An explicit dimension formula for Siegel cusp forms with respect to the non-split symplectic groups

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Abstract. We give an explicit dimension formula for the spaces of vector valued Siegel cusp forms of degree two with respect to a certain kind of arithmetic subgroups of the non-split $\mathbb{Q}$-forms of $Sp(2, \mathbb{R})$.

1. Introduction.

The purpose of this paper is to give an explicit dimension formula for the spaces of vector valued Siegel cusp forms of degree two with respect to a certain kind of discrete subgroups of the non-split $\mathbb{Q}$-forms of $Sp(2, \mathbb{R})$.

In general, the dimensions of the spaces of Siegel modular forms can be calculated, in principle, by Selberg trace formula or Riemann-Roch theorem if the weights are sufficiently large, but there are many difficulties in actual calculations. Our concern is explicit dimension formulae, i.e., elementary functions of weights which give numerical values of dimensions. Such formulae are very useful for determining explicit ring structures of Siegel modular forms. In addition, they can be used also for studying a possible correspondence between Siegel modular forms for different discrete subgroups by means of comparisons of dimension formulae.

Our main result is Theorem 3.1, which is shortly explained below. Let $B$ be an indefinite division quaternion algebra over $\mathbb{Q}$ with discriminant $D$. Let $\mathfrak{D}$ be the maximal order of $B$, which is unique up to conjugation. If we take a positive divisor $D_1$ of $D$ and put $D_2 := D/D_1$, then there is the unique maximal two-sided ideal $\mathfrak{A}$ of $\mathfrak{D}$ such that $\mathfrak{A} \otimes \mathbb{Z} \mathbb{Z}_p = \mathfrak{D}_p$ if $p \mid D_1$ or $p \nmid D$, and $\mathfrak{A} \otimes \mathbb{Z} \mathbb{Z}_p = \pi \mathfrak{D}_p$ if $p \mid D_2$, where $\pi$ is a prime element of $\mathfrak{D}_p$. We consider the unitary group of the quaternion hermitian space of rank two and denote by $\Gamma(D_1, D_2)$ the stabilizer of the maximal lattice $(\mathfrak{A}, \mathfrak{D})$, namely we define

$$\Gamma(D_1, D_2) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2; B) \mid g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^t \mathfrak{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ a, d \in \mathfrak{D}, \ b \in \mathfrak{A}^{-1}, c \in \mathfrak{A} \right\}.$$
This group can be regarded as a discrete subgroup of $Sp(2, \mathbb{R})$. Our main theorem (Theorem 3.1) is an explicit formula for dimension of $S_{k,j}(\Gamma(D_1, D_2))$, the space of vector valued Siegel cusp forms of weight $\det^k \otimes \text{Sym}_j$ with respect to $\Gamma(D_1, D_2)$ for $k \geq 5$. For example, we have

$$\dim_{\mathbb{C}} S_{k,0}(\Gamma(1, 6)) = \frac{4k^3 - 18k^2 + 696k - 1737 + (-1)^k \cdot 225}{1440} + \frac{[0, -1, 1; 3]_k}{9} + \frac{[1, 0, 0, -1; 4]_k}{4} + \frac{4[1, 0, 0, -1; 0; 5]_k}{5},$$

where $[a_0, \ldots, a_{m-1}; m]_k$ is the function of $k$ which takes the value $a_i$ if $k \equiv i \mod m$.

Explicit dimension formula for the spaces of Siegel cusp forms of degree two has been studied by many mathematicians. Among them, Arakawa [Ara75], [Ara81], Hashimoto [Has84] and Wakatsuki [Wak, Theorem 6.1] treated the non-split $\mathbb{Q}$-forms. Hashimoto [Has84] obtained an explicit dimension formula for scalar valued Siegel cusp forms for $\Gamma(D, 1)$ by using Selberg trace formula, and Wakatsuki [Wak, Theorem 6.1] generalized it to the vector valued Siegel cusp forms for $\Gamma(D, 1)$. Our main result of this paper (Theorem 3.1) is a generalization of [Has84] and [Wak, Theorem 6.1] to any $\Gamma(D_1, D_2)$. We obtain the result by essentially the same method as [Has84], but our situation is much more complicated.

We make some remarks on some known facts which are used to obtain our result. We divide $\Gamma = \Gamma(D_1, D_2)$ into disjoint union of four subsets $\Gamma^{(e)}$, $\Gamma^{(u)}$, $\Gamma^{(qu)}$ and $\Gamma^{(h)}$ as follows:

(i) $\Gamma^{(e)}$ consists of torsion elements of $\Gamma$.
(ii) $\Gamma^{(u)}$ consists of non-semi-simple elements of $\Gamma$ whose semi-simple factors are $1_4$ or $-1_4$.
(iii) $\Gamma^{(qu)}$ consists of non-semi-simple elements of $\Gamma$ whose semi-simple factors belong to $\Gamma^{(e)}$ other than $\pm 1_4$.
(iv) $\Gamma^{(h)}$ consists of the other elements of $\Gamma$ than the above three types.

We denote the contributions to dimension formula of each subset above by $I(\Gamma^{(e)})_{k,j}$, $I(\Gamma^{(u)})_{k,j}$, $I(\Gamma^{(qu)})_{k,j}$ and $I(\Gamma^{(h)})_{k,j}$. It is known that $I(\Gamma^{(h)})_{k,j} = 0$ and

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma) = I(\Gamma^{(e)})_{k,j} + I(\Gamma^{(u)})_{k,j} + I(\Gamma^{(qu)})_{k,j}.$$
that the formula for $I(\Gamma^{(e)})_{k,j}$ can be expressed adelically and can be reduced to local computation (cf. Theorem 4.1). We do not need to calculate the local data \(c_p(g, R_p, \Lambda_p)\) since they have been obtained in [HI80] and [HI83]. (Although $B$ is definite in the case of [HI80] and [HI83], there is no difference if being localized.) So, our main task is to combine the local data and determine $G$-conjugacy classes which appear in the first sum in Theorem 4.1. It is still a complicated work, and the details will be explained in Section 4. (The way of combining local data is different from that of [HI80] and [HI83] since $B$ is indefinite in our case.) On the other hand, as for the contributions $I(\Gamma^{(u)})_{k,j}$ and $I(\Gamma^{(qu)})_{k,j}$, we cannot reduce them to local calculations. Wakatsuki [Wak] gave an arithmetic formula for the contributions of them, but one still have to carry out detailed calculation to obtain an explicit formula. More precisely, we need to determine a complete system of representatives of $\Gamma$-conjugacy classes of “families” (cf. Proposition 5.7, 5.8, 5.9) and calculate some data for them. Arakawa has calculated the contribution $I(\Gamma^{(u)})_{k,j}$ in his master thesis [Ara75], but we prove it again in Section 5 by means of Wakatsuki’s formula (e-2) since [Ara75] was not published with enough generality. Hashimoto calculated the contribution $I(\Gamma^{(qu)})_{k,j}$ in the case where $D_1 = D$ and $D_2 = 1$ in [Has84]. We can calculate $I(\Gamma^{(qu)})_{k,j}$ also in the general case by almost the same way.

We organize this paper as follows. In Section 2, we will review Siegel cusp forms in Subsection 2.1, and give in Subsection 2.2 and 2.3 the precise definition of the discrete subgroup $\Gamma(D_1, D_2)$. In Section 3, we will state our main theorem (Theorem 3.1) which will be proved in Section 4 and 5. In Section 4, we evaluate the contribution $I(\Gamma^{(e)})_{k,j}$. First, we quote the formula of Hashimoto and Ibukiyama (Theorem 4.1), and then we evaluate $H_1, \ldots, H_{12}$ of Theorem 3.1 in Subsection 4.1–4.12. In Section 5, we evaluate the contribution $I(\Gamma^{(u)})_{k,j}$ and $I(\Gamma^{(qu)})_{k,j}$. We evaluate $I_1$, $I_2$ and $I_3$ of Theorem 3.1 in Subsection 5.1, 5.2 and 5.3 respectively. In Section 6, we give some numerical examples for our main theorem.

Finally, we want to refer to a possible application of our result. In the case where $B$ is definite, Ibukiyama has been studying a generalization of Eichler-Jacquet-Langlands correspondence to the case of $Sp(2)$ (cf. [Ibu85], [HI85], [Ibu07a]). He obtained some relations of dimension formulae and conjectured a correspondence of discrete subgroups. Moreover, he calculated some numerical examples of coincidence of Euler factors of $L$-functions in [Ibu84]. On the other hand, in our case $B$ is indefinite. The author constructed explicit generators of the graded ring of scalar valued Siegel modular forms for $\Gamma(1, 6)$ as an application of our dimension formula, which will appear in a forthcoming paper and can be used for calculating Hecke operators concretely. Our dimension formula and numerical calculations of Hecke eigenvalues will be eventually used for comparisons similar to the above.
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2. Preliminaries.

2.1. Siegel cusp forms.

Let $\text{Sp}(2; \mathbb{R})$ be the real symplectic group of degree two, i.e.

$$\text{Sp}(2; \mathbb{R}) = \left\{ g \in GL(4; \mathbb{R}) \mid g \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \cdot g^t = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} \right\}.$$ 

Let $\mathcal{H}_2$ be the Siegel upper half space of degree two, i.e.

$$\mathcal{H}_2 = \left\{ Z \in M(2; \mathbb{C}) \mid tZ = Z, \ \text{Im}(Z) \text{ is positive definite} \right\}.$$ 

The group $\text{Sp}(2; \mathbb{R})$ acts on $\mathcal{H}_2$ by

$$\gamma(Z) := (AZ + B)(CZ + D)^{-1}$$

for any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2; \mathbb{R})$ and $Z \in \mathcal{H}_2$.

Let $\Gamma$ be a discrete subgroup of $\text{Sp}(2; \mathbb{R})$ such that $\text{vol}(\Gamma \setminus \mathcal{H}_2) < \infty$. Let $\rho_{k,j} : GL(2; \mathbb{C}) \to GL(j + 1; \mathbb{C})$ be the irreducible rational representation of the signature $(j + k, k)$ for $k, j \in \mathbb{Z}_{\geq 0}$, i.e. $\rho_{k,j} = \det^k \otimes \text{Sym}_j$, where $\text{Sym}_j$ is the symmetric $j$-tensor representation of $GL(2; \mathbb{C})$. We denote by $S_{k,j}(\Gamma)$ the space of Siegel cusp forms of weight $\rho_{k,j}$ with respect to $\Gamma$, i.e. the space which consists of holomorphic function $f : \mathcal{H}_2 \to \mathbb{C}^{j+1}$ satisfying the following two conditions:

(i) $f(\gamma(Z)) = \rho_{k,j}(CZ + D)f(Z)$, for $\forall \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, $\forall Z \in \mathcal{H}_2$,

(ii) $|\rho_{k,j}(\text{Im}(Z)^{1/2})f(Z)|_{C^{j+1}}$ is bounded on $\mathcal{H}_2$.

where we define $|u|_{C^{j+1}} = (\text{tr} \bar{u})^{1/2}$ for $u \in C^{j+1}$. It is known that $S_{k,j}(\Gamma)$ is a finite dimensional $C$-vector space.

2.2. The non-split $Q$-forms of $Sp(2; R)$.

Let $B$ be an indefinite split quaternion algebra over $\mathbb{Q}$. We fix an isomorphism $B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M(2; \mathbb{R})$ and we identify $B$ with a subalgebra of $M(2; \mathbb{R})$. Let $D$ be a product of all prime numbers $p$ for which $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra. We call
$D$ the discriminant of $B$. Let $W$ be a left free $B$-module of rank 2. Let $f$ be a map on $W \times W$ to $B$ defined by

$$f(x, y) = x_1 y_2 + x_2 y_1,$$

where $\bar{\cdot}$ is the canonical involution of $B$. Any non-degenerate quaternion hermitian form on $W$ is equivalent to $f$. (cf. [Shi63]). Let $U(2; B)$ be the unitary group with respect to this hermitian space $(W, f)$, that is,

$$U(2; B) = \{ g \in GL(2; B) \mid f(xg, yg) = f(x, y) \text{ for } \forall x, y \in W \},$$

where $^t g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2; B) \otimes Q R$. It is known that $U(2; B) \otimes Q R$ is isomorphic to $Sp(2; R)$ by

$$\phi : U(2; B) \otimes Q R \rightarrow Sp(2; R),$$

$$\phi(g) = \begin{pmatrix} a_1 & a_2 & b_2 & -b_1 \\ a_3 & a_4 & b_4 & -b_3 \\ c_3 & c_4 & d_4 & -d_3 \\ -c_1 & -c_2 & -d_2 & d_1 \end{pmatrix}, \ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2; B) \otimes Q R$$

where $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$, $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$, $D = \begin{pmatrix} d_1 & d_2 \\ d_3 & d_4 \end{pmatrix} \in B \otimes Q R$. It is known that each $Q$-form of $Sp(2; R)$ can be obtained as $U(2; B)$ for some indefinite quaternion algebra $B$ (cf. [PR94]). If $B = M(2; Q)$, then $U(2; B)$ is isomorphic to $Sp(2; Q)$ by $\phi$. In this paper, we treat the case where $B$ is a division algebra.

2.3.

Let $\mathcal{O}$ be the maximal order of $B$, which is unique up to inner automorphisms. We fix a quaternion hermitian space $(W, f)$. Let $L$ be a left $\mathcal{O}$-lattice in $W$, that is, $L$ is a finitely generated $\mathbb{Z}$-module satisfying $L \otimes \mathbb{Z} Q = W$ and $aL \subset L$ for any $a \in \mathcal{O}$. We put

$$U(2; B)_L := \{ g \in U(2; B) \mid Lg = L \}.$$ 

Then it is a discrete subgroup of $Sp(2, R)$ such that $\text{vol}(U(2; B)_L \setminus H_2) < \infty$ by identifying it with its image by $\phi$ in $Sp(2; R)$.

The two-sided $\mathcal{O}$-ideal generated by the elements $f(x, y)$ for $x, y \in L$ is called the norm of $L$. We call $L$ a maximal lattice if $L$ is maximal among the left $\mathcal{O}$-
lattices having the same norm. For any maximal lattice \( L \) and any prime number \( p \), it is known by [Shi63] that

\[
L \otimes \mathbb{Z} \mathbb{Z} = \begin{cases} 
(\mathcal{O}_p, \mathcal{O}_p)g_p & \text{if } p \nmid D \\
(\mathcal{O}_p, \mathcal{O}_p) \text{ or } (\pi \mathcal{O}_p, \mathcal{O}_p)g_p & \text{if } p \mid D
\end{cases}
\]

for some \( g_p \in U(2; B) \otimes \mathbb{Q} \mathbb{Q} \), where \( \mathcal{O}_p := \mathcal{O} \otimes \mathbb{Z} p \) and \( \pi \) is a prime element of \( \mathcal{O}_p \). Hence there are exactly \( 2^s \) genera of maximal lattices in \( \mathcal{W} \) if \( D \) is a product of \( s \) prime numbers. We put \( D = D_1 D_2 \), where \( D_1, D_2 \in \mathbb{N} \) such that \( L \otimes \mathbb{Z} \mathbb{Z} = (\mathcal{O}_p, \mathcal{O}_p)g_p \) if \( p \mid D_1 \), and \( L \otimes \mathbb{Z} \mathbb{Z} = (\pi \mathcal{O}_p, \mathcal{O}_p)g_p \) if \( p \mid D_2 \), for some \( g_p \in U(2; B) \otimes \mathbb{Q} \mathbb{Q} \). It is known that if two maximal lattices \( L_1 \) and \( L_2 \) correspond to the same pair \((D_1, D_2)\), then \( L_1 \) and \( L_2 \) belong to the same class (i.e. \( L_1 = L_2g \) for some \( g \in U(2; B) \)) since \( B \) is indefinite, and therefore \( U(2; B)_{L_1} = U(2; B)_{L_2} \).

For simplicity, we put

\[
\Gamma(D_1, D_2) := U(2; B)_L
\]

for the maximal lattice \( L \) corresponding to the pair \((D_1, D_2)\).

3. Main result.

Our main result is Theorem 3.1 below. It is an explicit dimension formula of the spaces of Siegel cusp forms of weight \( \rho_{k,j} \) with respect to \( \Gamma(D_1, D_2) \) defined above. This formula is a generalization of [Has84] and [Wak, Theorem 6.1]. We prove Theorem 3.1 in Sections 4 and 5. We suppose that \( j \) is even. If \( j \) is odd, we have \( S_{k,j}(\Gamma(D_1, D_2)) = \{0\} \) for any \( k \) since \( \Gamma(D_1, D_2) \) contains \(-1_4\). For natural number \( m \) and \( n \), we denote by \([a_0, \ldots, a_{m-1}; m]_n \) the function on \( n \) which takes the value \( a_i \) if \( n \equiv i \mod m \). We define the set \( T(m; n) := \{ p \mid T \mid p \equiv m \mod n \} \) for \( T = D, D_1 \) or \( D_2 \).

