EVOLUTION FOR KHOVANOV POLYNOMIALS FOR FIGURE-EIGHT-LIKE FAMILY OF KNOTS

PETR DUNIN-BARKOWSKI, ALEKSANDR POPOLITOV, AND SVETLANA POPOLITOVA

Abstract. We look at how evolution method deforms, when one considers Khovanov polynomials instead of Jones polynomials. We do this for the figure-eight-like knots (also known as ‘double braid’ knots, see arXiv:1306.3197) – a two-parametric family of knots which “grows” from the figure-eight knot and contains both two-strand torus knots and twist knots. We prove that parameter space splits into four chambers, each with its own evolution, and two isolated points. Remarkably, the evolution in the Khovanov case features an extra eigenvalue, which drops out in the Jones ($t \to -1$) limit.

1. Introduction

One of the central questions in knot theory is whether two knots are topologically equivalent. Currently, the most effective way to determine this is to compute some polynomial knot invariant for each knot: if the invariants differ, knots are for sure not topologically equivalent. Complementary statement is not true: if invariants are the same it does not mean that the knots are equivalent. So, in knot theory one seeks stronger and stronger knot invariants that have the power to distinguish more and more knots.

Perhaps, the most widely known polynomial knot invariants (knot polynomials) are the so-called Jones [3] and HOMFLY [4, 5] polynomials. One of the ways to compute (and define) them is quite elementary – one just repeatedly applies skein relations

$$A \bigotimes - A^{-1} \bigotimes = (q - q^{-1})$$

and Reidemeister moves to express knot polynomial for a given planar diagram through polynomials for simpler diagrams.

A surprising and indirect consequence of the above naive definition is the existence of the so-called evolution method for both Jones and HOMFLY polynomials (see, for instance, [1, Introduction]; this story is also well-explained in [22]). This is the statement that, whenever an $m$-strand braid is present somewhere in the planar diagram $K$ of some knot, the knot polynomial $P$ (either Jones or HOMFLY) depends on the number $n$ of windings of the braid in quite a simple way

$$P^K(n) = \sum_{\lambda \vdash m} C^K(\lambda) \left((\pm)_{\lambda} q^{\text{power}(\lambda)}\right)^n$$

Here sum is over partitions $\lambda$ of the number of strands $m$ (i.e. over Young diagrams), and sign ‘$(\pm)_{\lambda}$’ and the exponent ‘power$(\lambda)$’ of an eigenvalue are universal; they depend only on partition $\lambda$ but not on the diagram $K$. In fact, they are simple combinatorial expressions of the shape of $\lambda$. The evolution coefficients $C^K(\lambda)$, on the contrary, depend both on the knot diagram $K$ and the partition $\lambda$. 
A subtle feature of the above formula is that number of windings $n$ can be arbitrary integer – the evolution coefficients $C_K^\lambda(\lambda)$ do not change as one goes from negative number of crossings (i.e. crossings in a different direction) to zero crossings to positive number of crossings in the braid. The formula seamlessly interpolates between these three cases. This feature is, in fact, a manifestation of another, more conceptual and deep, definition of Jones and HOMFLY polynomials – through the so-called Reshetikhin-Turaev (RT) formalism \cite{6}.

In the RT-formalism one associates a certain linear operator, the so-called $R$-matrix, to each positive crossing and to each negative crossing its inverse. The factors $\left((\pm)^q \text{power}(\lambda)\right)$ in the formula (2) are then naturally reinterpreted as eigenvalues of the $R$-matrix in the irreducible representation corresponding to the Young diagram $\lambda$. The knot polynomial itself is just a tensor contraction of several $R$-matrices, in the order dictated by the planar diagram $\mathcal{K}$.

RT-formalism allows one to establish a link between knot theory and physics: knot polynomials turn out to be Wilsonian averages in Chern-Simons theory (see, for instance, \cite{18} for a review). This is most easily seen in the so-called temporal gauge \cite{7}. The gauge theory point of view immediately leads to the generalization of both Jones and HOMFLY polynomials to the colored case, where one decorates the knot with some representation of the gauge group. The skein relation description is not available in the colored case and from the naive skein relations (1) it is not at all easy to guess, that the colored generalization should even exist.

