Dynamical Self-mass for Massive Quarks

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Abstract

We examine dynamical mass generation in QCD with large current mass quarks. A renormalization group analysis is performed to separate fermion self-mass into a dynamical and a kinematical part. It is shown that the energy scale of the Schwinger-Dyson (SD) equation and the effective gauge coupling are fixed by the current mass. The dynamical self-mass satisfies a homogeneous SD equation which has a trivial solution when the current mass exceeds a critical value. We therefore suggest that the quark condensate, as the function of the current mass, observes a local minimum around $\epsilon \Lambda_{QCD}$. 

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1 Introduction

Dynamical chiral symmetry breaking in QCD has been extensively studied [1]. The standard tool to study this problem is the Schwinger-Dyson (SD) equation, i.e. the fermion gap equation. Since one is usually interested in the chiral limit one studies the gap equation in the limit of the vanishing current quark mass. In the presence of a quark whose current mass is much larger than $\Lambda_{QCD}$, the nature of solutions to SD equation may exhibit non-analytic behavior in $m$. It has long been argued by Pagels [2] that if the current mass is large, it is possible that $m \neq 0$ does not belong to an analytic extension of $m = 0$. In this article we explore this scenario using the renormalization group equation (RGE) analysis.

Recently, Langfeld, Alkofer and Reinhardt [3] have reported on a numerical study of the Schwinger-Dyson equation for massive quarks in the background field of a classical vacuum solution. Besides the usual spontaneous chiral symmetry breaking solution for $m = 0$, they find that the total quark condensate as a function of the current mass exhibits a discontinuous drop at $m_c \simeq 70 \text{MeV}$. This is interesting because not much has been known about the behavior of quark condensates for quarks of large current mass.

However, in a recent conference report [4] we have discussed the problem of dynamical mass generation in QCD with large current mass ($m \gg \Lambda_{QCD}$). It was pointed out that using the renormalization group analysis one can separate unambiguously the quark self-mass (the part of the self-energy which commutes with $\gamma_5$) into a dynamical and a kinematical part. The dynamical part $B_D$ satisfies a homogeneous SD equation but with an effective coupling constant $\bar{g}^2(t) \simeq (\ln \frac{m}{\Lambda_{QCD}})^{-1}$ and hence when $m \gg \Lambda_{QCD}$ it describes weak coupling regime. As a consequence, the SD equation for $B_D$ can be solved quite reliably in a single effective gluon exchange and one finds that $B_D$ has only a trivial solution when $m$ exceeds a critical value $m_c$ which is of the order of $2.7 \Lambda_{QCD}$. The kinematical part $B_K$, however, is a non-singular function of the current mass and satisfies an inhomogeneous equation and hence is proportional to $m$. The total fermion condensate can be written similarly as a sum of a
dynamical and kinematical part which is expected to take a drop near current mass of the order of $m_c^1$.

The aim of this paper is to elaborate on our method which utilizes the renormalization group techniques. The dynamical self-mass may be understood as arising from an effective interaction with the gluons. It would be of interest to explore if the dip in condensate has any observable consequences [5].

2 The Renormalized SD Equation

The quark self-mass $B(p^2)$ satisfies the following SD equation (the gap equation) in the momentum space

$$B(p^2) = m + \frac{ig^2C_2(N)}{4} \text{Tr} \int \frac{d^4k}{(2\pi)^4} D^{\mu\nu}(p - k) \gamma_\mu S(k) \Gamma_\nu(p, k)$$

(1)

where $S(k)$, $D^{\mu\nu}(k)$ and $\Gamma_\nu(p, k)$ are the complete quark, gluon propagators and the proper vertex respectively. $m$ is the current mass; and $C_2(N) = \frac{N^2 - 1}{2N}$ for $SU(N)$. The quark propagator $S$ is related to $B$ by

$$S^{-1}(k) = k^2 + E(k) - B(k^2)$$

(2)

where $E(k)$ is the wave function correction factor. Before trying to solve (1), we must rewrite it in terms of renormalized quantities and specify a renormalization prescription. We take the point of view that the renormalization constants in the renormalized SD equation may be defined as in perturbation theory. We also require that after carrying out the renormalization prescription we get a finite SD equation. We adopt the dimensional regularization and mass independent renormalization scheme [6]. We define renormalized quantities as follows

$$S(p; g, m, \xi; \epsilon) = Z_F(g(\mu); \epsilon)S_R(p; g(\mu), m(\mu), \xi(\mu); \mu);$$

