On the monodromy group of confluenting linear equations

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Dedicated to Yu.S.Ilyashenko on the occasion of his 60th birthday.

Abstract

We consider a linear analytic ordinary differential equation with complex time having a nonresonant irregular singular point. We study it as a limit of a generic family of equations with confluenting Fuchsian singularities.

In 1984 V.I.Arnold asked the following question: is it true that some operators from the monodromy group of the perturbed (Fuchsian) equation tend to Stokes operators of the nonperturbed irregular equation? Another version of this question was also independently proposed by J.-P.Ramis in 1988.

We consider the case of Poincaré rank 1 only. We show (in dimension two) that generically no monodromy operator tends to a Stokes operator; on the other hand, in any dimension commutators of appropriate noninteger powers of the monodromy operators around singular points tend to Stokes operators.

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1 Introduction

1.1 Brief statements of results, the plan of the paper and the history

Consider a linear analytic ordinary differential equation

\[ \dot{z} = \frac{A(t)}{tk+1} z, \quad z \in \mathbb{C}^n, \quad |t| \leq 1, \quad k \in \mathbb{N} \quad (1.1) \]

with a nonresonant irregular singularity of order (the Poincaré rank) \( k \) at 0 (or briefly, an irregular equation). This means that \( A(t) \) is a holomorphic matrix function such that the matrix \( A(0) \) has distinct eigenvalues (denote them by \( \lambda_i \)). Then the matrix \( A(0) \) is diagonalizable, and without loss of generality we suppose that it is diagonal.

1.1 Definition Two equations of type (1.1) are analytically (formally) equivalent, if there exists a change \( z = H(t)w \) of the variable \( z \), where \( H(t) \) is a holomorphic invertible matrix function (respectively, a formal invertible matrix power series), that transforms one equation into the other.

The analytic classification of irregular equations (1.1) is well-known ([2], [3], [9], [10], [16]): the complete system of invariants for analytic classification consists of a formal normal form (1.4) and Stokes operators (1.6) defined in 1.2; the latters are linear operators acting in the solution space of (1.1) comparing appropriate “sectorial canonical solution bases”.

On the other hand, an irregular equation (1.1) can be regarded as a result of confluence of Fuchsian singular points (recall that a Fuchsian singular point of a linear equation is a first order pole of its right-hand side). Namely, consider a deformation

\[ \dot{z} = \frac{A(t, \varepsilon)}{f(t, \varepsilon)} z, \quad f(t, \varepsilon) = \prod_{i=0}^{k} (t - \alpha_i(\varepsilon)), \quad (1.2) \]

of equation (1.1) that splits the irregular singular point 0 of the nonperturbed equation into \( k+1 \) Fuchsian singularities \( \alpha_i(\varepsilon) \) of the perturbed equation, i.e., \( \alpha_i(\varepsilon) \neq \alpha_j(\varepsilon) \) for \( i \neq j \). The family (1.2) depends on a parameter \( \varepsilon \in \mathbb{R}_+ \cup 0, f(t, 0) \equiv t^{k+1}, A(t, 0) \equiv A(t) \).

The monodromy group of a Fuchsian equation acts linearly in its solution space by analytic extensions of solutions along closed loops. The analytic equivalence class of a generic Fuchsian
equation is completely determined by the local types of its singularities and the action of its monodromy group.

Everywhere below by \( M_i \) we denote the monodromy operator of the perturbed equation (1.2) along a loop going around the singular point \( \alpha_i \) (the choice of the corresponding loops will be specified later). The monodromy group of the perturbed equation is generated by appropriately chosen operators \( M_i \).

In 1984 V.I.Arnold proposed the following question. Consider a generic deformation (1.2). Is there an operator

\[
M_i^{d_i} \ldots M_i^{d_1}
\]

(1.3)

from the monodromy group of the perturbed equation that converges to a Stokes operator of the nonperturbed equation?

A version of this question was proposed independently by J.-P.Ramis in 1988.

It appears that already in the simplest case of dimension 2 and Poincaré rank \( k = 1 \) generically each operator from the monodromy group (except for that along a circuit (and its powers) around both singularities) tends to infinity (Theorem 4.6 in Section 4), so, no one tends to a Stokes operator.

In other terms, generically, no word (1.3) with \( d_i \in \mathbb{Z} \) tends to a Stokes operator. But if \( k = 1 \), then appropriate words (1.3) with noninteger exponents \( d_i \) tend to Stokes operators (Theorem 2.16 in 2.2).

The previous question and its nonlinear analogues were studied by J.-P.Ramis, B.Khesin, A.Duval, C.Zhang, J.Martinet, the author and others (see the historical overview in 1.3). It was proved by the author in [6] that appropriate branches of the eigenfunctions of the monodromy operators \( M_i \) of the perturbed equation tend to appropriate canonical solutions of the nonperturbed equation (Theorem 2.5 in 2.1). In the case of Poincaré rank \( k = 1 \) this implies (Corollary 2.6 in 2.1) that Stokes operators of the nonperturbed equation are limits of transition operators between appropriate eigenbases of the monodromy operators \( M_i \). This Corollary has a generalization for higher Poincaré rank [6].

The proofs of the results of the present paper are based on the previously mentioned results from [6], which are recalled in 2.1.

In 1.2 we recall the analytic classification of irregular equations (1.1) and the definitions of sectorial canonical solution bases and Stokes operators.

In 2.2 we state Theorem 2.16 on convergence of appropriate word (1.3) with noninteger exponents \( d_i \) to a Stokes operator in the case of Poincaré rank \( k = 1 \). Its proof is given in Section 3. The corresponding exponents \( d_i \) do not depend on the choice of deformation. In fact, in the case of the higher Poincaré rank \( k = 2 \) and \( n = 2 \) one can also prove a similar statement, but now the choice of the corresponding exponents \( d_i \) will depend on the choice of deformation. The latter case will be discussed in 2.3.

In Section 4 in the case, when \( k = 1, n = 2 \), for a typical nonperturbed equation (1.1) we prove the divergence of the operators from the monodromy group of the perturbed equation (except for the monodromy along a circuit around both singularities and its powers).

### 1.2 Analytic classification of irregular equations. Canonical solutions and Stokes operators

Let (1.1) be an irregular equation, \( \lambda_i, i = 1, \ldots, n \), be the eigenvalues of the corresponding matrix \( A(0) \).
One can ask the question: is it true that the variables $z = (z_1, \ldots, z_n)$ in the equation can be separated, more precisely, that (1.1) is analytically equivalent to a direct sum of one-dimensional linear equations, i.e., a linear equation with a diagonal matrix function in the right-hand side? Generically, the answer is “no”. At the same time any irregular equation (1.1) is formally equivalent to a unique direct sum of the type

$$\begin{align*}
\dot{w}_i &= \frac{b_i(t)}{t^{k+1}} w_i, \\
i &= 1, \ldots, n,
\end{align*}$$

where $b_i(t)$ are polynomials of degrees at most $k$, $b_i(0) = \lambda_i$. The normalizing series bringing (1.1) to (1.4) is unique up to left multiplication by constant diagonal matrix. The system (1.4) is called the formal normal form of (1.1) ([2], [3], [9], [10], [16]).

Generically the normalizing series diverges. At the same time there exists a finite covering $\bigcup_{j=0}^{N} S_j$ of a punctured neighborhood of zero in the $t$-line by radial sectors $S_j$ (i.e., those with the vertex at 0) that have the following property. There exists a unique change of variables $z = H_j(t)w$ over each $S_j$ that transforms (1.1) to (1.4), where $H_j(t)$ is an analytic invertible matrix function on $S_j$ that can be $C^\infty$-smoothly extended to the closure $\overline{S_j}$ of the sector so that its asymptotic Taylor series at 0 coincides with the normalizing series. The previous statement on existence and uniqueness of sectorial normalization holds in any good sector (see the two following Definitions); the covering consists of good sectors ([2], [3], [9], [10], [16]).

**Case** $k = 1$, $n = 2$, $\lambda_1 - \lambda_2 \in \mathbb{R}$.

**1.2 Definition** A sector in $\mathbb{C}$ with the vertex at 0 is said to be good, if it contains only one imaginary semiaxis $i\mathbb{R}_\pm$, and its closure does not contain the other one (see Fig.1).

**General case.**

**1.3 Definition** (see, e.g., [9]). Let $k \in \mathbb{N}$, $\Lambda = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ be a $n$-ple of distinct numbers, $t$ be the coordinate on $\mathbb{C}$. For a given pair $\lambda_i \neq \lambda_j$ the rays in $\mathbb{C}$ starting at 0 and forming the set $\text{Re}(\lambda_i - \lambda_j) = 0$ are called the $(k, \Lambda)$-imaginary dividing rays corresponding to the pair $(\lambda_i, \lambda_j)$. A radial sector is said to be $(k, \Lambda)$-good, if for any pair $(\lambda_i, \lambda_j)$, $j \neq i$, it contains exactly one corresponding imaginary dividing ray and so does its closure.

**1.4 Remark** In the case, when $k = 1$, $n = 2$, $\lambda_1 - \lambda_2 \in \mathbb{R}$, the imaginary dividing rays are the imaginary semiaxes, and the notions of good sector and $(k, \Lambda)$-good sector coincide.

