Research Article

Spectral Bounds for Polydiagonal Jacobi Matrix Operators

Arman Sahovic

Mathematics Department, Imperial College London, South Kensington, 180 Queen's Gate, London SW7 2AZ, UK

Correspondence should be addressed to Arman Sahovic; arman@ic.ac.uk

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The research on spectral inequalities for discrete Schrödinger operators has proved fruitful in the last decade. Indeed, several authors analysed the operator’s canonical relation to a tridiagonal Jacobi matrix operator. In this paper, we consider a generalisation of this relation with regard to connecting higher order Schrödinger-type operators with symmetric matrix operators with arbitrarily many nonzero diagonals above and below the main diagonal. We thus obtain spectral bounds for such matrices, similar in nature to the Lieb-Thirring inequalities.

1. Background

Let $W$ be the self-adjoint Jacobi matrix operator acting on $\ell^2(\mathbb{Z})$ as follows:

$$W = \begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots \\
\cdots & b_{-1} & a_{-1} & 0 & 0 \\
\cdots & a_{-1} & b_0 & a_0 & 0 \\
\cdots & 0 & a_0 & b_1 & a_1 \\
\cdots & 0 & 0 & a_1 & b_2 \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},$$

(1)

via

$$(W\varphi)(n) = a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1),$$

for $n \in \mathbb{Z},$

(2)

where $a_n > 0$ and $b_n \in \mathbb{R}$. This operator can be viewed as the one-dimensional discrete Schrödinger operator if $a_n = 1$ for all $n$. A variety of papers examined such operators; for example, we quote the work by Killip and Simon in [1], where they obtained sum rules for such Jacobi matrices. Additionally, Hundertmark and Simon in [2] were able to find spectral bounds for these operators. We thus state their result.

If $a_n \to 1, b_n \to 0$ rapidly enough, as $n \to \pm\infty$, the essential spectrum $\sigma_{es}(W)$ of $W$ is absolutely continuous and coincides with the interval $[-2,2]$ (see, e.g., [3]). Besides, $W$ may have simple eigenvalues $\{E_j^\pm\}_{j=1}^N$ where $N \in \mathbb{N} = \mathbb{N} \cup \{\infty\}$, and

$$E_1^+ > E_2^+ > \cdots > 2 > -2 > \cdots > E_2^- > E_1^-.$$

(3)

Indeed, in [2] the authors found the following.

**Theorem 1.** If $\{b_n\}_{n \in \mathbb{Z}}$, $\{a_n - 1\}_{n \in \mathbb{Z}} \in p^{y+1/2}(\mathbb{Z})$, $y \geq 1/2$, then

$$\sum_{j=1}^N |E_j^+ - 2|^y + \sum_{j=1}^N |E_j^- + 2|^y$$

$$\leq k_y \left[ \sum_{n=-\infty}^{\infty} |b_n|^{y+1/2} + 4 \sum_{n=-\infty}^{\infty} |a_n - 1|^{y+1/2} \right],$$

(4)

where

$$k_y = 2 \left(3^{1/2} \right) L^{cl}_{\gamma,1}, \quad L^{cl}_{\gamma,1} = \frac{\Gamma(y + 1)}{2\sqrt{\pi} \Gamma(y + 3/2)}.$$

(5)

The author (see [4]) then improved their result, achieving the smaller constant: $k_y = 3^{3/2} \pi L^{cl}_{\gamma,1}$, by translating a well-known method employed by Dolbeaut et al. in [5] to the discrete scenario. They, in turn, used a simple argument by Eden and Foias (see [6]) to obtain improved constants for Lieb-Thirring inequalities in one dimension.

The aim of this paper is to answer the natural question of whether these methods can be generalised to give bounds
for higher order Schrödinger-type operators and thus "poly-
diagonal" Jacobi-type matrix operators, which we will define
below.

2. Notation and Preliminary Material

For a sequence \( \{\psi(n)\}_{n \in \mathbb{Z}} \), let \( D \) and \( D^* \) be the difference 
operator and its adjoint, respectively, denoted by \( D\psi(n) = \psi(n+1) - \psi(n) \) 
and \( D^*\psi(n) = \psi(n) - \psi(n-1) \). We define the discrete one-dimensional Laplacian by \( \Delta_D := (D^* D)\psi(n) = \psi(n+1) - 2\psi(n) + \psi(n-1) \).

