DIMENSIONS OF IRREDUCIBLE MODULES OVER W-ALGEBRAS
AND GOLDIE RANKS

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Abstract. The main goal of this paper is to compute two related numerical invariants of a primitive ideal in the universal enveloping algebra of a semisimple Lie algebra. The first one, very classical, is the Goldie rank of an ideal. The second one is the dimension of an irreducible module corresponding to this ideal over an appropriate finite W-algebra. We concentrate on the integral central character case. We prove, modulo a conjecture, that in this case the two are equal. Our conjecture asserts that there is a one-dimensional module over the W-algebra with certain additional properties. The conjecture is proved for the classical types. Also, modulo the same conjecture, we compute certain scale factors introduced by Joseph, this allows to compute the Goldie ranks of the algebras of locally finite endomorphisms of simples in the BGG category \( \mathcal{O} \). This completes a program of computing Goldie ranks proposed by Joseph in the 80’s (for integral central characters and modulo our conjecture).

We also provide an essentially Kazhdan-Lusztig type formula for computing the characters of the irreducibles in the Brundan-Goodwin-Kleshchev category \( \mathcal{O} \) for a W-algebra again under the assumption that the central character is integral. In particular, this allows to compute the dimensions of the finite dimensional irreducible modules. The formula is based on a certain functor from an appropriate parabolic category \( \mathcal{O} \) to the W-algebra category \( \mathcal{O} \). This functor can be regarded as a generalization of functors previously constructed by Soergel and by Brundan-Kleshchev. We prove a number of properties of this functor including the quotient property and the double centralizer property.

We develop several side topics related to our generalized Soergel functor. For example, we discuss its analog for the category of Harish-Chandra modules. We also discuss generalizations to the case of categories \( \mathcal{O} \) over Dixmier algebras. The most interesting example of this situation comes from the theory of quantum groups: we prove that an algebra that is a mild quotient of Lusztig’s form of a quantum group at a root of unity is a Dixmier algebra. For this we check that the quantum Frobenius epimorphism splits.

Dedicated to Tony Joseph, on his 70th birthday.

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1. Introduction

1.1. Primitive ideals and Goldie ranks. The study of primitive ideals in universal enveloping algebras is a classical topic in Lie representation theory (recall that by a primitive ideal in an associative algebra one means the annihilator of a simple module; our base field is always an algebraically closed field $K$ of characteristic 0). The case when the Lie algebra is solvable is understood well: the classification of primitive ideals is known and relatively easy, see, for example, [D, Section 6]. A nice feature of that case is that all primitive ideals are completely prime, i.e., the quotients have no zero divisors, see [BGR, Section 5].

It seems that the most interesting case in the study of primitive ideals is when the Lie algebra $\mathfrak{g}$ under consideration is semisimple (partial results on a reduction of the general case to the semisimple one can be found in [Jo4, Section 4]). There was a lot of work on primitive ideals in the semisimple case and related topics in 70’s and in 80’s (by Barbasch, Duflo, Joseph, Lusztig, Vogan, to mention a few authors) that resulted, in particular, in a classification of the primitive ideals, for a survey, see [Jo4]. A basic result is a theorem of Duflo that says that any primitive ideal in the universal enveloping algebra $U(\mathfrak{g})$ has the form $J(\lambda) := \text{Ann}_{U(\mathfrak{g})} L(\lambda)$, where $L(\lambda)$ stands for the irreducible highest weight module with highest weight $\lambda - \rho$, as usual, $\rho$ denotes half the sum of all positive roots.

One of the classical problems about primitive ideals is to compute their Goldie ranks. Let us recall the definition of the Goldie rank. Take a prime noetherian algebra $\mathcal{A}$. According to the Goldie theorem, there is a full fraction algebra $\text{Frac}(\mathcal{A})$ of $\mathcal{A}$. The algebra $\text{Frac}(\mathcal{A})$ is
simple and so is the matrix algebra of rank $r$ over some skew-field. The number $r$ is called the Goldie rank of $A$ and is denoted by $\text{Grk}(A)$ or $\text{Grk}A$. Abusing the notation, by the Goldie rank of a primitive ideal $J$ of the universal enveloping algebra $U := U(g)$ one means the Goldie rank of the quotient $U/J$.

There are many results and constructions related to the computation of Goldie ranks due to Joseph, see, in particular, [Jo2]-[Jo7]. Some of them will be recalled in Subsection 5.1. Here we only going to recall one important construction – Joseph’s scale factors. Throughout the paper we basically only consider the case of integral central characters, i.e., we restrict ourselves to ideals $J(\lambda)$, where $\lambda$ is in the weight lattice $\Lambda$ of $g$.

Any integral weight $\lambda$ can be represented in the form $w\varrho$, where $w$ is an element in the Weyl group, and $\varrho \in \Lambda$ is dominant. This representation is unique if we require that $w\alpha < 0$ for all roots $\alpha$ with $\langle \varrho, \alpha \rangle = 0$ (in this case we say that $w$ and $\varrho$ are compatible).

Consider the algebra $L(L(w\varrho), L(w\varrho))$ of all $g$-finite linear endomorphisms of $L(w\varrho)$. According to Joseph, [Jo2, 2.5], this algebra is prime and noetherian. So one can define its Goldie rank. It turns out that the ratio

$$\frac{\text{Grk} L(L(w\varrho), L(w\varrho))}{\text{Grk}(U(g)/J(w\varrho))}$$

does not depend on $\varrho$ as long as $\varrho$ is a dominant integral weight compatible with $w$, [Jo2, 5.12]. This ratio, Joseph’s scale factor, is denoted by $z_w$. The number $z_w$ is an integer, [Jo3, 5.12]. It has many remarkable properties studied, for example, in [Jo5, Jo7]. One of these properties – a connection to Lusztig’s asymptotic Hecke algebra, [Jo5, 5.8], was an important motivation for the present project.

In [Jo3, 5.5] Joseph proposed a program of computing Goldie ranks. Knowing the scale factors is the first crucial step in this program. The second one is to prove that there are “sufficiently many” completely prime (=of Goldie rank 1) primitive ideals. For our purposes, this means that, for each special nilpotent orbit $O$, there is a completely prime primitive ideal $J$ with integral central character and $V(U/J) = \overline{O}$ (the terminology and notation will be explained in Subsections 1.2,1.6).

There is one relatively easy case in the problem of computing Goldie ranks: when $g$ is of type $A$, see [Jo1]. Namely, all scale factors equal 1, and there are sufficiently many completely prime ideals, so the problem of computing the Goldie ranks is settled. Also in [Jo1, 8.1,10], Joseph gets formulas for Goldie ranks. For recent developments here see [Br] (Theorem 1.6 in loc.cit. presents a combinatorial formula for the Goldie rank). In all other types the problem of computing Goldie ranks has been open.

In this paper we are going to state a conjecture, the affirmative answer to which will complete Joseph’s program, i.e., will yield both a formula for scale factors and the existence of sufficiently many completely prime primitive ideals. Then we will prove the conjecture for the B,C,D types.

Also the affirmative answer will provide a formula for the Goldie ranks. The formula, see Subsection 1.4, will compute the dimensions of finite dimensional irreducible modules over certain associative algebras known as W-algebras. To each primitive ideal, one can assign a collection of irreducible finite dimensional modules over the corresponding W-algebra, all of the modules have the same dimension. More details on W-algebras will be given in the next subsection. In the integral central character case, we will prove that the dimension coincides with the corresponding Goldie rank.

Let us summarize. We obtain the following results (in the integral character case):
(1) A formula for the scale factors $z_w$ (for classical types).
(2) The existence of sufficiently many completely prime primitive ideals (for classical types).
(3) A formula for the dimension of an irreducible $W$-algebra module corresponding to a primitive ideal (for all types).
(4) The coincidence of the dimension and the Goldie rank (for classical types).

Thanks to (4), (3) gives a formula for the Goldie rank. (1) and (2) are irrelevant for (3), but (2) is used to prove (4) and (1) is proved simultaneously with (4).

1.2. $W$-algebras and their finite dimensional irreducible representations. Now fix a nilpotent orbit $O$. From the data of $g$ and $O$ one can construct an associative algebra $W$ called a finite $W$-algebra (below we omit the adjective “finite”). In the full generality this was first done in [Pr2]. The properties of $W$ we need will be recalled in Sections 2, 3. For now, we will only need to know two things regarding $W$-algebras.

One is a result obtained in [Lo2] and describing a relationship between the sets $\text{Pr}_O(U)$ of all primitive ideals $J \subset U$ with associated variety $V(U/J)$ equal to $O$ and the set $\text{Irr}_{\text{fin}}(W)$ of all finite dimensional irreducible $W$-modules. The component group $A(=A(e))$ of the centralizer of $e \in O$ in the adjoint group $\text{Ad}(g)$ acts on $\text{Irr}_{\text{fin}}(W)$. This action is induced from a certain reductive group action on $W$ by algebra automorphisms. It turns out that there is a natural identification of the orbit space $\text{Irr}_{\text{fin}}(W)/A$ with $\text{Pr}_O(U)$, see [Lo2, 1.1]. Moreover, in [LO, Theorem 1.1], the author and V. Ostrik have computed the stabilizers in $A$ of irreducible $W$-modules with integral central character. We will need some details regarding this computation to state the main results of the present paper, so let us recall these details now.

Recall, see, for example, [Lu1, Chapter 5], that the Weyl group $W$ of $g$ splits into the union of subsets called two-sided cells. There is a bijection between the two-sided cells in $W$ and certain nilpotent orbits in $g$ called special. This bijection sends $c$ to the dense orbit in the associated variety $V(U/J)$, where $w \in c$ and $\rho$ is regular dominant, see [Ja, 14.15] or [CM, Theorem 10.3.3]. Until a further notice we will assume that $O$ is special. To each two-sided cell $c$ Lusztig in [Lu1, Chapter 13] assigned a finite group $\bar{A}$ that is naturally represented as a quotient of the component group $A$ of $O$.

Further, each two-sided cell splits into the union of subsets called left cells (and also into the union of subsets called right cells, those are obtained from the left ones by inversion, see [Lu1, Chapter 5]). For dominant regular $\rho \in \Lambda$, the map

$$w \mapsto J(w \rho): c \mapsto \{\text{primitive } J \text{ with central character } \rho \text{ and } V(U/J) = O\}$$

is surjective, its fibers are precisely the left cells, see e.g. [Ja, 14.15]. If $\rho$ is dominant but not regular, one needs to restrict to the left cells that are compatible with $\rho$ in the sense that any $\rho$ or equivalently some (see, for example, [LO, 6.2]) element is compatible with $\rho$. In [Lu2], to each left cell $c \subset c$ Lusztig assigned a subgroup $H_c \subset A$ defined up to conjugacy. The main result of [LO], Theorem 1.1, is that the $A$-orbit in $\text{Irr}_{\text{fin}}(W)$ lying over the primitive ideal $J(w \rho)$, where $w \in c$ is compatible with dominant $\rho$, is $A/H_c$.

We will need one more construction related to that result. This construction is a parametrization of elements in $c$ conjectured by Lusztig and established in [BFO]. In our language it can be stated as follows. Fix a regular dominant weight $\rho$. Let $Y$ be the set of all finite dimensional irreducible $W$-modules with central character $\rho$. By the results of [LO], recalled above, the group $\bar{A}$ acts on this set. In particular, we can consider the $\bar{A}$-equivariant sheaves
of finite dimensional vector spaces on the square $Y \times Y$. Irreducible sheaves are parameterized by triples $(x, y, \mathcal{V})$, where $x, y \in Y$ and $\mathcal{V}$ is an irreducible module over the stabilizer $\overline{A}(x,y)$ of the pair $(x,y)$. These triples are defined up to $\overline{A}$-conjugacy. Here $x$ is in the $\overline{A}$-orbit corresponding to $J(wg)$, while $y$ is in the orbit corresponding to $J(w^{-1}g)$.

The identification of $c$ with the set of triples $(x, y, \mathcal{V})$ (considered up to $\overline{A}$-conjugacy) in the previous paragraph comes from a certain functor established in [Lo 2] that maps the category of Harish-Chandra $U$-bimodules to a suitable category of equivariant bimodules over $\mathcal{W}$ (see Subsection 2.2 for a construction). Namely, we have a subquotient in the former category corresponding to $\mathcal{O}$, and $c$ parameterizes the simples in the subquotient. The functor is well-defined on the subquotient and defines its embedding into the category of finite dimensional equivariant $W$-bimodules. The triples naturally parameterize the simples in the image. This gives a required bijection, as was established in [LO, Section 7], see, in particular, Remark 7.7 there. We will elaborate on the bijection in Subsection 2.4. In type $A$, this bijection reduces to the RSK correspondence. Indeed, in type $A$ (for a regular integral character) the primitive ideals are parameterized by standard Young tableaux, and simple HC bimodules are parameterized by elements of $S_n$. In this language, the RSK correspondence sends a simple HC bimodule to its left and right annihilator, see, for example, [Ja, 5.22-25, 14.15].

Another fact about $W$-algebras we need is that they have categories $\mathcal{O}$ introduced by Brundan, Goodwin and Kleshchev in [BGK] that are analogous to the BGG categories $\mathcal{O}$ for $U$. To define such a category one needs to fix an element $e \in \mathcal{O}$ and, most importantly, an integral semisimple element $\theta$ centralizing $e$. A resulting category will be denoted by $\mathcal{O}^\theta(g,e)$. The category $\mathcal{O}^\theta(g,e)$ contains all finite dimensional simple modules with integral central characters (we restrict our attention to the integral blocks) and also has analogs of Verma modules labeled by irreducible modules over a smaller $W$-algebra (one for $\mathfrak{g}(\theta)$ and $e$). One can define the notion of a character (a graded dimension) for a module $N$ in that category $\mathcal{O}$, to be denoted by $\text{ch}_N$. As usual, the characters of Verma modules are computable, i.e., one can write the character of a Verma module starting from its label, see formula (1.1) below. So to compute the characters of simple modules (and hence dimensions of the finite dimensional ones) it is enough to determine the multiplicities of simples in Vermas. We would like to point out that in a relatively easy special case when $e$ is a principal element in some Levi subalgebra, the multiplicities are already known, see [Lo3, Section 4]. Our approach in the present paper builds on a construction from there.

To determine the multiplicities we will define an exact functor from the integral block of an appropriate parabolic category $\mathcal{O}$ for $\mathfrak{g}$ to the integral block of an “equivariant version” of the $W$-algebra category $\mathcal{O}$. This functor may be thought as a generalization of Soergel’s functor $\mathbb{V}$, [So], and so will be called a generalized Soergel functor. An analog of Soergel’s functor for parabolic categories $\mathcal{O}$ was studied by Stroppel in [St1],[St2]. In the special case when $\mathfrak{g}$ is of type $A$, Brundan and Kleshchev, [BK1], identified the target category for this functor with a subcategory in the category of modules over a $W$-algebra. We remark that the functor we consider, in general, is not isomorphic to Stroppel’s. Our functor will map Verma modules to some variant of Verma modules and will be a quotient functor onto its image. Using this functor we will relate the multiplicities in the $W$-algebra category $\mathcal{O}$ to those in a parabolic category $\mathcal{O}$. We will also study some other properties of $\mathbb{V}$ such as a relation to dualities and the double centralizer property.

1.3. Results on Goldie ranks. Now we are going to state the main results and conjectures of the present paper related to Goldie ranks. We start with a conjecture.
Conjecture 1.1. Let $\mathcal{O}$ be a special nilpotent orbit and suppose that $\mathcal{O}$ is not one of the following three exceptional orbits: $A_4 + A_1$ in $E_7$, $A_4 + A_1$, $E_6(a_1) + A_1$ in $E_8$ (we use the Bala-Carter notation, see [CM, 8.4]). Then there exists an $A$-stable 1-dimensional $\mathcal{W}$-module with integral central character. For the three exceptional orbits there is a 1-dimensional $\mathcal{W}$-module with integral central character.  

We remark that the primitive ideal corresponding to a 1-dimensional $\mathcal{W}$-module is automatically completely prime, see [Lo1, Proposition 3.4.6] (in fact, this also easily follows from the previous work of Moeglin, [Mo]).

Theorem 1.2. Conjecture 1.1 holds for all special orbits provided $\mathfrak{g}$ is classical.

Let us state an important corollary of Conjecture 1.1. Let $c$ be the two-sided cell corresponding to $\mathcal{O}$. Let $Y, \mathfrak{g}, A$ have the same meaning as in the Subsection 1.2. Pick $w \in c$ compatible with $\mathfrak{g}$. To $w$ assign a triple $(x, y, V)$ as explained in Subsections 1.2, 2.4 (for any regular dominant weight).

Theorem 1.3. Assume that Conjecture 1.1 holds for the pair $(\mathfrak{g}, \mathcal{O})$. Then

1. The Goldie rank of $J(w\mathfrak{g})$ coincides with the dimension of an irreducible $\mathcal{W}$-module in $Y$ whose $A$-orbit corresponds to $J(w\mathfrak{g})$ (all such modules differ by outer automorphisms, so their dimensions are the same).

2. $z_w = \dim Y \frac{|A_x|}{|A_{(x,y)}|}$.

Thanks to assertion (2) of Theorem 1.3, Conjecture 1.1 completes one of Joseph’s programs of computation of the Goldie ranks (for integral central character), see Subsection 5.1 for details.

Remark 1.4. For classical Lie algebras, the group $A$ is commutative, moreover, it is the sum of several copies of $\mathbb{Z}/2\mathbb{Z}$ (see, for example, [CM, Theorem 6.1.3]) and so $A_{(x,y)} = A_x \cap A_y$, and $\dim Y = 1$. In this case, one can recover all numbers appearing in the right hand side in (2) combinatorially starting from $w$, see [LO, 6.9]. We note however that we do not know how to recover $V$ itself (and $\dim V$) in the case when $A = S_3, S_4, S_5$ combinatorially starting from $w$, we refer the reader to [LO, 6.8] for the list of orbits with these component groups.

We also would like to remark that the formula in (2) is compatible with results on $z_w$ from [Jo7, 2.4.2.13].

Let us complete this subsection by describing previous results relating dimensions of irreducible $\mathcal{W}$-modules to Goldie ranks. The existence of such a relationship was conjectured by Premet in [Pr3, Question 5.1]. In [Lo1, Proposition 3.4.6], the author proved that the Goldie rank of an arbitrary primitive ideal $J$ does not exceed the dimension of the corresponding irreducible module. In [Pr4] Premet significantly improved that result by showing that the Goldie rank always divides the dimension. Moreover, he proved that the equality always holds in type $A$. However, in other classical types the equality does not need to hold (for ideals with non-integral central character). A counter-example is provided by the ideal $J(\rho/2)$ in type $C_n$ for $n$ large enough, see [Pr4, 1.3], for details. In [Br] Brundan reproved the equality of the Goldie rank and the dimension for type $A$.

\footnote{In a recent preprint of Premet, [Pr5], he proved that any $\mathcal{W}$-algebra admits an $A$-stable 1-dimensional representation, but the central character is generally not integral. For example, [LO] implies that, in the case of the three exceptional orbits, there are no $A$-stable finite dimensional representations with integral central character, so Premet’s representations automatically have non-integral central characters in these cases.}
1.4. Results on characters of simple $\mathcal{W}$-modules. We will need to fix some notation to state the main result that is the existence of a functor with certain properties.

Fix a nilpotent element $e \in g$. We include $e$ into a minimal Levi subalgebra $g_0$ so that $e$ is a distinguished nilpotent element in $g_0$. Choose Cartan and Borel subalgebras $h \subset b_0 \subset g_0$. Let $g = \bigoplus_{i \in \mathbb{Z}} g(i)$ stand for the eigendecomposition for $h$. We recall that a distinguished element is always even (see, for example, [CM, Theorem 8.2.3]) so $g_0(i) := g_0 \cap g(i)$ is zero when $i$ is odd. Form the $W$-algebra $W$ from $(g, e)$.

Pick an integral element $\theta \in \mathfrak{z}(g_0)$ such that $\mathfrak{z}(\theta) = g_0$. Consider the eigen-decomposition $g = \bigoplus_{i \in \mathbb{Z}} g_i$ for $\theta$. Let $b := b_0 \oplus g_{>0}$, where we set $g_{>0} := \bigoplus_{i > 0} g_i$, this is a Borel subalgebra in $g$. Further, set $p := g_0(>0) \oplus g_{>0}$. This is a parabolic subalgebra in $g$. Let $P$ denote the corresponding parabolic subgroup.

Let $A$ denote the weight lattice for $g$. For $\lambda \in A$ let $L_{00}(\lambda), L_0(\lambda), L(\lambda)$ denote the irreducible modules with highest weight $\lambda - \rho$ for $g_0(0), g_0$ and $g$, respectively.

We consider two categories: the sum of integral blocks of the parabolic category $O$ for $(g, p)$, to be denoted by $O^P$, and the category $\mathcal{O}$ for $g$, denoted by $\mathcal{O}^g(e, e)$. The definition of the latter will be recalled in Subsection 3.2. Let $\Lambda_p$ denote the subset of all weights $\lambda$ such that $\langle \lambda, \alpha^\vee \rangle > 0$ for all simple roots $\alpha$ of $g_0(0)$. The simples in $O^P$ are precisely $L(\lambda), \lambda \in \Lambda_p$. Let $\mathcal{W}^g$ denote the $W$-algebra for the pair $(g_0, e)$. The simples in $\mathcal{O}^g(e, e)$ are parameterized by the finite dimensional irreducible $\mathcal{W}^g$-modules, the simple corresponding to $N^{\sigma}_0$ will be denoted by $L^\theta_{\mathcal{W}}(N^{\sigma}_0)$. Further to every finite dimensional $\mathcal{W}^g$-module $N^{\sigma}_0$ we can assign a “Verma module” $\Delta^\theta_{\mathcal{W}}(N^{\sigma}_0)$.

One can compute the character of $\Delta^\theta_{\mathcal{W}}(N^{\sigma}_0)$ as follows. Consider the action of $\mathfrak{z}(g_0)$ on the centralizer $\mathfrak{z}(e)$ of $e$ in $g$. Let $\mu_1, \ldots, \mu_k$ be all weights (counted with multiplicities) of this action that are negative on $\theta$. Suppose that $\mathfrak{z}(g_0)$ acts on $N^{\sigma}_0$ with a single weight $\mu_0$. Then the character of $\Delta^\theta_{\mathcal{W}}(N^{\sigma}_0)$ equals

$$e^{\mu_0} \dim N^{\sigma}_0 \prod_{i=1}^{k} (1 - e^{\mu_i})^{-1}.$$

We remark that we will be dealing with modules $N^{\sigma}_0$ for which the dimension is easy to compute, see 1) below, generally, these modules will be reducible.

We will produce an exact functor $\mathbb{V} : O^P \rightarrow \mathcal{O}^g(e, e)$. The character computation for the simples with integral central character in $\mathcal{O}^g(e, e)$ will be based on the following two properties of $\mathbb{V}$.

1) $\mathbb{V}$ maps the parabolic Verma $\Delta_P(\lambda)$ to $\Delta^\theta_{\mathcal{W}}(N^{\sigma}_0)$, where $N^{\sigma}_0$ is a $\mathcal{W}^g$-module of dimension equal to $\dim L_{00}(\lambda)$. In particular, we have

$$\text{ch} \mathbb{V}(\Delta_P(\lambda)) = e^{\lambda - \rho} \dim L_{00}(\lambda) \prod_{i=1}^{k} (1 - e^{\mu_i})^{-1}.$$

2) One can also describe the image of $L(\lambda)$ under $\mathbb{V}$. By $\Lambda^\text{max,0}_P$ we denote the subset of $\Lambda_P$ consisting of all $\lambda$ such that the Gelfand-Kirillov (GK) dimension of $L_0(\lambda)$ equals to $\dim g_0(<0)$, this is the maximal possible GK dimension for an object in the parabolic category $O$ for $(g_0, p \cap g_0)$. If $\lambda \notin \Lambda^\text{max,0}_P$, then $\mathbb{V}(L(\lambda)) = 0$. If $\lambda \in \Lambda^\text{max,0}_P$, then the associated variety of the ideal $J_0(\lambda) = \text{Ann}_{U(g_0)} L_0(\lambda)$ is the closure of $G_0 e$. We have

$$\mathbb{V}(L(\lambda)) = \mathbb{V} \otimes \bigoplus_{N^{\sigma}_0} L^\theta_{\mathcal{W}}(N^{\sigma}_0).$$
Here the sum is taken over all irreducible \( W^0 \)-modules \( \mathcal{N}^0 \) lying over \( \mathcal{J}_0(\lambda) \), the description of the set of such \( \mathcal{N}^0 \) was recalled in Subsection 1.2, let \( n_\lambda \) be the number. When \( \mathfrak{g} \) is classical, then \( \mathcal{V} \) is just \( \mathbb{K} \). In general, \( \mathcal{V} \) is described as follows. We represent \( \lambda \) in the form \( w_{\mathfrak{g}_0} \), where \( \mathfrak{g}_0 \in \Lambda \) is dominant for \( \mathfrak{g}_0 \) and \( w \in W_0 \) is compatible with \( \mathfrak{g}_0 \). Then \( \mathcal{V} \) is the vector space from the triple \((x, y, \mathcal{V})\) assigned to \( w \) (viewed as an element in \( W_0 \)) in Subsection 1.2, see also Subsection 2.4.

It is not difficult to show (see Subsection 4.3.5) that \( \text{ch} L^\theta_{\mathcal{V}}(\mathcal{N}^0) \) does not depend on the choice of \( \mathcal{N}^0 \) in the \( A_0 \)-orbit (where \( A_0 \) denotes the component group for \( e \) viewed as an element in \( \mathfrak{g}_0 \)). Let us explain how to compute this character.

Define the integers \( c_{wu} \) by the equality \( L(w_{\mathfrak{g}_0}) = \sum_u c_{wu} \Delta_{P}(u_{\mathfrak{g}_0}) \) in the Grothendieck group \( K_0(\mathcal{O}_P^*) \) in the infinitesimal block of \( \mathcal{O}_P^* \) with generalized central character corresponding to \( \mathfrak{g}_0 \). Recall that \( K_0(\mathcal{O}_P^*) \) is a free abelian group with basis \( \Delta_{P}(u_{\mathfrak{g}_0}) \), where \( u \) runs over all elements in \( W_0 \) compatible with \( \mathfrak{g}_0 \) and such that \( u_{\mathfrak{g}_0} \in \Lambda_p \). Let us also point out that the numbers \( c_{wu} \) are known, they equal \( c_{wu}(1) \), where \( c_{wu}(q) \) is a parabolic Kazhdan-Lusztig polynomial.

Then we have

\[
(1.3) \quad n_{w_{\mathfrak{g}_0}} \text{dim} \mathcal{V} \cdot \text{ch} L^\theta_{\mathcal{V}}(\mathcal{N}^0) = \sum_u c_{wu} \text{ch} \mathcal{V}(\Delta_{P}(u_{\mathfrak{g}_0})).
\]

where \( \text{ch} \mathcal{V}(\Delta_{P}(u_{\mathfrak{g}_0})) \) can be computed by (1.2) and \( n_{w_{\mathfrak{g}_0}} \) is the number of the irreducible finite dimensional \( W^0 \)-modules lying over \( \mathcal{J}_0(w_{\mathfrak{g}_0}) \). As in Remark 1.4, when all simple summands of \( \mathfrak{g}_0 \) are classical, we can recover \( n_{w_{\mathfrak{g}_0}} \text{dim} \mathcal{V} = n_{w_{\mathfrak{g}_0}} \) combinatorially starting from \( w \).

To finish this discussion we would like to point out that the functor \( \mathcal{V} \) has other nice properties to be investigated in the present paper. We summarize them in Theorem 4.8.

Let us mention some special cases where the character formulas were known before. In type \( A \), they follow from the work of Brundan and Kleshchev, [BK1]. Somewhat more generally, if \( e \) is principal in a Levi subalgebra, then the character formulas follow from the results of [Lo3, Section 4]. We remark that both computations are based on using essentially opposite special cases of the functor \( \mathcal{V} \).

Another result related to ours is the main result of [BM]. There Bezrukavnikov and Mirkovic deal with simple non-restricted representations over semisimple Lie algebras in characteristic \( p \). More precisely, let \( \mathbb{F} \) be an algebraically closed field of characteristic \( p \gg 0 \). Since \( p \gg 0 \), there is a natural bijection between the nilpotent orbits in \( \mathfrak{g}_* \) and in \( \mathfrak{g}_*^\mathfrak{F} \), let \( \mathcal{O}_\mathfrak{F} \) denote the orbit in \( \mathfrak{g}_*^\mathfrak{F} \) corresponding to \( \mathcal{O} \). Bezrukavnikov and Mirkovic consider the category of all \( \mathfrak{g}_*^\mathfrak{F} \)-modules with trivial Harish-Chandra character and \( p \)-character \( e_\mathfrak{F} \in \mathcal{O}_\mathfrak{F} \) (we can view \( \mathcal{O}_\mathfrak{F} \) as an orbit in the Frobenius twist \( \mathfrak{g}_*^{(1)} \) because \( \mathfrak{g}_* \) and \( \mathfrak{g}_*^{(1)} \) are naturally identified). They identify the complexified \( K_0 \) of this category with \( H^* (\mathcal{B}_e) \), where \( \mathcal{B}_e \) stands for the Springer fiber (in characteristic 0). Then they consider a deformation \( H^* (\mathcal{B}_e) \) over \( \mathbb{K}[q^{\mathbb{Z}}] \) given by \( H^*_{\mathbb{K}^x} (\mathcal{B}_e) \) for a suitable \( \mathbb{K}^x \)-action on \( \mathcal{B}_e \) (so that \( q \) becomes the equivariant parameter for the \( \mathbb{K}^x \)-action). Then \( H^*_{\mathbb{K}^x} (\mathcal{B}_e) \) becomes an explicit module over the affine Hecke algebra \( \mathcal{H} \) for \( \mathbb{W} \). The \( \mathcal{H} \)-module \( H^*_{\mathbb{K}^x} (\mathcal{B}_e) \) comes equipped with a \( \mathbb{K}[q^{\mathbb{Z}}] \)-bilinear form and a semilinear bar-involution that can be written in terms of the \( \mathcal{H} \)-module structure. From these data one can define a canonical basis of \( H^*_{\mathbb{K}^x} (\mathcal{B}_e) \) in the sense of Kashiwara (the elements have to be fixed by the bar-involution and orthonormal modulo \( q \)). The classes of simple \( \mathfrak{g}_*^\mathfrak{F} \)-modules in \( H^* (\mathcal{B}_e) \) are then the specializations of the canonical basis elements at \( q = 1 \). If one knows the canonical basis elements, then one can write the dimension formulas.
for the simples. However it is unclear how to write those elements explicitly (with a possible exception of the case when \( e \) is principal in a Levi subalgebra but that case was not worked out either).

Now let us explain a connection of [BM] to our work. It follows from [Pr1, Theorem 1.1] that the central reduction of \( U(\mathfrak{g}_F) \) at the \( p \)-character \( e_F \) is isomorphic to the matrix algebra of rank \( p^{\dim O/2} \) over the analogous reduction of the W-algebra. So [BM] basically describes the classes of the simples over the latter algebra. For \( p \) large enough, one can take an irreducible finite dimensional representation of \( W \), reduce this representation (or rather its integral form; see, for example, [Pr4, Section 2] for technical details on this procedure) modulo \( p \) and get an irreducible finite dimensional representation for the W-algebra in characteristic \( p \) with \( p \)-character \( e_F \). However, a relatively small portion of simples in characteristic \( p \) is obtained in this way. So, comparing to [BM], we get explicit formulas but for a smaller number of modules.

Finally, let us point out that we get two ways to compute the Goldie ranks of primitive ideals: one via a more explicit formula coming from the character formulas in the W-algebra category \( O \) and one implementing a program of Joseph. It is completely unclear to the author why these formulas give the same result!

1.5. **Organization and content of the paper.** Let us describe the content of this paper, section by section.

Section 2 is preliminary and does not contain new results. There we explain (a slight ramification) of the definition of a W-algebra via a quantum slice construction that first appeared in [Lo1] somewhat implicitly and in [Lo5] in a more refined form. Next, in Subsection 2.2, we recall the main construction of [Lo2]: a functor \( \bullet \downarrow \) between the categories of Harish-Chandra bimodules for \( \mathcal{U} \) and for \( \mathcal{W} \). There are several reasons why we need this functor, for example, the functor \( \mathcal{V} \) mentioned above can be defined analogously to \( \bullet \downarrow \). Also \( \bullet \downarrow \) plays an important role both in the study of \( \mathcal{V} \) and in the computation of Goldie ranks. In Subsection 2.3 we recall the classification of finite dimensional irreducible \( \mathcal{W} \)-modules obtained in [LO]. Subsection 2.4 describes the images of certain simple Harish-Chandra \( \mathcal{U} \)-bimodules under \( \bullet \downarrow \). This gives rise to the generalized RSK correspondence mentioned in Subsection 1.2 and again plays an important role in the computation of characters and a crucial role in the computation of Goldie ranks. Subsection 2.5 briefly recalls some results on a parabolic induction for W-algebras obtained in [Lo4]. We will need these results to prove Conjecture 1.1 for classical groups in Subsection 5.4. Finally, large Subsection 2.6 deals with various isomorphisms, mostly of certain completions, that can be deduced from the quantum slice construction. These isomorphisms play an important role in the construction of \( \mathcal{V} \).

In Section 3 we introduce categories of modules that are involved in our construction and study of \( \mathcal{V} \). In the first subsection we recall some known facts about parabolic categories \( O \) including the Bernstein-Gelfand equivalence. The latter is a crucial tool to transfer the properties of \( \bullet \downarrow \) to those of \( \mathcal{V} \). In Subsection 3.2 we recall the definition of a category \( O \) for \( \mathcal{W} \) basically following [BGK]. We also recall the construction of Verma modules and explain how to compute their characters again following [BGK]. Next, we introduce the category of Whittaker \( \mathcal{U} \)-modules basically as it appeared in [Lo3] and recall an equivalence between W-algebra categories \( O \) and the Whittaker categories. In the final part of Subsection 3.2 we introduce a duality functor for a W-algebra category \( O \), the construction is pretty standard. Subsection 3.3 is new but is not very original. There we study the completed versions of W-algebra categories \( O \) and of Whittaker categories. A completed version of a W-algebra
category $\mathcal{O}$ is obtained from the usual one via completion. This is no longer so for Whittaker categories, a completed version is different from the usual one. Yet, we show that it still has analogs of Verma modules and is equivalent to a completed version of a category $\mathcal{O}$ for the $W$-algebra.

Section 4 is one of the two central sections of this paper. There we introduce a functor $\mathcal{V}$ and study its properties. In Subsection 4.1 we define a functor denoted by $\mathbf{\bullet}_{t,e}$ that generalizes $\mathbf{\bullet}_t$. Its source category is an appropriate category of $\mathcal{U}$-modules and the target category is a category of $W$-modules. After the functor is constructed we study some its general properties: we show that it is exact, intertwines tensor products with Harish-Chandra bimodules and study the question of existence of a right adjoint functor. In Subsection 4.2 we construct a functor $\mathcal{V} : \mathcal{O}_P^P \to \mathcal{O}^P(g,e)_W^R$ (the notation will be explained below). We provide three constructions and show that they are equivalent. The first construction is as $\mathbf{\bullet}_{t,e}$, while the other two constructions use the equivalences between the two versions (usual and completed) of the Whittaker category and the two versions of the category $\mathcal{O}$ for $\mathcal{W}$. In process of proving that the three constructions are equivalent we establish some important properties of $\mathcal{V}$, for example, its behavior on parabolic Verma modules and a relation with the duality functors. The last Subsection 4.3 establishes some further properties of $\mathcal{V}$. Most importantly, there we show that $\mathcal{V}$ is a quotient functor onto its image and has the double centralizer property. The properties of $\mathcal{V}$ including those needed for character formulas are summarized in Theorem 4.8. Further, under some restrictions on $P$ we show that $\mathcal{V}$ is 0-faithful, i.e. fully faithful on modules admitting a parabolic Verma filtration.

In Section 5 we deal with Goldie ranks. First, we recall some known results on them, almost entirely due to Joseph. In Subsection 5.2 we use the results quoted in Subsections 2.2,2.4 to prove some formula for the scale factors $z_w$. In Subsection 5.3 we complete the proof of Theorem 1.3 modulo Conjecture 1.1. Finally, in Subsection 5.4 we prove Conjecture 1.1 for the classical types. For this we first use a reduction procedure based on the parabolic induction for $W$-algebras. Then we deal with the three classical types (we do not consider type A) one by one, explaining type B in detail and then describing modifications to be made for types C and D.

Section 6 deals with three topics that are related to the functor $\mathcal{V}$. In Subsection 6.1 we discuss the functor $\mathbf{\bullet}_{t,e}$ for Harish-Chandra $(\mathfrak{g},K)$-modules. In Subsection 6.2 we study a different version of the functor $\mathbf{\bullet}_{t,e}$ for a parabolic category $\mathcal{O}$, one associated with the Richardson orbit. This functor was studied by Stroppel in [St1],[St2] but she did not relate the target category to $W$-algebras. We show that many properties of the functor $\mathcal{V}$ carry over to this situation. We also analyze conditions that guarantee the 0-faithfulness of our functor in type A. In the last subsection of Section 6 we extend (in a straightforward way) the definition of $\mathbf{\bullet}_{t,e}$ to the categories of modules over Dixmier algebras (i.e., algebras equipped with a homomorphism from $\mathcal{U}$ turning them into Harish Chandra bimodules). We are basically interested in two classes of Dixmier algebras: one coming from Lie superalgebras and the other from quantum groups at roots of unity: we show that the quantum Frobenius epimorphism splits turning (in fact, some mild quotient of) the Lusztig form of a quantum group into a Dixmier algebra.

