Extended QCD versus Skyrme-Faddeev Theory

W. S. Bae, Y.M. Cho, and S. W. Kimm
Department of Physics, College of Natural Sciences, Seoul National University
Seoul 151-742, Korea
ymcho@yongmin.snu.ac.kr

ABSTRACT

We discuss the physical impacts of the “Cho decomposition” (or the “Cho-Faddeev-Niemi-Shabanov decomposition”) of the non-Abelian gauge potential on QCD. We show how the decomposition makes a subtle but important modification on the non-Abelian dynamics, and present three physically equivalent quantization schemes of QCD which are consistent with the decomposition. In particular, we show that the decomposition enlarges the dynamical degrees of QCD by making the topological degrees of the non-Abelian gauge symmetry dynamical. Furthermore, with the decomposition we show that the Skyrme-Faddeev theory of non-linear sigma model and QCD have almost identical topological structures. In specific we show that an essential ingredient in both theories is the Wu-Yang type non-Abelian monopole, and that the Faddeev-Niemi knots of the Skyrme-Faddeev theory can actually be interpreted to describe the multiple vacua of the $SU(2)$ QCD. Finally we argue that the Skyrme-Faddeev theory is, just like QCD, a theory of confinement which confines the magnetic flux of the monopoles.

PACS numbers: 12.38.-t, 11.15.-q, 12.38.Aw, 11.10.Lm

I. INTRODUCTION

Recently Faddeev and Niemi have discovered three dimensional knot solutions in the Skyrme-Faddeev theory of non-linear sigma model [1], and made an interesting conjecture that the Skyrme-Faddeev action could be interpreted as an effective action for QCD in the low energy limit [2]. On the other hand, in the low energy limit QCD is believed to generate the confinement of color [3], which is supposed to take place through the monopole condensation [4,5]. This suggests that the Faddeev-Niemi conjecture and the confinement problem can not be discussed separately. To prove the Faddeev-Niemi conjecture one must construct the effective action of QCD from the first principles and produce the mass scale that the Skyrme-Faddeev action contains, which could demonstrate the dynamical symmetry breaking through the monopole condensation and triggers the confinement of color in QCD. This has been very difficult to do. Fortunately several authors have recently been able to argue that one can indeed derive a “generalized” Skyrme-Faddeev action from the effective action of QCD in the infra-red limit, at least in some approximation [6-8]. In fact there appears now a strong indication that the effective action of QCD does generate a monopole condensation, and a generalized Skyrme-Faddeev action could emerge as a crude approximation from the effective action near the vacuum condensation [9]. This strongly indicates that the Skyrme-Faddeev action is closely related to QCD in the infra-red limit.

In the discussion of the Faddeev-Niemi conjecture a gauge independent decomposition of the non-Abelian connection into the dual and valence potentials [4] has played an important role [6-8]. The purpose of this paper is to discuss the physical impacts of the decomposition on QCD in more detail. To do this we first clarify the considerable confusion and misunderstanding about the decomposition which has appeared in the literature recently [6-8]. With the clarification we show how the decomposition makes a subtle but important modification of the non-Abelian dynamics, and present three (physically equivalent) quantization schemes of QCD which are consistent with the decomposition. In particular we show that the decomposition enlarges the dynamical degrees of QCD by making the topological degrees of the non-Abelian gauge symmetry...
dynamical. This demonstrates that the decomposition has a deep impact on the non-Abelian dynamics.

With this we show how the decomposition allows us to reveal almost identical topological structures which exist between the Skyrme-Faddeev theory and QCD. Both theories have interesting topological structures. For example, the Skyrme-Faddeev theory has the topological knot solitons whose stability comes from the topological quantum number. This is because the non-linear sigma field of the knots defines the topological mapping \( \pi_3(S^2) \) from the (compactified) three-dimensional space \( S^3 \) to the internal space \( S^2 \). On the other hand it is well-known that the non-Abelian gauge theory has a similar topological structure. It allows infinitely many topologically distinct vacua which one can label with an integer, and that they are connected by the tunneling through the instantons \([10,11]\). In the \( SU(2) \) gauge theory the multiple vacua arise because the (time-independent) vacuum gauge potential could be interpreted to define the topological mapping \( \pi_3(S^3) \) from the (compactified) 3-dimensional space \( S^3 \) to the group space \( S^3 \), which could also be identified as the mapping \( \pi_3(S^2) \) through the Hopf fibering \([12,13]\). So one may wonder whether the topological structures in two theories have any common ground.

In this paper we show that indeed the two theories have almost identical topological structures. With the decomposition we establish that an essential ingredient in both theories is the Wu-Yang monopole, which plays an important role in both theories. Furthermore we demonstrate that the Faddeev-Niemi knots can actually be identified to describe the multiple vacua of \( SU(2) \) QCD.

With this observation we argue that the physical states of the Skyrme-Faddeev theory are not the monopoles, but the knots. This leads us to conjecture that the Skyrme-Faddeev theory (just like QCD) is another theory of confinement, where the non-linear self interaction of the theory confines the magnetic flux of the monopole-anti-monopole pairs to form the topologically stable knots. This confirms that the Skyrme-Faddeev non-linear sigma model is closely related to QCD more than one way.

II. ABELIAN PROJECTION AND VALENCE POTENTIAL

Our discussion is based on the gauge independent decomposition of the non-Abelian connection in terms of the restricted potential (i.e., the dual potential) of the maximal Abelian subgroup \( H \) of the gauge group \( G \) and the valence potential (i.e., the gauge covariant vector field) of the remaining \( G/H \) degrees \([4,5]\). Consider \( SU(2) \) QCD for simplicity. A natural way to make the decomposition is to introduce an isotriplet unit vector field \( \hat{n} \) which selects the “Abelian” direction (i.e., the color direction) at each space-time point, and to decompose the connection into the restricted potential (called the Abelian projection) \( \hat{A}_\mu \) which leaves \( \hat{n} \) invariant and the valence potential \( \vec{X}_\mu \) which forms a covariant vector field \([6,8]\),

\[
\begin{align*}
\vec{A}_\mu &= A_\mu \hat{n} - \frac{1}{g} \hat{n} \times A_\mu \hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu, \\
A_\mu &= \hat{n} \cdot \vec{A}_\mu, \\
\hat{n}^2 &= 1,
\end{align*}
\]

where \( A_\mu \) is the “electric” potential. Notice that the restricted potential is precisely the connection which leaves \( \hat{n} \) invariant under the parallel transport,

\[
\hat{D}_\mu \hat{n} = \partial_\mu \hat{n} + g \hat{A}_\mu \times \hat{n} = 0.
\]

Under the infinitesimal gauge transformation

\[
\delta \hat{n} = -\vec{\alpha} \times \hat{n}, \quad \delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha},
\]

one has
\[ \delta A_\mu = \frac{1}{g} \hat{n} \cdot \partial_\mu \hat{\alpha}, \quad \delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \hat{\alpha}, \]
\[ \delta \hat{X}_\mu = -\hat{\alpha} \times \hat{X}_\mu. \]  
(4)

This shows that \( \hat{A}_\mu \) by itself describes an \( SU(2) \) connection which enjoys the full \( SU(2) \) gauge degrees of freedom. Furthermore \( \hat{X}_\mu \) transforms covariantly under the gauge transformation. This confirms that our decomposition provides a gauge-independent decomposition of the non-Abelian potential into the restricted part \( \hat{A}_\mu \) and the gauge covariant part \( \hat{X}_\mu \).