Our next result is concerned with estimating those negative 
eigenvalues of \( \psi(n) \) of the discrete Schrödinger-
type operator and its adjoint, respectively, denoted by \( \Delta_D = \Delta \), 
where \( \Delta \) is the range of the above symbol, which can be found to be

\[
\Delta \Delta_D \psi(n) = \sum_{k=0}^{2\alpha_2} C_k (-1)^{k+\alpha} \psi(n - \alpha + k).
\] (7)

Furthermore, in order to identify our essential spectrum, 
we apply the discrete Fourier transform as follows:

\[
\mathcal{F} (\Delta_D \psi)(x) = \sum_{n \in \mathbb{Z}} e^{i\pi x} \left( \sum_{k=0}^{2\alpha_2} C_k (-1)^{k+\alpha} \psi(n - \alpha + k) \right),
\] (8)

which, after some rearrangement, yields

\[
\mathcal{F} (\Delta_D \psi)(x) = \left[ 2\alpha \cos ((x - k) x) \right] \times (\mathcal{F}(\psi)(x)).
\] (9)

The essential spectrum of the operator \( \Delta_D \) will thus be 
the range of the above symbol, which can be found to be

\( \zeta_{ess}(\Delta_D) = [0, 4^\alpha] \).

3. Main Results

We now let \( \{\psi_j\}_{j=1}^N \), \( N \in \mathbb{N} \), be the orthonormal system of eigensequences in \( \ell^2(\mathbb{Z}) \) 
corresponding to the negative 
eigenvalues \( \{\psi_j\}_{j=1}^N \) of the \( (2\alpha) \)th order discrete Schrödinger-
type operator as follows:

\[
(H_D^\alpha \psi_j)(n) := (\Delta_D^\alpha \psi_j)(n) - b_j \psi_j(n) = \epsilon_j \psi_j(n),
\] (10)

where \( j \in \{1, \ldots, N\} \) and we assume that \( b_j \geq 0 \) for all \( n \in \mathbb{Z} \).

Our next result is concerned with estimating those negative 
eigenvalues.

**Theorem 2.** Let \( b_n \geq 0 \), \( \{b_n\}_{n \in \mathbb{Z}} \in \ell^{1+1/2\alpha}(\mathbb{Z}) \), \( \gamma \geq 1 \). Then 
the negative eigenvalues \( \{\epsilon_j\}_{j=1}^N \) of the operator \( H_D^\alpha \) satisfy the inequality

\[
\sum_{j=1}^N |\epsilon_j|^\gamma \leq \eta_\gamma^\alpha \sum_{n \in \mathbb{Z}} |b_n|^{1+1/2\alpha},
\] (11)

where

\[
\eta_\gamma^\alpha := \frac{2\alpha}{(2\alpha + 1)^{1/2\alpha}} \frac{\Gamma((4\alpha + 1)/2\alpha)}{\Gamma(1/2\alpha)} \frac{\Gamma(\gamma + 1)}{\Gamma((\gamma + 1)/2\alpha)}.
\] (12)

**Remark 3.** As the discrete spectrum of \( H_D^\alpha \) lies in \([-\infty, 0] \) and 
\([4^\alpha, \infty] \), we shift our operator to the left by \( 4^\alpha \) and by analogy have an estimate for the positive 
eigenvalues of that operator, 
thus immediately obtaining Corollary 4.

**Corollary 4.** Let \( b_n \geq 0 \), \( \{b_n\}_{n \in \mathbb{Z}} \in \ell^{1+1/2\alpha}(\mathbb{Z}) \), \( \gamma \geq 1 \). Then 
the positive eigenvalues \( \{\epsilon_j\}_{j=1}^N \) of the operator \( \Delta_D - 4^\alpha + b \) satisfy the inequality:

\[
\sum_{j=1}^N \epsilon_j^\gamma \leq \eta_\gamma^\alpha \sum_{n \in \mathbb{Z}} |b_n|^{1+1/2\alpha},
\] (13)

Finally we will apply these results to obtain spectral bounds for the following operator.

