ON THE HEEGAARD FLOER HOMOLOGY OF $S^{3}_{-p/q}(K)$

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Abstract. Assume that the oriented 3-manifold $M = S^{3}_{-p/q}(K)$ is obtained by a rational surgery (with coefficient $-p/q < 0$) along an algebraic knot $K \subset S^{3}$. We compute the Heegaard Floer homology of $-M$ in terms of $p/q$ and the Alexander polynomial of $K$.

1. Introduction

In this article we compute the Heegaard Floer homology $HF^{+}(-M)$ (introduced by Ozsváth and Szabó [13]) for the oriented 3-manifold $M = S^{3}_{-p/q}(K)$ obtained by a negative rational surgery (with coefficient $-p/q$) along an algebraic knot $K \subset S^{3}$. In this case, since $H_{1}(M, \mathbb{Z}) = \mathbb{Z}_{p}$, the $spin^{c}$-structures $\{\sigma_{a}\}_{a}$ of $M$ can be parametrized by integers $a = 0, 1, \ldots, p - 1$. The main result of the article establishes $HF^{+}(-M, \sigma_{a})$ in terms of the integers $p, q, a$, and the Alexander polynomial $\Delta$ of $K \subset S^{3}$. Notice that the Alexander polynomial of an algebraic (any) knot is well-understood, it can be easily computed from most of the other invariants of the knot (e.g., in the present algebraic case, from Puiseux or Newton pairs, or from the semigroup associated with the corresponding local analytic germ). In particular, the input of the theorem is the simplest what one can hope. Since (in some sense) all the coefficients of $\Delta$ are effectively involved in the description of the Heegaard Floer homology, in fact, the result is optimal.

In the very recent manuscript [16], Ozsváth and Szabó computed $HF^{+}(S^{3}_{p}(K))$ – for any knot $K$ and any integer surgery coefficient $p$ – in terms of the filtered chain homotopy type of the Heegaard Floer complex associated with the pair $(S^{3}, K)$. Compared with this, our starting data, and also the description of $HF^{+}(-M)$, are simpler, and totally elementary; as a price for this we have to impose the ‘negativity restrictions’ for the surgery coefficient and for $K$.

The proof (and the structure of the article) is based on the results and constructions of [10] valid for plumbed 3-manifolds associated with some negative definite plumbing graphs – in fact, this also explains the source of our restrictions. Although [10] presents a precise algorithm how one should compute $HF^{+}$, its implementations in different situations sometimes is not straightforward. In the present case too, the proof and additional constructions run over many sections. In fact, with the present article, we also wish to advertise the efficiency, novelty and the power of [10]. This method, in fact, determines the ‘graded roots’ (some graded trees) associated with the plumbing graph of $M$, from which one can read easily the Heegaard Floer homology. The advantage of these graded roots is that (in all the cases known by the author) their structure reflects perfectly the corresponding geometrical construction which provides $M$. In particular, in many cases, from the topology of $M$ one can identify the roots rather conceptually.

Section 2 recalls the classical invariants of algebraic knots and connects them with the plumbing of $M$. The next section recalls the definition and first properties of graded roots – necessary to formulate the main theorem, which appears in section 4. Section 5 presents two relevant results of [10], as general principles to compute $HF^{+}$. (In fact, section 5 can serve as a general recipe for the
interested readers to compute the Heegaard Floer homology in different other cases as well.) The first result (which provides the absolute grading) is worked out for our present situation in section 6, the other (regarding the graded roots) in section 7. The last section contains some examples.

2. The manifold \( M = S^3_{-p/q}(K_f) \)

2.1. Review of algebraic knots. Let \( K_f \subset S^3 \) be the link of an irreducible complex plane curve singularity \( f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0) \); i.e. for \( \epsilon > 0 \) sufficiently small, write \( S^3 = \{ z \in \mathbb{C}^2 : |z| = \epsilon \} \) and take the transversal intersection \( K_f := \{ f = 0 \} \cap S^3 \rightarrow S^3 \). The natural orientations of \( S^3 \) and of the regular part of \( \{ f = 0 \} \) induces a natural orientation on \( K_f \). Since \( f \) is irreducible, \( K_f \approx S^3 \).

We will assume that \( \{ f = 0 \} \) is not smooth at the origin, i.e. \( K_f \subset S^3 \) is not the unknot. The isotopy type of \( K_f \subset S^3 \) is completely characterized by any of the following invariants listed below.

2.1.1. The Newton pairs of \( f \) consist of \( g \geq 1 \) pairs of integers \( \{(p_i, q_i)\}_{i=1}^g \), where \( p_i \geq 2, q_i \geq 1, q_1 > p_1 \) and \( \gcd(p_i, q_i) = 1 \).

2.1.2. In some topological constructions, it is preferable to replace the Newton pairs by the 'linking pairs' (or, the decorations of the splice diagram, cf. [9]) \( (p_i, a_i) \) \( \mathbb{C}_{i-1} \), where

\[
a_1 = q_1 \quad \text{and} \quad a_{i+1} = q_{i+1} + p_{i+1}a_i \quad \text{for} \quad i \geq 1.
\]

2.1.3. Let \( \Delta(t) \) be the Alexander polynomial of \( K_f \subset S^3 \), or equivalently, the characteristic polynomial of the algebraic monodromy acting on the first homology of the Milnor fiber of \( f \). It is normalized by \( \Delta(1) = 1 \). In terms of the \( (p_i, a_i) \) pairs it is

\[
\Delta(t) = \frac{(t^{p_1} - 1)(t^{p_3} - 1)\cdots (t^{p_g} - 1)(t - 1)}{(t^{a_1} - 1)(t^{a_3} - 1)\cdots (t^{a_g} - 1)(t - 1)}. \]

The degree of \( \Delta \), or equivalently, the first Betti number of the Milnor fiber of \( f \), is the Milnor number \( \mu \) of \( f \).

2.1.4. The semigroup \( \Gamma \) of the germ \( f \) is a sub-semigroup of \( \mathbb{N} \) defined by \( \Gamma := \{ i_0(f, h) : h \in O_{(\mathbb{C}^2,0)} \} \), where \( i_0(f, h) \) denotes the local intersection multiplicity of \( f \) and \( h \) (which equals the codimension of the ideal \( (f, h) \) in \( O_{(\mathbb{C}^2,0)} \)). For \( h \) invertible \( i_0(f, h) \) is zero, hence \( 0 \in \Gamma \).

It is known that \( \Gamma \) is generated by the integers \( \beta_0 = p_1p_2\cdots p_g, \beta_k = a_kp_{k+1}\cdots p_g \) for \( 1 \leq k \leq g - 1 \), and \( \beta_g = a_g \). Moreover, \( \#(\mathbb{N} \setminus \Gamma) \) is finite. Its cardinality is the delta-invariant \( \delta \) of \( f \), which in this case also equals \( \mu/2 \), cf. [9]. It is also known that \( \delta = \mu/2 \) equals the minimal Seifert genus of \( K_f \subset S^3 \).

The largest element of \( \mathbb{N} \setminus \Gamma \) is \( \mu - 1 \). In fact, for \( 0 \leq k \leq \mu - 1 \) one has: \( k \in \Gamma \) if and only if

\[
\mu - 1 - k \notin \Gamma.
\]

2.1.5. The embedded minimal good dual resolution graph of the germ \( f \) has the following shape

\[
G(f) : \quad \mathbb{N} \quad \mathbb{N} - 1 \quad K_f \quad v_0
\]

Above we emphasize only the vertices of degree one and three. The dash-line between two such vertices replaces a string \( \cdots \). The number of vertices of degree three is exactly \( g \). In general, any vertex \( v \) is decorated by the self-intersection of the corresponding irreducible exceptional divisor \( E_v \). In the above diagram we put only the decoration of the vertex \( v_0 \), which corresponds to the unique \((-1)\)-curve, and which also supports the strict transform of \( \{ f = 0 \} \). The other decorations are not really essential in our next discussions; for the description of the complete graph see e.g. [9] or [8], section 4.I.
The above diagram can also be identified with the plumbing graph of \((S^3, K_f)\). In this case the self-intersections are the corresponding Euler numbers of the \(S^1\)-bundles, all the surfaces used in the plumbing construction are \(S^2\)'s, and \(K_f\) is a generic fiber of the unique \((-1)\)-bundle.

The above resolution graph \(G(f)\) will be denoted by the following schematic diagram:

\[
\begin{array}{c}
\bar{G} \\
\bullet \\
v_0 \\
\end{array}
\begin{array}{c}
\rightarrow \\
-1 \\
\end{array}
\begin{array}{c}
K_f \\
\end{array}
\]

The polynomial \(\Delta(t)\) can also be deduced from \(G(f)\) by A’Campo’s formula \([1]\). Notice that (if we disregard the arrowhead) the graph \(\bar{G}\) can be blown down completely.

We emphasize again, the information codified in the following objects – isotopy class of \(K_f \subset S^3\), set of Newton pairs, set of linking pairs, Alexander polynomial, semigroup or the embedded resolution graph – are completely equivalent. This means that the polynomial \(Q\) introduced below can be deduced from any of them.

