SHARP FORM FOR IMPROVED MOSER-TRUDINGER INEQUALITY

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ABSTRACT. We prove that the improved Moser-Trudinger inequality with optimal coefficient $\alpha = 1/2$ holds for all functions on $S^2$ with zero moments.

1. INTRODUCTION

The standard Moser-Trudinger-Onofri inequality states ([10], [12]) that on the standard unit sphere $(S^2, g_0)$ with the induced metric $g_0$ from $R^3$, for any $u \in C^1(S^2)$,

$$\frac{1}{4\pi} \int_{S^2} e^{2u} \leq \exp\left\{ \frac{1}{4\pi} \int_{S^2} (|\nabla u|^2 + 2u) \right\},$$

and the equality holds if and only if $e^{2u}g$ is a metric of constant curvature. In the study of deforming metrics and prescribing curvatures on $S^2$, this inequality is often used to control the size and behavior of a new metric $e^{2u}g_0$ near a concentration point. With certain "balance" condition on the metric one would guess that if the metric concentrates, it should concentrate at more than one point. Thus it is reasonable to ask whether there is some small constant $\alpha \in (0, 1)$ and a constant $C_\alpha$ such that

$$\frac{1}{4\pi} \int_{S^2} e^{2u} \leq C_\alpha \exp\left\{ \frac{1}{4\pi} \int_{S^2} (2u) \right\}$$

holds for those functions satisfying certain balance condition.

It in fact was first observed by Moser in [11] that the above inequality holds for $\alpha = 1/2$ if $u(x)$ is symmetric with respect with the origin (that is: $u(x_1, x_2, x_3) = u(-x_1, -x_2, -x_3)$.) In general, Aubin [1] proved that if $u \in \Lambda := \{ f(x) \in H^1(S^2) : \int_{S^2} e^{2f} x_i = 0 \text{ for } i = 1, 2, 3 \}$, where $\{x_1, x_2, x_3\}$ are the standard coordinates in $R^3$, then for any given constant $\alpha \in (1/2, 1)$, there is a constant $C_\alpha$ such that

$$\frac{1}{4\pi} \int_{S^2} e^{2u} \leq C_\alpha \exp\left\{ \frac{1}{4\pi} \int_{S^2} (\alpha |\nabla u|^2 + 2u) \right\}.$$  

(1.1)

Later, in their study of prescribing curvature problem on $S^2$, Chang and Yang [3] were able to show that for $\alpha$ close to 1, the optimal constant for $C_\alpha$ in the above inequality is 1. On the other hand, using the standard bubbling sequence, one can see that Aubin’s inequality can not hold if $\alpha < 1/2$. Thus the immediate question is:

(I): Is there a constant $C_*$ so that

$$\frac{1}{4\pi} \int_{S^2} e^{2u} \leq C_* \exp\left\{ \frac{1}{4\pi} \int_{S^2} (\alpha |\nabla u|^2 + 2u) \right\}$$

holds for those functions satisfying certain balance condition.
holds for all \( u \in \Lambda \)?

If the answer to this question is affirmative, one may continue to ask

(II): what is the optimal constant \( C_\ast \)? Is it 1?

In this short note, we will give an affirmative answer to the first question. To answer the second question we need to solve a partial differential equation. So far we have no clue how to solve it. See more details in Remark at the end of this note.

Let
\[
O = \{ u \in H^1(S^2) : \int_{S^2} e^{2u} x_i = 0, \ i = 1, 2, 3 \text{ and } \int_{S^2} u = 0 \}.
\]

For any \( u \in O \), we define functional
\[
I(u) = \frac{1}{2} \int_{S^2} |\nabla u|^2 - \ln \int_{S^2} e^{2u}.
\]

We have the following main result.

**Theorem 1.** There exists constant \( C \in \mathbb{R} \), such that \( I(u) > C \) for all \( u \in O \).

Moreover \( \inf_{u \in O} I(u) \) is attained by some \( u \in O \).

