SO(5) non-Fermi liquid in a Coulomb box device

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Non-Fermi liquid (NFL) physics can be realized in quantum dot devices where competing interactions frustrate the exact screening of dot spin or charge degrees of freedom. We show that a standard nanodevice architecture, involving a dot coupled to both a quantum box and metallic leads, can host an exotic SO(5) symmetry Kondo effect, with entangled dot and box charge and spin. This NFL state is surprisingly robust to breaking channel and spin symmetry, but destabilized by particle-hole asymmetry. By tuning gate voltages, the SO(5) state evolves continuously to a spin and then "flavor" two-channel Kondo state. The expected experimental conductance signatures are highlighted.

Nanoelectronic circuit realizations of fundamental quantum impurity models allow the nontrivial physics associated with strong electron correlations to be probed via quantum transport measurements [1]. Quantum dot devices, in particular, can exhibit the Kondo effect at low temperatures [2]: a localized magnetic moment on the dot is dynamically screened by conduction electrons in the metallic leads. Single-dot devices can behave as single-electron transistors, with Kondo-enhanced spin-flip scattering strongly boosting the conductance between source and drain leads measured in experiments [3–5].

The conventional Kondo effect [6] involves a localized “impurity” spin-½ degree of freedom, coupled to a single effective channel of conduction electrons, and has SU(2) spin symmetry. However, the Kondo effect is also observed in more complex systems, such as coupled quantum dot devices [7, 8] and single-molecule transistors [9, 10], involving spin and orbital degrees of freedom. In such systems, it is possible to realize variants of the classic spin-½ single-channel Kondo paradigm; e.g. orbital [11], spin-1 [12, 13], and ferromagnetic [14] Kondo effects. In particular, the symmetry of the effective model is important in determining the low-energy physics. Kondo effects with SU(4) symmetry can be realized in double quantum dots [15, 16] and carbon nanotube dots [17, 18], and also have Fermi liquid (FL) ground states.

More exotic non-Fermi liquid (NFL) states can be realized in multi-channel systems, where competing interactions frustrate exact screening of the dot spin or charge degrees of freedom at special high-symmetry points [19]. This results in a residual dot entropy characteristic of fractionalized excitations, and anomalous conductance signatures [20, 21]. However, this kind of NFL physics is typically delicate, being found at the quantum critical point between more standard FL phases, and is unstable to relevant symmetry-breaking perturbations.

Experimentally, the major challenge to realize NFL Kondo physics in quantum dot devices is to prevent mixing between multiple conduction electron channels. Two prominent scenarios to achieve this utilize an interacting quantum box (“Coulomb box”) [22, 23]. The quantum box is a large quantum dot, hosting a macroscopically large number of electrons, but due to quantum confinement has a discrete level spacing δ and finite charging energy Ec. For δ < T < Ec the box effectively provides a continuum reservoir of conduction electrons, but also displays charge quantization [24].

Spin-two channel Kondo (s-2CK) physics can be realized in a device involving a small quantum dot coupled to a quantum box as well as metallic leads [22]. The low-energy effective model consists of a dot spin-½ exchange coupled to two conduction electron channels (leads and box), with mixing between the channels suppressed by the large box charging energy. Both channels compete to Kondo-screen the dot spin, resulting in an NFL state. Breaking channel or spin symmetry relieves the frustration and results in a standard FL state. This physics was realized experimentally in Refs. [25, 26].

By contrast, a charge-2CK (c-2CK) effect can be realized when a quantum box tuned to its charge degeneracy point is coupled to two leads, as proposed in Ref. [23] and realized experimentally in Refs. [27, 28]. In this case, the

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Figure 1. Right: Schematic of the device: a quantum dot coupled to a quantum box and source/drain leads. Left: NRG phase diagram spanned by dot and box gate voltages, Vd ∝ η/Ud and VB ∝ nB, showing the NFL line for various channel asymmetries Ls/Ld. SO(5) point located at nB = ± ½ and η = 0. Plotted for constant Ud = 0.3, Ec = 0.1 and tB = 0.12.
macroscopic box charge states play the role of a pseudospin impurity. Distinctive signatures of the resulting NFL state are observable in quantum transport [27–30].

In this Letter, we revisit the device of Refs. [22, 25, 26] but now examine the full phase diagram as function of dot and box gate voltages, which in turn control the dot and box occupancies – see Fig. 1. We show that the emergent SU(4) symmetry of the system arising when the dot hosts a local moment and the box is at its charge degeneracy point, is reduced to SO(5) at particle-hole symmetry. Although the SU(4) state is an FL [31], a novel NFL Kondo effect arises at the SO(5) point, in which both dot and box charge and spin are maximally entangled. We achieve a detailed understanding of this state using a combination of conformal field theory [32–34] (CFT), bosonization [35], and numerical renormalization group [36, 37] (NRG) techniques. Remarkably, the NFL physics at this point is robust to breaking channel and/or spin symmetry. Furthermore, we show that by tuning gate voltages, the SO(5) state evolves continuously into the more familiar s-2CK state of Refs. [22, 25, 26], and then into a “flavor”-2CK (f-2CK) effect when the dot local moment is lost but box charge fluctuations persist. The distinctive transport signatures associated with this physics are accessible in existing experimental setups.