We let \( W_\sigma \) be a polydiagonal self-adjoint Jacobi-type 
matrix operator as follows:

\[
W_\sigma := \begin{pmatrix}
\begin{array}{cccc}
\alpha & 0 & 0 & \cdots \\
0 & \alpha & 0 & \cdots \\
0 & 0 & \alpha & \cdots \\
\cdots & \cdots & \cdots & \cdots
\end{array}
\end{pmatrix},
\] (14)

viewed as an operator acting on \( \ell^2(\mathbb{Z}) \) as follows: for \( n \in \mathbb{Z} \), 
\( i \in [1, \ldots, \sigma] \),

\[
(W_\sigma \psi)(n) = \sum_{i=1}^{\sigma} a_n^i \psi(n - i) + b_n \psi(n) + \sum_{i=1}^{\sigma} a_n^i \psi(n + i)
\]

\[
= a_n^0 \psi(n - \sigma) + \cdots + a_n^{\sigma-1} \psi(n - 1) + b_n \psi(n) + a_n^1 \psi(n + 1) + \cdots + a_n^{\sigma} \psi(n + \sigma),
\] (15)

where \( a_n^0, b_n \in \mathbb{R} \), for all \( i \in [1, \ldots, \sigma] \). We denote \( (W_\sigma \{a_n^i\}, \ldots, a_n^{\sigma} \}, \{b_n\}) \psi(n) := \psi(n) \) where we understand \( [\cdot] \) to mean \( [\cdot]_{n \in \mathbb{Z}} \). We are then interested in perturbations of the following special case:

\[
(W_\sigma^\alpha \psi)(n) := \left( W_\sigma \left( \{a_n^i \equiv \omega_1 \}, \ldots, \{a_n^i \equiv \omega_\sigma \}, \{b_n = 0\} \right) \right) \psi(n),
\] (16)
where \( \omega_i := 2\sigma C_\sigma \omega_i \), and explicitly
\[
(W_0^0 \phi) (n) = \left( (\Delta_D - 2\sigma C_\sigma) \phi \right) (n)
= \sum_{k=0}^{2\sigma} C_k (-1)^{k+\sigma} (n - \sigma + k),
\]  
(17)
called the free Jacobi-type matrix of order \( \sigma \). In particular, we examine the case where \( W_\sigma - W_0 \) is compact. Thus in what follows we assume that our sequences tend to the operator coefficients rapidly enough; that is, \( d_n \to \omega, b_n \to 0 \), as \( n \to \pm \infty \). Then the essential spectrum \( \zeta_{ess} \) is given by \( \zeta_{ess}(W_\sigma) = \zeta_{ess}(W_0^0) = [2\sigma^2 C_\sigma, 4\sigma^2 - 2\sigma^2 C_\sigma] \) and \( W_\sigma \) may have simple eigenvalues \( \{ E_j \}_{j=1}^N \) where \( \{ E_j \}_{j=1}^N \) and
\[
E_1 > E_2 > \ldots > 4\sigma^2 - 2\sigma^2 C_\sigma > \ldots > E_m > E_1.
\]  
(18)

**Theorem 5.** Let \( \gamma \geq 1 \), \( \{ b_n \}_{n \in \mathbb{Z}} \), and \( \{ a_n - \omega \}_{n \in \mathbb{Z}} \in \ell^{(1+2\sigma)/2}(\mathbb{Z}) \) for all \( i \in \{1, \ldots, \sigma\} \). Then for the eigenvalues \( \{ E_j \}_{j=1}^N \) of the operator \( W_\sigma \) we have
\[
\sum_{j=1}^N |E_j + 2\sigma C_\sigma|^\gamma = \sum_{j=1}^N (E_j + (4\sigma^2 - 2\sigma^2 C_\sigma))^{\gamma/2}
\leq \gamma \left( \sum_{n \in \mathbb{Z}} |b_n|^{\gamma/2 + \sigma} + 4 \sum_{n \in \mathbb{Z}} \sum_{k=1}^\sigma (a_k - \omega_k)^{\gamma/2 + \sigma} \right),
\]  
where
\[
\gamma = 2\sigma(2\sigma + 1)^{\gamma/2 - 1} \Gamma(4\sigma + 1)/\Gamma(2\sigma + 1)\Gamma(\gamma + 1)/\gamma(\gamma + 1/2\sigma).
\]  
(19)

**4. Auxiliary Results**

We require the following discrete Kolmogorov-type inequality.

