SOME COMMENTS ON CONTINUOUS SYMMETRIES
OF AKNS HIERARCHY EQUATIONS AND THEIR
SOLUTIONS

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Abstract. We show that a well-known for NLS equation scaling
invariance and Galilean invariance property of its solutions can
be extended with appropriate modifications to the whole AKNS
hierarchy and its reduced and deformed versions.

Nonlinear Schrödinger equation; Hirota equation; AKNS hierarchy
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1. Introduction: Integrable systems described by AKNS
hierarchy

The k-th equation of the AKNS hierarchy results from the compati-
bility condition

\[(\Psi_x)_t = (\Psi_{tk})_x\]

of the following Lax system:

\[
\begin{cases}
\Psi_x = \mathfrak{M} \Psi, \\
\Psi_{tk} = \mathfrak{M}_k \Psi,
\end{cases}
\]

\[
\mathfrak{M} := \lambda J + \mathfrak{M}^0, \quad \mathfrak{M}_1 := 2\lambda \mathfrak{M} + \mathfrak{M}^0, \quad \mathfrak{M}_{k+1} := 2\lambda \mathfrak{M}_k + \mathfrak{M}^0_{k+1}, \quad k \geq 1
\]

\[
J := \begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix}, \quad \mathfrak{M}^0 := \begin{pmatrix}
0 & i\psi \\
-i\phi & 0
\end{pmatrix}.
\]

Equation (1) determines the form of the matrices \(\mathfrak{M}_k\), the recursion
relations between the off-diagonal elements of the matrices \(\mathfrak{M}^0_k\), and the relations between the diagonal and off-diagonal elements of these matrices:

\[
[J, \mathfrak{M}_1] = 2(\mathfrak{M}^0)_x, \quad [J, \mathfrak{M}^0_{k+1}] = 2(\mathfrak{M}^0_k)_x + 2[\mathfrak{M}^0_k, \mathfrak{M}^0], \quad k \geq 1.
\]

In particular,

\[
\mathfrak{M}_1^0 = \begin{pmatrix}
-i\psi \phi & -\psi_x \\
-\phi_x & i\psi \phi
\end{pmatrix}, \quad \mathfrak{M}_2^0 = \begin{pmatrix}
\psi_x \phi - \phi_x \psi & 2i\psi^2 \phi - i\psi_{xx} \\
-2i\phi^2 \psi + i\phi_{xx} & -\phi_x \psi - \psi_x \phi
\end{pmatrix}.
\]

The evolution equation, describing the dynamics of the matrix elements
of \(\mathfrak{M}^0\):

\[(\mathfrak{M}^0)_t = (\mathfrak{M}^0)_x + [\mathfrak{M}^0, \mathfrak{M}^0] = \frac{1}{2}[J, \mathfrak{M}^0_{k+1}],\]
also follows from (11). It is called the k-th member of the AKNS hierarchy. The complementary restriction \( \phi = -\psi^* \) transform the AKNS hierarchy to the reduced AKNS hierarchy (RAKNS hierarchy). Here we mainly restrict our discussion by considering scalar nonlinear PDE’s belonging to RAKNS hierarchy although all our results can be easily extended to the AKNS case. First member of the AKNS hierarchy is the coupled NLS system:

\[
\begin{cases}
  i\psi_{t_1} + \psi_{xx} - 2\psi^2\phi = 0, \\
  -i\phi_{t_1} + \psi_{xx} - 2\phi^2\psi = 0.
\end{cases}
\]

The first member of the RAKNS hierarchy is just a focusing NLS equation:

\( i\psi_{t_1} + \psi_{xx} + 2|\psi|^2\psi = 0. \)  \( (4) \)

Second member of AKNS hierarchy is a modified KdV (MKdV) system:

\[
\begin{cases}
  \psi_{t_2} + \psi_{xxxx} - 6\psi\phi\psi_x = 0, \\
  \phi_{t_2} + \phi_{xxxx} - 6\phi\phi_x = 0.
\end{cases}
\]

Its RAKNS counterpart is the scalar (complex) MKdV equation \( (5) \):

\[
\psi_{t_2} + \psi_{xxxx} + 6|\psi|^2\psi_x = 0.
\]

