Minimum Distances in Non-Trivial String Target Spaces

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ABSTRACT

The idea of minimum distance, familiar from $R \leftrightarrow 1/R$ duality when the string target space is a circle, is analyzed for less trivial geometries. The particular geometry studied is that of a blown-up quotient singularity within a Calabi-Yau space and mirror symmetry is used to perform the analysis. It is found that zero distances can appear but that in many cases this requires other distances within the same target space to be infinite. In other cases zero distances can occur without compensating infinite distances.
1 Introduction

Questions regarding very small distances in theories of quantum gravity inevitably run into conceptual difficulties. Classical general relativity is based on the concept of a space-time metric and so in this context one can be clear about one’s definition of length. In a theory of quantum gravity this issue is more model dependent. In models such as string theory, which to date appears to provide by far the most promising candidate for a consistent theory of quantum gravity, the metric does not appear as a fundamental concept. In order to make contact with classical gravity one analyzes the non-linear $\sigma$-model as an effective theory for the string theory. One may then retrieve the classical equations of general relativity \cite{1} in some low-energy limit.

Within the context of string theory therefore the most natural definition of length would appear to come from the non-linear $\sigma$-model. A conventional description of string theory might be as a theory based on 1 dimensional strings such that the action is given by the area of the world-sheet swept out. Our approach will be somewhat the reverse of this — one assumes string theory has a more fundamental definition than above and one defines area to be the object determining the action. This definition of area and thus distance is what is meant by saying that the string non-linear $\sigma$-model is used to define distance.

This approach of using the $\sigma$-model is effectively used in the well-known result \cite{2,3} concerning a string with a circle of radius $R$ as target space. Namely, such a theory is equivalent to a string on a circle of radius $\alpha'/R$. Thus the complete moduli space of strings on circles should only contain the interval $0 \leq R \leq \sqrt{\alpha'}$ or $\sqrt{\alpha'} \leq R \leq \infty$ but not both. Since we assume we want to make contact with classical physics at distance scales $\gg \sqrt{\alpha'}$ it is sensible to choose the latter. Thus we say that string theory “cuts off” distances $< \sqrt{\alpha'}$ since they do not appear anywhere in the moduli space.

This idea of a minimum distance of order $\sqrt{\alpha'}$ fits in well with other arguments both within the context of string theory and otherwise. A direct argument for a minimum distance in string theory along the lines of an uncertainty principle was made in \cite{4}. This was also in agreement with earlier ideas of \cite{5} and the results suggested by scattering strings at high energies studied in \cite{4}. One can also argue for minimum lengths to appear naturally in other forms of quantum gravity — see, for example, \cite{6} for a review of these ideas.

In \cite{8} more complicated string target spaces were considered with the condition of $N=(2,2)$ world-sheet supersymmetry imposed. For a compact smooth target space of definite metric (and vanishing “torsion”) this implies the Calabi-Yau condition \cite{9}. In the case of a circle (or a torus, for which similar results occur) the minimum distance appears due to non-trivial $\pi_1$, i.e., fundamental group, for the target space. The $R \leftrightarrow \alpha'/R$ symmetry appears due to an interchange of momentum modes and winding modes. In \cite{8}, and in this paper, the target spaces are simply-connected but have nontrivial $\pi_2$. Thus it is instantons.
rather than solitons which provide the interesting structure.

The approach taken in [8] was similar to the above method for finding the minimum radius of a circle. That is, one constructs the moduli space such that it contains theories corresponding to large distances and thus, presumably, classical general relativity and then one labels each point in the moduli space according to the size of target space it corresponds to. The minimum size is then the smallest value thus acquired in the entire moduli space. In the case of the circle there is only one distance that can be measured, namely the radius of the circle. In the case of a less trivial target space there are potentially many sizes corresponding to different parts of the target space that can be independently varied (at least classically). In [8] only one size at a time was measured while the other “independent” sizes were held at infinity. It was in this context that some zero distances were measured.

We are therefore now in contradiction with other work on minimum distances. All of the other works assumed that space looks, to a greater or lesser degree, locally flat. While this may appear quite reasonable within the context of classical general relativity, it is not at all clear that such an assumption is always valid in string theory. In particular, the perturbative analysis of the non-linear $\sigma$-model loop by loop gives conditions for conformal invariance similar to the analysis performed in [6]. It is known however that non-perturbative effects are very important in the non-linear $\sigma$-model and indeed it is precisely these effects we are studying in this paper. It should therefore not be a complete surprise if the results in [8] and this paper are not quite in agreement with a universal minimum length.

The main purpose of this paper will be to relax the constraint imposed in [8] of holding some sizes at infinity, i.e., we will ask the question as to whether zero distances can be measured within a target space of finite size.

By varying the various sizes with the target space, i.e., by varying various components of the Kähler form, various things might happen classically to result in some zero distance. Firstly, the “sides” of the target space may be brought together so that the resultant dimension of the target space decreases. An extreme example of this is shrinking the entire space down to a point. All the deformations of the Kähler form on a torus resulting in zero size are of this dimension-lowering type. It was found in [8], at least within the context of models of the type studied in [10], that string theory imposed a non-zero minimum distance on such deformations. Another possibility that can arise as the Kähler form is varied occurs when a subspace of the target space shrinks down to nothing but the rest of the target space keeps its dimensionality intact. It was found that in this case, again at least for the class of models considered, string theory never imposes a nonzero limit on such a process. Thus it might appear that the general philosophy of a minimum size or volume might by true in some form in the context of string theory but that it would be too naïve to state it in terms a simply as “string theory removes from all consideration distances $< \sqrt{\alpha'}$.

In this paper we wish to concentrate on a particular example of a subspace shrinking
down. This will consist of some divisor (i.e., complex codimension one space) being “blown-down” resulting in a singular target space where the singularity is locally of the form of a quotient singularity. String theory on this singular space, known as an “orbifold”, is fairly well-understood [11]. See, for example, [12] for a review of the stringy analysis of such a blowing-down procedure. It is common in the physics literature to assert that an orbifold can be written in the form $M/G$ for some smooth manifold $M$ and some finite group $G$. We will use the more general definition in which the singularities need only be written locally in the form of a quotient. Since we are only really interested in the properties of marginal operators in this paper and since the twisted marginal operators are localized around the quotient singularities, whether the quotient singularities are globally quotients should not be particularly important for our purposes.

