Bayesian Computing in the Statistics and Data Science Curriculum

Jim Albert* and Jingchen Hu**
*Department of Mathematics and Statistics, Bowling Green State University
**Department of Mathematics and Statistics, Vassar College

February 25, 2020

Abstract

Bayesian statistics has gained great momentum since the computational developments of the 1990s. Gradually, advances in Bayesian methodology and software have made Bayesian techniques much more accessible to applied statisticians and, in turn, have transformed Bayesian education at the undergraduate and graduate levels. In this article, we introduce the history behind Bayesian computing, discuss the important role of simulation, and lay out the foundation of Markov chain Monte Carlo. We further survey and weigh various options for implementing Bayesian computational methods in practice. We conclude with computing recommendations for different models of the modern Bayesian classroom, from introductory applied courses to advanced courses with more emphasis on theory.

Keywords: Bayesian computing, Bayesian education, JAGS, statistical computing, statistics education
1 Introduction

1.1 The Bayesian Paradigm

One attractive feature of the Bayesian paradigm is the clear algorithm for performing statistical inference. Suppose one collects an observation vector $y$ distributed according to a sampling density $f(y \mid \theta)$ depending on parameters $\theta$. Suppose that one’s prior beliefs about $\theta$ are stated in terms of a prior density $g(\theta)$. Once $y$ is observed, all inferences about the parameters are based on the posterior density $g(\theta \mid y)$ which is proportional to the product of the likelihood and the prior:

$$g(\theta \mid y) \propto f(y \mid \theta)g(\theta).$$

In addition, one is typically interested in predictions of future observations $\tilde{y}$. One learns about the location of this future data by means of the predictive density $p(\tilde{y} \mid y)$ obtained by integrating the sampling density $p(\tilde{y} \mid \theta)$ over the posterior density

$$p(\tilde{y} \mid y) = \int p(\tilde{y} \mid \theta)g(\theta \mid y)d\theta.$$

To obtain Bayesian point and interval estimates for $\theta$, one summarizes the posterior density in different ways. For example, the posterior mean of a particular parameter $\theta_j$ is expressible as the ratio of two integrals

$$E(\theta_j \mid y) = \frac{\int \theta_j f(y \mid \theta)g(\theta)d\theta}{\int f(y \mid \theta)g(\theta)d\theta}.$$

Interval estimates for the component $\theta_j$ are obtained by extracting specific percentiles, for example the 5th and 95th, from the marginal posterior distribution of $\theta_j$. Inference about a particular function of parameters $h(\theta)$ is obtainable by performing a transformation on the posterior density of $\theta$. Once this transformation is taken, one then obtains summaries of the marginal posterior density.

One impediment in teaching the Bayesian paradigm is the computational burden of posterior and predictive calculations. One aim of this paper is to provide a broad overview of computational strategies for teaching Bayesian thinking at the undergraduate and graduate levels. For each computational strategy, we present an illustration and describe situations where this particular strategy is helpful in teaching. A second aim of this paper is to
provide recommendations on Bayesian computation methods based on our experiences teaching Bayesian methods at the undergraduate and graduate levels.

1.2 A Selective History

In the 1960’s there was an active interest in the practice of Bayesian learning methods. Raiffa and Schlaifer (1961) was one of the early texts to describe the use of conjugate priors for exponential family distributions such as the normal, binomial, exponential, and Poisson. Other books provides descriptions of conjugate priors include Winkler (1972), Lee (1997) and Martz and Waller (1982).

In the 1960’s, due to the conceptual simplicity of Bayesian thinking, there were efforts to introduce Bayesian inference at a non-calculus level. One attractive way of introducing Bayes was to use discrete priors and use Bayes’ rule to update prior opinion. Blackwell (1969), Schmitt (1969), Phillips (1973) and Berry (1996) are examples of introductory statistics texts that present inference for standard sampling distributions from a Bayesian viewpoint using discrete priors.

There were several important developments in Bayesian computation in the 1980’s. Smith et al. (1985) describe a general method for approximating Bayesian integrals using adaptive quadrature methods. Tierney and Kadane (1986) describe accurate methods for approximating summaries of posterior distributions using Laplace expansions.

Statisticians become aware of Markov chain Monte Carlo (MCMC) methods through the seminal paper Gelfand and Smith (1990) that introduced Gibbs sampling for simulating from posterior distributions. At the same time, Gelfand et al. (1990) illustrate the application of Gibbs sampling for Bayesian fitting for a range of normal sampling models. Nice expositions of Gibbs sampling and the general Metropolis-Hastings algorithm are found in Casella and George (1992) and Chib and Greenberg (1995).