It is easy to observe that \( \inf_{u \in O} I(u) \leq 0 \). But it is not clear yet whether \( \inf_{u \in O} I(u) = 0 \) or not. On the other hand, to our surprise, we are able to show that if a minimizing sequence blows up (more details will be given later), then \( \inf_{u \in O} I(u) > 0 \) (through a dedicated asymptotic analysis). We thus obtain the existence of the extremal for \( \inf_{u \in O} I(u) \). Similar blow-up analysis is quite standard now (see, for example, [6] and [5]).

2. **Proof of the Theorem**

Let \( u \in O \). For any \( \epsilon \in (0, 1/2) \), we define a perturbed functional
\[
I_\epsilon(u) = \frac{1}{2(1-\epsilon)} \int_{S^2} |\nabla u|^2 - \ln \int_{S^2} e^{2u}.
\]

It follows from Aubin’s inequality \((1.1)\) that
\[
E_\epsilon = \inf_{u \in O} I_\epsilon(u) > -\infty.
\]

Further, one can show that the infimum is attained by some \( u_\epsilon \in O \). Thus \( u_\epsilon \) satisfies the following Euler-Lagrange equation:
\[
-\frac{1}{8\pi(1-\epsilon)} \Delta u_\epsilon = \frac{e^{2u_\epsilon}}{\int_{S^2} e^{2u_\epsilon}} - \frac{1}{4\pi} + e^{2u_\epsilon} \sum_{i=1}^{3} a_i x_i,
\]

where \( a_i \)'s are Langrange multipliers.

We claim: \( a_i = 0 \) for \( i = 1, 2, 3 \). The proof of this claim is along the same line as in [3].

Let \( v(x) \) be a solution to
\[
\Delta v + he^v = c \quad \text{on} \quad S^2.
\]

Kazdan and Warner (\cite{7}) showed that \( v \) satisfies
\[
\int_{S^2} e^v \nabla h \cdot \nabla x_i = (2-c) \int_{S^2} e^v h x_i.
\]
Let
\[ c = 4(1 - \epsilon), \quad h = 16\pi(1 - \epsilon) \left( \frac{1}{\int_{S^2} e^{2u_\epsilon}} + \sum_{i=1}^{3} a_i x_i \right), \quad v = 2u_\epsilon, \]
we have
\[ \int_{S^2} e^{2u_\epsilon} \nabla (\sum_{j=1}^{3} a_j x_j) \cdot \nabla x_i = (2 - 4(1 - \epsilon)) \int_{S^2} e^{2u_\epsilon} \left( \frac{1}{\int_{S^2} e^{2u_\epsilon}} + \sum_{j=1}^{3} a_j x_j \right) x_i. \]

Since \( \int_{S^2} e^{2u_\epsilon} x_i = 0 \), we know that
\[ \int_{S^2} e^{2u_\epsilon} \nabla (\sum_{j=1}^{3} a_j x_j) \cdot \nabla x_i = (2 - 4(1 - \epsilon)) \int_{S^2} e^{2u_\epsilon} \sum_{j=1}^{3} a_j x_j x_i. \]

Multiplying both sides by \( a_i \) and summing from \( i = 1 \) to \( 3 \), we obtain that
\[ \int_{S^2} e^{2u_\epsilon} |\nabla (\sum_{j=1}^{3} a_j x_j)|^2 = (2 - 4(1 - \epsilon)) \int_{S^2} e^{2u_\epsilon} \sum_{j=1}^{3} a_j x_j^2. \]

It follows from \( 2 - 4(1 - \epsilon) < 0 \) that \( a_i = 0, \ i = 1, 2, 3. \)

Therefore \( u_\epsilon \) satisfies
\[ -\Delta u_\epsilon = 8\pi(1 - \epsilon) \left( \frac{e^{2u_\epsilon}}{\int_{S^2} e^{2u_\epsilon}} - \frac{1}{4\pi} \right). \]

Recall that \( u_\epsilon \) is a minimizer of \( \inf_{u \in O} I_\epsilon(u) \), thus, if \( \int_{S^2} e^{2u_\epsilon} \) stays bounded as \( \epsilon \to 0, \int_{S^2} |\nabla u_\epsilon|^2 \leq C \). Then there exist a subsequence \( u_{\epsilon_n} \) converging to \( u_0 \) in \( H^1(S^2) \). Furthermore, \( u_0 \) is a minimizer for \( I(u) \) and Theorem 1 follows.

