HIGHLY SYMMETRIC POVMs
AND THEIR INFORMATIONAL POWER

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Abstract. We discuss the dependence of the Shannon entropy of rank-1 nor-
malized finite POVMs on the choice of the input state, looking for the states
that minimize this quantity. To distinguish the class of measurements where
the problem can be solved analytically, we introduce the notion of highly sym-
metric POVMs and classify them in dimension two (for qubits). In this case
we prove that the entropy is minimal (and hence the relative entropy is max-
imal), when the input state is orthogonal to one of the states constituting
a POVM. The method used in the proof, employing the Michel theory of crit-
ical points for group action, the Hermite interpolation and the structure of
invariant polynomials for unitary-antiunitary groups can also be applied in
higher dimensions and for other entropy-like functions.

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1. Introduction

Uncertainty is an intrinsic property of quantum physics: typically, a measurement of an observable can yield different results for two identically prepared states. This indeterminacy can be studied by considering the probability distribution of measurement outcomes given by the Born rule, and quantized by a number that characterizes the randomness of this distribution. The Shannon entropy is the most natural tool for this purpose. Obviously, the value of this quantity is determined by the choice of the initial state of the system before the measurement. When the number of possible measurement outcomes is finite and equals \( k \), it varies from 0, if the measurement outcome is determined, to \( \ln k \), if all outcomes are equiprobable.

If the measurement is represented by a normalized rank-1 positive-operator valued measure (POVM) on a \( d \)-dimensional complex Hilbert space, then the upper bound is achieved for the maximally mixed state \( I/d \). On the other hand, the Shannon entropy of measurement cannot be 0 unless the POVM is a projection valued measure (PVM) representing projection (Lüders-von Neumann) measurement with \( k = d \), since it is bounded from below by \( \ln (d/k) \). Thus in the general case the following questions arise: how to choose the input state to minimize the uncertainty of the measurement outcomes, and what is the minimum value of the Shannon entropy for the distribution of measurement results in this case? In the present paper we call this number the entropy of measurement.

The entropy of measurement has been widely studied by many authors since the 1960s [97], also in the context of entropic uncertainty principles [35], as well as in quantum information theory under the name of minimum output entropy of a quantum-classical channel [84]. Subtracting this quantity from \( \ln k \), we get the relative entropy of measurement (with respect to the uniform distribution), which may vary from 0 to \( \ln d \). In consequence, the optimization problem now reduces to finding its maximum value. Either way, we are looking for the ‘least quantum’ or ‘most classical’ states in the sense that the measurement of the system prepared in such a state gives the most defined results. Because of concavity of the entropy of measurement as a function of state, we know that the optimal states must be pure.

Like many other optimization problems where the Shannon function \( \eta(x) = -x \ln x \) is involved, the minimization of the entropy of measurement seems to be too difficult to be solved analytically in the general case. In fact, analytical solutions have been found so far only for a few two-dimensional (qubit) cases, where the Bloch vectors of POVM elements constitute an \( n \)-gon [81, 41, 3], a tetrahedron [69] or an octahedron [76, 25]. All these POVMs are symmetric (group covariant), but, as we shall see, symmetry alone is not enough to solve the problem analytically. However, for symmetric rank-1 POVMs the relative entropy of measurement gains an additional interpretation. It follows from [69], that it is equal to the informational power of measurement [3, 4], viz., the classical capacity of a quantum-classical channel generated by the POVM [49]. To distinguish the class of measurements for which the entropy minimization problem is feasible, we define highly symmetric
(HS) normalized rank-1 POVMs as the symmetric subsets of the state space without non-trivial factors of an equal or higher symmetry. The primary aim of this paper is to present a general method of attacking the minimization problem for such POVMs and to illustrate it, entirely solving the issue in the two-dimensional case.

In dimension two, we first classify all HS-POVMs, proving that their Bloch sphere representations must be either one of the five Platonic solids or the two quasiregular Archimedean solids (the cuboctahedron and icosidodecahedron), or belong to an infinite series of regular polygons. For such POVMs we show that their entropy is minimal (and so the relative entropy is maximal), when the input state is orthogonal to one of the states constituting a POVM. We present a unified proof of this fact for all eight cases, and for five of them (the cube, icosahedron, dodecahedron, cuboctahedron and icosidodecahedron) the result seems to be new.

The proof strategy is as follows. We consider a set $S = \{\rho_j : j = 1, \ldots, k\}$ contained in the space of pure states (one-dimensional projections) $\mathcal{P}(\mathbb{C}^d)$ representing a normalized rank-1 HS-POVM. The entropy of measurement $H$ is given by $H(\rho) = \sum_{j=1}^k \eta(p_j(\rho))$, where the probabilities of the measurement outcomes are $p_j(\rho) = (d/k) \text{tr}(\rho_j \rho)$ for $\rho \in \mathcal{P}(\mathbb{C}^d)$ and $j = 1, \ldots, k$. We start from analysing the group action of $\text{Sym}(S)$, the group of unitary-antiunitary symmetries of $S$, on $\mathcal{P}(\mathbb{C}^d)$. We identify points lying in the maximal stratum for this action, called inert states in physical literature. According to the Michel theory of critical orbits of group actions [65, 67], the elements of the maximal stratum, being critical points for the entropy of measurement $H$, which is a $\text{Sym}(S)$-invariant function, are natural candidates for the minimizers. Studying their character, we see that $H$ has local minima at the inert states $\rho_j^\perp$ ($j = 1, \ldots, k$) orthogonal to the elements of $S$. To prove that these minima are indeed global we look for a simpler (polynomial) $\text{Sym}(S)$-invariant function $P$ such that:

$$P(\rho) \leq H(\rho),$$

and

$$P(\rho) = H(\rho) \text{ at } \rho_j^\perp (j = 1, \ldots, k).$$

To construct such a polynomial function we define it as $P(\rho) = \sum_{j=1}^k p(\rho_j(\rho))$, for a polynomial $p$ being a suitable Hermite approximation of $\eta$ at values $p_j(\rho_i^\perp)$ ($j = 1, \ldots, k$) for some, and hence for all, $i = 1, \ldots, k$. Now it is enough to prove that these ‘suspicious’ points are global minimizers for $P$, which is an apparently easier task. Proving that $P$ has minimizers at $\rho_j^\perp$ ($j = 1, \ldots, k$), we use the fact that the structure of invariant polynomials for any finite subgroup of the projective unitary-antiunitary group is well known for $d = 2$. Employing a priori estimates for the degree of $p$, and hence for the degree of $P$, we can show that $P$ is either constant, which completes the proof, or it is a low degree polynomial function of known $\text{Sym}(S)$-invariant polynomials, which reduces the proof to a relatively easy algebraic problem. The following two points seem to be crucial to the proof: the form of the function $\eta$ that guarantees that the Hermite interpolation polynomial $p$ bounds $\eta$ from below, and the knowledge of the subgroups of unitary-antiunitary group that may act as symmetry groups of the sets representing HS-POVMs as well as their invariant polynomials.
The method used in the present paper to solve the minimization problem for the Shannon entropy in dimension two can be generalized to higher dimensions and to other functions with similar properties as $\eta$, such as power functions leading to the Rényi entropy or its variant, the Tsallis-Havrda-Charvát entropy [28], and even more general ‘information functionals’ considered in the same context in [16]. In particular, using this method one can prove that the entropy of a group covariant SIC-POVM in dimension three is minimal, when the input state is orthogonal to one of the three subspaces spanned by triples of linearly dependent vectors constituting a SIC-POVM [92].

The problem considered in the present paper has a well-known continuous counterpart: the minimization of the Wehrl entropy over all pure states, see Sec. 6.3, where the (approximate) quantum measurement is described by an infinite family of group coherent states generated by a unitary and irreducible action of a linear group on a highly symmetric fiducial vector representing the vacuum. More than thirty years ago Lieb [60], and quite recently Lieb & Solovej [61] proved for harmonic oscillator and spin coherent states, respectively, that the minimum value of the Wehrl entropy is attained, when the state before the measurement is also a coherent state. Surprisingly, an analogous theorem need not be true in the discrete case, since the entropy of measurement need not be minimal for the states constituting the POVM. This discrepancy requires further study.

In Sec. 6.2 we show that the minimization of the entropy of measurement is also closely related to entropic uncertainty principles [96]. Indeed, every such principle leads to a lower bound for the entropy of some measurement, and conversely, such bounds may yield new uncertainty principles for single or multiple measurements. Moreover, in Sec. 6.4 we reveal the connection between the entropy of measurement and the quantum dynamical entropy with respect to this measurement [86], the quantity introduced independently by different authors to analyse the results of consecutive quantum measurements interwind with a given unitary evolution.

The rest of this paper is organized as follows. In Sec. 2 we review some of the standard material on quantum states and measurements including the generalized Bloch representation. In Sec. 3 we analyse the general notion of highly symmetric sets in metric spaces, and in Sec. 4 we apply this universal notion to normalized rank-1 POVMs. Sec. 5 contains the classification of all HS-POVMs in dimension two. Sec. 6 provides a detailed exposition of entropy and relative entropy of quantum measurement as well as their relations to the notions of informational power and Wehrl entropy, and their connections with entropic uncertainty principles and quantum symbolic dynamics. In Sec. 7 we study local minima for the entropy of measurement in dimension two, and in Sec. 8 we use Hermite interpolation and group invariant polynomials techniques to derive our main theorem and to find the global minima in this case. Finally, in Sec. 9 we apply the obtained results to give a formula for the informational power of HS-POVMs in dimension two.
2. Quantum states and POVMs

In this section we collect all the necessary definitions and facts about quantum states and measurements that can be found, e.g., in [10] or [45]. Consider a quantum system for which the associated complex Hilbert space \( \mathcal{H} \) is finite dimensional, that is \( \mathcal{H} = \mathbb{C}^d \) for some \( d = 2, 3, \ldots \). The pure states of the system can be described as the elements of the complex projective space \( \mathbb{P} \mathcal{H} = \mathbb{C} \mathbb{P}^{d-1} \) endowed with the Fubini-Study (called also procrustean after Procrustes) Kähler metric given by \( D_{FS} ([\varphi], [\psi]) := \arccos \frac{|\langle \varphi | \psi \rangle|}{\| \varphi \| \| \psi \|} \) for \( \varphi, \psi \in \mathcal{H} \) [10, 59]. In this metric there is only one geodesic between two pure states unless they are maximally remote [59, Theorem 1]. We can also identify \( \mathbb{P} \mathcal{H} \) with the set \( \mathcal{P} (\mathcal{H}) \) of one-dimensional projections in \( \mathcal{H} \) by sending \([\varphi] \rightarrow P_\varphi := |\varphi \rangle \langle \varphi | / \langle \varphi | \varphi \rangle \), where \( |\varphi \rangle \langle \varphi | \) denotes the orthogonal projection operator onto the subspace generated by \( \varphi \in \mathcal{H} \) (Dirac notation). The transferred metric on \( \mathcal{P} (\mathcal{H}) \), also called the Fubini-Study metric, is given by \( D_{FS} (\rho, \sigma) := \arccos \sqrt{\text{tr} (\rho \sigma)} \) for \( \rho, \sigma \in \mathcal{P} (\mathcal{H}) \). By \( \mathcal{S} (\mathcal{H}) \) we denote the convex closure of \( \mathcal{P} (\mathcal{H}) \), that is the set of density (Hermitian, positive semi-definite, and trace one) operators on \( \mathcal{H} \), interpreted as mixed states of the system. Note that \( \dim_{\mathbb{R}} \mathcal{P} (\mathcal{H}) = 2d - 2 \) and \( \dim_{\mathbb{R}} \mathcal{S} (\mathcal{H}) = d^2 - 1 \). By \( m_{FS} \) we denote the unique unitarily invariant measure on \( \mathbb{C} \mathbb{P}^{d-1} \) or, equivalently, on \( \mathcal{P} (\mathbb{C}^d) \).

The mixed states can be also described as elements of a \((d^2 - 1)\)-dimensional real Hilbert space (in fact, a Lie algebra) \( \mathfrak{su} (d) \) of Hermitian traceless operators on \( \mathcal{H} \), endowed with the Hilbert-Schmidt product given by \( \langle \langle \sigma, \tau \rangle \rangle_{HS} := \text{tr} (\sigma \tau) \) for \( \sigma, \tau \in \mathfrak{su} (d) \). Namely, the map defined by \( b : \mathcal{S} (\mathcal{H}) \ni \rho \rightarrow \rho - I/d \in \mathfrak{su} (d) \) gives us an affine embedding (the generalized Bloch representation) of the set of mixed (resp. pure) states into the ball (resp. sphere) in \( \mathfrak{su} (d) \) of radius \( \sqrt{1 - d^{-1}} \), called the generalized Bloch ball (resp. the Bloch sphere). Only for \( d = 2 \) the map is onto, and for \( d > 2 \) its image (the Bloch vectors) constitute a ‘thick’ though proper subset of \((d^2 - 1)\)-dimensional ball, containing the (maximal) ball of radius \( 1/\sqrt{d (d - 1)} \) centered at 0. On the other hand, for \( d > 2 \), \( B (d) := b (\mathcal{P} (\mathcal{H})) \), the image of the space of pure states via \( b \), constitutes a ‘thin’ \((2d - 2)\)-dimensional submanifold of the \((d^2 - 2)\)-sphere. The metric spaces \( \mathcal{P} \mathcal{H}, D_{FS} \) and \( (B (d), D_B) \), where \( D_B \) is the great arc distance on the Bloch sphere, though non-isometric for \( d > 2 \), nevertheless are ordinally equivalent, as the distances \( D_{FS} \) and \( D_B \) are related by the formula \( D_B \left( b \left( |\varphi \rangle \langle \varphi | \right), b \left( |\psi \rangle \langle \psi | \right) \right) = \gamma \left( D_{FS} \left( |\varphi \rangle \langle \varphi |, |\psi \rangle \langle \psi | \right) \right) \) \( (\varphi, \psi \in \mathcal{H}) \), where a convex function \( \gamma : [0, \pi/2] \rightarrow \mathbb{R}^+ \) is given by \( \gamma (x) = \sqrt{1 - d^{-1}} \arccos \frac{x}{d-1} \) for \( 0 \leq x \leq \pi/2 \). In other words, scalar products of state vectors in \( \mathbb{C}^d \) and their images in \( \mathbb{R}^{d^2 - 1} \) fulfill the relation: \( |\langle \varphi | \psi \rangle|^2 = \langle b \left( |\varphi \rangle \langle \varphi | \right), b \left( |\psi \rangle \langle \psi | \right) \rangle_{HS} + 1/d \).

