A note on the Sine-Gordon expansion method and its applications

Nizhum Rahman
Email: nizhum.rahman@uq.edu.au

School of Mathematics and Physics, The University of Queensland, St Lucia, 4072, Australia
Faculty of Science and Information Technology, Daffodil International University Dhaka,1207, Bangladesh

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Abstract

The sine-Gordon expansion method; which is a transformation of the sine-Gordon equation has been applied to the potential-YTSF equation of dimension (3+1) and the reaction-diffusion equation. We obtain new solitons of this equation in the form hyperbolic, complex and trigonometric function by using this method. We plot 2D and 3D graphics of these solutions using symbolic software.

Keywords: Sine-Gordon expansion method; the potential-YTSF equation of dimension (3+1); the reaction-diffusion equation; trigonometric function solutions; hyperbolic function solutions; complex solutions.

1 Introduction

In recent decades, the nonlinear evolution equations (NLEEs) have become one of the vital topics of interest among the researchers from different field including physics, mathematics, engineering and biology [1–7]. The analytic solutions in the form of a travelling wave of the NLEEs play a crucial role for the development of scientific fields. A variety of effective methods have been constructed from the end of the 20th century by the researchers. Many of these methods have been used successfully in different NLEEs to obtain new solutions [8–32].

In this article we will apply the sine-Gordon expansion method [33] to the potential-YTSF equation of dimension (3+1) and the reaction-diffusion equation. Recently this method has been applied to only a few research articles [34–37]. The remaining part of the article is described as follows. The sine-Gordon expansion method has been stated in section 2. The next section is the applications of this method. Finally, we describe the necessity of this method briefly in the last section.

2 Method

The sine-Gordon equation (SGE) [33]

\[ \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = h^2 \sin(u) \]  

arises in many physical science applications [38,39], where \( h \) is a constant. By considering the moving coordinate as \( u(x, t) = U(\eta) \) where \( \eta = k(x - ct) \), after simplifying this equation, we get

\[ \frac{d^2 U}{d\eta^2} = \frac{h^2}{k^2(1 - c^2)} \sin(U) \]  

(2)
where $\eta$ and $c$ represent the width and velocity of the travelling waves respectively. After multiplying $dU/d\eta$ on both sides of Eq. (2) and integrating, we get,

$$\left[\left(\frac{U}{2}\right)\right]^2 = \frac{h^2}{k^2(1-c^2)} \sin^2\left(\frac{U}{2}\right),$$

where the constant of integration is considered to zero. Substituting $\omega(\eta) = \frac{U}{2}$ and $\frac{h^2}{k^2(1-c^2)} = 1$ into Eq.(3), we obtain

$$\omega' = \sin(\omega).$$

This is a modified formation of the SGE. We can write the solution of Eq. (4) as of the form

$$\sin[\omega(\eta)] = \frac{2r \exp(\eta)}{r^2 \exp(2\eta) + 1} = sech(\eta),$$

or

$$\cos[\omega(\eta)] = \frac{r^2 \exp(2\eta) - 1}{r^2 \exp(2\eta) + 1} = tanh(\eta),$$

with $r$ is a non-zero constant of integration.

The travelling wave solution $U(\eta)$ of the NLEEs of the form

$$Q(u, u_x, u_t, u_{xx}, u_{ttt}, ............) = 0,$$

can be written as

$$U(\eta) = \sum_{j=1}^{n} \tanh^{j-1}(\eta) [B_j \sech(\eta) + A_j \tanh(\eta)] + A_0.$$  

Using Eq. (5) and Eq. (6), the solutions of the Eq. (8) take the following form

$$U(\omega) = \sum_{j=1}^{n} \cos^{j-1}(\omega) [B_j \sin(\omega) + A_j \cos(\omega)] + A_0.$$  

We calculate the value of unknown $n$ by apply the homogeneous balance assumption. After letting the collection of the coefficients of $\sin^i(\omega) \cos^j(\omega)$ of equal power to be all 0 (zero), we get an algebraic system of equations. By find out the unknowns of the system and using Eq.(9), we get the solutions to Eq. (7) is the form of (8).

3 Applications

We can implement the sine-Gordon expansion method in many NLEEs. To demonstrate our method, we examine the method to the potential-YTSF equation of dimension (3+1) and the reaction-diffusion equation.

