POLYNOMIAL SCHUR AND
POLYNOMIAL DUNFORD-PETTIS PROPERTIES

JEFF FARMER AND WILLIAM B. JOHNSON

Abstract. A Banach space is polynomially Schur
if sequential convergence against analytic polynomials implies norm convergence.
Carne, Cole and Gamelin show that a space has this property and the Dunford-Pettis
property if and only if it is Schur. Herein is defined a reasonable generalization of
the Dunford–Pettis property using polynomials of a fixed homogeneity. It is shown,
for example, that
a Banach space will have the $P_N$ Dunford–Pettis property if and only if every
weakly compact $N$–homogeneous polynomial (in the sense of Ryan) on the space is
completely continuous. A certain geometric condition, involving estimates on spreading
models and
implied by nontrivial type,
is shown to be sufficient to imply that a space is polynomially Schur.

1. Introduction

The relationship between holomorphic functions defined on an infinite dimen-
sional Banach space and (geometric or topological) properties of the space has been
of recent interest (see, for example, [AAD], [ACG], [CCG], [CGJ], [F], [R 1]). As
in the one–dimensional case, holomorphic functions are defined in terms of Taylor
series, which in the infinite–dimensional case have terms consisting of homogeneous
analytic polynomials. Just as in the case of linear functionals (1–homogeneous poly-
nomials), one can consider properties of the topologies induced by the polynomials
on the space. In this paper we consider the properties which are analogous to the
Schur property and the Dunford–Pettis property; i.e., those obtained by replacing
weak sequential convergence with sequential convergence against an arbitrary $N$–
homogeneous analytic polynomial. We relate these properties to one another and
to the geometric property of type and the existence of certain
spreading models.

$X$ will be a complex infinite–dimensional Banach space. An $N$–homogeneous
analytic polynomial on $X$ is the restriction to the diagonal of an $N$–linear form
on the $N$–fold Cartesian product of $X$ with itself, or equivalently, a linear func-
tional on the $N$–fold projective tensor product of $X$ with itself. Indeed, given an
$N$–homogeneous analytic function $P$ on $X$, one obtains an $N$–linear form on $X$ by

1991 Mathematics Subject Classification. Primary 46B05, 46B20 Secondary 46G20.
This paper forms a portion of the first author’s doctoral dissertation written under the super-
vision of the second author.
The first author was supported in part by NSF Grant #DMS-9021369.
The second author was supported in part by NSF Grant #DMS-9003550.
This paper is in final form and no version of it will be submitted for publication elsewhere.

Typeset by A4AS-\TeX
taking the $N$th derivative and dividing by $N!$; the form is related to the polynomial
by the polarization formula:

$$A_P(x_1, \ldots, x_n) = \text{Avg}\{\epsilon_i = \pm 1\}\langle \Pi \epsilon_i \rangle P(\sum_{i=1}^n \epsilon_i x_i).$$

The form $A_P$ is clearly symmetric (invariant under permutations of the coordinates). Likewise any bounded symmetric $N$–linear form will give rise to an
$N$–homogeneous analytic polynomial. Such a form can be linearized by taking the
projective tensor product of $X$ with itself $N$ times and extending
the form to a linear functional on this tensor product. The subspace of symmetric
linear functionals is the dual of the symmetric $N$–fold projective tensor product,
which is a complemented subspace of the $N$–fold projective tensor product. The
projection is given by extending the following map linearly:

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_n) \rightarrow \frac{1}{n!} \sum_{\pi \in S_n} (x_{\pi_1} \otimes x_{\pi_2} \otimes \cdots \otimes x_{\pi_n}).$$

We denote the symmetric projective tensor product by $\hat{\otimes}_s^N X$. The $N$–linear form $A_P$ associated with $P$ can now be considered a linear functional on $\hat{\otimes}_s^N X$. The
supremum norm of the polynomial is related to that of the linear functional as
follows:

$$||P|| \leq ||A_P|| \leq \frac{N^N}{N!} ||P||.$$ 

If we call the space of polynomials $\mathcal{P}_N$ the above simply says that $\mathcal{P}_N$ is isomorphic to $(\hat{\otimes}_s^N X)^*$. Since for our purposes the index $N$ will be fixed, we will suppress
reference to this isomorphism and use the same label for a polynomial and its
associated symmetric linear functional. More details about the above relationships
may be obtained from [M] or [R 1]. We will study the topologies generated by these
polynomials, especially with respect to sequential convergence.