**Theorem 3.1.** If \( k \geq 5 \) and \( j \) is an even non-negative integer, then we have

\[
\dim_{\mathbb{C}} S_{k,j}(\Gamma(D_1, D_2)) = \sum_{i=1}^{12} H_i + \sum_{i=1}^3 I_i,
\]

where \( H_i \) and \( I_i \) are given as follows:

\[
H_1 = \frac{(j + 1)(k - 2)(j + k - 1)(j + 2k - 3)}{2^7 \cdot 3^3 \cdot 5} \cdot \prod_{p \mid D_1} (p - 1)(p^2 + 1) \cdot \prod_{p \mid D_2} (p^2 - 1)
\]
\[ H_2 = \frac{(-1)^k(j + k - 1)(k - 2)}{2^7 \cdot 3^2} \cdot \prod_{p \mid D} (p - 1)^2 \times \begin{cases} 7 & \text{if } 2 \nmid D_1, D_2 = 1 \\ 13 & \text{if } 2 \mid D_1, D_2 = 1 \\ 3 & \text{if } D_2 = 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ H_3 = \frac{((-1)^{j/2}(k - 2), -(j + k - 1), (-1)^{j/2+1}(k - 2), j + k - 1; 4)_k}{2^5 \cdot 3} \]
\[ \times \prod_{p \mid D_1} (p - 1) \left( 1 - \left( \frac{-1}{p} \right) \right) \times \begin{cases} 1 & \text{if } D_2 = 1 \\ 3 & \text{if } D_2 = 2 \\ 0 & \text{otherwise} \end{cases} \]

\[ H_4 = \frac{[j + k - 1, -(j + k - 1), 0; 3)_k + [k - 2, 0, -(k - 2); 3]_j + k}{2^3 \cdot 3^3} \]
\[ \times \prod_{p \mid D_1} (p - 1) \left( 1 - \left( \frac{-3}{p} \right) \right) \times \begin{cases} 1 & \text{if } D_2 = 1 \\ 8 & \text{if } D_2 = 3 \\ 0 & \text{otherwise} \end{cases} \]

\[ H_5 = 2^{-3} \cdot 3^{-2} \cdot \left( [-j + k - 1], -(j + k - 1), 0, j + k - 1, j + k - 1, 0; 6 \right)_k \]
\[ \quad + [k - 2, 0, -(k - 2), -(k - 2), 0, k - 2; 6]_j + k \]
\[ \times \prod_{p \mid D_1} (p - 1) \left( 1 - \left( \frac{-3}{p} \right) \right) \times \begin{cases} 1 & \text{if } D_2 = 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ H_6 = \sum_{n \mid 2D} A \prod_{p \mid n} (p - 1) \prod_{p \mid n} \left( 1 - \left( \frac{-1}{p} \right) \right) \prod_{p \mid n, p \not\mid D_2, p \neq 2} \left( \frac{p + 1}{2} \left( 1 - \left( \frac{-1}{p} \right) \right) \right) \cdot B \]

For each \( n \), \( A \) and \( B \) are defined as follows;

\[ A = \begin{cases} 2^{-7}3^{-1}(-1)^{k+j/2}(j + 1) & \text{if } n \text{ has odd numbers of prime divisors} \\ 2^{-7}3^{-1}(-1)^{j/2+2k - 3} & \text{if } n \text{ has even numbers of prime divisors} \end{cases} \]

If \( D_2 \) has a prime divisor \( p \) such that \( p \mid n \) and \( (-1/p) = -1 \), then \( B = 0 \), otherwise,

\[ B = \begin{cases} 5 & \text{if } 2 \mid D_1 \text{ and } 2 \mid n \\ 11 & \text{if } 2 \mid D_1 \text{ and } 2 \nmid n \\ 7 & \text{if } 2 \mid D_2 \text{ and } 2 \mid n \\ 9 & \text{if } 2 \mid D_2 \text{ and } 2 \nmid n \\ 3 & \text{if } 2 \nmid D \text{ and } 2 \mid n \\ 5 & \text{if } 2 \nmid D \text{ and } 2 \nmid n \end{cases} \]
\[H_7 = \sum_{n \mid 3D} A \prod_{p \mid n} (p - 1) \prod_{p \mid D_1 \atop p \neq 3} \left( 1 - \left( \frac{-3}{p} \right) \right) \prod_{p \mid D_2 \atop p \neq 3} \left( \frac{p + 1}{2} \left( 1 - \left( \frac{-3}{p} \right) \right) \right) \cdot B\]

For each \(n\), \(A\) and \(B\) are defined as follows:

\[A = \begin{cases} 
2^{-3}3^{-3}(j + 1)[0, 1, -1; 3]_{j+2k} & \text{if } n \text{ has odd numbers of prime divisors} \\
2^{-3}3^{-3}(j + 2k - 3)[1, -1, 0; 3]_j & \text{if } n \text{ has even numbers of prime divisors}
\end{cases} \]

If \(D_2\) has a prime divisor \(p\) such that \(p \mid n\) and \((-3/p) = -1\), then \(B = 0\), otherwise,

\[B = \begin{cases} 
1 & \text{if } 3 \mid D_1 \text{ and } 3 \nmid n \\
16 & \text{if } 3 \mid D_1 \text{ and } 3 \mid n \\
4 & \text{if } 3 \mid D_2 \text{ and } 3 \nmid n \\
10 & \text{if } 3 \mid D_2 \text{ and } 3 \mid n \\
1 & \text{if } 3 \nmid D \text{ and } 3 \mid n \\
4 & \text{if } 3 \mid D \text{ and } 3 \nmid n
\end{cases} \]

\[H_8 = \frac{C_1}{2^2 \cdot 3} \prod_{p \mid D} \left( 1 - \left( \frac{-1}{p} \right) \right) \left( 1 - \left( \frac{-3}{p} \right) \right) \times \begin{cases} 
1 & \text{if } D_2 = 1 \\
0 & \text{otherwise}
\end{cases} ,
\]

where we put

\[C_1 = \begin{cases} 
[1, 0, 0, -1, -1, -1, -1, 0, 0, 1, 1, 1; 12]_k & \text{if } j \equiv 0 \mod 12 \\
[-1, 1, 0, 1, 1, 0, 1, -1, 0, -1, -1, 0; 12]_k & \text{if } j \equiv 2 \mod 12 \\
[1, -1, 0, 0, -1, -1, -1, 1, 0, 0, 1, -1; 12]_k & \text{if } j \equiv 4 \mod 12 \\
[-1, 0, 0, -1, 1, -1, 1, 0, 0, 1, -1, 1; 12]_k & \text{if } j \equiv 6 \mod 12 \\
[1, 1, 0, 1, -1, 0, -1, -1, 0, -1, 1, 0; 12]_k & \text{if } j \equiv 8 \mod 12 \\
[-1, -1, 0, 0, 1, 1, 1, 1, 0, 0, -1, -1; 12]_k & \text{if } j \equiv 10 \mod 12
\end{cases} \]

\[H_9 = \frac{C_2}{2 \cdot 3^2} \times \prod_{p \mid D_1, p \neq 2} \left( 1 - \left( \frac{-3}{p} \right) \right)^2 \times \begin{cases} 
2 & \text{if } 2 \nmid D_1 \text{ and } D_2 = 1 \\
5 & \text{if } 2 \mid D_1 \text{ and } D_2 = 1 \\
3 & \text{if } 2 \nmid D_1 \text{ and } D_2 = 2 \\
0 & \text{otherwise}
\end{cases} ,
\]

where we put

\[C_2 = \begin{cases} 
[1, 0, 0, -1, 0, 0; 6]_k & \text{if } j \equiv 0 \mod 6 \\
[-1, 1, 0, 1, -1, 0; 6]_k & \text{if } j \equiv 2 \mod 6 \\
[0, -1, 0, 0, 1, 0; 6]_k & \text{if } j \equiv 4 \mod 6
\end{cases} .\]
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$$H_{10} = \frac{C_3}{2 \cdot 5} \times \prod_{p|D, p\neq 2} 2 \times \prod_{p \in D(4;5)} 2 \times \begin{cases} 
0 & \text{if } \bigcup_{i=1}^{3} D_1(i;5) \cup \bigcup_{i \in \{1,-1\}} D_2(i;5) \neq \emptyset \\
1 & \text{if } \bigcup_{i=1}^{3} D_1(i;5) \cup \bigcup_{i \in \{1,-1\}} D_2(i;5) = \emptyset \\
2 & \text{if } \bigcup_{i=1}^{3} D_1(i;5) \cup \bigcup_{i \in \{1,-1\}} D_2(i;5) = \emptyset 
\end{cases}$$

where we put

$$C_3 = \begin{cases} 
[1,0,0,-1,0;5]_k & \text{if } j \equiv 0 \mod 10 \\
[-1,1,0,0,0;5]_k & \text{if } j \equiv 2 \mod 10 \\
0 & \text{if } j \equiv 4 \mod 10 \\\n[0,0,0,1,-1;5]_k & \text{if } j \equiv 6 \mod 10 \\
[0,-1,0,0,1;5]_k & \text{if } j \equiv 8 \mod 10 
\end{cases}$$

$$H_{11} = \frac{C_4}{23} \times \prod_{p|D, p\neq 2} 2 \times \prod_{p \in D_1(7;8)} 2 \times \begin{cases} 
0 & \text{if } D(1;8) \cup D_2(7;8) \neq \emptyset \\
1 & \text{otherwise} 
\end{cases}$$

where we put

$$C_4 = \begin{cases} 
[1,0,0,-1;4]_k & \text{if } j \equiv 0 \mod 8 \\
[-1,1,0,0;4]_k & \text{if } j \equiv 2 \mod 8 \\
[-1,0,0,1;4]_k & \text{if } j \equiv 4 \mod 8 \\
[1,-1,0,0;4]_k & \text{if } j \equiv 6 \mod 8 
\end{cases}$$

$$H_{12} = 2^{-2}3^{-1}(-1)^{j/2+k}[1,-1,0;3]_j \times \prod_{p|D} 2 \times \prod_{p \in D_1(11;12)} 2 \times A + 2^{-2}3^{-1}(-1)^{j/2}[0,-1,1;3]_{j+2k} \times \prod_{p|D} 2 \times \prod_{p \in D_1(11;12)} 2 \times B,$$

where $A$ and $B$ are defined as follows.

(i) If $D(1;12) \cup D_2(11;12) \neq \emptyset$, then $A = B = 0$.

(ii) If $D(1;12) \cup D_2(11;12) = \emptyset$, then $A$ (resp. $B$) are given by the following table, depending on the conditions of $D, D_1$ and $D_2$. 

where case (I), (II) and (III) are given as follows:

\[
\begin{cases}
\text{(I)} & D_{1}(11; 12) = \emptyset \quad \text{and} \quad \#D(5; 12) = \text{even (resp. odd)} \\
\text{(II)} & D_{1}(11; 12) \neq \emptyset \\
\text{(III)} & D_{1}(11; 12) = \emptyset \quad \text{and} \quad \#D(5; 12) = \text{odd (resp. even)}
\end{cases}
\]

\[
I_1 = \frac{j+1}{2^3 \cdot 3} \prod_{p|D} (p-1)
\]

\[
I_2 = -\frac{(-1)^{j/2}}{2^3} \prod_{p|D} \left(1 - \left(\frac{-1}{p}\right)\right)
\]

\[
I_3 = -\frac{[1, -1, 0; 3]_j}{2 \cdot 3} \prod_{p|D} \left(1 - \left(\frac{-3}{p}\right)\right).
\]

4. The contribution of semi-simple conjugacy classes.

In this section, we evaluate \(I(\Gamma^{(e)})_{k,j}\), i.e. the contributions of torsion elements (cf. Section 1). The principal polynomials of torsion elements of \(G = U(2; B)\) are as follows, and each \(H_i\) in Theorem 3.1 means the contribution of \(\Gamma\)-conjugacy classes whose principal polynomials are of the form \(f_i(\pm x)\).

\[
f_1(x) = (x - 1)^4, \quad f_1(-x) = (x + 1)^4,
\]

\[
f_2(x) = (x - 1)^2(x + 1)^2,
\]

\[
f_3(x) = (x - 1)^2(x^2 + 1), \quad f_3(-x) = (x + 1)^2(x^2 + 1),
\]

\[
f_4(x) = (x - 1)^2(x^2 + x + 1), \quad f_4(-x) = (x + 1)^2(x^2 - x + 1),
\]
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\[ f_5(x) = (x - 1)^2(x^2 - x + 1), \quad f_5(-x) = (x + 1)^2(x^2 + x + 1), \]
\[ f_6(x) = (x^2 + 1)^2, \]
\[ f_7(x) = (x^2 + x + 1)^2, \quad f_7(-x) = (x^2 - x + 1)^2, \]
\[ f_8(x) = (x^2 + 1)(x^2 + x + 1), \quad f_8(-x) = (x^2 + 1)(x^2 - x + 1), \]
\[ f_9(x) = (x^2 + x + 1)(x^2 - x + 1), \]
\[ f_{10}(x) = x^4 + x^3 + x^2 + 1, \quad f_{10}(-x) = x^4 - x^3 + x^2 - x + 1, \]
\[ f_{11}(x) = x^4 + 1, \]
\[ f_{12}(x) = x^4 - x^2 + 1. \]

We will evaluate each \( H_i \) in Subsection 4.1–4.12. The method was developed by Hashimoto and Ibukiyama [Has80], [HI80], [HI82], [HI83], [Has83], [Has84].

We quote the formula for \( \text{I}(\Gamma^{(e)})_{k,j} \).

**Theorem 4.1.**

\[ \text{I}(\Gamma^{(e)})_{k,j} = c_{k,j} \sum_{\{g\}G} \text{J}_0'(g) \sum_{L_G(\Lambda)} M_G(\Lambda) \prod_p c_p(g, R_p, \Lambda_p), \]

where notations are as follows:

(i) The first sum is extended over the \( G \)-conjugacy classes \( \{g\}G \) of torsion elements of \( G \) which satisfies \( \{g\}G \cap \Gamma \neq \emptyset \).

(ii) The second sum is extended over the \( G \)-genus \( L_G(\Lambda) \) of \( \mathbb{Z} \)-orders in \( \mathbb{Z}(g) \) for each \( g \in G \). Here \( \mathbb{Z}(g) \) is the commutator algebra of \( g \) in \( M(2; B) \). The \( G \)-genus \( L_G(\Lambda) \) represented by a \( \mathbb{Z} \)-order \( \Lambda \) of \( \mathbb{Z}(g) \) consists of all \( \mathbb{Z} \)-orders in \( \mathbb{Z}(g) \) which are \( (\mathbb{Z}(g)^{\times} \cap G) \otimes_{\mathbb{Q}} \mathbb{Q}_p \)-conjugate to \( \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p \) for all \( p \).

(iii) The constant \( c_{k,j} \) for \( k, j \) are defined by

\[ c_{k,j} := 2^{-6} \pi^{-3}(k - 2)(j + k - 1)(j + 2k - 3). \]

(iv) We define \( J_0'(g) \) for each \( \{g\}G \) as follows. We put

\[ H_g^{k,j}(Z) := \text{tr} \left[ \rho_{k,j}(CZ + D)^{-1} \rho_{k,j} \left( \frac{g(Z) - Z}{2i} \right)^{-1} \rho_{k,j}(Y) \right], \]
\[ J_0(g) := \int_{C_0(g; S_p(2; R)) \setminus \mathbb{H}_2} H_g^{k,j}(\bar{Z}) d\bar{Z}, \]
where \( dZ = \det(Y)^{-3}dXdY, \) 
\( dX = dx_1dx_1dx_2, \) 
\( dY = dy_1dy_1dy_2 \) for 
\( Z = X + iY \in \mathfrak{h}_2, \) 
\( X = (\frac{x_1}{x_2} \frac{x_2}{x_2}), \) 
\( Y = (\frac{y_1}{y_2} \frac{y_2}{y_2}) \), and \( d\hat{Z} \) is an invariant measure on \( C_0(g; \text{Sp}(2; \mathbb{R})) \) \( \backslash \mathfrak{h}_2 \) induced from \( dZ \) and a Haar measure on \( C_0(g; \text{Sp}(2; \mathbb{R})) \). 

The definition of \( C_0(g; \text{Sp}(2; \mathbb{R})) \) is given for each \( g \) in each Subsection 4.1–4.12. We define \( J'_0(g) := J_0(g) \) if \( -14 \notin C_0(g; \text{Sp}(2; \mathbb{R})) \), and \( J'_0(g) := 2^{-1} \cdot J_0(g) \) if \( -14 \in C_0(g; \text{Sp}(2; \mathbb{R})) \).