\footnote{We have neglected the non-trivial topological gauge configurations. Inclusion of them may lead to a contribution from the gluon condensate. However, the essential feature of the behavior of the total quark condensate should not be substantially affected by this.}
\[ D^{\mu\nu}(p; g, m, \xi; \epsilon) = Z_A(g(\mu); \epsilon)D_R^{\mu\nu}(p; g(\mu), m(\mu), \xi(\mu); \mu); \]
\[ \Gamma_{\mu}(p; g, m, \xi; \epsilon) = Z_F^{-1}Z_A^{-1/2}\Gamma_R^{\mu}(p; g(\mu), m(\mu), \xi(\mu); \mu) \]
\[ B(p; g, m, \xi; \epsilon) = Z_F^{-1}B_R(p; g(\mu), m(\mu), \xi(\mu); \mu) \]  

where
\[ g(\epsilon) = Z_g(g(\mu); \epsilon)g(\mu); \]
\[ m(\epsilon) = Z_m(g(\mu); \epsilon)m(\mu); \]
\[ \xi(\epsilon) = Z_\xi(g(\mu); \epsilon)\xi(\mu). \]

The renormalized quantities are functions of \( g(\mu) \) and \( \epsilon \). \( \xi \) in (3) is the gauge parameter. Substituting (3) into (1) we obtain the renormalized integral equation (in Euclidean 4 \(-\epsilon\) dimensions)
\[ B^R(p; \mu) = Z_F(\epsilon)[Z_m(\epsilon)m(\mu) + Z_g^2(\epsilon)Z_A^{1/2}(\epsilon)g^2(\mu)C_2(N) \]
\[ \cdot \frac{\mu^\epsilon}{4-\epsilon} Tr \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \gamma^\mu D_R^{\mu\nu}(p-k; \mu)S_R(k; \mu)\Gamma_R^{\nu}(p, k; \mu)]. \]

In (4) we have written \( S_R(p; g(\mu), m(\mu), \xi(\mu); \mu) \) as \( S_R(p; \mu) \) for short.

Eq. (4) is in general not self-consistent since its right hand side contains divergent renormalization constants as \( \epsilon \to 0 \). It has been suggested by Johnson, Baker and Willey [7] that a suitable choice of the gauge parameter \( \xi \) (Landau gauge) would lead to a finite \( Z_F \) as \( \epsilon \to 0 \). Subsequently, it would be possible to require that the divergences arising from the integral cancel those in \( Z_m(\mu, \epsilon)m(\mu) \) and make (3) finite. Thus, a consistent solution to (3) should satisfy the condition of finiteness of \( Z_F(\mu, \epsilon) \) as \( \epsilon \to 0 \). We show below using RGE analysis that when the quark current mass is large, this condition can be approximately satisfied.
3 Renormalization Group Analysis

Equation (4) contains the exact propagators and vertex. If the current mass is small, \(B^R(p; g, m, \xi, \mu, \ldots)\) may be expanded in powers of \(m(\mu)\)

\[
\begin{align*}
B^R(p; g, m, \xi; \mu) &= \left[ B^R_0(p; g, 0, \xi; \mu) + m(\mu)B^R_1(p; g, 0, \xi; \mu) + m(\mu)^2B^R_2(p; g, 0, \xi; \mu) + \cdots \right]
\end{align*}
\]

where we call the first term \(B^R_0\) as the ‘dynamical’ part of \(B^R\) and other terms as the ‘kinematical’ part. A non-vanishing dynamical part signals the spontaneous chiral symmetry breaking. (5) is just the chiral perturbation expansion, which can only make sense when the series is convergent or \(m\) is small \((m \ll \Lambda_{QCD})\). We would like to develop a scheme applicable when \(m > \Lambda_{QCD}\), when the power series solution does not converge. A convenient method in this case is to apply RGE treatment. We may write (4) as a functional equation

\[
F(p; g(\mu), m(\mu), \xi(\mu); \mu) = 0.
\]

From (6) it follows that

\[
(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} + \gamma_m m \frac{\partial}{\partial m} + \delta(g) \xi \frac{\partial}{\partial \xi}) F = 0.
\]