### 2.2. Definition of the polynomial \(Q\).

One has the following identity connecting the Alexander polynomial \(\Delta\) and the semigroup \(\Gamma\), cf. \([6]\):

\[
\frac{\Delta(t)}{1-t} = \sum_{k \in \Gamma} t^k.
\]

Since \(\Delta(1) = 1\) and \(\Delta'(1) = \delta\) (use e.g. the formula of \([9, 8]\)), one gets

\[
\Delta(t) = 1 + \delta (t-1) + (t-1)^2 : Q(t)
\]

for some polynomial \(Q(t) = \sum_{i=0}^{\mu-2} \alpha_i t^i\) of degree \(\mu - 2\) with integral coefficients. In fact, all the coefficients \(\{\alpha_i\}_{i=0}^{\mu-2}\) are strict positive, and:

\[
\delta = \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_{\mu-2} = 1.
\]

Indeed, by the above identity, one has \(\delta + (t-1)Q(t) = \sum_{k \notin \Gamma} t^k\), or \(Q(t) = \sum_{k \notin \Gamma} t^{k-1} + \cdots + t + 1\). This shows that

\[
\alpha_i = \#\{k \notin \Gamma : k > i\}.
\]

### 2.3. The surgery 3-manifold \(M = S^3_{-p/q}(K)\).

In the sequel we fix an oriented algebraic knot \(K_f \subset S^3\) and we consider the oriented manifold \(M := S^3_{-p/q}(K_f)\) obtained by \(-p/q\)-surgery along \(K_f \subset S^3\), where \(p/q > 0\) is a positive rational number \((p > 0, \gcd(p, q) = 1)\).

It is easy to see (e.g. by Kirby calculus, see e.g. \([4]\)) that \(M\) can be represented by the following plumbing diagram (the symbols \(v_0, \ldots, v_s\) are the ‘names’ of the corresponding vertices, they are not really parts of the decoration of the diagram):

\[
\begin{array}{c}
\bar{G} \\
\bullet \\
v_0 \\
\end{array}
\begin{array}{c}
-1 \\
v_1 \\
v_2 \\
\vdots \\
v_{s-1} \\
v_s \\
\end{array}
\begin{array}{c}
\rightarrow \\
-k_1 - m_f \\
-k_2 \\
\cdots \\
-k_{s-1} \\
-k_s \\
\end{array}
\]

where \(k_1 \geq 1\) and \(k_j \geq 2\) \((2 \leq j \leq s)\) are integers determined by the continued fraction

\[
\frac{p}{q} = [k_1, k_2, \ldots, k_s] = k_1 - \frac{1}{k_2 - \frac{1}{\cdots - \frac{1}{k_s}}};
\]

and

\[
m_f = a_g p g,
\]

or equivalently, in the language of the embedded resolution graph of \(f\) (cf. \([2, 8]\)), \(m_f\) is the multiplicity of the pull back of the germ \(f\) along the \((-1)\)-curve. Topologically, e.g. if one uses Kirby
calculus and blows down all the vertices of $\bar{G}$, $m_f$ is a sum $\sum_j m_j^2$ of squares of a sequence of linking numbers $m_j$. In fact, in the language of plane curve singularities, this sequence of $m_j$’s is exactly the ‘multiplicity sequence’ of $f$.

Notice that if we would start with the unknot $K_f \subset S^3$, then $M = S^3_{-p/q}(K_f)$ would be the lens space $L(p, q)$ (in general, normalized with $0 < q < p$, hence $k_1 \geq 2$ as well). This case is completely solved, see e.g. [10], section 10. Therefore, in the sequel we will assume that $K_f$ is the unknot; and we accept the case $0 < p/q \leq 1$ (i.e. $k_1 = 1$) as well.

Take a point $* \in S^3 \setminus K_f$ and identify $S^3 \setminus *$ with $\mathbb{R}^3$. Then a Kirby diagram of $M$ is given by the knot $K_f \subset \mathbb{R}^3$ with decoration (surgery coefficient) $-p/q$. In particular, the Kirby diagram of the manifold $-M$ ($M$ with reversed orientation) – whose Heegaard Floer homology will be computed in the sequel – is given by the mirror image $K_f^m$ of $K_f$ (with respect to any plane in $\mathbb{R}^3$) with surgery coefficient $p/q$; i.e. $-M = S^3_{p/q}(K_f^m)$.

### 2.4. Some properties of the graph $G(M)$.

2.4.1. The plumbing graph $G(M)$ is a negative definite connected tree. For some particular plumbed 3-manifolds, with such a graph, the Heegaard Floer homology is determined in [13] and [10]. Notice that, in general, $G(M)$ may have many ‘bad vertices’ (in fact, their number is $\#\{1 \leq i < s : p_i < q_i\}$), hence [13] cannot be applied. Nevertheless, [10] works. Indeed, first recall (cf. [10]) that a graph is called almost rational, or AR, if it has at least one vertex with the following property: if one replaces the decoration (Euler-number) of the corresponding vertex by a smaller integer, then one gets a rational graph (in the sense of Artin [2]). Then the Heegaard Floer homology of a plumbed manifolds associated with AR-graphs can be determined by [10].

2.4.2. **Lemma.** $G(M)$ is an AR-graph.

**Proof.** There exists a particular family of rational graphs – the so called sandwiched graphs (associated with sandwiched singularities) – which are characterized as follows: they are subgraphs of (not necessarily minimal) connected negative definite plumbing graphs which can be completely blown down (cf. e.g. [13]). We will show that if we modify the $-1$ decoration of $v_0$ in $G(M)$ into $-2$ (let us call this new graph by $G(M)_{-2}$), we get a sandwiched graph. Indeed, consider the graph $G$. Blow up the $-1$ vertex $v_0$. The new decoration of $v_0$ will be $-2$, while a new $-1$ vertex is created. Then blow up this new vertex $(m_f + k_1 - 1)$ times. Then its new decoration will be $-m_f - k_1$, while it has $(m_f + k_1 - 1)$ neighbours which are all $-1$ curves. Fix one of them, and blow up $k_2 - 1$ times. If one continues this procedure, one gets a graph which contains $G(M)_{-2}$ as a subgraph. □

Using the results of [10], the above lemma has also the following consequence: the plumbed 3-manifold associated with $G(M)_{-2}$ is an $L$-space (in the sense of Ozsváth-Szabó, i.e. its reduces Heegaard Floer homology is trivial).

### 3. Graded roots

3.1. **Preliminary remark.** The Heegaard Floer homology of $M$ (in fact, of $-M$) will be computed in a combinatorial way from the graph $G(M)$. Following [10], we do this via some intermediate objects, the $d$ graded roots associated with $G(M)$, each corresponding to a spin-c-structure of $M$.

As we will see, the graded roots contain more information than the Heegaard Floer homology, a fact which hopefully will be more exploited in the future. On the other hand, compared with the homology, they have an enormous advantage: they can much-much easier be identified and described (because, probably, they preserve the geometric information of $M$ at more canonical level).

In this section we provide a short review of abstract graded roots; for more details, see [10].
3.2. Definition of a graded root \((R, \chi)\).

(1) Let \(R\) be an infinite tree with vertices \(V\) and edges \(E\). We denote by \([u, v]\) the edge with end-points \(u\) and \(v\). We say that \(R\) is a graded root with grading \(\chi : V \to \mathbb{Z}\) if

(a) \(\chi(u) - \chi(v) = \pm 1\) for any \([u, v] \in E\);

(b) \(\chi(u) > \min\{\chi(v), \chi(w)\}\) for any \([u, v], [u, w] \in E\);

(c) \(\chi\) is bounded below, \(\chi^{-1}(k)\) is finite for any \(k \in \mathbb{Z}\), and \(\#\chi^{-1}(k) = 1\) if \(k\) is sufficiently large.

(2) \(v \in V\) is a local minimum point of the graded root \((R, \chi)\) if \(\chi(v) < \chi(w)\) for any edge \([v, w]\).

Their set is denoted by \(V_{lm}\). In fact, \(V_{lm}\) coincides with the set of vertices with adjacent degree one.

(3) A geodesic path connecting two vertices is monotone if \(\chi\) restricted to the set of vertices on the path is strict monotone. If a vertex \(v\) can be connected by another vertex \(w\) by a monotone geodesic and \(\chi(v) > \chi(w)\), then we write \(v \succ w\). \(\succ\) is an ordering of \(V\). For any pair \(v, w \in V\) there is a unique \(\succ\)-minimal vertex \(\sup(v, w)\) which dominates both.

3.3. Examples. (1) For any integer \(n \in \mathbb{Z}\), let \(R_n\) be the tree with \(V = \{v^k\}_{k \geq n}\) and \(E = \{(v^k, v^{k+1})\}_{k \geq n}\). The grading is \(\chi(v^k) = k\).

(2) Let \(I\) be a finite index set. For each \(i \in I\) fix an integer \(n_i \in \mathbb{Z}\); and for each pair \(i, j \in I\) fix \(n_{ij} = n_{ji} \in \mathbb{Z}\) with the next properties: (i) \(n_i = n_i\); (ii) \(n_{ij} \geq \max\{n_{ij}, n_{jk}\}\); and (iii) \(n_{jk} \leq \max\{n_{ij}, n_{jk}\}\) for any \(i, j, k \in I\).

For any \(i \in I\) consider \(R_i := R_{n_i}\) with vertices \(\{v^k\}\) and edges \((\{v^k, v^{k+1}\})\), \((k \geq n_i)\). In the disjoint union \(\coprod R_i\), for any pair \((i, j)\), identify \(v^k\) and \(v^k\), \(v^{k+1}\) and \(v^{k+1}\), whenever \(k \geq n_{ij}\). Write \(\vec{v}^k\) for the class of \(v^k\). Then \(R := \coprod R_i/\sim\) is a graded root with \(\chi(\vec{v}^k) = k\).

Clearly \(V_{lm}\) is a subset of \(\{v^k\}_{i \in I}\). They are equal if in (ii) all the inequalities are strict. (Otherwise some indices are superfluous, and the corresponding \(R_i\)’s produce no contribution.)

(3) Any map \(\tau : \{0, \ldots, T_0\} \to \mathbb{Z}\) produces a starting data for construction (2). Indeed, set \(I = \{0, \ldots, T_0\}, n_i := \tau(i) (i \in I), n_{ij} := \max\{n_k : i \leq k \leq j\}\) for \(i \leq j\). Then \(\coprod R_i\) constructed in (2) using this data will be denoted by \((R_\tau, \chi_\tau)\).

To any graded root, one can associate a natural \(\mathbb{Z}[U]\)-module.