1.6. Notation and conventions. In this subsection we will list some notation and conventions used in this paper. They will be duplicated (and explained in more detail) below.

Our base field is an algebraically closed field $K$ of characteristic 0.
1.6.1. Algebras and groups. Let $G$ be a reductive algebraic group over $\mathbb{K}$ and $\mathfrak{g}$ be its Lie algebra. Fix a nilpotent element $e$ and include it into an $\mathfrak{sl}_2$-triple $(e, h, f)$. Let $\mathfrak{c}$ denote the $G$-orbit of $e$ (under the adjoint action). Fix a non-degenerate symmetric $G$-invariant form on $\mathfrak{g}$, say $(\cdot, \cdot)$, whose restriction to the rational form of a Cartan subalgebra is positive definite. Using this form we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$. We write $\chi$ for the image of $e$ under this identification. By $Q$ we denote the centralizer of $(e, h, f)$ in $G$. This is a reductive subgroup of $G$. We write $\mathfrak{g}(i)$ for the eigenspace of $\mathrm{ad} \ h := [h, \cdot]$ with eigenvalue $i$ and use the notation like $\mathfrak{g}(> 0)$ for $\bigoplus_{i > 0} \mathfrak{g}(i)$.

By $\mathcal{U}$ or $\mathcal{U}(\mathfrak{g})$ we denote the universal enveloping algebra of $\mathfrak{g}$. We consider the Slodowy slice $S = e + j_{\mathfrak{g}}(f) \subset \mathfrak{g} \cong \mathfrak{g}^*$, where $j_{\mathfrak{g}}(\bullet)$ stands for the centralizer in $\mathfrak{g}$. Inside $\mathfrak{g}^*$, the slice $S$ is realized as $\chi + [\mathfrak{g}, f]^{\perp}$, where the superscript $\perp$ indicates the annihilator. Let $\mathcal{W}$ denote the $W$-algebra of the pair $(\mathfrak{g}, e)$. Sometimes when we want to explicitly indicate the Lie algebra and the nilpotent element used to produce a $\mathcal{W}$-algebra, we use the notation like $\mathcal{U}(\mathfrak{g}, e)$ for the $\mathcal{W}$-algebra.

Set $V := [\mathfrak{g}, f]$. This is a symplectic vector space with form $\omega(x, y) = (e, [x, y])$. By $\mathcal{A}$ we denote the Weyl algebra of $V$, i.e., $\mathcal{A} = T(V)/(u \otimes v - v \otimes u - \omega(u, v))$.

We also will consider the “homogenizations” of the algebras above, they will be decorated with the subscript “$h$”.

In Subsection 2.6, Section 3 and Subsections 4.2.4.3 we will use the following notation. We will fix an integral element $\theta \in \mathfrak{g}$ centralizing $e, h, f$ such that the element $e$ is even in $\mathfrak{g}_0 := j_{\mathfrak{g}}(\theta)$. By $G_0$ we denote the connected subgroup of $G$ corresponding to $\mathfrak{g}_0$. We will consider the gradings by eigenspaces of $\mathrm{ad} \theta$: $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, V = \bigoplus_i \mathcal{V}_i, \mathcal{U} = \bigoplus_i \mathcal{U}_i, \mathcal{W} = \bigoplus_i \mathcal{W}_i$. We use the notation like $\mathcal{U} \geqslant \mathcal{W}$ similarly to the above. Further, we consider the algebras $\mathcal{U}^0 := \mathcal{U}_{\geqslant 0}/(\mathcal{U}_{\geqslant 0} \cap \mathcal{U} \mathcal{U}_{\geqslant 0}), \mathcal{W}^0, \mathcal{A}^0$ (to see that these spaces are actually algebras we notice that, for example, $\mathcal{U}_{\geqslant 0}$ is a subalgebra in $\mathcal{U}$, and $\mathcal{U}_{\geqslant 0} \cap \mathcal{U} \mathcal{U}_{\geqslant 0}$ is a two-sided ideal in $\mathcal{U}_{\geqslant 0}$).

We write $\mathfrak{m}$ for $\mathfrak{g}_0(- < 0) \oplus \mathfrak{g}_{\geqslant 0}$, $\mathfrak{m}$ for $\mathfrak{g}_0(- < 0) \oplus \mathfrak{g}_{< 0}$, $\mathfrak{p}$ for $\mathfrak{g}_0(\geqslant 0) \oplus \mathfrak{g}_{\geqslant 0}$, and $\mathfrak{t}$ for $j_{\mathfrak{g}}(\mathfrak{g}_0)$. Let $\mathcal{M}, \mathcal{M}, P, T$ be the corresponding connected subgroups of $G$ so that $\mathcal{M}, \mathcal{M}$ are unipotent, $P$ is parabolic, $T$ is a torus. Further we choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ and a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{b}_0 := \mathfrak{b} \cap \mathfrak{g}_0$ is a Borel subalgebra in $\mathfrak{g}_0$. Let $\Lambda \subset \mathfrak{h}^*$ denote the weight lattice of $G$ and $\Lambda^+$ be the subset of dominant weights. As usual, we write $\rho$ for half the sum of positive roots.

We write $\mathfrak{m}^0 := \mathfrak{m} \cap \mathfrak{g}_0, \mathfrak{m}^0 := \mathfrak{m} \cap \mathfrak{g}_0$. We consider the shift $\mathfrak{m}_x := \{x - (\chi, x) \mid x \in \mathfrak{m}\} \subset \mathfrak{g} \oplus \mathbb{K}$ and the similar shifts $\mathfrak{m}_x, \mathfrak{m}_x, \mathfrak{m}_x$. Further, we set $\mathfrak{v} := \mathfrak{m} \cap \mathcal{V}, \mathfrak{v} := \mathfrak{m} \cap \mathcal{V}$.

We consider the completions $\mathcal{U}^\wedge := \varprojlim_{n \to +\infty} \mathcal{U}/\mathcal{U}\mathfrak{m}_n, ^\wedge \mathcal{U} := \varprojlim_{n \to +\infty} \mathcal{U}/\mathfrak{m}_n^0 \mathcal{U}, \mathcal{A}^\wedge := \varprojlim_{n \to +\infty} \mathcal{A}/\mathcal{A} \mathfrak{v}_n, ^\wedge \mathcal{A} := \varprojlim_{n \to +\infty} \mathcal{A}/\mathfrak{v}_n^0 \mathcal{A}, ^\wedge \mathcal{W} := \varprojlim_{n \to +\infty} \mathcal{W}/\mathcal{W}\mathfrak{m}_n^0, ^\wedge \mathcal{W} := \varprojlim_{n \to +\infty} \mathcal{W}/\mathcal{W}\mathfrak{m}_n^0 \mathcal{W}$ and the similar completions of $\mathcal{U}^0, \mathcal{W}^0, \mathcal{A}^0$.

Finally, in this context, for $R$ we take the centralizer of $T \subset Q$ in $Q$. This is a reductive subgroup.

1.6.2. Categories and functors. Here we are going to explain some notation used mostly in Sections 3, 4.

We usually abbreviate “Harish-Chandra” as “HC”. We use the notation $\text{HC}(\mathcal{U})$ for the category of $\text{HC} \mathcal{U}$-bimodules, and $\text{HC}^G(\mathcal{W})$ for the category of $\mathcal{Q}$-equivariant $\mathcal{H} \mathcal{W}$-bimodules.

For an algebraic subgroup $K \subset G$ and a character $\nu$ of its Lie algebra $\mathfrak{k}$ we write $\mathcal{O}_\nu^K$ for the category of $(K, \nu)$-equivariant modules. We say that a $(\mathcal{U}, K)$-module $\mathcal{M}$ is $(K, \nu)$-equivariant if the structure map $\mathcal{U} \otimes \mathcal{M} \to \mathcal{M}$ is $K$-equivariant and the differential of the...
$K$-action coincides with the $\mathfrak{g}$-action given by $(x, m) \mapsto xm - \nu(x)m$. We write $\mathcal{O}_\nu^K$ for the full subcategory in $\hat{\mathcal{O}}_\nu^K$ consisting of finitely generated modules. When $\nu = 0$, we suppress the subscript.

In particular, we are going to consider the parabolic categories $\mathcal{O}_\nu^P$ for $(\mathfrak{g}, \mathfrak{p})$ and $\mathcal{O}_\nu^{\mathfrak{b}}$ for $(\mathfrak{g}_0, \mathfrak{p}_0)$. The parabolic Verma modules with highest weights $\lambda - \rho$ are denoted by $\Delta_P(\lambda), \Delta_{R_0}(\lambda)$. Their simple quotients are denoted by $L(\lambda), L_0(\lambda)$. We write $L_{00}(\lambda)$ for the irreducible $\mathfrak{g}_0(0)$-module with highest weight $\lambda - \rho$. The Verma modules in the whole BGG category $\mathcal{O}$ are denoted by $\Delta(\lambda)$.

The categories $\mathcal{O}$ for $W$ constructed from the element $\theta$ are denoted by $\mathcal{O}^\theta(\mathfrak{g}, e)_\nu$ (the usual version), $\mathcal{O}^\theta(\mathfrak{g}, e)_\nu^R$ (the $R$-equivariant version), $\hat{\mathcal{O}}^\theta(\mathfrak{g}, e)_\nu$ (the completed version). For the Whittaker categories we use the notation like $\hat{\text{Wh}}^\theta(\mathfrak{g}, e)_\nu$, etc. Again, when $\nu = 0$, we suppress it from the notation.

There are various functors between the categories under consideration. We write $\hat{\mathfrak{g}}$ for the weight completion functor, it maps a module of the form $\bigoplus t$ to the sum of all submodules that are objects in $\hat{\text{Wh}}^\theta$.

On some categories of interest, we have a “naive” duality functor to be denoted by $\hat{\mathcal{O}}^\theta(\mathfrak{g}, e)_\nu$. These are $\hat{\mathfrak{m}}_\lambda$-adic completions: $\mathcal{M}^\wedge := \lim_{\leftarrow n \to +\infty} \mathcal{M}/\hat{\mathfrak{m}}_\lambda^n\mathcal{M}$, this gives a functor, for instance, from $\mathcal{O}_\nu^P$ to $\hat{\text{Wh}}^\theta(\mathfrak{g}, e)_\nu^R$. Somewhat dually, we have a functor $\hat{\text{Wh}}_\nu$ that takes a $\mathcal{U}$-module and maps it to the sum of all submodules that are objects in $\hat{\text{Wh}}^\theta(\mathfrak{g}, e)_\nu$.

On some categories of interest, we have a “naive” duality functor to be denoted by $\hat{\mathcal{O}}^\theta(\mathfrak{g}, e)_\nu$. This functor is the composition of taking the restricted dual, to be denoted by $\mathcal{M} \mapsto \mathcal{M}^{(\ast)}$, and the twist by an anti-automorphism, say $\tau$, $\mathcal{M}^{(\ast)} \mapsto \tau \mathcal{M}^{(\ast)}$. We will also consider the homological duality functor that comes from $\text{RHom}$. This functor is denoted by $\mathcal{D}$.

Next, we have “Verma module” functors: $\Delta_W^0 : \mathcal{O}^\theta(\mathfrak{g}_0, e)_\nu \to \mathcal{O}^\theta(\mathfrak{g}, e)_\nu, \hat{\mathcal{O}}^\theta(\mathfrak{g}_0, e)_\nu^R \to \mathcal{O}^\theta(\mathfrak{g}, e)_\nu^R, \Delta^0 : \mathcal{U}^0\text{-mod} \to \mathcal{U}\text{-mod}, \Delta_D^0 : \mathcal{O}^\theta(\mathfrak{g}_0, e)_\nu \to \hat{\text{Wh}}^\theta(\mathfrak{g}, e)_\nu, \hat{\Delta}_W^0 : \mathcal{O}^\theta(\mathfrak{g}_0, e) \to \hat{\mathcal{O}}^\theta(\mathfrak{g}, e)$, etc. We will consider the quotients $\hat{L}_\nu^\lambda(\bullet)$ by the maximal submodule that does not intersect the highest weight subspace.

1.6.3. **Cells, primitive ideals, etc.** This notation is mostly used in Sections 4, 5. We write $W$ (resp., $W_0$) for the Weyl group of $\mathfrak{g}$ (resp., $\mathfrak{g}_0$) acting on the Cartan subalgebra $\mathfrak{h}$.

We usually denote a two-sided cell by $c$ and left cells by $c, c_1$, etc.. We write $\mathcal{O}_c$ for the special orbit corresponding to $c$. Let $\hat{A}$ denote the Lusztig quotient of the component group $A(\mathcal{O}_c)$ (i.e., the component group of the centralizer of some element of $\mathcal{O}_c$ in the adjoint group). By $H_c$ we denote the Lusztig subgroup of $\hat{A}$ corresponding to $c$. The corresponding objects for $\mathfrak{g}_0$ are decorated with the subscript “0”.

To an element $w$ we can assign a simple HC bimodule $\mathcal{M}_w$ of $\mathfrak{g}$-finite maps $\Delta(\rho) \to L(w \rho)$. We write $J(\lambda)$ for the annihilator of $L(\lambda)$ in $\mathcal{U}$. For a HC bimodule $\mathcal{M}$ we write $\mathcal{V}(\mathcal{M})$ for its associated variety that is a subvariety in $\mathfrak{g} \cong \mathfrak{g}^\ast$.

By $J_{P, \nu}$ we denote the annihilator of $\Delta_P(\nu + \rho)$ in $\mathcal{U}$. Here $\nu$ is a character of $\mathfrak{p}$.

To a left cell $c$ in $W$ one assigns its cell module, we denote it by $\mathcal{M}(c)$.

There will be some other notation related to Goldie ranks introduced in Section 5.

1.6.4. **Miscellaneous notation.** Below we present some other notation to be used in the paper.

$\mathcal{A}^{\text{opp}}$ the opposite algebra of $\mathcal{A}$. 
\( \otimes \) the completed tensor product of complete topological vector spaces/
modules.

\((a_1, \ldots, a_k)\) the two-sided ideal in an associative algebra generated by elements
\(a_1, \ldots, a_k\).

\(A^\wedge \chi\) the completion of a commutative (or “almost commutative”) algebra
\(A\) with respect to the maximal ideal of a point \(\chi \in \text{Spec}(A)\).

\(\text{Ann}_A(\mathcal{M})\) the annihilator of an \(A\)-module \(\mathcal{M}\) in an algebra \(A\).

\(D(X)\) the algebra of differential operators on a smooth variety \(X\).

\(G^\circ\) the connected component of unit in an algebraic group \(G\).

\((G, G)\) the derived subgroup of a group \(G\).

\(G_x\) the stabilizer of \(x\) in \(G\).

\(\text{Grk}(A)\) the Goldie rank of a prime Noetherian algebra \(A\).

\(\text{gr} \ A\) the associated graded vector space of a filtered vector space \(A\).

\(R_h(\mathcal{A}) := \bigoplus_{i \in \mathbb{Z}} h^i \mathcal{A}\) the Rees \([h]\)-module of a filtered vector space \(\mathcal{A}\).

\(S(V)\) the symmetric algebra of a vector space \(V\).

\(\mathcal{X}(H)\) the group of characters of an algebraic group \(H\).

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2. W-algebras and finite dimensional modules

2.1. W-algebras via a slice construction. Let \(G, \mathfrak{g}, e, h, f, (\cdot, \cdot), \chi := (e, \cdot), \mathcal{U}, Q, \mathcal{O}\) have
the same meaning as in Subsection 1.6. Further, let \(\tau\) be either an involutive anti-automorphism
of \(G\) with \(\tau(e) = e, \tau(f) = f, \tau(h) = -h\) or (for formal technical reasons) the identity. We
will mostly use \(\tau\) described in 2.6.1.

Let us recall the construction of a W-algebra associated to the pair \((\mathfrak{g}, e)\) (or, more pre-
cisely, to \(\mathfrak{g}\) and the \(\mathfrak{sl}_2\)-triple \(e, h, f\)). Consider the group \(\hat{Q} := \mathbb{Z}/2\mathbb{Z} \ltimes (Q \times \mathbb{K}^\times)\), where the
action of a nontrivial element \(\zeta \in \mathbb{Z}/2\mathbb{Z}\) on \(Q\) is induced by \(\tau\), while the action on \(\mathbb{K}^\times\) is
trivial. The group \(\hat{Q}\) acts on \(\mathfrak{g}\) and \(\mathfrak{g}^*\): for \(q \in Q, t \in \mathbb{K}^\times, x \in \mathfrak{g}, \alpha \in \mathfrak{g}^*\) we set:

\[
q.x = \text{Ad}_q(x), q.\alpha = \text{Ad}_q^*(\alpha), \quad t.x = t^2t^h.x, t.\alpha = t^{-2}t^h\alpha, \quad \zeta.x = \tau(x), \zeta.\alpha = \tau(\alpha).
\]

Here \(t \mapsto t^h\) stands for the one-parameter subgroup of \(G\) corresponding to the semisimple
element \(h\). We remark that the \(\hat{Q}\)-action fixes \(\chi\). Also the Slodowy slice \(S := \chi + [\mathfrak{g}, f]^\perp \subset \mathfrak{g}^*\)
is \(\hat{Q}\)-stable (under the isomorphism \(\mathfrak{g}^* \cong \mathfrak{g}\) given by the Killing form the Slodowy slice
becomes \(e + \delta_0(f)\), we would like to point out that the isomorphism is not \(\hat{Q}\)-equivariant
(because \(\hat{Q}\) rescales the Killing form). The restriction of the \(\hat{Q}\)-action to \(\mathbb{K}^\times\) is often called
the Kazhdan action.

Consider the universal enveloping algebra \(\mathcal{U}\) of \(\mathfrak{g}\). We endow this algebra with a “doubled”
PBW filtration \(F_0 \mathcal{U} = \mathbb{K} \subset F_1 \mathcal{U} \subset \ldots \subset \mathcal{U}\) such that \(F_i \mathcal{U}\) is the span of all monomials
of degree \(\leq i/2\). Let \(\mathcal{U}_h\) stand for the Rees algebra with respect to this filtration,
\(\mathcal{U}_h := \bigoplus_{i=0}^{\infty} F_i \mathcal{U} \cdot h^i\). On \(\mathcal{U}_h\) we have an action of the group \(\hat{Q} := \mathbb{Z}/2\mathbb{Z} \times \hat{Q}\) given as follows: the
action of \(\mathbb{Z}/2\mathbb{Z}\) on \(\mathfrak{g}\) is trivial and the action of \(\hat{Q}\) on \(\mathfrak{g}\) is as before, while \(\mathbb{Z}/2\mathbb{Z} \ltimes Q \subset \hat{Q}\)
acts trivially on \(h\), the other copy of \(\mathbb{Z}/2\mathbb{Z}\) acts on \(h\) by changing the sign, and \(t.h = th\) for
\(t \in \mathbb{K}^\times\).
We can view $\chi$ as a homomorphism $U_h \to K$ via $U_h \to S(\mathfrak{g}) \cong K$. Let $I_\chi, I_\chi$ denote the kernels of $\chi$ in $S(\mathfrak{g}), U_h$, respectively. Since $\chi$ is fixed by $Q$, we see that $I_\chi$ is $Q$-stable and therefore $\tilde{Q}$ acts on the completion $U_h^x := \lim_{n \to +\infty} U_h/I_{\chi}^n$.

Set $V := [\mathfrak{g}, f]$. We can view $V$ as a subspace in $T^*_\chi = I_\chi/I_{\chi}^2$ via the map $V \to I_\chi$ sending $v$ to $v - (\chi, v)$. We claim that there is an embedding $\iota : V \to I_{\chi}^x$ with the following properties:

(1) $\iota$ is $\tilde{Q}$-equivariant.

(2) The composition of $\iota$ with a natural projection $I_{\chi}^x \to I_\chi/I_{\chi}^2$ is the inclusion $V \subset I_\chi/I_{\chi}^2$.

(3) $[\iota(u), \iota(v)] = h^2 \omega(u, v)$ for all $u, v \in V$, where $\omega$ denotes the Kostant-Kirillov form on $V$, i.e. $\omega(u, v) = (e, [u, v])$.

See [Lo5, 2.1] for the proof (the construction there formally covers the $\mathbb{Z}/2\mathbb{Z} \times K^\times \times Q$-action, but the proof with $\tilde{Q}$ is similar). So $\iota$ extends to a homomorphism (actually an embedding) $A_h^\wedge \to U_h^x$, where $A_h := A_h(V)$ is the homogenized Weyl algebra

$$T(V)[h]/(u \otimes v - v \otimes u - h^2 \omega(u, v))$$

and the superscript $\wedge$ means the completion at 0, i.e., $A_h(V)^\wedge := \lim_{n \to +\infty} A_h(V)/J^n$, where we write $J$ for the ideal generated by $V$ and $h$.

Then the algebra $U_h^x$ decomposes into the tensor product

$$U_h^x = A_h^\wedge \otimes_K [h]] W_h^y,$$

where $W_h^y$ stands for the centralizer of $\iota(V)$ in $U_h^x$. This is shown analogously to the proof of [Lo2, Proposition 3.3.1]. We remark that $W_h = K[S]^{\wedge} \cong K[\mathfrak{g}^*]^{\wedge}/(V)$, this homomorphism factors through an isomorphism $W_h^y/(h) \cong K[S]^{\wedge}$ thanks to (2.1).

The subalgebra $W_h$ of all $K^\times$-finite elements (i.e., finite sums of semi-invariants) in $W_h^y$ (for the Kazhdan action) is dense in $W_h^y$ and $W_h/(h) = K[S]$. Those claims are direct consequences of two observations: that the isomorphism $W_h^y/(h) \cong K[S]^{\wedge}$ from the previous paragraph is $K^\times$-equivariant (by the construction) and that the $K^\times$-action on $S$ contracts $S$ to $\chi$.

By definition, $W := W_h/(h - 1)$. This is a filtered algebra that comes with a $Q \times \mathbb{Z}/2\mathbb{Z}$-action by automorphisms and an anti-automorphism (or the identity map) $\tau$, both preserve the filtration. Moreover, similarly to [Lo5, 2.1], the construction easily implies that there is a $Q$- and $\tau$-equivariant Lie algebra embedding (a quantum comoment map) $q \hookrightarrow W$ with image in $F_{\leq 2} W$ such that the adjoint $q$-action on $W$ coincides with the differential of the $Q$-action. In more detail, we can consider the natural inclusion $q \hookrightarrow U_h^x$, denote it by $\varphi_U$. Also we have a natural $\mathrm{Sp}(V)$-equivariant Lie algebra embedding $\mathfrak{sp}(V) \hookrightarrow A_h^\wedge$ (as a complement of $Kh$ in the degree 2 component of $A_h$). Composing this with the Lie algebra homomorphism $q \to \mathfrak{sp}(V)$, we get a map $\varphi_U : q \to A_h^\wedge$. As we have seen in [Lo5, 2.1], under the identification $U_h^x \cong A_h^\wedge \otimes_K [h]] W_h^y$, the map $\varphi_U$ decomposes into the sum $\varphi_A + \varphi_W$, with $\varphi_W$ being a $Q$-equivariant map $q \to W_h^y$ with image lying in degree 2 component, and, in particular, in $W_h$. The Lie algebra homomorphism $q \to W$ is obtained from $\varphi_W$ by taking the quotient by $h - 1$. The reason why this map is an embedding is as follows: the $Q^\wedge$-action on $K[S]$ and hence on $W$ has discrete kernel. Indeed, the kernel of the $Q$-action on $S$ has to act trivially on $f$ and on $j_h(e)$ and hence on the whole algebra $\mathfrak{g}$.
As in [Lo5, 2.1], all choices we have made differ by an automorphism of the form \[\exp\left(\frac{1}{\hbar} \text{ad} f\right),\]
where \(f \in \overline{I}_\chi^3\) and \(\frac{1}{\hbar} f\) is \(\hat{Q}\)-invariant. So, as a filtered algebra with a \(\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z} \times Q)\)-action and with a quantum comoment map \(q \rightarrow \mathcal{W}\), the algebra \(\mathcal{W}\) is independent of the choice of \(\iota\) up to an isomorphism.

An important property of \(\mathcal{W}\) is that its center is naturally identified with the center \(Z\) of \(\mathcal{U}\). This was first proved by Ginzburg and Premet, see [Lo2, 2.2] for details.

In fact, in Section 4 we will need a somewhat different choice of \(\iota\).

### 2.2. Functor \(\bullet_i\).

This is a functor from the category of Harish-Chandra \(\mathcal{U}\)-bimodules to the category of Harish-Chandra \(\mathcal{W}\)-bimodules introduced in [Lo2] (a closely related functor was constructed by Ginzburg in [Gi]). Let us recall the definitions of the categories, first.

By a Harish-Chandra bimodule over \(\mathcal{U}\) (relative to \(G\)), we mean a finitely generated \(\mathcal{U}\)-bimodule \(M\) such that the adjoint action of \(g,\ \text{ad}(x)m = xm - mx\), is locally finite and integrates to a \(G\)-action (the last condition is vacuous when the group is semisimple and simply connected). On such a bimodule one can introduce a good filtration, i.e., a \(G\)-stable filtration \(F_i M\) that is compatible with the algebra filtration \(F_i \mathcal{U}\) and such that the associated graded \(\text{gr} M\) is a finitely generated \(S(g)\)-module. Using this we can define the associated variety \(V(M)\) of \(M\) as the support of \(\text{gr} M\), this is a conical \(G\)-stable subvariety in \(g \cong g^*\) independent of the choice of a good filtration. The category of Harish-Chandra (HC, for short) bimodules will be denoted by \(HC(\mathcal{U})\).

By a \(Q\)-equivariant Harish-Chandra \(\mathcal{W}\)-bimodule we mean a \(\mathcal{W}\)-bimodule \(N\) equipped with a \(Q\)-action compatible with the \(\mathcal{W}\)-action on \(\mathcal{W}\) (in the sense that the structure map \(\mathcal{W} \otimes N \otimes \mathcal{W} \rightarrow N\) is \(Q\)-equivariant) and subject to the following conditions:

- there is a \(Q\)-stable filtration \(F_i N\) on \(N\) that is compatible with the algebra filtration on \(\mathcal{W}\),
- we have \([F_i \mathcal{W}, F_j N] \subset F_{i+j-2} N\) for all \(i\) and \(j\),
- \(\text{gr} N\) is a finitely generated \(K[S]\)-module,
- and the differential of the \(Q\)-action on \(N\) coincides with the the adjoint \(q\)-action.

The category of \(Q\)-equivariant HC \(\mathcal{W}\)-bimodules will be denoted by \(HC^Q(\mathcal{W})\).

A functor \(\bullet_i : HC(\mathcal{U}) \rightarrow HC^Q(\mathcal{W})\) was constructed in [Lo2, 3.3,3.4] using the same construction as was used to construct \(\mathcal{W}\) in the previous subsection. Namely, take \(\mathcal{M} \in HC(\mathcal{U})\). Form the Rees bimodule \(\mathcal{M}_h\) with respect to some good filtration. We can complete \(\mathcal{M}_h\) with respect to the (left or right, does not matter) \(I_\chi\)-adic topology. Denote the resulting \(\mathcal{U}_h^\wedge\times\)-bimodule by \(\mathcal{M}_h^\wedge\times\). This bimodule carries a \(Q \times K^\times\)-action compatible with a \(Q \times K^\times\)-action on \(\mathcal{U}_h^\wedge\times\). Then one can show that \(\mathcal{M}_h^\wedge\times\) splits into the completed tensor product \(A_h(V)^{\wedge_0} \otimes_{K[[[\hbar]]]} \mathcal{N}_h\), where \(\mathcal{N}_h\) is the centralizer of \(V\) in \(\mathcal{M}_h^\wedge\times\) and hence is a \(\mathcal{W}_h\)-bimodule. Then again the \(K^\times\)-finite part \(\mathcal{N}_h\) of \(\mathcal{N}_h'\) is dense. We set \(\mathcal{M}_i := \mathcal{N}_h/((\hbar - 1)\mathcal{N}_h).\) This can be shown to be canonically independent of the choice of a good filtration on \(\mathcal{M}\). So \(\bullet_i\) is a functor.

The functor \(\bullet_i : HC(\mathcal{U}) \rightarrow HC^Q(\mathcal{W})\) has the following properties, see [Lo2, Proposition 3.4.1,Theorem 4.4.1]:

(i) \(\bullet_i\) is exact.
(ii) With the choice of filtration as above, the sheaf \(\text{gr} \mathcal{M}_i\) on \(S\) is the restriction of the sheaf \(\text{gr} \mathcal{M}\) (on \(g^* = g\)) to \(S\) (this not explicitly stated in loc.cit. but is deduced directly from the construction).
(iii) In particular, consider the full subcategories $\text{HC}_\mathcal{O}(\mathcal{U}) \subset \text{HC}_\mathcal{W}(\mathcal{U}) \subset \text{HC}(\mathcal{U})$ consisting of all HC bimodules whose associated varieties are contained in the boundary $\partial \mathcal{O}$ and the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$. Then $\bullet_1$ annihilates $\text{HC}_\mathcal{O}(\mathcal{U})$ and sends $\text{HC}_\mathcal{W}(\mathcal{U})$ to finite dimensional $\mathcal{W}$-bimodules. In particular, $\bullet_1$ descends to the quotient category $\text{HC}_\mathcal{O}(\mathcal{U}) := \text{HC}_\mathcal{W}(\mathcal{U})/\text{HC}_\mathcal{O}(\mathcal{U})$.

(iv) For $\mathcal{M} \in \text{HC}_\mathcal{O}(\mathcal{U})$, the dimension of $\mathcal{M}_1$ coincides with the multiplicity $\text{mult}_\mathcal{O} \mathcal{M}$ of $\mathcal{M}$ on $\mathcal{O}$ (=generic rank of $\text{gr} \mathcal{M}$ on $\mathcal{O}$).

(v) There is a right adjoint functor $\bullet^\dagger : \text{HC}_\mathcal{W}^\dagger(\mathcal{W}) \to \text{HC}_\mathcal{O}(\mathcal{U})$. Moreover, its composition with the quotient $\text{HC}_\mathcal{W}(\mathcal{U}) \to \text{HC}_\mathcal{O}(\mathcal{U})$ is left inverse to $\bullet_1$. In other words, $\bullet^\dagger : \text{HC}_\mathcal{O}(\mathcal{U}) \to \text{HC}_\mathcal{W}^\dagger(\mathcal{W})$ is an equivalence onto its image.

(vi) Let $\mathcal{M} \in \text{HC}(\mathcal{U})$ and $\mathcal{N}$ be a $Q$-stable subbimodule of finite codimension in $\mathcal{M}_1$. Then there is a unique maximal subbimodule $\mathcal{M}' \subset \mathcal{M}$ with the property $\mathcal{M}'_1 = \mathcal{N}$. We automatically have $\mathcal{M}/\mathcal{M}' \in \text{HC}_\mathcal{W}(\mathcal{U})$.

(vii) The image of $\text{HC}_\mathcal{O}(\mathcal{U})$ under $\bullet_1$ is closed under taking subquotients (this is a direct corollary of (i) and (vi)).

2.3. Classification of finite dimensional irreducible modules. We are going to recall the classification of finite dimensional irreducible $\mathcal{W}$-modules with integral central characters (this notion makes sense because the centers of $\mathcal{U}$ and $\mathcal{W}$ are identified). This classification was obtained in [LO].

We start by recalling one of the main results from [Lo2]. By the construction of $\mathcal{W}$, the group $Q$ acts on $\mathcal{W}$ by algebra automorphisms. This gives rise to a $Q$-action on the set $\text{Irr}_{\text{fin}}(\mathcal{W})$ of isomorphism classes of finite dimensional irreducible $\mathcal{W}$-modules. Clearly, $Z(G)$ acts trivially. Also recall that the differential of the $Q$-action on $\mathcal{W}$ coincides with the adjoint action of $\mathfrak{q} \subset \mathcal{W}$. Therefore $\mathfrak{q}^\circ$ acts trivially on $\text{Irr}_{\text{fin}}(\mathcal{W})$ and so we get an action of the component group $A := Q/(Q^\circ Z(G))$ on that set. The orbit space $\text{Irr}_{\text{fin}}(\mathcal{W})/A$ gets naturally identified with the set

$$\text{Pr}_\mathcal{O}(\mathcal{U}) = \{\text{primitive } \mathcal{J}| V(\mathcal{U}/\mathcal{J}) = \overline{\mathcal{O}}\},$$

this is Premet’s conjecture, [Lo2, Conjecture 1.2.2] that is a corollary of [Lo2, Theorem 1.2.3]. Namely, to an $A$-orbit $N_1, \ldots, N_k$ we, first, assign $T := \bigcap_{i=1}^k \text{Ann}_\mathcal{W} N_i$. This intersection is a $Q$-stable ideal of finite codimension. Then we can apply property (vi) to $\mathcal{M} = \mathcal{U}$ (so that $\mathcal{M}_1 = \mathcal{W}$), and $\mathcal{N} := T$. The corresponding ideal $\mathcal{J} := \mathcal{M}' \subset \mathcal{U}$ can be seen to be primitive, and this is the ideal we need. The identification $\text{Irr}_{\text{fin}}(\mathcal{W})/A \cong \text{Pr}_\mathcal{O}(\mathcal{U})$ preserves the central characters under our identification of the centers of $\mathcal{U}$ and of $\mathcal{W}$, which follows from [Lo2, Theorem 3.3.1].

Now let us recall a result from [LO] that explains how to compute the $A$-orbit lying over a primitive ideal $\mathcal{J}$ in the case when $\mathcal{J}$ has integral central character. Below we use facts recalled in [LO, 6.1.6.2]. The existence of such $\mathcal{J}$ implies that $\mathcal{O}$ is special in the sense of Lusztig. So to $\mathcal{O}$ we can assign a two-sided cell, say $\mathfrak{c}$, that is a subset of $W$. To $\mathfrak{c}$ one assigns a subset $\text{Irr}_\mathfrak{c}^\circ(\mathcal{W})$ (called a family) in the set $\text{Irr}(\mathcal{W})$ of irreducible representations of $\mathcal{W}$, where, recall, $W$ denotes the Weyl group of $\mathfrak{g}$.

Let $Y^A$ denote the subset of $\text{Irr}_{\text{fin}}(\mathcal{W})$ consisting of all modules with integral central character. The $A$-action on $Y^A$ factors through a certain quotient $\tilde{A}$ of $A$ introduced by Lusztig in [Lu1]. To define this quotient consider the Springer $W \times A$-module $\text{Spr}(\mathcal{O})$. Consider its $W$-submodule $\text{Spr}(\mathcal{O})^\mathfrak{c}$ that is the sum of all irreducible $W$-submodules belonging to
Irref(W). Of course, this is also an A-submodule. The group \( \tilde{A} \) is the quotient of \( A \) by the kernel of the A-action on \( \text{Spr}(\mathcal{O})^c \), [Lu1, 13.1.3].

Now let us describe the stabilizers. Pick a left cell \( c \subseteq c \) and let \( \lambda \) be a dominant weight compatible with \( c \) (i.e., compatible with any \( w \in c \), this condition is independent of the choice of \( w \), see [LO, 6.2]). We can view \( \lambda \) as a point in \( \mathfrak{h}/W \) and hence as a central character for \( \mathcal{U} \). The set of the left cells in \( c \) compatible with \( \lambda \) is in a bijection with the set \( \text{Pr}_\mathbb{O}(\mathcal{U}) \) of the primitive ideals \( \mathcal{J} \) with central character \( \lambda \) and \( V(\mathcal{U}/\mathcal{J}) = \mathcal{O} \): to \( c \) we assign the ideal \( J(w\lambda) \), the annihilator of the irreducible highest weight module \( L(w\lambda) \) with highest weight \( w\lambda - \rho \), where \( w \in c \).

According to [LO, Theorem 1.1], the stabilizer of the orbit over \( J(w\lambda), w \in c \), is the subgroup \( H_c \subseteq \tilde{A} \) defined by Lusztig in [Lu2]. It can be described as follows. Consider the cell module \( M(c) \) associated to \( c \). Then \( \text{Hom}_W(M(c), \text{Spr}(\mathcal{O})) = \text{Hom}_W(M(c), \text{Spr}(\mathcal{O})^c) \) is an \( \tilde{A} \)-module. It turns out that there is a unique (up to conjugacy) subgroup \( H_c \subseteq \tilde{A} \) such that the \( \tilde{A} \)-modules \( \mathbb{Q}(\tilde{A}/H_c) \) and \( \text{Hom}_W(M(c), \text{Spr}(\mathcal{O})) \) are isomorphic. See [LO], Subsections 6.5-6.8, for explicit computations of \( H_c \) starting from \( M(c) \).

