THE BETTI NUMBERS FOR A FAMILY OF SOLVABLE LIE ALGEBRAS

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ABSTRACT. We give a characterization of symplectic quadratic Lie algebras that their Lie algebra of inner derivations has an invertible derivation. A family of symplectic quadratic Lie algebras is introduced to illustrate this situation. Finally, we calculate explicitly the Betti numbers of a family of solvable Lie algebras in two ways: using the cohomology of quadratic Lie algebras and applying a Pouseele’s result on extensions of the one-dimensional Lie algebra by Heisenberg Lie algebras.

0. INTRODUCTION

Let \( g \) be a complex Lie algebra endowed with a non-degenerate invariant symmetric bilinear form \( B \). \( \{X_1,\ldots,X_n\} \) be a basis of \( g \) and \( \{\omega_1,\ldots,\omega_n\} \) be its dual basis. Denote by \( \{Y_1,\ldots,Y_n\} \) the basis of \( g \) defined by \( B(Y_i,.) = \omega_i \), \( 1 \leq i \leq n \). Pinczon and Ushirobira discovered in [5] that the differential \( \partial \) on \( \bigwedge (g^*) \), the space of antisymmetric forms on \( g \), is given by \( \partial : \bigwedge (g^*) \to \bigwedge (g^*) \) where \( I \) is defined by:

\[
I(X,Y,Z) = B([X,Y],Z), \quad \forall X, Y, Z \in g
\]

and \( \{.,\} \) is the super Poisson bracket on \( \bigwedge (g^*) \) defined by

\[
\{\Omega,\Omega'\} = (-1)^{k+1} \sum_{i,j} B(Y_i, Y_j) t_{X_i}(\Omega) \wedge t_{X_j}(\Omega'), \forall \Omega, \Omega' \in \bigwedge^k (g^*), \Omega', \Omega \in \bigwedge (g^*).
\]

In Section 1, by using this, we detail a result of Medina and Revoy in [4] that there is an isomorphism between the second cohomology group \( H^2(g, \mathbb{C}) \) and \( \text{Der}_a(g)/\text{ad}(g) \) where \( \text{Der}_a(g) \) is the vector space of skew-symmetric derivations of \( g \) and \( \text{ad}(g) \) is its subspace of inner ones.

Involving in the well-known theorem by Jacobson on the invertibility of Lie algebra derivations that a Lie algebra over a field of characteristic zero is nilpotent if it admits an invertible derivation, we are interested in Lie algebras having an invertible derivation. We prove that the Lie algebra \( \text{ad}(g) \) of a symplectic quadratic Lie algebra has that property. In particular, we have the following (Proposition 1.5).

**THEOREM 1.** Let \((g,B,\omega)\) be a symplectic quadratic Lie algebra. Consider the mapping \( \mathcal{D} : \text{ad}(g) \to \text{ad}(g) \) defined by \( \mathcal{D}(\text{ad}(X)) = \text{ad}(\phi^{-1}(tx(\omega))) \) with \( \phi : g \to g^*, \phi(X) = B(X,.), \) then \( \mathcal{D} \) is an invertible derivation of \( \text{ad}(g) \).

The reader is referred to [2] for further information about symplectic quadratic Lie algebras. A family of such algebras is given to illustrate this situation.

In Section 2, motivated by Corollary 4.4 in [3], we give the Betti numbers for a family of solvable quadratic Lie algebras defined as follows. For each \( n \in \mathbb{N} \), let \( g_{2n+2} \) denote...
The Lie algebra with basis $\{X_0,\ldots, X_n, Y_0,\ldots, Y_n\}$ and non-zero Lie brackets $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$, $[X_i, Y_j] = X_0, 1 \leq i \leq n$. Denote by $B^k(\mathfrak{g}_{2n+2}) = B^k(\mathfrak{g}_{2n+2}, \mathbb{C})$, $Z^k(\mathfrak{g}_{2n+2}) = Z^k(\mathfrak{g}_{2n+2}, \mathbb{C})$, $H^k(\mathfrak{g}_{2n+2}) = H^k(\mathfrak{g}_{2n+2}, \mathbb{C})$ and $b_k = b_k(\mathfrak{g}_{2n+2}, \mathbb{C})$. By computing on super Poisson brackets, our second result is the following.

