A generalization of Laplace and Fourier transforms

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Abstract

In this note we propose a generalization of the Laplace and Fourier transforms which we call symmetric Laplace transform. It combines both the advantages of the Fourier and Laplace transforms. We give the definition of this generalization, some examples and basic properties. We also give the form of its inverse by using the theory of the Fourier transform. Finally, we apply the symmetric Laplace transform to a parabolic problem and to an ordinary differential equation.

Keywords Fourier transform, Laplace transform, functions of exponential order

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1 Introduction

In this note we will propose a generalization of the well known Fourier and Laplace transforms. Let us recall first the notion of the Fourier transform (see for example [2], [4]).

Definition 1 Let \( f : \mathbb{R} \to \mathbb{C} \) be a function. We define a new function \( F(s) \) with \( s \in \mathbb{R} \) by

\[
F(s) = \mathcal{F}(f)(s) = \int_{-\infty}^{\infty} f(t)e^{-ist}dt
\]

Here \( e^{-ix} = \cos x + i \sin x \).

A common condition in order a function \( f \) can be Fourier transformed is to be absolutely integrable, that is to satisfy the following definition,

Definition 2 A function \( f : \mathbb{R} \to \mathbb{C} \) is called absolutely integrable on \( \mathbb{R} \) when

\[
\int_{-\infty}^{\infty} |f(t)|dt < \infty
\]

The set of all absolutely integrable functions on \( \mathbb{R} \) is denoted by \( L^1(\mathbb{R}) \).

For an absolutely integrable function \( f \) the Fourier transform is well defined since

\[
|F(s)| = \left| \int_{-\infty}^{\infty} f(t)e^{-ist}dt \right| \leq \int_{-\infty}^{\infty} |f(t)e^{-ist}|dt = \int_{-\infty}^{\infty} |f(t)||e^{-ist}|dt
\]
But $|e^{-ist}| = 1$ so it follows that

$$|F(s)| \leq \int_{-\infty}^{\infty} |f(t)|dt < \infty$$

A disadvantage of the Fourier transform is that functions like the Heaviside function, which is the following

$$H(t) = \begin{cases} 
1, & \text{when } t \geq 0 \\
0, & \text{when } t < 0
\end{cases}$$

can not be Fourier transformed.

The Laplace transform can be applied to this kind of functions. Let us recall its definition, (see for example [7]),

**Definition 3** Let $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ and $s \in \mathbb{C}$. We define the following function

$$F(s) = \mathcal{L}(f)(s) = \int_{0}^{\infty} e^{-st} f(t)dt$$

where $s = x + iy$, if the integral exists.

It is well-known fact that the Laplace transform of the Heaviside function is the function $\frac{1}{s}$ for Re $s > 0$.

A well known condition for a function $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ so that can be Laplace transformed is to be of exponential order, that is

**Definition 4** We say that the function $f : \mathbb{R}^+ \rightarrow \mathbb{C}$ is of exponential order $a$ if there is some constant $M > 0$ and some $a > 0$ such that for some $t_0 \geq 0$, it hold that

$$|f(t)| \leq Me^{at}, \quad t \geq t_0$$

A similar definition for some $f : \mathbb{R} \rightarrow \mathbb{C}$ is the following.

**Definition 5** We say that the function $f : \mathbb{R} \rightarrow \mathbb{C}$ is of exponential order $a$ if there is some constant $M > 0$ and some $a > 0$ such that

$$|f(t)| \leq Me^{a|t|}, \quad t \in \mathbb{R}$$

A disadvantage of the Laplace transform is that it can not be applied to functions that are defined in all of $\mathbb{R}$, such as the following function,

$$g(t) = \begin{cases} 
1, & \text{when } t \geq 0 \\
-1, & \text{when } t < 0
\end{cases}$$

This function can not be Fourier transformed as well. So, we will propose a generalization of both previous transforms which we call the symmetric Laplace transform.
2 Symmetric Laplace Transform

We propose the following integral transform,

Definition 6 Let $f: \mathbb{R} \to \mathbb{C}$. We define the symmetric Laplace transform as follows,

$$SL(f)(x_1, x_2, y) = \int_{-\infty}^{\infty} e^{-x_1 H(t) + x_2 H(-t) - iy} f(t) dt$$

where $H(t) = \begin{cases} 1, & \text{when } t \geq 0 \\ 0, & \text{when } t < 0 \end{cases}$