1.3 Plan of the Paper

The general goal of this paper is to provide a broad perspective of the computational methods currently available in a Bayesian analysis and present guidance for the choice of method for introducing Bayesian thinking at both undergraduate and graduate levels. Section 2
summarizes various non-simulation approaches to Bayesian computing, such as the use of discrete prior distribution, normal approximations, and conjugate problems. Section 3 provides an introduction to simulation-based Bayesian computation and Section 4 reviews popular MCMC samplers including the Gibbs sampler, the Metropolis-Hasting algorithm, and the Hamiltonian Monte Carlo sampler. We believe that the Gibbs sampler and the Metropolis algorithm are attractive for introducing MCMC methods and we describe the advantages and disadvantages of each method from a pedagogical perspective. This section also reviews methods for coding MCMC samplers such as writing R functions, writing a model script to be used by another program, or using a wrapper function that implements MCMC fitting for specific Bayesian models. Section 5 concludes the paper with some summary remarks on effective Bayesian computational methods for students with different backgrounds.

2 Non-Simulation Approaches

2.1 Discrete Bayes

Define a model to be a particular characteristic of a population. A model can be a parameter such as a population mean or population proportion, or other population measures. Suppose one can construct a list of model values \( \{M_j\} \) with associated prior probabilities \( \{P(M_j)\} \). Let \( D \) represent an observation that can shed some light on the models. Then by Bayes’ rule, the posterior probability of model \( M_j \) is given by

\[
P(M_j \mid y) \propto P(M_j)L(M_j),
\]

where \( L(M_j) \), the likelihood, is the probability of the observed data \( D \) given the model value \( M_j \).

Testing for a disease

The discrete Bayes approach can be illustrated in the familiar testing for a disease example. A person is concerned that she has a rare disease found in one half of one percent (0.005) of the general population. She takes a blood test that will help determine if she has the
disease. Unfortunately, the blood test is not completely reliable. The chance of getting an incorrect positive result is 0.01 if she is free of the disease, and likewise the chance of an incorrect negative result is 0.01 if she does have the disease. Suppose the person takes the blood test and the result is positive – what is the probability she has the disease?

One way to present a discrete Bayes computation is by a Bayes’ table displayed in Table 1. One lists the possible disease alternatives in a “Model” column together with the initial probabilities in the “Prior” column. The data result is “positive test result” and the Likelihood column gives the conditional probability of this result, \( P(\text{result} | \text{model}) \), for each of the models. One can compute posterior probabilities of the two models in two steps. First, one computes the product of the prior probability and the likelihood for each model, and then one normalizes these products (by dividing each product by the sum of the products) to obtain the posterior probabilities.

Table 1: Bayes table representation of the testing of the disease example.

| Model                        | Prior | Likelihood | Product | Posterior |
|------------------------------|-------|------------|---------|-----------|
| Have Disease                 | 0.005 | 0.900      | 0.005   | 0.043     |
| Does Not Have Disease        | 0.995 | 0.100      | 0.100   | 0.957     |

Discrete approximation

One attractive way of learning about a continuous-valued parameter \( \theta \) from a Bayesian viewpoint is through a discrete Bayes approximation. For a given sampling model \( f(y | \theta) \) and prior density \( g(\theta) \), it may be difficult to compute summaries of the posterior density analytically. However, one has the approximation

\[
g(\theta | y) \propto g(\theta)f(y | \theta) \approx \sum_{j=1}^{N} g(\theta_j)f(y | \theta_j),
\]

where \( \theta_1, ..., \theta_N \) is a fine grid of \( N \) values of \( \theta \) that covers the range of values where the posterior density has most of its probability content.

From a computational perspective, this discrete approach is very appealing. In R, for example, one can implement Bayesian model by use of three vectors, one containing the
parameter values, one containing the prior values, and a third computing the likelihoods (facilitated by the availability of functions like `dnorm()` and `dbeta()` for common distributions).

Despite its computational simplicity, there are challenges in using discrete Bayes in practice. If a grid of values is used to approximate a continuous-valued posterior distribution, then one needs to make reasonable choices of the grid so one has an accurate approximation. Conceptually the discrete Bayes approach can be used for posteriors of multiple parameters. However, the number of posterior evaluations grows exponentially as a function of the number of parameters, therefore the use of the discrete Bayes approach may be limited to a small number of parameters.

### 2.2 Conjugate Analyses

For members of the one-parameter exponential family, there exists a conjugate Bayesian analysis where the prior and posterior densities have the same functional form. For example, if a sample $y_1, \ldots, y_n$ is taken from a Poisson distribution with mean $\lambda$, then a gamma prior is conjugate. If $\lambda$ is assigned a gamma prior with shape $\alpha$ and rate $\beta$ proportional to

$$g(\lambda) \propto \lambda^{\alpha-1} \exp(-\beta \lambda),$$

then the posterior density will also of the gamma functional form with updated parameters $\alpha_1 = \alpha + \sum_{j=1}^{n} y_j$ and $\beta_1 = \beta + n$.

There are several computational advantages to using a conjugate prior in a Bayesian analysis.

