New type of non-topological bubbling solutions to a competitive Chern-Simons model *

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Abstract

We investigate a non-Abelian Chern-Simons system, including the simple Lie algebras $A_2$ and $B_2$. In a previous work, we proved the existence of radial non-topological solutions with prescribed asymptotic behaviors via the degree theory. We also constructed a sequence of partially bubbling solutions, which blow up with only one component partially at infinity. In this paper, we construct a sequence of radial non-topological bubbling solutions of another type explicitly. One component of these bubbling solutions locally converge to a non-topological solution of the Chern-Simons-Higgs scalar equation, but both components blow up partially in different regions at infinity at the same time. This generalizes a recent work by Choe, Kim and the second author, where the $SU(3)$ case was studied.

1 Introduction

In this paper, we study a non-Abelian Chern-Simons system of rank 2:

$$\begin{align*}
\left( \Delta u_1 \right) - K \left( e^{u_1} \right) - K \left( e^{u_2} \right) = \left( \frac{4\pi N_1}{4\pi N_2} \delta_0 \right) \quad \text{in} \quad \mathbb{R}^2,
\end{align*}$$

where $N_1, N_2$ are non-negative integers, $\delta_0$ denotes the Dirac measure at 0, and $K = (a_{ij})$ is a $2 \times 2$ matrix satisfying

$$a_{11}, a_{22} > 0, \quad a_{12}, a_{21} < 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} > 0.$$  (1.2)

Clearly, system (1.1) could be considered as a perturbation of the following Liouville system

$$\begin{align*}
\left( \Delta u_1 \right) + K \left( e^{u_1} \right) - K \left( e^{u_2} \right) = \left( \frac{4\pi N_1}{4\pi N_2} \delta_0 \right) \quad \text{in} \quad \mathbb{R}^2.
\end{align*}$$

See [1, 2, 3]. Suppose that (1.2) holds, then system (1.1) is also called competitive in the literature. Henceafter, we also call that system (1.1) is competitive since $a_{12}, a_{21} < 0$, comparing to the cooperative case where $a_{12}, a_{21} > 0$.

In the last few decades, various Chern-Simons field theories [12] have been widely studied, largely motivated by their applications to the physics of high critical temperature superconductivity. Another interesting feature of Chern-Simons field theories is that it provides a gauge invariant mechanism of mass generation [11]. These Chern-Simons theories can be reduced to systems of nonlinear partial differential equations, which have posed many mathematically challenging problems to analysts. Our first motivation of system (1.1) comes from the relativistic non-Abelian self-dual Chern-Simons model, which was proposed by Kao and Lee [23] and Dunne [13, 14]. Following [13, 14], the relativistic non-Abelian self-dual Chern-Simons model is defined in the $(2 + 1)$ Minkowski space $\mathbb{R}^{1,2}$, and

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are complex-valued functions. Let

\[ D_+ \phi = 0, \]

\[ F_{+-} = \frac{1}{k^2} [\phi - [[\phi, \phi^\dagger], \phi^\dagger]], \]

where \( D_+ = D_1 - iD_2, k > 0, F_{+-} = \partial_+ A_+ - \partial_- A_+ + [A_+, A_-] \) with \( A_{\pm} = A_1 \pm iA_2, \partial_{\pm} = \partial_1 \pm i\partial_2 \) and \([., .]\) is the Lie bracket over \( G \). In [14], Dunne considered a simplified form of the self-dual system [14], in which the fields \( \phi \) and \( A \) are algebraically restricted:

\[ \phi = \sum_{a=1}^{r} \phi^a E_a, \]

where \( r \) is the rank of the gauge Lie algebra, \( E_a \) is the simple root step operator, and \( \phi^a \) are complex-valued functions. Let

\[ u_a = \log|\phi^a|, \quad a = 1, \ldots, r. \]

Then system (1.4) can be reduced to the following system of nonlinear partial differential equations

\[ \Delta u_a + \frac{1}{k^2} \left( \sum_{b=1}^r K_{ab} e^{u_b} - \sum_{b=1}^r \sum_{c=1}^r e^{u_b} K_{bc} e^{u_c} K_{ac} \right) = 4\pi \sum_{j=1}^{N_a} \delta_{p_j}, \quad a = 1, \ldots, r, \]

where \( K = (K_{ab}) \) is the Cartan matrix of a semi-simple Lie algebra, \( \{p_j^a\}_{j=1, \ldots, N_a} \) are zeros of \( \phi^a \) (\( a = 1, \ldots, r \)), and \( \delta_p \) denotes the Dirac measure concentrated at \( p \) in \( \mathbb{R}^2 \). See [29] for the derivation of (1.5) from (1.4). For example, there are three types of Cartan matrix of rank 2:

\[ A_2 (\text{i.e. SU}(3)) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \]

Let \((K^{-1})_{ab}\) denote the inverse of the matrix \( K \). Assume that

\[ \sum_{b=1}^r (K^{-1})_{ab} > 0, \quad a = 1, 2, \ldots, r. \]

A solution \( u = (u_1, \ldots, u_r) \) of (1.5) is called a topological solution if

\[ u_a(x) \rightarrow \ln \sum_{b=1}^r (K^{-1})_{ab} \text{ as } |x| \rightarrow +\infty, \quad a = 1, \ldots, r, \]

a solution \( u \) is called a non-topological solution if

\[ u_a(x) \rightarrow -\infty \text{ as } |x| \rightarrow +\infty, \quad a = 1, \ldots, r. \]

For any configuration \( p_j^a \) in \( \mathbb{R}^2 \), the existence of topological solutions to (1.5) was proved by Yang [29] in 1997. However, the existence question of non-topological solutions (and mixed-type solutions, see below) is much more difficult than the one for topological solutions, and has remained open for a long time. Only recently, with the help of the classification result in [24], the first existence result of non-topological solutions to (1.5) with \( K = A_2, B_2 \) and \( G_2 \) have been obtained by Ao, Wei and the second author [12] by a perturbation from the Liouville system (1.3). However, their results are still very limited toward understanding the non-topological solution structure.

In this paper, we focus on the radially symmetric solutions of (1.5) when all the vortices coincide at the origin. We only consider the rank 2 and competitive case, namely \( K \) is a
2 × 2 matrix satisfying (1.8). Moreover, we may assume, without loss of generality, that $\kappa = 1$. Then system (1.1) turns to be (1.11). In particular, when $K = A_2$, then (1.11) becomes the following $SU(3)$ Chern-Simons system
\[
\begin{cases}
\Delta u_1 + 2(e^{u_1} - 2e^{2u_1} + e^{u_1 + u_2}) - (e^{u_2} - 2e^{2u_2} + e^{u_1 + u_2}) = 4\pi N_1 \delta_0 & \text{in } \mathbb{R}^2, \\
\Delta u_2 + 2(e^{u_2} - 2e^{2u_2} + e^{u_1 + u_2}) - (e^{u_1} - 2e^{2u_1} + e^{u_1 + u_2}) = 4\pi N_2 \delta_0 & \text{in } \mathbb{R}^2.
\end{cases}
\] (1.8)

When $K = B_2$, then system (1.11) becomes the following $B_2$ Chern-Simons system
\[
\begin{cases}
\Delta u_1 + 2e^{u_1} - e^{u_2} - 4e^{2u_1} + 2e^{2u_2} = 4\pi N_1 \delta_0 & \text{in } \mathbb{R}^2, \\
\Delta u_2 + 2e^{u_2} - 2e^{u_1} - 4e^{2u_2} + e^{u_1 + u_2} + 4e^{2u_1} = 4\pi N_2 \delta_0 & \text{in } \mathbb{R}^2.
\end{cases}
\] (1.9)

As in [21], in order to simplify the expression of system (1.11), we consider the transformation
\[(u_1, u_2) \to \left( u_1 + \ln \frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, u_2 + \ln \frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \right),
\]
and let
\[(a_1, a_2) = \left( \frac{a_{12}(a_{11} - a_{21}) - a_{21}(a_{11} - a_{21})}{a_{11}a_{22} - a_{12}a_{21}}, \frac{a_{12}(a_{11} - a_{21}) - a_{21}(a_{11} - a_{21})}{a_{11}a_{22} - a_{12}a_{21}} \right).
\]
Clearly the assumption (1.2) gives $a_1 > 0$ and $a_2 > 0$. Then system (1.11) becomes
\[
\begin{cases}
\Delta u_1 + (1 + a_1)(e^{u_1} - (1 + a_1)e^{2u_1} + a_1e^{u_1 + u_2}) - a_1(e^{u_2} - (1 + a_2)e^{2u_2} + a_2e^{u_1 + u_2}) = 4\pi N_1 \delta_0 & \text{in } \mathbb{R}^2, \\
\Delta u_2 + (1 + a_2)(e^{u_2} - (1 + a_2)e^{2u_2} + a_2e^{u_1 + u_2}) - a_2(e^{u_1} - (1 + a_1)e^{2u_1} + a_1e^{u_1 + u_2}) = 4\pi N_2 \delta_0 & \text{in } \mathbb{R}^2.
\end{cases}
\] (1.10)

For the three types of Cartan matrix (1.6),
\[(a_1, a_2) = \begin{cases} (1, 1), & \text{if } K = A_2, \\ (2, 3), & \text{if } K = B_2, \\ (5, 9), & \text{if } K = G_2. \end{cases}
\] (1.11)

In particular, system (1.8) is invariant under the above transformation.

Clearly, to study system (1.1), we only need to consider system (1.11). It is more interesting to us that, when $(a_1, a_2)$ take some other special values but not (1.11), system (1.11) also arises in some other physical models, such as the Lozano-Marqués-Moreno-Schaposhnik model [25] and the Gudnason model [15, 16]. Lozano et al. [25] considered the bosonic sector of $N = 2$ supersymmetric Chern-Simons-Higgs theory when the gauge group is $U(1) \times SU(N)$ and has $N_f$ flavors of fundamental matter fields. They investigated so-called local $Z_N$ string-type solutions when $N_f = N$ and obtained a system of nonlinear differential equations (see [25], (19)-(22)) which, under a suitable change of variables and unknowns, can be transformed into (1.10) with $(a_1, a_2) = \left( \frac{k - 1}{N - 1}, \frac{(N - 1)(k - 1)}{k - 1} \right)$ and $k > 0$. If $k > 1$, then $a_1, a_2 > 0$ and $a_1 + a_2 = 1$. Gudnason [15, 16] considered a $N = 2$ supersymmetric Yang-Mills-Chern-Simons-Higgs theory with the general gauge group $G = U(1) \times G'$, where $G'$ is a non-Abelian simple Lie group represented by matrices. When the gauge group are $U(1) \times SO(2M)$ and $U(1) \times USp(2M)$, the so-called master equations are a system of nonlinear differential equations (see [15], (3.64)-(3.65), and [17], (2.1)-(2.2)).

Letting $M = 1$ and using a suitable transformation, this system turns to be (1.11) with
\[a_1 = a_2 = \frac{\beta - \alpha}{2\alpha} \quad \text{and} \quad \alpha^*, \beta^* > 0. \]
If $\beta^* > \alpha^*$, then $a_1 = a_2 > 0$. See [21] for these two transformations.

Therefore, it is worth for us to study system (1.11) with generic $a_1, a_2 > 0$ rather than only (1.11). As in [21], we easily see that a solution $(u_1, u_2)$ of (1.11) is a topological solution if $(u_1, u_2) \to (0, 0)$ as $|x| \to +\infty$; a non-topological solution if $(u_1, u_2) \to (-\infty, -\infty)$ as $|x| \to +\infty$; a mixed-type solution if $(u_1, u_2) \to (\ln \frac{1}{1 + a_1}, -\infty)$ or $(u_1, u_2) \to (-\infty, \ln \frac{1}{1 + a_2})$ as $|x| \to +\infty$.

Define a quadratic form $J : \mathbb{R}^2 \to \mathbb{R}$ by
\[
J(x, y) = \frac{a_2(1 + a_2)}{2} x^2 + a_1a_2xy + \frac{a_1(1 + a_1)}{2} y^2.
\] (1.12)
Recently, Huang and the second author made classifications of radially symmetric solutions for system \((1.1)\) in [20] [21]. Among other things, they proved the following interesting result.

**Theorem A.** [21] Let \(a_1, a_2 > 0\) and \((u_1, u_2) \neq (0, 0)\) be a radially symmetric solution of system \((1.10)\). Then both \(u_1 < 0\) and \(u_2 < 0\) in \(\mathbb{R}^2\), and one of the following conclusions holds.

(i) \((u_1, u_2)\) is a topological solution.

(ii) \((u_1, u_2)\) is a mixed-type solution.

(iii) \((u_1, u_2)\) is a non-topological solution and there exist constants \(\alpha_1, \alpha_2 > 1\) such that

\[
u_k(x) = -2\alpha_k \ln |x| + O(1) \quad \text{as} \quad |x| \to +\infty, \quad k = 1, 2. \tag{1.13}
\]

Consequently, \(e^{\alpha_1 u_1}, e^{\alpha_2 u_2} \in L^1(\mathbb{R}^2)\). Moreover, \((\alpha_1, \alpha_2)\) satisfies

\[
J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1). \tag{1.14}
\]

Inequality \((1.14)\) comes from the following Pohozaev identity (see [21]):

\[
J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1)
= \frac{1 + a_1 + a_2}{4} \int_0^\infty r \left[[a_2(1 + a_1)e^{2u_1} + a_1(1 + a_2)e^{2u_2} - 2a_1a_2e^{\alpha_1u_1 + \alpha_2u_2}] dr. \tag{1.15}
\]

Therefore, \((1.14)\) is a necessary condition for the existence of radially symmetric non-topological solutions satisfying the asymptotic condition \((1.13)\). After Theorem A, it is natural to consider the following question.

**Question:** Given \(\alpha_1, \alpha_2 > 1\) satisfying \((1.14)\). Is there a radially symmetric non-topological solution of system \((1.10)\) subject to the asymptotic condition \((1.13)\)?

If we let \(N_1 = N_2 = N\), \(a_1 = a_2\) and \(u_1 = u_2 = u\) in \((1.10)\), then system \((1.10)\) turns to be the following Chern-Simons-Higgs scalar equation

\[
\Delta u + e^u(1 - e^{-u}) = 4\pi N \delta_0 \quad \text{in} \quad \mathbb{R}^2. \tag{1.16}
\]

Equation \((1.16)\) is known as the \(SU(2)\) Chern-Simons equation for the Abelian case; see [10] [22]. For recent developments, we refer the reader to [5] [6] [7] and references therein. Remark that the Pohozaev identity plays an important role in studying non-topological solutions of \((1.16)\). Let \(\alpha \) be a radial non-topological solution of \((1.16)\) satisfying \(u(x) = -2\alpha \ln |x| + O(1)\) near \(\infty\). Then the Pohozaev identity gives

\[
(\alpha - 1)^2 - (N + 1)^2 = \frac{1}{2} \int_0^\infty re^{2u} dr > 0,
\]

which implies \(\alpha > N + 2\). In 2002, Chan, Fu and the second author [4] proved that the inequality \(\alpha > N + 2\) is also a sufficient condition for the existence of radial non-topological solutions satisfying \(u(x) = -2\alpha \ln |x| + O(1)\) near \(\infty\). However, as pointed out in [5] [9], this might not hold for system \((1.10)\). The reason is following: there might be a sequence of solutions \((u_{1,n}, u_{2,n})\) such that only one component blows up, but the other one does not, i.e., the so-called phenomena of partial blowup; see Theorem C for instance. As a result, only one of the \(L^\infty\) norms of \(e^{\alpha_{1,n}u_{1,n}}\) and \(e^{\alpha_{2,n}u_{2,n}}\) tends to 0 as \(n \to \infty\), which implies that the quantity \(J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1)\) might not converge to 0, namely it has a gap. Therefore, the inequality \((1.14)\) might not be a sufficient condition for the existence of radial non-topological solutions satisfying \((1.13)\).

In a previous work [5], we found a sufficient condition for the above question. To simplify the notations, in the sequel we denote

\[
A = (1 + a_1)(1 + a_2) \quad \text{and} \quad B = a_1a_2. \tag{1.17}
\]

Then \(A - B = 1 + a_1 + a_2 > 1\). As in [5], we define

\[
\Omega := \{ (\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 > 1 \text{ and } J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1) \}, \tag{1.18}
\]
\[ S := \{ (\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 > 0 \text{ and } (\alpha_1, \alpha_2) \text{ satisfies (1.20) - (1.23)} \}, \quad (1.19) \]

where

\[ (A - 2B)\alpha_2 - a_2(1 + a_2)\alpha_1 < a_2(1 + a_2)N_1 + AN_2 + 2(A - B), \quad (1.20) \]
\[ (A - 2B)\alpha_1 - a_1(1 + a_1)\alpha_2 < a_1(1 + a_1)N_2 + AN_1 + 2(A - B), \quad (1.21) \]
\[ (3A - 4B)\alpha_1 + \frac{1 + a_1}{a_2}(A - 2B)\alpha_2 > AN_1 + \frac{1 + a_1}{a_2}AN_2 + \left(4 + 2\frac{1 + a_1}{a_2}\right)(A - B), \quad (1.22) \]
\[ (3A - 4B)\alpha_2 + \frac{1 + a_2}{a_1}(A - 2B)\alpha_1 > AN_2 + \frac{1 + a_2}{a_1}AN_1 + \left(4 + 2\frac{1 + a_2}{a_1}\right)(A - B). \quad (1.23) \]

**Theorem B.** Assume that \( N_1, N_2 \) are non-negative integers and \( a_1, a_2 > 0 \) satisfy

\[ (1 + a_1)(1 + a_2) > \left(6 - 2\sqrt{5}\right)a_1a_2. \quad (1.24) \]

Let \( \Omega \) and \( S \) be defined in (1.18) - (1.19). Then \( S \cap \Omega \neq \emptyset \), and for any fixed \( (\alpha_1, \alpha_2) \in S \cap \Omega \), system (1.10) admits a radially symmetric non-topological solution \((u_1, u_2)\) satisfying the asymptotic condition (1.13).

Remark that \( S \cap \Omega \neq \emptyset \) if and only if \((\alpha_1, \alpha_2)\) satisfies (1.24), namely (1.24) is a necessary condition for Theorem B. For example, Theorem B can apply to the SU(3) system (1.8) and the B_2 system (1.9). The counterpart of Theorem B for the SU(3) system was firstly obtained by Choe, Kim and the second author [9], and Theorem B is a generalization of their result to the generic system (1.10). Applying Theorem B to the B_2 case, we see that if

\[ (\alpha_1, \alpha_2) \in S \cap \Omega = \{ (\alpha_1, \alpha_2) \mid \alpha_1 > N_1 + N_2 + 3 \text{ and } \alpha_2 > 2N_1 + N_2 + 4 \}, \quad (1.25) \]

then (1.9) has a radially symmetric non-topological solution satisfying (1.13).