We emphasize that the crucial element in our decomposition (1) is the restricted potential \( \hat{A}_\mu \). Once this part is identified, the expression follows immediately due to the fact that the connection space (the space of all gauge potentials) forms an affine space \[\mathbb{A}\]. Indeed the affine nature of the connection space guarantees that one can describe an arbitrary potential simply by adding a gauge-covariant piece \( \hat{X}_\mu \) to the restricted potential. Our decomposition (1), which has recently become known as the “Cho decomposition” \[\mathbb{C}\] or the “Cho-Faddeev-Niemi-Shabanov decomposition” \[\mathbb{F}\], was introduced long time ago in an attempt to demonstrate the monopole condensation in QCD \[\mathbb{G}\]. But only recently the importance of the decomposition in clarifying the non-Abelian dynamics has become appreciated by many authors \[\mathbb{H}\]. Indeed it is this decomposition which has played a crucial role to establish the possible connection between the Skyrme-Faddeev action and the effective action of QCD in the infra-red limit \[\mathbb{I}\], and the “Abelian dominance” in Wilson loops in QCD \[\mathbb{J}\].

To understand the physical meaning of our decomposition notice that the restricted potential \( \hat{A}_\mu \) actually has a dual structure. Indeed the field strength made of the restricted potential is decomposed as
\[ \hat{F}_{\mu\nu} = (F_{\mu\nu} + H_{\mu\nu})\hat{n}, \]
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad H_{\mu\nu} = -\frac{1}{2g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \hat{C}_\nu - \partial_\nu \hat{C}_\mu, \]
(5)

where \( \hat{C}_\mu \) is the “magnetic” potential \[\mathbb{K}\]. Notice that we can always introduce the magnetic potential (at least locally section-wise), because
\[ \partial_\mu \hat{H}_{\mu\nu} = 0 \quad (\hat{H}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} H_{\rho\sigma}). \]
(6)

This allows us to identify the non-Abelian monopole potential by
\[ \hat{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \]
(7)
in terms of which the magnetic field is expressed as
\[ \hat{H}_{\mu\nu} = \partial_\mu \hat{C}_\nu - \partial_\nu \hat{C}_\mu + g \hat{C}_\mu \times \hat{C}_\nu = -\frac{1}{g} \partial_\mu \hat{n} \times \partial_\nu \hat{n} = H_{\mu\nu} \hat{n}. \]
(8)

Another important feature of \( \hat{A}_\mu \) is that, as an \( SU(2) \) potential, it retains the full topological characteristics of the original non-Abelian potential. Clearly the isolated singularities of \( \hat{n} \) defines \( \pi_2(S^2) \) which describes the non-Abelian monopoles. Indeed \( \hat{A}_\mu \) with \( A_\mu = 0 \) and \( \hat{n} = \hat{r} \) (or equivalently, \( \hat{C}_\mu \) with \( \hat{n} = \hat{r} \)) describes precisely the Wu-Yang monopole \[\mathbb{L}\]. Besides, with the \( S^3 \) compactification of \( R^3 \), \( \hat{n} \) characterizes the Hopf invariant \( \pi_3(S^2) \cong \pi_3(S^3) \) which describes the topologically distinct vacua \[\mathbb{M}\]. This tells that the restricted gauge theory made of \( \hat{A}_\mu \) could describe the dual dynamics which should play an essential role in \( SU(2) \) QCD, which displays the full topological characters of the non-Abelian gauge theory \[\mathbb{N}\].

With (1) we have
\[ \tilde{F}_{\mu\nu} = \tilde{F}_{\mu\nu} + \partial_\mu \tilde{X}_\nu - \partial_\nu \tilde{X}_\mu + g \tilde{X}_\mu \times \tilde{X}_\nu, \]
(9)
\[ \hat{n} \cdot \hat{X}_\mu = 0, \quad \hat{n} \cdot \hat{D}_\mu \hat{X}_\nu = 0, \]  

so that the Yang-Mills Lagrangian is expressed as

\[ L = -\frac{1}{4} \hat{F}^2_{\mu\nu} = -\frac{1}{4} \hat{H}^2_{\mu\nu} - \frac{1}{4} (\hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu)^2 - \frac{g}{2} \hat{F}_{\mu\nu} \cdot (\hat{X}_\mu \times \hat{X}_\nu) \]
\[ - \frac{g}{4} (\hat{X}_\mu \times \hat{X}_\nu)^2 + \lambda (\hat{n}^2 - 1) + \lambda \mu \hat{n} \cdot \hat{X}_\mu, \]

where \( \lambda \) and \( \lambda \mu \) are the Lagrangian multipliers. (Notice that, since \( \hat{X}_\mu \) is covariant, one might be tempted to put in a mass term for the valence potential in the Lagrangian. As we will see later, however, this is not allowed because the mass term will spoil another type (the passive type) of gauge symmetry that the theory should satisfy. In any case this issue becomes irrelevant when the confinement sets in.) This shows that the Yang-Mills theory can be viewed as the restricted gauge theory made of the dual potential \( \hat{A}_\mu \), which has the valence gluon \( \hat{X}_\mu \) as its source \([4,5]\).

The equations of motion that one obtains from our Lagrangian by varying \( A_\mu, \hat{X}_\mu, \) and \( \hat{n} \), are given by

\[ \partial_\mu (\hat{F}_{\mu\nu} + \hat{H}_{\mu\nu} + \hat{X}_{\mu\nu}) = -g \hat{n} \cdot [\hat{X}_\mu \times (\hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu)], \]
\[ \hat{D}_\mu (\hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu) = g(\hat{F}_{\mu\nu} + \hat{H}_{\mu\nu} + \hat{X}_{\mu\nu}) \hat{n} \times \hat{X}_\mu. \]

where

\[ X_{\mu\nu} = g \hat{n} \cdot (\hat{X}_\mu \times \hat{X}_\nu). \]

Observe that the equation of motion (12) is identical to

\[ D_\mu \hat{F}_{\mu\nu} = 0. \]

In fact with (1) one could have obtained (12) directly from (14). This confirms that the decomposition does not change the dynamics of QCD at the classical level.

III. RE-INTERPRETATION OF NON-ABELIAN DYNAMICS

There is, however, a subtle but potentially disturbing feature in our Lagrangian. The decomposition (1) contains \( \hat{n} \) explicitly, which seems to imply that here we are introducing the \( \hat{n} \) field as an extra variable which was absent in QCD. In fact if one counts the independent degrees of our decomposition, one finds that on the left-hand side one has 12 degrees before the gauge fixing. But on the right-hand side one obviously has 14 [4 (for \( A_\mu \)) + 8 (for \( \hat{X}_\mu \)) + 2 (for \( \hat{n} \))] degrees. So it appears that with \( \hat{n} \) we have “two extra degrees” which we have to remove, if we could ever claim that our Lagrangian indeed describes QCD. This observation has created a considerable confusion in the literature, and recently has led many people to search for two extra constraints which can destroy the “two extra degrees” created by \( \hat{n} \) \([2,7,8]\).

In fact, some authors have advocated that one should impose the following extra constraint

\[ \hat{D}_\mu \hat{X}_\mu = 0, \]

or a similar constraint on \( \hat{X}_\mu \), to compensate the “two extra degrees” introduced by \( \hat{n} \) in our Lagrangian \([2,8]\). In the following we show that this is based on the misunderstanding of our decomposition, and argue that it is impossible to try to remove the “two extra degrees” imposing a constraint on \( \hat{X}_\mu \) by hand \([2,8]\). Indeed, the condition (15) is simply a necessary consistency
condition which one need to remove the unphysical degrees of $\vec{X}_\mu$ to keep it massless, but it has nothing to do with the “two extra degrees” introduced by $\hat{n}$. First, to see whether our Lagrangian really has two extra degrees, notice that the equations of motion (12) has no equation for $\hat{n}$ to satisfy. 