1. **Ease of specifying prior densities.** One challenge in implementing Bayes is the specification of a prior density to represent one’s opinion about the location of the parameter before sampling. If a conjugate prior is used, then the user only needs to specify a small number of hyperparameters that give the location and spread of the prior distribution.

2. **Ease of computing posterior means and variances.** With the use of a conjugate prior, the posterior means and posterior standard deviations are typically of closed form.
3. **Easy inference.** Exact summaries of the posterior density are available since the posterior has a familiar functional form. In our gamma-Poisson example, posterior probabilities can be found using the R function `pgamma()` and probability intervals can be found using gamma quantiles found using the R function `qgamma()`.

4. **Closed-form predictive densities.** Also due to the conjugate structure, exact analytical expressions exist for the predictive density. This will facilitate the construction of prediction intervals for future data.

### 2.3 Normal Approximation

An alternative computational method is based on approximating the posterior by a normal curve. (See Tierney and Kadane (1986) for a discussion of related approximations.) An algorithm such as Newton’s method can be used to find the posterior mode $\tilde{\theta}$, the value where the posterior density achieves its maximum value. Then one obtains the normal approximation

$$g(\theta \mid y) \approx \text{Normal}(\tilde{\theta}, V),$$

where the variance-covariance $V$ is estimated by the behavior of the posterior curve about the modal value.

From a computational perspective, this is an attractive approximation due to the nice properties of the multivariate normal distribution. For example, the marginal distributions of the posterior are approximated by normal curves, and posterior intervals are easily computed using normal quantiles.

### 2.4 Example: A Two-Group Logistic Model

To illustrate several computational approaches for summarizing a posterior distribution, consider the comparison of two proportions by a logistic model. Suppose a survey is given to a sample of male and female college students regarding their use of Facebook. Of $n_1$ men sampled, $y_1$ are frequent users of Facebook, and $y_2$ out of $n_2$ women sampled are frequent Facebook users. Let $p_1$ and $p_2$ denote respectively the proportions of college men and college women who are frequent users of Facebook. One can relate the proportions to
the gender variable by means of the following logistic model.

\[
\log \left( \frac{p_i}{1 - p_i} \right) = \beta_0 + \beta_1 I(Gender = "Female"), \ i = 1, 2,
\]

where \( I(\cdot) \) is the indicator function. In this model, the slope parameter \( \beta_1 \) represents the log odds ratio that measures the difference between the two proportions. Suppose that one believes a priori that the proportions of college men and college women who use Facebook frequently are similar in size. One represents this belief by assigning \( \beta_1 \) a Cauchy density with location 0 and scale 0.5. The intercept parameter \( \beta_0 \) is assigned a noninformative uniform density. The posterior density of \((\beta_0, \beta_1)\) is proportional to

\[
g(\beta_0, \beta_1 | data) \propto \frac{\exp(\beta_0 y_1)}{[1 + \exp(\beta_0)]^{n_1}} \frac{\exp((\beta_0 + \beta_1)y_2)}{[1 + \exp(\beta_0 + \beta_1)]^{n_2}} \frac{1}{0.25 + \beta_1^2}.
\]

Note that this posterior density is not a familiar density form, so some type of numerical method is required to summarize the posterior.

Suppose in the sample, one observes \( y_1 = 8 \) Facebook users in a sample of \( n_1 = 30 \) men and \( y_2 = 15 \) Facebook users in a sample of \( n_2 = 30 \) women. One can summarize this bivariate posterior density by use of a discrete approximation. By trial and error, one finds that the rectangle of values \((-2.5 < \beta_0 < 1, -1 < \beta_1 < 3)\) appears to cover most of the posterior probability. One approximates the bivariate density by a 30 by 30 grid of values displayed in Figure 1. By summing the discrete approximation over the “nuisance” parameter \( \beta_0 \), one obtains the marginal posterior density of \( \beta_1 \) which is displayed as the “exact” density in Figure 2.

An alternative approach is based on the normal approximation. By the use of the \texttt{laplace()} function in the \texttt{LearnBayes} package \cite{Albert2018} one finds that approximately

\[
\left[ \begin{array}{c} \beta_0 \\ \beta_1 \end{array} \right] \approx \text{Normal} \left( \begin{array}{c} -0.696 \\ 0.431 \end{array} , \begin{array}{cc} 0.137 & -0.126 \\ -0.126 & 0.239 \end{array} \right).
\]

One advantage of the multivariate normal approximation is that it gives normal approximations to the marginal posterior densities. In particular, it gives that the log odds ratio \( \beta_1 \) is approximately normal with mean 0.431 and standard deviation \( \sqrt{0.239} = 0.489 \). This approximation is displayed as the “Approx” curve in Figure 2. One observes that
Figure 1: Computation of the two-group logistic model posterior over a grid of values of $\beta_0$ and $\beta_1$.

