Kinetic theory for particle production

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Abstract

Recently, the phenomenological description of cosmological particle production processes in terms of effective viscous pressures has attracted some attention. Using a simple creation rate model we discuss the question to what extent this approach is compatible with the kinetic theory of a relativistic gas. We find the effective viscous pressure approach to be consistent with this model for homogeneous spacetimes but not for inhomogeneous ones.

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1 Introduction

It is well known [1 - 3] and widely used [4 - 10] that particle production processes in the expanding Universe may be phenomenologically described in terms of effective viscous pressures. This is due to the simple circumstance that any source term in the energy balance of a relativistic fluid may be formally rewritten in terms of an effective bulk viscosity. The advantage of this rewriting is obvious. While an energy momentum balance with a nonzero source term violates the integrability conditions of Einstein’s equations, the energy momentum balance for a viscous fluid does not. Consequently, if it is possible to mimic particle production processes consistently by effective viscous pressures, one is able to study the impact of these processes on the cosmological dynamics.

However, the question remains whether an energy momentum tensor that formally has been made divergence free by regarding the original source term as an effective viscous pressure of the cosmic medium, adequately describes the physics of the underlying process. Is the dynamical effect of a viscous pressure in any respect equivalent to the corresponding effect of particle production? There is no definite answer to these questions at the present state of knowledge.

What one would like to have is a microscopic justification of this effective viscous pressure approach. The production processes one has in mind are supposed to play a role at times of the order of the Planck time or during the reheating phase of inflationary scenarios. It is not unlikely that particle or string creation out of the quantum vacuum has considerably influenced the dynamics of the early Universe [11, 10, 8]. A real microscopic description on the quantum level, however, does not seem to be available in the near future, and therefore it may be useful to study this phenomenon as far as possible on different classical levels. Taking into account that the dynamics of a fluid in or close to equilibrium may be derived from kinetic theory, one may ask to what extent the phenomenological approach of regarding the effects of particle production to be in a sense equivalent to those of viscous pressures may be understood and backed up on the level of kinetic theory. It is this question that we want to address in the present paper. Specifically, we shall model particle production processes by a nonvanishing source term in a Boltzmann type equation and establish the link between this kinetic description and the effective viscous pressure approach on the fluid level. It will turn out that the simple phenomenological approach is compatible with the kinetic theory in homogeneous spaces but not in inhomogeneous ones.
One may be reluctant, of course, to assume the applicability of a kinetic approach in the early Universe. A gas, however, is the only system for which the correspondence between microscopic variables, governed by a distribution function, and phenomenological fluid quantities is sufficiently well understood. All considerations of this paper refer to a model universe for which the kinetic approach is assumed to be applicable. It is the hope that this idealized model nevertheless shares at least some of the basic features of our real Universe.

For conventional collisions the standard kinetic theory (see, e.g., [12]) is valid if the particles interact only weakly, i.e., for systems that are not too dense. On the other hand, we know from the theory of the strong interaction that because of the property of ‘asymptotic freedom’ the concept of weakly interacting particles is especially appropriate for very dense systems. This might indicate that beyond its conventional range of application, e.g., under the conditions of the early Universe, the hypothesis of weakly interacting particles on the level of kinetic theory may be not too misleading as well.

In section 2 the basic elements of kinetic theory needed in our study are introduced. In section 3 our main result is derived in the context of an ‘effective rate approximation’, modeled after relaxation time approximations for the Boltzmann collision term. In section 4 we summarize our conclusions. Units have been chosen so that \( c = k_B = 1 \).

## 2 General kinetic theory

Our starting point is the assumption that a change in the number of particles of a relativistic gas should manifest itself in a source term \( H \) on the level of kinetic theory.

The corresponding one-particle distribution function \( f = f(x, p) \) of a relativistic gas with varying particle number is supposed to obey the equation

\[
L[f] \equiv p^i f,_{,i} - \Gamma^l_{ikl} p^l p^i \frac{\partial f}{\partial p^i} = C[f] + H(x, p),
\]

where \( f(x, p) p^k n_k d\Sigma dP \) is the number of particles whose world lines intersect the hypersurface element \( n_k d\Sigma \) around \( x \), having 4-momenta in the range \( (p, p + dp) \); \( i, k, l ... = 0, 1, 2, 3 \).

