DYNAMICAL BEHAVIOR FOR THE SOLUTIONS OF THE NAVIER-STOKES EQUATION

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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Abstract. We study several quantitative properties of solutions to the incompressible Navier-Stokes equation in three and higher dimensions:

\begin{equation}
\begin{aligned}
& \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \\
& \text{div} u = 0, \\
& u(0, x) = u_0(x).
\end{aligned}
\end{equation}

More precisely, for the blow up mild solutions with initial data in \( L^\infty(\mathbb{R}^d) \) and \( H^{d/2-1}(\mathbb{R}^d) \), we obtain a concentration phenomenon and blowup profile decomposition respectively. On the other hand, if the Fourier support has the form \( \text{supp} \hat{u}_0 \subset \{ \xi \in \mathbb{R}^n : \xi_1 \geq L \} \) and \( \|u_0\|_\infty \ll L \) for some \( L > 0 \), then (1) has a unique global solution \( u \in C(\mathbb{R}^+, L^\infty) \). In 3D, we show the compactness of the set consisting of minimal-\( L^p \) singularity-generating initial data with \( 3 < p < \infty \), furthermore, if the mild solution with data in \( L^p(\mathbb{R}^3) \) blows up in a Type-I manner, we prove the existence of a blowup solution which is uniformly bounded in critical Besov spaces \( \dot{B}^{-1+6/p}_{p/2,\infty}(\mathbb{R}^3) \).

1. Introduction. We study the Cauchy problem of the incompressible Navier-Stokes equations (NS) with initial data in \( L^\infty(\mathbb{R}^d) \), \( d \geq 2 \):

\begin{equation}
\begin{aligned}
& \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla p = 0, \\
& \text{div} u = 0, \\
& u(0, x) = u_0(x),
\end{aligned}
\end{equation}

where \( u = (u_1, ..., u_d) \) denotes the flow velocity vector and \( p(t, x) \) describes the scalar pressure. \( \nabla = (\partial_1, ..., \partial_d) \), \( \Delta = \partial_1^2 + ... + \partial_d^2 \), \( u_0(x) = (u_0^1, ..., u_0^d) \) is a given velocity with \( \text{div} u_0 = 0 \). It is easy to see that (2) can be rewritten as the following equivalent form:

\begin{equation}
\begin{aligned}
& \frac{\partial u}{\partial t} - \Delta u + P \text{div}(u \otimes u) = 0, \\
& u(0, x) = u_0(x),
\end{aligned}
\end{equation}

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where $\mathbb{P} = I - \nabla \Delta^{-1} \text{div}$ is the projection operator onto the divergence free vector fields. The solution $u$ of (NS) formally satisfies
\[ \frac{1}{2} \|u(t)\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \|\nabla u(\tau)\|^2_{L^2(\mathbb{R}^d)} d\tau = \frac{1}{2} \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{4} \]

It is known that (2) is essentially equivalent to the following integral equation:
\[ u(t) = e^{t \Delta} u_0 - \int_0^t e^{(t-\tau) \Delta} \mathbb{P} \text{div}(u \otimes u)(\tau) d\tau, \tag{5} \]

and the solution of (5) is said to be a mild solution. Note that (2) is scaling invariant in the following sense: if $u$ solves (2), so does $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$ and $p_\lambda(t, x) = \lambda^d p(\lambda^2 t, \lambda x)$ with initial data $\lambda u_0(\lambda x)$. A function space $X$ defined in $\mathbb{R}^d$ is said to be a critical space for (2) if the norms of $u_\lambda(0, \cdot) \in X$ are equivalent for all $\lambda > 0$ (i.e., $\|u_\lambda(0, \cdot)\|_X \sim \|u_0\|_X$). It is easy to see that $L^d$ and $H^{d/2-1}$ are critical spaces for (NS).

For the sake of convenience, we will denote by $NS(u_0)$ the solution of (2) (or simply denote it by $u$ if there is no confusion), and by $T(u_0)$ the supremum of all $T > 0$ so that the solution $NS(u_0)$ exists in the time interval $[0, T]$.

Many years ago, Leray [43] showed that (NS) in 3D has at least one weak solution and he mentioned certain necessary blowup conditions for the weak solutions:
\[ \|NS(u_0)(t)\|_{L^p} \geq (T(u_0) - t)^{-\left(1-d/p\right)/2}, \quad d < p \leq \infty. \tag{6} \]

The existence of the mild solution in $L^p$ was established by Kato in [35] and the blowup rate (6) in all spatial dimensions was recovered by Giga [28] for the mild solution in $C([0, T(u_0)]; L^p)$ with $d < p < \infty$. In the 3D case, the blowup of Leray solution in the critical space $L^3(\mathbb{R}^3)$ was first considered by Escauriaza, Seregin and Sverak [20] and they proved $\limsup_{t \to T(u_0)} \|NS(u_0)(t)\|_{L^3} = \infty$ if $T(u_0) < \infty$, similar results for mild solution in critical spaces $\dot{H}^{1/2}$ and $L^3$ were obtained in [36, 24] via the profile decomposition arguments developed by Kenig and Merle [37] together with the backward uniqueness in [20]. Seregin [54] further proved that $\limsup_{t \to T(u_0)} \|NS(u_0)(t)\|_{L^3} = \infty$ if $T(u_0) < \infty$. Blowup results in $L^d$ for higher spatial dimensions were obtained in Dong and Du [19] by following the approach in [20]. Recently, some generalizations for the blowup rates in $H^s$ ($3/2 \leq s \leq 5/2$) were obtained in [52, 18],
\[ \|NS(u_0)(t)\|_{H^s} \geq c(T(u_0) - t)^{-\left(s-1/2\right)/2}, \quad d = 3. \tag{7} \]

Some other kind of blowing up criteria can be found in Kozono, Ogawa and Taniuchi [41] and references therein.

On the other hand, there are some works which have been devoted to generalizing the initial data in some larger spaces; cf. [4, 15, 16, 17, 30, 31, 33, 40, 47, 63] and references therein. For the initial data in critical Besov type spaces, Cannone [15], Planchon [47] and Chemin [16] obtained global solutions in 3D for small data in critical Besov spaces $B^{3/p-1}_{p,q}$ for all $p < \infty$, $q \leq \infty$. Bourgain and Pavlovic [10] showed the ill-posedness (i.e., the solution map $u_0 \rightarrow u$ is discontinuous) of (NS) in $B^{-1}_{\infty,\infty}$, Germain [26] proved that the solution map of (NS) is not $C^2$ in $B^{-1}_{\infty,q}$, for any $q > 2$, Yoneda [64] showed that the solution map is discontinuous in $\dot{B}^{-1}_{\infty,q}$ for any $q > 2$, and Wang [62] finally proved that (NS) is ill-posed in critical Besov spaces $B^{-1}_{\infty,q}$, $1 \leq q \leq 2$. So, (NS) is ill-posed in all critical Besov spaces $B^{-1}_{\infty,q}$, $1 \leq q \leq \infty$. Up to now, noticing the embedding $B^{-1+4/d/p}_{p,q} \subset BMO^{-1} (p < \infty)$, the
known largest critical space for which (NS) is globally well posed for small initial data is $BMO^{-1}$, see Koch and Tataru [40].

Before stating our main result, we first give some notations. $C \geq 1$, $c \leq 1$ will denote constants which can be different at different places, we will use $A \lesssim B$ to denote $A \leq CB$, $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. $Q_{\delta,T}$ denotes the space time strip $\mathbb{R}^3 \times (\delta,T)$ and $Q_T := Q_0,T$. We denote by $L^p = L^p(\mathbb{R}^d)$ the Lebesgue space on which the norm is written as $\| \cdot \|$. $\| \cdot \|_{\text{strip}R}$ denote constants which can be different at different places, we will use $A \lesssim B$. Data is known largest critical space for which (NS) is globally well posed for small initial data.

For $s \in \mathbb{R}$, the function $\| f \|_{H^s}$ is defined by $\| f \|_{H^s} := \| (\lambda \Delta)^{s/2} f \|_2$ and $H^s = L^2 \cap \dot{H}^s$ for $s \geq 0$. $\| f \|_{L^p_{x,t}}$ is defined by

$$\| f \|_{L^p_{x,t}} := \sup_{x_0 \in \mathbb{R}^d} \| f \|_{L^p(B(x_0,1))}.$$ 

For the space-time norm, $\| f \|_{L^p_t L^q_x} := \| f \|_{L^p_t L^q_x((0,T); \mathbb{R}^d)}$. Finally, let us write for any $\rho > 0$,

$$\mathbb{R}^d_{+,\rho} := \{ \xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d : \xi_1 \geq \rho \}.$$ 

The standard iteration sequence for (NS) is defined in the following way:

$$u^{(n+1)}(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \text{div}(u^{(n)} \otimes u^{(n)}) \, d\tau, \quad u^{(0)} = 0.$$ 

We will mainly consider the concentration behavior of the blowup solutions and obtain a global well-posedness result in $L^\infty$. The well-posedness of (NS) in $L^p$ with $d < p < \infty$ was established in Giga [28]:

**Theorem 1.1.** Let $d < p < \infty$, $u_0 \in L^p(\mathbb{R}^d)$ be a divergence free vector field. Then there exists a time $T(u_0)$ and a unique solution $\text{NS}(u_0)$ such that the solution belongs to $X_T(\text{NS}(u_0)) := C([0, T(u_0)), L^p(\mathbb{R}^d)) \cap L_t^{p(1+2/d)}((0, T(u_0)), L_x^{2(1+2/d)})$. Moreover, if $T(u_0) < \infty$, then a constant $c_0 > 0$ exists such that for any $0 \leq t < T(u_0)$,

$$(T(u_0) - t) \| u(t) \|_{L^p}^p \geq c_0, \quad \sigma_p := \frac{2}{1 - d/p}.$$ 

In Theorem 1.1 the left case is $p = \infty$. The Cauchy problem of the Navier-Stokes equations in $L^\infty$ and in $BUC$ spaces is studied by Cannone, Meyer [13, 14], Giga et al. [29] and they proved a unique existence of a local-in-time solution in $L^\infty$ and in $BUC$ spaces. In [29], the authors also obtained the smoothness of the solutions. We will obtain a concentration phenomena of the blowup solutions with initial data only in $L^\infty$. The first main result of this paper is

**Theorem 1.2.** Let $u_0 \in L^\infty(\mathbb{R}^d)$, div$u_0 = 0$. Then there exists a $T(u_0) > 0$ such that (5) has a unique solution $\text{NS}(u_0) \in L^\infty([0, T(u_0)) \times \mathbb{R}^d)$. If $T(u_0) < \infty$, then

$$\omega(t) := \| \text{NS}(u_0)(t) \|_{L^\infty} \gtrsim (T(u_0) - t)^{-1/2}, \quad 0 \leq t < T(u_0).$$ 

For any $1 \leq p \leq \infty$, there exist $x_n \in \mathbb{R}^d$ and $t_n \not\rightarrow T(u_0) < \infty$ such that

$$\| \text{NS}(u_0)(t_n) \|_{L^p([-x_n, -\omega(t_n)^{-1}])} \gtrsim \omega(t_n)^{1-d/p}.$$ 

Moreover, if supp $\widehat{u_0} \subset \mathbb{R}^d_{+,\rho}$ for some $\rho > 0$ and there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$4C \| u^{(n_0)} \|_{L^\infty_{x,t}([0,\infty))} + \| u^{(n_0+1)} \|_{L^\infty_{x,t}([0,\infty))} \leq (n_0 + 1) \rho$$ 

for some $C \gg 1$, then (5) has a unique global solution $\text{NS}(u_0) \in C(\mathbb{R}_+, L^\infty)$.
Let \( \{x_{j,n}\} \subset \mathbb{R}^d \) and \( \{\lambda_{j,n}\} \subset (0, \infty) \) be two sequences. \((\lambda_{j,n}, x_{j,n})_{n=1}^{\infty} (j \in \mathbb{N})\) are said to be orthogonal sequences of scales and cores, if for any \( j_1 \neq j_2, j_1, j_2 \in \mathbb{N}, \) one has that
\[
\lim_{n \to \infty} \left( \frac{\lambda_{j_1,n}}{\lambda_{j_2,n}} + \frac{|x_{j_1,n} - x_{j_2,n}|}{\lambda_{j_1,n}} \right) = \infty. \tag{14}
\]
It is known that \( \lim \sup_{t \to T(u_0)} \|NS(u_0)(t)\|_{H^{d/2-1}} = \infty \) if \( T(u_0) < \infty \) (cf. [20, 36, 24, 54, 19]). In the next result we describe the blowup profiles for the blowing up solutions.

**Theorem 1.3.** Let \( u_0 \in H^{d/2-1}, \) \( \text{div} u_0 = 0 \) and \( NS(u_0) \in C([0, T(u_0)); H^{d/2-1}) \) be the solution of (5) which blows up at \( T(u_0) < \infty. \) Let \( t_n \not\to T(u_0) \) satisfy
\[
\lim_{n \to \infty} \|NS(u_0)(t_n)\|_{H^{d/2-1}} = \infty.
\]
Then there exist \( \alpha_n, \{\phi_j\}, \{\lambda_j(t_n)\} \) and \( \{x_j(t_n)\} \) with \( \lim_{n \to \infty} \alpha_n = +\infty, \) \( \phi_j \in H^{d/2-1}, \) \( \lim_{n \to \infty} \lambda_j(t_n) = 0, \) \( x_j(t_n) \in \mathbb{R}^d, \) \( (\lambda_j(t_n), x_j(t_n)) \) are orthogonal, such that \( NS(u_0)(t_n) \) can be decomposed into the following profiles:
\[
NS(u_0)(t_n) = \alpha_n \left( \sum_{j=1}^{J} \frac{1}{\lambda_j(t_n)} \phi_j \left( \frac{x - x_j(t_n)}{\lambda_j(t_n)} \right) + r_n^J \right),
\tag{15}
\]
where \( r_n^J \) is a reminder of \( NS(u_0)(t_n)/\alpha_n \) that satisfies \( \lim_{J \to \infty} \lim_{\alpha_n \to \infty} \|r_n^J\|_d = 0, \) moreover, we have \( \alpha_n = ||NS(u_0)(t_n)||_{H^{d/2-1}}, \lambda_j(t_n) \leq \alpha_n^{-2/(d-2)} \) and in particular, for \( p \geq 2, \)
\[
\sup_{\xi \in \mathbb{R}^d} ||NS(u_0)(t_n)||_{L^p(\{|\xi| \leq \lambda_j(t_n)\})} \gtrsim \lambda_j(t_n)^{d/p-1} ||NS(u_0)(t_n)||_{H^{d/2-1}}. \tag{16}
\]

**Remark 1.** Theorems 1.2 and 1.3 need several remarks.

(i) Noticing that \( \omega(t) \geq c (T(u_0) - t)^{-1/2}, \) we have for any \( p \geq d, \)
\[
||NS(u_0)(t_n)||_{L^p(\{|x - x_0| \leq (T(u_0) - t_n)^{1/2}\})} \gtrsim (T(u_0) - t_n)^{-(1-d/p)/2}.
\tag{17}
\]
Taking \( p = d \) in (17), we find that
\[
||NS(u_0)(t_n)||_{L^d(\{|x - x_0| \leq (T(u_0) - t_n)^{1/2}\})} \gtrsim 1,
\tag{18}
\]
which implies that the solution has a concentration in a very small ball with radius less than or equals to \( C\sqrt{T(u_0) - t_n} \) in \( L^d. \)

(ii) Under some technical assumptions, the concentration result (12) has some connections with the \( \epsilon \)-regularity criterion of the Navier-Stokes equation in three dimensions. Indeed, let \( u \) be the solution associated with initial data \( u_0 \in L^\infty(\mathbb{R}^3) \) which blows up at a finite time \( T, \) assume further that \( u \) has a singular point at \( x_0 := (x_0, T), \) and \( u, p \) form a pair of suitable weak solution in domain \( B(x_0, a) \times (t_0, T). \) In view of \( \epsilon \)-regularity criterion (see Lemma A.2), we can find some \( t_k > t_0 \) sufficiently close to \( T, \) such that for some absolute constant \( \epsilon_0, \)
\[
\frac{1}{r^2} \int_{T-r^2}^{T} \int_{B(0,r)} |u|^3 + |p|^2 dxdt > \epsilon_0
\]
holds for any \( 0 < r < r_* = \sqrt{T - t_k} < a. \) Repeating the decomposition of pressure procedure as in [54], one can find for \( r \ll r_* \),
\[
\frac{1}{r^2} \int_{T-r^2}^{T} \int_{B(0,\sqrt{T-t_k})} |u|^3 dxdt > \frac{\epsilon_0}{2}.
\]
Thus
\[ \sup_{T-r^2 < \tau < T} \int_{B(x_0, \sqrt{T-t_k})} |u(\tau)|^3 \, dx > \frac{\epsilon_0}{2} . \]
Now one can assert that there exists some \( \tau_k \) with \( t_k \leq \tau_k \leq T \), such that
\[ \int_{B(x_0, \sqrt{T-t_k})} |u(\tau_k)|^3 \, dx > \frac{\epsilon_0}{2} . \]
By Hölder, for \( p \geq 3 \),
\[ \| u(\tau_k) \|_{L^p(|x-x_0| \leq \sqrt{T-t_k})} \gtrsim (T-t_k)^{-\frac{3}{2} (1+\frac{1}{p})}. \]
We point that for the blowup solution with initial data in \( L^\infty(\mathbb{R}^3) \), it is unclear to us whether there is singularity at infinity, and in case of the finite singular point, it is also unknown whether the solution form a suitable weak solution around the singular point, in contrast with \( L^p(\mathbb{R}^3) \) case, \( 3 < p < \infty \), see Remark 6 and Proposition A.7. The concentration phenomena for the dispersive equations were also studied in recent years. For example, let us consider the energy critical NLS in 3D:
\[ iu_t + \Delta u - |u|^4 u = 0, \quad u(0, x) = u_0(x). \]
Bourgain [9] considered the blowup concentration in the critical space and he obtained that for the solution \( u \) of NLS, if \( u \) blows up at \( T < \infty \) and \( \lim_{t_n \to T} \| \nabla u(t_n) \|_2 = \infty \), then
\[ \sup_{x_0 \in \mathbb{R}^3} \| \nabla u(t_n) \|_{L^2(|x-x_0| \leq \sqrt{T-t_n})} \gtrsim 1. \]  \( (19) \)
Comparing (18) with (19), we see that the concentration phenomena in the critical spaces are quite similar for Navier-Stokes and NLS.

(iii) Taking \( p = d \) in (16) and noticing that \( \lim_{n \to \infty} \| NS(u_0)(t_n) \|_{H^{4/2-1}} = \infty \), we have
\[ \sup_{\xi \in \mathbb{R}^d} \| NS(u_0)(t_n) \|_{L^d(|\cdot-\xi| \leq \lambda_j(t_n))} \gtrsim \| u(t_n) \|_{H^{4/2-1}}, \]  \( (20) \)
which means that a very large potential norm is concentrated in a very small ball with radius less than or equals to \( C \| NS(u_0)(t_n) \|^{-2/(d-2)} \). However, it is not very clear for us how to unify the concentration phenomena of (18) and (20).

(iv) In the blowup profile decomposition (15), noticing that \( \lambda_j(t_n) \to 0 \) as \( t_n \nearrow T(u_0) \), we see that concentration blowup is the only way in all of the possible blowing up manners. However, it seems difficult to show that \( \alpha_n r_n^J \) is a reminder of \( NS(u_0)(t_n) \).

(v) Taking \( n_0 = 0 \) in (13), we see that condition (13) can be substituted by the following condition:
\[ 4C \| u_0 \|_\infty \leq \rho. \]  \( (21) \)
Noticing that \( \hat{u}_0 \) is supported in \( \mathbb{R}_{+\rho} \), we see that condition (21) contains a class of large data in \( L^\infty \) if \( \rho \gg 1 \) which are out of the control of \( B_{p,q}^{-1+d/p} \) with \( p < \infty \).

Following [20, 36, 24], we see that, for the initial data \( u_0 \) in the critical spaces \( X = H^{1/2}(\mathbb{R}^3), \quad L^3(\mathbb{R}^3) \), the solution \( NS(u_0) \) enjoys the property
\[ T(u_0) < \infty \quad \implies \quad \limsup_{t \to T(u_0)} \| NS(u_0) \|_X = \infty. \]
On the other hand, it is also interesting to characterize the set of minimal initial data that gives rise to a blow up solution. In case of $X = \dot{H}^{1/2}(\mathbb{R}^3)$, let us denote

$$\rho_{1/2} = \inf \{ ||u_0||_{\dot{H}^{1/2}} \mid u_0 \in \dot{H}^{1/2}, \, T(u_0) < \infty \}.$$ 

and

$$\mathcal{M}_{1/2} = \{ u_0 \in \dot{H}^{1/2} : ||u_0||_{\dot{H}^{1/2}} = \rho_{1/2}, \, T(u_0) < \infty \}.$$ 

Rusin and Sverak [53] obtained that $\mathcal{M}_{1/2}$ is nonempty and compact up to dilation and translation provided $\rho_{1/2} < \infty$, see also [36] for a profile decomposition method proof. While for $X = L^3(\mathbb{R}^3)$, a lack of compactness is the main obstacle, Hao and Sverak [34] used a splitting argument and energy method to overcome this defect and get a similar conclusion. In [24], the existence of minimal initial data in critical Besov space $\dot{B}^{1/2}_{p,q}(\mathbb{R}^d)$ with $1 < p < \infty$, $\infty < q < \infty$ obtained by using the profile decomposition, which seems to be more natural, as in [53, 34], Caffarelli-Kohn-Nirenberg small regularity theorem is used and in general, it doesn’t hold in higher dimension. However, in [24], the compactness result is not recovered in the original space, viz $L^d$ and $\dot{B}^{1/2}_{p,q}$ respectively.

One can further ask what happens if $X$ is a sub-critical spaces, say $X = \dot{H}^s(\mathbb{R}^3)$, with $1/2 < s < 3/2$, or $X = L^p(\mathbb{R}^3)$, $p > 3$. E. Poulon [50] considered such a kind of question and she introduced

$$L_c^\ast := \inf \left\{ \limsup_{t \to T(u_0)} (T(u_0) - t) ||NS(u_0)(t)||_{\dot{H}^s}^{\sigma_s} \mid u_0 \in \dot{H}^s \text{ and } T(u_0) < \infty \right\}.$$ 

where $\sigma_s = 2/(s - 1/2)$. If $L_c < \infty$, the author proved that $L_c^\ast$ can be attainable for some $u_0 \in \dot{H}^s$ and the corresponding solution $NS(u_0)$ is uniformly bounded in critical Besov space $\dot{B}^{1/2}_{2,\infty}(\mathbb{R}^3)$. In addition, let

$$E_c^\ast := \inf \left\{ T(u_0) ||u_0||_{\dot{H}^s}^{\sigma_s} \mid u_0 \in \dot{H}^s \text{ and } T(u_0) < \infty \right\},$$

If $E_c < \infty$, in [51], she also showed $E_c^\ast$ can be reached at some $u_0 \in \dot{H}^s$ and the corresponding $\dot{H}^s$-minimal singularity-generating data set (for any $u_0$ in the set, fix $T(u_0) = T_\ast$) is compact modulo translations. Motivated by these results, we can also define similar critical minimal quantities adapted to the $L^p(\mathbb{R}^3)$ scale with $3 < p < \infty$ ($\sigma_p = 2/(1 - 3/p)$). Indeed, let us denote

$$M_c^\ast := \inf \left\{ \limsup_{t \to T(u_0)} (T(u_0) - t) ||NS(u_0)(t)||_{L^p}^{\sigma_p} : u_0 \in L^p \text{ and } T(u_0) < \infty \right\}.$$ 

For the $L^p$-minimal singularity-generating data, denoting

$$A_c^\ast := \inf \{ T(u_0) ||u_0||_{L^p}^{\sigma_p} \mid u_0 \in L^p(\mathbb{R}^3) \text{ and } T(u_0) < \infty \},$$

and

$$\mathcal{M}_p(T_\ast) := \{ u_0 \in L^p(\mathbb{R}^3) \mid T(u_0) = T_\ast, ||u_0||_p = r_p(T_\ast) \},$$

where $r_p(T_\ast) := A_c/T_\ast^{1/\sigma_p}$. Apparently, for $u_0 \in \mathcal{M}_p(T_\ast)$, it holds $T(u_0)||u_0||_{L^p}^{\sigma_p} = A_c^\ast$. If $M_c < \infty$, Giga’s Theorem 1.1 implies that there exists $u_0 \in L^p$ such that the solution to (NS) blows up at finite time $T(u_0)$ with rate

$$||NS(u_0)(t)||_p \sim (T(u_0) - t)^{-1+3(p)/2}, \quad 0 < t < T(u_0).$$

Such a kind of solution is said to be type-I blow up solution (cf. [39]). We have the following result:
Theorem 1.4. Let \( d = 3, 3 < p < \infty, M_{c}^{sp} < \infty \). Then there exists a \( \Phi_{0} \in L^{p} \cap B_{p,2}^{-1+6/p} \) such that \( \Phi := NS(\Phi_{0}) \) blows up at time 1, and satisfies
\[
\sup_{\tau < \tau < 0} (1 - \tau)\|\Phi(\tau)\|_{p}^{\sigma_{p}} = \limsup_{\tau \to 1} (1 - \tau)\|\Phi(\tau)\|_{p}^{\sigma_{p}} = M_{c}^{\sigma_{p}}.
\]
Moreover, \( \Phi \) lies in Besov spaces \( \dot{B}_{p,2}^{-1+6/p} \), and
\[
\sup_{\tau < \tau < 0} \|\Phi(\tau)\|_{\dot{B}_{p,2}^{-1+6/p}} < \infty.
\]

The next result concerns the compactness of the \( L^{p} \)-minimal singularity-generating initial data set, which reads as follows:

Theorem 1.5. Let \( d = 3, 3 < p < \infty \) and assume \( \mathcal{A}_{p}^{r} < \infty \). Then the set \( \mathcal{M}_{p}(T_{s}) \) is nonempty and compact modulo translations, more precisely, for any sequence \( \{u_{n}, n\} \subset \mathcal{M}_{p}(T_{s}) \), there exist a subsequence still denoted by \( n \), a sequence \( \{x_{n}\} \subset \mathbb{R}^{3} \) and function \( V \in \mathcal{M}_{p}(T_{s}) \), such that
\[
\lim_{n \to \infty} \|u_{n}(x + x_{n}) - V(x)\|_{L^{p}(\mathbb{R}^{3})} = 0.
\]

Remark 2. We point out that by replacing the standard \( L^{p} \) norm with an equivalent wavelet norm (see [38]), one can also obtain that \( \mathcal{M}_{p}(T_{s}) \neq \emptyset \) using \( L^{p} \) profile decomposition together with Theorem 4.2, but as in [24], it is not clear whether the compactness result can be recovered in \( L^{p}(\mathbb{R}^{3}) \), in contrast with the \( H^{s}(\mathbb{R}^{3}) \) scale, where the Hilbert structure plays a crucial role in the orthogonality property of the corresponding profile decomposition.

