New connections between moving curves and soliton equations

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Abstract

Lamb has identified a certain class of moving space curves with soliton equations. We show that there are two other classes of curve evolution that may be so identified. Hence three distinct classes of curve evolution are associated with a given integrable equation. The nonlinear Schrödinger equation is used to illustrate this explicitly.

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1 Introduction

Over two decades ago, Lamb \cite{1} presented a formalism which showed that certain special types of motion of space curves can be mapped to completely integrable, soliton-supporting \cite{2}, nonlinear partial differential equations (NLPDE) such as the nonlinear Schrödinger (NLS) equation, the sine-Gordon equation, the Hirota equation, etc., indicating that the corresponding curve motions are also integrable. This formalism arose as an extension of Hasimoto’s earlier work \cite{3}, which had established a connection between the equation of motion of a vortex filament regarded as a moving space curve, and the NLS. Since all soliton equations \cite{2} possess similar characteristics such a Lax pair, solvability by the inverse scattering transform method and an infinite number of constants of motion, the above mapping to the NLS provided a motivation for a general formulation to investigate the interesting curve evolutions that get associated with soliton-supporting equations.

Central to the Lamb formulation is the introduction of the following complex function $\psi$, called the Hasimoto function, defined \cite{3} as

$$
\psi(s, u) = K \exp \left[ i \int \tau ds \right].
$$

(1.1)

Here $s$ is the arc length of the curve and $u$ the time. $K(s, u)$ and $\tau(s, u)$ denote, respectively, the curvature and torsion of the moving curve. It is this function $\psi$ that satisfies various integrable equations in Lamb’s work, and is used frequently in the study of various aspects of curve-dynamics \cite{4}.

It is clear that by comparing the functional form of $\psi$ (Eq. (1.1) with a soliton solution of a given integrable equation for $\psi$, one can identify $K$ and $\tau$, and hence obtain the associated moving curve parameters that correspond to an integrable, shape-preserving curve motion. This in turn
unravels a certain special geometric structure of the given soliton-bearing
NLPDE. In this Letter, we show that two other complex functions defined
by \( \Phi(s,u) = \tau \exp \left[ i \int K ds \right] \) and \( \chi = (K - i\tau) \), also arise from the basic
curve evolution equations, just as naturally as the Hasimoto function \( \psi \) does.
We demonstrate that these can also satisfy various soliton equations. This in
turn leads to the result that each such integrable equation is associated with
not just one, but three distinct classes of space curve motion, and therefore
has a much richer geometric structure than hitherto envisaged. We illustrate
this by using the NLS as an example.

2 Moving space curves

A space curve embedded in three-dimensions may be described using the
usual Frenet-Serret equations \([5]\) for the orthonormal triad of unit vectors
made up of the tangent \( t \), normal \( n \) and the binormal \( b \):

\[ t_s = K n; \quad n_s = -K t + \tau b; \quad b_s = -\tau n. \quad (2.1) \]

Here, \( s \) stands for the arc length of the curve. The subscript \( s \) denotes \( (\partial/\partial s) \).
\( K \) and \( \tau \) are the curvature and torsion of the curve.

If the curve moves with time \( u \), then all quantities in Eq. (2.1) become
functions of both \( s \) and \( u \). To describe the time evolution of the triad \((t,n,b)\)
we write down the following set of equations \([4]\):

\[ t_u = g n + h b; \quad n_u = -g t + \tau_0 b; \quad b_u = -h t - \tau_0 n. \quad (2.2) \]

Here, the subscript \( u \) denotes \( (\partial/\partial u) \). As is clear, the parameters \( g, h \) and \( \tau_0 \)
which determine the motion of the curve are also functions of both \( s \) and \( u \).

