Proportional Budget Allocations: A Systematization

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Abstract

We contribute to the programme of lifting proportionality axioms from the multi-winner voting setting to participatory budgeting. We define novel proportionality axioms for participatory budgeting and test them on known proportionality-driven rules such as Phragmén and Rule X. We investigate logical implications among old and new axioms and provide a systematic overview of proportionality criteria in participatory budgeting.

1 Introduction

First introduced in Porto Alegre, Brazil in 1988, Participatory budgeting (PB) is a democratic budgeting practice in which citizens are consulted, through some voting method, on how to best allocate a given budget to public projects. The practice is attracting increasing attention from both democracy practitioners worldwide and researchers, among others within computational social choice [Aziz and Shah, 2021].

The axiomatic study of PB has formulated a number of criteria for the desirable behaviour of participatory budgeting methods, or PB rules. Special attention has been dedicated to forms of ‘fairness’ or ‘proportionality’. Intuitively, one may want a PB rule to output a division of the available budget over the projects that ‘reflects’ divisions in the voters’ preferences. A variety of proportionality axioms has been proposed in recent literature, and this paper provides a first systematization of the axiomatic landscape of proportionality in PB and its special case of multi-winner voting (MWV, [Faliszewski et al., 2017]), or committee selection, that is, a PB setting where all projects (referred to as candidates) have identical cost.

State of the art The current understanding of proportionality in PB is rooted in proportional MWV [Skowron et al., 2017; Peters, 2018]. A key fairness axiom in MWV is justified representation (JR) [Aziz et al., 2017]. In short, JR requires that if a large enough group of voters agrees about a candidate, there is at least one candidate in the chosen committee that at least one of the group members approves. Proportionality requirements have been added with the axioms of extended justified representation (EJR) [Aziz et al., 2017] and proportional justified representation (PJR) [Sánchez-Fernández et al., 2017], following the intuition that if a larger group agrees about more candidates, they should be represented by more candidates in the winning committee. Aziz et al. [2018] generalize these concepts from MWV to PB within an approval voting framework. A related fairness concept is the core [Fain et al., 2016; Brandl et al., 2020; Peters and Skowron, 2020]. A set of projects, or bundle, is a core bundle if there is no subset of agents who can afford a different bundle (with their own share of the total budget) where every agent in that subset gets more utility than in the chosen bundle.

In [Peters and Skowron, 2020], two MWV rules both claiming to guarantee proportionality –Proportional Approval Voting (PAV) and Phragmén’s rule (or simply Phragmén)– are analysed, and shown to focus on different types of proportionality. PAV induces a fair distribution of welfare, so every group of agents gets a utility proportional to its size, while Phragmén can be seen as inducing a fair distribution of power: every group of agents has an influence on the proposed budget proportional to its size. The authors introduce two new proportionality axioms for MWV: priceability and laminar proportionality (LP), and a new rule, Rule X, that is similar to both PAV and Phragmén, but satisfies both new axioms and the aforementioned EJR. Finally, work in MWV has also started exploring the logical relations between the proportionality axioms proposed in the literature. For instance, Peters et al. [2021] show that, in MWV, the core implies EJR, PJR, and JR.

Moving to PB, research has focused on assessing the extent to which the above MWV axioms and rules can be meaningfully generalised to PB. Peters et al. [2020] generalise Rule X and EJR to PB, and show that, even in this context, Rule X satisfies EJR. The PB variant of Rule X is also shown to satisfy an approximation of the core and the axiom of priceability for PB. PAV is generalised to PB too, and shown to fail EJR when the projects’ unit-cost assumption is dropped.

Contribution We pursue three objectives with this paper. First, we continue extending proportionality axioms from MWV to PB, and propose novel generalizations of PJR and LP (Definitions 6 and 9). Second, we complete the assessment of Phragmén and Rule X in the PB setting with respect to the above notions of proportionality (Table 3). Third, we provide an overview of the logical relations between propor-
Figure 1: Relations among proportionality axioms in PB. Dashed lines indicate relations that only hold in MWV. Arrows are labelled either by results in this paper or by the paper where they have been proven. Some of the implications only hold under certain conditions or restrictions: laminar instances (Definition 8), and unanimity affordability (u-afford, Definition 10). Transitive arrows are omitted. The transitive closure of the diagram is complete: absence of arrows denotes the existence of a counterexample.

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2 Preliminaries

2.1 The Participatory Budgeting (PB) Problem

We denote the set of projects (or candidates) by \( C = \{ c_1, c_2, \ldots, c_m \} \) and the set of voters by \( N = \{ v_1, v_2, \ldots, v_n \} \). Each voter \( i \) comes with a function \( u_i : C \to [0,1] \). In contrast, in the special case of approval-based PB, the utility function is restricted to two values: \( u_i : C \to \{0,1\} \), determining voter \( i \)'s approval set \( A_i = \{ c \in C : u_i(c) = 1 \} \). The utility of a set of projects \( T \subseteq C \) for a set of voters \( S \subseteq N \) is defined additively as: \( u_S(T) = \sum_{i \in S} \sum_{c \in T} u_i(c) \). A profile \( P \) is a vector of the utility functions of all voters: \( P = (u_1, \ldots, u_n) \). A function cost: \( C \to \mathbb{Q}_+ \) assigns a cost to every project. The cost of a set of projects \( T \) is given by \( \text{cost}(T) = \sum_{c \in T} \text{cost}(c) \). The total budget is denoted by \( l \). If \( l \) is not mentioned, it is equal to 1. Hence, an election instance (also called a PB-instance) \( E = (N, C, \text{cost}, P, l) \) consists of a set of voters \( N \), a set of projects \( C \), a cost function, a profile \( P \), and a budget \( l \). If all else is clear in context, we abbreviate this to \( E = (P, l) \). With \( C(P) \) we denote the set of all projects occurring in \( P \).

A voting rule \( R \) maps an election instance \( E \) to a winning bundle. We name approval-PB-instance a PB-instance in which, for all \( i \in N \), \( u_i \) is restricted to \( \{0,1\} \). Approval-based voting rules always take as input an approval-PB-instance. The special case of an approval-PB-instance in which all projects have the same cost is called a MWV-instance. In line with existing literature, in the MWV setting, we refer to a bundle as a committee.

Table 1: The profile used in Examples 1 and 2. Each column contains the utilities per project of a voter. The budget \( l = 1 \).

### Table 1

| project | cost | utilities | Winning bundles |
|---------|------|-----------|----------------|
| \( c_1 \) | 0.4  | 0.1 0.7 0.1 0 | ✓ ✓ |
| \( c_2 \) | 0.3  | 0.3 0.4 0.4 | ✓ ✓ |
| \( c_3 \) | 0.2  | 0.1 0.2 0.4 0.4 | ✓ ✓ |
| \( c_4 \) | 0.35 | 0   0.4 0.2 1   | ✓ ✓ |

2.2 Voting Rules

We focus on three rules inspired by proportionality concerns: Phragmén, proportional approval voting (PAV) and Rule X. Below, we recall the generalizations of Phragmén, PAV, and Rule X to PB introduced by Peters et al. [2020]. Note that Phragmén and PAV are approval-based voting rules and that all three rules are not resolute and may return a tie.

**Phragmén** Every voter gets money continuously over time at the same rate. At the first moment \( l \) when there is a group of voters \( S \) who all approve a not-yet-selected project \( c \), and who together have cost\( (c) \) units of currency, the rule adds \( c \) to the bundle and asks the voters from \( S \) to pay the cost of \( c \) (i.e., the rule resets the balance of each voter from \( S \)), while the others keep their so-far earned money. The process stops when it would select a project which would overshoot the budget.

**PAV** The winning bundle \( W \) of PAV is the bundle with \( \text{cost}(W) \leq l \) that maximises the score \( \text{PAV-score}(W) = \sum_{i \in N} \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{\text{cost}(W)} \right) \).

**Rule X** The rule starts by giving each voter an equal fraction of the budget. In case of a budget of 1, each of the \( n \) voters gets \( \frac{1}{n} \) unit of currency. We start with an empty bundle \( W = \emptyset \) and sequentially add projects to \( W \). To add a project \( c \) to \( W \), the voters have to pay for \( c \). Write \( p_i(c) \) for the amount that voter \( i \) pays for \( c \); we will need that \( \sum_{i \in N} p_i(c) = \text{cost}(c) \). Let \( p_i(W) = \sum_{c \in W} p_i(c) \leq \frac{1}{n} \) be the total amount voter \( i \) has paid so far. For \( \rho \geq 0 \), we say that a project \( c \notin W \) is \( \rho \)-affordable if \( \sum_{i \in N} \min \left( \frac{1}{\text{cost}(c)} - p_i(W), u_i(c) \cdot \rho \right) = \text{cost}(c) \). The rule iteratively selects a project \( c \notin W \) that is \( \rho \)-affordable for a minimum \( \rho \). Individual payments are given by \( p_i(c) = \min \left( \frac{1}{\text{cost}(c)} - p_i(W), u_i(c) \cdot \rho \right) \). If no project is \( \rho \)-affordable for any \( \rho \), Rule X terminates and returns \( W \).

**Example 1.** Consider the profile in Table 1. To get an approval profile needed for Phragmén and PAV, we use a threshold of 0.3: voters approve a project when their utility for it is at least 0.3. The approved projects are shaded. Phragmén will first select \( c_2 \) at time \( t = 0.1 \), which leaves voter \( v_3 \) with 0.1 units of currency. Then at \( t = 0.275 \), \( v_2 \) and \( v_4 \) can together buy \( c_4 \), which leaves \( v_1 \) with 0.175 and \( v_3 \) with 0.275. After adding \( c_4 \), the rule ends since both remaining projects are not affordable and outputs \( W = \{ c_2, c_3 \} \). For PAV, we compute the PAV-score of four sets: PAV-score\( \{ c_1, c_2 \} = 3.5 \), PAV-score\( \{ c_1, c_3 \} = 4 \), PAV-score\( \{ c_2, c_3 \} = 4.5 \), and PAV-score\( \{ c_2, c_4 \} = 4 \) (clearly, smaller sets have a lower score and larger sets are not affordable). Hence, PAV will select \( W = \{ c_2, c_3 \} \). Rule X starts by giving ev-
every voter $\frac{1}{c}$ unit of currency. First, $c_4$ is $\rho$-affordable for $\rho = \frac{0.35}{10} \approx 0.219$, then, for $\rho = \frac{0.4}{10} \approx 0.222$, $c_1$ is $\rho$-affordable. After selecting $c_1$, the sum of the remaining amounts of all voters is 0.25, so the other projects are not $\rho$-affordable for any $\rho$. Hence, Rule X returns $W = \{c_1, c_4\}$.

2.3 Known Proportionality Axioms

We recall axioms for PB from Peters et al. [2020] that generalize known MWV axioms. We start with the axioms of core and extended justified representation (EJR).

Definition 1 (Core). For a given PB-instance $E = (N, C, cost, P, l)$, a bundle $W$ is in the core if for every $S \subseteq N$ and $T \subseteq C$ with $|S| \geq \frac{\text{cost}(T)}{n}$ there exists $i \in S$ such that $u_i(W) \geq u_i(T)$. A voting rule $R$ satisfies the core property if for each PB-instance $E$ the winning bundle $R(E)$ is in the core.