**1.5 Remark** The ratio $\frac{w_i}{w_j}(t)$ of solutions of equations from (1.4) tends to either zero or infinity, as $t$ tends to zero along a ray distinct from the imaginary dividing rays corresponding to the pair $(\lambda_i, \lambda_j)$. Its limit changes exactly when the ray under consideration jumps over one of the latter imaginary dividing rays.

We consider a covering $\bigcup_{j=0}^{N} S_j$ of a punctured neighborhood of zero by good (or $(k, \Lambda)$-good) sectors numerated counterclockwise and put $S_{N+1} = S_0$. The standard splitting of the normal form (1.4) into the direct sum of one-dimensional equations defines a canonical base
in its solutions space (uniquely up to multiplication of the base functions by constants) with a diagonal fundamental matrix. Denote the latter fundamental matrix by

\[ W(t) = \text{diag}(w_1, \ldots, w_n). \]

Together with the normalizing changes \( H_j \) in \( S_j \), it defines the canonical bases \((f_{j1}, \ldots, f_{jn})\) in the solution space of (1.1) in the sectors \( S_j \) with the fundamental matrices

\[ Z^j(t) = H_j(t)W(t), \quad j = 0, \ldots, N + 1, \] (1.5)

where for any \( j = 0, \ldots, N \) the branch ("with the index \( j + 1 \)") of the fundamental matrix \( W(t) \) in \( S_{j+1} \) is obtained from that in \( S_j \) by the counterclockwise analytic extension for any \( j = 0, \ldots, N \). (We put \( S_{N+1} = S_0 \). The corresponding branch of \( W \) "with the index \( N + 1 \)" is obtained from that "with the index \( 0 \)" by the right multiplication by the monodromy matrix of the formal normal form (1.4).) In a connected component of the intersection \( S_j \cap S_{j+1} \) there are two canonical solution bases coming from \( S_j \) and \( S_{j+1} \). Generically, they do not coincide. The transition between them is defined by a constant matrix \( C_j \):

\[ Z^{j+1}(t) = Z^j(t)C_j. \] (1.6)

The transition operators (matrices \( C_j \)) are called \textit{Stokes operators (matrices)} (see [2], [3], [9], [10], [16]). The nontriviality of Stokes operators yields the obstruction to analytic equivalence of (1.1) and its formal normal form (1.4).

\textbf{1.6 Remark} Stokes matrices (1.6) are well-defined up to simultaneous conjugation by one and the same diagonal matrix.

\textbf{1.7 Example} Let \( k = 1, n = 2 \). In this case without loss of generality we assume that \( \lambda_1 - \lambda_2 \in \mathbb{R}_+ \) (one can achieve this by linear change of the time variable). Then the above covering consists of two sectors \( S_0 \) and \( S_1 \) (Figure 1). The former contains the positive imaginary semiaxis and its closure does not contain the negative one; the latter has the same properties with respect to the negative (respectively, positive) imaginary semiaxis. There are two components of the intersection \( S_0 \cap S_1 \). So, in this case we have a pair of Stokes operators. \textit{The Stokes matrices (1.6) are unipotent}: the one corresponding to the left intersection component is lower-triangular; the other one is upper-triangular ([2], [3], [9], [10], [16]).

\textbf{1.8 Remark} Stokes operators of an irregular equation (1.1) with a diagonal matrix in the right-hand side are identity operators. In this case (1.1) is analytically equivalent to its formal normal form. In general, \textit{two irregular equations are analytically equivalent, if and only if they have the same formal normal form and the corresponding Stokes matrix tuples are obtained from each other by simultaneous conjugation by one and the same diagonal matrix}, cf. the previous Remark. Thus, formal normal form and Stokes matrix tuple taken up to the previous conjugation present the complete system of invariants for analytic classification of irregular equations (see [2], [3], [9], [10], [16]).
1.3 Historical overview

Earlier in 1919 R. Garnier [5] had studied some particular deformations of some class of linear equations with nonresonant irregular singularity. He obtained some analytic classification invariants for these equations by studying their deformations. The complete system of analytic classification invariants (Stokes operators and formal normal form) for general irregular differential equations was obtained later in 1970-th years in the papers by Jurkat, Lutz, Peyerimhoff [10], Sibuya [16] and Balser, Jurkat, Lutz [3]. Later Jurkat, Lutz and Peyerimhoff had extended their results to some resonant cases [11]. It is well-known that the monodromy operators of a linear ordinary differential equation belong to its Galois group (see [9], [14]). In 1985 J.-P.Ramis have proved that the Stokes operators also belong to the Galois group ([14], see also [9]). In 1989 he considered the classical confluenting family of hypergeometric equations and proved convergence of appropriate branches of monodromy eigenfunctions of the perturbed equation to canonical solutions of the nonperturbed one by direct calculation [15]. In the late 1980-th years B.Khesin also proved a version of this statement, but his result was not published. In 1991 A.Duval [4] proved this statement for the biconfluenting family of hypergeometric equations (where the nonperturbed equation is equivalent to Bessel equation) by direct calculation. In 1994 C. Zhang [17] had obtained the expression of Garnier’s invariants via Stokes operators (for the class of irregular equations considered by Garnier).

The conjecture saying that Stokes operators are limit transition operators between monodromy eigenbases of the perturbed equation was firstly proposed by A.A.Bolibrukh in 1996. It was proved by the author in [6]. Later this result was extended to a generic resonant case [8].

Nonlinear analogues of the previous statements for parabolic mappings (i.e., one-dimensional conformal mappings tangent to identity) and their Écalle-Voronin moduli, saddle-node singularities of two-dimensional holomorphic vector fields and their Martinet-Ramis invariants (sectorial central manifolds in higher dimensions) were obtained by the author in [7]. Generalizations and other versions of the statement on parabolic mappings were recently
obtained in the joint paper [12] by P.Mardesic, R.Roussarie, C.Rousseau, and in two unpublished joint papers by the following authors: 1) X.Buff and Tan Lei; 2) A.Douady, Francisco Estrada, P.Sentenac.

A particular case of the result from [7] concerning parabolic mappings (analogous to the previously mentioned statements on linear equations) was obtained by J.Martinet [13].

2 Main results. Stokes operators and limit monodromy

Everywhere below (whenever the contrary is not specified) we consider that the (nonperturbed) irregular equation under consideration has Poincaré rank $k = 1$. In the present Section we recall the statements from [6] expressing the Stokes operators as limit transition operators between monodromy eigenbases of the confluenting Fuchsian equation (Theorems 2.5, 2.11 and Corollary 2.6 in Subsection 2.1). In 2.2 we state the results expressing the Stokes operators as limits of some words (1.3) of noninteger powers of monodromy operators (Theorem 2.16). In 2.3 we discuss the extension of these results to the case of higher Poincaré rank in dimension two.

2.1 Stokes operators as limit transition operators between monodromy eigenbases

We formulate the result from the title of the Subsection firstly in the case, when $k = 1, n = 2$, and then in the general case.

Case $n = 2, k = 1$. Let $\lambda_i, i = 1, 2$, be the eigenvalues of the matrix $A(0)$. Without loss of generality we assume that $\lambda_1 - \lambda_2 \in \mathbb{R}_+$: one can achieve this by linear change of the time variable.

We consider a deformation of (1.1),

$$
\dot{z} = \frac{A(t, \varepsilon)}{f(t, \varepsilon)} z, \quad f(t, \varepsilon) = (t - \alpha_0(\varepsilon))(t - \alpha_1(\varepsilon)), \quad f(t, 0) \equiv t^2, \quad A(t, 0) = A(t),
$$

(2.1)

where $A(t, \varepsilon)$ and $f(t, \varepsilon)$ depend continuously on a parameter $\varepsilon \geq 0$ so that $\alpha_0(\varepsilon) \neq \alpha_1(\varepsilon)$ for $\varepsilon > 0$. Without loss of generality we assume that $\alpha_0 + \alpha_1 \equiv 0$. We formulate the statement from the title of the Subsection for a generic deformation (2.1), see the following Definition.

2.1 Definition A family of quadratic polynomials $f(t, \varepsilon)$ depending continuously on a non-negative parameter $\varepsilon$, $f(t, 0) \equiv t^2$, with roots $\alpha_i(\varepsilon), i = 0, 1, \alpha_0 + \alpha_1 \equiv 0$, is said to be generic, if $\alpha_0(\varepsilon) \neq \alpha_1(\varepsilon)$ for $\varepsilon \neq 0$, and the line passing through $\alpha_0(\varepsilon)$ and $\alpha_1(\varepsilon)$ intersects the real axis by angle bounded away from 0 uniformly in $\varepsilon$. A family (2.1) of linear equations with $n = 2, k = 1, \lambda_1 - \lambda_2 \in \mathbb{R}_+$ is said to be generic, if so is the corresponding family of polynomials $f(t, \varepsilon)$.

2.2 Definition (see, e.g., [2]). A singular point $t_0$ of a linear analytic ordinary differential equation $\dot{z} = \frac{B(t)}{z(t_0)} z$ is said to be Fuchsian, if it is a first order pole of the right-hand side (i.e., the corresponding matrix function $B(t)$ is holomorphic at $t_0$). The characteristic numbers of a Fuchsian singularity are the eigenvalues of the corresponding residue matrix $B(t_0)$ (which are equal to the logarithms divided by $2\pi i$ of the eigenvalues of the corresponding monodromy operator).
2.3 Remark A family (2.1) of linear equations is generic, if and only if the difference of the characteristic numbers at $\alpha_0(\varepsilon)$ (or equivalently, at $\alpha_1(\varepsilon)$) of the perturbed equation is not real for small $\varepsilon$ and moreover has argument bounded away from $\pi \mathbb{Z}$ uniformly in $\varepsilon$ small enough. The latter condition implies that the monodromy operator of the perturbed equation around each singular point $\alpha_i$ has distinct eigenvalues (moreover, their modules are distinct), and hence, a well-defined eigenbase in the solution space (for small $\varepsilon$).