2.4. Semisimple HC bimodules. Now fix a finite set \( \Lambda' \) of dominant weights such that the pairwise differences of the elements of \( \Lambda' \) lie in the root lattice and such that there is a regular element \( \varrho \in \Lambda' \). Consider the subcategory \( \text{HC}_{\tilde{A}}(\mathcal{U})_{\Lambda'}^{ss} \subset \text{HC}_{\tilde{A}}(\mathcal{U}) \) consisting of all semisimple objects with left and right central characters lying in \( \Lambda' \). According to [LO, Theorem 7.4, Remark 7.7], this category is isomorphic to the category \( \text{Coh}^{\tilde{A}}(Y^{\Lambda'} \times Y^{\Lambda'}) \) of \( \tilde{A} \)-equivariant sheaves of finite dimensional vector spaces on \( Y^{\Lambda'} \times Y^{\Lambda'} \), where \( Y^{\Lambda'} \) is the set of finite dimensional irreducible \( \mathcal{W} \)-modules with central character in \( \Lambda' \). Irreducible objects in the latter category are parameterized by the triples \( (x, y, \mathcal{V}) \), where \( x, y \in Y^{\Lambda'} \) and \( \mathcal{V} \) is an irreducible \( \tilde{A}(x, y) \)-module, a triple is defined up to an \( \tilde{A} \)-conjugacy. Namely, the support of an irreducible sheaf is a single orbit and we take \( (x, y) \) from this orbit, for \( \mathcal{V} \) we take the fiber of the sheaf at \( (x, y) \). So any irreducible Harish-Chandra bimodule in \( \text{HC}_{\tilde{A}}(\mathcal{U})_{\Lambda} \) gets mapped to some triple \( (x, y, \mathcal{V}) \). We say that this triple corresponds to this bimodule (or to \( (w, \lambda) \) if the bimodule is \( M_w(\lambda) := L(\Delta(\varrho), L(w\lambda)) \) or just to \( w \) if \( \lambda = \varrho \); the triple does not depend on the choice of \( \varrho \)). [Lo2, Theorem 1.3.1] implies that \( x, y \) lie over the left and right annihilators of \( M_w(\lambda) \), the ideals \( J(w\lambda), J(w^{-1}\varrho) \), respectively. Moreover, the construction in [LO] implies that, being an idempotent object in \( \text{HC}_{\tilde{A}}(\mathcal{U})_{\Lambda'}^{ss} \), the quotient \( \mathcal{U}/J(w\lambda) \) gets mapped to the sheaf supported on the diagonal of the \( \tilde{A} \)-orbit corresponding to \( J(w\lambda) \), whose fiber is the trivial module. In particular, if \( d \) is the Duflo involution in \( c_w \), the left cell containing \( w \), then the triple corresponding to \( d \) has the form \( (x, x, \text{triv}) \). This is because \( M_d \) coincides with \( \mathcal{U}/J(d\varrho) \) in \( \text{HC}_{\tilde{A}}(\mathcal{U}) \). To see that we first recall that, by the definition of \( d \) given in [Jo2, 3.3.3.4], \( L(d\varrho) \) is the socle in \( \Delta(\varrho)/J(w\varrho)\Delta(\varrho) \) and the GK dimension of \( L(d\varrho) \) is bigger than that of \( [\Delta(\varrho)/J(w\varrho)\Delta(\varrho)]/L(d\varrho) \). Under the Bernstein-Gelfand equivalence, \( \Delta(\varrho)/J(w\varrho)\Delta(\varrho) \) corresponds precisely to \( \mathcal{U}/J(d\varrho) \) and the equality \( M_d = \mathcal{U}/J(d\varrho) \) in \( \text{HC}_{\tilde{A}}(\mathcal{U}) \) follows.

A important corollary from [LO] is a formula for the multiplicity of an irreducible object \( \mathcal{M} \) in \( \text{HC}_{\tilde{A}}(\mathcal{U})_{\Lambda} \), see Remark 7.7 and formula (7.1) in loc. cit. Namely, let \( (x, y, \mathcal{V}) \) be a triple corresponding to \( \mathcal{M} \). Then we have

\[
(2.2) \quad \text{mult}_{\mathcal{O}}(\mathcal{M}) = d_x d_y \frac{|\tilde{A}|}{|A(x, y)|} \dim \mathcal{V}.
\]
Here $d_x, d_y$ are the dimensions of irreducible $\mathcal{W}$-modules lying over the left and the right annihilators of $\mathcal{M}$. This formula will be one of the crucial tools to relate the Goldie ranks and the dimensions of $\mathcal{W}$-irreducibles.

2.5. **Parabolic induction.** Recall the Lusztig-Spaltenstein induction, [LS]. Take a Levi subalgebra $\mathfrak{g} \subset \mathfrak{g}$ and a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$. One can construct a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ (called *induced* from $\mathcal{O}$) from this pair as follows. Pick a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ with Levi subalgebra $\mathfrak{g}$. Let $\mathfrak{n}$ stand for the maximal nilpotent subalgebra of $\mathfrak{p}$. For $\mathcal{O}$ we take a unique dense orbit in $G(\mathcal{O} + \mathfrak{n})$. It turns out that $\mathcal{O}$ does not depend on the choice of $\mathfrak{p}$. The codimension of $\mathcal{O}$ in $\mathfrak{g}$ coincides with the codimension of $\mathcal{O}$ in $\mathfrak{g}$. The intersection of $\mathcal{O}$ with $\mathcal{O} + \mathfrak{n}$ is a single $P$-orbit, see [LS, Theorem 1.3].

Let $\mathcal{W}$ denote the $W$-algebra of the pair $(\mathfrak{g}, \mathcal{O})$. In [Lo4, Section 6] we have constructed a dimension preserving exact functor $\varsigma : \mathcal{W}^{\prime}-\text{mod}_{\text{fin}} \rightarrow \mathcal{W}-\text{mod}_{\text{fin}}$ between the categories of finite dimensional modules. This functor depends on the choice of $P$. Namely, see [Lo4, 6.3], there is a completion $\mathcal{W}'$ of $\mathcal{W}$ such that any finite dimensional $\mathcal{W}$-module extends to $\mathcal{W}'$ and an embedding $\Xi : \mathcal{W} \hookrightarrow \mathcal{W}'$. The functor under consideration is just the pull-back from $\mathcal{W}'$ to $\mathcal{W}$. Furthermore, we can choose $e \in \mathcal{O} \cap (\mathcal{O} + \mathfrak{n})$ in such a way that a reductive part $Q$ of the centralizer $Z_P(e)$ lies in the Levi subgroup $G$ of $P$ corresponding to $\mathfrak{g}$. The group $Q$ acts on $\mathcal{W}$ by automorphisms, the action extends to $\mathcal{W}'$, and the embedding $\mathcal{W} \hookrightarrow \mathcal{W}'$ is $Q$-equivariant (the latter can be deduced directly from the construction of $\Xi$ in [Lo4, Theorem 6.3.2]).

2.6. **Isomorphisms of completions.**

2.6.1. **Setting.** Let us fix a setting that will be used until Section 5.

Let $e \in \mathfrak{g}$ be a nilpotent element. We include $e$ into a Levi subalgebra $\mathfrak{g}_0$ so that $e$ is an even nilpotent element in $\mathfrak{g}_0$. For example, this is always the case when $e$ is distinguished in $\mathfrak{g}_0$, equivalently, $\mathfrak{g}_0$ is a minimal Levi subalgebra containing $e$, see, for example, [CM, Theorem 8.2.3]. So such $\mathfrak{g}_0$ always exists.

Choose an $\mathfrak{sl}_2$-triple $(e, h, f)$ in $\mathfrak{g}_0$. Choose Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b}_0 \subset \mathfrak{g}_0$ in such a way that $h \in \mathfrak{h}$ and is a dominant (for $\mathfrak{g}_0$) element there. Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$ stand for the eigendecomposition for $h$.

Pick an integral element $\theta \in \mathfrak{z}(\mathfrak{g}_0)$ such that $\mathfrak{z}_\mathfrak{g}(\theta) = \mathfrak{g}_0$. Consider the eigen-decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$. Set $\mathfrak{b} := \mathfrak{b}_0 \oplus \mathfrak{g}_{>0}$, where $\mathfrak{g}_{>0} := \bigoplus_{i>0} \mathfrak{g}_i$, clearly, $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g}$. Further, set $\mathfrak{p} := \mathfrak{g}_0(\geq 0) \oplus \mathfrak{g}_{>0}$. This is a parabolic subalgebra in $\mathfrak{g}$ containing $\mathfrak{b}$. Let $P$ denote the corresponding parabolic subgroup of $G$.

Let $\sigma$ be the anti-involution of $\mathfrak{g}$ defined as follows: $\sigma|_\mathfrak{h} = \text{id}, \sigma(e_i) = f_i, \sigma(f_i) = e_i$. We claim that one can choose $e$ and $f$ (still in the same orbit $\mathcal{O}$) in such a way that $h \in \mathfrak{h}$ is still dominant for $\mathfrak{g}_0$, $\sigma(e) = f$ and hence $\sigma(f) = e$. We remark that $h$ is fixed by $\sigma$ because $\sigma$ is the identity on $\mathfrak{h}$. So a result of Antonyan, [A], implies that $e$ is conjugate to a $\sigma$-invariant, say $e'$. The element $e'$ can be included into an $\mathfrak{sl}_2$-triple $(e', h', f')$ in $\mathfrak{g}_0$ with $\sigma(e') = e', \sigma(f') = f', \sigma(h') = -h'$. In other words, on the $\mathfrak{sl}_2$-subalgebra with standard basis $e', h', f'$ the anti-automorphism $\sigma$ acts as the transposition with respect to the anti-diagonal. It is conjugate to the usual transposition. So we can find the $\mathfrak{sl}_2$-triple $e'', h'', f''$ in that $\mathfrak{sl}_2$-subalgebra with $\sigma(e'') = f''$. Now we can replace $(e, h, f)$ with $(e'', h'', f'')$. Further, we can conjugate such $h$ to $\mathfrak{h}$ by an element of $\text{Ad}(\mathfrak{g}_0^\sigma)$. The Cartan space and the Weyl group of the symmetric pair $(\mathfrak{g}_0, \mathfrak{g}_0^\sigma)$ are just $\mathfrak{h}$ and $W$ (clearly, $\mathfrak{h} \subset \mathfrak{g}^\sigma$ and the claim about the
Weyl group can be checked for $\mathfrak{g} = \mathfrak{sl}_2$, where it is straightforward). So we can assume that $h$ is dominant and we are done. Clearly, $\sigma$ lifts to $G$.

Let $n$ be the image of $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ under the homomorphism $\text{SL}_2 \to G$ corresponding to the $\mathfrak{sl}_2$-triple $(e, h, f)$. The matrix is symmetric and therefore $\sigma(n) = n$. It follows that $\tau : g \mapsto n\sigma(g)n^{-1}$ is involutary. This is our choice of $\tau$ from now on and until Section 6.

Recall that we write $\mathfrak{t}$ for the center $\mathfrak{j}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$. Let $T$ stand for the torus in $G$ corresponding to $\mathfrak{t}$ and $R$ be the centralizer of $T$ in $Q$.

In the next two parts we establish certain isomorphisms of various algebras. In the rest of the paper we will always assume that the algebras are identified as explained below in this subsection.

2.6.2. Right-handed completions. Define a subalgebra $\mathfrak{m} \subset \mathfrak{g}$ by $\mathfrak{m} := \mathfrak{g}_0(<0) \oplus \mathfrak{g}_{>0}$ (in [Lo3] this subalgebra was denoted by $\tilde{\mathfrak{m}}$, while the notation $\mathfrak{m}$ was only used in the case $\mathfrak{g} = \mathfrak{g}_0$, but we want to simplify the notation here). Let us point out that $\chi$ is a character of $\mathfrak{m}$. Also we consider the shift of $\mathfrak{m}$, the subspace $\mathfrak{m}_\chi := \{x - (\chi, x), x \in \mathfrak{m}\} \subset \mathfrak{g} \oplus \mathbb{K}$.

We will need a completion of $\mathcal{U}$ considered in [Lo3, Section 5]: $\mathcal{U}^\wedge := \varprojlim_{t \to \infty} \mathcal{U}/\mathfrak{m}_\chi^n$. This is a topological algebra as explained in [Lo1, 3.2].

We can decompose $\mathcal{U}^\wedge$ into a completed tensor product as follows. As we have noticed in [Lo3, 5.1], $\mathfrak{v} := \mathfrak{m} \cap V$ is a lagrangian subspace in $V$ (note that the notation used there was different). Thanks to the embedding $\mathfrak{g} \hookrightarrow \mathcal{W}$ we can view $\theta$ as an element of $\mathcal{W}$ and consider the eigen-decomposition $\mathcal{W} = \bigoplus_{i \in \mathbb{Z}} \mathcal{W}_i$. Then we set $\mathcal{W}^\wedge := \varprojlim_{t \to \infty} \mathcal{W}/\mathcal{W}\mathcal{W}_{>0}, \mathcal{A}^\wedge := \varprojlim_{t \to \infty} \mathcal{A}/\mathcal{A}\mathcal{V}^\wedge$. We have seen in [Lo3, Section 5] (and, in a bit different setting, in [Lo1, Sections 3.2.3.3]), the decomposition $\mathcal{U}_h^\wedge = \mathcal{A}_h^\wedge \hat{\otimes}_{\mathbb{K}[\mathfrak{g}]} \mathcal{W}_h^\wedge$ gives rise to an isomorphism $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$, where we write $\mathcal{A}(\mathcal{W})^\wedge$ for $\mathcal{A}^\wedge \hat{\otimes} \mathcal{W}^\wedge$, of course, $\mathcal{A}(\mathcal{W})^\wedge$ is the completion of $\mathcal{A} \hat{\otimes} \mathcal{W}$ with respect to the left ideals $\mathcal{A} \hat{\otimes} \mathcal{W}(\mathfrak{v} \otimes 1 + 1 \otimes \mathcal{W}_{>0})$. The isomorphism $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$ maps the left ideal $\mathcal{U}^\wedge \mathfrak{m}_\chi$ to $\mathcal{A}(\mathcal{W})^\wedge(\mathfrak{v} \otimes 1 + 1 \otimes \mathcal{W}_{>0})$.

Let us recall how the isomorphism is constructed, see [Lo3, Section 5]. We embed $\mathbb{K}^\times$ into $\mathbb{K}^\times \times Q$ with differential $(1, -N\theta)$ for $N$ large enough. Then we can consider the subalgebras $$(\mathcal{U}_h^\wedge)_{\text{fin}}, (\mathcal{A}_h(V)^\wedge \hat{\otimes}_{\mathbb{K}[\mathfrak{g}]} \mathcal{W}_h^\wedge)_{\text{fin}}$$ of $\mathbb{K}^\times$-finite vectors for this copy of $\mathbb{K}^\times$. These algebras are isomorphic because the isomorphism $\mathcal{U}_h^\wedge \cong \mathcal{A}_h(V)^\wedge \hat{\otimes}_{\mathbb{K}[\mathfrak{g}]} \mathcal{W}_h^\wedge$ is $\mathbb{K}^\times$-equivariant. We mod out $h - 1$ and get isomorphic algebras

$$(2.3) \quad \mathcal{U}^\wedge := (\mathcal{U}_h^\wedge)_{\text{fin}}/(h - 1), \mathcal{A}(\mathcal{W})^\wedge := (\mathcal{A}_h(V)^\wedge \hat{\otimes}_{\mathbb{K}[\mathfrak{g}]} \mathcal{W}_h^\wedge)_{\text{fin}}/(h - 1).$$

They are embedded into $\mathcal{U}^\wedge, \mathcal{A}(\mathcal{W})^\wedge$, [Lo3, Proposition 5.1, Lemma 5.3]. Then the isomorphism $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$ extends by continuity to an isomorphism $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$ that maps $\mathcal{U}^\wedge \mathfrak{m}_\chi$ to $\mathcal{A}(\mathcal{W})^\wedge(\mathfrak{v} \otimes 1 + 1 \otimes \mathcal{W}_{>0})$, [Lo3, Lemma 5.3].

The isomorphism $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$ induces another isomorphism. Namely, we write $\mathcal{U}^0$ for $\mathcal{U}(\mathfrak{g}_0)$. Also we set $\mathcal{W}_{>0} := \bigoplus_{i \geq 0} \mathcal{W}_i$ and $\mathcal{W}^0 := \mathcal{W}_{>0}/(\mathcal{W}_{>0} \cap \mathcal{W}\mathcal{W}_{>0}) = \mathcal{W}_0/(\mathcal{W}_0 \cap \mathcal{W}_{<0}\mathcal{W}_{>0})$. As we have noticed in [Lo3, Section 5], the identification $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$ induces an isomorphism of $\mathcal{W}^0$ and $\mathcal{U}(\mathfrak{g}_0, e) := [\mathcal{U}^0/\mathfrak{m}_\chi^0]^{\text{alg}}$, where $\mathfrak{m}^0 := \mathfrak{m} \cap \mathfrak{g}_0$. The algebra $\mathcal{U}(\mathfrak{g}_0, e)$ (a W-algebra, as defined by Premet in [Pr2]) is identified with the W-algebra for $\mathfrak{g}_0$ in our sense via an isomorphism of completions analogous to $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$ but taken for $\mathfrak{g}_0$ and not for $\mathfrak{g}$.
The isomorphism $U(\mathfrak{g}_0, e) \xrightarrow{\sim} W^0$ is $R$-equivariant but does not intertwine the quantum comoment maps $r \mapsto U(\mathfrak{g}_0, e), W^0$. Here the quantum comoment map $r \mapsto U(\mathfrak{g}_0, e)$ is obtained as the composition $r \mapsto U^0 \mapsto U^0/\mathfrak{m}_r^0$, while the quantum comoment map $r \mapsto W^0$ is the composition $r \mapsto W \mapsto W/\mathcal{W}_{<0}$. Instead the isomorphism induces a shift on $r$ by a certain character $\delta$, as in [Lo3, Remark 5.5], i.e., it maps $\xi \in r \mapsto U(\mathfrak{g}_0, e)$ to $\xi - \langle \delta, \xi \rangle$. Our setting is a bit different from loc. cit., as we consider a larger Lie algebra here. So we are going to provide details.

Let us write $t_d, t_W, t_k$ for the quantum comoment maps to the corresponding algebras (this differs a bit from the conventions of Subsection 2.1) so that under the isomorphism $U^0 \cong A(W)$, we have $t_d = t_A + t_W$. A key observation is that the map $r \mapsto [A/\mathfrak{m}_r]^\mathfrak{p}$ induced by $t_d$ is the character $\delta$ that, by definition, equals a half of the character of $\Lambda^{\text{top}} \mathfrak{p}^*$ (see [Lo3, Remark 5.5] for a computation). In particular, the restriction of $\delta$ to $r$ is the same character as in loc. cit. (we remark that $t$ is naturally represented as a direct summand of $r$). So the map $r \mapsto W^0 = [A(W)/A(W)^\mathfrak{p}]^\mathfrak{p}$ induced by $t_d$ equals $t_W + \delta$. This implies the claim in the previous paragraph.

2.6.3. Left-handed completions. Set $\bar{m} := \tau(m) = \mathfrak{g}_{<0} \oplus \mathfrak{g}_0(\leq 0)$. Define the completion $\hat{U} := \lim_{\leftarrow n} U/\mathfrak{m}_r^\infty U$. Also set $\bar{V} := \bar{m} \cap V$, this is again a lagrangian subspace in $V$. Set $\hat{A} := \lim_{\leftarrow n} A/\mathfrak{b}_0^\infty A$, $\hat{W} := \lim_{\leftarrow n} W/W_{<0}^\infty W$, where $W_{<0} := \bigoplus_{t < 0} W_t$. Twisting the isomorphism $U^0 \cong A(W)^\wedge$ with $\tau$, we get an isomorphism $\hat{U} \cong \hat{A}(W) := \hat{A} \hat{\otimes} \hat{W}$.

Again, below we will need several isomorphisms induced by $\hat{U} \cong \hat{A}(W)$. We can form the analogous completions $\hat{U}^0$ of $U^0$, $\hat{A}^0$ of $A^0 := A(V)^0$, where $V^0 := \mathfrak{g}_0 \cap V = [\mathfrak{g}_0, f]$. Also we can consider the eigen-spaces $\hat{U}_\xi, \hat{A}_\xi, \hat{W}_\xi$ for the action of $\text{ad}(\theta)$ (in the case of $\hat{A}$ rather of the corresponding one-dimensional torus). Then we can define the subalgebras $\hat{U}_{\xi < 0} := \bigoplus_{\xi < 0} \hat{U}_\xi, \hat{A}_{\xi < 0}, \hat{W}_{\xi < 0}$ and their ideals $\hat{U}_{\xi < 0}, \hat{A}_{\xi < 0}, \hat{W}_{\xi < 0}$ similarly to $W_{>0}, W_{>0}$. We claim that

$$\hat{U}_{\xi < 0}/(\hat{U}_{\xi < 0} \cap \hat{U}_{\xi < 0} \hat{U}) = \hat{U}_0/(\hat{U}_0 \cap \hat{U}_0 \hat{U}_0)$$

is naturally identified with $\hat{U}^0$ and the similar equalities hold for the other two algebras (in the $W$-case we have, by definition, $\hat{W}^0 := W^0$).

We are going to prove the isomorphism in the $U$-case, the other two cases are similar. The algebra $\hat{U}$ can be realized “explicitly” as follows. Choose a basis $x_1, \ldots, x_a, y_1, \ldots, y_b, z_1, \ldots, z_c, w_1, \ldots, w_d$ of $\mathfrak{g}_x := \{x - \langle x, x \rangle | x \in \mathfrak{g}\}$ such that

- $x_1, \ldots, x_a$ are weight vectors for $\theta$ with negative weights, $y_1, \ldots, y_b, z_1, \ldots, z_c \in \mathfrak{g}_0$,
- while $w_1, \ldots, w_d$ are weight vectors for $\theta$ with positive weights.

Then $\hat{U}$ consists of all infinite sums $\sum_{\alpha, \beta, \gamma, \delta} n_{\alpha, \beta, \gamma, \delta} x^\alpha y^\beta z^\gamma w^\delta$, where $\alpha = (\alpha_1, \ldots, \alpha_a), x^\alpha := x_1^{\alpha_1} \ldots x_a^{\alpha_a}$ etc., subject to the condition that, for any given $\alpha, \beta$, only finitely many coefficients $n_{\alpha, \beta}$ are nonzero. The product is induced (=extended by continuity) from $U$.

The quotient $\hat{U}/\hat{U}_0 \hat{U}$ consists of the infinite sums of the form $\sum_{\alpha, \beta, \gamma} n_{\alpha, \beta, \gamma} x^\alpha y^\beta z^\gamma$ with the same finiteness condition as above. So $\hat{U}^0$ consists of the sums $\sum_{\beta, \gamma} n_{\beta, \gamma} y^\beta z^\gamma$ (with product induced from $U$ or, equivalently, $U^0$) and is naturally identified with $\hat{U}^0$.

The algebra $\hat{A}(W)_0 / (\hat{A}(W)_0 \cap \hat{A}(W)_{<0} \hat{A}(W)_{>0})$ is naturally identified with $\hat{A}^0(W)_0 = \hat{A} \hat{\otimes} W^0$. It follows that the isomorphism $\hat{U} \cong \hat{A}(W)$ induces an isomorphism $\hat{U}^0 \cong \hat{A}^0(W)$. Moreover, it also induces an isomorphism $\hat{U}/\hat{U}_0 \hat{U} \cong \hat{A}(W)/\hat{A}(W) \hat{A}(W)_{>0}$ that is linear both over $\hat{U} \cong \hat{A}(W)$ (acting on the left) and over $\hat{U}^0 \cong \hat{A}^0(W)$ (acting on the right).
Finally, similarly to 2.6.2, the isomorphism $\wedge U \cong \wedge A(W)$ induces an isomorphism of $U(e, g_0) := [U^0/m^0 U^0]^M$ and $W^0$. Again, this isomorphism is $R$-equivariant but does not intertwine the embeddings of $t$. Rather it again induces a shift by $\tau(\delta)$. The elements $\delta$ and $\tau(\delta)$ are different but they agree on $t$ (because $\tau$ is the identity on $t$) and their difference is the character of $R$ on $\bigwedge_{\text{top}} g_0(<0)^*$. 

3. Categories

3.1. Parabolic category $\mathcal{O}$.

3.1.1. Definition. Recall that $G$ denotes a connected reductive algebraic group and that we have fixed a parabolic subgroup $P \subset G$. Fix a character $\nu$ of $\mathfrak{p}$. Let $\mathcal{O}_\nu^P$ denote the full subcategory in the category of $(\mathfrak{g}, P)$-modules consisting of all modules $\mathcal{M}$ where the $\nu$-shifted $\mathfrak{p}$-action, i.e., $(x, m) \mapsto xm - \nu(x)m_1$ is locally finite and integrates to the action of $P$. We remark that the categories $\mathcal{O}_\nu^P$ and $\hat{\mathcal{O}}_\nu^P$ are naturally equivalent provided $\nu' - \nu$ is a character of $P$.

Inside $\hat{\mathcal{O}}_\nu^P$ we consider the full subcategory $\mathcal{O}_\nu^L$ of all modules where all weight spaces (for $\mathfrak{z}(l)$, where $l$ is a Levi subalgebra of $P$, or, equivalently, for a Cartan subalgebra $\mathfrak{h}$) are finite dimensional and where the center of $\mathcal{U}$ acts with finitely many eigen-characters. Equivalently, $\mathcal{O}_\nu^P$ consists of all finitely generated modules in $\hat{\mathcal{O}}_\nu^P$. It is known that all modules in $\mathcal{O}_\nu^P$ have finite length.

Consider the category $\mathcal{O}_\nu^L$ of all finite dimensional $l$- and $L$-modules, where the differential of the $L$-action coincides with the $\nu$-shifted $l$-action. We have the induction functor $\Delta_P : \mathcal{O}_\nu^L \to \mathcal{O}_\nu^P, \mathcal{M}_0 \to \mathcal{U} \otimes_{\mathcal{U}(\mathfrak{p})} \mathcal{M}_0$. For the irreducible $l$-module $L_l(\lambda)$ with highest weight $\lambda - \rho$, we write $\Delta_P(\lambda)$ for $\Delta_P(L_l(\lambda))$, this is, of course, a parabolic Verma module. It has a unique irreducible quotient to be denoted by $L(\lambda)$.

3.1.2. Completed version. We consider a category $\hat{\mathcal{O}}_\nu^P$ consisting of all topological $\mathcal{U}$- and $P$-modules $\mathcal{M}$ satisfying the following conditions:

(1) The weights of $\mathfrak{z}(l)$ in $\mathcal{M}$ are bounded from above in the sense that there is an element $\theta$ in $\mathfrak{z}(l)$ such that $\text{ad} \theta$ has only positive integral eigenvalues on the nilpotent radical of $\mathfrak{p}$ and all eigenvalues of $\theta$ on $\mathcal{M}$ are bounded from above.

(2) Any $\mathfrak{z}(l)$-weight space is finite dimensional and the center of $\mathcal{U}$ acts with finitely many eigen-characters.

(3) The $\nu$-shifted $l$-action on any weight space coincides with the differential of the $L$-action.

(4) $\mathcal{M}$ (considered as a topological $\mathcal{U}$-module) is the direct product of its $\mathfrak{z}(l)$-weight subspaces.

We have a completion functor $\mathcal{M} \mapsto \hat{\mathcal{M}} : \mathcal{O}_\nu^P \to \hat{\mathcal{O}}_\nu^P$ that sends $\mathcal{M} = \bigoplus_\mu \mathcal{M}_\mu$, where $\mathcal{M}_\mu$ is a weight space corresponding to $\mu \in \mathfrak{z}(l)^*$ to $\prod_\mu \mathcal{M}_\mu$. This functor is an equivalence of categories.

3.1.3. Right-handed versions. We will also consider the analogs $\mathcal{O}_\nu^{P,r}, \hat{\mathcal{O}}_\nu^{P,r}$ of $\mathcal{O}_\nu^P, \hat{\mathcal{O}}_\nu^P$ consisting of right modules (we impose the condition that the $\nu$-shifted right action of $\mathfrak{p}$ integrates to a right action of $P$).
3.1.4. Duality. We have a contravariant duality functor $\bullet^\vee : \mathcal{O}_\nu^P \to \mathcal{O}_\nu^P$. Namely, recall the anti-automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$, see 2.6.1. In particular, it sends $P$ to the opposite parabolic subgroup $P^-$. For $\mathcal{M} = \bigoplus_{\nu \in \mathcal{V}} \mathcal{M}_\nu$, the restricted dual $\mathcal{M}^{(\nu)} := \bigoplus_{\nu} \mathcal{M}_\nu^* \subset \mathcal{M}_\nu$ is a right $\mathcal{U}$-module that lies in $\mathcal{O}_\nu^P$.

3.1.5. Bernstein-Gelfand equivalence. Here we are going to recall the classical Bernstein-Gelfand equivalence relating the sum of suitable blocks of the full BGG category $\mathcal{O}$ to a certain category $\mathcal{HC}(\mathcal{U})^\theta$ of Harish-Chandra bimodules. Then we introduce its parabolic analog. This analog should be known but we did not find any reference.

Let $\mathcal{HC}(\mathcal{U})^\theta$ denote the category of all HC bimodules with right central character $\varrho$. Assume that $\varrho$ is strictly dominant meaning that $(\varrho, \alpha^\vee) \notin \mathbb{Z}_{\leq 0}$ for any positive root $\alpha$. Then the functor $X \mapsto X \otimes_{\mathcal{U}} \Delta(\varrho)$ is an equivalence $\mathcal{HC}(\mathcal{U})^\theta \to \mathcal{O}_\varrho$, see [BG, 5.9]. The quasi-inverse equivalence is given by $\mathcal{M} \mapsto L(\Delta(\varrho), \mathcal{M})$, where $L(\bullet, \bullet)$ denotes the space of $\mathfrak{g}$-finite maps.

Now let us proceed to a parabolic analog of this. Suppose we are given a parabolic category $\mathcal{O}_\nu^P$. Adding a suitable character of $P$ to $\nu$, we may assume that $\nu + \rho$ is strictly dominant (we remark that $\nu$ is 0 on the roots of $\mathfrak{l}$). Let $\mathcal{J}_{P,\nu}$ denote the annihilator of $\Delta_P(\nu + \rho)$ in $\mathcal{U}$. Consider the subcategory $\mathcal{HC}(\mathcal{U})^{\mathcal{J}_{P,\nu}}$ of $\mathcal{HC}(\mathcal{U})^{\nu + \rho}$ consisting of all bimodules annihilated by $\mathcal{J}_{P,\nu}$ on the right. Since $\Delta_P(\nu + \rho) = \Delta(\nu + \rho)/\mathcal{J}_{P,\nu} \Delta(\nu + \rho)$, this functor can be written as $X \mapsto X \otimes_{\mathcal{U}} \Delta_P(\nu + \rho)$. So the Bernstein-Gelfand equivalence restricts to a functor $\mathcal{HC}(\mathcal{U})^{\mathcal{J}_{P,\nu}} \to \mathcal{O}_\nu^P$.

For reader’s convenience, let us recall the proof of the equality $\Delta_P(\nu + \rho) = \Delta(\nu + \rho)/\mathcal{J}_{P,\nu} \Delta(\nu + \rho)$. First, we have a natural epimorphism $\Delta(\nu + \rho) \to \Delta_P(\nu + \rho)$ that factors through $\Delta(\nu + \rho)/\mathcal{J}_{P,\nu} \Delta(\nu + \rho)$. The latter corresponds to $\mathcal{U}/\mathcal{J}_{P,\nu}$ under the Bernstein-Gelfand equivalence. The ideal $\mathcal{J}_{P,\nu}$ is primitive. Indeed, $\Delta_P(\nu + \rho)$ is isomorphic to $\mathcal{U}(\mathfrak{g}_{>0})$ as a $\mathcal{U}(\mathfrak{g}_{<0})$-module. So its GK multiplicity is 1 and hence the socle of $\Delta_P(\nu + \rho)$ is simple. The latter proves that $\mathcal{J}_{P,\nu}$ is primitive. Further, any proper quotient of $\mathcal{U}/\mathcal{J}_{P,\nu}$ has GK dimension smaller than that of $\mathcal{U}/\mathcal{J}_{P,\nu}$ (equal to $\dim \mathfrak{g} - \dim \mathfrak{l}$). It follows that $\Delta(\nu + \rho)/\mathcal{J}_{P,\nu} \Delta(\nu + \rho)$ has simple socle and the quotient by this socle has GK dimension less than $\frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{l})$. Since the GK dimension of $\Delta_P(\nu + \rho)$ equals $\frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{l})$, the required equality follows.

Since $\Delta_P(\nu)$ is the largest quotient of $\Delta(\nu)$ lying in $\mathcal{O}_\nu^P$, we see that $\mathcal{M} \mapsto L(\Delta(\nu + \rho), \mathcal{M}) = L(\Delta_P(\nu + \rho), \mathcal{M}) : \mathcal{O}_\nu^P \to \mathcal{HC}(\mathcal{U})^{\mathcal{J}_{P,\nu}}$ is a quasi-inverse functor to $X \mapsto X \otimes_{\mathcal{U}} \Delta_P(\nu + \rho)$.

Summarizing, the functor $X \mapsto X \otimes_{\mathcal{U}} \Delta_P(\nu + \rho)$ defines an equivalence $\mathcal{HC}(\mathcal{U})^{\mathcal{J}_{P,\nu}} \to \mathcal{O}_\nu^P$ with quasi-inverse equivalence $\mathcal{M} \mapsto L(\Delta_P(\nu + \rho), \mathcal{M})$.

Now suppose that $\varrho$ is integral dominant, $w$ is compatible with $\varrho$, $L(w\varrho)$ lies $\mathcal{O}_\nu^P$ and has Gelfand-Kirillov dimension $\frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{l})$. Then $w$ lies in the right cell equal to $c^{-1}$, where $c$ is the left cell corresponding to the primitive ideal $\mathcal{J}_{P,\nu}$. This is a direct corollary of the parabolic Bernstein-Gelfand equivalence. Conversely, if $w \in c^{-1}$ is compatible with $\varrho$, we see that $L(w\varrho) \in \mathcal{O}_\nu^P$ and that $L(w\varrho)$ has GK dimension $\frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{l})$.

3.2. Category $\mathcal{O}$ for $\mathcal{W}$ and Whittaker modules I: non-completed version.

3.2.1. Categories $\tilde{\mathcal{O}}^\theta(\mathfrak{g}, e), \tilde{\mathcal{O}}^\theta(\mathfrak{g}, e)$. We will define the full categories $\tilde{\mathcal{O}}^\theta(\mathfrak{g}, e)_\nu \subset \tilde{\mathcal{O}}^\theta(\mathfrak{g}, e)_\nu$ in the category of left $\mathcal{W}$-modules. Namely, recall that $\mathfrak{q}$ and hence $\mathfrak{t} = \mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{q}$ are naturally included into $\mathcal{W}$.

Let $\tilde{\mathcal{O}}^\theta(\mathfrak{g}, e)$ stand for the category of all $\mathcal{W}$-modules $\mathcal{N}$ such that
(1) \( t \) acts on \( \mathcal{N} \) diagonally and with eigenvalues lying in \( \mathfrak{X}(P) + \nu + \delta \).
(2) The collection of integers \( \langle \mu - \nu - \delta, \theta \rangle \), where \( \mu \) is a \( t \)-weight of \( \mathcal{N} \), is bounded from above.

Below we write \( \mathcal{N}_\mu \) for the weight space with weight \( \mu + \delta \).

By definition, the subcategory \( \mathcal{O}^\theta(g, e)_\nu \subseteq \bar{\mathcal{O}}^\theta(g, e)_\nu \) consists of all modules with finite dimensional weight spaces and with finitely many eigen-characters for the action of the center of \( W \). Equivalently, \( \mathcal{O}^\theta(g, e)_\nu \subseteq \bar{\mathcal{O}}^\theta(g, e)_\nu \) consists of all finitely generated modules. This is, basically, the category that appeared in [BGK, 4.4] (they considered the case when \( \theta \) is generic in \( \mathfrak{q} \) but the general case is completely analogous). In particular, every object in \( \mathcal{O}^\theta(g, e)_\nu \) has finite length.

To an object \( \mathcal{N} \in \mathcal{O}^\theta(g, e)_\nu \) we can assign its formal character \( \text{ch}(\mathcal{N}) := \sum_{\mu}(\dim \mathcal{N}_\mu)e^\mu \), where the summation is taken over \( \mu \in t^* \).

3.2.2. Verma modules and their characters. Recall (see 2.6.2) that \( \mathcal{W}^0 \) is identified with the \( W \)-algebra \( U(g_0, e) \) for \( g_0 \). The category \( \mathcal{O}^\theta(g_0, e)_\nu \) is just the category of finite dimensional \( U(g_0, e) \)-modules, where \( t \) acts diagonalizable with weights in \( \mathfrak{X}(P) + \nu \).

We have the induction functor (to be called a Verma functor) \( \Delta^\theta_W : \bar{\mathcal{O}}^\theta(g_0, e)_\nu \to \bar{\mathcal{O}}^\theta(g, e)_\nu \) that maps \( \mathcal{N} \in \mathcal{O}^\theta(g_0, e)_\nu \) to \( \mathcal{W} \otimes_{\mathcal{W}^0} \mathcal{N} \), where we view \( \mathcal{N} \) as a \( \mathcal{W} \otimes \mathcal{W}^0 \)-module via the projection \( \mathcal{W} \otimes \mathcal{W}^0 \to \mathcal{W}^0 \). Of course, \( \Delta^\theta_W \) maps \( \mathcal{O}^\theta(g_0, e)_\nu \) to \( \mathcal{O}^\theta(g, e)_\nu \).