Third member of AKNS hierarchy is a following coupled system of nonlinear PDE’s:

\[
\begin{cases}
  i\psi_{t_3} - \psi_{xxxx} + 8\psi\phi\psi_{xx} + 2\psi^2\phi_{xx} + 6\psi^2\phi - 4\psi\psi_x\phi_x - 6\psi^3\phi^2 = 0, \\
  -i\phi_{t_3} + \phi_{xxxx} + 8\phi\phi_{xx} + 2\phi^2\psi_{xx} + 6\phi\phi_x - 6\psi^2\phi^3 = 0.
\end{cases}
\]

Its RAKNS counterpart for \( t_3 = -t \) becomes a well known Lakshmanan-Poroezian-Daniel equation (LPD equation) \( (6) \):

\[
i\psi_t + \psi_{xxxx} + 8|\psi|^2\psi_{xx} + 2\psi^2\psi_{xx}^* + 6\psi^2\psi_x + 4\psi|\psi_x|^2 + 6|\psi|^4\psi = 0.
\]

Higher order equations are rarely considered by physicists. For completeness we display here also a 4-th member of RAKNS hierarchy (RAKNS-4 equation) for \( t_4 = -t \):

\[
\psi_t + \psi_{5x} + 10|\psi|^2\psi_{xxx} + 20\psi_{xxx}\psi_x\psi^* + 10(|\psi_x|^2\psi_x + 30|\psi|^4\psi_x = 0, \]  \( (7) \)

and the 5-th member of RAKNS hierarchy (RAKNS-5 equation) with \( t_5 = t \):

\[
i\psi_t + \psi_{6x} + 12|\psi|^2\psi_{xxxx} + 2\psi^2\psi_{xxxx} + 30\psi_{xxxx}\psi_x\psi^* + 18\psi_{xxxx}\psi^* + 8\psi_x\psi\psi_{xxxx}^* + 50\psi_{xxx}\psi_{xx} + 20\psi_{xx}\psi^* + 22\psi_{xxx}|\psi|^2\psi_x + 20|\psi|^2\psi_{xx}^* + 20|\psi|^2\psi_x^* + 10\psi^3(\psi_x^*)^2 + 70|\psi|^2|\psi_x|^2\psi + 20|\psi|^6\psi = 0.
\]  \( (8) \)

\(^1\)Recall that the usual mKdV reads \( \psi_{t_4} + \psi_{xxx} + 6\psi^2\psi_x = 0, \) and only its real solutions are of physical interest.

\(^2\)Surprisingly it contains less terms with respect to \( (6) \).
Below we use for all equations of the RAKNS hierarchy the following shortened notations:

\[ i\psi_t + H_1(\psi) = 0, \quad \psi_t + H_2(\psi) = 0, \quad -i\psi_t + H_3(\psi) = 0, \]
\[ -\psi_t + H_4(\psi) = 0, \quad i\psi_t + H_5(\psi) = 0, \]

etc or equivalently

\[ \psi_t^k = i^{k+1}H_k(\psi). \]

The key property of the considered equations is the existence of the joint solutions

\[ \Psi(x, t_1, \ldots, t_k, \ldots), \]

satisfying all the AKNS equations simultaneously. When all times except one selected \( t_j \) are fixed this is still the solution to the \( j \)-th equation of the hierarchy where the other times play the role of parameters. Formal solutions of this kind are provided by the Sato theory and include the infinite number of times. See for instance [4] for an elementary introduction to the Sato theory.

We restrict our consideration by the case of any finite number of times. In this case the related common solutions are the finite-gap solutions of the fixed genus or their degenerate cases which are very diversified [5]. If the order of the considered equation is less than a number of phases of the solution variables belonging to the higher order phases plays a role of free parameters of the solution. It is worthwhile to mention that when the hierarchic order of equation \( n \) is greater than a number of phases \( k \) then \( \Psi \)-function either is independent from \( t_n \), or \( t_n \) enters into the lower order phases \( T_j \) in a linear way.

