A $q$-ANALogue FOR EULER’S EVALUATIONS OF THE RIEmann ZETA FUNCTION

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Abstract. We provide a $q$-analogue of Euler’s formula for $\zeta(2k)$ for $k \in \mathbb{Z}^+$. Our main results are stated in Theorems 3.1 and 3.2 below. The result generalizes a recent result of Z.W. Sun who obtained $q$-analogues of $\zeta(2) = \pi^2 / 6$ and $\zeta(4) = \pi^4 / 90$.

1. Introduction

Recently, Sun obtained a very nice $q$-analogue of Euler’s formula $\zeta(2) = \pi^2 / 6$. Motivated by this, the author obtained the $q$-analogue of $\zeta(4)$ and noted that it was simultaneously and independently obtained by Sun [9]. The author then obtained the $q$-analogue of $\zeta(6)$ in [4] but realized that this transition to the $q$-analogue of $\zeta(6)$ is more difficult as compared to $\zeta(2)$ and $\zeta(4)$. This difficulty arises due to an extra term that shows up in the identity; however in the limit as $q \to 1^-$, this term $\to 0$. Thus it is necessary to study the $q$-analogue of Euler’s celebrated formula

$$\zeta(2k) = \frac{(-1)^{k+1} \log^{2k} B_{2k} \pi^{2k}}{2(2k)!}$$

for all $k \in \mathbb{Z}^+$. We will see shortly that this requires a consideration of two cases: $k$ even and $k$ odd separately (see Theorems 3.1 and 3.2 below). We also mention here that Zudilin [10] and Krattenthaler-Rivoal-Zudilin [5] have studied the Diophantine properties of $q$-zeta values, the sums appearing in the left-hand side of Theorems 3.1 and 3.2.

2. Notations

For a positive integer $k$ we use the following standard notations. Let $B_{2k}$ denote the $2k$th Bernoulli number. Let $\{\binom{n}{k}\}$ denote a Stirling number of the second kind, which is the number of ways of partitioning a set of $n$ objects into $k$ non-empty subsets. Let the complex upper half-plane be denoted by $\mathcal{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$ and let $SL_2(\mathbb{Z})$ denote the full modular group which is defined to be the set of all $2 \times 2$ matrices with integer entries and determinant one. Also let $\Gamma_0(4)$ denote the well-known principal congruence subgroup of $SL_2(\mathbb{Z})$ defined by

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{4} \right\},$$

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Finally, let $\mathcal{M}_{2k}(\Gamma_0(4))$ be the vector space of all weight $2k$ modular forms over $\Gamma_0(4)$ and $\mathcal{S}_{2k}(\Gamma_0(4))$ denote the subspace of $\mathcal{M}_{2k}(\Gamma_0(4))$ of all weight $2k$ cusp forms over $\Gamma_0(4)$. Let $q = e^{2\pi i \tau}$ where $\tau \in \mathcal{H}$. We define the Dedekind eta function, a well-known modular form of weight $1/2$, by

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

For a thorough treatment on modular forms, the interested readers should consult [5, 7]. Finally, we denote the $n$th triangular number $T_n$ by

$$T_n = \frac{n(n+1)}{2}, \quad n = 1, 2, 3, \ldots.$$

and the corresponding generating function by

$$\psi(q) = \sum_{n=1}^{\infty} q^{T_n}.$$

Also let $d_k$ be given by

$$d_k = \frac{(-16)^k B_{2k}(4^k - 1)}{8k} \in \mathbb{Q}.$$ 

3. Main theorems

**Theorem 3.1.** Let $k \geq 2$ be an even integer. For a complex number $q$ with $|q| < 1$ we have

$$\sum_{n=0}^{\infty} \frac{q^{2n+1} P_{2k-2}(q^{2n+1})}{(1 - q^{2n+1})^{2k}} - T_{2k}(\tau/2) = q^{k/2} d_k \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{4k}}{(1 - q^{2n-1})^{4k}}$$

where

$$P_{2k-2}(z) = \sum_{l=1}^{2k-1} (-1)^l b_k(l) z^{l-1}$$

is a polynomial of degree $(2k - 2)$ with integer coefficients

$$b_k(l) = \sum_{m=0}^{2k-1} (-1)^m a_k(m) \binom{2k - m - 1}{l}$$

where the $a_k(m)$ are defined by

$$a_k(m) = \sum_{j=0}^{2k-1} j! (-1)^j \left\{ \binom{2k - 1}{j} \right\} \left\{ \binom{j}{m} \right\}.$$ 

and $T_{2k}(\tau) \in \mathcal{S}_{2k}(\Gamma_0(4))$, thus $T_{2k}(\tau/2) \to 0$ as $q \to 1$, where the limit is taken from inside the unit disk. In other words, Theorem 3.1 gives a $q$-analogue of (1.1) for $\zeta(2k)$ with $k$ even.