Choosing $x_1 = x_2 = x$ and $s = x + iy$ we obtain

$$SL(f(t))(s) = L(f(t))(s) + L(f(-t)(s))$$

Obviously, if $f(t) = 0$ for $t < 0$ then the symmetric Laplace transform coincides with the usual Laplace transform because $L(f(-t))(s) = 0$. Choosing, $x_1 = x_2 = 0$ then the symmetric Laplace transform coincides with the Fourier transform. Therefore, the symmetric Laplace transform generalizes both the Fourier and the Laplace transforms.

Let us give three examples of the symmetric Laplace transform.

Example 1 We will evaluate the symmetric Laplace transform of the function

$$f(t) = \begin{cases} 1, & \text{when } t \geq 0 \\ -1, & \text{when } t < 0 \end{cases}$$

We have that

$$SL(f)(x_1, x_2) = \int_{-\infty}^{\infty} e^{(-x_1 H(t) + x_2 H(-t) - iy) t} f(t) dt$$

$$= \int_{0}^{\infty} e^{-(x_1 + iy) t} dt - \int_{-\infty}^{0} e^{-(x_2 - iy) t} dt$$

$$= \frac{1}{x_1 + iy} + \frac{1}{x_2 - iy}, \quad \text{when } x_1 > 0, x_2 > 0, y \in \mathbb{R}$$

Choosing $x_1 = x_2 = x$ and $s = x + iy$ then

$$SL(f(t))(s) = \frac{1}{s} - \frac{1}{s}, \quad \text{when } s = x + iy, \quad x > 0, y \in \mathbb{R}$$

Example 2 We will compute the symmetric Laplace transform of the function $f(t) = 1$ with $t \in \mathbb{R}$. We have that

$$SL(f)(x_1, x_2) = \int_{-\infty}^{\infty} e^{(-x_1 H(t) + x_2 H(-t) - iy) t} dt$$

$$= \int_{0}^{\infty} e^{-(x_1 + iy) t} dt + \int_{0}^{\infty} e^{-(x_2 - iy) t} dt$$

$$= \frac{1}{x_1 + iy} + \frac{1}{x_2 - iy}, \quad \text{for } x_1 > 0, x_2 > 0, y \in \mathbb{R}$$

Choosing $x_1 = x_2 = x$ and $s = x + iy$ it follows that

$$SL(f)(s) = \frac{1}{s} + \frac{1}{s}, \quad x > 0, y \in \mathbb{R}$$
**Example 3** We will compute the symmetric Laplace transform of the following function

\[
f(t) = \begin{cases} 
sin xt, & \text{when } t \geq 0 \\
cos xt, & \text{when } t < 0 
\end{cases}
\]

We have that

\[
SL(f)(x_1, x_2, y) = \int_{-\infty}^{\infty} e^{(-x_1H(t)+x_2H(-t)-iy)t} f(t) dt = \int_0^{\infty} e^{-(x_1+iy)t} \sin xt \, dt + \int_{-\infty}^{0} e^{(x_2-iy)t} \cos xt \, dt \\
= \frac{x}{(x_1 + iy)^2 + x^2} + \frac{x_2 - iy}{(x_2 - iy)^2 + x^2}
\]

for \( x_1 > 0, \ x_2 > 0, \ y \in \mathbb{R} \). Choosing \( x_1 = x_2 = z \) and \( s = z + iy \) we obtain that

\[
SL(f)(s) = \frac{x}{s^2 + x^2} + \frac{s}{s^2 + x^2}, \quad \text{when } \text{Re } s = z > 0, \ y \in \mathbb{R}
\]

We will also compute the symmetric Laplace transform of the function

\[
f(t) = \begin{cases} 
cos xt, & \text{when } t \geq 0 \\
\sin xt, & \text{when } t < 0 
\end{cases}
\]

Choosing \( x_1 = x_2 = z \) and \( s = z + iy \) we obtain

\[
SL(f)(x, y) = \mathcal{L}(\cos xt)(s) + \mathcal{L}(\sin(-xt))(\overline{s}) \\
= \frac{s}{s^2 + x^2} - \frac{x}{\overline{s}^2 + x^2}, \quad \text{Re } s = z > 0, \ y \in \mathbb{R}
\]

### 3 The Inverse of the Symmetric Laplace Transform

In order to prove that the symmetric Laplace transform is invertible we will use the following theorem, (see theorem 19.3, page 248, \[1\]).

