QCD sum rules analysis of the rare $B_c \rightarrow X\nu\bar{\nu}$ decays

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Abstract

Taking into account the gluon correction contributions to the correlation function, the form factors relevant to the rare $B_c \rightarrow X\nu\bar{\nu}$ decays are calculated in the framework of the three point QCD sum rules, where $X$ stands for axial vector particle, $AV(D_{s1})$, and vector particles, $V(D^*, D_{s1}^*)$. The total decay width as well as the branching ratio of these decays are evaluated using the $q^2$ dependent expressions of the form factors. A comparison of our results with the predictions of the relativistic constituent quark model is presented.

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1 Introduction

The discovery of the $B_c$ meson by the CDF detector at the Fermi Lab in $p\bar{p}$ collisions via the decay mode $B_c \rightarrow J/\psi l^\pm \nu$ at $\sqrt{s} = 1.8$ TeV [1] has illustrated the possibility of the experimental study of the charm-beauty systems and has produced considerable interest in its spectroscopy. This meson constitutes a very rich laboratory since with the luminosity values of $L = 10^{34} \text{cm}^{-2}\text{s}^{-1}$ and $\sqrt{s} = 14$ TeV at LHC, the number of $B_c^{\pm}$ mesons is expected to be about $10^8 \sim 10^{10}$ per year [2, 3]. This will provide a good opportunity to study not only some rare $B_c$ decays, but also CP violation, T violation and polarization asymmetries.

The long-lived heavy quarkonium, $B_c$, is the only meson containing two heavy quarks with different charge and flavours ($b$ and $c$), in which its decay properties are expected to be different from that of flavour neutral mesons and this can produce a significant progress in the study of heavy quark dynamics. The $B_c$ system is the lowest bound state of two heavy quarks with open flavour. Such states have no annihilation decay modes due to the electromagnetic and strong interactions since the excited levels of $\bar{b}c$ lie below the threshold of decay into the pair of heavy $B$ and $D$ mesons, so this meson decays weakly. Many parameters enter in the description of weak decays of this meson. In particular, measuring the branching ratios of such decays provide a new framework for more precise calculation of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements $V_{tq}$ ($q = d, s, b$), leptonic decay constants, quark masses and mixing angles. Also, a study of this meson can be used as constrains on the physics beyond the Standard Model. Indeed, more collection of hadrons containing heavy quarks provides more accuracy and confidence in the understanding of QCD dynamics (for details about the physics of the $B_c$ meson see [4]).

In present work, the $B_c \rightarrow (D^*, D_{s1}^*)\nu \bar{\nu}$ and $B_c \rightarrow D_{s1} \nu \bar{\nu}$ transitions are investigated in the framework of the three point QCD sum rules. Theoretical calculation of the amplitudes for these decays is particularly reliable, owing to the absence of long-distance interactions that affect charged-lepton channels $B_c \rightarrow X l^+ l^-$. The rare $B_c \rightarrow (D^*, D_{s1}^*)\nu \bar{\nu}$ and $B_c \rightarrow D_{s1} \nu \bar{\nu}$ decays are proceeded by flavour changing neutral current (FCNC) transitions of $b \rightarrow s, d$. These transitions occur at loop level in the standard model (SM) and they are very sensitive to the physics beyond the SM, since some new particles might have contributions in the loops diagrams. New physics such as SUSY particles or a possible fourth generation could contribute to the penguin loop or box diagram and change the branching fractions [5]. The possibility of discovering light dark matter in $b \rightarrow s$ transitions with large missing momentum has been discussed in Ref. [6]. Note that, some possible $B_c$ decays such as $B_c \rightarrow \omega \gamma, B_c \rightarrow \rho^\pm \gamma, B_c \rightarrow K^{*+} \gamma, B_c \rightarrow B_u^{*} l^+ l^-, B_c \rightarrow B_u^\gamma$ and $B_c \rightarrow D_{s,d}^* \gamma$ have been previously studied in the framework of light-cone and three point QCD sum rules [7–11]. A larger set of exclusive nonleptonic and semileptonic decays of the $B_c$ meson, which have been studied within a relativistic constituent quark model can be found in Ref. [12]. Moreover, the $B_c \rightarrow (D^*, D_{s1}^*)\nu \bar{\nu}$ transitions have also been investigated in the framework of the relativistic constituent quark model (RCQM) [13].

The paper includes three sections. The calculation of the sum rules for the relevant form factors are presented in section II. In the sum rules expressions for the form factors, the light quark condensates don’t have any contributions, so, as first correction in the nonperturbative part of the correlator, the two gluon condensates contributions to the correlation function are taken into account. Section III contains numerical analysis, discussion and
comparison of the present work results with the predictions of the RCQM.

2 Sum rules for transition form factors of $B \to AV(V)\nu \bar{\nu}$

$[AV(V) = D_{s1}(D^*, D_s^*)]$

The $B \to X\nu\bar{\nu}$ process is described at quark level via the $b \to q\nu\bar{\nu}$ transition, $(q = d$ or $s)$ (see Fig. 1), in the SM and receives contributions from Z-penguin and box diagrams, where dominant contributions come from intermediate top quark. The explicit form of the effective Hamiltonian responsible for $b \to q\nu\bar{\nu}$ decays is described by only one Wilson coefficient, namely $C_{10}$,

$$H_{\text{eff}} = \frac{G_F \alpha}{2\pi\sqrt{2}} C_{10} \left( V_{tb} V_{tq}^* \right) \bar{q} \gamma^\alpha (1 - \gamma_5) b \left( \bar{\nu} \gamma^- (1 - \gamma_5) \nu \right), \quad (1)$$

where $G_F$ is the Fermi constant, $\alpha$ is the fine structure constant at the $Z$ mass scale and $V_{ij}$ are elements of the CKM matrix. The presence of only one operator in the effective Hamiltonian makes the $b \to q\nu\bar{\nu}$ process important, because the estimated theoretical uncertainty is related only to the value of the Wilson coefficient $C_{10}$. For more about the Wilson coefficients see [9, 13–15] and references therein. The amplitudes of the $B_c \to X\nu\bar{\nu}$ decays are obtained by sandwiching Eq. (1) between the initial and final meson states

$$M = \frac{G_F \alpha}{2\pi\sqrt{2}} C_{10} \left( V_{tb} V_{tq}^* \right) \bar{q} \gamma^\mu (1 - \gamma_5) b < X(p', \varepsilon) \mid \bar{q} \gamma^- (1 - \gamma_5) \nu \mid B_c(p) > . \quad (2)$$

Our aim is to calculate the matrix element $< X(p', \varepsilon) \mid \bar{q} \gamma^- (1 - \gamma_5) b \mid B_c(p) >$ appearing in Eq. (2). Both vector and axial vector part of the transition current $\bar{q} \gamma^- (1 - \gamma_5) b$
contribute to the matrix element discussed above. Considering the parity and Lorentz invariances, one can parameterized this matrix element in terms of the form factors in the following form:

\[
\langle V(p', \varepsilon) | \bar{q} \gamma_\mu b \rangle B_c(p) \rangle = -i \left[ \varepsilon^* \mu (m_{B_c} + m_V) A_1(q^2) - (\varepsilon^* q) \mathcal{P}_\mu \frac{A_2(q^2)}{m_{B_c} + m_V} \right.
\]

where \( \mathcal{P}_\mu = (p + p')_\mu \), \( q_\mu = (p - p')_\mu \) and \( \varepsilon^* \) is the polarization vector of the \( X \) mesons. To guarantee the finiteness of the results at \( q^2 = 0 \), it should be \( A_3(0)A'_3(0) = A_0(0)A'_0(0) \).