As a consequence, Theorem 1.5 combining with a stability property (Proposition B.6) can lead to the following conclusion.

Corollary 1. Let \( d = 3, 3 < p < \infty \), there exists a nondecreasing function \( F : [0, \mathcal{A}_{p}^{r}] \to \mathbb{R}^{+} \) with \( \lim_{r \to \mathcal{A}_{p}^{r}} F(r) = \infty \), such that for any initial data satisfying \( T(\Phi(0))\|u_{0}\|_{\mathcal{A}_{p}^{r}} = \mathcal{A}_{p}^{r} \), the resulting mild solution \( u := NS(\Phi_{0}) \) to (2) has an apriori uniform control, that is
\[
\|NS(\Phi_{0})\|_{\mathcal{X}_{p}^{0}} \leq \|u_{0}\|_{p}F(\|u_{0}\|_{\mathcal{A}_{p}^{r}}), \quad \forall T < T(\Phi(0)),
\]
where \( \mathcal{X}_{p}^{0} := L^{p/3}(0, T; L^{p/3}(\mathbb{R}^{3})) \cap C([0, T]; L^{p}(\mathbb{R}^{3})). \)

1.1. Besov spaces. Let \( \psi : \mathbb{R}^{d} \to [0, 1] \) be a smooth cut-off function which equals 1 on the closed ball \( B(0, 5/4) := \{\xi \in \mathbb{R}^{d} : |\xi| \leq 5/4\} \) and equals 0 outside the ball \( B(0, 3/2) \). Write
\[
\varphi(\xi) := \psi(\xi) - \psi(2\xi), \quad \varphi_{j}(\xi) = \varphi(2^{-j}\xi),
\]
\[\triangle_{j} := \mathcal{F}^{-1}\varphi_{j}\mathcal{F}, \quad j \in \mathbb{Z} \] are said to be the dyadic decomposition operators. One easily sees that \( \text{supp}\varphi_{j} \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1}) \). For convenience, we denote
\[
P_{\leq M}f := \mathcal{F}^{-1}\psi(\cdot /M)\mathcal{F}f, \quad P_{\geq M}f := f - P_{\leq M}f.
\]
The norms in homogeneous Besov spaces are defined as follows:
\[
\|f\|_{\dot{B}_{p,q}^{s}} = \left( \sum_{j=-\infty}^{+\infty} 2^{jsq}\|\triangle_{j}f\|_{p}^{q} \right) \left( \sum_{j=-\infty}^{+\infty} 2^{jsq}\|\triangle_{j}f\|_{p}^{q} \right)^{1/q} , \quad \|f\|_{\dot{B}_{p,\infty}^{s}} = \sup_{j \in \mathbb{Z}} 2^{js}\|\triangle_{j}f\|_{p}.
\]
Using the heat kernel, we have (see [5, 58])
\[
\|u\|_{\dot{B}_{p,q}^{s}} \sim \|t^{-\frac{s}{2}}e^{t\Delta}u\|_{L^{p}_{q}(\mathbb{R}^{3})}, \quad s < 0.
\]
where $e^{t\Delta}u := \mathcal{F}^{-1}e^{-t|\xi|^2}\mathcal{F}u$.

The rest of this paper is organized as follows. In Section 2, we consider the well-posedness and blowup concentration of (NS) in $L^\infty$ and prove Theorem 1.2. Using the profile decomposition techniques, in Section 3 we consider the blowup profile for the blowing up solution in $H^{d/2-1}$ and show Theorem 1.3. In Section 5 we will prove our Theorem 1.4, whose proof consists of two steps, constructing a critical solution in $L^p$ and $\dot{B}^{-1+6/p}_{p/2,\infty}$ separately. The proof of Theorem 1.4 relies upon a profile structure theorem, whose proof will be given in Sections 6 and 7. Section 8 and Section 9 are devoted to the proof of Theorem 1.5 and Corollary 1 respectively. Finally, in the Appendix, we list some facts on (NS) system and prove a perturbation result which is useful in obtaining the estimate of the remainder term in the profile structure theorem.

2. Initial data in $L^\infty(\mathbb{R}^d)$.

2.1. Local well-posedness and blowup analysis. We will frequently use the following Bernstein’s multiplier estimate (see [8, 60]):

**Lemma 2.1.** (Multiplier estimate). Let $L \in \mathbb{N}$, $L > n/2$, $\theta = n/2L$. We have
\[
\|\mathcal{F}^{-1}\rho\|_1 \lesssim \|\rho\|_2^{-\theta} \left( \sum_{i=1}^d \|\partial_i^L \rho\|_2^2 \right). 
\]

Recall that (see [16])\[\|\triangle e^{t\Delta}f\|_r \lesssim e^{-t2^{j-4}}\|f\|_r, \ 1 \leq r \leq \infty, \ j \in \mathbb{Z}. \tag{29}\]

Similarly, we have

**Lemma 2.2.** (Exponential decay). Let $1 \leq r \leq \infty$. We have\[\|\Delta_J (-\Delta)^{-1}\partial_{\lambda} \partial_{\mu} \partial_{\nu} e^{t\Delta}f\|_r \lesssim 2^j e^{-t2^{j-4}}\|f\|_r, \ j \in \mathbb{Z}, \ 1 \leq \lambda, \mu, \nu \leq d. \tag{30}\]

**Proof.** The idea follows from [16] (see also [60]). By Young’s inequality, we have\[\|\Delta_J (-\Delta)^{-1}\partial_{\lambda} \partial_{\mu} \partial_{\nu} f\|_r \lesssim \left\|\mathcal{F}^{-1}\left(\varphi_j \frac{\xi_{\lambda} \xi_{\mu} \xi_{\nu}}{\xi^2} e^{-t|\xi|^2}\right)\right\|_1 \|f\|_r \tag{31}\]

By scaling argument and Lemma 2.1, we have\[\left\|\mathcal{F}^{-1}\left(\varphi_j \frac{\xi_{\lambda} \xi_{\mu} \xi_{\nu}}{\xi^2} e^{-t|\xi|^2}\right)\right\|_1 = 2^j \left\|\mathcal{F}^{-1}\left(\varphi_j \frac{\xi_{\lambda} \xi_{\mu} \xi_{\nu}}{\xi^2} e^{-t2^j|\xi|^2}\right)\right\|_1 \lesssim 2^j e^{-t2^j-4}. \tag{32}\]

In view of (31) and (32), we immediately have (30).

For convenience, we denote \[(A_t f)(t) := \int_{t_0}^t e^{(t-\tau)\Delta} f(\tau) d\tau. \]

**Lemma 2.3.** (Decay of higher frequency). Assume that $\text{supp} \hat{f} \subset \{\xi : |\xi| \geq 2^{j_0}\}$. Then we have\[\|A_t \nabla f\|_{L^\infty([t_0, \infty) \times \mathbb{R}^d)} \lesssim 2^{-j_0}\|f\|_{L^\infty([t_0, \infty) \times \mathbb{R}^d)}. \]
Proof. By the dyadic decomposition, 
\[ f = \sum_{j \geq j_0} \Delta_j f. \]
Using Lemma 2.2, one sees that 
\[ \|\partial_t \mathcal{A} f \|_{L^\infty([0,\infty) \times \mathbb{R}^d)} \leq \sum_{j \geq j_0-1} \|\Delta_j \partial_t \mathcal{A} f \|_{L^\infty([0,\infty) \times \mathbb{R}^d)} \]
\[ \lesssim \sum_{j \geq j_0-1} 2^j \sup_{t \geq t_0} \int_{t_0}^t e^{-\tau} 2^{j\tau-4} \|f(\tau)\|_{L^\infty} d\tau \]
\[ \lesssim \sum_{j \geq j_0-1} 2^{-j} \sup_{t \geq t_0} \left(1 - e^{-\tau}\right) 2^{j\tau-4} \|f\|_{L^\infty([0,\infty) \times \mathbb{R}^d)} \]
\[ \lesssim 2^{-j_0} \|f\|_{L^\infty([0,\infty) \times \mathbb{R}^d)}. \]
The result follows. \[ \square \]

**Lemma 2.4.** (Short time estimates of lower frequency). Let \( j_0 \in \mathbb{Z} \). We have for any \( f \in L^\infty([t_0,t_1] \times \mathbb{R}^d) \),
\[ \|P_{\leq 2^{j_0}} \partial_t \mathcal{A} f \|_{L^\infty([0,\infty) \times \mathbb{R}^d)} \lesssim 2^{j_0}(t_1 - t_0) \|f\|_{L^\infty([0,\infty) \times \mathbb{R}^d)}. \]

**Proof.** Since \( e^{-\tau} |\xi|^2 \xi_j \xi_k \xi_l |\xi|^{-2} \) is in \( L^1(\mathbb{R}^d) \) for any \( t > 0 \), whose Fourier transform has an integral form, it follows that
\[ \mathcal{F}^{-1}(e^{-\tau} |\xi|^2 \xi_j \xi_k \xi_l |\xi|^{-2}) = \sum_{j \in \mathbb{Z}} \Delta_j \mathcal{F}^{-1}(e^{-\tau} |\xi|^2 \xi_j \xi_k \xi_l |\xi|^{-2}). \]
By Lemma 2.2 and (34), we have
\[ \|P_{\leq 2^{j_0}} \partial_t \mathcal{A} f \|_{L^\infty([0,\infty) \times \mathbb{R}^d)} \leq \sum_{j \leq j_0+1} \|\Delta_j \partial_t \mathcal{A} f \|_{L^\infty([0,\infty) \times \mathbb{R}^d)} \]
\[ \lesssim \sum_{j \leq j_0+1} 2^j \int_{t_0}^{t_1} \|f(\tau)\|_{L^\infty} d\tau \]
\[ \lesssim 2^{j_0}(t_1 - t_0) \|f\|_{L^\infty([0,\infty) \times \mathbb{R}^d)}, \]
which is the result, as desired. \[ \square \]

**Lemma 2.5.** (Local well posedness). Let \( u_0 \in L^\infty(\mathbb{R}^d) \) with \( \text{div} u_0 = 0 \). There exists a \( T > 0 \) such that (5) has a unique solution \( N \mathcal{S}(u_0) \in L^\infty([0,T) \times \mathbb{R}^d) \). If \( T < \infty \), then we have \( \lim_{t \rightarrow T} \|N \mathcal{S}(u_0)(t)\|_{L^\infty} = \infty \) and (11) holds true.

**Proof.** The proof of the local existence can be found in [29]. However, the blowup rate (11) is very important for our later purpose and we sketch the proof here. Put
\[ \mathcal{G} = \{ u \in L^\infty([0,t_0] \times \mathbb{R}^d) : \|u\|_{L^\infty([0,t_0] \times \mathbb{R}^d)} \leq 2C \|u_0\|_{L^\infty}, \} \]
with natural metric \( d(u,v) = \|u - v\|_{L^\infty([0,t_0] \times \mathbb{R}^d)} \). Let us consider the mapping
\[ \mathcal{G} : u(t) \rightarrow e^{t\Delta} u_0 - \mathcal{A}_0 \text{div}(u \otimes u). \]
\[ \mathcal{G} \mathcal{A} : L^\infty \rightarrow L^\infty, \]
and Lemmas (2.3) and (2.4), we have
\[ \|\mathcal{G} u\|_{L^\infty([0,t_0] \times \mathbb{R}^d)} \leq C\|u_0\|_{L^\infty} + C(2^{-j_0} + t_02^{-j_0})\|u\|_{L^\infty([0,t_0] \times \mathbb{R}^d)}^2. \]
Taking \( j_0 \) such that \( 2^{-2j_0} \sim t_0 \), one has that
\[ \|\mathcal{G} u\|_{L^\infty([0,t_0] \times \mathbb{R}^d)} \leq C\|u_0\|_{L^\infty} + C\sqrt{t_0}\|u\|_{L^\infty([0,t_0] \times \mathbb{R}^d)}^2. \]
Similarly,

\[ \| \mathcal{F} u - \mathcal{F} v \|_{L^\infty([0,t_0] \times \mathbb{R}^d)} \leq C \sqrt{t_0} \left( \sum_{u \neq v} \| u \|_{L^\infty([0,t_0] \times \mathbb{R}^d)} \right) \| u - v \|_{L^\infty([0,t_0] \times \mathbb{R}^d)}. \]

Further, one can choose \( t_0 \) verifying

\[ \sqrt{t_0} = 1/(8C^2\|u_0\|_\infty). \]

We easily see that \( \mathcal{F} \) is a contraction mapping from \( \mathcal{D} \) into itself. So, \( \mathcal{F} \) has a unique fixed point in \( \mathcal{D} \), which is a solution of (5). It is easy to see that \( u \in C((0,T];L^\infty) \) (see [29], for instance). The solution can be extended exactly in the same way as above. Indeed, considering the mapping:

\[ \mathcal{R}_t: u(t) \to e^{(t-t_0)\Delta} u(t_0) - \mathcal{A}_0 \mathcal{P} \text{ div}(u \otimes u), \]  

one has that

\[ \| \mathcal{R}_t u \|_{L^\infty([t_0,t_1] \times \mathbb{R}^d)} \leq C\|u(t_0)\|_\infty + C\sqrt{t_1-t_0} \| u \|_{L^\infty([t_0,t_1] \times \mathbb{R}^d)}. \]  

Similarly,

\[ \| \mathcal{R}_t u - \mathcal{R}_t v \|_{L^\infty([t_0,t_1] \times \mathbb{R}^d)} \leq C\sqrt{t_1-t_0} \left( \sum_{u \neq v} \| u \|_{L^\infty([t_0,t_1] \times \mathbb{R}^d)} \right) \| u - v \|_{L^\infty([t_0,t_1] \times \mathbb{R}^d)}. \]

So, we can extend the solution from \([0,t_0]\) to \([t_0,t_1]\) if

\[ \sqrt{t_1-t_0} = 1/(8C^2\|u(t_0)\|_\infty). \]

Repeating the procedure as above, we can extend the solution step by step to \([t_1,t_2],[t_2,t_3],...\) if

\[ \sqrt{t_i-t_{i-1}} = 1/(8C^2\|u(t_{i-1})\|_\infty), \quad i = 0,1,2,... \]

Now let us assume that \( t_i \nearrow T \). If \( T < \infty \), we easily see that

\[ \limsup_{t \to T} \| NS(u_0) (t) \|_\infty = \infty. \]

Moreover, we can show (11) holds true. Assume for a contrary that there exist two sequences \( s_n \nearrow T \) and \( c_n \searrow 0 \) satisfying

\[ \| NS(u_0)(s_n) \|_\infty \leq c_n(T-s_n)^{-1/2}, \quad n = 1,2,... \]  

Observing the integral equation

\[ u(t) = e^{(t-s_n)\Delta} NS(u_0)(s_n) - \mathcal{A}_{s_n} \mathcal{P} \text{ div}(u \otimes u), \]

similarly as in (36), we have for any \( s_n \leq s < T \),

\[ \| u \|_{L^\infty([s_n,s] \times \mathbb{R}^d)} \leq C\| NS(u_0)(s_n) \|_\infty + C\sqrt{s-s_n} \| u \|_{L^\infty([s_n,s] \times \mathbb{R}^d)} \]

\[ \leq Cc_n(s-s_n)^{-1/2} + C\sqrt{s-s_n} \| u \|_{L^\infty([s_n,s] \times \mathbb{R}^d)}. \]  

(38) implies that

\[ \sqrt{s-s_n} \| u \|_{L^\infty([s_n,s] \times \mathbb{R}^d)} \leq Cc_n + C(s-s_n) \| u \|_{L^\infty([s_n,s] \times \mathbb{R}^d)}^2. \]  

(39) follows from (39) that

\[ \sqrt{s-s_n} \| u \|_{L^\infty([s_n,s] \times \mathbb{R}^d)} \geq 1/2C, \quad \text{or} \quad \sqrt{s-s_n} \| u \|_{L^\infty([s_n,s] \times \mathbb{R}^d)} \leq 2Cc_n. \]  

Taking \( c_n \ll 1 \) and \( s \to T \), we see that (40) contradicts with (37) or with the fact that \( \limsup_{t \to T} \| NS(u_0)(t) \|_\infty = \infty. \)
Lemma 2.6. (Concentration) Let $T < \infty$ and $NS(u_0) \in L^\infty([0, T) \times \mathbb{R}^d)$ be the solution of (5) obtained in Lemma 2.5. Then for any $1 \leq p < \infty$, there exist two sequences $\{x_n\}$ and $\{\tau_n\}$ of $x_n \in \mathbb{R}^d$ and $\tau_n \nearrow T$ satisfying
\[
\|NS(u_0)(\tau_n)\|_{L^p([x_n - \rho(\tau_n)^{d-1}) \leq \omega(\tau_n)^{-1/d/p}, \omega(t) = \|NS(u_0)(t)\|_\infty. \tag{41}
\]

Proof. Put $\tau_0 = 0$. Since $\lim_{t \to T} \omega(t) = \infty$, we can find a $t_1 > \tau_0$ such that
\[
\omega(t_1) \geq 200C^2\omega(\tau_0).
\]
We can further find a $\tau_1 \in (\tau_0, t_1]$ verifying
\[
\omega(\tau_1) \geq \frac{1}{2} \sup_{t \in [\tau_0, t_1]} \omega(t).
\]
Then we have
\[
\omega(\tau_1) \geq \max \left\{ 100C^2\omega(\tau_0), \frac{1}{2} \sup_{t \in [\tau_0, \tau_1]} \omega(t) \right\}.
\]
Repeating this procedure, one can choose a monotone sequence $\{\tau_n\}$ verifying
\[
\omega(\tau_n) \geq \max \left\{ 100C^2\omega(\tau_{n-1}), \frac{1}{2} \sup_{t \in [\tau_{n-1}, \tau_n]} \omega(t) \right\}.
\]
Since $\omega(\tau_n) \to \infty$, we have $\tau_n \nearrow T$.

Claim. For simplicity, we write $u := NS(u_0)$. We have
\[
\|P_{\leq 100C\omega(\tau_n)}u(\tau_n)\|_\infty \geq \frac{1}{2} \omega(\tau_n). \tag{42}
\]
In fact, if (42) does not hold, then one has that
\[
\|P_{> 100C\omega(\tau_n)}u(\tau_n)\|_\infty \geq \frac{1}{2} \omega(\tau_n). \tag{43}
\]
Let us consider the integral equation
\[
u(t) = e^{(t-\tau_{n-1})\Delta}u(\tau_{n-1}) - \mathcal{A}_{\tau_{n-1}}\mathbb{P} \operatorname{div}(\nabla u) u).
\]
It follows from Lemma 2.3 and (43) that
\[
\omega(\tau_n) \leq 2\|P_{> 100C\omega(\tau_n)}u(\tau_n)\|_\infty
\]
\[
\leq 2C\|P_{> 100C\omega(\tau_n)}u(\tau_{n-1})\|_\infty
\]
\[
+ 2\|P_{> 100C\omega(\tau_n)}\mathcal{A}_{\tau_{n-1}}\mathbb{P} \operatorname{div}(\nabla u) u\|_{L^\infty([\tau_{n-1}, \tau_n] \times \mathbb{R}^d)}
\]
\[
\leq 2C^2\|u(\tau_{n-1})\|_\infty + \frac{1}{50}\omega(\tau_n)\|u\|_{L^\infty([\tau_{n-1}, \tau_n] \times \mathbb{R}^d)}^2
\]
\[
\leq \frac{1}{50}\omega(\tau_n) + \frac{4}{50}\omega(\tau_n) < \frac{1}{2}\omega(\tau_n).
\]
A contradiction! So, we have (42). For convenience, we write
\[
\beta(t) := 100C\omega(t) = 100C\|u(t)\|_\infty. \tag{44}
\]
Now we prove (41). Let $r_n = M\beta(\tau_n)^{-1}$ with $M$ specified later, there exists $x_n \in \mathbb{R}^d$ such that
\[
\|P_{\leq \beta(\tau_n)}u(\tau_n]\|_\infty \leq 2\beta(\tau_n)^d \left| \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\psi)(\beta(\tau_n)(x_n - y)) u(\tau_n, y) dy \right|
\]
\[
\leq 2\beta(\tau_n)^d \left| \int_{B(x_n, r_n)} (\mathcal{F}^{-1}\psi)(\beta(\tau_n)(x_n - y)) u(\tau_n, y) dy \right|
\]
Then we have

$$+ 2β(τ_n)^d \left| \int_{B(x_n, r_n)^c} (\mathcal{F}^{-1} ψ)(β(τ_n)(x_n - y)) u(τ_n, y) dy \right|$$

$$:= I + II.$$ 

By Hölder’s inequality,

$$II \leq 2\| \mathcal{F}^{-1} ψ \|_{L^1(|·| \geq M)} \| u(τ_n) \|_{∞} \leq 4\| \mathcal{F}^{-1} ψ \|_{L^1(|·| \geq M)} \| P_{≤ β(τ_n)} u(τ_n) \|_{∞}.$$ 

Taking a suitable $M \gg 1$, one easily sees that

$$II \leq \frac{1}{4} \| P_{≤ β(τ_n)} u(τ_n) \|_{∞}. \tag{45}$$

So, it follows from (45) and Hölder’s inequality that

$$\| P_{≤ β(τ_n)} u(τ_n) \|_{∞} \leq 4β(τ_n)^d \left| \int_{B(x_n, r_n)} (\mathcal{F}^{-1} ψ)(β(τ_n)(x_n - y)) u(τ_n, y) dy \right|$$

$$\lesssim β(τ_n)^d/p \| u(τ_n) \|_{L^p(|x_n - ·| \leq Mβ(τ_n)^{-1})}. \tag{46}$$

So, from (42) and (46) we immediately have the result, as desired.

\[\boxempty\]

2.2. Global well-posedness. For convenience, we denote

$$\mathcal{B}(u, v) = \int_0^t e^{(t-τ)Δ} \mathbb{P} \text{div}(u ⊗ v)(τ) dτ. \tag{47}$$

By Lemma 2.3, we have

Lemma 2.7. (Decay with frequency superposition). Let $\hat{u}_i \subset \mathbb{R}^d_{+, ρ_i}, i = 1, 2$. Then we have $\text{supp } \hat{u}_i \subset \mathbb{R}^d_{+, ρ_1 + ρ_2}$ and

$$\| \mathcal{B}(u_1, u_2) \|_{L^∞_{τ,τ+i Δ}} \leq C(ρ_1 + ρ_2)^{-1} \prod_{i = 1}^2 \| u_i \|_{L^∞_{τ,τ+i Δ}}.$$ 

Let us recall the iteration sequence

$$u^{(0)} = 0, \quad u^{(1)}(t) = e^{tΔ} u_0,$$

$$u^{(n+1)}(t) = e^{tΔ} u_0 - \mathcal{B}(u^{(n)}, u^{(n)}), \quad n = 1, 2, ... . \tag{48}$$

Lemma 2.8. (Frequency superposition of higher iteration terms). Let $\hat{u}_0 \subset \mathbb{R}^d_{+, ρ}$. Then we have

$$\text{supp } \hat{u}_0 \subset \mathbb{R}^d_{+, ρ}, \quad \text{supp } (u^{(n+1)} - u^{(n)}) \subset \mathbb{R}^d_{+,(n+1)ρ}. \tag{49}$$

Proof. By (48), $\text{supp } \hat{u}_0 \subset \mathbb{R}^d_{+, ρ}$ and induction we see that $\text{supp } \hat{u}^{(n)} \subset \mathbb{R}^d_{+, ρ}$. Let us observe that

$$(u^{(n+1)} - u^{(n)}) = \mathcal{B}(u^{(n-1)} - u^{(n)}, u^{(n)}) + \mathcal{B}(u^{(n-1)}, u^{(n-1)} - u^{(n)}). \tag{50}$$

and

$$(u^{(2)} - u^{(1)}) = -\mathcal{B}(e^{tΔ} u_0, e^{tΔ} u_0). \tag{51}$$

By (50) and supp $\hat{u}_0 \subset \mathbb{R}^d_{+, ρ}$, we see that $\text{supp } (u^{(2)} - u^{(1)}) \subset \mathbb{R}^d_{+, 2ρ}$. By supp $\hat{u}^{(n)} \subset \mathbb{R}^d_{+, ρ}$ and induction, it follows from (50) that (49) holds true. \[\boxempty\]
Lemma 2.9. (Compactness of Iteration). Let supp $\hat{u}_0 \subset \mathbb{R}^d_{+,\rho}$. Assume that there exists $n_0 \in \mathbb{N}$ such that
\[
M_0 := \|u^{(n_0)}\|_{L^\infty_{x,t \in [0,\infty)}} + \|u^{(n_0+1)}\|_{L^\infty_{x,t \in [0,\infty)}} \leq \frac{(n_0 + 1)\rho}{8C}. \tag{52}
\]
Then $\{u^{(n)}\}$ is a Cauchy sequence in $L^\infty_{x,t \in [0,\infty)}$.