In [5], we proved Theorem B via the Leray-Schauder degree theory. To do this, we proved a uniform boundedness result for radial solutions satisfying (1.13) when \((\alpha_1, \alpha_2) \in S \cap \Omega\). Then a natural question is whether the set \( S \cap \Omega \) is the optimal range of \((\alpha_1, \alpha_2)\) for the existence of radial solutions satisfying (1.13). This question has not been settled yet. However, in the same paper, we also proved the existence of partially bubbling solutions along some part of \( \partial (S \cap \Omega) \). Denote \( B(0, R) := \{ x \in \mathbb{R}^2 \mid |x| < R \} \).

**Theorem C.** Assume that \( N_1, N_2 \) are non-negative integers and \( a_1, a_2 > 0 \) satisfy

\[ (1 + a_1)(1 + a_2) > 2a_1a_2. \quad (1.26) \]

Let \((\alpha_1, \alpha_2) \in \Omega \) satisfy \( \alpha_1 \neq \alpha_2 \) and

\[ (A - 2B)\alpha_2 - a_2(1 + a_2)\alpha_1 = a_2(1 + a_2)N_1 + AN_2 + 2(A - B). \quad (1.27) \]

Then system (1.10) admits a sequence of radial non-topological bubbling solutions \((u_{1,n}, u_{2,n})\) such that \( \sup_{\mathbb{R}^2} u_{2,n} \to -\infty \) as \( n \to \infty \). Furthermore, there exists an intersection point \( R_{1,n} \gg 1 \) of \( u_{1,n} \) and \( u_{2,n} \) such that:

(i) \( u_{1,n} \to U \) in \( C_{loc}^2(B(0, R_{1,n})) \), where \( U \) is the unique radial solution of

\[
\begin{cases}
\Delta U + (1 + a_1)e^{U} - (1 + a_1)^2e^{-U} = 4\pi N_1\delta_0 & \text{in } \mathbb{R}^2, \\
U(x) = -2\gamma \ln |x| + O(1) & \text{as } |x| \to \infty
\end{cases}
\]

with \( \gamma = \alpha_1 + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) \). Besides,

\[ \lim_{n \to \infty} \int_{R_{1,n}} r e^{u_{1,n}} dr = 0. \quad (1.29) \]
(ii) there exists \((\alpha_{1,n}, \alpha_{2,n}) \in \Omega\) such that
\[
 u_{k,n}(r) = -2\alpha_{k,n} \ln r + O(1) \quad \text{as} \quad r \to \infty, \quad k = 1, 2,
\]
and \((\alpha_{1,n}, \alpha_{2,n}) \to (\alpha_1, \alpha_2)\) as \(n \to \infty\).

Theorem C proves the existence of bubbling solutions along the boundary of \((1.20)\). For these bubbling solutions, only the second component blows up. We call this type of bubbling solutions of type II. Inspired by Theorem B, there may exists another type of bubbling solutions along the boundary of \((1.22)\) (or \((1.23)\)), which we call of type II. But for type II, the estimate \((1.29)\) no longer holds, which means that both components of bubbling solutions blow up at infinity, namely the asymptotics of type II are more complicated. The main result of this paper is to prove the existence of bubbling solutions of type II.

**Theorem 1.1.** Assume that \(a_1, a_2 > 0\) satisfy
\[
3(1 + a_1)(1 + a_2) - 4a_1a_2 > 0 \tag{1.30}
\]
and \(N_1, N_2\) are non-negative integers satisfying
\[
(A - 4B)(N_1 + 1) < 2a_1(1 + a_1)(N_2 + 1) \quad \text{if} \quad A - 4B > 0. \tag{1.31}
\]
Let \((\alpha_1, \alpha_2)\) satisfy \(\alpha_1 \geq 1, \alpha_2 > 1\) and
\[
(3A - 4B)\alpha_1 + \frac{1 + a_1}{a_2}(A - 2B)\alpha_2 = A N_1 + \frac{1 + a_1}{a_2} A N_2 + \left(4 + 2 \frac{1 + a_1}{a_2}\right)(A - B), \tag{1.32}
\]
\[
4B - A (\alpha_1 - 1) + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) - (N_1 + 1) > 0. \tag{1.33}
\]
Then system \((1.10)\) admits a sequence of radial non-topological bubbling solutions \((u_{1,n}, u_{2,n})\) such that \(\sup_{\mathbb{R}^2} u_{2,n} \to -\infty\) as \(n \to \infty\). Furthermore, there exist two intersection points \(R_{3,n} \gg R_{1,n} \gg 1\) of \(u_{1,n}\) and \(u_{2,n}\) such that:

(i) \(u_{1,n} \to U\) in \(C^2_{loc}(B(0, R_{3,n}))\) as \(\varepsilon \to 0\), where \(U\) is the unique radial solution of \((1.28)\) with
\[
\gamma = \frac{4B - A}{A} (\alpha_1 - 1) + \frac{2a_1}{1 + a_2} (\alpha_2 - 1) + 1. \tag{1.34}
\]

(ii) \(\int_{R_{1,n}}^\infty r e^{u_{1,n}} dr \to 0, \int_{R_{3,n}}^\infty r e^{u_{2,n}} dr \to 0\) and
\[
\int_{R_{1,n}}^{R_{3,n}} r e^{u_{2,n}} dr \to \frac{4}{1 + a_2} \left(\frac{a_2}{1 + a_1} (\gamma + N_1) + N_2 + 1\right),
\]
\[
\int_{R_{1,n}}^{R_{3,n}} r e^{u_{1,n}} dr \to \frac{4}{1 + a_1} (\alpha_1 - 1) \tag{1.35}
\]
as \(n \to \infty\);

(iii) there exists \((\alpha_{1,n}, \alpha_{2,n}) \in \Omega\) such that
\[
 u_{k,n}(r) = -2\alpha_{k,n} \ln r + O(1) \quad \text{as} \quad r \to \infty, \quad k = 1, 2,
\]
and \((\alpha_{1,n}, \alpha_{2,n}) \to (\alpha_1, \alpha_2)\) as \(n \to \infty\).

**Remark 1.1.** For the SU(3) case, we have \((a_1, a_2) = (1, 1)\). Then it is easy to check that the range of \((a_1, a_2)\) given in Theorem \((1.1)\) is exactly
\[
2a_1 + a_2 = N_1 + 2N_2 + 6 \quad \text{and} \quad 1 \leq a_1 < N_2 + 2. \tag{1.36}
\]

We remark that the counterpart of Theorem \((1.1)\) for the SU(3) system \((1.5)\) was firstly proved by Choe, Kim and the second author \((10)\) under the following assumption
\[
2a_1 + a_2 = N_1 + 2N_2 + 6 \quad \text{and} \quad 1 < a_1 < N_2 + 2. \tag{1.37}
\]
Theorem 1.1 improves their result on two aspects. First, for the SU(3) case, Theorem 1.1 covers the special case $\alpha_1 = 1$ which was not considered in [77]. Remark that the case $\alpha_1 = 1$ is different from the case $\alpha_1 > 1$. We can see from (1.29) that the case $\alpha_1 > 1$ does not satisfy (1.29), namely the bubbling solutions are of type II. However, the case $\alpha_1 = 1$ satisfies (1.29), namely the bubbling solutions are of type I just as in Theorem C. This phenomena is reasonable, because the intersection point of line (1.27) with line (1.32), which exists provided $A > 2B$, is exactly

$$(\alpha_1, \alpha_2) = \left(1, \frac{a_2(1 + a_2)}{A - 2B}(N_1 + 1) + \frac{A}{A - 2B}(N_2 + 1) + 1\right).$$

Observe that $\alpha_1 > 1$ was assumed in Theorem C, so our study of the case $\alpha_1 = 1$ is also a complement of Theorem C. Second, Theorem 1.1 generalizes their result to the generic system (1.14). Theorem 1.1 indicates that, there exist bubbling solutions of type II along the boundary of (1.24). This, together with Theorem C, shows that the set $S \cap \Omega$ is an optimal range in view of the degree theory.

Remark 1.2. For the SU(3) system (1.3), we still do not know whether the set $S \cap \Omega$ is the optimal range for the existence of non-topological solutions, but we tend to believe so in view of Theorems B, C and 1.1; see [4] [10]. However, the generic system (1.14) is more involved than the SU(3) system (1.3). One example is the $G_2$ case where $(\alpha_1, \alpha_2) = (5, 9)$ and so $3A - 4B = 0$. Therefore, none of Theorems B, C and 1.1 can be applied to the $G_2$ case, and understanding the non-topological solution structure for the $G_2$ case remains open. Another example is the $B_2$ case, where $(\alpha_1, \alpha_2) = (2, 3)$ and so $A - 2B = 0$. Then it is easy to check that the range of $(\alpha_1, \alpha_2)$ given in Theorem 1.1 is exactly

$$\alpha_1 = N_1 + N_2 + 3 \quad \text{and} \quad \alpha_2 > 1. \quad (1.38)$$

From here, we conclude that the set $S \cap \Omega$ (see (1.29)) given in Theorem B is not the optimal range for the existence of non-topological solutions to the $B_2$ system (1.14).

Remark 1.3. Theorem B cannot be applied to the case $A \leq (6 - 2\sqrt{5})B$. Therefore, Theorem 1.1 also gives the first existence result of radial non-topological solutions for the case $\frac{4}{3}B < A \leq (6 - 2\sqrt{5})B$.

Remark 1.4. Clearly, assumptions $\alpha_1 \geq 1, \alpha_2 > 1$ are necessary for Theorem 1.1. As mentioned before, $\gamma > N_1 + 2$ is a necessary and sufficient condition for the existence of radial solutions for (1.28). Therefore, (1.29) indicates that (1.36) is a necessary condition for Theorem 1.1. In fact, (1.36) is also needed to guarantee that $(\alpha_1, \alpha_2)$ satisfies inequality (1.14) (see Lemma 2.2 below), which is obviously necessary by the Pohozaev identity. On the other hand, assumptions (1.30) - (1.31) are also necessary conditions for Theorem 1.1 because they are the necessary and sufficient condition to guarantee $\{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 1, \alpha_2 > 1 \text{ and satisfy (1.32) or (1.33)}\} \neq \emptyset$; see Lemma 2.1 below.

Theorem 1.1 will be proved via the shooting method in Section 2.

2 Shooting argument

In this section, we will prove Theorem 1.1 by constructing bubbling solutions via the shooting method. In the sequel, we assume that $a_1, a_2 > 0$ satisfy

$$3(1 + a_1)(1 + a_2) - 4a_1a_2 > 0. \quad (2.1)$$

Recall the notations $A, B$ in (1.17). Assume that $N_1, N_2$ are non-negative integers satisfying

$$(A - 4B)(N_1 + 1) < 2a_1(1 + a_1)(N_2 + 1) \text{ if } A - 4B > 0. \quad (2.2)$$

Define

$$\Sigma := \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 1, \alpha_2 > 1, g(\alpha_1, \alpha_2) = 0, h(\alpha_1, \alpha_2) > 0\}, \quad (2.3)$$

where

$$g(\alpha_1, \alpha_2) := (3A - 4B)\alpha_1 + \frac{1 + a_1}{a_2}(A - 2B)\alpha_2 - AN_1$$
\[-\frac{1 + a_1}{a_2}AN_2 - \left(4 + 2\frac{1 + a_1}{a_2}\right)(A - B),\]  
\(h(\alpha_1, \alpha_2) := \frac{4B - A}{A}(\alpha_1 - 1) + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) - (N_1 + 1).\)  
(2.4)  
(2.5)

**Lemma 2.1.** \(\Sigma \neq \emptyset\) if and only if (2.1), (2.2) hold.

**Proof.** Denote \(\tilde{N}_k = N_k + 1\) for \(k = 1, 2\). Clearly, \(\Sigma \neq \emptyset\) is equivalent to
\[\tilde{\Sigma} := \{ (x, y) | x \geq 0, y > 0, \tilde{g}(x, y) = 0, \tilde{h}(x, y) > 0 \} \neq \emptyset,\]
where
\[\tilde{g}(x, y) := (3A - 4B)x + \frac{1 + a_1}{a_2}(A - 2B)y - A\tilde{N}_1 - \frac{1 + a_1}{a_2}AN_2,\]
\[\tilde{h}(x, y) := \frac{4B - A}{A}x + \frac{2a_1}{1 + a_2}y - \tilde{N}_1.\]

If \(3A - 4B \leq 0\), then \(\{ (x, y) | x \geq 0, y > 0, \tilde{g}(x, y) = 0 \} = \emptyset\). Therefore, (2.1) is a necessary condition to guarantee \(\tilde{\Sigma} \neq \emptyset\). In the following, we always assume that (2.1) holds. Then it is trivial to see that \(\tilde{\Sigma} \neq \emptyset\) in the case \(A - 2B \leq 0\). Consider the remaining case \(A - 2B > 0\).

Observe that the intersection point of \(\tilde{g}(x, y) = 0\) with the \(y\)-axis is \((0, \frac{a_2(1 + a_2)\tilde{N}_1 + AN_2}{A - 2B})\).

If \(A - 4B \leq 0\), a direct computation shows that
\[\tilde{h} \left(0, \frac{a_2(1 + a_2)\tilde{N}_1 + AN_2}{A - 2B}\right) > 0\]  
(2.6)
holds automatically, which implies \(\tilde{\Sigma} \neq \emptyset\). If \(A - 4B > 0\), it is easy to see that \(\tilde{\Sigma} \neq \emptyset\) if and only if (2.6) holds, which is just equivalent to (2.2). This completes the proof. \(\square\)

In the sequel, we fix any \((\alpha_1, \alpha_2) \in \Sigma\). We will prove the existence of bubbling solutions near \((\alpha_1, \alpha_2)\) just as stated in Theorem 1.1.

Inspired by the blowup analysis in our previous work [5], we define
\[\gamma := \frac{4B - A}{A}(\alpha_1 - 1) + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) + 1.\]  
(2.7)

Then \(h(\alpha_1, \alpha_2) > 0\) gives
\[\gamma > N_1 + 2.\]  
(2.8)

Clearly, \(g(\alpha_1, \alpha_2) = 0\) and (2.7) give
\[\alpha_1 = -\frac{A - 2B}{A}(\gamma - 1) + \frac{2B}{A}(N_1 + 1) + \frac{2a_1}{1 + a_2}(N_2 + 1) + 1,\]  
(2.9)
\[\alpha_2 = \frac{a_2}{1 + a_1} \frac{3A - 4B}{A}(\gamma - 1) + \frac{2a_1}{1 + a_2} \frac{A - 4B}{A}(N_1 + 1) + \frac{A - 4B}{A}(N_2 + 1) + 1.\]  
(2.10)

By \(\alpha_1 \geq 1\) we obtain
\[\gamma \leq 1 + \frac{2B}{A - 2B}(N_1 + 1) + \frac{2a_1(1 + a_1)}{A - 2B}(N_2 + 1) \text{ if } A > 2B.\]  
(2.11)

As in Lemma 2.1 for convenience, we always denote
\[\tilde{\gamma} = \gamma - 1, \quad \tilde{\alpha}_k = \alpha_k - 1 \text{ and } \tilde{N}_k = N_k + 1, \quad k = 1, 2.\]  
(2.12)

**Lemma 2.2.** \(h(\alpha_1, \alpha_2) > 0\) implies \(J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1)\).

**Proof.** By the definition (1.12) of \(J\), a direct computation shows
\[J(x, y) = J(-x, -y)\]
\[= J \left( x, -\frac{2a_2}{1 + a_1}x - y \right) = J \left( -x, \frac{2a_2}{1 + a_1}x + y \right)\]
\[= J \left( -x, \frac{2a_2}{1 + a_1}x + y \right).

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and it is easy to check that
\[ V^2(J) = F(\alpha, \beta, \gamma) = \frac{2B - A}{A} \tilde{\gamma} + 2B A N_1 + \frac{2a_1}{1 + a_1} \tilde{N}_2, \quad \tilde{\alpha}_2 = \frac{2a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2 - \frac{2a_2}{1 + a_1} \tilde{\alpha}_1, \]
we can derive
\[
\begin{align*}
J(\tilde{\alpha}_1, \tilde{\alpha}_2) &= J \left( \tilde{\alpha}_1, \frac{2a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2 - \frac{2a_2}{1 + a_1} \tilde{\alpha}_1 \right) \\
&= J \left( \tilde{\alpha}_1, -\frac{2a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) - \tilde{N}_2 \right) \\
&= J \left( \tilde{\alpha}_1, -\frac{2a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2 \right) \\
&= J \left( \tilde{\gamma}, \frac{a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2 \right) \\
&= J(\tilde{N}_1, \tilde{N}_2) + \frac{a_2 (1 + a_1 + a_2)}{2(1 + a_1)} (\tilde{\gamma}^2 - \tilde{N}_1^2). \end{align*}
\]
By \( \gamma > N_1 + 2 \), we conclude \( J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1) \).

