EVASIVE PROPERTIES OF SPARSE GRAPHS AND SOME LINEAR EQUATIONS IN PRIMES

IGOR E. SHPARLINSKI

Abstract. We give an unconditional version of a conditional, on the Extended Riemann Hypothesis, result of L. Babai, A. Banerjee, R. Kulkarni and V. Naik (2010) on the evasiveness of sparse graphs.

1. Introduction

A Boolean function of $m$ variables is called evasive if its deterministic query (decision-tree) complexity is $m$.

A graph property $\mathcal{P}_n$ of $n$-vertex graphs is a collection of graphs on the vertex set $\{1, \ldots, n\}$ that is invariant under relabelling of the vertices. A property $\mathcal{P}_n$ is called monotone if it is preserved under the deletion of edges. The trivial graph properties are the empty set and the set of all graphs. We say that $\mathcal{P}_n$ is evasive if the Boolean function on

$$m = \frac{n(n - 1)}{2}$$

variables, deciding whether an $n$-vertex graph given by the adjacency matrix belongs to $\mathcal{P}_n$, is evasive.

The famous Karp Conjecture asserts that any monotone nontrivial graph property is evasive, see [1, Theorem 1.4 (b)] and references therein.

Towards this conjecture, Babai, Banerjee, Kulkarni and Naik [1, Theorem 1.4 (b)] have shown that, under the Extended Riemann Hypothesis, for any fixed $\varepsilon > 0$, any nontrivial monotone property of graphs on $n$ vertices with at most $n^{5/4 - \varepsilon}$ edges is evasive for a sufficiently large $n$.

The unconditional result of [1, Theorem 1.4 (c)] is much weaker, and applies to graphs with at most $cn \log n$ edges (for some absolute constant $c > 0$).

Furthermore, under the so-called Chowla Conjecture about the smallest prime in an arithmetic progression (which goes far beyond of what the Extended Riemann Hypothesis immediately implies), Babai, Banerjee, Kulkarni and Naik [1, Theorem 1.4 (a)] show that for any fixed
\( \varepsilon > 0 \), any nontrivial monotone property of graphs on \( n \) vertices with at most \( n^{3/2-\varepsilon} \) edges is evasive for a sufficiently large \( n \).

These estimates rely on some results about the distribution of primes in arithmetic progressions. Here we show that the Bombieri-Vinogradov theorem, see [8, Theorem 17.1], is sufficient to replace the Extended Riemann Hypothesis and so we obtain the same result unconditionally which improves [1, Theorem 1.4 (c)] that gives the evasiveness for graphs on \( n \) vertices with at most \( n \log n \) edges.

**Theorem 1.** There is a function \( f(n) = n^{5/4+o(1)} \) such that any nontrivial monotone property of graphs on \( n \) vertices, with at most \( f(n) \) edges, is evasive for a sufficiently large \( n \).

Furthermore, we show that using a different approach, based on a result of Balog and Sárközy [3] about prime divisors of sum-sets (which in turn is based on sieve methods), one can obtain much stronger estimates that hold for almost all \( n \).

We need to introduce some notation. For an integer \( k \), we use \( P(k) \) to denote the largest prime divisor of \( k \) (we also set \( P(1) = 0 \)).

**Theorem 2.** Assume that for some real positive \( \alpha < 1 \) and \( A \) we have
\[
\#\{r \leq z : \text{r prime, } P(r-1) > r^{\alpha}\} \geq A \frac{z}{\log z}
\]
as \( z \to \infty \). Then for any positive \( \gamma \leq \alpha \) there is a constant \( c(\alpha, \gamma, A) > 0 \) that depends only on \( \alpha \), \( \gamma \) and \( A \) such that for all but at most \( O\left(x^{\max\{0.2\gamma-1\}}(\log x)^4\right) \) integers \( n \leq x \) any nontrivial monotone property of graphs on \( n \) vertices with at most \( c(\alpha, \gamma, A)n^{1+\gamma} \) edges, is evasive.

The standard heuristic suggests that the condition of Theorem 2 holds with any \( \alpha < 1 \). Unconditionally, by a result of Baker and Harman [2], it is known that we can take
\[\alpha = 0.677\]
for some \( A > 0 \). Thus, with \( \gamma = \alpha = 0.677 \) we derive:

**Corollary 3.** There is an absolute constant \( c > 0 \), such that for all but at most \( O(x^{0.354}(\log x)^4) \) integers \( n \leq x \) any nontrivial monotone property of graphs on \( n \) vertices, with at most \( cn^{1.677} \) edges, is evasive.