**Theorem 1** If the function \( f \) is absolutely integrable on \( \mathbb{R} \) and such that

\[
\int_{-\infty}^{\infty} f(t) e^{-ist} dt = 0, \quad \text{for every } s \in \mathbb{R}
\]

then \( f(t) = 0 \) for every \( t \in \mathbb{R} \).

Let us define the set of all the continuous functions \( f \) which are also such that the function

\[
g(t) = e^{-(x_1H(t)+x_2H(-t)t} f(t)
\]

is absolutely integrable for \( (x_1, x_2) \in U \subseteq \mathbb{R}^2 \) with \( U \) chosen appropriately for the given function \( f \). We denote this set by \( \mathcal{C}_U SL \). For these functions the symmetric Laplace transform is well defined because \( SL(f)(s) = \mathcal{F}(g)(s) \) for \( f \in \mathcal{C}_U SL \).

The symmetric Laplace transform

\[
SL : \mathcal{C}_U SL \rightarrow \text{im } SL
\]
is a linear transform, recalling the linearity of the integral. The kernel of this transform contains only the zero element. Indeed, if
\[ \int_{-\infty}^{\infty} g(t)e^{-ist}dt = 0, \quad \text{for every } s \in \mathbb{R} \]
then \( g(t) = 0 \) for all \( t \in \mathbb{R} \) and therefore \( f(t) = 0 \) for all \( t \in \mathbb{R} \). That means that the symmetric Laplace transform is a 1-1 linear transform and so invertible. The inverse of this transform is also linear. These useful conclusions come from basic linear algebra (see for example [3], [5] and [6]).

However, if we choose \( s \in \mathbb{R}^+ \) (and not \( s \in \mathbb{C} \)) in the symmetric Laplace transform then it follows that is not invertible. To see this let us compute the symmetric Laplace transform (with \( s \in \mathbb{R}^+ \)) of the continuous function \( f(t) = t \) with \( t \in \mathbb{R} \). We have
\[ \mathcal{SL}(f)(s) = \int_0^\infty te^{-st}dt + \int_{-\infty}^{0} te^{st}dt, \quad s > 0 \]
It follows that \( \mathcal{SL}(f)(s) = 0 \) and thus the kernel of the transform contains at least one non-zero continuous function. That means that is not invertible. Therefore, it is important to choose \( s \in \mathbb{C} \) and not in \( \mathbb{R}^+ \).

In the next theorem we will see the actual form of the inverse of the symmetric Laplace transform if we assume further that is \( f \) is piecewise smooth.

**Theorem 2** Let \( f : \mathbb{R} \to \mathbb{C} \) a piecewise smooth function and such that the function \( g(t) = e^{-(x_1H(t)+x_2H(-t))t}f(t) \) is absolutely integrable on \( \mathbb{R} \) when \( (x_1, x_2) \in U \subseteq \mathbb{R}^2 \). If \( F(x_1, x_2, y) \) is the symmetric Laplace transform of \( f \) then
\[ \lim_{A \to \infty} \frac{1}{2\pi A} \int_{-A}^{A} F(x_1, x_2, y)e^{i(x_1H(t)-x_2H(-t)+iy)t}dy = \frac{1}{2} (f(t+) + f(t-)), \quad (x_1, x_2) \in U \]
for every \( t \in \mathbb{R} \).

**Proof.** We will use the form of the inverse Fourier transform of the function \( g \) (see Theorem 7.3, page 169 of [2]).