We define the $\mathcal{P}_N$–weak topology on $X$ to be the topology generated by the all
of the homogeneous analytic polynomials of degree less than or equal to $N$; that is,
a net $\{x_n\}$ converges to $x$ in the $\mathcal{P}_N$–weak topology if, for every $M \leq N$, for every
$M$–homogeneous analytic polynomial $P$, $P(x_n) \rightarrow P(x)$. Note that for $N = 1$ this
is the usual weak topology, and that for $M > N$ the $\mathcal{P}_M$–weak topology is finer
than the $\mathcal{P}_N$–weak one. We call the weak polynomial topology the topology which
is generated by the union of $\mathcal{P}_N$ for all $N \in Z_+$. In analogy to the Schur property,
we say a space is $\mathcal{P}_N$–Schur if whenever $P(x_n) \rightarrow 0$ for all $P \in \mathcal{P}_N$ then $x_n$ is
norm null. If $P(x_n) \rightarrow 0$ for all $P \in \mathcal{P}_N$ for all $N$ implies that $x_n$ is norm null,
then we say $X$ is $\mathcal{P}$–Schur. It is evident (multiply linear functionals) that every
$\mathcal{P}_N$–Schur space is $\mathcal{P}$–Schur and that every Schur space is $\mathcal{P}_N$–Schur for every $N$
(and $\mathcal{P}$–Schur).

These topologies were introduced in [R 1], and the weak polynomial topology also
appeared in [CCG] which considered relations between the Dunford-Pettis property,
the Schur property and the $\mathcal{P}$–Schur property (in the terminology of [CCG], “$X$
is $\mathcal{P}$–Schur ” = “$X$ is a $\Lambda$–space”).

Let $\theta: X \rightarrow \hat{\otimes}_s^N X$ by $\theta(x) = x \otimes \cdots \otimes x$ (N times) and define $\theta(X) = \Delta_N(X)$.

This set is a non-convex, norm-closed subset of $\hat{\otimes}_s^N X$ with the property that
$\lambda x \in \Delta_N(X)$ whenever $x \in \Delta_N(X)$. 

Now $\theta$ is a continuous $N-$homogeneous function, which is the (nonlinear) preadjoint of the isomorphism between $P_N$ and $\left(\hat{\otimes}_N X\right)^\ast$. Notice that $\theta$ is an $N-$to-one map; we have

$$\theta^{-1}(x \otimes \cdots \otimes x) = \{e^{2\pi i n x}|n \in \mathbb{Z}\}$$

(Use separating functionals to the $N$th power to show equality.) We reserve the symbol $\theta_N$ for this function.

If $P(x_\alpha) \to P(x)$ for all $P \in P_N(X)$ then the net need not converge against polynomials in $P_M$ for all $M < N$, but since $P(x) = P(y)$ for all $P \in P_N$ implies that $\hat{\Delta}$ is a complex $N$th root of unity, if also $x_\alpha \to x$ weakly, then $x_\alpha$

$P_N-$weakly converges to $x$. Thus in practice it is easy to pass from convergence against all $N-$homogeneous polynomials to $P_N-$weak convergence.

Although for any one polynomial $P$, $P(x - x_\alpha) \to 0$ and $P(x_\alpha) \to P(x)$ are not in general equivalent, the following known fact is useful.

**Lemma 1.1.** $x_\alpha \to x$ in the $P_N-$weak topology, if and only if $x - x_\alpha \to 0$ in the $P_N-$weak topology.

**Proof (sketch).** Let $P$ be an $N$-homogeneous polynomial and let $x_\alpha \to x$ in the $P_N-$weak topology. Then, letting $A_P$ be the $N-$linear form associated with $P$ we have

$$P(x - x_\alpha) = \sum_{i=1}^{N} (-1)^i \binom{N}{i} A_P(x, x, \ldots, x_a, \ldots, x)$$

where in each term $x$ appears $i$ times and $x_\alpha$ appears $N-i$ times. Since convergence in the $P_N-$weak topology implies convergence against any polynomial of lesser homogeneity, we consider each term as an $(N-i)-$homogeneous polynomial ($x$ being fixed), to see that the sum indeed converges to zero. The converse is obtained, using the same expansion, by induction on $N$.

### 2. Polynomial Dunford–Pettis Spaces

One result of [CCG] is that a Banach space is Schur if and only if it is polynomially Schur and has the Dunford–Pettis property. We can obtain an analogous result for polynomials of fixed homogeneity by defining an appropriately analogous Dunford-Pettis property. Our first task is to adapt Lemma 7.3 of [CCG] for our purposes.