To define EJR, we first introduce $(\alpha, T)$-cohesiveness:

Definition 2 ($(\alpha, T)$-cohesiveness). A group of voters $S$ is $(\alpha, T)$-cohesive for $\alpha : C \to [0, 1]$ and $T \subseteq C$, if $|S| \geq \frac{\alpha(T)}{n}$ and if it holds that $u_i(c) \geq \alpha(c)$ for every voter $i \in S$ and each project $c \in T$.

Definition 3 (Extended justified representation (EJR)). A rule $R$ satisfies extended justified representation (EJR) if for each PB-instance $E$ and each $(\alpha, T)$-cohesive group of voters $S$, there is a voter $i \in S$ such that $u_i(R(E)) \geq \sum_{c \in T} \alpha(c)$.

We now turn to priceability (also from Peters et al. [2020]). The notion of price system needs to be introduced first.

Definition 4 (Price systems). A price system is a pair $ps = (b, (p_i)_{i \in N})$ where $b \geq 1$ is the initial budget, and for each voter $i \in N$, there is a payment function $p_i : C \to \mathbb{R}$ such that (1) a voter can only pay for projects she gets at least some utility from: if $u_i(c) = 0$, then $p_i(c) = 0$ for each $i \in N$ and $c \in C$, and (2) each voter can spend the same budget of $\frac{b}{n}$ units of money: $\sum_{c \in C} p_i(c) \leq \frac{b}{n}$ for each $i \in N$.

Definition 5 (Priceability). A rule $R$ satisfies priceability (is priceable) if for each PB-instance $E$, there exists a price system $ps = (b, (p_i)_{i \in N})$ that supports $R(E)$, i.e. (1) for each $c \in R(E)$, the sum of the payments for $c$ equals its price: $\sum_{i \in N} p_i(c) = cost(c)$; (2) no project outside of the winning bundle gets any payment: for all $c \notin R(E)$, $\sum_{i \in N} p_i(c) = 0$; and (3) there exists no non-selected project whose supporters in total have a remaining unsubt bank budget of more than its cost: for all $c \notin R(E)$, $\sum_{i \in N, s.t. u_i(c) > 0} \left(\frac{b}{n} - \sum_{c' \in R(E)} p_i(c')\right) \leq cost(c)$.

The above properties are defined for PB. Throughout the paper, when needing to refer to the MWV specialization of a PB axiom, we will assume the axiom is defined on MWV-instances. We say that a rule satisfies an axiom in MWV-instances if, for all MWV-instances, the rule’s winning bundle satisfies the axiom. Observe also that, since Phragmén

3 Two Novel Axioms for PB

3.1 Proportional Justified Representation in PB

Sánchez-Fernández et al. [2017] define proportional justified representation for MWV (we refer to it as MWV-PJR here). We generalize this axiom to PB based on the generalisation of EJR provided by Peters et al. [2020]. Two steps are involved: dropping the unit-cost assumption, and allowing arbitrary utilities instead of just approval ones.

Definition 6 (Proportional justified representation (PJR)). A rule $R$ satisfies proportional justified representation (PJR) if for each PB-instance $E$ and each $(\alpha, T)$-cohesive group of voters $S$,

$$\sum_{c \in R(E)} \left(\max_{i \in S} u_i(c)\right) \geq \sum_{c \in T} \alpha(c). \quad (1)$$

The intuition is that, in the winning bundle, for each cohesive group $S$ (cohesive in that $S$ agrees to a certain degree about the set of projects $T$) there should be enough projects to which at least one voter in $S$ assigns enough utility.

Example 2. Consider the profile in Table 1, with the bundle $W = \{c_2, c_3\}$ as selected by PAV. Now consider the group $S = \{v_1, v_2\}$. Probably both voters in $S$ are happy that $c_2$ is selected, but they would both get more utility from $c_1$ than from the selected $c_2$ or $c_3$. Also, if each of them would get their share of the total budget ($\frac{1}{4}$), they could together afford the set $T = \{c_1\}$. Intuitively then, $W$ is not a fair bundle considering voters $v_1$ and $v_2$. Let us look at it more formally. $S$ is $(\alpha, T)$-cohesive for $\alpha(c_1) = 0.7$ $(\alpha(c_2), \alpha(c_3)$ and $\alpha(c_4)$ are arbitrary) since the voters in $S$ can afford $T$ with their share of the budget, and for both of them $u(c_1) \geq \alpha(c_1)$. However, Equation 1 is not satisfied: $\sum_{c \in W} \left(\max_{i \in S} u_i(c)\right) = 0.4 + 0.2 = 0.6$, while $\sum_{c \in T} \alpha(c) = 0.7$. This shows that in the given election instance, the bundle $W$ does not satisfy PJR.

Our definition of PJR for PB is rather different from its MWV variant. It requires some work to show that the proposed definition reduces to the definition of MWV-PJR under unit-cost assumption and approval preferences. First of all, let us recall the definition of MWV-PJR:

Definition 7 (MWV-PJR [Sánchez-Fernández et al., 2017]). An approval based voting rule $R$ satisfies MWV-PJR if for every ballot profile $P$ and committee size $k$, the rule outputs a committee $W = R(P, k)$ s.t.: for every $\ell \leq k$ and every $\ell$-cohesive set of voters $S \subseteq N$, it holds that $|W \cap (\cup_{i \in S} A_i)| \geq \ell$, where a set $S$ is $\ell$-cohesive if $|S| \geq \ell \frac{n}{k}$ and $|\cap_{i \in S} A_i| \geq \ell$.

We will need the following lemma:

Lemma 1. Let $E = (N, C, cost, P, l)$ be an approval-PB-instance where for all projects $c, cost(c) = \frac{1}{k}$. Then: (a) for given $\alpha : C \to [0, 1]$ and $T \subseteq C$, any group of voters $S \subseteq N$ that is $(\alpha, T)$-cohesive is also $\ell$-cohesive for $\ell = |T|$ with $T' = \{c \in T : \alpha(c) > 0\}$; and (b) for every group $S$ that is $\ell$-cohesive there are $T \subseteq C$ with $|T| = \ell$ and $\alpha : C \to [0, 1]$ with $\alpha(c) = 1$ for all $c \in C$, such that $S$ is $(\alpha, T)$-cohesive.
Theorem 2. MWV-PJR and PJR are equivalent in MWV-instances.

Proof Sketch. We show that in every MWV-instance, a rule that satisfies the new definition of PJR also satisfies MWV-PJR and a rule that satisfies MWV-PJR also satisfies PJR.

PJR $\Rightarrow$ MWV-PJR: Assume that a rule $\mathcal{R}$ satisfies PJR, and take an arbitrary MWV-instance $E$. Because $E$ satisfies the assumptions of unit-cost and approval based voting, the fact that $\mathcal{R}$ satisfies PJR boils down to: for any $S, C \subseteq N$ with $|S| \geq |T| \cdot \frac{1}{2}$ and for which $\forall c \in C$ with $\alpha(c) \geq \alpha(c)$ for all $i \in S$ and for all $c \in T$, it is the case that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$. Take arbitrary $S \subseteq N$ and $\ell \leq k$ and suppose that $S$ is $\ell$-cohesive. According to Lemma 1(b), there are $T \subseteq C$ with $|T| = \ell$ and $\alpha : C \rightarrow [0, 1]$ with $\alpha(c) = 1$ for all $c \in C$, such that $S$ is $(\alpha, T)$-cohesive. Because $\mathcal{R}$ satisfies PJR, this implies that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$. However, because of our choice of $T$ and $\alpha$, we know that $\sum_{c \in T} \alpha(c) = |T| = \ell$, so $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell$, which shows that $\mathcal{R}$ satisfies MWV-PJR.

MWV-PJR $\Rightarrow$ PJR: Assume that a rule $\mathcal{R}$ satisfies MWV-PJR. Take arbitrary MWV-instance $E$, and suppose that a group $S$ is $(\alpha, T)$-cohesive. Then according to Lemma 1(a), when we take $T' = \{c \in T : \alpha(c) > 0\}$, $S$ is $\ell$-cohesive for $\ell = |T'|$. Because $\mathcal{R}$ satisfies MWV-PJR, it follows that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell = |T'|$. By definition of $T'$, $|T'| \geq \sum_{c \in T} \alpha(c) = \sum_{c \in T} \alpha(c)$, so $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$, which is what we had to prove.

Remark 1. To our knowledge, besides Definition 6, the only generalization of MWV-PJR to PB is from Aziz et al. [2018]. They define an axiom called Strong-BPJRL that requires the following: For a budget $l$, a bundle $W$ is Strong-BPJRL if for all $\ell \in [1, l]$ there does not exist a set of voters $S \subseteq N$ with $|S| \geq \ell \cdot 2^l$, such that $\text{cost}(\cup_{i \in S} A_i) \geq \ell$ but $\text{cost}(\cup_{i \in S} A_i) \cap W < \ell$. It is possible to generalise this definition further to allow arbitrary utilities instead of approval votes. However, note that the requirement in this definition is not that for every $\ell$-cohesive $S$ the utility of the projects they all approve that are selected is at least $\ell$, but rather the cost of this set of projects. Although this is indeed a generalisation of MWV-PJR, as is shown by Aziz et al. [2018], we consider that the aim of PJR is to ensure a certain level of utility for every group of voters, rather than a certain cost. Definition 6 is equivalent to Strong-BPJRL when assuming that a project’s cost is directly proportional to a voter’s utility from it.

3.2 Laminar Proportionality in PB

The basic idea of LP for MWV [Peters and Skowron, 2020] is that if we know about a strict separation between different parties, we can divide the chosen projects proportionally over the parties. We generalize this notion to PB in the approval voting setting by taking the budget $l$ instead of the bundle size $k$, and by using the cost of each project instead of unit-cost.

Definition 8 (Laminar PB-instances). An approval-PB-instance $(P, l)$ is laminar if either: (1) $P$ is unanimous and $\text{cost}(C(P)) \geq l$; (2) there is a project $c \in C(P)$ such that $c \in A_i$ for all $A_i \in P$, the profile $P_{-c}$ is not unanimous and the instance $(P_{-c}, l - \text{cost}(c))$ is laminar (with $P_{-c} = (A_1 \setminus \{c\}, \ldots, A_n \setminus \{c\})$); or (3) There are two laminar PB-instances $(P_1, l_1)$ and $(P_2, l_2)$ with $C(P_1) \cap C(P_2) = \emptyset$ and $|P_1| \cdot l_2 = |P_2| \cdot l_1$ such that $P = P_1 + P_2$ and $l = l_1 + l_2$.

Example 3. The instance $P$ in Table 2 associated with the budget $l = 10$ is laminar. The instance $P_1$ with $v_1$ and $v_2$ and projects $c_1, c_2$, and $c_3$ with limit $l_1 = 6$ satisfies Definition 8.1, as does the instance $P_2$ with only voter $v_3$ and projects $c_4$ and $c_5$ and limit $l_2 = 3$. Those two instances can be added by Definition 8.3 since $|P_1| \cdot l_2 = 2 \cdot 3 = 6 = |P_2| \cdot l_1$. Then $c_6$ can be added by Definition 8.2 to get $P$ with limit $l = 6 + 3 + \text{cost}(c_6) = 10$.