The functor \( \Delta^\theta_W \) is right exact. It has a right adjoint functor that maps \( \mathcal{N} \) to the annihilator \( \mathcal{W}^0 \mathcal{W} \mathcal{N} \) of \( \mathcal{W} \mathcal{N} \). It turns out that the functor \( \Delta^\theta_W \) is exact and, moreover, one can compute the character of \( \Delta^\theta_W(\mathcal{N}) \).

Namely, consider the eigen-decomposition \( \mathfrak{j}_\theta(e) = \bigoplus_{i \in \mathbb{Z}} \mathfrak{j}_\theta(e)i \), with respect to \( \text{ad} \theta \) and set \( \mathfrak{j}_\theta(e)_0 = \bigoplus_{i < 0} \mathfrak{j}_\theta(e)i \). Pick a basis \( f_1, \ldots, f_k \) of \( \mathfrak{j}_\theta(e)_0 \) consisting of weight vectors for \( t \). Recall that \( \text{gr} \mathcal{W} = \mathbb{K}[S] = S(\mathfrak{j}_\theta(e)) \). Lift the elements \( f_1, \ldots, f_k \) to \( t \)-weight vectors \( \tilde{f}_1, \ldots, \tilde{f}_k \) in \( \mathcal{W} \).

The following is a straightforward generalization of assertion (1) of [BGK, Theorem 4.5].

**Proposition 3.1.** Let \( v_1, \ldots, v_m \) be a basis in \( \mathcal{N}^0 \). The elements \( \tilde{f}_1^{n_1} \cdots \tilde{f}_k^{n_k} v_i \), where \( i = 1, \ldots, m \), and \( n_j \in \mathbb{Z}_{\geq 0} \), form a basis in \( \Delta^\theta_W(\mathcal{N}^0) \).

**Corollary 3.2.** Suppose that \( t \) acts on \( \mathcal{N}^0 \) with a single weight, say \( \mu_0 \), and let \( \mu_1, \ldots, \mu_k \) be the weights of \( \tilde{f}_1, \ldots, \tilde{f}_k \), respectively. Then

\[
\text{ch} \Delta^\theta_W(\mathcal{N}^0) = e^{\mu_0} \dim \mathcal{N}^0 \prod_{i=1}^k (1 - e^{\mu_i})^{-1}.
\]

**Corollary 3.3.** The functor \( \Delta^\theta_W \) is exact.

Also we remark that, for \( \mathcal{N}^0 \in \mathcal{O}^\theta(g_0, e)_\nu \), \( \mathcal{N}^0 \) is naturally embedded into \( \Delta^\theta_W(\mathcal{N}^0) \). There is the maximal submodule of \( \Delta^\theta_W(\mathcal{N}^0) \) that does not intersect \( \mathcal{N}^0 \), the quotient will be denoted by \( L^\theta_W(\mathcal{N}^0) \). The modules \( L^\theta_W(\mathcal{N}^0) \), for simple \( \mathcal{N}^0 \), form a complete list of simple objects in \( \mathcal{O}^\theta(g, e)_\nu \). Since any object in \( \mathcal{O}^\theta(g, e)_\nu \) has finite length, we see that \( \mathcal{O}^\theta(g, e)_\nu \) coincides with the Serre subcategory of \( \bar{\mathcal{O}}^\theta(g, e)_\nu \) generated by \( \Delta^\theta_W(\mathcal{N}^0) \).

3.2.3. Categories \( \bar{\mathfrak{Wh}}^\theta(g, e), \mathfrak{Wh}^\theta(g, e) \). Recall the subalgebra \( \mathfrak{m} = g_{>0} \oplus g_0(< 0) \) and its character \( \chi \). Consider the full subcategory \( \bar{\mathfrak{Wh}}^\theta(g, e)_\nu \) in the category of left \( \mathcal{U} \)-modules consisting of all modules \( \mathcal{M} \) such that

(1) The shift \( \mathfrak{m}_\chi = \{ \xi - \langle \xi, \chi \rangle, \xi \in \mathfrak{m} \} \) acts locally nilpotently on \( \mathcal{M} \).
(2) \(t\) acts diagonalizably on \(M\) with weights in \(\mathfrak{X}(P) + \nu\).

One can define an analog of a Verma module in \(\tilde{\text{Wh}}^\theta(\mathfrak{g}, e)\). Namely, take \(N^0 \in \tilde{\mathcal{O}}^\theta(\mathfrak{g}_0, e)_\nu\). Then we have Skryabin’s equivalence (see 3.2.4 below) \(S^0 : \tilde{\mathcal{O}}^\theta(\mathfrak{g}_0, e)_\nu \simeq \tilde{\text{Wh}}^\theta(\mathfrak{g}_0, e)_\nu\). We set \(\Delta^\theta_\mu(N^0) := U \otimes_{U(\mathfrak{g}_0)} S_0(N^0)\). The functor \(\Delta^\theta_\mu\) admits a right adjoint, the functor \(M \mapsto M^m\).

By definition, for \(Wh^\theta(\mathfrak{g}_0, e)_\nu\) we take the Serre subcategory of \(\text{Wh}^\theta(\mathfrak{g}_0, e)_\nu\) generated by the modules \(\Delta^\theta_\mu(N^0)\) with \(N^0 \in \mathcal{O}^\theta(\mathfrak{g}_0, e)_\nu\).

### 3.2.4. Equivalences.

In [Lo3] we have produced an equivalence \(\mathcal{K} : \tilde{\text{Wh}}^\theta(\mathfrak{g}, e)_\nu \rightarrow \tilde{\mathcal{O}}^\theta(\mathfrak{g}, e)_\nu\). In the case when \(\mathfrak{g} = \mathfrak{g}_0\), the equivalence \(\mathcal{K}\) becomes an equivalence introduced by Skryabin in an appendix to [Pr2].

To construct \(\mathcal{K}\), recall the isomorphism \(U^\wedge \cong \mathbb{A}(\mathcal{W})^\wedge\) from 2.6.2. The category \(\tilde{\text{Wh}}^\theta(\mathfrak{g}, e)_\nu\) is nothing else but the category of topological \(U^\wedge\)-modules with respect to the discrete topology, where \(t\) acts diagonalizably with weights in \(\mathfrak{X}(P) + \nu\). In particular, we can view \(M \in \tilde{\text{Wh}}^\theta(\mathfrak{g}, e)_\nu\) as a module over \(\mathbb{A}(\mathcal{W})^\wedge\). Then we define \(\mathcal{K}(M)\) as \(M^\wedge\), where the lagrangian subspace \(\mathfrak{v} \subset V\) was introduced in 2.6.2. A quasi-inverse functor is given by \(\mathcal{N} \mapsto \mathbb{K}[\mathfrak{v}] \otimes \mathcal{N}\). Here \(\mathbb{A}\) acts on \(\mathbb{K}[\mathfrak{v}]\) via the identification \(\mathbb{A} \cong D(\mathfrak{v})\), or equivalently, \(\mathbb{K}[\mathfrak{v}] = \mathbb{A}/\mathfrak{a}^\mathfrak{v}\).

According to [Lo3, Theorem 4.1], the functor \(\mathcal{K}\) intertwines the functors \(\Delta^\theta_\nu, \Delta^\theta_{\nu'}\). It follows that \(\mathcal{K}\) induces an equivalence of \(\text{Wh}^\theta(\mathfrak{g}, e)_\nu\) and \(\mathcal{O}^\theta(\mathfrak{g}, e)_\nu\) because both these subcategories are defined in terms of \(\Delta^\theta_\cdot\).

### 3.2.5. Equivariant version.

Set \(R := Q \cap G_0\), this is a reductive subgroup in \(G_0\) (and a maximal reductive subgroup of \(Q \cap P\)). We can consider the \(R\)-equivariant versions of the categories under consideration. For example, by an \(R\)-equivariant object in \(\mathcal{O}^\theta(\mathfrak{g}, e)_\nu\) we mean a module \(\mathcal{N} \in \mathcal{O}^\theta(\mathfrak{g}, e)_\nu\) equipped with an action of \(R\) that makes the structure map \(\mathcal{W} \times \mathcal{N} \rightarrow \mathcal{N}\) into an \(R\)-equivariant map and such that the differential of the \(R\)-action coincides with the action of \(\mathfrak{r} \subset \mathcal{W}\) (shifted by \(\nu + \delta\); for Whittaker categories we just consider a shift by \(\nu\)). The \(R\)-equivariant categories will be denoted by \(\mathcal{O}^\theta(\mathfrak{g}, e)_{R\nu}\), etc. Since the isomorphism \(U^\wedge \cong \mathbb{A}(\mathcal{W})^\wedge\) is \(R\)-equivariant, we see that \(\mathcal{K}\) upgrades to an equivalence of equivariant categories. Let us also mention that the Verma module functors are lifted to the equivariant categories, i.e., for, say, \(N^0 \in \mathcal{O}^\theta(\mathfrak{g}_0, e)^R\), the Verma module \(\Delta^\theta_{\nu'}(N^0)\) has a natural \(R\)-equivariant structure.

### 3.2.6. Duality for \(\mathcal{O}^\theta(\mathfrak{g}, e)\).

Here we are going to define a contravariant involutive equivalence \(\mathcal{O}^\theta(\mathfrak{g}, e)_\nu \rightarrow \mathcal{O}^\theta(\mathfrak{g}, e)_{\nu'}\). We are going to use conventions of 2.6.1. In particular, \(t \subset \mathfrak{h}\).

To \(\mathcal{N} \in \mathcal{O}^\theta(\mathfrak{g}, e)_{\nu}\) we can assign its restricted dual \(\mathcal{N}^{\nu} = \bigoplus_{\mu \in \mathfrak{t}^*} \mathcal{N}^*_{\mu} \subset \mathcal{N}^*\). By the construction, the anti-automorphism \(\tau\) is the identity on \(\mathfrak{t}\). The twist \(\mathcal{N}^{\nu} := \tau\mathcal{N}^{\nu}\) is therefore an object in \(\mathcal{O}^\theta(\mathfrak{g}, e)_{\nu'}\). Since \(\tau^2 = 1\), we see that \(\cdot^{\nu}\) is involutive.

Also it follows directly from the construction that \(\text{ch}(\mathcal{N}) = \text{ch}(\mathcal{N}^{\nu})\).

A pairing \(\mathcal{N}_1 \times \mathcal{N}_2 \rightarrow \mathbb{K}\) is called contravariant (or \(\tau\)-contravariant) if \(\langle \tau(a)n_1, n_2 \rangle = \langle n_1, an_2 \rangle\) for all \(a \in \mathcal{W}, n_1, n_2 \in \mathcal{N}_i, i = 1, 2\). For example, there is a natural contravariant pairing \(\mathcal{N}^{\nu} \times \mathcal{N} \rightarrow \mathbb{K}\).

The following lemma characterizes the module \(\mathcal{N}^{\nu}\) up to an isomorphism.
Lemma 3.4. Let \( N \in \mathcal{O}^\theta(g, e)_\nu, N_1 \in \hat{\mathcal{O}}^\theta(g, e)_\nu \). Suppose that there is a contravariant pairing \( (\cdot, \cdot) : N \times N_1 \to \mathbb{K} \) with zero left and right kernels. Then there is a unique isomorphism \( N^\vee \cong N_1^\vee \) that intertwines the pairing \( N \times N \to \mathbb{K} \) with a natural pairing \( N^\vee \times N \to \mathbb{K} \).

The proof is straightforward.

Let us finish by noting that \( \bullet^\vee \) naturally upgrades to an equivalence of \( R \)-equivariant categories.

3.3. Category \( \mathcal{O} \) for \( \mathcal{W} \) and Whittaker modules II: completed version.

3.3.1. Categories \( \hat{\mathcal{O}}^\theta(g, e), \mathcal{O}^\theta(g, e) \). By definition, the category \( \hat{\mathcal{O}}^\theta(g, e)_\nu \) consists of all \( \mathcal{W} \)-modules \( \mathcal{M} \) satisfying the following conditions:

1. \( \mathcal{M} \) is complete and separated with respect to the \( \mathcal{W}_{<0} \)-adic topology.
2. The eigenvalues of \( t \) on \( \mathcal{M} \) are in \( \mathfrak{X}(P) + \nu + \delta \) and are bounded from above (in the sense that \( (\theta, \mu - \nu - \delta) \) is uniformly bounded for all weights \( \mu \) of \( \mathcal{M} \)).
3. \( \mathcal{M} \) (considered as a topological \( \mathcal{W} \)-module with respect to the \( \mathcal{W}_{<0} \)-adic topology) is the direct product of its \( t \)-weight subspaces (where the latter is considered with the direct product topology).

Inside \( \hat{\mathcal{O}}^\theta(g, e)_\nu \), we consider the full subcategory \( \hat{\mathcal{O}}^\theta(g, e)_\nu \) consisting of all modules with finite dimensional weight spaces and with finitely many eigen-characters for the action of the center of \( \mathcal{W} \). We again have a completion functor \( N \mapsto \hat{N} : \mathcal{O}^\theta(g, e)_\nu \to \hat{\mathcal{O}}^\theta(g, e)_\nu \) that sends \( \mathcal{N} = \bigoplus \mathcal{N}_\mu \) to \( \prod \mathcal{N}_\mu \). The only claim that one needs to check in order to verify \( \hat{\mathcal{N}} \in \hat{\mathcal{O}}^\theta(g, e)_\nu \) is that the \( \mathcal{W}_{<0} \)-adic topology on \( \hat{\mathcal{N}} \) is complete and separated. But this is straightforward from the decomposition into the product of weight spaces.

Lemma 3.5. The functor \( \bullet \) coincides with the functor of \( \mathcal{W}_{<0} \)-adic completion, \( N \mapsto \lim_{\nu \to +\infty} \hat{N}/\mathcal{W}_{<0}N \) and is an equivalence of categories.

Proof. Let us check the claim about an equivalence. For \( \mathcal{N}' \in \hat{\mathcal{O}}^\theta(g, e)_\nu \), let \( \mathcal{N}'_{fin} \) denote the subspace of \( t \)-finite elements. In other words, \( \mathcal{N}'_{fin} = \bigoplus \mathcal{N}_\mu \). Clearly, \( \mathcal{N}'_{fin} \) is an object of \( \mathcal{O}^\theta(g, e)_\nu \) and the functor \( \bullet_{fin} \) is quasi-inverse to \( \bullet \).

Now let us check that \( \bullet \) coincides with the \( \mathcal{W}_{<0} \)-adic completion functor. Clearly, \( \hat{\mathcal{N}} \) is complete in the \( \mathcal{W}_{<0} \)-adic topology so we have a natural map \( \lim_{\nu \to +\infty} \mathcal{N}/\mathcal{W}_{<0}\mathcal{N} \to \hat{\mathcal{N}} \). But \( \mathcal{N} \) is generated by finitely many weight vectors as a \( \mathcal{W}_{<0} \)-module. This implies that the homomorphism is actually an isomorphism. \( \square \)

We have a Verma functor \( \hat{\Delta}^\theta_W : \hat{\mathcal{O}}^\theta(g_0, e)_\nu \to \hat{\mathcal{O}}^\theta(g, e)_\nu \) given by \( \mathcal{N}^0 \mapsto \hat{\Delta}^\theta_W(\mathcal{N}^0) \). On the other hand, (this follows, for example, from Lemma 3.5)

\[
\hat{\Delta}^\theta_W(\mathcal{N}^0) = [^\wedge \mathcal{W} / ^\wedge \mathcal{W} ^\wedge \mathcal{W}_{>0}] \hat{\mathcal{O}}_{\mathcal{W}^0} \mathcal{N}^0.
\]

Thanks to the equivalence \( \hat{\mathcal{O}}^\theta(g, e)_\nu \cong \mathcal{O}^\theta(g, e)_\nu \) and the claim that the latter is generated by the Verma modules, we see that \( \hat{\mathcal{O}}^\theta(g, e)_\nu \) coincides with the Serre subcategory of \( \hat{\mathcal{O}}^\theta(g, e)_\nu \) generated by \( \hat{\Delta}^\theta_W(\mathcal{N}^0) \) with \( \mathcal{N}^0 \in \mathcal{O}^\theta(g_0, e)_\nu \). We will write \( \hat{\Delta}^\theta_W(\mathcal{N}^0) \) for \( \hat{\Delta}^\theta_W(\mathcal{N}^0) \) in the case when \( \mathcal{N}^0 \in \mathcal{O}^\theta(g_0, e)_\nu \).
3.3.2. Category $\hat{\text{Wh}}^{\theta}(g, e)$ and equivalence $\hat{\mathcal{K}}$. Now we are going to define a category $\hat{\text{Wh}}^{\theta}(g, e)_\nu$ and show that it is equivalent to $\hat{\mathcal{O}}^{\theta}(g, e)_\nu$.

Recall the subalgebra $\hat{\mathfrak{m}} := g_{<0} \oplus g_0(<0) = \tau(\mathfrak{m})$. Let $\hat{\text{Wh}}^{\theta}(g, e)_\nu$ consist of all $\hat{\mathcal{U}}$-modules $\mathcal{M}$ such that

1. The $\hat{\mathfrak{m}}\chi$-adic topology on $\mathcal{M}$ is separated and complete.
2. The weights of $\mathfrak{t}$ on $\mathcal{M}$ are integral after the $\nu$-shift and are bounded from above.
3. As a topological module, $\mathcal{M}$ is the direct product of its $t$-weight spaces.

We are going to construct an equivalence $\hat{\mathcal{K}} : \hat{\text{Wh}}^{\theta}(g, e)_\nu \to \hat{\mathcal{O}}^{\theta}(g, e)_\nu$. This functor will be produced as a restriction of an equivalence between the categories $\hat{\wedge}\mathcal{U}$-$\text{csMod}$ of $\hat{\wedge}\mathcal{U}$-modules that are complete and separated in the $\hat{\mathfrak{m}}\chi$-adic topology and the category $\hat{\wedge}\mathcal{W}$-$\text{csMod}$ of $\hat{\wedge}\mathcal{W}$-modules that are complete and separated in the $\mathcal{W}_{<0}$-adic topology.

Recall the isomorphism $\hat{\wedge}\mathcal{U} \cong \hat{\wedge}\mathcal{A}(\mathcal{W})$. Under this isomorphism, an equivalence $\hat{\mathcal{K}} : \hat{\wedge}\mathcal{U}$-$\text{csMod} \to \hat{\wedge}\mathcal{W}$-$\text{csMod}$ is given by taking the $\mathfrak{b}$-coinvariants. A quasi-inverse equivalence looks as follows. Choose a lagrangian subspace $\hat{\mathfrak{b}}^* \subset \mathcal{V}$ complimentary to $\mathfrak{b}$. Then $\mathbb{K}[\hat{\mathfrak{b}}^*] = \hat{\wedge}\mathcal{A}/\hat{\wedge}\mathcal{A}\mathfrak{b}^*$ is a $\hat{\wedge}\mathcal{A}$-module. A quasi-inverse equivalence sends $\mathcal{N}$ to $\mathbb{K}[\hat{\mathfrak{b}}^*] \otimes \mathcal{N}$.

The claim that the two functors are quasi-inverse to each other reduces to the following lemma.

**Lemma 3.6.** Let $U$ be a finite dimensional vector space and $M$ be a module over the algebra $D(U)$ of differential operators on $U$. Suppose that $M$ is complete and separated with respect to the $U^*$-adic topology. Then $M = \mathbb{K}[U] \otimes M_0$, where $M_0$ is a complete separated topological vector space ($D(U)$ acts on the first factor).

The proof is very similar to (and is a slight generalization of) the proof of [LO, Lemma 5.14] but we will provide it for reader’s convenience.

**Proof.** By induction, compare with *loc.cit.*, we reduce the proof to the case when $\dim U = 1$. Let $p$ be a basis vector in $U$ and $q \in U^*$ be the dual basis vector so that $[p, q] = 1$. Set $r := \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} q^i p^i$, this is a well-defined element of the completion $\hat{\wedge}D(U)$. Moreover, we have $pr = 0, rq = 0$ and $\sum_{i=0}^{\infty} \frac{1}{i!} q^i r^i = 1$ in $\hat{\wedge}D(U)$. It follows that $m = \sum_{i=0}^{\infty} \frac{1}{i!} q^i r^i m$ for every $m \in M$. But this just says that $M = \mathbb{K}[[q]] \otimes r(M)$ and $r(M)$ coincides with the annihilator of $p$ in $M$ (and is naturally isomorphic to $M/qM$).

Now let us show that $\hat{\mathcal{K}}$ restricts to an equivalence $\hat{\text{Wh}}^{\theta}(g, e) \cong \hat{\mathcal{O}}^{\theta}(g, e)$. The functor $\hat{\mathcal{K}}^{-1}$ is essentially just taking the tensor product with $\mathbb{K}[\hat{\mathfrak{b}}^*]$. All eigenvalues of $\theta$ on $\hat{\mathfrak{b}}$ are non-positive integers by the construction of $\hat{\mathfrak{b}}$. So the eigen-values of $\theta$ on $\mathbb{K}[\hat{\mathfrak{b}}^*]$ are non-positive integers as well. It follows that properties (1)-(3) in the definitions of the categories are equivalent for $\mathcal{N}$ and $\mathbb{K}[\hat{\mathfrak{b}}^*] \otimes \mathcal{N}$. This shows that $\hat{\mathcal{K}}^{-1}$ is an equivalence between $\hat{\mathcal{O}}^{\theta}(g, e)$ and $\hat{\text{Wh}}^{\theta}(g, e)$.

3.3.3. Verma functor for $\hat{\text{Wh}}^{\theta}(g, e)$ and definition of $\hat{\text{Wh}}^{\theta}(g, e)$. Once again, we have a Verma functor $\hat{\Delta}_t^{\theta} : \hat{\mathcal{O}}^{\theta}(g_0, e)_\nu \to \hat{\text{Wh}}^{\theta}(g, e)_\nu$:

$$\hat{\Delta}_t^{\theta}(\mathcal{N}^0) := \hat{\mathcal{U}}/\hat{\mathcal{U}}^t \hat{\mathcal{U}}_{>0} \otimes_{\hat{\mathcal{U}}^t \hat{\Delta}^0_0} \hat{\mathcal{S}}_0(\mathcal{N}^0).$$

Here $\hat{\mathcal{S}}_0$ is the $g_0$-counterpart of the equivalence $\hat{\mathcal{K}}^{-1}$. It is straightforward to check that $\hat{\Delta}_t^{\theta}(\mathcal{N}^0)$ is indeed an object of $\hat{\text{Wh}}^{\theta}(g, e)_\nu$. 

We write $\hat{\Delta}_g^0(\mathcal{N}^0)$ instead of $\hat{\Delta}_g^0(\mathcal{N}^0)$ when $\mathcal{N}^0 \in \mathcal{O}^g(\mathfrak{g}_0, e)_\nu$. We define the full subcategory $\hat{\mathcal{W}}h^g(\mathfrak{g}, e)_\nu \subset \hat{\mathcal{W}}h^g(\mathfrak{g}, e)_\nu$ as the Serre span of $\hat{\Delta}_g^0(\mathcal{N}^0)$ with $\mathcal{N}^0 \in \mathcal{O}^g(\mathfrak{g}_0, e)_\nu$.

We claim that $\hat{\mathcal{K}}^{-1}$ (and hence $\hat{\mathcal{K}}$) intertwines the functors $\hat{\Delta}_g^0$ and $\hat{\Delta}_g^0$. We have the following equality of modules over $^\wedge A(W)$,

$$\hat{\Delta}_g^0(\mathcal{N}^0) = \left( [^\wedge A(A(W)/^\wedge A(A(W))^\wedge A(A(W)_{>0})] \otimes_{^\wedge A(\chi_0)} \left( [^\wedge A/^\wedge A(A_{>0})] \otimes_{^\wedge A([\mathfrak{g}_0^0])} \left( [^\wedge A/\mathcal{W}^\wedge A(W)_{>0}] \otimes_{\mathcal{W}^\wedge A(W)_0} \mathcal{N}^0 \right) = \mathbb{K}[[\mathfrak{g}_0^0]] \hat{\Delta}_g^0(\mathcal{N}^0).$$

But this equality precisely means that $\hat{\mathcal{K}}^{-1}(\hat{\Delta}_g^0(\mathcal{N}^0)) = \hat{\Delta}_g^0(\mathcal{N}^0)$.

In particular, it follows that $\hat{\mathcal{K}}$ restricts to an equivalence of $\hat{\mathcal{W}}h^g(\mathfrak{g}, e)_\nu$ and $\hat{\mathcal{O}}^g(\mathfrak{g}, e)_\nu$, we write $\hat{\mathcal{K}}$ for the restriction.

We remark that we do not know any direct equivalence between $\hat{\mathcal{W}}h^g(\mathfrak{g}, e)_\nu$ and $\hat{\mathcal{O}}^g(\mathfrak{g}, e)_\nu$.

3.3.4. Equivariant versions. Again, we can consider the equivariant versions $\hat{\mathcal{O}}^g(\mathfrak{g}, e)_\nu^R$, $\hat{\mathcal{W}}h^g(\mathfrak{g}, e)_\nu^R$ of $\hat{\mathcal{O}}^g(\mathfrak{g}, e)_\nu$, $\hat{\mathcal{W}}h^g(\mathfrak{g}, e)_\nu$ and can upgrade $\hat{\mathcal{K}}$ to an equivalence of these categories. We remark that in the completed setting the differential of an $\mathcal{R}$-action still makes sense so the definition of an $\mathcal{R}$-equivariant object carries over to this setting.

4. Generalized Soergel functor

4.1. Functor $\otimes_{\mathfrak{k}, e}$.

4.1.1. Category $\mathcal{O}_\nu^K$. Recall that we have fixed an $\mathfrak{sl}_2$-triple $e, h, f \in \mathfrak{g}$. Next, pick a connected algebraic subgroup $K \subset G$ such that

- $\mathfrak{u} := \mathfrak{k} \cap V$ is a lagrangian subspace in $V := [\mathfrak{g}, f]$ (here $\mathfrak{k}$ denotes the Lie algebra of $K$),
- $\chi$ vanishes on $\mathfrak{k}$
- and $h \in \mathfrak{k}$.

Let $\mathcal{R}$ denote a maximal reductive subgroup of the intersection $Q \cap K$.

Let us provide an example of this situation that is of most interest for us.

Let $\theta, \mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \mathfrak{p}$ be as in 2.6.1. Then we set $K := P$. Let us check that our conditions are satisfied. Clearly, $\chi$ vanishes on $\mathfrak{p}$ and so it remains to prove that $\mathfrak{u}$ is lagrangian. First, it is easy to see that $\omega \chi$ vanishes on $\mathfrak{g}$ and hence on $\mathfrak{k} \cap V$. Let us compute the dimension of $\mathfrak{k} \cap V$. We have $\mathfrak{k} \cap V = (\mathfrak{g}_0(\geq 0) \cap [\mathfrak{g}_0, f]) \oplus [\mathfrak{g}_0, f]$. The first summand is isomorphic to $[\mathfrak{g}_0, f]/\mathfrak{g}_0(< 0)$ and so its dimension is $\dim \mathfrak{g}_0(< 0) = \frac{1}{2} \dim [\mathfrak{g}_0, f]$. The $\mathfrak{sl}_2$-modules $\mathfrak{g}_0$ and $\mathfrak{g}_0$ are dual to each other and hence are isomorphic. So $\dim [\mathfrak{g}_0, f] = \frac{1}{2} (\dim [\mathfrak{g}, f] - \dim [\mathfrak{g}, f])$. We get $\dim \mathfrak{k} \cap V = \frac{1}{2} \dim [\mathfrak{g}, f] = \frac{1}{2} \dim V$.

Two other examples of $\hat{\mathcal{K}}$ will be introduced in Section 6.

4.1.2. Choice of $\mathfrak{t} : V \to \mathfrak{I}_\chi$. Now we are going to prove that there is a $\mathbb{Z}/2\mathbb{Z} \times \mathcal{R} \times \mathbb{K}^\times$-equivariant embedding $\mathfrak{t} : V \to \mathfrak{I}_\chi$ with properties (2)-(3) from Subsection 2.1 such that the image of $\mathfrak{u}$ lies in $\mathfrak{U}_{\mathfrak{h}}^{\mathbb{K}^\times}$ (let us recall that here $\mathbb{Z}/2\mathbb{Z} \times \mathbb{K}^\times$ is a factor of $\mathcal{Q}$ and $\mathcal{R}$ is viewed as a subgroup of $Q$, see Subsection 2.1, the $\mathbb{Z}/2\mathbb{Z} \times \mathcal{R} \times \mathbb{K}^\times$-actions are the restrictions of the $\mathcal{Q}$-actions). Recall that we already have an embedding $V \hookrightarrow \mathfrak{I}_\chi$ that satisfies (1)-(3) from Subsection 2.1 but we do not know yet the claim about the image of $\mathfrak{u}$. 
Let us choose a basis in \( g_\chi := \{ x - (\chi, x), x \in g \} \) as follows. Let \( x_1, \ldots, x_n \) be a basis in \( u \). Let \( y_1, \ldots, y_k \in V \) be such that \( x_1, \ldots, x_n, y_1, \ldots, y_k \) is a Darboux basis, i.e., for the symplectic form \( \omega \) on \( V \) (given by \( \omega(u, v) = (\chi, [u, v]) \)) we have \( \omega(x_i, x_j) = \omega(y_i, y_j) = 0, \omega(y_i, x_j) = \delta_{ij} \). Further, complete \( x_1, \ldots, x_n, y_1, \ldots, y_k \) to a basis in \( g \) with vectors \( z_1, \ldots, z_m \in \mathfrak{g}(e) \cap \mathfrak{f}, w_1, \ldots, w_{m'} \in \mathfrak{g}(e) \). We view \( x_i, y_j, z_j, w_k \) as elements of \( g_\chi \) via the isomorphism \( x \mapsto x - (\chi, x) \). Being the completion of \( \mathbb{K}[g_\chi] \) at the origin, \( \mathbb{K}[g]^\chi \) coincides with \( \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_k, z_1, \ldots, z_m, w_1, \ldots, w_{m'}] \) as an algebra and \( U_h^\chi \) coincides with \( \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_k, z_1, \ldots, z_m, w_1, \ldots, w_{m'}, h] \) as a vector space.

**Lemma 4.1.** There are elements \( \tilde{x}_i, \tilde{y}_i, \tilde{z}_j, \tilde{w}_k \in U_h^\chi \) with \( i = 1, \ldots, n, j = 1, \ldots, m, k = 1, \ldots, m' \) such that

(i) \( \tilde{x}_i - x_i, \tilde{y}_i - y_i, \tilde{z}_j - z_j, \tilde{w}_k - w_k \in \tilde{I}_\chi^\ell \),

(ii) \( [\tilde{x}_i, \tilde{x}_j] = [\tilde{y}_i, \tilde{y}_j] = 0, [\tilde{x}_i, \tilde{y}_j] = \delta_{ij} h^2 \), and the \( \tilde{x} \)'s and \( \tilde{y} \)'s commute with \( \tilde{z} \)'s and \( \tilde{w} \)'s.

(iii) \( \tilde{x}_i, \tilde{z}_j \in U_h^\chi \mathfrak{f} \).

(iv) The map \( \iota : V \to \tilde{I}_\chi \) given by \( x_i \mapsto \tilde{x}_i, y_i \mapsto \tilde{y}_i \) is \( \mathbb{Z}/2\mathbb{Z} \times R \times \mathbb{K}^\times \)-equivariant.

**Proof.** We will construct such elements order by order. More precisely, set \( \tilde{x}_i^{(1)} := x_i, \tilde{y}_i^{(1)} := y_i \). For each positive integer \( \ell > 1 \), we will produce elements \( x_i^{(\ell)}, y_i^{(\ell)}, z_j^{(\ell)}, w_k^{(\ell)} \) with the following properties:

(i) \( x_i^{(\ell)} - x_i^{(\ell-1)}, y_i^{(\ell)} - y_i^{(\ell-1)}, z_j^{(\ell)} - z_j^{(\ell-1)}, w_k^{(\ell)} - w_k^{(\ell-1)} \in \tilde{I}_\chi^\ell \),

(ii) The commutation relations for the elements \( \tilde{x}_i, \tilde{y}_i, \tilde{z}_j, \tilde{w}_k \) specified above hold for the elements \( x_i^{(\ell)}, y_i^{(\ell)}, z_j^{(\ell)}, w_k^{(\ell)} \) modulo \( \tilde{I}_\chi^\ell \).

(iii) \( x_i^{(\ell)}, z_j^{(\ell)} \in U_h^\chi \mathfrak{f} \).

(iv) The map \( \iota^{(\ell)} : V \to \tilde{I}_\chi \) given by \( x_i \mapsto x_i^{(\ell)}, y_i \mapsto y_i^{(\ell)} \) is \( \mathbb{Z}/2\mathbb{Z} \times R \times \mathbb{K}^\times \)-equivariant.

We remark, modulo (i\((\ell-1))-\(iv^{(\ell-1)}\)), the conditions (i\((\ell)))-(iii\((\ell))

- are preserved by averaging over \( \mathbb{Z}/2\mathbb{Z} \times R \times \mathbb{K}^\times \),
- are preserved by adding summands from \( \tilde{I}_\chi^{\ell+1} \) (to \( y_i^{(\ell)}, w_k^{(\ell)} \)) or summands from \( U_h^\chi \mathfrak{f} \cap \tilde{I}_\chi^{\ell+1} \) (to \( x_i^{(\ell)}, z_j^{(\ell)} \)),
- and cut an affine subspace in

\[
\left( U_h^\chi / \tilde{I}_\chi^{\ell+1} \right)^{\oplus n+m'} \oplus \left( U_h^\chi \mathfrak{f} / [\tilde{I}_\chi^{\ell+1} \cap U_h^\chi \mathfrak{f}] \right)^{\oplus n+m}.
\]

and so we can automatically assume that (iv\((\ell))\) holds as well.

Set \( y_i^{(1)} := y_i^{(1-1)} \). Construct an element \( \tilde{x}_i^{(1)} \) as follows. Set \( a := y_i^{(1)} - 1 \) so that \( a \in \tilde{I}_\chi^{(1-1)} \). Expand \( a \) in the form \( a = \sum_{i=0}^{\infty} f_i \cdot (x_1^{(1-1)})^i \), where each \( f_i \) is a series in \( x_2^{(1-1)}, \ldots, x_n^{(1-1)}, y_1^{(1-1)}, \ldots, y_k^{(1-1)}, z_1^{(1-1)}, \ldots, z_m^{(1-1)}, w_1^{(1-1)}, \ldots, w_{m'}^{(1-1)} \) (with the variables written in this order, although this does not really matter). For \( i = 0, \ldots, \ell - 1 \) we have \( f_i \in \tilde{I}_\chi^{\ell-i-1} \). Set \( \tilde{x}_i^{(1-1)} := x_i^{(1-1)} - \int a \, dx_i^{(1-1)} \), where \( \int a \, dx_i^{(1-1)} := \sum_{i=0}^{\infty} f_i \cdot (x_1^{(1-1)})^i \). We remark that \( \int a \, dx_i^{(1-1)} \in \tilde{I}_\chi^{\ell} \cap U_h^\chi \mathfrak{f} \). We have

\[
\frac{1}{h^2} [y_1^{(1)}, \tilde{x}_1^{(1)}] = 1 + a - \sum_{i=0}^{\infty} \frac{1}{(i + 1) h^2} [y_1^{(1)}, f_i] (x_1^{(1-1)})^i + \sum_{i=0}^{\infty} f_i \cdot \frac{1}{(i + 1) h^2} [y_1^{(1)}, (x_1^{(1-1)})^i+1]
\]
We have $\frac{1}{\ell}[y_1^{(\ell)}, f_i] \in \tilde{I}_\chi^{\ell-1-i}$ because of the form of $f_i$. So the third summand of the right hand side is in $\tilde{I}_\chi^\ell$. On the other hand, $\frac{1}{(1+i)+\ell^2}[y_1^{(\ell)}, (x_1^{(\ell-1)})^{i+1}]$ is congruent to $(\tilde{x}_1^{(\ell-1)})^i$ modulo $\tilde{I}_\chi^\ell$. From here we deduce that $\frac{1}{\ell}[y_1^{(\ell)}, \tilde{x}_1^{(\ell)}] - 1 \in \tilde{I}_\chi^\ell$.

We can actually continue the above procedure and get $\frac{1}{\ell}[y_1^{(\ell)}, \tilde{x}_1^{(\ell)}] = 1$ (and $x_1^{(\ell)}$ is still in $U_h^{\wedge \chi}\mathfrak{t}$). We make this choice of $\tilde{x}_1^{(\ell)}$ but this is not our final choice for $\tilde{x}_1$ because we need to guarantee the equivariance. We remark that $K[[y_1^{(\ell)}, \tilde{x}_1^{(\ell)}, h]] \otimes_{K[[h]]} Z = U_h^{\wedge \chi}$, where $Z$ stands for the centralizer of $y_1^{(\ell)}, \tilde{x}_1^{(\ell)}$ in $U_h^{\wedge \chi}$, analogously to the proof of [Lo2, Proposition 3.3.1].

We can similarly correct other basis elements $b = \tilde{x}_i^{(\ell-1)}, y_i^{(\ell-1)}$, $i > 1$, $\tilde{z}_i^{(\ell-1)}, \tilde{w}_k^{(\ell-1)}$ by elements from $\tilde{I}_\chi^\ell$ so that (i) and (ii) still hold and the bracket of $b$ with $y_1^{(\ell)}$ vanishes.