\[
dP = A(p) \delta \left( p^2 p_i + m^2 \right) dP_4 \]

is the volume element on the mass shell \( p^i p_i = -m^2 \) in the momentum space. \( A(p) = 2 \), if \( p^i \) is future directed and \( A(p) = 0 \) otherwise; \( dP_4 = \sqrt{-g} dp^1 dp^2 dp^3 dp^4 \).
$C[f]$ is the Boltzmann collision term. Its specific structure discussed e.g. by Ehlers [12] will not be relevant for our considerations. Following Israel and Stewart [13] we shall only require that (i) $C$ is a local function of the distribution function, i.e., independent of derivatives of $f$, (ii) $C$ is consistent with conservation of 4-momentum and number of particles, and (iii) $C$ yields a nonnegative expression for the entropy production and does not vanish unless $f$ has the form of a local equilibrium distribution (see [13]).

The term $H(x,p)$ on the r.h.s. of (1) takes into account the fact that the number of particles whose world lines intersect a given hypersurface element within a certain range of momenta may additionally change due to creation or decay processes, supposedly of quantum origin. On the level of classical kinetic theory we shall regard this term as a given input quantity. Later we shall give an example for the possible functional structure of $H(x,p)$.

By the splitting of the r.h.s of eq.(1) into $C$ and $H$ we have separated the collisional from the creation (decay) events. In this setting collisions are not accompanied by creation or annihilation processes. In other words, once created, the interactions between the particle are both energy-momentum and number preserving. Situations like these might be typical, e.g., for the creation of relativistic particles out of a decaying scalar field during the reheating phase of standard inflationary universes [8].

For a vanishing $H$ eq.(1) reduces to the familiar Boltzmann equation (see, e.g., [12 - 14]). In the latter case there exist elaborate solution techniques that allow to characterize both the equilibrium and nonequilibrium phenomena, connected with entropy production. In the present paper we shall not study the entropy production due to collisions within the fluid represented by the term $C[f]$ on the r.h.s. of (1), although the corresponding effects may be additionally included at any stage of our presentation in an obvious way. Instead, we shall focus here on the entropy production due to the source term $H(x,p)$ in equation (1).

The particle number flow 4-vector $N^i$ and the energy momentum tensor $T^{ik}$ are defined in a standard way (see, e.g., [12]) as

\[ N^i = \int dP p^i f(x,p) , \quad (2) \]

and

\[ T^{ik} = \int dP p^i p^k f(x,p) . \quad (3) \]

The integrals in (2) and (3) and throughout this paper are integrals over the entire mass shell $p^i p_i = -m^2$. The entropy flow vector $S^a$ is given by [12,
where bosons and fermions are characterized by \( \varepsilon = 1 \) and \( \varepsilon = -1 \), respectively. For \( \varepsilon = 0 \) one recovers the classical Maxwell-Boltzmann case.

Using the general relationship \([15]\)

\[
\left[ \int p^a \ldots p^a p^b f dP \right]_{ab} = \int p^a \ldots p^a L[f] dP
\]

and eq.(1) we find

\[
N^a_{\alpha} = \int (C[f] + H) \, dP ,
\]

\[
T^a_{\alpha k} = \int p^a (C[f] + H) \, dP ,
\]

and

\[
S^a_{\alpha} = -\int \ln \left( \frac{f}{1 + \varepsilon f} \right) (C[f] + H) \, dP .
\]

In collisional equilibrium, which we shall assume from now on, \( \ln \left[ f / (1 + \varepsilon f) \right] \) in (8) is a linear combination of the collision invariants 1 and \( p^a \). The corresponding equilibrium distribution function becomes (see, e.g., \([12]\))

\[
f^0 (x, p) = \frac{1}{\exp [-\alpha - \beta_a p^a] - \varepsilon} ,
\]

where \( \alpha = \alpha (x) \) and \( \beta_a \) is a timelike vector that depends on \( x \) only. Inserting the equilibrium function into eq.(1) one gets

\[
\left[ p^a \alpha_{\alpha a} + \beta_{(a;b)} p^a p^b \right] f^0 \left( 1 + \varepsilon f^0 \right) = H (x, p) .
\]

It is well known \([12, 13]\) that for \( H = 0 \) this equation, which characterizes the ‘global equilibrium’, admits solutions only for very special cases in which \( \alpha = const \) and \( \beta_a \) is a timelike Killing vector. Below we shall discuss the corresponding conditions for a specific functional form of \( H \) in the classical case.

