are called local weak equivalences, local projective fibrations and local projective cofibrations, respectively. We denote the corresponding homotopy category by $\text{hosPre}$. Note that the local weak equivalences are precisely those maps $F \to G$ in $\text{sPre}$ such that the induced map of stalks $F_x \to G_x$ is a weak equivalence in $\text{sS}$ for every point $x$ in $T$.

Dugger, Hollander and Isaksen showed that the fibrations in the local projective model structure on $\text{sPre}$ have a nice characterization (see [6] §§3+7). Let $\mathcal{U} \to X$ be a hypercover in $\text{sPre}$ and let $F$ be a projective fibrant simplicial presheaf. Since each $\mathcal{U}_n$ is a coproduct of representables, we can form a product of simplicial sets $\prod_a F((U_n)^a)$ where $a$ ranges over the representable summands of $\mathcal{U}_n$. The simplicial
structure of \( U \) defines a cosimplicial diagram in \( sS \)
\[
\prod_a \mathcal{F}(U^a_0) \Rightarrow \prod_a \mathcal{F}(U^a_1) \Rightarrow \cdots
\]
The homotopy limit of this diagram is denoted by \( \operatorname{holim}_\Delta \mathcal{F}(U) \).
Following [6, Definition 4.3] we say that a simplicial presheaf \( F \) satisfies descent for a hypercover \( U \to X \) if there is a projective fibrant replacement \( F \to F' \) such that the natural map
\[
(7) \quad F'(X) \to \operatorname{holim}_\Delta F'(U)
\]
is a weak equivalence. It is easy to see that if \( F \) satisfies descent for a hypercover \( U \to X \), then the map \( (7) \) is a weak equivalence for every projective fibrant replacement \( F \to F' \). By [6, Corollary 7.1], the local projective fibrant objects in \( s\text{Pre} \) are exactly those simplicial presheaves which are projective fibrant and satisfy descent with respect to all hypercovers \( U \to X \). For our final applications we will need the following facts whose proofs can be found in [22].

**Lemma 2.1.** Let \( F \) be a simplicial presheaf that satisfies descent with respect to all hypercovers. Then every fibrant replacement \( F \to F_f \) in the local projective model structure is an objectwise weak equivalence, i.e., for every object \( X \in T \) the map
\[
F(X) \to F_f(X)
\]
is a weak equivalence of simplicial sets.

**Proposition 2.2.** Let \( F \) be a simplicial presheaf that satisfies descent with respect to all hypercovers and let \( X \) be an object of \( T \). Then, for every projective fibrant replacement \( g : F \to F' \), the natural map
\[
\operatorname{Hom}_{s\text{Pre}}(X, F) \to \pi_0(F'(X))
\]
is a bijection.

2.2. Homotopy pullbacks of simplicial presheaves. We briefly recall the construction of homotopy pullbacks in \( s\text{Pre} \) (see [11, §13.3] for more details) and will then show that its local and global versions are homotopy equivalent.

Let \( s\text{Pre} \) be equipped with any of the above model structures. We fix a functorial factorization \( E \) of every map \( f : X \to Y \) into
\[
X \xrightarrow{i_f} E(f) \xrightarrow{p_f} Y
\]
where \( i_f \) is an acyclic cofibration and \( p_f \) is a fibration. The *homotopy pullback* of the diagram \( X \xrightarrow{f} Z \xrightarrow{g} Y \) is the pullback of \( E(f) \xrightarrow{p_f} Z \xrightarrow{p_g} E(g) \). The homotopy pullback satisfies the following invariance. If we have the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow & & \downarrow \uparrow g \\
X' & \xrightarrow{f'} & Z'
\end{array}
\]
in which the three vertical maps are weak equivalences, then the induced map of homotopy pullbacks
\[
E(f) \times_Z E(g) \to E(f') \times_Z E'(g')
\]
is a weak equivalence as well.
We will need the following fact about pulling back local weak equivalences along maps which are merely projective fibrations.

**Lemma 2.3.** Let \( f : \mathcal{X} \to \mathcal{Z} \) be a projective fibration and \( g : \mathcal{Y} \to \mathcal{Z} \) be a local weak equivalence in \( \mathbf{sPre} \). Then the induced map \( f' : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X} \) is a local weak equivalence as well.

**Proof.** For a point \( x \) of \( \mathbf{sPre} \), and a map \( h : \mathcal{Y} \to \mathcal{W} \) of simplicial presheaves, let \( h_x : \mathcal{Y}_x \to \mathcal{W}_x \) denote the induced map of stalks at \( x \). With the notation of the lemma, we need to show that \( f'_x \) is a weak equivalence in \( \mathbf{sS} \) for every point \( x \) in \( \mathbf{T} \). Since \( x \) preserves finite limits, we have \( (\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_x = \mathcal{X}_x \times_{\mathcal{Z}_x} \mathcal{Y}_x \) and \( f'_x \) equals the induced map in the corresponding pullback diagram in \( \mathbf{sS} \). Since the standard model structure on \( \mathbf{sS} \) is right proper, it thus suffices to show that \( g_x \) is a Kan fibration. But every map in \( \mathbf{sPre} \) which is merely projective fibration.

Given a point \( x \) of \( \mathbf{T} \), we need to check that the map of sets induced by \( g \)

\[
\text{Hom}_{\mathbf{sS}}(\Delta[n], \mathcal{Y}_x) \to \text{Hom}_{\mathbf{sS}}(\Lambda_k[n], \mathcal{Y}_x) \times_{\text{Hom}_{\mathbf{sS}}(\Lambda_k[n], \mathcal{Z}_x)} \text{Hom}_{\mathbf{sS}}(\Delta[n], \mathcal{Z}_x)
\]

is surjective for all \( 1 \leq n \) and \( 0 \leq k \leq n \). Now for a simplicial presheaf \( \mathcal{V} \) and a finite simplicial set \( K \), we can consider the functor \( \mathcal{V} \mapsto \text{Hom}_{\mathbf{sS}}(K, \mathcal{V}(X)) \) as a presheaf of sets on \( \mathbf{T} \). We denote this presheaf by \( \text{Hom}(K, \mathcal{V}) \). The stalk of this presheaf at \( x \) is exactly the set \( \text{Hom}_{\mathbf{sS}}(K, \mathcal{V}_x) \). Hence the map \( \mathcal{Y}_x \) is surjective if and only if the map of presheaves of sets

\[
\text{Hom}(\Delta[n], \mathcal{Y}) \to \text{Hom}(\Lambda_k[n], \mathcal{Y}) \times_{\text{Hom}(\Lambda_k[n], \mathcal{Z})} \text{Hom}(\Delta[n], \mathcal{Z})
\]

induces a surjective map of stalks at \( x \). But, since \( g \) is a projective fibration, \( g(X) \) is a Kan fibration for every object \( X \in \mathbf{T} \). Hence, by the definition of \( \text{Hom}(-, -) \), the induced map

\[
\text{Hom}(\Delta[n], \mathcal{Y})(X) \to \text{Hom}(\Lambda_k[n], \mathcal{Y})(X) \times_{\text{Hom}(\Lambda_k[n], \mathcal{Z})(X)} \text{Hom}(\Delta[n], \mathcal{Z})(X)
\]

is surjective. Since forming stalks preserves objectwise epimorphisms, this implies that \( g_x \) is a Kan fibration. \( \square \)

The following result is probably a well-known fact. We include its proof for completeness and lack of a reference.

**Lemma 2.4.** The homotopy pullback of a diagram in \( \mathbf{sPre} \) in the projective model structure is stalkwise weakly equivalent to the homotopy pullback of the diagram in the local projective model structure.