In a mass-independent renormalization scheme, \(\beta(g), \gamma(g), \) and \(\delta(g)\) depend only on \(g(\mu)\). Introducing \(t\) by

\[
t = \ln \frac{m_P}{\mu}
\]

with \(m_P = m(\mu = m_P)\) and observing that \(F\) is homogeneous of order 1 in \(p, m\) and \(\mu\), we have

\[
\left( \frac{\partial}{\partial t} + p \frac{\partial}{\partial p} + \mu \frac{\partial}{\partial \mu} - 1 \right) F(p; g, e^t m', \xi; \mu) = 0.
\]
In (9) \( m'(\mu) = e^{-t}m(\mu) \). From (7) and (9), it follows that \( F \) satisfies the following RGE:

\[
\begin{align*}
(- \frac{\partial}{\partial t} - p \frac{\partial}{\partial p} & + \beta(g) \frac{\partial}{\partial g} + \gamma m' \frac{\partial}{\partial m} \\
& + \delta(g) \xi \frac{\partial}{\partial \xi} + 1) F(p; g, e^t m', \xi; \mu) = 0.
\end{align*}
\]  

(10)

From (10) it follows that

\[
F(p; g, e^t m', \xi; \mu) = e^t F(e^{-t} p; \bar{g}(t), \bar{m}'(t), \bar{\xi}(t); \mu)
\]

(11)

where the effective parameters \( \bar{g}(t), \bar{m}'(t), \) and \( \bar{\xi}(t) \) are defined by

\[
t = \int_g^{\bar{g}(t)} \frac{dx}{\beta(x)},
\]

(12)

\[
\bar{m}'(t) = m'(\mu) \exp \int_g^{\bar{g}(t)} dx \frac{\gamma m(x)}{\beta(x)}.
\]

(13)

and

\[
\bar{\xi}(t) = \xi(\mu) \exp \int_g^{\bar{g}(t)} dx \frac{\delta(x)}{\beta(x)}.
\]

(14)

It should be noted that even though \( t \) is a function of \( \mu \), the effective coupling \( \bar{g}(t) \) depends on \( m_P \) only. An explicit calculation of (12)-(14) gives the following results in one-loop approximation in the minimum subtraction scheme

\[
\alpha(m_P) \equiv \frac{\bar{g}^2(t)}{4\pi} = \frac{2\pi}{\beta_0 \ln \frac{m_p}{\Lambda_{QCD}}},
\]

(15)

\[
\bar{m}'(t) = e^{-t}m(\mu) \exp \int_g^{\bar{g}(t)} dx \frac{\gamma m(x)}{\beta(x)} = e^{-t} \bar{m}(t) = e^{-t}m(\mu = m_P) = \mu,
\]

(16)

and

\[
\bar{\xi}(t) = 1 - \frac{1}{\xi_0 (\ln \frac{m_P}{\Lambda_{QCD}})^d}.
\]

(17)
where $\beta_0 = \frac{33-2n_f}{3}$, $d_\xi = \frac{39-4n_f}{2(33-2n_f)}$, and $\xi_0$ is a constant of integration. It is interesting to observe that $\bar{m}'(t)$ is just the scale parameter $\mu$. This arises because we have defined $t$ by the on-shell current mass $m_P$.

The physical consequence of the RGE analysis follows from equation (11). Instead of solving the SD equation (1) with a large current mass $m$ directly by approximation procedures for $\Gamma_\nu$ and $D^{\mu\nu}$, we can solve, equivalently an effective SD equation governed by a much smaller mass $\mu (= \bar{m}'(t))$ and running coupling and gauge parameters. A suitable approximation to these are given in (15)-(17). It is seen from (17) that when $m_P \gg \Lambda_{QCD}$, $\bar{\xi}(t) \simeq 1$ (Landau gauge) and $\bar{g}(t) \to 0$ as $m_P \to \infty$. Thus a large current mass justifies the single gluon exchange approximation for the effective SD equation and ensures the compatibility of the condition $Z_F(\bar{g}, \bar{\xi}, \epsilon) \to 1 + O(\bar{g}^2)$ as $\bar{\xi} \to 1$, whereas, it would have been hard to justify such an approximation in (1).

As a consequence of these results, the renormalized Green’s functions $D^{\mu\nu}_R$ and $\Gamma^\mu_R$ may be approximated by

$$D^{\mu\nu}_R(p; \bar{g}, \mu, \bar{\xi}; \mu) = \frac{\delta^{\mu\nu} - p^\mu p^\nu/p^2}{p^2} + O(\bar{g}^2);$$
$$\Gamma^\mu_R(p; \bar{g}, \mu, \bar{\xi}; \mu) = \gamma^\mu + O(\bar{g}^2).$$