Our goal now is to show that we can modify the elements $\tilde{b}$ (again by adding elements from $\tilde{I}_\chi^\ell$) so that (ii) holds for the brackets with $y_1^{(\ell)}, x_1^{(\ell)}$, (i) holds for $b$, and (iii) holds if $b = \tilde{x}_i^{(\ell-1)}, \tilde{z}_i^{(\ell-1)}$. If $b = \tilde{y}_i^{(\ell-1)}$, then $b$ can be corrected similarly to $\tilde{x}_1^{(\ell-1)}$ above. Now let us consider the case when $b = \tilde{x}_i^{(\ell-1)}, \tilde{z}_i^{(\ell-1)}$: a priori it is unclear why the correction of $b$ commuting with $\tilde{x}_1^{(\ell)}$ produced by a construction similar to the above still lies in $U_h^{\wedge \chi}\mathfrak{t}$.

Since $[\tilde{y}_1^{(\ell)}, b] = 0$ we have $b = \sum_{i=0}^\infty (\tilde{y}_1^{(\ell)})^ib_i$ with $b_i \in Z$. We also know that $\frac{1}{\ell}[\tilde{x}_1^{(\ell)}, b] \in \tilde{I}_\chi^{\ell-1}$. This means that $b_i \in \tilde{I}_\chi^{\ell-i}$ for $i = 1, \ldots, \ell - 1$. In particular, $b - b_0 \in \tilde{I}_\chi^{\ell-1}$. We can take $b_0$ for a lift of $b$ if we know that $b_0 \in U_h^{\wedge \chi}\mathfrak{t}$. Let us check that $(\tilde{y}_1^{(\ell)})^ib_i \in U_h^{\wedge \chi}\mathfrak{t}$ for all $i$. First of all, let us point out that $\frac{1}{\ell}[U_h^{\wedge \chi}\mathfrak{t}, U_h^{\wedge \chi}\mathfrak{t}] \subset U_h^{\wedge \chi}\mathfrak{t}$, which follows from the observation that $\mathfrak{t}$ is a subalgebra of $\mathfrak{g}$. Consider the operator $E := -\frac{1}{\ell}[\tilde{y}_1^{(\ell)}, \cdot]$. It preserves the centralizer of $\tilde{y}_1^{(\ell)}$ and also $U_h^{\wedge \chi}\mathfrak{t}$ and sends an element of the form $\sum_{i=0}^\infty (\tilde{y}_1^{(\ell)})^ib_i$ to $\sum_{i=0}^\infty i(\tilde{y}_1^{(\ell)})^ib_i$. Also $U_h^{\wedge \chi}\mathfrak{t}$ is closed in the $\tilde{I}_\chi$-adic topology, see [Lo2, Lemma 2.4.4] for a more general result. Since all elements $E^ib_i$ are in $U_h^{\wedge \chi}\mathfrak{t}$, we see that indeed $(\tilde{y}_1^{(\ell)})^ib_i \in U_h^{\wedge \chi}\mathfrak{t}$ for any $i$.

In this way we achieve that (i) and (iii) are still satisfied and all basis elements commute with $\tilde{y}_1^{(\ell)}, \tilde{x}_1^{(\ell)}$. We can proceed in the same way with modified $y_2^{(\ell-1)}, x_2^{(\ell-1)}$, etc. We achieve that (i) and (iii) are satisfied and then average the map $\iota^{(\ell)}$ with respect to $\mathbb{Z}/2\mathbb{Z} \times R \times \mathbb{K}^\times$. 

Also if $\mathfrak{t}$ is $\tau$-stable, then, using the same argument as before, we see that $\iota$ can be made, in addition, $\tau$-stable. One can also show that, although $\mathfrak{t} = \mathfrak{p}$ is not $\tau$-stable, one can choose $\tau$-stable $\iota$, but we will not need that result.

In general, it seems that we cannot make $\iota$ both $Q$-equivariant (as in Subsection 2.1) and mapping $\mathfrak{u}$ to $U_h^{\wedge \chi}\mathfrak{t}$. Having such a property is desirable: this would imply that the functor $\bullet_4$ from Subsection 2.2 defined using $\iota$ maps HC $\mathcal{U}$-bimodules to $Q$-equivariant bimodules (a priori, we only get $R$-equivariance). However, it turns out that the bimodules in the image of $\bullet_4$ defined using $\iota$ have a $Q$-equivariant structure for our present choice of $\iota$ as well. Let us explain why this is the case. We still have an action of $\tilde{Q}$ on $\mathcal{W}_h^{\wedge \chi}$ that restricts to the $\mathbb{Z}/2\mathbb{Z} \times R \times \mathbb{K}^\times$-action coming from the splitting constructed from our present choice of $\iota$. This is because any two $\mathbb{Z}/2\mathbb{Z} \times R \times \mathbb{K}^\times$-equivariant $\iota$’s differ by an automorphism of the form $\exp(\frac{1}{\hbar}\text{ad}(a))$, where $\frac{1}{\hbar}a$ is a $\mathbb{Z}/2\mathbb{Z} \times R \times \mathbb{K}^\times$-invariant element and $a \in \tilde{I}_\chi^3$ as in Subsection 2.1. The target category for $\bullet_4$ is still HC$^Q(\mathcal{W})$ because the transformation $\exp(\frac{1}{\hbar}\text{ad}(a))$ acts on $\mathcal{M}_h^{\wedge \chi}$ for any HC bimodule $\mathcal{M}$. 


4.1.3. **Construction of $\bullet_{\tau,e}$**. Pick a module $M \in \mathcal{O}_\nu^K$. We can choose a $K$-stable increasing exhaustive filtration $F_0 M \subset F_1 M \subset \ldots$ on $M$ such that this filtration is compatible with the filtration on $U$ and $\text{gr} M$ is a finitely generated $S(\mathfrak{g})$-module. In fact, since the filtration is $K$-stable, $\text{gr} M$ is a $S(\mathfrak{g} / \mathfrak{k}) = \mathbb{K}[\mathfrak{t}^\perp]$-module. Consider the Rees $U_h$-module $M_h := \bigoplus_{i=0}^\infty F_i M h^i$.

The space $\mathbb{K}[u, h]$ has a natural structure of a $\mathbb{A}_h^\wedge$-module, $\mathbb{K}[u, h] = \mathbb{A}_h^\wedge / \mathbb{A}_h^\wedge u$ (an element $u \in u$ acts by $h^2 \partial_u$).

**Lemma 4.2.** Let $M'_h$ be the annihilator of $u$ in $M_h^\wedge$. The natural homomorphism

$$\mathbb{K}[u, h] \otimes_{\mathbb{K}[u]} M'_h \to M_h^\wedge$$

is bijective.

**Proof.** The proof is similar to that of Lemma 3.6 and to other related statements such as [Lo2, Proposition 3.3.1] but we provide a proof for reader’s convenience.

The kernel of the homomorphism is an $h$-saturated (meaning that the quotient is $\mathbb{K}[h]$-flat) $\mathbb{A}_h^\wedge$-submodule in $\mathbb{K}[u, h] \otimes_{\mathbb{K}[u]} M'_h$. Any such submodule can be shown to intersect $M'_h$ by an argument similar to [Lo1, Lemma 3.4.3]. It follows that the kernel is zero. It remains to prove that the homomorphism is surjective.

Recall that, thanks to the choice of a filtration on $M$, we have $\mathfrak{t} M_h \subset h^2 M_h$. It follows that $u M_h^\wedge \subset h^2 M_h^\wedge$.

Choose a basis $x_1, \ldots, x_m$ in $u$ and vectors $y_1, \ldots, y_m \in V$ such that $y_1, \ldots, y_m, x_1, \ldots, x_m$ form a Darboux basis of $V$. For each $i = 1, \ldots, m$ and any $v \in M_h^\wedge$, the sum $\rho_i(v) := \sum_{j=0}^{\infty} (-1)^j y_i^j x_i^j v$ converges. Moreover, this sum is annihilated by $x_i$. Also, by the construction, $v - v_0 = y_i v'$, where $v_0 := \rho_i(v)$, for some $v' \in M_h^\wedge$. We can repeat the same argument with $v'$ and get a decomposition $v - v_0 - y_i v_1 = y_i^2 v_0$. Repeating this procedure we represent $v$ as the infinite sum $\sum_{j=0}^{\infty} y_i^j v_j$ with $x_i v_j = 0$ for all $j$. The element $v_j$ in this expression has to be given by $v_j = \rho_i(\frac{y_i}{x_i x_i^j} x_i^j v)$. The operator $\rho_i$ commutes with $x_i, y_i, \rho_i$ for $i' \neq i$. So we can first decompose $v$ into the sum $\sum_{j=0}^{\infty} y_i^j v_j$, then decompose each $v_j$ into the sum $\sum_{k=0}^{\infty} y_i^k v_{jk}$ as above. But now each $v_{jk}$ is annihilated by both $x_1, x_2$. Proceeding in this way, we get a decomposition of $v$ as of an element in $\mathbb{K}[u, h] \otimes_{\mathbb{K}[u]} M'_h$. \hfill \square

With this lemma, we can construct the functor $\bullet_{\tau,e}$ completely analogously to $\bullet_\tau$. Namely, let $\mathcal{N}_h$ denote the subspace of $K^\wedge$-stable elements in $M_h$. Again, similarly to [Lo2, Proposition 3.3.1], using the fact that $W_h$ is positively graded, one shows that $\mathcal{N}_h^K = M'_h$. Then we set $M_{\tau,e} := \mathcal{N}_h / (h - 1) N_h$. This is a finitely generated $W$-module that is (canonically) independent of the choice of a good filtration on $M$ (as in the proof of a similar claim for $\bullet_\tau$ in [Lo2, 3.4]).

The group $R$ naturally acts on $M_{\tau,e}$ and this action is rational. However, the differential of the $R$-action coincides with the action of $\tau \subset W$ only up to a shift by $\nu + \delta_K$, where $\delta_K$ is half the character of $K$ on $\Lambda^\top u^\ast$, as in the end of 2.6.2, for the reason similar to that situation. In particular, when $K = P$, we have $\delta_P = \tau(\delta)$. This is because $u$ and $\bar{u}$ are complimentary lagrangian subspaces in $V$ so that the characters of the $R$-action on $\Lambda^\top u^\ast \cong \Lambda^\top \bar{u}$ coincide.

We also can consider the right-handed analog $\mathcal{O}_\nu^K_{s'}$ of $\mathcal{O}_\nu^K$ in the category of right $U$-modules. A straightforward ramification of the construction above produces a functor $\bullet_{\tau,e}'$ from $\mathcal{O}_\nu^K_{s'}$ to the category of $R$-equivariant right $W$-modules.
4.1.4. General properties of $\bullet_{t,e}$. It follows from the construction that the functor $\bullet_{t,e} : \mathcal{O}^K_\nu \to \mathcal{W}$ is exact. Here $\mathcal{W}$ is the category of finitely generated $\mathcal{W}$-modules that are $R$-equivariant after the shift of the $\tau$-action by $\nu + \delta_K$. Indeed, if we have an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of modules in $\mathcal{O}^K_\nu$, then we can choose a good filtration on $M_2$ and induce good filtrations on $M_1, M_3$. The sequence $0 \to M_{1h} \to M_{2h} \to M_{3h} \to 0$ becomes exact. We deduce the claim about exactness of $\bullet_{t,e}$ from the exactness of the completion functor.

Further, for $M \in \mathcal{O}^K_\nu$ and a Harish-Chandra bimodule $X$ we have $X \otimes U(M) \in \mathcal{O}^K_\nu$ and we have a natural isomorphism $(X \otimes U(M)_{t,e}) \sim X \otimes \mathcal{W} M_{t,e}$. This is proved by tracking the constructions of $\bullet_t$ and $\bullet_{t,e}$, compare with the proof of [Lo2, Proposition 3.4.1,(2)]. In particular, if $\mathcal{O}$ is an irreducible component of $V(U/Ann_U(M))$, then $\mathcal{M}_{t,e}$ is finite dimensional. Indeed, in this case $Ann_U(M)_t$ has finite codimension in $\mathcal{W}$, and $Ann_U(M)_t M_{t,e} = 0$.

Next, since $\mathcal{W}$ is a filtered algebra with $gr W = K[S]$, one can define the associated variety $V(N)$ of a finitely generated $\mathcal{W}$-module $N$, this will be a conical (with respect to the Kazhdan action) subvariety of $S$. Tracking the construction of $\bullet_{t,e}$, we have $V(M_{t,e}) = V(M) \cap S$.

Further, we claim that there is a functor $\bullet^{t,e} : \mathcal{W}$-mod$^R_\nu \to \mathcal{O}^K_\nu$ with the property that $Hom_U(\bullet^{t,e},?) \cong Hom_{\mathcal{W},R}(\bullet_{t,e},?)$. This functor is constructed similarly to the functor $\bullet^t$ in [Lo2, 3.3.3.4]. Namely, we pick $\mathcal{N} \in \mathcal{W}$-mod$^R_\nu$. Choose some good filtration and form the Rees module $\mathcal{N}_h$. Then set $\mathcal{N}^t := K[[\nu, \delta]] \otimes_K \mathcal{N}^\times$. Choose the maximal subspace $\mathcal{N}_h$, where $K^\times$ acts locally finitely and the action of $\delta$ shifted by $\nu$ (i.e., given by $(x, m) \mapsto \frac{1}{h} xm - (\nu, x)m$ integrates to a $K$-action. This subspace is $g$- and $R$-stable and hence we have two $R$-actions: one obtained by restricting the action from $\mathcal{N}_h$ and the other restricted from a $K$-action. Similarly to [Lo2, 3.3], the structure map $g \otimes \mathcal{N}_h \to \mathcal{N}_h$ is $R$-equivariant for both actions and the restrictions of the two $R$-actions to $R^\circ$ coincide. As in [Lo2, 3.2.3.3], we deduce that the difference of the two actions is an $R/R^\circ$-action commuting with $g$. We set $\mathcal{N}^{t,e} := \mathcal{M}^{R/R^\circ}_h/(\delta - 1)$. This is an object in $\mathcal{O}^K_\nu$. We remark that $\mathcal{N}^{t,e}$ comes equipped with a filtration (induced from the grading on $\mathcal{M}^{R/R^\circ}_h$). By the construction, for any $\mathcal{L} \in \mathcal{O}^K_\nu$ and any good filtration on $\mathcal{L}$, we have $Hom_V(R_h(\mathcal{L}_{1,e}), \mathcal{N}_h)^R \cong Hom_U(\mathcal{L}_h, M)$. (an isomorphism of $K[[\nu, \delta]]$-modules with $K^\times$-action). We have $Hom_V(\mathcal{L}_{1,e}, \mathcal{N})^R \cong Hom_V(R_h(\mathcal{L}_{1,e}), \mathcal{N}_h)^R/(\delta - 1)$ as any homomorphism $\mathcal{L}_{1,e} \to \mathcal{N}$ can be made filtration preserving after a shift of a filtration on $\mathcal{L}_{1,e}$. Here we only use that the filtration on $\mathcal{L}_{1,e}$ is good. Similarly, $Hom_U(\mathcal{L}_h, M)/(\delta - 1) \cong Hom_U(\mathcal{L}, \mathcal{N}^{t,e})$. So $\mathcal{N}^{t,e}$ comes equipped with a filtration (induced from the grading on $\mathcal{M}^{R/R^\circ}_h$).

For a conical $K$-stable subvariety $Z \subset \mathfrak{t}^\perp$ we can form the full subcategory $\mathcal{O}^K_{\nu, Z} \subset \mathcal{O}^K_\nu$ consisting of all modules $M$ with $V(M) \subset Z$.

Until the end of the subsection we make an additional assumption. Namely, we assume that $\mathfrak{t}^\perp \cap \mathfrak{O}$ has finitely many $K$-orbits. Set $Y := Ke$. This is an irreducible component of $\mathfrak{t}^\perp \cap \mathfrak{O}$ and its dimension equals $\frac{1}{2} \dim V$.

Consider the subvariety $Z := \bigcup_{\mathfrak{O}'} \mathfrak{O}^\perp \cap \mathfrak{t}^\perp$, where the union is taken over all nilpotent orbits $\mathfrak{O}'$ such that $\mathfrak{O} \not\subset \mathfrak{O}'$. We can form the subcategories $\mathcal{O}^K_{\nu, Z}, \mathcal{O}^K_{\nu, Z, Y} \subset \mathcal{O}^K_\nu$ and their quotient $\mathcal{O}^K_{\nu, Y}$. The subcategory $\mathcal{O}^K_{\nu, Z}$ is precisely the kernel of $\bullet_t$, while $\mathcal{O}^K_{\nu, Z, Y}$ is the preimage of the category of finite dimensional representations. In particular, the functor $\bullet_{t,e}$ descends to a functor from $\mathcal{O}^K_{\nu, Y}$ to the category $\mathcal{W}$-mod$^R_\nu$ of finite dimensional $\nu$-shifted $R$-equivariant $\mathcal{W}$-modules.
Moreover, the quotient $\overline{\mathcal{V}}$ separated in the $\mathcal{V}$ \(\in\mathcal{W}\)-mod. Then there is a unique maximal submodule \(M\in\mathcal{O}_\mathcal{V}\). Namely, pick \(M\in\mathcal{O}_\mathcal{V}\) such that the codimension. Then there is a unique maximal submodule $M'$ \(\subset\mathcal{M}_{t,e}\) be an $R$-stable submodule of finite codimension. Then there is a unique maximal submodule $M' \subset M$ such that $M'_{t,e} = N$. Moreover, the quotient $M/M'$ lies in $\mathcal{O}_{\mathcal{V}}$. The module $M'$ coincides with the preimage of $N'_{t,e} \subset (\mathcal{M}_{t,e})_{t,e}$ under the adjunction morphism $\mathcal{M} \to (\mathcal{M}_{t,e})_{t,e}$. The proof repeats that of [Lo2, Theorem 4.1.1].

We remark however that sometimes the codimension condition is not necessary: for $\mathcal{O}_\mathcal{V}$ the conclusions of the two previous paragraphs are still true, as we will see in Subsection 4.3.

4.2. Functor $\mathcal{V}$.

4.2.1. Three definitions. Our goal is to define a functor $\mathcal{V} : \mathcal{O}_\mathcal{V} \to \mathcal{O}_\mathcal{V}(\mathfrak{g}, e)^R$. We start by giving three different definitions. Below in this subsection we will see that all three functors are isomorphic.

We claim that $\mathcal{M}_{t,e} \in \mathcal{O}_\mathcal{V}(\mathfrak{g}, e)_{\nu}$ for $\mathcal{M} \in \mathcal{O}_\mathcal{V}$. By the construction, the action of $r \subset R$ integrates to $R$ after the $\nu+\delta$-shift. The other claims that we need to check (that the weight spaces are finite dimensional and the weights are bounded by above) will follow from similar claims about $\text{gr} \mathcal{M}_{t,e}$. In turn, those will follow if we show that the one-parametric subgroup $\mathbb{K}^\times \to Q$ with differential $\theta$ contracts $\mathcal{V}(\mathcal{M}_{t,e})$ to $e$. But $\mathcal{V}(\mathcal{M}_{t,e}) = \mathcal{V}(\mathcal{M}) \cap S$. Since $\mathcal{V}(\mathcal{M}) \subset p^+$ and $p^+ = (\mathfrak{g}_0 \cap p^+) \oplus \mathfrak{g}>0$, we only need to check that $(\mathfrak{g}_0 \cap p^+) \cap (e + \mathfrak{g}_0(f)) = \{e\})$. Since $\mathfrak{g}_0 \cap p^+ = \mathfrak{g}_0(>0)$ and $\mathfrak{g}_0(f) \subset \mathfrak{g}_0(\leq 0)$, our claim follows. So we can set $\mathcal{V}_1(\mathcal{M}) := \mathcal{M}_{t,e}$.

Let us now define $\mathcal{V}_2$. Pick $\mathcal{M} \in \mathcal{O}_\mathcal{V}$. Set $\mathcal{M}^\wedge := \varprojlim_{n \to +\infty} \mathcal{M}/\overline{m}_n^N \mathcal{M}$. We claim that $\mathcal{M}^\wedge = \prod_{\mu} \mathcal{M}_\mu^\wedge$, where $\mathfrak{m}^\wedge$ is the analog of $\mathfrak{m}^\wedge$ for $\mathfrak{g}_0$. Indeed, $\prod_{\mu} \mathcal{M}_\mu^\wedge$ is complete and separated in the $\overline{m}_\mathcal{M}$-adic topology. This gives a map $i : \mathcal{M}^\wedge \to \prod_{\mu} \mathcal{M}_\mu^\wedge$. Let us point out that the $U$-module $\prod_{\mu} \mathcal{M}_\mu$ coincides with the $\mathfrak{g}_{<0}$-adic completion on $\mathcal{M}$. So we get a map $i' : \prod_{\mu} \mathcal{M}_\mu^\wedge \to \mathcal{M}^\wedge$ that is the inverse of $i'$ because $i, i'$ are the identity on $\mathcal{M}$ that is dense in both modules.

So $\mathcal{M}^\wedge$ is an object in $\mathcal{W}_h(\mathfrak{g}, e)^R$. Therefore $\hat{\mathcal{K}}(\mathcal{M}^\wedge)_{\text{fin}} \in \hat{\mathcal{O}}^\theta(\mathfrak{g}, e)^R$. We set $\mathcal{V}_2(\mathcal{M}) := \hat{\mathcal{K}}(\mathcal{M}^\wedge)_{\text{fin}}$ so that $\mathcal{V}_2$ is a functor $\mathcal{O}_\mathcal{V} \to \hat{\mathcal{O}}^\theta(\mathfrak{g}, e)^R$. Below we will see that the image is actually in $\mathcal{O}_\mathcal{V}(\mathfrak{g}, e)^R$.

Finally, let us define a functor $\mathcal{V}_3$. Again, pick $\mathcal{M} \in \mathcal{O}_\mathcal{V}$. Consider the completion $\hat{\mathcal{M}}$. This is an object in $\hat{\mathcal{O}}^\theta$. Recall the element $n \in G$ defined in 2.6.1, it is the image of $\left(\begin{array}{c} 0 \\ i \\ 0 \end{array}\right) \in \text{SL}_2(\mathbb{K})$ under the homomorphism $\text{SL}_2(\mathbb{K}) \to G$ induced by the $\mathfrak{sl}_2$-triple $(e, h, f)$. We twist the $\mathfrak{g}$-action on $\hat{\mathcal{M}}$ with $\text{Ad}(n^{-1})$, we get an object $n^{-1}\hat{\mathcal{M}} \in \hat{\mathcal{O}}^\theta_{\nu}$. The nilradical of the parabolic subgroup $n^{-1}Pn$ coincides with $\mathfrak{m}$. For a module $\mathcal{M}' \in \hat{\mathcal{O}}^\theta_{\nu}$ we let $\mathcal{W}_h(\mathcal{M}')$ be the subspace in $\mathcal{M}'$ spanned by all $t$-weight vectors that are nilpotent for the action of $\mathfrak{m}_0^\wedge$ (and automatically nilpotent for $\mathfrak{g}_{<0}$). Hence $\mathcal{W}_h(\mathcal{M}') \in \mathcal{W}_h^\theta_{\nu}$. So $\mathcal{V}_3(\mathcal{M}) := \mathcal{K}(\mathcal{W}_h(n^{-1}\hat{\mathcal{M}}))$ is an object of $\hat{\mathcal{O}}^\theta(\mathfrak{g}, e)^R$.

Below in this subsection we will prove that the three functors $\mathcal{V}_i$ are isomorphic. The scheme of a proof is as follows. First, we show that $\mathcal{V}_1 \cong \mathcal{V}_2$, this is quite easy. After that,
it remains to prove that $\mathcal{V}_2 \cong \mathcal{V}_3$. We start proving this by showing that $\mathcal{V}_3$ is dual to $\mathcal{V}_2$, i.e. $\mathcal{V}_3(\bullet) \cong \mathcal{V}_2(\bullet^\vee)$. Next, we show that $\mathcal{V}_i(X \otimes_{\mathcal{U}} \mathcal{M}) \cong X_{\dagger} \otimes_{\mathcal{W}} \mathcal{V}_i(\mathcal{M})$ for all $i$ (a bi-functorial isomorphism). After that, we show that the functors $\mathcal{V}_i$ basically intertwine the Verma module functors. Finally, we use the Bernstein-Gelfand equivalence to establish an isomorphism $\mathcal{V}_2 \cong \mathcal{V}_3$.

### 4.2.2. Isomorphism of $\mathcal{V}_1$ and $\mathcal{V}_2$

We use the identification $\mathcal{U}^\vee \cong \mathcal{A}(\mathcal{W})$ to view $\mathcal{M}^\wedge = \mathcal{U} \otimes_{\mathcal{U}} \mathcal{M}$ as a module over $\mathcal{A}(\mathcal{W})$. This module is isomorphic to $\mathbb{K}[\hat{\mathfrak{g}}]/\mathfrak{n}$, where $\mathfrak{n}$ is a $\mathcal{W}$-module. Taking $\theta$-finite elements in $\mathfrak{n}$, we get $\mathcal{V}_2(\mathcal{M})$. Recall that by $\mathfrak{g}^+$ we mean any lagrangian subspace in $\mathcal{V}$ complimentary to $\mathfrak{g}$, in particular, we can take $\mathfrak{g}^+ := \mathcal{V}(V \cap \mathfrak{p})$.

We can define $\bullet_{\hat{\mathfrak{g}}}$ on the dehomogenized level (we still need to fix a good filtration on $\mathcal{M}$). Namely, we can set $\mathcal{M}_{\mathfrak{g}} = \mathcal{A}(\mathcal{W})_{\mathfrak{g}} \otimes_{\mathcal{A}(\mathcal{W})} (\mathbb{K}[\mathfrak{u}] \otimes \mathcal{M})$. Recall that by $\mathcal{M}_{\mathfrak{g}}$ we get a natural non-degenerate contravariant pairing between $\mathcal{M}^\vee$ and $\mathcal{M}$ and then taking $\theta$-finite vectors (in fact, the latter is not necessary). We get

$$
\mathcal{M}^\wedge = \mathcal{A}(\mathcal{W})^\wedge \otimes_{\mathcal{A}(\mathcal{W})} \mathcal{M}^\vee = \mathcal{A}(\mathcal{W})^\wedge \otimes_{\mathcal{A}(\mathcal{W})} (\mathcal{M})_{\mathfrak{g}} \otimes \mathcal{M}_{\mathfrak{g}} = \mathbb{K}[\mathfrak{u}] \otimes \mathcal{M}_{\mathfrak{g}}.
$$

So $\mathcal{N} \cong \mathcal{M}_{\mathfrak{g}}$ and hence $\mathcal{V}_2(\mathcal{M}) \cong \mathcal{M}_{\mathfrak{g}}$. Tracking the construction, we see that it is functorial.

It remains to prove that $\mathcal{V}_3 \cong \mathcal{V}_2$. The proof will occupy the rest of the subsection, its various parts will also be used later.

### 4.2.3. $\mathcal{V}_2, \mathcal{V}_3$ and duality

Here we are going to show that $\mathcal{V}_2(\bullet^\vee)^\vee \cong \mathcal{V}_3(\bullet)$. In particular, this will imply that $\mathcal{V}_3$ is exact and its image is in $\mathcal{O}_0(\mathfrak{g}, e)^R_\nu$.

We are going to show that there is a natural non-degenerate pairing between $\mathcal{V}_3(\mathcal{M})$ and $\mathcal{V}_2(\mathcal{M}^\vee)$. As we have seen, Lemma 3.4, this implies the existence of a functorial isomorphism $\mathcal{V}_3(\mathcal{M}) \cong \mathcal{V}_2(\mathcal{M}^\vee)^\vee$.

Recall that we have a $\sigma$-contravariant pairing $\mathcal{M}^\vee \times \hat{\mathcal{M}} \to \mathbb{K}$. It can be regarded a $\tau$-contravariant pairing $\mathcal{M}^\vee \times \mathcal{M}^\wedge$ that identifies $\mathcal{M}^\wedge$ with the full dual of $\mathcal{M}^\vee$. It follows that there is a $\tau$-contravariant pairing

$$
(\mathcal{M}^\vee)^\wedge \times \text{Wh}_{\nu}(\mathcal{M}^\wedge) \to \mathbb{K}
$$

that identifies $\text{Wh}_{\nu}(\mathcal{M})$ with the continuous dual of $(\mathcal{M}^\vee)^\wedge$. Recall that the identifications $\mathcal{U}^\wedge \cong \mathcal{A}(\mathcal{W})^\wedge$ and $\mathcal{U}^\vee \cong \mathcal{A}(\mathcal{W})^\vee$ are obtained from one another by a $\tau$-twist. So (4.1) gives rise to a non-degenerate (in the sense that both left and right kernels are zero) contravariant pairing between $(\mathcal{M}^\vee)^\wedge / \mathfrak{g}(\mathcal{M}^\wedge)^\wedge$ and $\text{Wh}_{\nu}(\mathcal{M}^\wedge)^\wedge$. The latter module is $\mathcal{V}_3(\mathcal{M})$. The former module is $\mathcal{V}_2(\mathcal{M}^\vee)$. So we get a natural non-degenerate contravariant pairing between $\mathcal{V}_2(\mathcal{M})$ and $\mathcal{V}_3(\mathcal{M})$. An isomorphism $\mathcal{V}_3(\bullet) \cong \mathcal{V}_2(\bullet^\vee)^\vee$ is therefore proved.
4.2.4. Products with Harish-Chandra bimodules. We claim that all three functors \( V_i \) satisfy 
\( V_i(X \otimes U, \mathcal{M}) \cong X \otimes W V_i(\mathcal{M}) \) (functorially in \( X \) and \( \mathcal{M} \)), where \( X \) is a Harish-Chandra bimodule. We have already established this property for \( V_1 \) in 4.1.4. For \( V_2 \), the property follows from the already proved isomorphism \( V_1 \cong V_2 \). So it remains to prove the functorial isomorphism for \( V_3 \).

First of all, we claim that there is a natural transformation \( X_0 \otimes W V_3(\mathcal{M}) \rightarrow V_3(X \otimes U, \mathcal{M}) \).

There is a natural homomorphism \( X \otimes U \cong X \otimes_{W} V_3(\mathcal{M}) \) extending \( X \otimes U \rightarrow X \otimes (X \otimes U, \mathcal{M}) \). This isomorphism restricts to a functorial homomorphism

\[
X \otimes U W_\nu(n^{-1}\hat{M}) \rightarrow W_\nu(n^{-1}(X \otimes U, \mathcal{M})).
\]

But according to [LO, Theorem 5.11], \( \mathcal{K}(X \otimes U, \mathcal{M}) \cong X_0 \otimes W \mathcal{K}(\mathcal{M}) \) for any \( \mathcal{M} \in \text{Wh}(\mathfrak{g}, e)_\nu \) and this homomorphism is bifunctorial. Applying this to \( \mathcal{M} := W_\nu(n^{-1}\hat{M}) \) and using (4.2), we get

\[
X_0 \otimes W V_3(\mathcal{M}) \cong X_0 \otimes W \mathcal{K}(\mathcal{M}) = \mathcal{K}(X \otimes U, \mathcal{M}) \cong \mathcal{K} \circ W_\nu(n^{-1}(X \otimes U, \mathcal{M})) \equiv V_3(X \otimes U, \mathcal{M}).
\]

The resulting homomorphism \( X_0 \otimes W V_3(\mathcal{M}) \rightarrow V_3(X \otimes U, \mathcal{M}) \) is bi-functorial and provides a natural transformation of interest.

4.2.3 shows that \( V_3 \) is exact. Thanks to the 5-lemma, it is enough to show that the natural transformation is an isomorphism for \( X := L \otimes U \), where \( L \) is a finite dimensional \( G \)-module (here \( \mathfrak{g} \) acts on the right in a naive way, while the left action is given by \( x(l \otimes m) = x.l \otimes m + l \otimes xm \)). It is clear that \( L \otimes \hat{\mathcal{M}} = L \otimes \check{\mathcal{M}}. \) Thanks to [LO, Theorem 5.11], it is enough to show that taking the tensor product with \( L \) commutes with taking \( \text{Wh}_\nu \). This easily reduces to the claim that the conditions for \( x \in L \otimes \mathcal{M} \) being a generalized eigenvector with zero eigenvalue are equivalent for the following actions:

- the diagonal \( \mathfrak{m}_\chi \)-action,
- the \( \mathfrak{m} \times \mathfrak{m}_\chi \)-action,
- the \( \mathfrak{m}_\chi \)-action on the second factor.

This observation is a formal corollary of \( \dim L < \infty \) and \( \mathfrak{m} \) acting on \( L \) by nilpotent endomorphisms. Here \( \mathcal{M} \) is an arbitrary \( \mathfrak{g} \)-module.

So we have proved that \( X_0 \otimes W V_3(\mathcal{M}) \cong V_3(X \otimes U, \mathcal{M}) \).

4.2.5. Images of induced modules. Let \( V_i^0 \) be the functor defined analogously to \( V_i \) for \( \mathfrak{g}_0 \), where \( i = 2, 3 \). Consider the parabolic category \( \mathcal{O}_\nu^0 \) for \( (\mathfrak{g}_0, P_0 := P \cap G_0) \). We have the induction functor \( \Delta^0 : \mathcal{U}^0 \text{-Mod} \rightarrow \mathcal{U} \text{-Mod}, \Delta^0(\mathcal{M}^0) := \mathcal{U} \otimes_{U(\mathfrak{g}_0)} \mathcal{M}^0 \) that maps \( \mathcal{O}_\nu^0 \) to \( \mathcal{O}_\nu^P \). Our goal now is to show that the functors \( V_i(\Delta^0(\bullet)), \Delta^0_T(V_i^0(\bullet)) : \mathcal{O}_\nu^0 \rightarrow \mathcal{O}_\theta^\mu(\mathfrak{g}, e)_\nu^R \) are isomorphic. This boils down to checking an isomorphism of bifunctors

\[
\text{Hom}_{\mathcal{O}_\theta^\mu(\mathfrak{g}, e)_\nu^R}(V_i(\Delta^0(\bullet)), ?) \cong \text{Hom}_{\mathcal{O}_\theta^\mu(\mathfrak{g}, e)_\nu^R}(\Delta^0_T(V_i^0(\bullet)), ?).
\]

Consider the case \( i = 2 \) first. We have \( V_2(\mathcal{M}) = \mathcal{K}(\mathcal{M}^\wedge)_{\text{fin}} \). Both \( \bullet_{\text{fin}} : \hat{\mathcal{O}}_\nu^0(\mathfrak{g}, e)_\nu^R \rightarrow \mathcal{O}_\theta^\mu(\mathfrak{g}, e)_\nu^R \) and \( \hat{\mathcal{K}} : \text{Wh}_\nu^0(\mathfrak{g}, e)_\nu^R \rightarrow \hat{\mathcal{O}}_\theta^\mu(\mathfrak{g}, e)_\nu^R \) are category equivalences. So, for \( \mathcal{M} \in \mathcal{O}_\nu^0, \mathcal{N} \in \mathcal{O}_\theta^\mu(\mathfrak{g}, e)_\nu^R \), we have a bifunctorial isomorphism

\[
\text{Hom}_{\mathcal{O}_\theta^\mu(\mathfrak{g}, e)_\nu^R}(V_2(\mathcal{M}), \mathcal{N}) \cong \text{Hom}_{\text{Wh}_\nu^0(\mathfrak{g}, e)_\nu^R}(\mathcal{M}^\wedge, \hat{\mathcal{K}}^{-1}(\hat{\mathcal{N}})).
\]

Clearly, \( \Delta^0(\mathcal{M}^0)^\wedge = \mathcal{U} \otimes U \Delta^0(\mathcal{M}^0) = \mathcal{U} \otimes_{U(\mathfrak{g}_0)} \mathcal{M}^0 = [\mathcal{U}/\mathcal{U}_0] \otimes_{U(\mathfrak{g}_0)} \mathcal{M}^0 \cong \Delta^0_T(V_i^0(\mathcal{M}^0)) \). Recall that \( \hat{\mathcal{K}} \) intertwines the functors \( \Delta^0_T, \hat{\mathcal{O}}_\theta^\mu, \) while \( \bullet_{\text{fin}} \) intertwines \( \Delta^0_T \).
and $\Delta^0_{\hat{W}}$. So

\begin{equation}
\text{Hom}_{\text{Wh}}(\Delta^0(\mathcal{M}^0), \hat{K}^{-1}(N)) \simeq \text{Hom}_{\text{O}}(\Delta^0_0(\mathcal{V}_2^0(\mathcal{M}^0)), N).
\end{equation}

Combining (4.3) and (4.4), we see that $\mathcal{V}_2(\Delta^0(\mathcal{M}^0)) \simeq \Delta^0_{\hat{W}}(\mathcal{V}_2^0(\mathcal{M}^0))$.