3.4. Notations. Consider the \(\mathbb{Z}[U]\)-module \(\mathbb{Z}[U, U^{-1}]\), and (following \[15\]) denote by \(\mathcal{T}_0^+\) its quotient by the submodule \(U \cdot \mathbb{Z}[U]\). It is a \(\mathbb{Z}[U]\)-module with grading \(\deg(U^{-h}) = 2h\). Similarly, for any \(n \geq 1\), define the \(\mathbb{Z}[U]\)-module \(\mathcal{T}_0(n)\) as the quotient of \(\mathbb{Z}[U, U^{-n}, U^{-n-2}, \ldots, U, 1]\) by \(U \cdot \mathbb{Z}[U]\) (with the same grading). Hence, \(\mathcal{T}_0(n)\), as a \(\mathbb{Z}\)-module, is the free \(\mathbb{Z}\)-module \(\mathbb{Z}[1, U, U^{-1}, u^{-n-1}]\) (generated by \(1, U^{-1}, \ldots, U^{-n-1}\)), and has finite \(\mathbb{Z}\)-rank \(n\).

More generally, fix an arbitrary \(Q\)-graded \(\mathbb{Z}[U]\)-module \(P\) with \(h\)-homogeneous elements \(P_h\). Then for any \(r \in \mathbb{Q}\) we denote by \(P[r]\) the same module graded in such a way that \(P[r]_{h+r} = P_h\). Then set \(\mathcal{T}_0^+ := \mathcal{T}_0^+|_{r}\) and \(\mathcal{T}_0(n) := \mathcal{T}_0(n)|_{r}\).

3.5. Definition. The \(\mathbb{Z}[U]\)-module associated with a graded root. For any graded root \((R, \chi)\), let \(\mathcal{H}(R, \chi)\) be the set of functions \(\phi : V \to \mathcal{T}_0^+\) with the following property: whenever \([v, w] \in E\) with \(\chi(v) < \chi(w)\), then

\[ U \cdot \phi(v) = \phi(w).\]

Then \(\mathcal{H}(R)\) is a \(\mathbb{Z}[U]\)-module via \((U \phi)(v) = U \cdot \phi(v)\). Moreover, \(\mathcal{H}(R)\) has a grading: \(\phi \in \mathcal{H}(R)\) is homogeneous of degree \(h \in \mathbb{Z}\) if for each \(v \in V\) with \(\phi(v) \neq 0\), \(\phi(v) \in \mathcal{T}_0^+\) is homogeneous of degree \(h - 2\chi(v)\).

In fact, \(\mathcal{H}(R, \chi)\) can also be computed as follows (cf. (3.6) in \[15\]):

3.6. Proposition. Let \((R, \chi)\) be a graded root. We order \(V_{lm}\) as follows. The first element \(v_1\) is an arbitrary vertex with \(\chi(v_1) = \min_v \chi(v)\). If \(v_1, \ldots, v_k\) is already determined, and \(J_k := \{v_1, \ldots, v_k\} \subset \mathcal{V}_{lm}\), then let \(v_{k+1}\) be an arbitrary vertex in \(\mathcal{V}_{lm} \setminus J_k\) with \(\chi(v_{k+1}) = \min_{v \in \mathcal{V}_{lm} \setminus J_k} \chi(v)\). Let \(w_{k+1} \in V\)
be the unique $\succ$-minimal vertex of $R$ which dominates both $v_{k+1}$, and at least one vertex from $J_k$. Then one has the following isomorphism of $\mathbb{Z}[U]$-modules
\[
\mathbb{H}(R, \chi) = \mathcal{T}_{2\chi(v_1)}^+ \oplus \oplus_{k \geq 2} \mathcal{T}_{2\chi(v_k)}(\chi(w_k) - \chi(v_k)).
\]
In particular, with the notations $m := \min_v \chi(v)$ and $\mathbb{H}_{\text{red}}(R, \chi) := \oplus_{k \geq 2} \mathcal{T}_{2\chi(v_k)}(\chi(w_k) - \chi(v_k))$, one has a direct sum decomposition of graded $\mathbb{Z}[U]$-modules:
\[
\mathbb{H}(R, \chi) = \mathcal{T}_{2m}^+ \oplus \mathbb{H}_{\text{red}}(R, \chi).
\]

Although the above proposition is elementary and in any concrete example is very easy to apply, in some general situation can be rather ‘inconvenient’ to write down the summands. E.g., in the case of $L(3,3)$, it is more natural to provide the integers $\tau(i)$ (which eventually have some geometric meaning) than to run the above algorithm 3.6. Therefore, sometimes we prefer to stop at the level of $(R, \chi)$, but the interested reader can always complete the picture with 3.6.

Proposition 3.6 for $(R_r, \chi_r)$ gives the following:

3.7. Corollary. (See (3.8) in [17]) Let $(R_r, \chi_r)$ be a graded root associated with a function $\tau : \{0, \ldots, T_0\} \rightarrow \mathbb{Z}$, cf. [17,3], which satisfies $\tau(1) > \tau(0) = 0$. Then the $\mathbb{Z}$-rank of $\mathbb{H}_{\text{red}}(R_r, \chi_r)$ is:
\[
\text{rank}_{\mathbb{Z}} \mathbb{H}_{\text{red}}(R_r) = \min_{i \geq 0} \tau(i) + \sum_{i \geq 0} \max\{\tau(i) - \tau(i+1), 0\}.
\]
Moreover, the summand $\mathcal{T}_{2m}^+$ of $\mathbb{H}(T_r, \chi_r)$ has index $m = \min_{i \geq 0} \tau(i) = \min_v \chi_r(v)$.

4. The Heegaard Floer homology $HF^+(M)$

4.1. Preliminaries. For any oriented rational homology 3-sphere $M$ the Heegaard Floer homology $HF^+(M)$ was introduced by Ozsváth and Szabó in [13] (cf. also with their long list of articles). In fact, $HF^+(M)$ is a $\mathbb{Z}[U]$-module with a $\mathbb{Q}$-grading compatible with the $\mathbb{Z}[U]$-action, where $\deg(U) = -2$. Additionally, $HF^+(M)$ also has an (absolute) $\mathbb{Z}_2$-grading: $HF^{+}_{\text{even}}(M)$, respectively $HF^{+}_{\text{odd}}(M)$, denote the part of $HF^+(M)$ with the corresponding parity.

Moreover, $HF^+(M)$ has a natural direct sum decomposition of $\mathbb{Z}[U]$-modules (compatible with all the gradings) corresponding to the spin$^c$-structures of $M$:
\[
HF^+(M) = \oplus_{\sigma \in \text{Spin}^c(M)} HF^+(M, \sigma).
\]
For any spin$^c$-structure $\sigma$, one has a graded $\mathbb{Z}[U]$-module isomorphism
\[
HF^+(M, \sigma) = \mathcal{T}_{d(M, \sigma)}^+ \oplus HF^{\text{red}}(M, \sigma),
\]
where $HF^{\text{red}}(M, \sigma)$ has a finite $\mathbb{Z}$-rank and an induced (absolute) $\mathbb{Z}_2$-grading (and $d(M, \sigma)$ can also be defined via this isomorphism). One also considers
\[
\chi(HF^+(M, \sigma)) := \text{rank}_{\mathbb{Z}} HF^{\text{red,even}}(M, \sigma) - \text{rank}_{\mathbb{Z}} HF^{\text{red,odd}}(M, \sigma).
\]
Then one recovers the (modified) Seiberg-Witten topological invariant of $(M, \sigma)$ (see [17]) via
\[
\text{sw}^{\text{OSW}}(M, \sigma) := \chi(HF^+(M, \sigma)) - \frac{d(M, \sigma)}{2}.
\]
With respect to the change of orientation the above invariants behave as follows:
\[
d(M, \sigma) = -d(-M, \sigma) \quad \text{and} \quad \chi(HF^+(M, \sigma)) = -\chi(HF^+(-M, \sigma)).
\]
Here one has a natural identification of the spin$^c$ structures of $M$ and $-M$. Notice also that one can recover $HF^+(M, \sigma)$ from $HF^+(-M, \sigma)$ via (7.3) [13] and (1.1) [13].
4.2. Assume now that $M = S^3_{-p/q}(K_f)$ as above. Then $H := H_1(M, \mathbb{Z}) = \mathbb{Z}_p$. There is natural identification of the spin$^c$-structures of $M$ with the classes of $\mathbb{Z}_p$ or with the integers $a = 0, 1, \ldots, p-1$, $a = 0$ corresponding to the 'canonical' spin$^c$-structure (cf. \([11]\)). For the precise identification, see \([5.2]\) and \([6.3\).

Let $\sigma_a$ be the spin$^c$-structure associated with $a$.

Next theorem provides a purely combinatorial description of $HF^+(-M)$ from $G(M)$. More precisely, $HF^+(-M, \sigma_a)$ will be expressed in terms of the delta-invariant $\delta$, the coefficients $\{\alpha_i\}$ of the polynomial $Q$ (cf. \([22]\)), and the integers $p, q$ and $a$. The symbol $s(q, p)$ denotes the Dedekind sum (see e.g. \([3.3]\)). The integer $q'$ is uniquely determined by $1 \leq q' \leq p$ and $qq' \equiv 1 \pmod{p}$.

4.3. Theorem. For each fixed $a = 0, 1, \ldots, p-1$, corresponding to the $p$ different spin$^c$-structures of $M$ one defines the following objects:

- $t_a := \left[ \frac{(2\delta-1)q-a-1}{p} \right]$;
- $r_a := 3s(q, p) + 2\sum_{j=1}^{a} \left\{ j \frac{q'}{p} \right\} - \frac{(1+2a)(p-1)}{2p} + \delta \left( 1 - \frac{q+1}{p} \right) + \frac{\delta^2 q}{p} - \frac{2\delta a}{p}$;
- a function $\tau_a : \{0, 1, \ldots, 2t_a + 2\} \rightarrow \mathbb{Z}$ by

\[
\left\{ \begin{array}{ll}
\tau_a(2t) = t(1 - \delta) + \sum_{j=0}^{t-1} \left[ \frac{jp+q}{q} \right], & (t = 0, \ldots, t_a + 1);
\tau_a(2t + 1) = \tau_a(2t + 2) + \alpha_{\lfloor (tp+a)/q \rfloor}, & (t = 0, \ldots, t_a).
\end{array} \right.
\]

- and the graded root $(R_{\tau_a}, \chi_{\tau_a})$ associated with $\tau_a$.