Even though the Lagrangian (11) contains $\hat{n}$ explicitly, the variation with respect to $\hat{n}$ does not create any new equation of motion for $\hat{n}$ at all. This means that our $\hat{n}$ can not be interpreted as a dynamical variable. This, of course, is what one should have expected. It must have been clear from (3) that our $\hat{n}$ is a gauge artifact which could be removed completely with a gauge transformation, at least locally section-wise. So, even with $\hat{n}$ in the Lagrangian, we have no extra degrees to remove, at least at the classical level. As for the other 12 variables $A_\mu$ and $\vec{X}_\mu$, they are obviously the legitimate dynamical variables of the $SU(2)$ gauge theory, as the above equation of motion testifies. Secondly, the fact that the connection space forms an affine space tells that $\vec{X}_\mu$ must be completely arbitrary (except for the constraint $\hat{n} \cdot \vec{X}_\mu = 0$). This means that by adding any covariant $\vec{X}_\mu$ (which is orthogonal to $\hat{n}$) to $A_\mu$, we can always obtain a legitimate potential $\vec{A}_\mu$. In other word, putting a constraint on $\vec{X}_\mu$ amounts to putting a constraint on $\vec{A}_\mu$ itself, which should only come from a gauge fixing. Thirdly, the condition (15) is nothing but the consistency condition for a vector field theory. In fact, when $\vec{X}_\mu$ becomes massive, the constraint (15) actually follows automatically from our equation of motion (12) as a consistency condition. Indeed it should be obvious that for any consistent theory of a vector field one need this constraint, just like one need the well-known 5 constraints in the Pauli-Fierz equation of massive spin two field for the consistency [17]. Furthermore, when $\vec{X}_\mu$ remains massless, one has to impose a “gauge condition” to suppress the unphysical degrees of $\vec{X}_\mu$. So any constraint on $\vec{X}_\mu$ should come from a gauge fixing, and (15) should be interpreted as a gauge condition which one need to remove the unphysical degrees of $\vec{X}_\mu$. Indeed it does become a gauge condition, when one tries to quantize our Lagrangian with the background field method with $\vec{A}_\mu$ as the background gauge potential, as we will show in the following. Finally, it must be obvious from (4) that the topological field $\hat{n}$ becomes an essential ingredient of the restricted potential $A_\mu$, which has the full non-Abelian gauge degrees of freedom even without $\vec{X}_\mu$. So $\hat{n}$ plays a fundamental role already in the restricted gauge theory in the absence of $\vec{X}_\mu$, and should be viewed completely independent of $\vec{X}_\mu$. This tells that it is futile to try to remove the “two extra degrees” created by $\hat{n}$ from our Lagrangian by imposing the constraint (15), or any other constraint on $\vec{X}_\mu$, by hand.

Nevertheless it should also be evident that our Lagrangian has indeed two extra degrees. To see this suppose we remove $\hat{n}$ completely with a gauge transformation. In this case $\hat{n}$ disappears, but it re-enters in a different form as the magnetic potential [18], when (and only when) $\hat{n}$ contains the topological singularities which define the topological quantum number $\pi_2(S^2)$. To demonstrate this, introduce the $SU(2)$ matrix element $S$ and parameterize it with $\alpha, \beta,$ and $\gamma$,

$$S = \exp(-t_3\gamma)\exp(-t_2\alpha)\exp(-t_3\beta),$$

where $t_i$ are the adjoint representation of the $SU(2)$ generators, and let

$$\hat{n}_i = S^{-1}\hat{e}_i \quad (i = 1, 2, 3),$$

$$\hat{e}_1 = (1, 0, 0), \quad \hat{e}_2 = (0, 1, 0), \quad \hat{e}_3 = (0, 0, 1),$$

$$\hat{n} = \hat{n}_3 = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha).$$  \hspace{1cm} (16)

Then under the gauge transformation $S$, one has

$$\hat{n} \rightarrow \hat{e}_3,$$

$$\vec{A}_\mu \rightarrow (A_\mu + \vec{C}_\mu)\hat{e}_3,$$

$$\vec{F}_{\mu\nu} \rightarrow (F_{\mu\nu} + H_{\mu\nu})\hat{e}_3,$$  \hspace{1cm} (17)

where
\[ \tilde{C}_\mu = \frac{1}{g} (\cos \alpha \partial_\mu \beta + \partial_\mu \gamma) = -\frac{1}{g} \hat{n}_1 \cdot \partial_\mu \hat{n}_2. \]  

This shows that, in this magnetic gauge where \( \hat{n} \) disappears completely, it is replaced by \( \tilde{C}_\mu \) which describes precisely the Dirac's monopole potential around the isolated singularities of \( \hat{n} \). This tells that our \( \hat{n} \) describes the topological degrees, not the local (i.e., dynamical) degrees, of the non-Abelian gauge theory. This is why the Lagrangian (11) has no equation of motion for \( \hat{n} \).

The “equation of motion” for our \( \hat{n} \) comes from an unexpected quarter. Remember that we are dealing with a gauge theory, and we have to impose a gauge condition. Since in our decomposition \( \vec{X}_\mu \) becomes a gauge covariant charged source, we can impose a gauge condition to the restricted potential \( \hat{A}_\mu \). We may choose the following gauge condition,

\[ \partial_\mu \hat{A}_\mu = (\partial_\mu A_\mu) \hat{n} + A_\mu \partial_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial^2 \hat{n} = 0. \] (19)

This could be re-expressed equivalently as two independent conditions

\[ \partial_\mu A_\mu = 0, \] (20)

and

\[ \hat{n} \times \partial^2 \hat{n} - gA_\mu \partial_\mu \hat{n} = 0. \] (21)

Now, observe that the last equation looks like a perfect equation of motion for \( \hat{n} \). In fact we can obtain exactly the same equation by varying \( \hat{n} \), if we include a mass term for \( \hat{A}_\mu \) in the Lagrangian (11). The reason is that the mass term of \( \hat{A}_\mu \) breaks the gauge symmetry, so that one can no longer remove \( \hat{n} \) with a gauge transformation. This symmetry breaking generates the above equation of motion for \( \hat{n} \), and makes it dynamical. This tells that one can make the topological field \( \hat{n} \) dynamical by imposing a gauge condition. This is not surprising, because we have already shown that in the magnetic gauge where \( \hat{n} \) disappears completely, the topological field is transformed into the magnetic potential which can be treated as dynamical.

The lesson that we learn from the above discussion is unmistakable. The constraint (15) is just a consistency condition that one need to remove the unphysical degrees of \( \vec{X}_\mu \) to make it transverse and keep it massless, which does not remove the extra degrees created by \( \hat{n} \) at all. Furthermore, the topological field \( \hat{n} \) becomes dynamical with the gauge fixing. This raises a very interesting question: How many degrees do we have in \( SU(2) \) QCD? The popular wisdom tells us that we have 6 degrees of freedom after the gauge fixing. But with our decomposition (1) we have two more degrees after the gauge fixing, even with the constraint (15). This seems to suggest that our decomposition does modify QCD in a subtle but important way. How can we implement this idea and modify QCD to make it consistent with our decomposition? To show how, we will discuss three different but equivalent methods to quantize the Lagrangian (11) with the decomposition in the following.