**Proof.** Let \( \mathcal{X} \xrightarrow{f} \mathcal{Z} \xleftarrow{g} \mathcal{Y} \) be a diagram in \( \mathbf{sPre} \). Consider the diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\text{proj}} & \mathcal{Z} \\
\downarrow{\text{proj}} & & \downarrow{\text{proj}} \\
\mathcal{E}_{\text{proj}}(f) & \xrightarrow{\text{proj}} & \mathcal{E}_{\text{proj}}(g) \\
\downarrow{\text{loc}} & & \downarrow{\text{loc}} \\
\mathcal{E}_{\text{loc}}(f) & \xrightarrow{\text{proj}} & \mathcal{E}_{\text{loc}}(g)
\end{array}
\]
where the superscripts \textit{proj} and \textit{loc} indicate that we work in the projective and local projective model structure, respectively. We denote the projective homotopy pullback of the initial diagram by \( \mathcal{P}^{\text{proj}} = \mathcal{E}^{\text{proj}(f)} \times_{\mathcal{Z}} \mathcal{E}^{\text{proj}(g)} \). The vertical maps \( i^{\text{proj}}_f \) and \( i^{\text{proj}}_g \) are projective acyclic cofibrations. Since left Bousfield localization does not change cofibrations and since objectwise weak equivalences are in particular local weak equivalences, \( i^{\text{proj}}_f \) and \( i^{\text{proj}}_g \) are local acyclic cofibrations as well. Hence the composition of the vertical maps in (11) are local acyclic cofibrations. Thus, \( \mathcal{E}^{\text{loc}(p^{\text{proj}}_f)} \times_{\mathcal{Z}} \mathcal{E}^{\text{loc}(p^{\text{proj}}_g)} \) computes the local homotopy pullback \( \mathcal{P}^{\text{loc}} \) of \( \mathcal{X} \xrightarrow{f} \mathcal{Z} \xleftarrow{g} \mathcal{Y} \). Hence we need to show that the induced map
\[
\mathcal{E}^{\text{proj}(f)} \times_{\mathcal{Z}} \mathcal{E}^{\text{proj}(g)} \to \mathcal{E}^{\text{loc}(p^{\text{proj}}_f)} \times_{\mathcal{Z}} \mathcal{E}^{\text{loc}(p^{\text{proj}}_g)}
\]
(12) is a local weak equivalence. This map equals the composition
\[
\mathcal{E}^{\text{proj}(f)} \times_{\mathcal{Z}} \mathcal{E}^{\text{proj}(g)} \to \mathcal{E}^{\text{loc}(p^{\text{proj}}_f)} \times_{\mathcal{Z}} \mathcal{E}^{\text{proj}(g)} \to \mathcal{E}^{\text{loc}(p^{\text{proj}}_f)} \times_{\mathcal{Z}} \mathcal{E}^{\text{loc}(p^{\text{proj}}_g)}.
\]
Hence in order to show that (12) is a local weak equivalence, it suffices to show that the those two maps in (11) are both local weak equivalences. For this, we have to show that the pullback of a local weak equivalence along a projective fibration is again a local weak equivalence which has been checked in Lemma 2.3. \( \square \)

\textbf{Remark 2.5.} The result of Lemma 2.4 does not depend on the fact that we use the projective model structure. The same proof (after replacing the superscript \textit{proj} with \textit{inj}) would also work if we used the injective model structure on \textit{sPre}. More precisely, the homotopy pullback of a diagram in \textit{sPre} in the injective model structure is stalkwise weakly equivalent to the homotopy pullback of the diagram in the local injective model structure.

This implies that we can calculate the set of homotopy classes of maps into a homotopy pullback via global sections in the following way.

\textbf{Proposition 2.6.} Let \( \mathcal{X} \xrightarrow{f} \mathcal{Z} \xleftarrow{g} \mathcal{Y} \) be a diagram in \textit{sPre}, and let \( \mathcal{P} \) denote the homotopy pullback of this diagram in the local projective model structure. We assume that all three simplicial presheaves \( \mathcal{X} \), \( \mathcal{Y} \), and \( \mathcal{Z} \) satisfy descent for all hypercovers. For an object \( X \in \mathbf{T} \), let \( Q(X) \) denote the homotopy pullback in \textit{sS} of the diagram of simplicial sets \( \mathcal{X}(X) \xrightarrow{f(X)} \mathcal{Z}(X) \xleftarrow{g(X)} \mathcal{Y}(X) \). Then there is a natural bijection for every \( X \in \mathbf{T} \)
\[
\text{Hom}_{\text{ho}\textit{sPre}}(\mathcal{X}, \mathcal{P}) \cong \pi_0(Q(X)).
\]

\textit{Proof.} Let \( \mathcal{X} \mapsto \mathcal{X}' \) be a functorial projective fibrant replacement in \textit{sPre}. The invariance property of homotopy pullbacks implies that the homotopy pullback of \( \mathcal{X} \xrightarrow{f} \mathcal{Z} \xleftarrow{g} \mathcal{Y} \) is stalkwise equivalent to the homotopy pullback of the induced diagram \( \mathcal{X}' \xrightarrow{f'} \mathcal{Z}' \xleftarrow{g'} \mathcal{Y}' \). But it also implies that, for every \( X \in \mathbf{T} \), \( Q(X) \) is equivalent to the homotopy pullback \( Q'(X) \) of \( \mathcal{X}'(X) \xrightarrow{f'(X)} \mathcal{Z}'(X) \xleftarrow{g'(X)} \mathcal{Y}'(X) \) in \textit{sS}. Hence we can assume from now on that \( \mathcal{X} \), \( \mathcal{Y} \) and \( \mathcal{Z} \) are also projective fibrant. Now consider the diagram
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{f} & \mathcal{Z} & \xleftarrow{g} & \mathcal{Y} \\
\mathcal{E}^{\text{proj}(f)} & \xrightarrow{p^{\text{proj}}_f} & \mathcal{Z} & \xleftarrow{p^{\text{proj}}_g} & \mathcal{E}^{\text{proj}(g)}
\end{array}
\]
where \( E^{proj} \) is a functorial replacement in the projective model structure as before. Let \( P^{proj} = E^{proj}(f) \times Z E^{proj}(g) \) denote the homotopy pullback of \( f : X \to Z \to Y \) calculated in the projective model structure. By definition of pullbacks in \( sPre \), we have \( P^{proj}(X) = E^{proj}(f)(X) \times_{Z(X)} E^{proj}(g)(X) \) for every \( X \in T \). The invariance property of homotopy pullbacks implies that \( P^{proj}(X) \) is equivalent to the homotopy pullback \( Q(X) \) of the diagram \( X(f(X), Z(X) \to g(X)) \) \( Y(X) \) in \( sS \). (In fact, we could compute \( Q(X) \) as \( P^{proj}(X) \).) Moreover, since \( X, Y, Z \) satisfy descent for all hypercovers and since homotopy pullbacks commute with homotopy limits in \( sS \), we see that \( P^{proj} \) satisfies descent for all hypercovers as well. By Lemma 2.1 this implies that \( P^{proj} \) is local projective fibrant. Finally, by Lemma 2.4 \( P^{proj} \) is equivalent to the homotopy pullback in the local projective model structure. Hence, by Proposition 2.2 for every \( X \in T \), there are natural bijections

\[
\text{Hom}_{sPre}(X, P) \cong \text{Hom}_{sPre}(X, P^{proj}) \cong \pi_0(P^{proj}(X)) \cong \pi_0(Q(X)).
\]

\[\square\]

3. Logarithmic Hodge filtered function spaces

In this section we construct spaces which represent logarithmic Hodge filtered cohomology groups. In particular, we will show that we can represent elements in logarithmic Hodge filtered complex bordism groups as triples consisting of a class in complex bordism, a holomorphic form with suitable coefficients and a homotopy that connects both in an appropriate sense. This description will be applied in the next section.