Models and mappings. – The device illustrated in Fig. 1 is described by the Hamiltonian $H = H_0 + H_{hyb} + \sum \gamma_k H_{hyb}^\gamma$, with $\gamma = Ls$, Ld, B for the source/drain leads and box, respectively. $H_0 = \sum_{\gamma,k,\sigma} \epsilon_{\gamma,k} \sigma \hat{c}_{\gamma,k\sigma}^\dagger \hat{c}_{\gamma,k\sigma}$ describes the three conduction electron reservoirs, while

$$H_B = E_C \left( \hat{N}_B - N_0 - n_B \right)^2,$$

$$H_d = \sum_{\sigma} \epsilon_d d_{\sigma}^\dagger d_{\sigma} + U_d d_{\uparrow}^\dagger d_{\uparrow} + U_d d_{\downarrow}^\dagger d_{\downarrow},$$

describe the box Coulomb interaction and the dot. The dot is tunnel-coupled to the leads and box via $H_{hyb} = \sum_{k,\sigma} \left( t_{\gamma,k} d_{\gamma}^\dagger \sigma \hat{c}_{\gamma,k\sigma} + \text{H.c.} \right)$. Here, $\sigma = \uparrow, \downarrow$ denotes (real) spin, and $d_{\sigma}$ or $c_{\alpha\sigma}$ are operators for the dot or conduction electrons, respectively. $\hat{N}_B = \sum_{k,\sigma} \hat{c}_{Bk\sigma}^\dagger \hat{c}_{Bk\sigma}$ is the total number operator for the box electrons. The dot and box occupations are controllable by gate voltages $V_d \propto \eta$ and $V_B \propto n_B$, respectively. For simplicity we now take equivalent conduction electron baths $\epsilon_{\gamma,k} \equiv \epsilon_k$ with a constant density of states $\nu$ defined inside a band of width $D = 1$, such that $\epsilon_k \sim k$ at low energies. We define $t_{\gamma}^2 = \sum_k |t_{\gamma,k}|^2$ and $t_L^2 = t_{LS}^2 + t_{LD}^2$.

Following Ref. [38], we incorporate the box interaction term, Eq. 1, into the hybridization,

$$H_B + H_{hyb}^B \rightarrow E_C \left( \hat{T}^{\pm} - n_B \right)^2 + \sum_{k,\sigma} \left( t_{Bk\sigma} d_{\sigma}^\dagger \hat{c}_{Bk\sigma} \hat{T}^{\pm} + \text{H.c.} \right),$$

where $\hat{T}^{\pm} = \sum_{\nu} \left| N_B \pm 1 \right\rangle \left\langle N_B \right|$ are ladder operators for the box charge, and $\hat{T}^z = \sum_{\nu} \left( N_B - N_0 \right) \left| N_B \right\rangle \left\langle N_B \right|$. Note that the model possesses the symmetry $n_B \rightarrow n_B \pm 1$. Particle-hole (ph) asymmetry is controlled by $n_B$ and $\eta$; the model is invariant to replacing $n_B \rightarrow -n_B$ and $\eta \rightarrow -\eta$, related by a ph transformation. Exact ph symmetry arises at $\eta = 0$ for any integer or half-integer $n_B$.

For the NRG calculations presented here, only a finite number of charge states around the reference $N_0$ are required to obtain converged results [36, 37, 39].

Spin-2CK regime. – For large box charging energy $E_C$ and deep in the dot and box Coulomb blockade regime (near the point $\eta = 0$ and $n_B = 0$), the dot hosts an effective spin-$\frac{1}{2}$ local moment, and the box has a well-defined number of electrons $N_0$. At low temperatures $T \ll E_C, U_d$ virtual charge fluctuations on the dot and box due to $H_{hyb}$ generate the spin-flip scattering responsible for the Kondo effect. However, finite $E_C$ blocks charge transfer between the leads and box, giving rise to a frustration of Kondo screening and the possibility of NFL physics [22]. In this regime, a standard Schrieffer-Wolff transformation (SWT) yields the s-2CK model [19, 22],

$$H_{s-2CK} = H_0 + \hat{S}_d \cdot \left( \hat{J}_L \hat{S}_L + \hat{J}_B \hat{S}_B \right),$$

where $\hat{S}_d$ is a spin-$\frac{1}{2}$ operator for the dot, while $\hat{S}_{\alpha=L,B} = \frac{1}{2} \sum_{\sigma,\sigma'} \epsilon_{\sigma,\sigma'} \sigma \sigma' c_{\alpha\sigma}^\dagger c_{\alpha\sigma'}$ with $c_{\alpha\sigma} = \frac{1}{2} \sum_k t_{Bk\sigma} c_{Bk\sigma}$ and $c_{\alpha\sigma} = \frac{1}{2} \sum_k \left( t_{Lk\sigma} c_{Lk\sigma} + t_{Lk\sigma} c_{Lk\sigma} \right)$ the local conduction electron orbitals at the dot position, and where

$$\hat{J}_L = \frac{8t_L^2}{U_d} \left[ 1 - \frac{(2n)^2}{(2n')^2} \right]^{-1}; \hat{J}_B = \frac{8t_B^2}{U_d} \left[ 1 - \frac{(2n')^2}{(2n'')^2} \right]^{-1},$$

with $U_d' = U_d + 2E_C$ and $n' = n + 2E_C n_B$. Deep in the s-2CK regime, NFL physics arises when $\hat{J}_L = \hat{J}_B$. For given physical device parameters $U_d, E_C, t_{Ls}, t_{LB}$, Eq. 4 implies the existence of two NFL lines in the $(n_B, \eta)$ plane related by the symmetry $\eta \rightarrow -\eta$ and $n_B \rightarrow -n_B$, see Fig. 1. NFL physics can therefore be accessed by tuning the gate voltages $V_d \propto \eta$ and $V_B \propto n_B$, as demonstrated experimentally in this regime in Refs. [25, 26]. At the ph symmetric point $\eta = n_B = 0$, s-2CK arises for $t_{LB} = \zeta_L$ with $\zeta^2 \approx 1 + 2E_C/U_d$. Although this NFL state is robust to ph asymmetry, it is destabilized by channel asymmetry $\hat{J}_L \neq \hat{J}_B$ or spin asymmetry $B \neq 0$ [40].