As indicated above, the various sizes within a single target space are classically independent. The question we will be addressing therefore lies in the realms of quantum geometry. This should come as no surprise since our problem involves small distances which is where we expect classical geometry to break down, at least within the context of string theory. As we shall see the minimum size of one part of the target space does generally depend on some of the other sizes within the target.

In section 2 we will review how to build examples of the desired moduli space using mirror symmetry. For this reason our discussion will allow us to analyze only models with an $N=(2,2)$ superconformal symmetry. We will also review the means by which the mirror map may be used to label each point in the moduli space by the size of target space to which it corresponds.

In section 3 we will perform this labeling by solving a system of hypergeometric equations. Due to the technicality of this process we will concentrate on a couple of examples rather than attempt any general discussion of a solution. Finally in section 4 we will discuss the meaning of the results.

2 Building the Moduli Space

In this section we will construct moduli spaces of conformal field theories. This should be thought of as the stringy analogue of the moduli space of Ricci-flat metrics for vacuum solutions of classical general relativity. Little technology currently exists for building the moduli space of a generic conformal field theory but in the case of $N=(2,2)$ superconformal field theories the situation is far more tractable. If one further imposes that the central charge have value $c = 3d$ for $d \in \mathbb{Z}$ and that for any NS field in the theory, $Q, \bar{Q} \in \mathbb{Z}$, where $Q$ and $\bar{Q}$ are the left and right-moving $U(1)$ charges respectively, then there is a general belief that this conformal field theory should be equivalent to a non-linear $\sigma$-model with a
target space of a Calabi-Yau $d$-fold or some generalized notion thereof. The most promising generalized notion is probably the infrared fixed point of the gauged linear $\sigma$-model with Landau-Ginzburg-type potential introduced in [13]. We will assume that our conformal field theory corresponds to some Calabi-Yau manifold $X$ or at least belongs to a phase in a moduli space which also contains a phase for $X$ in the sense of [13, 14].

Around any point in the moduli space, the tangent directions are given by marginal operators. For the theory in questions we can divide these marginal operators into two groups. One group gives deformations of complex structure of $X$ and the other gives deformation of the complexified Kähler form of $X$. Recall that the complexification of the Kähler form arises from the general form of the non-linear $\sigma$-model:

$$S = \frac{i}{4\pi\alpha'} \int \left\{ g_{ij}(\partial u^i \bar{\partial} u^j + \bar{\partial} u^i \partial u^j) - iB_{ij}(\partial u^i \bar{\partial} u^j - \bar{\partial} u^i \partial u^j) \right\} d^2 z,$$  

(1)

The field theory based on this action depends only the cohomology class of the real two-form given by $B_{ij}$. Assuming $h^{2,0}(X) = 0$, we construct $B + iJ \in H^2(X, \mathbb{C})$ as the complexified Kähler form where $J$ is the classical Kähler form derived from $g_{ij}$. An important observation is that, semi-classically, the field theory also obeys the symmetry

$$B + iJ \cong B + iJ + 4\pi^2 \alpha' L, \quad \forall L \in H^2(X, \mathbb{Z}).$$  

(2)

The deformations of complex structure have no quantum corrections. That is, the marginal operators corresponding to such deformations of the conformal field theory may be integrated along to build up a moduli space which is isomorphic to the classical moduli space of complex structures of $X$. The other marginal operators are different however. Instantons affect the form of this moduli space so that the classical moduli space and conformal field theory moduli space differ. See, for example, [15] for an account of the way instanton affects appear in this context. The classical moduli space for $B+iJ$ would be of the form $X \times (S^1)^{h^{1,1}}$ where $X$ is the classical moduli space of $J$, i.e., the Kähler cone. The cone-like structure appears since if $J$ is a valid Kähler form on $X$ then so is $\lambda J$ for $\lambda$ a positive real number. It is precisely this structure which should be modified if we expect minimum distances to appear — if $J$ is a valid Kähler form for the string target space it should not necessarily follow that $\lambda J$ is too.

The trick which allows us to construct the entire conformal field theory moduli space is that provided by mirror symmetry [16]. Namely there may be another space $Y$ on which a non-linear $\sigma$-model produces the same conformal field theory as that on $X$ except that the rôle of the marginal operators are interchanged: those giving deformations of complex structure on $Y$ give deformations of Kähler form on $X$ and visa versa. Thus to build the stringy version of the moduli space of Kähler forms on $X$ we simply build the moduli space of complex structures on $Y$ and assert it is isomorphic to the desired moduli space.
As an example we will consider the Calabi-Yau manifold studied at some length in [17]. Let the space \( \mathbb{P}^4_{2,2,2,1,1} \) be defined as \( (\mathbb{C}^5 - \{0\})/\mathbb{C}^* \) where the coordinates of \( \mathbb{C}^5 \) are \((X_0, X_1, \ldots, X_5)\) and the \( \mathbb{C}^* \)-action is given by

\[
(X_0, X_1, X_2, X_3, X_4) \mapsto (\lambda^2 X_0, \lambda^2 X_1, \lambda^2 X_2, \lambda X_3, \lambda X_4), \quad \lambda \in \mathbb{C}^*. \tag{3}
\]

This space has a quotient singularity locally of the form \( \mathbb{C}^2/\mathbb{Z}_2 \) along the \( \mathbb{P}^2 \), \( X_3 = X_4 = 0 \). This quotient singularity may be blown up with \( \mathbb{P}^1 \times \mathbb{P}^2 \) as exceptional divisor. Let \( X \) be the resolved hypersurface

\[
X_0^4 + X_1^4 + X_2^4 + X_3^8 + X_4^8 = 0 \tag{4}
\]

within the resolved \( \mathbb{P}^4_{2,2,2,1,1} \). \( X \) has \( h^{1,1} = 2 \), with one component given by the ambient \( \mathbb{P}^4_{2,2,2,1,1} \) and the other given by the exceptional divisor in the blow-up. The method of [16] tells us how to construct the mirror, \( Y \), as a blown-up orbifold of this. In order to study the moduli space \( M \) of complex structures of \( Y \), we need not concern ourselves with these latter blow-up modes. Instead we write down the most general form of defining equation for \( Y \) compatible with the orbifolding. This is

\[
W = a_0 X_0 X_1 X_2 X_3 X_4 + a_1 X_3^4 X_4^4 + a_2 X_0^4 + a_3 X_1^4 + a_4 X_2^4 + a_5 X_3^8 + a_6 X_4^8, \tag{5}
\]

where \( a_j \) are arbitrary complex numbers. Rescaling the \( X_i \) coordinates gives a 5-dimensional space of reparametrizations of this equation thus leaving the \( a_j \)'s to span a 2-dimensional moduli space in agreement with \( h^{1,1}(X) = 2 \). Actually, as explained in [14] this gives the moduli space the structure of a toric variety. This toric information also allows us to build the natural compactification, \( \mathcal{M} \), of \( M \) within the context of mirror symmetry by using the “secondary fan”. This procedure requires a fairly lengthy explanation and the reader is encouraged to consult [14] for a full account of this process.

