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ADMISSIBLE CONSTANTS FOR GENUS 2 CURVES

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ABSTRACT. S.-W. Zhang recently introduced a new adelic invariant \( \phi \) for curves of genus at least 2 over number fields and function fields. We calculate this invariant when the genus is equal to 2.

1. Introduction

Let \( X \) be a smooth projective geometrically connected curve of genus \( g \geq 2 \) over a field \( k \) which is either a number field or the function field of a curve over a field. Assume that \( X \) has semistable reduction over \( k \). For each place \( v \) of \( k \), let \( N_v \) be the usual local factor connected with the product formula for \( k \).

In a recent paper [11] S.-W. Zhang proves the following theorem:

**Theorem 1.1.** Let \( (\omega, \omega)_a \) be the admissible self-intersection of the relative dualizing sheaf of \( X \). Let \( \langle \Delta_\xi, \Delta_\xi \rangle \) be the height of the canonical Gross-Schoen cycle on \( X^3 \). Then the formula:

\[
(\omega, \omega)_a = \frac{2g-2}{2g+1} \left( \langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \phi(X_v) \log N_v \right)
\]

holds, where the \( \phi(X_v) \) are local invariants associated to \( X \otimes k_v \), defined as follows:

- if \( v \) is a non-archimedean place, then:
  \[
  \phi(X_v) = -\frac{1}{4} \delta(X_v) + \frac{1}{4} \int_{R(X_v)} g_v(x, x)((10g+2)\mu_v - \delta_{K_{X_v}}),
  \]
  where:
  - \( \delta(X_v) \) is the number of singular points on the special fiber of \( X \otimes k_v \),
  - \( R(X_v) \) is the reduction graph of \( X \otimes k_v \),
  - \( g_v \) is the Green’s function for the admissible metric \( \mu_v \) on \( R(X_v) \),
  - \( K_{X_v} \) is the canonical divisor on \( R(X_v) \).

In particular, \( \phi(X_v) = 0 \) if \( X \) has good reduction at \( v \);

- if \( v \) is an archimedean place, then:
  \[
  \phi(X_v) = \sum_{\ell} \frac{2}{\lambda_\ell} \sum_{m,n=1}^g \left| \int_{X(k_v)} \phi_\ell \omega_m \overline{\omega_n} \right|^2,
  \]
  where \( \phi_\ell \) are the normalized real eigenforms of the Arakelov Laplacian on \( X(k_v) \) with eigenvalues \( \lambda_\ell > 0 \), and \( (\omega_1, \ldots, \omega_g) \) is an orthonormal basis for the hermitian inner product \( (\omega, \eta) \mapsto \frac{i}{2} \int_{X(k_v)} \omega \overline{\eta} \) on the space of holomorphic differentials.

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Apart from giving an explicit connection between the two canonical invariants 
\((\omega, \omega)\) and \(\langle \Delta_\xi, \Delta_\xi \rangle\), Zhang’s theorem has a possible application to the effective Bogomolov conjecture, i.e., the question of giving effective positive lower bounds for \((\omega, \omega)\). Indeed, the height of the canonical Gross-Schoen cycle \(\langle \Delta_\xi, \Delta_\xi \rangle\) is known to be non-negative in the case of a function field in characteristic zero, and should be non-negative in general by a standard conjecture of Gillet-Soulé (op. cit., Section 2.4). Further, the invariant \(\varphi\) should be non-negative, and Zhang proposes, in the non-archimedean case, an explicit lower bound for it which is positive in the case of non-smooth reduction (op. cit., Conjecture 1.4.2). Note that it is clear from the definition that \(\varphi\) is non-negative in the archimedean case; in fact it is positive (op. cit., Remark after Proposition 2.5.3).