The (uncolored, or fundamental) Jones and HOMFLY polynomials have a homological generalization – Khovanov \cite{8} and Khovanov-Rozansky \cite{9,10} polynomials, respectively. The definition of these polynomials (especially the Khovanov-Rozansky one) seems much more elaborate than definition of their non-homological analogs – one needs to calculate the homology groups of a certain differential complex, built out of the planar diagram $\mathcal{K}$. While in the case of Khovanov polynomial at least the construction of the complex is well-understood \cite{11}, in the Khovanov-Rozansky case even explicit construction of the linear spaces in the complex presents difficulties (each space is a factor of an infinite-dimensional space of so-called foams over infinitely many relations \cite{12}). Explicit construction of maps between the spaces is even more difficult. Hence, one is tempted to search for alternative definitions for Khovanov and Khovanov-Rozansky polynomials \cite{13,14,15,16,17}, partly motivated by the empirical observation that answers for both Khovanov and Khovanov-Rozansky polynomials (in known examples) seem much simpler than their cumbersome definitions would lead one to expect.

A natural question on this path is whether the relevant generalization of the RT formalism exists for the Khovanov and Khovanov-Rozansky polynomials. This homological RT formalism is believed to be related to the so-called refined \cite{19} Chern-Simons theory, which is at the moment defined only for the simplest examples of knots.

Even more naively, one may wonder, whether something like evolution formula (2) exists for Khovanov and Khovanov-Rozansky polynomials. In \cite{1} this was investigated for the simplest case of torus knots. Surprisingly, it was observed that the symmetry between positive and negative crossings is broken – an essential difference with the Jones and HOMFLY case. Moreover, from studies of superpolynomials (the $N \to \infty$ stable component of Khovanov-Rozansky polynomials) for torus knots (see \cite{24}) one knows that for them the symmetry between positive and negative crossings is not broken, which makes the observed breaking at finite $N$ even more unexpected.

In this short note we look at what happens for a slightly more complicated family of knots (see Section 2). This family is inspired by the figure 8 knot (the first non-torus
We concentrate only on Khovanov polynomials and completely ignore Khovanov-Rozansky ones. First of all, Khovanov polynomials are easier to calculate: there is a computer program readily available in the “KnotTheory” package for Wolfram Mathematica, which is available on Katlas [20], though there are some caveats (see Section 4). Second of all, non-trivial new phenomena are seen already in Khovanov case, so this generalization is sufficiently interesting.

We find the following picture (see Section 3, Theorem 3.1). For some suitable subfamily (see Section 2) we observe that:

- evolution is preserved in the chambers on the parameter plane; there are also 2 isolated points, where the knot becomes two unlinked unknots,
- if one interprets the formulas as coming from some sort of RT-formalism, one concludes that fundamental $R$-matrix must have one more eigenvalue: $(-1) t q^3$, in addition to $t q^3$ and $q$. The coefficient in front of this eigenvalue is always proportional to $(t + 1)$ and vanishes when one goes to the Jones limit $t \to -1$,
- transitions between chambers are tricky: some coefficients in front of eigenvalues just get multiplied by some constants, while others re-glue in a more elaborate way.

We prove our evolution formulas (Theorem 3.1) in Section 5. The proof relies on the fact that all knots and links in the family we consider are alternating, and on the reconstruction theorem [21, Theorem 4.5] that allows one to completely determine Khovanov polynomial from the Jones polynomial for the alternating knot/link, provided the signature is known. The symmetry breaking in the evolution method is ultimately traced to the jumps in the signature.

Forward references throughout the introduction provide sufficient description of the organization of the paper, so “this paper is organized as follows” paragraph is really unnecessary.

Acknowledgements. We would like to thank A.Anokhina, A.Morozov, Y.Zenkevich and M.Khovanov for stimulating discussions. The research of A.P. is supported in part by Vetenskapsrådet under grant #2014-5517, by the STINT grant, by the grant “Geometry and Physics” from the Knut and Alice Wallenberg foundation and by RFBR grants 16-01-00291 and 18-31-20046 mol_a_ved. The research of P.D.-B. is supported in part by RFBR grants 18-01-00461 A and 18-31-20046 mol_a_ved.