**IV. QUANTIZATION OF EXTENDED QCD**

Certainly our decomposition leaves the Lagrangian invariant under the gauge transformation (3). But notice that the decomposition inevitably introduces a new type of gauge symmetry which is different from the gauge symmetry described by (3). This is because, for a given \( \vec{A}_\mu \), one can have infinitely many different decomposition described by (1), with different \( \hat{A}_\mu \) and \( \vec{X}_\mu \), by choosing different \( \hat{n} \). Equivalently, for a fixed \( \hat{n} \), one can have infinitely many different \( \vec{A}_\mu \) which are gauge-equivalent to each other. So it must be clear that with our decomposition we automatically have another type of gauge invariance which comes from different choices of
decomposition, which one has to deal with in the quantization. This type of extra gauge degree of freedom is what one encounters in the background field method. In the background field method one decomposes the field into two parts, the slow-varying classical part and the fluctuating quantum part, and deals with two type of gauge transformations, the one acting on the slow-varying fields and the other acting on the fluctuating fields, to obtain the effective action of the slow-varying fields. This doubling of the gauge degrees of freedom, of course, is an unavoidable consequence of the arbitrariness of the decomposition. In our case we can adopt a similar attitude and quantize the Lagrangian (11) in two steps, treating \( \hat{A}_\mu \) and \( \vec{X}_\mu \) separately.

With this strategy we introduce two types of gauge transformations, the active (background) gauge transformation and the passive (quantum) gauge transformation. Naturally we identify (3) as the active gauge transformation. As for the passive gauge transformation we choose

\[
\delta \hat{n} = 0, \quad \delta \hat{A}_\mu = \frac{1}{g} D_\mu \hat{\alpha},
\]

under which we have

\[
\delta \hat{A}_\mu = \frac{1}{g} \hat{n} \cdot D_\mu \hat{\alpha}, \quad \delta \hat{\alpha} = \frac{1}{g} (\hat{n} \cdot D_\mu \hat{\alpha}) \hat{n},
\]

\[
\delta \vec{X}_\mu = \frac{1}{g} [D_\mu \hat{\alpha} - (\hat{n} \cdot D_\mu \hat{\alpha}) \hat{n}].
\]

With this we can quantize \( \vec{X}_\mu \) first by fixing the passive gauge degrees of freedom. We can definitely choose (15) as the gauge fixing condition for the passive type gauge transformation (23), because under (23) the constraint (15) no longer transforms covariantly. In this case the Faddeev-Popov determinant corresponding to the gauge condition (15) is given by

\[
K_{ab} = \frac{\delta (\hat{D}_\mu \vec{X}_\mu) a}{\delta \alpha^b} = [(\hat{D}_\mu D_\mu)_{ab} - n_a (\hat{n} \cdot D_\mu)_{b} + g (\hat{n} \times \vec{X}_\mu)_{a}(\hat{n} \cdot D_\mu)_{b}].
\]

The determinant has a remarkable feature,

\[
n_a K_{ab} = K_{ab} n_b = 0.
\]

This is because the determinant (24) must be consistent with the constraints (2) and (10), so that we must have

\[
\delta (\hat{n} \cdot \hat{D}_\mu \vec{X}_\mu) = \hat{n} \cdot \delta (\hat{D}_\mu \vec{X}_\mu) = 0,
\]

\[
\hat{D}_\mu D_\mu \hat{n} = 0, \quad \hat{n} \cdot D_\mu \hat{n} = 0,
\]

when \( \hat{D}_\mu \vec{X}_\mu = 0 \). This means that the ghost fields which correspond to the determinant (24) should be orthogonal to \( \hat{n} \), and have only two (not three) degrees of freedom. With this gauge fixing of the passive type in mind, we now can make the gauge fixing of the active type gauge transformation (3) with the gauge condition (19). In this case the corresponding Faddeev-Popov determinant is given by

\[
M_{ab} = \frac{\delta (\partial_\mu \hat{A}_\mu) a}{\delta \alpha^b} = (\partial_\mu \hat{D}_\mu)_{ab}.
\]

From this we obtain the following generating functional

\[
W \{ j, j_\mu, j_\mu \} = \int \mathcal{D}\hat{n} \mathcal{D}A_\mu \mathcal{D}\vec{X}_\mu \mathcal{D}\eta_\perp \mathcal{D}\eta^*_\perp \mathcal{D} \bar{c} \mathcal{D} \bar{c}^* \exp \{ i \int \mathcal{L}_{eff} d^4 x \},
\]

\[
\mathcal{L}_{eff} = -\frac{1}{4} \hat{F}_{\mu \nu}^2 - \frac{1}{2 \xi_1} (\partial_\mu \hat{A}_\mu)^2 + \bar{c}^* \partial_\mu \hat{D}_\mu \bar{c} + \lambda (\hat{n}^2 - 1)
\]
where \( \tilde{j}, j_\mu, \tilde{j}_\mu \) are the external sources of \( \tilde{n}, A_\mu, \tilde{X}_\mu \), and \( \tilde{c}, \tilde{c}^*, \tilde{\eta}_\perp, \tilde{\eta}_\perp^* \) are the ghost fields of the active and passive gauge transformations (Remember that here \( \tilde{\eta}_\perp \) and \( \tilde{\eta}_\perp^* \) are orthogonal to \( \tilde{n} \)). This clearly shows how we can implement the consistency condition (15) as a legitimate gauge condition. But we emphasize again that in this quantization the topological field \( \tilde{n} \) becomes a real dynamical field of QCD, even with the gauge fixing (15). This is because, after the gauge fixing, the effective Lagrangian becomes a non-trivial function of \( \tilde{n} \).

Another way to quantize the theory is by making a slightly different decomposition. Remember that in our decomposition (1) we have defined \( A_\mu = \tilde{n} \cdot \tilde{A}_\mu \), which gives the constraint \( \tilde{n} \cdot \tilde{X}_\mu = 0 \). We can relax this condition by dividing \( A_\mu \) again into two parts, and let

\[
A_\mu = B_\mu + W_\mu, \quad \tilde{A}_\mu = \tilde{A}_\mu + \tilde{W}_\mu,
\]

\[
\tilde{A}_\mu = B_\mu \tilde{n} - \frac{1}{g} \tilde{n} \times \partial_\mu \tilde{n}, \quad \tilde{W}_\mu = \tilde{X}_\mu + W_\mu \tilde{n}.
\]

Notice that in this decomposition \( \tilde{W}_\mu \) need no longer be orthogonal to \( \tilde{n} \). Now, we can introduce the following active and passive gauge transformations for this decomposition

\[
\delta \tilde{n} = -\tilde{\alpha} \times \tilde{n}, \quad \delta \tilde{A}_\mu = \frac{1}{g} \tilde{D}_\mu \tilde{\alpha}, \quad \delta \tilde{W}_\mu = -\tilde{\alpha} \times \tilde{W}_\mu,
\]

\[
\delta \tilde{n} = 0, \quad \delta \tilde{A}_\mu = 0, \quad \delta \tilde{W}_\mu = \frac{1}{g} D_\mu \tilde{\alpha}.
\]

Notice that both recover the original gauge transformation,

\[
\delta \tilde{A}_\mu = \frac{1}{g} D_\mu \tilde{\alpha}.
\]

With this we can quantize \( \tilde{A}_\mu \) and \( \tilde{W}_\mu \) with the gauge conditions

\[
\partial_\mu \tilde{A}_\mu = 0, \quad \tilde{D}_\mu \tilde{W}_\mu = 0.
\]