**THEOREM 2.** The $k^{th}$ Betti numbers of $\mathfrak{g}_{2n+2}$ are given as follows:

1. If $k$ is even then one has
   
   $$b_k = \left\lfloor \left( \frac{n}{2} \right) \left( \frac{n}{2} \right) - \left( \frac{n}{k+1} \right) \left( \frac{n}{k-1} \right) \right\rfloor.$$

2. If $k$ is odd then one has
   
   - if $k < n+1$ then
   
     $$b_k = \left( \frac{n}{k-1} \right) \left( \frac{n}{k-1} \right) - \left( \frac{n}{k} \right) \left( \frac{n}{k} \right),$$

   - if $k = n+1$ then
   
     $$b_{n+1} = 2 \left( \frac{n}{n} \right) \left( \frac{n}{n} \right) - 2 \left( \frac{n+1}{n+1} \right) \left( \frac{n+1}{n+1} \right),$$

   - if $k > n+1$ then
   
     $$b_k = \left( \frac{n}{k-1} \right) \left( \frac{n}{k-1} \right) - \left( \frac{n}{k+1} \right) \left( \frac{n}{k+1} \right).$$

Our method is direct and different from the Pouseele’s method given in [6] that we shall recall in Appendix 1. In the Pouseele’s method, the Betti numbers of the 2n + 1-dimensional Lie algebra $\mathfrak{g}$ defined by $[x, x_i] = x_i$ and $[y, y_i] = -y_i$ for all $1 \leq i \leq n$.

Other results of Betti numbers for some families of nilpotent Lie algebras, we refer the reader to [1], [5] or [7].

1. **A Characterization of Symplectic Quadratic Lie Algebras**

Let $\mathfrak{g}$ be a complex Lie algebra endowed with a non-degenerate invariant symmetric bilinear form $B$. In this case, we call the pair $(\mathfrak{g}, B)$ a quadratic Lie algebra. Denote by $\text{Der}_u(\mathfrak{g})$ the vector space of skew-symmetric derivations of $\mathfrak{g}$, that is the space of derivations $D$ satisfying $B(D(X), Y) = -B(X, D(Y))$ for all $X, Y \in \mathfrak{g}$, then $\text{Der}_u(\mathfrak{g})$ is a Lie subalgebra of $\text{Der}(\mathfrak{g})$.

**Proposition 1.1.** There exists a Lie algebra isomorphism $T$ between $\text{Der}_u(\mathfrak{g})$ and the space $\{\Omega \in \wedge^2(\mathfrak{g}^*) \mid \{I, \Omega\} = 0\}$. This isomorphism induces an isomorphism from $\text{ad}(\mathfrak{g})$ onto $t_\mathfrak{g}(I) = \{t_\mathfrak{g}(I) \in \wedge^2(\mathfrak{g}^*) \mid X \in \mathfrak{g}\}$.

**Proof.** Let $D \in \text{Der}_u(\mathfrak{g})$ and set $\Omega \in \wedge^2(\mathfrak{g}^*)$ by $\Omega(X, Y) = B(D(X), Y)$ for all $X, Y \in \mathfrak{g}$. Then $D$ is a derivation of $\mathfrak{g}$ if and only if

$$\Omega([X, Y], Z) + \Omega([Y, Z], X) + \Omega([Z, X], Y) = 0$$

for all $X, Y, Z \in \mathfrak{g}$. It means $\{I, \Omega\} = 0$. Define the map $T$ from $\text{Der}_u(\mathfrak{g})$ onto $\{\Omega \in \wedge^2(\mathfrak{g}^*) \mid \{I, \Omega\} = 0\}$ by $T(D) = \Omega$ then $T$ is a one-to-one correspondence.
Definition 1.4. If Corollary 1.3. That means set $T(\mathfrak{g})$. Indeed, set $\Omega = T(D), \Omega' = T(D')$ and fix an orthonormal basis $\{X_j\}_{j=1}^n$ of $\mathfrak{g}$. One has

\[
\{\Omega, \Omega'\}(X,Y) = -\left(\sum_{j=1}^n t_{X_j}(\Omega) \wedge t_{X_j}(\Omega')\right)(X,Y)
\]

\[
= -\sum_{j=1}^n \left(\Omega(X_j, X)\Omega'(X_j, Y) - \Omega(X_j, Y)\Omega'(X_j, X)\right)
\]

\[
= -\sum_{j=1}^n B\left(B(D(X_j), X)D'(X_j) - B(D'(X_j), X)D(X_j), Y\right))
\]

\[
= -\sum_{j=1}^n B\left(D'(D(X)) - D(D'(X)),Y\right)) = -B([D', D](X), Y).
\]

That means $T([D, D']) = \{T(D), T(D')\}$ and then $T$ is a Lie algebra isomorphism.

If $D = \text{ad}(X_0)$ then $T(D)(X, Y) = B([X_0, X], Y) = I(X_0, Y, Z) = t_{X_0}(I)(X, Y)$. Therefore, $T(D) = t_{X_0}(I)$. □

Corollary 1.2. $\{t_X(I), t_Y(I)\} = t_{[X,Y]}(I)$.

Corollary 1.3. \[ \text{The cohomology group } H^2(\mathfrak{g}, \mathbb{C}) \simeq \text{Der}_\alpha(\mathfrak{g}, \mathfrak{B})/\text{ad}(\mathfrak{g}). \]

Definition 1.4. A non-degenerate skew-symmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called a symplectic structure on $\mathfrak{g}$ if it satisfies

\[
\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0
\]

for all $X, Y, Z \in \mathfrak{g}$.

