Performance Indicator for MIMO MMSE Receivers in the Presence of Channel Estimation Error

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Abstract—We present the derivation of post-processing SNR for Minimum-Mean-Squared-Error (MMSE) receivers with imperfect channel estimates, and show that it is an accurate indicator of the error rate performance of MIMO systems in the presence of channel estimation error. Simulation results show the tightness of the analysis.

Index Terms—MIMO, MMSE receiver, post-processing SNR.

I. INTRODUCTION

THE key component of a Multiple-Input-Multiple-Output (MIMO) communication system in terms of performance and complexity is the MIMO detector, which is used for separating independent data streams at the receiver. The Maximum Likelihood (ML) detectors achieve the optimal error rate performance. However, these types of detectors, including the near-optimal sphere decoder and its variants, are usually not suitable for practical systems due to their high complexity. Linear detectors, such as Zero-Forcing (ZF) and MMSE, achieve suboptimal performance, however, they are widely used in practical systems due to their low complexity implementations. Among linear receivers, MMSE is the optimal solution and seems to be the mainstream implementation choice due to its superior performance over ZF detectors.

Perfect channel state information (CSI) is usually assumed in the literature when simulating or analyzing the performance of linear detectors [1, 2]. However, in practice the channel estimates are inherently noisy. Important work [3, 4] has characterized the error rate performance of ZF receivers in the presence of channel estimation error. Nevertheless, less is known for the case of MMSE detectors in practical scenarios. For ZF and MMSE receivers, the joint effect of phase noise and channel estimation error is considered in [5] and the performance is analyzed in terms of the degradation in signal-to-noise-plus-interference-ratio (SINR) without expressing the closed form performance indicators or error rate analysis. The SINR derivations for the MMSE case in [5] are done only for low SNR region. In both [3] and [5], channel estimation error variance is assumed to be constant for all SNRs. This is not realistic approach for packet based or bursty communication systems as the channel estimation error is in fact a function of the SNR. In this letter, we analyze the MMSE receivers in the presence of channel estimation error, and derive a closed form post-processing SNR expression, which provides an accurate estimate of the error rate performance. The error rate performance is investigated for both the constant channel estimation error variance case and the case with a realistic channel estimation algorithm where the estimation error variance is clearly dependent on the channel SNR. We believe that it is a very useful tool for throughput prediction in link adaptation protocols and for error rate analysis in general. Accuracy of the analytical results is verified through simulations.

II. SYSTEM MODEL AND DERIVATIONS

We consider a MIMO system where the transmitter is equipped with \(N_t\) antennas, and the receiver uses \(N_r\) antennas. The \(N_r \times 1\) received signal vector \(y\) can be expressed as

\[
y = Hx + n
\]  

where \(x\) is the transmitted signal vector, \(H\) is the \(N_r \times N_t\) channel matrix, and \(n\) is the \(N_r \times 1\) additive Gaussian noise vector with zero mean and covariance matrix \(E[nn^H] = N_0 I\). We assume an uncorrelated Rayleigh flat channel, i.e. entries of \(H\) are i.i.d. zero mean circularly symmetric complex Gaussians (ZMCSCG) with unit variance, and the signal energy at each transmit antenna is assumed to be equal to \(E_s\).

The receiver can estimate the transmitted signal vector by applying the MMSE detector to the received signal, \(\hat{x} = Wy = WHx + Wn\). Using the orthogonality principle [6], the MMSE detector \(W\) is derived as

\[
W = \left[H^H H + \frac{N_0}{E_s} I\right]^{-1} H^H
\]  

At the output of the MMSE detector, the residual signal plus interference from other spatial streams is well approximated as Gaussian [7] and the post-processing SNR (PPSNR) of \(k^{th}\) spatial stream is calculated as

\[
\gamma_k = \frac{E_s \left(WH\right)_{k,k}^2}{E_s \sum_{l \neq k} \left(WH\right)_{k,l}^2 + N_0 \left(WW^H\right)_{k,k}}
\]  

The PPSNR is a good indicator for the error rate performance of MIMO systems, and therefore employed in link adaptation algorithms to predict the uncoded error rate [7]. Since the output of the MMSE detector is Gaussian, the bit error rate of a specific modulation can be calculated by simply plugging the PPSNR value into the AWGN error rate formula of the modulation. The same technique is also used for theoretical derivation of error rate performance in fading channels.