On requiring the compatibility conditions

\[ t_{su} = t_{us}; \quad n_{su} = n_{us}; \quad b_{su} = b_{us}, \quad (2.3) \]
a short calculation using Eqs. (2.1) and (2.2) leads to

\[ K_u = (g_s - \tau h) \quad \tau_u = (\tau_0)_s + Kh \quad h_s = (K\tau_0 - \tau g). \quad (2.4) \]

3 Formulation (I) using the Hasimoto function \( \psi \).

In Lamb’s formulation [1], which will be referred to hereafter as formulation (I), the second and third equations of the set (2.1) are combined to yield

\[ (n + i b)_s + i \tau (n + i b) = -Kt. \quad (3.1) \]

This immediately suggests the definition of a certain complex vector

\[ N = (n + i b) \exp [i \int \tau ds]. \quad (3.2) \]

Differentiating Eq. (3.2) with respect to \( s \) and using Eq. (3.1), we get

\[ N_s = -K \exp [i \int \tau ds] t. \quad (3.3) \]

Thus the Hasimoto function \( \psi \) (Eq. (1.1)) appears in a natural fashion in the above equation. Using the definitions of \( N \) and \( \psi \) given in Eqs. (3.2) and (1.1) respectively, Eqs. (2.1) can be written in the form

\[ t_s = \frac{1}{2} (\psi^* N + \psi N^*) \quad ; \quad N_s = -\psi t. \quad (3.4) \]

Next, Eqs. (2.2) take on the form

\[ t_u = -\frac{1}{2} (\gamma_1^* N + \gamma_1 N^*) \quad ; \quad N_u = iR_1 N + \gamma_1 t, \quad (3.5) \]

where

\[ \gamma_1 = -(g + ih) \exp [i \int \tau ds] \quad ; \quad R_1 = (\int \tau_u ds - \tau_0) = \int Kh ds. \quad (3.6) \]
Here we have used the second equation in the set (2.4) to write the last equality in (3.6). We use the subscript 1 here, to denote formulation (I). From Eqs. (3.4) and (3.5), setting $N_{su} = N_{us}$, and equating the coefficients of $t$ and $N$, respectively, we get
\begin{align}
\psi_u + \gamma_{1s} - iR_1\psi &= 0; \tag{3.7} \\
R_{1s} &= \frac{i}{2}(\gamma_{1}^*\psi - \gamma_{1s}^*\psi). \tag{3.8}
\end{align}

Interestingly, as noted by Lamb [1], the structure of Eqs. (3.7) and (3.8) which arose from compatibility conditions on curve evolution, suggests a possible relationship with soliton-bearing equations, via the Ablowitz-Kaup-Newell-Segur (AKNS) formalism [2]. This is seen as follows: It is well known that for a class of soliton-bearing equations with a function $q(s,u)$ as the dependent variable, the Lax pair $L$ and $M$ in the AKNS formalism are given by:
\begin{align}
L \quad y &= \begin{pmatrix} i \frac{\partial}{\partial s} & -iq \\ -iq^* & -i \frac{\partial}{\partial s} \end{pmatrix} y = \zeta \quad y \tag{3.9} \\
i \quad \frac{\partial y}{\partial u} &= \begin{pmatrix} A(s,u,\zeta) & B(s,u,\zeta) \\ -B^*(s,u,\zeta) & -A(s,u,\zeta) \end{pmatrix} y = M \quad y \tag{3.10}
\end{align}

Here, the eigenfunction $y$ is the column vector $(y_1 \quad y_2)^T$ and $\zeta$ is the eigenvalue parameter. Requiring $y_{su} = y_{us}$, Eqs. (3.9) and (3.10) lead to the following AKNS compatibility conditions [2]:
\begin{align}
q_u &= 2Aq + B_s + 2i\zeta B \tag{3.11} \\
A_s &= (Bq^* - B^*q) \tag{3.12}
\end{align}

Equations (3.11) and (3.12) are identical in form to Eqs. (3.7) and (3.8) provided the following identifications are made:
\begin{align}
q &= \psi/2; \quad A = iR_1; \quad B = -\gamma_{1}/2; \quad \zeta = 0. \tag{3.13}
\end{align}
Thus the curve evolution equations (2.1) and (2.2) imply AKNS equations, with \( \zeta = 0 \). Now, \( \gamma_1 \) and \( R_1 \) are given in Eqs. (3.6). It was shown by Lamb [1] that for appropriate choices of \( \gamma_1 \) as a function of \( \psi \) and its derivatives, \( R_1 \) can be found from Eq. (3.8), and substituted in Eq. (3.7) to yield integrable equations for \( \psi \).