Since \( \gamma > N_1 + 2 \), by [4, Theorem 2.1], there is a unique radially symmetric solution \( U \) of the Chern-Simons-Higgs equation
\[
\begin{align*}
\Delta U + (1 + a_1) e^U - (1 + a_1)^2 e^{2U} &= 4\pi N_1 \delta_0 \quad \text{in } \mathbb{R}^2, \\
U(x) &= -2\gamma \ln |x| + O(1) \quad \text{as } |x| \to \infty. \tag{2.14}
\end{align*}
\]
Moreover, \( U < -\ln(1 + a_1) \) in \( \mathbb{R}^2 \) and
\[
\int_0^\infty r \left[ (1 + a_1) e^U - (1 + a_1)^2 e^{2U} \right] dr = 2(\gamma + N_1), \tag{2.15}
\]
Let \( V(|x|) = V(x) := U(x) - 2N_1 \ln |x| \), then \( V(0) := \lim_{r \to 0} V(r) \) is well defined; see [4]. To use the shooting method, we consider an initial problem of system (1.10) in a radial variable. Denote
\[
F_k(r) := (1 + a_k) e^{2u_k(r)} - e^{u_k(r)} - a_k e^{u_1(r) + u_2(r)}, \quad k = 1, 2,
\]
for convenience. Clearly
\[
|F_k| \leq (1 + a_k) e^{u_k} \leq 1 + a_k \quad \text{whenever } u_1, u_2 \leq 0, \tag{2.16}
\]
and it is easy to check that
\[
\begin{align*}
F_k &< \frac{1}{2} e^{u_k} < 0 \quad \text{if } u_k < -\ln (1 + a_1 + a_2), \\
F_{3-k} &< F_k < 0 \quad \text{if } u_k < u_{3-k} < -\ln (1 + a_1 + a_2), \tag{2.17}
\end{align*}
\]
We study the following initial problem
\[
\begin{align*}
u''(r) + \frac{1}{r} u'(r) &= (1 + a_1) F_1(r) - a_1 F_2(r), \quad r > 0, \\
u''(r) + \frac{1}{r} u'(r) &= (1 + a_2) F_2(r) - a_2 F_1(r), \quad r > 0, \\
u_1(r) &= 2N_1 \ln r + V(0) + o(1), \quad r \to 0, \\
u_2(r) &= 2N_2 \ln r + \ln \epsilon + o(1), \quad r \to 0,
\end{align*}
\]
where \( \epsilon \in (0, 1) \). Clearly, the solution of (2.15) depends on \( \epsilon \) and we denote it by \( (u_{1, \epsilon}, u_{2, \epsilon}) \). Consequently, \( F_k(r) = F_{k, \epsilon}(r) \) also depends on \( \epsilon \). For the sake of convenience, we will omit the subscript \( \epsilon \) frequently since there is no confusion arising. The main result of this section is following, and Theorem [11] is a direct corollary.
Theorem 2.1. Assume \(u_{1, \varepsilon} - u_{2, \varepsilon}\) and fix any \((\alpha_1, \alpha_2) \in \Sigma\). Then there exists sufficiently small \(\varepsilon_0 > 0\) such that for any \(\varepsilon < \varepsilon_0\), system \(2.18\) has an entire solution \((u_{1, \varepsilon}, u_{2, \varepsilon})\). Furthermore, there exist two intersection points \(R_{1, \varepsilon} \gg R_{2, \varepsilon} \gg 1\) of \(u_{1, \varepsilon}\) and \(u_{2, \varepsilon}\) such that:

(i) \(u_{1, \varepsilon} \to U\) in \(C^2_{\text{loc}}(B(0, R_{1, \varepsilon}))\) and \(\sup u_{2, \varepsilon} \to -\infty\) as \(\varepsilon \to 0\);

(ii) \(\int_{R_{1, \varepsilon}}^{R_{2, \varepsilon}} e^{u_{1, \varepsilon}} dr \to 0\), \(\int_{R_{1, \varepsilon}}^{\infty} e^{u_{2, \varepsilon}} dr \to 0\) and

\[
\begin{align*}
\int_{R_{1, \varepsilon}}^{R_{2, \varepsilon}} e^{u_{1, \varepsilon}} dr & \to \frac{2}{1 + a_2} \left(\frac{2a_2}{1 + a_1} (\gamma + \hat{N}_1) + 2\hat{N}_2\right), \\
\int_{R_{1, \varepsilon}}^{\infty} e^{u_{2, \varepsilon}} dr & \to \frac{4}{1 + a_1} (\alpha_1 - 1)
\end{align*}
\]

as \(\varepsilon \to 0\);

(iii) there exists \((\alpha_{1, \varepsilon}, \alpha_{2, \varepsilon}) \in \Omega\) such that

\[
u_{k, \varepsilon}(r) = -2\alpha_{k, \varepsilon} \ln r + O(1) \quad \text{as} \quad r \to \infty, \quad k = 1, 2,
\]

and \((\alpha_{1, \varepsilon}, \alpha_{2, \varepsilon}) \to (\alpha_1, \alpha_2)\) as \(\varepsilon \to 0\).

In the rest of this section, we give the proof of Theorem 2.1, which is quite long and delicate, and we divide it into several lemmas. The basic strategy is similar to that in [10] where the \(SU(3)\) case was studied. However, as pointed out in [5, 18, 21], the generic situation poses new analytical difficulties compared to the \(SU(3)\) case. For example, in [10], they used many helpful inequalities, which hold in the \(SU(3)\) case because of \((\alpha_1, \alpha_2) = (1, 1)\) but can not hold for generic \(a_1, a_2 > 0\). This requires us to develop general ideas to avoid using this kind of inequalities. It is interesting that our general idea turns out to be somewhat more simple. This is reasonable in view of mathematics. Roughly speaking, people usually use a more special method when the problem is more special. When the problem is more general, people need to develop a more general method, the idea of which might be more natural and simple.

In the sequel, we denote positive constants independent of \(\varepsilon\) (possibly different in different places) by \(C, C_0, C_1, \cdots\). Let \(\delta\) be a constant such that

\[
0 < \delta < \frac{\min\{1, a_1, a_2, B\} \cdot \min\{1, \alpha_1, \alpha_2, \gamma - (N_1 + 2)\}}{100(1 + a_1 + a_2)^4},
\]

where \(\alpha_1 := \alpha_1 = a_1 - 1\) if \(\alpha_1 > 1\) and \(\alpha_1 := 1\) if \(\alpha_1 = 1\). Then by (2.14)-(2.15), we can fix a constant \(R_0 > 1\) large enough such that

\[
R_0 e^{U(R_0)} + |R_0 U'(R_0) + 2\gamma| < \delta^3, \quad \int_{R_0}^{\infty} e^{U(r)} dr < \delta^3,
\]

\[
\int_0^{R_0} r e^U - (1 + a_1) e^{2U} dr - \frac{2(\gamma + N_1)}{1 + a_1} < \delta^3.
\]

(2.20)

Repeating the argument of [5 Lemma 5.2], we can prove the existence of small \(\varepsilon_1 > 0\) such that for each \(\varepsilon \in (0, \varepsilon_1), \) problem (2.18) admits a solution \((u_{1, \varepsilon}, u_{2, \varepsilon})\) on \([0, R_0]\) which satisfies:

(1) Both \(u_{1, \varepsilon} < 0\) and \(u_{2, \varepsilon} < 0\) on \([0, R_0]\), \(u_{2, \varepsilon}(R_0) < u_{1, \varepsilon}(R_0) < 2 \ln \delta\) and \(R_0 e^{u_{1, \varepsilon}(R_0)} < \delta\).

(2) \(|R_0 u_{1, \varepsilon}'(R_0) + 2\gamma| < \delta^2\) and \(|R_0 u_{2, \varepsilon}'(R_0) - \frac{2a_2}{1 + a_1}(\gamma + N_1) - 2N_2| < \delta^2\).

(3) \(u_{1, \varepsilon} \to U\) \(\to 0\) and \(u_{2, \varepsilon} \to -\infty\) uniformly on \([0, R_0]\) as \(\varepsilon \to 0\).

Furthermore, by following the argument of [5 Lemma 5.3], for each \(\varepsilon \in (0, \varepsilon_1), \) there exists \(R_1 = R_{1, \varepsilon} \gg R_0\) such that

(4) \(u_{1, \varepsilon}(R_{1, \varepsilon}) = u_{2, \varepsilon}(R_{1, \varepsilon}), \quad u_{2, \varepsilon} < u_{1, \varepsilon} < 2 \ln \delta\) on \([R_0, R_{1, \varepsilon}]\) and

\[
\begin{cases}
\left|ru_{1, \varepsilon}'(r) + 2\gamma\right| < \delta, \\
\left|ru_{2, \varepsilon}'(r) - \frac{2a_2}{1 + a_1}(\gamma + N_1) - 2N_2\right| < \delta, \quad \forall r \in [R_0, R_{1, \varepsilon}].
\end{cases}
\]

(2.21)
In particular, together with (2.19), we have
\[ ru_1'(r) < -2\gamma + \delta < -4 \] for any \( r \in [R_0, R_1, \varepsilon] \).

**Lemma 2.3.** Let \( \varepsilon \to 0 \), then \( R_1, \varepsilon \to \infty \) and \( R_1, \varepsilon e^{u_k(R_1, \varepsilon)} \to 0 \) for \( k = 1, 2 \). Furthermore,
\[ R_1, \varepsilon u_1, \varepsilon (R_1, \varepsilon) = -2\gamma + o(1), \quad R_1, \varepsilon u_2, \varepsilon (R_1, \varepsilon) = \frac{2a_2}{1 + a_1}(\gamma + N_1) + 2N_2 + o(1). \]

**Proof.** By property (3) above, \( u_1(R_0) - u_2(R_0) \to \infty \) as \( \varepsilon \to 0 \). Since (2.21) gives \( |ru_1'(r)| \leq C \) on \([R_0, R_1]\), we easily obtain \( R_1 \to \infty \) as \( \varepsilon \to 0 \). Then it follows from (2.22) and property (1) that
\[ R_1^2 e^{u_2(R_1)} = R_1^2 e^{u_1(R_1)} < \frac{1}{R_1^2} R_0^2 e^{u_1(R_0)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \]

By (2.21) again, we know that \( u_2'(r) > 0 \) for \( r \in [R_0, R_1] \). Then by property (3), we have
\[ \int_0^{R_1} r e^{u_2(r)} dr = o(1) + \int_0^{R_1} r e^{u_2(r)} dr \leq o(1) + R_1^2 e^{u_2(R_1)} = o(1) \]
as \( \varepsilon \to 0 \). Moreover, \( e^{u_2} \to 0 \) uniformly on \([0, R_1]\), which implies
\[ |(1 + a_1)e^{u_1(r) + u_2(r)} + F_2(r)| \leq 2(A - B)e^{u_2(r)} \to 0 \quad \text{uniformly on} \quad [0, R_1]. \]

Recall that
\[ u_1'(r) + u_1'(r) = [(1 + a_1)^2 e^{u_1} - (1 + a_1)e^{u_1}] - a_1 [(1 + a_1) e^{u_1 + u_2} + F_2]. \]

By the standard continuous dependence on data in the ODE theory and \( R_1 \to \infty \) as \( \varepsilon \to 0 \), we conclude that \( |u_1 - U| \to 0 \) uniformly on any compact subset \( K \subset \subset [0, \infty) \).

This, together with \( e^{u_1(r)} \leq r^{-4} R_0^2 e^{u_1(R_0)} \) for \( r \in [R_0, R_1] \), easily yields that
\[ \lim_{\varepsilon \to 0} \int_0^{R_1} r e^{u_1} - (1 + a_1)e^{u_1} dr = \int_0^{\infty} r e^U - (1 + a_1)e^U dr = \frac{2(\gamma + N_1)}{1 + a_1}. \]

Moreover,
\[ \lim_{\varepsilon \to 0} \int_0^{R_1} r (e^{u_2} + e^{u_1 + u_2}) dr = 0. \]

Consequently, by integrating (2.18) over \((0, R_1)\), we obtain
\[ R_1 u_1'(R_1) = 2N_1 - (1 + a_1) \int_0^{R_1} r \left[ e^{u_1} - (1 + a_1)e^{u_1} \right] dr + o(1) = -2\gamma + o(1), \]
\[ R_1 u_2'(R_1) = 2N_2 + 2\int_0^{R_1} r \left[ e^{u_1} - (1 + a_1)e^{u_1} \right] dr + o(1) \]
\[ = \frac{2a_2}{1 + a_1}(\gamma + N_1) + 2N_2 + o(1) \]
as \( \varepsilon \to 0 \). This completes the proof. \( \square \)

For each \( \varepsilon \in (0, \varepsilon_1) \), we define
\[ R^* = R^*_\varepsilon := \sup \{ r > R_1, \varepsilon \mid u_1, \varepsilon < 0, u_2, \varepsilon < 0 \quad \text{on} \quad [0, r] \}. \]

Our final goal is to prove \( R^*_\varepsilon = \infty \) provided \( \varepsilon > 0 \) sufficiently small. To this goal, we define
\[ R_2 = R_2, \varepsilon := \sup \{ r \in (R_1, \varepsilon, R^*_\varepsilon) \mid u_2, \varepsilon > 0 \quad \text{on} \quad [R_1, \varepsilon, r] \}. \]

First, we recall the following Pohozaev identity (c.f. [21 Lemma 7.2] or [5] Lemma 2.2)
\begin{align*}
\frac{d}{dr} \left\{ J(ru_1'(r) + 2, ru_2'(r) + 2) + (1 + a_1 + a_2)r^2 \left[ a_2 e^{u_1} + a_1 e^{u_2} - \frac{a_2(1 + a_1)}{2} e^{2u_1} \right] 
- \frac{a_1(1 + a_2)}{2} e^{2u_2} + a_1 a_2 e^{u_1 + u_2} \right\} \\
= (1 + a_1 + a_2)r \left[ a_2(1 + a_1)e^{2u_1} + a_1(1 + a_2)e^{2u_2} - 2a_1 a_2 e^{u_1 + u_2} \right].
\end{align*} 

where \( J \) is defined in (1.12).
Lemma 2.4. There exists a small $\epsilon_2 \in (0, \epsilon_1)$ such that for each $\epsilon \in (0, \epsilon_2)$, $u_{1, \epsilon} < u_{2, \epsilon}$ on $(R_{1, \epsilon}, R_{2, \epsilon})$.

Proof. Assume by contradiction that there exist a sequence $\epsilon_n \downarrow 0$ and $r_n \in (R_{1, \epsilon_n}, R_{2, \epsilon_n})$ such that $u_{1, \epsilon_n}(r_n) = u_{2, \epsilon_n}(r_n)$ and $u_{1, \epsilon_n} < u_{2, \epsilon_n}$ on $(R_{1, \epsilon_n}, r_n)$. Clearly $u_{1, \epsilon_n}(r_n) \geq u_{2, \epsilon_n}(r_n) > 0$ for all $n$. We will omit the subscript $\epsilon_n$ in the following argument for convenience.

Step 1. We claim that $u_2(r_n) \to -\infty$ as $\epsilon_n \downarrow 0$.

If not, we may assume, up to a subsequence, that $u_2(r_n) \geq c_0$ for some constant $c_0 < \ln \delta$. Recall $u_2(R_1) \to -\infty$. For $n$ large, there exist $b_n, d_n \in (R_1, r_n)$ such that $b_n < d_n$, $u_2(d_n) = c_0 - 1$, $u_2(b_n) = c_0 - 2$, $u_2 \leq c_0 - 1$ on $[R_1, d_n]$ and $u_2 \geq c_0 - 2$ on $[b_n, d_n]$. Consequently, $u_1 < u_2 \leq c_0 - 1 < \ln \delta$ on $(R_1, d_n)$ and so (2.24) gives $F_2 < F_1 < 0$ on $(R_1, d_n)$, which implies

$$(ru'_2(r))' = r[(1 + a_2)F_2 - a_2F_1] < rF_2 < \frac{1}{2}re^{u_2(r)} \text{ on } (R_1, d_n).$$

This, together with Lemma 2.3, yields $0 \leq ru'_2(r) \leq R_1u'_2(R_1) \leq C$ uniformly on $[R_1, d_n]$. Then by the mean value theorem, there exists $\epsilon_n \in (b_n, d_n)$ such that

$$\frac{1}{d_n - b_n} = \frac{u_2(d_n) - u_2(b_n)}{d_n - b_n} = u'_2(\epsilon_n) \leq \frac{C}{R_1},$$

which implies $d_n - b_n \to \infty$. Consequently,

$$0 \leq d_n u'_2(d_n) = b_n u'_2(b_n) + \int_{b_n}^{d_n} (ru'_2)'dr \leq R_1 u'_2(R_1) \leq \frac{1}{2} \int_{b_n}^{d_n} re^{u_2(r)}dr$$

$$\leq C - \frac{1}{2}e^{c_0-2} \int_{b_n}^{d_n} rdr \to -\infty \text{ as } n \to \infty,$$

a contradiction.

Step 2. We claim the existence of constant $C > 0$ independent of $\epsilon_n$ such that

$$u_{k, \epsilon}(r) + 2 \ln r \leq C \text{ uniformly for } r \in [R_1, r_n], \; k = 1, 2. \quad (2.25)$$

By Step 1, we may assume $u_1 < u_2 \leq u_2(r_n) < \ln \delta$ on $(R_1, r_n)$ for all $n$. Then $[(1 + a_2)r u'_1 + a_1 ru'_2]' = (A - B)r F_1 \leq -\frac{A - B}{4}re^{u_1}$ and $[a_2ru'_1 + (1 + a_1)ru'_2]' = (A - B)r F_2 \leq -\frac{A - B}{4}re^{u_2}$ on $[R_1, r_n]$, which imply

$$0 \leq (1 + a_2)r_n u'_1(r_n) + a_1 r_n u'_2(r_n)$$

$$\leq (1 + a_2)R_1 u'_1(R_1) + a_1 R_1 u'_2(R_1) - \frac{A - B}{2} \int_{R_1}^{r_n} re^{u_1}dr,$$

$$0 \leq a_2 r_n u'_2(r_n) + (1 + a_1)r_n u'_2(r_n)$$

$$\leq a_2 R_1 u'_1(R_1) + (1 + a_1) R_1 u'_2(R_1) - \frac{A - B}{2} \int_{R_1}^{r_n} re^{u_2}dr.$$

By this and Lemma 2.3, we obtain

$$\int_{R_1}^{r_n} r(e^{u_1} + e^{u_2})dr \leq C \text{ for all } n. \quad (2.26)$$

Since $e^{u_1} \leq e^{u_2} \leq e^{u_2(r_n)} \to 0$ on $[R_1, r_n]$, it follows that

$$\lim_{n \to \infty} \int_{R_1}^{r_n} r(e^{2u_1} + e^{2u_2} + e^{u_1 + u_2})dr = 0. \quad (2.27)$$

Combining (2.27) with Lemma 2.3, we integrate the Pohozaev identity (2.23) over $(R_1, r)$ for any $r \in (R_1, r_n)$, which yields that

$$\frac{A - B}{2}r^2(a_2 e^{u_1} + a_1 e^{u_2}) \leq J(r u'_1(r) + 2, ru'_2(r) + 2)$$
We claim the existence of constant (2.24), we can prove that $0 - x = x - n$.