The singularities of the perturbed equation from a generic family have imaginary parts of constant (and opposite) signs (by definition). Without loss of generality everywhere below we consider that

$$\text{Im} \alpha_0 > 0, \text{ Im} \alpha_1 < 0,$$

see Fig. 2.

2.4 Definition Let (2.1) be a generic family of linear equations (see the previous Definition) whose singularity families satisfy the previous inequalities. Let $S_j$, $j = 0, 1$, be a pair of good sectors in the $t$- line (see Definition 1.2) such that for any $\varepsilon$ small enough $\alpha_j(\varepsilon) \in S_j$, $j = 0, 1$, $i\mathbb{R}_+ \subset S_0$, $i\mathbb{R}_- \subset S_1$ (see Fig. 1). The sector $S_j$ is said to be the sector associated to the singularity family $\alpha_j$, $j = 0, 1$.

We show that appropriate branches of the eigenfunctions of the monodromy operator $M_i$ around $\alpha_i$ of the perturbed equation converge to canonical solutions of the nonperturbed equation in the corresponding sector $S_i$. This will imply the statement from the title of the Subsection.

To formulate the latter statement precisely, consider the auxiliary domain

$$S_i' = S_i \setminus [\alpha_0(\varepsilon), \alpha_1(\varepsilon)],$$

which is simply-connected, and the canonical branches of the monodromy eigenfunctions on the domain $S_i'$. In more details, consider a small circle going around $\alpha_i$ and take a base point on it outside the segment $[\alpha_0(\varepsilon), \alpha_1(\varepsilon)]$. In the space of local solutions of the perturbed equation at the base point consider the monodromy operator $M_i$ acting by the analytic extension of a solution along the circle from the base point to itself in the counterclockwise direction. The eigenfunctions of $M_i$ have well-defined branches (up to multiplication by
constants) in the corresponding disc with the segment \([\alpha_0(\varepsilon), \alpha_1(\varepsilon)]\) deleted. Their immediate analytic extension yields their canonical branches on \(S_i'\). In other terms, we identify the space of local solutions with the space of solutions on \(S_i'\) by immediate analytic extension, consider \(M_i\) as an operator acting in the latter space and take its eigenfunctions.

The canonical basic solutions of the nonperturbed equation are numerated by the indices 1 and 2, which correspond to the eigenvalues \(\lambda_1, \lambda_2\) of \(A(0)\). To state the results previously mentioned, let us define the analogous numeration of the monodromy eigenfunctions at \(\alpha_i(\varepsilon)\). The monodromy eigenfunctions are numerated by the characteristic numbers (see Definition 2.2) of the corresponding singularity. The latters are proportional to the eigenvalues of the matrix \(A(\alpha_i(\varepsilon), \varepsilon)\), which tend to \(\lambda_1\) and \(\lambda_2\), as \(\varepsilon \to 0\). This induces the numeration of the monodromy eigenfunctions by the indices 1 and 2 corresponding to the limit eigenvalues \(\lambda_1\) and \(\lambda_2\).

2.5 Theorem (see [6]). Let (2.1) be a generic family of linear ordinary differential equations (see Definition 2.1), \(\alpha_i(\varepsilon)\) be its singularity family, \(S_i\) be the corresponding sector (see the previous Definition), \(S_i'\) be the domain (2.2). Consider the eigenbase on \(S_i'\) of the monodromy operator of the perturbed equation around \(\alpha_i(\varepsilon)\). The appropriately normalized eigenbase (by multiplication of the basic functions by constants) converges to the canonical solution base (1.5) on \(S_i\) of the nonperturbed equation.

2.6 Corollary (see [6]). Let (2.1) be a generic linear equation family (see Definition 2.1), \(\alpha_i\) be its singularity families, \(S_i\) be the corresponding sectors (see the previous Definition) chosen to cover a punctured neighborhood of zero, \(S_i'\) be the corresponding domains (2.2). Let \(C_0, C_1\) be the corresponding Stokes matrices (1.6) of the nonperturbed equation in the left (respectively, right) component of the intersection \(S_0 \cap S_1\). Consider the eigenbase on \(S_i'\) of the monodromy operator of the perturbed equation around \(\alpha_i(\varepsilon)\). Denote by \(Z_i^0(t)\) the fundamental matrix of this eigenbase. Let \(C_0(\varepsilon) (C_1(\varepsilon))\) be the transition matrix between the monodromy eigenbases \(Z_i^0(t), i = 0, 1, \) in the left (respectively, right) component of the intersection \(S_0' \cap S_1'\):

\[
Z_i^1(t) = Z_i^0(t)C_0(\varepsilon) \text{ for } \text{Re} t < 0; \quad Z_i^0(t) = Z_i^1(t)C_1(\varepsilon) \text{ for } \text{Re} t > 0. \tag{2.3}
\]

For any \(j = 0, 1\) and appropriately normalized monodromy eigenbases \(Z_i^0, i = 0, 1\) (the normalization of \(Z_i^0\) (only) depends on the choice of \(j\)) \(C_j(\varepsilon) \to C_j\), as \(\varepsilon \to 0\).

Case \(k = 1, n\) is arbitrary. To state the analogues of Theorem 2.5 and Corollary 2.6 in this more general case, let us firstly extend the notions of a generic family of linear equations and a sector associated to a singularity family.

2.7 Definition Let \(n, k \in \mathbb{N}, n \geq 2, \Lambda = (\lambda_1, \ldots, \lambda_n)\) be a set of \(n\) distinct complex numbers, \(\lambda_i \neq \lambda_j\) be a pair of them. A ray in \(\mathbb{C}\) starting at 0 is called a \((k, \Lambda)-\text{real dividing ray}\) associated to the pair \((\lambda_i, \lambda_j)\), if for any \(t\) lying in this ray \(\text{Im} \frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_k} = 0\). (Or equivalently, it is a ray bisecting an angle between two neighbor imaginary dividing rays (see Definition 1.3) associated to \((\lambda_i, \lambda_j)\)).

2.8 Definition Let (1.1) be an irregular equation with \(k = 1\), \(\Lambda\) be the vector of eigenvalues of the corresponding matrix \(A(0)\). Let (2.1) be its deformation depending continuously on a
nonnegative parameter $\varepsilon$, $f(t,0) \equiv t^2$, $\alpha_0 + \alpha_1 \equiv 0$. The family (2.1) is said to be generic, if $\alpha_0(\varepsilon) \neq \alpha_1(\varepsilon)$ for $\varepsilon \neq 0$, and the line passing through $\alpha_0(\varepsilon)$ and $\alpha_1(\varepsilon)$ intersects each $(k, \Lambda)$-real dividing ray by angle bounded away from 0 uniformly in $\varepsilon$.

**2.9 Definition** Let (2.1) be a generic family (see the previous Definition), $\Lambda$ be the corresponding eigenvalue tuple of $A(0) = A(0,0)$. Let $\alpha_0, \alpha_1$ be the corresponding singular point families, $V_i$ be the half-plane (depending on $\varepsilon$) containing $\alpha_i$ and bounded by the symmetry line of the segment $[\alpha_0, \alpha_1]$. The sector associated to $\alpha_i$ is a $(1, \Lambda)$-good sector (see Definition 1.3) independent on $\varepsilon$ that contains $V_i$ for any $\varepsilon$ small enough.

**2.10 Remark** In the previous Definition the sectors $S_0, S_1$ associated to $\alpha_0, \alpha_1$ respectively cover a punctured neighborhood of zero, so, the nonperturbed equation has a pair of Stokes operators $(C_0, C_1)$ associated to this covering.

**2.11 Theorem** (see [6]). Let (2.1) be a generic family of linear equations (see Definition 2.8), $S_0, S_1, S_0', S_1'$ be respectively the corresponding associated sectors (see the previous Definition) and the domains (2.2). Then the statement of the previous Theorem remains valid. The same is true for Corollary 2.6. In more details, consider the eigenbases $Z_\varepsilon$ on $S_i'$ of the monodromy operators around the singular points $\alpha_i(\varepsilon)$ of the perturbed equation, $i = 0, 1$. Let $C_0(\varepsilon), C_1(\varepsilon)$ be the transition matrices (2.3) between them in the connected components of the intersection $S_0' \cap S_1'$. Let $C_0, C_1$ be the Stokes matrices of the nonperturbed equation in the corresponding limit connected components of the intersection $S_0 \cap S_1$. Then for any $j = 0, 1$ and appropriately normalized monodromy eigenbases $Z_\varepsilon$ (the normalization of $Z_\varepsilon$ (only) depends on the choice of $j$) $C_j(\varepsilon) \to C_j$, as $\varepsilon \to 0$.

**2.2 Stokes operators as limits of commutators of appropriate powers of the monodromy operators**

The Stokes and monodromy operators act in different linear spaces: in the solution spaces of the nonperturbed (respectively, perturbed) equations. To formulate the statement from the title of the Subsection, let us firstly identify these solution spaces and specify the loops defining the monodromy operators.