Alternatively, once the quantities \( \gamma_1 \) and \( R_1 \) are identified so that Eqs. (3.7) and (3.8) take on the form of an integrable NLPDE for \( \psi \), it is possible to find the corresponding Lax pair, by working directly with equations (3.4) for \( t_s \) and \( N_s \), along with equations (3.5) for \( t_u \) and \( N_u \): This is done [1] by considering the constraint \( t_l^2 + n_l^2 + b_l^2 = 1 \), where the subscript \( l = 1, 2, 3 \) is used to represent the three components of the vectors concerned. One then defines a complex function

\[
f_l^{(1)} = (n_l + ib_l)/(1 - t_l) = (1 + t_l)/(n_l - ib_l),
\]

along with its two other counterparts \( f_l^{(2)} \) and \( f_l^{(3)} \), obtained by cyclically changing \( t, n \) and \( b \) in Eq. (3.14). It can then be shown that all the three functions \( f_l^{(\alpha)}, \alpha = 1, 2, 3 \) satisfy appropriate Riccati equations. The corresponding Lax pair in each case can then be obtained by setting \( f^{(\alpha)} = y_2^{(\alpha)}/y_1^{(\alpha)} \), and identifying the corresponding eigenfunction to be the column vector \((y_1^{(\alpha)} y_2^{(\alpha)})^T\). A short calculation shows that only one of the Lax pairs thus obtained has the AKNS form of Eqs. (3.9) and (3.10), with entries as in (3.13).

We paranthetically remark that it is possible to introduce a non-zero eigenvalue parameter \( \zeta \) in this formalism, by means of a suitable gauge transformation of \( \psi \) in Eq. (3.7). This in turn can be related to a certain gauge freedom [2] in the choice of the orientation of the axes in the plane perpendicular to the tangent of the space curve. The details of this and its
ramifications will be reported when completed.

4 New formulations (II) and (III) using functions $\Phi$ and $\chi$

As already noted, in Lamb’s formulation or formulation (I), one proceeds by first combining the second and third equations of the Frenet-Serret set of equations (2.1), leading to the appearance of $\psi$. In this section, we consider the other two possibilities: Formulation (II), that combines the first and second equations of (2.1), and formulation (III) that combines the first and third equations. As we shall see, these formulations lead to the appearance of two other complex functions $\Phi$ and $\chi$ respectively, in a natural fashion.

**Formulation (II):** Combining the first two equations in Eqs. (2.1), we get

\[
(n - it)_s + iK(n - it) = \tau b \tag{4.1}
\]

This suggests the definition of a second complex vector

\[
M = (n - it) \exp[i \int Kds] \tag{4.2}
\]

Differentiating Eq. (4.2) with respect to $s$ and using Eq. (4.1), we get

\[
M_s = \tau \exp[i \int Kds]b. \]

Thus a second complex function

\[
\Phi(s, u) = \tau \exp[i \int Kds] \tag{4.3}
\]

appears in a natural fashion in this case. Using the definitions of $M$ and $\Phi$ given in Eqs. (4.2) and (4.3) in the basic equations (2.1) and (2.2), and repeating the steps of Sec.3, we obtain the following counterparts of Eqs. (3.4),(3.5) and (3.6):

\[
M_s = \Phi b; \quad b_s = -\frac{1}{2}(\Phi^*M + \Phi M^*), \tag{4.4}
\]
\[ \mathbf{M}_u = iR_2 \mathbf{M} - \gamma_2 b \quad ; \quad b_u = \frac{1}{2}(\gamma_2^* \mathbf{M} + \gamma_2 \mathbf{M}^*) \quad (4.5) \]

where

\[ \gamma_2 = - (\tau_0 - ih) \exp[i \int K ds] \quad ; \quad R_2 = (\int K u ds - g) = - \int \tau h ds. \quad (4.6) \]

We have used the first equation in the set (2.4) to write the last equality in (4.6). The subscript 2 is used on \( \gamma \) and \( R \) to indicate that these correspond to formulation (II). From Eqs. (4.4) and (4.5), setting \( \mathbf{M}_{su} = \mathbf{M}_{us} \), and equating the coefficients of \( b \) and \( \mathbf{M} \), respectively, we get

\[ \Phi_u + \gamma_2 s - iR_2 \Phi = 0; \quad (4.7) \]
\[ R_{2s} = \frac{i}{2}(\gamma_2^* \Phi - \gamma_2^* \Phi). \quad (4.8) \]