The form factor \( A_3(q^2)(A'_3(q^2)) \) can be written as a linear combination of \( A_1(A'_1) \) and \( A_2(A'_2) \) in the following way:

\[
A_3(q^2)(A'_3(q^2)) = \frac{m_{B_c} + m_V(m_{AV})}{2m_V(m_{AV})} A_1(q^2)(A'_1(q^2)) - \frac{m_{B_c} - m_V(m_{AV})}{2m_V(m_{AV})} A_2(q^2)(A'_2(q^2)) .
\]

Therefore, we need to calculate the form factors \( V(V') \), \( A_1(A'_1) \) and \( A_2(A'_2) \). In order to obtain the sum rules expressions for these form factors, we consider the following three-point correlation function:

\[
\Pi^{\nu \alpha}_{\mu \nu} = \int \, d^4 x d^4 y e^{-ipx} e^{iq'y} \langle 0 | \mathcal{T} \{ J_{X\nu}(y) J_{\nu}(0)^{\nu \alpha} J_{Bc}(x) \} | 0 \rangle ,
\]

where \( J_{X\nu}(y) = \bar{q} \gamma_\nu q \), \( J_{AV\nu}(y) = \bar{q} \gamma_5 \gamma_\nu q \) (\( q = s, d \)) and \( J_{Bc}(x) = i \bar{q} \gamma_5 b \) are the interpolating current of the \( V \), \( AV \) and \( B_c \) mesons, respectively. \( J_\mu^\nu = \bar{q} \gamma_\mu b \) and \( J_\mu^\alpha = \bar{q} \gamma_\mu \gamma_5 b \) are the vector and axial vector part of the transition current.

From the general philosophy of the QCD sum rules, we calculate the above correlation function in two languages. First, in the hadron languages, the results of the correlator give us the phenomenological or physical part and the QCD or theoretical part of this correlator are obtained in the quark gluon languages. The sum rules for the form factors can be obtained by equating the coefficient of the corresponding structure from these two parts and applying double Borel transformation with respect to the momentums of the initial
and final meson states to eliminate the contributions coming from the higher states and continuum.

To calculate the phenomenological part of the correlator given in Eq. (8), two complete sets of intermediate states with the same quantum numbers as the currents $J_V$ and $J_{B_c}$ are inserted, respectively. As a result of this procedure, we get the following representation of the above-mentioned correlator:

$$
\Pi^{v,a}_{\mu\nu}(p^2, p'^2, q^2) = 
\left< 0 \mid J_{X\nu} \mid X(p', \varepsilon) > X(p, \varepsilon) \mid J^{v,a}_{\mu} \mid B_c(p) > B_c(p) \mid J^{\dagger}_{B_c} \mid 0 \right> \over (p'^2 - m_X^2)(p^2 - m_{B_c}^2) + \ldots
$$

(9)

where $\ldots$ represent contributions coming from higher states and continuum. The matrix elements in Eq. (9) are defined in the standard way as:

$$
< 0 \mid J'_X \mid X(p') = f_X m_X \varepsilon'\nu , \quad < 0 \mid J_{B_c} \mid B_c(p) = \frac{i f_{B_c} m_{B_c}^2}{m_b + m_c},
$$

(10)

where $f_X$ and $f_{B_c}$ are the leptonic decay constants of $X$ and $B_c$ mesons, respectively. Using Eqs. (3-6) and Eq. (10) in Eq. (9) and performing summation over the polarization of $X$ meson, we get the following result for the physical part:

$$
\Pi^{v,a}_{\mu\nu}(p^2, p'^2, q^2) = \frac{\frac{f_{B_c} f_{V(AV)} m_{B_c}^2 m_{V(AV)}}{(m_b + m_c)(p'^2 - m_{B_c}^2)(p^2 - m_{V(AV)}^2)}}{2V(V')} \frac{2V}{m_{B_c} + m_{V(AV)}} \left\{ \frac{i \epsilon_{\nu\mu\alpha\beta} p^\alpha p'^\beta}{m_{V(AV)}} \right\} \\
- \frac{1}{m_{B_c} + m_{V(AV)}} \mathcal{P}_\mu \left( - q_\nu + \frac{p' q (P - q)_\nu}{2m_{V(AV)}^2} \right) A_2(A'_2) \\
+ \frac{2m_{V(AV)}}{q^2} q_\mu \left( - q_\nu + \frac{p' q (P - q)_\nu}{2m_{V(AV)}^2} \right) (A_3(A'_3) - A_0(A'_0)) \right).
$$

(11)

The coefficients of the Lorentz structures $i \epsilon_{\nu\mu\alpha\beta} p^\alpha p'^\beta$, $g_{\mu\nu}$ and $\mathcal{P}_\mu q_\nu$ give the expressions for the form factors $V(V')$, $A_1(A'_1)$ and $A_2(A'_2)$, respectively. The correlation function can be written in terms of the Lorentz structures in the following form:

$$
\Pi^{v,a}_{\mu\nu}(p^2, p'^2, q^2) = \Pi V \epsilon_{\nu\mu\alpha\beta} p^\alpha p'^\beta + \Pi A_1 g_{\mu\nu} + \Pi A_2 \mathcal{P}_\nu q_\mu + \ldots
$$

(12)

To calculate the QCD side of correlation function, on the other side, we evaluate the three–point correlator by the help of the operator product expansion (OPE) in the deep Euclidean region $p^2 \ll (m_b + m_c)^2$, $p'^2 \ll (m_b^2 + m_q^2)$. For this aim, we write each $\Pi_{i(i')} [i(i')]$ stands for $V(V')$, $A_1(A'_1)$ and $A_2(A'_2)$] function in terms of the perturbative and nonperturbative parts as:

$$
\Pi_{i(i')}(p^2, p'^2, q^2) = \Pi^{per}_{i(i')}(p^2, p'^2, q^2) + \Pi^{non-per}_{i(i')}(p^2, p'^2, q^2),
$$

(13)
where \( \langle \bar{q}q \rangle \) and \( \langle G^2 \rangle \) denotes the light quark and two gluon condensates, respectively. For the perturbative part, we calculate the bare loop diagram (Fig. 1a), however, diagrams b, c, d in Fig. 1 are correspond to the light quark condensates contributing to the correlation function. In principle, the light quark condensate diagrams give contributions to the correlation function, but applying double Borel transformations kill their contributions, so as first nonperturbative correction, we consider the gluon condensate diagrams (see Fig. 2a, b, c, d, e, f).

Using the double dispersion representation, the bare-loop contribution is written as

\[
\Pi_{i(i')}^{\text{per}} = -\frac{1}{(2\pi)^2} \int ds' \int ds \frac{\rho_{i(i')}^{\text{per}}(s,s',Q^2)}{(s-p^2)(s'-p^2)} + \text{subtraction terms},
\]

where \( Q^2 = -q^2 \). The spectral densities \( \rho_{i(i')}^{\text{per}}(s,s',Q^2) \) are calculated by the help of the Gutkovsky rule, i.e., we replace the propagators with Dirac–delta functions

\[
\frac{1}{p^2 - m^2} \to -2i\pi \delta(p^2 - m^2),
\]

implying that all quarks are real and integration region in Eq. (14) is obtained by requiring that the argument of three delta vanish, simultaneously. This condition leads to the following inequality

\[
-1 \leq \frac{2ss' + (s+s' + Q^2)(m_b^2 - m_c^2 - s) + 2s(m_c^2 - m_q^2)}{\lambda^{1/2}(s,s',-Q^2)\lambda^{1/2}(m_b^2,m_c^2,s)} \leq +1,
\]

where \( \lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \). From this inequality, one can express \( s \) in terms of \( s' \) i.e. \( f(s') \) in the \( s-s' \) plane.