With (9), the balances (8) and (9) reduce to

\[
N^a_{\alpha} = \int H dP ,
\]

and

\[
T^a_{\alpha k} = \int p^a H dP ,
\]
respectively.
Neither \( N^a \) nor \( T^{ak} \) are conserved. The nonconservation of the number of particles, the phenomenon under consideration here, is accompanied by the existence of a source term in the energy-momentum balance as well. A nonconserved energy-momentum tensor \( T^{ak} \), however, is incompatible with Einstein’s field equations. But one may raise the question whether it is possible to rewrite eq. (12) as

\[
T^{ak}_{;k} - \int p^a H \, dP \equiv \dot{T}^{ak}_{;k} = 0,
\]

with an effective energy-momentum tensor \( \dot{T}^{ik} \) that is conserved and, consequently, is a suitable quantity in Einstein’s equations. Is is basically this problem that we are going to investigate below.

In collisional equilibrium there is entropy production only due to the source term \( H \). From eq. (8) we obtain

\[
S^a_{;a} = - \int H(x, p) \ln \left( \frac{f^0}{1 + \varepsilon f^0} \right) \, dP,
\]

implying

\[
S^a_{;a} = -\alpha N^a_{;a} - \beta_a T_{ab}^{;b}.
\]

Condition (14) may be imposed or not. The first possibility corresponds to the ‘global equilibrium’ case for \( H = 0 \), the second one to the ‘local equilibrium’ case. In collisional equilibrium with \( f \) replaced by \( f^0 \) in (2), (8) and (4), \( N^a, T^{ab} \) and \( S^a \) may be split with respect to the unique 4-velocity \( u^a \) according to

\[
N^a = nu^a,
\]

and

\[
T^{ab} = \rho u^a u^b + p h^{ab},
\]

with the spatial projection tensor \( h^{ab} = g^{ab} + u^a u^b \) and

\[
S^a = nsu^a,
\]

where \( n \) is the particle number density, \( \rho \) is the energy density, \( p \) is the equilibrium pressure and \( s \) is the entropy per particle. The exact integral expressions for \( n, \rho, p \) and \( s \) are given by the formulae (177) - (180) in [12]. According to our general setting described above we assumed in establishing (15) - (18) that the source term \( H \) does not affect the collisional equilibrium.
Using (14) and defining
\[ \Gamma \equiv \frac{1}{n} \int H(x,p) \, dP , \]  
(eq.(11) becomes
\[ \dot{n} + \Theta n = n \Gamma . \]  
It is obvious that \( \Gamma \) is the particle production rate. Similarly, with the decomposition (17) and the abbreviation
\[ t^a \equiv -\int p^a H(x,p) \, dP , \]
the balance equations for energy and momentum may be written as
\[ \dot{\rho} + \Theta (\rho + p) - u_a t^a = 0 , \]
and
\[ (\rho + p) \dot{u}^a + p_n h_{an} + h_{an} t^a = 0 , \]
respectively, where \( \Theta = u_{ia}^a \) is the fluid expansion.

From the Gibbs equation
\[ T \, ds = d\frac{\rho}{n} + pd\frac{1}{n} \]  
(24)
together with (20) and (22) we find
\[ nT \dot{s} = u_a t^a - (\rho + p) \Gamma . \]  
(25)
Inserting (11) and (12) with (19) and (21) into (13), the entropy production density is
\[ S_{m;n}^m = -\alpha n \Gamma + \beta_a t^a . \]  
(26)
Using [13]
\[ S^m = p\beta^m - \alpha N^m - \beta_a T^ma , \]
which follows from (4) with (3), as well as \( \beta_a = \beta u_a \), and (18), one finds [12]
\[ n \dot{s} = \beta (\rho + p) - \alpha n , \]  
(28)
and (26) may be rewritten to yield
\[ S_{m;n}^m = n \Gamma s + n \dot{s} , \]  
(29)
where we have used (25) and \( \beta \equiv T^{-1} \). The latter expression for the entropy production density coincides with the result that follows from differentiating (18). The first term on the r.h.s. of (29) describes the entropy production due to the enlargement of the phase space. The second one takes into account the contribution due to a possible change in the entropy per particle.

Henceforth we shall assume, that all particles are amenable to a perfect fluid description immediately after their creation. In this case the entropy per particle does not change, i.e., \( \dot{s} = 0 \). All particles are created with a fixed entropy. Of course, it is always possible to include dissipative effects in a standard way [16].