\(^1(\ldots)_{k,l}\) denotes the \((k, l)^{th}\) entry of the matrix.
This definition of PPSNR holds if the channel is perfectly known at the receiver. However, in practice, the channel matrix has to be estimated by the receiver, and the estimated channel is inherently noisy in practical systems. We model the estimated channel matrix as

\[ \hat{H} = H + \Delta H \]  

(4)

where \( \Delta H \) denotes the estimation error matrix which is uncorrelated with \( H \), and its entries are ZMCSG with variance \( \sigma^2 \). The quality of channel estimation is captured by \( \sigma^2 \), which can be appropriately estimated depending on the channel estimation method. We assume that each block (packet), that undergoes a specific channel realization, \( \hat{H} \), observes a different realization of \( \Delta H \) at the receiver. This situation occurs in packet based communication systems like 802.11n where the channel is estimated on a per packet basis.

**A. PPSNR derivation for practical systems**

In this section, we derive the PPSNR for practical MIMO systems which observe channel estimation error. The receiver uses the estimated channel \( \hat{H} \) to calculate the MMSE detector as

\[ \hat{W} = \left( (H + \Delta H)^H (H + \Delta H) + \frac{N_0}{E_s} I \right)^{-1} (H + \Delta H)^H \]  

(5)

We write the imperfect MMSE solution as \( \hat{W} = W + \Delta W \). Now, the MMSE estimate of the signal vector becomes

\[ \hat{x} = (W + \Delta W)y = \frac{WHx + 2\Delta WHx + Wn + \Delta Wn}{signal \ post-detection \ noise} \]  

(6)

We observe that there are additional interference and noise terms caused by \( \Delta W \), and denote the post detection noise as \( \hat{n} = \Delta WHx + Wn + \Delta Wn \). With this definition for the post detection noise, the PPSNR of the \( k \)th spatial stream in the presence of channel estimation error can be expressed as

\[ \gamma_k = \frac{E_s \left| (WH)_{k,k} \right|^2}{E_s \sum_{l \neq k} \left| (WH)_{k,l} \right|^2 + \left( E \left[ \hat{n} \hat{n}^H \right] \right)_{k,k}} \]  

(7)

where we replaced the original noise covariance in (3) with the covariance of \( \hat{n} \), which is calculated as

\[ E \left[ \hat{n} \hat{n}^H \right] = E \left[ \Delta WHx \Delta H^HH \Delta W^H \right] + E \left[ Wn \hat{n}^H W^H \right] + E \left[ \Delta Wn \hat{n}^H W^H \right] + E \left[ \Delta Wn \Delta W^H \right] \]  

(8)

In order to calculate the terms in (8), we need to first derive \( \Delta W \). For small \( \sigma^2 \), the \( \Delta H^H \Delta H \) term in (5) becomes negligible compared to others. Hence, we can rewrite (5) as

\[ \hat{W} \approx \left( H^H H + \frac{N_0}{E_s} I + H^H \Delta H + \Delta H^H H \right)^{-1} (H + \Delta H)^H \]  

(9)

which can be further simplified using the matrix approximation \( (P + \varepsilon^2 Q)^{-1} \approx P^{-1} - \varepsilon^2 P^{-1} Q P^{-1} \) for small \( \varepsilon^2 \). Let us also define \( K = \left( H^H H + \frac{N_0}{E_s} I \right)^{-1} \) for brevity and simplify (9) as

\[ \hat{W} \approx K - K \left( H^H \Delta H + \Delta H^H H \right) K \left( H + \Delta H \right)^H \]  

(10)

Finally, the desired error matrix becomes

\[ \Delta W \approx -K \left( H^H \Delta H + \Delta H^H H \right) KH^H + K \Delta H^H \]  

(12)

Using the above approximation, we can now calculate the terms in (8). We first note that the third and fourth terms in (8) are zero since \( E \left[ \Delta W \right] \approx 0 \). The second terms is \( E \left[ \Delta W n \hat{n}^H W^H \right] = N_0 W W^H \), and the first term becomes \( E \left[ \Delta W H x \Delta H^HH^H \Delta W^H \right] = E_s E \left[ \Delta W H H \Delta W^H \right] \). Below, we calculate the first and last terms in (8) by plugging the error matrix (12) into (8).