Let (2.1) be a generic family of linear equations (in the sense of some of Definitions 2.1 and 2.8). Take a "base point" $t_0$ in the unit disc punctured at 0.

**2.12 Remark** The space of local solutions of a linear equation at a nonsingular point $t_0 \in \mathbb{C}$ is identified with the space of initial conditions at $t_0$ (which is common for the nonperturbed and the perturbed equations). This identifies the solution spaces of the latters. The space thus obtained will be denoted by $H_{t_0}$.

**2.13 Remark** Let (1.1) be an irregular equation with $k = 1$, $\Lambda$ be the eigenvalue tuple of the corresponding matrix $A(0)$. Let $S_0, S_1$ be $(1, \Lambda)$-good sectors covering a punctured neighborhood of zero in the $t$-line. Let $C_0, C_1$ be the Stokes operators (1.6) corresponding to the connected components of their intersection. Each operator $C_i$ is well-defined in the space $H_{t_0}$ of local solutions of (1.1) at any point $t_0$ lying in the corresponding component of the intersection $S_0 \cap S_1$.

Now let us define the monodromy operators acting in the previous space $H_{t_0}$.
2.14 Definition Let (2.1) be a generic family of linear equations (in the sense of one of Definitions 2.1 and 2.8), \( \alpha_i(\varepsilon) \), \( i = 0, 1 \), be its singularity families. Fix a point \( t_0 \) (independent on \( \varepsilon \)) disjoint from the line passing through \( \alpha_0(\varepsilon) \) and \( \alpha_1(\varepsilon) \) for any \( \varepsilon \). Let \( l_i \) be a small circle centered at \( \alpha_i(\varepsilon) \) whose closed disc is disjoint from \( -\alpha_i(\varepsilon) \). Let \( \alpha_i(\varepsilon) = [t_0, a_i] \cap l_i \), the segment \( [t_0, a_i] \) be oriented from \( t_0 \) to \( a_i \). Consider the closed path \( \psi_i = [t_0, a_i] \circ l_i \circ [t_0, a_i]^{-1}, i = 0, 1 \), which starts and ends at \( t_0 \) (in the case, when \( k = 1, n = 2, \lambda_1 - \lambda_2 \in \mathbb{R} \), we choose \( t_0 \in \mathbb{R} \), see Fig.3). Define \( M_i : H_{t_0} \to H_{t_0} \) to be the corresponding monodromy operator of the perturbed equation.

![Figure 3](image)

We show that commutators of appropriate noninteger powers of the operators \( M_i \) (see the following Definition) tend to the Stokes operators.

2.15 Definition Let \( d \in \mathbb{R}, M : H \to H \) be a linear operator in a finite-dimensional linear space having distinct eigenvalues. The \( d \)-th power of \( M \) is the operator having the same eigenlines, as \( M \), whose corresponding eigenvalues are some values of the \( d \)-th powers of those of \( M \).

Let \( S_0, S_1 \) be sectors in \( \mathbb{C} \) with vertex at 0 covering a punctured neighborhood of 0. Their left (right) intersection component is the component of their intersection crossed while going from \( S_0 \) to \( S_1 \) in the counterclockwise (respectively, clockwise) direction.

2.16 Theorem Let (2.1) be a generic family of linear equations (in the sense of one of Definitions 2.1 and 2.8), \( \alpha_i(\varepsilon) \), \( i = 0, 1 \), be its singularity families. Let \( S_i \), \( i = 0, 1 \), be the corresponding associated sectors (see Definitions 2.4 or 2.9 respectively) forming a covering of a punctured neighborhood of zero, \( C_0, C_1 \) be the Stokes operators (1.6) of the nonperturbed equation corresponding to the left (respectively, right) component of the intersection \( S_0 \cap S_1 \) (see the previous paragraph). Let \( t_0 \) be a fixed point of unit disc lying in the left component of the intersection \( S_0 \cap S_1 \), \( H_{t_0} \) be the corresponding local solution space (see Remark 2.12). Then the operator \( C_0 \) (\( C_1 \)) acts in the space \( H_{t_0} \) (respectively, \( H_{-t_0} \), see Remark 2.13)
$M_i : H_{\pm t_0} \to H_{\pm t_0}$ be the corresponding monodromy operators from Definition 2.14. Then for any pair of numbers $d_0, d_1 > 0$ such that $d_0 + d_1 < 1$

$$M_1^{-d_1} M_0^{d_0} M_1^{d_1} M_0^{-d_0} \to C_0 \text{ in the space } H_{t_0},$$

$$M_0^{-d_0} M_1^{d_1} M_0^{d_0} M_1^{-d_1} \to C_1 \text{ in the space } H_{-t_0}, \text{ as } \varepsilon \to 0.$$ 

Theorem 2.16 is proved in Section 3.

### 2.3 The case of higher Poincaré rank

Theorem 2.5 on convergence of the monodromy eigenbases to canonical solution bases is stated and proved in [6] for arbitrary irregular equation (for arbitrary Poincaré rank and dimension). It holds for any generic family (1.2) defined to satisfy the following conditions: 1) $\sum \alpha_i \equiv 0,$

$$f'(0,0) \neq 0 \text{ (then } \alpha_i(\varepsilon) = a_i \varepsilon \frac{1}{\lambda} (1 + o(1)), \text{ where the points } a_i \text{ form a regular polygon centered at } 0);$$

2) no one of the previous points $a_i$ lies in a real dividing ray (see Definition 2.7), in other terms, no radial ray of $\alpha_i$ tends to a real dividing ray. To each singularity family $\alpha$ we put into correspondence a $(k, \Lambda)$-good sector $S$ (similarly to Definition 2.9) so that the canonical branches in $S_\varepsilon' = S \setminus \cup_{i=0}^{k}[0, \alpha_i(\varepsilon)]$ of the corresponding monodromy eigenfunctions converge to the canonical solutions of the nonperturbed equation on $S$. In the case of higher Poincaré rank for some pairs of neighbor singularities of the perturbed equation the corresponding sectors cannot not be chosen intersected; then the corresponding transition operator between the monodromy eigenbases tends to a product of Stokes operators. Each Stokes matrix is contained in some of the previous limit products, and its elements are expressed as polynomials in the elements of the corresponding limit product. On the other hand, in dimension two there are always two pairs of neighbor singularity families such that for each singularity pair the corresponding sectors may be chosen intersected. Then the transition operator between the corresponding appropriately normalized monodromy eigenbases tends to the Stokes operator of the nonperturbed equation corresponding to the intersection of the sectors.

### 2.17 Example

Consider the case, when $k = n = 2$. Then the perturbed equation has three singularities, and the number of $(2, \Lambda)$-good sectors covering a punctured neighborhood of zero is equal to 4. One can prove the following version of Theorem 2.16.

Consider a generic deformation (1.2) of an irregular equation (1.1) with $k = n = 2$. Let $\alpha_0, \alpha_1$ be a pair of singularity families numerated counterclockwise and corresponding to intersected sectors (denote the latters by $S_0$ and $S_1$ respectively). Let $t_0 \in \mathbb{C} \setminus 0$ be a fixed (base) point lying between the radial rays of $\alpha_0(\varepsilon), \alpha_1(\varepsilon)$ for all $\varepsilon$. Let $M_0, M_1$ be the corresponding monodromy operators (see Definition 2.14). Let $C$ be the Stokes operator corresponding to the intersection $S_0 \cap S_1$. Then for appropriate $d_0, d_1 \in \mathbb{R} \setminus 0$ (depending on the family of equations)

$$M_1^{-d_1} M_0^{d_0} M_1^{d_1} M_0^{-d_0} \to C, \text{ as } \varepsilon \to 0.$$ 

More precisely, there exist $s_i \in \mathbb{N}$, $l_0, l_1 > 0$ (depending on the family of equations but not on $\varepsilon$) such that the previous statement holds whenever $d_0, d_1$ satisfy the following system of inequalities:

$$(-1)^{s_i}d_i > 0, \quad i = 0, 1$$

$$l_0d_0 + l_1d_1 < 1.$$
2.18 Remark The previous coefficients $l_i$ depend on how close the radial rays of $\alpha_i$, $i = 0, 1$, approach the real dividing rays: if the minimal angle between the radial ray of $\alpha_i$ and some real dividing ray is small, then the corresponding coefficient $l_i$ should be chosen large enough (hence, the corresponding exponent $d_i$ should be taken small enough).

The author believes that the previous statements extend to the general case of arbitrary Poincaré rank and dimension.

3 Convergence of the commutators to Stokes operators. Proof of Theorem 2.16

Firstly we prove Theorem 2.16 in the case, when $k = 1$, $n = 2$. Its proof for the case of $k = 1$ and arbitrary $n$ is analogous: the modifications needed will be discussed in Subsection 3.4

Thus, from now on we consider that $k = 1$, $n = 2$, until the contrary will be specified. Without loss of generality we assume that $\Lambda = (\lambda_1, \lambda_2) = (1, -1)$.

3.1 Properties of the monodromy and the transition operators. The plan of the proof of Theorem 2.16

Let us prove the convergence of the first commutator from Theorem 2.16; the proof of the convergence of the second commutator is analogous.

Thus, from now on we assume that the base point $t_0$ lies in the left component of the intersection $S_0 \cap S_1$, and one can put $t_0 = \frac{-1}{2}$.