SO(5) Kondo. – At $n_B = \frac{1}{2}$, the box states with $N_0$ and $(N_0 + 1)$ electrons are exactly degenerate. Neglecting other box charge states (which are at least $2E_C$ higher in energy), we may define charge pseudospin-$\frac{1}{2}$ operators $\hat{T}_B^+ = \left| N_0 + 1 \right\rangle \left\langle N_0 \right|, \hat{T}_B^- = \left( \hat{T}_B^+ \right)^\dagger$, and $\hat{T}_B^z = \frac{1}{2} \left( \left| N_0 + 1 \right\rangle \left\langle N_0 + 1 \right| - \left| N_0 \right\rangle \left\langle N_0 \right| \right)$. The charge pseudospin is flipped by electronic tunneling between the dot and box. The effective low-energy effective model is obtained by projecting onto the dot spin and box pseudospin sectors using a generalized SWT. We now consider explicitly the special point with ph symmetry $(n_B = \frac{1}{2})$.
and \( \eta = 0 \) and channel symmetry \((J_l = J_B \equiv J, \) which implies \( t_B = \xi t_L \) with \( \xi^2 \simeq 1 + 2E_C/U' \)), whence \([41, 42]\)

\[
H_{\text{eff}} = H_0 + J \tilde{S}_d \cdot \left( c^\dagger \hat{\sigma}^b e + V \hat{T}^\dagger_B \left( c^\dagger \hat{\tau}^e \right) c \right) + Q L \tilde{S}_d \cdot \left( c^\dagger \hat{\sigma} \left( \hat{\tau}^+ \hat{T}^\dagger_B + \hat{\tau}^- \hat{T}^+_B \right) c \right),
\]

(5)

where we have suppressed spin \( \sigma = \uparrow, \downarrow \) and channel \( \alpha = L, B \) labels for clarity, and with Pauli matrices \( \hat{\sigma}^a \) or \( \hat{\tau}^b \) acting in spin or channel space, respectively.

Although initially the coupling constants in Eq. 5 take different values, perturbative scaling \([43]\) shows that the model develops an emergent symmetry \( J = V = Q_L \) at an isotropic low-temperature fixed point. Then the RG equations reduce to \( dJ/dl = 3J^2 \), and we have a Kondo scale \( T^0_{\text{SO}(5)} \sim D \exp(-1/3\nu J) \).

The fixed point has an unusual \( \text{SO}(5) \) symmetry, which can be seen by writing Eq. 5 in the symmetric form,

\[
H_{\text{SO}(5)} = H_0 + J \sum_{A=1}^{10} J^A M^A,
\]

(6)

where \( J^A = c^\dagger T^A c \) (with \( c \equiv c_{\alpha \sigma} \) \( x = 0 \)) the conduction electron operators at the dot position as before) and \( M^A = f^\dagger T^A f \) in terms of a fermionic ‘impurity’ operator \( f \equiv f_{\alpha \sigma} \) which carries both ‘flavor’ and spin labels subject to the constraint \( f_{\sigma}^\dagger f_{\sigma} = 1 \) such that \( \tilde{S}_d = \frac{1}{2} f^\dagger \hat{\sigma}^a f \) and \( T^\dagger_B = \frac{1}{2} f^\dagger \hat{\tau}^b f \). Here, \( \left( T^A \right) \) are the ten non-zero generators of \( \text{SO}(5) \) \([44]\), \( T^{ab} = -T^{ba} \) (with \( a, b = 1...5 \)) satisfying the algebra \( T^{ab} T^{cd} = -it_{ab} T^{ad} T^{bd} - t_{bd} T^{ac} + t_{ad} T^{bc} \). The equivalence between Eqs. 5 and 6 is then established by,

\[
\hat{\sigma}^1 = T^{23}, \quad \hat{\sigma}^2 = T^{31}, \quad \hat{\sigma}^3 = T^{12}, \quad \hat{\sigma}^{a=1,2,3} \hat{\tau}^1 = T^{04}, \quad \hat{\sigma}^{a=1,2,3} \hat{\tau}^2 = T^{05}, \quad \hat{\sigma}^2 \hat{\tau}^3 = T^{45}.
\]

We applied the machinery of CFT \([32–34]\) to analyze the fixed point properties using the symmetry decomposition \( U(1)_x \times Z_2 \times \text{SO}(5)_1 \). Here, \( U(1) \) corresponds to the charge sector, \( Z_2 \) is an Ising model. The primary fields of the \( \text{SO}(5)_1 \) theory consist of the singlet \( 1 \) of scaling dimension 0, the spinor \( 4 \) of scaling dimension \( \frac{2}{5} \) and the vector \( 5 \) of scaling dimension \( \frac{1}{2} \) \([45]\). The strong coupling fixed point describing the \( \text{SO}(5) \) fixed point can be obtained by fusion with the spinor \( 4 \), under which the impurity transforms.

The energies \( E \) and degeneracies \( \# \) of the resulting finite size spectrum characterize the fixed point. We find \([39]\) \( \left( E, \# \right) = (0, 2); (\frac{1}{5}, 4); (\frac{1}{2}, 10); (\frac{2}{5}, 12); (1, 26); \ldots \), consistent with our NRG results, and establishing the new \( \text{SO}(5) \) fixed point as NFL. Interestingly, this spectrum is identical to that of the standard s-2CK model \([33]\). The entropy at the fixed point \( S = \log g \) is given in terms of the modular S-matrix \([32–34]\), yielding a \( \frac{1}{2} \ln(2) \) entropy, consistent with NRG, see Fig. 2; again reminiscent of s-2CK.