3.1. Hodge filtration on forms and Eilenberg-MacLance spaces. Let \( C^* \) be a cochain complex of presheaves of abelian groups on \( T \). For any given \( n \), we denote by \( C^*[n] \) the cochain complex given in degree \( q \) by \( C^*[n] := C^q + n \). The differential on \( C^*[n] \) is the one of \( C^* \) multiplied by \((-1)^n\). The hypercohomology \( H^*(U, C^*) \) of an object \( U \) of \( T \) with coefficients in \( C^* \) is the graded group of morphisms \( \text{Hom}(\mathbb{Z}[U], aC^*) \) in the derived category of cochain complexes of sheaves on \( T \), where \( aC^* \) denotes the complex of associated sheaves of \( C^* \). We will denote by \( K(C^*, n) \) the Eilenberg-MacLane simplicial presheaf corresponding to \( C^*[-n] \). The following result is a version of Verdier’s hypercovering theorem due to Ken Brown in [3, Theorem 2].

Proposition 3.1. (3 Theorem 2), see also [19, 15] Let \( C^* \) be a cochain complex of presheaves of abelian groups on \( T \). Then for any integer \( n \) and any object \( U \) of \( T \) one has a canonical isomorphism

\[
H^n(U; C^*) \cong \text{Hom}_{sPre}(T)(U, K(C^*, n)).
\]

Now let \( T \) be the site \( Sm \) of smooth complex varieties with the Nisnevich topology. In view of Proposition 3.1 we would like to find a simplicial presheaf which represents Hodge filtered complex cohomology. In order to obtain a good filtration on holomorphic forms requires a compact variety. By the work of Hironaka, we know that every smooth complex variety does have a nice compactification. Following Deligne [4] and Beilinson [1], we will use this fact and construct simplicial presheaves on \( Sm \) whose global sections are isomorphic to the Hodge filtered cohomology groups of \( X \).
Let $\mathbf{Sm}$ be the category whose objects are smooth compactifications, i.e., pairs $(X, \overline{X}) = (X \subset \overline{X})$ consisting of a smooth variety $X$ embedded as an open subset of a projective variety $\overline{X}$ and having the property that $\overline{X} - X$ is a normal crossing divisor which is the union of smooth divisors. A map from $(X, \overline{X})$ to $(Y, \overline{Y})$ is a commutative diagram

$$
\begin{array}{ccc}
X & \rightarrow & \overline{X} \\
\downarrow & & \downarrow \\
Y & \rightarrow & \overline{Y}.
\end{array}
$$

By Hironaka’s theorem [10], every smooth variety over $\mathbb{C}$ admits a smooth compactification. Moreover, for a given smooth variety $X$, the category $C(X)$ of all smooth compactifications of $X$ is filtered (see [4]).

The forgetful functor $u : \mathbf{Sm} \to \mathbf{Sm}$

$$(X, \overline{X}) \mapsto X$$

induces a pair of adjoint functors on the categories of simplicial pre sheaves

$$\iota^* : \mathbf{sPre}(\mathbf{Sm}) \leftrightarrow \mathbf{sPre}(\mathbf{Sm}) : \iota_*.$$

The left adjoint $\iota^*$ is given by sending a simplicial presheaf $F$ on $\mathbf{Sm}$ to the simplicial presheaf

$$X \mapsto \iota^* F(X) = \varinjlim_{C(X)} F(\overline{X}).$$

For a smooth complex variety $X$, let $\overline{X}$ be a smooth compactification of $X$ and let $D := \overline{X} - X$ denote the complement of $X$. Let $\Omega^1_X(D)$ be the locally free submodule of $j_*\Omega^1_X$ generated by $\Omega^1_X$ and by $\frac{dz_i}{z_i}$ where $z_i$ is a local equation for an irreducible local component of $D$. The sheaf $\Omega^p_X(D)$ of meromorphic $p$-forms on $\overline{X}$ with at most logarithmic poles along $D$ is defined to be the locally free sub-sheaf $\bigwedge^p \Omega^1_X(D)$ of $j_*\Omega^p_X$. The Hodge filtration on the complex cohomology of $X$ can be defined as the image

$$F^p H^n(X; \mathbb{C}) := \text{Im} \left( H^n(\overline{X}; \Omega^{\geq p}_X(D)) \to H^n(X; \mathbb{C}) \right).$$

This definition is independent of the compactification $\overline{X}$ (see [4]).

We denote by $\Omega^*_{\overline{X}}$ the presheaf of differential graded $\mathbb{C}$-algebras on $\mathbf{Sm}$ that sends a pair $X \subset \overline{X}$ with $D := \overline{X} - X$ to $\Omega^*_{\overline{X}}(D)(X)$. For any given integer $p \geq 0$, we denote by $\Omega^\geq p_{\overline{X}}$ the presheaf on $\mathbf{Sm}$ that sends a pair $X \subset \overline{X}$ to $\Omega^\geq p_X(D)(\overline{X})$.

Let

$$\Omega^\geq p_X(D) \to A^\geq p_X(D)$$

be any resolution by cohomologically trivial sheaves which is functorial in $X \subset \overline{X}$ and which induces a commutative diagram

$$
\begin{array}{ccc}
\Omega^\geq p_X(D)(\overline{X}) & \longrightarrow & \Omega^\geq p_X(X) \\
\downarrow & & \downarrow \\
A^\geq p_X(D)(\overline{X}) & \longrightarrow & A^\geq p_X(X)
\end{array}
$$
where $A^*\mathbb{Z}^p$ denotes a functorial resolution by cohomologically trivial sheaves of $\Omega_{X}^*\mathbb{Z}^p$. For example, $A^2\mathbb{Z}^p(D)$ and $A_{X}^*\mathbb{Z}^p$ could be the Godement resolutions ([11 3.2.3]) or the logarithmic Dolbeault resolution ([20 8]). Even though $A^*_{X}$ and $A_{X}^*\mathbb{Z}^p(D)$ are double complexes, we will only consider their total complexes.

We denote the presheaf of complexes on $\overline{\text{Sm}}$ that sends a pair $(X, \overline{X})$ to $A^*\mathbb{Z}^p(D)(\overline{X})$ by $F^pA^*$, and let

$$\Pi^{\geq p} \to F^pA^*$$

be the associated map of complexes of presheaves on $\overline{\text{Sm}}$.

Now let $V_\ast$ be an evenly graded $\mathbb{C}$-algebra such that each $V_{2j}$ is a finite dimensional complex vector space. We will write

$$F^pH^n(X; V_\ast) := \bigoplus_j F^{p+j}H^{n+2j}(X; V_{2j})$$

for the graded Hodge filtered cohomology groups. The functor $X \mapsto F^pH^n(X; V_\ast)$ is representable in $\text{hosPre}(\text{Sm})$ in the following way.

For $(X, \overline{X}) \in \overline{\text{Sm}}$ and $j \in \mathbb{Z}$, let $F^{p+j}\tilde{A}^*(\overline{X}; V_{2j})[-2j]$ denote the corresponding complex with coefficients in $V_{2j}$ shifted by degree $2j$. We write

$$F^p\tilde{A}^*(\overline{X}; V_\ast) = \bigoplus_j F^{p+j}\tilde{A}^*(\overline{X}; V_{2j})[-2j].$$

Let $F^p\tilde{A}^*(V_\ast)$ denote the corresponding presheaf on $\overline{\text{Sm}}$.