In this case the corresponding Faddeev-Popov determinants are given by

\[
M_{ab} = \frac{\delta(\partial_\mu \tilde{A}_\mu)}{\delta \alpha^b} = (\partial_\mu \tilde{D}_\mu)_{ab}, \quad K_{ab} = \frac{\delta(\tilde{D}_\mu \tilde{W}_\mu)}{\delta \alpha^b} = (\tilde{D}_\mu D_\mu)_{ab}.
\]

From this we obtain the following generating functional

\[
W\{\tilde{j}, j_\mu, \tilde{j}_\mu\} = \int D\tilde{n} D\tilde{B}_\mu D\tilde{W}_\mu D\tilde{X}_\mu D\tilde{\eta} D\tilde{\eta}^* D\tilde{c} D\tilde{c}^* \delta(W_\mu - W_\mu^{(0)}) \exp\left\{ i \int L_{eff} \, d^4 x \right\},
\]

\[
L_{eff} = \frac{1}{4} F_{\mu \nu}^2 - \frac{1}{2 \xi_2} (\partial_\mu \tilde{A}_\mu)^2 + \tilde{c}^* \partial_\mu \tilde{D}_\mu \tilde{c} + \lambda (\tilde{n}^2 - 1)
\]

\[
- \frac{1}{4} (\tilde{D}_\mu \tilde{W}_\mu - \tilde{D}_\mu \tilde{\eta}_\perp)^2 - \frac{1}{2} F_{\mu \nu} \cdot \tilde{D}_\mu \tilde{W}_\nu - \frac{g}{2} \tilde{D}_\mu \tilde{W}_\nu \cdot (\tilde{W}_\mu \times \tilde{W}_\nu) - \frac{g^2}{2} F_{\mu \nu} \cdot (\tilde{W}_\mu \times \tilde{W}_\nu)
\]

\[
- \frac{g^2}{4} (\tilde{W}_\mu \times \tilde{W}_\nu)^2 - \frac{1}{2 \xi_2} (\tilde{D}_\mu \tilde{W}_\mu)^2 + \tilde{\eta}^* \tilde{D}_\mu \tilde{D}_\mu \tilde{\eta} + \lambda \tilde{n} \cdot \tilde{X}_\mu + \tilde{n} \cdot \tilde{j} + (B_\mu + W_\mu) j_\mu + \tilde{X}_\mu \cdot \tilde{j}_\mu.
\]
Notice that here we have inserted the delta-function gauge condition \( \delta(W_\mu - W_\mu^{(0)}) \) to fix the gauge degrees of freedom created by the decomposition \( A_\mu = B_\mu + W_\mu \). In principle here \( W_\mu^{(0)} \) can be any constant field, but we might choose \( W_\mu^{(0)} = 0 \) for simplicity. With this understanding the above generating functional reduces to

\[
W\{j, j, \hat{n}_\mu\} = \int D\hat{n} DB \mu D\vec{X}_\mu D\vec{n} D\vec{c} D\vec{c}^* \exp\left[ i \int L_{eff} dx \right],
\]

\[
L_{eff} = -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{2\xi_1} (\partial_\mu \hat{A}_\mu)^2 + \vec{c}^* \partial_\mu \hat{D}_\mu \vec{c} + \lambda (\hat{n}^2 - 1)
- \frac{1}{4} (\hat{D}_\mu \vec{X}_\mu - \hat{D}_\nu \vec{X}_\mu)^2 - \frac{g}{2} \hat{F}_{\mu\nu} \cdot (\vec{X}_\mu \times \vec{X}_\nu) - \frac{g^2}{4} (\vec{X}_\mu \times \vec{X}_\nu)^2
- \frac{1}{2\xi_2} (\hat{D}_\mu \vec{X}_\mu)^2 + \vec{n}^* \hat{D}_\mu \hat{D}_\mu \vec{n} + \lambda \hat{n} \cdot \vec{X}_\mu
+ \hat{n} \cdot j + B_\mu \hat{A}_\mu + \vec{X}_\mu \cdot \hat{n}_\mu.
\]

The difference in the two schemes is that in the second scheme the ghost fields \( \vec{n} \) and \( \vec{n}^* \) become isotriplets, and the ghost interaction of the passive type becomes simpler. This is because here \( \hat{D}_\mu \hat{W}_\mu \) need no longer be orthogonal to \( \hat{n} \).

Observe that, with the identification of \( B_\mu \) as \( A_\mu \), the effective Lagrangian (36) is what we would have obtained with the previous quantization, had we identified the passive gauge transformation by

\[
\delta \hat{A}_\mu = 0, \quad \delta \vec{X}_\mu = \frac{1}{g} \hat{D}_\mu \alpha,
\]

but not by (23). Strictly speaking, this identification is inconsistent with the definition \( A_\mu = \hat{n} \cdot \hat{A}_\mu \) and (22). This was why we did not adopt this gauge transformation in our previous quantization. But the lesson that we learn from the above analysis is that the above passive gauge transformation (37) could actually be acceptable and justifiable after all.

To be complete we will now discuss an alternative (a third) quantization. Remember that the new gauge degrees of freedom originates from the arbitrariness of the decomposition for a fixed \( \vec{A}_\mu \). Since the theory must be independent of the decomposition, we could quantize the theory just like in the perturbative QCD with the gauge condition

\[
\partial_\mu \vec{A}_\mu = \partial_\mu \hat{A}_\mu + \partial_\mu \vec{X}_\mu = 0,
\]

and take into account the arbitrariness of the decomposition with the following mathematical identity,

\[
\int D\hat{n} \det \left( \frac{\delta (\hat{D}_\mu \vec{X}_\mu)}{\delta \hat{n}_a} \right) \delta (\hat{D}_\mu \vec{X}_\mu) = 1.
\]

For a fixed \( \vec{A}_\mu \) we can calculate the determinant with the observation

\[
J_{ab} = \frac{\delta (\hat{D}_\mu \vec{X}_\mu)_a}{\delta \hat{n}_b} = -\frac{(\hat{D}_\mu \delta \hat{A}_\mu)_a}{\delta \hat{n}_b}.
\]

But an important point here is that we must take into account the following constraints

\[
\delta (\hat{n} \cdot \hat{D}_\mu \vec{X}_\mu) = \hat{n} \cdot \delta (\hat{D}_\mu \vec{X}_\mu) = 0, \quad \hat{n} \cdot \delta \hat{n} = 0,
\]

in the evaluation of the determinant, which make

\[
n_a J_{ab} = J_{ab} n_b = 0.
\]
So, here again the ghost fields corresponding to the determinant should be orthogonal to \( \hat{n} \) and have only two degrees. Now, the Lorentz gauge (38) is a perfect gauge for us, but here we will in stead choose a slightly different gauge condition

\[
\tilde{D}_\mu \tilde{A}_\mu = \partial_\mu \tilde{A}_\mu + \tilde{D}_\mu \tilde{X}_\mu = 0, \tag{43}
\]

for the purpose of comparison with the previous quantization schemes. In this case the corresponding determinant is given by

\[
M_{ab} = \frac{\delta(\tilde{D}_\mu \tilde{A}_\mu)_a}{\delta \alpha^b} = (\tilde{D}_\mu D_\mu)_{ab}.
\]