With a measurement of the system with a finite number \( k \) of possible outcomes one can associate a positive operator valued measure (POVM) defining the probabilities of the outcomes. A finite POVM is an ensemble of positive non-zero Hermitian operators \( \Pi_j \) \( (j = 1, \ldots, k) \) on \( \mathcal{H} \) that sum to the identity operator, i.e. \( \sum_{j=1}^k \Pi_j = \mathbb{I} \). If the state of the system before the measurement (the input state) is \( \rho \), then the probability \( p_j (\rho) \) of the \( j \)-th outcome is given by the Born rule,
measurement is \( \rho \). In general situation, there is an infinite number of completely positive maps (measurement instruments in the sense of Davies and Lewis [32]) describing conditional state changes due to the measurement and producing the same measurement statistics. They are given by the solutions of the set of equations \( \Pi_j = A_j^* A_j \) \((j = 1, \ldots, k)\), where \( A_j \) are bounded operators on \( \mathcal{H} \). If \( \rho \) is the input state and the measurement outcome is \( j \), then the state of the system after the measurement is \( \rho_j^{post} = A_j \rho A_j^*/\rho_j(\rho) \).

A special class of POVMs are normalized rank-1 POVMs, where \( \Pi_j \) \((j = 1, \ldots, k)\) are rank-1 operators and \( \text{tr}(\Pi_j) = \text{const}(j) = d/k \). Necessarily, \( k \geq d \) in this case, and there exists an ensemble of pure states \( \rho_j \in \mathcal{P}(\mathcal{H}) \) \((j = 1, \ldots, k)\) such that \( \Pi_j = (d/k) \rho_j \). Thus, \( \sum_{j=1}^{k} \rho_j = (k/d) \mathbb{1} \), and so a normalized rank-1 POVM can be also defined as a (multi-) set of points in \( \mathcal{P}(\mathcal{H}) \) that constitutes a uniform (or normalized) tight frame in \( \mathcal{P}(\mathcal{H}) \) [37, 9, 19], that is an ensemble that fulfills \( \sum_{j=1}^{k} \text{tr}(\rho_j \sigma) = k/d \) for every \( \sigma \in \mathcal{P}(\mathcal{H}) \). In this case we shall say that \( \rho_j \) \((j = 1, \ldots, k)\) constitute a POVM. Equivalently, we can define normalized rank-1 POVMs as complex 1-designs, where by a complex projective \( t \)-design \((t \in \mathbb{N})\) we mean an ensemble \( \{\rho_j : j = 1, \ldots, k\} \) such that

\[
(1) \quad \frac{1}{k^2} \sum_{j,m=1}^{k} f(\text{tr}(\rho_j \rho_m)) = \int_{\mathcal{P}(\mathbb{C}^d)} \int_{\mathcal{P}(\mathbb{C}^d)} f(\text{tr} (\rho \sigma)) \, dm_{FS}(\rho) \, dm_{FS}(\sigma)
\]

for every \( f : \mathbb{R} \to \mathbb{R} \) polynomial of degree \( t \) or less [82]. The equality \( \sum_{j=1}^{k} \rho_j = (k/d) \mathbb{1} \) is in turn equivalent to \( \sum_{j=1}^{k} b(\rho_j) = 0 \), which gives the following simple characterization of normalized rank-1 POVMs in the language of Bloch vectors:

**Proposition 1.** The generalized Bloch representation gives a one-to-one correspondence between finite normalized rank-1 POVMs and finite (multi-) sets of points in \( B(d) \) with its center of mass at 0.

The probabilities of the measurement outcomes in the generalized Bloch representation take the form \( p_j(\rho) = (d/k) \text{tr}(\rho_j \rho) = (d \cdot \langle \langle b(\rho_j), b(\rho) \rangle \rangle_{HS} + 1)/k \) for \( \rho \in \mathcal{P}(\mathbb{C}^d) \) and \( j = 1, \ldots, k \).

3. **Symmetric, resolving and highly symmetric sets in metric spaces**

In this section we present a framework to investigate the concept of symmetry in metric spaces. Let us start from general definition. Let \( S \) be a subset of a metric space \((X, r)\). By Sym \((S)\) we denote the group of symmetries of \( S \), that is, the group of all isometries leaving \( S \) invariant. We call \( S \) symmetric if Sym \((S)\) acts transitively on \( S \), i.e., for every \( x, y \in S \) there exists \( f \in \text{Sym}(S) \) such that \( f(x) = y \).

We say that \( S \) is a resolving set (metric basis) if and only if \( r(a, x) = r(b, x) \) for every \( x \in S \) implies \( a = b \), for \( a, b \in X \). The following proposition belongs to folklore:
Proposition 2. If $S$ is symmetric and resolving, then $f|_S = g|_S$ implies $f = g$ for every $f, g \in \text{Sym}(S)$. Moreover, if $S$ is finite, then $\text{Sym}(S)$ is finite.

Proof. Let $f, g \in \text{Sym}(S)$, $f|_S = g|_S$, and $a \in X$. Then, for every $x \in S$ we have $r(fa,x) = r(a,f^{-1}x) = r(a,g^{-1}x) = r(ga,x)$. Hence $fa = ga$. Now, if $|S| = k$, then $\text{Sym}(S)$ is a subgroup of the symmetric group $S_k$, and so is finite. □

To single out sets of higher symmetry we have to recall some notions from the general theory of group action, see e.g. [38]. Let $G$ be a group acting on $X$. For $x \in X$ we define its orbit as $Gx := \{gx : g \in G\}$ and its stabilizer (or isotropy subgroup) $G_x$ as the set of elements in $G$ that fix $x$, i.e. $G_x := \{h \in G : hx = x\}$. Obviously, two points lying on the same orbit have conjugate stabilizers, since $G_{gx} = gG_xg^{-1}$ for $x \in X$ and $g \in G$. The points of $X$ with the same stabilizers up to a conjugacy are said to be of the same isotropy type, which is a measure of symmetry of points (orbits). The points of the same isotropy type as $x$ forms the orbit stratum $\Sigma_x$. The decomposition of $X$ into orbit strata is called the orbit stratification. Clearly, it induces a stratification of the orbit space $X/G$. The natural partial order on the set of all conjugacy classes of subgroups of $G$ induces the order on the set of strata, namely, $\Sigma_x \prec \Sigma_y$ if and only if there exists $g \in G$ such that $G_x \subset gG_yg^{-1}$ for $x, y \in X$, so that the maximal strata consist of points with maximal stabilizers.

Assume now that $S$ is symmetric and consider the action of the group $\text{Sym}(S)$ on $X$. Clearly, the whole set $S$ is contained in one orbit and hence in one stratum. We shall say that $S$ is highly symmetric if and only if this stratum is maximal. A symmetric set is highly symmetric if and only if it has not a non-trivial factor of an equal or higher symmetry:

Proposition 3. Let $S \subset X$ be symmetric. Then $S$ is highly symmetric if and only if every $\text{Sym}(S)$-equivariant map $h : S \to X$ (i.e. such that $gh(x) = h(gx)$ for every $g \in \text{Sym}(S)$ and for some (and hence all) $x \in S$) is one-to-one.

Proof. If $|S| = 1$, then the proposition is trivial. Assume that $|S| \geq 2$ and put $G := \text{Sym}(S)$. If $S$ is not highly symmetric, then there exist $x \in S = Gx$ and $y \notin S$ such that $G_x \subsetneq G_y$. Put $h(gx) = g(y)$ for $g \in G$. Clearly $h$ is not one-to-one, since otherwise $G_y \subset G_x$, which is a contradiction. On the other hand, take $h : S \to X$ such that $gh(x) = h(gx)$ for all $x \in S$ and $g \in \text{Sym}(S)$. If there exist $x \in S$ and $g \in \text{Sym}(S)$ such that $x \neq gx$ and $h(x) = h(gx)$, then we have $G_x \subset G_{h(x)}$ and $g \in G_{h(x)} \backslash G_x$, a contradiction. □

It is interesting that an analogous idea was explored almost fifty years ago by Zajtz who defined so called primitive geometric objects in quite similar manner as highly symmetric sets defined above and proved the fact parallel to Proposition [3][102, Theorem 1].
4. Symmetric, Informationally Complete and Highly Symmetric Normalized Rank-1 POVMs

To apply these general definitions to normalized rank-1 POVMs, note that from the celebrated Wigner theorem \[98\] it follows that for every separable Hilbert space \( \mathcal{H} \) the group of isometries of \((\mathcal{P}(\mathcal{H}), D_{FS})\) (quantum symmetries) is isomorphic to the projective unitary-antiunitary group \( \text{PUA}(\mathcal{H}) \), consisting of unitary and antiunitary transformations of \( \mathcal{H} \) defined up to phase factors, see also \[20, 21, 54, 39\]. To be more precise, each such isometry is given by the map \( \sigma_U : \mathcal{P}(\mathcal{H}) \ni \rho \mapsto U\rho U^* \in \mathcal{P}(\mathcal{H}) \) for a unitary or antiunitary \( U \), and two such isometries coincide if and only if the corresponding transformations differ only by a phase. Equivalence classes of unitary isometries form a normal subgroup of \( \text{PUA}(\mathcal{H}) \) of index 2, namely the projective unitary group \( \text{PU}(\mathcal{H}) \). Clearly, every such isometry can be uniquely extended to a continuous affine map on \( S(\mathcal{H}) \).

If \( \mathcal{H} = \mathbb{C}^d \), then the generalized Bloch representation gives a one-to-one correspondence between the compact group \( \text{PUA}(d) \) and the group of isometries of the unit sphere in \((d^2-1)\)-dimensional real vector space \( \mathfrak{su}(d) \) endowed with the Hilbert-Schmidt product, whose action leaves the set of Bloch vectors \( B(\mathcal{H}) \) invariant. This correspondence is given by \([U] \rightarrow \{ \rho \rightarrow U\rho U^* : \rho \in \mathfrak{su}(d) \}\) for \( U \in \text{PUA}(d) \) \[5\]. Hence \( \text{PUA}(d) \) is isomorphic to a subgroup of the orthogonal group \( O(d^2-1) \). Moreover, \( m_{FS} \) is the unique \( \text{PUA}(d) \)-invariant measure on \( \mathcal{P}(\mathbb{C}^d) \cong \mathbb{CP}^{d-1} \). In particular, for \( d = 2 \), we have \( \text{PUA}(2) \cong O(3) \), and so all quantum symmetries of qubit states can be interpreted as rotations (for unitary symmetries, as \( \text{PU}(2) \cong SO(3) \)), reflections or rotoreflections of the three dimensional Euclidean space.

Taking this into account we can transfer the notions of symmetry and high symmetry from \( \mathcal{P}(\mathbb{C}^d) \) to finite normalized rank-1 POVMs in \( \mathbb{C}^d \). Let \( \Pi = (\Pi_j)_{j=1,...,k} \) be a finite normalized rank-1 POVM in \( \mathbb{C}^d \) and \( S \) be a corresponding set of pure quantum states. We say that

- \( \Pi \) is a symmetric POVM \( \iff S \) is symmetric in \((\mathcal{P}(\mathbb{C}^d), D_{FS})\);
- \( \Pi \) is a highly symmetric POVM (HS-POVM) \( \iff S \) is highly symmetric in \((\mathcal{P}(\mathbb{C}^d), D_{FS})\).

For finite normalized rank-1 measurements symmetric POVMs coincide with group covariant POVMs introduced by Holevo \[47\] and studied since then by many authors. We say that a measurement \( \Pi = (\Pi_{ij})_{j=1,...,k} \) is \( G \)-covariant for a group \( G \) if and only if there exists \( G \ni g \rightarrow \sigma_U \in \text{PUA}(d) \), a projective unitary-antiunitary representation of \( G \) (i.e. a homomorphism from \( G \) to \( \text{PUA}(d) \)), and a surjection \( s : G \rightarrow \{1,...,k\} \) such that \( \sigma_{U_g}(\Pi_{sh}) = U_g\Pi_{sh}U_g^* = \Pi_{sg} \) for all \( g,h \in G \). Let \( \Pi \) be a finite normalized rank-1 POVM in \( \mathbb{C}^d \) and \( S \) be a corresponding set of pure quantum states. It is clear that a symmetric finite normalized rank-1 POVM is \( \text{Sym}(S) \)-covariant, and, conversely, if a finite normalized rank-1 POVM is \( G \)-covariant, then \( (\sigma_U)_{g \in G} \) is a subgroup of the group of isometries of
We call the representation irreducible if and only if \( I/d \) is the only element of \( S(\mathbb{C}^d) \) invariant under action of the representation. It follows from the version of Schur’s lemma for unitary-antiunitary maps \([36, \text{Theorem II}]\) that this definition coincides with the classical one. Irreducibility of the representation can be also equivalently expressed as follows: for any pure state \( \rho \in \mathcal{P}(\mathbb{C}^d) \) its orbit under the action of the representation generates a rank-1 \( G \)-covariant POVM, i.e., 

\[
\frac{1}{|G|} \sum_{g \in G} \sigma_{U_g}(\rho) = I/d,
\]

see also \([93]\).