Let $K(F^p\tilde{A}^*(V_\ast), n)$ be the associated Eilenberg-MacLane simplicial presheaf. Note that (16) induces an isomorphism

$$K(F^p\tilde{A}^*(V_\ast), n) \cong \bigvee_j K(F^{p+j}\tilde{A}^*(V_{2j}), n + 2j).$$

Now [22 Theorem 3.5] implies that, for every smooth complex variety $X$, there is a natural isomorphism

$$\text{Hom}_{\text{hosPre}(\text{Sm})}(X, u^*K(F^p\tilde{A}^*(V_\ast), n)) \cong F^pH^n(X; V_\ast).$$

A crucial fact for the proof of [22 Theorem 3.5] is that the simplicial presheaf $u^*K(F^p\tilde{A}^*(V_\ast), n)$ satisfies Nisnevich descent. This implies that any projective fibrant replacement of $u^*K(F^p\tilde{A}^*(V_\ast), n)$ is already local projective fibrant. As a consequence we obtain that, for every smooth complex variety $X$, there is a natural isomorphism

$$\pi_0(u^*K(F^p\tilde{A}^*(V_\ast), n))(X) \cong F^pH^n(X; V_\ast).$$

Finally, we point out that, for every $n$ and $p$, the map of presheaves of complexes

$$F^p\tilde{A}^*(V_\ast)[-n] \to A^*(V_\ast)[-n].$$

induces a morphism of simplicial presheaves

$$u^*K(F^p\tilde{A}^*(V_\ast), n) \to K(A^*(V_\ast), n).$$
3.2. **The singular functor for complex manifolds.** As a short digression, we need to consider simplicial presheaves on complex manifolds as well. In this subsection, we let $\mathbf{T}$ be the category $\mathsf{Man}_C$ of complex manifolds and holomorphic maps. We consider it as a site with the Grothendieck topology defined by open coverings. We will use the singular functor as a fibrant replacement in the local projective model structure on $\mathbf{sPre} = \mathbf{sPre}(\mathsf{Man}_C)$.

Let $\Delta^n$ be the standard topological $n$-simplex

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} | 0 \leq t_j \leq 1, \sum t_j = 1\}.$$

For topological spaces $Y$ and $Z$, the singular function complex $\text{Sing}^*_Z(Y)$ is the simplicial set whose $n$-simplices are continuous maps $f: Z \times \Delta^n \to Y$.

We denote the simplicial presheaf $\mathbf{M} \mapsto \text{Sing}^*_Z(\mathbf{M}, Y) =: \text{Sing}^*_Z(\mathbf{M})$ on $\mathsf{Man}_C$ by $\text{Sing}^*_Z$. Since, for any CW-complex $Y$, $\text{Sing}^*_Z(Y)$ satisfies descent, the criterion of [6] implies that $\text{Sing}^*_Z$ is a fibrant object in the local projective model structure on $\mathbf{sPre}$ (see [12, Lemma 2.3]).

Let $V^*$ be an evenly graded complex vector space, and let $K(V^*, n)$ be an associated Eilenberg-MacLane space in the category of CW-complexes. Then the simplicial presheaf $\text{Sing}^*_Z(K(V^*, n))$ represents the functor of cocycles in $\mathbf{sPre}$, i.e., for every $\mathbf{M} \in \mathsf{Man}_C$ there is a natural isomorphism of abelian groups

$$\pi_0(\text{Sing}^*_Z(K(V^*, n))\mathbf{M}) \cong \text{Hom}_{\mathbf{sPre}}(\mathbf{M}, \text{Sing}^*_Z(K(V^*, n))).$$

Moreover, since $\mathbf{M}$ is a representable presheaf, we have a natural bijection of sets

$$\text{Hom}_{\mathbf{sPre}}(\mathbf{M}, \text{Sing}^*_Z(K(V^*, n))) = \text{Sing}^*_0(K(V^*, n))\mathbf{M}.$$

Moreover, $\mathbf{M}$ is a cofibrant object in the local projective model structure on $\mathbf{sPre}$. Hence there is a natural bijection

$$\pi_0(\text{Sing}^*_Z(K(V^*, n))\mathbf{M}) \cong \text{Hom}_{\mathbf{ho}sPre}(\mathbf{M}, \text{Sing}^*_Z(K(V^*, n))\mathbf{M}).$$

3.3. **Logarithmic Hodge filtered function spaces.** In this subsection we will work with both sites, $\mathsf{Sm}_C$ and $\mathsf{Man}_C$. For $X \in \mathsf{Sm}_C$, we denote by $X_{\text{an}}$ the associated complex manifold whose underlying set is $X(\mathbb{C})$. This defines a functor $\rho^{-1}: \mathsf{Sm}_C \to \mathsf{Man}_C$, $X \mapsto \rho^{-1}(X) := X_{\text{an}}$.

Composition with $\rho^{-1}$ induces a functor $\rho_*: \mathbf{sPre}(\mathsf{Man}_C) \to \mathbf{sPre}(\mathsf{Sm}_C)$.

Note that $\rho_*$ is the right adjoint in a Quillen pair of functors between the corresponding local projective model structures.

We can now construct logarithmic Hodge filtered spaces whose global sections yield generalized Hodge filtered cohomology groups. The idea to define Hodge filtered spaces is similar to the way that the differential function spaces were defined for presheaves on the category of smooth manifolds in [13].

Let $n, p$ be integers and $\mathcal{V}_*$ an evenly-graded complex vector space. Let $Y$ be a CW-complex and let $t \in Z^n(Y; \mathcal{V}_*)$ by a cocycle on $Y$. A cocycle corresponds to a map of CW-complexes

$$Y \to K(\mathcal{V}_*, n).$$
It induces a map of simplicial presheaves on $\text{Man}_C$

$$\rho_* \text{Sing}_* Y \to \text{Sing}_* K(\mathcal{V}_*, n).$$

Let $| \cdot |$ denote geometric realization of simplicial set. Using the canonical map $K(\mathcal{V}_*, n) \to |K(A^*(\mathcal{V}_*), n)|$ we can form the following diagram in $\text{sPre}(\text{Sm}_C)$

$$\begin{array}{ccc}
\rho_* \text{Sing}_* Y & \xrightarrow{\iota^*} & \rho_* |K(A^*(\mathcal{V}_*), n)| \\
\downarrow \rho_* & & \downarrow \rho_* \\
u^* K(F^p \check{A}^*(\mathcal{V}_*), n) & \xrightarrow{\iota^*} & \rho_* |K(A^*(\mathcal{V}_*), n)|.
\end{array}$$

**Definition 3.2.** We define the *logarithmic Hodge filtered function space* $(Y(p), \iota, n)$ to be the homotopy pullback of (23) in $\text{sPre}(\text{Sm}_C)$.

Note that $(Y(p), \iota, n)$ depends on $\iota$ only up to homotopy, i.e., if $\iota'$ is another cocycle which represents the same cohomology class as $\iota$ then $(Y(p), \iota, n)$ and $(Y(p), \iota', n)$ are equivalent.

**Remark 3.3.** Let us contemplate a little more on diagram (23). For a complex manifold $M$, let $Z^n(M \times \Delta^k; \mathcal{V}_*)$ be the simplicial abelian group whose group of $k$-simplices is given by the $n$-cocycles on $M \times \Delta^k$ with coefficients in $\mathcal{V}_*$. We denote the corresponding simplicial presheaf $M \mapsto Z^n(M \times \Delta^k; \mathcal{V}_*)$
on $\text{Man}_C$ by $Z^n(- \times \Delta^k; \mathcal{V}_*)$. Our chosen cocycle $\iota$ determines a map of simplicial presheaves

$$\text{Sing}_* Y \to Z^n(- \times \Delta^k; \mathcal{V}_*), \quad f \mapsto \iota^* f,$$

given by taking the pullback along $\iota$. Let $I$ denote the map given by integration of forms

$$I: F^p A^{n+2j}(X; \mathcal{V}_{2j}) \to C^{n+2j}(X; \mathcal{V}_{2j}), \quad \eta \mapsto (\sigma \mapsto \int_{\Delta_{n+2j}} \sigma^* \eta).$$

We can form the diagram of simplicial presheaves

$$\begin{array}{ccc}
\rho_* \text{Sing}_* Y & \xrightarrow{\iota^*} & \rho_* Z^n(- \times \Delta^k; \mathcal{V}_*) \\
\downarrow \rho_* & & \downarrow \rho_* \\
u^* K(F^p \check{A}^*(\mathcal{V}_*), n) & \xrightarrow{\iota^*} & \rho_* |K(A^*(\mathcal{V}_*), n)|.
\end{array}$$

The map

$$\text{Sing}_* K(\mathcal{V}_*, n)(M) \to Z^n(M \times \Delta^k; \mathcal{V}_*)$$

given by pulling back a fundamental cocycle in $Z^n(K(\mathcal{V}_*, n); \mathcal{V}_*)$ is a simplicial homotopy equivalence (see e.g. [13, Proposition A.12]). Hence the homotopy pullback of (24) is homotopy equivalent to the homotopy pullback of (23).