**Lemma 6.** For a sequence \( \phi \in \ell^2(\mathbb{Z}) \), and for \( n > k \geq 1 \), we have the following inequality:
\[
\| D^k \phi \|_{\ell^2(\mathbb{Z})} \leq \| \phi \|_{\ell^2(\mathbb{Z})}^{1-k/n} \| D^k \phi \|_{\ell^2(\mathbb{Z})}^{k/n}.
\]  
(20)

**Proof.** We proceed by induction, where we note that the initial case, \( k = 1, n = 2 \), holds true as the inequality
\[
\| D \phi \|_{\ell^2(\mathbb{Z})} \leq \| \phi \|_{\ell^2(\mathbb{Z})}^{1/2} \| D \phi \|_{\ell^2(\mathbb{Z})}^{1/2}
\]  
(21)
is in fact the simple inequality found by Copson in [7]. This case in turn, if used repeatedly, shows that the inequality holds true for all \( k, n \), if \( n = k+1 \). We then take the inductive step on the variable \( n \). Hence we assume that we have the required inequality for \( k < n \leq m \), given a fixed \( k \), and proceed to prove the statement for \( n = m + 1 \). Thus
\[
\| D^m \phi \|_{\ell^2(\mathbb{Z})}^2 = \langle D^m \phi, D^m \phi \rangle = \langle D^* D^m \phi, D^{m-1} \phi \rangle
\leq \| D^{m+1} \phi \|_{\ell^2(\mathbb{Z})} \| D^{m-1} \phi \|_{\ell^2(\mathbb{Z})},
\]  
(22)

We thus apply our induction hypothesis and set \( k = m - 1 \) and \( n = m \) as follows:
\[
\| D^m \phi \|_{\ell^2(\mathbb{Z})}^{2} \leq \| D^{m+1} \phi \|_{\ell^2(\mathbb{Z})} \| D^m \phi \|_{\ell^2(\mathbb{Z})}^{m-1/m} \| D^{m-1} \phi \|_{\ell^2(\mathbb{Z})}^{1/m}
\Rightarrow \| D^m \phi \|_{\ell^2(\mathbb{Z})} \leq \| D^{m+1} \phi \|_{\ell^2(\mathbb{Z})}^{m/(m+1)} \| D^{m-1} \phi \|_{\ell^2(\mathbb{Z})}^{1/(m+1)}.
\]  
(23)

We now return to the induction hypothesis as follows:
\[
\| D^k \phi \|_{\ell^2(\mathbb{Z})} \leq \| \phi \|_{\ell^2(\mathbb{Z})}^{1-k/n} \| D^k \phi \|_{\ell^2(\mathbb{Z})}^{k/n}
\]  
(24)

**Proposition 7.** For a sequence \( \phi \in \ell^2(\mathbb{Z}) \), we have for any \( \sigma \in \mathbb{N} \)
\[
\| \phi \|_{\ell^2(\mathbb{Z})} \leq \| \phi \|_{\ell^2(\mathbb{Z})}^{1-\sigma/n} \| D^\sigma \phi \|_{\ell^2(\mathbb{Z})}^{\sigma/n}.
\]  
(25)

**Proof.** First we use Lemma 6 with \( k = 1, n = \sigma \) as follows:
\[
\| D \phi \|_{\ell^2(\mathbb{Z})} \leq \| \phi \|_{\ell^2(\mathbb{Z})}^{1-\sigma} \| D^\sigma \phi \|_{\ell^2(\mathbb{Z})}^\sigma,
\]  
(26)

and we apply this estimate to the well-known discrete Agmon inequality (see [4]):
\[
\| \phi \|_{\ell^2(\mathbb{Z})} \leq \| \phi \|_{\ell^2(\mathbb{Z})}^{1-\sigma} \| D^\sigma \phi \|_{\ell^2(\mathbb{Z})}^\sigma
\]  
(27)

We are now equipped to prove an Agmon-Kolmogorov-type inequality.