With this we obtain the following generating functional

\[
W(\hat{j}, \hat{j}_\mu, \hat{j}_\mu) = \int \mathcal{D}\hat{n} \mathcal{D}A_\mu \mathcal{D}\hat{X}_\mu \mathcal{D}\hat{\eta}_\perp \mathcal{D}\hat{\eta}_\perp^* \mathcal{D}\hat{c}^* \mathcal{D}\hat{e}^* \exp[i \int \mathcal{L}_{eff} d^4 x],
\]

\[
\mathcal{L}_{eff} = -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{4} (\hat{D}_\mu \hat{X}_\nu - \hat{D}_\nu \hat{X}_\mu)^2 - \frac{g}{2} \hat{F}_{\mu\nu} \cdot (\hat{X}_\mu \times \hat{X}_\nu) - \frac{g^2}{4} (\hat{X}_\mu \times \hat{X}_\nu)^2 - \frac{1}{2 \xi_1} (\partial_\mu \hat{A}_\mu)^2 + \hat{e}^* \hat{D}_\mu \hat{e} - \frac{1}{2 \xi_2} (\hat{D}_\mu \hat{X}_\mu)^2 + \hat{\eta}_\perp^* \cdot (\partial_\mu \hat{n} + g \hat{X}_\mu \times \hat{n}) (\hat{A}_\mu + \hat{X}_\mu) \cdot \hat{\eta}_\perp
\]
\[
+ \hat{\eta}_\perp^* \cdot \hat{A}_\mu (\partial_\mu + \hat{D}_\mu) \hat{n}_\perp - (\hat{X}_\mu \cdot \partial_\mu \hat{n}) (\hat{\eta}_\perp^* \cdot \hat{\eta}_\perp) + \frac{1}{g} \hat{\eta}_\perp^* \cdot (\partial^2 \hat{n} \times \hat{\eta}_\perp - \hat{n} \times \partial^2 \hat{\eta}_\perp)
\]
\[
+ \lambda (\hat{n}^2 - 1) + \lambda_{\mu} \hat{n} \cdot \hat{X}_\mu + \hat{n} \cdot \hat{j} + \hat{A}_\mu \hat{j}_\mu + \hat{X}_\mu \cdot \hat{j}_\mu.
\]

Again notice that, although the ghost interaction of \( \hat{\eta}_\perp \) and \( \hat{\eta}_\perp^* \) is introduced to remove the extra gauge degrees of freedom caused by different choices of \( \hat{n} \), it does not remove the topological field \( \hat{n} \) from the theory. Obviously the three quantization schemes are based on different choices of decomposition. But from the physical point of view they should be equivalent to each other, and thus describe the same non-Abelian dynamics. It would be very nice to have a rigorous proof of the equivalence of the above three quantization schemes.

The modified theory obtained with the decomposition (1) has been called the restricted QCD (without \( \hat{X}_\mu \)), or the extended QCD (with \( \hat{X}_\mu \)). We emphasize that, even without \( \hat{X}_\mu \), the restricted QCD described by

\[
\mathcal{L}_{eff} = -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{2 \xi} (\partial_\mu \hat{A}_\mu)^2 + \hat{e}^* \partial_\mu \hat{D}_\mu \hat{e} + \lambda (\hat{n}^2 - 1), \tag{46}
\]

makes a non-trivial self-consistent theory. It has a full \( SU(2) \) gauge degrees of freedom with the non-Abelian monopole as an essential ingredient, and describes a very interesting dual dynamics of its own.

The serious question now is whether this modification induced by the decomposition (1) describes the same physics or not. We believe that this is so. In fact, we believe that the conventional QCD makes sense only when one expands it perturbatively around the trivial vacuum, but becomes incomplete otherwise in the sense that it does not properly take into account its topological structures. For instance, it can not define the topological charge of the Wu-Yang monopole, in spite of the fact that the theory obviously has the topological monopole. Moreover it has never been able to define the color direction in a gauge independent way. In other words, it has never been able to define the conserved color charge which is gauge invariant which is supposed to be confined. In comparison in our extended QCD the topological field \( \hat{n} \) allows us to define not only the monopole charge with the mapping \( \pi_2(S^2) \), but also the gauge independent color direction and gauge invariant color charge uniquely. In fact we can easily obtain the gauge invariant conserved color charge from our equation of motion (12). For these reasons we believe that only our extended QCD is able to describe the dynamics of the non-Abelian gauge theory unambiguously.
Before we leave this section it is worth to remark that our decomposition (1) can actually “Abelianize” (or more precisely “dualize”) the non-Abelian dynamics. To see this let

\[ \vec{X}_\mu = X_1^\mu \hat{n}_1 + X_2^\mu \hat{n}_2, \]

\[ (X_1^\mu = \hat{n}_1 \cdot \vec{X}_\mu, \quad X_2^\mu = \hat{n}_2 \cdot \vec{X}_\mu) \]

and find

\[ \hat{D}_\mu \vec{X}_\nu = \left[ \partial_\mu X_1^\nu \right. - g(A_\mu + \tilde{C}_\mu)X_2^\nu \left. \right] \hat{n}_1 + \left[ \partial_\mu X_2^\nu + g(A_\mu + \tilde{C}_\mu)X_1^\nu \right] \hat{n}_2. \] \quad (47)

So with

\[ A_\mu = A_\mu + \tilde{C}_\mu, \quad X_\mu = \frac{1}{\sqrt{2}}(X_1^\mu + iX_2^\mu), \] \quad (48)

one could express the Lagrangian explicitly in terms of the dual potential \( B_\mu \) and the complex vector field \( X_\mu \),

\[ L = -\frac{1}{4}(F_{\mu\nu} + H_{\mu\nu})^2 - \frac{1}{2} \left| \hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu \right|^2 + ig(F_{\mu\nu} + H_{\mu\nu})X_\mu^* X_\nu - \frac{1}{2} g^2 [(X_\mu^* X_\mu)^2 - (X_\nu^* X_\nu)^2], \] \quad (49)

where now

\[ \hat{D}_\mu X_\nu = (\partial_\mu + igA_\mu)X_\nu. \]

Clearly this describes an Abelian gauge theory coupled to the charged vector field \( X_\mu \). But the important point here is that the Abelian potential \( A_\mu \) is given by the sum of the electric and magnetic potentials \( A_\mu + \tilde{C}_\mu \). In this Abelian form the equation of motion (12) is re-expressed as

\[ \partial_\mu (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) = ig[X_\mu^*(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu) - X_\mu(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu)^*], \]

\[ \hat{D}_\mu(\hat{D}_\mu X_\nu - \hat{D}_\nu X_\mu) = igX_\mu(F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}), \] \quad (50)

where now

\[ X_{\mu\nu} = -ig(X_\mu^* X_\nu - X_\nu^* X_\mu). \]

This shows that one can indeed Abelianize the non-Abelian theory with our decomposition. But notice that here we have never fixed the gauge to obtain this Abelian formalism, and one might ask how the non-Abelian gauge symmetry is realized in this “Abelian” theory. To discuss this let

\[ \hat{\alpha} = \alpha_1 \hat{n}_1 + \alpha_2 \hat{n}_2 + \theta \hat{n}, \quad \hat{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n} = -C_\mu^1 \hat{n}_1 - C_\mu^2 \hat{n}_2, \]

\[ \alpha = \frac{1}{\sqrt{2}}(\alpha_1 + i \alpha_2), \quad C_\mu = \frac{1}{\sqrt{2}}(C_\mu^1 + i C_\mu^2). \] \quad (51)

Then the Lagrangian (49) is invariant not only under the active gauge transformation (3) described by

\[ \delta A_\mu = \frac{1}{g} \partial_\mu \theta - i(C_\mu^* \alpha - C_\mu \alpha^*), \quad \delta \tilde{C}_\mu = -\delta A_\mu, \]

\[ \delta X_\mu = 0, \] \quad (52)

but also under the passive gauge transformation (23) described by
\[
\delta A_\mu = \frac{1}{g} \partial_\mu \theta - i(X_\mu^\alpha - X_\mu^{\alpha*}), \quad \delta \tilde{C}_\mu = 0,
\]
\[
\delta X_\mu = \frac{1}{g} \hat{D}_\mu \alpha - i \theta X_\mu.
\] (53)

This tells that the “Abelian” theory not only retains the original gauge symmetry, but actually has an enlarged (both the active and passive) gauge symmetries. So the only change in this “Abelian” formulation is that here the topological field \( \hat{n} \) is replaced by the magnetic potential \( \tilde{C}_\mu \). But we emphasize that this is not the “naive” Abelianization of the \( SU(2) \) gauge theory. The difference is that here the Abelian gauge group is actually made of \( U(1)_c \otimes U(1)_m \), so that the theory becomes a dual gauge theory where the magnetic potential plays the crucial role [4,5]. This must be obvious from (52) and (53).