However, differences from the standard s-2CK picture can be seen in dynamical quantities such as the local susceptibilities and conductance – see middle and bottom panels in Fig. 2. Since the impurity spin \( \vec{S} \) and pseudospin \( \vec{T} \) operators are absorbed into the conduction electrons at the strong coupling fixed point, they must transform among the 10 generators of \( \text{SO}(5) \). But such fields occur only as descendents in \( \text{SO}(5)_1 \), and so spin-spin correlation functions appear FL-like, \( \chi_{\text{loc}}^\sigma(\omega) \sim \omega \) (similarly for flavor susceptibility). This contrasts to the regular \( k \)-channel Kondo effect: the spin \( SU(2)_k \) theory contains a vector field which transforms as the 3 components of the impurity spin, with scaling dimension \( \frac{2}{2k+1} \), which leads to anomalous NFL properties in the spin susceptibility \( \chi_{\text{loc}}^\sigma \sim \omega^{1-2k} \). At the \( \text{SO}(5) \) point, we find leading linear behavior of the conductance \( G(T) - G(0) \sim T \) from NRG. This contrasts to standard s-2CK conductance which approaches its fixed point value as \( \sqrt{T} \) \([22, 46–48]\), or the \( T^2 \) FL conductance of 1CK \([2]\).
To gain further insight, we expand on the bosonization and refermionization techniques [49] developed by Emery and Kivelson (EK) for the s-2CK model [35], and include the coupling to the flavour degree of freedom. This method allows us to express an anisotropic version of Eq. 5 in terms of local fermions $d \propto S^-$ and $a \propto T^-$, relating to impurity spin and flavor degrees of freedom, with corresponding Majorana operators $d_+ = \frac{1}{\sqrt{2i}}(d^+ + d)$ and $d_- = \frac{1}{\sqrt{2i}}(d^+ - d)$, and similarly for $a$, as well as for a 1D bulk fermionic field denoted $\psi_s(x)$, with Majorana components $\chi_+ = \frac{\psi_s^{(0)} + \psi_{sf}^{(0)}}{\sqrt{2}}$, $\chi_\perp = \frac{\psi_s^{(0)} - \psi_{sf}^{(0)}}{\sqrt{2}}$. The resulting EK Hamiltonian takes the form,

$$H_{EK} = H_0 + iJ_\perp d_- \chi_+ - 2iQ_\perp d_+ a_- .$$

Details of the derivation are given in the Supplemental Material [39]. The $J_\perp$ term is the usual EK form of the s-2CK interaction. The spin-flavour coupling term $Q_\perp$ couples and gaps out the pair $d_-$ and $a_-$. Unlike in the s-2CK model where $d_-$ remains decoupled, here we see that it is $a_+$ that is free at the SO(5) fixed point, and is responsible for the $\frac{1}{2} \ln(2)$ residual entropy.

**Stability of SO(5) Kondo.** We consider the effect of symmetry-breaking perturbations at the SO(5) point.

Channel asymmetry, corresponding to $t_B \neq \xi t_L$ in the bare model, generates an extra term in Eq. 5 given by $\delta H_{ch} = J_\perp \tilde{S}_z \cdot (c^\dagger \sigma_2 \tau^\dagger c)$ with $J_\perp \propto J_B - J_L$. As in the EK mapping of the s-2CK model, this becomes $\delta H_{ch} = -iJ_\perp d_+ \chi_-$. In contrast to the s-2CK model, the SO(5) point is robust to detuning away from channel symmetry. This is because the $d_+$ Majorana involved in $J_-$ is already gapped out by the spin-flavor coupling $Q_\perp$ in Eq. 7. This is confirmed by NRG [39].

Breaking spin symmetry by applying a dot magnetic field $\delta H_s = BS^z \tilde{S}_z$ is similarly irrelevant at the SO(5) fixed point. NFL physics is therefore robust to $B$, also confirmed by NRG [39].

Keeping $n_B = \frac{1}{2}$, ph symmetry is broken by $\eta \neq 0$. Performing the SWT yields an additional contribution to Eq. 5 of the form [42] $\delta H_{ph} = \frac{1}{2} \tilde{V}_\perp \sum_{a=1,2} \tilde{T}_{B}^a (c_a \sigma^\dagger \tau^a c_a) + Q \sum_{a=1,2,3} \tilde{T}_{B}^a (c_a \sigma^\dagger \tau^a c_a) \tilde{V}_\perp \tilde{Q}_2 \tilde{Q}_c \sim \eta$. This perturbation contains an additional 5 generators, which together with the 10 from SO(5) form the defining representation of SU(4). Indeed, under RG the system flows to a fully isotropic SU(4) FL fixed point, as discussed in Refs. [16, 31, 41], with zero residual entropy. Breaking ph symmetry therefore destabilizes the NFL SO(5) fixed point, with an emergent FL crossover scale $T^* \sim \eta^2$ [39].

Unusually then, lowering the symmetry of the bare model by introducing finite $\eta$ leads to a low-energy SU(4) fixed point with higher symmetry than the SO(5) fixed point obtained at $\eta = 0$. Applying the EK mapping, we obtain [39] $\delta H_{ph} = -i\tilde{V}_\perp a_+ \chi_-$. This is an RG relevant term with scaling dimension $\frac{1}{2}$, the previously free $a_+$ Majorana is now coupled to the $\chi_-$ field, quenching the $\frac{1}{2} \ln(2)$ entropy and leading to an FL state, with $\chi_\perp \propto \omega$ and $G(T) \sim G(0) \sim T^2$ [50].