Fix any $\theta \in [2.5, 3] \setminus \{2 - x_1, 2 - x_2\}$. Since $r_n u'_1(r_n) > 0$ and (2.22) gives $R_1 u'_1(R_1) < -4$, there exists $t_n \in (R_1, r_n)$ such that $t_n u'_1(t_n) = -\theta$ and $r u'_2(r) < -\theta$ for all $r \in [R_1, t_n]$. We claim the existence of constant $C$ independent of $\epsilon_n$ such that

$$ u_2(t_n) + 2 \ln t_n \geq C $$

for $n$ sufficiently large. (2.29)

Observe that $r^\theta u_1(r)$ is decreasing for $r \in [R_1, t_n]$, which implies

$$ \int^{t_n}_{R_1} r e^{-u_1(r)} \, dr \leq R_1^\theta e^{u_1(R_1)} \int^{t_n}_{R_1} r^{1-\theta} dr \leq \frac{R_1^2 e^{u_1(R_1)}}{\theta - 2} \to 0 \text{ as } n \to \infty. $$

Since $[(1 + a_2) r u'_1 + a_1 ru'_2]' = (A - B) r F_1$ and $|F_1| \leq (1 + a_1) e^{u_1}$, we can obtain

$$(1 + a_2) r u'_1(t_n) + a_1 t_n u'_2(t_n) = (1 + a_2) R_1 u'_1(R_1) + a_1 R_1 u'_2(R_1) + o(1) \quad (2.30)$$

as $n \to \infty$. This, together with Lemma 2.3 shows that

$$(1 + a_2)(2 - \theta) + a_1(2 + \lim_{n \to \infty} t_n u'_2(t_n)) = (1 + a_2)x_1 + a_1 y_1 \quad (2.31)$$

Since $2 - \theta \notin \{x_1, x_2\}$, it follows that $J(2 - \theta, 2 + \lim_{n \to \infty} t_n u'_2(t_n)) - J(x_1, y_1) \neq 0$. Recall from (2.24) that $t_n^2 e^{u_1(t_n)}(t_n) \leq C t_n^2 \to 0$ as $n \to \infty$ for $1 \leq i, j \leq 2$. Then by (2.27) and Lemma 2.3, we easily deduce via integrating the Pohozaev identity (2.23) over $(R_1, t_n)$ that

$$(A - B) t_n^2 (a_2 e^{u_1(t_n)} + a_1 e^{u_2(t_n)})
+ J(2 + R_1 u'_1(R_1), 2 + R_1 u'_2(R_1)) - J(2 + t_n u'_1(t_n), 2 + t_n u'_2(t_n)) + o(1)
= J(x_1, y_1) - J(2 - \theta, 2 + \lim_{n \to \infty} t_n u'_2(t_n)) \neq 0 \quad (2.32)$$

as $n \to \infty$. This proves (2.29) since $u_1(t_n) < u_2(t_n)$.

Step 4. For $k = 1, 2$, we consider the scaled functions

$$ u_{k,n}(r) := u_k, e_n(t nr) + 2 \ln t_n, \quad \frac{R_1, e_n}{t_n} \leq r \leq \frac{r_n}{t_n}. $$

Then $(\hat{u}_{1,n}, \hat{u}_{2,n})$ satisfies

$$
\hat{u}_{1,n}' + \frac{1}{2} \hat{u}_{1,n} = (1 + a_1) \left( \left( 1 + a_1 \right) t_n^{-2} e^{2u_{1,n} - e^{u_{1,n}} - a_1 t_n^{-2} e^{u_{1,n} + u_{2,n}}} - a_1 \left( 1 + a_2 \right) t_n^{-2} e^{2u_{2,n} - e^{u_{2,n}} - a_2 t_n^{-2} e^{u_{1,n} + u_{2,n}}} \right) \quad (2.33)
$$

By (2.20) and (2.29), we see that

$$ |\hat{u}_{2,n}(1)| \leq C \text{ for sufficiently large } n. $$

Step 5. We claim that $\frac{R_1, e_n}{t_n} \to 0$ and $u_{1, e_n}(t_n) - u_{1, e_n}(R_1, e_n) \to -\infty$ as $n \to \infty$.

Assume by contradiction that up to a subsequence, $t_n / R_1 \leq C$ for all $n$. Similarly as (2.24) we can prove that $0 \leq ru'_2(r) \leq R_1 u'_2(R_1) \leq C$ uniformly on $[R_1, r_n]$. Consequently,

$$ u_2(r) - u_2(R_1) \leq C \ln \frac{r}{R_1 \leq C \text{ uniformly for } r \in [R_1, t_n].$$
which implies from Lemma 2.5 that
\[\int_{R_1}^{t_n} reu_1 \, dr \leq \int_{R_1}^{t_n} reu_2 \, dr \leq R_1^2 e^{u_2(R_1) + C} \left[ \left( \frac{t_n}{R_1} \right)^2 - 1 \right] \to 0\]
as \(n \to \infty\). Recalling the first equation in (2.13) and \(|F_k| \leq (1 + a_k)e^{u_k}\), we obtain
\[-\theta = t_n u_1'(t_n) \leq R_1 u_1'(R_1) + \int_{R_1}^{t_n} r [(1 + a_1)|F_1| + a_1 |F_2|] \, dr = -2\gamma + o(1)\]as \(n \to \infty\), a contradiction with \(\theta \leq 3\) and \(\gamma > N_1 + 2\). This proves \(R_1/t_n \to 0\) as \(n \to \infty\). Consequently, we deduce from \(ru_1'(r) \leq -\theta\) on \([R_1, t_n]\) that \(u_1(t_n) - u_1(R_1) \leq -\theta \ln \frac{t_n}{R_1} \to -\infty\) as \(n \to \infty\).

**Step 6.** We claim that \(\frac{t_n}{R_1} \to \infty\) as \(n \to \infty\).

By (2.23), (2.24) and the Pohozaev identity (2.25), it follows that
\[J(ru_1'(r) + 2, ru_2'(r) + 2) = J(R_1 u_1'(R_1) + 2, R_1 u_2'(R_1) + 2) + O(1) \leq C\]
uniformly for \(r \in [R_1, r_n]\). Hence
\[|ru_1'(r)| + |ru_2'(r)| \leq C\]uniformly for \(r \in [R_1, r_n]\). (2.34)

Then by \(u_2(r_n) - u_2(R_1) > 0\) and Step 5, we have
\[C \ln \frac{t_n}{R_1} \geq \int_{t_n}^{r_n} u_1'(r) \, dr = u_1(r_n) - u_1(t_n) = u_2(r_n) - u_2(R_1) + u_1(R_1) - u_1(t_n) \to \infty\]
as \(n \to \infty\). This proves the claim.

**Step 7.** We conclude the proof by obtaining a contradiction.

By \(u_2(t_n) > u_2(R_1) = u_1(R_1)\), (2.33) and Step 5, we have
\[\hat{u}_{1,n}(1) = u_1(t_n) - u_2(t_n) + \hat{u}_{2,n}(1) \leq u_1(t_n) - u_1(R_1) + C \to -\infty\]
as \(n \to \infty\). Combining this with (2.33) and (2.34), we conclude that \(\hat{u}_{2,n}\) is uniformly bounded in \(C_{loc}([0, \infty))\), while \(\hat{u}_{1,n} \to -\infty\) uniformly on any compact subset \(K \subset (0, \infty)\) as \(n \to \infty\). Up to a subsequence, we may assume that \(\hat{u}_{2,n} \to \hat{u}\) in \(C_{loc}^2((0, \infty))\), where \(\hat{u}\) satisfies
\[\hat{u}'' + \frac{1}{r} \hat{u}' = -(1 + a_1) e^{\hat{u}}\]for \(r \in (0, \infty)\),
\[\int_0^\infty e^{\hat{u}} \, dr \leq \liminf_{n \to \infty} \int_{R_1}^{r_n} e^{u_2} \, dr < +\infty\.
Recalling \(u_2'(r) > 0\) on \((R_1, r_n)\), we easily conclude that \(\hat{u}'(r) > 0\) for any \(r \to \infty\), namely \(\hat{u}\) is increasing on \((0, \infty)\), which contradicts to \(\int_0^\infty reu_1 \, dr < \infty\). This completes the proof. \(\square\)

**Lemma 2.5.** There exists a small \(\varepsilon_3 \in (0, \varepsilon_2)\) such that for each \(\varepsilon \in (0, \varepsilon_3)\), there holds
\[R_{2,\varepsilon} < \infty, u_{2,\varepsilon}'(R_{2,\varepsilon}) = 0, u_{1,\varepsilon}'(R_{2,\varepsilon}) < u_{2,\varepsilon}'(R_{2,\varepsilon}) < \ln \delta, |u_{2,\varepsilon}'(R_{2,\varepsilon}) + 2 \ln R_{2,\varepsilon}| \leq C\]and
\[u_{k,\varepsilon}(r) + 2 \ln r \leq C\]uniformly for \(r \in [R_{1,\varepsilon}, R_{2,\varepsilon}]\), \(k = 1, 2\). (2.35)

Furthermore, \(\frac{R_{2,\varepsilon}}{R_{1,\varepsilon}} \to \infty\) as \(\varepsilon \to 0\).

**Proof.** Step 1. We claim that \(R_{2,\varepsilon} < \infty, u_{2,\varepsilon}'(R_{2,\varepsilon}) = 0, u_{1,\varepsilon}'(R_{2,\varepsilon}) < u_{2,\varepsilon}'(R_{2,\varepsilon}) < \ln \delta\) for \(\varepsilon > 0\) sufficiently small.

Lemma 2.4 shows \(u_1 < u_2\) on \((R_1, R_2)\). By repeating Step 1 of Lemma 2.4, we can prove
\[\sup_{R_1 \leq r < R_2} u_2(r) \to -\infty\]as \(\varepsilon \to 0\). (2.36)

So \(u_1 < u_2 < \ln \delta - 1\) on \((R_1, R_2)\) for \(\varepsilon > 0\) small enough. Then (2.21) holds for any \(r \in (R_1, R_2)\). Recalling \(u_2 > 0\) on \([R_1, R_2]\), we have for any \(r \in (R_1, R_2)\) that
\[-C \leq ru_2'(r) - R_1 u_2'(R_1) \leq \frac{1}{2} \int_{R_1}^{r} te^{u_2(t)} \, dt \leq -\frac{1}{4} e^{u_2(R_1)}(r^2 - R_1^2).\]
Letting \( r \uparrow R_2 \), it follows that \( R_2 < \infty \) and so \( u'_2(R_2) = 0 \). The proof of Lemma 2.4 also yields \( u_1(R_2) < u_2(R_2) \) for \( \epsilon > 0 \) sufficiently small.

**Step 2.** We claim that (2.35) holds provided \( \epsilon > 0 \) sufficiently small.

In fact, since \( u_1 < u_2 < \ln \delta \) on \( (R_1, R_2) \), (2.34) also implies \( \int_{R_1}^{R_2} r(e^{u_1} + e^{u_2})dr \leq C \).

This, together with (2.36), gives

\[
\lim_{\epsilon \to 0} \int_{R_1}^{R_2} r(e^{2u_1} + e^{2u_2} + e^{u_1+u_2})dr = 0. \tag{2.38}
\]

The rest argument is the same as Step 2 of Lemma 2.4.

**Step 3.** We claim that \( \frac{R_2 \epsilon}{R_2^2} \to \infty \) as \( \epsilon \to 0 \).

Recalling \( ru'_2(r) \leq R_1 u'_2(R_1) \leq C \) for \( r \in [R_1, R_2] \), we have \( u_2(r) \leq u_2(R_1) + C \ln \frac{r}{R_1} \) for all \( r \in [R_1, R_2] \). Consequently,

\[
-R_1 u'_2(R_1) = \int_{R_1}^{R_2} (ru'_2(r))'dr = \int_{R_1}^{R_2} r[(1 + a_2)F_2 - a_2F_1]dr
\geq (1 + a_2) \int_{R_1}^{R_2} rF_2dr \geq -(1 + a_2)^2 \int_{R_1}^{R_2} re^{u_2(r)}dr
\geq -(1 + a_2)^2 R_1^2 e^{u_2(R_1)} \left[ \frac{(R_2^2 - R_1^2)^{2C+2}}{2C+2} - 1 \right].
\]

This proves the claim since Lemma 2.3 gives \( R_1^2 e^{u_2(R_1)} \to 0 \) and \( R_1 u'_2(R_1) \to C > 0 \) as \( \epsilon \to 0 \).

**Step 4.** We prove the existence of constant \( C \) independent of \( \epsilon \) such that \( u_2, \epsilon(R_2, \epsilon) + 2 \ln R_2, \epsilon \geq C \) provided \( \epsilon > 0 \) sufficiently small.

Assume by contradiction that there exists a sequence \( \epsilon_n \) such that \( u_2, \epsilon_n(R_2, \epsilon_n) + 2 \ln R_2, \epsilon_n \to -\infty \) as \( n \to \infty \). We will omit the subscript \( \epsilon_n \) for convenience. Since \( u_2 \) is increasing on \( [R_1, R_2] \), we have

\[
r^2 e^{u_1(r)} \leq r^2 e^{u_2(r)} \leq R_2^2 e^{u_2(R_2)} \to 0 \text{ for any } r \in [R_1, R_2]. \tag{2.39}
\]

We consider two cases separately.

**Case 1.** Up to a subsequence, \( \sup_{[R_1, R_2]} ru'_2(r) \leq 2.5 \) for all \( n \).

Then \( r^2 e^{u_1(r)} \) is decreasing on \( [R_1, R_2] \), which implies

\[
\int_{R_1}^{R_2} re^{u_1}dr \leq 2 R_2^2 e^{u_1(R_1)} \to 0 \text{ as } n \to \infty. \tag{2.40}
\]

Let \((x_1, y_1)\) and \((x_2, y_2)\) be in Step 3 of Lemma 2.4. Clearly \( y_1 > 2 \). Fix any \( \theta \in (0, \frac{2 - \sqrt{2}}{\sqrt{2}}) \setminus \{y_2 - 2\} \). Since \( u'_2(R_2) = 0 \) and \( u'_2(R_1) = \frac{2a_2}{1 + a_2}(\gamma + N_1) + 2N_2 + o(1) = y_1 - 2 + o(1) > \frac{2a_2}{\sqrt{2}} \)
for \( n \) large, there exists \( t_n \in (R_1, R_2) \) such that \( t_n u'_2(t_n) = \theta \). By (2.40) we see that (2.34) holds as \( n \to \infty \), which implies

\[
(1 + a_2)(2 + \lim_{n \to \infty} t_n u'_1(t_n)) + a_1(2 + \theta) = (1 + a_2)x_1 + a_1 y_1.
\]

On the other hand, by (2.34) and (2.39), we can prove via the Pohozaev identity (2.23) that (compare with (2.31))

\[
J \left( 2 + \lim_{n \to \infty} t_n u'_1(t_n), 2 + \theta \right) - J(x_1, y_1) = 0.
\]

That is, \( (2 + \lim_{n \to \infty} t_n u'_1(t_n), 2 + \theta) \) is also a solution of (2.28), which yields a contradiction with \( 2 + \theta \notin \{y_1, y_2\} \). So Case 1 is impossible.

**Case 2.** Up to a subsequence, \( \sup_{[R_1, R_2]} ru'_2(r) > 2.5 \) for all \( n \).

In this case, since \( R_1 u'_1(R_1) < 0 \) by (2.22), we can repeat the argument of Step 3 in Lemma 2.4 to obtain the existence of \( t_n \in (R_1, R_2) \) such that (2.29) holds. Since \( u_2 \) is increasing on \([R_1, R_2]\), so

\[
C \leq u_2(t_n) + 2 \ln t_n \leq u_2(R_2) + 2 \ln R_2 \to -\infty
\]
as \( n \to \infty \), also a contradiction. So Case 2 is also impossible. This completes the proof. \( \square \)
Lemma 2.6 implies $R_{2, \varepsilon} < R^e_{\varepsilon}$ for each $\varepsilon \in (0, \varepsilon_3)$. Consider the following scaled functions:

$$
\bar{u}_k(r) = \bar{u}_{k, \varepsilon}(r) := u_{k, \varepsilon}(R_{2, \varepsilon} r) + 2 \ln R_{2, \varepsilon} \quad \text{for} \quad k = 1, 2, \quad \varepsilon \in (0, \varepsilon_3),
$$

(2.41)

where \( \frac{R_{1, \varepsilon}}{R_{2, \varepsilon}} \leq r \leq 1 \). Then \( (\bar{u}_1, \bar{u}_2) \) satisfies

$$
\begin{align*}
\bar{u}'_1 + \frac{1}{r} \bar{u}'_1 &= (1 + a_1) \left( (1 + a_1) R_{2, \varepsilon}^{-2} e^{2a_1} - e\bar{u}_1 - a_1 R_{2, \varepsilon}^{-2} e^{a_1 + \bar{u}_1} \right) \\
&\quad - a_1 \left( (1 + a_2) R_{2, \varepsilon}^{-2} e^{2a_2} - e\bar{u}_2 - a_2 R_{2, \varepsilon}^{-2} e^{a_2 + \bar{u}_2} \right) , \\
\bar{u}'_2 + \frac{1}{r} \bar{u}'_2 &= (1 + a_2) \left( (1 + a_1) R_{2, \varepsilon}^{-2} e^{2a_1} - e\bar{u}_2 - a_2 R_{2, \varepsilon}^{-2} e^{a_1 + \bar{u}_1} \right) \\
&\quad - a_2 \left( (1 + a_2) R_{2, \varepsilon}^{-2} e^{2a_2} - e\bar{u}_2 - a_2 R_{2, \varepsilon}^{-2} e^{a_2 + \bar{u}_2} \right) .
\end{align*}
$$

(2.42)

Moreover, \(|\bar{u}_2(1)| \leq C\) uniformly for \(\varepsilon\) by Lemma 2.6. Now we claim that

$$
\lim_{\varepsilon \to 0} \bar{u}_1(1) = -\infty.
$$

(2.43)

In fact, since \(\bar{u}_1 \leq \bar{u}_2 \leq \bar{u}_2(1) \leq C\) on \([R_1/R_2, 1]\), it follows from (2.42) that \(|(\bar{u}_2')'| \leq C\) uniformly for \(r \in [R_1/R_2, 1]\) and \(k = 1, 2, \varepsilon \). Consequently,

$$
r \bar{u}'_1(r) \leq R_{1, \varepsilon} \bar{u}_1 \left( \frac{R_1}{R_2} \right) + Cr = R_1 u'_1(R_1) + Cr \leq -4 + Cr,
$$

and so \(\bar{u}_1(r) \geq -4 \ln r + \bar{u}_1(1) - C\) for any \(r \in [R_1/R_2, 1]\). Recalling \(\int_{R_1}^{R_2} r e^{u_{1, \varepsilon}} dr \leq C\) uniformly for all \(\varepsilon \in (0, \varepsilon_3)\) by Step 2 of Lemma 2.6 we have

$$
C \geq \int_{R_1}^{R_2} r e^{u_{1, \varepsilon}} dr = \int_{\frac{R_1}{R_2}}^{1} r e^{\bar{u}_1} dr \geq \frac{1}{2} \left( \frac{R_2}{R_1} \right)^2 e^{\bar{u}_1(1) - C}.
$$

This proves (2.43) since \(R_2/R_1 \to \infty\) as \(\varepsilon \to 0\).

Again by \(|(\bar{u}_2')'| \leq C\) on \([R_1/R_2, 1]\) for \(k = 1, 2, \varepsilon \), it follows that \(\bar{u}_1 \to -\infty\) uniformly on any compact subset \(K \subset \subset (0, 1]\) as \(\varepsilon \to 0\) and \(\bar{u}_2\) is uniformly bounded in \(C_{\text{loc}}((0, 1])\).