Finally, taking \( \gamma = 1/2 \) and \( \alpha = 0.677 \) in Theorem 2, we obtain an unconditional version of the bound of [1, Theorem 1.4 (a)] however with a small exceptional set.
Corollary 4. There is an absolute constant $c > 0$, such that for all but at most $O((\log x)^4)$ integers $n \leq x$ any nontrivial monotone property of graphs on $n$ vertices, with at most $cn^{3/2}$ edges, is evasive.

We note that in [1, Theorem 1.4 (a)] the bound of Corollary 4 (in a slightly weaker form $n^{3/2-\varepsilon}$ for any $\varepsilon > 0$) is established for all sufficiently large $n$ under the so-called Chowla Conjecture. However proving this conjecture seems to be far beyond the capabilities of the modern number theory.

2. Preparations

Throughout the paper, the implied constants in the symbols ‘$O$’, ‘$\ll$’ and ‘$\gg$’ may occasionally, where obvious, depend on the small real parameter $\varepsilon > 0$ and are absolute otherwise. We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are all equivalent to the assertion that the inequality $|U| \leq c|V|$ holds for some constant $c > 0$.

Our main technical tool is the following result obtained and used in [1, Section 5]. For an integer $n \geq 1$ we define the function

$$f(n) = \max_{(k,p,q,r) \in \mathcal{W}_n} \min \{ p^2k, pk, qr \},$$

where the maximum is taken over the set $\mathcal{W}_n$ of all quadruples $(k,p,q,r)$ of integers $k \geq 1$ and primes $p$, $q$, $r$ with

$$n = kp + r \quad \text{and} \quad r \equiv 1 \pmod{q}.$$  

Lemma 5. There is an absolute constant $c > 0$ such that any nontrivial monotone properties of graphs on $n$ vertices with at most $cf(n)$ edges is evasive for a sufficiently large $n$.

In [1, Section 5] individual results about the distribution of primes in arithmetic progressions, have been obtained to get lower bounds on $f(n)$ and thus on the evasiveness of sparse graphs.

Here we use several results about the distribution of primes in arithmetic on average to improve the estimates from [1, Section 5].

For integers $m > a \geq 0$ and a real $y > 0$, let

$$\psi(y; m, a) = \sum_{n \leq y} \Lambda(y),$$

where, as usual, $\Lambda$ denotes the von Mangoldt function given by

$$\Lambda(n) = \begin{cases} 
\log p & \text{if } n \text{ is a power of the prime } p, \\
0 & \text{if } n \text{ is not a prime power}.
\end{cases}$$

We also use $\varphi(m)$ to denote the Euler function of $m$. 

We now recall (a somewhat simplified) version of the Bombieri-Vinogradov theorem, see [8, Theorem 17.1].

**Lemma 6.** For every $A > 0$ there exists $B$ such that for any real $z > 1$,
\[
\sum_{m \leq z/(\log z)^B} \max_{y \leq z} \max_{\gcd(a,m)=1} \left| \frac{\psi(y; m, a) - y}{\varphi(m)} \right| \ll \frac{z}{(\log z)^A}.
\]

Finally, by a straightforward modification of a result of Balog and Sárközy [3, Theorem 2] (which in the original formulation applies to $P(a + b)$ rather than to $P(a - b)$) we have:

**Lemma 7.** There is an absolute constant $c > 0$ such that for any sets $A, B \subseteq \{1, \ldots, N\}$ with
\[
\#A \#B \geq cN \log N^2
\]
we have
\[
\max_{a \in A, b \in B} P(a - b) \gg \frac{(\#A \#B)^{1/2}}{\log N}.
\]

We recall that when both sets $A$ and $B$ are large (of cardinalities of order $N$) an improvement of Lemma 7 is given by Sárközy and Stewart [10], see also a survey of related results given by Stewart [12].