Proof. By (50), we have for any $n > n_0$,
\[
\|u^{(n+1)} - u^{(n)}\|_{L^\infty_{x,t \in [0,\infty)}} = \|B(u^{(n)} - u^{(n-1)}, u^{(n)})\|_{L^\infty_{x,t \in [0,\infty)}} + \|B(u^{(n-1)}, u^{(n)} - u^{(n-1)})\|_{L^\infty_{x,t \in [0,\infty)}}. \tag{53}
\]
By Lemmas 2.7 and 2.8, it follows from (53) that
\[
\|B(u^{(n)} - u^{(n-1)}, u^{(n)})\|_{L^\infty_{x,t \in [0,\infty)}} \leq \frac{C}{(n+1)\rho} \|u^{(n)}\|_{L^\infty_{x,t \in [0,\infty)}} \|u^{(n)} - u^{(n-1)}\|_{L^\infty_{x,t \in [0,\infty)}}. \tag{54}
\]
Using a similar way, one can estimate the second term in (53). So, in view of (53) and (54) we have
\[
\|u^{(n+1)} - u^{(n)}\|_{L^\infty_{x,t \in [0,\infty)}} \leq \frac{C}{(n+1)\rho} \left( \|u^{(n)}\|_{L^\infty_{x,t \in [0,\infty)}} + \|u^{(n-1)}\|_{L^\infty_{x,t \in [0,\infty)}} \right) \times \|u^{(n)} - u^{(n-1)}\|_{L^\infty_{x,t \in [0,\infty)}}. \tag{55}
\]
Repeating the procedure as in (55), one can obtain that for any $n > n_0$,
\[
\|u^{(n+1)} - u^{(n)}\|_{L^\infty_{x,t \in \mathbb{R}^+}} \leq \frac{C^n - n_0(n_0 + 1)!}{\rho^{n-n_0}(n+1)!} \prod_{k=n_0}^{n-1} \left( \|u^{(k+1)}\|_{L^\infty_{x,t \in \mathbb{R}^+}} + \|u^{(k)}\|_{L^\infty_{x,t \in \mathbb{R}^+}} \right) \times \|u^{(n+1)} - u^{(n_0)}\|_{L^\infty_{x,t \in \mathbb{R}^+}}. \tag{56}
\]
Now we show the following
Claim. $\{u^{(n)}\}_{n > n_0+1}$ is bounded in $L^\infty_{x,t \in [0,\infty)}$ and $\|u^{(n)}\|_{L^\infty_{x,t \in [0,\infty)}} \leq 2M_0$, $n > n_0 + 1$.
We prove the Claim. Taking $n = n_0 + 1$ in (55), we have from (52) that
\[
\|u^{(n_0+2)} - u^{(n_0+1)}\|_{L^\infty_{x,t \in [0,\infty)}} \leq \frac{CM_0}{(n_0 + 1)\rho} \frac{M_0}{2} \tag{57}
\]
Hence, we have $\|u^{(n_0+2)}\|_{L^\infty_{x,t \in [0,\infty)}} \leq 2M_0$. Now let us assume the following induction assumption holds true:
\[
\|u^{(m)}\|_{L^\infty_{x,t \in [0,\infty)}} \leq 2M_0, \quad n_0 + 2 \leq m \leq n. \tag{58}
\]
We show that (58) also holds for $m = n + 1$. Applying (56), one has that for any $\ell = n_0 + 1, \ldots, n$,
\[
\|u^{(\ell+1)} - u^{(\ell)}\|_{L^\infty_{x,t \in \mathbb{R}^+}} \leq \frac{C^{\ell-n_0}(n_0 + 1)!}{\rho^{\ell-n_0}(\ell + 1)!} \prod_{k=n_0}^{\ell-1} \left( \|u^{(k+1)}\|_{L^\infty_{x,t \in \mathbb{R}^+}} + \|u^{(k)}\|_{L^\infty_{x,t \in \mathbb{R}^+}} \right) \times \|u^{(\ell+1)} - u^{(n_0)}\|_{L^\infty_{x,t \in \mathbb{R}^+}} \leq \frac{C^{\ell-n_0}(4M_0)^{\ell-n_0}(n_0 + 1)!}{\rho^{\ell-n_0}(\ell + 1)!} M_0 \leq \frac{M_0}{2^{\ell-n_0}}. \tag{59}
\]
We have
\[ \|u^{(n+1)} - u^{(n_0+1)}\|_{L^\infty_{x,t}([0,\infty])} \leq \sum_{\ell=n_0+1}^{n} \|u^{(\ell+1)} - u^{(\ell)}\|_{L^\infty_{x,t}([0,\infty])} \]
\[ \leq \sum_{\ell=n_0+1}^{n} \frac{M_0}{2^\ell - n_0} \leq M_0. \]
It follows that \( \|u^{(n+1)}\|_{L^\infty_{x,t}([0,\infty])} \leq 2M_0. \)

Finally, we show that \( \{u^{(n)}\} \) is a Cauchy sequence. Again, in view of (59), we have for any \( n > m \gg n_0, \)
\[ \|u^{(n+1)} - u^{(m+1)}\|_{L^\infty_{x,t}([0,\infty])} \leq \sum_{\ell=m+1}^{n} \|u^{(\ell+1)} - u^{(\ell)}\|_{L^\infty_{x,t}([0,\infty])} \]
\[ \leq \sum_{\ell=m+1}^{n} \frac{M_0}{2^\ell - n_0} \leq \frac{M_0}{2^{m-n_0}} \to 0, \ m \to \infty. \]

Therefore, \( \{u^{(n)}\} \) is a Cauchy sequence in \( L^\infty_{x,t}([0,\infty]) \). \( \square \)

2.3. Some remarks on the global well-posedness. Recall that in [15, 47, 16, 10, 26, 64, 62] it was shown that:

**Theorem 2.10.** Let \( 1 \leq p < \infty, \ 1 \leq q \leq \infty. \) Then (NS) is globally well-posed in \( \dot{B}^{-1+d/p}_{p,q} \) for sufficiently small data. Moreover, (NS) is ill posed in \( \dot{B}^{-1}_{\infty,q} \), i.e., the solution map \( u_0 \to u \) is discontinuous.

However, Theorem 1.2 implies the well-posedness for a class of large data in \( \dot{B}^{-1+d/p}_{p,q} \). Let \( \varphi \) be as in (25), \( \widetilde{\varphi}(\xi) = \varphi(\xi)\chi_{\{1/2 \leq \xi \leq 2\}}(\xi) \)
and
\[ \widetilde{u}_0^1(\xi) = c2^{L(1-d)} \prod_{i=1}^{d} \tilde{\varphi}(2^{-L} \xi_i), \quad \widetilde{u}_0^2 = -\frac{\xi_1}{\xi_2} \widetilde{u}_0^1(\xi), \quad \widetilde{u}_0^3 = \ldots = \widetilde{u}_0^d = 0, \]
where \( c < 1 \) is a small constant. It is easy to see that \( \text{div}\widetilde{u}_0 = 0. \) We have,
\[ \|\widetilde{u}_0^1\|_\infty \sim 2^L, \quad \|\widetilde{u}_0^2\|_\infty \sim 2^L. \]
It follows from Theorem 1.2 that for the initial data as in (60), (NS) is globally well-posed in \( L^\infty. \) Also, we see that \( \|u_0\|_{\dot{B}_{\infty,q}^{-1}} \ll 1 \) and \( \|u_0\|_{\dot{B}_{p,q}^{-1+d/p}} \sim 2^{dL/p}, \) which is arbitrarily large in \( \dot{B}_{p,q}^{-1+d/p}. \)

3. Blowup profile. If we consider the profile decomposition in \( \dot{H}^s, \) the orthogonal condition (14) can be replaced by

\[ \text{either} \quad \lim_{n \to \infty} \ln \left| \frac{\lambda_{j_1,n}}{\lambda_{j_2,n}} \right| = \infty, \quad \text{or} \quad \frac{\lambda_{j_1,n}}{\lambda_{j_2,n}} \equiv 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{|x_{j_1,n} - x_{j_2,n}|}{\lambda_{j_1,n}} = \infty. \quad (61) \]

In this section we always assume that \( u_0 \in \dot{H}^{d/2-1} \) and \( \text{div}u_0 = 0. \) Let the solution of (NS) blow up at \( T > 0. \) According to the results in [19, 20, 36, 54], we see that \( \lim_{n \to \infty} |N\text{S}(u_0)(\tau_n)|_{\dot{H}^{d/2-1}} = \infty. \) Taking \( \tau_n \nrightarrow T \) so that
\[ \lim_{n \to \infty} |N\text{S}(u_0)(\tau_n)|_{\dot{H}^{d/2-1}} = \infty. \quad (62) \]

\footnote{We denote by \( \chi_E \) the characteristic function on \( E. \)}
For convenience, we denote for any $\lambda_{j,n} > 0$, $x_{j,n} \in \mathbb{R}^d$,
\[ \Lambda_{j,n}^\alpha f(x) := \frac{1}{\lambda_{j,n}^\alpha} f \left( \frac{x - x_{j,n}}{\lambda_{j,n}} \right), \quad \Lambda_{j,n} f = \Lambda_{j,n}^1 f(x). \quad (63) \]

Let us recall the profile decomposition for a bounded sequence in $\dot{H}^{d/2 - 1}$ (see [27]).

**Theorem 3.1.** Let $u_{0,n}$ be a bounded sequence of divergence-free vector fields in $H^{d/2 - 1}$. Then there exists $\{x_{j,n}\} \subset \mathbb{R}^d$ and $\{\lambda_{j,n}\} \subset (0, \infty)$ which are orthogonal in the sense of (61), and a sequence of divergence-free vector fields $\{\phi_j\} \subset \dot{H}^{d/2 - 1}$ such that
\[ u_{0,n}(x) = \sum_{j=1}^J \Lambda_{j,n} \phi_j(x) + \omega_n^J(x), \]
where $\omega_n^J$ is a reminder in the sense that
\[ \lim_{J \to \infty} \limsup_{n \to \infty} \|\omega_n^J(x)\|_d = 0. \quad (64) \]
Moreover, we have
\[ \|u_{0,n}(x)\|_{\dot{H}^{d/2 - 1}}^2 = \sum_{j=1}^J \|\phi_j\|_{\dot{H}^{d/2 - 1}}^2 + \|\omega_n^J\|_{\dot{H}^{d/2 - 1}}^2 + \epsilon_n^J, \quad \lim_{n \to \infty} \epsilon_n^J = 0, \quad \forall \ J. \quad (65) \]

Using the conservation law (4) in $L^2$, we can further show that

**Lemma 3.2.** (Weak convergence). Let $\{\text{NS}(u_0)(\tau_n)\}$ be the solution sequence satisfying (62), $v(\tau_n) := \text{NS}(u_0)(\tau_n)/\|\text{NS}(u_0)(\tau_n)\|_{\dot{H}^{d/2 - 1}}$. Then $v(\tau_n)$ is weakly convergent to zero in $H^{d/2 - 1}$, $L^d$.

**Proof.** In view of (4) and $\lim_{n \to \infty} \|\text{NS}(u_0)(\tau_n)\|_{\dot{H}^{d/2 - 1}} = \infty$, we have $\|v(\tau_n)\|_2 \to 0$ as $n \to \infty$. This implies the weak convergent limit of $\{v(\tau_n)\}$ is zero in $H^{d/2 - 1}$, $L^d$. \hfill $\Box$

Lemma 3.2 indicates that, if we consider the profile decomposition of $v(\tau_n)$, it weakly vanishes in $L^d(\mathbb{R}^d)$ as $\tau_n \nearrow T$. In view of Theorem 3.1, $v(\tau_n)$ has a profile decomposition in $H^{d/2 - 1}$:
\[ v(\tau_n, x) = \sum_{j=1}^J \Lambda_{j,n} \phi_j(x) + \omega_n^J(x), \quad (66) \]
where $\omega_n^J$ satisfies (64) and (65). Let $j_0 \in \mathbb{N}$. For any $J \geq j_0$, one has (66). As $\phi_{j_0} \in H^{d/2 - 1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d)$, Hölder’s inequality gives
\[ \|f\|_{L^p(|\cdot| \leq R_0)} \leq R_0^{d(1/p - 1/q)} \|f\|_{L^q(|\cdot| \leq R_0)}, \quad q \geq p. \quad (67) \]
So we can take a $R_0 > 0$ such that
\[ \|\phi_{j_0}\|_{L^2(|\cdot| \leq R_0)} = \frac{1}{2} \min\{\|\phi_{j_0}\|_2, \ 1\}. \]
Since $\lim_{J \to \infty} \limsup_{n \to \infty} \|\omega_n^J(x)\|_d = 0$, we can choose sufficiently large $J_0, N_0 \in \mathbb{N}$ such that
\[ \|\omega_n^J(x)\|_d < \varepsilon \leq \frac{\|\phi_{j_0}\|_{L^2(|\cdot| \leq R_0)}}{100R_0^{d/2 - 1}}, \quad \forall J \geq J_0, \ n \geq N_0. \quad (68) \]
Lemma 3.3. (Vanishing $\lambda_{j,n}$). Let $\{NS(u_0)(\tau_n)\}$ be the solution sequence satisfying (62), $v(\tau_n) = NS(u_0)(\tau_n)/\|NS(u_0)(\tau_n)\|_{H^{d/2-1}}$. Let $v(\tau_n)$ has the profile decomposition as in (66) with $J \geq j_0$. Then we have

$$\lambda_{j_0,n}^{d/2-1} \leq \frac{4\|u_0\|_2}{\min\{\|\phi_{j_0}\|_2, 1\} \|NS(u_0)(\tau_n)\|_{H^{d/2-1}}}, \quad n \gg 1.$$ 

In particular, we have $\lambda_{j_0,n} \to 0$ as $n \to \infty$.

Proof. We can assume that the reminder term $\omega_n^d$ satisfying (68). Otherwise we can choose another profile decomposition with $J \geq j_0$. Let us write $\varphi_{j_0} = \phi_{j_0}X(\cdot|\cdot \leq R_0)$. Observing that

$$\langle A_{j,n}, A_{j_0,n}g \rangle = \frac{\lambda_{j,n}^{d-1}}{\lambda_{j,n}} \int g(y) f \left( \frac{\lambda_{j,n} |y + x_{j,n} - x_{j,n}|}{\lambda_{j,n}} \right) dy. \quad (69)$$

Taking the inner product of (66) and $A_{j_0,n}\varphi_{j_0}$, we have

$$\langle v(\tau_n), A_{j_0,n}\varphi_{j_0} \rangle = \sum_{j=1}^J \langle A_{j,n}\phi_j, A_{j_0,n}\varphi_{j_0} \rangle + \langle \omega_n^d, A_{j_0,n}\varphi_{j_0} \rangle. \quad (70)$$

In view of (69), we have

$$\langle A_{j_0,n}\phi_j, A_{j_0,n}\varphi_{j_0} \rangle = \lambda_{j_0,n}^{d-2} \|\varphi_{j_0}\|_2^2 = \lambda_{j_0,n}^{d-2} \|\phi_j\|_{L^2(|y| \leq R_0)}^2. \quad (71)$$

By Hölder’s inequality, (64) and (67),

$$\|\omega_n^d, A_{j_0,n}\varphi_{j_0}\|_d \leq \|\omega_n^d\|_d \|A_{j_0,n}\varphi_{j_0}\|_d (d-1) \leq \varepsilon R_0^{d/2-1} \lambda_{j_0,n}^{d-2} \|\varphi_{j_0}\|_{L^2(|y| \leq R_0)} \quad \quad (72)$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^d$, one can choose $\varphi_j \in C_0^\infty(\mathbb{R}^d)$ satisfying

$$\|\varphi_j - \varphi_j\|_d < \varepsilon/J, \quad j \neq j_0, \quad j = 1, \ldots, J. \quad \quad (73)$$

We have

$$|\langle A_{j,n}\phi_j, A_{j_0,n}\varphi_{j_0} \rangle| \leq \|A_{j,n}(\phi_j - \varphi_j)\|_d \|A_{j_0,n}\varphi_{j_0}\|_d (d-1) \leq \varepsilon J^{-1} \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \|\varphi_{j_0}\|_{L^2(|y| \leq R_0)}. \quad \quad (74)$$

First, we estimate $I$. By Hölder’s inequality, we have

$$I \leq \|A_{j,n}(\phi_j - \varphi_j)\|_d \|A_{j_0,n}\varphi_{j_0}\|_d (d-1) \leq \|\varphi_j - \varphi_j\|_d \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \|\varphi_{j_0}\|_2 \leq \varepsilon J^{-1} \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \|\varphi_{j_0}\|_{L^2(|y| \leq R_0)}. \quad \quad (75)$$

Next, we estimate $II$ and divide the proof into the following three cases.

Case 1. $\lambda_{j_0,n} = \lambda_{j,n}$ and $\lim_{n \to \infty} |x_{j_0,n} - x_{j,n}|/\lambda_{j,n} = \infty$. Since $\varphi_j \in C_0^\infty(\mathbb{R}^d)$, and one can also approximate $\varphi_{j_0}$ by a $C_0^\infty(\mathbb{R}^d)$ function, in view of (69) we see that $II \to 0$ if $n$ is sufficiently large.

Case 2. $\lim_{n \to \infty} \lambda_{j_0,n}/\lambda_{j,n} = 0$. Assume that $\text{supp } \varphi_j \subset B(0, R_1)$. For convenience, we denote $a_{j,n} = (x_{j_0,n} - x_{j,n})/\lambda_{j_0,n}$. We have from (69), Hölder’s inequality and (67) that

$$II = \frac{\lambda_{j_0,n}^{d-1}}{\lambda_{j,n}} \int \varphi_{j_0}(y) \varphi_j \left( \frac{\lambda_{j,n}}{\lambda_{j,n}} (y + a_{j,n}) \right) dy \leq \frac{\lambda_{j_0,n}^{d-1}}{\lambda_{j,n}} \|\varphi_{j_0}\|_{L^2(d-1)} \|\varphi_j\|_{L^2(d-1)} \|\varphi_{j_0}(\cdot + a_{j,n})\| \leq \varepsilon J^{-1} \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \|\varphi_{j_0}\|_{L^2(|y| \leq R_0)}. \quad \quad (76)$$
\[ \leq \lambda_{j,n}^{d-2} R_0^{d/2-1} \left( \frac{\lambda_{j,n} R_1}{\lambda_{j,n} R_0} \right)^{d/2-1} \| \phi_{j,0} \|_{L^2(|\cdot| \leq R_0)} \| \varphi_j \|_d. \]  

(76)

In view of \( \lim_{n \to \infty} \lambda_{j,n}/\lambda_{j_0,n} = 0 \), we see that

\[ \left( \frac{\lambda_{j,n} R_1}{\lambda_{j,n} R_0} \right)^{d/2-1} \| \varphi_j \|_d \leq \varepsilon/J, \quad j = 1, \ldots, J, \ j \neq j_0 \]  

(77)

if \( n \) is sufficiently large. Hence, it follows from (76) and (77) that

\[ II \leq \varepsilon J^{-1} \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \| \phi_{j,0} \|_{L^2(|\cdot| \leq R_0)}, \quad n \gg 1. \]  

(78)

**Case 3.** \( \lim_{n \to \infty} \lambda_{j_0,n}/\lambda_{j,n} = 0 \). Still assume that \( \text{supp} \ \varphi_j \subset B(0, R_1) \) and 
\[ b_{j,n} = (x_{j,n} - x_{j_0,n})/\lambda_{j,n}. \]  
Similarly as in Case 2, we have from (69), Hölder’s inequality and (67) that

\[ II = \lambda_{j_0,n}^{d-1} \int \varphi_j(y) \varphi_{j_0} \left( \frac{\lambda_{j,n}}{\lambda_{j_0,n}} (y + b_{j,n}) \right) dy \]  

\[ \leq \lambda_{j_0,n}^{d-1} \| \varphi_j \|_{L^d(|\cdot| \leq \lambda_{j_0,n} R_0/\lambda_{j,n})} \| \varphi_{j_0} \|_{L^d(|\cdot| \leq \lambda_{j_0,n} R_0/\lambda_{j,n})} \]  

\[ \leq \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \| \phi_{j_0} \|_{L^2(|\cdot| \leq R_0)} \| \varphi_j \|_{L^2(|\cdot| \leq \lambda_{j_0,n} R_0/\lambda_{j,n})}. \]  

(79)

Noticing that for any \( j = 1, \ldots, J \) and \( j \neq j_0 \), \( \varphi_j \in L^d(\mathbb{R}^d) \), in view of the absolute continuity of the integration, \( \lim_{n \to \infty} \lambda_{j_0,n}/\lambda_{j,n} = 0 \) implies that

\[ \| \varphi_j \|_{L^d(|\cdot| \leq \lambda_{j_0,n} R_0/\lambda_{j,n})} \leq \varepsilon/J, \quad n \gg 1. \]  

(80)

if \( n \) is sufficiently large. Hence, it follows from (79) and (80) that for \( 1 \leq j \leq J, \ j \neq j_0 \),

\[ II \leq \varepsilon J^{-1} \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \| \phi_{j_0} \|_{L^2(|\cdot| \leq R_0)}, \quad n \gg 1. \]  

(81)

Collecting (74), (75), (78) and (81), one has that,

\[ \sum_{j \neq j_0} |(A_{j,n} \varphi_j, A_{j_0,n} \varphi_{j_0})| \leq 2 \varepsilon \lambda_{j_0,n}^{d-2} R_0^{d/2-1} \| \phi_{j_0} \|_{L^2(|\cdot| \leq R_0)}, \quad n \gg 1. \]  

(82)

Now, using (70)–(72) and (82), we have

\[ |u(\tau_n), A_{j_0,n} \varphi_{j_0}| \geq \lambda_{j_0,n}^{d-2} \left( \| \varphi_{j_0} \|_{L^2(|\cdot| \leq R_0)}^2 - 3 \varepsilon R_0^{d/2-1} \| \phi_{j_0} \|_{L^2(|\cdot| \leq R_0)} \right) \]  

\[ \geq \frac{1}{2} \lambda_{j_0,n}^{d-2} \| \phi_{j_0} \|_{L^2(|\cdot| \leq R_0)}^2. \]  

(83)

Using the conservation (4), we have

\[ \| u_0 \|_2 \| A_{j_0,n} \varphi_{j_0} \|_2 \geq \| NS(u_0)(\tau_n) \|_2 \| A_{j_0,n} \varphi_{j_0} \|_2 \]  

\[ \geq \frac{1}{2} \lambda_{j_0,n}^{d-2} \| NS(u_0)(\tau_n) \|_{H^{d/2-1}} \| \phi_{j_0} \|_{L^2(|\cdot| \leq R_0)}^2. \]  

Hence,

\[ \lambda_{j_0,n}^{d/2-1} \leq \frac{4 \| u_0 \|_2}{\min\{ \| \phi_{j_0} \|_2, 1 \} \| NS(u_0)(\tau_n) \|_{H^{d/2-1}}}, \quad n \gg 1. \]  