The general result required in this context is as follows. Let there be \( n \) “homogeneous” coordinates \( X_i, i = 0, \ldots, n - 1 \) and let there be \( N \) terms in the general defining equation for \( Y \) with coefficients \( a_j, j = 0, \ldots, N - 1 \). The moduli space can then be built as a toric variety as a compactification:

\[
\mathcal{M} \supset (\mathbb{C}^*)^N / (\mathbb{C}^*)^n \cong (\mathbb{C}^*)^{N-n}. \tag{6}
\]

The dimension of the moduli space will thus be \( N - n \). This may be equal to, or less than \( h^{1,1}(X) \). The latter case arises when independent elements of homology of \( X \) cannot be written in terms of independent elements of homology of the ambient space. In this case we will not be studying the entire moduli space of complexified Kähler forms of \( X \) but only the “toric part”. The compact space \( \mathcal{M} \) is described by coordinate patches each having
coordinates $z_l, l = 1, \ldots, N - n$ given by some $(\mathbb{C}^*)^n$-invariant product (or quotient) of the $a_j$’s. The precise $(\mathbb{C}^*)^n$-invariant product used for each coordinate in each patch dictates how the patches are sewn together and is given by the data of the secondary fan — one cone for each coordinate patch.

Let us now try to identify each point in $\mathcal{M}$ with a particular $X$. Each origin of each coordinate patch marks a special point in $\mathcal{M}$ called a “limit point”. The number of limit points is thus equal to the number of coordinate patches used to build $\mathcal{M}$ which is equal to the number of cones in the secondary fan. The work of [13] and [14, 18, 19] may then be used to identify one of these limit points with the conformal field theory associated to the non-linear $\sigma$-model defined on the large radius limit of $X$. To be more precise, let us expand the Kähler form

$$B + iJ = \sum_l (B + iJ)_l e_l,$$  \hspace{1cm} (7)

where $e_l$ are positive generators of $H^2(X, 4\pi^2\alpha'\mathbb{Z})$. That is, $\int_C e_l$, is positive for any (one complex dimensional) curve in $X$. It is then claimed [19] that one of the coordinate patches within $\mathcal{M}$ given by $z_l(0)$ specifies $X$ as

$$(B + iJ)_l = \frac{1}{2\pi i} \log z_l(0) + O(z_l(0), z_l(0)^2, \ldots),$$ \hspace{1cm} (8)

near the origin $z_l(0) = 0, l = 1, \ldots, N - n$. The origin itself thus corresponds to the limit of $X$ with all components of $J \to \infty$. This is the “large radius limit”.

By making this identification we have thus ensured that large distances are included in our moduli space. In some respects this is similar to the step of choosing the interval $\sqrt{\alpha'} \leq R \leq \infty$ rather than $0 \leq R \leq \sqrt{\alpha'}$ in the definition of the stringy moduli space of a circle.

Now all we need to do is to extend the definition of $(B + iJ)_l$ away from the large radius limit over the whole of $\mathcal{M}$. To attempt to do this we use the result conjectured in [20] and proven in [21]. This result states that local geometry of the moduli space of $N=2$ theories dictates that there exist elements of homology $\gamma_0$ and $\gamma_l$ on $Y$ such that

$$(B + iJ)_l = \int_{\gamma_l} \Omega \int_{\gamma_0} \Omega,$$ \hspace{1cm} (9)

where $\Omega$ is a highest holomorphic form on $Y$. The choice of which cycles are used as $\gamma_0$ and $\gamma_l$ is governed by the global geometry of the moduli space as discussed in [22]. The periods in (3) (i.e., the numerator and denominator in the right hand side) may be evaluated as in
In this paper, as in [8], we will proceed in a slightly indirect manner. We find some differential equations satisfied by \( \int \gamma \Omega \) and then use (8) to choose which solutions we require. This system of differential equations is known as the “hypergeometric system” to which we now turn our attention.

### 3 The Hypergeometric System

Following the construction of [10], a hypersurface in a \((n - 1)\)-dimensional toric variety may be represented by a set of \( N \) points \( \mathcal{A} \) in a hyperplane in \( \mathbb{R}^n \). Each point in \( \mathcal{A} \) may be thought of as representing each monomial in some constraint such as (5) where the more generalized notion of homogeneous coordinate [24] may be used. The coordinates of each point are given by the degrees to which each homogeneous coordinate is raised. A lattice structure is also imposed within the space \( \mathbb{R}^n \) such that the point set \( \mathcal{A} \) is the intersection of this lattice with some convex polytope. Let us now define new coordinates based on this lattice and let \( \alpha_j \in \mathcal{A} \) have coordinates \( \alpha_{ji} \) for \( i = 0, \ldots, n - 1 \). The hypergeometric system [25] is formed by the set of differential operators

\[
Z_i = \left( \sum_{j=0}^{N-1} \alpha_{ji} a_j \frac{\partial}{\partial a_j} \right) - \beta_i
\]

\[
\Box_l = \prod_{m_{ij} > 0} \left( \frac{\partial}{\partial a_j} \right)^{m_{ij}} - \prod_{m_{ij} < 0} \left( \frac{\partial}{\partial a_j} \right)^{-m_{ij}},
\]

where \( i = 0, \ldots, n - 1 \) and \( l = 1, \ldots, N - n \) labels a relationship

\[
\sum_{j=0}^{N-1} m_{ij} \alpha_{ji} = 0, \quad \forall i.
\]