Our definition of highly symmetric POVMs resembles the definition of highly symmetric frames introduced by Broome and Waldron \([13, 14, 94]\). However, they consider subsets of \( \mathbb{C}^d \) rather than \( \mathbb{C}P^{d-1} \) and unitary symmetries rather than projective unitary-antiunitary symmetries.

The next proposition clarifies the relations between the properties of the set of pure states constituting a finite normalized rank-1 POVM and the properties of its Bloch representation. We call a normalized rank-1 POVM \( \Pi = (\Pi_j)_{j=1, \ldots, k} \) informationally complete (resp. purely informationally complete) if and only if the probabilities \( p_j(\rho) \) \( (j = 1, \ldots, k) \) determine uniquely every input state \( \rho \in S(\mathbb{C}^d) \) (resp. \( \mathcal{P}(\mathbb{C}^d) \)). The following result provides necessary and sufficient conditions for informational completeness and purely informational completeness:

**Proposition 4.** Let \( \Pi = (\Pi_j)_{j=1, \ldots, k} \) be a finite normalized rank-1 POVM in \( \mathbb{C}^d \) and \( S := \{\rho_j : j = 1, \ldots, k\} \) be a corresponding set of pure quantum states, i.e., \( \rho_j \in \mathcal{P}(\mathbb{C}^d) \) and \( \Pi_j = (d/k)\rho_j \) for \( j = 1, \ldots, k \). Let us consider the following properties:

1. \( S \) is a complex projective 2-design;
2. \( b(S) \) is a tight frame in \( \mathfrak{su}(d) \);
3. \( \Pi \) is informationally complete;
4. \( b(S) \) generates \( \mathfrak{su}(d) \);
5. \( b(S) \) is a frame in \( \mathfrak{su}(d) \);
6. \( \Pi \) is purely informationally complete;
7. \( S \) is a resolving set in \( (\mathcal{P}(\mathbb{C}^d), D_{FS}) \);
8. \( b(S) \) is a resolving set in \( (B(d), D_B) \).

Then \( (a) \iff (b) \iff (c) \iff (d) \iff (e) \iff (f) \iff (g) \iff (h) \). Moreover, if \( d = 2 \), then \( (f) \implies (c) \).

**Proof.** It is obvious that \( (b) \implies (e) \) and \( (c) \implies (f) \). The proof of \( (a) \iff (b) \) can be found in \([82, \text{Proposition 13}]\), and \( (c) \iff (d) \) in \([45, \text{Proposition 3.51}]\). It is well known that in finite dimensional spaces frames are generating sets, hence \( (d) \iff (e) \). Furthermore, \( (f) \iff (g) \iff (h) \) follows from the fact that the distances \( D_{FS} \) and \( D_B \) are ordinally equivalent, and from the equality \( \text{tr}(\rho\sigma) = \cos^2 D_{FS}(\rho, \sigma) \) for \( \rho, \sigma \in \mathcal{P}(\mathbb{C}^d) \). Moreover, for \( d = 2 \) the notions of purely informational completeness and informational completeness coincide \([44]\). \( \square \)
Note that (c) does not imply (b), even if $S$ is symmetric and $d = 2$. To show this, consider $S \subset P(C^2)$ such that $b(S) = \{2^{-1/2}(e_1 + e_2), 2^{-1/2}(-e_1 + e_3)\}$, where $\{e_1, e_2, e_3\}$ is any orthonormal basis of $su(2)$. Then $b(S)$ is a tetragonal disphenoid with the antiprismatic symmetry group $D_{2d}$. Clearly, $b(S)$ is a frame in $su(2)$, but not tight. On the other hand, one can prove $(c) \Rightarrow (b)$, under the additional assumption that the natural action of $\text{Sym}(S)$ on $su(d)$ is irreducible, applying [93, Theorem 6.3]. Moreover, as we shall see in the next section, all the conditions above are equivalent if $S$ is highly symmetric and $d = 2$.

5. Classification of highly symmetric POVMs in dimension two

**Theorem 1.** There are only eight types of HS-POVMs in two dimensions, seven exceptional informationally complete HS-POVM represented in $\mathbb{R}^3$ by five Platonic solids (the tetrahedron, cube, octahedron, icosahedron and dodecahedron) and two Archimedean solids (the cuboctahedron and icosidodecahedron), and an infinite series of non informationally complete HS-POVMs represented in $\mathbb{R}^3$ by regular polygons, including digon.

**Proof.** Let $S = \{\rho_j : j = 1, \ldots, k\} \subset P(C^2) \simeq \mathbb{C}P$ constitutes a HS-POVM, and let $B := b(S) \subset S^2$. Put $G := \text{Sym}(B)$. Then it follows for Proposition 4 that either $B$ is contained in a proper (one- or two-dimensional) subspace of $\mathbb{R}^3$, or the POVM is informationally complete and, according to Proposition 2, $G$ is finite.

If $G$ is infinite, then necessarily the stabilizer of any element $x \in B$ has to be infinite. As the only linear isometries of $\mathbb{R}^3$ leaving possibly $x$ invariant are either rotations about the axis $l_x$ through $x$, or reflections in any plane containing $l_x$, the stabilizer $G_x$ has to contain an infinite subgroup of rotations about $l_x$. Thus the orbit of any point beyond $l_x$ under $G$ must be infinite. In consequence, $B = \{-x, x\}$, and $G = D_{\infty h} \simeq O(2) \times C_2$.

If $G$ is finite, it must be one of the point groups, i.e., finite subgroups of $O(3)$. The complete characterization of such subgroups has been known for very long time [3]: there exist seven infinite families of axial (or prismatic) groups $C_n, C_{nv}, C_{nh}, S_{2n}, D_{n}, D_{nd}$ and $D_{nh}$, as well as seven additional polyhedral (or spherical) groups: $T$ (chiral tetrahedral), $T_d$ (full tetrahedral), $T_h$ (pyritohedral), $O$ (chiral octahedral), $O_h$ (full octahedral), $I$ (chiral icosahedral) and $I_h$ (full icosahedral). Analysing their standard action on $S^2$ (see e.g. [74, 62, 67, 103, 72]), one can find in all cases the orbits with maximal stabilizers. Gathering this information together, we get all highly symmetric finite subsets of $S^2$, and so all HS-POVMs in two dimensions. These sets are listed in Tab. 1 together with their symmetry groups and the stabilizers of their elements with respect to these symmetry groups. For all but the first two types of HS-POVMs, the symmetry group $G$ is a polyhedral group, and so it acts irreducibly on $\mathbb{R}^3$. Hence, $B$ must be a tight frame in all these cases.
convex hull of the orbit | cardinality of the orbit | group | stabilizer
-----|---------------|-----|-----
digon | 2 | $D_{\infty h}$ | $C_{\infty v}$
regular n-gon $(n \geq 3)$ | $n$ | $D_{n h}$ | $C_{2 v}$
tetrahedron | 4 | $T_d$ | $C_{3 v}$
octahedron | 6 | $O_h$ | $C_{4 v}$
cube | 8 | $O_h$ | $C_{3 v}$
cuboctahedron | 12 | $O_h$ | $C_{2 v}$

Tab. 1. HS-POVMs in dimension two, with their cardinalities, symmetry groups and stabilizers of elements (in Schoenflies notation).

Classification of all finite symmetric subsets of $S^2$ and, in consequence, all symmetric normalized rank-1 POVMs in two dimensions, is of course more complicated than for highly symmetric case. In particular, the number of such non-isometric subsets is uncountable. However, since each symmetric subset generates a vertex-transitive polyhedron in three-dimensional Euclidean space, the task reduces to classifying such polyhedra, which was done by Robertson and Carter in the 1970s, see [74, 75, 73, 26]. They proved that the transitive polyhedra in $\mathbb{R}^3$ can be parameterized (up to isometry) by metric space (with the Hausdorff distance under the action of Euclidean isometries related closely to the Gromov-Hausdorff distance, see [64]), which is a two-dimensional CW-complex with 0-cells corresponding exactly to highly symmetric subsets of $S^2$.

Note that not only ‘regular polygonal’ POVMs (e.g. the trine or ‘Mercedes-Benz’ measurement for $n = 3$; the ‘Chrysler’ measurement for $n = 5$), but also ‘Platonic solid’ POVMs have been considered earlier by several authors in various contexts, see for instance [18, 22, 33, 15].

6. Entropy and relative entropy of measurement

6.1. Definition. Let $\Pi = (\Pi_j)_{j=1,...,k}$ be a finite POVM in $\mathbb{C}^d$. We shall look for the most ‘classical’ or ‘coherent’ quantum states, i.e. for the states that minimize the uncertainty of the outcomes of the measurement. This uncertainty can be measured by the quantity called the entropy of measurement given by

$$H(\rho, \Pi) := \sum_{j=1}^{k} \eta \left( p_j (\rho, \Pi) \right),$$

for $\rho \in \mathcal{S} (\mathbb{C}^d)$, where the probability $p_j (\rho, \Pi)$ of the $j$-th outcome ($j = 1, \ldots, k$) is given by $p_j (\rho, \Pi) := \text{tr} (\rho \Pi_j)$, and the Shannon entropy function $\eta : [0, 1] \to \mathbb{R}^+$ by $\eta (x) := -x \ln x$ for $x > 0$, and $\eta (0) := 0$. (In the sequel, we shall use
frequently the identity \( \eta(xy) = \eta(x)y + \eta(y)x \), \( x, y \in [0, 1] \). Thus, the entropy of measurement \( H(\rho, \Pi) \) is just the Boltzmann-Shannon entropy of the probability distribution of the measurement outcomes, assuming that the state of the system before the measurement was \( \rho \). This quantity (as well as its continuous analogue) has been considered by many authors, first in the 1960s under the name of Ingarden-Urbanik entropy or \( A \)-entropy, then, since the 1980s, in the context of entropic uncertainty principles \[35, 57, 63, 96\], and also quite recently for more general statistical theories \[88, 85\]. Wilde called it the Shannon entropy of POVM \[99\].

It is also equal to the minimum output entropy of the quantum-classical channel \( \Phi : S(\mathbb{C}^d) \rightarrow S(\mathbb{C}^k) \) generated by \( \Pi \) and given by \( \Phi(\rho) = \sum_{j=1}^{k} \text{tr}(\rho \Pi_j) |e_j\rangle \langle e_j| \), where \( (|e_j\rangle)_{k \sum_{j=1}^{k}} \) is any orthonormal basis in \( \mathbb{C}^k \) \[84\]. For a history of this notion see \[97\] and \[6\]. The function \( H(\cdot, \Pi) : S(\mathbb{C}^d) \rightarrow \mathbb{R} \) is continuous and concave. In consequence, it attains minima in the set of pure states.

It is easy to show that the entropy of measurement has to fulfill the following bounds \[59\] Sec. 2.3:

\[
S(\rho) + \ln(k/d) \leq H(\rho, \Pi) \leq \ln k,
\]

where \( S(\rho) \) is the von Neumann entropy of the state \( \rho \). The upper bound is achieved for the maximally mixed state \( \rho_* := I/d \). Moreover for \( \rho \in S(\mathbb{C}^d) \), \( S(\rho) = \min H(\rho, \Pi) \), where the minimum is taken over all normalized rank-1 POVMs \( \Pi \), see e.g. \[99\] Sec. 11.1.2.

It is sometimes much more convenient to work with the relative entropy of measurement (with respect to the uniform distribution) \[43\] p. 67 that measures non-uniformity of the distribution of the measurement outcomes and is given by

\[
\tilde{H}(\rho, \Pi) := \ln k - H(\rho, \Pi),
\]

and to look for the states that maximize this quantity. Clearly, it follows from \[7\] that the relative entropy of measurement is bounded from below by 0, and from above by the relative von Neumann entropy of the state \( \rho \) with respect to the maximally mixed state \( I/d \):

\[
0 \leq \tilde{H}(\rho, \Pi) \leq S(\rho/\rho_*) \leq \ln d.
\]

The problem of minimizing entropy (and so maximizing relative entropy) is strictly connected with the problem of maximization of the mutual information between ensembles of initial states (classical-quantum states) and the POVM \( \Pi \).