Substituting (18) into the effective SD equation as given in (11), we arrive at the RGE improved effective SD equation to $O(\bar{g}^2)$

$$B(p^2) = Z_m(\bar{g}; \epsilon)\mu + 3\bar{g}^2 C_2(N) \int \frac{d^{1-\epsilon}k}{(2\pi)^4} \frac{(2\pi^\mu)^\epsilon}{(p-k)^2} \frac{B(k^2)}{k^2} + O(\bar{g}^4),$$

where we have used the notation $B(p^2) \equiv B_R(p; \bar{g}, \mu, \bar{\xi}; \mu)$. Finally, from a solution to (19), we can reconstruct $B_R(p; g(\mu), m(\mu), \xi(\mu); \mu)$ defined in Eq. (4) from the following equation, which is easily derived

$$B_R(p; g(\mu), \epsilon' m(\mu), \xi(\mu); \mu)$$
$$= \epsilon' \exp[- \int_0^{g(t)} \frac{\gamma_F(x)}{\beta(x)}] B_R(e^{-t}p; \bar{g}(t), \mu, \bar{\xi}(t); \mu).$$
Now in (19) the inhomogeneous term contains a mass of the order of $\Lambda_{QCD}$, we can thus use a chiral perturbation theory to expand $B^R(p; \bar{g}(t), m_0, \bar{\xi}(t); \mu)$ as power series in $m_0$ (in our case, of course, $m_0 = \mu$. We use a different symbol to make this decomposition more obvious)

$$B^R(p; \bar{g}(t), m_0, \bar{\xi}(t); \mu) = B^R_{\text{dyn}}(p; \bar{g}(t), 0, \bar{\xi}(t); \mu) + m_0 B^R_{\text{kin}}(p; \bar{g}(t), m_0, \bar{\xi}(t); \mu)$$

(21)

where we have called the term independent of $m_0$ as the dynamical self-mass and have lumped the rest as the kinematical part. In this way of splitting, it is obvious that the kinematical part is a regular function of the $m_0$ and it goes to zero as $m_0 \to 0$. But what is crucial is that both $B_{\text{dyn}}$ and $B_{\text{kin}}$ depend on $m_P$ in a non-analytical way through $\bar{g}(t)$.

Substituting (21) into (20) we obtain the following structure for the self-mass for a heavy quark (taking $\gamma_F = 0$)

$$B^R(p; \bar{g}(\mu), m(\mu), \xi(\mu); \mu) = e^t B_{\text{dyn}}(e^{-t}p; \bar{g}(t)) + m(\mu) \exp(- \int_0^t \bar{g}^{(t)} dx \frac{\gamma_m(x)}{\beta(x)}) B_{\text{kin}}(e^{-t}p; \bar{g}(t), m_0).$$

(22)

It is to be emphasized again that the separation of the self-mass into a dynamical and a kinematical part is not the usual power series expansion in current mass. In fact, substituting (21) into (19) we derive an effective homogeneous SD equation for $B_{\text{dyn}}$

$$B_{\text{dyn}}(p; \bar{g}(t)) = 3 \bar{g}^2(t) C_2(N) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2} \frac{B_{\text{dyn}}(k^2)}{k^2 + B_{\text{dyn}}^2(k^2)} + O(\bar{g}^4)$$

(23)

while $B_{\text{kin}}$ satisfies

$$B_{\text{kin}}(p; \bar{g}(t)) = 1 + 3 \bar{g}^2 C_2(N) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2} \frac{B_{\text{kin}}(k^2)}{(k^2 + B_{\text{kin}}^2(k^2))^2} (k^2 - 3 B_{\text{dyn}}^2).$$

(24)

Both (23) and (24) have been studied in the context of chiral symmetry breaking [9].
4 Solutions and Discussions

The effective SD equations for the heavy quark differ from those for the light quark where the chiral symmetry breaking is of concern. In the latter case the gauge coupling constant is a function of the momentum transfer. In the infrared range the coupling becomes arbitrarily large and a non-trivial solution exists for light quarks. In (23) and (24) the coupling constant is fixed and allows us to seek for a solution in the wider range (not just the asymptotic solution in the limit \( p^2 \to \infty \)). However, a non-trivial solution to (23) should not be referred to as the signal of chiral phase transition.