**Theorem 3.2.** Let $k \geq 1$ be an odd integer. For a complex number $q$ with $|q| < 1$ we have

$$\sum_{n=0}^{\infty} \frac{q^{2n+1} P_{4k-2}(q^{2n+1})}{(1 - q^{2(2n+1)})^{2k}} - T_{2k}(\tau) = q^k d_k \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{4k}}{(1 - q^{4n-2})^{4k}}$$

and $T_{2k}(\tau) \in \mathcal{S}_{2k}(\Gamma_0(4))$, thus $T_{2k}(\tau/2) \to 0$ as $q \to 1$, where the limit is taken from inside the unit disk. In other words, Theorem 3.2 gives a $q$-analogue of (1.1) for $\zeta(2k)$ with $k$ odd.
where
\[
P_{4k-2}^\sigma(z) = (1 + z)^{2k} P_{2k-2}^\sigma(z) - 2^{2k-1} z P_{2k-2}^\sigma(z^2)
\]
is a polynomial of degree \((4k - 2)\) with integer coefficients and where \(P_{2k-2}^\sigma(z)\) is
the polynomial defined in (3.2) and \(T_{2k}(\tau) \in \mathcal{S}_{2k}(\Gamma_0(4))\), thus \(T_{2k}(\tau) \to 0\) as \(q \to 1\),
where the limit is taken from inside the unit disk. In other words, Theorem 3.2
gives a \(q\)-analogue of (1.1) for \(\zeta(2k)\) with \(k\) odd.

Two remarks:

(1) Note that we are using (4.5) of Theorem 3.2 (see below) to prove Theorems
3.1 and 3.2. Clearly the left-hand of (4.5) is \(q^k\) times a function of \(q^2\). If
the right-hand side of (4.5) also turns out to be a function of \(q^2\), we can
replace \(q \to \sqrt{q}\) without affecting our results. This happens to be the case
in Theorem 3.1 where we obtain expressions in \(q^2\) for both the sum
and product, thereby giving us \(T_{2k}(\tau/2)\) in (3.1). However we do not obtain
such expressions in \(q^2\) on both sides of (3.5) in Theorem 3.2 and thus we get
\(T_{2k}(\tau)\) instead of \(T_{2k}(\tau/2)\). However numerical calculations for
\(k = 1, 3, 5\) suggest that we are likely to get expressions involving \(q^2\)
for the sum in (3.5) so that we can replace \(q \to \sqrt{q}\), thereby getting \(T_{2k}(\tau/2)\) in (3.5).

(2) The cusp form \(T_{2k}(\tau)\) in Theorems 3.1 3.2 is well-defined and uniquely
determined by the difference of a \(q\)-series and a \(q\)-product as follows:
\[
T_{2k}(\tau) = \begin{cases} 
\sum_{n=0}^{\infty} \frac{2^{2k-1} q^{4n+2} P_{2k-2}^\sigma(q^{4n+2})}{(1 - q^{4n+2})^{2k}} - q^k \prod_{n=1}^{\infty} \frac{1 - q^{4n})^{4k}}{(1 - q^{4n})^{4k}} & (k \text{ even}) \\
\sum_{n=0}^{\infty} \frac{q^{2n+1} P_{4k-2}^\sigma(q^{2n+1})}{(1 - q^{4n+2})^{2k}} - q^k \prod_{n=1}^{\infty} \frac{1 - q^{4n})^{4k}}{(1 - q^{4n-2})^{4k}} & (k \text{ odd})
\end{cases}
\]

4. SOME USEFUL LEMMAS

We next state an important theorem which follows from Jacobi triple product
identity, originally proved by Gauss (see [2], p.10, Cor. 1.3.4 and notes in p.23).