3.1 Definition Consider a linear diagonalizable operator acting on $\mathbb{C}^2$ with eigenvalues of distinct modules. Its projective multiplier is the ratio of its eigenvalue with the lower module over that with the higher module. Its projectivization is the Möbius transformation $\mathbb{C} \to \overline{\mathbb{C}}$ induced by its action and the tautological projection $\mathbb{C}^2 \setminus 0 \to \mathbb{P}^1 = \mathbb{C}$.

3.2 Remark In the conditions of the previous Definition the projectivization is a hyperbolic transformation (see [1] and Definition 4.6 in Section 4); in particular, it has an attracting fixed point. The projective multiplier is well-defined and its module is always less than 1. It is equal to the multiplier of the projectivization at its attracting fixed point.

Let us write down the monodromy operators in the eigenbase of $M_0$ (which converges to the canonical solution base of the nonperturbed equation on $S_0$). Then the matrix of $M_0$ is diagonal: denote it

$$\Lambda_0(\varepsilon) = \text{diag}(\lambda_{01}, \lambda_{02})(\varepsilon).$$

By Corollary 2.6, the matrix of $M_1$ is

$$M_1 = C(\varepsilon)\Lambda_1(\varepsilon)C^{-1}(\varepsilon), \quad C(\varepsilon) \to C_0, \quad \Lambda_1(\varepsilon) = \text{diag}(\lambda_{11}, \lambda_{12})(\varepsilon). \quad (3.1)$$

The transition matrix $C(\varepsilon)$ tends to the Stokes matrix $C_0$, which is lower-triangular. Thus, the upper-triangular element of $C(\varepsilon)$ (denoted by $u(\varepsilon)$) tends to 0.

First of all we find the asymptotics of the eigenvalues $\lambda_{ij}$ of $M_i$: 

3.3 Proposition Let (2.1) be a generic family of linear equations (see Definition 2.1), \( t_0 = -\frac{1}{2}, \) \( M_i \) be the monodromy operators of the perturbed equation from Definition 2.14, \( f_{i1,\varepsilon}, f_{i2,\varepsilon} \) be their basic eigenfunctions, \( \lambda_{i1}, \lambda_{i2} \) be the corresponding eigenvalues. Then

\[
\lambda_{01}, \lambda_{12} \to \infty, \ \lambda_{02}, \lambda_{11} \to 0,
\]

\[
\ln \lambda_{01} = -(1 + o(1)) \ln \lambda_{02} = -(1 + o(1)) \ln \lambda_{11} = (1 + o(1)) \ln \lambda_{12}, \ \text{as} \ \varepsilon \to 0.
\]

3.4 Corollary In the conditions of the previous Proposition the projective multipliers of \( M_0 \) and \( M_1 \) are equal respectively to

\[
\mu_0 = \frac{\lambda_{02}}{\lambda_{01}}, \ \mu_1 = \frac{\lambda_{11}}{\lambda_{12}}, \ \mu_i \to 0, \ \text{as} \ \varepsilon \to 0,
\]

\[
\ln \mu_0 = (1 + o(1)) \ln \mu_1, \ \text{as} \ \varepsilon \to 0.
\]

Proof of Proposition 3.3. It follows from definition that \( \ln \lambda_{01} = (1 + o(1)) \frac{2\pi i}{\alpha_0 - \alpha_1} \). The real part of the right-hand side of the previous formula is positive and tends to infinity (since \( \text{Im}(\alpha_0 - \alpha_1) > 0 \) by assumption, and \( \alpha_i \to 0 \)), which implies that \( \lambda_{01} \to \infty \). The similar formulas written for all the \( \lambda_{ij} \) prove the rest of the statements of the Proposition. \( \square \)

Let \( d_0, d_1 > 0, \ d_0 + d_1 < 1, \)

\[
\tilde{M}_i = M_i^{d_i}, \ \tilde{\Lambda}_i = \Lambda_i^{d_i}, \ i = 0, 1.
\]

We prove that

\[
\tilde{M}_1^{-1} \tilde{M}_0 \tilde{M}_1 \tilde{M}_0^{-1} \to C_0.
\]

By definition and (3.1), the matrix of the previous commutator in the eigenbase of \( M_0 \) is

\[
\tilde{M}_1^{-1} \tilde{M}_0 \tilde{M}_1 \tilde{M}_0^{-1} = C(\varepsilon) \tilde{\Lambda}_1^{-1} C^{-1}(\varepsilon) \tilde{\Lambda}_0 C(\varepsilon) \tilde{\Lambda}_1 C^{-1}(\varepsilon) \tilde{\Lambda}_0^{-1}.
\] (3.2)

Let \( u(\varepsilon) \) be the upper-triangular element of the transition matrix \( C(\varepsilon), \mu_1(\varepsilon) \) be the projective multiplier of \( M_1 \). For the proof of the convergence to \( C_0 = \lim C(\varepsilon) \) of the previous commutator we firstly prove that

\[
u = O(\mu_1), \ \text{as} \ \varepsilon \to 0.
\] (3.3)

More precisely, we show in the next Subsection that \( \nu = -(c_1 + o(1))\mu_1 \), where \( c_1 \) is the upper-triangular element of the other Stokes matrix \( C_1 \).

Let \( \nu_0 = \mu_0^{d_0}, \nu_1 = \mu_1^{d_1} \) be the projective multipliers of the operators \( \tilde{M}_0, \tilde{M}_1 \) respectively. Formula (3.3) together with the previous Corollary and the condition \( d_0 + d_1 < 1 \) imply that

\[
u(\varepsilon) = o(\nu_0 \nu_1), \ \text{as} \ \varepsilon \to 0.
\] (3.4)

Using (3.4) we show (in the next Lemma proved in 3.3) that if we eliminate subsequently the terms \( C^\pm(\varepsilon) \) in (3.2) (from the right to the left) except for the left \( C(\varepsilon) \), then on each step the asymptotics of the modified expression (3.2) remains the same: the modified expression can be obtained from the initial one by composing it with an operator tending to the identity. At the last step the final modified expression will be just \( C(\varepsilon) \), which tends to \( C_0 \). This will prove the convergence of (3.2) to \( C_0 \).

Now the convergence of the commutator (3.2) is implied by the following Lemma (modulo (3.3)).
3.5 Lemma Let \( \tilde{M}_0, \tilde{M}_1 \) be two families of two-dimensional complex diagonalizable linear operators depending on a positive parameter \( \varepsilon \). Let \( \Lambda_i = (\lambda_{i1}, \lambda_{i2}) \) be the (diagonal) matrices of \( \tilde{M}_i \), \( i = 0, 1 \), in their eigenbases. Let \( C(\varepsilon) \) be the transition matrix between the eigenbases, more precisely, in the eigenbase of \( \tilde{M}_0 \) one have \( \tilde{M}_1 = C(\varepsilon)\tilde{A}_1C^{-1}(\varepsilon) \). Let the eigenbases converge to some bases in the space so that the corresponding transition matrix \( C(\varepsilon) \) tends to a unipotent lower-triangular matrix (denoted by \( C_0 \)). Let

\[
\nu_0 = \frac{\lambda_{02}}{\lambda_{01}} \to 0, \quad \nu_1 = \frac{\lambda_{11}}{\lambda_{12}} \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

(3.5)

Let \( \nu_0, \nu_1 \) and the upper-triangular element \( u(\varepsilon) \) of the matrix \( C(\varepsilon) \) satisfy (3.4). Then

\[
\tilde{M}_1^{-1}\tilde{M}_0\tilde{M}_1\tilde{M}_0^{-1} \to C_0, \quad \text{as} \quad \varepsilon \to 0.
\]

The Lemma will be proved in Subsection 3.3.

Proof of Theorem 2.16 modulo (3.3) and Lemma 3.5. The operators \( \tilde{M}_i = M_i^d \) satisfy the conditions of the previous Lemma: all the conditions follow from the previous Proposition, Corollary and (3.3). This together with the Lemma implies the convergence of the first commutator in Theorem 2.16. This proves Theorem 2.16 modulo (3.3) and Lemma 3.5.

\( \square \)

3.2 The upper-triangular element of the transition matrix. Proof of (3.3)

We prove the following more precise version of (3.3).