Since \( \lim_{n \to \infty} \| u(\tau_n) \|_{H^{d/2-1}} = \infty \), we easily see that \( \lim_{n \to \infty} \lambda_{j_0,n} = 0 \). \qed
Lemma 3.4. (Blowup profile). Let \( \{NS(u_0)(\tau_n)\} \) be as in Lemma 3.3. There exist \( \{\alpha_n\}, \{\lambda_j(\tau_n)\}, \{\phi_j\} \) and \( \{x_j(\tau_n)\} \) with \( \alpha_n \to +\infty, \lambda_j(\tau_n) \to 0, \phi_j \in H^{d/2-1} \) and \( x_j(\tau_n) \in \mathbb{R}^d \) such that \( u(\tau_n) \) can be decomposed in the following way:

\[
NS(u_0)(\tau_n) = \alpha_n \left( \sum_{j=1}^{J} \frac{1}{\lambda_j(\tau_n)} \phi_j \left( \frac{x - x_j(\tau_n)}{\lambda_j(\tau_n)} \right) + \omega_n' \right).
\]

where \( \omega_n' \) satisfies \( \lim_{\tau_n \to \infty} \limsup_{n \to \infty} ||\omega_n'||_d = 0 \) and moreover,

\[
\alpha_n \sim ||NS(u_0)(\tau_n)||_{H^{d/2-1}}, \quad \lambda_j(\tau_n) \lesssim ||NS(u_0)(\tau_n)||_{H^{d/2-1}}^{-2/(d-2)},
\]

and

\[
\sup_{\xi \in \mathbb{R}^d} ||NS(u_0)(\tau_n)||_{L^p(|\cdot - \xi| \leq \lambda_j(\tau_n))} \gtrsim \lambda_j(\tau_n)^{d/p-1} ||NS(u_0)(\tau_n)||_{H^{d/2-1}}.
\]

Proof. Let us observe that

\[
|v(\tau_n)|, \Lambda_{\lambda_0,n} \varphi_{\lambda_0} \leq ||v(\tau_n)||_{L^2(\cdot \leq \lambda_0,n)} \lambda_0^{d/2-1} ||\varphi_{\lambda_0}||_2,
\]

we have from (83) that

\[
||NS(u_0)(\tau_n)||_{L^2(\cdot \leq \lambda_0,n)} \geq \frac{1}{4} \lambda_0^{d/2-1} ||\varphi_{\lambda_0}||_2 ||NS(u_0)(\tau_n)||_{H^{d/2-1}}. \tag{84}
\]

Again, in view of (67), one has that for any \( 2 \leq p \leq \infty, \)

\[
||NS(u_0)(\tau_n)||_{L^p(\cdot \leq \lambda_0,n)} \leq (R_0 \lambda_{\lambda_0,n})^{d/2-d/p} ||NS(u_0)(\tau_n)||_{L^p(|\cdot \leq \lambda_0,n)}. \tag{85}
\]

Combining (84) and (85),

\[
||NS(u_0)(\tau_n)||_{L^p(\cdot \leq \lambda_0,n)} \geq \frac{1}{4} R_0^{d/p-d/2} \lambda_{\lambda_0,n}^{d/p-1} ||\varphi_{\lambda_0}||_2 ||NS(u_0)(\tau_n)||_{H^{d/2-1}}.
\]

Taking \( x_j,n = x_j(\tau_n), \lambda_j(\tau_n) = \lambda_{j,n}, \) we see that the result follows from the profile decomposition in Theorem 3.1. \(\Box\)

4. Profile decomposition in \( L^p. \) First, let us recall the following theorem concerning the profile decomposition of bounded sequence in \( L^p(\mathbb{R}^d). \)

Theorem 4.1 \((38)\). Let \( 2 \leq p < q, r \leq \infty, \sigma_r,p := d(1/r - 1/p) \) be a bounded sequence in \( L^p(\mathbb{R}^d), \) there exists a sequence of profiles \( \{\phi_j\} \subset L^p(\mathbb{R}^d) \) and orthogonal sequences of scales and cores \( \{(\lambda_j,n,x_j,n)\} \subset (j \in \mathbb{N}), \) such that up to an extraction in \( n, \)

\[
f_n = \sum_{j=1}^{J} \Lambda_{\lambda_j,n}^{d/p} \phi_j + \psi_n', \tag{86}
\]

and the following properties hold:

(i) The remainder term \( \psi_n' \) satisfies the smallness condition:

\[
\lim_{J \to \infty} \left( \limsup_{n \to \infty} ||\psi_n'||_{\sigma_r,p} \right) = 0. \tag{87}
\]

(ii) The profiles satisfy the following inequality:

\[
\sum_{j=1}^{\infty} ||\phi_j||_p \lesssim \liminf_{n \to \infty} ||f_n||_p, \tag{88}
\]
and for each integer $J,$

$$
\|\psi_n^J\|_p \lesssim \|f_n\|_p + o(1) \quad \text{as} \quad n \to \infty.
$$

(89)

Using the diagonal method, we can assume that, up to an extraction, $\lim_{n \to \infty} \lambda_{j,n}$ exists for all $j \in \mathbb{N}$ (whose limit can be $+\infty$). We further denote

$$
\mathcal{J}_j = \{ j \in \mathbb{N} : \lim_{n \to \infty} \lambda_{j,n} \neq 0, \infty \}, \quad \mathcal{J}_1 = \mathcal{J}_j \cap \{ j \leq J \},
$$

$$
\mathcal{J}_0(J) = \{ j \in \mathbb{N} : \lim_{n \to \infty} \lambda_{j,n} = 0, j \leq J \},
$$

$$
\mathcal{J}_\infty(J) = \{ j \in \mathbb{N} : \lim_{n \to \infty} \lambda_{j,n} = \infty, j \leq J \},
$$

$$
\mathcal{J}_1^*(J) = \mathcal{J}_0(J) \cup \mathcal{J}_\infty(J).
$$

For $j \in \mathcal{J}_1,$ we shall simply call $\lambda_{j,n}$ constant scale. Now $f_n$ can be rewritten as

$$
f_n = \sum_{j \in \mathcal{J}_1(J)} \lambda_{j,n}^{d/p} \phi_j + \sum_{j \in \mathcal{J}_1^*(J)} \lambda_{j,n}^{d/p} \phi_j + \psi_n^J.
$$

Denote

$$
g_\eta(x) := g(x) \chi_{\{ \frac{1}{n} \leq |g(x)| \leq \eta \}}, \quad g_\eta^J(x) := g(x) - g_\eta(x).
$$

Next, we show that, if the scales $\lambda_{j,n}$ are very small or very large, then the corresponding profiles $\lambda_{j,n}^{d/p} \phi_j$ are very small in appropriate function spaces.

**Proposition 1.** Let $p, q, r, s, r_p$ be as in Theorem 4.1, then

1. $\lim_{n \to \infty} \left\| \sum_{j \in \mathcal{J}_0(J)} \lambda_{j,n}^{d/p} \phi_j \right\|_{p_1} = 0 \quad \text{for any} \quad p_1 < p \quad \text{and fixed} \quad J, \eta \geq 1.$

2. $\lim_{n \to \infty} \left\| \sum_{j \in \mathcal{J}_\infty(J)} \lambda_{j,n}^{d/p} \phi_j \right\|_{p_2} = 0 \quad \text{for any} \quad p_2 > p \quad \text{and fixed} \quad J, \eta \geq 1.$

3. $\lim_{J \to \infty} \lim_{n \to \infty} \sup_{\eta \to \infty} \left\| \sum_{j \in \mathcal{J}_1^*_J} \lambda_{j,n}^{d/p} \phi_{j,\eta^c} + \psi_n^J \right\|_{\dot{B}^s_{r,q}^p} = 0.

**Proof.** First, we prove (1). Let $p > p_1, J, \eta$ fixed, then

$$
\left\| \sum_{j \in \mathcal{J}_0(J)} \lambda_{j,n}^{d/p} \phi_j \right\|_{p_1} \leq \sum_{j \in \mathcal{J}_0(J)} \lambda_{j,n}^{d(\frac{1}{p} - \frac{1}{p_1})} \|\phi_j\|_{p_1}
$$

$$
\leq \sum_{j \in \mathcal{J}_0(J)} \lambda_{j,n}^{d \frac{1}{p} \frac{1}{p_1}} \frac{1}{\eta^r} \|\phi_j\|_{p_1}^p \longrightarrow 0 \quad \text{as} \quad n \to \infty.
$$

The proof of (2) is similar to (1), and the details of the proof are omitted.

Finally, we prove (3). By the assumptions on $p, q, r$ and Proposition B.1, we can easily see $L^p \hookrightarrow \dot{B}^s_{r,q}^p,$ thus

$$
\left\| \sum_{j \in \mathcal{J}_1^*(J)} \lambda_{j,n}^{d/p} \phi_{j,\eta^c} + \psi_n^J \right\|_{\dot{B}^s_{r,q}^p} \lesssim \left\| \sum_{j \in \mathcal{J}_1^*(J)} \lambda_{j,n}^{d/p} \phi_{j,\eta^c} \right\|_p + \|\psi_n^J\|_{\dot{B}^s_{r,q}^p}
$$

$$
\leq \sum_{j \in \mathcal{J}_1^*(J)} \|\phi_{j,\eta^c}\|_p + \|\psi_n^J\|_{\dot{B}^s_{r,q}^p}.
$$

By the Lebesgue dominated convergence theorem and the definition of $\phi_{j,\eta^c},$ we see that

$$
\lim_{\eta \to \infty} \limsup_{n \to \infty} \sum_{j \in \mathcal{J}_1^*(J)} \|\phi_{j,\eta^c}\|_p = 0.
$$

So, we have the result, as desired. \qed
Now, let us denote
\[ \lim_{n \to \infty} \lambda_{j,n} = \lambda_j, \quad A_j \phi_j = \frac{1}{\lambda_j^{3/p}} \phi_j\left(\frac{\cdot}{\lambda_j}\right), \quad j \in \mathbb{J}_1. \]  

(90)

Proposition 1 indicates that the scales with indices in \( J_0 \cup J_\infty \) may not take the major roles to the Navier-Stokes evolution, as they can be considered to be the error term in some function spaces. In fact, we have the following result, note that from now on, the space dimension \( d = 3 \).

**Theorem 4.2** ((NS) evolution of bounded \( L^p \) initial data). Fix \( 3 < p < \infty \), let \((u_{0,n})_{n \geq 1}\) be a bounded sequence of divergence-free vector fields in \( L^p(\mathbb{R}^d) \), whose profile decomposition is given in the following form as in Theorem 4.1:

\[ u_{0,n}(x) = \sum_{j=1}^{J} A_{j,n}^{3/p} \phi_j(x) + \psi_n^j(x). \]

We have the following conclusions.

(i) The existence time of the Navier-Stokes solution associated with initial data \( u_{0,n} \) satisfies

\[ \liminf_{n \to 1} T(u_{0,n}) \geq \tilde{T} := \inf_{j \in \mathbb{J}_1} T(A_j \phi_j), \]

and \( T(u_{0,n}) = \infty \) if \( \mathbb{J}_1 \) is empty.

(ii) There exists a \( J_0 \in \mathbb{N} \) and \( N(J) \in \mathbb{N} \), up to an extraction on \( n \), such that

\[ NS(u_{0,n}) = \sum_{j \in J_1(J)} NS(A_{j,n}^{3/p} \phi_j) + e^{t\Delta} \left( \sum_{j \in J_1(J)} A_{j,n}^{3/p} \phi_j(x) + \psi_n^j(x) \right) + R_n^J \]

is well-defined for \( J > J_0 \), \( n > N(J) \) and \( t < \tilde{T} \), moreover, for any \( T < \tilde{T} \),

\[ \lim_{J \to \infty} \limsup_{n \to \infty} \| R_n^J \|_{X_p^0} = 0, \quad X_p^0 := C([0,T]; L^p_x) \cap L^{3p}_x((0,T); L^{3p}_x). \]

(iii) The solution satisfies the orthogonality property in the sense

\[ \| NS(u_{0,n})(t) \|_p^p = \sum_{j \in J_1(J)} \| NS(A_j \phi_j)(t) \|_p^p + e^{t\Delta} \left( \sum_{j \in J_1(J)} A_{j,n}^{3/p} \phi_j + \psi_n^j \right) \|_p^p + \gamma_n^j(t), \]

(94)

with \( \lim_{J \to \infty} \limsup_{n \to \infty} |\gamma_n^j(t)| = 0 \) for each time \( 0 < t < \tilde{T} \).

**Remark 3.** \( \tilde{T} \) is attainable, provided \( \mathbb{J}_1 \) is nonempty, i.e. there exists some \( j_0 \in \mathbb{J}_1 \) such that \( \tilde{T} = T(A_{j_0} \phi_{j_0}) \). In fact, this is a simple consequence of (10) and (88).

**Remark 4.** By (88), we know \( \lim_{j \to \infty} \| \phi_j \|_p = 0 \), so for any fixed \( T < \tilde{T} \), there exists an index \( J_c = J_c(T) \), such that \( T \| \phi_j \|_p^{\sigma_p} < c_0, \ j \geq J_c \), where \( c_0 \) is given in (10), then by the fixed point argument in proving Theorem 1.1, we see that \( \| NS(\phi_j) \|_{X_p^0} \leq 2C \| \phi_j \|_p, \ j \geq J_c \). For the details one can refer to Appendix B.

The proof of Theorem 4.2 will be given in Section 6 and Section 7.
5. Construction of blowup solution with bounded Besov norm. Denote 

\[ D_c = \{ \text{NS}(u_0) : T(u_0) < \infty, \limsup_{t \to T(u_0)} (T(u_0) - t)\|\text{NS}(u_0)(t)\|_{L_p}^\sigma = M_c^\sigma \}. \]

In this section, we will prove Theorem 1.4 via Theorem 4.2. The proof is separated into two steps: Firstly, we show there is a solution \( \text{NS}(u_0) \in D_c \); Secondly, using the smoothing effect of the bilinear term of the Navier-Stokes equation, we can construct a new solution which belongs to both \( L^p \) and \( B^{-1+6/p}_{p/2,\infty} \). The argument follows the same idea as that in [50].

5.1. Existence of solution in subcritical \( L^p \) space.

**Proposition 2.** Let \( M_c^\sigma < \infty \). Then \( D_c \) is nonempty.

**Proof.** By the definition of \( M_c^\sigma \), there exists a sequence of \( (u_{0,n})_{n \geq 1} \), such that \( T(u_{0,n}) < \infty \) and

\[ \limsup_{t \to T(u_{0,n})} (T(u_{0,n}) - t)\|\text{NS}(u_{0,n})(t)\|_{L_p}^\sigma < M_c^\sigma + \frac{1}{2n}. \]

Further, one can find a time sequence \( t_n (T(u_{0,n}) - t_n \xrightarrow{n \to \infty} 0) \) verifying

\[ (T(u_{0,n}) - t)\|\text{NS}(u_{0,n})(t)\|_{L_p}^\sigma < M_c^\sigma + 1/n \quad \text{for all} \quad t_n \leq t < T(u_{0,n}). \]  \((95)\)

For simplicity, we write \( u_{0,n} = T(u_{0,n}) - t_n \) and define a rescaled initial data \( v_{0,n} \)

\[ v_{0,n} = \mu_n \|\text{NS}(u_{0,n})(t_n)\|_{L_p}^\sigma. \]

Obviously from (95), \( v_{0,n}\|_{L_p}^\sigma = \mu_n \|\text{NS}(u_{0,n})(t_n)\|_{L_p}^\sigma < M_c^\sigma + 1/n \), by uniqueness, the associated solution of (NS) is

\[ \text{NS}(v_{0,n})(\tau, x) = \mu_n^{\frac{1}{n}} \text{NS}(u_{0,n})(t_n + \mu_n \tau, \mu_n^n x), \]

which is defined on time interval \([0, 1]\). Again, it follows from (95) that

\[ (1 - \tau)\|\text{NS}(v_{0,n})(\tau)\|_{L_p}^\sigma \leq M_c^\sigma + 1/n. \]

Then the desired result is a direct consequence of the following Lemma 5.1. \( \square \)

**Lemma 5.1.** Let \( (\epsilon_n)_{n \geq 1} \) be a decreasing and vanishing sequence, \( (v_{0,n})_{n \geq 1} \) is a bounded sequence in \( L^p(\mathbb{R}^3) \) and satisfies

\[ T(v_{0,n}) = 1, \quad \sup_{0 < \tau < 1} (1 - \tau)\|\text{NS}(v_{0,n})(\tau)\|_{L_p}^\sigma \leq M_c^\sigma + \epsilon_n. \]  \((96)\)

Then the following conclusions hold (passing to a subsequence if necessary)

- In the profile decomposition of \( v_{0,n} \), there exists a unique scale profile \( \Lambda_{k_0} \phi_{k_0} \) such that \( \text{NS}(\Lambda_{k_0} \phi_{k_0}) \) blows up at time 1.
- \( \text{NS}(\Lambda_{k_0} \phi_{k_0}) \in D_c \) and

\[ \sup_{0 < \tau < 1} (1 - \tau)\|\text{NS}(\Lambda_{k_0} \phi_{k_0})(\tau)\|_{L_p}^\sigma = \limsup_{\tau \to 1} (1 - \tau)\|\text{NS}(\Lambda_{k_0} \phi_{k_0})(\tau)\|_{L_p}^\sigma = M_c^\sigma. \]

**Proof.** Assume the profile decomposition of \( v_{0,n} \) has the following form

\[ v_{0,n}(x) = \sum_{j \in J_j} \Lambda_{j,n}^{3/p} \phi_j + \sum_{j \in J_j} \Lambda_{j,n}^{3/p} \phi_j + \psi_n. \]

Recall that \( \tilde{T} = \inf_{j \in J_j} T(\Lambda_j \phi_j) \), by Remark 3, we assume \( \tilde{T} = T(\Lambda_{k_0} \phi_{k_0}) \) and divide the proof into three steps:
Let Proposition 3.

(i) We show that $\bar{T} = T(\Lambda_k \phi_{k_0}) = 1$. In view of (91) in Theorem 4.2, one has that $\bar{T} \leq 1$. Assume $\bar{T} < 1$, for any $\tau < \bar{T}$, by (94),

$$\|NS(v_{0,\tau}(\tau))\|_p^p \geq \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p^p - |\gamma^f(\tau)|.$$  

Multiplying $(T(\Lambda_k \phi_{k_0}) - \tau)\frac{\rho}{\tau}$ on both sides, and using the assumption (96), one has that

$$(T(\Lambda_k \phi_{k_0}) - \tau)\frac{\rho}{\tau} (M_c^\sigma + \epsilon_n) \geq \{(T(\Lambda_k \phi_{k_0}) - \tau)\frac{\rho}{\tau} \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p\}^p - |\gamma^f(\tau)|.$$

Taking limit with regard to $n$ and $J$ on both sides, we have

$$(T(\Lambda_k \phi_{k_0}) - \tau)\frac{\rho}{\tau} M_c^\sigma \geq \{(T(\Lambda_k \phi_{k_0}) - \tau)\frac{\rho}{\tau} \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p\}^p.$$  

Noting that $\tau < \bar{T} = T(\Lambda_k \phi_{k_0})$, so we have

$$\frac{(\bar{T} - \tau)\rho}{(1 - T)^p} M_c^\sigma \geq \{(T(\Lambda_k \phi_{k_0}) - \tau)\frac{\rho}{\tau} \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p\}^p.$$  

The left hand side of (97) can be arbitrarily small if we take $\tau$ sufficiently close to $\bar{T}$, however, the right hand side of (97) has a lower bound as in Theorem 1.1, which leads to a contradiction. Hence $\bar{T} = T(\Lambda_k \phi_{k_0}) = 1$.

(ii) We prove that $NS(\Lambda_k \phi_{k_0}) \in D_c$. It follows from the definition of $M_c^\sigma$ and (97) that

$$\limsup_{\tau \rightarrow 1} (1 - \tau) \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p^\sigma = \sup_{0 < \tau < 1} (1 - \tau) \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p^\sigma = M_c^\sigma.$$  

(iii) Constant scale profile blowing up at time 1 is unique. We also prove it by contradiction. Assume there is another constant scale profile $\Lambda_j \phi_{j_0}$ which blows up at time 1. Using (94) again, we find

$$\|NS(v_{0,\tau}(\tau))\|_p^p \geq \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p^p + \|NS(\Lambda_j \phi_{j_0}(\tau))\|_p^p - |\gamma^f(\tau)|.$$  

Repeating the the argument in (i), we can deduce that

$$M_c^\sigma \geq \{(1 - \tau)\frac{\rho}{\sigma} \|NS(\Lambda_k \phi_{k_0}(\tau))\|_p\}^p + \{(1 - \tau)\frac{\rho}{\sigma} \|NS(\Lambda_j \phi_{j_0}(\tau))\|_p\}^p.$$  

Due to $NS(\Lambda_k \phi_{k_0}) \in D_c$, it should hold $NS(\Lambda_j \phi_{j_0}) = 0$, so $\phi_{j_0} = 0$. The proof of Lemma 5.1 is finished.

5.2. Existence of solution in critical Besov space. In general, we could not expect that the linear term $e^{t\Delta} u_0$ belongs to $\dot{B}_{p/2,\infty}^{-1+6/p}$, provided we only assume the initial data $u_0 \in L^p$. To get the desired result, we need to use the regularization effect of the bilinear term $\mathcal{R}(u,u)$. In fact, based on our assumption, the following proposition holds.

**Proposition 3.** Let $u_0$ be in $L^p$, with $3 < p < \infty$, the associated Navier-Stokes solution $u(t) := NS(u_0)(t)$ satisfies a minimal blow up condition, i.e.

$$T(u_0) < \infty, \quad M^{\sigma_p} := \sup_{0 < t < T(u_0)} (T(u_0) - t) \|NS(u_0)(t)\|_p^\sigma < \infty, \quad \sigma_p = \frac{2}{1 - 3/p}.$$
then we have
\[
\sup_{0 < t < T(u_0)} \| \mathcal{B}(u, u)(t) \|_{H^{-\frac{1+6}{p}, \infty}} < \infty. \tag{98}
\]

Furthermore, if \( 0 < s < 1 - 3/p \), then
\[
\sup_{0 < t < T(u_0)} (T(u_0) - t)^{\frac{1}{2} (s+1-3/p)} \| \mathcal{B}(u, u)(t) \|_{B^s_{p, \infty}} < \infty. \tag{99}
\]

**Proof.** First, we prove (98). It follows from (29) that
\[
\| \triangle_j \mathcal{B}(u, u) \|_{L^p/2} \lesssim \int_0^t 2^j \| \mathcal{B}(u, u) \|_{L^p/2} \, ds,
\]
\[
\lesssim \int_0^t 2^j \| \mathcal{B}(u, u) \|_{L^p/2} \| u(s) \|_{L^p}^2 \, ds.
\]

Since \( u(t) \) is a blowup solution of type-I,
\[
\| u(t) \|_{L^p}^2 \leq \frac{M^2}{(T(u_0) - t)^{1-3/p}}.
\]

Hence
\[
2^{j\frac{6}{p}} \int_0^t \mathcal{B}(u, u)(t) \| \triangle_j \mathcal{B}(u, u) \|_{L^p/2} \lesssim \int_0^t 2^j \frac{e^{-c(t-s)^2 \tau}}{(T(u_0) - s)^{1-3/p}} \frac{M^2}{(T(u_0) - s)^{1-3/p}} \, ds \lesssim M^2.
\]

Let us observe that
\[
2^{j\frac{6}{p}} \int_0^t \chi(T(u_0)-s \leq 2^{-2j}) e^{-c(t-s)^2 \tau} \frac{M^2}{(T(u_0) - s)^{1-3/p}} \, ds \lesssim M^2.
\]

and
\[
2^{j\frac{6}{p}} \int_0^t \chi(T(u_0)-s \geq 2^{-2j}) e^{-c(t-s)^2 \tau} \frac{M^2}{(T(u_0) - s)^{1-3/p}} \, ds \lesssim 2^{2j} M^2 \int_0^t e^{-c(t-s)^2 \tau} \, ds \lesssim M^2.
\]

(98) is obtained.

Next, we prove (99), whose proof can be divided into two cases.