From now on, we assume that up to a subsequence \( \int_{S^2} e^{2u_\epsilon} \to \infty \) as \( \epsilon \to 0 \), and will derive an contradiction. For simplicity, we shall not distinguish a subsequence \( \{ \epsilon_n \} \) from the original \( \{ \epsilon \} \).

Let \( v_\epsilon = 2u_\epsilon - \ln \int_{S^2} e^{2u_\epsilon} \). Then \( v_\epsilon \) satisfies \( \int_{S^2} e^{v_\epsilon} = 1, \ v_\epsilon := \int v_\epsilon \to -\infty \) as \( \epsilon \to 0 \), and
\[ -\Delta v_\epsilon = 16\pi(1 - \epsilon)(e^{v_\epsilon} - \frac{1}{4\pi}). \]

We first have the \( L^q \) estimate for \( v_\epsilon \) for any \( q \in [1, 2)\):
\[ \| \nabla v_\epsilon \|_{L^q(S^2)} < C_q. \]

In fact, for any \( \varphi \in W^{1, q/(q-1)}(S^2) \) with \( \int_{S^2} \varphi = 0 \) and \( \| \varphi \|_{W^{1, q/(q-1)}(S^2)} = 1 \) (thus \( \varphi \in L^\infty(S^2) \)),
\[ \left| \int_{S^2} \nabla v_\epsilon \nabla \varphi \right| = \left| \int_{S^2} \Delta v_\epsilon \varphi \right| = 16\pi(1 - \epsilon) \left| \int_{S^2} (e^{v_\epsilon} - \frac{1}{4\pi}) \varphi \right| \leq C. \]

Since \( \int_{S^2} e^{v_\epsilon} = 1, \ 16\pi(1 - \epsilon)e^{v_\epsilon} \) converges in measure to \( d\mu \), a positive measure on \( S^2 \), that is \( \int_{S^2} e^{v_\epsilon} \psi \to \int_{S^2} \psi d\mu \), for any \( \psi \in C^0(S^2) \). Let
\[ R = \{ x \in S^2 : \exists \psi \in C^0(S^2), 0 \leq \psi \leq 1, \psi \equiv 1 \text{ around } x, \text{ s.t. } \int \psi d\mu < 4\pi \} \]
be the set of “regular points”, and
\[ S = \{ x \in S^2 : \exists x_n \in S^2 \text{ and } \{ \epsilon_n \}, \text{ s.t. } \lim_{n \to \infty} x_n = x, \text{ and } \lim_{n \to \infty} v_{\epsilon_n}(x_n) \to \infty \}, \]
be the set of “blow-up points”. We need the following lemma of Brezis and Merle [2] to initiate our analysis.

**Lemma 1. (Brezis-Merle Lemma)** Suppose \( u \) satisfies \(-\Delta u = f\) on a bounded domain \( \Omega \subset \mathbb{R}^2 \) and \( u|_{\partial \Omega} = 0 \) then for any \( \delta \in (0, 4\pi) \) there exists a constant \( c(\delta) \) such that

\[
\int_\Omega \exp \left( \frac{(4\pi - \delta)|u|}{\|f\|_{L^1(\Omega)}} \right) \leq c(\delta).
\]

Although Brezis-Merle Lemma was originally proved for a bounded domain \( \Omega \subset \mathbb{R}^2 \), the same result also holds on any domain \( \Omega \subset \mathbb{S}^2 \).

Using the above lemma, we derive the following property for regular points.

**Lemma 2.** For any \( x_0 \in R \), there exists \( r > 0 \), such that \( v_\epsilon - v_\epsilon^\alpha \) is bounded in \( L^\infty(B_r(x_0)) \) uniformly in \( \epsilon \), where \( v_\epsilon^\alpha = \alpha v + B_r(x_0) \) is the ball centered at \( x_0 \) with radius \( r \).