**Evolution of NFL line.** We now explore the evolution of the NFL state in the $(n_B, \eta)$ plane. For $n_B \neq \frac{1}{2}$, ph symmetry is broken, generating $\delta H_{ph}$, and also flavor degeneracy is removed, generating an effective flavor field $\delta H_{fl} = B_L T_B^0 = -iB_L a_+ a_-$, with $B_L = E_C(1 - 2n_B)$. Both effects might be expected to destabilize the NFL. However, we shall see that an NFL line extends away from the SO(5) point at $n_B = \frac{1}{2}$ and $\eta = 0$. Combining the flavour field $-iB_L a_+ a_-$ with the spin-flavour coupling $-2iQ_\perp d_+ a_-$, we define a new local Majorana basis $d'_+ = d_+ \cos \theta + a_+ \sin \theta$ and $a'_+ = a_+ \cos \theta - d_+ \sin \theta$ with $\tan \theta = \frac{B_L}{2Q_\perp}$. The Hamiltonian with additional channel- and ph-asymmetry perturbations, $H_{EK} + \delta H_{fl} + \delta H_{ch} + \delta H_{ph}$ remains of the same form as with $B_L = 0$ in the new Majorana basis, only taking modified coupling constants $\tilde{Q}_\perp, \tilde{J}_-, \tilde{V}_\perp$. For the relevant perturbation, $\delta H_{ph} = -i\tilde{V}_\perp a_+ \chi_-$, we find $\tilde{V}_\perp = V_\perp \cos \theta - J_- \sin \theta$. Importantly, we see that $a'_+$ can be completely decoupled to yield the NFL fixed point when $\tilde{V}_\perp = 0$. Near the point $\eta = 0$ and $n_B = \frac{1}{2}$, this happens along the line $2Q_\perp V_\perp = J_- B_L$, a result we have confirmed using NRG. Although ph asymmetry destabilizes the SO(5) NFL fixed point, the two sources of ph asymmetry from $\eta$ and $n_B$ can “cancel out”. Along the resulting NFL line, the free Majorana contributing the $\frac{1}{2} \ln(2)$ residual entropy is the rotated $a'_+$. Moving away from $n_B = \frac{1}{2}$, the free Majorana smoothly transforms from being flavor-like to spin-like.

**NRG phase diagram.** Finally, we examine numerically the full phase diagram in the plane $(n_B, \eta)$ using NRG. Returning to Fig. 1 we see that the s-2CK effect at $n_B = 0$ and the SO(5) Kondo effect at $n_B = \frac{1}{2}$ are continuously connected for all $t_B/t_L < \zeta$ (red and black lines). Interestingly, the NFL lines continue into an unexpected region of the phase diagram with $|\eta/|U_\perp| > \frac{1}{2}$, where the dot no longer hosts a local moment and f-2CK pertains. These NFL lines diverge with $\eta \to \pm \infty$ as $n_B \to \pm \frac{1}{2}$. Moving along an entire NFL line from $n_B = +\frac{1}{2}$ to $-\frac{1}{2}$, the spin and flavour susceptibilities $\chi_s, \chi_f$ (see [39]) show a continuous crossover from spin-flavour SO(5) Kondo to s-2CK and ultimately f-2CK. While both spin and flavor fluctuations are important at SO(5), flavor (spin) fluctuations are both important; no pure s-2CK or f-2CK effect can be realized.

For $t_B/t_L < \zeta$, the NFL line instead terminates at a point $(n_B^*, \eta^*)$, beyond which the condition $\tilde{V}_\perp = 0$ can no longer be satisfied (green and pink lines, Fig. 1). Spin and flavor fluctuations are both important; no pure s-2CK or f-2CK effect can be realized.

The NFL line with $t_B/t_L = \zeta$ (blue line) is special since it is invariant to the ph transformation $n_B \to -n_B$ and $\eta \to -\eta$. It therefore smoothly connects SO(5) NFL points at all half-odd-integer $n_B$.

**Conclusions.** We revisit a classic model describing
quantum dot/box experiments used to probe NFL physics, uncovering a rich range of new physics, including a novel spin-flavor SO(5) Kondo effect. We study the evolution of the NFL line as a function of dot and box gate voltages using a combination of analytical and numerical techniques, showing that the well-known s-2CK effect can continuously transform into the f-2CK or SO(5) Kondo effects. Distinctive experimental signatures of this new physics should be observable in conductance [50].

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S-1. SUPPLEMENTARY DATA

Figure S1 shows the impurity contribution to the total entropy $S_{\text{imp}}(T)$ vs temperature $T$ obtained by NRG at and in the vicinity of the SO(5) symmetric point. (a) channel asymmetry, $t_B/t_L \neq \xi$; (b) spin asymmetry due to a dot magnetic field, $B \neq 0$; (c) chiral asymmetry due to $\eta \neq 0$. At $n_B = \frac{1}{2}$, NFL physics is robust to channel and spin asymmetry, but unstable to chiral symmetry breaking.

Figure S2 shows the $T = \omega = 0$ spin and flavor susceptibilities (black and red lines) of a system with fixed $t_B/t_L > \zeta$ as a function of $n_B$ along the entire NFL line (black curves in Fig. 1 in the main text). Deep in the box Coulomb blockade regime, flavor fluctuations are suppressed by the box charging energy $E_C$, as seen from the small values of $\chi^{f}_{\text{loc}}$. However, the spin susceptibility is enhanced, characteristic of the s-2CK effect, with $T_K \chi^{s}_{\text{loc}} = \text{const}$. (panel c) over a wide range of $n_B$. Close to $n_B = \frac{1}{2}$, the flavor susceptibility $\chi^{f}_{\text{loc}}$ is strongly enhanced. However, we find $\chi^{s}_{\text{loc}}(0) \sim \sqrt{\frac{1}{2} - n_B^2}$, consistent with $\chi^{s}_{\text{loc}}(\omega) \sim \omega$, while $\chi^{f}_{\text{loc}}$ remains finite (panel b). Strong dot charge fluctuations between one and two electron states at $\eta/U_d = -\frac{1}{2}$ give rise to enhanced spin and flavor fluctuations, $\chi^{s}_{\text{loc}} = \chi^{f}_{\text{loc}}$. This situation arises at the point with the maximum Kondo temperature $T_K$ (blue line, panel a). Decreasing $n_B$ further towards $-\frac{1}{2}$ we see enhancement (suppression) of the flavor (spin) susceptibility, with $T_K \chi^{f}_{\text{loc}} \to \text{const}$ and $T_K \chi^{s}_{\text{loc}} \to 0$ as $n_B \to -\frac{1}{2}$, indicative of a crossover from s-2CK to f-2CK.