Let \( V = \{p_i, \tau_i\}_{i=1}^{l} \), where \( p_i \geq 0 \) are \( a \text{ priori} \) probabilities of density matrices \( \tau_i \in S(\mathbb{C}^d) \), where \( i = 1, \ldots, l \), and \( \sum_{i=1}^{l} p_i = 1 \). Then the mutual information between \( V \) and \( \Pi \) is given by:

\[
I(V, \Pi) := I(P) := \sum_{i=1}^{l} \eta \left( \sum_{j=1}^{k} P_{ij} \right) + \sum_{j=1}^{k} \eta \left( \sum_{i=1}^{l} P_{ij} \right) - \sum_{i=1}^{l} \sum_{j=1}^{k} \eta(P_{ij}),
\]

where \( P_{ij} := p_i \text{tr}(\tau_i \Pi_j) \) for \( i = 1, \ldots, l \) and \( j = 1, \ldots, k \). The problem of maximization of \( I(V, \Pi) \) consists of two dual aspects \[4, 48, 50\]: maximization over all
possible measurements, providing the ensemble \( V \) is given, see e.g. [46, 31, 79, 91], and (less explored) maximization over ensembles, when the POVM \( \Pi \) is fixed [3, 69]. In the former case, the maximum is called accessible information. In the latter case, Oreshkov et al showed that there exists an ensemble consisting of pure states only maximizing the mutual information [69]. Moreover, they proved that for a symmetric POVM one can find a maximizer consisting of equiprobable elements of the orbit of a pure state under the action of the symmetry group of the set of pure states \( S \) constituting POVM. Simple calculations lead us to the observation that for every such ensemble of the form \( V(\rho) := \frac{1}{|\text{Sym}(S)|} \sigma(\rho) \), where \( \rho \in S \) we have

\[
I(V(\rho), \Pi) = \tilde{H}(\rho, \Pi).
\]

Hence, these two maximization problems: of the relative entropy over mixed or pure states and of the mutual information over ensembles are equivalent in this case, i.e.

\[
\max_{V\text{-ensemble}} I(V, \Pi) = \max_{\rho \in S(\mathbb{C}^d)} \tilde{H}(\rho, \Pi) = \max_{\rho \in \mathcal{P}(\mathbb{C}^d)} \tilde{H}(\rho, \Pi).
\]

Dall’Arno et al [3, 4] introduced the name informational power of \( \Pi \) for this maximum and denoted it by \( W(\Pi) \). In [49] this quantity was identified as the classical capacity of quantum-classical channel. We shall compute its value for all HS-POVMs in dimension two in Sec. 9.

6.2. Relation to entropic uncertainty principles. The entropic uncertainty principles form another area of research, related closely to entropy minimization. In fact, any lower bound for the entropy of measurement can be regarded as an entropic uncertainty relation for single measurement [57]. Moreover, combining \( m \) normalized rank-1 POVMs \( \Pi^i = (\Pi^i_j)_{j=1,\ldots,k_i} \) \((i = 1, \ldots, m)\) we obtain another normalized rank-1 POVM \( \Pi := (\frac{1}{m} \Pi^i_j)_{j=1,\ldots,k_i} \). Now, from an entropic uncertainty principle for \((\Pi^i)_{i=1,\ldots,m}\) written in the form \( \frac{1}{m} \sum_{i=1,\ldots,m} H(\rho, \Pi^i) \geq C > 0 \) [96, p. 3] we get automatically a lower bound for entropy of \( \Pi \), namely

\[
H(\rho, \Pi) = \frac{1}{m} \sum_{i=1,\ldots,m} H(\rho, \Pi^i) + \ln m \geq C + \ln m
\]

for \( \rho \in \mathcal{P}(\mathbb{C}^d) \).

To be more specific, assume now that \( \Pi^i_j = (d/k) \rho^i_j \), where \( \rho^i_j = |\varphi^i_j\rangle \langle \varphi^i_j| \in \mathcal{P}(\mathbb{C}^d) \), denoting their Bloch vectors by \( x^i_j := b(\rho^i_j) \in \mathfrak{su}(d) \simeq \mathbb{R}^{d^2-1} \) for \( j = 1, \ldots, k, i = 1, 2 \). The Krishna-Parthasarathy entropic uncertainty principle [57, Corollary 2.6] gives us

\[
\ln \frac{2k}{d} - \frac{1}{2} \ln \left( \max_{j,l=1,\ldots,k} \langle x^1_j, x^2_l \rangle_{HS} + 1/d \right) = \ln \frac{2k}{d} - \ln \max_{j,l=1,\ldots,k} |\langle \varphi^1_j | \varphi^2_l \rangle| \leq H(\rho, \Pi)
\]
for \( \rho \in \mathcal{P}(\mathbb{C}^d) \). In consequence, we get an upper bound for relative entropy

\[
\tilde{H}(\rho, \Pi) \leq \ln d + \frac{1}{2} \ln \left( \max_{j,l=1,\ldots,k} \langle x_j^1, x_l^2 \rangle_{HS} + 1/d \right)
\]

\[
= \ln d + \frac{1}{2} \ln \left( (1 - d^{-1}) \left( \max_{j,l=1,\ldots,k} \cos (2\theta_{jl}) \right) + d^{-1} \right),
\]

where \( \theta_{jl} := \angle (x_j^1, x_l^2) / 2 \) for \( j, l = 1, \ldots, k \). As this upper bound does not depend on the input state \( \rho \), it gives us also an upper bound for the informational power of \( \Pi \).

If \( d = 2 \), this inequality takes a simple form

\[
\tilde{H}(\rho, \Pi) \leq \ln 2 + \ln \max_{j,l=1,\ldots,k} |\cos \theta_{jl}|.
\]

We may use this bound, e.g. for the ‘rectangle’ POVM analysed in Sec. 7 that can be treated as the aggregation of two pairs of antipodal points on the sphere representing two PVM measurements. In this case we deduce from (12) that \( \tilde{H} \leq \ln 2 + \ln \max (|\sin (\alpha/2)|, |\cos (\alpha/2)|) \), where \( \alpha \) is the measure of the angle between the diagonals of the rectangle. In particular, for the ‘square’ POVM we get \( \tilde{H} \leq \frac{1}{2} \ln 2 \). As we shall see in Sec. 9 this bound is actually reached for each of four states constituting the POVM and represented by the vertices of the square.

### 6.3. Relation to Wehrl entropy minimization.

Let \( \Pi = (\Pi_j)_{j=1,\ldots,k} \) be a finite normalized rank-1 POVM in \( \mathbb{C}^d \) and \( S := \{\rho_j : j = 1, \ldots, k\} \) be a corresponding (multi-)set of pure quantum states, i.e., \( \rho_j \in \mathcal{P}(\mathbb{C}^d) \) and \( \Pi_j = (d/k) \rho_j \) for \( j = 1, \ldots, k \). Then we get after simple calculations

\[
H(\rho, \Pi) = \frac{d}{k} \sum_{j=1}^k \eta(\text{tr} (\rho \rho_j)) - \ln (d/k)
\]

and so

\[
\tilde{H}(\rho, \Pi) = \ln d - \frac{d}{k} \sum_{j=1}^k \eta(\text{tr} (\rho \rho_j))
\]

for \( \rho \in \mathcal{P}(\mathbb{C}^d) \). Assume now, that \( \Pi \) is symmetric (group covariant) and put \( G := \text{Sym}(S) \). Then for each \( \tau \in S \) we have \( S = \{g\tau : g \in G\} \) and

\[
\tilde{H}(\rho, \Pi) = \ln d - \frac{d}{|G|} \sum_{g \in G} \eta(\text{tr} (\rho (g\tau)))
\]

\[
= \ln d - \frac{d}{|S|} \sum_{[g] \in G/G_{\tau}} \eta(\text{tr} (\rho (g\tau)))
\]

for \( \rho \in \mathcal{P}(\mathbb{C}^d) \). The same formulae are true for any subgroup of \( \text{Sym}(S) \) acting transitively on \( S \). Note that the behaviour of the functions \( H(\cdot, \Pi), \tilde{H}(\cdot, \Pi) : \mathcal{P}(\mathbb{C}^d) \to \mathbb{R}^+ \) depends only on the choice of the pure state \( \tau \) called a fiducial or
reference state. Moreover, observe that both functions are $G$-invariant, as for $\rho \in \mathcal{P}(\mathbb{C}^d)$ we have

$$\tilde{H}(\rho, \Pi) = \ln d - \frac{d}{|G|} \sum_{g \in G} \eta(\text{tr}(\tau(g\rho))).$$

Now, one can readily see that the relative entropy of symmetric POVM is closely related to the semi-classical quantum entropy introduced in 1978 by Wehrl for the harmonic oscillator coherent states \[97\] and named later after him. The definition was generalised later by Schroeck \[80\], who analysed its basic properties. Let $G$ be a compact topological group acting unitarily and irreducibly on $\mathcal{P}(\mathbb{C}^d)$. Fixing reference state $\tau \in \mathcal{P}(\mathbb{C}^d)$ we get the family of states $(g\tau)_{g \in G}$ called (generalized or group) coherent states \[71, 1\] that fulfills the identity:

$$\int_{G/G_\tau} g\tau d\mu([g]) = 1,$$

where $\mu$ is the $G$-invariant measure on $G/G_\tau$ such that $\mu(G/G_\tau) = d$. Then for $\rho \in \mathcal{S}(\mathbb{C}^d)$ we define the generalized Wehrl entropy of $\rho$ by

$$S_{\text{Wehrl}}(\rho) := \int_{G/G_\tau} \eta(\text{tr}(\tau(g\rho))) d\mu([g]_{G/G_\tau}).$$

It is just the Boltzmann-Gibbs entropy for the density function on $(G/G_\tau, \mu)$ called the Husimi function of $\rho$ and given by $G/G_\tau \ni [g]_{G/G_\tau} \mapsto \text{tr}(\tau(g\rho)) \in \mathbb{R}^+$ that represents the probability density of the results of an approximate coherent states measurement (or in other words continuous POVM) \[30, 17\]. Then the relative Boltzmann-Gibbs entropy of the Husimi distribution of $\rho$ with respect to the Husimi distribution of the maximally mixed state $\rho_*$, that is the constant density on $(G/G_\tau, \mu)$ equal $1/d$, given by

$$S_{\text{Wehrl}}(\rho|\rho_*):= \ln d - S_{\text{Wehrl}}(\rho)$$

is a continuous analogue of $\tilde{H}(\cdot, \Pi)$ given by \[15\]. What is more, the relative entropy of measurement is just a special case of such transformed Wehrl entropy, when we consider the discrete coherent states (i.e. POVM) generated by a finite group. On the other hand, the entropy of measurement $H(\cdot, \Pi)$ has no continuous analogue, as it may diverge to infinity, where $k \to \infty$. In principle, to define coherent states we can use an arbitrary reference state. However, to obtain coherent states with sensible properties one has to choose the reference state $\tau$ to be the vacuum state, that is the state with maximal symmetry with respect to $G$ \[56, 71\] Sec. 2.4.

To investigate the Wehrl entropy it is enough to require that $G$ should be locally compact. In fact, Wehrl defined this quantity for the harmonic oscillator coherent states, where $G$ is the Heisenberg-Weyl group $H_4$ acting on projective (infinite dimensional and separable) Hilbert space, $G_\tau \simeq U(1) \times U(1)$, and $G/G_\tau \simeq \mathbb{C}$. This notion was generalized by Lieb \[60\] to spin (Bloch) coherent states, with $G = SU(2)$ acting on $\mathbb{CP}^{d-1}$ ($d \geq 2$), $G_\tau \simeq U(1)$ and $G/G_\tau \simeq S^2$. In this paper Lieb proved that for harmonic oscillator coherent states the minimum value of the Wehrl entropy is attained for coherent states themselves. (It follows from the group invariance that this quantity is the same for each coherent state.) He also
conjectured that the statement is true for spin coherent states, but, despite many partial results, the problem, called the *Lieb conjecture*, had remained open for next thirty five years until it was finally proved by Lieb himself and by Solovej in 2012 [61]. They also expressed the hope that the same result holds for SU($N$) coherent states for arbitrary $N \in \mathbb{N}$, or even for any compact connected semisimple Lie group (the *generalized Lieb conjecture*), see also [42, 88]. Bandyopadhyay received recently some partial results in this direction for $G = SU(1,1)$ coherent states [7], where $G \tau \simeq U(1)$ and $G/G \tau$ is the hyperbolic plane.

For finite groups and covariant POVMs the minimization of Wehrl entropy is equivalent to the maximization of the relative entropy of measurement, which is in turn equivalent to the minimization of the entropy of measurement. Consequently, one could expect that the entropy of measurement should be minimal for the states constituting the POVM that are already known to be critical as inert states. We shall see in Sec. 8 that this need not be always the case. In particular, it is not true for the tetrahedral POVM or in the situation where the states constituting a POVM form a regular $n$-gon for an odd $n$. Thus, it is conceivable that to prove the generalized Lieb conjecture some additional assumptions will be necessary.

### 6.4. Relation to quantum dynamical entropy

As in the preceding section, let $\Pi = (\Pi_j)_{j=1,...,k}$ be a finite normalized rank-1 POVM in $\mathbb{C}^d$ and let $S = \{\rho_j : j = 1,\ldots,k\}$ be a corresponding (multi-)set of pure quantum states. Set $\rho_j = |\varphi_j\rangle\langle \varphi_j|$, where $\varphi_j \in \mathbb{C}^d$, $||\varphi_j|| = 1$. Assume that successive measurements described by $\Pi$ are performed on an evolving quantum system and that the motion of the system between two subsequent measurements is governed by a unitary matrix $U$. Clearly, the sequence of measurements introduces a nonunitary evolution and the complete dynamics of the system can be described by a quantum Markovian stochastic process, see [86].