Let us proceed to $\mathcal{V}_3$. We are going to use a similar argument. We have

\begin{equation}
\text{Hom}_{\text{O}}(\mathcal{V}_3(\mathcal{M}), N) \simeq \text{Hom}_{\text{Wh}}(\mathcal{W}(n^{-1}\hat{M}), \hat{K}^{-1}(N)).
\end{equation}

We have a natural map

\begin{equation}
\Delta^0(\mathcal{W}(n^{-1}\hat{M})) \rightarrow \mathcal{W}(\Delta^0(n^{-1}\hat{M}))
\end{equation}

induced by the inclusion $\mathcal{W}(n^{-1}\hat{M}) \rightarrow \mathcal{W}(\Delta^0(n^{-1}\hat{M}))$ (that comes from the inclusion $n^{-1}\hat{M} \subset \Delta^0(n^{-1}\hat{M})$; here $\mathcal{W}(\hat{M})$ is an analog of $\mathcal{W}$ for $\mathfrak{g}_0$). Assume for a moment that (6.6) is an isomorphism. The functors $\mathcal{M} \mapsto n^{-1}\Delta^0(\mathcal{M})$, $\Delta^0(\mathcal{M}) \rightarrow \Delta^0(n^{-1}\hat{M})$ are isomorphic via $a \otimes m \mapsto \text{Ad}(n)a \otimes m$. As in the case of $\mathcal{V}_2$, $\Delta^0(\mathcal{W}(n^{-1}\hat{M})) \simeq \Delta^0_{\hat{W}}(\mathcal{V}_3^0(\mathcal{M}^0))$ and, combining (4.5) with our assumption on (4.6), we see that

\begin{equation}
\text{Hom}_{\text{O}}(\mathcal{V}_3(\Delta^0(\mathcal{M}^0)), N) \simeq \text{Hom}_{\text{O}}(\Delta^0_{\hat{W}}(\mathcal{V}_3^0(\mathcal{M}^0)), N).
\end{equation}

Hence $\mathcal{V}_3(\Delta^0(\mathcal{M}^0)) \simeq \Delta^0_{\hat{W}}(\mathcal{V}_3^0(\mathcal{M}^0))$.

So it remains to show that (4.6) is an isomorphism. We start by showing that it is injective.

The composition of (4.6) with the inclusion $\mathcal{W}(n^{-1}\hat{M}) \subset \Delta^0(\mathcal{M})$ is injective.

The resulting map $\Delta^0(\mathcal{W}(n^{-1}\hat{M})) \rightarrow \Delta^0(n^{-1}\hat{M})$ is the composition of $\Delta^0(\mathcal{W}(n^{-1}\hat{M})) \rightarrow \Delta^0(n^{-1}\hat{M})$ and $\Delta^0(\mathcal{M}) \rightarrow \Delta^0(n^{-1}\hat{M})$. Both maps are embeddings. So (4.6) is injective.

To show that (4.6) is an isomorphism it is enough to check that the characters of the images of both modules under $\mathcal{K}$ are the same. The argument above shows that the coincidence of characters is equivalent to saying that the characters of $\mathcal{V}_3(\Delta^0(\mathcal{M}^0))$ and $\Delta^0(n^{-1}\hat{M})$ are the same. Since both functors $\mathcal{V}_3 \circ \Delta^0$, $\Delta^0_{\hat{W}} \circ \mathcal{V}_3$ are exact it is enough to consider the case when $\mathcal{M}^0$ is simple. Recall that $\mathcal{V}_3(\bullet) \simeq \mathcal{V}_2(\bullet)^{\mathcal{V}_3}$, so $\mathcal{V}_3 \circ \Delta^0(\bullet) \simeq \mathcal{V}_2(\Delta^0(\bullet))^{\mathcal{V}_3}$, where $\Delta^0(\bullet) := \Delta^0(\mathcal{M})$. The classes of $\Delta^0(\mathcal{M})$, $\Delta^0(\mathcal{W})$ in the Grothendieck group of $\mathcal{O}_\nu$ coincide because the duality preserves the simples. Since $\mathcal{V}_2$ is exact, the characters of $\mathcal{V}_2 \circ \Delta^0(\mathcal{M})$ and of $\mathcal{V}_2 \circ \Delta^0(\mathcal{W})$ coincide. Also the duality $\mathcal{M}^{\mathcal{V}_3}$ for the W-algebra does not change the character, see 3.2.6. We see that the characters of $\mathcal{V}_2(\Delta^0(\mathcal{M}))$ and of $\mathcal{V}_3(\Delta^0(\mathcal{M}))$ are the same. Now we claim that the characters of $\Delta^0_{\hat{W}}(\mathcal{V}_3^0(\mathcal{M}^0))$, $\Delta^0_{\hat{W}}(\mathcal{V}_3^0(\mathcal{M}^0))$ are the same, this will finish the proof. To show the coincidence of those characters one needs to show that $\mathcal{V}_2(\mathcal{M}^{\mathcal{O}_\nu})$, $\mathcal{V}_3(\mathcal{M}^{\mathcal{O}_\nu})$ are isomorphic as $\mathcal{O}_\nu$-modules. This again follows from $\mathcal{V}_2(\mathcal{M}^{\mathcal{O}_\nu}) \simeq \mathcal{V}_3(\mathcal{M}^{\mathcal{O}_\nu})$. Indeed, by construction, $\mathcal{V}_2(\mathcal{M}^{\mathcal{O}_\nu})$ does not change the $\mathcal{O}_\nu$-module structure.

4.2.6. Isomorphism of $\mathcal{V}_2$ and $\mathcal{V}_3$. First, we will show that the images of a certain parabolic Verma module under $\mathcal{V}_2$ and $\mathcal{V}_3$ are isomorphic. Then we will use 4.2.4 and the Bernstein-Gelfand equivalence recalled in 3.1.5 to show that $\mathcal{V}_2 \simeq \mathcal{V}_3$.

The parabolic Verma module we are going to consider is $\Delta_P(\nu + \rho)$. Recall that $\mathcal{J}_{P,\nu}$ denotes its annihilator in $\mathcal{U}$.

Clearly, $\Delta_P(\nu + \rho) = \Delta^0(\Delta_P(\nu + \rho))$. Thanks to 4.2.5, it is enough to show that $\Delta^0_P(\Delta_P(\nu + \rho)) \simeq \mathcal{V}_3^0(\Delta_P(\nu + \rho))$. So it is enough to assume that $\frak{g} = \frak{g}_0$.

In this case, the dimension of $\mathcal{V}_2(\Delta_P(\nu + \rho)) \simeq \mathcal{V}_1(\Delta_P(\nu + \rho))$ equals to the multiplicity of $\Delta_P(\nu + \rho)$ on $P\mathfrak{e}$, the dense orbit of $P$ in $\mathfrak{p}^\perp$. As we have recalled in 3.1.5, this multiplicity
equals 1. Since the duality does not change the multiplicity (it fixes all simples), 4.2.3 implies that \( \dim V_3(\Delta_P(\nu + \rho)) = 1. \)

So to show the isomorphism it remains to prove that the annihilators of both modules coincide. The ideal \( J := J_{P,\nu} \) coincides with the kernel of the epimorphism \( U \to D_\nu(G/P) \), where the target algebra is the algebra of \( \nu \)-twisted differential operators on \( G/P \). So \( J \) is prime (and even completely prime) and hence primitive, and \( V(U/J) = \overline{\mathcal{O}}. \) Moreover, the element \( e \) is even and hence the morphism \( T^*(G/P) \to \overline{\mathcal{O}} \) is birational. It follows that the multiplicity of \( U/J = D_\nu(G/P) \) on \( \overline{\mathcal{O}} \) is 1. So \( J_1 \) is an ideal of codimension 1 in \( \mathcal{W}. \) It annihilates \( V_2(\Delta_P(\nu + \rho)) \) because of the isomorphism \( \mathcal{W}_1 \cong V_3. \) Let us show that \( J_1 \) annihilates \( V_3(\Delta_P(\nu + \rho)). \) We recall that \( V_3(\Delta_P(\nu + \rho)) \cong \mathcal{K}(\text{Wh}_\nu(\hat{\Delta}_P(\nu + \rho))). \) The ideal \( J \) annihilates \( n^{-1}\hat{\Delta}_P(\nu + \rho) \) and hence \( \text{Wh}_\nu(\hat{\Delta}_P(\nu + \rho)) \). Thanks to [LO, Theorem 5.11], \( V_3(\Delta_P(\nu + \rho)) \cong \mathcal{K}(\text{Wh}_\nu(\hat{\Delta}_P(\nu + \rho))) \) is annihilated by \( J_1. \)

The proof of the isomorphism \( V_3(\Delta_P(\nu + \rho)) \cong V_2(\Delta_P(\nu + \rho)) \) is now complete (for \( \mathfrak{g} = \mathfrak{g}_0 \) and hence for general \( \mathfrak{g} \), as well).

**Remark 4.3.** Let \( c_\rho \) be the left cell corresponding to the primitive ideal \( J_{P,0}. \) Since \( \text{mult}_0(U/J_{P,0}) = 1, \) formula (2.2) together with the observation that \( U/J_{P,0} \) corresponds to the triple of the form \( (x, x, \text{triv}) \), see Subsection 2.4, imply that the Lusztig subgroup \( H_{c_\rho} \) coincides with \( \overline{A}. \)

Now we are ready to complete the proof of an isomorphism \( V_2 \cong V_3. \) Recall the parabolic Bernstein-Gelfand equivalence \( X \mapsto X \otimes_U \Delta_P(\nu + \rho) : \mathcal{HC}(U)^{J_{P,\nu}} \to \mathcal{O}_\nu^P. \) Under the identification of \( \mathcal{O}^P_\nu \) with \( \mathcal{HC}(U)^{J_{P,\nu}}, \) thanks to 4.2.4, we have \( V_3(X \otimes_U \Delta_P(\nu + \rho)) \cong V_3(X \otimes_{\mathcal{W}} V_3(\Delta_P(\nu + \rho))). \) Since \( V_3(\Delta_P(\nu + \rho)) \cong V_2(\Delta_P(\nu + \rho)), \) we are finally done.

From now on, all three isomorphic functors will be denoted by \( V. \)

### 4.3. Further properties of \( V. \) Here we will show that \( V \) is a quotient onto its image and identify the modules annihilated by \( V. \) The next important property we are going to prove is that \( V \) satisfies the double centralizer property, i.e., is fully faithful on projective objects. Then we are going to establish a sufficient condition for \( V \) to be fully faithful on standardly (=parabolic Verma) filtered objects. Finally, we summarize the properties we have proved in Theorem 4.8.

#### 4.3.1. Quotient property.

**Proposition 4.4.** The following is true.

1. The modules killed by \( V \) are precisely those whose all weight spaces for \( \mathfrak{t} \) have GK dimension less than \( \dim \mathfrak{g}_0(< 0) \) (the maximal GK dimension of a module in \( \mathcal{O}_\nu^P). \)

Let \( \ker V \) denote the full subcategory of such modules.

2. \( V : \mathcal{O}^P_\nu \to \mathcal{O}_\nu^\mathfrak{g} e \) admits a right adjoint functor to be denoted by \( V^*. \)

3. \( V^* \) is a left inverse of the functor \( \mathcal{O}_\nu^\mathfrak{g} e \mathfrak{g} \mathfrak{e} \mathfrak{l} \) induced by \( V. \)

4. The image of \( V \) is closed under taking subquotients. Equivalently, if \( N \) is a subobject of \( V(M) \), then \( V(V^*(N)) = N. \)

Let \( \text{im} V \subset \mathcal{O}_\nu^\mathfrak{g} e \) be the essential image. By (4), \( \text{im} V \) is an abelian category. Now (3) implies that \( V \) induces an equivalence of \( \mathcal{O}^P_\nu / \ker V \cong \text{im} V \) of abelian categories. In other words, \( V \) is a quotient functor onto its image.

**Proof.** First, let us consider the case \( \mathfrak{g} = \mathfrak{g}_0. \)
Recall the identification \( O^\nu_\nu \cong HC(U)^J \), where we set \( J := J_{P,\nu} \), and that under this identification the functor \( V \) becomes \( X \mapsto X^\dagger \otimes_W N^r \), where \( N^r \) is a unique 1-dimensional \( \mathcal{V} \)-module annihilated by \( J_t \). The functor \( Y \mapsto Y \otimes_W N^r \) is an equivalence between the category \( HC^{Q}_{fin}(W)^J_t \) of the bimodules annihilated by \( J_t \) from the right and the category of \( Q \)-equivariant finite dimensional \( \mathcal{W} \)-modules. Under the identification \( O^\nu_\nu \cong HC(U)^J \), modules with non-maximal GK dimension in \( O^\nu_\nu \) correspond to \( HC(U)^J \cap HC_{\nu}(U) \) because this identification preserves the left annihilators. That all such modules in \( O^\nu_\nu \) are annihilated by \( V \) follows from (iii) in Subsection 2.2, that no other module gets killed follows from (iv) there. This shows (1).

Let us prove (2). This will follow if we show that the functor \( \bullet^\dagger \) from (v) in Subsection 2.2 maps \( HC^{Q}_{fin}(W)^J_t \) to \( HC(U)^J \). Let us, first, show that \( J \) coincides with the kernel \( (J_t)^{tu} \) of \( U \to (W/J_t)^{\dagger} \). For this, we note that \( J \subset (J_t)^{tu} \), \( V(U/J) = V(U/(J_t)^{tu}) = \overline{U} \) and, by (v) of Subsection 2.2, \( V((J_t)^{tu}/J) \subset \partial U \). Since \( J \) is primitive, see 3.1.5, this implies the equality \( J = (J_t)^{tu} \) thanks to [BoKr, Corollary 3.6].

So if \( B \in HC^{Q}_{fin}(W) \) is annihilated by \( J_t \) from the right, then, by the construction of \( \bullet^\dagger \) in [Lo2, 3.3.3.4], \( B^\dagger \) is annihilated by \( (J_t)^{tu} \). So \( \bullet^\dagger : HC^{Q}_{fin}(W) \to HC_{\nu}(U) \) restricts to \( HC^{Q}_{fin}(W)^J_t \to HC(U)^J \). It follows that \( V \) possesses a right adjoint functor.

(3) now follows from (v) in Subsection 2.2, while (4) follows from (vii).

Now proceed to the case of a general \( \theta \). Using the realization of \( V \) as \( V_2 \), we see that it is enough to prove the direct analogs of (1)-(4) for \( F : \mathcal{M} \mapsto \mathcal{M}^\wedge : O^\nu_\nu \to \hat{W}^\theta(\mathfrak{g}, e)^R_\nu \). As we have noted in 4.2.1, \( \mathcal{M}^\wedge = \prod_{\mu \in \mathfrak{t}} \mathcal{M}_\mu^\wedge \), where \( \mathcal{M}_\mu \) is the \( \mu \)-weight space for the action of \( \mathfrak{t} \). (1) follows now from (1) for \( V^\theta \) already established above in this proof (an object in \( O^\nu_\nu \) is annihilated by \( V^\theta \) if and only if its support does not contain \( e \); this is equivalent to the condition on the GK dimension).

Let us prove (2), i.e., that there is a right adjoint functor \( \mathcal{G} \) for \( F \). First recall that the functor \( F^0 : \mathcal{M}^0 \mapsto \mathcal{M}^0_{\mathcal{U}^0} \) has a right adjoint functor because it becomes \( V^0 \) under a suitable category equivalence. The right adjoint functor is realized as follows. For \( \mathcal{N}^0 \in \hat{W}^\theta(\mathfrak{g}_0, e)^R_\nu \) let \( \mathcal{G}^0(\mathcal{N}^0) \) be the sum of all submodules of \( \mathcal{N}^0 \) belonging to \( O^\nu_\nu \). Similarly to 4.1.4, on \( \mathcal{G}^0(\mathcal{N}^0) \) we have two \( R \)-actions, one restricted from \( \mathcal{N}^0 \) and one coming from the \( P_0 \)-action. They agree on \( R^e \) and their difference is an \( R/R^e \)-action commuting with \( \mathfrak{g}_0 \). Let \( \mathcal{G}^0(\mathcal{N}^0) \) denote the subspace of \( R/R^e \)-invariants in \( \mathcal{G}^0(\mathcal{N}^0) \). From the construction,

\[
\text{Hom}_{O^\nu_\nu}(\mathcal{M}^0, \mathcal{G}^0(\mathcal{N}^0)) \cong \text{Hom}_{O^\nu_\nu}(\mathcal{M}^0, \mathcal{N}^0).
\]

So \( \mathcal{G}^0 \) is indeed a right adjoint functor to \( F^0 \).

Now take \( \mathcal{N}^r \in \hat{W}^\theta(\mathfrak{g}, e)^R_\nu \). Then its weight subspaces \( \mathcal{N}_\mu^r \) are objects of \( \hat{W}^\theta(\mathfrak{g}, e)^R_\nu \). Clearly, \( \bigoplus \mathcal{N}_\mu^r \subset \mathcal{N}^r \) is \( \mathfrak{g} \)-stable. Further, it is easy to see that \( \bigoplus \mathcal{G}^0(\mathcal{N}_\mu^r) \) is a \( \mathfrak{g} \)-submodule. The action map \( \mathfrak{g} \times \bigoplus \mathcal{G}^0(\mathcal{N}_\mu^r) \to \bigoplus \mathcal{G}^0(\mathcal{N}_\mu^r) \) is equivariant with respect to both \( R \)-actions. So \( \mathcal{G}(\mathcal{N}) := \bigoplus \mathcal{G}(\mathcal{N}_\mu^r) \) is a \( \mathfrak{g} \)-submodule in \( \mathcal{N} \). It follows easily from the construction that \( \text{Hom}_{\mathcal{U}^0}(\mathcal{M}^\wedge, \mathcal{N}^r)^R_\nu \cong \text{Hom}_{\mathcal{U}^0}(\mathcal{M}, \mathcal{G}(\mathcal{N})) \). Since all \( \mathcal{U}^0 \)-modules \( \mathcal{G}(\mathcal{N})_\mu = \mathcal{G}^0(\mathcal{N}_\mu^r) \) are in \( O^\nu_\nu \) (because \( \mathcal{G}^0 \) is right adjoint to \( F^0 \)), all \( \mathfrak{h} \)-weight spaces in \( \mathcal{G}(\mathcal{N}) \) are finite dimensional. Since the center of \( \mathcal{U} \) acts on \( \mathcal{N} \) with finitely many many eigen-characters, the same is true also for \( \mathcal{G}(\mathcal{N}) \). Therefore \( \mathcal{G}(\mathcal{N}) \in O^\nu_\nu \). So \( \mathcal{G} \) is a right adjoint functor for \( F \) and (2) is proved.

Let us prove (3) that amounts to showing that the kernel and the cokernel of the adjunction homomorphism \( \mathcal{M} \mapsto \mathcal{G} \circ F(\mathcal{M}) \) lie in the kernel of \( F \). But this homomorphism has the
form $\bigoplus_{\mu} \mathcal{M}_{\mu} \to \bigoplus_{\mu} \mathcal{G}^0(\mathcal{F}^0(\mathcal{M}_{\mu}))$, where the maps $\mathcal{M}_{\mu} \to \mathcal{G}^0(\mathcal{F}^0(\mathcal{M}_{\mu}))$ are the adjunction maps. Our claim follows from (3) in the case $\mathfrak{g} = \mathfrak{g}_0$, that was proved above.

Let us prove that the image of $\mathcal{F}$ is closed under taking subquotients (equivalently, taking subobjects). Namely, let us take a subobject $\mathcal{N}' \subset \mathcal{N}$ (in the category $\hat{\mathcal{W}}_{\theta}(\mathfrak{g}, e)^{\mathfrak{g}_0}$). It has the form $\prod_{\mu \in \tau} \mathcal{N}'_{\mu}$. So $\mathcal{G}(\mathcal{N}') = \bigoplus_{\mu} \mathcal{G}^0(\mathcal{N}'_{\mu})$ and $\mathcal{F}(\mathcal{G}(\mathcal{N}')) = \prod_{\mu} \mathcal{F}^0(\mathcal{G}^0(\mathcal{N}'_{\mu}))$. But $\mathcal{F}^0(\mathcal{G}^0(\mathcal{N}'_{\mu})) = \mathcal{N}'_{\mu}$ by (4) in the case $\mathfrak{g} = \mathfrak{g}_0$. 

4.3.2. $\mathcal{V}$ vs homological duality. Recall that we have a derived equivalence $\mathcal{D}_{\mathcal{U}} : D^b(\mathcal{O}^R_{\nu}) \to D^b(\mathcal{O}^R_{-\nu})^{\mathfrak{g}_0}$, where $\mathcal{O}^R_{-\nu}$ was defined in 3.1.3, given by

$$\mathcal{D}_{\mathcal{U}}(\bullet) := \text{RHom}(\bullet, \mathcal{U})[\text{dim } \mathfrak{p}].$$

The proof can be found, for example, in [GGOR, 4.1], the techniques there can be applied to our case as explained in [GGOR, footnote 1].

Clearly, if $\mathcal{N}$ is a finitely generated $R$-equivariant left $\mathcal{W}$-module, then $\text{Hom}_{\mathcal{W}}(\mathcal{N}, \mathcal{W})$ is also finitely generated and $R$-equivariant (as a right $\mathcal{W}$-module). Therefore we can form a homological duality functor for the $\mathcal{W}$-algebra, $\mathcal{D}_{\mathcal{W}} : D^b(\mathcal{W} \text{-mod}^R_{\nu}) \to D^b(\mathcal{W}^{opp} \text{-mod}^R_{-\nu})^{opp}$, $\mathcal{D}_{\mathcal{W}}(\bullet) = \text{RHom}(\bullet, \mathcal{W})[\text{dim } \mathfrak{g}_0(e)_{\geq 0}]$.

We can define the analog of $\mathcal{V}_1(\bullet) = \bullet_{t,e}$ for the categories of right modules completely analogously to the above, see 4.1.3. This functor will be denoted by $\mathcal{V}^R$.

Then we have the following statement.

**Proposition 4.5.** The functors $\mathcal{V}^R(H^i(\mathcal{D}_{\mathcal{U}}(\bullet))), H^i(\mathcal{D}_{\mathcal{W}}(\mathcal{V}(\bullet))) : \mathcal{O}^R_{\nu} \to (\mathcal{W}^{opp} \text{-mod}^R_{-\nu})^{opp}$ are isomorphic.

In fact, the functors $\mathcal{V}^R(\mathcal{D}_{\mathcal{U}}(\bullet))$ and $\mathcal{D}_{\mathcal{W}}(\mathcal{V}(\bullet))$ (from $D^b(\mathcal{O}^R_{\nu})$ to the $(R, -\nu)$-equivariant derived category of $\mathcal{W}^{opp}$) are isomorphic but we do not want to provide a proof of this because we will not need this fact. To prove the isomorphism one needs a derived version of $\bullet_{t,e}$, see [BL, Remark 5.12].

**Proof.** Pick $\mathcal{M} \in \mathcal{O}^R_{\nu}$ and fix a good $G_0$-stable filtration of $\mathcal{M}$ so that $\text{gr } \mathcal{M}$ is a $G_0$-equivariant finitely generated $S(\mathfrak{g})$-module. Then we can pick a graded free $G_0$-equivariant resolution $\ldots \to A^1 \to A^0 \to \text{gr } \mathcal{M}$ and lift it to a free $G_0$-equivariant resolution $\ldots \to A^1 \to A^0 \to \mathcal{M}$ such that $A^i$ is the sum of several copies of $U$ each equipped with a shift of the PBW filtration and all differentials are strictly compatible with filtrations. Then we get a graded resolution $\ldots A^1_h \to A^0_h \to M_h$. Let us tensor the resolution with $\mathcal{U}^\wedge_x$. Since $\mathcal{U}^\wedge_x$ is a flat $\mathcal{U}_h$-module, see [Lo2, 2.4], we get an $R$-equivariant resolution of $\mathcal{M}^\wedge_x$:

$$\ldots A^1_h^{\wedge_x} \to A^0_h^{\wedge_x} \to M_h^{\wedge_x}.$$

There is also another way to obtain a free resolution of $\mathcal{M}^\wedge_x = \mathbb{K}[\mathfrak{u}, h] \hat{\otimes}_{\mathbb{K}[\mathfrak{h}]} \mathcal{N}_h^{\wedge_x}$, where $\mathcal{N}_h$ is the Rees module of $\mathcal{V}(\mathcal{M})$ (with the filtration induced from $\mathcal{M}$). We can produce an $R$-equivariant graded free resolution $\ldots \to A^0_h \to N_h$ as before and then complete at $x$. Also we can take the Koszul resolution for the $A_h^{\wedge_0}$-module $\mathbb{K}[\mathfrak{u}, h] = A_h^{\wedge_0}/A_h^{\wedge_0}u$. This resolution is obtained by taking the Koszul resolution $\ldots \to \mathbb{K}[\mathfrak{u}] \otimes \mathbb{K}[\mathfrak{u}] \to \mathbb{K}[\mathfrak{u}] \otimes \mathbb{K}[\mathfrak{u}] \to \mathbb{K}[\mathfrak{u}]$ and applying the functor $A_h^{\wedge_0} \otimes_{\mathbb{K}[\mathfrak{u}]} \bullet$ to it. Multiplying the resolutions of $\mathbb{K}[\mathfrak{u}, h]$ and $\mathcal{N}_h^{\wedge_x}$, we get another resolution of $\mathcal{M}_h^{\wedge_x} = \mathbb{K}[\mathfrak{u}, h] \hat{\otimes}_{\mathbb{K}[\mathfrak{h}]} \mathcal{N}_h^{\wedge_x}$. 
We are going to show that the cohomology of the dual of the first resolution produces
\(H^i(\mathbb{V}(\mathcal{D}_\mathcal{U}(\mathcal{M})))\) (in the same way as was used in 4.1.3 to pass from \(\mathcal{U}_h^\times\)-modules to \(\mathcal{W}\)-modules: we need to, first, factor out \(\mathbb{K}[u,h]\), take \(\mathbb{K}^\times\)-finite elements, then take \(R/R^\circ\)-invariants, and then mod out \(h-1\)}, while the cohomology of the dual of the second resolution similarly produce \(H^i(\mathcal{D}_\mathcal{V}(\mathbb{V}(\mathcal{M})))\).

Consider the resolution \(\ldots \to \mathcal{A}_h^2 \xrightarrow{d_1} \mathcal{A}_h^1 \xrightarrow{d_0} \mathcal{A}_h^0 \to \mathcal{A}_h^0\) The complex
\[(4.7) \quad \mathcal{A}_h^0 \xrightarrow{d_0} \mathcal{A}_h^1 \xrightarrow{d_1} \mathcal{A}_h^2 \ldots \]
of right modules computes \(\text{RHom}(\mathcal{M}_h, \mathcal{U}_h)\). The complex \(\mathcal{A}_h^{0,\lambda} \xrightarrow{d_0} \mathcal{A}_h^{1,\lambda} \xrightarrow{d_1} \mathcal{A}_h^{2,\lambda} \ldots \) computes \(\text{RHom}(\mathcal{M}_h^{\lambda}, \mathcal{U}_h^{\lambda})\). Its cohomology is obtained by completing the cohomology of \((4.7)\). This means that the cohomology of \(\mathbb{V}(\mathcal{D}_\mathcal{U}(\mathcal{M}))\) is obtained from the first resolution in the way explained in the previous paragraph. The proof for the second resolution is similar.

The isomorphisms of the two \(H^i\)'s we have constructed do not depend on the choice of a filtration on \(\mathcal{M}\) for the same reason as for \(\bullet_{t,e}\) to be a functor. Also the construction is functorial in \(\mathcal{M}\) for the standard homological algebra reasons (two free resolutions are homotopic to each other).

\(\square\)

4.3.3. **Double centralizer property.** Our goal here is to prove the double centralizer property: that \(\text{Hom}_{\mathcal{O}_\nu}((\mathcal{P}_1, \mathcal{P}_2)) \cong \text{Hom}_{\mathcal{O}_{\nu}(\mathit{\mathcal{g}},\mathit{\mathcal{e}})^{\mathbb{R}}}(\mathbb{V}(\mathcal{P}_1), \mathbb{V}(\mathcal{P}_2))\) for any projective objects \(\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{O}_\nu\).

Our proof closely follows that of [GGOR, Theorem 5.16].

The socle of \(\Delta_{\mathcal{P}}(\lambda)\) has maximal GK dimension and hence is not annihilated by \(\mathbb{V} = \bullet_{t,e}\). It follows that the natural homomorphism \(\mathcal{M} \to \mathbb{V}^*(\mathbb{V}(\mathcal{M}))\) is injective for any standardly filtered (=admitting a filtration such that the subsequent quotients are parabolic Verma modules) module \(\mathcal{M}\). Similarly, a costandard object (=dual Verma) has no simple quotients annihilated by \(\bullet_{t,e}\). Since the quotient \(\mathbb{V}^*(\mathbb{V}(\mathcal{M}))/\mathcal{M}\) is annihilated by \(\mathbb{V}\) for any \(\mathcal{M}\), we see that the natural morphism \(\text{Hom}_{\mathcal{O}_\nu}(\mathcal{M}_1, \mathcal{M}_2) \to \text{Hom}_{\mathcal{O}_{\nu}(\mathit{\mathcal{g}},\mathit{\mathcal{e}})^{\mathbb{R}}}(\mathbb{V}(\mathcal{M}_1), \mathbb{V}(\mathcal{M}_2))\) is an isomorphism provided \(\mathcal{M}_2\) is standardly filtered, while \(\mathcal{M}_1\) is costandardly filtered.

Similar results hold for \(\mathcal{O}_{\nu,r}^{\mathcal{P}}\) and the functor \(\mathbb{V}^*\). Pick a projective \(\mathcal{P} \in \mathcal{O}_{\nu}\). Then \(\mathcal{D}_\mathcal{U}(\mathcal{P})\) is a tilting object in \((\mathcal{O}_{\nu,r}^{\mathcal{P}})^{\mathbb{R}}\) (see [GGOR, Proposition 4.2]) or, equivalently, in \(\mathcal{O}_{\nu,r}^{\mathcal{P}}\). In particular, \(\mathcal{D}_\mathcal{U}(\mathcal{P})\) is a standardly filtered object in \(\mathcal{O}_{\nu,r}^{\mathcal{P}}\). Also if \(\mathcal{M}\) is a standardly filtered object in \(\mathcal{O}_{\nu}^{\mathcal{P}}\), then \(\mathcal{D}_\mathcal{U}(\mathcal{M})\) is a standardly filtered object in \((\mathcal{O}_{\nu,r}^{\mathcal{P}})^{\mathbb{R}}\) (again, see [GGOR, Proposition 4.2]) and so a costandardly filtered object in \(\mathcal{O}_{\nu,r}^{\mathcal{P}}\). So

\[\text{Hom}_{\mathcal{O}_{\nu,r}^{\mathcal{P}}}(\mathcal{M}, \mathcal{P}) \cong \text{Hom}_{\mathcal{O}_{\nu,r}^{\mathcal{P}}}(\mathcal{D}_\mathcal{U}(\mathcal{P}), \mathcal{D}_\mathcal{U}(\mathcal{M})) \cong \text{Hom}_{\mathcal{W}_{\mathbb{R},\mathcal{P}}^{\mathbb{R}}}(\mathbb{V} \circ \mathcal{D}_\mathcal{U}(\mathcal{P}), \mathbb{V} \circ \mathcal{D}_\mathcal{U}(\mathcal{M})).\]

As we have seen, \(H^i \circ \mathbb{V} \circ \mathcal{D}_\mathcal{U} \cong H^i \circ \mathcal{D}_\mathcal{W} \circ \mathbb{V}\). In particular, \(\mathcal{D}_\mathcal{W} \circ \mathbb{V}(\mathcal{P}), \mathcal{D}_\mathcal{W} \circ \mathbb{V}(\mathcal{M})\) are objects of \(\mathbb{W}_{\mathbb{R},\mathcal{P}}^{\mathbb{R}}\) isomorphic to \(\mathbb{V} \circ \mathcal{D}_\mathcal{U}(\mathcal{P}), \mathbb{V} \circ \mathcal{D}_\mathcal{U}(\mathcal{M})\). It follows that

\[\text{Hom}_{\mathcal{W}_{\mathbb{R},\mathcal{P}}^{\mathbb{R}}}(\mathbb{V} \circ \mathcal{D}_\mathcal{U}(\mathcal{P}), \mathbb{V} \circ \mathcal{D}_\mathcal{U}(\mathcal{M})) \cong \text{Hom}_{\mathcal{W}_{\mathbb{R},\mathcal{P}}^{\mathbb{R}}}(\mathcal{D}_\mathcal{W} \circ \mathbb{V}(\mathcal{P}), \mathcal{D}_\mathcal{W} \circ \mathbb{V}(\mathcal{M})).\]

But the right hand side is just \(\text{Hom}_{\mathbb{W}_{\mathbb{R},\mathcal{P}}^{\mathbb{R}}}(\mathbb{V}(\mathcal{M}), \mathbb{V}(\mathcal{P})) = \text{Hom}_{\mathcal{O}_{\nu}(\mathit{\mathcal{g}},\mathit{\mathcal{e}})^{\mathbb{R}}}(\mathbb{V}(\mathcal{M}), \mathbb{V}(\mathcal{P}))\). From this argument we deduce that the spaces \(\text{Hom}_{\mathcal{O}_{\nu,r}^{\mathcal{P}}}(\mathcal{M}, \mathcal{P})\) and \(\text{Hom}_{\mathcal{O}_{\nu}(\mathit{\mathcal{g}},\mathit{\mathcal{e}})^{\mathbb{R}}}(\mathbb{V}(\mathcal{M}), \mathbb{V}(\mathcal{P}))\), at least, have the same dimension. Since \(\mathcal{M}\) is standardly filtered, the former Hom is included into the latter, so they coincide. For \(\mathcal{M}\) we can take another projective, since any projective in \(\mathcal{O}_{\nu,r}^{\mathcal{P}}\) is standardly filtered.
4.3.4. 0-faithfulness. Our goal here is to prove that under a certain assumption on \( P \) the functor \( V \) satisfies an even stronger property that the double centralizer one, namely that \( V \) is 0-faithful meaning that \( \text{Hom}_{\mathcal{O}_P}(M_1, M_2) \cong \text{Hom}_{\mathcal{O}_P}(V(M_1), V(M_2)) \) for any standardly filtered objects \( M_1, M_2 \). The condition on \( P \) is that the complement to the open orbit of \( P_0 \) in \( g_0(>0) \) has codimension in \( (g_0(>0)) \) bigger than 1. We will elaborate on this condition in Subsection 6.2.

**Proposition 4.6.** Suppose that \( P \) satisfies the condition of the previous paragraph. Then the functor \( V \) is 0-faithful.

**Proof.** The 0-faithfulness condition is equivalent to \( G \circ F(M) \cong M \) for any standardly filtered \( M \in \mathcal{O}_P \), where \( F \) is the completion functor \( M \mapsto M^\wedge \) and \( G \) is the right adjoint constructed in the proof of Proposition 4.4.

First, we reduce to the \( g = g_0 \) case. Recall that \( F(M) = \prod_{\mu \in \mathfrak{t}^*} F^0(M_\mu), G(N) = \bigoplus_\mu G^0(N_\mu) \) and therefore

\[ G(F(M)) = \bigoplus_\mu G^0(F^0(M_\mu)). \tag{4.8} \]

So it is enough to check that \( G^0(F^0(M_\mu)) \cong M_\mu \) for any \( \mathfrak{t} \)-weight space \( M_\mu \) of \( M \).

We claim that if \( M \) is standardly filtered (in \( \mathcal{O}_\nu \)), then any \( \mathfrak{t} \)-weight space \( M_\mu \) is standardly filtered in \( \mathcal{O}_\nu^{\text{sl}} \). It is enough to check this when \( M \) is a parabolic Verma module. We have \( \Delta_P(\mu) = \Delta^0(\Delta_P(\mu)) \). So as a \( \mathfrak{g}_o \)-module, \( \Delta_P(\mu) = U(g_{<0}) \otimes \Delta_P(\mu) \). So any \( \mathfrak{t} \)-weight space of \( \Delta_P(\mu) \) is the tensor product of \( \Delta_P(\mu) \) with some finite dimensional \( G_0 \)-module and hence is standardly filtered.

The previous paragraph together with (4.8) reduces the proposition to the case when \( g_0 = \mathfrak{g} \) (and \( \mathfrak{e} \) is even). We consider that case until the end of the proof. We remark that, thanks the 5-lemma, in order to check the isomorphism \( M \cong G(F(M)) \) for any standardly filtered module \( M \), it suffices to consider the case when \( M \) is a parabolic Verma module.

Let us equip \( M = \Delta_P(\lambda) = U \otimes_{U(\mathfrak{g})} L_{00}(\lambda) \) with a good filtration induced from the trivial filtration on \( L_{00}(\lambda) \). Using the realization of \( V \) as \( \bullet_{\mathfrak{t}, \mathfrak{e}} \) and \( V^* \) as \( \bullet_{\mathfrak{t}, \mathfrak{e}}^* \), we see that it is enough to prove \( G_h(M^{\wedge x}) h = M_\mu \), where \( G_h \) is the composition of

- the functor \( \bullet_{\mathfrak{t}, \mathfrak{e}}^{\text{fin}} \) of taking the maximal subspace where the \( \nu \)-shifted \( \mathfrak{p} \)-action integrates to \( P \) and the Kazhdan action of \( K^\wedge \) is locally finite
- and the functor of taking \( R/R^\mathfrak{e} \)-invariants.

We remark that the orbit \( Pe \) is simply connected because it is the complement of a codimension 2 subvariety in an affine space. On the other hand, we have a covering \( P/Z_P(e)^{\wedge} \to Pe \) so \( Z_P(e)^{\wedge} = Z_P(e) \) and \( R/R^\mathfrak{e} \) is trivial. So \( G_h(\bullet) = \bullet_{\mathfrak{t}, \mathfrak{e}}^{\text{fin}} \). To show that \( M_\mu = (M^{\wedge x})_{\mathfrak{t}, \mathfrak{e}}^{\text{fin}} \) it is enough to check (compare to [Lo2, 3.3]) the analogous equality modulo \( h \): i.e., that \( M = (M^{\wedge x})_{\mathfrak{t}, \mathfrak{e}}^{\text{fin}} \), where \( M := M_h(h) \). Thanks to our choice of a good filtration on \( M \), we see that \( M \) is a free \( K[g(>0)] \)-module, say \( K[g(>0)]^{\oplus n} \). Then we can apply results of [Lo2, 3.2] to see that \( (M^{\wedge x})_{P, \mathfrak{t}, \mathfrak{e}}^{\text{fin}} = K[P e]^{\oplus n} \). The codimension condition implies that \( K[P e] = K[g(>0)] \) which yields \( M = (M^{\wedge x})_{\mathfrak{t}, \mathfrak{e}}^{\text{fin}} \).

