THE ALGEBRAIC REPRESENTATION FOR HIGH ORDER SOLUTION OF SASA-SATSUMA EQUATION

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This article is dedicated to the Professor Boling Guo
(academician of Chinese Academy of Sciences) on his 80th birthday

Abstract. In this paper, we reestablish the elementary Darboux transformation for Sasa-Satsuma equation with the aid of loop group method. Furthermore, the generalized Darboux transformation is given with the limit technique. As direct applications, we give the single solitonic solutions for the focusing and defocusing case. The general high order solution formulas with the determinant form are obtained through generalized DT and the formal series method.

1. Introduction. Kodama and Hasegawa [15, 16] proposed a higher-order nonlinear Schrödinger (NLS) equation:

\[ iu_T + \alpha_1 u_{XX} + \alpha_2 |u|^2 u + i(\beta_1 u_{XXX} + \beta_2 (|u|^2)u_X + \beta_3 (|u|^2)u) = 0 \]  

(1)

to model the propagation and interaction of the ultrashort pulses in the subpicosecond or femtosecond regime, where \( u \) represents the slowly varying envelope of the electric field \( X \) and \( T \) are, respectively, the normalized distance along the direction of the propagation and retarded time, \( \alpha_1, \alpha_2, \beta_1, \beta_2 \) and \( \beta_3 \) are the real parameters with respect to the group velocity dispersion, self-phase modulation, third-order dispersion, self-steepening and stimulated Raman scattering, respectively.

In general, the equation (1) is not completely integrable. While the coefficients satisfy the special demand, it becomes integrable. These are the Chen-Lee-Liu derivative NLS equation [3] \( (\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 0) \), the Kaup-Newell derivative NLS equation [13] \( (\beta_1 : \beta_2 : \beta_3 = 0 : 1 : 1) \), the Hirota equation [11] \( (\beta_1 : \beta_2 : \beta_3 = 1 : 6\sigma : 0) \) and the Sasa-Satsuma equation (SSE) [28] \( (\beta_1 : \beta_2 : \beta_3 = 1 : 6\sigma : 3\sigma) \):

\[ iu_T + \frac{1}{2} u_{XX} + \sigma |u|^2 u + \frac{i}{6\epsilon} \left( u_{XXX} + 6\sigma (|u|^2)u_X - 3\sigma (|u|^2)u \right) = 0, \]  

(2)

where \( \sigma = 1 \) represents the focusing case, \( \sigma = -1 \) represents the defocusing case. Introducing the variable transformations[28]

\[ u(X,T) = q(x,t) \exp \left\{ i \epsilon \left( x + \epsilon^2 t \right) \right\}, \quad T = 6 \epsilon t, \quad X = x + 3 \epsilon^2 t, \]  

(3)

2010 Mathematics Subject Classification. Primary: 37K10, 35Q55; Secondary: 35C08.

Key words and phrases. Darboux transformation, high order solution, Sasa-Satsuma equation.

This work is supported by National Natural Science Foundation of China (Contact No. 11401221) and Fundamental Research Funds for the Central Universities (Contact No. 2014ZB0034).

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then equation (2) could be rewritten as
\[ q_t + 6\sigma \left( |q|^2 q \right)_x - 3\sigma (|q|^2)_x q = 0. \] (4)

SSE (4) is one of a limited number of integrable models and has been a field of active research for the past two decades. Thanks to the integrability, the sophisticated soliton construct underlying this wave equation can, therefore, be achieved using an array of mathematical tools such as inverse scattering transform [22, 14, 17, 33], Darboux transformation (DT) [27, 30, 1, 4, 32, 25], Hirota bilinear method [6, 5, 26], and others. Although the DT for the SSE was given in the previous study, the reduction of DT is not complete. In this work, we reestablish the DT through the loop group method [29]. With this way, we can obtain the DT completely. On the other hand, since the spectral problem for SSE (4) is $3 \times 3$ and possesses deep reduction, the high order DT can not be obtained directly. Based on the elementary DT [21, 7] and analysis of spectral parameters, one can obtain the generalized DT [2, 9, 10, 19]. As direct applications, the single solitonic solutions, multi-solitonic solution and the high order solitonic solutions are given through this method.

In the previous literature, most of works concentrate on the soliton solution for SSE on the zero background [27, 28]. Recently, there are some works focus on the solutions of SSE with the non-vanishing background (NVBC). For instance: the breather solution [30, 1], W-shape soliton [32], rational W-shape soliton [34] and the twist rogue wave solution [4] are obtained for the focusing SSE in the recent literature. In this work, we construct the solitonic solution for both the focusing and defocusing SSE on the NVBC systematically. For the focusing case, when the spectral parameter is located in the image axis, we can obtain the soliton solution and resonant soliton solution, periodical solution, half periodical solution, rational W-shape soliton, resonant rational W-shape soliton, and their high order ones; when the spectral parameter is not located in the image and real axis, we can obtain the breather solution, resonant breather solution, rogue wave solution, composite rogue wave solution and their high order ones. For the defocusing case, when the spectral parameter is located on the segment of real axis, we can obtain the dark soliton, W-shape dark soliton; when the spectral parameter is not located on the real and image axis, we can obtain the breather solution and its high order ones.

The structure of this paper is organized in the following: In section 2, we give the DT and generalized DT for SSE through the loop group method. The reduction for DT was analyzed by spectral parameter. Based on the limit technique, we give the generalized DT which could be used to derive the high order solutions. In section 3, we give the classification of regular solitonic solution with NVBC based on the DT. In section 4, we give the high order solutions explicitly by the formal series method. Final section involves some discussions and conclusions.

2. Darboux transformation. In this section, we give the elementary DT and generalized DT of SSE. Actually, the spectral problem of SSE is a deep reduction for coupled NLSE. To obtain the elementary DT for SSE (4), we merely need to find the deep reduction for the DT of coupled NLSE.

2.1. Elementary DT and N-fold DT. The SSE (4) admits the following Lax pair
\[ \Phi_x = U(\lambda) \Phi, \quad U(\lambda) = i\lambda (\sigma_3 + I_3) + Q, \]
\[ \Phi_t = V(\lambda) \Phi, \quad V(\lambda) = 4i\lambda^3 (\sigma_3 + I_3) + 4\lambda^2 Q + V_1 \lambda + V_0, \] (5)
where
\[ V_1 = 2i[\sigma_3(Q^2 - Q_x) + b^2 I_3], \quad V_0 = (Q_x Q - QQ_x) + (2Q^3 - Q_{xx}), \]
\[ Q = \begin{bmatrix} 0 & iq & i\sigma q \\ iq & 0 & 0 \\ i\bar{q} & 0 & 0 \end{bmatrix}, \quad \sigma_3 = \text{diag}(1, -1, -1) \]

\( b \) is a real constant, \( I_3 \) is a 3 \( \times \) 3 identity matrix and the overbar represents the complex conjugation (similarly hereafter). Which is nothing but the third flow of deep reduction for coupled nonlinear Schrödinger (CNLS) hierarchy. To derive the DT for system (5), we need to use the symmetry relation.

The symmetry relation can be readily obtained as
\[ U^\dagger(\bar{\lambda}) = -\sigma U(\lambda), \quad V^\dagger(\bar{\lambda}) = -\sigma V(\lambda), \quad (6) \]
and
\[ KU(\lambda)K = U(-\lambda), \quad KV(\lambda)K = V(-\lambda), \quad (7) \]
where \( ^\dagger \) represents the Hermite conjugation and
\[ K = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \]

Through Lax pair equation (5), we can see that (4) is the third flow for the CNLS hierarchy with deep reduction. The elementary DT for CNLS hierarchy can be constructed as the following:
\[ T = I - \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \bar{\lambda}_1} P_1, \quad P_1 = |y_1\rangle\langle y_1|J \quad (8) \]
where
\[ |y_1\rangle = \Phi_1(x, t; \lambda_1), \quad \langle y_1| = |y_1\rangle^\dagger, \quad J = \text{diag}(1, \sigma, \sigma), \quad \Phi_1(x, t; \lambda_1) \text{ is a special solution for Lax pair equations (5) at } \lambda = \lambda_1. \]

**Theorem 2.1.** The DT converts system (5) into a new system
\[
\Phi_x[1] = U[1]\Phi[1], \\
\Phi_t[1] = V[1]\Phi[1],
\]
where \( U[1] = U(Q[1]; \lambda), \quad V[1] = V(Q[1]; \lambda), \)
\[ Q[1] = \begin{bmatrix} 0 & i\sigma q[1] & i\sigma q[1] \\ iq[1] & 0 & 0 \\ iq[1] & 0 & 0 \end{bmatrix} \]

and the Bäcklund transformation between old potential function and new one is
\[ q[1] = q + \frac{|y_{1,2}\rangle\langle y_{1,1}|}{\langle y_1|J|y_1\rangle/\langle 2(\lambda_1 - \bar{\lambda}_1) \rangle}, \quad (9) \]
here \( |y_{1,2}\rangle \) represents the second component of vector \( |y_1\rangle \), \( \langle y_{1,1}| \) represents the first component of vector \( \langle y_1\rangle \).

**Proof.** We merely need to verify that
\[
T_x T^{-1} + TU(\lambda) T^{-1} = U[1], \quad (10a) \\
T_t T^{-1} + TV(\lambda) T^{-1} = V[1]. \quad (10b)
\]
Firstly, we prove that the equation (10a). We use the residue analysis method to prove it. Expanding $T$ and $T^{-1}$ with the series $\lambda$ at $\lambda = \infty$:

\[
T = I - (\lambda_1 - \hat{\lambda}_1)P_1 \left( \frac{1}{\lambda} + \frac{\hat{\lambda}_1}{\lambda^2} + \frac{\hat{\lambda}_1^2}{\lambda^3} + \cdots \right),
\]

\[
T^{-1} = I + (\lambda_1 - \hat{\lambda}_1)P_1 \left( \frac{1}{\lambda} + \frac{\lambda_1}{\lambda^2} + \frac{\lambda_1^2}{\lambda^3} + \cdots \right).
\]

(11)

With the aid of above equations (11), we can obtain that

\[
F(x, t; \lambda) = T_1T^{-1} + TUT^{-1} - (i\sigma_3 + I_3)(Q[1]) = O(\lambda^{-1}).
\]

In what following, we verify that the matrix function $F(x, t; \lambda)$ is holomorphic in a compact Riemann surface $S^2 \subset \mathbb{C} \cup \{\infty\}$. Indeed, we have

\[
\text{Res}_{\lambda=\hat{\lambda}_1} F(x, t; \lambda) = (\hat{\lambda}_1 - \lambda_1) \left[ P_{1,x}T^{-1} + P_1U(\lambda)T^{-1} \right] |_{\lambda=\hat{\lambda}_1} = 0,
\]

and

\[
\text{Res}_{\lambda=\lambda_1} F(x, t; \lambda) = (\lambda_1 - \hat{\lambda}_1) \left[ -P_{1,x}P_1 + TU(\lambda)P_1 \right] |_{\lambda=\lambda_1} = 0.
\]

Thus $F(x, t; \lambda)$ is an holomorphic function in $S^2$. By the asymptotical behavior for $F(x, t; \lambda)$, we have $F(x, t; \lambda) = 0$. So the equation (10a) is proved.

In the following, we consider the equation (10b). The same procedure as above, we can prove

\[
G(x, t; \lambda) = T_1T^{-1} + TV(\lambda)T^{-1} - \hat{V} = 0,
\]

where

\[
\hat{V} = 4i\sigma_3\lambda^3 + 4Q[1]\lambda^2 + \hat{V}_1\lambda + \hat{V}_0,
\]

\[
\hat{V}_1 = 2i[\sigma_3(Q^2 - Q_x) + b^2I_3] + 4(\hat{\lambda}_1 - \lambda_1)[P_1, Q] + 4i(\hat{\lambda}_1 - \lambda_1)(\hat{\lambda}_1P_1\sigma_3 - \hat{\lambda}_1\sigma_3P_1),
\]

\[
\hat{V}_0 = (Q_xQ - QQ_x) - (Q_{xx} - 2Q^2) + 2i(\hat{\lambda}_1 - \lambda_1)[P_1, \sigma_3(Q^2 - Q_x)] + 4(\hat{\lambda}_1 - \lambda_1)(\hat{\lambda}_1P_1Q - \lambda_1QP_1) + 4i(\hat{\lambda}_1 - \lambda_1)(\hat{\lambda}_1^2P_1\sigma_3 - \hat{\lambda}_1^2\sigma_3P_1),
\]

and $[.,.]$ represents the commutator. The remaining is to verify that $\hat{V} = V[1]$. Indeed, since $F(x, t; \lambda) = G(x, t; \lambda) = 0$, then we have

\[
\frac{\partial}{\partial t} U[1] - \hat{V}_x + [U[1], \hat{V}] = 0.
\]

(12)

Comparing the coefficient of $\lambda$, we can obtain that

\[
\lambda^2 : 4 \frac{\partial}{\partial x} Q[1] = i[\sigma_3, \hat{V}_1],
\]

\[
\lambda^1 : \hat{V}_{1x} = [Q[1], \hat{V}_1] + i[\sigma_3, \hat{V}_0],
\]

\[
\lambda^0 : \frac{\partial}{\partial t} Q[1] - \hat{V}_{0,x} + [Q[1], \hat{V}_0] = 0.
\]

(13)

By the first equation of (13), we have

\[
\hat{V}_1^{off} = V_1(Q[1])^{off} = -2i\sigma_3 \frac{\partial}{\partial x} Q[1],
\]

where $^{off}$ represents the $(1, 2)$, $(1, 3)$ and $(2, 1)$, $(3, 1)$ elements for the matrix. On the other hand, through the second equation of (13), we have

\[
\hat{V}_{1,x}^{diag} = [Q[1], V_1(Q[1])^{off}].
\]
where $\text{diag}$ represents the other elements except the $\text{off}$ ones. So we have

$$\hat{V}_1^{\text{diag}} = 2i[\sigma_3 Q[1]^2 + b^2 I_3] + C(t).$$

Indeed, the expression of $C(t)$ is independent with $Q[1]$. Thus we take $P_1 = 0$, and $Q[1] = Q$. It follows that $C(t) = 0$. In a similar way, we can obtain $\hat{V}_0 = V_0(Q[1])$. This completes the proof.

**Definition 2.2.** Suppose the DT is $T = I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} \pi$, where $\pi$ is a projector operator, we call the geometric multiplicity of DT is $\min(\text{rank}(\ker(I - \pi)), \text{rank}(\ker(\pi)))$.