The structure of Eqs. (4.7) and (4.8) is the same as that obtained in Lamb’s formulation (see Eqs. (3.7) and (3.8)). Thus using the same steps as in the last section, these can also be cast in the form of AKNS compatibility conditions (3.11) and (3.12), with the following new identifications:

\[ q = \Phi/2; \quad A = iR_2; \quad B = -\gamma_2/2; \quad \zeta = 0. \quad (4.9) \]

This shows that, just as one does in Lamb’s formulation, here also we could take choices of \( \gamma_2 \), as an appropriate function of \( \Phi \) and its derivatives, find \( R_2 \) from (4.8), and substitute these expressions in Eq. (4.7) to obtain some of the well known integrable equations.

**Formulation (III):** Combining the first and third equations of (2.1), we get

\[ (t + ib)_s = (K - i\tau) n \quad (4.10) \]

This suggests the definition of a *third* complex vector

\[ \mathbf{P} = (t + ib) \quad (4.11) \]
This leads to $P_s = (K - i\tau)n$. Thus a third complex function

$$\chi(s, u) = (K - i\tau)$$

(4.12)

appears in this case. Using the above definitions of $P$ and $\chi$ given above in the basic equations (2.1) and (2.2), and proceeding the same way as in the previous formulations, we get

$$P_s = \chi n; \quad n_s = -\frac{1}{2}(\chi^* P + \chi P^*),$$

(4.13)

$$P_u = i R_3 P - \gamma_3 n; \quad n_u = \frac{1}{2}(\gamma_3^* P + \gamma_3 P^*)$$

(4.14)

where

$$\gamma_3 = -(g - i\tau_0); \quad R_3 = -h = -\int (K\tau_0 - \tau g) ds.$$  

(4.15)

Here we have used the last equation in the set (2.4) to write the last equality above, and the subscript 3 corresponds to formulation (III). From Eqs. (4.13) and (4.14), setting $P_{su} = P_{us}$, and equating the coefficients of $n$ and $P$, respectively, we get

$$\chi_u + \gamma_3 s - i R_3 \chi = 0;$$

(4.16)

$$R_{3s} = i \frac{1}{2}(\gamma_3 \chi^* - \gamma_3^* \chi).$$

(4.17)

Once again, Eqs. (4.16) and (4.17) have the same form as Lamb’s result (Eqs. (3.7) and (3.8)), as well as the AKNS compatibility conditions (3.11) and (3.12). Here, the identifications are

$$q = \chi/2; \quad A = i R_3; \quad B = -\gamma_3/2; \quad \zeta = 0.$$  

(4.18)

Thus using the same reasoning as before, for suitable choices of $\gamma_3$ as functions of $\chi$ and its derivatives, Eq. (4.16) for $\chi$ can take the form of known integrable equations.
Now, as is clearly seen from Eqs. (3.6), (4.6) and (4.15), the parameters $\gamma_n, n = 1, 2, 3$ that arise in the three formulations, correspond to three complex functions involving different combinations of the curve evolution parameters $g, h$ and $\tau_0$. Further, the corresponding complex functions $\psi, \Phi$ and $\chi$ that appear in the formulations are also different functions of $K$ and $\tau$. (See Eqs. (1.1), (4.3) and (4.12) respectively.) Thus it is clear that the three formulations describe three distinct classes of curve motion. In the next section, we discuss an application to show this explicitly.

5 An illustrative example: The nonlinear Schrödinger equation

Next, we consider the application of our results to the NLS. This example is appropriate not only because the NLS was one of the first integrable equations to appear in the context of curve motion \[3\], but also because it has applications \[2\] in various fields such as vortex filament motion, optical solitons, magnetic chain dynamics etc.