**Proposition 2.1.** The following are equivalent for any Banach space $X$, and any fixed positive integer $N$.

(i) Any polynomial on $X$ is $P_N-$weakly sequentially continuous.

(ii) If $\{x_n\}_{n=1}^\infty$ is a $P_N-$weakly null sequence in $X$ (i.e. if $x_k \otimes \cdots \otimes x_k$ ($N$ times) is weakly null), then $\{x_k \otimes \cdots \otimes x_k\}$ ($m$ times) is weakly null in $\otimes^m X$ for $m > N$.

(iii) For $m > N$ the function $\theta = \theta(N, m)$ which takes $\theta : \Delta_N(X) \to \Delta_m(X)$ by

$$\Delta_N(X) \ni x \otimes \cdots \otimes x \mapsto x \otimes \cdots \otimes x \text{ (m times)}$$

is weak to weak sequentially continuous.
Proof. The proof of these equivalences is an exercise, and can be adapted easily from the proof of Lemma 7.3 in [CCG].

For $n = 1$, the equivalent properties of 2.1 were shown to be implied by the Dunford–Pettis property. We will now define a polynomial Dunford–Pettis property which will imply the conditions of 2.1 for each positive integer.

We say that a space $X$ has the $P_N$ Dunford–Pettis property provided that it satisfies any of the equivalent conditions of Proposition 2.2.

**Proposition 2.2.** Let $X$ be a Banach space. For fixed $N$, the following are equivalent:

1. Whenever $\{P_n\}_{n=1}^{\infty}$ is a weakly null sequence of $N$–homogeneous polynomials (or equivalently, symmetric bounded $N$–linear forms on $X$) and $\{x_n\}_{n=1}^{\infty}$ converges $P_N$–weakly to $x$ in $X$, then $P_n(x_n) \to 0$.
2. Every weakly compact operator on $\hat{\otimes}_N X$ is completely continuous when restricted to $\Delta_N(X)$.
3. If $K$ is a weakly compact set in any Banach space $Y$, and $J$ is $P_N$–weakly compact in $X$, then $\theta_N(J) \otimes K$ is a weakly compact set in $(\hat{\otimes}_N X) \otimes Y$

where by $\theta_N(J) \otimes K$ we mean simply the set $\theta_N(J) \times K$, that is, the set of all $\theta_N(j) \otimes k$ with $j \in J$ and $k \in K$.

In the case $N = 1$ these conditions reduce to known equivalent statements of the classical Dunford–Pettis property; this proposition justifies the definition of the $P_N$ Dunford–Pettis property. Before giving the proof, we make the following remark.

R. Ryan in [R 2] considers $N$–homogeneous polynomials from $X \to Y$; as in the scalar case, we can equivalently consider linear operators from $\hat{\otimes}_N X$ to $Y$; such a polynomial is weakly compact if it maps bounded sets to weakly compact ones (i.e. if the associated linear operator is weakly compact). Ryan investigates some conditions which are equivalent to weak compactness of such polynomials. Using this definition it is easy to see that (ii) above is equivalent to

(ii') Every weakly compact $N$–homogeneous polynomial from $X$ to any Banach space $Y$ is completely continuous (on $X$).

Proof.

(i)$\Rightarrow$(iii) We want to show that $\theta_N(J) \times K$ is weakly compact in $\hat{\otimes}_N X \otimes Y$. Take a sequence $\theta_N(x_n) \otimes k_n$ in $\theta_N(J) \otimes K$; by hypothesis assume we have passed to a subsequence such that $\theta_N(x_n)$ and $k_n$ are weakly convergent to $\theta_N(x)$ and $k$ in $\hat{\otimes}_N X$ and $Y$, respectively. That is, $\theta_N(x_n) - \theta_N(x)$ and $k_n - k$ are weakly null. If

$$\phi \in (\hat{\otimes}_N X \otimes Y)^* \equiv B(Y, (\hat{\otimes}_N X)^*)$$

then $\phi(k_n - k)$ is weakly null by continuity and (i) applies. We then have

$$\langle \phi(k_n - k), \theta_N(x_n) - \theta_N(x) \rangle =$$

$$\langle \phi(k_n), \theta_N(x_n) \rangle - \langle \phi(k), \theta_N(x_n) \rangle - \langle \phi(k_n), \theta_N(x) \rangle + \langle \phi(k), \theta_N(x) \rangle$$
POLYNOMIAL SCHUR AND POLYNOMIAL DUNFORD–PETTIS PROPERTIES

We note that the condition (ii) is sharply stated with the following example.