Straightforward calculations lead to the following expressions for the spectral densities:

\[
\rho_V(s,s',Q^2) = N_c I_0(s,s',-Q^2) \left[ -4m_c + 4(m_b - m_c)E_1 + 4(m_q - m_c)E_2 \right],
\]

\[
\rho_{A_1}(s,s',Q^2) = N_c I_0(s,s',-Q^2) \left[ 8(m_c - m_b)D_1 - 4m_qm_cm_q + 4(m_q + m_b - m_c)m_c^2 - 2(m_q - m_c)\Delta - 2(m_b - m_c)\Delta' - 2m_cu \right],
\]

\[
\rho_{A_2}(s,s',Q^2) = 2N_c I_0(s,s',-Q^2) \left[ E_2m_b + D_3(m_b - m_c) + (E_1 - E_2)m_c + D_2(m_c - m_b) - E_2m_q \right],
\]

\[
\rho_{V'}(s,s',Q^2) = -N_c I_0(s,s',-Q^2) \left[ 4m_c + 4m_b - m_c \right] + 4E_2 \left( m_c + m_s \right),
\]

\[
\rho_{A_1'}(s,s',Q^2) = -2N_c I_0(s,s',-Q^2) \left[ 4D_1 (m_b - m_c) + \Delta' (m_b - m_c) - \Delta (m_c + m_s) + 2m_c^2 (m_c + m_s - m_b) - m_c (2m_bm_s - u) \right],
\]
where \( u = s + s' + Q^2, \Delta = s + m_c^2 - m_b^2, \Delta' = s' + m_c^2 - m_q^2 \) and \( N_c = 3 \) is the number of colors. The functions \( E_1, E_2, D_1, D_2, D_3 \) and \( I_0 \) are defined as:

\[
I_0(s, s', -Q^2) = \frac{1}{4\lambda^{1/2}(s, s', -Q^2)},
\]

\[
\lambda(s, s', -Q^2) = s^2 + s'^2 + Q^4 + 2sQ^2 + 2s'Q^2 - 2ss',
\]

\[
E_1 = \frac{1}{\lambda(s, s', -Q^2)}[2s'\Delta - \Delta'u],
\]

\[
E_2 = \frac{1}{\lambda(s, s', -Q^2)}[2s\Delta' - \Delta u],
\]

\[
D_1 = \frac{1}{2\lambda(s, s', -Q^2)}[\Delta'^2s + \Delta^2s' - 4m_c^2ss' - \Delta\Delta'u + m_c^2u^2],
\]

\[
D_2 = \frac{1}{\lambda^2(s, s', -Q^2)}[2\Delta'^2ss' + 6\Delta^2s'^2 - 8m_c^2ss'^2 - 6\Delta\Delta's'u + \Delta^2u^2 + 2m_c^2s'u^2],
\]

\[
D_3 = \frac{1}{\lambda^2(s, s', -Q^2)}[2\Delta'^2ss' + 6\Delta^2s'^2 - 8m_c^2ss'^2 - 6\Delta\Delta'su + \Delta^2u^2 + 2m_c^2su^2].
\]

The next step is to calculate the gluon condensate contributions to the correlation function (diagrams in Fig. 2). In this section, we proceed in the definition of the integrals appearing in evaluation of the gluon condensates contribution the same as in Refs. [9, 16]. The diagrams are calculated in the Fock–Schwinger fixed–point gauge [17–19]

\[
x^\mu G_\mu^a = 0 ,
\]

where \( G_\mu^a \) is the gluon field. In calculating the diagrams, the following type of integrals are appeared:

\[
I_0[a, b, c] = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m_b^2]^a [(p + k)^2 - m_c^2]^b [(p' + k)^2 - m_q^2]^c},
\]

\[
I_\mu[a, b, c] = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{[k^2 - m_b^2]^a [(p + k)^2 - m_c^2]^b [(p' + k)^2 - m_q^2]^c},
\]

\[
I_{\mu\nu}[a, b, c] = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu k_\nu}{[k^2 - m_b^2]^a [(p + k)^2 - m_c^2]^b [(p' + k)^2 - m_q^2]^c}.
\]
Figure 2: Gluon condensate contributions to $B_c \rightarrow X \nu \bar{\nu}$ transitions
Hear, \( k \) is the momentum of the spectator quark \( c \). In Schwinger representation for the propagators, i.e.,

\[
\frac{1}{p^2 + m^2} = \frac{1}{\Gamma(\alpha)} \int_0^\infty d\alpha \alpha^{n-1} e^{-\alpha(p^2 + m^2)} ,
\]

the integrals take the suitable form to apply the Borel transformations as

\[
\mathcal{B}_{p^2}(M^2) e^{-\alpha p^2} = \delta(1/M^2 - \alpha) .
\]

Performing all integrals and applying double Borel transformations with respect to \( p^2 \) and \( p'^2 \) the Borel transformed form of the integrals are obtained as

\[
\hat{I}_0(a, b, c) = \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{2-a-b}(M_2^2)^{2-a-c} \mathcal{U}_0(a + b + c - 4, 1 - c - b) ,
\]

\[
\hat{I}_\mu(a, b, c) = \frac{1}{2} [\hat{I}_1(a, b, c) + \hat{I}_2(a, b, c)] \mathcal{P}_\mu + \frac{1}{2} [\hat{I}_1(a, b, c) - \hat{I}_2(a, b, c)] q_\mu ,
\]

\[
\hat{I}_{\mu\nu}(a, b, c) = \hat{I}_0(a, b, c) g_{\mu\nu} + \frac{1}{4} (2\hat{I}_4 + \hat{I}_3 + \hat{I}_5) \mathcal{P}_\mu \mathcal{P}_\nu + \frac{1}{4} (-\hat{I}_5 + \hat{I}_3) \mathcal{P}_{\mu\nu} + \frac{1}{4} (-2\hat{I}_4 + \hat{I}_3) q_\mu q_\nu ,
\]

where

\[
\hat{I}_1(a, b, c) = i \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{2-a-b}(M_2^2)^{3-a-c} \mathcal{U}_0(a + b + c - 5, 1 - c - b) ,
\]

\[
\hat{I}_2(a, b, c) = i \frac{(-1)^{a+b+c-1}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{3-a-b}(M_2^2)^{2-a-c} \mathcal{U}_0(a + b + c - 5, 1 - c - b) ,
\]

\[
\hat{I}_3(a, b, c) = i \frac{(-1)^{a+b+c}}{32\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{2-a-b}(M_2^2)^{4-a-c} \mathcal{U}_0(a + b + c - 6, 1 - c - b) ,
\]

\[
\hat{I}_4(a, b, c) = i \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{2-a-b}(M_2^2)^{3-a-c} \mathcal{U}_0(a + b + c - 6, 1 - c - b) ,
\]

\[
\hat{I}_5(a, b, c) = i \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{4-a-b}(M_2^2)^{2-a-c} \mathcal{U}_0(a + b + c - 6, 1 - c - b) ,
\]

\[
\hat{I}_6(a, b, c) = i \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a)\Gamma(b)\Gamma(c)}(M_1^2)^{3-a-b}(M_2^2)^{3-a-c} \mathcal{U}_0(a + b + c - 6, 2 - c - b) .
\]

Here, hat in Eqs. (28) and (29) denotes the double Borel transformed form of integrals. \( M_1^2 \) and \( M_2^2 \) are the Borel parameters in the \( s \) and \( s' \) channels, respectively, and the function \( \mathcal{U}_0(\alpha, \beta) \) is defined in the following way

\[
\mathcal{U}_0(a, b) = \int_0^\infty dy (y + M_1^2 + M_2^2)^a y^b \exp \left[ - \frac{B_{-1}}{y} - B_0 - B_1 y \right] ,
\]

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where

\[
B_{-1} = \frac{1}{M_1^2 M_2^2} \left[ m_b^2 m_1^2 + m_c^2 m_1^2 + M_1^2 M_2^2 (m_b^2 + m_c^2 + Q^2) \right],
\]

\[
B_0 = \frac{1}{M_1^2 M_2^2} \left[ (m_b^2 + m_c^2) M_1^2 + M_1^2 (m_b^2 + m_c^2) \right],
\]

\[
B_1 = \frac{m_c^2}{M_1^2 M_2^2}.
\]

After lengthy calculations, the following results for the gluon condensate contributions are obtained:

\[
\Pi_{i(i')}^{(G^2)} = -i \left( \frac{\alpha_s}{\pi} G^2 \right) C_{i(i')} \frac{C_{i(i')}}{12},
\]

where the explicit expressions for \( C_{i(i')} \) and are given in appendix–A.