Since the function \( g(t) \) is absolutely integrable on \( \mathbb{R} \) then the Fourier transform is well defined and moreover \( F(g)(y) = \mathcal{LS}(f)(x_1, x_2, y) = F(x_1, x_2, y) \). Therefore, knowing the form of the inverse of the Fourier transform we obtain
\[ \lim_{A \to \infty} \frac{1}{2\pi A} \int_{-A}^{A} F(x_1, x_2, y)e^{iyt}dy = \frac{1}{2} (g(t+) + g(t-)) \]
For \( t > 0 \) we have that \( g(t+) = f(t+)e^{-x_1t} \) and \( g(t-) = f(t-)e^{-x_1t} \) while for \( t < 0 \) it holds that \( g(t+) = f(t+)e^{x_2t} \) and \( g(t-) = f(t-)e^{x_2t} \). For \( t = 0 \) we have \( g(0+) = f(0+) \) and \( g(0-) = f(0-) \) thus we can write \( g(t+) = f(t+)e^{-(x_1H(t)+x_2H(-t))t} \) and \( g(t-) = f(t-)e^{-(x_1H(t)+x_2H(-t))t} \) for \( t \in \mathbb{R} \). That is, it holds that
\[ \lim_{A \to \infty} \frac{1}{2\pi A} \int_{-A}^{A} F(x_1, x_2, y)e^{i(x_1H(t)-x_2H(-t)+iy)t}dy = \frac{1}{2} (f(t+) + f(t-)), \quad (x_1, x_2) \in U \]
for every \( t \in \mathbb{R} \).
Remark 1 As we can see the symmetric Laplace transform of a function \( f \), when we choose \( x_1 = x_2 \) (see [L]), is the sum of two functions, \( g_1(s) + g_2(\overline{s}) \), with \( s = x + iy \). The function \( g_1(s) \) is the Laplace transform of \( f(t) \) for \( t > 0 \) while the function \( g_2(\overline{s}) \) is the Laplace transform of \( f(-t) \) for \( t > 0 \). Therefore, if we want to find the inverse symmetric Laplace transform of a function, we should separate it into a sum of two functions. The first will contain the terms with \( s \) and the second the terms with \( \overline{s} \), say \( g_1(s) \) and \( g_2(\overline{s}) \). Next, we find the inverse of the function \( g_1(s) \) which is equal to \( f \) for \( t \geq 0 \) while the inverse of the function \( g_2(\overline{s}) \) is the function \( f(t) \) for \( t < 0 \). Consequently, we have that

\[
\begin{align*}
    f(t) &= \begin{cases} 
        \mathcal{L}^{-1}(g_1(s))(t), & \text{ when } t \geq 0 \\
        \mathcal{L}^{-1}(g_2(\overline{s}))(\overline{t}), & \text{ when } t < 0 
    \end{cases}
\end{align*}
\]

Example 4 We will find the inverse symmetric Laplace transform of the function

\[
\frac{1}{s^2} - \frac{1}{\overline{s}^2}
\]

Here \( g_1(s) = \frac{1}{s^2} \) and \( g_2(\overline{s}) = -\frac{1}{\overline{s}^2} \). Therefore the function \( f(t) \) is such that

\[
\begin{align*}
    f(t) &= \begin{cases} 
        \mathcal{L}^{-1}(g_1(s))(t), & \text{ when } t \geq 0 \\
        \mathcal{L}^{-1}(g_2(\overline{s}))(\overline{t}), & \text{ when } t < 0 
    \end{cases}
\end{align*}
\]

But, \( \mathcal{L}^{-1}(g_1(s))(t) = t \) and \( \mathcal{L}^{-1}(g_2(\overline{s}))(\overline{t}) = t \) therefore \( f(t) = t \) with \( t \in \mathbb{R} \).

4 Basic Properties of the Symmetric Laplace Transform

We will give some basic properties concerning the symmetric Laplace transform of the derivative of a function.

Theorem 3 Let \( f, f' : \mathbb{R} \to \mathbb{C} \) are continuous \( \mathbb{R} \) and that \( f \) is of exponential order \( a \). Then

\[
\mathcal{S}\mathcal{L}(f')(s) = s\mathcal{L}(f(t))(s) - \overline{s}\mathcal{L}(f(-t))(\overline{s})
\]

where \( s = x + iy \) and \( \text{Re } s = x > a, y \in \mathbb{R} \).