**Proposition 8.** Let \( \{ \psi_j \}_{j=1}^N \) be an orthonormal system of sequences in \( \ell^2(\mathbb{Z}) \); that is, \( \langle \psi_j, \psi_k \rangle = \delta_{jk} \), and let \( \rho(n) := \sum_{j=1}^N |\psi_j(n)|^2 \). Then
\[
\sum_{n \in \mathbb{Z}} \rho(2\sigma+1)(n) \leq \sum_{n=1}^N \sum_{k=1}^N \| D^\sigma \psi_j(n) \|^2.
\]  
(28)

**Proof.** Let \( \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in C^N \). By Proposition 7, we have
\[
\sum_{j=1}^N \xi_j^\sigma |\psi_j(n)|^2 \leq \sum_{j=1}^N \xi_j^\sigma \psi_j(n)^2 \leq \sum_{j=1}^N \xi_j^\sigma \psi_j(n)^2 = \sum_{j=1}^N \xi_j^\sigma \left( \psi_j(n) \right)^{(2\sigma-1)/2\sigma},
\]  
(29)

\[
\sum_{j=1}^N \xi_j^\sigma \psi_j(n) \leq \sum_{j=1}^N \xi_j^\sigma \psi_j(n)^2 = \sum_{j=1}^N \xi_j^\sigma \left( \psi_j(n) \right)^{(2\sigma-1)/2\sigma},
\]  
(30)

\[ \times \left( \sum_{j=1}^{N} \xi_j \overline{F}_j \left(D^\sigma \psi_j, D^\sigma \psi_k \right) \right)^{1/2\sigma} \]
\[ \leq \left( \sum_{j=1}^{N} \xi_j \right)^{(2\sigma - 1)/2\sigma} \times \left( \sum_{j=1}^{N} \xi_j \overline{F}_j \left(D^\sigma \psi_j, D^\sigma \psi_k \right) \right)^{1/2\sigma} \]  
(30)

Let \( \xi_j := \psi_j(n) \) and as \( \rho(n) = \sum_{j=1}^{N} |\psi_j(n)|^2 \),
\[ \rho^2(n) \leq \rho^{(2\sigma - 1)/2\sigma}(n) \]
\[ \times \left( \sum_{j=1}^{N} \psi_j(n) \overline{\psi}_j(n) \left(D^\sigma \psi_j, D^\sigma \psi_k \right) \right)^{1/2\sigma} \]
\[ \Rightarrow \rho^{2\sigma + 1}(n) \leq \sum_{j=1}^{N} \psi_j(n) \overline{\psi}_j(n) \left(D^\sigma \psi_j, D^\sigma \psi_k \right) \]
(31)
\[ \Rightarrow \sum_{n \in \mathbb{Z}} \psi_j(n) \overline{\psi}_j(n) \left(D^\sigma \psi_j, D^\sigma \psi_k \right) \]

5. Proof of Theorem 2

We take the inner product with \( \psi_j(n) \) on (10) and sum both sides of the equation with respect to \( j \). We obtain
\[ \sum_{j=1}^{N} e_j = \sum_{j=1}^{N} \left( \sum_{n \in \mathbb{Z}} |D^\sigma \psi_j(n)|^2 \right) - \sum_{j=1}^{N} \left( \sum_{n \in \mathbb{Z}} b^\sigma \psi_j(n)^2 \right) \]  
(32)

We now use Proposition 8 and apply the appropriate Hölder's inequality; that is,
\[ \sum_{j=1}^{N} e_j \geq \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^{N} |\psi_j(n)|^2 \right)^{2\sigma + 1}/(2\sigma + 1) \]
\[ - \left( \sum_{n \in \mathbb{Z}} b^{(2\sigma + 1)/2\sigma} \right)^{2\sigma / (2\sigma + 1)} \]
\[ \times \left( \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^{N} |\psi_j(n)|^2 \right)^{2\sigma + 1}/(2\sigma + 1) \right)^{1/(2\sigma + 1)} \]  
(33)

We define
\[ \chi := \left( \sum_{n \in \mathbb{Z}} \left( \sum_{j=1}^{N} |\psi_j(n)|^2 \right)^{2\sigma + 1}/(2\sigma + 1) \right)^{1/(2\sigma + 1)} \]
(34)