This normal approximation does not match the right skewness of the exact posterior density. However, it provides a reasonably accurate, easy to compute, approximation to the posterior density of interest.

3 Simulation Approaches

3.1 Overview

Simulation provides a general method for summarizing posterior distributions. Suppose one can simulate a sample of $m$ values $\theta^{(1)}, \ldots, \theta^{(m)}$ from the posterior density $g(\theta \mid y)$. Then one can compute various summaries of the posterior distribution by summarizing the corresponding simulated draws. For example, the posterior mean of $\theta$, $E(\theta \mid y)$, is approximated by the sample mean of simulated values

$$E(\theta \mid y) \approx \frac{\sum_{i=1}^{m} \theta^{(i)}}{m}.$$ 

If one wishes to construct a 90% credible interval for $\theta$, this interval is approximated by $(q_1, q_2)$, where $q_1$ and $q_2$ are respectively the 5th and 95th percentiles of the sample of simulated draws $\{\theta^{(i)}\}$.

Simulation can be applied for each of the computational approaches described in Section
2. If one approximates a continuous-valued posterior $g(\theta \mid y)$ by a discrete distribution, one can simulate from the posterior by taking a random sample with replacement from the values $\{\theta_j\}$ with probabilities proportion to $\{g(\theta_j)\}$. Conjugate models allow for convenient simulation of posterior densities. In the gamma-Poisson model of section 2.2, the R function `rgamma()` can be used to simulate a large number of parameter draws from the gamma posterior distribution. In this setting, it is not necessary to use simulation since exact gamma posterior summaries are available. However, this setting provides a good way to introduce the use of simulation in a Bayesian analysis and it is helpful to contrast simulation-based posterior summaries with the exact values. Last, if a normal distribution is used to approximate the posterior, then simulated draws from the normal can be used to perform inference. These simulated draws from the normal approximation are especially helpful in learning about a multi-parameter posterior distribution.

There are several attractive aspects of using simulation in a Bayesian analysis.

1. **Learning about functions of parameters.** Suppose one has a simulated sample from a multivariate posterior distribution $g(\theta \mid y)$. Then one is able to learn about the marginal posterior of any function $h(\theta)$ of interest by simply applying this function on the vector of simulated draws.

2. **Simulating the predictive density.** There is a straightforward algorithm for sim-
ulating from the predictive distribution of future observations. Let \( \tilde{y} \) represent future data, then one can simulate from the posterior predictive distribution \( f(\tilde{y} | y) \) in two steps. First, one simulates the parameter \( \theta \) from its posterior distribution, and then one simulates \( \tilde{y} \) from the sampling density \( f(\tilde{y} | \theta) \).

3. **Posterior predictive model checking.** A general way of checking the suitability of a Bayesian model is to explore if the observed data is consistent with replicated data simulated from the posterior predictive distribution. This model checking approach is practical given the ease of simulating replicated datasets.

### 3.2 Two-Group Logistic Model

Returning to our example, one may be interested in learning about the location of the probability of women favoring Facebook \( p_2 \). Using an MCMC method described in Section 4, one obtains a simulated sample of the posterior of the regression vector \( \beta = (\beta_0, \beta_1) \). Since \( p_2 = \exp(\beta_0 + \beta_1)/(1 + \exp(\beta_0 + \beta_1)) \), one can obtain a simulated sample from the marginal posterior density of \( p_2 \) by simply applying this function on the simulated sample from the posterior of \( \beta \). The top graph of Figure 3 displays a density estimate of the simulated sample of \( p_2 \).

Suppose one wishes to predict the number of Facebook users among a future sample of 20 college women. One can simulate from the posterior prediction of the number of female Facebook users \( \tilde{Y}_2 \) in two steps. First one simulates a draw from the posterior distribution of \( \beta \) and computes the probability \( p_2 \) – denote the simulated value by \( p_2^* \). Then one simulates a value of \( \tilde{Y}_2 \) by simulating a binomial random variable with parameters 20 and \( p_2^* \). The bottom graph of Figure 2 displays a bar graph of a simulated sample from the predictive distribution of \( \tilde{Y}_2 \).

One can directly simulate from the posterior distribution using the three computational methods described in Section 2. However, direct simulation is difficult to achieve in larger Bayesian models such as multilevel models with a large number of parameters. This motivates the use of MCMC samplers describe in Section 4.
4 MCMC

4.1 Introduction

In non-conjugate, multi-parameter Bayesian models, exact solutions to the posterior distribution are usually analytically unavailable. Therefore we need to rely on simulation-based computations for posterior estimation, and a popular class of computation techniques is called Markov chain Monte Carlo (MCMC).