(v) We define \( M_G(\Lambda) \) for each \( \{g\}_G \) and \( L_G(\Lambda) \) as follows. We decompose the group \( (Z(g)^{\times} \cap G)_A \) into the disjoint union

\[
(Z(g)^{\times} \cap G)_A = \bigsqcup_{i=1}^{h} (Z(g)^{\times} \cap G)_y (\Lambda^x_A \cap G_A),
\]

where \( \Lambda_A = \Lambda \otimes \mathbb{Z} \mathbf{Z}_A \). We put \( \Lambda_i = y_i\Lambda y_i^{-1} = \cap_p ((y_i)_p\Lambda_p(y_i)_p)^{-1} \cap Z(g) \) and define

\[
M_G(\Lambda) := \sum_{i=1}^{h} \text{vol}((\Lambda^x_A \cap G) \cap C_0(g; \text{Sp}(2; \mathbb{R}))).
\]

(vi) We define \( c_p(g, R_p, \Lambda_p) \) for each \( \{g\}_G, L_G(\Lambda) \) and \( p \) as

\[
c_p(g, R_p, \Lambda_p) = \sharp((Z(g)^{\times} \cap G)_p \cap M_p(g, R_p, \Lambda_p)/(R_p^{\times} \cap G_p)),
\]

where

\[
R_p := M(2; \mathfrak{D}_p) \text{ if } p \nmid D_2 \text{ and } R_p := \begin{pmatrix} \mathfrak{D}_p & \pi^{-1}\mathfrak{D}_p \\ \pi\mathfrak{D}_p & \mathfrak{D}_p \end{pmatrix} \text{ if } p \mid D_2,
\]

\[
M_p(g, R_p, \Lambda_p) := \left\{ x \in G_p \mid x^{-1}gx \in R_p, \text{ and } Z(g)_p \cap xR_px^{-1} \text{ is } (Z(g)^{\times} \cap G)_p \text{-conjugate to } \Lambda_p \right\}.
\]

If \( M_p(g, R_p, \Lambda_p) = \emptyset \), then we put \( c_p(g, R_p, \Lambda_p) = 0 \).

Remark 4.2. We give some remarks about Theorem 4.1.

(1) We do not need to calculate the local data \( c_p(g, R_p, \Lambda_p) \) since they have been obtained in \([HI80] \) and \([HI83] \). We have only to combine the data depending on the cases.

(2) We need to determine \( G \)-conjugacy classes which appear in the first sum in Theorem 4.1. It is known that \( \{g\}_G \cap \Gamma \neq \emptyset \) if and only if \( \{g\}_{G_p} \cap R_p \neq \emptyset \) for all \( p \). (cf. Theorem 1–3 in \([Has83] \)). We can obtain the result by using \([HI80, \text{ Section } 2] \), \([Has84] \) and the results of \( c_p \) mentioned above.
(3) The integral $J'_0(g)$ depends only on $Sp(2; \mathbb{R})$-conjugacy classes. Langlands [Lan63] gave a formula for $J_0(g)$. We can evaluate $J'_0(g)$ by applying explicit formulae in [Has83] and [Wak].

We will evaluate each $H_i$ in Subsection 4.1–4.12. We denote by $G[f_i]$ the set of torsion elements of $G$ whose principal polynomials are $f_i(x)$. For $i = 1, 3, 4, 5, 7, 8, 10$, we have only to evaluate the contribution of $G[f_i]$ and double it to obtain $H_i$ because the contribution of $g$ is equal to that of $-g$. We use the notation:

\[
\alpha(\theta_1, \theta_2) = \begin{pmatrix} \cos \theta_1 & 0 & \sin \theta_1 & 0 \\ 0 & \cos \theta_2 & 0 & \sin \theta_2 \\ -\sin \theta_1 & 0 & \cos \theta_1 & 0 \\ 0 & -\sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix}.
\]

4.1. The contribution $H_1$.

In this subsection, we consider the contribution of $\pm 1_4$. If $\gamma = \pm 1_4$, we have $C(\gamma; Sp(2; \mathbb{R})) = C_0(\gamma; Sp(2; \mathbb{R})) = Sp(2; \mathbb{R})$, $C(\gamma; \Gamma) = C_0(\gamma; \Gamma) = \Gamma$ and $H^k_j(\gamma, Z) = 1$, so $J'_0(\gamma) = (1/2) \int_{Sp(2; \mathbb{R}) \backslash H_2} d\hat{Z}$. Also, we have

\[
c_p(\gamma, R_p, \Lambda_p) = \begin{cases} 1 & \text{if } \Lambda_p \sim R_p \\ 0 & \text{otherwise.} \end{cases}
\]

Hence from Theorem 4.1 we have

\[
H_1 = 2^{-6} \pi^{-3} (k - 2)(j + k - 1)(j + 2k - 3) \cdot \text{vol}(Sp(2; \mathbb{R}) \backslash \mathfrak{h}_2) \cdot \text{vol}(\Gamma \backslash Sp(2; \mathbb{R})).
\]

We have only to multiple the value $H_1$ in the case of $D_2 = 1$ in [Wak] by $\prod_{p | D_2} (p + 1)/(p^2 + 1)$ because we have the indexes as follows:

\[
\begin{align*}
\left[ G_p \cap \left( \frac{\mathfrak{D}_p}{\mathfrak{D}_p}, \frac{\mathfrak{D}_p}{\pi \mathfrak{D}_p} \right) \right]^\times & : G_p \cap \left( \frac{\mathfrak{D}_p}{\pi \mathfrak{D}_p}, \frac{\mathfrak{D}_p}{\mathfrak{D}_p} \right)^\times = p + 1, \\
\left[ G_p \cap \left( \frac{\mathfrak{D}_p}{\pi \mathfrak{D}_p}, \frac{\pi^{-1} \mathfrak{D}_p}{\mathfrak{D}_p} \right) \right]^\times & : G_p \cap \left( \frac{\mathfrak{D}_p}{\pi \mathfrak{D}_p}, \frac{\mathfrak{D}_p}{\mathfrak{D}_p} \right)^\times = p^2 + 1.
\end{align*}
\]

Hence we obtain $H_1$ as in Theorem 3.1.

4.2. The contribution $H_2$.

In this subsection, we evaluate the contribution of $G[f_2]$, where $f_2(x) = (x - 1)^2(x + 1)^2$. The set $G[f_2]$ consists of only one $G$-conjugacy class repre-
sented by an element \( g \). We have \( Z(g) \simeq B \oplus B \). We fix \( g \) and this isomorphism until the end of this subsection. We put

\[
L := \{(x, y) \in \mathcal{O} \oplus \mathcal{O} \mid x - y \in \pi \mathcal{O}_2\},
\]

where \( \pi \) is a prime element of \( \mathcal{O}_2 \). We have the following proposition.

**Proposition 4.3.**

1. The class \( \{g\}_G \) appears in the first sum of Theorem 4.1, i.e. \( \{g\}_G \cap \Gamma \neq \emptyset \), if and only if \( D_2 = 1 \) or 2.
2. We assume \( D_2 = 2 \). If \( \Lambda \) is a \( \mathbb{Z} \)-order of \( \mathbb{Z}(g) \) belonging to the same \( G \)-genus as \( L \), then we have \( \prod_p c_p(g, R_p, \Lambda_p) = 1 \). If \( \Lambda \) does not belong to the same \( G \)-genus as \( L \), then \( \prod_p c_p(g, R_p, \Lambda_p) = 0 \).

**Proof.** We can prove (1) and the latter part of (2) easily by [HI83, Proposition 2.4]. If \( D_2 = 2 \) and \( \Lambda \) is a \( \mathbb{Z} \)-order of \( \mathbb{Z}(g) \) belonging to the same \( G \)-genus as \( L \), then it follows from [HI80, Proposition 13] and [HI83, Proposition 2.4] that \( c_p(g, R_p, \Lambda_p) = 1 \) for any \( p \).

From Proposition 4.3, we have \( H_2 = 0 \) if \( D_2 \neq 1, 2 \). In the case where \( D_2 = 1 \), \( H_2 \) has been evaluated in [Has84] and [Wak]. Hereafter, we assume \( D_2 = 2 \). From Proposition 4.3, we have

\[
H_2 = c_{k,j} \cdot J'_0(g) \cdot M_G(L).
\]

We see that \( g \) is \( Sp(2; \mathcal{R}) \)-conjugate to \( \alpha(\pi, 0) \) and

\[
C_0(\alpha(\pi, 0); Sp(2; \mathcal{R})) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix} \middle| \begin{array}{c} ad - bc = 1 \\ a'd' - b'c' = 1 \end{array} \right\},
\]

\[
J'_0(g) = \frac{1}{2} J_0(\alpha(\pi, 0)) = c_{k,j}^{-1} 2^{-7} \pi^{-4} (-1)^k (j + k - 1)(k - 2)
\]

if \( j \) is even. (cf. (b-5) in [Wak]). If we put \( L_0 = \mathcal{O} \oplus \mathcal{O} \), then we have

\[
M_G(L) = \text{vol}(L_0^\times \cap G) \backslash C(g; Sp(2; \mathcal{R}))
= \left[ L_0^\times \cap G : L_0^\times \cap G \right] \cdot \text{vol}(L_0^\times \cap G) \backslash C(g; Sp(2; \mathcal{R}))
= 3^{-1} \pi^4 \prod_{p\mid D} (p - 1)^2.
\]
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Hence we can obtain $H_2$ as in Theorem 3.1.

4.3. The contribution $H_3$.

In this subsection, we evaluate the contribution of $G[f_3]$, where $f_3(x) = (x - 1)^2(x^2 + 1)$. We have only to double it to obtain $H_3$. Note that $G[f_3] \neq \emptyset$ if and only if $(-1/p) \neq 1$ for any prime divisor $p$ of $D$. Hereafter, we assume that $G[f_3] \neq \emptyset$. The set $G[f_3]$ consists of two $G$-conjugacy classes represented by $g$ and $g^{-1}$ for an element $g$. We have $Z(g) \simeq B \oplus F$ with $F = Q(\sqrt{-1})$. We fix $g$ and this isomorphism until the end of this subsection. We put

$$L := \{(x, y) \in O' \oplus O' \mid x - y \in \pi O_2\},$$

where $O'$ is the ring of integers of $F$ and $\pi$ is a prime element of $O_2$. Then we have the following proposition.

**Proposition 4.4.**

1. The classes $\{g\}_G$ and $\{g^{-1}\}_G$ appear in the first sum of Theorem 4.1 if and only if $D_1^2 = 1$ or $2$.

2. We assume $D_2 = 2$. If $\Lambda$ is a $Z$-order of $Z(g)$ belonging to the same $G$-genus as $L$, then we have

$$\prod_p c_p(g, R_p, \Lambda_p) = \prod_p c_p(g^{-1}, R_p, \Lambda_p) = 2^{\sharp D_1(3;4)}.$$

If $\Lambda$ does not belong to the same $G$-genus as $L$, then $\prod_p c_p(g, R_p, \Lambda_p) = 0$.

**Proof.** We can prove (1) and the latter part of (2) easily by [HI83, Proposition 2.4]. If $D_2 = 2$ and $\Lambda$ is a $Z$-order of $Z(g)$ belonging to the same $G$-genus as $L$, then it follows from [HI80, Proposition 14] and [HI83, Proposition 2.4] that

$$c_p(g, R_p, \Lambda_p) = c_p(g^{-1}, R_p, \Lambda_p) = \begin{cases} 2 & \text{if } p \mid D_1 \text{ and } \left(\frac{-1}{p}\right) = -1 \\ 1 & \text{otherwise.} \end{cases}$$

From Proposition 4.4, we have $H_3 = 0$ if $D_2 \neq 1, 2$. In the case where $D_2 = 1$, $H_3$ has been evaluated in [Has84] and [Wak]. Hereafter, we assume $D_2 = 2$. From Proposition 4.4, we have
\[ H_3 = 2 \cdot c_{k,j} \cdot \sum_{\gamma \in \{g, g^{-1}\}} J'_0(\gamma) \cdot M_\gamma(L) \prod_p c_p(\gamma, R_p, L_p) \]
\[ = 2 \cdot c_{k,j} \cdot (J'_0(g) + J'_0(g^{-1})) \cdot M_g(L) \cdot 2^{2D_1(3;4)}. \]

We see that \( g \) and \( g^{-1} \) are \( Sp(2; \mathbb{R}) \)-conjugate to \( \alpha(\pi/2, 0) \) and \( \alpha(-\pi/2, 0) \) respectively and

\[ C_0 \left( \alpha \left( \frac{\pi}{2}, 0 \right), Sp(2; \mathbb{R}) \right) = C_0 \left( \alpha \left( -\frac{\pi}{2}, 0 \right), Sp(2; \mathbb{R}) \right) \]
\[ = \left\{ \begin{array}{c|c|c|c|c|c} 1 & 0 & 0 & 0 & & \\ 0 & a & 0 & b & & \\ 0 & 0 & 1 & 0 & & \\ 0 & c & 0 & d & & \end{array} \right\} \left| \begin{array}{c} ad - bc = 1 \end{array} \right\}. \]

\[ J'_0(g) + J'_0(g^{-1}) = c_{k,j}^{-1} \cdot 2^j \cdot \pi^2 \cdot \left[ (-1)^{j/2}(k - 2), -(j + k - 1), \right. \]
\[ \left( -1 \right)^{j/2+1}(k - 2), j + k - 1; 4 \left]_k \right. \]

(cf. (b-4) in [Wak]). If we put \( L_0 = \mathcal{O} \oplus \mathcal{O} \), then we have

\[ M_g(L) = \text{vol}((L^\times \cap G) \setminus C(g; Sp(2; \mathbb{R}))) \]
\[ = \left[ L_0^\times \cap G : L^\times \cap G \right] \cdot \text{vol}((L_0^\times \cap G) \setminus C(g; Sp(2; \mathbb{R}))) \]
\[ = 2^{-2} \pi^2 \prod_{p|D} (p - 1). \]

(cf. (3.10), (3.11) of [Has84]). Hence we can obtain \( H_3 \) as in Theorem 3.1.

4.4. The contribution \( H_4 \).

In this subsection, we evaluate the contribution of \( G[f_4] \), where \( f_4(x) = (x - 1)^2(x^2 + x + 1) \). We have only to double it to obtain \( H_4 \). Note that \( G[f_4] \neq \emptyset \) if and only if \( (-3/p) \neq 1 \) for any prime divisor \( p \) of \( D \). Hereafter, we assume that \( D \) does not have such a prime divisor. The set \( G[f_4] \) consists of two \( G \)-conjugacy classes represented by \( g \) and \( g^{-1} \) for an element \( g \). We have \( Z(g) \simeq B \oplus F \) with \( F = \mathbb{Q}(\sqrt{-3}) \). We fix \( g \) and this isomorphism until the end of this subsection. We put

\[ L := \{(x, y) \in \mathcal{O} \oplus \mathcal{O} \mid x - y \in \pi \mathcal{O}_3 \}, \]

where \( \mathcal{O} \) is the ring of integers of \( F \) and \( \pi \) is a prime element of \( \mathcal{O}_3 \). Then we have the following proposition.
Proposition 4.5.

(1) The classes \( \{g\}_G \) and \( \{g^{-1}\}_G \) appear in the first sum of Theorem 4.1 if and only if \( D_2 = 1 \) or \( 3 \).

(2) We assume \( D_2 = 3 \). If \( \Lambda \) is a \( \mathbb{Z} \)-order of \( \mathbb{Z}(g) \) belonging to the same \( G \)-genus as \( L \), then we have

\[
\prod_p c_p(g, R_p, \Lambda_p) = \prod_p c_p(g^{-1}, R_p, \Lambda_p) = 2^{\sharp D_1(2;3)}.
\]

If \( \Lambda \) does not belong to the same \( G \)-genus as \( L \), then \( \prod_p c_p(g, R_p, \Lambda_p) = 0 \).

**Proof.** We can prove (1) and the latter part of (2) easily by [HI83, Proposition 2.4]. If \( D_2 = 3 \) and \( \Lambda \) is a \( \mathbb{Z} \)-order of \( \mathbb{Z}(g) \) belonging to the same \( G \)-genus as \( L \), then it follows from [HI80, Proposition 14] and [HI83, Proposition 2.4] that

\[
c_p(g, R_p, \Lambda_p) = c_p(g^{-1}, R_p, \Lambda_p) = \begin{cases} 2 & \text{if } p \mid D_1 \text{ and } \left(\frac{-3}{p}\right) = -1 \\ 1 & \text{otherwise.} \end{cases}
\]

From Proposition 4.5, we have

\[
H_4 = 2 \cdot c_{k,j} \cdot \sum_{\gamma \in \{g, g^{-1}\}} J'_0(\gamma) \cdot M_\gamma(L) \prod_p c_p(\gamma, R_p, L_p)
\]

\[
= 2 \cdot c_{k,j} \cdot (J'_0(g) + J'_0(g^{-1})) \cdot M_g(L) \cdot 2^{\sharp D_1(2;3)}.
\]

We see that \( g \) and \( g^{-1} \) are \( Sp(2; \mathbb{R}) \)-conjugate to \( \alpha(2\pi/3, 0) \) and \( \alpha(-2\pi/3, 0) \) respectively and

\[
C_0 \left( \alpha \left( \frac{2\pi}{3}, 0 \right), Sp(2; \mathbb{R}) \right) = C_0 \left( \alpha \left( -\frac{2\pi}{3}, 0 \right), Sp(2; \mathbb{R}) \right)
\]

\[
= \begin{cases} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix} \mid ad - bc = 1 \end{cases},
\]

\[
J'_0(g) + J'_0(g^{-1}) = c_{k,j}^{-1} \cdot 2^{-3} \cdot 3^{-1} \cdot \pi^{-2}
\]

\[
\times \left\{ [j + k - 1, -(j + k - 1), 0; 3]_k \\
+ [k - 2, 0, -(k - 2); 3]_{j+k} \right\}
\]
(cf. (b-4) in [Wak]). If we put \( L_0 = \mathcal{O} \oplus \mathcal{O} \), then we have
\[
M_g(L) = \text{vol}((L^\times \cap G) \setminus C(g; Sp(2; R)))
\]
\[
= [L_0^\times \cap G : L^\times \cap G] \cdot \text{vol}((L_0^\times \cap G) \setminus C(g; Sp(2; R)))
\]
\[
= 2^2 \cdot 3^{-2} \cdot \pi^2 \prod_{p|D} (p - 1).
\]
(c.f. (3.10), (3.11) of [Has84]). Hence we can obtain \( H_4 \) as in Theorem 3.1.