(i) Let \( K \) be the weakly null sequence and \( \{x_n\}_{n=1}^{\infty} \) be the sequence in \( X \) such that \( x_n \) goes to zero in a weaker Hausdorff topology, namely that generated by considering the weak topology on the second co-ordinate. Since it is clear that \( x_n \) is basic (or norm null, in which case the argument is simpler). Apply (iii) to the sets

\[
K = \{T^*(\phi_n) - \psi\}_{n=1}^{\infty} \cup \{0\} \quad \text{and} \quad J = \{x_n\}_{n=1}^{\infty} \cup \{x\}
\]

to see that \( \theta_N(J) \otimes K \) is weakly compact in \( \hat{\otimes}_s^N X \). The sequence \( \theta_N(x_n) \otimes \{T^*(\phi_n) - \psi\} \) thus has a convergent subsequence (we pass to that). First we claim that this subsequence must go weakly to zero; indeed, it goes to zero in a weaker Hausdorff topology, namely that generated by the weak topology on the second co-ordinate. Since it is clear that \( \theta_N(x) \otimes \{T^*(\phi_n) - \psi\} \) is weakly null, we can conclude that

\[
w = \lim_n \left( |\theta_N(x_n) - \theta_N(x)| \otimes \{T^*(\phi_n) - \psi\} \right) = 0
\]

Now consider the functional associated with the identity operator; call it \( \Gamma \). We have

\[
0 = \lim_{n \to \infty} \Gamma \left( \{T^*(\phi_n) - \psi\} \otimes [\theta_N(x_n) - \theta_N(x)] \right)
\]

\[
= \lim_{n \to \infty} \left( \langle T^*(\phi_n) - \psi, \theta_N(x_n) - \theta_N(x) \rangle \right)
\]

\[
= \lim_{n \to \infty} \left( \langle T^*(\phi_n), \theta_N(x_n) - \theta_N(x) \rangle - \langle \psi, \theta_N(x_n) - \theta_N(x) \rangle \right)
\]

\[
= \lim_{n \to \infty} \langle \phi_n, T(\theta_N(x_n)) - T(\theta_N(x)) \rangle
\]

\[
= \lim_{n \to \infty} \|T(\theta_N(x_n)) - T(\theta_N(x))\|
\]

This gives (ii).

(ii)\( \Rightarrow \) (i) Let \( \{P_n\}_{n=1}^{\infty} \) be the weakly null sequence and \( \{x_n\}_{n=1}^{\infty} \) go \( \mathcal{P}_N \)-weakly to \( x \) and define a map \( T \) from \( \Delta_N(X) \) to \( c_0 \) by

\[
T(\theta_N(z)) = (P_n(z))_n \quad \forall z \in X
\]

The map \( T \) extends linearly (via the polarization formula) to all of \( \hat{\otimes}_s^N X \). Since \( T^*(e_n) = P_n \) goes weakly to 0 we see that the map \( T \) is weakly compact. Applying (ii) we get \( T(\theta_N(x_n)) \) going in the norm on \( c_0 \) to \( T(\theta_N(x)) \). But since the norm on \( c_0 \) is the sup norm, this gives (i) and completes the proof.

We note that the condition (ii) is sharply stated with the following example.
Example. We will see momentarily that \( l_2 \) is \( P_2 \)-Schur and therefore is \( P_2 \)-Dunford–Pettis. Consider the operator
\[
Id \otimes Q_1 : l_2 \hat{\otimes} l_2 \to c_0
\]
where \( Q_1 \) is the projection onto the first basis vector. Consider the symmetrized version, that is, restrict the operator to the symmetric tensor product, which is a complemented subspace. This operator is weakly compact and therefore completely continuous on \( \theta_2(l_2) \) by Proposition 2.2 but is clearly not completely continuous on the entire symmetric tensor product; consider the image of \( e_n \otimes e_1 + e_1 \otimes e_n \), for example, which is weakly null but whose image is the unit vector basis of \( c_0 \).

Proposition 2.3. If a Banach space is \( P_N \) Dunford–Pettis then it satisfies the equivalent conditions in Proposition 2.1.

Proof. We prove 2.2(i) implies 2.1(ii).