**Proof.** By definition, for any \( x_0 \in R \), there exists \( r > 0 \), such that \( \|v_\epsilon\|^p_{L^1(B_r(x_0))} < 4\pi. \) Let \( v_1^\epsilon \) be the unique solution to the following Dirichlet problem:

\[
\begin{cases}
-\Delta v_1^\epsilon = 16\pi(1 - \epsilon)v_\epsilon & \text{in } B_r(x_0), \\
v_1^\epsilon = 0 & \text{on } \partial B_r(x_0),
\end{cases}
\]

and \( v_\epsilon^2 = v_\epsilon - v_1^\epsilon - v_\epsilon^\alpha \). Applying Brezis-Merle Lemma to \( v_1^\epsilon \), we obtain that

\[\int_{B_r(x_0)} \epsilon p|v_1^\epsilon| < \infty, \text{ for some } p \in (1, 2). \]

Since \( \Delta v_\epsilon^2 = 4(1 - \epsilon) \), using \( L^p \) interior estimate, we have

\[
\|v_\epsilon^2\|^p_{L^\infty(B_{2r}(x_0))} \leq C\|v_\epsilon^2\|^\alpha_{W^{2,p}(B_{2r}(x_0))} \leq C\|v_\epsilon^2\|^\alpha_{L^p(B_{4r}(x_0))}
\]

\[
\leq C(\|v_\epsilon - v_\epsilon^\alpha\|^\alpha_{L^p(B_{4r}(x_0))} + \|v_1^\epsilon\|^\alpha_{L^p(B_{4r}(x_0))})
\]

\[
\leq C(\|\nabla v_\epsilon\|^\alpha_{L^p(S^2)} + \|v_1^\epsilon\|^\alpha_{L^p(B_{4r}(x_0))})
\]

\[
\leq C
\]

where we also use \([2, 3]\) and Poincaré inequality. Therefore

\[
\int_{B_{2r}(x_0)} \epsilon p|v_\epsilon| = \int_{B_{2r}(x_0)} \epsilon p|v_1^\epsilon| \cdot \epsilon p|v_\epsilon^\alpha| \leq C.
\]

It follows easily from interior \( L^p \) estimate (for \( p > 1 \)) that \( \|v_1^\epsilon\|^p_{L^\infty(B_{r}(x_0))} \leq C \).

Hence \( \|v_\epsilon - v_\epsilon^\alpha\|^\alpha_{L^\infty(B_{r}(x_0))} \leq C \).

Using the above lemma, we immediately get that \( S \subset S^2 \setminus R \). Therefore

\[
\#S \leq \#(S^2 \setminus R) \leq \frac{\int_{S^2} d\mu}{4\pi} \leq 4,
\]

where \( \#S \) is the cardinality of \( S \). Choose \( r \) small, so that for \( x \in S \), \( B_r(x) \) are disjoint. For any \( x \in S \), by definition, for \( \epsilon \) sufficiently small, \( v_\epsilon \) has a local maximum \( x_\epsilon \in B_r(x) \) with \( v_\epsilon(x_\epsilon) \to \infty \), and up to a subsequence \( x_\epsilon \to x \) as \( \epsilon \to 0 \).

Choose a normal coordinate system around \( x \) and define

\[
\varphi_\epsilon(x) = v_\epsilon(\tau_\epsilon^{-1}x + x_\epsilon) - \lambda_\epsilon,
\]

where \( \lambda_\epsilon = v_\epsilon(x_\epsilon) \), \( \tau_\epsilon = e^{\lambda_\epsilon/2} \) and we use \( \tau_\epsilon^{-1}x + x_\epsilon \) to represent \( \exp_{x_\epsilon}(\tau_\epsilon^{-1}x) \). For fixed \( R > 0 \), when \( \epsilon \) is sufficiently small, \( \varphi_\epsilon \) satisfies

\[
-\Delta \varphi_\epsilon = 16\pi(1 - \epsilon)(e^{\varphi_\epsilon} - \frac{1}{4\pi \tau_\epsilon^2}) \quad \text{in } B_{2R}(0) \subset \mathbb{R}^2.
\]
Lemma 3. For a fixed $R > 0$, $\varphi_\varepsilon$ is bounded in $B_R(0)$ uniformly in $\varepsilon$.