The latter inequality can be written as
\[ \chi^{2\sigma + 1} - c \chi \leq \sum_{j=1}^{N} e_j \] 
(35)

The LHS is maximal when
\[ \chi = \frac{1}{(2\sigma + 1)^{1/(2\sigma + 1)}} \left( \sum_{n \in \mathbb{Z}} b_n^{(2\sigma + 1)/2\sigma} \right)^{1/(2\sigma + 1)} \]  
(36)

Substituting this into (33), we obtain
\[ \sum_{j=1}^{N} e_j \geq \frac{1}{(2\sigma + 1)^{1/(2\sigma + 1)}} \sum_{n \in \mathbb{Z}} b_n^{(2\sigma + 1)/2\sigma} \]
\[ - \frac{1}{(2\sigma + 1)^{1/(2\sigma + 1)}} \sum_{n \in \mathbb{Z}} b_n^{(2\sigma + 1)/2\sigma} \]
\[ = -\frac{2\sigma}{(2\sigma + 1)^{2\sigma}} \sum_{n \in \mathbb{Z}} b_n^{(2\sigma + 1)/2\sigma} \] 
(37)

Therefore,
\[ \sum_{n \in \mathbb{Z}} b_n \leq \frac{2\sigma}{(2\sigma + 1)^{2\sigma}} \sum_{n \in \mathbb{Z}} b_n^{(2\sigma + 1)/2\sigma} \]  
(38)

We lift this bound now with regard to moments by using the standard Aizenman-Lieb procedure (see [8]). We let \( \{e_j(\tau)\}_{j=1}^{N} \) be the negative eigenvalues of the operator \( \Delta^\sigma_D - (b_n - \tau)_+ \). By the variational principle for the negative eigenvalues \( \{-|e_j|\}_{j=1}^{N} \) of the operator \( \Delta^\sigma_D - (b_n - \tau) \) we have
\[ (|e_j| - \tau)_+ \leq |e_j(\tau)| \]  
(39)

By this estimate, we find that
\[ \sum_{j=1}^{N} |e_j| \leq \frac{1}{\mathcal{B}(y - 1/2)} \int_{0}^{\infty} \tau^{2\sigma} \sum_{j=1}^{N} |e_j| - \tau, d\tau \]
\[ \leq \frac{1}{\mathcal{B}(y - 1/2)} \int_{0}^{\infty} \tau^{2\sigma} \sum_{j=1}^{N} |e_j(\tau)|, d\tau \]
\[ \leq \frac{2\sigma}{(2\sigma + 1)^{1/(2\sigma + 1)}} \mathcal{B}(y - 1/2) \]
\[ \times \int_{0}^{\infty} \tau^{2\sigma} \sum_{n \in \mathbb{Z}} (b_n - \tau), d\tau, \]
(40)

by (38) above, where \( \mathcal{B}(x, y) = \Gamma(x) \Gamma(y) / \Gamma(x + y) \) is the well-known Beta function. Thus, after a change of variable,
\[ \sum_{j=1}^{N} |e_j| \leq \frac{2\sigma}{(2\sigma + 1)^{1/(2\sigma + 1)}} \]
\[ \times \Gamma((4\sigma + 1)/2\sigma) \Gamma(y + 1) \]
\[ \Gamma(y + (2\sigma + 1)/2\sigma) \sum_{n \in \mathbb{Z}} b_n^{y + 1/2\sigma}, \]  
(41)

completing our proof.
6. Proof of Theorem 5

We have the following matrix bounds for square, \( m \times m \) matrices, as given in [2]. For \( a^m_i, \omega_m \in \mathbb{R} \), we have
\[
\begin{pmatrix}
-|a^m_i - \omega_m| & 0 & \cdots & 0 & \omega_m \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
\omega_m & 0 & \cdots & 0 & |a^m_m - \omega_m|
\end{pmatrix} 
\leq
\begin{pmatrix}
0 & 0 & \cdots & 0 & a^m_i \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
a^m_m & 0 & \cdots & 0 & 0
\end{pmatrix}
\leq
\begin{pmatrix}
|a^m_i - \omega_m| & 0 & \cdots & 0 & \omega_m \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
\omega_m & 0 & \cdots & 0 & |a^m_m - \omega_m|
\end{pmatrix}
\]