Proof. We have that

\[
\mathcal{S}\mathcal{L}(f')(s) = \int_0^\infty f'(t)e^{-st}dt + \int_{-\infty}^0 f'(t)e^{\overline{s}t}dt
\]

But

\[
\int_{-\infty}^0 f'(t)e^{\overline{s}t}dt = [f(t)e^{\overline{s}t}]_{-\infty}^0 - \overline{s}\int_{-\infty}^0 f(t)e^{\overline{s}t}dt = f(0-) - \overline{s}\mathcal{L}(f(-t))(\overline{s})
\]

and similarly

\[
\int_0^\infty f'(t)e^{-st}dt = [f(t)e^{-st}]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = -f(0+) + s\mathcal{L}(f(t))(s)
\]

Using the continuity of \( f \) we get the desired result. □

Similarly, we have the following result.
**Theorem 4** Suppose that $f, f', f''$ are continuous on $\mathbb{R}$ and that $f, f'$ are of exponential order $a$. Then

$$SL(f''(s)) = s^2 L(f(t))(s) + \bar{s}^2 L(f(-t))(\bar{s}) - f(0)(s + \bar{s})$$

where $s = x + iy$ and $x > a, y \in \mathbb{R}$.

More general result in this direction is the following using induction.

**Theorem 5** Let $f, f', \cdots, f^{(n)}$ are continuous on $\mathbb{R}$, except maybe at zero, while the functions $f, f', \cdots, f^{(n-1)}$ are of exponential order $a$. Then

$$SL(f^{(n)}(t))(s) = L(f^{(n)}(t))(s) + \sum_{k=0}^{n-1} (-s)^k f^{(k)}(0) \left(\frac{1}{s} - \frac{1}{\bar{s}}\right)$$

**Example 5** We will study the following parabolic problem,

\[
\begin{align*}
    u_{xx}(x, t) &= u_t(x, t), & x \in \mathbb{R}, & t > 0 \\
    u(x, 0) &= f(x), & x \in \mathbb{R} \\
    u(0, t) &= 0, & t > 0
\end{align*}
\]

Here $f$ is as follows

\[
f(x) = \begin{cases} 
1, & \text{when } x \geq 0 \\
-1, & \text{when } x < 0 
\end{cases}
\]

We will apply the symmetric Laplace transform on the $x$ variable to get

$$s^2 G(s, t) + \bar{s}^2 \tilde{G}(\bar{s}, t) = G_t(s, t) + \tilde{G}_t(\bar{s}, t)$$  \hspace{1cm} (2)

where $G(s, t) = L(u(x, t))$ and $\tilde{G}(s, t) = L(u(-x, t))$.

We apply also the symmetric Laplace transform to the condition $u(x, 0) = f(x)$ to get

$$G(s, 0) + \tilde{G}(\bar{s}, 0) = \frac{1}{s} - \frac{1}{\bar{s}}$$  \hspace{1cm} (3)

From (2) and (3) we get two ordinary differential equations which are

\[
\begin{align*}
    s^2 G(s, t) &= G_t(s, t), & G(s, 0) &= \frac{1}{s} \\
    \bar{s}^2 \tilde{G}(\bar{s}, t) &= \tilde{G}_t(\bar{s}, t), & \tilde{G}(\bar{s}, 0) &= -\frac{1}{s}
\end{align*}
\]

Their solutions are

\[
G(s, t) = \frac{1}{s} e^{s^2 t}, \quad \tilde{G}(\bar{s}, t) = -\frac{1}{s} e^{\bar{s}^2 t}
\]
Inverting the symmetric Laplace transform (using the convolution theorem of the usual Laplace transform) we arrive at
\[
\begin{align*}
  u(x, t) &= \begin{cases} 
    \text{erf} \left( \frac{x}{2\sqrt{t}} \right), & \text{when } x \geq 0 \\
    -\text{erf} \left( \frac{-x}{2\sqrt{t}} \right), & \text{when } x < 0
  \end{cases}
\end{align*}
\]

where \( \text{erf} \left( \frac{x}{2\sqrt{t}} \right) = \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x}{2\sqrt{t}}} e^{-u^2} du \). It is easy to show that the function \( u(x, t) \) satisfies our problem.

Next, assuming further that every solution of our parabolic problem should be bounded as \(|x| \to \infty\) while \( u_x(x, t) \to 0\) as \(|x| \to \infty\) we will prove that this problem admits a unique solution. Note that these extra conditions satisfied by the solution above.