Definition 9 (Laminar proportionality (LP)). A rule $\mathcal{R}$ satisfies laminar proportionality (LP) if for every laminar PB-instance $E = (P, l)$, $\mathcal{R}(E) = W$ where $W$ is a laminar proportional, i.e. (1) if $P$ is unanimous, then $W \subseteq C(P)$ (if everyone agrees, then part of the projects they agree on is chosen); (2) if there is a unanimously approved project $c$ s.t. $P_{-c} \cup \{c\}$ is laminar, then $W = W \cup \{c\}$ where $W$ is a bundle which is laminar proportion for $(P_{-c}, l - \text{cost}(c))$; or (3) If $P$ is the sum of laminar PB-instances $(P_1, l_1)$ and $(P_2, l_2)$, then $W = W_1 \cup W_2$ where $W_1$ is laminar proportional for $(P_1, l_1)$ and $W_2$ is laminar proportional for $(P_2, l_2)$.

It is trivial that in case of unit-cost and budget $k$, these definitions are equivalent to the corresponding MWV definitions.

Example 4. Elaborating on Example 3, bundle $W = \{c_1, c_2, c_4, c_6\}$ (grey in Table 2) is laminar proportional in that instance with a budget of $l = 10$. In $(P_1, l_1)$, $(c_1, c_2)$ is laminar proportional, as is $(c_3)$ in $(P_2, l_2)$. Hence, $(c_1, c_2, c_4)$ is laminar proportional in $(P_1 + P_2, l_1 + l_2)$, and $(c_1, c_2, c_4, c_6)$ is laminar proportional in $(P, l)$.

4 Proportionality Properties of Rules

As Table 3 illustrates, the literature already provides a full picture of the proportionality properties of PAV: it does not satisfy the core, priceability, or LP in MWV (and hence not in PB either). On MWV-instances, PAV satisfies PJR, but as shown in [Peters et al., 2020, Figure 2], not on PB-instances. We turn now to our analysis of Phragmén and Rule X.

Proposition 1. Phragmén satisfies PJR.

Proof. Assume towards a contradiction that there exist a group of voters $S \subseteq N$, a set of projects $T \subseteq C$, and a function $\alpha : C \rightarrow [0, 1]$ such that $S$ is $(\alpha, T)$-cohesive, and...
for this $S$, $\alpha$, and $T$, the winning bundle $W$ of Phragmén does not contain enough projects that voters from $S$ like enough: $\sum_{c \in W} (\max_{i \in S} u_i(c)) < \sum_{c \in T} \alpha(c)$. Note that in approval-PB-instances this boils down to $|W \cap \bigcup_{i \in S} A_i| < \sum_{c \in T} \alpha(c)$. Because of $(\alpha, T)$-cohesiveness, for every voter $i \in S$ and each project $c \in T$, $u_i(c) \geq \alpha(c)$, so either $\alpha(c) = 0$ or $c \in A_i$, and therefore $\sum_{c \in T} \alpha(c) \leq |T \cap \bigcup_{i \in S} A_i|$. We write $W'$ for $T \cap \bigcup_{i \in S} A_i$, and $W''$ for $W \cap \bigcup_{i \in S} A_i$. Hence, $|W'| < |T'| \leq |T|$. Let $t$ be the moment when the rule stops: a project $c$ is reached that would overshoot the budget. Clearly, cost($W$) + cost($c$) > 1, but cost($W''$) ≤ 1. Let $x$ be the amount of virtual money earned by all voters so far, so

$$t \cdot n = x = \text{cost}(W) + \text{cost}(c) + y,$$  \hspace{1cm} (2)

where $y \geq 0$ is the money that non-supporters of $c$ have earned in the meantime. Because $|W'| < |T'|$, there must be some project in $T$ that is not in $W$. The voters in $S$ together have earned $\frac{x}{n} \cdot |S|$, and because $S$ is $(\alpha, T)$-cohesives, $|S| \geq \text{cost}(T) \cdot n$, so

$$\frac{x}{n} \cdot |S| \geq \text{cost}(T) \cdot n \cdot \frac{x}{n} = \text{cost}(T) \cdot x.$$  \hspace{1cm} (3)

From (2) and the fact that cost($W$) + cost($c$) > 1, it follows that $x > 1 + y$ (and $x > 1$). From (3), it follows that the voters in $S$ have earned enough together at time $t$ to buy all projects from $T$ (and therefore from $T''$), but have not done so. Hence, either they have also paid for projects not in $T''$, or, if they only spent their money on projects in $T''$, $c$ must be in $T''$, i.e., they do have the virtual money to buy $T''$ but would overshoot the budget. In the first case, for every project not in $T''$ that members of $S$ pay for, $|W''|$ grows by one (since they can only pay for projects they approve). In order to keep $|W''| < |T''|$, the mean amount of money they have paid at time $t$ for such a project must be greater than the mean cost of projects in $T''$. Otherwise, the number of projects they would pay for (that they approve of and that are selected) would exceed the number of projects in $T''$. However, for each project not in $T''$ that voters from $S$ pay for, they should (as a group) pay less than the cost of any project from $T''$ not yet selected. Otherwise, they would have paid earlier for a cheaper project from $T''$. This is a contradiction. Hence, the voters from $S$ only spent their money on projects from $T''$, and $c \in T''$. Let us assume that $c$ is the last project from $T''$ that is not yet selected. Because the rule stops exactly when $c$ can be paid by its supporters, we know that at that point in time, the voters in $S$ have earned exactly cost($T''$) units of money, so $t \cdot |S| = \text{cost}(T'')$. Hence, the total amount of money earned at time $t$ is $x = t \cdot n = \text{cost}(T'') \cdot \frac{n}{|S|}$. Because $S$ is $(\alpha, T)$-cohesives, we know that cost($T''$) ≤ cost($T$) ≤ $\frac{|S|}{n}$, so

$$x = \text{cost}(T'') \cdot \frac{n}{|S|} = \frac{|S|}{n} \cdot \frac{n}{|S|} = 1.$$  \hspace{1cm} (4)

However, we also had that $x > 1 + y > 1$. This is a contradiction with (4), as desired.

**Proposition 2. Phragmén satisfies priceability.**

Recall that Peters and Skowron [2020] shows that Phragmén does not satisfy EJR in MWV-instances. Since the core implies EJR, Phragmén does not satisfy the core in MWV-instances either, and hence it does not satisfy the core or EJR in PB-instances either.

**Proposition 3. Phragmén and Rule X do not satisfy LP.**

Proof sketch. LP requires any affordable unanimously approved projects to be selected, but Phragmén and Rule X do not necessarily select those if their cost is high enough compared to the other projects.

**Proposition 4. Rule X satisfies PJR.**

Proof. In Peters et al. [2020] Rule X was shown to satisfy EJR. In Theorem 5, we will show that EJR implies PJR. Hence Rule X also satisfies PJR.

5 Relations Between Axioms

We study the logical relationships among the axioms we introduced and report on the results recapitulated in Figure 1.

5.1 Priceability, PJR, EJR, and the Core

We start by showing that PJR, EJR, and the core do not imply priceability in MWV-instances, even in laminar ones.

**Theorem 3. In laminar MWV-instances there exist bundles that satisfy PJR, EJR, or are in the core, but are not priceable.**

Since laminar MWV-instances are a specific type of MWV-instances, which are a specific type of PB-instances, the same result holds for MWV and PB.

In MWV-instances, every priceable bundle satisfies PJR, as is shown by Peters and Skowron, 2020, Prop. 1. This raises the question whether this relation is also present in the PB setting. We show now that this is not the case.

**Theorem 4. There are priceable bundles not satisfying PJR.**
Proof sketch. PJR is based on the utility of the voters being higher than some threshold \( \alpha(c) \), while priceability only discriminates between utilities of 0 and utilities above zero. Which value above zero a utility has does not make any difference in the property of being priceable. Therefore, it is possible to construct a PB-instance with a bundle that is priceable but does not satisfy PJR.

From this result, it follows that priceability neither implies EJR nor the core. Priceability would otherwise imply PJR, since the core implies EJR, and EJR implies PJR (Theorem 5). These results hold even for MWV (shown in appendix).

**Theorem 5. EJR implies PJR.**

**Proof.** Suppose that rule \( R \) satisfies EJR and take an \((\alpha, T)\)-cohesive group of voters \( S \) for some \( \alpha : T \to [0, 1], T \subseteq C \). Because \( R \) satisfies EJR, there is a voter \( i \in S \) such that \( u_i(R(E)) \geq \sum_{c \in T} \alpha(c) \). For this voter \( i \), \( \sum_{c \in R(E)} u_i(c) \geq \sum_{c \in T} \alpha(c) \). Hence, \( R \) satisfies PJR, since \( \sum_{c \in R(E)} \max_{c \in S} u_i(c) \geq \sum_{c \in T} \alpha(c) \).

**5.2 Laminar Proportionality and Priceability**

We first show that in MWV-instances, and therefore also in PB-instances, priceability does not imply LP.

**Theorem 6. In laminar MWV-instances there exist priceable bundles that are not laminar proportional.**

However it is worth reporting that for a class of price systems (the so-called balanced systems) it is possible to prove that the implication goes through (Appendix B.1). On the other hand, LP does imply priceability on laminar election instances even in the general PB setting.

**Theorem 7. LP implies priceability in laminar PB-instances.**

**Proof sketch.** By induction on the structure of laminar PB-instances. We show that for every bundle \( W \) that is laminar proportional for a laminar PB-instance \((P, l)\), there exists a price system \( ps = (b, \{p_i\}_{i \in N}) \) where \( b = \text{cost}(W) \).

Consider three cases. The first is the basis of the induction. If \( P \) is unanimous with \( \text{cost}(C(P)) \geq \ell \) and \( W \) is laminar proportional for \((P, l)\) (with \( \text{cost}(W) \leq \ell \)), then \( W \subseteq C(P) \), so voters can divide their budget over \( W \). With an initial \( b = \text{cost}(W) \), all and only the projects in \( W \) can be bought.

In the second case, a unanimously approved project \( c \in W \) such that \( W' = W \setminus \{c\} \) is laminar proportional. By the inductive hypothesis, we know that there exists a price system \( ps' \) with initial budget \( b' = \text{cost}(W') \). Because \( c \) is unanimously approved, all voters could pay for \( c \). We know that in \( ps' \), there was no project not in \( W' \) that was affordable to its supporters. If every voter got \( \text{cost}(c) \) more budget, to spend entirely on \( c, c \) would get funded and no voter would have more unspent budget than before. Also, the initial budget of every voter is now \( \frac{b'}{n} + \text{cost}(c) = \frac{\text{cost}(W') + \text{cost}(c)}{n} = \frac{\text{cost}(W)}{n} = \frac{b}{n} \) units of money, and the initial budget is \( b = \text{cost}(W) \) and all the individual payment functions stay the same. Because for every project \( c \) in \( W' \) the sum of the individual payments was equal to \( \text{cost}(c) \), this is also the case for every project in \( W \).