A symplectic structure $\omega$ on a quadratic Lie algebra $(\mathfrak{g}, B)$ is corresponding to a skew-symmetric invertible derivation $D$ defined by $\omega(X, Y) = B(D(X), Y)$, for all $X, Y \in \mathfrak{g}$. As above, a symplectic structure is exactly a non-degenerate 2-form $\omega$ satisfying $\{I, \omega\} = 0$. If $\mathfrak{g}$ has a such $\omega$ then we call $(\mathfrak{g}, B, \omega)$ a symplectic quadratic Lie algebra.

For symplectic quadratic Lie algebras, the reader can refer to [2] for more details. Here we give a following property.

Proposition 1.5. Let $(\mathfrak{g}, B, \omega)$ be a symplectic quadratic Lie algebra. Consider the mapping $\mathcal{D} : \text{ad}(\mathfrak{g}) \to \text{ad}(\mathfrak{g})$ defined by $\mathcal{D}(\text{ad}(X)) = \text{ad}(\phi^{-1}(t_X(\omega)))$ with $\phi : \mathfrak{g} \to \mathfrak{g}^*, \phi(X) = B(X, \cdot)$, then $\mathcal{D}$ is an invertible derivation of $\text{ad}(\mathfrak{g})$.

Proof. As above we have $\{I, \omega\} = 0$ and then $t_X(\{I, \omega\}) = 0$ for all $X \in \mathfrak{g}$. It implies $\{t_X(I), \omega\} = \{I, t_X(\omega)\}$ for all $X \in \mathfrak{g}$. Note that if $X$ is nonzero, since $\omega$ is non-degenerate then $t_X(\omega)$ is non-trivial. Set $Y = \phi^{-1}(t_X(\omega))$ then $\{I, t_X(\omega)\} = I_Y(I)$ and therefore this defines an inner derivation. Let $D$ be the derivation corresponding to $\omega$ then one has $[\text{ad}(X), D] = \text{ad}(Y)$.

Let $\text{ad}(X) \in \text{ad}(\mathfrak{g})$. Set $\alpha = \phi(X)$. Since $\omega$ is non-degenerate then there exists an element $Y \in \mathfrak{g}$ such that $\alpha = t_Y(\omega)$. In this case, $\mathcal{D}(\text{ad}(Y)) = \text{ad}(X)$. That means $\mathcal{D}$ onto and therefore it is bijective. □

Next, we give a family of symplectic quadratic Lie algebras that has been defined in [3] as follows.
Example 1.6. Let $p \in \mathbb{N} \setminus \{0\}$. We denote the Jordan block of size $p$ by $J_1 := (0)$ and for $p \geq 2$,
\[
J_p := \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix}.
\]

For $p \geq 2$, we consider $q = \mathbb{C}^{2p}$ with a basis $\{X_i, Y_i\}, 1 \leq i \leq p$, and equipped with a bilinear form $B$ satisfying $B(X_i, X_j) = B(Y_i, Y_j) = 0$ and $B(X_i, Y_j) = \delta_{ij}$. Let $C : q \to q$ with matrix
\[
C = \begin{pmatrix} J_p & 0 \\ 0 & -J_p \end{pmatrix}
\]
in the given basis. Then $C \in \mathfrak{o}(2p)$.

Let $\mathfrak{h} = \mathbb{C}^2$ and $\{X_0, Y_0\}$ be a basis of $\mathfrak{h}$. Define on the vector space $i_{2p} = q \oplus \mathfrak{h}$ the Lie bracket $[Y_0, X] = C(X) \cdot [X, Y] = B(C(X), Y)X_0$ and the bilinear form $B(X_0, Y_0) = 1$, $B(X_i, Y_0) = B(Y_0, Y_i) = B(X_0, X_i) = B(Y_0, X_i) = 0$ and $B(X, Y) = B(X, Y)$ for all $X, Y \in q$. So $i_{2p}$ is a nilpotent Lie algebra and it will be called a $2p + 2$-dimensional nilpotent Jordan-type Lie algebra.

Denote by $\{\alpha, \alpha_1, \ldots, \alpha_p, \beta, \beta_1, \ldots, \beta_p\}$ the dual basis of $\{X_0, \ldots, X_p, Y_0, \ldots, Y_p\}$ then $I = \beta \wedge \sum_{i=0}^{p-1} \alpha_i \wedge \beta_i$. In this case, we choose $\omega = \alpha \wedge \beta + \sum_{i=1}^{p} i \alpha_i \wedge \beta_i$ then $\{I, \omega\} = 0$ and therefore $(i_{2p}, B, \omega)$ is a symplectic quadratic Lie algebra. Notice that if we define $D(\text{ad}(Y_0)) = -\text{ad}(Y_0), D(\text{ad}(X_i)) = i\text{ad}(X_i)$ and $D(\text{ad}(Y_i)) = -i\text{ad}(Y_i)$ then $D$ is an invertible derivation of $\text{ad}(i_{2p})$.