4.5. The contribution \( H_5 \).

In this subsection, we consider the contribution of \( G[f_5] \), where \( f_5(x) = (x - 1)^2(x^2 - x + 1) \). We have only to double it to obtain \( H_5 \). The set \( G[f_5] \) consists of two \( G \)-conjugacy classes represented by \( g \) and \( g^{-1} \) for an element \( g \).

We see from [HI80, Proposition 14] and [HI83, Proposition 2.4] that the classes \( \{g \}_G \) and \( \{g^{-1} \}_G \) do not appear in the first sum of Theorem 4.1 if \( D_2 \neq 1 \). In the case where \( D_2 = 1 \), \( H_5 \) has been evaluated in [Has84] and [Wak].

4.6. The contribution \( H_6 \).

In this subsection, we consider the contribution of \( G[f_6] \), where \( f_6(x) = (x^2 + 1)^2 \). Note that \( G[f_6] = \emptyset \) if and only if \( D_2 \) has a prime divisor \( p \) with \((-1/p) = 1 \). Hereafter, we assume that \( D \) does not have such a prime divisor. Then there are infinitely many \( G \)-conjugacy classes in \( G[f_6] \). As in [Has84, Theorem 3.2 (i),(ii)], we have a correspondence between the set of \( G \)-conjugacy classes \( \{g \}_G \)'s in \( G[f_6] \) and the set of isomorphism classes of quaternion algebras \( Z_0(g) \)'s over \( \mathbb{Q} \) which are contained in \( B \otimes_{\mathbb{Q}} F \), with \( F = \mathbb{Q}(\sqrt{-1}) \). This correspondence is one-to-one according as \( Z_0(g) \) is definite or indefinite. We denote by \( D(Z_0(g)) \) the discriminant of \( Z_0(g) \). If \( Z_0(g) \) is definite, two \( G \)-conjugacy classes \( \{g \}_G \) and \( \{g^{-1} \}_G \) correspond to \( Z_0(g) \). In this case, \( g \) is \( Sp(2; R) \)-conjugate to \( \alpha(\pi/2, \pi/2) \) and
\[
J_0'(g) + J_0'(g^{-1}) = c_{k,j}^{-1} \cdot 2^{-2} \cdot (j + 1) \cdot (-1)^{k+j/2},
\]
if \( j \) is even (cf. (b-2) in [Wak]),
\[
M_G(\Lambda) = \frac{1}{48} \prod_{p|D(Z_0(g))} (p - 1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)}
\]
for a \( \mathbb{Z} \)-order \( \Lambda \) of \( Z(g) \) (cf. [HI80, Proposition 12]), where \( \mathcal{O}_0 \) is a maximal order of \( Z_0(g) \) and
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\[ d_p(\Lambda) = \left[ \Omega_0^\infty : (\Lambda \cap \mathbb{Z}_0(g))^\infty \right], \quad e_p(\Lambda) = \left[ \Lambda_p^\infty \cap G_p : \Omega_0^\infty \cdot \mathcal{O}_F^\infty \right]. \]

On the other hand, if \( \mathbb{Z}_0(g) \) is indefinite, then only one \( G \)-conjugacy class \( \{ g \}_G \) corresponds to \( \mathbb{Z}_0(g) \). In this case, \( g \) is \( \text{Sp}(2; \mathbb{R}) \)-conjugate to \( \alpha(\pi/2, -\pi/2) \) and

\[ J'_0(g) = c_{k,j}^{-1} \cdot 2^{-5} \cdot \pi^{-2} \cdot (j + 2k - 3) \cdot (-1)^{j/2}, \]

if \( j \) is even (cf. (b-3) in [Wak]),

\[ M_G(\Lambda) = \frac{\pi^2}{6} \prod_{p \mid D(\mathbb{Z}_0(g))} (p - 1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} \]

for a \( \mathbb{Z} \)-order \( \Lambda \) of \( \mathbb{Z}(g) \) (cf. [Has84, (3.16)]).

Proposition 4.6.

(1) The class \( \{ g \}_G \) appears in the first sum of Theorem 4.1 if and only if \( D(\mathbb{Z}_0(g)) \mid 2D \).

(2) (i) the case where \( 2 \mid D_1 \),

- If a \( G \)-conjugacy class \( \{ g \}_G \) satisfies \( 2 \mid D(\mathbb{Z}_0(g)) \), then two \( G \)-genus of \( \mathbb{Z} \)-orders of \( \mathbb{Z}(g) \) appear in the second sum of Theorem 4.1, and

\[ \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(\mathbb{Z}_0(g))} (p + 1) \cdot \frac{3}{2} \text{ (resp. 1)} \]

- If a \( G \)-conjugacy class \( \{ g \}_G \) satisfies \( 2 \nmid D(\mathbb{Z}_0(g)) \), then three \( G \)-genus of \( \mathbb{Z} \)-orders of \( \mathbb{Z}(g) \) appear in the second sum of Theorem 4.1, and

\[ \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(\mathbb{Z}_0(g))} (p + 1) \cdot \frac{3}{2} \text{ (resp. 1 and 3)} \]

(ii) the case where \( 2 \mid D_2 \),

- If a \( G \)-conjugacy class \( \{ g \}_G \) satisfies \( 2 \mid D(\mathbb{Z}_0(g)) \), then two \( G \)-genus of \( \mathbb{Z} \)-orders of \( \mathbb{Z}(g) \) appear in the second sum of Theorem 4.1, and

\[ \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(\mathbb{Z}_0(g))} (p + 1) \cdot \frac{1}{2} \text{ (resp. 3)} \]
• If a $G$-conjugacy class $\{g\}_G$ satisfies $2 \nmid D(Z_0(g))$, then two $G$-genus of $\mathbf{Z}$-orders of $Z(g)$ appear in the second sum of Theorem 4.1, and

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(Z_0(g))} (p + 1) \cdot 3 \text{ (resp. } 3/2)$$

(iii) the case where $2 \nmid D$,

• If a $G$-conjugacy class $\{g\}_G$ satisfies $2 \mid D(Z_0(g))$, then only one $G$-genus of $\mathbf{Z}$-orders of $Z(g)$ appear in the second sum of Theorem 4.1, and

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(Z_0(g))} (p + 1) \cdot \frac{3}{2}$$

• If a $G$-conjugacy class $\{g\}_G$ satisfies $2 \nmid D(Z_0(g))$, then two $G$-genus of $\mathbf{Z}$-orders of $Z(g)$ appear in the second sum of Theorem 4.1, and

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(Z_0(g))} (p + 1) \cdot \frac{3}{2} \text{ (resp. } 1)$$

(3) For any case and any $G$-genus, we have $\prod_p c_p(g, R_p, \lambda_p) = \prod_{p \mid D(Z_0(g))} 2$.

PROOF. We see from [HI80, Proposition 15 and 16] and [HI83, Proposition 2.5 and 2.6] that

\[ p \mid D_1 \implies \{g\}_G \cap R_p \neq \emptyset, \]
\[ p \nmid D_1 \text{ and } p \nmid Z_0(g) \implies \{g\}_G \cap R_p \neq \emptyset, \]
\[ p \nmid D_1, p \mid Z_0(g) \text{ and } p = 2 \implies \{g\}_G \cap R_p \neq \emptyset, \]
\[ p \nmid D_1, p \mid Z_0(g) \text{ and } p \neq 2 \implies \{g\}_G \cap R_p = \emptyset. \]

Hence we obtain (1) (cf. [Has80, Theorem 1-3]). Also we can obtain (2), (3) from the above four propositions. \( \square \)

4.7. The contribution $H_7$.

In this subsection, we consider the contribution of $G[f_7]$, where $f_7(x) = (x^2 + x + 1)^2$. We have only to double it to obtain $H_7$. Note that $G[f_7] = \emptyset$ if and only
if \( D_2 \) has a prime divisor \( p \) with \((-3/p) = 1\). Hereafter, we assume that \( D \) does not have such a prime divisor. We can use the same method as in the case of \( H_6 \), we have a correspondence between the set of \( G \)-conjugacy classes \( \{g\}_G \)'s in \( G[f_7] \) and the set of isomorphism classes of quaternion algebras \( Z_0(g) \)'s over \( Q \) which are contained in \( B \otimes_Q F \), with \( F = Q(\sqrt{-3}) \). If \( Z_0(g) \) is definite, then

\[
J'_0(g) + J'_0(g^{-1}) = c_{k,j}^{-1}3^{-1}(j + 1)[0, 1, -1; 3]_{j+2k},
\]

if \( j \) is even (cf. (b-2) in [Wak]), and

\[
M_G(\Lambda) = \frac{1}{72} \prod_{p | D(Z_0(g))} (p - 1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)}
\]

for a \( Z \)-order \( \Lambda \) of \( Z(g) \). If \( Z_0(g) \) is indefinite, then

\[
J'_0(g) = c_{k,j}^{-1}2^3 \cdot 3\pi^2(j + 2k - 3)[1, -1, 0; 3]_j,
\]

if \( j \) is even (cf. (b-3) in [Wak]), and

\[
M_G(\Lambda) = \frac{\pi^2}{32} \prod_{p | D(Z_0(g))} (p - 1) \prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)}.
\]

**Proposition 4.7.**

(1) *The class \( \{g\}_G \) appears in the first sum of Theorem 4.1 if and only if \( D(Z_0(g)) | 3D \).*

(2) *(i) the case where \( 3 | D_1 \),

*If a \( G \)-conjugacy class \( \{g\}_G \) satisfies \( 3 | D(Z_0(g)) \), then only one \( G \)-genus of \( Z \)-orders of \( Z(g) \) appears in the second sum of Theorem 4.1, and

\[
\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p | D(Z_0(g)) \atop p | D_2, \; p \neq 3} (p + 1) \cdot \frac{1}{2}
\]

*If a \( G \)-conjugacy class \( \{g\}_G \) satisfies \( 3 \nmid D(Z_0(g)) \), then two \( G \)-genus of \( Z \)-orders of \( Z(g) \) appear in the second sum of Theorem 4.1, and

\[
\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p | D(Z_0(g)) \atop p | D_2, \; p \neq 3} (p + 1) \cdot 2 \text{ (resp. 6)}
\]
(ii) the case where $3 | D_2$,

- If a $G$-conjugacy class $\{g\}_G$ satisfies $3 | D(Z_0(g))$, then only one $G$-genus of $Z$-orders of $Z(g)$ appears in the second sum of Theorem 4.1, and

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(Z_0(g))} (p+1) \cdot 1$$

- If a $G$-conjugacy class $\{g\}_G$ satisfies $3 \nmid D(Z_0(g))$, then only one $G$-genus of $Z$-orders of $Z(g)$ appears in the second sum of Theorem 4.1, and

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(Z_0(g))} (p+1) \cdot 1$$

(iii) the case where $3 \nmid D$,

- If a $G$-conjugacy class $\{g\}_G$ satisfies $3 \mid D(Z_0(g))$, then only one $G$-genus of $Z$-orders of $Z(g)$ appears in the second sum of Theorem 4.1, and

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(Z_0(g))} (p+1) \cdot \frac{1}{2}$$

- If a $G$-conjugacy class $\{g\}_G$ satisfies $3 \nmid D(Z_0(g))$, then only one $G$-genus of $Z$-orders of $Z(g)$ appears in the second sum of Theorem 4.1, and

$$\prod_p \frac{d_p(\Lambda)}{e_p(\Lambda)} = \prod_{p \mid D(Z_0(g))} (p+1) \cdot 1$$

(3) For any case and any $G$-genus, we have $\prod_p c_p(g, R_p, \lambda_p) = \prod_{p \mid D(Z_0(g))} 2$.

Proof. We can prove this by the same way as Proposition 4.6. \hfill \Box

4.8. The contribution $H_{f_8}$

In this subsection, we evaluate the contribution of $G[f_8]$, where $f_8(x) = (x^2 + 1)(x^2 + x + 1)$. We have only to double it to obtain $H_{f_8}$. We see from [HI83, Proposition 2.7] that no $G$-conjugacy classes corresponding to $f_8(\pm x)$ ap-
pear in Theorem 4.1 if $D_2 \neq 1$. In the cases where $D_2 = 1$, $H_K$ has been evaluated in [Has84] and [Wak].

4.9. The contribution $H_9$.

In this subsection, we evaluate the contribution of $G[f_9]$, where $f_9(x) = (x^2 + x + 1)(x^2 - x + 1)$. Note that $G[f_9] \neq \emptyset$ if and only if $(-3/p) \neq 1$ for any prime divisor $p$ of $D$. Hereafter, we assume that $G[f_9] \neq \emptyset$. We put $F := \mathbb{Q}(\sqrt{-3})$, then $Z(g) \cong F \oplus F$ for any $g$. We put

$$L := \{(x, y) \in \mathcal{O} \oplus \mathcal{O} \mid x - y \in 2\mathcal{O}\},$$

where $\mathcal{O}$ is the ring of integers of $F$, then we have the following proposition.

**Proposition 4.8.**

(1) If $D_2 \neq 1, 2$, then no $G$-conjugacy classes in $G[f_9]$ appear in the first sum of Theorem 4.1.

(2) If $D_2 = 2$, then the followings hold.

(i) The number of $G$-conjugacy classes in $G[f_9]$ which appear in the first sum of Theorem 4.1 is $4 \cdot 2^{\sharp D_1(2;3)}$.

(ii) Let $\{g\}_G$ be any one of them. If $\Lambda$ is a $\mathbb{Z}$-order of $Z(g)$ belonging to the same $G$-genus as $L$, then

$$\prod_p c_p(g, R_p, \Lambda_p) = 2 \cdot 2^{\sharp D_1(2;3)}.$$

If $\Lambda$ does not belong to the same $G$-genus as $L$, then $\prod_p c_p(g, R_p, \Lambda_p) = 0$.

**Proof.** We can obtain (1) and the latter part of (2 ii) from [HI83, Proposition 2.7]. For an element $g$ of $G[f_9]$, $g$ is $G_p$-conjugate to

$$\begin{cases} 
\gamma_p & \text{if } \left(\frac{-3}{p}\right) = 1 \\
\gamma_p \text{ or } \delta_p & \text{if } \left(\frac{-3}{p}\right) \neq 1
\end{cases}$$

for some elements $\gamma_p$ and $\delta_p \in G_p$. It follows from [HI80, Proposition 18] and [HI83, Proposition 2.7] that if $D_2 = 2$ and $\Lambda$ is a $\mathbb{Z}$-order of $Z(g)$ belonging to the same $G$-genus as $L$, then $c_p$ is as in the following table for each prime number $p$ satisfying the first column:
Also, \( g \) is \( \text{Sp}(2; \mathbb{R}) \)-conjugate to \( g_1 := \alpha(\pi/3, 2\pi/3), \ g_1^{-1} = \alpha(-\pi/3, -2\pi/3), \ g_2 := \alpha(\pi/3, -2\pi/3), \) and \( g_2^{-1} = \alpha(-\pi/3, 2\pi/3). \) Since \( g^2 \) belongs to \( G[f_2], \) \( g \) is \( \text{Sp}(2; \mathbb{R}) \)-conjugate to \( g_2 \) or \( g_2^{-1} \) if \( Z_0(g^2) \) is definite. We take all combinations of \( G_p \)-conjugations, and also we take \( \text{Sp}(2; \mathbb{R}) \)-conjugation out of “\( g_1 \) or \( g_1^{-1} \)” or “\( g_2 \) or \( g_2^{-1} \),” according as \( Z_0(g^2) \) is indefinite or definite. Then \( G \)-conjugacy class is determined uniquely for them by Hasse principle ([Has80, Theorem 1-2]). □

We see from Proposition 4.8 that \( H_9 = 0 \) if \( D_2 \neq 1, 2. \) In the case where \( D_2 = 1, \) \( H_2 \) has been evaluated in [Has84] and [Wak]. Hereafter, we assume \( D_2 = 2. \) We obtain from Proposition 4.8 that

\[
H_9 = c_{k, j} \cdot \sum_{\{g\} \in C} J'_0(g) \cdot M_G(L) \cdot \prod_p c_p(g, R_p, L_p),
\]

\[
\sum_{\{g\} \in C} J'_0(g) = (J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1})) \cdot 2^{\#G_1(2; 3)}.
\]

We have \( C_0(g; \text{Sp}(2; \mathbb{R})) = \{14\} \) for any \( g, \) and

\[
J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1}) = c^{-1}_{k, j} \cdot \begin{cases} [1, 0, 0, -1, 0, 0; 6]_k & \text{if } j \equiv 0 \text{ mod } 6 \\ [-1, 1, 0, 1, -1, 0; 6]_k & \text{if } j \equiv 2 \text{ mod } 6 \\ [0, -1, 0, 0, 1, 0; 6]_k & \text{if } j \equiv 4 \text{ mod } 6 \end{cases}
\]

(cf. (b-1) in [Wak]). We have

\[
M_G(L) = \frac{1}{12}, \quad \text{(cf. (3.21) in [Has84])}
\]

\[
\prod_p c_p(g, R_p, L_p) = 2 \cdot 2^{\#G_1(2; 3)}
\]
for any \( g \). Hence we can obtain \( H_9 \) as in Theorem 3.1.