In the third case, the profile consists of two laminar profiles: \( P = P_1 + P_2 \) and \( l = l_1 + l_2 \), where \( W_1 \) and \( W_2 \) are laminar proportional for respectively \((P_1, l_1)\), \((P_2, l_2)\). Take \( W = W_1 \cup W_2 \), which is by definition laminar proportional for \((P, l)\). By the inductive hypothesis, there exist price systems \( ps_1 = (b_1, \{p_{1,i}\}_{i \in N}) \) and \( ps_2 = (b_2, \{p_{2,i}\}_{i \in N}) \) with initial budgets \( b_1 = \text{cost}(W_1) \) and \( b_2 = \text{cost}(W_2) \), and that \( |P_1| \cdot l_2 = |P_2| \cdot l_1 \). We can now define a price system \( ps \) that supports \( W \) as follows: \( ps = (b, \{p_i\}_{i \in N}) \) with \( b = \text{cost}(W) = b_1 + b_2 \), and for all voters \( i \in N \), \( p_i(c) = p_{1,i}(c) + p_{2,i}(c) \), where \( p_{1,i} \) and \( p_{2,i} \) are extended versions of respectively \( p_{1,i} \) and \( p_{2,i} \) that yield zero for the projects that those are not defined for. It is easy to check that \( ps \) supports \( W \) according to the criteria from Def. 5.

**5.3 Laminar Proportionality and the Core**

**Theorem 8. In laminar MWV-instances, there exist bundles that satisfy PJR, EJR, or are in the core, but do not satisfy LP.**

**Theorem 9. There exist laminar proportional bundles that do not satisfy PJR, EJR, or are not in the core.**

Theorem 9 holds because of instances where there is one relatively expensive unanimously approved project, that is included in any laminar proportional bundle, but where there are many cheap projects that can be satisfactory enough for a group of voters. A full counterexample is provided in the appendix. We show now that under certain restrictions, laminar proportional bundles are in the core (and hence also satisfy EJR and PJR). In general, a set of projects is in the core if for every group of voters \( S \subseteq N \) and set of projects \( T \subseteq C \) such that \( S \) can afford \( T \) with their share of the budget there is a voter \( i \in S \) such that \( u_i(W) \geq u_i(T) \). We will show that if for any unanimously approved project \( c \in W \) either \( c \) is part of \( T \) or if there is some project in \( T \) that costs at least as much as \( c \), then there is such a voter \( i \in S \) with \( u_i(W) \geq u_i(T) \). We define a property of bundles called unanimity-affordability (u-afford):

**Definition 10.** A bundle \( T \) is u-affordable w.r.t. instance \((P, l)\) whenever for any unanimously approved project \( c \in C(P) \) there exists \( t \in T \) s.t. \( \text{cost}(t) \geq \text{cost}(c) \).

Since \( \text{cost}(c) \geq \text{cost}(c) \), the definition is also satisfied if \( c \in T \). We will show that in laminar PB-instances, laminar proportional bundles satisfy the core subject to u-afford:

**Theorem 10. Laminar proportional bundles satisfy the core subject to u-afford.**

Since the core implies EJR and PJR (as showed by Peters et al. [2020] and Theorem 5), any laminar profile also satisfies EJR and PJR under the same restrictions. Note that in MWV-instances there always is a project in \( T \) that has cost of at least \( \text{cost}(c) \), so our restriction is always met. Hence, there, laminar bundles are in the core, and so satisfy EJR and PJR.

**Corollary 10.1.** In MWV-instances, LP implies PJR, EJR, and the core.

**6 Conclusions and Future Work**

Our study completes the picture of the logical relations between key proportionality axioms in PB. With respect to the
main proportionality-inspired rules, we showed that price-ability and PB generalizations of PJR and LP do not discriminate between Phragmén and RuleX (unlike EJR).

We focused on the basic PB framework. How proportionality axioms behave in richer settings (e.g., diversity constraints [Bredereck et al., 2018; Lang and Skowron, 2018; Aziz, 2019], project groups [Jain et al., 2020], negative attitudes [Talmon and Page, 2021], or resource types [Aziz and Shah, 2021] is a natural avenue of future research.

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A Proofs of Theorems

Theorem 2

MWV-PJR and PJR are equivalent in MWV-instances.

In order to show that given definition (Definition 6) is indeed a generalisation of MWV-PJR, we apply it to the multi-winner voting setting and show that MWV-PJR and PJR are equivalent in MWV-instances. Recall the definition of MWV-PJR.

Definition 11 (Proportional justified representation for MWV (MWV-PJR) [Sánchez-Fernández et al., 2017]). An approval based voting rule $\mathcal{R}$ satisfies MWV-PJR if for every ballot profile $P$ and committee size $k$, the rule outputs a committee $W = \mathcal{R}(P,k)$ that satisfies:

For every $\ell \leq k$ and every $\ell$-cohesive set of voters $S \subseteq N$, it holds that $|W \cap (\cup_{i \in S} A_i)| \geq \ell$, where a set $S$ is $\ell$-cohesive if $|S| \geq \ell \cdot \frac{1}{n}$ and $|\cap_{i \in S} A_i| \geq \ell$.

We need the following lemma:

Lemma 11. Let $N$ be a set of agents, $C$ be a set of alternatives, let each project $c$ have a cost cost$(c) = \frac{1}{n}$, and let each voter $i$ have a utility $u_i(c) \in \{0,1\}$ for each project $c$. Define the approval set $A_i$ of voter $i$ as $A_i = \{c \in C : u_i(c) = 1\}$. Then

(a) for given $\alpha : C \to [0,1]$, and $T \subseteq C$, any group of voters $S$ that is $(\alpha, T)$-cohesive is also $\ell$-cohesive for $\ell = |T'|$ with $T' = \{c \in T : \alpha(c) > 0\}$, and

(b) for every group $S$ that is $\ell$-cohesive there is $T \subseteq C$ with $|T| = \ell$ and $\alpha : C \to [0,1]$ with $\alpha(c) = 1$ for all $c \in C$, such that $S$ is $(\alpha, T)$-cohesive.

Proof of Lemma 11. (a): Assume that $S \subseteq N$ is $(\alpha, T)$-cohesive for some $\alpha : C \to [0,1]$ and $T \subseteq C$. By definition, in the approval based multi-winner voting setting this means that $|S| \geq |T| \cdot \frac{1}{n}$ and that for all $i \in S$ and all $c \in T$, $u_i(c) \geq \alpha(c)$. Now take a subset $T'$ from $T$ of all the projects in $T$ that have at least some utility for all voters in $S$: $T' = \{c \in T : \alpha(c) > 0\}$. Because $u_i(c) \in \{0,1\}$ (i.e. voting is approval based), $\alpha(c) > 0$ and $u_i(c) \geq \alpha(c)$ imply that $u_i(c) = 1$. Hence, for all $i \in S$ and all $c \in T'$, $u_i(c) = 1$. We can rewrite this as $T' \subseteq \cap_{i \in S} A_i$, so $|T'| \leq |\cap_{i \in S} A_i|$. Now if we call $|T'| = \ell$, we have $|S| \geq \ell \cdot \frac{1}{n}$ and $|\cap_{i \in S} A_i| \geq \ell$, so $S$ is $\ell$-cohesive.

(b): Assume that $S \subseteq N$ is $\ell$-cohesive for some $\ell \leq k$. This means that $|S| \geq \ell \cdot \frac{1}{n}$ and $|\cap_{i \in S} A_i| \geq \ell$. We take $T \subseteq \cap_{i \in S} A_i$ with $|T| = \ell$ (which we can do because $\ell \leq |\cap_{i \in S} A_i|)$. Now take $\alpha : C \to [0,1]$ such that $\alpha(c) = 1$ for all $c \in C$. Then $|S| \geq |T| \cdot \frac{1}{n}$ and for all $i \in S$ and all $c \in T$, $u_i(c) = 1 = \alpha(c)$, so $S$ is $(\alpha, T)$-cohesive.

Proof of Theorem 2. We show that a rule that satisfies the new definition of PJR also satisfies the old MWV-PJR for every MWV-instance.

PJR $\Rightarrow$ MWV-PJR: Assume that a rule $\mathcal{R}$ satisfies PJR. Now take an arbitrary MWV-instance $E$ where all projects have unit-cost, and voting is approval based: the utility of a project for a voter is either 0 (when the voter does not approve the project) or 1 (when the voter does approve the project).

According to PJR, it is true that for all $S$, $\alpha : C \to [0,1]$, and $T$ with $T \subseteq C$, if $S$ is $(\alpha, T)$-cohesive ($|S| \geq \text{cost}(T) \cdot n$ and it holds that $u_i(c) \geq \alpha(c)$ for every voter $i \in S$ and each project $c \in T$), then

$$\sum_{c \in R(E)} \max_{i \in S} u_i(c) \geq \sum_{c \in T} \alpha(c).$$

Because $E$ satisfies the assumptions of unit-cost and approval based voting, this boils down to the following: for all $S$, $\alpha : C \to [0,1]$, and $T \subseteq C$ with $|S| \geq |T| \cdot \frac{1}{n}$ and for which $u_i(c) \geq \alpha(c)$ for all $i \in S$ and for all $c \in T$, it is the case that

$$|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c).$$

In order to show that $\mathcal{R}$ satisfies MWV-PJR, we have to prove that for all $S$ and $\ell \leq k$ where $S$ is $\ell$-cohesive ($|S| \geq \ell \cdot \frac{1}{n}$ and $|\cap_{i \in S} A_i| \geq \ell$),

$$|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell.$$ 

Take arbitrary $S \subseteq N$ and $\ell \leq k$ and suppose that $S$ is $\ell$-cohesive. According to Lemma 1(b), there are $T \subseteq C$ with $|T| = \ell$ and $\alpha : C \to [0,1]$ with $\alpha(c) = 1$ for all $c \in C$, such that $S$ is $(\alpha, T)$-cohesive. Because $\mathcal{R}$ satisfies PJR, this implies that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$. However, because of our choice of $T$ and $\alpha$, we know that $\sum_{c \in T} \alpha(c) = |T| = \ell$, so $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell$, which is what we had to prove.

We still have to prove that a rule that satisfies MWV-PJR also satisfies the newly defined PJR in MWV-elections:

MWV-PJR $\Rightarrow$ PJR: Assume that a rule $\mathcal{R}$ satisfies MWV-PJR: in every election instance $E$, for every $\ell$-cohesive group of voters $S$, $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell$. We want to prove that in every MWV-instance, any $(\alpha, T)$-cohesive group $S$ satisfies $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$. Take arbitrary such election instance $E$, and suppose that a group $S$ is $(\alpha, T)$-cohesive. Then according to Lemma 1(a), when we take $T' = \{c \in T : \alpha(c) > 0\}$, $S$ is $\ell$-cohesive for $\ell = |T'|$. Because $\mathcal{R}$ satisfies MWV-PJR, it follows that $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \ell = |T'|$. By definition of $T'$, $|T'| \geq \sum_{c \in T} \alpha(c) = \sum_{c \in T} \alpha(c)$, so $|\mathcal{R}(E) \cap (\cup_{i \in S} A_i)| \geq \sum_{c \in T} \alpha(c)$, which is what we had to prove.