We thus use this on each block of indices of \( W_\sigma \) as follows:
\[
W_\sigma \left( \{a^\sigma_i \equiv \omega_1\}, \dots, \{a^\sigma_i \equiv \omega_q\}, \{b^{(-)}_n\} \right) 
\leq W_\sigma \left( \{a^\sigma_i \equiv \omega_1\}, \{b_n\} \right) 
\leq W_\sigma \left( \{a^\sigma_i \equiv \omega_1\}, \{a^\sigma_i \equiv \omega_q\}, \{b^{(+)}_n\} \right) ,
\]
where \( b^{(+)}_n \) is given by
\[
b^{(+)}_n = b_n \pm \left( (|a^1_{n-1} - \omega_1| + |a^\sigma_i - \omega_1|) \right.
\]
\[
+ \cdots + (|a^\sigma_i - \omega_q| + |a^\sigma_i - \omega_q|) \right)
\]
that is,
\[
b^{(+)}_n = b_n \pm \left( \sum_{k=1}^\sigma |a^k_{n-k} - \omega_k| + |a^\sigma_i - \omega_k| \right)
\]

Now we relate these to our Schrödinger-type operators:
\[
\Delta^\sigma_D - 4^\sigma + b_n = W_\sigma \left( \{a^\sigma_i \equiv \omega_1\}, \dots, \{a^\sigma_i \equiv \omega_q\} \right) 
\leq W_\sigma \left( \{a^\sigma_i \equiv \omega_1\}, \{a^\sigma_i \equiv \omega_q\}, \{b^{(+)}_n\} \right) ,
\]
where \( \Delta^\sigma_D + b_n = W_\sigma ^0 + 2^\sigma C_\sigma + b_n \)
\[
= W_\sigma \left( \{a^\sigma_i \equiv \omega_1\}, \dots, \{a^\sigma_i \equiv \omega_q\}, \{b^{(+)}_n\} \right) ,
\]
\[
\Delta^\sigma_D + b_n = W_\sigma ^0 + 2^\sigma C_\sigma + b_n
\]
\[
= W_\sigma \left( \{a^\sigma_i \equiv \omega_1\}, \dots, \{a^\sigma_i \equiv \omega_q\}, \{b^{(+)}_n\} \right) ,
\]
\[
\sum_{i=1}^{2\sigma+1} \frac{q}{(2\sigma+1)} \leq (2\sigma+1)^{q-1} \left( \sum_{i=1}^{2\sigma+1} \alpha_i^q \right) .
\]
to each of (51) and (53), we have
\[
\left( (b_n)_\pm + \sum_{k=1}^\sigma \left| (a_{n-k}^k - \omega_k) + \left| a_n^k - \omega_k \right| \right)^{\gamma+1/2}\right)
\leq (2\sigma + 1)^{-\gamma(2\sigma - 1)} / 2\sigma \\
\times \left( (b_n)_{\pm}^{\gamma+1/2} + \sum_{k=1}^\sigma \left| (a_{n-k}^k - \omega_k)^{\gamma+1/2} + \left| a_n^k - \omega_k \right|^{\gamma+1/2} \right)^. \tag{55}
\]

Summing these two inequalities, we arrive at
\[
\sum_{j=1}^{N_\sigma} \left| E_j^\sigma - 2\sigma C_\sigma \right|^{\gamma} + \sum_{j=1}^{N_\sigma} \left| E_j^\sigma - (4\sigma - 2\sigma C_\sigma) \right|^{\gamma} \\
\leq \nu_\sigma^\gamma \left( \sum_{n \in \mathbb{Z}} \left| b_n \right|^{\gamma+1/2} + 4 \sum_{n \in \mathbb{Z}} \sum_{k=1}^\sigma \left| a_n^k - \omega_k \right|^{\gamma+1/2} \right), \tag{56}
\]
where
\[
\nu_\sigma^\gamma = 2\sigma(2\sigma + 1)^{-\gamma-2} \frac{\Gamma \left( \left( 4\sigma + 1 \right) / 2\sigma \right) \Gamma \left( \gamma + 1 \right)}{\Gamma \left( \gamma + (2\sigma + 1) / 2\sigma \right)}, \tag{57}
\]
and the proof of Theorem 5 is complete.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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