Besides \(\varphi(X_v)\), Zhang also considers the invariant \(\lambda(X_v)\) defined by:

\[
\lambda(X_v) = \frac{g-1}{6(2g+1)} \varphi(X_v) + \frac{1}{12} \left( \epsilon(X_v) + \delta(X_v) \right),
\]

where:

- if \(v\) is a non-archimedean place, the invariant \(\delta(X_v)\) is as above, and:
\[
\epsilon(X_v) = \int_{R(X_v)} g_v(x, x)((2g-2)\mu_v + \delta_{K_X,v}),
\]

- if \(v\) is an archimedean place, then:
\[
\delta(X_v) = \delta_F(X_v) - 4g \log(2\pi)
\]

with \(\delta_F(X_v)\) the Faltings delta-invariant of the compact Riemann surface \(X(\bar{k}_v)\), and \(\epsilon(X_v) = 0\).

The significance of this invariant is that if \(\deg \det R\pi_* \omega\) denotes the (non-normalized) geometric or Faltings height of \(X\) one has a simple expression:

\[
\deg \det R\pi_* \omega = \frac{g-1}{6(2g+1)} \langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \lambda(X_v) \log N_v
\]

for \(\deg \det R\pi_* \omega\), as follows from the Noether formula:

\[
12 \deg \det R\pi_* \omega = (\omega, \omega) + \sum_v (\epsilon(X_v) + \delta(X_v)) \log N_v.
\]

Now assume that \(X\) has genus \(g = 2\). Our purpose is to calculate the invariants \(\varphi(X_v)\) and \(\lambda(X_v)\) explicitly. For the \(\lambda\)-invariant we obtain:

- if \(v\) is non-archimedean, then:
\[
10\lambda(X_v) = \delta_0(X_v) + 2\delta_1(X_v),
\]

where \(\delta_0(X_v)\) is the number of non-separating nodes and \(\delta_1(X_v)\) is the number of separating nodes in the special fiber of \(X \otimes k_v\);

- if \(v\) is archimedean, then:
\[
10\lambda(X_v) = -20 \log(2\pi) - \log \|\Delta_2\|(X_v),
\]

where \(\|\Delta_2\|(X_v)\) is the normalized modular discriminant of the compact Riemann surface \(X(\bar{k}_v)\) (see below).

Thus, the \(\lambda(X_v)\) are precisely the well-known local invariants corresponding to the discriminant modular form of weight 10 \([6][9][10]\). In particular we have:

\[
\deg \det R\pi_* \omega = \sum_v \lambda(X_v) \log N_v
\]
and we recover the fact that the height of the canonical Gross-Schoen cycle vanishes for $X$.

### 2. The non-archimedean case

Let $k$ be a complete discretely valued field. Let $X$ be a smooth projective geometrically connected curve of genus 2 over $k$. Assume that $X$ has semistable reduction over $k$. In this section we give the invariants $\phi(X)$ and $\lambda(X)$ of $X$.

The proof of our result is based on the classification of the semistable fiber types in genus 2 and consists of a case-by-case analysis. The notation we employ for the various fiber types is as in [8]. We remark that there are no restrictions on the residue characteristic of $k$.

**Theorem 2.1.** The invariant $\phi(X)$ is given by the following table, depending on the type of the special fiber of the regular minimal model of $X$:

| Type | $\delta_0$ | $\delta_1$ | $\varepsilon$ | $\phi$          |
|------|------------|------------|---------------|-----------------|
| I    | 0          | 0          | 0             | 0               |
| II(a) | 0          | $a$        | $a$           | $a$             |
| III(a) | $a$        | 0          | $\frac{1}{6}a$ | $\frac{1}{12}a$ |
| IV(a,b) | $b$       | $a$        | $a + \frac{1}{6}b$ | $a + \frac{1}{12}b$ |
| V(a,b) | $a + b$    | 0          | $\frac{1}{6}(a + b)$ | $\frac{1}{12}(a + b)$ |
| VI(a,b,c) | $b + c$ | $a$        | $a + \frac{1}{6}(b + c)$ | $a + \frac{1}{12}(b + c)$ |
| VII(a,b,c) | $a + b + c$ | 0          | $\frac{1}{6}(a + b + c) + \frac{1}{6 \frac{abc}{ab+bc+ca}}$ | $\frac{1}{12}(a + b + c) + \frac{5}{12 \frac{abc}{ab+bc+ca}}$ |

For $\lambda(X)$ the formula:

$$10\lambda(X) = \delta_0(X) + 2\delta_1(X)$$

holds.