\[ E \left[ \Delta W H H \Delta W^H \right] \]

\[ \approx E \left[ KH^H \Delta H H^H H \Delta H^H H \Delta H^H H \Delta H^H \right] + E \left[ KH^H \Delta H H^H H \Delta H^H H \Delta H^H \right] - E \left[ KH^H \Delta H H^H H \Delta H^H H \Delta H^H \right] - E \left[ K \Delta H^H H H^H H \Delta H^H H \Delta H^H \right] - E \left[ K \Delta H^H H H^H H \Delta H^H H \Delta H^H \right] - E \left[ K \Delta H^H H H^H H \Delta H^H H \Delta H^H \right] - E \left[ K \Delta H^H H H^H H \Delta H^H H \Delta H^H \right] + E \left[ K \Delta H^H H H^H H \Delta H^H H \Delta H^H \right] \]  

(13)

It can be proven that \( E \left[ \Delta H A \Delta H \right] = E \left[ \Delta H^H \Delta H A \Delta H \right] = 0 \) for any deterministic matrix \( A \). Hence the second, third, fourth and seventh terms in (13) are zero. For the remaining terms we use the fact that \( E \left[ \Delta H A \Delta H \right] = \sigma^2 \text{tr} \left( A \right) I \), and obtain

\[ E \left[ \Delta W H H \Delta W^H \right] \]

\[ \approx \sigma^2 \text{tr} \left( KH^H H H^H H \Delta H^H H \Delta H^H \right) + \sigma^2 \text{tr} \left( HH^H H H^H H \Delta H^H H \Delta H^H \right) + \sigma^2 \text{tr} \left( HH^H H H^H H \Delta H^H H \Delta H^H \right) \]  

(14)

Similarly, the last term in (8), \( E \left[ \Delta W n \hat{n}^H W^H \right] \), can be computed following the same way.

\[ E \left[ \Delta W n \hat{n}^H W^H \right] \]

\[ = E \left[ \Delta W n \hat{n}^H W^H \right] \]  

(15)

\[ E \left[ \Delta W \Delta W^H \right] \]

\[ \approx \sigma^2 \text{tr} \left( KH^H H H^H H \Delta H^H H \Delta H^H \right) + \sigma^2 \text{tr} \left( HH^H H H^H H \Delta H^H H \Delta H^H \right) + \sigma^2 \text{tr} \left( HH^H H H^H H \Delta H^H H \Delta H^H \right) \]  

(16)

Finally, we plug \( E \left[ \hat{n} \hat{n}^H \right] \) into (7) and obtain the PPSNR in the presence of channel estimation error as (17).

The BER of the system in the presence of channel estimation error can be found simply by plugging \( \gamma_k \) as the
symbol SNR into the AWGN BER formulas. For example, the BER of \( k \)th stream for BPSK is 
\[
P_b^k = Q \left( \sqrt{2 \gamma_k} \right),
\]
and 
\[
P_b^k = \frac{1}{2} Q \left( \sqrt{\frac{2 \gamma_k}{\sigma_e^2}} \right) + \frac{1}{2} Q \left( 3 \sqrt{\frac{2 \gamma_k}{\sigma_e^2}} \right) - \frac{1}{2} Q \left( 5 \sqrt{\frac{2 \gamma_k}{\sigma_e^2}} \right)
\]
for gray-coded 16QAM.

III. RESULTS

In order to test the performance of the analysis, we simulated transmission of thousands of packets through uncorrelated Rayleigh flat fading channels. For each SNR point on the BER plots, we randomly generate 1000 i.i.d. realizations of the channel matrix \( \mathbf{H} \). For each specific realization of the channel, we transmit 500 packets each of which carries 2000 information symbols. We perform channel estimation for each packet as explained below in Case I.

Case I: In our simulations, we employed the maximum likelihood (ML) channel estimation (CE) algorithm, in which the channel estimate is obtained via training symbols that are known to the receiver. During the training phase, the \( N_t \times N_t \) training matrix \( \mathbf{X}_{tr} \) is transmitted where \( N_t \geq N_t \) is the number of training symbols. The \( N_r \times N_t \) received signal is 
\[
\mathbf{Y}_{tr} = \mathbf{H} \mathbf{X}_{tr} + \mathbf{W}
\]
where \( \mathbf{W} \) is the \( N_r \times N_t \) noise matrix. Then, the ML estimate of the channel is given as [8]
\[
\hat{\mathbf{H}} = \mathbf{Y}_{tr} \mathbf{X}_{tr}^H \left( \mathbf{X}_{tr} \mathbf{X}_{tr}^H \right)^{-1}
\]
(18)