The results of consecutive measurements are represented by finite strings of letters from a $k$-element alphabet. Probability of obtaining the string $(i_1,\ldots,i_n)$, where $i_j = 1,\ldots,k$ for $j = 1,\ldots,n$ and $n \in \mathbb{N}$ is then given by

$$
P_{i_1,...,i_n}(\rho) := p_{i_1}(\rho) \cdot \prod_{m=1}^{n-1} p_{i_m,i_{m+1}},
$$

where $\rho$ is the initial state of the system, $p_i(\rho) := (d/k) \text{tr}(\rho \rho_i)$ is the probability of obtaining $i$ in the first measurement, and $p_{ij} := (d/k) \text{tr}(U \rho_i U^\ast \rho_j) = (d/k) |\langle \varphi_i| U |\varphi_j\rangle|^2$ is the probability of getting $j$ as the result of the measurement, providing the result of the preceding measurement was $i$, for $i,j = 1,\ldots,k$ [86, 88]. The randomness of the measurement outcomes can be analysed with the help of *(quantum) dynamical entropy*, the quantity introduced for the Lüders-von Neumann measurement independently by Srinivas [89], Pechukas [70], Beck & Graudenz [8], and many others, see [86, p. 5685], then generalized by Życzkowski and one of the present authors (W.S.) to arbitrary classical or quantum measurements and instruments [86, 58, 87, 88], and recently rediscovered by Crutchfield and Wiesner under the name of quantum entropy rate [27].
The definition of (quantum) dynamical entropy of $U$ with respect to $\Pi$ mimics its classical counterpart, the Kolmogorov-Sinai entropy:

$$H(U, \Pi) := \lim_{n \to \infty} H_{n+1} - H_n = \lim_{n \to \infty} H_n/n,$$

where $H_n := \sum_{i_1, \ldots, i_n=1}^k \eta(P_{i_1, \ldots, i_n} (\rho_*))$ for $n \in \mathbb{N}$. The maximally mixed state $\rho_* = I/d$ plays here the role of the ‘stationary state’ for combined evolution. It is easy to show that the quantity is given by

$$H(U, \Pi) = 1/k \sum_{i,j=1}^k \eta((d/k) \text{tr} (U \rho_i U^* \rho_j))$$

$$= \ln (k/d) + d/k \sum_{i,j=1}^k \eta(\text{tr} (U \rho_i U^* \rho_j))$$

$$= \ln (k/d) + d/k \sum_{i,j=1}^k \eta(|\langle \varphi_i | U | \varphi_j \rangle|^2),$$

which is a special case of much more general integral entropy formula [88]. Using (13) and (21) we see that the dynamical entropy of $U$ is expressed as the mean entropy of measurement over output states transformed by $U$:

$$H(U, \Pi) = \frac{1}{k} \sum_{i=1}^k H(U \rho_i U^*, \Pi)$$

There are two sources of randomness in this model: the underlying unitary dynamics and the measurement process. The latter can be measured by the quantity $H_{meas}(\Pi) := H(I, \Pi)$ called (quantum) measurement entropy. From (22) we get

$$H_{meas}(\Pi) = \frac{1}{k} \sum_{i=1}^k H(\rho_i, \Pi).$$

If $\Pi$ is symmetric, then all the summands in (23) are the same. Hence, in this case, the measurement entropy $H_{meas}(\Pi)$ is equal to the entropy of measurement $H(\rho, \Pi)$, where the input state $\rho$ is one of the output states from $S$.

7. Local extrema of entropy for symmetric POVMs in dimension two

Using the Bloch sphere representation for states and normalized rank-1 POVMs (see Sec. 2) one can reduce the problems of entropy minimization and relative entropy maximization in dimension two to the problem of finding the global extrema of the corresponding function on $S^2$. Namely, let $\Pi = (\Pi_j)_{j=1, \ldots, k}$ be a normalized rank-1 POVM in $\mathbb{C}^2$ such that $\Pi_j = (2/k) \rho_j$, $\rho_j \in \mathcal{P}(\mathbb{C}^2)$, and let $B := \{v_j := 1, \ldots, k\}$, where $v_j := \sqrt{2b} (\rho_j) \in S^2$ ($j = 1, \ldots, k$). For $\rho \in \mathcal{P}(\mathbb{C}^2)$, $u := \sqrt{2b} (\rho) \in S^2$ we get from (13) and (14)

$$H_B(u) := H(\rho, \Pi) = \sum_{j=1}^k \eta \left( \frac{u \cdot v_j + 1}{k} \right) = \ln \frac{k}{2} + \frac{2}{k} \sum_{j=1}^k h(u \cdot v_j).$$
and

\[ \bar{H}_B(u) := \bar{H}(\rho, \Pi) = \ln 2 - \frac{2}{k} \sum_{j=1}^{k} h(u \cdot v_j), \]

where the dot denotes the Euclidean scalar product in \( \mathbb{R}^3 \), and the function \( h : [-1, 1] \to \mathbb{R}^+ \) is given by

\[ h(t) := \eta \left( \frac{t + 1}{2} \right) \]

for \(-1 \leq t \leq 1\). It is clear that the functions \( H_B \) and \( \bar{H}_B : S^2 \to \mathbb{R}^+ \) are of \( C^2 \) class (even analytic) except at the points antipodal to the points from \( B \). Despite the fact that the function \( h \) is non-differentiable at \(-1\), some standard calculations show that at these points the functions are of \( C^1 \) class but not twice differentiable.

Usually, the simplest way to find the entropy minimizers leads through the majorization technique [10, Ch. 2]. However, for every pair of orthogonal states \( \rho \) and \( \rho^\perp \) in \( \mathcal{P}(\mathbb{C}^2) \) we have \( (p_j(\rho, \Pi) + p_j(\rho^\perp, \Pi))/2 = 1/k \) for \( j = 1, \ldots, k \). Hence it follows that if the distribution of the measurement outcomes with the input state \( \rho \) majorizes that with the input state \( \rho^\perp \), they must be equivalent, and so the entropies at these points are equal. This fact reduces the chance that the minimization problem can be solved in full generality via majorization. On the other hand, these procedure can be used to reduce this problem to a two-dimensional situation, if the Bloch representation of POVM is already two-dimensional.

Namely, we show that if \( B \) is contained in a plane \( L \), then \( H_B \) defined by (24) attains global minima on this plane. There is no loss of generality in assuming that \( L = \mathbb{R}^2 \times \{0\} \). Put \( w := (0, 0, 1) \). Since \( H_B(w) = H_B(-w) = \ln k \), it is enough to prove that \( H_B(u) \geq H_B(\bar{u}) \) for \( u = (x, y, z) \neq w, -w \), where \( \bar{u} := (x^2 + y^2)^{-1}(x, y, 0) \) is the normalized projection of \( u \) onto \( L \). To this aim, it suffices to show that the probability vector \( (p_1, \ldots, p_k) \) is majorized by the vector \( (\bar{p}_1, \ldots, \bar{p}_k) \), where \( p_j := (1 + v_j \cdot u)/k \) and \( \bar{p}_j := (1 + v_j \cdot \bar{u})/k \) for \( j = 1, \ldots, k \). Set \( \alpha := x^2 + y^2 \). We have \( p_j = (1 + \alpha v_j \cdot \bar{u})/k = (1 + \alpha(k\bar{p}_j - 1))/k = \alpha \bar{p}_j + (1 - \alpha)/k \). Let \( \sigma : \{1, \ldots, k\} \to \{1, \ldots, k\} \) be a permutation such that \( (\bar{p}_{\sigma(1)}, \ldots, \bar{p}_{\sigma(k)}) \) is decreasing. Then also \( (p_{\sigma(1)}, \ldots, p_{\sigma(k)}) \) is decreasing and it is enough to prove that \( \sum_{j=1}^{m} \bar{p}_{\sigma(j)} \geq \sum_{j=1}^{m} p_{\sigma(j)} = \sum_{j=1}^{m} (\alpha \bar{p}_{\sigma(j)} + (1 - \alpha)/k) \), for every \( m = 1, \ldots, k \). But this is equivalent to \( \sum_{j=1}^{m} \bar{p}_{\sigma(j)} \geq m/k \), which is always true, since any probability vector majorizes \( (1/k, \ldots, 1/k) \).

For a symmetric POVM, there exists a finite group \( G \subset O(3) \) acting transitively on \( B \). It follows from (24) and (25) that \( H_B \) and \( \bar{H}_B \) are \( G \)-invariant functions on \( S^2 \) given by

\[ H_0(u) := H_B(u) = \ln \frac{|G|}{2} + \frac{2}{|G|} \sum_{g \in G} h(gu \cdot v) \]
and
\[
(28) \quad H_v(u) := H_B(u) = \ln 2 - \frac{2}{|G|} \sum_{g \in G} h(gu \cdot v),
\]
for \(u \in S^2\), where \(v \in B = \{gv : g \in G\}\) is the Bloch vector of an arbitrary fiducial state. This fact allows us to use the theory of solving symmetric variational problems developed by Louis Michel and others in the 1970s, and applied since then in many physical contexts, especially in analysing the spontaneous symmetry breaking phenomenon \cite{66}.

We start by quoting several theorems concerning smooth action of finite groups on finite-dimensional manifolds. They are usually formulated for compact Lie groups, but since finite groups are zero-dimensional Lie group, thus the results apply equally well in this case.

Let \(G\) be a finite group of \(C^1\) maps acting on a compact finite-dimensional manifold \(M\). In the set of strata consider the order \(\prec\) introduced in Sec. 3. Then

**Theorem** (Montgomery and Yang \cite{68, 67, 38}). The set of strata is finite. There exists a unique minimal stratum, comprising elements of trivial stabilizers, that is open and dense in \(M\), called **generic** or **principal**. For every \(x \in M\) the set \(\bigcup \{\Sigma_u : u \in M, \Sigma_x \preceq \Sigma_u\}\) is closed in \(M\); in particular, the maximal strata are closed.

The next result tells us, where we should look for the critical points of an invariant function, i.e. the points where its gradient vanishes: we have to focus on the maximal stratum of \(G\) action on \(M\).

**Theorem** (Michel \cite{65, 67}). Let \(F : M \to \mathbb{R}\) be a \(G\)-invariant \(C^1\) map, and let \(\Sigma\) be a maximal strata. Then

1. \(\Sigma\) contains some critical points of \(F\);
2. if \(\Sigma\) is finite, then all its elements are critical points of \(F\).

Such points are called **inert states** in physical literature; they are critical regardless of the exact form of \(F\), see e.g. \cite{100}. Of course, an invariant function can have other critical points than those guaranteed by the above theorem (non-inert states). However, we shall see that for highly symmetric POVMs in dimension two, the global minima of entropy function \(H_B\) lie always on maximal strata. Although Michel’s theorem indicates a special character of the points with maximal stabilizer, it does not give us any information about the nature of these critical points. In some cases we can apply the following result:

**Theorem** (Modern Purkiss Principle \cite{95}). Let \(F : M \to \mathbb{R}\) be a \(G\)-invariant \(C^2\) map and let \(u \in M\). Assume that the action of the linear isotropy group \(\{T_u h : h \in G_u\}\) on \(T_u M\) is irreducible. Then \(u\) is a critical point of \(F\), which is either degenerate (i.e. the Hessian of \(F\) is singular at \(u\)) or a local extremum of \(F\).
For a finite group $G$ acting on the sphere $S^2$ points lying on its rotation axes form the maximal strata, and so, it follows from Michel’s theorem that they are critical for entropy functions. In our case we can divide them into three categories depending on whether they are antipodal to the elements of the fiducial vector’s orbit (type I), and, if not, whether their stabilizers act irreducibly on the tangent space (type II) or not (type III). In the first two cases we can determine the character of critical point using the following proposition:

**Proposition 5.** Let $u \in S^2$ be a point lying on a rotation axis of the group $G$. Then:

(I) If there exist $g \in G$ such that $u = -gv$, then $u$ is a local minimizer (resp. maximizer) for $H_v$ (resp. $\tilde{H}_v$);

(II) If $u \neq -gv$ for every $g \in G$ and the linear isotropy group $\{T_u g : g \in G_u\}$ acts irreducibly on $T_u S^2$ (or, equivalently, $G_u$ contains a cyclic subgroup of order greater than 2), then:

(a) if

$$\frac{2}{|G/G_u|} \sum_{[h] \in G/G_u} (hu \cdot v) \ln(1 + hu \cdot v) > 1,$$

then $u$ is a local minimizer (resp. maximizer) for $H_v$ (resp. $\tilde{H}_v$),

(b) if

$$\frac{2}{|G/G_u|} \sum_{[h] \in G/G_u} (hu \cdot v) \ln(1 + hu \cdot v) < 1,$$

then $u$ is a local maximizer (resp. minimizer) for $H_v$ (resp. $\tilde{H}_v$).

To prove this result, it is enough, in case (II), to use the preceding theorem and to show that the second derivative of the entropy function restricted to some (and hence to every) geodesic (a great circle) is positive. In case (I) the situation is less clear, since the entropy function is not twice differentiable. For the details see Appendix.

If $\Pi$ is a HS-POVM, we can assume that $G$ is one of the following groups: $D_{nh}$, $T_d$, $O_h$ or $I_h$, and the Bloch vector of the fiducial vector $v$ lies in the maximal strata, consisting of points where the rotation axes of the group intersect the Bloch sphere. According to Michel’s theorem exactly the same points are critical for entropy function. For $D_{nh}$ group we have one $n$-fold and $n$ 2-fold rotation axes ($2n + 2$ points: a digon and two regular $n$-gons); for $T_d$ group: three 2-fold, four 3-fold rotation axes (14 points: an octahedron and two dual tetrahedra); for $O_h$ group: six 2-fold, four 3-fold, three 4-fold rotation axes (26 points: a cuboctahedron, a cube and an octahedron); for $I_h$ group: fifteen 2-fold, ten 3-fold, six 5-fold rotation axes (62 points: an icosidodecahedron, a dodecahedron and an icosahedron). The character of these singularities is described by the following proposition.
Proposition 6. In the situation above, singular points of type I are minima (resp. maxima), of type II maxima (resp. minima), and of type III saddle points for $H_B$ (resp. $\tilde{H}_B$).