Let \( \{x_n\}_{n=1}^\infty \) be a \( P_N \)-weakly null sequence in \( X \), i.e. \( \theta_N(x_n)_{n=1}^\infty \) is weakly null in \( \hat{\otimes}^N X \) and \( \theta_M(x_n)_{n=1}^\infty \) is also weakly null in \( \hat{\otimes}^MX \) whenever \( 1 \leq M < N \). Let \( m = N + 1, \phi \in (\hat{\otimes}^N X)^* \) and consider \( \phi \) as a linear operator from \( X \) to \( (\hat{\otimes}^N X)^* \). Since \( \{x_n\}_{n=1}^\infty \) is weakly null in \( X \), so is its image in \( (\hat{\otimes}^N X)^* \) under \( \phi \). Thus
\[
\langle \phi, \theta_N(x_n) \rangle = \langle \phi(x_n), \theta_N(x_n) \rangle \to 0
\]
by the first formulation of the \( P_N \) Dunford–Pettis property (notice that for this application it matters not whether \( \phi \) is symmetric). This proves the proposition for \( m = N + 1 \) and by induction (in an obvious way) for \( m = qN + 1 \) for \( q = 1, 2, \ldots \).

But we can also write an analogous proof for \( m = N + k \) for \( 2 \leq k < N \) and extend it by induction as well.

Proposition 2.4. Let \( X \) be a Banach space. For fixed \( N \), the following are equivalent:

(i) \( X \) is \( P_N \)-Schur.
(ii) \( X \) has the \( P_N \) Dunford–Pettis property and is \( P \)-Schur.
(iii) \( X \) satisfies (i)–(iii) of proposition 1.1 and is \( P \)-Schur.

Proof. (i)\( \Rightarrow \) (ii) requires only Lemma 1.1 and (ii)\( \Rightarrow \) (iii) is Proposition 2.3, so (iii)\( \Rightarrow \) (i) remains. Let \( \theta_N(x_n) \) be weakly null. Then \( \theta_M(x_n) \) is weakly null for all \( M \) by 2.1. But since \( X \) is \( P_N \)-Schur, \( x_n \) must go to 0 in norm.

It is of interest to note that if we are not in the context of the \( P \)-Schur property, the conditions of 2.1 are weaker than the \( P_N \) Dunford–Pettis property; \( T^* \), the original Tsirelson space (or, in fact any space having the approximation property with \( P_N(X) \) reflexive for all \( N \); see [F] for further discussion of such spaces) will satisfy 2.1 for all \( N \) but fail to be \( P_N \) Dunford–Pettis.

Examples. It is clear from the classical work of Pitt [P] that \( l_p \) spaces for \( (1 \leq p < \infty) \) are \( P_N \)-Schur for \( N \geq p \), and it is proved in [CCG] that \( L_p \) spaces \( (2 \leq p < \infty) \) are \( P \)-Schur (in fact \( P_N \)-Schur for \( N \geq p \)); we can thus conclude that they are \( P_N \) Dunford–Pettis. The space \( c_0 \) is Dunford–Pettis and therefore \( P_N \) Dunford–Pettis for every \( N \). This implies (for example) that \( l_3 \oplus c_0 \) is \( P_3 \)-Dunford–Pettis but not \( P_N \)-Schur for any \( N \). In the next section we discuss further exactly which spaces may be \( P_N \)-Schur.
In this section we will give some sufficient criteria for spaces to be Polynomially Schur. We will show, for example, that any space having non-trivial type is $P$–Schur and indeed is $P_N$–Schur for some $N$.

(Jaramillo and Prieto [JP] have independently shown that every superreflexive space is polynomially Schur). In particular, $L_p$ spaces are Polynomially Schur for all $1 < p < \infty$.

It is convenient to use the concept of a spreading model, the construction of which is due to Brunel and Sucheston [BS 1].