Lemma 4.1. For \(|q| < 1\) we have
\[
\psi(q) = \prod_{n=1}^{\infty} \frac{1 - q^{2n}}{(1 - q^{2n-1})}.
\]
Thus Lemma 4.1 yields
\[
\psi^{4k}(q) = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{4k}}{(1 - q^{2n-1})^{4k}} = \sum_{n=1}^{\infty} t_{4k}(n) q^n
\]
where \(t_{4k}(n)\) is the number of ways of representing a positive integer \(n\) as a sum of
4\(k\) triangular numbers.

Next, the following well-known result in [1] due to Atanosov et al. gives us an exact
formula for \(t_{4k}(n)\). Indeed, the authors show that \(t_{4k}(n)\) behaves when \(n\) becomes
large like $\sigma_{2k-1}^\#(2n + k)$, the modified divisor function defined by

\begin{equation}
\sigma_k^\#(n) := \sum_{d|n, n/d \text{ odd}} d^k \begin{cases} 
\sigma_k(n) & \text{when } n \text{ is odd,} \\
2^k \sigma_k^\# \left(\frac{n}{2}\right) & \text{when } n \text{ is even}
\end{cases}
\end{equation}

where $\sigma_k(n)$ is the $k$th divisor function defined as

\begin{equation}
\sigma_k(n) = \sum_{d|n} d^k.
\end{equation}

**Theorem 4.2.** Let $k \in \mathbb{N}$. Then

\begin{equation}
q^k \psi(4k^2) = \frac{1}{d_k}(H_{2k}(\tau) - T_{2k}(\tau))
\end{equation}

where $d_k$ is defined as in Theorem 3.1 and $T_{2k}(\tau) \in \mathcal{S}(\Gamma_0(4))$ and $H_{2k}(\tau)$ is an Eisenstein series of weight $2k$ on $\Gamma_0(4)$ defined by

\begin{equation}
H_{2k}(\tau) = \begin{cases} 
\sum_{n > 0, n \text{ even}} \sigma_{2k-1}^\#(n)q^n & \text{for } k \text{ even,} \\
\sum_{n > 0, n \text{ odd}} \sigma_{2k-1}^\#(n)q^n & \text{for } k \text{ odd.}
\end{cases}
\end{equation}

By comparing coefficients in (4.5), Atanosov et al. obtain the following expression for $t_{4k}(n)$ in [4] (Cor. 2.6, p.119):

\begin{equation}
t_{4k}(n) = \frac{1}{d_k}(\sigma_{2k-1}^\#(2n + k) - a(2n + k))
\end{equation}

where $T_{2k}(\tau) = \sum_{n=0}^{\infty} a(n)q^n \in \mathcal{S}_{2k}(\Gamma_0(4))$. Indeed, Theorem 4.2 follows easily from [6] where the authors detail a closed formula for $t_{4k}(n)$.

We next state an important theorem for the generating function transformation involving Stirling numbers. Let $F(z)$ denote the infinite geometric series

\begin{equation}
F(z) := \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}
\end{equation}

with $|z| < 1$. Then choosing $f_n = 1$ in Prop. 3.1, p.135 of [3] we obtain

**Proposition 4.3.** Let $l$ be a fixed positive integer. Then we have

\begin{equation}
\sum_{n=0}^{\infty} n^l z^n = \sum_{j=0}^{l} j! \binom{l}{j} \frac{z^j}{(1 - z)^{l+1}}.
\end{equation}

The proof follows by induction on $l$ in conjunction with the recurrence relation of Stirling numbers

\begin{equation}
\binom{n}{l} = l \binom{n - 1}{l - 1} + \binom{n - 1}{l}.
\end{equation}
5. Proofs of Theorems 3.1 and 3.2

Since \( \zeta(2k) = \frac{(-1)^{k+1}2^{2k}B_{2k}}{2(2k)!} \pi^{2k} \) has the following equivalent form

\[
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = \left( \frac{2^{2k} - 1}{2^{2k}} \right) \zeta(2k) = \frac{(-1)^{k+1}(4^{k} - 1)B_{2k}}{2(2k)!} \pi^{2k}
\]

(5.1)

it will be sufficient to get the \( q \)-analogue of (5.1). From the \( q \)-analogue of Euler’s Gamma function we know that

\[
\lim_{q \uparrow 1} (1 - q) \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^{2n-1})^{2}} = \frac{\pi}{2}
\]

(5.2)

so that from (5.2) we have

\[
\lim_{q \uparrow 1} (1 - q)^{2k} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{4k}}{(1 - q^{2n-1})^{4k}} = \frac{\pi^{2k}}{2^{2k}}
\]

(5.3)

where \( q \uparrow 1 \) indicates \( q \to 1 \) from within the unit disk. We treat Theorems 3.1 and 3.2 separately.