The proof of this fact is quite elementary. From Proposition 5(I) we deduce the character of singular points of type I. For type II it is enough to use Proposition 5(II). For type III one have to indicate two great circles such that the second derivatives along these curves have different sign. As we will not use this fact in the sequel, we omit the details.

Hence the points of type I are the natural candidates for minimizing $H_B$ (resp. maximizing $\tilde{H}_B$), and indeed, we will show in the next section that they are global minimizers (resp. maximizers). However, if a POVM is merely symmetric, the global extrema of entropy functions may also occur in other points. An example of this phenomenon can be found in [41], see also [12]. Let us consider a symmetric (but non-highly symmetric) POVM generated by the set of four Bloch vectors forming a rectangle $B = \{v_1, -v_1, v_2, -v_2\}$, where $v_1, v_2 \in S^2$, $v_1 \notin \{-v_2, v_2\}$, and $v_1 \cdot v_2 \neq 0$, with $\text{Sym}(B) \simeq D_{2h}$ having three mutually perpendicular rotation axes. In this way we get six vectors in $S^2$ that are necessarily critical for $H_B$ and $\tilde{H}_B$: two perpendicular both to $v_1$ and to $v_2$, and four lying in the plane generated by $v_1$ and $v_2$, proportional to $\pm v_1 \pm v_2$. The former are local maxima of $H_B$, and the latter either local minima or saddle points, depending on the value of the parameter $\alpha := \arccos(v_1 \cdot v_2) \in (0, \pi)$, $\alpha \neq \pi/2$. Let $\bar{\alpha} \approx 1.17056$ be a unique solution of the equation $\cos(\alpha) = (\tan(\alpha)) \ln(\tan(\alpha/2)) = -2$ in the interval $(0, \pi/2)$. In [41] the authors showed that for $\alpha \in (0, \bar{\alpha}]$ the function $H_B$ (resp. $\tilde{H}_B$) attains the global minimum (resp. maximum) at the points $\pm (v_1 + v_2)/(2|\cos(\alpha/2)|)$, whereas $\pm (v_1 - v_2)/(2|\sin(\alpha/2)|)$ are saddle points, and for $\alpha \in [\pi - \bar{\alpha}, \pi)$ the situation is reversed. However, for $\alpha \in (\bar{\alpha}, \pi - \bar{\alpha})$ all these inert states become saddles, and two pairs of new global minimizers emerge, lying symmetrically with respect to the old ones. The appearance of this pitchfork bifurcation phenomenon shows also that, in general, one cannot expect an analytic solution of the minimization problem in a merely symmetric case. This is why we restrict our attention to highly symmetric POVMs.

Note also that for highly symmetric POVMs we can use, instead of full symmetry group $\text{Sym}(B)$, any subgroup acting transitively on $B$, e.g., $C_n$ for the regular $n$-gon, $T$ for the tetrahedron, $O$ for the cuboctahedron, cube and octahedron, and $I$ for the icosidodecahedron, dodecahedron and icosahedron. They have the same rotation axes as the full symmetry groups.

8. Global minima of entropy for highly symmetric POVMs in dimension two

8.1. The minimization method based on the Hermite interpolation. In order to prove that the antipodal points to the Bloch vectors of POVM elements
are not only local but also global minimizers, we shall use a method based on the Hermite interpolation.

Consider a sequence of points \( a \leq t_1 < t_2 < \ldots < t_m \leq b \), a sequence of positive integers \( k_1, k_2, \ldots, k_m \), and a real valued function \( f \in C^D([a,b]) \), where \( D := k_1 + k_2 + \ldots + k_m \). We are looking for a polynomial \( p \) of degree less than \( D \) that agree with \( f \) at \( t_i \) up to a derivative of order \( k_i - 1 \) (for \( 1 \leq i \leq m \)), that is,

\[
p^{(k_i)}(t_i) = f^{(k_i)}(t_i), \quad 0 \leq k < k_i.
\]

The existence and uniqueness of such polynomial follows from the injectivity (and hence also the surjectivity) of a linear map \( \Phi : \mathbb{R}_{<D}[X] \to \mathbb{R}^D \) given by \( \Phi(p) := (p(t_1), p'(t_1), \ldots, p^{(k_1-1)}(t_1), \ldots, p(t_m), \ldots, p^{(k_m-1)}(t_m)) \). We will also use the following well-known formula for the error in Hermite interpolation \[90\text{, Sec. 2.1.5}]:

**Lemma 1.** For each \( t \in (a,b) \) there exists \( \xi \in (a,b) \) such that \( \min\{t, t_1\} < \xi < \max\{t, t_m\} \) and

\[
f(t) - p(t) = \frac{f^{(D)}(\xi)}{D!} \prod_{i=1}^{m}(t - t_i)^{k_i}.
\]

Now we apply this general method in our situation. We will interpolate the function \( h : [-1,1] \to \mathbb{R}_+ \) defined by \[25\], choosing the interpolation points from the set \( T := \{-g \cdot v | g \in G\} \subset [-1,1] \), where \( v \) is the Bloch vector representation of the fiducial vector and \(-v\) is supposed to be the Bloch vector of a global minimizer. We must distinguish two situations: either the inversion \(-I \in G\) (equivalently \(-v \in B\)) or not. The latter is the case for \( G = D_{nh} \) (for even \( n \)), \( O_h \), \( I_h \), and then \( 1 \in T \), the latter for \( G = D_{nh} \) (for odd \( n \)), \( T_d \), and then \( 1 \notin T \). After reordering the elements of \( T \) we obtain an increasing sequence \( \{t_i\}_{i=1}^{m} \), where \( m := |T| \). In particular, \( t_1 = -1 \). We are looking for a polynomial \( p_v \) that matches the values of \( h \) at all points from \( T \) and the values of \( h' \) at all points but \(-1 \) and, possibly, \( 1 \), if \( 1 \in T \), i.e. such that \[31\] holds for \( f = h \) with

\[
k_i := \begin{cases} 
1, & \text{if } t_i \in \{-1,1\} \\
2, & \text{otherwise}
\end{cases}.
\]

Then \( \deg p_v < D(v) := 2m - 2 \), if \( 1 \in T \), and \( \deg p_v < D(v) := 2m - 1 \), otherwise. Though \( h \) is not differentiable at \( t_1 \), we still can use \[32\] to estimate the interpolation error, as proof of Lemma \[1\] is based on repeated usage of Rolle’s Theorem.

If \( 1 \in T \), then \( t_m = 1 \) and we have

\[
\prod_{i=1}^{m}(t - t_i)^{k_i} = (t + 1)(t - 1) \prod_{i=2}^{m-1}(t - t_i)^2 \leq 0,
\]

for \( t \in [-1,1] \). Similarly, if \( 1 \notin T \), then \( t_m < 1 \) and

\[
\prod_{i=1}^{m}(t - t_i)^{k_i} = (t + 1) \prod_{i=2}^{m}(t - t_i)^2 \geq 0
\]
for \( t \in [-1,1] \). Moreover, inequalities above turn into equalities only for \( t \in T \). Furthermore, as all the derivatives of \( h \) of even order are strictly negative in \((-1,1)\) and these of odd order greater than 1 are strictly positive, we get

\[
(36) \quad h^{(D(t))}(\xi) = \begin{cases} 
  h^{(2m-2)}(\xi) < 0, & \text{if } 1 \in T \\
  h^{(2m-1)}(\xi) > 0, & \text{if } 1 \notin T
\end{cases}
\]

for each \( \xi \in (-1,1) \). Hence and from Lemma 1 the interpolating polynomial \( p_v \) fulfills \( p_v(t) = h(t) \) if and only if \( t \in T \) and it interpolates \( h \) from below, see the illustration of this for the octahedral POVM in Fig. 1.

\[\text{Figure 1. The cubic polynomial function } p_v \text{ (purple) interpolating } h \text{ (violet) from below for the octahedral measurement, with } t_1 = -1, t_2 = 0 \text{ and } t_3 = 1.\]

Let us define now a \( G \)-invariant polynomial function \( P_v : \mathbb{R}^3 \to \mathbb{R} \) replacing \( h \) in (27) by its interpolation polynomial \( p_v \), i.e.,

\[
(37) \quad P_v(u) := \ln \frac{|G|}{2} + \frac{2}{|G|} \sum_{g \in G} p_v(gv \cdot u)
\]

for \( u \in \mathbb{R}^3 \). Combining the above facts, we get

\[
(38) \quad H_v(u) = \ln \frac{|G|}{2} + \frac{2}{|G|} \sum_{g \in G} h(gv \cdot u) \geq \ln \frac{|G|}{2} + \frac{2}{|G|} \sum_{g \in G} p_v(gv \cdot u) = P_v(u)
\]

for \( u \in S^2 \), and \( H_v(-gv) = P_v(-gv) \) for \( g \in G \).

In consequence, now it is enough to show that \( -v \) is a global minimizer of \( P \) (and hence all the elements of its orbit \( \{-gv : g \in G\} \) are), because then we have \( H_v(u) \geq P_v(u) \geq P_v(-v) = H_v(-v) \) for all \( u \in S^2 \). This method of finding global minima was inspired by the one used in \[10\] for \( G = T_d \), where \( h \) is constant. Note, however, that a similar technique was used by Cohn, Kumar and Woo \[23, 24\] to solve the problem of potential energy minimization on the unit sphere. The whole idea can be traced back even further to \[101\] and \[2\].

Exactly the same method applies if we replace the Shannon function \( \eta \) by any concave function \( \theta : [0,1] \to \mathbb{R} \) such that \( \theta(0) = \theta(1) = 0 \) and the second derivative of \(-\theta\) is completely monotone, i.e., it has derivatives of all orders which alternate
in sign. To give an example of such a function assume that \( q \leq 2, \ q \neq 1 \) and 
\[
\theta_q (x) := (x - x^q) / (q - 1) \quad \text{for } x \geq 0.
\]
This leads to entropy functional given by 
\[
p = (p_1, \ldots, p_k) \to \sum_{j=1}^k \theta_q (p_j)
\]
and called Havrda-Charvát-Tsallis entropy and, in particular, for \( q = 2 \), linear entropy \([10, 28]\). Note, however, that since informationally complete HS-POVMs in dimension 2 are complex projective 2-designs, so the linear relative entropy is constant, i.e. independent on the choice of the input state, and equal to \( \ln 2 - 1/3 \).

Of course, the lower the degree of the interpolating polynomial \( p_v \) is, the easier it is to find the minima of \( P_v \), as \( \deg P_v \leq \deg p_v \). The last quantity in turn depends on the cardinality of \( T := \{-g v \cdot v | g \in G\} \), that can be calculated by analyzing double cosets of isotropy subgroups of any subgroup \( K \subset G \cap SO(3) \) acting transitively on \( B \), because \( T = \{-g v \cdot v | g \in K\} \) and for \( c, g \in K \), if \( c \) is in a double coset \( K_v g K_v \) or \( K_v g^{-1} K_v \), then \( c v \cdot v = g v \cdot v \). Hence \( |T| \leq n(v) := n_s(v) + \frac{1}{2} n_a(v) \), where \( n_s(v) \) is the number of self-inverse double cosets of \( K_v \), i.e., the cosets fulfilling \( K_v g K_v = K_v g^{-1} K_v \), and \( n_a(v) \) is the number of non self-inverse ones. Thus

\[
\text{(39)} \quad \deg p_v \leq \begin{cases} 2 n(v) - 3, & \text{if } -v \in K v \\ 2 n(v) - 2, & \text{if } -v \notin K v \end{cases}
\]

Moreover, for \( g \in K \), using the well-known formula for the cardinality of a double coset, see e.g. \([11, \text{Proposition 5.1.3]}\), we have \( |K_v g K_v| = |K_v| |K_v/ (K_v \cap K_g)| = |K_v| \) if \( g v = v \) or \( g v = -v \), and \( |K_v|^2 \), otherwise. Hence, if \( -v \in K v \), then \( |K v| |K_v| = |K| = 2 |K_v| + (n_s(v) - 2) |K_v|^2 + n_a(v) |K_v|^2 \), and so \( n_s(v) + n_a(v) = (|K v| - 2) / |K_v| + 2. \) Analogously, if \( -v \notin K v \), then we have \( n_s(v) + n_a(v) = (|K v| - 1) / |K_v| + 1. \) Using these facts and (39) we get finally

\[
\text{(40)} \quad \deg p_v \leq \begin{cases} \frac{|K v| - 2}{|K_v|} + n_s(v) - 1, & \text{if } -v \in K v \\ \frac{|K v| - 1}{|K_v|} + n_s(v) - 1, & \text{if } -v \notin K v \end{cases}
\]

Applying (40) to HS-POVMs in dimension two we get the following upper bounds for the degree of interpolating polynomials:

| \( K v \) | \( |K v| \) | \( K \) | \( K_v \) | \( n_s(v) \) | \( n_a(v) \) | \( \deg p_v \) |
|---|---|---|---|---|---|---|
| regular \( n \)-gon \((n\text{-even})\) | \( n \) | \( C_n \) | \( C_1 \) | \( n - 2 \) | \( 2 \) | \( n - 1 \) |
| regular \( n \)-gon \((n\text{-odd})\) | \( n \) | \( C_n \) | \( C_1 \) | \( n - 1 \) | \( 1 \) | \( n - 1 \) |
| tetrahedron | \( 4 \) | \( T \) | \( C_3 \) | \( 0 \) | \( 2 \) | \( 2 \) |
| octahedron | \( 6 \) | \( O \) | \( C_4 \) | \( 0 \) | \( 3 \) | \( 3 \) |
| cube | \( 8 \) | \( O \) | \( C_3 \) | \( 0 \) | \( 4 \) | \( 5 \) |
| cuboctahedron | \( 12 \) | \( O \) | \( C_2 \) | \( 4 \) | \( 3 \) | \( 7 \) |
| icosahedron | \( 12 \) | \( I \) | \( C_5 \) | \( 0 \) | \( 4 \) | \( 5 \) |
| dodecahedron | \( 20 \) | \( I \) | \( C_3 \) | \( 4 \) | \( 4 \) | \( 9 \) |
| icosidodecahedron | \( 30 \) | \( I \) | \( C_2 \) | \( 14 \) | \( 2 \) | \( 15 \) |

Tab. 2. HS-POVMs in dimension two: an upper bound for the degree of interpolating polynomial.
To find global minimizers of $P_{n}$ we can express the polynomial in terms of primary and secondary invariants for the corresponding ring of $G$-invariant polynomials. In fact, as we will see in the next section, only the former will be used.