Proof. Let $\varphi_\varepsilon^{(1)}$ be the unique solution to

\[
\begin{cases}
\Delta \varphi_\varepsilon = 16\pi (1 - \varepsilon)(e^{\varphi_\varepsilon} - \frac{1}{4\pi\varepsilon^2}) & \text{in } B_{2R}(0) \subset \mathbb{R}^2, \\
\varphi_\varepsilon^{(1)}|_{\partial B_{2R}(0)} = 0.
\end{cases}
\]

Since $x_\varepsilon$ is a local maximum point of $v_\varepsilon(x)$, we have $\varphi_\varepsilon \leq \varphi_\varepsilon(0) = 0$ and $e^{\varphi_\varepsilon} \leq 1$. It follows that $||\varphi_\varepsilon^{(1)}||_{L^{\infty}} \leq C < +\infty$. Let $\varphi_\varepsilon^{(2)} = \varphi_\varepsilon - \varphi_\varepsilon^{(1)}$. Then $\varphi_\varepsilon^{(2)} \leq -\varphi_\varepsilon^{(1)} \leq C$. Since $2C - \varphi_\varepsilon^{(2)}(0) = 2C - \varphi_\varepsilon(0) + \varphi_\varepsilon^{(1)}(0) \leq 3C$,

Harnack’s inequality implies that $||2C - \varphi_\varepsilon^{(2)}||_{L^{\infty}} \leq \tilde{C}$ in $B(R)$. Hence $||\varphi_\varepsilon||_{L^{\infty}(R)} \leq C + \tilde{C}$.

Therefore, as $\varepsilon \to 0$,

\[
1 = \int_{S^2} e^{v_\varepsilon} \geq \sum_{x \in S} \int_{B_{\varepsilon}(x)} e^{v_\varepsilon} \geq \sum_{x \in S} \int_{B_{\varepsilon}(0)} e^{\varphi_\varepsilon} \to \sum_{x \in S} \frac{\pi R^2}{1 + 2\pi R^2}.
\]

It follows that $\# S \leq 2$ and $R = S^2 \setminus S$. Since $S$ is not empty and $\nu_\varepsilon \in O$, we obtain that $\# S = 2$ and $S = \{a, -a\}$ for some $a \in S^2$. Without loss of generality, we may assume that $S = \{n, s\}$, where $n$ and $s$ stand for the north pole and the south pole respectively.

For any compact domain $K \subset \subset S^2 \setminus S(= R)$, we know from Lemma 2 that $v_\varepsilon - v_\varepsilon^a$ is bounded in $L^{\infty}(K)$ uniformly in $\varepsilon$. Since $v_\varepsilon^a \to -\infty$, it follows from the standard elliptic estimate that

$v_\varepsilon - v_\varepsilon^a \to G(x)$ in $C^{1, \alpha}(K)$,

where $G(x)$ satisfies

\[
\begin{cases}
-\Delta G + 4 = 8\pi(\delta_n + \delta_s) & \text{on } S^2, \\
\int_{S^2} G = 0,
\end{cases}
\]

and $\delta_n, \delta_s$ are delta functions centered at the north pole and the south pole, respectively. It can be easily seen that

$G(x) = -4\ln \sin \theta - 4(1 - \ln 2),$

where $\theta$ is the angle between $x$ and $x_3$. 

Lemma 4.