According to (25) the case \( \dot{s} = 0 \) is equivalent to

\[
u_a t^a = (\rho + p) \Gamma ,
\]

relating the source term in the energy balance to that in the particle number balance. The entropy production density (29) reduces to

\[
S_{mn}^m = n \Gamma \dot{s} .
\]

In the perfect fluid case there is entropy production due to the increase in the number of particles, i.e., due to the enlargement of the phase space.

Introducing the quantity

\[
\pi \equiv -\frac{\nu_a t^a}{\Theta} ,
\]

it is possible to rewrite the energy balance (22) as

\[
\dot{\rho} + \Theta (\rho + p + \pi) = 0 ,
\]

i.e., \( \pi \) enters the energy balance in the same way as a bulk viscous pressure does. By (30) \( \pi \) is related to the particle production rate in the case \( \dot{s} = 0 \). It had been the hope that a rewriting like this, corresponding to the introduction of an effectively conserved energy-momentum tensor (see (13))

\[
\hat{T}^{ik} = \rho u^i u^k + (p + \pi) h^{ik} ,
\]

instead of the nonconserved quantity \( T^{ik} \) of (17), allowed one to study particle production processes in terms of effective viscous pressures. Assuming an energy-momentum tensor of the structure (34) is equivalent to map the entire source term in (13) onto an effective bulk viscous pressure. It should be mentioned that this is not the only way to take into account the influence
of particle number nonconserving processes on the cosmological dynamics. Gariel and Le Denmat [17] have developed a different approach that regards the particle production rate as a thermodynamic flux on its own.

The whole procedure leading to (33) is rather formal. It is unclear, e.g., whether \( h^a_{n} t^n \) in (24) will admit a splitting

\[
 h^a_{n} t^n = \pi \dot{u}^a + \pi, n h^a n ,
\]

that would lead to

\[
 (\rho + p + \pi) \dot{u}^a + (p + \pi), n h^a n = 0 .
\]

If it were generally true that particle production may be modelled by an effective bulk viscosity, the splitting (35) should be possible.

To clarify this problem, specific expressions for \( H (x, p) \) in (1) have to be investigated, while up to this point all relations are valid for any \( H \).

3 Effective rate approximation

3.1 The basic concept

As was mentioned earlier, the quantity \( H (x, p) \) is an input quantity on the level of classical kinetic theory. \( H \) is supposed to represent the net effect of certain quantum processes with variable particle numbers (see, e.g., [18]) at the interface to the classical (nonquantum) level of description. Lacking a better understanding of these processes we shall assume that their influence on the distribution function \( f (x, p) \) may be approximately described by a linear coupling to the latter:

\[
 H (x, p) = \zeta (x, p) f^0 (x, p) .
\]

Let us further assume that \( \zeta \) depends on the momenta \( p^a \) only linearly:

\[
 \zeta = - \frac{u_a p^a}{\tau (x)} + \nu (x) .
\]

\( \zeta \), or equivalently \( \nu \) and \( \tau \), characterize the rate of change of the distribution function due to the underlying processes with variable particle numbers. The restriction to a linear dependence of \( \zeta \) on the momentum is equivalent to the requirement that these processes couple to the particle number flow vector and to the energy momentum tensor in the balances for these quantities.
only, but not to higher moments of the distribution function. This ‘effective rate approximation’ is modeled after the relaxation time approximations for the Boltzmann collision term [19-21]. While the physical situations in both approaches are very different, their common feature is the simplified description of nonequilibrium phenomena by a linear equation for the distribution function in terms of some effective functions of space and time that characterize the relevant scales of the process under consideration. In the relaxation time approximation this process is determined by the rate at which the system relaxes to an equilibrium state. In the present case the corresponding quantity is the rate by which the number of particles changes.

Analogously to the relaxation time model of Maartens and Wolvaardt [21] it is possible to find an exact formal solution of (3) with (37) and (38) as well. Equation (1) may be rewritten as

\[ L(F) = 0. \] (39)

The formal solution is

\[ F(x(\eta), p(\eta)) = f^0(x(\eta), p(\eta)) \exp[-g(\eta)], \] (40)

with

\[ g(\eta) = \int_{\eta}^{\eta'} d\eta' \zeta(x(\eta'), p(\eta')) , \] (41)

\( \eta \) being the proper time.