In MCMC, one constructs a specific Markov chain to step through a high-dimensional posterior probability distribution. Informally, one is constructing a type of random walk that searches for locations where the posterior distribution has high probability content. Under general conditions, the Markov chain will approach, as the number of steps get large, an equilibrium distribution that is equivalent to the posterior distribution of interest.
4.2 Popular MCMC Samplers

Popular MCMC samplers for Bayesian inference include the Gibbs sampler, the Metropolis-Hastings algorithm, and the Hamiltonian Monte Carlo algorithm (HMC). A Gibbs sampler iteratively samples one parameter at a time given its full conditional posterior distribution, defined as the parameter’s distribution conditional on values of the remaining parameters. For problems where the full conditional posterior distributions are familiar distributions (Gelfand and Smith, 1990), the Gibbs sampler provides an attractive way of constructing a Markov chain. The family of Metropolis-Hastings algorithms provides a general way of implementing a Markov chain in situations where the full conditional posterior distributions are not recognizable. In a Metropolis-Hastings algorithm, a proposal distribution is used to select candidate simulated draws, and one decides to move to the candidate draw or remain at the current simulated value depending on an acceptance probability. The Metropolis algorithm is a special case of the Metropolis-Hastings algorithm that uses a symmetric proposal distribution. In the Metropolis algorithm, the user tunes the variance of the proposal distribution to achieve an MCMC chain that effectively explores the space of parameter values (Metropolis et al., 1953).

While conceptually intuitive and generally simple to implement, the Metropolis algorithm scales poorly with increasing dimension and complexity of the target posterior distribution. To exploit information about the geometry of the typical set of parameter draws, the HMC is designed for generating efficient draws of the posterior distribution for sufficiently well-behaved target posterior distributions (Neal, 2011; Betancourt, 2017). A popular MCMC software Stan (Carpenter et al., 2017) implements HMC to facilitate full Bayesian statistical inference with MCMC sampling.

4.3 MCMC diagnostics

It is important to note that an MCMC sampler will only converge to the target posterior distribution in theory and the collected MCMC draws are an approximation to the unknown joint posterior distribution. There are several natural questions to ask once an MCMC sampler is run. How quickly does the sampler need to run until one reaches the space where the posterior has most of its probability? How correlated are the successive
sampled values from a particular run of the sampler? How many iterations of the sampler need to be collected to obtain accurate estimates at posterior summaries of interest? The collection of diagnostic methods used to address these questions are called MCMC diagnostics \cite{Mengersen1999}. Some popular MCMC diagnostics methods include using traceplots, autocorrelation plots, computations of effective sample sizes and tests such as Gelman-Rubin and Geweke diagnostics procedures \cite{Gelman2013}.

4.4 What MCMC sampler to use in teaching?

We believe that students should be introduced to MCMC algorithms at an appropriate depth. Instead of using “black box” MCMC software for all non-conjugate, multi-parameter Bayesian models that they encounter, we advocate first introducing MCMC algorithms for relatively simple Bayesian models using self-written MCMC samplers. After the basic tenets of MCMC algorithms are learned, students can use MCMC software for models requiring more advanced MCMC techniques.

We now proceed to describe the pros and cons of the Gibbs sampler and the Metropolis algorithm from a pedagogical perspective. Although HMC is an appealing and useful MCMC algorithm for Bayesian computation, we focus our discussion on the Gibbs sampler and the Metropolis algorithm since the conceptual basis for these algorithms can be communicated to students with relatively modest mathematical and programming backgrounds.

4.4.1 The Gibbs sampler

The Gibbs sampler provides an automatic MCMC algorithm. Once students successfully derive and recognize the full conditional posterior distributions of each parameter in a posterior distribution, a Gibbs sampler is constructed by sampling parameter values by several lines of code in a programming language. Unlike the Metropolis algorithm, the Gibbs sampler will always accept the newly sampled parameter draw.

Despite being straightforward to understand and implement, students do need to correctly derive and recognize each parameter’s full conditional posterior distribution and doing so may be challenging in complicated models. In the process of developing a Gibbs
sampler, students will need to write out the joint likelihood function and prior distribution
of all parameters and derive the available full conditional posterior distributions. These
steps will deepen their understanding of the Bayesian process of deriving the posterior from
the likelihood and the prior.

The Gibbs sampler works for many common inference problems. For example, for
the normal sampling model where both parameters are unknown, the use of a conjugate
normal prior for the mean and a conjugate gamma prior for the precision produces a Gibbs
sampler with normal and gamma full conditional posterior distributions, respectively. The
Gibbs sampler is straightforward to construct for normal sampling models with hierarchical
normal priors and for a variety missing data problems and censored data problems \cite{Gelfand1990}. Gibbs sampler examples provide a rich set of Bayesian models and associated
inference problems.

On the other hand, one recognizes that the details of a Gibbs sampler can be understood
only for calculus-based undergraduate statistics courses and graduate statistics courses
where students are familiar with the posterior deviations for models with conjugate priors.
Since the Gibbs sampler algorithm rests on conditional distributions, it would be difficult
to communicate to students who do not have some experience with conditional probability
distributions. In addition, the Gibbs sampler may be slow to converge for some problems,
especially when the parameters are correlated. Also the Gibbs sampler does not work for
all models, such as a normal sampling model with non-conjugate priors chosen for the mean
and precision.