**Remark 3.4.** For $Y = K(\mathbb{Z}, n)$, we recover Deligne-Beilinson cohomology in the following way. Let $\iota: K(\mathbb{Z}, n) \to K(\mathbb{C}, n)$ be the map that is induced by the $(2\pi i)^p$-multiple of the inclusion $\mathbb{Z} \subset \mathbb{C}$. Then $K(\mathbb{Z}, n)(p) := (K(\mathbb{Z}, n)(p), \iota, n)$ represents Deligne-Beilinson cohomology in the homotopy category of $\text{sPre}(\text{Sm}_C)$ in the sense that there is a natural isomorphism

$$H^p_D(X; \mathbb{Z}(p)) \cong \text{Hom}_{\text{sPre}(\text{Sm}_C)}(X, K(\mathbb{Z}, n)(p)).$$
3.4. Hodge filtered spaces and spectra. Of particular interest is the case when $Y$ is a space in a spectrum. Let $E$ be a topological $\Omega$-spectrum and let $E_n$ be its $n$th space. We assume that $E$ is rationally even, i.e., $\pi_* E \otimes \mathbb{Q}$ is concentrated in even degrees. Let $\mathcal{V}_*$ be the evenly graded $\mathbb{C}$-vector space $\pi_* E \otimes \mathbb{C}$. Let $\tau : E \to E \wedge HC =: EC$

be a map of spectra which induces for every $n$ the map $\pi_{2n}(E) \xrightarrow{(2\pi i)^n} \pi_{2n}(EC)$ defined by multiplication by $(2\pi i)^n$ on homotopy groups. The choice of such a map is unique up to homotopy. For a given integer $p$, multiplication by $(2\pi i)^p$ on homotopy groups determines a map

$$E \xrightarrow{(2\pi i)^p \tau} E \wedge HC.$$ 

Let $E \wedge HC \to H(\pi_* E \otimes \mathbb{C})$ be a map that induces the isomorphism $\pi_*(E \wedge HC) \cong \pi_*(E \otimes \mathbb{C})$.

The composite with $(2\pi i)^p \tau$ defines a map of spectra $t^E : E \to H(\pi_* E \otimes \mathbb{C})$.

This map corresponds to a family of maps of spaces of the form $t_n : E_n \to K(V_n, n)$.

In other words, for each $n$, we have a cocycle $t_n \in Z^n(E_n; \mathcal{V}_n)$, and for various $n$ these cocycles are compatible with the structure maps of the spectrum $E$. We call $\iota$ a $p$-twisted fundamental cocycle of $E$.

For given $p$ and $\iota$ and each $n$, we can form the diagram in $\textbf{sPre}($Sm$_C)$

$$\begin{array}{ccc}
\rho_* \text{Sing}_* E_n & \xrightarrow{u^*} & \rho_* \text{Sing}_* |K(A^*(\mathcal{V}_*), n)| \\
\downarrow & & \downarrow \\
\rho_* \text{Sing}_* E_n(\mathbb{A}^n, n) & \xrightarrow{t_n} & \rho_* \text{Sing}_* |K(A^*(\mathcal{V}_*), n)|.
\end{array}$$

We will write $(E_n(p), \iota)$ for the homotopy pullback of (27) in $\textbf{sPre}($Sm$_C)$. Note that a different choice $\iota'$ of a $p$-twisted fundamental cocycle of $E$ yields a homotopy equivalent simplicial presheaf $(E_n(p), \iota')$. Therefore, we will often drop $\iota$ from the notation and write $E_n(p)$ for $(E_n(p), \iota)$.

**Definition 3.5.** According to our previous terminology, we call $E_n(p)$ the $n$th logarithmic Hodge filtered function space of $E$ (even though it is only unique up to homotopy equivalence).

**Remark 3.6.** For $X \in \textbf{Sm}_C$, let $E_{n, \log}(p)(X)$ denote the logarithmic Hodge filtered $E$-cohomology groups of $X$ as defined in [12, Definition 6.4]. It follows from the definition of $E_n(p)$ as a homotopy pullback of (27) that the groups $\text{Hom}_{\text{hosPre}}(X, E_n(p))$, for varying $n$, sit in long exact sequences analogous to the one of [12, Proposition 6.5]. This shows that we have a natural isomorphism

$$E_{\log}(p)(X) \cong \text{Hom}_{\text{hosPre}}(X, E_n(p)).$$
Alternatively, we could have remarked that \( E_n(p) \) is the \( n \)th space of the fibrant spectrum \( E_{\log}(p) \) of [12, §6].

3.5. **The case of smooth projective varieties.** For smooth projective varieties, we obtain a more concrete description of the global sections of a Hodge filtered space.

If \( X \) is a smooth projective variety, then \( X \) is an initial object in the filtered category \( C(X) \) of all smooth compactifications of \( X \). Hence the colimit that computes the value of \( u^*K(F^p\tilde{A}^*(V_\ast), n) \) at \( X \) reduces to

\[
u^*K(F^p\tilde{A}^*(V_\ast), n)(X) = K(F^pA^*(V_\ast), n)(X).
\]

Thus, for \( X \) projective, we have

\[
\text{Hom}_{\text{Pre}(\text{Sm})}(X, u^*K(F^p\tilde{A}^*(V_\ast), n)) \cong K(F^pA^*(X; V_\ast), n).
\]

Finally, in terms of homotopy classes of maps, isomorphism [19] just states the fact

\[
\pi_0K(F^pA^*(X; V_\ast), n) \cong F^pH^n(X; V_\ast).
\]

Now let \( E \) be a rationally even topological \( \Omega \)-spectrum together with the choice of a \( p \)-twisted fundamental cocycle \( \iota \). By Proposition 2.6, we can calculate the homotopy pullback of (27) objectwise. This implies that the space \( E_n(p)(X) \) is homotopy equivalent to the homotopy pullback the following diagram of simplicial sets

\[
\begin{array}{c}
\text{Sing}_*E_n(X) \\
\downarrow^\iota^*_n \\
K(F^pA^*(X; V_{2*}), n) \longrightarrow \text{Sing}_*K(A^*(V_{2*}), n)(X).
\end{array}
\]

By Remark 3.6, this implies that the logarithmic Hodge filtered \( E \)-cohomology group \( E^n_{\log}(p)(X) \) is isomorphic to the group of connected components of the space \( E_n(p)(X) \), i.e.,

\[
E^n_{\log}(p)(X) = \pi_0(E_n(p)(X)).
\]

**Remark 3.7.** We would like to rewrite diagram (30) in view of Remark 3.3. Let \( I \) denote again the map given by integration of forms. Following the argument in Remark 3.3, we see that \( E_n(p)(X) \) fits into the following homotopy cartesian square of simplicial sets

\[
\begin{array}{c}
E_n(p)(X) \\
\downarrow \\
K(F^pA^*(X; V_{2*}), n) \longrightarrow \text{Sing}_*K(A^*(V_{2*}), n)(X).
\end{array}
\]

An important consequence of these observations is that we can represent an element in \( E^n_{\log}(p)(X) \) as a triple

\[
q: X \rightarrow E_n, \quad \omega \in F^pA^n(X; V_{2*})_{cl}, \quad h \in C^{n-1}(X; V_{2*})
\]

such that \( \delta h = \iota^*_n q - I(\omega) \), where \( \delta \) denotes the differential in \( C^*(X; V_{2*}) \) and \( \omega \) is a closed form.
Remark 3.8. Recall from Remark [23] that for \( E = HZ \), i.e., \( E_n = K(Z, n) \),
\( K(Z, n)(p) \) represents Deligne-Beilinson cohomology in \text{hosPre}(\text{Sm}_C). \) The above observation then just rephrases the well-known fact (see e.g. \cite{20 §12.3.2} or \cite{8}) that an element in \( H^p_D(X; \mathbb{Z}(p)) \) can be represented (in the notation of \cite{20}) by a triple \((a_n^p, b_n^p, c_n^{p-1})\) where \( a_n^p \) is an integral singular cochain of degree \( n \), \( b_n^p \) is a form in \( F^pA^n(X) \), and \( c_n^{p-1} \) is a complex singular cochain of degree \( n - 1 \) such that \( \delta c_n^{p-1} = a_n^p - b_n^p \).