Let us indicate how the theorem is proved. Let $r$ be the effective resistance function on the reduction graph $R(X)$ of $X$, extended bilinearly to a pairing on $\text{Div}(R(X))$. By Corollary 2.4 of [2] the formula:

$$\phi(X) = -\frac{1}{4}(\delta_0(X) + \delta_1(X)) - \frac{3}{8}r(K,K) + 2\varepsilon(X)$$

holds, where $K$ is the canonical divisor on $R(X)$. The invariant $r(K,K)$ is calculated by viewing $R(X)$ as an electrical circuit. The invariant $\varepsilon$ is calculated on the basis of explicit expressions for the admissible measure and admissible Green’s function; see [7] and [8] for such computations. The results we find are as follows:
The values of $\varphi$ follow.

The formula for $\lambda(X)$ is verified for each case separately.

3. The archimedean case

Let $X$ be a compact and connected Riemann surface of genus 2. In this section we calculate the invariants $\varphi(X)$ and $\lambda(X)$ of $X$. Let $\text{Pic}(X)$ be the Picard variety of $X$, and for each integer $d$ denote by $\text{Pic}^d(X)$ the component of $\text{Pic}(X)$ of degree $d$. We have a canonical theta divisor $\Theta$ on $\text{Pic}^1(X)$, and a standard hermitian metric $\| \cdot \|$ on the line bundle $\mathcal{O}(\Theta)$ on $\text{Pic}^1(X)$. Let $\nu$ be its curvature form. We have:

$$\int_{\text{Pic}^1(X)} \nu^2 = \Theta^2 = 2.$$

Let $K$ be a canonical divisor on $X$, and let $P$ be the set of 10 points $P$ of $\text{Pic}^1(X) - \Theta$ such that $2P \equiv K$. Denote by $\| \theta \|$ the norm of the canonical section $\theta$ of $\mathcal{O}(\Theta)$. We let:

$$\| \Delta_2 \| (X) = 2^{-12} \prod_{P \in P} \| \theta \|^2 (P),$$

the normalized modular discriminant of $X$, and we let $\| H \| (X)$ be the invariant of $X$ defined by:

$$\log \| H \| (X) = \frac{1}{2} \int_{\text{Pic}^1(X)} \log \| \theta \| \nu^2.$$

These two invariants were introduced in [1].

**Theorem 3.1.** For the $\varphi$-invariant and the $\lambda$-invariant of $X$, the formulas:

$$\varphi(X) = -\frac{1}{2} \log \| \Delta_2 \| (X) + 10 \log \| H \| (X)$$

and

$$10\lambda(X) = -20 \log(2\pi) - \log \| \Delta_2 \| (X)$$

hold.

The key to the proof is the following lemma. Let $\Phi$ be the map:

$$X^2 \to \text{Pic}^1(X), \quad (x, y) \mapsto [2x - y].$$

**Lemma 3.2.** The map $\Phi$ is finite flat of degree 8.
Proof. Let $y \mapsto y'$ be the hyperelliptic involution of $X$. We have a commutative diagram:

$$
\begin{array}{ccc}
X^2 & \xrightarrow{\Phi} & \text{Pic}^1(X) \\
\alpha \downarrow & & \beta \downarrow \\
X^2 & \xrightarrow{\Phi'} & \text{Pic}^3(X)
\end{array}
$$

where $\alpha$ and $\beta$ are isomorphisms, with:

- $\alpha : X^2 \to X^2$, $(x, y) \mapsto (x, y')$
- $\Phi' : X^2 \to \text{Pic}^3(X)$, $(x, y) \mapsto [2x + y]$
- $\beta : \text{Pic}^3(X) \to \text{Pic}^1(X)$, $[D] \mapsto [D - K]$.