4.4.2 The Metropolis algorithm

The core of the Metropolis algorithm is the choice of the proposal distribution. At iteration
$s$, the Metropolis algorithm simulates a plausible new value of the parameter, $\theta^*$, from the
proposal density $J(\theta \mid \theta^{(s-1)})$, where $\theta^{(s-1)}$ is the accepted parameter draw at the previous
iteration. For the Metropolis algorithm, the proposal distribution is symmetric about the
current parameter draw, such as a uniform or a normal, and the acceptance probability is
found by evaluating the ratio of the posterior density at the proposed and current parameter
values. If the proposed value is more likely than the current value, then the Metropolis
algorithm moves to the proposed value. If not, then the algorithm moves the proposed value with a probability \( p \) smaller than one, and remains at the current value with probability \( 1 - p \).

One advantage of the Metropolis algorithm is that it is straightforward to program using popular programming languages. It is helpful for students to write their own scripts to confirm their own understanding of the sampling procedure. Conceptually the Metropolis algorithm is less challenging than Gibbs sampling as it consists of has three intuitive steps (1) propose, (2) compute the acceptance probability, and (3) move or stay. In this programming task, the student takes a random draw \( u \) from a uniform or a normal distribution, compares the draw to the calculated ratio \( r \) of densities, then determines whether or not to accept the proposal value. In this task, the student also understands that it is helpful to work with the logarithm scale for numerical stability and uses a counter of acceptances for monitoring the acceptance rate.

The use of the Metropolis algorithm naturally leads to discussions about MCMC diagnostics. One can use specific diagnostic procedures such as traceplots and autocorrelation plots to illustrate good and poor samplers by different choices for the width of the proposal region. The Metropolis algorithm works on a variety of Bayesian inference models, as it does not require a recognizable set of full conditional posterior distributions. Students’ exposure to the Metropolis algorithm not only enhances their statistical programming skills, but also deepens their overall understanding of MCMC.

On the negative side, the Metropolis algorithm requires tuning of the parameters in the proposal distribution, which typically requires one to set the width of the proposal region, a task that has to be done by trial and error. In addition, the Metropolis algorithm can be slow in exploring the posterior parameter space. If students are not comfortable with programming, then coding the Metropolis algorithm may take up too much that could be used instead to discuss statistical issues. If there is a programming issue, the instructor could supply code and the focus would be on using this code for particular Bayesian modeling problems.
4.5 Coding MCMC Samplers

To code MCMC samplers, students generally have three options: (1) write their own MCMC samplers by writing functions in R or another statistical programming languages, (2) write scripts to implement MCMC samplers through MCMC software such as Just Another Gibbs Sampler (JAGS) (Plummer et al., 2003), Bayesian inference Using Gibbs Sampling (BUGS) (Gilks et al., 1994) and Stan (Stan Development Team, 2018), or (3) use a wrapper function like `stan` in the `rstanarm` R package (Goodrich, 2019), similar to the `glm` function used in frequentist model estimations.

Our recommendations for coding MCMC are mostly based on our own teaching experience and the computing choices made by several Bayesian textbooks, including Bayesian Statistics and Marketing (Rossi et al., 2005), Bayesian Computation with R (Albert, 2009), Bayesian Data Analysis (Gelman et al., 2013), Bayesian Essentials with R (Marin and Robert, 2014), Doing Bayesian Analysis (Kruschke, 2015), Statistical Rethinking: A Bayesian Course with Examples in R and Stan (McElreath, 2016), Bayesian Statistical Methods (Reich and Ghosh, 2019), and Probability and Bayesian Modeling (Albert and Hu, 2019).

4.5.1 Writing one’s own MCMC samplers

In general, for students with some programming background, we believe they should gain experience with the Gibbs sampler and the Metropolis algorithm by writing their own scripts: Our experience suggests that for introductory level statistics courses for undergraduate students, the Metropolis algorithm is easy to understand and program and it naturally leads to discussions about MCMC diagnostics. For intermediate and advanced level undergraduate Bayesian courses with calculus and probability background and masters-level Bayesian courses, the Gibbs sampler is a good introduction to MCMC, and derivations of full conditional posterior distribution reinforce learning of the Bayesian paradigm.