For all $D$, some constant $\|v\|$, for all $o \epsilon$ and $v$

It follows from Theorem 0.2 in [9] that there exists some constant $C'$, such that

$$\left| e^{\lambda_1} \left(1 + 2\pi(1 - \epsilon)e^{\lambda_1}\text{dist}(n, x)^2\right)^2 \right| \leq C', \text{ for } x \in B_{\pi/4}(n)$$ and

$$\left| e^{\lambda_2} \left(1 + 2\pi(1 - \epsilon)e^{\lambda_2}\text{dist}(s, x)^2\right)^2 \right| \leq C', \text{ for } x \in B_{\pi/4}(s).$$

Since $v - v^* \to G$ in $C^{1,\alpha}(S^2 \setminus [B_{\pi/4}(n) \cup B_{\pi/4}(s)])$, we obtain that there exists some constant $C$, such that

$$|\lambda_1 - \lambda_2| \leq C,$$

for all $\epsilon$. Without loss of generality, we may assume that $\lambda_1 \geq \lambda_2$.

**Lemma 4.** For all $x \in \Omega$, $v_\epsilon(x) \geq G(x) + D_\epsilon + o_\epsilon(1)$, where

$$(2.5) \quad D_\epsilon = -\lambda_1 + 2 \ln \frac{R^2}{1 + 2\pi R^2} + 4(1 - \ln 2)$$

and $o_\epsilon(1)$ stands for some function that goes to $0$ as $\epsilon \to 0$.

**Proof.** In $B_1$, we have

$$G(x) = -4 \ln r - 4(1 - \ln 2) + o_\epsilon(1),$$

$$v_\epsilon(x) = \lambda_1 + 2 \ln \frac{1}{1 + 2\pi R^2} + o_\epsilon(1).$$

Hence,

$$(v_\epsilon - G)|_{\partial B_1} = \lambda_1 + 2 \ln \frac{1}{1 + 2\pi R^2} + 4 \ln \frac{R}{\tau_1} + 4(1 - \ln 2) + o_\epsilon(1)$$

$$= -\lambda_1 + 2 \ln \frac{R^2}{1 + 2\pi R^2} + 4(1 - \ln 2) + o_\epsilon(1)$$

$$= D_\epsilon + o_\epsilon(1).$$

Similarly

$$(v_\epsilon - G)|_{\partial B_2} = \lambda_2 + 2 \ln \frac{1}{1 + 2\pi R^2} + 4 \ln \frac{R}{\tau_2} + 4(1 - \ln 2) + o_\epsilon(1)$$

$$= -\lambda_2 + 2 \ln \frac{R^2}{1 + 2\pi R^2} + 4(1 - \ln 2) + o_\epsilon(1)$$

$$= \lambda_1 - \lambda_2 + D_\epsilon + o_\epsilon(1) \geq D_\epsilon + o_\epsilon(1).$$