Then the following facts hold:

$HF^+_{\text{odd}}(-M, \sigma_a) = 0$;

$HF^+_{\text{even}}(-M, \sigma_a) = \mathbb{H}(R_{\tau_a}, \chi_{\tau_a})[r_a]$;

$d(-M, \sigma_a) = 2 \cdot \min \tau_a + r_a$.

The proof is given in sections 5-6-7. We continue with some remarks and corollaries.

4.4. Remark. Notice that for any $t \in \{0, \ldots, t_a\}$, $\alpha_{\lfloor (tp+a)/q \rfloor}$ is strict positive, hence $\tau_a(2t + 1) > \tau_a(2t + 2)$. On the other hand, using properties of $\Gamma$ (see e.g. \([21.3]\) or \([7.0]\)), one can verify that the following identity also holds:

$\tau_a(2t + 1) = \tau_a(2t) + \#\{ \gamma \in \Gamma : \gamma \leq (tp + a)/q \}$.

Therefore, since $0 \in \Gamma$, one has $\tau_a(2t + 1) > \tau_a(2t)$. In particular, the above representation of the graded root is the most 'economical': all the values are essential, cf. also with \([3.3]\).

This also shows that $(R_{\tau_a}, \chi_{\tau_a})$ has exactly $t_a + 2$ local minimum points, and they correspond to the values $\tau_a(2t)$, $t = 0, 1, \ldots, t_a + 1$.

\([13]\) and \([14]\) imply the next two corollaries.

4.5. Corollary.

$sw^{OSz}(M, \sigma_a) = -sw^{OSz}(-M, \sigma_a) = \frac{r_a}{2} - \sum_{r \geq 0} \alpha_{\lfloor (tp+a)/q \rfloor}$.

Recall that $HF^+(-M, \sigma_a)$ is a $\mathbb{Z}[U]$-module. Denote by $\ker U(\sigma_a)$ (resp. by $\operatorname{coker} U(\sigma_a)$) the kernel (resp. the cokernel) of the $U$-action. They are finitely generated graded free $\mathbb{Z}$-module. Let $\mathbb{Z}_{(r)}$ denote a rank one free $\mathbb{Z}$-module whose grading is $r$. 
Since $B_{l-l} \in \text{Char}$ modulo 2 can be represented by an orbit $\mathbb{Z}_{l+1}$, one has a natural identification of the $L$-space. Notice that $H$ is the free $Z$-module of vertices of $M$, so $H$ admits a unique minimal $L$-structure corresponding to $a = 0$, $\mathbb{H}_{red}(R_0)$ is never trivial. In particular, $-M$ is never an $L$-space.

5.2. Characteristic elements. The set of characteristic elements are defined by

$$\text{Char} := \{ k \in L' : k(x) + (x, x) \in 2\mathbb{Z} \text{ for any } x \in L \}.$$

There is a natural action of $L$ on $\text{Char}$ by $x \cdot k := k + 2i(x)$ whose orbits are of type $k + 2i(L)$. Obviously, $H$ acts freely and transitively on the set of orbits by $[l'] \cdot (k + 2i(L)) := k + 2l' + 2i(L)$. One has a natural identification of the $\text{spin}^c$-structures $Spin^c(M)$ of $M$ with the set of orbits of $\text{Char}$ modulo $2L$; and this identification is compatible with the action of $H$ on both sets. Therefore, in the sequel, we think about $Spin^c(M)$ by this identification, hence any $\text{spin}^c$-structure of $M$ will be represented by an orbit $[k] := k + 2i(L) \subset \text{Char}$. (For more details, see e.g. [10].)

5.3. Remark. If $a \geq (2\delta - 1)q$, then $t_a = -1$, hence $R_{t_a} = R_0$ (cf. [10]). In particular, for such $a$, $\mathbb{H}_{red}(R_{t_a}) = 0$. For all the other $a$’s, $\mathbb{H}_{red}(R_{t_a})$ is not trivial. E.g., for the ‘canonical’ $\text{spin}^c$-structure corresponding to $a = 0$, $\mathbb{H}_{red}(R_0)$ is never trivial. In particular, $-M$ is never an $L$-space.

5.4. Corollary. $\ker U(\sigma_a) = \bigoplus_{0 \leq t \leq t_a + 1} \mathbb{Z}_{2(t_a + 2i + 1)}$, $\coker U(\sigma_a) = \bigoplus_{0 \leq t \leq t_a} \mathbb{Z}_{2(t_a + 2i + 1) + 2i - 2i}$, which depends essentially on the coefficients $\{\alpha_i\}_i$ of $Q$. Moreover:

$$\text{rank}_\mathbb{Z} \ker U = \sum_{a=0}^{p-1} (t_a + 2) = p + (2\delta - 1)q = p + \text{rank}_\mathbb{Z} \coker U.$$
For any fixed class (spin^c-structure) [k] we fix the distinguished representative \( k_r := K + 2l'[k] \) from [k]. Since \( l'[k] = 0 \), the distinguished representative in [K] is K itself.

5.3. The graded roots \((R_k, \chi_k)\). For any \( k \in \text{Char} \) one defines \( \chi_k : L \to \mathbb{Z} \) by
\[
\chi_k(x) := -(k + x, x)/2.
\]

For any \( n \in \mathbb{Z} \), one constructs a finite 1-dimensional simplicial complex \( \tilde{L}_{k, \leq n} \) as follows. Its 0-skeleton is \( L_{k, \leq n} := \{ x \in L : \chi_k(x) \leq n \} \). For each \( x \) and \( j \in J \) with \( x, x + b_j \in L_{k, \leq n} \), we consider a unique 1-simplex with endpoints at \( x \) and \( x + b_j \) (e.g., the segment \([x, x + b_j]\) in \( L \oplus \mathbb{R} \)). We denote the set of connected components of \( \tilde{L}_{k, \leq n} \) by \( \pi_0(\tilde{L}_{k, \leq n}) \). For any \( v \in \pi_0(\tilde{L}_{k, \leq n}) \), let \( C_v \) be the corresponding connected component of \( \tilde{L}_{k, \leq n} \).

Next, we define the graded root \((R_k, \chi_k)\) as follows. The vertices \( V(R_k) \) are \( \bigcup_{n \in \mathbb{Z}} \pi_0(\tilde{L}_{k, \leq n}) \). The grading \( V(R_k) \to \mathbb{Z} \), still denoted by \( \chi_k \), is \( \chi_k|\pi_0(\tilde{L}_{k, \leq n}) = n \).

If \( v_n \in \pi_0(\tilde{L}_{k, \leq n}) \) and \( v_{n+1} \in \pi_0(\tilde{L}_{k, \leq n+1}) \), and \( C_{v_n} \subset C_{v_{n+1}} \), then \([v_n, v_{n+1}]\) is an edge of \( R_k \).

All the edges \( E(R_k) \) of \( R_k \) are obtained in this way.

Theorems (4.8) and (8.3) of [10] say:

5.4. Theorem. Assume that \( G \) is an AR-graph. Then, for any \([k] \in \text{Spin}^c(M)\), \( HF^+_{\text{even}}(-M, [k]) = 0 \) and
\[
HF^+_{\text{even}}(-M, [k]) = \mathbb{H}(R_k, \chi_k)\left[ -\frac{k^2}{4} + \frac{\#J}{4} \right].
\]

In particular,
\[
d(-M, [k]) = -\max_{k' \in [k]} \frac{(k')^2 + \#J}{4} = -\frac{k^2}{4} + \frac{\#J}{4} + 2 \min \chi_k.
\]

We wish to emphasize that in most of the applications, the root \((R_k, \chi_k)\) is not determined by its definition but by an algorithm which will be described in the sequel. (In particular, the above definition, in concrete applications can be neglected.)

Next, we distinguish one of vertices of our AR-graph \( G \), which satisfies the definition of AR-graphs. By convention, it corresponds to the index \( 0 \in J \) (fixed for ever).

5.5. The definition of the sequence \( \{x_{[k]}(i)\}_{i \geq 0} \). We fix a class \([k]\). Then for any integer \( i \geq 0 \) there exists a unique cycle \( x_{[k]}(i) \in L \) with the following properties:
(a) \( \text{pr}_0(x_{[k]}(i)) = i \);
(b) \( (x_{[k]}(i) + l'_k, b_j) \leq 0 \) for any \( j \in J \setminus \{0\} \);
(c) \( x_{[k]}(i) \) is minimal (with respect the partial ordering \( \leq \)) with the properties (a) and (b).