4.5.2 Writing JAGS, BUGS and/or Stan scripts

The use of MCMC software programs JAGS, BUGS and Stan require the writing of a Bayesian model script including the specification of the sampling model and the prior. For example, the following JAGS script can be used to specify the two-group logistic model
discussed in Section 2.4. In this code the regression intercept and slope parameters are represented by the variables \texttt{beta0} and \texttt{beta1}, the binomial sampling is represented by the \texttt{dbin()} function, and the prior densities are specified by the \texttt{dt()} and \texttt{dnorm()} functions. (Note that the t prior \texttt{dt(0, 4, 1)} with location 4, scale 1, and degrees of freedom 1 is equivalent to the Cauchy prior specified in the example of Section 2.4.) The JAGS program requires the specification of proper priors and the \texttt{dnorm()} function is used with a small precision value 0.0001 to approximate the uniform prior for $\beta_0$ described in Section 2.4.

```r
modelString <- "
model {
  for (i in 1:2) {
    y[i] ~ dbin(p[i], n[i])
    logit(p[i]) <- beta0 + beta1*x[i]
  }
  beta1 ~ dt(0, 4, 1)
  beta0 ~ dnorm(0, 0.0001)
}
"
```

The use of a scripting language such as JAGS may be appropriate for undergraduate-level Bayesian courses without calculus and probability background, and masters-level and PhD-level Bayesian courses for students in non-statistics programs. For example, [Link and Barker (2009)] describes Bayesian modeling for ecology applications using JAGS as the primary computational tool.

The use of these scripting languages allows students to to implement posterior inference and MCMC samplers for more sophisticated Bayesian models such as multilevel models. Texts such as *Bayesian Statistical Methods* ([Reich and Ghosh, 2019](#)) and *Probability and Bayesian Modeling* ([Albert and Hu, 2019](#)) expose the students to self-written MCMC scripts for simpler Bayesian models and use JAGS to introduce more advanced Bayesian models. In non-statistics fields, *doing Bayesian Analysis* ([Kruschke, 2015](#)) illustrate the use of JAGS and Stan scripts with a primary focus on students in psychology, cognitive science, social sciences, clinical sciences, and consumer sciences in business.
4.5.3 Using wrapper functions in available packages

R functions such as \texttt{lm()} and \texttt{glm()} have facilitated the fitting of frequentist regression models. Similar functions such as \texttt{stan.glm()} in the \texttt{rstanarm} package and \texttt{brm()} in the \texttt{brms} package (Burkner 2019) provide MCMC sampling for Bayesian regression models. These approaches implement “black-box” MCMC sampling methods that are potentially attractive to introductory-level undergraduate courses and applied Bayesian courses in non-statistics fields. Also some statistics educators create their own Stan-based packages that help smooth and integrate students’ learning experience with computing tools. For example, \textit{Statistical Rethinking: A Bayesian Course with Examples in R and Stan} (McElreath 2016) is accompanied with the \texttt{rethinking} package (available on the author’s GitHub page), which includes various wrapper functions that facilitate students’ learning of advanced Bayesian regression models.

Other statistics educators have created their own R packages with the same goal of enhancing students’ learning experience with provided computing tools. \textit{Bayesian Statistics and Marketing} (Rossi et al. 2005) is accompanied with the \texttt{bayesm} package (Rossi 2019) and \textit{Bayesian Essentials with R} (Marin and Robert 2014) is accompanied with the \texttt{bayess} package (Robert and Marin 2013), both of which include various wrapper functions for estimating advanced Bayesian inference models.

5 Concluding Remarks

This paper has surveyed the wide range of computational methods available for instructors who wish to teach Bayesian methods at the undergraduate and graduate levels. For students with modest mathematical backgrounds, the use of discrete prior distributions is a helpful non-simulation approach for introducing the Bayesian paradigm in problems with a few parameters. Conjugate analyses are also helpful in communicating Bayesian thinking when applicable. Conjugate priors facilitate students’ learning and practice of prior assessment and how the prior and data are combined in a posterior analysis.

Estimation of multi-parameter Bayesian models is challenging, which motivates the use of MCMC simulation algorithms. For teaching the fundamental principles of MCMC, we
advocate introducing the Gibbs sampler to students with a calculus background and an understanding of conjugate priors. The derivation of a Gibbs sampler is a useful pedagogical exercise for these students. For introductory level courses for undergraduate students, we advocate introducing the Metropolis algorithm, as this MCMC sampler is easy to understand and naturally leads to discussions about MCMC diagnostics.

For estimating advanced Bayesian models, we believe the use of MCMC software programs such as JAGS, BUGS and Stan reinforce students’ learning, as writing a model script requires a clear understanding of the sampling model and the prior. The use of wrapper functions such as \texttt{stan.glm()} implement “black-box” MCMC sampling methods which are potentially attractive to introductory-level undergraduate courses and applied Bayesian courses in non-statistics fields.

References

Albert, J. (2009) \textit{Bayesian Computation with R}. Springer, 2 edn.

— (2018) Package learnbayes.

Albert, J. and Hu, J. (2019) \textit{Probability and Bayesian Modeling}. Texts in Statistical Science, CRC Press.

Berry, D. A. (1996) \textit{Statistics: A Bayesian Perspective}. Duxbury Press.