Applying double Borel transformations with respect to the \( p^2 (p^2 \to M_1^2) \) and \( p'^2 (p'^2 \to M_2^2) \) on the phenomenological as well as the perturbative parts of the correlation function and evaluating the physical and QCD parts of the correlator, the following sum rules for the form factors \( V, A_1 \) and \( A_2 \) are obtained:

\[
V(V') = -\frac{(m_b + m_c)(m_{Bc} + m_X)}{8\pi^2 f_{Bc}^2 m_B^2 f_X m_X} e^{m_{Bc}/M_1} e^{m_X/M_2}
\]

\[
\times \left\{ \int_{(m+q)^2}^{s_0} ds' \int_{f_-(s')}^{\min(s_0,f_+(s'))} ds \rho_{V(V')} (s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} - i \left( \frac{\alpha_s}{\pi} G^2 \right) \frac{C_{V(V')}}{12} \right\},
\]

\[
A_1(A_1') = -\frac{(m_b + m_c)}{4\pi^2 f_{Bc} m_B^2 f_X m_X (m_{Bc} + m_X)} e^{m_{Bc}/M_1} e^{m_X/M_2}
\]

\[
\times \left\{ \int_{(m+q)^2}^{s_0} ds' \int_{f_-(s')}^{\min(s_0,f_+(s'))} ds \rho_{A_1(A_1')} (s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} - i \left( \frac{\alpha_s}{\pi} G^2 \right) \frac{C_{A_1(A_1')}}{12} \right\},
\]

\[
A_2(A_2') = \frac{m_X (m_b + m_c)(m_{Bc} + m_X)}{\pi^2 f_{Bc} m_B^2 f_X (m_{Bc}^2 + 3m_X^2 + Q^2)} e^{m_{Bc}/M_1} e^{m_X/M_2}
\]

\[
\times \left\{ \int_{(m+q)^2}^{s_0} ds' \int_{f_-(s')}^{\min(s_0,f_+(s'))} ds \rho_{A_2(A_2')} (s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} - i \left( \frac{\alpha_s}{\pi} G^2 \right) \frac{C_{A_2(A_2')}}{12} \right\},
\]

where \( s_0 \) and \( s'_0 \) are the continuum thresholds in \( B_c \) and \( X \) channels, respectively and \( f_\pm (s') \) in the lower and upper limit of the integral over \( s \) are obtained from inequality \((16)\) with respect to \( s \), i.e., \( s = f_\pm (s') \). By \( \min(s_0, f_+(s')) \), for each value of the \( q^2 \), between \( s_0 \) and \( f_+ \), the smaller one is selected. In above equation, in order to subtract the contributions of the higher states and the continuum the quark-hadron duality assumption is also used, i.e., it is assumed that

\[
\rho_{\text{higher states}} (s, s') = \rho_{\text{OPE}} (s, s') \theta(s - s_0) \theta(s - s'_0).
\]

In physical side and perturbative part of the correlation function, we also use the following Borel transformations

\[
B_{p^2} \left\{ \frac{1}{m^2(s) - p^2} \right\} = e^{\frac{m^2(s)}{M_1^2}},
\]
\[
B_{p^2} \left\{ \frac{1}{m^2(s') - p^2} \right\} = e^{\frac{m^2(s')}{m^2}}.
\]

At the end of this section, we would like to present the differential decay width of \( B_c \to X\nu\bar{\nu} \) decays in terms of the form factors. Using the amplitude in Eq. (2), we obtain the following expressions for the differential decay width of these transitions.

\[
\frac{d\Gamma}{dq^2}(B_c \to V\nu\bar{\nu}) = \frac{G_F^2}{2\pi^3} |V_{tb}V^*_{tb}|^2 \lambda^{1/2}(1, r_V, t') m_{B_c}^3 |C_{10}|^2 \times \left[ 8\lambda(1, r_V, t')t' V^2 (1 + \sqrt{r_V})^2 + \frac{1}{r_V} \left[ \lambda(1, r_V, t')^2 \frac{A_2^2}{(1 + \sqrt{r_V})^2} \right. \right. \\
\left. \left. + \ (1 + \sqrt{r_V})^2(\lambda(1, r_V, t') + 12r_V t')A_1^2 - 2\lambda(1, r_V, t')(1 - r_V - t')Re(A_1 A_2) \right] \right],
\]

\[
(35)
\]

\[
\frac{d\Gamma}{dq^2}(B_c \to AV\nu\bar{\nu}) = \frac{G_F^2}{2\pi^3} |V_{tb}V^*_{tb}|^2 \lambda^{1/2}(1, r_V, t') m_{B_c}^3 |C_{10}|^2 \times \left[ 8\lambda(1, r_V, t')t' V^2 (1 + \sqrt{r_V})^2 + \frac{1}{r_V} \left[ \lambda(1, r_V, t')^2 \frac{A_2^2}{(1 + \sqrt{r_V})^2} \right. \right. \\
\left. \left. + \ (1 + \sqrt{r_V})^2(\lambda(1, r_V, t') + 12r_V t')A_1^2 - 2\lambda(1, r_V, t')(1 - r_V - t')Re(A_1 A_2) \right] \right],
\]

\[
(36)
\]

where, \( \lambda(1, r_V, t') \) and \( \lambda(1, r_V, t') \) are the usual triangle function with

\[
\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc,
\]

\[
(37)
\]

and

\[
r_V = \frac{m_{V}}{m_{B_c}}, \quad r_V = \frac{m_{AV}}{m_{B_c}}, \quad t' = -\frac{Q^2}{m_{B_c}^2}.
\]

\[
(38)
\]

The total decay widths are obtained from integration of Eqs. (35) and (36) on \( q^2 \) in the interval \( 0 < q^2 < (m_{B_c} - m_{V(AV)})^2 \).