3.2 The Maxwell-Boltzmann case corresponding to ‘global equilibrium’

For a classical gas with \( \varepsilon = 0 \) the specific ansatz (37) with (38) allows us to find conditions on the parameters \( \alpha \) and \( \beta^a \) in (3) that replace the so called global equilibrium conditions (\( \alpha = \text{const.}, \beta^a = \text{timelike Killing vector} \)) in the case \( H = 0 \).

Condition (10) with \( \varepsilon = 0 \) becomes

\[ p^a \alpha_a + \beta_{(a;b)}p^a p^b = E + \nu, \] (42)

where \( E = -u_a p^a \). Decomposing \( p^a \) according to \( p^a = Eu^a + \lambda e^a \), where \( e^a \) is a unit spatial vector, i.e., \( e^a e_a = 1, u^a e_a = 0 \), the mass shell condition \( p^a p_a = -m^2 \) is equivalent to \( \lambda^2 = E^2 - m^2 \). The conditions on \( \alpha \) turn out to be

\[ \dot{\alpha} = \frac{1}{\tau}, \quad h^a_b \alpha_a = 0, \] (43)

10
while $\beta_a$ obeys the equation for a conformal Killing vector

$$\beta_{(a;b)} = \Psi(x) g_{ab},$$

and

$$m^2 \beta_{(a;b)} u^a u^b = \nu,$$

implying

$$\Psi = -\frac{\nu}{m^2}.$$  

Different from the case $H = 0$ where $\alpha$ has to be constant in space and time, $\alpha$ is only spatially constant in the present case but changes along the fluid flow lines. From (45) it follows that $\nu = 0$ for $m = 0$. The condition $\nu = 0$ for $m > 0$, however, is equivalent to $\beta_{(a;b)} = 0$, i.e., the corresponding spacetime is stationary. Using $\beta_a = u_a/T$ in (44) yields

$$u_{a;b} + u_{b;a} - \left( \frac{T}{T} u_a + \frac{T}{T} u_b \right) = 2T \Psi g_{ab}. \quad (47)$$

By scalar multiplication with $u^a$ one obtains

$$\dot{u}_b + \frac{T}{T} u_b - \frac{T}{T} u_b = 2T \Psi u_b, \quad (48)$$

and projecting the latter equation in direction of $u^b$, the relation

$$\frac{T}{T} = -\Psi T, \quad (49)$$

results [22].

On the other hand, the trace of (47) is equivalent to

$$\Theta - \frac{T}{T} = 4T \Psi. \quad (50)$$

Combining (49) and (50) we find

$$\Theta = -3 \frac{T}{T}. \quad (51)$$

Projecting now (48) orthogonally to $u^a$ provides us with the following representation of the acceleration in terms of the spatial temperature gradient:

$$\dot{u}_a = -(\ln T)_b h^b_a. \quad (52)$$

This relation for a Maxwell-Boltzmann gas has an interesting consequence that will be discussed at the end of the following section.
3.3 The balance equations

Coming back now to the general case that encompasses quantum gases as well we realize that with (2), (37) and (38) the particle creation rate (19) is given by

\[ n\Gamma = -\frac{u_a}{\tau}N_a + Q(x) \]  (53)

for our model, where

\[ Q = \nu(x)M, \]  (54)

and \( M \) is the zeroth moment of the distribution function:

\[ M \equiv \int dP f(x, p). \]  (55)

With (53) the particle number balance eq. (20) may be written in terms of \( \tau \) and \( \nu \):

\[ \dot{n} + n\Theta = \frac{n}{\tau} + Q. \]  (56)

Similarly, (2), (3), (37), (38) and (21) yield

\[ -t^a = -\frac{u_aT^{ca}}{\tau(x)} + \nu(x)N^a \]  (57)

for the source term in the energy momentum balance, or, with (32),

\[ \Theta_{\pi} = -\frac{\rho}{\tau} - \nu n. \]  (58)

It is obvious from (53) and (57) that our effective rate approximation provides us with a definite coupling between the functions \( \tau \) and \( \nu \) that represent the microscopic production process on a macroscopic level, and the components of the particle number flow vector \( N^i \) and the energy momentum tensor \( T^{ik} \). The expression (38) for \( \zeta \) ensures that the source terms (53) and (57) depend on \( M, N^i \) and \( T^{ik} \) only, but not on higher moments of the distribution function.