Case 1. \( T(u_0) - t < 2^{-2j} \). By hypothesis, we see that
\[
(T(u_0) - t)^{\frac{1}{p}} \| \mathcal{B}(u, u)(t) \|_{L^p} \lesssim (T(u_0) - t)^{\frac{1}{p}} \| u(t) \|_{L^p} + (T(u_0) - t)^{\frac{1}{p}} \| \mathcal{B}(u, u)(t) \|_{L^p} \lesssim M + T(u_0)^{\frac{1}{p}} \| u_0 \|_{L^p} < \infty.
\]

For \( s > 0 \), noting \( \sigma_p = \frac{2}{1-3/p} \) and \( 2^j < (T(u_0) - t)^{-\frac{1}{2}} \),
\[
2^{j\sigma_p} \| \triangle_j \mathcal{B}(u, u)(t) \|_{L^p} \lesssim \frac{1}{(T(u_0) - t)^{\frac{1}{2} (s+1-3/p)}}.
\]

Case 2. \( T(u_0) - t \geq 2^{-2j} \). By Bernstein’s inequality and the assumption,
\[
\| \triangle_j \mathcal{B}(u, u)(t) \|_{L^p} \lesssim \int_0^t 2^j (1+3/p) e^{-c(t-s)^2 \tau} \frac{M^2}{(T(u_0) - \tau)^{1-3/p}} \, d\tau.
\]
Multiplying it by $2^{js}$ and using the fact that $\tau < t$ and $s - 1 + 3/p < 0$,
\[
2^{js}\|\triangle_j F(u, u)(t)\|_p \lesssim \int_0^t 2^{j(s+3/3/p)}_c(1-\tau)^{2^{j(s-1+3/3/p)}} \frac{M^2}{(T(u_0) - t)^{1-3/p}} \, d\tau,
\]
\[
\lesssim \frac{M^2}{(T(u_0) - t)^{1-3/p}} 2^{j(s-1+3/3/p)}.
\]

Combining the above two cases, we finally obtain (99).

Next we prove a lemma which allows us to exclude profiles without constant scales in the profile decomposition of a bounded sequence in $L^p(\mathbb{R}^3)$.

**Lemma 5.2.** Assume $3 < p < \infty, 0 < s < 1 - 3/p$, then the following statements hold:

1. Let $(f_n)_{n \geq 1}$ be a bounded sequence in $\dot{B}^{-1+6/p}_{p,\infty} \cap L^p$, s.t. $\limsup_{n \to \infty} \|f_n\|_{\dot{B}^0_{p,\infty}} > 0$, then there are no scales which tend to infinity in the $L^p$ profile decomposition of $f_n$.

2. Let $(f_n)_{n \geq 1}$ be a bounded sequence in $\dot{B}^{s}_{p,\infty} \cap L^p$, s.t. $\limsup_{n \to \infty} \|f_n\|_{\dot{B}^0_{p,\infty}} > 0$, then there are no scales which tend to zero in the $L^p$ profile decomposition of $f_n$.

*Proof.* First, we argue by contradiction to prove (1). Assume for a contrary, then there exists a scale $\lambda_{j,n} \xrightarrow{n \to \infty} \infty$. By Theorem 4.1, let the profile decomposition of $f_n$ be the following form

\[
f_n = \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{3/p}} \phi_j \left( \frac{x - x_{j,n}}{\lambda_{j,n}} \right) + \psi_n.
\]

It follows from $\limsup_{n \to \infty} \|f_n\|_{\dot{B}^0_{p,\infty}} > 0$ that $f_n \neq 0$. As we know, each $\phi_j$ is a weak limit of some rigid transformation of $f_n$; more precisely, we have

\[
\lambda_{j,n}^{3/p} f_n (\lambda_{j,n} x + x_{j,n}) \rightharpoonup \phi_j,
\]

in the sense of tempered distribution. Without loss of generality, we assume $\phi_j \neq 0$ for each $j$. Observing

\[
\lambda_{j,n}^{3/p} \|f_n (\lambda_{j,n} x + x_{j,n})\|_{\dot{B}^{-1+6/p}_{p,2,\infty}} \equiv \lambda_{j,n}^{3/p-1} \|f_n\|_{\dot{B}^{-1+6/p}_{p,2,\infty}} \xrightarrow{n \to \infty} 0,
\]

due to the hypothesis on $f_n$ and $\lambda_{j,n} \xrightarrow{n \to \infty} \infty$. By lower semi-continuity, we get

\[
\|\phi_k\|_{\dot{B}^{-1+6/p}_{p,2,\infty}} \leq \liminf_{n \to \infty} \lambda_{k,n}^{3/p} \|f_n (\lambda_{k,n} x + x_{k,n})\|_{\dot{B}^{-1+6/p}_{p,2,\infty}} = 0,
\]

so $\phi_k = 0$, this contradicts to our assumption. The proof of (2) proceeds in a similar manner and the details are omitted.

*Proof of Theorem 1.4.* We only outline the main procedures in the proof of existence of a critical element in $\dot{B}^{-1+6/p}_{p,2,\infty}$, as the argument is quite similar to [50]. Indeed, we already have a solution $\Psi = NS(\Psi_0) \in D_\epsilon$ from Proposition 2, so there exists a time sequence $t_n \to T(\Psi_0)$ and $\epsilon_n \to 0$ such that

\[
\lim_{n \to \infty} (T(\Psi_0) - t_n) \|NS(\Psi_0)(t_n)\|_p^{\sigma_p} = M^{\sigma_p}_c,
\]
where

\[ (T\Psi_0 - t)\|\text{NS}(\Psi_0)(t)\|_{p}^{p} \leq M^{p} + \epsilon_{n} \quad \text{for any} \quad t_n \leq t < T(\Psi_0). \]  

(100)

As before, we define a rescaled initial data \( v_{0,n} \),

\[ v_{0,n}(x) = (T\Psi_0 - t_n)^{1/2} \text{NS}(\Psi_0)(t_n, (T\Psi_0 - t_n)^{1/2} x), \]

denote \( \tau_n = T\Psi_0 - t_n \) for short, we get

\[ v_{0,n}(x) = \tau_n^{1/2} e^{t_n \Delta} \Psi_0(\tau_n^{1/2} x) + \tau_n^{1/2} \mathcal{B}(\Psi_0, \Psi_0)(t_n, \tau_n^{1/2} x). \]

Due to the fact \( \tau_n \to 0 \), (100) and Proposition 3, we can easily get

\[ \rho_n := \tau_n^{1/2} \mathcal{B}(\Psi_0, \Psi_0)(t_n, \tau_n^{1/2} x) \in L^p \cap \dot{B}^{-1+6/p}_{p/2,\infty} \cap \dot{B}^{s}_{p,\infty}. \]

Next we take \( L^p \) profile decomposition of \( \rho_n \),

\[ \rho_n = \sum_{j=1}^{J} \Lambda_{j,n}^{3/p} \phi_j(x) + \psi_n^J, \]

then come back to \( v_{0,n} \)

\[ v_{0,n}(x) = \sum_{j=1}^{J} \Lambda_{j,n}^{3/p} \phi_j(x) + \psi_n^J + \tau_n^{1/2} e^{t_n \Delta} \Psi_0(\tau_n^{1/2} x). \]

In this circumstance, \( \psi_n^J + \tau_n^{1/2} e^{t_n \Delta} \Psi_0(\tau_n^{1/2} x) \) can be considered as the remainders, and according to Lemma 5.2, in the profile decomposition of \( \rho_n \), only profiles with constant scales are left, and each \( \phi_j(x) \) is the weak limit of \( \Lambda_{j,n}^{3/p} \rho_n(\lambda_{j,n} x + x_j, n) \), so all these profiles lies in \( L^p \cap \dot{B}^{-1+6/p}_{p/2,\infty} \cap \dot{B}^{s}_{p,\infty} \). Now it can be easily checked that \( (v_{0,n})_{n \geq 1} \) satisfy the assumptions of Lemma 5.1, hence there exists a unique scale profile \( \Lambda_0 \phi_0 \in L^p \cap \dot{B}^{-1+6/p}_{p/2,\infty} \) such that \( u(t, \tau) := \text{NS}(\Lambda_0 \phi_0) \in D_c \) and blows up at time 1. Moreover for any 0 < \( \tau < 1 \), we have

\[
\sup_{0 < \tau < 1} \|u(\tau)\|_{\dot{B}^{-1+6/p}_{p/2,\infty}} = \sup_{0 < \tau < 1} \|e^{-\Delta} \Lambda_0 \phi_0 - \mathcal{B}(u, u)(\tau)\|_{\dot{B}^{-1+6/p}_{p/2,\infty}} \lesssim \|\phi_0\|_{\dot{B}^{-1+6/p}_{p/2,\infty}} + \sup_{0 < \tau < 1} \|\mathcal{B}(u, u)(\tau)\|_{\dot{B}^{-1+6/p}_{p/2,\infty}} < \infty.
\]

where we have used Proposition 3. The proof is completed. \( \square \)

6. (NS) evolution of profile decomposition. This section is devoted to the proof of (i) and (ii) in Theorem 4.2. Let \( (u_{0,n})_{n \geq 1} \) be a bounded sequence in \( L^p \) and the corresponding profile decomposition is given below

\[
u_{0,n} = \sum_{j \in J_1(J)} \Lambda_{j,n}^{3/p} \phi_j + \sum_{j \in J_1^n(J)} \Lambda_{j,n}^{3/p} \phi_j + \psi_n^J,
\]

We look for an approximative solution to the genuine solution \( \text{NS}(u_{0,n}) \), let

\[
\text{NS}(u_{0,n}) = U_{n}^{\text{app}, J} + R_{n}^J,
\]

where

\[
U_{n}^{\text{app}, J} = \sum_{j \in J_1(J)} \text{NS}(\Lambda_{j,n}^{3/p} \phi_j) + e^{t \Delta} \left( \sum_{j \in J_1^n(J)} \Lambda_{j,n}^{3/p} \phi_j + \psi_n^J \right).
\]
Lemma 6.1. For any $\varepsilon > 0$ and Proposition B.6 in the appendix. We remark that by the lower semi-continuity let $T$ be a free vector field, which is defined on $[0, T_J]$. Then (1) and (ii) of Theorem 4.2 will be consequences of the following two lemmas and Proposition B.6 in the appendix. We remark that by the lower semi-continuity in Corollary 5.7, for arbitrary $\varepsilon > 0$, there exists a $N := N(\varepsilon)$, such that for any $\varepsilon > N$, $\tilde{T}_1 := \inf_{j \in J_1} T(\lambda_{j,n}^2) \geq \tilde{T} - \varepsilon$, here $\tilde{T} := \inf_{j \in J} T(\lambda_{j})$.

**Lemma 6.1.** For any $0 < T < \tilde{T} := \inf_{j \in J} T(\lambda_j)$, there exists a constant $C > 0$ such that, up to an extraction for $n \in \mathbb{N}$,

$$\lim_{J \to \infty} \lim_{n \to \infty} \sup \|U_n^{app,J}\|_{X_T^p} \leq C.$$  

**Lemma 6.2.** Let $3 < p < \infty$, $1/3(p) = 1/2 - 3/10p$, $T$ be as in Lemma 6.1. Then we have

$$\lim_{J \to \infty} \lim_{n \to \infty} \sup \|G_n^J\|_{L_T^{1/3(p)} L_T^{5/p,n}} = 0.$$  

Let us turn to the proof of Lemma 6.1 and Lemma 6.2. First, we give a lemma, which will be used in the sequel.

**Lemma 6.3.** Let $J \in \mathbb{N}$, $3 < p < \infty$, $r = 5p/3$. Let $\phi_j \in L^p$ and $(x_{j,n}, \lambda_{j,n})_{n \geq 1}$ be orthogonal scales and cores in the sense of (14), then for any $T \leq \inf_{j \in J_1} T_J$, $T_j < T(\lambda_j)$ is arbitrary. Then there exists $N(T, J)$ such that for any $n > N(T, J)$,

$$\left\| \sum_{j \in J_1(J)} NS(\lambda_{j,n}^3) \phi_j \right\|_{L_T^{1/3(p)} L_T^{5/p,n}} \leq \sum_{j \in J_1(J)} \left\| NS(\lambda_j^3) \phi_j \right\|_{L_T^{1/3(p)} L_T^{5/p,n}} + \varepsilon(J, n), \tag{101}$$

where $\lim_{n \to \infty} \varepsilon(J, n) = 0$. Furthermore, for any $0 \leq t \leq T$, up to an extraction for $n \in \mathbb{N}$,

$$\left\| \sum_{j \in J_1(J)} NS(\lambda_{j,n}^3) \phi_j(t) \right\|_{L_T\phi_j(t)}^p = \sum_{j \in J_1(J)} \left\| NS(\lambda_j^3) \phi_j(t) \right\|_{L_T\phi_j(t)}^p + \sigma_{J,n}(t), \tag{102}$$

where $\lim_{n \to \infty} \sup_{0 < t < T} |\sigma_{J,n}(t)| = 0$.

With the help of Lemma 6.3, we can verify Lemma 6.1.
Proof of Lemma 6.1. By the definition of $U^{app,J}_n$,
\[
\|U^{app,J}_n\|_{\mathcal{F}^r} \leq \left\| \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) \right\|_{L_T^\infty L_x^p} + \left\| e^{\Delta} \left( \sum_{j \in J_1(J)} \Lambda_{j,n}^{3/p} \phi_j + \psi_n^J \right) \right\|_{X_T^p}
:= I + II,
\]
where
\[
I = \left\| \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) \right\|_{L_T^\infty L_x^p} + \left\| \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) \right\|_{L_T^{\infty} L_x^{\frac{5p}{3}}}. \]

From (102) it follows that
\[
\left\| \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) \right\|_{L_T^\infty L_x^p} \leq \sum_{j \in J_1(J)} \|NS(\Lambda_j \phi_j)\|_{L_T^\infty L_x^p} + \|\sigma_{J_n}(t)\|_{L_T^\infty}. \tag{103}
\]
In view of Remark 4, we see that there exists a $J_c \in \mathbb{N}$ such that for $j \geq J_c$,
\[
\|NS(\Lambda_j \phi_j)\|_{L_T^\infty L_x^p} \leq 2C\|\phi_j\|_p.
\]
Therefore, the left hand side of (103) can be bounded by
\[
\sum_{j \in J_1(J_c)} \|NS(\Lambda_j \phi_j)\|_{L_T^\infty L_x^p} + C \sum_{j=1}^\infty \|\phi_j\|_p. \tag{104}
\]
as indicated by (88), $\sum_{j=1}^\infty \|\phi_j\|_p \leq \liminf_{n \to \infty} \|u_{0,n}\|_p$ and $T < \tilde{T}$, thus
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left\| \sum_{j \in J_1(J_c)} NS(\Lambda_{j,n}^{3/p} \phi_j) \right\|_{L_T^\infty L_x^p} < \infty. \tag{105}
\]
Using the same way as above, in view of (101) and $l^p \hookrightarrow l^{5p/3}$, we have
\[
\limsup_{n \to \infty} \left\| \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) \right\|_{L_T^{\infty} L_x^{\frac{5p}{3}}} \leq \sum_{j \geq 1} \|\phi_j\|_{l^p} + \sum_{j \in J_1(J_c)} \|NS(\Lambda_j \phi_j)\|_{L_T^{\infty} L_x^{\frac{5p}{3}}} < \infty. \tag{106}
\]
Therefore, we get the estimate of $I$. Now we estimate $II$. By the estimate of $e^{\Delta}$ and Theorem 4.1 we have
\[
II \lesssim \|u_{0,n}\|_p + \left\| \sum_{j \in J_1(J)} \Lambda_{j,n}^{3/p} \phi_j \right\|_p.
\]
It follows from (88) and the fact $\|\Lambda_{j,n}^{3/p} \phi_j\|_p = \|\phi_j\|_p$ that
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \left\| e^{\Delta} \left( \sum_{j \in J_1(J)} \Lambda_{j,n}^{3/p} \phi_j + \psi_n^J \right) \right\|_{X_T^p} \lesssim \sup_{n \geq 1} \|u_{0,n}\|_p < \infty. \tag{107}
\]
Combining the estimate on $I$ and $II$, we finally complete the proof of Lemma 6.1. \qed
Next, we prove Lemma 6.3 by following the idea in [24]. However, in the subcritical cases $\Lambda^{3/p}_{j,n}$ has no scaling invariance for the solutions of (NS), i.e., $NS(\Lambda^{3/p}_{j,n}u_0) \neq \Lambda^{3/p}_{j,n}NS(u_0)$.

**Proof of Lemma 6.3.** For $r = 5p/3$, we have
\[
\left\| \sum_{j \in J_1(J)} NS(\Lambda^{3/p}_{j,n} \phi_j) \right\|_{L^r_{x,v}L^1_{t}} = \left\| \sum_{j \in J_1(J)} NS(\Lambda^{3/p}_{j,n} \phi_j) \right\|_{L^1_{t}L^r_{x,v}} \\
\leq \sum_{j \in J_1(J)} \left\| NS(\Lambda^{3/p}_{j,n} \phi_j) \right\|_{L^1_{t}L^r_{x,v}} + \varepsilon(J,n),
\]
where
\[
\varepsilon(J,n) := \left\| \sum_{j \in J_1(J)} NS(\Lambda^{3/p}_{j,n} \phi_j) \right\| - \sum_{j \in J_1(J)} \left\| NS(\Lambda^{3/p}_{j,n} \phi_j) \right\| \leq C_{J,p} \sum_{j,k \in J_1(J), j \neq k} \left\| NS(\Lambda^{3/p}_{j,n} \phi_j) \right\|_{L^1_{t}L^r_{x,v}}.
\]

Here we have used an elementary inequality (cf., e.g.,[27])
\[
\left| \sum_{j=1}^{L} a_j \right|^m - \left| \sum_{j=1}^{L} |a_j|^m \right| \leq C_{L,m} \sum_{1 \leq j,k \leq L: j \neq k} |a_j| |a_k|^{m-1}, \quad 1 < m < \infty. \quad (108)
\]

In order to show the result, we divide the proof into the following three steps.

**Step 1.** For any $j \in J_1$ and $\phi_j \in L^p$, we show that
\[
\lim_{n \to \infty} \| \Lambda^{3/p}_{j,n} \phi_j - (\Lambda_j \phi_j)(\cdot - x_{j,n}) \|_p = 0. \quad (109)
\]
For any $\varepsilon > 0$, one can choose $\varphi_j \in C_0^\infty(\mathbb{R}^d)$ satisfies
\[
\| \varphi_j - \phi_j \|_p < \varepsilon.
\]
We have
\[
\| \Lambda^{3/p}_{j,n} \phi_j - (\Lambda_j \phi_j)(\cdot - x_{j,n}) \|_p \\
\leq \| \Lambda^{3/p}_{j,n} \varphi_j - (\Lambda_j \varphi_j)(\cdot - x_{j,n}) \|_p + \| \Lambda^{3/p}_{j,n} (\varphi_j - \phi_j) \|_p + \| \Lambda_j (\varphi_j - \phi_j) \|_p \\
\leq \| (\Lambda_j / \Lambda_{j,n})^{3/p} \varphi_j (\Lambda_j \cdot / \Lambda_{j,n}) - \varphi_j \|_p + 2\varepsilon. \quad (110)
\]
Noticing that $\lim_{n \to \infty} \lambda_j / \lambda_{j,n} = 1$, from the uniform continuity of $\varphi_j$ we can obtain that
\[
\lim_{n \to \infty} \| (\lambda_j / \Lambda_{j,n})^{3/p} \varphi_j (\lambda_j \cdot / \Lambda_{j,n}) - \varphi_j \|_p = 0. \quad (111)
\]
It follows from (110) and (111) that (109) holds true.

**Step 2.** For any $j \in J_1$ and $\phi_j \in L^p$, we show that
\[
\lim_{n \to \infty} \| NS(\Lambda^{3/p}_{j,n} \phi_j) - NS(\Lambda_j \phi_j)(\cdot - x_{j,n}) \|_{X^p_{x,v}} = 0. \quad (112)
\]
In fact, in view of the translation invariance of (NS), one can regard $x_{j,n} = 0$ in (112). Now, let $u^n := NS(\Lambda^{3/p}_{j,n} \phi_j) - NS(\Lambda_j \phi_j)(x)$, obviously it satisfies the
following perturbation equation
\[
\begin{aligned}
&\partial_t v^n + (v^n \cdot \nabla)v^n - \Delta v^n + (v^n \cdot \nabla)NS(\Lambda_j \phi_j) + (NS(\Lambda_j \phi_j) \cdot \nabla)v^n = -\nabla p, \\
&\text{div } v^n = 0,
\end{aligned}
\]
\[v^n|_{t=0} = \Lambda^{3/p}_j \phi_j - \Lambda_j \phi_j.\]

Applying Proposition B.6 and (109), we see that for sufficiently large \(n > N(T)\),
\[
\|v^n\|_{X^p_T} \lesssim \|\Lambda^{3/p}_{j,n} \phi_j - \Lambda_j \phi_j\|_p \exp\left(CT^{5p(1-3/p)/6}\|NS(\Lambda_j \phi_j)\|_{L^{5p/3}_x L^{5p/3}_t}^{5p/3}\right)
\]
\[:= \delta(n, T) \to 0 \text{ as } n \to \infty.\]

Hence, we have (112) and
\[
\|NS(\Lambda^{3/p}_{j,n} \phi_j)\|_{X^p_T} \leq \|NS(\Lambda_j \phi_j)\|_{X^p_T} + \delta(n, T). \tag{113}
\]

So, by (113) and the definition of \(X^p_T\), one can get
\[
\sum_{j \in J_1(J)} \|NS(\Lambda^{3/p}_{j,n} \phi_j)\|_{L^p_x L^5_t}^{r-1} \leq \sum_{j \in J_1(J)} \|NS(\Lambda_j \phi_j)\|_{L^p_x L^5_t}^{r-1} + \rho(n, J, T),
\]
where \(\rho(n, J, T) \to 0 \text{ as } n \to \infty.\)

\text{Step 3.} We show that
\[
\lim_{n \to \infty} \left\|\|NS(\Lambda^{3/p}_{j,n} \phi_j)\|_{L^p_x L^5_t}^{r-1} - \|NS(\Lambda^{3/p}_{k,n} \phi_k)\|_{L^p_x L^5_t}^{r-1}\right\|_{L^1_t L^1_x} = 0, \tag{114}
\]
for each \(j \neq k, j, k \in J_1(J).\) By (112), it suffices to show that
\[
\lim_{n \to \infty} \left\|\|NS(\Lambda_j \phi_j)\|_{L^p_x L^5_t}^{r-1} - \|NS(\Lambda_k \phi_k)\|_{L^p_x L^5_t}^{r-1}\right\|_{L^1_t L^1_x} = 0. \tag{115}
\]

Since \(C^\infty_0((0, T) \times \mathbb{R}^d)\) is dense in \(L^p_T L^5_x\), it suffices to show (115) by assuming that \(NS(\Lambda_j \phi_j) \in C^\infty_0((0, T) \times \mathbb{R}^d)\) for all \(j \in J_1(J).\) Noticing that \(|x_{j,n} - x_{k,n}| \to \infty,\) we see that the left hand side of (115) is zero when \(n\) is sufficiently large. Hence, we have (114).

Now (101) follows from the estimates of Steps 1–3. Next, we turn to prove (102).
\[
\left\|\sum_{j \in J_1(J)} NS(\Lambda^{3/p}_{j,n} \phi_j)\right\|_p := \left\|\sum_{j \in J_1(J)} NS(\Lambda_j \phi_j)(\cdot - x_{j,n})\right\|_p + \alpha_1(t, J, n).
\]
In view of (112), (103) and (104), we see that
\[
\lim_{n \to \infty} \sup_{0 \leq t \leq T} |\alpha_1(t, J, n)| = 0.
\]

Let us write
\[
\left\|\sum_{j \in J_1(J)} NS(\Lambda_j \phi_j)(\cdot - x_{j,n})\right\|_p := \left\|\sum_{j \in J_1(J)} NS(\Lambda_j \phi_j)\right\|_p + \alpha_2(t, J, n).
\]
It is easy to see that
\[
|\alpha_2(t, J, n)| \leq C_{J, p} \sum_{j \neq k, j, k \leq J_1(J)} \int |NS(\Lambda_j \phi_j)(\cdot - x_{j,n})||NS(\Lambda_k \phi_k)(\cdot - x_{k,n})|^{p-1} dx.
\]
Define
\[
h_n(t) = \int_{\mathbb{R}^3} |NS(\Lambda_j \phi_j)(\cdot - x_{j,n})||NS(\Lambda_k \phi_k)(\cdot - x_{k,n})|^{p-1} dx.
\]
Finally, applying the result of (147), we can bound the above term by
\[
\lim_{n \to \infty} \sup_{0 < t < T} |h_n(t)| = 0 \quad \text{for } j \neq k.
\]
Using the fact \( NS(\lambda_j, \phi_j) \in C([0, T], L^p) \), we can show that \( h_n(t) \) is uniformly bounded and equi-continuous. In addition, due to the orthogonality of cores, it can be easily checked that \( \lim_{n \to \infty} h_n(t) = 0 \), for each \( t \in [0, T] \). Now we apply the Arzelà-Ascoli Theorem, there exists subsequence \((n_k)_{k \geq 1}\) such that
\[
\lim_{n_k \to \infty} \sup_{0 < t < T} h_{n_k}(t) = 0.
\]

Lemma 6.3 is proved. \( \square \)

Let us turn our attention to the proof of Lemma 6.2, which shows the smallness of the forcing term. First we present the main inequalities that will be used, whose proof relies heavily on the space-time estimate of heat kernel.