2. The Betti numbers for a family of solvable quadratic Lie algebras

For each $n \in \mathbb{N}$, let $\mathfrak{g}_{2n+2}$ denote the Lie algebra with basis $\{X_0, \ldots, X_n, Y_0, \ldots, Y_n\}$ and non-zero Lie brackets $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$, $[X_0, X_i] = X_i, 1 \leq i \leq n$. Then $\mathfrak{g}$ is quadratic with invariant bilinear form $B$ given by $B(X_i, Y_j) = 1$, $0 \leq i \leq j \leq n$, zero otherwise.

Let $\{\alpha, \alpha_1, \ldots, \alpha_n, \beta, \beta_1, \ldots, \beta_n\}$ be the dual basis of $\{X_0, \ldots, X_n, Y_0, \ldots, Y_n\}$ and set $V = \text{span}\{\alpha_i\}, W = \text{span}\{\beta_i\}, 1 \leq i \leq n$. It is easy to check that the associated 3-form of $\mathfrak{g}_{2n+2}$:
\[
\Omega := \sum_{i=0}^{n} \alpha_i \wedge \beta_i.
\]

Denote by $\Omega_n := \sum_{i=1}^{n} \alpha_i \wedge \beta_i$ then one has
\[
B^2(\mathfrak{g}_{2n+2}) = \{tX(I) \mid X \in \mathfrak{g}_{2n+2}\} = \text{span}\{\beta \wedge \alpha_i \wedge \beta_i, \Omega_n \mid 1 \leq i \leq n\}.
\]

If $n = 1$ then by we can directly calculate that $H^2(\mathfrak{g}_4) = \{0\}$. If $n > 1$, we have the non-zero super Poisson brackets:
(i) $\{I, \alpha \wedge \alpha_i\} = \alpha_i \wedge \Omega_n - \alpha \wedge \beta \wedge \alpha_i$ and $\{I, \alpha \wedge \beta_i\} = \beta_i \wedge \Omega_n + \alpha \wedge \beta \wedge \beta_i$,
(ii) $\{I, \alpha \wedge \beta\} = I$, 
(iii) $\{I, \alpha_i \wedge \alpha_j\} = 2 \beta \wedge \alpha_i \wedge \alpha_j$ and $\{I, \beta_i \wedge \beta_j\} = -2 \beta \wedge \beta_i \wedge \beta_j$.

It results that $\mathfrak{z}^2(\mathfrak{g}_{2n+2}) = \text{span}\{\beta \wedge \alpha_i \wedge \beta_i, \alpha_i \wedge \beta_i \mid 1 \leq i, j \leq n\}$ and then the second cohomology group $H^2(\mathfrak{g}_{2n+2}) = \text{span}\{\alpha_i \wedge \beta_j\} / \text{span}\{\sum_{i=1}^{n} \alpha_i \wedge \beta_i\}$, where $1 \leq i, j \leq n$. So we recover the result of Medina and Revoy in [4] obtained by describing the space $\text{Der}_n(\mathfrak{g}_{2n+2})$ that $b_2 = n^2 - 1$.

To get the Betti numbers $b_k$ for $k \geq 3$, we need the following lemma.
**Lemma 2.1.** The map \( \{ \Omega_n \} : \Lambda^k(V) \otimes \Lambda^m(W) \to \Lambda^j(V) \otimes \Lambda^l(W) \) with \( k, m \geq 0 \) is a vector space isomorphism if \( k \neq m \) and \( \{ \Omega_n, \Lambda^k(V) \otimes \Lambda^l(W) \} = \{ 0 \} \).

**Proof.** We have \( \{ \Omega_n, \alpha_{t_1} \wedge \ldots \wedge \alpha_{t_k}, \beta_{t_1} \wedge \ldots \wedge \beta_{t_m} \} = k \alpha_{t_1} \wedge \ldots \wedge \alpha_{t_k}, \Omega_n, \beta_{t_1} \wedge \ldots \wedge \beta_{t_m} \} = -m \wedge \beta_{t_1} \wedge \ldots \wedge \beta_{t_m} \) and \( \{ \Omega_n, \alpha_{t_1} \wedge \ldots \wedge \alpha_{t_k} \wedge \beta_{t_1} \wedge \ldots \wedge \beta_{t_m} \} = (k - m) \alpha_{t_1} \wedge \ldots \wedge \alpha_{t_k} \wedge \beta_{t_1} \wedge \ldots \wedge \beta_{t_m} \) then the result follows.