5.6. Theorem. (9.3) of [10] Fix a spin^c-structure \([k]\). There exists an integer \( T_0 \) (which depend on \([k]\)) such that \( \chi_{k_r}(x_{[k]}(i + 1)) \geq \chi_{k_r}(x_{[k]}(i)) \) for any \( i \geq T_0 \). Moreover, the graded root associated with \( \tau_{[k]} : \{0, \ldots, T_0\} \to \mathbb{N} \) given by \( \tau_{[k]}(i) := \chi_{k_r}(x_{[k]}(i)) \) satisfies
\[
(R_{k_r}, \chi_{k_r}) = (R_{\tau_{[k]}}, \chi_{\tau_{[k]}}).
\]

5.7. Remark. In general, it is not easy to find the cycles \( x_{[k]}(i) \). Fortunately, one does not need all the coefficients of these cycles, only the values \( \chi_{k_r}(x_{[k]}(i)) \). In most of the cases they are computed inductively using the following equality (cf. (9.1)(c) [10]):
\[
\chi_{k_r}(x_{[k]}(i + 1)) = \chi_{k_r}(x_{[k]}(i)) + b_0.
\]

Since the right hand side is \( \chi_{k_r}(x_{[k]}(i)) + \chi_{k_r}(b_0) - (b_0, x_{[k]}(i)) \), basically one needs only the intersections \((b_0, x_{[k]}(i))\) for any \( i \).
6. The first part of the proof: $k_1^2 + \# J$

6.1. The index set $J$. According to the shape of the plumbing graph $G(M)$, the index set of its vertices is $J = \mathcal{J} \cup \{1, \ldots, s\}$, where $\mathcal{J}$ is the index set of the vertices of $\bar{G}$, and the indices $\{1, \ldots, s\}$ correspond to $v_1, \ldots, v_s$. The distinguish vertex of $G(M)$ is $v_0$ corresponding to $0 \in \mathcal{J} \subset \mathcal{J}$. The base elements will also be denoted accordingly: $b_0, b_1, \ldots, b_s$; or $b_j$ for $j \in \mathcal{J}$.

6.2. The graph $\delta \Gamma$. As we already mentioned, the sub-graph $\bar{G}$ can be blown down. After this blow down, we get a graph which will be denoted as follows:

$\delta \Gamma: \begin{array}{c}
- \delta_1 \\
\vdots \\
- \delta_s
\end{array}$

We can think about this graph as the dual graph of a minimal resolution (of a normal surface singularity) obtained from a resolution with dual graph $G(M)$. In this sense, the new vertex $v_1$ is a rational curve with a singular point which has delta-invariant $\delta$; and the decorations $-k_i$ are the corresponding self-intersections. On the other hand, one can think about this graph also in the language of Kirby diagrams: $v_1$ represents the knot $K_\mathcal{J} \subset S^3$, the other vertices represent unknots, they are linked as usual with linking number one, and $-k_i$ are the corresponding surgery coefficients.

The point is that a large number of numerical invariants of the graph $G(M)$ – needed for the descriptions of section 5 – can already be determined from $\delta G$ in terms of $\delta$ and $\{k_i\}_i$ (for this comparison, see the second part of this section). The advantage of this is that the above graph $\delta G$ is the same as the graph of a lens space $L(p, q)$, provided that we disregard the decoration $\delta$.

In particular, its invariants can be computed by similar methods as those of lens spaces – in fact, in their computations we will even use the corresponding relations valid for lens spaces. This will be the subject of the first part of this section. Our model is section 10 of [10] where we run the algorithm of section 5 for lens spaces. The reader is invited to consult [loc. cit.] (and also verify that our present claims, verified in [loc. cit.] for case $q < p$, are valid for $q \geq p$ as well).

In fact, any invariant of the lattice (which does not involve $\delta$) equals the corresponding invariant of $L(p, q)$. But, formulas which involve $\delta$ (like the ‘adjunction formula’ determining the canonical characteristic element, or the ‘Riemann-Roch’ formula for $\chi_k$) depend essentially on $\delta$.

6.2.1. Notations. In order to make a distinction between invariants of the graphs $G(M)$ and $\delta G$, the invariants of the second one will have an extra $\delta$-decoration; e.g. $\bar{L}$ denotes the lattice of rank $s$ with base elements $\{b_i\}_{i=1}^s, \{\tilde{g}_i\}_{i=1}^s \in \bar{L}'$ are the dual bases, etc.

For any $1 \leq i \leq j \leq s$ we write the continued fraction $[k_i, \ldots, k_j]$ as a rational number $n_{ij}/d_{ij}$ with $n_{ij} > 0$ and $\gcd(n_{ij}, d_{ij}) = 1$. We also set $n_{i,i-1} := 1$ and $n_{ij} := 0$ for $j < i - 1$. In fact, one gets $d_{ij} = n_{i+1,j}$.

Clearly $n_{1s} = p$ and $n_{2s} = q$. We also write $q' := n_{1,s-1}$. One can verify (e.g. by induction) that $p \geq q' \geq 1$ (even if $p < q$), and $qq' \equiv 1 \pmod{p}$ (since $qq' \equiv p n_{2,s-1} + 1$).

6.2.2. The group $\tilde{H}$ and the elements $\tilde{t}_k$. Clearly $\tilde{H} = \bar{L}'/\bar{L} = \mathbb{Z}_p$, and $[\tilde{g}_s] = \tilde{g}_s + \bar{L}$ is one of its generators. Then $[\tilde{g}_j] = [n_{j+1,s} \tilde{g}_s]$ in $\tilde{H}$ ($1 \leq j \leq s$). The set of spin$^c$-structures is the set of orbits $\{-a \tilde{g}_s + \bar{L}\}_{0 \leq a < p}$ (we prefer to use this sign, since $-\tilde{g}_s > 0$). For any $0 \leq a < p$ write $\tilde{t}_{[-a \tilde{g}_s]} = -(a_1 \tilde{g}_1 + a_2 \tilde{g}_2 + \cdots + a_s \tilde{g}_s)$.

By [10] §10, the system $(a_1, \ldots, a_s)$ can be realized as the set of coefficients of a minimal vector $\tilde{t}_{[-a \tilde{g}_s]}$ for some $0 \leq a < p$ if and only if the entries satisfy the inequalities: $(SI)$

$$
\begin{cases}
\ a_1 \geq 0, \ldots, a_s \geq 0 \\
\ n_{i+1,s} a_1 + n_{i+2,s} a_{i+1} + \cdots + n_{is} a_{s-1} + a_s < n_{is} \text{ for any } 1 \leq i \leq s.
\end{cases}
$$

The integer $0 \leq a < p$ can be recovered from $(a_1, \ldots, a_s)$ (which satisfies (SI)) by

$$
\ a = n_{2s} a_1 + n_{3s} a_2 + \cdots + n_{ss} a_{s-1} + a_s.
$$
On the Heegaard Floer homology of $S^3_{-p/q}(K)$

Conversely, any $0 \leq a < p$ determines inductively the entries $a_1, \ldots, a_s$ by

$$a_i = \left[ \frac{a - \sum_{t=1}^{i-1} n_{t+1,s} a_t}{n_{i+1,s}} \right] \quad (1 \leq i \leq s).$$

6.2.3. **The canonical characteristic element.** Let $\delta\tilde{K}$ denote the canonical characteristic element associated with the graph $\delta G$ (see below). It is convenient to consider the canonical characteristic element $\tilde{K}$ of the graph of $L(p,q)$ (which is obtained from $\delta G$ by omitting $\delta$). The adjunction relations defining $\tilde{K}$ are the usual ones, but those which identify $\delta\tilde{K}$ are the following (see also [11, 3.3]):

$$(\delta\tilde{K}, \tilde{b}_j) - k_j + 2$$

should equal twice the delta-invariant of the vertex $v_j$, namely $2\delta$ for $j = 1$, and $0$ otherwise. Hence:

$$\delta\tilde{K} = \tilde{K} + 2\delta\tilde{g}_1.$$ 

In particular,

$$\delta\tilde{K}^2 + s = \tilde{K}^2 + 4\delta(\tilde{K}, \tilde{g}_1) + 4\delta^2(\tilde{g}_1, \tilde{g}_1).$$

$(\tilde{K}, \tilde{g}_1)$ is the $\tilde{b}_1$-coefficient of $\tilde{K}$, which equals (e.g. by [11, (5.2)]) $-1 + (q + 1)/p$. Similarly, $\tilde{g}_1^2 = \tilde{B}_{11}^{-1} = -n_{2s}/p = -q/p$. Notice also that (cf. (7.1) [11], or (10.4) of [10] – although this formula goes back to the work of Hirzebruch):

$$\tilde{K}^2 + s = \frac{2(p - 1)}{p} - 12 \cdot s(q,p),$$

where $s(q,p)$ denotes the **Dedekind sum**

$$s(q,p) = \sum_{i=0}^{p-1} \left( \left( \frac{i}{p} \right) \left( \frac{a_i}{p} \right) \right),$$

where $(x)$ is such that $x \in \mathbb{R} \setminus \mathbb{Z}$.

Therefore,

$$\delta\tilde{K}^2 + s = \frac{2(p - 1)}{p} - 12s(q,p) - 4\delta(1 - \frac{q + 1}{p}) - 4\delta^2 \frac{q}{p}.$$ 

The distinguished characteristic element $\tilde{k}_r$ of the orbit $-aq_s + \tilde{L}$ is

$$\tilde{k}_r = \delta\tilde{K} + 2\tilde{g}_{[-a\tilde{g}_s]}.$$ 

Therefore,

$$\tilde{k}_r^2 + s = \delta\tilde{K}^2 + s + 4 \cdot \left( \tilde{K} + \tilde{g}_{[-a\tilde{g}_s]}, \tilde{g}_{[-a\tilde{g}_s]} \right) + 8\delta \cdot \left( \tilde{g}_1, \tilde{g}_{[-a\tilde{g}_s]} \right).$$

By [10] (10.4.1) one has

$$(\tilde{K} + \tilde{g}_{[-a\tilde{g}_s]}, \tilde{g}_{[-a\tilde{g}_s]}) = \frac{a(p - 1)}{p} - 2 \sum_{j=1}^{a} \left\{ j\tilde{q} \right\}.$$ 

On the other hand, by the above formulas:

$$(\tilde{g}_1, \tilde{g}_{[-a\tilde{g}_s]}) = -\sum_{i=1}^{s} a_i (\tilde{g}_1, \tilde{g}_i) = \sum_{i=1}^{s} \frac{n_{i+1,s}}{p} a = \frac{a}{p}.$$ 

Summing all, one gets

$$-\frac{\tilde{k}_r^2}{4} + s = 3s(q,p) + 2 \sum_{j=1}^{a} \left\{ j\tilde{q} \right\} - \frac{(1 + 2a)(p - 1)}{2p} + \delta \left( 1 - \frac{q + 1}{p} \right) + \frac{\delta^2 q}{p} - \frac{2\delta a}{p}.$$
6.3. **Back to the graph** $G(M)$. Finally, we compare the invariants of the graphs $G(M)$ and $\delta G$. Some of the arguments are well-known, nevertheless for the convenience of the reader we provide the proofs.