Proposition 2

Phragmén satisfies priceability.

Proof. As Peters et al. [2020] mention, there is no obvious generalization of Phragmén to non-approval voting. For non-unit costs, it is however directly applicable. We can easily see that Phragmén for non-unit costs is still priceable (with the PB definition of priceability). We can construct a price system as follows: If the rule stopped at time $t$, let the initial budget of every voter be $t$, so the total initial budget is $b = t \cdot n$. Then every time that a project is selected by Phragmén and added to the winning bundle $W$, add the amount of money that a voter $i$ pays for it to $p_i(c)$. Clearly, in the price system constructed in this way, the cost for every project $c \in W$ is
paid, every voter has spent at most \( \frac{k}{n} \) units of money and pays only for voters she approves, and for each non-selected project \( c \notin W \), its supporters do not have enough money to buy it, because if they would have, \( c \) would have been added to \( W \) already.

**Proposition 3**

*Phragmén and Rule X do not satisfy LP.*

**Proof.** As a counterexample, we can use Table 4: Suppose the approval votes are as shown in Table 4, projects \( c_1, \ldots, c_5 \) have a cost of 0.1 and project \( c_6 \) has a cost of 0.7, and the total budget is 1. The profile is still a laminar PB-instance according to Definition 8. In this situation, Phragmén will return \( \{c_1, \ldots, c_5\} \): at time \( t = 0.05 \), \( v_1 \) and \( v_2 \) can buy \( c_1 \), then at \( t = 0.1 \), they can buy \( c_2 \) and \( v_3 \) can buy \( c_4 \), at \( t = 0.15 \), \( v_1 \) and \( v_2 \) can buy \( c_3 \) and at \( t = 0.2 \), \( v_3 \) can buy \( c_5 \). The remaining amount of budget is 0.5, so \( c_6 \) is not affordable anymore, and \( W = \{c_1, \ldots, c_5\} \) is returned. The same bundle will be returned for Rule X: each voter starts with a budget of \( \frac{1}{4} \). \( c_1, c_2, \) and \( c_3 \) are \( \rho \)-affordable for \( \rho = 0.05 \), \( c_4 \) and \( c_5 \) are \( \rho \)-affordable for \( \rho = 0.1 \) (whereas \( c_6 \) would only be \( \rho \)-affordable for \( \rho \geq \frac{0.7}{3} \) in the beginning), the remaining budget is not enough to buy \( c_6 \), so \( W \) will be returned. A laminar proportional bundle, however, would consist of \( c_6 \), two of \( \{c_1, c_2, c_3\} \) and one of \( \{c_4, c_5\} \).

**Theorems 3 and 8**

*In laminar MWV-instances there exist bundles that satisfy PJR or EJR or are in the core, but are not priceable.* (3)

*In laminar MWV-instances, there exist bundles that satisfy PJR, EJR, or are in the core, but do not satisfy LP.* (8)

**Proof.** We start by giving a counterexample (shown in Table 5) that shows that a bundle \( W \) in a laminar profile can satisfy PJR without being priceable. Suppose we have 3 voters, \( v_1, v_2, \) and \( v_3 \), and 5 projects \( c_1, \ldots, c_5 \), and that \( k = 4 \). Projects \( c_1 \) and \( c_2 \) are approved by voters \( v_1 \) and \( v_2 \), projects \( c_3 \) and \( c_4 \) are approved by voter \( v_3 \), and project \( c_5 \) is approved by all three voters. This election instance is laminar as can easily be checked, and the selected bundle \( W \), as indicated in grey in Table 5 satisfies PJR: for all \( \ell \leq 4 \) and all \( \ell \)-cohesive group of voters \( S \), it holds that \( |W \cap \bigcup_{i \in S} A_i| \geq \ell \). It is, however, not priceable: suppose, for a contradiction, that there is a price system \( ps \) that supports \( W \). The price \( p \) of this system has to be such that \( v_3 \) can pay for both \( c_3 \) and \( c_4 \) (because no other voter can pay for these projects), so \( p \leq 0.5 \). However, \( v_1 \) and \( v_2 \) together have a budget of 2, which they must spend only on \( c_1 \) and \( c_2 \). Hence, \( p > \frac{1}{2} \), because otherwise \( v_1 \) and \( v_2 \) together have enough unspent budget to pay for \( c_5 \) which is not in \( W \). Hence \( p \leq 0.5 \) and \( p > \frac{1}{2} \), which is a contradiction. Therefore, there is no price system that supports \( W \).

This counterexample also shows that a committee that satisfies EJR or in the core does not necessarily have to be priceable or laminar proportional. The committee in Table 5 does satisfy EJR and it is in the core, as can easily be checked, but it is neither priceable nor laminar proportional.

**Theorem 4**

*There exist priceable bundles that do not satisfy PJR.*

**Proof.** Take a PB-instance \( E \) with \( N = \{s_1, s_2, v_1, v_2, v_3\} \), \( C = \{t_1, t_2, c_1, c_2, c_3\} \), the cost of all projects is 0.2, \( \alpha(t_1) = \alpha(t_2) = 0.4 \), voters \( s_1 \) and \( s_2 \) have some utility only for the \( t \)-projects: \( u_{s_1}(t_1) = u_{s_1}(t_2) = 0.6 \), \( u_{s_1}(c) = 0 \) for \( c \in \{c_1, c_2, c_3\} \), \( u_{s_2} = u_{s_1} \), and voters \( v_1, v_2, \) and \( v_3 \) only have utility for the \( c \)-projects: for \( v \in \{v_1, v_2, v_3\} \), \( u_v(t_1) = u_v(t_2) = 0 \) and \( u_v(c) > 0 \) for \( c \in \{c_1, c_2, c_3\} \). Furthermore, define a pricesystem \( ps \) with \( b = 1 \) and \( p(c) = 0 \) for \( c \in \{c_1, c_2, c_3, t_1\} \), and \( p(t_2) = 0.1 \) for \( s \in \{s_1, s_2\} \), and with \( p(t) = 0 \) and \( p(c) = \frac{3}{5} \) for \( v \in \{v_1, v_2, v_3\} \), \( t \in \{t_1, t_2\} \), and \( c \in \{c_1, c_2, c_3\} \).

Now, let bundle \( W = \{t_2, c_1, c_2, c_3\} \) be the winning bundle of some election rule. We have:

1. A voter can only pay for projects she gets at least some utility form: if \( u_i(c) = 0 \), then \( p_i(c) = 0 \) for each \( i \in N \) and \( c \in C \);
2. Each voter can spend the same budget of \( \frac{b}{n} \) units of money: \( \sum_{c \in C} p_i(c) \leq \frac{b}{n} \) for each \( i \in N \);
3. For each \( c \in W \), the sum of the payments for \( c \) equals its price: \( \sum_{i \in N} p_i(c) = cost(c) \);
4. No project outside of the bundle gets any payment: for all \( c \notin W \), \( \sum_{i \in N} p_i(c) = 0 \);
5. There exists no non-selected project whose supporters in total have a remaining unspent budget of more than its cost: for all \( c \notin W \), \( \sum_{i \in N} p_i(c') \leq cost(c) \).

Hence, \( W \) is a priceable bundle. However, if we take \( S = \{s_1, s_2\} \) and \( T = \{t_1, t_2\} \), we have \(|S| = 2 = cost(T) \cdot n\).
and \( u_i(c) \geq \alpha(c) \) for all \( i \in S, c \in T \), so \( S \) is \((\alpha, T)\)-cohesive. Nevertheless, \( \sum_{c \in W}(\max_{i \in S} u_i(c)) = 0.6 < 0.8 = \sum_{c \in T} \alpha(c) \), so \( W \) does not satisfy PJR. Therefore, this counterexample shows that it is not the case that all priceable bundles satisfy PJR.

\[ \square \]

**Theorem 6**

In laminar MWV-instances, priceability does not imply LP.

**Proof.** Let us recall the definitions of laminar MWV-instances and laminar proportionality from Peters and Skowron [2020].

**Definition 12** (Laminar MWV-instances). An MW-instance \((P, k)\) is laminar if either:

1. \( P \) is unanimous and \( |C(P)| \geq k \).
2. There is a candidate \( c \in C(P) \) such that \( c \in A_i \) for all \( A_i \in P \), the profile \( P_{-c} \) is not unanimous and the instance \((P_{-c}, k-1)\) is laminar (with \( P_{-c} = (A_1 \setminus \{c\}, ..., A_n \setminus \{c\}) \)).
3. There are two laminar MWV-instances \((P_1, k_1)\) and \((P_2, k_2)\) with \( C(P_1) \cap C(P_2) = \emptyset \) and \(|P_1| - k_2 = |P_2| - k_1\) such that \( P = P_1 + P_2 \) and \( k = k_1 + k_2 \).

**Definition 13** (LP for MWV-instances). A rule \( R \) satisfies LP if for every laminar MWV-instance with ballot profile \( P \) and committee size \( k \), \( R(P, k) = W \) where \( W \) is a laminar proportional committee, i.e.:

1. If \( P \) is unanimous, then \( W \subseteq A_i \) for some \( i \in N \) (if everyone agrees, then part of the candidates they agree on is chosen).
2. If there is a unanimously approved candidate \( c \) s.t. \((P_{-c}, k-1)\) is laminar, then \( W = W' \cup \{c\} \) where \( W' \) is a committee which is laminar proportional for \((P_{-c}, k-1)\).
3. If \( P \) is the sum of laminar instances \((P_1, k_1)\) and \((P_2, k_2)\), then \( W = W_1 \cup W_2 \) where \( W_1 \) is laminar proportional for \((P_1, k_1)\) and \( W_2 \) is laminar proportional for \((P_2, k_2)\).

In order to show that priceability does not imply LP, we construct a counterexample. Consider a profile \( P \) as in Table 4, but where every project has unit cost. We have 3 voters, \( v_1, v_2, \) and \( v_3 \), and 6 projects \( c_1, ..., c_6 \). Projects \( c_1, c_2, \) and \( c_3 \) are approved by voters \( v_1 \) and \( v_2 \), projects \( c_4 \) and \( c_5 \) are approved by voter \( v_3 \), and project \( c_6 \) is approved by all three voters. This profile is laminar for \( k = 4 \), we can construct it according to the inductive Definition 12 by first concatenating \( \{c_1, c_2, c_3\} \) (\( k = 2 \)) and \( \{c_4, c_5\} \) (\( k = 1 \)) by Definition 12.3 and then adding \( c_6 \) by Definition 12.2, which results in \( k = 2 + 1 + 1 = 4 \).

The selected bundle \( W \), as indicated in grey, is not laminar proportional, because in order to be laminar proportional, \( c_6 \) would have to be included in \( W \). It is, however, priceable: we can construct a price system with price \( p = 0.65 \): \( v_1 \) and \( v_2 \) together pay for \( c_1, c_2 \), and \( c_3 \) and have \( 2 - 3 \cdot 0.65 = 0.05 \) left over, and \( v_3 \) pays for \( c_4 \) and has \( 1 - 0.65 = 0.35 \) left over. Hence, \( v_3 \) cannot pay for \( c_5 \) anymore, and all voters together have an unspent budget of 0.4, so they cannot pay for \( c_6 \).