3.6 Lemma Let (2.1) be a generic family of linear equations (see Definition 2.1), \( \alpha_i \) be its singularity families, \( S_i \) be the corresponding sectors (see Definition 2.4) chosen to cover a punctured neighborhood of zero, \( S'_i \) be the corresponding domains from (2.2). Let \( C_0, C_1 \) be the Stokes matrices (1.6) of the nonperturbed equation (corresponding to the left (respectively, right) component of the intersection \( S'_0 \cap S'_1 \)),

\[
C_0 = \begin{pmatrix} 1 & 0 \\ c_0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \quad \text{see Example 1.7.}
\]

Let \( M_i \) be the monodromy operator of the perturbed equation around \( \alpha_i(\varepsilon) \) acting in the space of solutions on \( S'_i \). Let \( Z'_i \) be the (fundamental matrix of) its eigenbase. Let \( C_0(\varepsilon) \) be the transition matrix (2.3) between the bases \( Z'_i \) in the left component of the intersection \( S'_0 \cap S'_1 \). Let the previous eigenbases be normalized to converge so that \( C(\varepsilon) \to C_0 \) (see Corollary 2.6):

\[
C_0(\varepsilon) = \begin{pmatrix} 1 + o(1) & u(\varepsilon) \\ c_0 + o(1) & 1 + o(1) \end{pmatrix}, \quad u(\varepsilon) \to 0.
\]

Let \( \lambda_{11}, \lambda_{12} \) be the eigenvalues of \( M_1 \), \( \mu_1 = \frac{\lambda_{11}}{\lambda_{12}} \) be the corresponding projective multiplier. Then the upper-triangular element \( u(\varepsilon) \) of the matrix \( C_0(\varepsilon) \) has the asymptotics

\[
u(\varepsilon) = (-c_1 + o(1))\mu_1^{-1}, \quad \text{as} \quad \varepsilon \to 0,
\]

(3.6)

where \( c_1 \) is the upper-triangular element of the Stokes matrix \( C_1 \).
**Proof** The transition matrix $C_0(\varepsilon), Z^1_\varepsilon = Z^0_\varepsilon C_0(\varepsilon)$, compares the monodromy eigenbases in the left component of the intersection $S'_0 \cap S'_1$, in particular, on a real interval in $\mathbb{R}_-$. It is not changed, when we extend the basic solutions analytically from $\mathbb{R}_-$ to $\mathbb{R}_+$ along the real line. Denote $Z^i_{\varepsilon,+}$ the corresponding branch on $\mathbb{R}_+$ of the extended fundamental matrix $Z^i_\varepsilon$, $i = 0, 1$. It follows from definition that $Z^1_{\varepsilon,+}$ is obtained from $Z^1_\varepsilon|_{\mathbb{R}_+}$ by applying the inverse monodromy operator $M_1^{-1}$:

$$Z^1_{\varepsilon,+} = Z^1_\varepsilon|_{S'_1} M_1^{-1}; \text{ the matrix } M_1 \text{ is diagonal.} \quad (3.7)$$

On the other hand, we can choose a renormalization of the eigenbase $Z^0_{\varepsilon,+}$ by multiplication of the basic solutions by constants (i.e., changing it to $Z^0_{\varepsilon,+} L(\varepsilon)$, $L(\varepsilon) = \text{diag}(l_1(\varepsilon), l_2(\varepsilon))$ is some family of diagonal matrices) so that in the right component of the intersection $S'_0 \cap S'_1$ the transition matrix $C_1(\varepsilon)$ between $Z^0_{\varepsilon,+} L(\varepsilon)$ and $Z^1_\varepsilon$ tends to the Stokes matrix $C_1$:

$$Z^0_{\varepsilon,+} L(\varepsilon) = Z^1_{\varepsilon}|_{S'_1} C_1(\varepsilon), \quad C_1(\varepsilon) \to C_1.$$  

By definition, $Z^1_{\varepsilon,+} = Z^0_{\varepsilon,+} C_0(\varepsilon)$. Substituting the latter and (3.7) to the previous formula yields

$$C_0(\varepsilon) = L(\varepsilon) C_1^{-1}(\varepsilon) M_1^{-1}. \quad (3.8)$$

The matrices $C_i(\varepsilon)$ tend to the Stokes matrices $C_i$, which are unipotent. The matrices $L(\varepsilon)$, $M_1$ are diagonal and depend on $\varepsilon$. This implies that

$$L(\varepsilon) = M_1(1 + o(1)), \quad \text{as } \varepsilon \to 0.$$

This together with (3.8) implies (3.6). \qed

### 3.3 Commutators of operators with asymptotically common eigenline.

**Proof of Lemma 3.5**

In the proof of Lemma 3.5 we use the following Proposition.

**3.7 Proposition** Let $C(\varepsilon)$ be a family of two-dimensional matrices depending on a parameter $\varepsilon \geq 0$ and converging to a unipotent lower-triangular matrix, as $\varepsilon \to 0$. Let $u = u(\varepsilon)$ be the upper-triangular element of $C(\varepsilon)$ (thus, $u(\varepsilon) \to 0$). Let $\Lambda(\varepsilon) = \text{diag}(\lambda_1(\varepsilon), \lambda_2(\varepsilon))$ be a family of diagonal matrices depending on $\varepsilon > 0$ such that

$$\nu = \frac{\lambda_1}{\lambda_2} \to 0, \quad u = o(\nu), \quad \text{as } \varepsilon \to 0. \quad (3.9)$$

Then

$$\Lambda^{-1}(\varepsilon) C(\varepsilon) \Lambda(\varepsilon) \to \text{Id}, \quad \text{as } \varepsilon \to 0. \quad (3.10)$$

**Proof** The diagonal elements of the matrix in (3.10) are equal to those of $C(\varepsilon)$, and thus, tend to 1. Its lower-triangular element tends to 0; it is equal to that of $C(\varepsilon)$ (which tends to a finite limit) times $\nu$ (which tends to 0 by (3.9)). Its upper-triangular element, which is equal to $u\nu^{-1}$, tends to 0 by (3.9). This proves (3.10). \qed

Consider the commutator (3.2):

$$C(\varepsilon) \Lambda^{-1}(\varepsilon) C^{-1}(\varepsilon) \Lambda_0 C(\varepsilon) \Lambda_1 C^{-1}(\varepsilon) \Lambda_0^{-1}. \quad (3.11)$$
Using the previous Proposition, we firstly "kill" the right $C^{-1}(\varepsilon)$: we show that expression (3.11) is equal to

$$C(\varepsilon)\tilde{\Lambda}_1^{-1}C^{-1}(\varepsilon)\tilde{\Lambda}_0 C(\varepsilon)\tilde{\Lambda}_1\tilde{\Lambda}_0^{-1}(Id + o(1)).$$

(3.12)

Then we kill similarly the right $C(\varepsilon)$ in (3.12) and the remaining $C^{-1}(\varepsilon)$. Finally we get that the initial commutator is equal to $C(\varepsilon)$ times the commutator of diagonal matrices (which is identity) times $(Id + o(1))$. This implies that (3.2) tends to $C_0 = \lim C(\varepsilon)$.

The first step: killing of $C^{-1}$. Let

$$Q(\varepsilon) = \tilde{\Lambda}_0 C^{-1}\tilde{\Lambda}_0^{-1}.$$

By definition, expression (3.11) is equal to

$$C(\varepsilon)\tilde{\Lambda}_1^{-1}C^{-1}(\varepsilon)\tilde{\Lambda}_0 C(\varepsilon)\tilde{\Lambda}_1\tilde{\Lambda}_0^{-1}Q(\varepsilon).$$

It suffices to show that $Q(\varepsilon) \to Id$. This follows from the previous Proposition applied to the families of matrices $C^{-1}(\varepsilon)$ and $\Lambda(\varepsilon) = \tilde{\Lambda}_0^{-1}(\varepsilon)$: these families satisfy the conditions of the previous Proposition. Indeed, by (3.5), $\nu = \nu_0 \to 0$. The upper-triangular element of $C^{-1} (\text{denoted by } \tilde{u})$ is $\tilde{u} = O(u) = o(\nu_0\nu_1) = o(\nu_0)$ by (3.4). This proves (3.9) for $\tilde{u}$. The conditions of the Proposition are checked. Thus, by (3.10), $Q(\varepsilon) \to Id$.

The second step: killing of the right $C$ in (3.12). It repeats the previous discussions with the families $C(\varepsilon)$ and $\Lambda(\varepsilon) = \tilde{\Lambda}_1\tilde{\Lambda}_0^{-1}$.

The third step: killing of the left $C^{-1}$. Done analogously by applying the previous Proposition to the matrix families $C^{-1}$ and $\tilde{\Lambda}_1$. Lemma 3.5 is proved.

3.4 Convergence of the commutators to Stokes operators: the higher-dimensional case

The proof of Theorem 2.16 in higher dimensions repeats that in the two-dimensional case with some changes specified below.

Let (2.1) be a generic family of equations, $\alpha_i$ be its singularity families, $i = 0, 1$, $S_i$, be the corresponding associated sectors. Consider their "left intersection component" that is crossed while going counterclockwise from $S_0$ to $S_1$. Let $t_0$ be a point lying in this component. Let $H_{t_0}$ be the corresponding local solution space, $M_i : H_{t_0} \to H_{t_0}$ be the corresponding monodromy operators (see Definition 2.14). Let $Z^0_\varepsilon$ be the eigenbase of the monodromy operator $M_0$, where the eigenfunctions are taken in the order of decreasing of the modules of the corresponding eigenvalues (it appears that these modules are really distinct, see the next Proposition). Let $Z^1_\varepsilon$ be that of $M_1$, and the order of the eigenfunctions coincide with the order of increasing of the modules of the eigenvalues. Let $C(\varepsilon)$ be the transition matrix between them:

$$Z^1_\varepsilon = Z^0_\varepsilon C(\varepsilon).$$

Let $C_0$, $C_1$ be the Stokes matrices of the nonperturbed equation in the left (respectively, right) connected component of the intersection $S_0 \cap S_1$.