**Lemma 2.6.** \(\lim_{\varepsilon \to 0} \int_{R_{1, \varepsilon}}^{R_{2, \varepsilon}} r e^{u_{1, \varepsilon}} dr = 0\). Consequently,

$$
\lim_{\varepsilon \to 0} R_{2, \varepsilon} u_{1, \varepsilon}'(R_{2, \varepsilon}) = -\frac{2A - B}{A} + \frac{2B}{A^2} N_1 + \frac{2a_1}{1 + a_2} N_2.
$$

(2.44)

*Proof.* Assume by contradiction that there exists a sequence \(\varepsilon_n \downarrow 0\) such that

$$
\lim_{n \to \infty} \int_{R_{1, \varepsilon_n}}^{R_{2, \varepsilon_n}} r e^{u_{1, \varepsilon_n}} dr > 0.
$$

(2.45)

Again, we will omit the subscript \(\varepsilon_n\) for convenience. We consider two cases separately.

**Case 1.** Up to a subsequence, \(\sup_{[R_{1, \varepsilon_n}/R_{2, \varepsilon_n}]} \bar{u}_2' \leq -2.5\) for all \(n\).

Then (2.40) holds, a contradiction with (2.45). So Case 1 is impossible.

**Case 2.** Up to a subsequence, \(\sup_{[R_{1, \varepsilon_n}/R_{2, \varepsilon_n}]} \bar{u}_2' \geq -2.5\) for all \(n\).

In this case, since \(R_1 u'_1(R_1) < -4\) by (2.22), we can repeat the argument of Step 3 in Lemma 2.4. In particular, there exist a constant \(\theta \in (2.5, 3)\) and a sequence \(t_n \in (R_1, R_2)\) such that \(t_n u'_1(t_n) = -\theta, \bar{u}_2'(r) < -\theta\) for \(r \in [R_1, t_n]\) and \(u_2(t_n) + 2 \ln t_n \geq C\) for \(n\) large. Then similarly as (2.40), we have

$$
\int_{R_1}^{t_n} r e^{u_1} dr \leq \frac{1}{\theta - 2} R_1^2 e^{-u_1(R_1)} \to 0 \quad \text{as} \quad n \to \infty.
$$

(2.46)

On the other hand, since \(\bar{u}_2 \leq \bar{u}_2(1) \leq C\) on \([R_1/R_2, 1]\), we have

$$
C \leq \frac{t_n^2}{R_2} e^{u_2(t_n)} = \left( \frac{t_n}{R_2} \right)^2 e^{u_2(t_n)} \leq C \left( \frac{t_n}{R_2} \right)^2 ,
$$

(2.47)
which implies $t_n/R_2 \geq C > 0$ for all $n$. Recalling that $\bar{u}_1 \to -\infty$ uniformly on $[C, 1]$, we conclude that
$$
\int_{R_n}^{R_2} re^{u_1} dr = \int_1^{R_2} re^{\bar{u}_1} dr \to 0 \quad \text{as} \quad n \to \infty.
$$
Combining this with (2.37), we obtain a contradiction with (2.45) again.

Therefore, $\lim_{\varepsilon \to 0} \int_{R_1}^{R_2} re^{u_1} dr = 0$. Consequently, similarly as (2.40), we have
$$
(1 + a_2) R_2 u_1'(R_2) + a_1 R_2 u_2'(R_2) = (1 + a_2) R_1 u_1'(R_1) + a_1 R_1 u_2'(R_1) + o(1).
$$
Then (2.44) follows directly from Lemma 2.3 and $u_2'(R_2) = 0$.

For each fixed $\varepsilon \in (0, \varepsilon_1)$, we define
$$
R_3 = R_{3, \varepsilon} := \sup \{ r \in [R_2, \varepsilon] \mid \bar{u}_1 < u_2 \text{ on } [R_2, \varepsilon] \}.
$$
Then $R_2 < R_3 \leq R^*$. If there exists $t \in (R_2, R_3)$ such that $u_2'(t) = 0$ and $u_2'(r) < 0$ for $r \in (R_2, t)$, then Lemma 2.3 yields $u_1 < u_2 < \ln \delta$ on $[R_2, t]$. Consequently, $F_2 < F_1 < 0$ on $[R_2, t]$ and so
$$
0 = tu_2'(t) - R_2 u_2'(R_2) - \int_{R_2}^{t} r[(1 + a_2)F_2 - a_2F_1]dr < 0,
$$
a contradiction. Therefore,
$$
u_2'(r) < 0 \quad \text{for any } r \in (R_2, R_3).
$$
(2.47)

Consider the scaled functions $\bar{u}_k$ defined in (2.41) for $k \in (R_1, R_2)$. By (2.41) and the definition of $R_3$, we have $\bar{u}_1(r) < \bar{u}_2(r) \leq \underline{u}_2(1) \leq C$ for all $r \in (R_2, R_3)$. This, together with (2.42), gives $|(\bar{u}_k')'(r)| \leq Cr$ for all $r \in (R_2, R_3)$. Consequently, $\bar{u}_1 \to -\infty$ uniformly on any compact subset $K \subset \subset (R_2, R_3)$ as $\varepsilon \to 0$ and $\bar{u}_2$ is uniformly bounded in $C_{loc}((R_1, R_3))$. Since $\bar{u}_2(1) = \underline{u}_1(1) \to \infty$ as $\varepsilon \to 0$, we conclude from the definition of $R_3$ that
$$
\lim_{\varepsilon \to 0} \frac{R_3}{R_2} = \infty.
$$
(2.48)

Then, for any constant $b > 1$, there holds
$$
\lim_{\varepsilon \to 0} \int_{R_1}^{bR_2} re^{u_1} dr = \lim_{\varepsilon \to 0} \int_{R_1}^{R_2} re^{u_1} dr + \lim_{\varepsilon \to 0} \int_{R_2}^{b} re^{\bar{u}_1} dr = 0.
$$
(2.49)

Lemma 2.7. $\bar{u}_{2, \varepsilon} \to \omega_2$ in $C_{loc}^0((0, \infty))$ as $\varepsilon \to 0$, where
$$
\omega_2(r) = \ln \frac{2D^2(D^2 - 4)r^{D-2}}{(1 + a_2)(D + 2 + (D - 2)r^{D-2})} \quad \text{for } r \in (0, \infty).
$$
(2.50)

Here $D := \frac{2a_2}{1 + a_2}(\gamma + N_1) + 2N_2 + 2$. Consequently,
$$
\int_0^\infty re^{-\omega} dr = \frac{2}{1 + a_2} D.
$$
(2.51)

Proof. Recall (2.43) and Lemma 2.3. By integrating the Pohozaev identity over $(R_1, R_2)$, we obtain
$$
J(R_2 u_1'(R_2) + 2, R_2 u_2'(R_2) + 2) + (A - B) R_2^2 \left[ a_2 e^{u_1(R_2)} + a_1 e^{u_2(R_2)} \right]
$$
$$
- \frac{a_2}{2} (1 + a_1) e^{u_1(R_2)} - \frac{a_1}{2} (1 + a_2) e^{u_2(R_2)} = a_1 A - B) R_2^2 e^{u_2(R_2)}
$$
$$
= J(R_1 u_1'(R_1) + 2, R_1 u_1'(R_1) + 2) + o(1), \quad \text{as } \varepsilon \to 0.
$$
Recall that $R_2^2 e^{u_2(R_2)} = e^{u_2(1)} \to 0$ and $R_2^2 e^{u_1(R_2) + u_2(R_2)} = R_2^2 e^{u_2(R_2)} + o(1) \to 0$ for $1 \leq i, j \leq 2$ as $\varepsilon \to 0$. Combining these with Lemma 2.3 and (2.44), we have
$$
a_1 A - B) R_2^2 e^{u_2(R_2)}
$$
Lemma 2.8. There exists a small \( \varepsilon_4 \in (0, \varepsilon_3) \) such that for each \( \varepsilon \in (0, \varepsilon_4) \), there holds \( R_3, \varepsilon < R_* \). Consequently, \( u_{1, \varepsilon}(R_{3, \varepsilon}) = u_{2, \varepsilon}(R_{3, \varepsilon}) \).

Proof. Assume by contradiction that there exists a sequence \( \varepsilon_n \downarrow 0 \) such that \( R_{3, \varepsilon_n} = R_{\varepsilon_n} \).

Since \( u_{1, \varepsilon_n} < u_{2, \varepsilon_n} \leq u_{2, \varepsilon_n}(R_{2, \varepsilon_n}) < \ln \delta \) on \( (R_{1, \varepsilon_n}, R_{3, \varepsilon_n}) \), we see from the definition of \( R_{\varepsilon_n} \) that \( R_{3, \varepsilon_n} = \infty \), namely \( u_{1, \varepsilon_n}, u_{2, \varepsilon_n} \) is an entire solution and \( u_{1, \varepsilon_n} < u_{2, \varepsilon_n} < \ln \delta \) on \( (R_{1, \varepsilon_n}, \infty) \). By Theorem A, we see that \( (u_{1, \varepsilon_n}, u_{2, \varepsilon_n}) \) is a non-topological solution and there exist constants \( \beta_{k, \varepsilon_n} > 1 \) such that \( u_{k, \varepsilon_n}(r) \rightarrow -2\beta_{k, \varepsilon_n} r \rightarrow \infty \) for \( k = 1, 2 \). Clearly, \( \beta_{2, \varepsilon_n} \leq \beta_{1, \varepsilon_n} \) for all \( n \). Again, we will omit the subscript \( \varepsilon_n \) for convenience.

By Lemma 2.7, we can fix a large constant \( b > 1 \) such that \( b^2 e^{\omega_2(b)} < \delta \) and \( \int_0^\infty r e^{\omega_2} dr < \delta/2 \). By the dominated convergence theorem,

\[
\int_{R_1}^{bR_2} r e^{\omega_2} dr = \int_{R_1}^{bR_2} r e^{\omega_2} dr \rightarrow \int_0^b r e^{\omega_2} dr \text{ as } n \rightarrow \infty.
\]

Recall (2.49), (2.51) and \( u_1 < u_2 \leq u_2(R_2) \rightarrow -\infty \) on \( (R_1, \infty) \). Then for \( n \) sufficiently large, we have

\[
|\bar{u}_2(b) - \omega_2(b)| < \delta, \quad \left| \int_{R_1}^{bR_2} r e^{\omega_2} dr - \frac{2}{1 + a_2} D \right| < \delta.
\]
\[(A - B)^2 \int_{R_1}^{bR_2} r(e^{u_1} + e^{2u_1} + e^{2u_2} + e^{u_1+u_2})dr < \delta.\]

Recalling (2.54) and (2.55), we have

\[bR_2 u'_1(bR_2) \geq R_1 u'_1(R_1) + a_1 \int_{R_1}^{bR_2} re^{u_2}dr\]

\[= - \int_{R_1}^{bR_2} r[(1 + a_1)e^{u_1} + a_1(1 + a_1)e^{u_1+u_2} + a_1(1 + a_2)e^{2u_2}]dr\]

\[\leq - 2\gamma - \delta + a_1 \left( \frac{2}{1 + a_2} D - \delta \right) - \delta\]

\[= -2 a - \frac{2B}{A} \gamma + \frac{4B}{A} N_1 + \frac{4a_1}{1 + a_2} (N_2 + 1) - (2 + a_1)\delta,\]

and

\[bR_2 u'_2(bR_2) \leq R_1 u'_2(R_1) - (1 + a_2) \int_{R_1}^{bR_2} re^{u_2}dr\]

\[= r[(1 + a_2)^2 e^{2u_2} + a_2 e^{u_1} + a_1 a_2 e^{u_1+u_2}]dr\]

\[\leq \frac{2a_2}{1 + a_1} (\gamma + N_1) + 2N_2 + \delta - (1 + a_2) \left( \frac{2}{1 + a_2} D - \delta \right) + \delta\]

\[= - \frac{2a_2}{1 + a_1} (\gamma + N_1) - 2N_2 - 4 + (3 + a_2)\delta\]

\[< -4 - \frac{4a_2}{1 + a_1} - 2(2 + a_1)\delta,\] \hspace{1cm} (2.54)

where we have used \(\frac{2a_2}{1 + a_1} (\gamma - 2) > 7(1 + a_1 + a_2)\delta\) (by (2.31)) to obtain the last inequality.

Recalling \(u_1 < u_2 \leq u_2(r_2) < \ln \delta\) on \([R_2, \infty)\), we have \(F_2 < F_1 < 0\) and so \((ru'_2)'(r) = r[(1 + a_2)F_2 - a_2 F_1] < 0\) on \([R_2, \infty)\). Consequently, \(ru'_2(r) \leq bR_2 u'_2(bR_2)\) for any \(r \geq bR_2\), which implies

\[-2\beta_2 \leq -4 - \frac{4a_2}{1 + a_1} - 2(2 + a_1)\delta.\] \hspace{1cm} (2.55)

On the other hand, by \(ru'_2(r) \leq bR_2 u'_2(bR_2) < -4\) for \(r \geq bR_2\), we also have

\[\int_{bR_2}^{\infty} re^{u_1}dr \leq \int_{bR_2}^{\infty} re^{u_2}dr \leq (bR_2)^4 e^{u_2(bR_2)} \int_{bR_2}^{\infty} r^{-3}dr\]

\[= \frac{1}{2} (bR_2)^2 e^{u_2(bR_2)} = \frac{1}{2} b^2 e^{\tilde{\omega}_2(b)} \leq \frac{1}{2} b^2 e^{\omega_2(b) + \delta} \leq \frac{1}{2} \delta e^\delta < \delta.\] \hspace{1cm} (2.56)

Consequently,

\[-2\beta_1 = bR_2 u'_1(bR_2) + \int_{bR_2}^{\infty} (ru'_1(r))'dr\]

\[\geq bR_2 u'_1(bR_2) - \int_{bR_2}^{\infty} r[(1 + a_1)e^{u_1} + a_1(1 + a_1)e^{u_1+u_2} + a_1(1 + a_2)e^{2u_2}]dr\]

\[\geq bR_2 u'_1(bR_2) - (2 + a_1)\delta\]

\[\geq -\frac{2A - 2B}{A} \gamma + \frac{4B}{A} N_1 + \frac{4a_1}{1 + a_2} (N_2 + 1) - 2(2 + a_1)\delta.\]

This, together with (2.55) and \(\beta_2 \leq \beta_1\), gives

\[\frac{A - 2B}{A} (\gamma - 1) \geq \frac{2B}{A} (N_1 + 1) + \frac{4a_1}{1 + a_2} (N_2 + 1) + 1 + \frac{2a_2}{1 + a_1},\]

However, \(\alpha_1 \geq 1\) and (2.4) give

\[\frac{A - 2B}{A} (\gamma - 1) \leq \frac{2B}{A} (N_1 + 1) + \frac{2a_1}{1 + a_2} (N_2 + 1),\]

which yields a contradiction. This completes the proof. \(\square\)
Lemma 2.9. \( \lim_{\varepsilon \to 0} \int_{R_1}^{R_3,e} re^{u_1,e} dr = 0, \) \( \lim_{\varepsilon \to 0} \int_{R_1}^{R_3} re^{u_2,e} dr = \frac{2}{1+\alpha_2} D \) and \( \lim_{\varepsilon \to 0} R_3^2 e^{u_k,e(R_3,e)} = 0 \) for \( k = 1, 2. \) Consequently,

\[
\begin{align*}
\lim_{\varepsilon \to 0} R_3 e^{u_1,e}(R_3,e) & = -\frac{2A - 2B}{A} \gamma + \frac{4B}{A} N_1 + \frac{4\alpha_1}{1+\alpha_2} (N_2 + 1) \geq -2, \quad (2.57) \\
\lim_{\varepsilon \to 0} R_3 e^{u_2,e}(R_3,e) & = -\frac{2\alpha_2}{1+\alpha_1} (\gamma + N_1) - 2N_2 - 4 < -4. \quad (2.58)
\end{align*}
\]

Furthermore, \( \lim_{\varepsilon \to 0} R_3 e^{u_1,e}(R_3,e) = -2 \) if and only if \( A - 2B > 0 \) and \( \alpha_1 = 1. \)

Proof. Given any \( \mu \in (0, \delta) \), there exists a large constant \( b_\mu > 1 \) such that \( b_\mu^2 e^{2\varepsilon(b_\mu)} < \mu \) and \( \int_{b_\mu}^{\infty} re^{2\varepsilon} dr < \mu/2. \) Then similarly as the argument of Lemma 2.3 we have for \( \varepsilon > 0 \) sufficiently small that

\[
|\bar{u}_2(b_\mu) - \omega_2(b_\mu)| < \mu, \quad \left| \int_{R_1}^{b_\mu R_2} re^{u_2} dr - \frac{2}{1+\alpha_2} D \right| < \mu,
\]

\[\begin{align*}
(A - B)^2 \int_{R_1}^{b_\mu R_2} \left( e^{u_1} + e^{2\varepsilon} + e^{u_2} + e^{u_1+u_2} \right) dr < \mu, \\
\left| R_1 u_2'(R_1) - \frac{2\alpha_2}{1+\alpha_1} (\gamma + N_1) - 2N_2 \right| < \mu.
\end{align*}\]

Consequently, by repeating the argument of (2.54)-(2.56), we can prove

\[
\int_{b_\mu R_2}^{R_3} re^{u_1} dr \leq \int_{b_\mu R_2}^{R_3} re^{u_2} dr \leq \frac{1}{2} b_\mu^2 e^{2\varepsilon(b_\mu)} < \mu,
\]

and

\[
R_3^2 e^{u_1(R_3,e)} = R_3^2 e^{u_2(R_3,e)} < (b_\mu R_2)^2 e^{2\varepsilon(b_\mu,R_2)} = b_\mu^2 e^{2\varepsilon(b_\mu)} \leq \frac{1}{2} b_\mu^2 e^{2\varepsilon(b_\mu)} < \mu,
\]

namely \( R_3^2 e^{u_k(e(R_3,e))} \leq 2\mu, \int_{R_1}^{R_3} re^{u_1} dr < 2\mu \) and \( \int_{R_1}^{R_3} re^{u_2} dr = \frac{2}{1+\alpha_2} D \) for \( \varepsilon > 0 \) sufficiently small. This proves

\[
\lim_{\varepsilon \to 0} R_3^2 e^{u_k(R_3,e)} = 0, \quad \lim_{\varepsilon \to 0} \int_{R_1}^{R_3} re^{u_1} dr = 0, \quad \lim_{\varepsilon \to 0} \int_{R_1}^{R_3} re^{u_2} dr = \frac{2}{1+\alpha_2} D. \quad (2.59)
\]