**Remark 4.7.** The proof of the double centralizer property can also be deduced from results of Stroppel, [St1],[St2, Theorem 10.1]. Namely, her results imply that \( V^0 \) (that is a quotient functor killing all simples with non-maximal GK dimension) has double centralizer property. We are going to deal with integral categories. Similarly to the proof of Proposition 4.6, it is enough to check that if \( P \) is projective in \( \mathcal{O}_P \), then all weight spaces \( P_\mu \) are projectives.
in $O_p^*$. If $P$ is a dominant Verma, then this is checked similarly to the parallel part of the proof of Proposition 4.6. In general, $P$ is a direct summand in the tensor product of a dominant Verma and a finite dimensional $g$-module. The claim that all $P_\mu$ are projective easily follows.

4.3.5. Summary of properties. Here is the summary of our results describing the properties of the functor $V$.

**Theorem 4.8.** There is an exact functor $V : O_p^* \to O^\theta(g,e)^R$ with the following properties:

(i) The essential image of $V$ is closed under taking subquotients and $V$ is a quotient onto its image. The functor $V$ annihilates precisely the modules whose all $t$-weight spaces have $GK$ dimension less than $\dim g_0(<0)$.

(ii) The functor $V$ intertwines the products with HC bimodules: $V(X \otimes_U M) \cong X_1 \otimes_W V(M)$.

(iii) Let $\Delta^0$ be the induction functor $U(g) \otimes_{U(g,g)} \bullet$. Then $V(\Delta^0(M^0)) \cong \Delta^0_V(V^0(M^0))$ for $M^0 \in O_p^*$. Here $V^0 : O_p^* \to O(g_0,e)^R$ is an analog of $V$ for $(g_0,e)$.

(iv) The dimension of $V^0(M^0)$ coincides with the multiplicity of $M_0$ in $P_0$. In particular, the character of $V(\Delta_p(\mu))$ equals $\dim L_0(\mu) e^{\mu} \prod_{i=1}^k (1-e^{\mu_i})^{-1}$. Here $\mu_1, \ldots, \mu_k$ are the weights of $t$ on $g_0(<0)$.

(v) $V$ commutes with the naive duality: $V(M^\vee) \cong (V(M))^\vee$.

(vi) The image of the simple $L(\mu) \in O_p^*$ under $V$ equals $L^0_V(V^0(L_0(\mu)))$. In the case when $\nu = 0$, the module $V^0(L_0(\mu))$ is computed as follows. Let $w$ be the element of the Weyl group $W_0$ corresponding to $\mu$. To $w$ we can assign the subgroup $H_0$ in the Lustzig quotient $A_0$ associated to $(g_0,e)$ and also an irreducible $H_0$-module $V$. Then $V^0(L_0(\mu))$ is the homogeneous bundle over $A_0/H_0$ with fiber $V \otimes N^0$, where $N^0$ is the irreducible $W^0$-module corresponding to the point $H_0 \in A_0/H_0$.

(vii) The functor $V$ has double centralizer property, i.e., is fully faithful on the projective objects.

(viii) Assume that the codimension of $g_0(>0) \setminus P_0 e$ in $g_0(>0)$ is bigger than 1. Then $V$ is 0-faithful.

Everything but (vi) has already been proved. Let us prove (vi). The description of $V^0(L_0(\mu))$ follows from [LO, Remark 7.7], (ii), the Bernstein-Gelfand equivalence, and the special case of (iii) describing $V^0(\Delta_{P_0}(\rho))$ from the proof of Proposition 4.4. Namely, let $X^0$ be the Harish-Chandra $U^0$-bimodule corresponding to $L_0(\mu)$ under the parabolic Bernstein-Gelfand equivalence from 3.1.5. We have $V^0(L_0(\mu)) \cong X^0 \otimes V^0(\Delta_{P_0}(\rho))$ by 4.2.4. But $V^0(\Delta_{P_0}(\rho))$ is the one-dimensional module $W^0/(J_{P_0})_{(\rho)}$ as we have seen in 4.2.6. The object $X^0$ is annihilated by $J_{P_0}$ from the right and so $V^0(L_0(\mu)) \cong X^0_{(\rho)}$ as a left $W^0$-module. Now we are in position to use [LO, Remark 7.7] and get the description of $V^0(L_0(\mu))$ in (vi).

To get the description of $V(L(\mu))$ one can argue as follows. The module $V(L(\mu))$ is a quotient of $V(\Delta(L(\mu))) \cong \Delta_V^0(V^0(L_0(\mu)))$. The object $V(L(\mu)) \in O^\theta(g,e)^R$ is simple thanks to (i). But $V^0(L_0(\mu))$ is a simple object in $O^\theta(g_0,e)^R$. So the object $\Delta^0_V(V^0(L_0(\mu))) \in O^\theta(g,e)^R$ has simple head, $L^0_V(V^0(L_0(\mu)))$, and therefore this head has to be isomorphic to $V(L(\mu))$.

We recall that it is parts (iv) and (vi) that allow us to compute the characters of the modules $L^0_V(N_0) \in O^\theta(g_0,e)$. We have mentioned already that these characters do not depend on the choice of $N_0$ in the $A_0(e) = R/R^e$-orbit. This is because the $A_0(e)$-action on
the classes of irreducibles in $\mathcal{O}^\theta(\mathfrak{g}_0, e)_\nu$ does not change the character and the functor $L^\theta_{\mathcal{W}}$ intertwines the $A_0(e)$-actions on the classes of simples in $\mathcal{O}^\theta(\mathfrak{g}_0, e)_\nu$ and in $\mathcal{O}^\theta(\mathfrak{g}, e)_\nu$.

Let us now summarize how the computation of the character $\text{ch} L^\theta_{\mathcal{W}}(N^0)$ works, step by step.

1) Let $W_0$ be the Weyl group of $\mathfrak{g}_0$. Let $\varrho_0$ be the $\mathfrak{g}_0$-dominant element representing the central character of $N^0$ and let $c^0_\varrho$ be the left cell in $W_0$ corresponding to $N^0$, see Subsection 2.3. The choice of a parabolic subalgebra $p_0 = \bigoplus_{i \geq 0} \mathfrak{g}_0(i)$ defines a right cell $c^l_\varrho$, see 3.1.5. Pick $w \in c^l_\varrho \cap c^0_\varrho$.

2) Decompose the class of $L(w\varrho_0)$ in $K_0(\mathcal{O}^p_\mathcal{W})$ via the classes of parabolic Verma modules:

$$L(w\varrho_0) = \sum_u c_{wu}^\Delta P(u\varrho_0).$$

(4.9)

Here $u$ runs over all elements in $W/W_{\varrho_0}$ such that $u\varrho_0$ is strictly dominant for $\mathfrak{g}_0(0)$. The numbers $c_{wu}$ are the parabolic Kazhdan-Lusztig coefficients.

3) Let $A_0$ be the Lusztig quotient for the two-sided cell in $W_0$ containing $c^l_\varrho, c^0_\varrho$ and $H_0$ be the subgroup of $A_0$ corresponding to the left cell $c^l_\varrho$. Then to $w$ viewed as an element in $W_0$ we can assign a pair $(x, \mathcal{V})$ of a point $x \in A_0/H_0$ and an irreducible $H_0$-module $\mathcal{V}$, see Subsection 2.4 (there we were dealing with triples $(x, y, \mathcal{V})$ but in the present situation $y$ is uniquely determined, see Remark 4.3).

4) Let $t$ stand for the center of $\mathfrak{g}_0$. Let $\mu_1, \ldots, \mu_k \in t^*$ denote the weights of $t$ in $\mathfrak{g}_{<0} \cap \mathfrak{z}_\mathfrak{g}(e)$ (counted with multiplicities). Then we have the following formula for $\text{ch} L^\theta_{\mathcal{W}}(N^0)$

$$\text{ch} L^\theta_{\mathcal{W}}(N^0) = \frac{|H_0|}{|A_0| \dim \mathcal{V}} \left( \sum_u c_{wu} e^{u\varrho_0 - \rho} \dim L_{00}(u\varrho_0) \right) \prod_{i=1}^k (1 - e^{\mu_i})^{-1}.$$ 

Here the range of summation is the same as in (4.9) and $L_{00}(u\varrho_0)$ is the irreducible finite dimensional $\mathfrak{g}_0(0)$-module with highest weight $u\varrho_0 - \rho$.

We remark that the module $L^\theta_{\mathcal{W}}(N^0)$ is finite dimensional if and only if $w\varrho_0$ has the form $w'\varrho$ for $w'$ lying in the two-sided cell corresponding to $\varnothing$ and compatible with a dominant weight $\varrho$.

5. Goldie ranks

In this section we fix a special orbit $\emptyset \subset \mathfrak{g}$ and take the $W$-algebra $\mathcal{W}$ for that orbit. Let $c$ be the two-sided cell corresponding to $\emptyset$. By $c_w$ we denote the left cell in $c$ containing an element $w$.

5.1. Reminders on Goldie ranks. In this subsection we will recall a few classical facts about Goldie ranks proved by Joseph.

First of all, following Joseph, we will consider some new algebras. Namely, consider the algebra $L(L(w\lambda), L(w\lambda))$ of $\mathfrak{g}$-finite linear endomorphisms of $L(w\lambda)$. It is known, see [Jo2, 2.5] that this algebra is prime and noetherian, so has Goldie rank. For a dominant integral weight $\lambda$ and compatible $w \in W$, let $g_w(\lambda)$ denote the Goldie rank of $L(L(w\lambda), L(w\lambda))$.

Each left cell $c$ has a unique distinguished involution $d_c$ called a Duflo involution. As Joseph proved in [Jo2, 3.4], for each dominant $\lambda$ compatible with $d_c$, we have

$$(5.1) \quad \text{Grk}(\mathcal{U}/J(d_c\lambda)) = g_{dc}(\lambda).$$

So the collection $g_w(\lambda)$ contains all Goldie ranks that we have originally wanted to compute.
According to [Jo2, Corollary 5.12] there is a polynomial \( q_w(\lambda) \) such that \( g_w(\lambda) = q_w(\lambda) \) for all dominant \( \lambda \) compatible with \( w \) (we remark that our notation here is different from Joseph’s; we write \( q_w(\lambda) \) for what Joseph in [Jo3] would denote \( \tilde{q}_w(\lambda) \); for Joseph \( q_w(\lambda) = g_w(w^{-1}\lambda) \)). In particular, we have \( \text{Grk}(U/J(w\lambda)) = p_w(\lambda) \), where \( p_w(\lambda) := q_{d_\lambda}(\lambda), w \in c \). The polynomials \( p_w(\lambda) \) are called Goldie rank polynomials. Furthermore, the quotient \( q_w(\lambda)/p_w(\lambda) \) is a positive integer independent of \( \lambda \). This positive integer is known as Joseph’s scale factor and is denoted by \( z_w \). In fact, below we will only need to know that \( z_w \) is a number bigger than or equal to 1 (the equality \( \text{Grk}(U/J(w\lambda)) \leq g_w(\lambda) \) is a consequence of the inclusion \( U/J(w\lambda) \subset L(L(w\lambda), L(w\lambda)) \) that is provided by the \( U \)-action on \( L(w\lambda) \) – such an inclusion implies the inequality, see [W]; but the claim that the ratio is independent of \( \lambda \) is not so easily seen). A crucial property of the polynomials \( p_w(\lambda) \) with \( w \in c \) is that their span is an irreducible \( W \)-submodule of \( \text{Irr}(W) \), see [Jo3, Theorem 5.5]. This submodule is isomorphic to the special \( W \)-module in \( \text{Irr}(W)^c \), [Ja, 14.15]. Moreover, if we choose elements \( w_1, \ldots, w_k \), one in each left cell of \( c \), then the polynomials \( p_{w_1}, \ldots, p_{w_k} \) form a basis in the submodule.

In [Jo3, Theorem 5.1] Joseph determined the polynomial \( p_w \) up to a scalar multiple. So to complete the Goldie ranks computation one needs to determine a collection of scalars, say \( s_c \), one for each left cell \( c \). As Joseph, basically, pointed out in [Jo3, Remark 1 in 5.5], if one knows the scale factors \( z_w \) for all \( w \in c \), then one can, in principle, determine the scalars \( s_c \) for all left cells \( c \subset c \) up to a common scalar multiple. Let us explain how this works.

In [Jo3, 5.5, Remark 1], Joseph finds a formula expressing \( y.q_w, y \in W, w \in c \) as a linear combination of elements \( q_{w'} \) with \( w' \in c \). The coefficients are expressed in terms of the multiplicities in the BGG category \( \mathcal{O} \) and so are known. If one knows the coefficients \( z_w \) for all \( w \in c \), then one can express \( y.p_{w_i} \) in terms of the \( p_{w_j}; j = 1, \ldots, k \), say

\[
(5.2) \quad y.p_{w_i} = \sum_{j=1}^{k} b^i_j(y)p_{w_j}.
\]

But the elements \( p_{w_j} \) are linearly independent and their span is an irreducible \( W \)-module so the basis \( (p_{w_j})_{i=1}^{k} \) is determined uniquely from (5.2) up to a scalar multiple.

That single multiple can be determined uniquely if one knows that there is \( w \in c \) such that \( p_w(\lambda) = 1 \) for some integral weight \( \lambda \). The latter happens if there are compatible \( w \in c \) and dominant integral \( \lambda \) such that \( J(w\lambda) \) is completely prime, i.e., \( \text{Grk}(U/J(w\lambda)) = 1 \). In fact, there is a conjecture of Joseph saying that for each \( w \in c \) there is \( \lambda \) with \( p_w(\lambda) = 1 \) (of course, generally, \( \lambda \) will not be dominant).

To finish this subsection let us mention that Joseph also had a conjecture computing the scale factors \( z_w \), see [Jo5, 5.3]. It is unclear to us whether his conjecture is compatible with Theorem 1.3.

5.2. Scale factors: Joseph vs Premet. Let \( w \in c \). The goal of this subsection is to provide a formula for Joseph’s scale factors \( z_w \), see Subsection 5.1, in terms of the triple \( x, y, \mathcal{V} \) corresponding to \( w \), see Subsection 2.4, and certain numbers that we call Premet’s
scale factors\footnote{The attribution to Premet is made because of his beautiful result saying that this scale factor is always integral. The first version of our proof below used that result. In fact – as was communicated to the author by Premet – one does not need that in the proof, it is enough to use the fact that the scale factor is bigger or equal than 1, which was established by the author. It is pleasant when somebody else understands your work better than you do...} that are given by

$$\operatorname{pr}_w(\lambda) = \frac{d_w(\lambda)}{p_w(\lambda)}.$$  

Here $\lambda$ is a dominant weight compatible with $w$, $p_w(\lambda)$ is the Goldie rank of $J(w\lambda)$, and $d_w(\lambda)$ is the dimension of the irreducible $W$-module lying over $J(w\lambda)$. Of course, $\operatorname{pr}_w(\lambda)$ depends only on the left cell $c_w$ containing $w$ and on $\lambda$ (below we will see that it is actually independent of $\lambda$).

Pick a regular element $\varrho \in \Lambda^+$. Recall that we represent $w \in c$ as a triple $(x, y, V)$, see Subsection 2.4. Below we write $\mathbb{K}$ for the trivial $\tilde{A}(x, y)$-module. Let $Y$ denote the subset of $Y^\Lambda$ consisting of all irreducibles with central character $\varrho$. Below for $x \in Y$ lying over a primitive ideal $\mathcal{J} = J(\varrho w)$ we write $d_x := d_w(\varrho), g_x := p_w(\varrho), \operatorname{pr}_x := \operatorname{pr}_w(\varrho)$.

**Proposition 5.1.** We have

$$z_w = \frac{\operatorname{pr}_x}{\operatorname{pr}_y} \cdot \frac{|\tilde{A}_y|}{|A_{x, y}|} \dim V.$$

**Proof.** Set

$$\mathcal{M}_w(\lambda) := L(\Delta(\varrho), L(w\lambda)), m_w(\lambda) := \text{mult}_0 \mathcal{M}_w(\lambda),$$

and $g_w(\lambda) := \text{Grk}(L(L(w\lambda), L(w\lambda)))$. Below we only consider $\lambda$ compatible with $w$.

**Lemma 5.2.** The ratio $\frac{m_w(\lambda)}{g_w(\lambda)}$ depends only on the right cell containing $w$ (equivalently, on $c_{w^{-1}}$).

**Proof of Lemma 5.2.** We have $m_w(\lambda) = e(\mathcal{M}_w(\lambda))/e(\mathbb{K}[0])$, where $e$ is the Gelfand-Kirillov multiplicity, for the definition of $e(\bullet)$ see, say, [Ja, Kapitel 8]. Now the result follows from [Ja, 12.5].

Below we write $m_w, g_w$ for $m_w(\varrho), g_w(\varrho)$. We write $z_{x, y, V}, m_{x, y, V},$ etc., instead of $z_w, m_w$ etc. if $w$ corresponds to a triple $(x, y, V)$.

By the definition of $z_w$ we have

$$z_{x, y, V} = \frac{g_{x, y, V}}{g_x}.$$  \hspace{1cm} (5.3)

(5.1) together with the observation that a Duflo involution corresponds to a triple of the form $(x, x, \mathbb{K})$, see Subsection 2.4, imply $g_x = g_{(x, x, \mathbb{K})}$. By Lemma 5.2, the ratio $\frac{m_{x, y, V}}{g_{x, y, V}}$ depends only on the left cell corresponding to $y$. In particular,

$$\frac{m_{x, y, V}}{g_{x, y, V}} = \frac{m_{y, y, \mathbb{K}}}{g_{y, y, \mathbb{K}}}.$$  \hspace{1cm} (5.4)

Plugging (2.2),(5.3) into (5.4) we get

$$\frac{d_x d_y |\tilde{A}| \dim V}{z_{x, y, V} g_x |\tilde{A}_{x, y}|} = \frac{d_y^2 |\tilde{A}|}{g_y |\tilde{A}_y|} \Rightarrow$$

$$z_{x, y, V} = \frac{d_x d_y |\tilde{A}| \dim V g_y |\tilde{A}_y|}{d_y^2 |\tilde{A}| g_x |\tilde{A}_{x, y}|} = \frac{\operatorname{pr}_x |\tilde{A}_y|}{\operatorname{pr}_y |\tilde{A}_{x, y}|} \dim V.$$
5.3. Proof of Theorem 1.3 modulo Conjecture 1.1.

**Proposition 5.3.** Let \( w, \lambda \) be compatible. Then \( pr_w(\lambda) \) depends only on the left cell containing \( w \).

**Proof.** The multiplicity \( \text{mult}_\mathcal{O}(\mathcal{U}/J(w\lambda)) \) is equal to some polynomial \( P_w(\lambda) \) and this polynomial is proportional to \( p_w(\lambda)^2 \), see [Ja, 12.7]. But, on the other hand, according to (2.2), we have \( \text{mult}_\mathcal{O}(\mathcal{U}/J(w\lambda)) = |\bar{A}/H_w|d_w(\lambda)^2 \). So \( d_w(\lambda) = \bar{p}_w(\lambda) \) for some polynomial \( \bar{p}_w(\lambda) \) proportional to \( p_w(\lambda) \). This implies the claim. 

For a cell \( c := c_w \) we will write \( pr_c := pr_w(\lambda) \).

**Proof of Theorem 1.3 modulo Conjecture 1.1.** Consider the case when the orbit \( \mathcal{O} \) is not one of the three exceptional orbits.

Suppose that \( x, y \in Y \) are such that \( \bar{A}_x \supset \bar{A}_y \) and let \( \dim \mathcal{V} = 1 \). We have \( |\bar{A}_y| = |\bar{A}_{x,y}| \) and so Proposition 5.1 implies \( z_w = \frac{pr_x}{pr_y} \). In particular, since \( z_w \geq 1 \), we have \( pr_x \geq pr_y \).

According to [Lo1, Proposition 3.4.6], \( pr_y \geq 1 \). So if \( pr_x = 1 \) and \( \bar{A}_x = \bar{A} \), then \( pr_y = 1 \) for all \( y \in Y \).

Thanks to Conjecture 1.1, there is a (possibly singular) dominant integral weight \( \lambda \) and \( w \in c \) as in Theorem 1.3 such that \( d_w(\lambda) = 1 \) (and hence \( pr_w(\lambda) = 1 \)) and the \( \bar{A} \)-orbit corresponding to \( J(w\lambda) \) is a single point. Then the \( \bar{A} \)-orbit corresponding to \( J(wg) \) is also a single point. Let \( c \) be the left cell containing \( w \). Then, thanks to Proposition 5.3, \( pr_c = pr_w(\lambda) = 1 \). Now it remains to use the result of the previous paragraph.

If \( \mathcal{O} \) is one of the three exceptional orbits, then \( \bar{A}_x = \{1\} \) for all \( x \in Y \), this follows from [LO, 6.7,Theorem 1.1]. One carries over the argument above to this case without any noticeable modifications. 

5.4. Proof of Theorem 1.2.

5.4.1. Reduction to weakly rigid orbits. We start the proof of Theorem 1.2 by introducing a certain induction procedure related to the Lusztig-Spaltenstein induction.

Let us define a certain class of special nilpotent orbits: weakly rigid ones. Then we will show that it is enough to prove Conjecture 1.1 for weakly rigid orbits only.

Recall that \( \mathcal{O} \) is called rigid if it cannot be induced from a nilpotent orbit in a proper Levi subalgebra.

**Definition 5.4.** We say that a special orbit \( \mathcal{O} \) is **strongly induced** if there is a proper parabolic subalgebra \( p \subset g \), Levi subalgebra \( g' \subset p \) and a nilpotent orbit \( \mathcal{O}' \subset g' \) such that \( \mathcal{O} \) is induced from \( \mathcal{O}' \) in the sense of Lusztig and Spaltenstein and moreover:

1. \( \mathcal{O} \) is special in \( g' \).
2. Pick \( e \in \mathcal{O} \cap (\mathcal{O} + n) \). Then the projection of \( Z_P(e) \) to the Lusztig quotient \( \bar{A} \) (independent of the choice of \( e \)) for \( \mathcal{O} \) is surjective.

We say that \( \mathcal{O} \) is **weakly rigid**, if it is not strongly induced.

We are going to use the parabolic induction for W-algebras recalled in Subsection 2.5. In particular, let \( N \) be a 1-dimensional \( W \)-module with integral central character. It follows from [Lo4, Corollary 6.3.3] that \( \varsigma(N) \) also has integral central character. Now assume in addition that \( N \) is \( A(e) \)-stable, where \( e \in \mathcal{O} \) is the projection of \( e \) along \( n \) and \( A(e) \) stands.

\(^3\)These are close to so called birationally rigid orbits.
for the component group of the centralizer of \( e \in G \). Then both \( N, \zeta(N) \) are \( Q \)-stable. So we see that if condition (2) holds, then \( \zeta(N) \) is \( A(e) \)-stable.

Summarizing, we see that it is enough to prove Conjecture 1.1 only for weakly rigid orbits. We will provide a complete proof only in type \( B \). Then we explain modification necessary for types \( C, D \).

5.4.2. Type \( B \). We start by recalling a few standard facts about nilpotent orbits in \( \mathfrak{so}_{2n+1} \).

First of all, nilpotent orbits in \( \mathfrak{so}_{2n+1} \) are parameterized by partitions \( \lambda \) of \( 2n + 1 \) having type \( B \), i.e., such that every even part appears with even multiplicity, see, for example, [CM, 5.1]. In general, for any partition \( \lambda \) of \( 2n + 1 \) there is the largest (with respect to the dominance) partition \( \lambda_B \) of type \( B \) smaller than or equal to \( \lambda \). The partition \( \lambda_B \) is called the \( B \)-\textit{collapse} of \( \lambda \).

A partition \( \lambda \) corresponds to a special orbit if and only if the transposed partition \( \lambda^t \) is again of type \( B \), see [CM, Proposition 6.3.7]. Explicitly, this means that there is an even number of odd parts between any two consecutive even parts or smaller than the smallest even part, but there is an odd number of odd parts larger than the largest even part.

Now let us recall what the Lusztig-Spaltenstein induction does on the level of partitions, see [CM, 7.3]. Any Levi subalgebra in \( \mathfrak{so}_{2n+1} \) has the form \( \mathfrak{gl}_{n_k} \times \mathfrak{gl}_{n_{k-1}} \times \ldots \times \mathfrak{gl}_{n_1} \times \mathfrak{so}_{2n_0+1} \) with \( \sum_{i=0}^k n_i = n \). The induction procedure is associative so it is enough to see what happens when we induce from an orbit \( \mathcal{O} = (0, \mathcal{O}_0) \subset \mathfrak{gl}_m \times \mathfrak{so}_{2(n-m)+1} \). Let \( \mu = (\mu_1, \ldots, \mu_l) \) with \( l \geq m + 1 \) be the partition of \( \mathcal{O}_0 \) (we add zero parts if necessary). Then the partition \( \lambda \) corresponding to the induced orbit \( \mathcal{O} \) is

(a) either \( (\mu_1 + 2, \ldots, \mu_m + 2, \mu_{m+1}, \ldots, \mu_l) \) if the latter orbit is of type \( B \),

(b) or \( (\mu_1 + 2, \ldots, \mu_{m-1} + 2, \mu_m + 1, \mu_{m+1} + 1, \mu_{m+2}, \ldots, \mu_l) \) otherwise.

In other words, \( \lambda \) is always the \( B \)-\textit{collapse} of the partition in (a).

**Lemma 5.5.** Suppose that the partition of \( \mathcal{O} \) is obtained as in (a). Then the conditions of Definition 5.4 are satisfied.

**Proof.** The claim that (1) is satisfied is straightforward from the combinatorial description of special orbits.

We will specify the choice of a parabolic \( P \) and then choose an element \( e \) as in (2) to compute \( Z_P(e) \) and see that actually \( Z_P(e) \) projects surjectively to \( A(e) \).

We represent \( \mathfrak{so}_{2n+1} \) as the Lie algebra of matrices that are skew-symmetric with respect to the main anti-diagonal (so that the symmetric form used to define \( \mathfrak{so}_{2n+1} \) is \( (x, y) = \sum_{i=1}^{2n+1} x_i y_{2n+2-i} \)). Choose the parabolic subalgebra \( \mathfrak{p} \) with Levi subalgebra \( \mathfrak{gl}_m \times \mathfrak{so}_{2(n-m)+1} \) in a standard way, i.e., \( \mathfrak{p} \) is the stabilizer of the span of the first \( m \) basis elements.

Let us specify an element \( e_0 \in \mathcal{O}_0 \). Consider the numbers \( \mu_1, \ldots, \mu_m \) and split them into \( q \) pairs of equal numbers and \( p \) pairwise different numbers. Then take the remaining numbers \( \mu_{m+1}, \ldots, \mu_l \) and do the same getting \( q' \) pairs and \( p' \) pairwise different numbers. E.g., for \( m = 6, \mu = (5^3, 4^2, 3^4, 2^2, 1^2) \) (as usual the superscripts are the multiplicities) we have \( q = 2 \) (with pairs \( (5, 5), (4, 4) \)), \( p = 2 \) (with numbers 5 and 3), \( q' = 3 \) (the pairs \( (3, 3), (2, 2) \) and \( (1, 1) \)) and \( p' = 1 \) (corresponding to 3).

Take the subspace \( \mathbb{K}^{2(n-m)+1} \subset \mathbb{K}^{2n+1} \), where \( \mathfrak{so}_{2(n-m)+1} \) acts, and represent it as a direct sum of subspaces

\[
\bigoplus_{i=1}^p V_i \oplus \bigoplus_{i=1}^q (U_i \oplus U_i^*) \oplus \bigoplus_{i=1}^{p'} V_i' \oplus \bigoplus_{i=1}^{q'} (U_i' \oplus U_i'^*),
\]
where \( V_1, \ldots, V_p, V'_1, \ldots, V'_p \) are orthogonal subspaces of dimensions equal to single \( \mu_i \)'s (5, 3, 3 in our example), and \( U_1, \ldots, U_q, U'_1, \ldots, U'_q \) are isotropic subspaces of dimensions equal to \( \mu_i \)'s from pairs (5, 4, 3, 2, 1 in our example), \( U_1^*, U'_1, \ldots, U_q^*, U'_q \) are dual isotropic subspaces. Below we denote by \( v_1(i), \ldots, v_{2d+1}(i) \) a basis in \( V_i \), where the form is written as above, by \( u_1(j), \ldots, u_d(j) \) a basis in \( U_j \), and by \( v_i^*(j), \ldots, u_q^*(j) \) the dual basis in \( U_j^* \). For \( \mu_i \)'s we denote the matrix unit sending \( v_n(i) \) to \( v_m(i) \). The notation \( u_m^*(i) \) has a similar meaning.

For \( e_0 \) we take

\[
\sum_{k=1}^{p} e_0(k) + \sum_{k=1}^{q} f_0(k) + \sum_{k=1}^{p'} e_0'(k) + \sum_{k=1}^{q'} f_0'(k).
\]

The element \( e_0(k) \) is given by the matrix \( \sum_{i=1}^{d} (v_i^i(k) - v_i^{i+d}(k)) \), where dim \( V_k = 2d + 1 \). The element \( f_0(k) \) is given by the matrix \( \sum_{i=1}^{d} u_i^i(k) - \sum_{i=1}^{d} u_i^{i+1}(k) \), where \( d = \text{dim} U_i \). The operators \( e_0'(k) \in \mathfrak{so}(V'_k), f_0'(k) \in \mathfrak{so}(U'_k + U''_k) \) are defined similarly. It is clear from the construction that \( e_0 \in \mathcal{O} \).

Now let us specify \( e \). Set \( \tilde{V}_i := V_i \oplus \mathbb{K}^2 \), where \( \mathbb{K}^2 \) is viewed as an orthogonal space with isotropic basis \( v_0(i), v_{2d+2}(i) \), and \( \tilde{U}_i := U_i \oplus \mathbb{K}^2 \), where \( \mathbb{K}^2 \) is viewed as an isotropic space with basis \( u_0(i), u_{d+1}(i) \). Then, of course

\[
\mathbb{K}^{2n} = \bigoplus_{k=1}^{p} \tilde{V}_k \oplus \bigoplus_{k=1}^{q} (\tilde{U}_k \oplus \tilde{U}'_k) \oplus \bigoplus_{k=1}^{p'} V'_k \oplus \bigoplus_{k=1}^{q'} (U'_k + U''_k).
\]

Now we set

\[
e := \sum_{k=1}^{p} e(k) + \sum_{k=1}^{q} f(k) + \sum_{k=1}^{p'} e'(k) + \sum_{k=1}^{q'} f'(k),
\]

where the matrices \( e(k), f(k) \) are defined as follows. We set \( e(k) := \sum_{i=0}^{d} (v_i^i(k) - v_i^{i+d}(k)) \). Further, \( f(k) = \sum_{i=0}^{d+1} u_i^i(k) - \sum_{i=0}^{d+1} u_i^{i+1}(k) \). It is clear that \( e \in \mathcal{O} \).

Let \( P \) be the maximal parabolic subgroup in \( O(2n + 1) \) (note that we take a disconnected group) stabilizing the \( m \)-dimensional subspace spanned by all basis vectors \( v_0(k), u_0(k), u_n(k) \). With this choice of \( P \) we have \( e \in e_0 + n \). Now let us produce elements in the Levi subgroup \( \text{GL}_m \times O_{2(n-m)+1} \) centralizing \( e \) whose images span the component group of \( Z_{O_{2n+1}}(e) \) (this will imply condition (2)). This component group is the sum of several copies of \( \mathbb{Z}/2\mathbb{Z} \), one for each different odd part of \( \lambda \), see, for example, [CM, Theorem 6.1.3]. More precisely, let \( \lambda = (\lambda_1^{n_1}, \lambda_2^{n_2}, \ldots, \lambda_k^{n_k}) \). Then the reductive part of \( Z_{O_{n}}(e) \) is \( \prod_{i=1}^{k} G_{n_i} \), where \( G_{n_i} \) means \( O_{n_i} \) if \( \lambda_i \) is odd, and \( S_{p_{n_i}} \) if \( \lambda_i \) and then automatically \( n_i \) is even. We have one involution in the basis of the component group for each \( G_{n_i} = O_{n_i} \).

Let us produce elements \( g_i \in O(\tilde{V}_i) \cap (\text{GL}_m \times O_{2(n-m)+1}) \) centralizing \( e(i) \), and \( h_j \in O(\tilde{U}_j \oplus \tilde{U}'_j) \cap (\text{GL}_m \times O_{2(n-m)+1}) \) centralizing \( f(j) \). For \( g_i \) we just take \(-\text{id}_{\tilde{V}_j} \). If \( \text{dim} U_j \) is even, then we set \( h_j = \text{id}_{\tilde{U}_j} \). Finally, suppose \( \text{dim} U_j \) is odd. Then define \( h_j \) by \( h_j(u_k(j)) := \sqrt{-1} u_k(j) \). It is easy to see that \( h_j \) is defined in this way lies in the required subgroup and centralizes \( f(j) \).

From the description of the component group given above we see that the elements \( g_i, h_j \) generate the component group.
From Lemma 5.5 we deduce that the partition of a weakly rigid orbit has the form
\[(5.5) \quad \{(2k + 1)^{2d_{2k+1}-1}, (2k)^{2d_{2k}}, (2k - 1)^{2d_{2k-1}}, \ldots, 2^{2d_2}, 1^{2d_1}\},\]
where \(d_{2k+1}, d_{2k}, \ldots, d_1\) are positive integers. It seems that any partition like this indeed corresponds to a weakly rigid orbit, but we will not need this.

Let us produce an explicit \(\mu \in \Lambda\) such that \(\text{mult}_x U/J(\mu) = 1\). According to (2.2) for a corresponding irreducible \(W\)-module \(x\) we will have \(A_x = \Lambda, d_x = 1\) as needed in Conjecture 1.1. This will be a so called Arthur-Barbasch-Vogan weight, see, for example, [BV, (1.15)]. Let us recall how this weight is constructed in general.

Let us recall how this weight is constructed in general. 1.1. This will be a so called Arthur-Barbasch-Vogan weight, see, for example, [BV, (1.15)].

There is a duality for nilpotent orbits in \(g\) and in the Langlands dual Lie algebra \(g^\vee\) \((B_n\) and \(C_n\) are dual to each other, while all other simple algebras are self-dual). This duality (called Barbasch-Vogan-Spaltenstein duality) is an order reversing bijection between the sets of special orbits in \(g\) and in \(g^\vee\). Take a special orbit \(\mathcal{O} \subset g\) and let \(\mathcal{O}^\vee \subset g^\vee\) be the corresponding dual orbit. Let \((e^\vee, h^\vee, f^\vee)\) be the corresponding \(\mathfrak{sl}_2\)-triple. Recall that we have fixed Cartan and Borel subalgebras \(\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}\). Then we can take \(h^\vee := h^*\) for a Cartan subalgebra in \(g^\vee\) and also we have a preferred choice of a Borel subalgebra \(\mathfrak{b}^\vee \subset \mathfrak{g}^\vee\). Conjugating \(h^\vee\) we may assume that \(h^\vee\) is dominant in \(h^*\). This determines \(h^\vee\) uniquely. The weight of interest is \(\mu := \frac{1}{2} h^\vee\).

For the classical Lie algebras the duality between nilpotent orbits can be described on the level of partitions, see, for example, [CM, 6.3]. For example, take \(\mathcal{O} \subset \mathfrak{so}_{2n+1}\) and let \(\lambda\) be the corresponding partition. For a partition \(\mu\) of \(2n + 1\), let \(l(\mu)\) denote the partition of \(2n\) obtained from \(\mu\) by decreasing the smallest part of \(\mu\) by 1. Recall that the nilpotent orbits in \(\mathfrak{sp}_{2n}\) are parameterized by partitions of type \(C\), i.e., such that the multiplicity of each odd part is even. For any partition \(\mu^*\) of \(2n\) we can define its \(C\)-collapse \(\mu^*_C\) that is a partition of type \(C\) similarly to the \(B\)-collapse. Now the duality sends the orbit \(\mathcal{O}\) with partition \(\lambda\) to the orbit \(\mathcal{O}^\vee\) with partition \([l(\lambda^*])_C\), see, for instance, [McG1, Theorem 5.1], and the discussion after it. It is easy to see that, for \(\lambda\) of the form (5.5), the partition \([l(\lambda^*])_C\) consists of even parts and so the orbit \(\mathcal{O}^\vee\) is even, meaning that all eigenvalues of \(ad(h^\vee)\) are even. In particular, \(\frac{1}{2} h^\vee \in \Lambda^+\).

According to [BV, Proposition 5.10], \(V(U/J(\frac{1}{2} h^\vee)) = \mathcal{O}\). Further, [McG1, Corollary 5.19] implies that the multiplicity of \(U/J(\frac{1}{2} h^\vee)\) on \(\mathcal{O}\) is \(1\).