**Formulaion (I):** As Lamb has shown, the choice

$$\gamma_1 = -i\psi_s,$$

(5.1)

when used in Eqs. (3.8) leads to

$$R_1 = \frac{1}{2}|\psi|^2.$$

(5.2)

Substituting this in Eq(3.7) yields the NLS:

$$i\psi_u + \psi_{ss} + \frac{1}{2}|\psi|^2 \psi = 0.$$  

(5.3)

Using Eq. (1.1) in Eq. (5.1) and equating the resulting expression to $\gamma_1$ defined in (3.6), we get the following curve evolution parameters $g, h$ and $\tau_0$
for this case:

\[ g = -K\tau \ ; \ h = K_s \ ; \ \tau_0 = (h_s + \tau g)/K = (K_{ss}/K) - \tau^2 \] (5.4)

On the other hand, substituting the above expressions for \( g \) and \( h \) in the first equation of (2.2) leads to \( t_u = gn + h b = -K\tau n + K_s b \). By using Eqs. (2.1) and (2.2), it can be easily verified that the above equation implies \( t_u = t \times t_{ss} \). This is identical in form to the Landau-Lifshitz (LL) equation \( S_u = S \times S_{ss} \) for the time evolution of the spin vector \( S \) in a continuous Heisenberg ferromagnetic spin chain. In this formulation, \( t \), the tangent to the space curve gets identified with \( S \). The connection between the NLS and the LL equation is well known [8] by now.

**Formulation (II):** Here, the choice \( \gamma_2 = -i\Phi_s \) when used in Eqs.(4.8) leads to \( R_2 = \frac{1}{2}|\Phi|^2 \). Using this in Eq. (4.7) yields the following NLS for \( \Phi \):

\[ i\Phi_u + \Phi_{ss} + \frac{1}{2}|\Phi|^2\Phi = 0, \] (5.5)

which is identical in form to Eq. (5.3) obtained in formulation (I). However, \( \Phi = \tau \exp[i \int K ds] \) here. Using this to find the above expression for \( \gamma_2 \), and equating it to the expression for \( \gamma_2 \) defined in Eq. (4.6), we obtain the following curve evolution parameters:

\[ h = -\tau_s \ ; \ \tau_0 = -K \tau \ ; \ g = (K \tau_0 - h_s)/\tau = (\tau_{ss}/\tau) - K^2. \] (5.6)

Thus the above parameters \( g, h \) and \( \tau_0 \) which describe the curve motion in this formulation are quite different from those of formulation (I) (See Eqs. (5.4)). In addition, using the above expressions for \( h \) and \( \tau_0 \) in the third equation of Eqs. (2.2), a short calculation shows that for this curve, \( b_u = -b \times b_{ss} \).

Thus, in contrast to formulation (I), it is the *binormal* of the moving curve that satisfies the LL equation in this case.
Formulation (III): Proceeding as in the previous two cases, here the choice is $\gamma_3 = -i\chi_s$, implying $R_3 = \frac{1}{2}|\chi|^2$. This leads to

$$i\chi_u + \chi_{ss} + \frac{1}{2}|\chi|^2\chi = 0,$$

(5.7)

where $\chi = (K - i\tau)$. (See Eq. (4.12)). The curve evolution in this case is described by the three parameters

$$g = \tau_s ; \quad h = (1/2)(K^2 + \tau^2) ; \quad \tau_0 = -K_s. \quad (5.8)$$

Thus by direct comparison of Eqs. (5.8), (5.4) and (5.6), we see that the above expressions for $g$, $h$ and $\tau_0$, which describe the geometry of curve evolution in this case, are quite different from those obtained in formulations (I) and (II). Further, using the parameters given in Eq. (5.8), it is possible to show that the normal to this moving curve satisfies the LL equation in this case, i.e., $n_u = -n \times n_{ss}$. We paranthetically remark that the correspondence of this equation to the NLS equation has been noted by Hirota in a different context [10].

To demonstrate that the three formulations lead to three distinct space curve evolutions, each with a different set $K$, $\tau$, $g$, $h$ and $\tau_0$, we consider the simple example of a one-soliton solution $q$ of the NLS equation:

$$q = a_0 \text{sech} \xi \exp i V_e(s - V_e u)/2. \quad (5.9)$$

Here $V_e$ and $V_c$ denote, respectively, the envelope velocity and carrier velocity of the soliton. The amplitude $a_0 = [V_e(V_e - 2V_c)]^{1/2}$ and $\xi = (s - V_e u)(a_0/2)$. Also, $V_e(V_e - 2V_c) \geq 0$. Note that any two of the three parameters $a_0$, $V_e$ and $V_c$ can be taken to be independent. We have taken the case $V_c = 0$ for illustration, and obtained the expressions for $K$ and $\tau$ for this
solution, in the three formulations, by using $\psi$, $\Phi$ and $\chi$ defined in Eqs. (1.1), (4.3) and (4.12) respectively. These have been entered in Table 1. Using these, the corresponding time-evolution parameters $g$, $h$ and $\tau_0$ for the three formulations are computed from Eqs. (5.4), (5.6) and (5.8) respectively. These are given in Table 2.