By a straightforward computation on super Poisson brackets we have the following corollary.

**Corollary 2.2.** The restrictions of the differential \( \partial \) from \( \alpha \wedge \Lambda^i(V) \otimes \Lambda^j(W) \) onto \( \Omega_n \wedge \Lambda^i(V) \otimes \Lambda^j(W) \) and from \( \Lambda^i(V) \otimes \Lambda^j(W) \) onto \( \beta \wedge \Lambda^i(V) \otimes \Lambda^j(W) \) with \( i, j \geq 0 \) are vector space isomorphisms.

Let us now give the cases for which \( \ker(\partial) \) can be obtained. The following lemma is easy:

**Lemma 2.3.** We have \( \partial(\Lambda^i(V) \otimes \Lambda^j(W)) = \partial(\beta \wedge \Lambda^i(V) \otimes \Lambda^j(W)) = \{ 0 \} \) with \( i, j \geq 0 \).

Moreover, \( \partial(\Lambda^i(V) \otimes \Lambda^j(W)) \subset \partial(\Lambda^{i+1}(V) \otimes \Lambda^{j+1}(W)) \) for all \( i, j \geq 0 \) and \( i \neq j \).

By the reason shown in (i) and (ii) of Lemma 2.3, we set the map

\[
\phi_{k_1, k_2, n} : \Lambda^{k_1}(\alpha_1, \ldots, \alpha_n) \otimes \Lambda^{k_2}(\beta_1, \ldots, \beta_n) \to \Lambda^{k_1+1}(\alpha_1, \ldots, \alpha_n) \otimes \Lambda^{k_2+1}(\beta_1, \ldots, \beta_n)
\]

defined by \( \phi_{k_1, k_2, n}(\omega) = \Omega_n \wedge \omega \) then we have the following result.

**Proposition 2.4.**

(i) If \( k \) is even then

\[
\dim \ker(\partial_k) = \binom{n+1}{k} - \sum_{i=0}^{k-1} \binom{n}{k-i} - \dim \ker \phi_{k-1, k-2, n} \wedge \Omega_n.
\]

(ii) If \( k \) is odd then

\[
\dim \ker(\partial_k) = \dim \ker \phi_{k-1, k-2, n} + \sum_{i=0}^{k-1} \binom{n}{k-i} - \binom{n+1}{k-1}.
\]

Using the formula \( b_k(\mathfrak{g}_{2n+2}) = \dim \ker(\partial_k) + \dim \ker(\partial_{k-1}) - \binom{2n+2}{k-1} \), the binomial identity

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

and the formula

\[
\sum_{i=0}^{k} \binom{n}{k-i} = \binom{2n}{k}
\]

we obtain the following corollary.

**Corollary 2.5.** The \( k \)th Betti numbers of \( \mathfrak{g}_{2n+2} \) are given as follows:

(i) If \( k \) is even then

\[
b_k(\mathfrak{g}_{2n+2}) = \binom{n}{k} + 2 \dim \ker \phi_{k-1, k-2, n} - \binom{n}{k-2}.
\]
(ii) If $k$ is odd then

$$b_k(g_{2n+2}) = \left( \begin{array}{c} n \\ \frac{k-1}{2} \end{array} \right) + \dim \ker \phi_{k-1,k,n} + \dim \ker \phi_{k-3,k,n} - \left( \begin{array}{c} n \\ \frac{k-3}{2} \end{array} \right).$$

Hence, it remains to compute $\dim \ker (\phi_{k,k,n})$. Consider the power $\phi_{k,k,n}^m$ of the map $\phi_{k_1,k_2,n}$ and let

$$K(m,k_1,k_2,n) = \dim \ker (\phi_{k_1,k_2,n}^m)$$

then one has:

**Lemma 2.6.**

(i) The map

$$\theta_{k_1,k_2,n+1}^m : \ker (\phi_{k_1-1,k_2-1,n}^m) \oplus \ker (\phi_{k_1-1,k_2,n}^m) \oplus \ker (\phi_{k_1,k_2-1,n}^m)$$

defined by

$$\theta_{k_1,k_2,n+1}^m (\omega_1, \omega_2, \omega_3, \omega_4) = \alpha_{n+1} \wedge \beta_{n+1} \wedge \omega_1 + \alpha_{n+1} \wedge \omega_2 + \beta_{n+1} \wedge \omega_3 + \omega_4 - \frac{1}{m} \phi_{k_1-1,k_2-1,n}(\omega_1)$$

is a vector space isomorphism.

(ii) $K(m,k_1,k_2,n) = K(m+1,k_1-1,k_2-1,n-1) + K(m-1,k_1,k_2,n-1) + K(m,k_1,k_2-1,n-1) + (m-1,k_1,k_2,n-1)$.