**V. COMPARISON BETWEEN QCD AND SKYRME-FADDEEV THEORY**

Now, we review the physical content of the Skyrme-Faddeev theory and discuss the similarities between the Skyrme-Faddeev theory and QCD. To do this we start from the Skyrme-Faddeev Lagrangian

\[
\mathcal{L}_{SF} = -\frac{\mu^2}{2} (\partial_\mu \hat{n})^2 - \frac{g^2}{4} (\partial_\mu \hat{n} \times \partial_\nu \hat{n})^2,
\] (54)

which gives the following equation of motion

\[
\hat{n} \times \partial^2 \hat{n} - \frac{g^2}{\mu^2} (\partial_\mu H_{\mu\nu}) \partial_{\nu} \hat{n} = 0.
\] (55)

This allows the Faddeev-Niemi knot solutions [49]. But obviously the knots are the solitons, and probably can not describe the elementary object of the theory. To find the elementary ingredient of the theory, notice that the equation of motion (55) has another solution which is much simpler. Let the polar coordinates of \( R^3 \) be \((r, \theta, \phi)\) and let

\[
\hat{n} = \hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\] (56)

Clearly we have

\[
\partial^2 \hat{r} = -\frac{2}{r^2} \hat{r}, \quad \partial_\mu H_{\mu\nu} = 0,
\] (57)

so that (56) becomes a solution of (55), except at the origin. Furthermore, the magnetic field it creates is identical to that of the Wu-Yang monopole in the \( SU(2) \) gauge theory sitting at the origin. Of course this monopole is also topological, whose topological quantum number is given exactly by the same mapping \( \pi_2(S^2) \) defined by \( \hat{n} \) as the Wu-Yang monopole. This means that the non-linear sigma field \( \hat{n} \) in the Skyrme-Faddeev theory describes the non-Abelian monopole, and the Faddeev-Niemi knots are the magnetic flux tubes made of the monopole-anti-monopole pair. The interesting point here is that, unlike in the Abelian theory, these flux tubes are able to form the topologically stable knots due to the non-linear self-interaction. This tells that both the non-Abelian gauge theory and the Skyrme-Faddeev theory describe a non-trivial monopole dynamics. Notice that (56) forms a solution of (54) even without the non-linear interaction. This means that the non-linear sigma model in general should really be viewed as a theory of monopole, even without the non-linear self interaction.

With this we are ready to prove that the Faddeev-Niemi knots can actually describe the multiple vacua of the \( SU(2) \) QCD. To show that the knot solutions of (55) can indeed be used to describe the vacuum solutions of our equation of motion (12), notice first that any \( \hat{n} \) which describes the Faddeev-Niemi knots is smooth everywhere in \( R^3 \), and thus defines the mapping
\[ \pi_3(S^2) \]. Secondly, for any knot described by (55) we can always introduce the magnetic potential 
\[ \tilde{C}_\mu \], which is smooth everywhere in \( R^3 \) through (5).

Let \( \tilde{C}_\mu \) be the magnetic potential of the knot described by \( \hat{n} \),

\[ H_{\mu\nu} = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu = -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}). \]

Then the knot quantum number \( k \) of \( \pi_3(S^2) \) is given by \([1,13]\)

\[ k = \frac{g^2}{32\pi^2} \int \epsilon_{ijk} \tilde{C}_i H_{jk} d^3x. \quad (i, j, k = 1, 2, 3) \quad (58) \]

Now, consider the following static potential in \( SU(2) \) QCD which is smooth everywhere in \( R^3 \),

\[ \hat{A}_\mu = -\tilde{C}_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \quad \vec{X}_\mu = 0, \quad (59) \]

where \( \tilde{C}_\mu \) and \( \hat{n} \) are given by the knot solution. It must be clear from (5) that this potential produces a vanishing field strength, and thus forms a vacuum solution of our equation of motion (12). Furthermore, in this case the vacuum quantum number \( m \) of the potential (59) which describes \( \pi_3(S^3) \) is defined by \([10,11]\)

\[ m = \frac{g^2}{96\pi^2} \int \epsilon_{ijk} \hat{A}_i \cdot (\hat{A}_j \times \hat{A}_k) d^3x. \quad (60) \]

But this is exactly the same mapping which defines the knot quantum number (58) with the Hopf fibering \([12]\). In fact, with (59) one can easily show that (60) reduces to (58). This proves that indeed the Faddeev-Niemi knots can be viewed to describe the multiple vacua of \( SU(2) \) QCD, and the topological quantum number of the knots becomes nothing but the vacuum quantum number. Actually we can simply claim that the knots describe the multiple vacua of the restricted QCD, because (59) obviously forms the vacuum solutions of the restricted QCD. We emphasize that the vacuum solutions (59) become nothing but the multiple vacuum solutions of a spontaneously broken \( SU(2) \) gauge theory that we proposed long time ago, if we identify \( \hat{n} \) as the normalized isotriplet (Higgs) scalar field \( \hat{\phi} \) in the spontaneously broken theory \([12]\). In this sense one can also claim that the Faddeev-Niemi knots also describe the multiple vacua of the spontaneously broken \( SU(2) \) gauge theory.

We have shown that the non-linear sigma model in general is a theory of monopoles. An important question then is what are the physical states of the Skyrme-Faddeev theory. Most probably the monopole is not likely to be a physical state, because it has an infinite energy (Remember here that the monopole solution (56) describes a classical state, not a quantum state). This tells that the Skyrme-Faddeev theory, just like QCD, is probably a theory of confinement, in which the non-linear self interaction of the monopoles confines the long range magnetic flux of the monopoles. In this view the Faddeev-Niemi knots can be viewed as the confined “glueball states” of the non-linear sigma model which are made of the monopole-anti-monopole pairs. The new feature here is that these “glueball states” (unlike those in QCD) have the topological quantum number which makes them topologically stable.

**VI. DISCUSSION**

The existence of the gauge independent reparametrization (1) of the non-Abelian potential in terms of the restricted potential \( \hat{A}_\mu \) and the valence potential \( \vec{X}_\mu \) has been known for more than twenty years \([4,5]\), but its physical significance have become appreciated only recently \([2,6]\). In this paper we have discussed the physical impacts of our reparametrization (1) on QCD. We have discussed how the reparametrization modifies the non-Abelian dynamics, and
presented three quantization schemes of the extended QCD. In particular, we have shown that the reparametrization enlarges the dynamical degrees of QCD by making the topological field \( \hat{n} \) dynamical upon quantization. Furthermore, with the reparametrization, we have demonstrated that the Skyrme-Faddeev theory of non-linear sigma model and SU(2) QCD have almost identical topological structures. In particular we have shown that both are not only theories of monopoles but also theories of confinement, where the monopole plays a crucial role in their dynamics. Together with the Faddeev-Niemi conjecture the parallel between the two theories is indeed remarkable.