It was shown that the optimal training signal has the property of \( \mathbf{X}_{tr} \mathbf{X}_{tr}^H = E_{\text{tx}} \mathbf{I} \). When this orthogonal training signal is employed, the entries of \( \Delta \mathbf{H} \) are i.i.d. with \( CN(0, \sigma_e^2) \), and the channel estimation noise variance \( \sigma_e^2 \) is 
\[
\sigma_e^2 = \frac{\sigma_n^2}{N_r E_{\text{tx}}/N_t}
\]
[9]. The estimation error in this case is caused by the AWGN in this case.

The following training signal, which is taken from 802.11n standard [9], was employed in the simulations. \( \mathbf{X}_{tr} = \sqrt{E_{\text{tx}}} \mathbf{P} \) where \( \mathbf{P} \) is the submatrix formed by first \( N_t \) rows and first \( N_t \) columns of the bigger matrix \( \mathbf{P} \), i.e. \( \mathbf{P} = \mathbf{P} \left[ 1: N_t ; 1: N_t \right] \).

\[
\mathbf{P} = \begin{bmatrix}
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 \\
-1 & 1 & 1 & 1
\end{bmatrix}
\]
(19)

It should be noted that with this choice of the training matrix, the ML channel estimation at the receiver becomes a very simple operation since the matrix inversion, \( \left( \mathbf{X}_{tr} \mathbf{X}_{tr}^H \right)^{-1} \), is now a trivial operation.

For each channel instance, analytical BER results are obtained by using the PPSNR derived in the previous section. Then, these BERs are averaged over all realizations of the channel.

First thing to notice in Fig. 1 and Fig. 2 is that for \( \sigma_e^2 = 0 \), simulation and analysis curves exactly match. Performance is significantly degraded for the systems experiencing channel estimation errors. This is particularly evident for the 4 \( \times \) 4 configurations.

We present the simulation results for BPSK in Fig. 1 and 16QAM in Fig. 2 with 1 \( \times \) 4, 2 \( \times \) 4, 4 \( \times \) 4 MIMO configurations. \( N_t = 4 \) is used in all the simulations. The case of \( \sigma_e^2 = 0 \), i.e. perfect channel estimation, is also included in the results.

As it can be seen in Fig. 1 and Fig. 2 our analysis gives a very tight approximation of the real performance. For BPSK 4 \( \times \) 4, and all of the 16QAM configurations the analysis results exactly match the simulated performances. For BPSK 1 \( \times \) 4, and 2 \( \times \) 4 configurations the analysis results are upper-bounds to the real performance at high SNR, however, they are still very close to the real performances. The analysis results become tighter for higher order modulations and higher order MIMO configurations. This is because of the fact that the Gaussian assumption, which is made for the post-detection noise, is more valid at higher order modulations and MIMO configurations. At low SNRs, the total post detection noise \( \mathbf{n} \) is dominated by the additive white Gaussian noise component \( \mathbf{n} \) therefore the assumption is valid even for lower configurations. However, at high SNRs the residual interference components...
In this letter, we present the analysis of post-processing SNR for practical MIMO MMSE receivers which experience imperfect channel estimation. Performance of MMSE receivers in the presence of channel estimation error is investigated and shown to be accurately estimated via analytical results. We verified the tightness of the analytical results via simulations. Besides the theoretical contributions, we believe that our closed form PPSNR expression can be useful for link adaptation purposes in real MIMO systems. There exist link adaptation algorithms \cite{7,10} based on PPSNR, however perfect CSI is always assumed which might lead to incorrect prediction of the throughput. More accurate prediction can be achieved using the results presented in this paper.

IV. CONCLUSION

In this letter, we presented the analysis of post-processing SNR for practical MIMO MMSE receivers which experience imperfect channel estimation. Performance of MMSE receivers in the presence of channel estimation error is investigated and shown to be accurately estimated via analytical results. We verified the tightness of the analytical results via simulations. Besides the theoretical contributions, we believe that our closed form PPSNR expression can be useful for link adaptation purposes in real MIMO systems. There exist link adaptation algorithms \cite{7,10} based on PPSNR, however perfect CSI is always assumed which might lead to incorrect prediction of the throughput. More accurate prediction can be achieved using the results presented in this paper.

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