4. A generalized Abel-Jacobi invariant

In this section, we will work on the site \( \text{Sm} = \text{Sm}_C \) with the Nisnevich topology. We will always assume that \( X \) is a smooth projective complex variety. Let \( n \) be the dimension of \( X \) and \( p \) a fixed integer \( \geq 0 \).

4.1. Cobordism, Jacobians, and Hodge structures.\) We first recall the definition of the generalized Jacobian we mentioned in the introduction (see \cite{12} §4.3). Let \( MU \) be the Thom spectrum representing complex cobordism. By the construction of the spaces \( MU_n(p) \) as a homotopy pullback, we deduce that the groups \( MU^n_{\log}(p)(X) \) sit in a long exact sequence of the form

\[
\ldots \to H^{n-1}(X; \pi_1 MU \otimes \mathbb{C}) \to MU^n_{\log}(p)(X) \to MU^n(X) \oplus F^pH^n(X; \pi_* MU \otimes \mathbb{C}) \to H^n(X, \pi_2 MU \otimes \mathbb{C}) \to \ldots
\]

For \( n = 2p \), this induces the bottom short exact sequence of diagram \( \text{[1]} \)

\[
0 \to J_{MU}^{2p-1}(X) \to MU^{2p}_{\log}(p)(X) \to \text{Hdg}_{SU}^{2p}(X) \to 0.
\]

The group \( \text{Hdg}_{SU}^{2p}(X) \) is the subgroup of elements whose image in \( H^{2p}(X; \mathbb{C}) \) are Hodge classes. The group on the left is defined as

\[
J_{MU}^{2p-1}(X) := MU^{2p-1}(X) \otimes \mathbb{C} / (F^pH^{2p-1}(X; \pi_* MU \otimes \mathbb{C}) + MU^{2p-1}(X)).
\]

The complex points of \( X \) have the homotopy type of a finite complex and each group \( MU^k(X) \) is finitely generated over \( \mathbb{Z} \). Hence we may consider \( MU^k(X) \) as a Hodge structure with the following filtration on \( MU^k(X) \otimes_\mathbb{Z} \mathbb{C} \). Using the isomorphism

\[
MU^k(X) \otimes_\mathbb{Z} \mathbb{C} \cong \bigoplus_{j\geq 0} H^{k+2j}(X; \mathbb{C}) \otimes_\mathbb{Z} \pi_{2j}MU
\]

we set

\[
F^iMU^k(X) \otimes_\mathbb{Z} \mathbb{C} := \bigoplus_{j\geq 0} F^{i+j}H^{k+2j}(X; \mathbb{C}) \otimes_\mathbb{Z} \pi_{2j}MU.
\]

We may also consider the decomposition of \( MU^k(X) \otimes_\mathbb{C} \) into the direct sum of the groups

\[
MU^{p,q}(X) := \bigoplus_{j\geq 0} H^{p+j,q+j}(X; \mathbb{C}) \otimes_\mathbb{Z} \pi_{2j}MU
\]

where \( p, q \) run over all integers such that \( p + q = k \).

The next lemma follows immediately from the definitions.

Lemma 4.1. The group \( \text{Hdg}_{SU}^{2p}(X) \) can be identified with the elements in \( MU^{2p}(X) \) whose image in \( MU^{2p}(X) \otimes \mathbb{C} \) lies in \( MU^{p-p}(X) \).
Then we may also interpret the generalized intermediate Jacobian \( J^p_{MU}(X) \) as the Jacobian associated to the Hodge structure of weight \( 2p - 1 \) on \( MU^{2p-1}(X) \):

\[
J^p_{MU}(X) = MU^{2p-1}(X) \otimes_{\mathbb{Z}} \mathbb{C}/(F^pMU^{2p-1}(X)_{\mathbb{C}} \oplus MU^{2p-1}(X)).
\]

Moreover, the canonical map \( MU \to H\mathbb{Z} \) induces a map of Hodge structures

\[
(MU^{2p-1}(X), F^pMU^{2p-1}(X)_{\mathbb{C}}) \to (H^{2p-1}(X; \mathbb{Z}), F^pH^{2p-1}(X; \mathbb{C}))
\]

which induces the natural map

\[
J^p_{MU}(X) \to J^{p-1}(X).
\]

4.2. The generalized Abel-Jacobi map given by integration over forms.

We are now going to discuss the map

\[
\Phi_{MU} : \Omega^p_{\text{top}}(X) \to J^p_{MU}(X).
\]

We start with a very brief reminder of the definition of algebraic cobordism \( \Omega^*(X) \) of Levine and Morel [17]. Let \( \mathcal{M}^p(X) \) be the set of isomorphism classes over \( X \) of projective morphisms

\[
f : Y \to X
\]

with \( Y \in \text{Sm} \). The set \( \mathcal{M}^p(X) \) is a commutative monoid under disjoint union of domains. Let \( \mathcal{M}^*(X) \) be the direct sum over all codimensions and let \( \mathcal{M}^*(X)^+ \) be the graded group completion of \( \mathcal{M}^*(X) \). The elements in \( \mathcal{M}^*(X)^+ \) will be called (algebraic) cobordism cycles.

Then, if \( T^* \) is an oriented cohomology theory on \( \text{Sm} \), there is a universal transformation in the sense that one obtains a homomorphism

\[
\mathcal{M}^p(X)^+ \to T^p(X), \quad f : Y \to X \mapsto f_*(1) = [f]_T \in T^p(X)
\]

which is natural in \( X \).

In order to turn also \( \mathcal{M}^*(X)^+ \) into an oriented cohomology theory, one has to impose suitable relations. By the work of Levine and Pandharipande [18], these relations have a very nice geometric interpretation. They consider the subgroup \( \mathcal{R}^*(X) \subset \mathcal{M}^*(X)^+ \) generated by all double point relations over \( X \) as described in [18] [0.3]. Then Levine and Pandharipande show in [18] Theorem 1] that

\[
\Omega^*(X) = \mathcal{M}^*(X)^+ / \mathcal{R}^*(X).
\]

Next, we recall from [26] Proposition 7.5] the isomorphism

\[
\pi^{n-k+1}(H^{2n-2k+1}(X; \mathbb{C}) = \frac{F^{n-k+1}A^{2n-2k+1}(X) \cap \text{Ker} \ d}{dF^{n-k+1}A^{2n-2k}(X)}
\]

where \( A^*(X) \) denotes the complex of smooth forms on \( X \) and \( k \geq 0 \) an integer.

Let \( \psi \in F^{n-k+1}H^{2n-2k+1}(X; \mathbb{C}) \) and \( \eta \in F^{n-k+1}A^{2n-2k+1}(X) \cap \text{Ker} \ d \) be a form representing \( \psi \). Assume we are given a compact manifold \( M \) of real dimension \( 2n - 2k + 1 \) together with a continuous map \( f : M \to X(\mathbb{C}) \). Then recall that the integral of \( \psi \) over \( M \) is defined by

\[
\int_M \psi := \int_M f^* \eta \in \mathbb{C}.
\]

The left hand side is well-defined. For, if \( \eta' \) is another form representing \( \psi \), then [68] shows that \( \eta - \eta' \) equals \( d\zeta \) for some form \( \zeta \in F^{n-k+1}A^{2n-2k}(X) \). Then

\[
\int_M \pi^*(d\zeta) = \int_M df^* \zeta = \int_{\partial M} f^* \zeta
\]
by Stokes’ formula. But this integral is zero because of the type of $\zeta$.