There are two natural morphisms connecting the corresponding lattices $L$ and $\tilde{L}$. The first one, $\pi_* : L \to \tilde{L}$, is defined by $\pi_*(b_j) = 0$ for any $j \in J$, while $\pi_*(b_i) = \tilde{b}_i$ for $i = 1, \ldots, s$.

In order to define the second morphism, we need an additional construction.

Let $Z_f := \sum_{j \in J} m_j b_j$ be the cycle supported by $G$ which satisfies

$$(Z_f + b_1, b_j) = 0 \text{ for } j \in J.$$ 

Since $\det \hat{G} = \pm 1$, this system has a unique integral solution $\{m_j\}_J$. In fact, in terms of the diagram $G(f)$ (cf. [24]), $m_j$ is the vanishing order of the pull back of $f$ along the corresponding irreducible exceptional divisor. E.g., $m_0 = m_f$.

Then one defines $\pi^* : \tilde{L} \to L$ by $\pi^*(\tilde{b}_j) = b_j$ for $j \geq 2$ and $\pi^*(\tilde{b}_1) = Z_f + b_1$. By the very definition follows the "projection formula":

$$\left( \pi^*(\tilde{l}), l \right) = \left( \tilde{l}, \pi_*(l) \right) \text{ for } \tilde{l} \in \tilde{L}, l \in L.$$ 

### 6.3.1. The group $H$.

$\pi_*$ has a natural extension to $L_Q \to \tilde{L}_Q$, and $\pi_*(L') \subset \tilde{L'}$. Therefore, $\pi_*$ induces a group morphism $H \to \tilde{H}$. Its surjectivity follows from $\pi_* \pi^* = 1$. In order to prove injectivity, consider an $l' \in L'$ with $\pi_*(l') \in \tilde{L}$. Then $\pi^* \pi_*(l')$ is an element of $L$. Since $\pi^* \pi_*(l') - l'$ is supported by $\hat{G}$, and $\det \hat{G} = \pm 1$, one gets that $\pi^* \pi_*(l') - l' \in L$ as well. Hence $l' \in L$, and $\pi_*$ induces an isomorphism $H \to \tilde{H} = \mathbb{Z}_p$, and $H$ is generated by $g_s$.

In the sequel, the set of spin-structures of $M$ will be identified with the set of orbits $\{-ag_s + L\}_{0 \leq a < p}$. We denote by $\sigma_s$ that spin-structure which correspond to the orbit $-ag_s + L$.

### 6.3.2. The element $l'_{[-ag_s]}$.

We claim that $l'_{[-ag_s]} = \pi^*(l'_{[-ag_s]})$. Indeed, $l' := \pi^*(l'_{[-ag_s]}) \in S_Q$ by the projection formula. Next, we verify the minimality of $l'$. Notice that any $l \in L$ with $l \geq 0$ can be written in the form $\pi^*(x) + \tilde{x}$, where $\tilde{x} \geq 0$ and $\tilde{x}$ is supported by $\hat{G}$. Assume that for such $\pi^*(x) + \tilde{x} \geq 0$ one has:

$$l' - \pi^*(x) - \tilde{x}, b_j) \leq 0 \text{ for any } j \in J.$$ 

Since $(\pi^*(y), b_j) = 0$ for any $j \in \tilde{J}$, one gets that $(-\tilde{x}, b_j) \leq 0$ for any $j \in \tilde{J}$. Since $\hat{G}$ is negative definite, it follows that $\tilde{x} \leq 0$. Using this, $\ast$ for $j = 1, \ldots, s$ gives

$$0 \geq (l' - \pi^*(x), b_j) + (-\tilde{x}, b_j) \geq (l' - \pi^*(x), b_j) + (l'_{[-ag_s]} - \tilde{x}, b_j).$$

hence, by the minimality of $l'_{[-ag_s]}$, one has $\tilde{x} = 0$. But then $\tilde{x} \geq 0$ and $\tilde{x} \leq 0$ implies $\tilde{x} = 0$ as well.

### 6.3.3. Claim: $\pi_*(K) = \delta \tilde{K}$.

First notice that by the projection formula:

$$\pi_*(g_j) = \begin{cases} m_j \tilde{g}_1 & \text{if } j \in \tilde{J} \\ \tilde{g}_j & \text{if } j = 1, \ldots, s. \end{cases}$$

(where $Z_f = \sum_{j \in \tilde{J}} m_j b_j$ as above). The adjunction relations for $G(M)$ can be rewritten as

$$K = -\sum_{j \in \tilde{J}} b_j - \sum_{j \in J} (2 - s_j) g_j,$$

where $s_j$ is the adjacent degree of the vertex $j$ in $G(M)$. They also satisfy (cf. [11]):

$$\sum_{j \in \tilde{J}} (2 - s_j) m_j = 1 - \mu = 1 - 2\delta.$$

Then, taking $\pi_*$ one gets $\pi_*(K) = \tilde{K} + 2\delta \tilde{g}_1 = \delta \tilde{K}$. 

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6.3.4. **Claim:** $K^2 + \# \mathcal{J} = \delta \tilde{K}^2 + s$. Indeed, set $K_\mathcal{G} := K - \pi^*(\delta \tilde{K})$. Then $K_\mathcal{G}$ is supported by $G$ and satisfies the adjunction relations $(K_\mathcal{G}, b_i) = -e_j - 2$ for any $j \in \mathcal{J}$, hence it is the canonical cycle of $\bar{G}$. Moreover, since $\bar{G}$ can be blown down (and since by an elementary blow up of a smooth point $K^2$ decreases by 1), one has $K_\mathcal{G}^2 = -(\# \mathcal{J} - s)$. By projection formula $\pi^*(\delta \tilde{K})$ is orthogonal to $K_\mathcal{G}$, hence

$$K^2 = (\pi^*(\delta \tilde{K}) + K_\mathcal{G})^2 = \delta \tilde{K}^2 + K_\mathcal{G}^2 = \delta \tilde{K}^2 - (\# \mathcal{J} - s).$$

Finally, this claim, via the projection formula, shows for any $spin^c$-structure that

$$k^2 + \# \mathcal{J} = \tilde{k}^2 + s.$$

6.4. **Example. Integer surgery.** Assume that $q = 1$, i.e. $M = S^3_{-p}(K_f)$. Then $q' = 1$ as well, and

$$-\frac{k^2 + \# \mathcal{J}}{4} = -\frac{\tilde{k}^2 + s}{4} = \frac{(p + 2\delta - 2 - 2a)^2 - p}{4p}.$$

6.5. **Example. (1/q)-surgery.** Assume that $p = 1$, i.e. $M = S^3_{-1/q}(K_f)$. Then again $q' = 1$. There is only one $spin^c$-structure corresponding to $a = 0$, and

$$-\frac{K^2 + \# \mathcal{J}}{4} = -\frac{\tilde{K}^2 + s}{4} = q\delta(\delta - 1).$$

7. The second part of the proof: $(R_{\tau(k)}) \cdot \chi_{\tau(k)}$

The construction of the cycles $\tau(k)(i)$ is given in two steps. The first step provides similar cycles associated with the graph $\bar{G}$, and it is based on the combinatorial properties of the graph $G(f)$ (involving also some techniques of plane curve singularities). In particular, we prefer to think about $\bar{G}$ as the dual graph of irreducible exceptional curves obtained by repeated blow ups of $(\mathbb{C}^2, 0)$. We will use the notation $S := \{y : (y, b_j) \leq 0, \ j \in \mathcal{J}\}$.

7.1. **The cycles $\{y(i)\}_{i \geq 0}$.** Since $\bar{G}$ is negative definite, (7.6) of [10] guarantees, for any $i \geq 0$, the existence of a positive cycle $y(i) \geq 0$ (supported by $\bar{G}$) with the following properties:

(a) $\text{pr}_0(y(i)) = i$;
(b) $(y(i), b_j) \leq 0$ for any $j \in \mathcal{J} \setminus \{0\}$;
(c) $y(i)$ is minimal (with respect the partial ordering $\leq$) with the properties (a) and (b).

E.g., $y(0) = 0$. Although there is very precise algorithm which determines all the cycles $y(i)$ (see e.g. the proof of [7.2.1] or [10], we are not interested in all the coefficients of $y(i)$. But what we really have to know is the set of intersection numbers $(y(i), b_0)$ (cf. 6.7). Let $Z_f$ be the divisorial part (supported by $\bar{G}$) of the germ $f$, cf. [6.8] which satisfies

$$(Z_f, b_j) = \begin{cases} 
0 & \text{if } j \in \mathcal{J} \setminus \{0\}; \\
-1 & \text{if } j = 0.
\end{cases}$$

Recall also that $\Gamma \subset \mathbb{N}$ denotes the semigroup of $f$, see [2.1].

7.2. **Proposition.** (a) If $i = tm_f + i_0$ with $t \geq 0$ and $0 \leq i_0 < m_f$, then $y(i) = tZ_f + y(i_0)$;
(b) For any $i < m_f$ one has

$$(y(i), b_0) = \begin{cases} 
1 & \text{if } i \notin \Gamma; \\
0 & \text{if } i \in \Gamma.
\end{cases}$$

Proof. (a) Clearly, $y'(i) := tZ_f + y(i_0)$ satisfies (a)-(b), we need to verify (c). But if for some $y \geq 0$, $y'(i) - y$ satisfies (a)-(b), then $y(i_0) - y$ would also satisfy (a)-(b) for $i_0$, hence $y = 0$ by minimality of $y(i_0)$. Part (b) will follow from the next sequence of lemmas. □

The first lemma was proved and used intensively in [10] as a general principle of rational graphs. For the convenience of the reader, we sketch its proof, for more details, see [loc. cit.].
7.2.1. **Lemma.** \((y(i), b_0) \leq 1\) for any \(i \geq 0\).