We may tie the labeling of these relationships in with the coordinate patches on \( \mathcal{M} \) we used in section [2]. If we write

\[
z_l = \prod_{j=0}^{N-1} a_j^{m_{ij}}
\]

then the \((\mathbb{C}^*)^n\) invariance of \( z_l \) is equivalent to the condition (14). Now we look for a function \( \Phi(a_0, a_1, \ldots, a_{N-1}) \) such that

\[
Z_i \Phi = \Box_l \Phi = 0, \quad \forall i, l.
\]

Let us further require that our coordinates for the lattice are chosen such that the hyperplane condition is imposed by \( \alpha_{j0} = 1 \) for any \( j \) and let the unique point [10] within the interior
of the polytope be $\alpha_0$ and have coordinates $\alpha_{0i} = 0$ for $i > 0$. It was then shown in [26] that the periods $\int_\gamma \Omega$ satisfy (13) for $\beta_0 = -1$ and $\beta_i = 0$ for $i > 0$.

It was shown in [25] that a solution of (13) may be written

$$\Phi_\gamma = \left( \prod_{j=0}^{N-1} a_j^\gamma \right) \sum_{\{P_l\} \in \mathbb{Z}^{N-n}} \frac{\prod_{l=1}^{N-n} z_l^{P_l}}{\prod_{j=0}^{N-1} \Gamma \left( \sum_{l=1}^{N-n} m_{lj} P_l + \gamma_j + 1 \right)} , \quad (14)$$

where $\gamma$ is a vector in $\mathbb{R}^N$ such that

$$\sum_{j=0}^{N-1} \gamma_j \alpha_{ji} = \beta_i . \quad (15)$$

In fact it is a simple matter to show that (14) formally satisfies (13). Treating $\Phi_\gamma$ as a sum of terms over the lattice $\mathbb{Z}^{N-n}$, one can show that $Z_l \Phi_\gamma$ vanishes individually at each lattice site. The two terms in $\Box_l$ in (10) act on $\Phi_\gamma$ to produce the same expression except shifted by some lattice vector. Thus the sum over the whole lattice vanishes. The only property of the gamma function used to show this result is $\Gamma(x + 1) = x \Gamma(x)$.

It is clear that for a generic $\gamma$, the expression (14) will have zero radius of convergence for $z_l$ and is thus somewhat useless. If $\gamma$ is tuned to the right value however, the fact that $\Gamma(x)$ acquires poles for $x = 0, -1, -2, -3, \ldots$ may be used to reduce the sum over all $P_l$ to only $P_l \geq 0$. One would then expect (14) to converge for sufficiently small $|z_l|$. In the generic case this is how one generates all the generalized hypergeometric series which solve the hypergeometric system, i.e., all solutions may be written in the form $\Phi_\gamma$ for a suitable set of $\gamma$.

If $\alpha_{ji}$ and $\beta_i$ satisfy certain “resonance” conditions [25] then not all the solutions of the hypergeometric system may be written in the form (14). As luck would have it, for the case we are considering, these parameters resonate sufficiently badly that none of the solutions may be written in this form. As we now explain however we may still use the expression (14) as a starting point for writing down all the required solutions. This issue was also studied in [27]. Our method is similar but will use Barnes-type integrals which directly allow the solutions to be analytically continued.

Firstly note that the expression

$$\Gamma(x) \Gamma(1-x) = \frac{\pi}{\sin(\pi x)} , \quad (16)$$
may be used to take gamma functions from the denominator into the numerator in (14). Suppose $M$ is an integer $\geq 0$. One may then write

$$\frac{1}{\Gamma(-M + \epsilon)} \simeq \Gamma(M + 1)(-1)^M \epsilon,$$  

for small $\epsilon$. Ignoring for a moment the $\epsilon$ factor on the right-hand side, in the limit $\epsilon \to 0$, this identity may be used to remove some poles from the denominator in (14). One can argue that the $\epsilon$ factor on the right side of (17) contributes only an overall scale to the solution and can thus be ignored, even though it’s zero! If the reader is skeptical of this manipulation, they should substitute the expression thus obtained back into the hypergeometric system to verify that it is indeed a solution.

Let us now illustrate this procedure by applying it to the example in (5). Denoting $z_1$ and $z_2$ by $x$ and $y$ for simplicity, analysis of the secondary fan shows that

$$x = \frac{a_5 a_6}{a_1^2}, \quad y = \frac{a_1 a_2 a_3 a_4}{a_4^2},$$  

are the coordinates whose origin corresponds to the large radius limit. Now, if we use the value $\gamma = (-1, 0, 0, \ldots)$ in (14) we have

$$a_0^{-1} \sum_{M,N} \frac{x^M y^N}{\Gamma(-4N) \Gamma^2(M + 1) \Gamma^3(N + 1) \Gamma(N - 2M + 1)}.$$  

This is zero since every term in the sum vanishes. Using the above trick however we may move $\Gamma(-4N)$ up into the numerator giving

$$\Phi_0 = a_0^{-1} \sum_{N \geq 0, 0 \leq M \leq N/2} \frac{\Gamma(4N + 1) x^M y^N}{\Gamma^2(M + 1) \Gamma^3(N + 1) \Gamma(N - 2M + 1)}.$$  

This is the unique solution (up to an overall scale) of the hypergeometric system which is regular at $(x, y) = (0, 0)$. No other value of $\gamma$ gives a convergent series and so we must obtain all our solutions from this single value of $\gamma$. We may write (20) as a Barnes integral:

$$\Phi_0 = a_0^{-1} \sum_{M \geq 0} \frac{x^M}{\Gamma^2(M + 1)} \frac{1}{2\pi i} \int_C \frac{\Gamma(4s + 1) \Gamma(2M - s)}{\Gamma^3(s + 1)} (e^{-\pi i y})^s ds,$$  

where $C$ is a contour running to the left of $\text{Re}(s) = 0$ as shown below (the poles are shown
for the $M = 0$ term — more generally poles will not occur from $s = 0, \ldots, 2M - 1$:

\begin{equation}
C
\end{equation}

We will discuss conditions for convergence of this integral below. One may think of obtaining (21) from (20) by moving $\Gamma(N - 2M + 1)$ up from the denominator into the numerator. This introduces poles which are rendered finite by the contour integral. The contour is chosen so that it may be completed to the right to enclose the correct values of $s$ to contribute to the sum. We may use this idea further to generate more solutions.