2. The figure-eight-like knots

We consider the following family of planar diagrams, inspired by the figure-8 knot

\[ \begin{array}{c}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure8knot.png}
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{modified_figure8knot.png}
\end{array}
\end{array} \]

i.e. we allow arbitrary 2-strand braid to be inserted in place of higher or lower pair of crossings of the figure eight knot. This family was also considered in [22], where it was
called “double braid” family of knots, and where in particular a conjecture was made about explicit form of colored HOMFLY polynomials for this family in any symmetric representation \([r]\).

This family is interesting to consider, since it includes both torus and non-torus knots (so we will not observe something that is specific just to torus case). It is also two-parametric, which allows us to see, how evolutions w.r.t the two braid-winding parameters \(a\) and \(b\) interplay. Moreover, it was instrumental in finding an explicit relation between so-called inclusive and exclusive Racah matrices (see [23]).

Depending on the parity of the braid parameters \(a\) and \(b\), strands are oriented differently (so \(a\)- and \(b\)-braids can be both parallel and antiparallel). Moreover, in case both \(a\) and \(b\) are odd the diagram is a link, so orientations of its two components can be chosen independently.

In what follows we choose a particular orientation of strands with

\[ a \]

\[ b \]

With this choice \(a\) can be arbitrary integer, while \(b\) can only be odd. On the example of the \(a\)-braid we are able to see what happens when we add a single crossing to the diagram. If the representation theory is at all applicable to Khovanov polynomials, then in some sense we must have, that in the \(a\)-braid there is the representation \([1] \otimes [1] = [1, 1] \oplus [2]\) running, while in the \(b\)-braid there is the representation \([1] \otimes [1] = \emptyset \oplus \text{adjoint}\) running. Of course, it would be interesting in future to consider a family of knots that allows to add single crossings to more that one of its braids and see whether this produces some interesting effects.

3. The phase diagram

We find (see Section 5 for a proof) that the space of parameters splits into 4 regions and 2 isolated points
The isolated points are (1, 1) and (−1, −1) and in what follows we will refer to the four regions as UL, UR, LL and LR for upper-left, upper-right, lower-left and lower-right, respectively.

Dependence of Khovanov polynomial on the number of braid windings \(a\) and \(b\) is quite remarkable:

**Theorem 3.1.** In each of the regions UL, UR, LL and LR dependence of the Khovanov polynomial on the braid parameters \(a\) and \(b\) is consistent with the evolution method

\[
K_{\text{UL}}^{b} = \left( (1)^b (tq^2)^b \right) \left( \begin{array}{ccc} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \end{array} \right) \left( \begin{array}{c} q^a \\ (-tq^3)^{-a} \end{array} \right)
\]

but the matrices of coefficients \(M\) are different among the four regions

\[
M_{UL} = \left( \begin{array}{ccc} \frac{q^8 t^3 + q^8 t^3 - q^8 t^2 - q^8 t^1 + q^2 t^1}{(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} & \frac{-q^8 t^2 - q^2 t^1}{(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1)} & 0 \\ \frac{2q^8 t^3 + q^8 t^3 - q^8 t^2 - q^8 t^1 + q^2 t^1}{2(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} & -q^8 t^2 - q^2 t^1 & -q^8 t^2 - q^2 t^1 \\ 0 & \frac{-q^8 t^2 - q^2 t^1}{2(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} & 0 \end{array} \right)
\]

\[
M_{UR} = \left( \begin{array}{ccc} \frac{-q^8 t^4 - q^8 t^3 - q^8 t^2 - 2q^8 t^1 + 2q^8 t^1}{q^8 (q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} & \frac{-q^8 t^4 - q^8 t^3 - q^8 t^2 - 2q^8 t^1 + 2q^8 t^1}{2(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} & \frac{-q^8 t^4 - q^8 t^3 - q^8 t^2 - 2q^8 t^1 + 2q^8 t^1}{2(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} \\ \frac{2q^8 t^4 - q^8 t^3 - q^8 t^2 - 2q^8 t^1 + 2q^8 t^1}{q^8 (q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} & -q^8 t^4 - q^8 t^3 - q^8 t^2 - 2q^8 t^1 + 2q^8 t^1 & \frac{-q^8 t^4 - q^8 t^3 - q^8 t^2 - 2q^8 t^1 + 2q^8 t^1}{2(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} \\ 0 & \frac{-q^8 t^4 - q^8 t^3 - q^8 t^2 - 2q^8 t^1 + 2q^8 t^1}{2(q^2 t^1 - 1)^3 (q^2 t^1 + q^2 t^1 + q^2 t^1 + 1)} & 0 \end{array} \right)
\]

We give a proof of this theorem in Section 5.