### 3. Proof of Theorem 1

Let us fix some $\varepsilon > 0$, and consider the products $m = pq$ where $p$ and $q$ are distinct primes from the interval $[n^{1/4-\varepsilon}, 2n^{1/4-\varepsilon}]$. Clearly for some constant $c > 0$ there are at least $M_1 \geq cn^{1/2-2\varepsilon}/(\log x)^2$ such values of $m$. On the other hand, by Lemma 6 applied with $A = 3$, we see that the number $M_2$ of $m \in [n^{1/2-2\varepsilon}, 4n^{1/2-2\varepsilon}]$ with
\[
\max_{y \leq n/2} \max_{\gcd(a,m)=1} \left| \frac{\psi(y; m, a) - y}{\varphi(m)} \right| \geq \frac{n}{10m}
\]
satisfies
\[
M_2 \frac{n}{4n^{1/2-2\varepsilon}} \ll \frac{n}{(\log n)^3},
\]
or
\[
M_2 \ll \frac{4n^{1/2-2\varepsilon}}{(\log n)^3}.
\]
Hence $M_2 < M_1$ for a sufficiently large $n$. We now choose any two distinct primes $p, q \in [n^{1/4-\varepsilon}, 2n^{1/4-\varepsilon}]$ such that for $m = pq$ we have
\[
\max_{y \leq n/2} \max_{\gcd(a,m)=1} \left| \frac{\psi(y; m, a) - y}{\varphi(m)} \right| < \frac{n}{10m}.
\]
In particular, if for these \( p \) and \( q \) we define \( a \in [0, m - 1] \) by the congruences
\[
a \equiv n \pmod{p} \quad \text{and} \quad a \equiv 1 \pmod{q}
\]
we have
\[
\psi(n/2; m, a) - \psi(n/4; m, a) \geq \frac{n}{4\phi(m)} - \frac{n}{5m} > 0.
\]
Thus there is a prime \( r \in [n/4, n/2] \) with \( r \equiv a \pmod{pq} \). Setting \( k = (n-r)/p \), we obtain a representation of the form (2) which implies that for the function (1)
\[
f(n) \gg n^{3/4-\varepsilon}.
\]
Since \( \varepsilon \) is arbitrary, by Lemma 5 the result now follows.

4. PROOF OF THEOREM 2

Clearly it is enough to show that the result holds for all but possibly \( O((\log x)^3) \) integers \( n \in [x/2, x] \).

From the definition of \( \alpha \) we see that there is a constant \( c_0 > 0 \) such that for the set
\[
R = \{ r \in [c_0x, x/4] : r \text{ prime, } P(r - 1) > r^{\alpha} \}
\]
we have
\[
(3) \quad \#R \gg x/\log x.
\]
Assume that for an integer \( n \in [x/2, x] \) there is with \( r \in R \) with \( P(n-r) \geq n^{\gamma} \). Taking \( p = P(n-r) \), \( q = P(r-1) \) and writing \( n = pk + r \), we see that \( pk > n/2 \). Thus for the function (1) we have
\[
f(n) \gg \min \{ np, nr, r^{1+\alpha} \} \gg n^{1+\gamma}.
\]
Let \( E \) be the set of remaining integers \( n \in [x/2, x] \) for which for all \( r \in R \) we have \( P(n-r) \leq n^{\gamma} \). We see from Lemma 7 applied with \( A = E \) and \( B = R \), that for any \( \varepsilon > 0 \) we have either
\[
\#E \#R \leq x(\log x)^2
\]
or
\[
\frac{(\#E \#R)^{1/2}}{\log x} \ll x^{\gamma}.
\]
Thus, recalling (3), we see that
\[
\#E \ll x^{\max\{0,2\gamma-1\}(\log x)^3}
\]
and the result now follows.
5. Comments

Clearly the exponent $5/4$ in Theorem 1 comes from the limit $z^{1/2+o(1)}$ of averaging in the Bombieri-Vinogradov theorem, see Lemma 6. However under the Elliott-Halberstam conjecture, which essentially asserts that the averaging in Lemma 6 can be extended up to $z^{1-\varepsilon}$ for any fixed $\varepsilon > 0$, see [8, Section 17.1], allows to replace $5/4$ with $3/2$. This is the same result as the one obtained in [1] under the Chowla conjecture. Note that the Chowla conjecture applies to individual progressions and thus may be more difficult to establish than the Elliott-Halberstam conjecture. Furthermore, under the Elliott-Halberstam conjecture, one can take any $\alpha < 1$ in Theorem 2.

Finally, we recall that there are stronger versions of this results due to Bombieri, Friedlander and Iwaniec [4] and Mikawa [9], see also a recent result of Fourvry [7]. Unfortunately all these results require some restrictions on the residues classes $a$ in $\psi(y,m,a)$ to which they apply. This makes them difficult to use for our purpose.

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Department of Computing, Macquarie University, Sydney, NSW 2109, Australia

E-mail address: igor.shparlinski@mq.edu.au