S-2. DETAILS OF NRG CALCULATIONS

Following Ref. 1 we implement with NRG the model $H_{\text{LS}} = \sum_{\alpha=L,B}(H^\alpha_0 + H^\alpha_{\text{hyb}}) + H_B + H_d$, where

$$H^0_0 = \sum_{k,\sigma} \sum_{\alpha = L,B} \epsilon_{\alpha k}^\sigma c_{\alpha k}^{\sigma\dagger} c_{\alpha k}^\sigma,$$  
(S1)

$$H_B = E_C \left( \hat{T}^z - n_B \right)^2,$$  
(S2)

$$H_d = \sum_{\sigma} \epsilon_d d_{L,\sigma}^\dagger d_{L,\sigma} + U_d d_{L,\uparrow} d_{L,\downarrow} + U_d d_{R,\uparrow} d_{R,\downarrow},$$  
(S3)

$$H^L_{\text{hyb}} = \sum_{k,\sigma} (t_{L k} c_{L k}^{\dagger} c_{L k} + \text{H.c.}),$$  
(S4)

$$H^B_{\text{hyb}} = \sum_{k,\sigma} (t_{B k} c_{B k}^{\dagger} c_{B k} + \text{H.c.}),$$  
(S5)

where $\hat{T}^z = \sum_{n = -m}^m n |n\rangle \langle n|$. At $T = 0$, we find

$$G(T) = \sum_{n = -m}^m n |n\rangle \langle n|, \quad \hat{G}^+ = (\hat{G}^+)^\dagger$$  

in terms of the box charge states $|n\rangle$. In practice we take $m = 2$, which yields well-converged low-temperature results for $E_C = 0.1$ and $t^2 = \sum_{k} |t_{\alpha k}|^2 \sim O(10^{-2})$ as used here. For simplicity we take equivalent lead and box bands, with a flat conduction electron density $\nu$ within bands of halfwidth $D = 1$. We define $\eta = \epsilon_d + 1/2 U_d$ as before.

The free conduction electron bands are discretized logarithmically using $A = 2.5$, and $N_\nu \sim 8000$ states are retained at every step of the iterative diagonalization procedure. We utilized the ‘interleaved NRG’ method on the generalized Wilson chain, exploiting conserved total charge and spin projection. Dynamical quantities are calculated using the full density matrix NRG method.

S-3. CONDUCTANCE

In the ‘split-lead’ geometry utilized in the experiments of Refs. 5 and 6 and depicted in Fig. 1 of the main paper, the free lead Hamiltonian is decomposed as $H^L_0 = \sum_{\gamma=Ls,Ld} \sum_{k,\sigma} t_{\gamma k} c_{\gamma k}^{\dagger} c_{\gamma k}$ into source and drain components, while $H^L_{\text{hyb}} = \sum_{\gamma=Ls,Ld} \sum_{k,\sigma} (t_{\gamma k} d_{\gamma k}^{\dagger} c_{\gamma k} + \text{H.c.})$, and with $t^2 = \sum_{k} |t_{\gamma k}|^2$. A voltage bias, $V_{\text{bias}}$, is applied symmetrically to the leads, $H_{\text{bias}} = \frac{e}{2} V_{\text{bias}} (N_{Ls} - N_{Ld})$ with $N_{\nu} = \sum_{\gamma} c_{\gamma k}^{\dagger} c_{\gamma k}$, and we measure the current into the drain lead, $I_{Ld} = \langle \frac{d}{dt} N_{Ld} \rangle$. The linear-response differential conductance,

$$G = \lim_{V_{\text{bias}} \to 0} \frac{dI_{Ld}}{dV_{\text{bias}}},$$  
(S6)

is related to the equilibrium dot Green’s function $G_{d\sigma}(\omega,T) \equiv \langle \langle d_{\sigma} d_{\sigma}^{\dagger} \rangle \rangle$ via the Meir-Wingreen formula,

$$G(T) = G_0 \int_{-\infty}^{\infty} \frac{df(\omega)}{d\omega} \left[ \pi \nu t^2_{Ld} \text{Im} G_{d\sigma}(\omega,T) \right],$$  
(S7)

where $f(\omega)$ is the Fermi function, and $G_0 = 2e^2/h \times 4t_{sL}^2 T^2_{Ld}/t^2_{dd}$. We obtain the dot Green’s function $G_{d\sigma}(\omega,T)$ as an entire function of (real) frequency $\omega$ at a given temperature $T$ using NRG as above.