Suppose that the problem admits at least two solutions, the \( u_1, u_2 \). Then their difference \( v = u_1 - u_2 \) will satisfy the following problem
\[
\begin{align*}
  v_{xx}(x, t) &= v_t(x, t), \quad x \in \mathbb{R}, \quad t > 0 \\
  v(x, 0) &= 0, \quad x \in \mathbb{R} \\
  v(0, t) &= 0, \quad t > 0 \\
  v(x, t) & \text{ bounded as } |x| \to \infty, \quad v_x(x, t) \to 0 \text{ as } |x| \to \infty
\end{align*}
\]

Multiply now the equation by \( v(x, t) \) and integrate over \((0, t)\). On the right hand side of the equation we get
\[
\int_{0}^{t} v_t(x, y)v(x, y)dy = v^2(x, t) - \int_{0}^{t} v(x, y)v_t(x, y)dy
\]

Therefore
\[
\int_{0}^{t} v_t(x, y)v(x, y)dy = \frac{1}{2} v^2(x, t)
\]

Next, we integrate over \( \mathbb{R} \) to get on the left side of the equation,
\[
\int_{-\infty}^{\infty} \int_{0}^{t} v_{xx}(y, r)v(y, r)dydr = \int_{0}^{t} \int_{-\infty}^{\infty} v_{xx}(y, r)v(y, r)dydr
\]
\[
= \int_{0}^{t} \left( v_x(y, r)v(y, r) \bigg|_{y=\infty}^{y=-\infty} \right) - \int_{-\infty}^{\infty} v^2_x(y, r)dydr
\]
\[
= - \int_{0}^{t} \int_{-\infty}^{\infty} v^2_x(y, r)dydr
\]

Therefore, it holds that
\[
\frac{1}{2} \int_{-\infty}^{\infty} v^2(y, t)dy + \int_{0}^{t} \int_{-\infty}^{\infty} v^2_x(y, r)dydr = 0
\]

which is true only when \( v(x, t) = 0 \) for every \( x \in \mathbb{R} \) and \( t > 0 \), that is \( u_1(x, t) = u_2(x, t) \), therefore the problem has a unique solution.
Example 6 We will evaluate the solution of the following ordinary differential equation
\[
y''(t) + y(t) = f(t) = \begin{cases} 
e^t, & t \geq 0 \\
1, & t < 0 \end{cases}, \quad t \in \mathbb{R}
y(0) = 0
\]

Obviously, we can not apply the Fourier transform neither the usual Laplace transform. We can apply the symmetric Laplace transform to get,
\[
s^2 \mathcal{L}(y(t))(s) + \overline{s^2 \mathcal{L}(y(-t))(\overline{s})} + \mathcal{L}(y(t))(s) + \mathcal{L}(y(-t))(\overline{s}) = \frac{1}{s-1} + \frac{1}{\overline{s}}, \quad \text{Re } s > 1
\]

Equating similar terms, we get
\[
\mathcal{L}(y(t))(s) = \frac{1}{2} \frac{1}{s-1} - \frac{1}{2} \frac{s}{s^2 + 1} - \frac{1}{2} \frac{1}{s^2 + 1}
\]
\[
\mathcal{L}(y(-t))(\overline{s}) = \frac{1}{\overline{s}} - \frac{s}{s^2 + 1}
\]
Computing the inverse transforms we arrive at
\[
y(t) = \begin{cases} 
\frac{1}{2} e^t - \frac{1}{2} \cos t - \frac{1}{2} \sin t, & t \geq 0 \\
1 - \cos t, & t < 0
\end{cases}
\]
It is easy to see that \(y(t)\) satisfy the above ode.

References

[1] J. Bak - D. J. Newman, *Complex Analysis*, Springer, 1997.
[2] R. J. Beerends-H. G. ter Morsche - J. C. van der Berg - E. M. van de Vrie, *Fourier and Laplace Transforms*, Cambridge University Press, 2003.
[3] S. Berberian, Linear Algebra, Oxford Science Publications, 1992.
[4] Phil Dyke, *An Introduction to Laplace Transforms and Fourier Series*, Springer, 2001.
[5] S. Lang, *Introduction to Linear Algebra*, Springer, 1986.
[6] R. Larson - D. Falvo, *Elementary Linear Algebra*, Houghton Mifflin Harcourt Publishing Company, 2009.
[7] J. L. Schiff, *The Laplace Transform*, Springer, 1999.