2. MIXED RAKNS EQUATIONS

Together with individual members of AKNS hierarchy the mixed equations, corresponding to some selection of finite number of times which are identified, and denoted by the single letter \( t \) are also frequently considered, obviously leading also to the integrable nonlinear PDE’s having in their RHS the linear combinations of the RHS of the arbitrary chosen individual members of the hierarchy.

The most popular example of this kind is the Hirota integrable equation [6–10]:

\[ i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) = 0, \quad \alpha, \beta \in \mathbb{R}. \]

Obviously this equation admits the solutions of the form \( \Psi(x, \alpha t, -\beta t, \ldots, t_k) \), where \( \Psi(x, t_1, \ldots, t_k) \) is an arbitrary solution of the AKNS hierarchy. More complicated ”mixed” equation [11][12], involving a linear combination of first 3 members of the RAKNS hierarchy is

\[ i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) + \gamma_1 H_3(\psi) = 0. \]
Its solutions are given by $\Psi(x, \alpha t, -\beta t, -\gamma_1 t, \ldots, t_k)$. Next by the level of complexity mixed models are described by the following PDE’s:

\[(11) \quad i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) + \gamma_1 H_3(\psi) - i\gamma_2 H_4(\psi) = 0,\]

and

\[(12) \quad i\psi_t + \alpha H_1(\psi) - i\beta H_2(\psi) + \gamma_1 H_3(\psi) - i\gamma_2 H_4(\psi) + \gamma_3 H_5(\psi) = 0,\]

A particular case of (11) with $\beta = 0$, $\gamma_1 = 0$ was considered very recently in in [13]. In particular, in [13] were studied the 2-breathers solutions and its limit case – quasi-rational MRW solution of rank 2.

More generally, it is a well known fact that all equations of the form

\[(13) \quad \psi_t = \sum_{k \geq 1} i^k b_k H_k(\psi), \quad b_k \in \mathbb{R}.\]

are completely integrable and their stationary solutions are the algebro-geometric finite gap potentials expressed by means of Riemann theta functions of the hyperelliptic curves or their degenerate cases.

3. Deformed RAKNS hierarchy

Suppose that $t_k = \alpha_k(t)$ in AKNS or RAKNS hierarchy where $\alpha_k(t)$ are any (in general different for different k-s) differentiable functions. It is obvious that the change of variable $t_j = \alpha_j(t)$ transforms the second of equation (2) into the following linear equation for the matrix function $\Psi$ :

\[(14) \quad \Psi_t = \alpha'_j(t)V_j \Psi.\]

The compatibility condition of (14) with first equation of the system (2) now takes the form:

\[(15) \quad U_t + \alpha'_j(t) (- (V_j)_x + [U, V_j]) = 0,\]

and the j-th equation of thus obtained hierarchy, which we call deformed RAKNS hierarchy, reads:

\[(16) \quad \psi_t = i^k \alpha'_j(t) H_j\]

Now it is clear that the function

\[(17) \quad \Psi(x, \alpha_1(t), \ldots, \alpha_k(t))\]

furnishes the solutions of the nonlinear PDE

\[(18) \quad i\psi_t - \sum_{k \geq 1} i^{k+1} \alpha'_j(t) H_k = 0,\]

which contains the ”mixed” RAKNS equations obtained from the linear combinations of the functions of the RHS of RAKNS equations with constant coefficients $b_k$, as a special case where $\alpha_j(t) := b_k t + d_k$. 
and $b_k, d_k$ are any different real constants. The simple explanation of their nature given here, simply coming from the general change of

time variables in (the system ) (2) seems to be more transparent with respect to [13]. Our logic immediately furnish the large variety of solutions from generic solutions of the RAKNS hierarchy. It is also quite obvious that choosing $\alpha_j(t)$ with nonintersecting compact supports $\Delta_j$

we obtain the solutions of [13], which, when $t \in \Delta_j$, is the same as the solution the j-th equation of the deformed hierarchy:

$$\psi_t = i^k \alpha_j'(t) H_j(\psi).$$

4. Continuous symmetries of the RAKNS hierarchy

All RAKNS hierarchy equations are invariant both with respect to the space and time translations and with respect to the constant phase transformations $\psi(x, t) \rightarrow e^{i\varphi} \psi(x,t), \varphi \in \mathbb{R}$. Less trivial are the scaling invariance and the Galilean invariance properties depending on the chosen equation of hierarchy or their mixture. For instance, the scaling covariance property of the n-th member of the RAKNS hierarchy takes the form:

$$\psi_n(x, t) \rightarrow q \psi(qx, q^{n+1}t), \quad q > 0,$$

which means that the transformation above maps the solution $\psi(x, t)$ to the new solution of the same equation. In particular this means that the study of the solutions, having asymptotic magnitude $q$ when $x^2 + t^2 \rightarrow \infty$, can always be be reduced to the case $q = 1$.

The proposition formulated below extends the scaling invariance and Galilean invariance properties to the the whole RAKNS hierarchy and its mixed and times deformed versions discussed above.

Suppose $a, b \in \mathbb{R}$ and

$$X := ax + a \sum_{m \geq 1} \left( \frac{m + 1}{1} \right) (2b)^m t_m,$$

$$T_1 := a^2 t_1 + a^2 \sum_{m \geq 2} \left( \frac{m + 1}{2} \right) (2b)^{m-1} t_m,$$

$$T_2 := a^3 t_2 + a^3 \sum_{m \geq 3} \left( \frac{m + 1}{3} \right) (2b)^{m-2} t_m,$$

$$\ldots$$

**Theorem 1.** Suppose the function

$$\Psi(x, t_1, t_2, \ldots)$$

3The same equations written in a form slightly different from our presentation were first mentioned in [13] (to appear in Chaos).
is a solution of the AKNS hierarchy equations. Then
\[ \tilde{\Psi}(x, t_1, t_2, \ldots) = a \Psi(X, T_1, T_2, \ldots) \exp \left\{ -2ibx - i \sum_{m \geq 1} (2b)^{m+1} b_m t_m \right\} \]
a new solution of the same hierarchy.

Suppose now that \( b_k, a, b \in \mathbb{R}, \)
\[ X := ax + at \sum_{m \geq 1} \left( \frac{m + 1}{1} \right) (2b)^m b_m, \]
\[ T_1 := \left( b_1 + \sum_{m \geq 2} \left( \frac{m + 1}{2} \right) (2b)^{m-1} b_m \right) a^2 t, \]
\[ T_2 := \left( b_2 + \sum_{m \geq 3} \left( \frac{m + 1}{3} \right) (2b)^{m-2} b_m \right) a^3 t, \]
\[ \ldots \]

**Theorem 2.** Suppose that the function
\[ \Psi(x, b_1 t, b_2 t, \ldots) \]
satisfies the mixed RAKNS equation
\[ i \psi_t - \sum_{m \geq 1} b_k H_k(\psi) = 0. \]
Then the function
\[ \tilde{\Psi}(x, t) = a \Psi(X, T_1, T_2, \ldots) \exp \left\{ -2ibx - it \sum_{m \geq 1} (2b)^{m+1} b_m \right\} \]
is also a solution of (10).

In particular, assuming that \( \psi(x, \alpha t, -\beta t) \) is a solution of the Hirota equation (9), the function
\[ \tilde{\psi} = a \psi(ax + 4[\alpha - 3b\beta]abt, [\alpha - 6b\beta]a^2 t, -\beta a^3 t) \exp \{-2ibx - 4i(\alpha - 2b\beta)b^2 t\} \]
is again the solution of the same equation.

The transformation \( \psi \rightarrow \tilde{\psi} \)
\[ \tilde{\psi}(x, t) = a \psi(ax + 4abt, a^2 t) \exp \{-2ibx - 4ib^2 t\} \]
in this case describes a well-known composition of Galilean and a scaling transformation for the NLS equation.