**Proposition 4.** Let \( 3 < p < \infty \), \( 1/\beta(p) = 1/2 - 3/10p \).

(i) Let \( 1/m = 1/p - \epsilon, \ 0 < \epsilon \leq 2/5p \). We have
\[
\|ve^{t\Delta} u_0\|_{L^\beta(p)(L^p)} \leq CT^{(1-3/m)/2}\|u_0\|_{L^p} \|v\|_{L^{5p/3}L^{5p/3}}.
\]

(ii) Let \( 1/r = 1/p + \epsilon, \ 0 < \epsilon \leq \min\{1/3 - 1/p, 4/5p\} \). We have
\[
\|ve^{t\Delta} u_0\|_{L^\beta(p)(L^{5p/6})} \leq CT^{(1-3/r)/2}\|u_0\|_{L^r} \|v\|_{L^{5p/3}L^{5p/3}}.
\]

(iii) Let \( \sigma(p) = 4p/3, \ s(p) = 3(1/\sigma(p) - 1/2) \). We have
\[
\|ve^{t\Delta} u_0\|_{L^\beta(p)(L^{5p/6})} \leq CT^{(1-3/p)/2}\|u_0\|_{B^s(p), p} \|v\|_{L^{5p/3}L^{5p/3}}.
\]

**Proof.** First, we prove (i). Let \( \nu(p) \) satisfy \( 3/m = 9/5p + 2/\nu(p) \). It is easy to see that \( p < m \leq \min\{\nu(p), 5p/3\} \). Applying Hölder’s inequality, one has that
\[
\|ve^{t\Delta} u_0\|_{L^\beta(p)(L^p)} \leq T^{(1-3/m)/2}\|e^{t\Delta} u_0\|_{L^\beta(p)(L^{5p/3})} \|v\|_{L^{5p/3}L^{5p/3}},
\]
\[
\lesssim T^{(1-3/m)/2}\|u_0\|_{L^p} \|v\|_{L^{5p/3}L^{5p/3}},
\]
where we have used Proposition B.2.

Next, we show (ii). Let \( \rho(p) \) satisfy \( 2/\rho(p) + 9/5p = 3/r \). It follows from the hypothesis that \( 3 < r < 5p/3, \rho(p) \). Using the same way as in the proof of (i), we have
\[
\|ve^{t\Delta} u_0\|_{L^\beta(p)(L^{5p/6})} \leq T^{(1-3/r)/2}\|e^{t\Delta} u_0\|_{L^\beta(p)(L^{5p/3})} \|v\|_{L^{5p/3}L^{5p/3}},
\]
\[
\lesssim T^{(1-3/r)/2}\|u_0\|_{L^r} \|v\|_{L^{5p/3}L^{5p/3}}.
\]
Finally, we prove (iii). Let \( \eta(p) = 3(1/\sigma(p) - 3/5p), \ 2/\kappa(p) + 9/5p = 3/\eta(p). \) Obviously, \( \eta(p) > 0, \kappa(p) > \sigma(p) \). By Proposition B.1, we see that \( B^{\eta(p)}_{\sigma(p)} \to L^{5p/3} \). So, we have
\[
\|ve^{t\Delta} u_0\|_{L^\beta(p)(L^{5p/6})} \lesssim \|v(t)\|_{L^{5p/3}} \|e^{t\Delta} u_0\|_{B^{\eta(p)}_{\sigma(p)}} \|v\|_{L^{5p/3}L^{5p/3}},
\]
\[
\lesssim T^{(1-3/\eta(p))/2}\|e^{t\Delta} u_0\|_{L^\beta(p)(B^{\eta(p)}_{\sigma(p)})} \|v\|_{L^{5p/3}L^{5p/3}}.
\]
Finally, applying the result of (147), we can bound the above term by
\[
CT^{(1-3/p)/2}\|u_0\|_{B^{\eta(p)}_{\sigma(p)}, p} \|v\|_{L^{5p/3}L^{5p/3}}.
\]
A simple calculation shows \( \eta(p) - 2/\kappa(p) = 3(1/\sigma(p) - 1/p) = s_{\sigma(p),p} \). Hence we obtain the desired result.

**Proof of Lemma 6.2.** we rewrite \( G_n^J \) so as to make use of the smallness of profiles without constant scales. We make a cut-off on \( \phi_j \) and denote

\[
U_{n,\eta}^0 := \sum_{j \in J_0(J)} \Lambda_{j,n}^{3/p} \phi_{j,\eta}, \quad U_{n,\eta}^\infty := \sum_{j \notin J_\infty(J)} \Lambda_{j,n}^{3/p} \phi_{j,\eta},
\]

\[
\psi_{n,\eta}^J := \sum_{j \in J_1(J)} \Lambda_{j,n}^{3/p} \phi_{j,\eta} + \psi_n^J.
\]

Obviously

\[
U_{n,\eta}^\text{app,J} := \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) + e^{\Delta}(U_{n,\eta}^0 + U_{n,\eta}^\infty + \psi_{n,\eta}^J).
\]

It follows that

\[
G_n^J = G_n^{J,1} + G_n^{J,2} + G_n^{J,3} + G_n^{J,4},
\]

where

\[
G_n^{J,1} = (2 \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) + e^{\Delta}(U_{n,\eta}^0 + U_{n,\eta}^\infty + \psi_{n,\eta}^J)) \otimes e^{\Delta} U_{n,\eta}^0,
\]

\[
G_n^{J,2} = (2 \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) + e^{\Delta}(U_{n,\eta}^0 + U_{n,\eta}^\infty + \psi_{n,\eta}^J)) \otimes e^{\Delta} U_{n,\eta}^\infty,
\]

\[
G_n^{J,3} = (2 \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) + e^{\Delta}(U_{n,\eta}^0 + U_{n,\eta}^\infty + \psi_{n,\eta}^J)) \otimes e^{\Delta} \psi_{n,\eta}^J,
\]

\[
G_n^{J,4} = \sum_{j,k \in J_1(J); j \neq k} NS(\Lambda_{j,n}^{3/p} \phi_j) \otimes NS(\Lambda_{k,n}^{3/p} \phi_k).
\]

First, we consider the estimate of \( G_n^{J,1} \). Let \( r \) be as in (ii) of Proposition 4, we see that

\[
\|G_n^{J,1}\|_{L_T^{\beta(p)} L_x^{5/6}} \leq C T (1 - 3/r)^{1/2} \|U_{n,\eta}^0\|_{L_T^\infty}
\]

\[
\times \left\| 2 \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) + e^{\Delta}(U_{n,\eta}^0 + U_{n,\eta}^\infty + \psi_{n,\eta}^J) \right\|_{L_T^{5/3} L_x^{5/3}}.
\]

As a result of (106) and (107), one has that

\[
\lim_{J \to \infty} \lim_{n \to \infty} \left\| 2 \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j) + e^{\Delta}(U_{n,\eta}^0 + U_{n,\eta}^\infty + \psi_{n,\eta}^J) \right\|_{L_T^{5/3} L_x^{5/3}}
\]

\[
\leq \sum_{j \in J_1, j \leq j_\epsilon} \| NS(\Lambda_j \phi_j) \|_{L_T^{5/3} L_x^{5/3}} + \sup_{n \geq 1} \| u_{0,n} \|_p < \infty.
\]

In addition, \( r < p \), by (1) of Proposition 1,

\[
\lim_{n \to \infty} \| U_{n,\eta}^0 \|_{L_T^\infty} = 0.
\]

Therefore, we attain

\[
\lim_{J \to \infty} \lim_{n \to \infty} \| G_n^{J,1} \|_{L_T^{\beta(p)} L_x^{5/6}} = 0.
\]

Next, using (i) of Proposition 4 and (2) of Proposition 1, the estimate of \( G_n^{J,2} \) is quite similar to \( G_n^{J,1} \) and we omit the details.
Thirdly, we estimate $G_{n,\eta}^{J_3}$. Let $\sigma(p), s_{\sigma(p),p}$ be as in Proposition 4 (iii), so the following inequality holds,

$$
\|G_{n,\eta}^{J_3}\|_{L_t^{\gamma(p)} L_x^{\gamma(p)/6}} \leq C T^{1/2(1-3/p)} \|\psi_{n,\eta}^J\|_{B^r_{s_{\sigma(p),p}}(\sigma(p))} \times \left\| 2 \sum_{j \in J_1(J)} \right. 
+ \left. \right. 
\begin{array}{l}
\|NS(\Lambda_{j,n}^{3/p} \phi_j) + e^{t\Delta}(U_0^{n,\eta} + U_{n,\eta} + \psi_{n,\eta}^J)\|_{L_t^{5/3} L_x^{5/3}}.
\end{array}
$$

Fix $q = r = \sigma(p) > p, s_{r,p} = s_{\sigma(p),p}$ in Theorem 4.1, then it follows from Proposition 1 (3) that

$$
\lim_{J \to \infty} \limsup_{n \to \infty} \|\psi_{n,\eta}^J\|_{B^r_{s_{r,p},r}(\sigma(p))} = 0.
$$
so we acquire

$$
\lim_{J \to \infty} \limsup_{n \to \infty} \|G_{n,\eta}^{J_3}\|_{L_t^{\gamma(p)} L_x^{\gamma(p)/6}} = 0.
$$

Finally, the smallness of $G_{n,\eta}^{J_4}$ has been given as in Lemma 6.3. Combining the estimate on $G_{n,\eta}^{J_1}$, $G_{n,\eta}^{J_2}$, $G_{n,\eta}^{J_3}$ and $G_{n,\eta}^{J_4}$, we complete the proof of Lemma 6.2. □

7. Orthogonality of the profiles of $L^p$-solutions. This section is intended to prove the orthogonality property of the Navier-Stokes solutions, i.e. formula (94) in Theorem 4.2. Let us introduce some simplified notations first, define

$$
A_n^J : = \sum_{j \in J_1(J)} NS(\Lambda_{j,n}^{3/p} \phi_j), \quad A_n^J : = \sum_{j \in J_1(J)} NS(\Lambda_{j,n} \phi_j)(-x_{j,n}),
$$

$$
B_n^J : = e^{t\Delta} \left( \sum_{j \in J_1(J)} \Lambda_{j,n}^{3/p} \phi_j + \psi_{n}^J \right) = e^{t\Delta}(U_0^{n,\eta} + U_{n,\eta} + \psi_{n,\eta}^J),
$$

where $U_{0,n,\eta}, U_{n,\eta}, \psi_{n,\eta}^J$ are defined as (116). Recall that we have

$$
NS(u_{0,n}) = A_n^J + B_n^J + R_n^J.
$$

Proof of (94). Let $0 < t < \bar{T}$ be fixed, using inequality (108) again and the above expression of $NS(u_{0,n})$, we easily find

$$
\|\gamma_n^J(t)\|_p = \|NS(u_{0,n})\|_p - \sum_{j \in J_1(J)} \|NS(\Lambda_{j,n}^{3/p} \phi_j)\|_p - \|e^{t\Delta} (\sum_{j \in J_1(J)} \Lambda_{j,n}^{3/p} \phi_j + \psi_{n}^J)\|_p
$$

$$
\leq C_R \left(\int_{\mathbb{R}^3} |A_n^J|^p dx - \sum_{j \in J_1(J)} \int_{\mathbb{R}^3} |NS(\Lambda_{j,n}^{3/p} \phi_j)|^p dx \right)
+ \int_{\mathbb{R}^3} |R_n^J|^p dx + \int_{\mathbb{R}^3} |B_n^J|^p dx + |B_n^J| \cdot |R_n^J|^p-1 dx
+ \int_{\mathbb{R}^3} |A_n^{J-1}|^p |R_n^J| + |A_n^J| \cdot |R_n^J|^p-1 dx
+ \int_{\mathbb{R}^3} |A_n^{J+1}|^p |B_n^J| + |A_n^J| \cdot |B_n^J|^p-1 dx
\right).
$$

it follows from (102) and (93) that

$$
\lim_{n \to \infty} \int_{\mathbb{R}^3} |A_n^J|^p dx - \sum_{j \in J_1(J)} \int_{\mathbb{R}^3} |NS(\Lambda_{j,n}^{3/p} \phi_j)|^p dx = 0,
$$

$$
\lim_{J \to \infty} \limsup_{n \to \infty} \int_{\mathbb{R}^3} |R_n^J|^p dx = 0.
$$
Concerning the terms $|B_n^J||R_n^J|$ and $|B_n^J| \cdot |R_n^J|^{p-1}$, by Hölder’s inequality we have
\[
\int_{\mathbb{R}^3} (|B_n^J|^{p-1} |R_n^J| + |B_n^J| \cdot |R_n^J|^{p-1}) \, dx \leq \|B_n^J\|_p^{p-1} \|R_n^J\|_p + \|B_n^J\|_p \|R_n^J\|^{p-1}.
\]
By (107) and the estimate on $R_n^J$, one has that
\[
\lim_{J \to \infty} \lim_{n \to \infty} \sup_{\mathbb{R}^3} (|B_n^J|^{p-1} |R_n^J| + |B_n^J| \cdot |R_n^J|^{p-1}) \, dx = 0.
\]
We can deal with the terms $|A_n^J|^{p-1} |B_n^J|$ and $|A_n^J| \cdot |B_n^J|^{p-1}$ similarly. Thus we are only left with terms $|A_n^J|^{p-1} |B_n^J|$ and $|A_n^J| \cdot |B_n^J|^{p-1}$. Moreover, in view of Lemma 6.3, the desired result follows if we prove the following conclusion.

**Proposition 5.** Let $\tilde{A}_n^J, B_n^J$ be as above, $0 < t < \tilde{T}$ be fixed, then it holds
\[
\lim_{J \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^3} (|\tilde{A}_n^J|^{p-1} |B_n^J| + |\tilde{A}_n^J| \cdot |B_n^J|^{p-1}) \, dx = 0.
\]

**Proof.** Let $\epsilon > 0$ be arbitrary, as we know
\[
B_n^J = e^{t\Delta}(U_{n,\eta}^0 + U_{n,\eta}^\infty + \psi_{n,\eta}^J).
\]
For $|\tilde{A}_n^J|^{p-1} |B_n^J|$, it suffices to verify the smallness of
\[
|\tilde{A}_n^J|^{p-1} |e^{t\Delta}U_{n,\eta}^0|, \ |\tilde{A}_n^J|^{p-1} |e^{t\Delta}U_{n,\eta}^\infty| \text{ and } |\tilde{A}_n^J|^{p-1} |e^{t\Delta}\psi_{n,\eta}^J|.
\]
Let us estimate them separately.

First, we show that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} |\tilde{A}_n^J|^{p-1} |e^{t\Delta}U_{n,\eta}^0| \, dx = 0.
\]
For each $j \in J_1$, we approximate $NS(\Lambda_j \phi_j)$ by a smooth function with compact support in $L^p$ spaces, denote this function as $\Theta_{j,t}(x)$, so we see
\[
\int_{\mathbb{R}^3} |\tilde{A}_n^J|^{p-1} |e^{t\Delta}U_{n,\eta}^0| \, dx \leq C_{p,J} \sum_{j \in J_1(J)} \|NS(\Lambda_j \phi_j) - \Theta_{j,t}||U_{n,\eta}^0||_p
\]
\[
+ C_{p,J} \left( \sum_{j \in J_1(J)} \|\Theta_{j,t}||^{p-1}_{L^{a'}(\eta)} \right) ||U_{n,\eta}^0||_{L^a}.
\]
where we have used Hölder inequality in the last inequality and $1 < a < p$, $a'$ meets $1/a + 1/a' = 1$. So we have
\[
\int_{\mathbb{R}^3} |\tilde{A}_n^J|^{p-1} |e^{t\Delta}U_{n,\eta}^0| \, dx \to 0 \text{ as } n \to \infty.
\]
Similarly, one can show that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} |\tilde{A}_n^J|^{p-1} |e^{t\Delta}U_{n,\eta}^\infty| \, dx = 0.
\]
Finally, we prove that
\[
\lim_{J \to \infty} \lim_{\eta \to \infty} \lim_{n \to \infty} \int_{\mathbb{R}^3} |\tilde{A}_n^J|^{p-1} |e^{t\Delta}\psi_{n,\eta}^J| \, dx = 0.
\]
Assume $0 < t < T < \bar{T}$, let $\bar{J} > J_c$ be specified later, $J_c = J_c(T)$ is given by Remark 4, choose $J > \bar{J}$ sufficiently large. Write $A_n^J$ into two parts:

$$A_n^J = \sum_{j \in J_1, J \leq j \leq \bar{J}} NS(\Lambda_j \phi_j) + \sum_{j \in J_1, j < J} NS(\Lambda_j \phi_j).$$

By Hölder’s inequality,

$$\int_{\mathbb{R}^3} \left| \sum_{j \in J_1, j \leq J} NS(\Lambda_j \phi_j) \right|^{p-1} |e^{t \Delta} \psi_{n,\eta}^J| dx$$

$$\leq \left\| \sum_{j \in J_1, j \leq J} NS(\Lambda_j \phi_j) \right\|_p^{p-1} \|\psi_{n,\eta}^J\|_p$$

$$\leq \left( \sum_{j \in J_1, j \leq J} \|NS(\Lambda_j \phi_j)\|_p^p + \varepsilon(J, n) \right)^{\frac{p-1}{p}} \|\psi_{n,\eta}^J\|_p. \quad (117)$$

Here $\lim_{n \to \infty} \varepsilon(J, n) = 0$ and we have used (102) again. Moreover, for any $J, \eta$, we have

$$\limsup_{n \to \infty} \|\psi_{n,\eta}^J\|_p = \limsup_{n \to \infty} \left\| \sum_{j \in J_1(J)} A_{j, n}^J \phi_j^n + \psi_n^J \right\|_p,$$

$$\leq \limsup_{n \to \infty} \left\| \sum_{j \in J_1(J)} A_{j, n}^J \phi_j^n \right\|_p + \limsup_{n \to \infty} \|\psi_n^J\|_p.$$

Using (89), we see that there exists a $N_1 = N_1(J, \eta)$, such that for any $n > N_1$, $\|\psi_{n,\eta}^J\|_p \leq C \sup_{n \geq 1} \|u_0, n\|_p$. Now let $n > N_1$, one sees that (117) has an upper bound

$$C \left( \sum_{j \in J_1, J \leq j \leq \bar{J}} \|NS(\Lambda_j \phi_j)\|_p^p \right)^{\frac{p-1}{p}} + \varepsilon(J, n)^{\frac{p-1}{p}} \sup_{n \geq 1} \|u_0, n\|_p. \quad (118)$$

In view of Remark 4 and noticing that $\bar{J} > J_c$ and $L^p$ norm invariance of $\Lambda_j \phi_j$, we can further bound (118) by

$$C \left( \sum_{j \in J_1, J \leq j \leq \bar{J}} \|\phi_j^n\|_p^{p-1} \right)^{\frac{p-1}{p}} + \varepsilon(J, n)^{\frac{p-1}{p}} \sup_{n \geq 1} \|u_0, n\|_p.$$

Since $\sum_{j \geq 1} \|\phi_j^n\|_p$ is convergent, we now fix $\bar{J} = \bar{J}(\varepsilon)$ such that

$$C \sum_{j \in J_1, J \leq j \leq \bar{J}} \|NS(\Lambda_j \phi_j)\|_p^{p-1} \sup_{n \geq 1} \|u_0, n\|_p \leq \varepsilon. \quad (119)$$

Using the fact that $\lim_{n \to \infty} \varepsilon(J, n) = 0$, we conclude there exists a $N_2 = N_2(\varepsilon, J)$, such that for all $n > N_2$

$$C |\varepsilon(J, n)|^{\frac{p-1}{p}} \sup_{n \geq 1} \|u_0, n\|_p < \varepsilon. \quad (120)$$

Gathering the estimates (119) and (120), we see that for $n > \max\{N_1, N_2\}$

$$\int_{\mathbb{R}^3} \left| \sum_{j \in J_1(J) \geq J} NS(\Lambda_j \phi_j) \right|^{p-1} |e^{t \Delta} \psi_{n,\eta}^J| dx < 2\varepsilon. \quad (121)$$

Considering the estimate of $\sum_{j \in J_1(J)} NS(\Lambda_j \phi_j)$, we approximate each $NS(\Lambda_j \phi_j)$ by $\Theta_{j,t}(x)$ in $L^p$ spaces,

$$\int_{\mathbb{R}^3} \left| \sum_{j \in J_1(J)} NS(\Lambda_j \phi_j) \right|^{p-1} |e^{t \Delta} \psi_{n,\eta}^J| dx$$
\[
\leq C_p \left( \sum_{j \in J_1(J)} \| NS(\Lambda_j \phi_j) - \Theta_{j,t} \|_p \right)^{p-1} \| \psi_{n,\eta}^J \|_p \\
+ C_p t^{s_\sigma(p),p/2} \| \psi_{n,\eta}^J \|_{\dot{B}^s_{r,p},\infty} \left( \sum_{j \in J_1(J)} \| \Theta_{t,x} \|_{L^s(p-\sigma(p),p')} \right)^{p-1}.
\]

Here we have used Hölder inequality and equivalent definition of Besov spaces in deriving the last inequality. From the above, we know \( n > N_1, \| \psi_{n,\eta}^J \|_p \leq C \sup_{n \geq 1} \| u_{0,n} \|_p \), so we choose \( \Theta_{j,x} \) sufficiently close to \( NS(\Lambda_j \phi_j) \) so that

\[
C_p \left( \sum_{j \in J_1(J)} \| NS(\Lambda_j \phi_j) - \Theta_{j,t} \|_p \right)^{p-1} \| \psi_{n,\eta}^J \|_p < \epsilon. \tag{122}
\]

while by (87) and Proposition 1 (iii), note we already take \( q = r = \sigma(p), s_{r,p} = s_{\sigma(p),p} \) there,

\[
\lim_{j \to \infty} \lim_{n \to \infty} \limsup_{n \to \infty} \| \psi_{n,\eta}^J \|_{\dot{B}^s_{r,p},\sigma(p)} = 0.
\]

As \( J \) is independent of \( J, \eta, 0 < t < T \) is fixed, so \( \exists J(\epsilon), \forall J > J(\epsilon), \exists \eta(J, \epsilon), \forall \eta > \eta(J, \epsilon), \exists \hat{N} = \bar{N}(J, \epsilon), \) for \( \forall n > \hat{N} \)

\[
C_p t^{1/2s_\sigma(p),p} \| \psi_{n,\eta}^J \|_{\dot{B}^s_{r,p},\infty} \left( \sum_{j \in J_1(J)} \| \Theta_{t,x} \|_{L^s(p-\sigma(p),p')} \right)^{p-1} < \epsilon. \tag{123}
\]

Gathering (121)-(123), we know for any \( \epsilon > 0, \exists J(\epsilon), \forall J > J(\epsilon), \exists \eta(J, \epsilon), \forall \eta > \eta(J, \epsilon), \exists \hat{N} = \max \{ \bar{N}, N_1, N_2 \} \), s.t for \( \forall n > \hat{N} \)

\[
\int_{\mathbb{R}^3} |\tilde{A}_n^J|^{p-1} |e^{\lambda \Delta} \psi_{n,\eta}^J| \ dx < 4\epsilon.
\]

Using the same way as above, one can estimate \( |\tilde{A}_n^J| \cdot |B_n^J|^{p-1} \) and the details are omitted. \( \square \)

8. **Compactness of the set \( \mathcal{M}_p(T_*) \).** We will give a proof to Theorem 1.5 in this section. Now let the space dimension \( d = 3 \). First, it can be inferred from the definition of \( A_\epsilon^{\sigma,p} \) that there exists a sequence \( u_{0,n} \in L_p(\mathbb{R}^3) \) satisfying

\[
\lim_{n \to \infty} T(u_{0,n}) \| u_{0,n} \|_p^{p_r} = A_{\epsilon_r}^{p_r}, \quad T(u_{0,n}) < \infty. \tag{124}
\]

In view of Proposition A.7, the mild solution \( NS(u_{0,n}) \) develops a singularity at some \( z_n := (x_n, T(u_{0,n})) \). By the scaling and translation invariance of the (NS) system, one can assume \( x_n = 0, T(u_{0,n}) = T_* \). Indeed, let

\[
\tilde{u}_{0,n}(x) := \lambda_n u_{0,n}(\lambda_n x + x_n), \quad \lambda_n := \sqrt{\frac{T(u_{0,n})}{T_*}}.
\]

Now the desired properties follows and (124) holds for \( \tilde{u}_{0,n} \). Recalling the definition of \( r_p(T_*) \), we know from (124) that

\[
\lim_{n \to \infty} \| u_{0,n} \|_p = r_p(T_*). \tag{125}
\]

Let \( u^n := NS(u_{0,n}) \), in the sequel, we will perform local energy estimate for \( u^n \), which is divided into two parts, namely, local energy estimate near the initial time and away from the initial time.
8.1. Local energy estimate near initial time. The main result in this part is the following conclusion.