3.8 Proposition In the above conditions for any $\varepsilon$ small enough each monodromy operator $M_i$, $i = 0, 1$, has distinct eigenvalues (denote them $\lambda_{i1}, \ldots, \lambda_{in}$). Moreover, for any $j, k = 1, \ldots, n$, $j < k$, one has $\frac{\lambda_{i0}}{\lambda_{ik}} \to \infty$, $\frac{\lambda_{i1}}{\lambda_{ik}} \to 0$, as $\varepsilon \to 0$. Appropriately normalized eigenbases
\(Z_\varepsilon^0\) and \(Z_\varepsilon^1\) converge to canonical solution bases of the nonperturbed equation in \(S_0\) and \(S_1\) respectively. Let \(\lambda_1, \ldots, \lambda_n\) be the eigenvalues of the matrix \(A(0)\) numerated in the order of increasing of the values \(\text{Re} \frac{\lambda_j}{\varepsilon_{0j}(\varepsilon)}\). The numeration of each (converging) monodromy eigenbase corresponds to the numeration of the limit canonical solution base by the previous eigenvalues \(\lambda_j\).

In dimension two the Proposition follows from Proposition 3.3. In higher dimension its proof is analogous to that of Proposition 3.3.

The Stokes matrices \(C_0\) and \(C_1\) are lower- (respectively, upper-) triangular. This is implied by the last statement of the previous Proposition and the following well-known fact.

3.9 Proposition (see, e.g., [9]). Let \(k, n \in \mathbb{N}, n \geq 2\), \((1.1)\) be an irregular equation, \(\Lambda = (\lambda_1,\ldots,\lambda_n)\) be the eigenvalues of the corresponding matrix \(A(0)\). Let \(S_0, S_1\) be a pair of intersected \((k, \Lambda)\)-good sectors. Let there exist a \(t \in S_0 \cap S_1\) such that the sequence of the values \(\text{Re} \frac{\lambda_j}{\varepsilon}, j = 1, \ldots, n\), increases. Consider the canonical sectorial solution bases of \((1.1)\) numerated by \(\lambda_j\). Then the Stokes matrix of \((1.1)\) in the connected component containing \(t\) of the intersection \(S_0 \cap S_1\) is lower-triangular. In the case of the inverse order of the eigenvalues it is upper-triangular.

The point \(t = i\alpha_0(\varepsilon)\) satisfies the conditions of the previous Proposition with \(k = 1\) (the last statement of Proposition 3.8). Hence, by Proposition 3.9, the Stokes matrix \(C_0\) is lower-triangular and \(C_1\) is upper-triangular.

Let

\[ C(\varepsilon) = (C_{ij}(\varepsilon)), \quad C_1 = C_1^{-1} = (C_{1,ij}). \]

Formula (3.6) of Lemma 3.6 extends to higher dimension as follows:

\[ C_{jk}(\varepsilon) = (C'_{1,jk} + o(1)) (\frac{\tilde{\lambda}_{ij}}{\lambda_{1k}}), \quad \text{as } \varepsilon \to 0. \quad (3.13) \]

The proof of (3.13) repeats that of (3.6) in Subsection 3.2.

Let \(d_0, d_1 > 0, d_0 + d_1 < 1, \tilde{\lambda}_{ij} = \lambda_{ij}^d, i = 0, 1, j = 1, \ldots, n\). One has

\[ C_{jk}(\varepsilon) = O(\frac{\tilde{\lambda}_{ij}}{\lambda_{1k} \lambda_{0j}}), \quad \text{as } \varepsilon \to 0. \quad (3.14) \]

Formula (3.14) follows from (3.13), the inequality \(d_0 + d_1 < 1\) and the asymptotic formula

\(\ln \lambda_{0j} = -(1 + o(1)) \ln \lambda_{1j}, j = 1, \ldots, n\), which is proved analogously to Proposition 3.3. As at the end of Subsection 3.1, Theorem 2.16 is implied by (3.14) and the following higher-dimensional analogue of Lemma 3.5.

3.10 Lemma Let \(\tilde{M}_0, \tilde{M}_1\) be two families of \(n\)-dimensional diagonalizable linear operators depending on a positive parameter \(\varepsilon\). Let \(\tilde{\Lambda}_i = \text{diag}(\lambda_{1i}, \ldots, \lambda_{ni})\) be the (diagonal) matrices of \(\tilde{M}_i\) in their eigenbases. Let \(C(\varepsilon) = (C_{jk}(x))\) be the transition matrix between their eigenbases, more precisely, in the eigenbase of \(\tilde{M}_0\) the matrix of \(\tilde{M}_1\) is \(C(\varepsilon)\tilde{\Lambda}_1C^{-1}(\varepsilon)\). Let the eigenbases converge to some bases in the space so that the transition matrix \(C(\varepsilon)\) converges to a unipotent lower-triangular matrix (denoted by \(C_0\)). Let for any \(j < k, j, k = 1, \ldots, n\),

\[ \frac{\lambda_{0j}}{\lambda_{0k}} \to \infty, \quad \frac{\lambda_{1j}}{\lambda_{1k}} \to 0, \quad \text{as } \varepsilon \to 0. \]
Let in addition the asymptotic formula (3.14) hold. Then
\[ \tilde{M}_1^{-1}\tilde{M}_0\tilde{M}_1\tilde{M}_0^{-1} \to C_0, \text{ as } \varepsilon \to 0. \]

The proof of the Lemma repeats that of Lemma 3.5 with obvious changes.

4 Generic divergence of monodromy operators along degenerating loops

In the present Section we consider only two-dimensional irregular equations with Poincaré rank \( k = 1 \) and their generic deformations. As before, without loss of generality we assume that \( \lambda_1 - \lambda_2 \in \mathbb{R}_+ \), \( \text{Im} \alpha_0 > 0 \), \( \text{Im} \alpha_1 < 0 \).

Let \( (2.1) \) be a generic family of linear equations, \( \alpha_i, i = 0, 1, \) be its singularity families, \( S_0, S_1 \) be the corresponding associated sectors forming a covering of a punctured neighborhood of 0. Let \( t_0 \in \mathbb{R}_- \) be arbitrary fixed base point independent on \( \varepsilon \). Let \( M_0 = M_0(\varepsilon) \), \( M_1 = M_1(\varepsilon) \) be the corresponding monodromy operators of the perturbed equation (see Definition 2.14).

Consider the circle centered at 0 and passing through \( t_0 \) with the counterclockwise orientation (it bounds a disc containing both singularities of the perturbed equation for any small \( \varepsilon \)). The monodromy operator along the previous circle is called the complete monodromy.

4.1 Remark The complete monodromy of the perturbed equation in a generic family \( (2.1) \) converges to the monodromy of the nonperturbed equation along the counterclockwise circuit. In the previous conditions the complete monodromy is equal to \( M_0M_1 \).

In the present Section we state and prove the Theorem saying that for any generic deformation \( (2.1) \) of a typical equation \( (1.1) \) (see the next Definition) each word \( (1.3) \) with integer exponents tends to infinity in \( GL_n \), except for the powers \( (M_0M_1)^k \) of the complete monodromy.

4.1 The statement of the divergence Theorem

4.2 Definition Let \( (1.1) \) be an irregular equation, as at the beginning of the paper, \( t_0 \in \mathbb{C} \setminus 0 \) be arbitrary fixed base point, \( M : H_{t_0} \to H_{t_0} \) be the counterclockwise monodromy operator around zero. Consider some branches at \( t_0 \) of all the sectorial canonical solutions of \( (1.1) \) as elements of \( H_{t_0} \) and take the collection of the complex lines in \( H_{t_0} \) generated by them. The equation is said to be typical, if for any \( k \in \mathbb{Z} \setminus 0 \) no line from the previous collection is transformed by \( M^k \) to another line from the same collection.

4.3 Remark The definition of typical equation does not depend on the choice of the base point and the branches of the canonical solutions. The condition that an equation \( (1.1) \) is typical is equivalent to a countable number of polynomial inequalities on the formal monodromy eigenvalues and the elements of the Stokes matrices.

4.4 Remark Both Stokes operators of a typical equation are nontrivial.

Each monodromy operator word \( (1.3) \) can be rewritten as
\[ M_{j_n}^{s_n} \cdots M_{j_1}^{s_1}, \text{ } s_i = \pm 1, \text{ } M_{j_k}^{s_k} M_{j_{k-1}}^{s_{k-1}} \neq 1 \text{ for any } k = 2, \ldots, n, \quad (4.1) \]
where $n = \sum_{i=1}^{l} |d_i|$ in the notations of (1.3). We consider those words (4.1) that do not coincide literally with powers of the complete monodromy.

**4.5 Definition** A word (4.1) is said to be *reduced*, if it does not coincide (literally) with neither $M_0M_1\ldots M_0M_1$, nor $M_1^{-1}M_0^{-1}\ldots M_1^{-1}M_0^{-1}$.

**4.6 Theorem** Let (1.1) be a typical equation (see Definition 4.2), (2.1) be its generic deformation, $t_0$, $M_0, M_1 : H_{t_0} \to H_{t_0}$, be as at the beginning of the Section. Then any monodromy operator given by a reduced word (4.1) tends to infinity (together with its projectivization, see Definition 3.1), as $\varepsilon \to 0$.

**4.2 Projectivization. The scheme of the proof of Theorem 4.6**

Instead of invertible linear operators $\mathbb{C}^2 \to \mathbb{C}^2$ we will consider their projectivizations, which are Möbius transformations $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Denote $m_0, m_1$ the projectivizations of the monodromy operators $M_0, M_1 : \mathbb{C}^2 \to \mathbb{C}^2$. We show that any reduced word $\tilde{m} = m_{s_0}^{s_n} \ldots m_{s_1}^{s_1}$ tends to infinity in the Möbius group, as $\varepsilon \to 0$. This will prove the Theorem.