Finite versions of Ramsey’s Theorem allow that given any property of $n$-tuples of elements from a sequence, one can pass to a subsequence with the property that all $n$-tuples formed from the subsequence share the property or else all fail it. By using the size of the norm of a sum of $n$ elements as the property one can, by repeatedly applying the theorem, approximately stabilize the norm (to within any desired $\epsilon_n$) of any finite combination as long as many of the beginning terms are thrown away. More precisely we have the following fact (see [B] or [BS 1]):

**Proposition 3.1.** Let $(f_n)$ be a bounded sequence with no norm-Cauchy subsequence in a Banach space $X$. Then there exists a subsequence $(e_n)$ of $(x_n)$ and a norm $L$ on the vector space $S$ of finite sequences of scalars such that

$$\forall \epsilon > 0 \quad \forall a \in S \quad \exists k \in N \text{ s.t. } \forall k < k_1 < k_2 < \cdots < k_M$$

we have

$$\left\| \sum a_i e_{k_i} \right\| - L(a) < \epsilon$$

The completion of $[e_i]$ (call it $F$) under the norm $L$ is called a spreading model for the sequence $(e_n)$. The reason for the terminology is that the sequence $(e_n)$ is invariant under spreading with respect to the norm $F$, that is to say, for every finite sequence of scalars $(a_i)$ and every subsequence $\sigma$ of the natural numbers

$$\left\| \sum_{i=1}^{M} a_i e_{i} \right\|_F = \left\| \sum_{i=1}^{M} a_i e_{\sigma(i)} \right\|_F$$

Thus any norm estimate satisfied by sequences in the spreading model will be approximately satisfied for sequences of finite length to any desired degree provided we go out far enough in the sequence $(x_n)$. If the original sequence was weakly null then the resulting sequence will be unconditional; that is to say, we have the following (Lemma 2 of [B], or see [BS 2]):

**Proposition 3.2.** If $(x_n)$ is weakly null, then the sequence $(e_n)$ is unconditional in $F$ with unconditional constant at most 2.

Now we are ready to state the criterion.

**Theorem 3.3.** Suppose a Banach space $X$ has the property that for every normalized weakly null sequence $\{y_n\}$ in $X$ there exists a subsequence and a sequence $\{f_n\}$ in $X^*$ biorthogonal to it which has an (unconditional) spreading model with
an upper $p$-estimate for some $p > 1$. Then $X$ is $\mathcal{P}$–Schur. If the same $p$ works for every such sequence and $N > p'$, then the space $X$ is $\mathcal{P}_N$–Schur.

We know of no space which is $\mathcal{P}$–Schur but which fails the above property. The space $(l_3 \oplus l_4 \oplus l_5 \oplus \cdots)_2$ is easily seen to be $\mathcal{P}$–Schur although it fails cotype (and hence type and superreflexivity), yet is reflexive; it does satisfy the hypothesis of 3.3. A Schur space satisfies the hypothesis vacuously.

Proof. We will pass to subsequences and relabel without mercy. Start with any bounded sequence in $X$ which is not norm null; we need to find a polynomial which is bounded away from zero on a subsequence. By Rosenthal’s theorem [D, Chapter XI] there is either a weakly Cauchy subsequence or a subsequence equivalent to the unit vector basis of $l_1$. Since the unit vector basis of $l_1$ is not weakly null, we are done in this case. For the same reason (linear functionals are polynomials) we are finished if our weakly Cauchy sequence is not weakly null. So we have reduced to the case of a bounded weakly null sequence which is not norm null and can apply the hypothesis to that sequence (it is purely formal that the “normalized” condition can be replaced by “bounded”).

Let $\{y_n, f_n\}$ be the biorthogonal system obtained, and assume with no loss of generality that the $f_n$ are normalized. By the definition of a spreading model we know that for any $c > 0$ we can find a constant $C$ so that for every $M$ we have

$$\left\| \sum_{i=c \log M}^{M} f_i \right\| \leq C \left( \sum_{i=c \log M}^{M} \| f_i \|^p \right)^{\frac{1}{p}}.$$

This is because once we have a spreading model we may always improve the stability estimates by passing to a subsequence. Choosing $c$ small enough so that $m = c \log M \leq M^{\frac{p}{r}}$ for all $M \in \mathcal{N}$, and letting $(a_i)$ be scalars of modulus less than or equal to 1, we obtain

$$\left\| \sum_{i=1}^{M} a_i f_i \right\| \leq \sum_{i=1}^{m-1} \| f_i \| + \left\| \sum_{i=m}^{M} a_i f_i \right\| \leq m + C \left( \sum_{i=m}^{M} |a_i|^p \right)^{\frac{1}{p}} \leq C_1 M^{\frac{p}{r}}.$$

Now by general Banach lattice techniques (see Lemma 3.4 below), we know that for any $r < p$ (we choose $r$ so that $p' < r' < N$) we can get a different $C$ so that for any sequence $(a_i)$ we will have

$$\left\| \sum_{i=1}^{M} a_i f_i \right\| \leq C \left( \sum_{i=1}^{M} |a_i|^{r'} \right)^{\frac{1}{r'}}.$$