Let $M$ be a closed weakly complex manifold $M$ of real dimension $2n - 2p + 1$. Let $[M]_{\mu U} \in MU_{2n-2p+1}(X)$ be the fundamental bordism class of $M$. We denote its image in $MU_{2n-2p+1}(X) \otimes \mathbb{C}$ also by $[M]_{\mu U}$. Using the canonical isomorphism

$$MU_{2n-2p+1}(X) \otimes \mathbb{C} \xrightarrow{\sim} \bigoplus_{j \geq 0} H_{2n-2p+1-2j}(X; \mathbb{C}) \otimes \mathbb{Z} \pi_2jMU$$

we see that the class $[M]_{\mu U}$ with complex coefficients splits into a sum of homology classes with coefficients in $\pi_*MU$. More precisely, there are homology classes $\mu_j(M) \in H_{2n-2p+1-2j}(X; \mathbb{C})$ and bordism classes $b_j(M) \in \pi_2jMU$ such that via the isomorphism

$$[M]_{\mu U} = \sum_j \mu_j(M) \otimes b_j(M) \in \bigoplus_{j \geq 0} H_{2n-2p+1-2j}(X; \mathbb{C}) \otimes \mathbb{Z} \pi_2jMU.$$ (41)

In fact, by the work of Thom [23] we can even assume that each $\mu_j(M)$ is represented by the (complex) fundamental class of a manifold. More precisely, for each $j$, there is a (real) manifold $M_j$ of dimension $2n - 2p + 1 - 2j$ together with a continuous map $f_j: M_j \to X(\mathbb{C})$ such that

$$(f_j)_*([M_j]) = \mu_j(M) \in H_{2n-2p+1-2j}(X; \mathbb{C}).$$ (42)

Now let

$$\Psi = (\psi_j)_j \in \bigoplus_j F^{-p+1-j} H^{2n-2p+1-2j}(X; \mathbb{C})$$

be a tuple of cohomology classes and

$$\eta_j)_j \in \bigoplus_j F^{-p+1-j} A^{2n-2p+1-j}(X) \cap \text{Ker} d$$ (44)

be a tuple of forms such that each $\eta_j$ represents $\psi_j$. We define the $p$-twisted integral of $\Psi$ over the bordism class of $M$ to be the element

$$\int_M \Psi = \sum_j \left( \frac{1}{(2\pi i)^{p+j}} \int_{M_j} f_j^* \eta_j \right) \otimes b_j(M) \in \mathbb{C} \otimes \mathbb{Z} \pi_*MU.$$ (45)

One can show as above that this element in $\pi_*MU \otimes \mathbb{C}$ does not depend on the choice of the representatives $\eta_j$. Moreover, since the integral of a form over a homology class $\mu$ is independent of the continuous chain representing $\mu$, we see that $\int_Y \Psi$ does not depend on the choice of the $M_j$.

Hence we obtain a well-defined map

$$MU_{2n-2p+1}(X) \to \bigoplus_j F^{-p+1-j} H^{2n-2p+1-2j}(X; \mathbb{C})^* \otimes \pi_2jMU.$$ (46)

Let $(f: Y \to X) \in MP(X)^+$ be a generator of the algebraic cobordism of $X$ which lies in the kernel of $\varphi_{MU}$. This means that the fundamental bordism class $[Y(\mathbb{C})]_{\mu U}$ vanishes in $MU_{2n-2p+1}(X)$. Hence there is a compact weakly complex manifold $W$ of real dimension $2n - 2p + 1$ with boundary together with a continuous map $\pi: W \to X(\mathbb{C})$ such that $Y(\mathbb{C})$ is the boundary of $W$. Moreover, we also know that the image of $[Y(\mathbb{C})]_{\mu U}$ vanishes in

$$MU_{2n-2p}(X) \otimes \mathbb{C} \xrightarrow{\sim} \bigoplus_j H_{2n-2p-2j}(X; \mathbb{C}) \otimes \mathbb{Z} \pi_2jMU.$$ (47)
As we remarked before, under this isomorphism the class $[Y(\mathbb{C})]_{MU}$ of $Y$ in $MU_{2n-2p}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ splits into a sum

$$[Y(\mathbb{C})]_{MU} = \sum_j [\gamma_j] \otimes b_j \in \bigoplus_j H_{2n-2p-2j}(X; \mathbb{C}) \otimes \pi_{2j}MU$$

with $\gamma_j \in H_{2n-2p-2j}(X; \mathbb{C})$ and $b_j \in \pi_{2j}MU$. (Of course, both the $\gamma_j$’s and the $b_j$’s depend on $Y$ but we prefer to drop $Y$ in the notation.) Now, since $W$ yields a bordism relation for $Y(\mathbb{C})$, it also induces a relation in homology under the map $\Phi$. This means that for each $\gamma_j$ there is a differentiable chain represented by a manifold $\gamma_j: \Gamma_j \to X(\mathbb{C})$ of real dimension $2n - 2p - 2j + 1$ whose boundary is $\gamma_j$.

For a tuple of cohomology classes $\Psi = (\psi_j)_j$ as in (43) with representing forms $\eta_j$ as in (44), we can define the integral of $Y$ over $\Psi$ to be the element

$$\int_Y \Psi := \sum_j \left( \frac{1}{(2\pi i)^{p+1}} \int_{\Gamma_j} g_j^* \eta_j \right) \otimes b_j \in \mathbb{C} \otimes_{\mathbb{Z}} \pi_{2*}MU. \tag{48}$$

We need to understand how this integral depends on the choice of $W$. Let $W'$ be another weakly complex manifold together with a continuous maps $\pi': W' \to X(\mathbb{C})$ such that $\partial W' = Y(\mathbb{C})$. Then $W - W'$ has no boundary and $W - W' \to X(\mathbb{C})$ defines an element in $MU_{2n-2p+1}(X)$. This shows that the difference between the integrals computed via $W$ and $W'$, respectively, lies in the image of $\Phi$. Hence $\int_Y \Psi$ is well-defined modulo the image of $MU_{2n-2p+1}(X)$.

This shows that, using the notation for the dual

$$F^{n-p+1}MU_{2n-2p+1}(X) = \bigoplus_j F^{n-p+1-j}H^{2n-2p-2j}(X; \mathbb{C})^* \otimes \pi_{2j}MU,$$

we get a well-defined map

$$\Phi_{MU}: \mathcal{M}^{p}(X)^{+} \to F^{n-p+1}MU_{2n-2p+1}(X)_{\mathbb{C}}/MU_{2n-2p+1}(X)$$

$$\Phi_{MU}: Y \to \int_Y \Psi. \tag{49}$$

4.3. Comparing maps. Our goal is to show that the map $\Phi_{MU}$ we have just constructed descends to the map $\Phi_{MU}$ introduced in diagram (4). In a first step, we need to give a different description of the generalized Jacobian. Recall that $J_{MU}^{2p-1}(X)$ is the quotient

$$MU^{2p-1}(X)/\left(F^{p}MU^{2p-1}(X)_{\mathbb{C}} + MU^{2p-1}(X)\right).$$

**Lemma 4.2.** Let $X$ be a smooth projective complex variety of dimension $n$ and $p$ an integer. Then there is a natural isomorphism

$$J_{MU}^{2p-1}(X) \cong F^{n-p+1}MU_{2n-2p+1}(X)_{\mathbb{C}}/MU_{2n-2p+1}(X). \tag{50}$$

**Proof.** Via the natural isomorphism $MU^{2p-1}(X)_{\mathbb{C}} = \bigoplus_j H^{2p-1+2j}(X; \mathbb{C}) \otimes \pi_{2j}MU$, we obtain that the quotient

$$MU^{2p-1}(X)_{\mathbb{C}}/F^{p}MU^{2p-1}(X)_{\mathbb{C}} \tag{51}$$

is naturally isomorphic to

$$\bigoplus_j H^{2p-1+2j}(X; \mathbb{C}) \otimes \pi_{2j}MU)/\bigoplus_j F^{p+j}H^{2p-1+2j}(X; \mathbb{C}) \otimes \pi_{2j}MU. \tag{52}$$
Poincaré duality for complex cohomology shows that the latter quotient is naturally isomorphic to the direct sum
\[
\bigoplus_j F^{n-p+1-j} H^{2n-2p+1-2j}(X; \mathbb{C})^* \otimes \pi_{2j} MU.
\]