\( \square \)

**Theorem 7**

LP implies priceability.

**Proof.** We construct an inductive proof on the structure of laminar PB-instances, very similar to the proof for the unit-cost case, to prove that for every bundle \( W \) that is laminar proportional for a laminar PB-instance \((P, l)\), where \( P \) is the list of approval sets of the voters and \( l \) is the budget, there exists a price system \( ps = (b, (p_i)_{i \in N}) \) where \( b = cost(W) \).

**Basis:** If \( P \) is unanimous with \( cost(C(P)) \geq 1 \) and \( W \) is laminar proportional for \((P, l)\) (with \( cost(W) \leq l \)), then \( W \subseteq C(P) \), so the voters can just divide their budget over the projects in \( W \). If we set the initial budget to be \( b = cost(W) \), every voter can spend \( \frac{1}{n} \cdot \frac{cost(W)}{n} \). We can now let every voter spend \( \frac{cost(c)}{n} \) on every project \( c \in W \), so every project \( c \in W \) gets exactly \( cost(c) \). Then every voter spends in total \( \sum_{c \in W} \frac{cost(c)}{n} = \frac{cost(W)}{n} \), so does not have anything left to spend on other projects.

**Inductive Hypothesis:** Suppose that \((P', l'), (P_1, l_1)\), and \((P_2, l_2)\) are laminar PB-instances, bundles \( W', W_1, \) and \( W_2 \) are laminar proportional for respectively \((P', l')\), \((P_1, l_1)\), and \((P_2, l_2)\), and suppose that for \( W' \) there exists a price system \( ps' \) with initial budget \( b' = cost(W') \). For \( W_1 \) there exists a price system \( ps_1 \) with initial budget \( b_1 = cost(W_1) \) and for \( W_2 \) there exists a price system \( ps_2 \) with initial budget \( b_2 = cost(W_2) \). Furthermore, suppose that \( P'\) is not unanimous, that \( C(P_1) \cap C(P_2) = \emptyset \) and that \(|P_1| \cdot l_2 = |P_2| \cdot l_1\).

**Inductive step:**

- There is a unanimously approved project \( c \) such that \( P = P'_c \), where \( P'_c = (A_1 \cup \{c\}, ..., A_n \cup \{c\}) \) (case 2 of Definition 8). Suppose that \( W \) is laminar proportional for \((P', l' + cost(c))\), then \( W = W' \cup \{c\} \). By the inductive hypothesis, there exists a price system \( ps' \) for \( W' \) with initial budget \( b' = cost(W') \). Because \( c \) is unanimously approved, in theory all voters can pay for \( c \). We know that in \( ps' \), there was no project that was not in \( W' \) for which its supporters together had enough (more than its cost) unspent budget. If we would give every voter \( \frac{cost(c)}{n} \) more budget, which we let them spend entirely on \( c \), \( c \) will get enough money and no voter will have more unspent budget than they had before. Also, the initial budget of every voter is now \( \frac{b' + cost(c)}{n} = \frac{cost(W')}{n} + \frac{cost(c)}{n} = \frac{cost(W)}{n} = \frac{1}{n} \) units of money, and the initial budget is \( b = cost(W) \) and all the individual payment functions stay the same. Because for every project \( c \) in \( W' \) the sum of the individual payments was equal to \( cost(c) \), this is also the case for every project in \( W'. \)

Formally we define the price system \( ps \) for the instance \((P, l)\) as follows: \( ps = (b, (p_i)_{i \in N}) \) with \( b = cost(W) \) and \( p_i : C \rightarrow [0, 1] \) such that \( p_i(c) = \frac{cost(c)}{n} \) and \( p_i(d) = \frac{cost(d)}{n} \) for all \( d \in T \setminus C_i \) and \( i \in N \).
$p'_i(d)$ for all other projects $d \in C(P)$, where $p'_i$ is the payment function of voter $i$ in the price system $ps'$.

To show that this is indeed a valid price system that supports $W'$, we look at the five points of the definition of a price system that supports a bundle:

1. Voters only pay for projects they get at least some utility from because they did so in $ps'$, and the only project which they now pay for that they did not pay for before is $c$, which is unanimously approved, so has some utility for all $i \in N$.

2. All voters $i \in N$ have an initial budget of $\frac{b_i}{n}$:
   \[
   \sum_{d \in C} p_i(d) = \sum_{d \in C} p'_i(d) + \frac{\text{cost}(c)}{n} \leq \frac{b'}{n} + \frac{\text{cost}(c)}{n} = \frac{b}{n},
   \]
   where (7) follows from the inductive hypothesis: because $ps'$ is a price system that supports $W'$, the sum of the payments of voter $i$ for the items in $W'$ is smaller than or equal to $\frac{b_i}{n}$. Equation 8 holds because the new budget $b$ is defined as $b = \text{cost}(W') = \text{cost}(W' \cap \{c\}) = b' + \text{cost}(c)$.

3. For each selected project $d \in W$, if $d \neq c$ the sum of the payments is
   \[
   \sum_{i \in N} p_i(d) = \sum_{i \in N} p'_i(d) = \text{cost}(d).
   \]
   This follows from the inductive hypothesis: because $ps'$ is a price system that supports $W'$, the sum of the payments of all voters for $d$ equals its cost. For $c$, $\sum_{i \in N} p_i(c) = n \cdot \frac{\text{cost}(c)}{n} = \text{cost}(c)$.

4. For any non-selected project $d \notin W$, $\sum_{i \in N} p_i(d) = \sum_{i \in N} p'_i(d) = 0$.

5. For any project outside of the bundle $d \notin W$, its supporters do not have a remaining unspent budget of more than cost($c$):
   \[
   \sum_{i \in N \text{ for which } u_i(d) > 0} (b - \sum_{e \in W = W' \cup \{c\}} p_i(e)) \leq \sum_{i \in N: u_i(d) > 0} (b - p_i(c) - \sum_{e \in W'} p_i(e)) \leq \sum_{i \in N: u_i(d) > 0} \left( \frac{\text{cost}(W)}{n} - \frac{\text{cost}(c)}{n} - \sum_{e \in W'} p'_i(e) \right) \leq \sum_{i \in N: u_i(d) > 0} \left( b' - \sum_{e \in W'} p'_i(e) \right) \leq \text{cost}(d),
   \]
   so there is no non-selected project whose supporters in total have a remaining unspent budget of more than its cost. Equation (15) follows from the inductive hypothesis because $ps'$ is a price system that supports $W'$, so satisfies Definition 5.5, the other equations are just rewritings of the formula.

Hence, $ps$ is indeed a valid price system that supports bundle $W$.

- $P = P_1 + P_2$ and $l = l_1 + l_2$ (case 3 of Definition 8).
  Take $W = W_1 \cup W_2$, which is by definition laminar proportional for $(P, l)$. We have to show that $W$ is priceable for this election instance. Note that there are no overlapping projects between $P_1$ and $P_2$, there is no voter in $P_1$ that gets any utility from a project from $C(P_2)$, and no voter in $P_2$ that gets any utility from a project from $C(P_1)$. By the inductive hypothesis, there exists a price system $ps_{1}$ = $(b_1, \{p_{1,i}\}_{i \in N})$ for $W_1$ with initial budget $b_1$ = $\text{cost}(W_1)$, and for $W_2$, there exists a price system $ps_2 = (b_2, \{p_{2,i}\}_{i \in N})$ with $b_2$ = $\text{cost}(W_2)$. Also by the inductive hypothesis, $|P_1| \cdot l_2 = |P_2| \cdot l_1$.

We can now define a price system $ps$ that supports $W$ as follows:

\[
ps = (b, \{p_{i}\}_{i \in N}) \text{ with } b = \text{cost}(W) = b_1 + b_2, \text{ and for all voters } i \in N,
\]

\[
\begin{align*}
&\text{for } p_{1,i} \text{ and } p_{2,i} \text{ that yield zero for the candidates that are not defined for:} \\
p_{1,i} &\in (P_1) \text{ and } i \in P_1, \\
p_{2,i} &\in (P_2) \text{ or } i \in P_2.
\end{align*}
\]

Again, we show that this is a valid price system that supports $W$ by looking at the five points of the definition:

1. We know that $ps_1$ is a valid price system that supports $W_1$, so for voters $i \in P_1$ and projects $c \in W_1$, if $p_{1,i}(c) > 0$, then $u_{i,c} > 0$, so $c > A_i$. Analogously, for voters $i \in P_2$ and $c \in W_2$ if $p_{2,i}(c) > 0$, then $u_{i,c} > 0$, so $c > A_i$. Suppose $p_{1,i}(c) > 0$. If $c \in (P_1)$, then $p_{1,i}(c) = p_{1,i,c}$, so $i \in P_1$ because there is no voter in $P_2$ that approves a project from $C(P_1)$ and vice versa. Hence, for $c \in (P_1)$, if $p_{1,i}(c) > 0$, then $u_{i,c} > 0$ and $c \in A_i$. Similarly, we can argue that for $c \in (P_2)$, if $p_{1,i}(c) > 0$, then $u_{i,c} > 0$ and $c \in A_i$. Because $P = P_1 + P_2$, $C(P) = C(P_1) \cup C(P_2)$, so for all $c \in (P)$, if $p_{1,i}(c) > 0$ then $c \in A_i$, so $u_{i,c} > 0$.

2. $\sum_{c \in (P)} p_i(c) = \sum_{c \in C(P)} p'_{1,i}(c) + p'_{2,i}(c)$. We already saw that voters from $P_1$ do not pay for projects from $C(P_2)$ and vice versa. Hence, if $i \in P_1$, then $\sum_{c \in C(P)} p_i(c) = \sum_{c \in C(P)} p'_{1,i}(c) \leq \frac{b_i}{n}$ by the inductive hypothesis (because $ps_1$ is a valid price system with initial budget $b_1$). Furthermore, we have $\frac{b_1}{n} \leq \frac{b_1 + b_2}{n} = \frac{b}{n}$, so
Theorem 9

There exist laminar proportional bundles that do not satisfy PJR, EJR, or are not in the core.

Proof. We will prove this theorem by giving a counterexample. Consider a situation with $N = \{v_1, v_2, v_3, v_4\}$, a unanimously approved project $c$ with $\cost(c) = 1$, a set of 8 projects $T = \{t_1, \ldots, t_8\}$ that cost $\frac{1}{3}$ each and are all approved by $v_1, v_2, v_3,$ and $v_4$, and a set of 4 projects $\{x_1, x_2, x_3, x_4\}$ that also cost $\frac{1}{3}$ and are approved by $v_4$. This profile is shown in Figure 2.