4.10. The contribution \( H_{10} \).

In this subsection, we evaluate the contribution of \( G[f_{10}] \), where \( f_{10}(x) = x^4 + x^3 + x^2 + x + 1 \). We have only to double it to obtain \( H_{10} \). Note that if \( D(1; 5) \neq \emptyset \), then \( G[f_{10}] = \emptyset \). Hereafter, we assume \( D(1; 5) = \emptyset \). We have \( Z(g) = \mathbb{Q}(g) \simeq \mathbb{Q}(\zeta_5) \) for any \( g \). We have the following proposition.

**Proposition 4.9.**

(1) If \( D_1(2; 5) \sqcup D_1(3; 5) \sqcup D_2(4; 5) \neq \emptyset \), then no \( G \)-conjugacy classes in \( G[f_{10}] \) appear in the first sum of Theorem 4.1.

(2) If \( D_1(2; 5) \sqcup D_1(3; 5) \sqcup D_2(4; 5) = \emptyset \), then the followings hold.

(i) The number of \( G \)-conjugacy classes in \( G[f_{10}] \) which appear in the first sum of Theorem 4.1 is \( 4 \cdot 2^{D_1(4; 5)} \).

(ii) Let \( \{g\}_G \) be any one of them. If \( \Lambda \) is a \( \mathbb{Z} \)-order of \( Z(g) \) belonging to the same \( G \)-genus as \( \mathcal{O} \), where \( \mathcal{O} \) is the ring of integers of \( Z(g) \), then

\[
\prod_p c_p(g, R_p, \Lambda_p) = 2^{2D_1(4; 5)} \cdot 2^{2D_2(2; 5)} \cdot 2^{2D_2(3; 5)}.
\]

If \( \Lambda \) does not belong to the same \( G \)-genus as \( \mathcal{O} \), then \( \prod_p c_p(g, R_p, \Lambda_p) = 0 \).

**Proof.** We can obtain (1) and the latter part of (2 ii) from [HI80, Proposition 19] and [HI83, Proposition 2.8]. For any element \( g \) of \( G[f_{10}] \), \( g \) is \( G_p \)-conjugate to

\[
\begin{cases}
\gamma_p & \text{if } p \notin D_1(4; 5) \\
\gamma_p \text{ or } \delta_p & \text{if } p \in D_1(4; 5)
\end{cases}
\]

for some elements \( \gamma_p \) and \( \delta_p \in G_p \). It follows from [HI80, Proposition 19] and [HI83, Proposition 2.8] that if \( D_1(2; 5) \sqcup D_1(3; 5) \sqcup D_2(4; 5) = \emptyset \) and \( \Lambda \) is a \( \mathbb{Z} \)-order of \( Z(g) \) belonging to the same \( G \)-genus as \( \mathcal{O} \), then \( c_p \) is as in the following table for each prime number \( p \) satisfying the first column:
Also, $g$ is $Sp(2; \mathbb{R})$-conjugate to $g_1 := \alpha(2\pi/5, 4\pi/5)$, $g_1^{-1} = \alpha(-2\pi/5, -4\pi/5)$, $g_2 := \alpha(2\pi/5, -4\pi/5)$, and $g_2^{-1} = \alpha(-2\pi/5, 4\pi/5)$. We can take all combinations of $Sp(2; \mathbb{R})$-conjugation and $G_p$-conjugations. \(\square\)

We see from Proposition 4.9 that $H_{10} = 0$ if $D_1(2; 5) \sqcup D_1(3; 5) \sqcup D_2(4; 5) \neq \emptyset$. In the other cases, we obtain from Proposition 4.9 that

$$H_{10} = c_{k,j} \cdot \sum_{\{g\} \subset \mathcal{G}} J'_0(g) \cdot M_{\mathcal{G}}(\mathcal{G}) \cdot \prod_p c_p(g, R_p, \mathcal{O}_p),$$

$$\sum_{(g) \subset \mathcal{G}} J'_0(g) = \left( J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1}) \right) \cdot 2^{2D_1(4; 5)}.$$

We have $C_0(g; Sp(2; \mathbb{R})) = \{14\}$ for any $g$, and

$$J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1}) = \begin{cases} [1, 0, 0, -1, 0; 5]_k & \text{if } j \equiv 0 \text{ mod } 10 \\ [-1, 1, 0, 0, 0; 5]_k & \text{if } j \equiv 2 \text{ mod } 10 \\ 0 & \text{if } j \equiv 4 \text{ mod } 10 \\ [0, 0, 0, 1, -1; 5]_k & \text{if } j \equiv 6 \text{ mod } 10 \\ [0, -1, 0, 0, 1; 5]_k & \text{if } j \equiv 8 \text{ mod } 10 \end{cases}.$$ (cf. (b-1) in [Wak]). We have
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\[ M_G(\mathcal{O}) = \frac{1}{10}, \quad \text{(cf. (3.23) in [Has84])} \]

\[ \prod_p c_p(g, R_p, \mathcal{O}_p) = 2^{\sharp D_1(4;5)} \cdot 2^{\sharp D_2(2;5)} \cdot 2^{\sharp D_2(3;5)} \]

for any \( g \). Hence we can obtain \( H_{10} \) as in Theorem 3.1.

4.11. The contribution \( H_{11} \).

In this subsection, we evaluate the contribution of \( G[f_{11}] \), where \( f_{11}(x) = x^4 + 1 \). Note that if \( D(1;8) \neq \emptyset \), then \( G[f_{11}] = \emptyset \). Hereafter, we assume \( D(1;8) = \emptyset \). We have \( Z(g) = Q(g) \simeq Q(\zeta_8) \) for any \( g \). We have the following proposition.

**Proposition 4.10.**

1. If \( D_2(7;8) \neq \emptyset \), then no \( G \)-conjugacy classes of \( G[f_{11}] \) appear in the first sum of Theorem 4.1.
2. If \( D_2(7;8) = \emptyset \), then the followings hold.
   
   (i) The number of \( G \)-conjugacy classes in \( G[f_{11}] \) which appear in the first sum of Theorem 4.1 is \( 4 \cdot 2^{\sharp D_1(7;8)} \).
   
   (ii) Let \( \{g\}_G \) be any one of them. If \( \Lambda \) is a \( \mathbb{Z} \)-order of \( Z(g) \) belonging to the same \( G \)-genus as \( \mathcal{O} \), where \( \mathcal{O} \) is the ring of integers of \( Z(g) \), then

\[ \prod_p c_p(g, R_p, \Lambda_p) = \prod_{p \parallel D} 2. \]

If \( \Lambda \) does not belong to the same \( G \)-genus as \( \mathcal{O} \), then \( \prod_p c_p(g, R_p, \Lambda_p) = 0 \).

**Proof.** We can prove (1) and the latter part of (2 ii) easily by [HI83, Proposition 2.9]. For any element \( g \) of \( G[f_{11}] \), \( g \) is \( G_p \)-conjugate to

\[
\begin{cases}
  \gamma_p & \text{if } p \equiv 1, 3 \text{ or } 5 \text{ mod } 8 \\
  \gamma_p \text{ or } \delta_p & \text{if } p = 2 \text{ or } p \equiv 7 \text{ mod } 8
\end{cases}
\]

for some elements \( \gamma_p \) and \( \delta_p \in G_p \). It follows from [HI80, Proposition 20] and [HI83, Proposition 2.9] that if \( \Lambda \) is a \( \mathbb{Z} \)-order of \( Z(g) \) belonging to the same \( G \)-genus as \( \mathcal{O} \), then \( c_p \) is as in the following table for each prime number \( p \) satisfying the first column:
(i) If $p \mid D_1$, then we have

|       | $c_p(\gamma_p, R_p, \mathcal{O}_p)$ | $c_p(\delta_p, R_p, \mathcal{O}_p)$ |
|-------|---------------------------------|---------------------------------|
| $p \equiv 3 \mod 8$ | 2 | $\times$ |
| $p \equiv 5 \mod 8$ | 2 | $\times$ |
| $p \equiv 7 \mod 8$ | 2 | 2 |
| $p = 2$ | 1 | 1 |

(ii) If $p \mid D_2$, then we have

|       | $c_p(\gamma_p, R_p, \mathcal{O}_p)$ | $c_p(\delta_p, R_p, \mathcal{O}_p)$ |
|-------|---------------------------------|---------------------------------|
| $p \equiv 3 \mod 8$ | 2 | $\times$ |
| $p \equiv 5 \mod 8$ | 2 | $\times$ |
| $p \equiv 7 \mod 8$ | 0 | 0 |
| $p = 2$ | 1 | 1 |

(iii) If $p \nmid D$, then we have

|       | $c_p(\gamma_p, R_p, \mathcal{O}_p)$ | $c_p(\delta_p, R_p, \mathcal{O}_p)$ |
|-------|---------------------------------|---------------------------------|
| $p \equiv 1 \mod 8$ | 1 | $\times$ |
| $p \equiv 3 \mod 8$ | 1 | $\times$ |
| $p \equiv 5 \mod 8$ | 1 | $\times$ |
| $p \equiv 7 \mod 8$ | 1 | 0 |
| $p = 2$ | 1 | 1 |

Also, $g$ is $Sp(2; \mathcal{R})$-conjugate to $g_1 := \alpha(\pi/4, 3\pi/4)$, $g_1^{-1} = \alpha(-\pi/4, -3\pi/4)$, $g_2 := \alpha(\pi/4, -3\pi/4)$, or $g_2^{-1} = \alpha(-\pi/4, 3\pi/4)$. Since $g^2$ belongs to $G[f_0]$, $g$ is $Sp(2; \mathcal{R})$-conjugate to $g_1$ or $g_1^{-1}$ if $Z_0(g^2)$ is indefinite, and $g$ is $Sp(2; \mathcal{R})$-conjugate to $g_2$ or $g_2^{-1}$ if $Z_0(g^2)$ is definite. We take all combinations of $G_p$-conjugacy classes for all $p$, and also take $Sp(2; \mathcal{R})$-conjugation out of “$g_1$ or $g_1^{-1}$” or “$g_2$ or $g_2^{-1}$”, according as $Z_0(g^2)$ is indefinite or definite. □

We see from Proposition 4.10 that $H_{11} = 0$ if $D_2(7; 8) \neq \emptyset$. Hereafter, we assume $D_2(7; 8) = \emptyset$. We obtain from Proposition 4.10 that

$$H_{11} = c_{k, j} \cdot \sum_{\{g\} \alpha} J'_0(g) \cdot M_G(\mathcal{O}) \cdot \prod_p c_p(g, R_p, \mathcal{O}_p),$$
\[
\sum_{g \in G} J'_0(g) = \left( J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1}) \right) \cdot 2^\sharp D_1(7;8). 
\]

We have \( C_0(g; Sp(2; \mathbb{R})) = \{14\} \) for any \( g \), and

\[
J'_0(g_1) + J'_0(g_1^{-1}) + J'_0(g_2) + J'_0(g_2^{-1}) = \begin{cases} 
[1, 0, 0, -1; 4]_k & \text{if } j \equiv 0 \mod 8 \\
[-1, 1, 0, 0; 4]_k & \text{if } j \equiv 2 \mod 8 \\
[-1, 0, 0, 4]_k & \text{if } j \equiv 4 \mod 8 \\
[1, -1, 0, 0; 4]_k & \text{if } j \equiv 6 \mod 8 
\end{cases}.
\]

(cf. (b-1) in [Wak]). We have

\[
M_G(\mathcal{O}) = \frac{1}{8}, \quad \text{(cf. (3.25) in [Has84])}
\]

\[
\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_{p \mid D \atop p \neq 2} 2
\]

for any \( g \). Hence we can obtain \( H_{11} \) as in Theorem 3.1.

### 4.12. The contribution \( H_{12} \).

In this subsection, we evaluate the contribution of \( G[f_{12}] \), where \( f_{12}(x) = x^4 - x^2 + 1 \). Note that \( G[f_{12}] = \emptyset \) if and only if \( D(1; 12) \neq \emptyset \). Hereafter, we assume \( D(1; 12) = \emptyset \). We see from [HI80, Proposition 21] and [HI83, Proposition 2.10] that if \( D_2(11; 12) \neq \emptyset \), then no \( G \)-conjugacy classes of \( G[f_{12}] \) appear in the first sum of Theorem 4.1. Hereafter, we assume that \( D_2(11; 12) = \emptyset \).

The set \( G[f_{12}] \) consists of four \( Sp(2; \mathbb{R}) \)-conjugacy classes represented by

\[
\begin{align*}
& h := \alpha(\pi/6, 5\pi/6), \quad h^{-1} = \alpha(-\pi/6, -5\pi/6), \\
& h' := \alpha(\pi/6, -5\pi/6), \quad h'^{-1} = \alpha(-\pi/6, 5\pi/6).
\end{align*}
\]

We have \( C_0(g; Sp(2; \mathbb{R})) = \{14\} \) for any \( g \in G[f_{12}] \) and

\[
\begin{align*}
& J'_0(h) + J'_0(h^{-1}) = (-1)^{j/2+k} \cdot [1, -1, 0; 3]_{j+2k}, \\
& J'_0(h') + J'_0(h'^{-1}) = (-1)^{j/2} \cdot [0, -1, 1; 3]_{j+2k}.
\end{align*}
\]

(cf. (b-1) in [Wak]). Only one \( G \)-genus represented by \( \mathcal{O} \), where \( \mathcal{O} \) is the ring of integers of \( Z(g) \cong \mathbb{Q}(\zeta_{12}) \), appears in the second sum of Theorem 4.1 and \( M_G(\mathcal{O}) = 1/12 \) (cf. (3.27) in [Has84]). If \( g \) is an element of \( G[f_{12}] \), then \( g^2 \) belongs to \( G[f_7] \). We can obtain the following proposition from [HI80, Proposition 21] and [HI83, Proposition 2.10].
Proposition 4.11.

(i) the case where $D_1(11; 12) = \emptyset$ and $\sharp D(5; 12)$ is even (resp. odd)

Two $G$-conjugacy classes $\{g\}_G$ and $\{g^{-1}\}_G$ appear in the first sum of Theorem 4.1. They are $\text{Sp}(2; R)$-conjugate to $h'$ and $h'^{-1}$ (resp. $h$ and $h^{-1}$) respectively, and

$$\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = \prod_{p \in D(5; 12)} 2 \cdot \prod_{p \in D(7; 12)} 2.$$ 

(ii) the case where $2 \mid D_2$ and $3 \nmid D_1$:

Two $G$-conjugacy classes $\{g\}_G$ and $\{g^{-1}\}_G$ appear in the first sum of Theorem 4.1. They are $\text{Sp}(2; R)$-conjugate to $h$ and $h^{-1}$ (resp. $h'$ and $h'^{-1}$) respectively, and

$$\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = \prod_{p \in D(5; 12)} 2 \cdot \prod_{p \in D(7; 12)} 2.$$ 

(iii) the case where $2 \nmid D_1$ and $3 \mid D_1$:

Four $G$-conjugacy classes appear in the first sum of Theorem 4.1. They are $\text{Sp}(2; R)$-conjugate to $h, h^{-1}, h', h'^{-1}$ respectively. If $g$ is $\text{Sp}(2; R)$-conjugate to $h$ (resp. $h'$), then

$$\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = \prod_{p \in D(5; 12)} 2 \cdot \prod_{p \in D(7; 12)} 2.$$ 

If $g$ is $\text{Sp}(2; R)$-conjugate to $h'$ (resp. $h$), then

$$\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = 2 \cdot \prod_{p \in D(5; 12)} 2 \cdot \prod_{p \in D(7; 12)} 2.$$ 

(iv) the case where $2 \mid D_1$ and $3 \mid D_1$:

Four $G$-conjugacy classes appear in the first sum of Theorem 4.1. They are $\text{Sp}(2; R)$-conjugate to $h, h^{-1}, h', h'^{-1}$ respectively. If $g$ is $\text{Sp}(2; R)$-conjugate to $h$ (resp. $h'$), then

$$\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = 2 \cdot \prod_{p \in D(5; 12)} 2 \cdot \prod_{p \in D(7; 12)} 2.$$ 

If $g$ is $\text{Sp}(2; R)$-conjugate to $h'$ (resp. $h$), then

$$\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = 2 \cdot \prod_{p \in D(5; 12)} 2 \cdot \prod_{p \in D(7; 12)} 2.$$
If $g$ is $\text{Sp}(2; \mathbb{R})$-conjugate to $h'$ (resp. $h$), then

$$
\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2.
$$

(v) the case where $2 \mid D_1$ and $3 \nmid D_1$:

Four $G$-conjugacy classes appear in the first sum of Theorem 4.1. They are $\text{Sp}(2; \mathbb{R})$-conjugate to $h, h^{-1}, h', h'^{-1}$ respectively.