It is obviously that the geometric multiplicity of DT can not more than $[m]$, where $m$ is the order of spectral problem. For instance, the order of spectral problem (5) is three, then the geometric multiplicity of DT (8) must be one. In the following, we illustrate the meaning for geometric multiplicity of DT. Suppose we have two different DT for CNLS hierarchy with $\text{rank}(\ker(I - \pi)) = 1$ or 2. Firstly,

$$T_1 = I - \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \bar{\lambda}_1} \Phi_1 \Phi_1^\dagger J = \frac{\Phi_1 \Phi_1^\dagger J}{\Phi_1^\dagger J \Phi_1},$$

(14)

Since $T_1$ is gauge transformation, we can multiply $\lambda - \bar{\lambda}_1$ with $T_1$, and denote it as $\hat{T}_1$, i.e.

$$\hat{T}_1 = \lambda - \bar{\lambda}_1 - (\lambda_1 - \bar{\lambda}_1) \frac{\Phi_1 \Phi_1^\dagger J}{\Phi_1^\dagger J \Phi_1}.$$

(15)

By the properties $\hat{T}_1|_{\lambda = \lambda_1} = 0$ and $\hat{T}_1|_{\lambda = \bar{\lambda}_1} = 0$, we can obtain that

$$\hat{T}_1 = \lambda - MDM^{-1}, \ M = [\Phi_1, \Psi_1, \Psi_2], \ D = \text{diag}(\lambda_1, \bar{\lambda}_1, \bar{\lambda}_1).$$

On the other hand, suppose $\Phi(\lambda_1) = (i \lambda_1 \sigma_3 + Q) \Phi(\lambda_1)$, then we can obtain that

$$-\left[\Phi^\dagger(\lambda_1) J\right]_x = \left[\Phi^\dagger(\lambda_1) J\right] (i \bar{\lambda}_1 \sigma_3 + Q).$$

It follows that $\left[\Phi^\dagger(\lambda_1) J\Phi(\bar{\lambda}_1)\right]_x = 0$. So we have $\Phi^\dagger(\lambda_1) J\Phi(\bar{\lambda}_1) = \Phi^\dagger(\lambda_1; 0, 0) J\Phi(\bar{\lambda}_1; 0, 0)$. Moreover, we have

$$\left[\Phi^\dagger(\lambda_1; 0, 0)\right]^{-1} \Phi^\dagger(\lambda_1) J\Phi(\bar{\lambda}_1)[\Phi(\bar{\lambda}_1; 0, 0)]^{-1} J = I.$$

Denote

$$\Phi_1^\dagger = (\alpha_1, \alpha_2, \alpha_3)[\Phi^\dagger(\lambda_1; 0, 0)]^{-1} \Phi^\dagger(\lambda_1).$$

By the standard Schmidt orthogonalization procedure, we construct the unitary matrix

$$M = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

such that $M^\dagger M = I$. It follows that $M[\Phi^\dagger(\lambda_1; 0, 0)]^{-1} \Phi^\dagger(\lambda_1) J\Phi(\bar{\lambda}_1)[\Phi(\bar{\lambda}_1; 0, 0)]^{-1} J M^\dagger = I$. And denote $\Phi(\bar{\lambda}_1)[\Phi(\bar{\lambda}_1; 0, 0)]^{-1} J M^\dagger = \begin{bmatrix} \hat{\Phi}_1, \Psi_1, \Psi_2 \end{bmatrix}$. Then we can obtain that

$$\Phi_1^\dagger J \Psi_1 = 0, \ \Phi_1^\dagger J \Psi_2 = 0.$$

Thus $\hat{T}_1$ can be represented as

$$\hat{T}_1 = \lambda - \lambda_1 - (\bar{\lambda}_1 - \lambda_1) \begin{bmatrix} \Psi_1, \Psi_2 \end{bmatrix} \begin{bmatrix} \Psi_1^\dagger J \Psi_1, \Psi_2^\dagger J \Psi_2 \end{bmatrix}^{-1} \begin{bmatrix} \Psi_1^\dagger \Psi_2 \end{bmatrix} J.$$
Moreover, we have \( T_1 = \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1} \left( I - \frac{\bar{\lambda}_1}{\lambda - \lambda_1} \right) \). Since Darboux matrix \( T_1 \) is a gauge transformation, we can neglect the coefficient \( \frac{\lambda - \bar{\lambda}_1}{\lambda - \lambda_1} \). So the Darboux matrix \( T_1 \) is equivalent with \( \tilde{T}_1 = I - \frac{\bar{\lambda}_1}{\lambda - \lambda_1} \). But \( \text{rank}(I - \tilde{T}_1) = 2 \). This illustrates both cases are equivalent each other. This implies that we merely need to choose a special vector solution to construct elementary DT for SSE (4).

In what following, we consider the reduction for DT of CNLS hierarchy, such that

\[
K\bar{U}[1](\lambda)K = U[1](-\lambda), \quad K\bar{V}[1](\lambda)K = V[1](-\lambda).
\]

(16)

Indeed, it is readily to verify that, if DT satisfies the relation

\[
K\bar{T}(\lambda)K = T(-\lambda)
\]

(17)

then the new potential function keeps the symmetry relation (16).

We look for the reduction condition with two different cases. For the first case, when \( \lambda_1 + \bar{\lambda}_1 = 0 \). With this choice, we can find that the symmetry relation (17) can be reduced as

\[
K\bar{P}_1 K = P_1,
\]

i.e. \( |y_1\rangle = (\varphi_1, \psi_1, \phi_1)^T \) and \( \phi_k \varphi_k + \bar{\phi}_k \varphi_k = 0 \). For the second case: when \( \lambda_1 + \bar{\lambda}_1 \neq 0 \). With this choice, we can not reduce the elementary DT to satisfy symmetry relation (16). To look for the reduction, we must iterate the DT. We find the twice iterated DT

\[
T_2 = I - [\langle y_1|, K|y_1\rangle] M^{-1} D \left[ \begin{array}{c} \langle y_1 | \rangle \\ \langle K | y_1 \rangle \end{array} \right] J
\]

(18)

satisfies the relation (16), where

\[
M = \begin{bmatrix}
\frac{\langle y_1 | J | y_1 \rangle}{\lambda - \lambda_1} & \frac{\langle y_1 | J | y_1 \rangle}{2\lambda_1} \\
\frac{\langle y_1 | J | y_1 \rangle}{2\lambda_1} & \frac{\langle y_1 | J | y_1 \rangle}{\lambda - \bar{\lambda}_1}
\end{bmatrix}, \quad D = \text{diag} \left( \frac{1}{\lambda - \lambda_1}, \frac{1}{\lambda - \bar{\lambda}_1} \right).
\]

Based on the above two different kinds of reduction, we have the following \( N \)-fold DT for the system (5). We conclude it as the following theorem:

**Theorem 2.3.** Suppose we have \( n \) different solutions \( \Phi_k = (\varphi_k, \psi_k, \phi_k)^T \) for system (5) at \( \lambda = \lambda_k \in \mathbb{R} \) such that \( \psi_k \varphi_k + \bar{\phi}_k \varphi_k = 0 \) \((k = 1, 2, \cdots, n)\), and \( m \equiv N - n \) different solutions \( \Psi_l \) for system (5) at \( \lambda = \mu_l \not\in \mathbb{R} \) \((l = 1, 2, \cdots, m)\), then we have the following \( N \)-fold DT:

\[
T_N = I - Y M^{-1} D Y^T J,
\]

(19)

where

\[
Y = \left[ |y_1\rangle, |y_2\rangle, \cdots, |y_n\rangle, |z_1\rangle, K|z_1\rangle, |z_2\rangle, K|z_2\rangle, \cdots, |z_m\rangle, K|z_m\rangle \right], \\
D = \text{diag} \left( \frac{1}{\lambda - \lambda_1}, \frac{1}{\lambda - \lambda_2}, \cdots, \frac{1}{\lambda - \lambda_n}, \frac{1}{\lambda - \bar{\lambda}_1}, \frac{1}{\lambda + \mu_1}, \frac{1}{\lambda + \mu_2}, \cdots, \frac{1}{\lambda + \mu_m}, \frac{1}{\lambda + \mu_m} \right), \\
M = \begin{bmatrix}
M_{11} & M_{12} \\
-M_{12} & M_{22}
\end{bmatrix}.
\]
through taking limit from those spectral parameters. Thus we assume that

\[
Y \mid \sum_{i=1}^{n} \psi_{i} = 1
\]

Different solutions

and \( \psi_y = \Phi_k (k = 1, 2, \cdots, n) \), \( |z_j| = \Psi_j (j = 1, 2, \cdots, m) \). And the Bäcklund
transformation for SSE (4) can be represented as

\[
q[N] = q - 1 + \frac{\det(M - 2Y_1^T Y_2)}{\det(M)},
\]

where \( Y_i \) represents the \( i \)-th row of matrix \( Y \).

2.2. Generalized Darboux transformation. In this subsection, we consider the
generalized DT for the SSE (4). We use the limit technique to obtain the generalized
DT from above theorem.

Indeed, the \( N \)-fold DT is dependent on the spectral parameters \( \lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3, \cdots, \bar{\lambda}_m, \bar{\mu}_1, \bar{\mu}_2, \cdots, \bar{\mu}_m \). The generalized DT can be obtained
through taking limit from those spectral parameters. Thus we assume that

\[
\lambda_1, \lambda_{1+k_1} = \lambda_1 + \epsilon_{k_1}, \ k_1 = 1, 2, \cdots, n_1 - 1, \ n_1 \geq 1,
\]
\[
\lambda_2, \lambda_{1+n_1-k_2} = \lambda_2 + \epsilon_{k_2}, \ k_2 = 1, 2, \cdots, n_2 - 1, \ n_2 \geq 1,
\]
\[
\vdots
\]
\[
\lambda_i, \lambda_{1+n_1+n_2+\cdots+n_{i-1}-(1)+k_i} = \lambda_i + \epsilon_{k_i}, \ k_i = 1, 2, \cdots, n_i, \ n_i \geq 1,
\]
\[
\sum_{i=1}^{n} n_i = n, \text{ and}
\]
\[
\mu_1, \mu_{1+l_1} = \mu_1 + \epsilon_{l_1}, \ l_1 = 1, 2, \cdots, m_1 - 1, \ m_1 \geq 1,
\]
\[
\mu_2, \mu_{1+m_1+l_2} = \mu_2 + \epsilon_{l_2}, \ l_2 = 1, 2, \cdots, m_2 - 1, \ m_2 \geq 1,
\]
\[
\vdots
\]
\[
\mu_j, \mu_{1+m_1+m_2+\cdots+m_{j-1}-(1)+l_j} = \mu_j + \epsilon_{l_j}, \ l_j = 1, 2, \cdots, m_j - 1, \ m_j \geq 1,
\]
\[
\sum_{j=1}^{m} m_j = m. \text{ Taking limit } \epsilon_{k_i}, \epsilon_{l_j} \to 0, \text{ we can obtain that the following generalized DT:

**Theorem 2.4.** Suppose we have \( i \) different solutions \( \Phi_k = (\varphi_k, \psi_k, \phi_k)^T \) for system
(5) at \( \lambda = \lambda_k \in \mathbb{R} \) such that \( \psi_k \varphi_k + \phi_k \varphi_k = 0 (k = 1, 2, \cdots, i) \), and \( j \) different
solutions $\Psi_t$ for system (5) at $\lambda = \mu_l \notin \mathbb{R}$ ($l = 1, 2, \cdots, j$), then we have the following generalized DT:

$$T = I - YM^{-1}DY^\dagger J,$$

where

$$Y = [Y_1, Y_2, \cdots, Y_i, Z_1, Z_2, \cdots, Z_j],$$

$$D = \text{diag}(D_1, D_2, \cdots, D_i, E_1, E_2, \cdots, E_j),$$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ -M_{12}^\dagger & M_{22} \end{bmatrix},$$

and

$$Y_k = \begin{bmatrix} \{y_k, \frac{\partial \hat{y}_k}{\partial \epsilon_k}\}_{\epsilon_k=0}, \cdots, \frac{\partial^{n_k-1} \hat{y}_k}{\partial \epsilon_k^{n_k-1}} |_{\epsilon_k=0} \end{bmatrix},$$

$$Z_l = \begin{bmatrix} |z_l), K|z_l), \cdots, \frac{\partial^{m_l-1} |\tilde{z}_l)}{\partial \tilde{z}_l^{m_l-1}} |_{\tilde{z}_l=0} \end{bmatrix},$$

$$D_k = \begin{bmatrix} (\lambda - \bar{\lambda})^{-1} & (\lambda - \bar{\lambda})^{-2} & \cdots & (\lambda - \bar{\lambda})^{-n_k} \\
0 & (\lambda - \bar{\lambda})^{-1} & \cdots & (\lambda - \bar{\lambda})^{-1-n_k} \\
0 & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (\lambda - \bar{\lambda})^{-1} \end{bmatrix},$$

$$E_l = \begin{bmatrix} (\lambda - \bar{\mu})^{-1} & 0 & (\lambda - \bar{\mu})^{-2} & \cdots & (\lambda - \bar{\mu})^{-m_l} & 0 \\
0 & (\lambda + \mu_i)^{-1} & 0 & \cdots & (\lambda + \mu_i)^{-2} & \cdots & (\lambda + \mu_i)^{-m_l} \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & (\lambda - \bar{\mu})^{-2} & 0 & (\lambda + \mu_i)^{-2} \\
0 & 0 & 0 & \cdots & (\lambda - \bar{\mu})^{-1} & 0 & \cdots & (\lambda + \mu_i)^{-1} \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} F_{1,1} & \cdots & F_{1,i} \\
\vdots & \ddots & \vdots \\
F_{i,1} & \cdots & F_{i,i} \end{bmatrix},$$

$$M_{12} = \begin{bmatrix} G_{1,1} & \cdots & G_{1,j} \\
\vdots & \ddots & \vdots \\
G_{i,1} & \cdots & G_{i,j} \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} H_{1,1} & \cdots & H_{1,j} \\
\vdots & \ddots & \vdots \\
H_{j,1} & \cdots & H_{j,j} \end{bmatrix},$$

and

$$\frac{\langle \hat{y}_s | J | \hat{y}_t \rangle}{\lambda_t + \epsilon_t - \bar{\lambda_s} - \bar{\tau_s}} = \sum_{s_i=0}^\infty \sum_{t_j=0}^\infty F_{s_i,t_j}^{[s_i,t_j]} \tau_{s_i}^{s_i} \epsilon_{t_j}^{t_j},$$