Recall the following

**4.7 Definition** (see [1]). A Möbius transformation $m : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is said to be *hyperbolic*, if it has one repelling fixed point (then there is a unique attracting fixed point and each orbit except for the repeller tends to the attractor). A hyperbolic transformation with repeller $a$ and attractor $b$ will be presented as the picture at Fig.4

![Figure 4](image)

In the proof of the divergence of a reduced word $\tilde{m}$ of the projectivizations we use their following properties.

**4.8 Proposition** Let (2.1), $S_i$, $t_0$, $M_i$ be as at the beginning of the Section (the nonperturbed equation is not necessarily typical), $m_i$ be the projectivizations of $M_i$, $i = 0, 1$ (see Definition 3.1). Let $Z^i_\varepsilon = (f_{i1,\varepsilon}, f_{i2,\varepsilon})$ be the eigenbases of $M_i$, $(f_{1,1}, f_{1,2})$ be the sectorial canonical solution bases on $S_i$ of the nonperturbed equation, $i = 0, 1$. Let $p_{ij,\varepsilon}, p_{ij} \in \overline{\mathbb{C}}$ be the tautological projection images of $f_{ij,\varepsilon}$ and $f_{ij}$ respectively. Then $m_i$ are hyperbolic transformations (see the previous Definition) with fixed points $p_{ij,\varepsilon}$: $p_{02,\varepsilon}, p_{01,\varepsilon}$ are respectively the repelling and
attracting fixed points of $m_0$; $p_{11,\varepsilon}, p_{12,\varepsilon}$ are respectively the repelling and attracting fixed points of $m_1$.

**Proof** The Proposition follows from Proposition 3.3.

**4.9 Proposition** Let (2.1), $S_i$, $t_0$, $M_i$ be as at the beginning of the Section, $p_{ij}$, $p_{ij,\varepsilon}$ be the tautological projection images of the canonical basic solutions of the nonperturbed equation and the eigenfunctions of $M_i$ respectively (see the previous Proposition). Then

$$p_{02} = p_{12}, \quad p_{ij} = \lim_{\varepsilon \to 0} p_{ij,\varepsilon}, \text{ see Fig. 5a,b.} \quad (4.2)$$

**Proof** The statements on the limits in (4.2) follow from Theorem 2.5. The coincidence of $p_{02}$ and $p_{12}$ in (4.2) follows from the lower-triangularity of the Stokes matrix $C_0$ (see Example 1.7).

As it is shown below, Theorem 4.6 is implied by the two previous Propositions and the following Lemma.

**4.10 Lemma** Let $p_{02} = p_{12}, p_{01}, p_{11}$ be a given triple of distinct points in $\mathbb{C}$. Let $m_i = m_i(\varepsilon)$, $i = 0, 1$, be two families of hyperbolic Möbius transformations depending on a positive parameter $\varepsilon$, $\mu_0, \mu_1$ be the multipliers of their attractors. Let $p_{01,\varepsilon}, p_{12,\varepsilon}$ be respectively the attractors of $m_0$ and $m_1$, $p_{02,\varepsilon}, p_{11,\varepsilon}$ be their repellers. Let $p_{ij,\varepsilon} \to p_{ij}, \mu_i \to 0$, as $\varepsilon \to 0$. Let in addition the product $m_0 m_1$ converge to a Möbius transformation $m$ such that for any $k \in \mathbb{Z} \setminus 0$, $i = 0, 1$, $j = 1, 2$ the image $m^k p_{ij}$ coincide with no other $p_{ls}$. Then any reduced word $\bar{m} = m_{j2}^{s_n} \ldots m_{j1}^{s_1}$ (see Definition 4.5) tends to infinity.

The projectivizations of the monodromy operators satisfy the conditions of the Lemma. Indeed, the multipliers $\mu_i$ tend to 0 by Corollary 3.4. The convergence $p_{ij,\varepsilon} \to p_{ij}$ follows from the previous Proposition. The product $m_0 m_1$ tends to the projectivization (denoted by $m$) of the monodromy of the nonperturbed equation. The inequalities $m^k p_{ij} \neq p_{ls}$ follow from typicality (see Definition 4.2). This together with the Lemma proves divergence of $\bar{m}$. Theorem 4.6 is proved modulo the Lemma.
4.3 Divergence of word of projectivizations. Proof of Lemma 4.10

As it is shown below, Lemma 4.10 is implied by the following statement.

4.11 Lemma In the conditions of the previous Lemma let \( \tilde{m} = m_{j_n}^{s_n} \ldots m_{j_1}^{s_1} \) be a reduced word such that
\[
m_{j_n}^{s_n} m_{j_{n-1}}^{s_{n-1}} \neq (m_0 m_1)^{\pm 1}.
\] (4.3)
Let \( x \in \mathbb{C} \) be arbitrary point such that
\[
m^x x \neq p_{ij} \text{ for any } s = -n, \ldots, n, \ i = 0, 1, \ j = 1, 2; \ m = \lim_{\varepsilon \to 0} m_0 m_1.
\] (4.4)
Then the image \( \tilde{m} x \) converges to the limit of the attractor of \( m_{j_n}^{s_n} \).

Let us prove Lemma 4.10 modulo Lemma 4.11. If the reduced word \( \tilde{m} \) under consideration satisfies (4.3), then it tends to infinity. Indeed, by Lemma 4.11, the image \( \tilde{m} x \) of a generic \( x \) tends to the attractor of \( m_{j_n}^{s_n} \) hence, \( \tilde{m} \to \infty \). Otherwise, \( \tilde{m} = (m_0 m_1)^{k m'} \), where \( k \in \mathbb{Z} \setminus 0 \), \( m' \) is a word satisfying (4.3) of a length less than that of \( \tilde{m} \). The new word \( m' \) tends to infinity by the previous statement. The product \( m_0 m_1 \) in the previous expression for \( \tilde{m} \) has a finite limit. Hence, \( \tilde{m} \) tends to infinity as well. Lemma 4.10 is proved.

Proof of Lemma 4.11. In the proof of Lemma 4.11 we use the following obvious

4.12 Proposition Let \( m' = m'(\varepsilon) \) be a family of hyperbolic Möbius transformations depending on a positive parameter \( \varepsilon \), \( m' \to \infty \), as \( \varepsilon \to 0 \), so that the attractor and the repeller of \( m' \) tend to distinct limits (hence, the multiplier of the attractor tends to 0). Then for any point \( x \in \mathbb{C} \) distinct from the limit of the repeller its image \( m' x \) tends to the same limit, as the attractor. The same statement holds for arbitrary family \( x(\varepsilon) \) of points bounded away from the limit of the repeller.

We prove Lemma 4.11 by induction in the length \( n \) of the word. For \( n = 1 \) its statement is obvious. Suppose we have proved the Lemma for the words of any length less than a given \( n \). Let us prove it for a word \( \tilde{m} = m_{j_n}^{s_n} \ldots m_{j_1}^{s_1} \) of the length \( n \).

Without loss of generality we assume that \( m_{j_n}^{s_n} = m_0 \); the contrary case is treated analogously. Then by (4.3) and the inequality \( m_{j_n}^{s_n} m_{j_{n-1}}^{s_{n-1}} \neq 1 \) (see (4.1)),
\[
m_{j_{n-1}}^{s_{n-1}} \neq m_1, m_0^{-1}, \text{ thus, } m_{j_{n-1}}^{s_{n-1}} = m_0 \text{ or } m_1^{-1}.
\] (4.5)
Consider the word
\[
m' = m_{j_{n-1}}^{s_{n-1}} \ldots m_{j_1}^{s_1} = m_{j_n}^{-s_n} \tilde{m}.
\]
Suppose firstly that it satisfies (4.3). Then for any \( x \) satisfying (4.4) \( m' x \) tends to the limit (denoted \( p_{ij} \)) of the attractor of \( m_{j_{n-1}}^{s_{n-1}} \) by the induction hypothesis. The latter attractor can be either \( p_{01,\varepsilon} \) or \( p_{11,\varepsilon} \), which are the attractor of \( m_0 \) and the repeller of \( m_1 \) respectively. This follows from (4.5). Hence, the limit \( p_{ij} \) is either \( p_{01} \), or \( p_{11} \); in both cases it does not coincide with the limit \( p_{02} \) of the repeller of \( m_0 \). Therefore, the image \( m_0 p_{ij} \) (and hence, \( m_0(m' x) = \tilde{m} x \)) tends to the same limit, as the attractor of \( m_0 \) (by the previous Proposition).

Now suppose that the word \( m' \) does not satisfy (4.3). Then
\[
m' = (m_0 m_1)^{k m''},
\]
where $k \in \mathbb{Z} \setminus 0$, and $m''$ is a word satisfying (4.3) of a length less than that of $m'$. The induction hypothesis applied to $m''$ implies that for any $x$ satisfying (4.4) its image $m''x$ tends to the limit of the attractor of the last left element of the word $m''$, thus, to some $p_{ij}$. Therefore, $m'x \to m^k p_{ij}$. The image $m^k p_{ij}$ coincides with no $p_{sl}$ (in particular, with the limit $p_{02}$ of the repeller of $m_0$). This follows from the last condition of Lemma 4.10. Therefore, $m_0 m^k p_{ij}$ tends to the limit of the attractor of $m_0$ (and so does $m_0 m'x = \tilde{m}x$ by the previous Proposition). The induction step is over. Lemma 4.11 is proved. The proof of Lemma 4.10 is completed.

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