**Proposition 6.** There exists a time $\bar{T} < \min\{1, T_\ast\}$ which depends only on $p, T_\ast$, such that for any $0 < t < \bar{T}$, we have

$$
sup_{0 < t < \bar{T}} \sup_{x_0 \in \mathbb{R}^3} \|u^n(t)\|_{L^2(B(x_0, 1))}^2 + \sup_{x_0 \in \mathbb{R}^3} \int_0^\bar{T} \int_{B(x_0, 1)} |\nabla u^n|^2 dx dt \leq c,
$$

where $c$ depends only on $p, T_\ast, 0 < \kappa(p) < 1$ depends on $p$ only, $p^n$ is the associated pressure, $c^n_{x_0}$ is given by

$$
c^n_{x_0} = \frac{1}{4\pi} \int_{|y - x_0| \geq 2} K(x_0 - y) : u^n(y) \otimes u^n(y) dy, \quad K(x) = \nabla^2 \left( \frac{1}{|x|} \right).
$$

**Proof.** The argument is essentially the same as that in [54]. First, splitting the solution into two parts $u^n = u_n^1 + u_n^2$, where $u_n^1 = e^{\Delta t} u_{0,n}$ is a solution to the heat equation and $u_n^2$ solves the following perturbation equation on $\mathbb{R}^3 \times (0, \bar{T})$,

$$
\partial_t u_n^2 - \Delta u_n^2 + u_n \cdot \nabla u_n + \nabla p^n = 0, \quad \text{div} u_n^2 = 0, \quad u_n^2(x, 0) = 0.
$$

One then estimates the local energy estimate of $u_n^1$ and $u_n^2$ separately. By (125), we can assume $\|u_{0,n}\|_p \leq 2r_p(T_\ast), \forall n \geq 1$. First it is quite obvious that

$$
\|u_n^1\|_{L^\infty(\mathbb{R}^3; L^p(\mathbb{R}^3))} \leq \|u_{0,n}\|_{L^p} \leq 2r_p(T_\ast), \tag{127}
$$

then, using the equation $u_n^1$ solves, it is not hard to get that for any $T \leq 1$,

$$
\sup_{0 < t < T} \sup_{x_0 \in \mathbb{R}^3} \|u_n^1(t)\|_{L^2(B(x_0, 1))}^2 + \sup_{x_0 \in \mathbb{R}^3} \int_0^T \int_{B(x_0, 1)} |\nabla u_n^1|^2 dx dt \leq c,
$$

where $c$ has the required dependence. The estimate of $u_n^2$ is a little complicated, but it seems simpler in our case because of the subcriticality of the initial data, we do not intend to give details, but emphasize that (127) is enough to get

$$
\sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B(x_0, 3/2)} |p^n - c^n_{x_0}|^2 dx dt \leq c \left[ \gamma(t) + \int_0^t 1 + \alpha^2(s) ds \right], \tag{128}
$$

and

$$
\alpha(t) + \beta(t) \leq c \left[ t^{\kappa(p)} + \int_0^t 1 + \alpha(s)(1 + \|u_n^1\|_{L^p(B(x, 1))}^r) + \alpha^3(s) \right],
$$

where $c$ relies upon $p, T_\ast$ only, $\kappa(p) < 1$ depends on $p$, $r$ meets $2/r + 3/p = 1$ and

$$
\alpha(t) := \sup_{x_0 \in \mathbb{R}^3} \|u_n^2(t)\|_{L^2(B(x_0, 1))}, \quad \beta(t) := \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B(x_0, 1)} |\nabla u_n^2(x, \tau)|^2 dx d\tau,
$$

$$
\gamma(t) := \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B(x_0, 1)} |u_n^2(x, \tau)|^3 dx d\tau.
$$

Consequently, one can use a standard bootstrap argument together with the estimate of $u_n^1$ and the interpolation inequality below

$$
\gamma(t) \lesssim \left( \int_0^t \alpha^2(s) ds \right)^{\frac{1}{2}} \left( \beta(t) + \int_0^t \alpha(s) ds \right)^{\frac{1}{2}}, \tag{129}
$$

where $\alpha(s)$, $\beta(s)$, and $\gamma(s)$ are positive solutions to the system

$$
\begin{align*}
\frac{d\alpha}{ds} &= \alpha^2 + \beta, \\
\frac{d\beta}{ds} &= \frac{\alpha^3}{\gamma}, \\
\gamma &= \left( \int_0^s \alpha^2(s) ds \right)^{\frac{1}{2}} \left( \beta + \int_0^s \alpha(s) ds \right)^{\frac{1}{2}}.
\end{align*}
$$

Thus, there exists a time $\bar{T} = \min\{\bar{T}_1, \bar{T}_2, \bar{T}_3\}$ such that for all $0 < t < \bar{T}$, we have

$$
\|u^n(t)\|_{L^\infty(B(x_0, 1))} \leq c, \quad \|\nabla u^n(t)\|_{L^2(B(x_0, 1))} \leq c, \quad \text{and} \quad \|u_n^1(t)\|_{L^2(B(x_0, 1))} \leq c.
$$

This completes the proof of Proposition 6.
to conclude the proof, for more details, we refer reader to [54].

8.2. Local energy estimate for large time. We shall obtain local energy estimate for the solution sequence \( \{u^n\} \) on domain \( \mathbb{R}^3 \times [T, T_*] \), thus leading to the fact that \((u^n, p^n)\) forms a pair of local energy solution, whose definition is given in the next subsection. To begin with, we introduce weak \( L^p \) solution, which is only a minor modification of the weak Calderon solution introduced in [2].

**Definition 8.1.** Let \( 3 < p < q < \infty, u_0 \in L^p(\mathbb{R}^3) \) be divergence free, we say a distribution \( u \) is a weak \( L^p \) solution to (2) on \( Q_T \) subject to the decomposition

\[
u = U + V, \quad T < T(V_0),
\]

where \( V \) is the mild solution associated with initial data \( V_0 \in L^q(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \), \( U_0 \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \), \( U \in L^\infty_1 L^2_2(Q_T) \cap L^2_1 H^1(Q_T) \) satisfies the conditions below:

(i) \( U \) solves the equation

\[
\partial_t U - \Delta U + U \cdot \nabla U + V \cdot \nabla U + U \cdot \nabla V + \nabla P = 0, \quad \text{div} U = 0,
\]

in the sense of distribution on \( Q_T \) with some \( P \in L^1_3 L^{3/2}_2(Q_T) \) + \( L^1 T \) + \( L^2(T) \); (ii) \( (U, P) \) verifies the local energy estimate with lower order terms:

\[
\partial_t |U|^2 + 2 \nabla |U|^2 \leq \Delta |U|^2 - \text{div}((|U|^2 + 2P)U)
\]

\[
- \text{div}(|U|^2 V) - 2U \text{div}(V \otimes U)
\]

holds distributionally for non-negative test functions supported in \( Q_T \);

(iii) \((u, p)\) forms a pair of suitable weak solution\(^2\) on any bounded domain of \( Q_T \);

(iv) \( U \) is weakly continuous in \( L^2(\mathbb{R}^3) \), and strongly continuous in \( L^2(\mathbb{R}^3) \) at \( t = 0 \);

(v) \( U \) meets the global energy inequality

\[
\int_{\mathbb{R}^3} |U(x, t_1)|^2 dx + 2 \int_{t_0}^{t_1} \int_{\mathbb{R}^3} |\nabla U|^2 dx dt
\]

\[
\leq \int_{\mathbb{R}^3} |U(x, t_0)|^2 dx + 2 \int_{t_0}^{t_1} \int_{\mathbb{R}^3} V \otimes U : \nabla U dx dt
\]

for a.e. \( t_0 > 0 \) and all \( t_1 > t_0 \) including \( t_0 = 0 \).

In the above statement, \( U \text{div}(V \otimes U) \) is the distribution

\[
\langle U \text{div}(V \otimes U), \phi \rangle := - \int V \otimes U : (U \otimes \nabla \phi + \phi \nabla U).
\]

One may ask whether weak \( L^p \) solution exists or not, the answer is affirmative by virtue of results in [2].

**Proposition 7.** Let \( 3 < p < q < \infty, u_0 \in L^p(\mathbb{R}^3) \), then there exists at least one weak \( L^p \) solution to (2) on \( Q_T \) subject to the decomposition \( u_0 = V_0 + U_0 \), where \( T < T(V_0), V_0 \in L^q \cap L^p, U_0 \in L^2 \cap L^p \).

**Proposition 8.** Let \( u_0 \in L^p(\mathbb{R}^3), u \) is a weak \( L^p \) solution on \( Q_T \), then \( u \) coincides with the mild solution \( NS(u_0) \) on the time interval \( [0, \min\{T, T(u_0)\}) \).

**Remark 5.** Proposition 8 is a weak-strong uniqueness result for weak \( L^p \) solution, which can be proved by adapting the standard energy method. We choose to omit the details.

\(^2\)see Definition A.1.
Now we proceed to estimate $u^n$. Our idea is to construct a weak $L_p$ solution $\hat{u}^n$ which coincides with the mild solution $u^n$ on the time interval $[0,T_*)$, then $u^n$ is endowed with the same properties as $\hat{u}^n$, which is sufficient for our purpose. Let $u_{0,n} = u_{0,1} + u_{0,2,n}^2$ with

$$u_{0,1} := \mathbb{P}(u_{0,n}|t_{[0,n]|\leq \lambda}), \quad u_{0,2,n} = u_{0,n} - u_{0,1}^2,$$

(133)

$I_A$ is the characteristic function on the set $A$. It is clear that for $3 < p < q < \infty$,

$$\|u_{0,1}\|_q^q \leq C(\lambda^{q-p})^p \|u_{0,n}\|_p^p \leq C2^{p}(\lambda^{q-p}r_p^p(T_*)), \quad \|u_{0,2,n}\|_2^2 \leq C(\lambda^{2-p})\|u_{0,n}\|_p^p.$$

By Proposition 7, one can construct a weak $L_p$ solution $\hat{u}^n = V^n + U^n$ corresponding to the decomposition $u_{0,n} = u_{0,1} + u_{0,2,n}$ on some $Q_{T_n}$. In view of local Cauchy theory for subcritical $L^q$ initial data, we can choose $T_n = 2T_*$ provided $\lambda$ is sufficiently small. Moreover, by Theorem A.5,

$$\sup_{0<t<T_n} t^{k/2}\|\nabla^k V^n\|_q \leq C_k\|u_{0,n}\|_q, \quad k = 0, 1.$$  

(134)

In addition, $U^n$ meets the global energy estimate (132), utilizing Gronwall inequality, one can see

$$\int_{\mathbb{R}^3} |U^n(x,t)|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\nabla U^n(x,s)|^2 dxds \leq \|u_{0,1}\|_q^q \exp\left( C \int_0^T \|V^n(s)\|_q^q ds \right),$$

(135)

holds for any $0 < t < T_n = 2T_*$, $r$ is such that $2/r + 3/q = 1$. Gathering results (134) and (135), we can see for any $\delta > 0$,

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{\delta < t < T_n} \|\hat{u}^n\|_{L^2(B(x_0,1))}^2 + \sup_{x_0 \in \mathbb{R}^3} \int_{B(x_0,1)} \int_0^{T_n} |\nabla u^n|^2 dxdt \leq c,$$

with $c$ independent of $n$. Now it follows Proposition 8 that

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{\tilde{T} < t < T_*} \|u^n\|_{L^2(B(x_0,1))}^2 + \sup_{x_0 \in \mathbb{R}^3} \int_{B(x_0,1)} \int_{\tilde{T}}^{T_*} |\nabla u^n|^2 dxdt \leq c,$$

where $\tilde{T}$ is defined in Proposition 6. Then one can appeal to (128) and (129) to infer

$$\sup_{x_0 \in \mathbb{R}^3} \int_{\tilde{T}}^{T_*} \int_{B(x_0,3/2)} |p^n - c_{x_0}^n|^2 dxdt \leq c,$$

with constant $c$ independent of $n$. In summary, collecting the estimates near and away from initial time, one can have

**Proposition 9.** There exists a constant $c$, independent of $n$, such that

$$\sup_{0 < t < T_*} \sup_{x_0 \in \mathbb{R}^3} \|u^n(t)\|_{L^2(B(x_0,1))}^2 + \sup_{x_0 \in \mathbb{R}^3} \int_0^{T_*} \int_{B(x_0,1)} |\nabla u^n|^2 dxdt \leq c,$$

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^{T_*} \int_{B(x_0,3/2)} |p^n - c_{x_0}^n|^2 dxdt \leq c,$$

where $c_{x_0}^n$ is the same as that in Proposition 6.

**Remark 6.** Let $3 < p < \infty$, based on the above conclusion, it can be inferred that the mild solution $(NS(u_0), P)$ to (2) forms a pair of local energy solution (see Definition 8.2) on $\mathbb{R}^3 \times (0, T(u_0))$, $P$ is the associated pressure.
8.3. Limiting procedure and proof of Theorem 1.5. We shall show the limit of \((u^n, p^n)\) forms a pair of local energy solution, exactly as that did in [54]. To proceed, we first recall the important notion-local energy solution, which was first introduced by Lemarie-Rieusset, see [44], and extensively used in [53, 34, 56, 54]. The following version is from [54].

**Definition 8.2.** We call a pair of functions \(u\) and \(p\) defined in \(Q_T\) a local energy solution to Cauchy problem (2) if they satisfy the following conditions:

1. \(u \in L^\infty((0, T); L^2_{\text{unif}}), \nabla u \in L^2((0, T); L^2_{\text{unif}})\), \(p \in L^{3/2}((0, T); L^{3/2}_{\text{loc}}(\mathbb{R}^3))\);
2. the function \(t \mapsto \int_{\mathbb{R}^3} u(x, t) \cdot \phi(x) dx\) is continuous on \([0, T]\) for any compactly supported function \(\phi \in L^2(\mathbb{R}^3)\), and for any compact set \(K\), \(\|u(t) - u_0\|_{L^2(K)} \to 0\) as \(t \to 0^+\);
3. \(u\) and \(p\) meet (2) in the sense of distributions and fulfill the local energy inequality, i.e.

\[
\partial_t |u|^2 + 2|\nabla u|^2 \leq \Delta |u|^2 - \text{div}((|u|^2 + 2p)u),
\]

holds distributionally for all non-negative test functions supported in \(Q_T\);
4. for any \(x_0 \in \mathbb{R}^3\), there exists a function \(c_{x_0} \in L^{3/2}(0, T)\) such that

\[
p_{x_0}(x, t) = p(x, t) - c_{x_0}(t) = p^1_{x_0}(x, t) + p^2_{x_0}(x, t),
\]

for \((x, t) \in B(x_0, 3/2) \times (0, T)\), where

\[
p^1_{x_0}(x, t) = -\frac{1}{3} |u|^2 + \frac{1}{4\pi} \int_{B(x_0, 2)} K(x - y) : u(y, t) \otimes u(y, t) dy,
\]

\[
p^2_{x_0}(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3 \setminus B(x_0, 2)} (K(x - y) - K(x_0 - y)) : u(y, t) \otimes u(y, t) dy
\]

and \(K(x) = \nabla^2(1/|x|)\).

The next result shows the uniqueness of local energy solution under some additional assumptions, for a proof, see [45].

**Theorem 8.3.** Suppose \((u, p)\) and \((v, q)\) are two local energy solutions on \(Q_T\) with the same initial data \(u_0 \in L^2_{\text{unif}}\), assume further \(v \in L^2(0, T; L^\infty)\), then \(u = v\) on \(Q_T\) almost everywhere.

With these preparations, we now turn to the proof of Theorem 1.5.

**Proof of Theorem 1.5.** As \(u_0, n\) is bounded in \(L^p(\mathbb{R}^3)\), without loss of generality, we can assume \(u_0, n \rightharpoonup v_0\) weakly in \(L^p\). By Proposition 9, (126) and properties of \(u^{n, 1}\), one can argue similarly as [54] to deduce that there exists a subsequence which is still denoted by \(n\), and functions \(v, q\), such that for any \(R > 0\),

\[u^n \rightharpoonup v\]

strongly in \(L^3(B(0, R) \times (0, T_\star))\), and for the pressure, it holds for any \(M \in \mathbb{N}\), there exists a subsequence \(c^M_\star \in L^{3/2}(0, T_\star)\), verifying

\[\tilde{p}^M_\star = p^n - c^M_\star \rightharpoonup q\]

weakly in \(L^{3/2}(0, T_\star; L^{3/2}(B(0, M)))\). Furthermore, \((v, q)\) constitutes a local energy solution on \(\mathbb{R}^3 \times (0, T_\star)\) associated with \(v_0\). As we already assume that \(u^n\) develops a singular point at \(z_0 = (0, T_\star)\), applying Proposition A.4, one sees \(v\) is also singular.
at $z_0$. Besides, according to Theorem 8.3, the mild solution $NS(v_0)$ coincides with $v$ for some time, therefore
\[
T(v_0) \leq T_*, \text{ and } T(v_0)\|v_0\|_{L^p}^p \geq A^*_p,
\]
while
\[
\|v_0\|_{L^p}^p \leq \liminf_{n \to \infty} \|v_{0,n}\|_{L^p}^p = \frac{A^*_p}{T_*}.
\]
so $T(v_0) = T_*$ and $\|v_0\|_p = r_p(T_*)$, as a byproduct, the uniform convexity of $L^p$ space finally leads to
\[
\lim_{n \to \infty} \|u_{0,n} - v_0\|_p = 0.
\]
Up to now, the proof of Theorem 1.5 is completed.

In the above proof, local energy solution and uniqueness play important roles, recently the notion of weak solution with infinite energy has again been extended by [55], where global weak $L^3$ solution is introduced, an interesting and direct application is to reprove the existence of $L^3$- minimal singularity-generating data, compactness result and blowup criterion in $L^3(R^3)$, instead of using local energy solution, see also other kinds of extensions of weak solutions in [7, 2]. We would also like to give an alternative proof of Theorem 1.5 by using weak $L^p$ solution in the limit procedure. First, we show that weak $L^p$ solution is stable with respect to weak convergence.

**Theorem 8.4.** Assume $u_{0,n} \to v_0$ weakly in $L^p$, $(\hat{u}^n, \hat{q}^n)$ is the weak $L^p$ solution subject to $u_{0,n} = u_{0,n}^1 + u_{0,n}^2$, on $Q_T$ with some $T > 0$, $u_{0,n}^1$ is the same as that in (133), then there exist a subsequence (still denoted by $n$) and functions $v, q$, such that
\[
\hat{u}^n \to v \text{ strongly in } L^3_{loc}(Q_T), \quad \hat{q}^n \to q \text{ weakly in } L^{3/2}_{loc}(Q_T).
\]
Moreover $(v, q)$ forms a weak $L^p$ solution to (2) on $Q_T$ with initial data $v_0$.

**Proof.** Note
\[
\|u_{0,n}\|_{L^q}^q \leq C\lambda^{2-p}\|u_{0,n}\|_p^p \leq C2^p\lambda^{2-p}r_p^p(T_*),
\]
as explained before, we can always assume $T > T_*$ if $\lambda$ is taken sufficiently small, then fix such $\lambda$. Let $\hat{u}^n = V^n + U^n$, where $V^n = NS(u_{0,n}^1)$, $U^n$ solves (130) with initial data $u_{0,n}^2$. Next, we treat the convergence of $V^n$ and $U^n$ separately, the proof goes along the same lines as [2].

**1. Convergence of $V^n$.** As a result of Theorem A.5, we know
\[
\sup_{0 < t < T} t^{j + \frac{1}{2}} \|\partial_j^k \nabla^k V^n\|_q \leq c\|u_{0,n}^1\|_{L^r}, \quad j, k \in \{0, 1\}. \tag{136}
\]
Obviously, $\nabla V^n \in L^p_t L^q_x(Q_T)$ with any $1 \leq \alpha < 2$. In addition, $V^n$ meets
\[
\partial_t V^n = \Delta V^n - P\text{div}(V^n \otimes V^n),
\]
Then for any test function $\psi(x) \in C^\infty_{0}(R^3)$, it holds
\[
|\langle \partial_t V^n, \psi \rangle| \lesssim \|V^n\|_q \|
abla \psi\|_{q'} + \|V^n\|_{L^q}^2 \|
abla \psi\|_{(q/2)'} \lesssim (\|V^n\|_q^2 + \|V^n\|_q^2) \|
abla \psi\|_{W^2,q'}.
\]
where $p'$ stands for the conjugate number of $p$, $1/p + 1/p' = 1$. So we have
\[
\partial_t V^n \in L^\infty(0, T; W^{-2,q'}(R^3)).
\]
Here $W^{-2,q'}(\Omega)$ is the dual space of $W^{2,q}_0(\Omega)$. Now for any $R > 0$, making use of Aubin-Lions Lemma\(^3\), one can have

$$V^n \to V \text{ strongly in } L^\alpha(0, T; L^q(B(R))) \cap C([0, T]; W^{-2,q'}(B(R))),$$

Specifically,

$$V^n(t) \to V(t)$$

distributionally for all $t \in [0, T]$. Besides,

$$V^n \to V \text{ weakly-* in } L^\infty_t L^q_x(Q_T),$$

By interpolation, one can see $V^n(t) \to V(t)$ strongly in $L^\beta(0, T; L^q(B(R))$ for any $\alpha \leq \beta < \infty, R > 0$. Let $V_0$ be the weak limit of $u^2_{0,n}$, by the convergence properties, it is not hard to know that $V$ solves (NS) with initial data $V_0$, and $V \in L^\infty_t L^2_x(Q_T)$, so $V$ coincides with the mild solution $NS(V_0)$, as $V \in L^2_t L^2_{unif,x}$, see [44]. By Calderon-Zygmund estimate, the pressure $Q^n$ associated with $V^n$ converges weakly in $L^{3/2}({Q}_T)$ to the pressure $Q$ associated with $V$.

2. Convergence of $U^n$. Let $U_0 \in L^2 \cap L^p$ be the weak limit of $u^2_{0,n}$ in $L^2 \cap L^p$. For $0 < t < T$, each $U^n$ satisfies the global energy inequality,

$$\int_{\mathbb{R}^3} |U^n(x, t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla U^n|^2 \, dx \, ds \leq \int_{\mathbb{R}^3} |u^2_{0,n}|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} V^n \otimes U^n : \nabla U^n \, dx \, ds.$$

Noticing that

$$\int_0^t \int_{\mathbb{R}^3} V^n \otimes U^n : \nabla U^n \, dx \, ds \leq C \int_0^t \|V^n\|_{H^r}^r \|U^n\|_{H^2}^2 \, ds + \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |\nabla U^n|^2 \, dx \, ds,$$

where $r$ is such that $2/r + 3/q = 1$. Now applying Gronwall inequality, one can get

$$\int_{\mathbb{R}^3} |U^n(x, t)|^2 \, dx + \int_0^t \int_{\mathbb{R}^3} |\nabla U^n(x, s)|^2 \, dx \, ds \leq \|u^2_{0,n}\|_2^2 \exp \left( C \int_0^t \|V^n(s)\|_q^q \, ds \right).$$

Noting that

$$\|u^2_{0,n}\|_2^2 \leq C \lambda^{2-p} \|u_{0,n}\|_p^p, \quad \|u_{0,n}\|_{L^p} \leq 2 \lambda \|T_s\|,$$

and (136), we know that $U^n$ is uniformly bounded in the energy space. What’s more, one can also show that

$$\|\partial_t U^n\|_{L^2(0, T; H^{-3/2}(\mathbb{R}^3))} \leq c,$$

where $H^{-s}(\Omega)$ is the dual of $H^s_0(\Omega)$, $c$ is independent of $n$. In view of Aubin-Lions lemma and boundedness of $U^n$ and Calderon-Zygmund estimates, we can assert there exist functions $U, P$ and a subsequence in $n$ still denoted by $n$, such that

$$U^n \to U \text{ weakly-* in } L^\infty_t L^2_x(Q_T), \quad \nabla U^n \to \nabla U \text{ weakly in } L^2_t L^2_x(Q_T),$$

$$U^n \to U \text{ strongly in } L^3_{loc}(Q_T), \quad P^n \to P \text{ weakly in } L^{3/2}_{loc}(Q_T),$$

and for all $t \in [0, T]$,

$$U^n(t) \to U(t) \text{ weakly in } L^2(\mathbb{R}^3).$$

\(^3\)see e.g. [57]
As \( U^n \) solves (130) and satisfies local energy estimate (131), so does \( U \), due to the convergence properties described above. Then one can argue exactly as [2] to deduce the global energy estimate from the local energy inequality, that is for almost all \( t_0 > 0 \) and all \( t_1 > t_0 \),

\[
\int_{\mathbb{R}^3} |U(x,t_1)|^2 \, dx + 2 \int_{t_0}^{t_1} \int_{\mathbb{R}^3} |\nabla U|^2 \, dx \, dt \\
\leq \int_{\mathbb{R}^3} |U(x,t_0)|^2 \, dx + 2 \int_{t_0}^{t_1} \int_{\mathbb{R}^3} V \otimes U : \nabla U \, dx \, dt.
\]

Let \( v = U + V, q = Q + P \), in order to verify that \( (v,q) \) is a weak \( L^p \) solution on \( Q_T \), it remains to show that the above global energy inequality can start from \( t_0 = 0 \), which together with (137) implies the strong continuity of \( v \) in \( L^2 \) at \( t = 0 \).