The fact that the space of parameters splits into chambers with phase transitions between them is striking by itself, however, evolution formula (3) has another notable feature: there are three eigenvalues for the evolution w.r.t \(a\)-parameter, whereas in the HOMFLY (and Jones) case there were only two. What is the meaning of the third eigenvalue in the relevant \(t\)-deformation of the representation theory (where, naively, from the decomposition \([1] \otimes [1] = [2] \oplus [1, 1]\) one expects two eigenvalues) is an interesting question for future research.

The coefficient in front of extra eigenvalue turns out to be always proportional to \((t + 1)\), so it never contributes to the Jones \((t \rightarrow -1)\) limit.

One immediately sees that some matrix elements in (4) transform quite simply when one goes between the chambers – they are just multiplied by some scalars. Namely, one has the following relations

\[
M_{UR,1,2} = (-t^{-1}) M_{UL,1,2}; \quad M_{UL,1,2} = M_{LR,1,2} = (-t^{-1}) M_{LL,1,2}
\]

\[
M_{UR,2,1} = (-t^{-1}) M_{UL,2,1}; \quad M_{UL,2,1} = M_{LR,2,1} = (-t^{-1}) M_{LL,2,1}
\]

\[
M_{UR,2,3} = q^2 M_{UL,2,3}; \quad M_{UL,2,3} = M_{LR,2,3} = q^2 M_{LL,2,3}
\]

Note that the scalar factor is the same when going from LL-chamber to LR or UL and when going from LR or UL to UR. At the moment we don’t know why is this so and whether it is always so.

On the other hand, coefficients \(M_{1,1}\) and \(M_{2,2}\) transform in a more complicated way. If one plots the (Laurent) polynomials that result from multiplication of the \(M_{1,1}\) matrix elements by \((q^2 t - 1)^2 (q^2 t + q)\) on the \((q, t)\) Newton plane one sees the following picture
(when two points are drawn close to the same integer point that means that they are both at this integer point, just are drawn this way so as not to clash)

One of the possible ways of how different monomials “flow” when one goes from LL-chamber to LR-chamber to UR-chamber is denoted with arrows. It’s interesting that this flow features multiplication by same coefficients $q^2$ and $-t^{-1}$ that appear in phase transition of the less complicated matrix elements $M_{1,2}$, $M_{2,1}$ and $M_{1,3}$. The picture of the Newton plane for the elements $M_{2,2}$ is similar. It is not clear, what is the generic rule of transformation of the matrix elements of the evolution method – one needs to study more complicated families of knots to even make a guess.

Transitions between regions, where evolution is valid, seem to happen when something “essential” happens to the planar diagram as a result of adding/removing a particular crossing.

For instance, when we go from the point $(1, 1)$ to the point $(2, 1)$ the knot goes from two unknots which are not mutually linked to one unknot

and in transition from $(1, 3)$ to $(0, 3)$ Hopf link goes into unknot

it is not at all clear where and when the phase transitions occur in general – one needs to study more examples.
4. Caveats in using KnotTheory package to calculate Khovanov polynomials

Even though the program “Kh” that is included in the “KnotTheory” package for Wolfram Mathematica calculates most Khovanov polynomials very fast and answers are correct, it somehow makes mistakes for the diagrams that contain a small number of crossings.

For instance, it incorrectly calculates Khovanov polynomial of an unknot whose diagram contains just one crossing. Since we are no experts in Java, in which the program is actually written, and cannot debug it, we’ve used a workaround: we inserted an extra two-strand braid in our diagram, that contained alternating positive and negative crossings. Thanks to the second Reidemeister move such planar diagram is equivalent to the original diagram, so the Khovanov polynomial should not change. But it does have more crossings and the “Kh” program does not err on it.

5. Proof

In this section we prove the Theorem 3.1.

First of all, note, that all knots and links in family that we consider are alternating.

Second, for alternating knots and links, there is a theorem ([21, Theorem 4.5]) that allows one to express Khovanov polynomial through the Jones one. For completeness, we reproduce it here.