If $g$ is $\text{Sp}(2; \mathbb{R})$-conjugate to $h'$ (resp. $h$), then

$$
\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = 2 \cdot \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2.
$$

(vi) the case where $2 \mid D_1$ and $3 \mid D_1$:

Eight $G$-conjugacy classes appear in the first sum of Theorem 4.1. Each two of them are $\text{Sp}(2; \mathbb{R})$-conjugate to $h, h^{-1}, h', h'^{-1}$ respectively.

If $g$ is $\text{Sp}(2; \mathbb{R})$-conjugate to $h'$ (resp. $h$), then

$$
\prod_p c_p(g, R_p, \mathcal{O}_p) = \prod_p c_p(g^{-1}, R_p, \mathcal{O}_p) = \begin{cases} 
2 \cdot \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \\
2 \cdot \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \\
2 \cdot \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \\
4 \cdot \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 
\end{cases}
$$

(II) the case where $D_1(11;12) \neq \emptyset$

(i) the case where $2 \nmid D_1$ and $3 \nmid D_1$:

The number of $G$-conjugacy classes which appear in the first sum of Theorem 4.1 is $2^{D_1(11;12)+1}$. They are $\text{Sp}(2; \mathbb{R})$-conjugate to $h, h^{-1}, h', h'^{-1}$. All of them satisfy
\[ \prod_p c_p(g, R_p, \sigma_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2. \]

(ii) the case where \(2 \mid D_1\) and \(3 \nmid D_1\) or \(2 \nmid D_1\) and \(3 \mid D_1\):

The number of \(G\)-conjugacy classes which appear in the first sum of Theorem 4.1 is \(2^{\sharp D_1(11;12)+2}\). They are \(Sp(2; R)\)-conjugate to \(h, h^{-1}, h', h'^{-1}\). In each case, \(2^{\sharp D_1(11;12) - 1}\) \(G\)-conjugacy classes satisfy

\[ \prod_p c_p(g, R_p, \sigma_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2 \cdot 2, \]

\(2^{\sharp D_1(11;12) - 1}\) \(G\)-conjugacy classes satisfy

\[ \prod_p c_p(g, R_p, \sigma_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2. \]

(iii) the case where \(2 \mid D_1\) and \(3 \mid D_1\):

The number of \(G\)-conjugacy classes which appear in the first sum of Theorem 4.1 is \(2^{\sharp D_1(11;12)+3}\). They are \(Sp(2; R)\)-conjugate to \(h, h^{-1}, h', h'^{-1}\). In each case, \(2^{\sharp D_1(11;12) - 1}\) \(G\)-conjugacy classes satisfy

\[ \prod_p c_p(g, R_p, \sigma_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2 \cdot 2^2, \]

\(2^{\sharp D_1(11;12)}\) \(G\)-conjugacy classes satisfy

\[ \prod_p c_p(g, R_p, \sigma_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2 \cdot 2, \]

\(2^{\sharp D_1(11;12) - 1}\) \(G\)-conjugacy classes satisfy

\[ \prod_p c_p(g, R_p, \sigma_p) = \prod_{p \in D(5;12)} 2 \cdot \prod_{p \in D(7;12)} 2 \cdot \prod_{p \in D_1(11;12)} 2. \]

5. The contribution of non-semi-simple conjugacy classes.

In this section, we evaluate \(I(\Gamma^{(u)})_{k,j}\) and \(I(\Gamma^{(qu)})_{k,j}\), i.e. the contributions of non-semi-simple conjugacy classes (cf. Section 1). We prove \(I_1\), \(I_2\) and \(I_3\) of Theorem 3.1. Since the class number of \(\mathfrak{O}\) is one, any maximal two-sided ideal \(\mathfrak{A}\) can be written as \(\mathfrak{A} = \mathfrak{O}\pi = \pi\mathfrak{O}\) for some \(\pi \in \mathfrak{O}\). By taking conjugation by
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\((\pi_{00}, \pi_{00}) \in G\), we may regard \(\Gamma = G \cap (\mathbb{D}, \mathfrak{a} \overline{\mathfrak{a}})\).

We put

\[
P = \left\{ \left( \begin{array}{cc} a & 0 \\ \overline{a}^{-1} & 1 \end{array} \right) \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \left| a \in B^\times, b \in B^0 \right. \right\}.
\]

Then \(P\) is the unique parabolic subgroup of \(G\) up to \(G\)-conjugation since we consider the case where \(B \neq M_2(Q)\) in this paper. We can prove that \(\Gamma \setminus \mathfrak{f}_2\) has only one 0-dimensional cusp, up to equivalence, in the same way as \([\text{Ara81}, \text{Proposition 2}]\). Arakawa proved Lemma 5.1 below in his master thesis \([\text{Ara75}, \text{Proposition 7}]\).

**Lemma 5.1.** We have \(G = P \cdot \Gamma\).

**Proof.** We take any \(\gamma = (a, b) \in G\). There are some \(\gamma, \delta \in \mathfrak{D}\) such that \(c^{-1}d = \pi^{-1}\gamma^{-1}\delta\). We can assume that there are some \(u, v \in \mathfrak{D}\) such that \(\gamma u + \delta v = 1\). If we put

\[
\tau := \left( \begin{array}{cc} \overline{\pi} & -\overline{\pi}^{-1}uv\gamma \pi \\ \gamma \pi & \overline{\pi}^{-1}u(1 - v\delta) \end{array} \right).
\]

then we have \(\tau \in \Gamma\) and \(\sigma \tau^{-1} \in P\). \(\square\)

By using Lemma 5.1, we can prove Proposition 5.2 below in the same way that Hashimoto proved it when \(D_2 = 1\) in \([\text{Has84}, \text{Lemma 1.2}]\).

**Proposition 5.2.** If \(\gamma\) is an element of \(\Gamma^{(u)} \sqcup \Gamma^{(qu)}\), then \(\gamma\) is \(\Gamma\)-conjugate to an element of the form:

\[
\gamma(a, b) = \left( \begin{array}{cc} a & 0 \\ 0 & a \end{array} \right) \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right),
\]

where \(a \in \mathfrak{D}^\times\) is a root of unity and \(b \in \mathfrak{D}^0 - \{0\}\).

If \(\gamma \in \Gamma^{(u)}\), then \(a = \pm 1\) and the principal polynomial of \(\gamma\) is \(f_1(x) = (x - 1)^4\) or \(f_1(-x)\). We put \(I_1 = I(\Gamma^{(u)})_{k,j}\). If \(\gamma \in \Gamma^{(qu)}\), then \(a\) is a primitive 4-th, 3-rd, or 6-th root of unity and the principal polynomial of \(\gamma\) is \(f_6(x) = (x^2 + 1)^2\), \(f_7(x) = (x^2 + x + 1)^2\) or \(f_7(-x)\) respectively. We denote by \(I_2\) (resp. \(I_3\)) the contribution of elements of \(\Gamma^{(qu)}\) whose principal polynomial is \(f_6(x)\) (resp. \(f_7(\pm x)\)). We evaluate \(I_1\) in Subsection 5.1, and \(I_2\) and \(I_3\) in Subsection 5.2 and 5.3. We use the notation
\[
\gamma(\theta, t) := \begin{pmatrix}
\cos \theta & \sin \theta & 0 & 0 \\
-\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & -\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
1 & 0 & t & 0 \\
0 & 1 & 0 & t \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

We summarize some lemmas which are used in Subsection 5.2. These lemmas were proved in the case of \(D_2 = 1\) by Hashimoto [Has84]. The following lemmas are easy generalizations of them and can be proved in the almost same method, so we omit the proof. Let \(a\) be a primitive 3-rd or 4-th or 6-th root of unity. We put \(F := \mathbb{Q}(a)\) and denote by \(O_F\) the ring of integers of \(F\) and by \(d\) the discriminant of \(F\). \(F\) is isomorphic to \(\mathbb{Q}(\sqrt{-1})\) or \(\mathbb{Q}(\sqrt{-3})\).

**Lemma 5.3.** Let \(\gamma\) be an element of \(\Gamma\) of the form \(\gamma(a, b)\) in Proposition 5.2. Then we have

1. If \(\beta\) is an element of \(B^\times\) such that \(\beta a = \overline{a}\beta\), then we have \(B = F \oplus F\beta\).
2. If we express \(b \in \mathfrak{A}^0\) as \(b = x\sqrt{d} + y\beta\) \((x \in \mathbb{Q}, y \in F)\), then the Jordan decomposition \(\gamma(a, b) = \gamma_s \cdot \gamma_u\) is given by

\[
\gamma_s = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & y\beta \\ 0 & 1 \end{pmatrix}, \quad \gamma_u = \begin{pmatrix} 1 & x\sqrt{d} \\ 0 & 1 \end{pmatrix}.
\]

**Lemma 5.4.** If we put, for a fixed \(a\) as above,

\[
C(a) := \{x^{-1}ax \mid x \in B^\times\},
\]

then we have

\[
\#((C(a) \cap \mathfrak{D})/ \sim_{\mathfrak{D}^\times}) = \prod_{p|D} \left(1 - \left(\frac{F}{p}\right)\right).
\]

**Lemma 5.5.** Let \(\gamma_i = \gamma(a_i, b_i)\) \((i = 1, 2)\) be two elements of \(\Gamma^{qu}\) of the form of Proposition 5.2. If \(\gamma_1\) and \(\gamma_2\) are \(\Gamma\)-conjugate, then \(a_1\) and \(a_2\) are \(\mathfrak{D}^\times\)-conjugate.

**Lemma 5.6.** Let \(\gamma_i = \gamma(a_i, b_i)\) \((i = 1, 2)\) be two elements of \(\Gamma^{(qu)}\) of the form of Proposition 5.2. We put

\[
L_\mathfrak{A}(a) := \{a^{-1}za - z \mid z \in \mathfrak{A}^0\}.
\]

Then \(\gamma(a, b_1)\) and \(\gamma(a, b_2)\) are \(\Gamma\)-conjugate if and only if \(b_1 - b_2 \in L_\mathfrak{A}(a)\).
5.1. The contribution $I_1$.

In this subsection, we evaluate the contribution $I_1$. We define the following four subsets of $\Gamma$:

$F_1 := \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \mid S \in SM_2(\mathbf{R}), \det S \neq 0, S \text{ : definite} \right\} \cap \Gamma,$

$F_2 := \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \mid S \in SM_2(\mathbf{R}), \det S \neq 0, S \text{ : indefinite}, -\det S \notin (\mathbb{Q}^\times)^2 \right\} \cap \Gamma,$

$F_3 := \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \mid S \in SM_2(\mathbf{R}), \det S \neq 0, S \text{ : indefinite}, -\det S \in (\mathbb{Q}^\times)^2 \right\} \cap \Gamma,$

$F_4 := \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \mid S \in SM_2(\mathbf{R}), \det S = 0 \right\} \cap \Gamma,$

where we denote by $SM_2(\mathbf{R})$ the set of all symmetric matrices of degree 2 over $\mathbf{R}$. We denote by $C(u)$ the set of all $\Gamma$-conjugacy classes of $\Gamma(u)$. We can prove the following proposition by Proposition 5.2.

**Proposition 5.7.** We can decompose $C(u)$ as

$$C(u) = \bigcup_{i=1}^{4} \left( \bigcup_{\gamma \in F_i/\sim \Gamma} \{\gamma\}_\Gamma \right),$$

where $F_i/\sim \Gamma$ denotes a complete system of representatives of $\Gamma$-conjugacy classes of $F_i$ and $\{\gamma\}_\Gamma$ denotes the $\Gamma$-conjugacy class represented by $\gamma$.

**Proof.** Take an arbitrary $\{\gamma'\}_\Gamma \in C_u$. By Proposition 5.2, we have some $x \in \Gamma$ such that $x^{-1}\gamma'x = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{R}^0 - \{0\}$. Identifying $x^{-1}\gamma'x$ and its image by $\phi$ in $Sp_2(\mathbf{R})$, it is contained in some $F_i$, so we have

$$\{\gamma'\}_\Gamma = \{x^{-1}\gamma'x\}_\Gamma \subseteq \bigcup_{\gamma \in F_i/\sim \Gamma} \{\gamma\}_\Gamma.$$

However, especially in the case of our $\Gamma$, we have $F_3 = F_4 = \emptyset$ and

$$I_1 = c_{k,j} \cdot \sum_{i=1}^{2} \text{vol}\left( C_0(\gamma_i; \Gamma) \setminus C_0(\gamma_i; Sp(2; \mathbf{R})) \right)$$

$$\cdot \lim_{s \to +0} \sum_{\gamma' \in F_i/\sim} \frac{J_0(\gamma'; s)}{C(\gamma'; \Gamma) : \pm C_0(\gamma'; \Gamma)},$$
where $\gamma_i$ is an any element of $F_i$ (cf. [Wak, Theorem 3.1]). By using the formula of [Wak, (e-2), (e-3)], we have

$$
\lim_{s \to +0} \sum_{\gamma' \in F_i/\sim} \frac{J_0(\gamma'; s)}{[C(\gamma'; \Gamma) : \pm C_0(\gamma'; \Gamma)]} = \begin{cases} 
\frac{c_{k,j}^{-1}}{2^{2\pi}} \cdot \frac{j+1}{[\Gamma : \Gamma_+]} \cdot \frac{1}{\text{vol}(\tilde{\Gamma}_+ \setminus \mathcal{F}_1)} \cdot \frac{\text{vol}(L)}{\text{vol}(L_i)} & i = 1 \\
0 & i = 2 
\end{cases}
$$

Here, we define the notations as follows.

We define a lattice $L$ in $SM_2(R)$ by

$$
\left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mid X \in L \right\} = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \mid S \in SM_2(R) \right\} \cap \Gamma.
$$

We put

$$
C_0(\gamma_1; Sp(2; R)) = \left\{ \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \mid S \in SM_2(R) \right\}.
$$

$$
C_0(\gamma_1; \Gamma) = C_0(\gamma_1; Sp(2; R)) \cap \Gamma = \left\{ \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \mid X \in L \right\}
$$

and

$$
\text{vol}(L) := \text{vol}(C_0(\gamma_1; \Gamma) \setminus C_0(\gamma_1; G(R))) = \int_{L \setminus SM_2(R)} dx_{11} dx_{12} dx_{22}
$$

for $\left( \begin{array}{cc} x_{11} & x_{12} \\ x_{12} & x_{22} \end{array} \right) \in SM_2(R)$. We put

$$
\tilde{\Gamma} = \left\{ \begin{pmatrix} x & 0 \\ 0 & \bar{x}^{-1} \end{pmatrix} \mid x \in B^\times \right\} \cap \Gamma, \quad \tilde{\Gamma}_+ = \left\{ \begin{pmatrix} x & 0 \\ 0 & \bar{x}^{-1} \end{pmatrix} \in \tilde{\Gamma} \mid \bar{x} > 0 \right\}.
$$

We can identify $\tilde{\Gamma}_+$ as the subgroup of $GL_+(2; R) = \{ g \in GL(2; R) \mid \det(g) > 0 \}$ and we define

$$
\text{vol}(\tilde{\Gamma}_+ \setminus \mathcal{F}_1) = \int_{\tilde{\Gamma}_+ \setminus \mathcal{F}_1} y^{-2} dy dx
$$
for $x + iy \in \mathcal{H}_1$, where $\mathcal{H}_1$ is the upper half plane $\{z \in \mathbb{C} | \text{Im}(z) > 0\}$.