3.1 The potential-YTSF equation of dimension (3+1)

We consider the potential-YTSF equation of dimension (3+1) in the form

$$-4 u_{xt} + u_{xxxx} + 4 u_x u_{xx} + 2 u_{xx} u_z + 3 u_{yy} = 0,$$

which arise in many physical problems and have been solved in different ways by the researchers. The moving coordinate

$$u(x, y, z, t) = U(\eta), \quad \eta = x + y + z - ct,$$

allows to convert Eq.(10) in to an ODE

$$U'''' + 6U'U'' + (4c + 3)U'' = 0,$$
integrating it with respect to $\eta$, we acquire

$$U''' + 3(U')^2 + (4c + 3)U' = 0,$$  \hspace{1cm} (12)

where integrating constant is set to zero. Setting $U' = V$, we have

$$V'' + 3V^2 + (4c + 3)V = 0.$$  \hspace{1cm} (13)

Applying homogeneous balance principle between $V''$ and $V^2$ in Eq. (13) based on Eq. (9) we obtain $n + 2 = n + n$, which implies $n = 2$. Now we can write Eq. (9) as

$$V(\omega) = B_1 \sin(\omega) + A_1 \cos(\omega) + B_2 \sin(\omega) \cos(\omega) + A_2 \cos^2(\omega) + A_0$$  \hspace{1cm} (14)

and

$$V'' = -B_1 \sin^3(\omega) + B_1 \cos^2(\omega) \sin(\omega) - 2A_1 \sin^2(\omega) \cos(\omega) - 5B_2 \cos(\omega) \sin^3(\omega) + B_2 \cos^3(\omega) \sin(\omega) + 2A_2 \sin^4(\omega) - 4A_2 \cos^2(\omega) \sin^2(\omega).$$  \hspace{1cm} (15)

By substituting Eq. (14) and Eq. (15) into Eq. (13) and equaling all polynomials with same degree to zero, we get a system of equation as below.

constant: $4vA_0 + 3A_0^2 + 3B_1^2 + 3A_0 + 2A_2 = 0$

$\cos(\omega): 4vA_1 + 6A_0A_1 + 6B_1B_2 + A_1 = 0$

$\sin(\omega): 4vB_1 + 6A_0B_1 + 2B_1 = 0$

$\cos^2(\omega): 4vA_2 + 6A_0A_2 + 3A_1^2 - 3B_1^2 + 3B_2^2 - 5A_2 = 0$

$\sin(\omega) \cos(\omega): 4vB_2 + 6A_0B_2 + 6A_1B_1 - 2B_20 = 0$

$\cos^3(\omega): 6A_1A_2 - 6B_1B_2 + 2A_1 = 0$

$\sin(\omega) \cos^2(\omega): 6A_1B_2 + 6A_2B_1 + 2B_1 = 0$

$\cos^4(\omega): 3A_2^2 - 3B_2^2 + 6A_2 = 0$

$\sin(\omega) \cos^3(\omega): 6A_2B_2 + 6B_2 = 0$

After solving the above system, we find the traveling wave solution $U(\eta)$ to Eq. (10) in the form of \textcircled{8}.

Case-1

$$c = -7/4, \ A_0 = 2, \ A_1 = 0, \ B_1 = 0, \ A_2 = -2, \ B_2 = 0,$$

which gives:

$$u(x, y, z, t) = U(\eta) = 2 \tanh(\eta), \text{ with } \eta = x + y + z + \frac{7}{4}t.$$  \hspace{1cm} (16)

Figure 1: The graphical representation of Eq. (16) (3D on the left side and 2D on the right side).
Case-2

c = 1/4, A_0 = 2/3, A_1 = 0, B_1 = 0, A_2 = -2, B_2 = 0,

which gives:

\[ u(x, y, z, t) = U(\eta) = 2 \tanh(\eta) - \frac{4}{3} \eta, \text{ with } \eta = x + y + z - \frac{1}{4} t. \]  (17)

Figure 2: The graphical representation of Eq. (17) (3D on the left side and 2D on the right side).

Case-3

c = -1, A_0 = 1, A_1 = 0, B_1 = 0, A_2 = -1, B_2 = \pm i,

which gives:

\[ u(x, y, z, t) = U(\eta) = \tanh(\eta) \pm i \text{sech}(\eta), \text{ with } \eta = x + y + z + t. \]  (18)

Figure 3: The graphical representation of Eq. (18) (3D on the left side and 2D on the right side).

Case-4

c = -1/2, A_0 = 2/3, A_1 = 0, B_1 = 0, A_2 = -1, B_2 = \pm i,
which gives:

\[
    u(x, y, z, t) = U(\eta) = \tanh(\eta) \pm i \text{sech}(\eta) - \frac{1}{3} \eta, \text{ with } \eta = x + y + z + \frac{1}{2} t. \tag{19}
\]

\[\text{Figure 4: The graphical representation of Eq. (19) (3D on the left side and 2D on the right side).}\]