Moreover, we can map \(MU_{2n-2p+1}(X)\) into \((53)\) via the natural map \(\text{tr}\). Finally, Poincaré duality for complex cobordism tells us that the image of \(MU_{2n-2p+1}(X)\) in \((53)\) agrees with the image of \(MU^{2p-1}(X)\) in \((51)\). This proves the assertion. \(\Box\)

Let \((f: Y \to X)\) be a generator in \(\mathcal{M}^p(X)^+\). As remarked in \((50)\), the maps \(\varphi_{MU}\) and \(\varphi_{MU_{log}}\) are induced by natural maps
\[
\varphi_{MU}: \mathcal{M}^p(X)^+ \to MU^{2p}(X), \quad \varphi_{MU_{log}}: \mathcal{M}^p(X)^+ \to MU^{2p}(p)(X),
\]
These maps are given by \(\varphi_{MU}(f) = [f([\mathbb{C}])]_{MU}\) and \(\varphi_{MU_{log}}(f) = [f]_{MU_{log}}\), respectively. Let \(\mathcal{M}^{p}_{top}(X)\) denote the subgroup of elements in \(\mathcal{M}^p(X)^+\) which are mapped to zero under \(\varphi_{MU}\). The map \(\Phi_{MU}\) in \((41)\) is then induced by a map
\[
(54) \quad \Phi_{MU}: \mathcal{M}^{p}_{top}(X) \to J^{2p-1}_{MU}(X).
\]

**Theorem 4.3.** Using isomorphism \((50)\), the map \(\Phi'_{MU}\) given by \((49)\) can be identified with the map \(\Phi_{MU}\) in \((51)\).

**Proof.** To prove the assertion we need to understand the element
\[
\varphi_{MU_{log}}(f) = [f]_{MU_{log}}
\]
under the assumption that \((f: Y \to X) \in \mathcal{M}^p(X)^+\) is an element in \(\mathcal{M}^{p}_{top}(X)\). By \((52)\), we can represent \(\Phi_{MU}(f)\) by a triple \((q, \omega, 0)\) with
\[
q: X(\mathbb{C}) \to MU_{2p}, \quad \omega \in F^p A^{2p}(X; \mathbb{C} \otimes \pi_{2s} MU)_{cl}
\]
such that \(q^* q = I(\omega)\). Recall that, via the Pontryagin-Thom construction, the cobordism class of the map \(q\) corresponds to the image \([f([\mathbb{C}])]\) of \(f\) in \(MU^{2p}(X)\). Hence, in a first step, we represent the cobordism class of \(q\) by the continuous map \(f([\mathbb{C}])\). By assumption, the cobordism class of \(f([\mathbb{C}])\) is zero in \(MU^{2p}(X)\). This means that there is a compact weakly complex manifold \(W\) of real dimension \(2n-2p+1\) together with a continuous map \(\pi: W \to X(\mathbb{C})\) such that \(Y(\mathbb{C})\) is the boundary of \(W\).

This implies that, if we write \(\text{cl}_H(Y)\) for the image of \(f: Y(\mathbb{C}) \to X(\mathbb{C})\) in \(Z^{2p}(X; \mathbb{C} \otimes \pi_{2s} MU)\), the cohomology class of \(\text{cl}_H(Y)\) in \(H^{2p}(X; \mathbb{C} \otimes \pi_{2s} MU)\) vanishes as well. Hence there is a cochain \(\omega' \in C^{2p-1}(X; \mathbb{C} \otimes \pi_{2s} MU)\) such that \(d\omega' = \omega\). On the other hand, the assumption implies there is a form \(\omega' \in F^p A^{2p-1}(X; \mathbb{C} \otimes \pi_{2s} MU)\) such that \(d\omega' = \omega\). This shows that the triple \((f, \omega, 0)\) in \(MU_{2p}(p)(X)\) is homotopic to the triple \((0, 0, \omega - I(\omega'))\), where \(\omega' - I(\omega')\) is an element in \(C^{2p-1}(X; \mathbb{C} \otimes \pi_{2s} MU)\).

Hence, in order to show that \(\Phi_{MU}\) equals \(\Phi'_{MU}\) under isomorphism \((50)\), we need to determine the distribution which is defined by \(\omega' - I(\omega')\) under the Poincaré pairing. Let us start with \(\omega'\). Recall that \(\omega'\) is actually a finite sum \(\sum_j \omega'_j \otimes b'_j\) with classes \(b'_j \in \pi_{2j} MU\) and forms
\[
\omega'_j \in F^{p+j} A^{2p-1+2j}(X; \mathbb{C}).
\]
For each \(j\), the image of \(\omega'_j\) under the Poincaré pairing is the map
\[
(55) \quad F^{n-p+1-j} A^{2n-2p+1-2j}(X; \mathbb{C}) \cap \text{Ker} d \to \mathbb{C}
\]
which sends a form \( \eta_j \) to the integral
\[
\int_{X(C)} \omega_j \wedge \eta_j \in \mathbb{C}.
\]
Since the wedge-product \( \omega_j \wedge \eta_j \) is a form of type \((p', 2n - p')\) with \( p' \geq n + 1 \) and \( X(C) \) is a complex manifold of complex dimension \( n \), this integral vanishes. Hence this map is zero, and the distribution defined by \( \omega' \) vanishes.

It remains to understand the distribution corresponding to \( c' \). Recall that the existence of \( c' \) was implied by the fact that the bordism class of \( Y(C) \) is zero, i.e., \( c' \) is induced by the compact weakly complex manifold \( W \) of real dimension \( 2n - 2p + 1 \) together with a continuous map \( g: W \to X(C) \) such that \( Y(C) \) is the boundary of \( W \). Hence, the distribution determined by \( c' \) is exactly the distribution given by the pair \((W, Y)\) which we defined in the discussion leading to (48). This shows that \( \Phi_{MU}(f) \) equals \( \Phi'_{MU}(f) \) under the identification of (50).

\[\square\]

**Corollary 4.4.** The map \( \Phi'_{MU} \) induces a natural homomorphism
\[
\Omega^p(X) \to F^{n-p+1}MU^{2n-2p+1}(X)/MU_{2n-2p+1}(X)
\]
which agrees with the map \( \Phi_{MU} \) in (4) under the isomorphism (50).

**Proof.** We know from (4) that \( \Phi_{MU} \) induces a map on \( \Omega^p(X) \) which is a quotient of \( M^p(X)^+ \). Since \( \Phi'_{MU} \) agrees with \( \Phi_{MU} \) on \( M^p(X)^+ \) under the isomorphism (50), this shows that \( \Phi'_{MU} \) descends to a map on \( \Omega^p(X) \) which agrees with \( \Phi_{MU} \). \[\square\]

**Remark 4.5.** Using known examples of non-trivial elements in the Griffiths group, it is not difficult to produce examples of elements in \( \Omega^p_{top}(X) \) which also lie in the kernel of \( \theta: \Omega^p(X) \to CH^p(X) \) (see [12, §7.3]). This shows that the new Abel-Jacobi invariant is able to detect elements in \( \Omega^p_{top}(X) \) which the classical invariant would not see. Nevertheless, we do not yet have an example of the following type. We would like to find an element in \( \Omega^p_{top}(X) \) which is in the kernel of \( \Phi \) but not in the kernel of \( \theta \) and not in the kernel of \( \Phi_{MU} \). This is a much more difficult task which requires a better understanding of the kernel of the map
\[
MU^{2p-1}(X) \otimes \mathbb{R}/\mathbb{Z} \to H^{2p-1}(X; \mathbb{R}/\mathbb{Z}).
\]

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