It follows that we have

$$I_1 = \frac{j + 1}{2^{3\pi}} \cdot \frac{\text{vol}(\tilde{\Gamma}_+ \backslash \mathcal{H}_1)}{[\tilde{\Gamma} : \tilde{\Gamma}_+]}. $$

Noting that $\tilde{\Gamma}$ and $\tilde{\Gamma}_+$ are independent on a choice of pairs $(D_1, D_2)$ for a fixed $D$, we see that the value $I_1$ is also independent on it. Hence we have

$$I_1 = 2^{-3}3^{-1}(j + 1) \prod_{p \mid D} (p - 1),$$

which is the same value as in [Wak, Theorem 6.1].

5.2. The contribution $I_2$.

In this section, we evaluate the contribution $I_2$. Let $\gamma$ be an element of $\Gamma^{(qu)}$ whose principal polynomial is $f_6(x) = (x^2 + 1)^2$. Then $\gamma$ is $\text{Sp}(2; \mathbb{R})$-conjugate to an element of the form

$$\gamma \left( \frac{\pi}{2}, s \right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(cf. Proposition 5.2) and corresponds to (1-3) of [Wak].

We denote by $C_6$ the set of all $\Gamma$-conjugacy classes of $\Gamma^{(qu)}$ whose principal polynomial is $f_6(x)$. Then we have the following proposition:

**Proposition 5.8.** We can decompose $C_6$ into disjoint union of $4N$ subsets as

$$C_6 = \bigcup_{i=1}^{N} \bigcup_{j=1}^{4} \bigcup_{\gamma \in F_{i,j}} \{\{\gamma\}\Gamma\},$$

where $N := \prod_{p \mid D} (1 - (-1/p))$ and $F_{i,j}$ is defined as follows.

Let $a_1, \ldots, a_N$ be a complete system of $\mathcal{O}^\times$-conjugacy classes of elements of $\mathcal{O}$ of order 4. (cf. Lemma 5.4). There exist some $x_i \in \mathbb{Q}_{>0}$ and $\beta_i \in \mathcal{O}^0$ depending on each $a_i$ such that $F_{i,j}$'s are given as one of the following four cases. Here we put
\[ \delta(a_i, \gamma_1, \gamma_2) := \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 1 & \gamma_1 \beta_i \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \gamma_2 \sigma_i a_i \\ 0 & 1 \end{pmatrix}, \]

where the symbol “\( \cdot \)” means the Jordan decomposition.

Case 1:

\[ F_{i,1} = \{ \delta(a_i, 0, l) \mid l \in \mathbb{Z} - \{0\} \}, \quad F_{i,2} = \{ \delta(a_i, 1, l) \mid l \in \mathbb{Z} - \{0\} \} \]
\[ F_{i,3} = \{ \delta(a_i, a_i, l) \mid l \in \mathbb{Z} - \{0\} \}, \quad F_{i,4} = \{ \delta(a_i, 1 + a_i, l) \mid l \in \mathbb{Z} - \{0\} \} \]

All elements of \( F_{i,j} \) are conjugate to \( \gamma(\pi/2, l) \) in \( Sp(2, \mathbb{R}) \).

Case 2:

\[ F_{i,1} = \{ \delta(a_i, 0, 2l) \mid l \in \mathbb{Z} - \{0\} \}, \quad F_{i,2} = \{ \delta(a_i, 1, 2l) \mid l \in \mathbb{Z} - \{0\} \} \]
\[ F_{i,3} = \left\{ \delta \left( a_i, \frac{1}{2} a_i, 2l + 1 \right) \mid l \in \mathbb{Z} \right\}, \quad F_{i,4} = \left\{ \delta \left( a_i, 1 + \frac{1}{2} a_i, 2l + 1 \right) \mid l \in \mathbb{Z} \right\} \]

All elements of \( F_{i,1} \) and \( F_{i,2} \) are conjugate to \( \gamma(\pi/2, l) \) in \( Sp(2, \mathbb{R}) \). All elements of \( F_{i,3} \) and \( F_{i,4} \) are conjugate to \( \gamma(\pi/2, l + (1/2)) \) in \( Sp(2, \mathbb{R}) \).

Case 3:

\[ F_{i,1} = \{ \delta(a_i, 0, 2l) \mid l \in \mathbb{Z} - \{0\} \}, \quad F_{i,2} = \{ \delta(a_i, a_i, 2l) \mid l \in \mathbb{Z} - \{0\} \} \]
\[ F_{i,3} = \left\{ \delta \left( a_i, \frac{1}{2} a_i, 2l + 1 \right) \mid l \in \mathbb{Z} \right\}, \quad F_{i,4} = \left\{ \delta \left( a_i, \frac{1}{2} + a_i, 2l + 1 \right) \mid l \in \mathbb{Z} \right\} \]

All elements of \( F_{i,1} \) and \( F_{i,2} \) are conjugate to \( \gamma(\pi/2, l) \) in \( Sp(2, \mathbb{R}) \). All elements of \( F_{i,3} \) and \( F_{i,4} \) are conjugate to \( \gamma(\pi/2, l + (1/2)) \) in \( Sp(2, \mathbb{R}) \).

Case 4:

\[ F_{i,1} = \{ \delta(a_i, 0, 2l) \mid l \in \mathbb{Z} - \{0\} \}, \quad F_{i,2} = \{ \delta(a_i, 1, 2l) \mid l \in \mathbb{Z} - \{0\} \} \]
\[ F_{i,3} = \left\{ \delta \left( a_i, \frac{1}{2} + \frac{1}{2} a_i, 2l + 1 \right) \mid l \in \mathbb{Z} \right\}, \quad F_{i,4} = \left\{ \delta \left( a_i, \frac{1}{2} + \frac{3}{2} a_i, 2l + 1 \right) \mid l \in \mathbb{Z} \right\} \]

All elements of \( F_{i,1} \) and \( F_{i,2} \) are conjugate to \( \gamma(\pi/2, l) \) in \( Sp(2, \mathbb{R}) \). All elements of \( F_{i,3} \) and \( F_{i,4} \) are conjugate to \( \gamma(\pi/2, l + (1/2)) \) in \( Sp(2, \mathbb{R}) \).

**Proof.** We take an arbitrary \( \{\gamma\}_\Gamma \in C_6 \). By Proposition 5.2, we have
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\[ \gamma \sim \Gamma \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \]

for some \( a \in \mathcal{O} \) of order 4 and \( b \in \mathfrak{A}^0 - \{0\} \). By taking \( \Gamma \)-conjugation, we may have \( a = a_i \) for some \( i \in \{1, \ldots, N\} \). Hence we have

\[ C_6 = \bigcup_{i=1}^{N} X_i, \quad X_i = \left\{ \left\{ \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \left| b \in \mathfrak{A}^0 - \{0\} \right. \right\}. \]

For each \( X_i \), we simply put \( a = a_i \). By Lemma 5.6, each \( X_i \) can be decomposed as

\[ X_i = \bigsqcup_{b \in \mathfrak{A}^0/L_\mathfrak{A}(a)} \left\{ \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \right\}. \]

We can describe the structure of \( \mathfrak{A}^0/L_\mathfrak{A}(a) \) by the same way as Hashimoto [Has84] as follows. From Proposition 2.5 of [Has84], we have

\[ \mathfrak{D}^0 = \begin{cases} \mathbb{Z} \cdot \frac{a + \beta}{2} + \mathcal{O}_F \beta & \text{if } 2 \nmid D, \\ \mathbb{Z} \cdot a + \mathcal{O}_F \beta & \text{if } 2 \mid D \end{cases} \]

for some \( \beta \). So we have \( \mathfrak{D}^0 \cap F^\perp = \mathcal{O}_F \beta \) and \( \mathfrak{O}^0 \cap F^\perp \) is a \( \mathcal{O}_F \)-submodule of \( \mathfrak{D}^0 \cap F^\perp \). Since \( \mathcal{O}_F \) is P.I.D. and \( \mathfrak{D}^0 \cap F^\perp \) is a free \( \mathcal{O}_F \)-module of rank 1, \( \mathfrak{A}^0 \cap F^\perp \) is also a free \( \mathcal{O}_F \)-module of rank 1. So we can write \( \mathfrak{A}^0 \cap F^\perp = \mathcal{O}_F \beta' \) with some \( \beta' \). Since \( \mathfrak{A}^0/(\mathfrak{A}^0 \cap F^\perp) \) is a torsion-free \( \mathbb{Z} \)-module, \( \mathfrak{A}^0 \cap F^\perp \) is a direct summand of \( \mathfrak{A}^0 \), that is, there exists some sub\( \mathbb{Z} \)-module \( M \) of \( \mathfrak{A}^0 \) and we can write \( \mathfrak{A}^0 = M \oplus (\mathfrak{A}^0 \cap F^\perp) \).

The \( \mathbb{Z} \)-module \( M \) is free of rank 1. A basis of \( M \) can be expressed as the form: \( xa + y\beta' \) (\( x \in \mathbb{Q} - \{0\}, \ y \in F \)) because we have \( B^0 = Qa + F\beta' \) with \( \beta' \) mentioned above. So we can take \( \rho_1 := xa + y\beta', \ \rho_2 := \beta' \) and \( \rho_3 := a\beta' \) as a basis of \( \mathfrak{A}^0 \). From the relation \(-2y\beta' = a^{-1}\rho_1 a - \rho_1 \in L_\mathfrak{A}(a) \subset \mathfrak{A}^0 \cap F^\perp = \mathcal{O}_F \beta' \), we have \( 2y \in \mathcal{O}_F = \mathbb{Z} + \mathbb{Z}a \). We divide the situation into two cases according as \( y \in \mathcal{O}_F \) of \( \notin \mathcal{O}_F \).

(i) The case of \( y \in \mathcal{O}_F \). We can write \( \rho_1 = xa + y_1\beta' + y_2a\beta' \) with some \( y_1, y_2 \in \mathbb{Z} \). So by replacing \( \rho_1, \rho_1 = xa, \ \rho_2 = \beta', \ \rho_3 = a\beta' \) forms a basis of \( \mathfrak{A}^0 \), that is

\[ \mathfrak{A}^0 = \mathbb{Z}xa \oplus \mathbb{Z}\beta' \oplus \mathbb{Z}a\beta'. \]
The case of \( y \not\in \mathcal{O}_F \). We have \( y = y_1 + y_2 a, \) \( 2y_1, 2y_2 \in \mathbb{Z} \) and \( \mathfrak{A}^0 = \mathbb{Z} \cdot (xa + y_1 \beta' + y_1 a\beta') \oplus \mathbb{Z} \beta' \oplus \mathbb{Z} a\beta' \). So \( \mathfrak{A}^0 \) is one of the following three cases:

Case (ii a) \( \mathfrak{A}^0 = \mathbb{Z} \cdot \left( xa + \frac{1}{2} a\beta' \right) \oplus \mathbb{Z} \beta' \oplus \mathbb{Z} a\beta' \)

Case (ii b) \( \mathfrak{A}^0 = \mathbb{Z} \cdot \left( xa + \frac{1}{2} \beta' \right) \oplus \mathbb{Z} \beta' \oplus \mathbb{Z} a\beta' \)

Case (ii c) \( \mathfrak{A}^0 = \mathbb{Z} \cdot \left( xa + \frac{1}{2} \beta' + \frac{1}{2} a\beta' \right) \oplus \mathbb{Z} \beta' \oplus \mathbb{Z} a\beta' \).

In each case, the structure of \( L_{\mathfrak{A}}(a) \) and \( \mathfrak{A}^0 / L_{\mathfrak{A}}(a) \) are given as follows:

Case (i):

\[
L_{\mathfrak{A}}(a) = \{ 2m\beta' + 2na\beta' \mid m, n \in \mathbb{Z} \},
\]

\[
\mathfrak{A}^0 / L_{\mathfrak{A}}(a) = \{ lxa \mid l \in \mathbb{Z} \} \sqcup \{ lxa + \beta' \mid l \in \mathbb{Z} \} \sqcup \{ lxa + a\beta' \mid l \in \mathbb{Z} \}
\]

\[
\sqcup \{ lxa + \beta' + a\beta' \mid l \in \mathbb{Z} \}.
\]

Case (ii a):

\[
L_{\mathfrak{A}}(a) = \{ 2m\beta' + na\beta' \mid m, n \in \mathbb{Z} \},
\]

\[
\mathfrak{A}^0 / L_{\mathfrak{A}}(a) = \{ lxa \mid l \in 2\mathbb{Z} \} \sqcup \{ lxa + \beta' \mid l \in 2\mathbb{Z} \}
\]

\[
\sqcup \left\{ lxa + \frac{1}{2} a\beta' \mid l \in 2\mathbb{Z} + 1 \right\} \sqcup \left\{ lxa + \beta' + \frac{1}{2} a\beta' \mid l \in 2\mathbb{Z} + 1 \right\}.
\]

Case (ii b):

\[
L_{\mathfrak{A}}(a) = \{ m\beta' + 2na\beta' \mid m, n \in \mathbb{Z} \},
\]

\[
\mathfrak{A}^0 / L_{\mathfrak{A}}(a) = \{ lxa \mid l \in 2\mathbb{Z} \} \sqcup \{ lxa + a\beta' \mid l \in 2\mathbb{Z} \}
\]

\[
\sqcup \left\{ lxa + \frac{1}{2} \beta' \mid l \in 2\mathbb{Z} + 1 \right\} \sqcup \left\{ lxa + \frac{1}{2} \beta' + a\beta' \mid l \in 2\mathbb{Z} + 1 \right\}.
\]

Case (ii c):

\[
L_{\mathfrak{A}}(a) = \{ m\beta' + na\beta' \mid m, n \in \mathbb{Z} \},
\]
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\[ \mathfrak{X}^0/L_{\mathfrak{X}}(a) = \{lxa \mid l \in 2\mathbb{Z}\} \sqcup \{lxa + \beta' \mid l \in 2\mathbb{Z}\} \]

\[ \sqcup \left\{ \frac{1}{2} lxa + \frac{1}{2} a \beta' \mid l \in 2\mathbb{Z} + 1 \right\} \]

\[ \sqcup \left\{ \frac{1}{2} lxa + \frac{3}{2} a \beta' \mid l \in 2\mathbb{Z} + 1 \right\}. \]

Thus we have completed the proof of Proposition 5.8. \(\square\)

The sets \(F_{i,l}'s\) are called families in \([Has83],[Has84],[Wak]\), etc. For each \(F_{i,l}\), there exist \(g_{i,l} \in \text{Sp}(2; \mathbb{R})\) and \(\lambda \in \mathbb{R}\) with \(0 \leq \lambda_{i,l} < 1\), such that

\[ F_{i,l} = g_{i,l} \left\{ \begin{array}{c}
0 1 0 0 \\
-1 0 0 0 \\
0 0 0 1 \\
0 0 -1 0 \end{array} \right\} \left( \begin{array}{c}
1 0 n + \lambda_{i,l} \\
0 1 0 \\
0 0 1 \\
0 0 0 \end{array} \right) \left| \begin{array}{c}
n \in \mathbb{Z} \\
 n + \lambda_{i,l} \neq 0 \end{array} \right\} g_{i,l}^{-1}. \]

We define

\[ C(F_{i,l}; \text{Sp}(2; \mathbb{R})) := g_{i,l} \left\{ \begin{array}{c}
\cos \theta & \sin \theta & 0 & 0 \\
-\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta \\
0 & 0 & -\sin \theta & \cos \theta \end{array} \right\} \left( \begin{array}{c}
1 0 t 0 \\
0 1 0 t \\
0 0 1 0 \\
0 0 0 1 \end{array} \right) \left| \begin{array}{c}
\theta, t \in \mathbb{R} \end{array} \right\} g_{i,l}^{-1}, \]

\[ C_0(F_{i,l}; \text{Sp}(2; \mathbb{R})) := g_{i,l} \left\{ \begin{array}{c}
1 0 t 0 \\
0 1 0 t \\
0 0 1 0 \\
0 0 0 1 \end{array} \right\} \left| \begin{array}{c}
t \in \mathbb{R} \end{array} \right\} g_{i,l}^{-1}, \]

\[ C(F_{i,l}; \Gamma) := C(F_{i,l}; \text{Sp}(2; \mathbb{R})) \cup \Gamma, \]

\[ C_0(F_{i,l}; \Gamma) := C_0(F_{i,l}; \text{Sp}(2; \mathbb{R})) \cup \Gamma. \]

Then, from (f-3) in \([Wak]\), we have

\[ I_2 = \sum_{i=1}^{N} \sum_{l=1}^{4} \frac{1}{2} \cdot \frac{\text{vol}(C_0(F_{i,l}; \Gamma) \setminus C_0(F_{i,l}; \text{Sp}(2; \mathbb{R})))}{[C(F_{i,l}; \Gamma) : \pm C_0(F_{i,l}; \Gamma)]} \cdot (-2^{-3}(-1)^{j/2}) \cdot (1 - \sqrt{-1} \cot^* \pi \lambda_{i,l}), \]

where we put
\[
\cot^* \pi \lambda := \begin{cases} 
0 & \text{if } \lambda = 0, \\
\cot \pi \lambda & \text{if } 0 \leq \lambda < 1.
\end{cases}
\]

We can verify that

\[
C(F_{i,l}; \Gamma) = g_{i,l} \begin{cases} 
\pm \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \pm \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & t \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\
\end{cases} \begin{pmatrix} 1 & 0 & t \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
C_0(F_{i,l}; \Gamma) = g_{i,l} \begin{cases} 
\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & t \\ 0 & 0 & 1 \end{pmatrix},
\end{cases}
\]

and

\[
\text{vol}(C_0(F_{i,l}; \Gamma) \setminus C_0(F_{i,l}; \text{Sp}(2; \mathbb{R}))) = 1,
\]

\[
[C(F_{i,l}; \Gamma) : \pm C_0(F_{i,l}; \Gamma)] = 2.
\]

Hence we have \( I_2 = -4N \cdot 2^{-5}(-1)^{3/2} \).