Now, given $y \in B$ choose $(a_i) \in l_r$ of $l_r$ norm one to norm the sequence $(f_i(y))_{i=1}^{M}$ in $l_{p'}$. We then have that

$$\left\| \left( \sum_{i=1}^{M} f_i(y)^{N} \right)^{\frac{1}{p'}} \leq \left( \sum_{i=1}^{M} |a_i|^{r'} \right)^{\frac{1}{r'}} \left( \sum_{i=1}^{M} |f_i(y)|^{r'} \right)^{\frac{1}{r}} \leq \left( \sum_{i=1}^{M} a_i f_i(y) \right)^{\frac{1}{p'}} \leq C \left( \sum_{i=1}^{M} |a_i|^{r'} \right)^{\frac{1}{r'}} \leq C_1 M^{\frac{p}{r'}}.$$

Let $\left( l_3 \oplus l_4 \oplus l_5 \oplus \cdots \right)_2$ be easily seen to be $\mathcal{P}$–Schur although it fails cotype (and hence type and superreflexivity), yet is reflexive; it does satisfy the hypothesis of 3.3. A Schur space satisfies the hypothesis vacuously.
\[
\sum_{i=1}^{M} a_i f_i \leq C \left( \sum_{i=1}^{M} |a_i|^r \right)^{\frac{1}{r}} = C
\]

which says that
\[
P(y) = \sum_{i=1}^{\infty} f_i(y)^N \leq C^N \quad \text{hence} \quad \|P\| \leq C^N.
\]

But we know that \( P(y_n) = f_n(y_n)^N \) is bounded away from zero because the sequence \((y_n, f_n)\) was biorthogonal. Therefore no sequence bounded away from 0 in \( X \) can be polynomially null, and so \( X \) is \( \mathcal{P} \)-Schur. If there is a uniform value for \( p \) then \( X \) is \( \mathcal{P}_N \)-Schur for \( N > p' \).

It remains for us to prove the following standard fact.

**Lemma 3.4.** Suppose that \( \{x_i\} \) is a normalized sequence in a Banach space satisfying
\[
\left\| \sum_{i \in B} a_i x_i \right\| \leq C |B|^{\frac{1}{p}} \max_{i \in B} |a_i|
\]
for all scalars \((a_i)\), and all finite subsets \( B \) of the natural numbers. Then for any \( 1 < r < p \) there exists a constant \( D \) so that for all \((a_i)\) and all \( M \)
\[
\left\| \sum_{i=1}^{M} a_i x_i \right\| \leq D \left( \sum_{i=1}^{M} |a_i|^r \right)^{\frac{1}{r}} .
\]

**Proof.** We use the convention that \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \frac{1}{r} + \frac{1}{s} = 1 \). Given the scalars, assume by homogeneity that \( \sum_{i=1}^{M} |a_i|^r = 1 \). Define
\[
B_n = \{ i \mid 2^{-n-1} < |a_i| \leq 2^{-n} \}
\]
and write
\[
\left\| \sum_{i=1}^{M} a_i x_i \right\| \leq \sum_{n} \left\| \sum_{i \in B_n} a_i x_i \right\| \leq C \sum_{n} 2^{-n} |B_n|^{\frac{1}{p}} r .
\]

Now just compute:
\[
\sum_{n} 2^{-n} |B_n|^{\frac{1}{p}} = \sum_{n} 2^{-n} \cdot 2^{n(\frac{1}{p} - 1)} |B_n|^{\frac{1}{p}} \leq \left( \sum_{n} 2^{-nr} |B_n| \right)^{\frac{1}{p}} \left( \sum_{n} 2^{nr(\frac{1}{p} - 1)} \right)^{\frac{1}{p}}
\]
\[
= 2^{\frac{1}{p}} \left( \sum_{n} 2^{-(n+1)r} |B_n| \right)^{\frac{1}{p}} D(p, r) \leq 2^{\frac{1}{p}} \left( \sum_{i=1}^{M} |a_i|^r \right)^{\frac{1}{p}} D(p, r)
\]
\[
= 2^{\frac{1}{r}} D(p, r) \left( \sum_{i=1}^{M} |a_i|^r \right)^{\frac{1}{p}} .
\]

The condition in Theorem 3.3 is rather hard to check. However, a much simpler criterion is sufficient.
**Theorem 3.5.** Suppose the dual space of \( X, X^* \), has type \( p > 1 \). Then for every normalized weakly null sequence \( \{ y_n \} \) in \( X \) there exists a subsequence and a sequence \( \{ x_n \} \) in \( X^* \) biorthogonal to it which has an upper \( p \)-estimate. In particular, \( X \) satisfies the hypothesis of 3.3, and thus is \( \mathcal{P}_N - \text{Schur} \) for all \( N > p \).