Eq. (23) is equivalent to the following differential equation

\[
x^2 \frac{d^2 B_{\text{dyn}}(x)}{dx^2} + 2x \frac{dB_{\text{dyn}}(x)}{dx} + \frac{x B_{\text{dyn}}(x)}{x + B_{\text{dyn}}(x)} = 0
\]  

(25)
together with the boundary conditions \((x = p^2)\)

\[
\frac{dB_{\text{dyn}}(x)}{dx} + B_{\text{dyn}}(x) \bigg|_{x=\Delta} = 0; \quad x^2 \frac{dB_{\text{dyn}}(x)}{dx} \bigg|_{x=\delta} = 0
\]  

(26)
where \(\Delta \to \infty\) and \(\delta \to 0\) and \(\lambda = 3g^2C_2(N)/16\pi^2\). We are interested in finding not so much in existence of non-trivial solution to (23) but in determining when the trivial solution \(B_{\text{dyn}}(x) = 0\) is in fact a stable solution. To examine the stability of the trivial solution, let us linearize (23) about \(B_{\text{dyn}} = \varepsilon(x)\). We find

\[
x^2 \varepsilon(x) + 2x \varepsilon(x) + \lambda x \varepsilon(x) = 0
\]  

(27)
with

\[
x \varepsilon(x) + \varepsilon(x) \big|_{x=\Delta} = 0; \quad x^2 \varepsilon(x) \big|_{x=\delta} = 0
\]  

(28)
whose general solution is

\[
\varepsilon(x) = c_1 x^{\rho^+} + c_2 x^{\rho^-}
\]  

(29)
where

\[
\rho_{\pm} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4\lambda}
\]  

(30)
$c_1$ and $c_2$ are to be determined by the boundary conditions. If $c_1$ and $c_2$ are non-zero, then the solution $\varepsilon(x) \to \infty$ as $x \to 0$ and thus the trivial solution is not stable. Thus stability requires $c_1 = c_2 = 0$. Substituting (29) into the b.c. (28), we find the following condition for the trivial solution to be stable:

$$\frac{1 + \sqrt{1 - 4\lambda}}{1 - \sqrt{1 - 4\lambda}} = (\frac{\delta}{\Delta})^{\frac{\alpha_c}{2}}. \quad (31)$$

If $\lambda \leq 1/4$, a solution to (31) is not possible since the left hand side of (31) is finite while the right hand side tends to zero in the limit $\delta \to 0$ and $\Delta \to \infty$. If $\lambda > 0$, $\sqrt{1 - 4\lambda}$ is purely imaginary, then (31) is a transcendental equation for $\lambda \ [10]$. For fixed $\Delta$ and $\delta$, it has an infinite set of solutions for $\lambda$ which becomes dense over the whole domain $\lambda > 1/4$ as $\Delta/\delta$ becomes large. Hence the critical point is $\lambda_c = 1/4$ or $\alpha_c = \pi/4$. For $\lambda \leq \lambda_c$ we have a stable trivial solution $B_{dyn}(x) = 0$. The critical current mass is

$$m_c = \Lambda_{QCD} \exp \frac{2\pi}{\beta_0 \alpha_c} \simeq e\Lambda_{QCD}. \quad (32)$$

Implicit in the analysis above is the assumption that there are no other stable solutions.

The solution to (24) for $B_{kin}$ can be derived by the iteration process while $B_{dyn} = 0$ must be substituted into (24). To the first order one has $B_{kin} = 1 + O(\bar{g}^2(t))$ and the kinematical part of total self-mass reads from (22)

$$B_R^K(p; g(\mu), m(\mu), \xi(\mu); \mu) \cong m(\mu)(b \ln \frac{m}{\mu})^{-c/b}(1 + O(\bar{g}^2)) \quad (m_P \gg \Lambda_{QCD}) \quad (33)$$

where $c = 3C_2(N)/8\pi^2$ and $b = \beta_0/8\pi^2$.

We have concerned ourselves with the self-mass for the heavy quark. Having assumed the chiral phase transition for light quarks, we may draw some conclusions for the self-mass of the massive quark as the function of its current mass. In the range of $m \approx 0$, the $B_D^R$ dominates. As $m$ becomes large, the kinematical part grows almost linearly and the dynamical part changes slowly. At $m_P = m_c$, the dynamical part drops to zero and the total self-mass will exhibit a local minimum. As $m$ gets even larger, the self-mass is completely governed by
the kinematical part (except for the possible contributions from the gluon condensate) and eventually blows up as shown in (33) when $m \to \infty$. The quark condensate defined as

\begin{align}
\langle \bar{q}q \rangle &= -\int \frac{d^4k}{(2\pi)^4} Tr S_F(k) \\
&= -\int \frac{d^4k}{(2\pi)^4} Tr \frac{B^R_K}{k^2 - (B^R_K + B^R_D)^2} - \int \frac{d^4k}{(2\pi)^4} Tr \frac{B^R_D}{k^2 - (B^R_K + B^R_D)^2} \\
&= \langle \bar{q}q \rangle_D + \langle \bar{q}q \rangle_K
\end{align}

(34)

is expected to observe the same drop when $B^R_D = 0$. 

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