With (32) and (58) the energy balance (22) becomes

\[ \dot{\rho} + \Theta (\rho + p) = \frac{\rho}{\tau} + \nu n. \]  (59)

From (25) with (53) and (57) it follows that \( s \) may be expressed in terms of \( \tau \) and \( \nu \):

\[ nT\dot{s} = \nu n - \frac{\rho + p}{n} Q - \frac{p}{\tau}. \]  (60)
Applying all the steps following (25) until (29) with (53) for $\Gamma$ and (57) for $t^\alpha$ we arrive at

$$S_{m}^{m} = -\frac{u_{i}S^{i}}{\tau} + Qs + n\dot{s}.$$ (61)

By virtue of (18) this is equivalent to (29) with $\Gamma$ from (53).

Focusing on the perfect fluid case $\dot{s} = 0$ again, we find that with (60) and (54) the general relation (30) in our specific model reduces to

$$\nu n = \frac{P}{\tau \left(1 - \frac{\rho + P}{\rho M} M_{n}^{n} \right)}.$$ (62)

This equation provides us with a relation between the functions $\nu$ and $\tau$ in the ansatz (38) for $\zeta(x,p)$. At this stage it becomes clear why it is necessary to include a function $\nu$ in (38), although intuitively one might have started with $\nu = 0$. In the latter case the condition $\dot{s} = 0$ seems to require $p = 0$, i.e., our effective rate approximation seems to apply to a pressureless medium (dust) only if $\tau$ is assumed to be given. But $p = 0$ exactly is an unphysical limiting case (see the integral expression (179) for $p$ in [12]). Consequently, $\nu = 0$ and $\dot{s} = 0$ are only compatible for $\tau^{-1} = 0$. There is no particle production at all in this case.

For a Maxwell-Boltzmann gas ($\varepsilon = 0$) a further statement is possible if eq. (45) is imposed. A vanishing $m$ in (45) necessarily implies $\nu = 0$. The obvious conclusion is that massless particles cannot be produced under this requirement. Since the relation (51) already holds for massless particles with $H = 0$ [21], this result is hardly surprising. There is no corresponding restriction, however, in the local equilibrium-like case.

Combining (53) and (54) with (62), the production rate $\Gamma$ may be written in terms of $\tau$ as

$$\Gamma = \frac{1}{\tau} \frac{1 - \frac{\rho M}{\rho n M_{n}^{n}}}{1 - \frac{\rho + P}{\rho M} M_{n}^{n}}.$$ (63)

With (58) and (12) the effective viscous pressure $\pi$ in the energy balance (33) is given in terms of $\tau$ as well:

$$\pi = -\frac{\rho + P}{\Theta \tau} \frac{1 - \frac{\rho M}{\rho n M_{n}^{n}}}{1 - \frac{\rho + P}{\rho M} M_{n}^{n}}.$$ (64)

Only for $p << \rho$ the quantity $\Gamma$ approaches $\tau^{-1}$. For $\dot{s} = 0$ the expression (31) for the entropy production density reduces to (91) with $\Gamma$ given by (63).

We conclude that as far as the balances of particle number, energy and entropy are concerned, the description of particle production in terms of an
effective viscous pressure is backed up by our effective rate approximation of kinetic theory. This is no longer true, however, if the momentum balance (23) comes into play. Eqs. (16), (17) and (57) imply

\[ h^m_{\alpha} \epsilon^\alpha = 0. \]  

(65)

The momentum balance is completely unaffected by the particle production rate. Consequently, in this respect the particle production is \textit{not} equivalent to an effective viscous pressure. While the simple analogy holds in the homogeneous case, the matter is more subtle if there are nonvanishing spatial gradients and the fluid motion is no longer geodesic. If one wants to use effective viscous pressures nevertheless to characterize particle production processes this is consistent only under the additional assumption that the effective viscous pressure terms cancel in the momentum balance. It is obvious that this condition is equivalent to

\[ \pi \dot{u}^\alpha = - \pi_n h^{\alpha n}, \]  

(66)

since (23) in this case reduces to

\[ (\rho + p) \dot{u}^\alpha = -p_n h^{\alpha n}. \]  

(67)

Using this equation to eliminate \( \dot{u}^\alpha \) from (66) one gets

\[ \frac{\pi_n}{\pi} h^{\alpha n} = \frac{p_n}{\rho + p} h^{\alpha n}. \]  

(68)

While this condition is empty in a homogeneous spacetime it restricts a possible spatial dependence of \( \pi \). Especially, it states that a spatially independent \( p \) necessarily implies a spatially independent \( \pi \), i.e., a homogeneous creation rate. In other words, it is impossible to have a homogeneous equilibrium pressure and at the same time an effective viscous pressure due to particle production that has a nonvanishing spatial gradient.