We find $\pi \nu t^2_{Ld} \text{Im} G_{d\sigma}(0,0) = \frac{1}{2}$, implying $G(0) = \frac{1}{2} G_0$, everywhere along the NFL line. Deep in the Coulomb blockade regime of the box where the standard spin-2CK effect is observed, $G(T) - G(0) \sim -\sqrt{T}/T_K G_0$. At
Figure S1. Dot contribution to the total entropy $S_{\text{imp}}(T)$ vs $T$ obtained by NRG at and in the vicinity of the SO(5) symmetric point. The SO(5) point for $n_B = \frac{1}{2}$, $\eta = 0$, $U_d = 0.3$, $E_C = 0.1$, $t_L = 0.085$, and $t_B = t_L = 0.1$ is shown as the bold red lines in each panel: the entropy flows from ln(4) for free impurity spin and flavor degrees of freedom, to $\frac{1}{2} \ln(2)$ characteristic of the NFL SO(5) fixed point, on the scale of the Kondo temperature $T_K^{\text{SO(5)}} \approx 10^{-4}$. (a) Channel symmetry breaking, parametrized by $\lambda = \xi^2 t_L^2/t_B^2 = 1$, 1.03, for the red, blue, and black lines respectively, with fixed $t_L^2(1+\lambda) = 0.02$. NFL physics is seen to be robust to this perturbation, with $S_{\text{imp}}(0) = \frac{1}{2} \ln(2)$ in all cases. Strong channel asymmetry generates a two-stage Kondo effect with first-stage single-channel screening on the scale of $T_K$ of the spin by the more strongly coupled channel, followed by two-channel overscreening of the flavor degrees of freedom on the scale of $T_K^d$. (b) Spin symmetry breaking, due to a dot magnetic field $B/T_K^{\text{SO(5)}} = 0$, 1, 3.3, 10 for the red, green, blue and black lines respectively. For magnetic fields $B/T_K^{\text{SO(5)}} \gg 1$, the dot spin is effectively frozen, suppressing spin-Kondo screening. The free flavor degree of freedom (giving a ln(2) entropy contribution) is then two-channel overscreened below $T_K^d$. (c) Particle-hole symmetry breaking, due to $\eta = \epsilon_d + \frac{1}{4} U_d = 0$, $10^{-4}$, $10^{-3}$, $10^{-2}$, $10^{-1}$ for the red, pink, green, blue, and black lines respectively. Ph asymmetry is seen to destabilize the NFL fixed point, generating a second Fermi liquid (FL) scale $T^* \sim \eta^2$, below which the residual impurity entropy is quenched, $S_{\text{imp}}(0) = 0$, at an SU(4) symmetric fixed point. With $T^* \ll T_K^{\text{SO(5)}}$, this FL crossover from SO(5) to SU(4) Kondo takes a universal form.

$n_B = \frac{1}{2}$ and $\eta = 0$ where spin and flavor fluctuations both play an important role in driving the Kondo effect, we obtain $G(T) - G(0) \sim \pm \sqrt{T/T_K G_0}$, with the + or − sign, respectively, for channel asymmetry $t_B > t_L$ or $t_B < t_L$. This is due to a two-stage Kondo effect in the strongly channel asymmetric case. However, when precisely at the SO(5) point, where $t_B = t_L$, the square-root behavior vanishes, leaving a leading linear temperature dependence, $G(T) - G(0) \sim (T/T_K) G_0$ (see Fig. 2 of the main paper). This is reminiscent of the behavior of conductance in the two-impurity Kondo model.

At the SO(5) point, the full conductance lineshape is a universal function of $T/T_K$, and is entirely characteristic of the novel SO(5) Kondo effect. This should be measurable in experiment (note also that the Kondo temperature itself is enhanced at the SO(5) point).

**Figure S2.** Evolution along the NFL line as a function of $n_B$ for fixed $t_B = 0.12$, $t_L = 0.08$, $U_d = 0.3$, $E_C = 0.1$ (along the black curves in Fig. 1 in the main text). (a) $T = \omega = 0$ susceptibilities $\chi_{\text{loc}}^r$ (black), $\chi_{\text{loc}}^f$ (red), and Kondo temperature $T_K$ (blue). (b) Same near $n_B = 0.5$ on a log-log plot vs $\frac{1}{2} - n_B$. (c) $T_K \chi_{\text{loc}}^r$ (black dashed) and $T_K \chi_{\text{loc}}^f$ (red dashed).

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**S-4. CFT TREATMENT OF THE SO(5) FIXED POINT**

In this appendix we derive the finite size spectrum at the SO(5) symmetric fixed point using CFT. For a review of CFT methods for Kondo problems see Ref. 9.

We employ a $U(1)_c \times Z_2 \times \text{SO(5)}$ conformal embedding. Here, $U(1)$ stands for the charge sector, whose primary fields are labelled by integer charges $Q$ and have scaling dimension $\frac{Q^2}{5}$; $Z_2$ is the Ising model consisting of three primary fields: the identity field of scaling dimension 0, the spin field $\sigma$ of scaling dimension $\frac{1}{16}$, and the fermion field $\epsilon$ with scaling dimension $1/2$, which satisfy the fusion rules $\sigma \times \sigma = 1 + \epsilon$, $\sigma \times \epsilon = \sigma$; primary fields of the SO(5) theory are labelled by SO(5) representations, consisting of the singlet (1) of scaling dimension 0, the spinor (4) of scaling dimension $\frac{2}{16}$ and the vector (5) of scaling dimension $1/2$. The SO(5) theory can also be thought of as 5 Ising models, and its fusion rules follow by identifying the fields (1), (4), (5) with 1, $\sigma, \epsilon$. We note that other representations of SO(5)
appear as descendants of the above primary fields, for example $J^A = c^T A c$ which transforms under the 10 dimensional representation, similar to the impurity spin and flavour degrees of freedom, is a (Kac-Moody) descendant of the singlet (1).

| Table I. Finite size spectrum. |
|-----------------------------|
| **Free spectrum** | **Fusion with (4)** |
| $Q$ Ising SO(5) $| E$ | $Q$ Ising SO(5) | $E - \frac{3}{16}$ |
| 0 | 1 | 0 | 0 | 1 | (4) |
| $\pm 1$ | $\sigma$ | (4) | $\pm 1$ | $\sigma$ | (5) |
| $\pm 2$ | $\epsilon$ | (1) | $\pm 2$ | $\epsilon$ | (4) |
| $\pm 2$ | 1 | (5) | $\pm 2$ | 1 | (4) |
| 0 | $\epsilon$ | (5) | 0 | $\epsilon$ | (4) |