**New geometries associated with a soliton solution of NLS:** We now present the geometrical consequences that arise from the three formulations, at the curve level. Let the moving curve be described by the vector $\mathbf{r}(s, u)$. Thus the tangent $\mathbf{t}$ appearing in the Frenet-Serret equations (2.1) is defined as $\mathbf{t} = \mathbf{r}_s(s, u)$. According to the fundamental theorem of curves [5], smooth functions $K (> 0)$ and $\tau$ define a curve $\mathbf{r}(s, u)$ uniquely, modulo orientation in space. Now, from the expressions for $K$ and $\tau$ given in Table 1, we see that formulation (I) yields a moving space curve with constant torsion $\tau$ but a space-time varying curvature $K$. On the other hand, the moving curve obtained using formulation (II) has a constant $K$ but varying $\tau$. In formulation (III), both $K$ and $\tau$ are space-time dependent. Since the three moving curves have different $(K, \tau)$ parameters, our results explicitly illustrate that they are indeed *geometrically distinct*. Further, since they are all gauge equivalent to the NLS, they correspond to integrable curve evolution.

Now, any moving curve arising from formulation (I) is well known to be the solution of the following ”localized induction (LI) equation” [3] for a vortex filament: $\mathbf{r}_u = \mathbf{r}_s \times \mathbf{r}_{ss}$. This is because the LI equation can be easily shown to imply the LL equation for the tangent $\mathbf{t}$ that appears in formulation (I). In contrast, in formulations (II) and (III), the LL equation is satisfied by $\mathbf{b}$ and $\mathbf{n}$ respectively, and hence the corresponding $\mathbf{r}(s, u)$ does not obey the LI equation any more. Thus the curves arising from the two latter formulations are distinct from that associated with the LI equation. It
is of course, not at all obvious whether analogs of the LI equation (i.e., a PDE at the curve level) can be written down for these two formulations. However, irrespective of this, it turns out that there is a procedure to construct the corresponding moving curve $r(s, u)$ itself, for each of these formulations. The results of this lengthy calculation will be presented elsewhere.

Clearly, the dynamics of these new curves which get associated with the LL equations for $n$ and $b$ respectively, would also be form-preserving curves, just as Hasimoto’s vortex filament was. These may find applications in fields other than fluid dynamics. For instance, a phenomenological modelling of the motion of an interface (regarded as a curve) could be one possible application.

6 Conclusions

In this Letter, we have found new connections between moving space curves and solitons using two natural extensions of Lamb’s formulation. Our unified procedure demonstrates that a specific evolution of an integrable equation can actually be associated with three different classes of space curve motion, the class discussed by Lamb being one of these. As described in Sec. 3, the corresponding Lax pairs in the three cases can also be constructed. (See Eq. (3.14) and the discussion below it [9].)

Application to the NLS is discussed. Our analysis shows that the class of curves that are solutions of the well known ”localized induction equation” for a vortex filament is but one of the three classes of moving curves that can be associated with the NLS, i.e., the class connected to Lamb’s formulation. There are (as we have shown in general) two more classes, and in order to understand these new geometric structures of the NLS more explicitly, we have determined the various curve parameters for each of the three moving
curves associated with a one-soliton solution (5.9) of the NLS and displayed them in Tables 1 and 2. While the curve corresponding to Lamb’s class has a constant torsion but varying curvature, the second curve has a constant curvature, but varying torsion. For the third curve, neither of these parameters is a constant. As this example shows, these three are clearly distinct curves. It would indeed be instructive to apply our results to other integrable equations as well, and find the three associated geometric structures.