Alternatively, the restriction on \( \pi \) may be interpreted geometrically. For a comoving observer the spatial part of the 4-acceleration becomes

\[ \ddot{u}^\mu = \Gamma^\mu_{\nu 0} \left( u^0 \right)^2, \quad (\mu, \nu, ... = 1, 2, 3). \]  

(69)

With

\[ u^0 = \frac{1}{\sqrt{-g_{00}}}, \]  

(70)
and for the rotation free case with
\[ \Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\nu} g_{00,\nu}, \] (71)
we have,
\[ \dot{u}^\mu = \frac{1}{2} g^{\mu\nu} \left[ \ln (-g_{00}) \right]_{,\nu}. \] (72)

On the other hand, (66) is equivalent to
\[ \dot{u}^\mu = -g^{\mu\nu} \left[ \ln (-\pi) \right]_{,\nu}. \] (73)

Consequently, the spatial dependence of \( \pi \) is completely determined by the spatial dependence of \( g_{00} \):
\[ -\pi = b(t) \sqrt{-g_{00}}, \] (74)
where \( b \) is an arbitrary function of the time. Using the relation (30) with (32), \( b \) may be expressed in terms of \( \Gamma \):
\[ b(t) = \frac{(\rho + p) \Gamma}{\Theta} \sqrt{-g_{00}}, \] (75)
or in terms of \( \tau \) if one uses (63).

Further relations hold in the Maxwell-Boltzmann case \( (\varepsilon = 0) \) corresponding to global equilibrium. Combining (52) and (67) we find
\[ \frac{T}{T} h_{\alpha}^{\beta} = \frac{p_{,\alpha}}{\rho + p} h_{\alpha}^{\beta}, \] (76)
additionally to (68). Likewise (52) together with (72) provides us with
\[ (T \sqrt{-g_{00}})_{,\nu} = 0 \] (77)
for a comoving observer. The latter formula for a Maxwell-Boltzmann gas replaces the Tolman relation \( (T \sqrt{-g_{00}})_{,n} = 0 \) if particles with \( m > 0 \) are produced.

### 3.4 Stephani universes and particle production

Finally, we shall briefly discuss our results concerning the effective viscous pressure in connection with recent attempts [23] to find a physical understanding of exact inhomogeneous solutions of Einstein’s field equations with
irrotational, shear-free, perfect fluid sources (Stephani-Barnes family [24, 25]).

Since there do not generally exist physically realistic equations of state for the latter family, Sussman [23] suggested a reinterpretation of these solutions replacing the perfect fluid source by a fluid with an isotropic bulk stress. The main advantage of this procedure lies in the introduction of an additional degree of freedom that might be helpful in finding reasonable equations of state.

We assume the energy-momentum tensor to be given by (34) where $\pi$ is to describe particle production processes in the manner discussed so far. While (34) was derived for a bounded system with a finite particle number we shall follow here the common procedure and apply a quantity like this to describe that part of the early Universe that developed into its presently visible part.

In comoving coordinates the metric of the Stephani-Barnes family is [22, 25]

$$ds^2 = -\left(\frac{L_0/L}{(\Theta/3)^2}\right)^2 dt^2 + L^{-2}(dx^2 + dy^2 + dz^2),$$

(78)

where $\Theta = \Theta(t)$ and $L = L(t, x^\alpha)$.