Since $\Delta (v_\epsilon - G) \leq 0$ in $\Omega$ and $(v_\epsilon - G)|_{\partial \Omega} \geq D_\epsilon + o_\epsilon(1)$, Lemma 4 follows from the maximum principle. \qed
We are now ready to estimate \( E_{\epsilon} = I_{\epsilon}(u_{\epsilon}) \).
\[
\int_{S^2} |\nabla v_{\epsilon}|^2 = \left( \int_{B_1} + \int_{B_2} + \int_{\Omega} \right) |\nabla v_{\epsilon}|^2 := I_1 + I_2 + I_3.
\]
Since the behavior of \( v_{\epsilon} \) near the north pole can be described by the behavior of \( \phi_{\epsilon} \) in \( B_R(0) \subset \mathbb{R}^2 \) and \( \phi_{\epsilon} \to \phi_0 \) in \( C^{1,\alpha} \), we obtain that
\[
I_1 = \int_{B_1} |\nabla v_{\epsilon}|^2 = \int_{B_R(0)} \left| \nabla \left( 2 \ln \frac{1}{1 + 2\pi r^2} \right) \right|^2 dx + o_{\epsilon}(1)
\]
\[
= 128\pi^3 \int_0^R \frac{r^2 r dr}{(1 + 2\pi r^2)^2} = 16\pi(\ln(1 + 2\pi R^2) - 1) + o_{\epsilon}(1) + o_R(1),
\]
where \( o_R(1) \to 0 \) as \( R \to \infty \). Similarly \( I_2 = 16\pi(\ln(1 + 2\pi R^2) - 1) + o_{\epsilon}(1) + o_R(1) \).
For \( I_3 \) we have
\[
I_3 = \int_{\Omega} |\nabla v_{\epsilon}|^2 = -\int_{\Omega} v_{\epsilon} \Delta v_{\epsilon} + \int_{\partial\Omega} v_{\epsilon} \frac{\partial v_{\epsilon}}{\partial n}.
\]
Using Green’s formula, we obtain that
\[
-\int_{\Omega} v_{\epsilon} \Delta v_{\epsilon} = \int_{\Omega} \left( (-\Delta v_{\epsilon} + 4(1 - \epsilon))v_{\epsilon} - 4(1 - \epsilon)v_{\epsilon} \right)
\]
\[
\geq \int_{\Omega} \left( (-\Delta v_{\epsilon} + 4(1 - \epsilon))(G + D_{\epsilon}) - 4(1 - \epsilon)v_{\epsilon} + o_{\epsilon}(1) \right)
\]
\[
= -\int_{\Omega} \Delta v_{\epsilon}(G + D_{\epsilon}) + 4(1 - \epsilon) \int_{\Omega} (G + D_{\epsilon} - v_{\epsilon}) + o_{\epsilon}(1)
\]
\[
= -\int_{\Omega} v_{\epsilon} \Delta G + \int_{\partial\Omega} (v_{\epsilon} \frac{\partial G}{\partial n} - (G + D_{\epsilon}) \frac{\partial v_{\epsilon}}{\partial n})
\]
\[
+ 4(1 - \epsilon) \int_{\Omega} (G + D_{\epsilon} - v_{\epsilon}) + o_{\epsilon}(1).
\]
It follows from \( \Delta G = 4 \) on \( \Omega \), that
\[
I_3 \geq -4(2 - \epsilon) \int_{\Omega} v_{\epsilon} + 4(1 - \epsilon) \int_{\Omega} (G + D_{\epsilon}) + \int_{\partial\Omega} (v_{\epsilon} \frac{\partial G}{\partial n} + (v_{\epsilon} - G - D_{\epsilon}) \frac{\partial v_{\epsilon}}{\partial n}).
\]
We now estimate the terms in the right hand side of the above inequality. First it is easy to see that
\[
\int_{\Omega} G = \left( \int_{S^2} - \int_{B_1 \cup B_2} \right) G = 0 - \int_{B_1 \cup B_2} G = o_{\epsilon}(1)
\]
and
\[
\int_{\Omega} v_{\epsilon} = \int_{S^2} v_{\epsilon} - \int_{B_1 \cup B_2} v_{\epsilon} = 4\pi v_{\epsilon}^a + o_{\epsilon}(1).
\]
Since \( \lambda_{\epsilon} \cdot \text{vol}(B_i) = o_{\epsilon}(1), i = 1, 2 \), it follows from (2.4) and (2.5) that \( D_{\epsilon} \cdot \text{vol}(B_i) = o_{\epsilon}(1), i = 1, 2 \). Therefore
\[
\int_{\Omega} D_{\epsilon} = \int_{S^2} D_{\epsilon} - \int_{B_1 \cup B_2} D_{\epsilon} = 4\pi D_{\epsilon} + o_{\epsilon}(1).
\]
On $\partial B_i$ for $i = 1, 2$

$$G(x) = 4 \ln \frac{1}{r_{ei}} - 4(1 - \ln 2) + o_{\epsilon}(1),$$

$$\frac{\partial G}{\partial n} = -4 \frac{1}{r_{ei}} + o_{\epsilon}(1),$$

$$v_{\epsilon}(x) = \lambda_{ei} + 2 \ln \frac{1}{1 + 2\pi R^2} + o_{\epsilon}(1),$$

$$\frac{\partial v_{\epsilon}}{\partial n} = \left( -\frac{8\pi R}{1 + 2\pi R^2} + o_{\epsilon}(1) \right) r_{ei},$$