5.1. Proof of Theorem 3.1. Let \( k \geq 2 \) be an even integer. Then from (4.6) and (4.3) we have

\[
H_{2k}(\tau) = \sum_{n=1}^{\infty} \sigma_{2k-1}^{\#}(2n)q^{2n} = 2^{2k-1} \sum_{n=1}^{\infty} \sigma_{2k-1}^{\#}(n)q^{2n}.
\]

Using the definition of \( \sigma_{2k-1}^{\#}(n) \) in the expression above we obtain

\[
H_{2k}(\tau) = 2^{2k-1} \sum_{n=1}^{\infty} \left( \sum_{d|n} d^{2k-1} \right) z^{n}
\]

(5.4)

We wish to find a polynomial \( Q_{2k-1}^{\#}(z) \) such that the expression in parentheses in the right-hand equation of (5.4) can be written as

\[
\frac{Q_{2k-1}^{\#}(q^{2(2i+1)})}{(1 - q^{2(2i+1)})^{2k}} = \sum_{j=0}^{\infty} (j + 1)^{2k-1} q^{2(j+1)/2i+1}.
\]

(5.5)

For notational simplicity let us write \( z = q^{2(2i+1)} \) so that (5.5) can be rewritten as

\[
\frac{Q_{2k-1}^{\#}(z)}{(1 - z)^{2k}} = \sum_{j=0}^{\infty} (j + 1)^{2k-1} z^{j+1}.
\]

(5.6)

**Lemma 5.1.** \( Q_{2k-1}^{\#}(z) \) is a polynomial of degree \((2k - 1)\) with integer coefficients.
Proof. The right hand side of (5.6) can be identified with the left-hand side of (4.9) so that using Proposition 4.3 we can rewrite the right-hand side of (5.6) as

\begin{equation}
Q_{2k-1}(z) = \sum_{j=0}^{2k-1} j! \binom{2k-1}{j} \frac{z^j}{(1-z)^{j+1}}
\end{equation}

Noting that \( z = 1 - (1-z) \) we use binomial expansion in \( z^j = (1-(1-z))^j \) followed by rearrangements of the sums above to obtain

\begin{equation}
Q_{2k-1}(z) = \sum_{m=0}^{2k-1} (-1)^m a_k(m) \frac{1}{(1-z)^{m+1}}
\end{equation}

where \( a_k(m) \) is defined by

\begin{equation}
a_k(m) = \sum_{j=0}^{2k-1} (-1)^j j! \binom{2k-1}{j} \binom{j}{m}.
\end{equation}

Note that in going from the second step to the third step in (5.8) we used the fact that \( \binom{j}{m} = 0 \) if \( m > j \) and hence we are able to interchange the sums over \( m \) and \( j \) above. Thus multiplying both sides of (5.8) by \( (1-z)^{2k} \) and using the binomial expansion yields

\begin{equation}
Q_{2k-1}(z) = \sum_{l=0}^{2k-1} (-1)^l b_k(l) z^l,
\end{equation}

where the \( b_k(l) \) are defined by

\begin{equation}
b_k(l) = \sum_{m=0}^{2k-1} (-1)^m a_k(m) \binom{2k-m-1}{l},
\end{equation}

which establishes Lemma 5.1. \( \square \)
We also note from (5.6) that
\[ Q_{2k-1}(0) = 0. \]
Thus we define the polynomial
\[ P_{2k-2}(z) = \sum_{l=1}^{2k-1} (-1)^l b_k(l) z^{l-1}. \]