Recalling \( u_1 < u_2 \leq u_2(R_2) \to -\infty \) on \((R_1, R_3)\) as \( \varepsilon \to 0 \), we have

\[
\lim_{\varepsilon \to 0} \int_{R_1}^{R_3} r (e^{2\varepsilon} + e^{u_2} + e^{u_1+u_2}) dr = 0.
\]

Consequently, we integrate (2.18) over \((R_1, R_3)\) to derive

\[
\begin{align*}
\lim_{\varepsilon \to 0} R_3 u_1'(R_3) & = \lim_{\varepsilon \to 0} \left[ R_1 u_1'(R_1) + \alpha_1 \int_{R_1}^{R_3} re^{u_2} dr \right] = -2\gamma + \frac{2\alpha_1}{1+\alpha_2} D \\
& = -\frac{2A - 2B}{A} \gamma + \frac{4B}{A} N_1 + \frac{4\alpha_1}{1+\alpha_2} (N_2 + 1) \geq -2, \quad (by \quad (2.11))
\end{align*}
\]

\[
\begin{align*}
\lim_{\varepsilon \to 0} R_3 u_2'(R_3) & = \lim_{\varepsilon \to 0} \left[ R_1 u_2'(R_1) - (1+\alpha_2) \int_{R_1}^{R_3} re^{u_2} dr \right] \\
& = \frac{2\alpha_2}{1+\alpha_1} (\gamma + N_1) + 2N_2 - 2D = -\frac{2\alpha_2}{1+\alpha_1} (\gamma + N_1) - 2N_2 - 4.
\end{align*}
\]

This completes the proof. \( \square \)

Fix any constant \( \vartheta \in (0, \delta) \) such that

\[
\vartheta = 0 \quad \text{if} \quad \alpha_1 > 1 \quad \text{and} \quad \vartheta > 0 \quad \text{if} \quad \alpha_1 = 1. \quad (2.60)
\]

Then by Lemma 2.4, there exists \( \varepsilon_5 \in (0, \varepsilon_4) \) such that \( R_3^2 e^{u_1,e(R_3,e)} < \delta \) and \( R_3 e^{u_1,e(R_3,e)} > -2 - \vartheta > R_3 e^{u_2,e(R_3,e)} + 1 \) for any \( \varepsilon \in (0, \varepsilon_5). \) For each \( \varepsilon \in (0, \varepsilon_5), \) we define

\[
\begin{align*}
R_4 & = R_{4,\varepsilon} := \sup \{ r \in [R_3,\varepsilon] \mid ru_1,e(r) > -2 - \vartheta \} , \\
R_5 & = R_{5,\varepsilon} := \sup \{ r \in (R_3,\varepsilon) \mid u_1,e > u_2,e \text{ on } (R_3,e, r) \}.
\end{align*}
\]

Clearly, \( R_{4,\varepsilon}, R_{5,\varepsilon} > R_3,e \) for all \( \varepsilon \in (0, \varepsilon_5). \)
Lemma 2.10. There exists a small $\varepsilon_0 \in (0, \varepsilon)$ such that for each $\varepsilon \in (0, \varepsilon_0)$, $u_{2, \varepsilon} < u_{1, \varepsilon} < \ln \delta$ on $(R_{3, \varepsilon}, R_{5, \varepsilon})$, $ru_1'(r)$ is strictly decreasing on $(R_{3, \varepsilon}, R_{5, \varepsilon})$ and $R_{4, \varepsilon} \leq R_{5, \varepsilon}$.

In particular, if $\alpha_1 > 1$, then $R_{4, \varepsilon} < R_{5, \varepsilon}$, namely $R_{4, \varepsilon}u_1'(R_{4, \varepsilon}) = -2$, $ru_1'(r) > -2$ on $[R_{3, \varepsilon}, R_{4, \varepsilon})$ and $ru_1'(r) < -2$ on $(R_{4, \varepsilon}, R_{5, \varepsilon})$.

**Proof.** Step 1. We claim that

$$\sup_{[R_{3, \varepsilon}, R_{5, \varepsilon})} u_{1, \varepsilon} \to -\infty \text{ as } \varepsilon \to 0. \quad (2.61)$$

Suppose by contradiction that there exist a sequence $\varepsilon_n \downarrow 0$ and a constant $c_0 < \ln \delta$ such that $\sup_{[R_{3, \varepsilon_n}, R_{5, \varepsilon_n})} u_{1, \varepsilon_n} \geq c_0$ for all $n$. We will omit the subscript $\varepsilon_n$ for convenience. The following proof is similar to Step 1 of Lemma 2.4. For $n$ large, there exist $b_n, d_n \in (R_3, R_5)$ such that $b_n < d_n$, $u_1(d_n) = c_0 - 1$, $u_1(b_n) = c_0 - 2$, $u_1 \leq c_0 - 1$ on $[R_3, d_n]$ and $u_1 \geq c_0 - 2$ on $[b_n, d_n]$. Clearly, $u_1'(d_n) \geq 0$, $u_2 < u_1 < \ln \delta$ and so $F_1 < F_2 < 0$ on $(R_3, d_n)$, which implies

$$(ru_1'(r))' = r[(1 + a_1)F_1 - a_1F_2] < rF_1 < -\frac{1}{2}ru_1'(r) \text{ on } (R_3, d_n). \quad (2.62)$$

Then $0 \leq ru_1'(r) \leq R_3u_1'(R_3) \leq C$ for any $r \in [R_3, d_n]$, which yields $d_n - b_n \to \infty$. Consequently,

$$0 \leq d_nu_1'(d_n) = b_nu_1'(b_n) + \int_{b_n}^{d_n} (ru_1')' dr \leq R_3u_1'(R_3) - \frac{1}{2} \int_{b_n}^{d_n} re^{u_1} dr$$

$$\leq C - \frac{1}{2}e^{c_0 - 2} \int_{b_n}^{d_n} r dr \to -\infty \text{ as } n \to \infty,$$

a contradiction.

Step 2. By Step 1, for $\varepsilon > 0$ sufficiently small, we have $u_2 < u_1 < \ln \delta$ on $(R_3, R_5)$, which implies that $(2.62)$ holds on $(R_3, R_5)$ and so $ru_1'(r)$ is strictly decreasing on $(R_3, R_5)$.

Step 3. We prove that $R_{4, \varepsilon} \leq R_{5, \varepsilon}$ for $\varepsilon > 0$ sufficiently small.

Assume by contradiction that there exists a sequence $\varepsilon_n \downarrow 0$ such that $R_{5, \varepsilon_n} < R_{4, \varepsilon_n}$. Again, we omit the subscript $\varepsilon_n$ for convenience. Consequently, $R_5 < \infty$ and $ru_1'(r) > -2 + \vartheta$ for $r \in (R_3, R_5)$. Since $a_2ru_1' + (1 + a_1)ru_2'' = (A - B)rF_2 < 0$ on $[R_3, R_5]$, we have

$$ru_2''(r) \leq \frac{a_2}{1 + a_1} R_3u_1'(R_3) + 2 + \vartheta + R_3u_2''(R_3) =: l_n \text{ uniformly for } r \in [R_3, R_5].$$

Recalling $(2.12)$ and $(2.57)$–$(2.58)$, it is easy to see that

$$l := \lim_{n \to \infty} l_n = \frac{4a_2}{1 + a_1} \frac{B - A}{A} \gamma - \frac{2a_2}{1 + a_1} \frac{A - 2B}{A} \bar{N}_1 - 2 \frac{A - 2B}{A} \bar{N}_2 + 2 + \frac{a_2\vartheta}{1 + a_1}.$$ 

We claim $l < -2 - \delta$. Recall $3A - 4B > 0$, $\bar{N}_1 > 1$, $\vartheta < \delta$ and $(2.19)$. Clearly $l < -2 - \frac{a_2}{1 + a_1} \frac{A - B}{A} \gamma < -2 - \delta$ if $A \geq 2B$. Let us consider the remaining case $\frac{1}{2}B < A < 2B$.

By $a_2 > 1$ and $(2.40)$, we easily obtain

$$\frac{a_2}{1 + a_1} \bar{N}_1 > \frac{a_2}{1 + a_1} \frac{4B - A}{A} \bar{N}_1 + \frac{4B - A}{3A - 4B} \bar{N}_2$$

$$> \frac{a_2}{1 + a_1} \frac{2B - A}{A} \bar{N}_1 + \frac{2B - A}{A - B} \bar{N}_2, \quad (2.63)$$

where we have used $\frac{4B - A}{3A - 4B} > \frac{2B - A}{A - B}$. So we also get $l < -2 - \frac{a_2}{1 + a_1} \frac{A - B}{A} \gamma < -2 - \delta$.

Hence, for a sufficiently large, $ru_2''(r) \leq l_n < -2 - \vartheta < ru_1''(r)$ for all $r \in [R_3, R_5]$, namely $u_2 - u_1$ is strictly decreasing on $[R_3, R_5]$, which contradicts to $u_2(R_3) - u_1(R_3) = 0$.

Step 4. Let $\alpha_1 > 1$, then $\vartheta = 0$. We prove that $R_4 \leq R_5$ for $\varepsilon > 0$ sufficiently small.

Assume by contradiction that $R_4 = \infty$ for some $\varepsilon > 0$ sufficiently small. Then $R_5 = \infty$ and so $ru_1'(r) > -2$ for all $r \geq R_3$, which implies $\int_{R_3}^{\infty} re^{u_1} dr = \infty$. On the other hand, Step 2 shows that $(2.62)$ holds on $(R_3, \infty)$, so

$$\infty = \frac{1}{2} \int_{R_3}^{\infty} re^{u_1} dr \leq \limsup_{r \to \infty} \left[-tu_1'(t)\right]_{R_3} \leq R_3u_1'(R_3) + 2 \leq C,$$

a contradiction. So $R_4 < \infty$ for $\varepsilon > 0$ sufficiently small. Then by repeating the argument of Step 3, we finally conclude that $R_4 < R_5$ for $\varepsilon > 0$ sufficiently small. $\square$
Lemma 2.11. For each $\varepsilon \in (0, \varepsilon_6)$, $u_{k,\varepsilon}(r) + 2\ln r \leq C$ uniformly for $r \in [R_3, R_5, \varepsilon]$ and $k = 1, 2$. Furthermore, $\lim_{\varepsilon \to 0} \int_{R_3}^{R_4} r e^{u_{k,\varepsilon}} \, dr = 0$ and
\[
\lim_{\varepsilon \to 0} \int_{R_3}^{R_4} \left[A_2 \frac{A - B}{A} \gamma - 2 a_2 \frac{A - 2 B}{A} N_1 - 2 \frac{A - 2 B}{1 + a_2} N_2 - 4 \frac{A - B}{1 + a_2} \right] \, dr = 0
\]
using (2.61) for all $r \in [R_3, R_5, \varepsilon]$.

Proof. Step 1. For each $\varepsilon \in (0, \varepsilon_6)$, we claim that
\[
u_k(r) + 2\ln r \leq C \quad \text{uniformly for } r \in [R_3, R_5].
\]
Lemma 2.10 shows that $u_2 < u_1 < \ln \delta$ and $(ru_1')^2 < -\frac{1}{2} te^{u_1}$ on $[R_3, R_5]$. Hence
\[-2 - \vartheta < ru_1'(r) \leq R_3u_2'(R_3) - 1 \leq \frac{1}{2} \int_{R_3}^{R_4} r e^{u_1} \, dr, \quad \forall r \in (R_3, R_4),
\]
which implies
\[
\int_{R_3}^{R_4} r (e^{u_1} + e^{u_2}) \, dr \leq 2 \int_{R_3}^{R_4} re^{u_1} \, dr \leq C.
\]
This, together with (2.61), gives
\[
\lim_{\varepsilon \to 0} \int_{R_3}^{R_4} r \left[e^{u_1} + e^{u_2} + e^{u_1 + u_2}\right] \, dr = 0.
\]
Then by repeating the argument of Step 2 in Lemma 2.4, we have $u_2(r) + 2\ln r \leq C$ for all $r \in (R_3, R_4)$ and $k = 1, 2$. If $R_4 = R_5$, we are done. If $R_4 < R_5$, by $ru_1'(r) < -2 - \vartheta$ for $r \in (R_4, R_5)$, we conclude that
\[
u_2(r) + 2\ln r < u_1(r) + 2\ln r \leq u_1(R_4) + 2\ln R_4 \leq C
\]
for all $r \in [R_4, R_5]$. This proves (2.65).

Step 2. Recalling (2.10) and $\alpha_2 = \alpha_2 - 1 > 0$, we claim that for $\varepsilon > 0$ sufficiently small,
\[
u_2(r) \leq -2 - \frac{3}{2} \alpha_2 \quad \text{uniformly for } r \in [R_3, R_5].
\]
By (2.65) we have
\[
\lim_{\varepsilon \to 0} \int_{R_3}^{R_4} r \left[e^{u_1} + e^{u_2} + e^{u_1 + u_2}\right] \, dr = 0.
\]
Consequently, by integrating the Pohozaev identity (2.23) over $[R_3, r]$, we obtain
\[
J(ru_1'(r) + 2, ru_2'(r) + 2) \leq J(ru_1'(r) + 2, ru_2'(r) + 2) + (A - B)^2 \left[a_2 e^{u_1}
+ a_1 e^{u_2} - a_2 \frac{1}{2} \alpha_1 e^{u_1} - a_2 \frac{1}{2} \alpha_2 e^{u_1 + u_2}\right]
= J(R_3 u_1'(R_3) + 2, R_3 u_2'(R_3) + 2) + o(1)
\]
uniformly for all $r \in [R_3, R_5]$ as $\varepsilon \to 0$. On the other hand, since $a_2 ru_1'(1 + a_1 ru_2)' = (A - B) F_2 < 0$ on $[R_3, R_5]$, we have for any $r \in [R_3, R_5]$ that
\[
ru_1'(r) + 2 + (1 + a_1)(ru_2'(r) + 2)
\leq a_2 R_3 u_1'(R_3) + 2 + (1 + a_1)(R_3 u_2'(R_3) + 2) =: \eta_k.
\]
Recalling (2.12) and (2.57)-(2.58), we have
\[
\eta := \lim_{\varepsilon \to 0} \eta_k = -4a_2 \frac{A - B}{A} \gamma - 2 a_2 \frac{A - 2 B}{A} N_1 - 2 \frac{A - 2 B}{1 + a_2} N_2.
\]
Using (2.68) if $A < 2B$, we easily see $\eta < 0$. Hence $\eta$ < 0 for $\varepsilon > 0$ sufficiently small. Note from (2.12) that

$$J(x, y) = \frac{a_2(A - B)}{2(1 + a_1)} x^2 + \frac{a_1}{2(1 + a_1)} [a_2x + (1 + a_1)y]^2.$$ 

This, together with (2.68)–(2.69) and $\eta < 0$, easily yields

$$[ru_1'(r) + 2]^2 \leq [R_3u_1'(R_3) + 2]^2 + o(1)$$

and so $ru_1'(r) \geq -R_3u_1'(R_3)e^2o(1)$ uniformly for any $r \in [R_3, R_5]$ as $\varepsilon \to 0$. Substituting this inequality into (2.69) and recalling (2.57)–(2.68), we finally obtain

$$ru_1(r) \leq \frac{2a_2}{1 + a_1} R_3u_1'(R_3) + R_3u_2'(R_3) + \frac{4a_2}{1 + a_1} + o(1)$$

$$= \frac{2a_2}{1 + a_1} \frac{3A - 4B}{A} - 2\alpha(u_1) + \frac{2a_2}{1 + a_1} \left( - \frac{2A - 4B}{A} \right) \tilde{N}_1 - 2 - o(1)$$

$$= -2\alpha + 2 + o(1) \ (\text{by (2.10)})$$

uniformly for any $r \in [R_3, R_5]$ as $\varepsilon \to 0$. This proves (2.66).

**Step 3.** We prove $\lim_{\varepsilon \to 0} \int_{R_3}^{R_5} e^{-u_2'} dr = 0$ and (2.64).

By (2.69), $r^{2\alpha} e^{-u_2'(r)}$ is strictly decreasing for $r \in [R_3, R_5]$, so Lemma 2.9 gives

$$\int_{R_3}^{R_5} r e^{-u_2'} dr \leq \frac{1}{\alpha_2} R_3^{\alpha} e^{-u_2(R_3)} \to 0 \text{ as } \varepsilon \to 0.$$ 

Then by integrating $[a_2ru_1' + (1 + a_1)r u_2'] = (A - B)rF_2$ over $[R_3, r]$ for any $r \in [R_3, R_5]$ and recalling $|F_2| \leq (1 + a_1) e^{u_2}$, we easily obtain (2.64). This completes the proof.

Now we consider the cases $\alpha_1 = 1$ and $\alpha_1 > 1$ separately.

**2.1 The case $\alpha_1 = 1$**

In this subsection, we consider the special case $\alpha_1 = 1$. Consequently, we see from (2.2), (2.9) and (2.12) that

$$A - 2B > 0 \text{ and } \tilde{\gamma} = \frac{2B}{A - 2B} \tilde{N}_1 + \frac{2a_1(1 + a_1)}{A - 2B} \tilde{N}_2.$$ 

(2.70)

The following lemma provides an evidence that this case is different from the case $\alpha_1 > 1$.

**Lemma 2.12.** There exists a small $\varepsilon_7 \in (0, \varepsilon_6)$ such that for each $\varepsilon \in (0, \varepsilon_7)$, $R_{4, \varepsilon} = R_{5, \varepsilon} = +\infty$.