**Proof.** We denote by \(Z_{\text{min}}\) Artin's minimal cycle associated with \(\tilde{G}\), i.e. the minimal cycle in \(S\). The cycle \(Z_{\text{min}}\) can be determined by the following (Lauret's) algorithm (72): Construct a computation sequence of cycles \(\{Z_i\}\) as follows. Set \(Z_0 := 0\), \(Z_1 := b_0\); if \(Z_i\) is already constructed, but for some \(j \in J\) one has \((Z_i, b_j) > 0\), then take \(Z_{i+1} := Z_i + b_j\). If \(Z_i \in S\) then stop and write \(Z_i = Z_{\text{min}}\).

Now, by a geometric genus computational algorithm, one gets for all rational graphs that whenever \((Z_i, b_j) > 0\) in the above algorithm, in fact one has the equality (73): \((Z_i, b_j) = 1\) (cf. [7]). Since \(\tilde{G}\) can be blown down, it is rational, hence (73) works.

The point is that \(y(i)\) can also be computed by a similar algorithm: Assume that \(y(i)\) is already determined. Then construct the sequence of cycles \(\{Z_i\}\) as follows. \(Z_0 := y(i), Z_1 := y(i) + b_0\), and if (for \(l \geq 1\)) \((Z_l, b_j) > 0\) for some \(j \in J \setminus \{0\}\), then take \(Z_{i+1} := Z_i + b_j\); otherwise stop and write \(Z_i = y(i + 1)\).

Then one can verify that any sequence connecting \(y(i)\) with \(y(i + 1)\) can be considered as part of a computation sequence associated with \(Z_{\text{min}}\), hence lemma follows by the above property (73). \(\square\)

7.2.2. **Lemma.** Fix an arbitrary \(i \geq 0\). Then \((y(i), b_0) \leq 0\) if and only if \(i \in \Gamma\).

**Proof.** \(i \Rightarrow \) \(i\) If \((y(i), b_0) \leq 0\) then \(y(i) \in S\). Since \(\tilde{G}\) is the dual graph of a modification of \((\mathbb{C}^2, 0)\), the cycle \(y(i)\) is the divisorial part \(Z_g\) of a holomorphic germ \(g \in \mathcal{O}(\mathbb{C}^2, 0)\). Since the intersection multiplicity \(i_0(f, g)\) of the germs \(f\) and \(g\) at \(0 \in \mathbb{C}^2\) is the multiplicity of \(Z_g\) along the exceptional divisor supporting the arrow of \(f\), one gets \(i_0(f, g) = \text{pr}_0(y(i)) = i\). But, by definition, \(i_0(f, g) \in \Gamma\).

\(i \Leftarrow \) \(i\) If \(i \in \Gamma\), then there exists \(g \in \mathcal{O}(\mathbb{C}^2, 0)\) with \(i_0(f, g) = i\); i.e. with \(\text{pr}_0(Z_g) = i\) (see above) and \(Z_g \in S\) (a general property of divisorial parts of holomorphic germs). Then the minimality of \(y(i)\) implies \(y(i) \leq Z_g\). This, and \(\text{pr}_0(y(i)) = \text{pr}_0(Z_g)\), and the fact that \(B\) off the diagonal has positive entries imply that \((b_0, y(i)) \leq (b_0, Z_g)\). But, since \(Z_g \in S\), \((b_0, Z_g) \leq 0\).

\(\square\)

Lemma 7.2.2 can be improved for \(i < m_f\):

7.2.3. **Lemma.** Assume that \(i < m_f\). Then \((y(i), b_0) = 0\) if and only if \(i \in \Gamma\).

**Proof.** Assume that \((y(i), b_0) < 0\) for some \(i\). Then, by a verification \(y(i) - Z_f \in S\), hence (since \(\tilde{G}\) is negative definite) \(y(i) - Z_f \geq 0\). In particular, \(i - m_f = \text{pr}_0(y(i) - Z_f) \geq 0\). \(\square\)

7.3. **Back to** \(G(M)\). We consider again the graph \(G(M)\). We fix an orbit \([k] = -aq_s + L\) for some \(0 \leq a < p\) (corresponding to the spin\(c\)-structure \(\sigma_a\) of \(M\)). In the sequel we will use freely the notations of the previous section. Additionally, we write for any \(j = 1, \ldots, s:\)

\[a_j' := n_{j+1,s}a_j + a_{j+2,s}a_{j+1} + \ldots + a_s.\]

E.g., by 123, \(a_j' = a\).

Consider now the sub-graph of \(G(M)\) with vertices \(v_0, \ldots, v_s\) and the \(s\) edges connecting them. We wish to identify the cycle \(z(i) := ib_0 + \sum_{j=1}^s u_je_j\) which has the properties of \(x(i)\) ‘restricted on this sub-graph’. More precisely:

7.4. **Proposition.** Fix \(i \geq 0\) and \(0 \leq a < p\).

(a) For \(j = 1, \ldots, s\) define the integers \(u_j\) inductively by:

\[u_1 := \left\lfloor \frac{iq - a}{p + qa} \right\rfloor, \quad u_j := \left\lfloor \frac{u_{j-1}n_{j+1,s} - a_j'}{n_{j,s}} \right\rfloor \quad (1 < j \leq s).\]

Then the cycle \(z(i) := ib_0 + \sum_{j=1}^s u_je_j\) is \(\geq 0\) and satisfies

\[(z(i) + l'_{[-a_s]}b_j) \leq 0 \quad \text{for any } j = 1, \ldots, s.\]

(b) Assume that the cycle \(z(i) := ib_0 + \sum_{j=1}^s a_jb_j\) is \(\geq 0\) and satisfies

\[(z(i) + l'_{[-a_s]}b_j) \leq 0 \quad \text{for any } j = 1, \ldots, s.\]
Corollary.\[\bar{u}_1 \geq \left[ \frac{iq - a}{p + q m_f} \right], \text{ and } \bar{u}_j \geq \left[ \frac{\bar{u}_{j-1} n_{j+1,s} - a'_j}{n_{j,s}} \right] \text{ for } 1 < j \leq s.\]

Proof. This property was used in a similar situation for the plumbing graph of a Seifert manifolds, applied for its ‘legs’, cf. 11.11 of [10]. For the convenience of the reader we sketch the proof.

(a) By the definition of $u_j$ one has $u_j n_{j,s} \geq u_{j-1} n_{j+1,s} - a'_j$. This is equivalent, via the identities $n_{j,s} = k_j n_{j+1,s} - n_{j+2,s}$ resp. $a'_j = n_{j+1,s} a_j + a_{j+1}$, with

$$u_j k_j - u_{j-1} + a_j \geq \frac{u_j n_{j+2,s} - a'_{j+1}}{n_{j+1,s}}.$$  

Using the definition of $u_{j+1}$, one gets $u_j k_j - u_{j-1} + a_j \geq u_{j+1}$, which is exactly $(z(i) + l'_[-a_{g,j}], b_j) \leq 0$. (b) can be proved by a descending induction, using the above sequence of arguments in reversed order. \[\square\]

7.5. Corollary. Fix $0 \leq a < p$ and write $i = tm_f + i_0$ for some $t \geq 0$ and $0 \leq i_0 < m_f$. Then

$$x_{[k]}(i) = t \cdot Z_f + y(i_0) + \left[ \frac{iq - a}{qm_f + p} \right] b_1 + \sum_{j \geq 2} u_j b_j.$$  

In particular,

$$\langle x_{[k]}(i), b_0 \rangle = -t + \left[ \frac{iq - a}{qm_f + p} \right] + \langle y(i_0), b_0 \rangle.$$  

Moreover:

$$\chi_{k_s}(x_{[k]}(i + 1)) - \chi_{k_s}(x_{[k]}(i)) = t + 1 - \left[ \frac{iq - a}{qm_f + p} \right] - \left\{ \begin{array}{cl} 1 & \text{if } i_0 \not\in \Gamma \\ 0 & \text{if } i_0 \in \Gamma. \end{array} \right.$$  

Proof. Since the subgraphs $\bar{G}$ and its complement are connected in $G(M)$ only through $v_0$, the two parts of $x_{[k]}(i)$ – one supported by $\bar{G}$, the other by its complement – have independent lives. But, for any $j \in \bar{J}$, $\langle x_{[k]}(i) + l'_[-k_{[k]}], b_j \rangle = (x_{[k]} + \pi^*(l'_[-k_{[k]}]), b_j) = (x_{[k]}, b_j)$. Hence $x_{[k]}(i)$ restricted on $\bar{G}$ should be exactly $y(i)$. Clearly, the restriction on the complement of $\bar{G}$ should be $z_{[k]}(i)$. Hence the first two statements follow. The last is the consequence of [10] together with the identity $\chi_{k_s}(b_0) = 1$, for $(l'_[-k_{[k]}], b_0) = (\pi^*(l'_[-k_{[k]}]), b_0) = 0$. \[\square\]