It suffices to prove that $\Phi'$ is finite flat of degree 8. Let $p : X^{(3)} \to \text{Pic}^3(X)$ be the natural map; then $p$ is a $\mathbb{P}^1$-bundle over $\text{Pic}^3(X)$, and $\Phi'$ has a natural injective lift to $X^{(3)}$. A point $D$ on $X^{(3)}$ is in the image of this lift if and only if $D$, when seen as an effective divisor on $X$, contains a point which is ramified for the morphism $X \to \mathbb{P}^1$ determined by the fiber $|D|$ of $p$ in which $D$ lies. Since every morphism $X \to \mathbb{P}^1$ associated to a $D$ on $X^{(3)}$ is ramified, the map $\Phi'$ is surjective. As every morphism $X \to \mathbb{P}^1$ associated to a $D$ on $X^{(3)}$ has only finitely many ramification points, the map $\Phi'$ is quasi-finite, hence finite since $\Phi'$ is proper. As $X^2$ and $\text{Pic}^3(X)$ are smooth and the fibers of $\Phi'$ are equidimensional, the map $\Phi'$ is flat.

By Riemann-Hurwitz the generic $X \to \mathbb{P}^1$ associated to a $D$ on $X^{(3)}$ has 8 simple ramification points. It follows that the degree of $\Phi'$ is 8. □

Let $G : X^2 \to \mathbb{R}$ be the Arakelov-Green’s function of $X$, and let $\Delta$ be the diagonal divisor on $X^2$. We have a canonical hermitian metric on the line bundle $\mathcal{O}(\Delta)$ on $X^2$ by putting $\|1\|(x, y) = G(x, y)$, where 1 is the canonical section of $\mathcal{O}(\Delta)$. Denote by $h_\Delta$ the curvature form of $\mathcal{O}(\Delta)$. We have:

$$
\int_{X^2} h_\Delta^2 = \Delta . \Delta = -2.
$$

Restricting $\mathcal{O}(\Delta)$ to a fiber of any of the two natural projections of $X^2$ onto $X$ and taking the curvature form we obtain the Arakelov $(1, 1)$-form $\mu$ on $X$. We have $\int_X \mu = 1$ and:

$$
\int_X \log G(x, y) \mu(x) = 0
$$

for each $y$ on $X$. Let $(\omega_1, \omega_2)$ be an orthonormal basis of $\mathcal{H}^0(X, \omega_X)$, the space of holomorphic differentials on $X$. We can write explicitly:

$$
h_\Delta(x, y) = \mu(x) + \mu(y) - i \sum_{k=1}^2 (\omega_k(x) \bar{\omega}_k(y) + \omega_k(y) \bar{\omega}_k(x))
$$

and:

$$
\mu(x) = \frac{i}{4} \sum_{k=1}^2 \omega_k(x) \bar{\omega}_k(x).
$$

By [11, Proposition 2.5.3] we have:

$$
\varphi(X) = \int_{X^2} \log G h_\Delta^2.
$$

We compute the integral using our results from [11] and [13]. Let $W$ be the divisor of Weierstrass points on $X$, and let $p_1 : X^2 \to X$ be the projection onto the first
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coordinate. The divisor \( W \) is reduced effective of degree 6. According to [3, p. 31] there exists a canonical isomorphism:
\[
\sigma : \Phi^* \mathcal{O}(\Theta) \xrightarrow{\cong} \mathcal{O}(2\Delta + p^*_1 W)
\]
of line bundles on \( X^2 \), identifying the canonical sections on both sides. In [4, Proposition 2.1] we proved that this isomorphism has a constant norm over \( X^2 \). Thus, the curvature forms on both sides are equal:
\[
\Phi^* \nu = 2h\Delta + 6\mu(x) \quad \text{on} \quad X^2.
\]