Theorem 5.1. [21] For an n component oriented nonsplit alternating link L with its components $S_1, \ldots, S_n$ and linking numbers $l_{jk}$ of $S_j$ and $S_k$,

\begin{equation}
Kh(L)(q, t) = q^{-\sigma(L)} \left( (q + q^{-1}) \left( \sum_{E \subset \{2, \ldots, n\}} (tq^2)^{\sum_{j \in E, k \notin E} 2l_{jk}} \right) + \left( q^{-1} + tq^2 \cdot q \right) Kh'(L)(tq^2) \right)
\end{equation}

for some polynomial $Kh'(L)$.

In our case link has at most two components. The signature, depending on the region of the parameter plane, equals

\begin{equation}
\sigma_{UR} = a - 2 \\
\sigma_{UL} = a \\
\sigma_{LR} = a \\
\sigma_{LL} = a + 2
\end{equation}

For even $a$ our diagram is a knot, whereas for odd $a$ it is a two-component link, with the components’ linking number $(b - a)/2$, so, for the sum over subsets $E \subset \{2, \ldots, n\}$ we can write

\begin{equation}
1 + \frac{1}{2} (1 - (-1)^a) (tq^2)^{b-a}
\end{equation}

So, for Khovanov and Jones polynomials in, say, UL region (one needs to adjust the value of the signature for the other regions) we can write

\begin{align}
Kh &= q^{-a} \left( (q + q^{-1}) \left( 1 + \frac{1}{2} (1 - (-1)^a) (tq^2)^{b-a} \right) + \left( q^{-1} + tq^2 \cdot q \right) Kh'(tq^2) \right) \\
J &= q^{-a} \left( (q + q^{-1}) \left( 1 + \frac{1}{2} (1 - (-1)^a) (-q^2)^{b-a} \right) + \left( q^{-1} - q^2 \cdot q \right) J'(-q^2) \right),
\end{align}
where polynomials $K'h'$ and $J'$ are actually equal, so if one considers $J'$ as a function of $q$ one can restore $K'h'$ by substituting $q \rightarrow q\sqrt{t}$.

Now, for the Jones polynomial we can obtain the following evolution on the whole parameter plane (using, for instance, RT-formalism)

$$J_{a=2l, b=2k+1} = (1)^b (-q^2)^b \, N \left( q^{-a} (-q^3)^{-a} \right)$$

Using formula (10) to first express $J'$ through $J$, then $K'h'$ through $J'$ and finally obtain $K'h$ through $a$ and $b$ one can straightforwardly check that, indeed, $K'h$ depends on $a$ and $b$ as in formulas (3) and (4). This completes the proof of the theorem.

6. Conclusion

In this paper we’ve built on the results of [1] by considering what happens to the evolution method for Khovanov polynomial for the simplest non-torus family of knots (see Section 2), that in particular includes two-strand and twist knots.

For this two-parametric family we found a peculiar chamber structure (see Section 3), which generalizes “mirror anomaly” observed in [1]. We also found an unexpected third eigenvalue of the (hypothetical) fundamental $R$-matrix, which drops out in the limit ($t \rightarrow -1$).

At the heart of the proof of our evolution formulas is the theorem [ that relates Khovanov polynomial for any alternating knot to its Jones polynomial. While computer experiments clearly show that evolution method extends beyond alternating knots, new ideas are required to extend the proof, even to the case of 2-strand braid insertion into arbitrary knot.

It is very interesting, what is the relevant generalization of the Reshetikhin-Turaev formalism. It should be capable of describing the observed chamber structure. The abrupt jumps of the evolution coefficients that occur when one goes between the chambers may hint that another deformation (in addition to $q$- and $t$-deformations) is required to smoothen these jumps out and embed the problem into the framework of the usual linear algebra. This indication is in accordance with recent developments of the representation theory of DIM algebra [2], where one more deformation parameter also begs to be introduced.

At the moment it is not clear where exactly does the simple evolution break down. There is only a vague idea that it breaks when “something interesting” happens to the planar diagram – either it goes from being disconnected to being connected, or it untwists in an unusual manner (see Section 3).

Study of more complicated families of knots is needed to clarify the situation. To perform such a study one needs better computer programs that allow to specify families of knots more conveniently (manually working out through all the odd-even cases for parameters and figuring orientations of strands is a bit tedious). We continue to work in this direction.