If we further move a $\Gamma(N + 1)$ from the denominator we obtain

\begin{equation}
\Phi_y = -a_0^{-1} \sum_{M \geq 0} \frac{x^M}{\Gamma^2(M + 1)} \frac{1}{2\pi i} \int_C \frac{\Gamma(4s + 1)\Gamma(-s)\Gamma(2M - s)}{\Gamma^2(s + 1)} y^s \, ds, \quad (23)
\end{equation}

along the same contour. Enclosing the contour to the right and taking residues we obtain

\begin{equation}
\Phi_y = a_0^{-1} \sum_{N \geq 0} \left\{ \sum_{M \leq N/2} \frac{\Gamma(4N + 1)x^M y^N}{\Gamma^2(M + 1)\Gamma^3(N + 1)\Gamma(N - 2M + 1)} \right. \left[ \log y + 4\Psi(4N + 1) - 3\Psi(N + 1) - \Psi(N - 2M + 1) \right] \right. \\
\left. \left. - \sum_{M > N/2} \frac{\Gamma(4N + 1)\Gamma(2M - N)x^M (-y)^N}{\Gamma^2(M + 1)\Gamma^3(N + 1)} \right\} \quad (24)
\end{equation}

This is also a solution of the hypergeometric system. The appearance of the logarithm is precisely what we required from (8). In fact (8), (9) and the behaviour of all the other solutions around the origin allow us to specify uniquely that

\begin{equation}
(B + iJ)_y = \frac{1}{2\pi i} \frac{\Phi_y}{\Phi_0}. \quad (25)
\end{equation}
The other component \((B + iJ)_x\) may be obtained similarly although this time more care must be taken to enclose the required poles with the contour. Let us consider

\[
\Phi_x = a_0^{-1} \sum_{N \geq 0} \frac{\Gamma(4N + 1)(-y)^N}{\Gamma^3(N + 1)} \frac{2}{2\pi i} \int_C \frac{\Gamma(2s - N)\Gamma(-s)}{\Gamma(s + 1)} (e^{-\pi_i x})^s ds,
\]

(26)

with the contour given by (e.g., \(N = 4\))

\[
\Phi_x = a_0^{-1} \sum_{N \geq 0} \left\{ \sum_{M \leq N/2} \frac{\Gamma(4N + 1)x^My^N}{\Gamma^2(M + 1)\Gamma^3(N + 1)\Gamma(N - 2M + 1)} \left[ \log x - \pi i 
\right.ight.

\[
+ 2\Psi(N - 2M + 1) - 2\Psi(M + 1) \right]\n
\[
+ 2 \sum_{M > N/2} \frac{\Gamma(4N + 1)\Gamma(2M - N)x^M(-y)^N}{\Gamma^2(M + 1)\Gamma^3(N + 1)} \right\}.
\]

(28)

We then see that

\[
(B + iJ)_x = \frac{1}{2\pi i} \left( \frac{\Phi_x}{\Phi_0} + \pi i \right).
\]

(29)

We have therefore found expressions giving the exact value of \(B + iJ\) at points in the moduli space \(\mathcal{M}\). Unfortunately the expressions (24) and (28) only lead to convergent series for sufficiently small \(|x|\) and \(|y|\) and cannot be used for the whole of \(\mathcal{M}\). In particular they are not valid for areas of \(\mathcal{M}\) where we would expect to find minimum distances. In order to find these minimum distances we analytically continue these hypergeometric functions and since we have written all the relevant expressions in terms of Barnes integrals this is a simple
matter. There is an issue that we have not paid sufficient attention to yet which is of great importance, namely that of branch cuts. In order to analytically continue our functions over \( \mathcal{M} \) we are required to specify the branch cuts — different branch cuts lead to different values of \( B + iJ \) associated to each point. Indeed, this property may lead one to assert that \( B + iJ \) does not make much sense outside the region of convergence of the above series. It turns out however that at least some branch cuts may be made naturally leading to a natural concept of \( B + iJ \) away from this region. These are the branch cuts which have already been made implicitly above.

The identification \((B + iJ)_l \cong (B + iJ)_{l+1}\) requires us to make some choice of interval for the \( B \)-field. We will cut so that \( 0 < B_1 < 1 \) (note that the cut \(-1 < B < 0\) was implicitly made in [8, 20] hence some minor differences in our formulæ here). This imposes

\[
0 < \arg \left\{ \frac{x}{y} \right\} < 2\pi. \tag{30}
\]

All of the above contour integrals are convergent for sufficiently small \(|x|\) and \(|y|\) and the condition (30). Assuming this is the only branch cut in the neighbourhood of the large radius limit, this fixes the cut. We analytically continue the above periods simply by completing the contour to the left rather than the right.

The region of \( \mathcal{M} \) that we are interested in probing is where zero distances appear. This happens for the exceptional divisor that grew out of the \( \mathbb{Z}_2 \)-quotient singularity in the ambient space. The phase of \( \mathcal{M} \) we wish to study is the “orbifold phase”. That is, where the limit point corresponds to \( X \) infinitely large with the \( \mathbb{Z}_2 \) singularity unresolved. The method of [14, 8] tells us that the patches in this region are given by

\[
\xi = \frac{a_1^2}{a_5a_6} = x^{-1}, \quad \eta = \frac{a_2a_2^2a_4a_5a_6}{a_6^8} = xy^2. \tag{31}
\]

The period \( \Phi_0 \) actually does not require analytical continuation, a change of variable gives

\[
\Phi_0 = a_0^{-1} \sum_{P,Q \in \mathbb{Z}/2\mathbb{Z} \geq 0} \frac{\Gamma(8Q+1)\xi^P\eta^Q}{\Gamma^3(2Q+1)\Gamma(2P+1)\Gamma^2(Q-P+1)}. \tag{32}
\]