Betancout, M. (2017) A conceptual introduction to Hamiltonian Monte Carlo. \textit{Preprint arXiv: 1701.02434}.

Blackwell, D. (1969) \textit{Basic Statistics}. New York: McGraw Hill.

Burkner, P. (2019) \textit{brms: Bayesian Regression Models using Stan}. R package version 2.10.0.

Carpenter, B., Gelman, A., Hoffman, M. D., Lee, D., Goodrich, B., Betancourt, M., Brubaker, M., Guo, J., Li, P. and Riddell, A. (2017) Stan: A probabilistic programming language. \textit{Journal of Statistical Software}, \textbf{76}.
Casella, G. and George, E. I. (1992) Explaining the Gibbs sampler. *The American Statistician, 46*, 167–174.

Chib, S. and Greenberg, E. (1995) Understanding the Metropolis-Hastings algorithm. *The American Statistician, 49*, 327–335.

Gelfand, A. E., Hills, S. E., Racine-Poon, A. and Smith, A. F. (1990) Illustration of Bayesian inference in normal data models using Gibbs sampling. *Journal of the American Statistical Association, 85*, 972–985.

Gelfand, A. E. and Smith, A. F. M. (1990) Sampling-based approaches to calculating marginal densities. *Journal of the American Statistical Association, 85*, 398–409.

Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A. and Rubin, D. B. (2013) *Bayesian Data Analysis*. Texts in Statistical Science, CRC Press.

Gilks, W. R., Thomas, A. and Spiegelhalter, D. J. (1994) A language and program for complex Bayesian modelling. *Journal of the Royal Statistical Society: Series D (The Statistician), 43*, 169–177.

Goodrich, B. (2019) *rstanarm: Bayesian Applied Regression Modeling via Stan*. R package version 2.19.2.

Kruschke, J. K. (2015) *Doing Bayesian Analysis: A Tutorial with R, JAGS, and Stan*. Academic Press.

Lee, P. M. (1997) *Bayesian statistics*. Arnold Publication.

Link, W. A. and Barker, R. J. (2009) *Bayesian inference: with ecological applications*. Academic Press.

Marin, J.-M. and Robert, C. P. (2014) *Bayesian Essentials with R*. Springer Texts in Statistics, Springer.

Martz, H. F. and Waller, R. (1982) Bayesian reliability analysis. *John Wiley & Sons, INC. McElreath, R. (2016) Statistical Rethinking: a Bayesian course with examples in R and Stan*. Texts in Statistical Science, CRC Press.

21
Mengersen, K. L., Robert, C. P. and Guihenneuc-Jouyaux, C. (1999) MCMC convergence diagnostics: a review. *Bayesian Statistics, 6*, 415–440.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H. and Teller, E. (1953) Equation of state calculations by fast computing machines. *Journal of Chemical Physics, 21*, 1087–1092.

Neal, R. M. (2011) MCMC using Hamiltonian dynamics. In *Handbook of Markov Chain Monte Carlo* (eds. S. Brooks, A. Gelman, G. L. Jones and X. L. Meng), 113–162. Chapman and Hall/CRC.

Phillips, L. D. (1973) *Bayesian Statistics for Social Scientists*. Thomas Nelson.

Plummer, M. et al. (2003) JAGS: A program for analysis of Bayesian graphical models using Gibbs sampling. In *Proceedings of the 3rd international workshop on distributed statistical computing*, vol. 124, 10. Vienna, Austria.

Raiffa, H. and Schlaifer, R. (1961) *Applied Statistical Decision Theory*. Division of Research, Graduate School of Business Administration, Harvard.

Reich, B. J. and Ghosh, S. K. (2019) *Bayesian Statistical Methods*. Texts in Statistical Science, CRC Press.

Robert, C. P. and Marin, J.-M. (2013) *bayess: Bayesian Essentials with R*. R package version 1.4.

Rossi, P. (2019) *bayesm: Bayesian Inference for Marketing/Micro-Econometrics*. R package version 3.1.

Rossi, P. E., Allenby, G. M. and McCulloch, R. (2005) *Bayesian Statistics and Marketing*. New York: Wiley.

Schmitt, S. (1969) *Measuring Uncertainty: An Elementary Introduction to Bayesian Statistics*. Reading, MA: Addison-Wesley.
Smith, A., Skene, A., Shaw, J., Naylor, J. and Dransfield, M. (1985) The implementation of the Bayesian paradigm. *Communications in Statistics-Theory and Methods*, **14**, 1079–1102.

Stan Development Team (2018) *Stan Modeling Language Users Guide and Reference Manual*. Version 2.18.0.

Tierney, L. and Kadane, J. B. (1986) Accurate approximations for posterior moments and marginal densities. *Journal of the American Statistical Association*, **81**, 82–86.

Winkler, R. L. (1972) *An Introduction to Bayesian Inference and Decision*. No. 519.2 W5.