8.2. Group invariant polynomials. The material of this subsection is taken from \textbf{34} Ch. 3 and \textbf{51}, see also \textbf{40}. Let $G$ be a finite subgroup of the general linear group $GL_n(\mathbb{R})$. By $\mathbb{R}[x_1, \ldots, x_n]^G$ we denote the ring of $G$-invariant real polynomials in $n$ variables. Its properties were studied by Hilbert and Noether at the beginning of twentieth century. In particular, they showed that $\mathbb{R}[x_1, \ldots, x_n]^G$ is finitely generated as an $\mathbb{R}$-algebra. Later, it was proven that it is possible to represent each $G$-invariant polynomial in the form $\sum_{j=1}^m P_j(\theta_1, \ldots, \theta_n) \eta_j$, where $\theta_1, \ldots, \theta_n$ are algebraically independent homogeneous $G$-invariant polynomials called primary invariants, forming so called homogenous system of parameters, $\eta_1 = 1, \ldots, \eta_m$ are $G$-invariant homogenous polynomials called secondary invariants, and $P_j (j = 1, \ldots, m)$ are elements from $\mathbb{R}[x_1, \ldots, x_n]$. Moreover, $\eta_1, \ldots, \eta_m$ can be chosen in such a way that they generate $\mathbb{R}[x_1, \ldots, x_n]^G$ as a free module over $\mathbb{R}[\theta_1, \ldots, \theta_n]$. Both sets of polynomials combined form so called integrity basis. Note that neither primary nor secondary invariants are uniquely determined. If $m = 1$, we call the basis regular and the group $G$ coregular. The invariant polynomial functions on $\mathbb{R}^n$ separate the $G$-orbits. In consequence, the map $\mathbb{R}^n/G \ni Gx \to (\theta_1(x), \ldots, \theta_n(x)) \in \mathbb{R}^{m-1}$ maps bijectively the orbit space onto an $n$-dimensional connected closed semialgebraic subset of $\mathbb{R}^{m-1}$. There is also a correspondence between the orbit stratification of $\mathbb{R}^n/G$ and the natural stratification of this semi-algebraic set into the primary strata, i.e., connected semialgebraic differentiable varieties. If $G \subseteq O(n)$ is a coregular group acting irreducibly on $\mathbb{R}^n$, we may assume that $\theta_1(x) = \sum_{i=1}^n x_i^2$ is a non-constant invariant polynomial of the lowest degree. Then the orbit map $\omega : S^{n-1}/G \ni Gx \to (\theta_2(x), \ldots, \theta_n(x)) \in \mathbb{R}^{m-1}$ is also one-to-one and its range is a semialgebraic $(n-1)$-dimensional set. In consequence, the minimizing of a $G$-invariant polynomial $P(x_1, \ldots, x_n)$ on $S^{n-1}$ is equivalent to the minimizing of the respective polynomial $P_1(\theta_1, \ldots, \theta_n)$ on the range of $\omega$. In the 1980s Abud and Sartori proposed a general procedure for finding the algebraic equations and inequalities defining this set and its strata, and thus also a general scheme for finding minima of $P_1$ on the range of the orbit map, see \textbf{74, 78}.

An element from $GL_n(\mathbb{R})$ is called a pseudo-reflection, if its fixed-points space has codimension one. The classical Chevalley-Shephard-Todd theorem says that every pseudo-reflection (i.e. generated by pseudo-reflections) group is coregular. As all the symmetry groups of polyhedra representing HS-POVMs in dimension two ($D_{nh}, T_d, O_h, I_h$) are pseudo-reflection groups, the interpolating polynomials can be expressed by their primary invariants listed below. Put $\rho := x^2 + y^2,$ $\gamma_n := \mathbb{R}(x + iy)^n,$ $I_2 := x^2 + y^2 + z^2,$ $I_3 := xyz,$ $I_4 := x^4 + y^4 + z^4,$ $I_5 := x^6 + y^6 + z^6,$ $I_6 := (\tau^2x^2 - z^2)(\tau^2y^2 - z^2)(\tau^2z^2 - x^2)$ and $I_{10} := (x + y + z)(x - y - z)(y - z - x)(z - y - x)(\tau^2x^2 - x^2)(\tau^2y^2 - y^2)(\tau^2z^2 - z^2)(\tau^2x^2 - z^2)$, where $\tau := (1 + \sqrt{5})/2$. 

\[ \quad \]
(the golden ratio). Note that the indices coincide with the degrees of invariant polynomials. Then (notation and results are taken from [51]) for the canonical representations of these groups, i.e., if coordinates $x, y$ and $z$ are so chosen that the origin is the fixed point for the group action and: the $x$ and $z$ axes are 2- and $n$-fold axes, respectively ($D_{nh}$); the 3-fold axes pass through vertices of a tetrahedron at $(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$ ($T_d$); $x, y$ and $z$ axes are the 4-fold axes ($O_h$); the 5-fold axes pass through the vertices of an icosahedron at $(±τ, ±1, 0), 0, ±τ, ±1)$, ($±1, 0, ±τ)$, we get the following primary invariants:

| group | primary invariants |
|-------|-------------------|
| $D_{nh}$ | $z^2, ρ, γ_n$ |
| $T_d$ | $I_2, I_3, I_4$ |
| $O_h$ | $I_2, I_4, I_6$ |
| $I_h$ | $I_2, I_6, I_{10}$ |

Tab. 3. Primary invariants for four point groups.

In [51] the stratification of the range of the orbit map is analytically described in all these cases.

8.3. The main theorem.

**Theorem 2.** For HS-POVMs in dimension two the points lying on the orbit of the point antipodal to the Bloch vector of the fiducial vector (that is the Bloch vector of the state orthogonal to the fiducial vector) are global minimizers (resp. maximizers) for the entropy of measurement (resp. the relative entropy of measurement).

**Proof.** To prove the main theorem we have to show that the antipodal points to the Bloch vectors of POVM elements, i.e., the points $\{-gv : g ∈ G\}$ are the global minima of the $G$-invariant polynomial $P_ν$ constructed in Sec. 8.1. We shall use the a priori estimates for $\deg P_ν$ that can be read from Tab. 2 and the primary invariants of $G$ listed in Tab. 3. We may exclude the trivial case when the HS-POVM in question is PVM represented by two antipodal points on the Bloch sphere (digon), as in this situation the minimal value of $H$ equal 0 is achieved at these points and the assertion follows. The proof is divided into four cases according to the symmetry group of the HS-POVM.

**Case I (prismatic symmetry)**

**Regular $n$-gon.** In Sec. 7 we showed that in this case it is enough to look for the global minimizers on the circle $S^1 := \{(x, y) ∈ \mathbb{R}^2 : x^2 + y^2 = 1\}$ containing the $n$-gon. Its symmetry group acts on the plane $z = 0$ as the dihedral group $D_n$, and so the interpolating polynomial $P_ν$ restricted to the circle $S^1$ can be expressed in terms of its primary invariants, i.e., $ρ = x^2 + y^2$ and $γ_n = \Re(x + iy)^n$. Since $\deg P_ν < n$, it follows that $P_ν|_S$ has to be a linear combination of $ρ^m, 0 ≤ 2m < n$, and hence constant.
Case II (tetrahedral symmetry)

**Tetrahedron.** This case is immediate, as \( \deg P_v \leq \deg p_v \leq 2 \), and so \( P_v \) has to be constant on the sphere \( S^2 \).

Case III (octahedral symmetry)

For \( O_h \) we have inert states at the \( O_h \)-orbits of the points: \( x_1 := (0,0,1) \) (vertices of an icosahedron), \( x_2 := \frac{1}{\sqrt{2}} (0,1,1) \) (vertices of a cuboctahedron), and \( x_3 := \frac{1}{\sqrt{3}} (1,1,1) \) (vertices of a cube). Using the Lagrange multipliers it is easy to check that these points are the only critical points for \( I_4 \) and \( I_6 \) restricted to the sphere \( S^2 \). By comparing the values of \( I_4 \) and \( I_6 \) (which are \( I_4 (x_1) = 1 \), \( I_4 (x_2) = 1/2 \), \( I_4 (x_3) = 1/3 \), \( I_6 (x_1) = 1 \), \( I_6 (x_2) = 1/4 \), \( I_6 (x_3) = 1/9 \)), we find that the points lying on the orbit of \( x_3 \) are global minimizers both for \( I_4 \) and \( I_6 \).

**Octahedron.** This case is straightforward, as for \( v = x_1 \) we have \( \deg P_v \leq \deg p_v \leq 3 \), and so \( P_v \) has to be constant on the sphere \( S^2 \).

**Cube.** In this case we have \( v = x_3 \) and \( \deg P_v \leq \deg p_v \leq 5 \). In consequence, \( P_v \) must be a linear combination of \( 1, I_2, I_4, \) and \( I_2^2 \). After the restriction to the sphere, \( P_v|_{S^2} \) can be expressed as \( A + B I_4 \), for some \( A, B \in \mathbb{R} \). Thus, all we need to know now is the sign of \( B \). Calculating the values of \( P_v \) in two points from different orbits (e.g. \( x_1 \) and \( x_3 \)) and solving the system of two linear equations we get \( B = 3/8 \ln(27/16) > 0 \). Thus the global minimizers for \( P_v \) are the same as for \( I_4 \), i.e., they lie on the orbit of \( v \) or, equivalently, \( -v \), as required.

**Cuboctahedron.** For the cuboctahedral measurement we have \( v = x_2 \) and \( \deg P_v \leq \deg p_v \leq 7 \). Consequently, \( P_v \leq 6 \) and \( P_v \) is a linear combination of \( 1, I_2, I_4, I_6, I_4 I_2, \) and \( I_2^2 \). Hence, after the restriction to the sphere \( S^2 \), we get \( P_v|_{S^2} = A + B I_4 + C I_6 \), for some \( A, B, C \in \mathbb{R} \). Put \( \beta := -B/3C \). Clearly, all inert states are critical for \( P_v|_{S^2} \) with \( P_v (x_1) = A + C (1 - 3\beta) \), \( P_v (x_2) = A + C (1 - 6\beta)/4 \), \( P_v (x_3) = A + C (1 - 9\beta)/9 \). One can show easily that they are only critical points unless \( 1/4 < \beta < 1/2 \). In this case one can find another critical points, namely the orbit of the point \( x_4 := (\sqrt{4\beta - 1}, \sqrt{1 - 2\beta}, \sqrt{1 - 2\beta}) \) with \( P_v (x_4) = C (1 - 9\beta + 24\beta^2 - 24\beta^3) \). To find \( B \) and \( C \), we need to calculate the values of \( P_v \) in three points from different orbits (e.g. \( x_1, x_2 \) and \( x_3 \)) and to solve the system of three linear equations. In this way we get \( B = \frac{260}{9} \ln 2 - 37 \ln 3 < 0 \), \( C = -\frac{364}{9} \ln 2 + 26 \ln 3 > 0 \) and \( \beta \approx 0.3775 \). Comparison of the values that \( P_v \) achieves at points \( x_1, x_2, x_3 \) and \( x_4 \) leads to the conclusion that the global minima are achieved for the vertices of cuboctahedron that form the orbit of \( v \) and also \(-v\).

Case IV (icosahedral symmetry)

The inert states for \( I_h \), that is the \( I_h \)-orbits of points: \( x_1 := (0,0,1) \) (vertices of an icosidodecahedron), \( x_5 := \frac{1}{\sqrt{4}} (0,1,1) \) (vertices of an icosahedron), and \( x_6 := \frac{1}{\sqrt{3}} (0,1,\tau) \) (vertices of a dodecahedron) are the only critical points for \( I_6^\prime \). They are, correspondingly, saddle, minimum, and maximum points with values: 0, \(-2+\sqrt{5})/5\), and \((2+\sqrt{5})/27\), respectively. For \( I_{10} \), the \( I_h \)-orbit of \( x_6 \) also coincides with the set of the global maxima, and we have local maxima at the \( I_h \)-orbit of
x_5 and saddle points at the orbit of x_1, but there are also non-inert critical points, namely sixty minima at the vertices of a truncated icosahedron, and sixty saddles at the vertices of an edge truncated truncated icosahedron, see [51, p. 26].

**Icosahedron.** This case is immediate, as v = x_5 and \( \deg P_v \leq \deg p_v \leq 5 \). Hence \( P_v \) restricted to \( S^2 \) is constant.

**Dodecahedron.** In this case v = x_6 and \( \deg P_v \leq \deg p_v \leq 9 \). Therefore \( P_v \) must be a linear combination of 1, \( I_2, I_2^2, I_6, I_6^2 \) and \( I_6 I_2 \). After restriction to \( S^2 \) we obtain \( P_v|_{S^2} = A + BI_6^9 \), for some \( A, B \in \mathbb{R} \). We can calculate \( B \) using the same method as in the cubical case. As it turns out to be negative (\( B \approx -0.06509 \)), the global minimizers coincide with the global maximizers for \( I_6^9 \), i.e., they are the vertices of the dodecahedron.