The bundle $W = \{c, t_1, \ldots, t_6, x_1, x_2\}$ as indicated in grey in the figure is laminar proportional for limit $l = \frac{1}{3}$ (which is also its cost). However, $S = \{v_1, v_2, v_3\}$ is a blocking coalition. $S$ can afford $T$: $|S| = 3 > \frac{1}{3} \cdot 4 = \frac{\cost(T)}{1} \cdot n$, and for any voter $i \in S$, $u_i(T) = 8 > 7 = u_i(W)$. Therefore, $W$ is not in the core.

Note that all laminar PB-instances are approval-PB-instances. In approval-PB-instances, the definitions of $(\alpha, T)$-cohesiveness and EJR simplify to the following [Peters et al., 2020]:

**Definition 14** ($T$-cohesiveness and EJR for approval-PB-instances). A group of voters $S$ is $T$-cohesive for $T \subseteq C$ if $T$ is affordable with their share of the budget and they all approve all projects in $T$: $|S| \geq \frac{\cost(T)}{1} \cdot n$ and $T \subseteq \cup_{i \in S} A_i$. A bundle $W$ satisfies approval-EJR if for all $T$-cohesive groups $S \subseteq N$, there is a voter $i$ in $S$ who approves at least as many projects in $W$ as in $T$: $|W \cap A_i| \geq |T|$.

In the same way we can restrict our definition of PJR to the approval-based setting, and obtain the following:

**Definition 15** (Approval-PJR). A bundle $W$ satisfies approval-PJR if for all $T$-cohesive groups $S \subseteq N$, the number of projects in $W$ that is approved by at least one of the voters in $S$ is larger than the number of projects in $T$: $|W \cap \cup_{i \in S} A_i| \geq |T|$.

Using the definitions for EJR (Def. 14) and PJR (Def. 15) for approval-PB-instances, we see that in the counterexample in Figure 2, the group $S$ is $T$-cohesive for given $T$, and there is no voter $i \in S$ such that $|W \cap A_i| \geq |T|$, neither is $|W \cap \cup_{i \in S} A_i| \geq |T|$. 

\[ \sum_{c \in C(P)} p_i(c) \leq \frac{1}{n} \]  

If $i \in P_2$, then $\sum_{c \in C(P)} p_i(c) = \sum_{c \in C(P)} p_{2,i}(c) \leq \frac{1}{n}$, in the same way.

3. For each selected project $c \in W$, the sum of its payments is $\sum_{i \in N} p_i(c) = \sum_{i \in N} (p_{1,i}(c) + p_{2,i}(c))$. For $c \in C(P_2)$ (with $x \in \{1, 2\}$) this is $\sum_{i \in N} p_{x,i}(c) = \cost(c)$. This follows from the inductive hypothesis that $p_{1}$ and $p_{2}$ are price systems that support $W_1$ and $W_2$, so the sum of payments of all voters in these systems for a selected project is equal to the cost of the project.

4. Because $W = W_1 \cup W_2$, any project that is not selected in the new bundle, $c \notin W$, was not selected in $W_1$ or $W_2$, so it did not get any payment there: for $c \notin C(P_1)$, $\sum_{i \in N} p_{x,i}(c) = 0$. Hence, it also does not get any payment in the new system: for $c \in C(P_2)$, $\sum_{i \in N} p_i(c) = \sum_{i \in N} p_{x,i}(c) = \sum_{i \in N} p_{x,i}(c) = 0$ (for $x \in \{1, 2\}$).

5. All non-selected projects are only supported by voters from their own ‘old’ system, who did not have in total a remaining unspent budget of more than its cost there, so neither will they have it now: Without loss of generality, assume that an non-selected project $c \notin W$ is part of $C(P_1)$. Then because $c \notin W$, we also have $c \notin W_1$, because if $c$ was in $W_1$, it would also have been in $W$. Because $p_{x}$ is a price system that supports $W_1$, we know that $\sum_{i \in N, u_i(c) \geq 0} (1 - \sum_{e \in W_1} p_{1,i}(e)) \leq \cost(c)$.

Hence we can conclude that LP implies priceability in laminar PB-instances.

\[ \sum_{i \in N} (1 - \sum_{e \in W_2} p_{1,i}(e)) \leq \cost(c) \]

\[ \sum_{i \in N, u_i(c) > 0} (1 - \sum_{e \in W_1 \cup W_2} p_{1,i}(e)) \leq \cost(c). \]

We can analogously show the same for $c \in C(P_2)$, so conclude that for all $c \in C(P) = C(P_1) \cup C(P_2)$, if $c \notin W$, 

\[ \sum_{i \in N, u_i(c) > 0} (1 - \sum_{e \in W} p_{i}(e)) \leq \cost(c) \]

By these five points, we have shown that $p_{i}$ is indeed a valid price system that supports bundle $W$.

We have shown by induction over laminar PB-instances that, if a bundle $W$ is laminar proportional in a laminar PB-instance $(P, l)$, it is also supported by a price system with with $b = \cost(W)$. Hence we can conclude that LP implies priceability in laminar PB-instances.\[\square\]
Theorem 10

Laminar proportional bundles satisfy the core subject to u-afford.

Proof. We will prove this by induction over the structure of laminar profiles.

Basis: For unanimous profiles $P$ (Definition 8.1), $W$ will consist of projects that are approved by every voter, so clearly for every group of voters $S \subseteq N$ and $T \subseteq C$ with $|S| \geq \frac{\text{cost}(T)}{n}$, there is some voter in $S$ (namely all voters in $S$) who approves at least as many projects in $W$ as in $T$. This is true even in the general situation, without the restriction of u-afford.

Inductive hypothesis: Suppose that $W', W_1$, and $W_2$ are arbitrary bundles in respective laminar instances $(P', l'), (P_1, l_1)$, and $(P_2, l_2)$ with $C(P_1) \cap C(P_2) = \emptyset$ and $|P_1| \cdot l_2 = |P_2| \cdot l_1$, that $W', W_1$, and $W_2$ are laminar proportional and are in the core subject to u-afford.

Inductive step:

- Suppose there is a unanimously approved project $c$ and $(P_{c, l} - \text{cost}(c)) = (P', l')$ is laminar, $W'$ is a laminar proportional bundle for $(P', l')$, and $W = W' \cup \{c\}$ (Definition 8.2).

Assume for a contradiction that $W$ is not in the core. Then there must exist a group of voters $S$ and a set of projects $T$ such that $|S| \geq \frac{\text{cost}(T)}{n}$, with $u_i(T) > u_i(W)$ for all voters $i$ in $S$. Because utilities are assumed to be additive and everyone approves project $c$, we know that for all voters $i \in S$, $u_i(T \setminus \{c\}) = u_i(T) - u_i(c)$ if $c \in T$ and $u_i(T \setminus \{c\}) = u_i(T)$ otherwise, and $u_i(W') = u_i(W) - u_i(c)$, so for all $i \in S$

$$u_i(T \setminus \{c\}) > u_i(W') \quad (17)$$

We distinguish two cases, with either $c \in T$ or $c \notin T$, and show that in both cases $W$ is in the core.

1. $c \in T$:

Because we have chosen $S$ and $T$ such that $\frac{|S|}{n} \geq \frac{\text{cost}(T)}{l}$ and because cost($T$) $\leq l$ (since by definition $|S| \leq n$), we find that

$$\frac{\text{cost}(T \setminus \{c\})}{l} = \frac{\text{cost}(T) - \text{cost}(c)}{l - \text{cost}(c)} \leq \frac{\text{cost}(T)}{l} \leq \frac{|S|}{n} \quad (18)$$

However, from the inductive hypothesis we know that $W'$ is in the core (subject to u-afford) in the election instance $(P', l')$, so for all $S' \subseteq N', T' \subseteq C'$ with $|S'| \geq \frac{\text{cost}(T')}{l'}$, there is a voter $i' \in S'$ with $u_{i'}(W') \geq u_{i'}(T')$. In this instance, we can take $T' = T \setminus \{c\}$ and $S' = S$. As shown above, $S$ can afford $T' \frac{\text{cost}(T')}{l'}$, so there is a voter $i' \in S'$ with $u_{i'}(W) \geq u_{i'}(T)$. This is a contradiction with 17, which proves that $W$ is indeed in the core (subject to u-afford) if $c \in T$.

2. $c \notin T$:

In this case, equation 18 does not hold anymore, because not necessarily $\frac{\text{cost}(T)}{l - \text{cost}(c)} \leq \frac{\text{cost}(T)}{l}$, in fact the first fraction is greater because $c$ has a positive cost. Now suppose that $S$ can afford $T$ and that every voter in $S$ prefers $T$ to $W \setminus \{c\}$, and even all voters in $S$ prefer $T \setminus \{t\}$, where $t$ is an arbitrary project in $T$, to $W \setminus \{c\}$, because we are in an approval voting setting (where the utility of a project a voter approves is 1 and the utility of all projects a voter does not approve is 0). However, according to our inductive hypothesis, if the voters in $S$ together could afford $T \setminus \{t\}$ in the situation where the budget is $l - \text{cost}(c)$, there would be a voter in $i \in S$ with $u_i(W \setminus \{c\}) \geq u_i(T \setminus \{t\})$, since $W \setminus \{c\} = W'$ is in the core (subject to u-afford) there. Hence, $S$ cannot afford $T \setminus \{t\}$ in the instance $(P', l - \text{cost}(c))$. We now have that

$$\frac{\text{cost}(T)}{l} \leq \frac{|S|}{n} \leq \frac{\text{cost}(T) - \text{cost}(t)}{l} \quad (19)$$

where $t$ was an arbitrary project in $T$, so all $t \in T$ must have a lower cost than $c$. Hence, under our restriction that there exists $t \in T$ with cost($t$) $\geq$ cost($c$), there is no $S$ that can block the winning bundle. Hence, if $c \notin T$, $W$ is in the core subject to u-afford.