### 3 Numerical analysis

The explicit expressions for the form factors \( V, A_1, A_2, V', A'_1 \) and \( A'_2 \) and \( \frac{d\Gamma}{dq^2}(B_c \to X\nu\bar{\nu}) \) indicate that the main input parameters entering to the expressions are Gluon condensate, Wilson coefficient \( C_{10} \), elements of the CKM matrix \( V_{tb}, V_{ts} \) and \( V_{td} \), leptonic decay constants; \( f_{BC}, f_{D^*}, f_{D^*_s} \) and \( f_{D_{s1}} \), Borel parameters \( M_1^2 \) and \( M_2^2 \), as well as the continuum thresholds \( s_0 \) and \( s'_0 \). For the numerical values of the Gluon condensate, leptonic decay constants, CKM matrix elements, Wilson coefficient and quark and meson masses \( < \frac{\alpha_s}{\pi} G^2 > = 0.012 \text{ GeV}^4 \) [20], \( C_{10} = -4.669 \) [21, 22], \( | V_{tb} | = 0.77^{+0.18}_{-0.24} \), \( | V_{ts} | = (40.6 \pm 2.7) \times 10^{-3} \), \( | V_{td} | = (7.4 \pm 0.8) \times 10^{-3} \) [23], \( f_{D^*} = 266 \pm 32 \text{ MeV} \) [24],
The expressions for the form factors contain also four auxiliary parameters: Borel mass squares $M_1^2$ and $M_2^2$ and continuum threshold $s_0$ and $s'_0$. These are not physical quantities, hence the physical quantities, form factors, must be independent of these auxiliary parameters. We should find the ”working regions” of these parameters, where the form factors are independent of them. The parameters $s_0$ and $s'_0$, which are the continuum thresholds of $B_c$ and $X$ mesons, respectively, are determined from the conditions that guarantees the sum rules to have the best stability in the allowed $M_1^2$ and $M_2^2$ region. The values of continuum thresholds calculated from the two–point QCD sum rules are taken to be $s_0 = 45 \text{ GeV}^2$ and $s'_0 = 8 \text{ GeV}^2$ [7, 20, 24]. The working regions for $M_1^2$ and $M_2^2$ are determined by requiring that from one side, the continuum and higher states contributions are effectively suppressed and on the other side the gluon condensate contribution is small, which guarantees that the contributions of higher dimensional operators are small. Both conditions are satisfied in the regions $10 \text{ GeV}^2 \leq M_1^2 \leq 25 \text{ GeV}^2$ and $5 \text{ GeV}^2 \leq M_2^2 \leq 15 \text{ GeV}^2$.

The values of the form factors at $q^2 = 0$ are shown in Table 1:

| $B_c \rightarrow D^* \nu \bar{\nu}$ | $B_c \rightarrow D_s^* \nu \bar{\nu}$ | $B_c \rightarrow D_{s1}(2460) \nu \bar{\nu}$ |
|-------------------------------------|-------------------------------------|-------------------------------------|
| $V(0)$ | $0.27 \pm 0.016$ | $0.29 \pm 0.017$ | $0.30 \pm 0.017$ |
| $A_1(0)$ | $0.12 \pm 0.012$ | $0.15 \pm 0.014$ | $0.13 \pm 0.012$ |
| $A_2(0)$ | $-0.018 \pm 0.0016$ | $-0.03 \pm 0.002$ | $-0.07 \pm 0.005$ |

Table 1: The values of the form factors at $q^2 = 0$, for $M_1^2 = 18 \text{ GeV}^2$, $M_2^2 = 8 \text{ GeV}^2$.

For obtaining the $q^2$ dependent expressions of the form factors, we should consider a range of $q^2$ where the sum rules can reliably be calculated. Our sum rules for the form factors are truncated at $(1.2 - 2) \text{ GeV}$ below the perturbative cut. In order to extend our results to the full physical region, i.e., the regions $0 \leq q^2 \leq 18 \text{ GeV}^2$, $0 \leq q^2 \leq 17.2 \text{ GeV}^2$ and $0 \leq q^2 \leq 14.4 \text{ GeV}^2$ for $B_c \rightarrow D^* \nu \bar{\nu}$, $B_c \rightarrow D_s^* \nu \bar{\nu}$ and $B_c \rightarrow D_{s1} \nu \bar{\nu}$, respectively, we look for parameterization of the form factors in such a way that this parameterization coincides with the sum rules prediction. Our numerical calculations shows that the best parameterization of the form factors with respect to $-Q^2$ are as follows:

$$f_i(-Q^2) = \frac{f_i(0)}{1 + \alpha \hat{q} + \beta \hat{q}^2},$$

where $\hat{q} = -Q^2/m_{B_c}^2$. The values of the parameters $f_i(0)$, $\alpha$ and $\beta$ are given in the Tables 2, 3 and 4 for $B_c \rightarrow D^* \nu \bar{\nu}$, $B_c \rightarrow D_s^* \nu \bar{\nu}$ and $B_c \rightarrow D_{s1} \nu \bar{\nu}$, respectively.

At the end of this section, we would like to present the value of the branching ratio of these decays. Taking into account the $q^2$ dependencies of the form factors and performing integration over $q^2$ in Eqs. (35), (36) in the whole physical region and using the total life time of $B_c$ meson, $\tau \simeq 0.46 \text{ ps}$ [32], the branching ratio of the $B_c \rightarrow X \nu \bar{\nu}$ decays are
Table 2: Parameters appearing in the form factors of the $B_c \rightarrow D^* \nu \bar{\nu}$ decay in a two-parameter fit, for $M_1^2 = 18 \, GeV^2$, $M_2^2 = 8 \, GeV^2$.

|       | f(0) | $\alpha$ | $\beta$ |
|-------|------|----------|--------|
| $V$   | 0.27 | -7.76    | 22.83  |
| $A_1$ | 0.12 | -8.37    | 22.29  |
| $A_2$ | -0.018 | -11.63   | 33.52  |

Table 3: Parameters appearing in the form factors of the $B_c \rightarrow D_s^* \nu \bar{\nu}$ decay in a two-parameter fit, for $M_1^2 = 18 \, GeV^2$, $M_2^2 = 8 \, GeV^2$.

|       | f(0) | $\alpha$ | $\beta$ |
|-------|------|----------|--------|
| $V$   | 0.29 | -3.17    | 9.90   |
| $A_1$ | 0.15 | -3.60    | 8.13   |
| $A_2$ | -0.03 | -3.29   | 10.85  |

obtained as presented in Table 5. This Table also encompasses a comparison of our results with the existing predictions of the relativistic constituent quark model (RCQM) [13]:

From this Table, we see a good consistency between our results and that of the relativistic constituent quark model.

In summary, we investigated the rare $B_c \rightarrow X \nu \bar{\nu}$ transition with $X$ been axial vector particle, $AV(D_{s1})$, and vector particles, $V(D^*, D_s^*)$ in the framework of the three point QCD sum rules. The $q^2$ dependent expressions for the form factors were calculated. The quark condensates contributions to the correlation function were zero, so we considered the gluon corrections to the correlation function as a first nonperturbative contributions. Finally, we calculated the total decay width and branching ratio of these decays and compared our results with the predictions of the quark model. Our results are in good agreement with the relativistic constituent quark model.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
 & $f(0)$ & $\alpha$ & $\beta$ \\
\hline
$V$ & 0.30 & -1.30 & 4.35 \\
$A_1$ & 0.13 & -2.40 & 4.43 \\
$A_2$ & -0.07 & -1.25 & 4.30 \\
\hline
\end{tabular}
\caption{Parameters appearing in the form factors of the $B_c \to D_{s1}\nu\bar{\nu}$ decay in a two-parameter fit, for $M_1^2 = 18$ GeV$^2$, $M_2^2 = 8$ GeV$^2$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
decay & our result & RCQM [13] \\
\hline
$Br(B_c \to D^*\nu\bar{\nu}) \times 10^{-8}$ & 5.23 $\pm$ 0.12 & 5.78 \\
$Br(B_c \to D_{s1}^*\nu\bar{\nu}) \times 10^{-6}$ & 1.34 $\pm$ 0.25 & 1.42 \\
$Br(B_c \to D_{s1}\nu\bar{\nu}) \times 10^{-6}$ & 1.73$\pm$0.10 & - \\
\hline
\end{tabular}
\caption{Our results for the branching ratios and their comparisons with the prediction of the relativistic constituent quark model (RCQM) [13].}
\end{table}

Appendix—A

In this appendix we give the explicit expressions of the coefficients of the gluon condensate which enter to the sum rules for the form factors $V$, $A_1$ and $A_2$, respectively.