It follows that

$$\int_{\partial \Omega} v_{\epsilon} \frac{\partial G}{\partial n} = -\left( \int_{\partial B_1} + \int_{\partial B_2} \right) v_{\epsilon} \frac{\partial G}{\partial n}$$

$$= 2\pi r_1 \left( \frac{4}{r_{e1}} + o_{\epsilon}(1) \right) \left( \lambda_{e1} + 2 \ln \frac{1}{1 + 2\pi R^2} + o_{\epsilon}(1) \right)$$

$$+ 2\pi r_2 \left( \frac{4}{r_{e2}} + o_{\epsilon}(1) \right) \left( \lambda_{e2} + 2 \ln \frac{1}{1 + 2\pi R^2} + o_{\epsilon}(1) \right)$$

$$= 8\pi (\lambda_{e1} + \lambda_{e2}) - 32\pi \ln(1 + 2\pi R^2) + o_{\epsilon}(1)$$

and

$$\int_{\partial \Omega} (v_{\epsilon} - G - D_{\epsilon}) \frac{\partial v_{\epsilon}}{\partial n} = -\left( \int_{\partial B_1} + \int_{\partial B_2} \right) (v_{\epsilon} - G - D_{\epsilon}) \frac{\partial v_{\epsilon}}{\partial n}$$

$$= 8\pi (\lambda_{e1} - \lambda_{e2}) + o_{\epsilon}(1) + o_R(1).$$

Using estimates of $I_1$, $I_2$ and $I_3$, we finally obtain that

$$\int_{S^2} |\nabla v_{\epsilon}|^2 = I_1 + I_2 + I_3 \geq 32\pi (\ln(1 + 2\pi R^2) - 1) - 4(2 - \epsilon)4\pi v_{\epsilon}^a$$

$$+ 8\pi (\lambda_{e1} + \lambda_{e2}) - 32\pi \ln(1 + 2\pi R^2) + 8\pi (\lambda_{e1} - \lambda_{e2})$$

$$+ 16\pi (1 - \epsilon) D_{\epsilon} + o_{\epsilon}(1) + o_R(1)$$

$$= -32\pi - 16\pi (2 - \epsilon) v_{\epsilon}^a + 8\pi (\lambda_{e1} + \lambda_{e2}) + 8\pi (\lambda_{e1} - \lambda_{e2})$$

$$+ 16\pi (1 - \epsilon) \left( -\lambda_{e1} + 2 \ln \frac{R^2}{1 + 2\pi R^2} + 4(1 - \ln 2) \right)$$

$$+ o_{\epsilon}(1) + o_R(1)$$

$$\geq 32\pi - 16\pi (2 - \epsilon) v_{\epsilon}^a - 32\pi \ln(8\pi) + o_{\epsilon}(1) + o_R(1).$$

It follows that

$$I_{\epsilon}(u_{\epsilon}) = \frac{1}{2(1 - \epsilon)} \int_{S^2} \frac{|\nabla u_{\epsilon}|^2}{4} - \ln \int_{S^2} e^{2u_{\epsilon}}$$

$$\geq 1 - \frac{2 - \epsilon}{2(1 - \epsilon)} v_{\epsilon}^a - \ln(8\pi) + \ln(4\pi) + v_{\epsilon}^a + o_{\epsilon}(1) + o_R(1)$$

$$\geq 1 - \ln 2 + o_{\epsilon}(1) + o_R(1).$$

Hence $I(u) \geq 1 - \ln 2$ for all $u \in O$. This contradicts with the fact that $I(0) = 0$. Therefore $\int_{S^2} e^{2u_{\epsilon}}$ stays bounded as $\epsilon \to 0$ and Theorem follows.
Remark 1. It is clear from the proof our main theorem that the extremal function $u \in O$ satisfies
\begin{equation}
-\Delta u = 8\pi \left( \frac{e^{2u}}{\int_{S^2} e^{2u}} - \frac{1}{4\pi} \right) \quad \text{on} \quad S^2.
\end{equation}

To answer question (II) in the introduction, one needs to answer whether $u = 0$ is the only solution to (2.6). We do not know the answer yet.

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