If we enclose the contour to the left in (26) and take residues we obtain

\[
\Phi_x = -\pi ia_0^{-1} \left\{ \sqrt{\xi} \sum_{M,N \geq 0} \frac{\Gamma(8N+1)\xi^M\eta^N}{\Gamma^2(N-M+\frac{1}{2})\Gamma^3(2N+1)\Gamma(2M+2)} \right. \\
+ \left. \sqrt{\eta} \sum_{M,N \geq 0} \frac{\Gamma(8N+5)\xi^M\eta^N}{\Gamma^2(N-M+\frac{3}{2})\Gamma^3(2N+2)\Gamma(2M+1)} \right\}. \tag{33}
\]
We may also write

\[
\Phi_x + 2\Phi_y = a_0^{-1} \sum_{N \geq 0, 0 \leq M \leq N/2} \frac{\Gamma(4N + 1) x^M y^N}{\Gamma^2(M + 1) \Gamma^3(N + 1) \Gamma(N - 2M + 1)} \left[ \log(xy^2) - \pi i \right. \\
\left. + 8\Psi(4N + 1) - 6\Psi(N + 1) - 2\Psi(M + 1) \right]
\]

\[
= a_0^{-1} \sum_{P, Q \in \mathbb{Z}/2 \geq 0, Q - P \in \mathbb{Z} \geq 0} \frac{\Gamma(8Q + 1) \xi^P \eta^Q}{\Gamma^3(2Q + 1) \Gamma(2P + 1) \Gamma(Q - P + 1)} \left[ \log \eta - \pi i \right. \\
\left. + 8\Psi(8Q + 1) - 6\Psi(2Q + 1) - 2\Psi(Q - P + 1) \right].
\]

(34)

Now we need to discuss branch cuts in this orbifold phase. There are branch points at the boundary of the region of convergence of the periods in the orbifold phase just as in the Calabi-Yau phase above. Thus the arguments of \(\xi\) and \(\eta\) should only be allowed to span an interval of \(2\pi\). It is easy to see that \(-2\pi < \arg\xi < 0\) from (31). For \(\eta\) we need to look at the discriminant locus which should be considered as a locus of branch points in some sense. One component of the discriminant is given by \([17]\) as

\[
(4^4 y - 1)^2 = 4^9 x y^2
\]

i.e., \((4^4 \sqrt{\eta \xi} - 1)^2 = 4^9 \eta\).

(35)

Thus for \(x = 0\) we have the branch point at \(y = 4^{-4}\). As \(x\) increases from zero this branch point will divide into two. The region of allowed \(\arg(\eta)\) will thus get squeezed into a smaller region between the branch cuts. Let \(x = Re^{\pi i}\) for a real positive number \(R\) and let \(R \to \infty\). One may follow the allowed region of \(\arg(\eta)\) to avoid the discriminant to find that \(\pi/2 < \arg(\eta) < 3\pi/2\). Applying \(\eta = x y^2\) we find

\[
-2\pi < \arg\xi < 0
\]

\[
2\pi < \arg\eta < 4\pi.
\]

(36)

Actually a simple guiding principle may be used for the general case. The discriminant locus is described by a large polynomial with integer (and thus real) coefficients. Thus if we originally have an allowed region of \(0 < \arg(z_l) < 2\pi\) close to the limit point then as we switch on values of \(z_k = R_k e^{\pi i}\) for the other parameters, the region of allowed \(\arg(z_l)\) must be squeezed symmetrically around \(\arg(z_l) = \pi\). Thus, if any region of the origin of one phase between cuts can be followed into a neighbouring region, it contains the direction
arg($z_l$) = $\pi$ for all $l$. We substitute this value into the change of variables from one phase to another to find the central value of arg($z'_l$) for the other phase.

Before discussing the interpretation of the above results it will be helpful to have another example in mind. For this we will look at the five parameter moduli space studied in [18, 14, 8]. In this case $X$ is the resolution of a hypersurface in $\mathbb{P}^4_{\{6,6,3,2,1\}}$ and $Y$ has general defining equation

$$a_0X_0X_1X_2X_3X_4 + a_1X_2^3X_4^9 + a_2X_3^6X_4^6 + a_3X_3^3X_4^{12} + a_4X_2^3X_3^3X_4^3 + a_5X_0^3 + a_6X_1^3 + a_7X_2^6 + a_8X_3^9 + a_9X_4^{18} = 0.$$ (37)

This has a remarkably rich phase structure [14]. The behaviour we are interested in concerns passage between Calabi-Yau phases and orbifold phases. The quotient singularity in question will be an isolated $\mathbb{Z}_3$ singularity. For the Calabi-Yau phase we have coordinates

$$z_1 = \frac{a_1a_2a_5a_6}{a_0a_9}$$
$$z_2 = \frac{a_2a_9}{a_3^2}$$
$$z_3 = \frac{a_3a_8}{a_2^2}$$
$$x = -\frac{a_1a_7a_8}{a_3^4}$$
$$y = \frac{a_2a_4}{a_1a_8}.$$ (38)

(See [8] for a discussion of the sign definition appearing above for $x$.) The period regular at the origin may then be computed as

$$\Phi_0 = \sum_{N,M,P_l \geq 0} \frac{\Gamma(3P_1 + 1)(-z_1)^{P_1}z_2^{P_2}z_3^{P_3}(-x)^My^N}{\Gamma(P_1 + M - N + 1)\Gamma(P_2 - 2P_3 + N + 1)\Gamma(P_1 - 2P_2 + P_3 + 1) \times \Gamma(N - 3M + 1)\Gamma(P_1 + 1)\Gamma(M + 1) \times \Gamma(P_3 + M - N + 1)\Gamma(P_2 - P_1 + 1)}.$$ (39)

In the above and from now on, we will drop the $a_0^{-1}$ factor common to all periods.