\begin{align*}
C_V &= -20 I_1(3, 2, 2)m_c^5 - 20 I_2(3, 2, 2)m_c^5 - 20 I_0(3, 2, 2)m_c^5 + 20 I_1(3, 2, 2)m_c^4 m_b \\
&+ 20 I_1(3, 2, 2)m_c^3 m_b^2 + 20 I_0(3, 2, 2)m_c^3 m_b^2 + 20 I_2(3, 2, 2)m_c^3 m_b^2 - 20 I_1(3, 2, 2)m_c^2 m_b^3 \\
&- 40 I_0(2, 2, 2)m_c^3 m_b^2 - 20 I_2(3, 1, 1)m_c^3 m_b^2 + 40 I_1^{[0, 1]}(3, 2, 2)m_c^3 m_b^2 - 20 I_0(3, 1, 2)m_c^3 m_b^2 \\
&+ 40 I_2^{[0, 1]}(3, 2, 2)m_c^3 - 60 I_2(4, 1, 1)m_c^3 - 40 I_1(2, 2, 2)m_c^3 + 20 I_2(3, 2, 1)m_c^3 \\
&- 40 I_2(2, 2, 2)m_c^3 - 60 I_0(4, 1, 1)m_c^3 + 40 I_0^{[0, 1]}(3, 2, 2)m_c^3 - 60 I_1(4, 1, 1)m_c^3 \\
&- 20 I_2(3, 2, 1)m_c^2 m_b + 60 I_1(4, 1, 1)m_c^2 m_b + 80 I_2(2, 3, 1)m_c^2 m_b + 20 I_0(3, 2, 1)m_c^2 m_b \\
&+ 40 I_1(2, 2, 2)m_c^2 m_b + 20 I_1(3, 2, 1)m_c^2 m_b - 40 I_1^{[0, 1]}(3, 2, 2)m_c^2 m_b + 40 I_1(2, 3, 1)m_c^2 m_b \\
&+ 120 I_0(1, 4, 1)m_c m_b^2 + 20 I_2^{[0, 1]}(3, 2, 2)m_c m_b^2 + 20 I_0^{[0, 1]}(3, 2, 2)m_c m_b^2 + 20 I_1^{[0, 1]}(3, 2, 2)m_c m_b^2 \\
&+ 120 I_1(1, 4, 1)m_c m_b^2 - 40 I_0(3, 2, 1)m_c m_b^2 - 40 I_2(3, 2, 1)m_c m_b^2 - 60 I_1(3, 2, 1)m_c m_b^2 \\
&- 120 I_1(1, 4, 1)m_c m_b^2 - 40 I_1(2, 3, 1)m_c m_b^2 + 40 I_1(3, 2, 1)m_c m_b^3 + 20 I_1(2, 2, 2)m_b^3 \\
&- 20 I_1^{[0, 1]}(3, 2, 2)m_b^3 + 40 I_1^{[0, 1]}(3, 1, 2)m_c + 40 I_1^{[0, 1]}(3, 2, 1)m_c + 40 I_2^{[0, 1]}(2, 2, 2)m_c \\
&+ 60 I_2^{[0, 1]}(3, 1, 2)m_c - 20 I_2^{[0, 2]}(3, 2, 2)m_c - 20 I_1(3, 1, 1)m_c - 20 I_1^{[0, 2]}(3, 2, 2)m_c \\
&+ 40 I_1^{[0, 1]}(2, 2, 2)m_c + 40 I_1^{[0, 1]}(3, 2, 1)m_c - 20 I_0(3, 1, 1)m_c + 20 I_2(3, 1, 1)m_c \\
&- 60 I_0(2, 1, 2)m_c - 40 I_1(1, 2, 2)m_c + 20 I_2^{[0, 1]}(3, 2, 1)m_c - 20 I_0^{[0, 2]}(3, 2, 2)m_c \\
&- 60 I_2(2, 1, 2)m_c + 60 I_0^{[0, 1]}(3, 1, 2)m_c - 40 I_2(1, 2, 2)m_c + 40 I_0(2, 2, 1)m_c
\end{align*}
\[ C_{A_1} = 10 I_0(3, 2, 2) m_c \epsilon m_b - 10 I_0(3, 2, 2) m_c^5 m_b^2 - 10 I_0(3, 2, 2) m_c^4 m_b^3 + 10 I_0(3, 2, 2) m_c^3 m_b^4 + 40 I_0(3, 2, 2) m_c^5 m_b + 10 I_0(3, 2, 2) m_c^4 m_b^2 - 40 I_0(3, 2, 2) m_c^3 m_b^3 - 40 I_0(3, 2, 2) m_c^2 m_b^4 - 40 I_0(3, 2, 2) m_c m_b^5 - 40 I_0(3, 2, 2) m_b^6 \]

\[ C_{A_2} = 10 I_3(4, 1, 2) m_c^5 + 10 I_2(3, 2, 2) m_c^5 + 20 I_2(4, 2, 1) m_c^5 + 10 I_2(4, 1, 2) m_c^5 \]
\[ -10 I_5(4, 1, 2)m_c^5 + 20 I_1(4, 2, 1)m_c^5 + 10 I_0(3, 2, 2)m_c^5 + 10 I_1(4, 1, 2)m_c^5 \\
+ 10 I_1(3, 2, 2)m_c^5 + 30 I_5(3, 3, 1)m_c^5 - 15 I_1(3, 3, 1)m_c^5 - 15 I_2(3, 3, 1)m_c^5 \\
- 30 I_5(4, 2, 1)m_c^5 - 30 I_3(3, 3, 1)m_c^5 + 30 I_3(4, 2, 1)m_c^5 - 10 I_5(3, 2, 2)m_c^4 m_b \\
+ 10 I_1(4, 2, 1)m_c^4 m_b - 10 I_5(4, 1, 2)m_c^4 m_b - 30 I_5(3, 3, 1)m_c^4 m_b + 80 I_2(2, 4, 1)m_c^4 m_b \\
- 10 I_2(2, 4, 1)m_c^4 m_b - 10 I_1(4, 2, 1)m_c^4 m_b + 10 I_3(4, 1, 2)m_c^4 m_b + 10 I_2(4, 2, 1)m_c^4 m_b \\
+ 30 I_3(3, 3, 1)m_c^4 m_b - 80 I_3(2, 4, 1)m_c^4 m_b + 10 I_5(3, 2, 2)m_c^4 m_b + 40 I_3(2, 4, 1)m_c^2 m_b^3 \\
- 40 I_5(2, 4, 1)m_c^2 m_b^3 - 20 I_3(2, 4, 1)m_b^5 + 20 I_5(2, 4, 1)m_b^5 + 20 I_1(2, 2, 2)m_c^3 \\
- 30 I_2(4, 1, 1)m_c^3 + 10 I_1(3, 2, 2)m_c^3 + 20 I_0(2, 2, 2)m_c^3 + 20 I_2(2, 2, 2)m_c^3 \\
- 15 I_0(3, 1, 2)m_c^3 - 10 I_1(3, 2, 2)m_c^3 - 30 I_1(4, 1, 1)m_c^3 + 40 I_1(3, 2, 1)m_c^3 \\
+ 20 I_2(3, 1, 2)m_c^3 + 40 I_2(3, 2, 1)m_c^3 - 30 I_1(2, 3, 1)m_c^3 - 30 I_2(2, 3, 1)m_c^3 \\
+ 20 I_0(3, 1, 2)m_c^3 - 20 I_1(1, 4, 1)m_c^2 m_b - 20 I_1(2, 4, 1)m_c^2 m_b + 20 I_2(3, 2, 1)m_c^2 m_b \\
- 5 I_0(2, 2, 2)m_c m_b + 40 I_5(3, 1, 2)m_c^2 m_b + 20 I_2(3, 2, 1)m_c^2 m_b + 10 I_0(3, 1, 2)m_c^2 m_b \\
- 40 I_3(2, 4, 1)m_c^2 m_b - 20 I_2(1, 4, 1)m_c^2 m_b + 20 I_1(3, 2, 1)m_c^2 m_b - 5 I_1(3, 2, 2)m_c m_b^2 \\
- 30 I_1(1, 4, 1)m_c m_b^2 + 5 I_1(3, 3, 2)m_c m_b^2 - 10 I_0(2, 2, 2)m_c m_b^2 - 30 I_2(1, 4, 1)m_c m_b^2 \\
+ 5 I_0(3, 1, 2)m_c m_b^2 - 40 I_5(3, 1, 2)m_c^3 + 40 I_3(2, 4, 1)m_b^3 - 5 I_0(2, 2, 2)m_b^3 \\
- 10 I_1(2, 4, 1)m_c m_b + 15 I_1(2, 1, 2)m_c + 20 I_2(2, 1, 2)m_c - 15 I_2(2, 1, 2)m_c \\
+ 15 I_2(2, 1, 2)m_c + 5 I_0(2, 1, 2)m_c + 5 I_1(1, 2, 2)m_c - 5 I_2(3, 2, 2)m_c \\
+ 10 I_2(3, 2, 1)m_c + 10 I_0(1, 2, 2)m_c - 15 I_2(3, 1, 1)m_c - 15 I_2(1, 3, 1)m_c \\
+ 5 I_2(2, 2, 2)m_c + 5 I_1(3, 2, 2)m_c - 5 I_0(3, 1, 2)m_c + 15 I_1(2, 3, 1)m_c \\
- 15 I_1(3, 1, 1)m_c + 20 I_1(2, 2, 1)m_c - 15 I_1(3, 3, 1)m_c + 10 I_2(2, 2, 2)m_c \\
+ 10 I_2(3, 1, 2)m_c - 10 I_1(3, 2, 1)m_c - 10 I_1(3, 1, 2)m_c + 10 I_2(2, 2, 1)m_c \\
- 25 I_0(2, 1, 2)m_b - 10 I_2(3, 1, 1)m_b + 40 I_5(1, 4, 1)m_b + 5 I_2(2, 2, 2)m_b \\
- 5 I_0(1, 2, 2)m_b + 40 I_3(2, 3, 1)m_b - 40 I_5(2, 3, 1)m_b + 20 I_5(2, 4, 1)m_b \\
+ 10 I_1(1, 4, 1)m_b - 5 I_5(2, 2, 2)m_b + 10 I_1(2, 2, 1)m_b - 40 I_3(1, 4, 1)m_b \\
- 20 I_3(2, 4, 1)m_b + 5 I_0(2, 2, 2)m_b ,
\]