**5.3. The contribution \( I_3 \).**

In this section, we evaluate the contribution \( I_3 \). We consider the contribution of elements whose principal polynomials are \( f_7(x) = (x^2 + x + 1)^2 \) and double it to obtain \( I_3 \). Let \( \gamma \) be an element of \( \Gamma^{(qu)} \) whose principal polynomial is \( f_7(x) \). Then \( \gamma \) is \( \text{Sp}(2; \mathbb{R}) \)-conjugate to an element of the form

\[
\gamma \left( \frac{2\pi}{3}, s \right) = \begin{pmatrix}
-1/2 & \sqrt{3}/2 & 0 & 0 \\
-\sqrt{3}/2 & -1/2 & 0 & 0 \\
0 & 0 & -1/2 & \sqrt{3}/2 \\
0 & 0 & -\sqrt{3}/2 & -1/2
\end{pmatrix} \begin{pmatrix} 1 & 0 & s & 0 \\ 0 & 1 & 0 & s \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

(cf. Proposition 5.2) and corresponds to (f-3) of [Wak].

We denote by \( C_7 \) the set of all \( \Gamma \)-conjugacy classes of \( \Gamma^{(qu)} \) whose principal polynomial is \( f_7(x) \). By the same way as Proposition 5.8, we can prove the following proposition:

**Proposition 5.9.** We can decompose \( C_7 \) into disjoint union of \( 3N \) subsets as
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\[ C_7 = \prod_{i=1}^{N} \prod_{l=1}^{3} \left( \bigcup_{\gamma \in F_{i,l}} \{ \gamma \} \right), \]

where \( N := \prod_{p \mid D} (1 - (-3/p)) \) and \( F_{i,j} \) is defined as follows.

Let \( a_1, \ldots, a_N \) be a complete system of \( \mathcal{O}^\times \)-conjugacy classes of elements of \( \mathcal{O} \) of order 3 (cf. Lemma 5.4). There exist some \( x_i \in \mathbb{Q}^>0 \) and \( \beta_i \in \mathcal{O}_0 \) depending on each \( a_i \) such that \( F_{i,j} \)'s are given as one of the following two cases. Here we put

\[ \delta(a_i, \gamma_1, \gamma_2) := \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \begin{pmatrix} 1 & \gamma_1 \beta_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \gamma_2 x_i \sqrt{-3} \\ 0 & 1 \end{pmatrix}, \]

where the symbol “ \( \cdot \) ” means the Jordan decomposition.

Case 1:

\[ F_{i,1} = \{ \delta(a_i, 0, n) \mid n \in \mathbb{Z} - \{0\} \}, \quad F_{i,2} = \{ \delta(a_i, 1, n) \mid n \in \mathbb{Z} - \{0\} \} \]
\[ F_{i,3} = \{ \delta(a_i, 2, n) \mid n \in \mathbb{Z} - \{0\} \}, \]

All elements of each \( F_{i,l} \) are \( \text{Sp}(2; \mathbb{R}) \)-conjugate to \( \gamma(2\pi/3, n) \).

Case 2:

\[ F_{i,1} = \{ \delta(a_i, 0, 3n) \mid n \in \mathbb{Z} - \{0\} \}, \quad F_{i,2} = \left\{ \delta \left( a_i, \frac{1 + 2a_i}{3}, 3n + 1 \right) \mid n \in \mathbb{Z} \right\} \]
\[ F_{i,3} = \left\{ \delta \left( a_i, \frac{2 + a_i}{3}, 3n + 2 \right) \mid l \in \mathbb{Z} \right\}, \]

All elements of each \( F_{i,l} \) are \( \text{Sp}(2; \mathbb{R}) \)-conjugate to \( \gamma(2\pi/3, n + (l - 1)/3) \).

For each \( F_{i,l} \), we define \( g_{i,l}, \lambda_{i,l}, C(F_{i,l}; \text{Sp}(2; \mathbb{R})), C_0(F_{i,l}; \text{Sp}(2; \mathbb{R})), C(F_{i,l}; \Gamma) \) and \( C_0(F_{i,l}; \Gamma) \) in the same way as in Subsection 5.2. Then, from (f-3) in [Wak], we have

\[ I_3 = \sum_{i=1}^{N} \sum_{l=1}^{3} \frac{\text{vol}(C_0(F_{i,l}; \Gamma) \setminus C_0(F_{i,l}; \text{Sp}(2; \mathbb{R})))}{[C(F_{i,l}; \Gamma) : \pm C_0(F_{i,l}; \Gamma)]} \cdot \left( -2^{-1}3^{-1}[1, -1, 0; 3]_j \right) \cdot \left( 1 - \sqrt{-1} \cot^* \pi \lambda_{i,l} \right). \]

We can verify that
\[
C(F_{i,l}; \Gamma) = g_{i,l} \left\{ \pm \gamma(\theta,t) \middle| \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, t \in \mathbb{Z} \right\} g_{i,l}^{-1},
\]
\[
C_0(F_{i,l}; \Gamma) = g_{i,l} \left\{ \gamma(0,t) \mid t \in \mathbb{Z} \right\} g_{i,l}^{-1}
\]

and

\[
\text{vol}(C_0(F_{i,l}; \Gamma) \setminus C_0(F_{i,l}; \text{Sp}(2; \mathbb{R}))) = 1, \quad [C(F_{i,l}; \Gamma) : \pm C_0(F_{i,l}; \Gamma)] = 3.
\]

Hence we have \( I_3 = -3N \cdot 2^{-1}3^{-2}[1, -1, 0; 3] \).

6. Numerical examples.

In this section, we give some numerical examples of \( \dim_{\mathbb{C}} S_{k,j}(\Gamma(D_1, D_2)) \) for various \( D_1, D_2 \). The tables for \( D = D_1 = 6, 10, 15 \) appeared in [Wak]. Our theorem can not be applied for \( k \leq 4 \). In the following tables, we formally substitute \( k \leq 4 \) in the formula of Theorem 3.1. Hashimoto conjectured that the dimension of \( S_{4,0}(\Gamma(D_1, 1)) \) (resp. \( S_{3,0}(\Gamma(D_1, 1)) \)) can be obtained by substituting \( k = 4 \) in Theorem 3.1 (resp. by substituting \( k = 3 \) and adding +1). (Conjecture 4.3, 4.4 in [Has84]. cf. [Ibu07b] in the split case).

(1) \( D = 2 \cdot 3 \)

(i) \( D_1 = 2 \cdot 3, D_2 = 1 \)

| \( j \) | 0 1 2 3 4 | 5 6 7 8 9 | 10 11 12 13 14 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 -1 0 -1 2 | 0 4 2 8 5 | 15 10 25 15 34 26 |
| 2 | -2 2 0 1 2 | 2 5 7 15 17 | 33 34 53 58 91 96 |
| 4 | 0 -1 0 2 4 | 6 14 19 35 42 67 | 77 114 126 179 200 |
| 6 | -2 -1 1 5 9 | 17 30 40 65 82 | 118 145 195 224 299 341 |
| 8 | -3 -2 2 7 19 | 27 49 67 106 131 | 188 223 298 346 448 514 |

(ii) \( D_1 = 3, D_2 = 2 \)

| \( j \) | 0 1 2 3 4 | 5 6 7 8 9 | 10 11 12 13 14 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | -1 -1 0 0 2 | 1 3 4 7 5 | 9 11 17 14 21 24 |
| 2 | 0 1 0 1 0 | 1 3 6 7 10 18 | 23 29 36 52 61 |
| 4 | 0 -1 0 1 2 | 2 7 12 19 23 36 | 48 65 75 100 122 |
| 6 | 0 0 1 5 6 | 11 19 29 39 51 72 | 93 116 140 180 214 |
| 8 | -1 -2 2 5 12 | 16 30 44 64 79 110 | 139 179 211 265 315 |
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#### (iii) $D_1 = 2$, $D_2 = 3$

| $j \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0                | -1 | -1 | 0 | 0 | 1 | 1 | 3 | 2 | 4 | 6 | 6 | 7 | 12 | 11 | 14 | 19 |
| 2                | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 3 | 4 | 7 | 10 | 14 | 18 | 25 | 31 | 39 |
| 4                | 1 | 0 | 0 | 2 | 1 | 3 | 7 | 8 | 13 | 20 | 24 | 34 | 45 | 53 | 69 | 86 |
| 6                | 0 | -1 | 1 | 3 | 2 | 8 | 12 | 16 | 25 | 36 | 43 | 60 | 77 | 92 | 115 | 143 |
| 8                | 0 | 0 | 2 | 3 | 9 | 13 | 21 | 30 | 43 | 56 | 75 | 94 | 119 | 146 | 178 | 212 |

#### (iv) $D_1 = 1$, $D_2 = 2 \cdot 3$

| $j \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0                | -1 | 0 | -1 | 1 | 2 | 2 | 2 | 3 | 4 | 6 | 6 | 8 | 8 | 11 | 13 | 18 |
| 2                | -1 | 2 | 0 | 0 | 0 | 0 | 1 | 2 | 2 | 4 | 5 | 9 | 10 | 15 | 18 | 22 |
| 4                | 1 | 0 | 0 | 1 | 1 | 1 | 4 | 5 | 7 | 11 | 15 | 19 | 26 | 32 | 40 | 50 |
| 6                | 0 | 0 | 1 | 3 | 1 | 6 | 7 | 11 | 17 | 21 | 27 | 38 | 46 | 58 | 70 | 86 |
| 8                | 0 | 0 | 2 | 1 | 8 | 8 | 12 | 19 | 27 | 34 | 47 | 56 | 72 | 89 | 109 | 127 |

#### (II) $D = 2 \cdot 5$

#### (i) $D_1 = 2 \cdot 5$, $D_2 = 1$

| $j \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0                | -1 | 0 | -1 | 1 | 2 | 3 | 9 | 12 | 28 | 39 | 82 | 99 | 170 | 185 | 285 | 316 |
| 2                | -2 | 3 | 0 | 3 | 9 | 12 | 28 | 39 | 82 | 99 | 170 | 185 | 285 | 316 | 470 | 513 |
| 4                | 0 | -3 | 0 | 8 | 23 | 33 | 76 | 99 | 180 | 227 | 346 | 408 | 587 | 675 | 926 | 1051 |
| 6                | -8 | -7 | 3 | 18 | 46 | 83 | 150 | 203 | 330 | 423 | 607 | 742 | 1004 | 1173 | 1534 | 1771 |
| 8                | -22 | -12 | 3 | 31 | 88 | 141 | 246 | 347 | 532 | 684 | 955 | 1157 | 1522 | 1805 | 2302 | 2669 |

#### (ii) $D_1 = 5$, $D_2 = 2$

| $j \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0                | 0 | -1 | 0 | -1 | 1 | 3 | 7 | 7 | 15 | 16 | 30 | 32 | 53 | 55 | 84 | 88 |
| 2                | -2 | 3 | 0 | 1 | 4 | 8 | 16 | 28 | 45 | 61 | 93 | 118 | 164 | 203 | 269 | 316 |
| 4                | 2 | -1 | 0 | 5 | 13 | 21 | 45 | 64 | 102 | 140 | 201 | 253 | 344 | 418 | 539 | 643 |
| 6                | -3 | -4 | 3 | 11 | 25 | 53 | 88 | 128 | 196 | 259 | 355 | 456 | 592 | 721 | 909 | 1079 |
| 8                | -12 | -5 | 3 | 17 | 53 | 88 | 146 | 218 | 315 | 415 | 564 | 706 | 905 | 1105 | 1367 | 1616 |

#### (iii) $D_1 = 2$, $D_2 = 5$

| $j \backslash k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| 0                | -1 | -1 | 0 | 0 | 2 | 2 | 4 | 5 | 8 | 10 | 14 | 17 | 23 | 28 | 35 | 42 |
| 2                | -1 | 2 | 0 | 0 | 2 | 4 | 5 | 12 | 16 | 24 | 35 | 47 | 60 | 81 | 100 | 124 |
| 4                | 2 | 0 | 0 | 2 | 4 | 7 | 16 | 24 | 36 | 53 | 73 | 96 | 127 | 160 | 200 | 247 |
| 6                | -1 | -1 | 3 | 7 | 10 | 25 | 35 | 53 | 78 | 106 | 137 | 184 | 229 | 285 | 352 | 426 |
| 8                | -3 | -1 | 3 | 6 | 23 | 35 | 57 | 86 | 122 | 161 | 218 | 275 | 347 | 430 | 524 | 626 |
(iv) $D_1 = 1$, $D_2 = 2 \cdot 5$

| j \ k | 0 1 2 3 4 | 5 6 7 8 9 10 11 12 13 14 15 |
|-------|------------|-----------------------------|
| 0     | -1 -1 0 0 2 | 3 4 5 7 9 12 14 18 21 26 31 |
| 2     | -1 2 0 0 1 | 2 3 7 9 14 20 28 35 48 59 73 |
| 4     | 2 0 0 1 2 | 3 9 13 20 30 42 55 74 93 117 145 |
| 6     | 0 0 3 6 7 | 17 23 34 50 66 85 114 141 175 215 260 |
| 8     | -1 0 3 4 16 | 22 35 53 75 98 133 166 210 260 317 377 |

(III) $D = 3 \cdot 5$

(i) $D_1 = 3 \cdot 5$, $D_2 = 1$

| j \ k | 0 1 2 3 4 | 5 6 7 8 9 10 11 12 13 14 15 |
|-------|------------|-----------------------------|
| 0     | -1 -1 1 0 9 | 8 34 29 86 85 183 178 331 318 536 531 |
| 2     | -3 3 0 7 30 | 52 117 170 311 405 640 775 1120 1324 1821 2100 |
| 4     | -29 28 1 28 | 149 298 431 703 934 1357 1694 2316 2789 3644 4283 |
| 6     | -79 -24 3 63 | 323 574 834 1281 1702 2373 2985 3936 4757 6044 7136 |
| 8     | -79 -54 6 119 | 575 979 1416 2091 2756 3752 4681 6044 7305 9117 10746 |

(ii) $D_1 = 5$, $D_2 = 3$

| j \ k | 0 1 2 3 4 | 5 6 7 8 9 10 11 12 13 14 15 |
|-------|------------|-----------------------------|
| 0     | -1 -1 1 1 3 | 6 15 17 30 50 63 86 126 150 194 254 |
| 2     | 0 2 0 4 9 | 24 44 75 115 172 239 327 429 555 699 869 |
| 4     | 3 -3 1 14 29 | 63 118 176 271 388 520 698 908 1134 1426 1751 |
| 6     | -8 -10 3 32 64 | 137 229 344 503 705 927 1219 1559 1935 2384 2909 |
| 8     | -24 -23 6 50 131 | 237 390 579 827 1121 1481 1899 2397 2960 3613 4343 |

(iii) $D_1 = 3$, $D_2 = 5$

| j \ k | 0 1 2 3 4 | 5 6 7 8 9 10 11 12 13 14 15 |
|-------|------------|-----------------------------|
| 0     | -1 -1 1 1 5 | 6 11 15 24 32 45 58 78 98 124 152 |
| 2     | 0 2 0 2 5 | 14 24 43 65 98 137 187 245 319 401 499 |
| 4     | 1 -1 1 6 17 | 35 64 102 153 218 300 398 516 654 816 1001 |
| 6     | -4 -2 3 20 42 | 83 133 206 295 409 543 711 901 1127 1384 1681 |
| 8     | -12 -11 6 30 79 | 139 228 337 481 649 859 1099 1387 1712 2089 2509 |

(iv) $D_1 = 1$, $D_2 = 3 \cdot 5$

| j \ k | 0 1 2 3 4 | 5 6 7 8 9 10 11 12 13 14 15 |
|-------|------------|-----------------------------|
| 0     | -1 -1 0 3 | 4 6 7 12 15 21 26 35 42 54 65 |
| 2     | -1 3 0 1 2 | 6 9 18 25 39 54 75 96 128 159 198 |
| 4     | 3 0 1 4 8 | 13 28 41 61 88 121 158 208 261 326 401 |
| 6     | -1 0 3 11 16 | 37 54 84 121 166 217 289 362 453 556 676 |
| 8     | -3 -2 6 11 38 | 57 93 138 197 260 350 441 558 689 841 1004 |
Dimension formula for Siegel cusp forms

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