$$F_{s_i,t_j}^{[s_i,t_j]} = \frac{1}{s_i!t_j! \partial \epsilon_{t_j} \partial \tau_{s_i}^{s_i}} \left( \frac{\langle \hat{y}_s | J | \hat{y}_t \rangle}{\lambda_t + \epsilon_t - \bar{\lambda_s} - \bar{\tau_s}} \right),$$

where

$$\lambda_t + \epsilon_t - \bar{\lambda_s} - \bar{\tau_s} \neq 0.$$
\[
\frac{\langle \varepsilon_s | J | \hat{y}_l \rangle}{\mu_t + \varepsilon_t - \lambda_s - \varepsilon_s} = \sum_{s_i=0}^{\infty} \sum_{t_j=0}^{\infty} G_{1:s,t}^{[s_i,t_j]} \varepsilon_s^{s_i} \varepsilon_t^{t_j},
\]

\[
G_{1:s,t}^{[s_i,t_j]} = \frac{1}{s_i!t_j!} \frac{\partial^{s_i+t_j}}{\partial \varepsilon_s^{s_i} \partial \varepsilon_t^{t_j}} \left( \frac{\langle \varepsilon_s | J | \hat{y}_l \rangle}{\mu_t + \varepsilon_t - \lambda_s - \varepsilon_s} \right),
\]

\[
-\frac{\langle \hat{y}_l | J | \xi_s \rangle}{\mu_t - \varepsilon_t - \lambda_s - \varepsilon_s} = \sum_{s_i=0}^{\infty} \sum_{t_j=0}^{\infty} G_{2:s,t}^{[s_i,t_j]} \varepsilon_s^{s_i} \varepsilon_t^{t_j},
\]

\[
G_{2:s,t}^{[s_i,t_j]} = \frac{1}{s_i!t_j!} \frac{\partial^{s_i+t_j}}{\partial \varepsilon_s^{s_i} \partial \varepsilon_t^{t_j}} \left( -\frac{\langle \hat{y}_l | J | \xi_s \rangle}{\mu_t - \varepsilon_t - \lambda_s - \varepsilon_s} \right),
\]

\[
\frac{\langle \varepsilon_s | J | \hat{z}_l \rangle}{\mu_t + \varepsilon_t - \mu_s - \varepsilon_s} = \sum_{s_i=0}^{\infty} \sum_{t_j=0}^{\infty} H_{1:s,t}^{[s_i,t_j]} \varepsilon_s^{s_i} \varepsilon_t^{t_j},
\]

\[
H_{1:s,t}^{[s_i,t_j]} = \frac{1}{s_i!t_j!} \frac{\partial^{s_i+t_j}}{\partial \varepsilon_s^{s_i} \partial \varepsilon_t^{t_j}} \left( \frac{\langle \varepsilon_s | J | \hat{z}_l \rangle}{\mu_t + \varepsilon_t - \mu_s - \varepsilon_s} \right),
\]

\[
-\frac{\langle \hat{z}_l | J | \xi_s \rangle}{\mu_t - \varepsilon_t - \mu_s - \varepsilon_s} = \sum_{s_i=0}^{\infty} \sum_{t_j=0}^{\infty} H_{2:s,t}^{[s_i,t_j]} \varepsilon_s^{s_i} \varepsilon_t^{t_j},
\]

\[
H_{2:s,t}^{[s_i,t_j]} = \frac{1}{s_i!t_j!} \frac{\partial^{s_i+t_j}}{\partial \varepsilon_s^{s_i} \partial \varepsilon_t^{t_j}} \left( -\frac{\langle \hat{z}_l | J | \xi_s \rangle}{\mu_t - \varepsilon_t - \mu_s - \varepsilon_s} \right),
\]

and

\[
F_{s,t} = \begin{bmatrix}
F_{s,t}^{[0,0]} & \cdots & F_{s,t}^{[0,n_i]} \\
\vdots & \ddots & \vdots \\
F_{s,t}^{[n_i,0]} & \cdots & F_{s,t}^{[n_i,n_i]}
\end{bmatrix},
\]

here \(1 \leq s, t \leq i,\)

\[
G_{s,t} = \begin{bmatrix}
G_{1:s,t}^{[0,0]} & \cdots & G_{1:s,t}^{[0,m_i]} & G_{1:s,t}^{[0,m_i]} \\
\vdots & \ddots & \vdots & \vdots \\
G_{1:s,t}^{[n_i,0]} & \cdots & G_{1:s,t}^{[n_i,m_i]} & G_{1:s,t}^{[n_i,m_i]}
\end{bmatrix},
\]

here \(1 \leq s \leq i, 1 \leq t \leq j,\)

\[
H_{s,t} = \begin{bmatrix}
H_{1:s,t}^{[0,0]} & \cdots & H_{1:s,t}^{[0,m_i]} & H_{1:s,t}^{[0,m_i]} \\
\vdots & \ddots & \vdots & \vdots \\
H_{1:s,t}^{[m_i,0]} & \cdots & H_{1:s,t}^{[m_i,m_i]} & H_{1:s,t}^{[m_i,m_i]}
\end{bmatrix},
\]

here \(1 \leq s, t \leq j,\) and \(|y_k| = \Phi_k (k = 1, 2, \cdots, i), |z_l| = \Psi_l (l = 1, 2, \cdots, j), |\hat{y}_l| = |y_k(\lambda_k + \varepsilon_k)|, |\hat{z}_l| = |z_l(\mu_l + \varepsilon_l)|.\) And the generalized Bäcklund transformation for SSE (4) can be represented as

\[
q[N] = q - 1 + \frac{\det(M - 2Y_1 Y_2)}{\det(M)},
\]

where \(Y_i\) represents the \(i\)-th row of matrix \(Y.\)
3. Localized wave solution and periodical solution. In this section, we use the DT to construct the soliton solution and periodical solution on the non-vanishing background. We depart from the seed solution $q[0] = ae^{ib[x+(b^2-6\sigma^2)t]}$. Since SSE (4) possesses the scaling symmetry, we can set $a = 1$ for any $a \neq 0$. Through the standard method to solve the linear differential equation [8], we can obtain the fundamental solution for Lax pair equation with $q = q[0]$ and $\lambda = \lambda_1$:
\[
\Phi_1 = D_1 L_1 M_1,
\]
where
\[
D_1 = \text{diag} \left( 1, e^{i\theta}, e^{-i\theta} \right), \quad \theta_1 = b[x+(b^2-6\sigma)t],
\]
\[
L_1 = \begin{bmatrix}
1 & 1 & 1 \\
\frac{1}{\chi_1+b} & \frac{1}{\chi_2+b} & \frac{1}{\chi_3+b} \\
\frac{1}{\chi_1-b} & \frac{1}{\chi_2-b} & \frac{1}{\chi_3-b}
\end{bmatrix}, \quad \text{if } b \neq 0; \quad L_1 = \begin{bmatrix}
1 & 1 & 0 \\
\frac{1}{\chi_1} & \frac{1}{\chi_2} & -1 \\
\frac{1}{\chi_1} & \frac{1}{\chi_2} & 1
\end{bmatrix}, \quad \text{if } b = 0;
\]
\[
M_1 = \text{diag} \left( e^{X_1}, e^{X_2}, e^{X_3} \right), \quad X_i = i\chi_i[x+(2\lambda_1\chi_i+b^2-4\sigma)t],
\]
and $\chi_i (i = 1, 2, 3)$ are three different roots for the following cubic equation
\[
\chi^3 - 2\lambda_1 \chi^2 - (b^2+2\sigma) \chi + 2b^2\lambda_1 = 0,
\]
if $b = 0$, then $\chi_3 = 0$.

Indeed, the classification of localized wave solution is determined by the cubic equation (24). So we give the classification of roots for equation (24). Set $\chi = i\alpha$, $\lambda_1 = i\beta$, then equation (24) becomes
\[
\alpha^3 - 2\beta \alpha^2 + (b^2+2\sigma) \alpha - 2b^2\beta = 0.
\]
Then the discriminant of equation (25) is
\[
\Delta = -64b^2\beta^4 + (-32b^4 + 160\sigma b^2 + 16)\beta^2 - 4b^6 - 24\sigma b^4 - 48b^2 - 32\sigma.
\]
If $\Delta = 0$, i.e. $\beta^2 = \frac{1}{32\sigma}\left[2b^4 + 10\sigma b^2 + 1 + \varrho(1-4\sigma b^2)^{3/2}\right]$, $\varrho = \pm 1$, then equation (25) possesses multiple root. If $\Delta > 0$ and $\beta \in \mathbb{R}$, then the equation (25) possesses three different real roots. If $\Delta < 0$ and $\beta \in \mathbb{R}$, then equation (25) possesses a pair of conjugation complex roots and a real root. Otherwise, for general $\beta$ equation (25) possesses three different complex roots.

**Proposition 1.** If $\sigma = 1$, then we have
- When $0 < |b| < 1/2$. If $|\beta| = |\beta\pm|$, where
  \[
  \beta\pm = \frac{\sqrt{2}}{4|b|}\sqrt{-2b^4 + 10b^2 + 1 \pm (1 - 4b^2)^{3/2}},
  \]
  it follows that equation (25) possesses multiple roots. If $\beta \in (\beta^-, \beta^+) \cup (-\beta^+, -\beta^-)$, then equation (25) possesses three distinct real roots. If $\beta \in \mathbb{R}/\{[\beta^-, \beta^+] \cup [-\beta^+, -\beta^-]\}$, equation (25) possesses a pair of conjugation complex roots and a real root. Otherwise, equation (25) possesses three different complex roots.

*When $|b| = 1/2$. If $\beta = \pm \frac{\sqrt{3}}{4}$, it follows that (25) possesses triple root $\alpha = -\frac{\sqrt{3}}{2}$. If $\beta \in \mathbb{R}/\{\pm \frac{3\sqrt{3}}{4}\}$, it follows that (25) possesses a pair of conjugation complex roots and a real root. Otherwise, equation (25) possesses three different complex roots.*
• When $|b| > 1/2$. If $\beta = \beta_\pm, \overline{\beta_\pm}$, where

$$\beta_\pm = \pm \frac{1}{4v} \left( \sqrt{2b(b^2 + 2)^{3/2} - 2b^4 + 10b^2 + 1 + i\sqrt{2b(b^2 + 2)^{3/2} + 2b^4 - 10b^2 - 1}} \right),$$

it follows that equation (25) possesses multiple roots. If $\beta \in \mathbb{R}$, then equation (25) possesses a pair of conjugation complex roots and a real root. Otherwise, equation (25) possesses three different complex roots.

**Proposition 2.** If $\sigma = -1$, then we have

• When $|b| = \sqrt{2}$. If $\beta = 0$, it follows that (25) possesses a tripe root $\alpha = 0$. If $\beta \in \mathbb{R}/\{0\}$, it follows that (25) possesses a pair of conjugation complex roots and a real root. Otherwise, equation (25) possesses three different complex roots.

• When $|b| \neq \sqrt{2}$. If $\beta = \pm i\sqrt{2} \sqrt{2b^4 + 10b^2 - 1 + \rho(1 + 4b^2)^{3/2}}$, it follows that (25) possesses multiple root. If $\beta \in \mathbb{R}$, it follows that (25) possesses a pair of conjugation complex roots and a real root. Otherwise, equation (25) possesses three different complex roots.

On the other hand, we need to analyze the positivity condition for the Hermite quadric form $\langle y_1 | J | y_1 \rangle$. For the focusing case, we can see that the quadric form $\langle y_1 | J | y_1 \rangle$ is positivity. We merely need to analyze the Hermite quadric form $\langle y_1 | J | y_1 \rangle$ for the defocusing case. We need to analyze the following Hermite quadric form

$$E_1 \equiv L_1^* J L_1 = 2i(\overline{\lambda}_1 - \lambda_1)C_1,$$

Indeed, the $C_1$ matrix is nothing but Cauchy matrix $c_{i,j} = \frac{1}{x_i + y_j}$ with $x_1 = \overline{\gamma}_1 = i\lambda_1$, $x_2 = \overline{\gamma}_2 = i\lambda_2$, $x_3 = \overline{\gamma}_3 = i\lambda_3$. The determinant of Cauchy matrix $C_1$ can be represented as

$$\det(C_1) = \prod_{i<j}^3 |x_i - x_j|^2 \prod_{i,j}^3 (x_i + \overline{x}_j).$$

So the sign of determinant $\det(C_1)$ is determined by $\prod_{i=1}^3 (x_i + \pi)$. Suppose $\text{Im}(\lambda_1) > 0$, so the eigenvalue of matrix $C_1$ is consistent with $E_1$. On the other hand, matrix $E_1$ has two negative eigenvalues and a positive one. So is the matrix $C_1$. To analyze the matrix $C_1$, we rearrange $x_i$ with the order $\text{Re}(x_1) \geq \text{Re}(x_2) \geq \text{Re}(x_3)$. Similar as Lemma 1 in ref. [18], we can conclude that $\text{Re}(x_1) > 0$ and $\text{Re}(x_2), \text{Re}(x_3) < 0$. Thus there exists an negative principal minor in matrix $C_1$. We would like to use the negative principal minor matrix to construct the non-singular solution for defocusing SSE (4).

If $b = 0$, then

$$E_1 \equiv L_1^* J L_1 = 2i(\overline{\lambda}_1 - \lambda_1)C_1,$$

Indeed, the $C_1$ matrix is nothing but Cauchy matrix $c_{i,j} = \frac{1}{x_i + y_j}$ with $x_1 = \overline{\gamma}_1 = i\lambda_1$, $x_2 = \overline{\gamma}_2 = i\lambda_2$, $x_3 = \overline{\gamma}_3 = i\lambda_3$. The determinant of Cauchy matrix $C_1$ can be represented as

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$$E_1 \equiv L_1^* J L_1 = 2i(\overline{\lambda}_1 - \lambda_1)C_1,$$

Indeed, the $C_1$ matrix is nothing but Cauchy matrix $c_{i,j} = \frac{1}{x_i + y_j}$ with $x_1 = \overline{\gamma}_1 = i\lambda_1$, $x_2 = \overline{\gamma}_2 = i\lambda_2$, $x_3 = \overline{\gamma}_3 = i\lambda_3$. The determinant of Cauchy matrix $C_1$ can be represented as

$$\det(C_1) = \prod_{i<j}^3 |x_i - x_j|^2 \prod_{i,j}^3 (x_i + \overline{x}_j).$$

So the sign of determinant $\det(C_1)$ is determined by $\prod_{i=1}^3 (x_i + \pi)$. Suppose $\text{Im}(\lambda_1) > 0$, so the eigenvalue of matrix $C_1$ is consistent with $E_1$. On the other hand, matrix $E_1$ has two negative eigenvalues and a positive one. So is the matrix $C_1$. To analyze the matrix $C_1$, we rearrange $x_i$ with the order $\text{Re}(x_1) \geq \text{Re}(x_2) \geq \text{Re}(x_3)$. Similar as Lemma 1 in ref. [18], we can conclude that $\text{Re}(x_1) > 0$ and $\text{Re}(x_2), \text{Re}(x_3) < 0$. Thus there exists an negative principal minor in matrix $C_1$. We would like to use the negative principal minor matrix to construct the non-singular solution for defocusing SSE (4).