\[ C_{V^*} = -20 I_1(3, 2, 2)m_c^5 - 20 I_2(3, 2, 2)m_c^5 - 20 I_0(3, 2, 2)m_c^5 + 20 I_1(3, 2, 2)m_c^4 m_b \\
+ 20 I_2(3, 2, 2)m_c^3 m_b^2 + 20 I_1(3, 2, 2)m_c m_b + 20 I_0(3, 2, 2)m_c^3 m_b^3 - 20 I_1(3, 2, 2)m_c^2 m_b^3 \\
- 40 I_0(2, 2, 2)m_c^3 + 40 I_0(3, 2, 2)m_c^3 - 40 I_2(2, 2, 2)m_c^3 - 40 I_1(2, 2, 2)m_c^3 \\
+ 40 I_1(3, 2, 2)m_c^3 - 60 I_0(4, 1, 1)m_c^3 + 40 I_0(3, 2, 2)m_c^3 - 60 I_2(4, 1, 1)m_c^3 \\
+ 20 I_2(3, 2, 2)m_c^3 - 20 I_0(3, 1, 2)m_c^3 - 60 I_1(4, 1, 1)m_c^3 + 60 I_3(3, 2, 1)m_c^3 \\
- 20 I_2(3, 2, 1)m_c^2 m_b + 40 I_1(3, 2, 2)m_c^2 m_b + 20 I_1(3, 2, 1)m_c^2 m_b + 60 I_1(4, 1, 1)m_c^2 m_b \\
+ 40 I_1(2, 3, 1)m_c^2 m_b + 80 I_0(2, 3, 1)m_c^2 m_b + 20 I_0(3, 2, 1)m_c^2 m_b + 40 I_1(2, 2, 2)m_c^2 m_b \\
- 60 I_1(3, 2, 1)m_c m_b^2 + 20 I_1(3, 2, 2)m_c m_b^2 + 20 I_0(3, 2, 2)m_c m_b^2 + 120 I_0(1, 4, 1)m_c m_b^2 \\
- 40 I_0(3, 2, 1)m_c m_b^2 - 40 I_2(3, 2, 1)m_c m_b^2 + 120 I_2(1, 4, 1)m_c m_b^2 + 120 I_1(1, 4, 1)m_c m_b^2 \\
+ 20 I_2(3, 2, 2)m_c m_b^2 - 120 I_1(1, 4, 1)m_c^3 - 20 I_0(3, 2, 2)m_b^3 + 40 I_1(3, 2, 1)m_b^3 }
\[C_A' = -10 I_0(3, 2, 2)m_6^6 m_b + 10 I_0(3, 2, 2)m_6^5 m_b^2 + 10 I_0(3, 2, 2)m_6^4 m_b^3 - 10 I_0(3, 2, 2)m_6^3 m_b^4 - 40 I_0(3, 2, 2)m_5^6 m_b + 10 I_0(3, 2, 2)m_5^5 m_b^2 + 10 I_0(3, 2, 2)m_5^4 m_b^3 - 10 I_0(3, 2, 2)m_5^3 m_b^4 - 40 I_0(3, 2, 2)m_4^6 m_b + 10 I_0(3, 2, 2)m_4^5 m_b^2 + 10 I_0(3, 2, 2)m_4^4 m_b^3 - 10 I_0(3, 2, 2)m_4^3 m_b^4 - 40 I_0(3, 2, 2)m_3^6 m_b + 10 I_0(3, 2, 2)m_3^5 m_b^2 + 10 I_0(3, 2, 2)m_3^4 m_b^3 - 10 I_0(3, 2, 2)m_3^3 m_b^4 - 40 I_0(3, 2, 2)m_2^6 m_b + 10 I_0(3, 2, 2)m_2^5 m_b^2 + 10 I_0(3, 2, 2)m_2^4 m_b^3 - 10 I_0(3, 2, 2)m_2^3 m_b^4 - 40 I_0(3, 2, 2)m_1^6 m_b + 10 I_0(3, 2, 2)m_1^5 m_b^2 + 10 I_0(3, 2, 2)m_1^4 m_b^3 - 10 I_0(3, 2, 2)m_1^3 m_b^4 - 40 I_0(3, 2, 2)m_0^6 m_b + 10 I_0(3, 2, 2)m_0^5 m_b^2 + 10 I_0(3, 2, 2)m_0^4 m_b^3 - 10 I_0(3, 2, 2)m_0^3 m_b^4\]
\[ C_{0_j} = -10 I_1 (3, 2, 2) c_e^5 - 10 I_2 (3, 2, 2) c_e^5 - 5 I_0 (3, 2, 2) c_e^5 + 10 I_5 (3, 2, 2) c_e^5 \]
\[-5 I_2 (3, 2, 2) m_c^4 m_b + 5 I_1 (3, 2, 2) m_c^4 m_b - 15 I_5 (3, 2, 2) m_c^4 m_b + 10 I_3 (3, 2, 2) m_c^4 m_b \]
\[+ 10 I_2 (3, 2, 2) m_c^3 m_b^2 + 10 I_1 (3, 2, 2) m_c^3 m_b^2 + 5 I_0 (3, 2, 2) m_c^3 m_b^2 - 10 I_5 (3, 2, 2) m_c^3 m_b^2 \]
\[+ 10 I_5 (3, 2, 2) m_c^3 m_b^2 - 10 I_3 (3, 2, 2) m_c^3 m_b^2 - 5 I_1 (3, 2, 2) m_c^3 m_b^2 + 5 I_2 (3, 2, 2) m_c^3 m_b^2 \]
\[+ 30 I_1 [0.1] (3, 2, 2) m_c^3 - 10 I_2 (3, 2, 2) m_c^3 + 10 I_2 (1, 2, 2) m_c^3 - 20 I_1 (3, 1, 2) m_c^3 \]
\[-5 I_0 (3, 1, 2) m_c^3 + 10 I_0 [0.1] (3, 2, 2) m_c^3 - 25 I_1 (3, 2, 1) m_c^3 + 20 I_0 [0.1] (3, 2, 2) m_c^3 \]
\[-15 I_2 (3, 2, 1) m_c^3 - 25 I_0 (3, 2, 1) m_c^3 - 20 I_0 [0.1] (3, 2, 2) m_c^3 - 15 I_0 (4, 1, 1) m_c^3 \]
\[-10 I_2 [0.1] (3, 2, 2) m_c^3 - 30 I_1 (4, 1, 1) m_c^3 - 20 I_2 (2, 2, 2) m_c^3 + 5 I_1 (2, 2, 2) m_c^3 \]