Generally, for a bulk viscous fluid the balances of energy and momentum are given by (33) and (36). If $\pi$ mimics the effect of matter creation then eq.(36) reduces to (67), implying (68). With

$$u^0 = \frac{\Theta}{3} \left(\frac{L_0}{L}\right)^{\frac{1}{3}},$$

(79)

and

$$u_\alpha = \left(\ln \frac{L_0}{L}\right)_{,\alpha},$$

(80)

(33) and (67) may be written as

$$\frac{1}{3} \frac{\rho_0}{L_{,0}/L} = -(\rho + p + \pi),$$

(81)

and

$$p_{,\alpha} = -(\rho + p) \left(\ln \frac{L_0}{L}\right)_{,\alpha},$$

(82)

respectively. According to the result for our effective rate approximation, $\pi$ enters the energy balance but not the momentum balance. At first sight this seems to be contradictory to Sussman’s procedure who replaced the $p$
of the perfect fluid source by \( p + \pi \) in both balance equations (his equations (3b) and (3c) in [23]). However, because of (68), there exists the additional relation
\[
\pi_{,\alpha} = -\pi \left( \ln \frac{L_0}{L} \right)_{,\alpha}.
\]
(D83)

Differentiating (81) with respect to \( x^\alpha \) and using (82) and (83) we find
\[
\frac{\partial}{\partial t}(\rho_{,\alpha} L^3) = 0,
\]
(84)
which is equivalent to
\[
L^3 \rho_{,\alpha} = f_{,\alpha},
\]
(85)
where \( f = f(x^\alpha) \) is an arbitrary function. Equation (85) is Sussman’s relation (3d) in [23]. Originally derived for a perfect fluid source, this result remains valid both if \( \pi \) is a ‘real’ bulk viscous pressure and if it mimics matter creation as in the present paper.

The basic intention of Sussman’s reinterpretation is to establish a ‘\( \gamma \)-law’ \( p = (\gamma - 1)\rho \) for the equilibrium pressure \( p \) and absorb all terms that do not fit into an equation of state like this into the quantity \( \pi \). With (84) the latter is then defined by
\[
\pi = -\gamma \rho - \frac{1}{3} \frac{\rho_0}{L_{,0}/L}.
\]
(86)
This is again a formal procedure and one has to clarify whether viscous pressures like this are physically meaningful. Let us consider as an example the subfamily of conformally flat (Petrov type-0) solutions, known as Stephani universes [24], with
\[
L = R^{-1}\left\{ 1 + \frac{1}{4} k(t)[(x - x_0(t))^2 + (y - y_0(t))^2 + (z - z_0(t))^2] \right\},
\]
(87)
where \( R(t), k(t), x_0(t), y_0(t) \) and \( z_0(t) \) are arbitrary functions of time. With a perfect fluid source, matter density and pressure are given by [26]
\[
\rho = 3C^2(t),
\]
(88)
and
\[
p = -3C^2(t) + 2CC_0 \frac{L}{L_{,0}},
\]
(89)
where \( C(t) \) is determined by

\[
k(t) = [C^2(t) - \frac{1}{9} \Theta^2(t)] R^2(t). \tag{90}
\]

The Stephani universes are solutions with a homogeneous \( \rho \) but with an inhomogeneous pressure for which, in general, there does not exist a ‘\( \gamma \)-law’. Applying Sussman’s procedure to the present case amounts to replacing \( p \) by \( p + \pi \) in eq.(89):

\[
p + \pi = -3C^2 + 2CC_0 \frac{L}{L_0}. \tag{91}
\]

Since \( \rho \) is homogeneous, establishing a ‘\( \gamma \)-law’ between \( p \) and \( \rho \) is equivalent to separate the homogenous part of the r.h.s of (91) and to relate all the inhomogeneities to \( \pi \). However, this is clearly inconsistent with (68), or (82) and (83) in our specific case.

We conclude that a physical interpretation of the Stephani universes with a ‘\( \gamma \)-law’ (\( \gamma = \text{constant} \)) and a viscous pressure that mimics particle production processes is impossible according to our ‘effective rate approximation’.

4 Concluding remarks

In the present paper we have investigated the question whether the widely used description of particle production processes in the early Universe in terms of effective viscous pressures is compatible with the kinetic theory of a simple quantum gas. Our conclusion is that this approach may be applied, provided the effective viscous pressure is subject to an additional condition which entirely fixes its spatial dependence. This condition is equivalent to the requirement that all effects due to particle production cancel in the momentum balance. The latter property is a consequence of a simple effective rate approximation to the source term in the Boltzmann equation that is supposed to describe particle production. While the condition on the spatial behaviour of the effective viscous pressure \( \pi \) is empty in a homogeneous universe, it excludes the possibility of models with an inhomogeneous \( \pi \) as long as the thermodynamic pressure \( p \) remains homogeneous. An explicit example has been given, showing that the Stephani universes are incompatible with a bulk viscous pressure associated to matter creation.

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