If $b = 0$, then

$$E_1 \equiv L_1^* J L_1 = 2i(\overline{\lambda}_1 - \lambda_1)C_1,$$
Similar as above, we merely consider the defocusing case \( \sigma = -1 \). Suppose \( \text{Im}(\chi_1) > \text{Im}(\chi_2) \), then we can obtain the negative submatrix

\[
\begin{bmatrix}
\frac{1}{i \chi_2 - i \lambda_1} & 0 \\
0 & \frac{-1}{i \lambda_1 - i \lambda_2}
\end{bmatrix}
\]

from the matrix \( E_1 (28) \).

Finally, the roots of cubic equation (24) can be obtained by Cardano’s formula. But the formula is rather complex. To avoid this problem, we introduce the following parameter transformation to solve the cubic equation (24) automatically:

**Proposition 3.**

- When \( b \neq 0 \), introducing the parameter transformation
  \[
  \lambda_1 = \frac{b (\kappa_1 - \sigma)}{2 (\kappa_1 + \sigma)} + \frac{\kappa_1^2 - 1}{4 b \kappa_1},
  \]
  one can verify that
  \[
  \chi_1 = \frac{k_1^2 - 1 + \sqrt{\Delta_1}}{4 b \kappa_1}, \quad \chi_2 = \frac{k_1^2 - 1 - \sqrt{\Delta_1}}{4 b \kappa_1}, \quad \chi_3 = \frac{b (\kappa_1 - \sigma)}{(\kappa_1 + \sigma)},
  \]
  satisfy the cubic equation (24) automatically, where
  \[
  \Delta_1 = \left( \kappa_1^2 + 4 b^2 \kappa_1 - 1 \right)^2 + 16 b^2 \kappa_1 (1 + \sigma \kappa_1).
  \]

- When \( b = 0 \), setting
  \[
  \lambda_1 = \frac{1}{\sqrt{2}} \left( \kappa_2 - \frac{\sigma}{\kappa_2} \right),
  \]
  then
  \[
  \chi_1 = \sqrt{2} \kappa_2, \quad \chi_2 = -\sqrt{2} \frac{\sigma}{\kappa_2}, \quad \chi_3 = 0,
  \]
  satisfy the cubic equation (24) automatically.

3.1. **Single localized wave solution and periodical solution.** We use two different DTs to construct the single solitonic and periodical solution respectively. Actually, the different types of solution are determined by the roots of characteristic equations (24). The single localized wave solution and periodical solution can be classified through the Table 1. According to the table, we can obtain a whole understanding about the exact solutions for the SSE on the NVBC. What should be pointed out that, most of the single localized wave solutions have been obtain in the previous literature. For the focusing case, the breather solution [1, 30], rogue wave solution[4], rational W-shape soliton [34] and degenerate resonant soliton on NVBC [32] have been obtained in the previous literature. For the defocusing case, the dark soliton and W-shape dark soliton (dark double-hump soliton) have been be obtain through bilinear method and symbolic computation [12]. Here we give the explicit expression of solution by DT, since they possess the different representation form. It is not readily to see that they are equivalent with each other directly, but we believe that they are equivalent since they possess the same dynamics.

(a). **Soliton and resonant soliton solution on the NVBC**

If the cubic equation (24) possesses three different pure image roots, then one obtains the W-shape soliton, anti-soliton and resonant soliton. The parameter conditions are given in **Proposition 1**. Choosing special solution

\[
|y_1\rangle = D_1 \begin{bmatrix}
\varphi_1 \\
\psi_{1,+} \\
\psi_{1,-}
\end{bmatrix}, \quad \begin{bmatrix}
\varphi_1 \\
\psi_{1,+} \\
\psi_{1,-}
\end{bmatrix} = L_1 M_1 \begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix}.
\]
In this case, one uses the first kind of DT to construct the solution. So one needs to verify the following condition

\[ \psi_1 + \overline{\psi}_1 + \overline{\psi}_1 \varphi_1 = 0, \lambda_1 \in \mathbb{R}, \]  

(31)

then we can obtain the parameters restriction \( c_1, c_2, c_3 \in \mathbb{R} \). Based on above analysis, this kind of solution merely appears in the focusing case. We can obtain the general soliton solution on the NVBC:

\[ q[1] = \left[ 1 - \frac{H}{M} \right] e^{ib[x + (b^2 - 6)t]}, \]  

(32)

where

\[ M = \sum_{i=1}^{3} \sum_{j=1}^{3} c_i c_j e^{X_i + X_j} \chi_i + \chi_j, \quad H = \sum_{i=1}^{3} \sum_{j=1}^{3} c_i c_j e^{X_i + X_j} \chi_i + \chi_j. \]

Here we merely analyze the case \( b \neq 0 \), the case \( b = 0 \) was given in literature [32]. In what following, we analyze the properties for above solutions. If \( c_3 = 0, c_1 c_2 \neq 0 \), then we can obtain a soliton solution on the plane wave background

\[ q[1] = \left[ \frac{e^{ib_1 + \alpha_1} + e^{ib_2 - \alpha_1} + 2\gamma_1 \beta_1}{e^{\alpha_1} + e^{-\alpha_1} + 4\gamma_1} \right] e^{ib[x + (b^2 - 6)t]} \]

where

\[ e^{ib_1} = \frac{b - \chi_1}{b + \chi_1}, \quad e^{ib_2} = \frac{b - \chi_2}{b + \chi_2}, \quad \alpha_1 = X_1 - X_2 + \ln \left| \frac{c_1}{c_2} \right| + \frac{1}{2} \ln \left( \frac{\chi_2}{\chi_1} \right), \]

\[ \beta_1 = \frac{b - \chi_1}{b + \chi_2} + \frac{b + \chi_2}{b - \chi_1}, \quad \gamma_1 = \text{sign} \left( \frac{c_1}{c_2} \right) \frac{\sqrt{\chi_1 \chi_2}}{\chi_1 + \chi_2}. \]

After simple calculation, we can obtain that the peak of soliton \( |q[1]|^2 \) is along the line

\[ i(\chi_1 - \chi_2)(x - vt) + \ln \left| \frac{c_1}{c_2} \right| + \frac{1}{2} \ln \left( \frac{\chi_2}{\chi_1} \right) = 0, \quad v_1 = -[2\lambda_1(\chi_1 + \chi_2) + b^2 - 4] \]
and the peak value is \( \left| \frac{e^{i\theta_1} + e^{i\theta_2} + 2\gamma_1\beta_1}{2 + 4\gamma_1} \right|^2 \). When \( c_1/c_2 > 0 \), we can obtain the bell-
shape soliton; while \( c_1/c_2 < 0 \), we can obtain the W-shape soliton. Similarly, we

can obtain the other cases \( c_2 = 0 \), \( c_1c_3 \neq 0 \) and \( c_1 = 0 \), \( c_2c_3 \neq 0 \).

Finally, we discuss the case \( c_1c_2c_3 \neq 0 \). We can obtain the resonant soliton solution

which is nothing but nonlinear superposition for above three solitons. Suppose \( \Im(\lambda_1) > 0 \), and \( \Im(\chi_1 + \chi_2) < \Im(\chi_1 + \chi_3) < \Im(\chi_2 + \chi_3) \), then \( v_1 < v_2 < v_3 \),

where

\[
\begin{align*}
v_1 &= -[2\lambda_1(\chi_1 + \chi_2) + b^2 - 4], \\
v_2 &= -[2\lambda_1(\chi_1 + \chi_3) + b^2 - 4], \\
v_3 &= -[2\lambda_1(\chi_2 + \chi_3) + b^2 - 4].
\end{align*}
\]

When \( t < \frac{\alpha_1 - \alpha_3}{\chi_3 - \chi_1} \), there are two solitons along the lines

\[
\begin{align*}
x - v_1t &= \alpha_1, \quad \alpha_1 = \left[ \ln \frac{c_1}{c_2} + \frac{1}{2} \ln \frac{\chi_2}{\chi_1} \right] i(\chi_2 - \chi_1), \\
x - v_3t &= \alpha_3, \quad \alpha_3 = \left[ \ln \frac{c_2}{c_3} + \frac{1}{2} \ln \frac{\chi_3}{\chi_2} \right] i(\chi_3 - \chi_2),
\end{align*}
\]

respectively. When above two solitons collision at \((x, t) = \left( \frac{\alpha_1 v_1 - \alpha_3 v_3}{\chi_3 - \chi_1}, \frac{\alpha_1 - \alpha_3}{\chi_3 - \chi_1} \right)\) approximately, they merge into a new soliton along the line

\[
x - v_2t = \alpha_2, \quad \alpha_2 = \left[ \ln \frac{c_3}{c_1} + \frac{1}{2} \ln \frac{\chi_1}{\chi_3} \right] i(\chi_1 - \chi_3).
\]

The height of three solitons are

\[
H_1 = \left| \frac{e^{i\theta_1} + e^{i\theta_2} + 2\gamma_1\beta_1}{2 + 4\gamma_1} \right|^2, \quad H_2 = \left| \frac{e^{i\theta_1} + e^{i\theta_3} + 2\gamma_2\beta_2}{2 + 4\gamma_2} \right|^2, \quad H_3 = \left| \frac{e^{i\theta_3} + e^{i\theta_2} + 2\gamma_3\beta_3}{2 + 4\gamma_3} \right|^2,
\]

respectively, where

\[
e^{i\theta_3} = \frac{b - \chi_3}{b + \chi_3}, \quad \beta_2 = \frac{b - \chi_1}{b + \chi_3} + \frac{b + \chi_2}{b - \chi_1}, \quad \beta_3 = \frac{b - \chi_1}{b + \chi_3} + \frac{b + \chi_2}{b - \chi_2},
\]

\[
\gamma_2 = \text{sgn} \left( \frac{c_1}{c_3} \right) \frac{\sqrt{\chi_1 \chi_3}}{\chi_1 + \chi_3}, \quad \gamma_3 = \text{sgn} \left( \frac{c_2}{c_3} \right) \frac{\sqrt{\chi_2 \chi_3}}{\chi_2 + \chi_3}.
\]

For instance, let \( \kappa_1 = \exp \left( \frac{\pi}{4} i \right) \), \( b = \frac{1}{4} \), by formula (29) and (30), it follows that

\[
\lambda_1 = \frac{(16 + 17\sqrt{2}) i}{8(2 + \sqrt{2})}, \quad \chi_1 = \frac{\sqrt{2}}{4(2 + \sqrt{2})},
\]

\[
\chi_2 = -i \left( \frac{1}{4} \sqrt{15 - 8\sqrt{2} - \sqrt{2}} \right), \quad \chi_3 = i \left( \frac{1}{4} \sqrt{15 - 8\sqrt{2} + \sqrt{2}} \right).
\]

Choosing parameters \( c_1 = -1, c_2 = c_3 = 1 \), we can obtain the two W-shape soliton

and a bell-shape soliton (Fig. 1). By formula (33), we can obtain that there are

two W-shape soliton with height 1.86 and 3.88 respectively, and a bell-shape soliton

with height 1.24.

(b) Periodical solution and half periodical solution

For the second case, if the characteristic equation (24) merely possesses one purely

image root, assume \( \chi_1 = -\bar{\chi}_2 \), then the restricted condition (31) gives the param-
eters restriction \( c_2 = c_1 \) and \( c_3 \in \mathbb{R} \). For the defocusing case, we can not obtain the
non-trivial regular solution through this kind of solution. If we choose the conjugate complex solution simultaneously, the solution would appear the singularity. So the periodical solution merely appear in the focusing case.

If $c_3 = 0$, then we can obtain the periodical solution

$$q[1] = \left[ 1 - \frac{1}{\chi_2} + \frac{1}{\chi_2} e^{2iX_{2,1}} + \frac{1}{\chi_2} e^{-2iX_{2,1}} \right] e^{i(b(x + (b^2 - 6)t))},$$

where $X_{2,1}$ is the image part of $X_2$. The period of $|q[1]|^2$ in $x$ direction is $\frac{\pi}{\lambda_{2,R}}$; and the period in $t$ direction is

$$\frac{\pi}{(b^2 - 4)\lambda_{2,R} - 4\lambda_{1,1}X_{2,R}X_{2,1}},$$

where $R$ and $I$ represent the real part and image part respectively.

If $c_3 \neq 0$, then we can obtain the semi-periodical solution. The expression for this solution is given in (32) with $\chi_1 = -\bar{\chi}_2$, $c_1 = \bar{c}_1$ and $c_3 \in \mathbb{R}$.

(c) Dark soliton

If $\lambda_1 = 0$, there is a pair of complex conjugation root $\chi_1 = \sqrt{2 - b^2}i$, $\chi_2 = -\sqrt{2 - b^2}i$ ($|b| < \sqrt{2}$) for cubic equation (24). The dark soliton solution merely exists in the defocusing case. Then we can construct the following dark soliton solution through the limit technique [20]:

$$q[1] = \left( \frac{b}{b + i\sqrt{2 - b^2}} + i\sqrt{2 - b^2} \tanh \left[ \frac{\sqrt{2 - b^2}}{b + i\sqrt{2 - b^2}} \left( x + (b^2 + 4) t \right) \right] \right) e^{i(bx + (b^3 + 4b)t)}. $$

The depth of dark soliton $|q[1]|^2$ is $1 - \frac{b^2}{2}$. And its hole is along the line $x + (b^2 + 4)t = 0$.