7.6. $\tau_a$. Corresponding to this and [5.4] we write $\tau_a(i) := \chi_{k_s}(x_{[k]}(i))$. With the notations of [6.3] notice that

$$\frac{iq - a}{qm_f + p} \leq t + 1,$$

hence $\tau_a(i + 1) - \tau_a(i) \geq -1$ for any $i$, and $= -1$ only in very special situations, namely if

$$\frac{(tm_f + i_0)q - a}{qm_f + p} > t \text{ and } i_0 \not\in \Gamma.$$  

In order to analyze when is this possible, we will consider sequences $Seq(t) := \{tm_f + i_0 : 0 \leq i_0 < m_f\}$ for fixed $t \geq 0$. In such a sequence, notice that the very last element of $\mathbb{N} \setminus \Gamma$, namely $\mu - 1 = 2 \delta - 1$ is strict smaller than $m_f - 1$ (a fact, which can be proved by induction over the number of Newton pairs), hence the complete set $\mathbb{N} \setminus \Gamma$ sits in $\{0, \ldots, m_f - 1\}$. Therefore, there exists in $Seq(t)$ an $i_0$ satisfying (1) if and only if

$$\frac{(tm_f + 2 \delta - 1)q - a}{qm_f + p} > t,$$
This is equivalent to \( t \leq t_a \), for \( t_a \) defined in Eq. 8.13. In other words, if \( i \geq T_0 := (t_0 + 1)m_f \), then \( \tau_a(i+1) \geq \tau_a(i) \), hence those values of \( \tau_a \) provide no contribution in the graded root (cf. Eq. 8.13. Moreover, for \( t \in \{0, \ldots, t_a\} \), in \( \text{Seq}(t) \) one has:

\[
\Delta(i_0) := \tau_a(tm_f + i_0 + 1) - \tau_a(tm_f + i_0) = \begin{cases} 
0 & \text{if } i_0 \leq (tp + a)/q, \quad \text{and } i_0 \not\in \Gamma; \\
+1 & \text{if } i_0 \leq (tp + a)/q, \quad \text{and } i_0 \in \Gamma; \\
-1 & \text{if } i_0 > (tp + a)/q, \quad \text{and } i_0 \not\in \Gamma; \\
0 & \text{if } i_0 > (tp + a)/q, \quad \text{and } i_0 \in \Gamma.
\end{cases}
\]

In particular, \( \Delta(i_0) \geq 0 \) for \( 0 \leq i_0 \leq (tp + a)/q \) with exactly
\[
A_t := \#\{\gamma \in \Gamma : \gamma \leq (tp + a)/q\}
\]
times taking the value +1, otherwise zero; and \( \Delta(i_0) \leq 0 \) for \( i_0 > (tp + a)/q \) with exactly
\[
B_t := \#\{\gamma \not\in \Gamma : \gamma > (tp + a)/q\}
\]
times taking the value −1, otherwise zero. Notice that both \( A_t \) and \( B_t \) are strict positive (since \( 0 \in \Gamma \), respectively \( 2\delta - 1 \not\in \Gamma \) and \( 2\delta - 1 > (tp + a)/q \). This shows that

\[
M_t := \max_{0 \leq i_0 < m_f} \tau_a(tm_f + i_0) = \tau_a(tm_f) + A_t = \tau_a((t + 1)m_f) + B_t,
\]

and

\[
M_t \geq \max\{\tau_a(tm_f), \tau_a(tm_f + m_f)\}.
\]

Therefore, the graded root associated with the values \( \{\tau_a(i)\}_{0 \leq i \leq (t_a+1)m_f} \) is the same as the graded root associated with the values

\[
\tau_a(0), M_0, \tau_a(m_f), M_1, \tau_a(2m_f), M_2, \ldots, \tau_a(t_am_f), M_{t_a}, \tau_a(t_am_f + m_f).
\]

Finally, notice, since \( \#\{\gamma \not\in \Gamma\} = \delta \), one has \( \delta - B_t = \#\{\gamma \not\in \Gamma : \gamma \leq (tp + a)/q\} \), hence \( \delta - B_t + A_t = [(tp + a)/q] + 1 \). Hence, by (2),

\[
\tau_a((t + 1)m_f) - \tau_a(tm_f) = \left\lceil \frac{tp + a}{q} \right\rceil + 1 - \delta.
\]

Since \( \tau_a(0) = 0 \), this gives \( \tau_a(tm_f) \) inductively. Notice also that \( B_t = \alpha_{[(tp+a)/q]} \).

Clearly, the graded root associated with \( \tau_a \) is the same as the graded root associated with \( \tau_{\hat{a}} : \{0, 1, 2, \ldots, 2t_a + 2\} \rightarrow \mathbb{Z} \), where \( \tau_{\hat{a}}(2t) := \tau_a(tm_f) \) and \( \tau_{\hat{a}}(2t + 1) := M_t \). This is the tau-function of Eq. 8.13 if we delete \( \sim \).

8. Examples

8.1. Example. Assume \( p = q = 1 \). In this case \( M \) is integral homology sphere; \( a = 0 \) and \( t_0 = 2\delta - 2 = \mu - 2 \). In particular, the rank of \( \text{ker} U \) is \( \mu \). Moreover, \( r_0 = \delta(\delta - 1) \) and \( \tau_0(2t) = t(t - 2\delta + 1)/2 \). The reader is invited to draw the graded root and verify that

\[
\text{HF}^+(\!-M\!) = T_0^+ \oplus T_0(\alpha_{\delta - 1}) \oplus \bigoplus_{i=1}^{\delta - 1} T_{i(i + 1)}(\alpha_{i - 1 + \delta})^2;
\]

\[
\text{ker} U = \bigoplus_{i=0}^{\delta - 1} \mathbb{Z}_{(i + 1)}^2.
\]

8.2. Example. More generally, assume only that \( p = 1 \). The Heegaard Floer homology of the unique \textit{spin}-structure is:

\[
\text{HF}^+(\!-M\!) = T_0^+ \oplus T_0(\alpha_{\delta - 1}) \oplus \bigoplus_{i=1}^{(\delta - 1)q} T_{i(i + 1)}((t/q)i + 1)(\alpha_{\delta - 1 + [i/q]})^2;
\]
8.3. **Remark.** Apparently in $S^3_{-p/q}(K)$ $HF^+(-M)$ contains less information than the polynomial $P$, the above formula involves only the coefficients $a_j$ for $j \geq \delta - 1$. But this is not the case. Indeed, since the Alexander polynomial $\Delta(t)$ is symmetric, one gets that

$$t^{\mu-2}P(1/t) - P(t) = \frac{\delta(1 + t^{\mu-1}) - (1 + t + \cdots + t^{\mu-1})}{t-1},$$

hence $\alpha_{\mu-2-i} - \alpha_i$ is a universal number. In particular, from $\{\alpha_j\}_{j \geq \delta - 1}$ one can recover $P$.

This shows that from $HF^+(-S^3_{-1/q}(K_f))$ one can recover both the integer $q$ and the isotopy type of $K_f \subset S^3$. It looks that similar result is valid for general surgery coefficients as well.

8.4. **Remark.** In the above example it is striking a $\mathbb{Z}_2$-symmetry of the graded root and of $HF^+(-M)$. We will explain this for $a = 0$.

The point is that if the canonical characteristic element has only integral coefficients, then all the theory associated with the integral cycles (the lattice $L$ in section 5) has a duality. This is happening for example if $p = 1$; or if $q = 1$ and $2\delta - 2$ is divisible by $p$. In our case, in these situations, the function $\tau_0$ is stable with respect to the $\mathbb{Z}_2$ action $i \mapsto 2t_0 + 2 - i$, i.e. $\tau_0(1) = \tau_0(2t_0 + 2 - 1)$. This induces a symmetry of the root and of the Heegaard Floer homology. [In singularity theory a resolution graph whose canonical characteristic element has only integral coefficients is called ‘numerical Gorenstein graph’.]

But, for general $p/q$, the graphs are not ‘numerical Gorenstein’. Even if they are, for general $a$, the symmetry may fail.

8.5. **Example.** Assume that $K_f \subset S^3$ is the torus knot $(4, 5)$ (i.e. $g = 1$ and $(p_1, q_1) = (4, 5)$). In this case $\delta = 6, \mu = 12, \Gamma$ is generated by 4 and 5, hence

$$P(t) = 6 + 5t + 4t^2 + 3t^3 + 3t^4 + 3t^5 + 2t^6 + t^7 + t^8 + t^9 + t^{10}.$$ 

Then for some $(p/q)$-surgeries the corresponding graded roots are presented in the next Figure. [Here we did not draw all the vertices, only those ones which are either local minimums or supremums of pairs of local minimums; in fact, exactly these ones are given by the function $\tau_a$. Also, the roots should be continued upward with one vertex in $\chi^{-1}(k)$ for each $k \geq 1$.]

Recall that when we compute the grading of $HF^+$, the value of $\tau_a$ is doubled, and then shifted by $r_a$. The corresponding values of $r_a$ in these five cases are:

$$30, \; 71/4, \; 49/4, \; \frac{(p + 10)^2 - p}{4p}, \; 60.$$ 

In particular, in the first and fifth case (i.e. when $p = 1$) one has $d(-M, \sigma_0) = 0$, as we expected; the Heegaard Floer homology are written in [\ref{8.1}] resp. [\ref{8.2}]. In the second and third cases (i.e. when $p/q = 2$ and $a = 0, 1$) the $\mathbb{Z}[U]$-modules are:

$$\left( \mathcal{T}_{-18}^+ \oplus \mathcal{T}_{-16}(2)^{\mathbb{Z}_2} \oplus \mathcal{T}_{-10}(1)^{\mathbb{Z}_2} \oplus \mathcal{T}_0(1)^{\mathbb{Z}_2} \right) [71/4];$$

$$\left( \mathcal{T}_{-12}^+ \oplus \mathcal{T}_{-12}(3) \oplus \mathcal{T}_{-8}(1)^{\mathbb{Z}_2} \oplus \mathcal{T}_0(1)^{\mathbb{Z}_2} \right) [49/4].$$

In the forth case it is

$$\mathcal{T}_{-10}^+ \oplus \mathcal{T}_0(1) \left[ \frac{(p + 10)^2 - p}{4p} \right].$$
\[ \tau_a(i) \]

\[
\begin{array}{cccc}
p/q = 1 & p/q = 2 & p/q = 2 & p \geq 11, q = 1 \\
a = 0 & a = 0 & a = 1 & a = 0 \\
p/q = 1/2 & a = 0 & & \\
\end{array}
\]

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