5.4.3. Type C. The proofs of Conjecture 1.1 in types \(C\) and \(D\) are very similar. So we will only explain the necessary modifications.

As we have already mentioned, the nilpotent orbits in type \(C\) are parameterized by partitions of type \(C\). A partition \(\lambda\) corresponds to a special orbit if and only if \(\lambda^*\) is again of type \(C\). Explicitly this means that there is an even number of odd parts between two consecutive even parts and also an even number of even parts larger than the largest odd part. On the level of partitions the Lusztig-Spaltenstein induction is described completely analogously to type \(B\). Also one can prove a direct analog of Lemma 5.5 and the proof basically repeats the original one. From here we see that the partition of a weakly rigid orbit has the form \((n^{d_n}, (n - 1)^{d_{n-1}}, \ldots, 1^{d_1})\), where \(d_1, \ldots, d_n\) are positive even integers.

On the level of partitions the duality is described as follows. For a partition \(\lambda\) of \(2n\) let \(r(\lambda)\) be the partition of \(2n + 1\) obtained from \(\lambda\) by increasing the largest part by 1. Now a partition \(\lambda\) of a special orbit \(\mathcal{O} \subset \mathfrak{sp}_{2n}\) is sent to \([r(\lambda^*])_B\), where, recall, the subscript \(B\) means the \(B\)-collapse. It is easy to see that all parts of \([r(\lambda^*])_B\) are odd, meaning that the corresponding orbit is even. So it remains to take the ideal \(J(\frac{1}{2} h^\vee)\) and use the results of Barbasch-Vogan and of McGovern, exactly as in type \(B\).
5.4.4. Type D. Nilpotent orbits in $\mathfrak{so}_{2n}$ are parameterized by partitions of $2n$ of type D: each even part occurs even number of times (in fact, this is only true for the $O_{2n}$-action, each partition with all parts even produces exactly two $SO_{2n}$-orbits, while all other partitions correspond to a single $SO_{2n}$-orbit). A partition $\lambda$ corresponds to a special orbit if and only if $\lambda'$ is of type C (not D!), i.e., there is an even number of odd parts between any two consecutive even parts, larger than the largest even part, and smaller than the smallest even part.

The Lusztig-Spaltenstein induction is described in the same way as above, and an analog of Lemma 5.5 with the same proof holds. From here we deduce that the partition of a weakly rigid orbit has the form $(n^{d_n}, (n-1)^{d_{n-1}}, \ldots, 1^{d_1})$, where $d_1, \ldots, d_n$ are positive even integers.

The dual orbit is even, the weight $\frac{1}{2}h^\vee$ is integral, and we are again done.

5.4.5. A few remarks towards the exceptional cases. We hope that for the exceptional Lie algebras the same strategy as above should work. Namely, one should be able to describe all weakly rigid orbits. Then, hopefully, the dual of a weakly rigid orbit will be even (this is true for all rigid orbits, as was checked in [BV, Proposition I]). Then one can hope that $\mathcal{U}/J(\frac{1}{2}h^\vee)$ will have multiplicity 1 on $\mathcal{O}$.

In [Lo4] we have developed some techniques to classify one-dimensional representations of $W$-algebras under certain conditions on the nilpotent element $e$. The most important condition is that the algebra $q = z_g(e, h, f)$ is semisimple. This condition is satisfied for all rigid orbits but not for all weakly rigid ones (it is peculiar that, in the classical types, a weakly rigid orbit is rigid precisely when $q$ is semisimple, this can be deduced from the classification of rigid orbits provided, for instance, in [CM, 7.3]). Another condition on $e$ that significantly simplifies the classification is that $e$ is principal in some Levi. This holds for all weakly rigid (special) orbits in classical algebras.

The classification result obtained in [Lo4, Section 5] (for elements $e$ subject to the conditions explained in the previous paragraph) takes the following form. It establishes a family $X$ of elements in $\mathfrak{h}^*$ (given by several conditions, the most implicit being that $V(\mathcal{U}/J(\lambda)) = \mathcal{O}$ for each $\lambda \in X$) such that $X$ is in bijection with 1-dimensional $\mathcal{W}$-modules in such a way that the primitive ideal corresponding to the 1-dimensional module attached to $\lambda \in X$ is $J(\lambda)$. For several rigid special elements the element $\frac{1}{2}h^\vee$ is in $X$, but for some it is not, which, however, does not mean that $J(\frac{1}{2}h^\vee)$ does not correspond to a 1-dimensional $\mathcal{W}$-module simply because the weight cannot be recovered from an ideal uniquely.

Finally, let us remark that even if $q$ is not semisimple, it is still sometimes possible to get some ramification of the classification result that will classify $A(e)$-stable 1-dimensional modules, see [Lo4, Corollary 5.2.2].

6. Supplements

6.1. Functor $\bullet_{t,e}$ for Harish-Chandra modules.

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$^4$As I learned from Jeffrey Adams there is only one weakly rigid orbit, where this is not true: it appears in type $E_8$. 
6.1.1. Setting. Our setting in this subsection is very different from the one in the main body of the paper. We fix an involutive antiautomorphism $\tau$ of $G$ (so that $x \mapsto \tau(x)^{-1}$ is an involutive automorphism). We set $K := \{ g \in G|\tau(g) = g^{-1}\}^0$, this is a reductive subgroup of $G$. Further set $s := g^0$ so that we have $g = \mathfrak{u} \oplus s$.

We pick $e \in s$. Then we can choose $h, f$ forming an $\mathfrak{sl}_2$-triple with $e$ in such a way that $h \in \mathfrak{u}$ and $f \in s$. It is known (and easy to check) that the intersection of $Ge$ with $s$ is a union of finitely many $K$-orbits, each being a lagrangian subvariety in $Ge$. As before, we take $V = [g, f]$. Then $u := \mathfrak{u} \cap V$ coincides with $[s, f]$ and is a lagrangian subspace in $V$. Further, $\chi = (e, \cdot)$ clearly vanishes on $\mathfrak{u}$. So the pair $(K, V)$ does satisfy the assumptions of 4.1.1.

So we can get a functor $\bullet : O^K \to \mathcal{W}$-$\text{mod}^R$. In fact, the functor is easier to construct then in the general case and we also can get some restriction on its image, it consists of “HC-modules for $(\mathcal{W}, \tau)$”. We are going to define those in the next part in a more general context.

6.1.2. Harish-Chandra modules for almost commutative algebras. Let $A$ be a unital associative algebra equipped with an exhaustive algebra filtration $\mathbb{K} = A_{\leq 0} \subset A_{\leq 1} \subset \ldots$. Assume that $A_{\leq i}A_{\leq j} \subset A_{\leq i+j-d}$ for some positive integer $d$. Finally, assume that $\text{gr} A$ is a finitely generated algebra.

Suppose that $A$ is equipped with an involutive anti automorphism $\tau$ that preserves the filtration. By definition, a HC $(A, \tau)$-module is an $A$-module $M$ that can be equipped with an increasing exhaustive filtration $\mathcal{M}_{\leq 0} \subset \mathcal{M}_{\leq 1} \subset \ldots$ that is compatible with the filtration on $A$ (in the sense that $A_{\leq i}\mathcal{M}_{\leq j} \subset \mathcal{M}_{\leq i+j}$) satisfying the following two additional conditions:

\begin{enumerate}[leftmargin=*,label=(\roman*)]
  \item $\text{gr} M$ is a finitely generated $\text{gr} A$-module.
  \item If $a \in A_{\leq i}$ satisfies $\tau(a) = -a$, then $a\mathcal{M}_{\leq j} \subset \mathcal{M}_{i+j-d}$.
\end{enumerate}

Let us consider two examples supporting this definition.

First, let $A = U$ and choose $\tau$ as in 6.1.1. In this example, we take the PBW filtration and so $d = 1$. We claim that a HC $(A, \tau)$-module is the same as a HC $(g, \mathfrak{u})$-module, i.e., a finitely generated module with locally finite action of $\mathfrak{u}$. Let $U^\tau$, resp. $U^{-\tau}$, denote the subspace of the elements $u \in U$ such that $\tau(u) = u$, resp. $\tau(u) = -u$. Then it is easy to see that $U_{\leq i}^{-\tau} \subset U_{i-1}\mathfrak{u} \oplus U_{\leq i-1}$. It follows that a HC $(g, \mathfrak{u})$-module satisfies (ii). Now (i) becomes a well-known claim that one has a good $\mathfrak{u}$-stable filtration on any HC $(g, \mathfrak{u})$-module. The claim that a HC $(A, \tau)$-module is HC as a $(g, \mathfrak{u})$-module is easy.

Let us proceed to the second special case. Let $B$ be a filtered algebra with an almost-commutativity condition analogous to the condition on $A$ above. Set $A := B \otimes B^{opp}$ and let $\tau$ be defined by $\tau(b_1 \otimes b_2) := b_2 \otimes b_1$. Of course, an $A$-module is just a $B$-bimodule. We claim that a HC $(A, \tau)$-module is the same thing as a HC $B$-bimodule in the sense of [Lo2, 2.5], i.e., a filtered bimodule $\mathcal{M}$ with $[B_{\leq i}, M_{\leq j}] \subset M_{i+j-d}$ such that $\text{gr} \mathcal{M}$ is a finitely generated $\text{gr} B$-module.

First, since $b \otimes 1 - 1 \otimes b$ lies in $A^{-\tau}$, we see that a HC $(A, \tau)$-module $\mathcal{M}$ is a HC $B$-bimodule (condition (ii) just says that $\text{gr} \mathcal{M}$ is finitely generated as a $\text{gr} B$-bimodule but thanks to (i) the left and right actions of $\text{gr} B$ on $\text{gr} \mathcal{M}$ coincide). On the other hand, the space $A^{-\tau}$ is the linear span of the elements of the form $b_1 \otimes b_2 - b_2 \otimes b_1$. Using this it is easy to see that a HC $B$-bimodule is a HC $(A, \tau)$-module.
According to [CM, 6.2], the condition $O$ have dim decreasing order. Then the orbit codimension condition in type A. Another functor $\bullet_t,e$. As we have mentioned, the construction of $\bullet_t,e$ is easier than in the general case: we do not need to fix $\iota : V \to I_\chi$ as in 4.1.2: any $R \times K^x \times Z/2Z$ and also $\tau$-equivariant $\iota$ works. The functor $\bullet_{t,e}$ is constructed by complete analogy with [Lo2, 3.3.3.4], a crucial thing to notice is that $(A^e_h)^{-\tau}M^h_\chi \subset h^2M^h_\chi$ because $(U^e_h)^{-\tau}M^h_\chi \subset h^2M^h_\chi$. It follows that $[s,f]M^e_h \subset h^2M^e_\chi$.

Also this description shows that, for a HC $(g, \mathfrak{t})$-module $M$, the image of $M$ under $\bullet_{t,e}$ is an $R$-equivariant HC $(V, \tau)$-module.

6.2. Another functor $\bullet_t$ for $O^P$.

6.2.1. $\bullet_{t,e}$ for a general parabolic category $O$. Now we again change our setting. Let $P$ be an arbitrary parabolic in $G$. Let $e$ be a Richardson element of $p^\perp$ meaning that $Pe$ is dense in $p^\perp$. Then we can choose $h, f$ forming an $\mathfrak{sl}_2$-triple in such a way that $h \in p$, see [Lo4, Lemma 6.1.3]. Let $V = [g, f]$. Since $\mathfrak{g}_0(e) \subset p$, see [LS, Theorem 1.3], it is easy to see that $V \cap p$ is a lagrangian subspace in $V$. So we can construct a functor $\bullet_{t,e}$ from the parabolic category $O^P$ to the category $W$-$\text{mod}_R^R$, where, recall, $R$ is a maximal reductive subgroup of $Z_P(e)$. The image of this functor consists of finite dimensional modules. We also would like to point out that $Q^o \subset P$ and so $Q^o \subset R$.

We remark that this functor $\bullet_{t,e}$ is a generalization of $\mathcal{V}$ in the case when $g = \mathfrak{g}_0$. It is not difficult to show that all results mentioned in Theorem 4.8 with a possible exception of (v) still hold for $\bullet_{t,e}$.

6.2.2. The orbit codimension condition in type A. We would like to analyze the condition $\text{codim}_{p^\perp} p^\perp \setminus Pe > 1$ in the case when $g = \mathfrak{sl}_n$. In this case a version of $\bullet_{t,e}$ was previously considered by Brundan and Kleshchev in [BK1],[BK2]. A special feature of type $A$ is that any nilpotent element admits a so called good even grading. Then one can replace $g(i)$ with the $i$th graded component with respect to this grading. All constructions of Sections 3,4 work for that modification. Brundan and Kleshchev studied the Whittaker coinvariant functor, which is just $\mathcal{V}_2$ in this special case. So our approach recovers many results from [BK1],[BK2].

Parabolic subalgebras in $\mathfrak{sl}_n$ are parameterized by compositions of $n$, i.e., ordered collections of positive integers summing to $n$. So let $p$ correspond to a composition $(s_1, \ldots, s_\ell)$. Brundan and Kleshchev checked that the Whittaker coinvariants functor is 0-faithful provided $s_1 > s_2 > \ldots > s_\ell$. The following proposition together with the previous subsection generalizes that.

**Proposition 6.1.** Let $e$ be a Richardson element for $P$. The condition $\text{codim}_{p^\perp} p^\perp \setminus Pe > 1$ holds provided all $s$'s are distinct.

**Proof.** Suppose that all $s_i$'s are distinct. We are going to prove that for any nilpotent orbit $O' \subset \overline{O}$ (this condition is necessary for $O'$ to intersect $p^\perp$), we have $\dim O' < \dim \overline{O} - 2$. Since $\dim \overline{O} \cap p^\perp \leq \dim O' \cap b^\perp = \frac{1}{2} \dim O'$, see, for instance, [CG, Theorem 3.3.7] for the last equality, the inequality $\dim O' < \dim \overline{O} - 2$ implies $\dim O' \cap p^\perp < \dim p^\perp - 1$. The number of nilpotent orbits is finite and so the previous inequality implies the required codimension condition.

Let $\lambda$ be the Young diagram with rows of lengths $\lambda_1 := s_1, \ldots, \lambda_\ell := s_\ell$ ordered in the decreasing order. Then $O$ is the orbit corresponding to the transposed diagram $\lambda^t$. We have $\dim O = n^2 - \sum \lambda_i^2$. Now let $\mu$ be a Young diagram such that $\mu^t$ corresponds to $O'$. According to [CM, 6.2], the condition $O' \subset \overline{O}$ is equivalent to $\lambda \leq \mu$ in the dominance
ordering, i.e., \( \sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i \) for all \( k \). In other words, there is a sequence \( \lambda^0 := \lambda, \lambda^1, \ldots, \lambda^m := \mu \) such that \( \lambda^i \) is obtained from \( \lambda^{i-1} \) by moving a box from a shorter row to a longer row. It is therefore enough to consider the case when \( \mu = \lambda^1 \). We need to show that \( \sum_i \mu_i^2 \geq \sum_i \lambda_i^2 + 4 \). There are indexes \( p < q \) such that \( \mu_i = \lambda_i \) if \( i \neq p, q \), \( \mu_p = \lambda_p + 1, \mu_q = \lambda_q - 1 \). We have

\[
\sum_i (\mu_i^2 - \lambda_i^2) = (\lambda_p + 1)^2 - \lambda_p^2 + (\lambda_q - 1)^2 - \lambda_q^2 = 2(\lambda_p - \lambda_q) + 2.
\]

Since \( \lambda_p > \lambda_q \) by our assumption, we see that \( 2(\lambda_p - \lambda_q) + 2 \geq 4 \), and we are done. \( \square \)

The 0-faithfulness of the Brundan-Kleshchev functor was a crucial ingredient in the proof of an equivalence between (the sum of certain blocks) of \( \mathcal{O}^P \) and a category \( \mathcal{O} \) for a suitable cyclotomic Rational Cherednik algebra in [GL, Theorem 6.9.1] (under the restriction that \( s_1 > \ldots > s_t \)). Using Proposition 6.1 one should be able to remove that restriction but we are not going to elaborate on this.

6.3. Dixmier algebras.

6.3.1. Functors \( \bullet_{t,e} \). By a Dixmier algebra one means a \( G \)-algebra \( \mathcal{A} \) equipped with a \( G \)-equivariant homomorphism \( \mathcal{U} \to \mathcal{A} \) that makes \( \mathcal{A} \) into a Harish-Chandra bimodule such that the differential of the \( G \)-action on \( \mathcal{A} \) coincides with the adjoint \( \mathfrak{g} \)-action. The algebra \( \mathcal{U} \) itself as well as any quotient of \( \mathcal{U} \) serve as examples of Dixmier algebras. Two more examples come from Lie superalgebras and from quantum groups at roots of 1, they are considered below. It is basically those two families of examples that motivate us to consider arbitrary Dixmier algebras.

Let \( K \) be as in \( 4.1.1 \). Let \( \mathcal{O}^K_{\nu}(\mathcal{A}) \) denote the category of \( \mathcal{A} \)-modules that lie in \( \mathcal{O}^K_{\nu} \) as \( \mathcal{U} \)-modules.

Fix a good algebra filtration \( F_\bullet \mathcal{A} \) on \( \mathcal{A} \), where, recall, good means that all \( F_i \mathcal{A} \) are \( G \)-stable and \( \text{gr} \mathcal{A} \) is a finitely generated \( S(\mathfrak{g}) = \text{gr} \mathcal{U} \)-module. The existence of a good algebra filtration is checked, for example, in [Lo2, Proof of Proposition 3.4.5]. In addition, we can assume that \( F_0 \mathcal{A} = F_1 \mathcal{A} = \mathbb{K} \) and that \( F_2 \mathcal{A} \) contains the image of \( \mathfrak{g} \). This insures that we have a filtered algebra homomorphism \( U(\mathfrak{g}) \to \mathcal{A} \). Consider the corresponding Rees algebra \( \mathcal{A}_h := \bigoplus_{i=0}^\infty (F_i \mathcal{A}) \mathfrak{h}^i \). This is a graded algebra, where \( \mathfrak{h} \) has degree 1. The corresponding \( \mathcal{W} \)-bimodule \( \mathcal{A}_h \) has a natural algebra structure, see [Lo2, 3.4].

Pick \( \mathcal{M} \in \mathcal{O}^K_{\nu}(\mathcal{A}) \). The \( R \)-equivariant \( \mathcal{W} \)-module \( \mathcal{M}_{t,e} \) has a natural \( \mathcal{A}_t \)-module structure. So we get an exact functor \( \bullet_{t,e} : \mathcal{O}^K_{\nu}(\mathcal{A}) \to \mathcal{A}_t\text{-mod}^R \). We have the forgetful functors \( \text{Fun}_U : \mathcal{O}^K_{\nu}(\mathcal{A}) \to \mathcal{O}^K_{\nu} \); \( \text{Fun}_W : \mathcal{A}_t\text{-mod}^R \to \mathcal{W}\text{-mod}^R \) and they intertwine the functors \( \bullet_{t,e} \), i.e., \( \text{Fun}_W(\bullet_{t,e}) = \text{Fun}_U(\bullet_{t,e}) \).

Let us make an observation that will be used later. There is a functor \( \bullet_{t,e} : \mathcal{A}_t\text{-mod}^R \to \mathcal{O}^K_{\nu}(\mathcal{A}) \), where \( \mathcal{O}^K_{\nu}(\mathcal{A}) \) is a category of not necessarily finitely generated \( \mathcal{A} \)-modules that are inductive limits of objects in \( \mathcal{O}^K_{\nu}(\mathcal{A}) \). This functor satisfies \( \text{Hom}_\mathcal{A}(\mathcal{M}, \mathcal{N}_{t,e}) = \text{Hom}_{\mathcal{A}_t,R}(\mathcal{M}_{t,e}, \mathcal{N}) \).

The functor is constructed completely analogously to \( \bullet_{t,e} \) in \( 4.1.4 \) and, in particular, does not depend on \( \mathcal{A} \) in the sense that the functors \( \bullet_{t,e} \) for \( \mathcal{A} \) and for \( \mathcal{U} \) are again intertwined by the forgetful functors.

6.3.2. Enveloping algebras of Lie superalgebras. Here is an interesting example of a Dixmier algebra \( \mathcal{A} \). Let \( \mathfrak{g} \) be a simple classical Lie superalgebra or a queer Lie superalgebra. In particular, the even part, denote it by \( \mathfrak{g} \), is a reductive Lie algebra. So the universal enveloping
(super)algebra $\mathcal{A} := U(\mathfrak{g})$ is a Dixmier algebra over $\mathcal{U} = U(\mathfrak{g})$. We also remark that $\mathcal{A}$ is a free left (or right) module over $\mathcal{U}$.

The algebra $\mathcal{A}_t$ may be thought as a W-algebra for the Lie superalgebra $\mathfrak{g}$. It should not be difficult to check that $\mathcal{A}_t$ is isomorphic to the Clifford algebra over the super W-algebra studied previously, see, for example, [Z],[BBG]. But we are not going to prove this here.

6.3.3. Quantum groups at roots of 1. Here we will show that the algebra that is basically the Lusztig form of a quantum group at a root of unity is a Dixmier algebra. More precisely we will show that the quantum Frobenius epimorphism splits.

Let us recall some generalities on quantum groups. We follow [Lu4].

We fix a Cartan matrix $A = (a_{ij})_{i,j=1}^n$ of finite type. Let $\mathfrak{g}$ be the corresponding finite dimensional semisimple Lie algebra. Let $D = \text{diag}(d_1, \ldots, d_n)$ be the matrix with coprime entries $d_i \in \{1, 2, 3\}$ such that $DA$ is symmetric. Let $v$ be an indeterminate. We write $[n]_d$ for the quantum integer $\frac{v^{d_i n} - v^{-d_i n}}{v^d_i - v^{-d_i}}$ and $[n]_d!$ for the corresponding quantum factorial. We then consider the quantum group $U_v(\mathfrak{g})$ that is a $\mathbb{K}(v)$-algebra generated by $E_i, F_i, K_i^{\pm 1}$ in a standard way, see, e.g., [Lu4, Section 1]. Inside we consider the $U_v$ then consider the quantum group $\mathcal{U}_v$.

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A standard way, see, e.g., [Lu4, Section 1]. Inside we consider the $U_v$ generated by the divided powers $E_i^{(N)} := \frac{E_i^N}{[N]_d!}$ and $F_i^{(N)} := \frac{F_i^N}{[N]_d!}$ and also $K_i^{\pm 1}$. It contains elements

$$\left( K_i; c \right)_t := \prod_{i=1}^t K_i v^{d_i(c-s+1)} - K_i^{-1} v^{-d_i(c-s+1)}$$

for the quantum integer $\frac{v^{d_i n} - v^{-d_i n}}{v^d_i - v^{-d_i}}$ and $[n]_d!$ for the corresponding quantum factorial. We then consider the quantum group $U_v(\mathfrak{g})$ that is a $\mathbb{K}(v)$-algebra generated by $E_i, F_i, K_i^{\pm 1}$ in a standard way, see, e.g., [Lu4, Section 1]. Inside we consider the $\mathbb{K}[v^{\pm 1}]$-subalgebra $\tilde{U}_v(\mathfrak{g})$ generated by the divided powers $E_i^{(N)} := \frac{E_i^N}{[N]_d!}$ and $F_i^{(N)} := \frac{F_i^N}{[N]_d!}$ and also $K_i^{\pm 1}$. It contains elements

$$\left( K_i; c \right)_t := \prod_{i=1}^t K_i v^{d_i(c-s+1)} - K_i^{-1} v^{-d_i(c-s+1)}$$

see [Lu4, Section 6]. Inside $\tilde{U}_v(\mathfrak{g})$ we consider subalgebras $\tilde{U}_v^+$, (resp., $\tilde{U}_v^-$ and $\tilde{U}_v^0$) generated by the elements $E_i^{(N)}$ (resp., $F_i^{(N)}$, and $K_i^{\pm 1}$, $(K_i; c)$). We have the triangular decomposition $\tilde{U}_v(\mathfrak{g}) = \tilde{U}_v^+ \otimes \tilde{U}_v^0 \otimes \tilde{U}_v^-.$

Pick an integer $\ell$ that is odd and, in the case when $\mathfrak{g}$ has a component of type $G_2$, coprime to 3. Let $e$ be an $\ell$th primitive root of 1. Let $\tilde{U}_v(e), \tilde{U}_v^e$ etc. denote the specializations of the corresponding algebras at $v = e$. The elements $K_i^e$ are central in $\tilde{U}_v(\mathfrak{g})$ and, moreover, one can show that $K_i^{2\ell} = 1$ in $\tilde{U}_v^0$. Let $\mathcal{A}$ denote the quotient of $\tilde{U}_v(\mathfrak{g})$ by $K_i^e - 1$ for $i = 1, \ldots, n$. We still have the triangular decomposition $\mathcal{A} = \mathcal{A}^- \otimes \mathcal{A}^0 \otimes \mathcal{A}^+$, and $\mathcal{A}^- = \tilde{U}_v^+(\mathfrak{g}), \mathcal{A}^+ = \tilde{U}_v^-(\mathfrak{g}).$

Also we have an epimorphism Fr$^* : \mathcal{A} \to \mathcal{U}$ called the quantum Frobenius epimorphism, see [Lu4, Section 8]. By construction, it maps $E_i^{(N)}, F_i^{(N)}$ to $e_i^{(N/\ell)}, j_i^{(N/\ell)}$ if $N$ is divisible by $\ell$ and to 0 else, and it sends $K_i$ to 1. Further, it maps $(K_i; e)$ to $h_e$.

Our goal is to prove the following proposition.

**Proposition 6.2.** There is a monomorphism $\iota : \mathcal{U} \to \mathcal{A}$ that is a section of Fr, meaning that Fr $\circ \iota = \text{id}$. Moreover, $\mathcal{A}$ becomes a Dixmier algebra with respect to this homomorphism. Finally, $\mathcal{A}$ is free over $\iota(\mathcal{U})$.

The proof of this proposition occupies the remainder of this subsection. First of all, let $u$ denote the small quantum group, the subalgebra of $\mathcal{A}$ generated by $K_i, E_i, F_i, i = 1, \ldots, n$. For $u \in u$ set $\delta_i(u) := E_i^{(\ell)} u - u E_i^{(\ell)}, \delta'_i(u) := F_i^{(\ell)} u - u F_i^{(\ell)}$. It turns out, see [Lu4, Section 8], that $\delta_i(u), \delta'_i(u) \in u$ and so $\delta_i, \delta'_i$ are derivations of $u$.

Also Lusztig in loc.cit. proved that the map $e_i \mapsto E_i^{(\ell)}$ extends to an algebra homomorphism $\mathcal{U}^+ \to \mathcal{A}^+$ and a similar claim is true for $\mathcal{U}^-$ and $\mathcal{A}^-.$

Consider the Lie subalgebra $\mathfrak{n}$ in $\mathcal{A}$ consisting of all elements $x \in \mathcal{A}$ with $[x, u] \subset u$. Of course, $u$ is an ideal in $\mathfrak{n}$, and $E_i^{(\ell)}, F_i^{(\ell)} \in \mathfrak{n}$.
The following lemma is crucial in the proof of Proposition 6.2.

**Lemma 6.3.** The assignment \( e_i \mapsto E_i^\ell, f_i \mapsto F_i^\ell, h_i \mapsto (K_i^\ell)^0 \) extends to a Lie algebra homomorphism \( g \to n/u \).

The proof is a computation and we omit it.

Since \( g \) is semisimple, any homomorphism \( g \to n/u \) lifts to a Lie algebra homomorphism \( \mathfrak{g} \to \mathfrak{n} \) and so to an algebra homomorphism \( \iota : U \to A \). We have \( \text{Fr} \circ \iota(x) = x \in K \) for \( x = e_i, f_i, h_i \), which implies that the difference is actually 0 (again, because \( g \) is semisimple). Also, since \( \iota(e_i) - E_i^\ell, \iota(f_i) - F_i^\ell, \iota(h_i) - (K_i^\ell)^0 \) extends to a Lie algebra homomorphism \( u \), we see that \( A \) is generated by \( u \) as a left (or right) \( U \)-module, while the adjoint \( g \)-action on \( A \) is locally finite. So \( A \) is a Dixmier algebra. The same argument shows that \( A \) is free over \( \iota(U) \).

The proof is a computation and we omit it. In particular, we have

\[
\mathcal{F}_U(\bullet_\iota^\ell,e) = \mathcal{F}_U(\bullet)_\iota^\ell,e, \mathcal{F}_U(\bullet_\iota^\ell,e) = \mathcal{F}_W(\bullet_\iota^\ell,e).
\]

It follows that \( \bullet_\iota^\ell,e \) is still a quotient functor onto its image annihilating precisely the modules with non-maximal GK dimension. Also the image is closed under taking subquotients.

Now let us prove that \( \bullet_\iota^\ell,e \) has the double centralizer property. The right adjoint functor to \( \mathcal{F}_U \), that is \( \text{Hom}_U(A, \bullet) \), is exact because, by our assumption, \( A \) is a free \( U \)-module. So if \( P \) is a projective object in \( \mathcal{O}_e^r(A) \), then \( \mathcal{F}_U(P) \) is projective in \( \mathcal{O}_e^r \). Thanks to (6.1), the morphism \( P \to (P_\iota^\ell,e)^\dagger,e \) does not depend on whether we consider \( P \) as a \( A \)-module or as a \( U \)-module. Thanks to the double centralizer property for \( U \), we see that the natural homomorphism \( P \to (P_\iota^\ell,e)^\dagger,e \) is an isomorphism, and this implies the double centralizer property for \( A \).

**References**

[A] L.V. Antonyan, *On the classification of homogeneous elements of \( Z_2 \)-graded semisimple Lie algebras.* Vestn. Mosk. Univ., Ser. 1, 1982, 29-34 (in Russian). English translation in: Mosc. Univ. Math. Bull. 37 (1982), 36-43.

[BV] D. Barbasch, D. Vogan, *Unipotent representations.* Ann. Math. 121(1985), 41-110.

[BG] J. Bernstein, S. Gelfand, *Tensor products of finite and infinite dimensional representations of semisimple Lie algebras.* Compositio Mathematica, 41(1980), n.2, p. 245-285.

[BFO] R. Bezrukavnikov, M. Finkelberg, V. Ostrik, *On tensor categories attached to cells in affine Weyl groups, III.* Israel J. Math. 170(2009), 207-294.

[BL] R. Bezrukavnikov, I. Losev, *Etingof conjecture for quantized quiver varieties.* arXiv:1309.1716.

[BM] R. Bezrukavnikov, I. Mirkovic, *Representations of semisimple Lie algebras in prime characteristic and noncommutative Springer resolution.* Ann. Math. 178 (2013), n.3, 835-919.

[BGR] W. Borho, P. Gabriel, R. Rentschler, *Primideale in Einhüllenden auflösbarer Lie-Algebren.* Lecture Notes in Mathematics, Vol. 357. Springer-Verlag, 1973.
[BoKr] W. Borho, H. Kraft. Über die Gelfand-Kirillov-Dimension. Math. Ann. 220(1976), 1-24.
[Bri] J. Brundan. Mœglin’s theorem and Goldie rank polynomials in Cartan type A. Compos. Math. 147 (2011), no. 6, 1741-1771.
[BBG] J. Brown, J. Brundan, S. Goodwin. Principal W-algebras for GL(m|n). Alg. Numb. Theory 7 (2013), 1849-1882.
[BGK] J. Brundan, S. Goodwin, A. Kleshchev. Highest weight theory for finite W-algebras. IMRN 2008, no. 15, Art. ID rnn051; arXiv:0801.1337.
[BK1] J. Brundan, A. Kleshchev. Representations of shifted Yangians and finite W-algebras. Mem. Amer. Math. Soc. 196 (2008), 107 pp.
[BK2] J. Brundan, A. Kleshchev. Schur-Weyl duality for higher levels. Selecta Math., 14(2008), 1-57.
[CG] N. Chriss, V. Ginzburg. Representation theory and complex geometry. Birkhäuser, 1997.
[CM] D. Collingwood, W. McGovern. Nilpotent orbits in semisimple Lie algebras. Chapman and Hall, London, 1993.
[D] J. Dixmier. Enveloping algebras. AMS 1977.
[Gi] V. Ginzburg. Harish-Chandra bimodules for quantized Slodowy slices, Repres. Theory 13(2009), 236-271.
[GGOR] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier, On the category O for rational Cherednik algebras. Invent. Math., 154 (2003), 617-651.
[GL] I. Gordon, I. Losev, On category O for cyclotomic Rational Cherednik algebras. arXiv:1109.2315. J. Eur. Math. Soc 16 (2014), 1017-1079.
[Ja] J.C. Jantzen. Einhüllende Algebren halbeinfacher Lie-Algebren. Ergebnisse der Math., Vol. 3, Springer, New York, Tokio etc., 1983.
[Jo1] A. Joseph. Kostants problem, Goldie rank and the Gelfand-Kirillov conjecture. Invent. Math. 56(1980), N3, 191-213.
[Jo2] A. Joseph. Goldie rank in the enveloping algebra of a semisimple Lie algebra, I. J. Algebra 65(1980), 269-283.
[Jo3] A. Joseph. Goldie rank in the enveloping algebra of a semisimple Lie algebra, II. J. Algebra 65(1980), 284-306.
[Jo4] A. Joseph. Primitive ideals in enveloping algebras. Proceedings of the International Congress of Mathematicians, 1983.
[Jo5] A. Joseph. On the cyclicity of vectors associated with Duflo involutions. in Lecture Notes in Mathematics, vol. 1243, 144-188, Springer-Verlag, Berlin, 1987.
[Jo6] A. Joseph. On the associated variety of a primitive ideal. J. Algebra, 93(1985), 509-523.
[Jo7] A. Joseph. A sum rule for the scale factors in the Goldie rank polynomials. J. Algebra, 118(1988), 276-311. Addendum in: J. Algebra, 118(1988), 312-321.
[Lo1] I. Losev. Quantized symplectic actions and W-algebras. J. Amer. Math. Soc 23(2010), 34-59.
[Lo2] I. Losev. Finite dimensional representations of W-algebras. Duke Math. J. 159(2011), n.1, 99-143.
[Lo3] I. Losev. On the structure of the category O for W-algebras. Séminaires et Congrès 24(2012), 351-368.
[Lo4] I. Losev. Parabolic induction and one-dimensional representations of W-algebras. Adv. Math. 226(2011), 6, 4841-4883.
[Lo5] I. Losev. Primitive ideals in W-algebras of type A. J. Algebra, 359 (2012), 80-88.
[LO] I. Losev, V. Ostrik. Classification of finite dimensional irreducible modules over W-algebras. Compos. Math. 150(2014), no.6, 1024-1076.
[Lu1] G. Lusztig. Characters of reductive groups over a finite field, Ann. Math. Studies 107, Princeton University Press (1984).
[Lu2] G. Lusztig. Leading coefficients of character values of Hecke algebras, Proc.Symp.Pure.Math.47(2), Amer.Math.Soc. 1987, 235-262.
[Lu3] G. Lusztig. Modular representations and quantum groups. in “Classical groups and related topics” (Beijing, 1987), Contemp. Math. 82(1989), 59-77.
[Lu4] G. Lusztig. Quantum groups at roots of 1. Geom. Dedicata, 35(1990), 89-114.
[LS] G. Lusztig, N. Spaltenstein. Induced unipotent classes. J. London Math. Soc. (2), 19(1979), 41-52.
[McG1] W. McGovern. Completely prime maximal ideals and quantization. Mem. Amer. Math. Soc. 519(1994).
[McG2] W. McGovern. The adjoint representation and the adjoint action. Encyclopaedia of mathematical sciences, 131. Invariant theory and algebraic transformation groups, II, Springer Verlag, Berlin, 2002.
[Mo] C. Moeglin, *Modèles de Whittaker et idéaux primitifs complètement premiers dans les algèbres enveloppantes II*, Math. Scand. 63 (1988), 5-35.

[Pr1] A. Premet, *Irreducible representations of Lie algebras of reductive groups and the Kac-Weisfeiler conjecture*, Invent. Math. 121(1995), 79-117.

[Pr2] A. Premet, *Special transverse slices and their enveloping algebras*, Adv. Math. 170(2002), 1-55.

[Pr3] A. Premet, *Enveloping algebras of Slodowy slices and the Joseph ideal*. J. Eur. Math. Soc, 9(2007), N3, 487-543.

[Pr4] A. Premet, *Enveloping algebras of Slodowy slices and Goldie rank*. Transform. Groups, 16(2011), N3, 857-888.

[Pr5] A. Premet, *Multiplicity-free primitive ideals associated with rigid nilpotent orbits*. Transform. Groups 19 (2014), no. 2, 569-641.

[So] W. Soergel, *Kategorie O, perverse Garben und Moduln den Koinvarianten zur Weyl-grouppe*. J. Amer. Math. Soc. 3(1990), 421-445.

[St1] C. Stroppel: *Der Kombinatorikfunktor V: Graduierte Kategorie O, Hauptserien und primitive Ideale*, Dissertation Universität Freiburg i. Br. (2001).

[St2] C. Stroppel. *Category O: quivers and endomorphisms of projectives*. Repres. Theory, 7(2003), 322-345.

[W] R.B. Warfield. *Prime ideals in ring extensions*. J. London Math. Soc. 28(1983), 453-460.

[Z] L. Zhao. *Finite W-superalgebras for queer Lie algebras*. Journal of Pure and Applied Algebra, 218(2014), no.7, 1184-1194.

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