(d) Breather solution

This kinds of solution can be derived by the second type DT with $\lambda_1 + \bar{\lambda}_1 \neq 0$. We can obtain the following breather solution

$$q[1] = \left[ 1 + \left( \frac{\phi_1 \bar{\psi}_1 - \bar{\psi}_1 \phi_1}{M_1} \right) M_1 - \left( \frac{\phi_1 \bar{M}_2 + \bar{\psi}_1 \bar{M}_2}{M_1} \right) M_1^2 \right] e^{i(b(x + (b^2 + 6\sigma)t))},$$

where $\sigma$ and $\bar{\sigma}$ represent the real part and image part respectively.
where

\[ M_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} c_i c_j e^{X_i + X_j} / \chi_j - \chi_i, \quad M_2 = \sum_{i=1}^{3} \sum_{j=1}^{3} c_i c_j e^{X_i + X_j} / \chi_j + \chi_i, \quad \varphi_1 = \sum_{i=1}^{3} c_i e^{X_i}, \]

\[ \psi_{1, \pm} = \sum_{i=1}^{3} c_i e^{X_i} / \chi_i \pm b, \quad \text{when} \quad b \neq 0; \]

\[ M_1 = \sum_{i=1}^{2} \sum_{j=1}^{2} c_i c_j e^{X_i + X_j} / \chi_j - \chi_i, \quad M_2 = \sum_{i=1}^{2} \sum_{j=1}^{2} c_i c_j e^{X_i + X_j} / \chi_j + \chi_i + \sigma c_3 e^{X_i} / 2\lambda_1, \quad \varphi_1 = \sum_{i=1}^{2} c_i e^{X_i}, \]

\[ \psi_{1, \pm} = \sum_{i=1}^{2} c_i e^{X_i} / \chi_i \pm b + c_3, \quad \text{when} \quad b = 0. \]

For the focusing case, there are two kinds of breather solution. The first kind of breather solution is given by choosing parameters \( c_1 c_2 \neq 0, c_3 = 0 \) or \( c_2 c_3 \neq 0, c_1 = 0 \) or \( c_1 c_3 \neq 0, c_2 = 0 \). This kind of breather solution has been obtain in [1]. The second kind of solution is given by choosing parameters \( c_1 c_2 c_3 \neq 0 \). This type of solution is the resonant breather.

For the defocusing case, we can obtain the breather solution by choosing parameters \( c_3 = 0, c_1 c_2 \neq 0, \text{Im}(\chi_1) < 0 \) and \( \text{Im}(\chi_2) < 0 \). We show the figure for the dark-breather solution (Fig. 2(a)).

![Figure 2](image_url)

**Figure 2.** (color online): (a) Dark breather. Parameters, \( k_1 = 2i, b = 1, c_1 = c_2 = 1 \). (b) W-dark soliton. Parameters, \( k_1 = -\frac{3}{2}, b = 1, \lambda_1 = -\frac{12}{120}, \chi_3 = -\frac{5}{24} + \frac{1}{24}i\sqrt{599}, c_1 = 1 \).

(f) **W-shape Dark soliton**

Through the limit technique, we can obtain the W-shape dark soliton solution. This type of dark soliton solution merely exist in the defocusing case. The expression can be represented as

\[ q[1] = \left[ 1 + \frac{L_2}{L_1} \right] e^{ib[x+(b^2+6)t]}, \]
Based on these consideration, we have different special solution $\Phi_3$. Multi-solitonic solution. Firstly, we give the multi-solitonic solution formula. Suppose we have $N$ different special solutions $\Phi_i$ for Lax pair (5), then $|y_i|$ can be constructed as

$$|y_i| = \sum_{k=1}^{3} c_{i,k} E_{i,k},$$

where

$$E_{i,k} = D_1 \begin{bmatrix} e^{X_{i,k}} \\ e^{X_{i,k}} \\ X_{i,k} + b \\ X_{i,k} - b \end{bmatrix}, \quad X_{i,k} = i\chi_{i,k}[x + (2\lambda_i\chi_{i,k} + b^2 - 4\sigma)t],$$

c_{i,k} are constants, $\chi_{i,k}$ are three different roots for cubic equation (24) at $\lambda = \lambda_i$, $i = 1, 2, \ldots, N$, $k = 1, 2, 3$. We discuss the solution formula with two different cases.

(a). Focusing case

For the focusing case, we have $n$ special solution with $\lambda_i \in i\mathbb{R}$. The parameters satisfy the following conditions: If $\chi_{i,k} \in i\mathbb{R}$, then $c_{i,k} \in \mathbb{R}$. If $\chi_{i,1} = -\chi_{i,2}, \chi_{i,3} \in i\mathbb{R}$, then $c_{i,1} = -c_{i,2}, c_{i,3} \in \mathbb{R}$. Another $N - n$ special solutions $\Phi_i$ with $\lambda_i \notin i\mathbb{R} \cup \mathbb{R}$. By the symmetry relation, we have $K\Phi_i$ is a solution for Lax pair (5) with $\lambda = -\bar{\lambda}_i$. Based on these consideration, we have

$$M_{i,j}^{[1]} = \frac{\langle y_i | y_j \rangle}{2(\lambda_j - \bar{\lambda}_i)} = \begin{cases} \sum_{k=1,l=1}^{3,3} \frac{c_{i,k}c_{j,l}e^{X_{i,k}+X_{j,l}}}{\chi_{j,l} - \chi_{i,k}}, & \text{if } b \neq 0; \\ \sum_{k=1,l=1}^{2,2} \frac{c_{i,k}c_{j,l}e^{X_{i,k}+X_{j,l}}}{\chi_{j,l} - \chi_{i,k}} + \frac{c_{i,k}c_{j,l}e^{X_{i,k}+X_{j,l}}}{\lambda_j - \bar{\lambda}_i}, & \text{if } b = 0, \end{cases}$$

c_1 > 0$ and $\lambda_1 \in \mathbb{R}/0$, $\chi_1 \notin \mathbb{R}$. The parameters must be satisfied the following conditions: if $|b| \geq \sqrt{2}$, then $\lambda_1 \in (-\lambda_+, -\lambda_-) \cup (\lambda_+, \lambda_-)$; where

$$\lambda_+ = \sqrt{\frac{2b^4 + 10b^2 - 1 + \sqrt{(4b^2 + 1)^3}}{2\sqrt{2}|b|}},$$

if $0 < |b| < \sqrt{2}$, then $\lambda_1 \in (-\lambda_0, \lambda_0)/\{0\}$, where

$$\lambda_0 = \sqrt{\frac{2b^4 + 10b^2 - 1 + \sqrt{(4b^2 + 1)^3}}{2\sqrt{2}|b|}}.$$
and

\[ M_{i,j}^{[2]} = \frac{\langle y_i | K | y_j \rangle}{2(-\lambda_j - \lambda_i)} = \begin{cases} \sum_{k=1, l=1}^{3, 3} c_{i,k} c_{j,l} e^{X_{i,k} + X_{j,l}} - \lambda_j l + \lambda_i k, & \text{if } b\neq 0; \\
\sum_{k=1, l=1}^{2, 2} c_{i,k} c_{j,l} e^{X_{i,k} + X_{j,l}} - \lambda_j l + \lambda_i k, & \text{if } b = 0. \end{cases} \]

On the other hand, we have

\[ Y_1 = [\varphi_1, \varphi_2, \ldots, \varphi_n, \varphi_{n+1}, \varphi_N], \quad Y_2 = [\psi_{1,1}, \psi_{2,1}, \ldots, \psi_{n,n}, \psi_{n+1,n}, \ldots, \psi_{N,n}, \psi_{N,n}]. \]

Then we have

\[ q[N] = \left[ \frac{\det(M - Y_1^T Y_2)}{\det(M)} \right] e^{ib[x+(b^2-6)t]}, \]

where

\[ M = \begin{bmatrix} M_{11} & M_{12} \\ -M_{12}^T & M_{22} \end{bmatrix}, \quad M_{11} = \begin{pmatrix} M_{i,j}^{[1]} \end{pmatrix}_{1 \leq i, j \leq n}, \]

\[ M_{12} = \begin{bmatrix} M_{1,1,n+1} & M_{1,2,n+1} & \cdots & M_{1,1,N} & M_{1,2,N} \\ M_{2,1,n+1} & M_{2,2,n+1} & \cdots & M_{2,1,N} & M_{2,2,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n,1,n+1} & M_{n,2,n+1} & \cdots & M_{n,1,N} & M_{n,2,N} \end{bmatrix}, \]

\[ M_{22} = \begin{bmatrix} M_{1,n+1,n+1} & M_{1,n+1,n+1} & \cdots & M_{1,n+1,1} & M_{1,n+1,2} \\ -M_{2,n+1,n+1} & -M_{2,n+1,n+1} & \cdots & -M_{2,n+1,1} & -M_{2,n+1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n,n+1,n+1} & M_{n,n+1,n+1} & \cdots & M_{n,n+1,1} & M_{n,n+1,2} \\ -M_{n,n+1,n+1} & -M_{n,n+1,n+1} & \cdots & -M_{n,n+1,1} & -M_{n,n+1,2} \end{bmatrix}. \]

For example: Taking \( n = 1 \) and \( N = 2 \), we can obtain that one resonant soliton and resonant breather solution by choosing special parameters (Fig. 3(a)). Similarly, one can obtain the periodical solution and breather solution pair solution.

**b. Defocusing case**

In what following, we consider the defocusing case: By above analysis, suppose

\[
\begin{align*}
|y_i| &= c_{i,2} E_{i,2} + c_{i,3} E_{i,3}, \quad i = 1, 2, \ldots, n, \\
|y_i| &= c_{i,1} E_{i,1} + c_{i,2}(\lambda_i - \lambda_i) E_{i,2}, \quad i = n + 1, n + 2, \ldots, N.
\end{align*}
\]

The parameters can be choosen as following: If \( b \neq 0 \), then

\[ \lambda_i \not\in \mathbb{R} \cup \mathbb{R}, \quad \text{Im}(\chi_{i,2}) < 0, \quad i = 1, 2, \ldots, n; \]

If \( b = 0 \), then \( \lambda_i \not\in \mathbb{R} \cup \mathbb{R}, \quad \text{Im}(\chi_{i,2}) < 0, \quad i = 1, 2, \ldots, n. \]

On the other hand, \( \lambda_i \in \mathbb{R} \) if \( \lambda_i = 0 \), then we merely need to choose one special solution, otherwise, if \( \lambda_i \in \mathbb{R} \setminus \{0\} \), then we need to use the symmetry
relation to choose another relative solution $R \Phi$, which satisfies $\lambda = -\lambda_i$, $\chi_{i,1} = \chi_{i,2}$, $c_{i,2}$ is appropriate parameter, $i = n + 1, n + 2, \cdots, N$.

Then we have

$$M^{[1]}_{i,j} = \frac{(y_i| J| y_j)}{2(\lambda_j - \lambda_i)} = \begin{cases} \sum_{k=2}^{3} \sum_{l=2}^{3} c_{i,k}^2 e^{|X_{i,k} + X_{j,l}|}, & b \neq 0; \\
\sum_{k=2}^{3} c_{i,k}^2 e^{|X_{i,k} + X_{j,j}|} - \sum_{k=2}^{3} c_{i,k}^2 e^{|X_{i,j} + X_{j,j}|}, & b = 0; \end{cases}$$

and

$$M^{[2]}_{i,j} = \frac{(y_i| J| K| y_j)}{2(-\lambda_j - \lambda_i)} = \begin{cases} \sum_{k=2}^{3} \sum_{l=2}^{3} c_{i,k}^2 e^{|X_{i,k} + X_{j,l}|}, & b \neq 0; \\
\sum_{k=2}^{3} c_{i,k}^2 e^{|X_{i,k} + X_{j,j}|}, & b = 0; \end{cases}$$

And

$$Y_1 = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\
1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\
\end{bmatrix} \text{ or }$$

$$Y_2 = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\
1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\
\end{bmatrix} \text{ or }$$

Then the solitonic formula can be presented as the following:

$$q[N] = \frac{\det(M - Y^T_1 Y_2)}{\det(M)} e^{ib[x+(b^2+6)t]},$$
where
\[
M = \begin{bmatrix}
M_{1,1}^{[1]} & M_{1,2}^{[2]} & \ldots & M_{1,N}^{[2]} & M_{1,N}^{[2]} \\
-M_{2,1}^{[1]} & -M_{1,2}^{[1]} & \ldots & -M_{2,N}^{[2]} & -M_{1,N}^{[2]} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
M_{N,1}^{[1]} & M_{N,2}^{[2]} & \ldots & M_{N,N}^{[1]} & M_{N,N}^{[2]} \\
-M_{N,1}^{[2]} & -M_{N,2}^{[1]} & \ldots & -M_{N,N}^{[2]} & -M_{N,N}^{[1]}
\end{bmatrix}, \quad \text{or}
\]
\[
\begin{bmatrix}
M_{1,1}^{[1]} & M_{1,2}^{[2]} & \ldots & M_{1,N-1}^{[2]} & M_{1,N-1}^{[2]} & M_{1,N}^{[1]} \\
-M_{2,1}^{[1]} & -M_{1,2}^{[1]} & \ldots & -M_{2,N-1}^{[2]} & -M_{1,N-1}^{[2]} & -M_{1,N}^{[1]} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
M_{N-1,1}^{[1]} & M_{N-1,2}^{[2]} & \ldots & M_{N-1,N-1}^{[1]} & M_{N-1,N-1}^{[1]} & M_{N-1,N}^{[1]} \\
-M_{N,1}^{[2]} & -M_{N,2}^{[1]} & \ldots & -M_{N,N-1}^{[2]} & -M_{N,N-1}^{[2]} & -M_{N,N}^{[1]}
\end{bmatrix}.
\]

There are two types of breather-dark soliton solution. One type is breather and single dark soliton pair solution. Another type is breather and W-shape dark soliton pair solution (Fig. 3b).

![Figure 3](image)

**Figure 3.** (color online): (a) Two-soliton-1: parameters: \( b = 0 \), \( \lambda_1 = \sqrt{2}(1 + \frac{3}{4}i) \), \( \lambda_2 = 2i \), \( \chi_{1,1} = \sqrt{2}(1 + i) \), \( \chi_{1,2} = \frac{\sqrt{2}}{2}(-1 + i) \), \( \chi_{2,1} = (2 + \sqrt{2})i \), \( \chi_{2,2} = (2 - \sqrt{2})i \), \( c_{1,i} = c_{2,i} = 1 \). (b) Two-soliton-2: parameters: \( b = 1 \), \( \lambda_1 = \frac{1}{2}(1 + i) \), \( \chi_{1,1} \approx -0.6930142150 - 0.1648875144i \), \( \chi_{1,2} \approx 0.9430142154 - 0.585112485i \), \( \chi_{1,3} = 1 + 2i \), \( \chi_{2,1} = \frac{181}{6840} + \frac{1}{360} \sqrt{129959} \), \( c_{1,2} = c_{1,3} = 1 \), and \( c_{2,1} = 1 \).