Squaring both sides of this identity we get:
\[
h^2\Delta = \frac{1}{4} \Phi^*(\nu^2) - 6h\Delta \mu(x),
\]
since \( \mu(x)^2 = 0 \). Denote by \( S(X) \) the norm of \( \sigma \). Then we have:
\[
2 \log G(x, y) + \sum \log G(x, w) = \log \|\theta\|(2x - y) + \log S(X)
\]
for generic \((x, y) \in X^2\), where \( w \) runs through the Weierstrass points of \( X \). By fixing \( y \) and integrating against \( \mu(x) \) on \( X \) we find that:
\[
\log S(X) = -\int_X \log \|\theta\|(2x - y) \mu(x).
\]

By integrating against \( h^2\Delta \) on \( X^2 \) we obtain:
\[
2\varphi(X) + \sum \int_X \log G(x, w) h^2_\Delta = -2 \log S(X) + \int_X \log \|\theta\|(2x - y) h^2_\Delta.
\]

As we have:
\[
h^2_\Delta = 2\mu(x)\mu(y) - \sum_{k,l=1}^2 (\omega_k(x)\bar{\omega}_l(x)\omega_k(y)\bar{\omega}_l(y) + \bar{\omega}_k(x)\omega_l(x)\omega_k(y)\bar{\omega}_l(y))
\]
it follows that:
\[
\int_X \log G(x, w) h^2_\Delta = 0
\]
for each \( w \) in \( W \) and hence we simply have:
\[
2\varphi(X) = -2 \log S(X) + \int_X \log \|\theta\|(2x - y) h^2_\Delta.
\]

Using our earlier expression for \( h^2_\Delta \) this becomes:
\[
2\varphi(X) = -2 \log S(X) + \int_X \log \|\theta\|(2x - y) \left( \frac{1}{4} \Phi^*(\nu^2) - 6h\Delta \mu(x) \right).
\]

It is easily verified that \( h_\Delta \mu(x) = h_\Delta \mu(y) = \mu(x)\mu(y) \) and hence:
\[
\int_X \log \|\theta\|(2x - y) h_\Delta \mu(x) = \int_X \log \|\theta\|(2x - y) \mu(x)\mu(y) = -\log S(X).
\]

From Lemma 3.2 it follows that:
\[
\int_X \log \|\theta\|(2x - y) \Phi^*(\nu^2) = 8 \int_{\text{Pic}^1(X)} \log \|\theta\| \nu^2 = 16 \log \|H\|(X).
\]

All in all we find:
\[
\varphi(X) = 2 \log S(X) + 2 \log \|H\|(X).
\]
Let $\delta_F(X)$ be the Faltings delta-invariant of $X$. According to [5, Corollary 1.7] the formula:

$$\log S(X) = -16 \log(2\pi) - \frac{5}{4} \log \|\Delta_2\|(X) - \delta_F(X)$$

holds, and in turn, according to [1, Proposition 4] we have:

$$\delta_F(X) = -16 \log(2\pi) - \log \|\Delta_2\|(X) - 4 \log \|H\|(X).$$

The formula:

$$\varphi(X) = -\frac{1}{2} \log \|\Delta_2\|(X) + 10 \log \|H\|(X)$$

follows.

By definition we have:

$$\lambda(X) = \frac{1}{30} \varphi(X) + \frac{1}{12} \delta_F(X) - \frac{2}{3} \log(2\pi)$$

so we obtain:

$$10\lambda(X) = -20 \log(2\pi) - \log \|\Delta_2\|(X)$$

by using [1, Proposition 4] once more.

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