In view of the fact that every space with non-trivial type also has a dual with some non-trivial type, we can state the following corollary.

**Corollary 3.6.** Suppose \( X \) has type. Then for some \( N \), \( X \) is \( \mathcal{P}_N - \text{Schur} \).

**Proof (of 3.5).** Recall that the fact that \( X^* \) has type \( p \) means that there is a constant \( T_p \) so that

\[
\int_0^1 \left| \sum_{i=1}^M r_i(t)x_i \right| dt \leq T_p \left( \sum_{i=1}^M \| x_i \|^p \right)^{\frac{1}{p}} \quad \forall M \in \mathbb{N}
\]

for any finite number of elements \( x_1, \ldots, x_n \) (where \( r_i \) are the Rademacher functions). Now suppose that \( \{ y_n \} \) is a normalized weakly null sequence, which we can assume is basic by passing to a subsequence. We can find a bounded sequence \( \{ x_n \} \) in \( X^* \) of functionals biorthogonal to \( \{ y_n \} \). Since \( l_1 \) is not embeddable in \( X^* \) we know by Rosenthal’s theorem that we can find a weakly Cauchy subsequence of \( \{ x_n \} \). Pass to the odd terms of \( \{ y_n \} \), relabel and replace \( \{ x_n \} \) with \( \{ x_{2n+1} - x_{2n} \} \). Then we have a biorthogonal system \( \{ x_n, y_n \} \) with \( \{ x_n \} \to 0 \) weakly. By proposition 3.1 and the remark following it we know that \( \{ x_n \} \) has an unconditional spreading model \( F \).

Now \( F \) is finitely representable in \( X^* \) (this means that given any finite dimensional subspace of \( F \) we can find a \( 1 + \epsilon \)-isomorphic copy of that subspace in \( X^* \), see again [B] or [BS 1]). Since the definition of type is local, \( F \) will also have type \( p \) with constant \( \leq T_p \). Since the basis \( e_n \) of \( F \) is unconditional with constant at most 2, it has an upper \( p \)-estimate with constant less than or equal to \( 2T_p \). Thus we have satisfied the hypothesis of Theorem 3.3.

**References**

[AAD] R. Alencar, R. Aron and S. Dineen, *A Reflexive space of holomorphic functions in infinitely many variables*, Proc. Amer. Math. Soc. 90 (1984), 407-411.

[ACG] R. Aron, B. Cole and T. Gamelin, *Spectra of algebras of analytic functions on a Banach space* (to appear).

[B] B. Beauzamy, *Banach–Saks properties and spreading models*, Math. Scand. 44 (1979), 357–384.

[BS 1] A. Brunel and L. Sucheston, *On \( B \)-convex Banach spaces*, Math. Systems Thy. 7 (1973), 294–299.

[BS 2] A. Brunel and L. Sucheston, *On \( J \)-convexity and some ergodic super-properties of Banach spaces*, Trans. Amer. Math. Soc. 204 (1975), 79–90.

[CCG] T. Carne, B. Cole and T. Gamelin, *A uniform algebra of analytic functions on a Banach space*, Proc. Amer. Math. Soc. 114 (1989), 639-659.

[CGJ] B. Cole, T. Gamelin, and W. B. Johnson, *Analytic disks in fibers over the unit ball of a Banach space* (to appear).

[D] J. Diestel, *Sequences and series in Banach spaces*, Springer–Verlag, New York, 1984.

[F] J. Farmer, *Polynomial reflexivity in Banach spaces* (to appear).

[JP] J. Jaramillo and Prieto, *Weak-polynomial convergence on a Banach space* (to appear).

[M] J. Mujica, *Complex analysis in Banach spaces*, Notas de Mathematica, vol. 120, North-Holland, Amsterdam, 1986.

[P] H. R. Pitt, *A note on bilinear forms*, J. London Math. Soc. 11 (1936), 174–180.
[R 1] R. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*, Thesis, Trinity College, Dublin, 1980.

[R 2] R. Ryan, *Weakly compact holomorphic mappings on Banach spaces*, Pac. J. Math. **131** (1988), 179-190.