4. **High order solutions.** In this section, we use the generalized DT and formal series method to derive the general high order solution formula with the algebraic
way. It should be pointed out that our formula without the differential for the spectral parameter.

4.1. General determinant formula for high order solitonic solution and high order breather.

Lemma 4.1. Suppose there exist three different roots $\chi_k^{[0]} (k = 1, 2, 3)$ for cubic equation (24) at $\lambda = \lambda_1^{[0]}$, then

$$\chi_k = \sum_{i=0}^{\infty} \chi_k^{[i]} e^i, \lambda_1 = \lambda_1^{[0]} + \epsilon$$

solve cubic equation (24), where $\epsilon$ is a small parameter; if $b \neq 0$,

$$\chi_k^{[1]} = \frac{2((\chi_k^{[0]})^3 - b^2)}{3(\chi_k^{[0]})^2 - 4\lambda_1^{[0]} \chi_k^{[0]} - b^2 - 2},$$

$$\chi_k^{[i]} = \frac{1}{3(\chi_k^{[0]})^2 - 4\lambda_1^{[0]} \chi_k^{[0]} - b^2 - 2} \times$$

$$\left[ - \sum_{l+m+n=i, \atop 0 \leq l,m,n \leq i-1} \chi_k^{[l]} \chi_k^{[m]} \chi_k^{[n]} + 2\lambda_1^{[0]} \sum_{l=1}^{i-1} \chi_k^{[l]} \chi_k^{[i-l]} + 2 \sum_{l=0}^{i} \chi_k^{[l]} \chi_k^{[i-l]} \right], \; i \geq 2;$$

if $b = 0$, then we have

$$\chi_k^{[1]} = \frac{1}{2(\chi_k^{[0]} - \lambda_1^{[0]})} \left( 2\chi_k^{[i-1]} - \sum_{l=1}^{i-1} \chi_k^{[l]} \chi_k^{[i-l]} \right), \; i \geq 1.$$

Specially, if $b \neq 0$, suppose $\lambda_1^{[0]} = \frac{b(\kappa_1 - 1)}{2(\kappa_1 + 1)} + \frac{\kappa_1^{-2} - 1}{4b\kappa_1}$, then we have

$$\chi_1^{[1]} = \frac{8b^2\kappa_1^2}{4b^2\kappa_1^2 + (\kappa_1^2 + 1)(\kappa_1 + 1)^2},$$

$$\chi_2^{[1]} = 1 + \frac{- (\kappa_1 - 1) \left[ 4b^2\kappa_1 + (\kappa_1 + 1)^2 \right] \delta - (\kappa_1 + 1) \left[ (\kappa_1^2 - 1)^2 - 16b^4\kappa_1^2 \right]}{(\kappa_1 - 1) \left[ 4b^2\kappa_1 - (\kappa_1 + 1)^2 \right] \delta - (\kappa_1 + 1) \delta^2},$$

$$\chi_3^{[1]} = 1 + \frac{- (\kappa_1 - 1) \left[ 4b^2\kappa_1 + (\kappa_1 + 1)^2 \right] \delta + (\kappa_1 + 1) \left[ (\kappa_1^2 - 1)^2 - 16b^4\kappa_1^2 \right]}{(\kappa_1 - 1) \left[ 4b^2\kappa_1 - (\kappa_1 + 1)^2 \right] \delta + (\kappa_1 + 1) \delta^2},$$

where $\delta = \sqrt{(4b^2\kappa_1 + \kappa_1^2 - 1)^2 + 16b^2\kappa_1 (\kappa_1 + 1)}$. If $b = 0$, suppose $\lambda_1^{[0]} = \frac{1}{\sqrt{2}} \left( \kappa_2 - \frac{1}{\kappa_2} \right)$, then we have

$$\chi_1^{[1]} = \frac{2\kappa_2}{\kappa_2^2 + 1}, \chi_2^{[1]} = \frac{2}{\kappa_2^2 + 1}.$$

In the following, we consider the expansion

$$X_k \equiv i\chi_k \left[ x + (2\lambda_1\chi_k + b^2 - 4) t \right] = \sum_{i=0}^{\infty} X_k^{[i]} e^i,$$
where
\[ X_k[0] = iX_k[0][x + (2\lambda_1^0 X_k[0] + b^2 - 4) t], \]
\[ X_k[i] = i \left[ X_k[i][x + \left( (b^2 - 4) X_k[i] + 2\lambda_1^0 \sum_{l,m=0\atop l+m=i}^{l+m=i} \chi[l,m] \chi[l,k] + 2 \sum_{l,m=1\atop l+m=i-1}^{l+m=i-1} \chi[l,m] \chi[l,k] \right] t \right], \quad i \geq 1. \]

Furthermore we have
\[ e^{X_k} = e^{X_k[0]} \sum_{i=0}^{\infty} S_i(X_k) e^i, \quad X_k = (X_k[1], X_k[2], \ldots). \]

The explicit expression of these polynomials can be are given by the elementary Schur polynomials
\[ S_0(X_k) = 1, \quad S_1(X_k) = X_k[1], \quad S_2(X_k) = X_k[2] + \frac{(X_k[1])^2}{2}, \]
\[ S_3(X_k) = X_k[3] + X_k[1] X_k[2] + \frac{(X_k[1])^3}{6}, \]
\[ S_i(X_k) = \sum_{l_1+2l_2+\ldots+sl_s=i} \frac{(X_k[1])^{l_1}(X_k[2])^{l_2} \ldots (X_k[s])^{l_s}}{l_1! l_2! \ldots l_s!}. \]

In what following, we give the following expansion:

**Lemma 4.2.**
\[ \frac{1}{\chi_k \pm b_1} = \sum_{i=0}^{\infty} \mu_{i, \pm}^0 e^i \]
where
\[ \mu_{i, \pm}^0 = \frac{1}{\chi_k \pm b_1}, \]
\[ \mu_{i, \pm}^j = \frac{-1}{\chi_k \pm b_1} \sum_{j=0}^{i-1} \mu_{j, \pm} \chi_k \chi_k^{[l]} \chi_k^{[j]}, \quad j \geq 1. \]

**Lemma 4.3.** Suppose we have
\[ \nu = \sum_{i=0}^{\infty} \nu[i] e^i, \quad \tau = \sum_{i=0}^{\infty} \tau[i] e^i, \]
then we can obtain that
\[ \frac{1}{\nu - \tau} = \sum_{i=0, j=0}^{\infty} w[i,j](\nu, \tau) \nu^i \tau^j, \]
where \( w[i,j](\nu, \tau) \) can be determined through the following way:
\[ w[0,0] = \frac{1}{\nu[0] - \tau[0]}, \]
\[ w[j,i-j] = \frac{1}{\nu[0] - \tau[0]} \left( \sum_{k=j}^{i-1} \nu[i-k] w[j,k-j] - \sum_{k=i-j}^{i-1} \tau[i-k] w[k-(i-j),i-j] \right), \]
\[i \geq 1, \ i \geq j \geq 0, \text{ recursively.}\]

**Proof.** Firstly, by the expansion
\[
\left[\sum_{i=0}^{\infty} (\nu_i \epsilon^i - \tau_i \epsilon^i)\right] \left[\sum_{j=0}^{\infty} \left(\sum_{j=0}^{i} w[j,i-j] \epsilon^j \epsilon^{i-j}\right)\right] = 1.
\]
Furthermore, we have
\[
\sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} \left(\sum_{j=0}^{k} w[j,k-j] \epsilon^j \epsilon^{k-j}\right)\right)\left(\nu[i-k] \epsilon^{i-k} - \tau[i-k] \epsilon^{i-k}\right) = \sum_{i=0}^{\infty} \left(\sum_{k=0}^{i} \left(\sum_{j=0}^{k} w[j,k-j] \epsilon^j \epsilon^{k-j}\right)\right) \epsilon^{i-j} = 1.
\]
It follows that the lemma is verified. \(\Box\)

**Lemma 4.4.** Suppose we have
\[Z = \sum_{i=0}^{\infty} Z[i] \epsilon^i, \ W = \sum_{i=0}^{\infty} W[i] \epsilon^i,
\]
then
\[e^{W+Z} \nu - \tau = \sum_{s=0}^{\infty} F^{[s,t]}(W, Z; \nu, \tau, \epsilon) \epsilon^s \epsilon^t,
\]
where \(W = (W[0], W[1], W[2], \cdots), \ Z = (Z[0], Z[1], Z[2], \cdots),\)
\[F^{[s,t]}(W, Z; \nu, \tau, \epsilon) = e^{Z[0] + W[0]} \sum_{n=0}^{s+t} \sum_{n+l=0}^{s-t-m} w[n,s+t-m-n] S_l(W) S_{n-l}(Z).
\]
**Proof.** Firstly, through the expansion
\[e^{Z+W} = e^{Z[0] + W[0]} \left[ \sum_{k=0}^{k} \left( \sum_{l=0}^{k} S_l(W) S_{k-l}(Z) \epsilon^l \epsilon^{k-l} \right) \right]
\]
together with
\[\frac{1}{\nu - \tau} = \sum_{k=0}^{k} \left( \sum_{l=0}^{k} w[l,k-l] \epsilon^l \epsilon^{k-l} \right),
\]
it follows that
\[
e^{Z + W} \nu^\tau = e^{Z_0 + W_0} \nu^\tau \left[ \sum_{k=0}^{\infty} \left( \sum_{t=0}^{k} S_1(W) S_{k-t}(Z) \varepsilon^t e^{-t} \right) \right] \left[ \sum_{k=0}^{\infty} \left( \sum_{t=0}^{k} W_{\nu_{[t,k-\ell]}} e^{-t} \right) \right]
\]
\[
= e^{Z_0 + W_0} \sum_{k=0}^{\infty} \left[ \sum_{m=0}^{\infty} \left( \sum_{l=0}^{k} S_1(W) S_{m-l}(Z) \varepsilon^l e^{m-l} \right) \right] \left( \sum_{n=0}^{\infty} W_{\nu_{[n,k-m-n]}} e^{-n} \right)
\]
\[
= e^{Z_0 + W_0} \sum_{k=0}^{\infty} \left[ \sum_{m=0}^{\infty} \left( \sum_{l=0}^{k} S_1(W) S_{m-l}(Z) \varepsilon^l e^{m-l} \right) \right] \left( \sum_{n=0}^{\infty} W_{\nu_{[n,k-m-n]}} e^{-n} \right)
\]
\[
= \sum_{k=0}^{\infty} \sum_{m=0}^{k} \sum_{n=0}^{k} \sum_{l=0}^{m} W_{\nu_{[n,k-m-n]}} S_1(W) S_{m-l}(Z) \varepsilon^l e^{m-l-s} \varepsilon^{-s} e^{k-s}.
\]
This completes the proof.

To obtain the general high order solution, we choose the special function
\[
|y_1\rangle = \sum_{k=1}^{3} d_k D_1 \left[ \begin{array}{c} e^{Y_k} \\ e^{Y_k} \\ \lambda_k + b \\ \lambda_k - b \end{array} \right] = D_1 \left[ \begin{array}{c} \psi_1+ \\ \psi_1- \\ \psi_1+ \\ \psi_1- \end{array} \right],
\]
where \( Y_k = \sum_{l=0}^{\infty} Y_k^{[l]} \ell_1^l \) and \( Y_k^{[l]} = X_k^{[l]} + e_k^{[l]} \), \( d_k = 0, 1 \).

Finally, based on above lemmas, we can obtain the following expansions:
\[
M_1 = \frac{\langle y_1 | y_1 \rangle}{2(\lambda_1 - \chi_1)} = \sum_{m=1}^{\infty} M_1^{[m,n]} \ell_1^{m-1} \ell_1^{n-1}
\]
where
\[
M_1^{[m,n]} = \begin{cases} 
\sum_{k=1}^{3} d_k d_l F_{k,l}^{[m-1,n-1]} & \text{if } b \neq 0; \\
\sum_{k=1}^{2} d_k d_l F_{k,l}^{[m-1,n-1]} + \frac{d_l^2 (-1)^{n-1}}{(\lambda_1^{[0]} - \lambda_1^{[0]})^{m+n-1}} \binom{m+n-2}{m-1}, & \text{if } b = 0,
\end{cases}
\]
and
\[
F_{k,l}^{[m-1,n-1]} = F_{k,l}^{[m-1,n-1]}(Y_k, Y_l; \chi_k, \chi_l), \quad Y_k = (Y_k^{[1]}, Y_k^{[2]}, \ldots).
\]
Similarly, we have
\[
M_2 = \frac{\langle y_1 | K | y_2 \rangle}{2(-\chi_2 - \chi_1)} = \sum_{m=1}^{\infty} M_2^{[m,n]} \ell_1^{m-1} \ell_2^{n-1}, \quad M_2^{[m,n]} = \lambda_2^{[0]} + \epsilon_2, \quad |y_2\rangle = |y_1(\epsilon_2)\rangle,
\]
Proposition 4. Suppose the semi-periodical solution is shown in figure (4 b).