Next proposition extends the transformations above to the case of the Deformed RAKNS hierarchy. Suppose \( \alpha_k(t), a, b \in \mathbb{R}, \) and
\[ X = ax + ax \sum_{m \geq 1} \left( \frac{m + 1}{1} \right) (2b)^m \alpha_m(t), \]
Theorem 3. Suppose that the function

\[ \Psi(x, \alpha_1(t), \alpha_2(t), \ldots) \]

is a solution of the deformed mixed RAKNS equation \[18\]. Then the function

\[ \tilde{\Psi}(x, t) = a \Psi(X, T_1, T_2, \ldots) \exp \left\{ -2ibx - i \sum_{m \geq 1} (2b)^{m+1} \alpha_m(t) \right\} \]

is also the solution of \[19\].

5. SOME PROPERTIES OF THE ALGEBRO-GEOMETRIC SOLUTIONS OF THE RAKNS HIERARCHY

Here we give only extremely brief outline of the well developed theory \[5\].

The theorems of the previous section follows immediately from the properties of the finite-gap solutions of RAKNS hierarchy but they are valid also for all possible solutions. For the first members of AKNS hierarchy this can be checked by straightforward calculation independent from the choice of the selected class of solutions. Somehow for the higher order equation the direct proof is very involved and for the algebro-geometric solutions it works for any genus for any equation of the hierarchy.

For AKNS hierarchy algebro-geometric solutions are parametrized by the moduli of the hyperelliptic spectral curve \( \Gamma = \{(w, \lambda)\} \) defined by the algebraic equation

\[ w^2 = \prod_{j=1}^{g+1} [(\lambda - \lambda_j)(\lambda - \lambda_j^*)], \quad \Im \lambda_j > 0, \]

and have the following form

\[ \psi(x, t_1, \ldots) = \frac{2K_0}{\rho} \frac{\Theta(Z) \Theta(U(x, t_1, \ldots) + Z - \Delta)}{\Theta(Z - \Delta) \Theta(U(x, t_1, \ldots) + Z)} \exp\{2i\Phi(x, t_1, \ldots)\}, \]

\[ \phi(x, t_1, \ldots) = 2\rho K_0 \frac{\Theta(Z - \Delta) \Theta(U(x, t_1, \ldots) + Z + \Delta)}{\Theta(Z) \Theta(U(x, t_1, \ldots) + Z)} \exp\{-2i\Phi(x, t_1, \ldots)\}. \]
Here \( \Theta \) is a multi-dimensional Riemann theta function defined on \( \Gamma \) (see [3]),

\[
\mathbf{U}(x, t_1, \ldots) = \mathbf{V}^1x + \sum_{j \geq 1} \mathbf{V}^{j+1}t_j, \quad \Phi(x, t_1, \ldots) = -K_1x - \sum_{j \geq 1} K_{j+1}t_j.
\]

Here above \( \mathbf{V}^j \) are vectors of the \( b \)-periods, and \( K_j \) are coefficients of asymptotics of the normalized Abelian integrals of the second kind \( \Omega_j \) (see [3, 13] for details):

\[
\oint_{a_m} d\Omega_j = 0, \quad \oint_{b_m} d\Omega_j = (\mathbf{V}^j)_m,
\]

\[
\Omega_j(\mathcal{P}) = \mp i \left( 2^{j-1} \lambda^j - K_j + O(\lambda^{-1}) \right), \quad \mathcal{P} \to \mathcal{P}^\pm,
\]

\[
\omega_0(\mathcal{P}) = \mp \left( \ln \lambda - \ln K_0 + O(\lambda^{-1}) \right), \quad \mathcal{P} \to \mathcal{P}^\pm,
\]

\[
w = \pm \left( \lambda^{g+1} + O(\lambda^g) \right), \quad \mathcal{P} \to \mathcal{P}^\pm.
\]

The affine transformation of \( \Gamma \)

\[
(23) \quad \tau : (w, \lambda) \to (\tilde{w}, \tilde{\lambda}), \quad \tilde{w} = a^{g+1}w, \quad \tilde{\lambda} = a\lambda + b.
\]

induces the related transformation of the abelian integrals:

\[
\tilde{\Omega}_j(\tilde{\mathcal{P}}) = \mp i \left( 2^{j-1} \tilde{\lambda}^j - \tilde{K}_j + O(\tilde{\lambda}^{-1}) \right), \quad \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}^\pm,
\]

\[
\tilde{\omega}_0(\tilde{\mathcal{P}}) = \mp \left( \ln \tilde{\lambda} - \ln \tilde{K}_0 + O(\tilde{\lambda}^{-1}) \right), \quad \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}^\pm,
\]

\[
\tilde{w} = \pm \left( \tilde{\lambda}^{g+1} + O(\tilde{\lambda}^g) \right), \quad \tilde{\mathcal{P}} \to \tilde{\mathcal{P}}^\pm.
\]