**Proof.**

(i) The map $\theta_{k_1,k_2,n+1}^m$ is clearly injective. To prove $\theta_{k_1,k_2,n+1}^m$ surjective, let us consider $\omega \in \wedge^{k_1}(\alpha_1, \ldots, \alpha_{n+1}) \otimes \wedge^{k_2}(\beta_1, \ldots, \beta_{n+1})$ such that $\Omega_{n+1}^m \wedge \omega = 0$. Observe that $\omega$ can be written in the form $\omega = \alpha_{n+1} \wedge \beta_{n+1} \wedge \omega_1 + \alpha_{n+1} \wedge \omega_2 + \beta_{n+1} \wedge \omega_3 + \omega_4$ where $\omega_i \in \wedge^{k_1}(\alpha_1, \ldots, \alpha_n) \otimes \wedge^{k_2}(\beta_1, \ldots, \beta_n)$, $\omega_2 \in \wedge^{k_1}(\beta_1, \ldots, \beta_n)$ and $\omega_4 \in \wedge^{k_2}(\alpha_1, \ldots, \alpha_n)$.

Moreover, $\wedge_{n+1}^m \wedge \omega = 0$ we obtain $\wedge_{n+1}^m \wedge \omega_2 = \wedge_{n+1}^m \wedge \omega_3 = \wedge_{n+1}^m \wedge \omega_4 = 0$.

(ii) The assertion (2) follows (1).

To calculate $K(m,k_1,k_2,n)$, we use the following boundary conditions from the definition of $\phi_{k_1,k_2,n}^m$ in which we assume $\phi_{k_1,k_2,n}^0$ is the identity map:

1. $K(0,k_1,k_2,n) = 0$ for all $k_1, k_2, n \geq 0$.
2. $K(m,0,0,n) = \begin{cases} 0, & \text{if } m \leq n, \\ 1, & \text{if } m > n. \end{cases}$
3. $K(m,0,1,n) = K(m,1,0,n) = \begin{cases} 0, & \text{if } m = 0 \text{ or } n > m, \\ n, & \text{if } 1 \leq n \leq m. \end{cases}$
(4) \( K(m, k_1, k_2, 0) = \begin{cases} 1, & \text{if } m \geq 1, k_1 = k_2 = 0, \\ 0, & \text{otherwise}. \end{cases} \)

By the condition (2) we extend \( K(m, k_1, k_2, n) = 0 \) for negative \( k_1 \) or \( k_2 \) and by the condition (1) we set the condition (5) by \( K(-m, k_1, k_2, n) = -K(m, k_1 - m, k_2 - m, n) \).

**Lemma 2.7.**

\[ K(m, k, n) = \sum_{p=0}^{m} \sum_{q=0}^{n} \binom{n}{p} \binom{n}{q} K(m + n - p - q, k - n + p, k - n + q, 0). \]

**Proof.** By induction on \( l \), we prove that

\[ K(m, k, n) = \sum_{p=0}^{m} \sum_{q=0}^{n} \binom{l}{p} \binom{1}{q} K(m + l - p - q, k - l + p, k - l + q, n - l). \]

Let \( l = n \) to get the lemma. \( \square \)

The Betti numbers of \( \mathfrak{g}_{2n+2} \) is in the case \( m = 1 \). By the conditions (4) and (5) we reduce the following.

**Corollary 2.8.**

\[ K(1, k, k, n) = \begin{cases} 0, & \text{if } k < \frac{2}{3} n, \\ \left( \binom{n}{k} - \binom{n}{k+1} \right), & \text{if } k \geq \frac{2}{3} n. \end{cases} \]

Finally, by applying this formula we obtain the Betti number of \( \mathfrak{g}_{2n+2} \) according to Corollary 2.5.

### 3. Appendix 1: Another Way to Get the Betti Numbers of \( \mathfrak{g}_{2n+2} \)

In this part, we shall give another way to get the Betti numbers of \( \mathfrak{g}_{2n+2} \). It is based on the following result.

**Proposition 3.1.** [6]

Let \( \mathfrak{g} \) be an extension of the one-dimensional Lie algebra \( \langle z \rangle \) by the Heisenberg Lie algebra \( \mathfrak{h}_{2n+1} \), for some \( n \),

\[ 1 \longrightarrow \mathfrak{h}_{2n+1} \longrightarrow \mathfrak{g} \longrightarrow \langle z \rangle \longrightarrow 0 \]

such that \( \mathfrak{g} \) acts trivially on the center \( \mathfrak{z} = \langle w \rangle \) of \( \mathfrak{h}_{2n+1} \). Let \( \mathfrak{f} = \mathfrak{g} / \mathfrak{z} \). Then

\[ b_k(\mathfrak{g}) = \begin{cases} b_k(\mathfrak{f}) & \text{for } k = 0 \text{ or } k = 1, \\ b_k(\mathfrak{f}) - b_{k-2}(\mathfrak{f}) & \text{for } 2 \leq k \leq n, \\ 2 [b_{n+1}(\mathfrak{f}) - b_{n-1}(\mathfrak{f})] & \text{for } k = n + 1, \\ b_{k-1}(\mathfrak{f}) - b_{k+1}(\mathfrak{f}) & \text{for } n + 2 \leq k \leq 2n, \\ b_{k-1}(\mathfrak{f}) & \text{for } k = 2n + 1 \text{ or } k = 2n + 2. \end{cases} \]