The dynamics of second order resonant soliton is shown where

\[
M_2^{[m,n]} = \begin{cases} 
3, & \text{if } b \neq 0; \\
2, & \text{if } b = 0,
\end{cases}
\]

and

\[
G_{k,l}^{[m-1,n-1]} = F^{[m-1,n-1]}(\mathcal{Y}_k, \mathcal{Y}_l; -\bar{\chi}_k, \bar{\chi}_l).
\]

Through the symmetry relation, we have

\[
M_3 = \frac{\langle y_2 | K | y_1 \rangle}{2(\lambda_1 + \lambda_2)} = -M_2 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\overline{M_2^{[n,m]}}) \epsilon_1^{m-1} \epsilon_2^{n-1},
\]

and

\[
M_4 = \frac{\langle \bar{y}_2 | \bar{y}_2 \rangle}{2(-\lambda_2 + \lambda_2)} = -M_1 = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (\overline{M_4^{[n,m]}}) \epsilon_2^{m-1} \epsilon_2^{n-1}.
\]

On the other hand, we have the following expansion:

\[
\varphi_1 = \sum_{n=1}^{\infty} \varphi_1^{[n]} \epsilon_1^{n-1}, \quad \psi_{1,\pm} = \sum_{n=1}^{\infty} \psi_{1,\pm}^{[n]} \epsilon_1^{n-1}
\]

where

\[
\varphi_1^{[n]} = \begin{cases} 
3, & \text{if } b \neq 0; \\
2, & \text{if } b = 0,
\end{cases}
\]

and

\[
\psi_{1,\pm}^{[n]} = \begin{cases} 
3, & \text{if } b \neq 0; \\
2, & \text{if } b = 0.
\end{cases}
\]

Proposition 4. Suppose \(\psi_1, \varphi_1 + \psi_1, -\varphi_1 = 0\), it follows that the high order solitonic solution can be represented as

\[
q[N] = \left[ \frac{\det(H_1)}{\det(M_1)} \right] e^{\theta \delta x + (b^2 - 6)t},
\]

where

\[
M_1 = \left( M_1^{[m,n]} \right)_{1 \leq m,n \leq N}, \quad H_1 = \left( H_1^{[m,n]} \right)_{1 \leq m,n \leq N}.
\]

Inserting the special parameters into above formula, we can obtain the second order solitonic solution. The dynamics of second order resonant soliton is shown in figure (4 a) with special parameters choosing. The dynamics of second order semi-periodical solution is shown in figure (4 b).
Proposition 5. The high order breather can be represented as
\[ q[N] = \left[ \frac{\det(M - H)}{\det(M)} \right] e^{ib[x + (b^2 - 6)t]}, \]
where
\[ M = \begin{bmatrix}
M_1^{[1,1]} & M_2^{[1,1]} & \cdots & M_1^{[1,N]} & M_2^{[1,N]} \\
-M_2^{[1,1]} & -M_1^{[1,1]} & \cdots & -M_2^{[N,1]} & -M_1^{[N,1]} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
M_1^{[N,1]} & M_2^{[N,1]} & \cdots & M_1^{[N,N]} & M_2^{[N,N]} \\
-M_2^{[1,N]} & -M_1^{[N,N]} & \cdots & -M_2^{[N,N]} & -M_1^{[N,N]}
\end{bmatrix}, \]
\[ H = \begin{bmatrix}
\varphi_1^{[1]} \psi_1^{[1]} & \varphi_1^{[1]} \psi_1^{[1]} & \cdots & \varphi_1^{[1]} \psi_1^{[N]} & \varphi_1^{[1]} \psi_1^{[N]} \\
-\varphi_1^{[1]} \psi_1^{[1]} & -\varphi_1^{[1]} \psi_1^{[1]} & \cdots & -\varphi_1^{[1]} \psi_1^{[N]} & -\varphi_1^{[1]} \psi_1^{[N]} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-\varphi_1^{[N]} \psi_1^{[1]} & -\varphi_1^{[N]} \psi_1^{[1]} & \cdots & -\varphi_1^{[N]} \psi_1^{[N]} & -\varphi_1^{[N]} \psi_1^{[N]}
\end{bmatrix}. \]

Inserting the special parameters into above formula, we can obtain the second order breather solution. The dynamics for this kind of high order solution is similar as the high order breather solution of classical NLSE (Fig. 5 a).

\[ \text{Figure 4. (color online):} \text{ (a): Second order resonant soliton: Parameters: } b = \frac{1}{4}, k_1 = \exp\{\frac{\pi}{4}\}, \epsilon_1^{[j]} = 0, i = 1, 2, 3, j = 0, 1; \text{ (b): Second order semi-periodical solution: Parameters: } b = 1, k_1 = \exp\{\frac{\pi}{4}\}, \epsilon_1^{[j]} = 0, i = 1, 2, 3, j = 0, 1. \]

4.2. General determinant formula for high order W-shape soliton-I, rogue wave solution and mixed case. When equation (24) possesses multiple roots,
 universally, we set \( b \) conveniently, we set \( b = \frac{1}{2}\sqrt{1 - \gamma^2} \), \( \gamma \neq 0, -1 \). If \( \lambda_1 = i\beta_1 \), then we have the double root \( \chi_1^0 = \frac{\delta_1}{2}, \chi_2^0 = \frac{\delta(1 - \gamma)}{2(1 + \gamma)} \), where \( \beta_1 = \frac{\delta(\gamma + 3)}{4(\gamma + 1)}, \delta = \sqrt{(\gamma + 1)(\gamma + 3)}. \)

**Lemma 4.5.** The asymptotical series

\[
\lambda_1 = i \left( \beta_1 + \rho_1 \epsilon^2 \right), \quad \rho_1 = \frac{\gamma \delta}{2(\gamma + 1)^2},
\]

and

\[
\chi_j = \sum_{i=0}^{+\infty} \chi_j^i \epsilon^i, \quad \chi_j^0 = \frac{i \delta}{2}, \chi_2^0 = \frac{i \delta (1 - \gamma)}{2(1 + \gamma)}, \quad j = 1, 2,
\]

solve the cubic equation (24). The coefficients \( \chi_1^i, \chi_2^i \) can be determined recursively. If \( \gamma \neq 1 \),

\[
\begin{align*}
\chi_1^i &= 1, \quad \chi_2^i = \frac{1}{\gamma \delta} \\
\chi_1^{i-1} &= \frac{2i + 1}{2i \delta} \left[ \sum_{k+l=m+1}^{\gamma, k+l, m \leq -2} \chi_l^k \chi_1^l \chi_1^m - \frac{\delta (\gamma + 3)}{2(\gamma + 1)} \sum_{i=2}^{\gamma, i \leq -2} \chi_1^i \chi_1^{i-1} - \frac{i \gamma \delta}{(\gamma + 1)^2} \sum_{i=0}^{\gamma, i \leq -2} \chi_1^i \chi_1^{i-2} \right], \quad i \geq 1 \\
\chi_2^i &= \frac{(\gamma + 1)}{\gamma \delta} \left[ \sum_{k+l=m+1}^{\gamma, k+l, m \leq -1} \chi_l^k \chi_2^l \chi_2^m - \frac{\delta (\gamma + 3)}{2(\gamma + 1)} \sum_{i=1}^{\gamma, i \leq -1} \chi_2^i \chi_2^{i-1} - \frac{i \gamma \delta}{(\gamma + 1)^2} \sum_{i=0}^{\gamma, i \leq -1} \chi_2^i \chi_2^{i-2} \right], \quad i \geq 2.
\end{align*}
\]

If \( \gamma = 1 \), then

\[
\begin{align*}
\chi_1^i &= 1, \quad \chi_2^i = 0, \quad i \geq 0, \\
\chi_1^{i-1} &= \sqrt{\frac{2}{3}} \chi_1^{i-2} + \frac{1}{2} \sum_{l=2}^{i-2} \chi_1^l \chi_1^{l-1}, \quad i \geq 2.
\end{align*}
\]
In the following, we consider the expansion

$$\chi_1^{[2]} = i \left( \frac{\gamma^2 + 2\gamma - 1}{2\gamma \delta} \right),$$

$$\chi_1^{[3]} = i \frac{\gamma^4 + 4\gamma^3 + 2\gamma^2 - 8\gamma + 5}{8\delta^2\gamma^2}.$$ 

Specially, we have

$$\chi_0^{[2]} = -4 i, \quad \chi_0^{[3]} = \frac{2}{(\delta + 2b)^2},$$

$$\chi_1^{[2]} = \frac{-\delta(\gamma^2 - 2\gamma - 1) \pm 2b(2\gamma + 1)}{2 [i(\gamma^3 + 4\gamma^2 + 3\gamma) \pm 2b\delta(\gamma + 2)] \gamma (\gamma + 1)},$$

$$\chi_1^{[3]} = \frac{\pm 2b(\gamma^4 - 4\gamma^3 - 14\gamma^2 + 5) \delta - (\gamma^4 - 4\gamma^3 - 14\gamma^2 + 16\gamma + 5)(\gamma + 1)^2}{8 [i(\gamma^2 + 2\gamma - 1) \pm 2b\delta] \gamma^2 \delta^2 (\gamma + 1)^2},$$

and

$$\mu_0^{[0]} = \frac{-2\delta}{i(\gamma^2 + 2\gamma - 3) \mp 2b\delta}, \quad \mu_0^{[2]} = \frac{2}{[i(\gamma^2 + 2\gamma - 1) \mp 2b\delta] \delta \gamma},$$

$\mu_1^{[0]} = \frac{2}{(\delta + 2b)^2}, \quad \mu_1^{[1]} = \frac{-4 i}{(\delta + 2b)^2},$
Since \( D_1 E_1(\epsilon_1) \) satisfies the Lax pair equations (5), then \( D_1 E_1(-\epsilon_1) \) also satisfies them. To obtain the general high order solution, we choose the general special solution

\[
|y_i| = D_1 \left( \frac{1}{2\epsilon_1} (E_1(\epsilon_1) - E_1(-\epsilon_1)) + d_2 E_2(\epsilon_1) \right) = D_1 \left[ \frac{\psi_1}{\psi_{1,+}} \right],
\]

where \( d_2 = 0, 1, \)

\[
E_1 = \begin{bmatrix}
\frac{e^{Y_1}}{\chi_1 + b} \\
\frac{e^{Y_1}}{-\chi_1 + b}
\end{bmatrix}, \quad Y_1 = \sum_{k=0}^{\infty} (X_1^{[k]} + e_1^{[k]})^k;
\]

and

\[
E_2 = \begin{bmatrix}
0 \\
-1 \\
1
\end{bmatrix}, \text{ if } b = 0; \quad E_2 = \begin{bmatrix}
e^{X_2} \\
e^{Y_2}
\end{bmatrix}, \text{ if } b \neq 0;
\]

Finally, we have

\[
M_1 = \frac{\langle y_1 | y_1 \rangle}{\lambda_1 - \lambda_2} = \sum_{m=1, n=1}^{\infty, \infty} M_{1,m,n}^2(2m-1)(2n-1),
\]

where

\[
M_{1,m,n}^2 = \begin{cases}
F_{1,1}^{[2m-1,2n-1]} + d_3 (F_{1,2}^{[m-1,2n-1]} - F_{1,1}^{[2m-1,n-1]}) + d_2^2 F_{2,2}^{[m-1,n-1]}, & \text{if } b \neq 0; \\
F_{1,1}^{[2m-1,2n-1]} + \frac{i d_2^2 \rho_1^{-1} \rho_1^{-1}}{(-\beta_0 - \beta_0)^{m+n-1}} \binom{m+n-1}{m-1}, & \text{if } b = 0,
\end{cases}
\]

and

\[
F_{1,1}^{[2m-1,2n-1]} = F_{1,1}^{[m-1,2n-1]}(X_1, Y_1, X_1, Y_1), \\
F_{1,2}^{[m-1,2n-1]} = F_{1,1}^{[m-1,2n-1]}(X_2, Y_2, X_2, Y_2), \\
F_{2,2}^{[m-1,n-1]} = F_{1,1}^{[m-1,n-1]}(X_2, Y_2, X_2, Y_2), \quad \forall_1 = \left( Y_1^{[1]}, Y_1^{[2]}, \ldots \right).
\]

Similarly, we have

\[
M_2 = \frac{\langle y_2 | K | y_2 \rangle}{2(\lambda_2 - \lambda_1)} = \sum_{m=1, n=1}^{\infty, \infty} M_{2,m,n}^2(2m-1)(2n-1), \quad |y_2| = |y_1(\epsilon_2)|, \quad \lambda_2 = \lambda_1(\epsilon_2),
\]

where

\[
M_{2,m,n}^2 = \begin{cases}
G_{1,1}^{[2m-1,2n-1]} + 2 d_3 G_{1,2}^{[m-1,2n-1]} + d_2 G_{2,2}^{[m-1,n-1]}, & \text{if } b \neq 0; \\
G_{1,1}^{[2m-1,2n-1]} + \frac{i d_2^2 (-\rho_1)^{m+n-2}}{(2\beta_1)^{m+n-1}} \binom{m+n-2}{n-1}, & \text{if } b = 0,
\end{cases}
\]
and
\[ G_{1,1}^{[2m-1,2n-1]} = F^{[2m-1,2n-1]}(\overline{\chi}_1, \chi_1; \xi_2, \xi_1), \]
\[ G_{1,2}^{[m-1,2n-1]} = F^{[m-1,2n-1]}(\overline{\chi}_1, \overline{\chi}_2; \xi_1, \chi_2), \]
\[ G_{2,2}^{[m-1,n-1]} = F^{[m-1,n-1]}(\overline{\chi}_2, \overline{\chi}_2; \chi_2, \chi_1). \]

Through the symmetry relation, we have
\[ M_3 = \frac{\langle y_2 | K | y_1 \rangle}{2(\lambda_1 + \lambda_2)} = -M_2 = \sum_{n=1,m=1}^{\infty, \infty} \left(-M_2^{[n,m]}\right) e^{2(n-1)} e^{2(1-n)}, \]
and
\[ M_4 = \frac{\langle y_1 | y_2 \rangle}{2(-\chi_2 + \lambda_2)} = -M_1 = \sum_{m=1,n=1}^{\infty, \infty} \left(-M_1^{[m,n]}\right) e^{2(m-1)} e^{2(1-n)}. \]

On the other hand, we have the following expansion:
\[ \varphi_1 = \sum_{n=1}^{\infty} \varphi_1^{[n]} e^{2(n-1)}, \quad \psi_{1,\pm} = \sum_{n=1}^{\infty} \psi_{1,\pm}^{[n]} e^{2(n-1)} \]
where
\[ \varphi_1^{[n]} = \left\{ \begin{array}{ll}
\left[ e^{Y_1^{[0]}} S_{2m-1}(Y_1) + d_2 e^{Y_2^{[0]}} S_{m-1}(\overline{X}_2) \right], & \text{if } b \neq 0; \\
\left[ e^{Y_1^{[0]}} S_{2m-1}(Y_1) \right], & \text{if } b = 0,
\end{array} \right. \]
and
\[ \psi_{1,\pm}^{[n]} = \left\{ \begin{array}{ll}
e^{Y_1^{[0]}} \sum_{k=0}^{2n-1} S_k(Y_1) \mu^{[2n-1-k]}_{1,\pm} \mp d_2 e^{Y_2^{[0]}} \sum_{k=0}^{n-1} S_k(\overline{X}_2) \mu_{2,\pm}^{[n-1-k]}, & \text{if } b \neq 0; \\
e^{Y_1^{[0]}} \sum_{k=0}^{2n-1} S_k(Y_1) \mu_{1,\pm}^{[2n-1-k]} \mp d_3 \delta_{1,n}, & \text{if } b = 0.
\end{array} \right. \]

**Proposition 6.** Suppose \( \lambda_1 \in \mathbb{R} \) and \( \epsilon_i^{[i]} \in \mathbb{R} \), then the high order W-shape soliton can be represented as
\[ q[N] = \left[ \frac{\det(H_1)}{\det(M)} \right] e^{i b [x + (b^2 - 6)t]}, \]
where
\[ M_1 = \left( M_1^{[m,n]} \right)_{1 \leq m,n \leq N}, \quad H_1 = \left( M_1^{[m,n]} - \varphi_1^{[n]} \psi_{1,1}^{[n]} \right)_{1 \leq m,n \leq N}. \]

Substituting the special parameters into above formula, we have the second order W-shape soliton. The dynamics is similar as derivative nonlinear Schrödinger equation [10].