The orbifold region is given by coordinates $z_1, z_2, z_3, \xi$ and $\eta$ where

$$\xi = -\frac{a_3}{a_1a_7a_8}$$
$$\eta = \frac{a_3^2a_7}{a_1^2a_8^2}.$$ (40)
For simplicity let us set $z_1 = z_2 = z_3 = 0$. This implies $\Phi_0 = 1$. We also impose the restrictions of (30). Choosing phases by $\xi = x^{-1}$ and $\eta = e^{-\pi i xy^2}$ and finding the allowed region of $\arg(y)$ from the above reasoning we obtain (36) again. Now we have

$$\Phi_x = \sum_{N \geq 0} \frac{(-y)^N}{\Gamma(N + 1)} 3 \frac{\Gamma(3s - N)\Gamma(-s)}{\Gamma^2(s - N + 1)} (e^{-\pi i x})^s ds,$$

with $C$ just to the left of $\text{Re}(s) = N$. This gives

$$\Phi_x = \log x - \pi i + 3 \sum_{N \geq 0} \frac{\Gamma(3M - N)}{\Gamma(M + 1)\Gamma(N + 1)\Gamma^2(M - N + 1)} x^M (-y)^N. \quad \text{(42)}$$

Completing to the left we obtain, to leading order

$$\Phi_x = -\frac{\Gamma(\frac{1}{3})}{\Gamma^2(\frac{1}{3})} e^{\pi i \xi^3} + 3 \frac{\Gamma(\frac{2}{3})}{\Gamma^2(1)} \eta^3 + \ldots \quad \text{(43)}$$

For the other period we use

$$\Phi_y = \sum_{M \geq 0} \frac{(-x)^M}{\Gamma(M + 1)} \frac{1}{2\pi i} \int_C \frac{\Gamma^2(s - M)}{\Gamma(s + 1)\Gamma(s - 3M + 1)} y^s ds,$$

where $C$ is given by (e.g., $M = 4$)

$$\Phi_y = \log y - \sum_{N \geq 0} \frac{\Gamma(3M - N)}{\Gamma(M + 1)\Gamma(N + 1)\Gamma^2(M - N + 1)} x^M (-y)^N. \quad \text{(46)}$$

This gives

$$\Phi_y = \log y - \sum_{N \geq 0} \frac{\Gamma(3M - N)}{\Gamma(M + 1)\Gamma(N + 1)\Gamma^2(M - N + 1)} x^M (-y)^N. \quad \text{(46)}$$
Thus
\[ \Phi_x + 3\Phi_y = \log \eta. \quad (47) \]

Asymptotic behaviour tells us that
\[
(B + iJ)_x = \frac{1}{2\pi i} (\Phi_x + \pi i) \\
(B + iJ)_y = \frac{1}{2\pi i} \Phi_y.
\quad (48)
\]

## 4 Interpretation

Now that we have mapped out the values of \( B + iJ \) around a region of moduli space where some potentially small distances appear let us look at the results. It will be simpler first to discuss the second example of section 3, i.e., the blow-up of the \( \mathbb{Z}_3 \)-quotient singularity. First it is easy to reproduce the result of [8] by inserting \( \xi = \eta = 0 \) into (43) to give \( \Phi_x = 0 \), i.e., \( J_x = 0 \) and \( B_x = \frac{1}{2} \). We also have from (47) that \( J_y = \infty \). This is interpreted as saying that this limit point in the moduli space corresponds to a space where the exceptional divisor (whose size is controlled by \( J_x \)) has zero size but that another size elsewhere in the target space (whose size is controlled by \( J_y \)) is infinite.

Although a zero size has appeared which one might consider unstringy, one should have expected this result from (global) orbifold theory. Saying that the exceptional divisor has zero size is like saying that the blow-up has not been performed, that is we still have a quotient singularity. We know from the arguments of [13] and [14] however that the point \( \xi = \eta = 0 \) is precisely where the large-radius orbifold conformal field theory should be. Thus our analysis of \( B + iJ \) fits in with the conformal field theory picture. This can be stated in terms of the commutativity of the following diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\text{\( \sigma \)-model}} & S_M \\
\downarrow/\!G \quad & & \downarrow/\!G \\
M/\!G & \xrightarrow{\text{"\( \sigma \)-model"}} & (S_M)/\!G
\end{array}
\quad (49)
\]

In the above diagram \( M \) represents a smooth Calabi-Yau manifold. This is associated with a conformal field theory \( S_M \) via the non-linear \( \sigma \)-model. The manifold admits a holonomy preserving symmetry \( G \) which can be modded out by to yield the orbifold \( M/\!G \). The conformal field theory \( S_M \) may also be modded out directly by adding in twisted sectors and then projecting out all non-\( G \)-invariant states. One may also apply the non-linear \( \sigma \)-model to the orbifold \( M/\!G \) by including the concept of twisted strings in the usual \( \sigma \)-model picture.
The latter modification is the reason for the quotation marks around “σ-model” in the diagram. Our statement about zero sized exceptional divisor may then be viewed in terms of the commutivity of (49). Since \( \eta = 0 \) may be considered as the large radius limit for \( M \), each link in the diagram should be well-defined and thus the diagram should indeed commute. Let us now address the question of commutivity when \( M \) is not at the large radius limit.

The form of (43) and the cuts (36) tell us that, at least for small \( \xi \) and \( \eta \), we have \( J_x \geq 0 \) with equality only achieved for \( \xi = \eta = 0 \). Thus \( J_y \) must be infinite to allow zero sized exceptional divisor. For a given small, nonzero \( \eta \) we have

\[
J_x \geq - \frac{3 \Gamma\left(\frac{2}{3}\right)}{2 \pi \Gamma^2\left(\frac{2}{3}\right)} \operatorname{Re}(\eta^{\frac{1}{3}}) = 0
\]

\[
J_y \simeq - \frac{1}{6 \pi} \log |\eta|.
\]

Thus for a given value of \( J_y \), i.e., \( |\eta| \), we achieve minimum \( J_x \) by hugging the branch cuts. As discussed in section \( \S \), the branch cut structure often warps as one moves out from the limit point but to leading order we may assume the cuts are fixed. Thus we may put \( \arg(\eta) = 2\pi \) in the above to obtain the asymptotic form of the limit on \( J_x \). Doing this we obtain

\[
J_x \geq \frac{3 \Gamma\left(\frac{2}{3}\right)}{4 \pi \Gamma^2\left(\frac{2}{3}\right)} e^{-2\pi J_y}, \quad \text{for } J_y \gg 1.
\]

This result embodies the true non-linear structure of the quantum Kähler cone of string theory. Indeed, one might argue that the term “cone” should be dropped in view of the non-linearity of (51).