We rewrite (5.14) using (5.12) as
\[ H_{2k}(\tau) = 2^{2k-1} \sum_{i=0}^{\infty} \frac{q^{2(2i+1)} P_{2k-2}(q^{2(2i+1)})}{(1 - q^{2(2i+1)})^{2k}}. \]

Now from (4.5) of Theorem 4.2 we have
\[ H_{2k}(\tau) - T_{2k}(\tau) = d_k q^k \psi^{4k}(q^2). \]
Thus from (4.2), (5.14) and (5.15) we get
\[ \sum_{n=0}^{\infty} 2^{2k-1} q^{2(2n+1)} P_{2k-2}(q^{2(2n+1)}) = d_k q^k \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{2k}}{(1 - q^{4n-2})^{2k}}. \]

Making the change of variable \( q \rightarrow \sqrt{q} \) in (5.16) we obtain
\[ \sum_{n=0}^{\infty} 2^{2k-1} q^{2n+1} P_{2k-2}(q^{2n+1}) = d_k q^{k/2} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{4k}}{(1 - q^{2n-1})^{4k}}. \]

On multiplying both sides of (5.7) by \( (1 - z)^{2k} \) we obtain
\[ Q_{2k-1}(z) = \sum_{j=0}^{2k-1} j! \binom{2k-1}{j} (1-z)^{2k-j-1}. \]

As \( z \rightarrow 1^{-} \), each summand in (5.18) vanishes except the term corresponding to \( j = 2k - 1 \). Thus we get
\[ \lim_{z \rightarrow 1^{-}} Q_{2k-1}(z) = (2k - 1)!. \]

In view of (5.19) and the fact that \( T_{2k}(\tau/2) \rightarrow 0 \) (cusp form), as \( q \rightarrow 1^{-} \), (5.17) gives
\[ \sum_{n=0}^{\infty} \frac{(2k-1)!}{(2n+1)^{2k}} = \frac{d_k \pi^{2k}}{2^{2k}}. \]

Using the definition of \( d_k \) we obtain identity (5.1). Thus Theorem 3.1 follows from all the above observations.
5.2. **Proof of Theorem 3.2.** Let \( k \geq 1 \) be an odd integer. Then from (4.3) and (4.6) we have

\[
H_{2k}(\tau) = \sum_{n > 0 \text{ } n \text{ odd}} \sigma^2_{2k-1}(n)q^n
\]

\[
= \sum_{n=0}^{\infty} \sigma^2_{2k-1}(2n+1)q^{2n+1}
\]

\[
= \sum_{n=1}^{\infty} \sigma^2_{2k-1}(n)q^n - \sum_{n=1}^{\infty} \sigma^2_{2k-1}(2n)q^{2n}
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{d|n \text{ even}} d^{2k-1} \right) q^n - 2^{2k-1} \sum_{n=1}^{\infty} \left( \sum_{d|n \text{ odd}} d^{2k-1} \right) q^{2n}
\]

(5.21)

In view of (5.9) and Lemma 5.1 we can rewrite (5.21) as

\[
H_{2k}(\tau) = \sum_{i=0}^{\infty} \frac{Q^o_{4k-1}(q^{2i+1})}{(1 - q^{2i+1})^{2k}} - 2^{2k-1} \sum_{i=0}^{\infty} \frac{Q^o_{2k-1}(q^{2i+1})}{(1 - q^{2i+1})^{2k}}
\]

(5.22)

where \( Q^o_{4k-1}(q^{2i+1}) \) is the polynomial in \( w = q^{2i+1} \) of degree \( 4k - 1 \) defined by

\[
Q^o_{4k-1}(w) = (1 + w)^{2k} Q^o_{2k-1}(w) - 2^{2k-1} Q^o_{2k-1}(w^2).
\]

Since \( Q^o_{2k-1}(0) = 0 \), in view of (5.6) and (5.23) we also have \( Q^o_{4k-1}(0) = 0 \). Therefore we define the polynomial \( P^o_{4k-2}(w) \) of degree \( 4k - 2 \) by

\[
Q^o_{4k-1}(w) = w(1 + w)^{2k} P^o_{2k-2}(w) - 2^{2k-1} w^2 P^o_{2k-2}(w^2)
\]

(5.24)

where

\[
P^o_{4k-2}(w) = (1 + w)^{2k} P^o_{2k-2}(w) - 2^{2k-1} w P^o_{2k-2}(w^2).
\]