The free fermion spectrum can be recovered by the gluing conditions shown in the left hand side part in Table I. As reviewed in Ref. 9, energy levels are obtained by adding up scaling dimensions in units of $2 \pi v_F / L$, where $L$ is the length of the effective 1D system and $v_F$ the Fermi velocity. We obtain the strong coupling fixed point by performing the EK transformation technique, originally used to solve the 2CK problem, allowing to express the Hamiltonian in a free fermion form as in the s-2CK model, as discussed extensively in the literature. The presence of 4 local Majorana fermions, simply accounts for the $2 \log 2$ entropy of the free fermion fixed point, due to both the spin and flavour impurity degrees of freedom $S$ and $T$. In these terms, the resulting Hamiltonian is:

$$H_{\text{EK}} = H_0 + iJ_+ J_- - 2Q_+ a_- \chi_+ d_- - 2iQ_+ a_+ d_-.$$  

(80)

The term $\propto J_+$ appears in the usual s-2CK model, which is a relevant perturbation to the free fermion fixed point $H_0$, leading to the absorption of the local Majorana $d_-$ into the Majorana field $\chi_+$. The last term $\propto Q_+$ gaps out the sector spanned by $d_+$ and $a_-$, meaning that we can exchange every appearance of the product $id_+a_-$ by its expectation value. This way, the term $\propto Q_+$ becomes the same as $J_+$, merely renormalizing it. Eq. 7 in the main text follows (with simplified notation $Q_+ \rightarrow Q_-$).

Symmetry breaking perturbations include channel asymmetry, magnetic field, and ph symmetry breaking. Channel asymmetry and magnetic field have the same EK form as in the s-2CK model, as discussed extensively in the literature. We focus on ph breaking terms which are unique to the spin-flavour model.

S-5. EMERY-KIVELSON HAMILTONIAN OF SPIN-FLAVOUR MODEL

Emery-Kivelson’s (EK) bosonization and renormalization technique, originally used to solve the 2CK problem, allows to express the Hamiltonian in a free fermion form. In this appendix we derive the EK form of the Hamiltonian of our spin-flavour Kondo model Eq. 5 and its various perturbations. We will outline the rigorous treatment of the s-2CK case and highlight the required additions involving the flavour impurity degree of freedom. (A similar derivation for the spin-flavour Kondo model appears in Ref. 13.)

Our starting point is the ph symmetric and channel symmetric Hamiltonian Eq. 5 in the main text with anisotropic bare coupling constants: (i) In the flavour sector the unperturbed model contains only a $V_z$ coupling and no a flavour-flip ($V_z$), since the latter breaks ph symmetry (treated below as a perturbation). (ii) In the spin sector we separate the SU(2) symmetric Kondo coupling $J$ into $J_z$ and $J_\perp$ components. (iii) Also the spin-flavour operator $Q_z$ in Eq. 5 in the main text is split into spin-flip part $Q_+^z$ and a non-spin-flip part $Q_\perp^z$.

The EK technique is applied to the Hamiltonian as follows: (1) Fermionic fields are transformed into bosonic fields $\phi_{\alpha \gamma}(x)$, according to the relation $\psi_{\alpha \gamma}(x) = F_{\alpha \gamma} e^{i\phi_{\alpha \gamma}}$ ($F_{\alpha \gamma}$ being fermionic Klein operators). (2) The new bosonic fields are re-expressed in a basis of charge, spin, flavour and spin-flavor degrees of freedom abbreviated as $\phi_A(x), (A = c, s, f, sf)$. (3) A unitary operator $U = \exp[i(S^z \phi_s + T^x \phi_f)]$ acting both in spin and flavor spaces is applied on the Hamiltonian. With a particular choice of the $z$-part of the spin and flavor Kondo couplings $J_z$ and $V_z$, (i) terms of the form $\partial_x \phi_s(0) S^z$ and $\partial_x \phi_f(0) T^2$ effectively drop out of the Hamiltonian, and (ii) vertex parts $c^{-i\phi_A}$ of both spin and flavour sectors cancel. To achieve both cancellations we must have specific bare couplings $J_z = V_z = V_F$. (4) Klein operators are expressed in the basis of $A = c, s, f, sf$. (5) The Hamiltonian is expressed in terms of new fermionic operators: $\psi_{\gamma}(x = 0) = F_{\gamma} e^{i\phi_{\gamma}}$, $d = F^\dagger S^-$, $a = F^\dagger T^-$. These can be used to define Majorana fermion fields evaluated at $x = 0$:

$$\chi_{A_+} = \frac{\psi_{A_+}^\dagger(0) + \psi_A(0)}{\sqrt{2}}, \quad \chi_{A_-} = \frac{\psi_{A_+}^\dagger(0) - \psi_A(0)}{\sqrt{2i}}.$$  

(99)

Similarly we define local Majoranas $a_\pm, d_\pm$ using the $a$ and $d$ operators for the flavour and spin impurity degrees of freedom, respectively. The presence of 4 local Majorana fermions, simply accounts for the $2 \log 2$ entropy of the free fermion fixed point, due to both the spin and flavour impurity degrees of freedom $S$ and $T$. In these terms, the resulting Hamiltonian is

$$H_{\text{EK}} = H_0 + iJ_+ d_- \chi_+ - 2Q_+ a_- \chi_+ d_- - 2iQ_+ a_+ d_-.$$  

(100)

The term $\propto J_+$ appears in the usual s-2CK model, which is a relevant perturbation to the free fermion fixed point $H_0$, leading to the absorption of the local Majorana $d_-$ into the Majorana field $\chi_+$. The last term $\propto Q_+$ gaps out the sector spanned by $d_+$ and $a_-$, meaning that we can exchange every appearance of the product $id_+a_-$ by its expectation value. This way, the term $\propto Q_+$ becomes the same as $J_+$, merely renormalizing it. Eq. 7 in the main text follows (with simplified notation $Q_+ \rightarrow Q_-$).
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