References

[1] A.Anokhina and A.Morozov, Are Khovanov-Rozansky polynomials consistent with evolution in the space of knots?, JHEP 1804 (2018) 066, arXiv:1802.09383
[2] Y. Zenkevich, *3d field theory, plane partitions and triple Macdonald polynomials*, arXiv:1712.10300
[3] V. F. R. Jones, *Invent. Math. 72 (1983) 1 Bull. AMS 12 (1985) 103 Ann. Math. 126 (1987) 335;*
[4] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millet, A. Ocneanu, *Bull. AMS. 12 (1985) 239;*
[5] J. H. Przytycki and K. P. Traczyk, *Kobe J. Math. 4 (1987) 115-139*
[6] N. Reshetikhin and V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. 127 (1990) 126
[7] A. Morozov and A. Smirnov, *Chern-Simons theory in the temporal gauge and knot invariants through the universal quantum R-matrix*, Nucl. Phys. B 835:284-313, 2010, arXiv:1001.2003
[8] M. Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000) 359-426
[9] M. Khovanov and L. Rozhansky, *Matrix factorizations and link homology*, Fund. Math. 199 (2008), math.QA/0401268
[10] M. Khovanov and L. Rozhansky, *Matrix factorizations and link homology II*, Geom. Topol. 12 (2008) 1387, math.QA/0505056
[11] D. Bar-Natan, *On Khovanov’s categorification of the Jones polynomial*, Algebraic and Geometric Topology, 2 (2002) 337-370, math/0201043
[12] L. H. Robert and E. Wagner *Symmetric Khovanov–Rozansky link homologies*, arXiv:1801.02244
[13] V. Dolotin and A. Morozov *Introduction to Khovanov Homologies. I. Unreduced Jones superpolynomial*, JHEP 1301 (2013) 065, arXiv:1208.4994
[14] V. Dolotin and A. Morozov *Introduction to Khovanov Homologies. II. Reduced Jones superpolynomials*, 2013 J. Phys.: Conf. Ser. 411 012013, arXiv:1209.5109
[15] V. Dolotin and A. Morozov *Introduction to Khovanov Homologies. III. A new and simple tensor-algebra construction of Khovanov-Rozansky invariants*, Nuclear Physics B878 (2014) 12-81, arXiv:1308.5759
[16] A. Anokhina and A. Morozov *Towards R-matrix construction of Khovanov-Rozansky polynomials. I. Primary T-deformation of HOMFLY*, JHEP07(2014)063, arXiv:1403.8087
[17] A. Anokhina *Towards formalization of the soliton counting technique for the Khovanov-Rozansky invariants in the deformed R-matrix approach*, arXiv:1710.07306
[18] A. Anokhina *On R-matrix approaches to knot invariants*, arXiv:1412.8444
[19] S. Arthamonov and Sh. Shakirov *Genus two generalization of A_1 spherical DAHA*, arXiv:1704.02947
[20] http://www.katlas.org
[21] E. S. Lee *An endomorphism of the Khovanov invariant*, arXiv:math/0210213
[22] A. Mironov, A. Morozov and And. Morozov *Evolution method and “differential hierarchy” of colored knot polynomials*, AIP Conf. Proc. 1562 (2013) 123; doi:10.1063/1.4828688; arXiv:1306.3197
[23] A. Mironov, A. Morozov, And. Morozov and A. Sleptsov *Racah matrices and hidden integrability in evolution of knots*, Phys. Lett. B760 (2016) 45-58; doi:10.1016/j.physletb.2016.06.041; arXiv:1605.04881
[24] P. Dunin-Barkowski, A. Mironov, A. Morozov, A. Sleptsov and A. Smirnov *Superpolynomials for toric knots from evolution induced by cut-and-join operators*, JHEP 03 (2013) 021; doi:10.1007/JHEP03(2013)021; arXiv:1106.4305
Higher School of Economics, Moscow, Russia
E-mail address: ptdbar@gmail.com

Department of Physics and Astronomy, Uppsala University, Box 516, SE-75120 Uppsala, Sweden.
E-mail address: popolit@gmail.com

Sub-department of Financial Management, MSTU “STANKIN”, Moscow, Russia
E-mail address: spopolitova@yandex.ru