### 3.2 The Reaction-Diffusion Equation

We consider the following form of the reaction-diffusion equation [46],

\[
    u_{tt} + \alpha u_{xx} + \beta u + \gamma u^3 = 0 \tag{20}
\]

where \(\alpha\), \(\beta\) and \(\gamma\) are constants (nonzero). The moving coordinate \(u(x, t) = U(\eta)\), where \(\eta = x - ct\) leads the Eq. (20) in an ordinary differential equation (ODE)

\[
    (\alpha + c^2) U'' + \beta U + \gamma U^3 = 0. \tag{21}
\]

Considering homogeneous balance between \(U''\) and \(U^3\) in Eq. (21) based on Eq. (9) we obtain \(n + 2 = n + n + n\), which implies \(n = 1\). Now we can write Eq. (9) as

\[
    U(\omega) = B_1 \sin(\omega) + A_1 \cos(\omega) + A_0. \tag{22}
\]

Using Eq. (22) into Eq. (21), and equaling all polynomials with same degree to get, we have a system of equation as below.

\[
\begin{align*}
\text{constant:} & \quad \gamma A_0^3 + 3 \gamma A_0 B_1^2 + \beta A_0 = 0 \\
\cos(\omega): & \quad 3 \gamma A_0^2 A_1 + 3 \gamma A_1 B_1^2 - 2 v^2 A_1 - 2 \alpha A_1 + \beta A_1 = 0 \\
\sin(\omega): & \quad 3 \gamma A_0^2 B_1 + \gamma B_1^3 - v^2 B_1 - \alpha B_1 + \beta B_1 = 0 \\
\cos^2(\omega): & \quad 3 \gamma A_0 A_1^2 - 3 \gamma A_0 B_1^2 = 0 \\
\sin(\omega) \cos(\omega): & \quad 6 \gamma A_0 A_1 B_1 = 0
\end{align*}
\]
\[
\begin{align*}
\cos^3(\omega) : & \quad \gamma A_1^3 - 3\gamma A_1 A_0^2 + 2\nu^2 A_1 + 2\alpha A_1 = 0 \\
\sin(\omega)\cos^2(\omega) : & \quad 3\gamma A_1^2 B_1 - \gamma B_1^3 + 2\nu^2 B_1 + 2\alpha B_1 = 0
\end{align*}
\]

After solving the above system, we find the traveling wave solution \( u(x, y, z, t) \) to Eq. (20) in the form of (8).

**Case-1**
\[
c = \pm \sqrt{\frac{\beta - 2\alpha}{2}}, \quad A_0 = 0, \quad A_1 = \pm \sqrt{\frac{-\beta}{\gamma}}, \quad B_1 = 0, \quad \alpha = \alpha, \quad \beta = \beta, \quad \gamma = \gamma,
\]
which gives:
\[
u(x, t) = U(\eta) = \pm \sqrt{\frac{-\beta}{\gamma}} \tanh(\eta), \text{ with } \eta = x \pm \sqrt{\frac{\beta - 2\alpha}{2}} t.
\]  

**Case-2**
\[
c = \pm \sqrt{-\beta - \alpha}, \quad A_0 = 0, \quad B_1 = \pm \sqrt{\frac{-2\beta}{\gamma}}, \quad A_1 = 0, \quad \alpha = \alpha, \quad \beta = \beta, \quad \gamma = \gamma,
\]
which gives:
\[
u(x, t) = U(\eta) = \pm \sqrt{\frac{-2\beta}{\gamma}} \text{sech}(\eta), \text{ with } \eta = x \pm \sqrt{-\beta - \alpha} t.
\]  

Figure 5: The graphical representation of Eq. (23) (3D on the left side and 2D on the right side).

**Figure 6:** The graphical representation of Eq. (24) (3D on the left side and 2D on the right side).
Case-3

\[ c = \pm \sqrt{2\beta - \alpha}, \ A_0 = 0, \ A_1 = \pm \sqrt{-\frac{\beta}{\gamma}}, \ B_1 = \pm \sqrt{\frac{\beta}{\gamma}}, \ \alpha = \alpha, \ \beta = \beta, \ \gamma = \gamma, \]

which gives:

\[ u(x, t) = U(\eta) = \pm \sqrt{-\frac{\beta}{\gamma}} \tanh(\eta) \pm \sqrt{\frac{\beta}{\gamma}} \text{sech}(\eta), \ \text{with} \ \eta = x \pm \sqrt{2\beta - \alpha}t. \]  \quad (25)

4 Conclusion

In this article, we have applied the Sine-Gordon expansion method for calculating new travelling wave solutions to the potential-YTSF equation of dimension (3+1) and the reaction-diffusion equation. We have found these solutions of the equation in the trigonometric, complex and hyperbolic function forms. This method is powerful and very efficient to finding travelling wave solutions to the NLEEs. We can solve various NLEEs by this method using any symbolic software.

Abbreviations

NLEEs, SGE and ODE.

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