Therefore,

\[
\tilde{\Omega}_j(\tilde{\mathcal{P}}) = \mp i \left( 2^{j-1}a^j \lambda^j + \sum_{m=1}^{j} 2^{j-1} \binom{j}{m} a^{j-m} b^m - \tilde{K}_j + O(\tilde{\lambda}^{-1}) \right),
\]

\[
\tilde{\omega}_0(\tilde{\mathcal{P}}) = \mp \left( \ln a + \ln a - \ln \tilde{K}_0 + O(\tilde{\lambda}^{-1}) \right)
\]

and

\[
\tilde{\mathbf{V}}^j = a^j \mathbf{V}^j + \sum_{m=1}^{j-1} 2^m \binom{j}{m} a^{j-m} b^m \mathbf{V}^{j-m},
\]

\[
\tilde{K}_j = K_j + 2^{j-1} b^j,
\]

\[
\tilde{K}_0 = aK_0.
\]

Let us consider the transformed solution

\[
(24) \quad \tilde{\psi}(x, t_1, t_2, \ldots) = \frac{2\tilde{K}_0 \Theta(\mathbf{Z}) \Theta(\tilde{\mathbf{U}}(x, t_1, \ldots) + \mathbf{Z} - \Delta)}{\rho \Theta(\mathbf{Z} - \Delta) \Theta(\tilde{\mathbf{U}}(x, t_1, \ldots) + \mathbf{Z})} \exp\{2i\tilde{\Phi}(x, t_1, \ldots)\},
\]

where

\[
\tilde{\mathbf{U}}(x, t_1, \ldots) = \tilde{\mathbf{V}}^1x + \sum_{j \geq 1} \tilde{\mathbf{V}}^{j+1}t_j, \quad \tilde{\Phi}(x, t_1, \ldots) = -\tilde{K}_1x - \sum_{j \geq 1} \tilde{K}_{j+1}t_j.
\]
Now it is easy to see that the solutions (24) and (4) are connected by the relation,

\[ \tilde{\psi}(x, t_1, t_2, \ldots) = a\psi(X, T_1, T_2, \ldots) \exp \left\{ -2ibx - i \sum_{m=1}^{\infty} (2b)^{m+1} t_m \right\}, \]

where \( X \) and \( T_j \) are defined by the formulas:

\[
X := ax + a \sum_{m=1}^{\infty} \left( \frac{m+1}{1} \right) (2b)^m t_m,
\]

\[
T_1 := a^2 t_1 + a^2 \sum_{m=2}^{\infty} \left( \frac{m+1}{2} \right) (2b)^{m-1} t_m,
\]

\[
T_2 := a^3 t_2 + a^3 \sum_{m=3}^{\infty} \left( \frac{m+1}{3} \right) (2b)^{m-2} t_m,
\]

\[
\ldots
\]

Relation (25) completes the proof of the Theorem 1 for the algebro-geometric solutions. Theorems 2 and 3 can be proved in a similar way.

6. Concluding remarks

We established above very important symmetry properties of solutions of the individual RAKNS hierarchy equations, or for their mixed and deformed versions, (Theorems 1-3). This was proved by us for generic algebro-geometric solutions of any finite genus. Therefore, the same properties are also valid for all degenerate cases of the algebro-geometric solutions, including in particular trigonometric, rational and quasi-rational solutions and a partially degenerate solutions as well. The direct calculation allows to prove the same statements\footnote{We checked it for the orders \( \leq 5 \).} for any sufficiently smooth solutions independently on their nature. Therefore we believe that the related properties are absolutely universal. It will be interesting to prove it in whole generality in a frame of Sato theory.

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