A straightforward consequence of our decomposition is the existence of the restricted QCD described by (49). What is remarkable with the restricted theory is that we can construct a non-Abelian gauge theory with much simpler gauge potential (i.e., with the restricted gauge potential), which nevertheless has the full topological characters of the non-Abelian gauge symmetry. The theory contains the non-Abelian monopole as an essential ingredient, and describes a very interesting dual dynamics of its own. So it must be obvious that the restricted theory should play a fundamental role in QCD. Furthermore we can construct a whole family of interesting new non-Abelian gauge theories by adding any gauge-covariant colored source to the restricted gauge theory.

We conclude with the following remarks.

1) Our analysis clarifies the physical content of the Skyrme-Faddeev theory. Indeed the physical content of the Skyrme-Faddeev theory of non-linear sigma model has never been clear. Faddeev and Niemi have shown that the theory is probably a theory of knots. But our analysis tells that it is a self-interacting theory of monopole. The interesting point here is that the non-linear interaction can actually screen (i.e., confine) the magnetic flux of the monopole, and force it to form a flux tube. Moreover, the topology of the theory can make the flux to form the stable knots. But what is really surprising is that, in spite of the fact that the Skyrme-Faddeev theory contains a non-trivial mass parameter, it admits a long range monopole solution. The elementary particle of the theory is the massless monopole, and the mass scale of the theory describes (not the mass of the monopole but) the penetration scale (i.e., the confinement scale) of the magnetic flux made of the monopole-anti-monopole pair. This is really remarkable.

2) Although the similarities between the Skyrme-Faddeev theory and QCD is striking, it should be emphasized that they are really different from the physical point of view. For example, the Faddeev-Niemi conjecture suggests that the knots might be interpreted to describe the glueball states in QCD. This is an overstatement. Although the two theories are the theories of confinement where the monopole dynamics plays the important role, their confinement mechanism is really different. In QCD the monopole condensation generates the confinement, but in Skyrme-Faddeev theory it is the self-interaction of the monopoles which creates the confinement. In this respect we emphasize that the Skyrme-Faddeev theory (unlike QCD) already has the confinement scale, and does not need any dynamical symmetry breaking to generate the confinement. Furthermore, the Faddeev-Niemi knots are the magnetic knots. But in QCD the gluons could form only the electric flux tubes after the monopole condensation. This means that, at the best, the Faddeev-Niemi knots can provide a dual description of the possible electric knots. If this is so, the really interesting question is whether QCD can actually allow any electric knot which is stable. Although this is an interesting issue worth further investigation, it would be difficult to establish the topological stability for the glueballs in QCD.

3) We hope that our analysis completely settles the controversy around the question whether one need any extra constraint to kill the topological degrees of \( \hat{n} \) with our decomposition (1) and (15). The condition (15) is nothing but a gauge condition which one need to remove the unphysical degrees of \( \hat{X}_\mu \) to keep it massless, which has nothing to do with the topological degrees of \( \hat{n} \). Furthermore, even with (15), the topological field \( \hat{n} \) becomes dynamical after the gauge fixing. As we have emphasized, any constraint for \( \hat{A}_\mu \) (and \( \hat{X}_\mu \)) should come from a gauge fixing, and we have shown how the condition (15) really becomes a gauge condition, especially in the quantization by the background field method. In fact with this gauge condition one could successfully calculate the effective action of QCD in the background field method. Remarkably the resulting effective action allows us to establish the monopole condensation at one loop level,
and to demonstrate that the monopole condensation is the true vacuum of QCD. Most importantly, the topological field already plays a fundamental role in the restricted QCD even without $\vec{X}_\mu$. This tells that any attempt to remove the topological field imposing a constraint on $\vec{X}_\mu$ is futile.

Note Added: The fact that the Faddeev-Niemi knots could be interpreted to describe the multiple vacua in $SU(2)$ QCD has also been re-discovered recently by van Baal and Wipf, who obtained an identical result in a different context. See P. van Baal and A. Wipf, hep-th/0105141, Phys. Lett. B, in press.

Acknowledgements

One of the authors (YMC) thanks L. Faddeev, and A. Niemi for the fruitful discussions, and Professor C. N. Yang for the continuous encouragements. The work is supported in part by Korea Research Foundation (Grant KRF-2000 -015-BP0072) and by the BK21 project of Ministry of Education.

[1] L. Faddeev and A. Niemi, Nature 387, 58 (1997); R. Battye and P. Sutcliffe, Phys. Rev. Lett. 81, 4798 (1998).
[2] L. Faddeev and A. Niemi, Phys. Rev. Lett. 82, 1624 (1999); Phys. Lett. B449, 214 (1999); B464, 90 (1999).
[3] Y. Nambu, Phys. Rev. D10, 4262 (1974); S. Mandelstam, Phys. Rep. 23C, 245 (1976); A. Polyakov, Nucl. Phys. B120, 429 (1977); G. ’t Hooft, Nucl. Phys. B190, 455 (1981).
[4] Y. M. Cho, Phys. Rev. D21, 1080 (1980); J. Korean Phys. Soc. 17, 266 (1984).
[5] Y. M. Cho, Phys. Rev. Lett. 46, 302 (1981); Phys. Rev. D23, 2415 (1981).
[6] E. Langman and A. Niemi, Phys. Lett. B463, 252 (1999).
[7] S. Shabanov, Phys. Lett. B458, 322 (1999); B463, 263 (1999).
[8] H. Gies, hep-th/0102026.
[9] Y. M. Cho and D. G. Pak, J. Korean Phys. Soc. 38, 151 (2001); Y. M. Cho, H. W. Lee, and D. G. Pak, hep-th/9905213, submitted to Phys. Lett. B.
[10] A. Belavin, A. Polyakov, A. Schwartz, and Y. Tyupkin, Phys. Lett. B59, 85 (1975).
[11] G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); C. Callan, R. Dashen, and D. G. Gross, Phys. Lett. B63, 334 (1976); O. Jahn, J. Phys. A33, 2997 (2000).
[12] Y. M. Cho, Phys. Lett. B81, 25 (1979).
[13] J. Whitehead, Proc. Nat. Acad. Sci. (London) 33, 117 (1947); G. Woo, J. Math. Phys. 18, 1756 (1974).
[14] Y. M. Cho, Phys. Rev. D62, 074009 (2000).
[15] T. T. Wu and C. N. Yang, Phys. Rev. D12, 3845 (1975).
[16] Y. M. Cho, Phys. Rev. Lett. 44, 1115 (1980); Phys. Lett. B115, 125 (1982); Y. M. Cho and D. Maison, Phys. Lett. B391, 360 (1997).
[17] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) A 173, 211 (1939); Y. M. Cho and S. W. Zoh, Phys. Rev. D 46, R2290 (1992); 3483 (1992).
[18] B. de Witt, Phys. Rev. 162, 1195 (1967); 1239 (1967).
[19] See for example, C. Itzikson and J. Zuber, Quantum Field Theory (McGraw-Hill) 1985; M. Peskin and D. Schroeder, An Introduction to Quantum Field Theory (Addison-Wesley) 1995; S. Weinberg, Quantum Theory of Fields (Cambridge Univ. Press) 1996.
[20] L. Faddeev and A. Niemi, hep-th/0101078.
[21] Y. M. Cho and G. D. Pak, in Proceedings of TMU-YALE Symposium on Dynamics of Gauge Fields, edited by T. Appelquist and H. Minakata (Universal Academy Press) Tokyo, 1999; hep-th/0006051, submitted to Phys. Rev. D.