**Proof.** Assume by contradiction that $R_{4, \varepsilon_n} < +\infty$ for some sequence $\varepsilon_n \downarrow 0$. We will omit the subscript $\varepsilon_n$ for convenience. Then $R_4 u_1'(R_4) = -2 - \vartheta$. This, together with (2.64) and (2.70), gives

$$\lim_{n \to \infty} R_4 u_2'(R_4) + 2 = \frac{4a_2}{1 + a_1} \frac{B - A}{A} \tilde{\gamma} - \frac{2a_2}{1 + a_1} \frac{A - 2B}{A} \tilde{N}_1 - 2 - 2B \tilde{N}_2 + \frac{a_2 \vartheta}{1 + a_1}$$

$$= - \frac{1 + a_2}{a_1} \tilde{\gamma} + \frac{a_2}{1 + a_1} \vartheta.$$ 

(2.71)

Similarly, by Lemma 2.9 we have $\lim_{n \to \infty} R_3 u_1'(R_3) + 2 = 0$ and

$$\lim_{n \to \infty} R_3 u_2'(R_3) + 2 = - \frac{2a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) - 2\tilde{N}_2 = - \frac{1 + a_2}{a_1} \tilde{\gamma}.$$ 

Since we have assumed $R_4 < \infty$, then by (2.65) and (2.66), we have $R^2 e^{-u_i'(R_4) + u_i'(R_4)} \leq CR_4^{-2} \to 0$ for $1 \leq i \leq 2$ and $R^2 e^{-u_2'(R_4)} < R^2 e^{-u_2'(R_5)} \to 0$ as $n \to \infty$. Combining these with (2.67), we can repeat the proof of Lemma 2.7 to obtain (similar to (2.52))

$$a_2(A - B) R^2 e^{-u_1'(R_4)}$$

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=J(R_3u'_1(R_3)+2,R_3u'_2(R_3)+2)−J(R_4u'_1(R_4)+2,R_4u'_2(R_4)+2)+o(1)

=J\left(0,−\frac{1+a_2}{a_1}\right)−J\left(−\vartheta,\frac{1+a_2}{a_1}\tilde{\gamma}+\frac{a_2}{1+a_1}\vartheta\right)+o(1)

=−\frac{a_2(A−B)}{2(1+a_1)}\vartheta^2+o(1) \text{ as } n \to \infty,

which yields a contradiction with \vartheta > 0.

Now we can finish the proof of Theorem \ref{thm1} for \(\alpha_1 = 1\).

**Completion of the proof of Theorem \ref{thm1} for \(\alpha_1 = 1\).** Let \(\varepsilon \in (0,\varepsilon_\gamma)\), then \(R_{4,\varepsilon} = R_3, \varepsilon = R_3^* = +\infty\). Since Lemma \ref{lem10} shows that \(u_{2,\varepsilon}(r) < u_{k,\varepsilon}(r) < \ln \delta\) for any \(r \in (R_3,+,\infty)\), we conclude that \((u_{1,\varepsilon},u_{2,\varepsilon})\) is an entire solution. By Theorem A, there exists \((\alpha_{1,\varepsilon},\alpha_{2,\varepsilon}) \in \Omega\) such that

\[ u_{k,\varepsilon}(r) = −2\alpha_{k,\varepsilon} \ln r + O(1) \text{ as } r \to \infty, \quad k = 1, 2. \]

Consequently, \(ru'_{k,\varepsilon}(r) = −2\alpha_{k,\varepsilon}\) as \(r \to \infty\). Then Lemma \ref{lem10} and the definition of \(R_{4,\varepsilon}\) yield \(-2 - \vartheta \leq −2\alpha_{1,\varepsilon} < R_{4,\varepsilon}u_{1,\varepsilon}'(R_{4,\varepsilon})\), namely

\[ 2 + \vartheta \geq \limsup_{\varepsilon \to 0} 2\alpha_{1,\varepsilon} \geq \liminf_{\varepsilon \to 0} 2\alpha_{1,\varepsilon} \geq −\lim_{\varepsilon \to 0} R_{3,\varepsilon}u_{1,\varepsilon}'(R_{3,\varepsilon}) = 2. \]

Since \(\vartheta \in (0,\delta)\) can be taken apriori arbitrary small, we conclude that

\[ \lim_{\varepsilon \to 0} \alpha_{1,\varepsilon} = \alpha_1 = 1. \]

This, together with \ref{eq2.71}, easily implies

\[ a_2 + (1 + a_1) \lim_{\varepsilon \to 0} \alpha_{2,\varepsilon} = 2a_2 \frac{A−B}{A} \tilde{\gamma} + a_2 \frac{A−2B}{1+a_2} \tilde{N}_1 + \frac{A−2B}{1+a_2} \tilde{N}_2 + 2 \frac{A−B}{1+a_2}. \]

so

\[ \lim_{\varepsilon \to 0} \alpha_{2,\varepsilon} = \frac{2a_2}{1+a_1} \frac{A−B}{A} \tilde{\gamma} + \frac{a_2}{1+a_1} \frac{A−2B}{1+a_2} \tilde{N}_1 + \frac{A−2B}{1+a_2} \tilde{N}_2 + 1 \]

\[ = \frac{1+a_2}{2a_1} \tilde{\gamma} + 1 \quad \text{(by \ref{eq2.71})} \]

\[ = \alpha_2. \quad \text{(by \ref{eq2.71})} \]

By Step 3 in the proof of Lemma \ref{lem11} we have \(\int_{R_{3,\varepsilon}}^\infty rF_2dr \to 0\) as \(\varepsilon \to 0\). Consequently, by integrating \((ru_{1,\varepsilon}') = (1 + a_1)rF_1 - a_1rF_2\) over \((R_{3,\varepsilon},+,\infty)\), we deduce from \(-2\alpha_{1,\varepsilon} < R_{3,\varepsilon}u_{1,\varepsilon}'(R_{3,\varepsilon}) \to 0\) that \(\int_{R_{3,\varepsilon}}^\infty rF_1dr \to 0\) as \(\varepsilon \to 0\). Then by \ref{lem14} we conclude that

\[ \lim_{\varepsilon \to 0} \int_{R_{3,\varepsilon}}^\infty re^{u_{1,\varepsilon}}dr = 0. \tag{2.72} \]

Note from \ref{lem14} and \ref{lem10} that \(u_{2,\varepsilon}(r) < 0\) for all \(r > R_{2,\varepsilon}\). Besides, Lemma \ref{lem3} shows that \(\sup_{0, R_{1,\varepsilon}} u_{2,\varepsilon} \to −\infty\). Combining these with \ref{eq2.66}, we conclude \(\sup_{R_{2,\varepsilon}} u_{2,\varepsilon} \to −\infty\) as \(\varepsilon \to 0\). This completes the proof.

**Remark 2.1.** In Theorem C where bubbling solutions of type I are constructed, we assumed \(\alpha_1 > 1\), which plays an essential role in the proof of Theorem C (see \ref{23}). In particular, the conclusion \((\alpha_{1,\varepsilon},\alpha_{2,\varepsilon}) \to (\alpha_1,\alpha_2)\) is a corollary of \ref{eq2.72} in the proof of Theorem C. However, for the case \(\alpha_1 = 1\) studied here, the idea used in Theorem C does not apply. Here we need to use an opposite argument: the conclusion \ref{eq2.72} is a corollary of \((\alpha_{1,\varepsilon},\alpha_{2,\varepsilon}) \to (\alpha_1,\alpha_2)\).
2.2 The case $\alpha_1 > 1$

In this subsection, we consider the generic case $\alpha_1 > 1$. Then $\vartheta = 0$ and $R_{4, \varepsilon} < R_{5, \varepsilon}$. The following lemma also provides an evidence that this case is different from the special case $\alpha_1 = 1$.

**Lemma 2.13.** Recalling $\alpha_1 > 1$ in (2.79), there holds (compare to (2.72))

$$\lim_{\varepsilon \to 0} \int_{R_{3, \varepsilon}} \varepsilon u_1 dr = \frac{4}{1 + a_1} \left( \frac{2B - A}{A} \tilde{\gamma} + \frac{2B}{A} \tilde{N}_1 + \frac{2a_1}{1 + a_2} \tilde{N}_2 \right) = \frac{4(\alpha_1 - 1)}{1 + a_1}. \quad (2.73)$$

**Proof.** Step 1. We claim that

$$\lim_{\varepsilon \to 0} R_{4, \varepsilon}^2 e^{u_1, \varepsilon(R_{4, \varepsilon})} = \frac{E^2}{2(1 + a_1)}, \quad (2.74)$$

where $E > 0$ is defined by

$$E := \lim_{\varepsilon \to 0} R_{3, \varepsilon} u'_1(R_{3, \varepsilon}) + 2 = 2 \frac{2B - A}{A} \tilde{\gamma} + \frac{4B}{A} \tilde{N}_1 + \frac{4a_1}{1 + a_2} \tilde{N}_2 = 2(\alpha_1 - 1). \quad (2.75)$$

Recalling (2.64) and $R_4 u'_1(R_4) = -2$, there holds

$$F := \lim_{\varepsilon \to 0} R_{3, \varepsilon} u_2(R_{3, \varepsilon}) + 2 = -\frac{4a_2}{1 + a_1} \frac{A - B}{A} \tilde{\gamma} - \frac{2a_2}{1 + a_1} \frac{A - 2B}{A} \tilde{N}_1 + \frac{2A - 2B}{A} \tilde{N}_2. \quad (2.76)$$

Again, (2.65) and (2.66) imply $R_{3, \varepsilon}^2 e^{u_i(R_{4, \varepsilon})} \leq CR_{4, \varepsilon}^{-2} \to 0$ for $1 \leq i, j \leq 2$ and

$$R_{3, \varepsilon}^2 e^{u_2(R_{4, \varepsilon})} < R_{4, \varepsilon}^2 e^{u_2(R_{4, \varepsilon})} \to 0 \text{ as } \varepsilon \to 0. \quad (2.77)$$

Combining these with (2.71), we can repeat the proof of Lemma 2.7 to obtain (similar to 2.59)

$$a_2(A - B) R_{4, \varepsilon}^2 e^{u_1(R_{4, \varepsilon})}$$

$$= J(R_{4, \varepsilon} u'_1(R_{4, \varepsilon}) + 2, R_{4, \varepsilon} u_2(R_{4, \varepsilon}) + 2) - J(R_{4, \varepsilon} u'_1(R_{4, \varepsilon}) + 2, R_{4, \varepsilon} u_2(R_{4, \varepsilon}) + 2) + o(1)
$$

$$= J \left( E, \frac{2a_1}{1 + a_2} \left( \tilde{\gamma} + \tilde{N}_1 \right) - 2 \tilde{N}_2 \right) - J(0, F) + o(1)
$$

$$= J \left( -2 \tilde{\gamma}, G \right) + J \left( 0, \frac{A - 2B}{A} G - \frac{2a_2}{1 + a_1} \tilde{\gamma} \right) + o(1)
$$

$$= J \left( 2 \tilde{\gamma}, G \right) - J \left( 0, \frac{A - 2B}{A} G - \frac{2a_2}{1 + a_1} \tilde{\gamma} \right) + o(1) \quad \text{(by (2.13))}
$$

$$= \frac{a_2}{2(1 + a_1)} \left( \frac{2a_1}{1 + a_2} G + 2 \tilde{\gamma} \right)^2 + o(1)
$$

$$= \frac{a_2}{2(1 + a_1)} E^2 + o(1) \text{ as } \varepsilon \to 0.
$$

Hence (2.74) holds.

Step 2. For $\varepsilon > 0$ sufficiently small, we consider the scaled functions

$$\tilde{u}_k(r) = \tilde{u}_k(r) = u_k(R_{4, \varepsilon} r) + 2 \ln R_{4, \varepsilon} \text{ for } k = 1, 2,$$

where $R_{4, \varepsilon} \leq r < R_{5, \varepsilon}$. Then $(\tilde{u}_1, \tilde{u}_2)$ satisfies

$$\begin{cases}
\frac{\partial}{\partial r} \tilde{u}_1' + \tilde{u}_1' = (1 + a_1) \left( (1 + a_1) R_{4, \varepsilon}^{-2} e^{2 \tilde{\gamma}} - e^{\tilde{\gamma}} - a_1 R_{4, \varepsilon}^{-2} e^{\tilde{\gamma} + \tilde{u}_1} \right) - a_1 \left( (1 + a_2) R_{4, \varepsilon}^{-2} e^{2 \tilde{\gamma}} - e^{\tilde{\gamma}} - a_2 R_{4, \varepsilon}^{-2} e^{\tilde{\gamma} + \tilde{u}_2} \right),
\end{cases}
$$

$$\begin{cases}
\frac{\partial}{\partial r} \tilde{u}_2' + \tilde{u}_2' = (1 + a_2) \left( (1 + a_2) R_{4, \varepsilon}^{-2} e^{2 \tilde{\gamma}} - e^{\tilde{\gamma}} - a_2 R_{4, \varepsilon}^{-2} e^{\tilde{\gamma} + \tilde{u}_2} \right) - a_2 \left( (1 + a_1) R_{4, \varepsilon}^{-2} e^{2 \tilde{\gamma}} - e^{\tilde{\gamma}} - a_1 R_{4, \varepsilon}^{-2} e^{\tilde{\gamma} + \tilde{u}_1} \right).
\end{cases}
$$

(2.77)
By Lemma 2.10, it is easy to see that $u_1(r) + 2\ln r \leq u_1(R_4) + 2\ln R_4$ for any $r \in [R_3, R_5]$, so
\[
\tilde{u}_2(r) + 2\ln r < \tilde{u}_1(r) + 2\ln r \leq \tilde{u}_1(1) = \ln \frac{E^2}{2(1 + a_1)} + o(1), \quad \forall \frac{R_3}{R_4} < r < \frac{R_5}{R_4}. \tag{2.78}
\]
Moreover, (2.76) gives $\tilde{\gamma}$ for some constant $\lambda > 1$. By Lemma 2.10, it is easy to see that $u_1(\omega) \leq u_1(\omega)$. Recalling (2.77), we have
\[
\tilde{u}_1(1) - \ln \left( R^2 e^{u_1(R_3)} \right) - 2\ln \frac{R_4}{R_3} = \tilde{u}_1(1) - \tilde{u}_1 \left( \frac{R_4}{R_3} \right) \leq C \left( \frac{R_4}{R_3} - 1 \right).
\]
Recalling $R^2 e^{u_1(R_3)} \rightarrow 0$ and (2.78), we conclude that $\frac{R_4}{R_3} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

By (2.78) and $R^2 e^{u_1} = e^{u_1} < 1$ for any $r \in \left( \frac{R_3}{R_4}, \frac{R_4}{R_3} \right)$, it is easy to deduce from (2.77) that $\|ru_1'(r)\| \leq C/r$ for all $r \in \left( \frac{R_3}{R_4}, \frac{R_4}{R_3} \right)$. Consequently, $\tilde{u}_1$ is uniformly bounded in $C_{loc}^2((0, \infty))$ and $\tilde{u}_2 \rightarrow -\infty$ uniformly on any compact subset $K \subset \subset (\frac{R_3}{R_4}, \frac{R_4}{R_3})$. This, together with the definition of $R_5$, yields $\frac{R_5}{R_4} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

**Step 4.** We claim that $\tilde{u}_1 \rightarrow \omega_1$ in $C_{loc}^2((0, \infty))$ as $\varepsilon \rightarrow 0$, where
\[
\omega_1(r) = \ln \frac{2E^2 r^{-2}}{(1 + a_1)(1 + rE)^2} \quad \text{for } r \in (0, \infty), \tag{2.79}
\]
and $E$ is seen in (2.76). Consequently,
\[
\int_0^{\infty} r e^{\omega_1} dr = \frac{2}{1 + a_1} E. \tag{2.80}
\]

By Step 3, up to a subsequence, we may assume that $\tilde{u}_1 \rightarrow \omega_1$ in $C_{loc}^2((0, \infty))$, where $\omega_1$ satisfies
\[
\begin{aligned}
\omega_1'' + \frac{1}{r} \omega_1' &= -(1 + a_1) e^{\omega_1}, \\
\omega_1(r) + 2\ln r &\leq \omega_1(1) = \ln \frac{E^2}{2(1 + a_1)} \quad \text{for } r \in (0, \infty) \tag{2.81}.
\end{aligned}
\]
Since $\omega_1'(1) = -2$ and $r\omega_1'(r)$ is strictly decreasing on $(0, \infty)$, it is easy to prove that $\int_0^{\infty} r e^{\omega_1} dr < \infty$. Recalling $-2 \leq \tilde{u}_1'(r) = R_4 u_1'(R_4 r) \leq R_4 u_1'(R_3) = E - 2 + o(1)$ for all $r \in [R_3/R_4, 1]$, we easily obtain
\[
2\gamma_1 := \lim_{r \rightarrow 0} r \omega_1'(r) \in (-2, E - 2).
\]

In conclusion, $\omega_1$ is a radial solution of the Liouville equation with singular sources
\[
\Delta v + (1 + a_1) e^v = 4\pi \gamma_1 \delta_0 \quad \text{in } \mathbb{R}^2, \quad \int_0^{\infty} r e^v dr < \infty.
\]
Since $\gamma_1 > -1$, again by the classification result due to Prajapat and Tarantello [29], there holds
\[
\omega_1(r) + \ln(1 + a_1) = \ln \frac{8\lambda(1 + \gamma_1)^{2\gamma_2}}{(1 + \lambda r^{2\gamma_2} + 2\gamma_2)^2}
\]
for some constant $\lambda > 0$. By $\omega_1'(1) = -2$ and $\omega_1(1) = \ln \frac{E^2}{2(1 + a_1)}$, a direct computation gives $\gamma_1 = E/2$ and $\lambda = 1$. This proves (2.79) and (2.80). Clearly, the above argument also shows that $\tilde{u}_1 \rightarrow \omega_1$ in $C_{loc}^2((0, \infty))$ as $\varepsilon \rightarrow 0$ (i.e. not only along a subsequence).