\[-10 I_1 (2, 3, 1) m_c^2 m_b - 5 I_2 (2, 2, 2) m_c^2 m_b - 20 I_0 (2, 3, 1) m_c^2 m_b + 20 I_5 [0.1] (3, 2, 2) m_c^2 m_b \]
\[-10 I_1 [0.1] (3, 2, 2) m_c^2 m_b - 20 I_3 [0.1] (3, 2, 2) m_c^2 m_b + 10 I_2 (3, 3, 1) m_c^2 m_b + 10 I_3 (3, 1, 2) m_c^2 m_b \]
\[+ 15 I_1 (4, 1, 1) m_c^2 m_b + 10 I_2 [0.1] (3, 2, 2) m_c^2 m_b + 5 I_0 (3, 2, 1) m_c^2 m_b + 20 I_1 (3, 2, 1) m_c^2 m_b \]
\[-15 I_2 (4, 1, 1) m_c^2 m_b - 10 I_2 (3, 1, 2) m_c^2 m_b + 10 I_2 (3, 2, 1) m_c^2 m_b - 10 I_2 (2, 2, 2) m_c^2 m_b \]
\[-20 I_1 (3, 2, 1) m_c^2 m_b + 30 I_0 (1, 4, 1) m_c^2 m_b - 10 I_1 (2, 2, 2) m_c^2 m_b + 10 I_3 [0.1] (3, 2, 2) m_c^2 m_b \]
\[+ 5 I_0 [0.1] (3, 2, 2) m_c^2 m_b - 15 I_0 (3, 2, 1) m_c^2 m_b - 5 I_0 [0.1] (3, 2, 2) m_c^2 m_b + 15 I_0 [0.1] (3, 2, 2) m_c^2 m_b \]
\[+ 60 I_1 (1, 4, 1) m_c^2 m_b^2 - 5 I_2 (3, 1, 2) m_c m_b^2 - 10 I_5 [0.1] (3, 2, 2) m_c m_b^2 + 5 I_1 (3, 1, 2) m_c m_b^2 \]
\[-10 I_2 (3, 2, 1) m_c m_b^3 + 5 I_2 [0.1] (3, 2, 2) m_c m_b^3 - 30 I_1 (1, 4, 1) m_c m_b^3 + 10 I_2 (2, 3, 1) m_c m_b^3 \]
\[-5 I_1 [0.1] (3, 2, 2) m_c m_b^3 + 10 I_1 (3, 2, 1) m_c m_b^3 + 10 I_5 [0.1] (3, 2, 2) m_c m_b^3 - 10 I_3 [0.1] (3, 2, 2) m_c m_b^3 \]
\[+ 30 I_2 (1, 4, 1) m_c m_b^3 - 10 I_2 (2, 3, 1) m_c m_b^3 + 20 I_3 [0.1] (2, 2, 2) m_c m_b - 20 I_0 (2, 2, 1) m_c \]
\[-5 I_0 [0.2] (3, 2, 2) m_c - 20 I_3 [0.1] (3, 2, 1) m_c - 15 I_5 [0.2] (3, 2, 2) m_c + 10 I_3 [0.1] (3, 1, 2) m_c \]
\[-10 I_3 [0.2] (3, 2, 2) m_c - 5 I_5 (3, 1, 1) m_c - 10 I_0 (3, 1, 1) m_c + 10 I_5 [0.2] (3, 2, 2) m_c \]
\[+ 30 I_1 [0.1] (3, 1, 2) m_c + 30 I_1 [0.1] (2, 2, 2) m_c + 20 I_3 [0.1] (3, 2, 1) m_c + 15 I_0 [0.1] (3, 2, 1) m_c \]
\[-25 I_1 (2, 2, 1) m_c - 20 I_5 [0.1] (2, 2, 2) m_c - 15 I_0 (2, 1, 2) m_c - 20 I_1 (2, 1, 2) m_c \]
\[-10 I_5 [0.1] (3, 1, 2) m_c + 15 I_0 [0.1] (3, 1, 2) m_c + 5 I_5 [0.2] (3, 2, 2) m_c + 5 I_1 (3, 1, 1) m_c \]
\[-15 I_5 [0.1] (3, 2, 1) m_c - 10 I_2 [0.1] (2, 2, 2) m_c - 10 I_2 (2, 1, 2) m_c + 35 I_1 [0.1] (3, 2, 1) m_c \]
\[-5 I_2 (2, 2, 1) m_c + 10 I_0 [0.1] (2, 2, 2) m_c - 10 I_1 [0.1] (2, 3, 1) m_c + 10 I_0 (1, 3, 1) m_c \]
\[+ 5 I_1 [0.2] (3, 2, 2) m_b - 10 I_3 [0.1] (2, 2, 2) m_b + 10 I_2 [0.1] (3, 2, 1) m_b + 15 I_2 (1, 2, 2) m_b \]
\[+ 20 I_2 (2, 2, 1) m_b + 10 I_3 [0.2] (3, 2, 2) m_b - 15 I_1 (1, 2, 2) m_b + 20 I_5 [0.1] (3, 1, 2) m_b \]
\[-5 I_5 [0.2] (3, 2, 2) m_b + 20 I_2 (2, 1, 2) m_b + 10 I_5 [0.1] (2, 3, 1) m_b - 20 I_3 [0.1] (3, 1, 2) m_b \]
\[-20 I_3 [0.1] (2, 3, 1) m_b + 10 I_5 [0.1] (2, 2, 2) m_b + 20 I_5 [0.1] (2, 3, 1) m_b + 10 I_2 [0.1] (3, 1, 2) m_b \]
\[-10 I_1^{[0,1]}(3, 1, 2)m_b + 10 I_5^{[0,1]}(3, 2, 1)m_b - 10 I_0(2, 2, 1)m_b - 10 I_1^{[0,1]}(3, 2, 1)m_b\]

where

\[
\hat{I}_n^{[i,j]}(a, b, c) = \left( M_1^2 \right)^i \left( M_2^2 \right)^j \frac{d^i}{d (M_1^2)^i} \frac{d^j}{d (M_2^2)^j} \left[ \left( M_1^2 \right)^i \left( M_2^2 \right)^j \tilde{I}_n(a, b, c) \right].
\]
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