**Icosidodecahedron.** The icosidodecahedral case (v = x_1) is the most complicated one. Since \( \deg P_v \leq \deg p_v \leq 15 \), and \( P_v \) must be a linear combination of polynomials 1, \( I_2, I_2^2, I_6, I_6^2, I_6^2 I_2, I_2^2, I_6 I_2, I_6^2 I_2, I_2^2 I_6, I_6 I_2^2 \), and \( I_6 I_2^2 \). Restriction to \( S^2 \) gives us: \( P_v|_{S^2} = A + BI_6^9 + CI_{10} + DI(I_6^9)^2 \), for some \( A, B, C, D \in \mathbb{R} \). Both of the polynomials \( I_6^9 \) and \( I_{10} \) take the value 0 at \( x_1 \), which is obviously a critical point for \( P_v|_{S^2} \). As we have conjectured that the vertices of the icosidodecahedron are the global minimizers of \( P_v|_{S^2} \), it is enough to prove that \( \tilde{P} := P_v|_{S^2} - A \) is nonnegative. We keep proceeding like in the previous cases to obtain formulae for \( B, C, D \).

The range \( \Omega \) of the orbit map \( \omega: S^2/I_h \ni I_h w \to (I_6^9(w), I_{10}(w)) \in \mathbb{R}^2 \) is the curvilinear triangle (see Fig. 2) defined by the following inequalities imposed on the coordinates \((\theta_1, \theta_2) \in \mathbb{R}^2\):

\[
0 \leq J_{15}^2 := 4\theta_2^2 - 8(3 + 4\tau)\theta_1 \theta_2 - 91(3 - 2\tau)\theta_1^3 - 4(5 + 8\tau)\theta_2^3 +
+ 159(1 - 2\tau)\theta_2^4 \theta_2 + 688(13 - 8\tau)\theta_1^4 + 325(1 + 2\tau)\theta_1 \theta_2^2 +
- 720(7 - 4\tau)\theta_1^3 \theta_2 - 1728(55 - 34\tau)\theta_1^5 - 25(11 + 18\tau)\theta_2^3,
\]

where \( J_{15}^2 \) is the only secondary invariant for the icosahedral group \( I_5 \).

Define \( P_1(\theta_1, \theta_2) := B \theta_1 + C \theta_2 + D \theta_1^2 \) for \((\theta_1, \theta_2) \in \Omega \). Then \( \tilde{P}(w) = P_1(\omega((I_h) w)) \) for w \( \in S^2 \). The level sets of \( P_1 \) are parabolas and the zero level parabola given by \( \theta_2 = - (B/C) \theta_1 - (D/C) \theta_1^2 \) (the purple curve in Fig. 2) divides the plane into two regions: \{\( P_1 \geq 0 \}\} and \{\( P_1 < 0 \}\}. Now, it is enough to show that the zero level set of \( P_1 \) meets with the zero level set of \( J_{15}^2 \) (the violet curve in Fig. 2), which defines the boundary of \( \Omega \) only at \((\theta_1, \theta_2) = (0, 0)\), since in this case \( P_1 \) has the same sign over the whole \( \Omega \), and, in consequence, \( \tilde{P} \) is positive on the whole unit sphere. This approach reduces the complexity of the problem by lowering the degree of a polynomial equation to be solved. In fact, now it is enough to show that the polynomial \( J_{15}^2 \) \((\theta_1, - (B/C) \theta_1 - (D/C) \theta_1^2) / \theta_1^2 \) of degree 4 has no real roots. This can be done by a standard method using Sturm’s theorem. \( \square \)
Figure 2. The zero level set for $P_1$ (purple) and for $J_{15}^2$ (violet).

9. INFORMATIONAL POWER

Surprisingly, the average value of relative entropy over all pure states does not depend on the measurement $\Pi$, but only on the dimension $d$. This can be proved using (14) and the formula (21) from Jones [52]. Namely, we have

$$\langle \bar{H}(\rho, \Pi) \rangle_{\rho \in \mathcal{P}(\mathbb{C}^d)} = \int_{\mathcal{P}(\mathbb{C}^d)} \left( \ln d - \frac{d}{k} \sum_{j=1}^{k} \eta(\text{tr}(\rho \rho_j)) \right) d_{FS}(\rho)$$

$$= \ln d - d \cdot \eta \left( \int_{\mathcal{P}(\mathbb{C}^d)} \eta(\text{tr}(\rho \rho_1)) d_{FS}(\rho) \right)$$

$$= \ln d - \sum_{j=2}^{d} \frac{1}{j} \rightarrow 1 - \gamma \quad (d \rightarrow \infty),$$

where $\gamma \approx 0.57722$ is the Euler-Mascheroni constant. This average is also equal to the maximum value (in dimension $d$) of entropy-like quantity called subentropy, providing the lower bound for accessible information [53] [29].

In particular, the average value of relative entropy is the same for every HS-POVM $\Pi$ in dimension two and equals $\ln 2 - 1/2 \approx 0.19315$. It follows from Theorem 2 and (28) that its maximal value, that is the informational power of $\Pi$, is given by the formula
where $G$ is any group acting transitively on the set of Bloch vectors representing $\Pi$. Note that the number of different summands in (43) is bounded by the number of self-inverse double cosets of $G_v$ plus half of the number of non self-inverse ones.

Applying the above formula to the $n$-gonal POVM we get

\begin{equation}
W(\Pi) = \ln 2 - \frac{2}{|G/G_v|} \sum_{[g] \in G/G_v} \eta \left( \frac{1 - g_v \cdot v}{2} \right),
\end{equation}

The approximate values of informational power for other HS-POVMs can be found in Tab. 4.

| convex hull of the orbit | informational power |
|--------------------------|---------------------|
| digon                    | 0.69315             |
| regular $n$-gon ($n \to \infty$) | 0.30685             |
| tetrahedron              | 0.28768             |
| octahedron               | 0.23105             |
| cube                     | 0.21576             |
| cuboctahedron            | 0.20273             |
| icosahedron              | 0.20189             |
| dodecahedron             | 0.19686             |
| icosidodecahedron        | 0.19486             |
| average value of relative entropy | 0.19315 |

Tab. 4. The approximate values of informational power (up to five digits) for different types of HS-POVMs in dimension two.

Comparing these values to the average value of relative entropy, we see that the larger is the number of elements in the HS-POVM, the flatter is the graph of $\tilde{H}$; see also Fig. 3, where the graphs in spherical coordinates are presented.
Figure 3. The relative entropy of highly symmetric qubit measurements, where their Bloch vectors form: a) an equilateral triangle; b) a regular pentagon; c) a tetrahedron; d) an octahedron; e) a cube; f) a cuboctahedron; g) an icosahedron; h) a dodecahedron; i) an icosidodecahedron. The rainbow-colors scale that ranges from red (maximum) to purple (minimum) is used.

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Appendix

Proof of Proposition 5. Fix any geodesic (i.e. a great circle) passing by \( u \). Let \( q \) be one of two vectors lying on the intersection of the plane orthogonal to \( u \) passing through 0 and the geodesic. As 0 is the only \( G \)-invariant vector in \( \mathbb{R}^3 \) and, at the same time, the only \( G_u \)-invariant element orthogonal to \( u \), we have \( \sum_{[h] \in G/G_u} hu = \sum_{g \in G_u} gq = 0 \). Consider a natural parametrisation of the great circle \( \gamma : (-\pi, \pi) \to S^2 \) (throwing away \( -u \)) given by \( \gamma(\delta) := (\sin \delta)q + (\cos \delta)u \) for \( \delta \in (-\pi, \pi) \), where \( \delta \) is the measure of the angle between vectors \( u \) and \( \gamma(\delta) \). Put \( w := \gamma(\delta) \). Then it follows from (27) that

\[
(H_v \circ \gamma)(\delta) = H_v(w)
\]

\[
= \ln |G| + \frac{1}{|G|} \sum_{g \in G} \eta(1 + gw \cdot v)
\]

\[
= \ln |G| + \frac{1}{|G|} \sum_{[h] \in G/G_u} \sum_{g \in G_u} \eta(1 + hgw \cdot v)
\]

\[
= \ln |G| + \frac{1}{|G|} \sum_{[h] \in G/G_u} \sum_{g \in G_u} \eta(1 + (\sin \delta)hgq \cdot v + (\cos \delta)hu \cdot v)
\]

\[
= \ln |G| + \frac{1}{|G/G_u|} \sum_{[h] \in G/G_u} f_h(\delta),
\]

where

\[
f_h(\delta) := \frac{1}{|G_u|} \sum_{g \in G_u} \eta(1 + (\sin \delta)hgq \cdot v + (\cos \delta)hu \cdot v).
\]

and so

\[
f'_h(\delta) = \frac{1}{|G_u|} \sum_{g \in G_u} \eta'(1 + (\sin \delta)hgq \cdot v + (\cos \delta)hu \cdot v) \times
\]

\[
\times ((\cos \delta)hgq \cdot v - (\sin \delta)hu \cdot v).
\]

In particular \( f'_h(0) = 0 \). Moreover,

\[
f''_h(\delta) = \frac{1}{|G_u|} \sum_{g \in G_u} \eta''(1 + (\sin \delta)hgq \cdot v + (\cos \delta)hu \cdot v) \times
\]

\[
\times ((\cos \delta)hgq \cdot v - (\sin \delta)hu \cdot v)^2 + \eta'(1 + (\sin \delta)hgq \cdot v + (\cos \delta)hu \cdot v)(-(\sin \delta)hgq \cdot v - (\cos \delta)hu \cdot v)).
\]

(1) Let \( hu = -v \) for some \( h \in G \). Then

\[
f'_h(\delta) = -(\ln(1 - \cos \delta) + 1) \sin \delta,
\]

and so

\[
f''_h(\delta) = -1 - (\cos \delta)(\ln(1 - \cos \delta) + 2).
\]
Moreover, there exists $c > 0$, such that the inequality $hu \cdot v \geq -1 + c$ holds for any $h \in G/G_u$, $[h] \neq [\tilde{h}]$. Now we can estimate $|f''_h(\delta)|$ as follows:

\[
|f''_h(\delta)| \leq \frac{1}{|G_u|} \sum_{g \in G_u} \left| \frac{((\cos \delta)hgq \cdot v - (\sin \delta)hu \cdot v)^2}{1 + hgw \cdot v} \right| + \frac{1}{|G_u|} \sum_{g \in G_u} |(\ln(1 + hgw \cdot v) + 1)(hgq \cdot v)|
\]

\[
\leq \frac{1}{|G_u|} \sum_{g \in G_u} \left( \frac{4}{|1 + hgw \cdot v|} + (\ln(1 + hgw \cdot v) + 1) \right)
\]

\[
\leq f(1 - \sin \delta + (c - 1) \cos \delta),
\]

for $|\delta| < \pi/2$, where $f(x) := \frac{4}{|x|} + |\ln x| + 1$ for $x > 0$.

Thus

\[
(H_v \circ \gamma)''(\delta) = \frac{1}{|G/G_u|} \left( f''_h(\delta) + \sum_{h \in G/G_u, h \neq \tilde{h}} f''_h(\delta) \right)
\]

\[
\geq g(\delta) \xrightarrow{\delta \to 0} +\infty.
\]

where

\[
g(\delta) := -1 + (\cos \delta)(\ln(1 - \cos \delta) + 2) + (|G/G_u| - 1)f(1 - \sin \delta + (c - 1) \cos \delta)
\]

for $\delta > 0$. In particular, there is $\varepsilon > 0$ such that $(H_v \circ \gamma)''(\delta) > 0$ for $|\delta| < \varepsilon$.

Hence, there is a neighbourhood $V \subset S^2$ of $u$ such that for any geodesic passing by $u$, $H_v$ is convex on its part contained in $V$ and has minimum at $u$. Consequently, $H_v(u) > H_v(w)$ for every $w \in V$, $w \neq u$, which completes the proof of (I).

(2) Assume that $u \neq -gv$ for every $g \in G$ and the linear isotropy group $\{T_u g : g \in G_u\}$ acts irreducibly on $T_u S^2$. Then

\[
f''_h(0) = \frac{1}{|G_u|} \sum_{g \in G_u} (\eta''(1 + hu \cdot v)(hgq \cdot v)^2 + \eta'(1 + hu \cdot v)(-hu \cdot v))
\]

\[
= \eta'(1 + hu \cdot v)(-hu \cdot v) + \eta''(1 + hu \cdot v)\frac{1}{|G_u|} \sum_{g \in G_u} (hgq \cdot v)^2
\]

\[
= (hu \cdot v)(\ln(1 + hu \cdot v) + 1) - \frac{1}{1 + hu \cdot v} \frac{1}{2} (1 - (hu \cdot v)^2)
\]

\[
= (hu \cdot v)(\ln(1 + hu \cdot v) + 2) - 1/2,
\]

where the last but one identity follows from the fact that $\{hgq : g \in G_u\}$ is a normalised tight frame in $S^2$ contained in the plane orthogonal to $hu$ for each $h \in G_u$. Thus we obtain

\[
(H_v \circ \gamma)''(0) = \frac{1}{|G/G_u|} \sum_{[h] \in G/G_u} (hu \cdot v)\ln(1 + hu \cdot v) - 1/2
\]

and (II) follows from the Modern Purkiss Principle. $\square$
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