Hence from (5.22), (5.24) and Theorem 4.2 we obtain

\[
H_{2k}(\tau) - T_{2k}(\tau) = d_k q^k \psi^{4k}(q^2),
\]

\[
= \sum_{n=0}^{\infty} \frac{q^{2n+1} P^o_{4k-2}(q^{2n+1})}{(1 - q^{2(2n+1)})^{2k}} - T_{2k}(\tau) = d_k q^k \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{2k}}{(1 - q^{4n-2})^{2k}}.
\]

(5.26)

Also as \( w \to 1^- \), (5.19) and (5.24) yield

\[
\lim_{w \to 1^-} Q^o_{4k-1}(w) = 2^{2k}(2k-1)! - 2^{2k-1}(2k-1)! = 2^{2k-1}(2k-1)!. \]

(5.27)
Thus on multiplying both sides of (5.20) by \((1-q^2)^{2k}\) and taking the limit as \(q \to 1\) from within the unit disk, we obtain the following using (5.27):

\[
2^{2k-1}(2k-1)! \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2k}} = \frac{d_k \pi^{2k}}{2^{2k}},
\]

Using definition of \(d_k\) we immediately obtain identity (5.1). Thus Theorem 3.2 follows from all of the above observations.

6. Explicit computations of \(P_{2k-2}(z)\) and \(P_{4k-2}(z)\) for different \(k\)

We used the Python programming language to compute the co-efficients \(a_k(m)\) and \(b_k(l)\) to determine the polynomials \(P_{2k-2}(z)\) and \(P_{4k-2}(z)\) for a few different values of \(k\). We will see that our results for \(k = 1, 2, 3\) tally with the results in [9] and [4].

6.1. Case \(k=1\) : Sun’s result. Since \(k = 1\) is odd we use (3.6) to get

\[
P_0^0(z) = (1 + z)^2 P_0^0(z) - 2z P_0^0(z),
\]

where we define \(P_0^0(z) = 1\). Therefore,

\[
P_2^0(z) = (1 + z)^2 - 2z = 1 + z^2.
\]

Thus (6.1) and (3.5) yield

\[
\sum_{n=1}^{\infty} \frac{q^{2n}(1 + q^{2(2n+1)})}{(1 - q^{2(2n+1)})^2} = \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^4}{(1 - q^{4n-2})^4},
\]

where, \(d_1 = 1\) and \(T_2(\tau) = 0\) (Table I, p.120, [1]). This is Theorem 1.1, (1.1), of [9] with \(q \to \sqrt{q}\) in (6.2).

6.2. Case \(k=2\). Here \(k = 2\) is even, so we use (3.2) to get

\[
P_2^0(z) = -b_2(1) + b_2(2)z - b_2(3)z^2
\]

where \(b_2(l), 1 \leq l \leq 3\) are given by (3.3). We need to evaluate \(a_2(0), a_2(1), a_2(2)\) and \(a_2(3)\). Using (3.1) we obtain

\[
a_2(0) = -1, \ a_2(1) = -7, \ a_2(2) = -12, \ a_2(3) = -6.
\]

Thus (3.3) yields

\[
b_2(1) = -1, \ b_2(2) = 4, \ b_2(3) = -1.
\]

Hence we obtain

\[
P_2^0(z) = 1 + 4z + z^2
\]

which when used in (3.1) yields the results in [9] and [4]. Here again \(T_4(\tau/2) = 0\) (Table I, p.120, [1]).
6.3. Case: $k=3$. Here we need to evaluate the coefficients $b_3(1), b_3(2), b_3(3), b_3(4), b_3(5)$ and the corresponding $a_3(0), a_3(1), a_3(2), a_3(3), a_3(4), a_3(5)$. Using (3.4) we get

$$a_3(0) = -1, \ a_3(1) = -31, \ a_3(2) = -180, \ a_3(3) = -390, \ a_3(4) = -360, \ a_3(5) = -120$$