When \( |b| \leq \frac{1}{2}, -1 < \gamma \leq 1 \), the W-shape rational solution can be constructed from above **proposition 6**:
\[ q[1] = \left[ 1 + \frac{2 b \gamma}{1 + \gamma} \left( X + \frac{1}{2} \right) - \frac{1}{1 + \gamma} \right] e^{i \left( b [x + (b^2 - 6)t] + \vartheta_1 \right)}, \]
where \( X = x + \left( -2 \beta_1 b + b^2 - 4 \right) t \), \( \exp(i \vartheta_1) = \frac{b - i d}{b + i d} \). The W-shape soliton was first obtained in [34] with a different form. The properties for W-shape soliton \(|q[1]|^2\) can be concluded as the following. The velocity for the W-shape soliton is
\[ v = \frac{3\gamma^2}{4} + 3\gamma + \frac{33}{4} \in (6, 12) \]. The value for peak is \((\gamma + 2)^2 \in (1, 9)\) monotonically increase at \(X = -\frac{1}{3}\). The value for holes is \(\frac{1}{3}(\gamma + 5)(1 - \gamma) \in [0, 1)\) monotonically decrease at \(X = -\frac{1 - \sqrt{3}}{6}\). When \(\gamma = 1\), the W-shape soliton degenerate as the rational traveling solution for mKdV equation.

**Proposition 7.** Suppose \(\lambda_1 \notin i\mathbb{R}\), then the high order rogue wave and composite rogue wave can be represented as

\[ q[N] = \left[ \frac{\det(M - H)}{\det(M)} \right] e^{ib|x| + (b^2 - 6)t}, \]

where

\[
M = \begin{bmatrix}
M_1^{1,1} & M_2^{1,1} & \cdots & M_1^{1,N} & M_2^{1,N} \\
-M_2^{1,1} & -M_1^{1,1} & \cdots & -M_2^{1,N} & -M_1^{1,N} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
M_1^{N,1} & M_2^{N,1} & \cdots & M_1^{N,N} & M_2^{N,N} \\
-M_2^{1,1} & -M_1^{1,1} & \cdots & -M_2^{1,N} & -M_1^{1,N}
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
\varphi_1^{[1]}\psi_1^{[1]} & \varphi_1^{[1]}\psi_1^{[-1]} & \cdots & \varphi_1^{[N]}\psi_1^{[1]} & \varphi_1^{[N]}\psi_1^{[-1]} \\
-\varphi_1^{[1]}\psi_1^{[1]} & -\varphi_1^{[1]}\psi_1^{[-1]} & \cdots & -\varphi_1^{[N]}\psi_1^{[1]} & -\varphi_1^{[N]}\psi_1^{[-1]} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\varphi_1^{[1]}\psi_1^{[1]} & \varphi_1^{[1]}\psi_1^{[-1]} & \cdots & \varphi_1^{[N]}\psi_1^{[1]} & \varphi_1^{[N]}\psi_1^{[-1]} \\
-\varphi_1^{[N]}\psi_1^{[1]} & -\varphi_1^{[N]}\psi_1^{[-1]} & \cdots & -\varphi_1^{[N]}\psi_1^{[1]} & -\varphi_1^{[N]}\psi_1^{[-1]}
\end{bmatrix}.
\]

When \(|b| > \frac{1}{2}\), set \(\gamma = i\chi\), then \(\chi = \xi + i\eta\), \(\delta = 2(\eta - i\xi)\), where

\[
\xi = \frac{\sqrt{3}}{4} \sqrt{\gamma^2 - 3 + \sqrt{(\gamma^2 + 1)(\gamma^2 + 3)}},
\]

\[
\eta = \frac{\sqrt{2}\sqrt{\gamma^2 - 3 + \sqrt{(\gamma^2 + 1)(\gamma^2 + 3)}} (\gamma^2 - 3 - \sqrt{(\gamma^2 + 1)(\gamma^2 + 3)})}{16\gamma}.
\]

Through the **proposition 7**, the first order rogue wave solution can be derived which is consistence with the twist rogue wave solution in [3]. Substituting the special parameters into above formula in **proposition 7**, we have the second order composite rogue wave solution. The dynamics is shown in Fig 6.

### 4.3. Determinant formula for High order W-shape soliton-II

Following a similar method as above two subsections, we can obtain the algebraic algorithm method for the high order W-shape soliton. When \(b = \frac{1}{2}\) and \(\lambda_1 = \frac{i3\sqrt{3}}{4}\), the cubic equation (24) possesses a tripe root \(\chi_1^{[0]} = \frac{3\sqrt{3}}{2}\). It follows that

\[ \lambda_1 = i\left(\frac{3\sqrt{3}}{4} + \frac{\epsilon^3}{2}\right), \quad \chi_1 = i\sum_{i=0}^{\infty} \alpha_i \epsilon^i, \]
Figure 6. (color online): Parameters $\gamma = \frac{7}{60}\sqrt{15}$, $z = x - \frac{2591}{320}t$, $\epsilon_1^{[0]} = \epsilon_1^{[1]} = 0$. (a): If $d_3 = 0$, then we can obtain the second order twist rogue wave solution. (b): If $d_3 = 1$, then we can obtain the second order composite rogue wave solution.

satisfies the cubic equation (24). The coefficients $\alpha_i$ can be obtained recursively:

$$
\alpha^{[0]} = \frac{\sqrt{3}}{2}, \quad \alpha^{[1]} = 1,
$$

$$
\alpha^{[i-2]} = \frac{1}{3} \left[ - \sum_{0 \leq k, l, m \leq i-3} \alpha^{[k]} \alpha^{[l]} \alpha^{[m]} + \frac{3\sqrt{3}}{2} \sum_{0 \leq k, l \leq i-3} \alpha^{[k]} \alpha^{[l]} + \sum_{k=0}^{i-3} \alpha^{[k]} \alpha^{[i-3-k]} \right],
$$

$$
i \geq 4.$$

Specially, we have

$$
\alpha^{[2]} = \frac{\sqrt{3}}{3}, \quad \alpha^{[3]} = \frac{1}{3}, \quad \alpha^{[4]} = \frac{2\sqrt{3}}{27}, \quad \alpha^{[5]} = \frac{1}{27}, \quad \alpha^{[6]} = 0, \quad \alpha^{[7]} = \frac{397}{1944}.
$$

On the other hand, we have the expansion

$$
X_1 \equiv i\chi_1 \left[ x + \left( 2\lambda_1\chi_1 - \frac{15}{4} \right) t \right] = \sum_{i=0}^{\infty} X_1^{[i]} E^i,
$$

where

$$
X_1^{[0]} = \frac{\sqrt{3}}{2} (-x + 6t), \quad X_1^{[1]} = \left( -x + \frac{33}{4} t \right), \quad X_1^{[2]} = \sqrt{3} \left( -\frac{1}{3} x + \frac{17}{4} t \right),
$$

$$
X_1^{[i]} = \alpha^{[i]} \left( -x + \frac{15}{4} t \right) + \left( \frac{3\sqrt{3}}{2} \sum_{j=0}^{i} \alpha^{[j]} \alpha^{[i-j]} + \sum_{j=0}^{i-3} \alpha^{[j]} \alpha^{[i-3-j]} \right) t, \quad i \geq 3.
Similarly, we have the following expansion:

$$E_1(\epsilon) = \begin{bmatrix} e^{Y_1} \\ e^{Y_1} \\ \chi_1 + \frac{1}{2} \\ e^{Y_1} \\ \chi_1 - \frac{1}{2} \end{bmatrix}, \quad Y_1 = X_1 + \sum_{l=0}^{\infty} \left( e_1^{[l]} \epsilon^{3l+1} + e_2^{[l]} \epsilon^{3l+2} \right) \equiv \sum_{l=0}^{\infty} Y_1^{[l]} \epsilon^l, \; e_1^{[l]}, e_2^{[l]} \in \mathbb{R}.$$ 

Through the symmetry relation, $D_1 E_1(\omega \epsilon)$ and $D_1 E_1(\omega^2 \epsilon)$ also satisfies them. Thus, we choose the special function

$$|y_1| = D_1 \left( \sum_{m=1}^{3} \frac{E_1(\omega^{m-1} \epsilon)}{3(\omega^{m-1} \epsilon)^n} \right) = D_1 \begin{bmatrix} \varphi_1 \\ \psi_{1,+} \\ \psi_{1,-} \end{bmatrix}, \; n = 1, 2.$$ 

Denote

$$F(\epsilon, \bar{\epsilon}) \equiv e^{Y_1 + \bar{Y}_1} \frac{e^{\bar{Y}_1}}{\chi_1 - \bar{\chi}_1} = e^{2Y_1^{[0]}} \sum_{i=0, j=0}^{+\infty, +\infty} F_{1,1}^{[i,j]} \epsilon^i \bar{\epsilon}^j,$$

where $Y_1^{[0]} = \frac{x}{2} \left( x - \frac{\bar{x}^2}{2} \right)$ and $F_{1,1}^{[i,j]} = F_{1,1}^{[i,j]}(Y_1, X_1; \chi_1, \bar{\chi}_1), \; Y_1 = \left( Y_1^{[1]}, Y_1^{[2]}, \cdots \right).$ 

In what following, we consider the expansion for the following function:

$$\frac{\langle y_1 | y_1 \rangle}{2(\lambda_1 - \bar{\lambda}_1)} = \left( \sum_{i=1}^{3} \sum_{j=1}^{3} e^{Y_1 + \bar{Y}_1} \frac{3(\omega^{j-1} \bar{\omega}^{i-1} \epsilon)^n (\chi_j - \bar{\chi}_i)}{9(\omega^{j-i} \bar{\omega}^{i-j} \epsilon)^n} \right) = e^{2Y_1^{[0]}} \sum_{i=1, j=1}^{+\infty, +\infty} M_{i,j} \epsilon^{3(i-j)},$$

where $\chi_2 = \chi_1(\omega \epsilon), \chi_3 = \chi_1(\omega^2 \epsilon), \; Y_2 = Y_1(\omega \epsilon), \; Y_3 = Y_1(\omega^2 \epsilon),$ 

$$M_{i,j} = \begin{cases} F_{1,1}^{[3-2,3-j-2]}, & n = 1; \\ F_{1,1}^{[3-1,3j-1]}, & n = 2. \end{cases}$$

On the other hand, we have the expansion

$$\frac{1}{\chi_1 + \frac{1}{2}} = \sum_{l=0}^{\infty} \mu^{[l]} \epsilon^l,$$

where

$$\mu^{[0]} = \frac{1}{2} - \frac{\sqrt{3}}{2} i,$$

$$\mu^{[l]} = - \left( \frac{\sqrt{3}}{2} + i \frac{1}{2} \right) \sum_{j=0}^{l-1} \mu^{[j]} \alpha^{[l-j]}, \; j \geq 1.$$ 

Similarly, we have the following expansion:

$$\varphi_1 = \sum_{k=1}^{3} \frac{e^{Y_k}}{3(\omega^{k-1} \epsilon)^n(\chi_k + \frac{1}{2})} = e^{Y_1^{[0]}} \sum_{i=1}^{\infty} S_{3(i-1)+n}(Y_1^{[i-1]} \epsilon^{3(i-1)},$$

$$\psi_1 = \sum_{k=1}^{3} \frac{\omega^{(k-1)} e^{Y_k}}{3(\omega^{k-1} \epsilon)^n(\chi_k + \frac{1}{2})} = e^{Y_1^{[0]}} \sum_{i=1}^{\infty} \left( \sum_{j=0}^{3(i-1)+n} S_j(\chi_1) \mu^{3(i-1)+n-j} \right) \epsilon^{3(i-1)}.$$
It follows that the high order W-shape soliton can be represented as
\[ q[N] = \left[ \frac{\det(H)}{\det(M)} \right] e^{\frac{i}{2}[x - \frac{3i}{2}t]}, \]
where
\[ M = \left( M^{[i,j]} \right)_{1 \leq i,j \leq N}, \]
\[ H = \left( M^{[i,j]} - \left[ S_{3(i-1)+n}(Y_1) \left( \sum_{k=0}^{3(j-1)+n} S_k(Y_1) \mu_{3(j-1)+n-k} \right) \right] \right)_{1 \leq i,j \leq N}. \]

There are two kinds of solutions for the high order W-shape soliton-II solution. One kind is \( n = 1 \), which corresponds to high order W-shape soliton-II solution with the figure 7(a). Another kind is \( n = 2 \), which corresponds to high order W-shape soliton-II solution with the figure 7(b).

\[ \text{Figure 7. (color online): (a): Second order W-shape soliton-I:} \]
\[ \text{Parameters, } e_1^{[0]} = e_2^{[0]} = 0, e_1^{[1]} = 40, z = x - \frac{33}{4}t. \] \( \text{(b): Second} \]
\[ \text{order W-shape soliton-II: Parameters } z = x - \frac{2991}{320}t, e_1^{[i]} = e_2^{[i]} = 0, \]
\[ i = 0, 1. \]

5. Conclusions and discussions. It is well known that, the DT is a powerful method to construct the exact solutions for the integrable equations. To the best of my knowledge, all the elementary function solutions could be derived from this method. It is naturally that how to classify or understand these solutions. The different dynamics corresponds to the different physical process. So it is necessary to classify these solutions through dynamical behavior.

In this work, we classify the single solitonic solutions through a characteristic equation (24). It follows that we give the general multi-solitonic solution formulas. To the best of our knowledge, it should be pointed out that the breather solution, dark soliton and W-shape dark soliton for defocusing SSE (4) were first derived through the DT method in this work.

On the other hand, based on the generalized DT and formal series method, we present an algebraic representation for high order solution of SSE model in detail.
Even more important, this method could be extended to the other integrable multicomponent model.

Note added: When this paper was accepted to be published, I noticed that Mu and Qin presented the high order rogue waves for the focusing Sasa-Satsuma model by the generalized Darboux transformation (or dressing transformation) [23], which were overlapped with a part of subsection 4.2 with a different form.

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Received July 2015; revised September 2016.

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