- Suppose that $(P, l)$ is the sum of $(P_1, l_1)$ and $(P_2, l_2)$, i.e., that $P = P_1 + P_2$ and $l = l_1 + l_2$, and that $W = W_1 \cup W_2$ (Definition 8.3). Assume for a contradiction that $W$ is not in the core. Then there must exist a group of voters $S$ and a set of projects $T$ such that $|S| \geq \frac{\text{cost}(T)}{n}$, with $u_i(T) > u_i(W)$ for all voters $i \in S$. Because $P_1$ and $P_2$ are strictly separated and voters can only approve projects from their own election instances, each voter only gets utility from the selected projects from his own instance, so we can divide $S$ into $S_1$ and $S_2$, and $T$ into $T_1$ and $T_2$ such that all voters from $S_1$ and projects from $T_1$ only occur in $P_1$ and all voters from $S_2$ and projects from $T_2$ only occur in $P_2$. Then we have that for all voters $i \in S_1$, $u_i(T_1) > u_i(W)$, and for all voters $i \in S_2$, $u_i(T_2) > u_i(W)$. From $|S| \geq \frac{\text{cost}(T)}{n}$, we have that

$$|S_1| + |S_2| \geq (\frac{\text{cost}(T_1)}{l_1} \cdot (n_1 + n_2) \quad (20)$$

Now, assume for a contradiction that both $|S_1| < \frac{\text{cost}(T_1)}{l_1}$ and $|S_2| < \frac{\text{cost}(T_2)}{l_2}$. Then from Equation 20 and the fact that $\frac{n_1}{l_1} = \frac{n_2}{l_2}$ (from the inductive
hypothesis) we have that:
\[
\frac{\text{cost}(T_1) \cdot n_1}{l_1} + \frac{\text{cost}(T_2) \cdot n_2}{l_2} > \frac{|S_1| + |S_2|}{l_2}
\]
\[
\frac{\text{cost}(T_1)}{l_1} + \frac{\text{cost}(T_2)}{l_2} \geq \frac{(\text{cost}(T_1) + \text{cost}(T_2)) \cdot (n_1 + n_2)}{l_1 + l_2}
\]
\[
|S_1| + |S_2| > \frac{n_1 + n_2}{l_1 + l_2}
\]
\[
\frac{n_2 \cdot (l_1 + l_2)}{l_2 \cdot (l_1 + l_2)} \geq \frac{n_1 + n_2}{l_1 + l_2}
\]
\[
l_1 \cdot n_2 + l_2 \cdot n_2 \geq l_1 \cdot n_1 + l_2 \cdot n_2
\]
which is clearly a contradiction. Hence, at least one of
\[
\frac{|S_1|}{l_1} \geq \frac{\text{cost}(T_1)}{l_1} \cdot (n_1), \quad \frac{|S_2|}{l_2} \geq \frac{\text{cost}(T_2)}{l_2} \cdot (n_2)
\]
\[\]must be true. Without loss of generality, assume that \( \frac{|S_1|}{l_1} \geq \frac{\text{cost}(T_1)}{l_1} \cdot (n_1) \). Then, since \( W_1 \) is a core solution in the instance \((P_1, l_1)\), there exists \( i \in S_1 \) such that \( u_i(W_1) \geq u_i(T_1) \). However, we already knew that for all voters \( i \in S_1, u_i(T_1) > u_i(W) \). Since for voters \( i \) from \( S_1 \),
\[
u_i(W_1) = u_i(W),
\]
this is a contradiction, that shows that \( W \) is in the core in the election instance \((P, l)\).

This completes the proof. \(\square\)

**B Additional Results**

**B.1 Relations Between Axioms**

**Balanced Stable Priceability**

Note that the counterexample in Table 4 that shows that priceability does not imply LP (in Appendix A, proof of Theorem 6)) is not efficient. By electing \( c_6 \) instead of \( c_3 \), no agent’s utility would decrease, but \( v_3 \)’s utility would increase. Maybe making it efficient would make it laminar proportional, because then unanimous candidates have to be selected. Note however that electing \( c_6 \) instead of \( c_4 \) would make it efficient and still keep it priceable (\( c_3 \) could just spend her money on \( c_6 \) instead of on \( c_4 \)), but still it is not laminar proportional because the instance without \( c_6 \) is not laminar proportional.

However, if the payments would have been equally divided over the voters that approve a candidate, it would have been laminar proportional. In Peters et al. [2021], a property that demands exactly this is defined: Balanced Stable Priceability (BSP). This property demands that a price system is balanced: voters that get utility from a candidate must all pay the same price for this candidate, and that the system is stable: there is no coalition of voters that wants to change their payments so that they get more utility (or pay less). As Peters et al. [2021] show, the bundles that satisfy BSP for a price \( p \) are the same as the bundles selected by a variant of Rule X, and because Rule X returns laminar proportional bundles in laminar profiles, probably BSP implies LP. We can show easily by induction over laminar profiles that this is indeed the case.

**Theorem 12. Balanced Stable Priceability implies LP in MWV-instances.**

**Proof.** We will give an inductive proof to show this.

**Basis:** For unanimous profiles with \(|C(P)| \geq k\), any candidate \( c \) that is in \( W \) gets at least some payment in the price system that supports \( W \), and hence is in \( A_i \) for some voter \( i \), and because \( P \) is unanimous, \( c \in A_i \) for all voters \( i \), so \( W \subseteq C(P) \).

**Inductive Hypothesis:** Assume laminar profiles \((P', k')\), \((P_2, k_2)\), and \((P_2, k_2)\) are laminar and respective bundles \( W' \), \( W_2 \), and \( W_2 \) are laminar proportional if they satisfy BSP, where \( P' \) is not unanimous.

**Inductive Step:**

- Suppose \( c \) is a unanimously approved candidate, such that the instance \((P', k') = (P - \{c\}, k - 1)\) and that \( P \) satisfies BSP in the instance \((P, k)\). Then, by stability, \( c \in W \). Assume for a contradiction that \( c \) would not be selected, then all voters together would rather pay for \( c \) and all give up one of the candidates they now pay for: then they would all get the same utility because they all approve \( c \), and would have to pay less because they can divide the price for \( c \) over them all. Hence \( c \) is selected in \( W \). Now the bundle \( W \) without \( c \), which we call to be the \( W' \) from the inductive hypothesis, still satisfies BSP, because every voter pays the same amount for \( c \) (because the price system is balanced), and hence we can just subtract the price they all pay for \( c \) from the total budget every voter gets. Then, by the inductive hypothesis, \( W' \) is laminar proportional, so \( W \) itself is laminar proportional as well.

- Suppose \((P, k)\) consists of two separate laminar MWV-instances \((P_1, k_1)\) and \((P_2, k_2)\). Define \( W_1 \) as the set of candidates in \( W \) from \( P_1 \), and \( W_2 \) as the set of candidates in \( W \) from \( P_2 \), so \( W = W_1 \cup W_2 \). If \( W \) satisfies BSP, then in the price system that witnesses this, voters from \( P_1 \) can only vote and pay for candidates in \( P_1 \), and voters from \( P_2 \) can only vote and pay for candidates in \( P_2 \), so we can split the price system to get a price system for both instances, which shows that both \( W_1 \) and \( W_2 \) satisfy BSP. According to the inductive hypothesis, then \( W_1 \) and \( W_2 \) are laminar proportional, so \( W \) is laminar proportional.

We have thus shown by induction over laminar profiles that if a winning bundle in a laminar profile satisfies BSP, it also satisfies LP. \(\square\)

Note however, that this result relies on all projects having unit-cost. In the general case with non-unit costs, it does not hold anymore. To show this, we first have to give a definition of BSP for PB-instances. Recall the definition of BSP for MWV-instances from Peters et al. [2021]. We still use approval votes (for compatibility with LP), but can have arbitrary costs. Requirement E1 stays the same. Requirement E5 changes a bit:

**Condition for Stability:** There exists no coalition of voters \( S \subseteq N \), no bundle \((W', \{u'_{i}\}_{i \in N})\) \((W' \subseteq C \setminus W)\) and no collections \( \{p'_{i}\}_{i \in S} \) and \( \{R_{i}\}_{i \in N} \) (with \( R_i \subseteq W \) for each \( i \in N \)) such that all the following hold:

1. For each \( c \in W' \): there exists a value \( \rho_c \) such that \( p'(c) = u'_{i}(c) \cdot \rho_c \).
2. For each \( c \in W' \): \( \sum_{i \in S} p'(c) > \text{cost}(c) \).
3. For each \( i \in S \): \( p_i(W \setminus R_i) + p'_i(W') \leq \frac{\lambda}{n} \).
Proof. Let S be a PB-instance with a bundle satisfying BSP but not LP. It should be read in the same way as Table 2.

4. For each $i \in S$: $(u_i(W \setminus R_i) + u'_i(W')) - (p_i(W \setminus R_i) + p'_i(W')) \geq (u_i(W) - p_i(W))$.

Now we can answer the question whether BSP still implies LP in laminar profiles without the unit-cost assumption. In MWV-instances, part of the reason why BSP implies LP is because the ‘stability’ part form BSP there requires that any unanimously approved candidate is selected in the winning bundle, since any voter would rather pay for a unanimously approved candidate than for a non-unanimously approved candidate they pay for now: a unanimous candidate costs less (costs are shared by all voters), and gives the same utility. In PB-instances, it will still give the same utility, but may not be affordable, or at least may not cost less. This is because it is possible that the unanimous candidate has a much higher cost than the other candidates. Hence, it is not the case that any bundle satisfying BSP in a laminar PB-instance is laminar proportional.

As an example, take the profile in Table 6, which is a laminar PB-instance with a budget of $\ell = 10$. The grey bundle $W = \{c_1, c_2, c_3, c_4\}$ is Balanced Stable Priceable: every voter has an initial budget of $2\frac{1}{2}$, $v_1$ and $v_2$ can pay for $c_1$ and $c_2$, and $v_3$ and $v_4$ for $c_3$ and $c_4$. In a MWV-instance, the bundle $\{c_2, c_4, c_5\}$ would be preferred, since all could share the cost for $c_5$, and still have a utility of 2. However, since $c_5$ has a higher cost here, it is not possible for any group of voters to afford $c_5$ and still have the same amount of utility. Hence, $W$ satisfies BSP, although it does not satisfy LP (since there is a non-selected unanimous candidate). One could argue about the fairness of the LP axiom here though: is there a clear reason why only selecting $c_5$ is more proportional than selecting $W$?

Priceability does not imply EJR or the core in MWV-instances

Theorem 13. Priceability does not imply EJR in MWV-instances.

Proof. Take an election instance $E$ with 3 voters $N = \{v_1, v_2, v_3\}$ and 6 candidates $C = \{c_1, \ldots, c_6\}$, and let $k = 3$. Let every voter approve four projects, namely one that only that voter approves and three that all three voters approve: $A_i = \{c_i, c_4, c_5, c_6\}$. Suppose the winning committee of the election is $W = \{c_1, c_2, c_3\}$. For this committee, there is a price system with $p = 1$ in which every voter $i$ pays the price of candidate $i$ and nothing else. The group $S = N$ is 3-cohesive, because $|S| = 3 = 3\frac{p}{1}$, and all three voters agree on the projects $\{c_4, c_5, c_6\}$. However, there is no voter in $S$ who approves 3 or more projects in $W$, so $W$ does not satisfy EJR.

Since the core implies EJR, this example also shows that priceable committees are not necessarily in the core.

B.2 Properties of Rules

PAV does not satisfy PJR.

As we mentioned in the main text, as shown in [Peters et al., 2020, Figure 2] that PAV does not give a proportional bundle without the unit cost assumption, and therefore PAV does not satisfy PJR. Here we give an exact argument why PAV does not satisfy approval-JJR (Definition 15).

Proposition 5. There exist approval-PB-instances where PAV does not satisfy PJR.

Proof. We use the example of Onetown from Peters et al. [2020]. The group of voters in Leftside are $T$-cohesive for $T = \{L_1, L_2, L_3\}$: they can with their share of the money afford all projects in $T$ and do all approve all projects in $T$. However, the amount of projects in the bundle $W$ that PAV returns that at least one of the voters in Leftside ($S$) approves of is $|W \cap_{i \in S} A_i| = 2$, which is less than the number of projects in $T$.

Phragmén satisfies the core and EJR in laminar MWV-instances.

From Theorem 10 follows the following: Since Phragmén satisfies LP in MWV-instances, in laminar MWV-instances the winning bundle of Phragmén will also satisfy the core and EJR.

Corollary 13.1. In laminar MWV-instances, Phragmén satisfies the core and EJR.