and using (3.3) we obtain

$$b_3(1) = -1, \ b_3(2) = 26, \ b_3(3) = -66, \ b_3(4) = 26, \ b_3(5) = -1.$$

Using these values in (6.4) we obtain

$$P_{16}^e(z) \overset{6.3}{=} (1 + z)^6 P_4^e(z) - 32 z P_4^e(z^2)$$

$$= (1 + z)^6(1 + 26 z + 66 z^2 + 26 z^3 + z^4) - 32 z(1 + 26 z^2 + 66 z^4 + 26 z^6 + z^8)$$

$$= z^{10} + 237 z^8 + 1682 z^6 + 1682 z^4 + 237 z^2 + 1$$

$$= (z^2 + 1)(z^8 + 236 z^6 + 1446 z^4 + 236 z^2 + 1)$$

which when used in (6.4) along with the change of variable $q \rightarrow \sqrt{7}$ gives us the result in (6.4). We note here that in (6.4) we obtained explicitly $T(\tau/2) = \phi^{12}(q)$ where $\phi(q) = \prod_{n=1}^{\infty} (1 - q^n)$ is the Euler's function.

6.4. Case: $k=4$. Here we use (3.4) and (3.3) to obtain

$$a_4(0) = -1, \ a_4(1) = -127, \ a_4(2) = -1932, \ a_4(3) = -10206, \ a_4(4) = -25200, \ a_4(5) = -31920, \ a_4(6) = -20160, \ a_4(7) = -5040$$

and

$$b_4(1) = -1, \ b_4(2) = 120, \ b_4(3) = -1191, \ b_4(4) = 2416, \ b_4(5) = -1191, \ b_4(6) = 120, b_4(7) = -1.$$

Thus we have

$$P_6^e(z) = z^6 + 120 z^5 + 1191 z^4 + 2416 z^3 + 1191 z^2 + 120 z + 1$$

which when used in (3.11) gives us the following $q$-analogue of $\zeta(8) = \pi^8/9450$

$$\sum_{n=0}^{\infty} \frac{q^{2n} P_6^e(q^{2n+1})}{(1 - q^{2n+1})^8} - T_8(\tau/2) = 136q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{16}}{(1 - q^{2n-1})^{16}}.$$

6.5. Case: $k=5$. Here we use (3.4) and (3.3) to obtain

$$a_5(0) = -1, \ a_5(1) = -511, \ a_5(2) = -18660, \ a_5(3) = -204630, \ a_5(4) = -1020600, \ a_5(5) = -2739240, \ a_5(6) = -4233600, \ a_5(7) = -3780000, \ a_5(8) = -1814400, \ a_5(9) = -362880$$

and

$$b_5(1) = -1, \ b_5(2) = 502, \ b_5(3) = -14608, \ b_5(4) = 88234, \ b_5(5) = -156190, \ b_5(6) = 88234, b_5(7) = -14608, \ b_5(8) = 502, b_5(9) = -1.$$
Thus from (3.6) we obtain the polynomial
\[ P_{16}(z) = (1 + z)^{10} P_8^e(z) - 1024 z P_8^e(z^2) \]
\[ = (1 + z)^{10} (z^5 + 502 z^7 + 14608 z^6 + 88234 z^5 + 156190 z^4 + 88234 z^3 \]
\[ + 14608 z^2 + 502 z + 1) - 512 z (z^{16} + 502 z^{14} + 14608 z^{12} + 88234 z^{10} \]
\[ + 156190 z^8 + 88234 z^6 + 14608 z^4 + 502 z^2 + 1) \]
\[ = (1 + z^2)(z^{16} + 19672 z^{14} + 1736668 z^{12} + 19971304 z^{10} + 49441990 z^8 \]
\[ + 19971304 z^6 + 1736668 z^4 + 19672 z^2 + 1). \]

Using this in (3.5) with \( q \to \sqrt{q} \) we obtain the following \( q \)-analogue of \( \zeta(10) = \frac{\pi^{10}}{93555} \)
\[ \sum_{n=0}^{\infty} \frac{q^n (1 + q^{2n+1}) S_8(q^{2n+1})}{(1 - q^{2n+1})^{10}} - T_{10}(\tau/2) = 2031616 q^2 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^{20}}{(1 - q^{2n-1})^{20}} \]
where
\[ S_8(z) = z^8 + 19672 z^7 + 1736668 z^6 + 19971304 z^5 + 49441990 z^4 \]
\[ + 19971304 z^3 + 1736668 z^2 + 19672 z + 1. \]

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