It is easy to see that \( \mathfrak{g}_{2n+2} \) is an extension of the one-dimensional Lie algebra \( \langle Y_0 \rangle \) by \( \mathfrak{h}_{2n+1} \). To calculate the Betti numbers of \( \mathfrak{g}_{2n+2} \) it needs to find the Betti numbers of the \( 2n + 1 \)-dimensional Lie algebra \( \mathfrak{f} \) with a basis \( \{ y, x_1, \ldots, x_n, y_1, \ldots, y_n \} \) and the Lie bracket

\[ [y, x_i] = x_i, \quad [y, y_i] = -y_i \]

for all \( 1 \leq i \leq n \).

Let \( \{ y^*, x^*_1, \ldots, x^*_n, y^*_1, \ldots, y^*_n \} \) be the dual basis of \( \{ y, x_1, \ldots, x_n, y_1, \ldots, y_n \} \).
Proposition 3.2.

(1) One has
\[ \partial_k \left( y^k \wedge \left( \bigwedge^{k-1}(x_1^*, \ldots, x_n^*, y_1^*, \ldots, y_n^*) \right) \right) = 0. \]

(2) Assume \( j + l = k \) then we have
- if \( j = l \) then
  \[ \partial_k \left( \bigwedge^j(x_1^*, \ldots, x_n^*) \otimes \bigwedge^l(y_1^*, \ldots, y_n^*) \right) = 0, \]
- if \( j \neq l \) then
  \[ \partial_k \left( \bigwedge^j(x_1^*, \ldots, x_n^*) \otimes \bigwedge^l(y_1^*, \ldots, y_n^*) \right) = y^k \wedge \left( \bigwedge^j(x_1^*, \ldots, x_n^*) \otimes \bigwedge^l(y_1^*, \ldots, y_n^*) \right). \]

Proof. The assertion (1) is obvious. For (2), we use the following computation:
\[ \partial_k \left( x_{i_1} \wedge \ldots \wedge x_{i_j} \wedge y_{r_1} \wedge \ldots \wedge y_{r_l} \right) = (j - k)y^k \wedge x_{i_1} \wedge \ldots \wedge x_{i_j} \wedge y_{r_1} \wedge \ldots \wedge y_{r_l} \]
for all \( 1 \leq i_1 < \ldots < i_j \leq n \) and \( 1 \leq r_1 < \ldots < r_l \leq n \).

It results the following corollary.

Corollary 3.3. The Betti numbers of \( \mathfrak{f} \) is given as follows:
\[ b_k(\mathfrak{f}) = \left( \begin{array}{c} n \\ \left[ \frac{k}{2} \right] \end{array} \right) \left( \begin{array}{c} n \\ \left[ \frac{k}{2} \right] \end{array} \right) \]
where \( [x] \) denotes the integer part of \( x \).

Applying this corollary, we have
\[ b_k(\mathfrak{g}_{2n+2}) = \left\{ \begin{array}{ll}
1 & \text{for } k = 0 \text{ or } k = 1, \\
\left( \begin{array}{c} n \\ \left[ \frac{k}{2} \right] \end{array} \right) \left( \begin{array}{c} n \\ \left[ \frac{k}{2} \right] \end{array} \right) & \text{for } 2 \leq k \leq n, \\
2 & \left( \begin{array}{c} n \\ \left[ \frac{k+1}{2} \right] \end{array} \right) - 2 & \text{for } k = n + 1, \\
\left( \begin{array}{c} n \\ \left[ \frac{k-1}{2} \right] \end{array} \right) \left( \begin{array}{c} n \\ \left[ \frac{k-1}{2} \right] \end{array} \right) & \text{for } n + 2 \leq k \leq 2n, \\
1 & \text{for } k = 2n + 1 \text{ or } k = 2n + 2.
\]

and then Theorem 2 is obtained.

4. APPENDIX 2: THE SECOND COHOMOLOGY GROUP OF A FAMILY OF NILPOTENT LIe ALGEBRAS

In this appendix, in the progress of our work, we give the second cohomology of a family of nilpotent Lie algebras that are double extensions of an Abelian Lie algebra (see \[ \] for more details about these Lie algebras).