3. The limit solution is a weak \( L^p \) solution. We already know that the weak solution \( U \) verifies

\[
\int_{\mathbb{R}^3} |U(x,t)|^2 \, dx + \int_{t_1}^{t} \int_{\mathbb{R}^3} |\nabla U(x,s)|^2 \, dx \, ds \\
\leq \int_{\mathbb{R}^3} |U(x,t_1)|^2 \, dx + 2 \int_{t_1}^{t} \int_{\mathbb{R}^3} V \otimes U : \nabla U \, dx \, ds
\]

for almost every \( t_1 > 0 \) and all \( t_1 < t < T \), and \( U \) solves (130) with initial data \( U_0 \). Choose some \( t_1 \) sufficiently small, as a result of Proposition 7, one can assert this weak solution \( U \) coincides with the strong solution \( W \) to (130) on the interval \([t_1,\bar{T})\) with some \( \bar{T} > 0 \). Because this uniqueness holds for almost every \( t_1 > 0 \) and \( W(0) = U(0) \), so for all \( t \in [0,\bar{T}), U = W \) a.e. \( x \in \mathbb{R}^3 \), Consequently, \( U \) meets the global energy inequality starting from \( t = 0 \), the proof is finished.

At present, we can prove the main result by using the weak \( L^p \) solution.

An alternative proof of Theorem 1.5. Let \( \{u_{0,n}\} \) be the sequence verifying (124), such that \( NS(u_{0,n}) \) has a singular point \( z_0 = (0,T_*) \). By extracting a subsequence, one can assume \( u_{0,n} \rightharpoonup v_0 \) weakly in \( L^p \). Now let \( \hat{u}^n, \hat{q}^n \) and \( v,q \) be the weak \( L^p \) solution described in Theorem 8.4. It can be inferred from Proposition 8 that \( \hat{u}^n \) also has a singular point at \( z_0 \), so does \( v \), due to Proposition A.4. Again, one can conclude that the mild solution \( NS(v_0) \) will also become singular at time \( T(v_0) \leq T_* \) by utilizing the uniqueness proposition, thus

\[
T(v_0)\|v_0\|^{\sigma_p}_p \geq A_{\sigma_p}^p.
\]

While weak convergence gives

\[
\|v_0\|^{\sigma_p}_p \leq \liminf_{n \to \infty} \|u_{0,n}\|_p = \frac{A_{\sigma_p}^p}{T_*}.
\]

Accordingly, we see

\[
T(v_0) = T_* \quad \|v_0\|_p = r_p(T_*).
\]

Noticing the uniform convexity of \( L^p \) spaces equipped with the standard norm, one has

\[
\lim_{n \to \infty} \|u_{0,n} - v_0\|_p = 0,
\]

which gives the desired result.
9. Uniform bounds for the minimal blowup solution. This section is devoted to proving Corollary 1, which in fact is a consequence of Theorem 1.5 and Proposition B.6. We mention that I. Gallagher [22] obtained a uniform bounds for mild solutions to (2) with initial data in \( \mathcal{B}_{1/2} = \{ u_0 \in \dot{H}^{1/2} \mid \| u_0 \|_{\dot{H}^{1/2}} < \rho_{1/2} \} \) in the sense that there exists a nondecreasing function \( F : [0, \rho_{1/2}) \to \mathbb{R}^+ \) such that
\[
\| NS(u_0) \|_{L^\infty(\mathbb{R}^+;\dot{H}^{1/2}) \cap L^2(\mathbb{R}^+;H^{3/2})} \leq F(\| u_0 \|_{\dot{H}^{1/2}})
\]
holds for any \( u_0 \in \mathcal{B}_{1/2} \). One can also refer to [51] for a similar conclusion in \( H^s(\mathbb{R}^3) \) with \( 1/2 < s < 3/2 \).

**Proof of Corollary 1.** The proof goes essentially the same as that in [51] by adapting some estimates adapted to the \( L^p \) scale, for completeness, we give the details. Noting that inequality (1) has scaling symmetry, one can ask \( \| u_0 \|_p = 1 \), then \( T(u_0) = A_p^p \). We now prove by contradiction. Let \( T_s = A_p^p \), there exists a sequence \( \{ w_{0,n} \} \subset \mathcal{M}_p(T_s) \) with \( T(w_{0,n}) = T_s \) such that
\[
\| NS(w_{0,n}) \|_{X_T^p} \geq n, \quad T < T_s, \quad \forall \ n \geq 1,
\] (138)
Besides, Theorem 1.5 infers there exist a \( w_0 \in \mathcal{M}_p(T_s) \) and a subsequence still denoted by \( n \) with the property
\[
\lim_{n \to \infty} \| w_{0,n} - w_0 \|_p = 0.
\]
On account of the stability estimate Proposition B.6, one sees for all \( n \geq N \), \( N \) is some sufficiently large number,
\[
\| NS(w_{0,n}) - NS(w_0) \|_{X_T^p} \leq C \| w_{0,n} - w_0 \|_p \exp \{ CT^5p^{1-3/p}/6 \| NS(w_0) \|_p^{5p/3 \| T^5p/3 L^{p/3 \|}(Q_T)} \}.
\]
Obeserving that the right hand side of the above inequality is uniformly bounded in \( n \), which implies the boundedness of \( NS(w_{0,n}) \) for all \( n \geq N \), thus we get a contradiction with hypothesis (138). The proof is completed.

**Appendix A. Some well-known facts on the Navier-Stokes system.** In this appendix, we shall list some conclusions appeared in the previous sections on the (NS) system below
\[
\begin{align*}
  u_t - \Delta u + u \cdot \nabla u + \nabla p &= 0, \quad \text{div} u = 0, \quad u(0,x) = u_0(x).
\end{align*}
\] (139)
Specifically, it includes \( \epsilon \)-regularity criterion for suitable weak solution, \( L^p \) local Cauchy theory for (NS) with \( 3 < p < \infty \) and a stability estimate. To begin with, we give the definition of suitable weak solution, which appeared first in [11], here the version is due to Lin [46].

**Definition A.1** (Suitable weak solution). Let \( D \) be an open set in \( \mathbb{R}^3 \), a pair \( u,p \) is called a suitable weak solution to (139) on the set \( \Omega := D \times (-T_1, T) \) if the following conditions are satisfied:

1. \( u \in L^\infty((-T_1,T),L^2(D)) \cap L^2(-T_1,T;W^1_2(D)) \), here \( W^1_2 \) denotes the usual Sobolev spaces, \( p \in L^{3/2}(\Omega) \);
2. The pair \( u,p \) satisfies (NS) in the sense of distributions on \( \Omega \);
3. The following local energy inequality
\[
\int_D \varphi(x,t) |u(x,t)|^2 \, dx + 2 \int_{-T_1}^T \int_D \varphi(x,s) |\nabla u(x,s)|^2 \, dx \, ds
\]
\[
\leq \int_{-T_1}^t \int_D (|u|^2 (\partial_t \varphi + \Delta \varphi) + u \cdot \nabla \varphi(|u|^2 + 2p)) \, dx \, ds
\]
holds for all \( t \in (-T_1, T) \) and for all non-negative functions \( \varphi \in C_0^\infty(\mathbb{R}^3 \times \mathbb{R}) \) vanishing in a neighborhood of the parabolic boundary \( \partial \Omega = D \times \{ t = -T_1 \} \cup \partial D \times (-T_1, T) \).

The importance of suitable weak solution can be revealed by the \( \varepsilon \)-regularity criterion below, which was first showed in [11], see also [20, 46, 42] for a different proof.

**Lemma A.2** (\( \varepsilon \)-regularity criterion). Let \( Q(r) = B(0, r) \times (-r^2, 0) \). There are absolute constants \( \varepsilon_0 \) and \( c_{0,k} > 0, k \geq 0 \), such that if \( (u, p) \) is a suitable weak solution on \( Q(1) \) with
\[
\int_{Q(1)} (|u|^2 + |p|^{2/3}) \, dx \, dt < \varepsilon_0,
\]
then \( u \) is smooth on \( Q(1/2) \) and for all \( k \geq 0 \),
\[
\sup_{Q(1/2)} |\nabla^k u(x, t)| < c_{0,k}.
\]

**Remark A.3.** In fact, \( k = 0 \) is proved in [11], see [39] for the case \( k \geq 1 \).

The next proposition deals with the stability of singular point for the suitable weak solution, see [53, 2] for reference.

**Proposition A.4** (Stability of singular point). Let \( (u_k, p_k) \) be a sequence of suitable weak solutions on \( Q(1) \) with \( u_k \to u \) in \( L^3(Q(1)) \) and \( p_k \to p \) in \( L^{3/2}(Q_1) \). Assume \( u_k \) is singular at \( (x_k, t_k) \in Q(1) \) and \( (x_k, t_k) \to (0, 0) \). Then \( u \) is singular at \( (0, 0) \). Here we say \( u \) is singular at some point \( z_0 = (x_0, t_0) \) provided for any \( r > 0 \), \( u \notin L^\infty(B(x_0, r) \times (0 - r^2, t_0)) \).

**Theorem A.5** (Local Cauchy theory in \( L^p \)). Let \( 3 < p < \infty \), \( u_0 \in L^p(\mathbb{R}^3) \), there exists a time \( T > C||u_0||_{L^p}^{-\sigma_p} \) and a unique mild solution \( NS(u_0) \in C([0, T]; L^p(\mathbb{R}^3)) \) to (139). Moreover
\[
\sup_{0 < t < T} t^{j + \frac{\sigma_p}{2}} \| \partial_t^j \nabla^k NS(u_0)(t) \|_p \leq C||u_0||_p, \quad j, k \in \{0, 1\},
\]
\( C \) depends on \( j, k, p \) and \( \sigma_p = 2/(1 - 3/p) \).

**Remark A.6.** By propagation, one can also get \( NS(u_0)(t) \in L^2(0, T; L^\infty(\mathbb{R}^3)) \).

The following conclusion is borrowed from [2], which asserts a finite blowup mild solution with initial data in \( L^p \) must develop a singular point at the maximal existence time.

**Proposition A.7** (Formation of singular point). Let \( u_0 \in L^p(\mathbb{R}^3) \) be a divergence free vector field, \( 3 < p < \infty \). Then the mild solution \( u = NS(u_0) \) in \( C([0, T_*]; L^p(\mathbb{R}^3)) \) must have a singular point at \( T_* \).

**Appendix B. Perturbation of the Navier-Stokes equation.** We will prove a result on the perturbed Navier-Stokes system in this section. First, let us recall two classical inclusions on Besov and Triebel-Lizorkin spaces, one can refer to Chapter 1 in [60] or Chapter 2 in [5] for details.
Proposition B.1. Assume \( s_1 < s_2, 1 \leq p_2 < p_1 < \infty, 1 \leq q, r \leq \infty, s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2}, \) then
\[
\hat{B}^s_{p_2,q}(\mathbb{R}^d) \hookrightarrow \hat{B}^s_{p_1,q}(\mathbb{R}^d), \quad \hat{F}^s_{p_2,q}(\mathbb{R}^d) \hookrightarrow \hat{F}^s_{p_1,q}(\mathbb{R}^d), \quad L^{p_1}(\mathbb{R}^d) = \hat{F}^0_{p_1,2}(\mathbb{R}^d).
\]
Between Besov and Triebel-Lizorkin spaces, for \( s \in \mathbb{R}, 1 \leq p < \infty, 1 \leq q \leq \infty, \) there holds
\[
\hat{B}^s_{p,p \wedge q} \hookrightarrow \hat{F}^s_{p,q} \hookrightarrow \hat{B}^s_{p,p \vee q}
\]
(140)
where \( p \wedge q = \min\{p, q\}, p \vee q = \max\{p, q\}. \)
Now let us recall some known estimates for the (NS) equation, which were essentially established or hidden in \([16, 28, 59, 47]\) and summarized in \([60]\).

Proposition B.2. Let \( a \geq 0, 1 \leq r \leq p \leq \infty, 0 < \lambda \leq \infty \) and \( 2/\gamma = a + n(1/r - 1/p). \) Then we have
\[
\|e^{t\Delta} f\|_{L^\gamma([0,\infty); \hat{B}^a_{p,\lambda})} \leq C\|f\|_{\hat{B}^{-a}_{p,\lambda}}.
\]
(141)
In addition, if \( \gamma \geq r > 1, \)
\[
\|e^{t\Delta} f\|_{L^\gamma([0,\infty); L^p)} \leq C\|f\|_{H^{-a}}.
\]
(142)
We remark that (142) is a consequence of (140), (141) and Proposition B.1.

Proposition B.3. Let \( 1 \leq r \leq p \leq \infty \) and \( 1 < \gamma, \gamma_1 < \infty \) satisfy
\[
\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{k}{2} + \frac{n}{2} \left( \frac{1}{p} - \frac{1}{r} \right) - 1, \quad \frac{k}{2} + \frac{n}{2} \left( \frac{1}{p} - \frac{1}{r} \right) < 1, \quad k = 0, 1.
\]
(143)
Then we have
\[
\|\nabla^k \mathcal{A}_0 f\|_{L^\gamma([t_0,\infty); L^r)} \lesssim \|f\|_{L^{\gamma_1}([t_0,\infty); L^r)}.
\]
(144)

Proposition B.4. Let \( 1 \leq r \leq \infty, 1 \leq q' \leq \lambda \leq \infty^4. \) Then
\[
\|\mathcal{A}_0 f\|_{L^{\infty}([t_0,\infty), \hat{B}^q_{r,\lambda})} \lesssim \|f\|_{L^{q'}([t_0,\infty), \hat{B}^{-2/q'}_{r,\lambda})}.
\]
(145)
As a direct consequence, we now give several estimates adapted to our needs.

Corollary B.5. Let \( s \in \mathbb{R}, 3 < p < \infty, 1 \leq q \leq \gamma \leq \infty, 1 \leq m \leq \infty, 0 < T < \infty, 1/\beta(p) = 1/2 - 3/10p, \) then we have
\[
\|e^{t\Delta} u_0\|_{L^p_T} \lesssim \|u_0\|_{L^p},
\]
(146)
\[
\|e^{t\Delta} u_0\|_{L^\gamma([0,\infty), \hat{B}^m_{r,q})} \lesssim \|u_0\|_{\hat{B}^m_{r,q}},
\]
(147)
\[
\|\mathcal{A}_0 \hat{\nabla} f\|_{L^p_T} \lesssim \|f\|_{L^{\beta(p)}_T L^5} \lesssim T^{(1-3/p)/2}\|f\|_{L^{5p/6}_T L^{5p/6}}.
\]
(148)
Proof. By Proposition B.3 and Hölder’s inequality,
\[
\|\mathcal{A}_0 \hat{\nabla} f\|_{L^{5p/6}_T L^5} \lesssim T^{(1-3/p)/2}\|f\|_{L^{5p/6}_T L^{5p/6}}.
\]
By Propositions B.4 and B.1, taking \( s(p) = 1 - 2/\beta(p) > 0, \) we see that
\[
\|\mathcal{A}_0 \hat{\nabla} f\|_{L^p_T L^5} \lesssim \|\mathcal{A}_0 \hat{\nabla} f\|_{L^{5p/6}_T \hat{B}^0_{5p/6,5p/6}} \lesssim \|f\|_{L^{5p/6}_T \hat{B}^0_{5p/6,5p/6}}.
\]
The result now follows if we note the embedding \( L^{5p/6} \hookrightarrow \hat{B}^0_{5p/6,5p/6}. \) \[\square\]

\(^4p'\) stands for the conjugate number of \( p, \) i.e. \( 1/p + 1/p' = 1.\)
Next, we perform the perturbation analysis for the Navier-Stokes equation. Our approach follows Theorem 3.1 in [23]. Similar conclusions in critical spaces can also be found in [24, 25].

**Proposition B.6.** Let \( 3 < p < \infty, w \in \mathcal{X}_T^p \) be vector fields in \( \mathbb{R}^3 \), \( f \in L_T^{\beta(p)} L_x^{5p/6} \) is a \( 3 \times 3 \) matrix function. \( \beta(p) \) is the same as Lemma 6.1, \( 0 < T < \infty, v \) satisfies the following Perturbed Navier-Stokes equation (PNS)

\[
\begin{aligned}
\partial_t v + (v \cdot \nabla)v - \Delta v + (v \cdot \nabla)w + (w \cdot \nabla)v &= -\nabla p - \nabla \cdot f, \\
\text{div } v &= 0, \\
v_{|t=0} &= v_0.
\end{aligned}
\]

Assume that there exist two constants \( \epsilon_0 \ll 1 \) and \( C \gg 1 \) such that

\[
\|v_0\|_p + \|f\|_{L_T^{\beta(p)} L_x^{5p/6}} \leq \epsilon_0 T^{-1(3/p)/2} \exp\left(-C T^{5p(1-3/p)/6} \|w\|_{L_T^{5p/3} L_x^{5p/3}}\right).
\]

Then \( v \) belongs to \( \mathcal{X}_T^p \), moreover we have

\[
\|v\|_{\mathcal{X}_T^p} \lesssim (\|v_0\|_p + \|f\|_{L_T^{\beta(p)} L_x^{5p/6}}) \exp\left(C T^{5p(1-3/p)/6} \|w\|_{L_T^{5p/3} L_x^{5p/3}}\right).
\]

**Proof.** The Duhamel formula of (PNS) is

\[
v = e^{t\Delta} v_0 - \mathcal{A}(v, v) - \mathcal{A}(v, w) - \mathcal{A}(w, v) - \mathcal{A}_0 \mathcal{P} \nabla \cdot f.
\]

For any time interval \((\alpha, \beta) \subset (0, T)\), we denote

\[
\mathcal{X}_{(\alpha, \beta)}^p := L^{5p/3}(\alpha, \beta, L_x^{5p/3}) \cap C([\alpha, \beta], L_x^p).
\]

It follows from Corollary B.5 that

\[
\|v\|_{\mathcal{X}_{(\alpha, \beta)}^p} \leq L(\|v(\alpha)\|_p + \|f\|_{L_T^{\beta(p)} L_x^{5p/6}}) + LT^{(1-3/p)/2}(\|v\|_{L_T^{5p/3}(\alpha, \beta, L_x^{5p/3})}^2 + \|v\|_{L_T^{5p/3}(\alpha, \beta, L_x^{5p/3})}) \|w\|_{L_T^{5p/3}(\alpha, \beta, L_x^{5p/3})} (149)
\]

for some constant \( L > 1 \). Based on this estimate, we can solve (PNS) locally and then extend to the maximal solution, i.e. solution with maximal existence time \( T(v_0) \). By the absolute continuity of Lebesgue integral, there exists \( N + 1 \) real numbers \((T_i)_{0 \leq i \leq N}\) such that \( T_0 = 0 \) and \( T_N = T \), satisfying \([0, T] = \bigcup_{i=0}^{N-1} [T_i, T_{i+1}]\) and

\[
\frac{1}{8L} \leq T^{(1-3/p)/2} \|w\|_{L_T^{5p/3}(T_i, T_{i+1}), L_x^{5p/3}} \leq \frac{1}{4L}, \quad \forall i \in \{0, \ldots, N-2\},
\]

\[
T^{(1-3/p)/2} \|w\|_{L_T^{5p/3}(T_{N-1}, T), L_x^{5p/3}} \leq \frac{1}{4L}. \quad (150)
\]

By virtue of (150), one sees that \( N \) satisfies

\[
\|w\|_{L_T^{5p/3} L_x^{5p/3}} \geq \sum_{i=0}^{N-2} \int_{T_i}^{T_{i+1}} \|w\|_{L_T^{5p/3} L_x^{5p/3}} dt \rightarrow (N - 1) \left( \frac{1}{8LT^{(1-3/p)/2}} \right)^{5p/3} . \quad (151)
\]

Now let us assume that \( \|v_0\|_p + \|f\|_{L_T^{5p/3} L_x^{5p/3}} \) is bounded by

\[
(8LT^{(1-3/p)/2}(2L)^5)^{-1} \exp \left[-(8LT^{(1-3/p)/2})^{5p/3} \|w\|_{L_T^{5p/3} L_x^{5p/3}} \ln(2L)\right]. \quad (152)
\]
By continuity in time, we define a maximal time \( \bar{T} < T(\nu_0) \) such that
\[
T^{(1-3/p)/2}\|v\|_{L^{5p/3}(0,T),L^{5p/3}} \leq \frac{1}{4L}.
\] (153)
If \( \bar{T} \geq T \), then by (149), we can get desired conclusion.
If \( \bar{T} < T \), then there exists \( k \in \{0,\ldots,N-1\} \), such that \( T_k \leq \bar{T} < T_{k+1} \). Inserting (150) and (153) into (149), we can get for any \( 1 \leq i < k \)
\[
\|v\|_{X^{T_i,T_{i+1}}} \leq L\|v(T_i)\|_p + \|f\|_{L^{5p/3}L^{5p/3}} + 2^{-1}\|v\|_{L^{5p/3}(T_i,T_{i+1}),L^{5p/3}}.
\] (154)
Since \( X^{T_i,T_{i+1}} = L^{5p/3}(T_i,T_{i+1}),L^{5p/3} \cap C([T_i,T_{i+1}],L^p) \), so we have
\[
\|v(T_{i+1})\|_p \leq \|v\|_{X^{T_i,T_{i+1}}} \leq 2L\|v(T_i)\|_p + \|f\|_{L^{5p/3}L^{5p/3}}.
\] (154)
Iterating the above inequality (154) with respect to \( i \), \( 1 \leq i < k \)
\[
\|v(T_i)\|_p \leq (2L)^{i+1}\|v_0\|_p + \|f\|_{L^{5p/3}L^{5p/3}}.
\] (155)
By (155) and (154), we see
\[
\|v\|_{X^{T_i,T_{i+1}}} \leq 2(2L)^{k+2}\|v_0\|_p + \|f\|_{L^{5p/3}L^{5p/3}}.
\]
Similarly we have
\[
\|v\|_{X^{T_{k},T_{k}}} \leq 2(2L)^{k+2}\|v_0\|_p + \|f\|_{L^{5p/3}L^{5p/3}}.
\]
Further,
\[
\|v\|_{L^{5p/3}L^{5p/3}} \leq \sum_{i=0}^{k-1} \|v\|_{L^{5p/3}(T_i,T_{i+1}),L^{5p/3}} + \|v\|_{L^{5p/3}(T_k,T),L^{5p/3}}
\leq (2L)^{N+4}\|v_0\|_p + \|f\|_{L^{5p/3}L^{5p/3}}.
\]
By (151) and (152), we have
\[
\|v_0\|_p + \|f\|_{L^{5p/3}L^{5p/3}} \leq \frac{1}{8L(2L)^{N+4}T^{1/2(1-3/p)}}.
\]
with this we finally obtain
\[
T^{1/2(1-3/p)}\|v\|_{L^{5p/3}L^{5p/3}} \leq 1/8L,
\]
which contradicts the maximality of \( \bar{T} \). Therefore, \( \bar{T} \geq T \), and using the above iteration, we have
\[
\|v\|_{X^{T_{k},T}} \leq (2L)^{N+4}\|v_0\|_p + \|f\|_{L^{5p/3}L^{5p/3}}.
\]
Together with the estimate (151), we can rewrite the above estimate as
\[
\|v\|_{X^{T_{k},T}} \leq C(\|v_0\|_p + \|f\|_{L^{5p/3}L^{5p/3}}) \exp (CT^{5p(1-3/p)/6}\|w\|_{L^{5p/3}L^{5p/3}}),
\]
for some constant \( C \gg 1 \). Hence we complete the proof. \( \square \)

As a direct consequence, we have the following corollary.

**Corollary B.7.** The mapping \( w_0 \mapsto T(w_0) \) is lower semi-continuous for \( (NS) \) with initial data in \( L^p \), \( 3 < p < \infty \), i.e. for arbitrary \( \varepsilon > 0 \), there exists a \( \eta > 0 \), such that if \( \|v_0\|_p < \eta \), then \( T(w_0 + \varepsilon) \geq T(w_0) - \varepsilon \).
Sketch Proof of Remark 4. Considering the map
$$\mathcal{T} : u(t) = e^{t\Delta}u_0 - \mathcal{R}(u,u),$$
by Corollary B.5, we have
$$\|\mathcal{T}u\|_{X^p_T} \leq \|e^{t\Delta}u_0\|_{X^p_T} + \|\mathcal{R}(u,u)\|_{X^p_T},$$
$$\lesssim \|u_0\|_p + T^{(1-3/p)/2}\|u\|_{L^{5p/6}_T L^{5p/6}_x},$$
$$\leq \|u_0\|_p + T^{(1-3/p)/2}\|u\|_{X^p_T}^2.$$

So, if
$$u \in \mathcal{D}_T := \{u \in X^p_T : \|u\|_{X^p_T} \leq 2C\|u_0\|_p\},$$
we can choose $T$ satisfying $T^{(1-3/p)/2}\|u_0\|_p \leq 1/4C^2$. It follows that $\mathcal{T}u \in \mathcal{D}_T$. Using contraction mapping principle, we obtain that (NS) has a unique solution $u \in \mathcal{D}_T$ and the above proof implies the result of Remark 4. 

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