We may now address the commutivity of the diagram (49) in the case of a non-trivial \( M \). The case we are looking at is not a global orbifold but, as explained earlier, this is not expected to be significant. The question we need to ask concerns the location of points in \( \mathcal{M} \) that correspond to the orbifolded conformal field theory. We claim that these are naturally identified with the locus \( \xi = 0 \) at least in the neighbourhood of \( \xi = \eta = 0 \) because of the order 3 monodromy of the periods around this curve in \( \mathcal{M} \). These then correspond to the minimum size \( J_x \) may obtain for a given \( \eta \). If \( \eta \neq 0 \) this size is nonzero. Thus (49) does not then commute. That is, if we go from \( M \) to \( M/G \) classically, naturally the exceptional divisor has zero size. If instead we go clockwise around the diagram via \( S_M \) and \( (S_M)/G \) we arrive at nonzero size for nonzero \( \eta \).

One might worry that (51) suggests some large scale non-local effects in the target space. If we ignore issues such as the dimension and signature of space-time we might consider the scenario in which the universe is shaped like the Calabi-Yau space in question. If \( J_y \) represents the size of the universe then we may measure this by “measuring” (in a way unspecified) the
minimum size of $J_x$. Thus one appears to make global observations concerning the size of the universe by only studying a very small part of it. Aside from the hopeless impracticality of such a suggestion — putting the order of the size of the observable universe in $J_y$ we would have to measure areas of order $10^{-(10^{100})} \alpha'$ for $J_x$ — this reasoning turns out to be flawed for reasons we will explain below.

Consider the neighbourhood of an isolated quotient singularity as we vary the sizes of the rest of the target space. In the large-radius limit of rest of the space, the neighbourhood of the quotient singularity will become flat except for the singularity itself — i.e., the space will be locally isometric to $\mathbb{C}^d/G$ for some $G$. Direct analysis of $\mathbb{C}^d/G$ always puts the orbifold point in the “right place”, i.e., the exceptional divisor is zero size for the orbifold [8, 12]. A better hypothesis at this point might then seem to be that exceptional divisors can only be shrunk down to zero size if the neighbourhood around the resultant singularity becomes flat in the process. The solves the apparent non-locality in the experiment to measure the size of the universe. What we were measuring would not actually be the size of the universe but the flatness of the space surrounding the exceptional divisor. It would be the assumptions about the global geometry of the universe which allowed us to infer its size.

An interesting case is that in which $M$ is a torus in (49). In this case, the neighbourhoods of the quotient singularities are always flat independent of the size of the original torus. It would be interesting to check explicitly to see if zero-sized exceptional set can be achieved for these cases. Unfortunately the construction of [10] does not cover this case. One may be able to use the construction of [28] however. One might also argue that when $M$ is a torus, each link in (49) is somehow exact and thus the diagram should commute.

Let us now look at the first example in section 3 in which a $\mathbb{Z}_2$-quotient singularity was analyzed. First recall that in [8] it was shown that the $\mathbb{Z}_2$-quotient singularity blow-up mode had the following form

$$(B + iJ)_l = \frac{1}{\pi i} \cosh^{-1} z_l^{-\frac{1}{2}},$$

where $(B + iJ)_l$ is the component of the Kähler form measuring the size of the blow-up, all other components of the Kähler forms are taken to be infinite and $z_l$ is the “algebraic” coordinate in $\mathbb{C}$ assumed to be in the form (12) so that $z_l = 0$ is the large blow-up limit, $z_l = \infty$ in the orbifold and normalized so that $z_l = 1$ lies on the discriminant. (To be more precise on this latter point, the whole rational curve given by varying $z_l$ including the point at infinity may lie in the discriminant. In this case another component of the discriminant will intersect this curve at $z_l = 1$.) Note that $J_l = 0$ for $z_l = 1$. This is equivalent to saying that $J_l = 0$ lies on the boundary of the Calabi-Yau phase. Thus skeptics who might suggest that we are only measuring zero distances in [8] and this paper because of our definition of distance in terms of analytical continuation should see that, in the $\mathbb{Z}_2$-quotient singularity case, zero distances appear without the need for analytical continuation.
special behaviour of the $\mathbb{Z}_2$ case can be seen by looking at the neighbourhood of the orbifold point in $\mathcal{M}$. In general, a blow-up mode $(B + iJ)_l$ will have finite monodromy around the orbifold point whose order we denote $K_l$. For the examples in this paper, $K_l = 2$ for the $\mathbb{Z}_2$-quotient singularity and $K_l = 3$ for the $\mathbb{Z}_3$-quotient singularity. More generally there will be more than one blow-up associated to a single singularity and the situation is more complicated. Still considering for the moment the case where all other components of the Kähler form are held at infinity, near the orbifold point, the behaviour is generally of the form $[12]:$

$$(B + iJ)_l = \frac{1}{2} + \text{Re} \pi i (\frac{1}{K_l} + \frac{1}{2}) z_l^{-\frac{1}{K_l}} + \ldots,$$

where $R$ is some positive real number and $0 < \arg(z_l) < 2\pi$. Thus $K_l > 2$, $J_l = 0 \Rightarrow z_l = \infty$. That is, zero size only appears at the orbifold point. For $K_l = 2$ however, following the branch cut out from the orbifold point maintains $J_l = 0$. Thus, zero size is easier to acquire in the case $K_l = 2$.

Considering now the case where the other components of the Kähler form are allowed finite values we see from (33) that $(B + iJ)_x$ is of the form (53) for $K_l = 2$ for both coordinates in $\mathcal{M}$. Thus the situation is similar to the above paragraph. The condition $J_x = 0$ may be maintained by following the branch cuts out from the orbifold point in $\mathcal{M}$ — in either the $\xi$ or the $\eta$ direction. This tells us that we may maintain zero size for the blow-up mode while having finite size for the rest of the Calabi-Yau manifold. This situation in thus different from the above. In the case of the first orbifold, to obtain zero size we required flatness for the target space metric (except at the singularities) but in the case given by non-zero $\eta$ for the first example in section 3 we do not.

In conclusion it appears difficult to make a clear statement about measuring zero distances within Calabi-Yau spaces. There is evidence that the whole space can never be shrunk to zero. Indeed any shrinking down which would lower the dimension of the target space appears to be ruled out. For the case of blowing-down an exception divisor to zero size, a sufficiently complicated topology will demand that some other part of the target space must become infinitely large. However, in simple cases where such a blow-down results in flatness or $K_l = 2$, this latter condition appears not to be necessary.

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