**Step 5.** We prove (2.76).
Given any \( \mu \in (0, \delta) \). By (2.34) - (2.36), there exist small constant \( b_\mu \in (0, 1) \) and large constant \( d_\mu > 1 \) such that

\[
b_\mu^2 e^{\alpha_1(b_\mu)} + d_\mu^2 e^{\alpha_1(d_\mu)} + \left| \int_{b_\mu}^{d_\mu} r e^{\alpha_1} dr - \frac{2E}{1 + a_1} \right| \leq \frac{\mu}{2},
\]

\[
b_\mu \alpha_1'(b_\mu) + 2 \geq \frac{2}{3} E, \quad d_\mu \alpha_1'(d_\mu) + 2 \leq -\frac{2}{3} E.
\]

Consequently, since \( \tilde{u}_1 \to \omega_1 \) in \( C^2([b_\mu, d_\mu]) \), there exists sufficiently small \( \epsilon_\mu > 0 \) such that for each \( \epsilon \in (0, \epsilon_\mu) \), we have

\[
b_\mu^2 e^{\alpha_1(b_\mu)} + d_\mu^2 e^{\alpha_1(d_\mu)} + \left| \int_{b_\mu}^{d_\mu} r e^{\alpha_1} dr - \frac{2E}{1 + a_1} \right| \leq \mu,
\]

\[
b_\mu \tilde{u}_1'(b_\mu) + 2 \geq \frac{1}{2} E, \quad d_\mu \tilde{u}_1'(d_\mu) + 2 \leq -\frac{1}{2} E.
\]

Recalling that \( r \tilde{u}_1'(r) = R_1 r \tilde{u}_1'(R_1 r) \) is strictly decreasing on \( \left( \frac{b_\mu}{R_1}, \frac{d_\mu}{R_1} \right) \), we obtain \( r \tilde{u}'(r) + 2 \leq -\frac{1}{2} E \) for all \( r \in [d_\mu, \frac{2E}{1 + a_1}] \), which implies that \( r^2 e^{\alpha_1(r)} \) decreases on \( [d_\mu, \frac{2E}{1 + a_1}] \) and so

\[
(R_1 r)^2 e^{\alpha_1(R_1 r)} = r^2 e^{\alpha_1(r)} \leq d_\mu^2 e^{\alpha_1(d_\mu)} < \mu \quad \text{for any } r \in \left[ d_\mu, \frac{R_5}{R_4} \right].
\]

Furthermore,

\[
\int_{d_\mu, R_4}^{R_5} r e^{\alpha_1} dr = \int_{d_\mu}^{R_4} r e^{\alpha_1} dr \leq \frac{2E}{1 + a_1} < \frac{2E}{1 + a_1}.
\]

Similarly, by \( r \tilde{u}'(r) + 2 \geq \frac{1}{2} E \) for all \( r \in [\frac{b_\mu}{R_3}, b_\mu] \), we can prove

\[
\int_{R_3}^{b_\mu R_4} r e^{\alpha_1} dr = \int_{R_3}^{b_\mu} r e^{\alpha_1} dr \leq \frac{2E}{1 + a_1} < \frac{2E}{1 + a_1}.
\]

Since \( \int_{b_\mu, R_4}^{R_5} r e^{\alpha_1} dr = \int_{b_\mu}^{R_4} r e^{\alpha_1} dr \), (2.35) follows immediately. This completes the proof.

\[\square\]

**Lemma 2.14.** Given any \( t_\epsilon \in (R_4, R_5, \epsilon) \) such that \( t_\epsilon < \infty \) and \( \frac{t_\epsilon}{R_4, \epsilon} \to \infty \) as \( \epsilon \to 0 \). Then \( \lim_{\epsilon \to 0} t_\epsilon^2 e^{\alpha_1(t_\epsilon)} = 0 \) for \( k = 1, 2 \) and

\[
\lim_{\epsilon \to 0} t_\epsilon u_1'(t_\epsilon) = -2\alpha_1, \quad \lim_{\epsilon \to 0} t_\epsilon u_2'(t_\epsilon) = -2\alpha_2,
\]

where \( \alpha_1, \alpha_2 \) are seen in (2.10) - (2.10).

**Proof.** Since \( \frac{t}{R_4} \to \infty \), by repeating the argument of Step 5 in Lemma 2.13 it is easy to prove that \( t^2 e^{\alpha_1(t)} \to 0 \) and

\[
\int_{R_3}^{t_\epsilon} e^{\alpha_1} dr \to \frac{2E}{1 + a_1} \quad \text{as } \epsilon \to 0.
\]

Recalling \( \int_{R_3}^{R_5} r e^{\alpha_2} dr \to 0 \) (see Lemma 2.11), (2.57), (2.57) - (2.58) and (2.75), we conclude

\[
\lim_{\epsilon \to 0} t_\epsilon u_1'(t) = \lim_{\epsilon \to 0} \left[ R_3 u_1'(R_3) - (1 + a_1) \int_{R_3}^{t} r e^{\alpha_1} dr \right] = -2\alpha_1,
\]

\[
\lim_{\epsilon \to 0} t_\epsilon u_2'(t) = \lim_{\epsilon \to 0} \left[ R_3 u_2'(R_3) + a_2 \int_{R_3}^{t} r e^{\alpha_1} dr \right]
\]

\[
= -\frac{2a_2}{1 + a_1} \left( \frac{3\gamma - \tilde{N}_1}{A} - 2\tilde{N}_2 - 2 + \frac{2a_2}{1 + a_1} \right)
\]

\[
= -\frac{2a_2}{1 + a_1} \left( \frac{3A - 4B}{A} - \frac{2a_2}{1 + a_1} \right) \tilde{N}_2 - 2
\]

\[= -2\alpha_2.
\]

This completes the proof.  \[\square\]
Lemma 2.15. There exists a small $\varepsilon_7 \in (0, \varepsilon_6)$ such that for each $\varepsilon \in (0, \varepsilon_7)$, there holds $ru_{2, \varepsilon}'(r) < -2 - \tilde{\alpha}_2$ for all $r \in [R_3, \varepsilon R]$. Consequently,
\begin{equation}
\lim_{\varepsilon \to 0} \int_{R_3}^{R_{\varepsilon}} re^{u_{2, \varepsilon}'} dr \leq \frac{1}{\tilde{\alpha}_2} \lim_{\varepsilon \to 0} R_{\varepsilon}^2 e^{u_{2, \varepsilon}(R_{\varepsilon})} = 0.
\end{equation}

Proof. Recall that $\tilde{\alpha}_2 = \alpha_2 + 1 > 0$ for $k = 1, 2$ and (2.76) gives that $ru_{2, \varepsilon}'(r) \leq -2 - \frac{2}{\tilde{\alpha}_2}$ for $r \in [R_3, \varepsilon R]$. Assume by contradiction that there exist a sequence $\varepsilon_n \downarrow 0$ and $t_n \in [R_3, \varepsilon_n]$, such that
\begin{equation}
ru_{2, \varepsilon}'(t_n) = -2 - \tilde{\alpha}_2 \quad \text{and} \quad ru_{2, \varepsilon}'(r) < -2 - \tilde{\alpha}_2 \quad \text{for} \quad r \in [R_3, t_n].
\end{equation}
We will omit the subscript $\varepsilon_n$ for convenience. Clearly $R_5 < t_n \to R^*$. Since $\frac{\delta_1}{R_5} \to \infty$, Lemma 2.14 yields $\lim_{n \to \infty} R_{\varepsilon_n}^2 e^{u_{\varepsilon_n}(R_{\varepsilon_n})} = 0$ for $k = 1, 2$
\begin{equation}
\lim_{n \to \infty} R_5 u_1'(R_5) = -2 - 2\tilde{\alpha}_1, \quad \lim_{n \to \infty} R_5 u_2'(R_5) = -2 - 2\tilde{\alpha}_2.
\end{equation}
Since $ru_{2}'(r) < -2 - \tilde{\alpha}_2$ for $r \in [R_3, t_n]$, we have
\begin{equation}
\int_{R_3}^{t_n} re^{u_{2}'} dr \leq \frac{1}{\tilde{\alpha}_2} R_{\varepsilon_n}^2 e^{u_{\varepsilon_n}(R_{\varepsilon_n})} \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}
Recall from (2.81) that $\sup_{[R_3, \varepsilon n]} u_1 \to -\infty$ as $n \to \infty$. Since $u_2 \leq u_1$ on $[R_3, R_5]$, we have $\sup_{[R_3, t_n]} u_2 \to -\infty$ as $n \to \infty$.

Step 1. We claim that $u_1' < 0$ on $[R_5, t_n]$ for $n$ sufficiently large. Consequently, $\sup_{[R_3, t_n]} u_1' \to -\infty$ as $n \to \infty$.

Suppose that, up to a subsequence, there exists $r_n \in [R_5, t_n]$ such that $u_1'(r_n) = 0$ and $u_1'(r) < 0$ for $r \in [R_5, r_n]$. Then for $n$ sufficiently large, $u_1 < 2\varepsilon$ and so $F_1 < 0$, $-F_2 < 2e^{u_2}$ on $[R_5, r_n]$, which implies that $(ru_1)' = r[(1 + a_1)F_1 - a_1 F_2] \leq 2a_1 e^{u_2}$ on $[R_5, r_n]$. Consequently,
\begin{equation}
0 - R_5 u_1'(R_5) = \int_{R_5}^{r_n} (ru_1)' dr \leq 2a_1 \int_{R_5}^{r_n} r e^{u_2} dr \leq 2a_1 \int_{R_5}^{t_n} r e^{u_2} dr \to 0
\end{equation}
as $n \to \infty$, which yields a contradiction with (2.83).

Step 2. We claim that $ru_1'(r) < -2 - \tilde{\alpha}_1$ on $[R_5, t_n]$ for $n$ large enough. Consequently,
\begin{equation}
\int_{R_5}^{t_n} r e^{u_1} dr \leq \frac{1}{\tilde{\alpha}_1} R_{\varepsilon_n}^2 e^{u_{\varepsilon_n}(R_{\varepsilon_n})} \to 0 \quad \text{as} \quad n \to \infty.
\end{equation}

By Step 1 and (2.83)-(2.84), we may take $n$ large enough such that $(ru_2)' \leq 2a_1 r e^{u_2}$ on $[R_5, t_n]$, $R_5 u_2'(R_5) < -2 - \frac{2}{\tilde{\alpha}_2}$ and $2a_1 \int_{R_5}^{t_n} r e^{u_2} dr \leq \frac{2}{\tilde{\alpha}_2}$. Then for any $r \in [R_5, t_n]$, we have
\begin{equation}
ru_1'(r) \leq R_5 u_1'(R_5) + 2a_1 \int_{R_5}^{t_n} r e^{u_2} dr < -2 - \tilde{\alpha}_1.
\end{equation}

Step 3. We complete the proof.

Similarly as Steps 1-2, we may take $n$ large enough such that $(ru_2)' \leq r[(1 + a_2)F_2 - a_2 F_1] \leq 2a_2 e^{u_1}$ on $[R_5, t_n]$, so we conclude from (2.83) and (2.85) that
\begin{equation}
-2 - \tilde{\alpha}_2 = t_n u_2'(t_n) \leq R_5 u_2'(R_5) + 2a_1 \int_{R_5}^{t_n} r e^{u_2} dr \to -2 - 2\tilde{\alpha}_2
\end{equation}
as $n \to \infty$, a contradiction with $\tilde{\alpha}_2 > 0$. This completes the proof.

We are now in a position to complete the proof of Theorem 2.4 for $\alpha_1 > 1$.

Completion of the proof of Theorem 2.4 for $\alpha_1 > 1$. For each $\varepsilon \in (0, \varepsilon_7)$, we take a number $t_\varepsilon \in (R_{1, \varepsilon}, R_{3, \varepsilon})$ such that $\frac{t_\varepsilon}{R_{3, \varepsilon}} \to \infty$ as $\varepsilon \to 0$. Then Lemma 2.19 gives $u_{2, \varepsilon}(t_\varepsilon) < u_{1, \varepsilon}(t_\varepsilon) \leq \ln \delta$. Moreover, by Lemma 2.14 we have
\begin{equation}
\lim_{\varepsilon \to 0} R_{3, \varepsilon}^2 e^{u_{2, \varepsilon}(t_\varepsilon)} = 0, \quad \lim_{\varepsilon \to 0} t_\varepsilon u_{1, \varepsilon}'(t_\varepsilon) = -2\alpha_1, \quad \lim_{\varepsilon \to 0} t_\varepsilon u_{2, \varepsilon}'(t_\varepsilon) = -2\alpha_2.
\end{equation}
Combining these with (2.84), we can repeat the argument of Steps 1-2 in Lemma 2.15 to conclude the existence of $\varepsilon_8 \in (0, \varepsilon_7)$ such that $ru_{k, \varepsilon}^\prime (r) < -2 - \alpha_2$ on $[t_\varepsilon, R_\varepsilon^\ast]$ for any $\varepsilon \in (0, \varepsilon_8)$. Consequently,

$$\int_{t_\varepsilon}^{R_\varepsilon^\ast} r e^{u_{1, \varepsilon}} dr \leq \frac{1}{\alpha_1} R_\varepsilon^2 e^{u_1 (t_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$  

(2.86)

Now we consider $\varepsilon \in (0, \varepsilon_8)$. Recall from Lemma 2.15 that $ru_{2, \varepsilon}^\prime (r) < -2 - \alpha_2$ on $[R_{k, \varepsilon}, R_\varepsilon^\ast]$. It turns out that both $u_{1, \varepsilon}$ and $u_{2, \varepsilon}$ decrease on $[t_\varepsilon, R_\varepsilon^\ast]$, which implies $u_{k, \varepsilon} (r) < u_{k, \varepsilon} (t_\varepsilon) < \ln \delta$ for any $r \in (t_\varepsilon, R_\varepsilon^\ast)$ and $k = 1, 2$. By the definition of $R_\varepsilon^\ast$, we conclude that $R_\varepsilon^\ast = \infty$, namely $(u_{1, \varepsilon}, u_{2, \varepsilon})$ is an entire solution for $\varepsilon \in (0, \varepsilon_8)$. By Theorem A, there exists $(\alpha_{1, \varepsilon}, \alpha_{2, \varepsilon}) \in \Omega$ such that

$$u_{k, \varepsilon} (r) = -2\alpha_{k, \varepsilon} \ln r + O(1) \text{ as } r \to \infty, \ k = 1, 2.$$

Recall that $|(ru_k')'| = |r(1 + a_k)F_k - akF_{3-k}| \leq C r(e^{u_{1, \varepsilon}} + e^{u_{2, \varepsilon}})$ for $k = 1, 2$. By (2.82), (2.86) and $R_\varepsilon^\ast = \infty$, we obtain

$$[-2\alpha_{k, \varepsilon} - t_\varepsilon u_{k, \varepsilon} (t_\varepsilon)] \leq C \int_{t_\varepsilon}^{\infty} r(e^{u_{1, \varepsilon}} + e^{u_{2, \varepsilon}}) dr \to 0 \text{ as } \varepsilon \to 0$$

for $k = 1, 2$. Therefore, $\lim_{\varepsilon \to 0} (\alpha_{1, \varepsilon}, \alpha_{2, \varepsilon}) = (\alpha_1, \alpha_2)$.

Note from (2.17) and Lemma 2.15 that $u_{2, \varepsilon}^\prime (r) < 0$ for all $r > R_{2, \varepsilon}$. Besides, Lemma 2.3 shows that $\sup_{[0, R_{1, \varepsilon}]} u_{2, \varepsilon} \to -\infty$. Combining these with (2.86), we conclude $\sup_{\mathbb{R}^2} u_{2, \varepsilon} \to -\infty$ as $\varepsilon \to 0$. This completes the proof. \(\square\)

References

[1] W. Ao, C.-S. Lin, J. Wei, On non-topological solutions of the $A_2$ and $B_2$ Chern-Simons system, Mem. Amer. Math. Soc., to appear.

[2] W. Ao, C.-S. Lin, J. Wei, On non-topological solutions of the $G_2$ Chern-Simons system, preprint.

[3] D. Chae, O. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory, Comm. Math. Phys., 215 (2000) 119-142.

[4] H. Chan, C. Fu, C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation, Comm. Math. Phys., 231 (2002) 189-221.

[5] Z. Chen, C.-S. Lin, Self-dual radial non-topological solutions to a competitive Chern-Simons model, preprint.

[6] K. Choe, Asymptotic behavior of condensate solutions in the Chern-Simons-Higgs theory, J. Math. Phys., 48 (2007) 103501-103517.

[7] K. Choe, Multiple existence results for the self-dual Chern-Simons-Higgs vortex equation, Comm. Partial Differ. Equ., 34 (2009) 1465-1507.

[8] K. Choe, N. Kim, C.-S. Lin, Existence of self-dual non-topological solutions in the Chern-Simons-Higgs model, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28 (2011) 837-852.

[9] K. Choe, N. Kim, C.-S. Lin, Self-dual symmetric nontopological solutions in the $SU(3)$ model in $\mathbb{R}^2$, Comm. Math. Phys., (2014) to appear.

[10] K. Choe, N. Kim, C.-S. Lin, New type of nontopological solutions in the $SU(3)$ Chern-Simons model in $\mathbb{R}^2$, preprint.

[11] S. Deser, R. Jackiw, S. Templeton, Topologically massive gauge theories, Ann. Phys. 140 (1982) 372.

[12] G. Dunne, Selfdual Chern-Simons theories, Lecture Notes in Physics, vol. 36, Springer-Verlag, (1995)

[13] G. Dunne, Vacuum mass spectra for $SU(N)$ self-dual Chern-Simons-Higgs systems, Nuclear Phys. B, 433 (1995) 333-348.

[14] G. Dunne, Mass degeneracies in self-dual models, Phys. Lett. B, 345 (1995) 452-457.
[15] S. Gudnason, Non-Abelian Chern-Simons vortices with generic gauge groups, Nuclear Phys. B, 821 (2009) 151-169.
[16] S. Gudnason, Fractional and semi-local non-Abelian Chern-Simons vortices, Nuclear Phys. B, 840 (2010) 160-185.
[17] X. Han, C.-S. Lin, G. Tarentello, Y. Yang, Chern-Simons vortices in the Gudnason model, J. Funct. Anal., 267 (2014) 678-726.
[18] X. Han, G. Tarantello, Doubly periodic self-dual vortices in a relativistic non-Abelian Chern-Simons model, Calc. Var. PDE., 49 (2014) 1149-1176.
[19] J. Hong, Y. Kim, P. Pac, Multivortex solutions of the Abelian Chern-Simons-Higgs theory, Phys. Rev. Lett., 64 (1990) 2230-2233.
[20] H. Huang, C.-S. Lin, On the entire radial solutions of the Chern-Simons SU(3) system, Comm. Math. Phys., 327 (2014) 815-848.
[21] H. Huang, C.-S. Lin, Classification of the entire radial self-dual solutions to non-Abelian Chern-Simons systems, J. Funct. Anal., 266 (2014) 6796-6841.
[22] R. Jackiw, E. Weinberg, Self-dual Chern-Simons vortices, Phys. Rev. Lett., 64 (1990) 2234-2237.
[23] H. Kao, K. Lee, Self-dual SU(3) Chern-Simons Higgs systems, Phys. Rev. D, 50 (1994) 6626-6632.
[24] C.-S. Lin, J. Wei, D. Ye, Classification and nondegeneracy of SU(n+1) Toda system with singular sources, Invent. Math., 190 (2012) 169-207.
[25] G. Lozano, D. Marqués, E. Moreno, F. Schaposnik, Non-Abelian Chern-Simons vortices, Phys. Lett. B, 654 (2007) 27-34.
[26] J. Prajapat, G. Tarantello, On a class of elliptic problems in $\mathbb{R}^2$: symmetry and uniqueness results, Proc. Roy. Soc. Edinburgh Sect. A, 131 (2001) 967-985.
[27] J. Spruck, Y. Yang, The existence of non-topological solutions in the self-dual Chern-Simons theory, Comm. Math. Phys., 149 (1992) 361-376.
[28] J. Spruck, Y. Yang, Topological solutions in the self-dual Chern-Simons theory: existence and approximation, Ann. Inst. H. Poincaré Anal. Non Linéaire, 12 (1995) 75-97.
[29] Y. Yang, The relativistic non-Abelian Chern-Simons equations, Comm. Math. Phys., 186 (1997) 199-218.