We conclude with a general remark on the possible connection between a singularity in the "velocities" (i.e., the time evolution parameters) of the geometric dynamics and that of the associated NLPDE. We have shown that the three complex functions \( \psi = K \exp [i \int \tau ds] \), \( \Phi = \tau \exp [i \int K ds] \) and \( \chi = (K - i\tau) = (K^2 + \tau^2)^{\frac{1}{2}} \exp [i \tan^{-1}(\tau/K)ds] \), satisfy the same NLPDE, and that three different moving curves get associated with it. Therefore if a certain solution of the NLPDE has a singularity at a point, then it implies that as the curve evolves, a corresponding singularity will appear in the curvature \( K \) of the first moving curve, in the torsion \( \tau \) of the second curve, and in the quantity \( (K^2 + \tau^2)^{\frac{1}{2}} \) for the third. Since these “velocities” \( g, h \) and \( \tau_0 \) will be certain functionals of \( K, \tau \) and their derivatives for the three curves, the corresponding singularity will be reflected in these velocities as well, and can be studied case by case.
References

[1] G.L. Lamb, J. Math. Phys. 18 (1977) 1654.

[2] See, for instance, M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, PA 1981.

[3] H. Hasimoto, J. Fluid. Mech. 51 (1972) 477.

[4] See, for e.g., I. Klapper, M. Tabor, J. Phys. A 27 (1994) 4919; K. Nakayama, H. Segur, M. Wadati, Phys. Rev. Lett. 69 (1992) 260; R.E. Goldstein, S. A. Langer, Phys. Rev. Lett. 75 (1995) 1094 and references therein.

[5] L.P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Dover, New York, 1960.

[6] Radha Balakrishnan, A. R. Bishop, R. Dandoloff, Phys. Rev. B 47 (1993) 3108; Phys. Rev. Lett. 64 (1990) 2107; Radha Balakrishnan, R. Blumenfeld, J. Math. Phys. 38 (1997) 5878.

[7] Radha Balakrishnan, A. R. Bishop, R. Dandoloff, Phys. Rev. B 47(1993) 5438.

[8] M. Lakshmanan, Phys. Lett. A 61 (1977) 53.

[9] We caution that the cases \( \alpha = 1, 2, 3 \) (see below Eq. (3.10)) should not be confused with the formulations (I), (II) and (III).

[10] R. Hirota, J. Phys. Soc. Jap. 51 (1982) 323. Here, \( \chi \) was used to illustrate that the LL equation (for \( n \)) and the NLS (for \( \chi \)) transform to the same bilinear form.
Table 1: Example: The curvature $K$ and torsion $\tau$ for the special soliton solution (Eq. (5.9)) of the nonlinear Schrödinger equation (NLS) for $\psi$, $\Phi$ and $\chi$ in the respective formulations (I), (II) and (III). This soliton has a vanishing carrier velocity and an envelope velocity $V_e$.

| Formulation | Solution of NLS | $K$ | $\tau$ |
|-------------|-----------------|-----|--------|
| I           | $\psi$          | $V_e \text{sech}\xi$ | $V_e/2$ |
| II          | $\Phi$          | $V_e/2$ | $V_e \text{sech}\xi$ |
| III         | $\chi$          | $V_e \text{sech}\xi \cos V_es$ | $-V_e \text{sech}\xi \sin V_es$ |

Table 2: The corresponding time evolution parameters $g$, $\tau_0$ and $h$ in the three formulations (Eqs. (5.4), (5.6) and (5.8)) for the example in Table 1.

| Formulation | $g$                  | $\tau_0$        | $h$                        |
|-------------|----------------------|-----------------|-----------------------------|
| I           | $(-V_e^2/2)\text{sech}\xi$ | $(-V_e^2/2)\text{sech}^2\xi$ | $(-V_e^2/2)\text{sech}\xi \tanh\xi$ |
| II          | $(-V_e^2/2)\text{sech}^2\xi$ | $(-V_e^2/2)\text{sech}\xi$ | $(V_e^2/2)\text{sech}\xi \tanh\xi$ |
| III         | $(V_e^2/2)\text{sech}\xi \times$ | $(V_e^2/2)\text{sech}\xi \times$ | $(V_e^2/2)\text{sech}^2\xi$ |
|             | $(\tanh\xi \cos V_es + 2\sin V_es)$ | $(\tanh\xi \cos V_es + 2\sin V_es)$ | $(V_e^2/2)\text{sech}^2\xi$ |