Let us denote \( \mathfrak{g}_{4n+2} \) a 2-nilpotent quadratic Lie algebra of dimension \( 4n + 2 \) spanned by \( \{X, X_1, \ldots, X_{2n}, Y, Y_1, \ldots, Y_{2n}\} \) where the Lie bracket is defined by \( [Y, Y_{2n-1}] = X_{2n}, [Y, Y_1] = -X_{2n-1}, \) \( [Y_{2n-1}, Y_1] = X \) and the bilinear form is given by \( B(X, Y) = B(X, Y) = 1, \) zero otherwise. Let \( \{\alpha, \alpha, \beta, \beta\} \) be the dual basis of \( \{X, X_1, Y, Y_1\} \). We can check that the associated 3-form \( I \) of \( \mathfrak{g}_{4n+2} \) is \( I = \beta \wedge \Omega \) where \( \Omega = \beta_1 \wedge \beta_2 + \beta_3 \wedge \beta_4 + \ldots + \beta_{2n-1} \wedge \beta_{2n} \). Therefore, it is easy to see that \( \iota_{\mathfrak{g}_{4n+2}}(I) = \text{span}\{\Omega, \beta \wedge \beta\} \) for all \( 1 \leq i \leq 2n \). We have the following proposition.
Proposition 4.1. \( \dim(H^2(g_{4n+2}, \mathbb{C})) = 8 \) if \( n = 1 \) and \( \dim(H^2(g_{4n+2}, \mathbb{C})) = 5n^2 + n \) if \( n > 1 \).

Proof. First we need describe \( \ker(\partial_2) \). Let \( V \) be the space spanned by \( \{\beta, \beta_1, ..., \beta_{2n}\} \) then \( \{I, \alpha\} = 0 \) for all \( \alpha \in V \oplus V \). By a straightforward computation, we have

1. \( \{I, \beta \land \alpha\} = \{I, \alpha_{2i-1} \land \beta_{2i}\} = \{I, \alpha_{2i} \land \beta_{2i-1}\} = 0 \),
2. \( \{I, \alpha \land \beta\} = I \),
3. \( \{I, \alpha \land \beta_{2i-1}\} = \beta_{2i-1} \land \Omega \), \( \{I, \alpha \land \beta_{2i}\} = \beta_{2i} \land \Omega \),
4. \( \{I, \alpha \land \beta_{2i-1}\} = \alpha_{2i-1} \land \Omega + \beta \land \beta_{2i} \land \alpha \), \( \{I, \alpha \land \beta_{2i}\} = \alpha_{2i} \land \Omega - \beta \land \beta_{2i-1} \land \alpha \),
5. \( \{I, \alpha_{2i-1} \land \alpha_{2i}\} = -\beta \land \beta_{2i} \land \alpha_{2j} - \beta \land \beta_{2j-1} \land \alpha_{2i-1}, \{I, \alpha_{2i} \land \alpha_{2j}\} = \beta \land \beta_{2i-1} \land \alpha_{2j} - \beta \land \beta_{2j-1} \land \alpha_{2i} \),
6. \( \{I, \alpha_{2i-1} \land \beta_{2j}\} = -\{I, \alpha_{2j-1} \land \beta_{2i}\} = -\beta \land \beta_{2i} \land \beta_{2j}, i \neq j \),
7. \( \{I, \alpha_{2i-1} \land \beta_{2j-1}\} = \{I, \alpha_{2j-1} \land \beta_{2i}\} = -\beta \land \beta_{2i} \land \beta_{2j-1} \),
8. \( \{I, \alpha_{2i} \land \beta_{2j-1}\} = -\{I, \alpha_{2j} \land \beta_{2i-1}\} = \beta \land \beta_{2i-1} \land \beta_{2j-1}, i \neq j \).

As a consequence, if \( n = 1 \) then it is direct that

\( \ker(\partial_2) = V \land V \oplus \span\{\beta \land \alpha_1, \beta \land \alpha_2, \alpha \land \beta - \alpha_1 \land \beta_1, \alpha_1 \land \beta_2, \alpha_1 \land \beta_1 - \alpha_2 \land \beta_2, \alpha_2 \land \beta_1\} \).

Therefore, we obtain \( \dim(H^2(g_{4n+2}, \mathbb{C})) = 8 \).

In the case \( n > 1 \) then \( \Omega \) is indecomposable. Hence,

\[
\ker(\partial_2) = V \land V \oplus \span\{\beta \land \alpha_{2i-1}, \beta \land \alpha_{2i}, \alpha \land \beta - \sum_{i=1}^{n} \alpha_{2i-1} \land \beta_{2i-1}, \alpha_{2i-1} \land \beta_{2j} + \alpha_{2j-1} \land \beta_{2i} - \beta_{2j} \land \alpha_{2i-1} + \alpha_{2j} \land \beta_{2i-1}\} \]

with \( 1 \leq i, j \leq n \) and it is easy to check that \( \dim(H^2(g_{4n+2}, \mathbb{C})) = 5n^2 + n \).

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